A Dynamical Approach to Efficient Eigenvalue Estimation in General Multiagent Networks

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Abstract

We propose a method to efficiently estimate the eigenvalues of any arbitrary, unknown network of interacting dynamical agents. The inputs to our estimation algorithm are measurements about the evolution of the outputs of a subset of agents (potentially one) during a finite time horizon; notably, we do not require knowledge of which agents are contributing to our measurements. We propose an efficient algorithm to exactly recover the eigenvalues corresponding directly to those modes that are recoverable from our measurements. We show how our technique can be applied to networks of multiagent systems with arbitrary dynamics in both continuous- and discrete-time. Finally, we illustrate our results with numerical simulations.

Key words: Multiagent networks; eigenvalue estimation; sparse estimation; spectral identification; Laplacian matrix

1 Introduction

The spectrum of a graph matrix describing a network of interacting dynamical agents provides a wealth of global information about the network structure and function; see, e.g., Fiedler (1973); Mohar et al. (1991); Merris (1994); Chung and Graham (1997); Preciado (2008); Mesbahi and Egerstedt (2010); Bullo (2019), and references therein. A particular example of interest is the Laplacian spectrum, which finds applications in multiagent coordination problems (Jadbabaie et al., 2003; Olfati-Saber et al., 2007), synchronization of oscillators (Pecora and Carroll, 1998; Dörfler et al., 2013), neuroscience (Becker et al., 2018), biology (Palsson, 2006), as well as several graph-theoretical problems, such as finding cuts (see Shi and Malik, 2000) or communities (see Von Luxburg, 2007) in graphs, among many others, as illustrated in Mohar (1997).

Due to its practical importance, numerous methods have been proposed to estimate the eigenvalues of a network of dynamical agents. For example, Kempe and McSherry (2008) proposed a distributed algorithm based on orthogonal iteration (see Golub and Van Loan, 2013) for computing higher-dimensional invariant subspaces. In the control literature, Franceschelli et al. (2013) define local interaction rules between agents such that the network response is a superposition of sinusoids oscillating at frequencies related to the Laplacian eigenvalues; however, this approach imposes a particular dynamics on the agents in the network, which is unrealistic in many scenarios. Aragues et al. (2014) proposed a distributed algorithm based on the power iteration for computing upper and lower bounds on the algebraic connectivity (i.e., the second smallest Laplacian eigenvalue). Leonados et al. (2019) proposed a distributed continuous-time dynamics over manifolds to compute the largest (or smallest) eigenvalues and eigenvectors of any graph. An approach by Kibangou et al. (2015) uses consensus optimization to deduce the spectrum of the Laplacian, but this requires a consensus algorithm to be run on the network separately from the dynamics. Using the Koopman operator, it has been shown that the spectrum of the Laplacian may be recovered using sparse local measurements, see Mauroy and Hendrickx (2017); Mesbahi and Mesbahi (2019); unfortunately, these methods require the system to be reset to known initial conditions multiple times.

We find in the literature several works more closely related to the techniques used in this paper. For example, an approach known as Prony’s method reconstructs the parameters of a uniformly sampled series of complex exponentials, which is used for spectral estimation and deconvolution, among other problems (see Potts and Tasche, 2010; Kunis et al., 2016). In contrast to our approach, Prony’s method only applies to symmetric ma-
traces, i.e., only for directed networks. In linear algebra we find the Newton-Girard equations (see, e.g., Herstein, 2006) which allows us to recover eigenvalues by analyzing symmetric polynomials of the traces of powers of the matrix. However, the traces of powers of matrices required represents a large amount of (centralized) data, which may not be feasible to collect in many applications. Using local structural information, Preciado and Jadbabaie (2013) computed the spectral moments of a graph in order to determine bounds on spectral properties of practical importance. A related method uses tools from probability theory to approximate the spectrum of a graph by counting the number of walks of length $k$ and then solving the classical moment problem, as in Preciado et al. (2013); Chen et al. (2020); Barreras et al. (2019). The latter approach requires only local measurements of walks, but provides only bounds on the support of the eigenvalue spectrum.

In this paper we present an approach to estimate the eigenvalues of any graph matrix, such as the Laplacian, corresponding to an unknown network of multingent systems using only a single finite sequence of measurements. The network structure of the system may be directed, and may incorporate edge weights, multi-edges, and self-loops. The finite sequence used as input to our spectral estimation algorithm can correspond to the output signal of a single agent, or to any weighted linear combination of outputs from a collection of agents; notably, our method requires no knowledge of which agents contribute to the measurements, nor does it require prior knowledge of the network topology or initial condition. Moreover, the length of the sequence of measurements required is at most twice the number of agents in the network, but fewer in practice. Our approach allows for the estimation of all recoverable eigenvalues from these sparse measurements, regardless of the (unknown) network structure. This approach requires no tuning of parameters, and may be applied in both discrete- and continuous-time to general multi-agent systems.

The remainder of this paper is structured as follows. We outline background and notation for our problem in Section 2. We introduce our approach on the simpler case of discrete-time Laplacian dynamics in Section 3. In Section 4 we present our results for discrete-time systems, and in Section 5 we describe our results in the continuous-time case. Finally, Section 6 illustrates our results via simulations in a variety of systems.

## 2 Background and Notation

Throughout this paper we use lower-case letters for scalars, lower-case bold letters for vectors, upper-case letters for matrices, and calligraphic letters for sets.

