ESTIMATES FOR GREEN’S FUNCTIONS OF ELLIPTIC EQUATIONS IN NON-DIVERGENCE FORM WITH CONTINUOUS COEFFICIENTS

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ABSTRACT. We present a new method for the existence and pointwise estimates of a Green’s function of non-divergence form elliptic operator with Dini mean oscillation coefficients. We also present a sharp comparison with the corresponding Green’s function for constant coefficients equations.

1. Introduction and main results

We consider a second-order elliptic operator $L$ in non-divergence form

$$Lu = a^{ij}(x)D_{ij}u,$$

(1.1)

where the coefficient $A := (a^{ij})$ are symmetric and satisfy the uniform ellipticity condition. Namely

$$a^{ij} = a^{ji}, \quad \lambda |\xi|^2 \leq a^{ij}(x)\xi^i \xi^j \leq \Lambda |\xi|^2,$$

(1.2)

for some positive constants $\lambda$ and $\Lambda$ in a domain $\Omega \subset \mathbb{R}^n$ with $n \geq 3$. Here and below, we use the usual summation convention over repeated indices.

In this article, we are concerned with construction and pointwise estimates for the Green’s function $G(x, y)$ of the non-divergent operator $L$ in $\Omega$. In a recent article [15], it is shown that if the coefficients $A$ is of Dini mean oscillation and the domain $\Omega$ is bounded and has $C^{2,\alpha}$ boundary, then the Green’s function exists and satisfies the pointwise bound

$$|G(x, y)| \leq C|x - y|^{2-n}.$$

(1.3)

Before proceeding further, let us introduce the definition of Dini mean oscillation. For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x, r)$ the Euclidean ball with radius $r$ centered at $x$, and write $\Omega(x, r) := \Omega \cap B(x, r)$. We denote

$$\omega_A(r, x) := \int_{\Omega(x, r)} |A(y) - \bar{A}_{\Omega(x, r)}| \, dy,$$

where $\bar{A}_{\Omega(x, r)} := \int_{\Omega(x, r)} A$,

and we write

$$\omega_A(r, D) := \sup_{x \in D} \omega_A(r, x) \quad \text{and} \quad \omega_A(r) = \omega_A(r, \bar{\Omega}).$$

(1.4)

We say that $A$ is of Dini mean oscillation in $\Omega$ if $\omega_A(r)$ satisfies the Dini condition; i.e.,

$$\int_0^1 \frac{\omega_A(t)}{t} \, dt < +\infty.$$

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It is clear that if $A$ is Dini continuous, then $A$ is of Dini mean oscillation. Also if $A$ is of Dini mean oscillation, then $A$ is uniformly continuous in $\Omega$ with its modulus of continuity controlled by $\omega_A$; see [15, Appendix]. However, a function of Dini mean oscillation is not necessarily Dini continuous; see [7] for an example.

The main result of [15] is interesting because unlike the Green’s function for uniformly elliptic operators in divergence form, the Green’s function for non-divergent elliptic operators does not necessarily enjoy the pointwise bound (1.3) even in the case when the coefficient $A$ is uniformly continuous; see [1]. It should be noted that the Dini mean oscillation condition is the weakest assumption in the literature that guarantees the pointwise bound (1.3). The proof in [15] is based on considering approximate Green’s functions (as in [13, 14]) and showing that they satisfy specific estimates, as well as a local $L^\infty$ estimate for solutions to the adjoint equation $L^*u = 0$, which is shown in [7, 8].

We should recall that the adjoint operator $L^*$ is given by

$$L^*u = D_{ij}(a^{ij}(x)u).$$

We should also mention that there are many papers in the literature dealing with the existence and estimates of Green’s functions or fundamental solutions of non-divergence form elliptic or parabolic operators with measurable or continuous coefficients; see e.g. [2, 3, 11, 12, 17, 9, 4].

In this article we give an alternative proof for the existence of Green’s function. More precisely, we construct Green’s function from that of the corresponding constant coefficients operator resulting from “freezing coefficients”. We shall use $L^p$ theory for the adjoint operator in this process. We then utilize the local $L^\infty$ estimates for adjoint solutions established in [7, 8] to get the pointwise bound (1.3) for the Green’s function. One prominent advantage of this approach is that it yields a sharp comparison with the Green’s function for constant coefficients operator. In particular, we shall show that

$$G(x_0, x) - G_0(x_0, x) = o(|x - x_0|^{2-n}) \text{ as } x \to x_0, \quad (1.5)$$

where $G_0$ is the Green’s function of the constant coefficient operator $L_0$ given by

$$L_0u := a^{ij}(x_0)D_{ij}u, \quad (1.6)$$

provided that the mean oscillation of $A$ satisfies so-called “double Dini condition” near $x_0$; that is, we have

$$\int_0^1 \frac{1}{s} \int_0^s \frac{\omega_A(t, \Omega(x_0, r_0))}{t} dt ds = \int_0^1 \frac{\omega_A(t, \Omega(x_0, r_0))}{t} \ln \frac{1}{t} dt < +\infty,$$

for some $r_0 > 0$. The asymptotic behavior (1.5) is well known for the Green’s functions for elliptic operators in divergence form with continuous coefficients; see [3]. However, in the non-divergence form setting, this is a new result and it is one of the novelties in our work. We now present our main theorem.

**Theorem 1.7.** Let $\Omega$ be a bounded $C^{2, \alpha}$ domain in $\mathbb{R}^n$ with $n \geq 3$. Assume the coefficient $A = (a^{ij})$ of the operator $L$ in (1.1) satisfies the uniform ellipticity condition (1.2) and is of Dini mean oscillation in $\Omega$. Then, there exists a unique Green’s function $G(x, y)$ of the
operator $L$ in $\Omega$ and it satisfies the pointwise estimate

$$|G(x, y)| \leq C|x - y|^{2-n},$$

where $C = C(n, \lambda, \Lambda, \omega_\Lambda)$. Moreover, if there is some $r_0 > 0$ such that $\omega_\Lambda(t, \Omega(x_0, r_0))$ satisfies double Dini condition

$$\int_0^1 \frac{1}{s} \int_0^s \omega_\Lambda(t, \Omega(x_0, r_0)) \frac{dt}{t} \, ds = \int_0^1 \omega_\Lambda(t, \Omega(x_0, r_0)) \ln \frac{1}{t} \, dt < +\infty,$$

then we have

$$\lim_{x \to x_0} |x - x_0|^{n-2} |G(x_0, x) - G_0(x_0, x)| = 0,$$

where $G_0$ is the Green’s function of the constant coefficient operator $L_0$ as in (1.6).

