Sine-Square Deformation and its Relevance to String Theory

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Abstract

Sine-square deformation, a recently found modulation of the coupling strength in certain statistical models, is discussed in the context of two-dimensional conformal field theories, with particular attention to open/closed string duality. This deformation is shown to be non-trivial and leads to a divergence in the worldsheet metric. The structure of the vacua of the deformed theory is also investigated. The approach advocated here may provide an understanding of string duality through the worldsheet dynamics.

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1 Introduction

In physics, the boundary condition is often treated as a secondary issue. Similar to the term itself, boundary conditions remain peripheral, never central. In string theory, however, the boundary condition plays a fundamental role. Certain boundary conditions of the worldsheet exhibit non-perturbative aspects of string theory through D-branes. They also distinguish between open and closed strings, which correspond to gauge theories and gravity, respectively. This short note concerns the boundary conditions in string theory.

In recent studies of a certain class of quantum systems, systems with closed and open boundary conditions were found to have identical vacua provided that the coupling constants of the open-boundary system are modulated in a way called *sine-square deformation* (SSD) [1]. In particular, SSD works for two-dimensional conformal field theories, which describe the worldsheets of string theory [2]. Therefore, the implications of this discovery to string theory are potentially vast.

The spatial modulation of the coupling constant is seldom investigated in condensed matter physics. However, such modulation may correspond to introducing a metric with non-trivial curvature. In this sense, the above-mentioned uncovering can be interpreted as an effect caused by the worldsheet metric. Thus, by investigating the effect of the SSD on the worldsheet, we may better understand the non-perturbative aspects of string theory such as D-branes or open/closed duality through interchanges of the boundary condition caused by the worldsheet metric. Specifically, if certain worldsheet metrics can alter the boundary condition, resulting in D-brane emission or transitions between open and closed strings, then non-perturbative aspects of string theory can be understood from
the dynamics of the worldsheet through its condensation. Although, the worldsheet metric can be gauged away in the perturbative treatment of string theory, the metric may couple to the dynamics when non-perturbative effects are incorporated.

The boundary condition, by nature, stipulates the development of a system, not the other way around. Once set up, a system only evolves within its predetermined boundary. Therefore, if non-perturbative effects of string theory are depicted in terms of boundary conditions, they remain unaltered throughout the system development. Here we explore a possibility that the condensation of the worldsheet metric effectively alter the boundary condition, thereby exhibiting non-perturbative effects of string dynamics. If this is the case, non-perturbative aspects of string theory can be understood in terms of worldsheet metric dynamics. In fact, one could argue that this has been somewhat achieved by the research through matrix models \[3\], in which the effects of D-branes were identified. Noting that the matrix models are nothing but the statistical mechanics of the discretized worldsheet, here we rather seek a continuum treatment of the world sheet based on the SSD.

In this note, we attempt to clarify the role of the world sheet metric in the non-perturabative dynamics of string theory. To this end, we explore the consequences of the SSD on the worldsheet. SSD is briefly overviewed in Section \[2\] and it is applied to conformal field theory in subsection \[3.1\]. In subsection \[3.2\] we verify that the SSD is actually a non-trivial transformation. A novel state in the deformed system is presented in subsection \[3.3\]. We examine the SSD in the Lagrangian formalism in Section \[4\]. Here, we reveal large divergence of the worldsheet metric. We conclude with notes and future perspectives in Section \[5\].
2 Sine-square deformation

First, we explain the SSD introduced by Gendiar, Krcmar and Nishino [1]. Consider a system of \( N \) quantum operators \( \sigma_n \). The operators are aligned one-dimensionally and each is connected to the next neighbors with the strength \( J_{n,n+1} \). The Hamiltonian of such a system is given by

\[
H_0 = -\sum J_{n,n+1} (\sigma_n \cdot \sigma_{n+1}).
\]  

(1)

The boundary condition of the system is governed by the configuration of the couplings \( J_{n,n+1} \). Setting \( J_{0,1} = J_{N,N+1} = 0 \) with \( J_{1,2} = J_{2,3} = \cdots = J_{N-1,N} \equiv J \), the system retains an open-boundary condition. On the other hand, if \( J_{N,1} = J_{1,2} = J_{2,3} = \cdots = J_{N-1,N} \equiv J \), a closed-boundary condition is imposed (Fig. 1).

