Scalar Insertions in Cusped Wilson Loops in the Ladders Limit of Planar $\mathcal{N} = 4$ SYM

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ABSTRACT: Compact expressions in terms of the Q-functions of the Quantum Spectral Curve are given for 3-cusped Wilson loops in the ladders limit of $\mathcal{N} = 4$ Super Yang-Mills with additional scalars inserted at a cusp between uncoupled arcs. This gives some further credence to the already natural and evidenced view that the Quantum Spectral Curve in fact pertains to a wide class of observables beyond the spectrum, as well as providing additional nonperturbative ladders limit results.
1 Introduction

The integrability approach to planar $\mathcal{N} = 4$ Super Yang-Mills excitingly allows for non-perturbative results in this interacting field theory — see [1] for a pedagogical introduction and [2] for a wider review. While the full theory remains unsolved, many advances have been made towards this end. The Thermodynamic Bethe Ansatz allowed for the cusp anomalous dimension to be determined in [3] and [4] after the full spectrum of anomalous dimensions for local operators was obtained in [5], [6] and [7]. In [8], the powerful hexagon bootstrap approach was investigated for the computation of correlation functions. One would hope that the Separation of Variables approach would allow access to full non-perturbative expressions for the structure constants in terms of the Q-functions which are solutions of the Quantum Spectral Curve equations. Some examples of non-perturbative correlation functions have been found in [9], [10], [11] and [12]. Recently, [13] carried out further non-perturbative work for correlation functions of specific operators.

The Quantum Spectral Curve was introduced in [14], where it was used to produce the spectrum of anomalous dimensions for all local single trace operators in the planar $\mathcal{N} = 4$ theory. These methods were extended in [15], where they were used to produce the cusp anomalous dimension. Among its numerous strengths, the QSC enjoys a relatively simple structure compared to the TBA equations and is highly suggestive from a holographic point of view.

Further progress was made in [9], where for the first time expressions in terms of the Q-functions were given for the structure constants associated to three-cusped Wilson loops in a particular double scaling limit of the theory. These expressions both provided a significant simplification and met the hope that the QSC approach could provide non-perturbative information on many more observables than solely the spectrum. The limit in question is of note for the fact that the only Feynman
diagrams that survive the limiting procedure are those consisting of loops containing an arbitrary number of non-crossing scalar propagators. The appearance of these diagrams gives rise to this limiting theory’s name, "ladders limit".

This limit was first fully defined in [16], but the relevant subset of ladder diagrams was studied in [17]. Exact computation was made possible by this relatively simple diagram structure, which allowed for a complete resummation through a Bethe-Salpeter equation. In this setup the Q-function’s relevance was manifested in [9] through a concise integral transform relating them to the eigenfunctions of the Schrödinger operator appearing in this resumming equation. [9] studied a similar set of correlators to those in [12], and the two sets of work agreed where they overlapped.

In this paper we use the methods of [9] to supplement their results with an additional set of exact structure constants. Specifically, we consider objects formed by concatenating three coplanar Wilson arcs and inserting additional scalars at one of the cusps. There are various restrictions on the scalar combinations entering on the arcs and at the cusp which we motivate and make clear in the following subsection after introducing necessary terminology. This object is a generalisation of previously considered correlators, and a hope is that knowledge of it will provide a starting point for accessing the most general case in which the combinations of scalars attached to each arc is completely arbitrary.

In short, our calculation boils down to identifying that when using the parametrisation of [9] every integral coming from each additional scalar propagator can be immediately carried out. This is because the propagator expression is the derivative of a generating function \( w \) that appears exponentiated in the aforementioned integral transform formula relating the Q-functions to solutions of the Bethe-Salpeter equation that resums the ladders. We expand on the various details of this statement throughout the paper.

1.1 Setup

The Maldacena-Wilson lines in \( \mathcal{N} = 4 \) SYM are given by

\[
W^y_x (\vec{n}) = \text{Pexp} \int_x^y (iA_\mu dx^\mu + \Phi^a n^a | dx) ,
\]

in which \( A^\mu \) denotes the gauge field, \( \Phi^a (a = 1, ..., 6) \) denotes the 6 scalars and \( \vec{n} \) is a unit 6-vector.

We restrict attention to the case where the path of integration is a circular arc (such arcs being related to straight lines by conformal transformations, a symmetry retained in the ladders limit).

