1. INTRODUCTION

The notion of a reachable set is a fundamental concept of control theory, which is both a subject and an object of studies. It is well known that, for linear systems, it is important and convenient to study the shape of a reachable set, i.e., consider this set up to an invertible linear transformation. Thus, the asymptotic behavior of shapes in the long run is much more clear than that of the reachable sets themselves. In the paper, we show that, for short motion time, the shapes of reachable sets also admit a complete asymptotic description. We study both the autonomous and non-autonomous cases.

2. PROBLEM STATEMENT

Consider the following linear control system:

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{V} = \mathbb{R}^n, \quad u \in U \subset \mathbb{U} = \mathbb{R}^m, \quad (1)$$

where the set $U$ is a centrally symmetric convex body in $\mathbb{R}^m$, i.e., $U$ is assumed to be a convex compact set with nonempty interior, and $U = -U$. For any $T > 0$, a reachable set $\mathcal{D}(T)$ is defined as the set $\{x(T)\}$ of the ends of time $T$ of all admissible trajectories of system (1) such that $x(0) = 0$.

We study the asymptotic behavior of the reachable sets $\mathcal{D}(T)$ as $T \to 0$. The problem is not as trivial as it may seem, since the obvious answer $\mathcal{D}(T) \to \{0\}$ is not good enough. Of course, the set $\mathcal{D}(T)$ is small for small $T$. Still, we can look at it through a kind of a microscope and see its shape. For instance, consider the simplest system (1) in which $A = 0$, $B = 1$, and $U$ is the unit ball $B_1$ in $\mathbb{R}^n$. We have $\mathcal{D}(T) = TB_1$, and the reachable set has the shape of a ball at any time $T > 0$.

There is a well developed mathematical concept of shapes [2]. We will cite some further details in Section 3. In our example, for all $T > 0$, the sets $\mathcal{D}(T)$ have the same shape, so that there exists a limit of the shapes as $T \to 0$, and this limit is the shape of a ball rather than zero.

Another, less trivial and more important, example is related to the system $x^{(n)} = u, |u| \leq 1$. In the coordinates $x_i = x^{(n-i)}$, the system takes the form

$$\dot{x}_i = x_{i+1}, \quad i \leq n - 1, \quad \dot{x}_n = u, \quad |u| \leq 1.$$  

For an admissible trajectory $x(t)$ and $T > 0$, we define a new admissible trajectory $\hat{x}(t) = T^{-n}x(Tt)$. In the coordinates introduced above, $x_i(t) = T^{-i-1}x(Tt)$. Let us define the matrix $\delta(T) = \text{diag}(T^{-n}, T^{-1-n}, ..., T^{-1})$. The map $x \mapsto x_T$ is bijective, which proves that $\delta(T)\mathcal{D}(T) = \mathcal{D}(1)$, so that the shapes of the reachable sets $\mathcal{D}(T)$ do not depend on $T$.

In this paper, we show that, for any completely controllable linear system, the shapes of reachable sets converge as time tends to zero. We will give an estimate for the rate of convergence. The paper can be regarded as the $T \to 0$ supplement of [2], where the existence of a limit shape as time goes to infinity was established.

3. SHAPES OF REACHABLE SETS

Consider the metric space $\mathbb{B}$ of central symmetric convex bodies with the Banach-Mazur distance $\rho$:

$$\rho(\Omega_1, \Omega_2) = \ln(t(\Omega_1, \Omega_2))(\Omega_2, \Omega_1)), \quad t(\Omega_1, \Omega_2) = \inf \{t \geq 1: t\Omega_1 \supset \Omega_2 \}. \quad (2)$$

The general linear group $GL(\mathbb{V})$ naturally acts on the space $\mathbb{B}$ by isometries. The quotient space $\mathbb{S}$ is called the space of shapes of central symmetric convex bodies, where the shape $\text{Sh}\Omega \in \mathbb{S}$ of a convex body $\Omega \in \mathbb{B}$ is the orbit $\text{Sh}\Omega = \{g\Omega: \det g \neq 0\}$ of the point $\Omega$ under the action of $GL(\mathbb{V})$. The Banach-Mazur quotient metric

$$\rho(\text{Sh}\Omega_1, \text{Sh}\Omega_2) = \inf_{g \in GL(\mathbb{V})} \rho(g\Omega_1, \Omega_2)$$

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**Birth of the Shape of a Reachable Set**

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makes $\Sigma$ into a compact metric space. In what follows, the convergence of the reachable sets $\mathcal{D}(T)$ and their shapes is understood in the sense of the Banach-Mazur metric. If $\rho(\Omega_1(T), \Omega_2(T)) \to 0$ as $T \to 0$, we say that the convex bodies $\Omega_1$ and $\Omega_2$ are asymptotically equal and write $\Omega_1(T) \sim \Omega_2(T)$. The asymptotic equivalence of shapes is defined in a similar way.