Now suppose that the couplings are configured so that they gradually vary. For example, in the open boundary system, we may reduce the strength of the couplings for the connections near both ends of the system (Fig. 2). We refer to this type of spatial coupling variation as modulation. Such modulation (illustrated in Figure 2) is motivated by the expectation that it reduces repercussions arising from the open boundaries. However, how far we should extend the
coupling modulation remains an interesting question.

Figure 2: Modulation of the coupling is expected to reduce the edge effect.

In [1], this step was boldly extended to the midpoint (see Fig. 3). This

Figure 3: Sine-square deformation of the coupling for the one-dimensional quan-
tum system.

modulation is expressed as

\[ J_{n,n+1} \equiv J \sin^2 \left( \frac{n}{N} \pi \right). \tag{2} \]

Note that the couplings \( J_{0,1} \) and \( J_{N,N+1} \) located at the both ends are retained at 0. Therefore, the system remains an open-boundary system, but its couplings are modulated by (2). For obvious reasons, modulation (2) is called sine-square deformation (SSD).

An astonishing feature of SSD is that it permits a ground state of the mod-
ulated system that is identical to the ground state of a closed-boundary system.
Namely, the ground state of a certain class of quantum operators coupled by (2) coincides with that of the system with \( J_{N,1} = J_{1,2} = J_{2,3} = \cdots = J_{N-1,N} \equiv J \). Given the apparently very different topologies of these configurations, this is a remarkable result.

This exact match between the ground states of the SSD system and that of the closed-boundary system is observed in spin-\( \frac{1}{2} \) XY spin-chain, 1D free fermion systems, 2D conformal field theories and 2D super conformal field theories [2]. This phenomenon is also expected in spin-\( \frac{1}{2} \) XXZ spin-chain [4], extended Hubbard model [5] and Kondo lattice model [6].

The underlying mechanism of the phenomenon can be understood as follows [7, 8]. In additions to the original Hamiltonian (1) under the closed boundary condition

\[
H_0 = -\sum_{n=1}^{N} J (\sigma_n \cdot \sigma_{n+1}), \quad \sigma_{N+1} \equiv \sigma_1, \quad (3)
\]

we introduce two “Hamiltonians”:

\[
H_\pm = -\sum_{n=1}^{N} e^{\pm 2\pi i n/N} J (\sigma_n \cdot \sigma_{n+1}). \quad (4)
\]

The SSD can then be formulated by replacing the original \( H_0 \) in (3) with the new Hamiltonian:

\[
H_{SSD} \equiv \frac{1}{2} H_0 - \frac{1}{4} (H_+ + H_-). \quad (5)
\]

Indeed,

\[
\frac{1}{2} H_0 - \frac{1}{4} (H_+ + H_-) = -\sum_{n=1}^{N} \frac{1}{2} \left( 1 - \frac{1}{2} e^{2\pi i n/N} - \frac{1}{2} e^{-2\pi i n/N} \right) J (\sigma_n \cdot \sigma_{n+1})
\]

\[
= -\sum_{n=1}^{N} \sin^2 \left( \frac{2\pi n}{N} \right) J (\sigma_n \cdot \sigma_{n+1}). \quad (6)
\]
The \( \sin^2 \left( 2\pi \frac{n}{N} \right) \) factor in (6) clearly implies an open boundary for the sine-square deformed Hamiltonian \( \mathcal{H}_{\text{SSD}} \). This openness detaches the coupling between the operators at both ends, \( \sigma_1 \) and \( \sigma_N \).

Denoting the ground state of the original Hamiltonian by \( |0\rangle \), we have

\[
\mathcal{H}_0|0\rangle = E_0|0\rangle,
\]

where \( E_0 \) is the ground energy. In certain systems, \( \mathcal{H}_\pm \) annihilates the ground state of the original Hamiltonian [2],

\[
\mathcal{H}_\pm|0\rangle = 0,
\]

yielding

\[
\mathcal{H}_{\text{SSD}}|0\rangle = \frac{1}{2} E_0|0\rangle.
\]

If the energy spectrum of \( \mathcal{H}_{\text{SSD}} \) can be shown to be bounded below as for 1D fermions and certain conformal field theories (CFTs), the ground state \( |0\rangle \) is obviously an exact ground state of \( \mathcal{H}_{\text{SSD}} \). In some cases, we can directly argue that \( \mathcal{H}_{\text{SSD}} \) has a unique ground state that corresponds to \( |0\rangle \) [7,8].

