LOCALIZATION PROPERTIES 
IN ONE DIMENSIONAL DISORDERED 
SUPERSYMMETRIC QUANTUM MECHANICS 

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ABSTRACT — A model of localization based on the Witten Hamiltonian 
of supersymmetric quantum mechanics is considered. The case where the 
superpotential \( \phi(x) \) is a random telegraph process is solved exactly. Both 
the localization length and the density of states are obtained analytically. 
A detailed study of the low energy behaviour is presented. Analytical and 
numerical results are presented in the case where the intervals over which \( \phi(x) \) 
is kept constant are distributed according to a broad distribution. Various 
applications of this model are considered.
I - INTRODUCTION

Since the pioneering work of Anderson [1], the localization problem in one dimension has been investigated extensively and is by now pretty well understood [2]. However, despite the large number of rigorous results, there are very few solvable models in the continuum for which one can compute exactly the density of states and the localization length. The purpose of this work is to present some new results for such a model which is defined by the one dimensional Schrödinger Hamiltonian

\[ H = -\frac{d^2}{dx^2} + \phi^2(x) + \phi'(x) \]  

(1.1)

This Hamiltonian, which was introduced by E. Witten as a toy model of supersymmetric quantum mechanics [3], has stimulated a number of interesting developments in the context of quantum mechanics. In particular it has provided exact solutions of the Schrödinger equation for a class of so called shape invariant potential [4]. It has also inspired a new method of semi-classical quantization [5,6]. In the context of disordered systems, i.e. when the superpotential \( \phi(x) \) is considered as random, one of the most interesting features of this model comes from its relation to the problem of classical diffusion of a particle in a one dimensional random medium. This correspondence [7], which has been exploited in great detail in [8], is based on the observation that the imaginary time Schrödinger equation

\[ \frac{\partial \psi}{\partial t} + H \psi = 0 \]  

(1.2)

can be cast into the Fokker-Planck equation

\[ \frac{\partial P}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\partial P}{\partial x} - 2\phi P \right) \]  

(1.3)

through the transformation

\[ \psi(x,t) = e^{-\int^x dy \ \phi(y)} \ P(x,t) \]

Alternatively, the diffusion process is described by the Langevin equation

\[ \dot{x}(t) = 2\phi(x) + \eta(t) \]  

(1.4)
where \( \eta(t) \) is a gaussian white noise such that
\[
\eta(t)\eta(t') = 2\delta(t - t') \quad (1.5)
\]
This correspondence enables one to express the transition probability \( P(x t | y 0) \) in terms of the eigenstates of \( H \). In the disordered case, namely when \( \phi(x) \) is random, the probability of returning to the starting site, averaged over the disorder, becomes
\[
<P(x t | x 0)> = \int_0^\infty e^{-Et} \rho(E) \, dE \quad (1.6)
\]
The long time behaviour of \( P \) is thus related to the low energy behaviour of the density of states \( \rho(E) \).

In the case of a white noise
\[
<\phi(x)> = \mu \sigma \quad (1.7)
\]
\[
<\phi(x)\phi(x')> - <\phi^2(x)> = \sigma \delta(x - x')
\]
each the localization length and the density of states have been computed exactly [8,9] and display unusual behaviours at low energy. Denoting by \( N(E) \) the integrated density of states, one obtains \( N(E) \propto \frac{1}{E \to 0 \ln^2 E} \) for \( \mu = 0 \), and \( N(E) \propto E^\mu \) for \( \mu > 0 \).

This implies that \(<P(x t | x 0)>\) will present an algebraic tail at large time which reflects the anomalous behaviour of the position \( x(t) \). Since the anomalous behaviour of the diffusion can be predicted qualitatively [10], the above correspondence can be used to understand physically the origin of the anomalous behaviour of the localization problem (see also [11,12]).

However a direct interpretation of these results without appealing to such a correspondence has until now been missing. One of the purposes of this paper is to fill this gap. In our analysis, a crucial role will be played by the zero energy states \( \psi_0(x) = \exp \int x \phi(t)dt \) and \( \psi_1(x) = \psi_0(x) \int x \frac{dy}{\psi_0(x)} \) which are exactly known for any realization of the disorder. This very unusual feature in the context of disordered systems can of course be traced back to the supersymmetric structure of the Hamiltonian.

In order to display this structure it is convenient to introduce the pair of Hamiltonians
\[
H_\pm = -\frac{d^2}{dx^2} + \phi^2(x) \pm \phi' \quad (1.8)
\]
They can be rewritten in the factorized form

\[
H_+ = Q^\dagger Q \quad \text{and} \quad H_- = QQ^\dagger , \quad \text{where} \quad Q = -\frac{d}{dx} + \phi(x) \quad (1.9)
\]

This implies \( H_+ \) and \( H_- \) have the same spectrum for \( E > 0 \) [3]. In the presence of disorder they are characterized by the same Lyapunov exponent and density of states. We will see that the possibility to treat them on the same footing will significantly simplify our analysis.

In all this work, the disorder is modeled by assuming that the potential \( \{\phi(x)\} \) is described by an ensemble of rectangular barriers with alternating heights \( \phi_0 \) and \( \phi_1 \) of random length. Most of our work deals with the case where \( \{\phi(x)\} \) is a random telegraph process [13]. The lengths of the intervals over which \( \phi(x) \) is constant are therefore distributed according to an exponential law \( p_i(\ell) = \theta(\ell) n_i \exp(-n_i \ell) \) for \( i = 0, 1 \). Besides its mathematical interest, this form of disorder is motivated by the fact that it can model certain quasi-one-dimensional structures with a piecewise constant order parameter [14]. Another advantage is that this type of disorder is less singular than a pure white noise process for which arbitrary jumps of the potential can occur. An interesting extension of this model corresponds to the case where the lengths of the barriers are chosen according to a broad distribution which behaves for large \( \ell \) as \( \ell^{-(1+\alpha)} \).

This paper is organized as follows. In part II we present a method to compute the density of states which is an adaptation of the one discussed by Benderskii and Pastur [15]. We then apply this method to the case where the disorder is a random telegraph process. Exact analytical expressions of the density of states and localization length are derived. This is followed by a careful analysis of the limiting behaviour for \( E \to 0 \). A physical interpretation of these results is given in part IV. Numerical simulations are presented in part V. Then the case where \( p(\ell) \) is a broad distribution is considered. Remarkably enough, both the density of states and localization length can be computed exactly when \( \alpha < 1 \) and \( \phi_1 = -\phi_0 \). Finally, in part VI we apply some of these results to the study of anomalous diffusion and discuss various applications.
II - THE PHASE FORMALISM FOR THE SUPERSYMMETRIC HAMILTONIANS $H_{\pm}$

The phase formalism is based on the following property of one-dimensional Hamiltonians: the integrated density of states per unit length $N(E)$ and the Lyapunov exponent $\gamma(E)$ are directly related to properties of the solution $\psi_E$ of the corresponding Schrödinger equation $H\psi_E = E\psi_E$ with a given logarithmic derivative at one point [16]. The number $N(E)$ of states of energy lower than $E$, per unit length, is equal to the number of nodes of the wave function $\psi_E$ per unit length, and the Lyapunov exponent $\gamma(E)$ measures the exponential growth rate of the envelope of the wave function $\psi_E$.

For any spatially homogenous disordered potential with short range correlations, $N(E)$ and $\gamma(E)$ are self-averaging quantities. This means that they are the same for all realizations of the disorder with probability one.

A convenient way to implement the phase formalism for the Schrödinger equations
\[
\begin{align*}
H_+ u_E &= E u_E \\
H_- v_E &= E v_E
\end{align*}
\]
(2.1)
is to introduce "polar" variables
\[
\begin{align*}
u_E(x) &= \rho_E(x) \cos \theta_E(x) \\
v_E(x) &= \rho_E(x) \sin \theta_E(x)
\end{align*}
\]
The phase $\theta_E(x)$ contains all the informations about the oscillations of the wave functions ($u_E$, $v_E$) and the modulus $\rho_E(x)$ represents the envelope of these wave functions. The new dynamical variables $\theta_E$ and $\rho_E$ will determine the value of $N(E)$ and $\gamma(E)$ respectively.

$N(E)$ is equal to the number of nodes of $u_E$ (or $v_E$) per unit length and can thus be written directly in terms of $\theta_E(x)$
\[
N(E) = \lim_{L \to \infty} \left( \frac{\theta_E(L) - \theta_E(0)}{\pi L} \right).
\]
(2.2)
Note that $[\theta_E(L) - \theta_E(0)]$ is the total phase accumulated on the interval $[0, L]$ and is therefore not an angle defined modulo $2\pi$. $\gamma(E)$ is the exponential growth rate of the envelope $\rho_E$ and reads therefore
\[
\gamma(E) = \lim_{L \to \infty} \frac{1}{L} \ln \left( \frac{\rho_E(L)}{\rho_E(0)} \right)
\]
(2.3)
Now in order to write the dynamical equations for $\theta_E$ and $\rho_E$ we have to separate the cases $E > 0$ and $E < 0$. From now on we will set $k = C|E|$.

1 — case $E = +k^2$

In this case, the factorized forms (1.9) allow to rewrite the system (2.1) as two coupled first order equations $Qu = kv$ and $Q^t v = ku$, which explicitly read

$$
\begin{align*}
\frac{du_E}{dx} &= \phi(x)u_E(x) - kv_E(x) \\
\frac{dv_E}{dx} &= -\phi(x)v_E(x) + ku_E(x)
\end{align*}
$$

(2.4)

This gives the following dynamical equations for the "polar" variables

$$
\begin{align*}
\frac{d\theta_E}{dx} &= k - \phi(x)\sin 2\theta_E(x) \\
\frac{d\ln \rho_E}{dx} &= \phi(x)\cos 2\theta_E(x)
\end{align*}
$$

(2.5)

Note that the equation for $\theta_E$ does not involve $\rho_E$.

Let us now present the direct resolution of these equations in $(\theta_E, \rho_E)$ for the special cases $E = 0$ and $E \to \infty$, before we turn to the general method to calculate $N(E)$ and $\gamma(E)$ for all $E > 0$.

a) case $E = 0$

For $k = 0$, the dynamical equations (2.5) read

$$
\begin{align*}
\frac{d\theta}{dx} &= -\phi(x)\sin 2\theta(x) \\
\frac{d\ln \rho}{dx} &= \phi(x)\cos 2\theta(x)
\end{align*}
$$

(2.6)

The phase $\theta(x)$ cannot grow more than $\pi/2$ on the interval $[0, L]$ since, for any $\{\phi(x)\}$, the velocity $\frac{d\theta}{dx}$ vanishes whenever $\theta = 0$ [modulo $\pi/2$]. Therefore (2.2) yields immediately $N(0) = 0$ in accordance with the positivity properties of $H_\pm$.

To get $\gamma(0)$, we can integrate the equation for $\theta(x)$

$$
\left|\frac{\tan \theta(L)}{\tan \theta(0)}\right| = e^{-2\int_0^L \phi(x)dx}
$$

(2.7)
and rewrite the equation for $\rho$ as
\[
\frac{d \ln \rho}{dx} = -\frac{d \theta}{dx} \cos 2\theta(x) \sin 2\theta(x) \tag{2.8}
\]
The integration gives
\[
\gamma(0) \equiv \lim_{L \to \infty} \left( \frac{\ln \rho(L) - \ln \rho(0)}{L} \right) = \lim_{L \to \infty} \frac{1}{2L} \ln \left| \frac{\sin 2\theta(0)}{\sin 2\theta(L)} \right| \tag{2.9}
\]
We now use (2.7); in the limit $L \to \infty$, the expression $\frac{1}{L} \int_0^L \phi(x)dx$ is simply the mean value of $\{\phi(x)\}$, that we will note $F_0 \equiv \langle \phi \rangle$ from now on. We obtain finally that the Lyapunov exponent at zero energy is equal to the absolute value of the average $F_0$ of $\{\phi(x)\}$
\[
\gamma(0) = |F_0| \tag{2.10}
\]
This simple result can also be recovered by starting from the two linearly independent solutions of $H_+\psi = 0$
\[
\begin{cases}
\psi_0(x) = e^{\int_0^x \phi(t)dt} \\
\psi_1(x) = \psi_0(x) \int_0^x \frac{dt}{\psi_0'(t)} = e^{\int_0^x \phi(t)dt} \int_0^x dy e^{-2 \int_0^y \phi(t)dt} \tag{2.11}
\end{cases}
\]
The asymptotic behaviour for large $x$
\[
\int_0^x \phi(t)dt \sim F_0 x [1 + o(1)] \]
allows to recover (2.10).

b) Limit $E \to \infty$

At high energy, (2.5) can be approximated by $\frac{d \theta}{dx} \sim k = CE$ and $N(E)$ behaves therefore asymptotically as the free integrated density of states $N(E) \sim \frac{CE}{E \to \infty \pi} \cdot$

At this order of approximation, the Lyapunov exponent vanishes
\[
\gamma(E) = \lim_{L \to \infty} \frac{1}{L} \int_0^L dx \phi(x) \cos[2\theta(x)] > \sim \lim_{E \to \infty} \lim_{L \to \infty} \langle \phi \rangle > \frac{1}{L} \int_0^L dx \cos(2kx) = 0
\]
To find the asymptotic behaviour of $\gamma(E)$ we must calculate the next order for $\theta(x)$

$$\theta(x) \sim kx - \int_0^x \phi(x') \sin(2kx') dx' \quad (2.12)$$

This allows to relate the high energy behaviour of $\gamma(E)$ to the two point correlation function of the process $\{\phi(x)\}$ [16]

$$\gamma(E) \sim \frac{1}{2} \int_0^\infty dx \cos(2kx) \left[ <\phi(x)\phi(0)> - <\phi>^2 \right] \quad (2.13)$$

This Fourier transform relation shows that the high energy decay of the Lyapunov exponent is directly related to the regularity of the random process $\{\phi(x)\}$.

