A POLYNOMIAL DEFINED BY THE $SL(2;\mathbb{C})$-REIDEMEISTER TORSION FOR A HOMOLOGY 3-SPHERE OBTAINED BY DEHN-SURGERY ALONG A TORUS KNOT

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Abstract. Let $M_n$ be a homology 3-sphere obtained by $\frac{1}{n}$-Dehn surgery along a $(p, q)$-torus knot. We consider a polynomial $\sigma_{(p,q,n)}(t)$ whose zeros are the inverses of the Reidemeister torsion of $M_n$ for $SL(2;\mathbb{C})$-irreducible representations. We give an explicit formula of this polynomial by using Tchebychev polynomials of the first kind. Further we also give a 3-term relations of these polynomials.

1. Introduction

Let $T(p, q)$ be a $(p, q)$-torus knot in $S^3$. Here $p, q$ are coprime and positive integers. Let $M_n$ be a homology 3-sphere obtained by $\frac{1}{n}$-Dehn surgery along $T(p, q)$. It is well known that $M_n$ is a Brieskorn homology 3-sphere $\Sigma(p, q, N)$ where we write $N$ for $\lfloor pgn + 1 \rfloor$. Here $\Sigma(p, q, N)$ is defined as

$$\{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1^p + z_2^q + z_3^N = 0, \ |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}.$$

In this paper we consider the Reidemeister torsion $\tau_\rho(M_n)$ of $M_n$ for an irreducible representation $\rho : \pi_1(M_n) \to SL(2;\mathbb{C})$.

In the 1980’s Johnson [11] gave an explicit formula for any non-trivial value of $\tau_\rho(M_n)$. Furthermore, he proposed to consider the polynomial whose zero set coincides with the set of all non-trivial values $\{1/\tau_\rho(M_n)\}$, which is denoted by $\sigma_{(2,3,n)}(t)$. Under some normalization of $\sigma_{(2,3,n)}(t)$, he gave a 3-term relation among $\sigma_{(2,3,n+1)}(t), \sigma_{(2,3,n)}(t)$ and $\sigma_{(2,3,n-1)}(t)$ by using Tchebychev polynomials of the first kind.

Recently in [5] we gave one generalization of the Johnson’s formula for a $(2p’, q)$-torus knot. Here $p’, q$ are coprime odd integers. In this paper, we show the formula for any torus knot $T(p, q)$.

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2. DEFINITION OF REIDMEISTER TORSION

First let us describe definitions and properties of the Reidemeister torsion for \( SL(2; \mathbb{C}) \)-representations. See Johnson [1], Kitano [2, 3] and Porti [7] for details.

Let \( b = (b_1, \cdots, b_d) \) and \( c = (c_1, \cdots, c_d) \) be two bases for a \( d \)-dimensional vector space \( W \) over \( \mathbb{C} \). Setting \( b_i = \sum_{j=1}^{d} p_{ij} c_j \), we obtain a nonsingular matrix \( P = (p_{ij}) \in GL(d; \mathbb{C}) \). Let \( [b/c] \) denote the determinant of \( P \).

Suppose \( C^* : 0 \to C_k \xrightarrow{\partial_k} \cdots \xrightarrow{\partial_2} \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0 \to 0 \) is an acyclic chain complex of finite dimensional vector spaces over \( \mathbb{C} \). We assume that a preferred basis \( c_i \) for \( C_i \) is given for each \( i \). That is, \( C^* \) is a based acyclic chain complex over \( \mathbb{C} \).

Choose any basis \( b_i \) for \( B_i = \text{Im}(\partial_{i+1}) \) and take a lift of it in \( C_{i+1} \), which is denoted by \( \tilde{b}_i \). Since \( B_i = Z_i = \text{Ker}\partial_i \), the basis \( b_i \) can serve as a basis for \( Z_i \). Furthermore since the sequence

\[
0 \to Z_i \to C_i \xrightarrow{\partial_i} B_{i-1} \to 0
\]

is exact, the vectors \( (b_i, \tilde{b}_{i-1}) \) form a basis for \( C_i \). Here \( \tilde{b}_{i-1} \) is a lift of \( b_{i-1} \) in \( C_i \). It is easily shown that \( [b_i, \tilde{b}_{i-1}/c_i] \) does not depend on a choice of a lift \( \tilde{b}_{i-1} \). Hence we can simply denote it by \( [b_i, b_{i-1}/c_i] \).

