On Conformal Vector Fields of a Class of Finsler Spaces

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Abstract

An \((\alpha, \beta)\)-metric is defined by a Riemannian metric \(\alpha\) and 1-form \(\beta\). In this paper, we characterize conformal vector fields of \((\alpha, \beta)\)-spaces by some PDEs. Further, we determine the local solutions of conformal vector fields of \((\alpha, \beta)\)-spaces in dimension \(n \geq 3\) under certain curvature conditions. In addition, we use certain conformal vector field to give a new proof to a known result on the local and global classifications for Randers metrics which are locally projectively flat and of isotropic S-curvature.

Keywords: Conformal vector field, \((\alpha, \beta)\)-space, Randers space, Projective flatness

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1 Introduction

Conformal vector fields (esp. Killing vector fields) play an important role in Riemann geometry. For a conformal vector field of a Riemannian space, its flow \(\varphi_t\) defines a conformal metric on the space for every \(t\), and in particular, a Killing vector field defines isometric transformations on the Riemannian space. It is shown that the local solutions of a conformal vector field can be determined on a Riemannian space \((M, g)\) if \((M, g)\) is of constant sectional curvature in dimension \(n \geq 3\) ([7], [10]), or more generally locally conformally flat in dimension \(n \geq 2\) ([14]), or under other curvature conditions ([15]). Conformal vector fields of Riemannian spaces are also applied in the study of some important problems in Finsler geometry. A Randers metric of isotropic S-curvature can be characterized equivalently by some conformal vector field \(W\) of a Riemannian metric \(h\) (\((h, W)\) is called the navigation data) ([13]). Then naturally some classifications for Randers metrics of isotropic S-curvature under certain curvature conditions are given (cf. [10], [14], [15]). Using the same navigation technique, Bao-Robles-Shen give the classification for Randers metrics of constant flag curvature by solving homothetic fields (being conformal in a special case) in a Riemannian space of constant sectional curvature ([2]). In [21], C. Yu introduces the method of using conformal vector fields of Riemannian spaces to solve the local structure of locally projectively flat \((\alpha, \beta)\)-metrics in dimension \(n \geq 3\) based on Shen’s result in [9]. Later on, in [12], the present author and Z. Shen first prove that square metrics of scalar flag curvature are locally projectively flat in dimension \(n \geq 3\), and then obtain the corresponding local and global classifications for such metrics by proving some properties of conformal vector fields in Riemannian spaces. In [16]–[20], it shows more interesting studies in Finsler geometry related to the applications of conformal vector fields in Riemannian spaces.

Because of the importance of conformal vector fields in Riemann geometry, it is natural to study the properties of conformal vector fields in Finsler geometry. In a similar way as that in Riemannian case, conformal vector fields in Finsler geometry are defined by the property of the flows generated by those vector fields ([6]). In [6], Mo-Huang define conformal vector fields in Finsler geometry and obtain the relation between the flag curvatures of two Finsler metric \(F\) and \(\tilde{F}\), where \(\tilde{F}\) is defined by \((F, V)\) under navigation technique for a homothetic
vector field $V$ of $F$. In [11], Shen-Xia study some curvature properties of conformal vector fields of Randers spaces and determine the local solutions of conformal vector fields in Randers spaces of isotropic flag curvature in dimension $n \geq 3$.

An $(\alpha, \beta)$-metric is defined by a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and a 1-form $\beta = b_i(x)y^i$ on a manifold $M$, which can be expressed in the following form:

$$F = \alpha \phi(s), \quad s = \beta/\alpha,$$

where $\phi(s)$ is a function satisfying certain conditions such that $F$ is regular. In [5], Kang characterizes conformal vector fields of $(\alpha, \beta)$-spaces by some PDEs in a special case. In this paper, we first give in the following theorem a complete characterization for conformal vector fields of $(\alpha, \beta)$-spaces.

**Theorem 1.1** Let $F = \alpha \phi(\beta/\alpha)$ be an $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$, where $\phi(s)$ satisfies $\phi(s) \neq \sqrt{1 + ks^2}$ for any constant $k$ and $\phi(0) = 1$. Let $V = V^i(x)\partial/\partial x^i$ be a vector field on $M$. Then $V$ is a conformal vector field of $(M, F)$ if and only if

$$V_{ij} + V_{ji} = -4\alpha c_{ij}, \quad V^j b_{ij} + b^i V_{ji} = -2c b_i,$$  \hspace{1cm} (1)

where $c = c(x)$ is a scalar function, $V_i$ and $b^i$ are defined by $V_i := a_{ij}V^j$ and $b^i := a^{ij}b_j$, and the covariant derivatives are taken with respect to the Levi-Civita connection of $\alpha$.

In [5], Kang proves the same result by assuming $\phi'(0) \neq 0$. In fact, Theorem 1.1 shows that the condition $\phi'(0) \neq 0$ is not necessary. When $\phi(s) = \sqrt{1 + ks^2}$, $F$ is essentially a Riemannian metric. To solve $V$ from (1), it needs further curvature conditions on $F$.

Now we consider the solutions of conformal vector fields in $(\alpha, \beta)$-spaces. An $(\alpha, \beta)$-metric $F = \alpha \phi(\beta/\alpha)$ is called of Randers type, if $\phi(s) = \sqrt{1 + ks^2}$, $F$ is essentially a Riemannian metric. To solve $V$ from (1), it needs further curvature conditions on $F$.

**Theorem 1.2** Let $F = \alpha \phi(\beta/\alpha)$ be a locally projectively flat $(\alpha, \beta)$-metric on an $n(\geq 3)$-dimensional manifold $M$, where $\phi(0) = 1$. Suppose $F$ is not of Randers type, or $F$ is additionally of isotropic S-curvature if $F = \alpha + \beta$ is a Randers metric. Let $V = V^i(x)\partial/\partial x^i$ be a conformal vector field of $(M, F)$. Then we have one of the following two cases:

(i) $V$ is given by

$$V^i = -2\tau x^i + q^i_k x^k + \gamma^i,$$  \hspace{1cm} (3)

where $\tau$ is a constant number, $\gamma = (\gamma^i)$ is a constant vector, and $Q = (q^i_k)$ is a constant skew-symmetric matrix satisfying $Qe = 0$. In this case, $\lambda = \mu = 0$ in (2), and $c$ in (1) is given by $c = \tau$, and so $V$ is a homothetic field.

