Quantization of Edge Currents along Magnetic Barriers and Magnetic Guides

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Abstract. We investigate the edge conductance of particles submitted to an Iwatsuka magnetic field, playing the role of a purely magnetic barrier. We also consider magnetic guides generated by generalized Iwatsuka potentials. In both cases, we prove quantization of the edge conductance. Next, we consider magnetic perturbations of such magnetic barriers or guides and prove stability of the quantized value of the edge conductance. Further, we establish a sum rule for edge conductances. Regularization within the context of disordered systems is discussed as well.

1. Introduction

Quantization of two-dimensional edge states appeared within the context of the quantum Hall effect in a seminal paper of Halperin [30]. Existence of quantum states flowing along edges has been studied mathematically in several recent works, e.g. [10,13,19–21]. Besides the existence of such states, the question of their quantization has been brought forth mathematically in [8,15,16,39] through the quantization of the so-called edge conductance, together with the issue of the equality between the edge conductance and the bulk conductance, also called the Hall conductance (concerning the quantization of the bulk conductance itself, see [2,4,6,7,27,28]). It is worth pointing out that in all these works, the edge is modeled by a confining electric potential or by a hard wall with Dirichlet boundary condition, a case that could be interpreted as an infinite electric potential in a half-plane. The perturbations considered were designed by electric potentials as well.

In this article we are interested in the same phenomena, but generated by purely magnetic effects. If in most situations considered here, we show that results obtained with electric potentials can also be generated by magnetic effects, we also investigate the quantization of currents carried by magnetic wave guides that have been recently studied in the physics literature [36,37].
Indeed, unlike the electric ones, these specific magnetic wave guides exhibit extended orbits called “snake orbits”. We prove that currents related to such extended states are quantized in the sense that the associated edge conductance is an integer.

Let us discuss more precisely the contents of this paper.

Magnetic walls are designed here by Iwatsuka magnetic fields [33], that are $y$-independent magnetic fields with a growing profile in the $x$-axis and of constant sign. As a matter of fact, the particle is subject to, say, a strong magnetic field on the right half plane, and to a weaker one on the left half plane. The extreme version of this model would be a magnetic field with strength $B_− > 0$ for $x < 0$ and $B_+ > B_−$ for $x > 0$, with $B_+ − B_−$ large enough. Due to the strength difference, the classical particle is localized on a circle of radius proportional to $1/B_+$ inside the right half plane, and proportional to $1/B_− > 1/B_+$ inside the left half plane; it is easy to be convinced that near the interface $x = 0$, there exist extended states which induce a current flowing in the $y$ direction (see for instance Fig. 6.1 in [12]); so that the interface $x = 0$ plays here the role of a “magnetic wall”.

The spectral interpretation of this fact is the absolutely continuous nature of the spectrum, as proven in [33]. However, the absolutely continuous spectrum does not shed light on existing currents flowing along the edge. We shall provide a simple computation of the edge conductance that validates this intuition, showing it is non-zero when considering energies above the first Landau level of a Landau Hamiltonian of magnetic intensity $B_−$; the edge conductance is actually quantized, in concordance with the physics of the quantum Hall effect and Halperin’s argument. We mention here that if the existence of edge states for the half-plane model has been proved in case of electric perturbations [13,21], by showing that band functions of the unperturbed system have a strictly positive derivative, and then using Mourre’s theory, a similar approach fails with the Iwatsuka potential for band functions may not be monotone. Nevertheless, our analysis goes through, because the edge conductance “computes” the net current, even if there are several channels of opposite sign.

A totally different situation is the one where the magnetic field profile is a $y$-independent monotone increasing function of non-constant sign. We will call such a Hamiltonian a generalized Iwatsuka Hamiltonian. The properties of such operators are nevertheless quite different from those of the standard Iwatsuka Hamiltonian. As mentioned above, there has been some recent attention in the physics literature for such quantum magnetic guides, since they exhibit interesting extended states called “snake-orbit” states [36,37]. We prove that currents carried by such states are quantized.

Secondly, the perturbations we consider are also of magnetic nature. To motivate the study of such magnetic perturbations, let us recall that relevant perturbations within the context of the quantum Hall effect are random perturbations (modelling impurities or defects of the sample), for the localized states they generate are responsible for the celebrated plateaux of the (bulk) Hall conductance. Occurrence of localized states which could arise from
random magnetic perturbations in dimension 2 in relation with quantum Hall systems, has been intensively studied in the physics literature over the past two decades (see e.g. [3,5,22,44]). Mathematically, the proof of the occurrence of Anderson localization due to random magnetic potentials only is not an easy task, and very few preliminary results are available: recently Ghribi et al. [29] proved localization for random magnetic perturbations of a periodic magnetic potential (see also [34] for a particular discrete model). Ueki [43] proved localization for some magnetic perturbation of the Landau Hamiltonian at the bottom of the spectrum (below the first Landau level). In the companion note [14], we provide an example that is relevant to the theory of the quantum Hall effect, namely a random magnetic perturbation of the Landau Hamiltonian with localized states at the edges of the first $J$ bands, $J \geq 1$ given.

As a preliminary result, we show that currents generated by Iwatsuka and generalized Iwatsuka Hamiltonians are quantized, and we compute the exact value of the edge conductance. Next, we prove that magnetic perturbations carried by magnetic fields compactly supported in the $x$ axis do not affect the edge conductance. In particular, if we consider a magnetic strip and a moderate magnetic field inside, then the net current flowing along these axes is zero, like in the electric case. Then we consider perturbations which do not vanish at infinity and provide a sum rule similar to the one obtained in [8]. Namely, the edge conductance of the perturbed system is the sum of the edge conductance of the magnetic confining potential and of the edge conductance of the system without magnetic wall defined by the Landau Hamiltonian of magnetic strength $B_-$ perturbed by the magnetic potential. This enables us to compute the edge conductance for the magnetically perturbed Hamiltonian when energies fall inside a gap of the unperturbed Landau Hamiltonian of magnetic strength $B_-$. To consider energies corresponding to localized states, one has to go one step further and regularize the trace that defines the edge conductance (see [8,16]) and use the localization properties of the model as provided by the theory of random Schrödinger operators [1,23]. As an illustration, we revisit the model considered in [14] and discuss the quantization of its (regularized) edge conductance in presence of an Iwatsuka confining wall.

Of course, it follows from [8] and the results of the present paper that any combination of electric and magnetic potentials defining the confining wall and the perturbation will work in the same way.

The paper is organized as follows. In Sect. 2 we state our main results. In Sect. 3 we gather properties of the generalized Iwatsuka Hamiltonian and prove the quantization of the edge conductance in the unperturbed case. In Sect. 4 we consider compact (in the $x$ axis) magnetic perturbations of magnetic barriers, and prove stability of the quantized value of the edge conductance. In Sect. 5, we consider non-vanishing magnetic perturbations and establish a sum rule for edge conductances. In Sect. 6 we discuss regularization of the edge conductance in the presence of disorder. Finally, in the Appendix, we gather for reader’s convenience some trace estimates used intensively in the main text.
2. Definitions and Results

2.1. Edge Conductance

Let $A \in L^2_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$. Define

$$H(A) := (i\nabla + A)^2$$

as the self-adjoint operator generated in $L^2(\mathbb{R}^2)$ by the closure of the quadratic form

$$\int_{\mathbb{R}^2} |i\nabla u + Au|^2 \, dx, \quad u \in C^\infty_0(\mathbb{R}^2).$$

If $A \in L^4_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ and $\text{div} \, A \in L^2_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$, then $H(A)$ is essentially self-adjoint on $C^\infty_0(\mathbb{R}^2)$ (see [35]).

We will say that the magnetic potential $A = (A_1, A_2)$ generates the magnetic field $B : \mathbb{R}^2 \to \mathbb{R}$ if

$$\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = B(x, y), \quad (x, y) \in \mathbb{R}^2. \tag{2.1}$$

The celebrated Landau Hamiltonian corresponds to the case $B(x, y) = B \neq 0$ constant, in which case the spectrum consists in the so called Landau levels $(2n+1)|B|$, $n \in \mathbb{N} := \{0, 1, 2, \ldots\}$. To fix notations, we will denote by $H_B$ the Landau Hamiltonian, and set

$$\mathbb{G}_0(B) = ]-\infty, |B|[, \quad \text{and} \quad \mathbb{G}_n(B) = ](2n-1)|B|, (2n+1)|B|[ \quad \text{for} \quad n \geq 1, \tag{2.2}$$

the (open) Landau gaps.

We will say that $f : \mathbb{R} \to [0, 1]$ is an increasing (resp. decreasing) switch function if $f \in C^\infty(\mathbb{R})$ is monotone, with a compactly supported derivative, $f \equiv 1$, (resp., $f \equiv 0$) on the right side of $\text{supp} \, f'$, and $f \equiv 0$, (resp., $f \equiv 1$) on the left one.

Definition 2.1. Let $\chi \in C^\infty(\mathbb{R}^2)$ be a $x$-translation invariant increasing switch function with $\text{supp} \, \chi' \subset \mathbb{R} \times [-\frac{1}{4}, \frac{1}{4}]$, and $g \in C^\infty(\mathbb{R})$ be a decreasing switch function with $\text{supp} \, g' \subset I = [a, b]$, a compact interval. The edge conductance of a Hamiltonian $H$ in the interval $I$ is defined as

$$\sigma_e^{(I)}(H) := -2\pi \text{tr} \, (g'(H)i[H, \chi]), \tag{2.3}$$

whenever the operator $g'(H)i[H, \chi]$ is trace-class, and hence the trace above is well-defined.

Detailed arguments for the physical motivation of this definition of the edge conductance could be found in [8,15,16,39]. Here we note only that although the above definition depends a priori on the choice of $g$ and $\chi$, results will not.

Note also that when the magnetic field is reversed $B \mapsto -B$, then the edge conductance is changed into its opposite, whenever it exists.

The factor $2\pi$ is introduced in order that conductance $\sigma_e^{(I)}(H)$ be integer-valued.
2.2. Generalized Iwatsuka Hamiltonians

A magnetic field will be called a generalized Iwatsuka magnetic field if the following conditions hold:

GIW.1 \( B \) depends only on the first coordinate, i.e. \( B(x, y) = B(x) \);

GIW.2 \( B \) is a monotone function of \( x \);

GIW.3 There exist two numbers \( B_-, B_+ \in \mathbb{R}\setminus\{0\} \) such that
\[
\lim_{x \to \pm\infty} B(x) = B_\pm.
\]

Depending on the context, we may further assume that the magnetic field \( B \) is in \( C^1(\mathbb{R}^2; \mathbb{R}) \).

We will call such a magnetic field a \((B_-, B_+)-\)magnetic field. Introduce the magnetic potential \( A^{(B_-, B_+)}_{GIW} = (A_1, A_2) \) with
\[
A_1 = 0, \quad A_2(x) := \int_0^x B(s)ds, \quad x \in \mathbb{R}. \tag{2.4}
\]

Set
\[
H(A^{(B_-, B_+)}_{GIW}) := -\frac{\partial^2}{\partial x^2} + \left(-i\frac{\partial}{\partial y} - \beta(x)\right)^2.
\]

Obviously, \( A^{(B_-, B_+)}_{GIW} \) generates \( B \). When \( B_-B_+ > 0 \), i.e. in the case considered in the original work by Akira Iwatsuka [33], we will use the shorter term Iwatsuka Hamiltonian, and will write \( A^{(B_-, B_+)}_I \) or simply \( A_I \). The case \( B_-B_+ < 0 \) corresponds to a magnetic wave guide.

