On the Classification of Homogeneous Hypersurfaces in Complex Space

A. V. Isaev

We discuss a family $M_n^t$, with $n \geq 2$, $t > 1$, of real hypersurfaces in a complex affine $n$-dimensional quadric arising in connection with the classification of homogeneous compact simply-connected real-analytic hypersurfaces in $\mathbb{C}^n$ due to Morimoto and Nagano. To finalize their classification, one needs to resolve the problem of the embeddability of $M_n^t$ in $\mathbb{C}^n$ for $n = 3, 7$. We show that $M_7^t$ is not embeddable in $\mathbb{C}^7$ for every $t$ and that $M_3^t$ is embeddable in $\mathbb{C}^3$ for all $1 < t < 1 + 10^{-6}$. As a consequence of our analysis of a map constructed by Ahern and Rudin, we also conjecture that the embeddability of $M_3^t$ takes place for all $1 < t < \sqrt{(2 + \sqrt{2})/3}$.

1 Introduction

For $n \geq 2$, consider the $n$-dimensional affine quadric in $\mathbb{C}^{n+1}$:

$$Q^n := \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : z_1^2 + \ldots + z_{n+1}^2 = 1\}. \quad (1.1)$$

The group $\text{SO}(n+1, \mathbb{R})$ acts on $Q^n$, and the orbits of this action are the sphere $S^n = Q^n \cap \mathbb{R}^{n+1}$ as well as the compact strongly pseudoconvex hypersurfaces

$$M_n^t := \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : |z_1|^2 + \ldots + |z_{n+1}|^2 = t\} \cap Q^n, \; t > 1. \quad (1.2)$$

These hypersurfaces play an important role in the classical paper [MN], where the authors set out to determine all compact simply-connected real-analytic hypersurfaces in $\mathbb{C}^n$ homogeneous under an action of a Lie group by CR-transformations. They showed that every such hypersurface is CR-equivalent to either the sphere $S^{2n-1}$ or, for $n = 3, 7$, to the manifold $M_t^n$ for some $t$. However, the question of the existence of a real-analytic CR-embedding of $M_t^n$ in $\mathbb{C}^n$ for $n = 3, 7$ was not clarified, thus the classification in these two dimensions was not fully completed.

In this paper, we first discuss the family $M_t^n$ for all $n \geq 2$, $t > 1$ (see Section 2). We observe that a necessary condition for the existence of a real-analytic CR-embedding of $M_t^n$ in $\mathbb{C}^n$ is the embeddability of the sphere

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$S^n$ in $\mathbb{C}^n$ as a totally real submanifold (see Corollary 2.3). The problem of the existence of a totally real embedding of $S^n$ in $\mathbb{C}^n$ was considered by Gromov (see [Gr1] and p. 193 in [Gr2]), Stout-Zame (see [SZ]), Ahern-Rudin (see [AR]), Forstnerič (see [F1]–[F3]). In particular, it has turned out that $S^n$ admits a totally real embedding in $\mathbb{C}^n$ only for $n = 3$, hence Corollary 2.3 implies that $M^n_t$ cannot be real-analytically CR-embedded in $\mathbb{C}^n$ for all $t$ if $n \neq 3$. For $n \neq 3, 7$ the non-embeddability of $M^n_t$ was established in [MN] by a different method, whereas for $n = 7$ it appears to be a new result (see Corollary 2.4). Furthermore, since $S^3$ is a totally real submanifold of $Q^3$, any real-analytic totally real embedding of $S^3$ into $\mathbb{C}^3$ extends to a biholomorphic map defined in a neighborhood of $S^3$ in $Q^3$. Owing to the fact that $M^3_t$ accumulate to $S^3$ as $t \to 1$, this observation implies that $M^3_t$ admits a real-analytic CR-embedding in $\mathbb{C}^3$ for all $t$ sufficiently close to 1. Thus, the classification of homogeneous compact simply-connected real-analytic hypersurfaces in complex dimension 3 is special as it includes manifolds other than the sphere $S^5$.