\section{CFT and sine-square deformation}

\subsection{\( SL(2,\mathbb{C}) \) invariant vacuum}

In this subsection, the SSD and its mechanism are further explained in the context of 2D CFT. Following [2], we first express the Hamiltonian of a CFT on a cylinder of circumference \( L \) in terms of the energy momentum tensor with the cylindrical
coordinate $w = \tau + ix = \frac{L}{2\pi} \log z$:

$$\mathcal{H}_0 = \int_0^L \frac{dx}{2\pi} \left( T_{\text{cyl}}(w) + \bar{T}_{\text{cyl}}(w) \right),$$  \hspace{1cm} (10)

where

$$T_{\text{cyl}}(w) = \left( \frac{2\pi}{L} \right)^2 \left[ T(z) z^2 - \frac{c}{24} \right].$$  \hspace{1cm} (11)

The energy momentum tensor $T(z)$ comprises Virasoro generators $L_n$ as $T(z) = \sum z^{-n-2} L_n$. Thus, the Hamiltonian of the CFT can also be expressed as

$$\mathcal{H}_0 = \frac{2\pi}{L} \left( L_0 + \bar{L}_0 \right) - \frac{\pi c}{6L}.  \hspace{1cm} (12)$$

As in the previous section, we introduce $\mathcal{H}_\pm$,

$$\mathcal{H}_\pm \equiv \int_0^L \frac{dx}{2\pi} \left( e^{\pm \frac{2\pi}{L} w} T_{\text{cyl}}(w) + e^{\mp \frac{2\pi}{L} \bar{w}} \bar{T}_{\text{cyl}}(\bar{w}) \right) = \frac{2\pi}{L} \left( L_{\pm 1} + \bar{L}_{\pm 1} \right).  \hspace{1cm} (13)$$

Note that, for 2d CFTs, $\mathcal{H}_\pm$ can be written as a linear combination of familiar Virasoro operators $L_{\pm 1}, \bar{L}_{\pm 1}$. $\mathcal{H}_{\text{SSD}}$ now reads

$$\mathcal{H}_{\text{SSD}} = \frac{1}{2} \mathcal{H}_0 - \frac{1}{4} (\mathcal{H}_+ + \mathcal{H}_-) = \frac{\pi}{L} \left( L_0 + \bar{L}_0 - \frac{L_1 + L_{-1} + \bar{L}_1 + \bar{L}_{-1}}{2} \right) - \frac{\pi c}{12L}.  \hspace{1cm} (14)$$

As for the ground state of 2D CFT, it is natural to assume the $SL(2, \mathbb{C})$ invariance. Denoting the $SL(2, \mathbb{C})$ invariant vacuum $|0\rangle$, we require that $|0\rangle$ is invariant under the global conformal transformations generated by $L_0, L_{\pm 1}, \bar{L}_0, \bar{L}_{\pm 1}$. From (12), it follows that

$$\mathcal{H}_0 |0\rangle = E_0 |0\rangle,$$  \hspace{1cm} (15)

with $E_0 = -\frac{\pi c}{6L}$. From (14), we observe that $|0\rangle$ is also a ground state of $\mathcal{H}_{\text{SSD}},$
with half the energy $E_0$

$$\mathcal{H}_{\text{SSD}}|0\rangle = \frac{E_0}{2}|0\rangle,$$  \hspace{1cm} (16)

because $|0\rangle$ is annihilated not only by $L_0$ and $\bar{L}_0$ but also by $L_{\pm 1}$ and $\bar{L}_{\pm 1}$. This analysis demonstrates the SSD mechanism in the more familiar setting of 2D conformal field theories.

