Faster Deterministic Approximation Algorithms for Correlation Clustering and Cluster Deletion

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Abstract

Correlation clustering is a framework for partitioning datasets based on pairwise similarity and dissimilarity scores, and has been used for diverse applications in bioinformatics, social network analysis, and computer vision. Although many approximation algorithms have been designed for this problem, the best theoretical results rely on obtaining lower bounds via expensive linear programming relaxations. In this paper we prove new relationships between correlation clustering problems and edge labeling problems related to the principle of strong triadic closure. We use these connections to develop new approximation algorithms for correlation clustering that have deterministic constant factor approximation guarantees and avoid the canonical linear programming relaxation. Our approach also extends to a variant of correlation clustering called cluster deletion, that strictly prohibits placing negative edges inside clusters. Our results include 4-approximation algorithms for cluster deletion and correlation clustering, based on simplified linear programs with far fewer constraints than the canonical relaxations. More importantly, we develop faster techniques that are purely combinatorial, based on computing maximal matchings in certain auxiliary graphs and hypergraphs. This leads to a combinatorial 6-approximation for complete unweighted correlation clustering, which is the best deterministic result for any method that does not rely on linear programming. We also present the first combinatorial constant factor approximation for cluster deletion.

1 Introduction

Clustering is a fundamental task in data mining and machine learning that seeks to partition a dataset into groups of related objects that are more similar to each other than they are to the rest of the dataset. Correlation clustering is a specific framework for this task that depends only on pairwise similarity and dissimilarity scores, rather than explicit representations for individual data objects. The simplest version of the problem can be cast as a partitioning objective on a complete signed graph, where the goal is to form clusters of nodes in a way that minimizes the number of disagreements, where a disagreement can either be a negative edge inside a cluster or a positive edge lying between clusters. Bansal, Blum, and Chawla [4] formalized and introduced to the problem to the theory community, and many subsequent approximation algorithms and hardness results have been developed for different variants of the problem [2, 9, 10, 14, 38]. In addition to extensive theoretical research, the problem has been applied in a wide variety of settings, including to image segmentation [24, 48], community detection [11, 43, 46], cross-lingual link detection [40], cancer mutation analysis [22], and detecting co-regulated genes based on expression profiles [6–8].

Despite significant previous research on both theoretical and applied aspects of correlation clustering, there still exists a wide gap between the best approximation algorithms and the most...
practical tools for this task. Many heuristic methods have been designed, but these come with no approximation guarantees for the NP-hard objective \[3, 5, 7, 17, 28, 35\]. Meanwhile, the best approximation algorithms for the problem rely on solving an expensive linear programming (LP) relaxation in order to obtain lower bounds for the objective. This is true both for the simplest unweighted version of the problem \[10\] as well as more general weighted variants \[2, 18, 23, 29, 31, 32, 41, 43\]. It is worth noting that when these lower bounds are computed in practice, they are useful for more than just designing approximation algorithms. In practice, linear programming lower bounds are often much closer to optimality than worst case theoretical results predict. If a good LP lower bound can be computed, the output of fast heuristic methods can be compared against the LP lower bound to obtain a posteriori approximation guarantees that are often very good in practice \[26, 39, 43, 48\]. However, despite some recent work on specialized solvers for these linear programs \[33, 37, 44\], these lower bounds can only be computed for medium-sized instances at best, and even then this can take a long time. There is therefore a need for faster approximation algorithms for correlation clustering, as well as an even more basic need to efficiently compute good lower bounds for the NP-hard objective in practice.

For the complete unweighted version of the problem, there is a fast randomized combinatorial algorithm that produces a three-approximation in expectation \[2\]. This method, commonly referred to as Pivot, iteratively selects a random unclustered node and places it with all of its unclustered positive neighbors. The procedure terminates when all nodes have been clustered. This approach can be made very fast \[12, 13, 30\], but is not without its limitations. First, the approach is designed for the complete unweighted case and does not extend as easily as LP-based techniques to other variants of correlation clustering. As one example, cluster deletion is a simple variant that strictly prohibits clustering two nodes together if they share a negative edge. Although constant-factor linear programming algorithms have been designed for cluster deletion \[31, 43, 47\], the standard Pivot technique does not even produce a feasible solution for this problem, and it is not clear how to adapt pivoting techniques to cluster deletion without first solving a linear program \[43\]. Another limitation of Pivot is that it only provides an expected approximation guarantee, and in the worst case can perform arbitrarily poorly. Although derandomization techniques have been developed, these rely again on solving the expensive canonical LP relaxation \[41\]. Finally, the proof of Pivot’s approximation guarantee relies only on an implicit lower bound that is not computed in practice, and therefore this method is used to compute a posteriori approximation guarantees.

The present work: efficient algorithms and practical lower bounds We provide significant steps in bridging the theory-practice gap in correlation clustering by designing practical techniques for computing lower bounds and faster corresponding approximation algorithms. Our results provide a useful trade-off between theoretically powerful but computationally expensive linear programming techniques, and the standard Pivot approach that is fast and combinatorial but does not generalize as easily and produces no explicit lower bounds. The algorithms we present completely avoid the expensive canonical LP relaxation, but nevertheless come with deterministic approximation guarantees and produce explicit lower bounds. In addition to new results for the widely-studied complete unweighted correlation clustering problem, we provide new lower bounds and approximation algorithms for cluster deletion.

Overview of techniques and algorithms. We prove our theoretical results by combining two major steps. We first of all prove new results on the relationship between correlation clustering and edge-labeling problems related to the principle of strong triadic closure \[16, 36\]. Strong triadic closure posits that two people in a social network will typically share at least a weak connection if they both share strong connections to a mutual friend. Similarities between these clustering
and labeling problems have been noted in previous work \cite{20, 25}. We strengthen known results by showing that optimal solutions to cluster deletion are always within a factor two of optimal solutions to a problem called minimum weakness strong triadic closure. We also prove a similar result for unweighted complete correlation clustering and a variant of this strong triadic closure problem. The second component of our theoretical results is to extend and apply deterministic pivoting strategies \cite{41} to round different lower bounds for strong triadic closure problems into approximate solutions for cluster deletion and correlation clustering. We provide the following new algorithms:

- We provide a deterministic combinatorial 4-approximation for the cluster deletion problem, which is the first constant factor approximation that does not depend on linear programming.
- We provide a deterministic combinatorial 6-approximation algorithm for complete unweighted correlation clustering. This is the best approximation factor obtained by any deterministic algorithm that does not depend on linear programming.
- We show how to obtain a 4-approximation for both problems by rounding linear programming relaxations with significantly fewer constraints than the canonical LP relaxations.

