Stress Tensor on Null Boundaries

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Abstract

Using the Brown-York prescription for the definition of quasilocal gravitational energy-momentum tensor on a boundary and also complete canonical structure on a null boundary which has been found recently [1], we propose a similar stress tensor on the null boundary. Then we exploit this stress tensor to compute the quasi-local energy and angular momentum for some well-known gravitational solutions. We have found that in addition to reference spacetime method for regularizing total energy, in the case of null boundary we can add a possible counterterm so avoiding embedding difficulties.
1 Introduction

In general relativity (GR), defining a local energy for gravitational field is problematic (for a review on the subject see [2]). The problem is rooted in the general covariance principle and the fact that the first derivative of metric can always vanish in a properly chosen local coordinate system. But in order to have a notion of local energy density with correct physical dimension, such quantity could be defined in terms of the metric and its first derivative. Also, a related notion to energy density in field theories is the action functional. In fact, the standard covariant action in GR, \textit{i.e.}, the Hilbert-Einstein (HE) action, contains second order derivatives of the metric, which is another consequence of the general covariance principle. As it is well-known, this action does not have a well-posed variational principle and needs to be complemented with additional terms defined on the boundaries of the spacetime. Brown and York [3] have pointed out that having an action with a well-posed variational principle,
by using Hamilton-Jacobi analysis, one can define the quasilocal energy of the system.

A careful variation of the HE action in addition to suggesting the proper boundary term (boundary action), to ensure a well-posed variational principle, also provides dynamical degrees of the theory as well as its canonical structure (see e.g., [1]). An important point, in finding the complete canonical structure by such a procedure, is the necessity of the condition that one should not suppress any degree of freedom beforehand by imposing restrictions. The proper boundary action complementing the HE action, when the boundary is timelike or spacelike, is the well-known Gibbons-Hawking-York (GHY) term [5,6]. Applying variational principle in GR for null boundaries has been a subject of investigation in recent years [7–10]. In these papers, the authors are either interested in finding the proper boundary terms on null boundaries, and therefore ignored those terms that are fixed by boundary conditions, or imposed some restrictions on variations in such a way that the resulting canonical structure is not complete. One of such restrictions is that the variations keep the character of the boundary unchanged. But one of the metric degrees of freedom is responsible for variations which alter the boundary character. This can be easily seen as follows: If the boundary is specified by $\phi = \text{const.}$ for some scalar field $\phi$, then the normal to the boundary is proportional to $\partial_a \phi$. For a null boundary, we have $\partial_a \phi \partial^a \phi = 0$, while for a general metric variation we find:

$$\partial_a \phi \partial^a \phi \rightarrow \partial_a \phi \partial^a \phi - \delta^{ab} \partial_a \phi \partial_b \phi.$$  

Therefore, unless we set variations of metric in direction of $\partial_a \phi$ to zero, the boundary doesn’t remain null. Recently we have shown that such variations appear in canonical structure [1]. In addition, it has been shown that, in order to preserve such variations, a general double-foliation framework is needed. The conjugate momentum to these variations is a scalar $\Xi$ (for definition see section 3 or [1]). In this article we will show that this scalar provides the quasilocal energy density on a null boundary. For the definition of quasilocal stress tensor on null boundary we follow the same method presented by Brown and York in the case of quasilocal stress tensor on timelike boundary which means that the derivative of the action with respect to the metric variations which are tangential to the boundary [3,11].

The content of the paper is organized as follows. In the next section we first briefly review the original Brown-York construction of quasi-local gravitational energy-momentum tensor. In section 3 we follow the same analysis for the case where instead of timelike boundary we have a null one. Finally, in section 4 we calculate the energy and angular momentum in various space times using the proposed stress tensor.
Throughout this work we have set $G = c = 1$. We do not use various indices for specification of different parts of spacetime and use Latin indices $\{a, b, c, \ldots \}$ everywhere; instead, different names are given to objects when defined on different structures. For example, we use $K_{ab}$ for extrinsic curvature of a spacelike hypersurface while $\chi_{ab}$ is preserved for extrinsic curvature of a timelike hypersurface.

2 The Brown-York Tensor

In this section we review the Brown-York [3] definition of gravitational stress-tensor on the boundary. To illustrate the main idea beyond Brown-York definition of quasi-local energy momentum, it is useful to start with an example in classical mechanics.

2.1 The Hamilton-Jacobi Method

Let $L(q(t), \dot{q}(t), t)$ be the classical Lagrangian for a particle. The action functional is

$$I[q(t)] = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t) \, dt,$$

for initial and final configuration, i.e. $(q_1, t_1)$ and $(q_2, t_2)$, respectively. A general variation of this action for a given history is:

$$\delta I[q(t)] = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q} \delta q - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} |_{t_1}^{t_2} - \left( \frac{\partial L}{\partial \dot{q}} \dot{q} - L \right) \delta t |_{t_1}^{t_2}. \quad (2)$$

Extremizing the action and imposing the boundary condition by fixing $q$ and $t$ at the end points, provides the equations of motion that are given by the first term in the integrand which are known as the Euler-Lagrange equations. Moreover, the Hamilton-Jacobi principal function $S(q_1, t_1; q_2, t_2)$ is defined as the value of the action for a solution $q(t)$ of the equation of motion from $(q_1, t_1)$ to $(q_2, t_2)$.

Thus, according to (2), the derivative of the principal function $S$ with respect to $q$, i.e. $\frac{\delta S}{\delta q}$, gives the canonical momenta $p = \frac{\partial L}{\partial \dot{q}}$ while the derivative with respect to $t$, i.e. $\frac{\delta S}{\delta t}$, yields the minus of energy: $H = \left( \frac{\partial L}{\partial \dot{q}} \dot{q} - L \right)$.

