EXISTENCE OF THE OPTIMUM FOR SHALLOW LAKE TYPE MODELS

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Abstract. We consider the optimal control problem associated with a general version of the well known shallow lake model, and we prove the existence of an optimum in the class $L^1_{\text{loc}}(0, +\infty)$. Any direct proof seems to be missing in the literature. Dealing with admissible controls that can be unbounded (even locally) is necessary in order to represent properly the concrete optimization problem; on the other hand, the non-compactness of the control space together with the infinite horizon setting prevents from having good a priori estimates - and this makes the existence problem considerably harder. We present an original method which is in a way opposite to the classical control theoretic approach used to solve finite horizon Mayer or Bolza problems. Synthetically, our method is based on the following scheme: i) two uniform localization lemmas providing, given $T \geq 1$ and a maximizing sequence of controls, another sequence of controls which is bounded in $L^\infty([0, T])$ and still maximizing. ii) A special diagonal procedure dealing with sequences which are not extracted one from the other. iii) A “standard” diagonal procedure. The optimum results to be locally bounded by construction.

Key words. Control, global optimization, non compact control space, uniform localization, convex-concave dynamics.

1. Introduction. In this work we examine the optimal control problem related to a general version of the Shallow Lake model, and we prove the existence of an optimum. In the last fifteen years, a literature about this model has grown up, but, in our knowledge, no direct existence proof has been provided up to now. The optimal control problem has been introduced in [13], and has been studied mostly via dynamic programming ([12]), or from the dynamical systems viewpoint (see e.g. [10], [11] and [13]). The latter approach consists in the analysis of the adjoint system that is obtained coupling the state equation with the adjoint equation given by the Pontryagin Maximum Principle. As it is well known, such principle provides conditions for optimality that in general are merely necessary.

The main technical difficulties in order to prove the existence of an optimum arise from the fact that good a priori estimates for the controls and for the states are missing, because of the infinite horizon setting and the unboundedness assumption on the set of admissible controls. Indeed, the intimate nature of the model requires that one may be allowed to choose a (locally integrable) control function that reaches arbitrarily large values in a finite time. Also controls that are arbitrarily near 0 are allowed, and this produces similar effects when the functional has logarithmic dependence on the control. In this context the application of any compactness result is not straightforward.

Funding: this research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

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Here we propose an original approach to the existence problem. In a sense, we proceed the opposite direction respect to what is done in the proof of some classical existence results for finite horizon problems such as Filippov-Cesari theorem. In the latter kind of proof, thanks to some a priori estimates, Ascoli-Arzelà theorem is applied in order to obtain an optimizing sequence of states converging to a candidate optimal state, which is proven to be almost everywhere differentiable; then some convexity assumption on the dynamics of the state equation allows to pointwise identify a control satisfying the instance of the state equation involving the candidate optimal state. Finally, such control is proven to be admissible by a measurable selection argument. This is what is essentially needed in the case of finite horizon Mayer problems; in the case of Bolza problems with coercive dependence of the integral functional on the control, the same scheme is, roughly speaking, applied to the couple $x_n, J_{int}(x_n, u_n)$, where $(x_n)_n$ is an optimizing sequence of states and $J_{int}(x, u)(t) = \int_{T_0}^t L(s, x(s), u(s)) \, ds$ is the integral part of the objective functional; in this case, after proving that the limit $x_*$ of $x_n$ has an admissible companion control $u_*$, one also has to prove that $u_*$ is in the proper relation with the limit of $J_{int}(x_n, u_n)$. For the details of the latter (complex) proof, see [8], Chapter III, § 5.

In other words, the classical control theoretic approach to the existence problem starts with the convergence of the states and associated functionals to some limit, and ends up with a control function giving those two limits the desired form; in particular no direct semi-continuity argument for the functional is used.

In our approach, dealing with a functional of the type

$$ J(u) = \int_0^{+\infty} e^{-\rho t} \left( \log u(t) - c x^2(t) \right) \, dt, $$

we consider an optimizing sequence of locally integrable controls $(u_n)_n$ and, in order to bypass the absence of a priori estimates, we prove two uniform localization lemmas (“from above” and “from below”). This way, for a fixed compact interval $[0, T]$, we are able to find a sequence $(\bar{u}_T^n)_n$ which is still optimizing and also uniformly bounded in $[0, T]$, by two quantities $N(T), \eta(T)$. By weak (relative) compactness we can extract a sequence $(\tilde{u}_T^T)_n$, weakly converging in $L^1([0, T])$. We repeat the process for bigger and bigger intervals, each time starting from the maximizing sequence we ended up with in the previous step.

In order to merge properly the local (weak) limits, the standard diagonal argument does not work, since we are in presence of two families of sequences which a priori are not extracted one from the other: the “barred” converging sequences and the “unbarred” sequences obtained by applying the uniform localization lemmas. For instance, $(u_{T+1}^T)_n$ will denote the sequence obtained by applying the lemmas to $(\bar{u}_T^n)_n$.
and to the interval $[0, T + 1]$.

Despite this, we can exploit a monotonicity property of the bound functions $N$ and $\eta$ provided by the uniform localization lemmas, in order to end up with a locally bounded optimizing sequence $(v_n)_n$ and a “pre-optimal” function $v$ such that $v_n \rightharpoonup v$ in $L^1([0, T])$ for every $T > 0$.

Then we prove the pointwise convergence of the states associated with $(v_n)_n$. Furthermore, another - standard - diagonal procedure is needed in order to extract from $(v_n)_n$ a sequence $(v_{n,n})_n$ such that $\log v_{n,n} \rightharpoonup \log u^*$ in every in $L^1([0, T])$, for a proper function $u^*$. This is eventually proven to be an admissible and optimal control, relying basically on dominated convergence combined with the following relations:

\[
x(\cdot; v_n) \to x(\cdot; v) \quad \text{pointwise in } [0, +\infty)
\]

\[
\log v_{n,n} \to \log u^* \quad \text{in } L^1([0, T]), \forall T > 0
\]

\[
u^* \leq v \quad \text{a.e. in } [0, +\infty),
\]

where $x(\cdot; u)$ denotes the trajectory associated with the control $u$.

These and other considerations serve as a semi-continuity argument and allow to conclude the proof.

The scheme

\[
\text{uniform localization lemmas } \xRightarrow{\text{"local" compactness}} \quad \text{two families diagonalization } \to \text{ one family diagonalization}
\]

can be considered a development and an improvement of the method introduced in [1] and may be hopefully generalized to a scheme for obtaining existence proofs, applicable to a wider class of infinite horizon optimal control problems with non-compact control space.

The model describes the dynamics of the accumulation of phosphorous in the ecosystem of a shallow lake, from an optimal control theory perspective. Precisely, the state equation expresses the (non-linear) relationship between the farming activities near the lake, which are responsible for the release of phosphorus, and the total amount of phosphorous in the water, depending also on the natural production and on the natural loss consisting of sedimentation, outflow and sequestration in other biomass. The objective functional that is to be maximized, represents the social benefit depending on the pollution released by the farming activities, and takes into account the trade-offs between the utility of the agricultural activities and the utility of a clear lake.
Following [13], we can assert that the essential dynamics of the eutrophication process can be modelled by the differential equation:

\[ \dot{P}(t) = -sP(t) + \frac{rP^2(t)}{m^2 + P^2(t)} + L(t), \]

where \( P \) is the amount of phosphorus in algae, \( L \) is the input of phosphorus (the “loading”), \( s \) is the rate of loss consisting of sedimentation, outflow and sequestration in other biomass, \( r \) is the maximum rate of internal loading and \( m \) is the anoxic level.

After a change of variable and of time scale, we consider the normalized equation

\[ \dot{x}(\tau) = -bx(\tau) + \frac{x^2(\tau)}{1 + x^2(\tau)} + u(\tau), \]

where \( x(\cdot) := \frac{P(\cdot)}{m}, u(\cdot) = \frac{L(\cdot)}{r} \) and \( b = \frac{sm}{r} \). Hence we see that the dynamics, as a function of the state, shows a convex-concave behaviour.

In an economical analysis, the dynamics of pollution must be considered together with the social benefit of the different interest groups operating in the lake system. The social benefit obviously depends both on the status of the water and on the intensity of agricultural activities near the lake, which in a way can be measured by the amount of phosphorous released in the water.

