MAXIMAL AMENABLE VON NEUMANN SUBALGEBRAS ARISING FROM MAXIMAL AMENABLE SUBGROUPS

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ABSTRACT. We provide a general criterion to deduce maximal amenability of von Neumann subalgebras $L\Lambda \subset L\Gamma$ arising from amenable subgroups $\Lambda$ of discrete countable groups $\Gamma$. The criterion is expressed in terms of $\Lambda$-invariant measures on some compact $\Gamma$-space. The strategy of proof is different from S. Popa’s approach to maximal amenability via central sequences [Po83], and relies on elementary computations in a crossed-product $C^*$-algebra.

INTRODUCTION

A separable von Neumann algebra $A \subset B(H)$ is said to be amenable if there exists a conditional expectation $E : B(H) \rightarrow A$, i.e. a projection with norm one. By W. Arveson’s extension theorem, this definition does not depend on the choice of the Hilbert space $H$ on which $A$ is represented.

Amenability has always played a central role in the study of von Neumann algebras. First it is a source of isomorphism, via the fundamental result of A. Connes [Co76] that amenable implies hyperfinite, and the uniqueness of the hyperfinite II$_1$-factor. Amenability is also at the core of the concepts of solidity and strong solidity defined in [Oz04, OP10a]. It is hence very natural to try to understand the maximal amenable subalgebras of a given finite von Neumann algebra.

In this direction, R. Kadison asked in the 1960’s the following question: is any maximal amenable subalgebra of a II$_1$-factor necessarily a factor? S. Popa solved this problem in [Po83], producing an example of a maximal amenable subalgebra of the free group factor $L\mathbb{F}_n$ which is abelian. The subalgebra in question is generated by one of the free generators of the free group $\mathbb{F}_n$. This striking result led to more questions, refining Kadison’s question: what if the ambient II$_1$-factor is McDuff? has property (T)? More generally can one provide concrete examples of maximal amenable subalgebras in a given II$_1$-factor? Some progress on this topic have been made recently.

By considering infinite tensor products of free group factors, J. Shen constructed in [Sh06] an abelian, maximal amenable subalgebra in a McDuff II$_1$-factor. In [CFRW10], it is proved that the subalgebra of the free group factor generated by the symmetric laplacian operator (the radial subalgebra) is maximal amenable. In [Ho14], C. Houdayer provided uncountably many non-isomorphic examples of abelian maximal amenable subalgebras in II$_1$-factors. Last year the authors showed in [BCT14] that any infinite maximal amenable subgroup in a hyperbolic group $\Gamma$ gives rise to a maximal amenable von Neumann subalgebra of $L\Gamma$. Note that hyperbolic groups can have property (T). This last result is a rigidity type result in the spirit of Popa’s approach to study von Neumann algebras in terms of their “building data”.

Question. Assume that $\Lambda < \Gamma$ is a maximal amenable subgroup. Under which conditions is $L\Lambda$ maximal amenable inside $L\Gamma$?

In this paper, we will prove the following general condition, providing a partial answer to the above question.

Research partially supported by ANR grant Neumann. R.B. is also partially supported by NSF Career Grant DMS 1253402.
Theorem A. Let $\Gamma$ be a countable group and $\Lambda < \Gamma$ be an amenable subgroup. Assume that there exists a continuous action $\Gamma \curvearrowright X$ on a compact metric space $X$ such that for any $\Lambda$-invariant probability measure $\mu$ on $X$ and any $g \notin \Lambda$, we have that $g \cdot \mu$ is singular with respect to $\mu$.

Then for any trace preserving action $\Gamma \curvearrowright (Q, \tau)$ on a finite amenable von Neumann algebra, $Q \rtimes \Lambda$ is maximal amenable inside $Q \rtimes \Gamma$.

The conclusion of the above proposition implies in particular that

- $L\Lambda$ is maximal amenable inside $L\Gamma$ (case where $Q = \mathbb{C}$);
- for any free pmp action on a probability space $\Gamma \curvearrowright (Y, \nu)$, the orbit equivalence relation $R(\Lambda \curvearrowright (Y, \nu))$ is maximal hyperfinite inside $R(\Gamma \curvearrowright (Y, \nu))$ (case where $Q = L^\infty(Y, \nu)$).

From Theorem A we deduce the following examples.

Corollary B. Assume that $\Lambda$ and $\Gamma$ are as follows. Then the conclusion of Theorem A holds.

1. $\Gamma$ is a hyperbolic group and $\Lambda$ is an infinite maximal amenable subgroup [BC14];
2. $\Lambda$ is any amenable group with an infinite index subgroup $\Lambda_0$, and $\Gamma = \Lambda \rtimes \Lambda_0 \Lambda'$, for some other group $\Lambda'$ containing $\Lambda_0$;
3. $\Gamma = \text{SL}_n(\mathbb{Z})$ and $\Lambda$ is the subgroup of upper triangular matrices.

Point (2) above was proved independently by B. Leary [Le14] for more general von Neumann algebras (not only group algebras).

Regarding the question of providing abelian, maximal amenable subalgebras in a given von Neumann algebra, we can prove the following. The result is not as explicit as the above examples, but it is quite general. We are grateful to Jesse Peterson for stimulating our interest in this question in the setting of lattices in Lie groups.