**Definition 2.1.** The torsion \( \tau(C_*) \) of a based chain complex \( C_* \) with \( \{c_i\} \) is given by the alternating product

\[
\tau(C_*) = \prod_{i=0}^{k} [b_i, b_{i-1}/c_i]^{(-1)^{i+1}}.
\]

**Remark 2.2.** It is easy to see that \( \tau(C_*) \) does not depend on choices of the bases \( \{b_0, \cdots, b_k\} \).

Now we apply this torsion invariant of chain complexes to geometric situations as follows. Let \( X \) be a finite CW-complex and \( \tilde{X} \) a universal covering of \( X \) with the lifted CW-complex structure. The fundamental group \( \pi_1 X \) acts on \( \tilde{X} \) from the right-hand side as deck transformations. We may assume that this action is free and cellular by taking a subdivision if we need. Then the chain complex \( C_*(\tilde{X}; \mathbb{Z}) \) has the structure of a chain complex of free \( \mathbb{Z}[\pi_1 X] \)-modules.

Let \( \rho : \pi_1 X \to SL(2; \mathbb{C}) \) be a representation. We denote the 2-dimensional vector space \( \mathbb{C}^2 \) by \( V \). Using the representation \( \rho \), \( V \) admits the structure of a \( \mathbb{Z}[\pi_1 X] \)-module and then we denote it by \( V_\rho \).
Define the chain complex $C_*(X; V_\rho)$ by $C_*(\hat{X}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1 X]} V_\rho$ and choose a preferred basis
\[ (\tilde{u}_1 \otimes e_1, \tilde{u}_1 \otimes e_2, \cdots, \tilde{u}_d \otimes e_1, \tilde{u}_d \otimes e_2) \]
of $C_i(X; V_\rho)$ where $\{e_1, e_2\}$ is a canonical basis of $V = \mathbb{C}^2$, $\{u_1, \cdots, u_d\}$ are the $i$-cells giving a basis of $C_i(\hat{X}; \mathbb{Z})$ and $\{\tilde{u}_1, \cdots, \tilde{u}_d\}$ are lifts of them on $\hat{X}$. Now we suppose that $C_*(X; V_\rho)$ is acyclic, namely all homology groups $H_*(X; V_\rho)$ are vanishing. In this case $\rho$ is called an acyclic representation.

**Definition 2.3.** Let $\rho : \pi_1(X) \to SL(2; \mathbb{C})$ be an acyclic representation. Then the Reidemeister torsion $\tau_\rho(X) \in \mathbb{C} \setminus \{0\}$ is defined by the torsion $\tau(C_*(X; V_\rho))$ of $C_*(X; V_\rho)$.

**Remark 2.4.**

1. We define $\tau_\rho(X) = 0$ for a non-acyclic representation $\rho$.
2. The definition of $\tau_\rho(X)$ depends on several choices. However it is well known that it is a piecewise linear invariant in the case of $SL(2; \mathbb{C})$-representations.

### 3. Johnson’s theory

Let $T(p, q) \subset S^3$ be a $(p, q)$-torus knot with coprime integers $p, q$. Now we write $M_n$ to a closed orientable 3-manifold obtained by a $\frac{1}{n}$-Dehn surgery along $T(p, q)$. Here the fundamental group of $S^3 \setminus T(p, q)$ has the presentation as follows:

\[ \pi_1(S^3 \setminus T(p, q)) = \langle x, y \mid x^p = y^q \rangle. \]

Furthermore $\pi_1(M_n)$ admits the presentation as follows;

\[ \pi_1(M_n) = \langle x, y \mid x^p = y^q, ml^n = 1 \rangle \]

where $m = x^{-r}y^s$ $(r, s \in \mathbb{Z}, ps - qr = 1)$ is a meridian of $T(p, q)$ and similarly $l = x^{-p}m^{pq} = y^{-q}m^{pq}$ is a longitude.