An important premonition is that in order to ensure that the above procedure works well, the variational principle must be well-posed beforehand. Indeed, a Lagrangian with no well-posed variational principle does not lead to the correct relations for momenta and energy of the system. For example, the action $S_1 = \int \frac{1}{2} m \dot{x}^2 dt$ leads to the equations of motion and correct momentum and energy for a free particle. However, by considering another action as $S_2 = S_1 - \int \frac{1}{2} m \frac{d}{dt}(x \dot{x}) dt = - \int \frac{1}{2} m x \ddot{x} dt$, one could see that, although the difference with the first action is a total derivative and the equations of motion are unchanged, by varying
this action one finds:

\[ \delta S_2 = - \int (m \ddot{x}) \delta x + \left[ \frac{1}{2} (\dot{x} \delta x - x \delta \dot{x}) \right]_{t_1}^{t_2}, \]  
(3)

which means that in order to get the equation of motion we need to determine both \( x \) and \( \dot{x} \) at both end points. This leads to inconsistency with second order equation of motion, because according to that we need to just fix the position at the ends. Therefore, for this action the variational principle is not well-posed\(^2\). As a consequence, the derivative of the principal function does not lead to the correct definition for momentum and energy. However, we must note that all total derivatives do not spoil the variational principle, e.g. changing the action by \( S \rightarrow S + \int_{t_1}^{t_2} \frac{dh}{dt} dt \) for arbitrary function \( h(q(t), t) \) is allowed. Hence, the action and the principal function are not unique; as a result, momenta and energy are also not exclusive. However, this arbitrariness can be fixed by choosing the zero point of energy, for example set the arbitrary function \( h \) so that it yields to zero energy for free particle at rest in the above example.

### 2.2 Stress tensor on timelike boundary

Having learned enough from the above simple mechanical example, let's begin with HE action in \( d \) dimension:

\[ S_{EH} = \frac{1}{16 \pi} \int d^d x \sqrt{-g} R, \]  
(4)

in which \( R \) is the Ricci scalar. By varying the action, if we consider the boundary segments to be either time-like or space-like, one gets:

\[ \delta S_{EH} = \frac{1}{16 \pi} \int_{\mathcal{M}} d^d x \sqrt{-g} G^{ab} \delta g_{ab} + \frac{1}{16 \pi} \sum_i [2 \delta \left( \int_{\mathcal{B}_i} d^{d-1} x \sqrt{|h|} K + \int_{\mathcal{C}_i} d^{d-2} x \sqrt{|q|} \vartheta q_{ab} \delta q_{ab} \right) ] + \int_{\mathcal{B}_i} d^{d-1} x \sqrt{|h|} (K^{ab} - K h^{ab}) \delta h_{ab} + \int_{\mathcal{C}_i} d^{d-2} x \sqrt{|q|} \vartheta q_{ab} \delta q_{ab}, \]  
(5)

where the sum is over each boundary segment \( \mathcal{B}_i \) and every corner \( \mathcal{C}_i \) at intersection of two neighboring segments of boundary. Moreover, \( K \) is the extrinsic curvature of each segment and \( \vartheta \) is the angle or the boost parameter between segments, depending on their character. In addition, \( \delta h_{ab} \) is the variation of metric on each boundary while \( \delta q_{ab} \) is the variation on each co-dimension two joints. For the details of the calculations and specially the method by which one can take into account the contribution of joints, see Refs. [8][10][11] or [1]. We

\(^2\)Of course this argument is valid if we are interested in Dirichlet or Neumann boundary conditions. By choosing a Robin boundary conditions one could revive the variational principle.
see that the variation of HE action includes both the metric and its normal derivative (the extrinsic curvature) on the boundary, thus, the variational principle is not well-posed for this action if we demand the Dirichlet boundary conditions \(3\). This variation also suggests the correct action with well-posed variational principle as:

\[
S = \frac{1}{16\pi} \int_{M} d^{d-1}x \sqrt{-g} \ R - \frac{1}{8\pi} \sum_{i} \left[ \int_{B_{i}} d^{d-1}x \sqrt{|h|} K + \int_{C_{i}} d^{d-2}x \sqrt{|q|} \ \vartheta \right].
\]

(6)

Now consider a region in space-time as in fig 1 which is generated by evolution of a spacelike surface \(\Sigma\) from \(\Sigma_{1}\) to \(\Sigma_{2}\) and a timelike boundary \(T\). The intersection of \(\Sigma\) leafs with \(T\) are co-dimension two surfaces \(S\) from \(S_{1}\) to \(S_{2}\). Now in this region of space-time we calculate the variation of the action (6), and impose the equations of motions, i.e. \(G_{ab} = 0\).

Then we may find:

\[
\delta S = \frac{1}{16\pi} \left[ \int_{\Sigma_{1}}^{\Sigma_{2}} d^{d-1}x \ P^{ab} \ \delta h_{ab} + \int_{T} d^{d-1}x \ \Pi^{ab} \ \delta \gamma_{ab} + \int_{S_{1}}^{S_{2}} d^{d-2}x \ \sqrt{|q|} \ \vartheta \ q^{ab} \ \delta q_{ab} \right].
\]

(7)

Here, the symbol \(\int_{\Sigma_{1}}^{\Sigma_{2}}\) is a shorthand for \(\int_{\Sigma_{2}} - \int_{\Sigma_{1}}\), \(h_{ab}\) is the metric on each spacelike boundary and \(\gamma_{ab}\) is the induced metric on the timelike boundary \(T\). \(P^{ab}\) and \(\Pi^{ab}\) are respectively

\footnote{For Neumann boundary condition in four dimensions, there is no need to any boundary term in the action to make a well-defined variational principle, see for example [12]. But usually in gravity one requires a Dirichlet boundary condition for which the metric on the boundary is fixed.}
the gravitational momenta of $\Sigma$ and $\mathcal{T}$:

\begin{align*}
P_{ab} &= \sqrt{h}(K_{ab} - K h_{ab})|_{\Sigma}, \\
\Pi_{ab} &= \sqrt{-\gamma}(\chi_{ab} - \chi \gamma_{ab})|_{\mathcal{T}},
\end{align*}

(8)

where $K_{ab}$ and $\chi_{ab}$ are their corresponding extrinsic curvatures. In the original work of Brown and York [3], they assumed the boundaries to be orthogonal, thus, the contribution of the joints was missing. However, after the work of Hayward [13], who emphasized the importance of this term for the variational principle to be well-posed, this term appeared in later works [11][14][15].