Next, in [AR] an explicit polynomial totally real embedding $f$ of $S^3$ into $\mathbb{C}^3$ was constructed, and in Section 3 we study the holomorphic continuation $F : \mathbb{C}^4 \to \mathbb{C}^3$ of $f$ in order to determine the interval of parameter values $1 < t < t_0$ for which the extended map yields an embedding of $M^3_t$ into $\mathbb{C}^3$. It turns out that the interval is rather small, namely $t_0 \leq \sqrt{(2 + \sqrt{2})/3} \approx 1.07$ (see Proposition 3.1). A lower bound for $t_0$ is given in Theorem 3.2, where we show that $t_0 \geq 1 + 10^{-6}$. Further, our discussion of the fibers of the map $F$ leads to the conjecture asserting that one can in fact take $t_0 = \sqrt{(2 + \sqrt{2})/3}$ (see Conjecture 3.4). For all other values of $t$ the problem of the embeddability of $M^3_t$ remains completely open.

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2 The family $M^n_t$

We start by discussing the family $M^n_t$ for all $n \geq 2$, $t > 1$. First of all, we note that the hypersurfaces in the family are all pairwise CR-nonequivalent (see Example 13.9 in [KZ]). Next, computing the isotropy subgroups of the $\text{SO}(n + 1, \mathbb{R})$-action on $M^n_t$, one observes that $M^n_t$ is diffeomorphic to $\text{SO}(n + 1, \mathbb{R})/\text{SO}(n - 1, \mathbb{R})$. On the other hand, from (1.1), (1.2) we see that

\[M^n_t = \left\{ x + iy \in \mathbb{C}^{n+1} : ||x|| = \sqrt{\frac{t+1}{2}}, ||y|| = \sqrt{\frac{t-1}{2}}, (x,y) = 0 \right\},\]
where \(x, y \in \mathbb{R}^{n+1}\), hence \(M^t_i\) is diffeomorphic to the tangent sphere bundle over \(S^n\). It then follows that \(\pi_1(M^t_i) = 0\) if \(n \geq 3\), \(\pi_1(M^2_t) \simeq \mathbb{Z}_2\), and that \(M^n_t\) accumulate to \(S^n\) as \(t \to 1\). Note also that \(M^2_t\) is a double cover of the following well-known homogeneous hypersurface in \(\mathbb{C}P^2\) discovered by É. Cartan (see [C]):

\[
\left\{ (\zeta_1 : \zeta_2 : \zeta_3) \in \mathbb{C}P^2 : |\zeta_1|^2 + |\zeta_2|^2 + |\zeta_3|^2 = t|\zeta_1^2 + \zeta_2^2 + \zeta_3^2| \right\}.
\]

For our arguments below we will utilize the strongly pseudoconvex Stein domains in \(Q^n\) bounded by \(M^n_t\):

\[
D^n_t := \{ (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : |z_1|^2 + \cdots + |z_{n+1}|^2 < t \} \cap Q^n, \ t > 1.
\]

The sphere \(S^n\) lies in \(D^n_t\) for all \(t\) and is a strong deformation retract of \(D^n_t\). Indeed, the following map from \(D^n_t \times [0, 1]\) to \(D^n_t\) is a strong deformation retraction of \(D^n_t\) to \(S^n\):

\[
(x + iy, s) \mapsto \frac{\sqrt{1 + s^2||y||^2}}{||x||}x + isy, \quad 0 \leq s \leq 1.
\]

In particular, \(D^n_t\) is simply-connected but not contractible.

**Remark 2.1** The domains \(D^n_t\) illustrate the relationship between the fundamental group of a smoothly bounded Stein domain and that of its (connected) boundary, which is important, for example, for the uniformization problem. Namely, if \(D\) is a smoothly bounded Stein domain, the fundamental groups \(\pi_1(D)\) and \(\pi_1(\partial D)\) are isomorphic in complex dimensions \(\geq 3\), whereas in dimension 2 there exists a surjective homomorphism \(\pi_1(\partial D) \to \pi_1(D)\) and the fundamental group of \(\partial D\) can be larger than that of \(D\) (see [NS] for a detailed discussion of these facts). Indeed, as we observed above, \(\pi_1(D^n_t) = 0\) for \(n \geq 2\), \(\pi_1(M^n_t) = 0\) for \(n \geq 3\), but \(\pi_1(M^2_t) \simeq \mathbb{Z}_2\). Examples of simply connected (in fact contractible) domains with non-simply connected boundaries exist even in the class of strongly pseudoconvex domains in \(\mathbb{C}^2\) although they are much harder to construct (see [Go]).

We will now turn to the problem of the CR-embeddability of \(M^n_t\) in \(\mathbb{C}^n\) and prove the following proposition.

**Proposition 2.2** Any real-analytic CR-embedding of \(M^n_t\) into \(\mathbb{C}^n\) extends to a biholomorphic mapping of \(D^n_t\) onto a domain in \(\mathbb{C}^n\).