**Remark 1.10.** As stated in [15], pointwise estimates for $D_i G(x, y)$ and $D^2_i G(x, y)$ are also available. They are obtained from (1.3) via local $L^\infty$ estimates for first and second derivatives of solutions to $Lu = 0$ as established in [7-8]. We only treat the case when $n \geq 3$ in this article and we refer to [6] for two dimensional case. In a separate paper [16], we construct the fundamental solution for parabolic equations in non-divergence form with Dini mean oscillation coefficients and establish Gaussian bounds for the fundamental solution.

## 2. Preliminary lemmas

In this section, we present some technical lemmas which will be used in the proof of Theorem 1.7. We need to consider the boundary value problem of the form

$$L^* v = \text{div}^2 g + f \text{ in } \Omega, \quad v = \frac{g^v \cdot \nu}{A^{ij} \cdot \nu} \text{ on } \partial \Omega,$$

(2.1)

where $g = (g^{ij})$ is an $n \times n$ matrix-valued function,

$$\text{div}^2 g := D_{ij} g^{ij},$$

and $\nu$ is the unit exterior normal vector of $\partial \Omega$. For $g \in L^p(\Omega)$ and $f \in L^p(\Omega)$, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, we say that $v$ in $L^p(\Omega)$ is an adjoint solution of (2.1) if $v$ satisfies

$$\int_\Omega v L u = \int_\Omega \text{tr}(g D^2 u) + \int_\Omega f u$$

(2.2)

for any $u$ in $W^{2, p'}(\Omega) \cap W^{1, p'}_0(\Omega)$.

**Lemma 2.3.** Let $1 < p < \infty$ and assume that $g \in L^p(\Omega)$ and $f \in L^p(\Omega)$. Then there exists a unique adjoint solution $u$ in $L^p(\Omega)$. Moreover, the following estimates holds.

$$\|u\|_{L^p(\Omega)} \leq C \left( \|g\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} \right),$$

where a constant $C$ depends on $\Omega$, $p$, $n$, $\lambda$, $\Lambda$, and $\omega_\Lambda$.

**Proof.** See [10] Lemma 2.6.

The proof of next lemma is implicitly given in the proof of [7, Theorem 1.10]. However, an estimate like (2.5) does not appear explicitly in the literature and we provide a proof in the Appendix for reader’s convenience. It should be emphasized that the lemma asserts that to get a local $L^\infty$ estimate of the solution $u$, only a local information on the data $g$ is needed.
Lemma 2.4. Let \( R_0 > 0 \) and \( g = (g^{ij}) \) be of Dini mean oscillation in \( B(x_0, R_0) \). Suppose \( u \) is an \( L^2 \) solution of
\[
L^* u = \text{div}^2 g \quad \text{in} \quad B(x_0, 2R),
\]
where \( 0 < R \leq \frac{1}{2} R_0 \). Then we have
\[
\|u\|_{L^\infty(B(x_0, R))} \leq C \left( \int_{B(x_0, 2R)} |u| + \int_0^R \frac{\omega_g(t, B(x_0, 2R))}{t} \, dt \right),
\]  
where \( C = C(n, \Lambda, \omega_A, R_0) \).

Proof. See Appendix.

The next lemma is an extension of Lemma 2.4 up to the \( C^{2,\alpha} \) boundary.

Lemma 2.6. Let \( \Omega \) be a bounded \( C^{2,\alpha} \) domain. Assume that \( g = (g^{ij}) \) are of Dini mean oscillation in \( \Omega \). Let \( u \in L^2(\Omega) \) be the solution of the adjoint problem
\[
L^* u = \text{div}^2 g \quad \text{in} \quad \Omega, \quad u = \frac{g^v \cdot \nu}{A^v \cdot \nu} \quad \text{on} \quad \partial \Omega.
\]
Then for \( x_0 \in \overline{\Omega} \) and \( 0 < R \leq \frac{1}{2} \text{diam} \, \Omega \), we have
\[
\|u\|_{L^\infty(\Omega(x_0, R))} \leq C \left( \int_{\Omega(x_0, 2R)} |u| + \int_0^R \frac{\omega_g(t, \Omega(x_0, 2R))}{t} \, dt \right),
\]  
where \( C = C(n, \Lambda, \omega_A, \Omega) \).

Proof. By flattening the boundary, it suffices to get an estimate in half balls that corresponds to (2.5). It is obtained by replicating the proof of [8, Lemma 2.26] in the same fashion as (2.5) is derived. We leave the details to the readers.

3. Proof of Theorem 1.7

The organization of the proof is as follows. In Sec. 3.1, we first construct the Green’s function \( G(x, y) \) for the adjoint operator. In Sec. 3.2 and 3.3, it will be shown that the adjoint Green’s function has the pointwise bound \( |G(x, y)| \leq C|x - y|^{2-n} \). In Sec. 3.4, we show that \( G(x, y) = G^*(y, x) \) becomes the Green’s function and thus it also has the pointwise bound \( |G(x, y)| \leq C|x - y|^{2-n} \). Finally, in Sec. 3.5, we establish the asymptotic formula (1.9).

3.1. Construction of adjoint Green’s function. Let \( x_0 \in \Omega \) be fixed and denote
\[
A_0 = A(x_0) \quad \text{and} \quad L_0 u := a^{ij}(x_0)D_{ij}u = \text{tr}(A_0 D^2 u).
\]
Let \( G_0(x, y) \) be the Green’s function for \( L_0 \) in \( \Omega \). Since \( L_0 \) is an elliptic operator with constant coefficients, the existence of \( G_0 \) as well as the following pointwise bound is well known.
\[
|G_0(x, y)| \leq C|x - y|^{2-n} \quad (x \neq y),
\]  
where \( C = C(n, \Lambda) \). Moreover, since \( A_0 \) is symmetric, we have \( L_0 = L_0^* \) and \( G_0 \) is also symmetric, i.e.,
\[
G_0(x, y) = G_0(y, x) \quad (x \neq y).
\]