If we ignore these joints and consider the boundaries to be orthogonal, then in an analogous way to the mechanical example one may define the gravitational canonical momentum as the derivative of the principal functions with respect to the induced metric on spacelike segments of the boundary and gravitational energy-momentum-stress tensor as the derivative with respect to the induced metric on timelike segment. In fact, the $\Pi_{ab}$ has the same expression as for ADM canonical momentum. Thus, the energy-momentum-stress tensor (or just the stress tensor for abbreviation) will be defined as:

\begin{equation}
T_{ab} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma_{ab}} = \frac{1}{8\pi}(\chi_{ab} - \chi \gamma_{ab}).
\end{equation}

(9)

To define conserved quantities, now we choose a spacelike co-dimension two surface $S$ in $\mathcal{T}$ with unit timelike normal $u^a$. Then, the metric $\gamma_{ab}$ is further decomposed as $\gamma_{ab} = q_{ab} - u^a u^b$. Here, $u^a$ defines local the flow of time in $\mathcal{T}$. Moreover, according to Brown-York [3] for an isometry in the boundary generated by the Killing vector $\xi^a$, the conserved charge associated to this symmetry is defined by:

\begin{equation}
Q_\xi = \int_S d^{d-2}x \sqrt{q} \ T_{ab} u^a \xi^b.
\end{equation}

(10)

In fact, there is some points regarding to the expression (9) and (10). The first one is that, as we pointed out before, these expressions are not unique. One can append a subtraction term $S_0$ to the boundary action without affecting variational problem when $S_0$ depends on fixed boundary data, $S_0 = S_0(h_{ab}, \gamma_{ab})$, which leads to ambiguities in the definitions of energy and momenta. According to Brown and York interpretation these ambiguities are consequence of freedom to choose the zero point of energy and redefine system momenta with canonical transformation [3][11]. On the other hand, Eq. (10) leads to infinities when
calculated for large spheres in general systems in spacetime. The Brown-York proposal is to choose subtraction term \( S_0 \) such that the modified action \( S - S_0 \) leads to zero energy for flat spacetime. Therefore, the zero point is chosen to be the flat spacetime while the scheme for other geometries is to embed their boundary in flat spacetime. Thus, the modified expression for the stress tensor will be

\[
T_{ab} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma_{ab}} - \frac{2}{\sqrt{-\gamma}} \frac{\delta S_0}{\delta \gamma_{ab}} = \frac{1}{8\pi} (\chi_{ab} - \chi_{\gamma ab}) - \frac{2}{\sqrt{-\gamma}} \frac{\delta S_0}{\delta \gamma_{ab}}.
\]  

(11)

As another remark, if we want to consider the effect of joints or non-orthogonal boundaries, we must care about the components of \( \delta \gamma_{ab} \) contained in \( \delta q_{ab} \) appeared in the last term of (7). Also, we must be careful about the question that with respect to which observer we desire to calculate the quasi-local quantities. Note that when the boundaries are non-orthogonal, the Eulerian observers orthogonal to \( \Sigma \) constant are different from the observers orthogonal to \( S \) constant in the boundary \( \mathcal{T} \). In these cases, a further decomposition of the induced metric \( \gamma_{ab} \) with the assistance of the vector \( u_a = N \nabla_a t \) is required, where \( t \) is a foliation of \( \mathcal{T} \). By this decomposition we get [11]:

\[
\delta \gamma_{ab} = \delta q_{ab} - \frac{2}{N} u(a) \delta V_b - u_a u_b \frac{\delta N}{N},
\]

where \( N \) and \( V^a \) are lapse and shifts of decomposition. The components of \( \frac{\delta S}{\delta \gamma_{ab}} \) can be calculated as

\[
\epsilon \equiv u^a u^b T_{ab} = \frac{1}{\sqrt{q}} \frac{\delta S}{\delta N},
\]

(12)

\[
j^a \equiv q^{ac} u^b T_{cb} = -\frac{1}{\sqrt{q}} \frac{\delta S}{\delta V_a},
\]

(13)

\[
s^{ab} \equiv q^{ac} q^{bd} T_{cd} = \frac{2}{N \sqrt{q}} \frac{\delta S}{\delta q_{ab}},
\]

(14)

which are known respectively as the quasilocal energy density, tangential momentum density and spatial stress. The details of calculations for different observers can be found in [11,15]. In these cases it has been shown that a double foliation of space-time is a natural setup for calculations as described in appendices of [11]. There, the relation between quantities, as measured by different observers, has been obtained. The relations between two sets of
The quasilocal energy density associated with the two surfaces \( S \) as seen by the observers orthogonal to \( \Sigma \) constant is [11]:

\[
\epsilon = \frac{1}{8\pi} \left( \gamma k + \gamma v l \right) - \frac{2}{\sqrt{-\gamma}} \frac{\delta S_0}{\delta \gamma_{ab}} u^a u^b,
\]

(17)

where \( k_{ab} = -q^c_\alpha q^d_\beta \nabla_c n_d \) and \( l_{ab} = -q^c_\alpha q^d_\beta \nabla_c u_d \) are defined so that \( k \) and \( l \) are their trace, respectively. In the case for which two observers are at rest with respect to each other i.e. \( v = 0 \) or, in other words, the boundaries are orthogonal, the quasilocal energy density becomes:

\[
\epsilon = \frac{1}{8\pi} (k - k_0),
\]

(18)

where \( k_0 \) is the corresponding extrinsic curvature as embedded in flat space. So, the total quasilocal energy on \( S \) becomes [3][11]:

\[
E = \frac{1}{8\pi} \int_S d^{d-2}x \sqrt{q} (k - k_0).
\]

(19)

Evaluating the above integral provides an expression for \( E \) as a function of \( r \), \( E = E(r) \). However, we must note that because of the Eq. (17), in general the form of this function varies for different observers. For example, in the case of Schwarzschild black hole the function for static, radially infalling and boosted observers has been obtained in [15].