2.3. Quantization of Edge Currents for Generalized Iwatsuka Hamiltonians

It turns out that the edge conductance can be explicitly computed for generalized Iwatsuka Hamiltonians.

**Theorem 2.2.** Let \( A^{(B_-, B_+)}_{GIW} \) be a generalized Iwatsuka potential. Let \( I \) be an interval such that for some integers \( n_-, n_+ \geq 0 \) we have
\[
I \subset \mathbb{G}_{n_-}(B_-) \cap \mathbb{G}_{n_+}(B_+). \tag{2.5}
\]

Then
\[
\sigma_e(I)(H(A^{(B_-, B_+)}_{GIW})) = (\text{sign } B_-)n_- - (\text{sign } B_+)n_+. \tag{2.6}
\]

**Corollary 2.3.** Assume \( 0 < B_- < B_+ \). Consider the Iwatsuka potential \( A^{(B_-, B_+)}_I \). Let \( I \subset \mathbb{G}_n(B_-) \cap (-\infty, B_+) \), for some integer \( n \geq 0 \). Then
\[
\sigma_e(I)(H(A^{(B_-, B_+)}_I)) = n. \tag{2.7}
\]

**Corollary 2.4.** Assume \( B_- < 0 < B_+ \), and consider the generalized Iwatsuka potential \( A^{(B_-, B_+)}_{GIW} \). Let \( I \) be an interval and \( n_-, n_+ \in \mathbb{N} \) such that \( I \subset \mathbb{G}_{n_-}(B_-) \cap \mathbb{G}_{n_+}(B_+) \). Then
\[
\sigma_e(I)(H(A^{(B_-, B_+)}_{GIW})) = -(n_- + n_+). \tag{2.8}
\]

In particular if \( B_- = -B_+ \) and \( I \subset \mathbb{G}_n(B_-) \), then \( \sigma_e(H(A^{(B_-, B_+)}_{GIW})) = -2n \).
Remark 2.5. Corollary 2.4 describes a purely magnetic phenomenon. With electric barriers, the net current is always zero [8, Corollary 1], as it is the case as well for magnetic barriers induced by magnetic fields of constant signs (see Corollary 2.8 below). If no magnetic field is present inside the strip, then currents are not quantized (the edge conductance is infinite).

Such extended states are called “snake-orbit” states.

In the particular case $B_- = -B_+$, it is interesting to note that the value of the edge conductance is exactly twice the value coming from the quantum Hall effect.

2.4. Stability of the Edge Conductance Under Magnetic Perturbations

We now state results concerning the stability of the edge conductance under purely magnetic perturbations. The first one asserts that magnetic perturbations supported on a strip in the $y$ direction do no affect the edge conductance.

Theorem 2.6. Let $A \in C^1(\mathbb{R}^2; \mathbb{R}^2)$. Assume that $a \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ is a magnetic potential compactly supported in the $x$-direction and polynomially bounded in the $y$-direction. Let $g$ be as in Definition 2.1, with supp$g' \subset I$. Then

$$g'(H(A + a))[H(A + a), \chi] - g'(H(A))[H(A), \chi] \in \mathcal{T}_1$$

where $\mathcal{T}_1$ denotes the trace class. Moreover, if $g'(H(A))[H(A), \chi] \in \mathcal{T}_1$, then

$$\sigma_e^{(\text{I})}(H(A + a)) = \sigma_e^{(\text{I})}(H(A)).$$

(2.10)

In particular, if $\mathcal{A}_{\text{GIw}}^{(B_-B_+)}$ generates a $(B_-B_+)$-magnetic field, and $I$ is as in (2.5), then

$$\sigma_e^{(\text{I})}(H(\mathcal{A}_{\text{GIw}}^{(B_-B_+)} + a)) = (\text{sign } B_-)n_- - (\text{sign } B_+)n_+.$$  

(2.11)

Remark 2.7. In [8, Theorem 1], the analog of the difference (2.9) is not only trace class but its trace automatically vanishes. This is not the case here since the velocity operators defined for $H(A)$ and $H(A + a)$ differ. Indeed,

$$g'(H(A + a))[H(A + a), \chi] - g'(H(A))[H(A), \chi]$$

$$= (g'(H(A + a) - g'(H(A)))[H(A), \chi]$$

$$+ g'(H(A + a))[H(A + a) - H(A), \chi].$$

(2.12)

The second term on the r.h.s. is due to the magnetic nature of the perturbation, and may lead to a non-trivial contribution to the current since a direct computation yields $2ig'(H(A + a))a_2\chi \in \mathcal{T}_1$. To cancel this extra term, we shall introduce a suitable gauge transform that will make $a_2$ vanish. To perform that gauge transform, we assume a bit more than in [8], namely, we assume that $g'(H(A))[H(A), \chi]$ (or equivalently $g'(H(A + a))[H(A + a), \chi]$)

is trace class.

Corollary 2.8. Let $\mathcal{A}_{\text{strip}} \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ be a vector potential generating a magnetic field $B \in C^2(\mathbb{R}^2, \mathbb{R})$ satisfying $B(x, y) = B(x) \geq B_0$ for all $(x, y) \in \{|x| \geq R_0\}$, $R_0 > 0$. Then for any magnetic potential $a \in C^2(\mathbb{R}^2, \mathbb{R}^2)$, polynomially bounded in the $y$-direction, supported on $\{|x| \leq R_0\}$, and for any closed interval $I \subset \mathbb{R}, B_0[$, we have $\sigma_e^{(\text{I})}(H(\mathcal{A}_{\text{strip}} + a)) = 0$. 
Remark 2.9. (i) If we perturb the operator $H(A_{\text{strip}})$ by a magnetic field supported on a strip $S$, then Proposition 4.2 below implies that there exists a potential $A$, which generates this magnetic field, and vanishes outside $S$, so that the hypotheses of Corollary 2.8 are satisfied.

(ii) The magnetic potentials of Corollary 2.8 can be produced by the superposition of two Iwatsuka-type potentials $A_{\text{lw}}^{(L)}, A_{\text{lw}}^{(R)}$, generating respectively a decreasing magnetic field $B^{(L)} = B^{(L)}(x)$ with upper limit $B^{(L)}_+ \geq B_0$ at $-\infty$, and an increasing magnetic field $B^{(R)} = B^{(R)}(x)$ with upper limit $B^{(R)}_+ \geq B_0$ at $+\infty$. Particles are then trapped in a magnetic strip created by these two magnetic barriers and thus can only travel along the axis of the strip. Corollary 2.8 asserts that no net current can flow in such a strip irrespective of the potential inside the strip. The situation is very different from the case of asymptotic values of $B$ of opposite sign described by Corollary 2.4, where the particle is constrained to a strip as well, but where a net current does exist, and we could talk about a quantum wave guide.

Our second theorem provides a sum rule which is similar to the one derived in [8]. We use the convenient notation

$$H(A^{(1)}, A^{(2)}) := H(A^{(0)} + A^{(1)} + A^{(2)}),$$

where $A^{(1)}$, resp., $A^{(2)}$, is supported in the half-plane $x < R_1$, resp., $x > R_2$, $R_1 < 0 < R_2$, and $A^{(0)} := \left(-\frac{\alpha}{2} \frac{x^2}{2} \right) B_-$ with $B_- > 0$ corresponds to a reference Landau potential. In particular, $H(0,0)$ is the Landau Hamiltonian with constant magnetic field $B_-$.  

**Theorem 2.10.** Let $I$ be a closed interval such that $I \cap \sigma(H(0,0)) = \emptyset$, $a_\alpha, a_\beta \in C^1(\mathbb{R}^2; \mathbb{R}^2)$ polynomially bounded. Suppose that $a_\alpha$ (resp., $a_\beta$), is supported in the half-plane $x < R_1$ (resp., $x > R_2$). Set

$$\mathcal{K}(a_\alpha, a_\beta) := g'(H(a_\alpha, a_\beta))i[H(a_\alpha, a_\beta), \chi] - g'(H(a_\alpha, 0))i[H(a_\alpha, 0), \chi]$$

$$- g'(H(0, a_\beta))i[H(0, a_\beta), \chi].$$

Then $\mathcal{K}(a_\alpha, a_\beta)$ is trace class. Moreover, if two out of the three terms of the r.h.s. of (2.14) are trace class, then $\text{tr}\mathcal{K}(a_\alpha, a_\beta) = 0$; in particular,

$$\sigma_e^{(I)}(H(a_\alpha, a_\beta)) = \sigma_e^{(I)}(H(a_\alpha, 0)) + \sigma_e^{(I)}(H(0, a_\beta)).$$

Moreover, let $A_{\text{lw}}^{(L)}, A_{\text{lw}}^{(R)}$ be left and right Iwatsuka-type potentials generating respectively a decreasing magnetic field $B^{(L)}(x)$ such that $B^{(L)}_+ \geq B_-$ for $x < R_1$, and an increasing magnetic field $B^{(R)}(x)$, such that $B^{(R)}_+ \geq B_-$ for $x > R_2$. Then the same result holds for

$$\mathcal{K}'(a_\alpha, a_\beta) := g'(H(a_\alpha, a_\beta))i[H(a_\alpha, a_\beta), \chi] - g'(H(a_\alpha, A_{\text{lw}}^{(R)}))i[H(a_\alpha, A_{\text{lw}}^{(R)}), \chi]$$

$$- g'(H(A_{\text{lw}}^{(L)}, a_\beta))i[H(A_{\text{lw}}^{(L)}, a_\beta), \chi] + g'(H(A_{\text{lw}}^{(L)}, A_{\text{lw}}^{(R)}))i[H(A_{\text{lw}}^{(L)}, A_{\text{lw}}^{(R)}), \chi].$$
In particular, if \( I \subset ]-\infty, \inf(B^{(L)}_+, B^{(R)}_+)\) and if two out of the first three terms on the r.h.s. of (2.16) are trace class, then

\[
\sigma_e^{(I)}(H(a_\alpha, a_\beta)) = \sigma_e^{(I)}(H(a_\alpha, A_{Iw}^{(R)})) + \sigma_e^{(I)}(H(A_{Iw}^{(L)}, a_\beta)). \tag{2.17}
\]

As a consequence of Corollary 2.3 and Theorem 2.10, we obtain a quantization of the edge conductance for magnetic perturbations of the Iwatsuka Hamiltonian, which in its turn implies the existence of edge states flowing in the \( y \) direction.

**Corollary 2.11.** Assume \( A_{Iw} \) generates a \((B_-, B_+)\)-magnetic field with \( B_+ \geq 3B_- > 0 \). Let \( a \in C^2(\{x \leq R_1\} \times \mathbb{R}) \) for some \( R_1 < \infty \) be so that \( ||a||_\infty \leq K_1\sqrt{B_-} \) and \( ||\text{div}a||_\infty \leq K_2B_- \), then there exists \( 0 < K_0 < \infty \) such that

\[
\sigma_e^{(I)}(H(a, A_{Iw})) = n, \quad n \in \mathbb{N},
\]

for \( B_- \) large enough and an interval \( I \subset ]-\infty, B_+\] satisfying

\[
I \subset (2n-1)B_+-K_0d_n(a, B_-), (2n+1)B_- - K_0d_n(a, B_-), \tag{2.18}
\]

if \( n \in \mathbb{N}^* := \{1, 2, \ldots\} \), or

\[
I \subset ]-\infty, B_- - K_0d_n(a, B_-)[, \tag{2.19}
\]

if \( n = 0 \), where \( d_n(a, B_-) = \max\{||\text{div}a||_\infty, ||a||_\infty\sqrt{(n+1)B_-}\} \).

