GEOMETRIC PREQUANTIZATION OF A MODIFIED
SEIBERG-WITTEN MODULI SPACE IN 2 DIMENSIONS

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Abstract. In this paper we consider a dimensional reduction of slightly modified Seiberg-Witten equations, the modification being a different choice of the Pauli matrices which go into defining the equations. We get interesting equations with a Higgs field, spinors and a connection. We show interesting solutions of these equations. Then we go on to show a family of symplectic structures on the moduli space of these equations which can be geometrically prequantized using the Quillen determinant line bundle.

1. Introduction

It is important to study the dimensional reductions of gauge theories for they sometimes possess beautiful symplectic or hyperKähler structures which can be geometrically quantised. [13], [14]. It is hoped that these Hilbert spaces of the quantizations could be used to produce invariants of 3 or 4 dimensional manifolds as in perhaps [42], [19], or perhaps could be used in Gromov-Witten theory [41].

In this paper we modify the Seiberg-Witten equations in \( \mathbb{R}^4 \) by choosing different \( I, J \) and \( K \) from the standard one and dimensionally reduce the equations to \( \mathbb{R}^2 \) and then finally patch them on the Riemann surface, much in the way it was done in [15] though the equations look different from those in [15]. Then we show that the moduli space is non-empty and in fact there are interesting solutions. Then we show there is a family of symplectic structures and geometrically prequantize them along the ideas of [13] and [14].

We must mention that in [15] and [16], the author had attempted to geometrically quantize the dimensional reduction of the Seiberg-Witten equations with a Higgs field. But there are some mistakes in these two papers which are to be rectified.

The present paper could be thought of as a modification and extension of the work done in [15] and [16].

Geometric prequantization has been described in the introductions of [13] and [14]. Let us just mention that it involves constructing a line bundle on the moduli space whose curvature is a symplectic form on the moduli space. In fact, we get a family of symplectic forms, parametrised by \( \psi_0 \), a section of a line bundle, which are quantised this way. For each of them we construct a Quillen determinant line bundle whose curvature is that symplectic form. Note that topologically all these line bundles are equivalent, since their Chern class is integral and does not vary. However holomorphically they may be distinct.

The equations in the Hitchin system involved a connection \( A \) and a Higgs field \( \Phi \). In the vortex equation, a connection \( A \) and one other field \( \Psi \) appeared. The equations we are dealing with in this paper are more complex and involve a connection \( A \), a Higgs field \( \Phi \) and two other fields \( \psi_1 \) and \( \psi_2 \). It would be interesting to
find an algebraic geometric interpretation of the latter moduli space like that of the Hitchin systems and the vortex moduli space. Also, as in the case of Chern-Simons theory and flat connections, [42], it would be interesting to find a Lagrangian theory in 3-dimensions whose quantization will lead naturally to the prequantization described in this paper. The Hilbert space of the quantization of the moduli space of flat connections turned out to be the space of conformal blocks in a certain conformal field theory, [32]. One could also try to answer the analogous question in the theory and flat connections, [42], it would be interesting to find a Lagrangian theory.

In this section we dimensionally reduce the modified Seiberg-Witten equations on \( \mathbb{R}^4 \) to \( \mathbb{R}^2 \) and define them over a compact Riemann surface \( M \).

### 2. Dimensional Reductions of the Seiberg-Witten Equations

In this section we dimensionally reduce the modified Seiberg-Witten equations on \( \mathbb{R}^4 \) to \( \mathbb{R}^2 \) and define them over a compact Riemann surface \( M \).

#### 2.1. The Seiberg-Witten equations on \( \mathbb{R}^4 \):

This is a brief description of the Seiberg-Witten equations on \( \mathbb{R}^4 \), [39], [1], [30]. Identify \( \mathbb{R}^4 \) with the quaternions \( \mathbb{H} \) (coordinates \( x = (x_1, x_2, x_3, x_4) \) identified with \( \zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \)) and let \( \{e_i, i = 1, 2, 3, 4\} \) be a basis for \( \mathbb{H} \). Fix the constant spin structure \( \Gamma : \mathbb{H} = T_\mathbb{H} \rightarrow \mathbb{C}^{4 \times 4} \), given by \( \Gamma(\zeta) = \begin{bmatrix} 0 & \gamma(\zeta)^* \\ \gamma(\zeta) & 0 \end{bmatrix} \), where

\[
\gamma(\zeta) = \begin{bmatrix} \zeta_1 - i\zeta_2 & \zeta_3 - i\zeta_4 \\ -\zeta_3 + i\zeta_4 & \zeta_1 - i\zeta_2 \end{bmatrix}.
\]

Thus \( \gamma(e_1) = Id, \gamma(e_2) = i, \gamma(e_3) = J, \gamma(e_4) = K \). Where

\[
I = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}, J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, K = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.
\]

Note: The choice of \( I, J, K \) is not standard. They don't satisfy the quaternionic algebra. This is our point of deviation from the Seiberg-Witten theory. For the standard choice see [15].

Recall that \( Spin^c(\mathbb{R}^4) = (Spin(\mathbb{R}^4) \times S^1)/\mathbb{Z}_2 \). Since \( Spin(\mathbb{R}^4) \) is a double cover of \( SO(4) \), a \( spin^c \) - connection involves a connection \( \omega \) on \( T_\mathbb{H} \) and a connection \( A = i \sum_{j=1}^4 A_j dx_j \in \Omega^1(\mathbb{T}, i\mathbb{R}) \) on the characteristic line bundle \( \mathbb{H} \times \mathbb{C} \) which arises from the \( S^1 \) factor (see [39], [30], [1] for more details). We set \( \omega = 0 \), which is equivalent to choosing the covariant derivative on the trivial tangent bundle to be \( d \). This is legitimate since we are on \( \mathbb{R}^4 \). The curvature 2-form of the connection \( A \) is given by \( F(A) = dA \in \Omega^2(\mathbb{H}, i\mathbb{R}) \). Consider the covariant derivative acting on \( \Psi \in C^\infty(\mathbb{H}, \mathbb{C}^4) \) (the positive spinor on \( \mathbb{R}^4 \)) induced by the connection \( A \) on \( \mathbb{H} \times \mathbb{C} : \nabla \Psi = (\frac{\partial}{\partial x_j} + iA_j)\Psi \). Then according to [39], the Seiberg-Witten equations for \( (A, \Psi) \) on \( \mathbb{R}^4 \) are equivalent to the equations:

\[
\begin{align*}
(SW1) & : \nabla_1 \Psi = i\nabla_2 \Psi + J\nabla_3 \Psi + K\nabla_4 \Psi, \\
(SW2a) & : F_{12} + F_{34} = \frac{i}{2} \Psi^* I \Psi = \frac{i}{2} (|\psi_1|^2 + |\psi_2|^2) = \frac{i}{2} \eta_1, \\
(SW2b) & : F_{13} + F_{24} = \frac{i}{2} \Psi^* J \Psi = -i(Im \psi_1 \psi_2) = \frac{i}{2} \eta_2, \\
(SW2c) & : F_{14} + F_{23} = \frac{i}{2} \Psi^* K \Psi = -Im \psi_1 \psi_2 = \frac{i}{2} \eta_3
\end{align*}
\]