The Wilson loop is then formed by concatenating three of these in a two dimensional plane as in figure 1. The expectation value of its trace is then taken:

\[
\langle \text{Tr} \ W^{x_3}_{x_1} (\vec{n}_{1,2}) \ W^{x_3}_{x_2} (\vec{n}_{2,3}) \ W^{x_3}_{x_3} (\vec{n}_{3,1}) \rangle.
\]

This quantity is parametrised by the three angles made at the intersections of the arcs \( \phi_i \) and the
value of each of the inner products $\vec{n}_{i-1,i} \cdot \vec{n}_{i,i+1} = \cos (\theta_i)$. It has the form of a 3 point correlator, and it is the structure constant thereof that is of interest. Analysis is restricted to the cases where the angles $\phi_i$ obey the triangle inequalities $\phi_1 + \phi_2 > \phi_3$, $\phi_2 + \phi_3 > \phi_1$, $\phi_3 + \phi_1 > \phi_2$ and additionally $0 < \phi_i < \pi$. Taken together these inequalities fix attention to cases where the intersections of the extensions of any two of the arcs lie outside of the cusped loop.

The ladders limit is a double scaling limit in which the ’t Hooft coupling $\hat{g}$ goes to zero with each $\hat{g}_i = \frac{2}{\hat{g}} e^{-i\theta_i/2}$ kept finite or zero. These $\hat{g}_i$ then provide us with three effective couplings. The sole Feynman diagrams relevant to the Wilson loops that survive this limit are those planar ones in which pairs of arcs are connected by scalar propagators (or no propagators). It is in this limit that a concise Bethe-Salpeter equation can be written to resum the ladders (i.e. the scalar propagators between arcs of which there are diagrams containing arbitrarily many, see figure 2).

It is worth explicitly mentioning a key property of the scalar propagator’s form: $\langle \phi^i \phi^j \rangle$ is proportional to $g^2 \delta^{ij}$, with $g$ the Yang-Mills coupling. Therefore for 6-vectors $\vec{r}$ and $\vec{k}$ we have that $\langle (r^a \phi^a) (k^b \phi^b) \rangle$ is proportional to $g^2 \delta^{ab} r^a k^b = g^2 \vec{r} \cdot \vec{k}$.

This ladders limit having been taken, one can see that a scalar propagator between the arcs meeting

**Figure 1.** The Wilson loop consisting of three circular arcs, with each arc’s respective field coupling indicated.
at $x_i$ comes with a factor of $\hat{g}_i^2$. There are then three cases, according to how many of these are nonzero:

- **HLL**, where only one $\hat{g}_i$ is nonzero.
- **HHL**, where exactly two $\hat{g}_i$ are nonzero.
- **HHH**, the most general case where the three $\hat{g}_i$ are arbitrary.

In the above $H$ and $L$ are respectively read as "Heavy" and "Light". One has that a Light cusp is one between arcs not connected by scalar propagators in the Feynman diagrams. This suggestive terminology is justified by the fact that the scaling dimensions of the cusps becomes large at strong coupling (see [9]), in agreement with the similar terminology for local operators and the relative sizes of their scaling dimensions.

### 1.2 Main result

Our main result is a set of expressions for HHL Wilson loops with an arbitrary number of additional scalar insertions at the L cusp (that is to say, the cusp between uncoupled arcs), which written in full is

$$\langle \text{Tr} \, W_{x_1}^{x_3} (\vec{n}_{1,2}) \, W_{x_2}^{x_3} (\vec{n}_{2,3}) \left( n_{2,3}^b \Phi^b (x_3) \right)^{m_2} \left( n_{3,1}^a \Phi^a (x_3) \right)^{m_1} \, W_{x_3}^{x_1} (\vec{n}_{3,1}) \rangle,$$

(1.3)

with $\vec{n}_{2,3} \cdot \vec{n}_{3,1} = 0$. The restriction on the combinations of scalars in the insertions is made for two reasons — one is that this retains the simple ladder structure of the diagrams (avoiding crossings between "rungs" of the ladders) and the other is that these combinations are those produced by acting on the cusp with [9]'s projection operators. We elucidate on the details of this latter statement in our conclusion, and for now press on with this restriction applied.

