On non-formality of a simply-connected symplectic 8-manifold

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Abstract. We show an alternative construction of the first example of a simply-connected compact symplectic non-formal 8-manifold given in [6]. We also give an alternative proof of its non-formality using higher order Massey products.

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INTRODUCTION

In [1, 2, 10] Babenko–Taimanov and Rudyak–Tralle give examples of non-formal simply-connected compact symplectic manifolds of any even dimension bigger than or equal to 10. Babenko and Taimanov raise the question of the existence of non-formal simply-connected compact symplectic manifolds of dimension 8, which cannot be constructed with their methods. In [6], it is constructed the first example of a simply-connected compact symplectic 8-dimensional manifold which is non-formal, thereby completing the solution to the question of existence of non-formal symplectic manifolds for all allowable dimensions. This example is constructed by starting with a suitable complex 8-dimensional compact nilmanifold $M$ which has a symplectic form (but is not Kähler). Then one quotients by a suitable action of the finite group $\mathbb{Z}_3$ acting symplectically and freely except at finitely many fixed points. This gives a symplectic orbifold $\hat{M} = M/\mathbb{Z}_3$, which is non-formal and simply-connected thanks to the choice of $\mathbb{Z}_3$-action. The last step is a process of symplectic resolution of singularities to get a smooth symplectic manifold. The symplectic resolution of isolated orbifold singularities has been described in detail in [4]. The non-formality of $\hat{M}$ is checked via a newly defined product in cohomology. This is a product of Massey type, which is called $a$-product, and it is discussed at length in [4].

The purpose of the present note is to give a new description of the symplectic orbifold $\hat{M}$ defined in [6]. The description presented here is in terms of real nilpotent Lie groups. Secondly, we prove the non-formality of $\hat{M}$ by using higher order Massey products instead of $a$-products. It remains thus open the question of the existence of a smooth 8-manifold with non-zero $a$-products but trivial (higher order) Massey products.
A NILMANIFOLD OF DIMENSION 6

Let $G$ be the simply connected nilpotent Lie group of dimension 6 defined by the structure equations

\[
\begin{align*}
  d\beta_i &= 0, \quad i = 1, 2 \\
  d\gamma_i &= 0, \quad i = 1, 2 \\
  d\eta_1 &= -\beta_1 \wedge \gamma_1 + \beta_2 \wedge \gamma_1 + \beta_1 \wedge \gamma_2 + 2\beta_2 \wedge \gamma_2, \\
  d\eta_2 &= 2\beta_1 \wedge \gamma_1 + \beta_2 \wedge \gamma_1 + \beta_1 \wedge \gamma_2 - \beta_2 \wedge \gamma_2,
\end{align*}
\]

(1)

where $\{\beta_i, \gamma_i, \eta_i; 1 \leq i \leq 2\}$ is a basis of the left invariant 1-forms on $G$. Because the structure constants are rational numbers, Mal’cev theorem \cite{7} implies the existence of a discrete subgroup $\Gamma$ of $G$ such that the quotient space $N = \Gamma \backslash G$ is compact.

Using Nomizu’s theorem \cite{9} we can compute the real cohomology of $N$. We get

\[
\begin{align*}
  H^0(N) &= \langle 1 \rangle, \\
  H^1(N) &= \langle [\beta_1], [\beta_2], [\gamma_1], [\gamma_2] \rangle, \\
  H^2(N) &= \langle [\beta_1 \wedge \beta_2], [\beta_1 \wedge \gamma_1], [\beta_1 \wedge \gamma_2], [\gamma_1 \wedge \gamma_2], [\beta_1 \wedge \eta_2 - \beta_2 \wedge \eta_1], [\gamma_1 \wedge \eta_2 - \gamma_2 \wedge \eta_1], \\
  &\quad [\beta_1 \wedge \eta_1 + \beta_1 \wedge \eta_2 + \beta_2 \wedge \eta_2], [\gamma_1 \wedge \eta_1 + \gamma_1 \wedge \eta_2 + \gamma_2 \wedge \eta_2]\rangle, \\
  H^3(N) &= \langle [\beta_1 \wedge \beta_2 \wedge \eta_1], [\beta_1 \wedge \beta_2 \wedge \gamma_1], [\beta_1 \wedge \beta_2 \wedge \gamma_2], [\gamma_1 \wedge \gamma_2 \wedge \eta_2], [\beta_1 \wedge \gamma_1 \wedge (\eta_1 + 2\eta_2)], [\beta_1 \wedge \gamma_1 \wedge \eta_1 - \beta_1 \wedge \gamma_2 \wedge \eta_1], [\beta_1 \wedge \gamma_2 \wedge \eta_2 - \beta_1 \wedge \gamma_2 \wedge \eta_2], \beta_1 \wedge \gamma_2 \wedge (\eta_2 + 2\eta_1), [\beta_2 \wedge \gamma_2 \wedge \eta_1 - \beta_2 \wedge \gamma_2 \wedge \eta_2], [\beta_2 \wedge \gamma_1 \wedge \eta_2 - \beta_2 \wedge \gamma_1 \wedge \eta_1]\rangle, \\
  H^4(N) &= \langle [\beta_1 \wedge \beta_2 \wedge \gamma_1 \wedge \eta_1], [\beta_1 \wedge \beta_2 \wedge \gamma_1 \wedge \eta_2], [\beta_1 \wedge \beta_2 \wedge \gamma_2 \wedge \eta_2], [\beta_1 \wedge \gamma_1 \wedge \gamma_2 \wedge \eta_2], [\beta_1 \wedge \gamma_1 \wedge \gamma_2 \wedge \eta_1 \wedge \eta_2], [\beta_1 \wedge \gamma_1 \wedge \gamma_2 \wedge \eta_1 \wedge \eta_2], [\beta_1 \wedge \gamma_2 \wedge \eta_1 \wedge \eta_2 + \beta_1 \wedge \gamma_1 \wedge \eta_1 \wedge \eta_2 + \beta_1 \wedge \gamma_1 \wedge \eta_1 \wedge \eta_2]\rangle, \\
  H^5(N) &= \langle [\beta_1 \wedge \beta_2 \wedge \gamma_1 \wedge \eta_1 \wedge \eta_2], [\beta_1 \wedge \beta_2 \wedge \gamma_2 \wedge \eta_1 \wedge \eta_2], [\beta_1 \wedge \gamma_1 \wedge \gamma_2 \wedge \eta_1 \wedge \eta_2], [\beta_1 \wedge \gamma_1 \wedge \gamma_2 \wedge \eta_1 \wedge \eta_2], [\beta_2 \wedge \gamma_1 \wedge \eta_2 \wedge \eta_1 \wedge \eta_2]\rangle, \\
  H^6(N) &= \langle [\beta_1 \wedge \beta_2 \wedge \gamma_1 \wedge \gamma_2 \wedge \eta_1 \wedge \eta_2]\rangle.
\end{align*}
\]

