A Chattering-Free Sliding Mode Filter Enhanced by First Order Derivative Feedforward

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ABSTRACT Noise reduction is one of the important issues for feedback control systems. Toward this problem, this paper proposes a new sliding mode filter, which is an improvement of Lv et al.’s parabolic sliding mode filter. The proposed filter employs a first-order derivative feed-forward term for increasing both tracking and noise attenuating performances. Its discrete-time algorithm is derived by applying the implicit-Euler discretization, and it realizes exact convergence without producing chattering in discrete-time implementation. The effectiveness of the proposed filter is evaluated through an open-loop and a closed-loop numerical examples.

INDEX TERMS Sliding mode filter, implicit Euler discretization, chattering avoidance, noise reduction, PD control, feedback control.

I. INTRODUCTION

Noise reduction is one of the important issues for feedback control systems. To remove noise, linear filters are widely applied owing to their simplicity. However, a large phase lag that is associated with a strong noise reduction may cause instability of the controlled systems.

Nonlinear filtering techniques have been researched for overcoming the disadvantages of linear methods. For example, Kalman filters [1]–[3] are widely applied for removing noise. However, they may diverge in the cases of large initial errors, and their performance depends on the accuracy of the system model. As another example, sliding mode technique [4]–[8], which was initially developed for control of nonlinear dynamic systems [9]–[13], based super-twisting observers [14]–[18] have been of significant research interest and have great practical applications for their theoretical robustness, finite-time convergence, and sufficient tracking performance. However, these observers tend to overshoot because of inherent converging behavior. In addition, the explicit-Euler discretization may degrade their advantages in digital controller implementation, particularly resulting in the chattering of the output [19]–[21].

High-order sliding mode observer [22]–[25] is useful for reducing chattering to some extent by increasing the relative degree. However, they cannot eliminate chattering completely, and their convergence speed is associated with the relative degree. Moreover, the effectiveness of sliding mode observers is also linked with the system model’s accuracy.

Sliding mode technique based model-free filters have been studied for smoothing signals without considering the system model. Jin et al. [26] proposed a sliding mode filter that possesses a parabolic-like sliding surface (PSMF). The system state of PSMF is directed to both sides of the sliding surface, while that of the conventional filter [27] is directed to only one side of the sliding surface. Thus, PSMF is less likely to overshoot than the conventional one. In addition, it is reported [28] that, compared with the filter [27], PSMF exhibits less phase lag and attenuates noise more effectively. Extensions of PSMF are addressed in the literature. For instance, Lv et al. [29] proposed a quick responding PSMF (PSMF-L) by including an exponentially converging term for accelerating the convergence without sacrificing the filtering performance. As another instance, Aung et al. [30] presented a filtering system (PSMF-A) that is an integration of a modification of PSMF and a low-pass filter for providing a smooth signal. However, in the abovementioned PSMFs, there is still a challenge for the trade-off between the effective
noise attenuation and the rapid response speed in the presence of strong noise contamination.

Toward these issues, this paper proposes a sliding mode filter, which is an improvement of PSMF-L, by introducing a first-order derivative feed-forward term. The proposed filter consists of two parts. One is PSMF-L that serves the estimator of the first-order derivative of the input signal, and the other one is a modification of PSMF-L that employs the estimation of PSMF-L as a feed-forward term. Its chattering-free discrete-time algorithm is derived by applying the implicit-Euler discretization, which is potentially effective in realizing chattering-free discrete-time sliding mode by keeping inherent advantages of the original continuous-time system [31]–[35]. The effectiveness of the proposed filter is evaluated through an open-loop and a closed-loop numerical examples.

The rest parts of the paper are organized as follows: Section II provides some mathematical preliminaries that are applied in the following sections. Section III briefly covers the related literature. Section IV proposes a new sliding mode filter, and Section V derives its chattering-free discrete-time algorithm by applying the implicit-Euler discretization. Section VI evaluates the performance of the proposed filter through numerical examples. Finally, Section VII gives the conclusion of the paper.

