Equal masses results choreographies $n$-body problems

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Abstract

We prove that equally spaced choreography solutions of a large class of $n$-body problems including the classical $n$-body problem and a subset of quasi-homogeneous $n$-body problems, have equal masses if the dimension of the space spanned by the point masses is $n-1$, $n-2$, or, if $n$ is odd, if the dimension is $n-3$. If $n$ is even and the dimension is $n-3$, then all masses with an odd label are equal and all masses with an even label are equal. Additionally, we prove that the same results hold true for any solution of $n+1$-body problems for which $n$ of the point masses behave like an equally spaced choreography and the $n+1$st point mass is fixed at the origin. Furthermore, we deduce that if the curve along which the point masses of a choreography move has an axis of symmetry, the masses have to be equal for $n=3$. Finally, we prove for the $n$-body problem in spaces of constant Gaussian curvature that for $n<6$ equally spaced choreography solutions have to have equal masses and that the same holds true for any solution to the $n+1$-body problem in spaces of constant Gaussian curvature for which $n$ of the point masses behave like an equally spaced choreography and the $n+1$st is fixed at a point.

1 Introduction

In this paper we will consider two types of $n$-body problems, the first of which being the problem of finding the orbits of point masses $q_1,...,q_n \in \mathbb{R}^d$, $d \in \mathbb{N}$, and respective masses $m_1 > 0,\ldots, m_n > 0$ as described by the system
of differential equations

\[ \ddot{q}_k = \sum_{j=1, j \neq k}^{n} m_j (q_j - q_k) f \left( \|q_j - q_k\|^2 \right), \quad k \in \{1, ..., n\}, \tag{1.1} \]

where \( \| \cdot \| \) is the Euclidean norm, \( f : \mathbb{R}_{>0} \to \mathbb{R}_{>0} \) is a positive-valued scalar function and \( \sqrt{x} f(x) \) is a decreasing function. The study of \( n \)-body problems of this type has applications to, for example, atomic physics, celestial mechanics, chemistry, crystallography, differential equations and dynamical systems (see for example [1]–[15], [24], [28], [33]–[47] and the references therein). The second \( n \)-body problem we will investigate is the \( n \)-body problem in spaces of constant Gaussian curvature, or curved \( n \)-body problem for short, which can be formulated as follows: Let \( \sigma = \pm 1 \). The \( n \)-body problem in spaces of constant Gaussian curvature is the problem of finding the dynamics of point masses

\[ q_1, ..., q_n \in M^2_{\sigma} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + \sigma x_3^2 = \sigma \}, \]

with respective masses \( m_1 > 0, ..., m_n > 0 \), determined by the system of differential equations

\[ \ddot{q}_k = \sum_{j=1, j \neq k}^{n} \frac{m_j (q_j - \sigma (q_k \odot q_j) q_k)}{(\sigma - \sigma (q_k \odot q_j)^2)^{\frac{3}{2}}} - \sigma (\dot{q}_k \odot \dot{q}_k) q_k, \quad k \in \{1, ..., n\}, \tag{1.2} \]

where for \( x, y \in M^2_{\sigma} \) the product \( \cdot \odot \cdot \) is defined as

\[ x \odot y = x_1 y_1 + x_2 y_2 + \sigma x_3 y_3. \]

While the curved \( n \)-body problem for \( n = 2 \) goes back as far as the 1830s, a working model for the \( n \geq 2 \) case was not found until 2008 by Diacu, Pérez-Chavela and Santoprete (see [25], [26] and [27]). This breakthrough then gave rise to further results for the \( n \geq 2 \) case in [16]–[23], [29], [30], [31] and [48]–[60]. See [23], [25], [26] and [27] for a historical overview. The study of the curved \( n \)-body problem has applications to for example geometric mechanics, Lie groups and algebras, non-Euclidean and differential geometry and stability theory, the theory of polytopes and topology (see for example [21]). A particular use of the curved \( n \)-body problem is that it may give information about the geometry of the universe. For example: Diacu, Pérez-Chavela and Santoprete showed that the configuration of the Sun,

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Jupiter and the Trojan asteroids cannot exist in curved space (see [25], [26]).

By a choreography, or choreographic solution to (1.1), or (1.2) we mean any solution $q_1, \ldots, q_n$ for which there exists a twice-continuously differentiable periodic vector-valued function $p(t)$ and constants $h_1, \ldots, h_n$ such that $q_k(t) = p(t + h_k)$, $k \in \{1, \ldots, n\}$. For ease of notation, we define $h_{k+Kn} = h_k + K\hat{P}$ for $k \in \{1, \ldots, n\}$, where $\hat{P}$ is the period of $p$, $K \in \mathbb{Z}$. If $h_{k+1} - h_k$ is independent of $k$, then we call a choreographic solution an equally spaced choreography, or equally spaced choreographic solution. Examples of choreographies are the well-known relative equilibrium where $n$ masses move along a circle, evenly distributed as if they were the vertices of a regular polygon (see for example [5], [9], [55] and [57] and the references therein), the famous figure eight, first discovered numerically by Moore (see [42]) and independently discovered and proved by Chenciner and Montgomery (see [11]), which gave rise to the discovery of additional families of choreographies (see for example [8], [32], [50], [39], [58] and the references therein) and numerically discovered choreographies for the curved $n$-body problem (see [41], [40]), of which the figure eight solution on $\mathbb{S}^2$ was analytically proven in [49]. Chenciner proved in [6] for general $n$ that for any equally spaced choreographic solution to the classical $n$-body problem, i.e. $f(x) = x^{-\frac{3}{2}}$ (though it should be noted that his proof does not depend on the choice of $f$), with unequal masses, the same choreographic solution solves the $n$-body problem if all masses are taken equal. He then coined equally spaced choreographic solutions with unequal masses perverse choreographic solutions and proved for $n < 6$ that such solutions do not exist if the choreography lies in a plane. He additionally stated that choreographies also exist in three-space and that the problem of their perversity is completely open. For the case that a choreographic solution is not assumed to be equally spaced, nothing is known regarding whether the masses are equal even for $n = 3$. In this paper we will investigate the existence of perverse choreographies in higher-dimensional spaces, in spaces of constant Gaussian curvature and prove that if the curve along which the point masses of a choreography move has an axis of symmetry, the masses have to be equal for $n = 3$. Specifically, we will prove the following results:

