On The Existence of Min-Max Minimal Surface of Genus $g \geq 2$

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Abstract

We prove an existence theorem similar to that of [5] [15] for min-max minimal surfaces of genus $g \geq 2$ by variational methods. We show that the min-max critical value for the area functional can be achieved by the bubbling limit of branched minimal surfaces with nodes of genus $g$ together with possibly finitely many branched minimal spheres. We also prove a strong convergence theorem similar to the classical mountain pass lemma. It is a further extension of the existence result in [5] [15].

1 Introduction

Existence theory of minimal surfaces and application to geometry and topology have been studied for a long time since the proof of classical Plateau Problem (see Chapter 4 of [3]) in 1931. There are lots of interesting results concerning the existence of area minimizing minimal surfaces in a given homotopy class. In particular, the existence theory for area-minimizing surfaces has been developed and applied to study topology for all genus in suitable senses (cf. [12] [13] [14] [6] etc.). Recently, existence theory for min-max minimal surfaces has attracted more interest, and has had nice applications and potential significance (cf. [2] [4] [5] [11]). One remarkable work was given by T. Colding and W. Minicozzi in [4] [5], where they constructed min-max minimal spheres and proved the finite time extinction for Ricci flow under certain topological conditions by studying the evolution of the area of the min-max minimal spheres. Motivated by that paper [5], the author studied the variational construction of min-max minimal tori in [15].
difference between spheres and surfaces of genus greater than zero is that the moduli space of conformal structures is nontrivial. The author developed a uniformization result in [15] to deal with this difficulty in the case of tori.

High genus cases are always very interesting (see [13][14] for application in the minimizing case). Recently, F. Marques and A. Neves gave an application of the min-max theory in the geometric measure theory setting(see [2]) to get some very interesting rigidity results on positive curved compact manifold. In this paper, we will extend the result in [5][15] to the high genus case($g \geq 2$). Since the moduli space of surfaces of genus larger than one is more complicated than that of the tori(genus one), we need a more delicate uniformization result. Besides this, we also need to extend the local replacement method and bubbling convergence theory in [5] to this case. Using notations introduced in Section 2.2 we summarize our main theorem as:

**Theorem 1.1** For any homotopically nontrivial path $\beta \in \Omega$, if $W > 0$, there exists a sequence $(\rho_n, \tau_n) \in [\beta]$, with $\max_{t \in [0,1]} E(\rho_n(t), \tau_n(t)) \to W$, and for any $\epsilon > 0$, there exists a large number $N > 0$ and a small number $\delta > 0$ such that if $n > N$, then for any $t \in (0,1)$ satisfying:

$$E(\rho_n(t), \tau_n(t)) > W - \delta, \quad (1)$$

there are a conformal harmonic map $u_0 : \Sigma_g \to N$ defined on the body of a genus $g$ Riemann surface with nodes $\Sigma_g^*$ and possibly finitely many harmonic sphere $u_i : S^2 \to N$, such that:

$$d_V(\rho_n(t), \bigcup u_i) \leq \epsilon. \quad (2)$$

Here the definition of Riemann surfaces with nodes is given in Section 5.1 and $d_V$ means varifold distance given in Appendix A in [5]. The theorem follows from Theorem 5.1 and Appendix A of [5].

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1In the case of tori, [7] also gave a method to deal with moduli space in an evolution setting.
2 Sketch of the variational methods

Now let us firstly review the method used by the author in [15]. In this method, we consider the area functional and energy functional simultaneously. Let $(N, h)$ be the target manifold. Consider the space of paths $\Omega = \{ \gamma(t) \in C^0([0, 1], C^0 \cap W^{1,2}(T^2_0, N)) \}$, where a path is a one parameter family of mappings $\gamma(t)$ from tori to the target manifold.

If we add certain degeneration constrains on the ends of all the paths, i.e. $\gamma(0), \gamma(1)$ are constant maps or maps to closed curves in $N$, we can define a min-max critical value $W = \inf_{\rho \in [\beta]} \max_{t \in [0,1]} \text{Area}(\rho(t))$ on a homotopy class of $\beta(t) \in \Omega$, where $\text{Area}(\cdot)$ is the area functional. Suppose the critical value is positive, i.e $W > 0$. A natural question is how to find the corresponding critical points. Classical 2-dimensional geometric variational methods are used to find the critical points. Firstly and naturally, take an area minimizing sequence of paths $\tilde{\gamma}_n(t) \in [\rho]$, such that $\lim_{n \to \infty} \max_{t \in [0,1]} \text{Area}(\tilde{\gamma}_n(t)) = W$.

Then we need to use the energy functional. Since energy functional depends not only on the mappings, but also on the conformal structures of the domain metrics, we need to module out the action of conformal group. We consider the following min-max critical value $W_E = \inf_{(\rho, \tau) \in ([\beta, \tau_0])} \max_{t \in [0,1]} E(\rho(t), \tau(t))$, where $E$ is the energy functional. In fact, $W_E = W$(See Section 3 in [15]). In order to module out conformal group action, we need to do reparametrizations on tori. Let $\tilde{g}_n(t) = \tilde{\gamma}_n(t)^*h$ be the pullback of the ambient metric, which may be degenerate. Using a uniformization result proved in [15] and a perturbation technique, $\tilde{g}_n(t)$ determines a continuous family of elements $\tau_n(t)$ in the Teichmüller space $T_1$ of genus one and a continuous isotopy family of diffeomorphisms $h_n(t) : (T^2, \tau_n(t)) \to (T^2, \tilde{g}_n(t))$, and if denoting $\gamma_n(t) = \tilde{\gamma}_n(t) \circ h_n(t)$, $E(\gamma_n(t), \tau_n(t)) - \text{Area}(\gamma_n(t)) \to 0$. After that, we do compactification to the sequences of path $\gamma_n(t)$ by the method of harmonic replacement developed by Colding and Minicozzi in [5]. Lastly, we combine the degeneration of conformal structures with the bubbling convergence firstly developed by Sack and Uhlenbeck in [12] to give a combined bubbling convergence for the compactified sequence of paths(See Theorem 5.1 in [15]). In the limit, we get some conformal harmonic maps from torus together with possibly some spheres or only harmonic spheres when degeneration happens. We can

2 They are also called sweep-outs in [5].

3 See [15] for details of the notations.

4 See also [5][7] for bubble convergence.
also get the energy identity (equation (45), (16) in [15]). In fact, we will achieve a strong deformation results for this specialized sequences (i.e. Theorem 1.1 of [15]).

Based on this method, let us describe the approach to high genus cases.

### 2.1 Teichmüller spaces of genus $g$ surfaces

Before going into the variational method, let us firstly describe the Teichmüller spaces $T_g$ and moduli spaces $\mathcal{M}_g$ of a genus $g$ surface $\Sigma_g$. We will summarize the following facts about Teichmüller spaces.

1°: Definition about Teichmüller spaces and Moduli spaces;
2°: Marked surface representation of Teichmüller spaces;
3°: Fuchsian model description for Teichmüller spaces;
4°: Quasi-conformal maps;
5°: Teichmüller mappings;

1°. Let $\text{Met}_g$ be the space of all the metrics on a topological surface $\Sigma_g$ of genus $g$. Denote $\text{Diff}(\Sigma_g)$ by the self diffeomorphism groups on $\Sigma_g$, and $\text{Diff}_0(\Sigma_g)$ the subgroup of $\text{Diff}(\Sigma_g)$ containing elements isotopy to the identity. Two metrics $ds^2$ and $(ds^2)'$ are said to be equivalent in the sense of moduli space, if there exists $w \in \text{Diff}(\Sigma_g)$, such that $w^*(ds^2)'$ is conformal to $ds^2$. Define all the equivalent classes to be the moduli space $\mathcal{M}_g = \text{Met}_g/\text{Diff}(\Sigma_g)$. Two metrics $ds^2$ and $(ds^2)'$ are said to be equivalent in the sense of Teichmüller space, if there exists $w \in \text{Diff}_0(\Sigma_g)$, such that $w^*(ds^2)'$ is conformal to $ds^2$. Define all the equivalent classes to be the Teichmüller space $T_g = \text{Met}_g/\text{Diff}_0(\Sigma_g)$. Here we are interested in the complex structure of the surfaces, obviously each $(\Sigma_g, ds^2)$ has a complex structure compatible with $ds^2$. Later on, we will use this complex structure without mentioning it.

2°. We firstly talk about the representation of Teichmüller spaces by the marked surfaces. We use the description in [9]. Given a fixed genus $g$ surface $\Sigma_0$, consider all the surfaces $(\Sigma, f)$, with $f : \Sigma_0 \to \Sigma$ a diffeomorphism. We say that $(\Sigma, f)$ and $(\Sigma', g)$ are equivalent in the sense of Teichmüller space, if $g \circ f^{-1} : \Sigma \to \Sigma'$ is homotopic to a conformal diffeomorphism from $\Sigma$ to $\Sigma'$. We call such a $f$ a marking, and $(\Sigma, f)$ a marked surface. The set of all equivalent classes of marked surfaces $\{(\Sigma, f)\}$ is another representation of the Teichmüller spaces $T_g$ of genus $g$ (Chap 1 of [9]).

3°. Let us talk about the Fuchsian model now. By uniformization theorem from complex analysis, all the closed surfaces $\Sigma_g$ with genus $g > 1$ have their universal covering space the upper half plane $\mathbb{H}$. The covering transformation group of $\pi : \mathbb{H} \to \Sigma_g$
is called Fuchsian group, which is denoted by $\Gamma$, and $(\Sigma_g, \Gamma)$ is called Fuchsian model. Usually, we also simply call $\Gamma$ a Fuchsian model. In the sense of complex analysis, the holomorphic diffeomorphism group of $\mathbb{H}$ is $PSL(2, \mathbb{R})$, so $\Gamma$ contains only linear fractional transformations with real coefficients, i.e., $\Gamma \subset PSL(2, \mathbb{R})$. If we consider the hyperbolic metric structure $(\mathbb{H}, ds^2_{-1})$, where $ds^2_{-1} = \frac{dx^2 + dy^2}{y^2}$, $\Gamma$ is constituted by isometries of $(\mathbb{H}, ds^2_{-1})$.

4°. We also need to talk about the quasi-conformal maps. Let $f : \Sigma \to \Sigma'$ be a diffeomorphism between two Riemann surfaces. Given local complex coordinates $(z, \bar{z})$, $(w, \bar{w})$ on $\Sigma$ and $\Sigma'$ respectively. Denote $f(z) = w \circ f \circ z$. Let

$$\mu = \frac{f_z}{f_{\bar{z}}}.$$  \hspace{1cm} (3)

It is easy to see that $|\mu|$ does not depend on the local complex coordinates. If $|\mu| \leq k < 1$, then we call such $f$ a quasi-conformal map.

Now let us combine the marked surface model with the quasi-conformal maps (see 5.1.2 of \cite{9}). Suppose $\Sigma_0$ is a fixed Riemann surface, with a Fuchsian group $\Gamma_0$. After some conjugation in $PSL(2, \mathbb{R})$, we can always assume $(0, 1, \infty)$ are fixed by some elements in $\Gamma_0$ (see chap 5 in \cite{9}). We call such $\Gamma_0$ a normalized Fuchsian group, and $(\Sigma_0, \Gamma_0)$ a normalized Fuchsian model. For any marked surface $(\Sigma, f)$, $f : \Sigma_0 \to \Sigma$ is always a quasi-conformal map (1.4.2 of \cite{9}). Now we lift the quasi-conformal map $f$ up to the upper half space $\mathbb{H}$ by the covering maps $\pi_0 : \mathbb{H} \to \Sigma_0$ and $\pi : \mathbb{H} \to \Sigma$ to get $\tilde{f} : \mathbb{H} \to \mathbb{H}$. After some $PSL(2, \mathbb{R})$ action on the target $\mathbb{H}$, we can assume that $\tilde{f}$ also fixes the three points $(0, 1, \infty)$ (We know the uniqueness of such quasi-conformal maps from \cite{9} and also from the following discussion). We call such maps $\tilde{f} : \mathbb{H} \to \mathbb{H}$ canonical quasi-conformal maps. By pulling over the normalized Fuchsian group $\Gamma_0$ on $\Sigma_0$ by $\tilde{f}$, we get another Fuchsian group $\Gamma_{\tilde{f}} = \tilde{f} \circ \Gamma_0 \circ \tilde{f}^{-1}$, such that $\Sigma = \mathbb{H}/\Gamma_{\tilde{f}}$. Now for such a marking $f$, we can define an injective homeomorphism:

$$\theta_{\tilde{f}} : \Gamma_0 \to PSL(2, \mathbb{R}),$$

where $\theta_{\tilde{f}}(\gamma) = \tilde{f} \circ \gamma \circ \tilde{f}^{-1}$. Lemma 5.1 in \cite{9} tells that $(\Sigma_1, f_1)$ and $(\Sigma_2, f_2)$ are equivalent in the sense of Teichmüller space, iff $\theta_{f_1} = \theta_{f_2}$. Now we can define the following set:

$$\mathcal{T}_g^2 = \{ \theta_{\tilde{f}} : \tilde{f} \text{ is a canonical quasiconformal map, such that } \theta_{\tilde{f}}(\Gamma_0) \text{ is a Fuchsian group for some genus } g \text{ surface.} \}$$  \hspace{1cm} (4)

\footnote{When $|\mu| = 0$, $f$ is holomorphic.}
Proposition 5.3 of [9] shows that $T_\#^2$ is identified with the Teichmüller space $T_g$. Later on, we will use this representation of Teichmüller space of genus $g$, and we will extend the quasi-conformal maps to more general settings, say, in the Sobolev spaces.

5°. We also need to talk about the Teichmüller mapping in a class of marked surfaces $[(\Sigma, f)]$, where $f : \Sigma_0 \to \Sigma$ is a diffeomorphism, and also a quasi-conformal map. By Theorem 5.9 in [9], there exists a unique a holomorphic quadratic differential $\phi$ on $\Sigma_0$ with $\|\phi\|_1 < 1$ and a unique quasi-conformal mapping $f_1 : \Sigma_0 \to \Sigma$ homotopic to $f$, and that the Beltrami coefficient $\mu_{f_1}$ of $f_1$ satisfies $\mu_{f_1} = \mu_{\phi}$, where

$$\mu_{\phi} \equiv \frac{||\phi||_1}{|\bar{\phi}|}.$$ (5)

We denote such a map by $f_\phi$ and call it Teichmüller mapping.