It is seen [1, 5] that the set of the conjugacy classes of the irreducible representations of $\pi_1(M_n)$ in $SL(2; \mathbb{C})$ is finite. Any conjugacy class can be represented by $\rho_{(a,b,k)} : \pi_1(M_n) \to SL(2; \mathbb{C})$ for some triple $(a, b, k)$ such that

1. $0 < a < p, 0 < b < q, a \equiv b \mod 2$,
2. $0 < k < N = \left| pqn + 1 \right|, k \equiv na \mod 2$,
3. $\text{tr}(\rho_{(a,b,k)}(x)) = 2 \cos \frac{a\pi}{p}$,
4. $\text{tr}(\rho_{(a,b,k)}(y)) = 2 \cos \frac{b\pi}{q}$,
5. $\text{tr}(\rho_{(a,b,k)}(m)) = 2 \cos \frac{k\pi}{N}$.

Furthermore Johnson computed $\tau_{\rho_{(a,b,k)}}(M_n)$ as follows.

**Theorem 3.1** (Johnson).

1. A representation $\rho_{(a,b,k)}$ is acyclic if and only if $a \equiv b \equiv 1$. 
For any acyclic representation $\rho_{(a,b,k)}$ with $a \equiv b \equiv 1$, then one has

$$\tau_{\rho_{(a,b,k)}}(M_n) = \frac{1}{2 \left(1 - \cos \frac{2\pi a}{p}\right) \left(1 - \cos \frac{2\pi b}{q}\right) \left(1 + \cos \frac{p q k \pi}{N}\right)}.$$  

4. Main theorem

In this section we give a formula of the torsion polynomial $\sigma_{(p,q,n)}(t)$ for $M_n = \Sigma(p, q, N)$ obtained by a $\frac{1}{n}$-Dehn surgery along $T(p, q)$. Now we define torsion polynomials as follows.

**Definition 4.1.** A one variable polynomial $\sigma_{(p,q,n)}(t)$ is called the torsion polynomial of $M_n$ if the zero set coincides with the set of all non trivial values $\{\tau_{\rho}(M_n) | \tau_{\rho}(M_n) \neq 0\}$ and it satisfies the following normalization condition as

$$\sigma_{(p,q,n)}(0) = \begin{cases} 
(-1)^{(N-1)(p-1)} \frac{1}{8} & p \text{ is even, } q \text{ is odd}, \\
(-1)^{(N-1)(p-1)} \frac{1}{8} & q \text{ is even, } q \text{ is odd}, \\
(-1)^{(N-1)(p-1)(q-1)} & p, q \text{ are odd, } n \text{ is even}, \\
(-1)^{(N-1)(p-1)(q-1)} \frac{1}{8} & p, q \text{ are odd, } n \text{ is odd} 
\end{cases}$$

where $N = |pqn + 1|$.

**Remark 4.2.**

1. For $M_0 = S^3$, the torsion polynomial $\sigma_{(p,q,0)}(t)$ is defined by $\sigma_{(p,q,0)}(t) = 1$.
2. In the case that $p = 2p'$ is even and $p'$ is odd, then this normalization condition coincides with the one in [5].

From here assume $n \neq 0$. Recall Johnson’s formula

$$\frac{1}{\tau_{\rho_{(a,b,k)}}(M_n)} = 2 \left(1 - \cos \frac{2\pi a}{p}\right) \left(1 - \cos \frac{2\pi b}{q}\right) \left(1 + \cos \frac{p q k \pi}{N}\right),$$

where $0 < a < p, 0 < b < q, a \equiv b \equiv 1 \mod 2, k \equiv n \mod 2$. Here by putting

$$C_{(p,q,a,b)} = \left(1 - \cos \frac{a\pi}{p}\right) \left(1 - \cos \frac{b\pi}{q}\right),$$

one has

$$\frac{1}{\tau_{\rho_{(a,b,k)}}(M_n)} = 4C_{(p,q,a,b)} \cdot \frac{1}{2} \left(1 + \cos \frac{p q k \pi}{N}\right).$$

Main result is the following.

