Dynamical-decoupling-protected nonadiabatic holonomic quantum computation

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The main obstacles to the realization of high-fidelity quantum gates are the control errors arising from inaccurate manipulation of a quantum system and the decoherence caused by the interaction between the quantum system and its environment. Nonadiabatic holonomic quantum computation allows for high-speed implementation of whole-geometric quantum gates, making quantum computation robust against control errors. Dynamical decoupling provides an effective method to protect quantum gates against environment-induced decoherence, regardless of collective decoherence or independent decoherence. In this paper, we put forward a protocol of nonadiabatic holonomic quantum computation protected by dynamical decoupling. Due to the combination of nonadiabatic holonomic quantum computation and dynamical decoupling, our protocol not only possesses the intrinsic robustness against control errors but also protects quantum gates against environment-induced decoherence.

I. INTRODUCTION

Quantum computation provides an effective solution to certain problems, such as factoring large integers [1] and searching unsorted data [2]. The implementation of circuit-based quantum computation relies on the ability to realize a universal set of high-fidelity quantum gates, including arbitrary one-qubit gates and a nontrivial two-qubit gate [3]. However, there are two main obstacles to the realization of high-fidelity quantum gates. One is the control errors arising from inaccurate manipulation of quantum systems. The other one is the decoherence caused by the interaction between the quantum system and its environment. Geometric phases are only dependent on evolution paths of quantum systems but independent of the evolution details, and therefore quantum computation based on geometric phases is robust against control errors.

In 1984, Berry found that a quantum system in a nondegenerate eigenstate undergoing adiabatic and cyclic evolution can acquire a geometric phase in addition to a dynamical phase [4]. The notion of geometric phases was then extended to quantum systems in degenerate eigenstates [5], in nonadiabatic evolution [6, 7], and in mixed states [8–10]. Until now, pure-state geometric phases have been used to realize quantum computation while mixed-state geometric phases have not. The early proposals [11–13] of geometric quantum computation are based on adiabatic Abelian geometric phases [4] and adiabatic non-Abelian geometric phases [5]. However, these proposals require quantum systems to undergo adiabatic evolution, which makes quantum systems evolve for a long time. To resolve this problem, nonadiabatic geometric quantum computation [14, 15] based on nonadiabatic Abelian geometric phases [6] and nonadiabatic holonomic quantum computation [16, 17] based on nonadiabatic non-Abelian geometric phases [7] were proposed. Compared with nonadiabatic geometric quantum computation that uses the geometric phase as one parameter of a quantum gate, nonadiabatic holonomic quantum computation uses the holonomic matrix itself as a quantum gate. This makes nonadiabatic holonomic quantum computation possess a whole-geometric property. Due to the merits of both geometric robustness and high-speed implementation without the limit of adiabatic evolution, nonadiabatic holonomic quantum computation has received increasing attention.

The first protocol of nonadiabatic holonomic quantum computation is based on a three-level quantum system driven by two resonant laser pulses [16]. It needs to combine two one-qubit gates to realize an arbitrary one-qubit gate. To simplify the operations, the single-shot protocol of nonadiabatic holonomic quantum computation [18, 19] and the single-loop protocol of nonadiabatic holonomic quantum computation [20] were proposed. The improved protocols allow us to realize an arbitrary one-qubit gate by a single-shot implementation and thus reduce the exposure time of nonadiabatic holonomic gates to error sources. To further shorten the exposure time of quantum gates to error sources, the path-shortening protocol of nonadiabatic holonomic quantum computation was put forward [21], where nonadiabatic holonomic gates can be realized based on a class of extended evolution paths that are shorter than the former ones. The key to realizing nonadiabatic holonomic quantum computation based on these protocols is to find the Hamiltonians that make the quantum system satisfy both the cyclic evolution condition and the parallel transport condition. Recently, a general approach of constructing Hamiltonians for nonadiabatic holonomic quantum computation was put forward [22]. By using this approach, one can easily find a Hamiltonian making the quantum system evolve along a desired path so that nonadiabatic holonomic gates can be realized with an economical evolution time. Up to now, a lot of works both in theories [23–38] and experiments [39–52] have contributed to nonadiabatic holonomic quantum computation.

