 Finite-Range Electromagnetic Interaction and Magnetic Charges: Spacetime Algebra or Algebra of Physical Space?

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A new finite-range electromagnetic (EM) theory containing both electric and magnetic charges constructed using two vector potentials \( A^\mu \) and \( Z^\mu \) is formulated in the spacetime algebra (STA) and in the algebra of the three-dimensional physical space (APS) formalisms. Lorentz, local gauge and EM duality invariances are discussed in detail in the APS formalism. Moreover, considerations about signature and dimensionality of spacetime are discussed. Finally, the two formulations are compared. STA and APS are equally powerful in formulating our model, but the presence of a global commuting unit pseudoscalar in the APS formulation and the consequent possibility of providing a geometric interpretation for the imaginary unit employed throughout classical and quantum physics lead us to prefer the APS approach.

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I. INTRODUCTION

Applications of Geometric Algebra (GA) to Maxwell’s theory of electromagnetism are known [1], [2], [3], [4], [5]. The layout of the paper is as follows. In Section II a brief introduction to the Algebra of Physical Space (APS) formalism is presented. In Section III, we construct the classical field theory of electric and magnetic charges where the electromagnetic interaction is mediated by non-zero mass photons. The theory is constructed using two vector potentials \( A^\mu \) for the electric charges and \( Z^\mu \) for the magnetic charges. Then, we formulate the theory in the STA and APS languages. In section IV we discuss Lorentz covariance of the Maxwell-Proca-Dirac system described by a single APS nonhomogeneous multivectorial equation. The loss of local gauge invariance in the Maxwell-Proca system is discussed followed by a consideration of the EM duality invariance in the Maxwell-Dirac system. In section V, general considerations about the signature and dimensionality of spacetime are carried out. Finally, in section VI, the two formulations are compared and we conclude that the lack of a global commuting pseudoscalar in the Dirac algebra \( \mathfrak{d}(1, 3) \) is one of the main deficiencies of the algebra. Furthermore, since the Pauli algebra \( \mathfrak{d}(3) \) has the same computational power and compactness of \( \mathfrak{d}(1, 3) \), the presence of a global commuting unit pseudoscalar in \( \mathfrak{d}(3) \) leads to the preference of such algebra in the formulation of the extended Maxwell’s theory. Finally it is emphasized that the formal identification of the unit pseudoscalar \( i_{\mathfrak{d}(3)} \) with the complex scalar \( i_{\mathcal{C}} \) opens up the possibility of providing a geometric interpretation for the unit imaginary employed not only in our work, but throughout physics in general.

II. OUTLINE OF ALGEBRA OF PHYSICAL SPACE

The basic idea in geometric algebra (GA) is that of uniting the inner and outer products into a single product, namely the geometric product. This product is associative and has the crucial feature of being invertible. The geometric product between two three-dimensional vectors \( \vec{a} \) and \( \vec{b} \) is defined by

\[
\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{b} + \vec{a} \wedge \vec{b},
\]

where \( \vec{a} \cdot \vec{b} \) is a scalar (a 0-grade multivector), while \( \vec{a} \wedge \vec{b} = i(\vec{a} \times \vec{b}) \) is a bivector (a grade-2 multivector). The quantity \( i \) is the unit pseudoscalar defined in (5), it is not the unit imaginary number \( i_{\mathcal{C}} \) usually employed in physics. The three-dimensional Euclidean space \( \mathbb{R}^3 \) is the place where classical physics takes place. Multiplying and adding vectors generate a geometric algebra denoted \( \mathfrak{d}(3) \). The whole algebra can be generated by a right-handed set of orthonormal
vectors \( \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \) satisfying the relation,
\[
\vec{e}_i \vec{e}_m = \vec{e}_i \cdot \vec{e}_m + \vec{e}_i \wedge \vec{e}_m = \delta_{im} + \varepsilon_{lmk} \vec{e}_k
\]
where \( i \equiv i_{cl(3)} \) is the unit three-dimensional pseudoscalar defined in (5). Equation (2) displays the same algebraic relations as Pauli's \( \sigma \)-matrices. Indeed, the Pauli matrices constitute a representation of the Clifford algebra \( cl(3) \), also called the Pauli algebra. The linear space \( cl(3) \) has dimension eight,
\[
\dim \, cl(3) = \sum_{k=0,1,2,3} \dim \, cl^{(k)}(3) = \sum_{k=0,1,2,3} \binom{3}{k} = 2^3 = 8
\]
where \( cl^{(k)}(3) \) is the \( \binom{3}{k} \)-dimensional subspace of \( cl(3) \) spanned by the \( k \)-grade multivectors in the algebra. A basis set for \( cl(3) \) is given by,
\[
B_{cl(3)} = \{1; \vec{e}_1, \vec{e}_2, \vec{e}_3; \vec{e}_1 \vec{e}_2 \vec{e}_3, \vec{e}_3 \vec{e}_1 \vec{e}_2, \vec{e}_1 \vec{e}_2 \vec{e}_3\}.
\]
Therefore, the GA for physical space is generated by a scalar, three vectors, three bivectors (area elements) and a trivector (volume element). The frame \( \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \) generates a unique unit trivector, the unit pseudoscalar
\[
i_{cl(3)} \equiv \vec{e}_1 \vec{e}_2 \vec{e}_3. \tag{5}
\]
The pseudoscalar \( i_{cl(3)} \) represents an oriented unit volume. It satisfies the following important relations,
\[
i^2 = -1, \quad i\bar{M} = \bar{M}i \quad \forall \bar{M} \in Cl(3), \quad i^\dagger = -i
\]
where \( i \equiv i_{cl(3)} \) and the involution \( \dagger \) is called reversion or Hermitian adjoint and it will be properly defined later. Within the Pauli algebra the operation of reversion plays the role of complex conjugation in \( \mathbb{C} \). Properties (6) lead to consider the possibility of identifying the \( i_{cl(3)} \) with the complex imaginary unit \( i_{\mathbb{C}} \equiv \sqrt{-1} \in \mathbb{C} \). Indeed, the hope that \( ic \), which figures so prominently in quantum mechanics could be given a geometric interpretation may be one of the main theoretical motivations behind exploring the power of the geometric algebra language. A general multivector \( \bar{M} \in cl(3) \) can be expanded as,
\[
\bar{M} = \sum_{k=0,1,2,3} \langle \bar{M} \rangle_k = \langle \bar{M} \rangle_0 + \langle \bar{M} \rangle_1 + \langle \bar{M} \rangle_2 + \langle \bar{M} \rangle_3
\]
\[
= \alpha + \vec{a} + i\vec{b} + i\beta = \text{scalar + vector + bivector + trivector.} \tag{7}
\]
The quantities \( \alpha \) and \( \beta \) are real scalars while \( \vec{a} = \vec{a} \cdot \vec{e}_k \) and \( \vec{b} = \vec{b} \cdot \vec{e}_k \) are vectors. The quantity \( \langle \bar{M} \rangle_k \) is the grade-\( k \) multivectorial part of the nonhomogeneous multivector \( \bar{M} \in cl(3) \). Identifying the unit pseudoscalar of \( cl(3) \) with the imaginary unit of \( \mathbb{C} \), the decomposition of \( \bar{M} \) has the formal algebraic structure of a "complex scalar" \( \alpha + i\beta \) added to an "complex vector" \( \vec{a} + i\vec{b} \). This idea is behind Baylis’s paravector approach to the geometric algebra of physical space [6] and [7]. Thus, a generic element \( \bar{M} \) of the Pauli algebra \( cl(3) \) can be written as
\[
\bar{M} = \langle \bar{M} \rangle_{cs} + \langle \bar{M} \rangle_{cv} = [\langle \bar{M} \rangle_{rs} + \langle \bar{M} \rangle_{is}] + [\langle \bar{M} \rangle_{rv} + \langle \bar{M} \rangle_{iv}] = M^0 + \bar{M} \tag{8}
\]
where \( \langle \bar{M} \rangle_{cs} \) is the sum of real and imaginary scalar parts,
\[
\langle \bar{M} \rangle_{cs} \equiv M^0 = \langle \bar{M} \rangle_{rs} + \langle \bar{M} \rangle_{is} \tag{9}
\]
while \( \langle \bar{M} \rangle_{cv} \) can be decomposed in real and imaginary vector parts,
\[
\langle \bar{M} \rangle_{cv} = \bar{M} = \langle \bar{M} \rangle_{rv} + \langle \bar{M} \rangle_{iv}. \tag{10}
\]
In this paper two involutions will be used, the reversion or Hermitian adjoint \( \dagger \) and the spatial reverse or Clifford conjugate \( \check{} \). For an arbitrary element multivector \( \bar{M} = \alpha + \vec{a} + i\vec{b} + i\beta \), these involutions are defined as,
\[
\bar{M}^\dagger = \alpha + \vec{a} - i\vec{b} - i\beta \quad \text{and} \quad \bar{M}^\check{} = \alpha - \vec{a} - i\vec{b} + i\beta. \tag{11}
\]
In the rest of the paper we will use the following notation \( \bar{M}^\dagger \equiv \bar{M}^\dagger \). Useful identities are,
\[
\langle M \rangle_{rs} = \frac{1}{4} \left[ M + M^\dagger + M^\check{} + (M^\check{})^\dagger \right], \quad \langle M \rangle_{rv} = \frac{1}{4} \left[ M^\dagger + (M^\check{})^\dagger - M - M^\check{} \right] \tag{12}
\]
the vector potential $A$ photons in presence of both electric and magnetic charges. We extend Maxwell’s theory using two vector potentials, strongly coupled non-abelian gauge theories and, in particular, quark confinement in QCD. To non-abelian theories would mean that the dual theory of weakly coupled monopoles could be used to understand