**Proof:** Let \(\varphi : M^n_t \to \mathbb{C}^n\) be a real-analytic CR-embedding. Then \(\varphi\) extends to a biholomorphic map between a neighborhood of \(M^n_t\) in \(Q^n\) and a neighborhood of the real-analytic strongly pseudoconvex hypersurface \(M := \varphi(M^n_t)\) in \(\mathbb{C}^n\). Further, \(\varphi\) extends to a holomorphic map from \(D^n_t\) to \(\mathbb{C}^n\) (which
follows, for instance, from results of [KR]), and we denote the extension of \( \varphi \) to a neighborhood of \( \overline{D^n_t} \) also by \( \varphi \).

Next, let \( D \) be the strongly pseudoconvex domain in \( \mathbb{C}^n \) bounded by \( M \) and \( \psi : M \to M^n_t \) the inverse of \( \varphi \) on \( M^n_t \). As before, the map \( \psi \) extends to a biholomorphic map between a neighborhood of \( M \) in \( \mathbb{C}^n \) and a neighborhood of \( M^n_t \) in \( Q^n \). Furthermore, \( \psi \) extends to a holomorphic map from \( D \) to \( \mathbb{C}^{n+1} \). We denote the resulting extension of \( \psi \) to a neighborhood of \( \overline{D} \) also by \( \psi \). Clearly, the range of \( \psi \) lies in \( Q^n \).

It now follows that \( \varphi(D^n_t) = D, \, \psi(D) = D^n_t \) and \( \varphi, \, \psi \) are the inverses of each other. In particular, \( \varphi \) maps \( D^n_t \) biholomorphically onto \( D \), which completes the proof. \( \square \)

As an immediate consequence of Proposition 2.2 we obtain the following result.

**Corollary 2.3** If for some \( t > 1 \) the manifold \( M^n_t \) is real-analytically CR-embeddable in \( \mathbb{C}^n \), then \( S^n \) admits a real-analytic totally real embedding in \( \mathbb{C}^n \).

It is not hard to see that for \( n \neq 3, 7 \) the sphere \( S^n \) does not admit a smooth totally real embedding in \( \mathbb{C}^n \) (see [Gr2], [SZ]). Indeed, multiplication by \( i \) establishes an isomorphism between the tangent and the normal bundles of any smooth totally real \( n \)-dimensional submanifold of \( \mathbb{C}^n \). On the other hand, the normal bundle to \( S^n \) induced by any smooth embedding in \( \mathbb{R}^{2n} \) is trivial (see Theorem 8.2 in [K]). Therefore, if \( S^n \) admits a smooth totally real embedding in \( \mathbb{C}^n \), it is parallelizable, which is impossible unless \( n = 3 \) or \( n = 7 \). Corollary 2.3 then yields that \( M^n_t \) cannot be real-analytically CR-embedded in \( \mathbb{C}^n \) for \( n \neq 3, 7 \). This last statement was obtained in [MN] by utilizing the facts that \( M^n_t \) is diffeomorphic to the sphere bundle over \( S^n \) and that a real-analytic embedding of \( M^n_t \) into \( \mathbb{C}^n \) induces a smooth embedding of the tangent bundle of \( S^n \) into \( \mathbb{R}^{2n} \), which again leads to the parallelizability of \( S^n \) (at least for \( n \geq 3 \)).

Further, the non-existence of a smooth totally real embedding of \( S^7 \) in \( \mathbb{C}^7 \) was first obtained in [SZ] by an argument relying on a result of [S], which states that any two smooth embeddings of \( S^n \) in \( \mathbb{R}^{2n} \) are regularly homotopic. Corollary 2.3 then yields:

**Corollary 2.4** No manifold in the family \( M^n_t \) admits a real-analytic CR-embedding in \( \mathbb{C}^7 \), hence every homogeneous compact simply-connected real-analytic hypersurface in \( \mathbb{C}^7 \) is CR-equivalent to \( S^{13} \).

