Dynamical generation of gauge and Higgs bosons in N=2 supersymmetric non-linear sigma-models

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Abstract

A four-dimensional N=2 supersymmetric non-linear sigma-model with the Eguchi-Hanson (ALE) target space and a non-vanishing central charge is rewritten to a classically equivalent and formally renormalizable gauged ‘linear’ sigma-model over a non-compact coset space in N=2 harmonic superspace by making use of an N=2 vector gauge superfield as the Lagrange multiplier. It is then demonstrated that the N=2 vector gauge multiplet becomes dynamical after taking into account one-loop corrections due to quantized hypermultiplets. This implies the appearance of a composite gauge boson, a composite chiral spinor doublet and a composite complex Higgs particle, all defined as the physical states associated with the propagating N=2 vector gauge superfield. The composite N=2 vector multiplet is further identified with the zero modes of a superstring ending on a D-6-brane. Some non-perturbative phenomena, such as the gauge symmetry enhancement for coincident D-6-branes and the Maldacena conjecture, turn out to be closely related to our NLSM via M-theory. Our results support a conjecture about the composite nature of superstrings ending on D-branes.

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1 Introduction

The idea that some of the ‘elementary’ particles, like a photon, Higgs or W bosons, may be composite is known in theoretical high-energy physics for many years, while it was proposed as a possible solution to many different problems in quantum field theory. For instance, the compositeness of photons was suggested long time ago, in order to resolve the ultra-violet problems of Quantum Electrodynamics related to the existence of Landau pole and the divergence of the effective coupling at high energies [1]. If the Higgs particles are to be interpreted as bound states, this would simply explain the experimental failure to observe them, since an acceptable scale for their compositeness is certainly much larger than any available energies. The compositeness of some of the vector bosons mediating weak or strong interactions was also proposed to accommodate the phenomenologically required gauge group $SU(3) \times SU(2) \times U(1)$ of the Standard Model (SM) in the maximally extended four-dimensional N=8 supergravity [2]. Gauging the internal symmetry of the N=8 supergravity merely produces $SO(8)$ as the gauge group which does not contain the SM gauge group as a subgroup [3]. However, since the scalar sector of the N=8 supergravity can be described by the non-compact non-linear sigma-model (NLSM) over the coset $E_7/SU(8)$ [4], assuming that its auxiliary gauge fields become dynamical in quantum theory would give rise to the gauge group $SU(8)$ which is big enough. Though the N=8 supergravity is no longer considered as the unifying quantum field theory because of its apparent non-renormalizability, its modern successor known under the name of M-theory [5] does, nevertheless, have the eleven-dimensional supergravity as the low-energy effective action, whose dimensional reduction down to four spacetime dimensions yields the N=8 supergravity. Moreover, the bound states arising in a system of the BPS-type extended classical solutions to the eleven-dimensional supergravity (known as branes) are known to play an important role in M-theory [5].

The quantum field-theoretical mechanisms of dynamical generation of composite particles are known in two or three spacetime dimensions [6, 7]. Unfortunately, little is known about the formation of bound states in quantized four-dimensional field theories (see, however, ref. [9]) or in M-theory (see, however, ref. [10]).

In the present paper I investigate the possible mechanism for a generation of composite N=2 vector multiplets in a four-dimensional N=2 supersymmetric NLSM with an ALE target space. The basic idea is to reformulate this classical NLSM to the renormalizable form given by the gauged ‘linear’ NLSM over a non-compact coset

\[^3\text{See also ref. [8] for an introduction.}\]
space, and then to take into account the one-loop quantum corrections due to the quantized hypermultiplets comprising fields of both positive and negative norm (cf. ref. [9]). The $N=2$ extended supersymmetry with a non-vanishing central charge plays the important role in our approach: on the one side, it implies an ALE hyper-Kähler geometry of the NLSM target space and the particular form of the associated scalar potential, whereas, on the other side, it automatically gives rise to many divergence cancellations which, otherwise, could destroy a consistency of the proposed theory. The technical power of $N=2$ harmonic superspace (HSS) allows us to take advantage of having the manifest $N=2$ extended supersymmetry in quantum perturbation theory, which significantly simplifies our calculations and makes them very transparent.

The paper is organized as follows: in sect. 2 we discuss some known general facts about superspace and complex geometry of supersymmetric 4d NLSM, which are going to be relevant in the next sections. In particular, we emphasize the relation between $N=2$ HSS and the hyper-Kähler geometry of $N=2$ NLSM, and the role of isometries in making this connection explicit. In sect. 3 we define several classically equivalent HSS forms of the $N=2$ NLSM with the Eguchi-Hanson metric and a non-vanishing central charge, and show their relation to a larger class of $N=2$ NLSM with multicentre (Gibbons-Hawking) metrics. In sect. 4 we quantize a coset (gauged) representation of the Eguchi-Hanson $N=2$ NLSM in HSS, and demonstrate a dynamical generation of an $N=2$ vector multiplet. A relation to M-theory and brane technology for $N=2$ supersymmetric quantum gauge field theories in 4d is discussed in sect. 5. Our conclusions and possible generalizations are outlined in sect. 6. A brief introduction into 4d, $N=2$ HSS is given in Appendix A. In Appendix B we collect some known facts about $N=2$ restricted chiral superfields and Fayet-Iliopoulos (FI) terms. A brief account of our main results is available in the electronically published Proceedings of the STRINGS’98 Conference [11].

2 Complex geometry of 4d NLSM and superspace

Let $x^\mu$, $\mu = 0, 1, 2, 3$, be the coordinates of a flat four-dimensional (4d) spacetime of signature $(+, -, -, -)$. By definition of the bosonic 4d NLSM, its real scalar fields $\phi^a(x^\mu)$, $a = 1, 2, \ldots, n$, themselves are to be considered as the coordinates of some (internal) NLSM target space $M$ of real dimension $n$. The standard action of the 4d bosonic NLSM reads

$$S_{\text{bosonic}}[\phi] = \frac{1}{2\kappa^2} \int d^4x \, g_{ab}(\phi) \partial_\mu \phi^a \partial^\mu \phi^b , \quad (2.1)$$
where the set of functions $g_{ab}(\phi)$ is called the NLSM metric. If the fields $\phi^a$ are chosen to be dimensionless, then the coupling constant $\kappa$ has to be of dimension of length, in order to make the action dimensionless (in units $\hbar = c = 1$). An important consequence of this fact is the non-renormalizability (by index) of the quantized field theory (2.1) with any non-flat NLSM metric (i.e. with a curved target NLSM space $\mathcal{M}$). It follows, for example, from the covariant background field method in application to the theory (2.1) that its one-loop on-shell counterterm includes some terms of the fourth-order in spacetime derivatives, while the field-dependent coefficient functions in front of these terms are essentially given by the NLSM curvature tensor squared (see e.g., ref. [12] for a review). Renormalizability thus requires the NLSM curvature to vanish, which just amounts to a flat NLSM metric. This clearly makes quantum 4d NLSM to be very different from their 2d renormalizable (in some generalised sense) counterparts whose coupling constant is dimensionless and whose one-loop one-shell counterterm in 2d is governed by the Ricci tensor not the curvature [12]. A supersymmetrization of 4d NLSM (see below) does not remove the non-renormalizability in 4d [13].

The action (2.1) is invariant under arbitrary (differentiable and invertible) field reparametrizations provided that $g_{ab}(\phi)$ transforms as a second-rank tensor. However, no conserved Noether current is associated with this symmetry, since the induced transformation of $g_{ab}(\phi)$ as the function of $\phi$ is generically different from the tensor transformation law. The only exception arises when a field diffeomorphism $\delta \phi = \xi$ is an isometry of $\mathcal{M}$, i.e. when the Lie derivative of metric vanishes, $\mathcal{L}_\xi g_{ab} = 0$.

The N=1 supersymmetrization of the theory (2.1) is straightforward in superspace [14]. Since the NLSM scalar fields are to belong to scalar N=1 supermultiplets which are described by chiral N=1 superfields $\Phi$, $\bar{D}_a \Phi = 0$, the NLSM geometry has to be complex. Moreover, on dimensional reasons, the most general NLSM action of the second-order in spacetime derivatives (in components) has to be governed in superspace by a real function of chiral superfields $\Phi$ and their conjugates (anti-chiral superfields $\bar{\Phi}$), i.e.

$$S[\Phi, \bar{\Phi}] = \frac{1}{\kappa^2} \int d^4x d^2\theta d^2\bar{\theta} \, K(\Phi, \bar{\Phi}) , \quad (2.2)$$

where the superspace measure is now of dimension two (in units of length), the coupling constant $\kappa$ is still of dimension one, and all chiral superfields are dimensionless.

After rewriting eq. (2.2) in components one finds a purely bosonic contribution of the form

$$S_{\text{bosonic}}[\phi, \bar{\phi}] = \frac{1}{\kappa^2} \int d^4 x \, g_{ij}(\phi, \bar{\phi}) \partial_\mu \phi^i \partial^\mu \bar{\phi}^j , \quad (2.3)$$
where $\phi^i$ are the leading complex scalar components of $\Phi^i$, $i = 1, 2, \ldots, m$. Hence, eq. (2.2) is just the N=1 supersymmetric extension of eq. (2.3) whose NLSM target space is a Kähler manifold of complex dimension $m$, with the function $K$ being the Kähler potential for the Kähler NLSM metric.

$$g_{i\bar{j}} = \frac{\partial^2 K}{\partial \phi^i \partial \bar{\phi}^j}.$$  \hspace{1cm} (2.4)

Eq. (2.2) thus provides us with the manifestly supersymmetric and universal description of all 4d, N=1 supersymmetric NLSMs in terms of a single non-holomorphic real potential $K(\Phi, \overline{\Phi})$. As is obvious from eq. (2.2), the Kähler potential is defined modulo Kähler gauge transformations,

$$K(\Phi, \overline{\Phi}) \rightarrow K(\Phi, \overline{\Phi}) + f(\Phi) + f(\overline{\Phi}),$$  \hspace{1cm} (2.5)

with the holomorphic gauge parameter $f(\Phi)$.

The extended supersymmetry in 4d NLSM is limited to N=2 since there exist no scalar supermultiplets beyond N=2 (by definition, all physical bosonic components of NLSM are scalars). The 4d, N=2 extended supersymmetry can be equivalently described as 6d, N=1 supersymmetry which has the same number (8) of supercharges.$^5$ An N=2 supersymmetric extension of a 4d NLSM (2.1) only exists if its metric is hyper-Kähler. This fact was initially established in components, by analyzing the restrictions imposed by extended supersymmetry on a Kähler potential$^6$. The 4d, N=2 scalar multiplet is called hypermultiplet. Though there exist many different off-shell versions of a hypermultiplet in the conventional N=2 superspace (see e.g., refs. for some earlier references, or a recent paper), the universal, manifestly N=2 supersymmetric formulation of the hypermultiplet is only possible in the N=2 harmonic superspace (HSS) introduced in ref. The HSS is an extension of the conventional N=2 superspace by a two-sphere $S^2 = SU(2)/U(1)$. Among the most important properties of HSS are (i) the existence of invariant subspace called analytic, and (ii) the description of $SU(2)$ tensor representations in terms of objects having definite $U(1)$ charge. Analytic N=2 superfields can be considered as the N=2 counterparts to N=1 chiral superfields, whereas the harmonic calculus can be efficiently performed by employing isospinor harmonics $u^+_{\pm}$ instead of the usual polar coordinates $(\varphi, \vartheta)$ on the sphere. Hypermultiplets in HSS are described by two basic types of unconstrained analytic superfields (usually denoted as $q$ and $\omega$), which are dual to

$^4$We ignore here possible global complications related to Kähler geometry.

$^5$See e.g., ref. for a discussion of supersymmetric NLSM in 6d.

$^6$A brief introduction into HSS is given in Appendix A.
each other (see eqs. (2.6) and (2.7) below) and can be chosen to be dimensionless. The physical components of a $q$-superfield comprise a complex scalar $SU(2)$-doublet and a Dirac spinor singlet. The physical components of an $\omega$–superfield comprise a real scalar singlet, a scalar $SU(2)$-triplet and a chiral spinor $SU(2)$ doublet. A $q$-superfield is complex and has $U(1)$ charge one, whereas the real $\omega$-superfield has vanishing $U(1)$ charge. The classical duality transformation between them reads

$$q^+_a = u^+_a \omega + u^+_a f^{++},$$  \hspace{1cm} (2.6)

where yet another $SU(2)_{PG}$ doublet $q^+_a = (q^+_a, \tilde{q}^+_a)$, $a = 1, 2$, and the auxiliary analytic complex superfield $f^{++}$, which plays the role of a Lagrange multiplier, have been introduced. Inverting eq. (2.6) yields

$$\omega = u^+_a q^a \quad \text{and} \quad f^{++} = -u^+_a q^{a+}.$$  \hspace{1cm} (2.7)

The hyper-Kähler manifold is a 4n-dimensional Riemannian manifold $\mathcal{M}$ whose holonomy group is a subgroup of $Sp(n)$. Directly imposing the hyper-Kähler condition on a Kähler potential results in the non-linear (Monge-Amperé) partial differential equation \cite{21}. The HSS offers a formal solution to this equation in the form of the most general $\mathcal{N}=2$ supersymmetric NLSM having the action

$$S[q, \tilde{q}] = \frac{1}{\kappa^2} \int d\zeta^{(-4)} du \mathcal{L}^{(+4)}(q^+, \tilde{q}^+, D^{++}q^+, D^{++}\tilde{q}^+; u^+\tilde{u})$$  \hspace{1cm} (2.8)

over the analytic subspace of HSS whose measure $d\zeta^{(-4)} du$ is of $U(1)$ charge $(-4)$. Here $D^{++}$ is the $\mathcal{N}=2$ covariant harmonic derivative of dimension zero and of $U(1)$ charge $(+2)$. The analytic function $\mathcal{L}^{(+4)}$ has to be of $U(1)$ charge $(+4)$ in order to compensate the opposite $U(1)$ charge of the measure, while it has to be of the first order in the derivatives $D^{++}q$ in order to guarantee the presence of the standard NLSM kinetic term (2.1) in the corresponding component NLSM action, i.e. without higher spacetime derivatives.

