Light Neutrinos without Heavy Mass Scales:  
A Higher-Dimensional Seesaw Mechanism

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Abstract

Recent theoretical developments have shown that extra spacetime dimensions can lower the fundamental GUT, Planck, and string scales. However, recent evidence for neutrino oscillations suggests the existence of light non-zero neutrino masses, which in turn suggests the need for a heavy mass scale via the seesaw mechanism. In this paper, we make several observations in this regard. First, we point out that allowing the right-handed neutrino to experience extra spacetime dimensions naturally permits the left-handed neutrino mass to be power-law suppressed relative to the masses of the other fermions. This occurs due to the power-law running of the neutrino Yukawa couplings, and therefore does not require a heavy scale for the right-handed neutrino. Second, we show that a higher-dimensional analogue of the seesaw mechanism may also be capable of generating naturally light neutrino masses without the introduction of a heavy mass scale. Third, we show that such a higher-dimensional seesaw mechanism may even be able to explain neutrino oscillations without neutrino masses, with oscillations induced indirectly via the masses of the Kaluza-Klein states. Fourth, we point out that even when higher-dimensional right-handed neutrinos are given a bare Majorana mass, the higher-dimensional seesaw mechanism surprisingly replaces this mass scale with the radius scale of the extra dimensions. Finally, we also discuss a possible new mechanism for inducing lepton-number violation by shifting the positions of D-branes in Type I string theory.

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1 Introduction

Recent theoretical developments have shown that extra spacetime dimensions have the potential to lower the fundamental GUT scale \([1]\), the fundamental Planck scale \([2]\), and the fundamental string scale \([3, 4, 5, 6]\). The extra dimensions that lower the GUT scale are “universal”, and are felt by all forces, both gauge and gravitational. Those that lower the Planck scale, by contrast, are felt only by the gravitational interaction. Together, both types of extra dimensions can conspire to lower the string scale. Indeed, by imagining extra spacetime dimensions of various types and sizes, it may even be possible to lower all of these scales to the TeV range, although this is probably only an interesting (and likely unrealistic) extrapolation. However, the important lesson from these developments is that the fundamental high energy scales of physics are not immutable, and that taking extra spacetime dimensions seriously as physical entities permits these energy scales to be lower (perhaps even substantially lower) than they are typically imagined to be on the basis of four-dimensional extrapolations from low-energy data. More recently, implications of these ideas have been considered in cosmology \([7, 8, 9, 10, 11]\), in radius stabilization \([8, 9]\), and even in potentially explaining the fermion mass hierarchy \([1, 12]\) and the properties of soft SUSY-breaking parameters \([13, 14]\). Earlier discussions of TeV-scale extra dimensions can also be found in Ref. \([15]\), and possible collider signatures of such extra dimensions are discussed in Refs. \([15, 16]\). General consequences of this new “brane world” picture of extra spacetime dimensions and reduced energy scales are also discussed in Ref. \([17]\).

At first glance, this situation may seem to suggest that there is no further need for high energy scales. However, as has recently been emphasized in Refs. \([18, 19, 20]\), low-energy neutrino data provide independent evidence for yet another high mass scale. Specifically, if neutrinos have light but non-zero masses (as suggested by recent SuperKamiokande data \([21]\)), then these masses are most naturally explained in the context of \(SO(10)\) unification via the seesaw mechanism \([22]\). However, the seesaw mechanism relies on the existence of a new heavy mass scale \(M\) associated with a right-handed neutrino singlet field \(N\). Indeed, in the simplest scenarios, light neutrino masses in the \(10^{-2}\) eV range imply that \(M\) should be of the same order of magnitude as the usual four-dimensional GUT scale \(\approx 10^{16}\) GeV. This therefore provides a further need for a high fundamental GUT scale. In Ref. \([20]\), this is referred to as “the third pillar of unification”, and we agree that this observation should not be taken lightly.

In this paper, we shall therefore consider how light neutrino masses may be generated without the introduction of heavy mass scales. Our goal is to suggest a number of higher-dimensional mechanisms which might permit naturally light neutrino masses to be generated. Our starting point is the observation that because the right-handed neutrino is a Standard-Model gauge singlet, it need not be restricted to a “brane” with respect to the full higher-dimensional space. It is therefore possible for this field
to experience extra spacetime dimensions and thereby accrue an infinite tower of Kaluza-Klein excitations. This then leads to a number of higher-dimensional mechanisms for suppressing the resulting neutrino mass without a heavy mass scale, and in this paper we shall make five specific observations.

- First, we point out that allowing the right-handed neutrino to feel extra spacetime dimensions naturally permits the resulting left-handed neutrino mass to be power-law suppressed relative to the masses of all of the other Standard-Model fermions. This occurs due to the power-law running of the neutrino Yukawa couplings, and can therefore drive the neutrino Yukawa couplings to extremely small values over a very short energy interval.

- Second, if the right-handed neutrino has a corresponding tower of Kaluza-Klein excitations, then the usual seesaw mechanism must be generalized to reflect mixings between the left-handed neutrino and the full tower of right-handed Kaluza-Klein states. We therefore examine some of the consequences of such a higher-dimensional seesaw mechanism, and show that such a higher-dimensional seesaw mechanism may also be capable of generating a naturally light neutrino Majorana mass without the introduction of a heavy right-handed neutrino mass scale.

- Third, we show that a higher-dimensional seesaw mechanism may even be able to explain neutrino oscillations without neutrino masses, with oscillations induced indirectly via the masses of the Kaluza-Klein states. This would therefore represent a radical departure from the usual four-dimensional situation in which neutrino oscillations are taken as evidence for neutrino masses.

- Fourth, we point out that even when the right-handed neutrino is given a bare Majorana mass, our higher-dimensional seesaw mechanism essentially replaces this mass scale with the radius scale of the extra spacetime dimensions. This replacement arises due to the cumulative effects of the Kaluza-Klein states, and is therefore also surprising from a naïve four-dimensional point of view.

- Finally, one of the crucial questions in explaining neutrino masses and oscillations is the violation of lepton number. We therefore propose, within the context of our higher-dimensional seesaw mechanism, a new mechanism for generating lepton-number violation. This method involves shifting the positions of D-branes in Type I string theory.

These five mechanisms all have, as their basic goal, the generation of light neutrino masses without the use of a heavy mass scale. We therefore propose these mechanisms in the expectation that they are likely to play an important role in any future systematic analysis of neutrino masses in theories with large extra spacetime dimensions.
2 Higher-dimensional mechanisms for light neutrino masses

2.1 Review: The usual seesaw mechanism

Let us begin by briefly reviewing the usual $SO(10)$ seesaw mechanism \cite{22}. We imagine that there exists a right-handed neutrino (henceforth denoted $N$), and that there exists a set of mass terms for $\mathcal{N} \equiv (\nu_L, N)$ of the form $\mathcal{N} M \mathcal{N}^T$, where

$$\mathcal{M} = \begin{pmatrix} 0 & m \\ m & M \end{pmatrix}. \quad (2.1)$$

Note that for the purposes of this discussion, we shall ignore possible non-diagonality in flavor indices. In this matrix, the entry $m$ arises as a standard Yukawa coupling resulting from electroweak symmetry breaking,

$$m \approx y_{\nu} \langle \phi \rangle, \quad (2.2)$$

where $\langle \phi \rangle \approx 246$ GeV is the electroweak Higgs VEV. Since the neutrino Yukawa coupling $y_{\nu}$ is presumed (on the basis of naturalness arguments) to be of order one, we expect $m \approx \mathcal{O}(10^2 \text{ GeV})$. The entry $M$, by contrast, is a Majorana mass for the right-handed singlet $N$, and is presumed to arise through the breaking of the $SO(10)$ GUT symmetry. Such a term can arise, for example, through the use of a large $SO(10)$ representation such as the 126 representation (the five-index totally antisymmetric tensor). Thus, in the usual scenario, we expect that $M \approx 10^{16}$ GeV. By diagonalizing the mass matrix (2.1), we then obtain the two mass eigenvalues

$$\lambda_{\pm} = \frac{1}{2} \left( M \pm \sqrt{M^2 + 4m^2} \right) \quad \Rightarrow \quad \lambda_- \approx -\frac{m^2}{M}, \quad \lambda_+ \approx M \quad (2.3)$$

to leading order in $m/M$. The physical light neutrino state is then interpreted as the linear combination corresponding to the mass eigenvalue $\lambda_-$, with mass $|\lambda_-|$. Thus, the presence of the heavy mass scale $M$ serves to suppress the neutrino mass so that it comes out substantially below the electroweak scale.

This scenario is very simple and elegant. In the context of string theory, however, certain difficulties may arise. The most pressing of these concerns the generation of the required Majorana mass $M$ for the right-handed neutrino $N$. As we remarked above, this is typically achieved in field theory through the use of a 126 representation. However, within the context of a wide class of string $SO(10)$ GUT models, it has been shown \cite{23, 24} that 126 representations generically do not arise. Other possibilities include simulating the effects of 126 representations through tensor products of smaller representations \cite{23}, but even this has been shown to be difficult within the context of string GUT models \cite{23}. 

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2.2 A higher-dimensional seesaw mechanism: General setup

Let us now consider how we might generate suppressed neutrino masses without the introduction of such a high mass scale. As we discussed in the Introduction, our goal is to lay out a number of alternatives within the context of theories with extra large spacetime dimensions. To this end, the first thing we notice is that unlike all of the other Standard-Model fermions, the right-handed neutrino $N$ is a Standard-Model singlet. Thus, this field need not necessarily be restricted to the “brane” that contains the remaining Standard-Model fermions — i.e., it is possible that this field experiences extra spacetime dimensions and thereby accrues an infinite tower of Kaluza-Klein excitations. For simplicity, we shall assume the appearance of one extra spacetime dimension of radius $R$, so that the mass of the $n^{\text{th}}$ Kaluza-Klein state $N(n)$ is given by

$$m_n \approx n/R, \quad n \in \mathbb{Z}.$$  \hfill (2.4)

For the purposes of this qualitative discussion, it will not be necessary to specify whether this extra dimension is “universal” (i.e., experienced by the Standard-Model gauge bosons and Higgs fields as well as by gravity), or only gravitational. Therefore we shall not need to specify whether $R^{-1} \gtrsim \mathcal{O}(\text{TeV})$, as is required in the first case, or $R^{-1} \lesssim \mathcal{O}(\text{TeV})$, as permitted in the second case. In either case, the important point is that $R^{-1}$ may be taken to be substantially below the usual four-dimensional GUT, Planck, or string scales. We also note that the following discussion continues to hold if more than one extra dimension are considered.

There are two immediate consequences of introducing a Kaluza-Klein tower for $N$. The first, as discussed in Ref. [1], is the power-law running that this induces for the Yukawa coupling $y_{\nu}$ through diagrams such as shown in Fig. 1. In such diagrams, the presence of an infinite tower of Kaluza-Klein states in the loop causes the evolution of the Yukawa coupling to accrue a power-law behavior which can drive the Yukawa coupling $y_{\nu}$ to extremely small values over a very short energy internal. Thus, we see that a Kaluza-Klein tower for the right-handed neutrino provides a natural way of suppressing the value of the Yukawa coupling $y_{\nu}$ and thereby suppressing $m$. Detailed calculations of the resulting neutrino masses would then proceed along the lines discussed in Refs. [1, 12].

The second observation\footnote{This possibility was also considered by S. Dimopoulos and J. March-Russell [24].} is that the coupling of the right-handed neutrino $N$ to the ordinary neutrino $\nu_L$ is automatically suppressed by a volume factor corresponding to the extra compactified dimension. Such a volume factor arises from the normalization of the wavefunction of the $N$ field in the compactified dimension, and will be discussed further below. This volume factor can also provide a natural mechanism for suppressing the Yukawa coupling and yielding a light neutrino mass.

Both of the above mechanisms suppress the neutrino mass by directly suppressing the value of $m$. However, as we shall now discuss, it may also be possible to suppress
Figure 1: Typical one-loop diagram that can induce power-law running of the neutrino Yukawa coupling as a result of Kaluza-Klein states for the right-handed neutrino field $N$. If only the right-handed neutrino $N$ experiences the extra dimensions, then the Yukawa coupling for the neutrino can be power-law suppressed relative to the Yukawa couplings for all other matter fields.

the neutrino mass via a higher-dimensional analogue of the seesaw mechanism.