Farmers have an interest in being able to increase the loading, so that the agricultural sector can grow without the need to invest in new technology in order to reduce emissions. On the other hand, groups such as fishermen, drinking water companies and any other industry making use of the water prefer a clear lake, and the same holds for people who use to spend leisure time in relation with the lake. It is assumed that a community or country, balancing these different interests, can agree on a welfare function of the form

\[ \log u - cx^2 \quad (c > 0), \]

in the sense that the lake has value as a “waste sink” for agriculture \( \log u \), where \( u \) is the input of phosphorous due to farming, and it provides ecological services that decrease with the total amount of phosphorus \( x \) as \(-cx^2\).

Here we focus on the case of monotone dynamics, as a first, fundamental step fore-shadowing further developments.

2. Boundedness of the value function.

**Definition 1.** For every \( x_0 \geq 0 \) and every \( u \in L^1_{\text{loc}}([0, +\infty)) \) the function \( t \to x(t; x_0, u) \) is the solution to the following Cauchy’s Problem:

\[
\begin{align*}
\dot{x}(t) &= F(x(t)) + u(t) \quad t \geq 0 \\
 x(t) &= x_0
\end{align*}
\]
in the unknown $x(\cdot)$, where $F$ has the following properties:

$F \in C^1(\mathbb{R}, \mathbb{R})$, $F' \leq 0$ in $\mathbb{R}$, $F(0) = 0$, $\lim_{x \to +\infty} F(x) = -\infty$, $\lim_{x \to +\infty} F'(x) := -l < 0$, there exist $\bar{x} > 0$ such that $F$ is convex in $[0, \bar{x}]$ and concave in $[\bar{x}, +\infty)$.

Moreover, we set $F'(0) < 0$.

For every $x_0 \geq 0$, the set of the admissible controls is:

$$\Lambda(x_0) := \{u \in L^1_{\text{loc}}([0, +\infty)) / u > 0 \text{ a.e. in } [0, +\infty)\}$$

and the objective functional is defined by

$$B(x_0; u) = \int_0^{+\infty} e^{-\rho t} \left[ \log u(t) - cx^2(t; x_0, u) \right] \, dt \quad \forall u \in \Lambda(x_0),$$

where $\rho$ and $c$ are positive constants.

The value function is

$$V(x_0) := \sup_{u \in \Lambda(x_0)} B(x_0; u).$$

Remark 2. The Cauchy's problem (2) has a unique global solution, since the dynamics $F(\cdot)$ has (globally) bounded derivative. We have

$$-b_0x \leq F(x) \leq -bx + M,$$

for some constants $b_0, b, M > 0$. This is easily proven setting $-b := -l + \epsilon$ for $\epsilon > 0$ sufficiently small, choose $b_0 = F'(0) \wedge -b$ and use the assumption $F' \to -l$ at $+\infty$ and the continuity of $F$.

Remark 3. Let $s_1, s_2 \geq 0$, $u_1, u_2 \in L^1_{\text{loc}}([0, +\infty), \mathbb{R})$ and $t_0 \geq 0$.
Set $x_1 = x(\cdot; s_1, u_1)$, $x_2 = x(\cdot; s_2, u_2)$ and define:

$$h(x_1, x_2)(\tau) := \begin{cases} \frac{F(x_1(\tau)) - F(x_2(\tau))}{x_1(\tau) - x_2(\tau)} & \text{if } x_1(\tau) \neq x_2(\tau) \\ F'(x_1(\tau)) & \text{if } x_1(\tau) = x_2(\tau). \end{cases}$$
Then $h(x_1, x_2)$ is continuous, $-b_0 \leq h \leq 0$ and the following relation holds:

$$\forall t \geq t_0 : x_1 (t) - x_2 (t) = \exp \left( \int_{t_0}^{t} h (x_1, x_2) (\tau) \, d\tau \right) (x_1 (t_0) - x_2 (t_0))$$

$$+ \int_{t_0}^{t} \exp \left( \int_{s}^{t} h (x_1, x_2) (\tau) \, d\tau \right) (u_1 (s) - u_2 (s)) \, ds.$$  (3)

In particular, taking $t_0 = 0$ and $s_1 = s_2$:

$$\forall t \geq 0 : x_1 (t) - x_2 (t) = \int_{0}^{t} \exp \left( \int_{s}^{t} h (x_1, x_2) (\tau) \, d\tau \right) (u_1 (s) - u_2 (s)) \, ds$$  (4)

Indeed, for every $t \geq t_0$:

$$\dot{x}_1 (t) - \dot{x}_2 (t) = F (x_1 (t)) - F (x_2 (t)) + u_1 (t) - u_2 (t)$$

$$= h (x_1, x_2) (t) [x_1 (t) - x_2 (t)] + u_1 (t) - u_2 (t).$$

Multiplying both sides of this equation by $\exp \left( - \int_{t_0}^{t} h (x_1, x_2) (\tau) \, d\tau \right)$ we obtain:

$$\frac{d}{dt} \left[ (x_1 (t) - x_2 (t)) \exp \left( - \int_{t_0}^{t} h (x_1, x_2) (\tau) \, d\tau \right) \right]$$

$$= \exp \left( - \int_{t_0}^{t} h (x_1, x_2) (\tau) \, d\tau \right) (u_1 (t) - u_2 (t)) \quad \forall t \geq t_0$$

Fix $t \geq t_0$ and integrate between $t_0$ and $t$; then (3) is easily obtained.

**Remark 4.** Relation (3) implies a well known comparison result, which in our case can be stated as follows.

Let $s_1, s_2 \geq 0$ and $u_1, u_2 \in \mathcal{L}^1_{loc} ([0, +\infty), \mathbb{R})$: then for every $t_0 \geq 0$ and every $t_1 \in (t_0, +\infty]$, if $u_1 \geq u_2$ almost everywhere in $[t_0, t_1]$ and $x (t_0; s_1, u_1) \geq x (t_0; s_2, u_2)$, then

$$x (t; s_1, u_1) \geq x (t; s_2, u_2) \quad \forall t \in [t_0, t_1].$$

Moreover another classical comparison result implies that for every $x_0 \geq 0$ and every $u \in \mathcal{L}^1_{loc} ([0, +\infty))$:

$$e^{-bt} \left( x_0 + \int_{0}^{t} e^{bs} u (s) \, ds \right) \leq x (t; x_0, u)$$

$$\leq e^{-bt} \left( x_0 + \int_{0}^{t} e^{bs} (M + u (s)) \, ds \right).$$  (5)
Remark 5. The objective functional is not constantly equal to $-\infty$. As a trivial example, consider the control $u \equiv 1 \in \Lambda (x_0)$. Then by (5):

$$0 \leq x(t; x_0, u) \leq e^{-bt}x_0 + (M + 1) \frac{1 - e^{-bt}}{b},$$

which implies

$$x^2(t) \leq \left(x_0^2 + \frac{(M + 1)^2}{b^2}\right)e^{-2bt} + 2(M + 1)\frac{x_0}{b}e^{-bt} + \frac{(M + 1)^2}{b^2}.$$ 

Hence

$$\mathcal{B}(u) = -c \int_0^{+\infty} e^{-\rho t}x^2(t; x_0, u) \, dt > -\infty.$$ 

Remark 6. Let $u \in \Lambda (x_0)$ and let $(u_n)_n \subseteq L^1([0, +\infty))$ be a sequence of simple functions such that $u_n \uparrow u$ pointwise in $[0, +\infty)$. Then

$$\mathcal{B}(u) \leq \liminf_{n \to +\infty} \mathcal{B}(u_n).$$

Indeed, for every $n \in \mathbb{N}$, $u_n > 0$ almost everywhere in $[0, +\infty)$, so $(e^{-\rho t} \log u_n(t))_n \subseteq L^1([0, +\infty))$ and $e^{-\rho t} \log u_n(t) \uparrow e^{-\rho t} \log u(t)$ for almost every $t \geq 0$. By monotone convergence we obtain:

$$\limsup_{n \to +\infty} [\mathcal{B}(u) - \mathcal{B}(u_n)] = \limsup_{n \to +\infty} \int_0^{+\infty} e^{-\rho t} \left[\log u(t) - \log u_n(t) - c(x^2(t) - x_n^2(t))\right] \, dt \leq \lim_{n \to +\infty} \int_0^{+\infty} e^{-\rho t} \left[\log u(t) - \log u_n(t)\right] \, dt = 0,$$

where the inequality holds since $0 \leq x_n \leq x$ for every $n \in \mathbb{N}$, by Remark 4.

