Finite Volume Method for Pricing European Call Option with Regime-switching Volatility

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Abstract. In this paper, we present a finite volume method for pricing European call option using Black-Scholes equation with regime-switching volatility. In the first step, we formulate the Black-Scholes equations with regime-switching volatility. We use a finite volume method based on fitted finite volume with spatial discretization and an implicit time stepping technique for the case. We show that the regime-switching scheme can revert to the non-switching Black Scholes equation, both in theoretical evidence and numerical simulations.

1. Introduction
Finite volume method, also called Box method is used in the numerical solution of fluid flow problems. But the application of finite volume method is not limited in problems of fluid. This method can also be used in financial mathematics problems for example at issue option. In solving the problems of option is usually used Black Scholes model as a mathematical model. In this paper we use black scholes model with regime switching because the model with constant volatility can not represent the actual stock price. In 2001 finite volume method was successfully used by Zvan in numerical problems on an options case. In 2006 Julien Berton conducted a study of a finite volume method, it was used to determine an American option price. From the research it was found that the scheme of a finite volume method to be more accurate and efficient than the two methods compared a Dynamic programming method and an ADI method. In 2008 Zhang and Wang conducted research on a finite volume method to determine the price of European options. In determining the option price Zhang and Wang use the Black-Scholes model which has 1 partial differential equation. From the results of his research, it was found that the scheme from a finite volume method in a space with an implicit scheme at a time is consistent and stable, so it is said to be convergent as the relevant financial solution. In this paper, we present a finite volume method in determining the price of European call option with regime-switching volatility as a measure of an uncertainty of stock movement. The option price is determined from Black-Scholes equations with regime-switching volatility. We shall show that the system matrix of discretization is an M-matrix.

2. European call option with regime-switching volatility
In this section, we will formulate a Black-Scholes model for European call option with regime-switching where the volatility moves randomly between two states. The Fluctuations of
underlying assets are assumed to follow the stochastic process described in the following equation:

\[ dS(t) = (\mu X_t - D)S(t)dt + \sigma X_t S(t) dW(t), \]  

(1)

These equation is used to formulate a Black-Scholes model with regime-switching volatility. Regime-switching is a process of jumping on several states that occur on the Markov chain. Let \( X_t \) is a markov chain with continuous time that describes the state changes that occur, the state of the economy grows and the economy is weak. Because the economic state consists of two states, the change of the option price has a different value for each initial position of each state. When the initial position is state 1, the possible state change is fixed in state 1 or changed to state 2. So if the chance of change from state 1 to 2 is \( \lambda_{12} dt \) then the chances of state 1 remain in state 1 is \( 1 - \lambda_{12} dt \). So we get the change of option price is:

\[
\begin{align*}
\frac{dV}{dt} &= \\
V_1(S(t+dt), t+dt) - V_1(S(t), t) &= dV_1 \\
V_2(S(t+dt), t+dt) - V_1(S(t), t) &= dV_2 + V_2 - V_1.
\end{align*}
\]

The option price changes for initial state state 1 are:

\[
\frac{dV}{dt} = \begin{cases} 
V_1(S(t+dt), t+dt) - V_1(S(t), t), & p = 1 - \lambda_{12} dt \\
V_2(S(t+dt), t+dt) - V_1(S(t), t), & p = \lambda_{12} dt
\end{cases}
\]

(2)

The possibility that occurs at state 2 is fixed at state 2 or changed to state 1. If the initial position of the economic state is state 2 then the probability state 2 turns to state 1 is \( \lambda_{21} dt \) and the probability of remaining in state 2 is \( 1 - \lambda_{21} dt \). So we obtain the change of option price is as follows:

\[
\begin{align*}
\frac{dV}{dt} &= \\
V_2(S(t+dt), t+dt) - V_2(S(t), t) &= dV_2 \\
V_1(S(t+dt), t+dt) - V_2(S(t), t) &= dV_1 + V_1 - V_2.
\end{align*}
\]

The option price changes for initial state 2 are:

\[
\begin{align*}
\frac{dV}{dt} &= \\
V_2(S(t+dt), t+dt) - V_2(S(t), t), & p = 1 - \lambda_{21} dt \\
V_1(S(t+dt), t+dt) - V_2(S(t), t), & p = \lambda_{21} dt
\end{align*}
\]