Denote the set of all holomorphic quadratic differentials on $\Sigma_0$ with one-norm $\| \cdot \|_1$ strictly less than one by $A_2(\Sigma_0)_1$. Given $\phi \in A_2(\Sigma_0)_1$, let $f_\phi$ be the unique Teichmüller mapping of the Beltrami coefficient $\mu_{\phi}$ in the homotopy class $id : \Sigma_0 \to \Sigma_0$. From Theorem 5.9 in [9], we know that the mapping

$$F : A_2(\Sigma_0)_1 \to T_\#^2,$$

is a homeomorphism, where $F(\phi) = \theta_{f_\phi}$ and $\tilde{f}_\phi : \mathbb{H} \to \mathbb{H}$ is the unique canonical quasi-conformal mapping lifted up with respect to $\Gamma_0$. By Riemann-Roch theorem, we know that $A_2(\Sigma_0)_1$ is homotopic to a $6g - 6$ dimension ball, and hence is $T_\#^2$. Later on, the topology on $T_g$ and $T_\#^2$ is identified by the topology on $A_2(\Sigma_0)_1$.

2.2 Some notation

Now let us set down the framework of the variational method. Given a Riemannian manifold $(N, h)$. Let $\Sigma_0$ be a fixed Riemann surface of genus $g > 1$ with a normalized Fuchsian group $\Gamma_0$. Denote elements in Teichmüller space $T_g$ by $\tau$. Let $\phi_\tau \in A_2(\Sigma_0)_1$ be the unique holomorphic quadratic differential on $\Sigma_0$ corresponding to $\tau$. Denote $f_\tau = f_{\phi_\tau}$ to be the unique Teichmüller mapping determined by the Beltrami coefficient $\mu_{\phi_\tau}$, and $\tilde{f}_\tau : \mathbb{H} \to \mathbb{H}$ the unique canonical quasi-conformal mapping lifted up with respect to $\Gamma_0$. By the results in the above section, we can view $\tau$ as an equivalent class of marked surfaces $[(\Sigma_\tau, f_\tau)]$ with normalized Fuchsian group $\Gamma_\tau = \theta_{f_\tau}(\Gamma_0)$, i.e. $\Sigma_\tau = \mathbb{H}/\Gamma_\tau$.

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*Here $||\phi||_1$ is the $L^1$ norm of $\phi$.**
Definition 2.1 The variational spaces are defined as
\[ \Omega = \{ \gamma(t) \in C^0([0, 1], C^0 \cap W^{1, 2}(\Sigma_0, N)) \}, \] 
and
\[ \bar{\Omega} = \{ (\gamma(t), \tau(t)) : \gamma(t) \in C^0([0, 1], C^0 \cap W^{1, 2}(\Sigma_\tau(t), N)), \tau(t) \in C^0([0, 1], T_\gamma) \}, \]
where \((\Sigma_\tau = \mathbb{H}/\Gamma_\tau, \Gamma_\tau)\) is the normalized Fuchsian model corresponding to \(\tau \in T_\gamma\). We always assume that the boundary mappings \(\gamma(0)\) and \(\gamma(1)\) are mapped onto close curves in \(N\).

Now let us talk about the continuity of \(\gamma(t) \in C^0([0, 1], C^0 \cap W^{1, 2}(\Sigma_\tau(t), N))\). Here we can view all the \(\gamma(t)\) as been defined on the upper half plane \(\mathbb{H}\) lifted up by \(\pi_{\tau(t)} : \mathbb{H} \to \Sigma_\tau(t)\), with the Fuchsian groups \(\Gamma_\tau(t)\) changing continuously w.r.t the parameter \(t\). The continuity of \(\gamma(t)\) w.r.t parameter \(t\) can be defined as mappings on compact subsets \(K\) of \(\mathbb{H}\) with the Poincaré metric, i.e. \(\gamma(t) \in C^0([0, 1], C^0 \cap W^{1, 2}(K, N))\). Another way to understand this is as follows. Let \(\phi_{\tau(t)}\) be the holomorphic quadratic differentials corresponding to \(\tau(t)\). The fact that \(\tau(t)\) change continuously w.r.t \(t\) is equivalent to that \(\phi_{\tau(t)}\) change continuously w.r.t \(t\) in \(A_2(\Sigma_0)\). Let \(f_{\tau(t)}\) be the Teichmüller mappings corresponding to \(\phi_{\tau(t)}\), then the canonical lift \(\tilde{f}_{\tau(t)} : \mathbb{H} \to \mathbb{H}\) change continuously in \(C^0_{loc} \cap W^{1, 2}(\mathbb{H}, \mathbb{H})\) by properties of quasi-conformal mapping. Using \(f_{\tau(t)}\) as special markings for a continuous family of elements in \(T_\gamma\), we can pull the path \(\gamma(t) : \Sigma_\tau(t) \to N\) back to \(\Sigma_0\), i.e. \(f_{\tau(t)}^*(\gamma(t)) = \gamma(t) \circ f_{\tau(t)} : \Sigma_0 \to N\). The continuity of \(\gamma(t)\) w.r.t \(t\) is defined as the continuity of the path \(f_{\tau(t)}^*(\gamma(t))\) w.r.t \(t\) on the same domain surface \(\Sigma_0\).

Next let us talk about the homotopy equivalence in \(\bar{\Omega}\). Consider two elements \(\{(\gamma_i(t), \tau_i(t)) : i = 1, 2\}\). They have different domains \(\Sigma_{\tau_i(t)}, i = 1, 2\) given by normalized Fuchsian models \(\Gamma_{\tau_i(t)}\). As above, we use Teichmüller mappings \(f_{\phi_{\tau_i(t)}} : \Sigma_0 \to \Sigma_{\tau_i(t)}, i = 1, 2\) to identify \(\Sigma_{\tau_i(t)}, i = 1, 2\) with \(\Sigma_0\), where \(\phi_{\tau_i(t)}\) are the holomorphic quadratic differentials corresponding to \(\tau_i(t), i = 1, 2\). Since \(T_\gamma\) is homotopic to a ball, \(\tau_1(t)\) and \(\tau_2(t)\) are always homotopic to each other. So \(\{(\gamma_1(t), \tau_1(t))\}\) are homotopic to \(\{(\gamma_2(t), \tau_2(t))\}\) if \(f_{\phi_{\tau_1(t)}}^* \gamma_1(t)\) are homotopic to \(f_{\phi_{\tau_2(t)}}^* \gamma_2(t)\).

Definition 2.2 Fix a homotopy class \([\beta] \subset \Omega\), and \(\tau_0\) a fixed element in \(T_\gamma\) given by \([(\Sigma_0, id)]\). For area functional, define
\[ W = \inf_{\rho \in [\beta]} \max_{t \in [0, 1]} Area(\rho(t)). \]
For energy functional, define
\[ W_E = \inf_{(\rho, \tau) \in ([\beta, \tau_0]) \in [0,1]} \max_{t \in [0,1]} E(\rho(t), \tau(t)). \] (9)

Remark 2.1 The definition of area and energy is referred to [10][12][5]. Later, we will show that \( W = W_E \) in Remark 3.2. We will mainly focus on the case when \( W > 0 \).

2.3 Sketch of the variational method

Now an interesting question is to find the critical points corresponding to \( W \). In fact, the critical points are achieved by some conformal harmonic mappings from surfaces degenerated from \( \Sigma_0 \) together with possibly some harmonic spheres. To achieve the critical points, we use geometric variational method. We take a minimizing sequence \( \{ \tilde{\gamma}_n(t) : n = 1, \cdots, \infty \} \subset [\beta] \subset \Omega \), such that
\[ \lim_{n \to \infty} \max_{t \in [0,1]} \text{Area}(\tilde{\gamma}_n(t)) = W. \]
In fact, by some mollification method, we can assume that \( \tilde{\gamma}_n(t) \) varies continuously in \( C^2 \)-class, i.e. \( \tilde{\gamma}_n(t) \in C^0([0,1], C^2(\Sigma_0, N)) \).

Then we would like to change to use the variational method of the energy functional \( E \) and hence work in \( \tilde{\Omega} \). We use the following three steps. **Firstly**, we do almost conformal reparametrizations to module out the conformal group action. Pull back the ambient metric \( \tilde{g}_n(t) = \tilde{\gamma}_n(t)^*h \). We want to show that \( \tilde{g}_n(t) \), which may be degenerate, determine a family of elements \( \tau_n(t) \in T_g \). Suppose that the corresponding normalized Fuchsian model and Teichmüller mappings are \( (\Sigma_{\tau_n(t)}, \Gamma_{\tau_n(t)}, f_{\tau_n(t)}) \), with \( \Gamma_{\tau_n(t)} = \theta_{f_{\tau_n(t)}}(\Gamma_0) \) and \( \Sigma_{\tau_n(t)} = \mathbb{H}/\Gamma_{\tau_n(t)} \). We want to find almost conformal parametrizations \( h_n(t) : \Sigma_{\tau_n(t)} \to (\Sigma_0, \tilde{g}_n(t)) \), such that the reparametrization \( (\gamma_n(t), \tau_n(t)) = (\tilde{\gamma}_n(h_n(t), t), \tau_n(t)) \in [(\tilde{\gamma}_n(t), \tau_0)] \) have energy close to area, i.e. \( E(\gamma_n(t), \tau_n(t)) - \text{Area}(\gamma_n(t)) \to 0 \). **Secondly**, we do compactification by deforming \( \gamma_n(t) \) to \( \rho_n(t) \). We use the local harmonic replacement method developed by Colding and Minicozzi in [5][15]. We make \( \rho_n(t) \) to be almost harmonic mappings to get bubbling compactness as in [12][5][15]. **Finally**, we discuss the degenerations of conformal structures of \( \tau_n(t) \). We will show that \( (\rho_n(t), \tau_n(t)) \) bubbling converge to several conformal harmonic mappings on surfaces degenerated from \( \Sigma_0 \) together with possibly some harmonic spheres, and we will show that the sum of the area is equal to \( W \).

In the following sections, we will discuss the three steps in details.

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8This is also the energy since the final targets are all conformal.
3 Conformal parametrization in the high genus case

In this section, we will do almost conformal reparametrization for the minimizing sequence \( \tilde{\gamma}_n(t) \in \Omega \). We can assume that \( \tilde{\gamma}_n(t) \) have better regularity.

Lemma 3.1 (Lemma D.1 of [5], Lemma 3.1 of [15]) Suppose \( \tilde{\gamma}_n(t) \) are chosen as in the above section, we can perturb them to get a new minimizing sequence in the same homotopy class \([\beta]\), such that if denoting them still as \( \tilde{\gamma}_n(t) \), \( \tilde{\gamma}_n(t) \in C^0([0,1], C^2(\Sigma_0, N)) \).

3.1 Summary of results on quasi-conformal mappings

Before going to the uniformization and reparametrization, we firstly summarize results of quasi-conformal mappings proved in [1][9] and the appendix of [15]. We will focus on the apriori estimates for the conformal diffeomorphisms between general metrics.

3.1.1 Results about quasi-conformal maps

We mainly refer to Ahlfors and Bers in [1] (see also Section 6.1 in [15]). They gave the existence and uniqueness of conformal diffeomorphism \( f^\mu : C_{|dz+\mu dz|^2} \to C_{dw+\nu dw} \) fixing three points \((0,1,\infty)\) for any \( L^\infty \) function \( \mu \) with \(|\mu| \leq k < 1 \) (see also Theorem 4.30 and Proposition 4.33 of [9]). We can such \( \mu \) Beltrami coefficient here.\(^9\) Such maps satisfy the following equation (see equation 57):

\[
f^\mu_{\bar{z}} = \mu(z) f^\mu_z.
\]

Define function space \( B_p(\mathbb{C}) = C^{1-\frac{2}{p}} \cap W^{1,p}_{\text{loc}}(\mathbb{C}) \), where \( p > 2 \) depends on the bound \( k \) of \(|\mu|\). Suppose \( \mu, \nu \in L^\infty(\mathbb{C}) \), and \(|\mu|, |\nu| \leq k \), with \( k < 1 \). Let \( f^\mu, f^\nu \) be the corresponding conformal homeomorphisms, then:

Lemma 3.2 (Lemma 16, Theorem 7, Lemma 17, Theorem 8 of [1], Lemma 6.2 of [15])

\[
d_{S^2}(f^\mu(z_1), f^\mu(z_2)) \leq cd_{S^2}(z_1, z_2)\alpha,
\]

\[
\|f^\mu_z\|_{L^p(B_R)} \leq c(R),
\]

\(^9\)We use \( \{z, \bar{z}\} \) and \( \{w, \bar{w}\} \) as complex coordinates.

\(^{10}\)This is different from that in Section 2.2 without invariance under Fuchsian group.
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\[ d_{S^2}(f^\mu(z), f^\nu(z)) \leq C\|\mu - \nu\|_\infty, \]  \hfill (13)

\[ \|(f^\mu - f^\nu)_z\|_{L^p(B_R)} \leq C(R)\|\mu - \nu\|_\infty. \]  \hfill (14)

Here \( d_{S^2} \) is the sphere distance, which is equivalent to the plane distance of \( \mathbb{C} \) on compact sets. \( \alpha = 1 - \frac{2}{p} \). \( B_R \) is a disk of radius \( R \) on \( \mathbb{C} \). All constants are uniformly bounded depending on \( k < 1 \).

### 3.1.2 Results about quasi-linear quasi-conformal maps

What we concern in our case are the conformal homeomorphisms \( h^\mu : \mathbb{C}_{dw=dw} \to \mathbb{C}_{|dz+\mu z|^2} \) fixing three points \((0, 1, \infty)\), which arise as the inverse mappings of those \( f^\mu \) of Ahlfors and Bers. In fact, suppose

\[ h^\mu(w) = (f^\mu)^{-1}(w), \]  \hfill (15)

then our mappings satisfy:

\[ \overline{h^\mu_w} = -\mu(h^\mu(w))\overline{h^\mu_w}. \]  \hfill (16)

Since the equation is quasi-linear (compared to linear equation 10), we call such \( h^\mu \) quasi-linear quasi-conformal maps.