By $\mathcal{N}=2$ supersymmetry eq. (2.8) thus uniquely determines, in principle, the component hyper-Kähler NLSM metric in terms of a single analytic function $\mathcal{L}^{(+4)}$. Their explicit relation is, however, highly non-trivial (and, in fact, not a 1-1 correspondence) since eq. (2.8) contains infinitely many auxiliary field components whose elimination requires solving infinitely many linear differential equations on the sphere altogether. This cumbersome procedure in HSS was only performed in a few special cases of $\mathcal{N}=2$ NLSM with four-dimensional hyper-Kähler target spaces \cite{22, 23, 24, 25}. Yet another caveat related to the infinite number of auxiliary fields is a considerable redundancy of the HSS description of an $\mathcal{N}=2$ NLSM, which exhibits itself in the existence of many
apparently different analytic HSS lagrangians leading to the same hyper-Kähler metric in components (see sect. 3 for some explicit examples). To make the things more tractable, let’s consider only those analytic lagrangians $\mathcal{L}^{(+4)}$ that have a well-defined kinetic term, i.e. are of the form

$$\mathcal{L}^{(+4)} = -\star q^+ D^{++} q^+ + \mathcal{K}^{(+4)}(\star q^+, q^+; u^\pm) ,$$

where the analytic potential $\mathcal{K}^{(+4)}$ is known as a hyper-Kähler potential [22, 23].

Eq. (2.9) naturally arises as the exact low-energy effective action (LEEA) for hypermultiplets in quantized N=2 supersymmetric gauge field theories [26, 27]. An explicit dependence of the function $\mathcal{K}$ upon harmonics signals the breaking of the internal $SU(2)$ symmetry rotating two spinor charges of N=2 supersymmetry. Since the duality relation (2.6) between $q$ and $\omega$ hypermultiplets involves harmonics, it may be useful to re-introduce a dependence upon both superfields into eq. (2.9) if it results in the absence of any explicit dependence upon harmonics. This is particularly relevant in the context of the hypermultiplet LEEA since the latter is normally dependent upon a dynamically generated real scale $\Lambda$ which is interpreted as the expectation value of some real Higgs hypermultiplet $\omega$,

$$\Lambda = \langle \omega \rangle = \text{const} > 0 .$$

As is clear from eqs. (2.8) and (2.9), a general hyper-Kähler metric does not have any isometries, and this is precisely the fact that makes its explicit construction via HSS to be so difficult. It is to be compared with a derivation of hyper-Kähler metrics from N=2 matter self-interaction in the conventional superspace [28, 29, 12], which is usually accompanied by duality transformations and leads to a presence of isometries in the hyper-Kähler metrics to be derived by using a finite number of auxiliary field components (see refs. [30, 19] for a HSS reformulation of off-shell versions of hypermultiplet with a finite number of auxiliary fields). It is therefore the absence of isometries in a hyper-Kähler metric that is apparently responsible for the failure to formulate a manifestly N=2 supersymmetric NLSM with the same metric in the conventional N=2 superspace, i.e. with a finite number of N=2 auxiliary fields. Though HSS is capable of providing such an N=2 NLSM formulation in principle, the elimination of HSS auxiliary fields to recover a component metric of the N=2 NLSM without isometries represents a fundamental technical problem.

We are mostly going to restrict ourselves to four-dimensional (euclidean) hyper-Kähler NLSM target spaces having at least one isometry. Since the Ricci tensor of
A hyper-Kähler metric vanishes, any hyper-Kähler space is an Einstein space. A study of four-dimensional (euclidean) Einstein spaces having an isometry (i.e. with a metric to be independent upon one coordinate ($\rho$) in some coordinates $\rho, y_i, i = 1, 2, 3$) allows one to distinguish the hyper-Kähler spaces among the Einstein spaces by the following form of metric \[31\]:

$$ds^2 = H(d\tilde{y})^2 + H^{-1}(d\rho + \tilde{C} \cdot d\tilde{y})^2,$$

where we have used the notation $\tilde{y} = \{y_i\}$ and $\tilde{\nabla} = \{\partial_i\}$. The vector function $\tilde{C}(y_i)$ is supposed to satisfy the first-order equation

$$\tilde{\nabla} \times \tilde{C} = \tilde{\nabla} H,$$

whereas the function $H(y_i)$ has to be harmonic, i.e. satisfy the Laplace equation

$$\Delta H(y) = 0$$

outside the origin $\tilde{y} = 0$. It is worth mentioning that the dummy coordinate $\rho$ should be periodic (of period $2\pi k, k \in \mathbb{Z}$) in order to avoid conical singularities in the metric (2.11).

An explicit relation between a harmonic function $H$ and a hyper-Kähler potential $K$ of the corresponding N=2 NLSM in HSS was established in ref. [24]. One needs just a single $q$-hypermultiplet, having four real bosonic physical components, in order to parameterize a four-dimensional hyper-Kähler NLSM target space, with an isometry being represented by a rigid $U(1)$ symmetry of the hyper-Kähler potential with respect to the hypermultiplet rotations

$$q^+ \rightarrow e^{i\alpha} q^+ , \quad \tilde{q}^+ \rightarrow e^{-i\alpha} \tilde{q}^+ .$$

This implies that the hyper-Kähler potential of $U(1)$ charge $(+4)$ is an analytic function of the invariant product $(q \tilde{q})$ of $U(1)$ charge $(+2)$, i.e. $K = K(q \tilde{q}, u)$. Hence, one has [24]

$$K^{(+4)} = \sum_{l=0}^{\infty} \xi^{(-2l)}(q \tilde{q})^{l+2} / (l+2) ,$$

where the harmonic-dependent ‘coefficients’ $\xi^{(-2l)}(u)$ have been introduced,

$$\xi^{(-2l)} = \xi^{(i_1 \cdots i_{2l})} u_{i_1}^{-} \cdots u_{i_{2l}}^{-} , \quad l = 1, 2, \ldots .$$

\[7\] A four-dimensional hyper-Kähler manifold can be equivalently characterized either as a complex Kähler and Ricci-flat (i.e Calabi-Yau) manifold, or as a real one with self-dual curvature.
The latter are subject to the reality condition
\[
\bar{\xi}^{(-2l)} = (-1)^l \xi^{(-2l)}.
\] (2.17)

A general solution to eq. (2.13) reads
\[
H = \text{const.} \frac{2r}{2} + \frac{U(\vec{y})}{2},
\] (2.18)
with the function \(U(\vec{y})\) being non-singular in the origin. Hence, the latter can be decomposed in terms of the standard momentum eigenfunctions \(Y_l^m(\vartheta, \varphi)\) depending upon the spherical coordinates \((r, \vartheta, \varphi)\) with \(r = \sqrt{\vec{y}^2}\) as follows:
\[
U(\vec{y}) = \sum_{l=0}^{+\infty} \sum_{m=-l}^{m+l} c_{lm} r^l Y_l^m(\vartheta, \varphi).
\] (2.19)

The one-to-one correspondence between the integration constants \(c_{lm}\) of eq. (2.19) and the hyper-Kähler potential coefficients of eq. (2.15) is given by
\[
\xi^{i_1=1, \ldots, i_{-m}=1, i_{l-m+1}=2, \ldots, i_{2l}=2} = \frac{c_{lm} (2l + 1)}{C (l + 1)}
\] (2.20)
where \(C\) is a normalization constant whose exact value is irrelevant for our purposes.

A physical meaning of the harmonic potential \(H\) is transparent for the solitonic (regular) multicentre hyper-Kähler metrics [31], which are defined by
\[
H(\vec{y}) = 1 + \sum_{A=1}^{p} \frac{|k_A|}{2 |\vec{y} - \vec{y}_A|}.
\] (2.21)

The corresponding solution (2.11) to the euclidean Einstein equations describes \(p \geq 1\) gravitational (Gibbons-Hawking) instantons, each having a topological charge \(k_A \in \mathbb{Z}\) and ‘sitting’ at a space point \(\vec{y}_A\). Since all these solutions are actually independent upon the fourth coordinate \(\rho\), they can also be interpreted as three-dimensional (static) multi-monopole solutions with \(4p\) moduli \(\{k_A, \vec{y}_A\}\). They are also known as the Kaluza-Klein (KK) monopoles in the literature [32, 33]. Though the HSS moduli \(\xi^{(i_1 \ldots i_{2l})}\) in the alternative HSS description of the same multi-monopole configuration have no direct physical interpretation and they are not independent at all, the HSS description itself in terms of the analytic hyper-Kähler potential (2.15) is manifestly non-singular. The latter is useful in M-theory and brane technology, when describing the gauge symmetry enhancement for coincident D-6-branes in non-singular terms (sect. 5). Note that the BPS nature of a KK monopole means that its mass is equal to its charge (in dimensionless units).
A non-vanishing central charge of N=2 supersymmetry algebra can be easily incorporated into the HSS formalism by modifying the harmonic covariant derivative $D^{++}$. It simply amounts to introducing into NLSM a minimal coupling of hypermultiplets with an N=2 abelian background gauge superfield having the constant N=2 superfield strength equal to the central charge (Appendix A). As was shown in refs. [25, 27], a non-vanishing central charge leads to the appearance of a non-trivial scalar potential in components, whose form is entirely determined by a hyper-Kähler metric of the kinetic NLSM terms. This fact will play an important role in the mechanism of dynamical generation of N=2 vector multiplets in N=2 NLSM (sect. 4).

3 N=2 NLSM with ALE metric

Let's take the harmonic potential (2.21) describing the two-centered ($p = 2$) monopole solution with equal charges ($k_A = 1, \vec{y}_1 = \vec{0}, \vec{y}_2 = \vec{\xi}$), and modify it by a constant $\lambda > 0$ as

$$H(\vec{y}) = \lambda + \frac{1}{2} \left\{ \frac{1}{|\vec{y} - \vec{0}|} + \frac{1}{|\vec{y} - \vec{\xi}|} \right\}. \quad (3.1)$$

The real vector $\vec{\xi}$ can be equally represented as an SU(2) triplet $\xi^{ij} = i\vec{\xi} \cdot \vec{\tau}^{ij}$ satisfying the reality condition

$$(\xi^{ij})^\dagger \equiv \xi^{ij}_l = \varepsilon_{il} \varepsilon_{jm} \xi^{lm} = \xi_{ij}, \quad (3.2)$$

where $\vec{\tau}$ are the usual $2 \times 2$ Pauli matrices. The hyper-Kähler metric defined by eqs. (2.11) and (3.1) is called the double Taub-NUT metric with a constant potential $\lambda$ at spacial infinity [34]. In accordance with the general results of sect. 2, the N=2 NLSM with the same target space metric is described by the HSS Lagrangian [24]

$$\mathcal{L}^{(+4)} = -\frac{1}{2} q^+_A D^{++} q^+_A - V^{++} \left( \varepsilon^{AB} q^+_A q^+_B + \xi^{++} \right) - \lambda \left( \sum_{A=1}^2 q^+_A q^+_A \right)^2, \quad (3.3)$$

where the N=2 vector gauge superfield $V^{++}$ has been introduced as a Lagrange multiplier, and $\xi^{++} = \xi^{ij} u^+_i u^+_j$. As is clear from eq. (3.3), this NLSM is invariant under the local $U(1)$ gauge symmetry

$$\delta q^+_1 = \Lambda q^+_2, \quad \delta q^+_2 = -\Lambda q^+_1, \quad \delta V^{++} = D^{++} \Lambda, \quad (3.4)$$

with the analytic HSS superfield parameter $\Lambda(\zeta, u)$. The rigid SU(2) automorphisms of N=2 supersymmetry algebra are obviously broken in eq. (3.3) to its abelian subgroup that leaves $\xi^{++}$ invariant. The extra (Pauli-Gürsey) symmetry SU(2)$_{PG}$ rotating $q$ and $\vec{q}$ is also broken in eq. (3.3) unless $\lambda \neq 0$. 

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Eq. (3.3) takes a particularly simple form in the limit $\xi \to 0$ where it reduces (after a superfield redefinition) to the well-known Taub-NUT NLSM action in HSS [22]. Similarly, in another limit $\lambda \to 0$, eq. (3.3) yields the N=2 NLSM with the Eguchi-Hanson (EH) metric [23]. In other words, the double Taub-NUT metric interpolates between the Taub-NUT and Eguchi-Hanson metrics [24], as was also explicitly demonstrated in ref. [24]. In both limits (Taub-NUT and Eguchi-Hanson), the metric has $U(2)$ isometry, whereas only a $U(1)$ isometry is left when both $\lambda \neq 0$ and $\xi \neq 0$.