We can give a more explicit description of the group $G$. As a differentiable manifold $G = \mathbb{R}^6$. The nilpotent Lie group structure of $G$ is given by the multiplication law

\[
m: \quad G \times G \quad \rightarrow \quad G \\
((y_1', y_2', z_1', z_2', v_1', v_2'), (y_1, y_2, z_1, z_2, v_1, v_2)) \quad \mapsto \quad (y_1 + y_1', y_2 + y_2', z_1 + z_1', z_2 + z_2', \\
v_1 + v_1' + (y_1' - y_2')z_1 - (y_1' - y_2')z_2, \\
v_2 + v_2' - (2y_1' + y_2')z_1 + (y_1' - y_2')z_2).
\]

(2)

We also need a discrete subgroup, which it could be taken to be $\mathbb{Z}^6 \subset G$. However, for later convenience, we shall take the subgroup

\[
\Gamma = \{(y_1, y_2, z_1, z_2, v_1, v_2) \in \mathbb{Z}^6 | v_1 \equiv v_2 \pmod{3}\} \subset G,
\]

and define the nilmanifold

\[
N = \Gamma \backslash G.
\]
In terms of a (global) system of coordinates $(y_1, y_2, z_1, z_2, v_1, v_2)$ for $G$, the 1–forms $\beta_i$, $\gamma_i$ and $\eta_i$, $1 \leq i \leq 2$, are given by

\[
\begin{align*}
\beta_i &= dy_i, \quad 1 \leq i \leq 2, \\
\gamma_i &= dz_i, \quad 1 \leq i \leq 2, \\
\eta_1 &= dv_1 - y_1dz_1 + y_2dz_1 + y_1dz_2 + 2y_2dz_2, \\
\eta_2 &= dv_2 + 2y_1dz_1 + y_2dz_1 + y_1dz_2 - y_2dz_2.
\end{align*}
\]

Note that $N$ is a principal torus bundle

\[
T^2 = \mathbb{Z}\langle(1,1), (3,0)\rangle \backslash \mathbb{R}^2 \hookrightarrow N \longrightarrow T^4 = \mathbb{Z}^4 \setminus \mathbb{R}^4,
\]

with the projection $(y_1, y_2, z_1, z_2, v_1, v_2) \mapsto (y_1, y_2, z_1, z_2)$.

The Lie group $G$ can be also described as follows. Consider the basis $\{\mu_i, v_i, \theta_i; 1 \leq i \leq 2\}$ of the left invariant 1–forms on $G$ given by

\[
\begin{align*}
\mu_1 &= \beta_1 + \frac{1 + \sqrt{3}}{2}\beta_2, \\
\mu_2 &= \beta_1 + \frac{1 - \sqrt{3}}{2}\beta_2, \\
v_1 &= \gamma_1 + \frac{1 + \sqrt{3}}{2}\gamma_2, \\
v_2 &= \gamma_1 + \frac{1 - \sqrt{3}}{2}\gamma_2, \\
\theta_1 &= \frac{2}{\sqrt{3}}\eta_1 + \frac{1}{\sqrt{3}}\eta_2, \\
\theta_2 &= \eta_2.
\end{align*}
\]

Hence, the structure equations can be rewritten as

\[
\begin{align*}
d\mu_i &= 0, \quad 1 \leq i \leq 2, \\
dv_i &= 0, \quad 1 \leq i \leq 2, \\
d\theta_1 &= \mu_1 \wedge v_1 - \mu_2 \wedge v_2, \\
d\theta_2 &= \mu_1 \wedge v_2 + \mu_2 \wedge v_1.
\end{align*}
\]

This means that $G$ is the complex Heisenberg group $H_{\mathbb{C}}$, that is, the complex nilpotent Lie group of complex matrices of the form

\[
\begin{pmatrix}
1 & u_2 & u_3 \\
0 & 1 & u_1 \\
0 & 0 & 1
\end{pmatrix}.
\]

