Gradient-Free Optimization for Non-Smooth Minimax Problems with Maximum Value of Adversarial Noise

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March 28, 2023

Abstract

This paper investigates zeroth-order methods for non-smooth convex-concave saddle point problems (with \(r\)-growth condition for duality gap). We assume that a black-box gradient-free oracle returns an inexact function value corrupted by an adversarial noise. In this work we prove that the standard zeroth-order version of the mirror descent method is optimal in terms of the oracle calls complexity and the maximum admissible noise.

Keywords: stochastic optimization, non-smooth optimization, gradient-free optimization, saddle point problems, minimax problems

1 Introduction

In this paper, we consider a stochastic non-smooth saddle point problem of the form

\[
\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y),
\]

where \(f(x, y) \triangleq \mathbb{E}_\xi [f(x, y, \xi)]\) is the expectation, w.r.t. \(\xi \in \Xi\), \(f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}\) is convex-concave and Lipschitz continuous, and \(\mathcal{X} \subseteq \mathbb{R}^{d_x}\), \(\mathcal{Y} \subseteq \mathbb{R}^{d_y}\) are convex compact sets. We
will denote \( z^* = (x^*, y^*) \) a solution of the problem (1). The standard interpretation of such min-max problems is the antagonistic game between a learner and an adversary, where the equilibria are the saddle points (Neumann, 1928). Now the interest in saddle point problems is renewed due to the popularity of generative adversarial networks (GANs), whose training involves solving min-max problems (Goodfellow et al., 2014; Chen et al., 2017).

Motivated by many applications in the field of reinforcement learning (Choromanski et al., 2018; Mania et al., 2018) and statistics, where only a black-box access to the function values of the objective is available, we consider zeroth-order oracle (also known as gradient-free oracle). Particularly, we mention the classical problem of adversarial multi-armed bandit (Flaxman et al., 2004; Bartlett et al., 2008; Bubeck and Cesa-Bianchi, 2012), where a learner receives a feedback given by the function evaluations from an adversary. Thus, zeroth-order methods (Conn et al., 2009) are the workhorse technique when the gradient information is prohibitively expensive or even not available and optimization is performed based only on the function evaluations.

**Related Work and Contribution.** Zeroth-order methods in the non-smooth setup were developed in a wide range of works (Polyak, 1987; Spall, 2003; Conn et al., 2009; Duchi et al., 2015; Shamir, 2017; Nesterov and Spokoiny, 2017; Gasnikov et al., 2017; Beznosikov et al., 2020; Gasnikov et al., 2022). Particularly, in (Shamir, 2017), an optimal algorithm was provided as an improvement to the work (Duchi et al., 2015) for a non-smooth case but Lipschitz continuous in stochastic convex optimization problems. However, this algorithm uses the exact function evaluations that can be infeasible in some applications. Indeed, an objective \( f(z, \xi) \) can be not directly observed but instead, its noisy approximation \( \varphi(z, \xi) \triangleq f(z, \xi) + \delta(z) \) can be queried, where \( \delta(z) \) is some adversarial noise. This noisy-corrupted setup was considered in many works (Polyak, 1987; Granichin and Polyak, 2003), however, such an algorithm that is optimal in terms of the number of oracle calls complexity and the maximum value of adversarial noise has not been proposed. For instance, in (Bayandina et al., 2018; Beznosikov et al., 2020), optimal algorithms in terms of oracle calls complexity were proposed, however, they are not optimal in terms of the maximum value of the noise. In papers (Risteski and Li, 2016; Vasin et al., 2021), algorithms are optimal in terms of the maximum value of the noise, however, they are not optimal in terms of the oracle calls complexity. In this paper, we provide an accurate analysis of a gradient-free version of the mirror descent method with an inexact oracle that shows that the method is optimal both in terms of the inexact oracle calls complexity and the maximum admissible noise. We consider two possible scenarios for the nature of the adversarial noise arising in different applications: the noise is bounded by a small value or is Lipschitz. Table 1 demonstrates our contribution by comparing our results with the existing optimal bounds. Finally, we consider the case when the objective satisfies the \( \gamma \)-growth condition and restate the results.

**Paper Organization.** This paper is organized as follows. In Section 2, we begin with background material, notation, and assumptions. In Section 3, we present the main results of the paper: the algorithm and the analysis of its convergence. In Section 4, we consider an additional assumption of \( \gamma \)-growth condition and restate the results. In Section 6, we provide
Table 1: Summary of the Contribution

| Paper                      | Problem          | Expectation or Large Deviation | Is the Noise Lipschitz? | Number of Oracle Calls | Maximum Noise       |
|----------------------------|------------------|--------------------------------|-------------------------|------------------------|---------------------|
| Bayandina et al. (2018)    | convex           | E                              |  \( \mathcal{X} \)     | \( d/\epsilon^2 \)     | \( \epsilon^2/d^{1/2} \) |
| Beznosikov et al. (2020)   | saddle point     | E                              |  \( \mathcal{X} \)     | \( d/\epsilon^2 \)     | \( \epsilon^2/d \)  |
| Vasin et al. (2021)        | convex           | E                              |  \( \mathcal{X} \)     | Poly \((d, 1/\epsilon)\) | \( \epsilon^2/\sqrt{d} \) |
| Risteski and Li (2016)     | convex           | E                              |  \( \mathcal{X} \)     | Poly \((d, 1/\epsilon)\) | \( \max\{\epsilon^2/\sqrt{d}, \epsilon/d\}\) |
| This work                  | saddle point     | E and \( P \)                  |  \( \mathcal{X} \)     | \( d/\epsilon^2 \)     | \( \epsilon^2/\sqrt{d} \) |
| This work                  | saddle point     | E and \( P \)                  |  \( \mathcal{X} \)     | \( d/\epsilon^2 \)     | \( \epsilon/\sqrt{d}(3) \) |

(1) The results obtained for saddle point problems are also valid for convex optimization problems.

(2) Notice, that this bound (up to a logarithmic factor) is also an upper bound for maximum possible value of noise. It is important to note, that \( \epsilon/d \) \( \ll \epsilon^2/\sqrt{d} \), when \( \epsilon^{-2} \ll d \). That is in the large-dimension regime, when subgradient method is better than center of gravity types methods (Nemirovskij and Yudin, 1983), the upper bound on the value of admissible noise (that allows one to solve the problem with accuracy \( \epsilon \)) will be \( \epsilon^2/\sqrt{d} \).

(3) This estimate is for the Lipschitz constant of the noise and it possibly is tight. However, we prove that the upper bound (possibly not tight) for the Lipschitz-noise constant is \( \epsilon \) (see Appendix B for details).

some of the key technical components used to prove the main results from Section 3.

2 Preliminaries: Notation, Setup and Assumptions

In this section, we give some key notation, background material and assumptions.

**Notation.** We use \( \langle x, y \rangle \triangleq \sum_{i=1}^{d} x_i y_i \) to define the inner product of \( x, y \in \mathbb{R}^d \), where \( x_i \) is the \( i \)-th component of \( x \). By norm \( \| \cdot \|_p \) we mean the \( \ell_p \)-norm. Then the dual norm of the norm \( \| \cdot \|_p \) is \( \| \lambda \|_q \triangleq \max\{\langle x, \lambda \rangle | \| x \|_p \leq 1 \} \). Operator \( \mathbb{E}[\cdot] \) is the full expectation and operator \( \mathbb{E}_\xi[\cdot] \) is the conditional expectation, w.r.t. \( \xi \).

**Setup.** Let us introduce the embedding space \( \mathcal{Z} \triangleq \mathcal{X} \times \mathcal{Y} \), and then some \( z \in \mathcal{Z} \) means \( z \triangleq (x, y) \), where \( x \in \mathcal{X}, y \in \mathcal{Y} \). On this embedding space, we introduce the \( \ell_p \)-norm and a prox-function \( \omega(z) \) compatible with this norm. Then we define the Bregman divergence associated with \( \omega(z) \) as

\[
V_z(v) \triangleq \omega(z) - \omega(v) - \langle \nabla \omega(v), z - v \rangle \geq \| z - v \|_p^2/2, \quad \text{for all } z, v \in \mathcal{Z}.
\]

We also introduce a prox-operator as follows

\[
\operatorname{Prox}_z(\xi) \triangleq \arg \min_{v \in \mathcal{Z}} (V_z(v) + \langle \xi, v \rangle), \quad \text{for all } z \in \mathcal{Z}.
\]

Finally, we denote the \( \omega \)-diameter of \( \mathcal{Z} \) by \( \mathcal{D} \triangleq \max_{z, v \in \mathcal{Z}} \sqrt{2V_z(v)} = \bar{O}\left(\max_{z, v \in \mathcal{Z}} \| z - v \|_p\right) \). Here \( \bar{O}(\cdot) \) is \( O(\cdot) \) up to a \( \sqrt{\log d} \)-factor.
Assumption 2.1 (Lischitz continuity of the objective). Function \( f(z, \xi) \) is \( M_2 \)-Lipschitz continuous in \( z \in \mathcal{Z} \) w.r.t. the \( \ell_2 \)-norm, i.e., for all \( z_1, z_2 \in \mathcal{Z} \) and \( \xi \in \Xi \),
\[
|f(z_1, \xi) - f(z_2, \xi)| \leq M_2(\xi) \|z_1 - z_2\|_2.
\]
Moreover, there exists a positive constant \( M_2 \) such that \( \mathbb{E}[M_2^2(\xi)] \leq M_2^2 \).

Assumption 2.2. For all \( z \in \mathcal{Z} \), it holds \( |\delta(z)| \leq \Delta \).

Assumption 2.3 (Lischitz continuity of the noise). Function \( \delta(z) \) is \( M_{2, \delta} \)-Lipschitz continuous in \( z \in \mathcal{Z} \) w.r.t. the \( \ell_2 \)-norm, i.e., for all \( z_1, z_2 \in \mathcal{Z} \),
\[
|\delta(z_1) - \delta(z_2)| \leq M_{2, \delta} \|z_1 - z_2\|_2.
\]

3 Main Results

In this section, we present one of the main results of the paper. For problem (1), we present an algorithm (see Algorithm 1) that is optimal in terms of the number of inexact zeroth-order oracle calls and the maximum adversarial noise. The algorithm is based on a gradient-free version of the stochastic mirror descent (SMD) (Ben-Tal and Nemirovski, 2013).

Black-box oracle. We assume that we can query zeroth-order oracle corrupted by an adversarial noise \( \delta(z) \):
\[
\varphi(z, \xi) \triangleq f(z, \xi) + \delta(z).
\]
(2)

We will consider two cases:
1. Noise \( \delta(z) \) is bounded (Assumption 2.2)
2. Noise \( \delta(z) \) is Lipschitz (Assumption 2.3)

Gradient approximation. The gradient of \( \varphi(z, \xi) \) from (2), w.r.t. \( z \), can be approximated by the function evaluations in two random points closed to \( z \). To do so, we define vector \( e \) picked uniformly at random from the Euclidean unit sphere \( \{e : \|e\|_2 = 1\} \). Let \( e \triangleq (e_x^\top, -e_y^\top)^\top \), where \( \dim(e_x) \triangleq d_x \), \( \dim(e_y) \triangleq d_y \) and \( \dim(e) \triangleq d = d_x + d_y \). Then the gradient of \( \varphi(z, \xi) \) can be estimated by the following approximation with a small variance (Shamir, 2017):
\[
g(z, \xi, e) = \frac{d}{2\tau} (\varphi(z + \tau e, \xi) - \varphi(z - \tau e, \xi)) \begin{pmatrix} e_x \\ -e_y \end{pmatrix},
\]
(3)
where \( \tau \) is some constant.
Randomized smoothing. For a non-smooth objective $f(z)$, we define the following function

$$f^\tau(z) \triangleq \mathbb{E}_\tilde{e} f(z + \tau \tilde{e}),$$

where $\tau > 0$ and $\tilde{e}$ is a vector picked uniformly at random from the Euclidean unit ball: $\{\tilde{e} : \|\tilde{e}\|_2 \leq 1\}$. Function $f^\tau(z)$ can be referred as a smooth approximation of $f(z)$. Here $f(z) \triangleq \mathbb{E}_z f(z, \xi)$. We notice that function $f^\tau(z)$ is introduced only for proof.

The next theorem presents the convergence rates for Algorithm 1 in terms of the expectation. For the convergence results in large deviation, we refer to the Appendix D.

**Theorem 3.1.** Let function $f(x,y,\xi)$ satisfy the Assumption 2.1. Then the following holds for $\epsilon_{\text{sad}} \triangleq f(\hat{x}^N,y^N) - f(x^*,y^N)$ where $\hat{x}^N \triangleq (\hat{x}^N,\hat{y}^N)$ is the output of Algorithm 1:

1. under Assumption 2.2 and learning rate $\gamma_k = \frac{D}{M_{\text{case}1}} \sqrt{\frac{2}{N}}$ with
   $$M_{\text{case}1}^2 \triangleq \mathcal{O} \left( da^2 M^2_2 + d^2 a^2 \Delta^2 \tau^{-2} \right)$$
   $$\mathbb{E} [\epsilon_{\text{sad}}] \leq M_{\text{case}1} D \sqrt{\frac{2}{N}} + \sqrt{dD} \tau^{-1} + 2\tau M_2. \quad (5)$$

2. under Assumption 2.3 and learning rate $\gamma_k = \frac{D}{M_{\text{case}2}} \sqrt{\frac{2}{N}}$ with
   $$M_{\text{case}2}^2 \triangleq \mathcal{O} \left( da^2 (M^2_2 + M^2_{2,\delta}) \right)$$
   $$\mathbb{E} [\epsilon_{\text{sad}}] \leq M_{\text{case}2} D \sqrt{\frac{2}{N}} + M_{2,\delta} \sqrt{dD} + 2\tau M_2, \quad (6)$$

where $\sqrt{\mathbb{E} [\|e\|^2_q]} = \mathcal{O} \left( \min\{q, \log d \} d^{2/q-1} \right) = a_q^2$ (Gorbunov et al., 2019b).

**Algorithm 1:** Zeroth-order SMD

```
Input: iteration number $N$
$z^1 \leftarrow \text{arg} \min_{z \in Z} d(z)$ for $k = 1, \ldots, N$
do  
    Sample $e^k, \xi^k$ independently
    Initialize $\gamma_k$
    Calculate $g(z^k, \xi^k, e^k)$ via (3)
    $z^{k+1} \leftarrow \text{Prox}_{\gamma_k}(g(z^k, \xi^k, e^k))$
end
Output:
$\hat{z}^N \leftarrow \left( \sum_{k=1}^N \gamma_k \right)^{-1} \sum_{k=1}^N \gamma_k z^k$
```

Algorithm 1 and its convergence analysis given in Theorem 3.1 are based on the same proximal setups for spaces $\mathcal{X}$ and $\mathcal{Y}$. For the case, when these setups are different we replace the proximal step $z^{k+1} \leftarrow \text{Prox}_{\gamma_k}(g(z^k, \xi^k, e^k))$ on the space $Z \triangleq \mathcal{X} \times \mathcal{Y}$ by two proximal steps on spaces $\mathcal{X}$ and $\mathcal{Y}$. The convergence analysis in this case is given in the Appendix E.

The next corollary presents our contribution.