If \( \{\mu_n\} \) are a sequence of Beltrami coefficients as above, such that \( \|\mu_n - \mu\|_{C^1} \to 0 \), and \( h^{\mu_n} \) satisfying 15 we have results similar to the above:

**Lemma 3.3** *(Lemma 6.3 of [15]*)

\[ d_{S^2}(h^{\mu_n}, h^\mu) \to 0, \]  \hfill (17)

\[ \|(h^{\mu_n} - h^\mu)_w\|_{L^p(B_R)} \to 0, \]  \hfill (18)

where \( p \) is given in the above section.

### 3.2 Uniformization for surfaces of genus \( g > 1 \)

Fix \( \Sigma_0 \) with normalized Fuchsian model \( \Gamma_0 \) as before. Denote \( \pi_0 \) to be the quotient map for \((\Sigma_0, \Gamma_0)\). Denote the Poincaré metric on \( \Sigma_0 \) by \( g_0 \). Given \( \tau \in \mathcal{T}_g \), let the corresponding normalized Fuchsian model be \((\mathbb{H}, \Gamma, \Sigma_\tau)\) as in Section 2.2. Let \( \pi_\tau : \mathbb{H} \to \Sigma_\tau \) be the quotient map, and \( f_\tau : \Sigma_0 \to \Sigma_\tau \) the Teichmüller mapping.
Proposition 3.1 Let $g$ be a $C^1$ metric on $\Sigma_0$. We can view $g$ as a metric on $\mathbb{H}$ by lifting up using $\pi_0$. Then there is a unique element $\tau \in \mathcal{T}_g$ with normalized Fuchsian model $(\Sigma_\tau, \Gamma_\tau)$, and a unique orientation preserving $C^1$ conformal diffeomorphism $h : \Sigma_\tau \to (\Sigma_0, g)$, such that $h$ is isotopic to $f_{\tau}^{-1}$, with the normalization that if lifting up to $\tilde{h} : \mathbb{H} \to \mathbb{H}$ by $\pi_\tau$ and $\pi_0$, $\tilde{h}^* (\Gamma_0) = \Gamma_\tau$. Furthermore, given $g(t)$ a family of $C^1$ metrics on $\Sigma_0$ which is continuous w.r.t $t$ in the $C^1$ class, i.e. $g(t) \in C^1([0, 1], C^1$-metrics), and $g(t) \geq \varepsilon g_0$ for some uniform $\varepsilon > 0$, let $(\tau(t), h(t))$ be the corresponding elements in $\mathcal{T}_g$ and normalized conformal diffeomorphisms, then $\tau(t)$ and $h(t)$ are continuously w.r.t $t$ in $\mathcal{T}_g$ and $C^0 \cap W^{1,2}(\Sigma_{\tau(t)}, \Sigma_0)$ respectively.

Remark 3.1 Here the space $C^0 \cap W^{1,2}(\Sigma_{\tau(t)}, \Sigma_0)$ have varying domain spaces $\Sigma_{\tau(t)}$, and the continuity is defined in Section 2.2.

We need the following result to prove the proposition. Let $g$ be a Riemannian metric on the complex plane $\mathbb{C}$.

Lemma 3.4 (Lemma 6.1 of [15]) In the complex coordinates $\{z, \bar{z}\}$, we can write $g = \lambda(z) |dz + \mu(z)d\bar{z}|^2$. Here $\lambda(z) > 0$, and $\mu(z)$ is complex function on the complex plane with $|\mu| < 1$. If $g \geq \varepsilon dzd\bar{z}$, there exists a $k = k(\varepsilon) < 1$, such that $|\mu| \leq k$. Furthermore, $\mu$ is a rational function of the components $g_{ij}(z)$, so if a family $g(t)$ is continuous w.r.t $t$ in the $C^1$ class, the corresponding $\mu(t)$ is also continuous continuous in the $C^1$ class.

Proof: (of Proposition 3.1). Firstly, let us show the existence of such mark $\tau \in \mathcal{T}_g$ and conformal homeomorphism $h$. Pulling $g$ back to $\mathbb{H}$ by $\pi_0$ and denote it still by $g$, then it is invariant under the $\Gamma_0$ group action. By Lemma 3.4, $g = \lambda(z) |dz + \mu(z)d\bar{z}|^2$, with $|\mu(z)| \leq k < 1$. Here $\mu$ is the Beltrami coefficient as in Section 3.1.1. We have a unique normalized quasi-conformal mapping $f^\mu : \mathbb{H}_{|dz + \mu d\bar{z}|^2} \to \mathbb{H}_{dwd\bar{w}}$(see also Proposition 4.33 of [9]). Now push forward the Fuchsian group $\Gamma_0$ under $f^\mu$. Since $f^\mu$ is a homeomorphism, we get another Fuchsian group $\Gamma_{f^\mu} = (f^\mu)_* (\Gamma_0) = \theta_{f^\mu} (\Gamma_0)$ on $\mathbb{H}_{d^2 d\bar{w}}$. This Fuchsian group gives a normalized Fuchsian model which represent an element in $\mathcal{T}_g$. Denote this element by $\tau$. Denoting $\Gamma_{f^\mu}$ by $\Gamma_\tau$, we get a Fuchsian model $\Sigma_\tau = \mathbb{H}/\Gamma_\tau$. Let $\pi_\tau : \mathbb{H} \to \Sigma_\tau$ be the quotient map, then after taking quotient of $f^\mu$ by $\pi_0$ and $\pi_\tau$, we get $f^\mu : \Sigma_0 \to \Sigma_{\tau}$. By the definition of quasi-conformal, this $f^\mu$ is conformal between $(\Sigma_0, |dz + \mu(z)d\bar{z}|^2)$ and $\Sigma_\tau$, and hence conformal between $(\Sigma_0, g)$ and $\Sigma_\tau$. So we take $h = (f^\mu)^{-1}$, then $h$ is a conformal homeomorphism between $\Sigma_\tau$.\footnote{We denote the quotient map still by $f^\mu$}
and $(\Sigma_0, g)$. The $C^{1,\frac{1}{2}}$ regularity of $h$ follows from Theorem 3.1.1 and Theorem 3.3.1 in [10]. By the definition of $f_\tau : \Sigma_0 \to \Sigma_\tau$, when pulling back to $\tilde{f}_\tau : \mathbb{H} \to \mathbb{H}$ by $\pi_0$ and $\pi_\tau$, $(\tilde{f}_\tau)_*(\Gamma_0) = \theta_{f_\tau}(\Gamma_0) = \Gamma_\tau$. So by Lemma 5.1 of [9], we know that $f_\tau$ is homotopic to $f^\mu$. So $h$ is homotopic to $f^{-1}_\tau$. The normalization of $\tilde{h}$, i.e. $h^*(\Gamma_0) = \Gamma_\tau$, comes trivially from the fact that $\Gamma_\tau = (f^\mu)_*(\Gamma_0)$ and $h = (f^\mu)^{-1}$. The uniqueness of such $\tau$ and $h$ follows from the uniqueness of $f^\mu$.

Now let us talk about the continuous dependence of $(\tau, h)$ on $\mu$. For a continuous family of $C^1$ metrics $g(t)$, after pulling back to $\mathbb{H}$ by $\pi_0$, $g(t) = \lambda(t) |dz + \mu(t) d\overline{z}|^2$, and is continuous w.r.t $t$ in the $C^1$ class. We have $|\mu(t)| \leq k(\epsilon) < 1$, and $\mu(t)$ continuous w.r.t $t$ in the $C^1$ class by Lemma 3.4. Let $f(t) = f^\mu(t)$ and $\tilde{h}(t) = (f(t))^{-1}$ as above.

Firstly, let us talk about the continuity of $\tau(t)$ w.r.t parameter $t$. Now the corresponding normalized Fuchsian model $\Gamma_{\tau(t)}$ is given by $(f^\mu(t))_*(\Gamma_0)$. Suppose the normalized generators for $\Gamma_0$ as in Section 2.5 of [9] are $\{\alpha_i^0, \beta_i^0\}_{i=1}^g$, where $\alpha_i^0$ has attractive fixed point at $1$ and $\beta_i^0$ has repelling and attractive fixed point at $0$ and $\infty$ respectively. Then clearly $\{\theta_{f^\mu(t)}(\alpha_i^0), \theta_{f^\mu(t)}(\beta_i^0)\}_{i=1}^g$ form the normalized generators for $\Gamma_{\tau(t)}$. Now

$$\theta_{f^\mu(t)}(\gamma) = f^\mu(t) \circ \gamma \circ (f^\mu(t))^{-1} = f^\mu(t) \circ \gamma \circ \tilde{h}(t). \quad (19)$$

Now by Lemma 3.1.1 and Lemma 3.1.2 $f^\mu(t)$ and $\tilde{h}(t)$ are continuous w.r.t parameter $t$ in $C^0$ class when acting on compact subsets of $\mathbb{C}$. So for fixed $\gamma \in \Gamma_0$, $\theta_{f^\mu(t)}(\gamma)$ is continuous w.r.t the parameter $t$, which means the coefficients of the linear fractional transformation corresponding to $\theta_{f^\mu(t)}(\gamma)$ are continuous functions of $t$. So the coefficients for $\{\theta_{f^\mu(t)}(\alpha_i^0), \theta_{f^\mu(t)}(\beta_i^0)\}_{i=1}^g$ are continuous functions of $t$. Now using the topology of Fricke Space as in Section 2.5 and Lemma 5.10 and Lemma 5.13 in [9], the corresponding elements $\tau(t) \in T_g$ are continuous w.r.t the parameter $t$ in the natural topology of $T_g$.

Next, let us show the continuity of $\tilde{h}(t)$. Now lift up to $\tilde{h}(t) : \mathbb{H}_{dud\overline{v}} \to \mathbb{H}_{d|z + \mu(t) d\overline{z}|^2}$, then $\tilde{h}(t) = (f^\mu(t))^{-1}$ is $\mu(t)$ quasi-linear quasi-conformal map as in Section 3.2.2 So by Lemma 3.3 we have the local $C^0 \cap W^{1,2}(\mathbb{H}, \mathbb{H})$ continuity of $\tilde{h}(t)$ w.r.t $t$, since $\mu(t)$ is continuous in $C^1$ w.r.t parameter $t$. It directly implies the continuity of $h(t) : \Sigma_{\tau(t)} \to \Sigma_0$ in the sense of Section 2.2, i.e. when restricting to compact subsets $K$ of $\mathbb{H}$, the lifted up mapping $\tilde{h}(t) \in C^0([0,1], C^0 \cap W^{1,2}(K, N))$. 

□
3.3 Construction of the conformal reparametrization

As above, we consider $\tilde{g}_n(t) = \tilde{\gamma}_n(t)^* h$, which is continuous w.r.t $t$ in the $C^1$ class by Lemma 3.1. Since $\tilde{g}_n(t)$ may be degenerate, let $g_n(t) = \tilde{g}_n(t) + \delta_n g_0$, where $g_0$ is the Poincaré metric of $\Sigma_0$, and $\delta_n$ arbitrarily small. Then $g_n(t)$ uniquely determines $\tau_n(t) \subset \mathcal{T}_g$ and conformal diffeomorphisms $h_n(t)$ by Proposition 3.1. We have the following result similar to Theorem 3.1 of [15].

**Theorem 3.1** Using the above notations, we have reparametrizations $(\gamma_n(t), \tau_n(t)) \in \tilde{\Omega}$ for $\tilde{\gamma}_n(t)$, i.e. $\gamma_n(t) = \tilde{\gamma}_n(h_n(t), t)$, such that $\gamma_n(t) \in [\tilde{\gamma}_n]$. And

$$E(\gamma_n(t), \tau_n(t)) - \text{Area}(\gamma_n(t)) \to 0,$$

for some sequence $\delta_n \to 0$ as $n \to \infty$.

**Proof:** We know that $h_n(t) : \Sigma_{\tau_n(t)} \to (\Sigma, g_n(t))$ are conformal diffeomorphisms. Let $\gamma_n(t) = \tilde{\gamma}_n(h_n(t), t) : \Sigma_{\tau_n(t)} \to N$ be the composition with the almost conformal parametrization. To show that $\gamma_n(t)$ is a path in $\Omega$, we only need to show the continuity. The continuity of $t \to \gamma_n(t)$ from $[0, 1]$ to $C^0 \cap W^{1,2}(\Sigma_{\tau_n(t)}, N)$ follows from the continuity of $t \to \tilde{\gamma}_n(t)$ in $C^2$ by Lemma 3.1 and that of $t \to h_n(t)$ in $C^0 \cap W^{1,2}(\Sigma_{\tau(t)}, \Sigma_0)$ by Proposition 3.1. Moreover $\gamma_n(t)$ is homotopic to $\tilde{\gamma}_n$. From our discussion of homotopy equivalence of mappings defined on different domains in Section 2.2, we view $\gamma_n(t)$ as mappings defined on $\Sigma_0$ by composing with $f_{\tau_n(t)} : \Sigma_0 \to \Sigma_{\tau_n(t)}$ and compare it to $\tilde{\gamma}_n(t)$. Since $h_n(t)$ are homotopic equivalent to $f_{\tau_n(t)}^{-1}$ by Proposition 3.1, $h_n(t) \circ f_{\tau_n(t)}$ is homotopic equivalent to identity map of $\Sigma_0$. While $\gamma_n$ are composition of $\tilde{\gamma}_n$ with $h_n(t)$, $\gamma_n \circ f_{\tau_n}$ is homotopic equivalent to $\tilde{\gamma}_n$, hence $\gamma_n \sim \tilde{\gamma}_n$.

