SHARP BOUNDS FOR $T$-HAAR MULTIPLIERS ON $L^2$

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Abstract. We show that if a weight $w \in C^d_{2t}$ and there is $q > 1$ such that $w^{2t} \in A^d_q$, then the $L^2$-norm of the $t$-Haar multiplier of complexity $(m, n)$ associated to $w$ depends on the square root of the $C^d_{2t}$-characteristic of $w$ times the square root $A^d_q$-characteristic of $w^{2t}$ times a constant that depends polynomially on the complexity. In particular, if $w \in C^d_{2t} \cap A^d_{\infty}$ then $w^{2t} \in A^d_q$ for some $q > 1$.

1. Introduction

Recently Tuomas Hytönen settled the $A_2$-conjecture [H]: for all Calderón-Zygmund integral singular operators $T$ in $\mathbb{R}^N$, weights $w \in A_p$, there is $C_{p,N,T} > 0$ such that,

$$\|Tf\|_{L^p(w)} \leq C_{p,N,T} \max\{1, \frac{1}{p-1}\} \|f\|_{L^p(w)}.$$

In his proof he developed and used a representation valid for any Calderón-Zygmund operator as an average of Haar shift operators of arbitrary complexity, paraproducts and their adjoints. See [L1, P4] for surveys of the $A_2$-conjecture. An important and hard part of the proof was to obtain bounds for Haar shifts operators that depended linearly in the $A_2$-characteristic and at most polynomially in the complexity.

In this paper we show that if a weight $w \in C^d_{2t} \cap A^d_{\infty}$, then the $L^2$-norm of the $t$-Haar multiplier of complexity $(m, n)$ associated to $w$ depends on the square root of the $C^d_{2t}$-characteristic of $w$ times the square root $A^d_q$-characteristic of $w^{2t}$ for some $q > 1$ depending on $t \in \mathbb{R}$ times a constant that depends polynomially on the complexity.

For $t \in \mathbb{R}$, $m, n \in \mathbb{N}$, and a weight $w$, the $t$-Haar multiplier of complexity $(m, n)$ was introduced in [MoP], and is defined formally by

$$T_{t,w}^{m,n} f(x) = \sum_{L \in D} \sum_{I \in \mathcal{D}_m(L), J \in \mathcal{D}_n(L)} \frac{\sqrt{|I| |J|}}{|L|} \frac{w^{t}(x)}{m_{L,w}} f^{*}(f, h_I) h_J(x),$$

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where $\mathcal{D}$ denotes the dyadic intervals, $|I|$ the length of interval $I$, $\mathcal{D}_m(L)$ denotes the dyadic subintervals of $L$ of length $2^{-m}|L|$, $h_I$ is a Haar function associated to $|I|$, and $\langle f, g \rangle$ denotes the $L^2$-inner product.

When $(m, n) = (0, 0)$ we denote the corresponding Haar multiplier by $T^t_w$, and, if in addition $t = 1$, simply $T_w$. The Haar multipliers $T_w$ are closely related to the resolvent of the dyadic paraproduct [P1], and appeared in the study of Sobolev spaces on Lipschitz curves [P3].

A necessary condition for the boundedness of $T^{m,n}_{w,t}$ on $L^2(\mathbb{R}^N)$ is that $w \in C^d_{2t}$, see [MoP], that is,

$$[w]_{C^d_{2t}} := \sup_{I \in \mathcal{D}} \left( \frac{1}{|I|} \int_I w^{2t}(x)dx \right) \left( \frac{1}{|I|} \int_I w(x)dx \right)^{-2t} < \infty.$$  

This condition is also sufficient for $t < 0$ and $t \geq 1/2$. Notice that for $0 \leq t < 1/2$ the condition $C^d_{2t}$ is always fulfilled; in this case, boundedness of $T^{m,n}_{w,t}$ is known when $w \in A^d_{\infty}$ [MoP, KP]. The first author showed in [Be, Chapter 5], that if $w \in C^d_{2t}$ and $w^{2t} \in A^d_q$ then the $L^2$-norm of $T^t_w$, is bounded by a constant times $[w]_{C^d_{2t}}^{1/2} [w^{2t}]_{A^d_q}^{1/2}$. Here we present a different proof of this result that holds for $t$-Haar multipliers of complexity $(m, n)$ with polynomial dependence on the complexity.

**Theorem 1.1.** Let $w \in C^d_{2t}$ and assume there is $q > 1$ such that $w^{2t} \in A^d_q$, then there is a constant $C_q > 0$ depending only on $q$, such that

$$\|T^{m,n}_{t,w} f\|_2 \leq C_q (m + n + 2)^3 [w]_{C^d_{2t}}^{1/2} [w^{2t}]_{A^d_q}^{1/2} \|f\|_2.$$  

When $w^{2t} \in A^d_q$, this was proved in [MoP].

Using known properties of weights we can replace the condition $w^{2t} \in A^d_q$, by what may seem to be a more natural condition $w \in C^d_{2t} \cap A^d_{\infty}$.

**Theorem 1.2.** Let $w \in C^d_{2t} \cap A^d_{\infty}$, then

(i) if $0 \leq 2t < 1$, there is $q > 1$ such that $w \in A^d_q$, then $w^{2t} \in A^d_q$, and

$$\|T^{m,n}_{t,w} f\|_2 \leq C_q (m + n + 2)^3 [w]_{A^d_q}^{1/2} \|f\|_2 \leq C_q (m + n + 2)^3 [w]_{A^d_q}^{1/2} \|f\|_2.$$  

(ii) If $2t \geq 1$ and $w \in A^d_p$ then for $q = 2t(p - 1) + 1$, $w^{2t} \in A^d_q$, and

$$\|T^{m,n}_{t,w} f\|_2 \leq C_q (m + n + 2)^3 [w]_{C^d_{2t}}^{1/2} [w^{2t}]_{A^d_q}^{1/2} \|f\|_2 \leq C_p (m + n + 2)^3 [w]_{C^d_{2t}}^{1/2} [w]_{A^d_q} \|f\|_2.$$  

(iii) If $t < 0$ then for $q = 1 - 2t$, $w^{2t} \in A^d_q$, and the bound becomes linear in the $C^d_{2t}$ characteristic of $w$,

$$\|T^{m,n}_{t,w} f\|_2 \leq C (m + n + 2)^3 [w]_{C^d_{2t}} \|f\|_2.$$
The result was known to be optimal when \( t = \pm 1/2 \) [Be, P2]. The bound in (ii) is not optimal since for \( t = 1 \), the \( L^2 \) norm of \( T_w \) is bounded by a constant times \( [w]_{C_2^d}^2 D(w) \), where \( D(w) \) is the doubling constant of \( w \), see [P2]. Here we get the larger norm \( C[w]_{C_2^d}^2[w]_{A_q} \).

