BEHAVIOR OF $R$-GROUPS FOR $p$-ADIC INNER FORMS OF QUASI-SPLIT SPECIAL UNITARY GROUPS

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For Freydoon, with deep gratitude for all he has done for us, and for the field.

Abstract. We study $R$-groups for $p$-adic inner forms of quasi-split special unitary groups. We prove Arthur’s conjecture, the isomorphism between the Knapp-Stein $R$-group and the Langlands-Arthur $R$-group, for quasi-split special unitary groups and their inner forms. Furthermore, we investigate the invariance of the Knapp-Stein $R$-group within $L$-packets and between inner forms. This work is applied to transferring known results in the second-named author’s earlier work for quasi-split special unitary groups to their non quasi-split inner forms.

1. Introduction

In the representation theory of $p$-adic groups, in particular, in the framework of the local Langlands correspondence for a connected reductive algebraic group $G$ over a $p$-adic field $F$, it is of great importance to study the reducibility of parabolically induced representations. This study yields information on constructing tempered $L$-packets of $G$ from discrete $L$-packets of its $F$-Levi subgroup. The determination of reducibility has been developing over decades via several approaches, for example, by means of harmonic analysis from investigation of poles or zeros of residues of intertwining operators, Plancherel measures, and local $L$-functions, [29, 30, 33]. The method we address here is in terms of the Knapp-Stein $R$-group, which provides a combinatorial description of the tempered dual of $G(F)$ as well as its elliptic tempered spectrum. Further, as conjectured by Arthur, the isomorphism of the Knapp-Stein and Langlands-Arthur $R$-groups, via the endoscopic $R$-group, plays a significant role in the comparison of trace formulas and the endoscopic classification of automorphic representations [3, 18, 23].

While there has been a great deal of progress on the theory of Knapp-Stein $R$-groups for $F$-quasi-split groups $G$, little is known for non $F$-quasi-split groups $G'$. In [9], we investigated the behavior of $R$-groups between $F$-inner forms of $SL_n$, and determined the Knapp-Stein $R$-groups for the non quasi-split inner form from those of the split $SL_n$. We also proved the Knapp-Stein $R$-group for $G'(F)$ embeds as a subgroup of the $R$-group for $SL_n$, and we characterized the quotient. Another approach to this case was carried out in [7] and an example was discovered for which the Knapp-Stein $R$-group for $G'(F)$ is strictly smaller than that of $SL_n$. In [10], we further showed the invariance of $R$-groups between $F$-inner forms of quasi-split classical groups $SO_{2n+1}$, $Sp_{2n}$, $SO_{2n}$, or $SO^*_{2n}$, and transferred all known, relevant facts developed by the second-named author in the quasi-split classical groups to their non quasi-split inner forms. Non quasi-split inner forms of $Sp_{4n}$ and $SO_{4n}$ are also treated in [14].

To study the Knapp-Stein $R$-group for a non quasi-split group $G'$ in general, as one may notice from our previous works [9, 10], it is natural that we investigate its behavior between $G$ and $G'$ and transfer the developed theories regarding the Knapp-Stein $R$-group from $G$ to $G'$. This is due to the notion of Langlands functoriality. Since $G'$ is induced from a new Galois action twisted by a Galois 1-cocycle in the inner automorphisms of $G$, their $L$-groups are isomorphic over the Galois group and the set of $L$-parameters for $G'$ is contained in that for $G$. In this paper, combining this strategy and the restriction method, we study the Knapp-Stein $R$-group for non quasi-split $F$-inner forms of quasi-split special unitary groups.

More precisely, we fix a quadratic extension $E$ of a $p$-adic field $F$ of characteristic zero. Let $G_n = SU_n$ be a quasi-split special unitary group over $F$ with respect to $E/F$ and let $G'_n$ be its non quasi-split inner form.

2010 Mathematics Subject Classification. Primary 22E50; Secondary 22E35.
over $F$. A simple consequence from the Satake classification or a computation of Galois cohomology reduces our study to the case when $n$ is even. In fact, there is a unique non quasi-split $F$-inner form $G_n'$, up to $F$-isomorphism (see Section 3). For the rest of the introduction, we assume that $n$ is even, unless otherwise stated. Let $M'$ be an $F$-Levi subgroup of $G_n'$, which is an $F$-inner form of an $F$-Levi subgroup $M$ of $G_n$. Then, $M = M \cap G_n$ and $M' = M' \cap G_n'$, where $M$ is an $F$-Levi subgroup of a quasi-split unitary group $G_n = U_n$ over $F$ with respect to $E/F$ and $M'$ is an $F$-Levi subgroup of a non quasi-split $F$-inner form $G_n'$ of $G_n$. We shall use $G$ for the group $G(F)$ of $F$-points of any connected reductive algebraic group $G$ over $F$.

Given an elliptic tempered $L$-parameter $\phi \in \Phi_{\text{disc}}(M)$, by [22, Théorème 8.1], we have a lifting $\varphi \in \Phi_{\text{disc}}(M)$ commuting with the natural projection $\hat{M} \to \hat{M}$, where $\hat{M}$ and $\hat{M}$ respectively denote the connected components of the $L$-groups of $M$ and $\hat{M}$ (see Section 2 for the details). Restricting the $L$-packet $\Pi_{\varphi}(M)$ constructed by Rogawski [26] and Mok [23] (see Section 5.1), we construct an $L$-packet $\Pi_{\phi}(M)$ as the set of isomorphism classes of irreducible constituents in the restriction from $M$ to $\hat{M}$. All the arguments used for $\Pi$ can be applied to $\Pi'$, as in Kaletha-Minguez-Shin-White [18] and the details are described in Section 6.1. For any $\sigma \in \Pi_{\phi}(M)$, and $\sigma' \in \Pi_{\phi}(M')$, we prove

$$R_\sigma \simeq R_{\phi,\sigma} \text{ and } R_{\sigma'} \simeq R_{\phi,\sigma'}.$$ 

In each of the above isomorphisms, the left side is the Knapp-Stein $R$-group, and the right side is the Langlands-Arthur $R$-group, defined in Section 2.2. This is known as Arthur’s conjecture, predicted in [1], for $G$ and $G'$ (See Theorem 6.1). In the course of the proofs, we apply some known results about $R$-groups for $G_n$ and $G_n'$ in [12, 18, 23], which are recalled in Section 5. We also investigate and utilize some relationships between identity components of the centralizers of the images of $\varphi$ and $\phi$ in $\hat{G}$ and $\hat{G}$ (see Section 3.3 and Lemma 6.2).

We further study the invariance of the Knapp-Stein $R$-groups for $G_n$ and $G_n'$. Namely, given $\sigma_1, \sigma_2 \in \Pi_{\phi}(M)$, and $\sigma', \sigma'' \in \Pi_{\phi}(M')$, we prove $R_{\sigma_1} \simeq R_{\sigma_2}$, and $R_{\sigma'} \simeq R_{\sigma''}$ (Theorem 6.6). Moreover, given $\sigma \in \Pi_{\phi}(M)$ and $\sigma' \in \Pi_{\phi}(M')$, we prove $R_\sigma \simeq R_{\sigma'}$ (Theorem 6.8). The crucial idea is to study the stabilizers $W(\sigma)$ and $W(\sigma')$. Theorem 6.3 shows

$$(1.1) \quad W(\sigma) \simeq \{ w \in W_M : "\Sigma \simeq \Sigma \lambda \text{ for some } \lambda \in (M/M)^D \},$$

where $W_M$ is the Weyl group of $M$ in $G$, $\Sigma \in \Pi_{\phi}(M)$ is a lifting of $\sigma$, and $(M/M)^D$ is the group of continuous characters on $M$ which are trivial on $M$. The same is true for $W(\sigma')$. The isomorphism (1.1) stems from the group structure of $M$ and $M'$ and the description of irreducible representations of $M$ and $M'$, which are discovered in [12, Section 2] and explained in Section 6.3.

As an application, a combination of Theorem 6.8 and the second-named author’s earlier result [12, Theorem 3.7] shows that the Knapp-Stein $R$-group, $R_{\sigma'}$, for non quasi-split inner forms $G'$ can be expressed in terms of a subgroup in $R_{\sigma'}$ and $Z^d$ for some integer $d$. It follows that $R_{\sigma'}$ is in general non-abelian (see Remark 6.9).

In Section 2, we recall basic notation and background, provide the detailed group structure of $G_n$, $G_n'$, $G_n''$, and their $F$-Levi subgroups, and study relations in their $L$-groups. In Section 4, we describe Weyl group actions on Levi subgroups and their representations. In Section 5, we revisit the theory of $R$-groups for $G_n$ and $G_n'$ based on [12, 18, 23]. In Section 6, we prove the Arthur’s conjecture for $G_n$ and $G_n'$ and the invariance of their $R$-groups.

Acknowledgements. K. Choiy was partially supported by an AMS-Simons Travel Grant. D. Goldberg was partially supported by Simons Foundation Collaboration Grant 279153.
Fix a minimal $F$-parabolic subgroup $P_0$ of $G$ with Levi decomposition $P_0 = M_0N_0$, where $M_0$ denotes a Levi factor and $N_0$ denotes the unipotent radical. We denote by $A_0$ the split component of $M_0$, that is, the maximal $F$-split torus in the center of $M_0$, and by $\Delta$ the set of simple roots of $A_0$ in $N_0$. We say an $F$-parabolic subgroup $P$ of $G$ to be standard if it contains $P_0$.