For the white noise case $<\phi(x)\phi(x')> = <\phi>^2 + \sigma \delta(x - x')$ we recover the result [8]

$$\gamma_{WN}(E) \sim \frac{\sigma}{2} \quad (2.14)$$

This behaviour at high energy is rather pathological and very particular to white noise. Indeed for an exponentially correlated noise

$$<\phi(x)\phi(x')> = <\phi>^2 + \frac{\lambda \sigma}{2} e^{-\lambda|x-x'|} \quad (2.15)$$

which can be viewed as a regularization of the white noise case as long as $\lambda$ is finite, we obtain a vanishing Lyapunov exponent

$$\gamma(E) \sim \left( \frac{\sigma^2}{8} \right) \frac{1}{E} \quad (2.16)$$

c) General method to calculate $N(E)$ and $\gamma(E)$ for all $E > 0$

We have seen that the dynamical equations (2.5) could be integrated directly for the particular values $E = 0$ and $E \to \infty$ in order to get $N(E)$ and $\gamma(E)$. This is obviously not possible in general. We therefore need a more powerful method [15]. We can use the identity $\delta[f(x)] = \sum_{x_i \text{ zero of } f} \delta(x - x_i) \left| \frac{1}{f'(x_i)} \right|$ to rewrite the number $N_L(E)$ of nodes of $\cos \theta_E(x)$ in the interval $[0, L]$

$$N_L(E) = \int_0^L dx \sum_{x_i \text{ solution of } \cos \theta(x)} \delta(x - x_i) = k \int_0^L dx \delta[\cos \theta(x)]$$

$$= k \int_0^L dx \sum_{n \in \mathbb{Z}} \delta[\theta(x) - \frac{\pi}{2} - n\pi] \quad (2.17)$$
The integrated density of states $N(E)$ then reads \[15\]
\[
N(E) = \lim_{L \to \infty} \left( \frac{N_L(E)}{L} \right) = kP_{eq} \left( \frac{\pi}{2} \right)
\] (2.18)
where $P_{eq}(\theta)$ is the stationary distribution of the reduced phase $\theta$ defined modulo $\pi$.

Similarly the Lyapunov exponent can be rewritten as \[17\]
\[
\gamma(E) = \lim_{L \to \infty} \frac{1}{L} \int_0^L dx \, \phi(x) \cos 2\theta(x) = \langle \phi(x) \cos 2\theta(x) \rangle
\] (2.19)
where the brackets $\langle \cdots \rangle$ denote the mean value taken over the joint probability of the process $\phi$ and of the reduced phase $\theta$. We will use these results in section III to calculate explicitly $N(E)$ and $\gamma(E)$ for a specific model of disorder.

\textbf{2 — case $E = -k^2$}

We have already seen that $N(0) = 0$ so that $N(E)$ vanishes identically for $E \in ]-\infty, 0]$. It is nevertheless interesting to study the Lyapunov exponent $\gamma(E)$ in this unphysical region since it is relevant for the problem of classical diffusion in the random force field $\{\phi(x)\}$.

For $E < 0$, the factorized forms (1.9) allow to rewrite the system (2.1) as two coupled first order equations $Qu = kv$ and $Q^\dagger v = -ku$, which explicitly read
\[
\begin{cases}
\frac{du_E}{dx} = \phi(x)u_E(x) - kv_E(x) \\
\frac{dv_E}{dx} = -\phi(x)v_E(x) - ku_E(x)
\end{cases}
\] (2.20)
where $k = C-E$.

The dynamical equations for the polar coordinates then read
\[
\begin{cases}
\frac{d\theta_E}{dx} = -k \cos 2\theta_E(x) - \phi(x) \sin 2\theta_E(x) \\
\frac{d\ln \rho_E}{dx} = -k \sin 2\theta_E(x) + \phi(x) \cos 2\theta_E(x)
\end{cases}
\] (2.21)
Note the differences with (2.4) and (2.5) respectively.
We recover immediately that \( N(E) = 0 \) for \( E < 0 \) by noting that the accumulated phase \( \theta(x) \) will be trapped in an interval of length \( \leq \frac{\pi}{2} \). Indeed for any \( \{ \phi(x) \} \) the velocity \( \frac{d\theta}{dx} \) is positive whenever \( \theta = \frac{\pi}{2} \) [modulo \( \pi \)], and negative whenever \( \theta = 0 \) [modulo \( \pi \)].

Still we need to study the stationary distribution \( P_{eq}(\theta) \) in order to calculate \( \gamma(E) \) along the lines explained in II 1 c).

III - APPLICATION TO A MODEL OF RECTANGULAR BARRIERS OF RANDOM LENGTHS

1 — Description of the model

Let us now consider a model [15] where \( \phi(x) \) takes alternatively two values \( \phi_0 \) and \( \phi_1 \) on intervals whose lengths are positive independent random variables (Fig.1), distributed according to the following probability densities respectively

\[
\begin{align*}
    f_0(l) &= \theta(l) n_0 e^{-n_0 l} \\
    f_1(h) &= \theta(h) n_1 e^{-n_1 h}
\end{align*}
\]

This choice for \( f_0 \) and \( f_1 \) is in fact the only one that makes the process \( \{ \phi(x) \} \) Markovian. This property enables us to write differential equations for the probability \( p_0(x) \) to have \( \phi(x) = \phi_0 \) and the probability \( p_1(x) = 1 - p_0(x) \) to have \( \phi(x) = \phi_1 \).

\[
\begin{align*}
    \frac{\partial p_0}{\partial x} &= -n_0 p_0 + n_1 p_1 = n_1 - (n_0 + n_1) p_0 \\
    \frac{\partial p_1}{\partial x} &= -n_1 p_1 + n_0 p_0 = n_0 - (n_0 + n_1) p_1
\end{align*}
\]

The corresponding stationary solutions are simply

\[
\begin{align*}
    \lim_{x \to \infty} p_0(x) &= \frac{n_1}{n_0 + n_1} = \frac{<l>}{<l> + <h>} \\
    \lim_{x \to \infty} p_1(x) &= \frac{n_0}{n_0 + n_1} = \frac{<h>}{<l> + <h>}
\end{align*}
\]

where \( < l > = \int_0^\infty dl \ l f_0(l) = \frac{1}{n_0} \) is the mean length of intervals \( \{ \phi(x) = \phi_0 \} \) and \( < h > = \int_0^\infty dh \ h f_1(h) = \frac{1}{n_1} \) is the mean length of intervals \( \{ \phi(x) = \phi_1 \} \).
The mean value $F_0$ and the two point correlation function $G(x)$ of the process $\{\phi(x)\}$ read

\[
\begin{cases}
F_0 \equiv <\phi> = \phi_0 \frac{n_1}{n_0 + n_1} + \phi_1 \frac{n_0}{n_0 + n_1} \\
G(x) \equiv <\phi(x)\phi(0)> - <\phi>^2 = \frac{n_0 n_1}{(n_0 + n_1)^2} (\phi_1 - \phi_0)^2 e^{-(n_0 + n_1)|x|}
\end{cases}
\] (3.4)

The case $\phi_1 = -\phi_0$ where

\[
\begin{cases}
F_0 = \phi_0 \frac{n_1 - n_0}{n_0 + n_1} \\
G(x) = (\phi_0^2 - F_0^2) e^{-(n_0 + n_1)|x|}
\end{cases}
\] (3.5)

tends to the white noise process

\[
\begin{align*}
<\phi> &= F_0 \\
<\phi(x)\phi(0)> &= F_0^2 + \sigma \delta(x)
\end{align*}
\]
in the limit

\[
\begin{align*}
\phi_0 &\to \infty \\
n_0 &\to \infty \\
n_1 &\to \infty
\end{align*}
\] where

\[
\begin{align*}
F_0 &= \phi_0 \frac{n_1 - n_0}{n_0 + n_1} \\
\sigma &= 2 \frac{\phi_0^2}{n_0 + n_1}
\end{align*}
\]

remain constants. (3.6)

It will be also useful to define $\mu = \frac{F_0}{\sigma} = \frac{n_1 - n_0}{2\phi_0}$.

We will now see that this model is exactly soluble, even though it contains correlations, because it is a Markovian process.

Note that this two-step process can be easily generalized into a multi-step process [18].

2 — Application of the Phase Formalism for $E > 0$

As we have already stressed, $\{\phi(x)\}$ is a Markovian process. Since the accumulated phase $\theta(x)$ evolves for $E > 0$ according to $\frac{d\theta}{dx} = k - \phi(x) \sin 2\theta(x)$, we see that the pair $\{\phi(x), \theta(x)\}$ forms a two-dimensional Markov process.

Let us define

\[
\begin{align*}
\tilde{P}_0(\theta, x) \, d\theta &\equiv \text{the probability to have} & \begin{cases}
\theta(x) \in [\theta, \theta + d\theta] \\
\phi(x) = +\phi_0
\end{cases} \\
\tilde{P}_1(\theta, x) \, d\theta &\equiv \text{the probability to have} & \begin{cases}
\theta(x) \in [\theta, \theta + d\theta] \\
\phi(x) = +\phi_1
\end{cases}
\end{align*}
\]
We can write two coupled master equations for $\tilde{P}_0$ and $\tilde{P}_1$

$$\begin{cases}
\frac{\partial \tilde{P}_0}{\partial x} = -\frac{\partial}{\partial \theta}[(k - \phi_0 \sin 2\theta)\tilde{P}_0] - n_0 \tilde{P}_0 + n_1 \tilde{P}_1 \\
\frac{\partial \tilde{P}_1}{\partial x} = -\frac{\partial}{\partial \theta}[(k - \phi_1 \sin 2\theta)\tilde{P}_1] - n_1 \tilde{P}_1 + n_0 \tilde{P}_0
\end{cases}$$

(3.7)

We now express the probability distributions $P_0(\theta, x)$ and $P_1(\theta, x)$ of the reduced phase $\theta$, defined modulo $\pi$, in terms of the probability distributions $\tilde{P}_0(\theta, x)$ and $\tilde{P}_1(\theta, x)$ of the accumulated phase $\theta \in R$ [15]

$$P_i(\theta, x) = \sum_{n \in \mathbb{Z}} \tilde{P}_i(\theta + n\pi, x)$$

$P_0(\theta, x)$ and $P_1(\theta, x)$ are by definition $\pi$-periodic, and satisfy the same system (3.7) as $\tilde{P}_0$ and $\tilde{P}_1$. As $x \to \infty$, they converge respectively towards stationary solutions $P_0(\theta)$ and $P_1(\theta)$ of (3.7)

$$\begin{cases}
\frac{d}{d\theta}[(k - \phi_0 \sin 2\theta)P_0] + n_0 P_0(\theta) - n_1 P_1(\theta) = 0 \\
\frac{d}{d\theta}[(k - \phi_1 \sin 2\theta)P_1] - n_1 P_0(\theta) + n_0 P_1(\theta) = 0
\end{cases}$$

(3.8)

The sum of these two equations gives a simple relation between $P_0(\theta)$ and $P_1(\theta)$

$$(k - \phi_0 \sin 2\theta)P_0(\theta) + (k - \phi_1 \sin 2\theta)P_1(\theta) = C$$

(3.9)

where $C$ is a constant which can be evaluated at the point $\theta = \frac{\pi}{2}$; in fact it is exactly the density of states $N(E)$ according to (2.18)

$$C = k P_0 \left( \frac{\pi}{2} \right) + k P_1 \left( \frac{\pi}{2} \right) = k P_{eq} \left( \frac{\pi}{2} \right) = N(E)$$

(3.10)

The system (3.8) can now be rewritten as two decoupled equations for $P_0(\theta)$ and $P_1(\theta)$, containing the a priori unknown constant $N(E)$

$$\begin{cases}
(k - \phi_1 \sin 2\theta)\frac{d}{d\theta}[(k - \phi_0 \sin 2\theta)P_0] + (n_0 + n_1)[k - F_0 \sin 2\theta]P_0 = n_1 N(E) \\
(k - \phi_0 \sin 2\theta)\frac{d}{d\theta}[(k - \phi_1 \sin 2\theta)P_1] + (n_0 + n_1)[k - F_0 \sin 2\theta]P_1 = n_0 N(E)
\end{cases}$$

(3.11)

Finally $N(E)$ will be determined by the normalization conditions that must be imposed on the $\pi$-periodic solutions $P_0(\theta)$ and $P_1(\theta)$ of (3.11)

$$\begin{cases}
\int_0^\pi P_0(\theta)d\theta = \frac{n_1}{n_0 + n_1} \\
\int_0^\pi P_1(\theta)d\theta = \frac{n_0}{n_0 + n_1}
\end{cases}$$

(3.12)
where we have used (3.3).