Moreover, an important algebra of physical space vector that will be used in our formulation is the vector derivatives $\partial$ and $\bar{\partial} \overset{\text{def}}{=} \partial^\dagger$ defined by,

$$\bar{\partial} = \bar{e}_\mu \partial^\mu = e^{-1} \partial_t - \vec{\nabla} \quad \text{and,} \quad \partial = e^\mu \partial_\mu = e^{-1} \partial_t + \vec{\nabla}. \quad (14)$$

Finally, the d’Alambertian differential wave scalar operator $\Box_{\text{cl}(3)}$ in the APS formalism is,

$$\Box_{\text{cl}(3)} \overset{\text{def}}{=} \bar{\partial} \partial = \bar{e}_\mu \bar{e}^\nu \partial^\mu \partial_\nu = \partial^\mu \partial_\mu \equiv \partial^2 = e^{-2} \partial_t^2 - \vec{\nabla}^2. \quad (15)$$

It describes lightlike traveling waves and will be used to formulate the wave equations for the gauge fields $A^\mu$ and $Z^\mu$.

III. FINITE RANGE EM INTERACTION AND MAGNETIC CHARGES

Finite-range electrodynamics is electrodynamics with nonzero photon mass. It is fully compatible with experiments and it turns out that the photon mass has to be very small, less than $10^{-24}$ GeV or even less than $10^{-36}$ GeV. Magnetic monopoles were first introduced theoretically by Dirac in 1931 [8] and 1948 [9]. Moving magnetically charged particles can be detected by monitoring the current in a superconducting ring. In 1982 at Stanford, B. Cabrera detected a single event which could be ascribed to a magnetically charged particle with one Dirac unit of magnetic charge, a magnetic monopole [10]. There are basically three grounds for believing in the existence of magnetic monopoles: 1) their existence leads to the quantization of electricity; 2) a large class of theories that include electromagnetism as a subset predict magnetic monopoles as solitons [11], [12]; a generalization of the electromagnetic duality symmetry to non-abelian theories would mean that the dual theory of weakly coupled monopoles could be used to understand strongly coupled non-abelian gauge theories and, in particular, quark confinement in QCD.

A. Tensor Algebra Formalism: The Two Vector Potentials Formulation

We want to write down a Lagrangian density which describes electromagnetic interaction mediated by nonzero mass photons in presence of both electric and magnetic charges. We extend Maxwell’s theory using two vector potentials, the vector potential $A^\mu \equiv (A_0, \vec{A})$ for the electric charges and the vector potential $Z^\mu \equiv (Z_0, \vec{Z})$ for the magnetic charges. Within this elegant and symmetric description, electric and magnetic charges are considered both as gauge symmetries. We extend the formalism presented in [13] by considering the presence of Proca fields. In cgs units, the Lagrangian density describing such a Maxwell-Proca-Dirac (MPD) system is given by

$$L_{\text{MPD}} (A, Z) = L_{\text{MP}} (A) + L_D (Z) + L_{\text{int}} \quad (16)$$

where the the Lagrangian density $L_{\text{MP}}$ is the standard Maxwell-Proca term, $L_D$ describes the magnetic charge as a gauge symmetry and finally $L_{\text{int}}$ describes the coupling between the electric and the magnetic charge. In their explicit form these Lagrangian densities are,

$$L_{\text{MP}} (A) = \alpha_{FZ} F_{\mu\nu} F^{\mu\nu} + \alpha_{JA} J_\mu^e A^\mu + \alpha_{AZ} A_\mu A^\mu,$$

$$L_D (Z) = \alpha_{WZ} W_{\mu\nu} W^{\mu\nu} + \alpha_{JZ} J_\mu^m Z^\mu,$$

$$L_{\text{int}} = \alpha_{FW} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} W_{\rho\sigma} = 4 \alpha_{FW} \partial_\mu (\varepsilon^{\mu\nu\rho\sigma} A_\nu \partial_\rho Z_\sigma). \quad (17)$$

The electric four-current $J_\mu^e \equiv (c \rho_e, \vec{j}_e)$ and the magnetic four-current $J_\mu^m \equiv (c \rho_m, \vec{j}_m)$ are the sources of the electromagnetic field. For the sake of simplicity we did not make explicit the values of the coupling $\alpha$-coefficients. We use cgs units in this paper and, for instance, $\alpha_{FZ} = -\frac{1}{137}$, $\alpha_{JA} = -\frac{2}{3}$, $\alpha_{AZ} = \frac{m_e^2}{\pi}$ where $m_e = \frac{\hbar}{2\pi}$ is the inverse of the Compton length associated with the photon mass of the electric gauge field $A_\mu$. The field strength tensors $F_{\mu\nu}$ and $W_{\mu\nu}$ are defined in terms of the two vector potentials $A_\mu$ and $Z_\mu$.

$$F_{\mu\nu} \overset{\text{def}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu, \quad W_{\mu\nu} \overset{\text{def}}{=} \partial_\mu Z_\nu - \partial_\nu Z_\mu. \quad (18)$$
Variation of the density Lagrangian $\mathcal{L}_{MPD} (A, Z)$ with respect to $A^\mu$ and $Z^\mu$ leads to,

$$\frac{\partial \mathcal{L}_{MPD}}{\partial A_\mu} - \partial_\nu \left( \frac{\partial \mathcal{L}_{MPD}}{\partial (\partial_\nu A_\mu)} \right) = 0, \quad \frac{\partial \mathcal{L}_{MPD}}{\partial Z_\mu} - \partial_\nu \left( \frac{\partial \mathcal{L}_{MPD}}{\partial (\partial_\nu Z_\mu)} \right) = 0. \quad (19)$$

Finally, the assumption of working in the Lorenz gauge conditions, $\partial_\mu A^\mu = 0$ and $\partial_\mu Z^\mu = 0$, leads to the field equations,