Definition 7. A sequence $(u_n)_{n \in \mathbb{N}} \subseteq \Lambda (x_0)$ is said to be maximizing at $x_0$ if

$$\lim_{n \to +\infty} \mathcal{B}(x_0; u_n) = V(x_0).$$

Proposition 8. i) The value function $V : [0, +\infty) \to \mathbb{R}$ satisfies:

$$V(x_0) \leq \frac{1}{\rho} \log \left(\frac{\rho + b_0}{\sqrt{2ec}}\right) \quad \forall x_0 \geq 0.$$
ii) For every \( x_0 \geq 0 \), there exist constants \( K_1(x_0), K_2(x_0) > 0 \) such that, for every \( u \in \Lambda(x_0) \) belonging to a maximizing sequence:

\[
\int_0^{+\infty} e^{-\rho t} u(t) \, dt \leq K_1(x_0),
\]

\[
\int_0^{+\infty} e^{-\rho t} x(t;x_0,u)(t) \, dt \leq K_2(x_0).
\]

Hereinafter we will often use the following weaker estimate relative to a control \( u \in \Lambda(x_0) \) belonging to a maximizing sequence:

\[
\int_0^t u(s) \, ds < K_1(x_0) e^{\rho t} \quad \forall t \geq 0.
\]

**Proof.** i) Let \( x_0 \geq 0, u \in \Lambda(x_0), x = x(\cdot;x_0,u) \) and \( B(u) = B(x_0;u) \).

First assume that

\[
\int_0^{+\infty} u(t) \, dt, \quad \int_0^{+\infty} e^{-\rho t} u(t) \, dt < +\infty.
\]

We estimate the quantity

\[
\int_0^{+\infty} e^{-\rho t} x^2(t) \, dt
\]

in terms of the quantities in (9).

**From above:** by (5), we have for every \( t \geq 0 \):

\[
0 \leq x(t) \leq e^{-bt} x_0 + \frac{M}{b} + e^{-bt} \int_0^t e^{bs} u(s) \, ds.
\]

Hence:

\[
x^2(t) \leq e^{-bt} (x_0 \lor x_0^2) \left(1 + \frac{2M}{b}\right) + \frac{M^2}{b^2} + e^{-2bt} \left(\int_0^t e^{bs} u(s) \, ds\right)^2
\]

\[
+ 2 \left(x_0 \lor \frac{M}{b}\right) e^{-bt} \int_0^t e^{bs} u(s) \, ds.
\]

Focusing on the last two terms leads to the estimate

\[
\int_0^{+\infty} e^{-\rho t} e^{-2bt} \left(\int_0^t e^{bs} u(s) \, ds\right)^2 \, dt \leq \int_0^{+\infty} e^{-\rho t} \left(\int_0^t u(s) \, ds\right)^2 \, dt
\]

\[
\leq \frac{1}{\rho} \left(\int_0^{+\infty} u(s) \, ds\right)^2
\]
and
\[
\int_0^{+\infty} e^{-\rho t} e^{-bt} \int_0^t e^{bs} u(s) \, ds \, dt = \int_0^{+\infty} e^{bs} u(s) \int_s^{+\infty} e^{-(\rho+b)t} \, dt \, ds \\
= \frac{1}{\rho+b} \int_0^{+\infty} e^{bs} u(s) e^{-(\rho+b)s} \, ds \\
= \frac{1}{\rho+b} \int_0^{+\infty} e^{-\rho s} u(s) \, ds.
\]
(12)

By (10), (11) and (12) we see that there exists a constant \( L(b,x_0) \geq 0 \) such that
\[
\int_0^{+\infty} e^{-\rho t} x^2(t) \, dt \leq L(b,x_0) + \frac{1}{\rho} \left( \int_0^{+\infty} u(t) \, dt \right)^2 + 2 \left( x_0 \vee \frac{M}{b} \right) \frac{1}{\rho+b} \int_0^{+\infty} e^{-\rho t} u(t) \, dt.
\]
(13)

From below: again by (5):
\[
\forall t \geq 0 : x(t) \geq e^{-b_0 t} \left( x_0 + \int_0^t e^{b_0 s} u(s) \, ds \right) \\
\geq e^{-b_0 t} \int_0^t e^{b_0 s} u(s) \, ds.
\]
Hence, since \( t \to \rho e^{-\rho t} dt \) is a probability measure, we have by Jensen’s inequality:
\[
\int_0^{+\infty} e^{-\rho t} x^2(t) \, dt \geq \rho \left( \int_0^{+\infty} e^{-\rho t} x(t) \, dt \right)^2 \\
\geq \rho \left( \int_0^{+\infty} e^{-\rho t} e^{-b_0 t} \int_0^t e^{b_0 s} u(s) \, ds \, dt \right)^2 \\
= \frac{\rho}{(\rho+b_0)^2} \left( \int_0^{+\infty} e^{-\rho s} u(s) \, ds \right)^2
\]
(14)
and the last equality holds by (12).

The finiteness of the integrals in (9) implies that the application of Fubini’s Theorem in (12) and in (14) are appropriate.

Relation (14) allows us to write down the following estimate for \( B(u) \), using again
Jensen’s inequality (in relation with the concave function log):

\[
B(u) = \int_{0}^{+\infty} e^{-\rho t} \log(u(t)) \, dt - c \int_{0}^{+\infty} e^{-\rho t} x^2(t) \, dt
\]

\[
\leq \frac{1}{\rho} \log\left( \rho \int_{0}^{+\infty} e^{-\rho t} u(t) \, dt \right) - \frac{c}{\rho(\rho + b_0)^2} \left( \rho \int_{0}^{+\infty} e^{-\rho t} u(t) \, dt \right)^2
\]

(15)

\[
\leq \frac{1}{\rho} \max_{z > 0} \left( \log z - \frac{c}{(\rho + b_0)^2} z^2 \right) = \frac{1}{\rho} \left( \log \frac{\rho + b_0}{\sqrt{2c}} - \frac{1}{2} \right)
\]

(16)

\[
= \frac{1}{\rho} \log \left( \frac{\rho + b_0}{\sqrt{2c}} \right).
\]

(17)

This holds under condition (9). In the opposite case, that is to say \( \int_{0}^{+\infty} e^{-\rho t} u(t) \, dt = +\infty \), consider a sequence \( (u_n)_{n \in \mathbb{N}} \) like in Remark (6). Hence

\[
B(u) \leq \liminf_{n \to +\infty} B(u_n) \leq \liminf_{n \to +\infty} \frac{1}{\rho} \log\left( \rho \int_{0}^{+\infty} e^{-\rho t} u_n(t) \, dt \right) - \frac{c}{\rho(\rho + b_0)^2} \left( \rho \int_{0}^{+\infty} e^{-\rho t} u_n(t) \, dt \right)^2
\]

(18)

\[
= \lim_{z \to +\infty} \left( \frac{1}{\rho} \log z - \frac{c}{\rho(\rho + b_0)^2} z^2 \right) = -\infty,
\]

since \( \int_{0}^{+\infty} e^{-\rho t} u_n(t) \, dt \to \int_{0}^{+\infty} e^{-\rho t} u(t) \, dt \), by monotone convergence.

In the intermediate case, that is to say

\[
\int_{0}^{+\infty} e^{-\rho t} u(t) \, dt < +\infty, \quad \int_{0}^{+\infty} u(t) \, dt = +\infty,
\]

let again \( (u_n)_{n \in \mathbb{N}} \) be as in Remark (6). We have:

\[
B(u) \leq \liminf_{n \to +\infty} B(u_n) \leq \frac{1}{\rho} \log\left( \lim_{n \to +\infty} \rho \int_{0}^{+\infty} e^{-\rho t} u_n(t) \, dt \right) - \frac{c}{\rho(\rho + b_0)^2} \left( \lim_{n \to +\infty} \rho \int_{0}^{+\infty} e^{-\rho t} u_n(t) \, dt \right)^2
\]

\[
= \frac{1}{\rho} \log\left( \rho \int_{0}^{+\infty} e^{-\rho t} u(t) \, dt \right) - \frac{c}{\rho(\rho + b_0)^2} \left( \rho \int_{0}^{+\infty} e^{-\rho t} u(t) \, dt \right)^2
\]

\[
\leq \frac{1}{\rho} \log \left( \frac{\rho + b_0}{\sqrt{2c}} \right).
\]

(19)

Taking the sup among \( u \in \Lambda(x_0) \), we see that the same estimate holds for \( V(x_0) \).

ii) Suppose that \( u \) belongs to a maximizing sequence, and assume that \( B(u) >
We showed at point $i$ that if \( \int_0^{+\infty} e^{-\rho t} u(t) \, dt < +\infty \), then relation (15), holds. Thus in this case it must be

\[
\int_0^{+\infty} e^{-\rho t} u(t) \, dt \leq \frac{1}{\rho} \tilde{K}(x_0) =: K_1(x_0).
\]

The case \( \int_0^{+\infty} e^{-\rho t} u(t) \, dt = +\infty \) implies \( B(u) = -\infty \) by (18), and consequently must be excluded, since \( u \) belongs to a maximizing sequence (see Remark 5).