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1 All the results obtained in this paper (except Section 4) will have the same form for $\epsilon_{\text{sad}} \triangleq \max_{y \in \mathcal{Y}} f(\hat{x}^N,y) - \min_{x \in \mathcal{X}} f(x,\hat{y}^N)$, but this required more accurate analysis, based on special trick from (Juditsky et al., 2011; Beznosikov et al., 2020).
Corollary 3.2. Let function $f(x, y, \xi)$ satisfy the Assumption 2.1, and let $\tau$ be chosen as $\tau = O(\epsilon/M_2)$ in randomized smoothing (4), where $\epsilon$ is the desired accuracy to solve problem (1). If one of the two following statement is true

1. Assumption 2.2 holds and $\Delta = O\left(\frac{\epsilon^2}{D M_2 \sqrt{d}}\right)$

2. Assumption 2.3 holds and $M_{2,\delta} = O\left(\frac{\epsilon}{D \sqrt{d}}\right)$

then for the output $\hat{z}^N \triangleq (\hat{x}^N, \hat{y}^N)$ of Algorithm 1, it holds $\mathbb{E}[\ell_{\text{sad}}(\hat{z}^N)] \leq \epsilon$ after

$$N = O\left(da_q^2 M_2^2 D^2 / \epsilon^2\right)$$

iterations, where $\sqrt{\mathbb{E}\left[\|e\|_q^4\right]} = O\left(\min\{q, \log d\} d^{2/q - 1}\right) = a_q^2$ (Gorbunov et al., 2019b).

Next we clarify the Corollary 3.2 in the two following special setups: the $\ell_2$-norm and the $\ell_1$-norm in the following examples.

Example 3.3. Let $p = 2$, then $q = 2$ and $\sqrt{\mathbb{E}\left[\|e\|_2^4\right]} = 1$. Thus, $a_2^2 = 1$ and $D^2 = \max_{z,v \in Z} \|z - v\|_2^2$. Consequently, the number of iterations in the Corollary 3.2 can be rewritten as follows

$$N = O\left(d a_q^2 M_2^2 D^2 / \epsilon^2\right)$$

Example 3.4. Shamir (2017, Lemma 4) Let $p = 1$ then, $q = \infty$ and $\sqrt{\mathbb{E}\left[\|e\|_\infty^4\right]} = O\left(\frac{\log d}{\epsilon}\right)$. Thus, $a_\infty^2 = O\left(\frac{\log d}{\epsilon}\right)$ and $D^2 = O\left(\log d \max_{z,v \in Z} \|z - v\|_1^2\right)$. Consequently, the number of iterations in the Corollary 3.2 can be rewritten as follows

$$N = O\left(\frac{M_2^2 \log^2 d}{\epsilon^2} \max_{z,v \in Z} \|z - v\|_1^2\right).$$

Remark 1. Corollary 3.2 bound for the Lipschitz constant given in the Corollary 3.2 is probably not the upper bound. In the Appendix B, we provide a natural analysis to prove the bound from Corollary 3.2 being optimal but our arguments are based on the assumption that the points of the function evaluation were chosen regardless of the stochastic realizations $\xi_1, \xi_2, \xi_3, \ldots$. Unfortunately, that is not the case of Algorithm 1 and many other practical algorithms.

Proof of the Theorem 3.1. By the definition $z^{k+1} = \text{Prox}_{\gamma_k} \left(\gamma_k g(z^k, e^k, \xi^k)\right)$ we get (Ben-Tal and Nemirovski, 2013), for all $u \in Z$

$$\gamma_k \langle g(z^k, e^k, \xi^k), z^k - u \rangle \leq V_{z^k}(u) - V_{z^{k+1}}(u) + \gamma_k^2 \|g(z^k, e^k, \xi^k)\|_q^2 / 2.$$
Taking the conditional expectation w.r.t. $\xi$, $e$ and summing for $k = 1, \ldots, N$ we obtain, for all $u \in \mathcal{Z}$

$$
\sum_{k=1}^{N} \gamma_k \mathbb{E}_{e^k, \xi^k} \left[ \langle g(z^k, e^k, \xi^k), z^k - u \rangle \right] \leq V_1(u) + \sum_{k=1}^{N} \frac{\gamma_k^2}{2} \mathbb{E}_{e^k, \xi^k} \left[ \|g(z^k, e^k, \xi^k)\|^2 \right].
$$

(7)

**Step 1.**
For the second term in the r.h.s of inequality (7) we use Lemma 6.6 and obtain

1. under Assumption 2.2:

$$
\mathbb{E}_{e^k, \xi^k} \left[ \|g(z^k, \xi^k, e^k)\|^2_q \right] \leq c a^2 q d M_2^2 + d^2 a^2 q \Delta^2 \tau^{-2},
$$

(8)

2. under Assumption 2.3:

$$
\mathbb{E}_{e^k, \xi^k} \left[ \|g(z^k, \xi^k, e^k)\|^2_q \right] \leq c a^2 d (M_2^2 + M_2^2),
$$

(9)

where $c$ is some numerical constant and $\sqrt{\mathbb{E} \left[ \|e^k\|^4_q \right]} \leq a^2_q$.

**Step 2.**
For the l.h.s. of Eq. (7), we use Lemma 6.4

1. under Assumption 2.2

$$
\sum_{k=1}^{N} \gamma_k \mathbb{E}_{e^k, \xi^k} \left[ \langle g(z^k, e^k, \xi^k), z^k - u \rangle \right] \geq \sum_{k=1}^{N} \gamma_k \langle \nabla f^\tau(z^k), z^k - u \rangle
$$

$$
+ \sum_{k=1}^{N} \gamma_k \mathbb{E}_{e^k} \left[ \|d \Delta^{-1} e^k, z^k - u \| \right].
$$

(10)

2. under Assumption 2.3

$$
\sum_{k=1}^{N} \gamma_k \mathbb{E}_{e^k, \xi^k} \left[ \langle g(z^k, e^k, \xi^k), z^k - u \rangle \right] \geq \sum_{k=1}^{N} \gamma_k \langle \nabla f^\tau(z^k), z^k - u \rangle
$$

$$
+ \sum_{k=1}^{N} \gamma_k \mathbb{E}_{e^k} \left[ \|d M_2 e^k, z^k - u \| \right].
$$

(11)
For the first term of the r.h.s. of Eq.(10) and (11) we have
\[
\sum_{k=1}^{N} \gamma_k \langle \nabla f^\tau(z^k), z^k - u \rangle = \sum_{k=1}^{N} \gamma_k \left( \left\langle \left( -\nabla_x f^\tau(x^k, y^k), x^k - x \right), \left( y^k - y \right) \right\rangle - \left\langle \nabla_y f^\tau(x^k, y^k), y^k - y \right\rangle \right)
\]
\[
= \sum_{k=1}^{N} \gamma_k \left( \left\langle \nabla_x f^\tau(x^k, y^k), x^k - x \right\rangle \right)
\geq \sum_{k=1}^{N} \gamma_k \left( f^\tau(x^k, y^k) - f^\tau(x, y^k) \right)
\]
\[
= \sum_{k=1}^{N} \gamma_k \left( f^\tau(x^k, y) - f^\tau(x, y^k) \right),
\]
where \(u \triangleq (x^\top, y^\top)\). Then we use the fact function \(f^\tau(x, y)\) is convex in \(x\) and concave in \(y\) and obtain
\[
\frac{1}{\sum_{i=1}^{N} \gamma_k} \sum_{k=1}^{N} \gamma_k \left( f^\tau(x^k, y) - f^\tau(x, y^k) \right) \leq f^\tau \left( \frac{\sum_{k=1}^{N} \gamma_k x^k}{\sum_{k=1}^{N} \gamma_k}, y \right) - f^\tau \left( x, \frac{\sum_{k=1}^{N} \gamma_k y^k}{\sum_{k=1}^{N} \gamma_k} \right)
\]
\[
= f^\tau \left( \hat{x}^N, y \right) - f^\tau \left( x, \hat{y}^N \right),
\]
where \((\hat{x}^N, \hat{y}^N)\) is the output of the Algorithm 1. Using Eq. (13) for Eq. (12) we get
\[
\sum_{k=1}^{N} \gamma_k \langle \nabla f^\tau(z^k), z^k - u \rangle \geq \sum_{k=1}^{N} \gamma_k f^\tau \left( \hat{x}^N, y \right) - f^\tau \left( x, \hat{y}^N \right).
\]
Next we estimate the term \(\mathbb{E}_{e^k} \left[ \|e^k, z^k - u\| \right]\) in Eq. (10) and (11), by the Lemma 6.1
\[
\mathbb{E}_{e^k} \left[ \|e^k, z^k - u\| \right] \leq \|z^k - u\|_2/\sqrt{d}.
\]
Now we substitute Eq. (14) and (15) to Eq. (10) and (11), and get
1. under Assumption 2.2
\[
\sum_{k=1}^{N} \gamma_k \mathbb{E}_{e^k, \xi^k} \left[ g(z^k, e^k, \xi^k), z^k - u \right] \geq \sum_{k=1}^{N} \gamma_k f^\tau \left( \hat{x}^N, y \right) - f^\tau \left( x, \hat{y}^N \right)
- \sum_{k=1}^{N} \gamma_k \sqrt{d} \Delta \|z^k - u\|_2 \tau^{-1}.
\]
2. under Assumption 2.3
\[
\sum_{k=1}^{N} \gamma_k \mathbb{E}_{e^k, \xi^k} \left[ g(z^k, e^k, \xi^k), z^k - u \right] \geq \sum_{k=1}^{N} \gamma_k f^\tau \left( \hat{x}^N, y \right) - f^\tau \left( x, \hat{y}^N \right)
- \sum_{k=1}^{N} \gamma_k \sqrt{d} M_2, \delta \|z^k - u\|_2.
\]
Step 3. (under Assumption 2.2)
Now we combine Eq. (16) with Eq. (44) for Eq. (7) and obtain under Assumption 2.2 the following
\[
\sum_{k=1}^{N} \gamma_k f^T(\hat{x}^N, y) - f^T(x, \hat{y}^N) - \sum_{k=1}^{N} \gamma_k \sqrt{d\Delta} \|z^k - u\|_{2\tau}^{-1} \leq V_{z1}(u) + \sum_{k=1}^{N} \frac{\gamma_k^2}{2} \left( c \alpha^2 d M^2_2 + d^2 \alpha^2 \Delta^2 \tau^{-2} \right). 
\] (18)

Using Lemma 6.2 we obtain
\[
f^T(\hat{x}^N, y) - f^T(x, \hat{y}^N) \geq f(\hat{x}^N, y) - f(x, \hat{y}^N) - 2\tau M_2.
\]
Using this we can rewrite (18) as follows
\[
f(\hat{x}^N, y) - f(x, \hat{y}^N) \leq \frac{V_{z1}(u)}{\sum_{k=1}^{N} \gamma_k} + \frac{c \alpha^2 d M^2_2 + d^2 \alpha^2 \Delta^2 \tau^{-2} \sum_{k=1}^{N} \gamma_k^2}{2 \sum_{k=1}^{N} \gamma_k} + \sqrt{d\Delta \max_k \|z^k - u\|_{2\tau}^{-1} + 2\tau M_2}. 
\] (19)

For the r.h.s. of (19) we use the definition of the \(\omega\)-diameter of \(Z\):
\[
D \triangleq \max_{z \in Z} \sqrt{2V_z(v)} \text{ and estimate } \|z^k - u\|_2 \leq D \text{ for all } z^1, \ldots, z^k \text{ and all } u \in Z.
\]
Using this for (19) and taking \((x, y) = (x_*, y_*),\) we obtain
\[
f(\hat{x}^N, y_*) - f(x_*, \hat{y}^N) \leq \frac{D^2 + (c \alpha^2 d M^2_2 + d^2 \alpha^2 \Delta^2 \tau^{-2}) \sum_{k=1}^{N} \gamma_k^2 / 2 \sum_{k=1}^{N} \gamma_k}{\sum_{k=1}^{N} \gamma_k} + \sqrt{d\Delta D \tau^{-1} + 2\tau M_2}. 
\] (20)

Then we use the definition of the \(\omega\)-diameter of \(Z\): \(D \triangleq \max_{z \in Z} \sqrt{2V_z(v)}\) and estimate \(\|z^k - u\|_2 \leq D\) for all \(z^1, \ldots, z^k\) and all \(u \in Z\). Thus, taking the expectation of (20) and choosing learning rate \(\gamma_k = \frac{D}{M_{\text{case1}}} \sqrt{\frac{2}{N}}\) with \(M^2_{\text{case1}} \triangleq c \alpha^2 d M^2_2 + d^2 \alpha^2 \Delta^2 \tau^{-2}\) in Eq. (20) we get
\[
\mathbb{E} \left[ f(\hat{x}^N, y_*) - f(x_*, \hat{y}^N) \right] \leq M_{\text{case1}} D \sqrt{\frac{2}{N}} + \frac{\Delta D \sqrt{d}}{\tau} + 2\tau M_2.
\]

Step 4. (under Assumption 2.3)
Now we combine Eq. (17) with Eq. (9) for Eq. (7) and obtain under Assumption 2.3
\[
\sum_{k=1}^{N} \gamma_k f^T(\hat{x}^N, y) - f^T(x, \hat{y}^N) - \sum_{k=1}^{N} \gamma_k \sqrt{dM_{2,\delta}} \|z^k - u\|_2 \leq V_{z1}(u) + \sum_{k=1}^{N} \frac{\gamma_k^2}{2} c \alpha^2 d (M^2_2 + M^2_{2,\delta}). 
\] (21)
Using Lemma 6.2 we obtain
\[ f^r(\hat{x}^N, y) - f^r(x, \hat{y}^N) \leq f(\hat{x}^N, y) - f(x, y^N) - 2\tau M_2. \]

Using this we can rewrite (21) as follows
\[ f(\hat{x}^N, y) - f(x, y^N) \leq V_z(1) + \sum_{k=1}^{N} \frac{\gamma_k^2}{2} + \sqrt{dM_2, \delta} \max_k \| z^k - u \|_2 + 2\tau M_2. \]

(22)

For the r.h.s. of (22) we use the definition of the \( \omega \)-diameter of \( Z \):
\[ D \triangleq \max_{z, v \in Z} \sqrt{2V_z(v)} \] and estimate \( \| z^k - u \|_2 \leq D \) for all \( z^1, \ldots, Z^k \) and all \( u \in Z \). Using this for (22) and taking \((x, y) = (x^\star, y^\star)\), we obtain
\[ f(\hat{x}^N, y^\star) - f(x^\star, \hat{y}^N) \leq D^2 + \frac{ca^2_d(M_2^2 + M_2^2)}{\sum_{k=1}^{N} \gamma_k^2 / 2} + M_2, \delta \sqrt{d} \max_k \| z^k - u \|_2 + 2\tau M_2. \]

(23)

Then we use the definition of the \( \omega \)-diameter of \( Z \): \( D \triangleq \max_{z, v \in Z} \sqrt{2V_z(v)} \) and estimate \( \| z^k - u \|_2 \leq D \) for all \( z^1, \ldots, Z^k \) and all \( u \in Z \). Thus, taking the expectation of (20) and choosing learning rate \( \gamma_k = \frac{D}{M_\text{case}2} \sqrt{\frac{2}{N}} \) with \( M_\text{case}2 \triangleq cda^2_d(M_2^2 + M_2^2, \delta) \) in Eq. (23) we get
\[ \mathbb{E}[f(\hat{x}^N, y^\star) - f(x^\star, \hat{y}^N)] \leq M_\text{case}2 D \sqrt{2/N} + M_2, \delta D \sqrt{d} + 2\tau M_2. \]

4 Replots

In this section, we assume that we additionally have the \( r \)-growth condition for duality gap (see, (Shapiro et al., 2021) for convex optimization problems). For such a case, we apply the restart technique (Juditsky and Nesterov, 2014) to Algorithm 1

Assumption 4.1 \((r\text{-growth condition})\). There is \( r \geq 1 \) and \( \mu_r > 0 \) such that for all \( z = (x, y) \in Z \triangleq X \times Y \)
\[ \frac{\mu_r}{2} \| z - z^\star \|_p \leq f(x, y^\star) - f(x^\star, y), \]
where \((x^\star, y^\star)\) is a solution of problem (1).