We can get estimates as in Appendix D of [5] and the proof of Theorem 3.1 of [15]:

$$E(\gamma_n(t), \tau_n(t)) = E(h_n(t) : T_{\tau_n(t)}^2 \to (\Sigma_0, \tilde{g}_n(t))) \leq E(h_n(t) : \Sigma_{\tau_n(t)} \to (\Sigma_0, g_n(t)))$$

$$= \text{Area}(h_n(t) : \Sigma_{\tau_n(t)} \to (\Sigma_0, g_n(t)))$$

$$= \text{Area}(\Sigma_0, g_n(t)) + \int_{\Sigma_0} [\det(g_n(t))]^{\frac{1}{2}} \text{dvol}_0$$

$$= \int_{\Sigma_0} [\det(\tilde{g}_n(t)) + \delta_n \text{Tr}_{g_0}\tilde{g}_n(t) + C(\tilde{g}_n(t))\delta_n^2]^{\frac{1}{2}} \text{dvol}_0$$

$$\leq \text{Area}(\Sigma_0, \tilde{g}_n(t)) + C(\tilde{g}_n(t)) \sqrt{\delta_n}$$

$$= \text{Area}(\gamma_n(t) : \Sigma_0 \to N) + C(\tilde{\gamma}_n) \sqrt{\delta_n}.$$

(21)
The first and last equality follow from the definition of energy and area integral, and the second inequality is due to the fact \( \hat{g}_n(t) \leq g_n(t) \). Hence we have equation (20) if we choose \( \delta_n \to 0 \) depending only on \( \tilde{\gamma}_n \).

\[ \square \]

**Remark 3.2** By argument similar to Proposition 1.5 in [5] and Remark 3.2 in [15], the above theorem implies that \( \mathcal{W} = \mathcal{W}_E \).

### 4 Compactification for mappings

For each \( (\gamma_n(t), \tau_n(t)) \) gotten above, \( \tau_n(t) \) corresponds to a normalized Fuchsian model \( (\Sigma_{\tau_n(t)}, \Gamma_{\tau_n(t)}) \). We can also view \( \gamma_n(t) \) as been lifted up to \( \mathbb{H} \) by \( \pi_{\tau_n(t)} : \mathbb{H} \to \Sigma_{\tau_n(t)} \). Denote the lifted mappings again by \( \gamma_n(t) \), then \( \gamma_n(t) \) can be viewed as defined on the same domain \( \mathbb{H} \), i.e. \( \gamma_n(t) : \mathbb{H} \to N \), but invariant under different Fuchsian groups \( \Gamma_{\tau_n(t)} \) action, i.e. \( \forall \gamma \in \Gamma_{\tau_n(t)}, \gamma_n(t) \circ \gamma = \gamma_n(t) \). We can apply similar perturbation procedure to the lifted mappings as in [5][15].

Before doing such perturbations, let us firstly talk about collections of disjoint balls on \( \Sigma_{\tau} \). Here we use \( B = \bigcup_{i=1}^{n} B_i \) to denote a finite collection of disjoint geodesic balls on \( \Sigma_{\tau} \), with the radii of balls less than the injective radius \( r_{\Sigma_{\tau}} \) of \( \Sigma_{\tau} \). Taking a ball \( B \in B \) with radius \( r_B \), we would like to talk about a sub-geodesic ball with the same center but with the radius only a ratio \( \mu < 1 \) of \( r_B \), which we denote by \( \mu B \). Such a geodesic ball \( B \) with hyperbolic metric of curvature \(-1\) can always pulled back to the Poincaré disk \( (D, ds_{-1}^2 = \frac{|dx|^2}{1-|x|^2}) \), such that the center of \( B \) goes to the center of \( D \). Then \( B \) can be viewed as a disk \( B(0, r_B^0) \subset D \) with hyperbolic metric \( ds_{-1}^2 \), where \( r_B^0 \) is the Euclidean radius of the image of \( B \) and \( r_B = \int_0^{r_B^0} \frac{1}{\sqrt{1-t^2}} dt = \sin^{-1}(r_B^0) \). The hyperbolic metric is now conformal and uniformly equivalent to the Euclidean metric \( ds_0^2 = |dx|^2 \) on \( B \). Here *uniformly equivalent* means \( C^{-1} ds_0^2 \leq ds_{-1}^2 \leq C ds_0^2 \) for some constant \( C > 1 \). There exists a small number:

\[
 r_0 = \sin^{-1}(\frac{1}{2}),
\]

(22)

such that if we restrict the radius \( r_B \) of \( B \) to be less than \( r_0 \), we can choose the constant \( C = \frac{4}{3} \). Then if we consider \( \frac{1}{4} B \), then we know that in the Euclidean metric \( ds_0^2 \), the radius of \( \frac{1}{4} B \) is less than \( \frac{1}{2} r_B^0 \), i.e. \( \frac{1}{4} B \subset B(0, \frac{1}{2} r_B^0) \). Later on, we will always assume that geodesic balls have their radii bounded from above by \( r_0 \).
Lemma 4.1 Let $[\beta]$ and $W_E$ be as defined in definition 2.2. For any $(\gamma(t), \tau(t)) \in [\beta] \subset \tilde{\Omega}$ with $\max_{t \in [0,1]} E(\gamma(t), \tau(t)) - W_E \ll 1$, if $(\gamma(t), \tau(t))$ is not harmonic unless $\gamma(t)$ is a constant map, we can perturb $\gamma(t)$ to $\rho(t)$, such that $\rho(t) \in [\gamma]$ and $E(\rho(t), \tau(t)) \leq E(\gamma(t), \tau(t))$. Moreover for any $t$ such that $E(\gamma(t), \tau(t)) \geq \frac{1}{2} W_E$, $\rho(t)$ satisfy:

(*) For any finite collection of disjoint balls $\bigcup_i B_i$ on $\Sigma_{\tau t}$ with geodesic radius of each ball $B_i$ bounded above by the injective radius $r_{\Sigma_{\tau t}}$ and $r_0$, such that $E(\rho(t), \bigcup_i B_i) \leq \epsilon_0$, let $v$ be the energy minimizing harmonic map with the same boundary value as $\rho(t)$ on $\frac{1}{64} \bigcup_i B_i$, then we have:

$$\int_{\frac{1}{64} \bigcup_i B_i} |\nabla \rho(t) - \nabla v|^2 \leq \Psi \left( E(\gamma(t), \tau(t)) - E(\rho(t), \tau(t)) \right).$$

(23)

Here $\epsilon_0$ is some small constant, and $\Psi$ is a positive continuous function with $\Psi(0) = 0$.

Remark 4.1 We will mainly use the idea in the proof of Theorem 2.1 of [5] and the proof of Lemma 4.1 of [15]. As talked in the remarks following Lemma 4.1 of [15], we would need to show the continuity of local harmonic replacement and comparison of energy decrease of successive harmonic replacements. The continuity of harmonic replacement is a conformal invariant property, which we can handle by pulling every ball we care to the center of the Poincaré disk as above. For the comparison of the energy decrease, it turns out that what we really need to care is the analysis on a single ball. So we could do that by pulling the chosen ball to the center of the Poincaré disk again, without caring about the image of the other balls.

In the following three subsections, we will list the results about analysis of harmonic replacements on disks. Then we will give a result of comparison of harmonic replacements, where we will show a similar result to Lemma 3.11 of [5] and Lemma 4.2 of [15] by showing a proof with minor differences to the previous ones. At the end, we will give the deformation map $\gamma \to \rho$ by explicit constructions.

4.1 Results about harmonic replacements on disks

Here we summarize some known results of harmonic replacements on disks. Let $B_1$ be the unit disk in $\mathbb{R}^2$, and $N$ the ambient manifold.
Theorem 4.1 (Theorem 3.1 of [5]) There exists a small constant $\epsilon_1$ (depending only on $N$) such that for all maps $u, v \in W^{1,2}(B_1, N)$, if $v$ is weakly harmonic with the same boundary value as $u$, and $v$ has energy less than $\epsilon_1$, then we have:

$$\int_{B_1} |\nabla_0 u|^2 - \int_{B_1} |\nabla_0 v|^2 \geq \frac{1}{2} \int_{B_1} |\nabla_0 u - \nabla_0 v|^2. \quad (24)$$

Here we use $\nabla_0$ to denote the flat connection of $B_1$.

Remark 4.2 Although this theorem is formulated when we use the standard metric $ds_0^2 = dx^2 + dy^2$ on $B_1$, we can still have inequality (24) if we take another metric $ds^2$ on $B_1$ which is conformal to $ds_0^2$, since both sides of inequality (24) are conformal invariant. So if we take the standard hyperbolic metric $ds_{-1}^2$ on a small ball as talked in the beginning of Section 4, inequality (24) is still true only by changing the flat connection to the connection $\nabla$ of $ds_{-1}^2$.

Remark 4.3 As talked in Section 4.2 of [15], we can use the energy gap to control the $W^{1,2}$-norm difference between a mapping defined on the unit disk with its corresponding energy minimizing harmonic mapping with the same boundary data. This theorem also implies the uniqueness of energy minimizing harmonic maps with energy less than $\epsilon_1$ and fixed boundary values (Corollary 3.3 of [5]).

Based on this theorem, we have the following result which shows that perturbing mappings locally to energy minimizing harmonic mappings is a continuous functional. This is a combination of Corollary 4.1 and 4.2 of [15], so here we omit the proof.

Corollary 4.1 (Corollary 3.4 of [5], Corollary 4.1 and 4.2 of [15]) Let $\epsilon_1$ be given in the previous theorem. Suppose $u \in C^0(\overline{B_1}) \cap W^{1,2}(B_1)$ with energy $E(u) \leq \epsilon_1$, then there exists a unique energy minimizing harmonic map $v \in C^0(\overline{B_1}) \cap W^{1,2}(B_1)$ with the same boundary value as $u$. Set $\mathcal{M} = \{v \in C^0(\overline{B_1}) \cap W^{1,2}(B_1) : E(v) \leq \epsilon_1\}$. If we denote $v$ by $H(u)$, the map $H : \mathcal{M} \to \mathcal{M}$ is continuous w.r.t the norm on $C^0(\overline{B_1}) \cap W^{1,2}(B_1)$. Here the norm is the sum of $C^0(\overline{B_1})$-norm and $W^{1,2}(B_1)$-norm.

Suppose $u_1, u$ are defined on a ball $B_{1+\epsilon}$ with energy less than $\epsilon_1$. Suppose $u_1 \to u$ in $C^0(\overline{B_{1+\epsilon}}) \cap W^{1,2}(B_{1+\epsilon})$. Choose a sequence $r_i \to 1$, and let $w_i, v$ be the mappings which coincide with $u_i, u$ outside $r_iB_1$ and $B_1$ and are energy minimizing inside $r_iB_1$ and $B_1$ respectively. Then $w_i \to w$ in $C^0(\overline{B_{1+\epsilon}}) \cap W^{1,2}(B_{1+\epsilon})$. 
Remark 4.4 If we use geodesic ball $B_r$ of geodesic radius $r \leq r_0$ on $\Sigma_g$ with Poincaré metric, all the results of the above lemma hold. This is because that the Poincaré metric $ds_{-1}^2$ is conformal and uniformly equivalent to the flat metric $ds_0^2$, so harmonic maps w.r.t. $ds_0^2$ are also harmonic w.r.t. $ds_{-1}^2$, and $C^0$ and $W^{1,2}$ norms of a fixed map w.r.t. $ds_{-1}^2$ are uniformly equivalent to those w.r.t. $ds_0^2$.

4.2 Comparison results of successive harmonic replacements

Now we will give a comparison result for successive harmonic replacements by adapting Lemma 3.11 in [5] and Lemma 4.2 in [15]. Fix a mapping $u \in W^{1,2}(\Sigma_g,N)$. We still denote $\mathcal{B}$ as a finite collection of disjoint geodesic balls on $\Sigma_g$ as above. Given $\mu \in [0,1]$, denote $\mu \mathcal{B}$ to be the collection of geodesic balls with the same centers as $\mathcal{B}$, but with geodesic radii $\mu$ timing those corresponding ones of $\mathcal{B}$. Suppose that $u$ has small energy on a collection $\mathcal{B}$. We denote $H(u,\mathcal{B})$ to be the mapping which coincides with $u$ outside $\mathcal{B}$, but are the energy minimizing ones inside $\mathcal{B}$ with the same boundary values as $u$ on $\partial \mathcal{B}$. We call $H$ the harmonic replacement in the following. If $\mathcal{B}_1, \mathcal{B}_2$ are two such collections, we denote $H(u,\mathcal{B}_1, \mathcal{B}_2)$ to be $H(H(u,\mathcal{B}_1), \mathcal{B}_2)$. We have the following energy comparison results for $u$, $H(u,\mathcal{B}_1)$ and $H(u,\mathcal{B}_1, \mathcal{B}_2)$.

Lemma 4.2 Fix a Riemann surface $\Sigma_g$ with Poincaré metric, and a mapping $u \in C^0 \cap W^{1,2}(\Sigma_g,N)$. Let $\mathcal{B}_1$, $\mathcal{B}_2$ be two finite collections of disjoint geodesic balls on $\Sigma_g$ with radii less than the injective radius $r_{\Sigma_g}$ and $r_0$ as (22). If $E(u, \mathcal{B}_i) \leq \frac{1}{3} \epsilon_1$ for $i = 1, 2$, with $\epsilon_1$ given in Theorem 4.1, then there exists a constant $k$ depending on $N$, such that:

$$E(u) - E[H(u,\mathcal{B}_1, \mathcal{B}_2)] \geq k \left( E(u) - E[H(u,\frac{1}{4} \mathcal{B}_2)] \right)^2,$$

and for any $\mu \in [\frac{1}{64}, \frac{1}{4}]$,

$$\frac{1}{k} \left( E(u) - E[H(u,\mathcal{B}_1)] \right)^{\frac{1}{2}} + E(u) - E[H(u,4\mu\mathcal{B}_2)] \geq E[H(u,\mathcal{B}_1)] - E[H(u,\mathcal{B}_1,\mu\mathcal{B}_2)].$$

Remark 4.5 The proof is very similar to that of Lemma 4.2 of [15]. We will use the Euclidean metric which is conformal to the hyperbolic metric on each of the geodesic balls we are considering. Since the inequalities (25) and (26) are all conformal invariant, the proof in the Euclidean metrics implies that in hyperbolic metrics. By the energy minimizing properties, we can easily get the following inequality:

$$E(u) - E[H(u,\mathcal{B}_1, \mathcal{B}_2)] \geq E(u) - E[H(u,\frac{1}{4} \mathcal{B}_1)].$$
This is because that \( E[H(u, B_1, B_2)] \leq E[H(u, B_1)] \leq E[H(u, \frac{1}{4} B_1)] \). Combining the above inequalities, we get the comparison for energy of any two successive harmonic replacements by appropriately shrinking the radii.

We need the following lemma to construct comparison maps. This is a scaling invariant version.