If the interval \( I \) does not lie in a gap of the perturbed Hamiltonian anymore, we have to introduce a regularization of the edge conductance (see Sect. 6).

As a remark we note that similar results can be obtained for Hamiltonians mixing the point of view of [8] with purely electric potentials (wall and perturbation), and the one of this work that is purely magnetic potentials (wall and perturbation). We can indeed perturb an Iwatsuka Hamiltonian by an electric potential, or perturb a Hamiltonian with an electric confining potential by a magnetic potential. Proofs are then similar to those of [8] and those of the present article, the most technical case being the purely magnetic model.

### 3. Spectral Properties of the Generalized Iwatsuka Hamiltonians

Denote by \( \mathcal{F} \) the partial Fourier transform with respect to \( y \), i.e.

\[
(\mathcal{F}u)(x, k) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-iky} u(x, y) dy, \quad u \in L^2(\mathbb{R}^2). \tag{3.1}
\]

Then the generalized Iwatsuka Hamiltonian is unitarily equivalent to a direct integral of operators with discrete spectrum, i.e.

\[
\mathcal{F}H(A_{GIw}^{(B_-, B_+)})\mathcal{F}^* = \int_{\mathbb{R}} \oplus h(k) dk, \tag{3.2}
\]
where
\[ h(k) := -\frac{d^2}{dx^2} + (k - \beta(x))^2, \quad k \in \mathbb{R}. \quad (3.3) \]

For each \( k \in \mathbb{R} \) the spectrum of \( h(k) \) is discrete and simple. Let \( \{E_j(k)\}_{j \in \mathbb{N}} \) be the increasing sequence of the eigenvalues of the operator \( h(k) \), \( k \in \mathbb{R} \). By the Kato perturbation theory, \( E_j \) are real analytic functions. Evidently,
\[ \sigma(H(A_{GW}^{(A_{GW}^{(A_{GW}(B_-,B_+)})})) = \bigcup_{j=1}^{\infty} E_j(\mathbb{R}), \]

where \( \sigma(H(A_{GW}^{(A_{GW}^{(A_{GW}(B_-,B_+)})})) \) denotes the spectrum of \( H(A_{GW}^{(A_{GW}^{(A_{GW}(B_-,B_+)})}) \).

In the next proposition we summarize for further references several spectral properties of the Iwatsuka Hamiltonian.

**Proposition 3.1.** [33, Lemmas 2,3, 4.1] Pick \( 0 < B_- < B_+ \). Let \( A_{l_w}^{(A_{l_w}^{(A_{l_w}(B_-,B_+)})} \) be a \( (B_-,B_+) \) Iwatsuka potential, and \( \{E_j(k)\}_{j=1}^{\infty} \) be the eigenvalues defined above. Then we have
\[ (2j - 1)B_- \leq E_j(k) \leq (2j - 1)B_+, \quad j \in \mathbb{N}^*, \quad k \in \mathbb{R}, \quad (3.4) \]
and
\[ \lim_{k \to \pm \infty} E_j(k) = (2j - 1)B_\pm, \quad j \in \mathbb{N}^*. \quad (3.5) \]

As a consequence,
\[ \sigma(H(A_{l_w}^{(A_{l_w}^{(A_{l_w}(B_-,B_+)})})) = \bigcup_{j=1}^{\infty} [(2j - 1)B_-, (2j - 1)B_+]. \quad (3.6) \]

In particular, if \( B_+ \geq 3B_- \), then \( \sigma(H(A_{l_w}^{(A_{l_w}^{(A_{l_w}(B_-,B_+)})})) = [B_-, \infty[. \]

Assume now that \( B_- < 0 \) and \( B_+ > 0 \). Let \( \{\mu_j\}_{j=1}^{\infty} \) be the non-decreasing sequence of the eigenvalues of the operator
\[ \left(-\frac{d^2}{dx^2} + B_-^2 x^2\right) \oplus \left(-\frac{d^2}{dx^2} + B_+^2 x^2\right), \]
where \( -\frac{d^2}{dx^2} + B_-^2 x^2 \) are harmonic oscillators, self-adjoint in \( L^2(\mathbb{R}) \) and essentially self-adjoint on \( C_0^{\infty}(\mathbb{R}) \).

**Proposition 3.2.** Let \( B_- < 0 \) and \( B_+ > 0 \).
(i) For each \( j \in \mathbb{N}^* \) we have
\[ \lim_{k \to -\infty} E_j(k) = +\infty. \quad (3.7) \]
(ii) Assume moreover \( \lim_{x \to \pm \infty} B'(x) = 0 \). Then for each \( j \in \mathbb{N}^* \) we have
\[ \lim_{k \to -\infty} E_j(k) = \mu_j. \quad (3.8) \]
Proof. Relation (3.7) follows easily from the mini–max principle (see also the proof of [12, Theorem 6.6]).

The argument leading to (3.8) goes along the general lines of the proof of [12, Theorem 11.1] (see also the proof of [45, Proposition 3.6]). □

Now we are in position to prove Theorem 2.2. Arguing as in the proof of [8, Proposition 1],

we have

\[-2\pi \text{tr}(g'(H(A_{G\text{Iw}}^{B_-B_+})) i[H(A_{G\text{Iw}}^{B_-B_+}), \chi])\]

\[= -\sum_{j \in \mathbb{N}} \int_{\mathbb{R}} g'(E_j(k)) E'_j(k) dk \tag{3.9}\]

\[= \sum_{j \in \mathbb{N}} g(E_j(-\infty)) - g(E_j(+\infty)). \tag{3.10}\]

The result then follows from the spectral properties of generalized Iwatsuka Hamiltonians, namely, from (3.5) of Proposition 3.1 if \(B_-B_+ > 0\), and from Proposition 3.2 if \(B_-B_+ < 0\).

4. Perturbation by a Magnetic Potential Supported on a Strip

4.1. More on Magnetic Fields and Magnetic Potentials

This subsection contains well-known facts about the possibility to construct magnetic potentials \(A : \mathbb{R}^2 \rightarrow \mathbb{R}^2\) with prescribed properties, which generate given magnetic fields \(B : \mathbb{R}^2 \rightarrow \mathbb{R}\). In the first proposition we define a magnetic potential \(A\) in the so-called Poincaré gauge.

**Proposition 4.1.** [42, Eq. (8.154)] Let \(B \in C^1(\mathbb{R}^2; \mathbb{R})\). Then the potential

\[A = (A_1, A_2) = \left( -y \int_0^1 sB(sx, sy) ds, x \int_0^1 sB(sx, sy) ds \right), \quad (x, y) \in \mathbb{R}^2, \tag{4.1}\]

generates the magnetic field \(B\).

**Proposition 4.2.** Let \(B \in C^k(\mathbb{R}^2; \mathbb{R}), k \in \mathbb{N}^*,\) satisfy \(B(\mathbf{x}) = 0\) for \(\mathbf{x} = (x, y) \in \mathbb{R}^2\) with \(|x| \geq R_0, R_0 > 0\). Then there exists a magnetic potential \(A \in C^k(\mathbb{R}^2; \mathbb{R}^2)\) which generates \(B\), and vanishes identically on \(|(x, y) \in \mathbb{R}^2 | |x| \geq R_0\)\).

**Proof.** Pick any magnetic potential \(A \in C^k(\mathbb{R}^2; \mathbb{R}^2)\) which generates \(B\) (say, the potential appearing in (4.1)). Set

\[S_- := \{(x, y) \in \mathbb{R}^2 | x < -R_0\}, \quad S_+ := \{(x, y) \in \mathbb{R}^2 | x > R_0\}.\]

Since \(S_\pm\) are simply connected domains, and \(B\) identically vanishes on them, there exist functions \(F_\pm \in C^{k+1}(\mathbb{S}_\pm; \mathbb{R})\) such that

\[\nabla F_- = A \quad \text{on} \quad S_-, \quad \nabla F_+ = A \quad \text{on} \quad S_+.\]

1 Note that in Eqs. (3.5)–(3.6) of [8] there is a missing factor \(-\frac{1}{2\pi}\) at the r.h.s. of (3.6).
Then there exists an extension $F \in C^{k+1}(\mathbb{R}^2; \mathbb{R})$ such that
$$\mathcal{F} = F_- \text{ on } S_-, \quad \mathcal{F} = F_+ \text{ on } S_+.$$ 
On $\mathbb{R}^2$ define $\mathcal{A} := A - \nabla \mathcal{F}$. Evidently, the magnetic potential $\mathcal{A} \in C^k(\mathbb{R}^2; \mathbb{R}^2)$ generates $B$, and $\mathcal{A}(x) = 0$ for $x \in S_+ \cup S_-$. $\square$

4.2. Proof of Theorem 2.6 and Corollary 2.8

Lemma 4.3. Let $A \in C^2(\mathbb{R}^2; \mathbb{R}^2)$. Assume that $a \in C^1(\mathbb{R}^2; \mathbb{R}^2)$ is supported on the strip $[-x_0, x_0] \times \mathbb{R}$ and admits the bound $|a(x, y)| \leq C_a(y)^k$, for some $k \geq 0$ and $C_a < \infty$. Set
$$F(x, y) = -\int_0^u a_2(x, s) \, ds, \quad (x, y) \in \mathbb{R}^2. \quad (4.2)$$

Then we have
$$[H(A + a + \nabla F), \chi] = [H(A), \chi]. \quad (4.3)$$

Proof. Note that if $\tilde{a} = (\tilde{a}_1, \tilde{a}_2) \in C^1(\mathbb{R}^2, \mathbb{R}^2)$, we have
$$H(A + \tilde{a}) - H(A) = 2\tilde{a} \cdot (i \nabla + A) + i \text{div} \tilde{a} + |\tilde{a}|^2. \quad (4.4)$$

Since $\partial x \chi = 0$, a direct computation shows that
$$[H(A + \tilde{a}), \chi] - [H(A), \chi] = 2i\tilde{a} \cdot \nabla \chi = 2i\tilde{a}_2\chi'. \quad (4.5)$$

Therefore,
$$[H(A + \tilde{a}), \chi] - [H(A), \chi] = 0 \iff \tilde{a}_2\chi' = 0. \quad (4.6)$$

Applying (4.5)–(4.6) with $\tilde{a} = a + \nabla F$, and taking into account that in this case $\tilde{a}_2 = a_2 + \partial_y F = 0$ by (4.2), we obtain (4.3). $\square$

Remark 4.4. Note that $a + \nabla F$ is supported on $[-x_0, x_0] \times \mathbb{R}$, and $|a + \nabla F| \leq C_a(y)^{k+1}$, $(x, y) \in \mathbb{R}^2$.

Proposition 4.5. Let $A, a$ be as in Lemma 4.3. Then
$$(g'(H(A + a)) - g'(H(A))) [H(A), \chi] \in \mathcal{T}_1.$$ 

Moreover if $[H(A + a), \chi] = [H(A), \chi]$, then
$$\text{tr} (g'(H(A + a)) - g'(H(A))) [H(A), \chi] = 0. \quad (4.7)$$

Assuming for the moment the validity of Proposition 4.5, we provide the proof of Theorem 2.6.

