DISCRETE MODULE CATEGORIES AND OPERATIONS IN $K$-THEORY

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ABSTRACT. We study the categories of discrete modules for topological rings arising as the rings of operations in various kinds of topological $K$-theory. We prove that for these rings the discrete modules coincide with those modules which are locally finitely generated over the ground ring.

1. INTRODUCTION

For well-behaved coalgebras $C$, the category of $C$-comodules is isomorphic to the category of discrete modules over the dual algebra $A = C^*$. It is thus interesting to be able to characterize discrete modules. In [7] it was shown that the category of discrete modules over the algebra of operations in $p$-local $K$-theory is isomorphic to the category of locally finitely generated modules. In this paper we will show that the same is true for a family of related examples. The method used also gives a new and simpler proof in the original case.

We work over a commutative unital Noetherian ground ring $R$; in the applications $R$ will be the $p$-local integers $\mathbb{Z}_p$ for some prime $p$; we also consider $R = \mathbb{Z}$. We are interested in topological algebras $A$ over $R$ of the following form. There is an infinite family of elements $\{a_n \mid n \geq 0\}$ of $A$, a topological basis, such that $A$ consists of precisely the infinite sums $\{\sum_{n \geq 0} r_n a_n \mid r_n \in R\}$. The sets $A_m = \{\sum_{n \geq m} r_n a_n \mid r_n \in R\}$ are ideals of $A$ and $A$ is complete with respect to the filtration by these.

A simple example is provided by the power series ring $R[[t]]$ with the usual $t$-adic filtration. This example has many special features, of course, as the filtration ideals are principal and $(t^r)(t^s) = (t^{r+s})$. These features lead to nice properties of the category of discrete modules over a power series ring.

In contrast the examples we consider are much more complicated and the filtrations are not multiplicative. Nonetheless we establish conditions on $A$ under which some of these good properties are preserved. The examples arise as the linear duals of coalgebras having a particular form and we also formulate our conditions in terms of the coalgebra to which $A$ is dual.

This article is organized as follows. In Section 2 we define discrete modules and discuss some basic properties. Section 3 is devoted to comodules.
over a coalgebra $C$. In Proposition 3.7 we give conditions on $C$ under which the category of right $C$-comodules is isomorphic to the category of discrete left modules over the dual of $C$. This explains our interest in discrete modules. In Section 4 we introduce the notion of a regular coalgebra and provide a characterization of the units in the dual of such a coalgebra. In Section 5 we explain how examples of topological interest fit into this framework.

We define the notion of a module which is locally finitely generated over the ground ring in Section 6 and here we give our main technical result, Theorem 6.2, providing conditions on $A$ under which such locally finitely generated modules coincide with discrete modules. The conditions can be interpreted as requiring $A$ to have a certain resemblance to a power series ring. We also discuss a kind of dual result, by providing the corresponding conditions on the coalgebra to which $A$ is dual (Theorem 6.9). In Section 7 we do the further work necessary to show that the main technical results apply to various examples of topological interest. These results are collected together in Theorem 7.1.

This paper is based on work in the Ph.D. thesis of the first author [10], produced under the supervision of the second author.

2. DISCRETE MODULES

We work over a commutative unital Noetherian ground ring $R$. In the applications $R$ will be $\mathbb{Z}(p)$; we also consider $R = \mathbb{Z}$. All modules considered will be left modules and all tensor products will be over $R$. Let $A$ be a unital topological algebra over $R$ and write $A\text{Mod}$ for the category of $A$-modules. In this section we define and investigate discrete $A$-modules and the subcategory of $A\text{Mod}$ they form.

Our interest in discrete modules arises from the fact that, for sufficiently nice cohomology theories $E$, the $E$-homology of a space or spectrum is a discrete module over the algebra of operations of the theory. The discrete module category of the topological $\mathbb{Z}(p)$-algebra of degree zero stable operations in $p$-local $K$-theory was studied in [7]. This paper is motivated by the desire to extend the results of [7] to many related examples of topological interest.

**Definition 2.1.** Let $A$ be a topological $R$-algebra. An $A$-module $M$ is *discrete* if it is continuous when given the discrete topology, *i.e.* if the action map $A \times M \to M$ is continuous.

The category $A\text{Disc}$ of discrete $A$-modules is the full subcategory of $A\text{Mod}$ with objects the discrete $A$-modules.

Of course, $A\text{Disc}$ depends on the topology on $A$, but we leave it out of the notation. In general, $A$ will not itself be a discrete $A$-module.

It is straightforward to check that the category $A\text{Disc}$ is closed under submodules, quotients and direct sums. Hence it is a cocomplete abelian category.
We will consider those \( A \) whose topology is given by a filtration
\[ A = A_0 \supset A_1 \supset A_2 \supset \ldots \]
by (two-sided) ideals. We recall that such a filtration is Hausdorff if the map \( A \to \lim \leftarrow \frac{A}{A_n} \) is injective (i.e. \( \bigcap_{n \geq 0} A_n \) is zero) and complete if \( A \to \lim \leftarrow \frac{A}{A_n} \) is surjective.

The algebras we work with will be pro-finitely generated as modules over the ground ring in the sense of the following definition.

**Definition 2.2.** A topological \( R \)-algebra \( A \), with topology given by a filtration
\[ A = A_0 \supset A_1 \supset A_2 \supset \ldots , \]
is pro-finitely generated if it is complete and Hausdorff, and each \( A/A_n \) is a finitely generated \( R \)-module.

Such an \( R \)-algebra is an inverse limit of finitely generated \( R \)-modules.

In practice we characterize discrete modules by the following property.

**Lemma 2.3.** [7, 2.4]. An \( A \)-module \( M \) is discrete if and only if for every \( x \in M \) there is \( n \) with \( A_n x = 0 \). \( \square \)

For example, if \( I \) is an ideal of \( A \), then \( A/I \) is discrete if and only if there is some \( n \) such that \( A_n \subseteq I \).

From now on, we assume that the filtration of \( A \) is complete and Hausdorff and that each containment \( A_n \subset A_{n-1} \) is strict. The canonical examples of discrete modules are the quotients \( A/A_n \) for \( n \geq 0 \). Note that, if \( M \) is discrete, there is not necessarily an \( n \) such that \( A_n M = 0 \): consider \( \bigoplus_{n \geq 0} A/A_n \).

It is important to note that we do not assume that our filtration is multiplicative. The examples we are interested in arise naturally in topology; they are complete with respect to their filtration topologies, but they are not complete with respect to any multiplicative filtration. This feature makes them complicated rings and leads to possibly unexpected properties of the discrete module categories.

For example, discrete module categories are in general not closed under extensions in the corresponding module category. Take \( m, n \geq 0 \) and suppose that there is no \( k \) such that
\[ A_k \subseteq A_m A_n . \]
Then the exact sequence
\[ 0 \to A_n/A_m A_n \to A/A_m A_n \to A/A_n \to 0 , \]
of \( A \)-modules, where the maps are the natural inclusion and projection, shows that \( A/A_m A_n \) is a non-discrete extension of discrete modules.

In contrast, the prototypical example with a multiplicative filtration is the power series ring \( \mathbb{Z}[[t]] \) with its usual \( t \)-adic filtration. It is easy to check that the category \( \mathbb{Z}[[t]] \text{Disc} \) is closed under extensions in \( \mathbb{Z}[[t]] \text{Mod} \).
Definition 2.4. We define a functor
\[ \text{Disc}: A\text{Mod} \to A\text{Disc} \]
as follows. On objects, for \( M \) an \( A \)-module, define
\[ \text{Disc} M = \{ x \in M \mid A^n x = 0 \text{ for some } n \} \subseteq M, \]
the maximal discrete submodule of \( M \). On morphisms, for an \( A \)-linear map \( f: M \to N \), define \( \text{Disc} f \) by restricting the domain of \( f \) to \( \text{Disc} M \) and the codomain to \( \text{Disc} N \). (If \( x \in \text{Disc} M \) and \( A^n x = 0 \) then \( A^n f(x) = f(A^n x) = 0 \), so \( f|_{\text{Disc} M} \) lands in \( \text{Disc} N \).)