II. MATHEMATICAL PRELIMINARIES

In the rest part of this paper, the following defined functions are used:

\[

csgn(h) = \begin{cases} 
0 & \text{if } h = 0 \\
|h| & \text{if } h \neq 0 
\end{cases} 
\]

(1)

\[
gsgn(\alpha, h, \beta) = \begin{cases} 
\beta & \text{if } h > 0 \\
\alpha & \text{if } h < 0 \\
[\alpha, \beta] & \text{if } h = 0 
\end{cases} 
\]

(2)

and

\[
gsat(\alpha, h, \beta) = \begin{cases} 
\beta & \text{if } h > \beta \\
h & \text{if } h \in [\alpha, \beta] \\
\alpha & \text{if } h < \alpha, 
\end{cases} 
\]

(3)

where \( h, \alpha, \beta \in \mathbb{R} \) and \( \alpha < \beta \). When \( h = 0 \), the return value of the function \( gsgn(\cdot) \) is a set, while that of the function \( csgn(\cdot) \) is a scalar.

Moreover, the following equivalence is applied in the process of deriving a discrete-time algorithm of the proposed filter:

\[
\phi(h) - k = gsgn(gsgn(\alpha_1, m - h, \alpha_2), n - h, gsgn(\alpha_3, m - h, \alpha_4)) 
\]

\[
\iff 
\phi(h) - k = gsat(gsat(\alpha_1, \phi(m) - k, \alpha_2), \phi(n) - k, \phi(\alpha_3, \phi(m) - k, \alpha_4)).
\]

Here \( \phi(\cdot) \) is a monotonically increasing function with \( \phi(0) = 0 \), and \( k, m, n, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R} \), and \( \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \).

III. OVERVIEW OF PSMFS

Jin et al. [26] proposed the following sliding mode filter that possesses a parabolic-like sliding surface (PSMF):

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &\in -\Gamma gsgn(gsgn(-\lambda, x_2, -1), \sigma, gsgn(1, x_2, \lambda)) \\
y &= x_1,
\end{align*}
\]

(4)

where

\[
\sigma \triangleq 2\Gamma(x_1 - u) + |x_2| x_2. 
\]

(5)

Here, \( u \) is the input, \( y \) is the output, \( x_1 \) is the estimation of \( u \), \( x_2 \) is the estimation of \( \dot{u} \), and \( \Gamma > 0 \) and \( \lambda > 1 \) are constants. Fig. 1 illustrates the behavior of PSMF in the state space. It is shown that the state is attracted to the sliding surface with \( x_2 = |\lambda \Gamma| \) in the case of \( \sigma x_2 > 0 \), while it is directed to the sliding surface with \( x_2 = |\Gamma| \) in other cases. However, in the case of using a large \( \lambda \), PSMF becomes sensitive to noise, although increasing \( \lambda \) contributes to convergence speed and overshoot suppression.

![Figure 1. Behavior of PSMF, i.e., the filter (4), in the state space.](image)

Toward this problem, Lv et al. [29] proposed the following improvement of PSMF that includes an exponentially converging term for increasing convergence speed without sacrificing the filtering performance (PSMF-L):

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &\in -\Gamma gsgn(gsgn(-\lambda, x_2, -1), \sigma, gsgn(1, x_2, \lambda)) - k \sigma \\
y &= x_1,
\end{align*}
\]

(6)

where \( k > 0 \) is a constant. As shown in Fig. 2, the state of PSMF-L is directed to the sliding surface more sharply in the regions \( \sigma \neq 0 \). Its effectiveness over PSMF has been validated in [29].

IV. PSMF WITH FIRST ORDER DERIVATIVE FEEDFORWARD

One limitation of PSMF-L is that the finite-time reaching condition of \( x_1 = u \) may be destroyed in the case of \( \dot{u} \neq 0 \), resulting in increased phase leg. One possible remedy for
overcoming this problem is to feed forward the first-order derivative of \( u \) to the sliding surface \( \sigma \) of PSMF-L, which leads to the followings:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &\in - \Gamma \text{sgn}(\text{sgn}(-\lambda, x_2, -1), \sigma, \text{sgn}(1, x_2, \lambda)) - k \sigma \\
y &= x_1,
\end{align*}
\]

where

\[
\sigma \triangleq 2\Gamma (x_1 - u) + |x_2 - \dot{u}|(x_2 - \dot{u}).
\]

Fig. 3 illustrates the behavior of the filter (7) in the state space.

However, this modification introduces a side effect, i.e., when the input \( u \) is corrupted by high-frequency noise, the performance of the filter (7) is distorted and biased. This is because the newly added term \( \dot{u} \) amplifies the input noise.