**Theorem 1.1.** Let $q_1, \ldots, q_n$ be an equally spaced choreographic solution of (1.1). For $d = n - 1$, $d = n - 2$ and if $n$ is odd for $d = n - 3$, all masses are equal and if $n$ is even, then for $d = n - 3$ all masses with an odd label are equal and all masses with an even label are equal.
Corollary 1.2. Let \( q_1, \ldots, q_{n+1} \) be a solution of (1.1), where \( q_1, \ldots, q_n \) are an equally spaced choreography and \( q_{n+1} = 0 \). For \( d = n - 1 \), \( d = n - 2 \) and if \( n \) is odd for \( d = n - 3 \) and \( d = n - 4 \), all masses are equal and if \( n \) is even, then for \( d = n - 3 \) and for \( d = n - 4 \) all masses with an odd label are equal and all masses with an even label are equal.

Theorem 1.3. Let \( q_1, \ldots, q_n \) be a choreographic solution of (1.1). If \( n = 3 \) and the curve along which the masses move has an axis of symmetry, then the masses \( m_1, m_2, m_3 \) are all equal.

Corollary 1.4. Let \( q_1, \ldots, q_{n+1} \) be a solution of (1.1), where \( q_1, \ldots, q_n \) are a choreography and \( q_{n+1} = 0 \). If \( n = 3 \) and the curve along which the masses move has an axis of symmetry and \( q_{n+1} = 0 \), then the masses \( m_1, m_2, m_3 \) are all equal.

Theorem 1.5. Let \( q_1, \ldots, q_n \) be an equally spaced choreographic solution of (1.2). Then for \( n < 6 \) for \( \sigma = -1 \) all masses are equal and for \( n = 6 \) there exist at most two different values for the masses. For \( \sigma = 1 \) we have the same two results, provided the point masses do not move along a great circle.

Corollary 1.6. Let \( q_1, \ldots, q_{n+1} \) be a solution of (1.2), where \( q_1, \ldots, q_n \) are an equally spaced choreography and \( q_{n+1} = (0, 0, 1)^T \). Then for \( n < 6 \) for \( \sigma = -1 \) all masses are equal and for \( n = 6 \) there exist at most two different values for the masses. For \( \sigma = 1 \) we have the same two results, provided the point masses do not move along a great circle.

Remark 1.7. Solutions of the type discussed in Corollary 1.2, Corollary 1.3 and Corollary 1.6 exist: It is well-known that one can construct a solution of (1.1) and (1.2) where the \( q_1, \ldots, q_n \) behave as the vertices of a regular polygon rotating around \( q_{n+1} = 0 \) and \( q_{n+1} = (0, 0, 1)^T \) respectively.

Remark 1.8. The argument used in the proof of Theorem 1.1 to prove that for \( d = n - 3, n \) odd, all masses are equal, or take on at most two values for \( n \) even can be applied for the case that \( d = n - 4 \) and \( n \) even as well, (with \( n - 3 \) replaced with \( n - 4 \)) under the condition that we can choose a \( k \in \{1, \ldots, n-1\} \) in such a way that the number of linearly independent vectors in the linear combinations of (3.6) and (3.7) is the same as in the linear combinations of (3.4) and (3.5), in which case for \( n \) even and \( d = n - 4 \) all \( a_l \) in (3.1), \( l \neq \frac{n}{2} + 1 \) have to be zero. However, for \( n > 6 \) it is possible that there is no such \( k \) and excluding that possibility is nontrivial and likely requires further knowledge on the dynamics of the choreography.
The remainder of this paper is as follows: We will first formulate necessary notation and lemmas in section 2 and then prove Theorem 1.1 in section 3, Corollary 1.2 in section 4, Theorem 1.3 in section 5, Corollary 6, Theorem 1.5 in section 7 and Corollary 1.6 in section 8.

2 Background theory

Throughout this paper we will use the notation introduced in the previous section for choreographies. For ease of notation, we will additionally define \( m_{k+Kn} = m_k \) for \( i \in \{1, \ldots, n\} \), \( K \in \mathbb{Z} \) and if we deal with equally spaced choreographies, we will choose \( h_{k+1} - h_k = 1 \) and \( \hat{P} = n \), as we can always use a rescaling argument if a choreography is equally spaced and \( h_{k+1} - h_k \neq 1 \). Furthermore, we will assume that for any choreography solution of (1.1) we have that

\[
\sum_{k=1}^{n} m_k q_k = 0. \tag{2.1}
\]