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = \frac{4\pi}{e} J^\nu_e, \quad \partial_\mu W^{\mu\nu} = \frac{4\pi}{e} J^\nu_m. \quad (20)$$

In terms of the field strength tensors $F_{\mu\nu}$ and $W_{\mu\nu}$, the electric field $\vec{E}$ and the magnetic field $\vec{B}$ can be written as,

$$E_i = F_{i0} + \frac{1}{2} \varepsilon^{ijk} W_{jk} = F_{i0} - G_{i0} = -\partial_i A_0 - c^{-1} \partial_i A_i - \varepsilon_{ijk} \partial_j Z_k \quad (21)$$

and,

$$B_i = W_{i0} - \frac{1}{2} \varepsilon^{ijk} F_{jk} = W_{i0} + F_{i0} = -\partial_i Z_0 - c^{-1} \partial_i Z_i - \varepsilon_{ijk} \partial_j A_k \quad (22)$$

where $F^{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$ and $G^{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} G_{\gamma\delta}$ are the duals of $F^{\alpha\beta}$ and $G^{\alpha\beta}$. Therefore using (21) and (22), the generalized Maxwell’s equations in the covariant form (20) become,

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho_e - m_\gamma^2 A_0, \quad \vec{\nabla} \times \vec{B} = c^{-1} \partial_t \vec{E} + m_\gamma^2 \vec{A} = 4\pi c^{-1} \vec{j}_e \quad (23)$$

and,

$$\vec{\nabla} \cdot \vec{B} = 4\pi \rho_m, \quad \vec{\nabla} \times \vec{E} + c^{-1} \partial_t \vec{B} = -4\pi c^{-1} \vec{j}_m. \quad (24)$$

Finally, substituting equations (21) and (22) in (23) and (24) and using the Lorenz gauge conditions, we obtain the wave equations for the gauge fields $A^\mu$ and $Z^\mu$,

$$\left( \Box + m_\gamma^2 \right) A_\mu = \frac{4\pi}{e} J^\mu_e, \quad \Box Z_\mu = \frac{4\pi}{e} J^\mu_m \quad (25)$$

where $\Box \equiv \nabla^2 - c^{-2} \partial_t^2$ is the d’Alambertian differential wave operator. Equations (25) lead to conclude that finite-range electromagnetic interaction in presence of electric and magnetic charges allow for two four-vector potentials, a massive "electric" photon and an extra degree of freedom, a massless gauge boson, a "magnetic" photon. There is no experimental evidence of such a boson, however such a presence can be theoretically hidden by use of the Higgs mechanism. A basic difference between the formalism presented in this paper and Dirac’s formulation (singular vector potentials for electric charges) or Wu and Yang’s formulations (two non-singular vector potentials for electric charges, potentials related by a gauge transformation) of massless electrodynamics with magnetic monopoles is that the two vector potentials formulation would not lead to Dirac’s charge quantization condition. However, since there are alternative explanations of the charge quantization based on both Grand Unified gauge Theories (GUT) and Kaluza-Klein theories, this is not a problem. In standard massless electrodynamics the existence of the magnetic charge rests upon the Dirac quantization condition, $q_e q_m = \frac{1}{2} n \hbar c$, with $n \in \mathbb{Z}$. This condition makes the string attached to the monopole invisible and it can be obtained either with the help of angular momentum quantization or gauge invariance. Unfortunately, neither of these methods work in massive electrodynamics [14].

**B. STA Formalism: The Two Vector Potentials Formulation**

In a previous paper, the author employed STA formalism to extend Maxwell theory to the case of massive photons and magnetic monopoles using a singular vector potential for electric charges. In this paper, instead, a different approach is used, the two vector potential formulation.

Spacetime algebra is the geometric algebra of Minkowski spacetime. It is generated by four orthogonal basis vectors $\{\gamma_\mu\}_{\mu=0,\ldots,3}$ satisfying the relations

$$\gamma_\mu \cdot \gamma_\nu = \frac{1}{2} (\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) \equiv \eta_{\mu\nu} = \text{diag}(+ - - -); \quad \mu, \nu = 0,\ldots,3 \quad (26)$$
\[ \gamma_{\mu} \wedge \gamma_{\nu} = \frac{1}{2} (\gamma_{\mu} \gamma_{\nu} - \gamma_{\nu} \gamma_{\mu}) \equiv \gamma_{\mu\nu}. \]  

Equations (26) and (27) display the same algebraic relations as Dirac’s \( \gamma \)-matrices. Indeed, the Dirac matrices constitute a representation of the spacetime algebra. From (26) it is obvious that

\[ \gamma_0^2 = 1, \quad \gamma_0 \cdot \gamma_j = 0 \quad \text{and} \quad \gamma_j \cdot \gamma_k = -\delta_{jk}; \quad j, k = 1, 2, 3. \]  

A basis for this 16-dimensional spacetime Clifford algebra \( \mathfrak{cl}(1, 3) \) is given by

\[ \mathcal{B}_{\mathfrak{cl}(1, 3)} = \{1, \gamma_{\mu}, \gamma_{\mu} \wedge \gamma_{\nu}, i_{\mathfrak{cl}(1, 3)} \gamma_{\mu}, i_{\mathfrak{cl}(1, 3)} \gamma_{\nu}\}. \]

whose elements represent scalars, vectors, bivectors, trivectors and pseudoscalars respectively. In \( \mathfrak{cl}(1, 3) \) the highest-grade element, the unit pseudoscalar, is defined as,

\[ i_{\mathfrak{cl}(1, 3)} \overset{\text{def}}{=} \gamma_0 \gamma_1 \gamma_2 \gamma_3. \]

It represents an oriented unit four-dimensional volume element. The corresponding volume element is said to be right-handed because \( i_{\mathfrak{cl}(1, 3)} \) can be generated from a right-handed vector basis by the oriented product \( \gamma_0 \gamma_1 \gamma_2 \gamma_3 \). A general multi-vector \( M_{\mathfrak{cl}(1, 3)} \) of the spacetime algebra can be written as

\[ M_{\mathfrak{cl}(1, 3)} = \sum_{k=0}^{4} \langle M_{\mathfrak{cl}(1, 3)} \rangle_k = \alpha + a + B + i_{\mathfrak{cl}(1, 3)} b + i_{\mathfrak{cl}(1, 3)} \beta, \]

where \( \alpha \) and \( \beta \) are real scalars, \( a \) and \( b \) are real spacetime vectors and \( B \) is a bivector. Since \( \mathfrak{cl}(1, 3) \) is built on a linear space of even dimension \( (n = 4) \), \( i_{\mathfrak{cl}(1, 3)} \) anticommutes with odd-grade multivectors and commutes with even-grade elements of the algebra,

\[ i_{\mathfrak{cl}(1, 3)} M_{\mathfrak{cl}(1, 3)} = \pm M_{\mathfrak{cl}(1, 3)} i_{\mathfrak{cl}(1, 3)} \]

where the multivector \( M_{\mathfrak{cl}(1, 3)} \) is even for \( (+) \) and odd for \( (-) \). An important spacetime vector that is used in STA formalism is the vector derivative \( \nabla \), defined by

\[ \nabla \overset{\text{def}}{=} \gamma^\mu \partial_\mu \equiv \gamma^0 c^{-1} \partial_t + \gamma^j \partial_j. \]

By post-multiplying with \( \gamma^0 \), we obtain

\[ \nabla \gamma_0 = c^{-1} \partial_t + \gamma^j \gamma_0 \partial_j = c^{-1} \partial_t - \vec{\nabla}, \]

where \( \vec{\nabla} \) is the usual vector derivative defined in vector algebra. Similarly, multiplying the spacetime vector derivative by \( \gamma^0 \), we obtain

\[ \gamma_0 \nabla = c^{-1} \partial_t + \vec{\nabla}. \]