In order to track the input signal rapidly by reducing noise effectively, this section proposes the following new filter, which is further modification of the filter (7) (PSMF-M) by integrating PSMF-L:

\[
\begin{align*}
\dot{x}_{M1} &= x_M \\
\dot{x}_M &= - \Gamma \text{sgn}(\text{sgn}(-\lambda_M, x_M, -1), \sigma_M, \text{sgn}(1, x_M, \lambda_M)) - k_M\sigma_M, \\
y &= x_{M1},
\end{align*}
\]

where

\[
\sigma_L \triangleq 2\Gamma_L(x_{L1} - u) + |x_{L2}|x_{L2}
\]

and

\[
\sigma_M \triangleq 2\Gamma_M(x_{M1} - u) + |x_{M2} - x_{L2}|(x_{M2} - x_{L2}).
\]

Here, \( \Gamma_L > 0, \Gamma_M > 0, \lambda_L > 1, \lambda_M > 1, k_L > 0 \) and \( k_M > 0 \). In addition, the subscript “L” refers to PSMF-L, and the subscript “M” refers to PSMF-M. Fig. 4 illustrates the block diagram of the proposed filter. In the proposed filter, PSMF-L (i.e., (9a), (9b) and (10)) can be considered as a feed-forward estimation of \( \dot{u} \) for PSMF-M (i.e., (9c), (9d) and (11)). Owing to this idea, the proposed filter provides a rapid and smooth output signal even in the presence of noise, as will be demonstrated in Section VI.

### V. DISCRETE-TIME IMPLEMENTATION OF THE PROPOSED FILTER

In order to implement the proposed filter on digital controllers, the continuous-time expression (9) must be properly discretized. Since the chattering-free discrete-time algorithm of PSMF-L has been presented in [29] (also provided in Appendix B), this section only derives an implicit-Euler based discrete-time algorithm of PSMF-M.

Based on the implicit-Euler discretization [31]–[35], (9c), (9d) and (11) can be approximated as:

\[
\begin{align*}
\frac{x_{M1}(i) - x_{M1}(i - 1)}{T} &= x_{M2}(i) \\
\frac{x_{M2}(i) - x_{M2}(i - 1)}{T} &= - \Gamma_M \text{sgn}(\text{sgn}(-\lambda_M, x_{M2}(i), -1), \sigma_M(i), \text{sgn}(1, x_{M2}(i), \lambda_M)) - k_M\sigma_M(i),
\end{align*}
\]

and

\[
\sigma_M(i) = 2\Gamma_M(x_{M1}(i) - u(i)) + |x_{M2}(i) - x_{L2}(i)|(x_{M2}(i) - x_{L2}(i)).
\]
where $T$ is the sampling interval and $i$ is the discrete-time index. It should be noticed that unknown variables $\sigma_M(i)$ and $x_{M2}(i)$ are in the set-valued function $gsgn(.)$ of (12b), which in general cannot be solved directly by conventional methods.

For this problem, one can apply the following procedures. First, by solving (12a) with respect to $x_{M1}(i)$, one can obtain:

$$x_{M1}(i) = Tx_{M2}(i) + x_{M1}(i-1).$$

(14)

Then, by substituting (14), (13) can be rewritten as follows:

$$\sigma_M(i) = [x_{M2}(i) - x_{L2}(i)](x_{M2}(i) - x_{L2}(i)) + 2\Gamma_MT x_{M2}(i) + 2\Gamma_M(x_{M1}(i-1) - u(i)),

(15)$$

which can be seen as a function of $x_{M2}(i)$. Subsequently, (12b) can be rewritten as follows by using (15):

$$Z_M e_{M2}(i) e_{M2}(i) + H_M e_{M2}(i) - \Upsilon_M

= \Gamma_MT gsgn(gsgn(-\lambda_M, -x_{L2}(i) - e_{M2}(i), -1), -\sigma_M(i), gsgn(1, -x_{L2}(i) - e_{M2}(i), \lambda_M)),

(16)$$

where $e_{M2}(i)$ is an intermediate variable defined as follows:

$$e_{M2}(i) \triangleq x_{M2}(i) - x_{L2}(i),

(17)$$

and $Z_M$, $H_M$, and $\Upsilon_M$ are known variables defined as follows:

$$Z_M \triangleq k_M T,

(18)$$

$$H_M \triangleq 1 + 2k_M \Gamma_M T^2,

(19)$$

$$\Upsilon_M \triangleq x_{M2}(i-1) - 2k_M \Gamma_M T u(i) - 2k_M \Gamma_M x_{M1}(i-1) - H_M x_{L2}(i).

