Continued fractions of cubic Laurent series

Dzmitry Badziahin

November 15, 2024

Abstract

We construct continued fraction expansions for several families of the Laurent series in \(\mathbb{Q}[t^{-1}]\). To the best of the author’s knowledge, this is the first result of this kind since Gauss derived the continued fraction expansion for \((1 + t)^r, r \in \mathbb{Q}\) in 1813. As an application, we apply an analogue of the hypergeometric method to one of those families and derive non-trivial efficient lower bounds on the distance \(|x - \frac{p}{q}|\) between one of the real roots of \(3x^3 - 3tx^2 - 3ax + at, a, t \in \mathbb{Z}\) and any rational number, under relatively mild conditions on the parameters \(a\) and \(t\). We also show that every real cubic irrational \(x\) admits a (generalised) continued fraction expansion in a closed form that can be explicitly computed. Finally, we provide an infinite family of cubic irrationals \(x\) that have arbitrarily (but finitely) many better-than-expected rational approximations. That is, for any \(\tau < 3 + \frac{15\ln 2}{24} \approx 3.4332\ldots\) there exists \(c > 0\) such that the inequality \(\|qx\| < (H(x)^\tau q e^{\sqrt{\ln q}})^{-1}\) has many solutions in integer \(q\).

Keywords: cubic irrationals, continued fractions, continued fractions of Laurent series, effective rational approximations of algebraic numbers

Math Subject Classification 2020: 11J68, 11J70

1 Introduction

Let \(\xi \in \mathbb{R}\) be an algebraic number of degree \(d > 1\). The classical theorem of Liouville states that there exists a constant \(c > 0\) such that

\[ \left| \xi - \frac{p}{q} \right| > \frac{c}{q^d} \]

for all rational numbers \(p/q\). Moreover, the constant \(c = c(\xi)\) can be explicitly computed. Liouville used that result to construct the first explicit examples of transcendental numbers. Later, in a series of papers by Thue, Siegel, Dyson, Gelfond and finally Roth [16], the power of \(q\) in Liouville’s theorem was reduced to \(2 + \varepsilon\) for any \(\varepsilon > 0\). This result is sharp because for \(\varepsilon = 0\), according to the classical Dirichlet theorem, the opposite inequality holds for infinitely many integers \(p, q\).

While the Roth theorem is very powerful, one of its weaknesses is that it is ineffective: for any \(d \geq 3\) and \(\tau > 0\) it does not allow to construct the constants \(c > 0\) and \(q_0\) such that the inequality

\[ \left| \xi - \frac{p}{q} \right| > \frac{c}{q^{d-\tau}} \]

holds for all \(p/q \in \mathbb{Q}\) with \(q > q_0\).

Effective analogues of the Roth theorem were studied by many number theorists starting from the 1960s. However, they are still far from being optimal. One of the approaches is based
on Feldman’s refinement of the theory of linear forms in logarithms [9]. Its advantage is that it improves the estimate of Liouville for all algebraic numbers. However, this improvement is usually extremely tiny. For state-of-the-art results regarding this approach, we refer to the book of Bugeaud [6].

Another approach was introduced by Bombieri in 1982 [4]. He managed to make the technique of Dyson [8] from 1947 effective for an infinite class of number fields of large degree. Later, this technique was further developed by Bombieri himself, Mueller, Vaaler and Van der Poorten among others. For example, in [5] the authors considered cubic extensions of number fields. In particular, they show that for all $\varepsilon > 0$, the parameters $a, b \in \mathbb{Z}$ with $|a| > e^{1000}$ and $|a| > C|b|^{2+\varepsilon}$, and for one of the roots $\xi$ of the cubic equation $x^3 + ax + b = 0$, the inequality

$$|\xi - \frac{p}{q}| < q^{-\frac{2 \log(|a|^3)}{\log(|a|^3/|b|)}} - \frac{14}{(3 \log(|a|)^{1/3}) - \varepsilon}$$

has finitely many solutions. Moreover, all of them can be effectively found. However, huge bounds on the coefficient $a$, denominator $q$ and the implied constants make this result hardly applicable in practice. We end this paragraph with the work of Wakabayashi [19] who provided better effective bounds on rational approximations for many cubic algebraics, compared to [5].

Now we will dwell on the third approach which historically appeared first and is usually called a hypergeometric method. It was introduced in 1964 in the work of A. Baker [1] and was later improved by Chudnovsky [7]. This method provides much better lower bounds for the distance $|\xi - p/q|$ compared to the previous methods, but only for algebraic numbers of certain specific forms. The most thoroughly studied numbers are $(1 + \frac{a}{n})^r$ where $r \in \mathbb{Q}$ and $a, n \in \mathbb{Z}$ with $|a|$ considerably smaller than $n$. The most recent results about these functions are perhaps due to Bennett [2] and Voutier [17]. For example, Voutier showed that

$$|\sqrt{2} - \frac{p}{q}| > \frac{1}{4q^{2.4325}}$$

for all integer $p$ and $q$. In the last three decades, the hypergeometric method helped to achieve similar results about some other families of algebraic numbers, see for example [12, 18, 19].

The core idea of the hypergeometric method is to consider Padé approximants of an algebraic Taylor series $x(z) \in \mathbb{Q}[[z]]$ and show that their specialisations at $z = \frac{a}{n}$ provide a family of good rational approximations to the algebraic number $x(a/n)$. One of the ways to do that for $(1 + z)^r$ is to describe those approximants as the convergents of the following continued fraction which was already known by Gauss:

$$(1 + z)^r = \frac{1}{1 + \frac{-rz}{1 + \frac{(r + 1)z}{2 + \frac{(1-r)z}{2 + \ldots}}}}$$

Here its $(2k + 1)$’st partial quotient is $\frac{(k-r)z}{2k+1}$ and its $2k$’th partial quotient is $\frac{k(r+k)z}{2k}$. To the best of the author’s knowledge and to their surprise, it seems that no continued fraction expansions are known for any other algebraic series in $\mathbb{Q}[[z]]$ of degree at least 3, since the work of Gauss in 1813.

In this paper, instead of $\mathbb{Q}[[z]]$, we consider the space $\mathbb{Q}[t^{-1}]$ of the Laurent series. It admits a very similar theory of continued fractions to that in the classical case of real numbers. We make a brief introduction to it in Section 2 and refer to [14] for more details.

Below we present the list of several families of cubic algebraic Laurent series that admit the continued fraction expansions in closed form. There is no reason to believe that this list is
exhaustive. For each item there, \( x = x(t) \) is the unique Laurent series in \( \mathbb{Q}[\![t^{-1}]\!] \) that solves the corresponding cubic equation and satisfies \( \deg(x) > 0 \). The existence and uniqueness of such \( x \) is justified in Section 3 (Lemma 3). The partial quotients in continued fractions are overlined to indicate that the rule for them is periodic, while the partial quotients themselves are not. The term \( k \) in the notation indicates the periodic template’s number where we start counting from \( k = 0 \).

1. For \( 3x^3 - 3tx^2 - 9x + t = 0 \) the continued fraction of \( x \) is

\[
K \left[ \frac{(3k + 2)(3k + 4)}{t \ (2k + 3)t} \right]
\]

2. For \( 3x^3 - 3tx^2 + 9x - t = 0 \) the continued fraction of \( x \) is

\[
K \left[ \frac{(3k + 2)(3k + 4)}{t \ (-1)^{k+1}(2k + 3)t} \right]
\]

3. For \( x^3 - tx^2 - at = 0 \) where \( a \in \mathbb{Q} \) is a parameter, the continued fraction of \( x \) is

\[
K \left[ \frac{3(12k + 1)(3k + 1)a \ 3(12k + 5)(3k + 2)a \ 3(12k + 7)(6k + 5)a \ 3(12k + 11)(6k + 7)a}{t \ (8k + 3)t \ (8k + 5)t \ 2(8k + 7)t \ (8k + 9)t} \right]
\]

4. For \( x^3 - tx^2 - a = 0 \) where \( a \in \mathbb{Q} \) is a parameter, the continued fraction of \( x \) is

\[
K \left[ \frac{3(12k + 1)(3k + 1)a \ 3(12k + 5)(3k + 2)a \ 3(12k + 7)(6k + 5)a \ 3(12k + 11)(6k + 7)a}{t \ (8k + 3)t^2 \ (8k + 5)t \ 2(8k + 7)t^2 \ (8k + 9)t} \right]
\]

5. For \( 3x^3 - 3tx^2 - 3ax + at = 0 \) where \( a \in \mathbb{Q} \) is a parameter, the continued fraction of \( x \) is

\[
K \left[ \frac{2(3k + 1)a \ (6k + 1)a \ 2(3k + 2)a^2 \ (6k + 5)a^2}{t \ 3(4k + 1)t \ t \ 3(4k + 3)t(t^2 + 2a) \ t} \right]
\]

6. For \( x^3 + (t - 2)x^2 - 2(t - 2)x + 2(t - 2) = 0 \) the continued fraction of \( x \) is

\[
K \left[ \frac{2(6k + 1)(3k + 1) \ 6(4k + 1)(3k + 2) \ 3(4k + 5)(6k + 5)}{-t \ (-1)^k((4k + 1)t + 2k) \ (-1)^{k+1}(4k + 3)(t^2 + 3t - 1) \ (-1)^k((4k + 5)t + 2k + 3)} \right]
\]

Remark 1. In the above list, we provide one representative from the infinite equivalence class of continued fractions. Indeed, given one continued fraction of some algebraic series, we can construct continued fractions for many other series by simply appending or removing partial quotients at the front of the continued fraction or by replacing \( t \) with a polynomial \( T \in \mathbb{Q}[\![t]\!] \). We provide a rigorous definition of the equivalence relation in Section 2.

Remark 2. The second equation transforms to the first one if one makes the change of variables \( x \mapsto ix, t \mapsto it \) where \( i \) is the imaginary unit. Hence the first two series will become equivalent under the extended equivalence relation. On the other hand, if we specialise both series \( x \) for \( t \in \mathbb{Q} \), we obtain essentially different continued fractions of algebraic numbers. The situation with series \( \#3 \) and \( \#4 \) is somewhat similar. If instead of the constant parameter \( a \in \mathbb{Q} \) in \( \#4 \) one uses the linear function \( at \) and then makes cancelations of \( t \) from numerators and denominators of the resulting continued fraction, they come up with the continued fraction \( \#3 \).
Remark 3. One can check that the series \( x = (1 + t^{-1})^{1/3} \) from the hypergeometric approach described above is equivalent to the series \( \#2 \). Indeed, they transform one to another by the change of the variables \( t \mapsto \frac{t-3}{6} \) and \( x \mapsto \frac{1+x}{1-x} \).

In Section \[3\] we establish the continued fractions for algebraic series \( \#1 \) and \( \#5 \). Given Remark 2, that automatically establishes the continued fraction for series \( \#2 \) as well. The proofs for the other continued fractions are very similar. Therefore, we only provide the necessary data for them in Appendix \[A\]. An interested reader can substitute that data into the proofs from Subsection \[3.3\] and verify the remaining continued fractions.

The above continued fractions shed the light on rational approximations to cubic irrationals. First of all, in Section \[4\] we show that an analogue of the hypergeometric method applies to the series \( \#5 \) and gives a substantial effective improvement of Liouville’s estimate on \( ||\xi - \frac{p}{q}|| \) for the solutions of the equation \( 3x^3 - 3tx^2 - 3ax + at = 0 \) under certain relatively mild conditions on integers \( a \) and \( t \). For convenience, we use the notation \( ||\xi|| \) which means the distance from \( \xi \) to the nearest integer.

**Theorem 1** Let \( a, t \) be positive integers such that \( t^2 \geq 9a \) and \( \xi \) be the unique real solution of the equation \( 3x^3 - 3tx^2 - 3ax + at = 0 \) that satisfy \( \xi > \frac{t}{2} \). Define

\[
\tau_1 := \frac{4(3t^2 + a)}{\sqrt{\pi}}, \quad \tau_2 := \frac{105\sqrt{\pi}e^2a^4}{2\sqrt{\pi}t(t^2 + 2a)(3t^2 + a)},
\]

\[
c_2 = \frac{144(t^2 + a)^3}{c_1^2e^{2}}, \quad c_3 := \frac{c_1^2e^{2}t^4(t^2 + 2a)^2}{9a^3(t^2 + a)} \quad \text{and}
\]

\[
c_1 := \frac{1}{\sqrt{3e}} \cdot \exp\left( -\sum_{p \in \mathbb{P}} \frac{\ln p}{p(p - 1)} \right) \approx 0.16948.
\]

Assume that \( c_3 > e \). Then for all \( q \geq \frac{c_1}{2\tau_2} \) one has

\[
||q\xi|| > \frac{(\log c_3)^{1/2}}{6\tau_1 c_2^3(2\tau_2)} \cdot q^{\frac{\log c_4}{\log c_3}} \cdot \log(2\tau_2q) \cdot \frac{\log c_4}{\log c_3} \frac{1}{2q}. \quad (3)
\]

While the parameters in the theorem look complicated, it gives good non-trivial lower bounds on \( ||q\xi|| \). They become better than in the classical Liouville theorem if \( \log c_2 < 2\log c_3 \) or equivalently, \( c_2 < c_3^2 \). In terms of \( a \) and \( t \), the last inequality is

\[
11664a^{12}(t^2 + a)^5 < c_1^6e^6t^8(t^2 + 2a)^4.
\]

With software like Wolfram Mathematica, one can verify that this inequality is always satisfied as soon as \( t > 10.34a^2 \). To the best of the author’s knowledge, Theorem 1 is the first result of this type for the family of cubic irrationals with the minimal polynomial \( 3x^3 - 3tx^2 - 3ax + at \).

Also notice that as the parameter \( a \) is fixed and \( t \to \infty \), the fraction \( \frac{\log c_2}{\log c_3} \) approaches 1 and hence for all \( \varepsilon > 0 \) there exists \( t_0(\varepsilon) \) such that for all \( t > t_0 \) the solution \( \xi \) from Theorem 1 satisfies \( ||q\xi|| > q^{-1-\varepsilon} \). The computations of the lower bounds on \( ||q\xi|| \) for some values of \( t \) and \( a \) are provided in Table 1.

It is worth mentioning that the lower bounds in Theorem 1 are not best possible. Direct numerical computations of the first several hundred convergents \( p_n/q_n \) of the continued fraction \( \#2 \) suggest that better bounds should take place. For example, for \( a = 1 \) they indicate that even with \( t = 3 \) we should get a non-trivial lower bound on \( ||q\xi|| \). We provide those better numerical bounds in the last column of Table 1. They can be considered as the
For all \( x \gg 3 \):

\[
\begin{array}{|c|c|c|c|}
\hline
a & t & \text{Equation} & \text{For all } q \gg & \text{one has } ||q\xi|| \gg & \text{Numeric evidence} \\
\hline
1 & 3 & x^3 - 3x^2 - x + 1 = 0 & 29 & q^{-4.276} & q^{-1.57} \\
1 & 4 & 3x^3 - 12x^2 - 3x + 4 = 0 & 536 & q^{-3.166} & q^{-1.98} \\
1 & 11 & 3x^3 - 33x^2 - 3x + 11 = 0 & 27812480 & q^{-1.96} & q^{-1.26} \\
1 & 12 & x^3 - 12x^2 - x + 4 = 0 & 71929013 & q^{-1.92} & q^{-1.49} \\
1 & 30 & x^3 - 30x^2 - x + 10 = 0 & 1.67 \cdot 10^{12} & q^{-1.62} & q^{-1.35} \\
2 & 6 & x^3 - 6x^2 - 2x + 4 = 0 & 49 & q^{-5.81} & q^{-1.93} \\
2 & 42 & x^3 - 42x^2 - 2x + 28 = 0 & 6.6 \cdot 10^{10} & q^{-1.94} & q^{-1.35} \\
2 & 43 & 3x^3 - 126x^2 - 6x + 86 = 0 & 8.6 \cdot 10^{10} & q^{-1.98} & q^{-1.63} \\
3 & 7 & x^3 - 7x^2 - 3x + 7 = 0 & 5 & q^{-12.19} & q^{-1.46} \\
3 & 94 & x^3 - 94x^2 - 3x + 94 = 0 & 8 \cdot 10^{12} & q^{-1.97} & q^{-1.29} \\
3 & 95 & x^3 - 95x^2 - 3x + 95 = 0 & 9 \cdot 10^{12} & q^{-1.99} & q^{-1.15} \\
\hline
\end{array}
\]

Table 1: lower bounds for \( ||q\xi|| \)

limitation of our method. In fact, for some pairs \((a, t)\) the numerators and denominators of the convergents seem to be both divisible by a big power of two or three, which leads to substantially better estimates on \( \gcd(p_{4k+2}, q_{4k+2}) \) than in Lemma 11. But even without this phenomenon, some of the estimates in the proof of Theorem 1 do not look optimal (for example, one in (38) and probably (34), (35)).

We expect that an analogue of the hypergeometric method also applies to other cubic series from the list. We will explore that direction in further research. It is worth mentioning that every cubic series can be easily transformed to the series \( N4 \) by an appropriate Möbius map.

Another important consequence of the constructed continued fractions is that they provide cubic irrationals \( \xi \) with very good rational approximations \( p/q \). In other words, with help of these continued fractions one can construct infinitely many pairs \( (p/q, \xi) \in Q \times A_3 \) with very small distances \( |p/q - \xi| \) in terms of the heights of \( p/q \) and \( \xi \). Here, by \( A_3 \) we denote the set of all real cubic irrationals. As numerical computations suggest (we will talk about them in detail later), these distances are close to, if not one of, smallest possible among all pairs in \( Q \times A_3 \). This allows us to investigate the question of providing an effective uniform lower bound for \( ||q\xi|| \) that is satisfied for all cubic irrationals \( \xi \).

Denote by \( P_\xi(x) \) the minimal polynomial of \( \xi \) with integer coprime coefficients. In this paper, by the height of \( \xi \) we mean its naive height, i.e. the maximal absolute value of the coefficients of \( P_\xi \), and denote it by \( H(\xi) \). In Section 3 we prove the following result.

**Theorem 2** For any \( \tau < 3 + \frac{15 \ln 2}{24} \approx 3.4332 \ldots \) there exists an effectively computable constant \( c > 0 \) such that for all \( N_0 \in N \) there exist infinitely many cubic irrationals \( \xi \in K_3 \) such that the inequality

\[
||q\xi|| < \frac{1}{H(\xi)^\tau q e^{\sqrt{\ln q}}} (4)
\]

has more than \( N_0 \) solutions \( q \in N \).

Notice that, according to the Khintchine theorem, for almost all \( \xi \in \mathbb{R} \) and all but finitely many \( q \in \mathbb{Z} \) one must have \( ||q\xi|| \geq (\ln^{1+c} q)^{-1} \). Therefore Theorem 2 provides cubic irrationals with much better estimates on \( ||q\xi|| \) than what is expected, for many (but finitely many) values of \( q \).
We also provide heuristic evidence that the power $\tau$ in Theorem 2 is nearly best possible and formulate the following conjecture.

**Conjecture A** For any $\varepsilon > 0$ define $\tau_0 = 3 + \frac{2}{3} \ln 2 + \varepsilon \approx 3.462 + \varepsilon$. There exists an absolute constant $C = C(\varepsilon)$ such that for all real cubic irrationals $\xi$ all partial quotients $a_n$ satisfy

$$|a_n| \leq C n^{1+\varepsilon} H(\xi)^{\tau_0}.$$ 

In view of the classical estimates for the convergents: $q_n \leq \phi^{n-1}$ and $||q_n \xi|| < (a_{n+1}q_n)^{-1}$ where $\phi = \frac{\sqrt{5}+1}{2}$, we can reformulate Conjecture A in a more standard form:

**Conjecture A*** Let $\varepsilon > 0$, $\tau_0$ and $C$ be as in Conjecture A. There exists an effectively computable constant $c$ such that for all real cubic irrational numbers $\xi$ and all $q \in \mathbb{N}$,

$$||q \xi|| \geq \frac{c}{H(\xi)^{\tau_0} q (\ln q)^{1+\varepsilon}}.$$ 

Moreover, one can take $c = \frac{\ln^2 \phi}{C}$.