Finally, we notice that the spacetime vector derivative satisfies the following relation

\[ \Box_{\mathfrak{cl}(1, 3)} \overset{\text{def}}{=} (\gamma_0 \nabla) (\nabla \gamma_0) = c^{-2} \partial_t^2 - \vec{\nabla}^2, \]

where \( \Box_{\mathfrak{cl}(1, 3)} \) is the d’Alembert operator used in the description of lightlike traveling waves. The STA formulation of the fundamental equations of massive classical electrodynamics in presence of magnetic monopoles is,

\[ \nabla F_{\mathfrak{cl}(1, 3)} = 4 \pi c^{-1} (j_e - i_{\mathfrak{cl}(1, 3)} j_m) - m_e^2 A. \]

The field strength \( F_{\mathfrak{cl}(1, 3)} \) is the spacetime Faraday bivector given by,

\[ F_{\mathfrak{cl}(1, 3)} = \frac{1}{2} F_{\mathfrak{cl}(1, 3)}^{\mu\nu} \gamma_\mu \wedge \gamma_\nu = \vec{E} + i_{\mathfrak{cl}(1, 3)} \vec{B} \]

\[ = E^i \gamma_i \gamma_0 - B^1 \gamma_2 \gamma_3 - B^2 \gamma_3 \gamma_1 - B^3 \gamma_1 \gamma_2 \]

where \( F_{\mathfrak{cl}(1, 3)}^{\mu\nu} = \gamma^\mu \wedge \gamma^\nu \cdot F_{\mathfrak{cl}(1, 3)} \) are the components of \( F_{\mathfrak{cl}(1, 3)} \) in the \( \{\gamma^\mu\} \) frame. Notice that the electric and magnetic fields are expressed in terms of two and not one vector potential, namely, \( E_i = -\partial_t A_0 - c^{-1} \partial_i A_t - \varepsilon_{ijk} \partial_j Z_k \)
and \( B_i = -\partial_i Z_0 - c^{-1} \partial_i Z_i - \varepsilon_{ijk} \partial_j A_k \). Moreover, \( j_e \) and \( j_m \) are the electric and magnetic spacetime currents defined as,

\[
j_e \overset{\text{def}}{=} (j_e \cdot \gamma_0 + j_e \wedge \gamma_0) \gamma_0 = (c \rho_e + \tilde{\gamma}_e) \gamma_0 \tag{39}\]

and,

\[
j_m \overset{\text{def}}{=} (j_m \cdot \gamma_0 + j_m \wedge \gamma_0) \gamma_0 = (c \rho_m + \tilde{\gamma}_m) \gamma_0. \tag{40}\]

Moreover, the spacetime vector potential \( A \) is defined by,

\[
A \overset{\text{def}}{=} (A \cdot \gamma_0 + A \wedge \gamma_0) \gamma_0 = \left( A_0 + \tilde{\gamma} \right) \gamma_0. \tag{41}\]

Finally, notice that the spacetime algebra decomposition of multivectors is performed considering the different grade \(-r\) multivectorial components with \( 0 \leq r \leq 3 \) of an arbitrary element of \( \mathfrak{d}(1, 3) \). For instance, a 0-grade multivector is a scalar; a 1-grade multivector is a vector; a 2-grade multivector is a bivector; finally, a 3-grade multivector is a trivector. The STA multivectorial decomposition of the LHS of equation (37) is,

\[
\nabla F_{\mathfrak{c}(1, 3)} = \langle \nabla F_{\mathfrak{c}(1, 3)} \rangle_1 + \langle \nabla F_{\mathfrak{c}(1, 3)} \rangle_3 \tag{42}\]

where the vectorial and trivectorial components are

\[
\langle \nabla F_{\mathfrak{c}(1, 3)} \rangle_1 = \nabla \cdot F_{\mathfrak{c}(1, 3)} \quad \text{and} \quad \langle \nabla F_{\mathfrak{c}(1, 3)} \rangle_3 = \nabla \wedge F_{\mathfrak{c}(1, 3)}. \tag{43}\]

Equation (37) will be compared with its APS analog and special focus will be devoted to the different properties of pseudoscalars \( i_{\mathfrak{c}(1, 3)} \in \mathfrak{d}(1, 3) \) and \( i_{\mathfrak{c}(3)} \in \mathfrak{d}(3) \).

C. APS Formalism: The Two Vector Potentials Formulation

In this subsection, we show that the generalized Maxwell’s equations, relations (23) and (24) can be cast into a single Lorentz invariant APS equation given by,

\[
\partial F_{\mathfrak{c}(3)} = 4\pi e^{-1} (J_e + i_{\mathfrak{c}(3)} J_m) - m_c^2 A. \tag{44}\]

The physical space vector derivative is given by,

\[
\partial = \epsilon^\mu \partial_\mu \equiv c^{-1} \partial_t + \tilde{\nabla} \tag{45}\]

while the paravector currents and the paravector electromagnetic potential are defined by,

\[
J_e \overset{\text{def}}{=} c \rho_e - \tilde{J}_e, \quad J_m \overset{\text{def}}{=} c \rho_m - \tilde{J}_m, \quad A \overset{\text{def}}{=} A_0 - \tilde{A}. \tag{46}\]

The unit pseudoscalar in (44) is \( i_{\mathfrak{c}(3)} \overset{\text{def}}{=} \epsilon_1 \epsilon_2 \epsilon_3 \), the global commuting unit pseudoscalar of \( \mathfrak{d}(3) \). The field strength \( F_{\mathfrak{c}(3)} \) is the algebra of physical space biparavector given by,

\[
F_{\mathfrak{c}(3)} = \frac{1}{2} F_{\mathfrak{c}(3)}^{\mu \nu} \langle e_\mu \epsilon_\nu \rangle_\nu = \tilde{E} + i_{\mathfrak{c}(3)} \tilde{B}. \tag{47}\]

where \( F_{\mathfrak{c}(3)}^{\mu \nu} \) are the components of \( F_{\mathfrak{c}(3)} \) in the \( \{ e_\mu \} \) frame while \( E_i \) and \( B_i \) are defined in (21) and (22). The source of the electromagnetic field \( F_{\mathfrak{c}(3)} \) is given by the sum of a real paravector current, \( J_e \), and a pseudoparavector current, \( i J_m \). These two sources behave in a different way under the operation of parity inversion \(" * \) ,

\[
J_e \overset{\text{def}}{\sim} (J_e)^* = (J_e)^\dagger \tag{48}\]

and,

\[
i J_m \overset{\text{def}}{\sim} (i J_m)^* \equiv [(i J_m)^\dagger]^\dagger = -i (J_m)^\dagger. \tag{49}\]
Substituting (45) and (47) into the LHS of equation (44), we obtain

$$\partial F_{\text{el}(3)} \equiv \vec{\nabla} \cdot \vec{E} + i \vec{\nabla} \cdot \vec{B} + c^{-1} \partial_t \vec{E} - \vec{\nabla} \times \vec{B} + i \left( c^{-1} \partial_t \vec{B} + \vec{\nabla} \times \vec{E} \right)$$  \hspace{1cm} (50)

where

$$\partial F_{\text{el}(3)} = \langle \partial F_{\text{el}(3)} \rangle_{rs} + \langle \partial F_{\text{el}(3)} \rangle_{iv} + \langle \partial F_{\text{el}(3)} \rangle_{is} + \langle \partial F_{\text{el}(3)} \rangle_{iv} \cdot \hspace{1cm} (51)$$