Theorem 4.2. Let \( f(x, y, \xi) \) satisfy the Assumption 2.1. Then the following holds for \( \hat{\epsilon}_\text{sad} \triangleq f(\hat{x}^N, y^\star) - f(x^\star, \hat{y}^N) \), where \( \hat{z}^N \triangleq (\hat{x}^N, \hat{y}^N) \) is the output of Algorithm 1,
1. under Assumption 2.2 and learning rate \( \gamma_k = \frac{\sqrt{\mathbb{E}[V_{z_1}(z^*)]}}{M_{\text{case1}}} \sqrt{\frac{2}{N}} \) with
\[
M_{\text{case1}}^2 \triangleq O \left( da_q^2 M_2^2 + d^2 a_q^2 \Delta^2 r^{-2} \right)
\]
\[
\mathbb{E} [\hat{\epsilon}_{\text{sad}}] \leq \sqrt{\frac{2}{N}} M_{\text{case1}} \sqrt{\mathbb{E}[V_{z_1}(z^*)]} + \Delta D \sqrt{d} \tau^{-1} + 2 \tau M_2.
\] (24)

2. under Assumption 2.3 and learning rate \( \gamma_k = \frac{\sqrt{\mathbb{E}[V_{z_1}(z^*)]}}{M_{\text{case2}}} \sqrt{\frac{2}{N}} \) with
\[
M_{\text{case2}}^2 \triangleq O \left( da_q^2 \left( M_2^2 + M_2^2, \delta \right) \right)
\]
\[
\mathbb{E} [\hat{\epsilon}_{\text{sad}}] \leq \sqrt{\frac{2}{N}} M_{\text{case2}} \sqrt{\mathbb{E}[V_{z_1}(z^*)]} + M_{2,\delta} D \sqrt{d} + 2 \tau M_2,
\] (25)

where \( \sqrt{\mathbb{E}_e [\|e\|_4^4]} \leq a_q^2 \).

Moreover, let \( \tau \) be chosen as \( \tau = \mathcal{O} (\epsilon/M_2) \) in randomized smoothing (4), where \( \epsilon \) is the desired accuracy to solve problem (1). If one of the two following statement is true

1. Assumption 2.2 holds and \( \Delta = \mathcal{O} (\epsilon^2 D/\sqrt{d}) \)

2. Assumption 2.3 holds and \( M_{2,\delta} = \mathcal{O} \left( \frac{\epsilon}{D \sqrt{d}} \right) \)

then for the output \( \hat{z}^N \triangleq (\hat{x}^N, \hat{y}^N) \) of Algorithm 1, it holds \( \mathbb{E} \left[ \hat{\epsilon}_{\text{sad}}(\hat{z}^N) \right] \leq \epsilon \) after
\[
N = \mathcal{O} \left( da_q^2 M_2^2 D^2 / \epsilon^2 \right)
\]
iterations. If also Assumption 4.1 is satisfied for \( r \geq 2 \) we can apply restarts and to achieve \( \mathbb{E} [\hat{\epsilon}_{\text{sad}}] \leq \epsilon \) in \( N_{\text{acc}} \) iterations where \( N_{\text{acc}} \) is given by
\[
N_{\text{acc}} = \tilde{\mathcal{O}} \left( \frac{a_q^2 M_2^2 d}{\mu r^{2/2} \epsilon^{2/(r-1)/r}} \right).
\] (26)

**Proof of Theorem 4.2** We repeat the proof of Theorem 3.1, except that now \( z^1 \) can be chosen in a stochastic way. Moreover, now we use a rougher inequality instead of (15)
\[
\mathbb{E}[e_k [\langle e^k, z^k - u \rangle]] \leq D / \sqrt{d}.
\] (27)

**Step 1.** (under Assumption 2.2)

Taking the expectation in (19), choosing \( (x, y) = (x^*, y^*) \), and learning rate \( \gamma_k = \frac{\sqrt{\mathbb{E}[V_{z_1}(z^*)]}}{M_{\text{case1}}} \sqrt{\frac{2}{N}} \) with
\[
M_{\text{case1}}^2 \triangleq cda_q^2 M_2^2 + d^2 a_q^2 \Delta^2 r^{-2}
\] we get
\[
\mathbb{E} \left[ f (\hat{x}^N, y^*) - f (x^*, \hat{y}^N) \right] \leq \sqrt{\frac{2}{N} M_{\text{case1}} \sqrt{\mathbb{E}[V_{z_1}(z^*)]} + \frac{\sqrt{d} \Delta D}{\tau} + 2 \tau M_2.
\] (28)
**Step 2.** (under Assumption 2.3)

Taking the expectation in (22), choosing \((x, y) = (x^*, y^*)\), and learning rate \(\gamma_k = \frac{\sqrt{\mathbb{E}[V_{z_1}(z^*)]}}{M_{\text{case2}}} \sqrt{\frac{2}{N}}\) with \(M_{\text{case2}}^2 \triangleq cda_q^2(M_2^2 + M_{2,\delta}^2)\) we obtain

\[
\mathbb{E}[f(\hat{x}^N, y^*) - f(x^*, \hat{y}^N)] \leq \sqrt{\frac{2}{N}} M_{\text{case2}} \sqrt{\mathbb{E}[V_{z_1}(z^*)] + M_{2,\delta} \sqrt{dD} + 2\tau M_2}. \tag{29}
\]

**Step 3.** (Restarts)

Now let \(\tau\) be chosen as \(\tau = O\left(\frac{\epsilon}{M_2}\right)\), where \(\epsilon\) is the desired accuracy to solve problem (1). If one of the following statement holds

1. Assumption 2.2 holds and \(\Delta = O\left(\frac{\epsilon^2}{D M_2 \sqrt{d}}\right)\)
2. Assumption 2.3 holds and \(M_{2,\delta} = O\left(\frac{\epsilon}{D \sqrt{d}}\right)\)

then using (28) we obtain the convergence rate of the following form

\[
\mathbb{E}[f(\hat{x}^{N_1}, y^*) - f(x^*, \hat{y}^{N_1})] = \tilde{O}\left(\frac{a_q M_2 \sqrt{d}}{\sqrt{N_1}} \sqrt{\mathbb{E}[V_{z_1}(z^*)]}\right). \tag{30}
\]

In this step we will employ the restart technique that is a generalization of the technique proposed in (Juditsky and Nesterov, 2014).

For the l.h.s. of Eq. (30) we use the Assumption 4.1. For the r.h.s. of Eq. (30) we use \(V_{z_1}(z^*) = \tilde{O}(\|z^1 - z^*\|_p^2)\) from Gasnikov and Nesterov (2018, Remark 3)

\[
\frac{\mu_r}{2} \mathbb{E}[\|z^{N_1} - z^*\|_p^2] \leq \mathbb{E}[f(\hat{x}^{N_1}, y^*) - f(x^*, \hat{y}^{N_1})] = \tilde{O}\left(\frac{a_q M_2 \sqrt{d}}{\sqrt{N_1}} \sqrt{\mathbb{E}[\|z^1 - z^*\|_p^2]}\right). \tag{31}
\]

Then the l.h.s of Eq. (31) we use the Jensen inequality and get the following

\[
\frac{\mu_r}{2} \left(\mathbb{E}[\|z^{N_1} - z^*\|_p^2]\right)^{r/2} \leq \frac{\mu_r}{2} \mathbb{E}[\|z^{N_1} - z^*\|_p^r] \leq \mathbb{E}[f(\hat{x}^{N_1}, y^*) - f(x^*, \hat{y}^{N_1})] = \tilde{O}\left(\frac{a_q M_2 \sqrt{d}}{\sqrt{N_1}} \sqrt{\mathbb{E}[\|z^1 - z^*\|_p^r]}\right). \tag{32}
\]

Finally, let us introduce \(R_k \triangleq \sqrt{\mathbb{E}[\|z^{N_k} - z^*\|_p^2]}\) and \(R_0 \triangleq \sqrt{\mathbb{E}[\|z^1 - z^*\|_p^2]}\). Then we take \(N_1\) so as to halve the distance to the solution and get

\[
N_1 = \tilde{O}\left(\frac{a_q^2 M_2^2 d}{\mu_r^2 R_1^{2(r-1)}}\right). \tag{33}
\]
Next, after $N_1$ iterations, we restart the original method and set $z^1 = z^{N_1}$. We determine $N_2$ similarly: we halve the distance $R_1$ to the solution, and so on. Thus, after $k$ restarts, the total number of iterations will be

$$N_{acc} = N_1 + \cdots + N_k = \tilde{O}\left(\frac{2^{2(r-1)}a_q^2M_2^2d}{\mu_1^2R_0^{2(r-1)}} \left(1 + 2^{2(r-1)} + \ldots + 2^{2(k-1)(r-1)}\right)\right). \quad (33)$$

Now we need to determine the number of restarts. To do this, we fix the desired accuracy and using the inequality (31) we obtain

$$\mathbb{E}[\epsilon_{sad}] = \tilde{O}\left(\frac{\mu_r R_k^r}{2}\right) = \tilde{O}\left(\frac{a_q M_2 \sqrt{d}}{\sqrt{N_k}} R_{k-1}\right) = \tilde{O}\left(\frac{\mu_r R_0^r}{2^{kr}}\right) \leq \epsilon. \quad (34)$$

Then to fulfill this condition, one can choose $k = \log_2(\tilde{O}(\mu_r R_0^r/\epsilon))/r$ and using Eq. (33) we get the total number of iterations

$$N_{acc} = \tilde{O}\left(\frac{2^{2k(r-1)}a_q^2M_2^2d}{\mu_1^2R_0^{2(r-1)}}\right) = \tilde{O}\left(\frac{a_q^2 M_2^2 d}{\mu_1^{2/r} \epsilon^{2(r-1)/r}}\right).$$

\[\square\]

**Remark 2.** If in Theorem 4.2, we use a tighter inequality (15) instead of (27) (as in Theorem 3.1), then the estimations on the $\Delta$ and $M_{2,\delta}$ can be improved. Choosing $u = (x^*, y^*)$ we can provide exponentially decreasing sequence of $D_k = \mathbb{E}\|z_k - u\|_2$ in Eq. (30) and get

1. under Assumption 2.2 $\Delta \lesssim \frac{\mu_1^{1/r} \epsilon^{2-1/r}}{M_2 \sqrt{d}}$

2. under Assumption 2.3 $M_{2,\delta} \lesssim \frac{\mu_1^{1/r} \epsilon^{1-1/r}}{\sqrt{d}}$.

**Remark 3.** Based on high-probability bound we can obtain high-probability bound by using the same restart-technique (Juditsky and Nesterov, 2014). In the case $r = 2$ it is probably possible to improve this bound in $\log \epsilon^{-1}$ factor by using alternative algorithm (Harvey et al., 2019).

**Theorem 4.3.** Let function $f(x, y, \xi)$ satisfy the Assumption D.1 and Assumption 2.2 for some constant $c$. Let the learning rate of Algorithm 1 is chosen as $\gamma_k = \frac{D}{M} \sqrt{\frac{2}{N}}$, where $N$ is the number of iterations and $M^2 \triangleq dM_2^2 + d^2 \Delta^2 \tau^{-2}$. Then for the output $(\hat{x}^N, \hat{y}^N)$ of Algorithm 1, the following holds

$$P\left\{f(\hat{x}^N, \hat{y}^N) - f(x^*, y^*) \geq \left(4\Omega^2 + \frac{2\sqrt{2}}{c} + \frac{2}{\sqrt{2}}\right)\frac{DM}{\sqrt{N}} + \frac{\Omega D \sqrt{d\Delta}}{\tau} + 2\tau M_2 + \Omega \frac{2\sqrt{2}MD}{cN}\right\} \leq \exp\{-\Omega^2/12\} + \exp\{-3\sqrt{N}\Omega/4\} + 2\exp\{-\Omega^2/3\} + 6\exp\{-\Omega/4\}. \quad (35)$$
Let $\tau$ be chosen as $\tau = \mathcal{O}(\epsilon/M_2)$ in randomized smoothing (4), where $\epsilon$ is the desired accuracy to solve problem (1). If also Assumption 2.2 holds true with $\Delta = \mathcal{O}\left(\frac{\epsilon^2}{D M_2 \sqrt{d}}\right)$ then for the output $\hat{z}^N \triangleq (\hat{x}^N, \hat{y}^N)$ of Algorithm 1, it holds $\mathbb{P}\left\{\hat{\varepsilon}_{sad}(\hat{z}^N) \leq \epsilon \right\} \geq 1 - \sigma$ after

$$N = \mathcal{O}\left(d M_2^2 D^2 / \epsilon^2\right)$$

iterations. If moreover Assumption 4.1 is satisfied we can apply restarts and to achieve $\mathbb{P}\left\{\hat{\varepsilon}_{sad}(\hat{z}^N) \leq \epsilon \right\} \geq 1 - \sigma$ in $N_{\text{acc}}$ iterations where $N_{\text{acc}}$ is given by

$$N_{\text{acc}} = \tilde{O}\left(\frac{M_2^3 d}{\mu_p^2 \epsilon^2 (r-1)/r}\right).$$

**Proof.** We recall inequality (96) from Theorem D.2:

$$\mathbb{P}\left\{ f(\hat{x}^N, y^*) - f(x^*, \hat{y}^N) \geq \frac{4 \Omega^2 \sqrt{d} \|z^1 - z^*\|_p M_2 \sqrt{\sum_{k=1}^{N} \gamma_k^2}}{\sum_{k=1}^{N} \gamma_k} + \frac{2 M_2^2 d \sum_{k=1}^{N} \gamma_k^2}{c \sum_{k=1}^{N} \gamma_k} + \frac{\Omega D \sqrt{d} \Delta}{\tau} + \frac{2 M_2^2 d \sqrt{\sum_{k=1}^{N} \gamma_k^4}}{c \sum_{k=1}^{N} \gamma_k} + \frac{d^2 \Delta^2 \sum_{k=1}^{N} \gamma_k^2}{2 \tau^2 \sum_{k=1}^{N} \gamma_k} + \frac{\|z^1 - z^*\|_p^2}{\sum_{k=1}^{N} \gamma_k} + 2 \tau M_2 \right\} \leq \exp\{-\Omega^2/12\} + \exp\{-3\sqrt{N} \Omega / 4\} + 2 \exp\{-\Omega^2/3\} + 6 \exp\{-\Omega/4\}. \quad (37)$$

Choosing learning rate $\gamma_k = \frac{\|z^1(z^*)_p}{M} \sqrt{\frac{2}{N}}$ with $M^2 \triangleq d M_2^2 + d^2 \Delta^2 \tau^{-2}$ in Eq. (37) we obtain

$$\mathbb{P}\left\{ f(\hat{x}^N, y^*) - f(x^*, \hat{y}^N) \geq \frac{4 \Omega^2 \sqrt{d} \|z^1 - z^*\|_p M_2 \sqrt{\sum_{k=1}^{N} \gamma_k^2}}{\sqrt{N}} + \frac{2 \sqrt{2} \|z^1 - z^*\|_p M_2 d}{c M \sqrt{N}} + \frac{\Omega D \sqrt{d} \Delta}{\tau} + \frac{2 M_2 \sqrt{2} M_2^2 d \sqrt{\sum_{k=1}^{N} \gamma_k^4}}{c M \sqrt{N}} + \frac{2 \tau M_2 \sqrt{2} D \sqrt{\sum_{k=1}^{N} \gamma_k^2}}{2 \tau^2 M \sqrt{N}} + \frac{\|z^1 - z^*\|_p M}{\sqrt{2 N}} + 2 \tau M_2 \right\} \leq \exp\{-\Omega^2/12\} + \exp\{-3\sqrt{N} \Omega / 4\} + 2 \exp\{-\Omega^2/3\} + 6 \exp\{-\Omega/4\}. \quad (38)$$

Using the notation of $M$ we obtain:

$$\mathbb{P}\left\{ f(\hat{x}^N, y^*) - f(x^*, \hat{y}^N) \geq \left(4 \Omega^2 + \frac{2 \sqrt{2}}{c} + \frac{2}{\sqrt{2}}\right) \frac{\|z^1 - z^*\|_p M}{\sqrt{N}} + \frac{\Omega D \sqrt{d} \Delta}{\tau} + 2 \tau M_2 + \frac{2 \sqrt{2} M \|z^1 - z^*\|_p}{c N} \right\} \leq \exp\{-\Omega^2/12\} + \exp\{-3\sqrt{N} \Omega / 4\} + 2 \exp\{-\Omega^2/3\} + 6 \exp\{-\Omega/4\}. \quad (39)$$

**Step 3.** (Restarts)
In this step we will employ the restart technique that is generalization of technique proposed
We note that now the notation \( \tilde{O}_1 \) contains the factor \( \log \sigma_1^{-1} \).