It is straightforward to check that the functor \( \text{Disc} \) is right adjoint to the inclusion of \( A\text{Disc} \) into \( A\text{Mod} \).

3. Comodules

In this section we recall some results about coalgebras and comodules. Our main reference is [4]. Let \( C \) be an \( R \)-coalgebra, with comultiplication map \( \Delta : C \to C \otimes C \). We write \( \text{Comod}_C \) for the category of right \( C \)-comodules and comodule maps. If the coalgebra \( C \) is flat over the ground ring then \( C \)-comodules have good finiteness properties.

Proposition 3.1. If \( C \) is \( R \)-flat, then

1. [4, 3.16] every element of a right \( C \)-comodule \( M \) is contained in a subcomodule which is finitely generated as an \( R \)-module;
2. every element of \( C \) is contained in a subcoalgebra of \( C \) which is finitely generated as an \( R \)-module.

The algebras we are interested in arise as the duals of such coalgebras. We define the \( R \)-module \( C^* = R\text{Mod}(C, R) \), the \( R \)-linear dual. Denote the evaluation pairing \( C^* \otimes C \to R \) by \( \langle - , - \rangle \) and recall that if \( f: M \to N \) is a map of \( R \)-modules, the dual map \( f^*: N^* \to M^* \) is defined by
\[ \langle f^*(\varphi), m \rangle = \langle \varphi, f(m) \rangle \]
for \( \varphi \in N^* \) and \( m \in M \). Then \( C^* \) has the convolution product \( * \): if \( a_1, a_2 \in C^* \), \( c \in C \) and \( \Delta(c) = \sum c^{(1)}_i \otimes c^{(2)}_i \),
\[ \langle a_1 * a_2, c \rangle = \sum \langle a_1, c^{(1)}_i \rangle \langle a_2, c^{(2)}_i \rangle \]
This makes \( C^* \) into an \( R \)-algebra.

Lemma 3.2. Let \( C \) be a flat \( R \)-coalgebra. Then the dual \( A = C^* \) of \( C \) is a filtered \( R \)-algebra which is pro-finitely generated in the sense of Definition 2.2.

Proof. Because \( C \) is flat, by Proposition 3.1, we may write
\[ C = \bigcup_{\alpha \in I} C_\alpha, \]
where each $C_\alpha$ is an $R$-finitely generated subcoalgebra of $C$. Since $R$ is Noetherian, the algebra $C_\alpha^*$ is also $R$-finitely generated and, because the dual functor $(-)^*$ takes colimits (in this case, unions) to limits,

$$A \cong \lim_{\leftarrow} C_\alpha^*.$$  

Let $A_\alpha$ be the kernel of the restriction map $A = C^* \to C_\alpha^*$. Then $A$ is filtered by the ideals $A_\alpha$ and $A/A_\alpha \cong C_\alpha^*$. So $A$ is pro-finitely generated. □

Now let $M$ be a right $C$-comodule $M$, with coaction map $\rho_M : M \to M \otimes C$. Define a left $A$-action on $M$ by

$$(3.3) \quad ax = \sum_i \langle a, c_i \rangle x_i,$$

where $a \in A$, $x \in M$ and $\rho_M(x) = \sum x_i \otimes c_i$.

This makes $M$ a left $A$-module and if $f : M \to N$ is a map of right $C$-comodules, then $f$ is also a map of left $A$-modules when $M$ and $N$ are given the above action.

In particular, $C$ itself is an $A$-module and this action coincides with the usual action on a coalgebra by its dual. We define a functor

$$i : \text{Comod}_C \to \text{AMod},$$

where $i(M)$ is $M$ with the above $A$-action and $i(f) = f$. Note that $i$ is faithful. In general, it does not have other nice properties, unless we impose conditions on $C$.

Recall that $C$ is a subgenerator of $\text{Comod}_C$, that is, every $C$-comodule is a subcomodule of a quotient of a direct sum of copies of $C$. If $M$ is an $A$-module, define the category $\sigma[M]$ to be the full subcategory of $\text{AMod}$ subgenerated by $M$.

**Theorem 3.4.** [4, 4.3]. The following are equivalent.

1. $i(\text{Comod}_C)$ is a full subcategory of $\text{AMod}$.
2. $i(\text{Comod}_C) = \sigma[i(C)]$.
3. $C$ is a locally projective $R$-module. □

We refer to [4, 42.9] for the definition of a locally projective module. (This condition on modules lies between projectivity and flatness.)

From Theorem [3.4] we see that, if $C$ is locally projective as an $R$-module (and hence flat),

1. $i$ is an inclusion of categories

$$\text{Comod}_C \to \text{AMod};$$

2. $\text{Comod}_C$ has all kernels, cokernels and direct sums;
3. every element of a $C$-comodule is contained in an $R$-finitely generated $C'$-comodule;
4. $C$ may be written as a union $\bigcup C_\alpha$ of finitely generated subcoalgebras $C_\alpha$. 


It is also known when \( i \) is an equality.

**Proposition 3.5.** [4, 4.7]. Suppose \( C \) is locally projective. Then \( i(\text{Comod}_C) \) and \( A\text{Mod} \) are equal if and only if \( C \) is a finitely generated and projective \( R \)-module. \( \square \)

**Lemma 3.6.** Let \( C \) be an \( R \)-coalgebra with dual algebra \( A = C^\ast \). Suppose that \( C = \bigcup C_\alpha \) for finitely generated subcoalgebras \( C_\alpha \). Then \( i(C) \) is a discrete \( A \)-module. If in addition \( C \) is a locally projective \( R \)-module, then \( i(\text{Comod}_C) \) is a full subcategory of \( A\text{Disc} \).

**Proof.** If \( c \in C \), take \( \alpha \) such that \( c \in C_\alpha \). Then \( \Delta(c) = \sum c_1^{(1)} \otimes c_1^{(2)} \in C_\alpha \otimes C_\alpha \). If \( a \in A \),

\[
ac = \sum \langle a, c_1^{(2)} \rangle c_1^{(1)},
\]

so \( A_\alpha c = 0 \). Hence \( i(C) \) is a discrete \( A \)-module.

If \( C \) is a locally projective \( R \)-module, then by Theorem 3.4, \( i(\text{Comod}_C) \) is the full subcategory \( \sigma[i(C)] \) of \( A\text{Mod} \). Since \( A\text{Disc} \) has direct sums, kernels and cokernels, \( i(\text{Comod}_C) \) is a full subcategory of \( A\text{Disc} \). \( \square \)

**Proposition 3.7.** Let \( C \) be a locally projective \( R \)-coalgebra with dual algebra \( A \). Suppose that \( C = \bigcup C_\alpha \) for subcoalgebras \( C_\alpha \) which are finitely generated and projective \( R \)-modules. Then \( i \) is an isomorphism of categories

\[
\text{Comod}_C \cong A\text{Disc}.
\]

**Proof.** We have already observed that in this case \( i(\text{Comod}_C) \) is a full subcategory of \( A\text{Disc} \). Directly, if \( M \) is a right \( C \)-comodule, \( x \in M \) and \( \rho_M(x) = \sum x_i \otimes c_i \), we take \( \alpha \) such that all \( c_i \) are in \( C_\alpha \); then \( A_\alpha x = 0 \).

Conversely, take a discrete left \( A \)-module \( N \) and an \( \alpha \). Let

\[
N_\alpha = \{ y \in N \mid A_\alpha y = 0 \};
\]

then \( N_\alpha \) is a left \( A/A_\alpha \)-module. Since \( C_\alpha^\ast \cong A/A_\alpha \), Proposition 3.5 implies that \( \text{Comod}_{C_\alpha} \cong A/A_\alpha \text{Mod} \). Hence \( N_\alpha \) is a right \( C_\alpha \)-comodule and so a right \( C \)-comodule. Finally, since \( N = \bigcup N_\alpha \), \( N \) is a union of \( C \)-comodules and therefore a \( C \)-comodule itself.