In fact, in terms of the natural (complex) coordinate functions $(u_1, u_2, u_3)$ on $H_{\mathbb{C}}$, we have that the complex 1–forms

\[
\mu = du_1, \quad v = du_2, \quad \theta = du_3 - u_2du_1
\]

are left invariant and $d\mu = dv = 0$, $d\theta = \mu \wedge v$. Now, it is enough to take $\mu_1 = \Re(\mu)$, $\mu_2 = \Im(\mu)$, $v_1 = \Re(v)$, $v_2 = \Im(v)$, $\theta_1 = \Re(\theta)$, $\theta_2 = \Im(\theta)$ to recover equations (3), where $\Re(\mu)$ and $\Im(\mu)$ denote the real and the imaginary parts of $\mu$, respectively.

**Lemma 1** Let $\Lambda \subset \mathbb{C}$ be the lattice generated by 1 and $\zeta = e^{2\pi i/3}$, and consider the discrete subgroup $\Gamma_H \subset H_{\mathbb{C}}$ formed by the matrices in which $u_1, u_2, u_3 \in \Lambda$. Then there is a natural identification of $N = \Gamma \backslash G$ with the quotient $\Gamma_H \backslash H_{\mathbb{C}}$. 
Proof We have constructed above an isomorphism of Lie groups \( G \rightarrow H_C \), whose explicit equations are 
\[(y_1, y_2, z_1, z_2, v_1, v_2) \mapsto (u_1, u_2, u_3),\]
where
\[
u_1 = \left( y_1 + \frac{1 + \sqrt{3}}{2} y_2 \right) + i \left( y_1 + \frac{1 - \sqrt{3}}{2} y_2 \right),
\]
\[
u_2 = \left( z_1 + \frac{1 + \sqrt{3}}{2} z_2 \right) + i \left( z_1 + \frac{1 - \sqrt{3}}{2} z_2 \right),
\]
\[
u_3 = \frac{1}{\sqrt{3}} (2v_1 + v_2 + 3z_1y_2 + 3z_2y_1 + 3z_2y_2) + i (v_2 + 2z_1y_1 + z_2y_1 + z_1y_2 - z_2y_2).
\]
Note that the formula for \( u_3 \) can be deduced from
\[du_3 - u_2 du_1 = \theta = \left( \frac{2}{\sqrt{3}} \eta_1 + \frac{1}{\sqrt{3}} \eta_2 \right) + i \eta_2.
\]
Now the group \( \Gamma \subset G \) corresponds under this isomorphism to
\[\{(u_1, u_2, u_3) | u_1, u_2 \in \mathbb{Z} \left( 1 + i, \frac{1 + \sqrt{3}}{2}, \frac{1 - \sqrt{3}}{2} \right), u_3 \in \mathbb{Z} \langle 2\sqrt{3}, \sqrt{3}i \rangle \}.
\]
Using the isomorphism of Lie groups \( H_C \rightarrow H_C \) given by
\[\left( u_1, u_2, u_3 \right) \mapsto \left( u'_1, u'_2, u'_3 \right) = \left( \frac{u_1}{1 + i}, \frac{u_2}{1 + i}, \frac{u_3}{(1 + i)^2} \right),
\]
we get that \( u'_1, u'_2, u'_3 \in \Lambda = \mathbb{Z} \langle 1, \zeta \rangle \), which completes the proof. \( \square \)

Remark 2 If we had considered the discrete subgroup \( \mathbb{Z}^6 \subset G \) instead of \( \Gamma \subset G \), then we would not have obtained the fact \( u'_3 \in \Lambda \) in the proof of Lemma 1. Note that \( N = \Gamma \backslash G \rightarrow \mathbb{Z}^6 \backslash G \) is a 3 : 1 covering.

Under the identification \( N = \Gamma \backslash G \cong \Gamma_H \backslash H_C \), \( N \) becomes the principal torus bundle
\[T^2 = \Lambda \backslash \mathbb{C} \hookrightarrow N \twoheadrightarrow T^4 = \Lambda^2 \backslash \mathbb{C}^2,
\]
with the projection \( (u_1, u_2, u_3) \mapsto (u_1, u_2) \).

A SYMPLECTIC ORBIFOLD OF DIMENSION 8

We define the 8–dimensional compact nilmanifold \( M \) as the product
\[M = T^2 \times N.
\]
By Lemma 1, there is an isomorphism between $M$ and the manifold $(\Gamma_H \backslash H_C) \times (\Lambda \backslash \mathbb{C})$ studied in [6, Section 2] (we have to send the factor $T^2$ of $M$ to the factor $\Lambda \backslash \mathbb{C}$). Clearly, $M$ is a principal torus bundle