where \( \Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \), where by our convention \( F_{12} = i(\partial_2 A_1 - \partial_1 A_2) \) etc.
2.2. Dimensional Reduction to \( \mathbb{R}^2 \): Using the same method of dimensional reduction as in \([23]\), we get the general form of the reduced equations which contain the so-called Higgs field. Namely, impose the condition that none of the \( A_i \)'s and \( \Psi \) in \((SW1)\) and \((SW2)\) depend on \( x_3 \) and \( x_4 \), i.e. \( A_i = A_i(x_1, x_2) \), \( \Psi = \Psi(x_1, x_2) \) and set \( \phi_1 = -iA_3 \) and \( \phi_2 = -iA_4 \). The \((SW2)\) equations reduce to the following system on \( \mathbb{R}^2 \), \( F_{12} = \frac{1}{2} \eta_1 \), and two other equations which is as follows: \( \frac{\partial i\phi_1 - i\phi_2}{\partial z} = \frac{i}{2}(\eta_2 - i\eta_3) = 0 \), where \( F_{12} = \frac{1}{2}(\frac{\partial}{\partial x_2} - i\frac{\partial}{\partial y_2}). \) This is because \( 0 = F_{13} + F_{14} - i(F_{14} + F_{31}) = \partial_1 \phi_1 - \partial_2 \phi_2 - i(\partial_1 \phi_2 + \partial_2 \phi_1) = (\partial_1 - i\partial_2)(\phi_1 - i\phi_2).

Setting \( \phi_1 - i\phi_2 = \phi \) and recalling \( dx_2 \wedge dx_1 = -idz \wedge dz \bar{z} \) we rewrite the reduction of \((SW2)\) as the following two equations,

\[
\begin{align*}
(1) \quad F(A) &= -\frac{i}{2} |\psi_1|^2 + |\psi_2|^2 \wedge dx_1 \\
&= \frac{i}{2} |\psi_1|^2 + |\psi_2|^2 \wedge dz \wedge d\bar{z},
\end{align*}
\]

where \( \Phi = \Phi^{1,0} + \Phi^{0,1} = \phi dz - \phi d\bar{z} \in \Omega^1(\mathbb{R}^2, i\mathbb{R}) \) and \( \psi_1, \psi_2 \in C^\infty(\mathbb{R}^2, \mathbb{C}) \) are spinors on \( \mathbb{R}^2 \). Next consider the Dirac equation \((SW1)\):

\[
\nabla_1 \psi - i\nabla_2 \bar{\psi} - J(3\psi) - K \nabla_4 \psi = 0
\]

which is rewritten as

\[
\begin{bmatrix}
\partial_1 + iA_1 + \frac{\partial}{\partial y_2} - A_2 & -iA_3 - A_4 \\
\frac{\partial}{\partial x_2} + iA_1 + \frac{\partial}{\partial y_2} - A_2
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\psi_2
\end{bmatrix} = 0.
\]

Introducing \( A^{1,0} = \frac{1}{2}(A_1 - iA_2)dz \) and \( A^{0,1} = \frac{1}{2}(A_1 + iA_2)d\bar{z} \) where the total connection \( A^{1,0} + A^{0,1} = i(A_1 dx + A_2 dy) \), we can finally write it as

\[
\begin{bmatrix}
2(\bar{\psi} + A^{0,1}) & \bar{\phi} d\bar{z} \\
-\phi dz & 2(\bar{\psi} + A^{1,0})
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\psi_2
\end{bmatrix} = 0
\]

We call equations \((1)-(3)\) as the dimensionally reduced Seiberg-Witten equations over \( \mathbb{C} \).

2.3. The Dimensionally Reduced Equations on a Riemann surface. Let \( M \) be a compact Riemann surface of genus \( g \) with a conformal metric \( ds^2 = h^2 dz \wedge d\bar{z} \) and let \( \omega = ie^{2\sigma} h dz \wedge d\bar{z} \) be a real form. Let \( L \) be a line bundle with a Hermitian metric \( H \). Let \( \psi_1, \psi_2 \) be sections of the line bundle \( L \) i.e., \( \psi_1 \in \Gamma(M, L) \) and \( \psi_2 \in \Gamma(M, \bar{L}) \). \( L \) has a Hermitian metric \( H \) and thus we can define an inner product between two sections \( \psi \) and \( \tau \) as follows: \( \langle \psi, \tau \rangle = \langle e, e \rangle \) where \( e \) is a section of \( L \) then \( \langle \psi, \tau \rangle = f \bar{\psi} < e, e \rangle \in C^\infty(M) \). By abuse of notation we write \( \langle \psi, \tau \rangle = \psi H \tau \). This inner product will come in handy when defining the determinant line bundles. The norm \( |\psi|_H \in C^\infty(M) \). Let \( A^{1,0} + A^{0,1} \) be a unitary connection on \( L \), i.e. \( A^{1,0} = -A^{0,1} \), and \( \Phi = \Phi^{1,0} + \Phi^{0,1} = \phi dz - \phi d\bar{z} \in \Omega^1(M, i\mathbb{R}) \).

We will assume that \( \Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \) is not identically zero. We can rewrite the equations \((1)-(3)\) in an invariant form on \( M \) as follows:

\[
(1) \quad F(A) = i \frac{(|\psi_1|^2_H + |\psi_2|^2_H)}{2} \omega,
\]

\[
(2) \quad \bar{\psi} - \phi dz = 0,
\]

\[
(3) \quad F(A) = i \frac{(|\psi_1|^2_H + |\psi_2|^2_H)}{2} \omega.
\]
\[
\begin{bmatrix}
\bar{\partial} + A^{0,1} \\
-\frac{1}{2} \partial \partial \bar{\partial} \\
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\
\psi_2 \\
\end{bmatrix}
= 0.
\]

Let \( C = A \times \Gamma(M, L \oplus L) \times \mathcal{H} \), where \( A \) is the space of connections on a line bundle \( L, \Gamma(M, L \oplus L) \) the space of sections of the bundle \( L \oplus L \) and \( \mathcal{H} \) be \( \Omega^1(M, i\mathbb{R}) \), the space of Higgs fields. Then \( (A, \Psi = [\psi_1 \bar{\psi}_2], \Phi) \in C \). The gauge group \( G \) which is locally Maps \((M, U(1)) \) acts on \( B \) as \( (A, \Psi, \Phi) \mapsto (A + u^{-1}du, u^{-1}\Psi, \Phi) \) and leaves the space of solutions to (2.1) – (2.3) invariant. There are no fixed points of this action. Because a fixed point would mean that there is a connection \( A_0 \) such that \( A_0 + u^{-1}du = A_0 \) for all \( u \) in the gauge group. This is not possible. We assume throughout that \( \Psi \) is not identically zero. Note that we let \( \sigma \) also vary.

By taking quotient of the space of solutions by the gauge group we get the moduli space \( N \).

**Proposition 2.1.** The moduli space \( N \) is not empty for a compact (oriented) Riemann surface of genus \( g > 1 \).