(ii) $V$ is given by

$$V^i = 2\mu(\gamma, x)x^i + (1 - \mu|x|^2)\gamma^i + q^i_k x^k,$$  \hspace{1cm} (4)

where $\gamma = (\gamma^i)$ is a constant vector and $Q = (q^i_k)$ is a constant skew-symmetric matrix satisfying $\langle \gamma, c e \rangle = 0$ and $Qe = -2\lambda \gamma$. In this case, $c$ in (1) is given by $c = 0$, and so $V$ is a Killing field.
In Theorem 1.2 if \( \beta \) is parallel with respect to \( \alpha \), then \( \beta = 0 \) (\( F = \alpha \) is Riemannian), or \( \alpha \) is flat \((\beta \neq 0)\). In this case, when \( F = \alpha \) is Riemannian, the conformal vector field \( V \) can be locally determined completely (see \((12)\), or \((15)\) below) \((\text{cf.} \ [7] \ [10] \ [14] \ [15])\); if \( \alpha (= |y|) \) is flat \((\beta = (e, y) \neq 0)\), then \( V \) is determined by \((3)\).

It is known that a Randers metric \( F = \alpha + \beta \) is locally projectively flat if and only if \( \alpha \) is of constant sectional curvature and \( \beta \) is closed \((\text{11})\). In Theorem 1.2 if \( F = \alpha + \beta \) is not assumed to be of isotropic S-curvature additionally, it seems hard to determine the local solution of \( V \) from \((\text{11})\) although \( \alpha \) is of constant sectional curvature in \((\text{11})\), since \( \beta \) is only known to be closed. On the other hand, if \( F = \alpha + \beta \) is assumed to be of isotropic flag curvature in dimension \( n \geq 3 \), then the local solutions of \( V \) can be determined by \((\text{11})\) and the navigation technique \((\text{(11)})\).

The present author shows in \([20]\) that if \( F = \alpha + \kappa \beta + \epsilon \beta^2 / \alpha \) for constants \( \kappa, \epsilon \), then \( F \) is of scalar flag curvature if and only if \( F \) is locally projectively flat. We have the following corollary.

**Corollary 1.3** Let \( F = \alpha + \kappa \beta + \epsilon \beta^2 / \alpha \) for constants \( \kappa, \epsilon \) be an \((\alpha, \beta)\)-metric of scalar flag curvature on an \((n \geq 3)\)-dimensional manifold \( M \). Suppose \( V = V^i(x) \partial / \partial x^i \) is a conformal vector field of \((M, F)\). Then \( V \) can be written in the form \((5)\) or \((4)\).

In \([3]\), Cheng-Mo-Shen give the local and global classifications for Randers metrics which are locally projectively flat and of isotropic S-curvature. In the last section, we will give a new proof to such classifications by constructing a conformal vector field of a Riemannian space with constant sectional curvature.

## 2 Conformal vector fields

Let \( F \) be a Finsler metric on a manifold \( M \), and \( V = V^i(x) \partial / \partial x^i \) be a vector field on \( M \). \( V \) is called a conformal vector of \((M, F)\) if the flow \( \varphi_t \) generated by \( V \) satisfies

\[
F(\varphi_t(x), (\varphi_t)_*(y)) = e^{-2ct}F(x, y),
\]

where \( c = c(x) \) is a scalar function which is called a conformal factor. If \( c = \text{constant} \), then \( V \) is called a homothetic field; if \( c = 0 \), \( V \) is called a Killing field \((\text{6})\). Differentiating \((5)\) by \( t \) at \( t = 0 \) and using the property of a flow we obtain that \( V \) is a conformal vector field if and only if

\[
X_V(F) = -2cF,
\]

where

\[
X_V := V^i \frac{\partial}{\partial x^i} + y_i \frac{\partial V^j}{\partial x^i} \frac{\partial}{\partial y^j}
\]

is a vector field on \( TM \) \((\text{6})\). A direct computation shows that

\[
X_V(F^2) = 2V_{0;0},
\]

where the covariant derivative is taken with respect to Cartan, Berwal or Chern connection of \( F \), and \( T_0 := T_0 y^i \) as an example.

**Lemma 2.1** On a manifold \( M \), let \( \alpha = \sqrt{a_{ij}(x)y^iy^j} \) be a Riemannian metric, \( \beta = b_i(x)y^i \) be a 1-form, and \( V \) be a vector field. Then we have

\[
X_V(\alpha^2) = 2V_{0;0}, \quad X_V(\beta) = (V^j \frac{\partial b_i}{\partial x^j} + b_j \frac{\partial V^i}{\partial x^j} y^j, = (V^j b_{ij} + b^j V_{j;i}) y^i,
\]

(9)
where \( V_i := a_{ij}V^j \) and \( b^i := a^{ij}b_j \), and the covariant derivative is taken with respect to the Levi-Civita connection of \( \alpha \). By (8) and (9), it shows that \( V \) is a conformal vector field of \((M, \alpha)\) with a conformal factor \( c = c(x) \) if and only if

\[
V_{0,0} = -2\alpha c^2. \tag{10}
\]

In (10), \( V \) can be solved under certain curvature properties of \( \alpha \), for instance, (i) \( \alpha \) is of constant flag curvature in dimension \( n \geq 3 \) (10); (ii) \( \alpha \) is locally conformally flat in dimension \( n \geq 2 \) (14); (iii) \( \alpha \) is a product metric of Riemannian metrics of constant sectional curvature (15). In the following lemmas, we introduce some properties of conformal vector fields of Riemannian spaces, which will be used in the later discussions.

**Lemma 2.2** Let \( \alpha \) and \( V \) satisfy (10) on an \( n (\geq 3) \)-dimensional manifold \( M \).

(i) ([10]) Let \( \alpha \) be of constant sectional curvature \( \mu \). Then locally we have

\[
\alpha = \frac{\sqrt{(1 + \mu|x|^2)|y|^2 - \mu(x, y)^2}}{1 + \mu|x|^2},
\]

\[
V^i = -2(\lambda \sqrt{1 + \mu|x|^2} + (d, x))x^i + \frac{2|x|^2d^i}{1 + \sqrt{1 + \mu|x|^2}} + q^i x^k + \eta^i + \mu(\eta, x)x^i, \tag{11}
\]

\[
c = \frac{\lambda + (d, x)}{\sqrt{1 + \mu|x|^2}}, \tag{12}
\]

(ii) ([14]) Let \( \alpha \) be locally conformally flat. Then locally we have

\[
\alpha^2 = e^{\sigma(x)}|y|^2, \tag{14}
\]

\[
V^i = -2(\lambda + (d, x))x^i + |x|^2d^i + q^i x^k + \eta^i, \tag{15}
\]

\[
c = \lambda + (d, x) - \frac{1}{4}V^r\sigma_r, \quad (\sigma_i := \frac{\partial \sigma}{\partial x^i}), \tag{16}
\]

where \( \lambda \) is a constant number, \( d, \eta \) are constant vectors and \((q^i)\) is skew-symmetric.