This conjecture not only implies the Roth theorem for cubic irrationals but also the stronger conjecture of Lang [11] which says that $||q \xi|| \gg (q \ln q)^{1+\varepsilon}$ where the implied constant may depend on $\xi$. Conjecture A is out of reach by current methods and we believe it will be extremely hard to prove.

In Section 5, we provide heuristic arguments supporting the statement that all partial quotients $a_n$ which come from the discovered continued fractions are not bigger than

$$|a_n| \leq C n^2 H(\xi)^{3+\frac{2\ln 2}{2.88}},$$ 

where the constant $C$ can be explicitly computed. To further verify Conjecture A, we have numerically computed the continued fraction expansions of a wide set of algebraic numbers. For the sake of simplifying the code, we only considered numbers inside the interval $(0,1)$ and if a cubic equation has more than one root in that region, we considered only one of them. Therefore our search is not exhaustive. For all polynomials of height at most 2, we computed the partial quotients up to index 10000, for all $P$ with $H(P) \leq 5$ up to index 5000, for all $P$ with $H(P) \leq 10$ up to index 1000 and for all $P$ with $H(P) \leq 100$ up to index 50. As a result, we have found only 6 instances when the constant $C$ in (6) is bigger than 8. All of them are for $\xi$ with $H(\xi) \leq 7$. On top of that, there are only four more instances with $C > 2$. The largest discovered $C = 17.751 \ldots$ is for the root of $2x^3 + 2x^2 + 2x - 1 = 0$. Since the search was not exhaustive, we can not claim this is the right value for Conjecture A. However it is probably safe to write that the conjecture is satisfied for $C = 100$ and that for $C = 2$ it is possible to explicitly write down all its counterexamples. In Appendix 13 we list all discovered cubic irrationals with $C > 2$. We conclude with a conjecture, where all the parameters are made specific.

**Conjecture B** Let $\tau_3 = 3 + \frac{2\ln 2}{2.88} \approx 3.4814$. For all real cubic irrationals $\xi$ all partial quotients $a_n$ satisfy

$$|a_n| \leq 100 n^2 H(x)^{\tau_3}.$$ 

Also, for all $q \in \mathbb{Z}$ and all cubic irrationals $\xi$ one has

$$||q \xi|| \geq \frac{1}{440 H(\xi)^{\tau_3} q (\ln q)^2}.$$
As the last application, in Section 6 we demonstrate that every cubic irrational number admits a (generalised) continued fraction expansion in a closed form.

**Theorem 3** Let $\xi \in \mathbb{R}$ be a cubic algebraic number. Then there exists a Möbius transformation $\mu : x \mapsto \frac{ax+b}{cx+d}$ with $a, b, c, d \in \mathbb{Z}$ such that $\mu(\xi)$ enjoys the continued fraction expansion of the form \( [1, \ldots] \) which converges to $\mu(\xi)$.

Notice that if we have a continued fraction expansion $K\left[ \begin{array}{ccc} a_0 & \beta_1 & \beta_2 & \cdots \\ \beta_1 & a_1 & a_2 & \cdots \end{array} \right]$ of $y = \mu(x)$ then we can construct a continued fraction for $x = \mu^{-1}(y) = \frac{aw+v}{wz+ux}$ too as follows:

$$x = \frac{u}{w} + \frac{vw - uz}{wz + w^2a_0 + \frac{w^2\beta_1}{a_1 + \frac{\beta_2}{a_2 + \cdots}}}.$$ 

Hence we can provide a continued fraction expansion in the closed form for all cubic $\xi \in \mathbb{R}$.

## 2 Laurent series

Let $\mathbb{F}$ be a field. Consider the set $\mathbb{F}[[t^{-1}]]$ of the Laurent series together with the valuation: $\| \sum_{k=-d}^{\infty} c_k t^{-k} \| = d$, the biggest degree $d$ of $t$ having non-zero coefficient $c_{-d}$. Sometimes in the paper we call it the degree of a series because it matches the definition of the degree for polynomials. We also use the notation $[f]$ for the polynomial part of $f$, i.e. $[f] := \sum_{k=-d}^{0} c_k t^{-k}$. It is well known that in this setting the notion of continued fraction is well defined. In other words, every $f(t) \in \mathbb{F}[[t^{-1}]]$ can be written as

$$f(t) = [a_0(t), a_1(t), a_2(t), \ldots] = a_0(t) + \frac{1}{a_1(t) + \frac{1}{a_2(t) + \cdots}},$$

where $a_i(t) \in \mathbb{F}[t]$ are called partial quotients, and $\text{deg}(a_i) \geq 1$ for all $i \geq 1$. We refer the reader to a nice survey [14] for more properties of the continued fractions of Laurent series.

In this paper, we have $\mathbb{F} = \mathbb{Q}$ and then for a given $f \in \mathbb{Q}[[t^{-1}]]$ the partial quotients $a_i$ are polynomials with rational coefficients. It will be more convenient to renormalise this continued fraction by multiplying its numerators and denominators by appropriate integer numbers so that all the coefficients of $a_i$ are integer:

$$f(t) = \frac{1}{\beta_0} \left( a_0(t) + \frac{\beta_1}{a_1(t) + \frac{\beta_2}{a_2(t) + \cdots}} \right) =: K\left[ \begin{array}{ccc} \beta_0 & \beta_1 & \beta_2 & \cdots \\ \beta_1 & a_1(t) & a_2(t) & \cdots \end{array} \right],$$

where $\beta_i \in \mathbb{Z} \setminus \{0\}$ and $a_i(z) \in \mathbb{Z}[t]$ for $i \geq 1$. If $\beta_0 = 1$ then sometimes we omit $\beta_0$ in the notation and write $K\left[ \begin{array}{ccc} \beta_1 & \beta_2 & \cdots \\ \beta_1 & a_1(t) & a_2(t) & \cdots \end{array} \right]$.

By analogy with the classical continued fractions over $\mathbb{R}$, by $k$’th convergent of $f$ we denote the rational function

$$\frac{p_k(t)}{q_k(t)} := K\left[ \begin{array}{ccc} \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_k \\ a_0(t) & a_1(t) & a_2(t) & \cdots & a_k(t) \end{array} \right].$$

The convergents satisfy the following recurrent relation:

$$p_0(z) = a_0(z), \quad p_1(z) = a_0(z)a_1(z) + \beta_1, \quad p_n(z) = a_n(z)p_{n-1}(z) + \beta_n p_{n-2}(z),$$

(7)
\( q_0(z) = 1, \quad q_1(z) = a_1(z), \quad q_n(z) = a_n(z)q_{n-1}(z) + \beta_n q_{n-2}(z). \) \hfill (8)

The important property of convergents is that they are the best rational approximants of the series \( f \). That is, the analogue of the Lagrange’s theorem is true: \( p(t)/q(t) \in \mathbb{Q}(t) \) with coprime polynomials \( p \) and \( q \) is a convergent of \( f \in \mathbb{Q}[[t^{-1}]] \) if and only if \( \deg(f - p/q) < -2\deg(q) \).

By \( k' \)th full quotient of \( f \) we denote the continued fraction
\[
 f_{k}(t) := K \begin{bmatrix} \beta_k & \beta_{k+1} & \beta_{k+2} & \cdots \\ a_k(t) & a_{k+1}(t) & a_{k+2}(t) & \cdots \end{bmatrix},
\]

One can easily verify that the consecutive full quotients of \( f \) satisfy the relation:
\[
 f_{k+1} = \varphi_{a_k, \beta_k} \circ f_k := \frac{1}{\beta_k f_k - a_k}.
\] \hfill (9)

In this paper we consider algebraic Laurent series \( x(t) \), i.e. solutions of the equations of the form
\[
 b_d t^d + b_{d-1} t^{d-1} + \cdots + b_1 t + b_0 = 0
\] \hfill (10)

where \( b_0, b_1, \ldots, b_d \in \mathbb{Q}[t] \). By a modification of the Newton-Puiseux theorem, there exists a neighbourhood \( U_\infty \subset \mathbb{C} \) of infinity such that \( x \), considered as a Laurent series in \( \mathbb{C}[[t^{-1}]] \), converges in \( U_\infty \setminus \{\infty\} \) and the limit is a holomorphic function that we also write as \( x(t) \). If \( \deg x > 0 \) then it has a pole at infinity.

Consider the sequence \( p_k/q_k \) of convergents of \( x \). Since they are all in \( \mathbb{C}(t) \), they are all holomorphic in the whole space \( \mathbb{C} \). Suppose that, as holomorphic functions, \( p_k/q_k \) converge uniformly in some neighbourhood \( V_\infty \) of \( \infty \). Then \( f \) is also a holomorphic function in \( V_\infty \) and hence has a Laurent series expansion that is convergent in some neighbourhood \( V^*_\infty \subset V_\infty \) of infinity. But then it must coincide with the expansion of \( x \).

Finally, we can analytically continue both functions \( f \) and \( x \) and derive the following statement: as soon as the functions \( p_k/q_k \) converge uniformly in some connected neighbourhood \( U \) of \( \infty \), the limit \( \lim_{k \to \infty} \frac{p_k(t)}{q_k(t)} \) equals \( x(t) \), which is a solution of the equation (10). Note that we do not need the Laurent series of \( x \) to converge at \( t \) for this statement to take place. In other words, as soon as we compute the continued fraction expansion of \( x \in \mathbb{Q}[[t^{-1}]] \), for all \( t \in U \) its specialisation at \( t \) is also a continued fraction expansion of an algebraic number \( x(t) \).

If a continued fraction for some algebraic \( x \in \mathbb{Q}[[t^{-1}]] \) is constructed then one can immediately construct many more continued fractions by one of the following operations:

- append or remove several first partial quotients;
- replace the variable \( t \) by \( P(t) \) where \( P \in \mathbb{Q}[t] \).

Therefore, in order to consider only essentially different continued fractions, we define the equivalence relation between them. We say that two series \( x \) and \( y \) are equivalent if the continued fraction of one of them can be achieved from that of another one by a finite number of transformations: adding or removing one partial quotient at the beginning of the continued fraction and replacing the variable \( t \) by \( P(t) \).

One observes that if \( x = K \begin{bmatrix} \beta_1 & \beta_2 & \cdots \\ a_0 & a_1 & a_2 & \cdots \end{bmatrix} \) then the continued fraction of \( ux + v \) for \( u, v \in \mathbb{Z} \) can be written as \( K \begin{bmatrix} u \beta_1 & \beta_2 & \cdots \\ ua_0 + v & a_1 & a_2 & \cdots \end{bmatrix} \), i.e. \( x \sim ux + v \). It is also easy to verify that \( x \sim 1/x \). Indeed, that is obvious if \( \deg x \neq 0 \), i.e. \( a_0(t) = 0 \) or \( \deg a_0 \geq 1 \). In the case when \( a_0 = \text{const} \), one verifies that
\[
 \frac{1}{x} = K \begin{bmatrix} -\beta_1 & a_0 \beta_2 & \beta_3 & \cdots \\ 1/a_0 & a_0a_1 + \beta_2 & a_2 & a_3 & \cdots \end{bmatrix}.
\]
By combining the last two statements, we derive that if \( x \) and \( y \) are related by the Möbius transformation \( y = \frac{ax + b}{cx + d} \), \( a, b, c, d \in \mathbb{Z} \), \( ad - bc \neq 0 \), then \( x \sim y \). The other easy-to-spot condition for equivalent series is as follows: if the algebraic equation for \( x \) is \( F(x, t) = 0 \) and the one for \( y \) is \( F(y, P(t)) = 0 \) for some \( P \in \mathbb{Z}[t] \) then \( x \sim y \).

We finish this section with a well known result about transformations of a continued fraction that do not change its limit. Since we did not find a good reference for it, we add its proof here.

**Lemma 1** Let \( \mathbf{K} = \left[ \ldots \beta_{i-1} \beta_i \beta_{i+1} \ldots \right] \) be a continued fraction and \( A \neq 0 \) a rational number. Then the transformations

1. \( \beta_{i-1} \mapsto \beta_{i-1}/A, a_{i-1} \mapsto a_{i-1}/A, \beta_i \mapsto \beta_i/A \)
2. \( \beta_{i-1} \mapsto \beta_{i-1}/A, a_{i-1} \mapsto a_{i-1}/A, a_i \mapsto a_i A, \beta_{i+1} \mapsto \beta_{i+1} A \)

do not change the limit of the continued fraction. Moreover, the convergents \( p_i/q_i \) do not change under these transformations too.

**Proof.** Let \( (p_n/q_n)_{n \in \mathbb{Z}_{\geq 0}} \) be the convergents of the initial continued fraction and \( (p^*_n/q^*_n)_{n \in \mathbb{Z}_{\geq 0}} \) the convergents of the modified one. Then we have

\[
p^*_i = \frac{a_{i-1}}{A} p_{i-2} + \frac{\beta_{i-1}}{A} p_{i-3} = \frac{p_{i-1}}{A}.
\]

Analogously, \( q^*_i = \frac{q_{i-1}}{A} \), therefore \( p_{i-1}/q_{i-1} = p^*_i/q^*_i \).

Next, in the first case we have

\[
p^*_i = a_i p^*_{i-1} + \frac{\beta_i}{A} p_{i-2} = \frac{p_i}{A}.
\]

Analogously, we have that \( q^*_i = \frac{q_i}{A} \) and again \( p^*_i/q^*_i = p_i/q_i \). Then one can easily check that for all \( j > i \), \( p^*_j = \frac{p_j}{A} \) and \( q^*_j = \frac{q_j}{A} \) and the claim of the lemma is verified.

In the second case, we have \( p^*_i = a_i Ap^*_{i-1} + \beta_i p_{i-2} = p_i \) and analogously \( q^*_i = q_i \). Also, \( p^*_{i+1} = a_i+1 p^*_{i-1} + \beta_{i+1} p_{i-2} = p_{i+1} \) and the same is true for \( q^*_{i+1} = q_{i+1} \). Finally, since all other partial quotients remain unchanged, we have for all \( j > i + 1 \), \( p^*_j = p_j \) and \( q^*_j = q_j \) and the claim is verified.

\[ \Box \]

### 3 Constructing continued fractions

#### 3.1 Riccati equation

Let \( x(t) \in \mathbb{Q}[[t^{-1}]] \) be a cubic series with the minimal polynomial \( P(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 \in \mathbb{Q}[t][x] \). One of the main tools for constructing the continued fractions of algebraic series is to understand how the equation for \( x \) changes under the transform \( \varphi_{a, \beta} \) from \([1]\), where \( a \in \mathbb{Q}[t] \), \( \beta \in \mathbb{Q} \). While it is not difficult to compute the minimal polynomial of \( \varphi_{a, \beta}(x) \) for a given algebraic \( x \), after applying a numbers of such transforms, very quickly the coefficients of the minimal polynomials of the full quotients \( x_k(t) \) become very complicated. Their degrees usually grow to infinity and their heights grow rapidly too. It is very hard to recognise any pattern in them. Instead, we first notice that \( x(t) \) satisfies a Riccati differential
Comparison of terms at \( x \)

Therefore, \( x'(t) \) belongs to the same cubic extension of \( \mathbb{Q}(t) \) as \( x \), and we can decompose it in the basis \( 1, x, x^2 \). One can do that in the following way. Let \( x' \) satisfy the equation \( x' = u + vx + wx^2 \) where \( u, v, w \in \mathbb{Q}(t) \) are rational functions to be found. Then we write
\[
(b_1 + 2b_2x + 3b_3x^2)(u + vx + wx^2) + (b'_0 + b'_1x + b'_2x^2 + b'_3x^3) = (\beta + \alpha x)(b_0 + b_1x + b_2x^2 + b_3x^3).
\]

Comparison of terms at \( x^4 \) and \( x^3 \) gives
\[
\alpha = 3w \quad \text{and} \quad \beta = 3v + \frac{b'_3 - b_2w}{b_3}.
\]

Next, comparison of terms at \( 1, x \) and \( x^2 \) gives the system of linear equations
\[
\begin{pmatrix}
b_3b_1 & -3b_3b_0 & b_2b_0 \\
2b_3b_2 & -2b_2b_1 & b_3b_1 - 3b_3b_0 \\
3b_3^2 & -b_3b_2 & b_2^2 - 2b_3b_1
\end{pmatrix}
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix}
= \begin{pmatrix}
b_0b'_3 - b_3b'_0 \\
b_1b'_3 - b_3b'_1 \\
b_2b'_3 - b_3b'_2
\end{pmatrix}.
\]

The application of Cramer’s rule to this equation leads to the following proposition:

**Proposition 1** Let \( x(t) \in \mathbb{Q}((t^{-1})) \) be a solution of a cubic equation \( b_3x^3 + b_2x^2 + b_1x + b_0 = 0 \). Then \( x \) is a solution of the following Riccati differential equation:
\[
Dx' = A + Bx + Cx^2
\]

where \( A, B, C, D \in \mathbb{Q}[t] \) are computed by the following formulae:
\[
D = \det \begin{pmatrix}
b_1 & -3b_0 & b_2b_0 \\
2b_2 & -2b_1 & b_3b_1 - 3b_3b_0 \\
3b_3 & -2b & b_2^2 - 2b_3b_1
\end{pmatrix} = 4b_3b_1^2 + 4b_2b_0 + 27b_3b_0^2 - b_3^2b_1^2 - 18b_3b_2b_1b_0;
\]
\[
A = \frac{1}{b_3} \det \begin{pmatrix}
b_0b'_3 - b_3b'_0 & -3b_0 & b_2b_0 \\
b_1b'_3 - b_3b'_1 & -2b_1 & b_2b_1 - 3b_3b_0 \\
b_2b'_3 - b_3b'_2 & -b_2 & b_2^2 - 2b_3b_1
\end{pmatrix};
\]
\[
B = \frac{1}{b_3} \det \begin{pmatrix}
b_1 & b_0b'_3 - b_3b'_0 & b_2b_0 \\
2b_2 & b_1b'_3 - b_3b'_1 & b_2b_1 - 3b_3b_0 \\
3b_3 & b_2b'_3 - b_3b'_2 & b_2^2 - 2b_3b_1
\end{pmatrix};
\]
\[
C = \det \begin{pmatrix}
b_1 & -3b_0 & b_0b'_3 - b_3b'_0 \\
2b_2 & -2b_1 & b_1b'_3 - b_3b'_1 \\
3b_3 & -b_2 & b_2b'_3 - b_3b'_2
\end{pmatrix}.
\]

**Remark.** Notice that \(-D\) equals the discriminant of \( P(x) \).

The next step is to investigate how this equation changes under the transform \( \varphi_{a,b} \).

**Proposition 2** Let \( x \in \mathbb{Q}((t^{-1})) \) be the solution of the Riccati equation \( Dx' = A + Bx + Cx^2 \). Then \( y = \varphi_{a,b}(x) \) satisfies the equation \( \tilde{D}y' = \tilde{A} + \tilde{B}y + \tilde{C}y^2 \) where
\[
\begin{pmatrix}
\tilde{A} \\
\tilde{B} \\
\tilde{C} \\
\tilde{D}
\end{pmatrix} = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & -\beta & -2a & 0 \\
-\beta^2 & -\beta a & -a^2 & \beta a' \\
0 & 0 & 0 & \beta
\end{pmatrix} \begin{pmatrix}
A \\
B \\
C \\
D
\end{pmatrix}.
PROOF. Notice that
\[ \beta Dy' = \beta D \frac{d}{dt} \left( \frac{1}{\beta x - a} \right) = -\beta D (\beta x' - a') \]
\[ = -\beta^2 (A + Bx + Cx^2) + \beta Da'. \]

On the other hand,
\[ A' + B'y + C'y^2 = \frac{A'(\beta x - a)^2 + B'(\beta x - a) + C'}{(\beta x - a)^2}. \]

Equating the coefficients for $1, x$ and $x^2$ of the numerators of both expressions gives
\[ A' = -C, \quad B' = -\beta B - 2aC, \quad C' = -\beta^2 A - \beta aB - a^2 C + \beta a'D. \]

Then the Proposition follows immediately.