Similarly, substituting (46) into the RHS of equation (44), we obtain

$$4\pi c^{-1}(J_e + iJ_m) - m_\gamma^2 \Lambda = 4\pi \rho_e - m_\gamma^2 A_0 + i4\pi \rho_m + m_\gamma^2 \Lambda - 4\pi c^{-1}J_e - i4\pi c^{-1}J_m. \hspace{1cm} (52)$$

Naming the RHS of (44) ”$s$”, we obtain

$$s = \langle s \rangle_{rs} + \langle s \rangle_{is} + \langle s \rangle_{rv} + \langle s \rangle_{iv} \hspace{1cm} (53)$$

and the APS decomposition of equation (44) leads to the following four equations,

$$\langle \partial F_{\text{el}(3)} \rangle_{rs} = \langle s \rangle_{rs}, \langle \partial F_{\text{el}(3)} \rangle_{is} = \langle s \rangle_{is}, \langle \partial F_{\text{el}(3)} \rangle_{rv} = \langle s \rangle_{rv}, \langle \partial F_{\text{el}(3)} \rangle_{iv} = \langle s \rangle_{iv}. \hspace{1cm} (54)$$

Notice that the algebra of physical space decomposition of arbitrary multifaravers is performed by considering the \((rs, \text{real scalar}; is, \text{imaginary scalar}; rv, \text{real vector}; iv, \text{imaginary vector})\) real, imaginary, scalar and vectorial parts of elements of \(\text{el}(3)\). Equations (54) are the APS analog of the vector algebra formulation of generalized Maxwell’s equations describing finite range electromagnetic interaction in presence of electric and magnetic charges, equations (23) and (24).

**IV. LORENTZ, LOCAL GAUGE AND EM DUALITY INVARIANCES IN THE APS FORMALISM**

The study of spacetime and gauge symmetries is fundamental in the theoretical modelling of physical phenomena. The lack of an advanced geometrization program of physics leads to the impossibility of finding an adequate understanding of any potential link between spacetime and local gauge invariances. GA formalism seems to be most adequately suited for the search of such a link.

**A. Lorentz Covariance in the Maxwell-Proca-Dirac System**

We discuss the Lorentz covariance of the theory described by (44). A generic restricted unimodular Lorentz transformation \(\Lambda, \Lambda^\dagger = 1\), is specified by six independent parameters \(\vec{\eta}\) (Lorentz boost) and \(\vec{\theta}\) (rotations),

$$\Lambda \equiv e^{\frac{i}{2}\vec{\xi}} \equiv e^{\frac{i}{4} \epsilon_{\mu
u}(\epsilon_{\mu}\epsilon_{\nu})} \overset{\text{def}}{=} e^{\frac{i}{2}(\vec{\eta} - i\epsilon_{el(3)} \vec{\theta})} \hspace{1cm} (55)$$

where \(\vec{\xi}\) is a biparavector in the APS formalism. Transformations with \(\vec{\eta} = 0\) are pure rotations which, in addition to being unimodular, are also unitary, \(\Lambda^\dagger = \Lambda^{-1}\). Transformations with \(\vec{\theta} = 0\) describe pure Lorentz boosts which are unimodular and real (Hermitian), \(\Lambda^\dagger = \Lambda\). Under an arbitrary active Lorentz transformation (LT) \(\Lambda\), paravectors \(\vec{M}\) and \(\vec{M}^\dagger \overset{\text{def}}{=} \vec{M}^\dagger\) transform as,

$$\vec{M}^{\text{old}} = M^\mu \epsilon_{\mu} \overset{\text{LT}}{\rightarrow} \vec{M}^{\text{new}} = \Lambda \vec{M}^{\text{old}} \Lambda^\dagger = \Lambda_{\mu}^\nu M^\mu \epsilon_{\nu} \hspace{1cm} (56)$$

and,

$$\vec{M}^{\text{old}} = M_{\mu} \epsilon^{\mu} \overset{\text{LT}}{\rightarrow} \vec{M}^{\text{new}} = (\Lambda^\dagger)^{-1} \vec{M}^{\text{old}} \Lambda^{-1} = \Lambda_{\mu}^\nu M_{\mu} \epsilon^{\nu} \hspace{1cm} (57)$$

Using (56) and (57), we obtain the following LT for the RHS and LHS of equation (44),

$$\partial F_{\text{el}(3)} \overset{\text{LT}}{\rightarrow} (\Lambda^\dagger)^{-1} \partial F_{\text{el}(3)} \Lambda^{-1} \hspace{1cm} (58)$$

$$\overset{\text{LT}}{\rightarrow} (\Lambda^\dagger)^{-1} (J_e + i_{el(3)} J_m) \Lambda^{-1}, \Lambda \overset{\text{LT}}{\rightarrow} (\Lambda^\dagger)^{-1} \Lambda^{-1}. \hspace{1cm} (59)$$

The proof of Lorentz covariance of equation (44) becomes then straightforward.
B. Local Gauge Invariance in the Maxwell-Proca System

In absence of magnetic charges, the Maxwell-Proca theory is described by,

$$\partial F_{cl(3)} = 4\pi c^{-1}J_e - m^2 \Delta.$$  \hspace{1cm} (60)

The imaginary scalar and vectorial parts of $\partial F_{cl(3)}$ are absent,

$$\langle \partial F_{cl(3)} \rangle_{iv}^{(MP)} = 0, \langle \partial F_{cl(3)} \rangle_{is}^{(MP)} = 0.$$  \hspace{1cm} (61)

This leads to conclude that the absence of magnetic charges (pseudoparavector magnetic currents) removes the underlying imaginary structure of the APS equation (60), making it completely real. The application of the wave operator $\Box_{cl(3)} = \partial^2$ to $F_{cl(3)}$ and the requirement of the validity of the Lorenz gauge condition,

$$\partial \cdot A = \langle \partial A \rangle_v = c^{-1} \partial_t A_0 + \nabla \cdot \vec{A} = 0 \hspace{1cm} (62)$$

lead to the equation of charge conservation,

$$0 = \langle \Box_{cl(3)} F_{cl(3)} \rangle_s^{(MP)} = \langle \Box_{cl(3)} F_{cl(3)} \rangle_{rs}^{(MP)} = \partial_t \rho_e + \nabla \cdot \vec{j}_e.$$  \hspace{1cm} (63)

Notice that $\langle \Box_{cl(3)} F_{cl(3)} \rangle_{s}^{(MP)} = 0$ because $\Box_{cl(3)}$ is a scalar operator and $\langle F_{cl(3)} \rangle_{s} = 0$. It is worthwhile mentioning that Lorenz gauge condition must be satisfied in order to have charge conservation in massive classical electrodynamics. In the Lorenz gauge, $F_{cl(3)} = \partial A$, and equation (60) becomes,

$$\Box_{cl(3)} A = 4\pi c^{-1} J_e - m^2 \Delta.$$  \hspace{1cm} (64)

This equation is not invariant under local gauge transformation (LGT),

$$A^{old}_{\text{LGT}} \rightarrow A^{new} = A^{old} + 2\chi(x)$$  \hspace{1cm} (65)

where the gauge function $\chi(x)$ satisfies the wave equation $\Box_{cl(3)} \chi(x) = 0$. Local gauge invariance is lost in the Maxwell-Proca system.