Corollary C. Consider a lattice $\Gamma$ in a connected semi-simple real Lie group $G$ with finite center. Then $L\Gamma$ admits a maximal amenable subalgebra which is virtually abelian.

As we explain in Remark 2.8 if moreover $G$ has no compact factors and $\Gamma$ is torsion free and co-compact in $G$, then $L\Gamma$ admits an abelian maximal amenable subalgebra.

At this point, let us mention that all the former results on maximal amenability followed Popa’s strategy of proving the maximal amenability of $Q \subset M$ by studying $Q$-central sequences in $M$. Namely the inclusion $Q \subset M$ was usually shown to satisfy the so-called “asymptotic orthogonality property”. In contrast, our result relies on a new strategy, more specific to group von Neumann algebras, and completely different from Popa’s approach.

The general idea in our approach is the following. Assume that $\Gamma$ acts on some compact space $X$. Then the maximal amenable subgroups of $\Gamma$ are stabilizers of probability measures on $X$. In non-commutative terms, one can more generally say that amenable subalgebras of $L\Gamma$ centralize states on the reduced $C^*$-algebraic crossed-product $C(X) \rtimes \gamma, \Gamma$. The advantage of considering this crossed-product instead of some $B(H)$ is that it allows concrete computations. We will see at the end of this paper that this point of view also has a theoretical interest, providing new insight on solidity and strong solidity.

Acknowledgements. We first warmly thank Cyril Houdayer for important comments on an earlier version of this paper. We are grateful to Stefaan Vaes for providing us with a shortcut in the proof of Proposition 3.1 and to Jean François Quint for an helpful discussion about lattices in algebraic groups. We also thank Adrian Ioana, Jesse Peterson and Sorin Popa for various discussions and remarks.
Notation. Given a continuous action $\Gamma \rtimes X$ on a compact metric space, denote by Provenience $\probf(X)$ the set of $\Gamma$-invariant probability measures on $X$.

In this section, we prove Theorem A. To motivate the result, let us mention the following trivial observation.

**Lemma 1.1.** Consider a subgroup $\Lambda$ of a discrete countable group $\Gamma$. Then $\Lambda$ is maximal amenable inside $\Gamma$ if and only if there exists a continuous action $\Gamma \rtimes X$ on a compact metric space $X$ such that for any $\mu \in \probf_\Lambda(X)$ and $g \notin \Lambda$, $g \cdot \mu \neq \mu$.

**Proof.** If such a space $X$ exists, then $\Lambda$ is clearly maximal amenable.

Conversely, assume that $\Lambda$ is maximal amenable. For any $g \notin \Lambda$, the group $\langle \Lambda, g \rangle$ is not amenable: there exists a compact metric $\Gamma$-space $X_g$ such that $\probf_{\langle \Lambda, g \rangle}(X_g) = \emptyset$.

Then define the compact metric space $X := \Pi_{g \notin \Lambda} X_g$ on which $\Gamma$ acts diagonally. If $\mu \in \probf_\Lambda(X)$ and $g \notin \Lambda$ then the push-forward of $\mu$ on $X_g$ is $\Lambda$-invariant. So it is not $g$-invariant by definition of $X_g$. Therefore $\mu$ is not $g$-invariant.

**Definition 1.2.** Consider an amenable subgroup $\Lambda$ of a discrete countable group $\Gamma$. Suppose that $\Gamma$ acts continuously on the compact metric space $X$. We say that $\Lambda$ is singular in $\Gamma$ (with respect to $X$) if for any $\mu \in \probf_\Lambda(X)$ and $g \notin \Lambda$, we have $g \cdot \mu \perp \mu$.

**Theorem 1.3 (Theorem A).** Suppose that $\Gamma$ is a discrete countable group admitting an amenable, singular subgroup $\Lambda$. Then for any trace preserving action $\Gamma \rtimes (Q, \tau)$ on a finite amenable von Neumann algebra, $Q \rtimes \Lambda$ is maximal amenable inside $Q \rtimes \Gamma$.

**Proof.** We will denote by $M := Q \rtimes \Gamma$ and by $N := Q \rtimes \Lambda$. Consider an intermediate amenable von Neumann algebra $N \subset A \subset M$. We will show that $A = N$.

Suppose that $\Lambda$ is singular in $\Gamma$ with respect to $X$ and choose a faithful representation $C(X) \subset B(H)$. Consider the reduced crossed-product $C^*$-algebra

$$B_X(Q) := (C(X) \otimes \min Q) \rtimes \Gamma \subset B(H \otimes L^2(Q) \otimes \ell^2 \Gamma).$$

With this representation, we see that $(Q \cup \Gamma)^\prime \prime \subset B(H \otimes L^2(Q) \otimes \ell^2 \Gamma)$ is isomorphic to $M$

For notational convenience we will also identify $C(X)$ with $C(X) \otimes 1 \subset B_X(Q)$, so that $C(X)$ and $Q$ commute to each other inside $B_X(Q)$. Since $A$ is amenable, there exists an $A$-central state $\varphi : B(H \otimes L^2(Q) \otimes \ell^2 \Gamma) \to \mathbb{C}$ whose restriction to $M$ coincides with the trace.