For the l.h.s. of Eq. (40) we use Assumption 4.1 then with probability at least \( 1 - \sigma_1 \)

\[
\frac{\mu_r}{2} \left\| z^{N_1} - z^* \right\|_p^r \leq f \left( \hat{x}^{N_1}, y^* \right) - f \left( x^*, \hat{y}^{N_1} \right) = \tilde{O} \left( M_2 \sqrt{d} \frac{\|z^1 - z^*\|_p}{\sqrt{N_1}} \right).
\]  

(41)

Then taking \( N_1 \) so as to reduce the distance to the solution by half, we obtain

\[
N_1 = \tilde{O}_1 \left( \frac{M_2^2 d}{\mu_r^2 R_1^{2(r-1)}} \right).
\]

(42)

Next, after \( N_1 \) iterations, we restart the original method and set \( z^1 = z^{N_1} \). We determine \( N_2 \) from a similar condition for reducing the distance \( R_1 \) to the solution by a factor of 2, and so on. We remind that at each restart step \( i \in \{1,k\} \), the resulting formula (41) is valid only with probability \( 1 - \sigma_i \). Thus, we choose \( \sigma_i = \sigma/k \) and then by the union bound inequality all inequalities are satisfied simultaneously with probability at least \( 1 - \sigma \). We will determine the number of restarts further, but at this stage we use the fact that \( k \) depends on the accuracy only logarithmically, which entails that \( \forall i \in \{1,k\} \tilde{O}_i = \tilde{O} \).

Thus, after \( k \) of such restarts, the total number of iterations will be

\[
N_{acc} = N_1 + \cdots + N_k = \tilde{O} \left( \frac{2^{2(r-1)} M_2^2 d}{\mu_r^2 R_0^{2(r-1)}} \left( 1 + 2^{2(r-1)} + \cdots + 2^{2(k-1)(r-1)} \right) \right).
\]

(43)

It remains for us to determine the number of restarts, for this we fix the desired accuracy in terms of \( \mathbb{P} \{ \hat{\epsilon}_{sad}(\hat{z}^N) \leq \epsilon \} \geq 1 - \sigma \) and using the inequality (41) we obtain

\[
\hat{\epsilon}_{sad} = \tilde{O} \left( \frac{\mu_r R_k}{2} \right) = \frac{M_2 \sqrt{d} \tilde{O}(R_{k-1})}{\sqrt{N_k}} = \tilde{O} \left( \frac{\mu_r R_0^{r}}{2^k r} \right) \leq \epsilon.
\]

(43)

Then to fulfill this condition one can choose \( k = \log_2 (\tilde{O} (\mu_r R_0^r / \epsilon)) / r \) and using Eq. (33) we get the total number of iterations

\[
N_{acc} = \tilde{O} \left( \frac{2^{2k(r-1)} M_2^2 d}{\mu_r^2 R_0^{2(r-1)}} \right) = \tilde{O} \left( \frac{M_2^2 d}{\mu_r^2 \epsilon^{2(r-1)/r}} \right).
\]

\[\square\]
5 Infinite Noise Variance

When the second moment of the stochastic gradient $\nabla f(z, \xi)$ is unbounded the rate of convergence may changes dramatically, see the next section. For such a case, we consider a more general inequality for Assumptions 2.1. We suppose that there exists a positive constant $\tilde{M}_2$ such that for $M_2(\xi)$ Assumptions 2.1 the following holds

$$\mathbb{E} [M_2(\xi)^{1+\kappa}] \leq \tilde{M}_2^{1+\kappa},$$

where $\kappa \in (0, 1]$. From Shamir (2017, Lemmas 9 – 11) the following can be obtained

1. under Assumption 2.2:

$$\mathbb{E} \left[ \| g(z, \xi, e) \|_{q}^{1+\kappa} \right] \leq \tilde{c} a_q^2 d^{(1+\kappa)/2} \tilde{M}_2^{1+\kappa} + 2^{1+\kappa} d^{1+\kappa} a_q^2 \Delta^2 \tau^{-2} = \tilde{M}_{case1}^{1+\kappa}, \quad (44)$$

2. under Assumption 2.3:

$$\mathbb{E} \left[ \| g(z, \xi, e) \|_{q}^{1+\kappa} \right] \leq \tilde{c} a_q^2 d^{(1+\kappa)/2} (\tilde{M}_2^{1+\kappa} + M_{2,\delta}^{1+\kappa}) = \tilde{M}_{case2}^{1+\kappa}, \quad (45)$$

where $\tilde{c}$ is some numerical constant and $\sqrt{\mathbb{E} \left[ \| e \|_{q}^{2+2\kappa} \right]} \leq \tilde{a}_q^2$. As a particular case: $\tilde{a}_2^2 = 1$, $\tilde{a}_\infty^2 = \mathcal{O} \left( \frac{\log(1+\kappa)/2 d}{d^{(1+\kappa)/2}} \right)$.

Let us assume that $q \in [1 + \kappa, \infty)$, $1/p + 1/q = 1$ and prox-function determined by

$$\omega(x) = K_q^{1/\kappa} \frac{\kappa}{1+\kappa} || x \|_p^{1+\kappa} \text{ with } K_q = 10 \max \{1, (q-1)^{(1+\kappa)/2} \}.$$  

Based on (Vural et al., 2022) one can prove the first part of Theorem 4.2 with $M_2$ replaced by $\tilde{M}_2$, stepsize is chosen as

$$\gamma_k = \frac{((1+\kappa) V_{z_1}(z^*)/\kappa)^{1/\kappa}}{\tilde{M}_{case}} N^{-\frac{1}{1+\kappa}},$$

and the first terms in the r.h.s. of (24), (25) is determined as follows

$$\tilde{M}_{case} \left( \frac{1+\kappa}{\kappa} V_{z_1}(z^*) \right)^{\frac{\kappa}{1+\kappa}} N^{-\frac{\kappa}{1+\kappa}}.$$  

These results can be further generalized to $r-$growth condition for duality gap ($r \geq 2$), see Assumption 4.1.

6 Auxiliary Results for Theorem 3.1

This section presents auxiliary results to prove Theorem 3.1 from Section 3.
Lemma 6.1. Let vector $e$ be a random unit vector from the Euclidean unit sphere $\{e : \|e\|_2 = 1\}$. Then it holds for all $r \in \mathbb{R}^d$

$$\mathbb{E}_e [\|e, r\|] \leq \|r\|_2/\sqrt{d}.$$ 

Lemma 6.2. Let $f(z)$ be $M_2$-Lipschitz continuous function. Then for $f^\tau(z)$ from (4), it holds

$$\sup_{z \in \mathbb{Z}} |f^\tau(z) - f(z)| \leq \tau M_2.$$

Proof. By the definition of $f^\tau(z)$ from (4) and Assumption 2.1 we have

$$|f^\tau(z) - f(z)| = |\mathbb{E}_{e} [f(z + \tau e)] - f(z)| = \mathbb{E}_e [\|f(z + \tau e) - f(z)\|] \leq \mathbb{E} [M_2\|\tau e\|_2] = M_2 \tau.$$

Lemma 6.3. Function $f^\tau(z)$ is differentiable with the following gradient

$$\nabla f^\tau(z) = \mathbb{E}_e \left[ \frac{d}{\tau} f(z + \tau e) e \right].$$

Lemma 6.4. For $g(z, \xi, e)$ from (3) and $f^\tau(z)$ from (4), the following holds

1. under Assumption 2.2

$$\mathbb{E}_{\xi, e} [\langle g(z, \xi, e), r \rangle] \geq \langle \nabla f^\tau(z), r \rangle - d \Delta \tau^{-1} \mathbb{E}_e [\|e, r\|],$$

2. under Assumption 2.3

$$\mathbb{E}_{\xi, e} [\langle g(z, \xi, e), r \rangle] \geq \langle \nabla f^\tau(z), r \rangle - d M_2 \delta \mathbb{E}_e [\|e, r\|],$$

Proof. Let us consider

$$g(z, e, \xi) = \frac{d}{2\tau} (\varphi(z + \tau e, \xi) - \varphi(z - \tau e, \xi)) e$$

$$= \frac{d}{2\tau} (f(z + \tau e, \xi) + \delta(z + \tau e) - f(z - \tau e, \xi) - \delta(z - \tau e)) e$$

$$= \frac{d}{2\tau} ((f(z + \tau e, \xi) - f(z - \tau e, \xi)) e + (\delta(z + \tau e) - \delta(z - \tau e)) e).$$

Using this we have the following

$$\mathbb{E}_{\xi, e} [\langle g(z, \xi, e), r \rangle] = \frac{d}{2\tau} \mathbb{E}_{\xi, e} [\langle (f(z + \tau e, \xi) - f(z - \tau e, \xi)) e, r \rangle]$$

$$+ \frac{d}{2\tau} \mathbb{E}_e [\langle (\delta(z + \tau e) - \delta(z - \tau e)) e, r \rangle].$$

(46)
Taking the expectation, w.r.t. \(e\), from the first term of the r.h.s. of (46) and using the symmetry of the distribution of \(e\), we have

\[
\frac{d}{2\tau} \mathbb{E}_{\xi,e} \left[ \langle f(z + \tau e, \xi) - f(z - \tau e, \xi) \rangle e, r \right] = \frac{d}{2\tau} \mathbb{E}_{\xi,e} \left[ \langle f(z + \tau e, \xi) \rangle e, r \right] + \frac{d}{2\tau} \mathbb{E}_{\xi,e} \left[ \langle f(z - \tau e, \xi) \rangle e, r \right] \\
= \frac{d}{\tau} \mathbb{E}_e \left[ \langle \mathbb{E}_\xi [f(z + \tau e, \xi)] \rangle e, r \right] = \frac{d}{\tau} \mathbb{E}_e \left[ \langle f(z + \tau e) \rangle e, r \right] \\
= \langle \nabla f^r(z), r \rangle. / * \text{Lemma 6.3} */
\]

(47)

1. For the second term of the r.h.s. of (46) under the Assumption 2.2 we obtain

\[
\frac{d}{2\tau} \mathbb{E}_e \left[ \langle (\delta(z + \tau e) - \delta(z - \tau e)) \rangle e, r \right] \geq - \frac{d}{2\tau} 2\Delta \mathbb{E}_e \left[ \|\langle e, r \rangle\| \right] = - \frac{d\Delta}{\tau} \mathbb{E}_e \left[ \|\langle e, r \rangle\| \right].
\]

(48)

2. For the second term of the r.h.s. of (46) under the Assumption 2.3 we obtain

\[
\frac{d}{2\tau} \mathbb{E}_e \left[ \langle (\delta(z + \tau e) - \delta(z - \tau e)) \rangle e, r \right] \geq - \frac{d}{2\tau} M_2 \, \delta \mathbb{E}_e \left[ \|e\|_2 \|\langle e, r \rangle\| \right] = -dM_2 \, \delta \mathbb{E}_e \left[ \|\langle e, r \rangle\| \right].
\]

(49)

Using Eq. (47) and (48) (or Eq. (49)) for Eq. (46) we get the statement of the lemma.

\[\square\]

**Lemma 6.5.** Shamir (2017, Lemma 9) For any function \(f(e)\) which is \(M\)-Lipschitz w.r.t. the \(\ell_2\)-norm, it holds that if \(e\) is uniformly distributed on the Euclidean unit sphere, then

\[
\sqrt{\mathbb{E}[(f(e) - \mathbb{E}f(e))^4]} \leq cM_2^2/d
\]

for some numerical constant \(c\).

**Lemma 6.6.** For \(g(z, \xi, e)\) from (3), the following holds under Assumption 2.1

1. and Assumption 2.2

\[
\mathbb{E}_{\xi,e} \left[ \|g(z, \xi, e)\|^2_q \right] \leq ca_q^2 dM_2^2 + d^2 a_q^2 \Delta^2/\tau^2,
\]

2. and Assumption 2.3

\[
\mathbb{E}_{\xi,e} \left[ \|g(z, \xi, e)\|^2_q \right] \leq ca_q^2 (M_2^2 + M_2^2, \delta),
\]

where \(c\) is some numerical constant and \(\sqrt{\mathbb{E}[\|e\|^4_q]} \leq a_q^2\).
Proof. Let us consider

\[
\mathbb{E}_{\xi,e} \left[ \|g(z, \xi, e)\|^2_q \right] = \mathbb{E}_{\xi,e} \left[ \left\| \frac{d}{2\tau} (\varphi(z + \tau e, \xi) - \varphi(z - \tau e, \xi)) e \right\|_q^2 \right]
\]

\[
= \frac{d^2}{4\tau^2} \mathbb{E}_{\xi,e} \left[ \|e\|^2_q (f(z + \tau e, \xi) + \delta(z + \tau e) - f(z - \tau e, \xi) - \delta(z - \tau e))^2 \right]
\]

\[
\leq \frac{d^2}{2\tau^2} \left( \mathbb{E}_{\xi,e} \left[ \|e\|^2_q (f(z + \tau e, \xi) - f(z - \tau e, \xi))^2 \right] + \mathbb{E}_e \left[ \|e\|^2_q (\delta(z + \tau e) - \delta(z - \tau e))^2 \right] \right).
\]

(50)

For the first term in the r.h.s. of Eq. (50), the following holds with an arbitrary parameter \(\alpha\)

\[
\frac{d^2}{2\tau^2} \mathbb{E}_{\xi,e} \left[ \|e\|^2_q (f(z + \tau e, \xi) - f(z - \tau e, \xi))^2 \right]
\]

\[
= \frac{d^2}{2\tau^2} \mathbb{E}_{\xi,e} \left[ \|e\|^2_q ((f(z + \tau e, \xi) - \alpha) - (f(z - \tau e, \xi) - \alpha))^2 \right]
\]

\[
\leq \frac{d^2}{\tau^2} \mathbb{E}_{\xi,e} \left[ \|e\|^2_q (f(z + \tau e, \xi) - \alpha)^2 + (f(z - \tau e, \xi) - \alpha)^2 \right] \quad */ \forall a, b, (a - b)^2 \leq 2a^2 + 2b^2 /* / \n\]

\[
= \frac{d^2}{\tau^2} \left( \mathbb{E}_{\xi,e} \left[ \|e\|^2_q (f(z + \tau e, \xi) - \alpha)^2 \right] + \mathbb{E}_{\xi,e} \left[ \|e\|^2_q (f(z - \tau e, \xi) - \alpha)^2 \right] \right)
\]

\[
= \frac{2d^2}{\tau^2} \mathbb{E}_{\xi,e} \left[ \|e\|^2_q (f(z + \tau e, \xi) - \alpha)^2 \right]. \quad */ \text{the distribution of } e \text{ is symmetric } /* / \n\]

(51)