In contrast, it turns out that \( S^3 \) can be embedded into \( \mathbb{C}^3 \) by a real-analytic CR-map. The first proof of this fact was given in [AR], where an explicit example of an embedding was constructed. Since \( S^3 \) is a totally
real submanifold of $Q^3$, any real-analytic totally real embedding of $S^3$ into $\mathbb{C}^3$ extends to a biholomorphic map defined in a neighborhood of $S^3$ in $Q^3$. The manifolds $M^3_t$ accumulate to $S^3$ as $t \to 1$, hence $M^3_t$ admits a real-analytic CR-embedding in $\mathbb{C}^3$ for all $t$ sufficiently close to 1. This shows that, interestingly, the classification of homogeneous compact simply-connected real-analytic hypersurfaces in $\mathbb{C}^3$ includes manifolds other than $S^5$. The embedding found in [AR] is a polynomial map on $\mathbb{R}^4 \subset \mathbb{C}^4$, hence it has a (unique) holomorphic continuation to all of $\mathbb{C}^4$. We will study this extended map in the next section.

**Remark 2.5** Observe that every hypersurface $M^a_t$ is non-spherical. Indeed, otherwise by results of [NS] the universal cover of the domain $D^a_t$ would be biholomorphic to the unit ball $B^n \subset \mathbb{C}^n$. Since $D^a_t$ is simply-connected, this would imply that $D^a_t$ is biholomorphic to $B^n$, which is impossible since $D^a_t$ is not contractible. Now, the non-sphericity and homogeneity of the hypersurface $M^a_t$ yield that it has no umbilic points. Therefore, every manifold $M^3_t$ embeddable in $\mathbb{C}^3$ provides an example of a compact strongly pseudo-convex simply-connected hypersurface in $\mathbb{C}^3$ without umbilic points. Such hypersurfaces have been known before, but the arguments required to obtain non-umbilicity for them are much more involved than the one given above. For example, the proof in [W] of the fact that every generic ellipsoid in $\mathbb{C}^n$ for $n \geq 3$ has no umbilic points relies on the Chern-Moser theory (note for comparison that every ellipsoid in $\mathbb{C}^2$ has at least four umbilic points – see [HJ]).

## 3 The holomorphic continuation of the Ahern-Rudin map

In this section we study the holomorphic continuation of the totally real embedding of $S^3$ into $\mathbb{C}^3$ constructed in [AR]. Let $(z, w)$ be coordinates in $\mathbb{C}^2$ and let $S^3$ be realized in the standard way as the subset of $\mathbb{C}^2$ given by

$$S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}.$$ 

The Ahern-Rudin map is defined on all of $\mathbb{C}^2$ as follows:

$$f : \mathbb{C}^2 \to \mathbb{C}^3, \quad f(z, w) := (z, w, w\overline{z^2w} + iz\overline{z^2w}). \quad (3.1)$$

Now, consider $\mathbb{C}^4$ with coordinates $z_1, z_2, z_3, z_4$ and embed $\mathbb{C}^2$ into $\mathbb{C}^4$ as the totally real subspace $\mathbb{R}^4$:

$$(z, w) \mapsto (\text{Re } z, \text{Im } z, \text{Re } w, \text{Im } w).$$
Clearly, the push-forward of the polynomial map \( f \) extends from \( \mathbb{R}^4 \) to a holomorphic map \( F : \mathbb{C}^4 \to \mathbb{C}^3 \) by the formula

\[
F(z_1, z_2, z_3, z_4) := \left( z_1 + iz_2, z_3 + iz_4, (z_3 + iz_4)(z_1 - iz_2)(z_3 - iz_4)^2 + i(z_1 + iz_2)(z_1 - iz_2)^2(z_3 - iz_4) \right).
\]

It will be convenient for us to argue in the coordinates

\[
w_1 := z_1 + iz_2, \quad w_2 := z_1 - iz_2, \quad w_3 := z_3 + iz_4, \quad w_4 := z_3 - iz_4.
\]

(3.2)

In these coordinates the quadric \( Q^3 \) takes the form

\[
\{ (w_1, w_2, w_3, w_4) \in \mathbb{C}^4 : w_1w_2 + w_3w_4 = 1 \},
\]

(3.3)

the sphere \( S^3 \subset Q^3 \) the form

\[
\{ (w_1, w_2, w_3, w_4) \in \mathbb{C}^4 : w_2 = \bar{w}_1, \ w_4 = \bar{w}_3 \} \cap Q^3,
\]

(3.4)

the hypersurface \( M^3_t \subset Q^3 \) the form

\[
\{ (w_1, w_2, w_3, w_4) \in \mathbb{C}^4 : |w_1|^2 + |w_2|^2 + |w_3|^2 + |w_4|^2 = 2t \} \cap Q^3,
\]

(3.5)

and the map \( F \) the form

\[
(w_1, w_2, w_3, w_4) \mapsto (w_1, w_3, w_2w_3w_4^2 + iw_1w_2^2w_4).
\]