According to (2.19) the Lyapunov exponent can be expressed in terms of the stationary distributions $P_0(\theta)$ and $P_1(\theta)$

$$\gamma(E) = \langle \phi(x) \cos 2\theta(x) \rangle = \int_0^{\pi} d\theta \cos 2\theta[\phi_0 P_0(\theta) + \phi_1 P_1(\theta)]$$  \hspace{1cm} (3.13)

The determination of $N(E)$ and $\gamma(E)$ is now reduced to the resolution of the equations (3.11) for $P_0(\theta)$ and $P_0(\theta)$. The results are presented in appendix A. In the following we will only consider various limiting cases for $N(E)$ and $\gamma(E)$.

* Limit $E \to 0^+$

In order to determine the limiting behaviour of $N(E)$ for $E \to 0^+$, we have to study how the integrals $I_1$ and $I_2$ given in (A.4) diverge in this limit. It is convenient to set $\mu_0 = \frac{n_0}{2\phi_0}$, $\mu_1 = \frac{n_1}{2\phi_0}$ and $\mu = \mu_1 - \mu_0$.

- For $\mu > 0$, $I_1$ has the dominant divergent behaviour and $N(E)$ vanishes as $E^\mu$

$$N(E) \approx \frac{2\sigma}{\mu_0 \mu_1} \left[ \frac{\Gamma(\mu_1)}{\Gamma(\mu_0)\Gamma(\mu)} \right]^2 \left( \frac{E}{\phi_0^2} \right)^\mu$$  \hspace{1cm} (3.14)

- For $\mu < 0$, $I_2$ has the dominant divergent behaviour and $N(E)$ vanishes as $E^{\mu}$

$$N(E) \approx \frac{2\sigma}{\mu_0 \mu_1} \left[ \frac{\Gamma(\mu_0)}{\Gamma(\mu_1)\Gamma(\mu)} \right]^2 \left( \frac{E}{\phi_0^2} \right)^{|\mu|}$$  \hspace{1cm} (3.15)

- For $\mu = 0$ $I_1$ and $I_2$ diverge only logarithmically and therefore $N(E)$ vanishes very slowly in comparison to the cases $\mu \neq 0$

$$N(E) \propto \frac{1}{(\ln E)^2}$$  \hspace{1cm} (3.16)

In the next section (IV) it will be shown how these limiting behaviours of $N(E)$ can be understood through a qualitative analysis.
Let us now turn to the Lyapunov exponent $\gamma(E)$. We can study how the integrals $J_1$ and $J_2$ given in (A.6) diverge in the limit $E \to 0$. $J_1$ (resp. $J_2$) has the dominant divergent behaviour if $\mu > 0$ (resp. $\mu < 0$) and we recover the result (2.10)

$$\gamma(0) = |F_0|$$

The case $\mu = 0$ deserves special attention since $\gamma(E)$ vanishes as

$$\gamma(E) \propto \frac{1}{E \to 0 (-\ln E)}$$

This singularity corresponds to the singularity (3.16) found for the density of states $N(E)$ through the Thouless formula [2]. Note that for $E = 0$, the wavefunctions are not exponentially localized since $\gamma(0) \equiv \lim_{L \to \infty} \frac{1}{L} \ln \frac{\rho(L)}{\rho(0)} = 0$, but it can nevertheless be shown that $\ln \frac{\rho(L)}{\rho(0)}$ behaves asymptotically for large $L$ as $CL$ (See Section IV).

* Limit of white noise process for $\{\phi(x)\}$

$\{\phi(x)\}$ tends to a white noise process $\left( < \phi > = F_0 \right)$ $< \phi(x)\phi(x') > = F^2_0 + \sigma\delta(x-x')$ in the limit $\left( \begin{array}{l} \phi_0 \to \infty \\ n_0 \to \infty \\ n_1 \to \infty \end{array} \right)$ provided that $\left( \begin{array}{l} F_0 = \phi_0 \frac{n_1 - n_0}{n_1 + n_0} \\ \sigma = \frac{2}{n_0 + n_1} \phi_0^2 \end{array} \right)$ remains constants (3.6).

The parameter $\mu = \frac{n_1 - n_0}{2\phi_0} = \frac{F_0}{\sigma}$ also remains constant. In this limit we recover the results [8,9] for $0 < E < \infty$

$$\left\{ \begin{array}{l} N_{WN}(E) = \frac{2\sigma}{\pi^2} J^2_\mu(z) + N^2_\mu(z) \\ \gamma_{WN}(E) = -\sigma z \frac{d}{dz} \ln C J^2_\mu(z) + N^2_\mu(z) \end{array} \right.$$ (3.18)

where $z = \frac{CE}{\sigma}$ and $(J_\mu, N_\mu)$ are Bessel functions.

* Limit $E \to \infty$

In this limit, we recover of course that $N(E)$ behaves asymptotically as the free density of states (II 1 b))

$$N(E) \sim \frac{CE}{\pi}$$

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The Lyapunov exponent $\gamma(E)$ vanishes asymptotically as $\frac{1}{E}$ in accordance with the result (2.16) for exponentially correlated processes. The correspondance between (2.15) and (3.5) is the following:

$$\left\{ \begin{align*}
\lambda &= n_0 + n_1 \\
\lambda \sigma &= \frac{n_0 n_1}{(n_0 + n_1)^2} 8\phi_0^2
\end{align*} \right.$$

and

$$\gamma(E) \sim \frac{n_0 n_1}{n_0 + n_1} \left( \frac{\phi_0^2}{E} \right)$$

Let us give a straightforward derivation of this result for the Hamiltonian (See Fig.2)

$$H_+ = -\frac{d^2}{dx^2} + \phi^2 + \phi'(x)$$

$$= -\frac{d^2}{dx^2} + \phi_0^2 + \sum_n (-1)^n 2\phi_0 \delta(x - x_n)$$

where

$$\left\{ \begin{align*}
(x_{2n}) &\text{ are the points where } \phi(x) \text{ changes from } (-\phi_0) \text{ to } (+\phi_0) \\
(x_{2n+1}) &\text{ are the points where } \phi(x) \text{ changes from } (+\phi_0) \text{ to } (-\phi_0)
\end{align*} \right.$$

On the interval $[0, L]$ there will be typically $N$ attractive delta potentials and $N$ repulsive delta potentials with (3.3)

$$L = N(<l> + <h>) = N\left( \frac{1}{n_0} + \frac{1}{n_1} \right)$$

The elementary potentials $V_\pm = \phi_0^2 \pm 2\phi_0 \delta(x)$ have respectively the following transmission coefficients in the region $E > \phi_0^2$

$$t_\pm(E) = \frac{iCE - \phi_0^2}{iCE - \phi_0^2 \mp \phi_0}$$

In the high energy limit, we can neglect interferences between the various delta potentials and approximate the total transmission coefficient on $[0,L]$ by a product of elementary coefficients $t_\pm(E)$ [16]

$$\left| \frac{\psi(L)}{\psi(0)} \right| \sim t_+(E) t_-(E) |N(L)|^{N(L)}$$

where

$$N(L) \sim \frac{L}{\frac{1}{n_0} + \frac{1}{n_1}}$$

according to (3.20). With the help of (3.21) this yields

$$\gamma(E) = -\lim_{L \to \infty} \frac{1}{L} \ln \left| \frac{\psi(L)}{\psi(0)} \right|$$

$$\sim -\frac{1}{n_0 + n_1} \ln |t_+(E) t_-(E)|$$

$$\sim \frac{n_0 n_1}{n_0 + n_1} \left( \frac{\phi_0^2}{E} \right)$$

(3.23)
From the spectral properties of $H_\pm$ we know that $N(E)$ identically vanishes for $E < 0$. Nevertheless it is interesting to calculate the Lyapunov exponent $\gamma(E)$ in this region since it is directly relevant for the problem of classical diffusion [8]. According to the dynamical equations (2.21), the Lyapunov exponent $\gamma(E)$ reads for $E < 0$

$$\gamma(E) = \lim_{L \to \infty} \frac{1}{L} \int_0^L dx \left[ -k \sin 2\theta(x) + \phi(x) \cos 2\theta(x) \right]$$  \hspace{1cm} (3.24)

and the phase $\theta(x)$ evolves according to

$$\frac{d\theta}{dx} = -k \cos 2\theta(x) - \phi(x) \sin 2\theta(x)$$  \hspace{1cm} (3.25)

Here, there is no need to distinguish accumulated and reduced phase, according to the discussion following (2.21). As before (3.7), we can easily write Master Equations for $P_0(\theta, x)$ and $P_1(\theta, x)$

$$\begin{cases} 
\frac{\partial P_0}{\partial x} = -\frac{\partial}{\partial \theta} \left[ (-k \cos 2\theta - \phi_0 \sin 2\theta) P_0 \right] - n_0 P_0 + n_1 P_1 \\
\frac{\partial P_1}{\partial x} = -\frac{\partial}{\partial \theta} \left[ (-k \cos 2\theta + \phi_0 \sin 2\theta) P_1 \right] + n_0 P_0 - n_1 P_1
\end{cases}  \hspace{1cm} (3.26)$$

As $x \to \infty$, they converge respectively towards stationary solutions $P_0(\theta)$ and $P_1(\theta)$ of (3.26)

$$\begin{cases} 
\frac{d}{d\theta} \left[ (k \cos 2\theta + \phi_0 \sin 2\theta) P_0 \right] - n_0 P_0 + n_1 P_1 = 0 \\
\frac{d}{d\theta} \left[ (k \cos 2\theta - \phi_0 \sin 2\theta) P_1 \right] + n_0 P_0 - n_1 P_1 = 0
\end{cases}  \hspace{1cm} (3.27)$$

The sum of these two equations gives

$$(k \cos 2\theta + \phi_0 \sin 2\theta) P_0(\theta) + (k \cos 2\theta - \phi_0 \sin 2\theta) P_1(\theta) = \text{constant}  \hspace{1cm} (3.28)$$

This constant vanishes at the point $\theta = \frac{\pi}{2}$ since $k[P_0(\frac{\pi}{2}) + P_1(\frac{\pi}{2})] = N(E) = 0$. The system (3.26) can now easily be decoupled.

The solutions $P_0(\theta)$ and $P_1(\theta)$, and the calculation of $\gamma(E)$ in terms of these stationary distributions are given in Appendix B. Finally, we obtain after some transformations

$$\gamma(E) = \frac{n_1}{n_0 + n_1} \phi_0 C_1 + \beta^2 \frac{\frac{N(\beta^2)}{D(\beta^2)}}{2}  \hspace{1cm} (3.29)$$
\[
\begin{align*}
N(\beta^2) &= \int_0^1 dx \ x^{\nu_0-1} \left( \frac{1 - x}{x + \beta^2} \right)^{\nu_1} \frac{x^2 + \beta^2}{(1 - x)(x + \beta^2)}. \\
D(\beta^2) &= \int_0^1 dx \ x^{\nu_0-1} \left( \frac{1 - x}{x + \beta^2} \right)^{\nu_1}.
\end{align*}
\]

We have set \( \beta = \frac{k}{\phi_0} \), \( \nu_1 = \frac{n_1}{2\phi_0} \frac{1}{C^{1+\beta^2}} \) and \( \nu_0 = \frac{n_1}{2\phi_0} \frac{1}{C^{1+\beta^2}} \).