Note that by (1.1) we have that

\[
\sum_{k=1}^{n} m_k \dot{q}_k = \sum_{k=1}^{n} \sum_{j=1}^{n} m_k m_j (q_j - q_k) f \left( \|q_j - q_k\|^2 \right) = 0,
\]

so there exist constant vectors \( A \) and \( B \) such that \( \sum_{k=1}^{n} m_k q_k = At + B \). But for the \( q_k, k \in \{1, \ldots, n\} \) to lie on a closed, periodic curve we need that \( A = 0 \) and if \( B \neq 0 \), we replace the \( q_k, k \in \{1, \ldots, n\} \) in (1.1) with \( \hat{q}_k = q_k - \frac{1}{\sum_{j=1}^{n} m_j} B \) and work with \( \hat{q}_k \) instead. This result is well-known, but was quickly proven to make the paper as self-contained as possible. For ease of notation, we will define \( \Delta_j(t) = p(t+j) - p(t) \) for all \( j \in \mathbb{Z} \) and \( M = \sum_{j=1}^{n} m_j \). Additionally, to prove Theorem 1.1 and Theorem 1.5 the following well-known vectors will be helpful: Let \( \hat{e}_1, \ldots, \hat{e}_n \in \mathbb{C}^n \), where the \( j \)th component of \( \hat{e}_l \) is \( \frac{1}{\sqrt{n}} e^{2 \pi i (l-1)(j-1)} \), \( j \in \{1, \ldots, n\}, l \in \mathbb{Z} \). Let \( \lambda_l = e^{2 \pi i (l-1)/n} \). Note that if \( \hat{B} \) is the \( n \times n \) matrix for which the \( j \)th component of \( \hat{B}v \) is the \( j + 1 \)st component of \( v \) for all vectors \( v \in \mathbb{C}^n \), then \( \hat{B} \hat{e}_l = \lambda_l \hat{e}_l \). Additionally, note that the Euclidean inner product of \( \hat{e}_j \) and \( \hat{e}_l \) is zero for \( j \neq l \) and 1 for \( j = l \), making the...
vectors $\hat{e}_1, \ldots, \hat{e}_n$ an orthonormal basis of $\mathbb{C}^n$. Finally, if the curve with parametrisation $p(t)$ lies in the plane and is axisymmetric, we will write

$$p(-t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} p(t).$$

(2.2)

Next we will formulate the following lemmas needed to prove Theorem 1.1, Corollary 1.2, Theorem 1.3, Corollary 1.4, Theorem 1.5 and Corollary 1.6.

**Lemma 2.1.** Let $q_1, \ldots, q_n$ be an equally spaced choreography of (1.1). Then

$$0 = \sum_{j=1}^{n-1} \left( m_{k+j} - \frac{M}{n} \right) \Delta_j(t) f(\|\Delta_j(t)\|)$$

and

$$0 = \sum_{j=1}^{n-1} \left( m_{k+j} - \frac{M}{n} \right) \Delta_j(t), \quad k \in \{1, \ldots, n\}.$$

(2.3)

(2.4)

**Proof.** Substituting $q_k(t)$ with $p(t+k)$ and $q_j(t)$ with $p(t+j)$ in (1.1) and subsequently replacing $t+k$ with $t$ gives

$$\ddot{p}(t) = \sum_{j=1}^{n-1} m_j (p(t+j-k) - p(t)) f(\|p(t+j-k) - p(t)\|^2),$$

which can be rewritten as

$$\ddot{p}(t) = \sum_{j=1}^{n-1} m_{j+k} (p(t+j) - p(t)) f(\|p(t+j) - p(t)\|^2) = \sum_{j=1}^{n-1} m_{j+k} \Delta_j(t) f(\|\Delta_j(t)\|^2)$$

(2.5)

for any fixed value $k$. Summing both sides of (2.5) from 1 to $n$ with respect to $k$ then gives

$$n \ddot{p}(t) = \sum_{j=1}^{n-1} \left( \sum_{k=1}^{n} m_{j+k} \right) \Delta_j(t) f(\|\Delta_j(t)\|^2) = \sum_{j=1}^{n-1} M \Delta_j(t) f(\|\Delta_j(t)\|^2).$$

(2.6)

Dividing both sides of (2.6) by $n$ and subtracting the resulting equation from (2.5) then gives (2.3). Next, we will prove (2.4) by using that $\sum_{k=1}^{n} m_k q_k = 0$: 
Note that
\[ Mq_k = \sum_{j=1}^{n} m_j q_k - 0 = \sum_{j=1}^{n} m_j q_k - \sum_{j=1}^{n} m_j q_j = \sum_{j=1}^{n} m_j (q_k - q_j), \]
which, writing \( q_k(t) = p(t + k), \ q_j(t) = p(t + j) \) and replacing \( t + k \) with \( t \), can be rewritten as
\[ -Mp(t) = \sum_{j=1}^{n} m_j (p(t + j - k) - p(t)), \]
which, for any fixed \( k \), can be rewritten as
\[ -Mp(t) = \sum_{j=1}^{n-1} m_{j+k} (p(t + j) - p(t)) = \sum_{j=1}^{n-1} m_{j+k} \Delta_j(t). \tag{2.7} \]
Summing both sides of (2.7) with respect to \( i \) from 1 to \( n \) and dividing the resulting equation on both sides by \( n \) then gives
\[ -Mp(t) = \sum_{j=1}^{n-1} \frac{M}{n} \Delta_j(t). \tag{2.8} \]
Subtracting (2.8) from (2.7) then finally gives (2.4). This completes the proof. \( \square \)

**Remark 2.2.** Lemma 2.1 was proven by Chenciner in [6] for \( f(x) = x^{-\frac{3}{2}} \) and his proof works for general \( f \) as well. As technically speaking the proof in [6] was written down only for \( f(x) = x^{-\frac{3}{2}} \) and to make the paper self-contained, we have included the proof here.