As the last point, let us note that for the above relations, and for different observers, it
is always assumed that these observers do not move in null geodesics. In fact, null observers can not be contained in the above setup because the normal vectors (or four-velocities) are always normalized to unity. As the name indicates, due to the fact that the norm vanishes for null observers, normalization does not make sense in this case. In fact, here, the normal vectors or four-velocities are both normal and tangent to the hypersurface, the induced metric becomes degenerate, and the usual extrinsic curvature does not make sense.

In the next section we will use standard treatment for null hyper-surfaces. The resulting stress tensor can provide the quasi-local quantities as measured by null observers.

2.3 Asymptotic AdS spacetime and counterterm method

Finding proper subtraction term by embedding in reference spacetime is a difficult task. In fact, it is not possible to embed a boundary with an arbitrary intrinsic metric in the reference spacetime. In the case of asymptotically AdS spacetimes, there is an attractive proposal without necessity of embedding in reference spacetime as proposed in [16]. This approach is inspired from AdS/CFT duality [17,18] by interpreting divergences which appear in stress tensor when the boundary is moved to infinity as dual to standard ultraviolet divergences of quantum field theory; then, they can be removed by adding local counterterms to the boundary action. For instance for AdS\(_4\) the counterterm Lagrangian in the boundary proposed to be:

\[
L_{ct} = -\frac{2}{\ell} \sqrt{-\gamma}(1 - \frac{\ell^2}{4} R),
\]

where \(R\) is the scalar curvature of induced metric on the boundary, and \(\ell\) is the AdS radius. Adding this term to the usual GHY term in the timelike boundary, then variation of action yields to a regularized stress tensor such that energy becomes finite at the \(r \to \infty\) limit. The main advantage of this method, beside the fact that counterterms are covariant and do not spoil the variational principle, is the lack of embedding difficulties in BY method. However, these counterterms are known just for asymptotically AdS spaces. For asymptotic flat space, since there is no length scale \(\ell\), finding such counterterms is problematic. Also, the limit \(\ell \to \infty\) in Eq. (20) does not lead to a unique covariant expression [19].

3 Hamilton-Jacobi analysis on null boundary

In this section we are going to repeat the calculation of previous section when instead of a timelike boundary we have a null one. The corresponding spacetime region is illustrated in Fig.3. In order to calculate the variation of the action in this region we need to express
surface divergences in the variation of HE action in terms of geometric objects of spacelike surfaces, i.e. $\Sigma_1$, $\Sigma_2$ and the null boundary $\mathcal{N}$. Variation on $\Sigma_1$ and $\Sigma_2$ is similar to the previous section; however, for a null surface (because of degeneracy of induced metric and divergence of extrinsic curvature) the calculation is completely different. In this section, we first introduce the basic tools for general investigation of null boundaries (without any gauge fixing) by introducing a general double foliation. Then, we find canonical momenta in this hypersurface so that by varying the action with respect to the metric components on this hypersurface we find stress tensor on the boundary.

3.1 The set up

A segment $\mathcal{N}$ of the spacetime boundary, characterized by $\phi_0 = 0$, is called a null hypersurface if $\nabla_a \phi_0 \nabla^a \phi_0 = 0$. This feature of null boundary indicates that the normal vector to the null surface is also tangent to it. This property is the origin of some difficulties when dealing with such hypersurfaces; because, as a consequence, the induced metric becomes degenerate and therefore constructing a projector to the null surface just from its normal is not possible. One standard remedy to this problem is to introduce an auxiliary null vector $k^a$ which lays out of the hypersurface and therefore $\ell_a k^a \neq 0$, when $\ell_a$ is the null normal to the boundary, e.g., $\ell_a \propto \nabla_a \phi_0$ on the boundary. For more details about the geometry of null hypersurfaces we refer the interested reader to [20][21].

By defining $\ell_a$ as the normal to the null boundary, we introduce the auxiliary null form $k_a$ and take the normalization of these null forms to be everywhere as:

$$\ell_a \ell^a = 0 \ , \ k_a k^a = 0 \ \text{and} \ \ell_a k^a = -1.$$  

(21)
With the aid of $\ell_a$ and $k_a$, we can define the projector as:

$$q^a{}_b = \delta^a{}_b + \ell^a k_b + k^a \ell_b$$  \hspace{1cm} (22)

This projector is not in fact a projector on null surface; instead, it essentially projects space-time vectors onto the co-dimension two surface $S$, to which $\ell_a$ and $k_a$ are orthogonal.

A systematic approach to define these co-dimension two surfaces is to use a double-foliation by two scalar fields ($\phi_0, \phi_1$). The intersection of level surfaces of $\phi_0$ and $\phi_1$ are the co-dimension two surfaces $S$. In this foliation $\ell_a$ and $k_a$ can be expanded generally as:

$$\ell_a = A \nabla_a \phi_0 + B \nabla_a \phi_1$$  \hspace{1cm} (23)

$$k_a = C \nabla_a \phi_0 + D \nabla_a \phi_1$$  \hspace{1cm} (24)

Three out of four coefficients in the above expansions are determined by normalization conditions (21) and one remains free, due to the rescaling or boost gauge freedom ($\ell_a \rightarrow a \ell_a, k_a \rightarrow \frac{1}{a} k_a$). On the boundary, we set the coefficient $B = 0$, so that the boundary is a level surface of $\phi_0$ and we have $\ell_a \equiv A \nabla_a \phi_0$, and it is null because $\nabla_a \phi_0 \nabla^a \phi_0 \propto \ell^a \ell_a = 0$. Note that $\ell_a$ and $k_a$ are form fields defined in whole space-time and everywhere we have $\ell^2 = k^2 = 0$, whereas $\nabla_a \phi_0 \nabla^a \phi_0 = 0$ is satisfied just on the boundary.