Eq. (3.3) at $\lambda = 0$ takes the form of the $SU(2)_{PG}$-invariant minimal coupling between the two ‘matter’ FS-type hypermultiplets $q_+^A$ and an abelian N=2 vector gauge multiplet $V^{++}$ in the presence of a gauge-invariant (electric) Fayet-Iliopoulos term linear in $V^{++}$,

$$\mathcal{L}^{(+4)}(q_A, V) = -\frac{1}{2}q^+_A D^{++} q^+_A - V^{++} \left( \frac{1}{2} \varepsilon^{AB} q^+_A q^+_B + \xi^{++} \right).$$ (3.5)

This HSS Lagrangian can be rewritten after some algebra to the following (classically equivalent) form [23]:

$$\mathcal{L}^{(+4)}(q) = -\frac{1}{2}q^+ D^{++} q^+ + \frac{(\xi^{++})^2}{2(q^+ u_-)^2},$$ (3.6)

which determines the hyper-Kähler potential of the EH metric according to eq. (2.9). After the duality transformation (2.6), one arrives at the dual action [23]

$$\mathcal{L}^{(+4)}(\omega) = -\frac{1}{2}(D^{++} \omega)^2 + \frac{(\xi^{++})^2}{2\omega^2},$$ (3.7)

in terms of the single real $\omega$ superfield. Therefore, the N=2 supersymmetric NLSM Lagrangian in HSS for a given (in this case, Eguchi-Hanson) hyper-Kähler metric is not unique.

Let’s now define yet another gauge-invariant HSS action in terms of another two FS-type hypermultiplet superfields and an N=2 vector gauge $V^{++}$ superfield as

$$S_{EH}[q_1, q_2, V] = \int d\zeta^{(-4)} du \left[ -\frac{1}{\bar{q}_1^+} D^{++} q_1^+ \frac{1}{\bar{q}_2^+} D^{++} q_2^+ + V^{++} \xi^{++} \right],$$ (3.8)

where we have returned to canonical dimensions for all the superfields involved, and introduced the gauge-covariant harmonic derivative [20]

$$D^{++} = D^{++} + iV^{++},$$ (3.9)

thus extending the rigid $U(1)$ symmetry (2.14) of a free hypermultiplet action to the local analytic one. It is not difficult to check that the classical theory (3.8) is

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8 In units of mass one has $[q] = 1$, $[V] = 0$ and $[\xi] = +2.$
equivalent to that of eq. (3.5), e.g. by considering a gauge $q_2^+ = 0$ in eq. (3.8) and a gauge $q_1^+ = iq_1^+$ in eq. (3.5), up to rescaling by a factor of 2. However, in the form (3.8), the $SU(2)_{PC}$ invariance is no longer manifest. Moreover, the action (3.8) has the wrong sign in front of the kinetic term for the $q_2$ hypermultiplet that indicates its non-physical (ghost) nature. This also implies its anti-causal propagation and the wrong (negative) sign of the residue in the propagator of $q_2$ superfield (see the next sect. 4). It does not, however, make our theory (3.8) non-unitary since the $q_2^+$ hypermultiplet is a gauge degree of freedom, while the classical action (3.8) itself is dual to any of the manifestly unitary NLSM actions with the ALE (Eguchi-Hanson) target space in eqs. (3.6) and (3.7). The action (3.8) has the form of a non-compact gauged NLSM over the coset $SU(1,1)/U(1)$ parameterized by the FS-type hypermultiplets $q_A$ in the fundamental representation of $SU(1,1)$ whose $U(1)$ subgroup is gauged in HSS.

We are going to exploit the freedom of choosing a classical HSS Lagrangian with the on-shell Eguchi-Hanson metric and to take eq. (3.8) as our starting point for quantization. It is worth mentioning here that the minimal gauge interaction of hypermultiplets with N=2 vector multiplets is the only renormalizable type of N=2 supersymmetric field-theoretical interaction in four-dimensional spacetime [35]. The classical correspondence with the formally unitary (but non-renormalizable) NLSM actions (3.6) and (3.7) ensures unitarity in our theory, whereas the non-anomalous gauge Ward identities should take care of the gauge invariance after quantization. Our approach may be compared to the standard bosonic string theory where the 2d Nambu-Goto classical string action [36] is substituted by the 2d Polyakov string action [37]. The non-polynomial Nambu-Goto action has a clear geometrical interpretation as the area of a string world-sheet but it is formally non-renormalizable. One defines a quantized bosonic string theory (in the critical dimension) after replacing the Nambu-Goto action by the classically equivalent Polyakov action which has the 2d auxiliary metric as a Lagrange multiplier. In our case, however, we will not integrate over our Lagrange multiplier given by an N=2 vector gauge superfield in 4d. The ghost hypermultiplet will be integrated out in quantum theory (sect. 4).

The quantized theory (3.8) is, however, of little interest unless it is supplemented by an N=2 central charge $\hat{Z}$ giving BPS masses to hypermultiplets. A hypermultiplet of mass $m$ can be described in HSS via the extension [20]

$$\hat{D}^{++} \equiv D^{++} + i(\theta^a + \theta^+_a)\hat{Z} + i(\bar{\theta}^+\bar{\theta}^{\beta})\hat{Z}$$

(3.10)

of the flat harmonic derivative $D^{++}$, with $\hat{Z}$ being an operator. It is not difficult to verify (see, e.g. ref. [27]) that the free hypermultiplet equation of motion $\hat{D}^{++}q^+ = 0$
implies
\[ (\Box + Z\bar{Z}) q^+ = 0 , \] (3.11)
which allows us to identify \( Z\bar{Z} q^+ = m^2 q^+ \). Because of eq. (3.9), the modification (3.10) in the case of a single charged hypermultiplet amounts to adding a minimal coupling to the particular N=2 abelian vector gauge superfield background having the constant N=2 gauge superfield strength equal to the central charge value [28, 27] (see also Appendix A). We are going to use the original interpretation (3.10) of the central charge, by associating it to \( D^{++} \) not \( V^{++} \), since we can then introduce different masses for the hypermultiplets \( q_1^+ \) and \( q_2^+ \) in eq. (3.8) via eqs. (3.9) and (3.10).

N=2 central charges in 4d HSS can be generated from a 6d HSS by the use of the standard (Scherk-Schwarz) mechanism of dimensional reduction [39], where the derivatives with respect to extra space coordinates play the role of the central charge operators (see e.g., refs. [23, 27] for details). As was noticed in ref. [23], the six-dimensional notation may sometimes simplify the equations with implicit central charges. For example, the bosonic kinetic terms of the NLSM (3.8) to be rewritten to 6d, after elimination of the HSS auxiliary fields in components, are given by [23, 25]

\[
S_{\text{bosonic}}[\phi_{1a}^i, \phi_{2a}^i, V_\mu] = \frac{1}{2} \int d^6x \left\{ \left( D^\mu \phi_{1a}^i \right) \left( D_\mu \phi_{1a1} \right) - \left( D^\mu \phi_{2a}^i \right) \left( D_\mu \phi_{1a2} \right) \right. \\
+ \frac{1}{2} D_{ij} \left( \phi_{1a}^i \phi_{1a}^j - \phi_{1a}^i \phi_{2a}^j + \xi^{ij} \right) \right\},
\] (3.12)
where \( \mu = 0, 1, 2, 3, 4, 5 \), \( D_\mu = \partial_\mu + i V_\mu \), and \( D_{ij} \) is the scalar triplet of the auxiliary field components of the N=2 vector superfield \( V^{++} \) in a WZ-gauge.

We are now in a position to formulate our model by the following HSS action:

\[
S_{\text{ALE}}[q_1, q_2, V] = \int d\zeta^{(-4)} du \left\{ - \tilde{q}_1^+ \left( \tilde{D}_1^{++} + i V^{++} \right) q_1^+ \\
+ \tilde{q}_2^+ \left( \tilde{D}_2^{++} + i V^{++} \right) q_2^+ + V^{++} \xi^{++} \right\},
\] (3.13)
where
\[
\tilde{Z}\bar{Z} q_1^+ = m_1^2 q_1^+ , \quad \tilde{Z}\bar{Z} q_2^+ = m_2^2 q_2^+ .
\] (3.14)

It should be remembered that the mass parameters \( m_1^2 \) and \( m_2^2 \) introduced in eq. (3.14) do not represent physical masses. As is clear from eq. (3.12), the classical on-shell physical significance has only their difference
\[
m_2^2 - m_1^2 \equiv m^2 ,
\] (3.15)
which can be identified with the classical mass of the single physical hypermultiplet in the NLSM under consideration, after taking into account the constraint imposed
by the Lagrange multiplier $D_{ij}$ (cf. ref. [9]). Moreover, because of the presence of a FI term linear in $D_{ij}$ in the action (3.12), the auxiliary triplet $D^{ij}$ of $V^{++}$ may develop a non-trivial vacuum expectation value in quantum theory after taking into account quantum corrections due to quantized hypermultiplets. This will influence the physical mass values to be defined with respect to a ‘true’ vacuum.

Accordingly, we first have to examine in the next sect. 4 whether the auxiliary field components $D_{ij}$ get a non-trivial vacuum expectation value. The latter has to be constant in order to maintain 4d Lorentz invariance. A constant solution $\langle D_{ij} \rangle \neq 0$ is clearly consistent with the abelian gauge invariance in components (Appendix B).

4 Quantum theory

To quantize both hypermultiplets of the theory (3.8) in a manifestly N=2 supersymmetric way, we need a quantum perturbation theory in terms of analytic HSS superfields in four spacetime dimensions. The HSS Feynman rules for massless N=2 supersymmetric gauge field theories were first obtained in ref. [20]. For our purposes in subsect. 4.2, we use a massive hypermultiplet propagator which was first derived in ref. [40]. In subsect. 4.3 we employ its generalization depending upon a background FI term too [27]. A HSS propagator of the unphysical hypermultiplet has some important differences in comparison to the physical hypermultiplet propagator, which are discussed in subsect. 4.1 along the lines of ref. [9]. A manifestly N=2 supersymmetric derivation of the low-energy gauge effective action (LEEA) to be obtained by an integration over a single matter hypermultiplet minimally coupled to an N=2 abelian vector gauge superfield in HSS is discussed at length in ref. [38] (see also refs. [11, 42] for some earlier component results, and some recent papers [43, 44] about a relation between HSS and components). These results are used in subsect. 4.3 to argue for a dynamical generation of an N=2 vector multiplet in the 4d quantum field theory (3.8). Further evidence coming from M-theory and brane technology is discussed in the next sect. 5, whereas some generalizations are outlined in sect. 6.

4.1 HSS propagators for hypermultiplets

The physical HSS propagator for a massive FS-type hypermultiplet reads [40, 27]

$$i \langle q^+(1) \tilde{q}^+(2) \rangle_{\text{phys}} = \frac{-1}{\Box + m^2 - i0} (D^+_1)^4 (D^+_2)^4 e^{\nu_2 - \nu_1} \delta^{12} (Z_1 - Z_2) \frac{1}{(u_1^+ u_2^+)^3}, \quad (4.1)$$
where \( v \) is the so-called ‘bridge’ defined by the general rule
\[
\mathcal{D} = e^{-v} D e^{v}
\] (4.2)

between the manifestly analytic HSS derivatives \( \mathcal{D} \) and the covariantly analytic ones \( D \). In the case of the central charge background (3.10) one easily finds [27]

\[
v = i(\theta^+ \bar{\theta}^-) \hat{Z} + i(\bar{\theta}^+ \theta^-) \hat{Z} .
\] (4.3)

The Green function \( G^{(1,1)}(1|2)_{\text{phys}} \equiv i \langle q^+(1) \bar{q}^+(2) \rangle_{\text{phys}} \) satisfies the equation
\[
\hat{D}_1^{++} G^{(1,1)}_{\text{phys}}(1|2) = \delta^{(3,1)}_{\text{phys}}(1|2) ,
\] (4.4)

where the analytic HSS delta-function \( \delta^{(3,1)}_{\text{phys}}(1|2) \) has been introduced [20].

A causal (unitary) propagation is ensured in quantum field theory by adding a small negative imaginary part to the mass squared, \( m^2 \rightarrow m^2 - i\epsilon \), in the propagator (4.1) [15]. The same prescription automatically takes care of (i) the convergence of the path integral defining the generating functional of quantum Green’s functions in Minkowski spacetime and (ii) free interchange of integrations. A propagator of the non-physical hypermultiplet entering the action (3.8) with the wrong sign (and, hence, formally leading to negative norms of the corresponding ‘states’) is also of the form (4.1) but with the negative residue and an anti-causal \( i\epsilon \)-prescription (Fig. 1),

\[
i \langle q^+(1) \bar{q}^+(2) \rangle_{\text{nonphys}} = \frac{1}{\Box + m^2 + i0} (D^+_1)^4 (D^+_2)^4 e^{v_2 - v_1} \delta^{12}(Z_1 - Z_2) \frac{1}{(u_1^+ u_2^+)^3} .
\] (4.5)

It can only occur as an internal line inside Feynman graphs, similarly to the HSS ghost hypermultiplet propagators in N=2 supersymmetric gauge field theories considered in ref. [38]. Our quantized non-physical hypermultiplet has, however, bosonic statistics.

