A 2-DOF model of an elastic rocket structure excited by a follower force

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Abstract. We present a two degree of freedom model of an elastic rocket structure excited by the follower force given by the motor thrust that is supposed to be always in the direction of the tangent to the deformed shape of the device at its lower tip. The model comprises two massless rigid pinned bars, initially in vertical position, connected by rotational springs. Lumped masses and dampers are considered at the connections. The generalized coordinates are the angular displacements of the bars with respect to the vertical. We derive the equations of motion via Lagrange’s equations and simulate its time evolution using Runge-Kutta 4th order time step-by-step numerical integration algorithm. Results indicate possible occurrence of stable and unstable vibrations, such as limit cycles.

1 Introduction
Launcher vehicles (popularly known as rocket) are essential devices to carry loads from the surface of the Earth to some orbit around it, whatever is the current space mission. The vehicle, like any other physical body, is not absolutely rigid, so that the structural behavior, via excitation by external loads, tends to affect the flight dynamics. It is intended, in this paper, to develop a mathematical model of an elastic space rocket structure as a Beck’s column excited by a follower (or circulatory) force. This force represents the rocket motor thrust that should be always in the direction of the tangent to the structure deformed axis at the base of the vehicle. We present a simplified two degree of freedom rigid bars discrete model. Its system of two second order nonlinear ordinary differential equations of motion are derived via Lagrange’s energy method, allowing for a general understanding of the main characteristics of the problem. The proposed equations consider up to third order (cubic) inertia, stiffness and forcing terms. Among other rich nonlinear dynamic behavior of this model, depending on parameters and initial conditions choices, either stable or unstable limit cycle solutions are possible. The unstable solution is, of course, an interesting simple example of flutter instability.

In the field of aerospace engineering several phenomena of interest are non-linear, as, for example, those studied by Aeroelasticity. This area of study had its development begun between 1910-1930 as presented by [1]. Aeroelastic phenomena are studied in aircraft, missiles and launch vehicles. The historical perspective on analysis and control of aeroelastic responses can be found in [2]. In particular, aeroelastic models in launch vehicles, based on articulated beams have been used by several authors as introduction to the study of this phenomenon. In [3] linear studies are developed to consider the critical load in launch vehicles. In [4] this problem is analyzed from the topological point of view of structural stability. However, in order to obtain a more adequate analysis of the structural dynamics of a launch vehicle, it is necessary to introduce non-linearities in its mathematical model. It is by means of this non-linear modeling that the determination of the amplitudes of the post-critical stationary states becomes possible, as it affirms [5] and how the work of [6] deserves attention. Another nonlinear theory associated with aeroelasticity in launch vehicles can be found in [7]. Often the study of aeroelastic
phenomena in launch vehicles applies to the interaction of aerodynamic, elastic, propulsive and inertial forces acting on their structure. In particular, the events associated with the dynamic instability of harmonic and self-excited oscillatory nature occurring due to the modifying interaction between two or more distinct modes of vibration of a system and the introduction of external energy into that system by means of An agent or dynamic flow (force), according to the definitions found in [8], [9] and [10]. In the present study, this energy introduction is associated with the axial circulatory load at the base of the launch vehicle because of its propulsion. In [11] an analytical solution of the fourth-order partial differential equation of the Beck column is also presented. Some thought will also be given in the future to possible control strategies of the resulting vibrations, a crucial problem when real rocket motions are considered. In future further work, we will develop the exact analytical solution to the fourth order partial differential equation governing the motion of an elastic space rocket structure under follower force excitation.

2 The structural physical model and modeling hypothesis

Fig. 1 is the simplified physical model of the structure of a launcher vehicle. It is constructed of two rigid massless bars $AB$ and $BC$, $L_1$ and $L_2$ long, respectively, pinned to nodes $A$ and $B$. Displacements are restricted at point $A$. We consider lumped masses $M_1$, $M_2$ and $M_3$ attached to nodes $A$ and $B$ where torsional springs $k_1$ and $k_2$ provide elastic restoring forces. Then, viscous dampers of damping constants $\mu_1$ and $\mu_2$ are added to the joints of the system.

1. It is adopted $L_1 = L_2 = L$.
2. The bars are rigid and massless.
3. Lumped masses $M_1$, $M_2$, and $M_3$ represent the actual masses of half the bars connected to that point. If $2m$ is the mass of each bar, $M_1 = M_3 = m$ and $M_2 = 2m$.
4. The stiffness constants of the torsion springs of the system are considered to represent the stiffness of the constituent material of the bars. In particular, the relation $k_1 = k_2$ is adopted for this model. These springs are assumed to be linear, that is, the restoring moments they apply to the structure (elastic moments) are proportional to the angular displacements comprised between the longitudinal axis of each bar and the local vertical.
5. The viscous damping constants of the shock absorbers of the system are considered to represent the dissipative tendency of energy by the structure of the vehicle given its deformation. In particular, the relation $\mu_1 = \mu_2 = \mu$ is adopted for this model. The dampers are assumed to be linear, i.e. the dissipation moments they apply to the structure are proportional to the first derivatives of the angular displacements comprised between the longitudinal axis of each bar and the local vertical.
6. Motions are restricted to the $Axy$ plane.
7. Initially, only self-weight forces act. This is the fundamental static equilibrium configuration of the system, representing the vehicle at rest in its launch platform.