To prove this theorem we modify the argument in [MoP] that works when \( w \in A_{\infty}^d \) \((p = 2)\). In particular we need a couple of new \( A_p \)-weight lemmas that are proved using Bellman function techniques: the \( A_p \)-Little Lemma, and the \( \alpha\beta \)-Lemma.

A few open questions remain. In case (i) \( 0 < 2t < 1 \), is \( w^{2t} \in A_\infty \) a necessary condition for the boundedness of \( T_{m,n}^tw \)? Here we show is sufficient. Is it possible to get an estimate independent of \( q > 1 \) such that \( w^{2t} \in A_{q_1}^d \)? More specifically, can we replace \( C_q[w^{2t}]_{A_q^d} \) by \( CD(w) \)? Similarly in case (ii).

The paper is organized as follows. In Section 2 we provide the basic definitions and basic results that are used throughout this paper. In Section 3 we prove the lemmas that are essential for the main result. In Section 4 we prove the main estimate for the \( t \)-Haar multipliers with complexity \((m, n)\). In the Appendix we prove the \( A_p \)-Little Lemma.

## 2. Preliminaries

### 2.1. Weights, maximal function and dyadic intervals.

A weight \( w \) is a locally integrable function in \( \mathbb{R} \) positive almost everywhere. The \( w \)-measure of a measurable set \( E \), denoted by \( w(E) \), is \( w(E) = \int_E w(x)dx \). For a measure \( \sigma \), \( \sigma(E) = \int_E d\sigma \), and \( |E| \) stands for the Lebesgue measure of \( E \). We define \( m_{E}^\sigma f \) to be the integral average of \( f \) on \( E \), with respect to \( \sigma \),

\[
m_{E}^\sigma f := \frac{1}{\sigma(E)} \int_E f(x)d\sigma.
\]

When \( dx = d\sigma \) we simply write \( m_{E}f \), when \( d\sigma = v dx \) we write \( m_{E}^{v}f \).

Given a weight \( w \), a measurable function \( f : \mathbb{R}^N \to \mathbb{C} \) is in \( L^p(w) \) if and only if \( \|f\|_{L^p(w)} := \left( \int_{\mathbb{R}^N} |f(x)|^p w(x) dx \right)^{1/p} < \infty \).

For a weight \( v \) we define the weighted maximal function of \( f \) by

\[
(M_v f)(x) = \sup_{I,x \in I} m_{I}^{v}|f|
\]

where \( I \) is a cube in \( \mathbb{R}^N \) with sides parallel to the axis. The operator \( M_v \) is bounded in \( L^p(v) \) for all \( p > 1 \) and furthermore

\[
(2.1) \quad \|M_v f\|_{L^p(v)} \leq C p' \|f\|_{L^p(v)},
\]

where \( p' \) is the dual exponent of \( p \), that is \( 1/p + 1/p' = 1 \). A proof of this fact can be found in [CrMPz1]. When \( v = 1 \), \( M_v \) is the usual
Hardy-Littlewood maximal function, which we will denote by $M$. It is well-known that $M$ is bounded in $L^p(w)$ if and only if $w \in A_p$ [Mu].

The collection of all dyadic intervals, $\mathcal{D}$, is given by: $\mathcal{D} = \bigcup_{n\in\mathbb{Z}} \mathcal{D}_n$, where $\mathcal{D}_n := \{ I \subset \mathbb{R} : I = [k2^{-n}, (k+1)2^{-n}), \ k \in \mathbb{Z} \}$. For a dyadic interval $L$, let $\mathcal{D}(L)$ be the collection of its dyadic subintervals, $\mathcal{D}(L) := \{ I \subset L : I \in \mathcal{D} \}$, and let $\mathcal{D}_n(L)$ be the $n^{th}$-generation of dyadic subintervals of $L$, $\mathcal{D}_n(L) := \{ I \in \mathcal{D}(L) : |I| = 2^{-n}|L| \}$.

For every dyadic interval $I \in \mathcal{D}_n$ there is exactly one $\hat{I} \in \mathcal{D}_{n-1}$, such that $I \subset \hat{I}$, $\hat{I}$ is called the parent of $I$. Each dyadic interval $I$ in $\mathcal{D}_n$ has two children in $\mathcal{D}_{n+1}$, the right and left halves, denoted $I_+$ and $I_-$ respectively.

A weight $w$ is dyadic doubling if $w(\hat{I})/w(I) \leq C$ for all $I \in \mathcal{D}$. The smallest constant $C$ is called the doubling constant of $w$ and is denoted by $D(w)$. Note that $D(w) \geq 2$, and that in fact the ratio between the length of a child and the length of its parent is comparable to one, more precisely, $D(w)^{-1} \leq w(I)/w(\hat{I}) \leq 1 - D(w)^{-1}$.

2.2. Dyadic $A^d_p$, reverse Hölder $RH^d_p$ and $C^d_p$ classes. A weight $w$ is said to belong to the dyadic Muckenhoupt $A^d_p$-class if and only if

$$[w]_{A^d_p} := \sup_{I \in \mathcal{D}} (m_I w)(m_{\hat{I}} w^{1-p})^{p-1} < \infty, \quad \text{for} \quad 1 < p < \infty,$$

where $[w]_{A^d_p}$ is called the $A^d_p$-characteristic of the weight. If a weight is in $A^d_p$ then it is dyadic doubling. These classes are nested, $A^d_p \subset A^d_q$ for all $p \leq q$. The class $A^d_\infty$ is defined by $A^d_\infty := \bigcup_{p>1} A^d_p$.

