Stable/unstable slow integral manifolds in critical cases

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Abstract. The paper deals with the problem of a construction of global stable/unstable slow
integral manifolds of the singularly perturbed systems in critical cases. In addition to the
well-known critical cases a novel scenario of the stability change of the slow integral manifold
is considered. All three critical cases leading to the change of the stability are discussed via
the Hindmarsh-Rose dynamic model. It is shown that the suitable choice of the additional
parameters of the system yields the slow integral manifold with multiple change of its stability.

1. Introduction

The usual assumption of the singular perturbation theory is based on the fact, that the spectrum
of the main functional matrix of the fast subsystem lies outside the imaginary axis. However, in
many applications this condition is violated that implies the critical phenomena. A violation of
this condition can lead to a delaying effect of loss of stability. Phenomenon of the delay of the
loss of stability is based on the fact that the actual escape of the phase point from the position
of equilibrium, which lost its stability, does not occur immediately. Two scenarios for delaying
of the loss of stability are well-known [1, 2].

The first scenario is connected to the critical case, when one real eigenvalue of the linearized
fast subsystem passes through zero under a changing of the slow variables. This scenario is
associated with the canards or duck-trajectories [3]–[17]. The second scenario one can observe
in the critical case, when a pair of complex conjugate eigenvalues passes from the left open
complex half-plane to the right one [18, 19, 20].

In present paper the third critical case is considered, when the real parts as well as the
imaginary parts of a pair of complex conjugate eigenvalues become zero with the appearance of
multiple zero root and then by the birth of a pair of real eigenvalues of opposite signs. All three
scenarios of change of stability are discussed via the Hindmarsh-Rose model of nerve conduction
[21].

2. Loss of stability of integral manifolds of slow-fast systems

Consider the autonomous singularly perturbed system

\[
\frac{dx}{dt} = f(x, y, \mu, \varepsilon),
\]  

(1)
\[
\frac{d\varepsilon y}{dt} = g(x, y, \mu, \varepsilon),
\]

(2)

where \(x\) and \(y\) are vectors in Euclidean spaces, \(\varepsilon\) is a small positive parameter, \(\mu\) is a vector of parameters, vector-functions \(f\) and \(g\) are sufficiently smooth and their values are comparable to unity. The slow and fast subsystems are described by (1) and (2), respectively.

In many application it is necessary to consider the behaviour of the system as a whole rather than separate trajectories, investigating the system dynamics by methods of the integral manifolds theory (see, for example, [8, 10, 16] and references therein). Especially this approach is effective for analysis of dynamic models with unfixed initial (boundary) conditions and parameters.

**Definition 1** A smooth surface \(S_\varepsilon\) is called an integral manifold of the system (1), (2) if any trajectory of the system that has at least one point in common with \(S_\varepsilon\) lies entirely on \(S_\varepsilon\).

Among the integral manifolds we distinguish the invariant surfaces of slow motions whose dimension is equal to that of the slow subsystem, the so-called slow integral manifolds. The stability or instability of the slow integral manifold is defined by the stability or instability of its zero-order approximation (\(\varepsilon = 0\)), the so-called slow surface.

**Definition 2** The surface \(S\) described by the equation

\[
g(x, y, \mu, 0) = 0
\]

(3)
is called a slow surface. When the dimension of this surface is equal to one, it is called a slow curve.

Let a vector-function \(y = \varphi(x, \mu)\) is an isolated root of the equation (3).

**Definition 3** The subset of \(S\) is stable (or attractive) if the spectrum of the Jacobian matrix

\[
J = \frac{\partial g}{\partial y}(x, \varphi(x, \mu), \mu, 0)
\]

(4)
is located in the left open complex half-plane. If there is at least one eigenvalue of the Jacobian matrix (4) with a positive real part then the subset of the slow surface is unstable (or repulsive).

As noted above the slow surface can be considered as a zero-order approximation of the slow integral manifold, hence, in an \(\varepsilon\)-neighborhood of a stable (unstable) subset of the slow surface there exists a stable (unstable) slow integral manifold.

A slow integral manifold can change its stability in some specific cases. Some critical cases leading to the stability change is considered below. Introduction of the additional conditions allows us to construct of the stable/unstable slow integral manifolds in these critical cases.