A directed graph $G = (V, E)$ has node set $V$ and edge set $E$, where $(i, j) \in E$ means node $i$ has an edge directed toward node $j$. The graph $G$ may have self-loops, may have (possibly negative) edge weights, and may contain multi-edges.

| Symbol | Meaning |
|--------|---------|
| $I_n$  | $n \times n$ identity matrix |
| $\mathbb{R}$ | set of real numbers |
| $e_i$   | $i$-th vector in the canonical basis of $\mathbb{R}^n$ |
| $\mathcal{V}$ | node set, $\mathcal{V} = \{1, \ldots, n\}$ |
| $\mathcal{E}$ | edge set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ |
| $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ | graph with node set $\mathcal{V}$ and edge set $\mathcal{E}$ |
| $\otimes$ | Kronecker product |
| $\oplus$ | Direct sum |
| $\sigma(X) := \{\lambda_i\}_{i=1}^n$ | eigenvalue spectrum of matrix $X$ |
| $A(\mathcal{G})$ | adjacency matrix of $\mathcal{G}$, $[A]_{ij} \neq 0 \Rightarrow (i, j) \in \mathcal{E}$ |
| $D(\mathcal{G})$ | degree matrix of $\mathcal{G}$, $[D]_{ii} = \sum_{j=1}^n [A]_{ij}$ |

### 3 Discrete-Time Laplacian Dynamics

We begin our exploration with a simple exposition of a network of single integrators following a discrete-time (DT) Laplacian dynamics. In this context, we will present a methodology to estimate the eigenvalues of the Laplacian matrix of an undirected network from a finite sequence of measurements of our system; for full details of this case, see Hayhoe et al. (2019). In Section 4, we will extend this result to more general directed networks of discrete-time agents, and will consider the general continuous-time (CT) case in Section 5.

Consider the discrete-time dynamics of a collection of single integrators,

$$
\begin{align*}
x[k+1] &= \mathcal{L}x[k], \quad x[0] = x_0, \\
y[k] &= c^T x[k],
\end{align*}
$$

where $\mathcal{L} := D(\mathcal{G})^{-1}A(\mathcal{G})$ is the normalized Laplacian matrix of an unknown undirected graph $\mathcal{G}$, $k \in \mathbb{N}$, and $c, x_0$ are arbitrary (possibly unknown) vectors in $\mathbb{R}^n$. For example, we may have $c = e_i$ when we only observe the state of agent $i$, or $c = \sum_{i \in S \subseteq V} \beta_i e_i$ when we observe the weighted sum of the states of a subset $S$ of agents. Thus, the evolution of the output measurement is $y[k] = c^T \mathcal{L}^k x_0$. In what follows we propose an efficient algorithm to recover the eigenvalues of the normalized Laplacian matrix $\mathcal{L}$ from the output sequence $y[0], y[1], \ldots, y[2n-1]$.

Let $u_i$ and $w_i$ be the (unknown) $i$-th right and left eigenvectors of $\mathcal{L}$, respectively. Since $\mathcal{L}$ is always diagonalizable with real eigenvalues when $\mathcal{G}$ is undirected (see Chung and Graham, 1997), we have that $\mathcal{L} = U \Lambda W$, where
where \( \Lambda := \text{diag}(\lambda_1, \ldots, \lambda_n) \), \( U := [u_1, \ldots, u_n] \), and \( W := [w_1^T, \ldots, w_n^T] = U^{-1} \); hence,

\[
y[k] = (c^T U) \Lambda^k (W x_0) = \sum_{i=1}^{n} \omega_i \lambda_i^k, \tag{2}
\]

where \( \lambda_i \) is the \( i \)-th real eigenvalue of \( \mathcal{L} \), and the weights \( \omega_i \) are given by

\[
\omega_i := [c^T U]_i [W x_0] = c^T u_i w_i^T x_0. \tag{3}
\]

Notice that it is possible for \( \omega_i = 0 \) whenever \( c^T u_i = 0 \) or \( w_i^T x_0 = 0 \). If \( \omega_i = 0 \) for some index \( i \), then the \( i \)-th eigenvalue \( \lambda_i \) does not influence the output \( y[k] \) in (2); consequently, we will not be able to estimate \( \lambda_i \) from a finite sequence of outputs. However, if \( x_0 \) is randomly generated, then almost surely \( w_i^T x_0 \neq 0 \); hence, it is possible that \( \omega_i = 0 \) only for those eigenvalues \( \lambda_i \) for which \( c^T u_i = 0 \). Therefore, according to the Popov-Belevitch-Hautus (PBH) test (see Hespanha, 2018), those eigenvalues corresponding to unobservable eigenmodes of the Laplacian dynamics will have \( \omega_i = 0 \) and it will be impossible to recover them from our observations. However, there are other (observable) eigenvalues that our method will not be able to recover. In particular, it may be that for some repeated eigenvalue \( \lambda_i \), we have \( \sum_j: \lambda_j = \lambda_i, \omega_j = 0 \) and our method will not be able to recover \( \lambda_i \), since \( \lambda_i \) would not have an influence on the summation in (2). Hence, the recoverable eigenvalues are described by the set

\[
\mathcal{S}_\mathcal{L} := \{ \lambda_i \in \sigma(\mathcal{L}) : \sum_{j: \lambda_j = \lambda_i} \omega_j \neq 0 \},
\]

which almost surely coincides with the set of eigenvalues corresponding to observable eigenmodes in the PBH test for a random initial condition \( x_0 \). Below we formally state that the eigenvalues in this set are those that can be recovered by any algorithm using a sequence of measurements alone.