We shall now construct \( G^*(\cdot, x_0) \), Green’s function for \( L^* \) in \( \Omega \) with a pole at \( x_0 \). Formally, we would have
\[
L^* G^*(\cdot, x_0) = \delta(\cdot - x_0) \quad \text{in} \quad \Omega, \quad G^*(\cdot, x_0) = 0 \quad \text{on} \quad \partial \Omega.
\]
On the other hand, since \( L_0 = L_0^* \), we have
\[
L_0^* G_0(\cdot, x_0) = \delta(\cdot - x_0) \quad \text{in} \; \Omega, \quad G_0(\cdot, x_0) = 0 \quad \text{on} \; \partial \Omega.
\]
Therefore, if we set \( v = G^*(\cdot, x_0) - G_0(\cdot, x_0) \), then we would have \( v = 0 \) on \( \partial \Omega \) and
\[
L^* v = L^* G^*(\cdot, x_0) - L^* G_0(\cdot, x_0) + L_0^* G_0(\cdot, x_0) - L_0^* G_0(\cdot, x_0) = -(L^* - L_0^*) G_0(\cdot, x_0) = -\text{div}^2((A - A_0) G_0(\cdot, x_0)) \quad \text{in} \; \Omega,
\]
which lead us to consider the problem
\[
L^* v = \text{div}^2 g \quad \text{in} \; \Omega, \quad v = \frac{g^\nu \cdot v}{A^\nu \cdot v} \quad \text{on} \; \partial \Omega,
\]
where we denote
\[
g := -(A - A_0) G_0(\cdot, x_0).
\]
Notice that since \( g \) vanishes on \( \partial \Omega \), the boundary condition in (3.2) reads simply that \( v = 0 \) on \( \partial \Omega \).

**Lemma 3.3.** For \( g = -(A - A_0) G_0(\cdot, x_0) \), we have \( g \in L^p(\Omega) \) for all \( p \in \left[ 1, \frac{\omega}{n-2} \right) \).

**Proof.** First, observe that we have
\[
\int_{\Omega(x_0, r)} |A - A_0| \leq \int_{\Omega(x_0, r)} |A(x) - \bar{A}_{\Omega(x_0, r)}| \, dx + |\bar{A}_{\Omega(x_0, r)} - A(x_0)|
\]
\[
\leq \omega_A(r, x_0) + C \int_0^r \frac{\omega_A(t, x_0)}{t} \, dt \leq C \int_0^r \frac{\omega_A(t, x_0)}{t} \, dt,
\]
where we used (15) Appendix] in the second line. Next, using (3.1) and
\[
||A - A_0||_{L^\infty(\Omega)} \leq C,
\]
where \( C = C(n, \Lambda) \), we have for \( 1 \leq p < \frac{\omega}{n-2} \) that
\[
\int_\Omega |g|^p \leq \sum_{k=0}^\infty \int_{\Omega(x_0, 2^{-k})} |g|^p + \int_{\Omega(x_0, 1)} |g|^p \leq C \sum_{k=0}^\infty 2^{(n-2)p k} \int_{\Omega(x_0, 2^{-k})} |A - A_0| + C \int_\Omega |x - x_0|^{(2-n)p} \, dx
\]
\[
\leq C \sum_{k=0}^\infty 2^{(n-2)p k} \int_{\Omega(x_0, 2^{-k})} |A - A_0| + C(\text{diam} \, \Omega)^{(2-n)p+n}
\]
\[
\leq C \left( \int_0^1 \frac{\omega_A(t, x_0)}{t} \, dt \int_0^\infty 2^{(n-2)p k} \, k + C(\text{diam} \, \Omega)^{(2-n)p+n} < +\infty, \right.
\]
where we used (3.4) in the last line.

By Lemmas 2.3 and 3.3 we find that there exists a unique solution \( v \) of the problem (3.2) and \( v \in L^p(\Omega) \) for all \( p \in \left( 1, \frac{\omega}{n-2} \right) \).

Now, We claim that \( G^*(\cdot, x_0) \) defined as
\[
G^*(\cdot, x_0) := G_0(\cdot, x_0) + v
\]
becomes Green’s function of \( L^* \) in \( \Omega \) with a pole at \( x_0 \). Indeed, for any \( f \in L^{p'}(\Omega) \) with \( p' > \frac{n}{2} \), let \( u \in W^{2,p'}(\Omega) \cap W^{1,p}_0(\Omega) \) be the strong solution of
\[
Lu = f \quad \text{in} \; \Omega, \quad u = 0 \quad \text{on} \; \partial \Omega.
\]
Then, by (2.2) and (3.2), we have
\[\int_{\Omega} v f = \int_{\Omega} G_0(\cdot, x_0) L_0 u - \int_{\Omega} G_0(\cdot, x_0) f = u(x_0) - \int_{\Omega} G_0(\cdot, x_0) f,\]
where we use the fact that \(G_0\) is the Green’s function for the operator \(L_0\). Therefore, by (3.6), we have
\[u(x_0) = \int_{\Omega} G^*(\cdot, x_0) f,\]
which means that \(G^*(x, y)\) is the Green’s function for the adjoint operator \(L^*\). See [15] Remark 1.14. As a matter of fact, we proved the following.

**Proposition 3.8.** For \(p > \frac{n}{2}\) and \(f \in L^p(\Omega)\), if \(u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)\) be the strong solution of (3.7), then we have the representation formula
\[u(x) = \int_{\Omega} G^*(y, x) f(y) dy. \quad (3.9)\]

### 3.2. Pointwise estimates of adjoint Green’s functions

In this section, we shall establish
\[|G^*(x, x_0)| \leq C|x - x_0|^{2-n} \quad \text{in} \quad \Omega \setminus \{x_0\}. \quad (3.10)\]
Let \(v\) be as in (3.2) and define \(g_1\) and \(g_2\) by
\[g_1 = -\zeta(A - A_0)G_0(\cdot, x_0) \quad \text{and} \quad g_2 = -(1 - \zeta)(A - A_0)G_0(\cdot, x_0),\]
where \(\zeta\) is a smooth function on \(\mathbb{R}^n\) such that
\[0 \leq \zeta \leq 1, \quad \zeta = 0 \quad \text{in} \quad B(x_0, r), \quad \zeta = 1 \quad \text{in} \quad \mathbb{R}^n \setminus B(x_0, 2r), \quad |D\zeta| \leq 2/r, \quad \text{and} \quad r > 0 \quad \text{is to be fixed later.}\]