- In the white noise limit (3.6) we recover the result [8]

\[
\gamma_{\text{WN}}(E = -k^2) = -2\sigma z K'_{\mu}(z) K_{\mu}(z)
\]

where \( z = \frac{k}{\sigma} \) and \( K_{\mu} \) is the modified Bessel function.

- In the limit \( E \to 0 \), we recover (2.10)

\[
\gamma(0) = |F_0|
\]

In section (VI), we will use the expression (3.29) to obtain the velocity for the associated problem of classical diffusion in the random force field \( \{\phi(x)\} \).

**IV - LOW ENERGY STATES - A PHYSICAL PICTURE**

The purpose of this section is to provide a simple physical picture that accounts for the low energy part of the spectrum (3.14, 3.16). We will show that the basic mechanism is very different from the one which is responsible for the usual Lifchitz singularities. We have to consider separately the cases \( \mu = 0 \) and \( \mu > 0 \).

1) \( \mu = 0 \).

For a given realization of the disorder, the two independent solutions of \( H_+ \psi = 0 \) have been constructed above (2.11). In general, none of these solutions is normalizable on the whole line. Instead, if we consider the problem on a finite interval \([-R, R]\) with Dirichlet boundary conditions, then one can show that there exists a quasi zero mode (which is a linear combination of \( \psi_0 \) and \( \psi_1 \)) whose energy is exponentially small. One obtains with exponential accuracy [19]

\[
E_0(R) \sim \frac{1}{\int_{-R}^R dx \ \psi_0^2(x)} + \frac{1}{\int_{-R}^R dx \ \psi_1^2(x)} \frac{1}{\int_{-R}^R dx \ \psi_0^2(x)}
\]

\[
(4.1)
\]
Since the corresponding wave function has its support on \([-R, R]\), it is also a quasi zero mode on the whole line with same energy. In the presence of disorder, in order to estimate \(E_0\) we will replace \(\psi_0(x)\) by its typical value. When \(\{\phi(x)\}\) is a random telegraph process, the distribution of the random variable \(\int_0^x dy \, \phi(y)\) can be obtained exactly (see Appendix D). However, for large \(x\), one merely expects that \(\frac{1}{\sqrt{|x|}} \int_0^x \phi(y)dy\) will be distributed as a Gaussian process [20,21]. More precisely, for the case \(\phi_1 = -\phi_0\) and \(n_0 = n_1 = n\) the Central Limit theorem gives \(\psi_{0,typ} \sim e^{-\phi_0 C \frac{|x|}{n}}\). This gives

\[
E \propto R \rightarrow \infty \frac{1}{\sqrt{R}} e^{-\frac{2\phi_0 C |x|}{n}} \quad (4.2)
\]

Phrased differently, this means that a quasi zero mode of energy \(E\) has typically a spatial extension \(2R\) such that

\[
\ln E \sim -2\phi_0 C \frac{R}{n} \quad (4.3)
\]

Therefore the number of such states per unit length behaves typically as

\[
N(E) \sim \frac{1}{2R} \sim \frac{1}{\ln^2 E} \quad (4.4)
\]

This argument can be easily generalized to account for other types of correlations. For instance, if one assumes that the correlation function has a power law behaviour

\[
<(\phi(x) - \phi(y))^2> \sim |x-y|^\alpha \quad (\alpha > 0)
\]

\[
|x-y| \rightarrow \infty \quad (4.5)
\]

one can easily show along the same lines that

\[
N(E) \propto E \rightarrow 0 \frac{1}{(-\ln E)^{\frac{2}{2+\alpha}}} \quad (4.6)
\]

This last result is in agreement with the one obtained in [22] in the context of anomalous diffusion.

2) \(\mu > 0\).

For simplicity we consider the case where \(\phi_1 = -\phi_0\) with \(n_1 \gg n_0\). The profile of the potential is sketched on Fig. 2. Each time \(\phi(x)\) jumps from \(\phi_0\) to \(-\phi_0\) and then back from \(-\phi_0\) to \(\phi_0\), there appears a potential well \(V_-(x) = -2\phi_0 \delta(x)\) which is
then followed by a potential barrier $V_+(x) = 2\phi_0 \delta(x)$. Such a dipole like configuration can support localized states which account for the low energy behaviour of the density of states. Since $n_1 \gg n_0$ the average distance between such configurations is large. Therefore one can safely ignore interference effects and consider a single doublet of width $a$. The potential is given by

$$V(x) = \phi_0^2 - 2\phi_0 \delta(x) + 2\phi_0 \delta(x-a)$$  \hspace{1cm} (4.7)

An elementary calculation shows that this potential can support a bound state provided

$$\frac{E}{\phi_0^2} = e^{-2a\phi_0 C_1 - \frac{E}{\phi_0}}$$  \hspace{1cm} (4.8)

Here we deal with low energy states such that $0 < E \ll \phi_0^2$, this gives

$$E \simeq \phi_0^2 e^{-2a\phi_0}$$  \hspace{1cm} (4.9)

which is consistent with the previous inequality if $a\phi_0 \gg 1$. If we would just consider the attractive potential well

$$V(x) = \phi_0^2 - 2\phi_0 \delta(x)$$

this would lead to a zero energy state $\psi_0(x) \propto e^{-|x|\phi_0}$ (consistent with supersymmetry). It is the coupling of this state with the repulsive potential barrier $2\phi_0 \delta(x-a)$ that increases the energy up to $E = \phi_0^2 e^{-2a\phi_0}$. This picture is indeed confirmed by an elementary calculation to first order in perturbation theory. Since the width of the doublet is distributed according to the distribution

$$p(a) = \theta(a) \frac{n_1}{2\phi_0} e^{-\frac{n_1 a}{2\phi_0}}$$  \hspace{1cm} (4.10)

this implies that the number of such states per unit length is

$$N(E) \propto e^{-\frac{n_1}{2\phi_0} \ln \frac{\phi_0^2}{E}} = \left( \frac{E}{\phi_0^2} \right)^{\frac{n_1}{2\phi_0}}$$  \hspace{1cm} (4.11)

Since $\mu = \frac{n_1-n_0}{2\phi_0} \simeq \frac{n_1}{2\phi_0}$ this result is consistent with eq. (3.14).

The above discussion clearly illustrates that the basic mechanism which is responsible for the low energy behaviour is the existence of low energy states that would
have strictly zero energy if one could ignore the couplings between wells. This mechanism is very different from the one which is at work in the case of Lifchitz singularities. In this case, very small energy can occur only if large regions of space are free from impurities. This is just the opposite picture of what happens in our case.

This discussion can be generalized to the case $\phi_1 < 0 < \phi_0$ with $< \phi > \neq 0$. Assuming $0 < |\phi_1| < \phi_0$ and $n_0 = n_1 = n$, an exact calculation of the density of states gives again a power law behaviour

$$ N(E) \propto E^{\frac{n}{2} \left( \frac{1}{|\phi_1|} - \frac{1}{|\phi_0|} \right)} . \quad (4.12) $$

However the situation is completely different when $\phi_1 = 0$. In this case, a single doublet configuration of length $a \gg \frac{1}{\phi_0}$

$$ V(x) = \phi_0^2 [\theta(x-a) + \theta(-x)] - \phi_0 \delta(x) + \phi_0 \delta(x-a) \quad (4.13) $$

supports a low energy state

$$ E(a) \simeq \left( \frac{\pi}{2a} \right)^2 . \quad (4.14) $$

The dependence on the length $a$ is very different from (4.9), and (4.10) gives now a Lifchitz singularity

$$ N(E) \propto \exp \left( - \frac{\pi n_1}{2 \sqrt{E}} \right) . \quad (4.15) $$

This result is in agreement with an exact calculation of the density of states that we do not reproduce here.

For $0 < \phi_1 < \phi_0$, one can prove that $H_+$ is bounded from below by $\phi_1^2$; $N(E)$ starts now with a Lifchitz singularity at $\phi_1^2$

$$ N(E) = 0 \quad E < \phi_1^2 $$

$$ N(E) \propto \exp \left( - \frac{\pi n_1}{2 \sqrt{E - \phi_1^2}} \right) . \quad (4.16) $$

V - COMPUTER SIMULATIONS AND NUMERICAL CALCULATIONS
In this section, we consider various probability distributions of the lengths steps and discuss the density of states $N(E)$ and inverse localization length $\gamma(E)$ as obtained from simulations and/or numerical calculations.

The procedure to obtain the integrated density of states $N(E)$ by simulation for the model (III.1) is the following. We generate the random process $\{\phi\}$ with the laws (3.1) for the lengths of intervals where $\phi(x)$ is constant. Then for each energy $E$, we integrate the dynamical equation (2.5) for the total accumulated phase $\theta_E(x)$

$$\frac{d\theta_E}{dx} = k - \phi(x) \sin 2\theta_E(x)$$

and we finally compute $N(E)$ with (2.2)

$$N(E) = \lim_{L \to \infty} \left( \frac{\theta_E(L) - \theta_E(0)}{\pi L} \right).$$

$N(E)$ is a self averaging quantity and requires therefore only one configuration $\{\phi\}$ to be computed. We have generated typically $3 \times 10^5$ intervals, and we have checked with a great accuracy that the $N(E)$ obtained was indeed independent of the particular realization of the disorder $\{\phi\}$.

In order to have a better accuracy, we did not integrate the dynamical equation for $\theta(x)$ step by step, but we rather integrated it exactly on each interval where $\phi(x)$ is constant. For an interval of length $\ell$ on which $\phi(x) = +\phi_0$, the initial value $\theta_i$ and the final value $\theta_f$ of the phase $\theta(x)$ are related through

$$\ell = \int_{\theta_i}^{\theta_f} \frac{d\theta}{k - \phi_0 \sin 2\theta} \quad (5.1)$$

The integration requires to separate $k < \phi_0$ from $k > \phi_0$ (See appendix C).

In Fig. 3, we display $N(E)$ for $\phi_0 = -\phi_1 = 1$ and various values of $n = n_0 = n_1$. Notice that, as expected, $N(E)$ is a continuous function of $E$. However, it seems that, in general, its derivative is not.

For large $n$ (Fig.3a), $N(E)$ is close to the free case $\frac{CE}{\pi}$ except for very small $E$ (3.16). This is easily understood if we consider two successive steps of small lengths $(\approx \frac{1}{n})$ in
(2.5). After two steps the phase increase is

\[ d\theta_0 + d\theta_1 \approx (k - \phi_0 \sin 2\theta)dx_0 + (k + \phi_0 \sin 2\theta)dx_1 \approx kdx \]  

(5.2)

This leads to a quasi free density of states.

In contrast, small values of \( n \) (Fig.3d) lead to

i) a plateau \( N(E) \approx \frac{n}{2} \) if \( E < \phi_0^2 \)

ii) a parabolic behaviour \( N(E) \approx \frac{CE - \phi_0^2}{\pi} \) if \( E > \phi_0^2 \)

(5.3)

The behaviour i) can be explained by noting that, for small \( n \), \( \theta_E(L) - \theta_E(0) \sim N \pi^2 \).

Therefore after \( N \) steps, we get

\[ N(E) \approx \lim_{N,L \to \infty} \frac{1}{\pi L} \left( \frac{\pi}{2} N \right) = \frac{n}{2} \]  

(5.4)

For \( E > \phi_0^2 \), one can show (see Appendix C) that \( \theta(L) \approx \frac{\pi L}{\tau} \), hence we get ii)

\[ N(E) \approx \frac{1}{\tau} = \frac{CE - \phi_0^2}{\pi} \]

A quasi-free behaviour develops only for \( E > \phi_0^2 \).