Proof of Theorem 2.6. The fact that the operator defined in (2.9) is trace class follows from the decomposition (2.12) of Remark 2.7, the fact that $2ig'(H(A + a))a_2\chi' \in \mathcal{T}_1$, and the first part of Proposition 4.5.

Further, in order to prove (2.10) we introduce a gauge transform $\exp(iF)$, where $F$ is given by Lemma 4.3. By Proposition 4.5 applied to the perturbation $a + \nabla F$, the operator
$$g'(H(A + a + \nabla F))[H(A), \chi] = g'(H(A + a + \nabla F))[H(A + a + \nabla F), \chi]$$
is trace-class since $g'(H(A))[H(A), \chi]$ is by the hypotheses of Theorem 2.6.
Now, since $F$ and $\chi$ commute, we have
\begin{align}
g'(H(A + a))i[H(A + a), \chi] &= e^{-iF}(e^{iF}g'(H(A + a))e^{-iF}e^{iF}i[H(A + a), \chi]e^{-iF})e^{iF}, \\
&= e^{-iF}(g'(H(A + a + \nabla F))i[H(A + a + \nabla F), \chi])e^{iF},
\end{align}
which is trace class, so that $g'(H(A + a))i[H(A + a), \chi] \in T_1$. It follows, using the cyclicity of the trace, that
\begin{align}
tr g'(H(A + a))i[H(A + a), \chi] &= tr g'(H(A))i[H(A), \chi] \\
&= tr (g'(H(A + a + \nabla F)) - g'(H(A)))i[H(A), \chi] \\
&= 0,
\end{align}
by Lemma 4.3 and Proposition 4.5.

The rest of the section is devoted to the proofs of Proposition 4.5 and Corollary 2.8.

**Proof of Proposition 4.5.** Let $\varphi_r : \mathbb{R}^2 \to \mathbb{R}^2$ be the smooth cut-off function satisfying $\varphi_r(x, y) = \varphi_r(y)$, $\varphi_r = 1$ for $|y| \leq r - 1$ and $\varphi_r = 0$ for $|y| \geq r$, $r > 1$. We decompose $a = a\varphi_r + a(1 - \varphi_r) := a_{\leq r} + a_{> r}$. Then we have
\begin{align}
tr (g'(H(A + a)) - g'(H(A))) [H(A), \chi] &= tr (g'(H(A + a)) - g'(H(A + a_{\leq r}))) [H(A), \chi] \\
&\quad + tr (g'(H(A + a_{< r})) - g'(H(A))) [H(A), \chi].
\end{align}
First, we will show that
\begin{equation}
(g'(H(A + a_{\leq r})) - g'(H(A))) [H(A), \chi] \in T_1,
\end{equation}
and
\begin{equation}
tr (g'(H(A + a_{< r})) - g'(H(A))) [H(A), \chi] = 0.
\end{equation}
After that we will show that for some $C < \infty$ and $p \geq 1$,
\begin{equation}
\|(g'(H(A + a)) - g'(H(A + a_{\leq r}))) [H(A), \chi]\|_1 \leq Cr^{-p}
\end{equation}
for $r$ large enough.

We are thus left with the proof of (4.15)–(4.17). Let us first prove (4.15). By the Helffer-Sjöstrand functional calculus (see e.g. [32, Lemma B.2]), applied to function $G(x) := -\int_x^\infty g(s)ds$, we have
\begin{equation}
g'(H(A + a_{\leq r})) - g'(H(A)) = -\frac{2}{\pi} \int_{\mathbb{R}^2} \overline{\partial G(u + iv)}(R_1^3 - R_2^3)du dv,
\end{equation}
where $R_j = (H_j - z)^{-1}$, $j = 1, 2$, $z = u + iv$, and $H_1 := H(A)$, $H_2 := H(A + a_{\leq r})$. Put
\begin{equation}W := H_2 - H_1 = 2a_{\leq r} \cdot (i\nabla + A) + i \text{div} a_{\leq r} + |a_{\leq r}|^2.
\end{equation}
Note that due to the fact that $\mathcal{W}$ is a first-order differential operator, we need one extra power of the resolvents in comparison to [8]. Thus, we have to analyze the operator $(R^3_1 - R^3_2)[H(A), \chi]$. It is easy to check that
\begin{align*}
2(R^3_1 - R^3_2) &= R^3_1 \mathcal{W} R_2 R_1 + R_1 \mathcal{W} R^2_2 R_1 + R_1 \mathcal{W} R_2 R^2_1 \\
&\quad + R^2_1 R_1 \mathcal{W} R_2 + R_2 R^2_1 \mathcal{W} R_2 + R_2 R_1 \mathcal{W} R^2_2,
\end{align*}
which could be formally guessed by computing the derivative $\partial_z (R^3_1 - R^3_2) = \partial_z (R_1 \mathcal{W} R_2 R_1 + R_2 R_1 \mathcal{W} R_2)$. We have
\begin{align*}
2(R^3_1 - R^3_2)[H_1, \chi] &= (R^3_1 \mathcal{W} R_2 R_1 + R_1 \mathcal{W} R^2_2 R_1 + R_1 \mathcal{W} R^2_2 R_1) [H_1, \chi] \\
&\quad + (R^2_1 R_1 \mathcal{W} R_2 + R_2 R^2_1 \mathcal{W} R_2 + R_2 R_1 \mathcal{W} R^2_2) [H_2, \chi] \\
&\quad - (R^2_1 R_1 \mathcal{W} R_2 + R_2 R^2_1 \mathcal{W} R_2 + R_2 R_1 \mathcal{W} R^2_2) [\mathcal{W}, \chi].
\end{align*}
We recall that $a_{s, r}$ is compactly supported. Applying Corollary 7.6 and Lemma 7.1 (ii), and bearing in mind that $R_j [H_j, \chi] = \mathcal{O}(|v|^{-1})$, $j = 1, 2$, $v = \mathbb{S} z$, we find that the trace-class norms of the terms on the r.h.s. of (4.22)–(4.24) are of order $\mathcal{O}(|v|^{-n})$ with an appropriate $n \in \mathbb{N}^*$, which combined with (4.18) implies (4.15).

Next, we prove (4.16). Using the identities $R_j [H_j, \chi] R_j = [\chi, R_j], j = 1, 2$, undoing the commutators, and introducing obvious notations, we get
\begin{align*}
\text{tr} (R^3_1 \mathcal{W} R_2 R_1)[H_1, \chi] &= \text{tr} [\chi, R_1] R_1 \mathcal{W} R_2 =: I_1 + I_2, \\
\text{tr} (R_1 \mathcal{W} R^2_2 R_1)[H_1, \chi] &= \text{tr} [\chi, R_1] \mathcal{W} R^2_2 =: II_1 + II_2, \\
\text{tr} (R_1 \mathcal{W} R_2 R^2_1)[H_1, \chi] &= \text{tr} [\chi, R_1] \mathcal{W} R^2_2 R_1 =: III_1 + III_2, \\
\text{tr} (R^2_1 R_1 \mathcal{W} R_2)[H_2, \chi] &= \text{tr} [\chi, R_2] R_2 R_1 \mathcal{W} =: IV_1 + IV_2, \\
\text{tr} (R_2 R^2_1 \mathcal{W} R_2)[H_2, \chi] &= \text{tr} [\chi, R_2] R^2_1 \mathcal{W} =: V_1 + V_2, \\
\text{tr} (R_2 R_1 \mathcal{W} R^2_2)[H_2, \chi] &= \text{tr} [\chi, R_2] R_1 \mathcal{W} R_2 =: VI_1 + VI_2.
\end{align*}
Let $0 \leq \zeta_j \in C^0(\mathbb{R}^2), j = 0, 1$, satisfy $\zeta_0 a_{s, r} = a_{s, r}$ and $\zeta_1 \zeta_0 = \zeta_0$ on $\mathbb{R}^2$. Rearranging the terms (4.25)–(4.30) of $2 \text{tr} (R^3_1 - R^3_2)[H_1, \chi]$ and applying Lemmas 7.1 and 7.2, we get
\begin{align*}
I_1 + V_2 &= \text{tr} [\chi R^2_1 \mathcal{W}, R_2] - \text{tr} [\chi R^2_1 \mathcal{W}, \zeta_0 R_2] = 0, \\
II_1 + VI_2 &= \text{tr} [\chi R_1 \mathcal{W} R_2, R_2] \\
&= \text{tr} ([\chi R_1 \mathcal{W} R_2, \zeta_0 R_2] - [\chi R_1 \mathcal{W} R_2 H_2, \zeta_0] R_2, R_2]) = 0, \\
III_1 + I_2 &= \text{tr} [\chi R_1 \mathcal{W} R_2, R_1] \\
&= \text{tr} ([\chi R_1 \mathcal{W} R_2, \zeta_0 R_1] - [\chi R_1 \mathcal{W} R_2 H_2, \zeta_0] R_2, R_1]) = 0, \\
IV_1 + II_2 &= \text{tr} [\chi R^2_2, R_1 \mathcal{W} \chi] \\
&= \text{tr} ([\chi R_2 \zeta_0 R_2, R_1 \mathcal{W} \chi] - [\zeta_0 R_2 H_2, \zeta_1] \chi R^2_2, R_1 \mathcal{W} \chi]) = 0, \\
V_1 + III_2 &= \text{tr} [\chi R_2 R_1, R_1 \mathcal{W} \chi].
\end{align*}
\[= \text{tr} ([\chi R_2 \zeta_0 R_1, R_1 \mathcal{W} \chi]) \]
\[- [\zeta_0 R_2 [H_2, \zeta_1] \chi R_2 R_1, R_1 \mathcal{W} \chi]) = 0, \quad (4.39)\]
\[V I_1 + IV_2 = \text{tr}[\chi R_2 R_1 \mathcal{W}, R_2] = \text{tr}[\chi R_2 R_1 \mathcal{W}, \zeta_0 R_2] = 0. \quad (4.40)\]

Therefore,
\[(4.22) + (4.23) = 0. \quad (4.41)\]

To get (4.16) it remains to see that (4.24) = 0, but this is immediate because by assumption \([H(A + a) - H(A), \chi] = 0\), which readily implies that
\[\mathcal{W}, \chi] = -[H(A) - H(A + a_{<r}), \chi] \quad (4.42)\]
\[= -[H(A + a) - H(A + a_{<r}), \chi] \quad (4.43)\]
\[= -[2a_{\geq r} \cdot (i\nabla + A), \chi] = 0, \quad (4.44)\]

since the supports of \(\chi'\) and \(a_{\geq r}\) are disjoint.