It is straightforward to check that these procedures are inverse to each other. \( \square \)

### 4. Regular Coalgebras and Their Duals

In this section we will introduce the particular kind of coalgebra we work with. Recall that the rational polynomial and Laurent polynomial rings \( \mathbb{Q}[w] \) and \( \mathbb{Q}[w, w^{-1}] \) are \( \mathbb{Z} \)-coalgebras with comultiplication determined by making the indeterminate \( w \) grouplike. We will be interested in \( \mathbb{Z} \)-subcoalgebras of these; we restrict to those which are free \( R \)-modules, where \( R \) is either \( \mathbb{Z} \) or \( \mathbb{Z}_{(p)} \) for a prime \( p \).

If \( M \) is a free \( R \)-submodule of \( \mathbb{Q}[w^r] \) (or \( \mathbb{Q}[w^r, w^{-r}] \)) for some \( r > 1 \), an \( R \)-basis for \( M \) is called \textit{regular} if it consists of polynomials \( g_n \), for \( n \in \mathbb{N}_0 \) (or \( n \in \mathbb{Z} \)), where the degree of the polynomial \( g_n \) is \( r n \). (See [3 II.1.3].)
Definition 4.1. Let $C$ be an $R$-subcoalgebra of $\mathbb{Q}[w^r]$ for some $r \geq 1$. Then we call $C$ a regular $R$-coalgebra if

1. $C$ is a free $R$-module on a chosen regular basis $\{c_n(w) \mid n \geq 0\}$, where the degree of $c_n(w)$ is $rn$, and
2. $w^{rn} \in C$ for every $n \geq 0$.

Examples 4.2. (1) For $R = \mathbb{Z}$ or $\mathbb{Z}(p)$, $R[w^r]$ with basis $\{w^{rn} \mid n \geq 0\}$ is a regular coalgebra.

(2) Taking $R = \mathbb{Z}$ and $r = 1$, the free $R$-module $C$ with basis the binomial coefficient polynomials

$$\binom{w}{n} = \frac{w(w-1) \ldots (w-n+1)}{n!}$$

is a regular coalgebra. As is well-known, these polynomials form a basis for the ring of integer-valued polynomials:

$$C = \{f(w) \in \mathbb{Q}[w] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}.$$

Further examples related to $K$-theory will appear in Section 5.

Now let $C$ be a regular coalgebra with chosen regular basis $\{c_n(w) \mid n \geq 0\}$. As will become clear, our methods are heavily dependent on the properties of the specified basis elements; indeed this is why we have chosen to incorporate the choice of basis into the definition. (While there are nice basis-free descriptions of some of the objects we are interested in, we do not know how to prove our main results without specifying bases.)

Thus we introduce notation for the coefficients appearing in the basis elements, in the comultiplication $\Delta$ applied to them and when expressing powers of $w$ in terms of the basis elements.

Definition 4.3. For $k, n \geq 0$, define integers $\lambda_k^n$ and $D_n$ by writing $c_n(w)$ as

$$c_n(w) = \frac{1}{D_n} \sum_{k=0}^{n} \lambda_k^n w^{kr},$$

where $D_n > 0$ and the integers $\lambda_0^n, \ldots, \lambda_n^n$, $D$ have no non-trivial common factor.

Also define $\Lambda_k^n$ for $n, k \geq 0$ by

$$w^{kr} = \sum_{n=0}^{k} \Lambda_k^n c_n(w).$$

For $i, j, n \geq 0$, define $\Gamma_{i,j}^n$ by

$$\Delta(c_n(w)) = \sum_{i,j=0}^{n} \Gamma_{i,j}^n c_i(w) \otimes c_j(w).$$

Where necessary, we will write $\Gamma(C)_{i,j}^n$, etc., to specify the coalgebra we are referring to.
Next we consider the dual algebras of regular coalgebras. If \( C \) is a regular \( R \)-coalgebra, define the finite rank subcoalgebras

\[
C_n = \{ f(w) \in C \mid \deg f(w) \leq r_n \} = R\{c_0(w), \ldots, c_n(w)\},
\]

so that we have an increasing filtration \( C_0 \subset C_1 \subset C_2 \subset \ldots \). Denote the dual \( R \)-algebra of \( C \) by \( A \). It has a topological \( R \)-basis \( \{a_n \mid n \geq 0\} \), where \( a_n = c_n(w)^* \); i.e. \( A \) consists exactly of the infinite sums

\[
\sum_{n \geq 0} r_n a_n
\]

for \( r_n \in R \). For any \( i, j, n \geq 0 \), \( \langle a_i a_j, c_n(w) \rangle = \Gamma_{i,j}^n \), so the product of basis elements is given by

\[
a_i a_j = \sum_{n \geq 0} \Gamma_{i,j}^n a_n.
\]

The algebra \( A \) has the topology given by the decreasing filtration by the ideals

\[
A_n = \text{Ann } C_{n-1} = \{ a \in A \mid \langle a, f(w) \rangle = 0 \text{ whenever } f(w) \in C_{n-1} \}
\]

\[
= \{ \sum_{k \geq n} r_k a_k \mid r_k \in R \}.
\]

Thus \( A \) is pro-finitely generated in the sense of Definition 2.2.

**Example 4.4.** For \( C = R[w^r] \) as in Example 4.2(1), the dual \( A = C^* \) is the infinite product \( \prod_{i=0}^{\infty} R \), filtered by the ideals \( \prod_{i=n}^{\infty} R \).

The coalgebra \( D = C[w^{-r}] \) is the union of the finitely generated sub-coalgebras

\[
D_n = D \cap \mathbb{Q}\{w^{-[n/2]r}, \ldots, w^{[n/2]r}\}
\]

for \( n \geq 0 \). Denote the dual algebra of \( D \) by \( B \); it also has a topological \( R \)-basis and may be filtered by the ideals \( B_n = \text{Ann } D_{n-1} \).

For a regular coalgebra \( C \), we can make the isomorphism of Proposition 3.7 between \( C \)-comodules and discrete \( A \)-modules very explicit. The inverse functor \( i^{-1} : A_{\text{Disc}} \rightarrow \text{Comod}_C \) to \( i : \text{Comod}_C \rightarrow A_{\text{Disc}} \) is given explicitly on objects as follows. (On morphisms, of course, it is the identity.) If \( N \) is a discrete \( A \)-module, then \( i^{-1}(N) \) has the \( C \)-coaction \( \rho_{i^{-1}(N)} : i^{-1}(N) \rightarrow i^{-1}(N) \otimes_C \), where

\[
\rho_{i^{-1}(N)}(y) = \sum_{n \geq 0} a_n y \otimes c_n(w).
\]

This sum has only finitely many terms because \( N \) is discrete.

An element \( a \) of \( A \) is determined by the sequence \( \langle \langle a, w^{rn} \rangle \rangle_{n \geq 0} \) of elements of \( R \). We use this fact to characterize the units in \( A \).

**Lemma 4.5.** Let \( C \) be a regular \( R \)-coalgebra and \( A = C^* \). Set \( D = C[w^{-r}] \) and \( B = D^* \). Then
(1) An element \( a \) of \( A \) is a unit if and only if \( \langle a, w^m \rangle \in R^+ \) for each \( m \geq 0 \).

(2) An element \( b \) of \( B \) is a unit if and only if \( \langle b, w^m \rangle \in R^+ \) for each \( m \in \mathbb{Z} \).

Proof. (See [6, Theorem 3.8].) We do part (1); part (2) is similar. If \( a \in A^+ \) then clearly each \( \langle a, w^m \rangle \in R^+ \). Conversely, take \( a \in A \) with \( \langle a, w^m \rangle \in R^+ \) for each \( m \geq 0 \) and write \( a = \sum_{k \geq 0} r_k a_k \). Suppose inductively that we have \( s_0, \ldots, s_{i-1} \in R^+ \) such that

\[
(1) a(s_0 + s_1 a_1 + \cdots + s_{i-1} a_{i-1}) \in 1 + A_i.
\]

For any \( s_i \), the element \( a(s_0 + s_1 a_1 + \cdots + s_i a_i) \) is in \( 1 + A_i \) and has \( a_i \)-coefficient

\[
s_i \left( \sum_{k=0}^{i} r_k \Gamma^i_{k,i} \right) + \text{(terms not involving } s_i),
\]

so we may choose \( s_i \) such that \( a(s_0 + s_1 a_1 + \cdots + s_i a_i) \) is in \( 1 + A_{i+1} \) (i.e. such that the above displayed expression is zero) provided \( \sum_{k=0}^{i} r_k \Gamma^i_{k,i} \) is a unit.