Once again, we shall assume that the right-handed neutrino feels extra dimensions, while the left-handed neutrino $\nu_L$ does not. Specifically, in higher dimensions (e.g., in five dimensions, for concreteness), we consider a Dirac fermion $\Psi$, which in the Weyl basis can be decomposed into two two-component spinors: $\Psi = (\psi_1, \psi_2)^T$. When the extra spacetime dimension is compactified on a $\mathbb{Z}_2$ orbifold, it is natural for one of the two-component Weyl spinors, e.g., $\psi_1$, to be taken to be even under the $\mathbb{Z}_2$ action $y \to -y$, while the other spinor $\psi_2$ is taken to be odd. If the left-handed neutrino $\nu_L$ is restricted to a brane located at the orbifold fixed point $y = 0$, then $\psi_2$ vanishes at this point and so the most natural coupling is between $\nu_L$ and $\psi_1$. For generality, we will also include a possible “bare” Majorana mass $M_0$ for $\Psi$. This then results in a Lagrangian of the form

$$\mathcal{L} = \int d^4x \, dy \, M_s \left\{ \bar{\psi}_1 i \sigma^\mu \partial_\mu \psi_1 + \bar{\psi}_2 i \sigma^\mu \partial_\mu \psi_2 + \frac{1}{2} M_0 (\psi_1 \psi_1 + \psi_2 \psi_2 + \text{h.c.}) \right\}$$

$$+ \int d^4x \, \left\{ \bar{\nu}_L i \sigma^\mu D_\mu \nu_L + (\tilde{m}\nu_L \psi_1 |_{y=0} + \text{h.c.}) \right\}.$$  \hspace{1cm} (2.5)

Here $y$ is the coordinate of the extra compactified spacetime dimension, and $M_s$ is the mass scale of the higher-dimensional fundamental theory (e.g., a reduced Type I string scale). The first line represents the kinetic-energy term for the five-dimensional $\Psi$ field as well as the bare Majorana mass term $\frac{1}{2} M_0 \bar{\Psi} \Psi$. By contrast, the second line represents the kinetic energy of the four-dimensional two-component neutrino field $\nu_L$ as well as the coupling between $\nu_L$ and $\psi_1$. Note that in five dimensions, a bare Dirac mass term for $\Psi$ would not have been invariant under the action of the $\mathbb{Z}_2$ orbifold, since $\bar{\Psi} \Psi \sim \psi_1 \psi_2 + \text{h.c.}$.

Next, we compactify the Lagrangian (2.5) down to four dimensions by expanding the five-dimensional $\Psi$ field in Kaluza-Klein modes. Imposing the orbifold relations
\( \psi_{1,2}(-y) = \pm \psi_{1,2}(y) \) implies that our Kaluza-Klein decomposition takes the form

\[
\psi_1(x, y) = \frac{1}{\sqrt{2\pi R}} \sum_{n=0}^{\infty} \psi_1^{(n)}(x) \cos(ny/R) \\
\psi_2(x, y) = \frac{1}{\sqrt{2\pi R}} \sum_{n=1}^{\infty} \psi_2^{(n)}(x) \sin(ny/R) .
\]

(2.6)

For convenience, we shall also define the linear combinations \( N^{(n)} \equiv (\psi_1^{(n)} + \psi_2^{(n)})/\sqrt{2} \) and \( M^{(n)} \equiv (\psi_1^{(n)} - \psi_2^{(n)})/\sqrt{2} \) for all \( n > 0 \). Inserting (2.6) into (2.5) and integrating over the compactified dimension then yields

\[
\mathcal{L} = \int d^4x \left\{ \bar{\psi}_L i \gamma^\mu D_\mu \nu_L + \bar{\psi}_1^{(0)} i \sigma^\mu \partial_\mu \psi_1^{(0)} + \sum_{n=1}^{\infty} \left( \bar{N}^{(n)} i \sigma^\mu \partial_\mu N^{(n)} + \bar{M}^{(n)} i \sigma^\mu \partial_\mu M^{(n)} \right) \right. \\
+ \left\{ \frac{1}{2} M_0 \psi_1^{(0)} \bar{\psi}_1^{(0)} \right. + \left. \frac{1}{2} \sum_{n=1}^{\infty} \left( \left( M_0 + \frac{n}{R} \right) N^{(n)} N^{(n)} + \left( M_0 - \frac{n}{R} \right) M^{(n)} M^{(n)} \right) \right. \\
+ m \nu_L \psi_1^{(0)} \bar{\psi}_1^{(0)} \right. + \left. \sum_{n=1}^{\infty} \left( m_N^{(n)} \nu_L N^{(n)} + m_M^{(n)} \nu_L M^{(n)} \right) + \text{h.c.} \right\} .
\]

(2.7)

Here the first line gives the four-dimensional kinetic-energy terms, while the second line gives the Kaluza-Klein and Majorana mass terms. Note that the Kaluza-Klein masses \( n/R \) are replaced by \( (n_1 + in_2)/R \) in the case of two extra spacetime dimensions.\footnote{At first glance, it may seem surprising that a \textit{complex} Kaluza-Klein mass is generated for \( \delta \geq 2 \). However, only the modulus \( \sqrt{n_1^2 + n_2^2}/R \) is the physical mass. The complex phase arises because the fermionic Kaluza-Klein reduction can yield only mass terms which are linear in the Kaluza-Klein momenta \( n_i/R \).}

The third line of (2.7) describes the coupling between the four-dimensional neutrino \( \nu_L \) and the five-dimensional field \( \Psi \). Note that in obtaining this Lagrangian, it is necessary to rescale the individual \( \psi_1^{(0)} \), \( N^{(n)} \), and \( M^{(n)} \) Kaluza-Klein modes so that their four-dimensional kinetic-energy terms are canonically normalized. This then results in a suppression of the Dirac neutrino mass coupling \( \hat{m} \) by the factor \( (2\pi M_s R)^{1/2} \). In the third line, we have therefore simply defined the effective Dirac neutrino mass couplings \( m_N^{(n)} = m_M^{(n)} = m \) for all \( n \), where

\[
m \equiv \frac{\hat{m}}{\sqrt{2} \sqrt{\pi M_s R}} .
\]

(2.8)

For a general \( \mathbb{Z}_N \) orbifold and \( \delta \) extra dimensions, this volume factor \( (\pi R)^{1/2} \) generalizes to \( (2\pi R/N)^{\delta/2} \).

One important dimensionless number in our analysis will be the product \( mR \). Therefore, let us give a rough estimate. First, we note that for a \( \mathbb{Z}_2 \) orbifold, we have

\[
mR \sim \frac{1}{(2\pi)^{\delta/2}} \left( \frac{\hat{m}}{M_s} \right) \frac{1}{(M_s R)^{\delta/2-1}} .
\]

(2.9)
Regardless of whether the extra dimensions are “universal” or are felt only by gravity, we always have $M_s R > 1$; indeed, in the latter case we even have $M_s R \gg 1$. Likewise we expect $\dot{m}/M_s < 1$. Thus, for $\delta \geq 2$, we find that $mR \ll 1$ in all cases, with this approximation becoming particularly appropriate in the case of gravity-only extra dimensions. For example, in the case of gravity-only extra dimensions, we have $mR \leq O(10^{-3})$ for $\delta = 2$. Note that the $\delta = 1$ case is also of some interest.

Although the case of one gravity-only extra dimension is excluded experimentally if the Planck scale is pushed to the TeV-range, it nevertheless remains possible for the higher-dimensional $\Psi$ field to feel only some of the large extra dimensions. This would depend on the sector from which the $\Psi$ field originates in the Type I string theory. In such cases, $mR$ can be quite a bit larger. For example, with $n$ total extra dimensions and $\delta = 1$ (i.e., only one of these extra dimensions felt by the $\Psi$ field), we have $mR \geq O(1)$, with $mR$ ranging from $O(10^5)$ for $n = 2$ to $O(1)$ for $n = 6$. It is also possible (and indeed suggested [1, 8, 15]) that the total compactification space may be anisotropic, with some extra dimensions large and others small. This would then alter the above estimates significantly. Thus, although we shall often focus on the case $mR \ll 1$, we shall attempt to keep our discussion general.

Given the Lagrangian (2.7), we see that the Standard-Model neutrino $\nu_L$ will mix with the entire tower of Kaluza-Klein states of the higher-dimensional $\Psi$ field. Indeed, if we restrict our attention for the moment to the case of only one extra dimension for simplicity and define

$$\mathcal{N}^T \equiv (\nu_L, \psi_1^{(0)}, N^{(1)}, M^{(1)}, N^{(2)}, M^{(2)}, ...),$$

we see that the mass terms in the Lagrangian (2.7) take the form $\frac{1}{2}(\mathcal{N}^T\mathcal{M}\mathcal{N} + \text{h.c.})$ where the mass matrix is symmetric and takes the form

$$\mathcal{M} = \begin{pmatrix}
0 & m & m_N^{(1)} & m_M^{(1)} & m_N^{(2)} & m_M^{(2)} & \ldots \\
0 & m & m_N^{(1)} & m_M^{(1)} & m_N^{(2)} & m_M^{(2)} & \ldots \\
0 & 0 & m_0 + 1/R & 0 & 0 & 0 & \ldots \\
0 & 0 & m_0 - 1/R & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & m_0 + 2/R & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\end{pmatrix}$$

Before proceeding further, let us discuss the assumptions inherent in the form of this mass matrix. First, note that the entries $m, m_N^{(n)}, m_M^{(n)}$ reflect the coupling in (2.7) between the left-handed neutrino state (which feels only four spacetime dimensions) and the $\Psi$ field (which also feels the extra dimensions). As we have seen above, the condition $m = m_N^{(n)} = m_M^{(n)}$ results for the case of a straightforward coupling of $\nu_L$ (restricted to a brane located at $y = 0$) to the higher-dimensional $\psi_1$ field. However, as we shall discuss in Sect. 2.5, it is possible to consider more general brane
configurations in which the parameters \( m, m^{(n)}_N, \) and \( m^{(n)}_M \) are all unequal. We shall therefore leave these couplings completely general, as in (2.11). Of course, the value of these couplings is no longer to be associated with the value given in (2.2), since the presence of the extra dimensions alters the result given in (2.2) by an overall volume factor, as discussed above. Second, note that the remaining entries along the diagonal reflect the masses of the Kaluza-Klein modes of the \( \Psi \) field, as given in (2.4). The contribution from the bare Majorana mass \( M_0 \) is also included. Third, note that we have not introduced any additional off-diagonal non-zero entries in this mass matrix, for such non-zero entries would violate Kaluza-Klein momentum conservation. It might seem at first that conservation of Kaluza-Klein momentum would also forbid the couplings between the left-handed neutrino \( \nu_L \) and the excited Kaluza-Klein modes of the higher-dimensional \( \Psi \) field. However, the difference in this case is the fact that the left-handed neutrino is presumed not to feel the extra spacetime dimensions, and is therefore essentially restricted to a brane with respect to these extra dimensions. Kaluza-Klein momentum conservation therefore does not apply for such couplings because the presence of the brane breaks translational invariance in the compactified direction(s). Thus, we conclude that the most general form for the mass matrix is the one given in (2.11).

