Winding Number in String Field Theory

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Abstract

Motivated by the similarity between cubic string field theory (CSFT) and the Chern-Simons theory in three dimensions, we study the possibility of interpreting \( \mathcal{N} = (\pi^2/3) \int (U Q_B U^{-1})^3 \) as a kind of winding number in CSFT taking quantized values. In particular, we focus on the expression of \( \mathcal{N} \) as the integration of a BRST-exact quantity, \( \mathcal{N} = \int Q_B A \), which vanishes identically in naive treatments. For realizing non-trivial \( \mathcal{N} \), we need a regularization for divergences from the zero eigenvalue of the operator \( K \) in the \( K B c \) algebra. This regularization must at same time violate the BRST-exactness of the integrand of \( \mathcal{N} \). By adopting the regularization of shifting \( K \) by a positive infinitesimal, we obtain the desired value \( \mathcal{N}[(U_{tv})^{\pm 1}] = \mp 1 \) for \( U_{tv} \) corresponding to the tachyon vacuum. However, we find that \( \mathcal{N}[(U_{tv})^{\pm 2}] \) differs from \( \mp 2 \), the value expected from the additive law of \( \mathcal{N} \). This result may be understood from the fact that \( \Psi = U Q_B U^{-1} \) with \( U = (U_{tv})^{\pm 2} \) does not satisfy the CSFT EOM in the strong sense and hence is not truly a pure-gauge in our regularization.

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1 Introduction

Cubic open string field theory (CSFT) \cite{1} has strong resemblances in its algebraic structure with the Chern-Simons (CS) theory. In fact, the action and the gauge transformation of CSFT,

\begin{align}
S &= \frac{1}{g^2} \int \left( \frac{1}{2} \Psi \ast Q_B \Psi + \frac{1}{3} \Psi \ast \Psi \ast \Psi \right), \\
\delta \Psi &= Q_B \Lambda + \Psi \ast \Lambda - \Lambda \ast \Psi,
\end{align}

are obtained from the action of the CS theory,

\begin{align}
S_{CS} &= \frac{k}{2\pi} \int_M \text{tr} \left( \frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A \right),
\end{align}

and its gauge transformation by the following replacements:

\begin{align}
A &\rightarrow \Psi, \quad d \rightarrow Q_B, \quad \wedge \rightarrow \ast, \quad \int_M \text{tr} \rightarrow \int.
\end{align}

The invariance of the CSFT action (1.1) under the infinitesimal gauge transformation (1.2) is due to that the three basic operations \( Q_B, \ast \) and \( \int \) in CSFT enjoy the same algebraic properties as those of \( d, \wedge \) and \( \int_M \text{tr} \) in the CS theory:

\begin{align}
Q_B^2 &= 0, \quad (1.5) \\
Q_B(\Phi \ast \Sigma) &= (Q_B \Phi) \ast \Sigma + (-1)^{g(\Phi)} \Phi \ast (Q_B \Sigma), \quad (1.6) \\
(\Phi \ast \Sigma) \ast \Xi &= \Phi \ast (\Sigma \ast \Xi), \quad (1.7) \\
\int \Phi \ast \Sigma &= (-1)^{g(\Phi)g(\Sigma)} \int \Sigma \ast \Phi, \quad (1.8) \\
\int Q_B \Phi &= 0, \quad (1.9)
\end{align}

where \( \Phi, \Sigma \) and \( \Xi \) are arbitrary string fields and \( g(\Phi) \) is the ghost number of \( \Phi \).

Under the finite gauge transformation, \( A \rightarrow g(d + A)g^{-1} \), by a gauge group valued function \( g(x) \), the CS action (1.3) is transformed as

\begin{align}
S_{CS} \rightarrow S_{CS} - 2\pi k N[g],
\end{align}

where \( N[g] \) is the winding number of the mapping \( g(x) \) from the manifold \( M \) to the gauge group:

\begin{align}
N[g] &= \frac{1}{24\pi^2} \int_M \text{tr} \left( gdg^{-1} \right)^3.
\end{align}
Due to this property, the coefficient $k$ multiplying the CS action (1.3) is required to be an integer (the level of the theory).

The CSFT has quite the same property under a finite gauge transformation:

$$\Psi \rightarrow U (Q_B + \Psi) U^{-1},$$

where all the products should be regarded as the star product $\ast$, and $U$ is given by $U = e^{-\Lambda} = 1 - \Lambda + (1/2)\Lambda^2 - \cdots$ with 1 being the identity string field. Under (1.12), the CSFT action is transformed as

$$S \rightarrow S - \frac{1}{2\pi^2 g_0^2} \mathcal{N}[U],$$

with $\mathcal{N}[U]$ given by

$$\mathcal{N}[U] = \frac{\pi^2}{3} \int (U Q_B U^{-1})^3.$$

Recently, various translationally invariant exact solutions in CSFT have been constructed in the pure-gauge form $\Psi = U Q_B U^{-1}$ [2, 3, 4, 5, 6]. For such solutions, their energy density is given by $\mathcal{N}/(2\pi^2 g_0^2)$.

Then, several questions naturally arise. The first question would be whether $\mathcal{N}$ (1.14) has an interpretation as the “winding number” taking quantized values. If so, windings in what sense does $\mathcal{N}$ count? Certainly, $\mathcal{N}$ is a topological quantity invariant under a small deformation of $U$ similarly to $N$ (1.11). If $\mathcal{N}$ takes integer values in some unit, should the inverse of the open string coupling constant $1/g_0^2$ be quantized as in the case of the CS theory?

The purpose of this paper is to study whether we can really interpret $\mathcal{N}$ as a kind of winding number taking quantized values. In particular, we focus on the following expression of $\mathcal{N}$:

$$\mathcal{N} = \int Q_B A,$$

where the quantity $A$ is given explicitly in (2.1) One might think that the RHS of (1.15) is equal to zero since the integration of a BRST-exact quantity is usually regarded to vanish as given in (1.9). However, eq. (1.9) is not an axiom of CSFT but is an equation to be proved. We already know that $\mathcal{N}$ is non-vanishing for $U$ corresponding to the tachyon vacuum solution $\Psi = U Q_B U^{-1}$, and hence the RHS must also be so. In fact, we will see in Sec. 3.2 that the RHS of (1.15) can be non-vanishing due to singularities existent in $A$. We add that the formula (1.15) is practically useful for calculating $\mathcal{N}$ for various $U$’s. Finally, eq. (1.15) suggests us to

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*On the RHS of (1.13), we have omitted the term $-\frac{1}{2} \int Q_B [(Q_B U^{-1}) U \Psi]$ (this is the case also in (1.10) for the CS theory). This term is equal to zero if we can use the property (1.9), and, for a pure-gauge $\Psi$, it is nothing but the extra term $\Delta \mathcal{N}$ (4.2) in the additive law of $\mathcal{N}$.

†We put the space-time volume equal to one in this paper. The factor $\pi^2/3$ multiplying (1.14) has been chosen so that $\mathcal{N} = -1$ for the tachyon vacuum solution.
rewrite its RHS further as an integration of \( \mathcal{A} \) on the “boundary”: \( \int_M Q_B \mathcal{A} = \int_{\partial \mathcal{M}} \mathcal{A} \). It is an interesting problem (but is beyond the scope of this paper) to clarify whether such a formula exists, and if so, what the “manifold” \( \mathcal{M} \) and its “boundary” \( \partial \mathcal{M} \) are. This is important for understanding the topological meaning of \( \mathcal{N} \) in CSFT.