**Theorem 4.3.** The torsion polynomial of $M_n$ is given by

$$\sigma_{(p,q,n)}(t) = \prod_{(a,b)} Y_{(a,b)}(t)$$
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where

\[
Y_{(n,a,b)}(t) = \begin{cases} 
\frac{T_{N+1}(s) - T_{N-1}(s)}{2(s^2 - 1)} & (p \text{ or } q \text{ is even}, n > 0), \\
- \frac{T_{N+1}(s) - T_{N-1}(s)}{2(s^2 - 1)} & (p \text{ or } q \text{ is even}, n < 0), \\
\frac{T_{N+1}(s) - T_{N-1}(s)}{2(s^2 - 1)^2} & (p, q \text{ are odd}, n \text{ is even}, n > 0), \\
- \frac{T_{N+1}(s) - T_{N-1}(s)}{2(s^2 - 1)^2} & (p, q \text{ are odd}, n \text{ is even}, n < 0), \\
T_N(s) & (p, q, n \text{ are odd}).
\end{cases}
\]

Here

- \( T_t(x) \) is the \( l \)-th Tchebychev polynomial of the first kind.
- \( s = \sqrt{t} \).
- \( C_{(p,q,a,b)} = (1 - \cos \frac{a\pi}{p})(1 - \cos \frac{b\pi}{q}) \).
- a pair of integers \((a, b)\) is satisfying the following conditions:
  - \( 0 < a < p, 0 < b < q \),
  - \( a \equiv b \equiv 1 \mod 2 \).

**Remark 4.4.** Recall that the \( l \)-th Tchebychev polynomial \( T_t(x) \) is defined by \( T_t(\cos \theta) = \cos(l\theta) \).

**Proof.** We consider the following;

\[
X_n(x) = \begin{cases} 
\frac{T_{N+1}(x) - T_{N-1}(x)}{2(x^2 - 1)} & (n > 0), \\
- \frac{T_{N+1}(x) - T_{N-1}(x)}{2(x^2 - 1)} & (n < 0).
\end{cases}
\]

\[
X'_n(x) = T_N(x).
\]

First we assume \( p = 2p' \) is even. For the case that \( p' \) is odd, then it is proved in \([5]\). Then we suppose that \( p' \) is even. Here \( N = |2p'qn + 1| \) is always odd.

Case 1: \( p = 2p', p' \) is even and \( n > 0 \)

We modify one factor \((1 + \cos \frac{2p'k\pi}{N})\) of \( \frac{1}{\tau_p(M_n)} \) as follows. See \([5]\) for the proof.

**Lemma 4.5.** The set \( \{\cos \frac{2p'k\pi}{N} | 0 < k < N, k \equiv n \mod 2\} \) is equal to the set \( \{\cos \frac{2p'k\pi}{N} | 0 < k < \frac{N-1}{2}\} \).

Now we can modify

\[
\frac{1}{2} \left(1 + \cos \frac{2p'k\pi}{N}\right) = \frac{1}{2} \cdot 2 \cos^2 \frac{2p'k\pi}{2N} = \cos^2 \frac{p'k\pi}{N}.
\]
We put
\[ z_k = \cos \frac{p'k\pi}{N} \quad (1 \leq k \leq N - 1). \]
By the definition, it is seen
\[ z_{N-k} = \cos \frac{p'(N - k)\pi}{N} \]
\[ = \cos(p'\pi - \frac{p'k\pi}{N}) \]
\[ = z_k \]
because \( p' \) is even.
Therefore it is enough to consider only \( z_k \) \((1 \leq k \leq \frac{N-1}{2})\).
Now we substitute \( x = z_k \) to \( T_{N+1}(x) \). Then one has
\[ T_{N+1}(z_k) = \cos \left( (N + 1) \frac{p'k\pi}{N} \right) \]
\[ = \cos \frac{p'k\pi}{N} \]
\[ = z_k \]
and
\[ T_{N-1}(z_k) = \cos \left( (N - 1) \frac{p'k\pi}{N} \right) \]
\[ = \cos \frac{p'k\pi}{N} \]
\[ = z_k. \]
Hence it holds
\[ T_{N+1}(z_k) - T_{N-1}(z_k) = 0. \]
By properties of Tchebychev polynomials, it is seen that
\[ T_{N+1}(1) - T_{N-1}(1) = 0, \]
\[ T_{N+1}(-1) - T_{N-1}(-1) = 0. \]
We remark that the degree of \( X_n(x) = \frac{T_{N+1}(x) - T_{N-1}(x)}{2(x^2 - 1)} \) is \( N - 1 \) and \( z_1, \cdots, z_{\frac{N-1}{2}} \) are zeros. Because both of \( T_{N+1}(x) \) and \( T_{N-1}(x) \) are even functions, then \(-z_1, \cdots, -z_{\frac{N-1}{2}}\) are also zeros of \( X_n(x) \). Hence \( X_n(x) \) is a functions of \( x^2 \). Here by replacing \( x \) by \( \frac{\sqrt{t}}{2\sqrt{C_{(p,q,a,b)}}} \), the degree of \( Y_{(n,a,b)}(t) \) is \( \frac{N-1}{2} \), and the roots of \( Y_{(n,a,b)}(t) \) are
\[ 4C_{(p,q,a,b)}z_k^2 = 4C_{(p,q,a,b)} \cos^2 \frac{\pi k}{N} \quad \left( 0 < k < \frac{N - 1}{2} \right), \]
which are all non trivial values of \( \frac{1}{\tau_{(a,b)}(M_n)}. \)
Here we check the normalization condition. By the definition of \(Y_{(n,a,b)}(t)\) and properties of \(T_{N+1}(x), T_{N-1}(x)\), one has
\[
Y_{(n,a,b)}(0) = \frac{T_{N+1}(0) - T_{N-1}(0)}{2(0 - 1)} = \frac{(-1)^{\frac{N+1}{2}} - (-1)^{\frac{N-1}{2}}}{2} = (-1)^{\frac{N+1}{2}}.
\]
Hence it can be seen
\[
\sigma_{(p,q,n)}(0) = \prod_{(a,b)} (-1)^{\frac{N+1}{2}} = \prod_{(a,b)} \left( (-1)^{\frac{N+1}{2}} \right)^{\frac{N(p+1)}{4}} = (-1)^{(N-1)p(q-1)}.
\]
Therefore we obtain the formula.