(3)

then we formulate a portfolio of options. The value of the portfolio is given by the equation:

\[ \Pi = V - \Delta S \]  

(4)

Over a period of time, portfolio value changes to:

\[ d\Pi = dV - \Delta dS \]  

(5)
So that the change of option value with dividend payment is:

\[ d\Pi = dV - \Delta dS - \delta \Delta S dt \]  

(6)

From equation (6) a portfolio change can be determined for each initial positions of different states. for the initial position of state 1, we obtain:

\[
\begin{align*}
    d\Pi &= \begin{cases} 
    dV_1 - \Delta dS - DS\Delta dt, & p = 1 - \lambda_{12}dt \\
    dV_2 + V_2 - V_1 - \Delta dS - DS\Delta dt, & p = \lambda_{12}dt,
    \end{cases}
\end{align*}
\]

Then we formulate an expected value of \(d\Pi\) as follows:

\[ E[d\Pi] = dV_1 - \Delta dS - \delta S \Delta dt + \lambda_{12}dt(dV_2 - dV_1 + V_2 - V_1). \]

(7)

Next we formulate \(dV_1\) from Lemma ITO. And from equation (1) is obtained:

\[ (dS)^2 = (\mu - \delta)^2 S^2 dt^2 + 2(\mu - \delta) S^2 \sigma dtdW_t + \sigma^2 S^2 dW_t^2 \]

because \(dtdW_t = 0\) and \(dt^2 = 0\), then:

\[ (dS)^2 = \sigma^2 S^2 dt. \]

Lemma ITO

\[ dV_i = \frac{\partial V_i}{\partial S} S dW_t + \left[ \frac{\partial V_i}{\partial S} (\mu - \delta) S + \frac{\partial V_i}{\partial t} + \frac{1}{2} \frac{\partial^2 V_i}{\partial S^2} \sigma_i^2 S^2 \right] dt \]

(8)

by substituting equation (1) and (8) into equation (7). Furthermore, it is assumed that \(dW_tdW_t = 0\) and \(dt^2 = 0\), then we get:

\[
\begin{align*}
    d\Pi &= \sigma_1 S \left( \frac{\partial V_1}{\partial S} - \Delta \right) dW_t + \left[ \mu S \left( \frac{\partial V_1}{\partial S} - \Delta \right) - \delta S \left( \frac{\partial V_1}{\partial S} - \Delta \right) + \frac{1}{2} \sigma_1^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \frac{\partial V_1}{\partial t} + \lambda_{12} (V_2 - V_1) - \delta \Delta S \right] dt.
\end{align*}
\]

To eliminate random components in a random walk in order to get a riskless then selected

\[ \Delta = \frac{\partial V_1}{\partial S} \]

so we get,

\[ d\Pi = \left[ \frac{1}{2} \sigma_1^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \frac{\partial V_1}{\partial t} - \delta S \frac{\partial V_1}{\partial S} + \lambda_{12} (V_2 - V_1) \right] dt. \]

(9)

the same way as initial position state 1 we get the \(d\Pi\) value at initial position state 2 as follows:

\[ d\Pi = \left[ \frac{1}{2} \sigma_2^2 S^2 \frac{\partial^2 V_2}{\partial S^2} + \frac{\partial V_2}{\partial t} - \delta S \frac{\partial V_2}{\partial S} + \lambda_{21} (V_1 - V_2) \right] dt. \]

(10)
It is assumed that portfolio results are free arbitrage and obtain a risk free rate:

\[ d\Pi = \Pi r dt \]  

so we obtain the partial differential equation of Black-Scholes for initial position state 1 as follows:

\[ \frac{\partial V_1}{\partial t} + \frac{1}{2} \sigma_1^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + (r - \delta) S \frac{\partial V_1}{\partial S} - rV_1 = \lambda_{12}(V_1 - V_2). \]  

(11)

Similarly, the partial differential equation of Black-Scholes for initial position state 2 is

\[ \frac{\partial V_2}{\partial t} + \frac{1}{2} \sigma_2^2 S^2 \frac{\partial^2 V_2}{\partial S^2} + (r - \delta) S \frac{\partial V_2}{\partial S} - rV_2 = \lambda_{21}(V_2 - V_1). \]  

(12)

Furthermore, Equations are combined with boundary conditions to form a pair of partial differential equation (PDE) systems for an European call option with dividend and regime-switching as follows:

**state 1**

\[ \frac{\partial V_1}{\partial t} + \frac{1}{2} \sigma_1^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + (r - \delta) S \frac{\partial V_1}{\partial S} - rV_1 = \lambda_{12}(V_1 - V_2) \]

with boundary conditions:

\[ V_1(0, t) = 0 \]

\[ \lim_{S \to \infty} V_1(S, t) = S \]

\[ V_1(S, T) = \max \{ S - E, 0 \} \].