Gauge couplings of physical hypermultiplets to N=2 vector superfields also differ by minus sign from those of non-physical hypermultiplets. Hence, a Feynman graph

Fig. 1. The Wick rotations for the physical (a) and non-physical (b) fields.
for the non-physical fields has extra minus signs for every internal line and every vertex, when being compared to the same graph for the physical fields. Though all these signs mutually cancel in loop diagrams with the same number of vertices and internal lines, the difference in $\bar{\epsilon}$ prescription remains. It forces the non-physical poles in the complex $p_0$-plane to be on the other side of the real axis $[9]$. This amounts to the appearance of a relative minus sign for every non-physical loop compared to the same physical loop, because of the opposite Wick rotation in the momentum space for the non-physical fields (Fig. 1). In this respect, the quantized non-physical (or of negative-norm) fields behave like fermions or Pauli-Villars regulators, so that one may already expect UV-divergence cancellations in Feynman graphs between physical and non-physical loops. It happens to be the case indeed (see the next subsections).

4.2 Gauge LEENA and vacuum structure

We are now in a position to discuss the N=2 gauge low-energy effective action (LEEA) to be defined by a Gaussian integration over both hypermultiplets in eq. (3.8) and then expanding the result in powers of external momenta. The quantum effective action $\Gamma(V^{++})$ is formally defined in HSS by the one-loop formula

$$\Gamma(V^{++}) = i \text{Tr} \ln D_{\text{phys}}^{++} - i \text{Tr} \ln D_{\text{nonphys}}^{++}, \quad (4.6a)$$

or, in terms of the Green functions (4.4), as

$$\Gamma(V^{++}) = i \text{Tr} \frac{\delta^{(3,1)}_A + i V^{++}G_{\text{phys}}^{(1,1)}}{\delta^{(3,1)}_A + i V^{++}G_{\text{nonphys}}^{(1,1)}}. \quad (4.6b)$$

A supergraph calculation of the LEEA for a single physical hypermultiplet in HSS (Fig. 2) was already done in refs. [38, 46], so that we can use the known results here.

Because of the gauge invariance, $\Gamma(V^{++})$ can only depend upon the abelian N=2 gauge superfield strength $W$ and its conjugate $\bar{W}$, with both being defined.
by eq. (A.25). On dimensional reasons, the general structure of the gauge LEEA (modulo terms explicitly depending upon N=2 superspace derivatives of \(W\) or \(\bar{W}\)) is given by

\[
\Gamma[V^{++}] = \left[ \int d^4x d^4\theta \mathcal{F}(W) + \text{h.c.} \right] + \int d^4x d^4\theta d^4\bar{\theta} \mathcal{H}(W, \bar{W}) ,
\]

where the leading holomorphic term is known as the (perturbative) Seiberg-Witten LEEA [17] or the integrated N=2 supersymmetric (chiral) \(U(1)_R\) anomaly [11, 12, 43], whereas the second term is called the non-holomorphic (perturbative) next-to-leading-order correction [14]. It is worth mentioning that the real function \(\mathcal{H}(W, \bar{W})\) is subject to the gauge transformations

\[
\mathcal{H}(W, \bar{W}) \rightarrow \mathcal{H}(W, \bar{W}) + F(W) + \bar{F}(\bar{W}) ,
\]

with the N=2 chiral superfield parameter \(F(W)\), quite similarly to the Kähler transformations (2.5) in N=1 superspace. The HSS calculations [38, 44] yield

\[
\mathcal{F}(W)_{\text{phys}} = -\frac{1}{(8\pi)^2} W^2 \ln \frac{W^2}{m^2} ,
\]

and

\[
\mathcal{H}(W, \bar{W})_{\text{phys}} = \frac{1}{(16\pi)^2} \left( \ln \frac{W}{\Lambda} \right) \left( \ln \frac{\bar{W}}{\Lambda} \right) ,
\]

where \(\Lambda\) is an irrelevant parameter since the action (4.7) does not really depend on it because of eqs. (4.8) and (4.10) [48].

Eqs. (4.9) and (4.10) in our case (4.6) immediately imply that

\[
\Gamma[V^{++}]_{\text{LEEA}} = -\frac{1}{32\pi^2} \ln \left( \frac{m^2}{m_i^2} \right) \int d^4x d^4\theta W^2 \equiv -\frac{1}{2c_\theta} \int d^4x d^4\theta W^2 ,
\]

which is just a free action of the N=2 vector gauge superfield! Eq. (4.11) implies, in particular, the dynamical generation of the term quadratic in the auxiliary field \(D_{ij}\), which is accompanying the standard kinetic terms of the N=2 vector multiplet in the well-known component form (B.8) of eq. (4.11). Together with the FI-term in eq. (3.8), this now implies a non-vanishing vacuum expectation value \(\langle D_{ij} \rangle \neq 0\).

Some comments are in order.

Unlike the N=2 gauge LEEA for a single hypermultiplet (Fig. 2), our LEEA (4.6) is both infra-red (IR) and ultra-violet (UV) finite. IR divergences are obviously absent due to non-vanishing hypermultiplet masses acting as IR-regulators. As regards the UV divergences of the N=2 gauge LEEA for a physical hypermultiplet, the leading 2-point contribution in Fig. 2 is known to be the only divergent one (all the higher
n-point contributions in Fig. 2 are automatically UV finite on dimensional reasons) [38]. The holomorphic 2-point contribution to the N=2 gauge LEEA due to a single physical hypermultiplet reads [38]

$$\Gamma^{(2)}_{\text{phys}}[V] = -\frac{i}{2(2\pi)^4} \int d^4p d^8\theta du V^{++}(p, \theta, u) \Pi_{\text{phys}}(-p^2) V^{--}(-p, \theta, u) \ , \ (4.12)$$

where the (dimensionally regularized) one-loop structure function $$\Pi_{\text{phys}}(-p^2)$$ has been introduced (with $$\mu$$ as a renormalization scale),

$$\Pi_{\text{phys}}(-p^2) = \mu^{2\varepsilon} \int \frac{d^{4-2\varepsilon}l}{(2\pi)^{4-2\varepsilon}} \left\{ \frac{1}{[l^2 + m_1^2][(l-p)^2 + m_1^2]} - \frac{1}{[l^2 + m_2^2][(l-p)^2 + m_2^2]} \right\} \ . \ (4.13)$$

Eq. (4.13) is logarithmically UV-divergent in four spacetime dimensions ($$\varepsilon \to +0$$). This UV divergence is simultaneously the origin of the renormalization scale dependence of the renormalized low-energy effective action (4.7) via its holomorphic (anomalous) contribution. In our case (4.6), the UV divergence of the self-energy integral (4.13) cancels against the opposite UV divergence of the similar contribution to the LEEA due to the nonphysical hypermultiplet, viz.

$$\Pi(-p^2) \equiv \Pi_{\text{phys}}(-p^2) + \Pi_{\text{nonphys}}(-p^2)$$

$$= \int \frac{d^4l}{(2\pi)^4} \left\{ \frac{1}{[l^2 + m_1^2][(l-p)^2 + m_1^2]} - \frac{1}{[l^2 + m_2^2][(l-p)^2 + m_2^2]} \right\}$$

$$= \frac{1}{16\pi^2} \int_0^1 dx \ln \frac{m_1^2 + p^2 x(1-x)}{m_1^2 + p^2 x(1-x)} \ , \ (4.14)$$

where Feynman parameterization has been used to evaluate the momentum integral in the euclidean domain (we assume that $$m_2^2 > m_1^2$$). A continuation to Minkowski space entails changing the sign of $$p^2$$ — this explains our notation, $$\Pi(s)$$ and $$s = -p^2$$, above. The function $$\Pi(s)$$ is analytic in the cut $$s$$ plane whose analytic structure is best exhibited by dispersion relations [49].

One obtains eq. (4.11) from eqs. (4.12) and (4.14) in the low-energy limit $$p^2 \to 0$$. We took the vanishing momenta since we are first interested in finding a Poincaré- and gauge-invariant vacuum background solution. It can only be represented by a spacetime-independent N=2 vector gauge superfield strength $$\langle W \rangle$$ having the form

$$\langle W \rangle = \langle a \rangle + \frac{i}{2}(\theta^a \theta_{a\beta}) \langle D^{ij} \rangle \ , \ (4.15)$$

where merely constant vacuum expectation values of the bosonic scalar components of $$W$$ have been kept. We can assume that $$\langle a \rangle = 0$$ without a loss of generality since: (i) there is no equation on $$\langle a \rangle$$ at all, and (ii) a constant $$\langle a \rangle$$ would simply amount to
the equal shift of both hypermultiplet masses in the theory (3.8). Hence, we are left with the induced scalar potential

\[ V(\vec{D}) = \vec{\xi} \cdot \vec{D} - \frac{1}{2e_0^2} \vec{D}^2, \]  

in components, which has the only vacuum solution

\[ \langle \vec{D} \rangle = e_0^2 \vec{\xi} \neq 0. \]  

It also follows from eqs. (3.15), (4.11) and (4.17) that

\[ m_1^2 = \frac{m^2}{e^{16\pi^2/e_0^2} - 1}, \quad m_2^2 = \frac{m^2}{1 - e^{-16\pi^2/e_0^2}}. \]  

This simple exercise can also be repeated in HSS, by using the results of Appendix A. Varying the N=2 gauge effective action with respect to the abelian N=2 vector gauge superfield \( V^{++} \) in HSS yields the equation of motion (in vacuum)

\[ \frac{1}{e_0^2} (D^+)^4 \langle A^{--} \rangle = \xi^{++}, \]  

where the HSS potentials \( A^{--} \) and \( V^{++} \) are related via eq. (A.21). A Poincaré-invariant solution to eq. (4.19) reads

\[ \langle V^{++} \rangle = (\theta^+)^2(\bar{\theta}^+)^2e_0^2\xi^{--}, \]  

and it is equivalent to eq. (4.17) because of eq. (A.29).

Any other non-trivial N=2 gauge LEA (4.7) having the form different from that of eq. (4.11), i.e. with a non-quadratic holomorphic function \( F(W) \), does not admit a constant non-vanishing solution for \( \vec{D} \), because of the appearance of an extra equation \( \partial^3 F/\partial W^3|_{W=a} \vec{D}^2 = 0 \) \[50\]. A non-vanishing value of \( \langle \vec{D} \rangle \) implies the appearance of Goldstone fermions (see the second line of eq. (B.7) in Appendix B) which inhomogeneously transform under on-shell N=2 supersymmetry. In other words, the N=2 supersymmetry is spontaneously broken in our theory (3.8).

### 4.3 Dynamical generation of composite particles

As was shown in the preceding subsect. 4.2, quantum effects due to hypermultiplets lead to the appearance of the propagating (physical) abelian N=2 vector multiplet \( V^{++} \). In the classical theory (3.8), \( V^{++} \) is merely present as a (non-propagating)
Lagrange multiplier. Because of eqs. (4.12) and (4.14), the induced gauge coupling constant is momentum-dependent,

\[
\frac{1}{e_{\text{ind}}^2} = \frac{1}{16\pi^2} \int_0^1 dx \ln \frac{m_1^2 + p^2x(1-x)}{m_1^2 + p^2x(1-x)} = \frac{1}{e_0^2} + O(p^2/m^2) .
\] (4.21)

Notably, the UV finiteness enjoyed by our theory in four spacetime dimensions is also necessary for its consistency: if there were UV divergent contributions to \(e_{\text{ind}}^2\), they would have to be removed by the corresponding counterterm proportional to the \(N=2\) gauge action. The latter must, however, be absent in the bare action (3.8) since, otherwise, it would contradict the classical nature of \(V^{++}\) as a Lagrange multiplier.

In order to calculate the full gauge LEEA, one has to repeat a calculation of the HSS graphs depicted in Fig. 2 in terms of the new hypermultiplet propagators to be defined with respect to the ‘true’ vacuum with the non-vanishing FI term (4.17).