We are interested in the restriction of this state to the $C^*$-algebra $B_X(Q)$. More specifically, define

$$A_{\varphi}(Q) := \{x \in M \mid \varphi(xT) = \varphi(Tx), \forall T \in B_X(Q)\}.$$  

Obviously this set contains $A$. We claim that $\mathcal{U}(M) \cap A_{\varphi}(Q) \subset N$, which implies the proposition.

Take $u \in \mathcal{U}(M) \cap A_{\varphi}(Q)$ and $\varepsilon > 0$.

By density, one can find $u_0 \in M$ of the form $u_0 = \sum_{g \in F} a_g u_g$, with $F$ finite and non-zero elements $a_g \in Q$ for all $g \in F$, such that

$$\|u^* - u_0\|_2 < \varepsilon.$$  

Since $\varphi$ is $A$-central, there exists a $A$-invariant measure $\mu$ on $X$ such that $\varphi(f) = \int_X f d\mu$, for all $f \in C(X)$. Our assumption implies that for all $g \in F \setminus \Lambda$, the measures $\mu$ and $(g^{-1} \cdot \mu)$ are singular with respect to each other.

So there exist a compact set $K$ and an open set $V$ containing $K$ such that:

- $\mu(K) > 1 - \varepsilon$;
Proposition 2.1 is a direct consequence of the following classical result describing maximal amenable subgroups of a hyperbolic group $\Gamma$ in terms of the action $\Gamma \curvearrowright X$.

**Lemma 2.2.**

In particular, if $\Lambda < \Gamma$ is an infinite maximal amenable subgroup in a hyperbolic group $\Gamma$, then $\Lambda$ is singular in $\Gamma$ with respect to the action $\Gamma \curvearrowright X = \partial \Gamma$ on the Gromov boundary. In particular $\Lambda \Lambda$ is maximal amenable inside $\Lambda \Gamma$.

Remark 1.4. In the above proof, the introduction of the set $A_\varphi(Q)$ may seem artificial, since the same argument directly shows that the centralizer of $\varphi$ in $B(H \otimes L^2(Q) \otimes \ell^2 \Gamma)$ (and hence $Q$) is contained in $\Lambda \Lambda$. However, we think that the proof is more transparent when working within the crossed-product C$^*$-algebra. Also we will present some more motivation for this framework in Section 3.

2. Examples

2.1. Hyperbolic groups. As a first application of our criterion, we give a new proof of the main result of [BCT14]. Note that one can also recover the results from [BCT14] about relatively hyperbolic groups.

**Proposition 2.1 ([BCT14]).** If $\Lambda$ is any maximal amenable subgroup in a hyperbolic group $\Gamma$, then $\Lambda$ is singular in $\Gamma$ with respect to the action $\Gamma \curvearrowright X = \partial \Gamma$ on the Gromov boundary. In particular $\Lambda \Lambda$ is maximal amenable inside $\Lambda \Gamma$.

**Lemma 2.2.** Let $\Lambda < \Gamma$ be an infinite maximal amenable subgroup. The following facts are true.

(i) There exist two points $a, b \in \partial \Gamma$ such that $\Lambda = \text{Stab}_{\partial \Gamma}(\{a, b\})$.

(ii) Any $\Lambda$-invariant probability measure on $\partial \Gamma$ is of the form $t \delta_a + (1 - t) \delta_b$ for some $t \in [0, 1]$.

(iii) Any element $g \in \Gamma \setminus \Lambda$ is such that $g : \{a, b\} \cap \{a, b\} = \emptyset$. 


We have already used this lemma in [BC14] without giving a proof. Let us give a proof for completeness.

**Proof.** By [GdH90] Théorème 8.37, Λ is virtually cyclic. Denote by \( h \in \Lambda \) an element of infinite order. Then by [GdH90] Théorème 8.29, \( h \) is a hyperbolic element: \( h \) acts on \( \partial T \) with a north-south dynamics. Denote by \( a \) and \( b \) the attractive and repulsive points of \( h \).

(i) Take \( s \in \Lambda \). Then \( s h s^{-1} \) is a hyperbolic element with fixed points \( s \cdot a \) and \( s \cdot b \). Since \( h \) and \( s h s^{-1} \) generate an amenable group (contained in \( \Lambda \)), an easy application of ping-pong lemma gives that they have at least a fixed point in common, say \( a \). By [GdH90] Théorème 8.30 they also both fix \( b \). Hence \( \{ s \cdot a, s \cdot b \} = \{ a, b \} \) and so \( \Lambda \subset \text{Stab}_G(\{a, b\}) \). The equality follows from maximal amenability.

(ii) This is a consequence of the north-south dynamics action of \( h \in \Lambda \).