While some protocols tried to reduce the influence of decoherence by shorting the exposure time of quantum gates, another line of protecting quantum gates against decoherence is to use decoherence-mitigation methods. To make quantum gates robust against both control errors and decoherence, the combination of nonadiabatic holonomic quantum computation and decoherence-mitigation methods is a promising strategy. The first protocol of nonadiabatic holonomic quan-
tum computation in decoherence-free subspaces was put forward in Ref. [17]. Afterwards, a number of alternative protocols [53–56] and physical implementation schemes [57–61] were put forward. These proposals are based on the system-environment interaction being with some symmetry and they focus mainly on protecting nonadiabatic holonomic gates against collective decoherence, especially the collective dephasing.

In this paper, we propose a protocol of nonadiabatic holonomic quantum computation protected by dynamical decoupling. The decoherence-mitigation method used here is dynamical decoupling [62], and therefore there is no need to require system-environment interaction to have some symmetry. Due to the combination of nonadiabatic holonomic quantum computation and dynamical decoupling, our protocol not only possesses the intrinsic robustness against control errors but also protects quantum gates against environment-induced decoherence, regardless of collective decoherence or independent decoherence.

II. PHYSICAL MODEL

Before proceeding further, we briefly review the basic idea of nonadiabatic holonomic quantum computation [16, 17]. Consider an N-dimensional quantum system exposed to the Hamiltonian $H(t)$. Assume there is an $L$-dimensional subspace $S(t) = \text{Span}\{|\phi_k(t)\rangle\}_{k=1}^L$, where $|\phi_k(t)\rangle$ are orthonormal basis vectors and satisfy the Schrödinger equation $i\partial_t|\phi(t)\rangle = H(t)|\phi(t)\rangle$. If $|\phi_k(t)\rangle$ satisfy the following conditions

(i) $\sum_{k=1}^L \langle \phi_l(t) | \phi_k(t) \rangle = \sum_{k=1}^L | \phi_l(0) \rangle \langle \phi_k(0) | = 0, \quad k, l = 1, \ldots, L,$

then the unitary operator $U(\tau)$ with $|\phi_k(\tau)\rangle = U(\tau)|\phi_k(0)\rangle$ is a nonadiabatic holonomic gate acting on $S(0)$. Here, $\tau$ is the evolution period.

Let us now elucidate our physical model. We consider a quantum system consisting of $N$ physical qubits, which interact though the well-known XXZ coupling [63–68]. The Hamiltonian reads

$$H = \sum_{k<l} \left[ J_{kl}^{x} (\sigma_{kx}^{l} \sigma_{lx}^{l} + \sigma_{lx}^{l} \sigma_{kx}^{l}) + J_{kl}^{y} \sigma_{ky}^{l} \sigma_{ly}^{l} \right],$$

where $J_{kl}^{x}$ and $J_{kl}^{y}$ are the real-valued controllable coupling parameters and $\sigma_{m}^{\alpha}$ represent the Pauli $\alpha$ operators ($\alpha = x, y, z$) acting on the $m$th qubit ($m = k, l$). For the quantum system considered here, we assume that each physical qubit interacts independently with its environment. The interaction Hamiltonian reads

$$H_I = \sum_{k,m} \sigma_{k}^{\alpha} \otimes B_{k}^{\alpha},$$

where $B_{k}^{\alpha}$ is the environment operator corresponding to the system operator $\sigma_{k}^{\alpha}$. If $B_{k}^{\alpha}$ is independent of the qubit index $k$, the environment-induced decoherence reduces to collective decoherence. In particular, if the system operator is further taken as $\sigma_{z}^{l}$, then the collective decoherence yields collective dephasing. To protect nonadiabatic holonomic gates against collective dephasing, nonadiabatic holonomic quantum computation in decoherence-free subspaces was proposed [17], and to protect nonadiabatic holonomic gates against collective decoherence, nonadiabatic holonomic quantum computation in noiseless systems was proposed [54]. For the more complicated decoherence induced by the interaction $H_I$ in Eq. (3), dynamical decoupling provides an effective method to protect nonadiabatic holonomic gates against decoherence.