We find that (1.3) can be computed exactly. Ladder resummation about each cusp produces a divergence which is canonically normalised upon division by factors given in section 2.3. These factors were found in [9] by studying the loop with two cusps. Factors of $\hat{g}_1$ and $\hat{g}_2$ are left present in our expression — one is free to adopt a different normalisation in order to modify these, but such measures would not affect the factors that we focus on here. All dependence in the $\phi_i$ is captured by the other factors, as is all nontrivial $\theta_i$ dependence. Subject to this normalisation, we find that (1.3) equals

$$\hat{g}_1^{2m_1} \cdot \hat{g}_2^{2m_2} \cdot \hat{C} \cdot \left( \frac{|x_1 - x_2|}{|x_1 - x_3||x_2 - x_3|} \right)^{m_1 + m_2} \cdot \left( \frac{|x_1 - x_3|}{|x_2 - x_3||x_1 - x_2|} \right)^{\Delta_0^1} \cdot \left( \frac{|x_2 - x_3|}{|x_1 - x_2||x_1 - x_3|} \right)^{\Delta_0^1},$$

(1.4)

In this formula $\hat{C}$ is a term independent of the $x_i$, depending only on the angles $\phi_i$. We give a concise expression for $\hat{C}$ in terms of Q-functions below. Note the correct dependence on $x_1$, $x_2$ and $x_3$ for a three-point correlator in a conformal theory. The dimensions $\Delta_0^1$ are the same as those given
indeed our result agrees with theirs when \( m_1 \) and \( m_2 \) are taken to zero.

We adopt the shorthand \( n = m_1 + m_2 \), which is the only combination of these numbers relevant outside of the coupling factors. The structure constants \( \hat{C} \) of these quantities are found to have a very simple form when expressed in terms of the Q-functions and a succinct bracket \( \langle \circ \rangle \) to be defined:

\[
\hat{C} = \frac{\langle q_0^1 (u) q_0^2 (u) e^{-\phi_3 u} \rangle}{\sqrt{\langle (q_0^1)^2 \rangle} \sqrt{\langle (q_0^2)^2 \rangle}}.
\] (1.5)

The symbol \( q_0^i \) denotes the Q-function associated to cusp i (the subscript 0 refers to a ground state, to be defined precisely in section 2.1 but included here for definiteness). The bracket is defined for functions \( f(u) \sim u^n e^{\beta u} \) at large \( u \) by

\[
\langle f(u) \rangle \equiv \left( 2 \sin \left( \frac{\beta}{2} \right) \right)^a \int f(u) \frac{du}{2\pi i u},
\] (1.6)

in which \( \Gamma \) is a vertical contour of constant positive real part. The \( n = 0 \) expression is precisely that obtained for the HHL cusped loop in [9], of which (1.5) constitutes a generalisation.

### 2 Review of informing material

#### 2.1 Q-function machinery and ladder resummation

The appearance of the Q-functions here is explicitly realised through a relation they share with terms appearing in solutions \( G \) of the Bethe-Salpeter equation that resums the ladders for the loops with two cusps (see figure 2):

\[
G(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = \sum_k \frac{4F_k (\Lambda_1 - \Lambda_2) F_k (\Lambda_4 - \Lambda_3)}{||F_k||^2 \sqrt{\text{E}_k}} \sinh \left( \frac{\sqrt{\text{E}_k}}{2} (\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4) \right).
\] (2.1)

We take this result from [9]. A different approach based on an integro-differential equation for resumming ladders was investigated in [18]. In [9] the Bethe-Salpeter equation in question decouples in suitable variables into two equations, one having exponentials as solutions and the other being a Schrödinger equation with potential term equal to the scalar propagator. This Schrödinger equation was first derived in [17], and we do not write or describe it more fully here as it is not directly useful for the following material. This latter equation possesses a discrete and continuous spectrum of eigenvalues \( E_k \) with eigenfunctions \( F_k \), which is what is being summed/integrated over. Writing
Figure 2. Ladder resummation over part of the 2-cusped loop, defining $G(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4)$. Note that the expression for this is divergent for resummation up to either of the cusps.

$$\Delta_k = -\sqrt{-E_k},$$

the Q-functions’s explicit relations to each $F_k$ were first given in [9] through an integral transform:

$$F_k(z) = e^{-\Delta_k z/2} \int q_k(u) e^{iu\phi(z)} du du u.$$  \hfill (2.2)

