THE RATE OF CONVERGENCE OF THE WALK ON SPHERES
ALGORITHM

ILIA BINDER AND MARK BRAVERMAN

ABSTRACT. In this paper we examine the rate of convergence of one of the standard algorithms for
emulating exit probabilities of Brownian motion, the Walk on Spheres (WoS) algorithm. We obtain
the complete characterization of the rate of convergence of WoS in terms of the local geometry of
a domain.

1. INTRODUCTION

The harmonic measure on a bounded domain $\Omega \subset \mathbb{R}^d$ at $x \in \Omega$ can be described as an exit distribution of Brownian motion (see [GM04]). This measure plays an important role in various problems of Probability Theory, Geometric Function Theory, Dynamical Systems, Partial Differential Equations, as well as in a vast range of problems of Applied Mathematics. The problem of efficiently sampling from harmonic measure is therefore a key problem in Computational Mathematics.

One of the simplest and most commonly used methods for sampling from harmonic measure is the Walk on Spheres (WoS) algorithm. It was first proposed in 1956 by M. Muller in [Mul56]. Roughly speaking, the algorithm consists of replacing the Brownian Motion by a martingale $\{X_t : t \in \mathbb{Z}_{\geq 0}\}$, such that $X_0 = x$, and $X_t$ is uniformly distributed on a sphere centered at $X_{t-1}$ of a radius which is a certain proportion of the distance form $X_{t-1}$ to the boundary $\partial \Omega$ (see Section 1.1 for the precise definition).

It is not hard to see that it takes at most $O(1/\varepsilon^2)$ steps for the WoS process to reach an $\varepsilon$-neighborhood of $\partial \Omega$. However, in many situations, this rate of convergence is unsatisfactory. In particular, if we wanted to get $2^{-n}$-close to the boundary, it would take us a number of steps exponential in $n$. As it turns out, that, depending on the local geometry of the boundary of the domain, the rate of convergence is polynomial or even linear in $n$ (i.e. logarithmic in $1/\varepsilon$).

Logarithmic rate of convergence of the process $X_t$ to the boundary was established for convex domains by M. Motoo in [Mot59]. It was later generalized by G.A. Mikhailov in [Mih79] to planar domains satisfying any cone condition (i.e. at every point of the boundary there is a cone of certain fixed opening in the complement of the domain), as well for 3-dimensional domains satisfying a cone condition with large enough surface angle. See also [EKMS80] and [Mil95] for additional historical background and the use of the algorithm for solving various types of boundary value problems.

In our earlier work [BB07], we established polylogarithmic, but not logarithmic, upper bounds on the rate of convergence of WoS for planar domains, and for a restricted class of higher-dimensional domains. Unfortunately, the techniques of [BB07] do not generalize well to general domains in higher dimensions.

Our present results subsume all prior work on the rate of convergence of the WoS. We introduce an easily verified metric condition on the domain which provides tight bounds for the rates of convergence. We also show that the condition is tight.

2000 Mathematics Subject Classification. 60G42, 65C05, 31B25, 31B05.

Key words and phrases. Walk on Spheres algorithm, Harmonic measure, Potential Theory.

The first author was partially supported by NSERC Discovery grant 5810-2004-298433. This research was partially conducted during the period the second author was employed by the Clay Mathematics Institute as a Liftoff Fellow.
1.1. The Walk on Spheres algorithm. Let us now define the WoS. We would like to simulate a BM in a given bounded domain $\Omega$ until it gets $\varepsilon$-close to the boundary $\partial \Omega$. Of course one could simulate it using jumps of size $\delta$ in a random direction on each step, but this would require $O(1/\delta^2)$ steps. Since we must take $\delta = O(\varepsilon)$, this would also mean that the process may take $O(1/\varepsilon^2)$ steps to converge.

The idea of the WoS algorithm is very simple: we do not care about the path the BM takes, but only about the point at which it hits the boundary. Thus if we are currently at a point $X_t \in \Omega$ and we know that

$$d(X_t) := d(X_t, \partial \Omega) \geq r,$$

i.e. that $X_t$ is at least $r$-away from the boundary, then we can just jump $r/2$ units in a random direction from $X_t$ to a point $X_{t+1}$. To justify the jump we observe that a BM hitting the boundary would have to cross the sphere

$$S_t = \{ x : |x - X_t| = r/2 \}$$

at some point, and the first crossing location $X_{t+1}$ is distributed uniformly on the sphere. There is nothing special about a jump of $d(X_t)/2$ and it can be replaced with any $\beta d(X_t)$ where $0 < \beta < 1$. Let $\{\gamma_t\}, t \in \mathbb{Z}_{\geq 0}$ be a sequence of i.i.d. random variables each being a vector uniformly distributed on the unit sphere in $\mathbb{R}^d$. We could take, for example, $\gamma_t = \Gamma_t^d / |\Gamma_t^d|$, where $\Gamma_t^d$ is a normally distributed $d$-dimensional Gaussian variable. Then, schematically, the Walk on Spheres algorithm can be presented as follows:

```
WalkOnSpheres($X_0, \varepsilon$)
    $n := 0$;
    while $d(X_t) = d(X_t, \partial \Omega) > \varepsilon$ do
        compute $r_t$: a multiplicative estimate on $d(X_t)$ such that $\beta \cdot d(X_t) < r_t < d(X_t)$;
        $X_{t+1} := X_t + (r_t/2) \cdot \gamma_t$;
        $t := t + 1$;
    endwhile
    return $X_t$
```

Thus at each step of the algorithm we jump at least $\beta/2$ and at most $1/2$-fraction of the distance to the boundary in a random direction. An example of running the WoS algorithm in 2-d is illustrated on Figure 1.

![Figure 1](image.png)

**Figure 1.** An illustration of the WoS algorithm for $d = 2$: one step jump (a), and a possible run of the algorithm for several steps (b)

The proof of the convergence of the algorithm can be found, for example, in [GM04]. Moreover, it is not hard to see that the WoS process hits the $\varepsilon$-neighborhood of $\partial \Omega$ in $O(1/\varepsilon^2)$ steps. However,
in many situations, this rate of convergence is unsatisfactory. In particular, if we wanted to get $2^{-n}$-close to the boundary, it would take us a number of steps exponential in $n$. As it turns out, in many natural situations, the rate of convergence is polynomial or even linear in $n$ (i.e. logarithmic in $1/\varepsilon$). The object of the paper is to prove that this is the case, and give precise condition on when the faster convergence occurs.

While an actual implementation of the WoS would involve round-off errors introduced through an imperfect simulation, we will ignore those to simplify the presentation as they do not affect any of the main results. Thus the problem becomes purely that of analyzing the family of stochastic processes $\{X_t\}$ and their convergence speed to $\partial \Omega$.

1.2. Results. Let $H_\beta(K)$ denote the $\beta$-dimensional Hausdorff content of $K$.

$$ H_\beta(K) = \inf_{K \subset \bigcup B(x_j, r_j)} \sum r_j^\beta. $$

**Definition 1.** A domain $\Omega \subset \mathbb{R}^d$ is said to be $\alpha$-thick $0 \leq \alpha \leq d$ if there exists a constant $C > 0$ such that for every $x \in \partial \Omega$

$$ H^{d-\alpha}(B(x, r) \setminus \Omega) \geq Cr^{d-\alpha}, \quad r < 1 $$

Roughly speaking, $\alpha$-thick domains have complements of codimension $\alpha$, which are uniformly large at every scale at every boundary point.

We call the constant $c$ the thickness of the domain $\Omega$. It is not hard to see that the property of $\alpha$-thickness is monotone: an $\alpha$-thick domain is $\alpha'$-thick for $\alpha < \alpha' \leq d$.

Let us list some examples of $\alpha$-thick domains.

1. All $d$-dimensional domains are $d$-thick;
2. All bounded $d$-dimensional domains $\Omega$ such that the complement $\Omega^c$ is connected are $d-1$-thick;
3. All convex domains and all domains satisfying cone condition are $0$-thick;
4. All domains $\Omega$ that are bounded by a smooth hypersurface $\partial \Omega$ are $0$-thick.

It turns out that the $\alpha$-thickness of the domain is responsible for the rate of convergence of the WoS algorithm. This idea is formulated precisely in our Main Theorem.

**Theorem 2.** Let $\Omega$ be a bounded $\alpha$-thick domain in $\mathbb{R}^d$. Then the expected rate of convergence of the WoS from any $x \in \Omega$ until termination at distance $< \varepsilon$ to the boundary is given by the following table:

| $\alpha$   | Rate of convergence                                      |
|------------|----------------------------------------------------------|
| $\alpha < 2$ | $O(\log 1/\varepsilon)$                              |
| $\alpha = 2$ | $O(\log^2 1/\varepsilon)$                             |
| $\alpha > 2$ | $O((1/\varepsilon)^{2-4/\alpha})$                    |

The $O(\cdot)$ in the expressions above depends on the dimension $d$, on $\alpha$, on the thickness constant $C$ from Definition 1 and on $\beta > 0$ from the definition of the WoS. It does not depend directly on $\Omega$. Moreover, the rates of convergence above are tight. That is, for each $\alpha$ there is a family of $\alpha$-thick domains $\Omega^n_\alpha$ with some thickness $C$, such that the rate of convergence with $\varepsilon = 1/n$ on $\Omega^n_\alpha$ is asymptotically given by the formulas in (1).