We have shown that every solution of a cubic equation is also a solution of the corresponding Riccati equation. However, each cubic equation may have more than one solution $x \in \mathbb{C}[t^{-1}]$. In fact, by the Newton-Puiseux theorem it always has three solutions, counting multiplicities, in the extension $\mathbb{C}[t^{-1/6}]$. In further discussion, we will always assume that all the solutions are distinct, otherwise the cubic equation can be reduced to one of a smaller degree. Riccati equations also have multiple solutions. The aim of the next lemma is to link a given solution of the cubic equation with a solution of the corresponding Riccati one.

**Lemma 2** Let $x_1, x_2, x_3 \in \mathbb{C}[t^{-1/6}]$ be three distinct solutions of the cubic equation $b_0 + b_1x + b_2x^2 + b_3x^3 = 0$ where $b_0, b_1, b_2, b_3 \in \mathbb{C}[t]$. Suppose that $\deg x_1 \geq 1$ and $\deg x_2, \deg x_3 \leq 0$. Then $x_1$ is the only solution of the corresponding Riccati equation (i) with the property $\deg x \geq 1$.

**Proof.** Without loss of generality, assume that $\deg x_2 \geq \deg x_3$. Proposition 1 implies that all three series $x_1, x_2, x_3$ are the solutions of the Riccati equation (i). Let $x \in \mathbb{C}[t^{-1/6}]$ be any other solution of (i), distinct from the three solutions above. Then it is well known that
\[ K := \frac{(x_1 - x_3)(x_2 - x)}{(x_2 - x_3)(x_1 - x)} \]
is a constant. Then we have
\[ x = \frac{x_2(x_1 - x_3) - Kx_1(x_2 - x_3)}{(x_1 - x_3) - K(x_2 - x_3)} \]
The degree of the right hand side is at most $\deg x_2 \leq 0$.

**Lemma 3** Let $a \in \mathbb{Q}[t]$ and $\beta \in \mathbb{Q}$ such that $\deg a \geq 1$. If the Riccati equation (i) has the unique solution $x \in \mathbb{Q}((t^{-1}))$ such that $\deg x \geq 1$ then the Riccati equation $Dy' = \tilde{A} + \tilde{B}y + \tilde{C}y^2$ for $y = \varphi_{a, \beta}(x)$ has at most one solution $y$ with $\deg y \geq 1$.

**Proof.** One can verify that $\varphi_{a, \beta}$ is the bijection between the solutions of (i) and $Dy' = \tilde{A} + \tilde{B}y + \tilde{C}y^2$. Now, if $y_1, y_2$ are both the solutions of the latter equation with $\deg y_1, \deg y_2 \geq 1$ then $\deg \varphi_{a, \beta}^{-1}(y_i) = \deg \left( \left( \frac{1}{y_i} + a \right) / \beta \right) \geq 1$ for both $i = 1, 2$. By assumption, such a series must be unique, i.e. $\varphi_{a, \beta}^{-1}(y_1) = \varphi_{a, \beta}^{-1}(y_2)$ or equivalently $y_1 = y_2$.

After we verify that the equation (i) has the unique solution of degree at least 1, one can readily construct it by comparing the coefficients at the corresponding powers of $t$ in (i). In the next two lemmata we will cover two cases that will be needed later. However an analogous approach can be applied for any quadruple of polynomials $A, B, C$ and $D$. 

---

11
Lemma 4 Let $A, B, C, D \in \mathbb{Q}[t]$ be polynomials of degrees $s_a, s_b, s_c$ and $s_d$ respectively such that $A = \sum_{n=0}^{\infty} r_{a,n} t^n$, $B = \sum_{n=0}^{\infty} r_{b,n} t^n$, $C = \sum_{n=0}^{\infty} r_{c,n} t^n$, $D = \sum_{n=0}^{\infty} r_{d,n} t^n$. Here, by convention, $r_{a,i} = r_{b,i} = r_{c,i} = r_{d,i} = 0$ for all $i > s_a, i > s_b, i > s_c$ and $i > s_d$ respectively. Assume that $s_d = s_b + 1$, $s_b > s_c$, $s_b \geq s_a$ and $d r_{d,s_a} \neq r_{b,s_b}$ for all $d \in \mathbb{N}$ with $0 < d < 2(s_b - s_c)$. Let $x$ be a solution of the equation $(11)$ of degree at least one. Then its degree and leading coefficient are computed by the formulae

$$s_x = \deg x = s_b - s_c, \quad r_{x,s_x} = \frac{r_{d,s_d}(s_b - s_c) - r_{b,s_b}}{r_{c,s_c}}. \tag{16}$$

**Proof.** Let $s_x$ be the degree of $x$, i.e. $x = \sum_{n=-\infty}^{\infty} r_{x,n} t^n$ with $r_{x,s} \neq 0$. Then the degrees of the terms $Dx', Ax, Bx$ and $Cx^2$ are $s_d + s_x - 1$, $s_a$, $s_b + s_x$ and $s_c + 2s_x$ respectively. In order for $r_{x,s}$ to be nonzero, the maximum of those four values need to be achieved at at least two of them. For $s_x > s_b - s_c$ we have that

$$s_c + 2s_x > s_d + s_x - 1 = s_b + s_x > s_a$$

which is a contradiction. Therefore $s_x \leq s_b - s_c$.

Suppose now that $0 < s_x < s_b - s_c$. Then the comparison of the coefficients at $t^{s+s_b}$ in (11) gives $s r_{d,s_d} r_{x,s} = r_{b,s,b} r_{x,s}$. But that contradicts the conditions of the lemma and the fact $r_{x,s} \neq 0$.

Hence we have $s_x = s_b - s_c$. Comparison of the leading coefficients now gives $s_x r_{d,s_d} r_{x,s_x} = r_{b,s_b} r_{x,s_x} + r_{c,s_c} r_{x,s_x}^2$ and (16) immediately follows.

Note that we can continue the comparison of the coefficients at $t^{s+s_b}$ in (11) for $s = s_x - 1, s_x - 2, \ldots$. That will allow us to identify all the coefficients of the polynomial part of $x$. However, this process is quite technical and not very enlightening. Therefore we omit it in this paper.

Lemma 5 With the same notation as in Lemma 4, assume that $s_d > s_b + 1$, $s_d > s_c + 1$ and $s_d > s_a$. Assume that the equation (11) has the unique solution $x$ of degree at least 1. Then its degree and leading coefficient are computed by the formulae

$$s_x = s_d - s_c - 1, \quad r_{x,s_x} = \frac{(s_d - s_c - 1)r_{d,s_d}}{r_{c,s_c}}. \tag{17}$$

**Proof.** The proof is analogous to that of Lemma 4. We notice that if $\deg x = s_x > s_d - s_c - 1$ the degrees of each term in $Dx', Ax, Bx$ and $Cx^2$ compare as follows:

$$s_c + 2s_x > s_d + s_x - 1 = s_b + s_x; \quad s_c + 2s_x > s_a$$

which is impossible as the maximum must be attained at at least two of those values. For $s < s_d - s_c - 1$, we get

$$s_d + s - 1 > s_c + 2s, \quad s_d + s - 1 > s_a, \quad s_d + s - 1 > s_b + s.$$  

That is again impossible. Hence the only possibility is $s_x = s_d - s_c - 1$.

Now, comparing the leading terms gives us $s_x r_{d,s_d} r_{x,s_x} = r_{c,s_c} r_{x,s_x}^2$ and (17) follows.

We end this section by showing that all the cubic polynomials $\mathbb{N}$ 1–6 indeed have only one solution $x$ of positive degree. This is the straightforward implication of the following
Lemma 6 Suppose that \( b_0, b_1, b_2, b_3 \in \mathbb{Q}[t] \) satisfy \( \text{deg}(b_3) = 0, \text{deg}(b_2) = 1 \) and \( \text{deg}(b_0), \text{deg}(b_1) \leq 1 \). then the equation \( b_3 x^3 + b_2 x^2 + b_1 x + b_0 = 0 \) has exactly one solution of strictly positive degree.

Proof. Notice that if \( \text{deg} x > 0 \) then \( \text{deg}(b_0) \) and \( \text{deg}(b_1 x) \) are strictly smaller than \( \text{deg}(b_2 x^2) \). Suppose \( \text{deg}(x) = d > 0 \). Then the degrees of the monomials \( b_3 x^3 \) and \( b_2 x^2 \) are \( 3d \) and \( 1 + 2d \) respectively. Since the degrees of at least two monomials among \( b_3 x^3, b_2 x^2, b_1 x, b_0 \) must coincide, we derive \( d = 1 \).

Let \( x = \sum_{n=-1}^{\infty} r_n t^n \) and \( b_2 = \beta_1 t + \beta_2 \). Then expanding the equation \( b_3 x^3 + b_2 x^2 + b_1 x + b_0 = 0 \) and comparing the terms at \( t^3 \) gives \( r_1^2 (b_3 r_1 + \beta_1) = 0 \) or \( r_1 = -\beta_1 / b_3 \). Hence the term \( r_1 \) is uniquely determined.

Next, we consequently compare the coefficients at \( t^{3-k} \) where \( k = 1, 2, \ldots \) Each time we get a polynomial in \( r_1, \ldots, r_{1-k} \) where all terms except \( r_{1-k} \) are already determined in the previous steps. The terms involving \( r_{1-k} \) are \( 3b_3 r_1^2 r_{1-k} + 2\beta_1 r_1 r_{1-k} = -r_1 \beta_1 r_{1-k} \). In other words, we get a linear equation in \( r_{1-k} \) which has the unique solution. This implies that each term \( r_1, r_0, \ldots \) of \( x \) is uniquely determined and therefore there is exactly one solution \( x \) with \( \text{deg} x > 0 \).

\[ \Box \]

3.2 The solution of \( 3x^3 - 3tx^2 - 9x + t = 0 \)

As the first step, we use Proposition 1 to transfer this equation to Riccati one. However, there is another, less technical way to derive the Riccati equation for this particular cubic equation: by differentiating \( y = (1 + \tau^{-1})^{1/3} \) one can derive the Riccati equation for \( y \) and then apply the transform \( x = \frac{1+y}{1-y} \) and \( t = \frac{-\tau^{-3}}{6} \) which maps the series \( y(\tau) \) to \( x(t) \). We leave the details to an interested reader and proceed with the formulae (12)–(15):

\[ D = -4 \cdot 3^7 \cdot 4 \cdot 3^3 t^4 + 3^5 t^2 - 3^6 t^2 - 2 \cdot 3^6 t^2 = -108(t^2 + 9)^2; \]
\[ A = \frac{1}{3} \begin{pmatrix} -3 & -3t & -3t^2 \\ 0 & 18 & 18t \\ 9 & 3t & 9t^2 + 54 \end{pmatrix} = -108(t^2 + 9); \]
\[ B = \frac{1}{3} \begin{pmatrix} -9 & -3 & -3t^2 \\ -6t & 0 & 18t \\ 9 & 9 & 9t^2 + 54 \end{pmatrix} = 0; \]
\[ C = \begin{pmatrix} -9 & -3t & -3 \\ -6 & 18 & 0 \\ 9 & 3t & 9 \end{pmatrix} = -108(t^2 + 9). \]

We now divide both sides of the equation \( Dx' = Ax + Cx^2 \) by \(-108(t^2 + 9)\) and end up with the Riccati equation:

\[ (t^2 + 9)x' = 1 + x^2. \]

We already know that the initial cubic equation has exactly one solution \( x \) with \( \text{deg} x \geq 1 \) and the other two solutions \( y, z \) satisfy \( \text{deg} y, \text{deg} z \leq 0 \). Therefore by Lemma 2 \( x \) is the only solution of the Riccati equation with positive degree. Hence we can freely apply Lemmata 4 and 5 for this and subsequent equations.

We have \( s_d = 2, s_a = 0, s_b = -\infty \) and \( s_c = 0 \), therefore Lemma 5 is applicable. Together with comparing the coefficients of (18) at \( t \), it gives \([x] = t \) and \( x_1(t) = \frac{1}{x-t} = \varphi_{t,1}(x) \).
Proposition 2 states that $x_1(t)$ satisfies the equation $D_1x'_1 = A_1 + B_1x_1 + C_1x_1^2$ where
\[
\begin{pmatrix}
A_1 \\
B_1 \\
C_1 \\
D_1
\end{pmatrix} = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & -1 & -2t & 0 \\
-1 & -t & -t^2 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 \\
0 \\
1 \\
t^2 + 9
\end{pmatrix} = \begin{pmatrix}
-1 \\
-2t \\
8 \\
t^2 + 9
\end{pmatrix}.
\]

Notice that $A_1$ and $B_1$ are constants, $C_1$ is an odd linear function and $D_1$ is an even quadratic function. We will show that the solution $x$ of any Riccati equation with these properties enjoys an easily described continued fraction expansion. This result may be of independent interest.

**Proposition 3** Let $u_1, u_2, u_3, v_1, v_2 \in \mathbb{Q}$ be such that

1. $u_2, u_3, v_1 \neq 0$;
2. $iv_1 \neq u_2$ for all $i \in \mathbb{N}$;
3. $(i^2v_1 - iv_2)v_2 \neq u_1u_3$ for all $i \in \mathbb{N}$.

Let $x \in \mathbb{Q}((t^{-1}))$ be a solution of the equation $(v_1t^2 + v_2)x' = u_1 + u_2tx + u_3x^2$ such that $\deg x \geq 1$. Then $x$ is given by the formula
\[
x(t) = K \begin{bmatrix}
u_3 & \beta_1 & \beta_2 & \cdots \\
\alpha_0t & \alpha_1t & \alpha_2t & \cdots
\end{bmatrix}
\]
where $\alpha_i = (2i + 1)v_1 - u_2$ and $\beta_i = (i^2v_1 - iv_2)v_2 - u_1u_3$.

**Proof.** Let $x(t)$ have the continued fraction expansion $K \begin{bmatrix}
u_3 & \beta_1 & \beta_2 & \cdots \\
\alpha_0t & \alpha_1t & \alpha_2t & \cdots
\end{bmatrix}$ and $x_i(t)$ be the corresponding quotients $K \begin{bmatrix}
\beta_i & \beta_{i+1} & \beta_{i+2} & \cdots \\
\alpha_{i+1}t & \alpha_{i+2}t & \cdots
\end{bmatrix}$. Denote $u_3$ by $\beta_0$. Then by Proposition 2 and Lemma 4 to prove by induction that for all $i \in \mathbb{N}$ one has $D_i = D = v_1t^2 + v_2$, $A_i = -1$ ($i > 0$), $B_i = (u_2 - 2iv_1)t$, $C_i = (i^2v_1 - iv_2)v_2 - u_1u_3$ and $\alpha_i = \alpha_it$.

We start with $x(t)$. Since $\deg x > 0$ and $u_2 \neq v_1$, all the conditions of Lemma 4 are satisfied and it implies that $\deg x = 1$ and the leading coefficient $b_1$ of $x$ is $\frac{v_1 - u_2}{u_3}$. Comparing the coefficients for $t$ in the Riccati equation gives that $b_0 = 0$ and hence $[x] = \frac{v_1 - u_2}{u_3}t$. Since $v_1 \neq u_2$ and $u_3 \neq 0$, we can take $a_0 = (v_1 - u_2)t$ and $\beta_0 = u_3$. Then Proposition 2 gives

\[
\begin{pmatrix}
A_1 \\
B_1 \\
C_1 \\
D_1
\end{pmatrix} = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & -u_3 & -2(v_1 - u_2)t & 0 \\
u_3 - u_3(v_1 - u_2)t & -u_3(v_1 - u_2)t^2 & (v_1 - u_2)u_3 \\
0 & 0 & 0 & u_3
\end{pmatrix} \begin{pmatrix}
u_1 \\
u_2t \\
u_3 \\
(v_1 - u_2)t + v_2
\end{pmatrix}
\]

\[
= \begin{pmatrix}
u_3(u_2 - 2v_1)t \\
u_3((v_1 - u_2)v_2 - u_1u_3) \\
u_3(v_1t^2 + v_2)
\end{pmatrix}.
\]

By dividing all terms of the resulting Riccati equation by $u_3$ we achieve the base of induction.

Now assume that the assumptions are satisfied for $i \in \mathbb{N}$ and verify them for $i + 1$. Since the equation $(v_1t^2 + v_2)x'_i = -1 + (u_2 - 2iv_1)t x_i + ((i^2v_1 - iv_2)v_2 - u_1u_3)x_i^2$ for $x_i$ is of the same
form as for the initial series $x$ and all three conditions for its coefficients are satisfied, we use the same arguments as in the proof of the base of induction and get $a_i = (v_1 - (u_2 - 2i v_1))t = ((2i + 1)v_1 - u_2)t$ and $\beta_i = (i^2 v_1 - (2i - 2)u_2)v_2 - u_1 u_3$. Then Proposition 2 gives that the coefficients of the Riccati equation for $\varphi_{a_i, \beta_i}(y_i)$ are: $A_{i+1} = -1, B_{i+1} = (u_2 - 2i v_1 - 2v_1)t = (u_2 - 2(i+1) v_1)t; C_{i+1} = (v_1 - (u_2 - 2i v_1))v_2 + (i^2 v_1 - (2i - 2)u_2)v_2 - u_1 u_3 = ((i + 1)^2 v_1 - (i + 1)u_2)v_2 - u_1 u_3$. The inductive step is verified.

We now substitute the values $u_1 = -1, u_2 = -2, u_3 = 8, v_1 = 1, v_2 = 9$ into Proposition 3 and get $a_{i+1} = (2i + 1 + 2)t = (2(i + 1) + 1)t, \beta_{i+1} = 9(i^2 + 2i) + 8 = 9(i + 1)^2 - 1$. The continued fraction $\varphi_{1}$ is established.

### 3.3 The solution of $3x^3 - 3tx^2 - 3ax + at = 0$

As in the previous case, we first transfer the algebraic equation for $x$ into the Riccati one. Equations (12) - (15) give us

$$D = -4 \cdot 3^4 a^3 - 4 \cdot 3^3 a t^4 + 3^5 a^2 t^2 - 3^4 a^2 t^2 - 2 \cdot 3^5 a t^2 = -108(3a^3 + 3a^2 t^2 + at^4)$$

$$A = \frac{1}{3} \begin{pmatrix} -3a & -3at & -3at^2 \\ 0 & 6a & 0 \\ 9 & 3t & 9t^2 + 18a \end{pmatrix} = -108a^3;$$

$$B = \frac{1}{3} \begin{pmatrix} -3a & -3a & -3at^2 \\ -6t & 0 & 0 \\ 9 & 9 & 9t^2 + 18a \end{pmatrix} = -108a^2t;$$

$$C = \begin{pmatrix} -3a & -3at & -3a \\ -6t & 6a & 0 \\ 9 & 3t & 9 \end{pmatrix} = -108at^2.$$

Then divide all parts of the equation by $-108a$ and get $(3a^2 + 3at^2 + t^4)x' = a^2 + atx + t^2x^2$. Application of Lemma 5 together with the comparison of the coefficients at $t^3$ give $[x] = t$ and $x_1(t) = \varphi_{t,1}(x)$. We apply Proposition 2 to compute the coefficients of the equation $D_1 x_1' = A_1 + B_1 x_1 + C_1 x_1^2$:

$$\begin{pmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & -2t & 0 \\ -1 & -t & -t^2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a^2 \\ at \\ t^2 \\ t^4 + 3at^2 + 3a^2 \end{pmatrix} = \begin{pmatrix} -t^2 \\ -2t^3 - at \\ 2at^2 + 2a^2 \\ t^4 + 3at^2 + 3a^2 \end{pmatrix}.$$  

**Remark.** Notice that $A_1, C_1$ and $D_1$ are even polynomials in $t$, while $B_1$ is odd. Application of Lemma 4 together with the comparison of the coefficients at $t^3$ gives that $[x_1]$ is a constant multiple of $t$. Then a quick observation of the formula in Proposition 2 tells us that the coefficients $A_2, C_2, D_2$ are still even, while $B_2$ is odd. From these observation one can deduce the following interesting fact: if the coefficients $A, C$ of the Riccati equation are even quadratic polynomials, $D$ is even polynomial of degree 4 and $B$ is odd polynomial of degree 3 and all partial quotients of the continued fraction of the solution $x$ are linear then they are all constant multiples of $t$.