Here the function $w_\phi$, which is central to our calculation, is given by

$$e^{iu\phi(z)} = \frac{\cosh \left(\frac{z-i\phi}{2}\right)}{\cosh \left(\frac{z+i\phi}{2}\right)}.$$ \hfill (2.3)

The symbol $q_k$ denotes the solution to the Quantum Spectral Curve equations at $k^{th}$ excitation. Specifically, the Baxter equation which the Q-functions solve possesses several other solutions for different energy levels. Their relevance to the cusped Wilson loop is manifested in [9], where they enter into the structure constants for cusped loops acted on by exciting projection operators.

The relation (2.2) is invertible, and the Q-functions formed in this manner indeed obey the Baxter equation coming from the Quantum Spectral Curve formalism (this calculation is carried out in full in [9]). Specifically, they are the solutions with large $u$ asymptotics $q(u) \sim u^\Delta e^{i\phi u}$. 

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2.2 Parametrisation

Before carrying out our calculation we must also review the parametrisation of [9]. This involves an explicit coordinatisation of the arcs with useful highlighted properties and a transfer function for changing direction on the arcs.

We adopt the complex coordinates \( x_i = (\Re(z_i), \Im(z_i), 0, 0) \) and the notation \( z_{ij} = |z_i - z_j| \). The directed arcs connecting \( x_1 = (\Re(z_1), \Im(z_1), 0, 0) \) to \( x_2 \) and \( x_3 \) respectively are then the images of

\[
\begin{align*}
\zeta_{13}(t) &= z_1 - \frac{z_{12}z_{13}e^t}{e^t z_{12} + \frac{i}{2 \sin(\phi_1)} z_{23} (1 - e^t) \left( -e^{-i\phi_1} + e^{i(\phi_2 - \phi_3)} \right)}, \\
\zeta_{12}(s) &= z_1 - \frac{z_{12}z_{13}e^s}{e^s z_{13} + \frac{i}{2 \sin(\phi_1)} z_{23} (1 - e^s) \left( -e^{i\phi_1} + e^{i(\phi_2 - \phi_3)} \right)}. 
\end{align*}
\]

(2.4)

Directed arcs connecting \( x_a \) and \( x_b \) are in the same way the images of the curves \( \vec{x}_{ab} = (\Re(\zeta_{ab}(k)), \Im(\zeta_{ab}(k), 0, 0), k \in (-\infty, 0). \zeta_{ab} \) is obtained from (2.4) via cyclic permutation of the indices. The principle merit of this parametrisation is how simple the propagator is:

\[
\begin{align*}
|\dot{x}_{12}(s)| |\dot{x}_{13}(t)| &= \frac{1}{2} \cosh (s - t - \delta x_1) + \cos (\phi_1), \\
\delta x_1 &= \log \left( \frac{\sin \left( \frac{\phi_1 - \phi_2 + \phi_3}{2} \right)}{\sin \left( \frac{\phi_1 + \phi_2 - \phi_3}{2} \right)} \right). 
\end{align*}
\]

(2.5)

(2.6)

In [9], this particular expression necessarily had the same form as the potential in the resumming Schrödinger equation, which allowed for neat cancellations to take place in their deriving of the Q-function expression for the HHL correlator. In our calculation these propagator factors appear once under each integral and handily are the derivative of the function \( w_{ij} \) appearing in the relation (2.2), allowing the integrals to be straightforwardly carried out. This is essentially all that there is to our calculation.

Since we will be resumming ladders in both directions along the \( x_1 \) to \( x_2 \) arc there is need for a function that relates the two different parametrisations (\( \vec{x}_{12} \) and \( \vec{x}_{21} \)) for a given point on the arc, as in \( \zeta_{12}(s) = \zeta_{21}(T_{12}(s)) \). Such a (surprisingly simple) function is given by

\[
e^{T_{12}(s)} = \frac{(1 - e^t)}{1 - e^{\cos(\phi_1) - \cos(\phi_1 + \phi_2)}}. 
\]