The rate of convergence cannot be better than $O(\log 1/\varepsilon)$ since at each step of the WoS, the distance of $X_t$ to the boundary $\partial \Omega$ decreases by at most a multiplicative constant. An intuitive explanation to the phase transition phenomenon occurring at $\alpha = 2$, is that a BM in $\mathbb{R}^d$ almost surely “misses” sets of co-dimension $> 2$, while hitting sets of co-dimension $< 2$ with positive probability.

It is worth noting that the main result in [BB07] is the special case $\alpha = d = 2$ of the theorem.

The following corollaries are implied directly by the Theorem 2.
Corollary 3.  
(1) Since any planar domain is 2-thick, the WoS converges in \( O(\log^2 1/\varepsilon) \) steps;  
(2) since any planar domain with connected exterior is 1-thick, the WoS converges in \( O(\log 1/\varepsilon) \) steps;  
(3) since any domain in \( \mathbb{R}^d \) is \( d \)-thick, for \( d \geq 3 \) the WoS converges in \( O((1/\varepsilon)^{2-4/d}) \) steps;  
(4) since any 3-dimensional domain with connected exterior is 2-thick, the WoS converges in \( O(\log^2 1/\varepsilon) \) steps;  
(5) since for any \( d \geq 4 \), any \( d \)-dimensional domain with connected exterior is \( d - 1 \)-thick, the WoS converges in \( O((1/\varepsilon)^{2-4/(d-1)}) \) steps;  
(6) since any domain bounded by a smooth hypersurface is 0-thick, the WoS converges in \( O(\log 1/\varepsilon) \) steps.

The rest of the paper is organized as follows. In Section 2 we construct the auxiliary boundary barrier measures and the energy functions. Using these functions, we prove the upper estimates of Theorem 2. More technical estimates on the energy function are done in Section 3. Finally, in Section 4 we present examples of \( \alpha \)-thick domains with the slow rate of convergence of the WoS process.

2. Upper bounds: energy functions

2.1. Construction of an auxiliary measure. In this section we will construct a family of measures near boundary points of an \( \alpha \)-thick domain. These measures will be used to construct energy functions, which, in turn, play crucial role in the proof of Theorem 2.

Lemma 4. There exists a constant \( c = c(\alpha, d, C) \) such that for any \( \alpha \)-thick domain \( \Omega \) with thickness \( C \) in \( \mathbb{R}^d \) and for any \( x \in \partial \Omega \), one can find a Borel measure \( \mu_x \) which satisfies the following conditions:

1. \( \text{supp}(\mu_x) \cap \Omega = \emptyset \), or, equivalently, \( \mu_x(\Omega) = 0 \);
2. for any \( y \in \mathbb{R}^d \) and \( r > 0 \), \( \mu_x(B(y, r)) \leq r^{d-\alpha} \);
3. for any \( r < 1 \), \( \mu_x(B(x, r)) \geq c \cdot r^{d-\alpha} \).

With a slight abuse of notation, we will also refer to the constant \( c \) from this lemma as the thickness of the domain.

Proof. The proof of the Lemma follows the standard reasoning that can be found in, say, Chapter II of [Car67].

Let us consider the dyadic grid selected so that the point \( x \) has coordinates \((1/3, 1/3, \ldots, 1/3)\). For an integer \( d \)-multi-index \( \gamma = (\gamma_1, \ldots, \gamma_d) \), let \( D_{k, \gamma} \) be the cube

\[
\{(x_1, \ldots, x_d) : \gamma_n 2^{-k} \leq x_n < (\gamma_n + 1) 2^{-k}, \ n = 1, \ldots, d\}.
\]

Let \( D_k(x) \) be the unique dyadic cube of the size \( 2^{-k} \) which contains \( x \). Note that \( x \) is always at distance \( 2^{-k}/3 \) from the boundary of \( D_k(x) \).

We will construct inductively the sequence of measures \( \nu_n \). They will satisfy the following properties:

a. \( \text{supp} \nu_n \cap \Omega = \emptyset \),

b. \( \nu_n(D_{k, \gamma}) \leq H^{d-\alpha}(D_{k, \gamma} \setminus \Omega) \) for \( 1 \leq k \leq n \)

c. \( \nu_n(D_k(x)) = H^{d-\alpha}(D_k(x) \setminus \Omega) \) for \( 1 \leq k \leq n \)

Let \( \nu_1 \) be a delta measure in a point of \( D_1(x) \setminus \Omega \) with the total mass \( H^{d-\alpha}(D_1(x)) \). It clearly satisfies all of our assumptions.

Assume now that the measure \( \nu_n \) has already been constructed. The measure \( \nu_{n+1} \) will be a sum of delta-measures on the points outside of \( \Omega \) lying in the cubes from the \( n+1 \)-st dyadic generation, such that \( \nu_{n+1}(D_{n, \gamma}) = \nu_n(D_{n, \gamma}) \) for all \( \gamma \) (so \( \nu_{n+1} \) will be obtained from \( \nu_n \) by re-distributing the
latter over the cubes of the \((n + 1)\)-st generation). Thus the measure \(\nu_{n+1}\) would automatically satisfy the second and the third condition for \(k \leq n\).

To construct \(\nu_{n+1}\), we use the following rule.

First, we set \(\nu_{n+1}(D_{n+1}(x)) = H^{d-\alpha}(D_{n+1}(x) \setminus \Omega)\). The measure \(\nu_{n+1}\) clearly satisfies our condition \((\mathbb{C})\) on \(D_{n+1}(x)\).

Second, for any other dyadic cubes \(D_{n+1,\gamma} \subset D_n(x)\), we assign the mass

\[
\nu_{n+1}(D_{n+1,\gamma}) = H^{d-\alpha}(D_{n+1,\gamma} \setminus \Omega) \frac{(\nu_n(D_n(x)) - \nu_{n+1}(D_{n+1}(x)))}{\sum_{D_{n+1,\delta} \subset D_n, D_{n+1,\delta} \neq D_{n+1}(x)} H^{d-\alpha}(D_{n+1,\delta} \setminus \Omega)},
\]

so that \(\nu_n(D_n(x)) = \nu_{n+1}(D_n(x))\). By sub-additivity of the Hausdorff content,

\[
\sum_{D_{n+1,\delta} \subset D_n} H^{d-\alpha}(D_{n+1,\delta} \setminus \Omega) \geq H^{d-\alpha}(D_n(x)),
\]

and hence

\[
\nu_{n+1}(D_{n+1,\gamma}) \leq H^{d-\alpha}(D_{n+1,\gamma} \setminus \Omega)
\]

for \(D_{n+1,\gamma} \subset D_n(x)\).

Finally, for any other dyadic cubes from \((n + 1)\)-st generation, we set

\[
\nu_{n+1}(D_{n+1,\gamma}) = H^{d-\alpha}(D_{n+1,\gamma} \setminus \Omega) \frac{\nu_n(D_n)}{\sum_{D_{n+1,\delta} \subset D_n} H^{d-\alpha}(D_{n+1,\delta} \setminus \Omega)},
\]

where \(D_n\) is the unique cube from the \(n\)-th dyadic generation containing \(D_{n+1,\gamma}\). Using the sub-additivity of the Hausdorff content, as above, we get the estimate \((\mathbb{D})\) for all cubes of the \((n + 1)\)-st generation. The construction again satisfies \(\nu_{n+1}(D_n) = \nu_n(D_n)\).

Let now \(\nu\) be any weak* limit point of the sequence \(\nu_n\). \(\nu\) is still supported outside of \(\Omega\). By the second property of the measures \(\nu_n\),

\[
\nu(D_{k,\gamma}) \leq \sum_{D_{k,\delta} \cap D_{k,\gamma} \neq \emptyset} H^{d-\alpha}(D_{k,\delta} \setminus \Omega) \leq 3^d H^{d-\alpha}(D_{k,\gamma}) < 3^d (\sqrt{d})^d 2^{-k(d-\alpha)}
\]

for all \(k\). Using the third property of the measures \(\nu_n\), the \(\alpha\)-thickness of \(\Omega\), and the fact that \(D_k(x)\) contains the ball of the radius \(2^{-k}/3\), we get that

\[
\nu(D_k(x)) \geq H^{d-\alpha}(D_k(x) \setminus \Omega) \geq c 3^{-d} 2^{-k(d-\alpha)}
\]

for any \(k\).

Every ball can be covered by certain \((d\)-dependent\) number of dyadic cubes of comparable size, so \((\mathbb{E})\) implies that \(\nu(B(y, r)) \leq r^{d-\alpha}\). Every ball centered at \(x\) also contains a dyadic cube of comparable (again, \(d\)-dependent) size, hence by \((\mathbb{F})\), \(\nu(B(x, r)) \geq r^{d-\alpha}\). Now we can set \(\mu_x\) to be an appropriately normalized measure \(\nu\).