The Match-Flip-Pivot strategy. The central contribution of our paper is our strategy for developing combinatorial approximation guarantees for correlation clustering and cluster deletion. Our strategy works in three basic steps. We first obtain lower bounds by computing maximal matchings in either an auxiliary graph (for cluster deletion) or an auxiliary 3-uniform hypergraph (for correlation clustering). We use the results of our matching to determine a set of edges whose sign should be flipped to create a new graph. Finally we prove that a deterministic pivoting procedure on the resulting graph yields approximation guarantees for the original problem. The fact that this Match-Flip-Pivot strategy applies to cluster deletion as well as unweighted correlation clustering indicates that it is more flexible and generalizable than the standard combinatorial PIVOT procedure. Exploring how to use this strategy to develop deterministic and combinatorial algorithms for other weighted variants is a promising direction for future work. We include a number of additional open questions for future research at the end of the manuscript.

2 Correlation Clustering Preliminaries

We begin with technical definitions, terminology, and notation for correlation clustering, as well as key algorithmic primitives developed in previous work that we build upon in this paper.

2.1 Correlation Clustering Objectives

The most general weighted version of correlation clustering is defined by a node set $V$ with $n = |V|$ and positive and negative weighted edge sets $W^+$ and $W^-$. Each pair of nodes $(i, j) \in V \times V$ is associated with a positive edge weight $w_{ij}^+ \in W^+$ and a negative edge weight $w_{ij}^- \in W^-$. The goal is to obtain a clustering that correlates as much as possible with the edge weights, which can be found by minimizing the following weight of disagreements:

$$\text{minimize } \sum_{i < j} w_{ij}^+ x_{ij} + w_{ij}^- (1 - x_{ij}), \quad (1)$$

where $x_{ij} = 0$ if nodes $i$ and $j$ are clustered together and $x_{ij} = 1$ if they are separated. In other words, a penalty of $w_{ij}^+$ is applied for separating $i$ and $j$, and a penalty of $w_{ij}^-$ is applied if they are
placed together. An alternative objective for correlation clustering is to maximize the weight of agreements, which is the same at optimality but is different from the perspective of approximations. Throughout this paper, we focus on the minimization variant of all correlation clustering problems.

Correlation clustering can be cast as an integer linear program (ILP) by optimizing objective function (1) subject to \(O(n^3)\) constraints of the form \(x_{ij} \leq x_{ik} + x_{jk}\) for all triplets \((i, j, k)\). This objective is NP-hard even for the simple unweighted case, but if the binary constraint \(x_{ij} \in \{0, 1\}\) is relaxed to linear constraints \(0 \leq x_{ij} \leq 1\), the result is the canonical linear programming relaxation of the problem, which can be solved in polynomial time:

\[
\begin{align*}
\min \quad & \sum_{i<j} w_{ij}^+ x_{ij} + w_{ij}^- (1 - x_{ij}) \\
\text{such that} \quad & x_{jk} + x_{ik} \geq x_{ij} \text{ for } i,j,k \\
& 0 \leq x_{ij} \leq 1.
\end{align*}
\]

Many approximation algorithms rely on solving and rounding this LP. An \(O(\log n)\) approximation can be obtained for the general case using LP-rounding [9, 14], and improved results exist for various other special weighted variants that lie somewhere in between unweighted and general weighted instances [1, 2, 23, 31, 41, 43, 45].

**Cluster editing** Minimizing disagreements in a complete unweighted signed graph [4] is the widely-studied special case of objective (1) where \((w_{ij}^+, w_{ij}^-) \in \{(0, 1), (1, 0)\}\) for each node pair \((i, j)\). This is equivalent to a problem called *cluster editing*: given a graph \(G = (V, E)\), find the minimum number of edges to add or remove in order to convert \(G\) into a disjoint union of cliques [6, 34]. Alternatively, this means clustering \(G\) in a way that minimizes the number of *mistakes*—the number of edges that cross between clusters plus the number of non-adjacent node pairs inside clusters.

Bansal, Blum, and Chawla [4] gave the first constant factor approximation for this problem, though it involved a very large constant factor. Charikar, Guruswami, and Wirth [9] later showed how to improve the approximation to 4 using the canonical LP relaxation. Ailon, Charikar, and Newman [2] improved the rounding technique to produce a 2.5 approximation. The current best approximation factor is 2.06 [10], which also uses the canonical LP. This result is nearly tight, as the LP relaxation has an integrality gap of 2 [9].

When proving new results for complete unweighted correlation clustering, we will focus on the *cluster editing* view of this problem—minimize non-edges inside clusters and edges between clusters. This simplifies our exposition in several places and best highlights the relationship to *cluster deletion* and other edge labeling problems we will consider.

**Cluster deletion** Cluster deletion [34] seeks to convert a graph \(G = (V, E)\) into a disjoint union of cliques by deleting the smallest number of edges. This can be viewed as an instance of correlation clustering (1) where \((w_{ij}^+, w_{ij}^-) = (1, 0)\) for adjacent nodes in \(G\), and \((w_{ij}^+, w_{ij}^-) = (0, \infty)\) for non-adjacent nodes. In this special case, the problem permits a canonical linear programming relaxation with fewer constraints than the standard correlation clustering LP, since we have \(x_{ij} = 1\) for node \((i, j) \notin E\). The updated LP is given by

\[
\begin{align*}
\min \quad & \sum_{(i,j) \in E} x_{ij} \\
\text{such that} \quad & x_{jk} + x_{ik} \geq 1 \quad \text{if } (i, k) \in E, (j, k) \in E, \text{ and } (i, j) \notin E \\
& x_{jk} + x_{ik} \geq x_{ij} \quad \text{if } (i, j, k) \text{ forms a triangle in } G \\
& x_{ik} + x_{ij} \geq x_{jk} \\
& 0 \leq x_{ij} \leq 1.
\end{align*}
\]
Although this LP has fewer constraints, it can still contain $O(n^3)$ of them. Charikar, Guruswami, and Wirth [9] provided the first constant factor approximation, which had an approximation factor of 4. The results of van Zuylen and Williamson [41] for constrained correlation clustering implied an improved approximation factor of 3. The best approximation algorithm for cluster deletion is the 2-approximation given by Veldt, Gleich, and Wirth [43]. All of these and other constant factor approximations [15, 31] for cluster deletion rely on solving the canonical LP relaxation (3).

The integrality gap for the LP is 2, which is realized by applying it to a star graph, so the 2-approximation is tight. Given that improved approximation factors using the canonical relaxation are impossible, a natural open question is whether we can obtain constant factor approximations using lower bounds that are easier to compute. We answer that question affirmatively in this paper by providing a combinatorial 4-approximation algorithm for cluster deletion.

### 2.2 Algorithmic Primitives for Correlation Clustering

We review key primitives for obtaining approximation algorithms for correlation clustering that are based on pivoting techniques and open wedge packings.

**Open wedge packings** Given a graph $G = (V,E)$, an open wedge is defined to be a triplet of nodes that include exactly two edges. Formally, a triplet of nodes $(i, j, k)$ is an open wedge centered at $j$ if $(i, j) \in E$ and $(j, k) \in E$ but $(i, k) \not\in E$. Let $W$ denote the set of node triplets that define open wedges in $G$, and $W_k \subseteq W$ denote the subset of open wedges centered at $k$.