Given an $F$-parabolic subgroup $P$ with Levi decomposition $P = MN$, there exist a subset $\Theta \subseteq \Delta$ such that $M$ equals $M_0$, the Levi subgroup generated by $\Theta$. Note that $M \supseteq M_0$ and $N \subseteq N_0$. We write $A_{M_0}$ for the split component $A_M$ of $M = M_0$. It follows that $A_M$ equals the identity component $(\cap_{\alpha \in \ker \alpha})^0$ in $A_0$, so that $M = Z_G(A_M)$, where $Z_G(A_M)$ is the centralizer of $A_M$ in $G$. We refer the reader to [6, Proposition 20.4] and [34, Section 15.1] for the details.

We let $\Phi(P, A_M)$ denote the set of reduced roots of $P$ with respect to $A_M$. Denote by $W_M = W(G, A_M) := N_G(A_M)/Z_G(A_M)$ the Weyl group of $A_M$ in $G$, where $N_G(A_M)$ is the normalizer of $A_M$ in $G$. For simplicity, we write $A_0 = A_{M_0}$.

For any topological group $H$, we write $Z(H)$ for the center of $H$. Denote by $H^o$ the identity component of $H$ and by $\pi_0(H)$ the group $H/H^o$ of connected components of $H$. We write $H^D$ for the group, $\text{Hom}(H, C^\times)$, of all continuous characters and by $1$ the trivial character. We say a character is unitary if its image is in the unit circle $S^1 \subset C^\times$. For any Galois module $J$, we denote by $H^i(F, J) := H^i(\text{Gal}(\bar{F}/F), J(\bar{F}))$ the Galois cohomology of $J$ for $i \in \mathbb{N}$.

Let $G$ be a connected reductive algebraic group over $F$. We denote by $\text{Irr}(G)$ the set of isomorphism classes of irreducible admissible complex representations of $G$. If there is no confusion, we do not make a distinction between each isomorphism class and its representative. For any $\sigma \in \text{Irr}(M)$, we write $\iota_{G,M}(\sigma)$ for the normalized (twisted by $\delta_P$) induced representation, where $\delta_P$ denotes the modulus character of $P$.

We denote by $\Pi_{\text{disc}}(G)$ and $\Pi_{\text{temp}}(G)$ the subsets of $\text{Irr}(G)$ which respectively consist of discrete series and tempered representations, where, a discrete series representation is an irreducible, admissible, unitary representation whose matrix coefficients are square-integrable modulo the center of $G$, that is, in $L^2(G/Z(G))$, and a tempered representation is an irreducible, admissible, unitary representation whose matrix coefficients are in $L^{2+\epsilon}(G/Z(G))$ for all $\epsilon > 0$. It is clear that $\Pi_{\text{disc}}(G) \subset \Pi_{\text{temp}}(G)$.

We denote by $W_F$ the Weil group of $F$ and by $\Gamma$ the absolute Galois group $\text{Gal}(\bar{F}/F)$. By fixing $\Gamma$-invariant splitting data, the $L$-group of $G$ is defined as a semi-direct product $L^G := \hat{G} \rtimes W_F$ (see [5, Section 2]). Following [5, Section 8.2], an $L$-parameter for $G$ is an admissible homomorphism

$$\phi : W_F \times SL_2(C) \to L^G.$$  

Two $L$-parameters are said to be equivalent if they are conjugate by $\hat{G}$. We denote by $\Phi(G)$ the set of equivalence classes of $L$-parameters for $G$.

We denote by $Z_G(\hat{G})$ the centralizer of the image of $\phi$ in $\hat{G}$. The center of $L^G$ is the $\Gamma$-invariant group $Z(\hat{G})^\Gamma$. Note that $C_\phi \supseteq Z(\hat{G})^\Gamma$. We say that $\phi$ is elliptic if the quotient group $C_\phi(\hat{G})/Z(\hat{G})^\Gamma$ is finite, and $\phi$ is tempered if $\phi(W_F)$ is bounded. We denote by $\Phi_{\text{disc}}(G)$ and $\Phi_{\text{temp}}(G)$ the subset of $\Phi(G)$ which respectively consist of elliptic and tempered $L$-parameters of $G$. We set $\Phi_{\text{disc}}(G) = \Phi_{\text{disc}}(G) \cap \Phi_{\text{temp}}(G)$.

The local Langlands conjecture for $G$ predicts that there is a surjective finite-to-one map from $\text{Irr}(G)$ to $\Phi(G)$. Given $\phi \in \Phi(G)$, we write $\Pi_\phi(G)$ for the $L$-packet attached to $\phi$, and then the local Langlands conjecture implies that

$$\text{Irr}(G) = \bigsqcup_{\phi \in \Phi(G)} \Pi_\phi(G).$$  

It is expected that $\Phi_{\text{disc}}(G)$ and $\Phi_{\text{temp}}(G)$ respectively parameterize $\Pi_{\text{disc}}(G)$ and $\Pi_{\text{temp}}(G)$.

Given two connected reductive algebraic groups $G$ and $G'$ over $F$, $G'$ is said to be an $F$-inner form of $G$ with respect to an $F$-isomorphism $\varphi : G' \to G$ if $\varphi \circ \tau(\varphi)^{-1}$ is an inner automorphism ($g \mapsto xgx^{-1}$) defined over $\bar{F}$ for all $\tau \in \text{Gal}(\bar{F}/F)$ (see [5, 2.4(3)]) or [21, p.280]). If there is no confusion, we often omit the references to $F$ and $\varphi$. It is well known from [21, p.280] that there is a bijection between $H^1(F, G_{\text{ad}})$ and the set of isomorphism classes of $F$-inner forms of $G$, where $G_{\text{ad}} := G/Z(G)$. We note that, when $G$ and $G'$ are inner forms of each other, we have $L^G \simeq L^{G'}$ [5, Section 2.4(3)].
2.2. \textit{\textbf{R}}-groups. We review the definitions of Knapp-Stein, Langlands-Arthur, and endoscopic \textit{\textbf{R}}-groups. Let \( \mathbf{G} \) be a connected reductive algebraic group over \( F \), and let \( \mathbf{M} \) be an \( F \)-Levi subgroup of \( \mathbf{G} \). Given \( \sigma \in \text{Irr}(\mathbf{M}) \) and \( w \in W_M \), we write \( ^w\sigma \) for the representation given by \( ^w\sigma (x) = \sigma (w^{-1}xw) \). Note that, for the purpose of studying \( \text{R-groups} \), we do not distinguish the representative of \( w \), since the isomorphism class of \( ^w\sigma \) is independent of the choices of representatives in \( G \) of \( w \in W_M \). Assume that \( \sigma \) lies in \( \Pi_{\text{disc}}(M) \), we define the stabilizer of \( \sigma \) in \( W_M \)

\[ W(\sigma) := \{ w \in W_M : ^w\sigma \simeq \sigma \}. \]

Write \( \Delta'_{\alpha} \) for \( \{ \alpha \in \Phi(P, A_M) : \mu_{\alpha}(\sigma) = 0 \} \), where \( \mu_{\alpha}(\sigma) \) is the rank one Plancherel measure for \( \sigma \) attached to \( \alpha \) [11, p.1108]. The \textit{Knapp-Stein R-group} is defined by

\[ R_{\sigma} := \{ w \in W(\sigma) : wo > 0, \forall \alpha \in \Delta'_{\alpha} \}. \]

We denote by \( W^0_\sigma \) the subgroup of \( W(\sigma) \), generated by the reflections in the roots of \( \Delta'_{\alpha} \). Then, for any \( \sigma \in \Pi_{\text{disc}}(M) \), we have

\[ W(\sigma) = R(\sigma) \rtimes W^0_\sigma, \]

which yields another description of the Knapp-Stein \textit{R-group}

\[ R(\sigma) \simeq W(\sigma)/W^0_\sigma. \]

We refer to [19, 31, 32] for details.

Given an \( L \)-parameter \( \phi \in \Phi(\mathbf{M}) \), we also consider \( \phi \) as an \( L \)-parameter for \( G \) via the inclusion \( \widehat{M} \hookrightarrow \widehat{G} \). Fix a maximal torus \( T_\phi \) in \( C_\phi(\widehat{G})^0 \). We set

\[ W^0_\phi := N_{C_\phi(\widehat{G})^0}(T_\phi)/Z_{C_\phi(\widehat{G})^0}(T_\phi), \quad W_\phi := N_{C_\phi(\widehat{G})}(T_\phi)/Z_{C_\phi(\widehat{G})}(T_\phi). \]

The \textit{endoscopic R-group} \( R_\phi \) is defined as follows

\[ R_\phi := W_\phi/W^0_\phi. \]

We identify \( W_\phi \) with a subgroup of \( W_M \) (see [1, p.45]). For \( \sigma \in \Pi_{\phi}(M) \), we set

\[ W^0_{\phi, \sigma} := W^0_\phi \cap W(\sigma), \quad W_{\phi, \sigma} := W_\phi \cap W(\sigma). \]

The \textit{Langlands-Arthur R-group} \( R_{\phi, \sigma} \) is defined as follows

\[ R_{\phi, \sigma} := W_{\phi, \sigma}/W^0_{\phi, \sigma}. \]

3. \textbf{Structure theory of Levi subgroups}

We discuss the group structure of \( U_n, SU_n \), their \( F \)-inner forms, and their \( F \)-Levi subgroups. We mainly refer to [12, 13, 18, 23, 27]. Fix a quadratic extension \( E/F \) with \( \cdot \) the non-trivial Galois element. For any positive integer \( n \), we let

\[ J_n := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \text{Mat}_{n \times n}(\mathbb{Z}). \]