For launch vehicles, it is common to consider a fixed reference coordinate system on its nose (the end of the warhead), with which one can define the vehicle's static margin. In this study, the coordinate system is fixed at $A$, as shown in Fig. 1. This system is non-inertial for a solo observer but is assumed to be inertial in the frame of the vehicle and as soon as it travels with it. For this reason, an articulated support in $A$ and elastic link between this support and the vehicle is assumed.

3 Physical loading model

Let $F$ be a follower (circulatory) non-conservative force applied to $C$, in the direction of bar $BC$. This force models the rocket’s thrust force due to combustion gases expansion at the motors in the basis of the vehicle. We do not consider, in this model, its dependence on time.

The action of force $F$ applied to $C$ excites the system to depart from its fundamental equilibrium position. The problem is now similar to an excited inverted double-pendulum with elastic properties. The equations of motion could be derived via Newton’s second law vector approach, but this method is
quite cumbersome in this case. Thus, a Lagrangian scalar energy scheme is preferable. Our generalized coordinates are angular displacements $\theta_1$ and $\theta_2$ of bars $AB$ and $BC$, computed from their original vertical equilibrium positions. They are, of course, implicitly time dependent, that is, $\theta_1 = \theta_1(t)$ and $\theta_2 = \theta_2(t)$. We denote $\theta_1(t) \equiv q_1(t)$ and $\theta_2(t) \equiv q_2(t)$. Nonzero values represent the vehicle in flight conditions as represented in Fig. 1.

Fig. 1: (left) System motion under follower force $\mathbf{F}$; (right) rocket appearance under follower force $\mathbf{F}$.

4 Mathematical model

4.1 Kinematics

4.1.1 Position vectors of the lumped masses

$$r_1 = 0$$

$$r_2 = L\left(\sin q_1 \hat{i} - \cos q_1 \hat{j}\right)$$

$$r_3 = L\left[\left(\sin q_1 + \sin q_2\right) \hat{i} - \left(\cos q_1 + \cos q_2\right) \hat{j}\right]$$

4.1.2 Velocity vectors of the lumped masses

$$r_1 = 0$$

$$r_2 = L\left(q_1 \cos q_1 \hat{i} + q_1 \sin q_1 \hat{j}\right)$$

$$r_3 = L\left[q_1 \cos q_1 + q_2 \cos q_2 \hat{i} + \left(q_1 \sin q_1 + q_2 \sin q_2\right) \hat{j}\right]$$

4.1.3 Approximations

We adopt a third order truncated polynomial McLaurin approximations to the sinusoidal functions.

$$\cos q_i \approx 1 - \frac{q_i^2}{2}, \quad \sin q_i \approx q_i - \frac{q_i^3}{6}$$
4.2 Energy computation
In the following equations, we neglect terms of higher order than third.

4.2.1 Kinetic energy

\[ T \approx \frac{1}{2} m L^2 \left[ 3q_1^2 + q_2^2 + 2\left( q_1q_2 - q_1q_2 \frac{q_1^2}{2} - q_1q_2 \frac{q_2^2}{2} + q_1q_2q_1q_2 \right) \right] \tag{8} \]

4.2.2 Total potential energy

\[ V = U - W = \frac{1}{2} k \left( 2q_1^2 + q_2^2 - 2q_1q_2 \right) = \frac{1}{2} mg L \left( 3q_1^2 + q_2^2 \right) \tag{9} \]

where \( U \) is the strain energy of the springs and \( W \) is the work of the conservative forces (masses weight).

4.3 Derivation of the equations of motion

Next, we apply Euler-Lagrange equations to the Lagrangian functional \( \mathcal{L} = T - V \):

\[ \frac{d}{dt} \left( \frac{\partial}{\partial q_i} \mathcal{L} \right) - \frac{\partial}{\partial q_i} \mathcal{L} = F_i^{nc} = \sum_{j=1}^{N} f_j^{nc} \cdot \frac{\partial}{\partial q_i} r_j \tag{10} \]

where the non-conservative generalized forces are in the left-hand term. The detailed mathematical deductions of this study can be found in [5], [12] and [13].

4.4 Generalized non-conservative forces

The generalized non-conservative forces, due to the follower force \( F \) are:

\[ f_1^{nc} = f_2^{nc} = 0, \quad f_3^{nc} = F \left( -\sin q_3 \hat{j} + \cos q_3 \hat{j} \right) \tag{11} \]

\[ F_1^{nc} \cong -FL \left( -q_1 + q_2 + \frac{q_1^3}{6} - \frac{q_2^3}{6} + \frac{q_1q_2^2}{2} - \frac{q_2q_1^2}{2} \right), \quad F_2^{nc} = 0 \tag{12} \]

where \( F \) is the scalar value of the follower force, considered time independent.