In cluster editing, the presence or absence of an edge between nodes $i$ and $j$ is interpreted as a preference for being clustered together or apart. Observe that any way of clustering the three nodes in an open wedge will result in a “disagreement,” i.e., a violation of at least one of these preferences. Thus, the size of any pair-disjoint set of open wedges provides a lower bound on the optimal cluster editing solution. Here, a pair-disjoint set is a set of wedges $W' \subseteq W$ such that each pair of distinct nodes $(i, j) \in V \times V$ shows up in at most one wedge in $W'$. This type of lower bound was used to develop the first constant factor approximation algorithm for complete unweighted correlation clustering [4]. For cluster deletion, it suffices to find an edge-disjoint set of open wedges to lower bound the optimal solution. This is because even if two wedges share a pair of non-adjacent nodes $(i, j)$, we must delete at least one edge in each of the wedges.

**Pivoting procedures** Many algorithms for correlation clustering rely on some form of a pivoting procedure that recursively selects a pivot node and clusters it with its positive neighbors in a signed graph. If applied to an unsigned graph $G = (V,E)$, this simply means clustering a pivot node with its neighbors in $E$. If nodes are selected uniformly at random, this produces a 3-approximation in expectation for cluster editing [2]. The proof of the approximation guarantee relies on a careful argument about an implicit packing of open wedges, though no explicit open wedge packing lower bound is computed in practice. This procedure does not apply to cluster deletion, since clustering a node with its neighbors is not guaranteed to produce a clique.

Applying Pivot directly to a weighted graph typically yields poor results, but this procedure can be successfully used as a step in more sophisticated algorithms for correlation clustering. One technique is to solve a linear programming relaxation for a weighted instance $(V, W^+, W^-)$, and then use the LP output to derive a new unweighted complete graph on the same edge set. When done carefully, applying Pivot to the derived graph can yield approximation guarantees for different variants of correlation clustering [41, 43, 45]. This strategy was formalized in the work of van

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1This has also been referred to as a bad triplet or bad triangle in the context of correlation clustering problems.
Algorithm 1 DeterministicPivot(V, W^+, W^-, {b_{ij}}, \hat{E})

Input: Correlation clustering instance (V, W^+, W^-), budgets \{b_{ij}\}, derived graph \hat{G} = (V, \hat{E})
Output: Clustering \mathcal{C} = \text{DeterministicPivot}(V, W^+, W^-, \{b_{ij}\}, \hat{E})

for \( k \in V \) do
\[ T^+_k = \{(i, j) \in \hat{E} : (j, k) \notin \hat{E}, (i, k) \in \hat{E}\} \]
\[ T^-_k = \{(i, j) \notin \hat{E} : (j, k) \in \hat{E}, (i, k) \in \hat{E}\} \]
\[ P_k = \frac{\sum_{(i, j) \in T^+_k} w^+_{ij} + \sum_{(i, j) \in T^-_k} w^-_{ij}}{\sum_{(i, j) \in T^+_k \cup T^-_k} b_{ij}} \]
end for
\[ p = \text{argmin}_{k \in V} P_k \] // select pivot
\[ S = \{v \in V : (p, v) \in \hat{E}\} \] // form cluster
\[ T = V \setminus S \] // nodes that remain unclustered
\[ W^+_T = \{w^+_{ij} : i \in T, j \in T\} \] // remaining weights, edges, and budgets
\[ W^-_T = \{w^-_{ij} : i \in T, j \in T\} \]
\[ \hat{E}_T = \{(i, j) \in \hat{E} : i \in T, j \in T\} \]
\[ \mathcal{B}_T = \{b_{ij} : i \in T, j \in T\} \]
15: Return clustering \( \mathcal{C} = \{S, \text{DeterministicPivot}(T, W^+_T, W^-_T, \mathcal{B}_T, \hat{E}_T)\} \)

Zuylen and Williamson [41]. We extract a key algorithmic strategy for deterministic pivoting algorithms from the work of these authors, and present it as Algorithm 1. This method takes in a weighted instance of correlation clustering \((V, W^+, W^-)\), a set of “budgets” \(\{b_{ij}\}_{ij \in V \times V}\), and applies a deterministic pivoting procedure to a derived graph \(\hat{G} = (V, \hat{E})\), with pivot choices guided by the problem weights and budgets. The success of this algorithm depends entirely on the choice of budgets and how the derived graph is constructed. The following theorem provides a set of conditions which, when satisfied, provide a useful bound on the objective score of Algorithm 1. This can be used as a building block in designing approximation algorithms for different variants of correlation clustering.

**Theorem 2.1.** (Theorem 3.1 in [41].) Let \((V, W^+, W^-)\) define a weighted instance of correlation clustering (1), and \(b_{ij}\) define the budget for node pair \((i, j) \in V \times V\). Assume that for some \(\alpha > 0\), there is a graph \(\hat{G} = (V, \hat{E})\) satisfying the following two properties:

1. For all \((i, j) \in \hat{E}\), we have \(w^-_{ij} \leq \alpha b_{ij}\), and for all \((i, j) \notin \hat{E}\), we have \(w^+_{ij} \leq \alpha b_{ij}\).

2. If \((i, j, k)\) is an open wedge centered at \(j\) in \(\hat{G}\), we have \(w^+_{ij} + w^+_{jk} + w^-_{ik} \leq \alpha (b_{ij} + b_{jk} + b_{ik})\).

Then \(\text{DeterministicPivot}(V, W^+, W^-, \{b_{ij}\}, \hat{E})\) will return a clustering with weight of disagreements bounded above by \(\alpha \sum_{i < j} b_{ij}\), and applying \(\text{Pivot}\) to \(\hat{G} = (V, \hat{E})\) with uniform random pivots will return produce a clustering with expected weight of disagreements bounded by \(\alpha \sum_{i < j} b_{ij}\).

In Algorithm 1, \(T^+_k\) denotes the set of edges in the derived graph that are cut, and \(T^-_k\) the set of non-edges that end up inside a cluster, when we pivot on a node \(k\). The proof of the bound in Theorem 2.1 relies on charging these types of “mistakes” to open wedges in the derived graph, which can in turn be charged to the budgets \(b_{ij}\) using property 2 in the theorem. By setting the budgets \(b_{ij}\) to be the contribution of a node pair \((i, j)\) to the LP relaxation (2), i.e., \(b_{ij} = w^+_{ij} x_{ij} + w^-_{ij} (1 - x_{ij})\), van Zuylen and Williamson showed how to obtain a derandomized 3-approximate \(\text{Pivot}\) method for complete unweighted correlation clustering. However, the bottleneck of this approach is solving the LP relaxation.
3 Strong Triadic Closure Preliminaries

Strong triadic closure [16,19] is the sociological principle that two people are likely to share at least a weak connection if they both share strong connections to a mutual friend. This is used as a guiding principle for social network analysis, and is the foundation for certain edge labeling problems [20,25,36].