Dynamical decoupling operates by applying a periodic sequence of fast and strong symmetrizing pulses to quantum systems to suppress the effect of undesired system-environment interaction. For the system-environment interaction in Eq. (3), we can use a periodic sequence with the decoupling operations $[\otimes_{k=1}^{N} I_{k}, \otimes_{k=1}^{N} \sigma_{z}^{k}, \otimes_{k=1}^{N} \sigma_{x}^{k}, \otimes_{k=1}^{N} \sigma_{y}^{k}]$ to suppress its effect. The corresponding unitary operator over a period of time reads

$$U_I = \left[ \left( \otimes_{k=1}^{N} \sigma_{x}^{k} \right) e^{-iH_I \tau} \left( \otimes_{k=1}^{N} \sigma_{x}^{k} \right) \right] \left[ \left( \otimes_{k=1}^{N} \sigma_{y}^{k} \right) e^{-iH_I \tau} \left( \otimes_{k=1}^{N} \sigma_{y}^{k} \right) \right] \left[ \left( \otimes_{k=1}^{N} \sigma_{z}^{k} \right) e^{-iH_I \tau} \left( \otimes_{k=1}^{N} \sigma_{z}^{k} \right) \right] \left[ \left( \otimes_{k=1}^{N} \sigma_{y}^{k} \right) e^{iH_I \tau} \left( \otimes_{k=1}^{N} \sigma_{y}^{k} \right) \right] \left[ \left( \otimes_{k=1}^{N} \sigma_{z}^{k} \right) e^{iH_I \tau} \left( \otimes_{k=1}^{N} \sigma_{z}^{k} \right) \right]$$

where $\tau$ is the duration time of pulse intervals and $I_{k}$ is the identity operator acting on the $k$th qubit. This result indicates that up to the first-order term $O(\tau)$, the system-environment interaction can be completely eliminated by using a decoupling pulse sequence.

To realize dynamical-decoupling-protected nonadiabatic holonomic quantum computation, the decoupling pulse sequence needs to be inserted into the native dynamical evo-
lution of the quantum system. Therefore, we need to properly choose the Hamiltonian that not only makes the decoupling pulse sequence compatible with the dynamical evolution but also keeps the cyclic evolution condition as well as the parallel transport condition valid. To this end, we choose the Hamiltonian in Eq. (2), which commutes with the decoupling operations. In this case, we can realize the desired evolution protected by dynamical decoupling.

### III. IMPLEMENTATION

To perform dynamical-decoupling-protected nonadiabatic holonomic quantum computation, we need to realize a universal set of quantum gates, including arbitrary one-qubit gates and a nontrivial two-qubit gate.

First, we realize an arbitrary one-qubit gate. To complete our realization, we utilize three physical qubits to encode a logical qubit. The specific encoding is

\[
|0\rangle_L = |001\rangle, \quad |1\rangle_L = |010\rangle.
\]

Meanwhile, we use \(|a\rangle = |100\rangle\) as an auxiliary state. In this case, we can apply the periodic sequence with decoupling operations \(\{\sigma^x_{k=1} I_k, \sigma^y_{k=1} I_k, \sigma^z_{k=1} I_k, \sigma^x_{k=1} \sigma^y_{k=1} I_k, \sigma^y_{k=1} \sigma^z_{k=1} I_k, \sigma^z_{k=1} \sigma^x_{k=1} I_k\}\) to the quantum system to protect quantum information against decoherence.

To realize nonadiabatic holonomic gates, we set the nonzero parameters of the Hamiltonian in Eq. (2) to be

\[
J_{12}^2 = -J(t) \cos \phi \frac{\theta_1}{2}, \quad J_{13}^3 = \frac{J(t)}{2} \cos \phi \frac{\theta_1}{2}, \quad J_{23}^1 = J(t) \sin \phi_1,
\]

where \(J(t)\) is a time-dependent parameter, and \(\phi_1\) and \(\theta_1\) are time-independent parameters. In this case, the Hamiltonian reads

\[
H_1(t) = \frac{J(t)}{2} \cos \phi_1 \left[ -\cos \frac{\theta_1}{2} (\sigma^x_1 \sigma^y_2 + \sigma^y_1 \sigma^x_2) + \sin \frac{\theta_1}{2} (\sigma^x_1 \sigma^y_3 + \sigma^y_1 \sigma^x_3) \right] + J(t) \sin \phi_1 \sigma^z_2 \sigma^z_3.
\]