\(\square\)

2.2. Energy Function of optimal growth. The heart of the proof of the upper bounds in Theorem \((\mathbb{F})\) is the construction of a subharmonic function with optimal growth at the boundary, the Energy Function \(U\) on \(\Omega\). We will construct \(U(x)\) so that it is “small” in the interior of \(\Omega\), and grows to \(\infty\) as \(x\) approaches the boundary \(\partial \Omega\). The \(\alpha\)-thickness of the domain allows us to establish that the value of \(U(X_t)\) grows in expectation as the WoS progresses. Thus after a certain number of steps \(U(X_t)\) will be large in expectation which would imply that \(X_t\) is close to \(\partial \Omega\) with high probability.
The construction of the function is based on the notion of a Riesz potential. For a finite Borel measure \( \mu \) on \( \mathbb{R}^d \), and \( \alpha < d \), the \( \alpha \)-Riesz potential of the measure \( \mu \) is defined by

\[
U^\mu_\alpha(x) = \frac{1}{d - \alpha} \int \frac{d\mu(z)}{|z - x|^{d - \alpha}}.
\]

For \( \alpha = d \), the \( d \)-Riesz potential is defined by

\[
U^\mu_\alpha(x) = \int \log \frac{1}{|z - x|} \, d\mu(z).
\]

The value \( U^\mu_\alpha(x) = \infty \) is allowed when the integral diverges.

An important special case is the case of \( \alpha = 2 \), the so-called Newton potential. We will denote \( U^\mu_2 \) simply by \( U^\mu \). In this case the expression under the integral is harmonic in \( \mathbb{R}^d \). It is well known (e.g. see [Lan72]) that the function \( U^\mu \) is superharmonic on \( \mathbb{R}^d \), and harmonic outside of \( \text{supp} \, \mu \).

More generally, outside of the support \( \mu \), we have the identity

\[
\Delta U^\mu_\alpha(y) = (d - \alpha + 2)(2 - \alpha)U^\mu_{\alpha-2}(y).
\]

It shows that for \( 0 < \alpha < 2 \), the function \( U^\mu_\alpha \) is subharmonic outside of \( \text{supp} \, \mu \).

The following important technical identity, which easily follows from Fubini’s Theorem and substitution, relates the local behavior of the measure \( \mu \) and the growth of its potential \( U^\mu_\alpha \). For \( \alpha < d \), we have

\[
U^\mu_\alpha(y) = \frac{1}{d - \alpha} \int_0^\infty \mu(B(y, t^{-1/(d-\alpha)})) \, dt = \int_0^\infty \frac{\mu(B(y, r))}{r^{d-\alpha+1}} \, dr,
\]

and for \( \alpha = d \),

\[
U^\mu_\alpha(y) = \int_{-\infty}^\infty \mu(B(y, e^{-t})) \, dt = \int_0^\infty \frac{\mu(B(y, r))}{r^d} \, dr.
\]

Let us now fix an \( \alpha \)-thick domain \( \Omega \subset B(0, 1) \subset \mathbb{R}^d \). Let us consider the set \( \mathcal{M} \) of all Borel measures \( \mu \) supported inside \( B(0, 2) \) and outside of \( \Omega \) (i.e. \( \mu(\Omega) = 0 \)), satisfying the following condition:

\[
\text{for any } y \in \mathbb{R}^d \text{ and } r > 0, \mu(B(y, r)) \leq r^{d-\alpha}
\]

Let us now introduce the Energy Function \( U(y) \). Recall that \( U^\mu(y) := U^\mu_2(y) \).

\[
U(y) := \begin{cases} 
\sup_{\mu \in \mathcal{M}} U^\mu_\alpha(y), & \text{when } \alpha \leq 2 \\
\sup_{\mu \in \mathcal{M}} U^\mu(y), & \text{when } \alpha \geq 2.
\end{cases}
\]

Since the set \( \mathcal{M} \) is weakly*-compact, for every \( y \in \partial \Omega \) there exists a measure maximizing the potential in \( \Omega \) at the point \( y \).

Let us summarize the properties of \( U(y) \) in the following claim. The proof uses the identities (8) and (9). Recall that \( d(y) = \text{dist}(y, \partial \Omega) \).

**Claim 5.** Let \( \Omega \) be an \( \alpha \)-thick domain. Then

1. \( U(y) \) is subharmonic in \( \Omega \).
2. For \( \alpha \leq 2 \), \( U(y) \leq \log \frac{2}{d(y)} \) for all \( y \in \Omega \).
3. For \( \alpha > 2 \), \( U(y) \leq \frac{1}{\alpha - 2} d(y)^{2-\alpha} \) for all \( y \in \Omega \).

**Proof.** Let \( \alpha \leq 2 \), \( y \in \Omega \) and \( \mu \in \mathcal{M} \). Equations (8) and (9) imply that

\[
U^\mu_\alpha(y) \leq \int_{d(y)}^2 \frac{1}{t} \, dt = \log \frac{2}{d(y)}.
\]
Similarly, for $\alpha > 2$, we will use the harmonic potential $U^\mu$. Let $\alpha > 2$, $y \in \Omega$ and $\mu \in \mathcal{M}$. Once again, (10), $\text{supp} \, \mu \cap \Omega = \emptyset$, and the equations (8) imply that

$$U^\mu(y) \leq \int_{d(y)}^2 \frac{1}{t^{\alpha-1}} \, dt \leq \frac{1}{(\alpha-2)d(y)^{\alpha-2}}.$$  

By equations (12) and (13), $U(y)$ is a supremum of a locally bounded family of subharmonic functions. Thus $U(y)$ is subharmonic.

The second and third statements of the claim follow directly from (12) and (13) respectively.

Let $X_t$ be the WoS process initiated at some point $X_0 = y \in \Omega$. Let us define a new process $U_t = U(X_t)$, the value of the energy function at the $t$-th step of the process. Note that because $U$ is subharmonic, $U_t$ is a submartingale, that is $E[U_{t+1}|U_t] \geq U_t$.

For the rest of the section let $n = 1/\varepsilon$. Claim 5 immediately implies that a large value of $U_t$ will guarantee the closeness to the boundary. More specifically,

**Claim 6.** For $\alpha \leq 2$, if $U_t > \log 2n$ then $d(X_t) < 1/n$.

For $\alpha > 2$, $U_t > (\alpha-2)n^{\alpha-2}$ implies $d(X_t) < 1/n$.

The proof of Theorem 2 relies on finer lower bounds on the function $U$, which would guarantee the optimal rate of boundary convergence. We prove the bounds in the next section. These bounds depend heavily on the value of $\alpha$. We first give a probabilistic proof of the upper bounds in Theorem 2 and then prove the finer estimates on $U$ in Section 3.

### 2.3. Logarithmic convergence: the case $\alpha < 2$

In the heart of the proof for this case lies the following strong estimate on the behavior of the Riesz potentials near the boundary.

**Lemma 7.** For any $\alpha < 2$ and $c > 0$, there exist two constants $\delta$ and $\eta$, such that the following holds.

Let $\Omega$ be an $\alpha$-thick domain in $\mathbb{R}^d$ with thickness $c$. Let $y \in \Omega$ and $x \in \partial \Omega$ be the closest point to $y$. Let $\mu \in \mathcal{M}$.

Then either

$$U(z) > U^\mu_{\alpha}(z) + 1 \text{ whenever } \delta/4 \cdot d(y) < |z - x| < \delta \cdot d(y).$$

or

$$\mu(B(y, 2d(y))) \geq \eta d(y)^{d-\alpha}.$$  

The lemma is established in Section 5.

Note that after $k = O(|\log \delta|)$ steps of the WoS process,

$$\delta/4 \cdot d(X_t) < |X_{t+k} - x| < \delta \cdot d(X_t)$$

with a certain probability $p$.

where $x$ is the point of $\partial \Omega$ that is closest to $X_t$, and $p > 0$ depends only on $\beta$ and the dimension $d$.

Let us fix $X_t$ and take the measure $\mu \in \mathcal{M}$ maximizing the value of $U^\mu_{\alpha}(X_t)$. By the preceding observation, in the first case in Lemma 7, the subharmonicity of $U$ implies that the expectation of $U_{t+k}$, conditioned on $U_t$, will increase by some definite constant.

On the other hand, using the identity (7) and the $\alpha$-thickness of $\Omega$, one can see that the Laplacian of $U^\mu_{\alpha}$ is large near the point $X_t$ in the second case of Lemma 7. Thus, since large Laplacian leads to a fast build-up of mean values, we have the above-mentioned increase by a constant after the first step. We arrive at the following estimate, which shows that $U_t$ grows at least linearly in expectation.

**Lemma 8.** There are constants $L$ and $k$, depending only on $c$, $\beta$, and $\alpha$, such that

$$E[(U_{t+k} - U_t)|U_t] > L.$$  

7
A detailed proof of the lemma can be found in Section 3.2. Lemma 8 implies that $E[U_t] > tL/k + U_0$. Since $d(X_t) \geq (1 - \beta)^t d(X_0)$, Claim 6 implies that $U_t \leq U_0 + t \log(1 - \beta) + \log 2$. This implies that $U_t > U_0 + tL/2k$ with probability at least $P$, where $P$ depends only on $\beta$. This, together with Claim 6, implies the necessary upper bound in the case $\alpha < 2$.

2.4. Polylogarithmic convergence: the case $\alpha = 2$. In the case $\alpha = 2$ the steady linear growth of $U_t$ given by Lemma 8 no longer holds. In fact, the only thing that generally holds in this case is the submartingale property $E(U_{t+1} | U_t) \geq U_t$. We are able to overcome this difficulty by showing that the submartingale process $\{U_t\}$ has a deviation bounded from below by a constant at every step. To this end it suffices to show that $U_t$ can grow by some $\eta$ with a non-negligible probability. We use the following estimate on the energy function (established in Section 3.1).