If \( \alpha \) is of constant sectional curvature \( \mu \), it is also locally conformally flat which can be taken as the form (14) with

\[
\sigma = \ln \frac{4}{(1 + \mu|x|^2)^2}, \tag{17}
\]

and then \( V \) can be written in the form (15) and \( c \) in (16) becomes

\[
c = \frac{\lambda(1 - \mu|x|^2) + (\mu \eta + d, x)}{1 + \mu|x|^2}. \tag{18}
\]

**Lemma 2.3** Let \( \alpha \) and \( V \) satisfy (11) on an \( n (\geq 2) \)-dimensional manifold \( M \), where \( \alpha \) is of constant sectional curvature \( \mu \) and \( V_0 \) is a closed 1-form.

(i) ([21]) If \( \alpha \) takes the local form (11), then \( V \) and \( c \) are given by

\[
V^i = \sqrt{1 + \mu|x|^2}(\lambda x^i + e^i), \quad c = \frac{-\lambda + \mu(\eta, x)}{2\sqrt{1 + \mu|x|^2}}, \tag{19}
\]

where \( \lambda \) is a constant and \( e = (e^i) \) is a constant vector.
(ii) If $\alpha$ takes the local form \( (14) \) with $\sigma$ being given by \( (17) \), then $V$ and $c$ are given by

\[
V^i = -2(\lambda + \mu(e, x)x^i) + (1 + \mu |x|^2)e_i, \quad c = \frac{\lambda(1 - \mu |x|^2) + 2\mu(e, x)}{1 + \mu |x|^2},
\]

where $\lambda$ is a constant and $e = (e^i)$ is a constant vector.

Proof: We prove the case (ii). It shows in \( (14) \) that if $\alpha$ takes the local form \( (14) \), then $V$ satisfies \( (10) \) if and only if

\[
\frac{\partial V^i}{\partial x^j} + \frac{\partial V^j}{\partial x^i} = 0 \quad (\forall i \neq j), \quad \frac{\partial V^i}{\partial x^i} = \frac{\partial V^j}{\partial x^j} \quad (\forall i, j).
\]

In this case, $c$ is given by

\[
c(x) = -\frac{1}{4}[2\frac{\partial V^i}{\partial x^i} + V^r \sigma_r], \quad \forall \text{ fixed index } i.
\]

Now let $\alpha$ take the local form \( (14) \) with $\sigma$ being given by \( (17) \).

If $n \geq 3$, then it follows from Lemma 2.2 that $V$ is given by \( (15) \). Since $V_0$ is closed, it is easy to show that $Q = 0$ and $d = \mu \eta$ in \( (15) \). Thus we get $V$ in \( (20) \) (put $\eta = e$), and \( (15) \) or \( (22) \) implies $c$ in \( (20) \).

If $n = 2$, then since $\partial V^1/\partial x^2 = -\partial V^2/\partial x^1$ from \( (21) \), it follows that $V_0$ is closed if and only if

\[
\frac{\partial V^2}{\partial x^1} = \frac{2\mu(V^2 x^1 - V^1 x^2)}{1 + \mu |x|^2}.
\]

Then solving the system \( (21) \) and \( (23) \), yields $V$ in \( (20) \). Q.E.D.

Lemma 2.4 Let $\alpha$ and $V$ satisfy \( (10) \) on an $n$-dimensional manifold $M$, where $\alpha$ is of constant sectional curvature $\mu$. Define

\[
C := ||\nabla c||^2_\alpha + \mu c^2,
\]

where $\nabla c$ is the gradient of $c$ with respect to $\alpha$.

(i) $C$ is a constant in $n \geq 3$. If additionally $V_0$ is closed, then $C$ is a constant in $n \geq 2$.

(ii) \( (20) \) \( (n \geq 2) \) Let $M$ be compact without boundary. Then $c = 0$ if $\mu \leq 0$.

Proof: It has been proved in \( (10) \) that for $n \geq 3$, $c$ satisfies

\[
c_{i;j} + \mu c_{i;j} = 0,
\]

where $c_i := c_{x^i}$. Using \( (24) \), it shows in \( (14) \) \( (20) \) that $f$ is a constant in $n \geq 3$. Here we have to prove that \( (24) \) still holds in $n \geq 2$ if $V_0$ is closed. By \( (10) \) and Ricci identities, we have \( (10) \)

\[
V_{k;i;j} = 2(c_k h_{ij} - c_i h_{jk} - c_j h_{ik}) - V_m R_{j k i}^m,
\]

where $R$ is the Riemann curvature tensor of $\alpha$. Since $V_0$ is closed ($V_{i;j} = V_{j;i}$), \( (25) \) is equivalent to

\[
2(c_k h_{ij} - c_i h_{jk}) - V_m R_{j k i}^m = 0, \quad V_{k;i;j} + 2c_j h_{ik} = 0.
\]

By the first equation in \( (24) \), we get

\[
2(c_k h_{ij} - c_i h_{jk}) = (\mu V_k - V_j) h_{ij},
\]

which are equivalent to

\[
(2c_k - \mu V_k) h_{ij} = (2c_i - \mu V_i) h_{jk}.
\]

Contracting the above by $h^{jk}$ we obtain $2c_i = \mu V_i$. Thus \( (24) \) holds. Q.E.D.
3 Proof of Theorem 1.1

To prove Theorem 1.1 we first show the following known lemma by Kangin [5], and we will also give a simple proof here.

Lemma 3.1 Let $F = \alpha \phi(\beta/\alpha)$ be an $(\alpha, \beta)$-metric and $V = V^i(x)\partial/\partial x^i$ be a vector field on an $n$-dimensional manifold $M$. Then $V$ is a conformal vector field of $(M, F)$ with a conformal factor $c = c(x)$ if and only if

$$\phi - s\phi'(V^i b_{ji} + b^i V_{ij})y^j = -2c\phi \alpha^2,$$

where the covariant derivative is taken with respect to $\alpha$.

Proof: Eq. (6) is equivalent to

$$X_V(F^2) = -4cF^2.$$ (28)

A direct computation shows that

$$X_V(F^2) = \phi^2 X_V(\alpha^2) + 2\alpha^2 \phi' \frac{\alpha X_V(\beta) - \beta X_V(\alpha)}{\alpha^2} = \phi(\phi - s\phi') X_V(\alpha^2) + 2\alpha\phi' X_V(\beta).$$

Now plugging (9) into the above equation and using (28), we obtain (27). Q.E.D.

