Mass ratio of W and Z bosons in SU(5) gauge-Higgs unification

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Abstract

We study the mass ratio of W and Z bosons in an SU(5) gauge-Higgs unification model with the left-right symmetry remaining locally for boundary conditions and with no scalar fields except for the extra-dimensional component of gauge bosons. We find that there are a variety of boundary conditions to yield distinct values of the mass ratio.
1 Introduction

The gauge-Higgs unification is a candidate of electroweak symmetry breaking [1, 2]. The four-dimensional Higgs field is identified with a part of the extra-dimensional component of gauge bosons in higher dimensions. The bosonic ingredients of the four-dimensional system are gauge bosons and scalar fields so that potentially it may have a phase transition as in the Coleman-Weinberg model for the scalar quantum electrodynamics. Because the four-dimensional scalar fields are originally higher-dimensional gauge fields, the mass terms are not locally generated but can appear as non-local effects via a Wilson line phase. This Wilson line phase is dynamical as an Aharonov-Bohm phase, where the possible phase transition is dynamical.

If the gauge-Higgs unification is a scenario beyond the standard model, a simple setup is a model in which the Higgs doublet for SU(2)$_L$ is included as a part of the extra-dimensional component of higher-dimensional gauge fields. In the standard model, the Higgs sector has the custodial symmetry SU(2)$_D$. In order that this property is taken into account in the gauge-Higgs unification, the left-right symmetry SU(2)$_L \times$ SU(2)$_R$ might be required for compatibility with experimental data. Here, it would be simple to treat such a left-right symmetry as a gauge symmetry. A gauge-Higgs unification model as an extension of the electroweak standard model can be a gauge theory with a group larger than SU(2)$_L \times$ SU(2)$_R$. The large group should be broken to U(1) for photon by a Wilson line phase and by boundary conditions with respect to extra dimensions.

It is nontrivial whether the mass ratio of W and Z bosons is correctly produced in a specified model even when an algebraic structure such as groups and multiplets is prepared. The standard model with a scalar field in the fundamental representation has degrees of freedom to adjust the mass ratio. For no scalar fields except for the extra-dimensional component of higher-dimensional gauge fields in the gauge-Higgs unification, it should be clarified what the value of the mass ratio of W and Z bosons can be.

In this paper, we study the mass ratio of W and Z bosons in an SU(5) gauge-Higgs unification model in the Randall-Sundrum warped spacetime. The group SU(5) is broken by boundary conditions consistently with higher-dimensional gauge transformation to SO(5) at one boundary and to [SU(2)$\times$U(1)]$_L \times$ [SU(2)$\times$U(1)]$_R$ at the other boundary. Locally left-right symmetry remains. The group with the overlapping of Neumann condition for the two boundary conditions is SU(2)$\times$U(1). When a nonzero Wilson line phase is developed, the SU(2)$\times$U(1) is broken to U(1). The rank of the gauge group is reduced at the levels of SU(5)$\rightarrow$SO(5) and SU(2)$\times$U(1)$\rightarrow$U(1). In this model, there are no scalar fields except for the extra-dimensional component of gauge bosons, that are required for the symmetry breaking of SU(5) to U(1). We work in two cases of boundary conditions for the rank reduction SU(5)$\rightarrow$SO(5). One is on the TeV brane and the other is on the Planck brane. We find that in the toy model there are deviations of the values of the mass ratio of W and Z masses from the experimental value. On the other hand, the values are obtained differently in the two cases. This shows that there are a variety of boundary conditions to yield distinct values of the mass ratio, as a room to build a realistic model for symmetry breaking without additional scalar fields. We will discuss possible prospects to improve the model.

The paper is organized as follows. In Sec. 2 we present our model including fields and action integrals. The mass ratios for the case with the rank reductions on the Plank brane and on the TeV brane are given in Sec. 3 and Sec. 4 respectively. We conclude in Sec. 5 with some remarks. Details of calculations and formulas are shown in appendices.
2 Model

The model is defined in the Randall-Sundrum warped spacetime whose metric is given by [3]

\[ ds^2 = e^{-2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \]  
\[ (2.1) \]

where \( \eta_{\mu\nu} = \text{diag}(-1,1,1,1) \), \( \sigma(y) = \sigma(y + 2L) \), and \( \sigma(y) = k|y| \) for \( |y| \leq L \). The fundamental region in the fifth dimension is given by \( 0 \leq y \leq L \). The Planck brane and the TeV brane are located by \( y = 1 \) and \( y = L \), respectively. The bulk region \( 0 < y < L \) is an anti-de Sitter spacetime with the cosmological constant \( \Lambda = -6k^2 \).

We consider an SU(5) gauge theory in the gauge-Higgs unification. The SU(5) symmetry is broken to SO(5) on one boundary and to \([SU(2) \times U(1)]_L \times [SU(2) \times U(1)]_R\) on the other boundary. The symmetry at this stage is broken by boundary conditions consistently with the five-dimensional gauge transformation [4]-[7]. The SU(5) five-dimensional gauge fields are \( A_M = \sum_{i=1}^{24} A_M^i T_i \). The generators are written as \( T_i = \frac{1}{\sqrt{2}} (T_i - T_{2i}) \), \( T_9 = \frac{1}{\sqrt{2}} (T_9 - T_{18}) \), and \( A_i \) is defined in a similar way to Gell-Mann matrices for SU(3) [4]. The generator of \([SU(2) \times U(1)]_L \times [SU(2) \times U(1)]_R\) is given by \( T_1, T_2, T_3, T_8, T_{15}, T_{19}, T_{22}, T_{23}, T_{24} \). The generators of SO(5) are given by

\[
\begin{align*}
T_1 &= \frac{1}{\sqrt{2}} (T_1 - T_{22}), & T_2 &= \frac{1}{\sqrt{2}} (T_2 + T_{23}), & T_3 &= \frac{1}{\sqrt{2}} (T_4 - T_{13}), & T_4 &= \frac{1}{\sqrt{2}} (T_5 - T_{14}), \\
T_5 &= \frac{1}{\sqrt{2}} (T_6 - T_{20}), & T_6 &= \frac{1}{\sqrt{2}} (T_7 - T_{21}), & T_7 &= \frac{1}{\sqrt{2}} (T_{11} - T_{16}), & T_8 &= \frac{1}{\sqrt{2}} (T_{12} - T_{17}), \\
T_9 &= \frac{1}{\sqrt{2}} (T_3 + \frac{\sqrt{6}}{4} T_{15} - \frac{\sqrt{6}}{4} T_{21}), & T_{10} &= \frac{1}{\sqrt{2}} (\frac{\sqrt{6}}{3} T_8 + 5\frac{\sqrt{6}}{12} T_{15} + \frac{\sqrt{6}}{4} T_{24}).
\end{align*}
\]
\[ (2.2) \]

The overlapping generators of \([SU(2) \times U(1)]_L \times [SU(2) \times U(1)]_R\) and SO(5) are \( T_1, T_2, T_3, T_8, T_{10}, T_{18, T_{19}} \) which form the SU(2)×U(1) algebra. The SU(2)×U(1) part of the gauge fields are written as

\[
A_\mu^T T_1 + A_\mu^2 T_2 + A_\mu^3 T_3 + A_\mu^4 T_8 + A_\mu^5 T_{10} = \frac{1}{2} \begin{pmatrix} U_\mu & 0 \\ 0 & -U_\mu \end{pmatrix}.
\]
\[ (2.3) \]

Here \( U_\mu = \frac{1}{\sqrt{2}} (W_\mu^a \sigma^a + B_\mu 1_2) \), with \( W_\mu^1 \equiv A_\mu^T, W_\mu^2 \equiv A_\mu^2, W_\mu^3 \equiv A_\mu^3, W_\mu^4 \equiv A_\mu^4, W_\mu^5 \equiv A_\mu^5, B_\mu \equiv A_\mu^1 \). Hereafter we will call the boundary for SU(5)→[SU(2)×U(1)]_L×[SU(2)×U(1)]_R the left-right symmetric boundary and the boundary for SU(5)→SO(5) the rank-reducing boundary. At the left-right symmetric boundary, the four-dimensional scalar fields, \( A_y \) have Neumann boundary condition for \( T_4, \ldots, T_7, T_9, \ldots, T_{14}, T_{16}, \ldots, T_{21} \). At the rank-reducing boundary, \( A_y \) have Neumann boundary condition for \( T_{15}, \ldots, T_{20}, T_9, T_{10}, T_{18}, T_{19} \). The overlapping Neumann condition of both boundaries are given by \( T_{15}, \ldots, T_{20}, T_9, T_{10}, T_{18}, T_{19} \). The Higgs doublet \( H \) is included as

\[
\sum_{i=13}^{16} A^i_y T_i = \frac{1}{2} \begin{pmatrix} H^+ & H \\ H^T & H^T \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \left( \begin{array}{c} A_\mu^{15} - iA_\mu^{16} \\ A_\mu^{15} + iA_\mu^{16} \end{array} \right).
\]
\[ (2.4) \]

In identifying the spectrum of particles and their wave functions, the conformal coordinate \( z = \exp^{\sigma(y)} \) for the fifth dimension is useful, with which the metric becomes

\[
ds^2 = \frac{1}{z^2} \left\{ \eta_{\mu\nu} dx^\mu dx^\nu + \frac{dz^2}{k^2} \right\}, \quad z = \exp^{ky}.
\]
\[ (2.5) \]

\*Matrix forms of generators are explicitly summarized in Ref. [7]. Some bases to represent SO(5) are given in Appendix [A].
The fundamental region $0 \leq y \leq L$ is mapped to $1 \leq z \leq z_L = e^{kL}$. Here $\partial_y = k z \partial_z$ and $\lambda_y = k z \lambda_z$. The SU(5) gauge fields are split into classical and quantum parts $A_M = A_M^q + A_M^q$, where $A_M^q = 0$ and $A_M^q = k z A_M^c$. The quadratic action for the SU(5) gauge fields is

$$S_{\text{bulk}}^\text{gauge} = \int d^2x \frac{dz}{k z} \left[ \text{tr} \left\{ \eta^{\mu
u} A_\mu^q (\square + k^2 P_4) A_\nu^q + k^2 A_\mu^q (\square + k^2 P_4) A_\mu^q \right\} \right], \quad (2.6)$$

for a gauge $\xi = 1$ analogous to the 't Hooft-Feynman gauge in four dimensions. The differential operators are expressed as $\square = \eta^{\mu\nu} \partial_\mu \partial_\nu$, $P_4 = z D_z^{-1} (1/z) D_z^+$, $P_z = D_z z D_z^-(1/z)$, where $D_M A_N^q = \partial_M A_N^q - ig_A [A_M^q, A_N^q]$. The $z$-dependence of the Higgs mode are given by $\sqrt{2/(k(z_L^2 - 1))} z$.

The Wilson line phase $\theta_H$ is given by $\exp \{ i\theta_H (2T_{19}) \} = \exp \{ i g_A \int_z^{z_L} dz \langle A_z \rangle \}$ so that $\theta_H = \frac{1}{2} g_A \theta \sqrt{(z_L^2 - 1)/(2k)}$. With a gauge transformation, a new basis in which the background field vanishes, $\tilde{A}_z^c = 0$ can be taken $[8, 9]$. In the twisted basis, gauge fields are defined as $\tilde{A}_M = \Omega A_M^q \Omega^{-1}$. Here $\Omega(z) = \exp \{ i g_A \int_z^{z_L} dz \langle A_z \rangle \} = \exp \{ i \varphi(z) (2T_{19}) \}$ and $\varphi(z) = \theta_H (z_L^2 - z^2)/(z_L^2 - 1)$. The twist matrix $\Omega$ is written in a matrix form as

$$\Omega(z) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \frac{c-1}{2} & \frac{i z}{\sqrt{2}} & 0 & \frac{c-1}{2} \\
0 & \frac{i z}{\sqrt{2}} & c & 0 & \frac{i z}{\sqrt{2}} \\
0 & 0 & 0 & 1 & 0 \\
0 & \frac{c-1}{2} & \frac{i z}{\sqrt{2}} & 0 & \frac{c-1}{2} 
\end{pmatrix}, \quad \text{with} \quad \begin{cases}
s \equiv \sin(\theta(z)), \\
c \equiv \cos(\theta(z)).
\end{cases} \quad (2.7)$$

For $\theta_H = 0$, $\Omega(z) = 1_5$. The twist matrix is unitary, $\Omega^{-1} = \Omega^\dagger$. The SU(5) gauge bosons are written in the basis in terms of $T_i$ and in the basis with $T_7^+ = \sum_{i=1}^{24} A_M^q T_i$ and $A_M^q = \sum_{i=1}^{24} A_M^q T_i + A_M^{q9} T_9 + A_M^{q10} T_{10} + A_M^{q18} T_{18} + A_M^{q19} T_{19}$, respectively. The twisted fields are given in the same way by $\tilde{A}_M = \sum_{i=1}^{24} \tilde{A}_M^q T_i = \sum_{i=1}^{24} \tilde{A}_M^q T_i + \tilde{A}_M^{q9} T_9 + \tilde{A}_M^{q10} T_{10} + \tilde{A}_M^{q18} T_{18} + \tilde{A}_M^{q19} T_{19}$. The component fields are related to each other under the basis transformations as $A_M^{q1} \lambda_1 + A_M^{q2} \lambda_2 + \cdots = \Omega^{-1} (\tilde{A}_M^q \lambda_1 + \tilde{A}_M^q \lambda_2 + \cdots) \Omega$ and $A_M^{q7} \lambda_7 + A_M^{q8} \lambda_8 + \cdots = \Omega^{-1} (\tilde{A}_M^q \lambda_7 + \tilde{A}_M^q \lambda_8 + \cdots) \Omega$. The mixing of generators with the twist matrix $\Omega$ is summarized in Appendix A. Basis transformations for gauge fields can be read from the mixing of generators with $\Omega$. The relations between component fields for $\tilde{A}_M^q \leftrightarrow A_M^q$, $\tilde{A}_M^7 \leftrightarrow A_M^7$, $\tilde{A}_M^8 \leftrightarrow A_M^8$, $\tilde{A}_M^9 \leftrightarrow A_M^9$, $\tilde{A}_M^i \leftrightarrow A_M^i$, $\tilde{A}_M^7 \leftrightarrow A_M^7$, $\tilde{A}_M^8 \leftrightarrow A_M^8$, $\tilde{A}_M^9 \leftrightarrow A_M^9$ are given in Appendix B.