**Lemma 2.3.** Consider any choreography of (1.1) for which the curve given by \( p(t) \) has an axis of symmetry. Then for all \( l, k \in \{1, ..., n\}, l \neq k \) we have that
\[ (m_k - m_l)(p(t + h_l - h_k) - p(t))f \left( \|p(t + h_l - h_k) - p(t)\|^2 \right) \]
\[ = \sum_{j=1}^{n} m_j \left( (p(t + h_j - h_k) - p(t))f \left( \|p(t + h_j - h_k) - p(t)\|^2 \right) \right. \]
\[ - (p(t - (h_j - h_l)) - p(t))f \left( \|p(t - (h_j - h_l)) - p(t)\|^2 \right)), \tag{2.9} \]
we have
\[
(m_k - m_\ell)(p(s + (h_\ell - h_k)) - p(s))f \left( \|p(s + (h_\ell - h_k)) - p(s)\|^2 \right)
\]
\[
= \sum_{j=1, j \neq k, \ell}^{n} m_j \left( (p(s + (h_j - h_k)) - p(s + (h_\ell - h_k)))f \left( \|p(s + (h_j - h_k)) - p(s + (h_\ell - h_k))\|^2 \right)
\]
\]
\[
- (p(s + (h_\ell - h_j)) - p(s + (h_\ell - h_k)))f \left( \|p(s + (h_\ell - h_j)) - p(s + (h_\ell - h_k))\|^2 \right)
\]
\]
\[
= \sum_{j=1, j \neq k, \ell}^{n} m_j \left( (p(s + (h_j - h_k)) - p(s + (h_\ell - h_k))) - (p(t - (h_j - h_\ell)) - p(t)) \right)
\]
\[
= \sum_{j=1, j \neq k, \ell}^{n} m_j \left( (p(s + (h_j - h_k)) - p(s + (h_\ell - h_k))) - (p(s + (h_\ell - h_j)) - p(s + (h_\ell - h_k))) \right)
\]
\[
(2.10)
\]

we have
\[
(m_k - m_\ell)(p(t + h_\ell - h_k) - p(t))
\]
\[
= \sum_{j=1, j \neq k, \ell}^{n} m_j \left( (p(t + h_j - h_k) - p(t)) - (p(t - (h_j - h_\ell)) - p(t)) \right)
\]
\[
= \sum_{j=1, j \neq k, \ell}^{n} m_j \left( (p(s + (h_j - h_k)) - p(s + (h_\ell - h_k))) - (p(s + (h_\ell - h_j)) - p(s + (h_\ell - h_k))) \right)
\]
\[
(2.11)
\]

and we have
\[
(m_k - m_\ell)(p(s + (h_\ell - h_k)) - p(s))
\]
\[
= \sum_{j=1, j \neq k, \ell}^{n} m_j \left( (p(s + (h_j - h_k)) - p(s + (h_\ell - h_k))) - (p(s + (h_\ell - h_j)) - p(s + (h_\ell - h_k))) \right)
\]
\[
(2.12)
\]

Proof. Substituting the \(q_k\) and \(q_j\) with \(p(t + h_k)\) and \(p(t + h_j)\) respectively in (1.1), we get that
\[
\ddot{p}(t + h_k) = \sum_{j=1, j \neq k}^{n} m_j (p(t + h_j) - p(t + h_k))f \left( \|p(t + h_j) - p(t + h_k)\|^2 \right),
\]
which holds for all \(t \in \mathbb{R}\). As such we may replace \(t + h_k\) with \(t\) to obtain
\[
\ddot{p}(t) = \sum_{j=1, j \neq k}^{n} m_j (p(t + h_j - h_k) - p(t))f \left( \|p(t + h_j - h_k) - p(t)\|^2 \right).
\]
\[
(2.13)
\]
Let $s = -t$. Substituting $t$ with $-s$ in (2.13) and using (2.2) we get

\[
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (-1)^2 \frac{d^2}{ds^2} p(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sum_{j=1, j \neq i}^n m_j (p(s - h_j + h_k) - p(t)) f \left( \|p(s - h_j + h_k) - p(s)\|^2 \right)
\]

and multiplying both sides of (2.14) from the left with \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \) gives

\[
\ddot{p}(s) = \sum_{j=1, j \neq k}^n m_j (p(s - h_j + h_k) - p(t)) f \left( \|p(s - h_j + h_k) - p(s)\|^2 \right).
\]

(2.15)

Replacing $s$ with $t$ and $k$ with $l$, $l \in \{1, \ldots, n\}$ in (2.15) and subtracting the resulting equation from (2.13) gives

\[
0 = (m_l - m_k)(p(t + h_l - h_k) - p(t)) f \left( \|p(t + h_l - h_k) - p(t)\|^2 \right) + \sum_{j=1, j \neq k, l}^n m_j ((p(t + h_j - h_k) - p(t)) f \left( \|p(t + h_j - h_k) - p(t)\|^2 \right) - (p(t - (h_j - h_l)) - p(t)) f \left( \|p(t - (h_j - h_l)) - p(t)\|^2 \right)),
\]

(2.16)

which proves (2.9). Replacing $t$ with $-s - (h_l - h_k)$ in (2.16) and multiplying both sides of the resulting equation from the left with \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) and using (2.2) gives

\[
0 = (m_l - m_k)(p(s + (h_l - h_k)) - p(s)) f \left( \|p(s + (h_l - h_k)) - p(s)\|^2 \right) + \sum_{j=1, j \neq k, l}^n m_j ((p(s + (h_j - h_k)) - p(s + (h_l - h_k))) f \left( \|p(s + (h_j - h_k)) - p(s + (h_l - h_k))\|^2 \right) - (p(s + (h_l - h_j)) - p(s + (h_l - h_k))) f \left( \|p(s + (h_l - h_j)) - p(s + (h_l - h_k))\|^2 \right)),
\]