In order to simplify (27), we choose a special coordinate system $(s, y^a)$ at a fixed point on a manifold as usually used. Fix an arbitrary point $x \in M$ and take an orthogonal basis $\{e_i\}$ at $x$ such that

$$\alpha = \sqrt{\sum_{i=1}^n (y^i)^2}, \quad \beta = by^1.$$ (29)

It follows from $\beta = s\alpha$ that

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad \left(\bar{\alpha} := \sqrt{\sum_{a=2}^n (y^a)^2}\right).$$

Then if we change coordinates $(y^i)$ to $(s, y^a)$, we get

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}} \bar{\alpha}.$$ (30)

Let

$$\bar{V}_{0:0} := V_{a;0} y^a, \quad \bar{V}_{1:0} := V_{1:a} y^a, \quad \bar{V}_{0:1} := V_{a;1} y^a, \quad \bar{b}_{0:1} := b_{a;1} y^a.$$ (31)

Note that under the coordinate $(s, y^a)$, we have $b_1 = b, \bar{b}_0 = 0$, but generally $\bar{b}_{0:1} \neq 0$.

Under the coordinate $(s, y^a)$, (27) can be rewritten equivalently, which is shown in the following lemma (see [5]).

Lemma 3.2 Eq. (27) $\iff$

$$0 = \bar{V}_{0:0} + 2c\bar{\alpha}^2,$$ (29)

$$0 = [(\bar{V}_{1:0} + \bar{V}_{0:1}) s^2 - b^2 \bar{V}_{1:0} - b V^j b_{0:1}] \phi' - (\bar{V}_{1:0} + \bar{V}_{0:1}) s \phi,$$ (30)

$$0 = [(V_{1:1} + 2c)(b^2 - s^2) + b V^j b_{1:1}] \phi' + (V_{1:1} + 2c) s \phi.$$ (31)
Now using (29)–(31), we will start our proof of Theorem 1.1

**Step A.** Firstly, we deal with (30). Let
\[ \phi = a_0 + \sum_{i=1}^{\infty} a_i s^i, \quad a_0 = 1. \]

Plugging the above Taylor series into (30) we obtain a power series \( \sum_k p_k s^k = 0 \). It is easily seen that the coefficient \( p_k \) of \( s^k \) is given by
\[
   p_k = (k - 2)a_{k-1}(\bar{V}_{1;0} + \bar{V}_{0;1}) - (k + 1)a_{k+1}b(b\bar{V}_{1;0} + V'\bar{b}_{0;i}),
\]
where we put \( a_i = 0 \) if \( i < 0 \). We have all \( p_k = 0 \).

Since \( \phi(s) \neq \sqrt{1 + 2a_2s^2} \), in all \( a_{2i+1} \)'s or \( a_{2i} \)'s there exists some minimal \( m \) such that
\[
   a_{2m+1} \neq 0, \quad (m \geq 0); \quad \text{or} \quad a_{2m} \neq C_m^m(2a_2)^m, \quad (m \geq 2),
\]
where \( C_m^m \) are the generalized combination coefficients. We will show in the following that in anyone of the two cases in (33), we always obtain (34) and (35) below. Conversely, if (34) and (35) below hold, then it is easy to verify that (30) holds identically for any \( \phi(s) \).

**Case I.** Assume \( a_{2m+1} \neq 0 \) in (33) for \( m \geq 0 \). Then plugging \( a_{2m} = 0 \) into (32) yields
\[
   b\bar{V}_{1;0} + V'\bar{b}_{0;i} = 0.
\]
Now plug (34) into \( p_1 = 0 \) (see (32)) and then we get
\[
   \bar{V}_{1;0} + \bar{V}_{0;1} = 0.
\]

**Case II.** Assume \( a_{2m} \neq C_m^m(2a_2)^m \) in (33) for \( m \geq 2 \). We have two subcases.

**Case IIa.** Assume \( a_2 \neq 0 \). Then by \( p_1 = 0 \) (see (32)), we have
\[
   V'\bar{b}_{0;i} + b\bar{V}_{1;0} = -\frac{\bar{V}_{1;0} + \bar{V}_{0;1}}{2a_2}.
\]
Plugging \( a_{2m-2} = C_{m-1}^m(2a_2)^{m-1} \) and (36) into \( p_{2m-1} = 0 \) (see (32)), and using
\[
   C_{m-1}^m = \frac{m}{\frac{1}{2} - m} \cdot C_m^m,
\]
we obtain
\[
   ma_2^{-1} [a_{2m} - C_m^m(2a_2)^m] (\bar{V}_{1;0} + \bar{V}_{0;1}) = 0,
\]
which implies (33). Then by (36), we get (34).

**Case IIb.** Assume \( a_2 = 0 \). Then by \( p_1 = 0 \) (see (32)) and \( a_2 = 0 \), we have (35). Plugging (35) into \( p_{2m-1} = 0 \) (see (32)), and using \( a_{2m} \neq 0 \), we have (34).

**Step B.** We deal with (31). For this case, we will show an almost completely same process in computation. Plugging the Taylor series of \( \phi(s) \) into (31) we obtain a power series \( \sum_k p_k s^k = 0 \). Similar to (32), we get
\[
   p_k = (k - 2)a_{k-1}(-2c + V_{1;1}) - (k + 1)a_{k+1}b(-2bc - bV_{1;1} - V'\bar{b}_{1;i}),
\]
where we put \(a_i = 0\) if \(i < 0\). We have all \(p_k = 0\).

Since \(\phi(s) \neq \sqrt{1 + 2a_2s^2}\), we have (33). We will show in the following that in both cases of (33), we always have (40) and (41) below. Conversely, if (40) and (41) below hold, then it is easy to verify that (31) holds identically for any \(\phi(s)\).