Finally, we prove (4.17). Due to the Helffer-Sjöstrand formula (4.18), we have to control
\[\| (R_3^2 - R_2^2)[H(A), \chi] \|_1, \quad (4.49)\]
with \(R_3(z) := (H(A + a) - z)^{-1}\). The resolvent identity yields
\[R_3^2 - R_2^2 = R_2^2 \mathcal{W}_r R_2 + R_3^2 \mathcal{W}_r R_2^2 + R_3 \mathcal{W}_r R_2^3, \quad (4.45)\]

where
\[
\mathcal{W}_r := H(A + a_{<r}) - H(A + a) \\
= -(2a_{\geq r} \cdot (i\nabla + A) + idiv(A_{\geq r}) + |a|^2 - |a_{<r}|^2).
\]

Note that \(\mathcal{W}_r \equiv 0\) whenever \(|y| \leq r - 1\), so that we write
\[
\mathcal{W}_r = \sum_{(x,y) \in \mathbb{Z}^2 \cap [-x_0 - 1, x_0 + 1] \times [-r + 2, r - 2]^c} 1_{(x_1, y_1)} \mathcal{W}_r \quad (4.46)\]
\[
\mathcal{W}_r = \sum_{(x,y) \in \mathbb{Z}^2 \cap [-x_0 - 1, x_0 + 1] \times [-r + 2, r - 2]^c} \mathcal{W}_r 1_{(x_1, y_1)} \quad (4.47)\]

where \(1_{(x,y)}\) stands for a smooth characteristic function of the cube of side length one and centered at \((x, y) \in \mathbb{R}^2\) such that \(\sum_{(x,y) \in \mathbb{Z}^2} 1_{(x,y)} = 1\). Similarly,
\[\|H(A), \chi]\| = \sum_{x \in \mathbb{Z}} |H(A), \chi| 1_{(x_2, 0)}. \quad (4.48)\]

For the moment fix \(x_1, y_1, x_2 \in \mathbb{Z}\), and introduce the short-hand notations \(\zeta_0 := 1_{(x_1, y_1)}\), and \(\zeta := 1_{(x_2,0)}\). Let \(\zeta_j\) be non-negative smooth compactly supported functions such that \(\zeta_j \zeta_{j-1} = \zeta_{j-1}\) on \(\mathbb{R}^2\), \(j = 1, 2, 3\). Then we have
\[\| R_3^2 1_{(x_1, y_1)} \mathcal{W}_r R_2 [H(A), \chi] 1_{(x_2, 0)} \| \leq \| R_3^2 \zeta_0 \mathcal{W}_r \|_1 \| \zeta_1 R_2 [H(A), \chi] \zeta \|. \quad (4.49)\]

Next,
\[\| R_3^2 1_{(x_1, y_1)} \mathcal{W}_r R_2 [H(A), \chi] 1_{(x_2, 0)} \| \leq \| R_3^2 \zeta_0 \mathcal{W}_r R_2 \|_1 (\| \zeta_1 R_2 [H(A), \chi] \zeta \| + \| \zeta_2 [H(A), \zeta_1] R_2^2 [H(A), \chi] \zeta \|), \quad (4.50)\]
and
\[
\|R_3 1_{(x_1, y_1)} \mathcal{W}_r R^2_2 [H(A), \chi] 1_{(x_2, 0)}\|_1 \leq \|R_3 \zeta_0 \mathcal{W}_r R_2\|_1 (\|\zeta_1 R_2 [H(A), \chi]\| + \|\zeta_2 [H(A), \zeta_1] R^2_2 [H(A), \chi]\|) + \|\zeta_3 [H(A), \zeta_2] R^3_2 [H(A), \chi]\|. \tag{4.53}
\]

Assume now that $z$ is in a compact subset of $\mathbb{C}$, and $\Im z \neq 0$. Applying Proposition 7.4 and estimate (7.20) below, we find that there exists a constant $c_1$ independent of $x_1, y_1, x_2 \in \mathbb{Z}$, and $z$, such that the trace-class norms
\[
\|R^3_3 \zeta_0 \mathcal{W}_r\|_1, \quad \|R^2_3 \zeta_0 \mathcal{W}_r R_2\|_1, \quad \|R_3 \zeta_0 \mathcal{W}_r R_2 [H(A), \zeta_1] R_2\|_1,
\]
appearing on the r.h.s. of (4.49)–(4.54) are upper bounded by $c_1 |\Im z|^{-3}$. On the other hand, making use of estimates of Combes-Thomas type (see [11, 24]), we find that there exists a constant $c_2 > 0$ independent of $x_1, y_1, x_2 \in \mathbb{Z}$, and $z$ such that the operator norms
\[
\|\zeta_1 R_2 [H(A), \chi]\|, \quad \|\zeta_2 [H(A), \zeta_1] R^2_2 [H(A), \chi]\|, \quad \|\zeta_3 [H(A), \zeta_2] R^3_2 [H(A), \chi]\|,
\]
appearing on the r.h.s. of (4.49)–(4.54) are upper bounded by
\[
c_2 |\Im z|^{-1} \exp (-c_2 |\Im z| (|x_1 - x_2| + |y_1|)).
\]
Taking into account these estimates, bearing into mind the representations (4.45), (4.47), and (4.48), and arguing as in the proof of [8, Lemma 2], we easily obtain (4.17).

**Proof of Corollary 2.8.** We introduce a modified strip confining potential $\tilde{A}_{\text{strip}}$ generating a magnetic field $\tilde{B} \in C^2(\mathbb{R}^2; \mathbb{R})$ which satisfies $\tilde{B}(x, y) \geq B_0$ for all $(x, y) \in \mathbb{R}^2$ and $\tilde{B}(x) = B(x)$ on $\{|x| \geq R_0\}$. Since the operator $H(\tilde{A}_{\text{strip}}) - \tilde{B}$ is non-negative (see e.g. [18]), we have $\inf \sigma(H(\tilde{A}_{\text{strip}})) \geq B_0$. As a consequence, $\sigma^{(I)}_e(H(\tilde{A}_{\text{strip}})) = 0$. Since the magnetic field $B - \tilde{B}$ is supported on a strip, Proposition 4.2 implies the existence of a magnetic potential $A \in C^2(\mathbb{R}^2; \mathbb{R}^2)$ which generates the magnetic field $B - \tilde{B}$, and is supported on the strip $\{|x| \leq R_0\}$. Applying Theorem 2.6, we find that
\[
\sigma^{(I)}_e(H(\tilde{A}_{\text{strip}} + A)) = \sigma^{(I)}_e(H(\tilde{A}_{\text{strip}})) = 0.
\]
Since the potentials $\tilde{A}_{\text{strip}} + A$ and $A_{\text{strip}}$ generate the same magnetic field $B$, the operators $H(\tilde{A}_{\text{strip}} + A)$ and $H(A_{\text{strip}})$ are unitarily equivalent under an appropriate gauge transform. Therefore,
\[
\sigma^{(I)}_e(H(A_{\text{strip}})) = \sigma^{(I)}_e(H(\tilde{A}_{\text{strip}} + A)) = 0.
\]
Finally, applying Theorem 2.6 once more, we find that
\[
\sigma^{(I)}_e(H(A_{\text{strip}} + a)) = \sigma^{(I)}_e(H(A_{\text{strip}})) = 0.
\]
5. Sum Rule for Magnetic Perturbations

The aim of this section is to prove Theorem 2.10 and Corollary 2.11.

Proof of Theorem 2.10. It is enough to prove the first part of the statement (the one concerning $K(a_\alpha, a_\beta)$), for the second part will follow from the relation

$$K'(a_\alpha, a_\beta) = K(a_\alpha, a_\beta) - K(a_\alpha, \mathcal{A}_w^{(R)}) - K(\mathcal{A}_w^{(L)}, a_\beta) + K(\mathcal{A}_w^{(L)}, \mathcal{A}_w^{(R)}), \quad (5.1)$$

where we used that $\text{tr} g(H(\mathcal{A}_w^{(L)}, \mathcal{A}_w^{(R)}))[H(\mathcal{A}_w^{(L)}, \mathcal{A}_w^{(R)}), \chi] = 0$ by Corollary 2.8.

We set $K := K(a_\alpha, a_\beta)$ and $K_r := K(a_\alpha, a_\beta_r)$, where $a_\beta_r := a_\beta \varphi_r$ and $\varphi_r$ is a smooth characteristic function of the region $x \leq r$. By Theorem 2.6, we have $K_r \in T_1$ and $\text{tr} K_r = 0$. It is thus enough to prove polynomial decay in $r$ of $\|K - K_r\|_1$. We set

$$Q_{(a,b)}^\chi := i[H(a,b), \chi]. \quad (5.2)$$

Note that although $H(a,b)$ is non-linear in $a, b$, the commutator $Q_{(a,b)}^\chi$ is linear, that is for arbitrary $a, b, a', b'$ in $C^1$,

$$Q_{(a,b)}^\chi - Q_{(a',b')}^\chi = Q_{(a-a',b-b')}^\chi. \quad (5.3)$$

We get

$$K - K_r = g(H(a_\alpha, a_\beta))Q_{(a_\alpha, a_\beta)}^\chi - g(H(a_\alpha, a_\beta_r))Q_{(a_\alpha, a_\beta_r)}^\chi - g'(H(0, a_\beta))Q_{(0, a_\beta)}^\chi + g'(H(0, a_\beta_r))Q_{(0, a_\beta_r)}^\chi \quad (5.4)$$

$$= (g(H(a_\alpha, a_\beta)) - g(H(a_\alpha, a_\beta_r)))Q_{(a_\alpha, a_\beta_r)}^\chi \quad (5.5)$$

$$= (g'(H(0, a_\beta)) - g'(H(0, a_\beta_r)))Q_{(0, a_\beta_r)}^\chi \quad (5.6)$$

$$+ (g'(H(a_\alpha, a_\beta)) - g'(H(0, a_\beta)))Q_{(a_\alpha, a_\beta_r)}^\chi \quad (5.7)$$

$$= (g'(H(0, a_\beta)) - g'(H(0, a_\beta_r)))Q_{(0, a_\beta_r)}^\chi \quad (5.8)$$

where we used (5.3). The term in (5.9) is evaluated as in [8], while (5.10) and (5.11) are new terms coming from the magnetic nature of the perturbation.

We first show that (5.10) and (5.11) satisfy a bound of the type (4.17). Notice, from the very definition of commutators $Q_{(a,b)}^\chi$, that if $a, b$ given are supported on the closed region $\Gamma$, then so is the operator $Q_{(a,b)}^\chi$, in the sense that $(1 - \chi_{\Gamma})Q_{(a,b)}^\chi = (Q_{(a,b)}^\chi)|_{\mathbb{R}^2 \backslash \Gamma} \equiv 0$, $\chi_{\Gamma}$ being the characteristic function of $\Gamma$.

Let us first consider (5.10). We use the Helffer-Sjöstrand formula (4.18), and then proceed as in the proof of Proposition 4.5, Eqs. (4.45) and below,
but this time with
\[ W_r := H(0, a_{\beta_r}) - H(0, a_\beta) \]  
\[ = \sum_{(x_1, y_1) \in \mathbb{Z}^2 \cap \{ x_1 \geq r-1 \}} 1(x_1, y_1) W_r \]  
\[ = \sum_{(x_1, y_1) \in \mathbb{Z}^2 \cap \{ x_1 \geq r-1 \}} W_r 1(x_1, y_1), \]  
and
\[ Q_{(a_\alpha, 0)}^x = \sum_{x_2 \in \mathbb{Z} \setminus \{ x_2 \leq 1 \}} Q_{(a_\alpha, 0)}^x 1(x_2, 0), \]

instead of (4.47) and (4.48). The proof of (5.11) is similar but now we consider  
\[ W := H(\alpha_\alpha, a_\beta) - H(0, a_\beta) \]
which we decompose over boxes centered at points \((x_1, y_1) \in \mathbb{Z}^2 \cap \{ x_1 \leq 1 \}\), while we decompose \(Q_{(0, a_\beta - a_{\beta_r})}^x\) over boxes centered at points \((x_2, y_2) \in \mathbb{Z}^2 \cap \{ x_2 \geq r - 1, y_2 = 0 \}\).