**Theorem 1** Given the sequence of observations \( y[k]_{k=0}^{2n-1} \) from the system in (5), define the following Hankel matrix

\[
Y := \begin{bmatrix}
y[0] & y[1] & \cdots & y[n-1] \\
y[1] & y[2] & \cdots & y[n] \\
\vdots & \vdots & \ddots & \vdots \\
y[n-1] & y[n] & \cdots & y[2n-2]
\end{bmatrix}. \tag{4}
\]

The rank of the Hankel matrix \( Y \) satisfies

\[
r := \text{rk}(Y) = |\mathcal{S}_\mathcal{L}| \leq n.
\]

The observable eigenvalues of \( \mathcal{L} \) are roots of the polynomial

\[
p_\mathcal{L}(x) = x^r + \alpha_{r-1} x^{r-1} + \cdots + \alpha_1 x + \alpha_0,
\]

where the coefficients \( \alpha_0, \ldots, \alpha_{r-1} \) are given by

\[
\begin{bmatrix}
\alpha_0 \\
\alpha_1 \\
\vdots \\
\alpha_{r-1}
\end{bmatrix} = \begin{bmatrix}
y[0] & y[1] & \cdots & y[r-1] \\
y[1] & y[2] & \cdots & y[r] \\
\vdots & \vdots & \ddots & \vdots \\
y[r-1] & y[r] & \cdots & y[2r-2]
\end{bmatrix}^{-1} \begin{bmatrix}
y[r] \\
y[r+1] \\
\vdots \\
y[2r-1]
\end{bmatrix}.
\]

**PROOF.** See Hayhoe et al. (2019).

In what follows, we will build upon this result to demonstrate results for any algorithm (possibly weighted and/or directed) network, in both discrete- and continuous-time.

## 4 Spectral Estimation for Discrete-Time Dynamics

Let \( G \) be any graph matrix whose sparsity pattern describes the connections of an arbitrary (unknown) graph \( \mathcal{G} \) with \( n \) nodes. The graph \( \mathcal{G} \) may be directed, may have self-loops, and may be weighted. Consider the discrete-time dynamics of a collection of single integrators,

\[
x[k+1] = G x[k], \quad x[0] = x_0, \\
y[k] = c^T x[k],
\]

where \( k \in \mathbb{N} \), and \( c, x_0 \) are arbitrary (possibly unknown) vectors in \( \mathbb{R}^n \). We may view our approach as a decentralized estimation problem when \( c = e_i \), wherein agent \( i \) is attempting to estimate the eigenvalues of \( \mathcal{G} \) when it can observe its own output. More generally, we may observe the weighted sum of the states of a subset \( \mathcal{S} \) of agents; hence, \( c = \sum_{i \in \mathcal{S} \subseteq V} \beta_i e_i \), which may correspond to a group of agents collectively estimating the spectrum of \( \mathcal{G} \) using a weighted linear combination of their outputs using (possibly unknown) weights \( \{\beta_i\}_{i \in \mathcal{S}} \).

Define the Jordan decomposition of \( G \) as

\[
G = V J V^{-1} = V \begin{bmatrix}
J_1 & 0 & \cdots & 0 \\
0 & J_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & J_d
\end{bmatrix} V^{-1},
\]

where \( J_i, \ i \in \{1, \ldots, d\}, \) is the \( m_i \times m_i \) Jordan block associated with the \( i \)-th eigenvalue \( \lambda_i \). Note that there
may be multiple Jordan blocks associated with a single

eigenvalue; hence, it may be that \( \lambda_i = \lambda_j \) for some \( i, j \in \{1, \ldots, d\} \). We thus also define the largest block size for
each unique eigenvalue \( \lambda_i \) as \( \hat{m}_i := \max_j \lambda_j = \lambda_i \) \( m_j \). Taking

powers of the matrix \( G \), we obtain

\[
G^k = (V J V^{-1})^k = V J^k V^{-1},
\]

where the \( m_i \times m_i \) exponentiated Jordan block \( J_i^k \) is

\[
J_i^k = \begin{bmatrix}
\lambda_i^k & (k) \lambda_i^{k-1} & \cdots & (k) \lambda_i^2 & (k) \lambda_i^1 \\
(k) \lambda_i^{k-1} & \lambda_i^k & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
(k) \lambda_i^1 & (k) \lambda_i^2 & \cdots & \lambda_i^k
\end{bmatrix},
\]

with entries being zero if the associated exponent is neg-

ative. For \( i \in \{1, \ldots, d\} \), let \( V^T V_i \) and \( V^{-1} x_0 \) denote the \( m_i \)-dimensional \( i \)-th blocks of \( V^T V \) and \( V^{-1} x_0 \), respectively, associated with Jordan block matrix \( J_i \).

Hence, for any graph matrix \( G \) of an arbitrary graph \( G \), the observations from our system (5) can be written as

\[
y[k] = (V^T V) J^k (V^{-1} x_0)
\]

For an eigenvalue \( \lambda_i \in \mathcal{S}_G \), we define the largest index

with nonzero total weight as

\[
\hat{m}_i := 1 + \max \left\{ s = 0, \ldots, \hat{m}_i - 1 : \bar{\omega}_i^{(s)} \neq 0 \right\}.
\]

Define the set of indices corresponding to unique recover-

able eigenvalues as \( \mathcal{I} := \left\{ i \in \{1, \ldots, n\} : \lambda_i \in \mathcal{S}_G \right\} \). We may thus rewrite the observations from (7) as

\[
y[k] = \sum_{i \in \mathcal{I}} \sum_{s=0}^{\hat{m}_i-1} \bar{\omega}_i^{(s)} \lambda_i^{k-s}.
\]

The Lemma below relates these quantities to the eigen-

values that we may recover from the observations

\( (y[k])_{k=0}^{2n-1} \).

**Lemma 2** The eigenvalues which may be recovered from

the sequence of measurements \( (y[k])_{k=0}^{2n} \), for any finite \( l \), are exactly those in \( \mathcal{S}_G \).

**PROOF.** See Appendix A.

In what follows, we will propose a computationally effi-
cient methodology to recover the eigenvalues in \( \mathcal{S}_G \) using

the sequence \( (y[k])_{k=0}^{2n-1} \). Towards that goal, we define the Hankel matrix of observations

\[
H := \begin{bmatrix}
y[0] & y[1] & \cdots & y[n-1] \\
y[1] & y[2] & \cdots & y[n] \\
\vdots & \vdots & \ddots & \vdots \\
y[n-1] & y[n] & \cdots & y[2n-2]
\end{bmatrix}.
\]

The following result relates the rank of this Hankel matrix to the largest recoverable Jordan blocks of \( G \).