Applying the Cauchy–Schwarz inequality for Eq. (51) and using \(\sqrt{\mathbb{E} \left[ \|e\|^4_q \right]} \leq a_q^2\) we obtain

\[
\frac{d^2}{\tau^2} \mathbb{E}_{\xi,e} \left[ \|e\|^2_q (f(z + \tau e, \xi) - \alpha)^2 \right] \leq \frac{d^2}{\tau^2} \mathbb{E}_\xi \left[ \sqrt{\mathbb{E} \left[ \|e\|^4_q \right]} \sqrt{\mathbb{E}_e \left[ f(z + \tau e, \xi) - \alpha \right]^4} \right]
\]

\[
\leq \frac{d^2a_q^2}{\tau^2} \mathbb{E}_\xi \left[ \sqrt{\mathbb{E}_e \left[ f(z + \tau e, \xi) - \alpha \right]^4} \right]. \quad (52)
\]

Then we assume \(\alpha = \mathbb{E}_e [f(z + \tau e, \xi)]\) and use Lemma 6.5 along with the fact that \(f(z + \tau e, \xi)\) is \(\tau M_2(\xi)\)-Lipschitz, w.r.t. \(e\) in terms of the \(\ell_2\)-norm. Thus for Eq. (52), it holds

\[
\frac{d^2a_q^2}{\tau^2} \mathbb{E}_\xi \left[ \sqrt{\mathbb{E}_e \left[ f(z + \tau e, \xi) - \alpha \right]^4} \right] \leq \frac{d^2a_q^2}{\tau^2} \cdot c \frac{\tau^2 \mathbb{E}_e M_2^2(\xi)}{d} = cda_q^2M_2^2. \quad (53)
\]

1. Under the Assumption 2.2 for the second term in the r.h.s. of Eq. (50), the following holds

\[
\frac{d^2}{4\tau^2} \mathbb{E}_e \left[ \|e\|^2_q (\delta(z + \tau e) - \delta(z - \tau e))^2 \right] \leq \frac{d^2\Delta^2}{\tau^2} \mathbb{E} \left[ \|e\|^2_q \right] \leq \frac{d^2a_q^2\Delta^2}{\tau^2}. \quad (54)
\]

2. Under the Assumption 2.3 for the second term in the r.h.s. of Eq. (50), the following holds
with an arbitrary parameter $\beta$

$$\frac{d^2}{4\tau^2} \mathbb{E}_e [\|e\|_q^2 (\delta(z + \tau e) - \delta(z - \tau e))^2]$$

$$= \frac{d^2}{2\tau^2} \mathbb{E} [\|e\|_q^2 ((\delta(z + \tau e) - \beta) - (\delta(z - \tau e) - \beta))^2]$$

$$\leq \frac{d^2}{\tau^2} \mathbb{E} [\|e\|_q^2 ((\delta(z + \tau e) - \beta)^2 + (\delta(z - \tau e) - \beta)^2)] \quad \forall a, b, (a - b)^2 \leq 2a^2 + 2b^2 \ast /
$$

$$= \frac{d^2}{\tau^2} (\mathbb{E} [\|e\|_q^2 (\delta(z + \tau e) - \beta)^2] + \mathbb{E} [\|e\|_q^2 (\delta(z - \tau e) - \beta)^2])$$

$$= \frac{d^2}{\tau^2} \mathbb{E} [\|e\|_q^2 (\delta(z + \tau e) - \beta)^2]. \quad \ast \text{ the distribution of } e \text{ is symmetric } \ast / \quad (55)$$

and for the second term in the Eq. (50), we get

$$\frac{d^2}{\tau^2} \mathbb{E}_e [\|e\|_q^2 (\delta(z + \tau e) - \beta)^2] \leq \frac{d^2}{\tau^2} \sqrt{\mathbb{E} [\|e\|_q^4]} \sqrt{\mathbb{E}_e [(\delta(z + \tau e) - \beta)^4]}$$

$$\leq \frac{d^2 a_q^2}{\tau^2} \sqrt{\mathbb{E}_e [(\delta(z + \tau e) - \beta)^4]} \quad (56)$$

Then we assume $\beta = \mathbb{E}_e [\delta(z + \tau e)]$ and use Lemma 6.5 together with the fact that $\delta(z + \tau e)$ is $\tau M_{2,\delta}$-Lipschitz continuous, w.r.t. $e$ in terms of the $\ell_2$-norm. Thus, Eq. (56) can be rewritten as

$$\frac{d^2 a_q^2}{\tau^2} \sqrt{\mathbb{E}_e [(\delta(z + \tau e) - \beta)^4]} \leq \frac{d^2 a_q^2}{\tau^2} \cdot c \frac{\tau^2 M_{2,\delta}^2}{d} \leq c a_q^2 d M_{2,\delta}^2. \quad (57)$$

Using Eq. (53) and (54) (or Eq. (57)) for Eq. (50), we get the statement of the theorem. \qed

### Conclusion

In this paper, we demonstrate how to solve non-smooth stochastic convex-concave saddle point problems with two-point gradient-free oracle. In the Euclidean proximal setup, we obtain oracle complexity bound proportional to $d/\epsilon^2$ that is optimal. We also generalize this result for an arbitrary proximal setup and obtain a tight upper bound on maximal level of additive adversary noise in oracle calls proportional to $\epsilon^2/\sqrt{d}$. We generalize this result for the class of saddle point problems satisfying $\gamma$-growth condition for duality gap and get a bound proportional to $d/\epsilon^{2(\gamma - 1)/\gamma}$ for the oracle complexity in the Euclidean proximal setup with $\sim \epsilon^2/\sqrt{d}$ maximal level of additive adversary noise in oracle calls.

### Acknowledgments

This work was supported by a grant for research centers in the field of artificial intelligence, provided by the Analytical Center for the Government of the Russian Federation in accordance
with the subsidy agreement (agreement identifier 000000D730321P5Q0002 ) and the agreement with the Ivannikov Institute for System Programming of the Russian Academy of Sciences dated November 2, 2021 No. 70-2021-00142.

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In this section, we give some possible generalizations for our results. For simplicity, we consider a non-stochastic non-smooth convex optimization problem in the Euclidean proximal setup

\[
\min_{x \in X \subseteq \mathbb{R}^d} f(x),
\]

where \( X \subseteq \mathbb{R}^d \) is a convex set and \( f \) is \( M \)-Lipschitz continuous. In this problem, we replace the non-smooth objective \( f(x) \) by its smooth approximation \( f^\tau(x) \) defined in (4), i.e.: \( f^\tau(x) \triangleq \mathbb{E}_\tilde{e} f(x + \tau \tilde{e}) \), where \( \tau > 0 \) is some constant and \( \tilde{e} \) is a vector picked uniformly at random from the Euclidean unit ball \( \{ \tilde{e} : \| \tilde{e} \|_2 \leq 1 \} \). From (Duchi et al., 2015) it follows that

\[
f(x) \leq f^\tau(x) \leq f(x) + \tau M. \tag{59}
\]

Let the objective \( f(x) \) can be not directly observed but instead, its noisy approximation \( \varphi(x) \triangleq f(x) + \delta(x) \) can be queried, where \( \delta(x) \) is some adversarial noise such that \( \| \delta(x) \| \leq \Delta \). Similarly to (3) we can estimate the gradient of \( \varphi(x) \) d by the following gradient-free approximation

\[
g(x,e) = \frac{d}{2\tau} (\varphi(x + \tau e) - \varphi(x - \tau e)) e,
\]

A The Basic Idea and Possible Generalization

In this section, we give some possible generalizations for our results. For simplicity, we consider a non-stochastic non-smooth convex optimization problem in the Euclidean proximal setup

\[
\min_{x \in X \subseteq \mathbb{R}^d} f(x),
\]

where \( X \subseteq \mathbb{R}^d \) is a convex set and \( f \) is \( M \)-Lipschitz continuous. In this problem, we replace the non-smooth objective \( f(x) \) by its smooth approximation \( f^\tau(x) \) defined in (4), i.e.: \( f^\tau(x) \triangleq \mathbb{E}_\tilde{e} f(x + \tau \tilde{e}) \), where \( \tau > 0 \) is some constant and \( \tilde{e} \) is a vector picked uniformly at random from the Euclidean unit ball \( \{ \tilde{e} : \| \tilde{e} \|_2 \leq 1 \} \). From (Duchi et al., 2015) it follows that

\[
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Let the objective \( f(x) \) can be not directly observed but instead, its noisy approximation \( \varphi(x) \triangleq f(x) + \delta(x) \) can be queried, where \( \delta(x) \) is some adversarial noise such that \( \| \delta(x) \| \leq \Delta \). Similarly to (3) we can estimate the gradient of \( \varphi(x) \) d by the following gradient-free approximation

\[
g(x,e) = \frac{d}{2\tau} (\varphi(x + \tau e) - \varphi(x - \tau e)) e,
\]
Due to (Gasnikov et al., 2017) (see also Lemma 6.1) for all $r \in \mathbb{R}^d$
\[ \mathbb{E}_e [(g(x, e) - \nabla f^\tau(x), r)] \lesssim \sqrt{d}\Delta \|r\|_2^{-1} \] (60)
and (Shamir, 2017; Beznosikov et al., 2020)
\[ \mathbb{E}_e \|g(x, e) - \mathbb{E}_e \nabla f^\tau(x, e)\|_2^2 \simeq \mathbb{E}_e \|g(x, e)\|_2^2 \lesssim dM^2 + d^2\Delta^2\tau^{-2}, \] (61)
where $e$ is a random vector uniformly distributed on the Euclidean unit sphere \{ $e : \|e\|_2 = 1$ \}.
Bound (60) is $\sqrt{d}$ better than in the bounds from (Beznosikov et al., 2020; Akhavan et al., 2020). Moreover, for $M_{2,\delta}$-Lipschitz noise, Eq. (60) can be rewritten as follows
\[ \mathbb{E}_e [(g(x, e) - \nabla f^\tau(x), r)] \lesssim \sqrt{d}M_{2,\delta}\|r\|_2. \] (62)

We will say that an algorithm $A$ (with $g(x, e)$ oracle) is robust for $f^\tau$ if the bias in the l.h.s. of (60) does not accumulate over method iterations. That is, if for $A$ with $\Delta = 0$
\[ \mathbb{E} f^\tau(x^N) - \min_{x \in \mathcal{X}} f^\tau(x) \leq \Theta_A(N), \] then with $\Delta > 0$ and (variance control) $d^2\Delta^2\tau^{-2} \lesssim dM^2$, see (61):
\[ \mathbb{E} f^\tau(x^N) - \min_{x \in \mathcal{X}} f^\tau(x) = O \left( \Theta_A(N) + \sqrt{d}\Delta \mathcal{D}\tau^{-1} \right), \] (63)
where $\mathcal{D}$ is a diameter of $\mathcal{X}$. Here $N$ should be taken such that the first term of the r.h.s. of Eq. (63) is not smaller than the second one. Similar definition can be made for (62). Many known methods are robust, e.g., stochastic versions of mirror descent, mirror prox, gradient method and fast gradient method are robust (Juditsky and Nemirovski, 2012; d’Aspremont, 2008; Cohen et al., 2018; Dvinskikh et al., 2020; Dvinskikh and Gasnikov, 2021; Gorbunov et al., 2019a).

Now we explain how to obtain the bound on the level of noise $\Delta$.

**Approximation.** To approximate non-smooth function $f(x)$ by smooth function $f^\tau(x)$, constant $\tau$ should be taken as follows: $\tau = \frac{\epsilon}{2M}$ (see (59)).

**Variance control.** To control the variance and obtain the second moment of the stochastic gradient with $\Delta > 0$ and $\Delta = 0$ of the same order, $\Delta$ should be taken not bigger that (see (61)): $\Delta \lesssim \frac{\epsilon M}{\sqrt{d}}$.

**Bias.** From (63) we will have more restrictive condition on the level of noise: $\Delta \lesssim \frac{\epsilon M}{\mathcal{D}\sqrt{d}}$. Combining the bias condition and approximation condition leads to the following bound on the $\Delta$:
\[ \Delta \lesssim \frac{\epsilon^2}{DM\sqrt{d}}. \] (64)

For Lipschitz noise and for saddle point problems, the same bound holds by the same reasoning.

More interestingly, as stochastic (batched) versions of fast gradient method and mirror prox are also robust (Gorbunov et al., 2019a; Stonyakin et al., 2021) and $f^\tau$ has $\sqrt{dM/\tau}$-Lipschitz gradient (Duchi et al., 2012; Gasnikov et al., 2022), we can improve the results of this paper by using parallelized smoothing technique from (Gasnikov et al., 2022). For instance, for non-smooth convex optimization problem (58), the gradient-free method from (Gasnikov et al., 2022) (with optimal number of oracle calls and the best known number of consistent iterations) provides also the highest possible level of noise, given by (64).
B “Upper” bound in the case of Lipschitz noise

Now let us consider a stochastic convex optimization problem of the form

$$\min_{x \in X} F(x) \triangleq E_\xi f(x, \xi),$$  \hfill (65)

where $X \subseteq \mathbb{R}^d$ is a convex set, and for all $\xi$, $f(x, \xi)$ is convex in $x \in X$ and satisfies Assumption 2.1. The empirical counterpart of this problem (65) is

$$\min_{x \in X} \hat{F}(x) \triangleq \frac{1}{N} \sum_{k=1}^N f(x, \xi^k).$$  \hfill (66)

The exact solution ($\epsilon/2$-solution) of (66) is an $\epsilon$-solution of (65) if the sample size $N$ is taken as follows (Shapiro and Nemirovski, 2005; Feldman, 2016)

$$N = \tilde{\Omega} \left( dM_2^2D_2^2/\epsilon^2 \right),$$  \hfill (67)

where $D_2$ is the diameter of $X$ in the $\ell_2$-norm. This lower bound is tight (Shapiro and Nemirovski, 2005; Shapiro et al., 2021).

On the other hand, $\hat{F}(x)$ from (66) can be considered as an inexact zeroth-order oracle for $F(x)$ from (65). If $\delta(x) = F(x) - \hat{F}(x)$ then in $\text{Poly}(d, 1/\epsilon)$ points $y, x$ with probability $1 - \beta$ the following holds

$$|\delta(y) - \delta(x)| = \mathcal{O} \left( \frac{M_2\|y - x\|_2}{\sqrt{N}} \ln \left( \frac{\text{Poly}(d, 1/\epsilon)}{\beta} \right) \right) = \tilde{\mathcal{O}} \left( \frac{M_2\|y - x\|_2}{\sqrt{N}} \right),$$

i.e., $\delta(x)$ is a Lipschitz function with Lipschitz constant

$$M_{2,\delta} = \tilde{\mathcal{O}} \left( M_2/\sqrt{N} \right).$$  \hfill (68)

Let us assume there exists a zeroth-order algorithm that can solve (65) with accuracy $\epsilon$ in $\text{Poly}(d, 1/\epsilon)$ oracle calls, where an oracle returns an inexact value of $F(x)$ with a noise that has the following Lipschitz constant

$$M_{2,\delta} \gg \frac{\epsilon}{D_2\sqrt{d}}.$$

We can use this algorithm to solve problem (65) with $N$ determines from (see (68))

$$\frac{M_2}{\sqrt{N}} \gg \frac{\epsilon}{D_2\sqrt{d}},$$

i.e.

$$N \ll dM_2^2D_2^2/\epsilon^2,$$

that contradicts the lower bound (67).
Thereby it is impossible in general to solve with accuracy $\epsilon$ (in function value) Lipschitz convex optimization problem via Poly $(d, 1/\epsilon)$ inexact zero-order oracle calls if Lipschitz constant of noise is greater than 
\[ \epsilon / (D_2 \sqrt{d}). \]  
(69)

Unfortunately, we obtain this upper bound assuming that Poly $(d, 1/\epsilon)$ points were chosen regardless of $\{\xi^k\}_{k=1}^N$. That is not the case for the most of practical algorithms, in particular, considered above. But the dependence of these points from $\{\xi^k\}_{k=1}^N$ is significantly weakened by randomization we use in zero-order methods. So we may expect that nevertheless this upper bound still takes place.