(20)$$

It should be noticed that there are still unknown variables $e_{M2}(i)$ and $\sigma_M(i)$ in the set-value function $gsgn(.)$ of (16), which can be further simplified as follows:

$$Z_M e_{M2}(i) e_{M2}(i) + H_M e_{M2}(i) - \Upsilon_M

= \Gamma_MT gsgn(gsgn(-\lambda_M, -x_{L2}(i) - e_{M2}(i), -1), x_{M2}^\dagger(i) - x_{M2}(i), gsgn(1, -x_{L2}(i) - e_{M2}(i), \lambda_M)).

(21)$$

where $x_{M2}^\dagger(i)$ is the value of $x_{M2}(i)$ that satisfies $\sigma_M(i) = 0$ at each discrete-time step. It should be mentioned that, since $\sigma_M(i)$ is an increasing function of $x_{M2}(i)$ and $\sigma_M(x_{M2}^\dagger(i)) = 0$, $\sigma_M(i)$ and $x_{M2}(i) - x_{M2}^\dagger(i)$ have the same sign (i.e., whether positive or negative).

After that, for further simplification, let us define an intermediate variable $\sigma_M^*(i)$ as follows:

$$\sigma_M^*(i) \triangleq 2(x_{M1}(i) - u(i)) + |x_{M2}(i)| x_{M2}(i)

(22)$$

Similarly, $\sigma_M^*(i)$ can be seen as an increasing function of $x_{M2}(i)$. Then, as shown in Fig. 5, the parabolic curve $\sigma_M(i) = 0$ can be obtained by shifting the parabolic shape $\sigma_M^*(i) = 0$ upward $x_{L2}(i)$ units. Thus, one can obtain the following relation:

$$x_{M2}(i) - x_{M2}^\dagger(i) = x_{M2}(i) - x_{L2}(i) - x_{M2}(i)

= e_{M2}(i) - x_{M2}^\dagger(i),

(23)$$

where $x_{M2}(i)$ is the value of $x_{M2}(i)$ that satisfies $\sigma_M^*(x_{M2}^*(i)) = 0$, i.e.,:

$$x_{M2}^*(i) = csgn(x_{M1}(i-1) - u(i))

\times \left( \Gamma_MT - \sqrt{\frac{H_M}{\Gamma_M^2} T^2 + 2\Gamma_M |x_{M1}(i-1) - u(i)|}. \right)

(24)$$

Therefore, (21) can be rewritten as follows:

$$\phi_M(\psi_{M2}(i)) - \Upsilon_M(i)

= gsgn(gsgn(-\lambda_M \Gamma_M T, -x_{L2}(i) - e_{M2}(i), -\Gamma_M T), x_{M2}^*(i) - e_{M2}(i), gsgn(\Gamma_M T, -x_{L2}(i) - e_{M2}(i), \lambda_M \Gamma_M T)),

(25)$$

where $\phi_M(\psi_{M2}(i)) = Z_M e_{M2}(i) e_{M2}(i) + H_M e_{M2}(i)$. One can observe that $\phi_M(\psi_{M2}(i))$ is an increasing function of $e_{M2}(i)$, and $\phi_M(\psi_{M2}(i)) \in (-\infty, \infty)$ and $\phi_M(0) = 0$. Now, by applying the Proposition that is provided in the Appendix A, the unknown variable $e_{M2}(i)$ that is in the set-valued function $gsgn(.)$ of (25) can be moved out as follows:

$$\phi_M(\psi_{M2}(i)) - \Upsilon_M(i)

= gsat(gs(\lambda_M \Gamma_M T, \phi_M(-x_{L2}(i)) - \Upsilon_M(i), -\Gamma_M T), \phi_M(x_{M2}^*(i)) - \Upsilon_M(i),

\psi_{M2}(\phi_M(-x_{L2}(i)) - \Upsilon_M(i), \lambda_M \Gamma_M T)).