(3.6)

We will study the map \( F \) in order to obtain some evidence regarding the values of \( t \) for which the manifold \( M^3_t \) is embeddable in \( \mathbb{C}^3 \). Clearly, \( F \) defines an embedding of \( M^3_t \) if its restriction \( \tilde{F} := F|_{Q^3} \) is non-degenerate and injective on \( M^3_t \), therefore it is important to investigate the non-degeneracy and injectivity properties of \( \tilde{F} \). First of all, observe that \( F \) has maximal rank at every point of \( Q^3 \) since its Jacobian matrix is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
iw_2^2w_4 & w_3w_4^2 + 2iw_1w_2w_4 & w_2w_4^2 & 2w_3w_4w_4 + iw_1w_2^2 \\
\end{pmatrix},
\]

and the entries \( w_3w_4^2 + 2iw_1w_2w_4, 2w_3w_4w_4 + iw_1w_2^2 \) cannot simultaneously vanish on \( Q^3 \). However, as the following proposition shows, most manifolds \( M^3_t \) contain points at which the restricted map \( \tilde{F} \) degenerates.

**Proposition 3.1** The map \( \tilde{F} \) degenerates at some point of \( M^3_t \) if and only if

\[
t \geq \sqrt{(2 + \sqrt{2})/3}.
\]
Proof: Observe that \(|w_1| + |w_3| > 0\) on \(Q^3\). For \(w_1 \neq 0\) we choose \(w_1, w_3, w_4\) as local coordinates on \(Q^3\) and write the third component of \(\tilde{F}\) as

\[
\varphi := \frac{1 - w_3 w_4}{w_1} (i w_4 + (1 - i) w_3 w_4^2)
\]

(see (3.3), (3.6)). Then the Jacobian \(J_{\tilde{F}}\) of \(\tilde{F}\) is equal to

\[
\frac{\partial \varphi}{\partial w_4} = \frac{(3i - 3w_3^2 w_4^2 + (2 - 4i) w_3 w_4 + i w_1)}{w_1},
\]

(3.7)

hence it vanishes if and only if

\[
w_3 w_4 = \frac{3 \pm \sqrt{2} - i}{6}.
\]

At such points we have

\[
|w_1|^2 + |w_2|^2 + |w_3|^2 + |w_4|^2 = |w_1|^2 + \frac{2 \pm \sqrt{2}}{6|w_1|^2} + |w_3|^2 + \frac{2 \pm \sqrt{2}}{6|w_3|^2}.
\]

(3.8)

Analogously, if \(w_3 \neq 0\), we choose \(w_1, w_2, w_3\) as local coordinates on \(Q^3\) and write the third component of \(\tilde{F}\) as

\[
\psi := \frac{1 - w_1 w_2}{w_3} (w_2 + (i - 1) w_1 w_2^2)
\]

(see (3.3), (3.6)). Then

\[
J_{\tilde{F}} = -\frac{\partial \psi}{\partial w_2} = \frac{-(3 - 3i)w_1^2 w_2^2 + (2i - 4) w_1 w_2 + 1}{w_3},
\]

which vanishes if and only if

\[
w_1 w_2 = \frac{3 \pm \sqrt{2} + i}{6}.
\]

Hence for all points of degeneracy of \(\tilde{F}\) we have \(w_1 \neq 0, w_3 \neq 0\), and therefore such points are described as the zeroes of \(\partial \varphi / \partial w_4\) or, equivalently, as the zeroes of \(\partial \psi / \partial w_2\).

Now, investigating the behavior of the function

\[
x + \frac{2 + \sqrt{2}}{6x} + y + \frac{2 - \sqrt{2}}{6y}
\]

for \(x, y > 0\), one can deduce from (3.8) that \(J_{\tilde{F}}\) vanishes at a point of \(M^3_t\) if and only if \(t \geq \sqrt{(2 + \sqrt{2})/3}\) as claimed. \(\Box\)

As we noted above, any real-analytic totally real embedding of \(S^3\) in \(C^3\) yields an embedding of \(M^3_t\) for all \(t\) sufficiently close to 1. Define

\[
t_0 := \sup\{t : \tilde{F}|_{M^3_s} \text{ is an embedding for all } 1 < s \leq t\}.
\]

Proposition 3.1 implies \(t_0 \leq \sqrt{(2 + \sqrt{2})/3}\). We will now give a lower bound for \(t_0\).
THEOREM 3.2  One has $t_0 \geq 1 + 10^{-6}$.