(2.17)

which proves (2.10).
Note that by (2.1) we have that
\[ -Mq_k = \sum_{j=1}^{n} m_j(-q_k) + 0 = -\sum_{j=1}^{n} m_j q_k + \sum_{j=1}^{n} m_j q_j = \sum_{j=1}^{n} m_j(q_j - q_k) \]
and if we replace \( q_k(t) \) with \( p(t + h_k) \) for all \( k \in \{1, ..., n\} \) in (2.18) and replace \( t + h_k \) with \( t \) in the resulting equation, then we get
\[ -Mp(t) = \sum_{j=1}^{n} m_j(p(t + h_j - h_k) - p(t)). \] (2.19)

Note that the right hand side of (2.19) is exactly (2.13) with \( f \) replaced by 1 and \( \ddot{p}(t) \) replaced by \(-Mp(t)\). Repeating the steps that led from (2.13) to (2.16) and (2.17) but starting with (2.19) instead then proves (2.11) and (2.12). This completes the proof. 

**Lemma 2.4.** Let \( q_1, ..., q_n \) be an equally spaced choreographic solution of (1.2). Then
\[ 0 = \sum_{j=1}^{n-1} \left( m_j + \frac{M}{n} \right) \left( p(t + j) - \sigma(p(t + j) \circ p(t))p(t) \right) \frac{1}{(\sigma - \sigma(p(t + j) \circ p(t))^2)^{\frac{3}{2}}} \] (2.20)
for all \( k \in \{1, ..., n\} \).

**Proof.** Let \( q_1, ..., q_n \) be an equally spaced choreographic solution of (1.2). Then by (1.2) we have that
\[ \ddot{p}(t + k) = \sum_{j=1, j \neq k}^{n} \frac{m_j(p(t + j) - \sigma(p(t + j) \circ p(t + k))p(t + k))}{(\sigma - \sigma(p(t + j) \circ p(t + k))^2)^{\frac{3}{2}}} \]
\[ - \sigma(\ddot{p}(t + k) \circ \dot{p}(t + k))p(t + k), \ k \in \{1, ..., n\}. \] (2.21)

Let \( s = t + k \). Then we can rewrite (2.21) as
\[ \ddot{p}(s) = \sum_{j=1, j \neq k}^{n} \frac{m_j(p(s + j) - \sigma(p(s + j - k) \circ p(s))p(s))}{(\sigma - \sigma(p(s + j - k) \circ p(s))^2)^{\frac{3}{2}}} - \sigma(\ddot{p}(s) \circ \dot{p}(s))p(s) \]
\[ = \sum_{j=1}^{n-1} \frac{m_j(p(s + j) - \sigma(p(s + j) \circ p(s))p(s))}{(\sigma - \sigma(p(s + j) \circ p(s))^2)^{\frac{3}{2}}} - \sigma(\ddot{p}(s) \circ \dot{p}(s))p(s). \] (2.22)
Summing both sides of (2.22) from 1 to $n$ with respect to $k$ and subsequently dividing both sides by $n$ gives

$$
\dot{p}(s) = \sum_{j=1}^{n-1} \frac{M_n}{n} \left( p(s + j) - \sigma(p(s + j) \odot p(s)) \right) \frac{p(s)}{(\sigma - \sigma(p(s + j) \odot p(s))^2)^{\frac{1}{2}}} - \sigma(\dot{p}(s) \odot \dot{p}(s))p(s).
$$

(2.23)

Subtracting (2.23) from (2.22) and replacing $s$ with $t$ then finally gives

$$
0 = \sum_{j=1}^{n-1} \left( m_{j+k} - \frac{M_n}{n} \right) \left( p(t + j) - \sigma(p(t + j) \odot p(t))p(t) \right) \frac{p(t)}{(\sigma - \sigma(p(t + j) \odot p(t))^2)^{\frac{1}{2}}}
$$

for all $k \in \{1, \ldots, n\}$. This completes the proof. \qed

3 Proof of Theorem 1.1

By Lemma 2.1 we have that

$$
0 = \sum_{j=1}^{n-1} \left( m_{k+j} - \frac{M_n}{n} \right) \Delta_j(t)f(\|\Delta_j(t)\|)
$$

and

$$
0 = \sum_{j=1}^{n-1} \left( m_{k+j} - \frac{M_n}{n} \right) \Delta_j(t).
$$

If $d = n - 1$, then there exists a subset $S$ of $[0, n]$ of Lebesgue measure $n$ for which the vectors $\Delta_1(t), \ldots, \Delta_{n-1}(t)$ are linearly independent for all $t \in S$, which means that by both (2.3) and (2.4) we have that $m_{j+k} = \frac{M_n}{n}$ for all $k$, $j \in \mathbb{Z}$, which proves that for $d = n - 1$ all masses are equal. For $d = n - 2$, there exists an $S \subset [0, n]$, such that for all $t \in S$ we have that the vectors $\Delta_1(t), \ldots, \Delta_{n-1}(t)$ span an $n - 2$-dimensional space. There therefore exists an $r \in \{1, \ldots, n - 1\}$ such that

$$
- \left( m_{k+r} - \frac{M_n}{n} \right) \Delta_r(t)f(\|\Delta_r(t)\|) = \sum_{j=1}^{n-1} \left( m_{k+j} - \frac{M_n}{n} \right) \Delta_j(t)f(\|\Delta_j(t)\|)
$$

and

$$
- \left( m_{k+r} - \frac{M_n}{n} \right) \Delta_r(t) = \sum_{j=1}^{n-1} \left( m_{k+j} - \frac{M_n}{n} \right) \Delta_j(t),
$$