In this double foliation framework, the spacetime metric becomes:

$$g_{ab} dx^a dx^b = H_{ij} d\phi^i d\phi^j + q_{AB}(d\sigma^A + \beta^A_i d\phi^i)(d\sigma^B + \beta^B_j d\phi^j),$$  \hspace{1cm} (25)

in which $\{i,j\} \in \{0,1\}$ whereas $\{A,B\} \in \{2,\ldots,d-1\}$. Here $\sigma^A$ are coordinates on co-dimension two surface $S$ and $\beta^A_i$ are shift vectors. The normal metric $H_{ij}$ consists of lapse functions as

$$H_{ij} = -\begin{pmatrix} 2AC & BC + AD \\ BC + AD & 2BD \end{pmatrix}.$$  \hspace{1cm} (26)

By covariant differentiation of vectors $\ell_a$ and $k_a$ and projecting them in different directions, using $q^a_b, \ell^a$ and $k^a$, we can define the following geometric objects from $\nabla_a \ell_b$ and $\nabla_a k_b$:

$$\nabla_a \ell_b = -\Theta_{ab} - \omega_a \ell_b - \ell_a \eta_b - k_a a_b + \kappa k_a \ell_b - \kappa \ell_a \ell_b$$  \hspace{1cm} (27)

$$\nabla_a k_b = -\Xi_{ab} + \omega_a k_b - k_a \eta_b - \ell_a a_b - \kappa k_a k_b + \kappa \ell_a k_b$$  \hspace{1cm} (28)
These relations are generalizations of the relation $\nabla a n_b = -K_{ab} + n_a n_b$ to current case where decomposition has been done with two null vectors. The definitions are as follows:

\[ \Theta_{ab} = -q^c a q^d b \nabla a \ell_b \quad , \quad \Xi_{ab} = -q^c a q^d b \nabla a k_b, \]
\[ \eta_a = q^c a k^b \nabla b \ell_c \quad , \quad \bar{\eta}_a = q^c a \ell^b \nabla b k_c, \]
\[ \omega_a = q^c a k^b \nabla c \ell_b = -q^c a \ell^b \nabla c k_b, \]
\[ a_a = q^c a \ell^b \nabla b \ell_c \quad , \quad \bar{a}_a = q^c a k^b \nabla b k_c, \]
\[ \kappa = \ell^a k^b \nabla a \ell_b = -\ell^a \ell^b \nabla a k_b, \quad \bar{\kappa} = k^a \ell^b \nabla a k_b = -k^a k^b \nabla a \ell_b, \]

where $\Theta_{ab}$ and $\Xi_{ab}$ are extrinsic curvatures of $S$ while $\omega_a$, $\eta_a$ and $\bar{\eta}_a$ are twists. In addition, $a_a$ and $\bar{a}_a$ are tangent accelerations of $\ell^a$ and $k^a$ to $S$, respectively. Moreover, $\kappa$ and $\bar{\kappa}$ are in-affinity parameters.$^4$

As a side remark, let us point out that this general double foliation described above is different from double null foliation as considered by various authors [22, 23]. Although, the double null foliation is useful for initial value problem as shown by Sachs [24], one could show that for variational principle it suffers from having a partial gauge fixing condition which is a disadvantage [25]. This is because requiring that the level sets of a coordinate $\phi$ being null, fixes one of the metric components:

\[ g^{\phi \phi} = g^{-1}(d\phi, d\phi) = 0. \]

The double null folation is a special case of the above set up if we set $B = C = 0$ in whole spacetime. If fact, we will see in the next section that the variation of such components is important for finding complete canonical structure, and more importantly the canonical momenta of such variations is the quasilocal energy density of system.

### 3.2 Variation of Hilbert-Einstein action

The main point in calculating the variations on a null surface is that the operators $\delta$ for variations and the covariant derivative $\nabla$ are defined in spacetime, whereas the condition for boundary, in the above setup $B = 0$, is valid only on the null boundary. Thus, we must first apply these operators and then impose the condition $B = 0$. For example consider variation of the vector $\ell_a$ given by Eq. [23]. By variation of this vector on the boundary, one finds $\delta \ell_a = \delta A \nabla_a \phi_0 + \delta B \nabla_a \phi_1$. On the other hand, if we set $B$ to zero first and then vary the

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$^4$The quantity $\kappa$ is called surface gravity in the case of a black hole horizon null surface.
equation, we find that $\delta \ell_a = \delta A \nabla_a \phi_0$. This variation is not valid, because variation of one degrees of freedom in metric has been killed in this relation, i.e. $\delta B \propto \ell^a \ell^b \delta g_{ab} = 0$. This variation is responsible for taking out the boundary from being null. To avoid losing any degree of freedom, in the following we will not set $B = 0$ until the end of calculations.

We consider a hypersurface to be a leaf of one of the foliations, let it be $\phi_0 = \text{const}$. Variation of HE action on such hypersurface in the double foliation determined by eqs. (23) to (25) leads to [1]:

$$
\delta S_{HE} = \frac{1}{8\pi} \delta \left( \int_{\mathcal{N}} d^{d-1}x \sqrt{q} \left[ D(\Theta + \kappa) - B(\Xi + \bar{\kappa}) \right] + \int_{S_1}^{S_2} d^{d-2}x \sqrt{q} \ln D \sqrt{H} \right) 
$$

$$
+ \frac{1}{16\pi} \int_{\mathcal{N}} d^{d-1}x \sqrt{q} \left[ D(\Theta^{ab} - q^{ab} (\Theta + \kappa)) - B(\Xi^{ab} - q^{ab}(\Xi + \bar{\kappa})) \right] \delta q_{ab} 
$$

$$
+ 2 \omega^a \delta \beta_1 a - 2\Xi \delta B + 2\Theta \delta D + \frac{1}{16\pi} \int_{S_1}^{S_2} d^{d-2}x \sqrt{q}(\ln D \sqrt{H}q^{ab})\delta q_{ab},
$$

(30)

where $H$ is the determinant of $H_{ij}$ defined above. The first line as a total variation suggests appropriate boundary term that must be subtracted in order to have a well-posed variational principle. The total variation term in the boundary integral, i.e.

$$
\sqrt{q}[D(\Theta + \kappa) - B(\Xi + \bar{\kappa})],
$$

(31)

has a nice geometric interpretation. It can be rewritten as $\sqrt{- g} v_a \mathcal{K}^a$, where $v_a = \nabla_a \phi_0 = \frac{1}{\sqrt{H}}(D\ell_a - Bk_a)$ is the normal to the hyperface, and $\mathcal{K}^a = -(k^a \nabla_b \ell_c + l^a \nabla_b k_c)$ is defined such that $\mathcal{K}^a$ is its trace on two last indices.