Let us mention here the correspondent of (1.15), \( \mathcal{N} = \int_M dG \), in the CS side. The two-form \( G \) is given in terms of the Lie algebra valued function \( \phi(x) \) in \( g(x) = e^{i\phi(x)} \), and \( \phi(x) \) has, in general, singularities in \( M \). We can evaluate \( \mathcal{N} \) as \( \mathcal{N} = \int_{\partial \mathcal{M}} G \) with \( \partial \mathcal{M} \) being the singular points of \( \phi(x) \). Let us take the simplest example of the \( SU(2) \) gauge group and the hedgehog type \( g(x) \) on \( M = S^3 \); \( g(x) = \exp (i f(r) \hat{x} \cdot \tau) \) with \( r = |x| \) and \( \hat{x} = x/r \). The regularity of \( g(x) \) at the origin and the infinity requests that \( f(0) \) and \( f(\infty) \) be integer multiples of \( \pi \). For this \( g(x) \), we have

\[
N = \int d^3x \nabla \cdot \left[ \frac{2f - \sin 2f}{8\pi^2 r^2} \hat{x} \right] = \frac{1}{\pi} (f(\infty) - f(0)).
\] (1.16)

In the rest of this section, let us explain in some detail the results of our analysis. As we mentioned below (1.14), exact classical solutions, including the tachyon vacuum solution, have been constructed in the pure-gauge form \( \Psi = U Q_B U^{-1} \). Their construction is most concisely given in terms of the \( KBc \) algebra [4, 5], which we summarize in Appendix A. The points concerning (1.15) is that there appears \( 1/K \) on its RHS and that the eigenvalue of the operator \( K \) is non-negative and, in particular, \( K \) has a zero eigenvalue. Furthermore, the existence of the zero eigenvalue of \( K \) endangers the simple identity \( K(1/K) = 1 \) and hence the validity of algebraic manipulations in the \( KBc \) algebra if we use the Schwinger parametrization for \( 1/K \):

\[
K \frac{1}{K} = K \int_0^{\infty} dt e^{-tK} = 1 - e^{-\infty K}.
\] (1.17)

It is a subtle problem whether the last term can be dropped if \( K \) has a zero eigenvalue. Therefore, some kind of regularization for the zero eigenvalue needs to be introduced for calculating (or defining) (1.15) in a well-defined manner. In this paper, we adopt the regularization of shifting \( K \) by an infinitesimal positive constant \( \varepsilon \):

\[
K \to K_\varepsilon = K + \varepsilon.
\] (1.18)

Namely, we lift the eigenvalues of \( K \) by \( \varepsilon \). In the end of the calculation, we take the limit \( \varepsilon \to +0 \). We call this regularization “\( K_\varepsilon \)-regularization” hereafter. There may be other kinds of regularizations for (1.15). However, introducing the upper cutoff to the integration of the Schwinger parameter for \( 1/K \) does not help making (1.15) non-vanishing.

\[\text{\footnotesize{‡Though there seems to be no rigorous proof for this fact, concrete calculations of various correlators support it.}}\]
If we apply the $K_\varepsilon$-regularization to (1.15), BRST-exactness of the integrand is violated by $O(\varepsilon)^\text{§}$. We find that this does in fact leads to a non-vanishing and the expected value of $\mathcal{N}$ for the tachyon vacuum solution. More concretely, we have $\mathcal{N} = \int (Q_B A)_{K \rightarrow K_\varepsilon} = \varepsilon \int \mathcal{W} = \varepsilon \times O(1/\varepsilon) = -1$, where $\mathcal{W}$ depends on $\varepsilon$ through $K_\varepsilon$.

We also examine $\mathcal{N}$ (1.15) for other $U$’s than that of the tachyon vacuum for testing whether $\mathcal{N}$ is really quantized. For this purpose we note the relation

$$\mathcal{N}[UV] = \mathcal{N}[U] + \mathcal{N}[V] + \int Q_B (\cdots),$$

(1.19)

where $(\cdots)$ in the last term is given in terms of $U$ and $V$ (see (4.2) for a precise expression). This equation implies the additive property of $\mathcal{N}$, if we can discard the last term given as the integration of a BRST-exact quantity. Therefore, for $(U_{tv})^n$ $(n = \pm 1, \pm 2, \ldots)$ with $U_{tv}$ describing the tachyon vacuum, we would have $\mathcal{N}[(U_{tv})^n] = n \mathcal{N}[U_{tv}] = -n$. We calculate $\mathcal{N}[(U_{tv})^n]$ for $n = \pm 1$ and $\pm 2$ in the $K_\varepsilon$-regularization to find that

$$\mathcal{N}[(U_{tv})^n] = \begin{cases} 2 - 2\pi^2 & (n = -2) \\ 1 & (n = -1) \\ -1 & (n = 1; \text{tachyon vacuum}) \\ -2 + 2\pi^2 & (n = 2) \end{cases}.$$

(1.20)

Namely, we get the expected result for $n = -1$, but the results for $n = \pm 2$ are anomalous and signal the violation of the additive property of $\mathcal{N}$. We calculate the last term of (1.19) in the $K_\varepsilon$-regularization and find that it is non-vanishing and accounts for the anomalous part of (1.20) for $n = \pm 2$. Does our result (1.20) give a counterexample to our expectation that $\mathcal{N}$ is quantized? Our answer would be no. We examine whether the classical solution $\Psi_n = (U_{tv})^n Q_B (U_{tv})^{-n}]_{K \rightarrow K_\varepsilon}$ in the $K_\varepsilon$-regularization satisfies the EOM in the strong sense, namely, whether $\int \Psi_n * (Q_B \Psi_n + \Psi_n * \Psi_n) = 0$ holds. In other words, we test whether $\Psi_n$ is really a pure-gauge. We find that the EOM in the strong sense holds for $n = \pm 1$, but it is violated for $n = \pm 2$. Therefore, our result (1.20) for $n = \pm 2$ cannot be regarded as a counterexample.

The organization of the rest of this paper is as follows. In Sec. 2, we derive (1.15) for a generic $U$, and its concrete expression in the $KBc$ algebra. In Sec. 3, we calculate $\mathcal{N}$ for $U_{tv}$ of the tachyon vacuum in the $K_\varepsilon$-regularization. Then, in Sec. 4, we present our analysis on $\mathcal{N}$, its additive law and the EOM for $(U_{tv})^n$. The final section (Sec. 5) is devoted to a summary and discussions on future problems. In Appendix A, we summarize the $KBc$ algebra and the correlators used in the text. In Appendix B, we present another way of calculating (1.15) for the tachyon vacuum.

\text{§}In our $K_\varepsilon$-regularization, we first evaluate the operations of $Q_B$, and then replace all $K$ in the resultant expression with $K_\varepsilon$. 

4
2 \( \mathcal{N} \) as the integration of a BRST-exact quantity

In this section, we first show (1.15), namely, that \( \mathcal{N} \) (1.14) is given as the integration of a BRST-exact quantity \( Q_B A \). Then, we derive the expression of \( Q_B A \) for \( U = 1 - F(K) B_c F(K) \), which has been used for constructions of classical solutions in the \( KBc \) algebra. In the rest this paper, we put the open string coupling constant \( g_o \) equal to one. We often omit * for the string field product unless confusion occurs.