A weight $w$ is said to belong to the dyadic reverse Hölder $RH^d_p$-class if and only if

$$[w]_{RH^d_p} := \sup_{I \in \mathcal{D}} (m_I w^p)(m_{\hat{I}} w^{-1}) < \infty, \quad \text{for} \quad 1 < p < \infty,$$

where $[w]_{RH^d_p}$ is called the $RH^d_p$-characteristic of the weight. If a weight is in $RH^d_p$ then it is not necessarily dyadic doubling (in the non-dyadic setting reverse Hölder weights are always doubling). Also these classes are nested, $RH^d_p \subset RH^d_q$ for all $p \geq q$. The class $RH^d_1$ is defined by $RH^d_1 := \bigcup_{p>1} RH^d_p$. In the non-dyadic setting $A_\infty = RH^d_1$. In the dyadic setting the collection of dyadic doubling weights in $RH^d_1$ is $A^d_\infty$, hence $A^d_\infty$ is a proper subset of $RH^d_1$. See [BeRez] for some recent and very interesting results relating these classes.

The following are well-known properties of weights (see [JN]) for (ii):

Lemma 2.1. The following hold
• If $0 \leq s \leq 1$ and $w \in A^d_\infty$ then $w^s \in A^s_\infty$. More precisely, if $p > 1$ and $w \in A^d_p$ then $w^s \in A^s_p$, and $[w^s]_{A^d_p} \leq [w]_{A^d_p}^{s}$. \\
• If $s, q > 1$ then $w \in RH^d_s \cap A^d_q$ if and only if $w^s \in A^{s(q-1)+1}_s$. Moreover $[w^s]_{A^{s(q-1)+1}_s} \leq [w]_{RH^d_s}^s [w]_{A^d_q}^s$, $[w]_{A^d_q}^s \leq [w^s]_{A^{s(q-1)+1}_s}$, and $[w]_{RH^d_s} \leq [w^s]_{A^{s(q-1)+1}_s}$. \\
• If $p > 1$, and $1/p + 1/p' = 1$, then $w \in A^d_p$ if and only if $w^{-1/p-1} \in A^d_{p'}$. Moreover $[w]_{A^d_p} = [w^{-1/p-1}]_{A^d_{p'}}^{p-1}$.

The following property can be found in [GaRu],

Lemma 2.2. If $w \in RH^d_s \cap A^d_q$ then for all $E \subset B$,

$$(|E|/|B|)^{q} [w]_{A^d_q}^{-1} \leq w(E)/w(B) \leq (|E|/|B|)^{1-\frac{q}{p}} [w]_{RH^d_s}.$$

In particular $D(w) \leq 2^q [w]_{A^d_q}$.

A weight $w$ satisfies the $C^d_s$-condition, for $s \in \mathbb{R}$, if

$$[w]_{C^d_s} := \sup_{I \in \mathcal{D}} \left( m_I w^s \right) \left( m_I w \right)^{-s} < \infty.$$

The quantity defined above is called the $C^d_s$-characteristic of $w$. The class of weights $C^d_s$ was defined in [KP]. Let us analyze this definition. For $0 \leq s \leq 1$, we have that any weight satisfies the condition with $C^d_s$-characteristic 1, this is just a consequence of Hörder’s Inequality (for $s = 0, 1$ is trivial). When $s > 1$, the condition is analogous to the dyadic reverse Hölder condition and $[w]_{C^d_s}^{1/s} = [w]_{RH^d_s}$. For $s < 0$, we have that $w \in C^d_s$ if and only if $w \in A^d_{1-1/s}$, moreover $[w]_{C^d_s} = [w]_{A^d_{1-1/s}}^{-s}$.

Lemma 2.3. If $w \in C^d_s \cap A^d_\infty$ then the following hold

• For all $0 \leq s \leq 1$, there is a $p > 1$ such that $w^s \in A^s_p$.
• If $s > 1$ then there is $q > 1$ such that $w^s \in A^{s(q-1)+1}_s$.
• If $s < 0$ then $w^s \in A^{s-1}_{1-s}$.

The proof of this lemma is a direct application of Lemma 2.1 item by item.

2.3. Weighted Haar functions. For a given weight $v$ and an interval $I$ define the weighted Haar function as

$$h^v_I(x) = \frac{1}{v(I)} \left( \sqrt{\frac{v(I_-)}{v(I_+)}} \chi_{I_+}(x) - \sqrt{\frac{v(I_+)}{v(I_-)}} \chi_{I_-}(x) \right),$$

where $\chi_I(x)$ is the characteristic function of the interval $I$. 
If \( v \) is the Lebesgue measure on \( \mathbb{R} \), we will denote the Haar function simply by \( h_I \). It is a simple exercise to verify that the weighted and unweighted Haar functions are related linearly as follows,

**Proposition 2.4.** For any weight \( v \), there are numbers \( \alpha_I^v, \beta_I^v \) such that

\[
    h_I(x) = \alpha_I^v h^v_I(x) + \beta_I^v \chi_I(x)/\sqrt{|I|}
\]

where (i) \( |\alpha_I^v| \leq \sqrt{m_I v} \), (ii) \( |\beta_I^v| \leq |\Delta_I v|/m_I v \), \( \Delta_I v := m_{I^+} v - m_{I^-} v \).

The family \( \{h^v_I\}_{I \in \mathcal{D}} \) is an orthonormal system in \( L^2(v) \), with inner product \( \langle f, g \rangle_v := \int_{\mathbb{R}} f(x) g(x) v(x) \, dx \).

2.4. **Carleson sequences.** If \( v \) is a weight, a positive sequence \( \{\alpha_I^v\}_{I \in \mathcal{D}} \) is called a \( v \)-Carleson sequence with intensity \( B \) if for all \( J \in \mathcal{D} \),

\[
    \sum_{I \in \mathcal{D}(J)} \lambda_I \leq B m_J v.
\]

When \( v = 1 \) we call a sequence satisfying (2.3) for all \( J \in \mathcal{D} \) a Carleson sequence with intensity \( B \).