Now,

\[
\Delta(c_n(w)) = \frac{1}{D_n} \sum_{k=0}^{n} \lambda^r_k w^{kr} \otimes w^{kr}
\]

\[
= \frac{1}{D_n} \sum_{k=0}^{n} \lambda^r_k \sum_{r,s=0}^{k} \Lambda^k_r \Lambda^k_s c_r(w) \otimes c_s(w)
\]

\[
= \sum_{r,s=0}^{n} \left( \frac{\sum_{k=0}^{n} \Lambda^k_r \Lambda^k_s \lambda^r_k \lambda^s_n}{D_n} \right) c_r(w) \otimes c_s(w),
\]

so, in particular, \( \Gamma^i_{n,i} = \Lambda^i_n \Lambda^i_i = D_i \) for any \( i \) and \( n \) (since all other terms are zero and \( \Lambda^i_i = D_i \)). Hence

\[
\langle a, w^r \rangle = \sum_{k \geq 0} r_k \langle a_k, w^r \rangle = \sum_{k=0}^{i} r_k \sum_{j=0}^{i} \Lambda^j_{j,i} \langle a_k, c_j(w) \rangle
\]

\[
= \sum_{k=0}^{i} r_k \Lambda^i_k = \sum_{k=0}^{i} r_k \Gamma^i_{k,i},
\]

as required. \( \square \)

5. \( K \)-THEORY COOPERATIONS AND OPERATIONS

In this section we will describe several examples from topology fitting into the framework of Section 4. For those unfamiliar with the topological context of our examples we give a brief description.

In algebraic topology we work with generalized cohomology theories. Such a theory \( E^* \) is a well-behaved functor from the homotopy category of
spaces (or spectra) to the category of graded abelian groups. An important example is provided by periodic complex $K$-theory $K^*$. The operations of the theory $E^*$ are the natural transformations of the functor. We will focus on those operations which are stable, meaning that they are compatible with the suspension functor in a standard way. Such stable operations, by the Yoneda Lemma, are given by $E^*(E) = [E, E]$, the homotopy classes of maps from $E$ to itself, where $E$ is a spectrum representing the cohomology theory $E^*$. In our examples, we will focus on the degree zero stable operations, given by $E^0(E)$. This has an algebra structure in which multiplication is given by composition of operations.

One can also consider $E_*(E) = \pi_*(E \wedge E)$, the so-called cooperations of the theory $E$, and $E_0(E) = \pi_0(E \wedge E)$, the degree zero cooperations. In favourable cases, such as those we study, $E^0(E)$ is the linear dual of $E_0(E)$.

We will take $E$ to be one of various cohomology theories related to periodic complex $K$-theory $K^*$. The coalgebras we study are the degree zero cooperations $E_0(E)$ for such $E$. The comultiplication is dual to composition of operations.

Our notation for the theories we use is as follows. We let $K_0(p)$ denote the periodic complex $K$-theory spectrum with $p$-local coefficients; $k_0(p)$ is the corresponding connective spectrum. For an odd prime $p$, it was proved by Adams in [1] Lecture 4] that these spectra split as wedges of suspensions of simpler spectra:

$$K_0(p) \cong \bigvee_{i=0}^{p-2} \Sigma^{2i}G, \quad k_0(p) \cong \bigvee_{i=0}^{p-2} \Sigma^{2i}g.$$ 

The spectra $G$ and $g$ appearing in these splittings are known as the Adams summands of $p$-local periodic and connective $K$-theory respectively. We write $KO$ and $ko$ for periodic and connective real $K$-theory respectively, and $KO_{(2)}$, $ko_{(2)}$ for their 2-localizations. Let $R = \mathbb{Z}$ or $\mathbb{Z}(p)$ as appropriate.

Writing $F$ for any of the periodic theories and $f$ for the corresponding connective theory, we have that the degree zero stable operation algebra $F^0(F)$ is the $R$-linear dual of the degree zero cooperations $F_0(F)$. Similarly, $f^0(f)$ is the $R$-linear dual of $F_0(f)$.

Rational Laurent polynomials arise as the degree zero cooperations for rational periodic $K$-theory:

$$K_0(K) \otimes \mathbb{Q} \cong \mathbb{Q}[w, w^{-1}].$$

Here $w$ can be thought of as $uv^{-1}$ where $K_*(K) \otimes \mathbb{Q} \cong \mathbb{Q}[u, v, u^{-1}, v^{-1}]$ with $u, v \in K_2(K)$.

To simplify notation somewhat we will write, for example, $K_0(K)_{(p)}$ for $K_{(p)}(K)_{(p)}$ and $K^0(K)_{(p)}$ for $K_{(p)}^0(K)_{(p)}$. In the case of operations, this is an abuse of notation, in the sense that $K_{(p)}^0(K)_{(p)}$ is isomorphic to a completed tensor product $K^0(K)\hat{\otimes}\mathbb{Z}(p)$, not to $K^0(K)\otimes\mathbb{Z}(p)$.

All of our examples of cooperations are torsion-free and can naturally be viewed as subalgebras of $\mathbb{Q}[w, w^{-1}]$. In the cooperations world the relation
between $F_0(F)$ and $F_0(f)$ is given by $F_0(F) = F_0(f)[w^{-r}]$ for suitable $r$ (namely $r = 1$ for $F = K_0(p)$, $r = 2$ for $F = KO_{(2)}$ and $r = p - 1$ for $F = G_0$).

We will see that our examples are regular coalgebras in the sense of the previous section. In order to describe their bases we need some notation for polynomials.  

**Definition 5.1.** If $n \geq 0$ and $(z_i)_{i \geq 1}$ is a sequence of elements of $\mathbb{Q}$, define the polynomial

$$
\theta_n(X; (z_i)) = \prod_{i=1}^{n} (X - z_i).
$$

Also for any $z \in \mathbb{Q}$, write $\theta_n(X; z)$ for $\theta_n(X; (z^{i-1})) = (X - 1)(X - z) \ldots (X - z^{n-1})$.

We now summarize some known results giving regular bases for cooperation coalgebras. In all of our examples, bases can be expressed in terms of the above $\theta$-polynomials for suitable choices of the sequence $(z_i)$. See also [12] where these examples are discussed in terms of integer-valued polynomials.

**Theorem 5.2.**

1. [5, Prop. 3]. Let $p$ be an odd prime and let $q$ be primitive mod $p^2$. Then $K_0(k)(p)$ is a regular $\mathbb{Z}(p)$-coalgebra with $\mathbb{Z}(p)$-basis

$$
f_n(w) = \frac{\theta_n(w; q)}{\theta_n(q^n; q)}, \quad \text{for } n \geq 0.
$$

2. [6, 4.2]. Let $p$ be an odd prime, let $q$ be primitive mod $p^2$ and let $\hat{q} = q^{p-1}$. Then $G_0(g)$ is a regular $\mathbb{Z}(p)$-coalgebra with $\mathbb{Z}(p)$-basis

$$
\hat{f}_n(w) = \frac{\theta_n(w^{p-1}; \hat{q})}{\theta_n(q^{p-1}; \hat{q})}, \quad \text{for } n \geq 0.
$$

3. [6, 9.2]. $KO_0(ko)(2)$ is a regular $\mathbb{Z}(2)$-coalgebra with $\mathbb{Z}(2)$-basis

$$
h_n(w) = \frac{\theta_n(w^2; 9)}{\theta_n(9^n; 9)}, \quad \text{for } n \geq 0.
$$

4. [5, Prop. 20] $K_0(k)(2)$ is a regular $\mathbb{Z}(2)$-coalgebra with $\mathbb{Z}(2)$-basis

$$
f^{(2)}_{2m}(w) = h_m(w)
$$

and

$$
f^{(2)}_{2m+1}(w) = \frac{3^m - w}{2.3^m} h_m(w)
$$

for $m \geq 0$. □

The dual topological bases can all be written as polynomials in Adams operations, which arise in this context as evaluation maps.