This proves relation (6).

In order to prove (7), observe that by (5) we have:

\[
\int_0^{+\infty} e^{-\rho t} x(t) \, dt \leq \int_0^{+\infty} e^{-\rho t} \left\{ e^{-bt} x_0 + \int_0^t e^{b(s-t)} (1+u(s)) \, ds \right\} \, dt
\]

\[
= x_0 \int_0^{+\infty} e^{-(\rho+b)t} \, dt + \int_0^{+\infty} e^{-(\rho+b)t} \int_0^t e^{bs} \, ds \, dt
\]

\[
+ \int_0^{+\infty} e^{-(\rho+b)t} \int_0^t e^{bs} u(s) \, ds \, dt
\]

\[
= \frac{x_0}{\rho+b} + \int_0^{+\infty} \int_s^{+\infty} e^{-(\rho+b)t} \, dt \, ds
\]

\[
+ \int_0^{+\infty} u(s) e^{bs} \int_s^{+\infty} e^{-(\rho+b)t} \, dt \, ds
\]

\[
= \frac{x_0}{\rho+b} + \frac{1}{\rho(\rho+b)} + \frac{1}{\rho+b} \int_0^{+\infty} e^{-\rho t} u(t) \, dt
\]

\[
\leq \frac{x_0}{\rho+b} + \frac{1}{\rho(\rho+b)} + \frac{K_1(x_0)}{\rho+b} =: K_2(x_0)
\]

3. Uniform localization lemmas.

**Lemma 9.** There exists a function \( N : [0, +\infty)^2 \to (0, +\infty) \), continuous and strictly increasing in the second variable, such that: for every \( x_0, T > 0 \) and for every \( u \in \Lambda(x_0) \) belonging to a maximizing sequence, there exists a control \( \tilde{u}^T \in \Lambda(x_0) \)
satisfying:

\[ B(x_0; \tilde{u}^T) \geq B(x_0; u) \]

\[ \tilde{u}^T = u \land N(x_0, T) \quad \text{a. e. in } [0, T]. \]

In particular, the norm \( \| \tilde{u}^T \|_{L^\infty([0,T])} \) is bounded above by a quantity which does not depend on the original control \( u \).

Moreover, the state \( x(\cdot; \tilde{u}^T, x_0) \) associated with the control \( \tilde{u}^T \) satisfies

\[ x(\cdot; \tilde{u}^T, x_0) \leq x(\cdot; u, x_0). \]

Eventually, the bound function \( N \) satisfies:

\[ \lim_{T \to +\infty} T e^{-\rho T} \log N(x_0, T) = 0. \]

Proof. Fix \( x_0 \) and \( T \geq 0 \). The equation

\[ \log \beta + \beta b_0 = -T b_0, \quad \beta > 0 \]

has a unique solution, which is strictly less than 1. Call this solution \( \beta_T \), and define

\[ N(x_0, T) := K(x_0) \beta_T^{-2} e^{2\rho(T+\beta_T)}, \]

where \( K(x_0) = K_1(x_0) \lor 1 \) and \( K_1(x_0) \) is the constant introduced in Proposition 8.

Now fix \( u \in \Lambda(x_0) \) such that \( u \) belongs to a maximizing sequence. If \( u \leq N(x_0, T) \) almost everywhere in \([0, T]\), then set \( \tilde{u}^T := u \), and the proof is over.

If there exists a non-negligible subset of \([0, T]\) in which \( u > N(x_0, T) \) then define

\[ \tilde{I} := \int_0^T (u(t) - u(t) \land N(x_0, T)) \, dt \]

\[ \tilde{u}^T := u \land N(x_0, T) \cdot \chi_{[0,T]} + (u + \tilde{I}) \cdot \chi_{(T,T+\beta_T]} + u \cdot \chi_{(T+\beta_T, +\infty)}. \]

Obviously \( \tilde{u}^T \in \Lambda(x_0) \), since \( u \in \Lambda(x_0) \) and \( N(x_0, T) > 0 \).

First we prove that

\[ 0 \leq x(\cdot; \tilde{u}^T, x_0) \leq x(\cdot; u, x_0) \quad \text{in } [0, +\infty) \]

Clearly \( x(\cdot; \tilde{u}^T, x_0) \geq 0 \), by the admissibility of \( \tilde{u}^T \). For simplicity of notation we set \( N = N(x_0, T), \tilde{x}_T = x(\cdot; \tilde{u}^T, x_0) \) and \( x = x(\cdot; u, x_0) \).

Obviously \( \ddot{x}_T \leq x \) in \([0, T]\), by Remark 4.

Fix \( t \in (T, T + \beta_T] \), and set \( h := h(\dot{x}_T, x) \), like in Remark 3. Hence:
\[ \tilde{x}_T(t) - x(t) = \int_0^T \exp \left( \int_s^t h \, d\tau \right) (u(s) \wedge N - u(s)) \, ds + \tilde{I} \int_T^t \exp \left( \int_s^t h \, d\tau \right) \, ds. \]

The first addend is estimated in the following way:

\[ \int_0^T \exp \left( \int_s^t h \, d\tau \right) (u(s) \wedge N - u(s)) \, ds \leq \int_0^T e^{(s-t)\beta_0} (u(s) \wedge N - u(s)) \, ds \]
\[ \leq e^{-t\beta_0} \int_0^T (u(s) \wedge N - u(s)) \, ds \]
\[ \leq e^{-(T+\beta)\beta_0} \int_0^T (u(s) \wedge N - u(s)) \, ds \]
\[ = -\tilde{I} e^{-(T+\beta)\beta_0}. \]

Since \( h \leq 0 \), the second addend is estimated from above by \( \tilde{I} \beta_T \).

Thus we obtain:

\[ \tilde{x}_T(t) - x(t) \leq \tilde{I} \left( \beta_T - e^{-(T+\beta_T)\beta_0} \right), \]

and the last quantity is zero, by definition of \( \beta_T \).

This implies that \( \tilde{x}_T \leq x \) also in \( (T + \beta_T, +\infty) \), again by Remark 4. Hence, relation (23) holds.

Now we estimate the "logarithmic" part of the difference between \( B(x_0; \tilde{u}^T) \) and \( B(x_0; u) \). By the concavity of log, we have:

\[ \int_0^{+\infty} e^{-\rho t} (\log \tilde{u}^T(t) - \log u(t)) \, dt \]
\[ = \int_0^T e^{-\rho t} \{ \log (u(t) \wedge N) - \log u(t) \} \, dt \]
\[ + \int_T^{T+\beta_T} e^{-\rho t} \left\{ \log \left( u(t) + \tilde{I} \right) - \log u(t) \right\} \, dt \]
\[ \geq \int_0^T e^{-\rho t} (u(t) \wedge N)^{-1} \{ u(t) \wedge N - u(t) \} \, dt \]
\[ + \tilde{I} \int_T^{T+\beta_T} e^{-\rho t} (u(t) + \tilde{I})^{-1} \, dt \]
\[
\frac{1}{N} \int_0^T e^{-\rho t} \{ u(t) \wedge N - u(t) \} \, dt
+ \int_T^{T+\beta T} e^{-\rho t} \left( u(t) + \tilde{I} \right)^{-1} \, dt
\geq \frac{1}{N} \int_0^T (u(t) \wedge N - u(t)) \, dt
+ \int_T^{T+\beta T} e^{-\rho t} \left( u(t) + \tilde{I} \right)^{-1} \, dt
= \tilde{I} \left( \int_T^{T+\beta T} e^{-\rho t} \left( u(t) + \tilde{I} \right)^{-1} \, dt - \frac{1}{N} \right).\]

Moreover, by Jensen’s inequality:

\[
\int_T^{T+\beta T} e^{-\rho t} \left( u(t) + \tilde{I} \right)^{-1} \, dt \geq e^{-\rho(T+\beta T)} \int_T^{T+\beta T} \left( u(t) + \tilde{I} \right)^{-1} \, dt
\geq \beta_T^2 e^{-\rho(T+\beta T)} \frac{1}{\int_T^{T+\beta T} \left( u(t) + \tilde{I} \right) \, dt}
\geq \beta_T^2 e^{-\rho(T+\beta T)} \frac{1}{\int_T^{T+\beta T} u(t) \, dt + \tilde{I}}
\geq \beta_T^2 e^{-\rho(T+\beta T)} \frac{1}{\int_0^{T+\beta T} u(t) \, dt}
\]

where the penultimate inequality holds since \( \beta_T < 1 \).