(2.7)
2.3 Normalisation

The last thing to mention is the normalisation, which again is the same as in [9]. There is a divergence when the ladders around a cusp are resummed, which we normalise by cutting an $\epsilon$-circle around each cusp. This amounts to having $s_i$ and $t_i$ run from $-\Lambda_{s_i}$ and $-\Lambda_{t_i}$ respectively, with

$$
\Lambda_{s_i} = \log \left( \frac{x_{12} x_{13} \sin \left( \phi_1 \right)}{x_{23} \epsilon \sin \left( \frac{1}{2} \left( \phi_1 - \phi_2 + \phi_3 \right) \right)} \right), \quad \Lambda_{t_i} = \Lambda_{s_i} + \delta x_1, \quad (2.8)
$$

and the other cutoffs defined by cyclic permutation of the indices. Normalisation is then obtained by dividing by $\frac{1}{\left| x_1 - x_2 \right|^2}$.

3 Scalar insertions at the L cusp

In this section we carry out the calculation explicitly. The leading order Feynman diagrams relevant to (1.3) take the form of $n$ propagators from $x_3$ to points along the arc connecting $x_1$ and $x_2$ and a resummation of ladders around the cusps at $x_1$ and $x_2$ up to the points where propagators from $x_3$ meet the arc (see figure 3).

Apart from the factor $\hat{g}_1^{2m_1} \cdot \hat{g}_2^{2m_2}$ which we suppress, resumming all of these diagrams amounts to the following integral:

$$
C = \frac{1}{\left| \frac{d_1(10)}{ds} \right|^n} \cdot \int_{-\infty}^{0} \frac{ds_1}{\cosh \left( s_1 - \delta x_1 \right) + \cos \left( \phi_1 \right)} \cdot G_1 \left( \Lambda_{t_1}, \Lambda_{t_i}, s - \delta x_1, 0 \right)
\times \int_{s_1}^{0} \frac{ds_2}{\cosh \left( s_2 - \delta x_1 \right) + \cos \left( \phi_1 \right)} \cdots \int_{s_{n-1}}^{0} \frac{ds_{n-1}}{\cosh \left( s_{n-1} - \delta x_1 \right) + \cos \left( \phi_1 \right)}
\times \int_{s_{n-1}}^{0} \frac{ds_n}{\cosh \left( s_n - \delta x_1 \right) + \cos \left( \phi_1 \right)} \cdot G_2 \left( \Lambda_{t_2}, \Lambda_{t_2}, -\delta x_2, T_{12} \left( s_n \right) \right). \quad (3.1)
$$

Note that although this expression and the informing parametrisation are not symmetric with respect to the points $x_1$ and $x_2$ our final expression will be. $G_1$ and $G_2$ are the ladder resummations about cusp 1 and 2 as in (2.1). We have incorporated the necessary shifts $\delta x_1$ and $\delta x_2$ so as to ensure that our propagators are properly placed, and stress that we have not yet divided by the renormalising factors. Each of the factors under the integration measures come from the propagators, along with the factor sitting outside the first integral (explicitly, these factors are introduced when using (2.5) to replace the primal coordinate-free propagator expressions with those pertaining to the parametrisation).
Figure 3. Leading order diagram for the cusp with insertions. Blue and red dashed lines are resummed up to the blue and red solid lines respectively, and the solid lines (one red, one blue and \( n - 2 \) green) are the propagators between the light cusp 3 and the arc opposite it.

In the small or large \( \Lambda \) limit, the dominant contributions from (2.1) are the terms containing the ground state function \( F_0 \) multiplied by the leading exponential term in \( \sinh \). Singling these out and collecting prefactors gives

\[
C = \frac{4}{|2 \frac{d\tilde{\Omega}(0)}{ds}|^n} \cdot \frac{F_0^1(0) F_0^2(0)}{||F_0^1||^2||F_0^1||^2 (-\Delta_0^1) (-\Delta_0^2)} \cdot \exp \left( -\Lambda_1 \Delta_0^1 - \Lambda_2 \Delta_0^2 \right)

\times \int_{-\infty}^0 ds_1 \frac{d s_1}{\cosh (s_1 - \delta x_1) + \cos (\phi_1)} \cdot F_0^1 (s_1 - \delta x_1) \cdot \exp \left( \frac{\Delta_0^1}{2} (\delta x_1 - s_1) \right)

\times \int_{s_1}^0 ds_2 \frac{d s_2}{\cosh (s_2 - \delta x_1) + \cos (\phi_1)} \cdot \cdots \cdot \int_{s_{n-2}}^0 \frac{d s_{n-1}}{\cosh (s_{n-1} - \delta x_1) + \cos (\phi_1)}