**state 2**

\[ \frac{\partial V_2}{\partial t} + \frac{1}{2} \sigma_2^2 S^2 \frac{\partial^2 V_2}{\partial S^2} + (r - \delta) S \frac{\partial V_2}{\partial S} - rV_2 = \lambda_{21}(V_2 - V_1) \]

with boundary conditions:

\[ V_2(0, t) = 0 \]

\[ \lim_{S \to \infty} V_2(S, t) = S \]

\[ V_2(S, T) = \max \{ S - E, 0 \} \].

3. **Finite volume method**

In this section, we consider a numerical approximation of the problem using a finite volume method based on a fitted finite volume. We will determine the finite volume method for a Black-Scholes equation of the European call option with regime-switching volatility.

3.1. **Spatial discretization**

Divide \( I \) into two parts \( I_i \) and \( J_i \). The first part consists of \( N \) sub-intervals \( I_i = (S_i, S_{i+1}) \), \( i = 0, 1, 2, \ldots, N - 1 \) with \( 0 = S_0 < S_1 < \ldots < S_N = S_{\text{max}} \). The second part \( J_i \) consist of \( J_i = (S_{i-1/2}, S_{i+1/2}) \) where \( S_{i-1/2} = (S_{i-1} + S_i)/2 \), \( S_{i+1/2} = (S_i + S_{i+1})/2 \), \( S_0 = S_{-1/2} \) and \( S_{i+1/2} = S_{N+1} \).
\textbf{State 1}

\[
\frac{\partial V_i}{\partial \tau} = \frac{1}{2} \sigma_1^2 S^2 \frac{\partial^2 V_i}{\partial S^2} + (r - \delta) S \frac{\partial V_i}{\partial S} - r V_i + \lambda_1 (V_2 - V_i)
\]

\[
\frac{\partial V_i}{\partial \tau} = \frac{\partial}{\partial S} \left[ a S^2 \frac{\partial V_i}{\partial S} + b S V_i \right] + c V_i + \lambda_1 V_2
\]

where \(a = \frac{1}{2} \sigma_1^2, b = r - \delta - \sigma_1^2, c = -b + r - \lambda_1\).

Integrating over \(J_i\):

\[
\int_{J_i} \frac{\partial V_i}{\partial \tau} dS = \int_{J_i} \frac{\partial V_i}{\partial S} \left[ a S^2 \frac{\partial V_i}{\partial S} + b S V_i \right] dS + \int_{J_i} c V_i dS + \int_{J_i} \lambda_1 V_2 dS
\]

let \(x_i = S_{i+1/2} - S_{i-1/2}\), we get:

\[
\frac{\partial V_i}{\partial \tau} x_i = S_{i+1/2} \rho(V_i) \bigg|_{S_{i+1/2}} - S_{i-1/2} \rho(V_i) \bigg|_{S_{i-1/2}} + c V_i x_i + \lambda_1 V_2 x_i
\]

where

\[
\rho(V_i) = a S \frac{\partial V_i}{\partial S} + b V_i
\]

Then we need to derive an approximation of the continuous flux \(\rho(V_i)\). This discussion is divided into two cases for \(i\) and \(i = 0, 1\) and \(i = 2, 3, \ldots, N - 1\), respectively.