**Case I.** Assume \(a_{2m+1} \neq 0\) in (33) for \(m \geq 0\). Then plugging \(a_{2m} = 0\) into (38) yields

\[
V^i b_{1;i} = b(-2c - V_{1;1}).
\]  

(39)

Now plug (38) into \(p_1 = 0\) (see (38)) and then we get

\[
V_{1;1} = -2c.
\]  

(40)

Then by (40), (39) becomes

\[
V^i b_{1;i} = 0.
\]  

(41)

**Case II.** Assume \(a_{2m} \neq C_m^{m}(2a_2)^m\) in (33) for \(m \geq 2\). We have two subcases.

**Case IIa.** Assume \(a_2 \neq 0\). Then by \(p_1 = 0\) (see (38)), we have

\[
V^i b_{1;i} = (b + \frac{1}{2ba_2})(-2c - V_{1;1}).
\]  

(42)

Plugging \(a_{2m-2} = C_{m-1}^{m-1}(2a_2)^{m-1}\) and (42) into \(p_{2m-1} = 0\) (see (38)), and using

\[
C_{m-1}^{m-1} = \frac{m}{2} \cdot C_{m}^{m},
\]

we obtain

\[
ma_2^{-1}[a_{2m} - C_m^{m}(2a_2)^m](-2c - V_{1;1}) = 0,
\]  

(43)

which implies (40). Then by (40), we get (41).

**Case IIb.** Assume \(a_2 = 0\). Then by \(p_1 = 0\) (see (38)) and \(a_2 = 0\), we have (40). Plugging (40) into \(p_{2m-1} = 0\) (see (38)), and using \(a_{2m} \neq 0\), we have (41).

To sum up, by the proofs in Step A and Step B above, we finally have (34), (35), (40) and (41). Now it is easy to see that (29), (35), (40) and (41) imply the first equation in (1), and (34), (40) and (41) imply the second equation in (1). Q.E.D.

## 4 Projectively flat \((\alpha, \beta)\)-metrics

To prove Theorem 1.2 we first show the following two key lemmas.

**Lemma 4.1** \([22, 23, 24]\) Let \(F = \alpha \phi(s), s = \beta/\alpha\), be an \(n(\geq 3)\)-dimensional \((\alpha, \beta)\)-metric, where \(\phi(0) = 1\). Suppose that \(\beta\) is not parallel with respect to \(\alpha\) and \(F\) is not of Randers type. Then \(F\) is locally projectively flat if and only if \(\phi(s)\) satisfies

\[
\{1 + (k_1 + k_3)s^2 + k_2s^4\}\phi''(s) = (k_1 + k_2s^2)\{\phi(s) - s\phi'(s)\},
\]

and \(\alpha, \beta\) are determined by

\[
h = \sqrt{u(b^2)\alpha^2 + v(b^2)\beta^2}, \quad \rho = w(b^2)\beta, \quad (b^2 := ||\beta||_\alpha^2),
\]  

(44)
where $k_1, k_2, k_3$ are constants satisfying $k_2 \neq k_1 k_3$, $h$ is a Riemann metric of constant sectional curvature $\mu$ and $\rho$ is a closed conformal 1-form, and $u = u(t) \neq 0, v = v(t), w = w(t) \neq 0$ satisfy the following ODEs:

\[
\begin{align*}
u' &= \frac{v - k_1 u}{1 + (k_1 + k_3)t + k_2 t^2}, \quad (45) \\
v' &= \frac{u(k_2 u - k_3 v - 2 k_1 v) + 2 v^2}{u[1 + (k_1 + k_3)t + k_2 t^2]}, \quad (46) \\
w' &= \frac{w(3v - k_3 u - 2k_1 v)}{2u[1 + (k_1 + k_3)t + k_2 t^2]}, \quad (47)
\end{align*}
\]

**Remark 4.2** We can have different suitable choices of $u, v, w$ satisfying (45)–(47). A suitable choice of the triple $(u, v, w)$ can be taken as (27).

\[
\begin{align*}
u &= e^{2\chi}, \quad v = (k_1 + k_3 + k_2 t^2) u, \quad w = \sqrt{1 + (k_1 + k_3)b^2 + k_2 b^4} e^\chi,
\end{align*}
\]

where $\chi$ is defined by

\[2\chi := \int_0^{b^2} \frac{k_2 t + k_3}{1 + (k_1 + k_3)t + k_2 t^2} dt.
\]

**Lemma 4.3** Let $\alpha = \sqrt{a_{ij} y^i y^j}$ be a Riemann metric and $\beta = b_i y^i$ be a 1-form and $V = V^i \partial / \partial x^i$ be a vector field on an $n$-dimensional manifold $M$. Define a Riemann metric $h = \sqrt{h_{ij}(x) y^i y^j}$ and a 1-form $\rho = p_i(x) y^i$ by

\[h = \sqrt{u(b^2)\alpha^2 + v(b^2)\beta^2}, \quad \rho = w(b^2)\beta, \quad (b^2 := ||\beta||_2^2),
\]

where $u = u(t) \neq 0, v = v(t), w = w(t) \neq 0$ are arbitrary smooth functions. Then $\alpha, \beta$ and $V$ satisfy (11) if and only if

\[V_{0|0} = -2c h^2, \quad V^i p_{ij} + p^j V_{j|i} = -2 c \rho_i,
\]

where $p^i := h^{ij} p_j, \quad V^i := h_{ij} V^j$ and the covariant derivative is taken with respect to the Levi-Civita connection of $h$.

**Proof:** Assume (11) holds. Let $X_V$ be defined by (7). We have

\[
\begin{align*}X_V(b^2) &= 2V^i b^i b_{ij}, \\
&= 2(-2c b^2 - b^i b_i V_{ij}), \quad (\text{by the second equation of } (11)) \\
&= 0, \quad (\text{by the first equation of } (11)).
\end{align*}
\]

By (11) and (10) we get

\[
X_V(\alpha^2) = -4 c \alpha^2, \quad X_V(\beta) = -2 c \beta.
\]

Therefore, it follows from (48), (50) and (51) that

\[
\begin{align*}X_V(h^2) &= u'(b^2) X_V(b^2)\alpha^2 + v'(b^2) X_V(b^2)\beta^2 + u(b^2) X_V(\alpha^2) + v(b^2) X_V(\beta^2) \\
&= u(b^2) X_V(\alpha^2) + v(b^2) X_V(\beta^2) \\
&= -4 c [u(b^2)\alpha^2 + v(b^2)\beta^2] \\
&= -4 c h^2, \quad (52)
\end{align*}
\]

\[
\begin{align*}X_V(\rho) &= w'(b^2) X_V(b^2)\beta + w(b^2) X_V(\beta) = w(b^2) X_V(\beta) \\
&= -2 c w(b^2)\beta = -2 c \rho.
\end{align*}
\]

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Now (19) follows directly from (52), (53) and Lemma 2.1. Conversely, assume (49) holds. Then we can prove that (11) holds in a similar way as above, since we can express $\alpha$ and $\beta$ in terms of $h$, $\rho$ and the norm of $\rho$ with respect to $h$. This completes the proof of the lemma. Q.E.D.