3.1 Strong Triadic Closure Labeling Objectives

If every edge in $E$ is labeled as either weak or as strong, we say this is a strong triadic closure labeling if at least one of the edges in each open wedge is weak. The rationale is that if both edges in a wedge are strong ties, we would expect the wedge to be closed because of strong triadic closure. The minimum weakness strong triadic closure (MinSTC) problem [36] seeks a strong triadic closure labeling with the minimum number of weak edges. We can cast this problem as a binary linear program:

$$\begin{align*}
\min & \quad \sum_{(i,j) \in E} z_{ij} \\
\text{s.t.} & \quad z_{jk} + z_{ik} \geq 1 \quad \text{if } (i,j,k) \in W_k \\
& \quad z_{ij} \in \{0,1\} \quad \text{for all } (i,j) \in E.
\end{align*}$$

If we set $z_{uv} = 1$, this indicates that $(u,v) \in E$ is labeled as a weak connection.

A variation of this objective called MinSTC+ additionally allows one to satisfy strong triadic close by adding weak edges between non-adjacent nodes. This is equivalent to viewing certain non-edges as weak connections that were simply not observed in the network. The binary linear program formulation for this problem is:

$$\begin{align*}
\min & \quad \sum_{i<j} z_{ij} \\
\text{s.t.} & \quad z_{jk} + z_{ik} + z_{ij} \geq 1 \quad \text{if } (i,j,k) \in W \\
& \quad z_{ij} \in \{0,1\} \quad \text{for all } (i,j) \in V \times V.
\end{align*}$$

If $(i,j) \in E$ and $z_{ij} = 1$, this again corresponds to labeling the edge as weak. If $(i,j) \notin E$ and $z_{ij} = 1$, this means we add a new edge between $i$ and $j$. A feasible solution to MinSTC+ is therefore a set of new edges $E'$ and a subset of edges $E_W \subseteq E$ that we will label as weak. The goal is to minimize $|E'| + |E_W|$. We assume all edges $E'$ are weak, to ensure we do not introduce new open wedges that violate strong triadic closure. We refer to a feasible solution $(E', E_L)$ for MinSTC+ as an STC+ labeling.

Sintos and Tsasparas [36] introduced both MinSTC and MinSTC+, and proved that they are both NP-hard. These authors provided a 2-approximation for MinSTC and an $O(\log |V|)$ approximation for MinSTC+ by reducing them to vertex cover problems in an auxiliary graph and hypergraph respectively. Next we review these reductions, as we build on them in our work.

3.2 Reductions for Strong Triadic Closure Labeling

The 2-approximation for MinSTC is obtained by approximately solving vertex cover in the Gallai graph [27] of $G = (V,E)$. The Gallai graph $\mathcal{G}$ is defined by introducing a node $v_{ij}$ for each edge $(i,j) \in E$, and placing an edge between two nodes $v_{jk}$ and $v_{ik}$ if $(i,j,k)$ is an open wedge centered at $k$ in $G$. The edges in $\mathcal{G}$ are therefore in one-to-one correspondence the open wedges of $G$. If we solve the vertex cover problem in $\mathcal{G}$, placing a node $v_{ij}$ in the cover can be viewed as labeling the edge $(i,j) \in E$ as weak. Since all edges in $\mathcal{G}$ are therefore at least one node in any vertex cover, this means that all open wedges in $G$ will have at least on weak edge. Applying the 2-approximation for vertex cover [42] yields the same approximation for MinSTC.
Objective (5) can instead be viewed as a vertex cover problem in a three-uniform hypergraph \( \mathcal{H} = (V_{\mathcal{H}}, E_{\mathcal{H}}) \). We define a node \( v_{ij} \in V_E \) for every pair of distinct nodes \((i, j) \in V \times V\) in the original graph \( G = (V, E) \), and we introduce a hyperedge \( w_{ijk} = \{v_{ij}, v_{ik}, v_{jk}\} \in E_{\mathcal{H}} \) whenever \((i, j, k)\) is an open wedge in \( G \). We will refer to \( \mathcal{H} \) as the open wedge hypergraph. Applying a greedy algorithm for hitting set yields an \( O(\log |V|) \) approximation guarantee [36]; a standard matching-based algorithm for vertex cover in 3-uniform hypergraphs produces a 3-approximation [21].

3.3 Connections to Correlation Clustering Objectives

The relationship between \( \text{minSTC} \) and cluster deletion can be readily seen by noting that the binary linear program (4) becomes the binary linear program for cluster deletion if we add the constraints \( z_{ij} + z_{jk} \geq z_{jk} \) for all permutations of nodes \( i, j, k \) when \((i, j, k)\) is a triangle in \( G \) (see constraint set in (3)). In other words, any feasible solution for cluster deletion will produce a feasible solution for \( \text{minSTC} \) by labeling all the deleted edges as weak. Similarly, the constraints in the binary program for \( \text{minSTC}+ \) (5) can be seen as a subset of the triangle inequality constraints in the binary program for cluster editing, once a change of variables is applied.

The observation that \( \text{minSTC} \) lower bounds cluster deletion, and \( \text{minSTC}+ \) lower bounds cluster editing, has already been noted in previous work [20, 21, 25]. The optimal solutions to cluster deletion and \( \text{minSTC} \) are known to coincide for co-graphs [25], though there exist concrete examples to confirm that in general they are not the same problem [20]. It is unknown if \( \text{minSTC}+ \) and cluster editing always coincide, or if there exist examples where they differ, despite some recent effort to answer this question [21]. Our work will expand on the relationship between strong triadic closure labeling problems and correlation clustering by proving upper bounds on the maximum difference between each labeling objective and its corresponding clustering problem.

4 Approximation Algorithms for Cluster Editing

We now present several different lower bounds for cluster editing (i.e., complete unweighted correlation clustering) and corresponding new approximation algorithms.

4.1 Rounding an STC Relaxation for Cluster Editing

We begin by developing a new approximation algorithm for cluster editing by rounding the LP relaxation of \( \text{minSTC}+ \) (5), obtained by replacing the constraint \( z_{ij} \in \{0, 1\} \) with linear constraints \( 0 \leq z_{ij} \leq 1 \). We first apply a convenient change of variables:

\[
x_{ij} = \begin{cases} 
  z_{ij} & \text{if } (i, j) \in E \\
  1 - z_{ij} & \text{if } (i, j) \notin E.
\end{cases}
\]

This leads to a linear program that can readily be seen as a relaxation of the canonical LP (2):

\[
\min \sum_{(i,j) \in E} x_{ij} + \sum_{(i,j) \notin E} (1 - x_{ij}) \\
\text{s.t.} \quad x_{ij} \leq x_{jk} + x_{ik} \quad \text{if } (i, j, k) \in W_k \\
\quad 0 \leq x_{ij} \leq 1 \quad \text{for all } (i, j) \in V \times V.
\]

(6)

The number of constraints in this reduced LP is bounded above by \( O(\sum_{v \in V} d_v^2) \), where \( d_v \) is the degree of node \( v \in V \). This is seen by summing the maximum possible number of open wedges centered at each node. This bound will typically be much smaller than \( O(n^3) \), the number of
Algorithm 2 Rounding the minSTC+ LP relaxation.