The twisted fields are decomposed into four-dimensional and extra-dimensional parts with mode functions as

$$\tilde{A}_{\mu}^a(x,z) = \sum_n h_{A,a}^n(z) A_{\mu}^n(x). \quad (2.8)$$

Here $n$ includes classification for $W$, $Z$ and photon. From Eq. (2.6), the bulk equations for twisted fields force the mode functions of the gauge bosons to $h_{A,a}^n(z) \propto z J_1(\lambda_n z)$, $z Y_1(\lambda_n z)$. The mass eigenvalues are given by $k \lambda_n$ where the subscript $n$ will be omitted if no confusion arises. At $y = L$, the twist matrix is $\Omega(z_L) = 1_5$ with which $\tilde{A}_M = A_M^q$. The basic function for Neumann condition at $y = 0$ is $C(z; \lambda) = \frac{5}{4} \lambda z z_L F_{1,0}(z \lambda, \lambda z_L)$. The basic function for Dirichlet condition at $y = 0$ is $S(z; \lambda) = -\frac{5}{4} \lambda z z_L F_{1,0}(z \lambda, \lambda z_L)$. The functions $C(z; \lambda)$ and $S(z; \lambda)$ satisfy the property $(S'C - C'S)(z; \lambda) = \lambda z$. The other constants of the mode functions are determined by the boundary conditions at $z = 1$ and by the normalization.
The mass eigenvalue equations for gauge bosons are obtained from the boundary conditions at \( z = 1 \). For the boundary conditions to yield symmetry breaking SU(5)→SO(5) and SU(5)→[SU(2)×U(1)]_L×[SU(2)×U(1)]_R, the common group SU(2)×U(1) is assigned as Neumann condition for both boundaries. The corresponding SU(2)×U(1) gauge bosons would be massless for vanishing Wilson line phase. When the Wilson line phase \( \theta_H \) is nonzero, a part of these gauge bosons acquire their masses. The Wilson line phase is a dynamical phase and it should be fixed dynamically. On the other hand, the value \( \theta_H \) depends on field contents. Our interest is in possible values for the mass ratio of \( W \) and \( Z \) bosons in the present model that has less parameters to adjust. As we will show explicitly, the possibilities of allowed values of mass ratio are narrowed from the structure with respect to \( \theta_H \) without knowing details of the dynamical fixing of \( \theta_H \). We will not treat how the value of \( \theta_H \) is fixed further. We examine possible values of the ratio of the masses by taking into account the \( \theta_H \)-dependence of mass eigenvalue equations and the effective potential.

### 3 Gauge bosons for Planck-brane rank reduction

In this section, we analyze the mass ratio of gauge bosons by placing the rank-reducing boundary for the Planck brane and the left-right symmetric boundary for the TeV brane. The boundary conditions are given as Neumann condition for \( A_{\mu}^{q_1}, \ldots, A_{\mu}^{q_{24}} \) at \( y = 0 \) and Neumann condition for \( A_{\mu}^{q_1}, A_{\mu}^{q_2}, A_{\mu}^{q_3}, A_{\mu}^{q_5}, A_{\mu}^{q_{15}}, A_{\mu}^{q_{22}}, A_{\mu}^{q_{23}}, A_{\mu}^{q_{24}} \) for \( y = L \).

For the boundary conditions at \( y = L \) where \( \Omega(z_L) = 1_5 \) and \( \hat{A}_M = A_M^q \), the mode functions have the forms

\[
\begin{align*}
\hat{h}^i_{A,n}(z) &= C^i_{A,n}C(z; \lambda_n), \\
\hat{h}^\dagger_{A,n}(z) &= C^i_{A,n}S(z; \lambda_n),
\end{align*}
\]

where \( i = 1, 2, 3, 8, 15, 22, 23, 24 \) and \( \hat{i} = 4, \ldots, 7, 9, \ldots, 14, 16, \ldots, 21 \). The fundamental functions \( C(z; \lambda_n) \) and \( S(z; \lambda_n) \) are defined below Eq. (2.8) and coefficients are denoted as \( C^i_{A,n} \) and \( C^\dagger_{A,n} \). In the basis with \( T_7 \), the gauge bosons are written as

\[
\begin{pmatrix}
A_{\mu}^{q_1} \\
A_{\mu}^{q_2} \\
A_{\mu}^{q_3} \\
A_{\mu}^{q_5} \\
A_{\mu}^{q_{15}} \\
A_{\mu}^{q_{22}} \\
A_{\mu}^{q_{23}} \\
A_{\mu}^{q_{24}}
\end{pmatrix} = \begin{pmatrix}
\frac{1+c}{2} & \frac{s}{\sqrt{2}} & \frac{1-c}{2} & \frac{s}{\sqrt{2}} & \frac{c-1}{2} & \frac{-1-c}{2} \\
\frac{-s}{\sqrt{2}} & c & \frac{-s}{\sqrt{2}} & c & \frac{-s}{\sqrt{2}} & \frac{-c}{\sqrt{2}} \\
\frac{1-c}{2} & \frac{-s}{\sqrt{2}} & \frac{1+c}{2} & \frac{s}{\sqrt{2}} & \frac{-c}{\sqrt{2}} & \frac{-s}{\sqrt{2}} \\
\frac{-s}{\sqrt{2}} & \frac{s}{\sqrt{2}} & \frac{c}{2} & \frac{s}{\sqrt{2}} & \frac{-c-1}{2} & \frac{c+1}{2} \\
\frac{1-c}{2} & \frac{s}{\sqrt{2}} & \frac{s}{\sqrt{2}} & \frac{c}{2} & \frac{-c}{\sqrt{2}} & \frac{s}{\sqrt{2}} \\
\frac{-s}{\sqrt{2}} & \frac{s}{\sqrt{2}} & \frac{c}{2} & \frac{s}{\sqrt{2}} & \frac{-c}{\sqrt{2}} & \frac{s}{\sqrt{2}} \\
\frac{-s}{\sqrt{2}} & \frac{s}{\sqrt{2}} & \frac{c}{2} & \frac{s}{\sqrt{2}} & \frac{-c}{\sqrt{2}} & \frac{s}{\sqrt{2}} \\
\frac{-s}{\sqrt{2}} & \frac{s}{\sqrt{2}} & \frac{c}{2} & \frac{s}{\sqrt{2}} & \frac{-c}{\sqrt{2}} & \frac{s}{\sqrt{2}}
\end{pmatrix}
\begin{pmatrix}
\hat{A}_{\mu}^{(n)}(x) \\
\sqrt{2}
\end{pmatrix},
\]

for the components \( (1, 4, 7, 11, 14, 17) \). Here the components \( (1, 4, 7, 11, 14, 17) \) for \( A_M^{q_5} \) are linked to the components \( (1, 5, 11, 14, 16, 22) \) for \( \hat{A}_M \). The mode expansion for all the other components is summarized in Appendix [C]. The components \( (1, 5, 11, 14, 16, 22) \) for \( \hat{A}_M \) are a real part of the components for charged gauge bosons. The boundary conditions at \( y = 0 \) are

\[
0 = \begin{pmatrix}
\frac{1+c}{2} & \frac{s}{\sqrt{2}} & \frac{1-c}{2} & \frac{s}{\sqrt{2}} & \frac{c-1}{2} & \frac{-1-c}{2} \\
\frac{-s}{\sqrt{2}} & c & \frac{-s}{\sqrt{2}} & c & \frac{-s}{\sqrt{2}} & \frac{-c}{\sqrt{2}} \\
\frac{1-c}{2} & \frac{-s}{\sqrt{2}} & \frac{1+c}{2} & \frac{s}{\sqrt{2}} & \frac{-c}{\sqrt{2}} & \frac{-s}{\sqrt{2}} \\
\frac{-s}{\sqrt{2}} & \frac{s}{\sqrt{2}} & \frac{c}{2} & \frac{s}{\sqrt{2}} & \frac{-c-1}{2} & \frac{c+1}{2} \\
\frac{1-c}{2} & \frac{s}{\sqrt{2}} & \frac{s}{\sqrt{2}} & \frac{c}{2} & \frac{-c}{\sqrt{2}} & \frac{s}{\sqrt{2}} \\
\frac{-s}{\sqrt{2}} & \frac{s}{\sqrt{2}} & \frac{c}{2} & \frac{s}{\sqrt{2}} & \frac{-c}{\sqrt{2}} & \frac{s}{\sqrt{2}} \\
\frac{-s}{\sqrt{2}} & \frac{s}{\sqrt{2}} & \frac{c}{2} & \frac{s}{\sqrt{2}} & \frac{-c}{\sqrt{2}} & \frac{s}{\sqrt{2}} \\
\frac{-s}{\sqrt{2}} & \frac{s}{\sqrt{2}} & \frac{c}{2} & \frac{s}{\sqrt{2}} & \frac{-c}{\sqrt{2}} & \frac{s}{\sqrt{2}}
\end{pmatrix}
\begin{pmatrix}
C^1_{A,n} \\
C^5_{A,n} \\
C^1_{A,n} \\
C^1_{A,n} \\
C^1_{A,n} \\
C^1_{A,n} \\
C^1_{A,n} \\
C^1_{A,n}
\end{pmatrix},
\]
Here some abbreviations have been employed for $C = C(z = 1; \lambda_n)$, $S = S(z = 1; \lambda_n)$, $C' = (dC/dz)(z = 1; \lambda_n)$ and $S' = (dS/dz)(z = 1; \lambda_n)$. At $z = 1$, $s = \sin \theta_H$ and $c = \cos \theta_H$. The condition that the determinant of the matrix in Eq. (3.2) vanishes means

$$SS' [2C'S + (1 + c^2)(S'C - C'S)] [2C'S + s^2(S'C - C'S)] = 0. \tag{3.3}$$

With the relation $(S'C - C'S)|_{z=1} = \lambda$, this equation is

$$SS' [2C'S + (1 + c^2)\lambda] [2C'S + s^2\lambda] = 0. \tag{3.4}$$

If the Wilson line phase were zero, a zero mode exists for the boundary condition $C'S = 0$. This mode corresponds to one real component of $W$ boson. For a nonzero Wilson line phase, $W$ boson acquires a mass determined by $2C'S + s^2\lambda = 0$. The product of the basic functions is approximated as $2C'S \approx -\lambda^3 kLe^{2kL}$ for $\lambda z_L \ll 1$ so that $W$ boson has the mass $m_W = k\lambda W \approx \sqrt{k/Le^{-kL}}|s|$. The mass depends on $\theta_H$ via a sine function. From Eq. (3.4), the components $(1, 5, 11, 14, 16, 22)$ for $\tilde{A}_M$ include the mode with the boundary condition $2CS' + (1 + c^2)\lambda = 0$. This mode has the mass $m_{W'} = k\lambda W' \approx \sqrt{k/Le^{-kL}}\sqrt{1+c^2}$. If $|s|$ is order $O(0.1 \sim 1)$, $m_{W'}$ can be of the same order as $m_W$. Then $m_{W'}$ needs to be made larger. If $|s|$ is so small that $m_W \ll m_{W'}$ at low energy only $W$ boson survives. For a very small $\theta_H$, $W$ boson without extra fields is simply obtained. The mode for $SS' = 0$ corresponds to massive Kaluza-Klein (KK) mode.