Proof: For $0 < \varepsilon < 1$ define

$$U_\varepsilon := \{(w_1, w_2, w_3, w_4) \in \mathbb{C}^4 : |w_2 - \overline{w}_1| < \varepsilon, |w_4 - \overline{w}_3| < \varepsilon\} \cap Q^3.$$  

Clearly, $U_\varepsilon$ is a neighborhood of $S^3$ in $Q^3$ (see (3.4)). We will find $\varepsilon$ such that $\overline{F}$ is biholomorphic on $U_\varepsilon$. Writing any point in $Q^3$ as

$$W = (w_1, \overline{w}_1 + \mu, w_3, \overline{w}_3 + \eta),$$

from (3.3) we observe

$$|w_1|^2 + |w_3|^2 + \mu w_1 + \eta w_3 = 1,$$

which implies

$$|w_1|^2 + |\overline{w}_1 + \mu|^2 + |w_3|^3 + |\overline{w}_3 + \eta|^2 = 2 + |\mu|^2 + |\eta|^2. \quad (3.10)$$

Clearly, $W$ lies in $U_\varepsilon$ if and only if $|\mu| < \varepsilon, |\eta| < \varepsilon$. It then follows from (3.5), (3.10) that $M^3_1 \subset U_\varepsilon$ for all $1 < t < 1 + \varepsilon^2/2$. Below we will see that one can take $\varepsilon = \sqrt{2} \cdot 10^{-3}$, which will then imply the theorem.

In order to choose $\varepsilon$, we study the fibers of the map $\overline{F}$. Let two points $W = (w_1, w_2, w_3, w_4)$ and $\overline{W} = (\overline{w}_1, \overline{w}_2, \overline{w}_3, \overline{w}_4)$ lie in $Q^3$ and assume that $\overline{F}(W) = \overline{F}(\overline{W})$. Then from (3.6) one immediately has $\hat{w}_1 = w_1, \hat{w}_3 = w_3$ and

$$\hat{w}_2 w_3 \overline{w}_4^2 + i w_1 \overline{w}_2^2 \overline{w}_4 = w_2 w_3 w_4^2 + i w_1 w_2^2 w_4. \quad (3.11)$$

If $w_1 = 0$ or $w_3 = 0$, then (3.11) implies $\hat{W} = W$, so we suppose from now on that $w_1 \neq 0$ and $w_3 \neq 0$. Then, using (3.3) we substitute

$$w_2 = \frac{1 - w_3 w_4}{w_1}, \quad \hat{w}_2 = \frac{1 - w_3 \hat{w}_4}{w_1} \quad (3.12)$$

into (3.11) and simplifying the resulting expression obtain

$$(\hat{w}_4 - w_4) \left[(i - 1)w_3^2 \hat{w}_4^2 + ((1 - 2i)w_3 + (i - 1)w_3^2 w_4) \hat{w}_4 + \right.$$

$$\left. (i + (1 - 2i)w_3 w_4 + (i - 1)w_3^2 w_4^2) \right] = 0. \quad (3.13)$$

We treat identity (3.13) as an equation with respect to $\hat{w}_4$. The solution $\hat{w}_4 = w_4$ leads to the point $W$ (see (3.12)). Further, the quadratic equation

$$(i - 1)w_3^2 \hat{w}_4^2 + ((1 - 2i)w_3 + (i - 1)w_3^2 w_4) \hat{w}_4 +$$

$$\left( i + (1 - 2i)w_3 w_4 + (i - 1)w_3^2 w_4^2 \right) = 0. \quad (3.14)$$
has the following solutions:

\[
\hat{w}_4 = \frac{2i - 1 + (1 - i)w_3w_4 + \sqrt{6iw_3^2w_4^2 - (2 + 6i)w_3w_4 + 1}}{(2i - 2)w_3}. \tag{3.15}
\]

Our goal now is to choose \(\varepsilon\) so that for \(W \in U_\varepsilon\) neither of the points \(\hat{W}\) defined by solutions (3.15) lies in \(U_\varepsilon\). We write \(w_2 = \bar{w}_1 + \mu, w_4 = \bar{w}_3 + \eta, \hat{w}_4 = \bar{w}_3 + \hat{\eta}\) and show that \(\varepsilon\) can be taken so that if \(|\mu| < \varepsilon, |\eta| < \varepsilon\), then \(|\hat{\eta}| \geq \varepsilon\).