C. EM Duality Invariance in the Maxwell-Dirac System

The conventional massless classical electrodynamics in presence of magnetic charges (Maxwell-Dirac System) is described by,

$$\partial F_{cl(3)} = 4\pi c^{-1} (J_e + i_{cl(3)} J_m).$$  \hspace{1cm} (66)

The application of the wave operator $\Box_{cl(1,3)}$ to $F_{cl(3)}$ leads to the equation of electric and magnetic charge conservation,

$$0 = \langle \Box_{cl(3)} F_{cl(3)} \rangle_s^{(MD)} = \langle \Box_{cl(3)} F_{cl(3)} \rangle_{rs}^{(MD)} + \langle \Box_{cl(3)} F_{cl(3)} \rangle_{is}^{(MD)}$$

$$= 4\pi c^{-1} \left[ (\partial_t \rho_e + \nabla \cdot \vec{j}_e) + i_{cl(3)} \left( \partial_t \rho_m + \nabla \cdot \vec{j}_m \right) \right]$$  \hspace{1cm} (67)

It is worthwhile emphasizing that the magnetic charges satisfy the same form of the continuity equation as the electric charges,

$$\partial_t \rho_e + \nabla \cdot \vec{j}_e = 0, \hspace{0.5cm} \text{and} \hspace{0.5cm} \partial_t \rho_m + \nabla \cdot \vec{j}_m = 0$$  \hspace{1cm} (68)

but the electric charge conservation has its origin in setting equal to zero the real part of $\langle \Box_{cl(3)} F_{cl(3)} \rangle_{s}^{(MD)}$, while the magnetic charge conservation arises from the request that the imaginary part of $\langle \Box_{cl(3)} F_{cl(3)} \rangle_{s}^{(MD)}$ equals zero. In the Maxwell-Proca system there is no imaginary scalar part of $\Box_{cl(3)} F_{cl(3)}$. It is interesting to study the EM duality
invariance in the APS formalism. Considering a duality rotation (DR) of arbitrary real angle \( \theta \), we obtain that the electromagnetic biparavector \( F^{\text{cl}}(3) \) transforms as,

\[
F^{\text{old}}_{\text{cl}(3)} \xrightarrow{\text{DR}} F^{\text{new}}_{\text{cl}(3)} = F^{\text{old}}_{\text{cl}(3)} e^{-i \epsilon(3) \theta}.
\]

(69)

For the paravectorial electric and magnetic currents \( J^e \) and \( J^m \), we obtain

\[
J^{\text{old}}_{e} \xrightarrow{\text{DR}} J^{\text{new}}_{e} = J^{\text{old}}_{e} \cos \theta + J^{\text{old}}_{m} \sin \theta, \quad J^{\text{old}}_{m} \xrightarrow{\text{DR}} J^{\text{new}}_{m} = -J^{\text{old}}_{e} \sin \theta + J^{\text{old}}_{m} \cos \theta.
\]

(70)

Considering the complex electromagnetic paravector current \( J \equiv J^e + i \epsilon(3) J^m \), we determine that its duality transformation law is,

\[
J^{\text{old}} \xrightarrow{\text{DR}} J^{\text{new}} = J^{\text{old}} e^{-i \epsilon(3) \theta}.
\]

(71)

The electromagnetic duality invariance of equation (66) becomes then straightforward.

**D. The APS Analog of the Lorentz Force**

Finally, let us consider the APS analog of the Lorentz force on a magnetic charge \( q^m \) and electric charge \( q^e \) with four velocity \( \bar{u} = \gamma (1 + \frac{\vec{v}}{c}) \) where \( \gamma = \left[ 1 - \frac{(\vec{v})^2}{c^2} \right]^{-\frac{1}{2}} \). Notice that,

\[
F_{\epsilon(3)} \bar{u} = \frac{\gamma}{c} \left[ \vec{E} \cdot \vec{v} + i \epsilon(3) \vec{B} \cdot \vec{v} \right] + \frac{\gamma}{c} \left[ (\vec{E} + \vec{v} \times \vec{B}) + i \epsilon(3) \left( \vec{B} - \vec{v} \times \vec{E} \right) \right].
\]

(72)

The electromagnetic force on \( q^e \) is then,

\[
\overline{F}^e = \frac{d}{d \tau} \bar{p}^e = q^e \langle F_{\epsilon(3)} \bar{u} \rangle_{\overline{\nu}} = \gamma q^e \left( \frac{\vec{E}}{c} + \frac{\vec{v}}{c} \times \vec{B} \right)
\]

(73)

while the force acting on the magnetic charge is,

\[
\overline{F}^m = \frac{d}{d \tau} \bar{p}^m = q^m \langle F_{\epsilon(3)} \bar{u} \rangle_{\overline{\nu}} = \gamma q^m \left( \frac{\vec{B}}{c} - \frac{\vec{v}}{c} \times \vec{E} \right).
\]

(74)

Equation (74) can be derived from (73) under a DR with \( \theta = \frac{\pi}{2} \), where \( q^e \xrightarrow{\text{DR}} q^m, \vec{E} \xrightarrow{\text{DR}} \vec{B} \) and \( \vec{B} \xrightarrow{\text{DR}} -\vec{E} \).

**V. SIGNATURE AND DIMENSION OF SPACETIME**

In this paper, the concept of spacetime and that of paravector space have been used to extend Maxwell’s theory to the case of massive photons and magnetic monopoles where the electric and magnetic charges are considered both as gauge symmetries. In this section, considerations about the signature and the dimensionality of spacetime are carried out. Furthermore, the possibility that APS formalism has to accommodate the GA formalism of a 4D spacetime with arbitrary signature is considered.

**A. Signature and Dimension in GA: General Considerations**

The Geometric (Clifford) Algebra of a given \( n \)-dimensional linear space \( V = \mathbb{R}^{p+q} \) endowed with a symmetric bilinear form \( \eta \),

\[
\eta : (e_\mu, e_\nu) \in \mathbb{R}^{p+q} \times \mathbb{R}^{p+q} \rightarrow \mathbb{R}, \quad \eta (e_\mu, e_\nu) = \eta_{\mu\nu}
\]

(75)

depends not only on the dimension of \( V \) but also on the signature \( s \) of \( \eta \), \( s = p - q \) where \( p \) is the number of basis vectors with positive norm and \( q \) enumerates the basis vectors with negative norm. In the GA formalism, the metric structure of the space whose geometric algebra is built, reflects the properties of the unit pseudoscalar of the algebra. Indeed, the existence of a pseudoscalar is equivalent to the existence of a metric. For instance, in spaces of positive
definite metric, the pseudoscalar has magnitude $|\mathbf{i}| = 1$ while the value of $i^2$ depends only on the dimension of space as $i^2 = (-1)^{n(n-1)/2}$. The real geometric Clifford algebras $\mathcal{cl}(p, q)$ and $\mathcal{cl}(p', q')$ with $p + q = p' + q' = n$ are in general not isomorphic. In particular, $\mathcal{cl}(p, q)$ and $\mathcal{cl}(q, p)$ are not isomorphic. Therefore change of signature may lead to different Clifford algebras. Physical theories formulated with Clifford algebra are therefore potentially inequivalent pending the underlying choice of signature. Finally, it is worthwhile emphasizing that given the real Clifford algebra of a quadratic space with a given signature, it is possible to define new products, vees and tilt products, such that they simulate the Clifford product of a quadratic space with another signature different from the original one [15], [16].