For the components (2, 3, 8, 12, 13, 18) for $A_M^4$, the boundary conditions at $y = 0$ are

$$0 = \begin{pmatrix}
\frac{1+c}{2}C' & -\frac{s}{\sqrt{2}}s' & -\frac{1-c}{2}C' & -\frac{s}{\sqrt{2}}s' & \frac{c-1}{2}C' & -\frac{1+c}{2}C' \\
\frac{s}{\sqrt{2}}s' & \frac{1-c}{2}C' & -\frac{s}{\sqrt{2}}s' & -\frac{1+c}{2}C' & \frac{s}{\sqrt{2}}s' & -\frac{1+c}{2}C' \\
\frac{1+c}{2}C' & -\frac{s}{\sqrt{2}}s' & -\frac{1-c}{2}C' & -\frac{s}{\sqrt{2}}s' & \frac{c-1}{2}C' & -\frac{1+c}{2}C' \\
\frac{s}{\sqrt{2}}s' & \frac{1-c}{2}C' & -\frac{s}{\sqrt{2}}s' & -\frac{1+c}{2}C' & \frac{s}{\sqrt{2}}s' & -\frac{1+c}{2}C' \\
\frac{s}{\sqrt{2}}s' & \frac{1-c}{2}C' & -\frac{s}{\sqrt{2}}s' & -\frac{1+c}{2}C' & \frac{s}{\sqrt{2}}s' & -\frac{1+c}{2}C' \\
\frac{1+c}{2}C' & -\frac{s}{\sqrt{2}}s' & -\frac{1-c}{2}C' & -\frac{s}{\sqrt{2}}s' & \frac{c-1}{2}C' & -\frac{1+c}{2}C'
\end{pmatrix} \begin{pmatrix}
C_{A,n}^2 \\
C_{A,n}^4 \\
C_{A,n}^{12} \\
C_{A,n}^{17} \\
C_{A,n}^{23} \\
C_{A,n}^{24}
\end{pmatrix}. \tag{3.5}
$$

Here the components (2, 3, 8, 12, 13, 18) for $A_M^4$ are linked to the components (2, 4, 12, 13, 17, 23) for $\tilde{A}_M$. These components give the other real component of $W$ boson. The condition that the determinant vanishes is the same as Eq. (3.3). The charged $W$ boson is composed of the components $(1, 5, 11, 14, 16, 22)$ and $(2, 4, 12, 13, 17, 23)$ for $A_M$. The charged $W'$ boson is also composed of the components $(1, 5, 11, 14, 16, 22)$ and $(2, 4, 12, 13, 17, 23)$ for $A_M$. For the components (5, 19, 15), the boundary conditions at $y = 0$ are

$$0 = \begin{pmatrix}
cS' & \sqrt{2}sS' & -cS' \\
-sS & \sqrt{2}sS' & sS \\
S & 0 & S
\end{pmatrix} \begin{pmatrix}
C_{A,n}^6 \\
C_{A,n}^{19} \\
C_{A,n}^{20}
\end{pmatrix}. \tag{3.6}
$$

Here the components (5, 19, 15) for $A_M^4$ are linked to the components (6,19,20) for $\tilde{A}_M$. The condition that the determinant vanishes means $S'S^2 = 0$. This corresponds to massive KK mode. For the components (9,10) for $A_M^4$ and $\tilde{A}_M$, the boundary conditions at $y = 0$ are $S = 0$ which are massive KK mode.

For the components (6, 9, 10, 16, 19, 20, 18), the boundary conditions at $y = 0$ are

$$0 = E(C_{A,n}^3, C_{A,n}^7, C_{A,n}^8, C_{A,n}^{15}, C_{A,n}^{18}, C_{A,n}^{21}, C_{A,n}^{24})^T. \tag{3.7}$$
Here the components \((\bar{6}, \bar{9}, \bar{10}, \bar{16}, \bar{19}, \bar{20}, 18)\) for \(A_M^q\) are linked to the components \((3, 7, 8, 15, 18, 21, 24)\) for \(\tilde{A}_M\) and the definition of \(E\) is given in Appendix D. The components \((3, 7, 8, 15, 18, 21, 24)\) for \(\tilde{A}_M\) are the mode of neutral gauge bosons. The condition that the determinant of the matrix \(E\) vanishes gives a complicated equation with \(\sin \theta_H\) and \(\cos \theta_H\). We consider typical cases for \(\theta_H\): a small \(|s|\) and \(s = 1\). For a small \(|s|\), it is simple to obtain \(W\) boson without extra light fields. The other case \(s = 1\) can be a typical value for the dynamical phase as described in the following, although extra fields need to be made heavier. In the present model, scalar fields appear from only the extra-dimensional component of gauge bosons. For zero \(\theta_H\), the potential \(V(\theta_H = 0)\) vanishes. Because the \(\theta_H\) is a phase, physics is periodic for \(\theta_H \rightarrow \theta_H + 2\pi\) so that \(V(0) = V(2\pi) = 0\). A typical phase transition with a stationary point for a nonzero \(s\) could have the potential minimum at \(\theta_H = \pi/2\). An example of such a symmetry breaking was given in a realistic model \([10]\). Then \(s = 1\).

For a small \(|s|\) and \(c = 1 - s^2/2 + \mathcal{O}(s^4)\), the condition \(\det E = 0\) is obtained as

\[
0 = 2gCC' S \left[ C' S + s^2 \frac{f}{2g} \lambda \right] \left[ C' S + \lambda + s^2 \left( \frac{f - 2b}{2g} \right) \lambda \right] + \mathcal{O}(s^4),
\]

(3.8)

where \(g \equiv 6149\sqrt{2} + 3177\sqrt{5}, \ f \equiv 773\sqrt{2} + 539\sqrt{5}\) and \(b \equiv 406\sqrt{2} + 2099\sqrt{5}\). The mode for \(C' S = 0\) is massless for the lowest level and corresponds to photon. The mode of \(Z\) boson is obtained for \(C' S + s^2 \frac{L}{2g} \lambda = 0\). The \(Z\) boson mass is given by \(m_Z = k\lambda_Z \approx \sqrt{k/Le^{-kL}}|s|\sqrt{f/g}\). Together with the mass of \(W\) boson for the components \((1, 2, 4, 5, 11, \ldots, 14, 16, 17, 22, 23)\) for \(\tilde{A}_M\), this means the mass ratio \(m_W^2/m_Z^2 = g/f\). Because \(g/f > 1\), for these boundary conditions and \(|s|\), \(W\) boson is amount to be heavier than \(Z\) boson.

For \(s = 1\), the condition \(\det E = 0\) means

\[
0 = CC' S(C' S + \lambda)(13C' S + 18\lambda).
\]

(3.9)

The mode for \(C' S = 0\) corresponds to photon. The mode of \(Z\) boson appears from \(C' S + \lambda = 0\). The mass is given by \(m_Z = k\lambda \approx \sqrt{2k/Le^{-kL}}\). The mass of the mode for \(13C' S + 18\lambda = 0\) is \(m_{Z'} = \sqrt{36k/(13L)e^{-kL}}\). It needs to be made heavier. Together with the mass of \(W\) boson, the mass ratio is given by \(m_W^2/m_Z^2 = 1/2\) for \(s = 1\). In this case, \(Z\) boson becomes heavier than \(W\) boson although there still remains a deviation of the mass ratio from the experimental value. As a characteristic aspect, it has been found to depend on \(\theta_H\) which \(W\) boson is heavier of \(Z\) boson and \(W\) boson in the present context. In next section, it will be shown that the mass ratio can change for the same value of \(\theta_H\) and a simple exchange of the assignment of boundary conditions.

## 4 Gauge bosons for TeV-brane rank reduction

In this section, we analyze the masses of gauge bosons interchanging the places of the rank-reducing boundary and the left-right symmetric boundary in Section 3. The boundary conditions are given as Neumann condition for \(A^1_\mu, A^2_\mu, A^3_\mu, A^{08}_\mu, A^{15}_\mu, A^{22}_\mu, A^{23}_\mu, A^{24}_\mu\) at \(y = 0\) and Neumann condition for \(A^T_\mu, \ldots, A^{10}_\mu\) for \(y = L\).

For the boundary conditions at \(y = L\) where \(\Omega(z_L) = 1_5\) and \(\tilde{A}_M = A^q_M\), the mode functions have the forms

\[
h_{A,n}^i(z) = C_{A,n}^i C(z; \lambda_n), \quad h_{A,n}^i(z) = C_{A,n}^i S(z; \lambda_n),
\]

(4.1)
where \( i = \overline{1}, \ldots, \overline{10} \) and \( \hat{i} = \overline{1}, \ldots, \overline{20} \). In the basis in terms of \( T_i \), the gauge bosons are written as

\[
\begin{pmatrix}
A_{\mu}^{q1} \\
A_{\mu}^{q5} \\
A_{\mu}^{q11} \\
A_{\mu}^{q14} \\
A_{\mu}^{q16} \\
A_{\mu}^{q22}
\end{pmatrix} = \begin{pmatrix}
\frac{c+1}{2} & s & -\frac{c-1}{2} & s & \frac{c+1}{2} & s \\
-\frac{s}{2} & c & \frac{s}{2} & -c & \frac{s}{2} & c \\
-\frac{c+1}{2} & s & -\frac{c-1}{2} & s & \frac{c+1}{2} & s \\
-\frac{s}{2} & -c & \frac{s}{2} & c & \frac{s}{2} & c \\
-\frac{c+1}{2} & s & -\frac{c-1}{2} & s & \frac{c+1}{2} & s \\
-\frac{s}{2} & c & -\frac{s}{2} & c & \frac{s}{2} & c
\end{pmatrix}
\begin{pmatrix}
C_{A,n}^{n} C(z; \lambda_n) \\
C_{A,n}^{q1} C(z; \lambda_n) \\
C_{A,n}^{q1} C(z; \lambda_n) \\
C_{A,n}^{q1} S(z; \lambda_n) \\
C_{A,n}^{q1} S(z; \lambda_n) \\
C_{A,n}^{q1} S(z; \lambda_n)
\end{pmatrix} \frac{A_{\mu}^{(n)}(x)}{\sqrt{2}},
\]

for the components \((1, 5, 11, 14, 16, 22)\) for \( A_{\mu}^{q} \) are linked to the components \((\overline{1}, \overline{4}, \overline{7}, \overline{11}, \overline{14}, \overline{17})\) for \( \hat{A}_{\mu} \). The mode expansion for all the other components are summarized in Appendix C. The components \((\overline{1}, \overline{4}, \overline{7}, \overline{11}, \overline{14}, \overline{17})\) for \( \hat{A}_{\mu} \) are a real part of the components for charged gauge bosons. The boundary conditions at \( y = 0 \) are

\[
0 = \begin{pmatrix}
\frac{c+1}{2} C' & -\frac{s}{\sqrt{2}} C' & -\frac{c-1}{2} C' & \frac{c+1}{2} S' & -\frac{s}{\sqrt{2}} S' & \frac{c+1}{2} S' \\
-\frac{s}{\sqrt{2}} C & c C & \frac{s}{\sqrt{2}} C & -\frac{s}{\sqrt{2}} S & c S & -\frac{s}{\sqrt{2}} S \\
-\frac{c+1}{2} C' & \frac{s}{\sqrt{2}} C' & -\frac{c-1}{2} C' & \frac{c+1}{2} S' & -\frac{s}{\sqrt{2}} S' & \frac{c+1}{2} S' \\
-\frac{s}{\sqrt{2}} C & -c C & \frac{s}{\sqrt{2}} C & \frac{s}{\sqrt{2}} S & c S & \frac{s}{\sqrt{2}} S \\
-\frac{c+1}{2} C' & \frac{s}{\sqrt{2}} C' & -\frac{c-1}{2} C' & \frac{c+1}{2} S' & -\frac{s}{\sqrt{2}} S' & \frac{c+1}{2} S' \\
-\frac{s}{\sqrt{2}} C & -c C & \frac{s}{\sqrt{2}} C & \frac{s}{\sqrt{2}} S & c S & \frac{s}{\sqrt{2}} S
\end{pmatrix}
\begin{pmatrix}
C_{A,n}^{n} \\
C_{A,n}^{q1} \\
C_{A,n}^{q1} \\
C_{A,n}^{q1} \\
C_{A,n}^{q1} \\
C_{A,n}^{q1}
\end{pmatrix} \frac{A_{\mu}^{(n)}(x)}{\sqrt{2}},
\]

The condition that the determinant vanishes means

\[
CS \left[ (C'S + S'C)^2 - c^4(C'S - S'C)^2 \right] = 0.
\]