$$T^2 \hookrightarrow M \xrightarrow{\pi} T^6.$$  

Let $(x_1, x_2)$ be the Lie algebra coordinates for $T^2$, so that $(x_1, x_2, y_1, y_2, z_1, z_2, v_1, v_2)$ are coordinates for the Lie algebra $\mathbb{R}^2 \times G$ of $M$. Then $\pi(x_1, x_2, y_1, y_2, z_1, z_2, v_1, v_2) = (x_1, x_2, y_1, z_1, z_2)$. A basis for the left invariant (closed) 1–forms on $T^2$ is given as \{\$a_1, a_2\$, where $a_1 = dx_1$ and $a_2 = dx_2$. Then \{\$a_i, b_i, \gamma_i; 1 \leq i \leq \$2\$\} constitutes a (global) basis for the left invariant 1–forms on $M$. Note that \{\$a_i, b_i, \gamma_i; 1 \leq i \leq \$2\$\} is a basis for the left invariant closed 1–forms on the base $T^6$. (We use the same notation for the differential forms on $T^6$ and their pullbacks to $M$.) Using the computation of the cohomology of $N$, we get that the Betti numbers of $M$ are: $b_0(M) = b_8(M) = 1$, $b_1(M) = b_7(M) = 6$, $b_2(M) = b_6(M) = 17$, $b_3(M) = b_5(M) = 30$, $b_4(M) = 36$. In particular, $\chi(M) = 0$, as for any nilmanifold.

Consider the action of the finite group $\mathbb{Z}_3$ on $\mathbb{R}^2$ given by

$$\rho(x_1, x_2) = (-x_1 - x_2, x_1),$$

for $(x_1, x_2) \in \mathbb{R}^2$, $\rho$ being the generator of $\mathbb{Z}_3$. Clearly $\rho(\mathbb{Z}^2) = \mathbb{Z}^2$, and so $\rho$ defines an action of $\mathbb{Z}_3$ on the 2-torus $T^2 = \mathbb{Z}^2 \backslash \mathbb{R}^2$ with 3 fixed points: $(0, 0)$, $(\frac{1}{3}, \frac{1}{3})$, and $(\frac{2}{3}, \frac{2}{3})$. The quotient space $T^2/\mathbb{Z}_3$ is the orbifold 2–sphere $S^2$ with 3 points of multiplicity 3. Let $x_1, x_2$ denote the natural coordinate functions on $\mathbb{R}^2$. Then the 1–forms $dx_1$, $dx_2$ satisfy $\rho^*(dx_1) = -dx_1 - dx_2$ and $\rho^*(dx_2) = dx_1$, hence $\rho^*(-dx_1 - dx_2) = dx_2$. Thus, we can take the 1–forms $a_1$ and $a_2$ on $T^2$ such that

$$\rho^*(a_1) = -a_1 - a_2, \quad \rho^*(a_2) = a_1. \quad (4)$$

Define the following action of $\mathbb{Z}_3$ on $M$, given, at the level of Lie groups, by $\rho : \mathbb{R}^2 \times \mathbb{R}^6 \longrightarrow \mathbb{R}^2 \times \mathbb{R}^6$,

$$\rho(x_1, x_2, y_1, y_2, z_1, z_2, v_1, v_2) = (-x_1 - x_2, x_1, -y_1 - y_2, y_1, -z_1 - z_2, z_1, -v_1 - v_2, v_1).$$

Note that $m(\rho(p'), \rho(p)) = \rho(m(p', p))$, for all $p, p' \in G$, where $m$ is the multiplication map (2) for $G$. Also $\Gamma \subset G$ is stable by $\rho$ since

$$v_1 \equiv v_2 \pmod{3} \implies -v_1 - v_2 \equiv v_1 \pmod{3}.$$ 

Therefore there is a induced map $\rho : M \to M$, and this covers the action $\rho : T^6 \to T^6$ on the 6–torus $T^6 = T^2 \times T^2 \times T^2$ (defined as the action $\rho$ on each of the three factors simultaneously). The action of $\rho$ on the fiber $T^2 = \mathbb{Z}\langle(1, 1), (3, 0)\rangle$ has also 3 fixed points: $(0, 0)$, $(1, 0)$ and $(2, 0)$. Hence there are $3^4 = 81$ fixed points on $M$.

**Remark 3** Under the isomorphism $M \cong (\Gamma_H \backslash H_C) \times (\Lambda \backslash \mathbb{C})$, we have that the action of $\rho$ becomes $\rho(u_1, u_2, u_3) = (\xi u_1, \xi u_2, \xi u_3)$, where $\xi = e^{2\pi i/3}$. Composing the isomorphism of Lemma 1 with the conjugation $(u_1, u_2, u_3) \mapsto (v_1, v_2, v_3) = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$ (which is an isomorphism of Lie groups $H_C \to H_C$ leaving $\Gamma_H$ invariant), we have that the action of $\rho$ becomes $\rho(v_1, v_2, v_3) = (\xi v_1, \xi v_2, \xi^2 v_3)$. This is the action used in [6].
We take the basis \( \{ \alpha_i, \beta_i, \gamma_i, \eta_i; 1 \leq i \leq 2 \} \) of the 1–forms on \( M \) considered above. The 1–forms \( dy_i, dz_i, dv_i, 1 \leq i \leq 2 \), on \( G \) satisfy the following conditions similar to (4): 
\[
\rho^*(dy_1) = -dy_1 - dy_2, \quad \rho^*(dy_2) = dy_1, \quad \rho^*(dz_1) = -dz_1 - dz_2, \quad \rho^*(dz_2) = dz_1,
\]
\[
\rho^*(dv_1) = -dv_1 - dv_2, \quad \rho^*(dv_2) = dv_1.
\]
So
\[
\rho^*(\alpha_1) = -\alpha_1 - \alpha_2, \quad \rho^*(\alpha_2) = \alpha_1, \\
\rho^*(\beta_1) = -\beta_1 - \beta_2, \quad \rho^*(\beta_2) = \beta_1, \\
\rho^*(\gamma_1) = -\gamma_1 - \gamma_2, \quad \rho^*(\gamma_2) = \gamma_1, \\
\rho^*(\eta_1) = -\eta_1 - \eta_2, \quad \rho^*(\eta_2) = \eta_1.
\]