2.1. The case of a zero root

Consider the system (1), (2) with scalar parameter \(\mu\), for which an equilibrium of the fast equation (2) becomes unstable with transition of one real eigenvalue of (4) through zero when the slow variables are changed. This means that the slow integral manifold of the system loses its stability when the slow variables reach a so-called breakdown surface [10, 16].

**Definition 4** The subset of \(S\) given by

\[
\det \left| \frac{\partial g}{\partial y}(x, y, \mu, 0) \right| = 0
\]

(5)
determines a breakdown surface (breakdown, turning or jump points in the scalar case).
The presence of the additional scalar parameter in the differential system provides the possibility of gluing the stable and unstable slow integral manifolds at one point of the breakdown surface to form a single trajectory, the canard.

**Definition 5** A canard is a trajectory of a singularly perturbed system of differential equations if it follows at first a stable integral manifold, and then an unstable one. In both cases the length of the trajectory is more than infinitesimally small.

The technique of a canard construction by a gluing the stable and unstable slow integral manifolds at one point of the breakdown surface was first proposed in [4, 5]. The mathematical justification of this approach for the case when the slow integral manifolds of the system (1), (2) can be represented as $y = h(x, \mu, \varepsilon) = \varphi(x, \mu) + O(\varepsilon)$, $\varepsilon \to 0$, where $h(x, \mu, \varepsilon)$ is a sufficiently smooth function of $\varepsilon$, and $x$ and $y$ are scalar, one can find in [7, 17].

The canards and corresponding values of the parameter $\mu$ allow asymptotic expansions in powers of the small parameter $\varepsilon$. Near the slow curve the canards are exponentially close, and have the same asymptotic expansion in powers of $\varepsilon$. An analogous assertion is true for corresponding parameter values. Namely, any two values of the parameter $\mu$ for which canards exist have the same asymptotic expansions, and the difference between them is given by $\exp(-1/c\varepsilon)$, where $c$ is some positive number. In this sight we can state the uniqueness for the canard (and the corresponding parameter value) of a planar system.

But in the case dim $x \geq 2$ the situation is essentially another: if the differential system has a canard then it has one-parameter family of canards at once, and a choice of a value of an additional parameter $\mu$ means a selection of a point on the breakdown surface at which the stable and unstable integral manifolds are glued. The mathematical justification of this fact for the case dim $x \geq 2$, dim $y = 1$ one can find in [6, 10].

Consider a more general case, when both slow and fast variables are vectors. Let the autonomous system of ordinary differential equations with a small positive parameter $\varepsilon$ and an additional scalar parameter $\mu$ for the variables $x, y, z$ be reduced by the standard procedure of elimination of the independent variable $t$ to the nonautonomous form

$$
\frac{dy}{dx} = Y(x, y, z_1, z_2, \varepsilon), \ y \in \mathbb{R}^n, \ x \in \mathbb{R};
$$

$$
\varepsilon \frac{dz_1}{dx} = 2x z_1 + \mu + Z_1(x, y, z_1, z_2, \varepsilon), \ z_1 \in \mathbb{R}, \ |z_1| \leq r;
$$

$$
\varepsilon \frac{dz_2}{dx} = A(x) z_2 + \mu B + Z_2(x, y, z_1, z_2, \varepsilon), \ z_2 \in \mathbb{R}^n, \ |z_2| \leq r.
$$

Here $A(x)$ is a bounded matrix satisfying the Lipschitz condition, its eigenvalues $\lambda_i(x)$ satisfy the condition

$$
\text{Re} \lambda_i(x) \leq -2\beta < 0 \ (i = 1, 2, \ldots, m),
$$

$\mu$ is a constant satisfying

$$
|\mu| \leq \varepsilon^2 K,
$$

$B$ is a constant vector, and $Y, Z_1,$ and $Z_2$ are continuous functions satisfying the inequalities

$$
\|Y(x, y, z_1, z_2, \varepsilon)\| \leq N, \quad (10)
$$

$$
|Z_1(x, y, z_1, z_2, \varepsilon)| \leq M (\varepsilon^2 + \varepsilon \|z\| + \|z\|^2), \quad (11)
$$

$$
|Z_2(x, y, z_1, z_2, \varepsilon)| \leq M (\varepsilon^2 + \varepsilon \|z\| + \|z\|^2), \quad (12)
$$

$$
\|Y(x, y, z_1, z_2, \varepsilon) - Y(x, \bar{y}, \bar{z}_1, \bar{z}_2, \varepsilon)\| \leq M (\|y - \bar{y}\| + \|z - \bar{z}\|), \quad (13)
$$
It continues to move along the unstable pair of eigenvalues through the imaginary axis when the slow variables are changed.