\textbf{Case I.} For \(i = 2, 3, \ldots, N - 1\)

Approximation of \(\rho(V_i)\) at the point \(S_{i-1/2}\) and \(S_{i+1/2}\) of the interval \(I_i\). Let us consider the following two-point boundary value problem:

at the point \(S_{i+1/2}\)

\[
(a_{i+1/2} S V_i' + b_{i+1/2} V_i)' = 0, S \in I
\]

\[
V_1(S_i) = V_i i, V_1(S_{i+1}) = V_i + 1
\]

and at the point \(S_{i-1/2}\)

\[
(a_{i-1/2} S V_i' + b_{i-1/2} V_i)' = 0, S \in I
\]

\[
V_1(S_i) = V_i i, V_1(S_{i-1}) = V_i - 1
\]

Solving the problem analytically, we get: at the point \(S_{i+1/2}\)

\[
\rho_i(V_i) = b_{i+1/2} \frac{V_i S_i^{k_i} - V_{i+1} S_i^{k_i}}{S_i^{k_i} - S_{i+1}^{k_i}}
\]

and at the point \(S_{i-1/2}\)

\[
\rho_i(V_i) = b_{i-1/2} \frac{V_i S_i^{k_i} - V_{i-1} S_i^{k_i}}{S_i^{k_i} - S_{i-1}^{k_i}}
\]

where

\[
k_i = \frac{b_{i+1/2}}{a_{i+1/2}}
\]
Case II. For $i = 0, 1$ Note that the analysis in Case I does not apply to approximation of the flux on $(0, S_1)$ because the equation is degenerated for $S \to 0$. In this case we can consider in the following form

\begin{align}
(a_{1/2}SV'_i + b_{1/2}V_i)' &= c, \; S \in I \\
V_i(0) &= V_i0, V_i(S_1) = V_i1
\end{align}

Solving these equation, we get

$$
\rho_i(V_1) = \frac{1}{2}((a_{1/2} + b_{1/2})V_1 - (a_{1/2} - b_{1/2})V_0)
$$

$$
V_1 = V_10 + \frac{S}{S_1}(V_11 - V_10)
$$

substituting (10), (11), (15) and (16), we obtain for $i = 1$

$$
\frac{\partial V_1}{\partial \tau} x_i = S_{3/2}b_{3/2} \left[ \frac{V_i S_{k1}^{i1} - V_{i+1} S_{k1}^{i+1}}{S_{k1}^{i1} - S_{k1}^{i+1}} \right] - S_{1/2}b_{1/2} \left[ \frac{V_i S_{k1}^{i1} - V_{i-1} S_{k1}^{i-1}}{S_{k1}^{i1} - S_{k1}^{i-1}} \right] + cx_i + x_i\lambda_{12}V_2
$$

So we get

$$
\frac{\partial V_1}{\partial \tau} = \xi_1 V_10 + \psi_1 V_11 + \phi_1 V_12 + \lambda_{12} V_21
$$

where

$$
\xi_1 = \frac{S_1(a_{1/2} - b_{1/2})}{4\xi_1}
$$

$$
\psi_1 = \frac{-S_1(a_{1/2} + b_{1/2})}{4x_i} - \frac{S_{3/2}b_{3/2}S_{k1}^{i1}}{x_i(S_{k1}^{i2} - S_{k1}^{i1})} - c
$$

$$
\phi_1 = \frac{S_{3/2}b_{3/2}S_{k1}^{i1}}{x_i(S_{k1}^{i2} - S_{k1}^{i1})}
$$

and for $i=2,3,...N-1$, we obtain,

$$
\frac{\partial V_i}{\partial \tau} = \xi_i V_{i-1} + \psi_i V_i + \phi_i V_{i+1} + \lambda_{12} V_{2i}
$$

where

$$
\xi_i = \frac{S_{i-1/2}b_{i-1/2}S_{k1}^{i1}}{x_i(S_{k1}^{i1} - S_{k1}^{i-1})}
$$

$$
\psi_i = \frac{-S_{i-1/2}b_{i-1/2}S_{k1}^{i1}}{x_i(S_{k1}^{i1} - S_{k1}^{i-1})} - \frac{S_{i+1/2}b_{i+1/2}S_{k1}^{i1}}{x_i(S_{k1}^{i1} - S_{k1}^{i+1})} - c
$$

$$
\phi_i = \frac{S_{i+1/2}b_{i+1/2}S_{k1}^{i1}}{x_i(S_{k1}^{i1} - S_{k1}^{i+1})}
$$