Next, we will show by induction the following formulae for the coefficients $a_i, b_i, c_i$ and the partial quotients $a_i, \beta_i$ of $x_i$:

$$A_{4k+4} = -(12k + 1)t^2 - 18ka, B_{4k+1} = -(12k + 2)t^3 - (12k + 1)at, C_{4k+1} = (6k + 2)at^2 + (6k + 2)a^2, (20)$$

$$a_{4k+1} = 3(4k + 1)t, \beta_{4k+1} = (6k + 2)a;$$
\[ A_{4k+2} = -t^2 - a, \quad B_{4k+2} = -(12k + 4)t^3 - (12k + 5)at, \quad C_{4k+2} = (6k + 1)(12k + 5)at^2 + 9(6k + 1)a^2; \]  
\[ a_{4k+2} = t, \quad b_{4k+2} = (6k + 1)a; \]  
\[ A_{4k+3} = -(12k + 5)t^2 - 9(2k + 1)a, \quad B_{4k+3} = -(12k + 6)t^2 - (24k + 13)at, \quad C_{4k+3} = (6k + 4)a^2; \]  
\[ a_{4k+3} = 3(4k + 3)(t^3 + 2at), \quad \beta_{4k+3} = (6k + 4)a^2. \]  
\[ A_{4k+4} = -(12k + 12)t^3 - (24k + 23)at, \quad C_{4k+4} = (6k + 5)a^2((12k + 13)t^2 + (18k + 18)a); \]  
\[ a_{4k+4} = t, \quad \beta_{4k+4} = (6k + 5)a^2. \]

Indeed, already computed values of \( A_i, B_i, C_i \) constitute the base of induction. Suppose now that for all \( 1 \leq i \leq 4k + 1 \) the values of \( A_i, B_i \) and \( C_i \) are given by the formulae above. Then Lemma [4] is applicable for \( x_{4k+1} \) and \( [16] \) gives \( x_{4k+1} = \frac{12k+3}{(6k+2)a}t \), thus we choose \( a_i = 3(4k + 1)t, \beta_i = (6k + 2)a \). Next, Proposition [2] infers

\[
\begin{pmatrix}
0 & 0 & -1 & 0 & 0 \\
0 & -(6k + 2)a & -6(4k + 1)t & 0 & 0 \\
-(6k + 2)a^2 & -(6k + 2)(4k + 1)at & -9(4k + 1)t^2 & 3(4k + 1)(6k + 2)a \\
0 & 0 & 0 & (6k + 2)a \\
\end{pmatrix}
\begin{pmatrix}
-12k + 3t^2 - 18ka \\
-(12k + 2)t^3 - (12k + 1)at \\
(6k + 2)a(t^2 + 6k + 2)a^2 \\
t^4 + 3at^2 + 3a^2 \\
\end{pmatrix}
\]

After division of both sides of the new equation by \((6k + 2)a\) we get the values \( A_{4k+2}, B_{4k+2} \) and \( C_{4k+2} \) as in \([21]\). Then application of Lemma \([3]\) gives \( x_{4k+2} = \frac{12k+5}{(6k+1)(12k+5)a}t \) and we choose \( a_{4k+2} = t, \beta_{4k+2} = (6k + 1)a \).

We apply Proposition \([2]\) once again to get

\[
\begin{pmatrix}
0 & 0 & -1 & 0 & 0 \\
0 & -(6k + 1)a & -2t & 0 & 0 \\
-(6k + 1)a^2 & -(6k + 1)(4k + 1)at & -t^2 & (6k + 1)a \\
0 & 0 & 0 & (6k + 1)a \\
\end{pmatrix}
\begin{pmatrix}
-12k + 4t^2 - (12k + 5)at \\
(6k + 1)(12k + 5)at^2 + 9(6k + 1)(2k + 1)a^2 \\
t^4 + 3at^2 + 3a^2 \\
\end{pmatrix}
\]

Dividing both sides of the new equation by \((6k + 1)a\) gives the values \( A_{4k+3}, B_{4k+3} \) and \( C_{4k+3} \) from \([22]\).

We apply Lemma \([4]\) to get that \( x_{4k+3} \) is a cubic polynomial with the leading coefficient \( \frac{3+12k+6}{(6k+4)a^2} \). After the comparison of the coefficients at both sides of the Riccati equation for \( t^5, t^4, t^3 \), we get that

\[ x_{4k+3} = \frac{3(4k + 3)}{(6k + 4)a^2}(t^3 + 2at) \]

and therefore we choose \( a_{4k+3} = 3(4k + 3)(t^3 + 2at), \beta_{4k+3} = (6k + 4)a^2 \).

We apply Proposition \([2]\) the third time

\[
\begin{pmatrix}
0 & 0 & -1 & 0 & 0 \\
0 & -(6k + 4)a^2 & -6(4k + 3)(t^3 + 2at) & 0 & 0 \\
-(6k + 4)^2a^4 & -(6k + 4)(12k + 9)a^2(t^3 + 2at) & -(12k + 9)^2(t^3 + 2at)^2 & 3(4k + 3)(6k + 4)a^2(3t^2 + 2a) \\
0 & 0 & 0 & (6k + 5)a^2 \\
\end{pmatrix}
\begin{pmatrix}
-6(4k + 3)(t^3 + 2at) \\
-(6k + 4)a^2 \\
(6k + 4)a^2((12k + 12)t^3 + (24k + 23)at) \\
(6k + 4)a^2(t^4 + 3at^2 + 3a^2) \\
\end{pmatrix}
\]

Dividing both sides of the Riccati equation by \((6k + 4)a^2\) gives the values \( A_{4k+4}, B_{4k+4} \) and \( C_{4k+4} \) from \([23]\).
Lemma 4 infers that \( x_{4k+4} = \frac{12k+13}{(6k+5)(12k+13)a^2} \) and we choose \( a_{4k+4} = t \).

\( \beta_{4k+4} = (6k+5)a^2 \).

We apply Proposition 2 the last time to get

\[
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & -6k+5a^2 & -2t & 0 \\
-(6k+5)^2a^2 & -(6k+5)a^2t & -(6k+5)a^2 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
-1 \\
-(6k+5)(12k+13)a^2t^2 + 18(6k+5)(k+1)a^3 \\
(6k+5)(12k+13)a^2t^2 + 18(6k+5)(k+1)a^3 \\
t^4 + 3at^2 + 3a^2 \\
\end{pmatrix}
\]

After division of both sides of the resulting equation by \((6k+5)a^2\), we derive the values \( A_{4k+5}, B_{4k+5} \) and \( C_{4k+5} \) from (20). That finishes the inductive step.

4 Application to rational approximations of cubic irrationals

In this section we prove Theorem 1. But first of all, we need to separate the root of the cubic equation \( x_5 \) whose continued fraction is (2) from the other two.

Lemma 7 Let \( t \in \mathbb{C} \) be such that \(|t|^2 > 9|\alpha| > 0\). Then the equation

\[ 3x^3 - 3tx^2 - 3ax + at = 0 \] (24)

has exactly one root \( \xi \) such that \(|\xi| > |\alpha|^{1/2}\). Moreover, if the continued fraction \( x_5 \) converges then it converges to \( \xi \).

PROOF. Let \( a \) be fixed. The leading term of the series \( x(t) \) from (2) is \( t \). Therefore, \( \lim_{t \to \infty} |x(t)| = \infty \). Moreover, for \( t > T_0 \), where \( T_0 \) is the radius of convergence of \( x(t) \), this function is analytic and therefore continuous. Hence, the continued fraction (2) of \( x(t) \) corresponds to a root of the equation \( 3x^3 - 3tx^2 - 3ax + at = 0 \) which continuously tends to infinity as \( t \to \infty \).

Let \( x \) be any complex number that satisfies \(|x| = |\alpha|^{1/2}\). We have

\[ |3x^3 - 3ax| < 3|\alpha|^{1/2} \cdot 2|a| = 6|a|^{3/2}. \]

On the other hand, \(|3tx^2 - at| \geq |t| \cdot 2|a| > 6|a|^{3/2}\). Therefore the number of roots of (24) inside \(|x| < |\alpha|^{1/2}\) is the same as one for \( 3tx^2 - at = 0 \), i.e. it is two. We thus get that there is only one root \( \xi \) of (24) outside that circle. Then, as \( t \to \infty \), this root \( \xi(t) \) continuously tends to infinity, because the other two cannot leave the circle \(|x| < |\alpha|^{1/2}\).

As the next step, we show that the continued fraction (2) uniformly converges in some neighbourhood \( U_\infty \subseteq \mathbb{C} \) of infinity. As discussed in Section 2 that will imply that the continued fraction (2), computed at \( t \in U_\infty \), will converge to the solution \( x(t) \in \mathbb{C} \) of a cubic equation \( x_5 \).

Lemma 8 Let \( a \in \mathbb{C} \) be fixed. Then the continued fraction (2) uniformly converges for all \( t \in U_\infty = \{ t \in \mathbb{C} : |t| \geq 12|a|^{2} \} \).

PROOF. For each \( k \in \mathbb{Z} \) we simultaneously divide the values \( \beta_{4k+1}, \beta_{4k+2} \) and \( a_{4k+1} \) by \( k \). and also divide \( \beta_{4k+3}, \beta_{4k+4} \) and \( a_{4k+3} \) by \( k \). As was verified in Lemma 4 such
transformation do not change the convergence and the limit of the continued fraction. For a modified continued fraction, one can readily verify that \( \forall t \in U_\infty \),

\[
3(4 + k^{-1})|t| > 2(3 + k^{-1})|a| + 2; \quad |t| > (6 + k^{-1})|a| + 2;
\]

\[
3(4 + 3k^{-1})|t| \cdot |t^2 + 2a| > 2(3 + 2k^{-1})|a^2| + 2; \quad |t| > (6 + 5k^{-1})|a^2| + 2.
\]

Then the easy modification of Pringsheim convergence criteria [10, Theorem 4.35] implies that

\[
K = \begin{pmatrix} 2(3 + k^{-1})a & (6 + k^{-1})a & 2(3 + 2k^{-1})a^2 & (6 + 5k^{-1})a^2 \\ 0 & 3(4 + k^{-1})t & t & 3(4 + 3k^{-1})t(t^2 + 2a) \end{pmatrix}
\]

uniformly converges in \( U_\infty \).

We will now have a closer look at the convergence of (2) in an open neighbourhood of \( \mathbb{R}_{\geq 1} \). We will see that the continued fraction uniformly converges in this region and therefore it can be analytically extended from \( U_\infty \) to contain \( \mathbb{R}_{\geq 1} \). For the rest of this section we assume that \( a \) is a positive integer and \( t \in \mathbb{R}_{\geq 1} \).

Let \( \xi = x(t, a) \) be the cubic irrational number from Lemma 7. Let \( p_n/q_n = p_n(t, a)/q_n(t, a) \) be the \( n \)th convergent of the continued fraction (2). We want to estimate the size of \( |\xi - p_n/q_n| \). We will do that for values of \( n \) of the form \( n = 4k + 2 \), \( k \in \mathbb{Z} \).

Denote

\[
S_k := \begin{pmatrix} p_{4k+2} & q_{4k+2} \\ p_{4k+1} & q_{4k+1} \end{pmatrix}; \quad T_k := \begin{pmatrix} p_{4k+2} & q_{4k+2} \\ p_{4k-2} & q_{4k-2} \end{pmatrix}; \quad U_k := \begin{pmatrix} p_{4k-1} & q_{4k-1} \\ p_{4k-2} & q_{4k-2} \end{pmatrix}.
\]

From (2) and the recurrent formulae for convergents we get that \( S_{k+1} = A_k S_k \) where

\[
A_k = \begin{pmatrix} t & (6k + 7)a \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3(4k + 5)t & (6k + 8)a \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t & (6k + 5)a^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3(4k + 3)t(t^2 + 2a) & (6k + 4)a^2 \\ 1 & 0 \end{pmatrix}.
\]  \hspace{1cm} (25)

For convenience, we denote the four matrices in the product above by \( C_{4k+1}, C_{4k+2}, C_{4k+3} \) and \( C_{4k+4} \) so that \( A_k = C_{4k+4} C_{4k+3} C_{4k+2} C_{4k+1} \). Then, the relation between \( T_{k+1} \) and \( S_k \) is

\[
T_{k+1} = \begin{pmatrix} a_{k11} & a_{k12} \\ 1 & 0 \end{pmatrix} S_k
\]  \hspace{1cm} (26)

where \( a_{k11} \) and \( a_{k12} \) are the corresponding entries of the matrix \( A_k \). One then computes (we used Wolfram Mathematika for convenience)

\[
a_{k11} = 9(4k + 3)(4k + 5)t^2(a^2 + a)(a^2 + 2a) + 3(4k + 5)(6k + 5)a^2t^2 + (6k + 5)(6k + 7)a^2,
\]

\[
a_{k12} = 6(4k + 5)(3k + 2)ta^2(t^2 + a).
\]  \hspace{1cm} (27)

Another application of the recurrent formula for convergents and (2) gives \( S_k = C_{4k} C_{4k-1} C_{4k-2} U_k \) or

\[
U_k = \frac{-1}{(6k - 1)(6k + 1)(6k + 2)a^4} B_k S_k =: d^{-1} B_k S_k
\]

where

\[
B_k = \begin{pmatrix} 0 & -(6k - 1)a^2 \\ -1 & t \end{pmatrix} \begin{pmatrix} 0 & -(6k + 2)a \\ -1 & 3(4k + 1)t \end{pmatrix} \begin{pmatrix} 0 & -(6k + 1)a \\ -1 & t \end{pmatrix}.
\]

Now we can get the relation between \( T_k \) and \( S_k \):

\[
T_k = \begin{pmatrix} 1 & 0 \\ b_{k21}/d & b_{k22}/d \end{pmatrix} S_k.
\]  \hspace{1cm} (28)
where \( b_{k21} \) and \( b_{k22} \) are the corresponding entries of the matrix \( B_k \). The computations show that
\[
b_{k21} = -(12k + 3)t^2 - (6k + 2)a, \quad b_{k22} = 3(4k + 1)t(t^2 + a). \tag{29}
\]

Finally, (26) and (28) provide the relation between \( T_k \) and \( T_{k+1} \):
\[
T_{k+1} = \frac{d}{b_{k22}} \begin{pmatrix}
  a_{k11} & a_{k12} \\
  1 & 0
\end{pmatrix} \begin{pmatrix}
  b_{k22}/d & 0 \\
  -b_{k21}/d & 1
\end{pmatrix} T_k = \begin{pmatrix}
  a_{k11} - a_{k12} b_{k21}/b_{k22} \\
  1 & 0
\end{pmatrix} \begin{pmatrix}
  a_{k11} \\
  b_{k21}/b_{k22}
\end{pmatrix} T_k =: D_k T_k. \tag{30}
\]

Now we compute
\[
d a_{k12} = (4k + 5)(6k - 1)(6k + 1)(6k + 2)(6k + 4)a^6
\]

\[
\frac{b_{k22}}{4k + 1}. \tag{31}
\]

**Lemma 9** Let \( a \in \mathbb{Z} \) be positive and \( t \in \mathbb{R}_{>1} \). Then the denominators \( q_{4k+2} \) and \( q_{4k+6} \) satisfy the inequalities
\[
9(4k + 3)(4k + 5)t^2(t^2 + a)(t^2 + 2a)q_{4k+2} < q_{4k+6} < 9(4k + 3)(4k + 5)(t^2 + a)^3 q_{4k+2} \tag{32}
\]

**Proof.** First of all, since all the entries of the matrices \( C_i \) are positive, we immediately get that \( q_n > 0 \) for all \( n \in \mathbb{N} \). The first inequality follows immediately from (26) and (27) and for the second observation we observe that \( t^2(t^2 + a)(t^2 + 2a) = (t^2 + a)^3 - t^2a^2 - a^3 \) and therefore (27) can be rewritten as
\[
a_{k11} = 9(4k + 3)(4k + 5)(t^2 + a)^3 - 6(4k + 5)(3k + 2)t^2a^2 - 4(27k^2 + 54k + 25)a^3. \tag{33}
\]

We also notice that \( q_{4k+2} = t q_{4k+1} + (6k + 1)a q_{4k} \), therefore (27) implies
\[
a_{k12} q_{4k+1} = 6(4k+5)(3k+2)a^2(t^2+a)(q_{4k+2} - (6k+1)a q_{4k}) < 6(4k+5)(3k+2)a^2(t^2+a)q_{4k+2}.
\]

Finally, we substitute the last inequality and (33) into (26) and get
\[
q_{4k+6} < (9(4k + 3)(4k + 5)(t^2 + a)^3 - (6k + 5)(6k + 8a^3)q_{4k+2},
\]
and the second inequality in (32) follows immediately.

Now we are ready to provide the lower and upper bounds for \( q_{4k+2} \). In view of (2), one can easily compute that \( q_2 = 3t^2 + a \). Also, the Stirling’s formula together with the evaluation of the Wallis integral gives that \( \forall k \in \mathbb{Z}_{>2}, \)
\[
2k \left( \frac{2k}{e} \right)^k < (2k + 1)!! < 4k \left( \frac{2k}{e} \right)^k.
\]

We can now get the estimates on the denominators \( q_{4k+2} \):
\[
q_{4k+2} < (3t^2 + a)(4k + 1)!!(9(t^2 + a)^3)^k < 8(3t^2 + a)kc_6^k \cdot k^{2k} \quad \text{and} \quad \tag{34}
\]
\[
q_{4k+2} > (3t^2 + a)(4k + 1)!!(9(t^2 + a)(t^2 + 2a))^k > 4(3t^2 + a)kc_7^k \cdot k^{2k}, \tag{35}
\]

where \( c_6 = 144(t^2 + a)^3/e^2 \) and \( c_7 = 144t^2(t^2 + a)(t^2 + 2a)/e^2. \)

In the next step, we estimate
\[
\frac{P_{4k+6}}{q_{4k+6}} - \frac{P_{4k+2}}{q_{4k+2}}.
\]
Lemma 10. For all \( a \in \mathbb{N} \) and \( t \geq 1 \) with \( a^3 < t^2(t^2 + a)(t^2 + 2a) \) the continued fraction (2) converges. Moreover, its limit \( \xi = x(t, a) \) satisfies

\[
\left| \frac{p_{4k+2}}{q_{4k+2}} - \frac{p_{4k+6}}{q_{4k+6}} \right| < \left| \xi - \frac{p_{4k+2}}{q_{4k+2}} \right| < 2 \left| \frac{p_{4k+2}}{q_{4k+2}} - \frac{p_{4k+6}}{q_{4k+6}} \right| < \tau_3 c_4^k,
\]

where

\[
\tau_3 = \frac{105\sqrt{3}e^2a^4}{8t(t^2 + 2a)(3t^2 + a)^2} \quad \text{and} \quad c_4 = \frac{a^6}{16t^4(t^2 + a)^2(t^2 + 2a)^2}.
\]

In particular, the continued fraction converges uniformly in \( t \).