We denote by \( \text{Res}_{E/F} \) the Weil restriction of scalars of \( E/F \) (see [35, Chapter 1] and [25, 2.1.2]). For \( g = (g_{ij}) \in \text{Res}_{E/F}GL_n \), we let \( \bar{g} = (\bar{g}_{ij}) \) and set \( \varepsilon(g) = J_n \cdot \bar{g} \cdot J_n^{-1} \), where \( g \mapsto \cdot g \) is the transpose.
3.1. $U_n$ and its inner forms. Let $\mathcal{G} = \mathcal{G}_n$ denote the quasi-split unitary group $U_n$ with respect to $E/F$ and $J_n$. Thus,

$$\mathcal{G} = \{ g \in \text{Res}_{E/F} \text{GL}_n : gJ_n \bar{g} = J_n \}$$

(see [13, Section 1]; our $J_n$ is the inverse of $u_n$ therein). We denote by $\mathcal{M}$ an $F$-Levi subgroup of $\mathcal{G}$. Then, $\mathcal{M}$ is of the form

$$(3.1) \quad \text{Res}_{E/F} \text{GL}_{n_1} \times \cdots \times \text{Res}_{E/F} \text{GL}_{n_k} \times \mathcal{G}_m,$$

where $\sum_{i=1}^k 2n_i + m = n$ with $n_i \geq 0$ and $m \geq 0$. By convention, we note that, $\mathcal{G}_0 = 1$ for $n$ even, $\mathcal{G}_1 = U_1$ for $n$ odd, and $\text{GL}_0 = 1$. The group of $F$-points, $\mathcal{A}_\mathcal{M}$, of the split component $\mathcal{A}_\mathcal{M}$ is of the form

$$(3.2) \quad \{ \text{diag}(x_1 I_{n_1}, \ldots, x_k I_{n_k}, I_m, x_k^{-1} I_{n_k}, \ldots, x_1^{-1} I_{n_1}) : x_i \in F^\times \}.$$

We let $\mathcal{G}' = \mathcal{G}_n'$ denote an $F$-inner form of $\mathcal{G}$. By the Satake classification in [27, Section 3.3], for $n$ odd, there is no non quasi-split $F$-inner form of $\mathcal{G}_n$. On the other hand, for $n$ even, there is a unique non quasi-split $F$-inner form $\mathcal{G}_n'$, up to $F$-isomorphism. The $\Gamma$-diagram of the connected, simply-connected, semi-simple type of such $\mathcal{G}'$ is

$$(3.3) \quad \text{(see the table in p.119 of [27])}.$$

In the diagram above, the arrow indicates the non-trivial Galois action. Furthermore, the black vertex indicates a root in the set of simple roots of a fixed minimal $F$-Levi subgroup $\mathcal{M}'_0$ of $\mathcal{G}'$. So, we remove only a subset (denoted by $\vartheta$) of (Gal($E/F$)-orbits) of white vertices to obtain an $F$-Levi subgroup $\mathcal{M}'$ (see [27, Section 2.2] and [5, Section I.3]). As discussed in Section 2.1, the $F$-Levi subgroup $\mathcal{M}'$, corresponding to $\Theta = \Delta \setminus \vartheta$, is the centralizer in $\mathcal{G}'$ of the split component $\mathcal{A}_\mathcal{M}' = (\cap_{\alpha \in \Theta} \text{ker} \alpha)^\circ$. Then, $\mathcal{M}'$ is of the form

$$(3.4) \quad \text{Res}_{E/F} \text{GL}_{n_1'} \times \cdots \times \text{Res}_{E/F} \text{GL}_{n_k'} \times \mathcal{G}_m',$$

where $\sum_{i=1}^k 2n_i' + m' = n$ with $n_i' \geq 0$ and $m' \geq 2$. Notice here that $m'$ is always even. Considering the forms (3.1) and (3.4), it is obvious that, if $n_i = n_i'$ for all $i, k = k'$, and $m = m'$, then $\mathcal{M}'$ is an $F$-inner form of $\mathcal{M}$. In this case, furthermore, two split components $\mathcal{A}_\mathcal{M}$ and $\mathcal{A}_\mathcal{M}'$ are isomorphic over $F$.

Remark 3.1. Since there is a bijection between $H^1(F, (\mathcal{G}_n)_{\text{ad}})$ and the set of isomorphism classes of $F$-inner forms of $\mathcal{G}_n$ (see Section 2.1), using the fact ([15, Lemma 1.2.1(ii) and p.657]) that

$$H^1(F, (\mathcal{G}_n)_{\text{ad}}) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } n \text{ is even,} \end{cases}$$

it follows that, when $n$ is odd, there is no non quasi-split $F$-inner form of $\mathcal{G}_n$, and when $n$ is even, there is a unique non quasi-split $F$-inner form, $\mathcal{G}_n'$, of $\mathcal{G}_n$ up to $F$-isomorphism, as we discussed above.

3.2. $SU_n$ and its inner forms. We let $G = G_n$ denote the quasi-split special unitary group $SU_n$ with respect to $E/F$ and $J_n$. Thus,

$$G = \mathcal{G} \cap \text{Res}_{E/F} \text{SL}_n = \{ g \in \text{Res}_{E/F} \text{SL}_n : gJ_n \bar{g} = J_n \}.$$

We denote by $\mathcal{M}$ an $F$-Levi subgroup of $G$. Then, $\mathcal{M}$ is of the form $\mathcal{M} \cap G$ and $\mathcal{A}_\mathcal{M}$ is of the form $\mathcal{A}_\mathcal{M} \cap G$.

Due to (3.2), the $F$-points $A_\mathcal{M}$ of the split component $\mathcal{A}_\mathcal{M}$ thus equals

$$\{ \text{diag}(x_1 I_{n_1}, \ldots, x_k I_{n_k}, I_m, x_k^{-1} I_{n_k}, \ldots, x_1^{-1} I_{n_1}) : x_i \in F^\times \}.$$

Especially, when $m \geq 2$, from (3.1) and [13, Lemma 2.8] we have a useful isomorphism

$$(3.5) \quad M \simeq (\text{GL}_{n_1}(E) \times \cdots \times \text{GL}_{n_k}(E)) \rtimes G_m,$$
where $\sum_{i=1}^{k} 2n_i + m = n$ with $n_i \geq 0$. More precisely, for $(g, h)$ with $g \in \text{GL}_{n_1}(E) \times \cdots \times \text{GL}_{n_k}(E)$ and $h \in G_m$, the isomorphism (3.5) from $(\text{GL}_{n_1}(E) \times \cdots \times \text{GL}_{n_k}(E)) \times G_m$ to $M$ is given by

$$(g, h) \mapsto \begin{pmatrix} g & \alpha_m(g)^{-1}h & \varepsilon(g) \end{pmatrix} \in \mathcal{M} \cap SU_n(F) = M,$$

where $\alpha_m(g) = \alpha_m(\det g)$ and for $a \in E^\times$,

$$(3.7) \quad \alpha_m(a) := \begin{pmatrix} a & I_{m-2} & \bar{a}^{-1} \end{pmatrix} \in \text{GL}_m(E).$$

By convention, we note that $G_0 = G_1 = 1$. Given $(g, h)$ with $g \in \text{GL}_{n_1}(E) \times \cdots \times \text{GL}_{n_k}(E)$ and $h \in G_m$, the action of $g$ on $h$ is given by

$$g \circ h = \alpha_m(g) \cdot h \cdot \alpha_m(g)^{-1}.$$  

Next, we let $G' = G'_n$ denote an $F$-inner form of $G$. As discussed in Section 3.1, for $n$ odd, there is no non quasi-split $F$-inner form of $G_n$. On the other hand, for $n$ even, there is a unique non quasi-split $F$-inner form $G'_n$, up to $F$-isomorphism, whose $\Gamma$-diagram is (3.3). Any $F$-Levi subgroup $M'$ of $G'$ is of the form $\mathcal{M}' \cap G'$. Thus, from (3.4) and (3.5), we have an isomorphism

$$(3.8) \quad M' \simeq (\text{GL}_{n'_1}(E) \times \cdots \times \text{GL}_{n'_k}(E)) \times G'_{m'},$$

where $\sum_{i=1}^{k'} 2n'_i + m' = n$ with $n'_i \geq 0$ and $m' \geq 2$. Notice as before that $m'$ is always even. To explain (3.8) more precisely, for $(g', h')$ with $g' \in \text{GL}_{n'_1}(E) \times \cdots \times \text{GL}_{n'_k}(E)$ and $h' \in G'_{m'}$, the isomorphism (3.8) from $(\text{GL}_{n'_1}(E) \times \cdots \times \text{GL}_{n'_k}(E)) \times G'_{m'}$ to $M'$ is given by

$$(g', h') \mapsto \begin{pmatrix} g' & \alpha_{m'}(g')^{-1}h' & \varepsilon(g') \end{pmatrix} \in \mathcal{M}' \cap G'_n = M',$$

where $\alpha_{m'}(g') = \alpha_{m'}(\det g')$ and for $a \in E^\times$,

$$\alpha_{m'}(a) := \begin{pmatrix} a & I_{m'-2} & \bar{a}^{-1} \end{pmatrix} \in \text{GL}_{m'}(E).$$

By convention, we note that, $G'_{0} = 1$. Given $(g', h') \in M'$ with $g' \in \text{GL}_{n'_1}(E) \times \cdots \times \text{GL}_{n'_k}(E)$ and $h' \in G'_{m'}$, the action of $g'$ on $h'$ is given by

$$g' \circ h' = \alpha_{m'}(g') \cdot h' \cdot \alpha_{m'}(g')^{-1}.$$  

Like the unitary case in Section 3.1, it is obvious from (3.5) and (3.8) that, if $n_i = n'_i$ for all $i, k = k'$, and $m = m'$, then $M'$ is an $F$-inner form of $M$. In this case, there is an $F$–isomorphism $A_M \simeq A_{M'}$ between two split components.