**Definition 5.3.** For $\beta \in \mathbb{Q}$ let the Adams operation $\Psi^\beta : \mathbb{Q}[w, w^{-1}] \to \mathbb{Q}$ be the evaluation map, $\Psi^\beta(f(w)) = f(\beta)$. 

We also write $\Psi^\beta$ for the restriction of this map to an $R$-subcoalgebra $C$ of $\mathbb{Q}[w, w^{-1}]$. For suitable choices of $\beta$ this evaluation map will take values in $R \subset \mathbb{Q}$ and so can be viewed as an element of the dual algebra to $C$.

**Theorem 5.4.**

1. [6, 2.2]. If $p$ is odd and $q$ is primitive mod $p^2$, 
   \[ \{ \theta_n(\Psi^g; q) \mid n \geq 0 \} \]
   is the dual topological $\mathbb{Z}_p$-basis for $k^0(k)_{(p)}$ to the basis $f_n(w)$ for $K_0(k)_{(p)}$.

2. [11]. If $p$ is odd, $q$ is primitive mod $p^2$ and $\hat{q} = q^{p-1}$, 
   \[ \{ \theta_n(\Psi^g; \hat{q}) \mid n \geq 0 \} \]
   is the dual topological $\mathbb{Z}_p$-basis for $g^0(g)$ to the basis $\hat{f}_n(w)$ for $G_0(G)$.

3. [6, 9.3]. The dual topological $\mathbb{Z}_2$-basis for $k^0(\text{ko})_{(2)}$ to the basis 
   \[ h_n(w) \] for $KO_0(\text{KO})_{(2)}$ is $\{ \theta_n(\Psi^3; 9) \mid n \geq 0 \}$. \[\square\]

Using the method of Adams and Clarke [2] one obtains the following bases for the periodic versions.

**Theorem 5.5.**

1. [5, Cor. 6]. If $p$ is an odd prime, then the Laurent polynomials $w^{-[n/2]} f_n(w)$, for $n \geq 0$, form a $\mathbb{Z}_p$-basis for $K_0(K)_{(p)}$.

2. If $p$ is an odd prime, 
   \[ \{ w^{-[n/2](p-1)} \hat{f}_n(w) \mid n \geq 0 \} \]
   is a $\mathbb{Z}_p$-basis for $G_0(G)$.

3. \[ \{ w^{-n} h_n(w) \mid n \geq 0 \} \]
   is a $\mathbb{Z}_2$-basis for $KO_0(\text{KO})_{(2)}$.

4. \[ \{ F_n^{(2)}(w) = w^{-[n/2]} f_n^{(2)}(w) \mid n \geq 0 \} \]
   is a $\mathbb{Z}_2$-basis for $K_0(K)_{(2)}$. \[\square\]

The next step is to give topological bases for the dual algebras of these periodic objects. Define the polynomials 

\[ \Theta_n(X; q) = \theta_n(X; (q^{-1})^{[n/2]}) \].

**Theorem 5.6.**

1. [6, 6.2]. The set $\{ \Theta_n(\Psi^g; q) \mid n \geq 0 \}$ is a topological $\mathbb{Z}_p$-basis for $K^0(K)_{(p)}$.

2. The set $\{ \Theta_n(\Psi^g; \hat{q}) \mid n \geq 0 \}$ is a topological $\mathbb{Z}_p$-basis for $G^0(G)$.

3. [6, 9.3]. The set $\{ \Theta_n(\Psi^3; 9) \mid n \geq 0 \}$ is a topological $\mathbb{Z}_2$-basis for $KO^0(\text{KO})_{(2)}$. \[\square\]

It is a bit more complicated to describe topological bases for $k^0(k)_{(2)}$ and $K^0(K)_{(2)}$. The former is done in [6, 8.2] and the latter can be done using Theorem 5.2 of [12], but this does not give the answers in any particularly nice form. Instead our strategy for these cases will be to work directly with the cooperations.
6. Locally finitely generated modules

In this section we study modules which are locally finitely generated over the ground ring \( R \). As we explain, it is easy to see that discrete modules are locally finitely generated over \( R \). We will give conditions under which the converse is true.

Consider an \( R \)-algebra \( A \) with a filtration \( A_\alpha \) satisfying the conditions of Definition 2.2 and a discrete \( A \)-module \( M \). We will consider only \( R = \mathbb{Z} \) and \( R = \mathbb{Z}(p) \) for \( p \) prime. If \( x \in M \), by Lemma 2.3, there is \( \alpha \) with \( A_\alpha x = 0 \). Then \( Ax = (A/A_\alpha)x \), so \( Ax \) is finitely generated as an \( R \)-module, as \( A/A_\alpha \) is.

**Definition 6.1.** A module \( M \) over an \( R \)-algebra \( A \) is locally finitely generated over \( R \) if \( Ax \) is a finitely generated \( R \)-module for every \( x \in M \). The category \( A_{\text{LFG}} \) is the full subcategory of \( A_{\text{Mod}} \) whose objects are the \( A \)-modules which are locally finitely generated over \( R \).

Of course, if \( A \) is itself finitely generated as an \( R \)-module then every \( A \)-module is locally finitely generated over \( R \). The definition is of interest in the case where \( A \) is not itself finitely generated over \( R \). In this case, \( A \) regarded as a module over itself is clearly not locally finitely generated.

The above discussion shows that \( A_{\text{Disc}} \subseteq A_{\text{LFG}} \). In this section we give our main technical result, Theorem 6.2, providing conditions under which these subcategories of \( A_{\text{Mod}} \) are equal. At the end of the section, Theorem 6.9 gives a version of the conditions formulated more directly in terms of the structure of a regular coalgebra to which \( A \) is dual.

In the next section, we show that our method applies to the algebras of stable operations of many variants of topological \( K \)-theory.

**Theorem 6.2.** Let \( p \) be a prime and \( A \) be a topological \( \mathbb{Z}(p) \)-algebra with a topological basis \( \{a_n \mid n \geq 0\} \). If for every \( l > 0 \) there is an infinite set \( N_l \subseteq \mathbb{N}_0 \) such that

1. \( 1 - a_{n-m} \) is a unit in \( A \) for every \( m, n \in N_l \) with \( m < n \), and
2. \( a_m a_n \equiv a_{m+n} \mod p^l \) whenever \( m \in N_l \) and \( n \in \mathbb{N}_0 \),

then \( A_{\text{Disc}} = A_{\text{LFG}} \).

Of course, if \( A = \mathbb{Z}[t] \) with basis \( \{t^n \mid n \geq 0\} \), then the conditions of the theorem are satisfied with \( N_l = \mathbb{N} \) for all \( l \).

We need some preliminary results before we prove Theorem 6.2. First, we can make a simplification. Suppose that \( M \) is an \( A \)-module which is locally finitely generated over \( R \) and let \( x \in M \). Then \( x \) is contained in an \( R \)-finitely generated \( A \)-submodule of \( M \), namely \( Ax \). If every \( Ax \) is discrete, then \( x \in Ax \subseteq \text{Disc} M \) for every \( x \in M \), so \( \text{Disc} M = M \) and \( M \) is discrete. Thus it is enough to show that every \( R \)-finitely generated \( A \)-module is discrete.

Let \( N \) be an \( R \)-finitely generated \( A \)-module. We consider separately the cases of \( N \) a free \( R \)-module of finite rank and \( N \) a finite torsion \( R \)-module.
and then prove a limited extension theorem to combine the two. Take first an \(N\) which is free over \(R\). Discreteness will follow from the fact that this is a ‘slender’ abelian group.