**Lemma 4.3** (Lemma 3.14 of [5]) There exists a \( \delta \) and a large constant \( C \) depending on \( N \), such that for any \( f, g \in C^0 \cap W^{1,2}(\partial B_R, N) \), if \( f, g \) are equal at some point on \( \partial B_R \), and:

\[
R \int_{\partial B_R} |f' - g'|^2 \leq \delta^2, \quad (28)
\]

we can find some \( \rho \in (0, \frac{1}{2} R] \), and a mapping \( w \in C^0 \cap W^{1,2}(B_R \setminus B_{R-\rho}, N) \) with \( w|_{B_R} = f, \ w|_{B_{R-\rho}} = g \), which satisfies estimates:

\[
\int_{B_R \setminus B_{R-\rho}} |\nabla w|^2 \leq C \left( R \int_{\partial B_R} |f'|^2 + |g'|^2 \right)^\frac{2}{7} \left( R \int_{\partial B_R} |f' - g'|^2 \right)^\frac{1}{7}. \quad (29)
\]

**Proof:** (of Lemma 4.2) Here we will adapt the proof of Lemma 4.2 in [15]. Since we assume that \( E(u, B_1) \leq \frac{1}{3} \epsilon_1 \), we know that \( u \) and \( H(u, B_1) \) have energy less than \( \frac{2}{3} \epsilon_1 \) on \( B_1 \lor B_2 \), so we can use energy gaps to control \( W^{1,2} \) norms difference by Theorem 4.1. Denote balls in \( B_1 \) by \( B^1 \), and balls in \( B_2 \) by \( B^2 \). We prove the two inequalities separately.

1° Inequality 25. We divide the second collection \( B_2 \) into two sub-collections \( B_2 = B_{2+} \lor B_{2-} \), where \( B_{2+} = \{ B^2_j : \frac{1}{4} B^2_j \subset B^1_\alpha \lor \frac{1}{4} B^2_j \cap B_1 = \emptyset \text{ for some } B^1_\alpha \subset B_1 \} \) and \( B_{2-} = B_2 \setminus B_{2+} \), and deal with them separately.

For collection \( B_{2+} \), we separate it into another two sub-collections \( \{ \frac{1}{4} B^2_j \cap B_1 = \emptyset \} \) and \( \{ \frac{1}{4} B^2_j \subset B^1_\alpha \} \). For balls \( \frac{1}{4} B^2_j \cap B_1 = \emptyset \), we can use the energy minimizing property of small energy harmonic maps as in remark 4.3 and similar arguments as inequality (18) and (19) in [15] to get,

\[
\sum_{\{ \frac{1}{4} B^2_j \cap B_1 = \emptyset \}} (E(u) - E[H(u, \frac{1}{4} B^2_j)]) \leq E(u) - E[H(u, B_1, \cup_{\frac{1}{4} B^2_j \cap B_1 = \emptyset} B^2_j)]. \quad (30)
\]

For balls \( \frac{1}{4} B^2_j \subset B^1_\alpha \), \( H(u, B_1, \frac{1}{4} B^2_j) = H(u, B_1) \). We denote \( u_1 = H(u, B_1) \). Using energy minimizing property of small energy harmonic maps again, and similar arguments
as inequality (20) and (21) of [15], we have,
\[
\int \bigcup_{\frac{1}{4}B_j^2 \subset B_1^0} \left| \nabla u \right|^2 - \left| \nabla H(u, \frac{1}{4}B_j^2) \right|^2 \leq \int \bigcup_{\frac{1}{4}B_j^2 \subset B_1^0} \left| \nabla u \right|^2 - \left| \nabla H(u, B_1, B_j^2) \right|^2 \\
\leq \int |\nabla u|^2 - |\nabla u_1|^2 + \int \bigcup_{\frac{1}{4}B_j^2 \subset B_1^0} \left| \nabla u_1 \right|^2 - \left| \nabla H(u, B_1, B_j^2) \right|^2
\]
(31)

The second "≤" of the above is gotten by adding a term \(\int \bigcup_{\frac{1}{4}B_j^2 \subset B_1^0} |\nabla u_1|^2\) and subtracting a same term after the first "≤". For the first term, using Theorem 4.1 and the following Remark 4.2 we have that \(\int |\nabla u|^2 - |\nabla u_1|^2 \leq \int |\nabla u - \nabla u_1|^2 \leq 4 (E(u) - E(u_1))\).

The second term is bounded from above by \(E(u_1) - E[H(u_1, \bigcup_{\frac{1}{4}B_j^2 \subset B_1^0} B_j^2)] \leq E(u) - E[H(u, B_1, \bigcup_{\frac{1}{4}B_j^2 \subset B_1^0} B_j^2)]\). So combining the above estimates together, we get inequality,
\[
E(u) - E[H(u, \frac{1}{4}B_2^+)] \leq C(E(u) - E[H(u, B_1, B_2^+)])
\]
(32)

Now let us consider the sub-collection \(B_2^-\). Here we deal with balls individually. Fix a \(B_j^2 \in B_2^-\), then \(\frac{1}{4}B_j^2 \cap B_1^1 \neq \emptyset\) for some \(B_1^1 \in B_1\), but \(\frac{1}{4}B_j^2\) does not belong to any \(B_1^1 \in B_1\). Using discussions about small geodesic balls in the beginning of Section 4, we can identify this \(B_j^2\) with a sub-disk centered at the origin of the Poincaré disk, and model it by \((B(0, r_0^B), \frac{ds_0^2}{1-|x|^2})\). Simply denote it by \(B_{r_0^B}\), and denote \(u_1 = H(u, B_1)\) as above. Lower subindex here is used to denote the radius of that ball w.r.t. \(ds_0^2\). Now let us construct an auxiliary comparison map. Using basic measure theory, there exists a subset of \([\frac{1}{4}r_0^B, r_0^B]\) with measure \(\frac{1}{30}r_0^B\), such that for any \(r\) in this subset, we have,
\[
\int_{\partial B_r} |\nabla u_1 - \nabla u|^2 \leq \frac{9}{r_0^B} \int_{\frac{1}{4}r_0^B} \int_{\partial B_r} |\nabla u_1 - \nabla u|^2 \leq \frac{9}{r} \int_{B_{\frac{1}{4}r_0^B}} |\nabla u_1 - \nabla u|^2,
\]
(33)
\[
\int |\nabla u_1|^2 + |\nabla u|^2 \leq \frac{9}{r_0^B} \int_{\frac{1}{4}r_0^B} \int_{\partial B_r} |\nabla u_1|^2 + |\nabla u|^2 \leq \frac{9}{r} \int_{B_{\frac{1}{4}r_0^B}} |\nabla u_1|^2 + |\nabla u|^2,
\]
(34)
where \(\nabla_0\) is the connection of \(ds_0^2\). By choosing \(\epsilon_1\) small enough, we can make \(r \int_{\partial B_r} |\nabla_0 u_1|^2 + |\nabla_0 u|^2 \leq \delta^2\) and \(r \int_{\partial B_r} |\nabla_0 u_1 - \nabla_0 u|^2 \leq \delta^2\) with \(\delta\) as in the above Lemma 4.3. Since \(\frac{1}{4}B_{r_0^B} \subset B_{\frac{1}{2}r_0^B}\) as discussed in the beginning of Section 4, and that \(B_{r_0^B} \in B_2^-\), \(B_{\frac{1}{2}r_0^B}\) and hence \(B_r\) must intersect a ball in \(B_1\) but is not contained in any ball of \(B_1\), so \(u\)
and $u_1$ must coincide at least one point on $\partial B_r$. So by Lemma 4.3 \( \exists \rho \in (0, \frac{1}{2}r) \) and \( \exists w \in C^0 \cap W^{1,2}(B_r \setminus B_{r-\rho}) \) with \( w|_{\partial B_r} = u_1|_{\partial B_r} \), \( w|_{\partial B_{r-\rho}} = u|_{\partial B_{r}} \), and:

\[
\int_{B_r \setminus B_{r-\rho}} |\nabla_0 w|^2 \leq C \left( \int_{\partial B_r} |\nabla_0 u_1 - \nabla_0 u|^2 \right)^{\frac{1}{2}} \left( \int_{\partial B_r} |\nabla_0 u_1|^2 + |\nabla_0 u|^2 \right)^{\frac{1}{2}} \\
\leq C \left( \int_{B_{r-\rho}^o \setminus B_{r-\rho}} |\nabla_0 u_1 - \nabla_0 u|^2 \right)^{\frac{1}{2}} \left( \int_{B_{r-\rho}^o \setminus B_{r-\rho}} |\nabla_0 u_1|^2 + |\nabla_0 u|^2 \right)^{\frac{1}{2}}.
\]

(35)

Now construct comparison map \( v \) on \( B_{r}^o \) such that:

\[
v = \begin{cases} 
  u_1 & \text{on } B_{r-\rho}^o \setminus B_r \\
  w & \text{on } B_r \setminus B_{r-\rho} \\
  H(u, B_r)(\frac{r}{r-\rho}) & \text{on } B_{r-\rho} 
\end{cases}
\]

In the last equation, we do a scaling w.r.t. the flat coordinates. Now \( E[H(u_1, B_{r-\rho}^o)] \leq E(v) \) on \( B_{r-\rho}^o \), since \( H(u_1, B_{r-\rho}^o) \) is the energy minimizing harmonic maps among maps with the same boundary values. So:

\[
\int_{B_{r-\rho}^o \setminus B_r} |\nabla_0 H(u_1, B_{r-\rho}^o)|^2 \leq \int_{B_{r-\rho}^o \setminus B_r} |\nabla_0 v|^2 \\
= \int_{B_r \setminus B_{r-\rho}} |\nabla_0 u_1|^2 + \int_{B_{r-\rho}^o \setminus B_r} |\nabla_0 w|^2 + \int_{B_{r-\rho}^o \setminus B_r} |\nabla_0 H(u, B_r)(\frac{r}{r-\rho})|^2 \\
= \int_{B_r \setminus B_{r-\rho}} |\nabla_0 w|^2 + \int_{B_{r-\rho}^o \setminus B_r} |\nabla_0 H(u, B_r)|^2.
\]

(36)

Now since \( \frac{1}{4}B_{r}^o \subset B_{\frac{3}{2}} \subset B_r \), we have:

\[
\int_{\frac{1}{4}B_{r}^o} |\nabla_0 u|^2 - \int_{\frac{1}{4}B_{r-\rho}^o} |\nabla_0 H(u, \frac{1}{4}B_{r}^o)|^2 \leq \int_{B_r} |\nabla_0 u|^2 - \int_{B_{r-\rho}^o} |\nabla_0 H(u, B_r)|^2 \\
\leq \int_{B_r} |\nabla_0 u|^2 - \int_{B_{r-\rho}^o} |\nabla_0 H(u_1, B_{r-\rho}^o)|^2 + \int_{B_{r-\rho}^o \setminus B_r} |\nabla_0 w|^2 + \int_{B_{r-\rho}^o \setminus B_r} |\nabla_0 u_1|^2 \\
\leq \int_{B_r \setminus B_{r-\rho}} |\nabla_0 w|^2 + \int_{B_{r-\rho}^o \setminus B_r} |\nabla_0 u|^2 - \int_{B_r} |\nabla_0 u_1|^2.
\]

(37)

Now we can use the conformal invariance for energy integral to change all the flat connection \( \nabla_0 \) and flat metric \( ds_0^2 \) to hyperbolic connection \( \nabla \) and hyperbolic metric \( ds^2_{-1} \). Summing the above inequality on all balls in \( B_{2-} \), and using Theorem 4.1 and the
following Remark 4.2 together with inequality 35, we can get the following inequality by similar arguments as those in inequalities (29) and (30) in [15]:

$$E(u) - E[H(u, \frac{1}{4} B_{2-}^-)] \leq C'(E(u) - E[H(u, B_1, B_2)])^{\frac{1}{2}}. \quad (38)$$

Combining inequalities on $B_{2+}$ and $B_{2-}$, we get the inequality 25.

2° Inequality (26) We divide $B_2$ into two disjoint sub-collections $B_{2+}$ and $B_{2-}$, with $B_{2+} = \{ B^2_j : \mu B^2_j \subset B^1_{1, \alpha} \text{ or } \mu B^2_j \cap B_1 = \emptyset \}$. For collection $B_{2+}$, similar method also gives:

$$E[H(u, B_1)] - E[H(u, B_1, \mu B_{2+})] \leq E(u) - E[H(u, 4 \mu B_{2+})]. \quad (39)$$

For subcollection $B_{2-}$, we use similar proof as above. Here we identify $4 \mu B^2_j$ with a sub-disk centered at the origin of the Poincaré disk again, and get an isometric representation $(B^0_{r_B}, ds^2_{-1})$. In the construction of $w$, we change the role of $u$ and $u_1$. Let the comparison map be,

$$v = \begin{cases} 
  u & \text{on } B_{r_B} \setminus B_r \\
  w & \text{on } B_r \setminus B_{r-\rho} \\
  H(u_1, B_r)(\frac{x}{r-\rho}) & \text{on } B_{r-\rho} 
\end{cases}.$$

We have $\int_{B_{r_B}} |\nabla_0 H(u, B_{r_B})|^2 \leq \int_{B_{r_B}} |\nabla_0 v|^2$ by the energy minimizing property. Since we have $\mu B^2_j = \frac{1}{4} B_{r_B} \subset B_{r_B}$, by argument similar to inequalities (34)(35) and (36) in [15], we get,

$$E(u_1) - E[H(u_1, \mu B_{2-})] \leq E(u) - E[H(u, 4 \mu B_{2-})] + C(E(u) - E(u_1))^{\frac{1}{2}}. \quad (40)$$

Combining results on $B_{2+}$ and $B_{2-}$, we get inequality 26.

\[\square\]

4.3 Construction of deformation maps

Let us talk about harmonic replacements on paths $\{\gamma(t), \tau(t)\} \in \hat{\Omega}$ now. The normalized Fuchsian models of $\tau(t)$ are given by $(\Sigma_{\tau(t)}, \Gamma_{\tau(t)})$, and denote the injective radius of $\Sigma_{\tau(t)}$ by $r_{\tau(t)}$. Firstly, let us point out where to do harmonic replacements. Fix a time parameter $t \in (0, 1)$. Suppose $B$ is a ball on $\Sigma_{\tau(t)}$, with radius $r_B < r_{\tau(t)}$. As discussed in the beginning of this section, we can view $\gamma(t)$ as been defined on the upper half plan $\mathbb{H}$ by lifting up using $\pi_{\tau(t)} : \mathbb{H} \to \Sigma_{\tau(t)}$. Since $\{\tau(t)\}$ is a compact set in
we can always pick one connected component of the pre-images \( \pi_{\tau(t)}^{-1}(B) \) inside a fix compact subset \( K \subset \mathbb{H} \). Denote that connected component still by \( B \), then obviously it has radius \( r_B \) w.r.t the hyperbolic metric \( ds^2 \) of \( \mathbb{H} \). Moreover \( B \) is a standard ball in \( \mathbb{H} \) w.r.t. the flat metric \( ds^2 \). By the continuity of \( \tau(t) \), for parameter \( |s - t| \ll 1 \), we have that the injective radius \( |r_{\tau(s)} - r_{\tau(t)}| \ll 1 \). So \( r_B < r_{\tau(s)} \), hence the image of this ball \( B \) under \( \pi_{\tau(s)}: \mathbb{H} \rightarrow \Sigma_{\tau(s)} \) is also a geodesic ball with radius less than the injective radius \( r_{\tau(s)} \) of \( \Sigma_s \). Denoting the image by \( B \) again, we want to do harmonic replacement simultaneously on \( B \subset \Sigma_s \) for \( |s - t| \ll 1 \).