5 Equations of motion

Although gravity has been considered in the process of mathematical modeling of the problem, present in the working terms of the conservative forces acting on the vehicle (weight forces), from now on, the gravitational effect on the dynamics of the system is ignored due to the comparison established with the Propulsion force. This situation models the vehicle’s flight at high altitudes, in which the Earth’s gravitational acceleration can be neglected for the purpose of simplifying the problem. In fact, as we know, the Euclidean metric norm of acceleration of Earth’s gravity decays with the square of the distance between the center of mass of the body under study (in this case the vehicle) and the center of mass of the Earth. It is an interesting fact, though not impressive, that the acceleration of gravity contributes to the dynamic stability of the system. In order to analyze the dynamics of the vehicle subjected to the loading from the propulsion, so that the effects of this one became more evident for study, the rigidity associated to the work of the conservative forces was suppressed.
These, it is considered. The results

\[
\begin{aligned}
\begin{bmatrix}
3 & 1 \\
1 & 1
\end{bmatrix}
&+
\begin{bmatrix}
0 & -\frac{1}{2}(q_1 - q_2)^2 \\
-\frac{1}{2}(q_1 - q_2)^2 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{q}_1(t) \\
\dot{q}_2(t)
\end{bmatrix}
+ 
\begin{bmatrix}
\dot{q}_1(q_1 - q_2) \\
-q_1(q_1 - q_2)
\end{bmatrix}
+ 
\mu
\begin{bmatrix}
2 & -1 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{q}_1(t) \\
\dot{q}_2(t)
\end{bmatrix}
+ 
c
\begin{bmatrix}
2 & -1 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
q_1(t) \\
q_2(t)
\end{bmatrix}
= 0
\end{aligned}
\]

(13)

Where

\[
c = \frac{k}{mL^2}, \quad \lambda = \frac{FL}{k}, \quad \mu = \frac{\dot{\mu}}{mL^2}
\]

(14)

6 Results

Next, computational simulations are presented, from the parameterized model of Eq. (13), for a case in which the launch vehicle model is stable, a case of limit cycle, and a case of flutter-type intactness. In the scope of this study, the notion of stability is associated with the limited output relationship - amplitude of the response, which is the generalized displacements of the model - for limited input, which is the follower force, of constant intensity in time. The simulations were obtained for the initial conditions \( \theta_1(t) = q_1(t) = 0.01 \text{ rad}, \theta_2(t) = q_2(t) = 0.01, \) \( \theta_3(t) = q_1(t) = 0 \text{ rad} \) and \( \theta_4(t) = q_3(t) = 0 \text{ rad} \). These conditions represent the axis of symmetry of the rocket slightly misaligned with the vertical reference direction. It is important to emphasize that instability in the example of post-critical steady regime of the limit cycle type is characteristic of the physical phenomenon and not of the integrator the 4th order Runge-Kutta. Several integration steps were tested in the simulations and the same dynamic temporal behavior of the physical system was observed.

6.1 A case of asymptotic stability

As an example of a case of stability of the system modeled under load action at its base, the arrangement \( m = 1 \text{ kg}, L = 1 \text{ m}, k = 1 \text{ Nm/rad}, \) \( \dot{\mu} = 0.1 \text{ kg/s kg/s} \) and \( P = 1 \text{ N} \) is considered. The results in the configuration are shown Fig. 2:

6.2 A case of post-critical stationary regime type limit cycle

As an example of a case of limit cycle of the system modeled under load action at its base, the arrangement \( m = 1 \text{ kg}, L = 1 \text{ m}, k = 1 \text{ Nm/rad}, \) \( \dot{\mu} = 2 \text{ kg/s} \) and \( P = 4 \text{ N} \) is considered. The results in the configuration are shown Fig. 3:

7 Conclusions and future research

We presented a two degree of freedom lumped parameters model of an elastic rocket structure excited by the follower force given by the motor thrust that is supposed to be always in the direction of the tangent to the deformed shape of the device at its lower tip. We derived the equations of motion via Lagrange’s equations and simulate its time evolution using Runge-Kutta 4th order time step-by-step numerical integration algorithm. Results indicate possible occurrence of stable and unstable vibrations, such as limit cycles. research on flutter instability is ongoing.

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Fig. 2: (above and left) Generalized displacements in stable condition; (above and right) rates of generalized displacements in stable condition; (below and left) phase plane of generalized displacement $q_1$ in stable condition; (below and right) phase plane of generalized displacement $q_2$ in stable condition.
Fig. 3: (above and left) generalized displacements in limit cycle case; (above and right) rates of generalized displacements in limit cycle case; (below and left) phase plane of generalized displacement $q_1$ in limit cycle case; (below and right) phase plane of generalized displacement $q_2$ in limit cycle case

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