**Lemma 3** The rank of \( H \) in (13) satisfies

\[
\text{rk}(H) = \sum_{i : \lambda_i \in \hat{\mathcal{S}}_G} \hat{m}_i,
\]

where \( \hat{m}_i \) is defined in (11).

**PROOF.** See Appendix B.

With this Lemma in hand, we present our main result on estimating the recoverable eigenvalues of \( G \).
Theorem 4 Given the sequence of observations \((y[k])_{k=0}^{2n-1}\) from the system in (5), consider the matrix \(H\) from (13) and denote its rank by \(r\). The recoverable eigenvalues of \(G\) are roots of the polynomial

\[
p_G(x) = x^r + \alpha_{r-1}x^{r-1} + \cdots + \alpha_1x + \alpha_0,
\]

where the coefficients \(\alpha_0, \ldots, \alpha_{r-1}\) are given by

\[
\begin{bmatrix}
\alpha_0 \\
\alpha_1 \\
\vdots \\
\alpha_{r-1}
\end{bmatrix} = \begin{bmatrix}
y[0] & y[1] & \cdots & y[r-1] \\
y[1] & y[2] & \cdots & y[r] \\
\vdots & \vdots & \ddots & \vdots \\
y[r-1] & y[r] & \cdots & y[2r-2]
\end{bmatrix}^{-1} \begin{bmatrix}
y[r] \\
y[r+1] \\
\vdots \\
y[2r-1]
\end{bmatrix}.
\]

Moreover, \(\lambda_i \in \mathcal{S}_G\) is a root of \(p_G(x)\) with multiplicity \(m_i\).

PROOF. See Appendix C.

While Theorem 4 makes use of \(2n\) observations \((y[k])_{k=0}^{2n-1}\), in practice, fewer observations may be required. Since at most \(r\) eigenvalues can be recovered, although \(r\) is unknown a priori, we can build a \(k \times k\) Hankel matrix of observations using the first \(2k\) observations from the system. Then we should stop taking observations whenever the rank of this Hankel matrix ceases to grow, since this occurs when \(k = r\), or when \(2n\) observations are obtained, whichever occurs first. In other words, at most \(2n\) observations are required to recover the eigenvalues of \(G\) which correspond to the recoverable modes of the dynamics, but in practice fewer may be used.

### 4.1 Network of Identical Discrete-Time Agents

In many applications, the network of interest will not only contain single integrators, but instead will consist of agents with more general dynamics. With this in mind, consider a network of \(n\) agents where each agent follows the dynamics \(x_i[k+1] = Ax_i[k] + u_i[k]\), where \(x_i\) is a \(d\)-dimensional vector of states, \(A\) is a known \(d \times d\) state transition matrix, and \(u_i[k]\) is an input consisting of a linear combination of the states of the neighboring agents of \(i\), i.e.,

\[
x_i[k+1] = Ax_i[k] + \sum_{j=1}^{n} g_{ij}x_j[k], \quad x_i[0] = x_{0i}\beta,
\]

\[
y[k] = \sum_{i=1}^{n} c_i \gamma^T x_i[k],
\]

where \(g_{ij} = [G]_{ij}, c_i = [c]_i, x_{0i} = [x_0]_i\). We assume that all agents start with the initial condition \(\beta\) weighted by \(x_{0i}\), and all individual observations are \(\gamma^T x_i[k]\) weighted by \(c_i\). Stacking the vectors of states in a large vector \(x = (x^T_1, \ldots, x^T_n)^T\), the dynamics can be written as

\[
x[k+1] = (I_n \otimes A + G \otimes I_d) x[k], \quad x[0] = x_0 \otimes \beta, \quad y[k] = (c \otimes \gamma)^T x[k].
\]

We assume the state matrix \(A\) of each agent is known, but the graph matrix \(G\) is unknown. We aim towards reconstructing the recoverable eigenvalue spectrum of \(G\) from a finite sequence of outputs. This result is summarized in the following theorem.

Theorem 5 Given the sequence of observations \((y[k])_{k=0}^{2n-1}\) from the system in (14), consider the Hankel matrix \(H\) defined in (13) and denote its rank by \(r\). The weighted sums of eigenvalues \(s_k := \sum_{i=1}^{d} \sum_{s=0}^{m_i-1} \omega_i^{(s)}(k) \lambda_i^{k-s}\) satisfy the following equality:

\[
[\begin{bmatrix}
s_0 \\
s_1 \\
\vdots \\
s_{2r-1}
\end{bmatrix} = \begin{bmatrix}
b_{0,0} \nu_0 & 0 & \cdots & 0 \\
b_{1,0} \nu_1 & b_{1,1} \nu_0 & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
b_{2r-1,0} \nu_{2r-1} & b_{2r-1,1} \nu_{2r-2} & \cdots & b_{2r-1,2r-1} \nu_{2r-2}
\end{bmatrix}^{-1} \begin{bmatrix}
y_0 \\
y_1 \\
\vdots \\
y_{2r-1}
\end{bmatrix}
\]

where \(\nu_{k-s} := \gamma^TA^{k-s} \beta, b_{k,s} := \binom{k}{s}\), and the matrix is invertible when \(\gamma^T \beta \neq 0\). Then, the recoverable eigenvalues of \(G\) are roots of the polynomial

\[
p_G(x) = x^r + \alpha_{r-1}x^{r-1} + \cdots + \alpha_1x + \alpha_0,
\]

where the coefficients \(\alpha_0, \ldots, \alpha_{r-1}\) satisfy

\[
\begin{bmatrix}
\alpha_0 \\
\alpha_1 \\
\vdots \\
\alpha_{r-1}
\end{bmatrix} = \begin{bmatrix}
s_0 & s_1 & \cdots & s_{r-1} \\
s_1 & s_2 & \cdots & s_r \\
\vdots & \vdots & \ddots & \vdots \\
s_{r-1} & s_r & \cdots & s_{2r-2}
\end{bmatrix}^{-1} \begin{bmatrix}
s_r \\
s_{r+1} \\
\vdots \\
s_{2r-1}
\end{bmatrix}.
\]

PROOF. See Appendix D.