**Proposition 2.5.** Let \( v \) be a weight, \( \{\lambda_I\}_{I \in \mathcal{D}} \) and \( \{\gamma_I\}_{I \in \mathcal{D}} \) be two \( v \)-Carleson sequences with intensities \( A \) and \( B \) respectively then for any \( c, d > 0 \) we have that

(i) \( \{c\lambda_I + d\gamma_I\}_{I \in \mathcal{D}} \) is a \( v \)-Carleson sequence with intensity \( cA + dB \).

(ii) \( \{\sqrt{\lambda_I} \sqrt{\gamma_I}\}_{I \in \mathcal{D}} \) is a \( v \)-Carleson sequence with intensity \( \sqrt{AB} \).

(iii) \( \{(c\sqrt{\lambda_I} + d\sqrt{\gamma_I})^2\}_{I \in \mathcal{D}} \) is a \( v \)-Carleson sequence with intensity \( 2c^2A + 2d^2B \).

The proof of these statements is quite simple, see [MoP].

3. **Main tools**

In this section, we state the lemmas and theorems necessary to get the estimate for the \( t \)-Haar multipliers of complexity \( (m, n) \).

3.1. **Carleson Lemmas.** The Weighted Carleson Lemma we present here is a variation in the spirit of other weighted Carleson embedding theorems that appeared before in the literature [NV, NTV1]. You can find a proof in [MoP].

**Lemma 3.1** (Weighted Carleson Lemma). Let \( v \) be a weight, then \( \{\alpha_L\}_{L \in \mathcal{D}} \) is a \( v \)-Carleson sequence with intensity \( B \) if and only if for all non-negative \( v \)-measurable functions \( F \) on the line,

\[
    \sum_{L \in \mathcal{D}} \alpha_L \inf_{x \in L} F(x) \leq B \int_{\mathbb{R}} F(x) v(x) \, dx.
\]
The following lemma we view as a finer replacement for Hölder's inequality: 

$$1 \leq (m_I v) (m_I w^{-1/(p-1)})^{p-1}. $$

**Lemma 3.2 (A_p-Little Lemma).** Let $v$ be a weight, such that $v^{-1/(p-1)}$ is a a weight as well, and let $\{\lambda_I\}_{I \in D}$ be a Carleson sequence with intensity $Q$ then $\{\lambda_I/(m_I v^{-1/(p-1)})^{p-1}\}_{I \in D}$ is a $v$-Carleson sequence with intensity $4Q$, that is for all $J \in D$,

$$\frac{1}{|J|} \sum_{I \in D(J)} \frac{\lambda_I}{m_I v^{-1/(p-1)}} \leq 4Q m_J v. $$

For $p = 2$ this was proved in [Be, Proposition 3.4], or [Be1, Proposition 2.1], using the same Bellman function as in the proof we present in the Appendix.

**Lemma 3.3 ([NV]).** Let $v$ be a weight such that $v^{-1/(p-1)}$ is also a weight. Let $\{\lambda_J\}_{J \in D}$ be a Carleson sequence with intensity $B$. Let $F$ be a non-negative measurable function on the line. Then

$$\sum_{J \in D} \frac{\lambda_J}{m_J v^{-1/(p-1)}} \inf_{x \in J} F(x) \leq C B \int_R F(x) v(x) \, dx. $$

Lemma 3.3 is an immediate consequence of Lemma 3.2, and the Weighted Carleson Lemma 3.1. Note that Lemma 3.2 can be deduced from Lemma 3.3 with $F(x) = \chi_J(x)$.

The following lemma, for $v = w^{-1}$, and for $\alpha = 1/4$ appeared in [Be], and for $0 < \alpha < 1/2$, in [NV]. With small modification in their proof, using the Bellman function $B(x, y) = x^\alpha y^\beta$ with domain of definition the first quadrant $x, y > 0$, we can accomplish the result below, for a complete proof see [Mo].

**Lemma 3.4. (\alpha\beta-Lemma)** Let $u, v$ be weights. Then for any $J \in D$ and any $\alpha, \beta \in (0, 1/2)$

$$\sum_{I \in D(J)} \frac{|\Delta_I u|^2}{m_I u^2} |I| (m_I u)^\alpha (m_I v)^\beta \leq C_{\alpha, \beta} (m_J u)^\alpha (m_J v)^\beta. $$

The constant $C_{\alpha, \beta} = 36/\min\{\alpha - 2\alpha^2, \beta - 2\beta^2\}$.

From this lemma we immediately deduce the following,

**Lemma 3.5.** Let $1 < q < \infty$, $w \in A^q$, then $\{\mu^{q, q}_{I}\}_{I \in D}$, where

$$\mu^{q, q}_{I} := (m_I w)^\alpha (m_I w^{-1/(q-1)})^{\alpha(q-1)} |I| \left( \frac{|\Delta_I w|^2}{m_I w^2} + \frac{|\Delta_I w^{-1/(q-1)}|^2}{m_I w^{-1/(q-1)}^2} \right), $$


is a Carleson sequence with Carleson intensity at most $C_\alpha[w]_{A_q}^\alpha$ for any $\alpha \in (0, \max\{1/2, 1/2(q-1)\})$. Moreover, $\{\nu^q_I \}_{I \in \mathcal{D}}$, where

$$\nu^q_I := (m_Iw)(m_Iw^{\frac{1}{q-1}})^{(q-1)}|I| \left( \frac{|\Delta_I w|^2}{(m_Iw)^2} + \frac{|\Delta_I w^{\frac{1}{q-1}}|^2}{(m_Iw)^2} \right)$$

is a Carleson sequence with Carleson intensity at most $C[w]_{A_q}$.

**Proof.** Set $u = w$, $v = w^{-\frac{1}{q-1}}$, $\beta = \alpha(q-1)$. By hypothesis $0 < \alpha < 1/2$ and also $0 < \alpha < 1/2(q-1)$ which implies that $0 < \beta < 1/2$, we can now use Lemma 3.4 to show that $\mu^{q,\alpha}_I$ is a Carleson sequence with intensity at most $c_\alpha[w]_{A_q}^{\alpha}$. For the second statement suffices to notice that $\nu^q_I \leq \mu^{q,\alpha}_I[w]_{A_q}^{1-\alpha} \alpha$ for all $I \in \mathcal{D}$, for some $\alpha \in (0, \max\{1/2, 1/2(q-1)\})$. \qed

A proof of this lemma for $q = 2$ that works on geometric doubling metric spaces can be found in [NV1, V]. In those papers $\alpha = 1/4$ can be used, and in that case the constant $C_\alpha$ can be replaced by 288.