Now we turn to the Lyapunov exponents and consider the ending angles \( \pi \)-periodic stationary distributions. \( W_0(\theta) \ d\theta \) (resp. \( W_1(\theta) \ d\theta \)) is the probability for the reduced phase to be in the interval \( [\theta, \theta + d\theta] \) when the system switches from state \( \phi_0 \) to state \( \phi_1 \) (resp. from \( \phi_1 \) to \( \phi_0 \)). For \( E > 0 \), (2.3) and (2.5) lead to

\[ \gamma(E) = -\frac{n_0 n_1}{2(n_0 + n_1)} \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln \left| \frac{CE - \phi_0 \sin 2\theta}{\sqrt{E - \phi_1 \sin 2\theta}} \right| \cdot [W_0(\theta) - W_1(\theta)]d\theta \]  

(5.5)

The stationarity condition implies the relationship

\[ W_0(\theta) = \int_0^\infty d\ell \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta' f_0(\ell) W_1(\theta') \delta(\theta - g(\ell, \theta')) \]  

(5.6)

(g deduced from (5.1)) and, after some algebra, the differential equations

\[ \frac{d}{d\theta}[W_0(\theta)(k - \phi_0 \sin 2\theta)] = n_0(W_1(\theta) - W_0(\theta)) \]  

(5.7)

\[ \frac{d}{d\theta}[W_1(\theta)(k - \phi_1 \sin 2\theta)] = n_1(W_0(\theta) - W_1(\theta)) \]
Comparison with (3.8) shows that distributions $W_{0,1}(\theta)$ and $P_{0,1}(\theta)$ are the same up to a normalization constant

$$P_{0,1} = \frac{n_{1,0}}{n_0 + n_1} W_{0,1} \quad (5.8)$$

In particular, we can demonstrate, via (5.8), the equivalence of the two calculations, (3.13) and (5.5), of the Lyapunov exponent $\gamma(E)$. (5.8) is somewhat unexpected and could be possibly related to the Poisson process we use for the step lengths.

Distributions $W_0(\theta)$ and $W_1(\theta)$ are plotted in Figs. 4 and 5 for two values of $n$. $W_0(\theta)$ (resp. $W_1(\theta)$) exhibits a peak at $\theta = \alpha_0$ (resp. $\theta = \alpha_1$), the magnitude of which is a decreasing function of $n$. For small $n$ (Fig.4), $W_0$ is essentially concentrated around $\alpha_0$. On the other hand, for large $n$ (Fig.5), $W_0$ becomes practically flat. (5.5) shows that, in this case, we must get $\gamma(E) \approx 0$ (free case). The small bump at $\theta = \alpha_1$ ($= \alpha_0 - \pi/2$ when $\phi_0 = -\phi_1$) is essentially a memory effect due to the preceding step ($\phi(x) = \phi_1$). With the aid of (3.13) and (5.5, 8), we could compute $\gamma(E)$ numerically as well as by simulations. Some results are displayed in Fig. 6 ($n_0 = n_1$) and Fig. 7 ($n_0 = 2n_1$). We clearly see that $\gamma(0) = |F_0|$ (2.10) and that $\gamma(E)$, with $E$ fixed, is a decreasing function of $n_0$ (at least for $E$ not too large). This is in agreement with the above discussion. The predicted asymptotic behaviour $\gamma(E) \approx E^{-1}$ (3.19) is reached for values that largely depend on $n_0$ and $n_1$. For instance, with $n_0 = 1 = 2n_1$, it is practically attained for $E \geq 2$. But when $n_0 = 2n_1 = 4$, values of $E \geq 20 - 30$ are needed.

Small $n_{0,1}$ values deserve a special comment. As already noticed, $W_0(W_1)$ distribution is peaked at $\alpha_0(\alpha_1)$. Approximating $P_{0,1}(\theta)$ by $\delta$-functions, (3.13) readily leads to ($\phi_0 = -\phi_1$)

$$\gamma(E) \approx C\phi_0^2 - E \quad , \quad E < \phi_0^2$$
$$\gamma(E) \approx 0 \quad , \quad E \geq \phi_0^2 \quad (5.9)$$

This is precisely what we get by means of computer simulations for $n_0 = 0.01 = 2n_1$. However, eq.(2.10) $\gamma(0) = |F_0|$ seems contradictory with (5.9). Investigating the limit $E \to 0^+$ numerically, we could see that very small $E$ values are needed ($\approx 10^{-3} - 10^{-4}$) to detect a decrease of $\gamma(E)$ towards its limit $|F_0|$. So, the behaviour of the Lyapunov exponent at small energy appears to be far from trivial.

We now consider step length distributions whose first moment diverges. More
precisely, exponential laws \( f_{0,1}(\ell) \) of eq.(3.1) are replaced by distributions who behave as

\[
\ell^{-(1+\alpha)} \quad \text{for} \quad \ell \to \infty \quad (0 < \alpha \leq 1) .
\] (5.10)

\( \alpha = 1 \) corresponds to a one-sided Cauchy law.

Detailed analytical calculations such as those performed in the preceding sections become hardly feasible. However, some simple characteristic features appear in the computer simulations. For instance, when \( \phi_0 = -\phi_1 \), the density of states, \( N(E) \), is given by

\[
N(E) \approx \theta(E - \phi_0^2) \frac{CE - \phi_0^2}{\pi} 
\] (5.11)

In fact, one can prove that result (5.11) is an exact one. Indeed, since the first moment of (5.10) is infinite, the sum \( L \), of \( N \) independent positive random variables all distributed according to (5.10) behaves, when \( N \to \infty \), like [23]

\[
L \sim N^{1/\alpha} , \quad \alpha < 1 \\
L \sim N \ln N , \quad \alpha = 1
\] (5.12)

More precisely, \( \frac{L}{N^{1/\alpha}} \) (or \( \frac{L}{N \ln N} \) if \( \alpha = 1 \)) is distributed according to a Lévy stable law.

Thus, for \( E < \phi_0^2 \), we get

\[
N(E) \lesssim \frac{1}{\pi} \cdot \frac{2^{N/\alpha}}{N^{1/\alpha}} \frac{C}{N} \to 0 \quad (\alpha < 1) \\
N(E) \lesssim \frac{1}{\pi} \cdot \frac{2^{N/\alpha}}{N \ln N} \frac{C}{N} \to 0 \quad (\alpha = 1)
\] (5.13)

Now we examine what happens for \( E > \phi_0^2 \) and call \( \ell_1, \ell_2, \ldots, \ell_N \) the lengths of the successive steps

\[
L = \ell_1 + \cdots + \ell_N \sim N^{1/\alpha} , \quad (\alpha < 1, \ N \to \infty)
\] (5.14)

According to (C.4), each \( \ell_i \) can be written as

\[
\ell_i = n_i \tau + \delta \ell_i , \\
0 < \delta \ell_i < \tau
\] (5.15)

and therefore

\[
L = \tau \left( \sum_i n_i \right) + \sum_i \delta \ell_i
\] (5.16)
The upper bound \( \sum_{i} \delta \ell_i < N \tau \) allows to neglect \( (\sum_{i} \delta \ell_i) \) in \( L \), and we finally get

\[
L = (n_1 + \cdots + n_N) \tau + O(N^{1/\alpha})
\]
\[
\theta(L) = (n_1 + \cdots + n_N) \pi + O(N^{1/\alpha})
\]
\[
N(E) = \frac{1}{\tau} = \frac{CE - \phi_0^2}{\pi} \quad E > \phi_0^2.
\]

Replacing \( N^{1/\alpha} \) by \( N \ln N \) in (5.14-17), we easily show that the final result (5.17) is still valid when \( \alpha = 1 \). Moreover, Thouless’s formula [2] implies that

\[
\gamma(E) = 0 \quad \text{when} \quad E > \phi_0^2
\]

Thus, the case \( \alpha < 1, \phi_0 = -\phi_1 \), is like a “free” one, up to the shift of \( \phi_0^2 \) in energy.

However, simulations with \( \phi_0 \neq -\phi_1 \) reserve some surprise. In Fig. 8, three different events drawn with the same law, \( \alpha = 0.5 \) are displayed. We now notice that \( N(E) \) is no longer a self-averaging quantity. In (5.14) splitting \( L \) into \( L_0 \) (and \( L_1 \)), the sum of all the steps lengths where \( \phi(x) = \phi_0 \) (or \( \phi_1 \)) and observing that \( L_0 \) and \( L_1 \) are both of the same order (\( N^{1/\alpha} \) or \( N \ln N \)), we get, in analogy with (5.14-17)

\[
N(E) = \frac{L_0 \theta(E - \phi_0^2)CE - \phi_0^2 + L_1 \theta(E - \phi_1^2)CE - \phi_1^2}{\pi(L_0 + L_1)}, \quad (5.19)
\]

in perfect agreement with our simulations. Of course, \( L_0 \) and \( L_1 \) now strongly depend on each drawing. It is interesting to point out that

i) \( N(E) \sim \frac{CE}{\pi} \) when \( E \to \infty \)

ii) \( \phi_0^2 = \phi_1^2 \) in (5.19) leads to \( N(E) = \frac{\theta(E - \phi_0^2)CE - \phi_0^2}{\pi} \), i.e. \( N(E) \) is again self-averaging.

To conclude this section, we mention that nothing special seems to happen when \( 1 < \alpha < 2 \). In particular, (5.11, 19) are not observed and the general behaviour of \( N(E) \) is quite similar to the one we obtained with the exponential law.

VI - SOME PHYSICAL APPLICATIONS

In this section we describe some physical problems to which the above model can be applied. As mentioned in the introduction, one of the most interesting application is the description of classical diffusion in a one dimensional random medium. In the
presence of a force $\phi(x)$, the probability density $P(x \mid y \, 0)$ satisfies the Fokker-Planck equation
\[
\frac{\partial P}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\partial P}{\partial x} - 2\phi(x)P \right)
\] (6.1)
supplemented by the initial condition $P(x \mid y \, 0) \to \delta(x - y)$

As we have already explained in the introduction, the transformation
\[
P(x \mid y \, 0) = e^{\int_y^x \phi(z) \, dz} G_S(x \mid y \, 0)
\] (6.2)
allows to rewrite (6.1) as the Schrödinger problem
\[
\begin{cases}
\frac{\partial G_S}{\partial t} = -H_+ G_S \\
G_S(x \mid y \, 0) \to \delta(x - y)
\end{cases}
\] (6.3)
where $H_+ = -\frac{d^2}{dx^2} + \phi^2(x) + \phi'(x)$ is the Hamiltonian whose properties have been discussed in the preceding sections.

The Laplace transform $\hat{G}_S(x, y; E) = \int_0^{\infty} dt \, e^{-Et} G_S(x \mid y \, 0)$ satisfies
\[
(E + H_+) \hat{G}_S(x, y; E) = \delta(x - y)
\] (6.4)
The Laplace transform $\hat{P}(x, y; E) = \int_0^{\infty} dt \, e^{-Et} P(x \mid y \, 0)$ therefore reads
\[
\hat{P}(x, y; E) = e^{\int_y^x \phi(z) \, dz} < x \mid \frac{1}{H_+ + E} \mid y >
\] (6.5)

When increasing the mean bias $< \phi >$, it is known that one obtains a succession of phases [10] characterized by different anomalous diffusive behaviour. The characterization of these phases requires the knowledge of the velocity and the diffusion constant. Here we just consider the velocity given by [8]
\[
\frac{1}{V} = \lim_{E \to 0^+} < \hat{P}(x, x; E) >
\] (6.6)

In order to express $V$ in terms of the localization problem, we consider the average of the resolvant at coinciding points
\[
\lim_{L \to \infty} \frac{1}{L} \int_0^L dx < x \mid \frac{1}{H_+ - \mathcal{E} - i\epsilon} \mid x > = -\frac{d}{d\mathcal{E}} \gamma(\mathcal{E}) + i\pi \rho(\mathcal{E})
\] (6.7)
For negative $E$, the density of states $\rho(E)$ vanishes and (6.7) reduces for $E > 0$ to

$$\lim_{L \to \infty} \frac{1}{L} \int_0^L dx < x | \frac{1}{H_+ + E} | x > = \frac{d}{dE} \gamma(-E)$$  \hspace{1cm} (6.8)

The velocity is therefore given in terms of the left derivative of the Lyapunov exponent at zero energy

$$\frac{1}{V} = \lim_{E \to 0^+} < \hat{P}(x, x; E) >= \lim_{E \to 0^+} \frac{d}{dE} \gamma(-E)$$  \hspace{1cm} (6.9)

For the model of rectangular barriers of random lengths described in (III.1) we have obtained the Lyapunov exponent in the negative energy region in (3.29). It follows that the velocity displays a transition at $\mu = 1$

$$\begin{cases} 
V = 0 \text{ if } |\mu| < 1 \\
V = \frac{2\sigma(|\mu| - 1)}{1 - \frac{\sigma^2}{\phi_0^2} |\mu|} \text{ if } |\mu| > 1
\end{cases}$$  \hspace{1cm} (6.10)

In the white noise limit (Eq. 3.6), we recover the result [8]

$$\begin{cases} 
V = 0 \text{ if } |\mu| < 1 \\
V = 2\sigma(|\mu| - 1) \text{ if } |\mu| > 1
\end{cases}$$  \hspace{1cm} (6.11)

It is interesting to rederive this result by performing a direct configurational average in (6.6). The zero energy Green function at coinciding points $P(x, x; 0^+)$ can be written for any configuration of the force field $\{\phi(x)\}$

$$P(x, x; 0^+) = 2 \int_0^{+\infty} d\xi \ e^{-2\int_x^\xi \phi(u)du}$$  \hspace{1cm} (6.12)

provided one assumes $F_0 > 0$ [8]. Averaging over the disorder $\{\phi(x)\}$ gives

$$\frac{1}{V} = 2 \int_0^{+\infty} dL < e^{-2\int_0^L \phi(u)du} >$$  \hspace{1cm} (6.13)

For the two-step model (III.1), $\phi(x)$ can only take two values ($\pm \phi_0$), therefore

$$\int_0^L \phi(u)du = \phi_0 \ A(L) - \phi_0[L - A(L)]$$  \hspace{1cm} (6.14)

where $A(L)$ is the random variable that measures the total length in the state $\phi(x) = +\phi_0$ during the interval $[0, L]$.  