2.1 Derivation of \( \mathcal{N} = \int Q_B A \)

Let us consider \( \Psi \) given in a pure-gauge form \( \Psi = U Q_B U^{-1} \). For this \( \Psi \), we introduce \( \Psi_s = U_s Q_B U_s^{-1} \) with \( U_s \) carrying a parameter \( s \) (\( 0 \leq s \leq 1 \)) and interpolating \( U \) and 1; \( U_{s=1} = U \) and \( U_{s=0} = 1 \). Then, we can show that \( A \) given by

\[
A = \pi^2 \int_0^1 ds \Psi_s * \frac{d\Psi_s}{ds},
\]

(2.1)

satisfies (1.15). The proof goes as follows:

\[
\frac{1}{\pi^2} \mathcal{N} = \frac{1}{3} \int \Psi^3 = \frac{1}{3} \int_0^1 ds \frac{d}{ds} \int \Psi^2_s = \int_0^1 ds \int \Psi_s * \frac{d\Psi_s}{ds} * \Psi_s
\]

\[
= \int_0^1 ds \int \left( \Psi_s * \frac{d}{ds} (\Psi_s)^2 - (\Psi_s)^2 * \frac{d}{ds} \Psi_s \right)
\]

\[
= - \int_0^1 ds \int \left( \Psi_s * \frac{d}{ds} Q_B \Psi_s - (Q_B \Psi_s) * \frac{d}{ds} \Psi_s \right) = \int_0^1 ds \int Q_B \left( \Psi_s * \frac{d\Psi_s}{ds} \right),
\]

(2.2)

where we have used the cyclicity (1.8) and the EOM satisfied by \( \Psi_s \): \( Q_B \Psi_s + (\Psi_s)^2 = 0 \).

2.2 \( \mathcal{N} \) in the \( KBc \) algebra

We wish to calculate \( \mathcal{N} \) given in the form (1.15) with \( A \) given by (2.1) for various \( U \), in particular, for \( U \) corresponding to the tachyon vacuum. This calculation will be carried out in Secs. 3 and 4. Here, as a preparation, we present a convenient expression of the RHS of (1.15). This is obtained by interchanging the order of the \( s \) and the CSFT integrations:

\[
\mathcal{N} = \pi^2 \int_0^1 ds \mathcal{B}(s),
\]

(2.3)

\[\text{This proof remains valid for } \Psi \text{ not restricted to the pure-gauge form if we can construct } \Psi_s \text{ satisfying the EOM for all } s.\]
with $\mathcal{B}(s)$ given again as the integration of a BRST-exact quantity:

$$\mathcal{B}(s) = \int Q_B \left( \Psi_s \ast \frac{d\Psi_s}{ds} \right).$$

(2.4)

By following the manipulation of (2.2) in the reverse way, we obtain another expression of $\mathcal{B}(s)$:

$$\mathcal{B}(s) = \int \frac{d\Psi_s}{ds} \ast \Psi_s \ast \Psi_s.$$  

(2.5)

It should be noted that the integrand of (2.5) is, though not manifest, BRST-exact.

Next, we present concrete expressions of $\mathcal{B}(s)$ for $U$ which has been adopted in the construction of classical solutions using the $KBc$ algebra [4, 5]:

$$U = 1 - F(K)BcF(K),$$

(2.6)

where $F(K)$ is a function of $K$, which should be carefully chosen to realize a non-trivial solution. The inverse of $U$ is

$$U^{-1} = 1 + \frac{F}{1 - F^2} BcF,$$

(2.7)

and the corresponding $\Psi$ is given by

$$\Psi = U Q_B U^{-1} = F cK \frac{1}{1 - F^2} BcF.$$  

(2.8)

For a given $F(K)$, we introduce an interpolating $F_s(K)$ which satisfies $F_{s=1} = F$ and $F_{s=0} = 0$. Then, $U_s$ and $\Psi_s$ are given by

$$U_s = 1 - F_s BcF_s, \quad \Psi_s = U_s Q_B U_s^{-1} = F_s cK \frac{1}{1 - F_s^2} BcF_s.$$  

(2.9)

For this $\Psi_s$, $\mathcal{B}(s)$ of (2.5) is calculated to give

$$\mathcal{B}(s) = \int \frac{d}{ds} \left( BcF_s^2 c \frac{K}{1 - F_s^2} \right) \left( BcF_s^2 c \frac{K}{1 - F_s^2} \right)^2 = \sum_{a=1}^{3} \mathcal{B}_a(s),$$

(2.10)

with

$$\mathcal{B}_1(s) = \int BcK \left( \frac{d}{ds} \frac{1}{1 - F_s^2} \right) \left[ c, F_s^2 \right] \frac{K}{1 - F_s^2} BcF_s^2 c \frac{K}{1 - F_s^2},$$

$$\mathcal{B}_2(s) = \int Bc \frac{K}{1 - F_s^2} \left[ F_s^2, c \right] \frac{K}{1 - F_s^2} BcF_s^2 c \frac{K}{1 - F_s^2} \frac{dF_s^2}{ds},$$

$$\mathcal{B}_3(s) = \int BcK \left( \frac{d}{ds} \frac{1}{1 - F_s^2} \right) \left[ F_s^2, c \right] \frac{K}{1 - F_s^2} BcF_s^2 c \frac{K F_s^2}{1 - F_s^2}. $$  

(2.11)
In this derivation, we have used the following identity due to the $KBc$ algebra:

$$\int BcA_1cA_2BcA_3cA_4BcA_5cA_6 = \int BcA_1A_2cA_3A_4cA_5cA_6 - \int BcA_1A_2A_3cA_4cA_5cA_6$$

$$- \int BcA_1A_2A_3A_4cA_5cA_6 + \int BcA_1A_2A_3cA_4A_5cA_6,$$  \hspace{1cm} (2.12)

where $A_k$ ($k = 1, \cdots, 6$) are arbitrary functions of $K$. Of course, we get the same result if we use (2.4) for $B(s)$. More elaborate manipulation using the $KBc$ algebra leads to a simpler expression:

$$B(s) = \tilde{B}_1(s) + \tilde{B}_2(s),$$  \hspace{1cm} (2.13)

with

$$\tilde{B}_1(s) = \int BcF^2cK \left\{ c \frac{K}{1 - F_s^2} \frac{dF^2}{ds} c \frac{K}{1 - F_s^2} - \frac{1}{1 - F_s^2} \frac{dF^2}{ds} cK \left[ cK, \frac{1}{1 - F_s^2} \right] \right\},$$

$$\tilde{B}_2(s) = - \int BcF^2cKc \frac{K}{(1 - F_s^2)^2} \frac{dF^2}{ds} cK.$$

(2.14)

$\tilde{B}_1(s)$ and $\tilde{B}_2(s)$ are separately given as integrations of BRST-exact quantities. Derivation of this expression is summarized in Appendix B.

3 \quad $\mathbf{N}$ for the tachyon vacuum

In this section, we evaluate $N$ (1.15) for $U$ representing the tachyon vacuum. In particular, we show that, by using the $K_\varepsilon$-regularization mentioned in the Introduction, $N$ reproduces the expected result $N = -1$.