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such that the vectors $\Delta_j(t), j \in \{1, ..., n-1\}, j \neq r,$ span an $n-2$-dimensional space for all $t \in S,$ which then means that for those $j$ we have that $(m_{k+j} - \frac{M}{n}) f(\|\Delta_k(t)\|) = (m_{k+j} - \frac{M}{n}) f(\|\Delta_j(t)\|)$ for all $k \in \mathbb{Z}, j \in \{1, ..., n-1\}, t \in S,$ so either there are $j$ for which $m_{k+j} - \frac{M}{n} = 0,$ or $f(\|\Delta_j(t)\|) = f(\|\Delta_k(t)\|)$ for all $j \in \{1, ..., n-1\},$ for all $t \in [0, n].$ But if $f(\|\Delta_j(t)\|) = f(\|\Delta_k(t)\|)$ for all $j \in \{1, ..., n-1\},$ for all $t \in [0, n],$ then because $f$ is a strictly decreasing and therefore bijective function, we have that $\|\Delta_j(t)\| = \|\Delta_k(t)\|$ for all $j \in \{1, ..., n-1\},$ for all $t \in [0, n],$ which means that, by using a suitable change of variables, we have that $\|q_k(t) - q_j(t)\|$ is the same function for all $k, j \in \{1, ..., n\}, k \neq j,$ for all $t \in S,$ which means that the $q_1, ..., q_n$ represent the vertices of an $n-1$-dimensional simplex, which contradicts that $d = n - 2.$ So all masses are equal. For $d = n - r,$ $r \geq 3,$ we have that there exists a set $S \subset [0, n]$ with Lebesgue measure $n$ and for which the vectors $\Delta_j(t), j \in \{1, ..., n-1\}$ span an $n-r$-dimensional space for all $t \in S.$ Additionally, note that because $\mathbf{\hat{e}}_1, ..., \mathbf{\hat{e}}_n$ span $\mathbb{C}^n,$ there exist constants $a_2, ..., a_n$ such that

\[
\begin{pmatrix}
m_{1+k} - \frac{M}{n} \\
m_{2+k} - \frac{M}{n} \\
\vdots \\
m_{n+k} - \frac{M}{n}
\end{pmatrix} = \sum_{l=2}^{n} a_l \lambda_l^k \mathbf{\hat{e}}_l. \tag{3.1}
\]

Note that by construction $\mathbf{\hat{e}}_1$ is orthogonal to $(m_{1+k} - \frac{M}{n}, m_{2+k} - \frac{M}{n}, ..., m_{n+k} - \frac{M}{n})^T,$ so we may exclude $\mathbf{\hat{e}}_1$ in the linear combination in (3.1). Again by Lemma 2.1, this means that

\[
0 = \sum_{j=1}^{n-1} \left( m_{k+j} - \frac{M}{n} \right) \Delta_j(t) f(\|\Delta_j(t)\|) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n-1} \left( \sum_{l=1}^{n} a_l \lambda_l^{k+j-1} \right) \Delta_j(t) f(\|\Delta_j(t)\|)
\]

and

\[
0 = \sum_{j=1}^{n-1} \left( m_{k+j} - \frac{M}{n} \right) \Delta_j(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n-1} \left( \sum_{l=1}^{n} a_l \lambda_l^{k+j-1} \right) \Delta_j(t),
\]

so

\[
0 = \sum_{l=1}^{n} a_l \lambda_l^k \sum_{j=1}^{n-1} \lambda_l^{j-1} \Delta_j(t) f(\|\Delta_j(t)\|) \quad \text{and} \quad 0 = \sum_{l=1}^{n} a_l \lambda_l^k \sum_{j=1}^{n-1} \lambda_l^{j-1} \Delta_j(t),
\]

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which mean by the linear independence of the $\lambda^k_l$ as functions of $k$ that for all $l \in \{1, ..., n\}$ we have that $a_l = 0$, or

$$0 = \sum_{j=1}^{n-1} \lambda^j_l \Delta_j(t) f(\|\Delta_j(t)\|)$$

(3.2)

and

$$0 = \sum_{j=1}^{n-1} \lambda^j_l \Delta_j(t).$$

(3.3)

If there are any $l$ for which $a_l \neq 0$, $l \neq \frac{n}{2} + 1 \mod n$ if $n$ is even, then as all terms in the sums on the right-hand side of (3.2) and (3.3) are real-valued, we have that

$$0 = \sum_{j=1}^{n-1} \lambda^j_{\pm l} \Delta_j(t) f(\|\Delta_j(t)\|)$$

(3.4)

and

$$0 = \sum_{j=1}^{n-1} \lambda^j_{\pm l} \Delta_j(t).$$

(3.5)

so for all $k \in \{1, ..., n\}$ we have that

$$0 = \lambda^k_{l-1} \sum_{j=1}^{n-1} \lambda^j_{l-1} \Delta_j(t) f(\|\Delta_j(t)\|) - \lambda^k_{l} \sum_{j=1}^{n-1} \lambda^j_{l} \Delta_j(t) f(\|\Delta_j(t)\|)$$

and

$$0 = \lambda^k_{l} \sum_{j=1}^{n-1} \lambda^j_{l} \Delta_j(t) - \lambda^k_{l-1} \sum_{j=1}^{n-1} \lambda^j_{l-1} \Delta_j(t),$$

so as $\lambda^k_{l-1} \lambda^j_{l-1} - \lambda^k_{l} \lambda^j_{l} \lambda^j_{l-1} = 2i \sin \frac{2\pi(l-1)}{n}(k - j)$ that means that