**Definition 6.3.** \([8, §94]\). Let

\[
P = \prod_{n \geq 0} \mathbb{Z}e_n
\]

for some generators \(e_n\). An abelian group \(G\) is **slender** if every linear map \(\eta: P \to G\) has \(\eta(e_n) = 0\) for all but finitely many \(n\).

In the literature, e.g. in \([8]\), this property is only considered for \(G\) torsion-free, but we can define it for every \(G\). It is relevant to us because \(A\) contains a subgroup isomorphic to \(P\), under the map \(e_n \mapsto a_n\), so any linear map out of \(A\) induces a linear map out of \(P\). In particular this applies to the map \(a \mapsto ax\) from \(A\) to \(M\), where \(M\) is an \(A\)-module and \(x \in M\).

**Lemma 6.4.** \([8, 94.2]\). If \(R = \mathbb{Z}\) or \(R = \mathbb{Z}(p)\), then a free \(R\)-module of finite rank is slender. \(\square\)

**Lemma 6.5.** If \(R = \mathbb{Z}\) or \(R = \mathbb{Z}(p)\) and \(C\) is a regular \(R\)-coalgebra, the \(R\)-linear map \(C \to A^*\) determined by

\[
c_n(w) \mapsto a_n^*
\]

is an isomorphism of abelian groups.

**Proof.** Certainly this map is injective. If \(\alpha: A \to \mathbb{Z}(p)\) is a linear map, then by slenderness of \(R\), \(\alpha(a_n) = 0\) for all but finitely many \(n\). Hence \(\alpha\) is a finite linear combination of the \(a_n^*\) and so lies in the image of \(C\). \(\square\)

Generally, we would expect \(A^*\) to be larger than \(A\), just as \(A = C^*\) is larger than \(C\). This odd property of slender groups solves the first part of our problem.

**Proposition 6.6.** An \(A\)-module \(N\) which is free of finite rank as an \(R\)-module is discrete.

**Proof.** Take \(x \in N\) and let \(x_1, \ldots, x_k\) be an \(R\)-basis for \(N\). Then if \(a \in A\),

\[
ax = \sum \gamma_i(a, x)x_i
\]

for some \(\gamma_i(a, x) \in R\). For \(i = 1, \ldots, k\), define \(R\)-linear maps

\[
g_i(x): A \to R
\]

by \(g_i(x)(a) = \gamma_i(a, x)\) and let \(g_i'(x)\) be the element of \(C\) corresponding to \(g_i(x)\) under the isomorphism of Lemma \([6,5]\). The \(g_i'(x)\) give us a map \(\rho: N \to N \otimes C\),

\[
\rho(x) = \sum x_i \otimes g_i'(x),
\]

which is \(R\)-linear because \(a(x + y) = ax + ay\). Such a map makes \(N\) a \(C\)-comodule if and only if the corresponding map (using the action \([5,3]\)) \(A \otimes N \to N\) is an \(A\)-action map. But this map is the given action of \(A\) on \(N\), so \(N\) is indeed a \(C\)-comodule. By Lemma \([5,6]\), \(N\) is discrete. \(\square\)
We cannot use the concept of slenderness to attack the case of finite \( A \)-modules, since no non-zero finite \( \mathbb{Z}(p) \)-module is slender. (One can see this by showing that for \( M \) a non-zero finite \( \mathbb{Z}(p) \)-module, \( \text{Hom}_{\mathbb{Z}}(P, M) \) is an uncountable set.)

The remainder of our approach is motivated by the case of \( \mathbb{Z}[t] \). If \( N \) is a \( \mathbb{Z} \)-finitely generated \( \mathbb{Z}[t] \)-module, we can use the classification of finitely generated abelian groups to write \( N \) as an extension of \( \mathbb{Z}[t] \)-modules

\[
0 \to T \to N \to F \to 0,
\]

where \( T \) is the torsion part of \( N \) and \( F \) is free over \( \mathbb{Z} \). By Proposition 6.6, \( F \) is discrete. If \( x \in T \) then since \( T \) is finite there must be \( m < n \) such that \( t^m x = t^n x \). Then \( (t^m - t^n)x = 0 \), so \( t^m (1 - t^{n-m})x = 0 \). Since \( 1 - t^{n-m} \) is a unit in \( \mathbb{Z}[t] \), it follows that \( t^m x = 0 \). So \( (t^m)x = 0 \) and therefore \( T \) is discrete. As discussed earlier, over \( \mathbb{Z}[t] \), discrete modules are closed under extensions, so \( N \) is discrete. The conditions of Theorem 6.2 allow us to generalize this argument.

**Lemma 6.7.** If \( A \) satisfies conditions (1) and (2) of Theorem 6.2, \( M \) is a finite \( A \)-module and \( p^\lambda M = 0 \), then there is \( m \in N_\lambda \) with \( a_m M = 0 \).

**Proof.** Let \( x \in M \). Since \( N_\lambda \) is infinite and \( M \) is finite, there must be \( m, n \in N_\lambda \) with \( m < n \) and \( a_m x = a_n x \), equivalently,

\[
(a_m - a_n)x = 0.
\]

We may factorize this as \( a_m (1 - a_{n-m})x = p^\theta x = 0 \) for some \( \theta \in A \) using (2); then, by (1), \( a_m x = 0 \).

Now if we have such an \( m_x \) for each \( x \in M \), let \( m \) be the largest. For any \( x, a_m x = a_{m-\lambda} a_m x = 0 \), because \( m_x \in N_\lambda \).

**Lemma 6.8.** If \( A \) satisfies condition (2) of Theorem 6.2, \( m \in N_\lambda \) and \( n \geq 0 \), then for any \( a \in A_{m+n} \) there is \( b \in a_m A_n \) such that \( p^j \) divides \( a - b \).

**Proof.** We may write

\[
a = \sum_{i \geq n} \lambda_{m+i} a_{m+i}
\]

for some \( \lambda_{m+i} \in \mathbb{Z}(p) \). By (2), for each \( i \) there is \( \theta_i \in A \) with

\[
a_{m+i} = a_m a_i + p^j \theta_i.
\]

Note that, since \( A_i \) is an ideal, \( \theta_i \in A_i \). Hence the infinite sum \( \sum_{i \geq n} \lambda_{m+i} \theta_i \) converges and therefore represents an element of \( A \).

Summing over \( i \), we get

\[
a = \sum_{i \geq n} \lambda_{m+i} (a_m a_i + p^j \theta_i)
\]

\[
= a_m \sum_{i \geq n} \lambda_{m+i} a_i + p^j \sum_{i \geq n} \lambda_{m+i} \theta_i,
\]

as required. \( \square \)
Proof of Theorem 6.2. We have reduced this to showing that a \( \mathbb{Z}_p \)-finitely generated \( A \)-module \( N \) is discrete. Consider the extension
\[
0 \to T \to N \to F \to 0
\]
of \( A \)-modules, where \( T \) is the \( p \)-torsion \( A \)-submodule of \( N \) and \( F = N/T \). By the classification of finitely generated \( \mathbb{Z}_p \)-modules \cite{9}, \( F \) is a free \( \mathbb{Z}_p \)-module of finite rank; hence it is discrete by Proposition 6.6. Take \( n \) with \( A_n F = 0 \). (Since \( F \) is \( \mathbb{Z}_p \)-finitely generated and hence \( A \)-finitely generated, there is such an \( n \).

Now take \( x \in N \). Since \( A_n(x + T) = 0 \) in \( F \), we must have \( A_n x \subseteq T \). By Lemma 6.7, there is \( m \in N_s \), where \( p^s T = 0 \), such that \( a_m T = 0 \); hence \( a_m A_n x = 0 \).

We want to show that \( A_{m+n} x = 0 \), so take \( a \in A_{m+n} \). By Lemma 6.8 there are \( b \in a_m A_n \) and \( \theta \in A \) such that \( a - b = p^s \theta \). Then \( ax = bx + p^s \theta x = p^s \theta x \), as \( a_m A_n x = 0 \). We also know that \( ax \in T \), because \( A_{m+n} x \subseteq A_n x \subseteq T \), so \( p^s ax = 0 \). This means \( p^s \theta x = 0 \), so \( \theta x \in T \). But then \( p^s \theta x = 0 \), i.e. \( ax = 0 \).