We note that \(g_1 \in L^{p_1}(\Omega)\) for any \(p_1 > \frac{2n}{n-2}\) and \(g_2 \in L^{p_2}(\Omega)\) for any \(p_2 < \frac{n}{n-2}\). Indeed, the computation in (3.5) reveals that for \(p_1 > \frac{n}{n-2}\) we have
\[\int_{\Omega} |g_1|^{p_1} \leq C \int_{\Omega \setminus B(x_0, r)} |x - x_0|^{(2-n)p_1} \leq C r^{(2-n)p_1+n} \quad (3.11)\]
and for \(p_2 < \frac{n}{n-2}\) we have
\[\|g_2\|_{L^{p_2}(\Omega)} \leq C \left( \int_0^{2r} \frac{\omega_A(t, x_0)}{t} dt \right)^{1/p_2} \|\nu\|_{L^{2-n+n/p_2}}. \quad (3.12)\]
Fix a \(p_1 \in (\frac{n}{n-2}, \infty)\) and let us write \(v = v_1 + v_2\), where \(v_1 \in L^{p_1}(\Omega)\) is the solution of
\[L^*v_1 = \text{div}^2 g_1 \quad \text{in} \quad \Omega, \quad v_1 = \frac{g_1 \cdot \nu}{A \cdot \nu} \quad \text{on} \quad \partial\Omega. \quad (3.13)\]
Then by Lemma 2.3 and (3.11), we have
\[\|v_1\|_{L^{p_1}(\Omega)} \leq C r^{2-n+n/p_1}. \quad (3.14)\]
On the other hand, note that \(v_2 = v - v_1\) satisfies
\[L^*v_2 = \text{div}^2 g_2 \quad \text{in} \quad \Omega, \quad v_2 = \frac{g_2 \cdot \nu}{A \cdot \nu} \quad \text{on} \quad \partial\Omega. \quad (3.15)\]
By Lemma 2.3 and (3.12), for \(p_2 \in (1, \frac{2n}{n-2})\), we have \(v_2 \in L^{p_2}(\Omega)\) with
\[\|v_2\|_{L^{p_2}(\Omega)} \leq C \left( \int_0^{2r} \frac{\omega_A(t, x_0)}{t} dt \right)^{1/p_2} r^{2-n+n/p_2}. \quad (3.16)\]
Now, for any fixed \(y_0 \in \Omega\) with \(y_0 \neq x_0\), we take
\[
r = \frac{1}{2}|y_0 - x_0|
\]
and estimate \(v_1(y_0)\) by using Lemma 2.6 as follows.
\[
|v_1(y_0)| \leq \|v_1\|_{L^p(\Omega(y_0, r))} \leq C \int_{\Omega(y_0, 2r)} |v_1| + C \int_0^r \frac{\omega_g(t, \Omega(y_0, 2r))}{t} dt. \tag{3.17}
\]
By Hölder’s inequality and (3.14), we have
\[
\int_{\Omega(y_0, 2r)} |v_1| \leq \left( \int_{\Omega(y_0, 2r)} |v_1|^p \right)^{1/p} \leq Cr^{-n/p}\|v_1\|_{L^p(\Omega)} \leq Cr^{2-n}. \tag{3.18}
\]

**Lemma 3.19.** Let \(\eta\) be a Lipschitz function on \(\mathbb{R}^n\) such that \(0 \leq \eta \leq 1\) and \(|D\eta| \leq 4/\delta\) for some \(\delta > 0\). Let
\[
g = -\eta(A - A_0)G_0(\cdot, x_0)
\]
and take \(y_0 \in \Omega \setminus \{x_0\}\) with \(r := \frac{1}{2}|x_0 - y_0| \leq \delta\). Then, we have for any \(t \in (0, r]\) that
\[
\omega_g(t, \Omega(y_0, 2r)) \leq C r^{2-n} \left( \omega_A(t, \Omega(x_0, 7r)) + \frac{t}{r} \int_0^t \frac{\omega_A(s, \Omega(x_0, 7s))}{s} ds \right),
\]
where \(C = C(n, \lambda, \Lambda, \Omega)\).

The above lemma, the proof of which is given in Section 3.3, yields that (take \(\eta = \zeta\) with \(\delta = r\))
\[
\int_0^r \frac{\omega_g(t, \Omega(y_0, 2r))}{t} dt \leq C r^{2-n} \left( \int_0^r \frac{\omega_A(t)}{t} dt + \frac{1}{r} \int_0^r \int_0^t \frac{\omega_A(s)}{s} ds dt \right)
\]
\[
\leq C r^{2-n} \int_0^r \frac{\omega_A(t)}{t} dt. \tag{3.20}
\]
Putting (3.20) back to (3.17) together with (3.18), we get
\[
|v_1(y_0)| \leq Cr^{2-n} \left(1 + \int_0^r \frac{\omega_A(t)}{t} dt \right). \tag{3.21}
\]

Next, we shall estimate \(v_2(y_0)\). Again, by Lemma 2.6 we have
\[
|v_2(y_0)| \leq C \int_{\Omega(y_0, 2r)} |v_2| + C \int_0^r \frac{\omega_g(t, \Omega(y_0, 2r))}{t} dt.
\]
Notice that \(g_2\) vanishes in \(\Omega(y_0, 3r)\) since \(B(x_0, 2r) \cap B(y_0, 3r) = \emptyset\). Therefore, we have \(\omega_{g_2}(t, \Omega(y_0, 2r)) = 0\) and thus
\[
|v_2(y_0)| \leq C \int_{\Omega(y_0, 2r)} |v_2| \leq C \left( \int_{\Omega(y_0, 2r)} |v_2|^p \right)^{1/p} \leq Cr^{-n/p}\|v_2\|_{L^p(\Omega)} \leq C \left( \int_0^{2r} \frac{\omega_A(t)}{t} dt \right)^{1/p} \left(r^{2-n}\right), \tag{3.22}
\]
where we used (3.16). Therefore, by using (3.21) and (3.22), and recalling that \(v = v_1 + v_2\) and \(r = \frac{1}{2}|x_0 - y_0|\), we have
\[
|v(y_0)| \leq C \left(1 + \int_0^{r|x_0 - y_0|} \frac{\omega_A(t)}{t} dt + \left( \int_0^{r|x_0 - y_0|} \frac{\omega_A(t)}{t} dt \right)^{1/2} \right) |x_0 - y_0|^{2-n}, \tag{3.23}
\]
where $C = C(n, \lambda, \Lambda, \omega_A, \Omega)$. Since
\[ G^*(y_0, x_0) = G_0(y_0, x_0) + v(y_0) \]
and $y_0 \in \Omega \setminus \{x_0\}$ is arbitrary, the desired estimate (3.10) follows from (3.1) and (3.23).