### 3.2 Non-triviality of $\mathcal{H}_{\text{SSD}}$

At this point, it would be reasonable to question the non-triviality of $\mathcal{H}_{\text{SSD}}$. This Hamiltonian differs from the original $\mathcal{H}_0$ only by the generators of the global conformal transformations, $L_{\pm 1}, \bar{L}_{\pm 1}$. Since the vacuum is assumed to be invariant under the $SL(2, \mathbb{C})$-transformations or global conformal transformations, a Hamiltonian or any other operator can be modified by the $SL(2, \mathbb{C})$-transformations with no physical consequences. In fact, we apply the following $SL(2, \mathbb{C})$-transformation to (the holomorphic part of) $\mathcal{H}_0$ to obtain

$$\exp\left(-a \frac{L_1 - L_{-1}}{2}\right) L_0 \exp\left(a \frac{L_1 - L_{-1}}{2}\right) = \cosh a L_0 - \sinh a \frac{L_1 + L_{-1}}{2}.$$  \hspace{1cm} (17)

The above result appears similar to the right-hand side of (14). If $\mathcal{H}_{\text{SSD}}$ can be obtained from $\mathcal{H}_0$ by the $SL(2, \mathbb{C})$-transformations, the matching of the ground states is trivial. Closer inspection reveals that this is not the case.

If the right-hand side of (17) accords with $\mathcal{H}_{\text{SSD}}$, we need to require $\cosh a = \sinh a$, which directly contradicts the identity $\cosh^2 a - \sinh^2 a = 1$. One may take the limit as $a \to \infty$ and suitably rescale; however, in any case, $\mathcal{H}_{\text{SSD}}$ and $\mathcal{H}_0$ are not connected through the ordinary $SL(2, \mathbb{C})$-transformation.

This result can also be generally confirmed by considering the two-dimensional
representations of the generators:

\[
L_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L_\pm \equiv \frac{L_1 \pm L_{-1}}{2} = \frac{1}{2} \begin{pmatrix} 0 & \pm i \\ i & 0 \end{pmatrix}.
\]

(18)

A SL(2, \mathbb{C}) group element is non-unitarily represented by the products of the exponents of the following generators:

\[
\exp(i\theta_0 L_0), \exp(i\theta_+ L_+), \exp(\theta_- L_-),
\]

(19)

which multiply to yield:

\[
\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \quad \text{where } |\alpha|^2 - |\beta|^2 = 1.
\]

(20)

In this representation, the SL(2, \mathbb{C}) group acts on \(L_0\) as follows

\[
\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}^{-1} L_0 \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} = \frac{1}{2} \begin{pmatrix} |\alpha|^2 + |\beta|^2 & 2\alpha^* \beta \\ -2\alpha \beta^* & -|\alpha|^2 - |\beta|^2 \end{pmatrix}.
\]

(21)

We now explore the parameter region in which the above expression can be expressed as a linear combination of \(L_0\) and \(L_+\);

\[
aL_0 - bL_+ = \frac{1}{2} \begin{pmatrix} a & -ib \\ -ib & -a \end{pmatrix}.
\]

(22)
Combining (20), (21) and (22), the following conditions are easily obtained:

\[
\begin{align*}
|\alpha|^2 + |\beta|^2 &= a \\
2\alpha^*\beta &= -2\alpha\beta^* = -ib \\
|\alpha|^2 - |\beta|^2 &= 1
\end{align*}
\]  
(23)

The left-hand side of the following identity inequality

\[|\alpha - i\beta|^2 \geq 0,\]  
(24)

can be expanded as

\[|\alpha - i\beta|^2 = (\alpha - i\beta)(\alpha^* + i\beta^*) = |\alpha|^2 + |\beta|^2 - i\alpha^*\beta + i\alpha\beta^*,\]  
(25)

yielding

\[a - b \geq 0,\]  
(26)

where we have used the first and second conditions in (23). The case of interest is \(a = b\), which implies that a \(SL(2, \mathbb{C})\) action on \(L_0\) or \(H_0\) yields \(H_{SSD}\) up to a normalization. However, (26) becomes an equality only when \(\alpha = i\beta\), which directly contradicts the third condition in (23). Therefore, we have proven by contradiction that \(SL(2, \mathbb{C})\) cannot act on \(H_0\) to yield \(H_{SSD}\).