3.1 \quad $\mathbf{B}(s)$ without regularization

As $F(K)$ corresponding to the tachyon vacuum, we choose [5]

$$F^2 = \frac{1}{1 + K},$$  \hspace{1cm} (3.1)

and as the interpolating $F_s^2$ we take simply

$$F_s^2 = sF^2 = \frac{s}{1 + K}.$$  \hspace{1cm} (3.2)

Let us consider calculating $B(s)$ given by (2.10) or (2.13). There appear in (2.11) and (2.14) quantities $1/(1 - s + K)^k$ ($k = 1, 2$) and $1/(1 + K)$; the former is from $1/(1 - F_s^2)^k$, while the latter is $F^2$ itself. For them we use the Schwinger parametrizations:

$$\frac{1}{(1 - s + K)^k} = \frac{1}{(k - 1)!} \int_0^\infty dt t^{k-1} e^{-t(1 - s + K)}, \quad \frac{1}{1 + K} = \int_0^\infty d\tilde{t} e^{-\tilde{t}(1 + K)},$$  \hspace{1cm} (3.3)
Then, we make a change of variables from $t_a$’s for $1/(1 - s + K)^k$ and $\tilde{t}_b$’s for $1/(1 + K)$ to $(x, y, z_1, z_2, \cdots)$ satisfying

$$\sum_a t_a = x, \quad \sum_b \tilde{t}_b = x \left( \frac{1}{y} - 1 \right), \quad (0 \leq x < \infty, \ 0 \leq y \leq 1). \quad (3.4)$$

The variables $(z_1, z_2, \cdots)$ are introduced for expressing $t_a$’s and $\tilde{t}_b$’s in such a way that they satisfy the constraints (3.4). An example which appears in the calculation of $B_1(s)$ in (2.11) is

$$\int B_c \frac{1}{(1 - s + K)^2} c \frac{1}{1 - s + K} c \frac{1}{1 + K} c K$$

$$= \int_0^\infty dx \int_0^1 dy e^{-x(1/y-s)} \frac{x^2}{y^2} \int_0^1 dz t_1 \left( -\frac{\partial}{\partial t_4} \right) G(t_1, t_2, \tilde{t}_3, t_4) \bigg|_{t_1 = xz, t_2 = x(1-z), \tilde{t}_3 = x(1/y-1), t_4 = 0}, \quad (3.5)$$

where $x^2/y^2$ is the Jacobian of the change of variables and $G(t_1, t_2, t_3, t_4)$ is given in terms of the correlator on the cylinder with circumference $\sum_{a=1}^4 t_a$ by (see Appendix A)

$$G(t_1, t_2, t_3, t_4) = \langle B_c(0)c(t_1)c(t_1 + t_2)c(t_1 + t_2 + t_3) \rangle_{t_1 + t_2 + t_3 + t_4}. \quad (3.6)$$

Carrying out this kind of calculation for the whole of (2.10) or (2.13), we find that $B(s)$ is given as an integration over $(x, y)$,

$$B(s) = \int_0^\infty dx \int_0^1 dy e^{-x(1/y-s)} H(x, y), \quad (3.7)$$

and moreover that $H(x, y)$ vanishes identically; $H(x, y) = 0$. This is consistent with the fact that $B(s)$ (2.4) is the integration of a BRST-exact quantity. Therefore, $\mathcal{N}$ given by (2.3) also vanishes. On the other hand, we know that $\mathcal{N} = -1$ from the direct calculation of the energy density of the tachyon vacuum solution. This contradiction will be resolved in the next subsection by introducing the $K_\varepsilon$-regularization.

### 3.2 $B(s)$ in the $K_\varepsilon$-regularization

As we explained in the Introduction, calculations of various correlators suggest that the eigenvalue of the operator $K$ is non-negative and, in particular, that there is a zero eigenvalue. This is also seen from a concrete calculation of (3.5); it is finite for $s < 1$, while it diverges at $s = 1$ (and also for $s > 1$). Therefore, in order to make $B(s)$ non-vanishing and obtain $\mathcal{N} = -1$ from (2.3), it seems necessary to introduce a regularization to $B(s)$ which extracts and at the same time regularize the divergent contribution of the zero eigenvalue of $K$ to $B(s)$. Without
regularization, the zero eigenvalue would be unseen due to the BRST-exactness of the integrand of \(B(s)\). Such a regularization must fulfill two requirements: First, it must regularize the divergence of each term in \(B(s)\), such as (3.5), at \(s = 1\) due to the zero eigenvalue. Second, it must violate the BRST-exactness of the integrand of \(B(s)\) (2.4). For example, introduction of the upper cutoff to the \(x\)-integration in (3.5) regularizes the divergence at \(s = 1\). However, it does not violate the BRST-exactness, and \(B(s)\) given by (3.7) remains zero in this regularization since \(H(x, y)\) remains unchanged from zero.

As a regularization which can also violate the BRST-exactness of the integrand of \(B(s)\) (2.4) and hence that of \(N\) (1.15), we adopt the \(K_\varepsilon\)-regularization (1.18) as we mentioned in the Introduction. This is to replace all \(K\)'s in the integrand of \(B(s)\) (2.4) (and that of \(N\) (1.15)) with \(K_\varepsilon = K + \varepsilon\). In particular, we must make the replacement \(K \rightarrow K_\varepsilon\) after calculating the operation of \(Q_B\). Namely,

\[
\left. B(s) \right|_{K_\varepsilon\text{ - reg.}} = \int \left[ Q_B \left( \Psi_s * \frac{d\Psi_s}{ds} \right) \right]_{K \rightarrow K_\varepsilon} = \int \left( \frac{d\Psi_s}{ds} * \Psi_s * \Psi_s \right)_{K \rightarrow K_\varepsilon}. ~\quad (3.8)
\]

For concrete calculations of the regularized \(B(s)\), we use (2.10) or (2.13) with all the \(K\)'s replaced with \(K_\varepsilon\).

Let us recalculate \(B(s)\) for the tachyon vacuum in the \(K_\varepsilon\)-regularization by using the expression (2.10). Note that each of \(B_a(s)\) \((a = 1, 2, 3)\) (2.11) is given in the form

\[
\int BcW_1cW_2cW_3cW_4,
\]

with \(W_k\) \((k = 1, 2, 3, 4)\) being functions of \(K\) and \(s\). For the present \(F_2^2(3.2)\), \(W_k\) is a rational function with its denominator consisting of \(1 - s + K\) and \(1 + K\). For each \(W_k\) we take the expression where the numerators do not contain any \(K\). For example,

\[
\frac{K}{1 - F_2^2} = K + s - \frac{s(1 - s)}{1 - s + K}, ~\quad (3.10)
\]

\[
K \left( \frac{d}{ds} \frac{1}{1 - F_2^2} \right) = 1 - \frac{1 - 2s}{1 - s + K} - \frac{s(1 - s)}{(1 - s + K)^2}. ~\quad (3.11)
\]

For later convenience, we call \(K\) which is not contained in the denominators (for example, the first \(K\) on the RHS of (3.10)) “bare \(K\” hereafter. In the \(K_\varepsilon\)-regularization, all the \(K\)'s are replaced by \(K_\varepsilon\). However, a bare \(K\) in \(W_{1,2,3}\) which is sandwiched between two \(c\)'s remains \(K\) owing to \(c^2 = 0\). For example, we have

\[
c \left. \frac{K}{1 - F_2^2} \right|_{K \rightarrow K_\varepsilon} = c \left( K + s - \frac{s(1 - s)}{1 - s + K_\varepsilon} \right) c. ~\quad (3.12)
\]
However, this is not the case for $W_4$ since it is not located between two $c$'s. Note that $W_4$ containing the bare $K$ is only $K/(1 - F_s^2)$ (3.10) which appears in $\mathcal{B}_1$. $W_4$ in $\mathcal{B}_2$ and $\mathcal{B}_3$ is given by

$$\frac{K}{1 - F_s^2} \frac{dF_s^2}{ds} = \frac{1}{s} \frac{K F_s^2}{1 - F_s^2} = 1 - \frac{1 - s}{1 - s + K^2}. \quad (3.13)$$

Therefore, the regularized $\mathcal{B}(s)$ consists of two parts corresponding to the replacement of the bare $K$ in $W_4$ by $K_{\varepsilon} = K + \varepsilon$. One is the part from this $\varepsilon$, and the other is all the rest. We call the former proportional to $\varepsilon$ “$\varepsilon$-term”, and the latter “non-$\varepsilon$-term”. In both the terms, all the $K$'s in the denominators are now replaced with $K_{\varepsilon}$.