### 3.2. Lift Lemma

Given a dyadic interval $L$, and weights $u, v$, we introduce a family of stopping time intervals $ST_L^m$ such that the averages of the weights over any stopping time interval $K \in ST_L^m$ are comparable to the averages on $L$, and $|K| \geq 2^m|L|$. This construction appeared in [NV] for the case $u = w, v = w^{-1}$. We also present a lemma that lifts $w$-Carleson sequences on intervals to $w$-Carleson sequences on “$m$-stopping intervals”. This was used in [NV] for a very specific choice of $m$-stopping time intervals $ST_L^m$.

**Lemma 3.6 (Lift Lemma [NV]).** Let $u$ and $v$ be weights, $L$ be a dyadic interval and $m, n$ be fixed positive integers. Let $ST_L^m$ be the collection of maximal stopping time intervals $K \in \mathcal{D}(L)$, where the stopping criteria are either (i) $|\Delta_K u|/m_Ku + |\Delta_K v|/m_Kv \geq 1/m + n + 2$, or (ii) $|K| = 2^{-m}|L|$. Then for any stopping interval $K \in ST_L^m$, $e^{-1}m_Ku \leq m_Ku \leq e_mL\mu$, and hence also $e^{-1}mLv \leq m_Kv \leq e_mLv$.

Note that the roles of $m$ and $n$ can be interchanged and we get the family $ST_L^n$ using the same stopping condition (i) and condition (ii) replaced by $|K| = 2^{-n}|L|$. Notice that $ST_L^n$ is a partition of $L$ in dyadic subintervals of length at least $2^{-m}|L|$. The following lemma lifts a $w$-Carleson sequence to $m$-stopping time intervals with comparable intensity. For the particular $m$-stopping time $ST_L^m$ given by the stopping criteria (i) and (ii) in Lemma 3.6, and $w = 1$, this appeared in [NV].
Lemma 3.7. For each $L \in \mathcal{D}$ let $ST_L^m$ be a partition of $L$ in dyadic subintervals of length at least $2^{-m}|L|$. Assume $\{\nu_I\}_{I \in \mathcal{D}}$ is a $w$-Carleson sequence with intensity at most $A$, let $\nu_L^m := \sum_{K \in ST_L^m} \nu_K$, then $\{\nu_L^m\}_{L \in \mathcal{D}}$ is a $w$-Carleson sequence with intensity at most $(m + 1)A$.

For proofs you can see [MoP].

3.3. Auxiliary quantities. For a weight $v$, and a locally integrable function $\phi$ we define the following quantities,

$$P_L^m \phi := \sum_{I \in \mathcal{D}_m(L)} |\langle \phi, h_I \rangle| \sqrt{|I|/|L|},$$

$$S_L^{v,m} \phi := \sum_{J \in \mathcal{D}_m(L)} |\langle \phi, h_J^v \rangle_v| \sqrt{m_J v} \sqrt{|J|/|L|},$$

$$R_L^{v,m} \phi := \sum_{J \in \mathcal{D}_m(L)} \frac{|\Delta_J^v|}{m_J^v} m_J(|\phi|v) |J|/\sqrt{|L|},$$

Let $w \in A_q^d$, $ST_L^m$ be an $m$-stopping time family of subintervals of $L$, $0 < \alpha < \max\{1/2, 1/2(q - 1)\}$, and $\{\mu_K = \mu_K^{q,\alpha}\}_{K \in \mathcal{D}}$ be the Carleson sequence with intensity $C_{\alpha}[w]_{A_q^d}$ defined in Lemma 3.5. For each $m > 0$, we introduce another sequence $\{\mu_L^m\}$, which is Carleson by Lemma 3.7:

$$\mu_L^m := \sum_{K \in ST_L^m} \mu_K^{q,\alpha} \text{ with intensity } C_{\alpha}(m + 1)[w]_{A_q^d}.$$

We will use the following estimates for $S_L^{v,m} \phi$ and $R_L^{v,m} \phi$, where $1 < p < 2$ will be dictated by the proof of the theorem.

$$S_L^{v,m} \phi \leq \left( \sum_{J \in \mathcal{D}_m(L)} |\langle \phi, h_J^v \rangle_v|^2 \right)^{1/2} (m_L v)^{1/2},$$

$$R_L^{v,m} \phi \leq C C_m (m_L v)^{(q-1)/2} (m_L v)^{1/2} \inf_{x \in L} (M_{w-1} |g|^p)(x)^{1/2} \sqrt{\mu_L^m},$$

See [NV] for the proof when $q = 2$, slight modification of their argument gives the estimate for $R_L^{v,m} \phi$. Estimating $P_L^m \phi$ is very simple:

$$(P_L^m \phi)^2 \leq \sum_{I \in \mathcal{D}_m(L)} |I|/|L| \sum_{I \in \mathcal{D}_m(L)} |\langle \phi, h_I \rangle|^2 = \sum_{I \in \mathcal{D}_m(L)} |\langle \phi, h_I \rangle|^2.$$

Remark 3.8. In [NV1], Nazarov and Volberg extend the results that they had in [NV] for Haar shifts to metric spaces with geometric doubling. Following the same modifications in the argument made from [NV] to [NV1], one could obtain the same result as in Theorem 4.1 on a metric space with geometric doubling, see [Mo1].
4. HAAR MULTIPLIERS

For a weight \( w, t \in \mathbb{R} \), and \( m, n \in \mathbb{N} \), a \( t \)-Haar multiplier of complexity \((m,n)\) is the operator defined as

\[
T_{t,w}^{m,n} f(x) := \sum_{L \in D} \sum_{I \in \mathcal{D}_n(L), J \in \mathcal{D}_m(L)} \sqrt{|I|/|J|} \left( \frac{w(x)}{m_l w} \right)^t (f, h_I) h_J(x).
\]