On a null boundary now we set $B = 0$; thus, we find the action with a well-posed variational principle to be

$$
S = \frac{1}{16\pi} \int_{\mathcal{M}} d^d x \sqrt{- g} R - \frac{1}{8\pi} \int_{\mathcal{N}} d^{d-1}x \sqrt{q}[D(\Theta + \kappa)] + \frac{1}{8\pi} \int_{S_1}^{S_2} d^{d-2}x \sqrt{q} \ln D \sqrt{H}.
$$

(32)

Let us remind that, as mentioned in previous subsection, there is a scaling boost gauge symmetry in null hypersurface description by two vectors $\ell_a$ and $k_a$. We can use this gauge freedom to set $D = 1$ in this gauge and by putting $B = 0$ we have $\sqrt{H} = A$. Therefore, the action becomes:

$$
S = \frac{1}{16\pi} \int_{\mathcal{M}} d^d x \sqrt{- g} R - \frac{1}{8\pi} \int_{\mathcal{N}} d^{d-1}x \sqrt{q}(\Theta + \kappa) + \frac{1}{8\pi} \int_{S_1}^{S_2} d^{d-2}x \sqrt{q} \ln A.
$$

(33)

\footnote{We note that the common Gibbons-Hawking-York term can be rewritten as: $\sqrt{|h|} K = \sqrt{- g} \nabla_a \phi (n^a K)$, with $K = -\nabla_a n^a$, in this sense, the above term is a generalization of GHY term for null surfaces.}
This is what found in [8,9]. In [7] the authors also have set the lapse $A = 1$, so the last term vanishes. Using the above action, the variation of the principal function on the null boundary becomes:

$$
\delta S = \frac{1}{16\pi} \int_{\cal B} d^{d-1}x \sqrt{q} \left( \left[ (\Theta^{ab} - q^{ab}(\Theta + \kappa)) \right] \delta q_{ab} + 2 \omega_a \delta \beta^a_1 - 2\Xi \delta B \right) 
$$

$$
+ \frac{1}{16\pi} \int_{S_1} d^{d-2}x \sqrt{q}(\ln A q^{ab}) \delta q_{ab}.
$$

(34)

Similar expression for canonical structure on the null boundary has been found in [9]. There, the authors considered only the variations that keeps the boundary to remain null, thus, the last term of the first line was missing in their analysis. Note that in this expression $\delta q_{ab}$, $\delta \beta_{1a}$ and $(-2\delta B)$ are all variations of the metric components tangential to the boundary as easily can be seen from the decomposition (25). This is an important point because, as in the non-null case, the variational principle tells us that for Dirichlet boundary condition we only need to fix the tangential metric components. Here also the number of degrees of freedom matches with the non-null case; for example, in four dimensions $\delta h_{ab}$ has six components while $(\delta q_{ab}, \delta \beta_{1a}, -2B)$ altogether have $3 + 2 + 1 = 6$ components.

### 3.3 The stress tensor

Having found the variation of action on the null boundary, we can use BY prescription to find the stress tensor on the boundary. In doing so we must differentiate the principal function in space-time region, illustrated in Fig 3, with respect to the metric component tangential to the null boundary segments $N$. Note that differentiating with respect to metric components in $\Sigma_1$ and $\Sigma_2$ gives the canonical momenta $P^{ab}$ similar to the previous section. According to the Eq.(25), the metric components tangential to $N$ are $(\delta q_{ab}, \delta \beta_{1a}, -2B)$. Note that, in contrast to previous timelike boundary presented in the last section, there is no induced metric on null surface; thus, we must differentiate with respect to each component of metric separately according to:

$$
\frac{2}{\sqrt{q}} \delta S|_N = \frac{2}{\sqrt{q}} (q^c_a q^d_b \delta S|_{q_{cd}} - q^c_a k^b_{(a}) \delta S|_{\beta^a_1 c} + k_a^b k_b^c \delta S|_{(-2B)}).
$$

(35)

On the other hand, the variational principle does not fix the boundary action completely; therefore, we can subtract any functional $S_0$ of fixed boundary data from the action (33).
Using the expression (34), hence, the stress tensor components read:

\[ \epsilon \equiv \ell^a \ell_b T_{ab} = -\frac{1}{\sqrt{q}} \delta S \frac{\delta S}{\delta B} = \frac{1}{8\pi} [\Xi - \Xi_0], \]  

(36)

\[ j^a \equiv q^{ac} \ell_b T_{cb} = \frac{1}{\sqrt{q}} \delta S \frac{\delta S}{\delta \beta_{1a}} = \frac{1}{8\pi} [\omega^a - \omega_0^a], \]  

(37)

\[ s^{ab} \equiv q^{ac} q^{bd} T_{cd} = \frac{2}{\sqrt{q}} \delta S \frac{\delta S}{\delta q_{ab}} = \frac{1}{8\pi} \left[ \Theta^{ab} - q^{ab} (\Theta + \kappa) - \frac{2}{\sqrt{q}} \delta S_0 \right], \]  

(38)

where \( \omega_0^a \) and \( \Xi_0 \) are defined as:

\[ \omega_0^a = \frac{1}{\sqrt{q}} \delta S_0 \frac{\delta S_0}{\delta \beta_{1a}} \quad \text{and} \quad \Xi_0 = -\frac{1}{\sqrt{q}} \delta S_0 \frac{\delta S_0}{\delta B}. \]

According to BY prescription, eqs. (36) to (38), may be used to define quasilocal energy density \( \epsilon \), tangential momentum density \( j^a \) and spatial stress \( s^{ab} \), respectively.