(26)$$

Then, the solution of (26) with respect to $e_{M2}(i)$ can be obtained as follows:

$$e_{M2}(i) = \frac{\psi_{M2}(\psi_M(i))(-H_M + \sqrt{H_M^2 + 4Z_M \Phi_M(i)})}{2Z_M},

(27)$$

where

$$\Phi_M(i) \triangleq \Upsilon_M(i) + gsat(gs(\lambda_M \Gamma_M T, \phi_M(-x_{L2}(i)) - \Upsilon_M(i), -\Upsilon_M(i), -\Upsilon_M(i), \phi_M(-x_{L2}(i)) - \Upsilon_M(i), \lambda_M \Gamma_M T)).

(28)$$

Finally, $x_{M2}(i)$ and $x_{M1}(i)$ are obtained as follows:

$$x_{M2}(i) = e_{M2}(i) + x_{L2}(i)

(29)$$
\[ x_{M1}(i) = T x_{M2}(i) + x_{M1}(i - 1). \]  

(30)

In conclusion, the above derivation process can be summarized as the following complete discrete-time algorithm of PSMF-M:

**Algorithm 1 PSMF-M**

**Input:**
- \( u(i) \): input signal
- \( x_{L2}(i) \): estimation of the change rate of the input \( u(i) \), i.e., change rate of the output \( x_{L1}(i) \) of PSMF-L
- \( Z_M = k_M T \): parameter
- \( H_M = 1 + 2 k_M \Gamma_M T^2 \): parameter

**Output:**
- \( x_{M1}(i) \): estimation of the input \( u(i) \)

**Procedure:**
1. \( x_{M2}^*(i) = \text{csgn}(x_{M1}(i - 1) - u(i)) \)
   \( (\Gamma_M T - \sqrt{T^2 + 2 \Gamma_M} x_{M1}(i - 1) - u(i)) \)
2. \( \Upsilon_M(i) = x_{M2}(i - 1) + 2 k_M \Gamma_M T u(i) - 2 k_M \Gamma_M T x_{M1}(i - 1) - H_M x_{L2}(i) \)
3. \( \phi_M(i) = Z_M x_{M2}^*(i) + x_{M2}(i) + H_M x_{M2}(i) \)
4. \( \psi_M(i) = \Upsilon_M(i) \phi_M(i) \)
5. \( \psi_M(i) = \Upsilon_M(i) \phi_M(i) \)
6. \( e_M^*(i) = \text{csgn}(\psi_M(i)) (1 - H_M) \)
7. \( e_M(i) = e_M^*(i) + x_{L2}(i) \)
8. \( x_{M1}(i) = T x_{M2}(i) + x_{M1}(i - 1) \)

Fig. 6 illustrates the step response of the proposed filter with a step input signal. As can be clearly observed in the enlarged views, the output of the proposed filter exactly reaches the desired value, and there is no chattering in its output. Such an advantage is owing to the use of implicit-Euler discretization.

**VI. NUMERICAL SIMULATION**

This section evaluates the performance of the proposed filter by comparing those of PSMF-L and PSMF-A through numerical examples. It should be mentioned here that the comparison considers the following two facts: (1) the proposed filter is an improvement of PSMF-L that is an extension of PSMF, and (2) PSMF-A is an extension of PSMF by also integrating another filter. The sampling interval \( T \) = 0.0001 s is used in the rest part of this section.

**A. OPEN-LOOP PERFORMANCE**

It is known that the noise removing ability is one of the most important issues for a filter. Toward this end, this subsection evaluates the open-loop performance of the proposed filter, i.e., the noise removing ability, by comparing those of PSMF-L and PSMF-A by using the following input signal:

\[ u(i) = u_s(i) + 0.0001 \varepsilon(i). \]

(31)

Here, \( u(i) \) is the input, \( u_s(i) \) is the signal component of \( u(i) \) given as follows:

\[ u_s(i) = 0.5 \sin(0.3(iT)^2), \]

and 0.0001\( \varepsilon(i) \) is the noise component, where \( \varepsilon(i) \sim \mathcal{N}(0, 1) \) is the unit-variance white Gaussian noise. In addition, the following measures are defined for evaluation:

\[ \omega_1(i) = \frac{y(i) - y(i - 1)}{T}, \]

\[ \omega_2(i) = \frac{u(i) - u_s(i - 1)}{T}, \]

\[ E_1(i) = u_s(i) - y(i), \]

\[ E_2(i) = \omega_1(i) - \omega_2(i). \]

\[ \text{AE}_{1, [1, 10]} \triangleq \frac{1}{90000} \sum_{i=10}^{10^4} |E_1(i)|, \]

\[ \text{AE}_{2, [1, 10]} \triangleq \frac{1}{90000} \sum_{i=10}^{10^4} |E_2(i)|, \]

where the subscript \([1, 10]\) denotes the time period of data used for evaluation, i.e., data between 1 s and 10 s.