In [MoP] it is shown that \( w \in C_d^{2t} \) is a necessary condition for boundedness of \( T_{w,t}^{m,n} \) in \( L^2(\mathbb{R}) \). It is also shown that the \( C_d^{2t} \)-condition is sufficient for a \( t \)-Haar multiplier of complexity \((m,n)\) to be bounded in \( L^2(\mathbb{R}) \) for most \( t \); this was proved in [KP] for the case \( m = n = 0 \). Here we are concerned not only with the boundedness but also with the dependence of the operator norm on the \( C_d^{2t} \)-constant. For \( T_{w}^{t} \) and \( t = 1, \pm 1/2 \) this was studied in [P2]. The first author [Be] was able to obtain estimates, under the additional condition on the weight \( w^{2t} \in A_q^d \) for some \( q > 1 \), for \( T_{w}^{t} \) and for all \( t \in \mathbb{R} \). Her results were generalized for \( T_{w,t}^{m,n} \) for all \( t \) when \( w^{2t} \in A_2^d \), see [MoP]. We will show that:

**Theorem 4.1.** Let \( t \) be a real number and \( w \) a weight such that \( w^{2t} \in A_q^d \) for some \( q > 1 \) (i.e. \( w^{2t} \in A_{\infty}^d \)), then

\[
\|T_{t,w}^{m,n} f\|_2 \leq C_q (m + n + 2)^3 [w]_{C_d^{2t}}^{1/2} [w^{2t}]_{A_q^d}^{1/2} \|f\|_2.
\]

Using Lemmas 2.1 and 2.3 we can refine the result as follows, where \( C_m^n = n + m + 2 \).

**Theorem 4.2.** Let \( t \in \mathbb{R}, w \in C^{2t} \)

(i) If \( 0 < 2t < 1 \) and \( w \in A_p^d \), then

\[
\|T_{t,w}^{m,n} f\|_2 \leq C_p (C_m^n)^3 [w]_{C_d^{2t}}^{1/2} [w^{2t}]_{A_p^d}^{1/2} \|f\|_2 \leq C_p (C_m^n)^3 [w]_{A_p^d} \|f\|_2.
\]

(ii) If \( t > 1 \) and \( w \in A_p^d \) then if \( q = 2t(p - 1) + 1 \)

\[
\|T_{t,w}^{m,n} f\|_2 \leq C_p (C_m^n)^3 [w]_{C_d^{2t}}^{1/2} [w^{2t}]_{A_q^d}^{1/2} \|f\|_2 \leq C_p (C_m^n)^3 [w]_{C_d^{2t}} [w]_{A_q^d} \|f\|_2.
\]

(iii) If \( t < 0 \) then

\[
\|T_{t,w}^{m,n} f\|_2 \leq C (C_m^n)^3 [w]_{C_d^{2t}} \|f\|_2 = C (C_m^n)^3 [w]_{A_{1-1/2t}^d} \|f\|_2.
\]

**Remark 4.3.** Throughout the proof a constant \( C_q \) will be a numerical constant depending only on the parameter \( q > 1 \) that may change from line to line.
Proof of Theorem 4.2. By Lemma 2.3 if \( w \in \mathcal{C}^d_2 \cap \mathcal{A}^d_q \), then there is \( q > 1 \) such that \( w^{2t} \in \mathcal{A}^d_q \), matching cases perfectly. Now use Theorem 4.1.

\[
\text{Proof of Theorem 4.1.} \text{ Fix } f, g \in L^2(\mathbb{R}). \text{ By duality, it is enough to show that}
\]
\[
|\langle T^{m,n}_{t,w} f, g \rangle| \leq C(m + n + 2)^3[w]_{\mathcal{C}^d_2}^{\frac{3}{2}} [w^{2t}]_{\mathcal{A}^d_q}^{\frac{1}{2}} \|f\|_2 \|g\|_2.
\]

The inner product on the left-hand-side can be expanded into a double sum, that we now estimate,
\[
|\langle T^{m,n}_{t,w} f, g \rangle| \leq \sum_{L \in \mathcal{D}} \sum_{I \in \mathcal{D}_m(L) : J \in \mathcal{D}_n(L)} \frac{\sqrt{|I||J|}}{|L|} \frac{\sqrt{m_j(w^{2t})}}{(m_L w)^t} |\langle f, h_I \rangle| |\langle gw^1, h_J \rangle|.
\]

Write \( h_J \) as a linear combination of a weighted Haar function and a characteristic function, \( h_J = \alpha_J h_J^{w^{2t}} + \beta_J \chi_J / \sqrt{|J|} \), where \( \alpha_J = \alpha_J^{w^{2t}}, \beta_J = \beta_J^{w^{2t}}, |\alpha_J| \leq \sqrt{m_j w^{2t}}, \) and \( |\beta_J| \leq |\Delta_j(w^{2t})| / m_j w^{2t} \). Now break into two terms to be estimated separately so that,
\[
|\langle T^{m,n}_{t,w} f, g \rangle| \leq \Sigma^{m,n}_1 + \Sigma^{m,n}_2,
\]
where
\[
\Sigma^{m,n}_1 := \sum_{L \in \mathcal{D}} \sum_{I \in \mathcal{D}_m(L) : J \in \mathcal{D}_n(L)} \frac{\sqrt{|I||J|}}{|L|} \frac{\sqrt{m_j(w^{2t})}}{(m_L w)^t} |\langle f, h_I \rangle| |\langle gw^1, h_J \rangle|,
\]
\[
\Sigma^{m,n}_2 := \sum_{L \in \mathcal{D}} \sum_{I \in \mathcal{D}_m(L) : J \in \mathcal{D}_n(L)} \frac{|J| \sqrt{|I|}}{|L|(m_L w)^t} \frac{|\Delta_j(w^{2t})|}{m_j(w^{2t})} |\langle f, h_I \rangle| m_J(|g| w^1).
\]

Let \( p = 2 - (C^m_n)^{-1} \) (note that \( 2 > p > 1 \), in fact is getting closer to \( 2 \) as \( m \) and \( n \) increase), and define as in (3.3), (3.4) and (3.5), the quantities \( P^m_n \phi, S^w_n \phi \) and \( R^w_n \phi \), we will use here the case \( v = w^{2t} \), for appropriate \( \phi \)s and corresponding estimates. Note that \( 1 < p < 2 \).