With the relation \((S'C - C'S)|_{z=1} = \lambda\), this equation is written as

\[
CS \left[ 2C'S + (1 + c^2) \lambda \right] \left[ 2C'S + s^2 \lambda \right] = 0.
\]

As in Eq. (3.1), the eigenvalue equation for \( W \) boson is \( m_W = k\lambda W \approx \sqrt{k/Le^{-kL}} |s| \). Similarly to Eq. (3.1), the components \((\overline{1}, \overline{4}, \overline{7}, \overline{11}, \overline{14}, \overline{17})\) for \( \hat{A}_M \) include the mode with the boundary condition \( 2C'S + (1 + c^2) \lambda = 0 \). For the components \((2, 4, 12, 13, 17, 23)\) for \( A_{\mu}^{q} \), the boundary conditions at \( y = 0 \) are

\[
0 = \begin{pmatrix}
\frac{c+1}{2} C' & -\frac{s}{\sqrt{2}} C' & -\frac{c-1}{2} C' & \frac{c+1}{2} S' & -\frac{s}{\sqrt{2}} S' & \frac{c+1}{2} S' \\
-\frac{s}{\sqrt{2}} C & c C & \frac{s}{\sqrt{2}} C & -\frac{s}{\sqrt{2}} S & c S & -\frac{s}{\sqrt{2}} S \\
\frac{c+1}{2} C' & \frac{s}{\sqrt{2}} C' & -\frac{c-1}{2} C' & \frac{c+1}{2} S' & -\frac{s}{\sqrt{2}} S' & \frac{c+1}{2} S' \\
\frac{s}{\sqrt{2}} C & -c C & \frac{s}{\sqrt{2}} C & \frac{s}{\sqrt{2}} S & c S & \frac{s}{\sqrt{2}} S \\
\frac{c+1}{2} C' & \frac{s}{\sqrt{2}} C' & -\frac{c-1}{2} C' & \frac{c+1}{2} S' & -\frac{s}{\sqrt{2}} S' & \frac{c+1}{2} S' \\
\frac{s}{\sqrt{2}} C & -c C & \frac{s}{\sqrt{2}} C & \frac{s}{\sqrt{2}} S & c S & \frac{s}{\sqrt{2}} S
\end{pmatrix}
\begin{pmatrix}
C_{A,n}^{n} \\
C_{A,n}^{q1} \\
C_{A,n}^{q1} \\
C_{A,n}^{q1} \\
C_{A,n}^{q1} \\
C_{A,n}^{q1}
\end{pmatrix} \frac{A_{\mu}^{(n)}(x)}{\sqrt{2}},
\]

Here the components \((2, 4, 12, 13, 17, 23)\) for \( A_{\mu}^{q} \) are linked to the components \((\overline{2}, \overline{3}, \overline{8}, \overline{12}, \overline{13}, \overline{18})\) for \( \hat{A}_M \). The condition that the determinant vanishes is the same as Eq. (4.4). The components \((\overline{2}, \overline{3}, \overline{8}, \overline{12}, \overline{13}, \overline{18})\) for \( \hat{A}_M \) give the other real component of \( W \) boson. For the components \((6, 19, 20)\) for \( A_{\mu}^{q} \), the boundary conditions at \( y = 0 \) are

\[
0 = \begin{pmatrix}
c C & s S & S \\
-\sqrt{2} s C & \sqrt{2} c S & 0 \\
-c C & -s S & S
\end{pmatrix}
\begin{pmatrix}
C_{A,n}^{15} \\
C_{A,n}^{19} \\
C_{A,n}^{A,n}
\end{pmatrix}.
\]
Here the components (6,19,20) for \( A_M^q \) are linked to the components \( \{5, 19, 20\} \) for \( \tilde{A}_M \). The condition that the determinant vanishes means \( CS^2 = 0 \). This corresponds to massive KK mode. For the components (9,10) for \( A_M^q \) and \( \tilde{A}_M \), the boundary conditions at \( y = 0 \) are \( S = 0 \) which are massive KK modes. For the components (3,7,8,15,18,21,24) for \( A_M^q \), the boundary conditions at \( y = 0 \) are

\[
0 = L( C_{A,n}^6, C_{A,n}^9, C_{A,n}^{10}, C_{A,n}^{16}, C_{A,n}^{19}, C_{A,n}^{20}, C_{A,n}^{18})^T. \tag{4.7}
\]

Here the components (3,7,8,15,18,21,24) for \( A_M^q \) are linked to the components \( \{6, 9, 10, 16, 19, 20\} \) for \( \tilde{A}_M \) and the definition of the matrix \( L \) is given in Appendix \( \text{[E]} \). The components \( \{6, 9, 10, 16, 19, 20\} \) for \( \tilde{A}_M \) are the mode of neutral gauge bosons. The condition that the determinant of \( L \) vanishes gives a complicated equation with \( \sin \theta_H \) and \( \cos \theta_H \). Similarly to analysis in Section \( \text{[3]} \) we consider typical cases for \( \theta_H \): a small \( |s| \) and \( s = 1 \). For a small \( |s| \) and \( c = 1 - s^2/2 + \mathcal{O}(s^4) \), the condition \( \det L = 0 \) means

\[
0 = C'SS'(16 - 5s^2)q \left[ C'S + \left( 1 + s^2 \frac{21q-p}{16q} \right) \lambda \right] [C'S + s^2 \lambda] + \mathcal{O}(s^4), \tag{4.8}
\]

where \( q \equiv 383 \sqrt{2} + 156 \sqrt{5} \) and \( p \equiv 24567 \sqrt{2} + 9884 \sqrt{5} \). The mode for \( C'S = 0 \) corresponds to photon. The mode of \( Z \) boson is for \( C'S + s^2 \lambda = 0 \). The mass of \( Z \) boson is \( m_Z = k \lambda_Z \approx \sqrt{2k/Le^{-kL}}|s| \). Together with the mass of \( W \) boson, the mass ratio becomes \( m_W^2/m_Z^2 = 1/2 \). By comparing with the result in Section \( \text{[3]} \) we find that the behavior of the mass ration significantly depends on the assignment of the place of the rank-reducing boundary. For \( s = 1 \), the condition \( \det L = 0 \) means \( S'C'S[C'S + \lambda]^2 = 0 \). The mode of \( C'S = 0 \) corresponds to photon. The mass of \( Z \) boson is \( m_Z = k \lambda_Z \approx \sqrt{2k/Le^{-kL}}|s| \). Then \( m_W^2/m_Z^2 = 1/2 \). For \( s = 1 \), the mass of \( W' \) boson is the same as the mass of \( Z \) mass, \( m_{Z'} = m_Z \). While in the present toy model there remains a deviation of the mass ration from the external value, the mass ration has been found to change for the assignment of boundary conditions and the value of the Wilson line phase that are essential elements in the gauge-Higgs unification.

Before closing the section, we would like to mention how gauge bosons are described with mode functions. As an explicit example, we focus on one real component of \( W \) boson for the component \( \{1, 4, 7, 11, 14, 17\} \) for \( \tilde{A}_M \). The boundary condition is given in Eq. (4.2). With the solution to eigenvectors for Eq. (4.2), the gauge boson is written as

\[
\frac{C_{A,n}^{1}}{1 + c} W_x(x) \left[ (1 + c)C(z; \lambda)T_T + (1 - c)C(z; \lambda)T_T + \sqrt{2s} \frac{C(z = 1; \lambda)}{S(z = 1; \lambda)}S(z; \lambda)T_{11} \right], \tag{4.9}
\]

where \( c = \cos \theta_H \) and \( s = \sin \theta_H \). The constant \( C_{A,n}^{1} \) is determined by the normalization. Details of a derivation are given in Appendix \( \text{[E]} \). For zero \( \theta_H \), \( W \) boson is massless and has only the component for \( T_T \) which is a part of the generators of the corresponding unbroken SU(2)\( \times \)U(1) as described in Eq. (2.3). For nonzero \( \theta_H \), \( W \) boson acquires its mass and is composed of the mixing of the three components \( T_T, T_T \) and \( T_{11} \). The other mode functions can be derived in a similar way.

5 Conclusion

We have studied the mass ratio of \( W \) and \( Z \) bosons in an SU(5) gauge-Higgs unification model with the Randall-Sundrum warped spacetime. The group SU(5) is broken to SO(5)
at one boundary and $[SU(2) \times U(1)]_L \times [SU(2) \times U(1)]_R$ at the other boundary by boundary conditions. The Higgs doublet for $SU(2)_L$ is included as the extra-dimensional component of five-dimensional SU(5) gauge bosons. For nonzero Wilson line phase, the overlapping $SU(2) \times U(1)$ is broken to $U(1)$. Additional scalar fields are not required for the breaking $SU(5) \rightarrow U(1)$. Because the starting group is a single group and there are no additional scalar fields, possible values of the mass ratio for $W$ and $Z$ bosons are restrictive. Instead of examining the potential for the Wilson line phase, we have analyzed the mass ratio for typical values of the Wilson line phase $\theta_H$.

When the rank-reducing boundary where SU(5) is broken to SO(5) is the Planck brane, the mass ratio is $m_W^2/m_Z^2 = (6149\sqrt{2} + 3177\sqrt{5})/(773\sqrt{2} + 539\sqrt{5})$ for $\theta_H \ll 1$ and $m_W^2/m_Z^2 = 1/2$ for $\theta_H = \pi/2$. When the rank-reducing boundary is the TeV brane, the mass ratio is $m_W^2/m_Z^2 = 1/2$ for both $\theta_H \ll 1$ and $\theta_H = \pi/2$. For $\theta_H = \pi/2$, there are gauge bosons whose masses are of the same order as $m_W$ and $m_Z$. These modes need to be made heavier. While the values of the mass ratio in this toy model deviate from the experimental value, they have been found to depend on the assignment of the boundary conditions even with the same starting group and with the same pattern of subgroups for the symmetry breaking. This gives a room to build a realistic model for symmetry breaking without additional scalar fields.

In order to produce the correct mass ratio, we would like to discuss possible prospects to improve the model. While our starting point is a single group, one might think that a direct product group is an alternative candidate. However, that the starting point has direct product groups does not seem to be a favorable choice in the gauge-Higgs unification because the mixing between groups as direct products does not seem to occur via boundary conditions and via the Wilson line phase without additional scalar fields. We have found that for a fixed single group SU(5) the assignment of boundary conditions can change the mass ratio. A way to improve the mass ratio might be to change a balance of the distribution for the pattern of symmetry breaking. As in the present model, a five-dimensional spacetime has two endpoints for a finite extra-dimensional space. Then the rank-reducing boundary and the left-right symmetric boundary are assigned piece by piece for each brane. In six dimensions, there are four end-points with respect to two finite extra-dimensional spaces. If the left-right symmetric boundary is assigned for three branes and the rank-reducing boundary is assigned for the other single brane, the resulting mass ratio could change so as to respect left-right symmetry in a wider region than in the five-dimensional case. The two extra dimensions also involve two dimensionful quantities. The difference of the scales of the two extra dimensions can modify the values of low-energy physical quantities such as quark and charged lepton masses \[11, 12\]. Thus the distribution for symmetry breaking and multiple dimensionful quantities might affect the mass ratio of $W$ and $Z$ bosons. This analysis is left for future work.