(iii) By [GdH90] Théorème 8.30], any element which fixes one of the points \( a \) or \( b \) is in \( \Lambda \). Take \( g \in \Gamma \) such that \( g \cdot a = b \). Assume first that there exists \( s \in \Lambda \) which exchanges \( a \) and \( b \). Then \( sg \) fixes \( a \) and so \( g \in \Lambda \). If all elements in \( \Lambda \) fix \( a \) and \( b \), then \( g s g^{-1} \) fixes \( b \) and \( g^{-1} sg \) fixes \( a \), for all \( s \in \Lambda \). In that case \( g \) normalizes \( \Lambda \) so \( g \in \Lambda \) by maximal amenability. \( \Box \)

### 2.2. Amalgamated free products and HNN extensions.

Using Bass-Serre theory, our criterion also applies for amalgamated free products.

**Proposition 2.3.** Let \( \Lambda_1 \) and \( \Lambda_2 \) be discrete groups (not necessarily finitely generated) with a common subgroup \( \Lambda_0 \). Put \( \Gamma := \Lambda_1 \ast_{\Lambda_0} \Lambda_2 \). If \( \Lambda_1 \) is amenable and the index \( [\Lambda_1 : \Lambda_0] = \infty \) then \( \Lambda_1 \) is singular in \( \Gamma \). In particular \( \Lambda \Lambda_1 \) is maximal amenable inside \( \Gamma \).

**Proof.** Let us first construct the compact \( \Gamma \)-space \( X \) for which we will verify the singularity property of \( \Lambda_1 \subset \Gamma \). Assume that \( \Gamma \) is as in the statement of Corollary 2.3 and consider the Bass-Serre tree \( T \) of \( \Gamma \). By definition the vertex set of \( T \) equals to \( V(T) := \Gamma / \Lambda_1 \cup \Gamma / \Lambda_2 \) and its edge set equals to \( E(T) := \Gamma / \Lambda_0 \), where the edge \( g \Lambda_0 \) relates \( g \Lambda_1 \) to \( g \Lambda_2 \). By assumption the vertex \( \Lambda_1 \) has infinitely many neighbours. In particular this tree is not locally finite. However it is a uniformly fine hyperbolic graph in the sense of [Bo12] Section 8] so one can still consider its visual boundary \( \partial T \) and define a compact topology on \( X := V(T) \cup \partial T \) as follows.

For \( x, y \in X \) denote by \( [x, y] \) the unique geodesic path between \( x \) and \( y \). If \( x \in X \) and \( A \subset V(T) \) is finite set of vertices, define

\[
M(x, A) := \{ y \in X \mid [x, y] \cap A = \emptyset \}.
\]

Then the family of sets \( M(x, A) \) with \( x \in X \), \( A \subset V(T) \) finite, forms an open basis of a compact (Hausdorff) topology on \( X \). See [Bo12] Section 8] for a proof or [Oz06] Section 2] for a short presentation of these facts.

Note that the action \( \Gamma \acts X \) is continuous for this topology.

To prove Proposition 2.3 it is enough to show that the only \( \Lambda_1 \)-invariant probability measure on \( X \) is the Dirac measure \( \delta_{\Lambda_1} \).

To that aim, assume that \( \mu \) is a \( \Lambda_1 \)-invariant probability measure on \( X \). Note that \( X \) is the disjoint union of \( \{ \Lambda_1 \} \) and the open sets \( \{ M(g \Lambda_2, \{ \Lambda_1 \}) \}_{g \in \Lambda_1} \). Because of the infinite index condition, there are infinitely many \( g \in \Lambda_1 \) such that the open sets \( M(g \Lambda_2, \{ \Lambda_1 \}) \) are disjoint. Since \( \Lambda_1 \) acts transitively on these sets, they must all have measure 0 and we are done. \( \Box \)

With the same proof we also get the following result.
Proposition 2.4. Assume that $\Gamma = \text{HNN}(\Lambda, \Lambda_0, \theta)$ is an HNN extension, where $\Lambda_0 < \Lambda$ and $\theta : \Lambda_0 \to \Lambda$ is an injective morphism. If $\Lambda$ is amenable and $[\Lambda : \Lambda_0] = [\Lambda : \theta(\Lambda_0)] = \infty$ then $\Lambda$ is singular in $\Gamma$.

In the finite index setting the result is false in general and the condition of Theorem \[\text{A}\] is never satisfied. For instance assume that $\Gamma = \text{BS}(m, n) = \langle a, t \mid t a t^{-1} = a^{m n} \rangle$ with $m, n \geq 2$, and that $\Lambda = \langle a \rangle$. Then the conjugacy action of $\Lambda$ on $\Gamma \setminus \Lambda$ admits a finite orbit. Namely, $tat^{-1}$ has an orbit with $m$ elements, and so the element $x := \sum_{k=0}^{m-1} a^k t a^{-k} \in \mathbb{C} \Gamma$ commutes with $\Lambda$. In this case, $\Lambda \Lambda$ is not even maximal abelian in $L \Gamma$. However, one can check using Lemma \[\text{[L]}\] that $\Lambda$ is maximal amenable inside $\Gamma$ as soon as $|m|, |n| \geq 3$ (but it is not true for $|n| = 2$ or $|m| = 2$).

2.3. Lattices in semi-simple groups. Finally, our criterion also allows to produce examples of a different kind, out of the (relatively)-hyperbolic world.

Proposition 2.5. For $n \geq 2$, put $\Gamma := \text{SL}_n(\mathbb{Z})$ and denote by $\Lambda$ the subgroup of upper triangular matrices in $\Gamma$. Then $\Lambda$ is singular in $\Gamma$. Moreover, $\Lambda A$ has a diffuse center.