We have following exact sequences of algebraic groups

$$1 \rightarrow G \rightarrow G \rightarrow U_1 \rightarrow 1,$$

and

$$1 \rightarrow G' \rightarrow G' \rightarrow U_1 \rightarrow 1.$$  

Applying Galois cohomology, since $H^1(F, G) = H^1(F, G') = 1$ due to [25, Theorem 6.4], we have following exact sequences of $F$-points

$$1 \rightarrow G \rightarrow G \rightarrow E^1 \rightarrow 1,$$

and

$$1 \rightarrow G' \rightarrow G' \rightarrow E^1 \rightarrow 1.$$  

where $E^1 = \{ x \in E : N_{E/K}(x) = x \bar{x} = 1 \}$. All above exact sequences are true for $F$-Levi subgroups. In particular, $H^1(F, M) = H^1(F, M') = 1$ since $H^1(F, M) \hookrightarrow H^1(F, G) = 1$ and $H^1(F, M') \hookrightarrow H^1(F, G') = 1$ (see [24, p.95, footnote], [21, p.270], and [9, Remark 2.5]). We further have $$\mathcal{G}_{der} = G_{der} = G, \quad \mathcal{G}'_{der} = G'_{der} = G', \quad \mathcal{M}_{der} = M_{der} \subset M, \quad \mathcal{M}'_{der} = M'_{der} \subset M'.
$$

### 3.3. $L$-groups

We describe $L$-groups of $\mathcal{G} = U_n, G = SU_n$, their inner forms $\mathcal{G}', G'$, and the $L$-groups of their $F$-Levi subgroups $\mathcal{M}, \mathcal{M}', \mathcal{M},$ and $\mathcal{M}'$. Furthermore, we explain a relationship between $L$-groups of $\mathcal{M}$ and $\mathcal{M}$, and investigate a connection between two $\Gamma$-split components $A_{\tilde{G}}$ and $A_{\tilde{M}}$ of $\tilde{M}$ and $\tilde{M}$, defined as follows

$$A_{\tilde{G}} := (Z(\tilde{M}))^\circ \quad \text{and} \quad A_{\tilde{M}} := (Z(\tilde{M}))^\circ.$$  

These will be used in Section 6. Based on [23], we set

$$L^G = \mathcal{G}' = GL_n(\mathbb{C}) \times W_F,$$

where $W_E$ acts trivially on $GL_n(\mathbb{C})$, and the action of $w_c \in W_F \setminus W_E$ on $\hat{g} \in GL_n(\mathbb{C})$ is given by

$$w_c(\hat{g}) = J_n^{-1} \hat{g}^{-1} J_n. \tag{3.9}$$

For $G$ and $G'$, the $L$-group is

$$L^G = L^G' = PGL_n(\mathbb{C}) \times W_F,$$

where $W_E$ acts trivially on $PGL_n(\mathbb{C})$, and the action of $w_c \in W_E \setminus W_E$ on $\hat{g} \in PGL_n(\mathbb{C})$ is given by (3.9).

Let $\mathcal{M}, \mathcal{M}', \mathcal{M}$, and $\mathcal{M}'$ be $F$-Levi subgroups of $\mathcal{G}, \mathcal{G}', G$, and $G'$, respectively, such that $\mathcal{M}'$ is an $F$-inner form of $\mathcal{M}$, and $\mathcal{M}'$ is an $F$-inner form of $\mathcal{M}$. We have

$$\tilde{M} = \tilde{M}' \subset \tilde{G} = \tilde{G}', \quad L^\mathcal{M} = L^M = \tilde{M} \times W_F, \quad \tilde{M}' \subset \tilde{G} = \tilde{G}', \quad L^M = L^M' = \tilde{M} \times W_F.$$  

Considering (3.1) and (3.4), we set

$$\mathcal{M} = \text{Res}_{E/F}(GL_{n_1}) \times \cdots \times \text{Res}_{E/F}(GL_{n_k}) \times \mathcal{G}_m,$$

$$\mathcal{M}' = \text{Res}_{E/F}(GL_{n_1}) \times \cdots \times \text{Res}_{E/F}(GL_{n_k}) \times \mathcal{G}'_m,$$

where $\mathcal{G}_m$ is an $F$-inner form of $\mathcal{G}_m$, $\sum_{i=1}^{k} 2n_i + m = n$ with $n_i \geq 0$, and $m \geq 0$. Then, we have

$$\tilde{\mathcal{M}} = \tilde{\mathcal{M}'} = (GL_{n_1}(\mathbb{C}) \times GL_{n_1}(\mathbb{C})) \times \cdots \times (GL_{n_k}(\mathbb{C}) \times GL_{n_k}(\mathbb{C})) \times GL_m(\mathbb{C}).$$

$W_E$ acts trivially on $\tilde{\mathcal{M}}$, and the action of $w_c \in W_F \setminus W_E$ on $((\hat{g}_{11}, \hat{g}_{12}), (\hat{g}_{21}, \hat{g}_{22}), \cdots, (\hat{g}_{k1}, \hat{g}_{k2}), \hat{h}) \in \tilde{\mathcal{M}}$ is given by

$$w_c((\hat{g}_{11}, \hat{g}_{12}), (\hat{g}_{21}, \hat{g}_{22}), \cdots, (\hat{g}_{k1}, \hat{g}_{k2}), \hat{h}) = ((\hat{g}_{12}, \hat{g}_{11}), (\hat{g}_{22}, \hat{g}_{21}), \cdots, (\hat{g}_{k2}, \hat{g}_{k1}), J_m^{-1} \hat{h}^{-1} J_m). \tag{3.10}$$

Next, we note that, since $\mathcal{M}_{der}$ and $\mathcal{M}'_{der}$ are simply connected, we have

$$\tilde{\mathcal{M}}_{ad} = (\tilde{\mathcal{M}})/Z(\tilde{\mathcal{M}}), \quad \tilde{M}_{ad} = (\tilde{M})/Z(\tilde{M}).$$

Further, from [20, (1.8.1) p.616], the exact sequence of algebraic groups

$$1 \to \mathcal{M} \to \tilde{\mathcal{M}} \to \tilde{M} \to 1$$

yields an exact sequence

$$1 \to \tilde{U}_1 = \mathbb{C}^\times \to Z(\tilde{M}) \to (\mathbb{C}^\times)^{2k+1} \to Z(\tilde{M}) \to 1. \tag{3.11}$$
We thus have the following commutative diagram of $L$-groups (cf., \[9, \text{Remark 2.4}\])

\[
\begin{array}{cccc}
1 & 1 \\
\downarrow & \downarrow \\
\widehat{U}_1 = \mathbb{C}^\times & \xrightarrow{\cong} \ker & 1 \\
\downarrow & \downarrow \\
Z(\widehat{M}) = (\mathbb{C}^\times)^{2k+1} & \rightarrow & \widehat{M} & \rightarrow \widehat{M}_{\text{der}} & 1 \\
\downarrow & \downarrow & \downarrow \\
Z(\widehat{M}) = (\mathbb{C}^\times)^{2k+1}/\mathbb{C}^\times & \rightarrow & \widehat{M} & \rightarrow \widehat{M}_{\text{der}} & 1 \\
\downarrow & \downarrow & \downarrow \\
1 & 1 & 1 & 1 \\
\end{array}
\]

The middle vertical exact sequence becomes

\[(3.12) \quad 1 \rightarrow \widehat{U}_1 = \mathbb{C}^\times \rightarrow \widehat{M} = \widehat{M}' \rightarrow \widehat{M} = \widehat{M}' \rightarrow 1.
\]

We also have

\[(3.13) \quad 1 \rightarrow \widehat{U}_1 = \mathbb{C}^\times \rightarrow \widehat{G} = \widehat{G}' \rightarrow \widehat{G} = \widehat{G}' \rightarrow 1.
\]

Note that $\widehat{U}_1$ in (3.12) and (3.13) equals $Z(\widehat{G}) = Z(\widehat{G}')$. Furthermore, $\widehat{U}_1$ is diagonally embedded into $\widehat{G}$ in (3.13) and the action of $W_F$ on $\widehat{U}_1 = \mathbb{C}^\times$ is obtained from (3.9). The action of $W_F$ can be also obtained from (3.10), since it is diagonally embedded into $Z(\widehat{M})$ in (3.11) as well. It thus follows that the subgroups of $\Gamma = \text{Gal}(\overline{F}/F)$-invariants satisfy

\[(3.14) \quad Z(\widehat{G})^\Gamma = Z(\widehat{G}')^\Gamma = \widehat{U}_1^\Gamma = \{\pm 1\}.
\]

Moreover, using the action of $W_F$ on $\widehat{M} = \widehat{M}'$ in (3.10) and the surjective map $\widehat{M} = \widehat{M}' \rightarrow \widehat{M} = \widehat{M}'$ in (3.12), the action of $W_F$ on $\widehat{M} = \widehat{M}'$ can be obtained.

From (3.10), we have

\[A_{\widehat{M}} := (Z(\widehat{M})^\Gamma)^\circ \simeq ((\mathbb{C}^\times)^k \times \{\pm 1\})^\circ = (\mathbb{C}^\times)^k.
\]

We note that

\[A_{\widehat{M}}/Z(\widehat{G})^\Gamma \simeq (A_{\widehat{M}} \cdot Z(\widehat{G}))/Z(\widehat{G}) \subset \widehat{G}.
\]

Then, we have the following lemmas.