4. MAIN RESULT: THE AUTONOMOUS CASE

We assume that system (1) is time-invariant and the Kalman controllability condition holds. The Kalman condition ensures that the reachable sets $\mathcal{D}(T)$ of system (1) are central symmetric convex bodies in $\mathbb{R}^n$.

**Theorem 1.** The shapes $\text{Sh} \mathcal{D}(T)$ have a limit $\text{Sh}_0$ as $T \to 0$. The Banach-Mazur distance $\rho(\text{Sh} \mathcal{D}(T), \text{Sh}_0)$ is $O(T)$.

This means that there exists a time independent convex body $\Omega$ such that

$$\mathcal{D}(T) \sim C(T)\Omega,$$

where $C(T)$ is an invertible matrix for all $T > 0$. The Banach-Mazur distance between the left- and right-hand sides of the last formula is $O(T)$.

Note that the initial reachable set $\mathcal{D}(0) = \{0\}$ does not belong to the space $\mathcal{B}$ of symmetric convex bodies. The Banach-Mazur distance between $\text{Sh} \mathcal{D}(T)$ and $\text{Sh} \mathcal{D}(0)$ equals infinity.

The proof of Theorem 1 is based upon the Brunovsky normal form and the following two easy lemmas.

**Lemma 1.** Suppose that a linear system

$$\dot{x} = \tilde{A}x + \tilde{B}u, \quad u \in U,$$

is obtained from system (1) by a gauge transformation.

Here, $\tilde{A} = C^{-1}AC$, $\tilde{B} = C^{-1}B$, and $C$ is an invertible matrix. Then $\text{Sh} \mathcal{D}(T) = \text{Sh} \mathcal{D}(T)$, where $\mathcal{D}(T)$ is the reachable set of system (4).

**Lemma 2.** Suppose that a linear system

$$\dot{x} = \tilde{A}x + \tilde{B}u, \quad u \in U$$

is obtained from system (1) by adding a linear feedback, that is, $\tilde{A} = A + BC$ and $\tilde{B} = B$. Then the Banach-Mazur distance $\rho(\mathcal{D}(T), \mathcal{D}(T))$ is $O(T)$ as $T \to 0$,

where $\mathcal{D}(T)$ and $\mathcal{D}(T)$ are the reachable sets of systems (5) and (1), respectively.

In view of Lemma 1, we conclude that applying gauge transformations does not change the shapes of reachable sets. Lemma 2 implies that the reachable set of system (1) is asymptotically equal to the reachable set of the system obtained by adding a linear feedback. By such transformations one can reduce the general system (1) to the Brunovsky normal form [1], where the matrices $A$ and $B$ are the direct sums $A = \bigoplus A_i$ and $B = \bigoplus B_i$, and the matrices $A_i$ and $B_i$ of sizes $n_i \times n_i$ and $n_i \times 1$, respectively, have the form

$$A_i = \begin{pmatrix} 0 & 1 \\ 0 & \ddots & \ddots \\ 0 & \cdots & 1 \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}. \ (6)$$

Note that the Brunovsky classification is closely related to the Grothendieck theorem [3] on a decomposition of vector bundles on $\mathbb{P}^1$ into a sum of linear bundles.

To the Brunovsky system (1), (6) we can relate the distinguished matrix function $\delta = \mathcal{D}_\delta$, where $\delta_i(T) = \text{diag}(T^{-\varepsilon}, T^{-\varepsilon + 1}, \ldots, T^{-1})$.

Note that $\delta A \delta^{-1} = T^{-1}A$, $\delta B = T^{-1}B$.

This immediately implies that, for fixed $T$ and $y = \delta x$, we have

$$\dot{y} = T^{-1}(Ay + Bu), \quad y(0) = 0. \quad (7)$$

Equation (7) reveals the geometric meaning of the matrix $\delta(T)$: the corresponding gauge transformation is equivalent to the application of the time scaling $t \mapsto \frac{1}{T}t$ to system (1), (6). Thus, we conclude that the shapes $\text{Sh} \mathcal{D}(T)$ of the reachable sets of the Brunovsky system do not depend on $T$.