We shall now proceed to study the physical implications of this mass matrix. Most of our attention will focus on the generation of a seesaw mechanism, just as in the usual four-dimensional case. This is important for the following reason. Let us imagine, for the moment, that the bare Majorana mass is absent, so that \( M_0 = 0 \). By itself, the usual four-dimensional seesaw mechanism between \( \nu_L \) and the zero-mode field \( \psi_1^{(0)} \) would then result in a \( 2 \times 2 \) matrix

\[
\begin{pmatrix}
0 & m \\
 m & 0
\end{pmatrix},
\]

(2.12)

which leads to the degenerate eigenvalues \( \lambda_{\pm} = \pm m \). These eigenvalues can be combined to form a Dirac mass for the neutrino. Comparing with (2.3), we see that this is the limit in which there is no seesaw at all — i.e., the two lightest states \( \nu_L \) and \( \psi_1^{(0)} \) remain degenerate, with neither becoming lighter or heavier than the other. However, as we shall see below, this need not remain the case in higher dimensions: the excited Kaluza-Klein states can induce a seesaw even if ground-state zero-mode itself does not. This will be important if we want to give the neutrino a Majorana mass, rather than merely a Dirac mass. As we shall see, the degree to which a true seesaw mechanism can be realized ultimately depends on the values of

\[
\left[ m_N^{(n)} \right]^2 - \left[ m_M^{(n)} \right]^2.
\]

(2.13)

We shall therefore begin by studying the physical implications of our mixing matrix (2.11) in a number of simple limits in order to elucidate its basic properties and consequences for the resulting neutrino mass.
2.3 A toy model: The case $M_0 = m^{(n)}_M = 0, \ m^{(n)}_N = m$

In order to most dramatically illustrate the possibilities of a higher-dimensional seesaw mechanism, let us begin by considering the extreme case of (2.11) in which $m^{(n)}_M = 0$ for all $n$. As we shall see, this will result in the strongest seesaw behavior — i.e., the maximal splitting between the light eigenvalues $\lambda_{\pm}$. Under the assumption that $m^{(n)}_M = 0$ for all $n$, the neutrino $\nu_L$ no longer couples to the $M^{(n)}$ modes, and hence they decouple from the problem. For simplicity, we shall also disregard the bare Majorana mass, setting $M_0 = 0$. This then results in the simplified Lagrangian

$$L = \int d^4x \left\{ \bar{\nu}_L i \bar{\sigma}^\mu D_\mu \nu_L + \sum_{n=0}^\infty N^{(n)}_L i \bar{\sigma}^\mu \partial_\mu N^{(n)} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{R_n}{m} N^{(n)}_L N^{(n)} + m \nu_L N^{(n)} + \text{h.c.} \right\}$$

(2.14)

where we have defined $N^{(0)}_L \equiv \psi^{(0)}_1$. We emphasize that in writing (2.14), we have neglected the terms containing the (decoupled) field $M^{(n)}$. Although it may seem that lepton number is apparently broken in the Kaluza-Klein mass terms for $N^{(n)}$, this is precisely compensated for by the analogous mass terms for $M^{(n)}$ as in (2.7). Thus, defining the reduced set of fields

$$\mathcal{N}^T \equiv (\nu_L, N^{(0)}, N^{(1)}, N^{(2)}, ...) \ ,$$

(2.15)

we see that the mass terms in the Lagrangian (2.14) correspond to the simplified mass matrix

$$M = \begin{pmatrix}
0 & m & m & m & m & m & \ldots \\
m & 0 & 0 & 0 & 0 & 0 & \ldots \\
m & 0 & 1/R & 0 & 0 & 0 & \ldots \\
m & 0 & 0 & 2/R & 0 & 0 & \ldots \\
m & 0 & 0 & 0 & 3/R & 0 & \ldots \\
m & 0 & 0 & 0 & 0 & 4/R & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$  

(2.16)

The remarkable feature of (2.16) is that the resulting “light” eigenvalues $\lambda_{\pm}$ are maximally split from each other, with a splitting that becomes infinitely great as the size of the matrix is taken to infinity. In particular, while one eigenvalue grows infinitely heavy as more and more of the Kaluza-Klein states participate in the mixing, the other becomes arbitrarily light. We will prove this statement analytically below, but let us first sketch how this happens in practice. Once again, we begin by focusing on only the upper-left $2 \times 2$ mixing sub-matrix between $\nu_L$ and $N^{(0)}$ alone. This is the same as the matrix (2.12), which produces two eigenvalues $\pm m$. Therefore both of the resulting mass eigenstates would have masses equal to $m$ — i.e., there would be no seesaw between $\nu_L$ and $N^{(0)}$. However, as we increase the size of this
matrix by adding the further rows and columns corresponding to the excited Kaluza-Klein states, we find that the cumulative effect of the excited Kaluza-Klein states is to pull the negative eigenvalue $-m$ further in the negative direction, but also to decrease the positive eigenvalue $+m$. Ultimately, as the dimensionality of this mass matrix is taken to infinity, the negative eigenvalue $-m$ falls all the way to negative infinity while the positive eigenvalue $+m$ falls all the way to zero. Note that each new row/column also introduces a new eigenvalue which, in the limit as the matrix becomes infinite-dimensional, simply remains fixed near $n/R$. Thus, we find that our infinite-dimensional matrix produces one zero eigenvalue and one infinite eigenvalue, with all other eigenvalues of size $R^{-1}$ or larger.

We shall now give an analytical derivation of the eigenvalues of this infinite-dimensional matrix. We begin by considering a matrix of finite size $(n+2) \times (n+2)$, so that the highest diagonal entry is $n/R$. This matrix will therefore have $(n+2)$ different eigenvalues; these consist of the $n$ different eigenvalues $\lambda_k$ ($k = 1, ..., n$) corresponding to the excited Kaluza-Klein states, as well as the two remaining “light” eigenvalues $\lambda_+$ and $\lambda_-$ whose values are respectively $\pm m$ in the special case $n = 0$, as discussed above. Our procedure will be to solve for the “light” eigenvalues $\lambda_\pm$ as a function of $n$, and to consider their behavior as $n \to \infty$.

We begin by considering the characteristic eigenvalue equation $\det(\mathcal{M} - \lambda I) = 0$. Given the mass matrix $\mathcal{M}$ in (2.16), this equation takes the exact analytic form

$$
\prod_{k=1}^{n} \left( \frac{k}{R} - \lambda \right) \left[ \lambda^2 - m^2 + \lambda m^2 R \sum_{k=1}^{n} \frac{1}{k - \lambda R} \right] = 0.
$$

(2.17)

However, since we know that we can always ultimately write this eigenvalue equation in the form

$$
\prod_{k=1}^{n} (\lambda_k - \lambda) (\lambda_+ - \lambda) (\lambda_- - \lambda) = 0,
$$

(2.18)

we see that we can obtain a number of different relations amongst the eigenvalues by considering the coefficients of various powers of $\lambda$ in (2.17). For example, the constant term $C_0$ (i.e., the coefficient of $\lambda^0$) gives the product of the eigenvalues, $\prod \lambda$, which is nothing but the determinant of $\mathcal{M}$. Likewise, the coefficient $C_1$ of the term linear in $\lambda$ is identified as the sum of the products of all possible subsets of $n+1$ of the eigenvalues, i.e., $C_1 = -\sum_{i_1,...,i_{n+1}} \lambda_{i_1} ... \lambda_{i_{n+1}}$, where the $i$-indices run over the set $\{1, 2, ..., n, +, -\}$. Similarly, the coefficient $C_{n+1}$ of the $\lambda^{n+1}$ term gives $(-1)^{n+1} \sum \lambda$, which is equivalently $(-1)^{n+1}$ times the trace of the matrix $\mathcal{M}$.

By examining the matrix $\mathcal{M}$ and the characteristic equation (2.17), it is easy to see that

$$
C_0 = -\frac{n! m^2}{R^n}, \quad C_1 = \frac{2n! m^2}{R^{n-1}} \sum_{k=1}^{n} \frac{1}{k}, \quad C_{n+1} = (-1)^{n+1} \frac{1}{R} \sum_{k=1}^{n} k.
$$

(2.19)

Moreover, for $mR \ll 1$, it is easy to show that the excited Kaluza-Klein eigenvalues behave as $\lambda_k \approx k/R + m^2 R/k + ...$ for large $n$. (This will be discussed further below.)
Using this information, we can then obtain various simultaneous equations for $\lambda_+$ and $\lambda_-$. For example, from the $C_0$ determinant relation we find

$$
\lambda_+ \lambda_- \approx -m^2 \left( 1 + m^2 R^2 \sum_{k=1}^{n} \frac{1}{k^2} + \ldots \right)^{-1} .
$$

Likewise, from the $C_1$ relation we find

$$
\lambda_+ + \lambda_- + R \lambda_+ \lambda_- \sum_{k=1}^{n} \frac{1}{k} \approx -2m^2 R \sum_{k=1}^{n} \frac{1}{k} ,
$$

and from the $C_{n+1}$ trace relation we find

$$
\lambda_+ + \lambda_- \approx -m^2 R \sum_{k=1}^{n} \frac{1}{k} .
$$

We can now solve any two of these equations simultaneously for the eigenvalues $\lambda_{\pm}$. In all cases, we obtain

$$
\lambda_{\pm} = \frac{1}{2} \left[ \mu \pm \sqrt{\mu^2 + 4m^2} \right] \quad (2.23)
$$

where we have defined the quantity

$$
\mu \equiv -m^2 R \sum_{k=1}^{n} \frac{1}{k} .
$$

Thus, comparing with the usual seesaw result in (2.3), we see that the entire tower of Kaluza-Klein states has generated a seesaw that splits the two lightest eigenvalues. We shall discuss the physical interpretation of $\mu$ below.

There are two limits that will be of interest to us, depending on the size of $\mu$. The first case, with $mR \ll 1$ and finite $n$, corresponds to $\mu \ll m$. In this case, we obtain the solutions

$$
\lambda_{\pm} = \pm m - \frac{1}{2} m^2 R \sum_{k=1}^{n} \frac{1}{k} . \quad (2.25)
$$

As discussed above, these are the two light eigenvalues that arise when only the lightest Kaluza-Klein states participate in the mixing. By contrast, the full “seesaw” limit arises as we take $n \to \infty$, with all Kaluza-Klein states included in the mixing. In this case, we have $\mu \gg m$, whereupon we find

$$
\lambda_+ \sim \frac{1}{R \sum_{k=1}^{n} 1/k} \sim \frac{1}{R \ln n} \quad (2.26)
$$

and

$$
\lambda_- \sim -m^2 R \sum_{k=1}^{n} \frac{1}{k} \sim -m^2 R \ln n . \quad (2.27)
$$
We thus see that \( \lambda_+ \) becomes arbitrarily light as \( n \to \infty \), while \( \lambda_- \) becomes arbitrarily heavy. It is interesting that the \( n \to \infty \) limit is capable of producing such an infinite splitting between these two eigenvalues — and with it an arbitrarily light eigenvalue — regardless of the intrinsic value of the radius \( R \).

Note that these results can also be seen directly from the characteristic equation (2.17), regardless of the value of \( mR \), by noticing that \( \lambda \to 0 \) is a solution of the term in square brackets. Specifically, near \( \lambda = 0 \), the sum \( \sum_{k=1}^{n} 1/(k - \lambda R) \) is analytic in \( \lambda \). Thus, performing a Taylor expansion about the origin and keeping only terms linear in \( \lambda \), we again find a solution that behaves like (2.26). This then implies the solution (2.27). This argument does not rely on the value of \( mR \), and thus we see that these solutions continue to hold regardless of the value of \( mR \). Similarly, note that the excited Kaluza-Klein eigenvalues given above, namely \( \lambda_k \approx k/R + m^2 R/k \), are valid only for finite \( n \). In the limit \( n \to \infty \), one finds that \( \lambda_k \sim k/R + 1/(R \ln n) \). This can be seen by substituting the value \( \lambda_k = k/R + c \) into the characteristic equation (2.17) and showing, in a fashion similar to that for the zero eigenvalue, that the term in square brackets vanishes if \( c \sim 1/(R \ln n) \).

Another useful way to derive these results is to consider the tree-level Feynman diagram in Fig. 2. For any given Kaluza-Klein state \( N^{(k)} \), this diagram can be interpreted as contributing to an individual seesaw between \( \nu_L \) and \( N^{(k)} \). In the limit \( m \ll 1/R \), the masses of the excited Kaluza-Klein states exceed the size of their Dirac couplings \( m \), and can be integrated out. Thus, by summing over all possible intermediate Kaluza-Klein states (i.e., by summing over all of the individual Kaluza-Klein seesaw contributions), we then generate a neutrino mass term \( \mu \nu_L \nu_L \),

\[
\mu = \sum_{k=1}^{n} m \frac{-1}{k/R} m \approx -m^2 R \ln n .
\]

This is the same quantity defined in (2.24). Once these massive Kaluza-Klein states are integrated out, the problem is reduced to an effective seesaw mechanism between \( \nu_L \) and \( N^{(0)} \), with mass matrix

\[
\begin{pmatrix}
\mu & m \\
m & 0
\end{pmatrix}.
\]
This then leads to the eigenvalues given in (2.23). We also note, in passing, that Fig. 2 also gives a contribution to the wavefunction renormalization of the left-handed neutrino. The corresponding Kaluza-Klein summation is finite for $\delta = 1$, logarithmically divergent for $\delta = 2$, and power-law divergent for $\delta > 2$. As we shall discuss, however, the Kaluza-Klein sum must be truncated at the string scale, and therefore the final result is a finite renormalization factor which we shall implicitly disregard.