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The velocity $V$ can thus be expressed in terms of the probability density $\mathcal{P}_L(x)dx = Pr(x < A(L) < x + dx)$. One obtains

$$\frac{1}{V} = 2 \int_0^\infty dL < e^{2\phi_0 L - 4\phi_0 A(L)} >$$

$$= 2 \int_0^\infty dL e^{2\phi_0 L} \int_0^L dx e^{-4\phi_0 x} \mathcal{P}_L(x) . \tag{6.15}$$

From the expression of $\mathcal{P}_L(x)$ given in Appendix D one recover (6.10) after some tedious calculations.

Another interesting application of this model is the fact that it can be used to discuss the spectral properties of the two dimensional Euclidean Dirac operator in a special type of random magnetic field. Consider the two dimensional Euclidean Dirac operator

$$i\mathcal{D} = i\sigma_1 \left( \frac{\partial}{\partial t} + iA_t \right) + i\sigma_2 \left( \frac{\partial}{\partial x} + iA_x \right) \tag{6.16}$$

coupled to the external field

$$\begin{cases}
A_t = f(x) \\
A_x = 0
\end{cases} \tag{6.17}$$

If we set $\psi(x, t) = \exp i\omega t \chi(x)$ where $\chi(x) = (v(x), u(x))$ is a two component spinor, the eigenvalue equation

$$i\mathcal{D}\psi = k\psi . \tag{6.18}$$

then reads

$$\begin{cases}
\frac{du}{dx} - u\phi = kv \\
-\frac{dv}{dx} - v\phi = ku
\end{cases} \tag{6.19}$$

where

$$\phi(x) = f(x) + \omega \tag{6.20}$$

After decoupling one obtains

$$\begin{pmatrix}
H_- & 0 \\
0 & H_+
\end{pmatrix} \chi = k^2 \chi \tag{6.21}$$

The knowledge of the density of states of the one dimensional model therefore permits to evaluate the density of states in the quasi one dimensional random magnetic field

$$B = \frac{\partial A_t}{\partial x} - \frac{\partial A_x}{\partial t} = f'(x) \tag{6.22}$$
In a previous paper [24] we have shown that the low energy part of $\rho(E)$ is enhanced with respect to the free case. Although the analytical expressions are of course model dependent, the enhancement effect is expected to occur on very general grounds.

The type of disorder that we have introduced in section III can be motivated by the following picture. Imagine that the potential $A_t(x)$ is created by a line of dipoles of same strength distributed randomly on the $x$ axis with alternating signs $+2f_0, -2f_0, +2f_0, -2f_0 \cdots$. The resulting potential $f(x) = -f_0 \sum_i \epsilon(x - y_i)$ is an ensemble of square functions such that each sample function can only take the value $\pm f_0$. If one assumes further that the probability to find $m$ dipoles on an interval of length $x$ is given by the Poisson process

$$p(m, x) = \frac{(nx)^m}{m!} \exp(-nx)$$

(6.23)

then $f(x)$ is a random telegraph process. After some calculations, very similar to the one presented in [24] one obtains

$$N(E) = \frac{1}{\pi} \int_{-\infty}^{+\infty} N(\phi_0 = f_0 + \omega, \phi_1 = -f_0 + \omega) \, d\omega \propto \frac{1}{E \to 0^+} \left(-\ln \frac{E}{f_0^2}\right)^3$$

(6.24)

Similar logarithmic behaviours have been obtained by a variational method in the context of the Schwinger model [25]. However a direct comparison with our result is not possible because the gauge field measure is not the same.

Interestingly enough, this two dimensional Dirac operator can also model other systems. For instance it arises in the study of the electronic properties of polyacetylene. In this context $\phi(x)$ is proportional to the dimerization pattern of the carbon-hydrogen chain [26,27]. Introduction of the disorder can of course influence the nature of the ground state [28]. It can also model certain layered structures with a piecewise constant order parameter. Application of these ideas to Fermi superfluids was recently discussed [14].

Another potential application of this model is in the field of dynamical systems perturbed by noise. A large amount of works have been devoted to the study of two
dimensional linear stochastic systems of the form

\[ \dot{x}(t) = Ax(t) + \eta(t)Bx(t) \]  \hspace{1cm} (6.25)

where A and B are 2x2 matrices, \( \eta(t) \) a random noise and \( x(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \) is a two dimensional vector. In order to understand how the linear system \( \dot{x}(t) = Ax(t) \) is perturbed by the noise in the asymptotic regime \( t \to \infty \) one introduces the Lyapunov exponent

\[ \lambda = \lim_{t \to \infty} \frac{1}{t} \log ||x(t)|| \]  \hspace{1cm} (6.26)

which characterizes the stability of the solution and the rotation number

\[ \rho = \lim_{t \to \infty} \frac{1}{t} \arctg \frac{x_2(t)}{x_1(t)} \]  \hspace{1cm} (6.27)

Although there are general theorems regarding the existence of such limits, in most cases one cannot compute them analytically. Therefore one has to resort to perturbative methods (for instance small noise expansions) [29].

The Dirac equation (6.19) can be written in this canonical form with the identification

\[ A = CE \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]  \hspace{1cm} (6.28)

A more physical picture is to consider that \( x(t) = \begin{pmatrix} p \\ q \end{pmatrix} \) characterizes the two dimensional phase space of an harmonic oscillator perturbed by a random telegraph process. This linear system (6.25) can be put into an hamiltonian form with

\[ H = \frac{CE}{2} (p^2 + q^2) - \phi(t)pq \]  \hspace{1cm} (6.28)

From the discussion in section III it follows that \( \lambda \) and \( \rho \) are nothing but the inverse localization length and density of states of the localization problem.

**VII - CONCLUSION**

In this work we have studied a model of localisation based on the Witten Hamiltonian of supersymmetric quantum mechanics. This model exhibits interesting behaviour
of the density of states at low energy. We have shown how this behaviour can be understood through simple physical arguments. In the case $\mu = 0$, it is the existence of quasi-zero modes that is responsible for the logarithmic singularity, whereas for $\mu \neq 0$ it is the formation of low energy bound states that explains the power law behaviour. These mechanisms rely of course crucially on the supersymmetric structure of the potential. The resulting expressions are qualitatively similar to the ones that had been obtained before in the case of a white noise. In that respect, the existence of a finite correlation length does not make much difference. In contrast, when the lengths of the barriers are drawn with a distribution with infinite moments, we have demonstrated, analytically as well as numerically, that the density of states develops a very different behaviour. A similar observation was recently made in the context of anomalous diffusion. It was shown in [30] that the asymptotic regime can be qualitatively different in the presence of correlated transfert rates (see also [12]). An extension of this model in higher dimension is certainly worth pursuing. In particular it would be interesting to extend the zero mode analysis to the case of the random magnetic field problem.

**APPENDIX A : CALCULATION OF $N(E)$ AND $\gamma(E)$ FOR $E > 0$**

The determination of $N(E)$ and $\gamma(E)$ is reduced to the resolution of the equations (3.11) for $P_0(\theta)$ and $P_0(\theta)$, which contain the expressions $(k - \phi_0 \sin 2\theta)$ and $(k - \phi_1 \sin 2\theta)$. The solutions will be different if these expressions can vanish or not for some values of $\theta$, and it is therefore necessary to separate the cases $E > \phi_0^2$ from $E < \phi_0^2$ and $E < \phi_1^2$ from $E > \phi_1^2$.

In order to simplify the discussion, we will set $\phi_1 = -\phi_0$, in which case the spectrum is only divided into two regions $0 < E < \phi_0^2$ and $E > \phi_0^2$.

\begin{align*}
a) & \text{ Results for } 0 < E < \phi_0^2 \left( \begin{array}{l}
\text{in the case } \\
\begin{cases} 
\phi_0 > 0 \\
\phi_1 = -\phi_0
\end{cases}
\end{array} \right) \\
& \text{We introduce the angle } \alpha \in [0, \frac{\pi}{4}] \text{ such that } \sin 2\alpha = C \frac{E}{\phi_0} \text{ and the function}
\end{align*}
\[ q(\theta) = \left| \frac{\cos(\theta-\alpha)}{\sin(\theta+\alpha)} \right|^{\nu_1} \left| \frac{\sin(\theta-\alpha)}{\cos(\theta+\alpha)} \right|^{\nu_0} \] where \( \nu_1 = \frac{n_1}{2\phi_0 \cos 2\alpha} \) and \( \nu_0 = \frac{n_0}{2\phi_0 \cos 2\alpha} \).

In this region of the spectrum the resolution of (3.11) requires some care since the expressions \( k \pm \phi_0 \sin 2\theta \) can vanish and this is why the \( \pi \)-periodic solutions \( P_0(\theta) \) and \( P_1(\theta) \) are defined by two different expressions on the intervals \([\alpha, \alpha + \pi/2] \) and \([\alpha + \pi/2, \alpha + \pi]\):

\[
\begin{aligned}
\alpha \leq \theta \leq \pi/2 + \alpha & \quad P_0(\theta) = \frac{n_1}{2\phi_0} N(E) \frac{q(\theta)}{\cos(\alpha+\theta) \sin(\alpha-\theta)} \int_{\pi/2}^{\pi/2+\alpha} \frac{dt}{\cos(\alpha-t) \sin(\alpha+t) q(t)} \\
\pi/2 + \alpha \leq \theta \leq \alpha + \pi & \quad P_0(\theta) = \frac{n_1}{2\phi_0} N(E) \frac{q(\theta)}{\cos(\alpha+\theta) \sin(\alpha-\theta)} \int_{e_\pi}^{\pi/2+\alpha} \frac{dt}{\cos(\alpha-t) \sin(\alpha+t) q(t)} \quad (A.1)
\end{aligned}
\]

\[
\begin{aligned}
\alpha \leq \theta \leq \pi/2 + \alpha & \quad P_1(\theta) = \frac{n_0}{2\phi_0} N(E) \frac{q(\theta)}{\cos(\alpha-\theta) \sin(\alpha+\theta)} \int_{\pi/2}^{\pi/2+\alpha} \frac{dt}{\cos(\alpha-t) \sin(\alpha+t) q(t)} \\
\pi/2 + \alpha \leq \theta \leq \alpha + \pi & \quad P_1(\theta) = \frac{n_0}{2\phi_0} N(E) \frac{q(\theta)}{\cos(\alpha-\theta) \sin(\alpha+\theta)} \int_{\pi}^{\pi/2+\alpha} \frac{dt}{\cos(\alpha-t) \sin(\alpha+t) q(t)} \quad (A.2)
\end{aligned}
\]

The functions \( P_0(\theta) \) and \( P_1(\theta) \) are presented on Figs. 4 and 5 for a certain choice of the parameters \( \phi_0, n_0, n_1 \).

The normalization condition \( \int_{\alpha}^{\alpha+\pi} P_0(\theta) d\theta = \frac{2\phi_0}{n_0+n_1} \) gives the density of states \( N(E) \)

\[ N(E) = \frac{2\sigma}{I_1 + I_2} \quad (A.3) \]

where \( I_1 \) and \( I_2 \) are double integrals that read after some transformations

\[
\begin{aligned}
I_1 &= (1 + \beta^2) \int_{0}^{\infty} dx \left[ \frac{x^{2\nu_0-1}}{(x + \beta^2)^{\nu_1}} \right] \frac{1}{|x - 1|^{\nu_0}} \frac{1}{(x - 1)} \int_{1}^{x} dy \left[ \frac{(y + \beta^2)^{\nu_1-1}}{y^{\nu_0}} \right] |y - 1|^{\nu_0} \\
I_2 &= (1 + \beta^2) \int_{0}^{\infty} dx \left[ \frac{x^{\nu_1}}{(x + \beta^2)^{\nu_0+1}} \right] \frac{1}{|x - 1|^{\nu_1}} \int_{1}^{x} dy \left[ \frac{(y + \beta^2)^{\nu_0}}{y^{\nu_1+1}} \right] |y - 1|^{\nu_1} \quad (A.4)
\end{aligned}
\]

We have set \( \sigma = \frac{2\phi_0^2}{n_0+n_1} \) and \( \beta = \tan 2\alpha = \frac{C_E}{C_{\phi_0^2} - E} \).