### 3.3. Proof of Lemma 3.19

For $\bar{\omega} \in \Omega(y_0, 2r)$ and $0 < t \leq r$, we have
\[
\omega_g(t, \bar{x}) = \int_{\Omega(t, \bar{x})} \left| (A - A_0)G_0(\cdot, x_0) \eta - \|A - A_0\|_{\Omega(t, \bar{x})} G_0(\cdot, x_0) \eta \right| \leq \int_{\Omega(t, \bar{x})} \left| (A - A_0)G_0(\cdot, x_0) \eta - \|A - A_0\|_{\Omega(t, \bar{x})} G_0(\cdot, x_0) \eta \right| \]
\[
\quad + \int_{\Omega(t, \bar{x})} \left| (A - A_0)G_0(\cdot, x_0) \eta - \|A - A_0\|_{\Omega(t, \bar{x})} G_0(\cdot, x_0) \eta \right| =: I + II.
\]

Observe that we have $\text{dist}(x_0, \Omega(\bar{x}, t)) \geq 2r$ and thus for $x, y \in \Omega(\bar{x}, t)$, we have
\[
|G_0(x, x_0) - G_0(y, x_0)| \leq C r^{-n},
\]
where $C = C(n, \lambda, \Lambda, \Omega)$. Since $\text{dist}(x_0, \Omega(\bar{x}, t)) \geq 2r$, by using (3.1) we obtain
\[
I \leq \int_{\Omega(t, \bar{x})} \left| (A - A_0) - \|A - A_0\|_{\Omega(t, \bar{x})} \right| |G_0(\cdot, x_0)|
\]
\[
\leq \int_{\Omega(t, \bar{x})} C r^{-n} |A - \bar{A}|_{\Omega(t, \bar{x})} \leq C r^{-n} \omega_A(t, \bar{x}).
\] (3.25)

Also, we have
\[
II \leq \int_{\Omega(t, \bar{x})} \left( \int_{\Omega(t, \bar{x})} |A(y) - A(x_0)| \left| G_0(x, x_0) \eta(x) - G_0(y, x_0) \eta(y) \right| dy \right) dx
\]
\[
\leq \int_{\Omega(t, \bar{x})} \left( \int_{\Omega(t, \bar{x})} |A(y) - A(x_0)| \left| G_0(x, x_0) \eta(x) - G_0(y, x_0) \eta(y) \right| dy \right) dx.
\] (3.26)

By using (3.1), (3.24), $|D\eta| \leq 4/\delta$, and $r \leq \delta$, we have for $x, y \in \Omega(\bar{x}, t)$ that
\[
|G_0(x, x_0) \eta(x) - G_0(y, x_0) \eta(y)| \leq |G_0(x, x_0) - G_0(y, x_0)| |\eta(x)| + |G_0(y, x_0)| |\eta(x) - \eta(y)|
\]
\[
\leq C r^{1-n} + C r^{-n} t/\delta \leq C r^{1-n}.
\] (3.27)

Plugging (3.27) into (3.26), we obtain
\[
II \leq C r^{1-n} \int_{\Omega(t, \bar{x})} |A(y) - A(x_0)| dy.
\]

We claim that
\[
\int_{\Omega(t, \bar{x})} |A(y) - A(x_0)| dy \leq C \left( \frac{\omega_A(t, \Omega(x_0, 7r))}{t} + \int_0^t \frac{\omega_A(s, \Omega(x_0, 7r))}{s} ds \right),
\] (3.28)
where $C = C(n, \lambda, \Lambda, \Omega)$. Let us take the claim granted for now. Then, we have
\[
II \leq C r^{1-n} \left( \frac{\omega_A(t, \Omega(x_0, 7r))}{t} + \int_0^t \frac{\omega_A(s, \Omega(x_0, 7r))}{s} ds \right).
\] (3.29)
Combining (3.25) and (3.29), we have (recall \( t \leq r \))

\[
\omega_g(t, \bar{x}) \leq I + II \leq Cr^{2-n} \left(\omega_A(t, \Omega(x_0, 7r)) + \frac{t}{r} \int_0^t \frac{\omega_A(s, \Omega(x_0, 7r))}{s} \, ds\right).
\]

The lemma is proved by taking supremum over \( \bar{x} \in \Omega(y_0, 2r) \).

It remains to prove the claim (3.28). Notice that we can choose a sequence of points \( x_1, x_2, \ldots, x_N \) in \( \Omega(x_0, 7r) \) with \( x_N = \bar{x} \) such that each line segment \([x_i-1, x_i]\) lies in \( \Omega \) and \(|x_i-1 - x_0| \leq t\) for \( i = 1, \ldots, N \). Moreover, there exists a constant \( C = C(\Omega) \) independent of \( t \) and \( r \) such that

\[
N \leq Cr/t.
\]

Then by using triangle inequalities, we have

\[
|A(y) - A(x_0)| \leq |A(y) - \bar{A}_{\Omega(x_i, t)}| + \sum_{i=1}^N |\bar{A}_{\Omega(x_i, t)} - \bar{A}_{\Omega(x_{i-1}, t)}| + |\bar{A}_{\Omega(x_0, t)} - A(x_0)|.
\]

Note that by [15 Appendix], we have

\[
|A(y) - \bar{A}_{\Omega(x_i, t)}| \leq C t \int_0^t \frac{\omega_A(s, r)}{s} \, ds,
\]

\[
|\bar{A}_{\Omega(x_i, t)} - \bar{A}_{\Omega(x_{i-1}, t)}| \leq C \omega_A(t, \Omega(x_0, 7r)), \quad i = 1, \ldots, N.
\]

Using (3.32) and averaging the inequality (3.31) over \( y \in \Omega(\bar{x}, t) \), we obtain

\[
\int_{\Omega(\bar{x}, t)} |A(y) - A(x_0)| \, dy \leq \omega_A(t, \bar{x}) + C t \omega_A(t, \Omega(x_0, 7r)) + C t \int_0^t \frac{\omega_A(s, r)}{s} \, ds.
\]

Then (3.28) follows from the above inequality and (3.30). \( \blacksquare \)

3.4. Construction and symmetry relation for Green’s function. In this section, we shall prove that The function \( G(x, y) \) given by

\[
G(x, y) = G^*(y, x), \quad \forall \ x, y \in \Omega, \ x \neq y,
\]

is the Green’s function for the operator \( L \) in \( \Omega \). Then in light of (3.10), we see that the Green’s function \( G(x, y) \) has the pointwise bound (3.13).