Proof. Put $G = \text{GL}_n(\mathbb{R})$ and denote by $P < G$ be the subgroup of upper triangular matrices, so that $\Lambda = \Gamma \cap P$. We will show that $\Lambda < \Gamma$ is singular with respect to the action on the homogeneous space $B = G / P$. It is enough to prove that the unique $\Lambda$-invariant probability measure on $B$ is the Dirac mass on $[P]$. Fix $\mu \in \text{Prob}\_\Lambda(P)$.

Denote by $N < P$ the subgroup of unipotent matrices and put $\Lambda_0 := \Gamma \cap N$. Then Proposition 2.6 implies that the support of $\mu$ is pointwise fixed by the Zariski closure of $\Lambda_0$, namely $N$. So we are left to check that $N$ has only one fixed point on $B$. Note that a point $g[P] \in B$ is fixed by $N$ if and only if $g^{-1}Ng \subset P$. So let us take $g \in G$ such that $g^{-1}Ng \subset P$ and show that $g \in P$.

For a matrix $h \in G$, we denote with $\sigma(h)$ the spectrum of $h$. Observe that given $p \in P$ we have

$$p \in N \text{ if and only if } \sigma(p) = \{1\}.$$ 

Since the spectrum is conjugacy invariant and $g^{-1}ng \in P$ for all $n \in N$, we have that $g^{-1}ng$ commutes, and this is even an equality because the vector spaces $N$ and $g^{-1}Ng$ have the same dimension. But a simple induction shows that the normalizer of $N$ in $G$ is $P$, so $g \in P$, as wanted.

For the moreover part, denote by $I$ the identity matrix and by $E_{1,n}$ the matrix with 0 entries except for the entry row 1/column $n$ which is equal to 1. A simple calculation shows that the $\Lambda$-conjugacy class of $I + E_{1,n}$ is contained in $\{I \pm E_{1,n}\}$. Therefore the center of $\Lambda A$ contains the element $u + u^*$, where $u$ is the unitary in $\Lambda A$ corresponding to the element $I + E_{1,n} \in \Lambda$. Note that $I + E_{1,n}$ has infinite order, so $u$ generates a copy of $L \mathbb{Z}$. Finally $u + u^*$ generates a subalgebra of index 2, which implies that $\Lambda A$ has diffuse center.

Of course, the example given in the above proposition is not abelian (unless $n = 2$). We now turn to the question of existence of abelian, maximal amenable subalgebras in von Neumann algebras associated with lattices in semi-simple Lie groups.

Proposition 2.6. Consider a lattice $\Gamma$ in a connected semi-simple real algebraic Lie group $G$ with finite center. Then there exists a virtually abelian subgroup $\Lambda$ in $\Gamma$ which is singular in $\Gamma$.

Proof. Before starting the proof, let us fix some notation. Denote with $d$ the real rank of $G$ and let $G = KAN$ be an Iwasawa decomposition of $G$, so that $K$ is a maximal compact subgroup, $A \cong \mathbb{R}^d$ and $N$ is nilpotent. Denote with $M$ the centralizer of $A$ in $K$. By Theorem 2.8 there exists an abelian subgroup $H \subset MA$ (a so-called Cartan subgroup) such
that $H \cap \Gamma$ is cocompact in $H$. Moreover $H$ contains $A$, so it is co-compact in $MA$. Therefore $\Lambda_0 := MA \cap \Gamma$ is a co-compact lattice in $MA$ and it contains the virtually abelian subgroup $\Gamma \cap H$ as a finite index subgroup.

Let $P = MAN$ be a minimal parabolic subgroup. We will show that the normalizer $\Lambda := N_\Gamma(\Lambda_0)$ is singular in $\Gamma$ with respect to the action $\Gamma \curvearrowright G/P$. Consider a measure $\mu \in \text{Prob}_A(G/P)$.

**Claim 1.** $\mu$ is supported on the set $F := \{ x \in G/P \mid ax = x, \forall a \in A \}$.

The measure $\mu$ is $\Lambda_0$-invariant. Put $\tilde{\mu} := \int_M (g \cdot \mu) dg$, where $dg$ denote the Haar probability measure on $M$. Given an element $ma \in \Lambda_0$, with $m \in M, a \in A$, we see that

$$a \cdot \tilde{\mu} = \int_M (ag \cdot \mu) dg = \int_M (ga \cdot \mu) dg = \int_M (gma \cdot \mu) dg = \tilde{\mu}$$

Hence $\tilde{\mu}$ is invariant under the projection of $\Lambda_0$ on $A$. But this projection is a lattice in $A$. Applying [Mo79, Proposition 2.6], we deduce that $\tilde{\mu}$ is supported on $F$. Since $M$ commutes with $A$, the set $F$ is globally $M$-invariant: $g \cdot \mu(F) = \mu(F)$ for all $g \in M$. Hence $1 = \tilde{\mu}(F) = \mu(F)$, as claimed.