**Proof.** Let us take the line bundle \( L \) to be the tangent bundle of a compact (oriented) Riemann surface of genus \( g > 1 \). We take the connection \( A = A^{1,0} + A^{0,1} = \partial \ln(e^\sigma h) - \bar{\partial} \ln(e^\sigma h) \), (page 77). Let us take \( \Phi = 0 \). The second equation is solved naturally. The third equation becomes

\[
(\bar{\partial} + A^{0,1})\bar{\psi}_2 = 0
\]

where \( A^{0,1} = \bar{\partial} \ln(e^\sigma h) \), etc. These two equations imply

\[
\bar{\partial} \ln(e^\sigma h \psi_1) = 0
\]

\[
\bar{\partial} \ln(e^\sigma h \bar{\psi}_2) = 0
\]

or in otherwords, \( \ln(e^\sigma h \psi_1) = f(z) \) and \( \ln(e^\sigma h \bar{\psi}_2) = g(z) \). Thus we get a whole family of solutions

\[
\psi_1 = e^{f(z)} e^{-\sigma h^{-1}}
\]

\[
\bar{\psi}_2 = e^{g(z)} e^{-\sigma h^{-1}}
\]

Next the first equation becomes

\[
F(A) = dA = -\frac{1}{2} \Delta \ln(e^\sigma h) dz \wedge d\bar{z} = K(e^\sigma h) e^{2\sigma h^2} dz \wedge d\bar{z}
\]

\[
= \frac{1}{2} (|\psi_1|^2_H + |\psi_2|^2_H) \omega
\]

\[
= \frac{-1}{2} (|\psi_1|^2_H + |\psi_2|^2_H) e^{2\sigma h^2} dz \wedge d\bar{z}
\]

which implies that \( K(e^\sigma h) = \frac{1}{2} (|\psi_1|^2_H + |\psi_2|^2_H) \). This always has a solution \( \sigma \) for a compact (oriented) genus \( g > 1 \) surfaces, see [10]. \( \square \)

**Proposition 2.2.** There exists global solutions with \( \Phi \neq 0 \) identically and \( \psi_1 \) and \( \psi_2 \) not equal to zero identically, on a compact oriented Riemann surface of genus \( g > 1 \).
Proof. Let $X_1$ be a torus with a puncture where the puncture looks like a long cylinder with negative Gaussian curvature at the root of the cylinder and zero Gaussian curvature at the end. Let $X_2$ be a long cylinder with zero Gaussian curvature everywhere. Let $X_3$ be an identical copy of $X_1$. Consider a Riemann surface $X$ which is obtained by gluing $X_1$, $X_2$ and $X_3$ where the gluing occurs along the flat ends of the cylinders and $X_2$ is in the middle of $X_1$ and $X_3$. In other words, to $X_1$ we glue $X_2$ along the flat ends of the cylinders, and then to $X_2$ we glue $X_3$ along the flat ends of the cylinders.

Let the line bundle $L$ be the tangent bundle on the surface $X$ which is flat on the middle cylinder and non-flat at the torus ends. Thus let $ψ_1$ and $ψ_2$ be defined on $X_1$ such that they are non-zero on most of $X_1$ and decay very fast at zero at the end of the cylinder of $X_1$ such that the first derivatives are also zero at the end of the cylinder. By Proposition 2.3 (where we take $K_0$ to be negative near the root of the cylinder and zero at the end of the cylinder) there is a solution $σ$ to the equation:

$$F(σ) = K(e^σ) e^{2σ} dz \wedge d\bar{z} = \frac{1}{2} (|ψ_1|^2_H + |ψ_2|^2_H) \omega$$

where recall $ω = ie^{2σ} dz \wedge d\bar{z}$ and $K(e^σ)$ is negative everywhere on $X_1$ except on the cylinder where it is $K_0$, which is negative at the root of the cylinder and zero at the end of the cylinder. This is possible since $X_1$ is a hyperbolic Riemann surface with puncture. Proposition 2.3. As mentioned before we take $ψ_1$ and $ψ_2$ to be decaying to zero fast enough at the end of the cylinder of $X_1$. This is possible by suitable choice of $f(z)$ and $g(z)$, (notation as in proposition (2.1). We take $Φ = 0$ on $X_1$.

On $X_2$ we take $ψ_1 = ψ_2 = 0$, i.e. $F(A) = 0$ (which is possible since $X_2$ is a flat cylinder) and $Φ^{0,1} = c(\bar{z}) d\bar{z}$ where $c(\bar{z})$ decays to zero fast enough so that its first derivatives are zero at the two ends of the cylinder $X_2$.

On $X_3$ we have a solution exactly as in $X_1$ with $Φ = 0$ but $ψ_1$ and $ψ_2$ non-zero.

In the two cylindrical regions where the three solutions are glued all of $Φ, ψ_1$ and $ψ_2$ are zero and their derivatives are also zero (since $f$ and $g$ and $c$ decay to zero very fast) – so that the equations which involve first derivatives are satisfied identically.

Thus on $X$, a compact oriented Riemann surface with genus $g = 2$, we have constructed a solution with $Φ, ψ_1, ψ_2$ non-identically zero.

By repeating this process, we can get the result on any genus $g > 1$ surface – because we can construct these by adding torus with a cylindrical puncture to a genus $g − 1$ surface with a long cylindrical puncture and on any of the hyperbolic pieces we can have solution as in $X_1$ and $Φ ≠ 0$ in the middle flat cylinders. □

**Proposition 2.3.** Let us consider the moduli space $N$. Suppose $(A, Ψ, Φ)$ is a point on the moduli space such that $Ψ$ is not identically zero. The (virtual) dimension of $N$ is $2g + 2c_1(L) + 2$

If $Φ = 0$ then (i) if $ψ_1$ and $ψ_2$ are not identically zero, then the dimension is $2c_1(L) + 2$ and (ii) if $ψ_1 ≡ 0$ then the dimension is $g + c_1(L) + 1$.

**Proof.** To calculate the dimension of $N$ let $S$ be the solution space to (2.1) – (2.3). Consider the tangent space $T_p S$ at a point $p = (A, Ψ, Φ) ∈ S$, which is defined by the linearization of equations (2.1) – (2.3). Let $X = (α, β, γ) ∈ T_p S$, where.
where $\alpha \in \Omega^1(M, i\mathbb{R})$ and $\beta = \left[ \begin{array}{c} \beta_1 \\ \beta_2 \end{array} \right] \in \Gamma(M, L \oplus L)$, and $\gamma \in H$. The linearizations of the equations are as follows

\[(2.1)' \quad d\alpha = i(\psi_1, \beta_1 >_H + \psi_2, \beta_2 >_H)\omega, \]

\[(2.2)' \quad \partial\gamma^{0,1} = 0 \]

\[(2.3)' \quad \left[ \begin{array}{c} \bar{\partial} + A^{0,1} \\ -\frac{i}{2}\partial d\bar{z} \end{array} \right] \left[ \begin{array}{c} \beta_1 \\ \beta_2 \end{array} \right] + \left[ \begin{array}{c} \frac{a^{0,1}}{2} \\ -\frac{1}{2}\gamma^{0,1} \end{array} \right] \left[ \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right] = 0. \]