**Input:** Cluster editing instance \( G = (V, E) \)

**Output:** Clustering \( \hat{C} \) of \( G \).

Solve LP-relaxation of (6)

Set \( \hat{E} \leftarrow \{(i, j) \in V \times V : x_{ij} < 1/2\} \)

5: For \( (i, j) \in E \), \((w_{ij}^+, w_{ij}^-) = (1, 0), \) and \( b_{ij} = x_{ij} \)

For \( (i, j) \notin E \), \((w_{ij}^+, w_{ij}^-) = (0, 1), \) and \( b_{ij} = 1 - x_{ij} \)

Return DeterministicPivot\((V, \{w_{ij}^+\}, \{w_{ij}^-\}, \{b_{ij}\}, \hat{E})\)

Constraints in the canonical relaxation. For constant degree graphs, the number of constraints will be \( O(n) \). In general, we have an upper bound of \( O(mn) \), where \( m = |E| \). This linear program and its dual were previously considered implicitly in the proof of the expected approximation guarantee for the standard Pivot algorithm \([2]\). We show that by solving this linear program explicitly and carefully rounding the output, we can obtain a deterministic approximation algorithm that avoid the expensive canonical LP relaxation. Note that solving the LP also provides the benefit of being able to compare the output of the algorithm (or any algorithm) against a specific lower bound to obtain improved a posteriori approximation guarantees. Pseudocode for our method is given in Algorithm 2. To prove its approximation guarantee, we show how to construct a graph from the output of LP (6) that satisfies the conditions of Theorem 2.1 with \( \alpha = 4 \).

**Theorem 4.1.** Algorithm 2 is a deterministic 4-approximation algorithm for cluster editing.

**Proof.** To prove the result we must check that Theorem 2.1 is satisfied with \( \alpha = 4 \). For cluster editing, the weights are given by

\[
(w_{ij}^+, w_{ij}^-) = \begin{cases} 
(1, 0) & \text{if } (i, j) \in E \\
(0, 1) & \text{if } (i, j) \notin E,
\end{cases}
\]  

(7)

and the budgets defined by the LP relaxation are given by

\[
b_{ij} = \begin{cases} 
x_{ij} & \text{if } (i, j) \in E \\
1 - x_{ij} & \text{if } (i, j) \notin E.
\end{cases}
\]  

(8)

Considering the way \( \hat{G} \) is constructed in Algorithm 2, the conditions in Theorem 2.1 translate to the following:

1. If \( x_{ij} < 1/2 \), we have \( w_{ij}^- \leq 4b_{ij} \), and if \( x_{ij} \geq 1/2 \), then \( w_{ij}^+ \leq 4b_{ij} \).

2. If \( x_{ij} < 1/2 \) and \( x_{jk} < 1/2 \) but \( x_{ik} \geq 1/2 \), then

\[
w_{ij}^+ + w_{jk}^+ + w_{ik}^- \leq 4(b_{ij} + b_{jk} + b_{ik}).
\]  

(9)

Condition 1 is straightforward to check by considering the definitions of edge weights (7) and budgets (8). We can prove the second condition by case analysis, considering separately whether each node pair \((i, j), (i, k), \) and \((j, k)\) is an edge or not in original graph \( G = (V, E) \). Regardless of the case, we have the following bounds, based on the assumption that \((i, j, k)\) is an open wedge centered at \( j \) in \( \hat{G} \):

\[
1 - x_{ij} > 1/2, \quad 1 - x_{jk} > 1/2, \quad x_{ik} \geq 1/2.
\]  

(10)
We summarize all of the cases in succinct tabular format, where we state whether each edge is in $E$ or not, and then give lower bounds on the right hand side of inequality (9) to show it is greater than the left hand side in each case. We have ordered cases so that moving from one row to the next changes the edge status of only one node pair at a time, making it easy to quickly see changes in the left and right hand sides of the inequality (9) for the corresponding budgets and weights. Several of the bounds we list for the right hand side of (9) could be tightened further, but this would not lead to an improved overall approximation guarantee.

| Is the edge in $E$? | Right side of (9) | Left side of (9) | Explanation | Note |
|---------------------|-------------------|------------------|-------------|------|
| $(i, j)$ | $(j, k)$ | $(i, k)$ | $4(b_{ij} + b_{jk} + b_{ik})$ | $w^+_{ij} + w^+_{jk} + w^-_{ik}$ |
| Y | Y | Y | $4(x_{ij} + x_{jk} + x_{ik}) \geq 4x_{ik} \geq 2$ | $2 = 1 + 1 + 0$ | $x_{ik} \geq 1/2$ |
| Y | Y | N | $4(x_{ij} + x_{jk} + 1 - x_{ik}) \geq 4$ | $3 = 1 + 1 + 1$ | $x_{ij} + x_{jk} - x_{ik} \geq 0$ (LP constraint) |
| Y | N | N | $4(x_{ij} + 1 - x_{jk} + 1 - x_{ik}) > 2$ | $2 = 1 + 0 + 1$ | $1 - x_{jk} > 1/2$ |
| Y | N | Y | $4(x_{ij} + 1 - x_{jk} + x_{ik}) \geq 2$ | $1 = 1 + 0 + 0$ | $x_{ik} \geq 1/2$ |
| N | N | Y | $4(1 - x_{ij} + 1 - x_{jk} + x_{ik}) \geq 0$ | $0 = 0 + 0 + 0$ | zero left side |
| N | Y | Y | $4(1 - x_{ij} + x_{jk} + x_{ik}) \geq 2$ | $1 = 0 + 1 + 0$ | $x_{ik} \geq 1/2$ |
| N | Y | N | $4(1 - x_{ij} + x_{jk} + 1 - x_{ik}) > 2$ | $2 = 0 + 1 + 1$ | $1 - x_{ij} > 1/2$ |
| N | N | N | $4(1 - x_{ij} + 1 - x_{jk} + 1 - x_{ik}) > 2$ | $1 = 0 + 0 + 1$ | $1 - x_{ij} > 1/2$ |

The approximation guarantee of Algorithm 2 matches the guarantee of the algorithm given by Charikar, Guruswami, and Wirth [9], which requires solving the full canonical relaxation of cluster editing. The best approximation factor that uses the canonical LP relaxation still has a better guarantee of 2.06 [10], but at a significant increase in computational cost. Our algorithm represents a useful trade-off between runtime and approximation guarantee.

4.2 Rounding Feasible STC Solutions for Cluster Editing

Next we explore bounds for cluster editing that can be obtained by rounding approximate feasible solutions to minSTC+, instead of rounding fractional lower bounds. We start with a generic algorithm (Algorithm 3) that rounds a feasible solution ($E', E_W$) for objective (5) (i.e., an STC+ labeling) into a feasible solution for cluster editing, with provable guarantees. The first step of this algorithm is to flip edges in the original graph $G = (V, E)$, meaning that we convert some non-adjacent node pairs into edges $E'$, and we delete edges $E_W$ that were previously in $E$. We then run a deterministic procedure on the new graph, and prove that the number of mistakes made can be bounded in terms of the number of flipped edges.