First, we find that the non-$\varepsilon$-term is equal to zero. To see this, note that the effect of the replacement $K \to K_{\varepsilon}$ in a denominator is to multiply its Schwinger parametrization by $e^{-\varepsilon t}$. Since the sum of all the Schwinger parameters is equal to $x/y$ (see (3.4)), we find that the whole of the non-$\varepsilon$-term is given simply by (3.7) with $e^{-x(1/y-s)}$ replaced by $e^{-x((1+s)/y-s)}$. The non-$\varepsilon$-term vanishes since the function $H(x, y)$ remains unchanged from the $\varepsilon = 0$ case and is equal to zero.

Therefore, we have only to calculate the $\varepsilon$-term. As we explained above, only $\mathcal{B}_1$ contributes to the $\varepsilon$-term. Denoting $\mathcal{B}(s)$ in the $K_{\varepsilon}$-regularization by $\mathcal{B}_{\varepsilon}(s)$, we have

$$\mathcal{B}_{\varepsilon}(s) = \varepsilon s^2 \left\{ (1 - s)^2 \int c \frac{1}{1 - s + K_{\varepsilon}} c \frac{1}{1 - s + K_{\varepsilon}} c \frac{1}{1 + K_{\varepsilon}} - \int c \left[ \frac{1}{1 - s + K_{\varepsilon}} - \frac{1 - s}{(1 - s + K_{\varepsilon})^2} \right] cK_{\varepsilon} \frac{1}{1 + K_{\varepsilon}} \right\}

= \varepsilon s^2 \int_0^\infty dx \int_0^1 dy e^{-x((1+s)/y-s)} \left\{ (1 - s)^2 \frac{x^2}{z^2} \int_0^1 dz G_{\varepsilon}(t_1, t_2, t_3) \right|_{t_1=x, t_2=1-x, t_3=x(1/y-1)}

+ \frac{x}{z^2} [1 - (1-s)t_1] \frac{\partial}{\partial t_2} G_{\varepsilon}(t_1, t_2, t_3) \right|_{t_1=x, t_2=0, t_3=x(1/y-1)} \right\}

= \varepsilon s^2 \int_0^\infty dx \int_0^1 dy e^{-x((1+s)/y-s)} f(x, y, s), \quad (3.14)$$

where $G_{\varepsilon}$ is the $ccc$ correlator on the cylinder (see (A.7)),

$$G_{\varepsilon}(t_1, t_2, t_3) = \langle c(0)c(t_1)c(t_1 + t_2) \rangle_{t_1+t_2+t_3} = G(t_1, t_2, t_3, t_4 = 0), \quad (3.15)$$

and the function $f(x, y, s)$ is given by

$$f(x, y, s) = \frac{x^3}{2\pi^3 y^6} \left\{ \pi (1-s)^2 x^2 y \cos \pi y - [(1 - s)^2 x^2 - 2\pi^2 (1 - s) x y^2 + 2\pi^2 y^2] \sin \pi y \right\}. \quad (3.16)$$

In the first expression of (3.14), we have used (A.1) to eliminate $B$.  

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It seems difficult to carry out explicitly the \((x, y)\) integrations in (3.14) to obtain an analytic expression of \(B_\varepsilon(s)\) for a finite \(\varepsilon\). However, we can exactly evaluate \(\mathcal{N}\) (2.3) by carrying out first the \(s\) integration:

\[
\mathcal{N} = \lim_{\varepsilon \to 0} \pi^2 \int_0^1 ds B_\varepsilon(s) = -\lim_{\varepsilon \to 0} \frac{1}{(1 + \varepsilon)^3} = -1. \tag{3.17}
\]

This is the desired result for the tachyon vacuum. Next, let us consider \(B_\varepsilon(s)\) itself given by (3.14). First, we see that \(\lim_{\varepsilon \to 0} B_\varepsilon(s) = 0\) for \(s < 1\). This is understood from the facts that (3.14) is multiplied by \(\varepsilon\), and that the denominator \(1 - s + K_\varepsilon\) is positive definite for \(s < 1\) even when \(\varepsilon = 0\). In order to obtain the expression of \(B_\varepsilon(s)\) near \(s = 1\) for an infinitesimal \(\varepsilon\), we make a change of variables from \((x, y)\) to \((\xi, \eta)\) defined by

\[
x = \frac{\xi}{\varepsilon}, \quad y = \left(1 + \frac{\varepsilon}{1 + \eta}\right)^{-1}. \tag{3.18}
\]

In addition, since we are interested in \(s \simeq 1\), we use, instead of \(s\), the variable \(w\):

\[
s = 1 - \varepsilon w. \tag{3.19}
\]

In terms of the new variables, the exponential function in the last expression of (3.14) is simply given by \(e^{-\xi(y + w + 1)}\). Then, we obtain

\[
B_\varepsilon(s) = \frac{\varepsilon s^2}{1 + \varepsilon} \int_0^\infty d\xi \int_0^\infty d\eta e^{-\xi(y + w + 1)} y^2 f(x, y, s) = -\frac{3s^2 w^2}{\pi^2 \varepsilon(1 + w)^4} = -\frac{s^2}{\pi^2 (1 - s + \varepsilon)^4} \to -\frac{1}{\pi^2} \delta(1 - s), \tag{3.20}
\]

where we have used

\[
y^2 f(x, y, s) = -\frac{\eta w^2 \xi^5}{2\pi^2 \varepsilon^2} + O(1/\varepsilon). \tag{3.21}
\]

In obtaining the final expression of (3.20), we used that the \(\varepsilon \to +0\) limit of

\[
\delta_\varepsilon(1 - s) = \frac{3\varepsilon(1 - s)^2}{(1 - s + \varepsilon)^4}, \tag{3.22}
\]

can be identified as a delta function \(\delta(1 - s)\) in the interval \(0 \leq s \leq 1\) since it satisfies

\[
\lim_{\varepsilon \to +0} \delta_\varepsilon(1 - s) = 0 \quad (s < 1), \quad \lim_{\varepsilon \to +0} \int_0^1 ds \delta_\varepsilon(1 - s) = 1. \tag{3.23}
\]

Our result (3.20) implies that \(\mathcal{N}\) (2.3) given as the integration of \(\mathcal{B}(s)\) has a contribution only at \(s = 1\). This reconfirms our earlier expectation that it is the zero eigenvalue of \(K\) that makes \(\mathcal{N}\) non-trivial.
4 Additivity of \( \mathcal{N} \) and the EOM