The Green function of a physical hypermultiplet in a generic \(N=2\) vector superfield background \(\hat{V}^{++}\) satisfies the defining equation

\[
\mathcal{D}_{1+}^{(1,1)} G_{\text{phys},\hat{V}}^{(1,1)}(1|2) = \delta_A^{(3,1)}(1|2) ,
\] (4.22)

whose solution can be formally written down in the form

\[
G_{\text{phys},\hat{V}}^{(1,1)}(1|2) = \frac{-1}{\Box_{\text{cov}} - i0}(D^+_1)^4 (D^+_2)^4 e^{\hat{V}_2 - \hat{V}_1} \delta^{12}(Z_1 - Z_2) \frac{1}{(u_1 u_2)^3} ,
\] (4.23)

where the covariantly constant ‘bridge’ \(e^{-\hat{V}}\) and the covariant d’Alambertian \(\Box_{\text{cov}}\) in the analytic HSS have been introduced [38, 27]. The defining equation for the ‘bridge’ reads

\[
\mathcal{D}^{++} e^{-\hat{V}} = (\mathcal{D}^{++} + i\hat{V}^{++}) e^{-\hat{V}} = 0 ,
\] (4.24)

whereas the defining equation for the covariant d’Alambertian is given by

\[
-\frac{1}{2} (D^+_1)^4 (D^-_2)^4 \Phi^{(p)} = \Box_{\text{cov}} \Phi^{(p)} ,
\] (4.25)

where \(\Phi^{(p)}\) is a HSS analytic superfield of (positive) \(U(1)\) charge \(p\). The definition (4.25) obviously implies that

\[
[D^+_\alpha, \Box_{\text{cov}}] = [\bar{D}^+_\alpha, \Box_{\text{cov}}] = 0 .
\] (4.26)

An explicit form of the operator \(\Box_{\text{cov}}\) in a generic background \(\hat{V}^{++}\) was calculated in ref. [38] in the covariantly analytic form

\[
\Box_{\text{cov. analytic}} = \mathcal{D}^\mu \mathcal{D}_\mu + \frac{i}{2}(\mathcal{D}^\alpha \mathcal{W})\mathcal{D}_\alpha^- + \frac{i}{2} (\bar{\mathcal{D}}^\alpha_+ \bar{W}) \bar{\mathcal{D}}^\alpha_- - \frac{1}{4}(\bar{\mathcal{D}}^\alpha_+ \mathcal{D}^- \mathcal{W})\mathcal{D}^- + \frac{1}{4}(\mathcal{D}^\alpha_- \mathcal{D}^\alpha_+ \mathcal{W}) + \bar{W} \mathcal{W} ,
\] (4.27)

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which is related to $\Box_{\text{cov}}$ via the ‘bridge’ transform, i.e.

$$\Box_{\text{cov}} = e^{-V} \Box_{\text{cov, analytic}} e^{V}.$$  \hspace{1cm} (4.28)

The particular form of the operator $\Box_{\text{cov}}$ in the spacetime-constant gauge-invariant background (4.15) reads (cf. ref. [27])

$$\Box_{\text{cov, const}} = \Box + \hat{Z} \hat{Z} + \frac{i}{2} \xi^{+-} \left[ (\theta^{+}\theta^-)^2 + \frac{1}{4} \xi^{++}(\theta^{+}\theta^-)^2(\bar{\theta}^{+}\bar{\theta}^-) \right]
+ \left[ \frac{i}{2} \xi^{++}(\theta^- \bar{\theta}^-) + \frac{i}{2} \xi^{--}(\theta^{+} \bar{\theta}^+ - \xi^{+-}(\theta^{+} \bar{\theta}^-) + \text{h.c.} \right]
+ \left[ \frac{1}{2} \xi^{++}(\theta^- D^-) - \frac{1}{2} \xi^{+-}(\theta^+ D^-) + \text{h.c.} \right] + \frac{1}{4} \xi^{++}D^{--}.$$  \hspace{1cm} (4.29)

The ‘bridge’ itself is given by

$$\mathcal{V}_{\text{const}} = \frac{1}{2} \xi^{--}(\theta^+ \theta^-)(\bar{\theta}^+ \bar{\theta}^-)^2 + \frac{1}{4} \xi^{++}(\theta^+ \theta^-)^2(\bar{\theta}^+ \bar{\theta}^-)
- \frac{1}{4} \xi^{+-} \left[ 2(\theta^+ \theta^-)(\bar{\theta}^+ \bar{\theta}^-) + (\theta^+ \theta^-)^2(\bar{\theta}^+ \bar{\theta}^-) \right] - \text{h.c.}.$$  \hspace{1cm} (4.30)

To get the non-physical hypermultiplet propagator, eq. (4.23) has to be changed according to subsect. 4.1.

The hypermultiplet propagator defined by eqs. (4.23), (4.29) and (4.30) seems to be too complicated, which prevented me from doing explicit perturbative calculations in HSS with the use of it now (this work is in progress). It is not even obvious to me whether the HSS methods remain to be technically superior in comparison to the conventional quantum perturbation theory in components (or in N=1 superfields) when N=2 supersymmetry is spontaneously broken and N=2 superfield propagators are manifestly $\theta$-dependent. Nevertheless, the qualitative picture remains to be the same as in the previous subsect. 4.2: the kinetic term of the N=2 vector gauge superfield is dynamically generated, with the induced (dimensionless and momentum-dependent) gauge coupling constant

$$e_{\text{ind}}^2(p) = e^2 + O(p^2/m^2).$$  \hspace{1cm} (4.31)

A momentum dependence of $e_{\text{ind}}^2$ is calculable, while its low-energy value $e^2$ is non-vanishing, being a function of the dimensionless ratio $m^2/\xi$. Indeed, the modification (4.29) of the box operator in the low-energy limit essentially amounts to a shift of the hypermultiplet mass, which is clearly the same for both (i.e. physical and non-physical) hypermultiplet propagators.

The dynamical generation of the whole N=2 vector multiplet implies, of course, the dynamical generation of all of its physical components, i.e. a complex scalar
which can be interpreted as a ‘Higgs’ particle, a chiral spinor doublet representing a complex ‘photino’, and a real ‘photon’. In the next sect. 5 we interpret the composite N=2 vector multiplet components as the zero modes of a superstring ending on a Dirichlet (D) 6-brane. The non-perturbative phenomenon of the gauge symmetry enhancement for coincident D-6-branes appears to be surprisingly connected to the perturbative field theory considerations above via M-theory.

5 Relation to M-theory and brane technology

An exact solution to the LEEA of N=2 super-QCD with \( N_c \) colors in spacetime \( R^{1,3}_1 \), can be identified with the LEEA of the effective (called N=2 MQCD) field theory defined in a single M-5-brane worldvolume given by the local product of \( R^{1,3}_1 \) and a hyperelliptic curve \( \Sigma_g \) of genus \( g = N_c - 1 \) \[5\]. The hyperelliptic curve \( \Sigma_g \) has to be holomorphically embedded into the hyper-Kähler four-dimensional multicentre Taub-NUT space \( Q_{mTN} \) associated with a multiple KK monopole. The identification of LEEA in these two apparently very different field theories (namely, the N=2 super-QCD in the Coulomb branch, on the one side, and the N=2 MQCD defined in the M-5-brane worldvolume, on the other side) is highly non-trivial, since the former is defined as the leading contribution to the quantum LEEA in a gauge field theory, whereas the latter is determined by classical M-5-brane dynamics or by eleven-dimensional (11d) supergravity equations of motion whose extended BPS solutions preserving some part of 11d supersymmetry are called M-theory branes.

5.1 Multiple KK monopole

The multiple KK monopole is a non-singular (solitonic) BPS solution to the classical equations of motion of 11d supergravity, with 11d spacetime being the product of the seven-dimensional (flat) Minkowski spacetime \( R^{1,6} \) and the four-dimensional euclidean multicentre Taub-NUT space \( Q_{mTN} \) \[32, 33, 54\]:

\[
d s^2_{[11]} = d x^m d x^n \eta_{mn} + d s^2_{[4]} , \quad F_{(4)} \equiv d A_{(3)} = 0 , \tag{5.1}
\]

where \( d s^2_{[4]} \) has been already defined in eq. (2.11), \( m = 0,1,2,3,7,8,9, \ y = \{ y_i \}, \ i = 4,5,6. \) The eleventh coordinate in 11d has been identified with the periodic coordinate \( \rho \) of the multi-Taub-NUT space (2.11) with the harmonic function \( H \) defined by eq. (2.21). The moduli \( (k_A, y_A) \) in eq. (2.21) are interpreted as charges

\[9\] See e.g., refs. \[72, 14\] for a review and ref. \[53\] for an introduction.
and locations of KK monopoles. For instance, the simple Taub-NUT space ($p = 1$) can be thought of as a non-trivial bundle (Hopf fibration) with the base $R^3$ and the fiber $S^1$ of magnetic charge $k$. In general ($p \geq 1$), there exist $p$ linearly independent normalizable self-dual harmonic 2-forms $\omega_A$ in $Q_{mTN}$, which satisfy the orthogonality condition

$$\frac{1}{(2\pi k)^2} \int_{Q_{mTN}} \omega_A \wedge \omega_B = \delta_{AB}. \quad (5.2)$$

Two adjacent KK monopoles are connected by a homology 2-sphere having poles at the positions of the monopoles. Near a singularity of $H$, the KK circle $S^1$ contracts to a point. A holomorphic embedding of the Seiberg-Witten spectral curve $\Sigma_g$ into the hyper-Kähler manifold $Q_{mTN}$ is the consequence of the BPS condition [56, 57]

$$\text{Area}_{\Sigma} = \left| \int_{\Sigma} \Omega_{\Sigma} \right|, \quad (5.3)$$

where $\Omega_{\Sigma}$ is the pullback of the Kähler (1,1) form $\Omega$ of $Q_{mTN}$ on $\Sigma_g$. In fact, any four-dimensional hyper-Kähler manifold possesses a holomorphic (2,0) form $\omega$, which is simply related to the Kähler form $\Omega$ as [34]

$$\Omega^2 = \omega \wedge \bar{\omega}. \quad (5.4)$$

The BPS states in M-theory, whose zero modes appear in the effective field theory defined in the M-5-brane worldvolume, correspond to the supermembranes (or M-2-branes) having minimal area (BPS!) and ending on the M-5-brane. The spacial topology of such M-2-brane determines the type of the corresponding N=2 supermultiplet in the effective (macroscopic) spacetime $R^{1,3}$: a cylinder ($Y$) leads to an N=2 vector multiplet, whereas a disc ($D$) gives rise to a hypermultiplet [56, 57]. Since the pullback $\omega_Y$ on $Y$ is closed [57], there exists a meromorphic differential $\lambda_{SW}$ satisfying the relations $\omega_Y = d\lambda_{SW}$ and

$$Z = \int_Y \omega_Y = \oint_{\partial Y} \lambda_{SW}, \quad (5.5)$$

where $Z$ is the central charge and $\partial Y \in \Sigma$. Hence, $\lambda_{SW}$ can be identified with the Seiberg-Witten differential which determines the N=2 gauge LEEA in $R^{1,3}$ [58].

### 5.2 Hypermultiplet LEEA from D-6-brane dynamics

As a result of KK compactification on the Seiberg-Witten curve $\Sigma_g$, the leading Nambu-Goto (NG) term (proportional to the M-5-brane worldvolume) in the effective (six-dimensional) M-5-brane action reduces in the low-energy approximation to
a four-dimensional scalar NLSM having the special Kähler geometry. This is enough to unambiguously restore the full Seiberg-Witten LEEA \cite{17} by $N = 2$ supersymmetrization of the special bosonic NLSM: one considers the NLSM complex scalars as the leading components of abelian $N=2$ vector multiplets in four spacetime dimensions, and then one deduces the Seiberg-Witten holomorphic potential $F$ out of the known special Kähler NLSM potential (see ref. \cite{16} for details)

$$K(\Phi, \bar{\Phi}) = \text{Im} \left( \Phi^i \frac{\partial F}{\partial \Phi^i} \right). \quad (5.6)$$

Being applied to a derivation of the hypermultiplet LEEA of $N = 2$ super-QCD in the Coulomb branch, brane technology suggests to dimensionally reduce the effective action of a D-6-brane (to be described in M-theory by a KK-monopole) down to four spacetime dimensions \cite{16}. In a static gauge for the D-6-brane, the induced metric in the brane worldvolume is given by

$$g_{\tilde{\mu} \tilde{\nu}} = \eta_{\tilde{\mu} \tilde{\nu}} + G_{mn} \partial_{\tilde{\mu}} y^m \partial_{\tilde{\nu}} y^n, \quad (5.7)$$

where $\tilde{\mu}, \tilde{\nu} = 0, 1, 2, 3, 7, 8, 9, m, n = 4, 5, 6, 10$, and $G_{mn}$ is the multicentre euclidean Taub-NUT metric. After expanding the NG-part of the D-6-brane effective action

$$S_{NG} = \int d^7 \xi \sqrt{-\det(g_{\tilde{\mu} \tilde{\nu}})} \quad (5.8)$$

up to the second order in the spacetime derivatives and performing a plain dimensional reduction from seven to four spacetime dimensions, one arrives at the hyper-Kähler NLSM

$$S[y] = \frac{1}{2} \int d^4 x G_{mn}(y) \partial_\mu y^m \partial^\mu y^n, \quad \mu = 0, 1, 2, 3, \quad (5.9)$$

whose $N=2$ supersymmetrization yields the full $N=2$ supersymmetric hypermultiplet LEEA, in agreement with the $N=2$ supersymmetric quantum field theory calculations in HSS \cite{27}.

### 5.3 Symmetry enhancement for two coincident D-6-branes

We are now in a position to discuss the symmetry enhancement in the case of two nearly coincident D-6-branes. The non-singular interpretation of D-6-branes in M-theory is based on the fact that the isolated singularities of the harmonic function (2.21) are merely the coordinate singularities of the eleven-dimensional metric (5.1), though they are truly singular with respect to the dimensionally reduced ten-dimensional metric which is associated with D-6-branes in the type-IIA picture. The
physical significance of the ten-dimensional metric singularities is now understood due to the illegitimate neglect of the KK modes related to the compactification circle \(S^1\), since these KK particles (also called D-0-branes) become massless near the D-6-brane core [54]. Their inclusion is equivalent to accounting for instanton corrections in the four-dimensional \(N=2\) supersymmetric gauge field theory.