One important point that emerges from the above discussion is that this higher-dimensional seesaw mechanism is \textit{inverted} relative to the usual four-dimensional one. This is clear, for example, upon comparing (2.29) with (2.1). Specifically, it is the neutrino $\nu_L$ that becomes heavier as more and more Kaluza-Klein states are included in the mixing, while the zero-mode $N^{(0)}$ becomes light. In other words, while $\mu$ serves as an effective seesaw scale for the light eigenstate $N^{(0)}$, the effective seesaw scale for the neutrino eigenstate is actually

$$M_{\text{eff}} = -\frac{m^2}{\mu} = \frac{1}{R \ln n}.$$ 

Thus, our seesaw mechanism rotates the states in a direction that is opposite to the direction that usually emerges in the four-dimensional case.

However, this may be useful for the following reason. Recall that the approach we have followed is one based on an effective field theory. If our true underlying theory is a string theory with mass scale $M_s$, then we expect our considerations to be valid only up to the mass scale $M_s$. This in turn means that the maximum number of Kaluza-Klein states which should enter into our considerations is $n_{\text{max}} \sim O(RM_s)$. Given this observation, it is natural to identify the result (2.27) with values of the neutrino masses suggested by the recent SuperKamiokande data. However, as we discussed in Sect. 2.2, there is tremendous variation in the size of $mR$ and in the resulting rescaled coupling $m$ defined in (2.8). Indeed, it generically appears to be difficult to arrange $m$ itself to be of the size suggested by the SuperKamiokande results. Thus, we now see that our seesaw mechanism plays an important role by offering the possibility of \textit{altering} this mass:

$$m \rightarrow m^2R\ln(RM_s).$$

(2.31)

In fact, in many instances $m$ itself is too \textit{small} to agree with these neutrino masses. The above substitution may therefore enable a useful \textit{enhancement} of the naive value of $m$. Of course, the phenomenological details of this mechanism ultimately depend on the sizes of $mR$ and $RM_s$, which in turn depend crucially on the geometry (and in particular the anisotropy) of the compactification manifold.

We also stress that in this section we have been considering only the illustrative toy model that emerges from the extreme limit in which we take $m_n^{(n)} = 0$ for all $n$. Nevertheless, many of the crucial features of this scenario will continue to hold in the subsequent scenarios that have a more natural realization in string theory.
2.4 The orbifold case: $m = m^{(n)}_N = m^{(n)}_M$, $M_0 = 0$

Let us now return to (2.11), and discuss the special case in which the Dirac couplings satisfy $m = m^{(n)}_N = m^{(n)}_M$ for all $n$. As discussed in Sect. 2.2, this corresponds to a straightforward orbifold coupling between the four-dimensional $\nu_L$ field and the higher-dimensional $\psi_1$ field. For simplicity, we shall again disregard the possible bare Majorana mass term, setting $M_0 = 0$. In this situation, our mass matrix (2.11) then becomes

$$
\mathcal{M} = \begin{pmatrix}
0 & m & m & m & m & \ldots \\
0 & m & m & m & m & \ldots \\
0 & m & m & m & m & \ldots \\
0 & m & m & m & m & \ldots \\
0 & m & m & m & m & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

(2.32)

The characteristic polynomial equation which determines the eigenvalues of this mass matrix then takes the form

$$
\prod_{k=1}^{\infty} \left( \frac{k^2}{R^2} - \lambda^2 \right) \left[ \lambda^2 - m^2 + 2 \lambda^2 m^2 R^2 \sum_{k=1}^{\infty} \frac{1}{k^2 - \lambda^2 R^2} \right] = 0,
$$

(2.33)

which is invariant under $\lambda \to -\lambda$. From this we immediately see that all eigenvalues are exactly degenerate, falling into pairs of opposite sign. This implies, in particular, that the two lightest eigenvalues are degenerate (combining to produce a Dirac mass for the neutrino), and that there is no seesaw behavior. Indeed, all of the resulting masses for the Kaluza-Klein eigenstates are Dirac as well. This is ultimately a consequence of the alternating signs in the diagonal entries of (2.32).

In order to solve this eigenvalue equation, it is convenient to note that $\lambda = k/R$ is never a solution (unless of course $m = 0$), as the cancellation that would occur in the first factor in (2.33) is offset by the divergence of the second factor. We are therefore free to disregard the first factor entirely, and focus on solutions for which the second factor vanishes. The summation in second factor can then be performed exactly, resulting in the transcendental equation

$$
\lambda R = \pi (mR)^2 \cot(\pi \lambda R).
$$

(2.34)

All of the eigenvalues can be determined from this equation, as functions of the product $mR$. The solutions are shown graphically in Fig. 3(a). We immediately see that in the limit $mR \to 0$ (corresponding to $m \to 0$), the eigenvalues are $k/R$, $k \in \mathbb{Z}$, with a double eigenvalue at $k = 0$. Conversely, in the limit $mR \to \infty$, the eigenvalues with $k > 0$ smoothly shift to $(k + \frac{1}{2})/R$, while those with $k < 0$ shift to $(k - \frac{1}{2})/R$ and the double zero eigenvalue splits towards the values $\pm 1/(2R)$. In order to derive general analytical expressions valid in the limit $mR \ll 1$, we can solve
Figure 3: (a) Eigenvalue solutions to (2.34), represented as those values of \( \lambda \) for which \( \cot(\pi \lambda R) \) intersects \( \lambda R/[\pi (mR)^2] \). We have taken the fixed value \( mR = 0.4 \) for this plot. The behavior of the eigenvalues as functions of \( mR \) can be determined graphically by changing the slope of the intersecting diagonal line. (b) The lightest eigenvalue (neutrino mass) \( \lambda_+ \) as a function of \( mR \). For \( mR \ll 1 \), we see that the curve is approximately linear, corresponding to \( \lambda_+ \approx m \). However, as \( mR \) increases, the neutrino mass increases non-linearly, ultimately reaching an asymptote at \( \lambda_+ R = 1/2 \) at which point the volume factor becomes irrelevant.

(2.34) iteratively by power-expanding the cotangent function. To order \( O(m^5 R^5) \), this gives the solutions

\[
\lambda_{\pm k} = \pm \frac{k}{R} + \frac{1}{2R(1 + \pi^2 m^2 R^2/3)} \left[ \mp k \pm \sqrt{k^2 + 4m^2 R^2(1 + \pi^2 m^2 R^2/3)} \right], \quad k \in \mathbb{Z},
\]

(2.35)

where \( \lambda_{\pm k} \) are the two eigenvalues at each Kaluza-Klein level \( k \). Note that this expression also includes the “light” eigenvalues \( \lambda_\pm \) at \( k = 0 \), as shown in Fig. 3(b).

Expanding to order \( O(m^5 R^5) \), we thus find

\[
\lambda_\pm = \pm m \left( 1 - \frac{\pi^2}{6} m^2 R^2 + \ldots \right), \quad \lambda_{\pm k} = \pm \frac{k}{R} \left( 1 + \frac{m^2 R^2}{k^2} - \frac{m^4 R^4}{k^4} + \ldots \right).
\]

(2.36)

Finally, it is also straightforward to explicitly solve for the light mass eigenstates \( |\tilde{\nu}_\pm \rangle \) corresponding to \( k = 0 \). To leading order in \( mR \), we find

\[
|\tilde{\nu}_\pm \rangle = \frac{1}{\sqrt{2}} \left\{ \left( 1 - \frac{\pi^2}{6} m^2 R^2 \right) |\nu_L \rangle \pm |\psi_1^{(0)} \rangle - mR \sum_{k=1}^{\infty} \frac{1}{k} \left[ |N^{(k)} \rangle - |M^{(k)} \rangle \right] \right\}.
\]

(2.37)
This implies that the overlap between the light mass eigenstates and the neutrino gauge eigenstate is generically less than half in this scenario. We will give an exact all-order solution for this eigenvector in Sect. 3.

Thus, we conclude that in this orbifold case with \( m_N^{(n)} = m_M^{(n)} = m \) for all \( n \), all eigenvalues remain degenerate and there is no seesaw mechanism.

### 2.5 Brane-shifting, lepton-number violation, and the induced seesaw

We have already seen in Sect. 2.3 that a maximal seesaw emerges in the case when \( m_N^{(n)} \) and \( m_M^{(n)} \) are unequal, and likewise we have seen in Sect. 2.4 that the seesaw is completely cancelled when \( m_N^{(n)} \) and \( m_M^{(n)} \) are precisely equal. This suggests (and we shall shortly verify) that the magnitude of the resulting seesaw is directly governed by the differences \( [m_N^{(n)}]^2 - [m_M^{(n)}]^2 \). It is therefore important, for the sake of our seesaw mechanism, to generate such non-zero differences. This is also important if we want to split the “light” eigenvalues from each other and thereby produce a Majorana neutrino mass rather than a Dirac neutrino mass.

When the extra spacetime dimension is compactified on a \( \mathbb{Z}_2 \) orbifold, we have already seen in Sect. 2.2 that it is natural for one of the two-component right-handed spinors, e.g., \( \psi_1 \), to be taken to be even under the \( \mathbb{Z}_2 \) action \( y \to -y \), while the other spinor \( \psi_2 \) is taken to be odd. If the left-handed neutrino \( \nu_L \) is restricted to a brane located at the orbifold fixed point \( y = 0 \), then \( \psi_2 \) vanishes at this point and so the most natural coupling is between \( \nu_L \) and \( \psi_1 \). This then implies \( m_N^{(n)} = m_M^{(n)} = m \) for all \( n > 0 \).

It is therefore natural to ask under what conditions the difference \( [m_N^{(n)}]^2 - [m_M^{(n)}]^2 \) can be non-zero. Specifically, given that the straightforward orbifold coupling gives a vanishing difference, one wonders whether there might exist a physical mechanism related to the orbifold that permits \( m_N^{(n)} \) and \( m_M^{(n)} \) to be unequal. Remarkably, however, such a mechanism exists within Type I string theory. Specifically, in Type I string theory, we have the freedom to shift the branes away from the orbifold fixed points \[27\], and under some restrictions, this shift can be done in a continuous way. This then permits couplings between \( \nu_L \) and more general combinations of \( \psi_1 \) and \( \psi_2 \). Thus, we see that brane-shifting provides us with a uniquely “stringy” method of breaking lepton number and generating a seesaw mechanism and corresponding Majorana neutrino mass.\[†\]

Let us now analyze this situation in more detail. Once again, we shall restrict our attention to the case of five dimensions, and imagine that the left-handed neutrino

\[†\] One might still wonder whether a coupling between \( \nu_L \) and \( \psi_2 \) can exist, given that such a term would not have been invariant under the original \( \mathbb{Z}_2 \) orbifold action. However, it is improper to impose the orbifold projection on the spectrum and interactions corresponding to states on those branes that have been shifted away from the orbifold fixed points. Instead, the orbifold action simply relates the wavefunctions of states on such a shifted brane to the wavefunctions of states on another, oppositely shifted brane. Thus, while the total theory still remains invariant under the orbifold projection, the states on the individual shifted brane need not.
field $\nu_L$ (along with the other Standard-Model fields) is restricted to a brane whose bulk coordinate $y$ is shifted away from the orbifold fixed point location $y = 0$ to a general bulk coordinate $y^*$. In such a situation, our generalized coupling between $\nu_L$ and the higher-dimensional $\Psi$ field takes the form $\hat{m}\hat{\nu}_L(\Psi + \Psi^c)|_{y^*} + \text{h.c.}$ Decomposing this into two-component spinors and performing the Kaluza-Klein reduction as in Sect. 2.2 then yields the coupling

$$
m\nu_L \left\{ \psi_1^{(0)} + \sqrt{2} \sum_{n=1}^{\infty} \left[ \cos(ny^*/R)\psi_1^{(n)} + \sin(ny^*/R)\psi_2^{(n)} \right] \right\} + \text{h.c.}
$$

where $m$ is defined in (2.8) and where

$$m_N^{(n)} \equiv m \left[ \cos(ny^*/R) + \sin(ny^*/R) \right],
$$

$$m_M^{(n)} \equiv m \left[ \cos(ny^*/R) - \sin(ny^*/R) \right].$$

We thus see that brane-shifting has eliminated the degeneracy between $m_N^{(n)}$ and $m_M^{(n)}$.