The sequence \( \{ \eta_I \}_{I \in \mathcal{D}} \) where
\[
\eta_I := (m_I w^{2t}) (m_I w^{\frac{2t}{q-1}})^{(q-1)} \left( \frac{|\Delta_I(w^{2t})|^2}{m_I w^{2t}} + \frac{|\Delta_I(w^{2t/(q-1)})|^2}{m_I w^{-2t/(q-1)}} \right) |I|,
\]
is a Carleson sequence with intensity \( C_q[w^{2t}]_{A^d_q} \) by Lemma 3.5. The sequence \( \{ \eta^m_I \}_{I \in \mathcal{D}} \) where
\[
\eta^m_I := \sum_{I \in \mathcal{S}^m T^m_L} \eta_I,
\]
and the stopping time \( \mathcal{S}T^m_L \) is defined as in Lemma 3.6 but with respect to the weights \( u = w^{2t}, v = w^{-2t/(q-1)} \), is a Carleson sequence with intensity \( C_q(m + 1)[w^{2t}]_{A^d_q} \) by Lemma 3.7.
Observe that on the one hand \( \langle gw^t, h_j^m w^2 \rangle = \langle gw^{-t}, h_j^m \rangle w^2 \), and on the other \( m_J(|g|w^t) = m_J(|gw^{-t}|w^2) \). Therefore,

\[
\Sigma_{3}^{m,n} = \sum_{L \in D} (m_L w)^{-t} S_{L}^{w^2, n}(gw^{-t}) P^m f,
\]

\[
\Sigma_{4}^{m,n} = \sum_{L \in D} (m_L w)^{-t} R_{L}^{w^2, n}(gw^{-t}) P^m f.
\]

Estimates (3.6) and (3.7) hold for \( S_{L}^{w^2, m}(gw^{-t}) \) and \( R_{L}^{w^2, m}(gw^{-t}) \) with \( v \) and \( \phi \) replaced by \( w^2 \) and \( gw^{-t} \):

\[
S_{L}^{w^2, n}(gw^{-t}) \leq (m_L w^2)^{\frac{1}{2}} \left( \sum_{J \in D_{m}(L)} |\langle gw^{-t}, h_j^m \rangle w^2|^2 \right)^{\frac{1}{2}},
\]

\[
R_{L}^{w^2, n}(gw^{-t}) \leq C C_{m}^{n}(m_L w^2)^{\frac{1}{2}} (m_L w^2)^{\frac{1}{2}} \frac{C_{m}^{n}}{2} F^\frac{1}{2}(x) \sqrt{\eta_{m}^{n}},
\]

where \( F(x) = \inf_{x \in L} (M_{w^2} |gw^{-t}|^p(x))^{\frac{1}{p}} \).

**Estimating \( \Sigma_{1}^{m,n} \):** Plug in the estimates for \( S_{L}^{w^2, n}(gw^{-t}) \) and \( P^m f \), observe that \( (m_L w^2)^{\frac{1}{2}} (m_L w)^t \leq [w]^{\frac{1}{2}} \), use the Cauchy-Schwarz inequality, to get

\[
\Sigma_{1}^{m,n} \leq \sum_{L \in D} [w]^{\frac{1}{2}} C_{2}^{d} \left( \sum_{J \in D_{m}(L)} |\langle gw^{-t}, h_j^m \rangle w^2|^2 \right)^{\frac{1}{2}} \left( \sum_{I \in D_{m}(L)} |\langle f, h_I \rangle|^2 \right)^{\frac{1}{2}}
\]

\[
\leq [w]^{\frac{1}{2}} C_{2}^{d} \|f\|_2 \left( \sum_{L \in D} \sum_{J \in D_{m}(L)} |\langle gw^{-t}, h_j^m \rangle w^2|^2 \right)^{\frac{1}{2}}
\]

\[
\leq [w]^{\frac{1}{2}} C_{2}^{d} \|f\|_2 \|gw^{-t}\|_{L^2(w^2)} = [w]^{\frac{1}{2}} C_{2}^{d} \|f\|_2 \|g\|_2.
\]

**Estimating \( \Sigma_{2}^{m,n} \):** Plug in the estimates for \( R_{L}^{w^2, n}(gw^{-t}) \) and \( P^m f \), where \( F(x) = (M_{w^2} |gw^{-t}|^p(x))^{\frac{1}{2}} \), use the Cauchy-Schwarz inequality and \( (m_L w^2)^{\frac{1}{2}} (m_L w)^t \leq [w]^{\frac{1}{2}} \) to get

\[
\Sigma_{2}^{m,n} \leq C C_{m}^{n}[w]^{\frac{1}{2}} C_{2}^{d} \|f\|_2 \left( \sum_{L \in D} \frac{\sqrt{\eta_{L}^{n}}}{(m_L w^2)^{\frac{1}{2}}} \inf_{x \in L} F(x) \right)^{\frac{1}{2}}.
\]

Now using Weighted Carleson Lemma 3.1 with \( \alpha_{L} = \eta_{L}^{n}/(m_L w^q)^{1-q} \) (which by Lemma 3.2 is a \( w^2 \)-Carleson sequence with intensity no larger than \( C_q(m+1)[w]_{A_{q}} \)), \( F(x) = (M_{w^2} |gw^{-t}|^p(x))^{\frac{1}{2}} \), and \( v = w^2 \),

\[
\Sigma_{2}^{m,n} \leq C_q(C_{m}^{n})^{2} [w]^{\frac{1}{2}} C_{2}^{d} [w^2]_{A_{q}} \|f\|_2 \|M_{w^2} |gw^{-t}|^p\|_{L^2(w^2)}^{\frac{1}{2}}.
\]
Using (2.1), that is the boundedness of $M_{w^2}$ in $L^{\frac{2}{p}}(w^{2\ell})$ for $2/p > 1$,
\[
\Sigma_2^{m,n} \leq C_q(C_m'')^2(2/p)' [w]^\frac{3}{2} [w^{2\ell}]_A^\frac{3}{2} \|f\|_2 \|gw^{-\epsilon}|p\|_L^\frac{p}{2}.
\]
\[
\leq C_q(C_m'')^3[w]^{\frac{3}{2}} [w^{2\ell}]_A^{\frac{3}{2}} \|f\|_2 \|g\|_2.
\]
Since $(2/p)' = 2/(2 - p) = 2C_m''$. The theorem is proved.