### 3.3 Another vacuum?

In subsection 3.1 the \(SL(2, \mathbb{C})\) invariant vacuum \(|0\rangle\) was shown to also constitute the lowest-energy \(\frac{1}{2}E_0\) eigenstate of \(H_{SSD}\). Thus, we may naturally seek other eigenstate of \(H_{SSD}\). For the original Hamiltonian \(H_0\), there exists a set of

\[1\text{Alexandros Kehagias has alerted the authors that } U_q(sl(2)) \text{ action on } H_0 \text{ might yield } H_{SSD}.\]
eigenstates corresponding to primary fields of CFT:

|\ h, \ ¯h \rangle \equiv \phi(0, 0)|0\rangle, \quad (27)

where $\phi$ is a primary field of dimensions $h$ and $\bar{h}$. Unlike $|0\rangle$, $|h, \bar{h}\rangle$ is not an eigenstate of $H_{\text{SSD}}$, but provides a useful starting point. Consider a state of the form

$$\sum_{n} a_{n} (L_{-1})^{n} |h\rangle, \quad (28)$$

where we have focused on the holomorphic part for simplicity. Simple calculation shows that for (28) to be an eigenstate of $H_{\text{SSD}},$

$$\left(L_{0} - \frac{L_{1} + L_{-1}}{2}\right) \sum_{n} a_{n} (L_{-1})^{n} |h\rangle = \alpha \sum_{n} a_{n} (L_{-1})^{n} |h\rangle, \quad (29)$$

the following recurrence relation for $a_{n}$ should hold:

$$(2(n + 1)h + n(n + 1)) a_{n+1} + 2(\alpha - n - h) a_{n} + a_{n-1} = 0. \quad (30)$$

A solution to the above recurrence relation is

$$a_{n} = \frac{1}{n!}, \quad \alpha = 0. \quad (31)$$

Thus, we appear to have identified vacuums other than $|0\rangle$ of the form

$$e^{L_{-1}}|h\rangle. \quad (32)$$
However, since the norm of (32) is divergent, a limiting process is required to properly define (32).  

4 SSD and strings

In the following exposition, we adapt the notation of [9]. The Lagrangian of the two-dimensional free bosonic field $\phi(x, t)$ is

$$\mathcal{L}_0 = \frac{1}{2} g \int dx \left\{ (\partial_t \phi)^2 - (\partial_x \phi)^2 \right\}, \quad (33)$$

where the non-dimensional normalization of the Lagrangian $g$ is left undetermined, for convenient comparison with different conventions. A common convention is $g = \frac{1}{4\pi}$. We consider the bosonic field $\phi(x, t)$ on a cylinder of circumference $L$ so that $\phi(x+L, t) = \phi(x, t)$. Then, the field is expressed by the following Fourier expansion:

$$\phi(x, t) = \sum_{n \in \mathbb{Z}} e^{2\pi inx/L} \phi_n, \quad (34)$$

where $\phi_n(t) \equiv \frac{1}{L} \int_0^L dx e^{-2\pi inx/L} \phi(x, t). \quad (35)$

In terms of the Fourier components $\phi_n(t)$, the Lagrangian is expressed as

$$\mathcal{L}_0 = \frac{1}{2} g \sum_n \left\{ \dot{\phi}_n \dot{\phi}_{-n} - \left( \frac{2\pi n}{L} \right)^2 \phi_n \phi_{-n} \right\}. \quad (36)$$

\footnote{After the completion of the manuscript, we learned from H. Katsura that there is another non-normalizable vacuum, which takes the form of $\sum_{n>1} L_{-n}|0\rangle$.}
Then the momentum conjugate to $\phi_n$ is

$$\pi_n = gL\dot{\phi}_n. \quad (37)$$

The Hamiltonian is obtained as

$$H_0 = \frac{1}{2gL} \sum_n \{\pi_n\pi_{-n} + (2\pi ng)^2 \phi_n\phi_{-n}\}. \quad (38)$$

Note that $\phi_n^\dagger = \phi_{-n}$ and $\pi_n^\dagger = \pi_{-n}$.

Introducing

$$a_n \equiv -i n \sqrt{\pi g} \phi_n + \frac{1}{\sqrt{4\pi g}} \pi_{-n}, \quad (39)$$

$$\bar{a}_n \equiv -i n \sqrt{\pi g} \phi_{-n} + \frac{1}{\sqrt{4\pi g}} \pi_n,$$

we obtain the following commutation relations from the canonical commutation relation $[\phi_n, \pi_m] = i\delta_{nm}$:

$$[a_n, a_m] = n\delta_{n+m}, \quad [\bar{a}_n, \bar{a}_m] = n\delta_{n+m}, \quad [a_n, \bar{a}_m] = 0. \quad (40)$$

In terms of $a_n, \bar{a}_n$, the Hamiltonian is expressed as

$$H_0 = \frac{2\pi}{L} \left\{ \sum_{n>0} a_{-n}a_n + \frac{1}{2}a_0^2 + \sum_{n>0} \bar{a}_{-n}\bar{a}_n + \frac{1}{2}\bar{a}_0^2 \right\}. \quad (41)$$

Note that $a_0 = \bar{a}_0 = \frac{\pi n}{\sqrt{4\pi g}}$.