To establish (3.33), first observe that \( G^*(\cdot, x_0) \) satisfies

\[
L^* G^*(\cdot, x_0) = 0 \text{ in } \Omega \setminus B(x_0, r) \text{ for any } r > 0.
\]

By [7, 8], we see that \( G^*(\cdot, x_0) \) is continuous in \( \Omega \setminus B(x_0, r) \) for any \( r > 0 \). Next, for \( y_0 \in \Omega \) and \( \epsilon > 0 \), let \( u = G_\epsilon(\cdot, y_0) \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \) be a unique strong solution of the problem (3.7) with \( f = \frac{1}{|B(y_0, \epsilon)|} \chi_{\Omega(y_0, \epsilon)} \). Then by (3.9), we have

\[
G_\epsilon(x_0, y_0) = \int_{\Omega(y_0, \epsilon)} G^*(y, x_0) \, dy.
\]

We conclude from (3.33) and (3.10) that for any \( x, y \in \Omega \) with \( x \neq y \), we have

\[
|G_\epsilon(x, y)| \leq C|x - y|^{2-n}, \quad \forall \epsilon \in (0, \epsilon_0|x - y|),
\]

which coincides with [15 Lemma 2.11]. With the above key estimate at hand, we can replicate the same argument as in [15] and construct the Green’s function \( G(x, y) \) for the operator \( L \) out of the family \( \{G_\epsilon(x, y)\} \). In particular, there is a sequence \( \{\epsilon_i\} \to 0 \) such that (see [15 (2.24)])

\[
G_\epsilon(x, y) \to G(x, y) \text{ uniformly on } \Omega \setminus B(y_0, r), \quad \forall r > 0.
\]
Then, by using the continuity of $G'(\cdot, x_0)$ away from $x_0$, we derive from (3.34) the desired identity (3.33).

3.5. Asymptotic behavior near a pole. In this section, we assume that condition (1.8) holds for some $r_0 > 0$. The estimate (3.23) leaves a room that we might be able to get an asymptotic behavior

$$G'(x, x_0) - G_0(x, x_0) = o(|x - x_0|^{2 - n}) \text{ as } x \to x_0.$$  

To see this, we closely follow the argument in Section 3.2. Let $v$ be as before in (3.2). For any $\epsilon > 0$, we can choose $\delta \in (0, \frac{1}{8} r_0]$ such that

$$\left( \int_0^\infty \int_0^\infty \frac{\omega_A(t, \Omega(x_0, r_0))}{t} \, dt \, ds \right)^{\frac{1}{2}} < \epsilon. \quad (3.35)$$

Let $\zeta$ be a smooth function on $\mathbb{R}^n$ such that

$$0 \leq \zeta \leq 1, \quad \zeta = 0 \text{ in } B(x_0, \delta), \quad \zeta = 1 \text{ in } \mathbb{R}^n \setminus B(x_0, \delta), \quad |D\zeta| \leq 4/\delta.$$  

We then define $g_1$ and $g_2$ by

$$g_1 = -\zeta(A - A_0)G_0(\cdot, x_0) \quad \text{and} \quad g_2 = -(1 - \zeta)(A - A_0)G_0(\cdot, x_0).$$

Then we have $g_1 \in L^p(\Omega)$ for $p > \frac{m}{n-2}$ and $g_2 \in L^{\frac{m}{p}}(\Omega)$. Indeed, we have

$$\int |g_1| \leq C \int |x - x_0|^{2-n} \, dx \leq C \delta^{(2-n)p+n} \quad (3.36)$$

and using (3.3), (3.4), and (3.35), we have

$$\int \Omega |g_2|^{\frac{m}{p}} = \sum_{k=0}^\infty \int_{\Omega(x_0, 2^{-k-1})} |g_2|^{\frac{m}{p}} \leq C \sum_{k=0}^\infty \int_{\Omega(x_0, 2^{-k})} |A - A_0| \leq C \int_0^\infty 2^{-k} \frac{\omega_A(t)}{t} \, dt \leq C \int_0^\infty \frac{\omega_A(t, x_0)}{t} \, dt \, ds \leq C e^{\frac{m}{p}}. \quad (3.37)$$

Let $v_1$ and $v_2$ be the solutions of the problems (3.13) and (3.15), respectively. Then, similar to (3.14) and (3.16), using (3.35) and (3.37) we obtain

$$\|v_1\|_{L^p(\Omega)} \leq C \delta^{(2-n)+\frac{m}{p}} \quad (p > \frac{m}{n-2}) \quad \text{and} \quad \|v_2\|_{L^{\frac{m}{p}}(\Omega)} \leq C e. \quad (3.38)$$

For $y_0 \in \Omega \setminus \{x_0\}$ with $|y_0 - x_0| \leq 5\delta$, we estimate $v_1(y_0)$ and $v_2(y_0)$ as follows. Set

$$r = \frac{1}{5}|y_0 - x_0|$$

and using Lemma (2.6), we have

$$|v_i(y_0)| \leq C \int_{\Omega(y_0, 2r)} |v_i| + C \int_0^r \frac{\omega_E(t, \Omega(y_0, 2r))}{t} \, dt, \quad i = 1, 2. \quad (3.39)$$

Using (3.38) together with Hölder’s inequalities, we have

$$\int_{\Omega(y_0, 2r)} |v_1| \leq C r^{\frac{m}{p}} \|v_1\|_{L^p(\Omega(y_0, 2r))} \leq C r^{\frac{m}{p}} \delta^{(2-n)+\frac{m}{p}} \quad (p > \frac{m}{n-2}),$$

$$\int_{\Omega(y_0, 2r)} |v_2| \leq C r^{2-n} \|v_2\|_{L^{\frac{m}{p}}(\Omega(y_0, 2r))} \leq C r^{2-n}. \quad (3.40)$$
On the other hand, applying Lemma 3.19 with \( \eta = \zeta \) and \( \eta = 1 - \zeta \), respectively, and using the fact that \( r \leq \delta \leq \frac{1}{2} r_0 \) for all \( t \in (0, r) \) we have

\[
\omega_g(t, \Omega(y_0, 2r)) \leq C r^{2-n} \left( \omega_A(t, \Omega(x_0, r_0)) + \frac{t}{r} \int_0^T \omega_A(s, \Omega(x_0, r_0)) \frac{ds}{s} \right), \quad i = 1, 2.
\]