Proof. Equation (31) together with (30) give

\[
\frac{p_{4k+2}}{q_{4k+2}} - \frac{p_{4k+6}}{q_{4k+6}} = \frac{(4k+5)(6k-1)(6k+1)(6k+2)(6k+4)q_{4k-2}q_{4k+2}p_{4k+2}}{(4k+1)q_{4k+2}q_{4k+6}q_{4k+2}} \left( \frac{p_{4k+2}}{q_{4k+2}} - \frac{p_{4k-2}}{q_{4k-2}} \right).
\]

In particular, that means that differences \( \frac{p_{4k+6}}{q_{4k+6}} - \frac{p_{4k+2}}{q_{4k+2}} \) share the same sign for all \( k \in \mathbb{N} \). By Lemma 9, the big fraction in the last equation is bounded from above by

\[
\frac{(4k+5)(6k-1)(6k+1)(6k+2)(6k+4)a^6}{(4k+1)\cdot 9^2(4k-1)(4k+1)(4k+3)(4k+5)(t^2(t^2 + a)(t^2 + 2a)^2)} < \frac{1}{2}.
\]

One computes \( \frac{p_{4k+2}}{q_{4k+2}} \) as the sum

\[
\frac{p_{4k+2}}{q_{4k+2}} = \frac{p_2}{q_2} + \sum_{i=1}^{k} \left( \frac{p_{4i+2}}{q_{4i+2}} - \frac{p_{4i-2}}{q_{4i-2}} \right).
\]

Since the terms in the summation decrease by a factor bigger than two, the right hand side has a limit as \( k \to \infty \) and moreover,

\[
\xi = \lim_{k \to \infty} \frac{p_{4k+2}}{q_{4k+2}} < \frac{p_2}{q_2} + 2 \left( \frac{p_6}{q_6} - \frac{p_2}{q_2} \right).
\]

Now, the first two inequalities in (36) readily follow from

\[
\xi = \frac{p_{4k+2}}{q_{4k+2}} + \sum_{i=k}^{\infty} \left( \frac{p_{4i+6}}{q_{4i+6}} - \frac{p_{4i+2}}{q_{4i+2}} \right).
\]

For the remaining inequality in (36), we use (31) to get

\[
p_{4k+6}q_{4k+2} - p_{4k+2}q_{4k+6} = \prod_{i=0}^{k} \det D_{i+1} \det T_1
\]

\[
= \frac{4k+5}{5} \cdot a^{6k} \prod_{i=1}^{k} (6i - 1)(6i + 1)(6i + 2)(6i + 4) \det T_1
\]

\[
= \frac{(4k+5)a^{6k}}{5 \cdot 2 \cdot 4} \cdot (6k + 1)(6k + 2)(6k + 4) \cdot \frac{(6k)!}{3^{2k}(2k)!} \det T_1.
\]

The last expression is bounded from above by

\[
< 126k^4a^{6k} \cdot \sqrt[3]{127k(6k/e)^{6k}} \cdot \frac{\sqrt{3\pi k}(6k/e)^{6k}}{3^{2k}(2k)!} \det T_1 = 126\sqrt{3}k^4 \cdot c_5^k \cdot k^{4k} \det T_1.
\]
where $c_5 = 6^4 a^6 / e^4$. To compute $\det T_1$, we use (26) which gives

$$\det T_1 = -60ta^2(t^2 + a) \det S_0.$$ 

Direct computations then reveal

$$p_1 = 3t^2 + 2a, \quad p_2 = 3t(t^2 + a), \quad q_1 = 3t, \quad q_2 = 3t^2 + a$$

and therefore $\det S_0 = -2a^2$. Combining all that information together, we get

$$\det T_{k+1} < 15120 \sqrt{3}a^4 (t^2 + a) k^4 \cdot c_5^k \cdot k^{4k}.$$ 

The second inequality in (36) implies

$$\left| \xi - \frac{p_{4k+2}}{q_{4k+2}} \right| < 2 \cdot \frac{\det T_{k+1}}{q_{4k+2} q_{4k+6}} < 2 \cdot \frac{15120 \sqrt{3}a^4 (t^2 + a)}{16(3t^2 + a)^2 c_7} \cdot \left( \frac{c_5}{c_7^2} \right)^k$$

$$= \frac{105 \sqrt{3} e^2 a^4}{8t(t^2 + 2a)(3t^2 + a)^2} \left( \frac{c_5}{c_7^2} \right)^k.$$ 

Finally, a direct computation implies that $c_5 / c_7^2 = c_4$ and hence the last inequality in (36) is verified.

Notice that if we do not restrict $t$ to positive real numbers, then the sequence of the convergents $\frac{p_{4k+2}}{q_{4k+2}}$ considered as functions of $t$ converges uniformly as soon as the absolute value of the fraction (37) together with the value of $|c_4|$ is strictly less than some absolute constant $\sigma < 1$. For all $t \in \mathbb{R}$ that satisfy the conditions of Lemma 10 there exists an open neighbourhood $U(t) \in \mathbb{C}$ of $t$ where those upper bounds are satisfied. Therefore, we can analytically continue the limit of the continued fraction (2) from $U_\infty$ to $U_\infty \cup \bigcup_t U(t)$.

This observation justifies that the continued fraction (2) converges to the root of the cubic equation $\mathbb{N}5$ for all positive integer $a$ and all $t \in \mathbb{R}_{>1}$ such that $t^2(t^2 + a)(t^2 + 2a) > a^3$. Notice that the last inequality follows from the condition $t^2 \geq 9a$ of Theorem 1.

In the further discussion we always assume that $t \in \mathbb{N}$. We then provide a series of good approximations to $\xi$ which will later be used to show that there are no too good rational approximations of $\xi$.

**Lemma 11** For all $t \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 2}$,

$$\gcd(p_{4k+2}, q_{4k+2}) \geq \prod_{\substack{p \leq 5 \quad \text{p odd} \quad \text{p} | 4m + 5}} p^\left\lfloor \frac{2k}{p^2} \right\rfloor. \quad (38)$$

**Proof.** Consider a positive integer $m$ and any integer divisor $p | 4m + 5$. To save the space we denote $r := t(t^2 + 2a)$. Then $m \equiv -5/4 \pmod{p}$ and one can check that for all integer $l$

$$C_{4(m+l)+4} = \begin{pmatrix} t & (6(m + l) + 7)a \\ 1 & 0 \end{pmatrix} \equiv \begin{pmatrix} t & (6l - 1/2)a \\ 1 & 0 \end{pmatrix} \pmod{p};$$
\[ C_{4(m+l)+3} = \begin{pmatrix} 3(4(m+l) + 5)t & (6(m+l) + 8)a \\ 1 & 0 \end{pmatrix} \equiv \begin{pmatrix} 12lt & (6l + 1/2)a \\ 1 & 0 \end{pmatrix} \pmod{p}; \]
\[ C_{4(m+l)+2} = \begin{pmatrix} t & (6(m+l) + 5)a^2 \\ 1 & 0 \end{pmatrix} \equiv \begin{pmatrix} t & (6l - 5/2)a^2 \\ 1 & 0 \end{pmatrix} \pmod{p}; \]
\[ C_{4(m+l)+1} = \begin{pmatrix} 3(4(m+l) + 3)t & (6(m+l) + 4)a^2 \\ 1 & 0 \end{pmatrix} \equiv \begin{pmatrix} (12l - 6)t & (6l - 7/2)a^2 \\ 1 & 0 \end{pmatrix} \pmod{p}. \]

We will show by induction by \( l \) that
\[ C_l := \prod_{k=4m+3+l}^{4m+3-l} C_k \equiv \begin{pmatrix} 0 & \Pi_{i=0}^{l-1} c_i \\ \Pi_{i=0}^{l-1} c_i & 0 \end{pmatrix} \pmod{p}, \quad \text{where} \quad \begin{cases} \ c_{4i} = -(6i - 1/2)a \\ \ c_{4i+1} = (6i + 5/2)a^2 \\ \ c_{4i+2} = -(6i + 7/2)a^2 \\ \ c_{4i+3} = (6i + 13/2)a \end{cases}. \]

Indeed, the base of induction \( l = 0 \) is straightforward and the inducional step is verified by the sequence of matrix multiplications:
\[ C_{4(m+j)+4} C_{4(m-j)+2} \equiv \begin{pmatrix} t & (6j + 1/2)a \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \Pi_{i=0}^{j+1} c_i \\ \Pi_{i=0}^{j+1} c_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \Pi_{i=0}^{j+1} c_i \\ \Pi_{i=0}^{j+1} c_i & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \Pi_{i=0}^{j+1} c_i \\ \Pi_{i=0}^{j+1} c_i & 0 \end{pmatrix}; \]
\[ C_{4(m+j)+5} C_{4m-j+1} C_{4(m-j)+1} \equiv \begin{pmatrix} (12j + 6)t & (6j + 5/2)a^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \Pi_{i=0}^{j+1} c_i \\ \Pi_{i=0}^{j+1} c_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \Pi_{i=0}^{j+1} c_i \\ \Pi_{i=0}^{j+1} c_i & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \Pi_{i=0}^{j+1} c_i \\ \Pi_{i=0}^{j+1} c_i & 0 \end{pmatrix}; \]
\[ C_{4(m+j)+6} C_{4m-j+2} C_{4(m-j)+1} \equiv \begin{pmatrix} (12j + 7/2)a^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \Pi_{i=0}^{j+1} c_i \\ \Pi_{i=0}^{j+1} c_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \Pi_{i=0}^{j+1} c_i \\ \Pi_{i=0}^{j+1} c_i & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \Pi_{i=0}^{j+1} c_i \\ \Pi_{i=0}^{j+1} c_i & 0 \end{pmatrix}; \]
\[ C_{4(m+j)+7} C_{4m-j+3} C_{4(m-j)+1} \equiv \begin{pmatrix} (12j + 11/2)a \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \Pi_{i=0}^{j+1} c_i \\ \Pi_{i=0}^{j+1} c_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \Pi_{i=0}^{j+1} c_i \\ \Pi_{i=0}^{j+1} c_i & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \Pi_{i=0}^{j+1} c_i \\ \Pi_{i=0}^{j+1} c_i & 0 \end{pmatrix}. \]

The last observation implies that, as soon as \( c_i \equiv 0 \pmod{p} \), the entries of \( C_l \) with indices 11 and 12, as well as all the entries of \( C_{l+1} \) are multiples of \( p \). Now, if \( p \) is of the form \( 4l + 1 \) then it divides \( 4m + 5 \) for all \( m = l - 1 \) + \( rp \) where \( r \) is an arbitrary integer. If on top of that \( 3 \mid p \) then either \( p = 4l + 1 = 12i + 5 \) for some integer \( i \) and hence \( p \mid c_{4i+1} \) or \( p = 12i + 13 \) and \( p \mid c_{4i+3} \). In both cases, we have that \( p \mid c_j \) for an integer \( j \in \left[ \frac{2i+1}{3}, \frac{2i+2}{3} \right] \) and hence the product of matrices \( C_{4p+2p} C_{4p+2p-1} \cdots C_{4p+1} \) has all its entries divisible by \( p \). Similarly, if \( p \) is of the form \( 4l + 3 \), then for any \( m = 3l + 1 + r p \) with \( r \in \mathbb{Z} \) we have \( p \mid 4m + 5 \). Also, if \( 3 \nmid p \) then \( p \mid c_j \) for either \( j = \frac{2i+2}{3} \) or \( j = \frac{2i+4}{3} \). It is easy to derive then that all the entries of the product \( C_{4p+4p} C_{4p+4p-1} \cdots C_{4p+2p+1} \) are divisible by \( p \).

In a similar way we can deal with any divisor \( p \) of \( 4m + 3 \). In that case we have \( m \equiv -3/4 \pmod{p} \) and the following congruences modulo \( p \) are satisfied
\[ C_{4(m+l)+4} \equiv \begin{pmatrix} t & (6l + 5/2)a \\ 1 & 0 \end{pmatrix}; \quad C_{4(m+l)+3} \equiv \begin{pmatrix} (12l + 6)t & (6l + 7/2)a \\ 1 & 0 \end{pmatrix}; \]
\[ C_{4(m+l)+2} \equiv \begin{pmatrix} t & (6l + 1/2)a^2 \\ 1 & 0 \end{pmatrix}; \quad C_{4(m+l)+1} \equiv \begin{pmatrix} 12lr & (6l - 1/2)a^2 \\ 1 & 0 \end{pmatrix}. \]

By induction by \( l \), one can verify that
\[ D_l := \prod_{k=4m+1+l}^{4m+1-l} C_k \equiv \begin{pmatrix} 0 & \Pi_{i=0}^{l} d_i \\ \Pi_{i=0}^{l} d_i & 0 \end{pmatrix} \pmod{p}, \quad \text{where} \quad \begin{cases} \ c_{4i} = -(6i + 1/2)a^2 \\ \ c_{4i+1} = (6i + 7/2)a \\ \ c_{4i+2} = -(6i + 5/2)a \\ \ c_{4i+3} = (6i + 11/2)a^2 \end{cases}. \]
Similarly as before, for \( p \) of the form \( 4l + 1 \) such that \( 3 \nmid p \) we have that all the entries of the product \( C_{4rp+4}C_{4rp+4p-1} \cdots C_{4rp+2p+1} \) are divisible by \( p \). And for \( p \) of the form \( 4l + 3 \), \( 3 \nmid p \), we have that all the entries of \( C_{4rp+2p}C_{4rp+2p-1} \cdots C_{4rp+1} \) are divisible by \( p \).

The upshot of the above arguments is that the product of matrices \( \prod_{i=k}^{l} C_k \) can be split in at least \( \left\lfloor \frac{k}{2p} \right\rfloor \) pieces that have all their entries divisible by \( p \). Hence all the entries of the whole product are divisible by \( p^{\left\lfloor \frac{k}{2p} \right\rfloor} \). After collecting the information for all primes \( p \geq 5 \) and taking into account that \( S_k = \prod_{i=4k}^{l} C_k S_0 \) we get (38).

For simplicity, denote the right hand side of (38) by \( g(k) \). We estimate this function by something nicer. We have

\[
2\sum_{i=1}^{\infty} \frac{2k}{\pi} \cdot 3\sum_{i=1}^{\infty} \frac{2k}{\pi} \cdot \prod_{p \in \mathbb{P}} p^{\frac{2k}{p}} \cdot \prod_{p \in \mathbb{P}} p^{\frac{2k}{p}} \geq (2k)! \geq \sqrt{4\pi k} \left( \frac{2k}{e} \right)^{2k}.
\]

Therefore

\[
g(k) \geq \sqrt{4\pi k}(c_1 k)^{2k}, \quad \text{where} \quad c_1 = \frac{1}{\sqrt{3e}} \cdot \exp\left(-\sum_{p \in \mathbb{P}} \frac{\ln p}{p(p-1)}\right) \approx 0.16948. \tag{39}
\]

Consider \( p_k := p_{4k+2}/\gcd(p_{4k+2},q_{4k+2}) \) and \( q_k := q_{4k+2}/\gcd(p_{4k+2},q_{4k+2}) \). Definitely, they are both integer numbers and Lemma 11 together with (39) and (34) give us

\[
q_k^* < \frac{4(3t^2 + a)\sqrt{k}}{\sqrt{\pi}} \left( \frac{c_0}{c_1} \right)^k := Q(k, t, a).
\]

Then with help of Lemma 10 we compute

\[
||q_k^*\xi|| \leq ||q_k^*\xi - P_k^*|| = q_k^* \left| \xi - \frac{P_k^*}{q_k^*} \right| < \frac{105\sqrt{3e^2a^4}\sqrt{k}}{2\sqrt{\pi} t(t^2 + a)(3t^2 + a)} \left( \frac{9a^6(t^2 + a)}{c_1^2e^2t^4(t^2 + a)^2} \right)^k =: R(k, t, a).
\]

Notice that by the definition of \( \tau_1, \tau_2, c_2 \) and \( c_3 \) in Theorem 1, we have \( Q(k, t, a) = \tau_1 \sqrt{kc_2}^k \) and \( R(k, t, a) = \tau_2 \sqrt{kc_3}^k \).

**Proof of Theorem 1** Consider an arbitrary \( q \geq \frac{1}{2R(1,t,a)} \). Notice that since \( c_3 > e \) then the sequence \( c_3^2/\sqrt{k} \) is strictly increasing for all \( k \geq 1 \). Therefore, there exists a unique \( k \geq 2 \) such that \( R(k, t, a) < \frac{1}{2q} \leq R(k - 1, t, a) \). Let \( p \in \mathbb{Z} \) be such that \( ||q\xi|| = ||q\xi - p|| \). Since two vectors \( (p_k^*, q_k^*) \) and \( (p_{k+1}^*, q_{k+1}^*) \) are linearly independent, at least one of them must be linearly independent with \( (p, q) \). Suppose that is \( (p_k^*, q_k^*) \). Then we estimate the absolute value of the following determinant:

\[
1 \leq \left| \begin{array}{c}
q \\
q_k^*
p_k^*
\end{array} \right| = \left| \begin{array}{c}
q \\
q_k^* - p_k^* \xi - p^* \xi
\end{array} \right| \leq qR(k, t, a) + ||q\xi||Q(k, t, a).
\]

Since \( qR(k, t, a) < \frac{1}{2q} \), we must have \( ||q\xi|| \geq \left( 2Q(k, t, a) \right)^{-1} \). Analogously, if \( (p, q) \) is linearly independent with \( (p_{k+1}^*, q_{k+1}^*) \), we have \( ||q\xi|| \geq \left( 2Q(k + 1, t, a) \right)^{-1} \). The latter lower bound is weaker. Now, we need to rewrite the right hand side of the inequality in terms of \( q \) rather than \( k \).

From \( \frac{1}{2q} \leq R(k - 1, t, a) \) we have that

\[
\frac{c_3^{k-1}}{2\tau_2 \sqrt{k - 1}} \leq q.
\]
We show that the last inequality implies

\[ k - 1 \leq \frac{\log(2\tau_2q) + \log \log(2\tau_2q)}{\log c_3}. \tag{40} \]

Indeed, if we take \( \kappa \) such that \( \kappa \log c_3 = \log(2\tau_2q) + \log \log(2\tau_2q) \) then \( c_3^2/\sqrt{\kappa} \geq 2\tau_2q \) is equivalent to

\[ \log \log(2\tau_2q) \geq \frac{1}{2} \log \left( \frac{\log(2\tau_2q) + \log \log(2\tau_2q)}{\log c_3} \right). \]

By taking exponents of both sides and after simplifications, we derive \( \log c_3 \cdot \log(2\tau_2q) \geq 1 + \log \log(2\tau_2q) \). In view of \( \log c_3 > 1 \), the last inequality is satisfied for all \( 2\tau_2q > 1 \).

Substitute that into \( ||q\xi|| \geq (2Q(k + 1, t, a))^{-1} \) and get

\[ ||q\xi|| \geq \frac{1}{2\tau_1 \sqrt{k + 1}^{c_2^{k+1}}} \geq \frac{(\log c_3)^{1/2}}{2\sqrt{3} \tau_1 c_2^{2}(\log(2\tau_2q) + \log \log(2\tau_2q))^{1/2}(2\tau_2q)^{\log c_2^{2}} \cdot q \cdot \log(2\tau_2q) - \log c_2^{2} - \frac{1}{2}}{6\tau_1 c_2^{2}(2\tau_2q)^{\log c_2^{2} - \frac{1}{2}}}. \]

5 Very good rational approximations to cubic irrationals

In this section we will look for the best possible rational approximations which can be derived from the continued fraction (2) and then prove Theorem 2.

Given \( n \in \mathbb{N} \), define \( d_2(n) \) to be the largest odd divisor of \( n \). For convenience, in the remaining part of this section we always have \( a = 1 \). We apply the following continued fraction transformations:

- divide \( a_1 \) and \( \beta_1 \) by \( d_2(\beta_1) \) and then multiply \( a_2 \) and \( \beta_3 \) by \( d_2(\beta_1) \);
- repeat the similar transformations for indices \( i = 3, \ldots, n \): divide the values \( a_i \) and \( \beta_i \) of a newly achieved continued fraction by \( d_2(\beta_i) \) and then multiply \( a_{i+1} \) and \( \beta_{i+2} \) by \( d_2(\beta_i) \).

By Lemma 11, they do not change the sequence of convergents of the continued fraction. Let \( \beta_i^* \) and \( \beta_i^* \) be the coefficients of the continued fraction after performing these transformations. One can verify that for all \( i, k \in \mathbb{Z}_{\geq 0}, 1 \leq i, 4k + 4 \leq n \) they satisfy

\[ \beta_i^* = \frac{\beta_i}{d_2(\beta_1)}; \; a_{4k+1}^* = \frac{a_{4k+1}}{d_2(3k+1)} \cdot \frac{\prod_{j=0}^{k-1}(6j+1)(6j+5)}{\prod_{j=0}^{k-1} d_2(3j+1)d_2(3j+2)}. \tag{41} \]

\[ a_{4k+2}^* = \frac{a_{4k+2}d_2(3k+1)}{(6k+1)} \cdot \frac{\prod_{j=0}^{k-1} d_2(3j+1)d_2(3j+2)}{\prod_{j=0}^{k-1} (6j+1)(6j+5)}. \tag{42} \]

\[ a_{4k+3}^* = a_{4k+3}(6k+1) \cdot \frac{\prod_{j=0}^{k-1} (6j+1)(6j+5)}{\prod_{j=0}^{k-1} d_2(3j+1)d_2(3j+2)}. \tag{43} \]

\[ a_{4k+4}^* = a_{4k+4} \cdot \frac{\prod_{j=0}^{k} d_2(3j+1)d_2(3j+2)}{\prod_{j=0}^{k} (6j+1)(6j+5)}. \tag{44} \]

Let \( t_k := \text{lcm}\{a \in \mathbb{Z} : 1 \leq a \leq 6k + 1; a \equiv \pm1 \pmod{6}\} \).
Lemma 12 With $t = t_k$, all values $a_i^*$ for $1 \leq i \leq 4k + 3$ are integer.