Formula (3.15) implies

\[
-8|w_3|^2 + 4 + i(24|w_3|^4 - 24|w_3|^2 + 4) + \\
\left[24i\eta|w_3|^2w_3 + 8i\eta^2w_3^2 - (4 + 12i)\eta w_3\right] = \\
-24i\hat{\eta}|w_3|^2w_3 - 8i(\hat{\eta}^2 + \eta\hat{\eta})w_3^2 + (4 + 12i)\hat{\eta}w_3. \tag{3.16}
\]

Observe that for any \(w_3\) one has

\[
\left|-8|w_3|^2 + 4 + i(24|w_3|^4 - 24|w_3|^2 + 4)\right| \geq \frac{4}{3}. \tag{3.17}
\]

Next, since \(\varepsilon < 1\), formula (3.9) yields

\[
|\bar{w}_1 + \frac{\mu}{2}|^2 + |\bar{w}_3 + \frac{\eta}{2}|^2 = 1 + \frac{|\mu|^2}{4} + \frac{|\eta|^2}{4} < 1 + \frac{\varepsilon^2}{2} < \frac{3}{2},
\]

which implies

\[
|w_3| < 2. \tag{3.18}
\]

Using (3.18), one can estimate the terms in square brackets in the left-hand side of (3.16) as follows:

\[
\left|24i\eta|w_3|^2w_3 + 8i\eta^2w_3^2 - (4 + 12i)\eta w_3\right| < 32\varepsilon^2 + 224\varepsilon. \tag{3.19}
\]

Similarly, using (3.18) for the right-hand side of (3.16) we have

\[
\left|-24i\hat{\eta}|w_3|^2w_3 - 8i(\hat{\eta}^2 + \eta\hat{\eta})w_3^2 + (4 + 12i)\hat{\eta}w_3\right| < 32|\hat{\eta}|^2 + 256|\hat{\eta}|. \tag{3.20}
\]

It follows from formulas (3.16), (3.17), (3.19), (3.20) that

\[
32|\hat{\eta}|^2 + 256|\hat{\eta}| > \frac{4}{3} - (32\varepsilon^2 + 224\varepsilon). \tag{3.21}
\]

Thus, in order to finalize the proof of the theorem, we need to choose \(\varepsilon\) so that inequality (3.21) implies \(|\hat{\eta}| \geq \varepsilon\). For example, let \(\varepsilon\) be such that

\[
32\varepsilon^2 + 224\varepsilon < \frac{1}{3}
\]
For instance, $\varepsilon = \sqrt{2} \cdot 10^{-3}$ satisfies this condition. Then from (3.21) one has

$$32|\hat{\eta}|^2 + 256|\eta| > 1,$$

which implies $|\hat{\eta}| > 0.003 > \varepsilon$ as required. The proof of the theorem is complete. □

**Remark 3.3** By experimenting with inequality (3.21) one can slightly improve the value of $\varepsilon$. However, the improved value is still of order $10^{-3}$, thus the lower bound for $t_0 - 1$ it leads to is still of order $10^{-6}$.

We finish the paper by making several observations regarding solutions (3.15) of equation (3.14) concentrating on the case when $W \in M^3_t$ for some $t < \sqrt{(2 + \sqrt{2})/3}$.

(i) It follows from (3.7) that $w_4$ is a solution of (3.14) if and only if $J_{\hat{F}}$ vanishes at the point $W$. Therefore, by Proposition 3.1, neither of the values in the right-hand side of (3.15) is equal to $w_4$ if $W \in M^3_t$ with $t < \sqrt{(2 + \sqrt{2})/3}$.

(ii) Arguing as in the proof of Proposition 3.1, one can see that the polynomial $6iw^2_3w^2_4 - (2 + 6i)w_3w_4 + 1$ vanishes at some point of $M^3_t$ if and only if $t \geq 2/\sqrt{3}$. Since $2/\sqrt{3} > \sqrt{(2 + \sqrt{2})/3}$, it follows that the right-hand side of (3.15) defines two distinct points $\hat{W}_1, \hat{W}_2$ if $W \in M^3_t$ with $t < \sqrt{(2 + \sqrt{2})/3}$. Combining this fact with (i), we see that for such $W$ the fiber of $\hat{F}$ over $\hat{F}(W)$ consists of exactly three points.