**Claim 2.** For all $x \in F$, we have $\text{Stab}_G(x) \cap \Gamma = \Lambda_0$.

To prove this claim take $x \in F$, written $x = gPg^{-1}$, so that $A \subset gPg^{-1}$. By [BT65, Theorem 4.15], we have that $gPg^{-1} = MAgNg^{-1}$. In particular $MA$ (and $\Lambda_0$) fixes $x$.

Take now $\gamma \in \Gamma \cap \text{Stab}_G(x) = \Gamma \cap gPg^{-1}$. Write $\gamma = man$ with $m \in M, a \in A$ and $n \in gNg^{-1}$. By [ZS], Proposition 8.2.4 there exists a sequence $(b_k)_k$ in $MA$ such that $b_kb_k^{-1}$ converges to the identity element $1_G$. Since $\Lambda_0$ is a uniform lattice in $MA$, there exists a subsequence $(b_{k_j})$ and a sequence $(c_j)_j \subset \Lambda_0$ such that $b_{k_j}c_j^{-1}$ converges in $MA$. It is easy to conclude that $c_jmc_j^{-1}$ converges to the identity. Now, $c_j\gamma c_j^{-1}$ lies in $\Gamma$ and we have

$$c_j\gamma c_j^{-1} = c_j(man)c_j^{-1} = (c_jmc_j^{-1})a(c_jnc_j^{-1}), \text{ for all } j.$$

But $c_jmc_j^{-1}$ belongs to the compact set $M$, so taking a subsequence if necessary, we see that $c_j\gamma c_j^{-1}$ converges to an element in $MA$. By discreteness of $\Gamma$, this implies that $c_j\gamma c_j^{-1} \in MA$ for $j$ large enough. Therefore $\gamma \in MA \cap \Gamma = \Lambda_0$ which proves Claim 2.

To prove that $\Lambda$ is singular in $\Gamma$, consider an element $g \in \Gamma$ such that $g \cdot \mu$ is not singular with respect to $\mu$. Then $g \cdot F \cap F \neq \emptyset$, so there exist two points $x, y \in F$ such that $y = gx$. This implies that $g\Lambda_0g^{-1}$ fixes $y$, while $g^{-1}\Lambda_0g$ fixes $x$. From Claim 2 we deduce that $g$ normalizes $\Lambda_0$, so that $\Lambda$ is indeed singular in $\Gamma$.

To complete the proof of the proposition, it remains to show that $\Lambda_0$ has finite index inside $\Lambda$, which will ensure that $\Lambda$ is virtually abelian.

Assume that $g \in \Gamma$ normalizes $\Lambda_0$. Since $A$ lies in the Zariski closure of $\Lambda_0$, we have $gAg^{-1} \subset MA$. But $MA$ has a unique maximal $\mathbb{R}$-split torus, $A$. So $gAg^{-1} = A$ and $g$ normalizes $A$. Moreover $MA$ coincides with the centralizer $Z_G(A)$ of $A$ in $G$. Now we have only to observe that the Weyl group $N_G(A)/Z_G(A)$ is finite, see [Kn96, Section VII.7, item 7.84] for instance.

**Remark 2.7.** For $\text{SL}_3(\mathbb{Z})$, let $\Lambda_0$ be the group generated by the following two commuting matrices:

$$
\begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & -16 \\
0 & 1 & 8 \\
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
81 & 4 & -4 \\
-36 & 17 & 68 \\
4 & -4 & -15 \\
\end{pmatrix}.
$$

Then $\Lambda_0$ has finite index in a singular subgroup of $\text{SL}_3(\mathbb{Z})$. We do not know whether $\Lambda_0$ itself is singular in $\text{SL}_3(\mathbb{Z})$. 


Remark 2.8. We remark here that whenever \( \Gamma \) is a co-compact and torsion free lattice of a real algebraic group \( G \) without compact factors, there exists a free abelian subgroup of \( \Gamma \) that is singular in \( \Gamma \). In fact the authors in [RS10] proved that under these hypothesis, the Cartan subgroup constructed in [PR03] \( H \subset G \) is such that \( \Lambda_0 := \Gamma \cap H \) is isomorphic to \( \mathbb{Z}^{\text{rk}(G)} \) and \( \Lambda_0 \) is malnormal in \( \Gamma \). So we can use this Cartan subgroup in the proof of the above proposition to get that \( \Lambda_0 \) has finite index in a singular subgroup of \( \Gamma \) and since \( \Lambda_0 \) is malnormal in \( \Gamma \), then we must have that \( \Lambda_0 \) is singular in \( \Gamma \).

3. Amenable subalgebras as stabilizers of measures on some compact space

As explained in the introduction, the key of the above results is to view maximal amenable subalgebras of a group von Neumann algebra \( \mathcal{L}(\Gamma) \) as centralizers of states on some reduced crossed-product \( C^* \)-algebra \( C(X) \rtimes \Gamma \). In this section we further develop this point of view and explain its link with more theoretical questions. What follows is largely inspired from the work of N. Ozawa on solidity [Oz04, Oz10].

Proposition 3.1. Assume that \( \Gamma \) is a countable discrete group which acts continuously on a compact space \( X \). Denote by \( B := C(X) \rtimes \Gamma \) the reduced crossed-product \( C^* \)-algebra. Consider a state \( \varphi \) on \( B \) which coincides on \( C^*_r(\Gamma) \) with the canonical trace \( \tau \). The following are true.