B. Does the choice of signature have physical relevance?

In this paper, we have considered a classical field theory, no quantum considerations have been carried out. Classical field theories such as electrodynamics and geometrodynamics cannot distinguish between the two Lorentzian signatures (+,−−−) and (−,+++). Einstein’s field equations do not impose any particular restriction on spacetime signature; in fact, they do not refer to signature at all. Electrodynamics and geometrodynamics can be cast in signature invariant form, covariant under signature change transformations, $\eta_{\mu\nu} \rightarrow -\eta_{\mu\nu}$. The choice of a metric $\eta_{\mu\nu}$ with signature $(p, q)$ or $(q, p)$ has no physical relevance. The origin of this may be found in the Lorentz group structure $SL(2, \mathbb{C})$, the double covering group locally isomorphic to $SO(1, 3)$ and to $SO(3, 1)$. The Lorentz group is briefly considered in our work. The situation in quantum mechanics is less clear. For instance, it seems that the sign of the metric is important string theory where spinors in curved background transforming under the double cover $Pin(p, q)$ of $O(p, q)$ are used in Polyakov path integral calculations [17]. However, classical electrodynamics or the extended Maxwell’s theory considered in this paper can distinguish between (+,−−−) and (++,++) signatures. Faraday’s law is not signature invariant,

$\nabla \times \vec{E} + c^{-1} \partial_t \vec{B} = 0$, (+,−−−), $\nabla \times \vec{E} - c^{-1} \partial_t \vec{B} = 0$, (++,++). (76)

Euclidean electrodynamics with signature (++,++) is ”just like” ordinary electrodynamics except for ”anti-Lenz” law [18], [19]. However this one change has far-reaching effects. It changes the equations from hyperbolic to elliptic, so there is no propagation with a finite speed in Euclidean spaces. It may be worthwhile emphasizing this since the importance of classical Euclidean field theory of source-free Maxwell equations minimally coupled to Einstein gravity is well known, especially in the study of black holes and magnetic monopoles [20], [21]. Applications of GA may be extended in a positive way in these areas.

C. Spacetime: Signature and Dimensionality

Two fundamental facts about spacetime are its Lorentzian signature and dimensionality $d = 4$, where the Lorentzian signature arises dynamically in quantum field theory [22]. Furthermore, group-theoretic arguments lead to conclude that it is only natural to have a 3+1 signature rather than a 4+0 or a 2+2 for its metric. A (4+0)-world has no meaningful dynamics [23]. Furthermore, for even or odd $d > 4$, only metrics with one time dimension are physically acceptable. Two time signatures are irrelevant from a physical point of view.

If $V$ is a vector space of dimension $n = 4$, as it is in this paper, then there are five different Clifford algebras depending on the signature: $\mathcal{cl}(4, 0)$, $\mathcal{cl}(3, 1)$, $\mathcal{cl}(2, 2)$, $\mathcal{cl}(1, 3)$, $\mathcal{cl}(0, 4)$. With the exception of $\mathcal{cl}(2, 2)$ the importance of the others in modern physics is more than obvious. General relativists use a Minkowski spacetime metric with $s = +2$. This involves the algebra $\mathcal{cl}(3, 1)$ where the spacelike vectors have positive norm. The algebras $\mathcal{cl}(1, 3)$ and $\mathcal{cl}(3, 1)$ are not isomorphic. Quantum field theorists prefer $\mathcal{cl}(1, 3)$ over $\mathcal{cl}(3, 1)$ because of the isomorphism $\mathcal{cl}(1, 3) \simeq \mathcal{cl}(4, 0)$, whereas $\mathcal{cl}(3, 1) \simeq \mathcal{cl}(2, 2)$.

In this paper, the spacetime algebra $\mathcal{cl}(1, 3)$ with signature (+,−−−) was used and the paravector space of signature (1, 3) was considered. Within such APS formalism,

$$\overline{\mathcal{M}} = M_0 + \overline{\mathcal{M}}, \quad \mathcal{M} \dagger = \overline{\mathcal{M}} \overline{\mathcal{M}} = M_0^2 - \overline{\mathcal{M}}^2.$$ (77)

If we had used $\mathcal{cl}(3, 1)$ with signature (++,++), then we would have considered a paravector space of signature (3, 1) where,

$$\overline{\mathcal{M}} = M_0 + \overline{\mathcal{M}}, \quad \mathcal{M} \dagger = \overline{\mathcal{M}} \overline{\mathcal{M}} = -M_0 + \overline{\mathcal{M}}, \quad \overline{\mathcal{M}} \overline{\mathcal{M}} = -M_0^2 + \overline{\mathcal{M}}^2.$$ (78)

The simple change of the overall sign on the definition of the quadratic form $\overline{\mathcal{M}} \overline{\mathcal{M}}$ allows the APS formalism to accommodate both possibilities, $\mathcal{cl}(1, 3) \simeq \mathcal{cl}(4, 0)$ and $\mathcal{cl}(3, 1) \simeq \mathcal{cl}(2, 2)$. Notice that to take account of the Lorentz
signature of the Minkowski spacetime, a general factor $\epsilon$ can be introduced to account for the overall sign difference between the two choices. This allows one to compare the two choices at any stage of the development,

$$\mathcal{M} = \epsilon \left( M_0^2 - M^2 \right).$$

The paravector space of signature $(1, 3)$ is generated when $\epsilon = +1$ corresponding to a Lorentz signature $(+++--)$ and a paravector space of signature $(3, 1)$ when $\epsilon = -1$ corresponding to a Lorentz signature $(++--)$.

VI. SPACETIME ALGEBRA OR ALGEBRA OF PHYSICAL SPACE?

The four-dimensional Minkowski spacetime with Lorentzian signature $(++--)$ can be represented by the paravector space in the three-dimensional space $\mathfrak{cl}(3)$ without any loss of generality. The $\mathfrak{cl}(3)$ Pauli algebra formalism used in this paper reproduces all standard spacetime and gauge invariances presented in our former paper [5] where the $\mathfrak{cl}(1, 3)$ STA formalism was employed. In both Clifford algebras $\mathfrak{cl}(1, 3)$ with $\dim \mathfrak{cl}(1, 3) = 16$ and $\mathfrak{cl}(3)$ with $\dim \mathfrak{cl}(3) = 8$, calculations can be performed in a compact coordinate-free manner. STA and APS are equally tools to describe massive classical electrodynamics with magnetic charges without selecting any specific choice of frames or set of coordinates which could obscure the physical content of the theory. The compactness of the formulation is evident in both cases,

$$\partial F_{\mathfrak{cl}(3)} = 4\pi e^{-1} \left( J_e + i_{\mathfrak{cl}(3)} J_m \right) - m^2 A$$

and,

$$\nabla F_{\mathfrak{cl}(1, 3)} = 4\pi e^{-1} \left( J_e - i_{\mathfrak{cl}(1, 3)} J_m \right) - m^2 A$$

Notice that $\mathfrak{cl}(1, 3)$ has twice the size of $\mathfrak{cl}(3)$ but both algebras lead to the same compactness. This can be explained noticing the doubleness played by elements of a given grade in the APS formalism. Lorentz scalars are $0$-grade multivectors in both algebras. However, $1$-grade multivectors in STA, spacetime vectors like $j_e$, $j_m$ and $A$ in (80) are homogeneous elements of grade $1$. In the APS formalism, $1$-grade multivectors, paravectors like $\mathbf{j}_e$, $\mathbf{j}_m$ and $A$ in (81) are nonhomogeneous elements which mix elements of grades 0 and 1. Spacetime vectors are real paravectors in APS. The vector part of the paravector is the usual spatial vector, and the scalar part is the time component. Time enters the Pauli algebra not as the new dimension of an enlarged linear space, but rather as the scalar part of an element of $\mathfrak{cl}(3)$. Finally, $2$-grade multivectors in STA, that is spacetime bivectors like $F_{\mathfrak{cl}(1, 3)}$ are homogeneous elements of grade $2$, while biparavectors like $F_{\mathfrak{cl}(3)}$ are nonhomogeneous elements which mix elements of grades $1$ and $2$. However, in the STA formalism of $\mathfrak{cl}(1, 3)$, the unit spacetime pseudoscalar $i_{\mathfrak{cl}(1, 3)} \overset{\text{def}}{=} \gamma_0 \gamma_1 \gamma_2 \gamma_3$ has negative square and commutes only with even-grade multivectors. Therefore it can be represented by the imaginary unit only for certain applications. The pseudoscalar $i_{\mathfrak{cl}(1, 3)}$ does not commute with spacetime vectors while it commutes with the elements of the six-dimensional subspace of $\mathfrak{cl}(1, 3)$ spanned by the set of bivectors (six is the number of independent parameters that define a generic restricted unimodular Lorentz transformation). The unit spacetime pseudoscalar $i_{\mathfrak{cl}(1, 3)}$ provides a natural complex structure for the set of bivectors of $\mathfrak{cl}(1, 3)$, $B_j = \gamma_j \gamma_0$ with $j = 1, 2, 3$,