**Lemma 9.** There exists a constant $\delta$, dependent only on the thickness $c$, such that the following holds. Let $\Omega$ be a 2-thick domain. Let $y \in \Omega$ and $x \in \partial \Omega$ be the closest point to $y$. Then

$$U(z) > U(y) + 1 \text{ whenever } |z - x| < \delta \cdot d(y).$$

Since the function $U$ is subharmonic, observation (16) implies the following estimate (see Section 3.2 for a proof).

**Lemma 10.** Let $\Omega$ be a 2-thick domain in $\mathbb{R}^d$. There are constants $k$ and $L$, depending only on the thickness $c$, the jump ratio $\beta$, and the dimension $d$, such that

$$E[(U_{t+k} - U_t)^2 | U_t] > L.$$ 

We can now use Lemma 10 to prove the upper bounds on the rate of convergence for the case $\alpha = 2$. Let us replace the submartingale $U_t$ by a stopped submartingale

$$V_t = \begin{cases} U_t, & t < T_n \\ U_{T_n}, & t \geq T_n \end{cases}$$

By the optional stopping time theorem (see [KS91]), $V_t$ is also a positive submartingale; $V_t \leq \log \frac{A}{n}$. This implies, in particular, that

$$E[V_t(V_{t+k} - V_t)] = E[E[V_t(V_{t+k} - V_t) | V_t]] \geq E[E[V_t(V_t - V_t) | V_t]] = 0$$

Lemma 10 implies that

$$E[(V_{t+k} - V_t)^2] > L \cdot P[T_n > t + k].$$

We are now in a position to establish the upper bounds for $\alpha = 2$.

**Proof of the upper bound from Theorem 2** for $\alpha = 2$. Assume first that for some $M$, $P[T_n > M \log^2 n] \geq 1/2$.

It means that for all $t \leq M \log^2 n - k$, $P[T_n \geq t + k] \geq 1/2$. This implies

$$E[V_{t+k}^2] = E[((V_{t+k} - V_t) + V_t)^2] = E[V_t^2] + E[(V_{t+k} - V_t)^2] + 2E[V_t(V_{t+k} - V_t)] \geq E[V_t^2] + L/2.$$ 

The last inequality follows from (18) and (19). Hence $E\left[V_{t+k}^2 / M \log^2 n \right] \geq L M \log^2 n / 2k$. Since $V_t \leq \log \frac{A}{n}$, this leads to a contradiction for large enough $M$. \qed
2.5. **Polynomial convergence: the case** $\alpha > 2$. For the case $\alpha > 2$, the required converse to Claim 6 is relatively simple.

**Lemma 11.** For $\alpha > 2$, and an $\alpha$-thick domain $\Omega$ in $\mathbb{R}^d$ with the thickness $c$,

$$U(y) \geq K \cdot d(y)^{2-\alpha}$$

for all $y \in \Omega$. Here the constant $K = K(c, \alpha)$ depends only on $c$ and $\alpha$.

The lemma is established in Section [3.1](#).

The idea of the proof of Theorem 2 in this case is now as follows. When the WoS is far from the boundary $\partial \Omega$ it makes fairly big steps and when it is close it makes small steps. There are not too many big steps because the number of big steps of length $> \varepsilon$ confined to $B(0,1)$ is bounded by $O(1/\varepsilon^2)$. On the other hand, there are not too many small steps, because a small step means that the WoS is very close to $\partial \Omega$, and should converge before an opportunity to make many more steps. More precisely, the number of “big” jumps is bounded by the following Claim.

**Claim 12.** Let $N(\varepsilon, T)$ be the number of the jumps in the WoS process before the time $t$ which are bigger then $\varepsilon$, i.e.

$$N(\varepsilon, T) = \# \{ t \leq T \mid |X_t - X_{t-1}| \geq \varepsilon \}.$$.

Then

$$P\left[ N(\varepsilon, T) > \frac{4}{\varepsilon^2} \right] < 1/4.$$.

**Proof.** Note now that because $X_t$ is a martingale, we have

\begin{equation}
1 \geq E[X_T^2] - X_0^2 = \sum_{k=1}^{T} (E[X_t^2] - E[X_{t-1}^2]) = \sum_{t=1}^{T} E[(X_t - X_{t-1})^2] = \sum_{t=1}^{T} E[\sum_{k=1}^T (X_t - X_{t-1})^2] \geq \varepsilon^2 E[N(\varepsilon, T)]
\end{equation}

The last equation implies the statement of the claim, by Tschebyshev inequality.

---

**Figure 2.** The regions $R_k$ from the proof of the upper bounds for $\alpha > 2$

To bound the number of small jumps, we denote by $R_0 \subset \Omega$ the $1/n$-neighborhood of $\partial \Omega$, and more generally, by

$$R_k := \{ x \in \Omega : 2^{k-1}/n < d(x, \partial \Omega) \leq 2^k/n \}$$.
(see Figure 2). Note that by Lemma 11 we have
\[2^{(k-1)(2-\alpha)}n^{\alpha-2} \geq U(y) \geq K2^{(k-1)(2-\alpha)}n^{\alpha-2}\]
for \(y \in R_k\). Using this fact we prove the following.

**Claim 13.** Denote by \(v_k\) the number of visits of \(X_t\) to \(R_k\) before the time \(T\) when \(X_t\) first hits the \(1/n\)-neighborhood of the boundary \(\partial\Omega\),

\[v_k = \#\{t < T : X_t \in R_k\} \]

Then
\[\mathbb{P}[v_k > C_2 \cdot 2^{k(\alpha-2)}M] < 1/4^M,\]
for some constant \(C_2 = C_2(c, d, \alpha, \beta)\) and for any \(M > 1\).

**Proof.** Suppose that at some point \(t, X_t \in R_k\). We estimate from below the probability that this is the last time the WoS visits \(R_k\).

First of all, with some probability \(p = p(\alpha, \beta, c_i) > 0\) and for some constant \(\eta, X_{t+\eta} \in R_{k+2\log c_i}\), i.e. the series of first jump brings us much closer to \(\partial\Omega\). Consider the subharmonic function
\[\Phi(y) = (2n)^{2-\alpha}U(y) - 2^{k(2-\alpha)}\]
in \(\Omega\). Then the process \(\Phi(X_{t+\tau})\) is a submartingale. We stop it at time \(t + \tau, \tau \geq \eta\), when either the WoS terminates or when \(X_{t+\tau} \in R_k\) (i.e. the process gets back to \(R_k\)), whichever comes first. If \(X_{t+\tau}\) is \(1/n\)-close to \(\partial\Omega\) (but not closer than \(1/2n\)), then \(\Phi(X_{t+\tau}) \leq 1\), by (21). If \(X_{t+\tau} \in R_k\), then, again by (21), \(\Phi(X_{t+\tau}) \leq 0\). Another application of (21) implies that if \(y \in R_{k+2\log c_i}\), then \(\Phi(y) > \gamma\) for some constant \(\gamma\). Thus the probability that the WoS terminates at \(X_{t+\tau}\) (i.e. we never visit \(R_k\) again) is at least
\[\mathbb{P}[X_{t+\tau} \notin R_k] \geq \mathbb{E}[\Phi(X_{t+\tau})] \geq \Phi(X_{t+\eta}) \geq p\gamma.\]
Thus the probability that the visit \(X_k\) to \(R_k\) is the last one is at least \(p \cdot \gamma\). The claim now follows from an estimate of the probability of having at least \(v_k\) returns to \(R_k\), each of them not being the last one. \(\square\)

Claims 12 and 13 together imply the upper bounds on the rate of convergence for \(\alpha > 2\).

**Proof of the upper bounds from Theorem 2 for \(\alpha > 2\).** By Claim 13 for any \(k\), we have that
\[\mathbb{P}[v_{k-s} > C_2 \cdot 2^{k(s-\alpha-2)} \cdot (3/2 + s/2)] < 1/4^{3/2+s/2} = (1/8) \cdot 2^{-s}.\]

Hence, by union bound \(v_{k-s} \leq C_2 \cdot 2^{k(s-\alpha-2)} \cdot (3/2 + s/2)\) for all \(s \geq 0\) with probability at least \(3/4\). Let \(k\) be such that \(2^k \approx n^{2/\alpha}\). Then, with the probability at least \(3/4\), we have the total number of jumps smaller than \(2^{k}/n\) bounded by
\[\sum_{s=0}^{k} v_{k-s} \leq \sum_{s=0}^{k} C_2 \cdot 2^{k(s-\alpha-2)} \cdot (3/2 + s/2) < 4C_2 \cdot 2^{k(\alpha-2)} \approx 4C_2 \cdot n^{2-4/\alpha}.\]
If we take \(N = (C_1 + 8C_2)n^{2-4/d}\) steps of the WoS, (22) implies that at least half the steps would be of magnitude at least \(2^{k}/n \approx n^{2/d-1}\), except with probability \(< 1/4\). Applying the estimate from Claim 12 with \(\varepsilon = 2^k/n\), we see that with the probability at least \(3/4\), \(N(2^k/n, t) \leq 4n^{2}/2^{2k} \approx 4n^{2-4/\alpha}\). Hence with probability \(\geq 1/2\) the WoS terminates after \(O(n^{2-4/\alpha})\) steps. \(\square\)

3. **Boundary behavior of the energy function**

In this section we prove the analytical estimates on the behavior of the energy function that have been used in Section 2.
3.1. **Estimating boundary growth.** We start with the easiest case \( \alpha > 2 \).