We turn now to (5.9). Again, in order to estimate
\[ \left\| g'(H(\alpha_\alpha, a_\beta)) - g'(H(\alpha_\alpha, a_{\beta_r})) - g'(H(0, a_\beta)) + g'(H(0, a_{\beta_r})) \right\|_1, \]
we bound, with obvious notations for the resolvents, the norm
\[ \left\| \left\{ \left( R_{(a_\alpha, a_\beta)}^3 - R_{(a_\alpha, a_{\beta_r})}^3 \right) - \left( R_{(0, a_\beta)}^3 - R_{(0, a_{\beta_r})}^3 \right) \right\} Q_{(a_\alpha, a_{\beta_r})}^x \right\|_1. \]

To do so, we utilize the resolvent identity (4.45) together with
\[ H(\alpha_\alpha, a_\beta) - H(\alpha_\alpha, a_{\beta_r}) = H(0, a_\beta) - H(0, a_{\beta_r}) =: -W_r, \]
and get
\[ \left( R_{(a_\alpha, a_\beta)}^3 - R_{(a_\alpha, a_{\beta_r})}^3 \right) - \left( R_{(0, a_\beta)}^3 - R_{(0, a_{\beta_r})}^3 \right) \]
\[ = \left( R_{(a_\alpha, a_\beta)}^3 W_r R_{(a_\alpha, a_{\beta_r})} + R_{(a_\alpha, a_\beta)}^3 W_r R_{(a_\alpha, a_{\beta_r})} + R_{(a_\alpha, a_\beta)}^3 W_r R_{(a_\alpha, a_{\beta_r})} \right) \]
\[ = \left( R_{(a_\alpha, a_\beta)}^3 W_r R_{(a_\alpha, a_{\beta_r})} + R_{(a_\alpha, a_\beta)}^3 W_r R_{(a_\alpha, a_{\beta_r})} + R_{(a_\alpha, a_\beta)}^3 W_r R_{(a_\alpha, a_{\beta_r})} \right) \]
\[ = \left( R_{(a_\alpha, a_\beta)}^3 W_r R_{(a_\alpha, a_{\beta_r})} + R_{(a_\alpha, a_\beta)}^3 W_r R_{(a_\alpha, a_{\beta_r})} + R_{(a_\alpha, a_\beta)}^3 W_r R_{(a_\alpha, a_{\beta_r})} \right) \]
\[ + \left( R_{(a_\alpha, a_\beta)}^3 W_r R_{(a_\alpha, a_{\beta_r})} + R_{(a_\alpha, a_\beta)}^3 W_r R_{(a_\alpha, a_{\beta_r})} + R_{(a_\alpha, a_\beta)}^3 W_r R_{(a_\alpha, a_{\beta_r})} \right) \]
\[ + \left( R_{(a_\alpha, a_\beta)}^3 W_r R_{(a_\alpha, a_{\beta_r})} + R_{(a_\alpha, a_\beta)}^3 W_r R_{(a_\alpha, a_{\beta_r})} + R_{(a_\alpha, a_\beta)}^3 W_r R_{(a_\alpha, a_{\beta_r})} \right). \]
Next, with $W_{a\beta} = H(0, a\beta) - H(a\alpha, a\beta)$ and $W_{a\beta r} = H(0, a\beta r) - H(a\alpha, a\beta r)$, we rewrite (5.22) using the resolvent identity:

$$R^3_{(a_\alpha, a_\beta)} W_r R_{(a_\alpha, a_\beta r)} - R^3_{(0, a_\beta)} W_r R_{(0, a_\beta r)}$$ (5.25)

$$= R^3_{(a_\alpha, a_\beta)} W_r R_{(a_\alpha, a_\beta r)} W_{a\beta r} R_{(0, a_\beta r)}$$

$$+ (R^3_{(a_\alpha, a_\beta)} - R^3_{(0, a_\beta)}) W_r R_{(0, a_\beta r)}$$ (5.26)

$$= R^3_{(a_\alpha, a_\beta)} W_r R_{(a_\alpha, a_\beta r)} W_{a\beta r} R_{(0, a_\beta r)}$$

$$+ R^3_{(a_\alpha, a_\beta)} W_{a\beta} R_{(0, a_\beta)} W_r R_{(0, a_\beta r)}$$ (5.27)

$$+ R^2_{(a_\alpha, a_\beta)} W_{a\beta} R^2_{(0, a_\beta)} W_r R_{(0, a_\beta r)}$$

$$+ R_{(a_\alpha, a_\beta)} W_{a\beta} R^3_{(0, a_\beta)} W_r R_{(0, a_\beta r)}.$$ (5.28)

As previously, $W_r$ is decomposed over boxes centered at $(x_2, y_2) \in \mathbb{Z}^2$ such that $x_2 \geq r - 1$, while both $W_{a\beta}$ and $W_{a\beta r}$ are decomposed over integers $(x_2, y_3)$'s such that $x_3 \leq 4$. Proceeding as above yields the result. We then apply the same argument as for (5.23) and (5.24). □

Corollary 2.11 is a direct consequence of Theorem 2.10 and Lemma 5.1 below, for it is enough to prove that $I \cap \sigma(H(a, 0)) = \emptyset$, which readily implies that $\sigma_e^1(H(A_{1\omega}, a_\beta)) = 0$.

**Lemma 5.1.** Let $H(A^{(0)})$ be the Landau Hamiltonian with constant magnetic field $B_-$. Let $a \in C^1(\mathbb{R}^2)$ be such that $\|a\|_{\infty} \leq K_1 \sqrt{B_-}$ and $\|\text{div} a\|_{\infty} \leq K_2 B_-$. Then there exists a constant $0 < K_0 < \infty$ such that we have

$$I \cap \sigma(H(A_0 + a)) = \emptyset$$

for $B_-$ large enough and for any interval $I$ satisfying

$$I \subset [2n - 1)B_- + K_0 d_n(a, B_-), (2n + 1)B_- - K_0 d_n(a, B_-)],$$ (5.29)

if $n \in \mathbb{N}^*$, or

$$I \subset ] - \infty, B_- - K_0 d_n(a, B_-)[$$ (5.30)

if $n = 0$, where $d_n(a, B_-) = \max(\|\text{div} a\|_{\infty}, \|a\|_{\infty} \sqrt{(n + 1)B_-})$, $n \in \mathbb{N}$.

**Proof.** We denote by $R_a$ (resp. $R_0$), the resolvent of $H(A_0 + a)$ (resp. $H(A_0)$), and set $W_a = H(A_0 + a) - H(A_0)$. We start with the resolvent identity

$$R_0(E) = R_a(E)(\text{id} - W_a R_0(E))$$ (5.31)

with $E \in I \subset \mathbb{R} \setminus \sigma(H(A_0))$. To show that $E$ belongs to the resolvent set of $H(A^{(0)} + a)$, it is enough to show that $\|W_a R_0(E)\| < 1$ so that $\text{id} - W_a R_0(E)$ is invertible. We have

$$\|W_a R_0(E)\| \leq \|(\text{div} a + |a|^2) R_0(E)\| + \|a \cdot (i\nabla + A^{(0)}) R_0(E)\|$$ (5.32)

$$\leq (\|\text{div} a\|_{\infty} + \|a\|_{\infty}^2) \|R_0(E)\| + \|a\|_{\infty} \|(i\nabla + A^{(0)}) R_0(E)\|.$$ (5.33)
We set \(d = \text{dist}(E, \sigma(H(0, 0)))\). We have, noting that \(d \leq \max(|E|, B_-)\),

\[
\| (i \nabla + A^{(0)}) R_0(E) \|^2 = \| R_0(E) H(A_0) R_0(E) \| \\
\leq \frac{1}{d} + \frac{\max(|E|, 0)}{d^2} \leq \frac{2 \max(|E|, B_-)}{d^2},
\]
so that

\[
\| W_a R_0(E) \| \leq \left( \| \text{diva} \|_\infty + \| a \|_\infty^2 \right) \frac{1}{d} + \| a \|_\infty \frac{\sqrt{2 \max(|E|, B_-)}}{d}. \tag{5.35}
\]

Assuming that \(E\) belongs to the \(n\)th band, \(n \in \mathbb{N}\), and \(d\) satisfies both \(d > 8\|a\|_\infty \sqrt{(n+1)B_-}\) and \(d > \frac{1}{2}(\|\text{diva}\|_\infty + \|a\|_\infty^2)\), we find that \(\| W_a R_0(E) \| < 1\). \(\square\)

As a final remark, we take advantage of the second sum rule \((2.17)\) to sketch an alternative proof of the existence of quantized currents in magnetic guides created by magnetic barriers of opposite signs.

We first specify a given reference decreasing switch function \(\theta\) such that \(\theta(x) = 1\) (resp. \(\theta(x) = -1\)), for \(x < q_-\) (resp. \(x > q_+\)), with some real numbers \(q_- \leq q_+\). Set \(B(x) = B \theta(\sqrt{B}x)\) and denote by \(\tilde{\beta}(x) = B \int_0^x \theta(\sqrt{B}s)ds\) the corresponding generalized Iwatsuka potential (recall \((2.4)\)). Note that \(B(x) = B\) (resp. \(B(x) = -B\)), when \(x < B^{-1/2}q_-\) (resp. \(x > B^{-1/2}q_+\)).

If \(q_- = q_+ = 0\) then we get a sharp interface and the magnetic potential is just \((0, -B|E|)\).

**Lemma 5.2.** Let \(A_{GIw}^{(B)} = (0, -\tilde{\beta})\) be the \((B, -B)\)-generalized Iwatsuka potential described above. Then there exists a constant \(c_0 > 0\) such that for all \(B > 0\) we have

\[
\inf \sigma(H(A_{GIw}^{(B)})) \geq c_0 B. \tag{5.36}
\]

**Proof.** Performing the partial Fourier transform \((3.1)\), we end up with the analysis of \(h(k) = -d^2/dx^2 + (k - \tilde{\beta}(x))^2\). Using dilations \((U \psi)(x) = \psi(x \sqrt{B})\), we see that \(U^{-1} h(k) U = \tilde{h}(k, B)\) with

\[
\tilde{h}(k, B) := -B \frac{d^2}{dx^2} + (k - \tilde{\beta}(x/\sqrt{B}))^2 \tag{5.37}
\]

\[
= B \left( -\frac{d^2}{dx^2} + \left( \frac{k}{\sqrt{B}} - \sqrt{B} \int_0^{x/\sqrt{B}} \theta(s \sqrt{B}) ds \right)^2 \right) \tag{5.38}
\]

\[
= B \left( -\frac{d^2}{dx^2} + \left( \frac{k}{\sqrt{B}} - \int_0^{x} \theta(s) ds \right)^2 \right) \tag{5.39}
\]

\[
= B \tilde{h}(kB^{-1/2}, 1) \tag{5.40}
\]
As a consequence, by the virtue of the direct decomposition (3.2), we get
\[
\inf \sigma(H(A^{(B)}_{GIw})) = \inf \left( \inf \sigma(h(k)) \right) = B \inf \left( \inf \sigma(\tilde{h}(k,1)) \right) \quad (5.41)
\]
\[
= B \inf \sigma(H(A^{(1)}_{GIw})). \quad (5.42)
\]
\[\square\]

Let \( B \) be large enough so that \( I \subset ] - \infty, c_0 B], \) where \( c_0 \) comes from Lemma 5.2. In particular, \( B_-, B_+ < B. \) Let \( A^{(L)}_{Iw} \) (resp., \( A^{(R)}_{Iw} \)), be a \((B, B_+)]\) (resp. \((-B_-, -B))\), Iwatsuka potential. Then (2.17) yields
\[
\sigma_I(e(H(A^{(-B_-, B_+)}_{GIw}))) = \sigma_I(e(H(A^{(B, B_+)}_{Iw}))) - \sigma_I(e(H(A^{(-B, B)}_{GIw}))) \quad (5.43)
\]
\[
\sigma_I(e(H(A^{(-B_-, B_+)}_{GIw}))) = \sigma_I(e(H(A^{(B_+, B_+)}_{Iw}))) - \sigma_I(e(H(A^{(-B, B)}_{GIw}))) \quad (5.44)
\]
\[
\sigma_I(e(H(A^{(-B_-, B_+)}_{GIw}))) = -n_+ - n_-, \quad (5.45)
\]
recovering Corollary 2.4.