Remark 4.4 In (48), take $u(b^2) = 1 - b^2$, $v(b^2) = b^2 - 1$, $w(b^2) = b^2 - 1$, and then $(h, \rho)$ is called the navigation data for a Randers metric $F = \alpha + \beta$ under navigation representation. The method used in the proof of Lemma 4.3 also gives a brief proof to Proposition 3.1 in [11] for the case of Randers metrics (cf. the long proof therein).

Let $F = \alpha \phi(\beta/\alpha)$ be an $(\alpha, \beta)$-metric (not of Randers type) satisfying the conditions in Theorem 1.2. Now we can start the proof of Theorem 1.2 in this case.

Condition (i): Suppose $\beta$ is not parallel with respect to $\alpha$.

Define $h$ and $\rho$ by (18) (or (11)), where $u = u(t) \neq 0, v = v(t), w = w(t) \neq 0$ are a suitable choice satisfying the ODEs (45)–(47). Then by Lemma 4.1, $h$ is a Riemann metric of constant sectional curvature $\mu$ and $\rho$ is a closed conformal 1-form. Since $V$ is a conformal vector field of $F$, $V$ satisfies (11) by Theorem 1.1. Then by Lemma 4.3, $V$ satisfies (49).

Since $h$ is of constant sectional curvature $\mu$ and $\rho$ is closed, $h$ and $\rho$ can be taken as the local form (2) by Lemma 2.3 (ii). Therefore, to prove Theorem 1.2 in this case, we only need to solve $V$ from (49) in the following proof with $h$ and $\rho$ being given by (2).

Since $V$ is a conformal vector field of $h$ by the first equation of (19), it follows from Lemma 2.2 (ii) that $V$ can be locally expressed as

$$V^i = -2(\tau + \langle \eta, x \rangle)x^i + |x|^2\eta^i + q^i_r x^r + \gamma^i,$$

where $\tau$ is a constant number, $\eta, \gamma$ are constant vectors and $Q = (q^i_r)$ is a skew-symmetric constant matrix. In this case, it follows from (18) that $c$ in (49) is given by

$$c = \frac{\tau(1 - \mu|x|^2) + (\mu\gamma + \eta, x)}{1 + \mu|x|^2},$$

The second equation in (49) is equivalent to

$$V^i_{\partial p_i \partial x^i} + p_j \frac{\partial V^j}{\partial x^i} = -2c p_i,$$

and $p_i$ in (2) are given by

$$p_i = \frac{4}{(1 + \mu|x|^2)^2}\left\{ -2(\lambda + \mu(e, x))x^i + (1 + \mu|x|^2)e^i \right\}$$

Now plugging (54), (55) and (57) into (56) yields an equivalent equation of a polynomial in $(x^i)$ of order four, in which every order must be zero. Then we respectively have (from order zero to order four)

$$Qe = -2\lambda\gamma,$$

$$(\langle \eta - \mu \gamma, e \rangle + 2\lambda\tau)x^i - (\eta^i + \mu \gamma^i)\langle e, x \rangle = 0,$$

$$[\lambda(\eta^i - \mu \gamma^i) - \mu q^i_k e^k]|x|^2 + \mu[2\lambda(\gamma, x) + 4\tau(e, x) - \langle e, Qx \rangle]x^i = 0,$$

$$[\mu(\langle \eta - \mu \gamma, e \rangle - 2\lambda\tau)x^i - \mu(e, x)(\eta^i + \mu \gamma^i)]|x|^2 + 2\mu(e, x)(\eta + \mu \gamma, x)x^i = 0,$$

$$\mu[(2\lambda\eta^i - \mu q^i_k e^k)|x|^2 - 2(2\lambda\eta - \mu Qe, x)x^i] = 0.$$
**Step A:** Assume $\mu = 0$. Then by (60) we have $\lambda = 0$ or $\eta = 0$. Since $\mu = 0$, (59) becomes
\[
\left(\langle \eta, e \rangle + 2\lambda \tau \right)x^i - \eta^i \langle e, x \rangle = 0.
\]
Plus (58), it is easy to see that (58)–(62) are equivalent to one of the following three cases:
\[
\begin{align*}
\lambda &= 0, \quad \eta = 0, \quad Qe = 0, \quad (\mu = 0); \quad (63) \\
\lambda &= 0, \quad e = 0, \quad (\mu = 0); \quad (64) \\
\eta &= 0, \quad \tau = 0, \quad Qe = -2\lambda \gamma, \quad (\mu = 0). \quad (65)
\end{align*}
\]

**Step B:** Assume $\mu \neq 0$. Then by (62) we get
\[
2\lambda \eta^i - \mu q^i_k e^k = 0, \quad 2\lambda \eta - \mu Qe = 0. \quad (66)
\]
Now plugging (58) into (66) yields
\[
\lambda = 0, \quad \text{or} \quad \eta = -\mu \gamma. \quad (67)
\]
Contracting (59) by $x^i$ we have
\[
\langle \eta - \mu \gamma, e \rangle + 2\lambda \tau = 0, \quad (\eta^i + \mu \gamma^i)e = 0. \quad (68)
\]

**Case I:** Assume $\lambda = 0$. Then by (68) we have
\[
e = 0, \quad \text{or} \quad \eta = -\mu \gamma \quad \text{and} \quad \langle \eta, e \rangle = \langle \gamma, e \rangle = 0. \quad (69)
\]

**Case Ia:** Assume $e = 0$. In this case, we have
\[
\lambda = 0, \quad e = 0, \quad (70)
\]
which implies (58)–(62) automatically hold.

**Case Ib:** Assume $\eta = -\mu \gamma$ ($\lambda \neq 0$). By (59) we have
\[
\langle \gamma, e \rangle = \frac{\lambda \tau}{\mu}. \quad (72)
\]
Plugging (58) and $\eta = -\mu \gamma$ into (71) yields
\[
\tau = 0, \quad \text{or} \quad e = 0. \quad (73)
\]
Now it follows from $\eta = -\mu \gamma$, (58), (72) and (73) that
\[
\eta = -\mu \gamma, \quad \tau = \langle \gamma, e \rangle = 0, \quad Qe = -2\lambda \gamma, \quad (74)
\]
which implies (58)–(62) automatically hold.
Summing up the above discussions, we have six conditions: (63)–(65), (70), (71) and (74). However, it is easy to show that: (i) (64) \( \Rightarrow \) (70); (ii) (71) \( \Rightarrow \) (74); (iii) (65) \( \Rightarrow \) (63) when \( \lambda = 0 \); (iv) (65) \( \Rightarrow \) (74) when \( \lambda \neq 0 \). Therefore, we have three independent conditions: (63), (70) and (74). But (70) implies \( F = \alpha \) is Riemannian since \( \beta = \rho = 0 \) by (48) and (57). So we finally obtain (63) and (74). Now plug (63) and (74) into (54) respectively, we obtain (3) and (4). Then by (55), we can get \( c \) in two cases.