Taking into account the quotient by the gauge group $G$, we arrive at the following sequence $C$

\[0 \to \Omega^0(M, i\mathbb{R}) \xrightarrow{d_1} \Omega^1(M, i\mathbb{R}) \oplus \Gamma(M, L) \oplus H \xrightarrow{d_2} \Omega^2(M, i\mathbb{R}) \oplus \Omega^2(M, \mathbb{C}) \oplus V \to 0, \]

where $L = L \oplus L$, $V = (L \otimes \Omega^{1,0}(M)) \oplus (L \otimes \Omega^{1,0}(M)) = \Gamma(M, \mathcal{L}) \oplus \mathcal{H}^{\partial} \oplus \Lambda^2(M, i\mathbb{R}) \oplus \Lambda^2(M, \mathbb{C}) \oplus V \to 0$.

\[d_1 = (d\bar{f}, -f\bar{\Psi}, 0), \quad d_2(\alpha, \left[ \begin{array}{c} \beta_1 \\ \beta_2 \end{array} \right], \gamma) = (A_1, B_1, C_1), \]

where $A_1 = d\alpha - it[\psi_1, \beta_1 >_H + \psi_2, \beta_2 >_H]\omega$,

\[B_1 = \partial\gamma^{0,1} \in \Omega^2(M, \mathbb{C}) \]

\[C_1 = \left[ \begin{array}{c} \bar{\partial} + A^{0,1} \\ -\frac{i}{2}\partial d\bar{z} \end{array} \right] \left[ \begin{array}{c} \beta_1 \\ \beta_2 \end{array} \right] + \left[ \begin{array}{c} \frac{a^{0,1}}{2} \\ -\frac{1}{2}\gamma^{0,1} \end{array} \right] \left[ \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right] \in V. \]

Clearly, $ind(C^t)$ does not depend on $t$. The complex $C^0$ (for $t = 0$) is

\[0 \to \Omega^0(M, i\mathbb{R}) \xrightarrow{d_1} \Omega^1(M, i\mathbb{R}) \oplus \Gamma(M, L) \oplus H \xrightarrow{d_2} \Omega^2(M, i\mathbb{R}) \oplus \Omega^2(M, \mathbb{C}) \oplus V \to 0 \]

where

\[d_1 = (d\bar{f}, 0, 0), \]

\[d_2(\alpha, \beta, \gamma) = (d\alpha, \partial\gamma^{0,1}, \mathcal{D}A\beta). \]
Here \( D_A = \begin{bmatrix} \bar{\partial} + A^{0,1} & 0 \\ 0 & \bar{\partial} + A^{0,1} \end{bmatrix} \).

\( C^0 \) decomposes into a direct sum of three complexes
\( (a) \ 0 \to \Omega^0(X, i\mathbb{R}) \xrightarrow{d} \Omega^1(M, i\mathbb{R}) \xrightarrow{d} \Omega^2(M, i\mathbb{R}) \to 0, \)
\( (b) \ 0 \to \Omega^1(M, i\mathbb{R}) \xrightarrow{\partial} \Omega^{1,1}(M, i\mathbb{R}) \to 0 \)
\( (c) \ 0 \to \Gamma(M, S) \xrightarrow{D_A} \Gamma(M, S') \to 0, \) where \( S = L \oplus L, S' = (L \otimes K) \oplus (L \otimes K). \)
\( \dim H^1(\text{complex } (a)) = 2g, \dim H^1(\text{complex } (b)) = 2g. \)

The complex (c) breaks into two complexes as follows
\( (c1) \ 0 \to \Gamma(M, L) \xrightarrow{\bar{\partial} + A^{0,1}} \Gamma(M, L \otimes K) \to 0, \)
\( (c2) \ 0 \to \Gamma(M, L) \xrightarrow{\bar{\partial} + A^{0,1}} \Gamma(M, L \otimes K) \to 0. \)

(c1) comes from the equation \((\bar{\partial} + A^{0,1})\beta_1 = 0 \) and (c2) is the equation \((\bar{\partial} + A^{0,1})\beta_2 = 0, \) which is holomorhicity of the sections \( \beta_1 \) and \( \beta_2 \) of \( L. \)

By Riemann Roch, the index of (c1) is \((c_1(L) - g + 1)\) and that of (c2) is \((c_1(L) - g + 1)\) and thus the sum is \(2g + 2g + 2c_1(L) - 2g + 2 + 2g + 2c_1(L) + 2.\)

If \( \Phi = 0 \) then case (i) if \( \psi_1 \) and \( \psi_2 \) are not identically zero, then the dimension is \(2c_1(L) + 2\) since complex (b) is missing case (ii) if \( \psi_1 \equiv 0 \) then the dimension is \(g + c_1(L) + 1\) since complex (b) and complex (c1) are missing.

\[ \square \]

3. FAMILY OF SYMPLECTIC STRUCTURES

In the next section we discuss a standard symplecton form and a variation of it which gives a whole family of symplectic structures.

Let \( C = \mathcal{A} \times \Gamma(M, L \oplus L) \times \mathcal{H} \) be the space on which equations (2.1) – (2.3) are imposed.

Let \( p = (A, \Psi, \Phi) \in C, X = (\alpha_1, \beta, \gamma_1), Y = (\alpha_2, \eta, \gamma_2) \in T_p C. \)

Let us define \( \langle \beta, \eta \rangle \Phi = \beta_1 H \eta_1 + \beta_2 H \eta_2. \)

Let \( * : \Omega^1 \to \Omega^1 \) is the Hodge star operator on \( M \) which acts as follows: \( * \langle \alpha^{1,0} \rangle = -i \alpha^{0,1}, \) and \( * \langle \alpha^{0,1} \rangle = i \alpha^{0,1}. \)

On \( C \) one can define a metric
\[ g(X, Y) = \int_M * \alpha_1 \wedge \alpha_2 + \int_M \Re \langle \beta, \eta \rangle \Phi + \int_M * \gamma_1 \wedge \gamma_2 \]

Note: if we take \( \alpha_1 = adz - \bar{a}d\bar{z} \) and \( \gamma_1 = cdz - \bar{c}d\bar{z} \) then it is easy to check that
\[ g(X, X) = 4 \int_M |a|^2 dx \wedge dy + \int_M (|\beta_1|^2 + |\beta_2|^2) dx \wedge dy + 4 \int_M |c|^2 dx \wedge dy \]
which is of definite sign.

Define an almost complex structure \( \mathcal{I} = \begin{bmatrix} * & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix} : T_p C \to T_p C \)

We define
\[ \Omega(X, Y) = -\int_M \alpha_1 \wedge \alpha_2 + \int_M \Re \langle I \beta, \eta \rangle \Phi - \int_M \gamma_1 \wedge \gamma_2 \]

where \( I = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \) such that \( g(IX, Y) = \Omega(X, Y). \) Moreover, we have the following:
Proposition 3.1. The metrics $g$, the symplectic form $\Omega$, and the almost complex structure $I$ are invariant under the gauge group action on $C$.

Proof. Let $p = (A, \Psi, \Phi) \in C$ and $u \in G$, where $u \cdot p = (A + u^{-1} du, u^{-1} \Psi, \Phi)$.

Then $u_\ast : T_p C \rightarrow T_{u \ast p} C$ is given by the mapping $(Id, u^{-1}, Id)$ and it is now easy to check that $g$ and $\Omega$ are invariant and $I$ commutes with $u_\ast$. \hfill $\square$

Proposition 3.2. The equation (2.1) can be realised as a moment map $\mu = 0$ with respect to the action of the gauge group and the symplectic form $\Omega$.