6. Application to Disordered Systems

The quantum Hall effect actually deals with disordered systems, for its famous plateaux are consequences of the existence of localized states. As noticed in [8,16], when adding a random potential and if the energy interval falls inside the localization phase, the definition of the edge conductance requires a regularization to make sense. This regularization encodes the localization properties of the disordered system, killing possible spurious currents.

We first go back to Definition 2.1 and extend it to regularized conductances. Let \( \chi, I, g \) be as in Definition 2.1. Following [8], a family \( \{J_r\}_{r>0} \) will be called a regularization for an Hamiltonian \( H \) and the interval \( I \) if the following conditions hold true
\[
(C1) : \|J_r\| = 1, \forall r > 0 \quad \text{and} \quad \forall \varphi \in E_H(I)L^2(\mathbb{R}^2), \quad \lim_{r \to \infty} J_r \varphi = \varphi; \quad (6.1)
\]
\[
(C2) : g'(H)i[H, \chi]J_r \in T_1, \forall r > 0, \quad \text{and there exists} \quad \lim_{r \to \infty} \text{tr}(g'(H)i[H, \chi]J_r) < \infty. \quad (6.2)
\]
For such a regularization we define the regularized edge conductance by
\[
\sigma_{e^{\text{reg.(I)}}}(H) := -2\pi \lim_{r \to \infty} \text{tr}(g'(H)i[H, \chi]J_r). \quad (6.3)
\]

From now on and for the rest of this section, let \( A_{Iw} = A^{(-B_-, B_+)}_{Iw} \) be an Iwatsuka potential as in Corollary 2.3, that is with \( 0 < B_- < B_+ \), and let \( I \) be an interval such that \( I \subset ](2n - 1)B_-, (2n + 1)B_- [\cap ] - \infty, B_+ [ \) for some integer \( n. \) It follows from Corollary 2.3 that \( \sigma_{e^{\text{reg.(I)}}}(H(A_{Iw})) = n. \)

Write \( A_{Iw} = A^{(0)} + A^{(R)}_{Iw} \) as in Theorem 2.10, and recall the notations in (2.13). Consider the pair \((H(a, A^{(R)}_{Iw}), H(a, 0))\), where \( a \in C^1(\mathbb{R}^2; \mathbb{R}^2) \) is
polynomially bounded and supported on some half-plane $x < R_1 < 0$. As an immediate consequence of Theorem 2.10, if $J_R$ regularizes one operator of this pair then it regularizes the second one, and one has

$$
\sigma^\text{reg,(I)}_{e}(H(A_{1w} + a)) = \sigma^\text{reg,(I)}_{e}(H(a, A_{1w}^{(R)})) = n + \sigma^\text{reg,(I)}_{e}(H(a, 0)). \quad (6.4)
$$

In particular, if $\sigma^\text{reg,(I)}_{e}(H(a, 0)) = 0$, then one has

$$
\sigma^\text{reg,(I)}_{e}(H(A_{1w} + a)) = n. \quad (6.5)
$$

We turn to disordered systems. We describe the model and then compute a regularized conductance along the lines described above. We define the random magnetic potential

$$
a_{\eta, \omega}(x, y) = \sum_{k=(k_1, k_2) \in \mathbb{Z}^2, k_1 < 0} \omega_k v_k(x, y), \quad (6.6)
$$

with $v_k(x, y) = (v_1(x - k_1), v_2(y - k_2))$, $v_1, v_2$ being two given $L^\infty$ compactly supported functions, and $\omega_k$ independent and identically distributed random variables supported on $[-1, 1]$, with common density $\rho_\eta(s) = C_\eta s^{-1} \exp(-s^2)\chi_{[-1,1]}(s)$, $\eta > 0$, where $C_\eta$ is such that $\int \rho_\eta(s)\,ds = 1$. The support of $\rho_\eta$ is $[-1, 1]$ for all $\eta > 0$, but as $\eta$ goes to zero the disorder becomes weaker, in the sense that for most $k$ the coupling constant $\omega_k$ will be very small. This model is the half-plane version of the perturbation considered in [14].

We set

$$
H_{B_-, \lambda, \eta, \omega} = (-i\nabla - A_0 - \lambda a_{\eta, \omega})^2,
$$

where $A_0$ generates a constant magnetic field of strength $B_-$ in the perpendicular direction. We denote by $\Sigma_{B, \lambda}$ the almost sure spectrum of $H_{B_-, \lambda, \omega, \eta}$ (it does not depend on $\eta > 0$ since by construction the support of $\rho_\eta$ is independent of $\eta > 0$). By Lemma 5.1, for $\lambda$ small enough, the spectrum of $H_{B_-, \lambda, \eta, \omega}$ is contained in disjoint intervals $[a_j(B_-, \lambda), b_j(B_-, \lambda)] \ni (2j - 1)B_-$, $j \in \mathbb{N}^*$. Thanks to the ergodicity in the $y$ direction, the spectrum is almost surely deterministic (see e.g. [17, Theorem 2], which can be extended to random perturbations of order 1 as considered here).

**Definition 6.1.** The region of strong dynamical localization for $H_{B_-, \lambda, \eta, \omega}$ is denoted by $\Xi_{(B_-, \lambda, \eta)}^{SDL} \subset \mathbb{R}$, and is defined as the set of $E \in \mathbb{R}$ such that there exists an interval $I \ni E$ satisfying

$$
\mathbb{E} \left\{ \sup_{t \in \mathbb{R}} \left\| \langle x \rangle \frac{\partial}{\partial t} e^{-itH_{B_-, \lambda, \eta, \omega}} \chi_I(H_{B_-, \lambda, \eta, \omega}) \tilde{\chi}_0 \right\|_2^2 \right\} < \infty, \quad (6.7)
$$

for any $p > 0$. Here $\| \cdot \|_2$ denotes the Hilbert-Schmidt norm, $\chi_I$ is the characteristic function of $I$, and $\tilde{\chi}_0$ is the characteristic function of the unit square centered at the origin.

Strong dynamical localization is known to characterize the so-called region of complete localization, where many localization properties turn out to be equivalent [25, 26]. In particular, this region coincides with the set of energies where the bootstrap multiscale analysis of [23] applies.
Theorem 6.2. Fix $J \in \mathbb{N}^*$. Let $H_{B_-,\lambda,\eta,\omega}$ be the Hamiltonian described above. Then there exists $\kappa_J > 0$ (depending on $B_-$ and $J$) and $\Lambda = \Lambda(B_-,J) > 0$, such that for any $\lambda \in ]0,\Lambda]$ and $\eta \in ]0,c_{B_-,J}\lambda|\log \lambda|^{-2}]$, for intervals $I_j$, $j = 1,\ldots,J$, satisfying
\begin{equation}
I_{j-1} \subset \Sigma_{B_-}\cap [a_j(B_-), (2j-1)B_- - \kappa_J \lambda^2],
\end{equation}
or
\begin{equation}
I_j \subset \Sigma_{B_-}\cap [(2j-1)B_- + \kappa_J \lambda^2, b_j(B_-,\lambda)],
\end{equation}
we have $I_j \subset \Xi_{SDL_{(B_-,\lambda,\eta)}}$, $\sigma_e^{reg,(I)}(H_{B_-,\lambda,\eta,\omega}) = 0$, and
\begin{equation}
\sigma_e^{reg,(I)}(H(A_{I\omega} + \lambda a_{\eta,\omega})) = j,
\end{equation}
where the regularization $J_r$ is given by
\begin{equation}
J_r = E_{H_{B_-,\lambda,\eta,\omega}}(I)1_{x \leq r}E_{H_{B_-,\lambda,\eta,\omega}}(I).
\end{equation}

Remark 6.3. 1) Any other regularization introduced in [8,9] can be used in place of (6.11), in particular a time average version of (6.11) which exploits directly the strong localization property described in Definition 6.1. 2) In [14] we construct explicit single site potential $\nu_k$ in (6.6), for which, for a given integer $J$, the $J$th first Landau levels of the Landau Hamiltonian $H(B_-)$ are shown to split into non-trivial intervals as $\lambda$ is turned on. As a consequence, the part of the spectrum of $H_{B_-,\lambda,\eta,\omega}$ where localization can be proven is not empty. It then follows from [17, Theorem 2] (extended to magnetic perturbations) that these intervals are also contained in the spectrum of the corresponding $H_{B_-,\lambda,\eta,\omega}$.

Proof of Theorem 6.2. The proof follows from [9,14]. Pick $\lambda, \eta$ as in the theorem, and $I_j$ satisfying (6.8) or (6.9). In [14], the authors show that the $\mathbb{Z}^2$-ergodic version of $H_{B_-,\lambda,\eta,\omega}$ exhibits strong dynamical localization in $I_j$. The same analysis holds true for (6.6) as well. Indeed, the Wegner estimate of [31] used in [14] holds the same (the same vector field can be used), and the initial condition is verified in the same way uniformly for all boxes of the initial scale. This comes from the fact that within the region where the magnetic perturbation is zero, localization holds for free at a given distance to the Landau levels. Next, a version of the bootstrap multiscale analysis of [23] for non-ergodic models is described in [38] and can be applied here.

By [9], $J_r$ regularizes $H_{B_-,\lambda,\eta,\omega}$ and $I_j$, and we have $\sigma_e^{reg,(I)}(H_{B_-,\lambda,\eta,\omega}) = 0$. Finally, (6.10) corresponds to (6.5).

Remark 6.4. 1) It is worth pointing out that the half plane potential given in (6.6) is relevant within our context where we deal with interface issues. In particular, in some situations, it is possible to observe edge currents “without edges”, meaning edge currents created by an interface random potential, as shown in [8,9]. Playing with the sum rule it is actually possible to show the quantization of the regularized edge conductance for models considered in [17], namely two different random electric potentials.
on the left and right half spaces, provided that the disorder difference is large. It can be extended to a high disorder electric potential and a small disorder magnetic potential. We cannot yet prove such a phenomenon for purely magnetic random potentials.

2) The value \( n \) in (6.10) is of course in agreement with the value of the (bulk) Hall conductance, as argued by Halperin [30]. Indeed, by extending [27] or [28] to random magnetic perturbations, the Hall conductance can be defined and computed for Fermi energies lying in the localized states region, and shown to be equal to the number of the highest Landau level lying below the Fermi level.

Since the regularization \( J_r \) in (6.11) (or any \( J_r \) defined in [8,9]) involves the operator \( H_{B-,-,\lambda,\eta,\omega} \) where the random potential is located on a half plane, it is designed to study the interface problem, and to compute directly the edge conductance. The equality with the bulk conductance is then a by-product of this computation if by other means the bulk conductance can be computed.