**Lemma 3.2.** The quotient $A_{\widehat{M}}/Z(\widehat{G})^\Gamma$ is connected.

**Proof.** Since $A_{\widehat{M}} = (Z(\widehat{M})^\Gamma)^\circ$ by definition and since $Z(\widehat{G}) \subset A_{\widehat{M}}$, we have

\[A_{\widehat{M}}/Z(\widehat{G})^\Gamma = (Z(\widehat{M})^\Gamma)^\circ / (Z(\widehat{G})^\Gamma \cap (Z(\widehat{M})^\Gamma)^\circ)
\]

which is isomorphic to $Z(\widehat{M})^\Gamma / Z(\widehat{G})^\Gamma$ by \[2, \text{Lemma 1.1}\]. Note that $Z(\widehat{M})^\Gamma / Z(\widehat{G})^\Gamma$ is connected due to the proof of \[2, \text{Lemma 1.1}\]. \(\square\)

From (3.11) and (3.14), we have the following exact sequence

\[1 \rightarrow A_{\widehat{M}}/Z(\widehat{G})^\Gamma \rightarrow Z(\widehat{M})^\Gamma \rightarrow H^1(F, Z(\widehat{G})).
\]

The following lemma proves that the embedding $A_{\widehat{M}}/Z(\widehat{G})^\Gamma \rightarrow Z(\widehat{M})^\Gamma$ is in fact an equality.

**Lemma 3.3.** With the notation above, we have

\[A_{\widehat{M}} = Z(\widehat{M})^\Gamma \text{ and } A_{\widehat{M}} = A_{\widehat{M}}/Z(\widehat{G})^\Gamma.
\]
Proof. We note from [2, Lemma 1.1] that

\[ Z(\hat{M})^G = Z(\hat{G})^G \cdot (Z(\hat{M})^G)^\circ. \]

Since \( Z(\hat{G}) = 1 \) and \( A_{\widehat{M}} = (Z(\hat{M})^G)^\circ \), the first equality is verified (hence, \( Z(\hat{M})^G \) is connected). We note that

\[ A_{\widehat{M}}/Z(\hat{G})^G \subset Z(\hat{M})^G = A_{\widehat{M}} \]

and \( A_{\widehat{M}}/Z(\hat{G})^G \) is connected by Lemma 3.2. Since \( A_{\widehat{M}} \) is a maximal torus in \( C_\phi^e \), and since \( A_{\widehat{M}}/Z(\hat{G})^G \) is also a torus having the same dimension with \( A_{\widehat{M}} \) (cf., Lemma 6.2 in Section 6.1), we have \( A_{\widehat{M}}/Z(\hat{G})^G = A_{\widehat{M}} \). □

4. Weyl group actions

For an \( F \)-Levi subgroup \( M \) of a connected reductive algebraic group \( G \), we recall from 2.1 that the Weyl groups are \( W_M = W(G, A_M) := N_G(A_M)/Z_G(A_M) \), \( W_{M'} = W(G', A_{M'}) := N_G(A_{M'})/Z_G(A_{M'}) \), and \( W_{\widehat{M}} = W(\hat{G}, A_{\widehat{M}}) := N_{\hat{G}}(A_{\widehat{M}})/Z_{\hat{G}}(A_{\widehat{M}}) \). Through the duality

\[ s_\alpha \mapsto s_\alpha^\vee \]

between simple reflections for \( \alpha \in \Delta \), we have

\[ W_M \simeq W_{\widehat{M}} \quad W_{M'} \simeq W_{\widehat{M}'} \]

We thus identify

\[ (4.1) \quad W_M = W_{\widehat{M}} = W_{\widehat{M}'} = W_{M'}. \]

4.1. On Levi subgroups. For simplicity, in Sections 4.1 and 4.2, we will write \( G \) for both quasi-split unitary groups \( U_n \) and its non quasi-split \( F \)-inner forms \( G' \), and \( G \) for both quasi-split special unitary groups \( SU_n \) and its non quasi-split \( F \)-inner forms \( G' \).

Let \( \mathcal{M} \) and \( \mathcal{M} \) be \( F \)-Levi subgroups of \( G \) and \( G \), respectively. We recall from Section 3 that

\[ \mathcal{M} \simeq \text{GL}_{n_1}(E) \times \cdots \times \text{GL}_{n_k}(E) \times G_m, \]

\[ M \simeq (\text{GL}_{n_1}(E) \times \cdots \times \text{GL}_{n_k}(E)) \times G_m, \]

where \( \sum_{i=1}^k 2n_i + m = n \) with \( n_i \geq 0 \) and \( m \geq 0 \). Notice here that \( m \geq 2 \) and is always even for non quasi-split inner forms. We describe the action of Weyl group on Levi subgroups \( \mathcal{M} \) and \( \mathcal{M} \) as well as irreducible representations of \( \mathcal{M} \) and \( \mathcal{M} \), based on the results in [12, 13]. We denote by \( S_k \) the symmetric group in \( k \) letters. From [12, 13], we have

\[ (4.2) \quad W_{\mathcal{M}} = W_M \subset S_k \times \mathbb{Z}_2^k. \]

More precisely, \( W_{\mathcal{M}} \simeq S \ltimes C \), where \( S = \langle (ij) \mid n_i = n_j \rangle \), and \( C = \mathbb{Z}_2^k \). For \( g \in \mathcal{M} \), write

\[ g = (g_1, \ldots, g_i, \ldots, g_j, \ldots, g_k, h) \in \text{GL}_{n_1}(E) \times \text{GL}_{n_2}(E) \times \cdots \times \text{GL}_{n_k}(E) \times G_m. \]

The permutation \( (ij) \) acts on \( g \in \mathcal{M} \) by

\[ (ij) : (g_1, \ldots, g_i, \ldots, g_j, \ldots, g_k, h) \mapsto (g_1, \ldots, g_j, \ldots, g_i, \ldots, g_k, h). \]

The finite 2-group \( \mathbb{Z}_2^k \) is generated by “block sign changes” \( C_i \) which acts on \( g \in \mathcal{M} \) by

\[ C_i : (g_1, \ldots, g_i, \ldots, g_k, h) \mapsto (g_1, \ldots, \varepsilon(g_i), \ldots, g_k, h), \]

where \( \varepsilon(g_i) = t_\varepsilon^{-1} \).
4.2. On representations of Levi subgroups. Set $\Sigma$ to be $\Sigma_1 \otimes \Sigma_2 \otimes \cdots \otimes \Sigma_k \otimes \Upsilon$. From (4.3) and (4.4), we have
\[
(ij)\Sigma = \Sigma_1 \otimes \cdots \otimes \Sigma_j \otimes \cdots \otimes \Sigma_i \otimes \cdots \otimes \Sigma_k \otimes \Upsilon;
\]
\[
C_i \Sigma = \Sigma_1 \otimes \cdots \otimes \varepsilon(\Sigma_i) \otimes \cdots \otimes \Sigma_k \otimes \Upsilon,
\]
where $\varepsilon(\Sigma_i)(g_i) = \Sigma_i(\varepsilon(g_i))$, and these describe the action of $W_M$ on $\Sigma$.

Now we turn to the case of $M$. Note that, since $G_0 = G_1 = 1$, $W_M$ acts on $M$ and an irreducible representation of $M$ in the same manner for $m = 0, 1$. In particular, when $m = 0$ (thus, $n$ is even), from the proof of [13, Lemma 2.4], $M$ is of the form
\[
M = \{ \begin{pmatrix} g & \varepsilon(g) \\ \varepsilon(g) & 1 \end{pmatrix} : \det(g) \det(\varepsilon(g)) = 1, \ g \in GL_{n_1}(E) \times GL_{n_2}(E) \times \cdots \times GL_{n_k}(E) \},
\]
which implies $M \simeq \{ g \in GL_{n_1}(E) \times GL_{n_2}(E) \times \cdots \times GL_{n_k}(E) : \det g \in F^\times \}$. When $m = 1$ (thus, $n$ is odd), since $M \simeq GL_{n_1}(E) \times GL_{n_2}(E) \times \cdots \times GL_{n_k}(E) \times U_1(F)$, we have\[
M \simeq GL_{n_1}(E) \times GL_{n_2}(E) \times \cdots \times GL_{n_k}(E).
\]
Note $U_1(F) = E^I = \{ x \in E : \bar{x}x = 1 \}$. Further, for any $\Sigma \simeq \Sigma_1 \otimes \Sigma_2 \otimes \cdots \otimes \Sigma_k \otimes \omega \in \text{Irr}(M)$, we have
\[
\text{Res}_{G_n}^{G_m} \simeq \Sigma_1 \otimes \Sigma_2 \otimes \cdots \otimes \Sigma_k \otimes \omega \varepsilon,
\]
which is always irreducible (see the proof of [13, Lemma 2.5]). Let us move to the case of $m \geq 2$. For $g \in M$, from (3.8), we write\[
g = (g_0, h) = (g_1, \ldots, g_i, \ldots, g_j, \ldots, g_k, h) \in (GL_{n_1}(E) \times GL_{n_2}(E) \times \cdots \times GL_{n_k}(E)) \times G_m.
\]
Denote $w g w^{-1}$ by $(w g_0, h)$ for $w \in W_M$. Following arguments in [13, p.353], the permutation $(i j)$ acts on $g \in M$
\[
(ij) : (g_1, \ldots, g_i, \ldots, g_j, \ldots, g_k, h) \mapsto (g_1, \ldots, g_j, \ldots, g_i, \ldots, g_k, \alpha_m(g_0)^{-1}\alpha_m((ij)g_0)h)
\]
\[
= (g_1, \ldots, g_j, \ldots, g_i, \ldots, g_k, h),
\]
since $\alpha_m(g_0)^{-1}\alpha_m((ij)g_0) = I_m$. That is, the permutation $(i j)$ acts trivially on $G_m$.