**Theorem 4.2.** Algorithm 3 returns a cluster editing solution with at most $2(|E'| + |E_W|)$ mistakes.

**Proof.** The sum of budgets is exactly $\sum_{i<j} b_{ij} = |E'| + |E_W|$, so the result holds if we can prove that the conditions of Theorem 2.1 are satisfied with $\alpha = 2$. We first need to check that

$$
(i, j) \in \hat{E} \implies w^-_{ij} \leq 2b_{ij} \quad (11)
$$

and

$$
(i, j) \notin \hat{E} \implies w^+_{ij} \leq 2b_{ij} \quad (12)
$$
Algorithm 3 \textsc{FlipPivot}(G, E', E_W)

\begin{itemize}
  \item \textbf{Input:} Graph \(G = (V, E)\) and STC+ label set \((E', E_W)\)
  \item \textbf{Output:} Clustering \(C\) of \(G\).
  \begin{itemize}
    \item Construct graph \(\hat{G} = (V, \hat{E})\) where \(\hat{E} = E' \cup (E - E_W)\)
  \end{itemize}
  \begin{itemize}
    \item Set budgets:
      \[ b_{ij} = \begin{cases} 
        1 & \text{if } (i, j) \in E_W \cup E' \\
        0 & \text{otherwise} 
      \end{cases} \]
  \end{itemize}
  \begin{itemize}
    \item 5: Set weights
      \[ (w_{ij}^+, w_{ij}^-) = \begin{cases} 
        (1, 0) & \text{if } (i, j) \in E \\
        (0, 1) & \text{if } (i, j) \notin E 
      \end{cases} \]
  \end{itemize}
  \begin{itemize}
    \item Return \(C = \textsc{DeterministicPivot}(V, \{w_{ij}^+\}, \{w_{ij}^-\}, \{b_{ij}\}, \hat{E})\)
  \end{itemize}
\end{itemize}

\textbf{Checking (11):} If \((i, j) \in \hat{E} \cap E\) then \(w_{ij}^- = 0 = b_{ij}\), and if \((i, j) \in \hat{E}\) but \((i, j) \notin E\), then \(b_{ij} = w_{ij}^- = 1\) since \((i, j)\) is a non-edge \((w_{ij}^- = 1)\) that was flipped \((b_{ij} = 1)\).

\textbf{Checking (12):} If \((i, j) \notin \hat{E}\) and \((i, j) \notin E\) then we have \(w_{ij}^+ = b_{ij} = 0\). If \((i, j) \notin \hat{E}\) and \((i, j) \in E\), then \(w_{ij}^- = 1 = b_{ij}\).

Next we confirm that if \((i, j, k)\) is an open wedge centered at \(j\) in \(\hat{G} = (V, \hat{E})\), then
\[ w_{ij}^+ + w_{jk}^+ + w_{ik}^- \leq 2(b_{ij} + b_{jk} + b_{ik}). \] (13)

Regardless of the edge structure of \((i, j, k)\) in the original graph \(G = (V, E)\), we must have
\[ b_{ij} + b_{jk} + b_{ik} + w_{ij}^+ + w_{jk}^+ + w_{ik}^- = 3. \] (14)

To see why, observe first of all that \((i, j) \in \hat{E}\), \((j, k) \in \hat{E}\), and \((i, k) \notin \hat{E}\), by our assumption that \((i, j, k)\) is an open wedge centered at \(j\) in \(G\). Consider node pair \((i, j)\): either this pair is an edge \((i, j) \in E\) (meaning \(w_{ij}^+ = 1\)) or it was flipped (meaning \(b_{ij} = 1\)) but not both. Therefore, \(b_{ij} + w_{ij}^- = 1\), and by the same argument we can show \(b_{jk} + w_{jk}^+ = b_{ik} + w_{ik}^- = 1\). This yields (14).

A key step in the proof is to realize that
\[ b_{ij} + b_{jk} + b_{ik} \geq 1. \] (15)

If instead we assume \(b_{ij} + b_{jk} + b_{ik} = 0\), this means that none of the edges were flipped, so \((i, j, k)\) is also an open wedge in the original graph \(G = (V, E)\). This contradicts the fact that \((E', E_W)\) is a strong triadic closure labeling. A strong triadic closure labeling would either add \((i, k)\) to the new edge set \(E'\), or label one of the edges as weak, which would subsequently lead to one node pair being flipped. Combining (15) and (14), we can see that
\[ w_{ij}^+ + w_{jk}^+ + w_{ik}^- \leq 2 = 2(1) \leq 2(b_{ij} + b_{jk} + b_{ik}). \]

As a corollary, we note the following strong relationship between \textsc{minSTC}+ and cluster editing.

\textbf{Corollary 4.3.} If \(A\) is an \(\alpha\)-approximation algorithm for \textsc{minSTC}+, running Algorithm 3 with an STC+ labeling \((E', E_W)\) returned by \(A\) will return a \(2\alpha\)-approximation for cluster editing. If \(\text{OPT}^+\) and \(\text{OPT}^{CE}\) are optimal solutions to STC+ and cluster editing, then
\[ \text{OPT}^+ \leq \text{OPT}^{CE} \leq 2\text{OPT}^+. \] (16)
Algorithm 4 MatchFlipPivot(G)

**Input:** Graph G = (V, E)

**Output:** Clustering C of G.

**Reduce:** Build open wedge hypergraph \( \mathcal{H} = (V_H, E_H) \) (Section 3.2)

**Match:** Find maximal matching \( \mathcal{M} \subseteq E_H \)

5: **Cover:** Set \( C = \{ v_{ij} \in V_H : v_{ij} \in w \text{ for some hyperedge } w \in \mathcal{M} \} \)

**STC+ Labeling:**

\[
E' = \{(i, j) \notin E : v_{ij} \in C\} \\
E_W = \{(i, j) \in E : v_{ij} \in C\}
\]

Return \( C = \text{FlipPivot}(G, E', E_W) \)

---

**Proof.** We have previously established that \( OPT^+ \leq OPT^{CE} \). If \((E', E_W)\) is the \( \alpha \)-approximate STC+ labeling returned by \( \mathcal{A} \) and \( B = |E'| + |E_W| \), then

\[
B \leq \alpha OPT^+ \leq 2OPT^{CE} \implies \frac{B}{\alpha} \leq OPT^{CE},
\]

which provides a lower bound on the optimal cluster editing solution. Using Theorem 4.2, we can find a cluster editing solution that makes at most \( 2B \) mistakes, which is within \( 2\alpha \) of the lower bound. If we solve \( \min \text{STC+} \) optimally, this mean \( \alpha = 1 \), so we get the bound in (16).