Theorem 1 Let the conditions (9)–(15) be satisfied. Then there are numbers \( \varepsilon_0 > 0 \) and \( q, \delta \) such that, for all \( \varepsilon \in (0, \varepsilon_0) \), there exist \( \mu = \mu^*(\varepsilon) \) and a canard corresponding to \( \mu^*(\varepsilon) \), which passes through the point with \( x = 0, y = y^* \) on the slow integral manifold.

Thus, a canard can be defined as an one-dimensional slow integral manifold of variable stability, which arise due to special choice of an additional parameter [8, 10].

2.2. The case of a pair of purely imaginary roots

Consider the analytical slow-fast system, obtaining from the system (1), (2) by the time scaling transformation,

\[
\frac{dx}{d\tau} = \varepsilon f(x, y, \mu, \varepsilon),
\]

\[
\frac{dy}{d\tau} = g(x, y, \mu, \varepsilon),
\]

for which a singular point of the equation of fast motions becomes unstable with transition of a pair of eigenvalues through the imaginary axis when the slow variables are changed.

As before, the slow integral manifold changes its stability, but unlike the previous case, the stability boundary is not the breakdown surface (curve or point) because the condition (5) is not fulfilled. This means that the trajectory of the system (16), (17) does not jump from the slow integral manifold immediately as in the previous case, it continues to move along the unstable
slow surface for a time of order $O(\varepsilon^{-1})$ after crossing the stability border. And this slow path of the trajectory along the unstable part of the slow surface has a distance of order $O(1)$ as $\varepsilon \to 0$. Only then the trajectory can jump from the slow integral manifold and the transition to the fast movement occurs. This delay phenomenon of loss of stability was firstly investigated via an example in [18] and in the general case was considered in [19].

Note that the trajectories described above behave like canards. However there are some difference between these two phenomena. Canards exist in systems with finite smoothness, while the delays considered above occur only in analytic systems. Further, canards are rare and they exist for an exponentially small interval of values of an additional parameter, whereas for a delay phenomenon of loss of stability it is not necessary to select parameters. In the case of delay a function describing the slow integral manifold has a jump discontinuity while in the canard’s case it has an infinite discontinuity. If we want to eliminate the trajectory jump from the slow unstable manifold by a constructing of the global stable/unstable slow integral manifold, we need to remove this discontinuity via a selection of a pair of additional functions.

Consider the slow-fast system autonomous system for the variables $x$, $y$, and $z$, which is reduced by the standard procedure of elimination of the independent variable $t$ to the nonautonomous form

$$\frac{dy}{dx} = \varepsilon Y(x, y, z, \varepsilon), \quad y \in \mathbb{R}^n, \quad x \in \mathbb{R};$$

$$\frac{dz}{dx} = A(x)z + \mu(y, \varepsilon) + Z(x, y, z, \mu(y, \varepsilon), \varepsilon), \quad z \in \mathbb{R}^2, \quad \|z\| \leq r;$$

where

$$A(x) = \begin{pmatrix} 2x & \nu \\ -\nu & 2x \end{pmatrix},$$

and vector-functions

$$\mu(y, \varepsilon) = \begin{pmatrix} \mu_1(y, \varepsilon) \\ \mu_2(y, \varepsilon) \end{pmatrix}, \quad Z(x, y, z, \mu(y, \varepsilon), \varepsilon) = \begin{pmatrix} Z_1(x, y, z, \mu_1(y, \varepsilon), \mu_2(y, \varepsilon), \varepsilon) \\ Z_2(x, y, z, \mu_1(y, \varepsilon), \mu_2(y, \varepsilon), \varepsilon) \end{pmatrix}$$

are continuous and satisfy the following conditions

$$\|Y(x, y, z, \varepsilon)\| \leq N,$$

$$\|Z(x, y, z, \mu(y, \varepsilon), \varepsilon)\| \leq M (\varepsilon + \varepsilon\|z\| + \|z\|^2),$$