The subtraction term \( S_0 \) is related to the zero point of energy and must be well chosen so that the stress tensor leads to finite energy for different systems and conventionally zero energy for Minkowski space time. The expression (36) is similar to the Eq. (18) for timelike boundary. \( \Xi \) is defined by \( \Xi = -q^{ab} \nabla_a k_b \) while \( k \) reads as \( k = -q^{ab} \nabla_a n_b \). The similarity is in the sense that for null boundary the only vector pointing out of the boundary is \( k_n \). For non-null boundaries, studied in the previous section, the normal to boundary was also pointing out of it; however, note that for null hypersurfaces the normal is within the boundary.

4 Quasi-local quantities

In this section we examine the proposed stress tensor by evaluating conserved charges of various space-times and then we compare it with the known results. Here, as to the timelike boundary, we propose the Quasi-local quantity to be:

\[ Q_\xi = \int_S d^{d-2}x \sqrt{q} T_{ab} \ell^a \xi^b. \]  

(39)

where \( \ell^a \) has the role of time flow on the boundary while \( \xi^a \) is a Killing vector which generates an isometry of the boundary. If \( \ell^a \) is the generator of time translation symmetry, then the total energy becomes:

\[ E = \int_S d^{d-2}x \sqrt{q} \epsilon, \]  

(40)
where the quasi-local energy density $\epsilon$ is defined by (36). When there is a rotational Killing vector $\zeta^{a}$, then its corresponding angular momentum is:

$$J = \int_{S} d^{d-2}x \sqrt{q} j_{a} \zeta^{a}. \quad (41)$$

In the following we will calculate the above quantities for some well-known gravitational solutions.

### 4.1 Minkowski space

The simplest example for investigation is Minkowski spacetime. In retarded-spherical coordinates the metric is as follows:

$$ds^{2} = -du^{2} - 2dudr + r^{2}d\Omega^{2} \quad (42)$$

where $d\Omega^{2} = \gamma_{AB} d\sigma^{A} d\sigma^{B}$ is the metric on a unit sphere. It is evident that $u = \text{const.}$ is a null surface. Comparing the above line element with Eq.(25) and using the foliation relations (23) and (24) yields $\ell_{a} = \nabla_{a} u$, $k_{a} = \nabla_{a} r + \frac{1}{2} \nabla_{a} u$ and $q_{AB} = r^{2} \gamma_{AB}$. From the definitions (29) one easily finds:

$$\Theta_{AB} = -2\Xi_{AB} = r\gamma_{AB}, \quad \omega_{A} = 0. \quad (43)$$

As a result $\Xi = -\frac{1}{r}$, and the integral $\int d\theta d\phi \sqrt{q} \Xi = -\int d\theta d\phi \ r \sin^{2} \theta$ will be infinite as $r \to \infty$. Subtracting $\Xi_{0}$ leads to zero energy for flat space as expected.

### 4.2 Schwarzschild black hole

Our next simple example is Schwarzschild black hole. In retarded Eddington-Finkelstein coordinates the metric is given by:

$$ds^{2} = -f(r)du^{2} - 2dudr + r^{2}d\Omega^{2} \quad (44)$$

In this case we have $\ell_{a} = \nabla_{a} u$ and $k_{a} = \frac{1}{2f(r)} \nabla_{a} u + \nabla_{a} r$. Here, by calculating the quasilocal energy density, we get:

$$\epsilon = \frac{1}{8\pi} [\Xi - \Xi_{0}] = \frac{1}{8\pi} [-\frac{f(r)}{r} + \frac{1}{r}]. \quad (45)$$

By replacing $f(r) = 1 - \frac{2M}{r}$, we obtain $\epsilon = \frac{M}{4\pi r^{2}}$. Thus the total energy becomes:

$$E = \int_{0}^{\pi} \int_{0}^{2\pi} d\theta d\phi \ r^{2} \sin \theta \ \epsilon = 4\pi r^{2} \left(\frac{M}{4\pi r^{2}}\right) = M. \quad (46)$$
It is interesting that the quasilocal energy, as calculated for the null observers in $\mathcal{N}$, is independent of the distance $r$ for which the energy is calculated. At first sight, it may seems strange because the usual BY expression for energy is just equal to the ADM mass at $r \to \infty$. However, notice that as mentioned previously, in general the function $E(r)$ is observer dependent. The in-dependency of energy to the distance also has been observed for boosted foliation of Schwarzschild black hole in [11].

4.3 Slow-rotating black hole

Here we examine conserved quantities for slow rotating Kerr black holes. Again we write the metric in the retarded Eddington-Finkelstein coordinates:

$$ds^2 = -f(r)du^2 - 2dudr + r^2d\Omega^2 + \frac{2J}{r} \sin^2 \theta \ d\theta \ d\phi + \frac{2J}{rf(r)} \sin^2 \theta \ dr \ d\phi$$

Comparing with metric (25) we arrive at:

$$A = 1, \quad B = 0, \quad C = \frac{1}{2f}, \quad D = 1, \quad \beta_1 = \frac{2J}{r} \sin^2 \theta, \quad \beta_2 = \frac{2J}{rf} \sin^2 \theta. \quad (48)$$

Up to the first power in $J$, one finds the same expression for energy as the Schwarzschild case. The angular momentum quantity is related to $\omega_a$, using its definition we find:

$$\xi^a \omega_a = \frac{3J \sin^2 \theta}{r^2}. \quad (49)$$

where $\xi = \partial_\phi$ is rotational Killing symmetry. Therefore, the total angular momentum becomes:

$$Q_\xi = \frac{1}{8\pi} \int_0^\pi \int_0^{2\pi} d\theta d\phi \ r^2 \sin \theta \frac{3J \sin^2 \theta}{r^2} = J \quad (50)$$