Then, similar to (3.20), we have

\[
\int_0^r \frac{\omega_g(t, \Omega(y_0, 2r))}{t} dt \leq C r^{2-n} \int_0^r \frac{\omega_A(t, \Omega(x_0, r_0))}{t} dt, \quad i = 1, 2. \tag{3.41}
\]

Now we substitute (3.40) and (3.41) back to (3.39) to obtain

\[
|v(y_0)| \leq |v_1(y_0)| + |v_2(y_0)| - C r^{2-n} \left( r^{n-2-\frac{\delta}{2(n-\frac{\delta}{2})}} + \epsilon + \int_0^r \frac{\omega_A(t, \Omega(x_0, r_0))}{t} dt \right) \quad (p > \frac{n}{n-2}). \tag{3.42}
\]

Note that \( p > \frac{n}{n-2} \) implies \( n - 2 - \frac{\delta}{2} > 0 \). From (3.42) and the fact that

\[v = G^* (\cdot, x_0) - G_0 (\cdot, x_0) \quad \text{and} \quad r = \frac{1}{2} |y_0 - x_0|,\]

we conclude that

\[
\lim_{x \to x_0} |x - x_0|^p |G^*(x, x_0) - G_0(x, x_0)| = 0
\]

since \( y_0 \in \Omega \setminus \{x_0\} \) and \( \epsilon \) are arbitrary. Since \( G_0 \) is symmetric, we obtain (1.9) from the above and (3.33).

4. Appendix: Proof of Lemma 2.4

Let us consider the quantity

\[\phi(x, r) := \inf_{q \in R} \left( \int_{B(x, r)} |u - q| \right)^2\]

for \( x \in B(x_0, \frac{1}{2} R) \) and \( 0 < r \leq \frac{1}{2} R \). We decompose \( u = v + w \), where \( w \in L^2(B(x, r)) \) is the solution of the problem

\[
\begin{array}{l}
L^*_u w = - \nabla^2 ((A - \bar{A})u) + \nabla^2 (g - \bar{g}) \quad \text{in} \quad B(x, r), \\\n\quad w = \frac{(g - \bar{g}) - (A - \bar{A})u}{\bar{A}^2} \quad \text{on} \quad \partial B(x, r),
\end{array}
\]

where we use the notation

\[\bar{A} = \bar{A}(x, r), \quad g = \bar{g}(x, r), \quad L^*_u w = \nabla^2 (\bar{A} w).
\]

By [7] Lemma 2.23, we have

\[
|\{ y \in B(x, r) : |w(y)| > t \}| \leq \frac{C}{t} \left( \|u\|_{L^\infty(B(x, r))} \int_{B(x, r)} |A - \bar{A}| \right) + \int_{B(x, r)} |g - \bar{g}|, \quad \text{where} \quad C = C(n, \lambda, A).
\]

This yields (see [7] pp. 422-423) and recall the notation (1.4)

\[
\left( \int_{B(x, r)} |w|^2 \right)^{\frac{1}{2}} \leq C \omega_A(r) \|u\|_{L^\infty(B(x, r))} + C \omega_g(r, x), \quad \text{(4.1)}
\]

where \( C = C(n, \lambda, A) \). On the other hand, note that \( v = u - w \) satisfies

\[L^*_v v = \nabla^2 (\bar{A} v) = \nabla(\bar{A} \nabla v) = 0 \quad \text{in} \quad B(x, r),\]
and so does \( v - q \) for any constant \( q \in \mathbb{R} \). By the interior estimates for elliptic equations with constant coefficients, we have

\[
\| Du \|_{L^\infty(B(x, r))} \leq C_0 r^{-1} \left( \int_{B(x, r)} |v - q|^2 \right)^{1/2},
\]

where \( C_0 = C_0(n, \lambda, \Lambda) > 0 \) is a constant. Let \( 0 < \kappa \leq \frac{1}{2} \) to be a number to be fixed later. Then, we have

\[
\left( \int_{B(x, r)} |v - \sigma_{B(x, r)}|^2 \right)^{1/2} \leq 2\kappa r \| Du \|_{L^\infty(B(x, r))} \leq 2 C_0 \kappa \left( \int_{B(x, r)} |v - q|^2 \right)^{1/2}. \tag{4.2}
\]

By using the decomposition \( u = v + w \), we obtain from (4.2) that

\[
\left( \int_{B(x, r)} |u - \sigma_{B(x, r)}|^2 \right)^{1/2} \leq 2 \left( \int_{B(x, r)} |v - \sigma_{B(x, r)}|^2 \right)^{1/2} + 2 \left( \int_{B(x, r)} |w|^2 \right)^{1/2} \leq 4 C_0 \kappa \left( \int_{B(x, r)} |v - q|^2 \right)^{1/2} + 2 \left( \int_{B(x, r)} |w|^2 \right)^{1/2} \leq 8 C_0 \kappa \left( \int_{B(x, r)} |u - q|^2 \right)^{1/2} + (2\kappa^{-2} + 2C_0\kappa) \left( \int_{B(x, r)} |\omega|^2 \right)^{1/2}.
\]

Since \( q \in \mathbb{R} \) is arbitrary, by using (4.1), we thus obtain

\[
\phi(x, \kappa r) \leq 8 C_0 \kappa \phi(x, r) + C \left( \omega_A(r) \| u \|_{L^\infty(B(x, r))} + \omega_B(r, x) \right),
\]

where \( C = C(n, \lambda, \Lambda, \kappa) \). Now we choose \( \kappa \) such that \( 8 C_0 \kappa = \frac{1}{2} \). Then we have

\[
\phi(x, \kappa r) \leq \frac{1}{2} \phi(x, r) + C \left( \omega_A(r) \| u \|_{L^\infty(B(x, r))} + \omega_B(r, x) \right),
\]

where \( C = C(n, \lambda, \Lambda) \). By iterating, for \( j = 1, 2, \ldots \), we get

\[
\phi(x, \kappa^j r) \leq 2^{-j} \phi(x, r) + C \| u \|_{L^\infty(B(x, \kappa^j r))} \sum_{i=1}^j 2^{1-i} \omega_A(k^{-j}r) + C \sum_{i=1}^j 2^{1-i} \omega_B(k^{-j}r, x).
\]