The finite 2-group $Z_2^n$ is generated by “block sign changes” $C_i$ which acts on $g \in M$
\[
C_i : (g_1, \ldots, g_i, \ldots, g_k, h) \mapsto (g_1, \ldots, \varepsilon(g_i), \ldots, g_k, \alpha_m(g_0)^{-1}\alpha_m(C_ig_0)h).
\]
Note from the definition of $\alpha_m$ in (3.7) that
\[
\alpha_m(g_0)^{-1}\alpha_m(C_ig_0) := \begin{pmatrix} (\det(g_i) \det(\varepsilon(g_i)))^{-1} & I_{m-2} \\ I_{m-2} & \det(g_i) \det(\varepsilon(g_i)) \end{pmatrix} \in SU_m(F).
\]

Given $\sigma \in \text{Irr}(M)$, fix a lift $\Sigma \in \text{Irr}(M)$. Set $\Sigma \in \text{Irr}(M)$ to be $\Sigma_1 \otimes \Sigma_2 \otimes \cdots \otimes \Sigma_k \otimes \Upsilon$, and $\Upsilon$ to be a lift in $\text{Irr}(G_m)$ with $\tau \mapsto \text{Res}_{G_m}^{G_n} \Upsilon$. Write $V_\sigma, V_\Sigma$, for $i = 1, \ldots, k$, $V_\tau$, and $V_\sigma$ for the corresponding representation $C$-vector spaces. Then, $V_\sigma$ is of the form
\[
V_{\Sigma_1} \otimes V_{\Sigma_2} \otimes \cdots \otimes V_{\Sigma_k} \otimes V_\tau.
\]
From [13, p. 353], for $g_0, h = (g_1, \ldots, g_i, \ldots, g_j, \ldots, g_k, h) \in (GL_{n_1}(E) \times GL_{n_2}(E) \times \cdots \times GL_{n_k}(E)) \times G_m$, the representation $\sigma$ acts on $(v_1, v_2, \ldots, v_k, v_0) \in V_\sigma$ as follows,
\[
\sigma(g_0, h)(v_1, v_2, \ldots, v_k, v_0) = \sigma_1(g_1)(v_1) \otimes \sigma_2(g_2)(v_2) \otimes \cdots \otimes \sigma_k(g_k)(v_k) \otimes \Upsilon(\alpha_m(g_0)^{-1})\tau(h)(v_0).
\]
From (4.5) and [13, p. 353], we have
\[
(ij)\sigma(g_0, h) = \sigma_1(g_1) \otimes \cdots \otimes \sigma_j(g_j) \otimes \cdots \otimes \sigma_k(g_k) \otimes \Upsilon(\alpha_m((ij)g_0)^{-1})\tau(\alpha_m(g_0)^{-1}\alpha_m((ij)g_0)h)
\]
\[
= \sigma_1(g_1) \otimes \cdots \otimes \sigma_j(g_j) \otimes \cdots \otimes \sigma_k(g_k) \otimes \Upsilon(\alpha_m(g_0)^{-1})\tau(h).
\]
Likewise, using (4.6), we have
\[ C_i \sigma(g_0, h) = \sigma_1(g_1) \cdots \sigma_k(g_k) \otimes \tau(\alpha_m(C_i g_0) \alpha_m(C_i g_0) h) \]
Therefore, \( W_M \) acts non-trivially only on \( \sigma_1, \sigma_2, \ldots, \sigma_k, \) but trivially on \( \tau. \)

5. Revisiting R–groups for \( U_n \) and their inner forms

Based on some known results in [4, 12, 18, 23] regarding R-groups for \( U_n \) and its \( F \)-inner form, we discuss Arthur's conjecture for \( U_n \) and its inner forms, behavior of R-groups within L-packets of \( U_n \) and its inner forms, and behavior of R-groups between \( U_n \) and its inner forms.

5.1. L-packets for \( U_n \) and its inner forms. Let \( G = G_n \) denote the quasi-split unitary group \( U_n \) with respect to \( E/F \) and \( J_n, \) and \( M \) an \( F \)-Levi subgroup of \( G. \) For our purpose of studying R-groups, we focus on \( \Phi_{\text{temp}}(G). \) In [23, Theorem 2.5.1.(b)], Mok generalized Rogawski's results ([26]) in the case of unitary groups in three variables as follows. There is a surjective finite-to-one map \( \Pi_{\text{temp}}(G) \rightarrow \Phi_{\text{temp}}(G), \)
and for \( \varphi \in \Phi_{\text{temp}}(G), \) the tempered L-packet \( \Pi_{\varphi}(G) \) is constructed. The same is true for an \( F \)-inner form \( G' \) of \( G \) by Kaletha-Minguez-Shin-White [18, Section 1.6.1].

Let \( M \) and \( M' \) be F-Levi subgroups of \( G \) and \( G', \) respectively, which are \( F \)-inner forms of each other. Then, from (3.1) and (3.4), we recall
\[ M \simeq \text{Res}_{E/F} \text{GL}_{n_1} \times \cdots \times \text{Res}_{E/F} \text{GL}_{n_k} \times G_m, \]
and for \( \varphi \in \Phi_{\text{temp}}(G), \) the tempered L-packet \( \Pi_{\varphi}(G) \) is constructed. The same is true for an \( F \)-inner form \( G' \) of \( G \) by Kaletha-Minguez-Shin-White [18, Section 1.6.1].

5.2. Invariance of R-groups between \( U_n \) and its inner forms.

Theorem 5.1. (Goldberg, [12, Theorem 3.4]) Let \( M \) be an F-Levi subgroup of \( G. \) Given \( \Sigma \simeq \Sigma_1 \otimes \Sigma_2 \otimes \cdots \otimes \Sigma_k \otimes Y \in \Pi_{\text{disc}}(M), \) we have
\[ R_\Sigma \simeq \mathbb{Z}_2^d, \]
where \( d \) is the number of inequivalent \( \Sigma_i \) such that the induced representation \( i\varphi_{n_i+m} : \text{GL}_{n_i}(E) \otimes G_m(\Sigma \otimes Y) \) is reducible.

Theorem 5.2. Let \( \varphi \in \Phi_{\text{disc}}(M) \) be given. Under the identity (4.1), for any \( \Sigma \in \Pi_{\varphi}(M) \) and \( \Sigma' \in \Pi_{\varphi}(M'), \) we have
\[ W(\Sigma) = W(\Sigma'), \]
and
\[ W_\Sigma = W_\varphi = W_{\Sigma'}. \]
Therefore, we have
\[ R_\Sigma \simeq R_{\varphi,\Sigma} \simeq R_{\varphi} \simeq R_{\varphi,\Sigma'} \simeq R_{\Sigma'}. \]
Corollary 5.3. Let \( \varphi \in \Phi_{\text{disc}}(\mathcal{M}) \) be given. For \( \Sigma \in \Pi_{\varphi}(\mathcal{M}) \) and \( \Sigma' \in \Pi_{\varphi}(\mathcal{M}') \), we have \( i_{\varphi,\mathcal{M}}(\Sigma) \) is irreducible if and only if \( i_{\varphi,\mathcal{M}'}(\Sigma') \) is irreducible.

Corollary 5.4. Let \( \mathcal{M}' \) be an \( F \)-Levi subgroup of \( \mathcal{G}' \). Given \( \Sigma' \simeq \Sigma'_1 \otimes \Sigma'_2 \otimes \cdots \otimes \Sigma'_k \otimes \Upsilon' \in \Pi_{\text{disc}}(\mathcal{M}') \), we have

\[
R_{\Sigma'} \simeq \mathbb{Z}_2^d,
\]

where \( d \) is the number of inequivalent \( \Sigma'_i \) such that the induced representation \( i_{\varphi,\mathcal{M}'}(\Sigma') \) is reducible.

Corollary 5.5. Suppose \( w \in R_{\Sigma'} \) and \( w = sc \), with \( s \in S \) and \( c \in C \). Then \( w = 1 \).

Proof. This is a consequence of [12, Lemma 3.2] and Theorem 5.2.

Let \( \Sigma' \in \Pi_{\text{disc}}(\mathcal{M}') \) be given. Denote by \( \mathbb{C}[R(\Sigma')]_\eta \) the group algebra of \( R(\Sigma') \) twisted by a 2-cocycle \( \eta \), and by \( C(\Sigma') \), known as the commuting algebra of \( i_{\varphi,\mathcal{M}'}(\Sigma') \), the algebra \( \text{End}_{\mathcal{G}'}(i_{\varphi,\mathcal{M}'}(\Sigma')) \) of \( \mathcal{G}' \)-endomorphisms of \( i_{\varphi,\mathcal{M}'}(\Sigma') \). Then, we have

\[
C(\Sigma') \simeq \mathbb{C}[R(\Sigma')]_\eta
\]
as group algebras (see [19, 31, 32]).

Proposition 5.6. With the above notation, we have

\[
C(\Sigma') \simeq \mathbb{C}[R_{\Sigma'}].
\]

Proof. The proof is similar to [12, Proposition 4.1], due to Theorem 5.2 and Corollaries 5.4 and 5.5.

As a consequence of Proposition 5.6, we have the following corollary.

Corollary 5.7. Let \( \Sigma' \in \Pi_{\text{disc}}(\mathcal{M}') \) be given. Then, each constituent of \( i_{\varphi,\mathcal{M}'}(\Sigma') \) appears with multiplicity one.

Proposition 5.8. Let \( \Sigma' \in \Pi_{\text{disc}}(\mathcal{M}') \) be given. Then, \( i_{\varphi,\mathcal{M}'}(\Sigma') \) has an elliptic constituent if and only if all constituents of \( i_{\varphi,\mathcal{M}'}(\Sigma') \) are elliptic if and only if

\[
R_{\Sigma'} \simeq \mathbb{Z}_2^k.
\]

Proof. The proof is similar to [12, Theorem 4.3], due to Corollary 5.5 and Proposition 5.6.

Corollary 5.9. Let \( \varphi \in \Phi_{\text{disc}}(\mathcal{M}) \) be given. For any \( \Sigma \in \Pi_{\varphi}(\mathcal{M}) \) and \( \Sigma' \in \Pi_{\varphi}(\mathcal{M}') \), there is an elliptic constituent in \( i_{\varphi,\mathcal{M}}(\Sigma) \) if and only if there is an elliptic constituent in \( i_{\varphi,\mathcal{M}'}(\Sigma') \).