As stated in the Introduction, we can easily derive the following identity for \( \mathcal{N} \) (1.14):

\[
\mathcal{N}[UV] = \mathcal{N}[U] + \mathcal{N}[V] + \Delta \mathcal{N},
\]

(4.1)

with \( \Delta \mathcal{N} \) given by

\[
\Delta \mathcal{N} = \pi^2 \int Q_B \left\{ (Q_B U^{-1}) UV (Q_B V^{-1}) \right\}.
\]

(4.2)

The same kind of equations as (4.1) and (4.2) hold for the winding number (1.11) in the CS theory. Eq. (4.1) leads to the additive law of \( \mathcal{N} \),

\[
\mathcal{N}[UV] = \mathcal{N}[U] + \mathcal{N}[V],
\]

(4.3)

if we can discard the last term \( \Delta \mathcal{N} \) which is the integration of a BRST-exact quantity. The additivity (4.3) means, in particular, that \((U_{tv})^n\) \((n = \pm 1, \pm 2, \ldots)\) with \(U_{tv}\) of the tachyon vacuum has an integer \( \mathcal{N} \):

\[
\mathcal{N}[(U_{tv})^n] = n \mathcal{N}[U_{tv}] = -n.
\]

(4.4)

In the above argument, we did not take the regularization into account. In this section, we will examine whether (4.4) really holds in our \( K_\varepsilon \)-regularization.

4.1 Calculation of \( \mathcal{N}[(U_{tv})^n] \)

In this subsection, we calculate \( \mathcal{N} \) for \( U = (U_{tv})^n \) in the \( K_\varepsilon \)-regularization. For this purpose, we first obtain \( F^2 \) corresponding to \((U_{tv})^n\). Let us start with a generic \( U \) given in the form (2.6). Then, using the \( KBc \) algebra, we find that

\[
U^n = 1 - \tilde{F}_n BcF, \quad (n = 0, \pm 1, \pm 2, \ldots),
\]

(4.5)

with \( \tilde{F}_n \) being a function of \( K \) only \((\tilde{F}_0 = 0, \tilde{F}_1 = F)\), and further that \( \tilde{F}_n \) satisfies the following recursion relation:

\[
\tilde{F}_{n+1} = F + \tilde{F}_n (1 - F^2).
\]

(4.6)

Though \( U^n \) given by (4.5) is not of the standard form (2.6), we can bring it into the standard form in terms of \( R_n \) which is a function of \( K \) only:

\[
R_n U^n R_n^{-1} = 1 - F_n BcF_n.
\]

(4.7)

Note that \( R_n U^n R_n^{-1} \) carries the same \( \mathcal{N} \) as \( U^n \) since \( R_n \) commutes with \( Q_B \). From (4.5) and (4.7), we have \( R_n \tilde{F}_n = F_n = FR_n^{-1} \) and hence \( F^2_n = F \tilde{F}_n \). Then, from the recursion relation (4.6), we obtain \( 1 - F^2_{n+1} = (1 - F^2)(1 - F^2_n) \) and therefore

\[
F^2_n = 1 - (1 - F^2)^n, \quad (n = 0, \pm 1, \pm 2, \ldots).
\]

(4.8)
In particular, by taking as $F^2$ that for the tachyon vacuum, (3.1), we obtain
\[ F_n^2 = 1 - \left( \frac{K}{1 + K} \right)^n. \] (4.9)
This coincides with $F^2$ proposed in [6] as an example giving $\mathcal{N} = -n$ from quite a different argument.

We have calculated $\mathcal{N}[\{(U_{tv})^n\}]$ by using $F_n^2$ given by (4.9) in the $K_\varepsilon$-regularization. As the interpolating $(F_n^2)_s$ with parameter $s$, we adopt (4.8) with $F^2$ replaced by the interpolating $F_s^2$ (3.2) for the tachyon vacuum:
\[ (F_n^2)_s = 1 - \left( \frac{1 - s + K}{1 + K} \right)^n. \] (4.10)

The calculations are almost the same as in the case of the tachyon vacuum except that, for a negative $n$, there also appear terms which contain $1/(1 - s + K)$ but no $1/(1 + K)$. For such terms the variable $y$ in (3.4) is unnecessary, and they are reduced to a single integration of the form
\[ \int_0^\infty dx e^{-(1-s)x} J(x). \] (4.11)
Now the total of $\mathcal{B}(s)$ is given as the sum of two types of integrations, (3.7) and (4.11). The point is that each of $H(x, y)$ and $J(x)$ separately vanishes before introducing the regularization since they separately come from the integration of a BRST-exact quantity. Therefore, in obtaining $\mathcal{B}(s)$ in the $K_\varepsilon$-regularization, we are allowed to take only the $\varepsilon$-terms as in the case of the tachyon vacuum explained in Sec. 3.2.

Our results for $n = \pm 1$ and $\pm 2$ are already given in (1.20). It shows that our expectation (4.4) does not hold except in the cases $n = \pm 1$. Even worse, $\mathcal{N}$ for $n = \pm 2$ are not integers. This result (1.20) implies that the additivity (4.3) is violated. We have examined the extra term $\Delta \mathcal{N}$ (4.2) in the $K_\varepsilon$-regularization, namely, $\Delta \mathcal{N}|_{K_\varepsilon-\text{reg.}} = \pi^2 \int (Q_B \{ \cdots \})_{K \rightarrow K_\varepsilon}$ with $U = V = (U_{tv})^{\pm 1}$ to confirm that it is non-vanishing and exactly accounts for the violation of the additivity $\mathcal{N}[(U_{tv})^{\pm 2}] = 2\mathcal{N}[(U_{tv})^{\pm 1}]$.

On the other hand, \[ \mathcal{N}[U^{-1}] = -\mathcal{N}[U], \] (4.12)
holds for any $U$, since we have $\Delta \mathcal{N}|_{K_\varepsilon-\text{reg.}} = \pi^2 \int (Q_B^2(U^{-1}Q_B U))_{K \rightarrow K_\varepsilon} = 0$ for (4.1) and (4.2) with $UV = 1$ owing to $Q_B^2 = 0$. Our result (1.20) is consistent with the property (4.12).

### 4.2 EOM in the strong sense

Let us check whether $\Psi = UQ_BU^{-1}$ with $U = (U_{tv})^n$ satisfies the EOM in the strong sense, \[ \int \Psi \ast (Q_B \Psi + \Psi^2) = 0, \] in the $K_\varepsilon$-regularization. Since this EOM is the same between $U = (U_{tv})^n$ and $R_n(U_{tv})^nR_n^{-1}$, we consider the latter with $F_n^2$ given by (4.9).
For this purpose, we prepare the expression of the EOM for a generic $U$ in the standard form (2.6). Let $\Psi_\varepsilon$ be the $K_\varepsilon$-regularized $\Psi$,

$$\Psi_\varepsilon = (UQ_BU^{-1})_{K \rightarrow K_\varepsilon} = F_\varepsilon c \frac{K_\varepsilon B}{1 - F_\varepsilon^2} c F_\varepsilon,$$  

(4.13)

with $F_\varepsilon = F(K_\varepsilon)$. Using the $KBc$ algebra, we find that the EOM of $\Psi_\varepsilon$ is reduced to an apparently of $O(\varepsilon)$ quantity:

$$Q_B \Psi_\varepsilon + \Psi_\varepsilon^* \Psi_\varepsilon = \varepsilon \times F_\varepsilon c \frac{K_\varepsilon}{1 - F_\varepsilon^2} c F_\varepsilon,$$  