We also observe in passing that an \( \alpha \)-approximation algorithm for vertex cover would imply a \( 2\alpha \)-approximation for cluster editing, since \( \min \text{STC+} \) can be reduced to vertex cover in an approximation preserving way. We can also use Theorem 4.2 to provide a new **deterministic** approximation algorithm for cluster editing.

**Corollary 4.4.** Algorithm 4 is a deterministic 6-approximation for cluster editing.

**Proof.** By construction, the minimum vertex cover in the 3-uniform hypergraph \( \mathcal{H} = (V_H, E_H) \) is equivalent to \( \min \text{STC+} \) on \( G = (V, E) \). The algorithm performs the standard steps to obtain a 3-approximation: find a maximal matching, and place all nodes from the matched edges in the vertex cover. This can be converted to an STC+ labeling that is a 3-approximation for \( \min \text{STC+} \), which can be fed to FlipPivot to produce a \( 2 \cdot 3 = 6 \) approximation for cluster editing.

Before moving on we discuss the significance of this theorem in the context of previous results. Better deterministic approximation results are possible by rounding the canonical LP relaxation for cluster editing [10]. Furthermore, a faster **randomized** 3-approximation algorithm is possible if we simply apply Pivot directly to \( G \). The significant feature of our approach is that, unlike these previous approaches, it is both deterministic and combinatorial, and provides the best approximation guarantee for any algorithm satisfying these two conditions. An added advantage of this approach over standard Pivot is that it produces an explicit lower on the optimal solution, which makes it possible to obtain improved a posteriori approximation guarantees in practice. Finally, we have shown a more general result in this section regarding the relationship between \( \min \text{STC+} \) and cluster editing. It is not known whether these two problems always coincide, but we have shown that they are always within a factor two of each other. Improved algorithms for \( \min \text{STC+} \) will directly imply new results for cluster editing.
Algorithm 5 Rounding the minSTC LP relaxation.

**Input:** Graph $G = (V, E)$
**Output:** Feasible cluster deletion clustering $C$ of $G$.

Solve LP (17)

Set $\hat{E} \leftarrow \{(i, j) \in V \times V : z_{ij} < 1/2\}$

5: For $(i, j) \in E$, $(w_{ij}^+, w_{ij}^-) = (1, 0)$, and $b_{ij} = z_{ij}$

For $(i, j) \notin E$, $(w_{ij}^+, w_{ij}^-) = (0, \infty)$ and $b_{ij} = 0$

Return DETERMINISTICPivot($V, \{w_{ij}^+, w_{ij}^-, b_{ij}\}, \hat{E}$)

## 5 Approximation Algorithms for Cluster Deletion

In this section we prove results for cluster deletion that are analogous to our results for cluster editing in the previous section.

### 5.1 Rounding an STC Relaxation for Cluster Deletion

Analogous to our results for cluster editing, we show how to round the following LP relaxation for minSTC to produce a 4-approximation for cluster deletion:

$$\min \sum_{(i,j) \in E} z_{ij}$$

s.t. $z_{jk} + z_{ik} \geq 1$ if $(i, j, k) \in W_k$

$1 \geq z_{ij} \geq 0$ for all $(i, j) \in E.$

(Pseudocode is given in Algorithm 5.)

**Theorem 5.1.** Algorithm 5 is a deterministic 4-approximation algorithm for cluster deletion.

**Proof.** We must check that Theorem 2.1 is satisfied with $\alpha = 4$. First of all, note that applying PIVOT to the derived graph $\hat{G} = (V, \hat{E})$, using any order of pivot choices, will produce a feasible instance for cluster deletion. To see why, observe that if $k$ is the pivot node and $i$ and $j$ are two of its neighbors in $\hat{G}$, then $z_{ki} < 1/2$ and $z_{kj} < 1/2$, which implies that $(i, j) \in E$. If $(i, j)$ were not an edge, then $(i, j, k)$ would be an open wedge and the LP relaxation would include the constraint $z_{ki} + z_{kj} \geq 1$. The conditions from Theorem 2.1 that we must check are:

1. If $(i, j) \in \hat{E}$, we have $w_{ij}^- \leq 4b_{ij}$, and if $(i, j) \notin \hat{E}$, then $w_{ij}^+ \leq 4b_{ij}$.

2. If $(i, j) \in \hat{E}$ and $(j, k) \in \hat{E}$ and $(i, k) \notin \hat{E}$, then

$$w_{ij}^+ + w_{jk}^+ + w_{ik}^- \leq 4(b_{ij} + b_{jk} + b_{ik}).$$

(18)

**Checking condition 1.** Observe that $(i, j) \in \hat{E} \implies (i, j) \in E \implies w_{ij}^- = 0 \leq 4b_{ij}$. Similarly, if $(i, j) \notin \hat{E}$ and $(i, j) \notin E$, then $w_{ij}^+ = b_{ij} = 0$. If $(i, j) \notin \hat{E}$ but $(i, j) \in E$, then $z_{ij} = b_{ij} \geq 1/2$ and so $w_{ij}^+ = 1 < 4b_{ij}$.

**Checking condition 2.** For condition 2, note that $(i, j) \in \hat{E} \subseteq E$ and $(j, k) \in \hat{E} \subseteq E$ imply that $z_{ij} + z_{jk} < 1$ and therefore $(i, j, k)$ is not an open wedge in $G$ and so $(i, k) \in E$. Since $(i, k) \notin \hat{E}$, we know $b_{ik} = z_{ik} \geq 1/2$. Overall, we have that

$$w_{ij}^+ + w_{jk}^+ + w_{ik}^- = 2 = 4 \cdot \frac{1}{2} \leq 4b_{ik} \leq 4(b_{ij} + b_{jk} + b_{ik}).$$

$\square$
Algorithm 6 FlipPivotCD(G, EW)

**Input:** Graph G = (V, E) and STC label set EW

**Output:** Feasible cluster deletion clustering C of G.

Construct graph \( \hat{G} = (V, \hat{E}) \) where \( \hat{E} = (E - EW) \)

Set budgets:

\[
b_{ij} = \begin{cases} 
1 & \text{if } (i, j) \in EW \\
0 & \text{otherwise}
\end{cases}
\]

5: Set weights

\[
(w_{ij}^+, w_{ij}^-) = \begin{cases} 
(1, 0) & \text{if } (i, j) \in E \\
(0, \infty) & \text{if } (i, j) \notin E
\end{cases}
\]

Return \( C = \text{DeterministicPivot}(V, \{w_{ij}^+\}, \{w_{ij}^-\}, \{b_{ij}\}, \hat{E}) \)

Charikar, Guruswami, and Wirth [9] showed that the canonical LP relaxations of cluster editing and cluster deletion can be rounded to produce 4-approximation algorithms for both problems. Combining the results of Theorem 4.1 and Theorem 5.1, we see that the same approximation guarantee is possible when rounding a relaxation with fewer constraints. Instead of having three constraints for every triplet of nodes in a graph, we only need to include one constraint for every open wedge.