When \( |s - t| \ll 1 \), let us pick up a continuous cutoff function \( \mu(s) \), such that \( \mu(t) = 1 \), and \( \mu(s) = 0 \) for \( |s - t| > \delta \) with \( \delta > 0 \) small enough. If we do harmonic replacements for \( \gamma(s) \) on balls \( \mu(s)B \), Corollary 4.1 and discussion in Remark 4.4 together with the definition of continuity of pathes directly imply that we get another continuous path in \( \tilde{\Omega} \). Similarly, we can continuously shrink the radii on balls \( \mu(s)B \) where we do harmonic replacements continuously to 0, so that the new path can be continuously deformed to the original one in \( \tilde{\Omega} \), which implies that they lie in the same homotopy class by the definition of homotopy equivalence in Section 2.2.

The strategy to construct the deformation map is to do harmonic replacement firstly on a collection of disjoint geodesic balls where the energy decrease is almost maximal, and then use Lemma 4.2 to get estimate of form (23) for any other harmonic replacements on collection of balls with small energy. For \( \sigma \in C^0 \cap W^{1,2}(\Sigma_{\tau}, N), \epsilon \in (0, \epsilon_1] \), define the maximal possible energy decrease as,

\[
e_{\epsilon, \sigma} = \sup_{B} \{ E(\sigma, \tau) - E[ H(\sigma, \frac{1}{4} B), \tau] \}, \tag{41}\]

where \( B \) are chosen as any finite collection of disjoint geodesic balls on \( \Sigma_{\tau} \) with radii less than \( r_0 \) as in (22) and the injective radius of \( \Sigma_{\tau} \), satisfying: \( E(\sigma, B) \leq \epsilon \). When \( \sigma \) is not harmonic, we always have that \( e_{\epsilon, \sigma} > 0 \). Now for a path \( (\sigma(t), \tau(t)) \in \tilde{\Omega} \), we have the following continuity property similar to Lemma 3.34 in [5] and Lemma 4.4 in [15].

**Lemma 4.4** \( \forall t \in (0, 1), \) if \( \sigma(t) \) is not harmonic, there exists a neighborhood \( I^t \subset (0, 1) \) of \( t \) depending on \( t, \epsilon \) and the path \( \sigma \), such that \( \forall s \in 2I^t \).

\[
e_{\frac{1}{2} \epsilon, \sigma(s)} \leq 2e_{\epsilon, \sigma(t)} \tag{42}\]

**Proof:** Since \( e_{\epsilon, \sigma(t)} > 0 \), the continuity of \( \sigma(s) \) implies that that there exists a neighborhood \( \tilde{I}^t \) of \( t \), such that \( \forall s \in 2\tilde{I}^t \), and for any finite collection of balls \( B \subset K \), where
K is a fixed compact subset of \( \mathbb{H} \),

\[
\frac{1}{2} \int_{\mathcal{B}} |\nabla \sigma(s) - \nabla \sigma(t)|^2 \leq \min \left\{ \frac{1}{4}e_{\epsilon, \sigma(t)}, \frac{1}{2}\epsilon \right\} ,
\]

where we view \( \sigma(s) \) as being lifted up to \( \mathbb{H} \).

Fix \( s \in 2\tilde{t} \). By definition \[14\] we can pick a finite collection of balls \( \mathcal{B} \subset \Sigma_{\tau(s)} \), such that \( E(\sigma(s), \mathcal{B}) \leq \frac{1}{2}\epsilon \) and \( E(\sigma(s)) - E[H(\sigma(s), \frac{1}{4}\mathcal{B})] \geq \frac{3}{2}e_{\epsilon, \sigma(s)} \). By taking the compact set \( K \subset \mathbb{H} \) large enough, we can always find a connected pre-image in \( K \) for each ball in \( \mathcal{B} \). Denote those connected pre-image balls by \( \tilde{\mathcal{B}} \) and \( \tilde{\Sigma} \).

Next, we will choose families of collections of disjoint geodesic balls corresponding to paths \( (\gamma(t), \tau(t)) \in \tilde{\Omega} \).

**Lemma 4.5** There exist a covering \( \{I^j : j = 1, \cdots, m\} \) for parameter space \([0, 1] \), and \( m \) collection of disjoint geodesic balls \( \mathcal{B}_j \subset \Sigma_{\tau(t_j)}, j = 1, \cdots, m \) having radii less than \( r_0 \) in \([22]\) and the injective radius \( r_{\tau(t_j)} \), together with continuous functions \( r_j : [0, 1] \rightarrow [0, 1], j = 1, \cdots, m \), satisfying:

1. Each \( r_j(t) \) is supported in \( I^j \);
2. For a fixed \( t \), at most two \( r_j \) are positive, and \( E(\gamma(t), r_j(t)\mathcal{B}_j) \leq \frac{1}{2}e_{\epsilon_1} \);
3. If \( t \in [0, 1] \), such that \( E(\gamma(t), \tau(t)) \geq \frac{1}{2}W \), there exists a \( j \), such that \( E(\gamma(t)) - E[H(\gamma(t), \frac{1}{4}r_j(t)\mathcal{B}_j)] \geq \frac{1}{8}e_{\epsilon_1, \gamma(t)} \).
The proof use continuity of the paths and $e_{\epsilon, \gamma(t)}$ together with a covering argument for the parameter space $[0, 1]$. It is similar to that of Lemma 3.39 in [5] and Lemma 4.5 in [15], so we left it for readers.

**Proof:** (of Lemma 4.1) The perturbation from $\gamma(t)$ to $\rho(t)$ is done by successive harmonic replacements on the collection of balls given in Lemma 4.3. Denote $\gamma^0(t) = \gamma(t)$, and $\gamma^k(t) = H(\gamma^{k-1}(t), r_k(t)B_k)$, for $k = 1, \ldots, m$. Then $\rho(t) = \gamma^m(t)$. Here we can shrink the length of each interval $I^i$, such that the harmonic replacements from $\gamma(t)$ to $\rho(t)$ keep the continuity of the paths as discussed in the beginning of this section. The homotopy equivalent of $\rho \in [\gamma]$ is also a consequence of the discussions in the beginning of this section. Since harmonic replacements decrease energy, we have $E(\rho(t)) \leq E(\gamma(t))$.

Now the property (*) comes from similar argument as in the proof of Lemma 4.1 of [15] which originate from the proof of Theorem 3.1 of [5]. For $t$ such that $E(\gamma(t), \tau(t)) \geq \frac{1}{2}V$, we deform $\gamma(t)$ to $\rho(t)$ by at most two harmonic replacements, with the possible middle one denoted by $\gamma^k(t)$. Now we focus on the case of two replacements, and the other case is similar and much easier. For any collection $\mathcal{B}$ with $E(\rho(t), \mathcal{B}) \leq \frac{1}{12} \epsilon_1$, we can assume that both $\gamma(t)$ and $\gamma^k(t)$ have energy less than $\frac{1}{8} \epsilon_1$ on $\mathcal{B}$, or inequality 23 is trivial. By property 3 of Lemma 4.5, at least one of the energy decrease from $\gamma(t)$ to $\rho(t)$ is bounded from below by $\frac{1}{8} e_{\epsilon, \gamma(t)}$, so we have from either inequality 25 of Lemma 4.2 or inequality 27 that:

$$E(\gamma(t)) - E(\rho(t)) \geq k \left( \frac{1}{8} e_{\epsilon, \gamma(t)} \right)^2. \quad (45)$$

Now using inequality 26 twice for $\mu = \frac{1}{64}$, $\frac{1}{16}$, we get:

$$E(\rho(t)) - E[H(\rho(t), \frac{1}{64} \mathcal{B})] \leq E(\gamma^k(t)) - E[H(\gamma^k(t), \frac{1}{64} \mathcal{B})] + \frac{1}{k} \{ E(\gamma^k(t)) - E(\rho(t)) \}^{\frac{1}{2}}$$

$$\leq E(\gamma(t)) - E[H(\gamma(t), \frac{1}{4} \mathcal{B})] + \frac{1}{k} \{ E(\gamma(t)) - E(\gamma^k(t)) \}^{\frac{1}{2}}$$

$$+ \frac{1}{k} \{ E(\gamma(t)) - E(\rho(t)) \}^{\frac{1}{2}} \leq e_{\epsilon, \gamma(t)} + C \{ E(\gamma(t)) - E(\rho(t)) \}^{\frac{1}{2}} \leq C \{ E(\gamma(t)) - E(\rho(t)) \}^{\frac{1}{2}}. \quad (46)$$

By taking $\epsilon_0 = \frac{1}{15} \epsilon_1$ and $\Psi$ a square root function, we can get inequality 26 by using Theorem 4.1 to change the left hand side to $W^{1,2}$ norm difference.
5 Convergence results

Here we talk about the convergence about our deformed sequences \( \{\rho_n(t), \tau_n(t)\}_{n=1}^{\infty} \). In Lemma 4.1, we need our sequence \( \{\gamma_n(t), \tau_n(t)\}_{n=1}^{\infty} \) to have no non-constant harmonic slices. We can achieve this by an argument similar to Remark 4.6 of [15]. In fact, we can modify the minimizing sequence \( \{\tilde{\gamma}_n(t)\}_{n=1}^{\infty} \) such that \( \tilde{\gamma}_n(t) \) are constant mappings on a small open neighborhood on \( \Sigma_0 \), without changing the area too much. By Theorem 3.1, \( \gamma_n(t) \) are gotten from \( \tilde{\gamma}_n(t) \) by composing with diffeomorphisms \( h_n(t) \), so \( \gamma_n(t) \) are also constant mappings on some small neighborhood. By the unique continuation of harmonic maps (Corollary 2.6.1 of [10]), we know that for any parameter \( t \), \( \gamma_n(t) \) could not be harmonic mapping unless constant. So we can apply Lemma 4.1.

We would also like to preserve the almost conformal property given in Theorem 3.1 after the deformation given by Lemma 4.1. Although we could not make sure that \( \rho_n(t) \) are still almost conformal for every parameter \( t \) after the deformation, we can prove similar results for the parameter \( t \) with \( E(\rho_n(t_n), \tau_n(t_n)) \) closed to the min-max critical value \( W \). The proof is almost the same as Lemma 5.1 of [15], so we omit the proof here. The result is as following.

**Lemma 5.1** Given a sequence of parameters \( \{t_n\}_{n=1}^{\infty} \), such that \( E(\rho_n(t_n), \tau_n(t_n)) \to W \), then we have

\[
E(\rho_n(t_n), \tau_n(t_n)) - \text{Area}(\rho_n(t_n)) \to 0. \tag{47}
\]

5.1 Degeneration of conformal structures

Let us talk about the compactification of moduli space \( \mathcal{M}_g \). Here we mainly refer to Appendix B of [9] and Chapter IV of [8].\(^{12}\) In fact, we will use hyperbolic metrics to represent elements in \( \mathcal{M}_g \) and its compactification. Firstly, let us talk about the representation of the moduli space \( \mathcal{M}_g \) and Teichmüller space \( \mathcal{T}_g \) by hyperbolic and complex structures. Fix a topological surface \( \Sigma_0 \) of genus \( g \geq 2 \). In fact, every metric on \( \Sigma_0 \) determines a complex structure \( j \). There exists a hyperbolic metric \( h \) compatible with \( j \). In fact, by uniformization theorem the covering projection \( \pi : \mathbb{H} \to (\Sigma_0, j) \) is holomorphic, and the deck transformation group acts isomorphically w.r.t. the

\(^{12}\)Section 4 of [16] also gives a nice summation in hyperbolic structures.
hyperbolic metric $ds^2_{-1}$. So we can get a hyperbolic metric $h$ on $\Sigma_0$ by pushing down $ds^2_{-1}$, and this metric is compatible to $j$ since $ds^2_{-1}$ is compatible to the standard complex structure on $\mathbb{H}$. Denote such a hyperbolic Riemman surface by a triple $(\Sigma_0, h, j)$. Two hyperbolic metrics on $\Sigma_0$ are conformal equivalent if and only if they are isomorphic to each other. So we can view $\mathcal{M}_g$ as the set of equivalent classes of $(\Sigma_0, h, j)$ up to isomorphisms, and $\mathcal{T}_g$ as the set of equivalent classes of $(\Sigma_0, h, j)$ up to isotopic isomorphisms.

Now we will introduce the concept of **Riemann surfaces with nodes**. The precise definition is given in Appendix B.2 of [9]. A compact connected Hausdorff space $\Sigma^*$ is called a **closed Riemann surface of genus** $g$ **with nodes** if the following conditions hold. 

1° Every point $p \in \Sigma^*$ either has a neighborhood homeomorphic to $\{z \in \mathbb{C} : |z| < 1\}$ or to $\{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |z_1| < 1, |z_2| < 1\}$, and in the second case we call $p$ a **node**. These complex coordinates give a complex structure $j$ on $\Sigma^*$ minus nodes. Since $\Sigma^*$ is compact, there are only finitely many nodes.  