Proof. Since each $a_i$ is an integer multiple of $t$, it is sufficient to show that all the values (41)–(44) still remain integer if $a_i$’s are replaced by $t$. Also notice that for $m \in \mathbb{Z}$, $m \geq 0$, the number $d_2(3m + 1)$ appears in the denominator of $a_{4m+1}^*$ for the first time and then occurs in the fractions for all $a_i^*$, $i \geq 4m + 1$. Analogously, $d_2(3m + 2)$ appears in the denominator of $a_{4m+3}^*$, $6m + 1$ appears in $a_{4m+2}$ and $6m + 5$ appears in $a_{4m+4}$ for the first time.

Let $p > 3$ be integer which is coprime with 6. First, assume that $p = 6m + 1$ for some integer $m$. We investigate, for what indices $i$ do multiples of $p$ appear in the denominators of $a_i^*$ for the first time.

- $(6l + 1)(6m + 1)$ appears at index $4(6ml + m + l) + 2$;
- $(6l + 5)(6m + 1)$ appears at index $4(6ml + 5m + l) + 4$;
- $d_2((3l + 1)(6m + 1))$ appears at index $4(6ml + 2m + l) + 1$;
- $d_2((3l + 2)(6m + 1))$ appears at index $4(6ml + 4m + l) + 3$.

Notice that the first two and the last two factors always appear at the opposite sides of the fractions (41)–(44). Obviously, $6m + m + l < 6m + 2m + l < 6m + 4m + l < 6m + 5m + l < 6m(l + 1) + m + (l + 1)$ for all integers $m > 0$, $l \geq 0$. Therefore, we conclude that the amount of multiples of $p$ in the numerators and denominators of each fraction (41)–(44) do not differ by more than 1.

The case $p = 6m + 5$ is considered analogously. We have

- $(6l + 1)(6m + 5)$ appears at index $4(6ml + m + 5l) + 4$;
- $(6l + 5)(6m + 5)$ appears at index $4(6ml + 5m + 5l + 4) + 2$;
- $d_2((3l + 1)(6m + 5))$ appears at index $4(6ml + 2m + 5l + 1) + 3$;
- $d_2((3l + 2)(6m + 5))$ appears at index $4(6ml + 4m + 5l + 3) + 1$.

Then one can easily check that again the amount of multiples of $p$ in the numerators and denominators of each fraction (41)–(44) do not differ by more than 1.

Consider a prime number $p > 3$. Let $d = d_{p,k}$ be the largest power of $p$ such that $p^d \leq 6k + 1$. We have shown that for all $i \in \mathbb{N}$, the number of multiples of $p$ in the denominators of (41)–(44) do not exceed the number of multiples of $p$ in the numerators by one. The same observation is true for multiples of $p^2, p^3, \ldots, p^d$. Finally, there are no multiples of $p^{d+1}$ in the fractions for $a_i^*$ with $i \leq 4k + 3$. Then one concludes that, after all cancelations, the power of $p$ in the denominators of (41)–(44) does not exceed $d$.

Finally, the fact that $t_k = \prod_{p \in \mathbb{P}, p > 3} p^{d_{p,k}}$ concludes the proof.

Notice that

$$t_k = \frac{e^{\psi(6k+1)}}{2 \log(6k+1) \log(6k+2)^3},$$

where $\psi(n)$ is the second Chebyshev function. Then, one can use $\psi(n) \approx n$ to provide a good estimate for $t_k$. In fact, for all $\epsilon > 0$ there exists $k_0 \in \mathbb{Z}$ such that for all $k > k_0$,

$$\frac{e^{(1-\epsilon)(6k+1)}}{(6k+1)^2} < t_k < \frac{6e^{(1+\epsilon)(6k+1)}}{(6k+1)^2}.$$  \hspace{1cm} (45)
For small values of \( \varepsilon \) the value of \( k_0 \) may become big and hard to locate. To make the bounds precise and effective (which is needed for Theorem 2 and Conjecture B) we use \( 0.96x \leq \psi(x) \leq 1.04x \) from [13] to get:

\[
\frac{e^{0.96(6k+1)}}{(6k+1)^2} < t_k < \frac{6e^{1.04(6k+1)}}{(6k+1)^2}.
\] (46)

The first inequality is satisfied for \( k \geq 2 \).

We have that for all \( i \leq 4k + 3 \) the values of \( a_i^* \) and \( \beta_i \) are positive integers, therefore \( q_i > q_i - 1 \). Moreover,

\[
a_i^* q_{i-1} = a_i^* q_{i-1} + \beta_i^* q_i - 2 < (a_i^* + \beta_i^*) q_{i-1}.
\]

For even values of \( i \) we have \( \beta_i^* = 1 \leq a_i^* \), while \( \beta_{4m+1}^* \leq 3m + 1 < 4m + 1 \leq a_{4m+1}^* \). Analogously, \( \beta_{4m+3}^* \leq 3m + 2 < 4m + 3 \leq a_{4m+3}^* \). Therefore we can simplify the upper bound for \( q_i \):

\[
q_i < 2a_i^* q_{i-1}.
\] (47)

The inequality \((a_i^* + \beta_i^*) q_{i-1} > q_i\), for \( i = m \leq k \) implies

\[
q_{4m+3} < \left( \frac{3(4m+3)}{6m+5} t_k^2 + 2 \right) \prod_{j=0}^{m} \frac{4(3j+1)(3j+2)}{d_2(3j+1)d_2(3j+2)} + \frac{6m+4}{d_2(3m+2)} q_{4m+2}.
\]

Denote the product of \( 4(3j+1)(3j+2)/d_2(3j+1)d_2(3j+2) \) in the formula above by \( P_m \). Notice that it is bigger than \( 4^m \) which is larger than \( 6m + 4 \) for \( m \geq 3 \). Then we transform the inequality to a simpler form

\[
q_{4m+3} < 8P_m t_k^3 q_{4m+2}.
\] (48)

For the opposite estimate between \( q_{4m+3} \) and \( q_{4m+2} \), we provide a lower bound on \( a_{4m+3}^* \):

\[
a_{4m+3}^* = \frac{3(4m+3)}{6m+5} t_k^2 + 2 \prod_{j=0}^{m} \frac{(6j+1)(6j+5)}{(6j+2)(6j+4)}.
\]

As \( m \) grows, the product in this expression decreases and converges to \( 0.577... \), therefore we get an estimate

\[
q_{4m+3} > a_{4m+3}^* q_{4m+2} > P_m t_k^3 q_{4m+2}.
\] (49)

As the next step, we estimate \( ||q_{4m+2}^2 |^\xi || \) for all \( m < k \). By Lemma 10 we have

\[
|\xi - \frac{p_{4m+2}}{q_{4m+2}}| < 2 \left| \frac{p_{4m+2}}{q_{4m+2}} - \frac{p_{4m+6}}{q_{4m+6}} \right|,
\]

We also derive

\[
\max_{4m+2 < i < 4m+5} \left| \frac{p_i}{q_i} - \frac{p_{i+1}}{q_{i+1}} \right| \leq \frac{\prod_{i=-1}^{m+4} \det C_i}{q_{4m+2} q_{4m+3}} \leq \prod_{i=1}^{m+6} \frac{\beta_i}{d_2(\beta_i)} \frac{q_{4m+2} q_{4m+3}}{q_{4m+4}} \leq \frac{2(3m+4)P_m}{d_2(3m+4)q_{4m+2} q_{4m+3}},
\]

where the matrices \( C_i^* \) are defined in (25) but with the new partial quotients \( a_i^*, \beta_i^* \) instead of \( a_i, \beta_i \). Together with (10), the last two inequalities yield

\[
|q_{4m+2}^2 - p_{4m+2}| < \frac{8(3m+4)}{d_2(3m+4) t_k^3 q_{4m+2}}.
\] (50)
Lemma 13  The value $P_m$ satisfies

$$P_m \leq (3m + 2)2^{4m+4}$$

Proof. We can compute $P_m$ by counting the number of even integers between 1 and $3m + 2$ that are not multiples of 3, then count the number of integers in the same set, that are multiples of 4, multiples of 8, etc. We stop when we reach the maximal power of two not exceeding $3m + 2$.

The number of multiples of $2^j$ among all numbers between 1 and $3m + 2$ is $\left\lfloor \frac{3m + 2}{2^j} \right\rfloor$. Among them, $\left\lfloor \frac{m}{2^j} \right\rfloor$ numbers are multiples of 3. For $j = 1$ we have

$$\left\lfloor \frac{3m + 2}{2^j} \right\rfloor - \left\lfloor \frac{m}{2^j} \right\rfloor = m + 1$$

and for bigger values of $j$,

$$\left\lfloor \frac{3m + 2}{2^j} \right\rfloor - \left\lfloor \frac{m}{2^j} \right\rfloor < \frac{m + 1}{2^{j-1}} + 1.$$ 

Combining these inequalities for all $j \in \{1, 2, \ldots, \log_2(3m + 2)\}$, we derive

$$\frac{P_m}{4^m} < 2^{(m+1)\sum_{j=0}^{m-1} 2^{-j} + \log_2(3m+2)} < (3m + 2)2^{2m+2}.$$ 

In the next lemma, we show that $p_{4m+2}$ and $q_{4m+2}$ are both multiples of a large power of 2. Therefore we can cancel it from $p_{4m+2}$ and $q_{4m+2}$ and hence significantly improve the inequality $\leq 50$.

Lemma 14  Let $t$ be odd and $a = 1$. Then all the entries of the matrix $A_{4n+3}A_{4n+2}A_{4n+1}A_{4n}$ are multiples of $2^7$, and all the entries of the matrix $A_{4n+7} \cdots A_{4n+1}A_{4n}$ are multiples of $2^{15}$.

Proof. Let $m$ be odd. Then both $6m + 4$ and $6m + 8$ are multiples of 2 but not 4. Then we use (25) to compute

$$C_{4m+2}C_{4m+1} \equiv \begin{pmatrix} t^4 + 2t^2 + 3 & 2t \\ t(t^2 + 2) & 2 \end{pmatrix} \equiv \begin{pmatrix} 2 & 2 \\ 1 \text{ or } 3 & 2 \end{pmatrix} \pmod{4}. $$

Analogously,

$$C_{4m+4}C_{4m+3} \equiv \begin{pmatrix} 3t^2 + 1 & 2t \\ 3t & 2 \end{pmatrix} \equiv \begin{pmatrix} 0 & 2 \\ 1 \text{ or } 3 & 2 \end{pmatrix} \pmod{4}. $$

After multiplying the last two equations we derive

$$A_m = \begin{pmatrix} 2u_m & 4v_m \\ 4w_m & 2t_m \end{pmatrix}, \quad (51)$$

where $u_m, v_m, t_m$ are odd integers and $w_m$ is an integer (odd or even).

If $m = 4n$, we have that $6m + 4$ is a multiple of 4 but not 8, and $6m + 8$ is a multiple of 8. Let $2^{\delta_m}$ be the largest power of 2 that divides $6m + 8$. Then we compute

$$C_{4m+2}C_{4m+1} = \begin{pmatrix} 8e_{m1} & 4e_{m2} \\ e_{m3} & 4e_{m4} \end{pmatrix}; \quad C_{4m+4}C_{4m+3} = \begin{pmatrix} 2e_{m5} & 2^{\delta_m}e_{m6} \\ e_{m7} & 2^{\delta_m}e_{m8} \end{pmatrix}.$$
where $e_{m2}, \ldots e_{m8}$ are some odd integers and $e_{m1}$ shares the parity with $n$. Therefore, in this case we get

$$A_m = \begin{pmatrix} 2^{\tau_m} u_m & 2^{3} v_m \\ 3^{\tau_m} u_m & 2^{2} t_m \end{pmatrix}, \quad \tau_m = \begin{cases} 3 & \text{if } \delta_m = 3; \\
5 & \text{if } \delta_m = 4; \\
4 & \text{if } \delta_n > 4 \end{cases}$$

(52)

where $v_m, w_m, t_m$ are odd integers, and $u_m$ is integer (not necessarily odd).

Consider the last case $m = 4n + 2$. Then we have that $6m + 8$ is a multiple of 4 but not 8, and $6m + 4$ is a multiple of 8. Let $q_{2n}$ be the largest power of 2 that divides $6n + 4$. We compute

$$C_{4m+2}C_{4m+1} = \begin{pmatrix} 4e_{m1} & 2^{\delta_m} e_{m2} \\ e_{m3} & 2^{\delta_m} e_{m4} \end{pmatrix} \quad \text{and} \quad C_{4m+4}C_{4m+3} = \begin{pmatrix} 2e_{m5} & 4e_{m6} \\ e_{m7} & 4e_{m8} \end{pmatrix},$$

where $e_{m1}, \ldots, e_{m8}$ are all odd integer numbers. Combined, these formulae infer

$$A_m = \begin{pmatrix} 4u_m & 2^{\delta_m+1} v_m \\ 8w_m & 2^{\delta_m} t_m \end{pmatrix},$$

(53)

where $u_m, v_m$ and $t_m$ are odd integers and $w_m$ is any integer.

As a final step, we notice that $\delta_4n = 3$ if and only if $\delta_{4n+2} > 3$. Then we combine (51), (52) and (53) to get

$$A_{4n+3}A_{4n+2}A_{4n+1}A_{4n} = \begin{pmatrix} 2^{\lambda_n} U_n & 2^{5} V_n \\ 2^{8} W_n & 2^{\theta_n} T_n \end{pmatrix}, \quad \lambda_n = \begin{cases} 7 & \text{if } \delta_{4n} = 3; \\
8 & \text{if } \delta_{4n} > 3. \end{cases} \quad \text{and} \quad \theta_n = \begin{cases} 7 & \text{if } \delta_{4n+2} = 3; \\
8 & \text{if } \delta_{4n+2} > 3. \end{cases}$$

where $U_n, V_n, W_n, T_n$ are some integer numbers. That verifies the first claim of the lemma.

For the second claim, we need to multiply two consecutive blocks $A_{4n+7}A_{4n+6}A_{4n+5}A_{4n+4}$ and $A_{4n+3}A_{4n+2}A_{4n+1}A_{4n}$ and notice that the consecutive values of $\lambda_n$ and $\theta_n$ alternate.

Lemma [14] implies that for all $n \in \mathbb{Z}$, $2^{15n}$ divides $p_{32n+1}, q_{32n+1}, p_{32n+2}$ and $q_{32n+2}$. Finally, we get that $q_{32n+3} = q_{32n+3} q_{32n+2} + q_{32n+2} + q_{32n+1}$ is also a multiple of $2^{15n}$. The same condition is satisfied for $p_{32n+3}$.

In further discussion, we specify $k$ to be divisible by 8, i.e. $k = 8k_0$. Then $q_{4k+2}$ and $p_{4k+2}$ are all multiples of $2^{15k_0}$, i.e. $q_{4k+2} = 2^{15k_0} q_{4k+2}^*$. Then for all $m < k_0$, (50) implies

$$\|q_{32m+2}^*\| < \frac{8(24m + 4)}{d_2(24m + 4) \ell_k^3 3^{20m} q_{32m+2}^*} = \frac{32}{4^{15m} \ell_k^3 q_{32m+2}^*},$$

(54)

Remark. Direct computations suggest that for infinitely many values of $m$ the powers of 2 which divide both $p_{32m+2}$ and $q_{32m+2}$ are around $2^{16m}$. Therefore, we believe that the power of 4 in the denominator of (54) can be replaced by $4^{16m}$. However, we were not able to prove it.

Proof of Theorem[2] Fix a small value $\varepsilon > 0$. In view of (45), consider $k_0$ large enough, so that $t_k < e^{48(1+\varepsilon)k_0}$ and $4^{(1-\varepsilon)k_0} > 32 \cdot 3^{4}$. Choose any $m$ in the range $(1-\varepsilon)k_0 < m < k_0$. Then for all $k_0$ large enough one has

$$4^{-15(1-\varepsilon)m} < e^{-48(1+\varepsilon)k_0} \frac{15 \ln 4 (1-\varepsilon)^2}{48(1+\varepsilon)} < t_k^{-\frac{15 \ln 2 (1-\varepsilon)^2}{24(1+\varepsilon)}}.$$
\[ (1 - \varepsilon)k_0 < m < k_0 \text{ one has} \]
\[ \|q_{32m+2}\| < \frac{1}{4^{14\varepsilon m} \cdot (3t_k)^7 d_{32m+2}^*}. \]

Next, we estimate \( q_{32m+2}^* \). By (17), \( q_{32m+2} \leq 2^{32m+2} \prod_{i=1}^{32m+2} a_i^* \). Then we use (11) - (14) to get
\[ q_{32m+2} < \frac{2^{32m+2} \prod_{j=1}^{32m+2} a_j}{(48m + 1) \prod_{m=0}^{8m-1} (6j + 1) (6j + 5)} \leq \frac{2^{32m+2} q_{q_{32m+2}} + 3 m + 2 q_{32m+2} + 32m + 1}{(48m + 1) \prod_{m=0}^{8m-1} (6j + 1) (6j + 5)}.
\]

Since \( t_k > 8m, 4j + 1 < \frac{2}{3}(6j + 5) \) and \( 4j + 3 < \frac{2}{3}(6j + 7) \), there exists an effectively computable constant \( C \) such that the last expression is bounded from above by \( C \cdot 2^{48m} t_k^{48m+2} (2m + 1) \).