When some parallel and similarly oriented D-branes coincide (this may happen in some special points of the moduli space of M-theory), it is accompanied by a gauge symmetry enhancement [59, 60]. Since the brane singularities become non-isolated in the coincidence limit, first, they have to be resolved by considering the branes to be separated by some distance \(\xi\). Then one takes the limit of small \(\xi\). In the case of two parallel D-6-branes one substitutes the harmonic function (3.1), describing the double-centered Taub-NUT metric with a constant potential \(\lambda\) at infinity, into eq. (5.1). Then it describes two parallel and similarly oriented M-theory KK monopoles with both centers on a line \(\vec{\xi}\) in the sixth direction, which dimensionally reduce to a double D-6-brane configuration in ten dimensions. The homology 2-sphere connecting two KK monopoles contracts to a point in the limit \(\xi \to 0\), which gives rise to a curvature singularity of the dimensionally reduced metric in ten dimensions. From the eleven-dimensional perspective, M-2-branes can wrap about the 2-sphere connecting the KK monopoles, while the energy of the wrapped M-2-brane is proportional to the area of the sphere [59]. When the sphere collapses, its area vanishes and, hence, the zero modes of the wrapped M-2-brane become massless, thus giving rise to an extra massless vector supermultiplet in the LEEA and, hence, the gauge symmetry enhancement

\[
U(1) \times U(1) \rightarrow U(2)
\]  

(5.10)

associated with the \(A_1\)-type singularity. A non-perturbative phenomenon of the gauge symmetry enhancement was first observed in a K3 compactification of M-theory due to collapsing 2-cycles of K3 on the basis of duality with the heterotic string compactifications [61, 62]. It is worth mentioning here that the geometry near a collapsing 2-cycle of K3 is the same as the geometry near two almost coincident parallel KK monopoles in M-theory [63], i.e. the corresponding hyper-Kähler metric is governed by the harmonic function (3.1) in the limit \(\xi \to 0\).

From the ten-dimensional viewpoint, the wrapped M-2-branes are just the 6-6 superstrings stretched between two D-6-branes, so that it is the zero modes of these 6-6 superstrings that become massless in the coincidence limit for D-6-branes (Fig. 3).

Each of the \(U(1)\) factors on the left-hand-side of eq. (5.10) is associated with a single D-6-brane, being related to a 6-6 superstring whose both ends are on this brane.
ends on the same brane (a), or on different branes (b).

(i.e. of type (a) in Fig. 3). The massless zero modes of this 6-6 superstring define an $U(1)$ gauge vector supermultiplet in the field theory LEEA describing the dynamics of small fluctuations about a D-6-brane. We can therefore identify this abelian vector supermultiplet with the composite vector supermultiplet dynamically generated from the hypermultiplet low-energy LEEA in the D-6-brane worldvolume (sect. 4).

Unlike the N=2 gauge LEEA, the exact hypermultiplet LEEA is entirely determined by its one-loop (perturbative) contribution having the form of the NLSM whose target space metric is equal to the KK-monopole metric [10]. Unlike the 6-6 superstrings of the type (a) in Fig. 3, the 6-6 superstrings of the type (b) (see Fig. 3) with their ends on different D-6-branes cannot be understood this way, being of truly non-perturbative origin. Indeed, our quantum field theory (3.8) becomes singular in the limit $\xi \to 0$. Accordingly, the non-abelian gauge symmetry enhancement (5.10) is beyond the scope of the hypermultiplet LEEA approach alone, which is apparently limited to a single D-brane worldvolume.

The hypermultiplet LEEA is obtained by a 4d spacetime N=2 supersymmetrization of the bosonic NLSM (5.9) whose hyper-Kähler metric (2.11) is fixed by the harmonic function (3.1). The abelian gauge symmetry in terms of the hypermultiplet LEEA metric amounts to the gauged isometries in the corresponding NLSM target space, while their gauging itself can be made manifest in HSS. As was already
mentioned in sect. 3, the N=2 supersymmetric NLSM with the double Taub-NUT metric (two KK monopoles) is equivalent to the one with the mixed (Eguchi-Hanson-Taub-NUT) metric. The corresponding HSS action is given by eq. (3.3) which makes it clear that the mixed hyper-Kähler NLSM metric does interpolate between the Eguchi-Hanson metric ($\lambda = 0$) and the Taub-NUT metric ($\xi = 0$), both having the maximal isometry group $U(2)$. The action of the $U(2)$ isometry is linear in both limiting cases, while it is even holomorphic in the second case. Within the HSS approach this internal symmetry enhancement $U(1) \rightarrow SU(2)$ can be understood either as a restoration of the $SU(2)_A$ automorphism symmetry of N=2 supersymmetry algebra in the Taub-NUT limit, or as a restoration of the $SU(2)_{PG}$ symmetry in the Eguchi-Hanson limit \[17\].

On the one side, the geometry of two almost coinciding D-6-branes near the origin $\vec{y} = 0$ can be approximated by the Eguchi-Hanson metric in M-theory since a finite asymptotical potential $\lambda$ in eq. (3.1) becomes irrelevant near the singularity $\vec{y} = 0$. The corresponding hypermultiplet LEEA (3.3) then reduces to our model (3.8) whose one-loop quantum fluctuations were investigated in the preceding sect. 4. On the other side, a D-6-brane naturally has in its worldvolume a massless abelian vector supermultiplet which can be understood as the Nambu-Goldstone mode associated with the 11d symmetries broken by the D-6-brane (BPS !) classical solution \[46\]. Therefore, the dynamical generation of an abelian N=2 vector multiplet in the quantized 4d field theory (3.8) is consistent with the effective classical dynamics of two nearly coincident D-6-branes.

It may be natural to treat all 6-6 superstrings (i.e. of both types (a) and (b) in Fig. 3) on equal footing, like their zero modes. It is then tempting to conjecture that any (T-dual) open superstring ending on a D-brane should be considered as a composite (bound state) of the D-brane physical degrees of freedom.

6 Conclusion and outlook

The 11d supergravity approximation to M-theory is only valid for well-separated KK monopoles. When KK monopoles coincide, their low-energy dynamics should be approximated by weakly coupled (perhaps, composite) superstrings propagating in the multi-Eguchi-Hanson (ALE) background \[33\]. The corresponding metric was found by Gibbons and Hawking in ref. \[64\], while it naturally originates as a particular limit of the multi-Taub-NUT (multicentre) metric, as we have already seen in the preceding sections in the case of two coincident KK monopoles.
When \( p \geq 2 \) D-6-branes coincide, i.e. all the moduli \( \vec{y}_A \) in the harmonic function (2.21) are set to be zero, an additive asymptotic potential \( \lambda \) can be always ignored near the core of these D-6-branes on top of each other. The multi-Eguchi-Hanson ALE space thus possesses an \( A_{p-1} \) simple singularity which implies the enhanced non-abelian gauge symmetry \( U(p) \) in the effective supersymmetric field theory defined in the common worldvolume of the coincident D-6-branes [60]. Indeed, the effective gauge field theory is supposed to be defined in the limit where gravity decouples. The 11d supergravity has a 3-form \( A^{(11)}_{B(3)} \) which is decomposed with respect to the product of the D-6-brane worldvolume \( R^{1,6} \) and the multi-Taub-NUT space \( Q_{mTN} \) as (cf. ref. [62])

\[
A^{(11)}_{B(3)} = \sum_{B=1}^{p} A^{[7]}_{B(1)} \wedge \omega^{[4]}_{B(2)},
\]

(6.1)

where the 2-forms \( \omega_B \) in \( Q_{mTN} \) have been introduced in subsect. 5.1, whereas \( A_B \) are \( p \) massless vectors (1-forms) in \( R^{1,6} \). In addition, there are \( 3p \) scalar fields associated with the translational zero modes (or moduli) \( \vec{y}_A \). Taken together, these vectors and scalars constitute the bosonic components of \( p \) massless vector supermultiplets in \( 1 + 6 \) dimensions, each having \( 8_B + 8_F \) on-shell components. The gauge group of the effective field theory (in the case of separated KK monopoles) is therefore given by \( U(1)^p \). Since the intersection matrix of 2-cycles in \( Q_{mTN} \) is known to be given by the Cartan matrix of \( A_{p-1} \), the abelian gauge symmetry \( U(1)^p \) should be enhanced to \( U(p) \) in the coincidence limit. The area of the 2-cycles vanishes in this limit, so that the M-2-branes wrapped around these 2-cycles lead to the additional massless vectors to be identified with the M-2-brane zero modes.

In the type-IIA picture, the 6-6 superstrings stretched between separated D-6-branes do not contribute to the effective LEEA in the Coulomb branch at all [51]. However, since the zero modes of these 6-6 superstrings become massless when the brane separation vanishes, they do contribute to the LEEA in our case, which may be called the non-abelian Coulomb branch. After a plain dimensional reduction from \( R^{1,6} \) to \( R^{1,3} \), the effective \( N=1 \) super-Yang-Mills theory in \( 1 + 6 \) dimensions gives rise to the \( N=4 \) super-Yang-Mills theory in \( 1 + 3 \) dimensions, which has the same number of on-shell physical components.

A (spacetime) four-dimensional \( N=2 \) supersymmetric NLSM having the four-dimensional multi-Eguchi-Hanson target space can be constructed, for example, in HSS by coupling \( p + 1 \) hypermultiplets in the fundamental representation of \( SU(1, p) \) whose Cartan subalgebra (CSA) \( u(1)^p \) is gauged by using \( p \) abelian \( N=2 \) vector gauge superfields entering the NLSM action as Lagrange multipliers in the presence of FI terms for all of them (cf. eq. (3.8) and ref. [23]). There is, however, a problem with
this approach because of the mismatch between the numbers of physical and non-
physical hypermultiplets (unless $p = 1$), which may lead to UV divergences in 4d
and, hence, quantum inconsistencies. A possible resolution may be just taking into
account more physical hypermultiplets. Above we only considered a single physical
hypermultiplet whose LEEA (NLSM) metric had a direct geometrical interpretation
in M-theory in terms of four-dimensional (KK) monopoles. However, more physical
hypermultiplets can appear in the field theory LEEA, e.g., after taking into account
the zero modes of the 4-6 superstrings stretched between D-4- and D-6-branes in
N=2 MQCD \cite{51} 10. The hypermultiplet LEEA, capable to dynamically generate a
non-abelian N=2 vector gauge multiplet, might be given by the N=2 supersymmetric
gauged NLSM over the non-compact coset $SU(p,p)/U(1)$ in HSS, whose $p$ physical
and $p$ non-physical hypermultiplets together are in some (other than fundamental)
representation of $SU(p,p)$, with the $U(1)$ subgroup being gauged by a non-abelian
N=2 vector gauge superfield. However, there is another caveat here since FI terms
only exist for abelian gauge groups. It is not clear to me how to formulate a corre-
sponding N=2 supersymmetric NLSM action, if any.

A unique consistent possibility seems to be the hypermultiplet LEEA having the
form of a gauged N=2 NLSM over the coset $SU(p,p)/U(1)^p$. The hypermultiplets
then belong to the fundamental representation of $SU(p,p)$ whose abelian subgroup
$U(1)^p$ is to be gauged. The latter is supposed to be associated with the CSA of
the full non-abelian gauge group $U(p)$ which only appears after taking into account
non-perturbative corrections due to D-6-branes.

The starting N=2 NLSM action in the analytic HSS is given by

$$S[q,V] = \int_{\text{analytic}} \left\{ \text{tr}_{\text{fund}} \left( \hat{q}^+ D^{++} q^+ \right) + \text{tr}_{\text{CSA}} \left( V^{++} \xi^{++} \right) \right\}, \quad (6.2)$$

where the $u(1)^p$ CSA-valued harmonic- and gauge-covariant derivative with central
charge, $D^{++} = D_{\tilde{Z}}^{++} + iV^{++}$, has been introduced. The FI terms in eq. (6.2) are also
necessary in order to get rid of a singularity which would appear in their absence.

It is straightforward to calculate the local part of the one-loop effective action
$i Tr \log D^{++}$ along the lines of sect. 4 in the LEEA approximation, with the latter
being defined by the condition $p^2 \ll \xi^2$ for all external momenta $p^\mu$. It should result
in a dynamical generation of an N=2 supersymmetric $U(1)^p$ gauge field theory, whose
unique N=2 supersymmetric non-abelian $U(p)$ extension is given by a standard N=2

\footnote{The M-5-brane discussed in the beginning of sect. 5 represents in 11d M-theory an intersecting
configuration of two NS 5-branes and $N_c$ D-4-branes in ten dimensions (=type-IIA picture).}
super-Yang-Mills (SYM) action. The latter reads in HSS as follows [65]:

\[
S_{N=2 \text{ SYM}}[V] = \frac{1}{g_{YM}^2} \int d^4x d^4\theta d^4\bar{\theta} \text{tr} \sum_{n=2}^{\infty} \frac{(-i)^n}{n} \int du_1 du_2 \cdots du_n \times \\
\times \frac{V^{++}(Z, u_1)V^{++}(Z, u_2) \cdots V^{++}(Z, u_n)}{(u_1^+ u_2^+)(u_2^+ u_3^+) \cdots (u_n^+ u_1^+)}.
\]

(6.3)

The induced ‘running’ gauge coupling constant takes the form (cf. sect. 4)

\[
\frac{1}{g_{YM}^2} = \frac{CA}{16\pi^2} \int_0^1 dx \ln \frac{|Z|^2 + \xi + p^2 x(1-x)}{|Z|^2 + p^2 x(1-x)},
\]

(6.4)

where we introduced the gauge group generators \(t_a\) in the fundamental representation of \(U(p)\), subject to \(\text{tr}(t_a t_b) = C_A \delta_{ab}\), the physical central charge \(Z\) and \(\xi = \sqrt{(\vec{\xi})^2} > 0\).