(4.14)

where $Q_B \Psi_\varepsilon$ is $\Psi_\varepsilon$ (4.13) acted by $Q_B$, and is not equal to $(Q_B \Psi)_{K \rightarrow K_\varepsilon}$. Using this for $F = F_n$ (4.9), a straightforward calculation gives

$$\int \Psi_\varepsilon^* (Q_B \Psi_\varepsilon + \Psi_\varepsilon^* \Psi_\varepsilon) = \varepsilon \int BcF_\varepsilon^2 c \frac{K_\varepsilon}{1 - F_\varepsilon^2} c F_\varepsilon \frac{K_\varepsilon}{1 - F_\varepsilon^2} c F_\varepsilon \rightarrow 0 \begin{cases} -6 & (n = -2) \\ 0 & (n = \pm 1) \\ 2 & (n = 2) \end{cases}.$$  

(4.15)

Our result that $\Psi_\varepsilon$ for $n = \pm 2$ does not satisfy the EOM in the strong sense implies that this $\Psi_\varepsilon$ cannot be regarded as a pure-gauge even in the limit $\varepsilon \rightarrow 0$. $N$ (1.14) is formally invariant under small deformations of $U$ owing to the fact that $\Psi = UQ_BU^{-1}$ is a pure-gauge. The violation of the EOM for $n = \pm 2$, (4.15), means that $N[(U_{tv})^{\pm 2}]$ is not such a stable quantity. Therefore, the anomalous (non-integer) values of $N$ presented in (1.20), $N[(U_{tv})^{\pm 2}] = \mp (2 - 2\pi^2)$, should not be taken as a counterexample to the quantization of $N$.

This hand-waving argument should, of course, be made more rigorous. In particular, we must clarify the relationships among the various requirements: the EOM, the additive law (4.3) of $N$, inertness of $N$ under deformations of $U$, and the quantization of $N$. Here, we examined the validity of the EOM,

$$\int O^* (Q_B \Psi_\varepsilon + \Psi_\varepsilon^* \Psi_\varepsilon) = 0,$$  

(4.16)

only for $O = \Psi_\varepsilon$. It is necessary to understand for what class of $O$ the EOM should hold in order for the requirements on $N$ to be valid.

Finally, a comment is in order concerning a simpler derivation of (4.15). In the above, it was evaluated without any approximation. However, the same result can be obtained by taking only the term with the least power of $K_\varepsilon$ in the Laurent series of each of the quantities $F_\varepsilon^2$ and $K_\varepsilon/(1 - F_\varepsilon^2)$ with respect to $K_\varepsilon$. For example, for $n = 2$, we have

$$F_\varepsilon^2 = 1 - K_\varepsilon^2 + O(K_\varepsilon^3), \quad \frac{K_\varepsilon}{1 - F_\varepsilon^2} = \frac{1}{K_\varepsilon} + O(K_\varepsilon^0),$$  

(4.17)

and (4.15) for $n = 2$ is reproduced by

$$\varepsilon \int BcK_\varepsilon^2 c \frac{1}{K_\varepsilon} c K_\varepsilon^2 c \frac{1}{K_\varepsilon} = 2.$$  

(4.18)
5 Summary and discussions

In this paper, motivated by the similarity between the CSFT and the CS theories, we pursued the possibility that $N$ (1.14) is interpreted as a kind of winding number in CSFT which is quantized to integer values. We especially focused on the expression (1.15) of $N$ as the integration of a BRST-exact quantity, which naively vanishes identically and manifests the topological nature of $N$. For realizing non-vanishing values of (1.15), we need to introduce a regularization for divergences arising from the zero eigenvalue of the operator $K$. This regularization must also cause an infinitesimal violation of the BRST-exactness of the integrand of $N$. As such a regularization, we proposed the $K_\varepsilon$-regularization (1.18) of shifting $K$ by a positive infinitesimal $\varepsilon$. Applying the $K_\varepsilon$-regularization to the calculation of (1.15) for $U = U_{tv}$ which represents the tachyon vacuum, we got the expected result $N[U_{tv}] = -1$. In this calculation, we found that the non-vanishing value of $N$ is realized by $\varepsilon \times (1/\varepsilon)$ with $\varepsilon$ from the violation of the BRST-exactness of the integrand and $1/\varepsilon$ from the zero eigenvalue of $K$. Then, we further studied $N$ for $U = (U_{tv})^n$ with $n = -2, -1, 2$. The additive law of $N[U]$ for the product of $U$ predicts that $N[(U_{tv})^n] = -n$. However, explicit calculations show that $N$ for $n = \pm 2$ are anomalous as given in (1.20). At the same time, we found that $\Psi = UQ_BU^{-1}$ with $U = (U_{tv})^{\pm 2}$ does not satisfy the EOM in the strong sense either. This implies that $UQ_BU^{-1}$ for such $U$ cannot be regarded as truly pure-gauge, and may explain the violation of the quantization of $N$.

This paper is a first step toward identifying $N$ as a winding number in CSFT and thereby unveiling the “topological structure” of CSFT. Our analysis is of course far from being complete and there remains many open questions to be answered. They include the followings:

- We attributed our unwelcome result that $N[U = (U_{tv})^{\pm 2}]$ take non-integer values to the breakdown of the EOM in the strong sense for $\Psi = UQ_BU^{-1}$. However, we do not know a precise connection between the two. We have to understand the relationships among the quantization of $N$, invariance of $N$ under small deformations of $U$, the additive law of $N$, and the EOM in the strong sense.

- In this paper, we proposed and used the $K_\varepsilon$-regularization. This regularization certainly regularizes the infinities arising from the zero eigenvalue of $K$ and, at the same time, violates the BRST-exactness of the integrand of $N$ (1.15), thus leading to the desired value $N[U_{tv}] = -1$. However, we do not know whether our $K_\varepsilon$-regularization is a fully satisfactory one. It might be that the non-integer values of $N[U]$ and the violation of EOM for $U = (U_{tv})^{\pm 2}$ are artifacts of the $K_\varepsilon$-regularization. We need to understand the basic principles that the regularization has to satisfy.
• Besides such general considerations as presented above, it is an interesting problem to construct $U$’s which give integer $N$ other than $\pm 1$, and at the same time, satisfy the EOM in the strong sense.

• The existence of $U$ with $N[U] < -1$ apparently implies a physically unwelcome fact that $\Psi = U Q_B U^{-1}$ represents a state with its energy density lower than that of the tachyon vacuum. We have to show that, if there exists such $U$, $\Psi = U Q_B U^{-1}$ never satisfies the EOM in the strong sense.

• In this paper, we calculated $N[(U_{tv})^{n}]$ only for $n = \pm 1, \pm 2$. It is an interesting technical problem to obtain its expression for a generic integer $n$.

• It is a challenging problem to evaluate $N$ as a “surface integration”, $N = \int_{M} Q_B A = \int_{\partial M} A$. For this we have to understand the meaning of the boundary $\partial M$ (which should be a set of singularities of $A$) as well as that of the original “manifold” $M$. We have to understand of course what kind of “windings” the quantity $N$ counts.

By resolving these problems, we wish to find fruitful structure of SFT which we still do not know.