Proof. Let $\zeta \in \Omega(M, i\mathbb{R})$ be the Lie algebra of the gauge group (the gauge group element being $u = e^\zeta$); It generates a vector field $X_\zeta$ on $C$ as follows:

$$X_\zeta(A, \Psi, \Phi) = (d\zeta, -\zeta \Psi, 0) \in T_p C, p = (A, \Psi, \Phi) \in C.$$ 

We show next that $X_\zeta$ is Hamiltonian. Namely, define $H_\zeta : C \rightarrow \mathbb{C}$ as follows:

$$H_\zeta(p) = \int_M \zeta \cdot (F_A - i \frac{(|\psi_1|^2_H + |\psi_2|^2_H)}{2}) \omega.$$ 

Then for $X = (\alpha, \beta, \gamma) \in T_p C$,

$$dH_\zeta(X) = \int_M d\zeta \alpha - i \int_M \zeta \text{Re}(\psi_1 \bar{H}_1 + \psi_2 \bar{H}_2) \omega$$

$$= \int_M (-d\zeta) \wedge \alpha + \int_M \text{Re} \left( -\zeta \left[ \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right] \right) \left[ \begin{array}{c} \bar{H}_1 \\ \bar{H}_2 \end{array} \right] >_H \omega$$

$$= \Omega(X_\zeta, X),$$

where we use that $\bar{\zeta} = -\zeta$.

Thus we can define the moment map $\mu : C \rightarrow \Omega^2(M, i\mathbb{R}) = G^*$ (the dual of the Lie algebra of the gauge group) to be

$$\mu(A, \Psi) = (F(A) - i \frac{(|\psi_1|^2_H + |\psi_2|^2_H)}{2}) \omega.$$ 

Thus equation (2.1)) is $\mu = 0$. \hfill $\square$

Lemma 3.3. Let $S$ be the solution spaces to equation (2.1) - (2.3), $X \in T_p S$. Then $IX \in T_p S$ if and only if $X$ is orthogonal to the gauge orbit $O_p = G \cdot p$.

Proof. Let $X_\zeta \in T_p O_p$, where $\zeta \in \Omega^0(M, i\mathbb{R})$, $g(X, X_\zeta) = -\Omega(IX, X_\zeta) = -\int_M \zeta \cdot d\mu(IX)$, and therefore $IX$ satisfies the linearization of equation (2.1) iff $d\mu(IX) = 0$, i.e., iff $g(X, X_\zeta) = 0$ for all $\zeta$. Second, it is easy to check that $IX$ satisfies the linearization of equation (2.2), (2.3) whenever $X$ does.

For instance the action of $I$ in the linearisation of equation (2.3) is

$$\left[ \begin{array}{c} \bar{H} + A_{0,1}^{0,0} \\ \frac{1}{2} \bar{H} \partial \bar{\zeta} + i \bar{\zeta} \partial \bar{\zeta} \\ \frac{1}{2} \partial \zeta \bar{\zeta} + A_{0,1}^{0,1} \end{array} \right] \left[ \begin{array}{c} i \beta_1 \\ i \beta_2 \\ i \alpha_1 + i \gamma_1 \end{array} \right] + \left[ \begin{array}{c} i \alpha_0^{0,1} \\ i \gamma_0^{0,1} \\ i \alpha_0^{0,1} \end{array} \right] \left[ \begin{array}{c} \psi_1 \\ \psi_2 \\ \psi_2 \end{array} \right] = 0$$ 

since the factor of $i$ comes out and the remaining equation is linearization of (2.3). (Note that the action of $I$ is $\beta_1 \rightarrow i \beta_1$, $\beta_2 \rightarrow i \beta_2$, $\alpha_0^{0,1} \rightarrow i \alpha_0^{0,1}$ and $\gamma_0^{0,1} \rightarrow i \gamma_0^{0,1}$.) \hfill $\square$

Theorem 3.4. $\mathcal{N}$ has a natural symplectic structure and an almost complex structure compatible with the symplectic form $\Omega$ and the metric $g$.

Proof. First we show that the almost complex structure descends to $\mathcal{N}$. Then using this and the symplectic quotient construction we will show that $\Omega$ gives a symplectic structure on $\mathcal{N}$. 


(a) To show that $\mathcal{I}$ descends as an almost complex structure we let $pr : S \to S/G = \mathcal{N}$ be the projection map and set $[p] = pr(p)$. Then we can naturally identify $T_{[p]}\mathcal{N}$ with the quotient space $T_pS/T_pO_p$, where $O_p = G \cdot p$ is the gauge orbit. Using the metric $g$ on $S$ we can realize $T_{[p]}\mathcal{N}$ as a subspace in $T_pS$ orthogonal to $T_pO_p$. Then by lemma 3.3 this subspace is invariant under $I$. Thus $I_{[p]} = I|_{T_p(O_p)\perp}$, gives the desired almost complex structure. This construction does not depend on the choice of $p$ since $\mathcal{I}$ is $G$-invariant.

(b) The symplectic structure $\Omega$ descends to $\mu^{-1}(0)/G$, (by proposition 3.2 and by the Marsden-Wienstein symplectic quotient construction, 22, 23, since the leaves of the characteristic foliation are the gauge orbits). Now, as a 2-form $\Omega$ descends to $\mathcal{N}$, due to proposition (3.1) so does the metric $g$. We check that equation (2.2), (2.3), does not give rise to new degeneracy of $\Omega$ (i.e. the only degeneracy of $\Omega$ is due to (2.1) but along gauge orbits). Thus $\Omega$ is symplectic on $\mathcal{N}$. Since $g$ and $\mathcal{I}$ descend to $\mathcal{N}$ the latter is symplectic and almost complex.

Choose a $\psi_0 \in \Gamma(M,L)$ such that its gauge equivalence class is fixed and $\psi_0 = 0$ on a set of measure zero on $M$. This $\psi_0$ has nothing to do with $\psi_1$ but we allow it to gauge transform as $u^{-1}\psi_0$ when $\psi_1$ gauge transforms to $u^{-1}\psi_1$. (This will be handy in defining the determinant line bundles).

Define a symplectic form on $\mathcal{C}$ as

$$\Omega_{\psi_0}(X,Y) = -\int_M \alpha_1 \wedge \alpha_2 + \int_M Re < I\beta, \eta > H \frac{\partial}{\partial \psi_0} \omega - \int_M \gamma_1 \wedge \gamma_2$$

$$= -\int_M \alpha_1 \wedge \alpha_2 + \frac{i}{2} \int_M [((\beta_1 H\eta_1 - \beta_1 H\eta_1) - (\beta_2 H\eta_2 - \beta_2 H\eta_2)] \frac{\partial}{\partial \psi_0} \omega - \int_M \gamma_1 \wedge \gamma_2$$

$|\psi_0|^2_H$ plays the role of a conformal rescaling of the volume form $\omega$ on $M$ which appears in $\Omega$, where we allow the conformal factor to have zeroes on sets of measure zero.

**Theorem 3.5.** $\Omega_{\psi_0}$ descends to $\mathcal{M}$ as a symplectic form.

**Proof.** Let $p = (A, \Psi, \Phi)$. It is easy to show that $\Omega_{\psi_0}$ is closed (this follows from the fact that on $\mathcal{C}$ it is a constant form – does not depend on $(A, \Psi, \Phi)$). We have to show it is non-degenerate.