As before, the effect of this generalized coupling on the mass matrix can be analyzed most conveniently by integrating out all the massive Kaluza-Klein states. This then gives rise to eigenvalues which again take the form (2.23), except with $\mu$ now given by

$$\mu = -m^2 R \sum_{k=1}^{n} \frac{1}{k} \sin \left( \frac{2ky^*}{R} \right).$$

Thus, we see that brane-shifting has succeeded in generating an effective seesaw between $\nu_L$ and $\psi_1^{(0)}$.

If the brane is at the orbifold fixed points $y^* = 0$ or $y^* = \pi R$, then $\mu = 0$ and we recover the previous result (2.36) for which the eigenvalues are degenerate. In this case, the eigenvalues are suppressed only by the wavefunction volume renormalization factors implicit in $m$. This also is the result obtained at the midpoint $y^* = \pi R/2$. However, at non-trivial values of $y^*$, we find that the eigenvalues are no longer degenerate, and instead experience a seesaw whose magnitude depends non-trivially on the value of $y^*$.

### 2.6 Bare Majorana masses: The case $M_0 \neq 0$

Let us now return to (2.11), and consider the case in which we include a fundamental higher-dimensional Majorana-type mass term into our analysis. As we discussed earlier, such lepton-number violating terms can be generated via $126$-type representations (when they arise), or through other effective non-renormalizable terms that may appear in the low-energy superpotential derived from a given string model.
Given that both the GUT scale and the string scale can be reduced to the TeV-range in Type I string models with large extra dimensions [1], it is natural to imagine that $M_0$ is close to the Type I string scale, which implies that $M_0 \gg 1/R$. However, we shall not make any approximations based on this assumption in what follows.

In order to exhibit the effect of such a bare Majorana mass on the resulting eigenvalues, we shall for simplicity consider the orbifold case with $m = m^{(n)}_N = m^{(n)}_M$ for all $n$. This then results in the mass matrix

$$M = \begin{pmatrix}
0 & m & m & m & m & m & \ldots \\
 m & M_0 & 0 & 0 & 0 & 0 & \ldots \\
 m & 0 & M_0 + 1/R & 0 & 0 & 0 & \ldots \\
 m & 0 & 0 & M_0 - 1/R & 0 & 0 & \ldots \\
 m & 0 & 0 & 0 & M_0 + 2/R & 0 & \ldots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}. \quad (2.41)$$

At this stage, however, it proves useful to define

$$k_0 \equiv [M_0 R], \quad \epsilon \equiv M_0 - \frac{k_0}{R} \quad (2.42)$$

where $[x]$ denotes the integer nearest to $x$. Thus, $\epsilon$ is the smallest diagonal entry in the mass matrix (2.41), corresponding to the excited Kaluza-Klein state $M^{(k_0)}$. In other words, we have $\epsilon \equiv M_0 \pmod{R^{-1}}$, satisfying $-\frac{1}{2} R^{-1} < \epsilon \leq \frac{1}{2} R^{-1}$. The remaining diagonal entries in the mass matrix can then be expressed as $\epsilon \pm k'/R$ where $k' \in \mathbb{Z}^+$. Upon suitably reordering the rows and columns of our mass matrix, we can therefore cast this matrix into the form

$$M = \begin{pmatrix}
0 & m & m & m & m & m & \ldots \\
 m & \epsilon & 0 & 0 & 0 & 0 & \ldots \\
 m & 0 & \epsilon + 1/R & 0 & 0 & 0 & \ldots \\
 m & 0 & 0 & \epsilon - 1/R & 0 & 0 & \ldots \\
 m & 0 & 0 & 0 & \epsilon + 2/R & 0 & \ldots \\
 m & 0 & 0 & 0 & 0 & \epsilon - 2/R & \ldots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}. \quad (2.43)$$

While this may look similar to our original mass matrix (2.41), the important consequence of this rearrangement is that the heavy mass scale $M_0$ has been replaced by the light mass scale $\epsilon$. Unlike $M_0$, we see that $|\epsilon| \lesssim O(R^{-1})$. Thus, the heavy Majorana mass scale $M_0$ completely decouples from the physics! Indeed, the value of $M_0$ enters the results only through its determinations of $k_0$ and the precise value of $\epsilon$. At first sight, this may seem counter-intuitive, because the (heavy) bare Majorana mass $M_0$ would naively appear to shift the ground-state energy of the Kaluza-Klein tower, and thereby induce a strong suppression for the neutrino mass. However, as we
have seen, the presence of the infinite tower of regularly-spaced Kaluza-Klein states ensures that only the value of $M_0$ modulo $R^{-1}$ plays a role.

The easiest way to solve (2.43) for the eigenvalues $\lambda_{\pm}$ is to use the same diagram as in Fig. 2, and integrate out the Kaluza-Klein modes. It turns out that there are two cases to consider, depending on the value of $\epsilon$. If $|\epsilon| \gg m$ (which can arise when $mR \ll 1$), then all of the Kaluza-Klein modes are extremely massive relative to $m$, and we can integrate them out to obtain an effective $\nu_L\nu_L$ mass term of size $\|\epsilon\| \ll m$:

$$m_{\nu} = m^2/\epsilon + m^2 \sum_{k'=1}^{\infty} \left( \frac{1}{\epsilon + k'/R} + \frac{1}{\epsilon - k'/R} \right) = \pi m^2 R \cot (\pi R \epsilon).$$

(2.44)

We shall discuss the special case $\epsilon = \frac{1}{2} R^{-1}$ in Sect. 4. Alternatively, if $|\epsilon| \gg m$, then the lightest Kaluza-Klein mode $M^{(k_0)}$ should not be integrated out, and we obtain an effective $\nu_L\nu_L$ mass term of size $\mu$, where

$$|\epsilon| \gg m : \quad \mu = -m^2 \sum_{k'=1}^{\infty} \left( \frac{1}{\epsilon + k'/R} + \frac{1}{\epsilon - k'/R} \right) = \frac{m^2}{\epsilon} - \pi m^2 R \cot (\pi R \epsilon).$$

(2.45)

Note that $\mu \to 0$ smoothly as $\epsilon \to 0$, with $\mu$ otherwise of size $O(m^2 R)$. Diagonalizing the final resulting $2 \times 2$ mass matrix between $\nu_L$ and $M^{(k_0)}$ in the presence of this mass term then yields the result

$$|\epsilon| \gg m : \quad \lambda_{\pm} = \frac{1}{2} \left[ (\mu + \epsilon) \pm \sqrt{(\mu - \epsilon)^2 + 4m^2} \right].$$

(2.46)

Thus, as $M_0 \to 0$ (or as $M_0 \to n/R$ where $n \in \mathbb{Z}$), we see that $\epsilon, \mu \to 0$, and we recover the eigenvalues given in (2.36).

\footnote{In (2.44), care needs to be taken with respect to the order in which the terms are introduced into the (apparently divergent) Kaluza-Klein summations. The only physically consistent organization of the terms that respects the symmetries of the Kaluza-Klein theory is to pair positive Kaluza-Klein modes with their corresponding negative Kaluza-Klein modes (or equivalently to pair the modes $N^{(k)}$ and $M^{(k)}$, which originally resulted from the algebraic decomposition of the fields $\psi_1$ and $\psi_2$). This pairing is therefore utilized in (2.44). It may seem at first that the reorganization of the Kaluza-Klein modes in passing from (2.41) to (2.43) would render (2.44) invalid. However, if we were to directly integrate out the modes in the original order corresponding to (2.41), we would obtain the same expression as (2.44) except with $\epsilon$ replaced by $M_0$. Because $M_0$ appears as the argument of a cotangent function in the result, this replacement is inconsequential. Thus, (2.44) remains correct as written. This conclusion will also hold for (2.45). Indeed, in general, we are free to reorder the rows and columns of this matrix without introducing any subtleties into the determination of the resulting eigenvalues. As a separate matter, we also note that integrating out the Kaluza-Klein states results in an overall sign which is opposite to that given in (2.44). However, interpreting this result as the physical neutrino mass allows us to disregard this sign.}
We therefore conclude that although we may have started with a bare Majorana mass $M_0 \gg R^{-1}$, in all cases the final neutrino mass remains of order $m^2 R$. Even though we might have expected a neutrino mass of order $m^2/M_0$ from the mixing between $\nu_L$ and the original zero-mode $\psi^{(0)}_1$, the contribution $m^2/M_0$ from the zero-mode is completely cancelled by the summation over the Kaluza-Klein tower, with the seesaw between $\nu_L$ and $M^{(k_0)}$ becoming dominant instead. It is this feature that causes the heavy scale $M_0$ to be effectively replaced by the radius $R^{-1}$, so that once again our effective seesaw scale is $M_{\text{eff}} \sim \mathcal{O}(R^{-1})$. Thus, we see that in a theory in which the heavy lepton experiences large extra dimensions, the radius — and not the bare lepton mass $M_0$ with which we started — plays the role of the heavy scale in the seesaw mechanism.

2.7 General case: $y^* \neq 0, M_0 \neq 0$

Finally, for completeness, we turn to the general case in which we include the effects of brane-shifting and bare Majorana masses simultaneously. Thus, we take $y^* \neq 0$ and $M_0 \neq 0$. As before, we define $\epsilon$ as in (2.42), and for simplicity we shall restrict our attention to the case where $|\epsilon| \gg m$. We can therefore integrate out all Kaluza-Klein modes, obtaining

$$m_\nu = \frac{m^2}{M_0} + \sum_{k=1}^{\infty} \left\{ \frac{m^2 [\cos(ky^*/R) + \sin(ky^*/R)]^2}{M_0 + k/R} + \frac{m^2 [\cos(ky^*/R) - \sin(ky^*/R)]^2}{M_0 - k/R} \right\}$$

$$= \pi m^2 R \left\{ \cot(\pi \epsilon) + \frac{\sin[(\pi - 2y^*/R)M_0 R]}{\sin(\pi M_0 R)} \right\}.$$  (2.47)

From this result, we see that when brane-shifting and bare Majorana masses are present simultaneously, the bare Majorana mass $M_0$ does not completely decouple from the final result in favor of $\epsilon$. Instead, a slight dependence on $M_0$ remains in the second term, due to the effects of the shifted brane. However, once again this effect is small (appearing only within the arguments of trigonometric functions), and the overall scale remains $m_\nu \sim \mathcal{O}(m^2 R)$. Thus, the effective seesaw scale in this case is again $M_{\text{eff}} \sim \mathcal{O}(R^{-1})$ rather than $M_0$.

3 Higher-dimensional neutrino oscillations

Let us now consider the implications of the above higher-dimensional seesaw scenarios for neutrino oscillations. Once again, we shall find that significant differences exist relative to the usual four-dimensional case.
3.1 Four-dimensional neutrino oscillations

Let us begin by recalling how neutrino oscillations arise in the usual four-dimensional case. We suppose, in all generality, that we have two sets of neutrinos, a set of gauge eigenstates $\nu_f$ and a set of mass eigenstates $\tilde{\nu}_i$ which are non-trivially related to each other through a unitary mixing matrix $U$:

$$\nu_f = \sum_i U_{fi} \tilde{\nu}_i .$$ 

(3.1)

Here the tilde indicates a mass eigenstate. This matrix $U$ is uniquely determined from the mass mixing matrix $\mathcal{M}$ involved in the seesaw mechanism, and is nothing but the inverse of the matrix of eigenvectors of $\mathcal{M}$. It is the fact that the gauge eigenstates are non-trivial combinations of the physical propagating mass eigenstates that causes the gauge eigenstates to oscillate as a function of time. Specifically, given such a mixing matrix $U$, we find that the probability of oscillation from $\nu_f$ to $\nu_{f'}$ after time $t$ is given by

$$P_{f \rightarrow f'}(t) = \sum_i |U_{fi}^* U_{f'i}|^2 + 2 \sum_{i>j} \text{Re} \left\{ U_{fi}^* U_{f'i}^* U_{fj}^* U_{f'j} \exp [i(E_j - E_i)t] \right\}$$

(3.2)

where $E_i \equiv (p^2 + m_i^2)^{1/2}$ is the energy of $\tilde{\nu}_i$. In the extreme relativistic limit for which we assume that all neutrinos have the same momentum $p \gg m_i$, we can approximate $E_j - E_i \approx (m_j^2 - m_i^2)/2p$. For $f \neq f'$, we thus see that this probability can be non-zero only if $m_i \neq m_j$ for some pair of mass eigenstates $(i, j)$ for which the appropriate matrix elements of $U$ are non-vanishing.