Fig. 7 and Fig. 8 show the performances of PSMF-L with \( \Gamma = 300, \lambda = 3, k = 10 \), PSMF-A with \( \Gamma = 200, \lambda = 3, \omega_1 = 100 \), and the proposed filter with \( \Gamma_L = 500, k_L = 100, \lambda_L = 3, \Gamma_M = 15, k_M = 800, \lambda_M = 1.2 \). In the figures, one can observe that the tracking errors of the proposed filter are the smallest among the three filters. That is, the performances of PSMF-L and PSMF-A cannot close to that of the proposed filter by adjusting their parameters. Thus, it can be concluded that the noise removing performance of the proposed filter is the best among the three filters.
FIGURE 7. Results of open-loop simulation. (PSMF-L: \( \Gamma = 300, \lambda = 3, k = 10 \); PSMF-A: \( \Gamma = 200, \lambda = 3, \omega_c = 100 \); the proposed filter: \( \Gamma_L = 500, k_L = 100, \lambda_L = 3, \lambda_M = 800, \lambda_M = 1.2 \).)

FIGURE 8. Comparison of the proposed filter with different parameters of PSMF-L and PSMF-A.

FIGURE 9. Illustration of variable-length pendulum used in the closed-loop simulation.

B. CLOSED-LOOP PERFORMANCE

It is true that a filter with improved open-loop performance does not always imply its usefulness in closed feedback control systems. Toward this issue, this subsection validates the effectiveness of the proposed filter through position feedback control of a variable-length pendulum, as shown in Fig. 9. Such a simulation has been widely conducted in literature, e.g., [36], [37]. In this system, a 1 kg mass moves without friction along the massless pendulum rod. The distance \( R \) between the mass and the axis point \( O \) is bounded, and it is immeasurable. Specifically, the motion of the system is described as follows:

\[
\begin{align*}
\nabla p(i) &= v(i) \\
\nabla v(i) &= -2\frac{\nabla R(i)}{R(i)} v(i) - \frac{g\sin(p(i))}{R(i)} + \frac{\tau(i)}{MR(i)^2},
\end{align*}
\] (39a, 39b)

where \( p(i) \) is the current angle, \( v(i) \) is the current angular velocity, \( g = 9.8 \) N/kg is the gravitational constant, and the symbol \( \nabla \) is the backward-Euler difference operator. In addition, \( \tau(i) \) is the torque that actuates the pendulum for tracking the desired angle \( p_d(i) \).

The following proportional-derivative (PD) controller without filter:

\[
\tau(i) = K_p(p_d(i) - p(i)) + K_d(\nabla p_d(i) - \nabla p(i)),
\] (40)

and the following PD controller with filter:

\[
\tau(i) = K_p(p_d(i) - y(i)) + K_d(\nabla p_d(i) - \nabla y(i))
\] (41)

are implemented. Here, it is assumed that the current angle \( p(i) \) is corrupted by white Gaussian noise 0.0001\( \epsilon(i) \) at the output of the system. In addition, desired angle \( p_d(i) \) and the immeasurable variable distance \( R(i) \) are given as follows:

\[
p_d(i) = 0.5\sin(0.5iT) + 0.5\cos(iT),
\] (42)

\[
R(i) = 0.5 + 0.1\sin(8iT) + 0.2\cos(4iT).
\] (43)

Fig. 10 shows the block diagrams of the pendulum system without and with filter.
FIGURE 10. Block diagrams of the control systems of the variable-length pendulum. Here, $K_p = 100$ and $K_d = 10$ are used for the PD controller.

FIGURE 11. Results of closed-loop simulation: comparisons among the cases of without filter, PSMF-L and the proposed filter.