\section*{APPENDIX}

\textbf{Proof of Lemma 3.2.} We will show this inequality using a Bellman function type method. Consider $B(u, v, l) := u - \frac{1}{(v^{p-1} + 1 + l)}$ defined on the domain $\mathbb{D} = \{(u, v, l) \in \mathbb{R}^3, u > 0, v > 0, uv^{p-1} > 1 \text{ and } 0 \leq l \leq 1\}$. Note that $\mathbb{D}$ is convex. Note that
\[
0 \leq B(u, v, l) \leq u \quad \text{for all } (u, v, l) \in \mathbb{D}
\]
and
\[
(\partial B/\partial l)(u, v, l) \geq 1/4v^{p-1} \quad \text{for all } (u, v, l) \in \mathbb{D}.
\]
and also $-(du, dv, dl)d^2B(du, dv, dl)^t$ is non-negative because, it equals
\[
-(du, dv, dl) \begin{pmatrix}
0 & 0 & 0 \\
0 & p(1 - p)^{\frac{v^{p-1}}{1 + l}} & (1 - p)^{\frac{v^{p-1}}{l(1)^3}} \\
0 & (1 - p)^{\frac{v^{p-1}}{l(1)^3}} & -2v^{1-p} \frac{v^{p-1}}{l^{1+1}}
\end{pmatrix} \begin{pmatrix}
du \\
dv \\
dl
\end{pmatrix}
\]
\[
= p(p - 1) \frac{v^{p-1}}{1 + l}(du)^2 + 2(p - 1) \frac{v^p}{(l + 1)^2} dudv + 2 \frac{v^{1-p}}{(l + 1)^3} (dv)^2 \geq 0,
\]
since all terms are positive for $p > 1$.

Now let us show that if $(u_-, v_-, l_-)$ and $(u_+, v_+, l_+)$ are in $\mathbb{D}$ and we define $(u_0, v_0, l) \in \mathbb{D}$ where $l$ is in between $l_+$ and $l_-$, $u_0 = (u_+ + u_-)/2$, $v_0 = (v_- + v_+)/2$, and $l_0 = (l_- + l_+)/2$, then
\[
B(u_0, v_0, l) - (B(u_-, v_-, l_-) + B(u_+, v_+, l_+))/2 \geq |l - l_0|/4v_0^{p-1}
\]
Write for $-1 \leq t \leq 1$, $u(t) = [(t+1)u_+ + (1-t)u_-]/2$, $v(t) = [(t+1)v_+ + (1-t)v_-]/2$, and $l(t) = [(t+1)l_+ + (1-t)l_-]/2$. Define $b(t) := B(u(t), v(t), l(t))$, then $b(0) = B(u_0, v_0, l_0)$, $b(1) = B(u_+, v_+, l_+)$, $b(-1) = B(u_-, v_-, l_-)$, $du/dt = (u_+ - u_-)/2$, $dv/dt = (v_+ - v_-)/2$ and $dl/dt = (l_+ - l_-)/2$. If $(u_+, v_+, l_+)$ and $(u_-, v_-, l_-)$ are in $\mathbb{D}$ then $(u(t), v(t), l(t))$ is also in $\mathbb{D}$ for all $|t| \leq 1$, since $\mathbb{D}$ is convex. It is a calculus exercise to show that
\[
b(0) - \frac{b(1) + b(-1)}{2} = \frac{-1}{2} \int_{-1}^1 (1 - |t|)b''(t)dt
\]
Also it is easy to check that 

\[-b''(t) = \left(\frac{du}{dt}, \frac{dv}{dt}, \frac{dl}{dt}\right) d^2B(\frac{du}{dt}, \frac{dv}{dt}, \frac{dl}{dt})^t.\]

By the Mean Value Theorem and (4.4),

\[B(u_0, v_0, l) - \frac{B(u_-, v_-, l_-) + B(u_+, v_+, l_+)}{2} = \frac{(l - l_0) \partial B(u_0, v_0, l') - \frac{1}{2} \int_{l-1}^{l-1} (1 - |t|)b''(t)dt \geq \frac{l - l_0}{4v_0^{-p-1}},}\]

where \(l'\) is a point between \(l\) and \(l_0 = (l_- + l_+)/2\).

Now we can use the Bellman function argument. Let \(u_+ = m_{J+}w\), \(u_- = m_{J-}w\), \(v_+ = m_{J+}w^{-1}\), \(v_- = m_{J-}w^{-1}\), \(l_+ = \frac{1}{|J+|Q} \sum_{I \in D(J_+)} \lambda_I\) and \(l_- = \frac{1}{|J+|Q} \sum_{I \notin D(J_+)} \lambda_I\). Thus \((u_-, v_-, l_-), (u_+, v_+, l_+) \in \mathbb{D}\) and \(u_0 = m_Jw, v_0 = m_Jw^{-1}\), and \(l_0 = \frac{1}{|J+|Q} \sum_{I \in D(J)} \lambda_I\). Then we can run the usual induction on scale arguments using the properties of the Bellman function,

\[|J|m_Jw \geq |J|B(u_0, v_0, l_0)\]

\[\geq |J|\frac{B(u_+, v_+, l_+)}{2} + |J|\frac{B(u_-, v_-, l_-)}{2} + \lambda_J/4Q(m_Jw^{-1})^{p-1}\]

\[= |J_+|B(u_+, v_+, l_+) + |J_-|B(u_-, v_-, l_-) + \lambda_J/4Q(m_Jw^{-1})^{p-1}\]

Iterating, we get

\[m_Jw \geq \frac{1}{4Q|J|} \sum_{I \in D(J)} \frac{\lambda_I}{(m_Jw^{-1/p-1})^{p-1}}.\]

\[\square\]

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