$$\|Y(x, y, z, \varepsilon) - Y(x, \tilde{y}, \tilde{z}, \varepsilon)\| \leq M (\|y - \tilde{y}\| + \|z - \tilde{z}\|).$$

$$\|Z(x, y, z, \mu(y, \varepsilon), \varepsilon) - Z(x, \tilde{y}, \tilde{z}, \mu(\tilde{y}, \varepsilon), \varepsilon)\| \leq M (\varepsilon + \varepsilon\|\tilde{z}\| + \|\tilde{z}\|^2)(\|y - \tilde{y}\| + \varepsilon\|\mu - \tilde{\mu}\|),$$

$$\|\mu(y, \varepsilon)\| \leq \varepsilon K, \quad \|\mu(y, \varepsilon) - \mu(\tilde{y}, \varepsilon)\| \leq \varepsilon L\|y - \tilde{y}\|,$$

where $M$, $N$, $K$, and $L$ are positive constants.

The following theorem holds [22, 23].

**Theorem 2** Let for (18), (19) the conditions (20)–(24) be satisfied. Then there are numbers $\varepsilon_0 > 0$ and $q, \delta$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, there exist a function $\mu = \mu^*(y, \varepsilon)$ and the stable/unstable slow integral manifold $z = h(x, y, \varepsilon)$ corresponding to $\mu^*(y, \varepsilon)$ and satisfying the inequalities

$$\|h(x, y, \varepsilon)\| \leq q, \quad \|h(x, y, \varepsilon) - h(x, \tilde{y}, \varepsilon)\| \leq \varepsilon\delta\|y - \tilde{y}\|.$$

Note that in the case of delayed stability loss considered in [18, 19, 20] the gluing of the stable and unstable parts of the one-dimensional slow integral manifold for the case $\dim y = 0$ requires two additional parameters. But if $\dim y \geq 1$ then a pair of functions is needed to construct the stable/unstable slow integral manifold.
2.3. The case of a multiple zero root

Consider a situation when the real parts as well as the imaginary parts of a pair of complex conjugate eigenvalues of the Jacobian matrix (4) become zero simultaneously. In this case a stable focus of the fast equation (2) can transform into a saddle. It should be noted that this bifurcation is among the non-robust. In such situation a canards-like trajectory can arise.

Let us consider this phenomenon via an example, proposed by V.A. Sobolev:

\[
\dot{x} = -1, \quad \varepsilon \dot{y} = z, \quad \varepsilon \dot{z} = -x(y + z) + f(x) + \mu, \quad (25)
\]

where \( f(x) = \alpha_0 x^3 + \alpha_1 x^2 + \alpha_2 x + \alpha_3 \), and \( \alpha_i \) \((i = 0, \ldots, 3)\) are constant.

The Jacobian matrix (4) of the system (25)

\[
J(x) = \begin{pmatrix} 0 & 1 \\ -x & -x \end{pmatrix},
\]

has the eigenvalues \( \lambda_{1,2} = -x \pm \sqrt{x^2 - 4x} \),

and for \( x \in (0, 4) \) the equilibrium of the fast subsystem of (25) is a stable focus, for \( x > 4 \) it is a stable node, and for \( x < 0 \) it is a saddle. Thus, for \( x > 0 \) the slow integral manifold of (25) is stable, while for \( x < 0 \) it is unstable.

For \( \mu = -\alpha_3 + \varepsilon^2 2\alpha_0 \) the system (25) has a canard-like trajectory

\[
\begin{align*}
  y &= \alpha_0 x^2 + \alpha_1 x + \alpha_2, \\
  z &= -\varepsilon(2\alpha_0 x + \alpha_1),
\end{align*}
\]

(26)

where \( \alpha_0 = \alpha_0, \quad \alpha_1 = \alpha_1 + \varepsilon 2\alpha_0, \quad \alpha_2 = \alpha_2 + \varepsilon \alpha_1 + \varepsilon^2 2\alpha_0. \)

This trajectory, corresponding to the exact solution of the system (25), is the global stable/unstable slow integral manifold. All other trajectories, starting from an initial point in the basin of attraction of the stable part, follows it and continues their movement along the unstable part of the slow integral manifold for a distance of order \( O(1) \) as \( \varepsilon \to 0 \), see Figure 1.