Finally, we use \( t_k < e^{49(1+\varepsilon)k_0} \), the fact that \( q_{32m+2} = 2^{15m} q_{32m+2}^* \) and \( (1 - \varepsilon)k_0 < m < k_0 \) to get that for \( k_0 \) large enough,
\[ C \cdot 2^{48m} \frac{e^{1+\varepsilon} m}{215m} < e^{48m} \]
\[ m > \sqrt{(1 - \varepsilon) \ln(q_{32m+2}^*)} \quad \implies \quad 4^{14\varepsilon m} > e^{48m} \cdot \ln(q_{32m+2}^*). \]

Choose \( k_0 \) to be large enough so that all the inequalities in the arguments above take place and on top of that there are at least \( N_0 \) values \( m \) in the range \( (1 - \varepsilon)k_0 < m < k_0 \). Then the height of a cubic irrational \( \xi = x(t_k, 1) \) from (2) is \( 3t_k \) and therefore the inequality (4) is satisfied for all \( q = q_{32m+2}^* \) and \( c = \frac{7e}{21} \sqrt{1-\varepsilon}. \)

In the end of this section we provide lower bounds for \( \|q_n\xi\| \) that support Conjecture B. Some of the arguments here will be heuristic or numerical. First of all, notice that among all continued fractions \( N1 - 6 \) only (2) has the partial quotients which are cubic in \( t \). The others are either linear or quadratic. Therefore, for \( \xi = x(t, a) \) from (2) and large \( t \), the expression \( \|q_{4k+2}\xi\| \) will more likely provide the smallest value in terms of the denominator \( q_{4k+2}^* \) among other convergents that come from those continued fractions. We also assume that \( a = 1 \) because bigger absolute values of \( a \) give bigger determinants of matrices \( C_n \) and hence bigger differences \( \left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| \). Finally, we assume that \( t = t_k \) because in that case, as was shown above, the continued fraction can be transformed in such a way that all the values \( \beta_i, i \leq 4k + 2 \) can be significantly reduced to \( \beta_i^* = \frac{\beta_i}{d_i(\beta_i)} \) and the numerators and denominators \( (p_{4k+2}, q_{4k+2}) \) of the convergents are divisible by a large power of two. On top of that, numeric computations suggest that such values of \( t \) provide the largest possible partial quotients \( a_{4k+2}. \)

From Lemma (12) we have that
\[ t_k \cdot \frac{\prod_{j=0}^{6j + 1} (6j + 1) (6j + 5)}{\prod_{j=0}^{3d_2(3j + 1) d_2(3j + 2)}} \]
is integer and therefore \( q_{4k+4} > q_{4k+3} > a_{4k+3} q_{4k+2} > (t_k^2 + 2) q_{4k+2} \). Now we estimate
\[ \max_{4k+3 \leq \ell \leq 4k+5} \left| \frac{p_{\ell} - p_{\ell+1}}{q_{\ell} q_{\ell+1}} \right| \leq \left| \frac{p_{4k+2} q_{4k+2} - p_{4k+3} q_{4k+3}}{q_{4k+2} q_{4k+3}} \right| \cdot | \det C_{4k+2} \det C_{4k+3} \det C_{4k+4} | \cdot \frac{q_{4k+2} q_{4k+4}}{q_{4k+2}} \]

29
In view of (2), for \( k \geq 2 \) the last expression is smaller than
\[
\left| \frac{p_{4k+2}}{q_{4k+2}} - \frac{p_{4k+3}}{q_{4k+3}} \right| \cdot \frac{(6k + 5) \cdot 2(3k + 4)(6k + 7)}{t_k^2 + 2} < \frac{1}{4} \left| \frac{p_{4k+2}}{q_{4k+2}} - \frac{p_{4k+3}}{q_{4k+3}} \right|
\]
The last inequality follows from (46). Finally, we use Lemma 10 to conclude that
\[
\left| \xi - \frac{p_{4k+2}}{q_{4k+2}} \right| \geq \left| \frac{p_{4k+2}}{q_{4k+2}} - \frac{p_{4k+6}}{q_{4k+6}} \right| \geq \left| \frac{p_{4k+2}}{q_{4k+2}} - \frac{p_{4k+3}}{q_{4k+3}} \right| - \left| \frac{p_{4k+3}}{q_{4k+3}} - \frac{p_{4k+6}}{q_{4k+6}} \right| \geq \frac{1}{4} \left| \frac{p_{4k+2}}{q_{4k+2}} - \frac{p_{4k+3}}{q_{4k+3}} \right|
\]
The right hand side is bounded from below by
\[
\frac{1}{4} \left| \frac{p_{4k+2}}{q_{4k+2}} - \frac{p_{4k+3}}{q_{4k+3}} \right| = \left| \prod_{i = -1}^{k+2} \det C_i \right| 4q_{4k+2}q_{4k+3} \geq \frac{P_k}{4q_{4k+2}q_{4k+3}}.
\]
We apply (48) to get the estimate
\[
\| q_{4k+2}\xi \| \geq \frac{P_k}{4 \cdot 8P_k t_k^2 q_{4k+2}} \gg \frac{1}{t_k^2 q_{4k+2}}.
\]
Since \( |p_{4k+2}q_{4k+3} - p_{4k+3}q_{4k+2}| = \beta^*_{4k+3} P_k \) is a power of 2, we have that \( \gcd(p_{4k+2}, q_{4k+2}) \) is also a power of two. Computations suggest that
\[
\gcd(p_{4k+2}, q_{4k+2}) \asymp 2^{2k},
\]
but we only manage to prove a weaker inequality \( \gcd(p_{32k+2}, q_{32k+2}) \ll k^{17k} \), by using Lemmata 13 and 14.

Denote by \( q_{4k+2}\) the ratio \( \frac{q_{4k+2}}{\gcd(p_{4k+2}, q_{4k+2})} \). We will provide the lower bounds for \( \| q_{4k+2}\xi \| \) based on the numerical evidence (stronger) and based on the rigorous proof (weaker). Numerically, from (56) and (57) we have
\[
\| q_{4k+2}\xi \| \gg \frac{1}{2^{4k} t_k^2 q_{4k+2}}.
\]
Then, (46) gives that \( 2^{4k} = e^{0.96(6k)} \frac{2^{10}2^{50}}{8^{36}36} \ll k^2 t_k^2 \frac{2^{10}2^{50}}{8^{36}36} \). Whence,
\[
\| q_{4k+2}\xi \| \gg \frac{1}{k^2 t_k^2 \frac{2^{10}2^{50}}{8^{36}36} q_{4k+2}}.
\]
Based on these arguments (not completely rigorous) and the assumption that \( p_{4k+2}/q_{4k+2}^* \) is a convergent of \( \xi \) of index comparable to \( k \), we formulate our Conjecture B.

Using (45) instead of (46) will result in a slightly better power \( 3 + \frac{2}{3} \ln 2(1 + \varepsilon) \) of \( t_k \) that supports Conjecture A. However, for very small \( \varepsilon \) the inequality will only be valid for huge values of \( k \). That makes the task of computing the precise value of the constant \( C \), as in Conjecture B, harder but theoretically possible.

6 Continued fractions for all cubic irrationals

In this section, for any real cubic irrational \( \xi \) we construct a Möbius transform \( \mu \) and a continued fraction of the form \( N \) that converges to \( \mu(\xi) \). This will confirm Theorem 3.

Consider an arbitrary cubic algebraic \( \xi \in \mathbb{R} \) with the minimal polynomial \( P_\xi(x) = b_3 x^3 + b_2 x^2 + b_1 x + b_0 \in \mathbb{Z}[x] \). One can easily check that the minimal polynomial of \( \eta := (3b_3 \xi + b_2)^{-1} \)
is of the form $B_3y^3 + B_2y^2 + 1 = 0$, where $B_3$ and $B_2$ are integer. Finally, the minimal polynomial of $\eta := B_3y_0$ is $y^3 + B_2y^2 + B_2^2 = 0$, which is of the form $N^4$ with $t = -B_2$ and $a = -B_2^2$. That immediately implies that for at least one root $\zeta$ of $P_\zeta(x)$, the number $\eta = \frac{B_3}{\zeta + B_2}$ is linked with the continued fraction expansion $[\zeta]$. However, firstly we want the continued fraction $[\zeta]$ to converge, but that is not guaranteed for an arbitrary $\eta$. Secondly, we want to cover all real roots of $P_\zeta(x)$, not just one of them. In this section we will resolve these two problems and provide a (generalised) continued fraction expansion in a closed form for all cubic irrationals $\xi \in \mathbb{R}$.

Let $\xi$ be a real root of the cubic polynomial $P(x)$. Consider $\eta \in \mathbb{R}$ such that $\xi = \frac{u\eta + v}{w\eta + x}$. We will find the conditions on $u, v, s, w \in \mathbb{Z}$ such that the coefficient at $y$ of the minimal polynomial $P_\eta(y)$ of $\eta$ equals zero. The minimal polynomial $P_\eta$ is

$$b_3(uy + v)^3 + b_2(uy + v)^2(sy + w) + b_1(uy + v)(sy + w)^2 + b_0(sy + w)^3 = 0. \quad (59)$$

That is a cubic polynomial in $y$ and its coefficient at $y$ is

$$3b_3uv^2 + 2b_2uvw + b_2v^2s + b_1uw^2 + 2b_1vsw + 3b_0sw^2.$$

Equating this coefficient to zero gives

$$u(3b_3v^2 + 2b_2vw + b_1w^2) = -s(3b_0w^2 + 2b_1vw + b_2v^2).$$

The solutions of this equation for arbitrary $v, w \in \mathbb{Z}$ are

$$s = 3b_3v^2 + 2b_2vw + b_1w^2 = w^2P_\xi'(\frac{v}{w}),$$

$$u = -v^2\frac{d}{dx}(x^3P_\xi(1/x))\big|_{x=w/v} = vwP_\xi'(\frac{v}{w}) - 3w^2P_\xi(\frac{v}{w}) \quad (60)$$

To make the notation shorter, we will write $P$ instead of $P(v/w)$ and $P'$ instead of $P'(v/w)$.

With $u$ and $s$ as in (60), the free coefficient of (59) is obviously $w^3P$. Now, we compute the coefficient of this polynomial at $y^2$:

$$3b_3v(wvP' - 3w^2P + 2b_2w^2vP'(vwP' - 3w^2P) + b_2w(vwP' - 3w^2P) + 2b_1w^3P'(vwP' - 3w^2P) + b_1w^4P'(vwP' - 3w^2P) + 3b_0w^5P'(vwP' - 3w^2P)$$

$$= (3w^5P)(P')^2 - (6w^5P')P' + 9(3b_3v + b_2v)w^4P^2$$

$$= 3w^3P((3b_3b_1 - b_2^2)v^2 + (9b_3b_0 - b_2b_1)vw + (3b_2b_0 - b_1^2)w^2) =: 3w^5PR(v/w). \quad (61)$$

We verify that $R$ is not the constant zero. Indeed, if that is the case, we must have $3b_3b_1 = b_2^2, 9b_3b_0 = b_2b_1$ and $3b_2b_0 = b_1^2$. The solutions of this system of equations are $b_0 = 27\beta, b_1 = 27\beta\gamma, b_2 = 9\beta\gamma^2$ and $b_3 = \beta\gamma^3$ for some $\beta, \gamma \in \mathbb{Q}$. However, in this case the polynomial $P_\zeta(x)$ has a root $\frac{-3}{\gamma}$ which contradicts the irreducibility of $P_\zeta$.

In a similar way, we compute the leading coefficient of (59):

$$b_3(vwP' - 3w^2P)^3 + b_2w^2P'(vwP' - 3w^2P)^2 + b_1w^4(P')^2(vwP' - 3w^2P) + b_0w^6(P')^3$$

$$= (w^6P)(P')^3 - (3w^6P')P'(P')^2 + 9(3b_3v + b_2v)w^5P'P^2 - 27w^6b_3P^3$$

$$= w^3P((-18b_3b_2b_1 + 2b_2^3 + 27b_3b_0)v^3 + (3b_2b_1 - 18b_3b_2^2 + 27b_3b_0)b_1^2)w^2$$

$$+ (18b_3b_0 - 3b_2b_1^2 - 27b_3b_1b_0)vw + (9b_2b_1b_0 - 2b_1^2 - 27b_3b_0^2)w^3 \quad (62)$$

Therefore the leading coefficient can be written as $w^6P(v/w)Q(v/w)$. The upshot is that the minimal polynomial of $\eta$ is $w^3Q(v/w)y^2 + 3w^2R(v/w)y^2 + 1$. Finally, we make the change of variables $y \mapsto -w^3Q(v/w)y$ and the minimal polynomial of the new value $\eta$ is

$$y^3 - 3w^2R(v/w)y^2 - (w^3Q(v/w))^2. \quad (63)$$
It is of the form \(\mathcal{N}4\) where \(t = 3w^2R(v/w)\) and \(a = (w^3Q(v/w))^2\).

To find the roots of \(Q\), we observe that the leading term of \(P_0\) equals zero if \(\frac{w}{s} = \xi\) or

\[
\frac{-b_2z^2 - 2b_1z - 3b_0}{3b_3z^2 + 2b_2z + b_1} = \xi, \quad z = \frac{v}{w}.
\]

This equation is quadratic in \(z\). It is easy to verify that \(z = \xi\) is its solution, which corresponds to the factor \(w^3P\) in (62). Then the other root is

\[
z(\xi) = -\frac{2(b_2\xi + b_1)}{3b_3\xi + b_2} - \xi.
\]  

(64)

In particular, this means that the number of real roots of the polynomial \(Q\) coincides with the number of them for \(P\), and \(z(x)\) is a bijection between the roots of \(P\) and \(Q\). Also notice that \(z(\xi) \neq \xi\) because otherwise the degree of \(x\) is at most 2.

**Lemma 15** Let \(a, t \in \mathbb{C}\) be such that \(|t| > 2|a|^{1/3}\). Then the equation

\[
x^3 - tx^2 - a = 0
\]

(65)

has exactly one root \(\xi\) that satisfies \(|\xi| > |a|^{1/3}\) and the continued fraction \(\mathcal{N}4\) corresponds to \(\xi\).

**Proof.** Let \(a\) be fixed. Since the leading term of the Laurent series \(x(t)\) from (1) is \(t\), as soon as \(x(t)\) converges, the limit tends to infinity as \(t \to \infty\). Moreover, for \(t > T_0\), where \(T_0\) is the radius of convergence of \(x(t)\), this function is analytic and therefore continuous. Hence, \(x(t)\) corresponds to the root of the equation in \(x^3 - tx^2 - a = 0\) which continuously tends to infinity as \(t \to \infty\).

Let \(x\) be any complex number that satisfies \(|x| = |a|^{1/3}\). Then we have \(|x^3 - a| \leq 2|a| < |tx^2|\). Therefore the number of roots of (65) inside \(|x| < |a|^{1/3}\) is the same as the number of roots of \(tx^2 = 0\) in the same region, i.e. it is two. We thus get that there is only one root \(\xi\) of (65) outside that circle. Then, as \(t \to \infty\), this root \(\xi(t)\) continuously tends to infinity, because the other two roots can not leave the circle \(|x| < |a|^{1/3}\) and the sum of three roots is \(t\).

\[\Box\]

**Lemma 16** Let \(\xi\) be a root of the polynomial \(P(x)\) and \(z = z(\xi)\) given by (64). Define \(d := |z - \xi|\). Suppose that integers \(v\) and \(w\) are such that \(3|R(v/w)| > 2|Q(v/w)|^{2/3}, \ |\frac{v}{w} - z| < \frac{d}{2}\) and \(\frac{d}{2|s| - w} > |w| \cdot |Q(v/w)|^{2/3}\) where \(u\) and \(s\) are given by (60). Then the continued fraction of (63) corresponds to the root \(\eta\) such that \(\xi = \frac{w\eta + v}{sw + w}\).

**Proof.** Solving \(\xi = \frac{w\eta + v}{sw + w}\) for \(\eta\) gives

\[
|\eta| = \frac{|v - w\xi|}{s|\xi| - u} > \frac{dw}{2|s\xi - u|} > \left|w^3Q\left(\frac{v}{w}\right)\right|^{2/3}
\]

The second inequality is due to the fact that \(|\frac{v}{w} - \xi| \geq |z - \xi| - |z - \frac{w}{w}| > \frac{d}{2}\). In view of (63), the right hand side of this inequality is \(|a|^{1/3}\). We also verify that the coefficients \(t\) and \(a\) satisfy \(|t| > 2|a|^{1/3}\). \(3|w^2R(v/w)| > 2|w^3Q(v/w)|^{2/3}\). Therefore, all the conditions of Lemma 15 are satisfied and \(\eta\) corresponds to the continued fraction (1) of (63). 

\[\Box\]
The last lemma shows that, as soon as we can find integer parameters $v$ and $w$ that satisfy the lemma’s conditions, we can provide a continued fraction for $\eta = e^{\frac{2\pi i}{p^4 k}}$ where $\xi$ is any given root of the cubic polynomial $P$. We still need to verify that there exist $v$ and $w$ such that the continued fraction (1) which corresponds to $\eta$, uniformly converges and hence equals $\eta$. In order to do that, we first provide the conditions on $a$ and $t$ that guarantee that the continued fraction (1) converge. Here we proceed in a similar way as for the one (2) in Section 5.

Denote

$$S_k := \begin{pmatrix} p_{4k} & q_{4k} \\ p_{4k-1} & q_{4k-1} \end{pmatrix}; \quad T_k := \begin{pmatrix} p_{4k} & q_{4k} \\ p_{4k-4} & q_{4k-4} \end{pmatrix}; \quad U_k := \begin{pmatrix} p_{4k-3} & q_{4k-3} \\ p_{4k-4} & q_{4k-4} \end{pmatrix}$$

From (1) and the recurrent formulae for convergents we get that

$$A_k = \begin{pmatrix} (8k+9)t & 3(12k+11)(6k+7)a \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2(8k+7)t^2 & 3(12k+7)(6k+5)a \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (8k+5)t & 3(12k+5)(3k+2)a \\ 1 & 0 \end{pmatrix}$$

(66)

We also denote the four matrices involved in the product above by $A_{4k+1}, A_{4k+2}, A_{4k+3}$ and $A_{4k+4}$ so that $A_k = A_{4k+4}A_{4k+3}A_{4k+2}A_{4k+1}$. Then the relation between $T_{k+1}$ and $S_k$ is

$$T_{k+1} = \begin{pmatrix} a_{k11} & a_{k12} \\ 1 & 0 \end{pmatrix} S_k$$

(67)

where $a_{k11}$ and $a_{k12}$ are the corresponding entries of the matrix $A_k$. One then computes

$$a_{k11} = 2(8k+3)(8k+5)(8k+7)(8k+9)t^6 + 18(8k+5)(8k+7)(36k^2 + 55k + 16)a^3 t^3 + 9(12k+5)(12k+11)(3k+2)(6k+7)a^2 ;$$

$$a_{k12} = 6(12k+1)(3k+1)(8k+7)at((8k+5)(8k+9)t^3 + 6(36k^2 + 63k+25)a).$$

Next, we relate $U_k$ and $S_k$: $S_k = C_{4k}C_{4k-1}C_{4k-2}U_k$ or

$$U_k = C_{4k}^{-1}C_{4k-1}^{-1}C_{4k}^{-1}S_k = \frac{B_k}{d} S_k,$$

where, by (1), $d = -27(12k-7)(12k-5)(12k-1)(3k-1)(6k-1)(6k+1)a^3$ and

$$B_k = \begin{pmatrix} 0 & -3(12k-7)(3k-1)a \\ 1 & (8k-3)t \end{pmatrix} \begin{pmatrix} 0 & -3(12k-5)(6k-1)a \\ -1 & 2(8k-1)t^2 \end{pmatrix} \begin{pmatrix} 0 & -3(12k-1)(6k+1)a \\ -1 & (8k+1)t \end{pmatrix}.$$ 

Then the relation between $T_k$ and $S_k$ is

$$T_k = \begin{pmatrix} 1 & 0 \\ b_{k21}/d & b_{k22}/d \end{pmatrix} S_k.$$

(69)

where $b_{k21}$ and $b_{k22}$ are the corresponding components of the matrix $B_k$. We compute

$$b_{k21} = -3(12k-5)(6k-1)a - 2(8k-3)(8k-1)t^3;$$

$$b_{k22} = 2(8k-1)t((8k-3)(8k+1)t^3 + 6(36k^2 - 9k-2)a) = : 2(8k-1)tp(k).$$

Finally, we use (69) and (67) to relate $T_k$ and $T_{k+1}$:

$$T_{k+1} = \begin{pmatrix} a_{k11} - a_{k12} b_{k21} & b_{k21} \\ a_{k12} b_{k22} & 0 \end{pmatrix} T_k.$$

(70)

We then compute

$$\frac{da_{k12}}{b_{k22}} = \frac{-81(12k+1)(12k-1)(12k-5)(12k-7)(3k-1)(3k+1)(6k-1)(6k+1)(8k+7)a^4 p(k+1)}{(8k-1)p(k)}.$$

(71)
Lemma 17 Let $a \in \mathbb{Z}$ be positive and $t$ satisfy $12a \leq |t|^3$. Then $q_{4k+4}$ and $q_{4k}$ satisfy the relation

$$q_{4k+4} > (8k + 3)(8k + 5)(8k + 7)(8k + 9)(t^3 + 2a)^2 q_{4k}.$$ 

If $t$ is positive then (67) together with (68) and $12a \leq t^3$ imply

$$q_{4k+4} > a_{k11}q_{4k} > (8k + 3)(8k + 5)(8k + 7)(8k + 9)(t^3 + 2a)^2 q_{4k}.$$ 

Assume now that $t$ is negative. In that case, we compare the terms of $a_{k11}$ and use $12a \leq |t|^3$ to get for $k \geq 1$

$$18(8k + 5)(8k + 7)(36k^2 + 55k + 16)a|t|^3 < (8k + 3)(8k + 5)(8k + 7)(8k + 9)t^6.$$ 

Therefore, $a_{k11} > (8k + 3)(8k + 5)(8k + 7)(8k + 9)t^6$. On top of that, by comparing the terms in $a_{k12}$, one can verify that $(8k + 5)(8k + 9)|t|^3 > 6(36k^2 + 63k + 25)a$ and therefore $a_{k12} > 0$. We combine the last two inequalities for $a_{k11}$ and $a_{k12}$ and get

$$q_{4k+4} = a_{k11}q_{4k} + a_{k12}q_{4k-4} > (8k + 3)(8k + 5)(8k + 7)(8k + 9)t^6 q_{4k}.$$ 

The lemma then follows from the fact that $|t^3| > |t^3 + 2a|$.