Now by Proposition 8 we can complete this estimate in the following way:

\[
\int_T^{T+\beta T} e^{-\rho t} \left( u(t) + \tilde{I} \right)^{-1} \, dt \geq K(x_0)^{-1} \beta_T^2 e^{-2\rho(T+\beta T)}
=: \alpha(x_0, T).
\]

Observe that, by definition, \( N(x_0, T) = \alpha(x_0, T)^{-1} \). Hence, joining (24) with (25) we obtain

\[
\int_0^{+\infty} e^{-\rho t} \{ \log \tilde{u}_T(t) - \log u(t) \} \, dt \geq \tilde{I} \left( \alpha(x_0, T) - \frac{1}{N(x_0, T)} \right) = 0.
\]

This implies, by (23):

\[
B(x_0; \tilde{u}_T) - B(x_0; u) = \int_0^{+\infty} e^{-\rho t} \{ \log \tilde{u}_T(t) - \log u(t) \} \, dt
- c \int_0^{+\infty} e^{-\rho t} \{ \tilde{x}_T(t) - x^2(t) \} \, dt
\geq 0.
\]
Finally we prove the monotonicity of $N(x_0, T)$ in $T$.

First observe that $T \rightarrow \beta_T$ is clearly a strictly decreasing function, since the function $\beta \rightarrow \log \beta + \beta b_0$ is strictly increasing, and remembering equation (21).

Moreover, the function $T \rightarrow T + \beta_T$ is strictly increasing. Indeed, set $f(x) := \log x + b_0x$ and let $\phi$ be the inverse of $f$. Then $\beta_T = \phi(-Tb_0)$, and:

$$
\frac{d}{dT} (T + \beta_T) = 1 - b_0 \phi'(-Tk) = 1 - \frac{b_0}{f'(\beta_T)} = 1 - \frac{b_0\beta_T}{1 + b_0\beta_T} > 0.
$$

This shows that $N(x_0, \cdot)$ is strictly increasing.

Finally observe that:

\begin{equation}
\beta_T \sim e^{-Tb_0} \quad \text{for} \quad T \rightarrow +\infty.
\end{equation}

Indeed, with $f$ defined as before, we have:

$$
\lim_{x \to 0^+} \frac{f(x)}{\log x} = 1.
$$

Hence $\phi(y) \sim e^y$ for $y \to -\infty$ and $\beta_T = \phi(-Tb_0) \sim e^{-Tb_0}$ for $T \to +\infty$.

It follows from (27) and (22), that:

$$
Te^{-\rho T} \log N(x_0, T) = Te^{-\rho T} \log K(x_0) + Te^{-\rho T} \log (\beta_T^{-2})
$$

$$
+ 2\rho Te^{-\rho T} (T + \beta_T)
$$

$$
\sim T e^{-\rho T} \log (\beta_T^{-2})
$$

$$
\sim 2T^2 e^{-\rho T} b_0 \quad \text{for} \quad T \to +\infty.
$$

This shows that (20) holds.

\[ \square \]

**Lemma 10.** There exists a function $\eta : [0, +\infty)^2 \to (0, +\infty)$, continuous and strictly decreasing in the second variable, with the following property:

i) $\eta(x_0, T) < N(x_0, T) \quad \forall T > 0$

where $N$ is the function defined in Lemma 9;

ii) for every $x_0 \geq 0$ and every $T \geq 1$, if $u \in \Lambda(x_0)$ belongs to a maximizing sequence, there exists $u^T \in \Lambda(x_0)$ such that

$$
B(x_0; u^T) \geq B(x_0; u)
$$

$$
u^T = (u \land N(x_0, T)) \lor \eta(x_0, T) \quad a. e. \text{ in } [0, T].
$$
In particular the norm $\|\log u^T\|_{L^\infty([0,T])}$ is bounded above by a quantity which does not depend on $u$.

**Proof.** Fix $x_0$ and $u$ as in the hypothesis, and set $x := x(\cdot; x_0, u)$. In order to define the function $\eta$, we preliminarily observe that there obviously exits a number $L(x_0) > \rho$ such that

$$e^{L(x_0) - \rho} - 2c\rho^{-1}e^{-L(x_0)} \geq 2cK_2(x_0).$$

A simple computation shows that the function $T \to e^{(L(x_0) - \rho)T} - 2c\rho^{-1}Te^{-L(x_0)T}$ is increasing if

$$L(x_0) > \rho + \frac{2c}{\rho},$$

Now we now choose $L(x_0)$ satisfying (28) and (29) and we define

$$\eta(x_0, T) := e^{-L(x_0)T}.$$

Relation $i)$ follows from the fact that $N(x_0, T) > 1$; moreover we have:

$$e^{(L(x_0) - \rho)T} - 2c\rho^{-1}Te^{-L(x_0)T} - 2cK_2(x_0) \geq 0 \quad \forall T \geq 1.$$

Now fix $T \geq 1$ and take $\tilde{u}^T$ as in Lemma 9. Define $u^T := \tilde{u}^T$ if $\tilde{u}^T \geq \eta(x_0, T)$ almost everywhere in $[0, T]$, and

$$u^T := (\tilde{u}^T \vee \eta(x_0, T)) \chi_{[0,T]} + \tilde{u}^T \chi_{(T, +\infty)}$$

if there exists a subset of $[0, T]$ of positive measure where $\tilde{u}^T < \eta(x_0, T)$. In this case define also

$$I := \int_0^T \left[ \tilde{u}^T(s) \vee \eta - \tilde{u}^T(s) \right] ds.$$

We show that

$$\mathcal{B}(x_0; u^T) - \mathcal{B}(x_0; \tilde{u}^T) \geq 0,$$

and the conclusion will follow from Lemma 9.

We provide two different estimates of the quantity $x(\cdot; x_0, u_T) - x(\cdot; x_0, \tilde{u}_T)$. Set $x_T = x(\cdot; x_0, u_T)$, $\tilde{x}_T = x(\cdot; x_0, \tilde{u}_T)$, $h = h(x_T, \tilde{x}_T)$, $\eta = \eta(x_0, T)$ and $N = N(x_0, T)$.
for simplicity of notation. Remembering that $h \leq 0$, we have, for every $t \in [0, T]$:

$$x_T(t) - \tilde{x}_T(t) = \int_0^t e^{\int_s^t h\, ds} [u^T(s) - \tilde{u}^T(s)] \, ds$$

$$\leq \int_0^T e^{\int_s^t h\, ds} [\tilde{u}^T(s) \vee \eta - \tilde{u}^T(s)] \, ds$$

$$\leq I.$$

The same estimate holds for $t > T$, since $u^T = \tilde{u}^T$ in $(T, +\infty)$. Hence:

$$x_T - \tilde{x}_T \leq I \quad \text{in } [0, +\infty). \quad (31)$$

Moreover, since $\eta > 0$:

$$I = \int_0^T [\tilde{u}^T(s) \vee \eta - \tilde{u}^T(s)] \, ds$$

$$= \int_{[0, T] \cap \{\tilde{u}^T \leq \eta\}} [\eta - \tilde{u}^T(s)] \, ds$$

$$\leq T \eta.$$

Hence

$$x_T - \tilde{x}_T \leq T \eta \quad \text{in } [0, +\infty). \quad (32)$$

By (31) and (32), using the convexity relation $x^2 - y^2 \leq 2x(y - x)$, we obtain:

$$c \int_0^{+\infty} e^{-\rho t} [x_T^2(t) - \tilde{x}_T^2(t)] \, dt \leq 2c \int_0^{+\infty} e^{-\rho t} x_T(t) [x_T(t) - \tilde{x}_T(t)] \, dt$$

$$\leq 2cI \int_0^{+\infty} e^{-\rho t} x_T(t) \, dt$$

$$= 2cI \int_0^{+\infty} e^{-\rho t} [x_T(t) - \tilde{x}_T(t)] \, dt$$

$$+ 2cI \int_0^{+\infty} e^{-\rho t} \tilde{x}_T(t) \, dt$$

$$\leq 2cIT \eta \int_0^{+\infty} e^{-\rho t} \, dt + 2cI \int_0^{+\infty} e^{-\rho t} x(t) \, dt$$