We note that

\[
\sum_{j=0}^\infty \sum_{i=1}^j 2^{1-i} \omega_A(k^{-j}r) = \sum_{i=1}^\infty \sum_{j=0}^\infty 2^{1-i} \omega_A(k^{-j}r) = \sum_{i=1}^\infty \sum_{j=0}^\infty \omega_A(k^j r) = 2 \sum_{i=0}^\infty \omega_A(k^i r) \leq C \int_{0}^{\infty} \frac{\omega_A(t)}{t} dt < +\infty,
\]

where \( C = C(\kappa) = C(n, \lambda, \Lambda) \) and we used \([7, \text{Lemma 2.7}]. \) A similar computation holds for \( \omega_B(r, x) \), and thus we obtain

\[
\sum_{j=0}^\infty \phi(x, \kappa^j r) \leq 2 \phi(x, r) + C \| u \|_{L^\infty(B(x, r))} \int_{0}^{\infty} \frac{\omega_A(t)}{t} dt + C \int_{0}^{\infty} \frac{\omega_B(t, x)}{t} dt. \tag{4.3}
\]

Now, let \( q_{x, r} \) be chosen so that

\[
\left( \int_{B(x, r)} |u - q_{x, r}|^2 \right)^{1/2} = \inf_{q \in \mathbb{R}} \left( \int_{B(x, r)} |u - q|^2 \right)^{1/2} = \phi(x, r). \tag{4.4}
\]
Since we have
\[ |q_{x,r} - q_{x,r^*}|^2 \leq |u(x) - q_{x,r}|^2 + |u(y) - q_{x,r^*}|^2, \]
for any \( r \), note that \( k \) taking the average over \( x \in B_{cr}(x) \) and then taking the square, we obtain
\[ |q_{x,r} - q_{x,r^*}| \leq 2k^{-2n} \phi(x, r) + 2\phi(x, kr) \leq 2k^{-2n} \left( \phi(x, r) + \phi(x, kr) \right). \]
Then, by iterating and using the triangle inequality
\[ |q_{x,k^nr} - q_{x,r}| \leq 4k^{-2n} \sum_{j=0}^{N} \phi(x, k^jr). \]
Therefore, by using the fact that \( q_{x,k^nr} \to u(x) \) as \( N \to \infty \) and (4.3), we have
\[ |u(x) - q_{x,r}| \leq 4k^{-2n} \sum_{j=0}^{\infty} \phi(x, k^jr) \]
\[ \leq C\phi(x, r) + C||u||_{L^\infty(B(x,r))} \int_0^r \frac{\omega_A(t)}{t} \, dt + C \int_0^r \frac{\omega_B(t, x)}{t} \, dt. \] (4.5)
By averaging the inequality
\[ |q_{x,r}|^2 \leq |u(x) - q_{x,r}|^2 + |u(y)|^2 \]
over \( y \in B(x, r) \), taking the square, and using (4.4) we get
\[ |q_{x,r}| \leq 2\phi(x, r) + 2\left( \int_{B(x,r)} |u|^2 \right)^{\frac{1}{2}} \leq 4r^{-n}||u||_{L^1(B(x,r))}. \]
Therefore, by combining the above with (4.5), we get
\[ |u(x)| \leq C r^{-n}||u||_{L^1(B(x,r))} + C||u||_{L^\infty(B(x,r))} \int_0^r \frac{\omega_A(t)}{t} \, dt + C \int_0^r \frac{\omega_B(t, x)}{t} \, dt. \]
Now, taking supremum for \( x \in B(\bar{x}, r) \), where \( \bar{x} \in B(x_0, \frac{2}{3}R) \) and \( 0 < r \leq \frac{1}{3}R \), we have
\[ ||u||_{L^\infty(B(\bar{x}, r))} \leq C r^{-n}||u||_{L^1(B(\bar{x}, r))} + C||u||_{L^\infty(B(\bar{x}, 2r))} \]
\[ \int_0^r \frac{\omega_A(t)}{t} \, dt + C \int_0^r \frac{\omega_B(t, B(\bar{x}, r))}{t} \, dt. \]
We fix \( r_0 = r_0(n, \lambda, L, \omega_A) \) such that
\[ C \int_0^{r_0} \frac{\omega_A(t)}{t} \, dt \leq \frac{1}{3^n}. \]
Then, we have for any \( \bar{x} \in B(x_0, \frac{3}{2}R) \) and \( 0 < r \leq \min(r_0, \frac{1}{3}R) \) that
\[ ||u||_{L^\infty(B(\bar{x}, r))} \leq 3^{-n}||u||_{L^\infty(B(\bar{x}, 2r))} + Cr^{-n}||u||_{L^1(B(\bar{x}, 2R))} + C \int_0^r \frac{\omega_B(t, B(x_0, 2R))}{t} \, dt. \] (4.6)
For \( k = 1, 2, \ldots \), denote
\[ r_k = \left( \frac{3}{2} - \frac{1}{2^k} \right) R. \]
Note that \( r_{k+1} - r_k = 2^{-k-1}R \) and \( r_1 = R \). For \( \bar{x} \in B(x_0, r_k) \) and \( r \leq 2^{-k-2}R \), we have
\( B(\bar{x}, 2r) \subset B(x_0, r_{k+1}) \). We take \( k_0 \geq 1 \) sufficiently large such that \( 2^{-k_0-3}R_0 \leq r_0 \). Note that \( k_0 = k_0(r_0, R_0) = k_0(n, \lambda, L, \omega_A, R_0) \). Then for any \( k \geq k_0 \), we have \( 2^{-k-2}R \leq r_0 \) and thus by taking \( r = 2^{-k-2}R \) in (4.6), we obtain
\[ ||u||_{L^\infty(B(x_0, r))} \leq 3^{-n}||u||_{L^\infty(B(x_0, 2r))} + C2^{k_0}R^{-n}||u||_{L^1(B(x_0, 2R))} + C \int_0^k \frac{\omega_B(t, B(x_0, 2R))}{t} \, dt. \]
Multiplying the above by $3^{-kn}$ and summing over $k = k_0, k_0 + 1, \ldots$, we get
\[
\sum_{k=k_0}^{\infty} 3^{-kn} \|u\|_{L^\infty(B(x_0, r_k))} \leq \sum_{k=k_0}^{\infty} 3^{-(k+1)n} \|u\|_{L^\infty(B(x_0, r_{k+1}))} + C R^{-n} \|t\|_{L^1(B(x_0, 2R))} + \int_0^R \frac{\omega_k(t, B(x_0, 2R))}{t} dt.
\]
Noting that $R = r_1 \leq r_{k_0}$, and shifting the index in the second sum, we get (2.5). ■

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