A different gauge symmetry enhancement pattern appears when \(p\) of D-6-branes come on top of an orientifold six-plane, which leads to the \(SO(2p)\) gauge symmetry [61]. The orientifold six-plane can be represented in M-theory by the (hyper-Kähler) Atiyah-Hitchin space [34] instead of a KK monopole. Indeed, far from the origin the Atiyah-Hitchin space has the topology \(R^3 \times S^1 / T_4\), i.e. it looks like \(Q_{mTN}\) whose points are now supposed to be identified under the action of the discrete symmetry \(T_4\) reversing signs of all four coordinates of \(Q_{mTN}\). This matches the definition of the orientifold six-plane according to ref. [63]. It is now straightforward to generalize our discussion to orthogonal gauge groups too.

To this end, let’s briefly consider the conformally invariant limit of the hypermultiplet LEEA corresponding to the case of \(N_c\) coincident D-6-branes (\(\xi \to 0\)) in \(N=2\) MQCD at large number of colors \(N_c\). The physical hypermultiplets in the 4d effective (\(N=2\) supersymmetric, by construction) field theory then form the adjoint representation of the gauge group, so that we arrive at a sum of the gauge-invariant hypermultiplet LEEA and the induced \(N=2\) SYM action (6.3),

\[
\int_{\text{analytic}} \text{tr}_{\text{ad}} \left( \bar{q}^+ D^{++} q^+ \right) + S_{N=2 \text{ SYM}} \equiv S_{N=4 \text{ SYM}},
\]

(6.5)

which is ‘almost’ the \(N=4\) supersymmetric Yang-Mills action in the 4d, \(N=2\) harmonic superspace, whose \(N=4\) supersymmetry is merely broken down to \(N=2\) by a non-vanishing central charge \(Z\). The induced \(N=4\) SYM coupling constant in the low-energy limit (\(p^2 = 0\)) but at large number \(N_c\) of colors (i.e. in the t’Hooft limit) satisfies the relation

\[
N_c g_{YM}^2 \sim \frac{|Z|^2}{|\xi|}.
\]

(6.6)

It should be compared to the recent conjecture of Maldacena [66]. He discussed a classical BPS solution describing a ‘plain’ M-5-brane in the particular limit (down
the ‘throat’) given by the product $AdS_7 \times S^4$ whose both radii are proportional to $N_c^{1/3}$. For large $N_c$ the Maldacena LEEA is given by a $(2,0)$ superconformally invariant gauge field theory in six dimensions, which is supposed to be dual to M-theory compactified on $AdS_7 \times S^4$. Down to four spacetime dimensions, Maldacena considered $N_c$ D-3-branes at large $N_c$ instead, and he argued that the N=4 SYM theory in the t’Hooft limit has to be dual to the IIB superstring theory compactified on $AdS_5 \times S^5$. The four-dimensional N=4 SYM theory is defined on the boundary of the $AdS_5$-space, with the correspondence

$$N_c g_{YM}^2 \sim (\alpha')^{-2} R_{AdS}^4 \quad \text{and} \quad g_{YM}^2 \sim g_{\text{string}}^2.$$ (6.7)

Note that the t’Hooft limit of large $N_c g_{YM}^2$ is equivalent to $|\xi/Z^2| \rightarrow 0$ in our approach. Hence, the conformal (t’Hooft-Maldacena) LEEA limit described by the ‘almost’ N=4 supersymmetric Yang-Mills theory can be deduced from the hypermultiplet LEEA near the singularity, after taking into account a non-vanishing physical central charge, a dynamical generation of the abelian N=2 vector gauge multiplets associated with the SCA of the gauge group, and a non-perturbative non-abelian gauge symmetry enhancement. A possible dynamical origin of the N=2 central charge is left as an open problem.

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Appendix A: N=2 harmonic superspace (HSS)

Four-dimensional field theories with N=2 extended supersymmetry can be formulated in the conventional N=2 extended superspace parameterized by the coordinates $Z_M = (x^\mu, \theta^\alpha_i, \bar{\theta}_{\dot{\alpha}i})$, $\mu = 0, 1, 2, 3$, $\alpha = 1, 2$, $i = 1, 2$, and $\bar{\theta}_{\dot{\alpha}i} = \bar{\theta}_{\dot{\alpha}i}$, in terms of constrained N=2 superfields \[67\]. Unfortunately, the standard constraints \[17, 68\],

$$\{D_i^\alpha, \bar{D}_{\dot{\alpha}j}\} = -2\delta^i_j D_{a\alpha}, \quad \{D_i^\alpha, \bar{D}^j_{\dot{\alpha}}\} = -2\varepsilon_{\alpha\beta}\varepsilon^{ij} \bar{W}, \quad \{\bar{D}_{\dot{\alpha}i}, \bar{D}_{\dot{\beta}j}\} = -2\varepsilon_{\alpha\beta}\varepsilon^{ij} W,$$

and

$$D_{(i}^{(j)} = \bar{D}^{(i}_{\alpha} q^{j)} = 0,$$

defining a (non-abelian) N=2 vector multiplet and a Fayet-Sohnius hypermultiplet, respectively, in the conventional N=2 superspace in terms of the gauge- and super-covariant (Lie algebra-valued) N=2 superspace derivatives

$$D_M \equiv (D_\mu, D_\alpha, \bar{D}_{\dot{\alpha}i}) = D_M + A_M,$$

do not have a manifestly holomorphic (or analytic) structure. Accordingly, they also do not have a simple solution in terms of unconstrained N=2 superfields which are needed for a manifestly supersymmetric quantization. The situation is even more dramatic for the FS hypermultiplet, since its defining equations (\(A.2\)) (in the absence of central charges) are merely on-shell constraints whereas the known off-shell formulations of a hypermultiplet in the conventional $N = 2$ superspace are either not universal (like an N=2 tensor multiplet \[29\]) or very cumbersome (like a relaxed hypermultiplet \[35\] or the generalized N=2 tensor multiplets \[18\]) so that their practical meaning is limited \[12\].

In the HSS formalism, the standard N=2 superspace is extended by adding bosonic variables (or ‘zweibeins’) $u^{\pm i}$ parameterizing a 2-sphere $S^2 \sim SU(2)/U(1)$. By using these extra variables one can make manifest the hidden analyticity structure of the standard N=2 superspace constraints (\(A.1\)) and (\(A.2\)) as well as to find their manifestly N=2 supersymmetric solutions in terms of unconstrained (analytic) N=2 superfields. The harmonic variables have the property

$$\begin{pmatrix} u^{+i} \\ u^{-i} \end{pmatrix} \in SU(2),$$

so that

$$u^{+i}u_i^- = 1, \quad u^{+i}u_i^+ = u^{-i}u_i^- = 0, \quad \text{and} \quad \bar{u}^{+i} = u_i^-.$$
Instead of using an explicit parameterization of the sphere $S^2$, it is convenient to deal with functions of zweibeins, that carry a definite $U(1)$ charge $U$ to be defined by $U(u_i^\pm) = \pm 1$, and use the following integration rules \cite{20}:

$$
\int du = 1 \, , \quad \int du u^{\pm (i_1 \ldots i_m} u^{-j_1 \ldots j_n)} = 0 \, , \quad \text{when } m + n > 0 \, . \quad (A.6)
$$

It is obvious that any integral over a $U(1)$-charged quantity vanishes.

The usual complex conjugation does not preserve analyticity. However, being combined with another (star) conjugation that only acts on $U(1)$ indices as $(u_i^\pm) = u_i^-$ and $(u_i^\pm)^* = -u_i^\pm$, it does preserve analyticity. One easily finds \cite{20}

$$
\begin{align*}
\left( u_i^\pm \right)^* &= -u_i^\pm \, , \quad \left( u_i^\pm \right)^* &= u_i^\pm . 
\end{align*}
$$

\quad (A.7)

The covariant derivatives with respect to the zweibeins, preserving the defining equations (A.4) and (A.5), are given by

$$
D^{++} = u^+i \frac{\partial}{\partial u^{-i}} , \quad D^{--} = u^{-i} \frac{\partial}{\partial u^{+i}} \, , \quad D^0 = u^+i \frac{\partial}{\partial u^{+i}} - u^{-i} \frac{\partial}{\partial u^{-i}} . \quad (A.8)
$$

It is easy to check that they satisfy the $SU(2)$ algebra,

$$
[D^{++}, D^{--}] = D^0 \, , \quad [D^0, D^{\pm\pm}] = \pm 2D^{\pm\pm} \, , \quad (A.9)
$$

and commute with the N=2 superspace derivatives (A.3). Eq. (A.9) is supposed to be added to the constraints (A.1) and (A.2).

The key feature of the N=2 HSS is the existence of the so-called analytic subspace parameterized by the coordinates

$$
(\zeta, u) = \left\{ x_{\alpha}^{(i)} = x^{\mu} - 2i \theta^i(\sigma^\mu \bar{\theta}) u_i^+ u_j^- \, , \quad \theta^+_{\alpha} = \theta^i_{\alpha} u_i^+ \, , \quad \bar{\theta}^i_{\alpha} = \bar{\theta}^i_{\alpha} u_i^+ \, , \quad u_i^\pm \right\} \, , \quad (A.10)
$$

which is invariant under N=2 supersymmetry and closed under the combined conjugation of eq. (A.7) \cite{20}. This allows one to define analytic superfields of any non-negative and integer $U(1)$ charge $q$ by the analyticity conditions

$$
D_{\alpha}^+ \phi^{(q)} = \bar{D}_{\alpha}^+ \phi^{(q)} = 0 \, , \quad \text{where } D^\pm_{\alpha} = D_{\alpha}^i u_i^\pm \text{ and } \bar{D}^\pm_{\alpha} = \bar{D}_{\alpha}^i u_i^\pm . \quad (A.11)
$$

The analytic measure reads $d\zeta^{(-4)}du \equiv d^4x_{\alpha}d^2\theta^+d^2\bar{\theta}^+du$. It carries the $U(1)$ charge $(-4)$, whereas the full neutral measure of $N = 2$ HSS is given by

$$
d^4xd^4\theta^+d^4\bar{\theta}du = d\zeta^{(-4)}du(D^+)^4 \, , \quad (A.12)
$$

where

$$
(D^+)^4 = (D^+)^2(D^+)^2 = \frac{1}{16} (D^{+\alpha} D^+_\alpha)(D^+^\alpha \bar{D}^+^\alpha) . \quad (A.13)
$$
In the analytic subspace, the harmonic derivative \( D^{++} \) reads
\[
D^{++}_{\text{analytic}} = D^{++} - 2i\theta^+ \sigma^\mu \tilde{\theta}^+ \partial_\mu ,
\] (A.14)
it preserves analyticity, and it allows one to integrate by parts. Both the original (central) basis and the analytic one can be used on equal footing in the HSS. In the main text and in what follows we omit the subscript \( \text{analytic} \) at the covariant derivatives in the analytic basis, in order to simplify our notation.

It is the advantage of the analytic N=2 HSS compared to the ordinary N=2 superspace that both an off-shell N=2 vector multiplet and an off-shell hypermultiplet can be introduced there on equal footing. There exist two off-shell hypermultiplet versions in HSS, which are dual to each other. The so-called \textit{Fayet-Sohnius-type} (FS) hypermultiplet is defined as an unconstrained complex analytic superfield \( q^+ \) of \( U(1) \)-charge \((+1)\), whereas its dual, called the \textit{Howe-Stelle-Townsend-type} (HST) hypermultiplet, is a real unconstrained analytic superfield \( \omega \) with the vanishing \( U(1) \)-charge.\footnote{It is worth mentioning here that both FS and HST multiplets were originally introduced in the \textit{conventional} N=2 superspace \cite{Fayet1980, Sohnius1983}, whereas we use the same names to denote N=2 \textit{harmonic} superfields, which are different off-shell but reduce to the FS and HST multiplets on-shell.} The on-shell physical components of the FS hypermultiplet comprise an \( SU(2) \) doublet of complex scalars and a Dirac spinor which is a singlet with respect to \( SU(2) \). The on-shell physical components of the HST hypermultiplet comprise a real singlet and a real triplet of scalars, and a doublet of chiral spinors. The FS hypermultiplets are natural for describing a charged N=2 matter, whereas the HST hypermultiplets are more appropriate for describing a neutral N=2 matter.

Similarly, an N=2 vector multiplet is described by an unconstrained analytic superfield \( V^{++} \) of the \( U(1) \)-charge \((+2)\). The \( V^{++} \) is real in the sense \( V^{++*} = V^{++} \), and it can be naturally introduced as a connection to the harmonic derivative \( D^{++} \).

A free FS hypermultiplet HSS action is given by (in canonical normalization)
\[
S[q] = -\int d\zeta^{(-4)} du \, \hat{\bar{q}}^+ D^{++} q^+ ,
\]
(A.15)
whereas its minimal coupling to an abelian N=2 gauge superfield reads
\[
S[q, V] = -\int d\zeta^{(-4)} du \, \hat{\bar{q}}^+ (D^{++} + iV^{++}) q^+ .
\]
(A.16)
It is not difficult to check, for example, that the free FS hypermultiplet equations of motion, \( D^{++} q^+ = 0 \), imply \( q^+ = q^+ (Z) u^+_1 \) and the (on-shell) Fayet-Sohnius constraints (A.2) in the conventional N=2 superspace,
\[
D^{(i)}_\alpha q^j (Z) = D^{(i)}_\alpha (Z) = 0 .
\]
(A.17)
Similarly, a free HSS action of the HST hypermultiplet is given by

\[ S[\omega] = -\frac{1}{2} \int d\zeta (-4) du (D^{++} \omega)^2, \quad (A.18) \]

and it is equivalent (dual) to the standard N=2 tensor (linear) multiplet action \[12\].