**Condition (ii):** Suppose \( \beta \) is parallel with respect to \( \alpha \).

Since \( F \) is locally projectively flat, \( \alpha \) is also locally projectively flat and then is of constant sectional curvature \( \mu \). If \( \beta = 0 \), then \( F = \alpha \) is Riemannian, which is excluded. If \( \beta \neq 0 \), then \( \mu = 0 \) and so \( F \) is flat-parallel (\( \alpha \) is flat and \( \beta \) is parallel). In this case, \( \beta \) is also a closed 1-form which is conformal with respect to \( \alpha \). By Theorem 1.1, we only need to solve \( V \) from (1), which is just the case (63), since we can put \( b_i \) in the form of the right hand side of (57) with \( \lambda = 0 \), and thus \( V \) is given by (3). Q.E.D.

5 **Projectively flat Randers metrics**

In this section, by constructing a conformal vector field in a Riemannian space of constant sectional curvature, we will first go on with the proof of Theorem 1.2 when \( F = \alpha + \beta \) is a locally projectively flat Randers metric of isotropic S-curvature. Then we will give a new proof to the known local and global classifications in [3] for Randers metrics which are locally projectively flat and of isotropic S-curvature.

**Lemma 5.1** Let \( F = \alpha + \beta \) be an \( n \)-dimensional Randers metric which is locally projectively flat and of isotropic S-curvature. Let \( \mu \) be the constant sectional curvature of \( \alpha \). Define a 1-form \( \rho = p_i(x) y^i \) by

\[
\rho = \frac{1}{\sqrt{1 - b^2}} \beta, \quad (b^2 := ||\beta||^2_\alpha).
\]

Then \( \rho \) is a closed 1-form which is conformal with respect to \( \alpha \).

**Proof:** \( F = \alpha + \beta \) is locally projectively flat if and only if \( \alpha \) is of constant sectional curvature \( \mu \) and \( \beta \) is closed ([1]), and so \( F \) is of isotropic S-curvature \( S = (n + 1)\tau F \) for some scalar function \( \tau = \tau(x) \) if and only if ([1])

\[
b_{i,j} = 2\tau(a_{ij} - b_i b_j),
\]

where the covariant derivative is taken with respect to the Levi-Civita connection of \( \alpha \). Under the deformation (75), it can be directly verified that (75) is equivalent to

\[
p_{||ij} = \frac{2\tau}{\sqrt{1 - b^2}} a_{ij} \quad (= -2c a_{ij}),
\]

where the covariant derivative is taken with respect to the Levi-Civita connection of \( \alpha \). Thus \( \rho \) is a closed 1-form which is conformal with respect to \( \alpha \). Q.E.D.

**Proof of Theorem 1.2:** Let \( F = \alpha + \beta \) be locally projectively flat and of isotropic S-curvature. Define \( h := \alpha \) and \( \rho \) by (75), which is a special case of (18) with \( u(t) = 1, v(t) = 0, w(t) = 1/\sqrt{1 - t} \), and then it follows from Lemma 5.1 that \( h \) is of constant sectional curvature \( \mu \) and \( \rho \) is a closed 1-form which is conformal with respect to \( h \). Thus we can express \( h \) and \( \rho \) locally by (2). Let \( V \) be a conformal vector fields of \( F \). Then \( V \) satisfies
by Theorem 1.1 and thus $V$ satisfies (49) by Lemma 4.3. Therefore, by the proof of Theorem 1.2 in Section 4 when $F$ (not of Randers type) is locally projectively flat, $V$ is determined by (3) or (4). Q.E.D.

Now we investigate a new proof to the following proposition, originally proved in [3].

**Proposition 5.2** Let $F = \alpha + \beta$ be a Randers metric on an $n$-dimensional manifold $M$. Suppose $F$ is locally projectively flat and of isotropic S-curvature $S = (n + 1)\tau F$. Let $\mu$ be the constant sectional curvature of $\alpha$.

(i) Let $\alpha$ take the local form (11). Then $\beta$ is given by

$$\beta = \frac{1}{\xi} \left\{ \frac{\lambda - \mu \langle e, x \rangle}{1 + \mu |x|^2} (x, y) + \langle e, y \rangle \right\},$$

(78)

where $\xi$ is defined as

$$\xi := \sqrt{[\lambda^2 + (1 + |e|^2)\mu] |x|^2 + (2\lambda - \mu \langle e, x \rangle)(e, x) + |e|^2 + 1},$$

(79)

and $\lambda$ is a constant and $e = (e^i) \in \mathbb{R}^n$ is a constant vector. In this case, $\tau$ and the scalar flag curvature $K$ of $F$ are given by

$$\tau = \frac{\lambda - \mu \langle e, x \rangle}{2\xi}, \quad K = \frac{3}{4}(\mu + 4\tau^2)\frac{\alpha - \beta}{\alpha + \beta} + \frac{\mu}{4},$$

(80)

(ii) Let $M$ be compact without boundary. Then

(iia) $(\mu < 0)$ $F$ is Riemannian.

(iib) $(\mu = 0)$ $F$ is flat-parallel.

(iic) $(\mu > 0)$ $\beta$ is given by

$$\beta = \frac{2c_0}{\sqrt{4\delta^2 + \mu^2 - 4\mu c^2}}, \quad (c_i := c_{x^i}),$$

(81)

where

$$c := -\frac{\tau}{\sqrt{1 - b^2}}, \quad \delta := \sqrt{||\nabla c||^2_{\alpha} + \mu c^2} = \text{constant.}$$

(82)

In this case, the scalar flag curvature $K$ satisfies

$$K = \frac{3}{4} \frac{\mu(4\delta^2 + \mu^2)(\alpha \sqrt{4\delta^2 + \mu^2 - 4\mu c^2 - 2c_0})}{(4\delta^2 + \mu^2 - 4\mu c^2)(\alpha \sqrt{4\delta^2 + \mu^2 - 4\mu c^2 + 2c_0})} + \frac{\mu}{4},$$

(83)

$$\frac{6\delta^2 + \mu^2 - 3\delta \sqrt{4\delta^2 + \mu^2}}{\mu} \leq K \leq \frac{6\delta^2 + \mu^2 + 3\delta \sqrt{4\delta^2 + \mu^2}}{\mu}.$$  