For arbitrary Poly $(d, 1/\epsilon)$ points (possibly that could depend on $\{\xi^k\}_{k=1}^N$) we can guarantee only (see (Shapiro et al., 2021))
\[ M_{2,\delta} = \tilde{O}(\sqrt{dM_2 / \sqrt{N}}). \]  
(70)
That is large than (68). Consequently, the upper bound (69) should be rewritten as 
\[ \epsilon / D_2. \]  
(71)

We believe that (71) is not tight upper bound, rather than (69). That is, there exists algorithm (see Algorithm 1) that can solve (65) with accuracy $\epsilon$ (in function value) via $\approx dM_2^2 D_2^2 / \epsilon^2$ inexact zero-order oracle calls if Lipschitz constant of noise is bounded from above by $M_{2,\delta} \lesssim \epsilon / (D_2 \sqrt{d})$. But there are no algorithms reaching the same accuracy $\epsilon$ by using Poly $(d, 1/\epsilon)$ inexact zero-order oracle calls if Lipschitz constant of noise is bounded from above by $M_{2,\delta} \gg \epsilon / (D_2 \sqrt{d})$, in particular, for $M_{2,\delta}$ given by (71).

Note, that (67) holds also if $f(x, \xi)$ has Lipschitz $x$-gradient (Feldman, 2016). So we can expect that obtained above lower bounds take place also for smooth problems. Known maximal level of noise bounds for concrete algorithms confirm this (Gasnikov et al., 2017; Berahas et al., 2021; Vasin et al., 2021).

C Intuition of Smoothing

In this section, we provide a motivation for smoothing a non-smooth function according to (4).

Subgradients of a non-smooth function cannot be estimated by the gradient-free approximation (3). Indeed, let us consider the following example.

**Example C.1.** Let us consider the one-dimensional ($d = 1$) deterministic function $f(x)$ from Figure 1 with zeros noise ($\delta = 0$) for simplicity. Then, gradient-free approximation (3) can be simplified as 
\[ g(x) = \frac{1}{2\tau} (f(x + \tau) - f(x - \tau)). \]  
(72)

Let us consider non-smooth function $f(x) = |x|$. For point $x > 0$ taken close to 0 we have 
\[ g(x) = \frac{x}{\tau}. \]
when \( x \in [-\tau, \tau] \). However, for all \( x > 0 \), \( \nabla f(x) = 1 \).

![Figure 1: Smooth approximation \( f^\tau(x) \) of a non-smooth function \( f(x) \)](image)

C.1 Differentiation of smooth approximation

For simplicity, let us proof Lemma 6.3 in the one dimensional case \((d = 1)\)

\[ f^\tau(x) \triangleq \mathbb{E}_{\tilde{e}} f(x + \tau \tilde{e}) = \frac{1}{2} \int_{-1}^{1} f(x + \tau \tilde{e}) d\tilde{e}, \]

where \( \tilde{e} \) is a random vector from segment \([-1, 1]\).

We get for the derivative of \( f \)

\[ (f^\tau(x))^' = \frac{1}{2} \int_{-1}^{1} f'_x(x + \tau \tilde{e}) d\tilde{e}. \]

With the change of variable \( y = x + \tau \tilde{e} \) we obtain

\[ (f^\tau(x))^'_x = \frac{1}{2\tau} \int_{x-\tau}^{x+\tau} f'_y(y) dy = \frac{d}{2\tau} (f(x + \tau) - f(x - \tau)) = \mathbb{E}_{e} \left[ \frac{d}{\tau} f(x + \nu e) \right], \]

where \( e \) is a random vector equal to 1 or -1 with probability \( 1/2 \).

D High-probability bound

**Assumption D.1** (Uniformly Lipschitz continuity of the objective). Function \( f(z, \xi) \) is uniformly \( M_2 \)-Lipschitz continuous in \( z \in \mathcal{Z} \) w.r.t. the \( \ell_2 \)-norm, i.e., for all \( z_1, z_2 \in \mathcal{Z} \) and \( \xi \in \Xi \),

\[ |f(z_1, \xi) - f(z_2, \xi)| \leq M_2 \|z_1 - z_2\|_2. \]

Let function \( f(x, y, \xi) \) satisfy the Assumption 2.1.
Theorem D.2. Let function $f(x,y,\xi)$ satisfy the Assumption D.1. Let the learning rate of Algorithm 1 is chosen as $\gamma_k = \frac{D}{M} \sqrt{\frac{2}{N}}$, where $N$ is the number of iterations and $M^2 \triangleq dM_2^2 + d^2 \Delta^2 \tau^{-2}$. Then for the output $(\hat{x}^N, \hat{y}^N)$ of Algorithm 1, under Assumption 2.2 for some constant $c$ the following holds

$$
\text{Prob}\left\{ \max_{y \in \mathcal{Y}} f(\hat{x}^N, y) - \min_{x \in \mathcal{X}} f(x, \hat{y}^N) \geq \left( 4 \Omega^2 + \frac{2\sqrt{2}}{c} + \frac{2}{\sqrt{2}} \right) \frac{\sqrt{V_z(u)M}}{\sqrt{N}} + \Omega \frac{2\sqrt{2}M \sqrt{V_z(u)}}{cN} + 2\tau M_2 + \frac{\Omega D \sqrt{d\Delta}}{\tau} \right\} \leq \exp\{-\Omega^2/12\} + \exp\{-3\sqrt{N} \Omega/4\} + 2 \exp\{-\Omega^2/3\} + 6 \exp\{-\Omega/4\},
$$

(73)

where $\epsilon_{sad} \triangleq \max_{y \in \mathcal{Y}} f(\hat{x}^N,y) - \min_{x \in \mathcal{X}} f(x,\hat{y}^N)$.

Proof of the Theorem D.2. By the definition $z^{k+1} = \text{Prox}_{z^k}(\gamma_k g(z^k,e^k,\xi^k))$ we get (Ben-Tal and Nemirovski, 2013), for all $u \in \mathcal{Z}$

$$
\gamma_k g(z^k,e^k,\xi^k), z^k - u \leq V_z(u) - V_{z^{k+1}}(u) + \gamma_k^2 \|g(z^k,e^k,\xi^k)\|^2_2 / 2
$$

Summing for $k = 1, \ldots, N$ we obtain, for all $u \in \mathcal{Z}$

$$
\sum_{k=1}^{N} \gamma_k g(z^k,e^k,\xi^k), z^k - u \leq V_z(u) + \sum_{k=1}^{N} \gamma_k^2 \|g(z^k,e^k,\xi^k)\|^2_2 / 2 \\
\leq V_z(u) + \sum_{k=1}^{N} \frac{\gamma_k^2}{2} \|g(z^k,e^k,\xi^k)\|^2_2, \quad (74)
$$

For the next steps we provide the new definition of gradient approximation of $\nabla f(z,\xi)$ following the Eq. (3):

$$
g_f(z, \xi, e) = \frac{d}{2\tau} (f(z + \tau e, \xi) - f(z - \tau e, \xi))(e_x, -e_y), \quad (75)
$$

Step 1.

In this step we will estimate the second term in the r.h.s of inequality (74). We can rewrite this term in the form :

$$
\|g(z^k,e^k,\xi^k)\|^2_2 = \frac{d^2}{4\tau^2} (f(z + \tau e, \xi) + \delta(z + \tau e) - f(z - \tau e, \xi) - \delta(z - \tau e))^2 \leq \\
\leq 2 \|g_f(z^k,e^k,\xi^k)\|^2_2 + \frac{d^2}{2\tau^2} (\delta(z + \tau e) - \delta(z - \tau e))^2 \quad (76)
$$

1. To estimate the first term in the r.h.s of Eq. (76) we invoke the Assumption D.1 and the Eq. (75). We observe that the function $g_f(z^k,e^k,\xi^k)$ is uniformly $M_2d$-Lipschitz
continious and \( \mathbb{E}_{e^k} g_f(z^k, e^k, \xi^k) = 0 \). Thus we can use Lemma 9 from (Shamir, 2017) and obtain for some constant \( c \):

\[
\text{Prob} \left\{ \| g_f(z^k, e^k, \xi^k) \|_2 > t \right\} \leq \exp \left\{ - \frac{ct^2}{M_2^2 d} \right\} \tag{77}
\]

Invoking the last inequality and Assumption D.1, the random variables

\[
\phi_k = \gamma_k^2 \| g_f(z^k, e^k, \xi^k) \|_2^2 \left( \sum_{k=1}^N \gamma_k \right)^{-1}
\]

satisfy the premise of case B of Lemma 2 from (Lan et al., 2012) with \( \sigma_k = \frac{2dM_2^2}{c} \gamma_k^2 \left( \sum_{k=1}^N \gamma_k \right)^{-1} :\)

\[
\mathbb{E}_{\mid t-1} \left[ \exp \{|\phi_k|/\sigma_k\} \right] = \mathbb{E}_{\mid t-1} \left[ \exp \{|\| g_f(z^k, e^k, \xi^k) \|_2^2 / 4M_2^2 d\} \right] =
\]

\[
= \int_0^\infty \text{Prob} \left\{ \exp \{|\| g_f(z^k, e^k, \xi^k) \|_2^2 / 4M_2^2 d\} \geq \bar{t} \right\} \, d\bar{t} \leq \int_0^1 1 \, d\bar{t} +
\]

\[
+ \int_1^\infty \text{Prob} \left\{ \exp \{|\| g_f(z^k, e^k, \xi^k) \|_2^2 / 4M_2^2 d\} \geq \bar{t} \right\} \, d\bar{t} \leq 1 + 1 \leq \exp\{1\}
\]

by (77): \( \leq \exp\left\{ -\frac{c\sigma_k^2 \ln \bar{t}}{M_2^2 d} \right\} \leq \left( \frac{1}{\bar{t}} \right)^{\frac{c\sigma_k^2}{M_2^2 d}} = \left( \frac{1}{\bar{t}} \right)^2 \)

Invoking case B of Lemma 2 from (Lan et al., 2012), we get

\[
\text{Prob} \left\{ \left| \sum_{k=1}^N \gamma_k^2 \| g_f(z^k, e^k, \xi^k) \|_2^2 \left( \sum_{k=1}^N \gamma_k \right)^{-1} \right| > \frac{2M_2^2 d \sum_{k=1}^N \gamma_k^2}{c \sum_{k=1}^N \gamma_k} + \Omega \frac{2M_2^2 d \sqrt{\sum_{k=1}^N \gamma_k^4}}{c \sum_{k=1}^N \gamma_k} \right\}
\]

\[
\leq \exp\{-\Omega^2 / 12\} + \exp\{-3\sqrt{N}\Omega / 4\} \tag{78}
\]

2. Under the Assumption 2.2 for the third term in the r.h.s. of Eq. (76), the following holds

\[
\frac{d^2}{2\tau^2} (\delta(z + \tau e) - \delta(z - \tau e))^2 \leq \frac{d^2 \Delta^2}{2\tau^2}. \tag{79}
\]

Thus using Eq.(78) and (79) we can estimate the second term in the r.h.s of inequality (74) as follow:

\[
\text{Prob} \left\{ \left| \sum_{k=1}^N \gamma_k^2 / 2 \| g(z^k, e^k, \xi^k) \|_2 \left( \sum_{k=1}^N \gamma_k \right)^{-1} \right| > \frac{2M_2^2 d \sum_{k=1}^N \gamma_k^2}{c \sum_{k=1}^N \gamma_k} + \Omega \frac{2M_2^2 d \sqrt{\sum_{k=1}^N \gamma_k^4}}{c \sum_{k=1}^N \gamma_k} +
\]

\[
+ \frac{d^2 \Delta^2 \sum_{k=1}^N \gamma_k^2}{2\tau^2 \sum_{k=1}^N \gamma_k} \right\} \leq \exp\{-\Omega^2 / 12\} + \exp\{-3\sqrt{N}\Omega / 4\} \tag{80}
\]

**Step 2.**

Using the notation (75) we rewrite the l.h.s. of Eq. (74) as following:
\[
\sum_{k=1}^{N} \gamma_k \langle g(z^k, e^k, \xi^k), z^k - u \rangle = \sum_{k=1}^{N} \gamma_k \langle g_f(z^k, e^k, \xi^k) - \nabla f^\tau(z^k), z^k - u \rangle + \\
+ \sum_{k=1}^{N} \gamma_k \langle \frac{d}{2\tau} (\delta(z^k + \tau e) - \delta(z^k - \tau e))e, z^k - u \rangle + \sum_{k=1}^{N} \gamma_k \langle \nabla f^\tau(z^k), z^k - u \rangle
\] (81)

1. For the first term in r.h.s. of Eq. (81), we provide the following notions:

\[\sigma_k = 2\gamma_k \sqrt{dM_2} \sqrt{V_{z^k}(u)}\Omega,\]
\[\phi_k(\xi_k) = \gamma_k \langle g_f(z^k, e^k, \xi^k) - \nabla f^\tau(z), z^k - u \rangle.\]

For applying the case A of Lemma 2 from (Lan et al., 2012) we need to estimate the module of function \(\gamma_k \langle g_f(z^k, e^k, \xi^k), z^k - u \rangle\). Using Assumption D.1 we obtain:

\[\left| \gamma_k \langle g_f(z^k, e^k, \xi^k), z^k - u \rangle \right| \leq \gamma_k dM_2 \left| \langle e, z^k - u \rangle \right|\]

\[\text{Ass.D.1} \leq \gamma_k dM_2 \left| \langle e, z^k - u \rangle \right|\] (82)

Now we need to estimate the term \(\left| \langle e, z^k - u \rangle \right|\). To do this, using the Poincaré’s lemma from (Lifshits, 2012) paragraph 6.3 we rewrite \(e\) in different form:

\[e \overset{D}{=} \frac{\eta}{\sqrt{\eta_1^2 + \cdots + \eta_d^2}},\]

where \(\eta = (\eta_1, \eta_2, \ldots, \eta_d)^T = \mathcal{N}(0, I_d)\). Using Lemma 1 (Laurent and Massart, 2000) it follows:

\[\text{Prob} \left\{ \sum_{k=1}^{d} |\eta_k|^2 \leq d - 2\sqrt{\Omega d} \right\} \leq \exp\{-\Omega\}.\] (84)

Using the definition of \(\eta\) we obtain:

\[\langle \eta, z^k - u \rangle \overset{D}{=} \mathcal{N} \left( 0, \|z^k - u\|_2^2 \right).\] (85)

Using Eq. (84), (85) and (83) we can estimate the r.h.s of Eq. (82):

\[\text{Prob} \left\{ \left| \langle e, z^k - u \rangle \right| > \frac{\Omega V_{z^k}(u)}{\sqrt{d}} \right\} \leq 3 \exp\{-\Omega/4\}.\] (86)

Moreover, using Eq. (82), Eq. (86) and the fact that \(V_{z^k}(u) = \mathcal{O}(V_{z^1}(u)) \forall k \geq 1\) from (Gorbunov et al., 2021) we have:
Prob \( \left\{ \exp \left\{ (\gamma_k \langle g_f(z^k, e^k, \xi^k) - \nabla f^\tau(z), z^k - u \rangle)^2 / \sigma_k^2 \right\} \geq \exp\{1\} \right\} \leq 3 \exp\{-\Omega/4\} \)

It follows by case A of Lemma 2 from (Lan et al., 2012) that

\[
\sum_{k=1}^N \gamma_k \langle \nabla f^\tau(z^k), z^k - u \rangle \geq - \sum_{k=1}^N \gamma_k \frac{d \Delta}{\tau} |\langle e, z^k - u \rangle| \quad (87)
\]