Furthermore, the following measures are defined for the quantitative evaluation:

$$\text{APE} \triangleq \frac{1}{1.8 \times 10^5} \sum_{i=2 \times 10^5}^{2 \times 10^5} |p_d(i) - p(i)|,$$  \hspace{1cm} (44)

$$\text{AT} \triangleq \frac{1}{1 \times 10^5} \sum_{i=2 \times 10^4}^{2 \times 10^5} |\tau(i)|,$$  \hspace{1cm} (45)

$$\text{ACT} \triangleq \frac{1}{1 \times 10^5} \sum_{i=2 \times 10^4}^{2 \times 10^5} \frac{|\tau(i) - \tau(i-1)|}{T},$$  \hspace{1cm} (46)

$$\text{OS} \triangleq \max_{i \in [3 \times 10^3, 5 \times 10^3]} (p(i) - p_d(i)).$$  \hspace{1cm} (47)
where APE, AT and ACT denotes the averaged values of position error, control torque and change rate of torque between time interval 2 s and 20 s, respectively, and OS is the overshoot value of the controlled position that is observed between 0 s and 2 s.

Fig. 11 - Fig. 16 show the results obtained without filter, with PSMF-L, PSMF-A and the proposed filter. One can observe that the case of without filter produces the largest position error and the most serious high-frequency vibration. The uses of the three filters improve the control performance. However, by comparing with the proposed filter, both PSMF-L and PSMF cannot reduce APE, AT, ACT and OS simultaneously to the levels of the proposed filter by adjusting their parameters. Thus, it can be said that, among the three filters, the proposed filter provides the most reliable feedback signal for improving control performance.

VII. CONCLUSION

The paper has proposed a new sliding mode filter, which is an improvement of PSMF-L, by employing a first-order derivative feed-forward term. Its discrete-time algorithm has been derived by applying the implicit-Euler discretization, and its discrete-time implementation does not produce chattering. The proposed filter provides a rapid and smooth output signal even in the presence of noise, and these advantages over previous methods have been validated through an open-loop and a closed-loop simulations.

APPENDIXES

APPENDIX A

Proposition:
The following two equations are equivalent to each other:

$$\phi(h) - k = \text{gsat}(\text{gsat}(\alpha_1, \phi(m) - k, \alpha_2), \phi(n) - k),$$

$$\phi(h) - k = gsgn(\text{gsat}(\alpha_1, \phi(m) - k, \alpha_2), \phi(n) - k),$$

where \(\phi(.)\) is a monotonically increasing function with \(\phi(0) = 0\), and \(k, m, n, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}\), and \(\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4\).

Proof:

In the case of \(n-h < 0\), (48) is equivalent to the followings:

\[(n-h < 0) \land ((\phi(h) - k = \alpha_1 \land m-h < 0) \lor \phi(h) - k = \alpha_2 \land m-h > 0)\]
\[
\forall (\phi(h) - k \in [\alpha_1, \alpha_2] \land m - h = 0)) \\
\Leftrightarrow (n - h < 0) \land (\phi(h) - k = \alpha_1 \land \phi(h) > \phi(m)) \\
\forall (\phi(h) - k = \alpha_2 \land \phi(h) < \phi(m)) \\
\forall (\phi(h) - k \in [\alpha_1, \alpha_2] \land \phi(h) = \phi(m))) \\
\Leftrightarrow (n - h < 0) \land ((\phi(h) - k = \alpha_1 \land \phi(h) - k < \alpha_1) \\
\forall (\phi(h) - k = \alpha_2 \land \phi(m) - k > \alpha_2) \\
\forall (\phi(h) - k = -k \land \phi(m) - k \in [\alpha_1, \alpha_2])) \\
\Leftrightarrow (\phi(n) < \phi(h) \land \phi(h) - k = \text{gsat}(\alpha_1, \phi(m) - k, \alpha_2)) \\
\Leftrightarrow (\phi(h) - k = \text{gsat}(\alpha_1, \phi(m) - k, \alpha_2) \\
\land \phi(n) - k < \text{gsat}(\alpha_1, \phi(m) - k, \alpha_2)).
\]