(84)

**Remark 5.3** Case (i) of Proposition 5.2 can be divided into three subcases further: (iia) if $\lambda = \mu = 0$, then $F$ is flat-parallel; (iib) if $\lambda^2 + (1 + |e|^2)\mu = 0$ with $\mu \neq 0$, then $\beta$ in (78) becomes

$$\beta = \sqrt{-\mu} \left\{ \pm \frac{\langle x, y \rangle}{1 + \mu |x|^2} \pm \frac{\langle e, y \rangle}{1 - \mu \langle e, y \rangle} \right\}, \quad (e = \frac{e}{\lambda});$$

(78)

(iic) if $\lambda^2 + (1 + |e|^2)\mu \neq 0$ ($\Leftrightarrow \mu + 4\tau^2 \neq 0$), then $\beta$ in (78) can be rewritten as

$$\beta = -\frac{2\tau_0}{\mu + 4\tau^2}.$$
Proof of Proposition 5.2: (i) Define $\rho$ by (75) and let $\alpha$ take the local form (11). Then Lemma 5.1 shows that $\rho$ is a closed 1-form which is conformal with respect to $\alpha$ satisfying (77). Now it follows from Lemma 2.3 (i) that
\begin{equation}
\rho = \left(\lambda - \mu \langle e, x \rangle \right) \langle x, y \rangle + \frac{\lambda^2 |x|^2 + 2\lambda \langle e, x \rangle - \mu \langle e, x \rangle^2}{1 + \mu |x|^2}.
\end{equation}

By (75) we have
\begin{equation}
p^2 := ||\rho||^2_\alpha = |e|^2 + \frac{\lambda^2 |x|^2 + 2\lambda \langle e, x \rangle - \mu \langle e, x \rangle^2}{1 + \mu |x|^2}.
\end{equation}

Next, plugging (85), (86) and (88) into (75) yields (78), and it follows from (77), (86)–(88) that $\tau$ is given by (80). Using (78) and $\tau$ in (80), it is easy to verify that
\begin{equation}
\tau_0 = -\frac{1}{2}(\mu + 4\tau^2)\beta.
\end{equation}

Since $F$ is projectively related to $\alpha$, it follows from (8) ((8.56)) that
\begin{equation}
K F^2 = \mu \alpha^2 + 3\left(\frac{\Psi}{2F}\right)^2 - \frac{\Psi}{2F},
\end{equation}

where by (70),
\begin{align*}
\Phi &:= b_{ij} y^i y^j = 2\tau (\alpha^2 - \beta^2), \\
\Psi &:= b_{ijkl} y^i y^j y^k = 2(\tau_0 - 4\tau^2)\beta (\alpha^2 - \beta^2).
\end{align*}

Then by (89) and (90), the scalar flag curvature $K$ is given by (80).

(ii) Now suppose $M$ is compact without boundary. Then Lemma 2.4 (ii) shows that $c = 0$ when $\mu \leq 0$. If $\mu < 0$, then by (87) we have $\lambda = 0$, $e = 0$. Thus $F = \alpha$ is Riemannian since $\beta = 0$ by (83) and (75). If $\mu = 0$, then by (87) we have $\lambda = 0$. So by (83) and (75), $\beta$ is parallel. Thus $F$ is flat-parallel. Finally, assume $\mu > 0$. By Lemma 2.4, we get
\begin{equation}
\tau^2 = \frac{\mu^2 c^2}{4\delta^2 + \mu^2 - 4\mu c^2}.
\end{equation}

Now plug (81) and (91) into $K$ in (80), and then we obtain $K$ given by (83). Using the expression of $K$ in (83) and
\begin{equation}
|\frac{c_0}{\alpha}| \leq ||\nabla c||_\alpha = \sqrt{\delta^2 - \mu c^2},
\end{equation}

we obtain
\begin{equation}
K_1 \leq K \leq K_2,
\end{equation}

where
\begin{equation}
K_1 := \frac{3\mu (4\delta^2 + \mu^2) (\sqrt{4\delta^2 + \mu^2 - 4\mu c^2} - 2\sqrt{\delta^2 - \mu c^2})}{4(4\delta^2 + \mu^2 - 4\mu c^2)(\sqrt{4\delta^2 + \mu^2 - 4\mu c^2} + 2\sqrt{\delta^2 - \mu c^2})} + \frac{\mu}{4},
\end{equation}

and
\[ K_2 := \frac{3\mu(4\delta^2 + \mu^2)}{4(4\delta^2 + \mu^2 - 4\mu c^2)(\sqrt{4\delta^2 + \mu^2 - 4\mu c^2} - 2\sqrt{\delta^2 - \mu c^2})} + \frac{\mu}{4}. \]  

Put

\[ A := \frac{\sqrt{\delta^2 - \mu c^2}}{\sqrt{4\delta^2 + \mu^2 - 4\mu c^2}}. \]

Then

\[ A \leq \text{Sup}_{x \in M} A = \frac{\delta}{\sqrt{4\delta^2 + \mu^2}}, \quad 1 - 2A > 0. \]  

We have

\[ K_1 = \frac{3(4\delta^2 + \mu^2)}{4\mu}(1 - 2A)^2 + \frac{\mu}{4}, \quad K_2 = \frac{3(4\delta^2 + \mu^2)}{4\mu}(1 + 2A)^2 + \frac{\mu}{4}. \]

It follows from (95) and (96) that

\[ K_1 \geq \frac{6\delta^2 + \mu^2 - 3\delta\sqrt{4\delta^2 + \mu^2}}{\mu}, \quad K_2 \leq \frac{6\delta^2 + \mu^2 + 3\delta\sqrt{4\delta^2 + \mu^2}}{\mu}. \]

Then (84) holds. Q.E.D.

**Remark 5.4** In [3], for \( \mu > 0 \), it puts

\[ f(x) := \frac{2\tau}{\sqrt{\mu + 4\tau^2}}, \quad \bar{\delta} := \sqrt{||\nabla f||_\alpha^2 + \mu f^2}. \]

It can be verified that

\[ \delta = \frac{\sqrt{4\bar{\delta}^2 + \mu^2}}{2\sqrt{\mu}} \bar{\delta}. \]

Then if we set \( \mu = 1 \), (84) becomes (see [3])

\[ \frac{2 - \bar{\delta}}{2(1 + \bar{\delta})} \leq K \leq \frac{2 + \bar{\delta}}{2(1 - \bar{\delta})}. \]

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