By using the basis \(|0\rangle_L, |1\rangle_L, |a\rangle\), this Hamiltonian can be recast as

\[
H_1(t) = \frac{J(t)}{2} \cos \phi_1 \left[ \sin \frac{\theta_1}{2} |a\rangle_L \langle 0| - \cos \frac{\theta_1}{2} |a\rangle_L \langle 1| + \text{H.c.} \right] + J(t) \sin \phi_1 (|a\rangle \langle a| - |0\rangle_L \langle 0| - |1\rangle_L \langle 1|),
\]

which can be further rewritten as

\[
H_1(t) = \frac{J(t)}{2} \cos \phi_1 \left[ \sin \frac{\theta_1}{2} |a\rangle_L \langle 0| - \cos \frac{\theta_1}{2} |a\rangle_L \langle 1| + \text{H.c.} \right] + 2J(t) \sin \phi_1 (|a\rangle \langle a| - |0\rangle_L \langle 0| + |1\rangle_L \langle 1|).
\]

It is noteworthy that \(|a\rangle \langle a| + |0\rangle_L \langle 0| + |1\rangle_L \langle 1|\) is an identity operator and thus \(-J(t) \sin \phi_1 (|a\rangle \langle a| + |0\rangle_L \langle 0| + |1\rangle_L \langle 1|)\) can only generate a global phase during evolution. This global phase does not affect the quantum gates and therefore the terms \(-J(t) \sin \phi_1 (|a\rangle \langle a| + |0\rangle_L \langle 0| + |1\rangle_L \langle 1|)\) in Eq. (9) can be ignored. If we introduce two orthonormal states,

\[
|d\rangle = \cos \frac{\theta_1}{2} |0\rangle_L + \sin \frac{\theta_1}{2} |1\rangle_L,
\]

\[
|b\rangle = \sin \frac{\theta_1}{2} |0\rangle_L - \cos \frac{\theta_1}{2} |1\rangle_L,
\]

the Hamiltonian then reads

\[
H_1(t) = J(t) \cos \phi_1 (|a\rangle \langle b| + |b\rangle \langle a|) + 2J(t) \sin \phi_1 |a\rangle \langle a|.
\]

The evolution operator corresponding to the above Hamiltonian can be written as \(U_1(t) = \exp [-i \int_0^t H_1(t')dt']\), which can be explicitly expressed as

\[
U_1(t) = |d\rangle \langle d| + e^{-i \int_0^t J(t')dt'} \sin \phi_1 (|a\rangle \langle b| + |b\rangle \langle a|) + e^{-i \int_0^t J(t')dt'} |b\rangle \langle b|.
\]

If the evolution time \(T\) is taken to satisfy

\[
\int_0^T J(t)dt = \pi,
\]

then the evolution operator is reduced to

\[
U_1(T) = |d\rangle \langle d| + e^{i \pi \sin \phi_1} |b\rangle \langle b| + e^{-i \pi \sin \phi_1} |a\rangle \langle a|.
\]

From Eqs. (12) and (14), we can see that a state initially prepared in the computational space \(S_1 = \text{Span}(|0\rangle_L, |1\rangle_L, |a\rangle\) will evolve outside \(S_1\) during \(t \in (0, T)\) and then return back to \(S_1\) at \(t = T\). Thus, the cyclic evolution condition (i) is satisfied. By using the commutation relation \([H_1(t), U_1(t)] = 0\), we can verify that \(\{\phi(t)|H_1(t)|\phi(t)\} = \{\phi(t)|H_1(t)|\phi(t)\}, \langle U_1(t)|\phi(t)\rangle = \langle \phi(t)|H_1(t)|\phi(t)\rangle = 0\), where \(|\phi(t)\rangle\) is an evolution state such that \(|\phi(t)\rangle = U_1(t)|\phi(t)\rangle\) with \(|\phi(t)\rangle \in S_1\). It indicates that the parallel transport condition (ii) is satisfied. Therefore, \(U_1(T)\) is a holonomic transformation. Acting on the computational space \(S_1\), the evolution operator \(U_1(T)\) is equivalent to

\[
U_1 = |d\rangle \langle d| + e^{-i \pi \sin \phi_1} |b\rangle \langle b|,
\]

which plays the role of a nonadiabatic holonomic gate.