When $a_n$'s and $\bar{a}_n$'s are treated as operators, they can form the following
Virasoro operators:

\[
L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} a_{n-m} a_m \quad (n \neq 0), \quad L_0 = \sum_{n>0} a_{-n} a_n + \frac{1}{2} a_0^2, \tag{42}
\]

\[
\bar{L}_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} \bar{a}_{n-m} \bar{a}_m \quad (n \neq 0), \quad \bar{L}_0 = \sum_{n>0} \bar{a}_{-n} \bar{a}_n + \frac{1}{2} \bar{a}_0^2. \tag{43}
\]

The Hamiltonian and Virasoro operators are related as follows:

\[
\mathcal{H}_0 = \frac{2\pi}{L} (L_0 + \bar{L}_0), \tag{44}
\]

up to a constant, which is irrelevant in the following discussion.

We can consider new terms if the Hamiltonian contains \(L_1\) and \(L_{-1}\). Equations (42) and (43) can be expressed in the form

\[
L_1 = \frac{1}{2} \sum_{m \in \mathbb{Z}} a_{1-m} a_m = \frac{1}{2} \sum_{m \in \mathbb{Z}} \left( -\sqrt{\pi} g i (1 - m) \phi_{1-m} + \frac{\pi_{m-1}}{4\pi g} \right) \left( -\sqrt{\pi} g i m \phi_m + \frac{\pi_{-m}}{4\pi g} \right), \tag{45}
\]

from which the following relation is easily observed:

\[
\frac{2\pi}{L} \left( L_1 + \bar{L}_1 + L_{-1} + \bar{L}_{-1} \right) = \frac{1}{2gL} \sum_{n \in \mathbb{Z}} \left\{ \pi_n \pi_{-(n+1)} + \pi_n \pi_{-(n-1)} \right\} + \left( 2\pi g \right)^2 n (n+1) \phi_n \phi_{(n+1)} + \left( 2\pi g \right)^2 n (n-1) \phi_n \phi_{-(n-1)} \right\}. \tag{46}
\]

We now proceed to evaluate the the Lagrangian expression corresponding to the deformed Hamiltonian \(\mathcal{H}_{\text{SSD}} \sim L_0 + \bar{L}_0 - \frac{1}{2} (L_1 + \bar{L}_1 + L_{-1} + \bar{L}_{-1})\).

We may reasonably expect a general form of the corresponding deformed Lagrangian, such as

\[
\mathcal{L} = \frac{1}{2} \int_0^L dx \{ (\partial_t \varphi) F(x) (\partial_t \varphi) - (\partial_x \varphi) G(x) (\partial_x \varphi) \}. \tag{47}
\]
Postulating the following forms for \( F(x) \) and \( G(x) \)

\[
F(x) = N \sum_{k \in \mathbb{Z}} r^{|k|} e^{2\pi ikx/L} \quad \text{and} \quad G(x) = 1 - \alpha \cos \frac{2\pi x}{L},
\]

we determine whether the deformed Lagrangian generates \( H_{SSD} \). In (48), the parameter \( \alpha \) represents deformation. The number \( r \) should be less than unity and may depend on the value of \( \alpha \). \( N \) is the normalization factor for \( G \) and may also depend on \( \alpha \). Since (48) should revert to the original Lagrangian when \( \alpha = 0 \), we expect that \( r \to 0 \) and \( N \to 1 \) as \( \alpha \to 0 \). The deformed Lagrangian, denoted \( L_\alpha \) as a reminder of the role of \( \alpha \), can be expressed in terms of the Fourier modes \( \dot{\phi} \)

\[
\mathcal{L}_\alpha = \frac{1}{2} \int_0^L dx \left\{ \left( \frac{\partial}{\partial t} \varphi \right) F(x) \left( \frac{\partial}{\partial t} \varphi \right) - \left( \frac{\partial}{\partial x} \varphi \right) G(x) \left( \frac{\partial}{\partial x} \varphi \right) \right\}
= \frac{gL}{2} \sum_{n,k} \dot{\phi}_n \dot{\phi}_{-n-k} N_r^{|k|}
- \frac{2\pi^2 g}{L} \left\{ n^2 \phi_n \phi_{-n} - \frac{\alpha}{2} (n (n+1) \phi_n \phi_{-n-1} + n (n-1) \phi_n \phi_{-n+1}) \right\}.
\]