Suppose there exists $(\alpha_1, \beta, \gamma_1) \in T_{[p]}(\mathcal{N})$ s.t.

$$\Omega_{\psi_0}((\alpha_2, \eta, \gamma_2), (\alpha_1, \beta, \gamma_1)) = 0$$

$\forall (\alpha_2, \eta, \gamma_2) \in T_{[p]}(\mathcal{N})$. Using the metric $G$ we identify $T_{[p]}\mathcal{N}$ with the subspace in $T_pS$, $G$-orthogonal to $T_pO_p$ (i.e. the tangent space to the moduli space is identified to the tangent space to solutions which are orthogonal to the gauge orbits, the orthogonality is with respect to the metric $G$.) Thus $(\alpha_1, \beta, \gamma_1)$, $(\alpha_2, \eta, \gamma_1)$ satisfy the linearization of equation (2.1), (2.2) and (2.3) and $G((\alpha_1, \beta, \gamma_1), X_\zeta) = 0$ and $G((\alpha_2, \eta, \gamma_1), X_\zeta) = 0$ for all $\zeta$. [33]
Since $d\mu((\alpha_1, \beta, \gamma_1)) = 0$ is precisely one of the equations saying that $(\alpha_1, \beta, \gamma_1) \in T_pS$. Thus $\mathcal{I}(\alpha_1, \beta, \gamma_1) \in T_p\mathcal{N}$, (since it is in $T_pS$ and $G$-orthogonal to gauge orbits).

Take $(\alpha_2, \eta, \gamma_1) = \mathcal{I}(\alpha_1, \beta, \gamma_1) = (\star \alpha_1, I\beta, \star \gamma_1)$. Then
\[
0 = \Omega_{\psi_0}(\mathcal{I}(\alpha_1, \beta, \gamma_1), (\alpha_1, \beta, \gamma_1)) = -\int_M (\star \alpha_1 \wedge \alpha_1) + \int_M Re < I(I\beta), \beta > \omega - \int_M (\star \gamma_1 \wedge \gamma_1)
\]
\[
= -2i \int_M |a|^2 dz \wedge d\bar{z} - i \int_M (|\beta_1|^2_M + |\beta_2|^2_M) |\psi_0|^2_M c^{2\sigma} h^2 dz \wedge d\bar{z}
\]
\[
-2i \int_M |c|^2 dz \wedge d\bar{z}
\]
where $\omega = i e^{2\sigma} h^2 dz \wedge d\bar{z}$ and $\alpha_1 = adz - \bar{a}d\bar{z} \in \Omega^1(M, i\mathbb{R})$ and $\star \alpha_1 = -i(adz + \bar{a}d\bar{z})$ and $\gamma_1 = cdz - \bar{c}d\bar{z}$. By the same sign of all the terms and the fact that $\psi_0$ has zero on a set of measure zero on $M$, $(\alpha_1, \beta, \gamma_1) = 0$ a.e. Thus $\Omega_{\psi_0}$ is symplectic. \hfill \Box

4. PREQUANTUM LINE BUNDLE

In this section we briefly review the Quillen construction of the determinant line bundle of the Cauchy Riemann operator $\bar{\partial}A = \bar{\partial} + A^{(0,1)}$, [37], which enables us to construct prequantum line bundle on the moduli space $\mathcal{N}$. First let us note that a connection $A$ on a $U(1)$-principal bundle induces a connection on any associated line bundle $L$. We will denote this connection also by $A$ since the same “Lie-algebra valued 1-form” $A$ (modulo representations) gives a covariant derivative operator enabling you to take derivatives of sections of $L$ [31], page 348. A very clear description of the determinant line bundle can be found in [37] and [3]. Here we mention the formula for the Quillen curvature of the determinant line bundle $\Lambda^{top}(\text{Ker}\bar{\partial}A)^* \otimes \Lambda^{top}(\text{Coker}\bar{\partial}A) = \det(\bar{\partial}A)$, given the canonical unitary connection $\nabla_Q$, induced by the Quillen metric, [37]. Recall that the affine space $A$ (notation as in [37]) is an infinite-dimensional Kähler manifold. Here each connection is identified with its $(0, 1)$ part which is the holomorphic part. Since the connection $A$ is unitary (i.e. $A = A^{(1,0)} + A^{(0,1)}$ s.t. $\bar{A}^{(1,0)} = -A^{(0,1)}$) this identification is easy. In fact, for every $A \in A$, $T'_A(A) = \Omega^{0,1}(M, i\mathbb{R})$ and the corresponding Kähler form is given by

\[
F(\alpha_1^{(0,1)}, \alpha_2^{(0,1)}) = \text{Re} \int_M (\alpha_1^{(0,1)} \wedge \star_1 \alpha_2^{(0,1)}) = -\frac{1}{2} \int_M \alpha_1 \wedge \alpha_2
\]

where $\alpha^{(0,1)}, \bar{\beta}^{(0,1)} \in \Omega^{0,1}(M, i\mathbb{R})$, $\alpha_i = \alpha_i^{(1,0)} + \alpha_i^{0,1}$ and $\star_1$ is the Hodge-star operator such that $\star_1(\alpha^{1,0}) = -\overline{\alpha^{0,1}} = \alpha^{0,1}$ and
\[ \ast_1(\alpha^{(0,1)}) = \bar{\alpha}^{(0,1)} - \alpha^{(1,0)} \] where we have used \( \alpha^{(0,1)}_i = -\alpha^{(1,0)}_i, \) \( i = 1, 2. \) Let \( \nabla_Q \) be the connection induced from the Quillen metric. Then the Quillen curvature of \( \det(\bar{\partial}_A) \) is

\[
\mathcal{F}(\nabla_Q) = \frac{i}{\pi} F = \frac{-i}{2\pi} \int_M (\alpha_1 \wedge \alpha_2).
\]

5. Prequantum bundle on the moduli space \( \mathcal{N} \)

First we note that to the connection \( A \) we can add any one form and still obtain a covariant derivative operator.

Let \( \omega = ie^{2\pi h^2}dz \wedge d\bar{z} \) where recall \( h \) is real. Let \( \theta = h\partial, \bar{\theta} = h\partial \) be 1-forms (\[\text{[21]}, \) page 28) such that \( \omega = i\theta \wedge \bar{\theta} = ie^{2\pi h^2}dz \wedge d\bar{z}. \) Let \( \psi_0 \) be the same section used to define \( \Omega_{\psi_0} \) whose gauge equivalence class is fixed, and which gauge transforms in the same way as \( \psi_1 \) and \( \bar{\psi}_2. \)

\( \psi_0 \) has zero on a set of measure zero on \( M. \) Note \( \psi_1 H \bar{\psi}_0 \) and \( \psi_2 H \psi_0 \) are smooth gauge invariant functions on \( M. \) Thus we define

\[
B_{\pm} = B^{(0,1)}_{\pm} + B^{(1,0)}_{\pm}
\]

such that

\[
B^{(0,1)}_{\pm} = \pm \bar{\psi}_2 H \psi_0 \theta - \psi_1 H \bar{\psi}_0 \theta,
\]

\[
B^{(1,0)}_{\pm} = \bar{\psi}_1 H \psi_0 \theta \mp \psi_2 H \psi_0 \theta.
\]

\( B_{\pm} \) are two unitary 1-forms we would like to add to the connection \( A \) to make another connection form. (Note that \( B^{(0,1)} = -B^{(1,0)}, \) as apt for unitary 1-forms.