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A SU(5) generators and Wilson line mixing

A.1 Base of SU(5) generators

The $\lambda_i$ are represented like the standard Gell-Mann matrices for SU(3). The matrices with barred indices such as $\lambda^\overline{7}$ are defined as

$$
\begin{align*}
(\lambda^\overline{7}) = G \begin{pmatrix} \lambda_1 \\ \lambda_{22} \end{pmatrix}, & \quad (\lambda_T) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} (\lambda_2 \\ \lambda_{23}), \\
(\lambda_3^\overline{3}) = G \begin{pmatrix} \lambda_4 \\ \lambda_{13} \end{pmatrix}, & \quad (\lambda_3^T) = G \begin{pmatrix} \lambda_5 \\ \lambda_{14} \end{pmatrix}, \quad (\lambda_5^\overline{5}) = G \begin{pmatrix} \lambda_6 \\ \lambda_{20} \end{pmatrix}, \\
(\lambda_5^\overline{7}) = G \begin{pmatrix} \lambda_7 \\ \lambda_{21} \end{pmatrix}, & \quad (\lambda_7^T) = G \begin{pmatrix} \lambda_{11} \\ \lambda_{16} \end{pmatrix}, \quad (\lambda_7^\overline{7}) = G \begin{pmatrix} \lambda_8 \\ \lambda_{18} \end{pmatrix}, \\
(\lambda_{10}^\overline{7}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & \frac{\sqrt{15}}{3} \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{15}}{4} & \frac{\sqrt{10}}{4} \\ \frac{\sqrt{6}}{4} & \frac{\sqrt{10}}{4} \end{pmatrix} \begin{pmatrix} \lambda_3 \\ \lambda_8 \\ \lambda_{15} \end{pmatrix}, \quad \lambda_9, \lambda_{10}, \lambda_{18}, \lambda_{19}.
\end{align*}
$$

where $G = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

When transformation matrices are denoted as $G$ collectively, the gauge bosons in a basis of matrices with barred indices are given by

$$
A^\overline{7}_j = \sum_i (G^T)_{j7} A_i, \quad \text{with} \quad A_i \lambda_i = A_i G_{ij} \lambda^\overline{7}_j = (G^T)_{j7} A_i \lambda^\overline{7}. \quad (A.2)
$$

The relation between $A^{q_i}$ and $A^{q^7}$ is explicitly given as follows:

$$
\begin{align*}
\begin{pmatrix} A^{q1} \\ A^{q5} \\ A^{q11} \\ A^{q14} \\ A^{q16} \\ A^{q22} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} A^{qT} \\ A^{q7} \\ A^{q11} \\ A^{q14} \\ A^{q16} \\ A^{q22} \end{pmatrix}, \\
\begin{pmatrix} A^{q2} \\ A^{q4} \\ A^{q12} \\ A^{q13} \\ A^{q17} \\ A^{q23} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} A^{q2} \\ A^{q4} \\ A^{q12} \\ A^{q13} \\ A^{q17} \\ A^{q23} \end{pmatrix}, \\
\begin{pmatrix} A^{q6} \\ A^{q19} \\ A^{q20} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} A^{q5} \\ A^{q19} \\ A^{q20} \end{pmatrix}, \quad A^{q9}, A^{q10}, \quad (A.5)
\end{align*}
$$
\[
\begin{pmatrix}
A_{q^3} \\
A_{q^7} \\
A_{q^8} \\
A_{q^{15}} \\
A_{q^{18}} \\
A_{q^{21}} \\
A_{q^{24}}
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & \frac{\sqrt{3}}{2} & 0 \\
0 & \frac{\sqrt{3}}{2} & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\
0 & \frac{\sqrt{10}}{4} & \frac{\sqrt{10}}{4} & 0 & \frac{\sqrt{10}}{4} & 0 & 0
\end{pmatrix} \begin{pmatrix}
A_{q^3} \\
A_{q^7} \\
A_{q^8} \\
A_{q^{15}} \\
A_{q^{18}} \\
A_{q^{21}} \\
A_{q^{24}}
\end{pmatrix}. \tag{A.6}
\]

### A.2 Mixing of SU(5) generators with \( \Omega \)

#### A.2.1 Mixing of \( \lambda_i \) and \( \Omega \)

The mixing of \( \lambda_i \) with \( \Omega \) is summarized as follows:

\[
\begin{pmatrix}
\Omega^{-1}\lambda_1 \Omega \\
\Omega^{-1}\lambda_5 \Omega \\
\Omega^{-1}\lambda_{16} \Omega
\end{pmatrix} = M_1 \begin{pmatrix}
\lambda_1 \\
\lambda_5 \\
\lambda_{16}
\end{pmatrix}, \quad \begin{pmatrix}
\Omega^{-1}\lambda_2 \Omega \\
\Omega^{-1}\lambda_4 \Omega \\
\Omega^{-1}\lambda_{17} \Omega
\end{pmatrix} = M_1^T \begin{pmatrix}
\lambda_2 \\
\lambda_4 \\
\lambda_{17}
\end{pmatrix}, \tag{A.7}
\]

\[
\begin{pmatrix}
\Omega^{-1}\lambda_6 \Omega \\
\Omega^{-1}\lambda_{19} \Omega \\
\Omega^{-1}\lambda_{20} \Omega
\end{pmatrix} = M_2 \begin{pmatrix}
\lambda_6 \\
\lambda_{19} \\
\lambda_{20}
\end{pmatrix}, \quad \Omega^{-1}\lambda_9 \Omega = \lambda_9, \quad \Omega^{-1}\lambda_{10} \Omega = \lambda_{10}, \tag{A.8}
\]

\[
\begin{pmatrix}
\Omega^{-1}\lambda_{11} \Omega \\
\Omega^{-1}\lambda_{14} \Omega \\
\Omega^{-1}\lambda_{22} \Omega
\end{pmatrix} = M_1^T \begin{pmatrix}
\lambda_{11} \\
\lambda_{14} \\
\lambda_{22}
\end{pmatrix}, \quad \begin{pmatrix}
\Omega^{-1}\lambda_{12} \Omega \\
\Omega^{-1}\lambda_{13} \Omega \\
\Omega^{-1}\lambda_{23} \Omega
\end{pmatrix} = M_2 \begin{pmatrix}
\lambda_{12} \\
\lambda_{13} \\
\lambda_{23}
\end{pmatrix}, \tag{A.9}
\]

\[
\begin{pmatrix}
\Omega^{-1}\lambda_3 \Omega \\
\Omega^{-1}\lambda_7 \Omega \\
\Omega^{-1}\lambda_8 \Omega \\
\Omega^{-1}\lambda_{15} \Omega \\
\Omega^{-1}\lambda_{18} \Omega \\
\Omega^{-1}\lambda_{21} \Omega \\
\Omega^{-1}\lambda_{24} \Omega
\end{pmatrix} = V \begin{pmatrix}
\lambda_3 \\
\lambda_7 \\
\lambda_8 \\
\lambda_{15} \\
\lambda_{18} \\
\lambda_{21} \\
\lambda_{24}
\end{pmatrix}, \quad V = \begin{pmatrix}
B_1 & F_1^T(-s) & B_2(s) \\
F_1(s) & 1 + \frac{(c+3)(c-1)}{48} & F_2^T(-s) \\
B_2^T(-s) & F_2^T(-s) & B_3
\end{pmatrix}. \tag{A.10}
\]

Here \( c = \cos(\theta(z)), s = \sin(\theta(z)) \) and

\[
M_1 = \begin{pmatrix}
\frac{c+1}{2} & \frac{s}{\sqrt{2}} & \frac{c-1}{2} \\
\frac{s}{\sqrt{2}} & c & \frac{s}{\sqrt{2}} \\
\frac{c-1}{2} & \frac{s}{\sqrt{2}} & \frac{c+1}{2}
\end{pmatrix}, \quad M_2 = \begin{pmatrix}
\frac{c+1}{2} & -\frac{s}{\sqrt{2}} & -\frac{c-1}{2} \\
\frac{s}{\sqrt{2}} & c & \frac{s}{\sqrt{2}} \\
-\frac{c-1}{2} & \frac{s}{\sqrt{2}} & \frac{c+1}{2}
\end{pmatrix}, \tag{A.11}
\]

\[
B_1 = \begin{pmatrix}
\frac{1}{2} \left\{ c + 1 + \left(\frac{c-1}{2}\right)^2 \right\} & \frac{(c+1)s}{2\sqrt{2}} & -\frac{(5c+7)(c-1)}{8\sqrt{3}} \\
\frac{(c+1)s}{2\sqrt{2}} & \frac{(c+1)(2c-1)}{8\sqrt{3}} & \frac{2(c+1)s}{2\sqrt{6}} \\
\frac{(5c+7)(c-1)}{8\sqrt{3}} & \frac{2(c+1)s}{2\sqrt{6}} & \frac{25c^2+2c-3}{24}
\end{pmatrix}, \tag{A.12}
\]

\[
B_2(s) = \begin{pmatrix}
\frac{c^2}{4} & -\frac{(c-1)s}{2\sqrt{2}} & \frac{\sqrt{10}(c-1)^2}{16} \\
\frac{c^2}{2\sqrt{2}} & \frac{(c-1)(2c+1)}{4\sqrt{3}} & \frac{\sqrt{5}(c-1)s}{4} \\
\frac{\sqrt{10}(c-1)^2}{16} & \frac{\sqrt{5}(c-1)s}{4} & \frac{5c^2+10c+1}{16}
\end{pmatrix}, \tag{A.13}
\]

\[
B_3 = \begin{pmatrix}
-\frac{c^2}{2} & \frac{c^2}{\sqrt{2}} & \frac{\sqrt{10}c^2}{8} \\
\frac{(2c-1)(c+1)}{4\sqrt{3}} & \frac{\sqrt{5}(c+1)s}{4} & \frac{5c^2+10c+1}{16}
\end{pmatrix}, \tag{A.14}
\]
\[ F_1(s) = \left( \frac{(c-1)^2}{8\sqrt{6}}, \frac{(c-1)s}{4\sqrt{3}}, \frac{(c-1)(5c+3)}{24\sqrt{2}} \right), \quad (A.15) \]

\[ F_2(s) = \left( \frac{s^2}{4\sqrt{6}} - \frac{(c+1)s}{4\sqrt{3}}, \frac{\sqrt{15}(c+3)(c-1)}{48} \right). \quad (A.16) \]

For \( c = 1 \) and \( s = 0 \), \( V = 1_7 \).

### A.2.2 Mixing of \( \lambda_i \) and \( \Omega \)

The mixing of \( \lambda_i \) with \( \Omega \) is summarized as follows:

\[
\begin{pmatrix}
\Omega^{-1} \lambda_7 \Omega \\
\Omega^{-1} \lambda_9 \Omega \\
\Omega^{-1} \lambda_{10} \Omega \\
\Omega^{-1} \lambda_{19} \Omega \\
\Omega^{-1} \lambda_{18} \Omega
\end{pmatrix} = \begin{pmatrix}
c - s \\
s & c
\end{pmatrix}
\begin{pmatrix}
\lambda_7 \\
\lambda_9 \\
\lambda_{10} \\
\lambda_{19} \\
\lambda_{18}
\end{pmatrix}, \quad (A.19)
\]

Here \( c = \cos(\theta(z)) \), \( s = \sin(\theta(z)) \) and

\[
U_1 = \begin{pmatrix}
\frac{2c^2-1}{(-\sqrt{2}+\sqrt{5})cs} & \frac{\sqrt{2}s(2+3c)}{26+3\sqrt{10}} & \frac{5\sqrt{2}s(-8+(14+3\sqrt{10})c)}{12(26+3\sqrt{10})} & \frac{0}{16} & \frac{\sqrt{2}}{16} \\
\frac{1}{16} & \frac{\sqrt{2}s(2+3c)}{8+4\sqrt{10}+(18-\sqrt{5})c} & \frac{5\sqrt{2}s(-8+(14+3\sqrt{10})c)}{(98+169\sqrt{10})(c-1)} & \frac{0}{16} & \frac{-s}{\sqrt{2}} \\
0 & \frac{2\sqrt{2}s(cc-9)}{26+3\sqrt{10}} & \frac{2\sqrt{2}s(10+4(2+\sqrt{5})c)}{18-\sqrt{10}+4(2+\sqrt{10})c} & \frac{0}{16} & c \\
0 & \frac{-\sqrt{2}}{16} & \frac{-s}{\sqrt{2}} & \frac{-s}{\sqrt{2}} & \frac{-s}{\sqrt{2}} \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad (A.21)
\]
\[ U_2 = \begin{pmatrix}
\frac{s(4-(20+3\sqrt{10})c)}{\sqrt{2}(26+3\sqrt{10})} & \frac{s(4\sqrt{10}+75c+71\sqrt{10}c)}{6(26+3\sqrt{10})} & \frac{c_s}{\sqrt{2}+\sqrt{5}}s^2 \\
\frac{5s(10-(26+3\sqrt{10})c)}{\sqrt{2}(26+3\sqrt{10})} & \frac{5s(10-(26+3\sqrt{10})c)}{\sqrt{2}(26+3\sqrt{10})} & \frac{s}{\sqrt{2}+\sqrt{5}}s^2 \\
\frac{c^2+3}{4} & -\frac{\sqrt{2}c}{4} & \frac{s^2}{4}
\end{pmatrix}
\]

For \( c = 1 \) and \( s = 0 \), \( U = 1_7 \).

## B Base transformation between fields

In this appendix, the base transformations between fluctuation fields and twisted fields are given. Fluctuation fields and twisted fields are related to each other through \( A_M^q = \Omega^{-1} A_M^i \). All the relations for component fields are derived from this equation. For \( A_M^q \) and \( A_M^i \), which correspond to generators \( T_i \), the component fields are included as \( A_M^q \lambda_i = \Omega^{-1} A_M^i \lambda_i \Omega = \tilde{A}_M^i \sum_j c_{ij} \lambda_j \) where \( c_{ij} \) are the coefficients for the mixing of \( \lambda_i \) with \( \Omega \). With the same \( c_{ij} \) being expressed as elements of a matrix, the relations for components are given by \( A_M^q = \sum_i A_M^i c_{ij} = \sum_i (c^T)_{ji} \tilde{A}_M^i \). Explicit equations are summarized in the following.