2° Let $\Sigma$ be $\Sigma^*$ minus nodes, and $\overline{\Sigma}$ the one point compactification of $\Sigma^*$. Every connected component $\Sigma_i$ of $\Sigma$, which we call it a **part of** $\Sigma^*$, is of type $(g_i, k_i)$, which means that $\Sigma_i$ is gotten by removing $k_i$ distinct points from a Riemman surface of genus $g_i$, and $2g_i - 2 + k_i > 0$. The second condition makes sure that $\Sigma_i$ is not homotopic to complex plane and cylinder, which means that $\Sigma_i$ has the universal cover $\mathbb{H}$. We call such a part $\Sigma_i$ having **signature** $(g_i, k_i)$.  

3° If $m$ and $k$ denote the numbers of nodes and parts of $\Sigma^*$, then the genus $g$ is given by $g = \sum_{i=1}^{k} g_i + m + 1 - k$. The last condition tells us that we can get a Riemman surface $\Sigma_0$ of genus $g$ from $\Sigma^*$ by opening each node.

Two Riemman surfaces with nodes $\Sigma^*_1$ and $\Sigma^*_2$ of genus $g$ are said to be **biholomorphically equivalent** if there exists a homeomorphism $f : \Sigma^*_1 \to \Sigma^*_2$ preserving nodes, such that $f$ is biholomorphic between parts $(\Sigma_1)_i$ and $(\Sigma_2)_i$ of $\Sigma^*_1$ and $\Sigma^*_2$ respectively. If we add the equivalent classes $[\Sigma^*]$ of Riemman surfaces with nodes of genus $g$ to the moduli space $\mathcal{M}_g$, we get a compactification $\hat{\mathcal{M}}_g$ of $\mathcal{M}_g$.

In fact, we are interested in the convergence of $[\Sigma_n] \to [\Sigma^*_\infty]$ of a sequence of elements in $\mathcal{M}_g$ to the boundary of $\hat{\mathcal{M}}_g$. We will describe the convergence by representing all the equivalent classes by hyperbolic structures. Now let us firstly talk about the hyperbolic representation of Riemman surfaces with nodes. Given a Riemann surface with nodes $\Sigma^*$, let $j$ be the complex structure on the body $\Sigma$ of $\Sigma^*$. On each part $\Sigma_i$, there

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13Later on, we will always use $\Sigma^*$ to denote surface with nodes, $\Sigma$ surface minus nodes, and $\overline{\Sigma}$ one points compactification of $\Sigma$.

14We refer to Appendix B.2 and B.3 of [9] for topology on $\hat{\mathcal{M}}_g$ and Theorem B.1 of [9] for compactness.
exists a complete hyperbolic metric \( h \) compatible with \( j \), with the nodes becoming cusps. So we use \((\Sigma^*, h, j)\) to denote a hyperbolic Riemann surface with nodes. A triple-connected Riemann surfaces with possibly degenerated boundaries is called a pair of pants. Fix a hyperbolic Riemann surface with nodes \((\Sigma^*, h, j)\), there exists the pair of pants decomposition\(^{15}\). It means that we can find a largest possible collection of pairwise disjoint, simply closed geodesics \( L = \{\gamma_i : i = 1 \cdots 3g - 3\} \) under the hyperbolic metric \( h \), with \( \gamma_i \) possibly degenerating to nodes, such that each connected component of \( \Sigma^* \setminus L \) is a pair of pants. Now we give a concept for convergence of a sequence of closed hyperbolic Riemann surfaces of genus \( g \) to a hyperbolic Riemann surface with nodes\(^{16}\).

**Definition 5.1** A sequence \( \{(\Sigma_n, h_n, j_n)\} \) of closed hyperbolic Riemann surfaces of genus \( g \) is said to converge to a hyperbolic Riemann surface with nodes \((\Sigma^*_\infty, h_\infty, j_\infty)\), if there exists a sequence of finite sets \( L_n = \{\gamma_i^n\}_{i=1}^{k_n} \subset \Sigma_n \) constituted by pairwise disjoint simply closed geodesics on \((\Sigma_n, h_n)\), with the number of elements \( k_n \) bounded by \( 0 \leq k \leq 3g - 3 \), and a sequence of continuous mappings \( \phi_n : \Sigma_n \to \Sigma^*_\infty \), satisfying the following conditions as \( n \to \infty \):

1° : \( \phi_n(\gamma_i^n) = p_i \), where \( p_i \) is a node on \( \Sigma^*_\infty \), and the length \( l(\gamma_i^n) \to 0 \).
2° : \( \phi_n : \Sigma_n \setminus L_n \to \Sigma_\infty \) is a diffeomorphism, where \( \Sigma_\infty \) is the body of \( \Sigma^*_\infty \).
3° : \( (\phi_n)_* h_n \to h_\infty \) in \( C^\infty_{\text{loc}}(\Sigma_\infty) \).
4° : \( (\phi_n)_* j_n \to j_\infty \) in \( C^\infty_{\text{loc}}(\Sigma_\infty) \).

Now using the hyperbolic description of convergence, we can summarize a version of the compactification \( \hat{\mathcal{M}}_g \) of \( \mathcal{M}_g \). We refer to Proposition 5.1 of Chap 4 in \cite{8} for a proof.

**Proposition 5.1** For any sequence \( \{(\Sigma_n, h_n, j_n)\}_{n=1}^\infty \), where each element \((\Sigma_n, h_n, j_n)\) represents an equivalent class in \( \mathcal{M}_g \), there exists a subsequence \( \{(\Sigma_{n'}', h_{n'}', j_{n'}')\} \) converging to a hyperbolic Riemann surface with nodes \((\Sigma^*_\infty, h_\infty, j_\infty)\), which represents an equivalent class in \( \hat{\mathcal{M}}_g \).

Besides the convergence results, we also have a detailed description of the geometry near the degenerating geodesics. We refer to Proposition 4.2 of Chap 4 in \cite{8} and Lemma 4.2 of \cite{16} for the following collar lemma.

\(^{15}\)See Section 3 of \cite{9} and Chap IV. of \cite{8} for detailed discussion of definitions and properties.

\(^{16}\)For general convergence of a sequence of Riemann surfaces with nodes to a fixed Riemann surface with nodes, see Page 71 of \cite{8}. 
Lemma 5.2 For any simply closed geodesic $\gamma$ with length $l(\gamma) = l$ in a hyperbolic surface $(\Sigma, h)$, there exists a collar neighborhood of $\gamma$, which is isomorphic to the following collar region in hyperbolic plane $\mathbb{H}$:

$$
\mathcal{C}(\gamma) = \{ z = re^{i\theta} \in \mathbb{H} : 1 \leq r \leq e^l, \ \theta_0(l) \leq \theta \leq \pi - \theta_0(l) \},
$$

with the circles $\{ r = 1 \}$ and $\{ r = e^l \}$ identified by the isometry $\Gamma_l : z \to e^l z$. Here $\theta_0(l) = \tan^{-1} \left( \sinh \left( \frac{l}{2} \right) \right)$, and $\gamma$ is isometric to $\{ z = re^{\frac{\pi}{2}i} \in i\mathbb{R} : 1 \leq r \leq e^l \}$.

Remark 5.1 In fact, this result follows from the proof of Lemma 1.6 of Chap 4 in [8]. They consider half of the collar, and they show that the collar region should be part of annuli $\{ re^{i\theta} : \theta_0 \leq \theta \leq \frac{\pi}{2}, 1 \leq r \leq y \}$. Instead of using polar coordinates $\{ r, \theta \}$, they use the length of boundary of the region $\{ re^{i\theta} : \theta_0 \leq \theta \leq \frac{\pi}{2}, r = 1 \}$ as parameter. It is easy to change back to polar coordinates and get our formulation above.

As stated in [15], we can give a explicit metric on the collar region by a conformal change of coordinates. Now, we can view the parameters $r$ and $\theta$ in [48] as azimuthal and vertical coordinates for a cylinder respectively. Under the following transformation:

$$
re^{i\theta} \to (t, \phi) = \left( \frac{2\pi}{l} \theta, \frac{2\pi}{l} \log(r) \right),
$$

where $l$ is the length of the center geodesic, the collar region $\mathcal{C}(\gamma)$ is changed to a cylinder

$$
\mathcal{C} = \{(t, \phi) : \frac{2\pi}{l} \theta_0 \leq t \leq \frac{2\pi}{l} (\pi - \theta_0), 0 \leq \phi \leq 2\pi \},
$$

and the hyperbolic metric $ds^2 = \frac{|dz|^2}{(1 - z\bar{z})^2}$ is expressed as $ds^2 = \left( \frac{l}{2\pi \sinh(\frac{l}{2})} \right)^2 (dt^2 + d\phi^2)$, which is conformal to the standard cylindrical metric $ds^2 = dt^2 + d\phi^2$. We can see that if the geodesic $\gamma$ shrink to a point, a conformally infinitely long cylinder will appear.

### 5.2 Convergence

Before talking about bubbling convergence of the sequence of paths $\{ \rho_n(t), \tau_n(t) \}_{n=1}^\infty$ gotten by the previous section, let us firstly clarify the concepts of convergence for a sequence $\{ \tau_n \}_{n=1}^\infty \subset \mathcal{T}_g$. Since the area and energy functionals are both conformally invariant, we can choose good representatives in the conformal classes of $\{ \tau_n \}_{n=1}^\infty$, or in another word, we would like to project $\mathcal{T}_g$ to $\mathcal{M}_g$, and use the compactification $\hat{\mathcal{M}}_g$ of $\mathcal{M}_g$ to discuss the convergence of $\{ \tau_n \}_{n=1}^\infty$. Here we use hyperbolic representatives as
Theorem 5.1 Let \( \{ (\rho_n(t), \tau_n(t)) \}_{n=1}^{\infty} \) be the sequence gotten by the perturbation from \( \{ (\gamma_n(t), \tau_n(t)) \}_{n=1}^{\infty} \) by Lemma 4.1. Then all min-max sequences \( \{ (\rho_n(t_n), \tau_n(t_n)) \}_{n=1}^{\infty} \) with \( E(\rho_n(t_n), \tau_n(t_n)) \) converge to \( W_E \), satisfy:

(*) For any finite collection of disjoint geodesic balls \( \bigcup_i B_i \) on \( \tau_{\tau_n(t_n)} \) with radii bounded as in Lemma 4.1, such that \( E(\rho_n(t_n), \bigcup_i B_i) \leq \epsilon_0 \), let \( v \) be the harmonic replacement of \( \rho_n(t_n) \) on \( \frac{1}{\epsilon_0} \bigcup_i B_i \). We have:

\[
\int_{\frac{1}{\epsilon_0} \bigcup_i B_i} |\nabla \rho_n(t_n) - \nabla v|^2 \rightarrow 0
\]  

By compactness Proposition 5.1, a subsequence of \( \{ \tau_n(t_n) \}_{n=1}^{\infty} \) converge to some \( \tau_\infty \) in \( \hat{M}_g \) up to a subsequence, which is achieved by the convergence of a sequence hyperbolic Riemann surfaces \( (\Sigma_n, h_n, j_n) \in \tau_n(t_n) \) to \( (\Sigma_\infty^*, h_\infty, j_\infty) \in \tau_\infty \) as in Definition 5.1. If we denote the one point compactification of \( \Sigma_\infty \) by \( \Sigma_\infty \), and \( j_\infty \) the extended complex structure, then there exist a conformal harmonic map \( u_0 : (\Sigma_\infty, j_\infty) \rightarrow N \) and possibly some harmonic spheres \( \{ u_i : S^2 \rightarrow N \mid i = 1, \cdots, l \} \), such that \( (\rho_n(t_n), (\Sigma_n, h_n, j_n)) \) bubbling converge to \( (u_0, u_1, \ldots, u_l) \), with:

\[
\lim_{n \rightarrow \infty} E(\rho_n(t_n), j_n) = E(u_0, j_\infty) + \sum_i E(u_i)
\]  

Remark 5.2 In fact, property (*) in the above theorem is scaling invariant, so we can apply the Sacks-Uhlenbeck bubbling convergence theory to \( \{ \rho_n(t_n) \} \). In fact, the left hand side of 5.1 is the min-max critical value \( W \), and the right side is the sum of areas since \( (u_0, u_1, \ldots, u_l) \) are all conformal, so we get the conclusion that the min-max critical value is achieved by the area of a set of minimal surfaces.

The proof is divided into several steps in the following sections.

\[\text{Appendix B.6 in \cite{5}}\] for more details about bubbling convergence.
5.2.1 Convergence on domains

Firstly we summarize the known facts of convergence of almost harmonic maps defined on a sequence of converging domains. Suppose \( \{(\Omega_n, h_n, j_n)\}_{n=1}^{\infty} \) is a sequence of two dimensional domains with metric \( h_n \) and compatible complex structure \( j_n \). We assume that \( (\Omega_n, h_n, j_n) \rightarrow (\Omega_\infty, h_\infty, j_\infty) \) in the following sense. For \( n \) large enough, there exist a sequence of diffeomorphisms \( \phi_n : \Omega_\infty \rightarrow \Omega_n \), such that the pull-back metrics and complex structures converge, i.e. \( (\phi_n)^* h_n \rightarrow h_\infty \) and \( (\phi_n)^* j_n \rightarrow j_\infty \) in \( C^3 \) on any compact subsets of \( \Omega_\infty \). Let \( \{u_n : (\Omega_n, h_n, j_n) \rightarrow N\}_{n=1}^{\infty} \) be a sequence of \( W^{1,2} \) almost harmonic maps satisfying the following condition:

\[ \int_{\Omega_n} |\nabla u_n - \nabla v|^2 \leq \delta(n) \rightarrow 0. \]

**Lemma 5.3** For a sequence \( \{u_n : (\Omega_n, h_n, j_n) \rightarrow N\}_{n=1}^{\infty} \) as above with \( E(u_n, j_n) \leq E_0 < \infty \), there exist finitely many points \( \{x_1, \ldots, x_k\} \subset \Omega_\infty \), a subsequence \( \{n'\} \) and a harmonic mapping \( u_\infty \in W^{1,2}(\Omega \setminus \{x_1, \ldots, x_k\}, N) \), such that for any compact subset \( K \subset \Omega_\infty \setminus \{x_1, \ldots, x_k\} \), the subsequence \( u_{n'} : (\phi_{n'}(K) \subset \Omega_{n'}, h_{n'}, j_{n'}) \rightarrow N \) converge in \( W^{1,2} \) to \( u_\infty \).