Of particular interest is the total probability that a given neutrino $\nu_f$ oscillates into any other state. This deficit probability is the complement of the probability that this neutrino is preserved, and from (3.2) this preservation probability can be easily evaluated as

$$P_{f \rightarrow f}(t) = \left| \sum_i |U_{fi}|^2 \exp(iE_i t) \right|^2 .$$

(3.3)

Note that these results apply not only to flavor oscillations (in which case we interpret the $f$ index as indicating flavor), but also to neutrino/anti-neutrino oscillations (in which case we identify $\nu_f = (\nu, N)$ for a fixed flavor) as well as general combinations of the two.

3.2 Higher-dimensional neutrino oscillations

The above formalism carries over directly into the higher-dimensional seesaw scenarios we presented in Sect. 2. For simplicity, let us focus first on neutrino/anti-neutrino oscillations, and consider the orbifold case discussed in Sect. 2.4. We have seen in Sect. 2 that in higher dimensions, it is natural to imagine a Kaluza-Klein tower for the right-handed neutrino, and that this automatically leads to a mixing
mass matrix of the form \((2.32)\). This then generates a set of corresponding mass eigenstates which we can denote
\[
\tilde{\mathcal{N}}^T \equiv (\tilde{\nu}_L, \tilde{\psi}_1^{(0)}, \tilde{N}^{(1)}, \tilde{M}^{(1)}, \tilde{N}^{(2)}, \tilde{M}^{(2)}, ...)
\] (3.4)
in analogy with \((2.10)\). Given these results, we see that \((3.2)\) and \((3.3)\) continue to hold; as before, we simply identify the matrix \(U\) in \((3.2)\) as the inverse of the matrix of eigenvectors of the mass matrix \(M\) given in \((2.32)\). Specifically, we write
\[
\mathcal{N} = U\tilde{\mathcal{N}}
\] (3.5)
where the gauge eigenstates \(\mathcal{N}\) are defined in \((2.10)\) and the mass eigenstates \(\tilde{\mathcal{N}}\) are defined in \((3.4)\). Note that since \(U\) is (by definition) the matrix that diagonalizes \(M\), the non-diagonality of \(M\) implies the non-diagonality of \(U\).

It turns out to be remarkably simple to obtain an exact result for this \(U\)-matrix in the case \((2.32)\), valid for all values of \(mR\). We find that \(U\) is given by
\[
U^\dagger = 
\begin{pmatrix}
U_+ \\
U_- \\
U_{+1} \\
U_{-1} \\
U_{+2} \\
U_{-2} \\
\vdots
\end{pmatrix}
\] (3.6)
where each individual row (i.e., each individual eigenvector) is given exactly by
\[
U_i \equiv \frac{1}{\sqrt{N_i}} \left(1, \frac{m}{\lambda_i}, \frac{m}{\lambda_i - 1/R}, \frac{m}{\lambda_i + 1/R}, \frac{m}{\lambda_i - 2/R}, \frac{m}{\lambda_i + 2/R}, ...\right).
\] (3.7)
Here \(\lambda_i\) are the mass eigenvalues which are the exact solutions to \((2.34)\), and likewise the normalization factor \(N_i\) in \((3.7)\) is given exactly by
\[
N_i = 1 + \frac{\pi^2 m^2 R^2}{\sin^2(\pi \lambda_i R)} = 1 + \pi^2 m^2 R^2 + \frac{\lambda_i^2}{m^2},
\] (3.8)
where we have used \((2.34)\) in the final equality. Note that approximate expressions for the eigenvalues \(\lambda_i\) are given for \(mR \ll 1\) in \((2.35)\) and \((2.36)\). Given the result \((3.6)\) for the \(U\)-matrix, it is then straightforward to show that
\[
U^\dagger M U = \text{diag}(\lambda_+, \lambda_-, \lambda_{+1}, \lambda_{-1}, \lambda_{+2}, \lambda_{-2}, ...),
\] (3.9)
thereby verifying that this \(U\)-matrix indeed diagonalizes \(M\).

Given this \(U\)-matrix, it is straightforward to calculate the corresponding probability of the neutrino gauge eigenstate \(\nu_L\) oscillating into any of the Kaluza-Klein excited
states \(\{\psi_0^{(0)}, N^{(k)}, M^{(k)}\}\), or conversely the probability that the neutrino \(\nu_L\) is preserved as a function of time. Using (3.3) and (3.7), we see that the latter probability is simply given by \[P_{\nu_L \rightarrow \nu_L}(t) = \left| \sum_i \frac{1}{N_i} \exp \left( \frac{i \lambda_i^2 t}{2p} \right) \right|^2. \tag{3.10}\]

The result is plotted in Fig. 4.

Figure 4: Higher-dimensional neutrino oscillations in the orbifold scenario discussed in Sect. 2.4. (a) The evolution of the probability sum in \((3.10)\) as more and more Kaluza-Klein states are included in the sum. We have taken \(mR = 0.4\). The flat line shows the contribution when only the degenerate zero-mode \(\lambda_\pm\) eigenvalues are included (no oscillations); the cosine shows the probability when the first excited Kaluza-Klein states are also included; and the irregular curve shows the interference that results when the second excited Kaluza-Klein states are also included. Note that the initial probability \(P(t = 0)\) approaches 1 as the full spectrum of Kaluza-Klein states is included. (b) The final result: the total probability that the gauge neutrino \(\nu_L\) is preserved as a function of time when all Kaluza-Klein states are included. The multi-component nature of the neutrino oscillation is reflected in the jagged shape of the oscillations, as well as in the fact that the resulting neutrino deficits and regenerations, though sizable, are never total.

One possible worry regarding such higher-dimensional oscillation scenarios might...
have initially seemed to be that because the neutrino oscillations take place within an infinite-state system whose mass eigenvalues are not commensurate, the resulting oscillations would tend to interfere destructively, thereby amounting to a neutrino “damping” without any possibility of neutrino regeneration. However, we now see from Fig. 4 that this is not the case, and we indeed continue to have oscillations with both neutrino deficits and neutrino regeneration. Thus, we see that while the multi-state oscillation has eliminated the formal periodicity of the oscillation probability as a function of time, the final result is still effectively periodic. This is the reflection of the fact that the dominant component of the oscillation is the simple two-state oscillation between the zero-mode neutrino states and the first excited Kaluza-Klein states. The “wavelength” of this oscillation is thus set by the lowest-lying eigenvalue difference $\lambda_{\pm 1}^2 - \lambda_1^2$. However, the striking signature of the multi-state nature of the oscillations is (in addition to their jagged profile) the fact that the neutrino never completely oscillates away or is restored — i.e., the neutrino deficits and regenerations are never total. This is therefore in strong contrast to the simpler case of two-component oscillations.

This result is not qualitatively affected when we consider the cases involving brane-shifting or bare Majorana masses. In the brane-shifted case, the result (3.7) continues to hold, where $m$ is replaced by $m^{(k)}_N$ for the components corresponding to $N^{(k)}$ and by $m^{(k)}_M$ for the components corresponding to $M^{(k)}$. Finally, we can also consider the case when a bare Majorana mass $M_0$ is included. In such situations, we find that the oscillation into the lowest Kaluza-Klein state is proportional to $m^2/\epsilon^2$, where $\epsilon \equiv M_0 \pmod{R^{-1}}$.

It is important to stress that there is also the usual possibility of flavor oscillations in addition to the oscillations into Kaluza-Klein states that we are discussing here. However, the above sorts of higher-dimensional oscillation scenarios easily generalize to the case of flavor oscillations: we simply introduce an additional flavor index, and imagine that our mass matrices are also non-diagonal in flavor space. Note that this last assumption is completely analogous to what must be assumed in the ordinary four-dimensional case. We then likewise find that the above neutrino/anti-neutrino oscillations can also indirectly induce flavor oscillations, with the flavor oscillations occurring indirectly through the masses and flavor mixings of the corresponding excited Kaluza-Klein states. In fact, such indirect flavor oscillations through excited Kaluza-Klein states can be viewed as the higher-dimensional analogue of the indirect flavor oscillations discussed in Ref. [28].

### 3.3 Comparison with experiment

Given the above results, the recent experimental detection of neutrino oscillations can be used to estimate the level spacings of the Kaluza-Klein states, which in turn permits us to estimate the size of the radius required. In the normal four-dimensional scenario, a neutrino mass difference of the order $\delta m^2 \sim 10^{-4}$ eV$^2$ is quoted [21].
as being sufficient to explain the oscillation observed at SuperKamiokande. In our higher-dimensional scenario, however, we have seen that this mass difference can be attributed not to the left-handed neutrinos, but to the Kaluza-Klein tower of $\tilde{N}^{(n)}$ and $\tilde{M}^{(n)}$ mass eigenstates whose masses are given by the eigenvalues \( \{\lambda_{\pm}, \lambda_{\pm k}\}, k \in \mathbb{Z}^+ \).

We have seen that for \( mR \ll 1 \), these eigenvalues are approximately given by \( \lambda_{\pm k} \approx k/R \); we have also seen above that the effective oscillation “wavelength” is set by the lowest-lying eigenvalue difference \( \lambda_{\pm 1} - \lambda_{\pm} \approx R^{-2} \). Thus, we can roughly associate \( \delta m^2 \) with \( R^{-2} \), obtaining the estimate \( R \approx 10^{-5} \) meters. Such an extra dimension would therefore be perfectly consistent with the scenario advocated in Ref. [2], which would in turn enable us to identify the extra dimension we have been discussing as one which only gravity (and our higher-dimensional $\Psi$ field) can experience. As discussed in Ref. [2], a “gravity-only” extra dimension of this size is believed to be consistent with all laboratory, astrophysical, and cosmological constraints.

Pursuing this line of reasoning a bit further, we may even use the results given in (2.9) and (3.3) in conjunction with the mixing-parameter bound \( \sin^2 2\theta > 0.82 \) given in Ref. [21]. If we associate the finite value of \( n \) with \( M_s R \) (as might be expected in an effective field-theory approach where we keep only the lowest excitations of the Kaluza-Klein tower), we can obtain a rough bound on the string scale \( M_s \lesssim \tilde{m} \sim 1 \) TeV for \( \delta = 2 \) and for a Yukawa coupling \( \sim \mathcal{O}(1) \) for oscillations into the first Kaluza-Klein state. Thus, it would appear that the experimental bound on the mixing rules out larger values of \( \delta \), so that only two extra dimensions felt by the higher-dimensional neutrino field are consistent with the SuperKamiokande results. However, more flexibility is allowed in the case of flavor oscillations, which we have not discussed here.

Of course, the above analysis is at best only qualitative. Ultimately, one would also like to take into account the data concerning both atmospheric and solar neutrinos. Likewise, one would also need to take into account the energy-dependence of the experimental signals. We leave this subject for future investigation.

4 Neutrino oscillations without neutrino masses

Finally, let us turn our attention to something far more speculative: the possibility of neutrino oscillations without neutrino masses.

In the usual four-dimensional seesaw mechanism, neutrino oscillations require (and therefore can be interpreted as the unique signature of) neutrino masses. Let us recall why this is the case for neutrino/anti-neutrino oscillations. In the usual seesaw mechanism, we are required to have a mass matrix of the form (2.1). Regardless of the Majorana mass \( M \) of the right-handed neutrino \( N \), the only way to achieve a massless neutrino in this scenario is to set \( m = 0 \). However, this then results in a diagonal mass matrix, so that the corresponding matrix \( U \) of eigenvectors is also diagonal. Therefore no oscillations are produced. Consequently, the only way to have
neutrino oscillations in this scenario is to have neutrino masses. This argument can also be extended to the case of flavor oscillations.

The crucial ingredient in the above four-dimensional argument is that matrices of the form (2.1) cannot have a vanishing mass eigenvalue without being diagonal. In higher dimensions, however, we have seen that our mass matrices are infinite-dimensional (corresponding to mixings between the infinite numbers of Kaluza-Klein modes), and therefore this constraint may be relaxed. This then would permit the possibility of neutrino oscillations without neutrino masses.