Introducing the following notation

\[
F_{nm} \equiv \sum_{k \in \mathbb{Z}} \delta_{n+m+k,0} N_r^{|k|},
\]

the kinetic part of the Lagrangian can be simply expressed as

\[
\frac{gL}{2} F_{nm} \dot{\phi}_n \dot{\phi}_m.
\]
From (51), it follows that the canonical conjugate momentum becomes

$$
\pi_n = gL \sum_m F_{nm} \dot{\phi}_n = gL \sum_k N r^{[k]} \dot{\phi}_{-n-k}.
$$
(52)

The last equality follows from the explicit definition of $F_{mn}$ [50].

Here we claim that

$$
\exists N, \exists r \quad F^{-1}_{nm} = \delta_{n,m} - \frac{\alpha}{2} \delta_{n-1,m} - \frac{\alpha}{2} \delta_{n+1,m}.
$$
(53)

In other words, $N$ and $r$ can be expressed in terms of $\alpha$ such that

$$
\frac{1}{gL} \sum_m \left( \delta_{n,m} - \frac{\alpha}{2} \delta_{n-1,m} - \frac{\alpha}{2} \delta_{n+1,m} \right) \pi_m = \dot{\phi}_n.
$$
(54)

To validate claims (53) or (54), we require that

$$
\frac{1}{gL} \sum_m \left( \delta_{n,m} - \frac{\alpha}{2} \delta_{n-1,m} - \frac{\alpha}{2} \delta_{n+1,m} \right) \pi_m
= \frac{1}{gL} \sum_{m,k} \left( \delta_{n,m} - \frac{\alpha}{2} \delta_{n-1,m} - \frac{\alpha}{2} \delta_{n+1,m} \right) gLN r^{[k]} \dot{\phi}_{-m-k}
= N \sum_{k \in \mathbb{Z}} \left\{ r^{[k]} \dot{\phi}_{n-k} - \frac{\alpha}{2} r^{[k]} \dot{\phi}_{n+1-k} - \frac{\alpha}{2} r^{[k]} \dot{\phi}_{n-1-k} \right\}
= N \sum_{k \in \mathbb{Z}} \left\{ r^{[k]} \dot{\phi}_{n-k} - \frac{\alpha}{2} r^{[k-1]} \dot{\phi}_{n-k} - \frac{\alpha}{2} r^{[k+1]} \dot{\phi}_{n-k} \right\}
$$

is identical to $\dot{\phi}_n$. This requirement can be met only under the following conditions:

$$
\begin{align*}
  r^k - \frac{\alpha}{2} r^{k-1} - \frac{\alpha}{2} r^{k+1} &= 0 \quad (k \geq 1) \\
  N(1 - \frac{\alpha}{2} r - \frac{\alpha}{2} r) &= 1 \quad (k = 1) \\
  r^{-k} - \frac{\alpha}{2} r^{-k+1} - \frac{\alpha}{2} r^{-k-1} &= 0 \quad (k \leq -1)
\end{align*}
$$
(55)
It is trivial to see that conditions (55) are satisfied if

\[ r - \frac{\alpha}{2} - \frac{\alpha}{2} r^2 = 0, \quad N = \frac{1}{1-\alpha r}. \] (56)

Solving the above quadratic equation and demanding that \( r \to 0 \) as \( \alpha \to 0 \), we find that the expressions

\[ r = \frac{1 - \sqrt{1-\alpha^2}}{\alpha}, \quad N = \frac{1}{\sqrt{1-\alpha^2}} \] (57)

validate claims (53) and (54). We assume that \( r \) and \( N \) satisfy (57) and that \( F_{mn} \) is accordingly determined from (50) in the following.