In the following, we demonstrate how to use dynamical decoupling to protect the dynamical evolution for the realization of nonadiabatic holonomic gates. We assume that the quantum system is coupled to its environment with the total Hamiltonian \(H(t) = H_1(t) + H_E + H_I\), where \(H_1(t)\) is the system Hamiltonian in Eq. (7), \(H_E\) is the environment Hamiltonian, and \(H_I\) is the interaction Hamiltonian in Eq. (3). If the decoupling operations \(\{\sigma^x_{k=1} I_k, \sigma^y_{k=1} I_k, \sigma^z_{k=1} I_k\}\) are applied
Finally, we demonstrate that an arbitrary one-qubit gate can be realized by using the nonadiabatic holonomic gate $U_1$. One can see that $U_1$ can be rewritten as

$$U_1 = e^{i\gamma_1/2} e^{-i\gamma_2 (\sin \theta_1 X + \cos \theta_1 Z)/2},$$

where $\gamma_1 = -(\pi + \pi \sin \phi_1)$, and $X$ and $Z$ are the Pauli $x$ and $z$ operator acting on $|0\rangle_L$ and $|1\rangle_L$, respectively. Ignoring a trivial global phase, we can obviously see that $U_1$ is a quantum gate with a rotation axis in the $x-z$ plane and the rotation angle $\gamma_1$. An arbitrary one-qubit can be realized by combining two such quantum gates about parallel axes in the plane. For example, $U_1$ is reduced to the quantum gate about the $x$ axis by setting $\theta_1 = \pi/2$ and about the $z$ axis by setting $\theta_1 = 0$. By combining these noncommuting one-qubit gates, an arbitrary one-qubit can be realized.

Second, we realize a nontrivial two-qubit gate. To this end, we use six physical qubits to encode two logical qubits. To make the two-logical-qubit encoding compatible with the one-logical-qubit encoding, we encode two-logical-qubit states as $|00\rangle_L = |001001\rangle$, $|01\rangle_L = |001010\rangle$, $|10\rangle_L = |010001\rangle$, $|11\rangle_L = |010100\rangle$. Meanwhile, we use $|a_1\rangle = |011000\rangle$ and $|a_2\rangle = |000011\rangle$ as auxiliary states. In this case, we can utilize decoupling operations $[\sigma^3_k, \sigma^3_k \otimes \sigma^3_k, \cdots, \sigma^3_k \otimes \sigma^3_k, \cdots, \sigma^3_k] \otimes \sigma^3_k$ to suppress the effect of the undesired system-environment interaction in Eq. (3).

To realize nonadiabatic holonomic gates, we take the nonzero parameters of the Hamiltonian in Eq. (2) as

$$J_{23}^x = -\frac{J_2(t)}{2} \cos \phi_2 \cos \theta_2,$$

$$J_{26}^x = \frac{J_2(t)}{2} \cos \phi_2 \sin \theta_2,$$

where $J_2(t)$ is time dependent, and $\phi_2$ and $\theta_2$ are time independent. In this case, the Hamiltonian can be expressed as

$$H_2(t) = \frac{J_2(t)}{2} \cos \phi_2 \left[ - \cos \frac{\theta_2}{2} (\sigma^2_{z} \sigma^3_{y} + \sigma^3_{z} \sigma^2_{y}) + \sin \frac{\theta_2}{2} (\sigma^2_{z} \sigma^3_{y} + \sigma^3_{z} \sigma^2_{y}) \right] + J_2(t) \sin \phi_2 \sigma^2_{z} \sigma^3_{y}.$$

By using the basis $\{|00\rangle_L, |01\rangle_L, |10\rangle_L, |11\rangle_L, |a_1\rangle, |a_2\rangle\}$, this Hamiltonian can be recast as

$$H_2(t) = J_2(t) \cos \phi_2 \left[ (\sin \frac{\theta_2}{2} |a_1\rangle_2 |00\rangle_L - \cos \frac{\theta_2}{2} |a_1\rangle_2 |01\rangle_L + \text{H.c.}) 
- \left( \cos \frac{\theta_2}{2} |a_2\rangle_2 |10\rangle_L - \sin \frac{\theta_2}{2} |a_2\rangle_2 |11\rangle_L + \text{H.c.} \right) 
+ J_2(t) \sin \phi_2 (|a_1\rangle_2 |a_1\rangle_2 |00\rangle_L |00\rangle_L - |01\rangle_2 |01\rangle_L) 
+ (|a_2\rangle_2 |a_2\rangle_2 |10\rangle_L |10\rangle_L - |11\rangle_2 |11\rangle_L), \right],$$