As a concrete example of this phenomenon, let us consider the results of Sect. 2.6, where we examined the consequences of introducing a bare Majorana mass $M_0$ for the higher-dimensional field $\Psi$. We showed, remarkably, that the overall scale of this Majorana mass completely decouples from the problem, and that only $\epsilon \equiv M_0 \pmod{R^{-1}}$ plays a role in determining the mass of the resulting neutrino mass eigenstate.

Let us now consider what happens in the special case that $\epsilon = \frac{1}{2} R^{-1}$. After further reordering of the rows and columns corresponding to the excited Kaluza-Klein states, the mass matrix (2.43) then takes the form

\[
\mathcal{M} = \begin{pmatrix}
0 & m & m & m & m & m & m & \ldots \\
m & 1/(2R) & 0 & 0 & 0 & 0 & 0 & \ldots \\
m & 0 & -1/(2R) & 0 & 0 & 0 & 0 & \ldots \\
m & 0 & 0 & 3/(2R) & 0 & 0 & 0 & \ldots \\
m & 0 & 0 & 0 & -3/(2R) & 0 & 0 & \ldots \\
m & 0 & 0 & 0 & 0 & 5/(2R) & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}.
\]

In order to obtain the corresponding neutrino mass, we note that for $\epsilon = \frac{1}{2} R^{-1}$, the assumption $mR \ll 1$ translates into $\epsilon \gg m$, whereupon the result (2.44) is valid. Thus, for $\epsilon = \frac{1}{2} R^{-1}$, we find the remarkable result that $m_{\nu} = 0$! In obtaining this result, one might worry that (2.44) is only approximate because it relies on the procedure of integrating out the Kaluza-Klein states rather than a full diagonalization of the corresponding mass matrix. However, it is straightforward to show that when $\epsilon = \frac{1}{2} R^{-1}$, the characteristic eigenvalue equation $\det(\mathcal{M} - \lambda I) = 0$ for the mass matrix (4.1) becomes

\[
\lambda R \prod_{k=1}^{\infty} \left( \lambda^2 R^2 - (k - \frac{1}{2})^2 \right) \left[ 1 - 2m^2 R^2 \sum_{k=1}^{\infty} \frac{1}{\lambda^2 R^2 - (k - 1/2)^2} \right] = 0. \quad (4.2)
\]

This has an exact trivial solution $\lambda = 0$, corresponding to an exactly massless neutrino. Thus, we conclude that $m_{\nu} = 0$ for $\epsilon = \frac{1}{2} R^{-1}$, regardless of the relative sizes of $m$ and $R$.

There is also another useful way to understand the emergence of this vanishing eigenvalue. For $m \neq 0$, it is straightforward to see that the value $\lambda = (k + \frac{1}{2})/R$ is not
Figure 5: Eigenvalue solutions to (4.3), represented as those values of $\lambda$ for which $-\tan(\pi \lambda R)$ intersects $\lambda R/\left[\pi (mR)^2\right]$. We have taken the fixed value $mR = 0.4$ for this plot. The behavior of the eigenvalues as functions of $mR$ can be determined graphically by changing the slope of the intersecting diagonal line. Regardless of the value of $mR$, we see that the zero eigenvalue is fixed and unique.

a solution of (4.2) because the cancellation of the first bracketed factor is offset by the divergence of the second bracketed factor. We can therefore restrict our attention to the second bracketed factor, and reduce (4.2) to the form

$$\lambda R = -\pi (mR)^2 \tan (\pi \lambda R) .$$

(4.3)

This equation is the analogue of (2.34), and upon plotting this condition graphically, we find the result shown in Fig. 5. Remarkably, this is effectively the same as Fig. 3(a) except that the cotangent curves of Fig. 3(a) have been translated horizontally by $\frac{1}{2} \lambda R$. We thus see that the effect of adding a bare Majorana mass term corresponding to $\epsilon = \frac{1}{2} R^{-1}$ is simply to shift the positions of the cotangent curves by exactly half of the oscillation period. This explains graphically why the zero eigenvalue emerges precisely in the case $\epsilon = \frac{1}{2} R^{-1}$. We also see from this figure that the zero eigenvalue is independent of the value of $mR$, and is unique.

Note that this graphical result is completely general: the effect of adding a general bare Majorana mass term $M_0$ is simply to shift the positions of the cotangent curves.
by an amount proportional to $M_0$. In fact, by changing the value of $M_0$, we see that it is possible to smoothly interpolate between the orbifold scenario discussed Sect. 2.4 and the scenario we are discussing here. This also provides another explanation of why only the value $\epsilon \equiv M_0 \text{(modulo } R^{-1})$ is relevant physically. The regular, repeating aspect of the infinite towers of Kaluza-Klein states is now manifested graphically in the periodic nature of the cotangent function.

We can also solve for the full spectrum of eigenvalues as a function of $mR$. Following the same steps as in Sect. 2.4, we find that the non-zero eigenvalues are identical to those given in (2.35) for $k \neq 0$, except with $k \to k - 1/2$. To order $O(m^5 R^5)$, this yields the non-zero eigenvalues

\[
\lambda_{\pm k} = \pm \frac{k - 1/2}{R} \left[ 1 + \frac{m^2 R^2}{(k - 1/2)^2} - \frac{m^4 R^4}{(k - 1/2)^4} + \ldots \right], \quad k > 0.
\]

(4.4)

It is also straightforward to calculate the exact neutrino mass eigenstate that corresponds to our vanishing eigenvalue. If we use the basis of gauge eigenstates corresponding to the mass matrix (4.1), and denote this basis as \{\nu_L, \hat{N}^{(1)}, \hat{M}^{(1)}, \hat{N}^{(2)}, \hat{M}^{(2)}, \ldots\}, we easily see that the neutrino mass eigenstate $|\tilde{\nu}_L\rangle$ is given by the (normalized) result

\[
|\tilde{\nu}_L\rangle = \frac{1}{\sqrt{1 + \pi^2 m^2 R^2}} \left\{ |\nu_L\rangle - mR \sum_{k=1}^{\infty} \frac{1}{k - 1/2} \left[ |\hat{N}^{(k)}\rangle - |\hat{M}^{(k)}\rangle \right] \right\}.
\]

(4.5)

This result is exact for all $mR$, and is essentially the analogue of (2.37) in which one replaces $k \to k - 1/2$. Also note that this neutrino mass eigenstate is primarily composed of the neutrino gauge eigenstate $\nu_L$, since $mR \ll 1$. Although this neutrino mass eigenstate also contains a small, non-trivial admixture of Kaluza-Klein states, the dominant component of our massless eigenstate is still the gauge-eigenstate neutrino $\nu_L$, as required phenomenologically. Nevertheless, this combined neutrino mass eigenstate is exactly massless in the limit that the full, infinite tower of Kaluza-Klein states participates in the mixing! We stress that this remarkable result is valid regardless of the value of neutrino Yukawa coupling $m$ or the radius scale $R^{-1}$ of the Kaluza-Klein states.

Given this result, it is straightforward to calculate the probability that the neutrino gauge eigenstate oscillates into any of the excited Kaluza-Klein states $\hat{N}^{(k)}$ or $\hat{M}^{(k)}$, or conversely the probability that the neutrino $\nu_L$ is preserved as a function of time. We find that the $U$-matrix corresponding to (4.4) once again takes the form (3.6), where now the individual rows (i.e., eigenvectors) are given exactly for all values of $mR$ by

\[
U_i \equiv \frac{1}{\sqrt{N_i}} \begin{pmatrix} 1, & \frac{m}{\lambda_i - 1/(2R)}, & \frac{m}{\lambda_i + 1/(2R)}, & \frac{m}{\lambda_i - 3/(2R)}, & \frac{m}{\lambda_i + 3/(2R)}, & \ldots \end{pmatrix}
\]

(4.6)
and the normalization factor $N_i$ in (4.6) is given by

$$N_i = 1 + \frac{\pi^2 m^2 R^2}{\cos^2(\pi \lambda R)} = 1 + \pi^2 m^2 R^2 + \frac{\lambda^2}{m^2}. \quad (4.7)$$

Here we have used (4.3) in the final equality. Substituting this result into (3.3), we then find the preservation probability shown in Fig. 6.

---

**Figure 6:** Higher-dimensional neutrino oscillations, even when the neutrino itself is massless. (a) The evolution of the probability sum in (4.10) as more and more Kaluza-Klein states are included in the sum. We have taken $mR = 0.4$. The flat line shows the contribution when only the massless neutrino is included (no oscillations); the cosine shows the probability when the first excited Kaluza-Klein states are also included; and the irregular curve shows the interference that results when the second excited Kaluza-Klein states are also included. Note that the initial probability $P(t = 0)$ approaches 1 as the full spectrum of Kaluza-Klein states is included. (b) The final result: the total probability that the gauge neutrino $\nu_L$ is preserved as a function of time when all Kaluza-Klein states are included. The multi-component nature of the neutrino oscillation is reflected in the jagged shape of the oscillations. Unlike the oscillation in Fig. 4(b), however, in this case the deficits are total even though the regenerations are not.

Fig. 6 provides an explicit verification that neutrino oscillations do indeed occur, even though the physical neutrino is exactly massless. Of course, this result is expected, because the mass matrix (4.1) is non-diagonal. Therefore the full mixing matrix $U$ that diagonalizes $M$ must also be non-diagonal. The fact that this $U$-matrix is non-diagonal then leads to the non-trivial mixings that produce oscillations.

Just as in Sect. 3, we observe that these higher-dimensional neutrino oscillations lead to neutrino deficits as well as neutrino regeneration, in a roughly periodic man-
ner. Specifically, there is no neutrino “damping” arising from the separate incommensurate Kaluza-Klein oscillations, as might have been feared. However, unlike the oscillation in Sect. 3, we see that in this scenario the deficits are total even though the regenerations are not. This could therefore serve as a potential experimental method of distinguishing between this scenario and that discussed in Sect. 2.4.

Thus, we conclude that the neutrino mass eigenstate oscillates into the entire tower of higher-dimensional Kaluza-Klein neutrinos, even though it has no mass of its own! Indeed, the masses of the right-handed Kaluza-Klein states themselves are sufficient to generate the desired oscillations indirectly. Although this mechanism applies for neutrino/anti-neutrino oscillations, it can also easily be generalized to accommodate flavor oscillations as well, even if \((\nu_e, \nu_\mu, \nu_\tau)\) are all taken to be massless.

One possible drawback to this scenario might initially seem to be that it requires the precise value \(\epsilon = \frac{1}{2} R^{-1}\), corresponding to a precise bare Majorana mass of the form \(M_0 = (n + \frac{1}{2})/R\) where \(n \in \mathbb{Z}\). This would therefore seem to require a precise fine-tuning. However, it turns out that bare Majorana masses of precisely this form emerge naturally from Scherk-Schwarz decompositions in string theory. Recall that our original five-dimensional Dirac spinor field \(\Psi\) is decomposed in the Weyl basis as \(\Psi = (\psi_1, \bar{\psi}_2)^T\), where \(\psi_1\) and \(\psi_2\) individually have the orbifold mode-expansions given in (2.6). However, let us consider performing a local rotation in \((\psi_1, \psi_2)\) space of the form

\[
\begin{pmatrix}
\bar{\psi}_1 \\
\bar{\psi}_2
\end{pmatrix} \equiv R
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
\]

where

\[
R \equiv \begin{pmatrix}
\cos(\omega y/R) & -\sin(\omega y/R) \\
\sin(\omega y/R) & \cos(\omega y/R)
\end{pmatrix}.
\]  (4.8)

Such a general rotation is allowed in field theory because it corresponds to a \(U(1)\) symmetry of the higher-dimensional theory. However, in string theory there are additional topological constraints (coming from the preservation of the form of the worldsheet supercurrent) that permit only discrete rotations. In particular, in a compactification from five to four dimensions, this restriction limits us to the only non-trivial possibility \(\omega = 1/2\). (The trivial case \(\omega = 0\) corresponds to the straightforward orbifold situation discussed in Sect. 2.4.) Taking \(\omega = 1/2\) then implies \(\psi_{1,2}(2\pi R) = -\psi_{1,2}(0)\), which shows that lepton number is broken globally (although not locally) as the spinor is taken around the compactified space. This is the result of the discrete “twist” induced by the Scherk-Schwarz \(R\)-matrix. After the Kaluza-Klein decomposition, this breaking of lepton number in turn induces a Majorana mass term with \(M_0 = \omega/R\) (modulo \(R^{-1}\)). Thus, we see that exactly the desired value of the Majorana mass emerges naturally from a Scherk-Schwarz decomposition, for reasons that are topological and hence do not require any fine-tuning.