$$0 = \sum_{j=1}^{n-1} \left( \sin \frac{2\pi(l-1)}{n}(k - j) \right) \Delta_j(t) f(\|\Delta_j(t)\|)$$

and

$$0 = \sum_{j=1}^{n-1} \left( \sin \frac{2\pi(l-1)}{n}(k - j) \right) \Delta_j(t).$$
for all \( k \in \{1, ..., n\} \). Particularly, for \( n \) even, this means that

\[
0 = \sum_{j=1}^{n-1} \left( \sin \frac{2\pi (l - 1)}{n} (k - j) \right) \Delta_j(t) f(\|\Delta_j(t)\|) \quad \text{and} \quad (3.6)
\]

\[
0 = \sum_{j=1}^{n-1} \left( \sin \frac{2\pi (l - 1)}{n} (k - j) \right) \Delta_j(t). \quad (3.7)
\]

for all \( k \in \{1, ..., n\} \).

So for \( d = n - 3 \) we have, reusing the argument we used for the \( d = n - 2 \) case, that there are two ways we can write a vector as a linear combination of \( n - 3 \) linearly independent vectors, unless \( \|\Delta_j(t)\| \) is independent of \( j \), in which case the point masses are the vertices of an \( n \)-simplex, which is a contradiction. This means that for \( d = n - 1, d = n - 2 \) and if \( n \) is odd for \( d = n - 3 \), all masses are equal (as the \( a_l \) are all zero) and if \( n \) is even for \( d = n - 3 \) all masses with an odd label are equal and all masses with an even label are equal (as we have not excluded the possibility that \( a_1 + \frac{n}{2} \neq 0 \)).

This completes the proof.

4 Proof of Corollary 1.2

If \( q_1, ..., q_n \) move along a curve like an equally spaced choreography and \( q_{n+1} = 0 \), then by (1.1) we have for \( k \neq n + 1 \) that

\[
\ddot{q}_k = \sum_{j=1}^{n} m_j (q_j - q_k) f \left( \|q_j - q_k\| \right) - m_{n+1} q_k f \left( \|q_k\| \right),
\]

which means that the same argument that gave (2.5) gives

\[
\ddot{p}(t) + m_{n+1} p(t) f \left( \|p(t)\| \right) = \sum_{j=1}^{n-1} m_{j+k} (p(t + j) - p(t)) f \left( \|p(t + j) - p(t)\| \right)
\]

\[
= \sum_{j=1}^{n-1} m_{j+k} \Delta_j(t) f \left( \|\Delta_j(t)\| \right). \quad (4.1)
\]
Summing both sides of (4.1) with respect to $k$ from 1 to $n$ and dividing both sides of the resulting equation by $n$ then gives

$$\ddot{p}(t) + m_{n+1}p(t)f \left( \|p(t)\|^2 \right) = \sum_{j=1}^{n-1} \frac{M}{n} \Delta_j(t)f \left( \|\Delta_j(t)\|^2 \right)$$ \hspace{1cm} (4.2)

and subtracting (4.2) from (4.1) then gives (2.3) again. As the formula for the center of mass is exactly the same as in the proof of Lemma 2.1, (2.4) holds true as well and as (2.3) and (2.4) generated the proof of Theorem 1.1, the proof of Theorem 1.3 proves Corollary 1.2 as well.

## 5 Proof of Theorem 1.3

Consider a choreography solution of (1.1) for $n = 3$ for which $p$ has an axis of symmetry. Then by Lemma 2.1 we have that for any $i, j, k \in \{1, 2, 3\}$, $i \neq j, i \neq k, j \neq k$,

$$(m_i - m_k)(p(t + h_k - h_i) - p(t))f \left( \|p(t + h_k - h_i) - p(t)\|^2 \right)$$

$$= m_j \left( (p(t + h_j - h_i) - p(t))f \left( \|p(t + h_j - h_i) - p(t)\|^2 \right) \right)$$

$$(5.1)$$

$$- (p(t - (h_j - h_k)) - p(t))f \left( \|p(t - (h_j - h_k)) - p(t)\|^2 \right),$$

and

$$(m_i - m_k)(p(t + h_k - h_i) - p(t))$$

$$= m_j \left( (p(t + h_j - h_i) - p(t)) - (p(t - (h_j - h_k)) - p(t)) \right)$$

$$= m_j \left( (p(t + h_j - h_i) - p(t)) - (p(t - (h_j - h_k)) - p(t)) \right)$$

$$= m_j \left( (p(t + h_j - h_i) - p(t)) - (p(t - (h_j - h_k)) - p(t)) \right).$$

If $p(t + h_j - h_i) - p(t)$ and $p(t - (h_j - h_k)) - p(t)$ are linearly independent, or pointing in the same direction, then for (5.1) and (5.3) to both be true, because $xf(x^2)$ is a decreasing function, we need that these vectors are the
same length, as otherwise \((m_i - m_k)(p(t + h_k - h_i) - p(t))\) has two different directions. By the same argument, working with \((5.2)\) and \((5.4)\) instead, we need that \(p(t + (h_j - h_i)) - p(t + (h_k - h_i))\) and \(p(t - (h_j - h_k)) - p(t + (h_k - h_i))\) have the same length if they are linearly independent, or pointing in the same direction. If for all \(t\) at least one of these conditions is not met, then it is easy to see that using the continuity of \(p\) all point masses lie on a straight line, which contradicts that they lie on a closed curve. Thus we find that \(p(t + (h_j - h_i))\) and \(p(t - (h_j - h_k))\) lie on a circle of radius \(\|p(t + (h_k - h_i)) - p(t)\|\) around \(p(t)\) and we find that \(p(t + (h_j - h_i))\) and \(p(t - (h_j - h_k))\) lie on a circle of radius \(\|p(t + (h_k - h_i)) - p(t)\|\) around \(p(t + (h_k - h_i))\). This is only possible if the vectors on the left-hand sides of \((6.1),(6.2),(6.3)\) and \((6.4)\) cancel out against each other, which means that as \(p(t + (h_k - h_i)) - p(t) \neq 0\), we have that \(m_i - m_k = 0\).