**Remark 4** If we define the 1–forms \( \alpha_3 = -\alpha_1 - \alpha_2, \beta_3 = -\beta_1 - \beta_2, \gamma_3 = -\gamma_1 - \gamma_2 \) and \( \eta_3 = -\eta_1 - \eta_2 \), then we have \( \rho^*(\alpha_1) = \alpha_3, \rho^*(\alpha_2) = \alpha_1, \rho^*(\alpha_3) = \alpha_2 \), and analogously for the others.

Define the quotient space \( \hat{M} = M/\mathbb{Z}_3 \), and denote by \( \varphi : M \to \hat{M} \) the projection. It is an orbifold, and it admits the structure of a symplectic orbifold (see [4] for a general discussion on symplectic orbifolds).

**Proposition 5** The 2–form \( \omega \) on \( M \) defined by
\[
\omega = \alpha_1 \wedge \alpha_2 + \eta_2 \wedge \beta_1 - \eta_1 \wedge \beta_2 + \gamma_1 \wedge \gamma_2
\]
is a \( \mathbb{Z}_3 \)-invariant symplectic form on \( M \). Therefore it induces \( \hat{\omega} \in \Omega^2_{\text{orb}}(\hat{M}) \), such that \( (\hat{M}, \hat{\omega}) \) is a symplectic orbifold.

**Proof** Clearly \( \omega^4 \neq 0 \). Using (5) we have that \( \rho^*(\omega) = (-\alpha_1 - \alpha_2) \wedge \alpha_1 + \eta_1 \wedge (-\beta_1 - \beta_2) + (\eta_1 + \eta_2) \wedge \beta_1 - (\gamma_1 - \gamma_2) \wedge \gamma_1 = \omega \), so \( \omega \) is \( \mathbb{Z}_3 \)-invariant. Finally,
\[
d\omega = d\eta_2 \wedge \beta_1 - d\eta_1 \wedge \beta_2 = (\beta_2 \wedge \gamma_1 - \beta_1 \wedge \gamma_2 + \gamma_2 \wedge \beta_1 - (\beta_1 \wedge \gamma_1 + \beta_1 \wedge \gamma_2) \wedge \beta_2 = 0.
\]

It can be seen (cf. proof of Proposition 2.3 in [6]) that \( \hat{M} \) is simply connected. Moreover, its cohomology can be computed using that
\[
H^* (\hat{M}) = H^* (M)^{\mathbb{Z}_3}.
\]
We get
\[
H^1(\hat{M}) = 0, \\
H^2(\hat{M}) = \langle [\alpha_1 \wedge \alpha_2], [\alpha_1 \wedge \beta_2 - \alpha_2 \wedge \beta_1], [\alpha_1 \wedge \beta_1 + \alpha_1 \wedge \beta_2 + \alpha_2 \wedge \beta_2], \\
[\alpha_1 \wedge \gamma_2 - \alpha_2 \wedge \gamma_1], [\alpha_1 \wedge \eta_1 + \alpha_1 \wedge \eta_2 + \alpha_2 \wedge \eta_2], [\beta_1 \wedge \beta_2], [\beta_1 \wedge \gamma_2 - \beta_2 \wedge \gamma_1], \\
[\beta_1 \wedge \eta_1 + \beta_1 \wedge \eta_2 + \beta_2 \wedge \eta_2], [\beta_1 \wedge \eta_2 - \beta_2 \wedge \eta_1], [\beta_1 \wedge \eta_1 + \beta_1 \wedge \eta_2 + \beta_2 \wedge \eta_2], \\
[\gamma_1 \wedge \gamma_2], [\gamma_1 \wedge \eta_2 - \gamma_2 \wedge \eta_1], [\gamma_1 \wedge \eta_1 + \gamma_1 \wedge \eta_2 + \gamma_2 \wedge \eta_2] \rangle, \\
H^3(\hat{M}) = 0.
\]
**Remark 6** The Euler characteristic of $\tilde{M}$ can be computed via the formula for finite group action quotients: let $\Pi$ be the cyclic group of order $n$, acting on a space $X$ almost freely. Then

$$\chi(X/\Pi) = \frac{1}{n}\chi(X) + \sum_p \left(1 - \frac{1}{\#\Pi_p}\right),$$

where $\Pi_p \subset \Pi$ is the isotropy group of $p \in X$. In our case $\chi(\tilde{M}) = \frac{1}{2}\chi(M) + 81(1 - \frac{1}{5}) = 54$.

Using this remark and the previous calculation, we get that $b_1(\tilde{M}) = b_2(\tilde{M}) = 0$, $b_2(\tilde{M}) = b_6(\tilde{M}) = 13$, $b_3(\tilde{M}) = b_5(\tilde{M}) = 0$ and $b_4(\tilde{M}) = 26$. Note that $M$ satisfies Poincaré duality since $H^*(\tilde{M}) = H^*(M)^{\mathbb{Z}_3}$ and $H^*(M)$ satisfies Poincaré duality.