### 5 Continuous-Time Dynamics

In the case of continuous-time dynamics, there are some subtle but important differences to that of discrete-time. Fortunately, similar results can still be derived in this domain, as we will describe in the following subsections.
5.1 Network of Single Integrators

We begin our exposition by considering the simple case of a network of coupled continuous-time single integrators:

\[
\dot{x}(t) = Gx(t), \quad x(0) = x_0, \\
y(t) = c^T x(t),
\]

where \( G \) is a graph matrix whose connectivity structure matches that of \( G \). We thus have \( y(t) = c^T e^{\lambda t} x_0 \). In practice, we take discrete samples of the output with an arbitrary period \( \tau > 0 \). Using the Jordan decomposition \( G = V J V^{-1} \), we have

\[
y_k := y(k\tau) = c^T V e^{Jk\tau} V^{-1} x_0 \\
= \sum_{s=0}^{d} \sum_{i=1}^{m_i-1} \omega_i(s)(k\tau)^s \frac{1}{s!}(e^{\lambda_i\tau})^k (15)
\]

with \( \omega_i(s) \) as defined in (8).

Similarly to the discrete-time case, the set of recoverable eigenvalues is \( S_G := \{ \lambda_i \in \sigma(G) : \exists s \text{ s.t. } \omega_i(s) \neq 0 \} \).

Note that, in this case, we can apply Theorem 4 to recover the quantities \( e^{\lambda_i\tau} \) from the finite sequence \( (y_k)_{k=0}^{2n-1} \). The eigenvalues of the graph matrix \( G \) corresponding to the recoverable modes of the system may then be recovered by taking a logarithm and dividing by \( \tau \).

5.2 Network of Identical Continuous-Time Agents

Similarly to the more general setting in Section 4.1, we consider the dynamics of a network of continuous-time agents, which can be described (in a compact form) as

\[
\dot{x}(t) = (I_n \otimes A + G \otimes I_d) x(t), \quad x(0) = x_0 \otimes \beta, \\
y(t) = (c \otimes \gamma)^T x(t).
\]

Hence, considering a sampling period \( \tau > 0 \), we have that

\[
y(k\tau) = (c \otimes \gamma)^T e^{(I_n \otimes A + G \otimes I_d)k\tau} (x_0 \otimes \beta) \\
= (c \otimes \gamma)^T e^{Gk\tau \otimes e^{Ak\tau}} (x_0 \otimes \beta) \\
= (c^T V e^{Jk\tau} V^{-1} x_0) (\gamma^T e^{Ak\tau} \beta) \\
= \nu_k \sum_{s=0}^{d} \sum_{i=1}^{m_i-1} \omega_i(s)(k\tau)^s \frac{1}{s!}(e^{\lambda_i\tau})^k,
\]

where \( \nu_k := \gamma^T e^{Ak\tau} \beta \), and \( \omega_i(s) \) is defined in (8), and the second equality follows by commutativity of the identity matrix and properties of the Kronecker product (see Petersen and Pedersen, 2012). Applying Theorem 5 followed by a logarithmic transformation, we again obtain the eigenvalues of the graph matrix \( G \).

6 Simulations

In this section we illustrate our results in both discrete- and continuous-time on simulated networks, where the underlying network structure is unknown to us. The evolution of the dynamics of these systems are simulated with an arbitrary random initial condition vector \( x_0 \) and an observability vector \( c \). Both \( x_0 \) and \( c \) are unknown to the algorithm. Then, we apply Theorem 4 to estimate the eigenvalues of \( G \) from the sequence of observations \( (y[k])_{k=0}^{2n-1} \) and compare our estimated eigenvalues against the true spectrum of the graph matrix \( G \).

Figure 1 shows the result of using Theorem 4 on the undirected, randomly generated 10-agent preferential attachment network shown in Figure 1(a) (see Barabási and Albert, 1999). We model each agent using a single integrator dynamics in discrete-time, as in (5). We assume that we only have access to the output of the integrator agent indicated in red in Fig 1(a). In Figure 1(b), we show the evolution of the output signal; as only one agent’s output is measured, this may be viewed as a decentralized eigenvalue estimation problem. Figure 1(c) compares both the true and estimated eigenvalues of \( G \). In this case there are 9 unique eigenvalues of \( G \) and all of these are perfectly recovered using a sequence of 20 measurements retrieved from a single agent.

In Figure 2 we apply our estimation approach followed by a logarithmic transformation on the 8-agent ring network shown in Figure 2(a), wherein the agents obey the continuous-time dynamics described in Section 5. The thickness of an edge in Figure 2(a) is proportional to that edge’s weight in the graph matrix \( G \), which are generated according to a Uniform[0, 1] distribution. In this case the output is a linear combination of the states of the two agents highlighted in Figure 2(a), and as such this can be viewed as a centralized eigenvalue estimation problem. Our realization of \( G \) renders an unstable system and the output grows exponentially, as shown in Figure 2(b). While this does have a mild effect on the accuracy of the estimated eigenvalues due to the inherent numerical instability of computational root-finding methods, Figure 2(c) shows that we are still able to recover most of the true spectrum of \( G \) with high accuracy.