$$\leq I \left( 2\frac{c}{\rho} T \eta + 2cK_2(x_0) \right),$$

where we also used (23) and (7) (the trajectory $x(\cdot)$ is associated with a control in a maximizing sequence).
Moreover:

$$\int_0^{+\infty} e^{-\rho t} \left( \log u^T(t) - \log \tilde{u}^T(t) \right) dt = \int_0^T e^{-\rho t} \left( \log (u^T(t) \vee \eta) - \log \tilde{u}^T(t) \right) dt$$

$$\geq \int_0^T e^{-\rho t} \frac{1}{\tilde{u}^T(t) \vee \eta} \left( \tilde{u}^T(t) \vee \eta - \tilde{u}^T(t) \right) dt$$

$$= \frac{1}{\eta} \int_0^T e^{-\rho t} \left( \tilde{u}^T(t) \vee \eta - \tilde{u}^T(t) \right) dt$$

$$\geq \frac{e^{-\rho T}}{\eta} I.$$ 

Joining the last two estimates leads to:

$$B \left( x_0; u^T \right) - B \left( x_0; \tilde{u}^T \right) = \int_0^{+\infty} e^{-\rho t} \left( \log u^T(t) - \log \tilde{u}^T(t) \right) dt$$

$$- c \int_0^{+\infty} e^{-\rho t} \left[ x_1^2(t) - \tilde{x}_1^2(t) \right] dt$$

$$\geq I \left( \frac{e^{-\rho T}}{\eta(x_0, T)} - 2c \frac{T}{\rho} \eta(x_0, T) - 2cK_2(x_0) \right)$$

$$= I \left( e^{(L(x_0)-\rho)T} - 2c \rho^{-1} T e^{-L(x_0)T} - 2cK_2(x_0) \right)$$

$$\geq 0,$$

where the last inequality holds by (30).

\[ \square \]

4. Diagonal procedures and functional convergence. From this point on, the initial state \( x_0 \geq 0 \) is to be considered fixed.

Lemma 11. There exists a sequence \( (v_n)_{n\in\mathbb{N}} \) and a function \( v \) in \( \Lambda \left( x_0 \right) \) such that:

\begin{align*}
\lim_{n \to +\infty} B \left( x_0; v_n \right) &= V \left( x_0 \right) \\
\lim_{n \to +\infty} v_n &= v \text{ in } L^1 \left( [0, T] \right) \quad \forall T > 0 \\\n\forall T \in \mathbb{N} &:\text{almost everywhere in } [0, T] : \forall n \geq T : \eta \left( x_0, T \right) \leq v, v_n \leq N \left( x_0, T \right)
\end{align*}

where \( N, \eta \) are the functions defined in Lemmas 9 and 10 .

Proof. Set \( B = B \left( x_0; \cdot \right) \) and fix \( (u_n)_{n\in\mathbb{N}} \) and such that

$$\lim_{n \to +\infty} B \left( u_n \right) = V \left( x_0 \right).$$

Set, for every \( n \in \mathbb{N} \), \( u^1_n \) as the function obtained by applying Lemma 10 to \( u_n \),
for $T = 1$. Then
\[
 u^1_n = (u_n \wedge N(x_0, 1)) \vee \eta(x_0, 1) \quad \text{a.e. in } [0, 1] \\
\mathcal{B}(u^1_n) \geq \mathcal{B}(u_n).
\]

Hence, as a consequence of the Dunford-Pettis criterion, there exists a subsequence $(u^1_n)_n$ of $(u_n)_n$ and a function $u^1 \in L^1([0,1])$ such that
\[
\mathbf{u}^1_n \rightharpoonup u^1 \text{ in } L^1([0,1]).
\]

Now apply Lemma 10 to the elements of the sequence $(u^1_n)_n$ in order to obtain a sequence $(u^2_n)_n$ satisfying, for every $n \in \mathbb{N}$:
\[
 u^2_n = (\mathbf{u}^1_n \wedge N(x_0, 2)) \vee \eta(x_0, 2) \quad \text{a.e. in } [0, 2] \\
\mathcal{B}(u^2_n) \geq \mathcal{B}(u^1_n).
\]

Take, again by Dunford-Pettis, $(\mathbf{u}^2_n)_n$ extracted from $(u^2_n)_n$ and a function $u^2 \in L^1([0,2])$ such that
\[
\mathbf{u}^2_n \rightharpoonup u^2 \text{ in } L^1([0,2]).
\]

Iterating this process we define families $(\mathbf{u}^T_n)_n$, $(u^T_n)_n$, $\sigma_T$ ($T \in \mathbb{N}$) such that the $\sigma_T$’s are strictly increasing with $\sigma_T \geq 1d$, satisfying for every $T, n \in \mathbb{N}$:
\[
\mathbf{u}^T_n = u^T_{\sigma_T(n)} \\
\mathbf{u}^T_n = (\mathbf{u}^{T-1}_n \wedge N(x_0, T)) \vee \eta(x_0, T) \quad \text{a.e. in } [0, T] \\
\mathcal{B}(u^T_n) \geq \mathcal{B}(\mathbf{u}^{T-1}_n) \\
\mathbf{u}^T_n \rightharpoonup u^T \text{ in } L^1([0,T]).
\]

Fix $T \in \mathbb{N}$. The sequence $(\mathbf{u}^T_n)_n$ coincides, almost everywhere in $[0, T-1]$, with a sequence that is extracted from $(\mathbf{u}^{T-1}_n)_n$. Indeed, for every $n \in \mathbb{N}$:
\[
\mathbf{u}^T_n = u^T_{\sigma_T(n)} \quad \text{a.e. in } [0,T] \\
\mathbf{u}^T_n \overset{a.e. in [0,T]}{=} \mathbf{u}^{T-1}_{\sigma_T(n)} \wedge N(x_0, T) \vee \eta(x_0, T) \\
\mathbf{u}^T_n \overset{a.e. in [0,T-1]}{=} \mathbf{u}^{T-1}_{\sigma_T(n)}.
\]

The last equality holds since applying recursively (in $T$) relation (37) together with relation (36) gives $\mathbf{u}^{T-1}_{\sigma_T(n)} \in [\eta(x_0, T-1), N(x_0, T-1)]$; then observe that by Lemmas 9 and 10 the function $\eta(x_0, \cdot)$ is decreasing and the function $N(x_0, \cdot)$ is increasing. Hence $u^{T-1} = u^T$ almost everywhere in $[0, T-1]$, by the essential uniqueness of the weak limit.
Hence, defining
\[ \forall t \geq 0 : v(t) := u^{[t]+1}(t) \]
we obtain \( v = u^T \) almost everywhere in \([0, T]\) and
\[ (40) \quad \forall T \in \mathbb{N} : \pi_n^T \to v \text{ in } L^1[0, T]. \]

Repeating the previous argument, we see that for every \( T, n \in \mathbb{N} \):
\[
\begin{align*}
\eta_n &\text{ a.e. in } [0, T-1] \quad \pi_{T-1}^{\sigma_T(n)} \\
\eta_n &\text{ a.e. in } [0, T-2] \quad \pi_{T-2}^{\sigma_{T-1} \circ \sigma_T(n)} \\
&\vdots \\
\eta_n &\text{ a.e. in } [0, T-j] \quad \pi_{T-j}^{\sigma_{T-j+1} \circ \cdots \circ \sigma_T(n)}.
\end{align*}
\]
Observe that \( \left( \pi_{T-j}^{\sigma_{T-j+1} \circ \cdots \circ \sigma_T(n)} \right)_n \) is a subsequence of \( \left( \pi_{n}^{T-j} \right)_n \) since the composition \( \sigma_{T-j+1} \circ \cdots \circ \sigma_T \) is strictly increasing and satisfies
\[ \sigma_{T-j+1} \circ \cdots \circ \sigma_T(n) \geq n \quad \forall n \in \mathbb{N}. \]

Hence, inverting the quantifiers “\( \forall n \in \mathbb{N} \)” and “a.e. in \([0, T-j]\)”, we see that \( \left( \pi_{n}^{T-j} \right)_n \) coincides, almost everywhere in \([0, T-j]\) with a subsequence of \( \left( \pi_{n}^{T-j} \right)_n \), for every \( T \in \mathbb{N} \) and \( j = 1, \ldots, T-1 \).