Note that \( B \) is gauge invariant, since \( \psi_1, \psi_2 \) and \( \psi_0 \) gauge transform in the same way. Note that \( A^{(0,1)} \pm B^{(0,1)} \) are the \( (0,1) \) parts of a connection defined by \( A \pm B = A^{(0,1)} \pm B^{(0,1)} + A^{(1,0)} \pm B^{(1,0)} \), where \( B \) can be one of \( B_{\pm} \).

**Definitions:** Let us denote by \( \mathcal{L}^+_{\pm} = \text{det}[\frac{1}{\sqrt{4}}(\bar{\partial} + A^{(0,1)} \pm B^{(0,1)}_{\pm})] \) two determinant bundles on the affine spaces \( \mathcal{J}^\pm = \{ A \in \mathcal{A}, \Psi \in \Gamma(M, L \oplus L) \} \) respectively. These affine spaces are isomorphic to \( \mathcal{A} \times \Gamma(M, L \oplus L) \times \Phi_0, \Phi_0 \) being a fixed Higgs field. We can extend it to all of \( \mathcal{C} = \mathcal{A} \times \Gamma(M, L \oplus L) \times \mathcal{H} \) by defining the fibers to be same for all \( \Phi. \)

Similarly define \( \mathcal{L}^\pm_0 = \text{det}[\frac{1}{\sqrt{4}}(\bar{\partial} + A^{(0,1)} \pm B^{(0,1)}_{\pm})] \)

Thus \( \mathcal{P}_{\psi_0} = \mathcal{L}^+_1 \otimes \mathcal{L}^-_1 \otimes \mathcal{L}^+_2 \otimes \mathcal{L}^-_2 \) well-defined line bundle on \( \mathcal{C}. \)

**Lemma 5.1.** \( \mathcal{P}_{\psi_0} \) is a well-defined line bundle over \( \mathcal{N} \subset \mathcal{C}/G, \) where \( G \) is the gauge group.

**Proof:** First consider the Cauchy-Riemann operators \( D = \frac{1}{\sqrt{4}}(\bar{\partial} + A^{(0,1)} + B^{(0,1)}_{\pm}). \)

Under gauge transformation \( D = [\frac{1}{\sqrt{4}}(\bar{\partial} + A^{(0,1)} + B^{(0,1)}_{\pm})] \rightarrow D_g = g[\frac{1}{\sqrt{4}}(\bar{\partial} + A^{(0,1)} + B^{(0,1)}_{\pm})] g^{-1}. \) We can show that the operators \( D \) and \( D_g \) have isomorphic kernel and cokernel and their corresponding Laplacians have the same spectrum and the eigenspaces are of the same dimension. Let \( \Delta \) denote the Laplacian corresponding to \( D \) and \( \Delta_g \) that corresponding to \( D_g. \) The Laplacian is \( \Delta = \bar{D}D \) where \( \bar{D} = [\frac{1}{\sqrt{4}}(\bar{\partial} + A^{(1,0)} + B^{(1,0)}_{\pm})], \) where recall \( A^{(1,0)} = -A^{(0,1)} \) and \( B^{(1,0)}_{\pm} = -B_{\pm}^{(0,1)} \). Note that \( \bar{D} \rightarrow \bar{D}_g = g\bar{D}g^{-1} \) under gauge transformation. Then \( \Delta_g = g\Delta g^{-1}. \) Thus
the isomorphism of eigenspaces is $s \to gs$. We describe here how to define the line bundle on the moduli space. Let $K^a(\Delta)$ be the direct sum of eigenspaces of the operator $\Delta$ of eigenvalues $< a$, over the open subset $U^a = \{ \frac{1}{\sqrt{2}}(A(0,1) + B_+^0) \} \setminus \text{Spec}\Delta$ of the affine space $M$. The determinant line bundle is defined using the exact sequence

$$0 \to \text{Ker}D \to K^a(\Delta) \to D(K^a(\Delta)) \to \text{Coker}D \to 0$$

Thus one identifies $\text{Spec}\Delta$ in the previous case. By extending this definition from $U^a$ to $V^a = \{(A, \Psi, \Phi)|a \notin \text{Spec}\Delta\}$, an open subset of $C$, we can define the fiber over the quotient space $V^a/G$ to be the equivalence class of fibers coming from the identifications of the isomorphism of eigenspaces is $s \to gs$. We describe here how to define the line bundle on $C$. Covering $C$ with open sets of the type $V^a$, we can define it on $C/G$. Then we can restrict it to $N \subset C/G$.

Similarly one can deal with the other cases of $\frac{1}{\sqrt{2}}(\bar{\partial} + A(0,1) \pm B_+^0)$. For instance, let $([A], [\Psi], [\Phi]) \in C/G$, where $[A], [\Psi], [\Phi]$ are gauge equivalence classes of $A, \Psi, \Phi$, respectively. Then associated to the equivalence class $([A], [\Psi], [\Phi])$ in the base space, there is an equivalence class of fibers coming from the identifications of $\det[\frac{1}{\sqrt{2}}(\bar{\partial} + A(0,1) - B_+^0)]$ with $\det[\frac{1}{\sqrt{2}}(\bar{\partial} + A(0,1) - B_+^0)g^{-1}]$ as mentioned in the previous case.

This way one can prove that $P_{\Psi_0}$ is well defined on $C/G$. Then we restrict it to $N \subset C/G$. 

Next, in a similar way, we define two other determinant line bundles. Recall $\Phi^{(1,0)} = -\Phi^{(0,1)}$. Let us denote by $\mathcal{M}_\pm = \text{det}[\frac{1}{\sqrt{2}}(\bar{\partial} + A(0,1)) \pm \Phi^{(0,1)}]$ a determinant bundle on $M = \{ \frac{1}{\sqrt{2}}(A(0,1) \pm \Phi^{(0,1)}) \} \subset C$, which is isomorphic to $A \times \mathcal{H}$. We can extend it to $C = A \times \Gamma(M, L \oplus L) \times \mathcal{H}$ by defining the fibers to be the same for all $\Psi$. Thus $\mathcal{M} = \mathcal{M}_+ \otimes \mathcal{M}_-$ well-defined line bundle on $C$.

This can be defined exactly in a similar way to $P_{\Psi_0}$ over the moduli space $N$.

[Note: The square root of 2 comes with the $\bar{\partial} + A(0,1)$-term alone.]

**Curvature and symplectic form:**

Let $p = (A, \Psi, \Phi) \in S$. Let $X, Y \in T_pN$. Since $T_pN$ can be identified with a subspace in $T_pS$ orthogonal to $T_pO_p$, if we write $X = (\alpha_1, \beta, \gamma_1)$ and $Y = (\alpha_2, \eta, \gamma_2)$, (notation as before) then $X, Y$ can be said to satisfy a) $X, Y \in T_pS$ and b) $X, Y$ are $G$-orthogonal to $T_pO_p$, the tangent space to the gauge orbit.