**NON-FORMALITY OF THE SYMPLECTIC ORBIFOLD**

Formality is a property of the rational homotopy type of a space which is of great importance in symplectic geometry. This is due to the fact that compact Kähler manifolds are formal [3] whilst there are compact symplectic manifolds which are non-formal [11, 8, 9]. A general discussion of the property of formality can be found in [11].

The non-formality of a space can be detected by means of Massey products. Let us recall its definition. The simplest type of Massey product is the triple (also known as ordinary) Massey product. Let $X$ be a smooth manifold and let $a_i \in H^{p_i}(X)$, $1 \leq i \leq 3$, be three cohomology classes such that $a_1 \cup a_2 = 0$ and $a_2 \cup a_3 = 0$. The (triple) Massey product of the classes $a_i$ is defined as the set

$$\langle a_1, a_2, a_3 \rangle = \{ [\alpha_1 \wedge \eta + (-1)^{p_1+1} \xi \wedge \alpha_3] \mid a_i = [\alpha_i], \alpha_1 \wedge \alpha_2 = d\xi, \alpha_2 \wedge \alpha_3 = d\eta \}$$

inside $H^{p_1+p_2+p_3-1}(X)$. We say that $\langle a_1, a_2, a_3 \rangle$ is trivial if $0 \in \langle a_1, a_2, a_3 \rangle$.

The definition of higher Massey products is as follows (see [8, 11]). The Massey product $\langle a_1, a_2, \ldots, a_t \rangle$, $a_i \in H^{p_i}(X)$, $1 \leq i \leq t$, $t \geq 3$, is defined if there are differential forms $\alpha_{i,j}$ on $X$, with $1 \leq i \leq j \leq t$, except for the case $(i, j) = (1, t)$, such that

$$a_i = [\alpha_{i,i}], \quad d\alpha_{i,j} = \sum_{k=i}^{j-1} \bar{\alpha}_{i,k} \wedge \alpha_{k+1,j}, \quad (6)$$

where $\bar{\alpha} = (-1)^{\deg(\alpha)} \alpha$. Then the Massey product is

$$\langle a_1, a_2, \ldots, a_t \rangle = \left\{ \left[ \sum_{k=1}^{t-1} \bar{\alpha}_{1,k} \wedge \alpha_{k+1,t} \right] \mid \alpha_{i,j} \text{ as in (6)} \right\} \subset H^{p_1+\cdots+p_t-(t-2)}(X).$$

We say that the Massey product is trivial if $0 \in \langle a_1, a_2, \ldots, a_t \rangle$. Note that for $\langle a_1, a_2, \ldots, a_t \rangle$ to be defined it is necessary that $\langle a_1, \ldots, a_{t-1} \rangle$ and $\langle a_2, \ldots, a_t \rangle$ are defined and trivial.

The existence of a non-trivial Massey product is an obstruction to formality, namely, if $X$ has a non-trivial Massey product then $X$ is non-formal.
In the case of an orbifold, Massey products are defined analogously but taking the forms to be \textit{orbifold forms} (see [4, Section 2]).

Now we want to prove the non-formality of the orbifold $\hat{M}$ constructed in the previous section. By the results of [11], $M$ is non-formal since it is a nilmanifold which is not a torus. We shall see that this property is inherited by the quotient space $\hat{M} = M/\mathbb{Z}_3$. For this, we study the Massey products on $\hat{M}$.

**Lemma 7** $\hat{M}$ has a non-trivial Massey product if and only if $M$ has a non-trivial Massey product with all cohomology classes $a_i \in H^*(M)$ being $\mathbb{Z}_3$-invariant cohomology classes.

**Proof** We shall do the case of triple Massey products, since the general case is similar. Suppose that $\langle a_1, a_2, a_3 \rangle$, $a_i \in H^p(M)$, $1 \leq i \leq 3$ is a non-trivial Massey product on $\hat{M}$. Let $a_i = [\alpha_i]$, where $\alpha_i \in \Omega^*_\text{orb}(M)$. We pull-back the cohomology classes $\alpha_i$ via $\phi^*: \Omega^*_\text{orb}(\hat{M}) \to \Omega^*(M)$ to get a Massey product $\langle [\phi^*\alpha_1], [\phi^*\alpha_2], [\phi^*\alpha_3] \rangle$. Suppose that this is trivial on $M$, then $\phi^*\alpha_1 \wedge \phi^*\alpha_2 = d\xi$, $\phi^*\alpha_2 \wedge \phi^*\alpha_3 = d\eta$, with $\xi, \eta \in \Omega^*(M)$, and $\phi^*\alpha_1 \wedge \eta + (-1)^{p_1+1}\xi \wedge \phi^*\alpha_3 = df$. Then $\tilde{\eta} = (\eta + \rho^*\eta + (\rho^*)^2\eta)/3$, $\tilde{\xi} = (\xi + \rho^*\xi + (\rho^*)^2\xi)/3$ and $\tilde{f} = (f + \rho^*\eta + (\rho^*)^2\eta)/3$ are $\mathbb{Z}_3$-invariant and $\phi^*\alpha_1 \wedge \tilde{\eta} + (-1)^{p_1+1}\tilde{\xi} \wedge \phi^*\alpha_3 = d\tilde{f}$. Writing $\tilde{\eta} = \phi^*\tilde{\eta}$, $\tilde{\xi} = \phi^*\tilde{\xi}$, $\tilde{f} = \phi^*\tilde{f}$, for $\eta, \xi, f \in \Omega^*_\text{orb}(M)$, we get $\alpha_1 \wedge \tilde{\eta} + (-1)^{p_1+1}\tilde{\xi} \wedge \alpha_3 = d\tilde{f}$, contradicting that $\langle a_1, a_2, a_3 \rangle$ is non-trivial.