According to (3.13) the Lyapunov exponent \( \gamma(E) \) reads

\[ \gamma(E) = \phi_0 \int_{\alpha}^{\pi+\alpha} d\theta \cos 2\theta[P_0(\theta) - P_1(\theta)] \]

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From the expressions (A.1) and (A.2) for the stationary distributions \( P_0(\theta) \) and \( P_1(\theta) \) in the interval \( 0 < E < \phi_0^2 \) we obtain after some transformations

\[
\gamma(E) = N(E)[J_1 - J_2] \tag{A.5}
\]

where

\[
J_1 = \frac{1}{2} \int_0^\infty dx \left( \frac{x^2 + \beta^2(2x - 1)}{x^2 + \beta^2} \right) \frac{1}{(x + \beta^2)^\nu_1} \left| \frac{1}{x} - 1 \right|^{\nu_0} \int_1^x dy (y + \beta^2)^\nu_1 \left| \frac{1}{y} - 1 \right|^{\nu_0} \left[ \frac{\nu_1}{x(x - 1) y + \beta^2} - \frac{\nu_0}{x + \beta^2} \right] y(y - 1)
\]

and \( J_2 \) is obtained from \( J_1 \) by the exchange between \( \nu_1 \) and \( \nu_0 \).

\[ \text{b) Results for } E > \phi_0^2 \]

Here we use the parameter \( \frac{1}{\gamma} = C \frac{E}{\phi_0^2} > 1 \) and the function \( \tilde{q}(\theta) = \exp \left( -\alpha_1 \arctan \left( \frac{\tan \theta + \gamma}{C1 - \gamma_2} \right) - \alpha_0 \arctan \left( \frac{\tan \theta - \gamma}{C1 - \gamma_2} \right) \right) \) with \( \alpha_1 = \frac{\gamma}{C1 - \gamma^2} \frac{n_1}{\phi_0} \) and \( \alpha_0 = \frac{\gamma}{C1 - \gamma^2} \frac{n_0}{\phi_0} \), and where \( \arctan \) stands for the principal determination of \( \tan^{-1} \) taking values in \([ -\frac{\pi}{2}, \frac{\pi}{2} ] \).

The solutions \( P_0(\theta) \) and \( P_1(\theta) \) of (3.11) read for \( \theta \in [ -\frac{\pi}{2}, \frac{\pi}{2} ] \)

\[
\begin{align*}
P_0(\theta) &= \frac{n_1}{\phi_0} N(E) \frac{\tilde{q}(\theta)}{\frac{1}{\gamma} - \sin 2\theta} \left[ \int_0^\theta \frac{dt}{(\frac{1}{\gamma} + \sin 2t)\tilde{q}(t)} + A \right] \\
P_1(\theta) &= \frac{n_0}{\phi_0} N(E) \frac{\tilde{q}(\theta)}{\frac{1}{\gamma} + \sin 2\theta} \left[ \int_0^\theta \frac{dt}{(\frac{1}{\gamma} - \sin 2t)\tilde{q}(t)} + B \right]
\end{align*}
\tag{A.7}
\]

where the constants \( A \) and \( B \) are chosen to ensure the \( \pi \) periodicity of \( P_0(\theta) \) and \( P_1(\theta) \)

\[
A = \frac{1}{1 - e^{-(\alpha_0 + \alpha_1)\pi}} \int_{-\frac{\pi}{2}}^0 \frac{dt}{(\frac{1}{\gamma} + \sin 2t)\tilde{q}(t)} + \frac{1}{e^{(\alpha_0 + \alpha_1)\pi} - 1} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dt}{(\frac{1}{\gamma} + \sin 2t)\tilde{q}(t)}
\]

and \( B \) is obtained from \( A \) by replacing \( (\frac{1}{\gamma} + \sin 2t) \) by \( (\frac{1}{\gamma} - \sin 2t) \) in the integrals.

\( N(E) \) is obtained by the normalization condition \( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta P_0(\theta) = \frac{n_1}{n_0 + n_1} \) and \( \gamma(E) \) by the expression

\[
\gamma(E) = \phi_0 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \cos 2\theta \left( P_0(\theta) - P_1(\theta) \right)
\]
Let us now consider the limit $E \to \phi_0^2$. $N(E)$ is given by two very different expressions for $E < \phi_0^2$ and $E > \phi_0^2$ but it is nevertheless continuous at the point $E = \phi_0^2$, as it should by definition of the integrated density of state $s$. $N(\phi_0^2 + 0) = N(\phi_0^2 - 0) = \frac{2\sigma}{I_1 + I_2}$ where

$$\begin{align*}
I_1 &= \int_0^\infty dx \ e^{-\frac{x}{\mu_1} + \mu_0 x} \ \int_x^\infty \frac{dy}{y} \ e^{\frac{\mu_1}{y} - \mu_0 y} \\
I_2 &= \int_0^\infty dx \ e^{\frac{x}{\mu_1} - \mu_0 x} \ \int_0^x \frac{dy}{y^2} \ e^{-\frac{\mu_1}{y} + \mu_0 y}
\end{align*}$$  \hspace{1cm} (A.8)

However one should not expect to have more than continuity of $N(E)$ at this point. In particular the derivative $\rho(E) = \frac{dN}{dE}$ has no reason to be continuous at this point (See V).

**APPENDIX B : THE LYAPUNOV EXPONENT FOR $E < 0$**

Here the dynamical equation for the phase $\theta$ is given in (2.21)

$$\frac{d\theta}{dx} = -k \cos 2\theta(x) - \phi(x) \sin 2\theta(x) \hspace{1cm} (B.1)$$

We have already seen in the discussion following (2.21) that $\theta(x)$ is trapped in an interval of the form $[-\frac{\pi}{2}, 0]$ [modulo $\pi$].

For the two step model \( \phi_1 = -\phi_0 \) defined in (III.1), a more accurate discussion shows that the phase $\theta$ is trapped in the interval $[\alpha_1, \alpha_0]$ [modulo $\pi$] where $\alpha_0 \in ]-\frac{\pi}{4}, 0[$ is the angle defined by

$$\sin 2\alpha_0 = \frac{-k}{Ck^2 + \phi_0^2} \hspace{1cm} (B.2)$$

and $\alpha_1 = -\frac{\pi}{2} - \alpha_0$.

Indeed on an interval where $\phi(x) = \phi_0$ the phase $\theta(x)$ is attracted towards $\alpha_0$ according to $\frac{d\theta}{dx} = -C\phi_0^2 + k^2 \sin(2\theta - 2\alpha_0)$ and on the interval $\phi(x) = -\phi_0$, the phase $\theta(x)$ is attracted towards $\alpha_1$ according to $\frac{d\theta}{dx} = -C\phi_0^2 + k^2 \sin(2\theta - 2\alpha_1)$

The stationary distributions $P_0(\theta)$ and $P_1(\theta)$ have therefore their support on the interval $[\alpha_1, \alpha_0]$.  

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Using the relation (3.28), the system (3.27) can be rewritten as two distinct equations for $P_0(\theta)$ and $P_1(\theta)$, whose solutions are for $\theta \in [\alpha_1, \alpha_0]$

\[
\begin{align*}
P_0(\theta) &= A \frac{q(\theta)}{\sin(2\theta - 2\alpha_0)} \\
P_1(\theta) &= A \frac{q(\theta)}{\sin(2\theta - 2\alpha_1)}
\end{align*}
\]

(B.3)

where

\[
\begin{align*}
q(\theta) &= |\tan(\theta - \alpha_0)|^{\nu_0} |\tan(\theta - \alpha_1)|^{\nu_1} \\
\nu_0 &= \frac{n_0}{2C\phi_0^2 + k^2} \\
\nu_1 &= \frac{n_1}{2C\phi_0^2 + k^2}
\end{align*}
\]

(B.4)

The constant $A$ is determined by the normalization condition

\[
\int_{\alpha_1}^{\alpha_0} d\theta \; P_0(\theta) = \frac{n_1}{n_0 + n_1}
\]

(B.5)

The expression (3.24) for the Lyapunov exponent $\gamma(E)$

\[
\gamma(E) = \lim_{L \to \infty} \frac{1}{L} \int_0^L \! dx \left[ -k \sin 2\theta(x) + \phi(x) \cos 2\theta(x) \right]
\]

can now be rewritten in terms of the stationary distributions $P_0(\theta)$ and $P_1(\theta)$

\[
\gamma(E) = \int_{\alpha_1}^{\alpha_0} \! d\theta \left[ -k \sin 2\theta(P_0(\theta) + P_1(\theta)) + \phi_0 \cos 2\theta(P_0(\theta) - P_1(\theta)) \right]
\]

\[
= C\phi_0^2 + k^2 \int_{\alpha_1}^{\alpha_0} \! d\theta \left[ \cos(2\theta - 2\alpha_0)P_0(\theta) + \cos(2\theta - 2\alpha_1)P_1(\theta) \right]
\]

(B.6)

APPENDIX C: PHASE EVOLUTION ON AN INTERVAL WHERE $\phi(x) = \phi_0$

The integration of (5.1) requires to separate $k < \phi_0$ from $k > \phi_0$.

• For $k < \phi_0$ we introduce $\alpha_0 \in ]0, \frac{\pi}{2}[$ satisfying $2\alpha_0 = \frac{k}{\phi_0}$. 

\[
\begin{align*}
\frac{d\theta}{dx} > 0 &\iff \theta \ [\text{modulo } \pi] \in ]-\frac{\pi}{2} - \alpha_0, \alpha_0 [ \\
\frac{d\theta}{dx} < 0 &\iff \theta \ [\text{modulo } \pi] \in ]\alpha_0, \frac{\pi}{2} - \alpha_0 [ 
\end{align*}
\]

Therefore $|\theta_f - \theta_i|$ cannot exceed $\pi$.

More precisely, if $(\theta_i + \alpha_0)$ belongs to the interval $]-\frac{\pi}{2} - n\pi, \frac{\pi}{2} + n\pi|$ with $n \in \mathbb{N}$ then
\((\theta_f + \alpha_0)\) stays within the same interval, and the corresponding tangents are related through

\[
\tan(\theta_f + \alpha_0) - \tan 2\alpha_0 = [\tan(\theta_i + \alpha_0) - \tan 2\alpha_0]e^{-2\ell \sqrt{\phi_0^2 - E}} \tag{C.1}
\]

- For \(k > \phi_0\) the increment \((\theta_f - \theta_i)\) is not limited.

The local primitive of the function involved in (5.1) reads

\[
\int_{\phi}^{\theta} \frac{d\theta}{k - \phi_0 \sin 2\theta} = \frac{1}{\sqrt{E - \phi_0^2}} \arctan \left( \frac{\tan \phi - \gamma_0}{\sqrt{1 - \gamma_0^2}} \right) \quad \text{where} \quad \gamma_0 = \frac{\phi_0}{k}
\]

An increment of \(\pi\) for the phase \(\theta\) requires therefore the length \(\tau\) given by

\[
\tau = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{k - \phi_0 \sin 2\theta} = \frac{\pi}{CE - \phi_0^2} \tag{C.2}
\]

If we denote by \(m \in \mathbb{N}\) the integer part of \(\left(\frac{L}{\tau}\right)\), the final phase reads

\[
\theta_f = \theta_i + m\pi + \varphi \tag{C.3}
\]

where the angle \(\varphi \in [0, \pi]\) is determined by

\[
\ell = m\tau + \frac{1}{\sqrt{E - \phi_0^2}} \left[ \arctan \left( \frac{\tan \theta_i - \gamma_0}{\sqrt{1 - \gamma_0^2}} \right) \right]_{\theta_i}^{\theta_i + \varphi} \tag{C.4}
\]

Here \(\arctan\) is a continuous determination of \(\tan^{-1}\), and therefore

\[
\left[ \arctan \left( \frac{\tan \theta_i - \gamma_0}{\sqrt{1 - \gamma_0^2}} \right) \right]_{\theta_i}^{\theta_i + \varphi} \in [0, \pi] \quad \text{for} \quad \varphi \in [0, \pi].
\]

**APPENDIX D : PROBABILITY DENSITY \(P_L(x)\)**

Let \(\{\phi(x)\}\) be a random process which alternates in 2 states \(\phi(x) = \phi_0\) and \(\phi(x) = -\phi_0\). We denote by \(l_n \ (h_n)\) the successive sojourn lengths in state \(\phi_0\) (resp. \(-\phi_0\)). We are interested in the distribution of the random variable \(A(L)\) which measures the total length in state \(\phi_0\) on the interval \([0, L]\) (see for instance [31]). The total distance in the state \(\phi(x) = -\phi_0\) on \([0, L]\) is then simply \([L - A(L)]\).