\times \int_{s_{n-1}}^0 \frac{d s_n}{\cosh (s_n - \delta x_1) + \cos (\phi_1)} \cdot F_0^2 (T_{12} (s_n) + \delta x_2) \cdot \exp \left( \frac{\Delta_0^2}{2} (\delta x_2 - T_{12} (s_n)) \right).
\]

(3.2)
Again, the superscripts on the $F$ pertain to which cusp they are relevant to. The ground state functions $F_0^i$ are even with $F_0^i(z) = F_0^i(-z)$, so in addition to (2.2) there is also another otherwise identical relationship with $z \to -z$. Using both of these relations to replace the instances of $F$ appearing under the integrals in (3.2) with the more enlightening Q-functions yields

$$
C = \frac{-4}{2\frac{d\xi_{13}(0)}{ds}} \cdot \frac{F_0^1(0) F_0^2(0)}{|F_0^1(0)|^2 |F_0^2(0)|^2 (\Delta_0^1 - \Delta_0^2)} \cdot \exp \left( -\Lambda_1 \Delta_0^1 - \Lambda_2 \Delta_0^2 \right) \\
\times \int_{-\infty}^{0} \frac{ds_1}{\cosh (s_1 - \delta x_1) + \cos (\phi_1)} \cdot \int_1^{0} q_0^1(u) e^{w_{\phi_1}(\delta x_1 - s_1)u} \frac{du}{u} \\
\times \int_{s_1}^{0} \frac{ds_2}{\cosh (s_2 - \delta x_1) + \cos (\phi_1)} \cdots \int_{s_{n-2}}^{0} \frac{ds_{n-1}}{\cosh (s_{n-1} - \delta x_1) + \cos (\phi_1)} \\
\times \int_{s_{n-1}}^{0} \frac{ds_n}{\cosh (s_n - \delta x_1) + \cos (\phi_1)} \cdot \int_{1}^{0} q_0^2(v) e^{w_{\phi_1}(-T_{12}(s) - \delta x_1)u} \frac{dv}{v} \cdot e^{\Delta_2^1 \delta x_2}. \quad (3.3)
$$

Things are simplified further by applying the relation between the cutoff parameters, $\Lambda_{ij} = \Lambda_{ij} + \delta x_i$, in the exponential prefactor. It is at this stage that another one of the strengths of the parametrisation comes into play, in the form of the surprising relation

$$
w_{\phi_2}(-\delta x_2 - T_{12}(s)) = w_{\phi_1}(s - \delta x_1) - \phi_3. \quad (3.4)
$$

Using (3.4) we can eliminate $T_{12}$ from our expression, leading to

$$
C = \frac{-4}{2\frac{d\xi_{13}(0)}{ds}} \cdot \frac{F_0^1(0) F_0^2(0)}{|F_0^1(0)|^2 |F_0^2(0)|^2 (\Delta_0^1 - \Delta_0^2)} \cdot \exp \left( -\Lambda_1 \Delta_0^1 - \Lambda_2 \Delta_0^2 \right) \\
\times \int_{-\infty}^{0} \frac{ds_1}{\cosh (s_1 - \delta x_1) + \cos (\phi_1)} \cdot \int_1^{0} q_0^1(u) e^{w_{\phi_1}(\delta x_1 - s_1)u} \frac{du}{u} \\
\times \int_{s_1}^{0} \frac{ds_2}{\cosh (s_2 - \delta x_1) + \cos (\phi_1)} \cdots \int_{s_{n-2}}^{0} \frac{ds_{n-1}}{\cosh (s_{n-1} - \delta x_1) + \cos (\phi_1)} \\
\times \int_{s_{n-1}}^{0} \frac{ds_n}{\cosh (s_n - \delta x_1) + \cos (\phi_1)} \cdot \int_{1}^{0} q_0^2(v) e^{w_{\phi_1}(s - \delta x_1)u} \frac{dv}{v} \cdot e^{\Delta_2^1 \delta x_2}. \quad (3.5)
$$