Conversely, suppose that $\langle a_1, a_2, a_3 \rangle$, $a_i \in H^p(M)^{\mathbb{Z}_3}$, $1 \leq i \leq 3$, is a non-trivial Massey product on $M$. Then we can represent $a_i = [\alpha_i]$ by $\mathbb{Z}_3$-invariant differential forms $\alpha_i \in \Omega^p(M)$. Let $\tilde{\alpha}_i$ be the induced form on $\hat{M}$. Then $\langle [\tilde{\alpha}_1], [\tilde{\alpha}_2], [\tilde{\alpha}_3] \rangle$ is a non-trivial Massey product on $\hat{M}$.

For if it were trivial then pulling-back by $\phi$, we would get $0 \in \langle \phi^*\tilde{\alpha}_1, \phi^*\tilde{\alpha}_2, \phi^*\tilde{\alpha}_3 \rangle = \langle a_1, a_2, a_3 \rangle$. □

In our case, all the triple and quintuple Massey products on $\hat{M}$ are trivial. For instance, for a Massey product of the form $\langle a_1, a_2, a_3 \rangle$, all $a_i$ should have even degree, since $H^1(M) = H^3(M) = H^5(M) = H^7(\hat{M}) = 0$. Therefore the degree of the cohomology classes in $\langle a_1, a_2, a_3 \rangle$ is odd, hence they are zero.

Since the dimension of $\hat{M}$ is 8, there is no room for sextuple Massey products or higher, since the degree of $\langle a_1, a_2, \ldots, a_6 \rangle$ is at least $s + 2$, as $\deg a_i \geq 2$. For $s = 6$, a sextuple Massey product of cohomology classes of degree 2 would live in the top degree cohomology. For computing an element of $\langle a_1, \ldots, a_6 \rangle$, we have to choose $\alpha_{i,j}$ in (6). But then adding a closed form $\phi$ with $a_1 \cup [\phi] = \lambda[M] \in H^8(M)$ to $a_{2,6}$ we can get another element of $\langle a_1, \ldots, a_6 \rangle$ which is the previous one plus $\lambda[M]$. For suitable $\lambda$ the we get $0 \in \langle a_1, \ldots, a_6 \rangle$.

The only possibility for checking the non-formality of $\hat{M}$ via Massey products is to get a non-trivial quadruple Massey product.

From now on, we will denote by the same symbol a $\mathbb{Z}_3$-invariant form on $M$ and that induced on $\hat{M}$. Notice that the 2 forms $\gamma_1 \wedge \gamma_2$, $\beta_1 \wedge \beta_2$ and $\alpha_1 \wedge \gamma_1 + \alpha_2 \wedge \gamma_1 + \alpha_3 \wedge \gamma_2$ are $\mathbb{Z}_3$-invariant forms on $M$, hence they descend to the quotient $M = M/\mathbb{Z}_3$. We have the following:
Proposition 8 The quadruple Massey product
\[ \langle [\gamma \land \gamma_2], [\beta_1 \land \beta_2], [\beta_1 \land \beta_2], [\alpha_1 \land \gamma_1 + \alpha_2 \land \gamma_1 + \alpha_2 \land \gamma_2] \rangle \]
is non-trivial on \( \hat{M} \). Therefore, the space \( \hat{M} \) is non-formal.