We end this section with a result corresponding to Theorem 6.2 under the assumption that we are given a regular coalgebra \( C \) or \( D = C[w^{-r}] \). This form of the result is less intuitive, but we will need it to deal with those applications in the next section where the dual topological basis is impractical to work with. We refer to Definition 4.3 for the coefficients \( \Lambda \) and \( \Gamma \) appearing in the statement.

Theorem 6.9. Let \( p \) be a prime. Let \( C \) be a regular \( \mathbb{Z}_p \)-coalgebra with specified basis \( \{ c_n(w) \mid n \geq 0 \} \) and let \( A = C^\ast \). If for every \( l > 0 \) there is an infinite set \( N_l \subseteq \mathbb{N}_0 \) such that

1. \( p \) divides \( \Lambda_{m-n}^j \) whenever \( m, n \in N_l \) with \( m < n \) and \( j \in \mathbb{N}_0 \), and
2. if \( m \in N_l \) and \( n \in \mathbb{N}_0 \), then \( \Gamma_{m+n}^l = 1 \) and \( p^j \) divides \( \Gamma_{m,n}^l \) whenever \( i \neq m + n \),

then \( A \text{Disc} = A \text{LFG} \).

If \( D = C[w^{-r}] \) has a \( \mathbb{Z}_p \)-basis \( F_n(w) \) which satisfies analogous conditions to those above and \( B = D^\ast \), then \( B \text{Disc} = B \text{LFG} \).

Proof. We prove the first statement; the second is similar. It is enough to check that these conditions imply those of Theorem 6.2. For any \( n > m \geq 0 \) and \( j \geq 0 \),
\[
\langle 1 - a_{n-m}, w^j \rangle = 1 - \left( a_{n-m}, \sum_{i=0}^j \Lambda_{j-f_i(w)}^i \right) = 1 - \Lambda_{n-m}^i,
\]
which, by (1), is in \( 1 + p\mathbb{Z}_p \subseteq \mathbb{Z}_p^\ast \) provided \( m, n \in N_l \). By Lemma 4.3(1), therefore, \( 1 - a_{n-m} \) is a unit, so condition (1) of Theorem 6.2 is satisfied. Since
\[
a_m a_n = \sum_{i=0}^\infty \Gamma_{m,n}^i a_i,
\]
condition (2) above immediately implies condition (2) of Theorem 6.2. □

7. Applications of the Theorems

In this section we apply our results on locally finitely generated modules to the algebras of operations described in Section 5. We will use Theorems 6.2 and 6.9 to prove the following result.

Theorem 7.1. Let $p$ be a prime. If $E$ is $K(p)$, $k(p)$, $G$, $g$, $KO(2)$ or $ko(2)$, then

$$E^0(E)\text{Disc} = E^0(E)LFG.$$  

Note that the cases of $K(2)$ and $k(2)$ are included in this statement; we will, however, have to prove them separately from $K(p)$ and $k(p)$ for $p$ odd. In each case we need to find suitable infinite subsets $N_l$ of $N_0$, one for each $l > 0$. We list them in the following table. In general we must discard small values of $n$ in order for condition (1) of Theorem 6.2 to be satisfied and impose a divisibility condition for condition (2) to be satisfied. For $K(2)$ and $k(2)$ we will, as noted before, use Theorem 6.9.
Let \( \psi \) be a sequence of rationals. Recall the polynomials \( \theta_n(X; (z_i)) \) from Definition 5.4. Note that, for any \( 0 \leq i \leq m \), the quotient

\[
\frac{\theta_m(X; (z_i))}{\theta_{m-i}(X; (z_i))} = \prod_{k=m-i+1}^{m} (X - z_k)
\]

is a polynomial in \( X \).

**Lemma 7.3.** Let \( (z_i)_{i \geq 1} \) be a sequence of rationals and, for brevity, write \( \theta_n(X) \) for \( \theta_n(X; (z_i)) \). Then, for any \( m, n \geq 0 \),

\[
\theta_{m+n}(X) = \theta_m(X)\theta_n(X) + \sum_{i=0}^{n-1} (z_{n-i} - z_{m+n-i}) \frac{\theta_n(X)}{\theta_{m-i}(X)} \theta_{m+n-i}(X).
\]

**Proof.** For \( 0 \leq i \leq n-1 \),

\[
\theta_{m+n-i}(X) = (X - z_{m+n-i})\theta_{m+n-i-1}(X)
\]

\[
= (X - z_{n-i})\theta_{m+n-i-1}(X) + (z_{n-i} - z_{m+n-i})\theta_{m+n-i-1}(X).
\]

Repeatedly apply this formula.

Next we give conditions which imply those of Theorem 6.2 specifically for the case where \( A \) is the dual of a regular coalgebra and has a topological basis of \( \theta \)-polynomials. We write \( \nu_p \) for \( p \)-adic valuation.

**Theorem 7.4.** Let \( A \) be the dual of a regular \( \mathbb{Z}_{(p)} \)-coalgebra \( C \) and suppose that \( A \) has a topological \( \mathbb{Z}_{(p)} \)-basis of the form \( \{a_n = \theta_n(\Psi^\beta, (z_i)) \mid n \geq 0\} \) for some \( \beta, z_i \in \mathbb{Z}_{(p)} \). If for every \( l > 0 \) there is an infinite set \( N_l \subseteq \mathbb{N}_0 \) such that

1. \( \theta_{n-m}(\beta^j, (z_i)) \in p\mathbb{Z}_{(p)} \) for every \( m, n \in N_l \) with \( m < n \) and \( j \in \mathbb{N}_0 \), and
(2) \( \nu_p(z_{n-i} - z_{m+n-i}) \geq 1 \) for \( 0 \leq i \leq n - 1 \) whenever \( m \in N_l \) and \( n \in \mathbb{N}_0 \).

then \( _A \text{Disc} = _A \text{LFG} \).

**Proof.** Since \( \langle 1 - a_{n-m}, w^{jr} \rangle = 1 - \theta_n - m(\beta^{jr}, (z_i)) \), using Lemma 4.5, we see that condition (1) here implies condition (1) of Theorem 6.2. It follows directly from Lemma 7.3 that condition (2) here implies condition (2) of Theorem 6.2. □

We now give the details of how to apply Theorem 7.4 to most of the connective theories.

**Proposition 7.5.** Let \( E \) be \( k_{(p)} \) for \( p \) an odd prime, \( g \) or \( ko(2) \). Consider the topological bases for \( A = E^0(E) \) given in Theorem 5.4 and let the sets \( N_l \) for \( l > 0 \) be as specified in Table 7.2. Then conditions (1) and (2) of Theorem 7.4 are satisfied.

**Proof.** The three theories fit into the framework of Theorem 7.4 with data as follows.

| \( E \) | \( r \) | \( \beta \) | \( z_i \) |
|---|---|---|---|
| \( k_{(p)} \) | 1 | \( q \) | \( q^{i-1} \) |
| \( g \) | \( p - 1 \) | \( q \) | \( \hat{q}^{i-1} \) |
| \( ko(2) \) | 2 | 3 | \( 9^{i-1} \) |

First consider \( E = k_{(p)} \). Let \( j \geq 0 \). Taking \( n \geq p - 1 \), there is some \( 0 \leq i \leq n - 1 \) such that \( p - 1 \) divides \( j - i \) and so \( p \) divides \( q^j - q^i \). Thus \( \theta_n(\beta^{jr}, (z_i)) = \prod_{i=0}^{n-1} (q^j - q^i) \in p\mathbb{Z}(p) \) and this shows condition (1) is satisfied.