(i) Given \( x \in \mathcal{L}(\Gamma) \) and \( T \in B \), one may define \( \varphi(Tx) \) by the formula \( \varphi(Tx) = \lim_n \varphi(x_nT) \), where \( (x_n)_n \) is any bounded sequence in \( C^*_r(\Gamma) \subset B \) converging strongly to \( x \). One can define similarly \( \varphi(Tx) \).

(ii) The set \( A_\varphi := \{ x \in \mathcal{L}(\Gamma), \varphi(xT) = \varphi(Tx), \forall T \in B \} \) is a von Neumann subalgebra of \( \mathcal{L}(\Gamma) \).

(iii) If the action is topologically amenable, then \( A_\varphi \) is amenable (and every maximal amenable subalgebra of \( \mathcal{L}(\Gamma) \) arises this way).

Proof. (i) Take \( x \in \mathcal{L}(\Gamma) \), \( T \in B \) and fix a bounded sequence \( (x_n)_n \subset C^*_r(\Gamma) \subset B \) which converges strongly to \( x \). Then \( (x_n)_n \) is a Cauchy sequence for the norm \( \| \cdot \|_2 \) associated with \( \tau \). By applying Cauchy-Schwarz inequality we deduce that \( (\varphi(x_nT))_n \) is a Cauchy sequence as well. Indeed, since \( \varphi | C^*_r(\Gamma) = \tau \), one has for every \( a, b \in C^*_r(\Gamma) \subset B \)

\[
|\varphi(aT) - \varphi(bT)| \leq \|a - b\|_2 \|T\|_2 \tag{3.1}
\]

Moreover equation (3.1) also shows that the limit of the sequence \( (\varphi(x_nT))_n \) does not depend on the choice of the sequence \( (x_n)_n \) converging to \( x \). We can thus define \( \varphi(xT) := \lim_n \varphi(x_nT) \).

(ii) Denote by \( (\pi_\varphi, H_\varphi, \xi_\varphi) \) the GNS triplet associated with \( \varphi \). Extend the state \( \varphi \) to a normal state on \( \tilde{B} := \pi_\varphi(B)' \) by the formula \( \varphi(x) = (x\xi_\varphi, \xi_\varphi) \). Note that \( \mathcal{L}(\Gamma) \subset \tilde{B} \). Then one checks that \( A_\varphi = \mathcal{L}(\Gamma) \cap c_\varphi(\tilde{B}) \), where \( c_\varphi(\tilde{B}) \) is the centralizer of \( \varphi \) in \( \tilde{B} \). So \( A_\varphi \) is a von Neumann algebra.

(iii) Assume that the action is amenable. Then \( B \) is nuclear, and so \( \tilde{B} := \pi_\varphi(B)' \) is amenable. Moreover, there exists a \( \varphi \)-preserving conditional expectation \( E : \tilde{B} \to A_\varphi \), because \( A_\varphi \) centralizes \( \varphi \). Hence \( A_\varphi \) is amenable. \( \square \)

Let us provide some applications of this proposition to solidity and strong solidity for bi-exact groups [Oz04, OP10a, OP10b, CS13].

Definition 3.2 ([BO08], Section 15.2). A discrete group \( \Gamma \) is bi-exact if there exists a compactification \( X \) of \( \Gamma \) such that

(i) the left translation action of \( \Gamma \) on itself extends to a continuous action \( \Gamma \curvearrowright X \) which is topologically amenable;
(2) the right translation action of $\Gamma$ on itself extends continuously to an action on $X$ which is trivial on the boundary $X \setminus \Gamma$.

For instance any hyperbolic group is bi-exact (because the Gromov compactification $\Delta \Gamma$ satisfies the above conditions).

Given a bi-exact group $\Gamma$, choose a compactification $X$ as in Definition 3.2. Since it is a compactification, we have inclusions $c_0(\Gamma) \subset C(X) \subset \ell^\infty(\Gamma)$. Denote by $\lambda$ and $\rho$ respectively the left and right regular representations of $\Gamma$ on $\ell^2 \Gamma$, and define

$$B_\Gamma := C^*(C(X) \cup \lambda(\Gamma)) \subset B(\ell^2 \Gamma).$$

By BO08 Proposition 5.1.3, $B_\Gamma$ is isomorphic to the reduced crossed product $C(X) \rtimes_{\text{reg}} \Gamma$ by the left action of $\Gamma$. Moreover condition 3.2(2) implies that $B_\Gamma$ commutes with $C^*_\rho(\Gamma)$ modulo compact operators:

$$[B_\Gamma, C^*_\rho(\Gamma)] \subset C^*(\lambda(\Gamma) \cdot [C(X), \rho(\Gamma)]) \subset C^*(\lambda(\Gamma) \cdot c_0(\Gamma)) \subset K(\ell^2(\Gamma)).$$

We now show how solidity and strong solidity results can be deduced from Proposition 3.4.

**Theorem 3.3 (Oz04).** If $\Gamma$ is bi-exact, then $L\Gamma$ is solid, meaning that the relative commutant of any diffuse subalgebra of $L\Gamma$ is amenable.