Another grace of the parametrisation is the fact that

$$
w^t_{\phi}(z) = \frac{-\sin (\phi_i)}{\cosh (z) + \cos (\phi_i)}, \quad (3.6)
$$
which allows us to straightforwardly carry out each integral over \( s_i \) for \( 2 \leq i \leq n \). Since \( w_{\phi_i} (-\delta x_1) = -\phi_2 + \phi_3 \) the terms that arise from evaluation of the primitives at \( s_j = 0 \) are exponentially suppressed in \( u \) (bearing in mind the triangle inequalities on the \( \phi_i \)), so vanish at \( \Re (v) = \infty \). The \( Q \)-functions have no poles outside of the imaginary axis so we can move the \( v \) contour to \( \Re (v) = \infty \) where this vanishing occurs. We therefore need only to retain terms arising from evaluation of the primitives at \( s_i = -\infty \) with the understanding that we take \( \Re (v) \rightarrow \infty \). This gives us

\[
C = \frac{-4}{2 \frac{d^{\xi_1(0)}}{ds}} \cdot \frac{F_0^1(0) F_0^2(0)}{|F_0^1||F_0^1| |F_1^0|^2 (-\Delta_0^1)} \cdot \exp \left(-\Lambda_1 \Delta_0^1 - \Lambda_2 \Delta_0^2\right) \\
\times \int_{-\infty}^{\infty} \frac{ds_1}{\cosh (s_1 - \delta x_1) + \cos (\phi_1)} \cdot \int q_0^1(u) e^{w_{\phi_1}(s_1 - \delta x_1) u} du \\
\times \left(\frac{1}{\sin (\phi_1)}\right)^{n-1} \int q_0^2(v) e^{w_{\phi_1}(s_1 - \delta x_1) v} e^{-\phi_2 u} dv. \tag{3.7}
\]

Note that \( w_{\phi}(z) \) is odd and so the integral over \( s_1 \) is carried out in much the same way except that a factor of \( \frac{1}{v} \) arises rather than simply \( \frac{1}{v} \). As \( z \rightarrow \infty \), \( w_{\phi_1}(z) \rightarrow \phi_1 \) and so

\[
C = \frac{-4}{2 \frac{d^{\xi_1(0)}}{ds}} \cdot \frac{F_0^1(0) F_0^2(0)}{|F_0^1||F_0^1| |F_1^0|^2 (-\Delta_0^1)} \cdot \exp \left(-\Lambda_1 \Delta_0^1 - \Lambda_2 \Delta_0^2\right) \\
\times \left(\frac{1}{\sin (\phi_1)}\right)^{n} \int \frac{du}{u} \int \frac{dv}{v^n} \cdot q_0^1(u) \cdot q_0^2(v) e^{-\phi_1 u v} u - v \left(e^{-(\phi_2 + \phi_3)(v-u)} - e^{\phi_1(v-u)}\right). \tag{3.8}
\]

The integrand actually has no pole at \( u = v \), and so the \( u \) and \( v \) contours can be moved over each other without concern. We therefore fix the \( v \) contour to be to the right of the \( u \) contour.

Additionally, the integral with integrand proportional to \( e^{-\phi_2 u} \) can then be discarded, as we are taking the \( v \) contour to \( \Re (v) = \infty \) where this integrand vanishes. The remaining integral over \( u \) is then straightforwardly carried out by closing the \( u \) contour with a semicircle in the right half of the complex plane and picking up the residue at \( u = v \):

\[
C = \frac{-4}{2 \frac{d^{\xi_1(0)}}{ds}} \cdot \frac{F_0^1(0) F_0^2(0)}{|F_0^1||F_0^1| |F_1^0|^2 (-\Delta_0^1)} \cdot \exp \left(-\Lambda_1 \Delta_0^1 - \Lambda_2 \Delta_0^2\right) \\
\times -2\pi i \left(\frac{1}{\sin (\phi_1)}\right)^{n} \int \frac{dv}{v^{n+1}} \cdot q_0^1(v) \cdot q_0^2(v) e^{-\phi_1 v}. \tag{3.9}
\]

We have, from differentiation of the coordinate functions as in (2.4), the first sighting of our correlator’s conformal dependence on \( x_3 \) in this analysis:
Also, we should note this relation for the norm of the wavefunctions from [9]:

\[
||F_0^k|| = 2\pi i \cdot \sqrt{\frac{2}{\Delta_0^2}} \sqrt{\int \left( \frac{q_0^k(u)}{u} \right)^2 \frac{du}{2\pi i}}.
\] (3.11)