This implies that for every \( T \in \mathbb{N} \) the sequence \( (v_n)_{n \geq T} \) defined by \( v_n := \pi_n^T \) coincides with a subsequence of \( \left( \pi_{n}^{T-j} \right)_n \), almost everywhere in \([0, T]\). Hence
\[ (41) \quad \forall T \in \mathbb{N} : \text{almost everywhere in } [0, T] : \]
\[ \forall n \geq T : \eta(x_0, T) \leq v_n \leq N(x_0, T). \]

and
\[ v_n \to v \text{ in } L^1([0, T]) \quad \forall T \in \mathbb{N}, \]
by (37) and (40).

The extension to every \( T > 0 \) is straightforward, so we obtain (34). Now fix \( T > 0 \); a well known property of the weak convergence implies that
\[ (42) \quad \liminf_{n \to +\infty} v_n(t) \leq v(t) \leq \limsup_{n \to +\infty} v_n(t) \text{ for almost every } t \in [0, T]. \]

Considering the intersection between the subsets of \([0, T]\) where relations (41) and (42) hold, we obtain (35).
In order to prove (33), observe that
\[ B(v_n) = B\left(u_{\sigma_n(n)}^n\right) \geq B\left(\pi_{\sigma_n(n)}^{n-1}\right) \]
\[ = B\left(u_{\sigma_{n-1}\circ\sigma_n(n)}^{n-1}\right) \geq \cdots \geq B\left(u_{\sigma_1\circ\cdots\circ\sigma_n(n)}^1\right) \]
\[ \geq \cdots \geq B\left(\sigma_{1\circ\cdots\circ\sigma_n(n)}^1\right). \]

Fix \( \epsilon > 0 \) and \( n_\epsilon \in \mathbb{N} \) such that \( V(x_0) - B(u_n) < \epsilon \) for \( n \geq n_\epsilon \); since \( \sigma_1 \circ \cdots \circ \sigma_m \geq Id \), we have
\[ V(x_0) - B(v_n) < \epsilon \quad \forall n \geq n_\epsilon. \]

**Proposition 12.** Let \( v_n (n \in \mathbb{N}) \) and \( v \) be as in Proposition 11, and let \( x_n := x(\cdot; x_0, v_n) \) and \( x := x(\cdot; x_0, v) \) be the associated trajectories starting at \( x_0 \). Then
\[ x_n \to x \text{ pointwise in } [0, \infty). \]

**Proof.** Fix \( T > 0 \). By (35) in Proposition 11 and by Remark 4, \( v \) is admissible and the following uniform estimate holds:
\[ |x - x_n| \leq x(\cdot; x_0, N(x_0, T)) \text{ in } [0, T], \forall n \in \mathbb{N}. \]

Now fix \( t \in [0, T] \) and \( n \in \mathbb{N} \). Subtracting the state equation for \( x \) from the state equation for \( x_n \), we obtain, for every \( s \in [0, t] \):
\[ x_n'(s) - \dot{x}(s) = F(x_n(s)) - F(x(s)) + v_n(s) - v(s) \]
\[ = h_n(s)[x_n(s) - x(s)] + v_n(s) - v(s), \]
where \( h_n := h(x_n, x) \) is the function defined in Remark 3.

Integrating both sides of this equation between 0 and \( t \), then taking absolute values leads to:
\[ |x_n(t) - x(t)| \leq \int_0^t |h_n(s)| |x_n(s) - x(s)| \, ds + \int_0^t [v_n(s) - v(s)] \, ds. \]

Observe that, for every \( s \in [0, t] \):
\[ |h_n(s)| |x_n(s) - x(s)| \leq b_0 x(s; x_0, N(x_0, T)), \]
by Remark 3 and by (43).

Since the function on the right hand side obviously belongs to \( L^1([0, t]) \), passing
to the limsup in (44) and remembering (34), we obtain by Dominated Convergence:

\[
\limsup_{n \to +\infty} |x_n(t) - x(t)| \leq \limsup_{n \to +\infty} \int_0^t |h_n(s)| \cdot |x_n(s) - x(s)| \, ds
\]

(45)

\[
= \int_0^t \limsup_{n \to +\infty} |h_n(s)| \cdot |x_n(s) - x(s)| \, ds
\]

\[
\leq b_0 \int_0^t \limsup_{n \to +\infty} |x_n(s) - x(s)| \, ds.
\]

Hence by Gronwall’s inequality:

\[
\limsup_{n \to +\infty} |x_n(t) - x(t)| = 0,
\]

for every \( t \in [0, T] \). This is equivalent to

\[
\lim_{n \to +\infty} x_n = x \quad \text{in } [0, T],
\]

which proves the thesis, since \( T > 0 \) is generic.

Lemma 13. Take \((v_n)_{n \in \mathbb{N}}\) and \(v\) as in Lemma 11. There exists a sequence

\((v_{n,n})_{n \in \mathbb{N}},\) extracted from \((v_n)_{n \in \mathbb{N}},\) and a function \(u_* \in \Lambda(x_0),\) satisfying, for every \(T > 0:\)

\[
\log v_{n,n} \rightharpoonup \log u_* \quad \text{in } L^1([0, T])
\]

(46)

\[
\eta(x_0, T) \leq u_* \leq N(x_0, T) \quad \text{a.e. in } [0, T].
\]

(47)

\[
0 \leq x(\cdot; x_0, u_*) \leq x(\cdot; x_0, v) \quad \text{in } [0, +\infty).
\]

(48)

Proof. We conduct “standard” diagonalization on the sequence \((\log v_n)_{n \in \mathbb{N}}\). Observe that this sequence, by (35), is also uniformly bounded in the \(L^\infty_{[0,1]}\) norm. Precisely, for any \(n \in \mathbb{N}:\)

\[
\log \eta(x_0, 1) \leq \log v_n \leq \log N(x_0, 1) \quad \text{a.e. in } [0, 1].
\]

Hence by the Dunford-Pettis criterion there exists a function \(f^1 \in L^1([0,1])\) and a sequence \((v_{n,1})_n\) extracted form \((v_n)\) such that

\[
\log v_{n,1} \rightharpoonup f^1 \quad \text{in } L^1([0,1]).
\]
Again by (35), \((v,1)\)_n satisfies, for every \(n \in \mathbb{N}\):

\[
\log \eta (x_0, 2) \leq \log v_{n,1} \leq \log N (x_0, 2) \quad \text{a.e. in } [0, 2];
\]

therefore there exist \(f^2 \in L^1 ([0, 2])\) and \((v,2)_n\) extracted from \((v,1)_n\) such that

\[
\log v_{n,2} \to f^2 \quad \text{in } L^1 ([0, 2]),
\]

and so on. This shows that there exists a function \(f \in L^1_{loc} ([0, +\infty))\) satisfying, together with the diagonal sequence \((v_{n,n})_n\), for every \(T > 0\):

\[
\log v_{n,n} \to f \quad \text{in } L^1 ([0, T])
\]

\[
\log \eta (x_0, T) \leq \log v_{n,n} \leq \log N (x_0, T) \quad \text{a.e. in } [0, T], \forall n \geq T.
\]

Define \(u_* := e^f\); then relations (46) and (47) are easy consequences of this definition and of the properties of the weak convergence.

In order to prove (48), we first observe that, obviously, \(x (\cdot; x_0, u_*) \geq 0\). Fix \(0 < t_0 < t_1 < T\) and let \(t_0\) be a Lebesgue point for both \(\log u_*\) and \(v\). By Jensen’s inequality we have, for every \(n \in \mathbb{N}\):

\[
\frac{\int_{t_0}^{t_1} \log v_{n,n} (s) \, ds}{t_1 - t_0} \leq \log \left( \frac{\int_{t_0}^{t_1} v_{n,n} (s) \, ds}{t_1 - t_0} \right);
\]

since \((v_{n,n})_n\) is a subsequence of \((v_n)_n\), passing to the limit for \(n \to +\infty\) in the previous relation, we obtain by (34) and (46):

\[
\frac{\int_{t_0}^{t_1} \log u_* (s) \, ds}{t_1 - t_0} \leq \log \left( \frac{\int_{t_0}^{t_1} v (s) \, ds}{t_1 - t_0} \right).
\]

Passing now to the limit for \(t_1 \to t_0\) yields to \(\log u_* (t_0) \leq \log v (t_0)\). By the Lebesgue Point Theorem, \(t_0\) is a generic element of a full measure subset of \([0, T]\). This implies (48), by Remark 4.