In addition, in the case of \(n - h > 0\), (48) is equivalent to the followings:
\[
(n - h > 0) \land ((\phi(h) - k = \alpha_3 \land m - h < 0) \\
\forall (\phi(h) - k = \alpha_4 \land m - h > 0) \\
\forall (\phi(h) - k \in [\alpha_3, \alpha_4] \land m - h = 0)) \\
\Leftrightarrow (n - h > 0) \land ((\phi(h) - k = \alpha_3 \land \phi(h) > \phi(m)) \\
\forall (\phi(h) - k = \alpha_4 \land \phi(h) < \phi(m)) \\
\forall (\phi(h) - k \in [\alpha_3, \alpha_4] \land \phi(h) = \phi(m))) \\
\Leftrightarrow (n - h > 0) \land ((\phi(h) - k = \alpha_3 \land \phi(m) - k < \alpha_3) \\
\forall (\phi(h) - k = \alpha_4 \land \phi(m) - k > \alpha_4) \\
\forall (\phi(h) - k = -k \land \phi(m) - k \in [\alpha_3, \alpha_4])) \\
\Leftrightarrow (\phi(n) > \phi(h)) \land \phi(h) - k = \text{gsat}(\alpha_3, \phi(m) - k, \alpha_4)) \\
\Leftrightarrow (\phi(h) - k = \text{gsat}(\alpha_3, \phi(m) - k, \alpha_4) \\
\land \phi(n) - k < \text{gsat}(\alpha_3, \phi(m) - k, \alpha_4)).
\]

Moreover, when \(n - h = 0\), (48) is equivalent to the followings:
\[
(n - h = 0) \land ((\phi(h) - k \in [\alpha_1, \alpha_3] \land m - h < 0) \\
\forall (\phi(h) - k \in [\alpha_2, \alpha_3] \land m - h > 0) \\
\forall (\phi(h) - k \in [\alpha_1, \alpha_4] \land m - h = 0)) \\
\Leftrightarrow (\phi(h) - k = \phi(n) - k \\
\land (\phi(h) - k \in [\alpha_1, \alpha_3] \land \phi(m) < \phi(h)) \\
\forall (\phi(h) - k \in [\alpha_2, \alpha_3] \land \phi(m) > \phi(h)) \\
\forall (\phi(h) - k \in [\alpha_1, \alpha_4] \land \phi(m) = \phi(h))) \\
\Leftrightarrow (\phi(h) - k = \phi(n) - k \\
\land (\phi(h) - k \in [\alpha_1, \alpha_3] \land \phi(m) < \phi(h)) \\
\forall (\phi(h) - k \in [\alpha_2, \alpha_3] \land \phi(m) > \phi(h)) \\
\forall (\phi(h) - k \in [\alpha_1, \alpha_4] \land \phi(m) = \phi(h)))
\]
\[ \varphi(h) - k = g_{\text{sat}}(\varphi(h), -\Gamma L T) - \phi(h) \]
\[ \Leftrightarrow \varphi(h) - k = g_{\text{sat}}(\varphi(h), -\Gamma L T) - \phi(h) \]
\[ \text{QED.} \]

**APPENDIX B**

In (9), the part of PSFM-L (i.e., (9a), (9b) and (10)) can be implemented by the following algorithm in discrete-time without producing chattering [29]:

**Algorithm 2 PSFM-L**

**Input:**
\[
u(i): \text{input signal} \\
Z_L = k_1 T: \text{parameter} \\
H_L = 1 + 2k_1 \Gamma L T^2: \text{parameter}
\]

**Output:**
\[ x_{\text{L2}}(i): \text{estimation of the input } u(i) \]

**Procedure:**
1. \[ x_{\text{L2}}(i) = \text{csgn}(x_{\text{L1}}(i - 1) - u(i)) \]
2. \[ \dot{\varphi}(i) = Z_L x_{\text{L2}}(i) x_{\text{L2}}(i) + H_L x_{\text{L2}}(i) \]
3. \[ \Psi_L(i) = \dot{\varphi}(i) - g_{\text{sat}}(-4L T, \varphi_L(i), -\Gamma L T), \Psi_L(i) - \phi(i), g_{\text{sat}}(\Gamma L T, \varphi_L(i), -\Gamma L T) \]
4. \[ x_{\text{L2}}(i) = \text{csgn}(\Psi_L(i)) \]
5. \[ H_L + (1/4)Z_L(\Psi_L(i)) \]
6. \[ x_{\text{L1}}(i) = T x_{\text{L2}}(i) + x_{\text{L1}}(i - 1) \]

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