Proof. This is a consequence of [12, Theorem 4.3] and Proposition 5.8.

6. Behaviour of \( R \)-groups for \( SU_n \) and its inner forms

In this section, we prove Arthur’s conjecture for both quasi-split special unitary groups \( SU_n \) and its inner forms. Furthermore, we study the behavior of \( R \)-groups within \( L \)-packets and between inner forms of \( SU_n \). We will use the notation in Sections 3, 4, and 5.

6.1. \( L \)-packets for \( SU_n \) and its inner forms. We discuss tempered \( L \)-packets of \( \mathcal{G} = SU_n \) and its \( F \)-inner form \( \mathcal{G}' \). It is natural to construct \( L \)-packets for \( \mathcal{G} \) and \( \mathcal{G}' \), by restricting \( L \)-packets for \( \mathcal{G} = U_n \) and its \( F \)-inner form \( \mathcal{G}' \) which has been done by Rogawski [20], Mok [23], and Kaletha-Minguez-Shin-White [18] (see Section 5.1). Thus, given \( \phi \in \Phi(\mathcal{G}) \), from [22, Théorème 8.1], there exists a lifting \( \varphi \in \Phi(\mathcal{G}) \) such that

\[
\phi = \varphi \circ pr,
\]

where \( pr \) is the projection \( \widehat{\mathcal{G}} \to \widehat{\mathcal{G}} \) in (3.13). Note that the homomorphism \( pr \) is compatible with \( \Gamma \)-actions on \( \widehat{\mathcal{G}} \) and \( \widehat{\mathcal{G}} \) (see Section 3.3) and the lifting \( \varphi \in \Phi(\mathcal{G}) \) is chosen uniquely up to a 1-cocycle of \( W_F \) in \( (\mathcal{G}/\mathcal{G}) \) (see [22, Section 7] and [7, Theorem 3.5.1]).
For our purpose, we are interested in $\phi \in \Phi_{\text{temp}}(G)$ and the lifting $\varphi$ lies in $\Phi_{\text{temp}}(G)$. Using the $L$-packet $\Pi_{\varphi}(G)$ for $\varphi \in \Phi_{\text{temp}}(G)$ in Section 5.1, we construct an $L$-packet $\Pi_{\phi}(G)$ for $\phi \in \Phi_{\text{temp}}(G)$ as the set of isomorphism classes of irreducible constituents in the restriction from $G$ to $G$ as follows:

$$\Pi_{\phi}(G) := \{ \sigma \mapsto \operatorname{Res}_G^G(\Sigma), \Sigma \in \Pi_{\varphi}(G) \}/ \sim.$$ 

Likewise, given $\phi \in \Phi_{\text{disc}}(M)$, we construct an $L$-packet $\Pi_{\phi}(M)$ for $\phi \in \Phi_{\text{disc}}(M)$ as follows:

$$\Pi_{\phi}(M) := \{ \sigma \mapsto \operatorname{Res}_M^M(\Sigma), \Sigma \in \Pi_{\varphi}(M) \}/ \sim,$$

where $\varphi$ lies in $\Phi_{\text{disc}}(M)$ such that $\phi = \varphi \circ pr$, with the projection $pr : \hat{M} \to \hat{M}$ in (3.12).

All the above arguments apply verbatim to $F$-inner forms $G'$ and their $F$-Levi subgroups $M'$.

6.2. Arthur conjecture for $SU_n$ and its inner forms. The purpose of the section is to prove Arthur’s conjecture, predicted in [1], for $G$ and $G'$.

**Theorem 6.1.** Given $\phi \in \Phi_{\text{disc}}(M)$, $\sigma \in \Pi_{\phi}(M)$, and $\sigma' \in \Pi_{\phi}(M')$, we have

$$R_{\sigma} \simeq R_{\phi,\sigma} \text{ and } R_{\sigma'} \simeq R_{\phi,\sigma'}.$$

The rest of the section is devoted to the proof of Theorem 6.1. Since all the following techniques apply to both $M$ and $M'$, we shall state the proof for $M$.

Let $\phi \in \Phi_{\text{disc}}(M)$ and $\sigma \in \Pi_{\phi}(M)$ be given. Identifying $W_\phi$ with a subgroup of $W_M$ (see Section 2.2), we note that $W(\sigma) \subset W_\phi$, which implies that

$$(6.1) \quad W(\sigma) = W_{\sigma,\phi} = W_\phi \cap W(\sigma).$$

Denote by $\varphi \in \Phi_{\text{disc}}(M)$ a lifting of $\phi$ as in Section 6.1. For any $\Sigma \in \Phi_{\varphi}(M)$, we have

$$W^\circ_\Sigma = W^\circ_{\sigma},$$

since the Plancherel measure is compatible with restriction (see [8, Proposition 2.4], [13, Lemma 2.3], for example) and $\Phi(P, A_M) = \Phi(P, A_M)$ (see Section 2.1). Using (5.1), we have

$$(6.2) \quad W_\phi^\circ = W_{\phi}^\circ.$$

To prove Theorem 6.1, from (6.1) and (6.2), it is thus enough to show that

$$(6.3) \quad W^\circ_{\phi} = W^\circ_{\varphi}.$$ 

Indeed, if so, then we have

$$W^\circ_{\sigma,\phi} \overset{\text{definition}}{=} W^\circ_{\phi} \cap W(\sigma) \overset{(6.3)}{=} W^\circ_{\phi} \cap W(\sigma) \overset{(6.2)}{=} W^\circ_{\sigma} \cap W(\sigma) = W^\circ_{\sigma}.$$ 

In what follows, we prove (6.3). From [23, Chapter 3.4] we set a maximal torus $T_{\varphi}$ in $C_{\varphi}(\hat{G})^\circ$ to be the identity component

$$A_{\hat{M}} = (Z(\hat{M})^\Gamma)^\circ$$

doing the $\Gamma$–invariants of the center $Z(\hat{M})$. So, we have $\hat{M} = Z_{\hat{G}}(T_{\varphi})$. Likewise, we set $T_\phi = A_{\hat{M}} \subset C_{\phi}(\hat{G})^\circ$, and $\hat{M} = Z_{\hat{G}}(T_\phi)$. Here, put

$$\hat{C}_{\varphi} := C_{\varphi}(\hat{G})^\circ / Z(\hat{G})^\Gamma, \quad \hat{T}_{\varphi} := T_{\varphi} / Z(\hat{G})^\Gamma.$$

Note that $\hat{C}_{\varphi} \subset C_{\phi}(\hat{G})^\circ \subset \hat{G}$ and $T_{\varphi} \subset T_\phi \subset \hat{M}$. Using an equality in [23, p.64], we have

$$W^\circ_{\varphi} \overset{\text{definition}}{=} N_{C_{\varphi}(\hat{G})^\circ}(T_{\varphi}) = N_{\hat{C}_{\varphi}}(T_{\varphi}).$$

Thus, it suffices to show that

$$N_{\hat{C}_{\varphi}}(T_{\varphi}) = N_{C_{\phi}(\hat{G})^\circ}(T_\phi).$$

With above notation, the following lemma holds.

**Lemma 6.2.**

$$\hat{C}_{\varphi} = C_{\phi}(\hat{G})^\circ$$
Proof. We first consider the following commutative diagram

\[
\begin{array}{ccccccccc}
1 & \rightarrow & Z(\widehat{G})^\Gamma & \rightarrow & C_{\varphi}(\widehat{G})^\circ & \rightarrow & \overline{\varphi} := C_{\varphi}(\widehat{G})^\circ / Z(\widehat{G})^\Gamma & \rightarrow & 1 \\
\| & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & Z(\widehat{G})^\Gamma & \rightarrow & C_{\varphi}(\widehat{G}) & \rightarrow & C_{\varphi}(\widehat{G}) / Z(\widehat{G})^\Gamma & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
S_{\varphi} := \pi_0(C_{\varphi}(\widehat{G})) & \rightarrow & S_{\varphi} := \pi_0(C_{\varphi}(\widehat{G})) & \rightarrow & 1.
\end{array}
\]

It follows from the right vertical exact sequence above that

\[
\overline{\varphi} = C_{\varphi}(\widehat{G})^\circ / Z(\widehat{G})^\Gamma \subset (C_{\varphi}(\widehat{G}) / Z(\widehat{G})^\Gamma)^\circ \subset C_{\varphi}(\widehat{G}) / Z(\widehat{G})^\Gamma.
\]

Note that the index \([\overline{\varphi} : C_{\varphi}(\widehat{G}) / Z(\widehat{G})^\Gamma]\) is finite and so is \([\overline{\varphi} : (C_{\varphi}(\widehat{G}) / Z(\widehat{G})^\Gamma)^\circ]\). Since \(\overline{\varphi}\) is connected due to the isomorphism \(C_{\varphi}(\widehat{G})^\circ / Z(\widehat{G})^\Gamma \simeq (C_{\varphi}(\widehat{G})^\circ / Z(\widehat{G})) / Z(\widehat{G})\), we have

\[
(6.4) \quad \overline{\varphi} = (C_{\varphi}(\widehat{G}) / Z(\widehat{G})^\Gamma)^\circ.
\]

From the proof of [7, Lemma 5.3.4] and the fact that \(C_{\varphi}(\widehat{G}) / Z(\widehat{G})^\Gamma \subset C_{\varphi}(\widehat{G})\), we have the exact sequence

\[
1 \rightarrow C_{\varphi}(\widehat{G}) / Z(\widehat{G})^\Gamma \rightarrow C_{\varphi}(\widehat{G}) \rightarrow X^\varphi(\varphi) \rightarrow 1,
\]

where \(X^\varphi(\varphi) := \{a \in H^1(W_{\overline{F}}, \mathbb{U}_1) : a\varphi \simeq \varphi\}\). Since \(X^\varphi(\varphi)\) is finite, we have

\[
(C_{\varphi}(\widehat{G}) / Z(\widehat{G})^\Gamma)^\circ = C_{\varphi}(\widehat{G})^\circ.
\]

Therefore, from (6.4), Lemma 6.2 is proved.