![Figure 1](image.png)

This result can be generalized.
Theorem 3 For the system (25) with

\[ f(x) = \alpha_0 x^k + \alpha_1 x^{k-1} + \ldots + \alpha_{k-1} x + \alpha_k, \]

where \( \alpha_i, (i = 0, \ldots, k) \) and \( k \) are constant, there is a number \( \varepsilon_0 > 0 \) such that, for all \( \varepsilon \in (0, \varepsilon_0) \), there exist \( \mu = \mu^*(\varepsilon) \) and the exact stable/unstable slow integral manifold

\[
\begin{cases}
  y = a_0 x^{k-1} + a_1 x^{k-2} + \ldots + a_{k-2} x + a_{k-1}, \\
  z = -\varepsilon (a_0 (k-1) x^{k-2} + a_1 (k-2) x^{k-3} + \ldots + 2a_{k-3} x + a_{k-2}),
\end{cases}
\]

corresponding to \( \mu^*(\varepsilon) \), where

\[
\begin{align*}
  \mu^*(\varepsilon) &= -\alpha_k + \varepsilon^2 2\alpha_{k-3}, \\
  a_0 &= \alpha_0, \\
  a_1 &= \alpha_1 + \varepsilon (k-1) \alpha_0, \\
  a_2 &= \alpha_2 + \varepsilon (k-2) \alpha_1, \\
  a_3 &= \alpha_3 + \varepsilon (k-3) \alpha_2 - \varepsilon^2 (k-1)(k-2) \alpha_0, \\
  a_4 &= \alpha_4 + \varepsilon (k-4) \alpha_3 - \varepsilon^2 (k-2)(k-3) \alpha_1, \\
  &\vdots \\
  a_{k-1} &= \alpha_{k-1} + \varepsilon \alpha_{k-2} - \varepsilon^2 6 \alpha_{k-4}.
\end{align*}
\]

3. Loss of stability of integral manifolds in the Hindmarsh-Rose model

The considered above scenarios of change of stability of the slow integral manifold occur in the Hindmarsh-Rose model. The model describes the basic properties of individual neurons, the generation of spikes and a constant level potential. In this model, Kirchhoff’s law is written for each ionic currents flowing through the cell membrane. The dimensionless form of the Hindmarsh-Rose model is

\[
\begin{align*}
  \varepsilon \dot{x} &= y - ax^3 + bx^2 - z + I, \\
  \varepsilon \dot{y} &= c - dx^2 - y, \\
  \dot{z} &= s(x - \alpha) - z,
\end{align*}
\]

where \( x \) is a transmembrane neuron potential, \( y \) and \( z \) are the characteristics of ionic currents dynamic, \( I \) is ambient current, other parameters reflect the physical features of the neurons. The typical values of the positive parameter are small [24].

The slow curve of system (27)–(29) is described by:

\[
\begin{cases}
  y - ax^3 + bx^2 - z + I = 0, \\
  c - dx^2 - y = 0.
\end{cases}
\]

The graph of the slow curve is shown on Figure 2. From (30) we get

\[ ax^3 + (d - b)x^2 + z - I - c = 0, \]

which allow us to investigate a projection of the slow curve on the XOZ-plane, see Figure 3.

The Jacobian matrix (4) for the system (27)–(29)

\[
J = \begin{pmatrix} -3ax^2 + 2bx & 1 \\ -2dx & -1 \end{pmatrix}
\]

has a characteristic equation

\[
\lambda^2 + \lambda (1 + 3ax^2 - 2bx) + 3ax^2 + 2(d - b)x = 0.
\]
Figure 2. The slow curve (30) of the system (27)–(29); \( a = 1, b = 3, c = 1, d = 5, I = 2.7 \)

Figure 3. The projection of the slow curve and the points of changing of stability on the XOZ–plane. The parameters’ values are the same as for Figure 2

The necessary condition of stability (which is also the sufficient in this case) for the polynomial (31) is

\[
\begin{align*}
3ax^2 - 2bx + 1 &> 0, \\
3ax^2 - 2bx + 2dx &> 0.
\end{align*}
\tag{32}
\]

From (32) we can determine the abscissas of the points of an expected change of stability of the slow curve:

\[
x_1 = \frac{2(b - d)}{3a}, \quad x_2 = 0, \quad x_{3,4} = \frac{b \pm \sqrt{b^2 - 3a}}{3a}.
\]

These points divide the slow curve into several parts, see Figure 3.