$$B_j \times B_k = \epsilon_{jkm} i_{\mathfrak{cl}(1, 3)} B_m, \quad i_{\mathfrak{cl}(1, 3)} B_j \times i_{\mathfrak{cl}(1, 3)} B_k = -\epsilon_{jkm} i_{\mathfrak{cl}(1, 3)} B_m, \quad i_{\mathfrak{cl}(1, 3)} B_j \times B_k = -\epsilon_{jkm} B_m.$$  

where " $\times$ " is the conventional commutator product defined in GA. This structure of the bivector algebra in the STA formalism leads to emphasize that there is a hidden complex structure in the Lorentz group. However, in the STA formalism, the hidden "complexity" of the Lorentz group is extended solely to the bivector algebra, not to the whole algebra. The algebra $\mathfrak{cl}(3)$ is more appealing than $\mathfrak{cl}(1, 3)$ in that the volume element of the algebra $i_{\mathfrak{cl}(3)} \overset{\text{def}}{=} \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3$ commutes with all elements of the algebra and squares to $-1$. Indeed, this circumstance appears for every Clifford algebra $\mathfrak{cl}(3+4n)$ with $n \in \mathbb{N}$. Therefore, in these cases, the unit pseudoscalar can be associated identically with the unit imaginary $i_{\mathfrak{c}} \in \mathbb{C}$ in all applications. In this paper, for instance, the unit pseudoscalar $i_{\mathfrak{cl}(3)}$ appearing in the Lorentz transformation $\Lambda = e^{\frac{1}{2} i_{\mathfrak{cl}(3)} \hat{\theta}}$ might be safely identified with $i_{\mathfrak{c}}$. The advantage of the Pauli algebra over the Minkowski spacetime approach is therefore substantial because the former formalism naturally includes an imaginary unit which commutes with all elements of the algebra and not just with the even-grade multivectors. More in general, the lack of a global commuting pseudoscalar $i_{\mathfrak{cl}(p, q)}$, regardless of the metric signature, is one of the main deficiencies of any Clifford algebras associated with an even dimensional space such as the four-dimensional space. Finally, it is worthwhile mentioning that comparisons of Clifford algebras are not new in the literature. In [24], for instance, special relativistic processes are modelled in the APS and STA formalisms.
VII. CONCLUSIONS

Maxwell’s theory of electromagnetism is extended to the case of magnetic monopoles and non-zero mass photons using two vector potentials, $A^\mu$ for electric charges and $Z^\mu$ for magnetic charges. This theory is then presented in the STA and APS formalisms. In both cases, a single nonhomogeneous multivectorial (multiparavectorial) equation describes the physical system. No reference to specific choices of frames or set of coordinates is assumed, therefore the physical content of the theory is not obscured. A detailed discussion about Lorentz, local gauge and EM duality invariances is considered in the APS formalism. General considerations about the signature and the dimensionality of spacetime were carried out. Finally the two formulations were compared and we conclude that the lack of a global commuting pseudoscalar in the Dirac algebra $\mathfrak{cl}(1, 3)$ is one of the main deficiencies of the algebra. Furthermore, since the APS formalism is able to accommodate both signatures and since the Pauli algebra $\mathfrak{cl}(3)$ has the same computational power and compactness of $\mathfrak{cl}(1, 3)$, the presence of a global commuting unit pseudoscalar in $\mathfrak{cl}(3)$ leads to the preference of such algebra in our work. Finally, the formal identification of the unit pseudoscalar $i_{\mathfrak{cl}(3)}$ with the imaginary unit $i_C$ leads to strengthen the possibility of providing a geometric interpretation for the unit imaginary and complex numbers employed throughout classical and quantum physics.

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IX. REFERENCES

1. D. Hestenes, "Vectors, spinors, and complex numbers in classical and quantum physics", Am. J. Phys., Vol 39 (1013), September 1971.
2. T. G. Vold, "An introduction to geometric calculus and its applications to electrodynamics", Am. J. Phys., Vol 61, No.6, June 1993.
3. B. Jancewicz, "Multivectors and Clifford Algebra in Electrodynamics", (World Scientific, Teanack, New Jersey, 1988).
4. W.E. Baylis, "Electrodynamics: A Modern Geometric Approach", Birkhauser, Boston 1998.
5. C. Cafaro and S. A. Ali, "The Spacetime Algebra Approach to Massive Classical Electrodynamics with Magnetic Monopoles", Adv. Appl. Cliff. Alg. 17, 23-36 (2007).
6. W. E. Baylis, "Geometry of Paravector Space with Applications to Relativistic Physics", Proceedings of the NATO Advanced Study Institute, ed. J. Byrnes (Kluwer, 2004).
7. W. E. Baylis and G. Jones, "The Pauli algebra approach to special relativity", J. Phys. A: Math. Gen. 22 (1989) 1-15.
8. P. A. M. Dirac, "Quantised singularities in the electromagnetic field", Proc. Roy. Soc. A133, 60 (1931).
9. P. A. M. Dirac, "The Theory of Magnetic Poles", Phys. Rev. 74, 817 (1948).
10. B. Cabrera, "First Results from a Superconductive Detector for Moving Magnetic Monopoles", Phys. Rev. Lett. 48, 1378 (1982).
11. G. t’Hooft, "Magnetic monopoles in unified gauge theories", Nucl. Phys. B79, 276 (1974).
12. A. Polyakov, "Particle spectrum in quantum field theory", JETP Lett. 20, 194 (1974).
13. D. Singleton, "Magnetic Charge as a "Hidden" Gauge Symmetry", Int. J. Theor. Phys. 34, 37-46 (1995).
14. A. Yu Ignatiev and G. C. Joshi, "Massive Electrodynamics and the Magnetic Monopoles", Phys. Rev. D 53, 984 (1995).
15. D. Miralles, J. M. Parra and J. Vaz Jr., "Signature Change and Clifford Algebras", Int. J. Theor. Phys. 40 (2001) 229-242.

16. P. Lounesto, "Clifford Algebras and Hestenes Spinors", Found. Phys. 23, 1203-1237 (1993).

17. S. Carlip and C. De Witt-Morette, "Where the Sign of the Metric Makes a Difference", Phys. Rev. Lett. 60, 1599-1601 (1988).

18. D. Brill, "Euclidean Maxwell-Einstein theory", arXiv:gr-qc/9209009. In: Topics on Quantum Gravity and Beyond: Essay in Honor of Louis Witten on His Retirement. F. Mansouri and J.J. Scanio, eds. (World Scientific: Singapore, 1993).

19. E. Zumpino, "A brief study of the transformation of Maxwell equations in Euclidean four-space", J. Math. Phys. 27 (1986) 1315-1318.

20. L. Witten, "Gravitation, An Introduction to Current Research", ed. L. Witten, John Wiley, New York (1962).

21. V. V. Varlamov, "Discrete Symmetries and Clifford Algebras", Int. J. Theor. Phys. 40 (2001) 769-805.

22. A. Carlini and J. Greensite, "Why is space-time Lorentzian?", Phys. Rev. D49 (1994) 866-878.

23. H. van Dam and Y. J. Ng, "Why 3+1 metric rather than 4+0 or 2+2?", Phys. Lett. B520 (2001) 159-162.

24. W. E. Baylis and G. Sobczyk, "Relativity in Clifford's Geometric Algebras of Space and Spacetime", Int. J. Theor. Phys. 43 (10), 2061-2079 (2004); arXiv: math-ph/0405026 (2004).