Thus, in this respect, the scenario that leads to an exactly massless neutrino is the Scherk-Schwarz “twisted” counterpart of the straightforward orbifold scenario of Sect. 2.4. Relative to the orbifold scenario, we see that this twisting introduces lepton-number violation in a natural way, brings the neutrino mass to zero, and also breaks the two-fold degeneracy for the lightest ground state — all while maintaining neutrino oscillations.
Note also that the masslessness of the neutrino mass eigenstate relies rather crucially on taking the full $n \to \infty$ limit in the above calculation. This might seem to go against the spirit of the effective field-theory approach we have been following wherein we would truncate the Kaluza-Klein sum at $n_{\text{max}} \sim \mathcal{O}(M_s R)$. Nevertheless, it is not unreasonable to expect that in the full underlying string theory, a similar mechanism might be implemented once all of the string states (not only Kaluza-Klein states, but also winding states and oscillator states) are properly included. This would represent a uniquely “stringy” behavior, not unlike the Hagedorn phenomenon \cite{29} which also emerges only when all string states are included.

Indeed, even within the effective field-theory approach that we have been following, the resulting neutrino mass is extraordinarily suppressed. To see this, let us imagine truncating our Kaluza-Klein levels at $n_{\text{max}} \sim \mathcal{O}(M_s R)$ where $M_s$ is the mass scale of the underlying (string) theory, and let us take $M_0 = \frac{1}{2} R^{-1}$ as suggested by the Scherk-Schwarz analysis above. In this case, the original mass matrix \eqref{2.41} takes the form

$$
\mathcal{M} = \begin{pmatrix}
0 & m & m & m & m & m & \cdots \\
m & 1/(2R) & 0 & 0 & 0 & 0 & \cdots \\
m & 0 & 3/(2R) & 0 & 0 & 0 & \cdots \\
m & 0 & 0 & -1/(2R) & 0 & 0 & \cdots \\
m & 0 & 0 & 0 & 5/(2R) & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

\hfill (4.9)

where we have not performed any reordering of the rows and columns corresponding to the excited Kaluza-Klein states. If we truncate the Kaluza-Klein states at a chosen value $n_{\text{max}}$, we see that we always have an unpaired diagonal element of size $(n_{\text{max}} + \frac{1}{2})/R$. Thus, when the excited Kaluza-Klein states are integrated out, this leaves a net contribution to the neutrino mass:

$$m_\nu \approx \frac{m^2 R}{n_{\text{max}} + 1/2} \approx \frac{m^2}{M_s}.
$$

\hfill (4.10)

If we imagine $m \lesssim R^{-1} \approx 10^{-2}$ eV (as would roughly be required to explain the neutrino oscillations) and $M_s \approx 10$ TeV, this gives rise to $m_\nu \approx 10^{-15}$ eV. Thus, we see that sufficiently sizable neutrino oscillations can be generated even with vanishingly small neutrino masses!

We conclude, then, that sizable neutrino oscillations can occur regardless of the actual mass of the neutrino, thanks to the indirect masses and mixings of the Kaluza-Klein states. Thus, if such a scenario can be realized within the context of a fully realistic string model, then the recent observations of neutrino oscillations can be re-interpreted not as providing evidence for neutrino masses, but rather as providing evidence for extra spacetime dimensions!
5 Conclusions, discussion, and open questions

The scenarios that we have outlined in this paper are certainly unorthodox, and so far they are only qualitative. Certainly we have not performed a detailed comparison to see if the wealth of existing experimental neutrino data can be accommodated or explained in this manner. Our goal, as we have stated throughout, has merely been to provide a number of qualitative mechanisms which are capable of yielding light neutrino masses without the ad hoc introduction of heavy mass scales. The important task of implementing these mechanisms within self-consistent string models remains. It also remains necessary to perform a detailed comparison with experimental data.

However, even at this preliminary stage, there are several theoretical and phenomenological challenges that these scenarios face. We would therefore like to conclude by discussing what some of these challenges are.

One important theoretical issue which we have not addressed concerns the dynamics of the branes to which the Standard-Model fields (but not the right-handed neutrino) are presumably restricted. This issue may ultimately play an important role in the seesaw mechanism because it has the potential to affect the form of our mass mixing matrices. Let us consider the matrix (2.11) for concreteness. As we discussed above, the non-zero entries along the first row/column reflect the coupling of the left-handed neutrino to the excited Kaluza-Klein modes of the right-handed neutrino field. Such couplings do not conserve momentum in the compactified directions, but are allowed because the presence of the brane to which the left-handed neutrinos are restricted breaks translational invariance in these directions.

However, while these sorts of couplings are permitted in the case of an infinitely rigid brane (as is typical in many standard treatments), in reality the brane can be expected to have a dynamics of its own. In such cases, the couplings between the fields on the brane and the fields in the bulk will become more complicated, and will presumably involve the fluctuation modes of the brane itself. A full analysis of this question is beyond the scope of this paper.

Likewise, in the general matrix (2.11), we have set all remaining off-diagonal entries to zero. As discussed in Sect. 2.2, this reflects Kaluza-Klein momentum conservation for couplings purely between fields in the bulk. However, this too is only an approximation: in a complete theory (such as a string theory), we can expect there to be higher-order couplings between different Kaluza-Klein modes of the bulk fields that arise indirectly through their momentum-violating couplings to fields on the brane. However, once again this is a higher-order effect which can be neglected at our level of approximation. Furthermore, even if such couplings are included, we do not expect our primary results to be significantly affected.

Turning to phenomenological considerations, we also find a number of outstanding questions. One such question involves decays such as $\pi \rightarrow \mu \bar{\nu}_\mu$. Experimental measurements of the muon momentum spectrum reveal a very sharp peak, confirming the fact that there is only one neutrino that carries away momentum in such a decay.
However, in the higher-dimensional seesaw mechanisms we have been discussing, the weak gauge charge of the gauge-eigenstate neutrino is ultimately distributed over a whole tower of excited Kaluza-Klein states. Therefore, the infinite tower of Kaluza-Klein states will, in principle, partake in such processes. Moreover, the lowest-lying Kaluza-Klein states can be relatively light. In such situations, the muon momentum spectrum would therefore be expected to be quasi-continuous rather than discrete. This therefore has the potential to severely constrain our higher-dimensional seesaw scenarios (or generally, any scenario in which the left-handed neutrino mixes with an infinite tower of Kaluza-Klein neutrinos). One redeeming feature of these scenarios, however, is the extreme suppression of the Kaluza-Klein admixture. For example, in (4.3) we have seen that the Kaluza-Klein components are suppressed by \( m_R \), where \( m_R \ll 1 \). Thus, these admixtures may be sufficiently small to evade these sorts of experimental bounds.

Another important phenomenological issue concerns the ultimate stability of the light (or vanishing) neutrino masses that are generated by these higher-dimensional seesaw mechanisms. Although these mechanisms naturally yield light neutrino masses, these masses must still be made stable against possible higher-order operators that can be generated in the full effective (string) theory. Discrete symmetries may be able to accomplish this, but we have not investigated this possibility in this paper. Nevertheless, it is interesting to note that any higher-order operators that tend to generate effective heavy neutrino Majorana mass terms will \emph{not} destabilize our results, for we have already seen in Sect. 2.6 that the effect of the infinite towers of Kaluza-Klein states is to eliminate the dependence on such an external Majorana mass scale \( M_0 \) (regardless of its origin), automatically replacing this scale with the new light scale \( \epsilon \equiv M_0 \) (modulo \( R^{-1} \)). Thus, this feature may also provide some protection against higher-order destabilizing effects. Moreover, in the particular case of the Scherk-Schwarz breaking of lepton number, we expect this breaking to vanish in the \( R \to \infty \) limit (with \( M_s \) assumed fixed). Therefore, any such higher-dimensional operators must be suppressed by powers of \( R \).

Thus, to summarize the main results of this paper, we have seen that there exist several higher-dimensional analogues of the usual seesaw mechanism in which a mixing between a left-handed neutrino and an infinite Kaluza-Klein tower of higher-dimensional bulk fields has the potential to produce a light neutrino mass eigenstate whose mass is suppressed relative to the other mass scales in the problem. Moreover, this occurs without the introduction of an arbitrary high mass scale. Even if such a bare Majorana mass scale is present, we have seen that the higher-dimensional seesaw mechanism with its summation over the Kaluza-Klein states essentially eliminates this scale in favor of the large radius of the extra spacetime dimensions. We also pointed out a possible new mechanism, involving brane shifting within Type I string theory, for generating lepton-number violation and thereby also inducing neutrino oscillations. Finally, we also proposed an explicit higher-dimensional seesaw mechanism in which neutrino oscillations occur even without neutrino masses. In
this case, the neutrino oscillations occur indirectly thanks to the masses and mixings of the Kaluza-Klein towers of bulk neutrinos or through the usual flavor oscillations (which we have not discussed here).

Furthermore, as we remarked above, our higher-dimensional seesaw scenarios are not restricted to the case of a single extra dimension. Indeed, it is straightforward to extend these sorts of scenarios to arbitrary numbers of extra dimensions, and similar results are obtained. Moreover, one may even consider different radii (and even different fields) for the different dimensions in order to explain different types of neutrino oscillations. This might therefore be capable of leading to a richer and more flexible neutrino phenomenology than is possible within the usual four-dimensional framework.

Thus, if these qualitative scenarios can be made to operate within the context of fully realistic models, they might provide higher-dimensional mechanisms for generating light (or even vanishing) neutrino masses without the need for an intrinsic heavy mass scale for right-handed neutrinos. However, as we have also seen, these sorts of scenarios face a number of outstanding open questions. It is important to stress that these problems are not specific to the scenarios that we have put forth here, but rather generically arise whenever the right-handed neutrino field is allowed to feel large extra spacetime dimensions and whenever its resulting Kaluza-Klein states can also couple to the left-handed neutrinos. Thus, the open questions that we have discussed above have a generality that transcends the specific mechanisms we have outlined, and will ultimately need to be addressed in any scenario utilizing higher-dimensional right-handed neutrinos.

Acknowledgments

We wish to thank K. Benakli, B. Campbell, C. Johnson, J. Lykken, R. Myers, Y. Nambu, J. Pati, A. Pomarol, G. Shiu, R. Shrock, S.-H.H. Tye, F. Wilczek, and A. Zaffaroni for discussions. We also wish to thank P. Ramond for originally encouraging us to consider the problem of neutrino masses in higher dimensions.

Note Added

After this paper originally appeared in November 1998, some readers apparently became confused regarding the violation of lepton number in the higher-dimensional seesaw mechanism presented in Sect. 2.3. We therefore reiterate that the kinetic-energy terms in (2.14) conserve lepton number, as required, since the apparent non-conservation from the kinetic term for the $N$ field is precisely cancelled by the kinetic term for the $M$ field. In other words, even though one Majorana component of each Kaluza-Klein mass term has decoupled in (2.14), the underlying Kaluza-Klein masses are indeed Dirac masses, as required. It is only the interaction between the left- and right-handed neutrinos which breaks lepton number, as necessary in order to obtain neutrino oscillations of the sort we have been discussing. Moreover, in order to avoid
possible future confusion, we would like to reiterate that our scenario of neutrino oscillations without neutrino masses (as discussed in Sect. 4) has all of the following features: the neutrino is exactly massless for all values of \( m_R \), as shown in Fig. 5; the massless eigenstate is unique and is primarily composed of the gauge neutrino \( \nu_L \) rather than the Kaluza-Klein modes, as shown in (4.3); the resulting oscillations are sizable and effectively periodic, as shown in Fig. 6; and this scenario can be realized directly in string theory through an orbifold Scherk-Schwarz compactification, as discussed below (4.8). In such a compactification, the breaking of lepton number is topological and fixed by the string boundary conditions.

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