### 5.2 Rounding Feasible STC Solutions for Cluster Deletion

We now consider how to round an approximate feasible solution to minSTC in order to produce approximate solutions to cluster deletion. Recall that a feasible solution to minSTC for a graph \( G = (V, E) \) is a set of edges \( EW \) which should be labeled weak to ensure all open wedges in \( G \) have at least one weak edge. Algorithm 6 provides a strategy for rounding any STC labeling \( EW \) into a feasible cluster deletion solution with a bound on the number of mistakes.

**Theorem 5.2.** Algorithm 6 returns a cluster deletion solution with at most \( 2|EW| \) mistakes.

**Proof.** We must first confirm that this approach produces a feasible solution to cluster deletion, meaning that all clusters returned are cliques in the original graph \( G = (V, E) \). Consider pivoting on any node \( j \) in the derived graph \( \hat{G} = (V, \hat{E}) \). If \( (j, k) \in \hat{E} \) and \( (i, j) \in \hat{E} \), this means neither of these edges were labeled weak, and so we must have \( (i,k) \in E \) or else strong triadic closure would be violated. Thus, pivoting on any node produces cliques.

The sum of budgets in Algorithm 6 is \( \sum_{i<j} b_{ij} = |EW| \). The result will follow if we can confirm that the conditions in Theorem 2.1 hold with \( \alpha = 2 \):

1. For all \( (i, j) \in \hat{E} \), we have \( w_{ij}^+ \leq 2b_{ij} \), and for all \( (i, j) \notin \hat{E} \), we have \( w_{ij}^+ \leq 2b_{ij} \).

2. If \( (i, j, k) \) is an open wedge centered at \( j \) in \( \hat{G} \), we have \( w_{ij}^+ + w_{jk}^+ + w_{ik}^- \leq 2(b_{ij} + b_{jk} + b_{ik}) \).

**Checking condition 1:** If \( (i, j) \in \hat{E} \), then \( (i, j) \in E \), so \( w_{ij}^- = 0 \leq 2b_{ij} \). If \( (i, j) \notin \hat{E} \) and \( (i, j) \notin E \), then have \( w_{ij}^- = 0 \leq 2b_{ij} \). If \( (i, j) \notin \hat{E} \) but \( (i, j) \in E \), then \( (i, j) \in EW \) and so \( b_{ij} = 1 \), and thus \( w_{ij}^+ = 1 = b_{ij} \).

**Checking condition 2:** If \( (i, j) \in \hat{E} \) and \( (j, k) \in \hat{E} \), then we must have \( (i, j) \in E \) or else there would be a violation of strong triadic closure. Since we are assuming in condition 2 that \( (i, j, k) \) is an
Algorithm 7 MatchFlipPivotCD(G)

**Input:** Graph \( G = (V, E) \)

**Output:** Feasible cluster deletion clustering of \( G \).

Reduce: Build Gallai graph \( \tilde{G} = (V_{\tilde{G}}, E_{\tilde{G}}) \) (Section 3.2)

Match: Find maximal matching \( M \subseteq E_{\tilde{G}} \)

5: **Cover:** Set \( C = \{v_{ij} \in V_{\tilde{G}}: v_{ij} \in w \text{ for some edge } w \in M\} \)

STC Labeling: \( E_W = \{(i, j) \in E: v_{ij} \in C\} \)

Return \( C = \text{MatchFlipPivotCD}(G, E_W) \)

open wedge centered at \( j \) in \( \tilde{G} \), the edge \((i, k) \in E_W\), and so \( b_{ik} = 1 \). Thus, we have

\[ w^+_{ij} + w^+_{jk} + w^-_{ik} = 2 = 2b_{ij}. \]

Therefore, the weight of mistakes (i.e., the number of deleted edges) resulting from running Algorithm 6 is \( \alpha \sum_{i<j} b_{ij} = 2|E_W| \). \( \square \)

We obtain the following corollary on the relationship between \text{minSTC} and cluster deletion. We omit the proof as it follows the same simple argument as (4.3).

**Corollary 5.3.** If \( \mathcal{A} \) is an \( \alpha \)-approximation algorithm for \text{minSTC}, running Algorithm 6 with an STC weak label set \( E_W \) returned by \( \mathcal{A} \) will return a \( 2\alpha \)-approximation for cluster deletion. If \( \text{OPT}^{\text{STC}} \) and \( \text{OPT}^{\text{CD}} \) are optimal solutions to \text{minSTC} and cluster deletion, then

\[ \text{OPT}^{\text{STC}} \leq \text{OPT}^{\text{CD}} \leq 2\text{OPT}^{\text{STC}}. \] (19)

Since there already exists a polynomial time 2-approximation for cluster deletion [43] and \text{minSTC} is NP-hard, we do not expect this corollary to produce a new best polynomial approximation algorithm. However, this result makes it possible to develop faster constant factor approximation algorithms for cluster deletion. In particular, we can easily obtain a 2-approximation for \text{minSTC} by finding a maximal matching in the Gallai graph of \( G = (V, E) \) and using it to get a 2-approximation for vertex cover in the Gallai graph. We can then use this in conjunction with Corollary 5.3 to obtain the first combinatorial approximation algorithm for cluster deletion. Pseudocode for this procedure is given in Algorithm 7, and we end with a summarizing corollary.

**Corollary 5.4.** Algorithm 7 provides a 4-approximation for cluster deletion.

If desired, we can also run a randomized pivot procedure on the reduced graph in Algorithm 7 which comes with an expected 4-approximation and is faster. Importantly, and unlike the case for cluster editing, we cannot simply run the standard Pivot procedure on a graph \( G = (V, E) \) when we are trying to solve cluster deletion, as it rarely even provides a feasible solution, let alone a solution with approximation guarantees.

6 Discussion and Open Questions

We have presented a number of new approximation algorithms for correlation clustering and cluster deletion by combining new connections to strong triadic closure [36] with extensions of deterministic pivoting techniques [41]. This opens up several interesting directions for future work. First of all, these results on alternative lower bounds for correlation clustering motivate further research
on understanding runtime and approximation tradeoffs in algorithms for cluster editing and cluster deletion. Is it possible to obtain further improved approximation guarantees with the lower bounds we have considered here? Alternatively, can we provide tightness results showing that our approaches for rounding different types of lower bounds are nearly tight?

In our work we have shown that optimal solutions to cluster deletion and \( \text{minSTC} \) are always within a factor of 2. Previous work has shown that there exist cases where their objectives differ by a factor of \( \frac{8}{7} \) [20]. Despite some semi-automated effort, we were unable to find examples with a larger ratio between optimal solutions. One question is whether we can tighten these bounds in either direction. A related open direction is to tighten the bounds on the difference between STC+ and cluster editing. Unlike the case for cluster deletion, we do not even known of any cases where these objectives differ. Finally, perhaps the most compelling direction for future work is to see whether our combinatorial MATCHFLIPPIVOT techniques can be used to obtain combinatorial approximation algorithms for other weighted variants of correlation clustering, whose only approximation algorithms currently rely on expensive LP relaxations [23, 31, 43].

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