**Lemma 11 (Section 2.5):** For \( \alpha > 2 \), and an \( \alpha \)-thick domain \( \Omega \) in \( \mathbb{R}^d \) with the thickness \( c \),

\[
U(y) \geq K \cdot d(y)^{2-\alpha}
\]

for all \( y \in \Omega \). Here the constant \( K = K(c, \alpha) \) depends only on \( c \) and \( \alpha \).

**Proof.** Let \( x \) be the closest to \( y \) point at \( \partial \Omega \), and let \( \mu = \mu_x \) be the corresponding measure from the definition of the \( \alpha \)-thick domains. Then, by the identity (8) and since \( B(x, r) \subset B(y, r + d(y)) \),

\[
U(y) \geq U^\mu_2(y) = \int_{d(y)}^2 \frac{\mu(B(y, r))}{r^{d-1}} dr \geq \int_{2d(y)}^2 \frac{\mu(B(x, r-d(x)))}{r^{d-1}} dr \geq c2^{\alpha-d} \int_{2d(y)}^2 t^{1-\alpha} \geq K \cdot d(y)^{2-\alpha}.
\]

\( \square \)

Unfortunately, in the case \( \alpha \leq 2 \) lower bounds of the type established in the proof of Lemma 11 are insufficient, and we will use finer estimates provided by the following construction.

Let \( y \) be a point in \( \Omega \), \( x \) be the point of \( \partial \Omega \) that is the closest to \( y \), and \( \mu \) be a measure in the class \( \mathcal{M} \). We construct a new measure \( \nu \in \mathcal{M} \), which we call the *amalgamation* of \( \mu \) at the point \( y \) in the following way.

Let measure \( \mu_1 \) be the measure \( \mu_x \) from Lemma 11 restricted to \( B(y, 2d(y)) \), \( \mu_2 = \mu_3 = 0 \), and for \( k \geq 4 \), let \( \mu_k \) be the measure \( \mu \) restricted to the \( d \)-dimensional annulus

\[
A_k = \{ w : 2^{k-1}d(y) \leq |w-y| \leq 2^kd(y) \}
\]

scaled by the factor \( 1 - \gamma_k := 1 - 2^{(4-k)(d-\alpha)} \). Let us also put \( \gamma_1 = \gamma_2 = \gamma_3 = 1 \). We define

\[
\nu := \sum_k \mu_k.
\]

The ingredients of the construction are illustrated on Figure 3.

![Figure 3. Construction of the amalgamation \( \nu = \sum_k \mu_k \)](image)
Let us now prove that \( \nu \in \mathcal{M} \). Consider any disk \( B(w, r) \). Let \( K \) be the largest number such that \( B(w, r) \) intersects \( A_K \). If \( B(w, r) \) does not intersect \( B(y, 2d(y)) \), the measure \( \nu \) is no greater than \( \mu \) on \( B(w, r) \), and thus \( \nu(B(w, r)) \leq r^{d-\alpha} \). If \( K \leq 3 \), \( \nu(B(w, r)) \leq \mu_x(B(w, r)) \leq r^{d-\alpha} \). For all other cases, \( r \geq 2^{K-3}d(y) \), which, by the choice of \( \gamma_K \), implies that \( \gamma_K r^{d-\alpha} \geq (2d(y))^{d-\alpha} \). Thus

\[
\nu(B(w, r)) \leq \mu_x(B(y, 2d(y))) + \mu(B(w, r)) - \sum_{k=1}^{K} \gamma_k \mu(B(w, r) \cap A_k) \leq (2d(y))^{d-\alpha} + (1 - \gamma_K)\mu(B(w, r)) \leq \gamma_K r^{d-\alpha} + (1 - \gamma_K)r^{d-\alpha} = r^{d-\alpha}.
\]

The second inequality follows from the fact that the sequence \( \{\gamma_k\} \) is non-increasing. We first apply the amalgamation construction to the case \( \alpha = 2 \).

**Lemma 9 (Section 2.3):** There exists a constant \( \delta \), dependent only on the thickness \( c \), such that the following holds. Let \( \Omega \) be a 2-thick domain. Let \( y \in \Omega \) and \( x \in \partial \Omega \) be the closest point to \( y \). Then

\[(24) \quad U(z) > U(y) + 1 \text{ whenever } |z - x| < \delta \cdot d(y). \]

**Proof.** Since \( \mathcal{M} \) is a compact set, \( U(y) = U^\mu(y) \) for some \( \mu \in \mathcal{M} \).

Let \( \mu_0 \) be the restriction of the measure \( \mu \) to \( B(0, 2) \setminus B(y, 2d(y)) \). By (8) and (10)

\[(25) \quad U^{\mu_0}(y) \geq U^\mu(y) - \log 2. \]

Let \( \nu \) be the amalgamation of \( \mu \) at \( y \). Next, we will show that

\[(26) \quad U^\nu(z) \geq U^{\mu_0}(z) - C_1 + c \cdot 2^{2-d} \log \frac{1}{\delta} \]

and

\[(27) \quad U^{\mu_0}(z) \geq U^{\mu_0}(y) - C_2 \]

whenever \( |z - x| < \delta d(y) \) for some constants \( C_1 \) and \( C_2 \) depending only on \( d \) and \( c \). These inequalities, together with (25), imply the statement of the lemma whenever \( \delta \) is sufficiently small (namely, when \( \log 1/\delta > 2^{d-2}(C_1 + C_2 + 1 + \log 2)/c \)).

To establish (26), let us note that for any \( k \) we have

\[\mu(A_1) + \mu(A_2) + \cdots + \mu(A_k) = \mu(B(y, 2^k d(y))) \leq (2^k d(y))^{d-2}.\]

By the Abel summation formula,

\[
\sum_k \gamma_k 2^k (2-d) \mu(A_k) \leq \sum_k d(y)^{d-2}(2^{d-2} \gamma_k - 1) - 2^{2-d} \leq 5(d(y))^{d-2}.
\]

This implies

\[(28) \quad \frac{1}{d-2} \sum_{k \geq 2} \gamma_k \int_{A_k} \frac{1}{|w - z|^{d-2}} d\mu_0(w) \leq \sum_{k \geq 2} \gamma_k \mu(A_k) (2^{k-2} d(y))^{2-d} \leq 5 \cdot 4^{d-2}. \]
Thus we obtain
\[
U^\nu(z) \geq \int_{2\delta d(y)}^{d(y)} \frac{\mu_x(B(z, r))}{r^{d-1}} \, dr + \sum_{k \geq 2} \int_0^\infty \frac{\mu_k(B(z, r))}{r^{d-1}} \, dr \geq \int_{2\delta d(y)}^{d(y)} \frac{\mu_x(B(z, r))}{r^{d-1}} \, dr + \int_0^\infty \frac{\mu_0(B(z, r))}{r^{d-1}} \, dr - \frac{1}{d-2} \sum_{k \geq 1} \gamma_k \int_{A_k} \frac{d\mu_0(w)}{w - z} \geq \int_{2\delta d(y)}^{d(y)} \frac{\mu_x(B(z, r))}{r^{d-1}} \, dr + \int_0^\infty \frac{\mu_0(B(z, r))}{r^{d-1}} \, dr - 5 \cdot 4^{d-2} \geq c \cdot 2^{2-d} \log \frac{1}{2\delta} + U^\nu_0(z) - 5 \cdot 4^{d-2},
\]
which implies (26).

To obtain (27), note that for any point \(w \in [y, z]\), for \(d > 2\) we have, by the estimate (10)
\[
|\nabla U^\nu_0(w)| \leq \frac{1}{d-2} \int \left| \nabla \frac{1}{|\xi - w|^{d-2}} \right| d\mu_0(\xi) = \int \frac{1}{|\xi - w|^{d-1}} d\mu_0(\xi) = (d - 1) \int_0^\infty \frac{\mu_0(B(w, r))}{r^{d}} \, dr = (d - 1) \int_0^\infty \frac{\mu_0(B(w, r))}{r^{d}} \, dr \leq (d - 1) \int_0^\infty \frac{r^{d-2}}{r^{d}} \, dr \leq \frac{A}{d(y)}
\]
for some constant \(A\) depending only on \(d\) and \(c\). The same inequality is derived similarly in the case \(d = 2\). This implies that
\[
U^\nu_0(z) - U^\nu_0(y) = \int_{[y, z]} \nabla U^\nu_0(w) \cdot dw \geq -|z - y| \frac{A}{d(y)} \geq -A,
\]
which is exactly the equation (27). \(\square\)

Another application of the amalgamation construction will establish the lower bounds required in the case \(\alpha < 2\).

**Lemma 7 (Section 2.3):** For any \(\alpha < 2\) and \(c > 0\), there exist two constants \(\delta\) and \(\eta\), such that the following holds.

Let \(\Omega\) be an \(\alpha\)-thick domain in \(\mathbb{R}^d\) with thickness \(c\). Let \(y \in \Omega\) and \(x \in \partial \Omega\) be the closest point to \(y\). Let \(\mu \in \mathcal{M}\).