Case 2: \(p = 2p'\) and \(n < 0\)
In this case we modify \(N = |2p'qn + 1| = 2p'|q|n| - 1\). By the same arguments, it is easy to see the claim of the theorem is proved.

Next assume both of \(p, q\) are odd integers.

Case 3: \(p, q\) are odd and \(n\) is even
If \(n\) is even, then \(N = |pqn + 1|\) is odd. Then the similar arguments in [5] work well. Then it can be proved.

Case 4: \(p, q\) are odd and \(n\) is odd
Suppose \(n\) is positive. First note that \(N = |pqn + 1|\) is even. We can modify one factor \((1 + \cos \frac{pk\pi}{N})\) of \(\frac{1}{\tau_p(M_n)}\) as follows. It is clear because \((q, N) = 1\).

**Lemma 4.6.** The set \(\{\cos \frac{pk\pi}{N} | 0 < k < N, k \equiv n \text{ mod } 2\}\) is equal to the set \(\{\cos \frac{b\pi}{N} | 0 < k < N, k \equiv 1 \text{ mod } 2\}\).

Now we can modify
\[
\frac{1}{2} \left( 1 + \cos \frac{pk\pi}{N} \right) = \frac{1}{2} \cdot 2 \cos^2 \frac{pk\pi}{2N} = \cos^2 \frac{pk\pi}{2N}.
\]
We put
\[
z'_k = \cos \frac{pk\pi}{2N} (1 \leq k \leq N - 1, k \equiv 1 \text{ mod } 2).
\]
Here we substitute $x = z'_k \ (1 \leq k \leq \frac{N-1}{2}, k \equiv 1 \mod 2)$ to $T_N(x)$. Then one has
\[
T_N(z'_k) = \cos \left( \frac{N(pk\pi)}{2N} \right) \\
= \cos \left( \frac{pk\pi}{2} \right) \\
= 0
\]
because $pk$ is odd.
Similarly it can be also seen that
\[
T_N(-z'_k) = 0.
\]
We mention that the degree of $X'_n(x) = T_N(x)$ is $N$ and $\pm z'_1, \cdots, \pm z'_{N-1}$ are the zeros. Because $X'_n(x)$ is a functions of $x^2$. Here by replacing $x$ by \(\frac{\sqrt{N}}{2\sqrt{C(p,q,a,b)}}\), Here it holds that its degree of $Y_{(n,a,b)}(t)$ is $\frac{N-1}{2}$, and the roots of $Y_{(n,a,b)}(t)$ are
\[
4C(p,q,a,b)z^2_k = 4C(p,q,a,b)\cos^2 \frac{\pi k}{N} \left( 0 < k < \frac{N - 1}{2} \right),
\]
which are all non trivial values of $\frac{1}{\tau_{(n,a,b)}(M_n)}$.
Finally we can check the normalization condition as follows. By the definition of $Y_{(n,a,b)}(t)$, one has
\[
Y_{(n,a,b)}(0) = T_N(0) \\
= (-1)^{\frac{N}{2}}
\]
and
\[
s_{(p,q,n)}(0) = \prod_{(a,b)} (-1)^{\frac{N}{2}} \\
= \left( (-1)^{\frac{N}{2}} \right)^{\frac{(p-1)(q-1)}{2}} \\
= (-1)^{\frac{N(p-1)(q-1)}{4}}.
\]
Therefore we obtain the formula.
In the case that $n$ is negative, then it can be proved by similar arguments. Therefore this completes the proof. \(\square\)