Lemma 18 Let $a \in \mathbb{Z}$ and $t \in \mathbb{R}$ be such that $0 < 12a \leq |t|^3$. Then the continued fraction (1) converges. Moreover, there exists $k_0 > 0$ such that for all $k > k_0$ its limit $\xi = x(t, a)$ satisfies

$$\left| \frac{p_{4k}}{q_{4k}} - \frac{p_{4k+4}}{q_{4k+4}} \right| < \left| \frac{\xi - p_{4k}}{q_{4k}} \right| < 2 \left| \frac{p_{4k}}{q_{4k}} - \frac{p_{4k+4}}{q_{4k+4}} \right|.$$ 

Proof. Equation (70) infers that

$$\frac{p_{4k+4}}{q_{4k+4}} - \frac{p_{4k}}{q_{4k}} = -\frac{d a_{k12}q_{4k}q_{4k-4}}{b_{k22}q_{4k+4}q_{4k}} \left( \frac{p_{4k}}{q_{4k}} - \frac{p_{4k-4}}{q_{4k-4}} \right).$$

(72)

In particular, since $d$ is negative and $a_{k12}$ and $b_{k22}$ are positive numbers, as was shown in the proof of the previous lemma, the differences $\frac{p_{4k+4}}{q_{4k+4}} - \frac{p_{4k}}{q_{4k}}$ share the same sign for all $k \geq 1$. Now we use Lemma 17 and (71) to estimate

$$-\frac{d a_{k11}q_{4k-4}}{b_{k22}q_{4k+4}} < \frac{81(12k - 1)(12k + 1)(12k - 5)(12k - 7)(9k^2 - 1)(36k^2 - 1)a^4}{(8k - 5)(8k - 3)(8k - 1)(8k + 1)(8k + 3)(8k + 5)(8k + 7)(8k + 9)(t^3 + 2a)^2} \frac{(8k + 7)p(k + 1)}{(8k - 1)p(k)}.$$ 

Since $p(k)$ is a quadratic polynomial in $k$, the second fraction in the last expression tends to one as $k \to \infty$. The first fraction tends to $\frac{3^{12}2^{10}}{2^{24}} \cdot \left( \frac{a}{t^3 + 2a} \right)^4$, which is strictly less than $\frac{1}{2}$ as soon as $5a < |t^3 + 2a|$.

For the rest of the proof we proceed analogously to the proof of Lemma 10. We write $\frac{p_{4k+4}}{q_{4k+4}}$ as the sum

$$\frac{p_{4k+4}}{q_{4k+4}} = \frac{p_0}{q_0} + \sum_{i=0}^{k} \left( \frac{p_{4i+4}}{q_{4i+4}} - \frac{p_{4i}}{q_{4i}} \right).$$

Since, starting from some $k \geq k_0$ the terms in the summation on the right hand side decay by a factor at least 2, the limit of $\frac{p_{4k+4}}{q_{4k+4}}$ as $k \to \infty$ exists, let’s call it $\xi$. Then the estimates on $|\xi - \frac{p_{4k}}{q_{4k}}|$ follow easily.

Now we are ready to complete the proof of Theorem 3. Let $\xi$ be a root of $P$. We need to construct the integers $v$ and $w$ such that the resulting coefficients $a$ and $t$ in (63)
satisfy all the conditions of Lemmata 17 and 18. Denote by $B$ the height of $\xi$, i.e. $B := \max\{|b_0|, |b_1|, |b_2|, |b_3|\}$.

We construct $v, w$ such that $v/w$ is very close to $z(\xi)$ which is defined in (64). By the Dirichlet theorem, there exist $v, w \in \mathbb{Z}$ such that $w$ is arbitrarily large and

$$|z - \frac{v}{w}| < \frac{1}{w^2}.$$ 

Since $z$ is a root of the polynomial $Q$ from (63), we get for $z$ close enough to $v/w$,

$$|w^3 Q(v/w)| \leq w^3 \cdot 2 |z - \frac{v}{w}| \cdot |Q'(z)| \leq 2 |Q'(z)| w.$$ 

Now we estimate $|\frac{u}{s} - \xi|$ where $u$ and $s$ are defined in (60).

$$\left|\frac{u}{s} - \xi\right| = |D(v/w) - D(z)|,$$

where $D(z)$ is the rational function $D(z) := \frac{-b_3 z^2 - 2b_2 z - 3b_1}{3b_2 z^2 + 2b_1 z + b_3}$. Since $z$ is cubic irrational, it is not a singularity of $D$. We choose $v/w$ close enough to $z$ so that the whole interval between $z$ and $v/w$ does not contain singularities of $D$. Moreover, one can check that $D'(z) \neq 0$, thus for $z$ close enough to $v/w$,

$$|D(v/w) - D(z)| < 2 \left|\frac{v}{w} - z\right| |D'(z)| < \frac{2 |D'(z)|}{w^2}.$$ 

(73)

Since $|s| = |u^2 P'(v/w)| \leq 6Bw^2|z|^2$, we get that

$$|u - s\xi| \leq 12B|z^2 \cdot D'(z)|$$

and therefore for $w$ large enough, we get

$$w |Q(v/w)|^{2/3} \leq \frac{4 |Q'(z)|^{2/3}}{w^{1/3}} \leq \frac{d}{24B |z^2 \cdot D'(z)|} \leq \frac{d}{2 |u - s\xi|},$$

where $d = |z - \xi|$.

Consider the polynomial $R$ from (61). Since it is not the constant zero and has degree at most 2, $R(z) \neq 0$. Therefore, for $v/w$ close enough to $z$ we have $|R(v/w)| > \frac{1}{2} |R(z)| > 0$. Therefore

$$|3R(v/w)| > \frac{3}{2} |R(z)|.$$ 

On the other hand,

$$|2Q(v/w)|^{2/3} < \left|\frac{4Q'(z)}{w^2}\right|^{2/3}.$$ 

Therefore, for large enough $w$ we get that $|3R(v/w)| > |2Q(v/w)|^{2/3}$.

The last condition to check is one in Lemma 18 which is written as $|12(w^3 Q(v/w))| < |3w^2 R(v/w)|^3$. This is indeed true because, for $w$ large enough,

$$|12(w^3 Q(v/w))|^2 < 12 \cdot 4(Q'(z))^2 w^2 < \frac{27}{8} |R(z)|^3 u^3 w^6 < |3w^2 R(v/w)|^3.$$ 

All the conditions of Lemmata 17 and 18 are satisfied and therefore $\eta$, which solves the equation $\xi = \frac{u\eta + v}{s\eta + w}$ admits a continued fraction expansion of the form (1). Moreover, that continued fraction converges to $\eta$. This finished the proof of Theorem 3.
A Formulae for coefficients of the remaining continued fractions

In Section 3 we only verified the continued fractions of the cubic Laurent series №1 and №5. As discussed in the Introduction, the continued fraction for the series №2 follows from the one for №1. The verification of the remaining continued fractions repeats all the steps in Subsection 3.3. Therefore, here we only provide the formulae for the coefficients $A_i, B_i, C_i$ and $D$ as well as $a_i, \beta_i$. An interested reader can then verify them by induction in the same way as for the algebraic series №5.

Algebraic series №3: $x^3 - tx^2 - at = 0$.

$$D = 4t^3 + 27at;$$

$$A_{4k+1} = -4t, \quad B_{4k+1} = -(32k + 8)t^2 - 9a, \quad C_{4k+1} = 12(12k + 1)(3k + 1)t,$$

$$a_{4k+1} = (8k + 3)t, \quad \beta_{4k+1} = 3(12k + 1)(3k + 1)a;$$

$$A_{4k+2} = -4t, \quad B_{4k+2} = -(32k + 16)t^2 + 9a, \quad C_{4k+2} = 12(12k + 5)(3k + 2)t;$$

$$a_{4k+2} = (8k + 5)t, \quad \beta_{4k+2} = 3(12k + 5)(3k + 2)a;$$

$$A_{4k+3} = -4t, \quad B_{4k+3} = -(32k + 24)t^2 - 9a, \quad C_{4k+3} = 6(12k + 7)(6k + 5)t;$$

$$a_{4k+3} = 2(8k + 7)t, \quad \beta_{4k+3} = 3(12k + 7)(6k + 5)a;$$

$$A_{4k+4} = -2t, \quad B_{4k+4} = -(32k + 32)t^2 + 9a, \quad C_{4k+4} = 12(12k + 11)(6k + 7)t;$$

$$a_{4k+4} = (8k + 9)t, \quad \beta_{4k+4} = 3(12k + 11)(6k + 7)a.$$

Algebraic series №4: $x^3 - tx^2 - a = 0$. As discussed in the Introduction, the continued fraction for this series is derived from that for the series №3.

Algebraic series №6: $x^3 + (t - 2)x^2 - 2(t - 2)x + 2(t - 2) = 0$.

$$D = (t - 2)(t^2 + 6t + 11);$$

$$A_{3k+1} = (-1)^{3k}((8k + 3)t + (20k + 1)), \quad B_{3k+1} = -2(4k + 1)^2t^2 - (96k^2 + 48k + 5)t - (112k^2 + 56k + 3),$$

$$C_{3k+1} = (-1)^{3k} \cdot 2(6k + 1)(3k + 1)((8k + 3)t + (20k + 9)), \quad D_{3k+1} = (4k + 1)D;$$

$$a_{3k+1} = (-1)^{3k+1}((4k + 1)t + 2k), \quad \beta_{3k+1} = 2(6k + 1)(3k + 1);$$

$$A_{3k+2} = (-1)^{3k+1}((8k + 3)t + (20k + 9)), \quad B_{3k+2} = -(4k + 1)((8k + 4)t^2 + (24k + 13)t - (8k + 3)),$$

$$C_{3k+2} = (-1)^{3k+1} \cdot 12(4k + 1)^2(3k + 2), \quad D_{3k+2} = (4k + 1)D;$$

$$a_{3k+2} = (-1)^{3k+1}(4k + 3)(t^2 + 3t - 1), \quad \beta_{3k+2} = 6(4k + 1)(3k + 2);$$

$$A_{3k+3} = (-1)^{3k} \cdot 2, \quad B_{3k+3} = -(8k + 8)t^2 - (24k + 23)t + (8k + 9),$$

$$C_{3k+3} = (-1)^{3k} \cdot 18k + 15)((8k + 9)t + (20k + 21)), \quad D_{3k+3} = D;$$

$$a_{3k+3} = (-1)^{3k}((4k + 5)t + (2k + 3)), \quad \beta_{3k+3} = 3(4k + 5)(6k + 5).$$
B Continued fractions of cubic irrationals with very large partial quotients

Here we present the list of cubic numbers $\xi \in \mathbb{R}$ whose continued fractions have at least one partial quotient $a_n$ such that

$$a_n \geq 2H(\xi)^{\tau_1} n^2,$$

$$\tau_1 = 3 + \frac{2 \ln 2}{2.88} \approx 3.4814...$$  (74)

If two or more equivalent numbers satisfy (74), we present only one of them with the largest value of $C := \frac{a_n}{H(\xi)^{\tau_1} n^2}$. Here, we say that $\xi$ and $\zeta$ are equivalent if their continued fractions eventually coincide.

This list is produced by a computer search of roots of cubic polynomials with small height. To simplify the algorithm, we only go through polynomials that take opposite signs at 0 and 1 and only consider their largest real root in the interval $(0, 1)$. The code was implemented in C++ computer language with NTL library for long arithmetic operations. It can be provided by request.

Under the above constraints, for all algebraic $\xi$ with $H(\xi) \leq 2$ the first 10000 partial quotients were computed; for all $\xi$ with $H(\xi) \leq 5$ the first 5000 partial quotients; for all $\xi$ with $H(\xi) \leq 10$ the first 1000 partial quotients and for all $\xi$ with $H(\xi) \leq 100$ the first 50 partial quotients were computed. These calculations took around 19 hours 49 minutes on one core of Ryzen 3700X CPU.

The resulting list is:

1. Root of $x^3 + x^2 + x - 1 = 0$ has $\xi = [0; 1, 1, 5, 4, 2, 305, ...]$. For $a_6(\xi)$ the value $C := \frac{a_6(x)}{6^2 \cdot H(\xi)^{\tau_1}}$ equals 8.472...

2. Root of $2x^3+2x-1 = 0$ has $\xi = [0; 2, 2, 1, 3, 1, 1, 1, 2, 1, 5, 456, 1, 30, 1, 3, 4, 29866, ...]$. For $a_{18}(\xi)$, $C = 8.253...$

3. Root of $2x^3 + 2x^2 + 2x - 1 = 0$ has

$$\xi = [0; 2, 1, 11, 2, 3, 1, 23, 2, 3, 1, 1337, 2, 8, 3, 2, 1, 7, 4, 2, 2, 87431, ...].$$

For $a_{21}(\xi)$, $C = 17.751...$

4. Root of $x^3 - 2x^2 - 3x + 1 = 0$ has

$$\xi = [0; 3, 2, 26, 1, 6, 3, 3, 1, 2, 4, 92, 24, 2, 3, 2, 4, 2, 1, 16, 40033, ...].$$

For $a_{20}(\xi)$, $C = 2.184...$

5. Root of $2x^3 - 2x^2 + 4x - 3 = 0$ has

$$\xi = [0; 1, 4, 3, 7, 4, 2, 30, 1, 8, 3, 1, 1, 1, 9, 2, 2, 1, 3, 22986, 2, 1, 32, 8, 2, 1, 8, 55, 1, 5, 2, 28, 1, 5, 1, 1501790, 1, 2, 1, 7, 6, 1, 5, 2, 1, 6, 2, 2, 1, 2, 1, 1, 3, 1, 3, 1, 2, 4, 3, 1, 35657, 1, 17, 2, 15, 1, 1, 2, 1, 1, 5, 3, 2, 1, 1, 7, 2, 1, 7, 1, 3, 25, 49405, 1, 1, 3, 1, 1, 4, 1, 2, 15, 1, 2, 83, 1, 162, 2, 1, 1, 2, 1, 53460, 1, 6, 4, 3, 4, 13, 5, 15, 6, 1, 4, 1, 4, 1, 1, 2, 1, 1, 16467250, ...].$$

For $a_{35}(\xi)$, $C = 9.828...$; for $a_{123}(\xi)$, $C = 8.726...$
6. Root of $7x^3 + 4x^2 - 4x - 6 = 0$ has
\[ \xi = [0; 1, 22, 1, 31, 2, 3, 1, 63, 1, 10, 1, 2, 1, 7, 1, 160905, 2, 1, 4, 58, 2, 2, 1, 2, 1, 7, 3, 1, 3, 1, 4, 3, 1, 47, 1, 214540, 1, 2, 9, 1, 45, 1, 3, 1, 48, 1, 21, 1, 9, 1, 8, 1, 2, 249610, 1, 1, 1, 1, 3, 1, 1, 1, 20, 1, 4, 19, 1, 2, 1, 1, 1, 3, 4, 1, 1, 1, 3, 345838, 1, 13, 1, 3, 3, 1, 1, 1, 9, 1, 11, 7, 23, 5, 13, 1, 374230, 31, 6, 2, 1, 2, 5, 3, 1, 1, 7, 4, 1, 37, 115270760, \ldots]. \]

For $a_{114}(\xi)$, $C = 10.134 \ldots$

7. Root of $14x^3 + 10x^2 + 8x - 5 = 0$ has
\[ \xi = [0; 2, 1, 2, 1, 1, 1, 1, 4, 1, 1, 1, 10, 24, 6, 2, 8, 436745, 2, 1, 1, 16, 1, 29, 2, 1, 2, 2, 1, 1, 3, 34, 2, 1, 3, 28534040, \ldots]. \]

For $a_{36}(\xi)$, $C = 2.252 \ldots$

8. Root of $11x^3 + 21x^2 + 24x - 30 = 0$ has
\[ \xi = [0; 1, 2, 5, 1095, 2, 1, 2, 5, 1, 8, 2, 5, 1, 14, 1, 1, 1, 2, 11, 11, 1, 2, 4, 1, 1, 1, 1, 9, 1, 6, 2, 829131361, \ldots]. \]

For $a_{35}(\xi)$, $C = 5.485 \ldots$

9. Root of $44x^3 + 42x^2 + 24x - 15 = 0$ has
\[ \xi = [0; 2, 1, 10, 547, 1, 2, 5, 2, 1, 17, 1, 11, 1, 6, 1, 4, 2, 1, 5, 23, 2, 1, 1, 1, 4, 3, 4, 1, 13, 1, 1658262722, \ldots]. \]

For $a_{31}(\xi)$, $C = 3.277 \ldots$

References

[1] A. Baker, *Rational approximations to certain algebraic numbers*, Proc. LMS *14* (1964), No 3, 385–398.

[2] M. A. Bennett, *Effective Measures of Irrationality for Certain Algebraic Numbers*, J. AustMS *62* (1997), 329–344.

[3] V. Beresnevich, F. Ramírez, S. Velani, *Metric Diophantine Approximation: Aspects of Recent Work*, In: “Dynamics and Analytic Number theory”, D. Badziahin, A. Gorodnik, N. Peyerimhoff (eds.), LMS Lecture Notes Series, 2016, 1–95.

[4] E. Bombieri, *On the Thue-Siegel-Dyson theorem*, Acta Math. *148* (1982), 255–296.

[5] E. Bombieri, A.J. van der Poorten, J. D. Vaaler, *Effective measures of irrationality for cubic extensions of number fields*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) *23* (1996), No 2, 211–248.

[6] Y. Bugeaud, “Linear forms in logarithms and applications”, European Mathematical Society, 2018.

[7] G. V. Chudnovsky, *On the method of Thue-Siegel*, Ann. of Math., II ser. *117* (1983), 325–382.

[8] F. Dyson, *The approximation to algebraic numbers by rationals*, Acta Math. *79* (1947), 225–240.
[9] N. I. Feldman, *Improved estimate for a linear form of the logarithms of algebraic numbers*. Mat. Sb. 77 (1968), 256–270 (in Russian). English translation in Math. USSR. Sb. 6 (1968), 393–406.

[10] W. B. Jones, W. J. Thron, “Continued fractions: Analytic theory and Applications”, Encyclopedia of Mathematics and Applications 11, Addison-Wesley, 1980, p. 92.

[11] S. Lang, “Number Theory III”, Encyclopedia of Mathematical Sciences 60, Springer-Verlag, New York, 1991, p. 214.

[12] G. Lettl, A. Pethö, P. Voutier, *Simple families of thue inequalities*. Trans. of AMS 351 (1999), No 5, 1871–1894.

[13] C. F. Osgood, *An Effective Lower Bound on the Diophantine Approximation of Algebraic Functions by Rational Functions*, Mathematika 20 (1973), 4–15.

[14] A. J. van der Poorten, *Formal Power Series and their Continued Fraction Expansion*, In: “Algorithmic Number Theory”, Lecture Notes in Computer Science 1423, 1998, 358–371.

[15] J. B. Rosser, L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. 6 (1962), 64–94.

[16] K. F. Roth, *Rational approximations to algebraic numbers*, Mathematika 2 (1955), 337–360.

[17] P. M. Voutier, *Rational approximations to \( \sqrt{2} \) and other algebraic numbers revisited*, J. de Théorie des Nombres de Bordeaux 19 (2007), No 1, 263–288.

[18] P. M. Voutier, *Thue’s Fundamental theorem, I: The general case*, Acta Arith. 143 (2010), No 2, 101–144.

[19] I. Wakabayashi, *Cubic Thue inequalities with negative discriminant*, J. Number Theory, 97 (2002), No 2, 225–251.

Dzmitry Badziahin
The University of Sydney
Camperdown 2006, NSW (Australia)
dzmitry.badziahin@sydney.edu.au