**Remark 5.3** The convergence of \( u_{n'} \) to \( u_\infty \) can be understood as the convergence after pulling \( u_{n'} \) back to \( \Omega_\infty \) by \( \phi_{n'} \). We call points \( \{x_1, \ldots, x_k\} \) energy concentration points. The proof of results similar to the above lemma is given in [12],[13], Appendix B.2 of [5] and the proof of Theorem 5.1 of [15]. In fact, step 1 of the proof of Theorem 5.1 in [15] almost directly gives the proof of the above lemma, so we omit it. By the Removable Singularity Theorem 3.6 of [12], we can extend \( u_\infty \) to a harmonic map on \( \Omega_\infty \).

5.2.2 Convergence on cylinders

Now based on the above lemma, the next step to study the convergence of \( \{(\rho_n, \tau_n)\}_{n=1}^{\infty} \) is to do rescaling near energy concentration points, and to consider regions near degenerating geodesics. In both of the cases which we will discuss in detail later, we need to

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19The infimum exists and is positive because of the convergence.
20In order to apply Sacks-Uhlenbeck bubbling convergence, we can pick \( \epsilon_0 < \epsilon_{SU} \), where \( \epsilon_{SU} \) is a small constant depending only on the ambient manifold \( N \) given in [12].
consider almost harmonic maps on long cylinders. We use $C_{t_1,t_2} = \{(t, \theta) \in \mathbb{R} \times S^1 : t_1 \leq t \leq t_2, \theta \in [0,2\pi]\}$ to denote a cylinder with length parameter between $t_1$ and $t_2$, and $h$ a metric on $C_{t_1,t_2}$ conformal to the standard metric $ds^2 = dt^2 + d\theta^2$. We denote $S_\varphi = \{(t, \theta) : t = t_0, \theta \in [0,2\pi]\}$ to be a slice of $C_{t_1,t_2}$. We say a sequence of cylinders $\{(C_{t_1^n,t_2^n}, h_n) : 1 \leq n < \infty\}$ converge to $(C_\infty = \mathbb{R} \times S^1, ds^2 = dt^2 + d\theta^2)$, if when we identify all the cylinders by the center slices $S_{\varphi_0}$ with $t_0^n = \frac{1}{2}(t_1^n + t_2^n)$, the metrics $h_n$ converges in $C^3$ to $ds^2$ on any compact subsets of $C_\infty$, i.e. when we choose $\phi_n : C_{t_1^n,t_2^n} \to C_\infty$, such that $\phi_n(t,\theta) = (t-t_0^n, \theta)$, then $(\phi_n)_* h_n \to ds^2$ in $C^3(K)$ for any compact subset $K \subset C_\infty$. Consider a sequence of almost harmonic maps defined on a sequence of converging cylinders $\{u_n : (C_{t_1^n,t_2^n}, h_n) \to N, n = 1, \ldots, \infty\}$ satisfying property (1) in the above section. By Lemma 5.3, they sub-converge to a harmonic map on $C_\infty$. Before talking about further results, we need to introduce another type of almost harmonic maps and a corresponding energy estimate.

**Definition 5.2** For $\nu > 0$, we call $u \in W^{1,2}((C_{r_1,r_2}, h), N)$ a $\nu$-almost harmonic map (Definition B.27 in [5]) if for any finite collection of disjoint geodesic balls $B$ in $(C_{r_1,r_2}, h)$ with radii bounded from above by the injective radius of $(C_{r_1,r_2}, h)$ and $r_0$ as in (222), there is an energy minimizing map $v : \frac{1}{64}B \to N$ with the same boundary value as $u$ such that:

$$\int_{\frac{1}{64}B} |\nabla u - \nabla v|^2 \leq \nu \int_{C_{r_1,r_2}} |\nabla u|^2. \tag{52}$$

This definition traces back to Definition B.27 in [5], but we modify it here to be adapted to our setting. Now a proof similar to that of Proposition B.29 of [5] gives a similar estimate as follows.

**Proposition 5.2** \(\forall \delta > 0, \) there exist small constants $\nu > 0$(depending on $h$, $\delta$ and $N$), $\epsilon_2 > 0$ and large constant $l \geq 1$(depending on $\delta$ and $N$), such that for any integer $m$, if $u$ is a $\nu$-almost harmonic map defined above on $(C_{-l-3l,3l}, h)$ with $E(u) \leq \epsilon_2$, then:

$$\int_{C_{-l,0}} |u_\theta|^2 \leq 7\delta \int_{C_{-l-3l,3l}} |\nabla u|^2. \tag{53}$$

Here $u_\theta$ means the differentiation w.r.t $\theta$.

Now we would like to give a more precise description of the convergence on cylinders.

\[\text{We can let } \epsilon_2 < \epsilon_{SU} \text{ as above again.}\]
Lemma 5.4 In the convergence of $u_n : (\mathcal{C}_{1,\epsilon^2_0}, h_n) \to N$ as discussed above, if $E(u_n) \leq \epsilon_2$ with $\epsilon_2$ given in the above proposition, then either $\liminf_{n \to \infty} E(u_n) = 0$, or $u_n$ must be uniformly un-conformal for $n$ large enough in the following sense, i.e. there exists a small number $\delta_0$, such that:

$$E(u_n) - \text{Area}(u_n) \geq \delta_0.$$  \hspace{1cm} (54)

Furthermore, if $\{u_n\}$ are almost conformal and $\liminf_{n \to \infty} E(u_n) \geq \epsilon_2$, then there exists a large fixed number $L > 0$, such that $E(\rho_n, \mathcal{C}_{r^2_0-L,r^2_0+L}) \geq \epsilon_2$, i.e. the energy must concentrate on some finite part of the cylinders.

Remark 5.4 This is a summarization of the results proved in step 5 of the proof of Theorem 5.1 in [12]. In fact, if $E(u_n) \leq \epsilon_2$ and $\liminf_{n \to \infty} E(u_n) > 0$, it is easy to show that $u_n$ is $\mu$–almost harmonic as in Definition 5.2 for $\mu$ small enough when $n$ is large enough. If we apply the estimate in Proposition 5.2, we get an upper bound for $\int_{\mathcal{C}_{ml,0}} |(u_n)_\theta|^2$. Then by computing the difference between the energy and area of $u_n$ as in equation 55 of [17], we will get the lower bound for $E(u_n) - \text{Area}(u_n)$. In the second case, we use contradiction argument. We will go back to the first case to get a sequence of almost harmonic mappings on long cylinders with energy bounded from above by $\epsilon_2$ and away from 0, which will lead to a contradiction to almost conformal property. We omit the detailed proof here and refer that to [15].

5.2.3 Proof of Theorem 5.1

Now we use the results summarized above to show the bubble convergence and energy identity of Theorem 5.1. Let us denote $\rho_n = \rho_n(t_n)$, and $\tau_n = \tau_n(t_n)$ in the following.

Step 1: bubble convergence on domain surfaces. In the convergence of $(\Sigma_n, h_n, j_n) \in \tau_n$ to $(\Sigma^*_\infty, h_\infty, j_\infty) \in \tau_\infty$, let us denote $\mathcal{L}_n$ to be the sets of geodesics and $\phi_n : \Sigma_n \to \Sigma^*_\infty$ the continuous mappings as in Definition 5.1. Now let us consider the sequence of almost harmonic maps $\{\rho_n : (\Sigma_n \setminus \mathcal{L}_n, h_n, j_n) \to N\}^\infty_{n=1}$ satisfying property (\ast) in Theorem 5.1. By Lemma 5.3, there exists a finite set of energy concentration points $\{x_1, \cdots, x_l\}$ on the body $\Sigma_\infty$ of $\Sigma^*_\infty$, and a subsequence which we still denote by $\rho_n$, that converge to a harmonic map $u_0 : \Sigma_\infty \to N$ in $W^{1,2}$ on any compact subsets of $\Sigma_n \setminus (\mathcal{L}_n \cup \phi_n^{-1}\{x_1, \cdots, x_l\})$. Denote $x_{n,i} = \phi_n^{-1}(x_i)$. Near each energy concentration point $x_{n,i}$, let $r_{n,i}$ be the smallest radii such that $E(\rho_n, B_{x_{n,i}, r_{n,i}}) = \epsilon_0$ with $\epsilon_0$ as in condition (\ast1), where $B_{x_{n,i}, r}$ denotes the hyperbolic geodesic balls centered at $x_{n,i}$ with radii...
When we rescale the metrics such that the center slice $S_r$ converges to the flat metric on any compact subset of the infinite long cylinder, we see that the metrics converge to the flat metric on any compact subset of the infinite long cylinder, as in the case of necks show that if we want to recover all the energy of $u_\infty$ on $\Sigma_n$, we need to study the behavior near degenerating geodesics. Near an energy concentration point, if we compare the energy limit $\lim_{n \to \infty} E(\rho_n, B(x_i, r_n))$ with the sum of the limit energy $E(u_0, B(x_i, r))$ and bubble energy $\lim_{n \to \infty} E(u_{n,i}, B_{r/r_{n,i}})$, we need to count the neck part, which is $\lim_{r \to 0, R \to \infty} \lim_{n \to \infty} E(\rho_n, B(x_i, r) \setminus B(x_i, r_{n,i}R))$. Here we refer to the step 4 of proof of Theorem 5.1 in [15] for details. Denote the annuli by $A(x_i, r, r_{n,i}) = B(x_i, r) \setminus B(x_i, r_{n,i}),$ and we call them necks. Under the change of coordinates $(r, \theta) \to (t, \theta) = (\log r, \theta)$, the annuli are changed to long cylinders $C_{r_1, r_2}$, with $r_1 = \ln(r_{n,i}R)$, $r_2 = \ln(r)$, and the hyperbolic metrics are $ds^2_1 = \frac{e^{2t}}{1-e^{2t}}(dt^2 + d\theta^2)$. When we rescale the metrics such that the center slice $S_{r_n}$ has length $2\pi$, it is easy to see that the metrics converge to the flat metric on any compact subset of the infinite long cylinder $\mathbb{R} \times S^1$. Since property (*) is invariant under scaling, we go back to the setting for the previous section. We will continue studying the convergence in this case after we introducing the behavior near degenerating geodesics.

Now let us see the behavior near degenerating geodesics $\gamma^n_i \in \mathcal{L}_n$. Similar arguments as in the case of necks show that if we want to recover all the energy of $\rho_n$ on $\Sigma_n$ from the limit $u_0$ and all the bubbles $u_n$, we need to consider the amount of energy on the collar neighborhoods $\mathcal{C}(\gamma^n_i)$ given by Lemma 5.2. As in equation (48) we use $(r, \theta)$ as parameters.
for the cylinder, and denote $C(\gamma_n^i, \theta_0)$ to be the sub-collar with $\theta_0 \leq \theta \leq \pi - \theta_0$. In fact, as $l_n = l(\gamma_n^i) \to 0$, we need to take care of the limit $\lim_{\theta_0 \to \frac{\pi}{2}} \lim_{n \to \infty} E(\rho_n, C(\gamma_n^i, \theta_0))$. Using the change of coordinates as given in the remark below Lemma 5.2, those collars can be viewed as a sequence of cylinders $C_{r_n^1, r_n^2}$ with $r_n^1 = \frac{2\pi}{l_n} \theta_0$, $r_n^2 = \frac{2\pi}{l_n} (\pi - \theta_0)$. If we rescale the hyperbolic metrics $ds^2_{-1} = (\frac{l_n}{2\pi \sin(\frac{\theta_n}{2})})^2 (dt^2 + d\phi^2)$ on $C_{r_n^1, r_n^2}$ such that the center slice $S_{(\frac{2\pi}{l_n})^2}$ has length $2\pi$, it is easy to see that those metrics converge to the flat metric on any compact subset of $\mathbb{R} \times S^1$, which goes back to the setting for the previous section again by the conformal invariance of property $(\ast)$.

Summarizing the above two paragraphs, we need to study the case of a sequence of almost harmonic maps defining on cylinders approximating the infinite long standard cylinder. If $\liminf_{n \to \infty} E(\rho_n, (C_{r_n^1, r_n^2}, ds^2_{-1})) = 0$, then we can discard this part in the energy identity 5.1 or since our sequence of maps are almost conformal by Lemma 5.1, $\liminf_{n} E(\rho_n, (C_{r_n^1, r_n^2}, ds^2_{-1})) \geq \epsilon_2$ by Lemma 5.4. Then there exists a large fixed number $L > 0$, such that $E(\rho_n, C_{r_n^1-L, r_n^1+L}) \geq \epsilon_2$ by Lemma 5.4 again. Now $(\rho_n, (C_{r_n^1}, ds^2_{-1}))$ converge in $W^{1,2}$ to a harmonic map $u_\infty : \mathbb{R} \times S^2 \to N$ on any compact subsets of $\mathbb{R} \times S^2$ minus possibly finite many energy concentration points by Lemma 5.3. We can repeat the above steps near energy concentration points again. Now in order to count all the energy, we need to consider sub-cylinders $C_{t_n-L_n, t_n+L_n} \subset C_{r_n^1, r_n^2}$ with $|t_n - t_0^n| \to \infty$ and $L_n \to \infty$. We need to show that $\lim_{n \to \infty} E(\rho_n, C_{t_n-L_n, t_n+L_n})$ is counted by some bubble maps. In fact, when we rescale the metrics such that the center slice $S_{t_n}$ of $C_{t_n-L_n, t_n+L_n}$ has length $2\pi$, the sequence of cylinders will converge to $\mathbb{R} \times S^1$ again as in the previous section. So we can repeat the steps again.

We can see that no energy loss will happen since once there are energy concentrated on long cylinders, they must be counted in the next bubbling step. We know that either $u_\infty : \mathbb{R} \times S^1 \to N$ is nontrivial, which can be extended to a harmonic map on $S^2$ by removable singularity theorem in [12], since $S^2$ is conformal to $\mathbb{R} \times S^1$, or some of the bubble maps near energy concentration points are nontrivial since $E(\rho_n, C_{r_n^1-L_n, r_n^1+L_n}) \geq \epsilon_2$. So each of such steps also takes away a fixed amount of energy, so we must stop in finite many steps. All such steps form the convergence in Theorem 5.1. Count all the energy of those finite many bubble maps, which are harmonic maps on spheres, we will get the energy identity 5.1. So we finish the proof.
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