Then either
\[
U(z) > U_\alpha^\nu(z) + 1 \text{ whenever } \delta/4 \cdot d(y) < |z - x| < \delta \cdot d(y).
\]

or
\[
\mu(B(y, 2d(y))) \geq 2d(y)^{-d-\alpha}
\]

**Proof.** Let \(\eta = \delta^{d-\alpha+2}\). Assume that \(\mu(B(y, 2d(y))) < \eta d(y)^{d-\alpha}\). Let \(\mu_0\) be the restriction of \(\mu\) to \(B(0, 2) \setminus B(y, 2d(y))\), as in the previous lemma. We have, by (5),
\[
U_\alpha^\mu(z) \geq \mu(B(y, 2d(y))) - \int_{2d(y)}^{2d(y)} \frac{\mu(B(z, r))}{r^{d-\alpha+1}} \, dr \geq \mu(B(y, 2d(y))) - 2d(y)\delta.
\]

On the other hand, the same reasoning as the proof of (26) above, gives
\[
U(z) \geq U_\alpha^\nu(z) - C_1 + c \cdot 2^{\alpha-d} \log \frac{1}{\delta}
\]
for some constant \(C_1\) depending only on \(d\), \(\alpha\), and \(c\). Here, as in (26), \(\nu\) is the amalgamation of \(\mu\) at \(y\).

Estimates (32) and (33) together imply the statement of the lemma. \(\square\)
3.2. The boundary drift of the WoS process: \( \alpha \leq 2 \). First we establish that the process \( U_t \) has the drift toward the boundary in the case \( \alpha < 2 \).

**Lemma 8 (Section 2.3):** There are constants \( L \) and \( k \), depending only on \( c, \beta, \) and \( \alpha \), such that
\[\mathbb{E}[(U_{t+k} - U_t)|U_t] > L.\]

**Proof.** Let us fix \( X_t \). By weak*-compactness of the set \( \mathcal{M} \), there exists a measure \( \mu \) such that \( U^\mu_0(X_t) = U(X_t) = U_t \). By Lemma 7 either
\[ U(z) > U^\mu_0(z) + 1 \text{ whenever } \delta/2 \cdot d(X_t) < |z - x| < \delta \cdot d(X_t). \]
where \( x \) is the closest to \( X_t \) point on \( \partial \Omega \), or
\[ \mu(B(X_t, 2d(X_t))) \geq \eta d(X_t)^{d-\alpha} \]
Let us start with the first case.
For some \( p > 0 \) dependent only on \( d \) and \( \beta \),
\[ (1 - \beta/2)^{-k} d(X_t)/2 < \mathbb{P}[[X_{t+k} - x] < (1 - \beta/2)^{-k} d(X_t)] > p^k. \]
Hence, for sufficiently large \( k \),
\[ \mathbb{P}[\delta/2 \cdot d(X_t) < |X_{t+k} - x| < \delta \cdot d(X_t)] > p^k. \]
Let us now observe that by subharmonicity of the functions \( U \) and \( U^\mu_0 \), the previous estimate, the fact that \( U \geq U^\mu_0 \) and the assumption \[ \mathbb{P}[(U_{t+k} - U_t) | X_t] = \mathbb{E}[(U^\mu_0(X_{t+k}) - U^\mu_0(X_t)) | X_t] + \mathbb{E}[U(X_{t+k}) - U^\mu_0(X_{t+k}) | X_t] \geq \mathbb{E}[U(X_{t+k}) - U^\mu_0(X_{t+k}) | X_t] \text{ and } \delta/2 \cdot d(X_t) < |X_{t+k} - x| < \delta \cdot d(X_t)] > p^k. \]
Since the value of \( X_t \) determines the value of \( U_t \), this establishes the statement of Lemma in the first case (with \( L = p^k \)).
Now let us consider the second case. By the Green formula, for a \( C^2 \)-smooth function \( u \),
\[ \mathbb{E}[u(X_{t+1}) | X_t] - u(X_t) = \int_{\beta d(X_t) S^d} u(y) dS(y) - u(X_t) = \int_0^{\beta d(X_t)} r^{1-d} \int_{B(x_t, r)} \Delta u(y) dV(y) dr \]
where \( S^d \) is the unit sphere in \( \mathbb{R}^d \) with the normalized Lebesgue measure \( S \), and \( dV \) is the volume element in \( \mathbb{R}^d \).
Note that by (3) and (7), for \( |y - X_t| \leq \beta d(X_t) \) we have
\[ \Delta U^\mu_0(y) = (d - \alpha + 2)(2 - \alpha) U^\mu_{\alpha-2}(y) =
\]
\[ (d - \alpha + 2)(2 - \alpha) \int_0^\infty \frac{\mu(B(y, r))}{r^{d-\alpha+3}} dr \geq
\]
\[ (d - \alpha + 2)(2 - \alpha) \mu(B(X_t, 2d(X_t))) \int_0^\infty \frac{1}{(2+\beta)d(X_t)} \frac{1}{r^{d-\alpha+3}} dr \geq
\]
\[ \eta(2 - \alpha) \frac{\mu(B(X_t, 2d(X_t)))}{(2 + \beta)d(X_t)} \geq C_1(d(X_t))^{-2}. \]
for some constant \( C_1 \) depending only on \( d, \alpha, \) and \( \beta \).
So, using (3), applied to \( u = U^\mu \) and (39), we get
\[ \mathbb{E}[U_{t+k} | X_t] - U_t \geq \mathbb{E}[U_{t+1} | X_t] - U_t \geq \mathbb{E}[U^\mu(X_{t+1}) | X_t] - U^\mu(X_t) \geq
\]
\[ \int_0^{\beta d(X_t)} r^{1-d} (C_2(d(X_t))^{-2} r^d) dr = C_2 ((d(X_t))^2) (d(X_t)^{-2} \beta^2/2 = L. \]
for some constants $C_2$ and $L$ depending only on $d$, $\alpha$, and $\beta$. Since again the value of $X_t$ determines the value of $U_t$, the Lemma follows.

Let us now turn to the case $\alpha = 2$.

**Lemma 10 (Section 2.4):** Let $\Omega$ be a 2-thick domain in $\mathbb{R}^d$. There are constants $k$ and $L$, depending only on the thickness $c$, the jump ratio $\beta$, and the dimension $d$, such that

$$E[(U_{t+k} - U_t)^2|U_t] > L.$$ 

**Proof.** Fix $X_t$. By Lemma 9 there exists a constant $\delta$, dependent only on $d$ and $c$, such that

$$U(y) > U(X_t) + 1 \text{ whenever } |y - x| < \delta \cdot d(X_t).$$

This implies that $||U_{t+k} - U_t||^2 > 1$ whenever $|X_{t+k} - x| < \delta d(X_t)$. Note that for some $p > 0$
dependent only on $d$ and $\beta$,

$$P[|X_{t+k} - x| < (1 - \beta/2)^{-k} d(X_t)] > p^k.$$ 

Hence, for sufficiently large $k$, $P[|X_{t+k} - x| < \delta d(X_t)] > p^k$, which, in turn, implies the statement of the lemma.

4. Lower bounds: examples

In this section we construct examples of $\alpha$-thick domains for which the bounds in Theorem 2 are tight. The main idea of the construction is as follows. We take a domain $A$ in $\mathbb{R}^d$, such as the unit ball or a cylinder. We remove a “thin” subset of points $C$ from $A$ to obtain $\Omega = A \setminus C$. The set $C$ can be thought of as the subset of the grid $(\gamma \mathbb{Z})^d$, for some small $\gamma > 0$. The set $C$ will be chosen so that it “separates” the origin from the boundary of $A$. We set $n = 1/\varepsilon$. We choose $\gamma$ so that the probability of the WoS originated at 0 hitting a $1/n$-neighborhood of $C$ before hitting the boundary of $A$ is $< 1/2$ (this means that $C$ is “thin”). Hence, with high probability, the WoS will reach $\partial A$ before terminating. However, in this case the WoS will have to “pass through” the set $C$, where its step magnitudes are bounded by $\gamma$. This will, in turn, yield an $\Omega(1/\gamma^2)$ bound on the convergence time. The analysis is more intricate in the case when $\alpha = 2$. In the case when $\alpha > 2$ is not an integer, a slight modification to this construction is needed, as will be described below.

4.1. Proof of the lower bound in the case $\alpha > 2$. In this section we will give an example of a “thin” $\alpha$-thick domain $\Omega_\alpha$ for which the WoS will likely take $\Omega(n^{2-4/\alpha})$ steps to converge within $\varepsilon = 1/n$ from the boundary $\partial \Omega_\alpha$. The domain $\Omega_\alpha$ will reside in $\mathbb{R}^d$, where $d = [\alpha] \geq 3$. It is easy to see that the examples in higher dimensions $d' > d$ can be constructed from $\Omega_\alpha$ by simply multiplying $\Omega_\alpha$ by $[-1,1]^{d'-d}$.