The probability density \(P_L(x)\) such that

\[
P_L(x)dx = Pr\{x < A(L) < x + dx\}
\]
naturally splits into four terms according to the values of $\phi$ at the end points $\phi(0) = \pm \phi_0$ and $\phi(L) = \pm \phi_0$. In obvious notations one has

$$\mathcal{P}_L(x) = \mathcal{P}_L^{++}(x) + \mathcal{P}_L^{+-}(x) + \mathcal{P}_L^{-+}(x) + \mathcal{P}_L^{--}(x) \quad (D1)$$

Let us consider the case $\phi(0) = \phi_0$, $\phi(L) = \phi_0$. A typical configuration of this type is represented in the following figure.

The probability density $\mathcal{P}_L^{++}(x)$ therefore reads

$$\mathcal{P}_L^{++}(x) dx = \Pr(\phi(0) = \phi_0) \times \sum_{N=0}^{\infty} \Pr \left( L - x - dx \leq \sum_{i=1}^{N} h_i \leq L - x \right) \Pr \left( \sum_{j=1}^{N} \ell_j \leq x \leq \sum_{j=1}^{N+1} \ell_j \right) \quad (D2)$$

The probability involved in this expression can be obtained easily. We have already seen that (3.3)

$$\Pr(\phi(0) = \phi_0) = \frac{n_1}{n_0 + n_1}$$

Each $\ell_i$ is distributed according to the probability density (3.1)

$$f_0(\ell) = \theta(l) n_0 e^{-n_0 \ell}$$

whose Fourier transform reads

$$\hat{f}_0(p) \equiv \int_{-\infty}^{+\infty} dx \ f_0(x) e^{-ipx} = \frac{n_0}{n_0 + ip}$$
The total length \( y = \sum_{i=1}^{N} \ell_i \) is a sum of independent identically distributed random variables. It is distributed with the probability density
\[
f_0^{\{N\}}(y) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ipy} \left[ \hat{\gamma}(p) \right]^{N} = \theta(y) n_0 \frac{(n_0y)^{N-1}}{(N-1)!} e^{-n_0y}
\] (D3)

The probability \( Q_0^{\{N\}}(x) \) to have \( \sum_{i=1}^{N} \ell_i \leq x \leq \sum_{i=1}^{N+1} \ell_i \) then reads
\[
Q_0^{\{N\}}(x) = Pr \left( \sum_{i=1}^{N} \ell_i \leq x \right) - Pr \left( \sum_{i=1}^{N+1} \ell_i \leq x \right) = \int_{0}^{x} dy \left[ f_0^{\{N\}}(y) - f_0^{\{N+1\}}(y) \right]
\]
\[
= \theta(x) \frac{(n_0x)^{N}}{N!} e^{-n_0x}.
\] (D4)

We also define \( f_1^{\{N\}}(x) \) and \( Q_1^{\{N\}}(x) \) by replacing \( n_0 \) by \( n_1 \)

\( f_1^{\{N\}}(x) \) is the probability density to have \( \sum_{i=1}^{N} h_i = x \)

\( Q_1^{\{N\}}(x) \) is the probability to have \( \sum_{i=1}^{N} h_i \leq x \leq \sum_{i=1}^{N+1} h_i \)

Eq. (D2) becomes
\[
P_0^{++}(x) = \frac{n_1}{n_0 + n_1} \sum_{N=0}^{\infty} f_1^{\{N\}}(L-x) Q_0^{\{N\}}(x)
\] (D5)

In a similar way we can compute \( P_L^{+-}(x) \) such that
\[
P_L^{+-}(x) dx = Pr \{ x < A(L) < x + dx \mid \phi(0) = \phi_0, \phi(L) = -\phi_0 \}
\]

by considering configurations represented in the following figure.
We get
\[ \mathcal{P}_L^{+ -}(x) dx = Pr(\phi(0) = \phi_0) \sum_{N=0}^{\infty} Pr \left[ x \leq \sum_{i=1}^{N+1} \ell_i \leq x + dx \right] Pr \left[ \sum_{i=1}^{N} h_i \leq L - x \leq \sum_{i=1}^{N+1} h_i \right] \]
\[ = \frac{n_1}{n_0 + n_1} \sum_{N=0}^{\infty} f_0^{\{N+1\}}(x) Q_1^{\{N\}}(L - x) \] (D6)

We can now obtain \( \mathcal{P}_L^{+ +}(x) \) [resp. \( \mathcal{P}_L^{- -}(x) \)] from \( \mathcal{P}_L^{+ -}(x) \) [resp. \( \mathcal{P}_L^{- +}(x) \)] by the simple exchanges \( n_0 \leftrightarrow n_1, x \leftrightarrow L - x \) so that the probability density \( \mathcal{P}_L(x) \) (A1) finally reads
\[ \mathcal{P}_L(x) = \frac{n_1}{n_0 + n_1} \sum_{N=0}^{\infty} \left[ f_1^{\{N\}}(L - x) Q_0^{\{N\}}(x) + f_0^{\{N+1\}}(x) Q_1^{\{N\}}(L - x) \right] \]
\[ + \frac{n_0}{n_0 + n_1} \sum_{N=0}^{\infty} \left[ f_0^{\{N\}}(x) Q_1^{\{N\}}(L - x) + f_1^{\{N+1\}}(L - x) Q_0^{\{N\}}(x) \right] \] (D7)

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REFERENCES

[1] P.W. Anderson, *Phys. Rev.* 109 (1958) 1492.

[2] B. Souillard, ”Les Houches”, XLVI, J. Souletie, J. Vanimenus and R. Stora, (1987).

[3] E. Witten, *Nucl. Phys.* B188 (1981) 513.

[4] R. Dutt, A. Khare and U.P. Sukhatme , *Am. J. Phys.* 59 (1991) 723.

[5] A. Comtet, A.D. Bandrauk and D.K. Campbell , *Phys. Lett.* 150B (1985) 159.

[6] A. Inomata and G. Junker , ”Proceedings of International Symposium on Advanced Topics in Quantum Physics” (J.Q.Liang, M.L.Wang, S.N.Qiao and D.C.Su Eds.) Science Press, Beijing, (1993) 61.
[7] N.G. Van Kampen, "Stochastic Processes in Physics and Chemistry", North Holland, Amsterdam (1981).

[8] J.P. Bouchaud, A. Comtet, A. Georges and P. Le Doussal, *Ann. Phys.* 201 (1990) 285.

[9] J.P. Bouchaud, A. Comtet, A. Georges and P. Le Doussal, *Europhysics. Lett.* 3 (6) (1987) 653.

[10] B. Derrida and Y. Pomeau, *Phys. Rev. Lett.* 48 (1982) 627.

[11] J.P. Bouchaud and A. Georges, *Phys. Rep.* 195 (1990) 127.

[12] G. Oshanin, S.F. Burlatsky, M. Moreau and B. Gaveau, *Chem. Phys.* 178 (1993).

[13] C.W. Gardiner, "Handbook of Stochastic Methods", Springer Verlag, New York (1982).

[14] D. Waxman and K.D. Ivanova-Moser, *Ann. Phys.* 226 (1993) 271.

[15] M.M. Benderskii and L.A. Pastur, *Sov. Phys. JETP* 30 (1970) 158.

[16] I.M. Lifshits, S.A. Gredeskul and L.A. Pastur, "Introduction to the theory of disordered systems", John Wiley and Sons, New York (1987).

[17] L.A. Pastur and E.P. Feld’man, *Sov. Phys. JETP* 40 (1974) 241.

[18] P. Erdos and Z. Domanski, "Analogies in Optics and Micro Electronics" (W.van Haeringen and D.Lenstra Eds.) Kluwer Academic Publishers (1990) 49.

[19] G. Barton, A.J. Bray and A.J. McKane, *Am. J. Phys.* 58 (1990) 751.
[20] E. Tossati, A. Vulpiani and M. Zannetti, *Physica A*. **A164** (1990) 705.

[21] G. Theodorou and M.H. Cohen, *Phys. Rev. B* **13** (1976) 4597 ; A. Bovier, *J.Stat.Phys.* **56** (1989) 645.

[22] J.P. Bouchaud, A. Comtet, A. Georges and P. Le Doussal, *J. Physique* **48** (1987) 1445.

[23] B.V. Gnedenko and A.N. Kolmogorov, "Limit Distributions for Sums of Independent variables", Addison-Wesley, Reading, MA, 1954.

[24] A. Comtet, A. Georges and P. Le Doussal, *Phys. Lett.* **B208** (1988) 487.

[25] A.V. Smilga, *Phys.Rev.* **D46** (1992) 5598.

[26] H. Takayama,Y.R. Lin-Liu and K. Maki, *Phys. Rev. B* **21** (1980) 2388.

[27] D.K. Campbell and A.R. Bishop, *Nucl. Phys.* **B200** (1982) 297.

[28] B.C. Xu and S.E. Trullinger, *Phys. Rev. Lett.* **57** (1986) 3113.

[29] K.A. Loparo and X. Feng, *Siam J. Appl. Math* **53** (1993) 283.

[30] C. Aslangul, N. Potier, P. Chvosta and D. Saint-James, *Phys.Rev.* **E47** (1993) 1610.

[31] S. Karlin and H.M. Taylor, "A first course in stochastic processes", Academic Press, New-York (1975)
FIGURE CAPTIONS

Fig. 1  The random force $\phi(x)$ takes alternatively two values $\phi_0$ and $\phi_1$ on intervals whose lengths $\ell'_i$ and $h'_i$ are independent random variables, distributed according to (3.1) unless otherwise stated.

Fig. 2  The Schrödinger potential ($\phi^2 + \phi'$) as a function of $x$ in the case $\phi_0 = -\phi_1$. Delta functions appear each time $\phi(x)$ jumps.

Fig. 3  Computer simulations of the average density of states $N(E)$ for $\phi_0 = -\phi_1 = 1$ and $n_0 = n_1 = n$, with respectively a) $n = 5$, b) $n = 1$, c) $n = 0.5$, d) $n = 0.1$. $N(E)$ appears to be an increasing function of $n$. In general, its derivative is not a continuous function of the energy; a discontinuity can occur at $E = \phi_0^2$.

Fig. 4  The $\pi$-periodic stationary distributions $W_0(\theta)$ and $W_1(\theta)$ are plotted, respectively in full curve and in dashed curve, for the case $\phi_0 = -\phi_1 = 1$, $E = 0.4$, $n_0 = n_1 = 2$. $W_0$ and $W_1(\theta)$ defined just before (5.5) are the same distributions as $P_0(\theta)$ and $P_1(\theta)$ defined just before (3.8) up to a normalization constant (5.8). Note the peak of $W_0(\theta)$ at $\theta = \alpha_0 \approx 0.34$, and the peak of $W_1(\theta)$ at $\theta = \alpha_1 = \alpha_0 - \frac{\pi}{2}$. For further explanations, see text.

Fig. 5  The same as Fig. 4 except for $n_0 = n_1 = 4$ instead of $n_0 = n_1 = 2$. The lengths of the intervals on which $\phi(x)$ is constant are shorter, and the peaks at $\alpha_0$ and $\alpha_1$ are therefore attenuated.

Fig. 6  Lyapunov exponents $\gamma(E)$ are plotted for $\phi_0 = -\phi_1 = 1$ and $n_0 = n_1 = n$, with respectively a) $n = 1$, b) $n = 2$, c) $n = 4$. For these three cases, $\gamma(E)$ vanishes at zero energy, but very slowly (3.17).

Fig. 7  The same as Fig. 6 except for $n_0 = 2n_1 = n$ and $\gamma(0) = |F_0| = 1/3$, according to (2.10).

Fig. 8  The density of states $N(E)$, for the case $\phi_0 = 1$ and $\phi_1 = -C2$, when the
steps lengths are drawn according to the same broad distribution, behaving as (5.10) with $\alpha = 0.5$, and whose first moment therefore diverges. Curves a) b) c) represent three different simulation events. Obviously, $N(E)$ is no longer self-averaging (5.19).