(iii) Using formula (3.15) one can determine, in principle, all values $t < \sqrt{(2 + \sqrt{2})/3}$ such that neither of $\hat{W}_1, \hat{W}_2$ lies in $M^3_t$ provided $W \in M^3_t$, which is equivalent to the injectivity of the map $\hat{F}$ on $M^3_t$. However, the computations required for this analysis are rather complicated, and we did not carry them out in full generality. These computations significantly simplify if, for instance, $w_4 = 0$. In this case formula (3.15) yields the solutions

$$\hat{w}_4 = \frac{1}{w_3}, \quad \hat{w}_4 = \frac{1 - i}{2w_3}.$$

Also, we have

$$|w_1|^2 + \frac{1}{|w_1|^2} + |w_3|^2 = 2t,$$

which implies $|w_3| < |w_1|$ since otherwise $t \geq \sqrt{2} > \sqrt{(2 + \sqrt{2})/3}$. Therefore, for each of the two points $\hat{W}_1, \hat{W}_2$ given by (3.22) one has

$$|w_1|^2 + |\hat{w}_2|^2 + |w_3|^2 + |\hat{w}_4|^2 > 2t,$$
hence neither of these points lies in $M^3_t$. We also arrived at the same conclusion when analyzing several other special cases. Thus, our calculations lead to the following conjecture.

**Conjecture 3.4** The restriction of $\tilde{F}$ to $M^3_t$ is an embedding for all parameter values $1 < t < \sqrt{(2 + \sqrt{2})/3}$.

(iv) We note that the non-injectivity of $\tilde{F}$ on $M^3_t$ is easy to see, for example, if $t \geq \sqrt{2}$. Indeed, let $u \neq 0$ be a real number satisfying

$$2u^2 + \frac{1}{u^2} = 2t,$$

and consider the following three distinct points in $Q^3$:

$$W_u := \left(u, \frac{1}{u}, u, 0\right), \quad W_u' := \left(u, 0, \frac{1}{u}\right), \quad W_u'' := \left(u, \frac{1+i}{2u}, u, \frac{1-i}{2u}\right). \quad (3.23)$$

Then $W_u, W_u', W_u'' \in M^3_t$ and $\tilde{F}(W_u) = \tilde{F}(W_u') = \tilde{F}(W_u'') = (u, u, 0)$. Since every fiber of $\tilde{F}$ contains at most three points, $W_u, W_u', W_u''$ form the complete fiber of $\tilde{F}$ over the point $(u, u, 0)$.

(v) In [AR] the authors in fact introduced not just the map $f$ (see (3.1)) but a class of maps of the form

$$g : \mathbb{C}^2 \to \mathbb{C}^3, \quad g(z,w) := (z, w, P(z, \bar{z}, w, \bar{w})).$$

Here $P$ is a harmonic polynomial given by

$$P = \left(\bar{z} \frac{\partial}{\partial w} - \bar{w} \frac{\partial}{\partial z}\right) \left(\sum_{j=1}^{m} \frac{1}{p_j(q_j + 1)} Q_j\right),$$

where $Q_j$ is a homogeneous harmonic complex-valued polynomial on $\mathbb{C}^2$ of total degree $p_j \geq 1$ in $z, w$ and total degree $q_j$ in $\bar{z}, \bar{w}$, such that the sum $Q := Q_1 + \ldots + Q_m$ does not vanish on $S^3$. Observe that although the third component of the map $f$ is not harmonic, it is obtained (up to a multiple) from the harmonic polynomial $w\bar{z}\bar{w} - zz^2\bar{w} + iz\bar{w}$ by replacing the term $iz\bar{w}$ with $i\bar{z}\bar{w}(|z|^2 + |w|^2)$ (these two expressions coincide on $S^3$). It is convenient to take $Q$ to be a polynomial in $|z|^2, |w|^2$ (as was done in [AR]), but in this case $P$ is divisible by $\bar{z}\bar{w}$, which implies that the holomorphic extension $G$ of the push-forward of $g$ to $\mathbb{R}^4$ is not injective on $M^3_t$ with $t \geq \sqrt{2}$. Indeed, writing $G$ in the coordinates $w_j$ defined in (3.2), for the points $W_u, W_u'$ introduced in (3.23) one has $G(W_u) = G(W_u') = (u, u, 0)$. Thus, one cannot obtain the embeddability of $M^3_t$ in $\mathbb{C}^3$ for $t \geq \sqrt{2}$ by utilizing any of the maps introduced in [AR] with $Q$ being a function of $|z|^2, |w|^2$ alone.
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Department of Mathematics  
The Australian National University  
Canberra, ACT 0200  
Australia  
e-mail: alexander.isaev@anu.edu.au