The set

$$\Omega_\alpha := (B(0,1)_{d-1} \times [-1,1]) \setminus S$$

is comprised of a $d$-dimensional cylinder with a set of points $S$ removed. Here $B(0,1)_{d-1}$ denotes the unit ball in $\mathbb{R}^{d-1}$. We take $A$ to be the “middle 1/3” shell of the $d$-dimensional cylinder:

$$A = \{z \in \mathbb{R}^{d-1} : 1/3 < |z| < 2/3\} \times \{x \in [-1,1] : 1/3 < |x| < 2/3\}.$$ 

Let $0 < \gamma \ll 1$ be the grid size that will be selected later. We consider the set $A_\gamma$ of gridpoints in $A$.

$$A_\gamma = (\gamma \mathbb{Z})^d \cap A.$$
Let $0 \leq \eta := d - \alpha < 1$. Denote by $C_\eta$ the $\eta$-dimensional Cantor set in the interval $[0, 1]$. It is obtained by removing the middle $\lambda$-fraction of the interval, then removing the middle $\lambda$-fraction of each subinterval etc. For the set $C_\eta$ to be $\eta$-dimensional, we choose $\lambda$ so that
\[
\eta = \frac{\log 2}{\log 2 - \log(1 - \lambda)}.
\]
In the special case when $\eta = 0$, we set $C_0 = \{0\}$. We now define the set $S$:
\[
S := A_\gamma + \{0\} \times \gamma C_\eta.
\]
In other words, $S$ is obtained by attaching a $\gamma$-scaled copy of $C_\eta$ to each gridpoint of $A_\gamma$. This completes the definition of the set $\Omega_\alpha = \left( B(0, 1)_{d-1} \times [-1, 1] \right) \setminus S$. Each point in $\partial \Omega_\alpha$ has an $\eta$-dimensional Cantor set in $\Omega_\alpha$ attached to it is captured by the following claim. Thus there is a universal constant $C \geq 1/16$ such that for every $\gamma$, the set $\Omega_\alpha$ is $\alpha$-thick with the thickness $C$.

The following two claims assert that for an appropriately chosen $\gamma$, the WoS originated at the origin $0 \in \mathbb{R}^d$ and terminated at the $1/n$ neighborhood of $\partial \Omega_\alpha$ is likely to hit the boundary of the external cylinder (as opposed to the neighborhood of $S$), and is likely to spend $\Omega(n^{2-4/\alpha})$ steps getting there.

**Claim 14.** If $\gamma > 8n^{2/\alpha - 1}$ then a WoS originated at 0 and terminated at the $1/n$-neighborhood of the boundary $\partial \Omega_\alpha$ will hit the boundary of the cylinder $B(0, 1)_{d-1} \times [-1, 1]$ with probability at least $3/4$.

**Proof.** It is not hard to see that we can choose a finite subset $P$ of points in $S$ such that $|P| < 2\gamma^{-\alpha} \cdot n^\beta$, and for every $x$ such that $d(x, S) < 1/n$ there is a $p \in P$ such that $|x - p| < 2/n$. Consider the harmonic function
\[
\Phi(x) := \sum_{y \in P} \frac{1}{|x - y|^{d-2}} > 0.
\]
Since the function $\Phi$ is harmonic, its application to the WoS process $X_t$ gives a martingale. Hence if $T$ is the stopping time of the process,
\[
\mathbb{E}[\Phi(X_T)] = \Phi(X_0) = \Phi(0) < 3^{d-2} \cdot |P| < 6\gamma^{-\alpha} \cdot n^\eta.
\]
On the other hand, if $d(X_T, S) < 1/n$, then there is a $y \in P$ with $|X_T - y| < 2/n$, and
\[
\Phi(X_T) \geq \frac{1}{|y - X_T|^{d-2}} > (n/2)^{d-2}.
\]
Hence the probability of $X_T$ being near $S$ is bounded by
\[
\frac{\mathbb{E}[\Phi(X_T)]}{(n/2)^{d-2}} < \frac{6\gamma^{-\alpha} \cdot n^\eta}{(n/2)^{d-2}} < \frac{2^{d+1}\gamma^{-\alpha}}{n^{\alpha-2}} < \frac{8\gamma^2}{4(\gamma n)^\alpha} < 1/4.
\]
The last inequality follows from the condition on $\gamma$. \hfill $\square$

**Claim 15.** There is a universal constant $\delta > 0$ such that for $\gamma$ as above, with probability at least $1/2$ the WoS takes at least $\delta(1/\gamma)^2$ steps to reach the boundary of the cylinder $B(0, 1)_{d-1} \times [-1, 1]$.

**Proof.** The proof is done analogously to the proof of Claim 14 below. \hfill $\square$

Hence the expected number of steps is at least
\[
\frac{\delta}{2} \cdot \left( \frac{1}{8n^{2/\alpha - 1}} \right)^2 = \Omega(n^{2-4/\alpha}),
\]
which completes the proof of the lower bound for Theorem 2 in the case when $\alpha > 2$. 

---

(continued on the next page)
4.2. Proof of the lower bound in the case \( \alpha = 2 \). We will now give an example of a two dimensional domain \( \Omega \) such that the expected convergence time of the WoS to a \( O(1/n) \)-neighborhood of \( \partial \Omega \) is \( \Omega(\log^2 n) \). By taking the \( d \)-dimensional domain \( \Omega_d = \Omega \times \mathbb{R}^{d-2} \) for \( d > 2 \), we obtain a lower bound of \( \Omega(\log^2 n) \) for 2-thick domains in \( \mathbb{R}^d \), proving the lower bound for \( \alpha = 2 \) in Theorem 2. The domain \( \Omega \) will consist of the unit disc in \( \mathbb{R}^2 \) with points from \( \Gamma \) removed from the “middle third” annulus of the disc. Hence the probability that the WoS terminates near a hole is less than \( 1/2 \). We define two regions \( \Omega \) to be the unit disc with points from \( \Gamma \) removed from the “middle third” annulus of the disc.

(43) \[ \Omega = B(0, 1) \setminus ((B(0, 2/3) \setminus B(0, 1/3)) \cap \Gamma). \]

The set \( \Omega \) is illustrated on Fig. 4(a).

We will show that a WoS originated at the origin \( X_0 = 0 \) terminated at time \( T \) to converge. It is immediate to see that the same lower bound holds for any point \( X_0 \in B(0, 1/3) \). We first observe the following:

Claim 16. With probability at least \( 7/8 \), a WoS originated at \( X_0 = 0 \) that runs until \( d(X_t) < 1/n \) terminates near the unit circle (and not near one of the holes).

Proof. Let \( \{a_i\}_{i=1}^k = B(0, 1) \setminus \Omega \) be the set of holes in \( \Omega \). Define the harmonic function

\[ \Phi(z) = \sum_{i=1}^k \log(2/|z - a_i|). \]

It is clear the \( \Phi(z) > 0 \) for all \( z \in B(0,1) \). For any point \( u \) in the \( 1/n \)-neighborhood of any of the holes, \( \Phi(u) > \log n \). On the other hand,

\[ \Phi(0) < k \cdot \log 6 < 2/\gamma^2 = (\log n)/8. \]

If \( X_t \) is the WoS process with \( X_0 = 0 \) terminated at time \( T \) when \( d(X_T, \partial \Omega) < 1/n \), then \( \Phi(X_t) \) is a martingale. Hence,

\[ (\log n)/8 > \Phi(X_0) = \mathbf{E}[\Phi(X_t)] > \mathbf{P}[X_t \text{ near a hole}] \cdot \log n. \]

Hence the probability that the WoS terminates near a hole is less than \( 1/8 \). \( \square \)

For simplicity, we will assume that at every step the process the WoS jumps exactly half way to the boundary \( \partial \Omega \).

To facilitate the analysis we replace the WoS process \( X_t \) on \( \Omega \) with the following process \( Y_t \). It evolves in exactly the same fashion as \( X_t \), except when \( Y_t \) is closer than \( 1/n \) to one of the holes in \( \Omega \). In this case, instead of terminating, the process makes a jump of \( 1/n \) in a direction selected uniformly at random. The process \( Y_t \) is guaranteed to terminate near the unit circle. We denote the termination time by \( T \). Further, we set \( Y_t = Y_T \) for \( t > T \). Note that if the process \( X_t \) does not terminate near one of the holes, then the process \( Y_t \) coincides with \( X_t \). Claim 16 implies that this happens with probability at least \( 7/8 \):

Claim 17. \( \mathbf{P}[X_t \text{ does not coinside with } Y_t] < 1/8 \).

We define two regions \( A \) and \( B \), \( B \subset A \subset \Omega \). We take \( A \) to be the union of discs with radius \( r = \gamma/4 \) around the holes in \( \Omega \). We take \( B \) to be the union of discs with radius \( r/2 \) around the same holes. The sets \( \Omega \), \( A \) and \( B \) are illustrated on Fig. 4(a).

Let time \( t_0 \) be the first time with \( |Y_t| > 1/2 \). Let \( t' \) be the first time afterward with either \( |Y_t| > 2/3 \) or \( |Y_t| < 1/3 \). Our goal is to show that with probability at least \( 3/4 \), \( |t_0 - t'| = \Omega(\log^2 n) \). We define a subprocess \( Z_t \) of \( Y_t \) as follows. Let \( \{s_i\}_{i=0}^k \) be a subsequence of times \( s \) between \( t_0 \) and \( t' \) such that \( Y_s \notin A \). We set \( Z_i = Y_{s_i} \). We further define \( \Delta_i = Z_i - Z_{i-1} \). An instance of the process
Figure 4. An illustration of the sets $\Omega$, $A$ and $B$ (a), and a possible sequence of jumps in the processes $\{Y_t\}$ and $\{Z_t\}$ (b).

$Z_t$ is illustrated on Fig. 4(b). Since $Y_t$ is a martingale, and $Z_t$ is defined by a stopping rule on $Y_t$, $Z_t$ is also a martingale, and

$E[\Delta_i | \Delta_1, \Delta_2, \ldots, \Delta_{i-1}] = 0.$

In addition, it is not hard to see from the definition of $Y_t$ that $|\Delta_i| < 4/\log^{1/2} n$ for all $i$. Our first claim is that the number $k$ of steps $Z_t$ is $\Omega(\log n)$.