If one rather puts the focus on the equality bulk-edge, then a regularization involving the localization properties of the \( \mathbb{Z}^2 \) ergodic bulk Hamiltonian is needed. Such an analysis is pulled through in [16] for the discrete magnetic Anderson model. The authors are indeed able to reconcile the edge and bulk points of view, showing that a priori their regularized edge conductance and the Hall conductance match. It is likely that such an analysis can be carried over to the context of the present paper. However it would require to extend the analysis of [16] to the continuous setting and to random magnetic potentials.

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7. Appendix: Trace Estimates

Let \( \mathcal{H} \) be a given separable Hilbert space. Denote by \( \mathcal{B} \) the class of bounded linear operators with norm \( \| \cdot \| \), acting in \( \mathcal{H} \), and by \( \mathcal{T}_p \), \( p \in [1,\infty[ \), the Schatten-von Neumann class of compact operators acting in \( \mathcal{H} \). We recall that \( \mathcal{T}_p \) is a Banach space with norm \( \| T \|_p := \left( \text{tr} (T^* T)^{p/2} \right)^{1/p} \). In particular, in coherence with our previous notations, \( \mathcal{T}_1 \) is the trace class, and \( \mathcal{T}_2 \) is the
Hilbert-Schmidt class. The following lemma contains some well-known properties of the Schatten-von Neumann spaces, used systematically in the proofs of our results.

**Lemma 7.1.** [40]

(i) Let $T \in \mathcal{T}_p$, $p \in [1, \infty[$. Then $T^* \in \mathcal{T}_p$, and we have

$$
\|T\|_p = \|T^*\|_p. 
$$

(7.1)

(ii) Let $T \in \mathcal{T}_p$, $p \in [1, \infty[$, and $Q \in \mathcal{B}$. Then $TQ \in \mathcal{T}_p$, and we have

$$
\|TQ\|_p \leq \|T\|_p \|Q\|. 
$$

(7.2)

(iii) Let $p_j \in [1, \infty]$, $j = 1, \ldots, n$, $p \in [1, \infty[$, and $\sum_{j=1}^n p_j^{-1} = p^{-1}$. Assume that $T_j \in \mathcal{T}_{p_j}$, $j = 1, \ldots, n$. Then $T := T_1 \ldots T_n \in \mathcal{T}_p$, and we have

$$
\|T\|_p \leq \prod_{j=1}^n \|T_j\|_{p_j}. 
$$

(7.3)

**Lemma 7.2.** [40]

(i) Let $T \in \mathcal{T}_1$, $Q \in \mathcal{B}$. Then we have

$$
\text{tr} \, TQ = \text{tr} \, QT. 
$$

(7.4)

(ii) Let $p \in [1, \infty[$, $q \in [1, \infty[$, $p^{-1} + q^{-1} = 1$. Assume that $T \in \mathcal{T}_p$, $Q \in \mathcal{T}_q$. Then (7.4) holds true again.

Our next lemma contains a simple condition which guarantees the inclusion $T \in \mathcal{T}_p$ for operators of the form $T = f(x)g(-i\nabla)$.

**Lemma 7.3.** [40, Theorem 4.1] Let $d \geq 1$, $p \in [2, \infty[$, $f, g \in L^p(\mathbb{R}^d)$. Set $T := f(x)g(-i\nabla)$. Then we have $T \in \mathcal{T}_p$, and

$$
\|T\|_p \leq (2\pi)^{-d/p} \|f\|_{L^p} \|g\|_{L^p}. 
$$

(7.5)

Assume that

$$
\beta \in L^\infty(\mathbb{R}^2; \mathbb{C}^2), \quad \text{div} \, \beta \in L^\infty(\mathbb{R}^2). 
$$

(7.6)

Define the operator

$$
\mathcal{L}_\beta u := \beta \cdot \nabla u, \quad u \in C_c^\infty(\mathbb{R}^2),
$$

and then close it in $L^2(\mathbb{R}^2)$.

**Proposition 7.4.** Let $A \in C^1(\mathbb{R}^2; \mathbb{R}^2)$, $z \in \mathbb{C} \setminus [0, \infty[$. Set $R_A(z) := (H(A) - z)^{-1}$.

(i) Assume that $\alpha \in L^2(\mathbb{R}^2)$. Then we have

$$
\alpha R_A(z) \in \mathcal{T}_2, \quad R_A(z) \alpha \in \mathcal{T}_2, 
$$

(7.7)

and there exists a constant $C_1$ independent of $z$, such that

$$
\|\alpha R_A(z)\|_2 \leq C_1 C_0(z), \quad \|R_A(z) \alpha\|_2 \leq C_1 C_0(z) 
$$

(7.8)

where

$$
C_0(z) := \sup_{\lambda \in [0, \infty[} \frac{\lambda + 1}{|\lambda - z|}. 
$$

(7.9)
(ii) Assume that \( \beta \) is compactly supported and satisfies (7.6). Then we have
\[
\mathcal{L}_\beta R_A(z) \in \mathcal{T}_4, \quad R_A(z)\mathcal{L}_\beta \in \mathcal{T}_4,
\]
and there exists a constant \( C_2 \) independent of \( z \), such that
\[
\|\mathcal{L}_\beta R_A(z)\|_4 \leq C_2 C_0(z), \quad \|R_A(z)\mathcal{L}_\beta\|_4 \leq C_2 C_0(z).
\]

**Proof.**

(i) By (7.1) it suffices to prove only the first estimate in (7.8), which follows immediately from
\[
\|\alpha R_A(z)\|_2 = \|\alpha R_A(-1)(H(A) + 1)R_A(z)\|_2 \leq C_0(z)\|\alpha R_A(-1)\|_2
\]
\[
\leq C_0(z)\|\alpha(-\Delta + 1)^{-1}\|_2 = \frac{C_0(z)}{2\pi} \left( \int_{\mathbb{R}^2} |\alpha|^2 dx \int_{\mathbb{R}^2} \frac{d\xi}{(|\xi|^2 + 1)} \right)^{1/2} < \infty.
\]

(7.12)

Note that the second inequality is a special case of the diamagnetic inequality of Hilbert-Schmidt operators (see e.g. [40, Theorem 2.13]), and the last equality just follows from the Parseval identity.

(ii) Since we have
\[
(R_A(z)\mathcal{L}_\beta)^* = (-\mathcal{L}_\beta - \text{div} \, \overline{\beta})R_A(z)
\]
and \( \text{div} \, \overline{\beta}R_A(z) \in \mathcal{T}_2 \subset \mathcal{T}_4 \) by (7.7), again it suffices to check only the first estimate in (7.11). As in the proof of (7.7) we have
\[
\|\mathcal{L}_\beta R_A(z)\|_4 \leq C_0(z)\|\mathcal{L}_\beta R_A(-1)\|_4.
\]

Further,
\[
\mathcal{L}_\beta R_A(-1) = i\beta \cdot (-i\nabla - A)R_A(-1) + i\beta \cdot AR_A(-1),
\]
and \( i\beta \cdot AR_A(-1) \in \mathcal{T}_2 \subset \mathcal{T}_4 \) by (7.7). Let \( 0 \leq \zeta_j \in C_0^\infty(\mathbb{R}^2), j = 0, 1, \) satisfy \( \zeta_0 = \beta, \zeta_1 \zeta_0 = 0 \) on \( \mathbb{R}^2 \). Since \( [R_A(-1), \zeta_0] = -R_A(-1)[H(A), \zeta_0] \)
\[
R_A(-1), \]
we have
\[
i\beta \cdot (-i\nabla - A)R_A(-1) = i\beta \cdot (-i\nabla - A)R_A(-1)\zeta_0
\]
\[
-\beta \cdot (-i\nabla - A)R_A(-1)\zeta_1[H(A), \zeta_0]R(-1).
\]

(7.14)

Note that the operator
\[
[H(A), \zeta_0]R_A(-1) = 2\nabla \zeta_0 \cdot (-\nabla + iA)R_A(-1) - \Delta \zeta_0 R_A(-1)
\]
is bounded. Therefore, it follows from (7.2), (7.13), (7.14), that it suffices to check
\[
\beta \cdot (-i\nabla - A)R_A(-1)\zeta \in \mathcal{T}_4
\]
with \( 0 \leq \zeta \in C_0^\infty(\mathbb{R}^2) \). The mini–max principle implies
\[
\|\beta \cdot (-i\nabla - A)R_A(-1)\zeta\|_4 \leq \|\beta\|_{L^\infty} \|R_A(-1)^{1/2}\zeta\|_4.
\]

(7.16)

On the other hand,
\[
\|R_A(-1)^{1/2}\zeta\|_4 = \|\zeta R_A(-1)^{1/2}\|_4
\]

(7.17)
by (7.1). The diamagnetic inequality for $\mathcal{T}_4$-operators (see e.g. [40, Theorem 2.13]) entails

$$\|\zeta R_A(\Delta)^{-1/2}\|_4 \leq \|\zeta (\Delta + 1)^{-1/2}\|_4,$$

and by (7.5) we obtain

$$\|\zeta (-\Delta + 1)^{-1/2}\|_4 \leq (2\pi)^{-1/2}\|\zeta\|_{L^4} \left( \int_{\mathbb{R}^2} \frac{d\xi}{(|\xi|^2 + 1)^2} \right)^{1/4}.$$ (7.19)

Putting together (7.16)–(7.19), we obtain (7.15), and hence (7.10) and (7.11).

Proof. By (7.1) it suffices to consider only the first two operators in (7.21). Introduce three functions $0 \leq \zeta_j(\mathbb{R}^2) \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, $j = 0, 1, 2$, such that $\zeta_0 \alpha = \alpha$, $\zeta_0 \beta = \beta$, $\zeta_0 \zeta_j - 1 = \zeta_{j-1}$, $j = 1, 2$. Note that $[R_j, \zeta_k] = -R_j[H_j, \zeta_k]R_j$, and

$$[H_j, \zeta_k] = 2\nabla \zeta_k \cdot (\nabla + iA(j)) - \Delta \zeta_k$$ (7.22)

with $j = 1, 2, 3$, and $k = 0, 1, 2$. Then we have

$$(\mathcal{L}_\beta + \alpha)R_j R_k R_l = (\mathcal{L}_\beta + \alpha)R_j \zeta_0 R_k \zeta_1 R_l$$

$$- (\mathcal{L}_\beta + \alpha)R_j \zeta_0 R_k [H_k, \zeta_1] R_l$$

$$- (\mathcal{L}_\beta + \alpha)R_j [H_j, \zeta_0] R_k \zeta_1 R_l$$

$$+ (\mathcal{L}_\beta + \alpha)R_j [H_j, \zeta_0] R_j [H_j, \zeta_1] R_j \zeta_2 R_k R_l$$

$$- (\mathcal{L}_\beta + \alpha)R_j [H_j, \zeta_0] R_j [H_j, \zeta_1] R_j [H_j, \zeta_2] R_j R_k R_l,$$

(7.23)

$$R_j (\mathcal{L}_\beta + \alpha)R_k R_l = R_j \zeta_0 (\mathcal{L}_\beta + \alpha)R_k \zeta_1 R_l - R_j \zeta_0 (\mathcal{L}_\beta + \alpha)R_k [H_k, \zeta_1] R_k R_l.$$ (7.24)

Taking into account Proposition 7.4, (7.22), as well as (7.3) with $p = 1$ and (7.2), we find that (7.23) and (7.24) imply that the operators in (7.21) are
trace-class, and their trace-class norms are bounded by $C_3|\Im z|^{-n}$ with suitable $n$ and $C_3$.

\[\square\]

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