A simple integration by parts provides the following decomposition of the objec-
tive functional:

$$\forall u \in \Lambda (x_0) : \mathcal{B} (x_0; u) = \int_0^{+\infty} e^{-pt} (\log u (t) - cx^2 (t)) \, dt$$

$$= \int_0^{+\infty} e^{-pt} \log u (t) \, dt - c \int_0^{+\infty} e^{-pt} x^2 (t) \, dt$$

$$= \lim_{T \to +\infty} e^{-pT} \int_0^T \log u (s) \, ds +$$

$$\rho \int_0^{+\infty} e^{-pt} \left( \int_0^t \log u (s) \, ds - \frac{c}{\rho} x^2 (t) \right) \, dt$$

$$=: \lim_{T \to +\infty} e^{-pT} \int_0^T \log u (t) \, dt + \mathcal{B}_1 (x_0; u)$$

where

$$\mathcal{B}_1 (x_0; u) := \rho \int_0^{+\infty} e^{-pt} \left( \int_0^t \log u (s) \, ds - \frac{c}{\rho} x^2 (t; x_0, u) \right) \, dt.$$  

With this notation, we prove the final step.

**Corollary 14.** The control $u_*$ defined in Lemma 13 is optimal at $x_0$, and

$$u_* \in L_\text{loc}^\infty ([0, +\infty)).$$

**Proof.** Obviously $u_* \in L_\text{loc}^\infty ([0, +\infty))$, by (47). Observe that, by Jensen’s inequality and by Proposition 8, for every $n \in \mathbb{N}$ and $t > 0$:

$$e^{-pt} \int_0^t \log v_{n,n} (s) \, ds \leq te^{-pt} \log \left( \frac{\int_0^t v_{n,n} (s) \, ds}{t} \right)$$

$$\leq te^{-pt} \log (K (x_0) e^{pt}) - te^{-pt} \log (t).$$  

(49)

This implies that $\lim_{t \to +\infty} e^{-pt} \int_0^t \log v_{n,n} (s) \, ds \leq 0$ and consequently

$$\mathcal{B} (x_0; v_{n,n}) \leq \mathcal{B}_1 (x_0; v_{n,n}).$$  

(50)

Moreover

$$\int_0^{+\infty} \left( te^{-pt} \log (K (x_0) e^{pt}) - te^{-pt} \log (t) \right) \, dt$$

$$\leq \int_0^1 te^{-pt} \log (K (x_0) e^{pt}) \, dt - \int_0^1 te^{-pt} \log (t) \, dt$$

$$+ \int_1^{+\infty} te^{-pt} \log (K (x_0) e^{pt}) \, dt < +\infty.$$  

(51)
Set \( x_{n,n} := x (\cdot; x_0, v_{n,n}), x := (\cdot; x_0, v) \) and \( x_* := (\cdot; x_0, u_*) \). Relations (49) and (51) imply that the hypotheses of Lemma 15 are satisfied for the integral

\[
\int_0^\infty e^{-\rho t} \left( \int_0^t \log v_{n,n} (s) \, ds - \frac{c}{\rho} x_{n,n}^2 (t) \right) \, dt.
\]

Combining this result with relations (50),(46), (48) and with Proposition 12 we obtain:

\[
V (x_0) = \lim_{n \to +\infty} B (x_0; v_{n,n}) \leq \lim_{n \to +\infty} B_1 (x_0; v_{n,n}) \\
= \rho \lim_{n \to +\infty} \int_0^\infty e^{-\rho t} \left( \int_0^t \log v_{n,n} (s) \, ds - \frac{c}{\rho} x_{n,n}^2 (t) \right) \, dt \\
\leq \rho \int_0^\infty e^{-\rho t} \limsup_{n \to +\infty} \left( \int_0^t \log v_{n,n} (s) \, ds - \frac{c}{\rho} x_{n,n}^2 (t) \right) \, dt \\
= \rho \int_0^\infty e^{-\rho t} \left( \int_0^t \log u_* (s) \, ds - \frac{c}{\rho} x_*^2 (t) \right) \, dt \\
\leq \rho \int_0^\infty e^{-\rho t} \left( \int_0^t \log u_* (s) \, ds - \frac{c}{\rho} x_*^2 (t) \right) \, dt \\
= B_1 (x_0; u_*).
\]

Finally observe that by (47), for every \( t \geq 0 \):

\[
te^{-\rho t} \log \eta (x_0, t + 1) \leq e^{-\rho t} \int_0^t \log u_* (s) \, ds \leq te^{-\rho t} \log N (x_0, t + 1),
\]

which implies that the estimated quantity vanishes for \( t \to +\infty \), since \( \eta (x_0, t) = e^{-L(x_0)t} \) and by (20).

Hence \( B_1 (x_0; u_*) = B (x_0; u_*) \), and this concludes the proof. \( \square \)

Appendix.

Lemma 15. Let \((E, \sigma, \mu)\) a measure space, \(f_n (n \in \mathbb{N})\) and \(g\) \(\mu\)-measurable functions in \(E\), \(F \subseteq E\) a full measure set such that:

\[
\forall n \in \mathbb{N}: f_n \leq g \quad \text{in} \quad F \\
\int_E g \, d\mu < +\infty.
\]

Then

\[
\limsup_{n \to +\infty} \int_E f_n \, d\mu \leq \int_E \limsup_{n \to +\infty} f_n \, d\mu.
\]
\textbf{Proof.} \textbf{Case I.} $\int_E g \, d\mu = -\infty$. Then

$$\limsup_{n \to +\infty} \int_E f_n \, d\mu = -\infty$$

and the thesis is trivially true.

\textbf{Case II.} $\int_E g \, d\mu \in (-\infty, +\infty)$

The sequence

$$a_n := g - \sup_{k \geq n} f_k$$

satisfies

$$0 \leq a_n \uparrow g - \limsup_{m \to +\infty} f_m \quad \text{in} \; F.$$ 

Hence by Monotone convergence:

\begin{equation}
\int_E \left( g - \sup_{k \geq n} f_k \right) \, d\mu = \int_E a_n \, d\mu \uparrow \int_E \left( g - \limsup_{m \to +\infty} f_m \right) \, d\mu.
\end{equation}

Observe that the quantities

$$\int_E \left( -\sup_{k \geq n} f_k \right) \, d\mu := \int_E \left( g - \sup_{k \geq n} f_k \right) \, d\mu - \int_E g \, d\mu$$

$$\int_E \left( -\limsup_{m \to +\infty} f_m \right) \, d\mu := \int_E \left( g - \limsup_{m \to +\infty} f_m \right) \, d\mu - \int_E g \, d\mu$$

make sense and belong to $(-\infty, +\infty]$. It follows from (52) that:

\begin{equation}
\lim_{n \to +\infty} \int_E \left( -\sup_{k \geq n} f_k \right) \, d\mu = \int_E \left( -\limsup_{m \to +\infty} f_m \right) \, d\mu.
\end{equation}

Indeed, if $\int_E \left( -\sup_{k \geq n_0} f_k \right) \, d\mu = +\infty$ for some $n_0 \in \mathbb{N}$, then both

$$\lim_{n \to +\infty} \int_E \left( -\sup_{k \geq n} f_k \right) \, d\mu$$

and

$$\int_E \left( -\limsup_{m \to +\infty} f_m \right) \, d\mu$$

are $+\infty$. If $\int_E \left( -\sup_{k \geq n} f_k \right) \, d\mu < +\infty$ for every $n \in \mathbb{N}$ and

$$\int_E \left( -\limsup_{m \to +\infty} f_m \right) \, d\mu < +\infty$$

then clearly (53) follows from (52), whilst in case

$$\int_E \left( -\limsup_{m \to +\infty} f_m \right) \, d\mu = +\infty$$

we have

$$+\infty = \int_E \left( g - \limsup_{m \to +\infty} f_m \right) \, d\mu = \lim_{n \to +\infty} \int_E \left( g - \sup_{k \geq n} f_k \right) \, d\mu$$

$$= \int_E g \, d\mu + \lim_{n \to +\infty} \int_E \left( -\sup_{k \geq n} f_k \right) \, d\mu$$

which implies

$$\lim_{n \to +\infty} \int_E \left( -\sup_{k \geq n} f_k \right) \, d\mu = +\infty.$$
It follows from (53) that

$$\inf_{n \in \mathbb{N}} \int_E \sup_{k \geq n} f_k d\mu = \int_E \limsup_{m \to +\infty} f_m d\mu.$$ 

Moreover, it is a consequence of the definition of sup that

$$\limsup_{m \to +\infty} \int_E f_m d\mu \leq \inf_{n \in \mathbb{N}} \int_E \sup_{k \geq n} f_k d\mu.$$ 

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