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Note added
The $K_\varepsilon$-regularization was also used in [7] and [8] in their analysis of the energy and the EOM of classical solutions. We would like to thank Ted Erler and Masaki Murata for correspondences.
A  **KBc algebra and correlators**

Here, we summarize the KBc algebra and the correlators which are used in the text. (See [4, 5] for details. In this paper, we follow the convention of [5].)

The elements of the KBc algebra satisfy

\[
[B, K] = 0, \quad \{B, c\} = 1, \quad B^2 = c^2 = 0,
\]

and

\[
Q_BB = K, \quad Q_BK = 0, \quad Q_Bc = cKc.
\]

Their ghost numbers are

\[
g(K) = 0, \quad g(B) = -1, \quad g(c) = 1.
\]

In the text, there appeared the following quantities:

\[
G(t_1, t_2, t_3, t_4) = \int Bce^{-t_1K} e^{-t_2K} c e^{-t_3K} c e^{-t_4K} = \langle Bc(0)c(t_1 + t_2)c(t_1 + t_2 + t_3+t_4) \rangle_{t_1+t_2+t_3+t_4}, \tag{A.4}
\]

\[
G_c(t_1, t_2, t_3) = \int c e^{-t_1K} e^{-t_2K} c e^{-t_3K} = \langle c(0)c(t_1 + t_2) \rangle_{t_1+t_2+t_3}. \tag{A.5}
\]

They are given in terms of the correlators on the cylinder with infinite length and the circumference \(\ell\):

\[
\langle Bc(z_1)c(z_2)c(z_3)c(z_4) \rangle_{\ell} = \left(\frac{\ell}{\pi}\right)^2 \left\{ -\frac{z_1}{\pi} \sin \left[\frac{\pi}{\ell}(z_2 - z_3)\right] \sin \left[\frac{\pi}{\ell}(z_2 - z_4)\right] \sin \left[\frac{\pi}{\ell}(z_3 - z_4)\right] \\
+ \frac{z_2}{\pi} \sin \left[\frac{\pi}{\ell}(z_1 - z_3)\right] \sin \left[\frac{\pi}{\ell}(z_1 - z_4)\right] \sin \left[\frac{\pi}{\ell}(z_3 - z_4)\right] \\
- \frac{z_3}{\pi} \sin \left[\frac{\pi}{\ell}(z_1 - z_2)\right] \sin \left[\frac{\pi}{\ell}(z_1 - z_4)\right] \sin \left[\frac{\pi}{\ell}(z_2 - z_4)\right] \\
+ \frac{z_4}{\pi} \sin \left[\frac{\pi}{\ell}(z_1 - z_2)\right] \sin \left[\frac{\pi}{\ell}(z_1 - z_3)\right] \sin \left[\frac{\pi}{\ell}(z_2 - z_3)\right]\right\}, \tag{A.6}
\]

\[
\langle c(z_1)c(z_2)c(z_3) \rangle_{\ell} = \left(\frac{\ell}{\pi}\right)^3 \sin \left[\frac{\pi}{\ell}(z_1 - z_2)\right] \sin \left[\frac{\pi}{\ell}(z_1 - z_3)\right] \sin \left[\frac{\pi}{\ell}(z_2 - z_3)\right]. \tag{A.7}
\]

B  **Eq. (2.13) and rederivation of (3.17)**

In this appendix, we outline the derivation of another and simpler expression (2.13) for \(B(s)\) and the calculation of \(\mathcal{N}\) for the tachyon vacuum using this expression.
First, $\mathcal{B}(s)$ is given in the following form:

$$
\mathcal{B}(s) = \int BcF_s^2 c \frac{K}{1-F_s^2} \left( \frac{d}{ds} BcF_s^2 c \frac{K}{1-F_s^2} \right) BcF_s^2 c \frac{K}{1-F_s^2}
$$

$$
= - \int Bc \left[ cK, F_s^2 \right] \frac{1}{1-F_s^2} \left( \frac{d}{ds} Bc \left[ cK, F_s^2 \right] \frac{1}{1-F_s^2} \right) Bc \left[ cK, F_s^2 \right] \frac{1}{1-F_s^2}
$$

$$
= - \int Bc \left[ cK, F_s^2 \right] \frac{1}{1-F_s^2} \left( \frac{d}{ds} \left[ cK, F_s^2 \right] \frac{1}{1-F_s^2} \right) \left[ cK, F_s^2 \right] \frac{1}{1-F_s^2}, \quad (B.1)
$$

where we have used $c^2 = 0$ at the second equality. The last expression is due to

$$
\int Bc \left[ cK, A_1 \right] A_2 Bc \left[ cK, A_3 \right] A_4 Bc \left[ cK, A_5 \right] A_6 = \int Bc \left[ cK, A_1 \right] A_2 \left[ cK, A_3 \right] A_4 \left[ cK, A_5 \right] A_6, \quad (B.2)
$$

which is valid for arbitrary $A_k$'s depending only on $K$. Then, using $(1-F_s^2)^{-1} \left[ cK, F_s^2 \right] (1-F_s^2)^{-1} = \left[ cK, (1-F_s^2)^{-1} \right]$, $\mathcal{B}(s)$ is further rewritten as follows:

$$
\mathcal{B} = \int BcF_s^2 cK \left\{ \frac{1}{1-F_s^2} \left[ cK, \frac{dF_s^2}{ds} \right] + \left[ cK, \frac{1}{1-F_s^2} \right] \frac{dF_s^2}{ds} \right\} \left[ cK, \frac{1}{1-F_s^2} \right]
$$

$$
= \int BcF_s^2 cK \left[ cK, \frac{1}{1-F_s^2} \frac{dF_s^2}{ds} \right] \left[ cK, \frac{1}{1-F_s^2} \right]. \quad (B.3)
$$

Expanding the commutators, we get (2.13). Each of $\mathcal{B}_1(s)$ and $\mathcal{B}_2(s)$ is given as the integration of a BRST-exact quantity:

$$
\tilde{\mathcal{B}}_1(s) = \int Q_B \left[ BcF_s^2 c \frac{K}{1-F_s^2} \frac{dF_s^2}{ds} \frac{K}{1-F_s^2} \right], \quad (B.4)
$$

$$
\tilde{\mathcal{B}}_2(s) = - \int Q_B \left[ BcF_s^2 c \frac{K}{(1-F_s^2)^2} \frac{dF_s^2}{ds} cK \right]. \quad (B.5)
$$

From this it is manifest that $\mathcal{B}(s)$ without regularization vanishes.

Then, let us consider evaluating $\mathcal{B}(s)$ using the expression (2.13) in the $K_\varepsilon$-regularization. As explained in Sec. 3.2, we have only to take the $\varepsilon$-term. Differently to the case of (2.10), all the terms in (2.13) contribute to the $\varepsilon$-term. We get

$$
\mathcal{B}_\varepsilon(s) = \varepsilon s^2 \left\{ (1-s)^2 \int c \frac{1}{1+K_\varepsilon} c \frac{1}{1-s+K_\varepsilon} c \frac{1}{1-s+K_\varepsilon}
$$

$$
- \int c \frac{1}{1+K_\varepsilon} cKc \left[ \frac{1}{1-s+K_\varepsilon} - \frac{1-s}{(1-s+K_\varepsilon)^2} \right] \right\}. \quad (B.6)
$$

This leads to exactly the same result as the final expression of (3.14).
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