Now, we return to the proof of Lemmas 1 and 2. Lemma 1 is obvious. Let us establish the truth of Lemma 2. Consider a trajectory $t \mapsto x(t)$ of system (1) and the corresponding trajectory $\tilde{x}(t)$ of (5). We have

$$\dot{x}(t) = A\tilde{x}(t) + Bu(t),$$

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + B(u(t) + C\tilde{x}(t)).$$

It is clear that $C\tilde{x}(t) = O(t)$, and, therefore, for all $t \leq T$, the control vector $u(t) = u(t) + C\tilde{x}(t)$ belongs to the set $(1 + \varepsilon)U$, where $\varepsilon = O(T)$. This means that $\mathcal{D}(T) \subset (1 + \varepsilon)\mathcal{D}(T)$. Since the relation between systems (1) and (5) is symmetric, we obtain in a similar way that $\mathcal{D}(T) \subset (1 + \varepsilon)\mathcal{D}(T)$. But this implies Lemma 2 in view of the definition of the Banach-Mazur distance (2).

5. THE NONAUTONOMOUS CASE

In fact, the same phenomenon of the existence of a limit shape takes place in the nonautonomous case. It suffices to introduce a kind of a genericity condition generalizing the Kalman one, which we used in the time-invariant case.

We study system (1), where the data $A$, $B$, and $U$ are now $C^n$-functions of time $t \geq 0$. First, by a standard trick, we make (1) into the time-invariant system

$$\dot{x} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

$$\dot{x} = A(t)x + B(t)u.$$
Second, consider the Lie algebra \( \mathcal{L} \) generated by the vector fields \((1, A(\tau)x)\) and \((0, B(\tau)u)\) in \( \mathbb{R} \times \mathcal{V} = \mathbb{R}^{n+1} \), where \( u \in \mathbb{R}^m \) is a constant vector. We define \( \mathcal{L}(\tau, x) \) as the set of the values at \((\tau, x)\) of all vector fields from \( \mathcal{L} \).

We use the following Kalman type condition as a standing assumption:

For each \((\tau, x) \in \mathbb{R} \times \mathcal{V} \), the set \( \mathcal{L}(\tau, x) \) coincides with the entire tangent space \( \mathbb{R}^{n+1} \). In other words,

\[
\dim \mathcal{L}(\tau, x) = n + 1.
\]

The gist of condition (8) is that the reachable set of the system is a body. It is well known that, in the time-invariant case, this assumption coincides with the Kalman controllability condition.

This condition can also be restated as follows. Consider the differential operator \( \tilde{A} = \frac{\partial}{\partial t} \tilde{A} \) on matrix functions of time \( C(t) : \mathbb{R}^m \to \mathbb{R}^n \) and form the infinite compound matrix \( K = [B \tilde{A} B ... \tilde{A}^k B ...] \). Then condition (8) is equivalent to \( \text{rank } K = n \).

**Theorem 2.** Let \( \mathcal{D}(T) \) be the reachable set of a non-autonomous system of the form (1) and the genericity condition (8) hold. Then the shapes \( Sh_0(T) \) have a limit \( Sh_0 \) as \( T \to 0 \). Moreover, the Banach-Mazur distance \( \rho(Sh_0(T), Sh_0) = O(T) \).

The proof of this theorem is based on a reduction to the case \( \tilde{A} = 0 \) and a subsequent study of the filtration \( F_k = \{ \xi \in \mathcal{V}^* : B(\tau)\xi = O(\tau^k) \} \) of the dual space \( \mathcal{V}^* \).

6. CONCLUSION

We hope that our results can be applied to more specific issues of control theory, e.g., for designing a feedback control that brings the system (1) to the equilibrium in finite time. The time-invariant case was studied in [5], where it was shown that, by means of standard linear algebra tools, one can design a bounded feedback control bringing a time-invariant linear system to the origin, provided that the initial state vector \( x \) is sufficiently small. Moreover, the required time \( \tau(x) \) of delivery is comparable with the absolutely minimal one \( \tau_{\text{min}}(x) \), which means that \( \frac{\tau(x)}{\tau_{\text{min}}(x)} \) is bounded as \( x \to 0 \). Possibly, in the nonautonomous case, a similar result can be obtained by using Theorem 2 instead of the Brunovsky normal form used in [5].

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