**Claim 18.** $P[k < 10^{-4} \log n] < 1/8$.

**Proof.** Denote $\ell = 10^{-4} \log n$. Then, by (44),

$$E[(Z_0 - Z_{\ell})^2] = E[(\Delta_1 + \Delta_2 + \ldots + \Delta_\ell)^2] = \sum_{j=1}^{\ell} E[\Delta_j^2] + \sum_{1 \leq i < j \leq \ell} E[\Delta_i \Delta_j] =$$

$$\sum_{j=1}^{\ell} E[\Delta_j^2] + \sum_{1 \leq i < j \leq \ell} E[\Delta_i \cdot E[\Delta_j | \Delta_i]] = \sum_{j=1}^{\ell} E[\Delta_j^2] < \ell \cdot 16/\log n < 1/288.$$

On the other hand, by definition, $|Z_0 - Z_k| > 1/6$, and $(Z_0 - Z_k)^2 > 1/36$. Thus,

$$P[k \leq \ell] = P[Z_{\ell} = Z_k] < (1/288)/(1/36) = 1/8.$$  

Thus the number of steps the process $Z_t$ takes is at least $10^{-4} \log n$ w.p. $> 7/8$. The process $Y_t$ consists of the steps of the process $Z_t$ plus, in addition, steps the process takes within the region.
A. We claim that once the process $Y_t$ enters the region $A$, it is expected to spend $\Omega(\log n)$ steps there. Moreover, the following holds.

**Claim 19.** Let $\eta > 2$. Then there is a $\theta > 0$ such that whenever $Y_t \in A$, if $s > t$ is the first time, conditioned on $Y_t$ such that $Y_s \notin A$, then

$$\mathbb{P}[s - t > \theta \log^2 n] > \eta/\log n,$$

for sufficiently large $n$.

**Proof.** Denote the hole in $\Omega$ that is closest to $Y_t$ by $x$. Given that $Y_t \in A$, there is some fixed probability $p > 0$ that $Y_{t+1} \in B$. In other words, $|Y_{t+1} - x| < r/2 = \gamma/8$. Consider the harmonic function

$$\Phi(z) = \log(r/|x - z|).$$

Let $t' > t + 1$ be the first time such that either $Y_{t'} \notin A$ (and thus $t' = s$), or $|Y_{t'} - x| < n^{-p/(5\eta)}$. If $Y_{t'} \notin A$, then $\Phi(Y_{t'}) < 0$. In the other case, $\Phi(Y_{t'}) < p \log n/(4\eta)$. Since $t'$ is a stopping time, the optional stopping time theorem applied to the martingale $\Phi(Y_{t+\tau})$ combined with the estimate $\Phi(Y_t) > 1/2$, gives

$$\mathbb{P}[|Y_{t'} - x| < n^{-p/(5\eta)}] > (1/2)/(p \log n/(4\eta)) > 2\eta/(p \log n).$$

To complete the argument, we claim that assuming $|Y_{t'} - x| < n^{-p/(5\eta)}$, it will take the process another $\Omega(\log^2 n)$ steps to escape $A$ with probability at least 1/2. We consider the process $\phi_t = \Phi(Y_{t+\tau})$ stopped at time $\tau_0$ when either $Y_{t+\tau_0}$ escapes $A$, or gets closer than distance $1/n$ from $x$. $\phi_t$ is a martingale. Moreover, it is not hard to see that $|\phi_0 - \phi_{\tau_0}| > p \log n/(6\eta)$, and $|\phi_i - \phi_{i+1}| < 1$ for all $i$. These two facts imply that

$$\mathbb{E}[\tau_0] > \sum_{i=1}^{\tau_0} \mathbb{E}[(\phi_i - \phi_{i-1})^2] = \mathbb{E}[(\phi_{\tau_0} - \phi_0)^2] > (p \log n/(6\eta))^2 = p^2 \log^2 n/(36\eta^2).$$

Tschebyshev inequality implies that $\theta = p^2/(72\eta^2)$ satisfies the statement of the claim. \qed

By Claim 18 we know that except with probability $< 1/8$ the walk will contain at least $\Omega(\log n)$ visits to $A$. It remains to use Claim 19 to show that at least one of these stays must be $\Omega(\log^2 n)$ long. Recall that $T$ is the stopping time of the process $Y_T$, and $k$ is the number of steps $Y_t$ takes outside of $A$.

**Claim 20.** Let $\alpha_1 = 10^{-4}$ from Claim 18. There is a constant $\alpha_2 > 0$ such that

$$\mathbb{P}[k > \alpha_1 \log n \text{ and } T < \alpha_2 \log^2 n] < 1/8.$$

**Proof.** For every $t$ such that $1/3 < |Y_t| < 2/3$ and $Y_t \notin A$, there is a probability $p_1 > 0$ such that either $Y_{t+1} \in A$ or $Y_{t+2} \in A$. By Claim 19 we can choose $\alpha_2 > 0$ such that whenever $Y_t \in A$, the process $Y_{t+\tau}$ does not escape $A$ for at least $\alpha_2 \log^2 n$ with probability at least $p_2 = 6/(\alpha_1 p_1 \log n)$. Hence for each $1/3 < |Y_t| < 2/3$ with $Y_t \notin A$, the probability that $Y_{t+1}$ or $Y_{t+2}$ enters $A$, and stays there for at least $\alpha_2 \log^2 n$ steps is at least $p_1 \cdot p_2 = 6/(\alpha_1 \log n)$. Since there are at least $k = \alpha_1 \log n$ $Y_t$’s satisfying $1/3 < |Y_t| < 2/3$ and $Y_t \notin A$, the probability that for neither one of them does $Y_{t+1}$ or $Y_{t+2}$ enter $A$, and stay there for at least $\alpha_2 \log^2 n$ steps is at most

$$(1 - 6/(\alpha_2 \log n))^{k/2} < (1 - 6/(\alpha_1 \log n))^{\alpha_1 \log n/2} < e^{-6/(\alpha_1 \log n)} \cdot (\alpha_1 \log n/2) = e^{-3} < 1/8.$$

\qed

Claims 17, 18 and 20 imply the following.
Claim 21. Let $X_t$ be the WoS process on the set $\Omega$ with $X_0 = 0$. Let $T'$ be its termination time. Then

$$P[T' > \alpha_2 \log^2 n] > 5/8,$$

where $\alpha_2 > 0$ is the constant from Claim 20. In particular, this implies that $E[T'] = \Omega(\log^2 n)$.

Proof. We know that $T' > \alpha_2 \log^2 n$ if the following three conditions hold: (C1) the process $X_t$ coincides with the process $Y_t$; (C2) the process $Y_t$ makes at least $k > \alpha_1 \log n$ steps outside of $A$ in the $\{z : 1/3 < |z| < 2/3\}$ annulus; and (C3) the stopping time $T$ of $Y_t$ satisfies $T > \alpha_2 \log^2 n$. In fact conditions (C1) and (C3) suffice. We have $P(C_1) < \frac{1}{8}$ by Claim 17, $P(C_2) < \frac{1}{8}$ by Claim 18, and $P(C_2 \cap C_3) < \frac{1}{8}$ by Claim 20. Here $\overline{C}$ denotes the complement of an event $C$. Hence

$$P(C_1) + P(C_2) + P(C_2 \cap C_3) < \frac{3}{8},$$

and

$$P[T' > \alpha_2 \log^2 n] = \frac{P(C_1 \cap C_2 \cap C_3)}{1 - P(C_1 \cup C_2 \cup C_3)} > \frac{5}{8}.$$

Claim 21 gives the lower bound for Theorem 2 in the case $\alpha = 2$.

REFERENCES

[BB07] I. Binder and M. Braverman. Derandomization of Euclidean Random Walks. LNCS, 4627:353–365, 2007.

[Car67] Lennart Carleson. Selected problems on exceptional sets. Van Nostrand Mathematical Studies, No. 13. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1967.

[EKMS80] B. S. Elepov, A. A. Kronberg, G. A. Mihailov, and K. K. Sabel’fel’d. Reshenie kraevykh zadach metodom Monte-Karlo. “Nauka” Sibirsk. Otdel., Novosibirsk, 1980.

[GM04] J. B. Garnett and D. E. Marshall. Harmonic Measure. Cambridge Univ Press, 2004.

[KS91] I. Karatzas and S. E. Shreve. Brownian Motion and Stochastic Calculus. Springer Verlag, 2 edition, 1991.

[Lan72] N. S. Landkof. Foundations of modern potential theory. Translated from the Russian by AP Doohovskoy, volume 180. 1972.

[Mih79] G. A. Mihailov. Estimation of the difficulty of simulating the process of “random walk on spheres” for some types of regions. Zh. Vychisl. Mat. i Mat. Fiz., 19(2):510–515, 558–559, 1979.

[Mil95] G. N. Milstein. Numerical Integration of Stochastic Differential Equations. Kluwer Academic Publishers, Dodrecht, 1995.

[Mot59] Minoru Motoo. Some evaluations for continuous Monte Carlo method by using Brownian hitting process. Ann. Inst. Statist. Math. Tokyo, 11:49–54, 1959.

[Mul56] M. E. Muller. Some continuous Monte Carlo methods for the Dirichlet problem. Ann. Math. Statist., 27:569–589, 1956.

ILIA BINDER, DEPT. OF MATHEMATICS, UNIVERSITY OF TORONTO.

MARK BRAVERMAN, MICROSOFT RESEARCH, NEW ENGLAND