Let $F_{\pm}$ denote the Quillen curvatures of the four determinant line bundles $\mathcal{L}^+_1, \mathcal{L}^+_2$, respectively, which are determinants of Cauchy-Riemann operators of the connections $\frac{1}{\sqrt{2}}(A(0,1) \pm B_+^0)$. In the curvature formula of Quillen the terms that will appear are $\frac{1}{\sqrt{2}}(\alpha_1 \pm b_\pm)$ and $\frac{1}{\sqrt{2}}(\alpha_2 \pm c_\pm)$ where $b_\pm = b_\pm^{(0,1)} + b_\pm^{(1,0)}$, $c_\pm = c_\pm^{(1,0)} + c_\pm^{(0,1)}$ such that

$$b_\pm^{(0,1)} = \pm \beta_\pm H \psi_0 \bar{\theta} - \beta_\pm H \bar{\psi}_0 \bar{\theta}$$
the Quillen curvature formula:

\[ b^{(1,0)}_\pm = \beta_1 H \psi_0 \theta \mp \beta_2 H \psi_0 \theta \]

\[ c^{(0,1)}_\pm = \pm \eta_2 H \bar{\psi}_0 \bar{\theta} - \eta_1 H \bar{\psi}_0 \bar{\theta} \]

\[ c^{(1,0)}_\pm = \eta_1 H \psi_0 \theta \mp \eta_2 H \psi_0 \theta \]

\[ \mathcal{F}_{L^\pm_1}(X,Y) = -\frac{i}{2\pi} \int_M \frac{1}{\sqrt{4}} (\alpha_1 \pm b_+ \wedge \frac{1}{\sqrt{4}}(\alpha_2 \pm c_+) \]

\[ = -\frac{i}{8\pi} \int_M \left[ \left( (\alpha_1 \wedge \alpha_2) \pm (b_+ \wedge \alpha_2) \pm (\alpha_1 \wedge c_+) + (b_+ \wedge c_+) \right) \right] \]

\[ \mathcal{F}_{L^\pm_2}(X,Y) = -\frac{i}{2\pi} \int_M \frac{1}{\sqrt{4}} (\alpha_1 \pm b_- \wedge \frac{1}{\sqrt{4}}(\alpha_2 \pm c_-) \]

\[ = -\frac{i}{8\pi} \int_M \left[ \left( (\alpha_1 \wedge \alpha_2) \pm (b_- \wedge \alpha_2) \pm (\alpha_1 \wedge c_-) + (b_- \wedge c_-) \right) \right] \]

One can easily compute that

\[ \mathcal{F}_{\psi\psi}(X,Y) = (\mathcal{F}_{L^+_1} + \mathcal{F}_{L^-_1} + \mathcal{F}_{L^+_2} + \mathcal{F}_{L^-_2})(X,Y) \]

\[ \mathcal{F}_{M^\pm}(X,Y) = -\frac{i}{2\pi} \int_M \left[ \left( \frac{\alpha_1 \wedge c_+}{\sqrt{2}} \pm \gamma_1 \right) \wedge \left( \frac{\alpha_2 \wedge c_+}{\sqrt{2}} \pm \gamma_2 \right) \right] \]

One can easily compute that

\[ \mathcal{F}_{M}(X,Y) = (\mathcal{F}_{M^+} + \mathcal{F}_{M^-})(X,Y) \]

\[ = -\frac{i}{2\pi} \int_M \left[ \left( (\alpha_1 \wedge \alpha_2) + 2(\gamma_1 \wedge \gamma_2) \right) \right] \omega \]

**Holomorphicity** Since in \[ A^{0,1} \pm B^{0,1}_\pm \], terms with \( \psi \) and \( \bar{\psi}_2 \) comes, i.e. under the action of \( \mathcal{L} \), \( \alpha_1^{0,1} \pm b_+^{0,1} \) goes to \( i(\alpha_1^{0,1} \pm b_+^{0,1}) \), and \( \alpha^{0,1} \pm \gamma^{0,1} \) goes to \( i(\alpha^{0,1} \pm \gamma^{0,1}) \), these line bundles are holomorphic.

Thus, we have the following theorem:

**Theorem 5.2.** \( Q_\psi = \mathcal{P}_\psi \otimes \mathcal{M} \) is a well-defined holomorphic line bundle on \( \mathcal{N} \) whose Quillen curvature is \( \frac{B}{\sqrt{2}} \Omega_\psi \). Thus \( Q_\psi \) is a prequantum bundle on \( \mathcal{N} \).
Remark:
As $\psi_0$ varies, the corresponding line bundles are all topologically equivalent since the curvature forms have to be of integral cohomology and that would be constant. Thus they have the same Chern class. Holomorphically they may differ.

6. ALTERNATIVE METHOD FOR THE PREQUANTIZATION

We fix the gauge equivalence class of the connection $A_0$, i.e. $A_0$ is a \textit{fixed} connection which gauge transforms like $A$ when $\Psi$ gauge transforms.

We define two determinant line bundles on the moduli space in the same way as before $T_\pm = \det(\bar{\partial} + A_0^{(0,1)} \pm B_\pm^{(0,1)})$ on $\mathcal{N} \subset \mathcal{C}/G$.

Let $T = T_+ \otimes T_-$
Then $\mathcal{F}_{T_+}(X,Y) = \frac{-i}{2\pi} \int_M (b_+ \wedge c_+)$ and $\mathcal{F}_{T_-}(X,Y) = \frac{-i}{2\pi} \int_M (b_- \wedge c_-)$.
Thus the curvature
\[
\mathcal{F}_T(X,Y) = \mathcal{F}_{T_+}(X,Y) + \mathcal{F}_{T_-}(X,Y)
= \frac{-i}{2\pi} \int_M (b_+ \wedge c_+ + b_- \wedge c_-)
= \frac{-i}{2\pi} \int_M -2i[(\beta_1 H \bar{\eta}_1 - \bar{\beta}_1 H \eta_1) - (\beta_2 H \bar{\eta}_2 - \bar{\beta}_2 H \eta_2)]|\psi_0|^2 \Omega
\]

Define $S_{\pm} = \det(\bar{\partial} + A^{(0,1)} \pm \Phi^{(0,1)})$ a determinant bundle on $\mathcal{N}$.
Let $S = S_+^2 \otimes S_-^2$.

\[
\mathcal{F}_{S_{\pm}}(X,Y) = \frac{-i}{2\pi} \int_M [(\alpha_1 \pm \gamma_1) \wedge (\alpha_2 \pm \gamma_2)]
\]

One can easily compute that
\[
\mathcal{F}_{S}(X,Y) = 2(\mathcal{F}_{S_+} + \mathcal{F}_{S_-})(X,Y)
= \frac{-i}{2\pi} \int_M [4(\alpha_1 \wedge \alpha_2) + 4(\gamma_1 \wedge \gamma_2)]\Omega
\]

It is easy to calculate that $\mathcal{D}_{\psi_0} = T \otimes S$ has curvature $\frac{2i}{\pi} \Omega_{\psi_0}$.
It is also a holomorphic line bundle.
Thus we have proved

\textbf{Theorem 6.1.} $\mathcal{D}_{\psi_0}$ is a holomorphic prequantum line bundle on $\mathcal{N}$ with curvature $\frac{2i}{\pi} \Omega_{\psi_0}$.

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