### B.1 \( A_M^q \leftrightarrow \tilde{A}_M^i \)

\[
\begin{pmatrix}
A_M^{q1} \\
A_M^{q5} \\
A_M^{q11} \\
A_M^{q14} \\
A_M^{q16} \\
A_M^{q22}
\end{pmatrix} = \begin{pmatrix}
\frac{c+1}{2} & \frac{s}{\sqrt{2}} & 0 & 0 & \frac{c-1}{2} & 0 \\
-\frac{\sqrt{2}}{2} & c & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 \\
0 & 0 & \frac{c+1}{2} & 0 & -\frac{s}{\sqrt{2}} & 0 \\
0 & 0 & \frac{s}{\sqrt{2}} & c & 0 & \frac{\sqrt{2}}{2} \\
\frac{c-1}{2} & \frac{s}{\sqrt{2}} & 0 & 0 & \frac{c+1}{2} & 0 \\
0 & 0 & -\frac{\sqrt{2}}{2} & 0 & -\frac{c-1}{2} & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{A}_M^1 \\
\tilde{A}_M^5 \\
\tilde{A}_M^{11} \\
\tilde{A}_M^{14} \\
\tilde{A}_M^{16} \\
\tilde{A}_M^{22}
\end{pmatrix},
\]

\[
\begin{pmatrix}
A_M^{q2} \\
A_M^{q12} \\
A_M^{q13} \\
A_M^{q17} \\
A_M^{q23}
\end{pmatrix} = \begin{pmatrix}
\frac{c+1}{2} & -\frac{s}{\sqrt{2}} & 0 & 0 & \frac{c-1}{2} & 0 \\
\frac{s}{\sqrt{2}} & c & 0 & 0 & \frac{s}{\sqrt{2}} & 0 \\
0 & 0 & \frac{c+1}{2} & 0 & \frac{s}{\sqrt{2}} & 0 \\
0 & 0 & -\frac{s}{\sqrt{2}} & c & 0 & \frac{s}{\sqrt{2}} \\
\frac{c-1}{2} & -\frac{s}{\sqrt{2}} & 0 & 0 & \frac{c+1}{2} & 0 \\
0 & 0 & -\frac{\sqrt{2}}{2} & 0 & -\frac{c-1}{2} & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{A}_M^2 \\
\tilde{A}_M^{12} \\
\tilde{A}_M^{13} \\
\tilde{A}_M^{17} \\
\tilde{A}_M^{23}
\end{pmatrix},
\]

\[
\begin{pmatrix}
A_M^{q6} \\
A_M^{q9} \\
A_M^{q19} \\
A_M^{q20}
\end{pmatrix} = \begin{pmatrix}
\frac{c+1}{2} & \frac{s}{\sqrt{2}} & -\frac{c-1}{2} & 0 \\
-\frac{s}{\sqrt{2}} & c & \frac{s}{\sqrt{2}} & \frac{\sqrt{2}}{2} \\
\frac{c-1}{2} & \frac{s}{\sqrt{2}} & 0 & \frac{c+1}{2} \\
-\frac{\sqrt{2}}{2} & 0 & -\frac{c-1}{2} & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{A}_M^6 \\
\tilde{A}_M^{19} \\
\tilde{A}_M^{20} \\
\tilde{A}_M^{24}
\end{pmatrix},
\]

\[
\begin{pmatrix}
\tilde{A}_M^{10} \\
\tilde{A}_M^{15} \\
\tilde{A}_M^{18} \\
\tilde{A}_M^{21}
\end{pmatrix} = V^T
\end{pmatrix}
\]

and \( A_M^{q0} = \tilde{A}_M^0, A_M^{q10} = \tilde{A}_M^{10} \). Here \( V \) is given in Eq. \( (A.10) \).
B.2 $A_M^{q_i} \leftrightarrow \mathcal{A}_M^{\tilde{q}_i}$

\[
\begin{align*}
\begin{pmatrix}
A_M^{\tilde{q}_i} \\
A_M^{q_1} \\
A_M^{q_2} \\
A_M^{q_3} \\
A_M^{q_4} \\
A_M^{q_5}
\end{pmatrix}
&=
\begin{pmatrix}
\frac{c+1}{2} & 0 & -\frac{c-1}{2} & 0 & -\frac{s}{\sqrt{2}} & 0 \\
0 & c & 0 & \frac{s}{\sqrt{2}} & 0 & \frac{s}{\sqrt{2}} \\
-\frac{c-1}{2} & 0 & \frac{c+1}{2} & 0 & \frac{s}{\sqrt{2}} & 0 \\
0 & -\frac{s}{\sqrt{2}} & 0 & \frac{c+1}{2} & 0 & \frac{c-1}{2} \\
\frac{s}{\sqrt{2}} & 0 & -\frac{s}{\sqrt{2}} & 0 & c & 0 \\
0 & -\frac{s}{\sqrt{2}} & 0 & c & 0 & \frac{c+1}{2}
\end{pmatrix}^T
\begin{pmatrix}
\mathcal{A}_M^{\tilde{q}_i} \\
\mathcal{A}_M^{q_1} \\
\mathcal{A}_M^{q_2} \\
\mathcal{A}_M^{q_3} \\
\mathcal{A}_M^{q_4} \\
\mathcal{A}_M^{q_5}
\end{pmatrix}, \\
\text{(B.4)}
\end{align*}
\]

\[
\begin{align*}
\begin{pmatrix}
A_M^{q_6} \\
A_M^{q_7} \\
A_M^{q_8} \\
A_M^{q_9} \\
A_M^{q_{10}} \\
A_M^{q_{11}}
\end{pmatrix}
&=
\begin{pmatrix}
\frac{c+1}{2} & 0 & -\frac{c-1}{2} & 0 & \frac{s}{\sqrt{2}} & 0 \\
0 & c & 0 & -\frac{s}{\sqrt{2}} & 0 & -\frac{s}{\sqrt{2}} \\
-\frac{c-1}{2} & 0 & \frac{c+1}{2} & 0 & -\frac{s}{\sqrt{2}} & 0 \\
0 & \frac{s}{\sqrt{2}} & 0 & \frac{c+1}{2} & 0 & \frac{c-1}{2} \\
-\frac{s}{\sqrt{2}} & 0 & \frac{s}{\sqrt{2}} & 0 & c & 0 \\
0 & \frac{s}{\sqrt{2}} & 0 & c & 0 & \frac{c+1}{2}
\end{pmatrix}^T
\begin{pmatrix}
\mathcal{A}_M^{q_6} \\
\mathcal{A}_M^{q_7} \\
\mathcal{A}_M^{q_8} \\
\mathcal{A}_M^{q_9} \\
\mathcal{A}_M^{q_{10}} \\
\mathcal{A}_M^{q_{11}}
\end{pmatrix}, \\
\text{(B.5)}
\end{align*}
\]

\[
\begin{align*}
\begin{pmatrix}
A_M^{q_6} \\
A_M^{q_7} \\
A_M^{q_8} \\
A_M^{q_9} \\
A_M^{q_{10}} \\
A_M^{q_{11}}
\end{pmatrix}
&=
\begin{pmatrix}
c & s & 0 \\
-s & c & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\mathcal{A}_M^{q_6} \\
\mathcal{A}_M^{q_7} \\
\mathcal{A}_M^{q_8} \\
\mathcal{A}_M^{q_9} \\
\mathcal{A}_M^{q_{10}} \\
\mathcal{A}_M^{q_{11}}
\end{pmatrix},
\text{(B.6)}
\end{align*}
\]

and $A_M^{q_6} = \mathcal{A}_M^{q_6}$, $A_M^{q_{10}} = \mathcal{A}_M^{q_{10}}$. Here $U$ is given in Eq. (A.20).

B.3 $A_M^{q_i} \leftrightarrow \mathcal{A}_M^{\tilde{q}_i}$

\[
\begin{align*}
\begin{pmatrix}
A_M^{q_1} \\
A_M^{q_5} \\
A_M^{q_9} \\
A_M^{q_{13}} \\
A_M^{q_{17}} \\
A_M^{q_{21}}
\end{pmatrix}
&= \frac{1}{\sqrt{2}}
\begin{pmatrix}
\frac{c+1}{2} & \frac{s}{\sqrt{2}} & -\frac{c-1}{2} & \frac{s}{\sqrt{2}} & \frac{c+1}{2} & \frac{s}{\sqrt{2} \\
-\frac{s}{\sqrt{2}} & c & \frac{s}{\sqrt{2}} & -\frac{s}{\sqrt{2}} & c & \frac{s}{\sqrt{2}} \\
\frac{c-1}{2} & -\frac{s}{\sqrt{2}} & \frac{c+1}{2} & -\frac{s}{\sqrt{2}} & \frac{c-1}{2} & -\frac{s}{\sqrt{2}} \\
\frac{s}{\sqrt{2}} & -\frac{s}{\sqrt{2}} & \frac{c+1}{2} & \frac{s}{\sqrt{2}} & \frac{c-1}{2} & \frac{s}{\sqrt{2}} \\
\frac{s}{\sqrt{2}} & \frac{s}{\sqrt{2}} & -\frac{s}{\sqrt{2}} & \frac{s}{\sqrt{2}} & c & \frac{s}{\sqrt{2}} \\
\frac{c-1}{2} & \frac{s}{\sqrt{2}} & \frac{s}{\sqrt{2}} & \frac{c+1}{2} & -\frac{s}{\sqrt{2}} & \frac{c-1}{2}
\end{pmatrix}
\begin{pmatrix}
A_M^{\tilde{q}_i} \\
A_M^{q_1} \\
A_M^{q_5} \\
A_M^{q_9} \\
A_M^{q_{13}} \\
A_M^{q_{17}} \\
A_M^{q_{21}}
\end{pmatrix}, \\
\text{(B.7)}
\end{align*}
\]

\[
\begin{align*}
\begin{pmatrix}
A_M^{q_2} \\
A_M^{q_6} \\
A_M^{q_{10}} \\
A_M^{q_{14}} \\
A_M^{q_{18}} \\
A_M^{q_{22}}
\end{pmatrix}
&= \frac{1}{\sqrt{2}}
\begin{pmatrix}
\frac{c+1}{2} & -\frac{s}{\sqrt{2}} & -\frac{c-1}{2} & -\frac{s}{\sqrt{2}} & -\frac{c+1}{2} & -\frac{s}{\sqrt{2} \\
\frac{s}{\sqrt{2}} & c & \frac{s}{\sqrt{2}} & c & \frac{s}{\sqrt{2}} & c \\
-\frac{c-1}{2} & \frac{s}{\sqrt{2}} & \frac{c+1}{2} & \frac{s}{\sqrt{2}} & \frac{c-1}{2} & \frac{s}{\sqrt{2}} \\
\frac{s}{\sqrt{2}} & -\frac{s}{\sqrt{2}} & \frac{c+1}{2} & -\frac{s}{\sqrt{2}} & \frac{c-1}{2} & -\frac{s}{\sqrt{2}} \\
\frac{s}{\sqrt{2}} & \frac{s}{\sqrt{2}} & -\frac{s}{\sqrt{2}} & \frac{s}{\sqrt{2}} & c & \frac{s}{\sqrt{2}} \\
\frac{c-1}{2} & \frac{s}{\sqrt{2}} & \frac{s}{\sqrt{2}} & \frac{c+1}{2} & -\frac{s}{\sqrt{2}} & \frac{c-1}{2}
\end{pmatrix}
\begin{pmatrix}
A_M^{\tilde{q}_i} \\
A_M^{q_2} \\
A_M^{q_6} \\
A_M^{q_{10}} \\
A_M^{q_{14}} \\
A_M^{q_{18}} \\
A_M^{q_{22}}
\end{pmatrix}, \\
\text{(B.8)}
\end{align*}
\]
\[
\begin{pmatrix}
A_{M}^{q_6} \\
A_{M}^{q_{10}} \\
A_{M}^{q_{20}} \\
A_{M}^{q_{24}}
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
c & s & 1 \\
-\sqrt{2}s & \sqrt{2}c & 0 \\
-c & -s & 1
\end{pmatrix}
\begin{pmatrix}
\tilde{A}_{M}^{q_6} \\
\tilde{A}_{M}^{q_{10}} \\
\tilde{A}_{M}^{q_{15}} \\
\tilde{A}_{M}^{q_{24}}
\end{pmatrix} = T \begin{pmatrix}
\tilde{A}_{M}^{q_6} \\
\tilde{A}_{M}^{q_{10}} \\
\tilde{A}_{M}^{q_{15}} \\
\tilde{A}_{M}^{q_{24}}
\end{pmatrix}
\]

and \(A_{M}^{q_6} = \tilde{A}_{M}^{q_6}, A_{M}^{q_{10}} = \tilde{A}_{M}^{q_{10}}\). Here

\[
T = \frac{1}{\sqrt{2}} \times \begin{pmatrix}
-\frac{c}{\sqrt{2}} & \frac{s}{\sqrt{2}} & \frac{3c^2}{4} & \frac{\sqrt{3}(1-c^2)}{2} \\
\frac{2c^2}{\sqrt{2}} - 1 & -\frac{s}{\sqrt{2}} & \frac{-3c^2}{4} & \frac{s}{2\sqrt{2}} \\
\frac{5cs}{\sqrt{6}} & \frac{5c^2}{\sqrt{6}} & \frac{5(1-c^2)}{2} & \frac{\sqrt{3}(c^2-1)}{2} \\
\frac{s}{\sqrt{2}} & \frac{c}{\sqrt{2}} & \frac{-s}{\sqrt{2}} & \frac{c}{\sqrt{2}} \end{pmatrix}
\]