**Proof.** Consider a sequence of unitaries $(u_n) \subset U(L\Gamma)$ which tends weakly to $0$. We will show that the von Neumann algebra $A$ of elements $x \in L\Gamma$ satisfying $\|[x, u_n]\|_2 \to 0$ is amenable.

Consider the state $\varphi$ on $B(\ell^2 \Gamma)$ defined by

$$\varphi(T) := \lim_{n \to \omega} \langle Tu_n, u_n \rangle,$$

where $\omega$ is a free ultrafilter on $\mathbb{N}$. Note that $\varphi|_{L\Gamma} = \tau = \varphi|_{\Gamma\Gamma}$ and that $\varphi$ vanishes on the compact operators because $u_n$ tends weakly to $0$. Applying Proposition 3.4, we get that $A_{\varphi} := \{x \in L\Gamma, \varphi(xT) = \varphi(Tx), \forall T \in B_\Gamma\}$ is an amenable von Neumann algebra. Let us show that $A \subset A_{\varphi}$.

Take $u \in U(A)$. By definition of $A$, we have for any $T \in B(H)$

$$\varphi(Tu) = \lim_n \langle Tu(u_n), u_n \rangle = \lim_n \langle T(u_nu), u_n \rangle = \varphi(TJ u^* J).$$

Similarly, we have $\varphi(uT) = \varphi(J u^* J T)$. Fix a bounded sequence $(x_k) \subset C^*_\rho(\Gamma)$ which converges strongly to $Ju^* J$. Since $\varphi|_{\Gamma\Gamma}$ is normal, the Cauchy-Schwarz inequality implies that

- $\lim_k \varphi(T x_k) = \varphi(T Ju^* J)$
- $\lim_k \varphi(x_k T) = \varphi(J u^* J T)$.

Now for each $k$, the operator $[T, x_k]$ is compact thanks to 3.2. Since $\varphi$ vanishes on compact operators we get

$$\varphi(u T) = \lim_k \varphi(x_k T) = \lim_k \varphi(T x_k) = \varphi(T Ju^* J) = \varphi(T u).$$

**Theorem 3.4 ([OP10a, CS13]).** If $\Gamma$ is biexact and weakly amenable (this is the case if $\Gamma$ is hyperbolic, [Oz08]) then $L\Gamma$ is strongly solid, in the sense that the normalizer of a diffuse amenable subalgebra of $L\Gamma$ is amenable.

Our proof still relies on the following weak compactness property due to Ozawa and Popa. The formulation is a combination of [OP10a] Theorem 3.5 and [Oz12] Theorem B with the characterization of weak compactness given in [OP10a, Proposition 3.2(4)].

**Theorem 3.5 ([OP10a, Oz12]).** Assume that $\Gamma$ is weakly amenable. Then for any amenable subalgebra $A$ of $L\Gamma$, there exists a state $\varphi$ on $B(L^2(M))$ such that
\[ \varphi(xT) = \varphi(Tx) \text{ for every } T \in B(L^2(M)) \text{ and } x \in A; \]
\[ \varphi(uJu^*J) = \varphi(TuJu^*J) \text{ for every } T \in B(L^2(M)) \text{ and } u \in N_M(A); \]
\[ \varphi(x) = \tau(x) = \varphi(Jx^*J) \text{ for every } x \in M. \]

The new part of the proof is the conclusion of strong solidity from the existence of such a state. It becomes extremely simple.

**Proof of Theorem 3.4.** Assume that \( A \) is a diffuse amenable subalgebra and consider a state \( \varphi \) on \( B(\ell^2 \Gamma) \) as in Theorem 4.5. By Proposition 5.1 it suffices to show that \( N_M(A) \subset A_\varphi := \{ x \in LT, \varphi(xT) = \varphi(Tx), \forall T \in B_T \} \).

First note that [OP10b, Lemma 3.3] implies that \( \varphi \) vanishes on compact operators because \( A \) is diffuse.

Take \( u \in N_M(A) \) and \( T \in B_T \). By definition of \( \varphi \), we have \( \varphi(uJu^*JTu^*J) = \varphi(Tu) \).

Fix a bounded sequence \( (x_k) \subset C^*_\rho(\Gamma) \) which converges strongly to \( Ju^*J \). Since \( \varphi|_{RT^*} \) is normal, the Cauchy-Schwarz inequality implies that

- \( \lim_k \varphi(uJu^*JTx_k) = \varphi(uJu^*JTu^*J) \)
- \( \lim_k \varphi(uJu^*Jx_kT) = \varphi(uT). \)

Now for each \( k \), the operator \( uJu^*JT \) is compact thanks to (3.2). Since \( \varphi \) vanishes on compact operators we get

\[ \varphi(u^T) = \lim_k \varphi(uJu^*JTx_kT) = \lim_k \varphi(uJu^*Jx_k) = \varphi(uJu^*JTu^*J) = \varphi(Tu). \]

Let us mention that one could also do a relative version of this strategy to prove relative strong solidity results. In particular, one could recover some of the results in [PV14a, PV14b] by applying this result on bi-exactness explicitly.

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