Tedious but straightforward algebra (including applying (2.8) and dividing by the two normalisation constants \(N_1^1\) and \(N_2^2\)) then leads to

\[
C_{\text{Renormalised}} = \left( \frac{|z_1 - z_2|}{|z_1 - z_3||z_2 - z_3|} \right)^n \cdot \left( \frac{|z_1 - z_3|}{|z_2 - z_3||z_1 - z_2|} \right)^{\Delta_0^2} \cdot \left( \frac{|z_2 - z_3|}{|z_1 - z_2||z_1 - z_3|} \right)^{\Delta_0^1}
\times \left( 2 \sin \left( \frac{\phi_1 + \phi_2 - \phi_3}{2} \right) \right)^{-\eta + \Delta_1^1 + \Delta_0^2} \cdot \left( 2 \sin (\phi_1) \right)^{-\Delta_0^1} \cdot \left( 2 \sin (\phi_2) \right)^{-\Delta_0^2}
\times \frac{2 \cdot 2\pi i}{||F_0^1||||F_0^2||} \int \frac{dv}{u^{\eta+1}} \cdot q_0^1(u) \cdot q_0^2(u) e^{-\phi_1 u}
\int \left( q_0^1 \right)^2 \left( q_0^2 \right)^2 e^{-\eta \phi_1 u}.
\] (3.12)

As it should, this has the form of a 3-point correlator in a conformal field theory. It should again be stated that we have suppressed the coupling factors \(g_1^{2m_1} \cdot g_2^{2m_2}\). One can replace the \(||F_0^1||\) in the above using (3.11). The Q-functions behave as \(u^{\Delta} e^{\phi u}\) for large \(u\), and so the structure constant \(\hat{C}\) is further simplified through use of the bracket (1.6) which encapsulates the extraneous factors:

\[
\hat{C} = \frac{\langle q_0^1(u) q_0^2(u) e^{-\phi_1 u} \rangle}{\sqrt{\langle (q_0^1)^2 \rangle \langle (q_0^2)^2 \rangle}}.
\] (3.13)

Note that this entire analysis can be repeated with each of the H cusps excited to level \(k\) and \(l\), and the result therein amounts to replacing each Q-function \(q_0^i\) in (3.13) with the corresponding excited Q-functions \(q_1^k\) and \(q_2^l\) as in (2.2). It should be born in mind that one also needs to use a different normalisation for these excited cusps as laid out in [9], but the form of the result is the same as (3.13) when using the bracket notation.
4 Conclusions

Equation (1.5) in [9] gives a simple expression for the derivative of the cusp anomalous dimension $\Delta$ with respect to the square of the coupling $\hat{g}$, which was interpreted as the structure constant of two cusps with a single insertion of the type considered here. Our result gives perfect agreement with this and generalises it by considering multiple insertions of this type.

There is a more general class of insertions to consider. These consist of derivatives of the scalar fields of various orders. Schematically these are $\phi', \phi'',\ldots$, et cetera and various products of these. Combinations may include such things as $(\phi')^2\phi''$, $(\phi')^3(\phi'')^2$, $(\phi')^3(\phi'')^2(\phi')^2$ and so on. The projection operators defined in [9] that excite a cusp act so as to produce specific sums of such insertions, which justifies some restriction of attention.

Some study of these insertions has been carried out, and expressions for any one of these are found relatively simply by the methods employed in this paper. They appear to be given by a bracket of two $Q$-functions, an exponential $\exp(-\phi_j u)$ and a function $P(u)$. The $P(u)$ are complicated polynomials in $(u\pm ni)^{-1}$ for various $n$, with $n$ an integer no larger than the highest order of derivative appearing. A succinct catch all expression for the $P(u)$ generated by the $k^{th}$ order projection operator is still lacking.

With or without such a formula, which may not even exist, one could still hope to make comparisons between expressions found in this way and those given in [9] for the HLL cusp with a single L cusp at $k^{th}$ excitation. There is a belief that the $Q$-functions at zero coupling and $k^{th}$ excitation should be related to the $k^{th}$ $P(u)$ by some involutory transformation. Progress towards an expression for the HHH structure constant, which remains an interesting open problem, may be achieved in this way.

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