4.4 Asymptotic flat spacetime and the Bondi mass

Here, we want to study the quasi-local quantities for gravitational theories in which the metric has an asymptotic flat space behaviour. In order to do that, we suppose the metric in the Bondi coordinates. In this gauge, the most general four-dimensional metric takes the form:

$$ds^2 = -UVdu^2 - 2Vdudr + q_{AB}(d\sigma^A + U^A du)(d\sigma^B + U^B du) \quad (51)$$
where $\partial_r \det(\frac{2AB}{r^2}) = 0$. By comparison the above equation with Eq. (25), we find:

$$A = V, \quad B = 0, \quad C = \frac{U}{2}, \quad D = 1, \quad \beta_0^A = U^A, \quad \beta_1^A = 0.$$  \hspace{1cm} (52)

The expressions for $\Theta_{AB}$ and $\Xi_{AB}$ in the metric (25) have been calculated explicitly in [1] and are:

$$\Theta_{AB} = -\frac{1}{2\sqrt{H}} (B \partial_0 q_{AB} - A \partial_1 q_{AB} - 2B D(A \beta_0^B) + 2A D(A \beta_1^B))$$  \hspace{1cm} (53)

$$\Xi_{AB} = -\frac{1}{2\sqrt{H}} (-D \partial_0 q_{AB} + C \partial_1 q_{AB} + 2D D(A \beta_0^B) - 2C D(A \beta_1^B))$$  \hspace{1cm} (54)

where $D$ is covariant derivative on two sphere $S$, compatible with the metric $q_{AB}$. Using standard asymptotic expansions [26]:

$$U = 1 - \frac{2m_B}{r} + O\left(\frac{1}{r^2}\right), \quad V = 1 + O\left(\frac{1}{r^2}\right)$$

$$\beta_0^A = \frac{W^A}{r^2} + O\left(\frac{1}{r^3}\right), \quad q_{AB} = r^2 \gamma_{AB} + O(r)$$

we can easily find the following leading terms for $\Xi$:

$$\Xi = -\frac{1}{r} + \frac{2m_B(u, \sigma^A)}{r^2} + \frac{D_A W^A}{r^2} + O\left(\frac{1}{r^3}\right)$$

By embedding in flat space we have the reference term $\Xi_0 = -\frac{1}{r}$. The term $D_A W^A$ is a total derivative on compact two sphere $S$ which vanishes by integration. Thus, finally the total energy becomes:

$$E = \frac{1}{8\pi} \int_S d^2x \sqrt{q} \epsilon = \frac{1}{4\pi} \int_S d\Omega \, m_B(u, \sigma^A)$$  \hspace{1cm} (55)

which is the expression known as Bondi mass.

4.5 A possible counterterm for asymptotic flat spacetime

We have seen that for asymptotic AdS space there is counterterm method for regularizing quasi-local quantities that has some advantages. The first benefit is that the dependency on the reference spacetime and mathematical difficulty of embedding is avoided. Furthermore, the counterterms have direct interpretation in the dual field theory and has important role for building the dictionary of AdS/CFT. Then the natural question is that: why there is no

\footnote{These relations are counterpart to the well-known relation for extrinsic curvature in $3+1$ decomposition: $K_{ij} = -\frac{1}{2\kappa}(\partial_i h_{ij} - 2D_i N_{ij})$}
similar counterterms for flat space? For flat space there is not a length scale counterpart to the AdS length scale; thus, we cannot have terms like: $\ell^{2n-1}R^n$. In addition, the counterterms should not spoil the variational principle so that the extrinsic curvature terms are forbidden. Therefore, on the time-like boundary we don’t have a viable candidate as counterterm in asymptotic flat spacetime. However, as we have seen in this paper, null boundaries are special. Consider the following term on the null boundary:

$$\alpha \int_N d^{d-1}x \sqrt{q} B \Theta$$

with arbitrary numerical coefficient $\alpha$. The first point is that adding such term is compatible with Dirichlet boundary conditions. Also, we must note that although the above term vanishes on the null boundary, its variation is non-zero. Therefore, it has a contribution to the stress tensor and energy though it is zero for on-shell action. Fortunately, we can set the coefficient $\alpha = -\frac{1}{2}$ so that its contribution to energy make the total finite. By adding this term to the boundary action, the quasilocal energy density becomes:

$$\epsilon = \frac{1}{8\pi} \left[ \Xi + \frac{1}{2} \Theta \right]$$

For asymptotic flat spacetime studied previously, using (53), $\Theta$ is easily computed and its leading terms is as:

$$\Theta = \frac{2}{r} + \mathcal{O}\left(\frac{1}{r^3}\right).$$

Therefore the expression (57) leads to correct total energy without needing to embedding and reference spacetime.

5 Conclusion and outlook

In this work we have extended the Brown-York prescription to the case of null boundaries. We have seen that the variation of one metric components is responsible for taking away the boundary from being null. In fact, it was shown that an exact derivation with this variation gives the expression for quasilocal energy density. In addition, the expressions for total energy and angular momentum were examined for some known spacetimes. Moreover, a special property of the calculated energy for null observers was found to be the in-dependency to radial distance. Furthermore, it was shown that in the case of null boundaries, there is a possible counterterm which is consistent with variational principle and can be added to
boundary action so that the total energy becomes finite, without necessity of embedding in reference spacetime.

Regarding the similarity with the standard AdS/CFT dictionary, one may interpret the introduced stress tensor as expectation value of stress tensor in dual field theory to flat space. Because, according to Penrose diagrams for asymptotic flat spacetimes, the null hypersurfaces $\mathcal{I}^+$ and $\mathcal{I}^-$ are regarded as the boundaries of spacetime; thus, one application of the stress tensor found in this paper may be flat holography.

Another application of the formalism presented in this article and [1] is to revisit the gravity in light-front. The usual investigation of gravity in light-front uses the double-null foliation of spacetime [27][28]. However, as noted above, the double-null foliation leads to partial gauge fixing of the metric. It is in contrast to field theory formulation in the light-front coordinate which no gauge fixing is required. As we have seen a general double foliation can preserve all degrees of freedom. Thus, it is motivating to revisit gravity in light-front.

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