(B.9)

\[
\begin{pmatrix}
A_{M}^{q_6} \\
A_{M}^{q_{10}} \\
A_{M}^{q_{20}} \\
A_{M}^{q_{24}}
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\frac{1+c}{2} & \frac{s}{2} & \frac{s}{2} & \frac{c-1}{2} \\
\frac{-s}{2} & \frac{1-c}{2} & \frac{-s}{2} & \frac{-c-1}{2} \\
\frac{1+c}{2} & \frac{-s}{2} & \frac{s}{2} & \frac{-c-1}{2} \\
\frac{-s}{2} & \frac{-c}{2} & \frac{s}{2} & \frac{c-1}{2}
\end{pmatrix}
\begin{pmatrix}
\tilde{A}_{M}^{q_6} \\
\tilde{A}_{M}^{q_{10}} \\
\tilde{A}_{M}^{q_{15}} \\
\tilde{A}_{M}^{q_{24}}
\end{pmatrix}
\]

(B.10)

B.4 \(A_{M}^{T} \leftrightarrow \tilde{A}_{M}^{T}\)

\[
\begin{pmatrix}
A_{M}^{T_4} \\
A_{M}^{T_{14}} \\
A_{M}^{T_{17}} \\
A_{M}^{T_{22}}
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\frac{1+c}{2} & \frac{s}{2} & \frac{s}{2} & \frac{c-1}{2} \\
\frac{-s}{2} & \frac{1-c}{2} & \frac{-s}{2} & \frac{-c-1}{2} \\
\frac{1+c}{2} & \frac{-s}{2} & \frac{s}{2} & \frac{-c-1}{2} \\
\frac{-s}{2} & \frac{-c}{2} & \frac{s}{2} & \frac{c-1}{2}
\end{pmatrix}
\begin{pmatrix}
\tilde{A}_{M}^{T_4} \\
\tilde{A}_{M}^{T_{14}} \\
\tilde{A}_{M}^{T_{17}} \\
\tilde{A}_{M}^{T_{22}}
\end{pmatrix}
\]

(B.11)

\[
\begin{pmatrix}
A_{M}^{T_2} \\
A_{M}^{T_{3}} \\
A_{M}^{T_{12}} \\
A_{M}^{T_{13}}
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\frac{1+c}{2} & \frac{s}{2} & \frac{s}{2} & \frac{c-1}{2} \\
\frac{-s}{2} & \frac{1-c}{2} & \frac{-s}{2} & \frac{-c-1}{2} \\
\frac{1+c}{2} & \frac{-s}{2} & \frac{s}{2} & \frac{-c-1}{2} \\
\frac{-s}{2} & \frac{-c}{2} & \frac{s}{2} & \frac{c-1}{2}
\end{pmatrix}
\begin{pmatrix}
\tilde{A}_{M}^{T_2} \\
\tilde{A}_{M}^{T_{3}} \\
\tilde{A}_{M}^{T_{12}} \\
\tilde{A}_{M}^{T_{13}}
\end{pmatrix}
\]

(B.12)

\[
\begin{pmatrix}
A_{M}^{T_1} \\
A_{M}^{T_{19}} \\
A_{M}^{T_{20}} \\
A_{M}^{T_{24}}
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\frac{c}{2} & \frac{\sqrt{2}s}{2} & -c \\
\frac{-s}{2} & \frac{\sqrt{2}c}{2} & s \\
1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\tilde{A}_{M}^{T_1} \\
\tilde{A}_{M}^{T_{19}} \\
\tilde{A}_{M}^{T_{20}} \\
\tilde{A}_{M}^{T_{24}}
\end{pmatrix}
\]

(B.13)
\[ \left( \begin{array}{c} A_{q6}^M \\ A_{q5}^M \\ A_{q4}^M \\ A_{q3}^M \\ A_{q2}^M \\ A_{q1}^M \end{array} \right) = \frac{1}{\sqrt{2}} J \left( \begin{array}{c} A_J^M \\ A_{J1}^M \\ A_{J2}^M \\ A_{J3}^M \end{array} \right), \quad J = (J_1, J_2, J_3), \quad (B.14) \]

and \( A_{q6}^M = \tilde{A}_6^M, \ A_{q5}^M = \tilde{A}_5^M \). Here

\[ J_1 = \left( \begin{array}{c} \frac{cs}{\sqrt{2}} \\ \frac{16(1+c)}{26+3\sqrt{10}} \\ -\frac{18+\sqrt{10}+4(2+\sqrt{10})c}{26+3\sqrt{10}} \\ \frac{633+96\sqrt{10}+6(-246+96\sqrt{10}+293c)}{1172} \\ -\frac{(1+c)(-25-11\sqrt{10}+15c+13\sqrt{10}c)}{2\sqrt{2}(26+3\sqrt{10})} \\ \frac{-5cs}{\sqrt{6}} \end{array} \right), \quad (B.15) \]

\[ J_2 = \left( \begin{array}{c} \frac{-\sqrt{5}c}{2} \\ \frac{\sqrt{5}}{8} \frac{6(c-1)}{18-\sqrt{10}+4(2+\sqrt{10})c} \frac{\sqrt{3}(26+3\sqrt{10})}{\sqrt{3}(26+3\sqrt{10})} \\ -\frac{4\sqrt{5}}{\sqrt{3}(26+3\sqrt{10})} + \frac{5(c-1)(118+21\sqrt{10}+5(26+3\sqrt{10})c^2)}{4\sqrt{3}(26+3\sqrt{10})} \\ \frac{5cs}{\sqrt{6}} + \frac{5(c-1)(118+21\sqrt{10}+5(26+3\sqrt{10})c^2)}{4\sqrt{3}(26+3\sqrt{10})} \\ \frac{1}{2\sqrt{6}} \end{array} \right), \quad (B.16) \]

\[ J_3 = \left( \begin{array}{c} 0 \\ 1 - \frac{c^2}{2} \frac{4\sqrt{5}}{2\sqrt{6}} \frac{1+5c}{\sqrt{6}} \frac{293}{293} \\ 0 \\ \frac{1+5c}{\sqrt{6}} \frac{293}{293} \\ \frac{1+5c}{\sqrt{6}} \frac{293}{293} \\ \frac{1+5c}{\sqrt{6}} \frac{293}{293} \end{array} \right), \quad (B.17) \]
C  Mode expansion of gauge bosons

In the case where the rank-reducing boundary is located at the Planck brane, gauge bosons are expanded in terms of fundamental functions as

\[
\begin{pmatrix}
A_{\mu}^T \\
A_{\mu}^{\tau T} \\
A_{\mu}^{\tau} \\
A_{\mu}^{\tau T} \bigg| \\
A_{\mu}^{\tau} \bigg|
\end{pmatrix}
= \begin{pmatrix}
\frac{1+c}{2} & \frac{s}{\sqrt{2}} & \frac{1-c}{2} & \frac{s}{\sqrt{2}} & \frac{c-1}{2} & \frac{1+c}{2} \\
-\frac{s}{\sqrt{2}} & c & -\frac{s}{\sqrt{2}} & -c & -\frac{s}{\sqrt{2}} & \frac{1+c}{2} \\
\frac{1+c}{2} & -\frac{s}{\sqrt{2}} & \frac{1+c}{2} & -\frac{s}{\sqrt{2}} & -\frac{s}{\sqrt{2}} & c-1 \\
\frac{1+c}{2} & \frac{s}{\sqrt{2}} & \frac{1+c}{2} & \frac{s}{\sqrt{2}} & -\frac{s}{\sqrt{2}} & c-1 \\
-\frac{s}{\sqrt{2}} & c & \frac{s}{\sqrt{2}} & -c & \frac{s}{\sqrt{2}} & \frac{c-1}{2} \\
\frac{c}{2} & \frac{s}{\sqrt{2}} & \frac{1+c}{2} & -\frac{s}{\sqrt{2}} & \frac{s}{\sqrt{2}} & \frac{c-1}{2}
\end{pmatrix}
\begin{pmatrix}
C_{A_{n}}^{1}C(z; \lambda_{n}) \\
C_{A_{n}}^{0}S(z; \lambda_{n}) \\
C_{A_{n}}^{11}S(z; \lambda_{n}) \\
C_{A_{n}}^{14}S(z; \lambda_{n}) \\
C_{A_{n}}^{16}S(z; \lambda_{n}) \\
C_{A_{n}}^{22}C(z; \lambda_{n})
\end{pmatrix}
\frac{A_{\mu}^{(n)}(x)}{\sqrt{2}},
\]

\[
\begin{pmatrix}
A_{\mu}^{T} \\
A_{\mu}^{\tau T} \\
A_{\mu}^{\tau} \\
A_{\mu}^{\tau T} \\
A_{\mu}^{\tau}
\end{pmatrix}
= \begin{pmatrix}
\frac{1+c}{2} & -\frac{s}{\sqrt{2}} & -\frac{s}{\sqrt{2}} & \frac{c-1}{2} & \frac{1+c}{2} \\
\frac{s}{\sqrt{2}} & c & -\frac{1+c}{2} & -\frac{s}{\sqrt{2}} & -\frac{s}{\sqrt{2}} \\
\frac{1+c}{2} & \frac{s}{\sqrt{2}} & \frac{1+c}{2} & -\frac{s}{\sqrt{2}} & -\frac{s}{\sqrt{2}} \\
\frac{1+c}{2} & \frac{s}{\sqrt{2}} & \frac{1+c}{2} & \frac{s}{\sqrt{2}} & -\frac{s}{\sqrt{2}} \\
\frac{s}{\sqrt{2}} & c & \frac{s}{\sqrt{2}} & -\frac{s}{\sqrt{2}} & \frac{s}{\sqrt{2}} \\
\frac{c}{2} & \frac{s}{\sqrt{2}} & \frac{1+c}{2} & -\frac{s}{\sqrt{2}} & \frac{s}{\sqrt{2}}
\end{pmatrix}
\begin{pmatrix}
C_{A_{n}}^{2}C(z; \lambda_{n}) \\
C_{A_{n}}^{1}S(z; \lambda_{n}) \\
C_{A_{n}}^{12}S(z; \lambda_{n}) \\
C_{A_{n}}^{13}S(z; \lambda_{n}) \\
C_{A_{n}}^{17}S(z; \lambda_{n}) \\
C_{A_{n}}^{23}C(z; \lambda_{n})
\end{pmatrix}
\frac{A_{\mu}^{(n)}(x)}{\sqrt{2}},
\]

\[
\begin{pmatrix}
A_{\mu}^{6} \\
A_{\mu}^{9} \\
A_{\mu}^{10} \\
A_{\mu}^{16} \\
A_{\mu}^{19} \\
A_{\mu}^{20} \\
A_{\mu}^{18}
\end{pmatrix}
= J \begin{pmatrix}
C_{A_{n}}^{3}C(z; \lambda_{n}) \\
C_{A_{n}}^{7}S(z; \lambda_{n}) \\
C_{A_{n}}^{8}C(z; \lambda_{n}) \\
C_{A_{n}}^{15}C(z; \lambda_{n}) \\
C_{A_{n}}^{18}S(z; \lambda_{n}) \\
C_{A_{n}}^{21}S(z; \lambda_{n}) \\
C_{A_{n}}^{24}C(z; \lambda_{n})
\end{pmatrix}
\frac{A_{\mu}^{(n)}(x)}{\sqrt{2}},
\quad \begin{cases}
A_{\mu}^{6} = C_{A_{n}}^{9}S(z; \lambda_{n})A_{\mu}^{(n)}(x), \\
A_{\mu}^{10} = C_{A_{n}}^{10}S(z; \lambda_{n})A_{\mu}^{(n)}(x).
\end{cases}
\]

where \( J \) is given in Eq. \( \text{(B.14)} \).