Proof First we see that
\[
(\gamma_1 \land \gamma_2) \land (\beta_1 \land \beta_2) = d\xi,
\]
\[
(\beta_1 \land \beta_2) \land (\alpha_1 \land \gamma_1 + \alpha_2 \land \gamma_1 + \alpha_2 \land \gamma_2) = d\zeta,
\]
where \( \xi \) and \( \zeta \) are the differential 3–forms on \( \hat{M} \) given by
\[
\xi = \frac{1}{6}(\gamma_1 \land (\beta_1 \land \eta_2 + \beta_2 \land \eta_2 + \beta_2 \land \eta_1) + \gamma_2 \land (\beta_1 \land \eta_2 + \beta_1 \land \eta_1 + \beta_2 \land \eta_1)),
\]
\[
\zeta = \frac{1}{3}(-\alpha_1 \land (\eta_2 \land \beta_1 + \eta_1 \land \beta_1 + \eta_1 \land \beta_2) + \alpha_2 \land (\eta_2 \land \beta_2 - \eta_1 \land \beta_1)).
\]
Therefore, the triple Massey products \( \langle [\gamma_1 \land \gamma_2], [\beta_1 \land \beta_2], [\beta_1 \land \beta_2] \rangle \) and \( \langle [\beta_1 \land \beta_2], [\beta_1 \land \beta_2], [\alpha_1 \land \gamma_1 + \alpha_2 \land \gamma_1 + \alpha_2 \land \gamma_2] \rangle \) are defined, and they are trivial because all the (triple) Massey products on \( \hat{M} \) are trivial. (Notice that the forms \( \xi \) and \( \zeta \) are \( \mathbb{Z}_3 \)-invariant on \( M \) and so descend to \( \hat{M} \).) Therefore, the quadruple Massey product \( \langle [\gamma_1 \land \gamma_2], [\beta_1 \land \beta_2], [\beta_1 \land \beta_2], [\alpha_1 \land \gamma_1 + \alpha_2 \land \gamma_1 + \alpha_2 \land \gamma_2] \rangle \) is defined on \( \hat{M} \). Moreover, it is trivial on \( M \) if and only if there are differential forms \( f_i \in \Omega^3(\hat{M}) \), \( 1 \leq i \leq 3 \), and \( g_j \in \Omega^4(\hat{M}) \), \( 1 \leq j \leq 2 \), such that
\[
(\gamma_1 \land \gamma_2) \land (\beta_1 \land \beta_2) = d(\xi + f_1),
\]
\[
(\beta_1 \land \beta_2) \land (\beta_1 \land \beta_2) = df_2,
\]
\[
(\beta_1 \land \beta_2) \land (\alpha_1 \land \gamma_1 + \alpha_2 \land \gamma_1 + \alpha_2 \land \gamma_2) = d(\zeta + f_3),
\]
\[
(\gamma_1 \land \gamma_2) \land f_2 - (\xi + f_1) \land (\beta_1 \land \beta_2) = dg_1,
\]
\[
(\beta_1 \land \beta_2) \land (\zeta + f_3) - f_2 \land (\alpha_1 \land \gamma_1 + \alpha_2 \land \gamma_1 + \alpha_2 \land \gamma_2) = dg_2,
\]
and the 6–form given by
\[
\Psi = -(\gamma_1 \land \gamma_2) \land g_2 - g_1 \land (\alpha_1 \land \gamma_1 + \alpha_2 \land \gamma_1 + \alpha_2 \land \gamma_2) + (\xi + f_1) \land (\zeta + f_3)
\]
defines the zero class in \( H^6(\hat{M}) \). Clearly \( f_1 \), \( f_2 \) and \( f_3 \) are closed 3–forms. Since \( H^3(\hat{M}) = 0 \), we can write \( f_1 = df_1' \), \( f_2 = df_2' \) and \( f_3 = df_3' \) for some differential 2–forms \( f_1' \), \( f_2' \) and \( f_3' \in \Omega^2(\hat{M}) \). Now, multiplying \( [\Psi] \) by the cohomology class \( [\sigma] \in H^2(\hat{M}) \), where \( \sigma = 2\alpha_1 \land \gamma_2 - \alpha_2 \land \gamma_1 + \alpha_1 \land \gamma_1 + \alpha_2 \land \gamma_2 \) we get
\[
\sigma \land \Psi = -\frac{1}{3}(\alpha_1 \land \alpha_2 \land \beta_1 \land \beta_2 \land \gamma_1 \land \gamma_2 \land \eta_1 \land \eta_2) + d(\sigma \land \xi \land f_1' + \sigma \land \zeta \land f_1' + \sigma \land \xi \land f_2' + \sigma \land \zeta \land f_2' + \sigma \land f_1' \land df_3').
\]
Hence, \( [2\alpha_1 \land \gamma_2 - \alpha_2 \land \gamma_1 + \alpha_1 \land \gamma_1 + \alpha_2 \land \gamma_2] \cup [\Psi] \neq 0 \), which implies that \( [\Psi] \) is non-zero in \( H^6(\hat{M}) \). This proves that the Massey product \( \langle [\gamma_1 \land \gamma_2], [\beta_1 \land \beta_2], [\beta_1 \land \beta_2], [\alpha_1 \land \gamma_1 + \alpha_2 \land \gamma_1 + \alpha_2 \land \gamma_2] \rangle \) is non-trivial, and so \( \hat{M} \) is non-formal. \( \square \)
Finally, there is a way to desingularize $(\tilde{M}, \tilde{\omega})$ to get a smooth symplectic manifold.

**Theorem 9** There is a smooth compact symplectic 8-manifold $(\tilde{M}, \tilde{\omega})$ which is simply-connected and non-formal.

**Proof** By [4, Theorem 3.3], there is a symplectic resolution $\pi : (\tilde{M}, \tilde{\omega}) \rightarrow (\hat{M}, \hat{\omega})$, which consists of a smooth symplectic manifold $(\tilde{M}, \tilde{\omega})$ and a map $\pi$ which is a diffeomorphism outside the singular points.

To prove the non-formality of $\tilde{M}$, we work as follows. All the forms of the proof of Proposition 8 can be defined on the resolution $\tilde{M}$. Take a $\mathbb{Z}_3$-equivariant map $\psi : M \rightarrow M$ which is the identity outside small balls around the fixed points, and contracts smaller balls onto the fixed points. Substitute the forms $\vartheta$, $\tau_i$, $\kappa$, $\xi$, ... by $\psi^* \vartheta$, $\psi^* \tau_i$, $\psi^* \kappa$, $\psi^* \xi$, ... Then the corresponding elements in the quadruple Massey product are non-zero, but these forms are zero in a neighbourhood of the fixed points. Therefore they define forms on $\tilde{M}$, by extending them by zero along the exceptional divisors $E_p = \pi^{-1}(p) \quad (p \in \hat{M} \quad \text{singular point})$. Now the proof of Proposition 8 works for $\tilde{M}$ with these forms.

Finally, the manifold $\tilde{M}$ is simply connected as it is proved in [6, Proposition 2.3] (basically, this follows from the simply-connectivity of $\hat{M}$).

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