Taking \( l > 0 \), \( n \geq 0 \) and \( m \in N_l \), we have \( q^{n-i-1} - q^{m+n-i-1} = q^{n-i-1}(1 - q^m) \) and since \( p^{l-1}(p - 1) \) divides \( m \), \( p^l \) divides \( 1 - q^m \). So condition (2) is satisfied.

Next we consider \( E = g \). For each \( n \geq 1 \) and \( j \geq 0 \), \( \theta_n(\hat{q}^j; \hat{q}) \) lies in \( p\mathbb{Z}(p) \), and condition (1) is satisfied. Since \( \nu_p(\hat{q}^{n-i-1} - \hat{q}^{m+n-i-1}) = \nu_p(1 - \hat{q}^m) = 1 + \nu_p(m) \), condition (2) is also satisfied.

Finally, let \( E = ko(2) \). Recall that

\[
\nu_2(3^i - 1) = \begin{cases} 
1 & \text{if } i \text{ is odd;} \\
2 + \nu_2(i) & \text{if } i \text{ is even.}
\end{cases}
\]

Thus \( \theta_n(9^i; 9) \in p\mathbb{Z}(p) \) for all \( n \geq 1 \) and condition (1) is satisfied. Also

\[
\nu_2(9^{n-i-1} - 9^{m+n-i-1}) = \nu_2(1 - 9^m) = 3 + \nu_2(m).
\]

For \( l = 1, 2 \), this is at least 3, which is enough. For \( l \geq 3 \), let \( m \in N_l \) and then we have chosen \( N_l \) such that \( 3 + \nu_2(m) \geq l \). So condition (2) is satisfied. □

Next we turn to the periodic versions. The complex periodic case \( K^0(K)_{(p)} \), for \( p \) odd, recovers [7, 3.2].
**Proposition 7.7.** Let $E$ be $K(p)$ for $p$ an odd prime, $G$ or $KO(2)$. Consider the topological bases for $A = E^0(E)$ given in Theorem 5.6 and let the sets $N_i$ for $l > 0$ be as specified in Table 7.2. Then conditions (1) and (2) of Theorem 7.4 are satisfied.

**Proof.** The three theories fit into the framework of Theorem 7.4 with data as follows.

| $E$ | $K(p)$ | $G$ | $KO(2)$ |
|-----|--------|-----|---------|
| $p$ | 1 $q$ | $q^{-1}[i/2]$ | $q^{-1}[i/2]$ |
| $r$ | $q^{-1}[i/2]$ | $p-1$ | $q^{-1}[i/2]$ |
| $z_i$ | $q^{-1}[i/2]$ | $2$ | $3$ |

First consider $E = K(p)$ and write $q_i = q^{-1}[i/2]$. For condition (1) it will be enough to show that $\theta_n(q^i; (q_i)) \in p\mathbb{Z}(p)$ for $j > 0$ and $n \geq p - 1$. Since $q$ generates $(\mathbb{Z}/p)^\times$, the difference $q^r - q_i$ is divisible by $p$ if and only if $r - (-1)^{i}[i/2]$ is divisible by $p - 1$. The set $\{(-1)^{i}[i/2] | i = 1, 2, \ldots, n\}$ consists of $n$ consecutive integers, so, since $n \geq p - 1$, at least one of the linear factors of $\theta_n(q^j; (q_i))$ is divisible by $p$.

For condition (2), take $l > 0$, $m \in N_i$ and $n \in \mathbb{N}_0$. We need to show that $p^l$ divides $q_{m+1} - q_{m+n+i}$ for $0 \leq i \leq n - 1$. Since $m$ is even, $q_{m+1} - q_{m+n+i} = q_{m+1} - (q^{(-1)^{n+i}}m/2)$. Since $q$ generates $(\mathbb{Z}/p)^\times$ and $p^{l-1}(p-1)$ divides $(-1)^{n+i}m/2$, we deduce that $p^l$ divides $1 - q^{(-1)^{n+i}m/2}$, as required.

Next consider $E = G$. Write $\hat{q}_i = q^{-1}[i/2]$. For each $n \geq 1$ and $j \in \mathbb{Z}$, $\theta_n(\hat{q}^i; (q_i)) \in p\mathbb{Z}(p)$, so condition (1) is satisfied. For condition (2), take $l > 0$ and $m \in N_i$. Then

$$\nu_p(\hat{q}_{m+1} - \hat{q}_{m+n+i}) = \nu_p(1 - \hat{q}^{m/2}) = 1 + \nu_p(m/2) \geq l,$$

as required.

Finally consider $E = KO(2)$ and write $q_i = 9^{-1}[i/2]$. By (7.6), we have $\theta_n(q^i; (q_i)) \in 2\mathbb{Z}(2)$ for every $n \geq 1$, so condition (1) is satisfied.

Take $l > 0$ and $m \in N_i$. Let

$$A_{m,n}^i = g^{-1}[i][m-n+i/2] - g^{-1}[i][m+n+i/2]$$

$$= g^{-1}[i][m-n+i/2] - g^{-1}[i][m-n+i/2]$$

since $m$ is even.

For condition (2) we need to show that $A_{m,n}^i$ is divisible by $2^l$ for $0 \leq i \leq m - 1$. Now $\nu_2(1 - g^{m/2}) = \nu_2(1 - 3^m) = 2 + \nu_2(m)$, using (7.6). For $l = 1, 2$, this is at least 3, which is enough; for $l \geq 3$, we have chosen $N_i$ such that $2 + \nu_2(m) \geq l$.

The last two examples we consider are the 2-local complex spectra, and for these we use Theorem 6.9.

**Proposition 7.8.** (1) Let $A = k^0(k)(2)$. Consider the coalgebra $K_0(k)(2)$ to which $A$ is dual, with basis $f^{(2)}_n(w)$ as given in Theorem 5.2. Let $N_i$ be
as specified in Table 7.2. Then conditions (1) and (2) of Theorem 6.9 are satisfied.

(2) Let $A = K^0(K)_{(2)}$. Consider the coalgebra $K^0(K)_{(2)}$ to which $A$ is dual, with basis $F_n^{(2)}(w)$ as given in Theorem 5.3. Let $N_l$ be as specified in Table 7.2. Then conditions (1) and (2) of Theorem 6.9 are satisfied.

Proof. We only give the details for the first part, since the proof of the second is very similar. Note that 

$$w f_{2m}^{(2)}(w) = 3^m f_{2m}^{(2)}(w) - 2.3^m f_{2m+1}^{(2)}(w).$$

Since $\Delta$ is a map of bialgebras,

$$\Delta(f_{2m+1}^{(2)}(w)) = \frac{1}{2.3^m} \Delta(3^m - w) \Delta(h_m(w))$$

$$= \frac{3^m}{2.3^m} \Gamma(KO_0(ko)_{(2)})^m_{i,j} h_i(w) \otimes h_j(w),$$

so we compute that

$$\Gamma(K_0(k)_{(2)})^{2m+1}_{2i,2j} = \frac{1 - 3^{i+j-m}}{2} \Gamma(KO_0(ko)_{(2)})^m_{i,j}$$

and

$$\Gamma(K_0(k)_{(2)})^{2m+1}_{2i,2j+1} = 3^{i+j-m} \Gamma(KO_0(ko)_{(2)})^m_{i,j}.$$

(We will not need the other cases.)

Write $\eta_n = \theta_n(\Psi^3; 9)$ for the basis elements given in Theorem 5.4. It follows from Theorem 7.3 and Proposition 7.5 that we have

1. if $m, n \in N_l$ and $j \in N_0$, then $\langle 1 - \eta_{n-m}, w^j \rangle \in 1 + 2\mathbb{Z}_{(2)}$; and
2. if $m \in N_l$ and $n \in N_0$, then $\eta_m \eta_n \equiv \eta_{m+n} \mod 2^l$.

The former is equivalent to condition (1) of Theorem 6.9 and, since

$$\eta_m \eta_n = \sum_{i \geq 0} \Gamma(KO_0(ko)_{(2)})^i_{m,n} \eta_i,$$

the latter is equivalent to condition (2) of Theorem 6.9 \hfill \Box

We have now completed the proof of Theorem 7.1.

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