In the case where the rank-reducing boundary is located at the TeV brane, gauge bosons are expanded in terms of fundamental functions as

\[
\begin{pmatrix}
A_{\mu}^{q_{1}} \\
A_{\mu}^{q_{5}} \\
A_{\mu}^{q_{11}} \\
A_{\mu}^{q_{14}} \\
A_{\mu}^{q_{16}} \\
A_{\mu}^{q_{22}}
\end{pmatrix}
= \begin{pmatrix}
\frac{c+1}{2} & \frac{s}{\sqrt{2}} & -\frac{c-1}{2} & \frac{s}{\sqrt{2}} & \frac{c-1}{2} & \frac{c+1}{2} \\
-\frac{s}{\sqrt{2}} & c & -\frac{s}{\sqrt{2}} & c & -\frac{s}{\sqrt{2}} & \frac{c-1}{2} \\
\frac{c+1}{2} & -\frac{s}{\sqrt{2}} & \frac{c+1}{2} & -\frac{s}{\sqrt{2}} & -\frac{s}{\sqrt{2}} & \frac{c+1}{2} \\
\frac{c+1}{2} & \frac{s}{\sqrt{2}} & \frac{c+1}{2} & \frac{s}{\sqrt{2}} & -\frac{s}{\sqrt{2}} & \frac{c+1}{2} \\
-\frac{s}{\sqrt{2}} & c & \frac{s}{\sqrt{2}} & -c & \frac{s}{\sqrt{2}} & \frac{c+1}{2} \\
\frac{c}{2} & \frac{s}{\sqrt{2}} & \frac{c+1}{2} & -\frac{s}{\sqrt{2}} & \frac{s}{\sqrt{2}} & \frac{c+1}{2}
\end{pmatrix}
\begin{pmatrix}
C_{A_{n}}^{q_{1}}C(z; \lambda_{n}) \\
C_{A_{n}}^{q_{5}}C(z; \lambda_{n}) \\
C_{A_{n}}^{q_{11}}C(z; \lambda_{n}) \\
C_{A_{n}}^{q_{14}}S(z; \lambda_{n}) \\
C_{A_{n}}^{q_{16}}S(z; \lambda_{n}) \\
C_{A_{n}}^{q_{22}}S(z; \lambda_{n})
\end{pmatrix}
\frac{A_{\mu}^{(n)}(x)}{\sqrt{2}},
\]

\[
\begin{pmatrix}
A_{\mu}^{q_{2}} \\
A_{\mu}^{q_{4}} \\
A_{\mu}^{q_{12}} \\
A_{\mu}^{q_{13}} \\
A_{\mu}^{q_{17}} \\
A_{\mu}^{q_{23}}
\end{pmatrix}
= \begin{pmatrix}
\frac{c+1}{2} & -\frac{s}{\sqrt{2}} & -\frac{c-1}{2} & \frac{s}{\sqrt{2}} & \frac{c-1}{2} & \frac{c+1}{2} \\
\frac{s}{\sqrt{2}} & c & -\frac{s}{\sqrt{2}} & c & -\frac{s}{\sqrt{2}} & \frac{c-1}{2} \\
\frac{c+1}{2} & \frac{s}{\sqrt{2}} & \frac{c+1}{2} & \frac{s}{\sqrt{2}} & -\frac{s}{\sqrt{2}} & \frac{c-1}{2} \\
\frac{c+1}{2} & \frac{s}{\sqrt{2}} & \frac{c+1}{2} & \frac{s}{\sqrt{2}} & -\frac{s}{\sqrt{2}} & \frac{c-1}{2} \\
\frac{s}{\sqrt{2}} & c & \frac{s}{\sqrt{2}} & -c & \frac{s}{\sqrt{2}} & \frac{c-1}{2} \\
\frac{c}{2} & \frac{s}{\sqrt{2}} & \frac{c+1}{2} & -\frac{s}{\sqrt{2}} & \frac{s}{\sqrt{2}} & \frac{c-1}{2}
\end{pmatrix}
\begin{pmatrix}
C_{A_{n}}^{q_{2}}C(z; \lambda_{n}) \\
C_{A_{n}}^{q_{4}}C(z; \lambda_{n}) \\
C_{A_{n}}^{q_{12}}C(z; \lambda_{n}) \\
C_{A_{n}}^{q_{13}}S(z; \lambda_{n}) \\
C_{A_{n}}^{q_{17}}S(z; \lambda_{n}) \\
C_{A_{n}}^{q_{23}}S(z; \lambda_{n})
\end{pmatrix}
\frac{A_{\mu}^{(n)}(x)}{\sqrt{2}},
\]
where $T$ is given in Eq. (B.10).

## D  Matrices for Planck-brane boundary conditions

The matrix $E$ employed in Eq. (3.7) is defined as $E = (E_1, E_2, E_3)$ with

\[
E_1 = \begin{pmatrix}
\frac{c^2}{2}C' & 16(1+c) \sqrt{26} \sqrt{10} \sqrt{C'} \\
-18 + \sqrt{10} + 4(2 + \sqrt{10})c & 26 + 3\sqrt{10} \\
633 + 96 \sqrt{10} + c(-246 + 96 \sqrt{10} + 293c) & 1172 \\
(1 + c)(-25 + 11 \sqrt{10} + 15c + 13 \sqrt{10}c) & 2\sqrt{2}(26 + 3\sqrt{10}) \\
\end{pmatrix},
\]

\[
E_2 = \begin{pmatrix}
-\frac{5cS}{\sqrt{6}}C' & \frac{2cS}{\sqrt{3}}C' \\
\frac{5cS}{\sqrt{6}}C' & -\frac{30cS}{\sqrt{3}}C' \\
\frac{5cS}{\sqrt{6}}C' & \frac{2cS}{\sqrt{3}}C' \\
\frac{5cS}{\sqrt{6}}C' & -\frac{30cS}{\sqrt{3}}C' \\
\end{pmatrix},
\]

\[
E_3 = \begin{pmatrix}
\sqrt{2}cS & \frac{1}{4}\sqrt{2}(1 + 5c)sS' \\
0 & \frac{293}{26 + 3\sqrt{10}}cS' \\
0 & \frac{293}{26 + 3\sqrt{10}}cS' \\\
\end{pmatrix},
\]

\[
(D.1)
\]
The matrix $L$ employed in Eq. (4.17) is defined as $L = (L_1 L_2)$ with

$$
L_1 = \begin{pmatrix}
-\frac{c_5}{\sqrt{2}} C'' & 405 + 348c - 11c^2 + (58 + 28c + 10c^2)\sqrt{10} C'' \\
(2e^2 - 1)C & -\frac{32(26 + 3\sqrt{10})}{\sqrt{2} + 3}\sqrt{10} C'' \\
\frac{5c_5}{\sqrt{6}} \frac{c_5}{c_5} C'' & -\frac{\sqrt{2} \sqrt{10}}{\sqrt{2} + 3}\sqrt{10} C'' \\
s & \frac{2c_5}{\sqrt{10}} C'' & \frac{2c_5}{\sqrt{10}} C'' \\
\sqrt{2} csc C & (1 - 2e^2) C & \frac{2c_5}{\sqrt{10}} C'' \\
(1 - 2e^2) C & \frac{1}{6} \frac{2c_5}{\sqrt{6}} C'' & \frac{1}{6} \frac{2c_5}{\sqrt{6}} C'' \\
\end{pmatrix},
$$

$$
L_2 = \begin{pmatrix}
\frac{1-c}{2} C' & -\frac{s}{2} S' & \frac{3+e^2}{4} S' & \frac{\sqrt{21} - c^2}{4} S' & \frac{s^2}{2} S' \\
\frac{e^2}{2} C' & \frac{c}{2} S & \frac{c}{2} S & \frac{c}{2} S & \frac{s}{2} S \\
\frac{c^2}{4} C' & \frac{s^2}{2} S & \frac{c^2}{4} S & \frac{c^2}{4} S & \frac{s^2}{2} S \\
0 & 0 & \frac{c}{2} S & \frac{c}{2} S & \frac{c}{2} S \\
\frac{s}{2} C' & \frac{c}{2} S & \frac{s}{2} S & \frac{c}{2} S & \frac{s}{2} S \\
\frac{s}{2} C' & \frac{c}{2} S & \frac{s}{2} S & \frac{c}{2} S & \frac{s}{2} S \\
\end{pmatrix}.
$$

(E.2)

## E A determination of mode functions

In this appendix, we give a derivation for determining the coefficients for the mode functions for a real part of $W$ boson by solving Eq. (4.2). Eq. (4.2) is simplified as

$$
0 = \begin{pmatrix}
(c + 1)C'' & 0 & (1 - c)C'' & 0 & \sqrt{2} s S' & 0 \\
0 & \sqrt{2} c C' & 0 & -s S & 0 & -s S \\
(1 - c)C & 0 & (c + 1)C & 0 & -\sqrt{2} s S' & 0 \\
-s C' & 0 & s C' & 0 & \sqrt{2} c S & 0 \\
0 & \sqrt{2} s C' & 0 & (c - 1) S & 0 & (c + 1) S' \\
0 & \sqrt{2} s C' & 0 & (c + 1) S' & 0 & (c - 1) S' \\
\end{pmatrix} \begin{pmatrix}
C_{A,n}^T \\
C_{A,n}^T \\
C_{A,n}^T \\
C_{A,n}^T \\
C_{A,n}^T \\
C_{A,n}^T \\
\end{pmatrix}.
$$

(E.1)

In this expression, it is seen that the components $(C_{A,n}^T, C_{A,n}^T, C_{A,n}^T)$ and the components $(C_{A,n}^T, C_{A,n}^T, C_{A,n}^T)$ are decoupled. Therefore we can calculate each set of the components independently. A part of the components $(C_{A,n}^T, C_{A,n}^T, C_{A,n}^T)$ in Eq. (E.1) are given by

$$
(c + 1) C'' C_{A,n}^T + (1 - c) C'' C_{A,n}^T + \sqrt{2} s S' C_{A,n}^T = 0, \\
(1 - c) C C_{A,n}^T + (c + 1) C C_{A,n}^T + \sqrt{2} s S C_{A,n}^T = 0, \\
-s C C_{A,n}^T + s C C_{A,n}^T + \sqrt{2} c S C_{A,n}^T = 0.
$$

(E.2) (E.3) (E.4)

From Eqs. (E.2) and (E.3), $C_{A,n}^T$ and $C_{A,n}^T$ are obtained as

$$
C_{A,n}^T = -\frac{2C'S + \lambda(1-c)C_{A,n}^T}{2C'S + \lambda(1+c)C_{A,n}^T}, \quad C_{A,n}^T = -\frac{\sqrt{2}c}{2C'S + \lambda(1+c)C_{A,n}^T}.
$$

(E.5)
The left-hand side in Eq. (E.4) is
\[-\frac{2C}{s} 2C'S + \lambda s^2 \frac{C^T}{2C'S + \lambda(1 + c)} C_{A,n}^T.\]  
(E.6)

This equation vanishes for \( W \) boson whose mass eigenvalue equation is \( 2C'S = -\lambda_W s^2 \).

For \( \lambda = \lambda_W \), the coefficients \( C_{A,n}^T \) and \( C_{A,n}^{\overline{T}} \) are
\[ C_{A,n}^T = \frac{1 - c}{1 + c} C_{A,n}, \quad C_{A,n}^{\overline{T}} = \frac{\sqrt{2} s C}{1 + c} C_{A,n}. \]  
(E.7)

A part of the components \( (C_{A,n}^T, C_{A,n}^{\overline{T}}, C_{A,n}^{\overline{T}}) \) in Eq. (E.1) are given by
\[
\begin{align*}
\sqrt{2} c CC_{A,n}^T - sSC_{A,n}^{\overline{T}} - sSC_{A,n}^{\overline{T}} &= 0, \\
\sqrt{2} sCC_{A,n}^T + (c - 1)SC_{A,n}^{\overline{T}} + (c + 1)SC_{A,n}^{\overline{T}} &= 0, \\
\sqrt{2} sC'C_{A,n}^T + (c + 1)SC_{A,n}^{\overline{T}} + (c - 1)SC_{A,n}^{\overline{T}} &= 0. 
\end{align*}
\]  
(E.8) (E.9) (E.10)

From Eqs. (E.9) and (E.10), \( C_{A,n}^T \) and \( C_{A,n}^{\overline{T}} \) are obtained as
\[ C_{A,n}^T = -\sqrt{2} c \frac{2SS'}{s} \frac{2C'S + (1 - c)\lambda}{2C'S + \lambda(1 + c)} C_{A,n}^{T}, \quad C_{A,n}^{\overline{T}} = \frac{2C'S + \lambda(1 + c)}{2C'S + \lambda(1 - c)} C_{A,n}^{\overline{T}}. \]  
(E.11)

The left-hand side in Eq. (E.8) is
\[-\frac{2S}{s} \frac{2C'S + \lambda(1 + c^2)}{2C'S + \lambda(1 - c)} C_{A,n}^{\overline{T}}. \]  
(E.12)

The condition that this equation vanishes means \( C_{A,n}^{\overline{T}} = 0 \). Combining this with Eq. (E.7), we obtain a real component of \( W \) boson as Eq. (4.9).
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