Using Eq. (92) for Eq. (91) we get

\[
\sum_{k=1}^N \gamma_k (f^\tau(x^k, y^k) - f^\tau(x, y^k)) \leq \frac{1}{\sum_{i=1}^N \gamma_k} \sum_{k=1}^N \gamma_k (f^\tau(x^k, y) - f^\tau(x, y)) \leq f^\tau \left( \begin{array}{c} \sum_{k=1}^N \gamma_k x^k \\ \sum_{k=1}^N \gamma_k y^k \end{array} \right) - f^\tau \left( \begin{array}{c} x, \sum_{k=1}^N \gamma_k y^k \\ \sum_{k=1}^N \gamma_k \end{array} \right) = f^\tau (\hat{x}^N, y) - f^\tau (x, \hat{y}^N) \quad (92)
\]

where \((\hat{x}^N, \hat{y}^N)\) is the output of the Algorithm 1. Using Eq. (92) for Eq. (91) we get

\[
\sum_{k=1}^N \gamma_k \langle \nabla f^\tau(z^k), z^k - u \rangle \geq \left( \sum_{k=1}^N \gamma_k \right) (f^\tau (\hat{x}^N, y) - f^\tau (x, \hat{y}^N)) \quad (93)
\]
Step 3.
Then we use this with Eq. (81), (89) and (93) for Eq. (74) and obtain
\[
f^\tau (\hat{x}^N, y) - f^\tau (x, \hat{y}^N) \leq \frac{V_{z1}(u)}{\sum_{k=1}^{N} \gamma_k} + \left(\sum_{k=1}^{N} \gamma_k\right)^{-1} \sum_{k=1}^{N} \frac{\gamma_k^2}{2} \|g(z^k, e^k, \xi^k)\|^2_2
\]
\[
- \left(\sum_{k=1}^{N} \gamma_k\right)^{-1} \sum_{k=1}^{N} \gamma_k (g_f(z^k, e^k, \xi^k) - \nabla f^\tau(z^k), z^k - u) + \left(\sum_{k=1}^{N} \gamma_k\right)^{-1} \sum_{k=1}^{N} \gamma_k \frac{d\Delta}{\tau} (e, z^k - u)
\]
(94)

For the l.h.s. of Eq. (94) we use Lemma 6.2 and obtain
\[
f^\tau (\hat{x}^N, y) - f^\tau (x, \hat{y}^N) \geq f (\hat{x}^N, y) - f (x, \hat{y}^N) - 2\tau M_2
\]
(95)

Using Eq. (80), (88), (90), (95) and taking the maximum in \((x, y) \in (\mathcal{X}, \mathcal{Y})\), we obtain we can rewrite Eq. (94) as follow:

\[
\text{Prob} \left\{ \max_{y \in \mathcal{Y}} f (\hat{x}^N, y) - \min_{x \in \mathcal{X}} f (x, \hat{y}^N) \geq \frac{4\Omega^2 \sqrt{d} \sqrt{\sum_{k=1}^{N} \gamma_k^2}}{M_2} + \frac{2\Omega \Delta}{\tau} \right\}
\]
\[
\leq \exp\{-\Omega^2/12\} + \exp\{-3\sqrt{N} \Omega/4\} + 2 \exp\{-\Omega^2/3\} + 6 \exp\{-\Omega/4\}
\]
(96)

Choosing the learning rate \(\gamma_k = \frac{\sqrt{V_{z1}(u)}}{M} \sqrt{\frac{2}{N}}\) with \(M^2 \triangleq M_2^2 + d\Delta^2 \tau^{-2}\) in Eq. (96) we obtain

\[
\text{Prob} \left\{ \max_{y \in \mathcal{Y}} f (\hat{x}^N, y) - \min_{x \in \mathcal{X}} f (x, \hat{y}^N) \geq \frac{4\Omega^2 \sqrt{d} \sqrt{\sum_{k=1}^{N} \gamma_k^2}}{M_2} + \frac{2\sqrt{2} \sqrt{\sum_{k=1}^{N} \gamma_k}}{cM \sqrt{N}} + \frac{\Omega \Delta}{\tau} \right\}
\]
\[
\leq \exp\{-\Omega^2/12\} + \exp\{-3\sqrt{N} \Omega/4\} + 2 \exp\{-\Omega^2/3\} + 6 \exp\{-\Omega/4\}
\]
(97)

Using the notation of \(M\) we obtain:

\[
\text{Prob} \left\{ \max_{y \in \mathcal{Y}} f (\hat{x}^N, y) - \min_{x \in \mathcal{X}} f (x, \hat{y}^N) \geq \left(4\Omega^2 + \frac{2\sqrt{2}}{cM} + \frac{2}{\sqrt{2}}\right) \frac{\sqrt{V_{z1}(u)}M}{\sqrt{N}} + \frac{\Omega \Delta}{\tau} \right\}
\]
\[
+ 2\tau M_2 + \frac{2\sqrt{2} M \sqrt{V_{z1}(u)}}{cN} \right\} \leq \exp\{-\Omega^2/12\} + \exp\{-3\sqrt{N} \Omega/4\} + 2 \exp\{-\Omega^2/3\} + 6 \exp\{-\Omega/4\}
\]
(98)
E Variable Separation

In this section, we assume proximal setups for spaces $\mathcal{X}$ and $\mathcal{Y}$ can be different. Thus, we replace the proximal step $z^{k+1} \leftarrow \text{Prox}_{x} \left(\gamma_{k}g(z^{k}, x^{k}, e^{k})\right)$ on the space $\mathcal{Z} \triangleq \mathcal{X} \times \mathcal{Y}$ from Algorithm 1 by two proximal steps on spaces $\mathcal{X}$ and $\mathcal{Y}$:

$$x^{k+1} \leftarrow \text{Prox}_{x} \left(\gamma_{k}g_{x}(z^{k}, x^{k}, e^{k})\right),$$

$$y^{k+1} \leftarrow \text{Prox}_{y} \left(\gamma_{k}g_{y}(z^{k}, x^{k}, e^{k})\right).$$

We also note that we have different Bregman divergence for each space $\mathcal{X}, \mathcal{Y}$ and the step is taken according to the corresponding divergence.

**Lemma E.1.** For $g(z, x, e)$ from (3), the following holds under Assumption 2.1

1. and Assumption 2.2

$$\mathbb{E} \left[\left\|g_{x}(z, x, e)\right\|^{2}_{q_{x}}\right] \leq c\alpha_{q_{x}}^{2} dM_{2}^{2} + d^{2}\alpha_{q_{x}}^{2} \Delta^{2}\tau^{-2},$$

2. and Assumption 2.3

$$\mathbb{E} \left[\left\|g_{x}(z, x, e)\right\|^{2}_{q_{x}}\right] \leq c\alpha_{q_{x}}^{2} \left(dM_{2}^{2} + M_{2,\delta}^{2}\right),$$

where $c$ is some numerical constant and $\sqrt{\mathbb{E}_{x}\left[\left\|e_{x}\right\|^{4}_{q_{x}}\right]} \leq \alpha_{q_{x}}^{2}$.

**Proof.**

$$\begin{align*}
\mathbb{E} \left[\left\|g_{x}(z, x, e)\right\|^{2}_{q_{x}}\right] &= \mathbb{E} \left[\left\|\frac{d}{2\tau} (\varphi(z + \tau e, x) - \varphi(z - \tau e, x)) e_{x}\right\|^{2}_{q_{x}}\right] \\
&= \frac{d^{2}}{4\tau^{2}} \mathbb{E} \left[\left\|e_{x}\right\|^{2}_{q_{x}} (f(z + \tau e, x) + \delta(z + \tau e) - f(z - \tau e, x) - \delta(z - \tau e))^{2}\right] \\
&= \frac{d^{2}}{4\tau^{2}} \left(\mathbb{E} \left[\left\|e_{x}\right\|^{2}_{q_{x}} (f(z + \tau e, x) - f(z - \tau e, x))^{2}\right] + \mathbb{E} \left[\left\|e_{x}\right\|^{2}_{q_{x}} (\delta(z + \tau e) - \delta(z - \tau e))^{2}\right]\right) \\
&= (99)
\end{align*}$$

For the first term in the r.h.s. of Eq. (99), the following holds with an arbitrary parameter $\alpha$

$$\begin{align*}
\frac{d^{2}}{4\tau^{2}} \mathbb{E} \left[\left\|e_{x}\right\|^{2}_{q_{x}} (f(z + \tau e, x) - f(z - \tau e, x))^{2}\right] \\
&= \frac{d^{2}}{2\tau^{2}} \mathbb{E} \left[\left\|e_{x}\right\|^{2}_{q_{x}} ((f(z + \tau e, x) - \alpha) - (f(z - \tau e, x) - \alpha))^{2}\right] \\
&\leq \frac{d^{2}}{2\tau^{2}} \mathbb{E} \left[\left\|e_{x}\right\|^{2}_{q_{x}} (f(z + \tau e, x) - \alpha)^{2} + (f(z - \tau e, x) - \alpha)^{2}\right] / * \forall a, b, (a - b)^{2} \leq 2a^{2} + 2b^{2} */ \\
&= \frac{d^{2}}{2\tau^{2}} \left(\mathbb{E} \left[\left\|e_{x}\right\|^{2}_{q_{x}} (f(z + \tau e, x) - \alpha)^{2}\right] + \mathbb{E} \left[\left\|e_{x}\right\|^{2}_{q_{x}} (f(z - \tau e, x) - \alpha)^{2}\right]\right) \\
&= \frac{d^{2}}{\tau^{2}} \mathbb{E} \left[\left\|e_{x}\right\|^{2}_{q_{x}} (f(z + \tau e, x) - \alpha)^{2}\right] / * \text{the distribution of } e \text{ is symmetric } */
\end{align*}$$

(100)
Applying the Cauchy–Schwarz inequality for Eq. (100) and using \( \sqrt{\mathbb{E}_e \left[ \|e_x\|^4_{q_x} \right]} \leq a^{2}_{q_x} \) we obtain
\[
\frac{d^2}{\tau^2} \mathbb{E} \left[ \|e_x\|^2_{q_x} \left( f(z + \tau e, \xi) - \alpha \right)^2 \right] \leq \frac{d^2}{\tau^2} \sqrt{\mathbb{E}_e \left[ \|e_x\|^4_{q_x} \right]} \sqrt{\mathbb{E} \left[ f(z + \tau e, \xi) - \alpha \right]}^2 \leq \frac{d^2 a^{2}_{q_x}}{\tau^2} \mathbb{E}_\xi \left[ \sqrt{\mathbb{E}_e \left[ f(z + \tau e, \xi) - \alpha \right]} \right].
\] (101)

Then we assume \( \alpha = \mathbb{E} [f(z + \tau e, \xi)], \beta = \mathbb{E} [\delta(z + \tau e)] \) and use Lemma 6.5 along with the fact that \( f(z + \tau e, \xi) \) is \( \tau M^2(\xi) \)-Lipschitz continuous, w.r.t. \( e \) in terms of the \( \ell_2 \)-norm. Thus, it holds for the Eq. (101)
\[
\frac{d^2 a^{2}_{q_x}}{\tau^2} \mathbb{E}_\xi \left[ \sqrt{\mathbb{E}_e \left[ f(z + \tau e, \xi) - \alpha \right]} \right] \leq \frac{d^2 a^{2}_{q_x}}{\tau^2} \sqrt{\mathbb{E}_e} \left[ \frac{\tau^2 M^2(\xi)}{d} \right] = cda^{2}_{q_x} M^2. \] (102)

1. Under the Assumption 2.2 for the second term in the r.h.s. of Eq. (99), the following holds
\[
\frac{d^2}{4\tau^2} \mathbb{E} \left[ \|e_x\|^2_{q_x} \left( \delta(z + \tau e) - \delta(z - \tau e) \right)^2 \right] \leq \frac{d^2 \Delta^2}{\tau^2} \mathbb{E} \left[ \|e_x\|^2_{q_x} \right] \leq \frac{d^2 a^{2}_{q_x} \Delta^2}{\tau^2}. \] (103)

2. Under the Assumption 2.3 for the second term in the r.h.s. of Eq. (99), the following holds with an arbitrary parameter \( \beta \)
\[
\frac{d^2}{4\tau^2} \mathbb{E} \left[ \|e_x\|^2_{q_x} \left( \delta(z + \tau e) - \delta(z - \tau e) \right)^2 \right] = \frac{d^2}{2\tau^2} \mathbb{E} \left[ \|e_x\|^2_{q_x} \left( \left( \delta(z + \tau e) - \beta \right) - \left( \delta(z - \tau e) - \beta \right) \right)^2 \right] \leq \frac{d^2}{\tau^2} \mathbb{E} \left[ \|e_x\|^2_{q_x} \left( \left( \delta(z + \tau e) - \beta \right)^2 + \left( \delta(z - \tau e) - \beta \right)^2 \right) \right] \quad \text{/} \ast \forall a, b, (a - b)^2 \leq 2a^2 + 2b^2 \ast / \nabla^2 \mathbb{E} \left[ \|e_x\|^2_{q_x} \delta(z + \tau e) - \beta^2 \right] + \mathbb{E} \left[ \|e_x\|^2_{q_x} \delta(z - \tau e) - \beta^2 \right] \nabla^2 \mathbb{E} \left[ \|e_x\|^2_{q_x} \delta(z + \tau e) - \beta^2 \right] \quad \text{/} \ast \text{ the distribution of } e \text{ is symmetric } \ast / \] (104)

and for the second term in the Eq. (99), we get
\[
\frac{d^2}{\tau^2} \mathbb{E} \left[ \|e_x\|^2_{q_x} \left( \delta(z + \tau e) - \beta \right)^2 \right] \leq \frac{d^2}{\tau^2} \sqrt{\mathbb{E}_e \left[ \|e_x\|^4_{q_x} \right]} \sqrt{\mathbb{E} \left[ \left( \delta(z + \tau e) - \beta \right)^4 \right]} \leq \frac{d^2 a^{2}_{q_x} \mathbb{E}_\xi \left[ \sqrt{\mathbb{E}_e \left[ \left( \delta(z + \tau e) - \beta \right)^4 \right]} \right]}{\tau^2}. \] (105)

Then we assume \( \beta = \mathbb{E} [\delta(z + \tau e)] \) and use Lemma 6.5 together with the fact that \( \delta(z + \tau e) \) is \( \tau M_{2,\delta} \)-Lipschitz continuous, w.r.t. \( e \) in terms of the \( \ell_2 \)-norm. Thus, Eq. (105) can be rewritten as
\[
\frac{d^2 a^{2}_{q_x}}{\tau^2} \mathbb{E}_\xi \left[ \sqrt{\mathbb{E}_e \left[ \left( \delta(z + \tau e) - \beta \right)^4 \right]} \right] \leq \frac{d^2 a^{2}_{q_x}}{\tau^2} \frac{\tau^2 M^2_{2,\delta}}{d} \leq ca^{2}_{q_x} d M^2_{2,\delta}. \] (106)
Using Eq. (102) and (103) (or Eq. (106)) for Eq. (99), we get the statement of the lemma.

\[ x^{k+1} = \text{Prox}_{x} \left( \gamma_k g_x(z^k, e^k, \xi^k) \right), \]
\[ y^{k+1} = \text{Prox}_{y} \left( \gamma_k g_y(z^k, e^k, \xi^k) \right) \]

we get (Ben-Tal and Nemirovski, 2013), for all \( u = (x, y) \in \mathcal{X} \times \mathcal{Y} \)

\[ \gamma_k \langle g_x(z^k, e^k, \xi^k), x^k - x \rangle \leq V_{x^k}(x) - V_{x^{k+1}}(x) - \gamma_k^2 \| g_x(z^k, e^k, \xi^k) \|_{q_x}^2 / 2 \]
\[ \gamma_k \langle g_y(z^k, e^k, \xi^k), y^k - y \rangle \leq V_{y^k}(y) - V_{y^{k+1}}(y) - \gamma_k^2 \| g_y(z^k, e^k, \xi^k) \|_{q_y}^2 / 2 \]

where \( V_x, V_y \) are Bregman divergences in \( \mathcal{X} \) and \( \mathcal{Y} \) spaces respectively. Summing for \( k = 1, \ldots, N \) and taking the expectation, we obtain, for all \( u \in \mathcal{Z} \)

\[ \mathbb{E} \left[ \sum_{k=1}^N \gamma_k \langle g_x(z^k, e^k, \xi^k), x^k - x \rangle \right] \leq V_{x^1}(x) - \mathbb{E} \left[ \sum_{k=1}^N \gamma_k^2 \| g_x(z^k, e^k, \xi^k) \|_{q_x}^2 / 2 \right] \]
\[ \leq D_x^2 + \sum_{k=1}^N \frac{\gamma_k^2}{2} \mathbb{E} \left[ \| g_x(z^k, e^k, \xi^k) \|_{q_x}^2 \right], \quad (107) \]
\[ \mathbb{E} \left[ \sum_{k=1}^N \gamma_k \langle g_y(z^k, e^k, \xi^k), y^k - y \rangle \right] \leq V_{y^1}(y) - \mathbb{E} \left[ \sum_{k=1}^N \gamma_k^2 \| g_y(z^k, e^k, \xi^k) \|_{q_y}^2 / 2 \right] \]
\[ \leq D_y^2 + \sum_{k=1}^N \frac{\gamma_k^2}{2} \mathbb{E} \left[ \| g_y(z^k, e^k, \xi^k) \|_{q_y}^2 \right], \quad (108) \]

where \( D_x = \max_{x, \xi \in \mathcal{X}} \sqrt{2V_x(\xi)} \), \( D_y = \max_{y, \xi \in \mathcal{Y}} \sqrt{2V_y(\xi)} \) are the diameters of \( \mathcal{X}, \mathcal{Y} \). In what follows, we deal only with the variable \( x \), taking into account the fact that the calculations for the other variable are similar.