which can be further recast as

$$H_2(t) = J_2(t) \cos \phi_2 \left[ (\sin \frac{\theta_2}{2} |a_1\rangle_2 |00\rangle_L - \cos \frac{\theta_2}{2} |a_1\rangle_2 |01\rangle_L + \text{H.c.}) 
- \left( \cos \frac{\theta_2}{2} |a_2\rangle_2 |10\rangle_L - \sin \frac{\theta_2}{2} |a_2\rangle_2 |11\rangle_L + \text{H.c.} \right) 
+ 2 J_2(t) \sin \phi_2 (|a_1\rangle_2 |a_1\rangle_2 + |a_2\rangle_2 |a_2\rangle_2) 
- J_2(t) \sin \phi_2 (|a_1\rangle_2 |a_1\rangle_2 + |a_2\rangle_2 |a_2\rangle_2) + |00\rangle_L |00\rangle_L 
+ |01\rangle_2 |01\rangle_2 + |10\rangle_2 |10\rangle_2 + |11\rangle_2 |11\rangle_2).$$
It is noteworthy that $|a_1\rangle\langle a_1| + |a_2\rangle\langle a_2| + \ket{00}_{LL}(\ket{00}_{LL}(\ket{01}_{LL}(0) + |11\rangle_{LL}(11)}$ is an identity operator and thus $-J_2(t)\sin\phi_2(|a_1\rangle\langle a_1| + |a_2\rangle\langle a_2| + \ket{00}_{LL}(\ket{00}_{LL}(\ket{01}_{LL}(0) + |10\rangle_{LL}(10) + |11\rangle_{LL}(11))$ can only generate a global phase during evolution. This global phase does not affect the quantum gates and therefore the terms $-J_2(t)\sin\phi_2(|a_1\rangle\langle a_1| + |a_2\rangle\langle a_2| + \ket{00}_{LL}(\ket{00}_{LL}(\ket{01}_{LL}(0) + |10\rangle_{LL}(10) + |11\rangle_{LL}(11))$ in Eq. (24) can be ignored. If we introduce four orthonormal states,

\[
\begin{align*}
|d_1\rangle &= \cos\frac{\theta_2}{2}|00\rangle_L + \sin\frac{\theta_2}{2}|01\rangle_L,
|b_1\rangle &= \sin\frac{\theta_2}{2}|00\rangle_L - \cos\frac{\theta_2}{2}|01\rangle_L,
|d_2\rangle &= \sin\frac{\theta_2}{2}|10\rangle_L + \cos\frac{\theta_2}{2}|11\rangle_L,
|b_2\rangle &= \cos\frac{\theta_2}{2}|10\rangle_L - \sin\frac{\theta_2}{2}|11\rangle_L.
\end{align*}
\]

the Hamiltonian can be further written as

\[
H_2(t) = J_2(t) \cos\phi_2(|a_1\rangle\langle a_1| + |b_1\rangle\langle b_1|) + 2J_2(t)\sin\phi_2|a_1\rangle\langle a_1| - J_2(t) \cos\phi_2(|a_2\rangle\langle a_2| + |b_2\rangle\langle b_2|) + 2J_2(t)\sin\phi_2|a_2\rangle\langle a_2|.
\]

The evolution operator corresponding to this Hamiltonian then reads $U_2(t) = \exp[-i \int_0^T H_2(t')dt']$, which can be explicitly expressed as

\[
U_2(t) = |d_1\rangle\langle d_1| + e^{-i \int_0^T J_2(t')dt' \sin\phi_2(|a_1\rangle\langle a_1| + |b_1\rangle\langle b_1|)}
\times e^{-i \int_0^T J_2(t')dt' \sin\phi_2(|a_1\rangle\langle a_1| + |b_1\rangle\langle b_1|)}
\times |d_2\rangle\langle d_2| + e^{-i \int_0^T J_2(t')dt' \sin\phi_2(|a_2\rangle\langle a_2| + |b_2\rangle\langle b_2|)}
\times e^{-i \int_0^T J_2(t')dt' \sin\phi_2(|a_2\rangle\langle a_2| + |b_2\rangle\langle b_2|)}.
\]

If the evolution period $T$ is taken to satisfy

\[
\int_0^T J_2(t)dt = \pi,
\]

the evolution operator is reduced to

\[
U_2(T) = |d_1\rangle\langle d_1| + e^{-i(\pi + \pi \sin\phi_2)}|b_1\rangle\langle b_1|
+ |d_2\rangle\langle d_2| + e^{-i(\pi + \pi \sin\phi_2)}|b_2\rangle\langle b_2|
+ e^{-i(\pi + \pi \sin\phi_2)}|a_1\rangle\langle a_1| + e^{-i(\pi + \pi \sin\phi_2)}|a_2\rangle\langle a_2|.
\]
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