A Proof of Lemma 2

By (12), it is clear that any term associated with a \( \lambda_i \notin S_G \) will be zero since necessarily \( \omega_i(s) = 0 \) for all \( s = 0, \ldots, m_i - 1 \), and so such a term will never appear in any observation \( y[k] \); therefore, the eigenvalues \( \lambda_i \notin S_G \) may not be recovered from any finite sequence of measurements. \( \square \)
Initial condition is randomly generated as $x_0$.

Fig. 2. 8-agent single integrator ring network in continuous-time, with sampling rate $\tau = 1$ and random initial condition $x_0 \sim \text{Uniform}[0,1]^n$. There are 9 unique eigenvalues of $G$ in this case, which are all recovered via our estimation approach.

\( \text{B Proof of Lemma 3} \)

Recall the set of indices corresponding to recoverable eigenvalues $\mathcal{I} = \{ i \in \{1, \ldots, n\} : \lambda_i \in \mathcal{S}_G \}$, and the total weights corresponding to each unique eigenvalue $\hat{\omega}_i^{(s)} = \sum_{j: \lambda_j = \lambda_i} \omega_j^{(s)}$ from (9). Now let $v_i := [1, \lambda_i, \lambda_i^2, \ldots, \lambda_i^{n-1}]$ and $b_k^0 := \binom{n}{k}$, recalling $\binom{n}{k} = 0$ for $n < k$, and then combining (6), (12), and (13) we obtain

$$H = \sum_{i \in \mathcal{I}} \sum_{s=0}^{\hat{m}_i-1} \frac{\omega_i^{(s)}}{s!} \frac{d^s}{d\lambda_i^s} (v_i v_i^\top)$$

where the derivative is taken element-wise to the entries of the matrix $v_i v_i^\top$. Notice that for all $s$ and any given $i, j \in \mathcal{I}$ the Hankel matrices $\frac{d^s}{d\lambda_i^s} (v_i v_i^\top)$ and $\frac{d^s}{d\lambda_j^s} (v_j v_j^\top)$ have orthogonal ranges since the $\lambda_i$ for $i \in \mathcal{I}$ are unique, and so $v_i$ and $v_j$ are linearly independent.

Let us now examine the ranks of the matrices $D_i^{(s)} := \frac{d^s}{d\lambda_i^s} (v_i v_i^\top)$ for a particular $i \in \mathcal{I}$. We will proceed via induction on $s$ to show that $\text{rk} \left( D_i^{(s)} \right) = s + 1$. For the base case of $s = 0$ we have $D_i^{(0)} = v_i v_i^\top$, which clearly has rank 1.

Now we assume $\text{rk} \left( D_i^{(s-1)} \right) = s$. The $j$-th column of
$D_i^{(s)}$ is of the form

$$D_i^{(s)} := \begin{bmatrix}
  b_{i}^{s-1} \lambda^{j-1-s} \\
  b_{i}^{s} \lambda^{-s} \\
  \vdots \\
  b_{i}^{s+n-2} \lambda^{j+n-2-s}
\end{bmatrix} = s! \lambda^{j-1-s} \begin{bmatrix}
  b_{i}^{s-1} \\
  b_{i}^{s} \\
  \vdots \\
  b_{i}^{s+n-2} \lambda^{n-1}
\end{bmatrix}. $$

Recall that $b_{i}^{k} = \binom{n}{i} = 0$ for $k < s$. By the leading-zero structure of $D_i^{(s)}$, wherein the first column has $s$ leading zeros followed by a nonzero value, the second has $s-1$ leading zeros followed by a nonzero value, all the way to the $s$-th column having a nonzero value in the first component, we can see that $\text{rk} \left( D_i^{(s)} \right) \geq s+1$. Now take any collection of $s+2$ columns of $D_i^{(s)}$, and we will show they must be linearly dependent. Via the identity

$$\binom{j}{s} - \binom{j-k}{s} = \binom{j-1}{s-1} + \binom{j-2}{s-1} + \cdots + \binom{j-k}{s-1}, $$

we may write

$$\frac{d_{i,j}^{(s)}}{\lambda^{j-1-s}} - \frac{d_{i,j-k}^{(s)}}{\lambda^{j-k-1-s}} = \frac{s!}{(s+1)!} \sum_{k=1}^{s} \frac{d_{i,j-k}^{(s-1)}}{\lambda^{j-k-1-s}}. $$

In other words, we may express the $j$-th and $j-k$-th columns of $D_i^{(s)}$ as a linear combination of exactly $k$ columns from $D_i^{(s-1)}$. Since we have a collection of $s+2$ columns of $D_i^{(s)}$, we need at least $s+1$ unique columns of $D_i^{(s-1)}$ to express linear combinations of our entire collection (in the case where the columns are sequential), but may need more. However, the rank of $D_i^{(s-1)}$ is $s$, so any collection of at least $s+1$ unique columns of $D_i^{(s-1)}$ must be linearly dependent; hence, our collection of $s+2$ columns of $D_i^{(s)}$ must be linearly dependent. Thus, $\text{rk} \left( D_i^{(s)} \right) = s+1$.