3.2. Time discretization

Let $\tau$ denotes points from $[0, T]$ such that $0 = \tau_0 < \tau_1 < ... < \tau_M = T$ and $\Delta \tau_n = \tau_n - \tau_{n-1} > 0$, where $M > 1$ is a positive integer. Applying the fully implicit time discretization to (35) for simplicity, we get

$$
\frac{V_i^{n+1} - V_i^n}{\Delta \tau_{n+1}} = \xi_i^{n+1}V_i^{n+1} + \psi_i^{n+1}V_i^{n+1} + \phi_i^{n+1}V_i^{n+1} + \lambda_{12} V_{2i}^{n+1}
$$
so that we obtain $V_1^n$ i is the solution of $S_i$ with time $\tau_n$ as follows:

$$V_1^n = (1 - \Delta \tau_{n+1} M^{n+1}) V_1^{n+1} - \Delta \tau_{n+1} R^{n+1}$$  \hspace{1cm} (40)$$

where

$$M^{n+1} = \begin{bmatrix}
\psi_1^{n+1} & \phi_1^{n+1} \\
\xi_2^{n+1} & \psi_2^{n+1} & \phi_2^{n+1} \\
& \ddots & \ddots \\
& & \psi_{N-2}^{n+1} & \phi_{N-2}^{n+1} \\
& & & \psi_{N-1}^{n+1} & \phi_{N-1}^{n+1}
\end{bmatrix}_{(N-1)\times(N-1)}$$

$$V_1^n = \begin{bmatrix}
V_1^n \\
V_1^{'n} \\
\vdots \\
V_1^{N-1}'
\end{bmatrix}, \quad R_{n+1} = \begin{bmatrix}
\xi_1^{n+1} V_1^{n+1} + \lambda_{12} V_2^{n+1} \\
\lambda_{12} V_2^{n+1} \\
\lambda_{12} V_2^{n} \\
\phi_{N-1}^{n+1} V_1^{n+1} + \lambda_{12} V_2^{n+1}
\end{bmatrix}$$

And in position state 2 we obtain,

$$V_2^n = (1 - \Delta \tau_{n+1} M^{n+1}) V_2^{n+1} - \Delta \tau_{n+1} R^{n+1}$$  \hspace{1cm} (41)$$

where:

$$M^{n+1} = \begin{bmatrix}
\psi_1^{n+1} & \phi_1^{n+1} \\
\xi_2^{n+1} & \psi_2^{n+1} & \phi_2^{n+1} \\
& \ddots & \ddots \\
& & \psi_{N-2}^{n+1} & \phi_{N-2}^{n+1} \\
& & & \psi_{N-1}^{n+1} & \phi_{N-1}^{n+1}
\end{bmatrix}_{(N-1)\times(N-1)}$$

$$V_2^n = \begin{bmatrix}
V_2^n \\
V_2^{'n} \\
\vdots \\
V_2^{N-1}'
\end{bmatrix}, \quad R_{n+1} = \begin{bmatrix}
\xi_1^{n+1} V_2^{n+1} + \lambda_{21} V_1^{n+1} \\
\lambda_{21} V_1^{n+1} \\
\lambda_{21} V_1^{n} \\
\phi_{N-1}^{n+1} V_1^{n+1} + \lambda_{21} V_1^{n+1}
\end{bmatrix}$$

4. Simulation and discussion

In this section, we conduct a computational process to examine our results. After we get two regime switching equations it will be done numerical simulation using MATLAB program. Let $V_1 = V_2$ to show the accuracy of models. So that the results are the same as conditions without regime switching and dividend. Then we will compare with the exact results of equations in the Black Scholes without regime-switching. We set up a parameter as follows: $T - t = 1, E = 100, r = 0.015, \sigma_1 = 0.2, \sigma_2 = 0.03, K = 100, S = 200, T = 0.25, \delta = 0.05, \lambda_{12} = 0.02$. It is seen that the settlement graph using finite volume method to European call option equations with
regime-switching is almost same or near with exact solutions if \( V_1 = V_2 \) where \( V_1 \) is an option value of \( \sigma_1 \) and \( V_1 \) with \( \sigma_2 \).

\[ \begin{align*}
\text{Figure 1.} \quad \text{simulation results and exact solutions}
\end{align*} \]

\[ \begin{align*}
\text{Figure 2.} \quad \text{simulation results and exact solutions}
\end{align*} \]

5. Conclusion
In this paper, we presented a finite volume method based on fitted finite volume method for the numerical solution of 2 Black-Scholes equations governing European call option with regime-switching volatility. We have shown that the numerical scheme results with M-system matrix. Numerical results were presented to demonstrate the usefulness of the method.

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