The Hamiltonian corresponding to \( L_\alpha \), which we denote \( \mathcal{H}_\alpha \), is now calculated as

\[
\mathcal{H}_\alpha = \sum_n \pi_n \dot{\phi}_n - L_\alpha \\
= \sum_{n,m} \pi_n \frac{F_{mn}^{-1}}{gL} \pi_m - \sum_{n,m} \frac{1}{2} F_{nm} \dot{\phi}_n \dot{\phi}_m \\
+ \sum_n \frac{2\pi^2 g}{L} \left\{ n^2 \phi_n \phi_{-n} - \frac{\alpha}{2} n (n+1) \phi_n \phi_{n-1} - \frac{\alpha}{2} n (n-1) \phi_n \phi_{n+1} \right\} \\
= \frac{1}{2gL} \sum_{n,m} \pi_n F_{nm}^{-1} \pi_m \\
+ \sum_n \frac{2\pi^2 g}{L} \left\{ n^2 \phi_n \phi_{-n} - \frac{\alpha}{2} n (n+1) \phi_n \phi_{n-1} - \frac{\alpha}{2} n (n-1) \phi_n \phi_{n+1} \right\} \\
= \frac{1}{2gL} \left[ \pi_n \pi_{-n} - \frac{\alpha}{2} \pi_n \pi_{-n+1} - \frac{\alpha}{2} \pi_n \pi_{-n-1} \\
+ (2\pi g)^2 n^2 \phi_n \phi_{-n} - \frac{\alpha}{2} (2\pi g)^2 n (n+1) \phi_n \phi_{n-1} - \frac{\alpha}{2} (2\pi g)^2 n (n-1) \phi_n \phi_{n+1} \right],
\]
which evaluates to

\[ \frac{2\pi}{L} \left( L_0 + \bar{L}_0 - \frac{\alpha}{2} \left( L_1 + \bar{L}_1 + L_{-1} + \bar{L}_{-1} \right) \right), \tag{58} \]

using (46), (38), and (44). Thus, \( H_\alpha \) varies from the original free Hamiltonian to a sine-square deformed Hamiltonian up to the overall factor \( \frac{1}{2} \) as \( \alpha \) is varied from 0 to 1. When \( \alpha = 1 \) in (57), \( r = 1 \) and (48) becomes

\[ F(x) = N \sum_{k \in \mathbb{Z}} e^{2\pi ikx/L} = N \delta(x) \quad \text{and} \quad G(x) = 1 - \cos \frac{2\pi x}{L}. \tag{59} \]

However, from (57), we note that \( N \) in (59) diverges as \( \alpha \) tends to unity.

Thus, we find that the \( g_{00} \) component of the worldsheet metric in the Lagrangian, \( F(x) \) severely diverges under the SSD. This result is expected because the change of the boundary condition is quite a singular event. If the metric is smooth, it can be transformed into a flat metric by diffeomorphism of the worldsheet. Nonetheless, such divergence impedes attempts at further analysis. A possible approach is to maintain \( \alpha \) away from unity. In this approach, \( \alpha \) serves as a regularization parameter. These analyses are left for future study.

## 5 Discussion

To better understand the relationships between open and closed strings, we investigated the SSD of string theory. We encountered strong divergence in the worldsheet metric of the sine-square deformed model. This divergence in the Lagrangian could be partly caused by the continuous treatment of the worldsheet. Therefore one may try to discretize the worldsheet itself besides the approach

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\(^{3}\)A similar in-between Hamiltonian with (58) was also discussed in [8].
proposed in the previous section. One such attempt could be achieved by the use of matrix models. Another might be an introduction of non-commutativity on the worldsheet. Non-commutative worldsheet had been considered in [10] and it had lead the deformed Virasoro algebra [11]. The point raised in the footnote in subsection 3.2 might also be relevant to this respect.

We also emphasize that the coupling constant of our analyzed system was spatially modulated. In statistical models, such modulations have not played a significant role. However, this is exactly what we would do when one introduces gravity to the model. In the context of string theory, this rather seems to be a natural option. In fact, [12] considered inhomogeneous XXX model, the particular case of XXZ model. The inhomogeneity in [12] differs from our analysis but it may have relevance in a wider sense. It is possible that a variety of spatial modulations introduced to statistical models (especially solvable ones) may reveal a rich structure and become essential in future studies of string theory.

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