The constraints (A.1) defining the N=2 super-Yang-Mills theory in the conventional N=2 superspace imply the existence of a (covariantly) chiral \[12\] and gauge-covariant N=2 SYM field strength \( W \) satisfying the reality condition (or the Bianchi ‘identity’)

\[ \mathcal{D}^\alpha (i \mathcal{D}_j) \omega = \bar{\mathcal{D}}_\alpha (i \mathcal{D}_j) \bar{\omega}, \quad (A.19) \]

which is a consequence of eq. (A.1).

An N=2 supersymmetric solution to the non-abelian N=2 SYM constraints (A.1) in the ordinary N=2 superspace is not known in an analytic form (see, however, ref. \[69\] for partial results). It is the N=2 HSS reformulation of the N=2 SYM theory that makes it possible. An exact non-abelian relation between the constrained, harmonic-independent superfield strength \( W \) and the unconstrained analytic (harmonic-dependent) superfield \( V^{++} \) is given in refs. \[20\], and it is highly non-linear and complicated. The abelian relation is simple, and it is given by

\[ W = \{ \mathcal{D}^+_{\alpha} , \mathcal{D}^0 \} = -(\bar{\mathcal{D}}^+)^2 A^{--} , \quad (A.20) \]

where the non-analytic harmonic superfield connection \( A^{--}(Z,u) \) to the derivative \( D^{--} \) has been introduced, \( \mathcal{D}^{--} = D^{--} + iA^{--} \).

As a consequence of the N=2 HSS abelian constraint \([\mathcal{D}^{++}, \mathcal{D}^{--}] = \mathcal{D}^0 = D^0\), the connection \( A^{--} \) satisfies the relation

\[ D^{++} A^{--} = D^{--} V^{++} , \quad (A.21) \]

whereas eq. (A.19) can be rewritten to the form

\[ (D^+)^2 W = (\bar{D}^+)^2 \bar{W}. \quad (A.22) \]

A solution to the \( A^{--} \) in terms of the analytic unconstrained superfield \( V^{++} \) easily follows from eq. (A.21) when using the identity \[21\]

\[ D_{1}^{++}(u_1^+ u_2^+)^{-2} = D_{1}^{--} \delta^{(2, -2)}(u_1, u_2) , \quad (A.23) \]

\[ ^{12}\text{A covariantly-chiral superfield can be transformed into a chiral superfield by field redefinition.} \]
where we have introduced the harmonic delta-function $\delta^{(2,-2)}(u_1, u_2)$ and the harmonic distribution $(u_1^+ u_2^+)^{-2}$ according to their definitions in ref. [20], hopefully, in the self-explaining notation. One finds [40]

$$A^{--}(z, u) = \int dv \frac{V^{++}(z, v)}{(u^+ v^+)^2},$$

(A.24)

and

$$W(z) = -\int du (\bar D^--)^2 V^{++}(z, u) , \quad \bar W(z) = -\int du (D^--)^2 V^{++}(z, u) , \quad \text{(A.25)}$$

by using an identity

$$u_i^+ = v_i^+(v^- u^+) - v_i^-(u^+ v^+) , \quad \text{(A.26)}$$

which is the obvious consequence of the definitions (A.5).

The free equations of motion of an N=2 vector multiplet are given by the vanishing analytic superfield

$$(D^+)^4 A^{--}(Z, u) = 0 , \quad \text{(A.27)}$$

while the corresponding action reads [40]

$$S[V] = -\frac{1}{2e^2} \int d^4x d^2\theta W^2 = -\frac{1}{2e^2} \int d^4x d^4\bar\theta d^4\bar\theta du V^{++}(Z, u) A^{--}(Z, u)$$

$$= -\frac{1}{2e^2} \int d^4x d^4\bar\theta d^4\bar\theta du_1 du_2 \frac{V^{++}(Z, u_1) V^{++}(Z, u_2)}{(u_1^+ u_2^+)^2} ,$$

(A.28)

where we have introduced an electromagnetic coupling constant $e$.

In a WZ-like gauge, the abelian analytic HSS prepotential $V^{++}$ amounts to the following explicit expression [20]:

$$V^{++}(x_A, \theta^+, \bar\theta^+, u) = \bar\theta^+ \theta^+ a(x_A) + \bar a(x_A) \theta^+ \theta^+ - 2i\theta^+ \sigma^\mu \bar\theta^+ V_\mu(x_A)$$

$$+ \bar\theta^+ \theta^+ \omega^i(\bar a(x_A)u_i^- + \theta^+ \theta^+ \bar\psi^i(\bar a(x_A)u_i^-)$$

$$+ \theta^+ \theta^+ \bar\theta^+ D^{(ij)}(x_A) u_i^- u_j^- ,$$

(A.29)

where $(a, \psi^i, V_\mu, D^{ij})$ are the usual N=2 vector multiplet components (see Appendix B for more details).

A hypermultiplet (BPS) mass can only come from the central charges in the N=2 supersymmetry algebra since, otherwise, the number of the massive hypermultiplet components has to be increased. The most natural way to introduce central charges $(Z, \bar Z)$ is to identify them with spontaneously broken $U(1)$ generators of dimensional reduction from six dimensions via the Scherk-Schwarz mechanism [27]. Indeed, after
being written down in six dimensions, eq. (A.14) implies an additional ‘connection’ term in the associated four-dimensional harmonic derivative,

\[ D^{++} = D^{++} + v^{++}, \quad \text{where} \quad v^{++} = i(\theta^+ \theta^+) \bar{Z} + i(\bar{\theta}^+ \bar{\theta}^+) Z. \quad (A.30) \]

Comparing eq. (A.30) with eqs. (A.16), (A.25) and (A.29) clearly shows that the N=2 central charges can be equivalently treated as the abelian N=2 vector superfield background with the covariantly constant chiral superfield strength \[ \langle W \rangle = \langle a \rangle = Z. \quad (A.31) \]

It is also worth mentioning here that introducing central charges into the algebra (A.1) of the N=2 superspace covariant derivatives implies corresponding charges in a similar N=2 supersymmetry algebra and, hence, in the N=2 supersymmetry transformation laws of N=2 superfields and their components. The HSS formalism automatically incorporates these changes via simple modifications of the HSS covariant derivatives. Non-vanishing N=2 central charges also break the rigid R-symmetry \[ \theta^i_{\alpha} \rightarrow e^{-i\gamma} \theta^i_{\alpha}, \quad \bar{\theta}^{\dot{a}i}_{\dot{\alpha}} \rightarrow e^{+i\gamma} \bar{\theta}^{\dot{a}i}_{\dot{\alpha}}, \quad (A.32) \]
of a massless N=2 supersymmetric field theory. This fact alone is responsible for a presence of anomalous (holomorphic) terms in the N=2 gauge low-energy effective action [38, 39].

**Appendix B: N=2 restricted chiral superfield**

In this Appendix we collect the known facts about components of the restricted N=2 chiral superfield [67] describing the abelian N=2 vector gauge superfield strength \( W \) in the main text. It simultaneously defines the rest of our notation.

The convenient realization of the supercovariant derivatives in the ordinary N=2 superspace (with vanishing central charge) is given by

\[ D^i_{\alpha} = \frac{\partial}{\partial \theta^i_{\alpha}} + i \theta^{\dot{a}i}_{\dot{\alpha}} \partial^{\dot{a}} \cdot, \quad \bar{D}^{\dot{a}i}_{\dot{\alpha}} = - \frac{\partial}{\partial \bar{\theta}^{\dot{a}i}_{\dot{\alpha}}} - i \bar{\theta}^{\dot{a}i}_{\dot{\alpha}} \partial_{\dot{a}} \cdot, \quad (B.1) \]

where we have used the standard two-component spinor notation, \( \partial_{\alpha} \cdot = \sigma^{\mu}_{\alpha} \partial_{\mu} \).

The restricted N=2 chiral superfield \( W \) is an off-shell irreducible N=2 superfield satisfying the constraints

\[ \bar{D}^{\dot{a}i}_{\dot{\alpha}} W = 0, \quad D^i W = \Box W. \quad (B.2) \]
The first constraint of eq. (B.2) is an N=2 generalization of the usual N=1 chirality condition, whereas the second one can be considered as a generalized reality condition \[67\] having no analogue in N=1 superspace. A solution to eq. (B.2) in N=2 chiral superspace \((y^\mu, \theta_i^\alpha)\) reads

\[
W(y, \theta) = a(y) + \theta_i^\alpha \psi_i^\alpha(y) - \frac{1}{2} \theta_i^\alpha \bar{\tau}^i_j \theta_j^\alpha \cdot \bar{D}(y) + \frac{i}{8} \theta_i^\alpha (\sigma_{\mu\nu})^\alpha_\beta \theta_j^\beta F_{\mu\nu}(y) - \frac{i}{2} \theta_i^\alpha \bar{\tau}^i_j \theta_j^\alpha \cdot \bar{D}(y) + \theta^4 \bar{a}(y) ,
\]

where we have introduced a complex scalar \(a\), a chiral spinor doublet \(\psi\), a real isovector \(\bar{D} = \frac{1}{2}(\bar{\tau}^i_j) D^i_j \equiv \frac{1}{2} \text{tr}(\bar{\tau} D)\), \(\text{tr}(\tau^m \tau^n) = 2\delta_{mn}\), and a real antisymmetric tensor \(F_{\mu\nu}\) as the field components of \(W\), while \(F_{\mu\nu}\) has to satisfy a spacetime constraint \[67\]

\[
\epsilon^{\mu\nu\lambda\rho} \partial_\nu F_{\lambda\rho} = 0 .
\]

Eq. (B.4) can be interpreted as the `Bianchi identity’ whose solution is given by

\[
F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu
\]

in terms of a vector gauge field \(V_\mu\) subject to gauge transformations \(\delta V_\mu = \partial_\mu \lambda\).

N=2 supersymmetry transformation laws for the components are easily obtained by imposing the scalar transformation law on the N=2 restricted chiral superfield (B.3) under chiral N=2 supertranslations

\[
\delta y^\mu = -i(\theta_i \sigma_{\mu i} \bar{\varepsilon}) , \quad \delta \theta^\alpha_i = \varepsilon_i^\alpha .
\]

One easily finds \[67\]

\[
\delta a = \varepsilon_i^\alpha \psi_i^\alpha ,
\]

\[
\delta \psi_i^\alpha = - \bar{\tau}^i_j \cdot \bar{D} \varepsilon_j^\alpha - i \partial_{\alpha \beta} a \varepsilon_i^\beta + \frac{i}{4} (\sigma_{\mu\nu} \varepsilon^\alpha ) a F_{\mu\nu} ,
\]

\[
\delta \bar{D} = - \frac{i}{2} (\varepsilon_{\alpha i} \bar{\tau}^i_j \partial_{\alpha \beta} \bar{\psi}_j^\beta ) + \text{h.c.} ,
\]

\[
\delta F_{\mu\nu} = - \partial_\mu(\varepsilon_i^\alpha \sigma_{\mu a} \bar{\psi}_i^\alpha ) + \partial_\nu(\varepsilon_i^\alpha \sigma_{\nu a} \bar{\psi}_i^\alpha ) + \text{h.c.}
\]

A free N=2 supersymmetric Maxwell Lagrangian is given by

\[
-\frac{1}{2e^2} \int d^4 \theta W^2 = \frac{1}{e^2} \lvert \partial_\mu a \rvert^2 + \frac{i}{4e^2} \bar{\psi}_i^\alpha \partial_{\alpha \beta} \bar{\psi}_j^\beta - \frac{1}{4e^2} F_{\mu\nu}^2 + \frac{1}{2e^2} \bar{D}^2 .
\]

Given the constraints (B.2), it is not difficult to verify that the N=2 superfield

\[
L^{ij} \equiv D^{ij} W
\]

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satisfies

\[ D_\alpha^{(i} L^{jk)} = D_\alpha^{(i} L^{jk)} = 0 , \quad (B.10) \]

and it is subject (up to a constant – see below) to the reality condition

\[ (L^{ij})^\dagger = \varepsilon_{ik} \varepsilon_{jl} L^{kl} , \quad \text{or, equivalently,} \quad \vec{L}^\dagger = \vec{L} . \quad (B.11) \]

Eqs. (B.10) and (B.11) are known as the defining constraints of an N=2 tensor multiplet superfield \( L^{ij} \) [29, 67]. The constraint (B.10) on \( W \) via eq. (B.9) is often taken as the substitute to the generalized reality condition in eq. (B.2) — see, for example, eqs. (A.19) and (A.22). However, it merely follows from eqs. (B.2) and (B.9) that \( \Box (\vec{L} - \vec{L}^\dagger) = 0 \) which generically implies that the harmonic function \( \text{Im} \vec{L} \) is a constant, \( \text{Im} \vec{L} = \vec{M} = \text{const} \). This constant \( \vec{M} \) then enters N=2 supersymmetry transformation laws, modifies the abelian constraints (A.19) as

\[ D^{ij} W - \bar{D}^{ij} \bar{W} = 4i M^{ij} , \quad (B.12) \]

and it can be interpreted as a ‘magnetic’ Fayet-Iliopoulos term [70]. It is not clear to me, however, how the magnetic FI term could be introduced into the theory (3.8) since the N=2 vector gauge superfield strength \( W \) defined by eq. (A.25) automatically satisfies eq. (B.12) with \( M^{ij} = 0 \).

\[ ^{13}\text{We assume that all the superfield components of } W \text{ are regular in spacetime.} \]

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