We will now examine the ranges of the matrices $D_i^{(s)}$ for a particular $i \in \mathcal{I}$. For $1 \leq s < j-k$ and $1 \leq k < j$ we have the identity

$$\binom{j}{s} - \binom{j-k}{s} = \binom{j-k}{s-k}. $$

Thus,

$$d_{i,j-k}^{(s-1)} = \frac{(s-k)!}{s!} \left[ d_{i,j}^{(s)} - \lambda_{i}^{k} d_{i,j-k}^{(s)} \right]. $$

In other words, we may write the $j-k$-th column of $D_i^{(s-1)}$ as a linear combination of the $j$-th and $j-k$-th columns of $D_i^{(s-1)}$ for $1 \leq k \leq s$ and $k < j \leq n$. Recall that $\text{rk} \left( D_i^{(s-k)} \right) = s+1-k$. Since we may write the first $s+1-k$ columns of $D_i^{(s-k)}$ as linear combinations of the columns of $D_i^{(s)}$, the same is true for all columns of $D_i^{(s-k)}$. Thus $\text{rg} \left( D_i^{(s-k)} \right) \subseteq \text{rg} \left( D_i^{(s)} \right)$ for $1 \leq k \leq s \leq \tilde{m}_i - 1$. Hence,

$$\text{rg} \left( H_{i} \right) = \text{rg} \left( \sum_{s=0}^{\tilde{m}_i - 1} \frac{\tilde{d}_{i,k}^{(s)}}{s!} D_i^{(s)} \right) = \text{rg} \left( D_i^{(\tilde{m}_i - 1)} \right) \Rightarrow \text{rk}(H_{i}) = \text{rk}(D_i^{(\tilde{m}_i - 1)}).$$

where $\tilde{m}_i$, as defined in (11), is the largest index with a nonzero total weight. Thus, the rank of $H_{i}$ is simply the largest $s$ for which $\tilde{d}_{i,k}^{(s)} \neq 0$, i.e., $\text{rk}(H_{i}) = \tilde{m}_i$.

Since for all $s$ and any $i \neq j \in \mathcal{I}$ the matrices $D_i^{(s)}$ and $D_j^{(s)}$ have orthogonal ranges, we have that $\text{rg} \left( H \right) = \text{rg} \left( \sum_{i \in \mathcal{I}} H_{i} \right) = \oplus_{i \in \mathcal{I}} \text{rg} \left( H_{i} \right)$, and hence $\text{rk}(H) = \sum_{i \in \mathcal{I}} \text{rk}(H_{i})$. Therefore, the rank of $H$ is equal to the sum of the sizes of the largest recoverable Jordan blocks for each unique eigenvalue, which is $\sum_{i \in \mathcal{I}} \tilde{m}_i$. □

### C Proof of Theorem 4

By Lemma 2, we know that at most we may recover all eigenvalues $\lambda_i \in \mathcal{S}_G$. As before, let $\mathcal{I} = \{i \in \{1, \ldots, n\} : \lambda_i \in \mathcal{S}_G \}$. By Lemma 3, we know that $\text{rk} \left( H \right) = \sum_{i \in \mathcal{I}} \tilde{m}_i$, which we denote by $r$. Define the following polynomial:

$$p_{G} \left( x \right) := \prod_{i \in \mathcal{I}} (x-\lambda_i) \tilde{m}_i = x^r + \alpha_{r-1} x^{r-1} + \cdots + \alpha_1 x + \alpha_0,$$

where $\tilde{m}_i$ is defined in (11). Notice that, since the eigenvalues are unknown, the coefficients of the polynomial are also unknown. In what follows, we propose an efficient technique to find these coefficients.

Let us calculate $p_{G}(J_i)$ for each $i \in \mathcal{I}$. Recall that there may be multiple Jordan blocks associated with a single eigenvalue, and that the Jordan block $J_i$ is of size $m_i \times m_i$. First consider the case that there exists some Jordan block $i$ such that $m_i = m_i$. By Cayley-Hamilton theorem we know $(J_i - \lambda_i I_{m_i})^{m_i} = \mathbf{0}_{m_i \times m_i}$, and so

$$p_{G}(J_i) = J_i^r + \alpha_{r-1} J_i^{r-1} + \cdots + \alpha_1 J_i + \alpha_0 = \mathbf{0}_{m_i \times m_i}. $$

Note from (6) that each upper diagonal of $J_i^r$ contains the same values. For ease of exposition, define $b_{i}^{k} = \binom{n}{i}$. Hence, since $J_i^r$ is of size $m_i \times m_i$, we in fact have $m_i$ separate equations (one per upper diagonal) of the form

$$b_{s}^{s} \lambda_{s}^{r} + \alpha_{r-1} b_{s}^{s-1} \lambda_{s}^{(r-1)} + \cdots + \alpha_{s+1} b_{s}^{s+1} \lambda_{s} + \alpha_{s} = 0,$$
for \( s \in \{0, \ldots, m_1 - 1\} \). If there is no Jordan block \( i \) such that \( m_1 = \tilde{m}_i \), then pick one such that \( m_1 > \tilde{m}_i \), and consider the first \( \tilde{m}_i \) upper diagonals of \( (J_i - \lambda_i I_{m_1})^{m_1} \), which will be zero. Multiplying the equations above by the corresponding total weights \( \tilde{\omega}^{(s)}_i \), some of which may be zero, we obtain for \( s \in \{0, \ldots, \tilde{m}_i - 1\} \)

\[
\tilde{\omega}^{(s)}_i \left( b_s^r \lambda_i^{-s} + \alpha_{r-1} b_s^{r-1} \lambda_i^{(r-s)-1} + \cdots + \alpha_s + b_s^{r+1} \lambda_i + \alpha_s \right) = 0.
\]

Summing all of these equations, noting that \( b_s^r = \binom{r}{s} = 0 \) for \( r < s \), defining \( \alpha_r = 1 \), we have

\[
\sum_{s=0}^{\tilde{m}_i - 1} \tilde{\omega}^{(s)}_i \left( b_s^r \lambda_i^{-s} + \alpha_{r-1} b_s^{r-1} \lambda_i^{(r-s)-1} + \cdots + \alpha_s + b_s^{r+1} \lambda_i + \alpha_s \right) = 0.
\]