Combining Lemmas 3.3 and 6.2, we have proved (6.3). This completes the proof of Theorem 6.1.

6.3. Invariance of R-groups within L-packets and between inner forms. Let \(\phi \in \Phi_{\text{disc}}(M)\) be given. In this section, we will discuss the behavior of Knapp-Stein R-groups within L-packets and between inner forms. We first provide the following theorem that is crucial to the invariance (Theorems 6.6 and 6.8).

Theorem 6.3. Fix a lift \(\varphi \in \Phi_{\text{disc}}(M)\) and \(\Sigma \in \Pi_{\varphi}(M)\). Then, we have

\[
W(\sigma) \simeq \{w \in W_M : \text{\#} \Sigma \simeq \Sigma \lambda \text{ for some } \lambda \in (M/M)^D\}.
\]

Proof. It suffice to consider the case of \(m \geq 2\), since the equality is already true for the case \(m = 1\) due to [13, Lemma 2.5]. Let \(m \geq 2\). We then recall from (3.5) that

\[
M \simeq (\text{GL}_m(E) \times \cdots \times \text{GL}_{m_k}(E)) \times \text{SU}_m(F).
\]

Since \(\Sigma\) is of the form

\[
\Sigma_1 \otimes \Sigma_2 \otimes \cdots \otimes \Sigma_k \otimes \Upsilon,
\]

we set \(\Sigma = \Sigma_0 \otimes \Upsilon\). Denote by \(\tau\) is an irreducible constituent in \(\text{Res}^U_{\text{SU}_m}(\Upsilon)\). Write \(V_{\Sigma}, V_{\Sigma_0} = \otimes_{i=1}^k V_{\Sigma_i}, V_\Upsilon, V_\tau\) for the corresponding representation \(\mathbb{C}\)-vector spaces.

Given \((g, h) \in M\) with \(g = (g_1, \cdots, g_k) \in \text{GL}_m(E) \times \cdots \times \text{GL}_{m_k}(E)\) and \(h \in \text{SU}_m(F)\), \(v_0 = v_1 \otimes \cdots \otimes v_k \in V_{\Sigma_0}\), and \(v_0 \in V_\tau\), we define a representation \(\sigma^*\) of \(M\)

\[
(6.5) \quad \sigma^*((g, h))(v_0 \otimes v_0) = \Sigma_0(g) v_0 \otimes \Upsilon(a_m(g)^{-1}) \tau(h) v_0.
\]

It turns out from [13, p.353] that \(\sigma^*\) is an irreducible constituent in \(\text{Res}^U_{\text{SU}_0}(\Sigma)\).

In what follows, using the idea in the proof of [13, Proposition 2.9], we will show

\[
(6.6) \quad W(\sigma^*) = \{w \in W_M : \text{\#} \Sigma \simeq \Sigma \lambda \text{ for some } \lambda \in (M/M)^D\}.
\]
The inclusion $\subset$ is obvious. Suppose that $\omega \Sigma \simeq \Sigma \lambda$ for some $\lambda \in (M/M)^D$. From (4.2), we set $w = sc$ with $s \in S_k$ and $c \in \mathbb{Z}_2^k$. From Sections 4.1 and 4.2, we have

$$W^\omega \Sigma \simeq \bigotimes_{i=1}^k \varepsilon_i(\Sigma_{s(i)}) \otimes \Upsilon,$$

where $\varepsilon_i$ is either $\varepsilon$ or trivial. Since $W^\omega \Sigma \simeq \Sigma \lambda$, we have $\varepsilon_i(\Sigma_{s(i)}) \simeq \Sigma_i \lambda$ for each $i$, and $\Upsilon \simeq \Sigma \lambda$. We fix intertwining maps $T_i : V_{\Sigma_{s(i)}} \to V_{\Sigma_i}$ with $\Sigma_i \lambda T_i = T_i \varepsilon_i(\Sigma_{s(i)})$, and $T_\lambda : V_T \to V_T$ with $\Upsilon \lambda T_\lambda = T_\lambda \Upsilon$. Write

$$T = \bigotimes_{i=1}^k T_i \otimes T_\lambda.$$

Note that $\Sigma T = T^\omega \Sigma$. Then, from the definition (6.5) of $\sigma^*$, we have

$$T^\omega \sigma^*(g, h)(v'_0 \otimes v_0) = \bigotimes_{i=1}^k T_i \varepsilon_i(\Sigma_{s(i)})(g_i)v_i \otimes T_\lambda \Upsilon(\alpha_m(w g)^{-1} \tau(\alpha_m(g)^{-1} \alpha_m(w g) h)v_0$$

$$\quad = \bigotimes_{i=1}^k T_i \varepsilon_i(\Sigma_{s(i)})(g_i)v_i \otimes T_\lambda \Upsilon(\alpha_m(g)^{-1}) \tau(h)v_0$$

$$\quad = \bigotimes_{i=1}^k \Sigma_i(g_i) \lambda(g_i) T_i(v_i) \otimes \Upsilon(\alpha_m(g)^{-1}) \lambda(\alpha_m(g)^{-1}) \tau(h) T_\lambda(v_0)$$

$$\quad = \sigma^* \lambda(g, h) T(v'_0 \otimes v_0)$$

$$\quad = \sigma^* T(v'_0 \otimes v_0),$$

since

$$\lambda(g, h) \overset{(3.6)}{=} \lambda\left(\begin{array}{c} g \\ \alpha_m(g)^{-1} h \\ \varepsilon(g) \end{array}\right) = 1.$$

Thus, we have $w \in W(\sigma^*)$, which implies (6.6). From the proof of [9, Theorem 4.19], we note

$$W(\sigma) \simeq W(\sigma^*),$$

since both are in the same restriction $\mathrm{Res}_{SU_n}^U (\Sigma)$. Therefore, we proved the theorem. \hfill \square

Remark 6.4. The difference between $W(\sigma)$ and $W(\Sigma)$ turns out to be whether the twisting by a character in $(M/M)^D$ is allowed or not. We also refer to [13, Theorem 3.7] for another description of this difference.

Remark 6.5. We note that Theorem 6.3 holds for $F$-inner forms $M'$ as well. To be precise, given $\sigma' \in \Pi_{\phi}(M')$, we fix $\Sigma' \in \Pi_{\phi}(M')$. Then, we have

$$W(\sigma') \simeq \{ w \in W_{M'} : \omega \Sigma' \simeq \Sigma' \lambda \text{ for some } \lambda \in (M'/M')^D \}.$$
Fix $\Sigma^1$ and $\Sigma^2$ in $\Pi_\phi(M)$ such that $\sigma_1 \hookrightarrow \Res_{SU_n}^U(\Sigma^1)$ and $\sigma_2 \hookrightarrow \Res_{SU_n}^U(\Sigma^2)$. Suppose that $w \Sigma^1 \cong \Sigma^1 \lambda$ for some $\lambda \in (M/M)^\times$. From (4.2), we write $w = sc$ with $s \in S_k$ and $c \in Z_k$. Set $\Sigma^1 \cong \Sigma^1_1 \oplus \Sigma^2_1 \oplus \cdots \oplus \Sigma^k_1 \otimes \Upsilon_1$. Then $\Sigma^1_i \cong \varepsilon_i(\Sigma^1_{c(i)}) \lambda$ for each $i$, and $\Upsilon_1 \cong \Upsilon_1 \lambda$. Set

$$\lambda' := \left\{ \begin{array}{ll}
\lambda, & \text{on } \GL_{n_1}(E) \times \cdots \times \GL_{n_k}(E), \\
1, & \text{on } U_m(F),
\end{array} \right.$$ 

which is a character on $M/M$. Moreover, $\lambda'$ satisfies

$$w \Sigma^2 \cong \Sigma^2 \lambda',$$

which implies, from Theorem 6.3, that $W(\sigma_1) \hookrightarrow W(\sigma_2)$. In the same manner, one can verify $W(\sigma_2) \hookrightarrow W(\sigma_1)$. Thus, we have $R_{\sigma_1} \cong R_{\sigma_2}$. Since the method is the same, we omit the proof for $R_{\sigma_1'} \cong R_{\sigma_2'}$.  

Remark 6.7. The trivial character 1 in the definition of $\lambda'$ can be replaced by a character $\lambda_0$ on $U_m(F)$ such that $\Upsilon_2 \cong \Upsilon_2 \lambda_0$, where $\Upsilon_2$ is the representation of $U_m(F)$ in the decomposition $\Sigma^2 \cong \Sigma^2_1 \oplus \Sigma^2_2 \oplus \cdots \oplus \Sigma^2_k \otimes \Upsilon_2$.

Theorem 6.8. Given $\sigma \in \Pi_{\phi}(M)$ and $\sigma' \in \Pi_{\phi}(M)$, we have

$$R_\sigma \cong R_{\sigma'}.$$ 

Proof. The proof is similar to Theorem 6.6.  

Remark 6.9. Due to [12, Theorem 3.7], Theorem 6.8 shows that the Knapp-Stein $R$-group $R_{\sigma'}$ is of the form

$$\Gamma_{\sigma'} \times \mathbb{Z}^d,$$

for some subgroup $\Gamma_{\sigma'}$ in $R_{\sigma'}$ defined in loc. cit and some integer $d$ in Corollary 5.4. This implies that $R_{\sigma'}$ is in general non-abelian from the argument in [12, Remark p.359].

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