**Remark 4.7.** By defining as $X_0(t) = 1$, it implies $Y_{(0,a,b)}(t) = 1$. Then the above statement is true for $n = 0$.

By direct computation, one obtains the following corollary.
Corollary 4.8. The degree \( \deg(\sigma_{(p,q,n)}(t)) \) is given by

\[
\deg(\sigma_{(p,q,n)}(t)) = \begin{cases} 
\frac{(N-1)p(q-1)}{8} & (p \text{ even, } q \text{ odd}), \\
\frac{(N-1)(p-1)q}{8} & (p \text{ odd, } q \text{ even}), \\
\frac{(N-1)(p-1)q}{8} & (p, q \text{ odd, } n \text{ even}), \\
\frac{N(p-1)(q-1)}{8} & (p, q \text{ odd, } n \text{ odd}).
\end{cases}
\]

We mention the 3-term relations. For each factor of \( Y_{(n,a,b)}(t) \) of \( \sigma_{(p,q,n)}(t) \), there exists the following relation.

**Proposition 4.9.**

1. Assume one of \( p \) and \( q \) is even. For any \( n \), it holds that

\[
Y_{(n+1,a,b)}(t) = D(t)Y_{(n,a,b)}(t) - Y_{(n-1,a,b)}(t)
\]

where \( D(t) = 2T_{pq} \left( \frac{\sqrt{\sigma}}{2\sqrt{\epsilon_{pq,a,b}}} \right) \).

2. Assume both of \( p, q \) are odd. For any \( n \), it holds that

\[
Y_{(n+2,a,b)}(t) = D(t)Y_{(n,a,b)}(t) - Y_{(n-2,a,b)}(t)
\]

where \( D(t) = 2T_{2pq} \left( \frac{\sqrt{\sigma}}{2\sqrt{\epsilon_{2pq,a,b}}} \right) \).

**Proof.** Here we need to consider \( N = |pqn + 1| \) is a function of \( n \in \mathbb{Z} \) for fixed \( p, q \). Then we write \( N(n) \) for \( N \) in this proof. The proof for the first case is essentially the same one for the 3-term relations [5]. We give the proof only for the second case. Recall the following property of Tchebychev polynomials

\[
2T_m(x)T_n(x) = T_{m+n}(x) + T_{m-n}(x)
\]

for any \( m, n \in \mathbb{Z} \).

Case 1: \( n \) is even

If \( n > 0 \) one has

\[
2T_{2pq}(x)X_n(x) = 2T_{2pq}(x)X_{N(n)+1}(x) - T_{N(n)-1}(x)
\]

\[
= T_{pq(n+1)+1+2pq}(x) + T_{pq(n+1)+1-2pq}(x) - (T_{pq(n+1)+1+2pq}(x) + T_{pq(n+1)+1-2pq}(x))
\]

\[
= T_{pq(n+2)+1+1}(x) - T_{pq(n+2)+1-1}(x) + T_{pq(n-2)+1+1}(x) - T_{pq(n-2)+1-1}(x)
\]

\[
= T_{N(n+2)+1}(x) - T_{N(n+2)-1}(x) + T_{N(n-2)+1}(x) - T_{N(n-2)-1}(x)
\]

\[
= X_{n+2}(x) + X_{n-2}(x).
\]
Therefore it can be seen that
\[ X_{n+2}(x) = 2T_{2pq}(x)X_{n}(x) - X_{n-2}(x) \]
and
\[ Y_{(n+2,a,b)}(t) = 2T_{2pq}\left(\frac{\sqrt{t}}{2\sqrt{C_{(2p,q,a,b)}}}\right) Y_{(n,a,b)}(t) - Y_{(n-2,a,b)}(t). \]
If \( n < 0 \), it can be also proved by the above argument.