**Step 1.**
For the second term in the r.h.s of inequality (107) we use Lemma E.1 and obtain

1. under Assumption 2.2
\[ \mathbb{E} \left[ \| g_x(z, \xi, e) \|_{q_x}^2 \right] \leq ca^{2}_q d M_2^2 + \frac{d^2 a^2 q_s \Delta^2}{\tau^2}, \quad (109) \]

2. under Assumption 2.3
\[ \mathbb{E} \left[ \| g_x(z, \xi, e) \|_{q_x}^2 \right] \leq ca^{2}_q d (M_2^2 + M_2^2), \quad (110) \]
where $c$ is some numerical constant and $\sqrt{\mathbb{E}_e[\|e_x\|_{q_e}^4]} \leq a_{q_e}^2$.

**Step 2.**

For the l.h.s. of Eq. (107), we use Lemma 6.4 under Assumption 2.2

1. under Assumption 2.2

$$
\mathbb{E} \left[ \sum_{k=1}^{N} \gamma_k \langle g_x(z^k, e^k, \xi^k), x^k - x \rangle \right] = \mathbb{E} \left[ \sum_{k=1}^{N} \gamma_k \langle \mathbb{E} \left[ g_x(z^k, e^k, \xi^k) \mid z^k \right], x^k - x \rangle \right] \\
\geq \mathbb{E} \left[ \sum_{k=1}^{N} \gamma_k \left\langle \nabla_x f^\tau(z^k) - \frac{d\Delta}{\tau} e_x^k, x^k - x \right\rangle \right]. \quad (111)
$$

2. under Assumption 2.3

$$
\mathbb{E} \left[ \sum_{k=1}^{N} \gamma_k \langle g_x(z^k, e^k, \xi^k), x^k - x \rangle \right] = \mathbb{E} \left[ \sum_{k=1}^{N} \gamma_k \langle \mathbb{E} \left[ g_x(z^k, e^k, \xi^k) \mid z^k \right], x^k - x \rangle \right] \\
\geq \mathbb{E} \left[ \sum_{k=1}^{N} \gamma_k \left\langle \nabla_x f^\tau(z^k) - dM_{2,\delta} e_x^k, x^k - x \right\rangle \right]. \quad (112)
$$

For the first term of the r.h.s. of Eq.(111) and (112) we have

$$
\mathbb{E} \left[ \sum_{k=1}^{N} \gamma_k \langle \nabla_x f^\tau(z^k), x^k - x \rangle \right] = \mathbb{E} \left[ \sum_{k=1}^{N} \gamma_k \langle \nabla_x f^\tau(x^k, y^k), x^k - x \rangle \right] \\
\geq \mathbb{E} \left[ \sum_{k=1}^{N} \gamma_k (f^\tau(x^k, y^k) - f^\tau(x, y^k)) \right]. \quad (113)
$$

By exactly the same calculations, we get the inequality for dual variable $y$

$$
\mathbb{E} \left[ \sum_{k=1}^{N} \gamma_k \langle \nabla_y f^\tau(z^k), y^k - y \rangle \right] = \mathbb{E} \left[ \sum_{k=1}^{N} \gamma_k \langle -\nabla_y f^\tau(x^k, y^k), y^k - y \rangle \right] \\
\geq \mathbb{E} \left[ \sum_{k=1}^{N} \gamma_k (-f^\tau(x^k, y^k) + f^\tau(x^k, y)) \right]. \quad (114)
$$

Summing up the inequalities (115) and (114) we have

$$
\mathbb{E} \left[ \sum_{k=1}^{N} \gamma_k \langle \nabla_x f^\tau(z^k), x^k - x \rangle \right] + \mathbb{E} \left[ \sum_{k=1}^{N} \gamma_k \langle \nabla_y f^\tau(z^k), y^k - y \rangle \right] \\
\geq \mathbb{E} \left[ \sum_{k=1}^{N} \gamma_k (f^\tau(x^k, y) - f^\tau(x, y^k)) \right]. \quad (115)
$$
Then we use the fact function $f^\tau(x, y)$ is convex in $x$ and concave in $y$ and obtain

\[
\mathbb{E} \left[ \frac{1}{\sum_{k=1}^{N} \gamma_k} \sum_{k=1}^{N} \gamma_k (f^\tau(x^k, y) - f^\tau(x, y^k)) \right]
\leq \mathbb{E} \left[ f^\tau \left( \frac{1}{\sum_{k=1}^{N} \gamma_k} \sum_{k=1}^{N} \gamma_k x^k, y \right) - f^\tau \left( x, \frac{1}{\sum_{k=1}^{N} \gamma_k} \sum_{k=1}^{N} \gamma_k y^k \right) \right]
\]

\[
= \mathbb{E} \left[ f^\tau(\hat{x}^N, y) - f^\tau(x, \hat{y}^N) \right], \tag{116}
\]

where $(\hat{x}^N, \hat{y}^N)$ is the output of the Algorithm. Using Eq. (116) for Eq. (115) we get

\[
\mathbb{E} \left[ \sum_{k=1}^{N} \gamma_k \langle \nabla f^\tau(z^k), z^k - u \rangle \right] \geq \sum_{k=1}^{N} \gamma_k \mathbb{E} \left[ f^\tau(\hat{x}^N, y) - f^\tau(x, \hat{y}^N) \right]. \tag{117}
\]

Next we estimate the term $\mathbb{E}_{e^k_x} [\langle e^k_x, x^k - x \rangle]$ in Eq. (111) and (112) by the Lemma 6.1

\[
\mathbb{E}_{e^k_x} [\langle e^k_x, x^k - x \rangle] \leq \frac{D_x}{\sqrt{d_x}}, \tag{118}
\]

and similarly

\[
\mathbb{E}_{e^k_y} [\langle e^k_y, y^k - y \rangle] \leq \frac{D_y}{\sqrt{d_y}}. \tag{119}
\]

Now we substitute Eq. (117), (119) and (118) to Eq. (111) and (112) summed with similar ones for the variable $y$, and get

1. under Assumption 2.2

\[
\mathbb{E} \left[ \sum_{k=1}^{N} \gamma_k \langle g(z^k, e^k, \xi^k), z^k - u \rangle \right] \geq \sum_{k=1}^{N} \gamma_k \mathbb{E} \left[ f^\tau(\hat{x}^N, y) - f^\tau(x, \hat{y}^N) \right] - \sum_{k=1}^{N} \gamma_k \Delta \left( \frac{dD_x}{\sqrt{d_x} \tau} + \frac{dD_y}{\sqrt{d_y} \tau} \right). \tag{120}
\]

2. under Assumption 2.3

\[
\mathbb{E} \left[ \sum_{k=1}^{N} \gamma_k \langle g(z^k, e^k, \xi^k), z^k - u \rangle \right] \geq \sum_{k=1}^{N} \gamma_k \mathbb{E} \left[ f^\tau(\hat{x}^N, y) - f^\tau(x, \hat{y}^N) \right] - \sum_{k=1}^{N} \gamma_k M_2 \Delta \left( \frac{dD_x}{\sqrt{d_x} \tau} + \frac{dD_y}{\sqrt{d_y} \tau} \right). \tag{121}
\]
we obtain

We can rewrite this as follows

\[ \text{Assumption 2.2} \]

For the l.h.s. of Eq. (123) we use Lemma 6.2 and obtain

\[ \text{Step 4. (under Assumption 2.3)} \]

Now we combine Eq. (121) with Eq. (109) for sum of Eq. (107), (108) and obtain under Assumption 2.3 the following

\[ \sum_{k=1}^{N} \gamma_k \mathbb{E} \left[ f^\tau (\hat{x}^N, y) - f^\tau (x, \hat{y}^N) \right] - \sum_{k=1}^{N} \gamma_k \Delta \left( \frac{dD_x}{\sqrt{d_x^\tau}} + \frac{dD_y}{\sqrt{d_y^\tau}} \right) \leq D^2 + \sum_{k=1}^{N} \frac{\gamma_k^2}{2} \left( c(a_{q_x}^2 + a_{q_y}^2) dM_2^2 + d^2(a_{q_x}^2 + a_{q_y}^2) \Delta^2 \right) . \] (122)

We can rewrite this as follows

\[ \mathbb{E} \left[ f^\tau (\hat{x}^N, y) - f^\tau (x, \hat{y}^N) \right] \leq \frac{D^2}{\sum_{k=1}^{N} \gamma_k} + \frac{c(a_{q_x}^2 + a_{q_y}^2) dM_2^2 + d^2(a_{q_x}^2 + a_{q_y}^2) \Delta^2 \tau^{-1}}{\sum_{k=1}^{N} \gamma_k} \sum_{k=1}^{N} \frac{\gamma_k}{2} \right) + \Delta \left( \frac{dD_x}{\sqrt{d_x^\tau}} + \frac{dD_y}{\sqrt{d_y^\tau}} \right) . \] (123)

For the l.h.s. of Eq. (123) we use Lemma 6.2 and obtain

\[ f^\tau (\hat{x}^N, y) - f^\tau (x, \hat{y}^N) \geq f (\hat{x}^N, y) - f (x, \hat{y}^N) - 2\tau M_2. \]

Using this for Eq. (123) and taking \((x, y) = (x_s, y_s)\) we obtain

\[ \mathbb{E} \left[ \max_{y \in Y} f (\hat{x}^N, y) - \min_{x \in X} f (x, \hat{y}^N) \right] \leq \frac{D^2}{\sum_{k=1}^{N} \gamma_k} + \frac{c(a_{q_x}^2 + a_{q_y}^2) dM_2^2 + d^2(a_{q_x}^2 + a_{q_y}^2) \Delta^2 \tau^{-1}}{\sum_{k=1}^{N} \gamma_k} \sum_{k=1}^{N} \frac{\gamma_k}{2} \right) + \Delta \left( \frac{dD_x}{\sqrt{d_x^\tau}} + \frac{dD_y}{\sqrt{d_y^\tau}} \right) + 2\tau M_2. \] (124)

Choosing the learning rate \(\gamma_k = \frac{D}{M_{\text{case1}}} \sqrt{2N} \) with \(M_{\text{case1}}^2 \triangleq (cdM_2^2 + d^2 \Delta^2 \tau^{-1}) (a_{q_x}^2 + a_{q_y}^2)\) in Eq. (124) we obtain

\[ \mathbb{E} \left[ \max_{y \in Y} f (\hat{x}^N, y) - \min_{x \in X} f (x, \hat{y}^N) \right] \leq \sqrt{\frac{2}{N}} M_{\text{case1}} D + \Delta \left( \frac{dD_x}{\sqrt{d_x^\tau}} + \frac{dD_y}{\sqrt{d_y^\tau}} \right) + 2\tau M_2. \]

Step 4. (under Assumption 2.3)

Now we combine Eq. (121) with Eq. (110) for sum of Eq. (107), (108) and obtain under Assumption 2.3

\[ \sum_{k=1}^{N} \gamma_k \mathbb{E} \left[ f^\tau (\hat{x}^N, y) - f^\tau (x, \hat{y}^N) \right] - \sum_{k=1}^{N} \gamma_k M_{2,\delta} \left( \frac{dD_x}{\sqrt{d_x^\tau}} + \frac{dD_y}{\sqrt{d_y^\tau}} \right) \leq D^2 + \sum_{k=1}^{N} \frac{\gamma_k^2}{2} c(a_{q_x}^2 + a_{q_y}^2) d(M_2^2 + M_{2,\delta}^2). \] (125)

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We can rewrite this as follows

\[
\mathbb{E} \left[ f^\tau (\hat{x}^N, y) - f^\tau (x, \hat{y}^N) \right] \leq \frac{D^2}{\sum_{k=1}^{N} \gamma_k} + \frac{c(a_{q_x}^2 + a_{q_y}^2)d(M_2^2 + M_{2,\delta}^2)}{\sum_{k=1}^{N} \gamma_k} \sum_{k=1}^{N} \frac{\gamma_k^2}{2}
\]

\[+ M_{2,\delta} \left( \frac{dD_x}{\sqrt{d_x \tau}} + \frac{dD_y}{\sqrt{d_y \tau}} \right). \tag{126} \]

For the l.h.s. of Eq. (126) we use Lemma 6.2 and obtain

\[
f^\tau (\hat{x}^N, y) - f^\tau (x, \hat{y}^N) \geq f (\hat{x}^N, y) - f (x, \hat{y}^N) - 2\tau M_2.
\]

Using this for Eq. (126) and taking \((x, y) = (x^*, y^*)\) we obtain

\[
\mathbb{E} \left[ f (\hat{x}^N, y^*) - f (x^*, \hat{y}^N) \right] \leq \frac{D^2 + c(a_{q_x}^2 + a_{q_y}^2)d(M_2^2 + M_{2,\delta}^2)}{\sum_{k=1}^{N} \gamma_k} \sum_{k=1}^{N} \frac{\gamma_k^2}{2}
\]

\[+ M_{2,\delta} \left( \frac{dD_x}{\sqrt{d_x \tau}} + \frac{dD_y}{\sqrt{d_y \tau}} \right) + 2\tau M_2. \tag{127} \]

Choosing the learning rate \(\gamma_k = \frac{D}{M_{\text{case2}}} \sqrt{\frac{2}{N}}\) with \(M_{\text{case2}}^2 \triangleq cd(a_{q_x}^2 + a_{q_y}^2)(M_2^2 + M_{2,\delta}^2)\) in Eq. (124) we obtain

\[
\mathbb{E} \left[ f (\hat{x}^N, y^*) - f (x^*, \hat{y}^N) \right] \leq \sqrt{\frac{2}{N}} M_{\text{case2}} D + M_{2,\delta} \left( \frac{dD_x}{\sqrt{d_x \tau}} + \frac{dD_y}{\sqrt{d_y \tau}} \right) + 2\tau M_2.
\]

\(\square\)