We check the sign of real part of eigenvalues of the matrix \( J \) for all parts to find out whether the region is stable or unstable. One of the two real eigenvalues of the Jacobian matrix of the fast system changes its sign at the points \( A_1 \) and \( A_2 \), and at these points the slow curve changes its stability. Thus, \( A_1 \) and \( A_2 \) are the jump points. The proper choice of an additional parameter of the system allows us to glue the stable and the unstable integral manifolds of the system that exist in the \( \varepsilon \)-neighborhood of the stable and unstable regions of the slow curve. As
a result of this gluing we obtain a canard [25]. However, there are other points of the stability’s change (see $B_1$ and $B_2$ on Figure 3), which are not jump points, because the trajectory does not immediately escape the slow integral manifold as soon as it reaches these points, compared to the previous case. One can find them by equating the real parts of the complex conjugate eigenvalues of $J$ to zero, i.e., from

\[
\begin{align*}
3ax^2 - 2bx + 1 &= 0, \\
3ax^2 - 2bx + 2dx &= 0.
\end{align*}
\]

The phenomenon of the delay of loss of stability occurs as the trajectory goes through these points.

In the case under consideration there are points $C_1$ (between points $A_2$ and $B_1$) and $C_2$ (to the right from the point $B_2$) on the slow curve, at which the pair of real eigenvalues of the Jacobian matrix $J$ becomes the pair of the complex conjugate eigenvalues.

It is possible to find the relations between parameters values for which the real and the imaginary parts of the pair of complex conjugate eigenvalues become equal to zero simultaneously, i.e.,

\[
\begin{align*}
3ax^2 - 2bx + 1 &= 0, \\
3ax^2 - 2bx + 2dx &= 0,
\end{align*}
\]

and a multiple zero root arises with the following emergence of the pair of real eigenvalues with the opposite sign. From (33) we obtain

\[
d = \frac{3a}{2 \left( b + \sqrt{b^2 - 3a} \right)},
\]

which corresponds to the case when the points $A_2$, $B_2$, and $C_1$ coincide, and the equilibrium of the fast subsystem of (27)–(29) switches from a stable focus to a saddle with decreasing $x$.

For (34) and the proper choice of the values of other parameters it is possible to construct the global slow integral manifold with multiple change of stability: we need one parameter, say $s$, to glue the stable and unstable slow integral manifolds at the point $A_1$ using the techniques of canards (see Figures 4 and 5), and one more, say $a$ or $I$, for gluing integral manifolds at the point $A_2$ with help the method described in section 2.3.

As result of these gluing procedures we get the slow integral manifold with multiple change of stability (see Figure 6) that looks like a canard cascade [16, 17, 26]. The difference between these two objects consists in that for a canard cascade we apply the canard technique only.

4. Conclusion

In this paper we outlined the approach for construction the stable/unstable slow integral manifolds of the slow-fast systems of ODE in the critical cases. Novel critical case, in addition to the previously studied cases, leading to the stability change of the slow integral manifold was investigated.

As an illustration of this approach the Hindmarsh-Rose model has been considered, in which all three critical cases of stability change occur. The crucial result of present investigation is that in this dynamic model it is possible to construct the slow integral manifold with multiple change of its stability.

It is important to emphasize that in the present paper we deal with the one-dimensional stable/unstable slow integral manifolds, not necessarily the canards, because these integral manifolds can consist of several trajectories at once (see, for for instance, example 8 in [10]).
Figure 4. The projections of the canard without head (red line) and the slow curve (blue line) of the system (27)–(29) under the condition (34) and $a = 1$, $b = 3$, $c = 1$, $\alpha = -1.2$, $I = 2.7$, $\varepsilon = 0.01$, $s = 3.0810445478558141214$

Figure 5. The projections of the canard with head (red line) and the slow curve (blue line) of the system (27)–(29) under the condition (34) and $a = 1$, $b = 3$, $c = 1$, $\alpha = -1.2$, $I = 2.7$, $\varepsilon = 0.01$, $s = 3.0810445478558141213$

Figure 6. The projections of the integral manifold with multiple change of stability (black line) and the slow curve (blue line) of the system (27)–(29)

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