Now, let us sum over all eigenvalues \( \lambda_i \in S_G \):

\[
\sum_{i \in I} \sum_{s=0}^{\tilde{m}_i - 1} \tilde{\omega}^{(s)}_i \sum_{l=0}^r \alpha_l \binom{l}{s} \lambda_i^{-l} = \sum_{l=0}^r \alpha_l \sum_{i \in I} \sum_{s=0}^{\tilde{m}_i - 1} \tilde{\omega}^{(s)}_i \binom{l}{s} \lambda_i^{-l-s} = \sum_{l=0}^r \alpha_l y_l = 0,
\]

by definition of the observations \( y_s \) from (7). Now, for \( k \in \{1, \ldots, r - 1\} \), let us examine the equations

\[
J^k p_g(J) = J^{r+k} + \alpha_{r-1} J^{r+k-1} + \cdots + \alpha_1 J^{k+1} + \alpha_0 J^k.
\]

Repeating the same process from above, we obtain for \( k \in \{1, \ldots, r - 1\} \)

\[
\sum_{i \in I} \sum_{s=0}^{\tilde{m}_i - 1} \tilde{\omega}^{(s)}_i \sum_{l=0}^r \alpha_l \binom{l+k}{s} \lambda_i^{l+k-s} = \sum_{l=0}^r \alpha_l \sum_{i \in I} \sum_{s=0}^{\tilde{m}_i - 1} \tilde{\omega}^{(s)}_i \binom{l+k}{s} \lambda_i^{l+k-s} = \sum_{l=0}^r \alpha_l y_l + k = 0.
\]

In summary, we have \( r \) equations of the form

\[
y_{r+k} + \alpha_{r-1} y_{r+k-1} + \cdots + \alpha_1 y_{k+1} + \alpha_0 y_k = 0,
\]

where \( k \in \{0, \ldots, r - 1\} \). In matrix form,

\[
\begin{bmatrix}
y_0 & y_1 & \cdots & y_{r-1} \\
y_1 & y_2 & \cdots & y_r \\
\vdots & \vdots & \ddots & \vdots \\
y_{r-1} & y_r & \cdots & y_{2r-2}
\end{bmatrix}
\begin{bmatrix}
\alpha_0 \\
\alpha_1 \\
\vdots \\
\alpha_{r-1}
\end{bmatrix}
= \begin{bmatrix}
y_r \\
y_{r+1} \\
\vdots \\
y_{2r-1}
\end{bmatrix}.
\]

By Lemma 3 we know \( rk(H) = r \) and hence we may find the values of the coefficients \( \alpha_0, \ldots, \alpha_{r-1} \) by a simple matrix inversion. Using these coefficients we can compute the roots of \( p_G \) to recover the eigenvalues of \( G \) that are in the set \( S_G \), i.e., those eigenvalues \( \lambda_i \) corresponding to the recoverable eigennodes of the dynamics. Moreover, the multiplicity of the root \( \lambda_i \) will be \( \tilde{m}_i \); hence, we recover \( \lambda_i \) with multiplicity of exactly \( \tilde{m}_i \).

\[ \square \]

### D Proof of Theorem 5

Considering the Jordan decomposition \( G = V J V^{-1} \), we have

\[
(I_n \otimes A + G \otimes I_d)^k = \left[ (V \otimes I_d) (I_n \otimes A + J \otimes I_d) (V^{-1} \otimes I_d) \right]^k = (V \otimes I_d) (I_n \otimes A + J \otimes I_d)^k (V^{-1} \otimes I_d) = (V \otimes I_d) \left[ \sum_{s=0}^k \binom{k}{s} (I_n \otimes A^{k-s}) (J^s \otimes I_d) \right] (V^{-1} \otimes I_d).
\]

Thus,

\[
y[k] = (c \otimes \gamma)^T (I_n \otimes A + G \otimes I_d)^k (x_0 \otimes \beta) = \sum_{s=0}^k \binom{k}{s} (c^T V \otimes \gamma^T) (I_n \otimes A^{k-s}) (J^s \otimes I_d) (V^{-1} x_0 \otimes \beta) = \sum_{s=0}^k \binom{k}{s} (c^T V J^s V^{-1} x_0) (\gamma^T A^{k-s} \beta).
\]

Hence, we obtain

\[
y[k] = \sum_{s=0}^k \binom{k}{s} \nu_{k-s} \sum_{i=1}^{m_1-1} \tilde{\omega}^{(s)}_i \binom{k}{s} \lambda_i^{k-s} = \sum_{s=0}^k \binom{k}{s} \nu_{k-s} \sigma_s, \quad (D.1)
\]

where \( \sigma_s = \sum_{i=1}^d \sum_{s=0}^{m_1-1} \tilde{\omega}^{(s)}_i \lambda_i^{k-s} \) and \( \nu_{k-s} = \gamma^T A^{k-s} \beta \). From the sequence \( (y[k])_{k=0}^{2r-1} \), we obtain a lower triangular system of linear equations that can be solved to find the sequence \( (m_k)_{k=0}^{2r-1} \). Specifically, if we collect \( 2r \) observations, with \( b_{k,s} = \binom{k}{s} \), we have that (D.1) for \( k = 0, \ldots, 2r - 1 \) results in

\[
\begin{bmatrix}
y_0 \\
y_1 \\
\vdots \\
y_{2r-1}
\end{bmatrix}
= \begin{bmatrix}
b_{0,0} \nu_0 & 0 & \cdots & 0 \\
b_{1,0} \nu_1 & b_{1,1} \nu_0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
b_{2r-1,0} \nu_{2r-1} & b_{2r-1,1} \nu_{2r-2} & \cdots & b_{2r-1,2r-1} \nu_0
\end{bmatrix}
\begin{bmatrix}
\sigma_0 \\
\sigma_1 \\
\vdots \\
\sigma_{2r-1}
\end{bmatrix}.
\]
As long as $\nu_i = \gamma^T \beta \neq 0$, the above matrix is full-rank. We may then recover the values $\sigma_i$ by a simple inversion, and apply Theorem 4 to find the eigenvalues of $G$. □

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