Case 2: \( n \) is odd
If \( n > 0 \), one has
\[ 2T_{2pq}(x)X'_{n}(x) = 2T_{2pq}(x)T_{N(n)}(x) \]
\[ = T_{pq(n+1)2pq}(x) + T_{pq(n+1-2pq}(x) \]
\[ = T_{pq(n+2)+1}(x) + T_{pq(n-2)+1}(x) \]
\[ = T'_{N(n+2)}(x) + T_{N(n-2)}(x) \]
\[ = X'_{n+2}(x) + X'_{n-2}(x). \]
Therefore it can be seen that
\[ X'_{n+2}(x) = 2T_{2pq}(x)X'_{n}(x) - X'_{n-2}(x) \]
and
\[ Y_{(n+2,a,b)}(t) = 2T_{2pq}\left(\frac{\sqrt{t}}{2\sqrt{C_{(2p,q,a,b)}}}\right) Y_{(n,a,b)}(t) - Y_{(n-2,a,b)}(t). \]
If \( n < 0 \), it can be also proved.
This completes the proof of this proposition. \( \square \)

5. Examples

Finally we give some examples.

**Example 5.1.** Put \( p = 4, q = 3 \). Now \( N = |12n + 1| \). In this case \((a, b) = (1, 1), (3, 1)\). By applying the main theorem, one has
\[ \sigma_{(4,3,-1)}(t) = 34359738368t^{10} - 77309411328t^{9} + 66840428544t^{8} \]
\[ - 28655484928t^{7} + 6677331968t^{6} - 882900992t^{5} + 66371584t^{4} \]
\[ - 2723840t^{3} + 55680t^{2} - 480t + 1. \]
\[ \sigma_{(4,3,0)}(t) = 1. \]
\[ \sigma_{(4,3,1)}(t) = 439804651104t^{12} - 12094627905536t^{11} + 13434657701888t^{10} \]
\[ - 7859790151680t^{9} + 2670664351744t^{8} - 552909930496t^{7} \]
\[ + 71319945216t^{6} - 5727322112t^{5} + 278757376t^{4} \]
\[ - 7741440t^{3} + 110208t^{2} - 672t + 1. \]
Example 5.2. Put $p = 3, q = 5$. Now $N = |15n + 1|$. In this case $(a, b) = (1, 1), (1, 3)$. For any odd number $n$, one has

$$
\sigma_{(3,5,0)}(t) = Y_{(n,1,1)}(t)Y_{(n,1,3)}(t)
= T_N \left( \frac{\sqrt{t}}{2 \sqrt{C_{(3,5,1,1)}}} \right) Y_N \left( \frac{\sqrt{t}}{2 \sqrt{C_{(3,5,1,3)}}} \right).
$$

By applying the main theorem, we obtain

$$
\sigma_{(3,5,-1)}(t) = 18014398509481984t^{14} - 47287796087390208t^{13} + 51721026970583040t^{12}
- 3084789822883456t^{11} + 1108500135333068t^{10} - 2520389888507904t^9
+ 372923420377088t^8 - 36436086620160t^7 + 2352597696512t^6
- 98837200896 + 260502322t^4 - 40341504t^3 + 329280t^2 - 1176t + 11.
\sigma_{(3,5,0)}(t) = 1.
\sigma_{(3,5,1)}(t) = 4611686018427387904t^{16} - 13835058055282163712t^{15}
+ 17726168133330272256t^{14} - 12754194144713244672t^{13}
+ 5718164151876976640t^{12} - 1682516673287946240t^{11}
+ 334779300425236480t^{10} - 45872724622442496t^9
+ 4367893693202432t^8 + 288911712583680t^7
+ 13126896451584r^6 - 399582953472t^5
+ 7798652928t^4 - 90832896t^3 + 563200t^2 - 1536t + 1.
$$

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