6 Proof of Corollary \([1.4]\)

If \(q_1,\ldots, q_n\) move along a curve like a choreography and \(q_{n+1} = 0\), then, as in the proof of Corollary \([1.2]\) by \((1.1)\) we have for \(k \neq n + 1\) that

\[
\ddot{q}_k = \sum_{\substack{j=1 \atop j \neq k}}^n m_j (q_j - q_k) f \left( \|q_j - q_k\|^2 \right) - m_{n+1} q_k f \left( \|q_k\|^2 \right). \tag{6.1}
\]

Using that \(q_j(t) = p(t + h_j)\) for \(j \leq n + 1\) and defining \(s = t + h_k\), \((6.1)\) can be rewritten as

\[
\ddot{p}(s) + m_{n+1} p(t) f \left( \|p(s)\|^2 \right) = \sum_{\substack{j=1 \atop j \neq k}}^n m_j (p(s + h_j - h_k) - p(s)) f \left( \|p(s + h_j - h_k) - p(s)\|^2 \right). \tag{6.2}
\]

Replacing \(s\) with \(-t\) in \((6.2)\) and subtracting the resulting equation from \((6.2)\), using that \(p(-u) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p(u)\) for any \(u \in \mathbb{R}\) gives

\[
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left( \ddot{p}(t) + m_{n+1} p(t) f \left( \|p(t)\|^2 \right) \right)
= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sum_{\substack{j=1 \atop j \neq k}}^n m_j (p(t - (h_j - h_k)) - p(t)) f \left( \|p(t - (h_j - h_k)) - p(t)\|^2 \right). \tag{6.3}
\]
Multiplying (6.3) on both sides with \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \) from the left and subtracting (6.3) from (6.2) gives (2.9), after which the proof of Lemma 2.3 can be followed to the letter to obtain (2.14), (2.15) and (2.16), after which we can repeat the proof of Theorem 1.3 to obtain our result. This completes the proof.

7 Proof of Theorem 1.5

Let \( q_1, \ldots, q_n \) be an equally spaced choreographic solution of (1.2). Then by Lemma 2.4 we have that

\[
0 = \sum_{j=1}^{n-1} \left( m_{j+k} - \frac{M}{n} \right) \frac{(p(t + j) - \sigma(p(t + j) \odot p(t))p(t))}{(\sigma - \sigma(p(t + j) \odot p(t))^2}^2
\]

for all \( k \in \mathbb{Z} \). We again note, as in the proof of Theorem 1.1, that because \( \hat{e}_1, \ldots, \hat{e}_n \) span \( \mathbb{C}^n \), there exist constants \( a_2, \ldots, a_n \) such that

\[
\begin{pmatrix}
(m_{1+k} - \frac{M}{n}) \\
(m_{2+k} - \frac{M}{n}) \\
\vdots \\
(m_{n+k} - \frac{M}{n})
\end{pmatrix} = \sum_{l=2}^{n} a_l \lambda_l^j \hat{e}_l. \tag{7.1}
\]

Again, note that by construction \( \hat{e}_1 \) is orthogonal to \( (m_{1+k} - \frac{M}{n}, m_{2+k} - \frac{M}{n}, \ldots, m_{n+k} - \frac{M}{n})^T \), so we may exclude \( \hat{e}_1 \) in the linear combination in (7.1). Finally, analogous to the proof of Theorem 1.1 we find by (7.1) that proving that \( m_{j+k} = \frac{M}{n} \) for all \( j, k \in \mathbb{Z} \) is equivalent to proving that \( a_l = 0 \) for all \( l \in \{2, \ldots, n\} \), which can be done by proving that

\[
0 \neq \sum_{j=1}^{n-1} \lambda_l^{-1} \frac{(p(t + j) - \sigma(p(t + j) \odot p(t))p(t))}{(\sigma - \sigma(p(t + j) \odot p(t))^2}^2 \tag{7.2}
\]

for all \( l \in \{2, \ldots, n\} \).

In order to prove that we will show that for an \( l \in \{2, \ldots, n\} \) for which (7.2) does not hold, the corresponding configuration of the point masses either does not exist, or lies on a great circle for all \( t \). For such an \( l \), we have by (7.2) that

\[
0 = \sum_{j=1}^{n-1} \lambda_l^{-1} \frac{p(t + j)}{(\sigma - \sigma(p(t + j) \odot p(t))^2}^2 - p(t) \sum_{j=1}^{n-1} \lambda_l^{-1} \frac{\sigma(p(t + j) \odot p(t))}{(\sigma - \sigma(p(t + j) \odot p(t))^2}^2. \tag{7.3}
\]
for all $k \in \{1, \ldots, n\}$.

Let $F(u, s) = (\sigma - \sigma(p(u) \circ p(s)))^{-\frac{3}{2}}$ for all $u, s \in \mathbb{R}$ and let

$$C_l(t) = \sum_{j=1}^{n-1} \lambda_l^{j-1} \sigma(p(t+j) \circ p(t)),$$

Replacing $t$ with $t+1$ in (7.3) gives

$$0 = \sum_{j=1}^{n-2} \lambda_l^{j-1} p(t+j+1) F(t+j+1, t+1) + p(t) \lambda_l^{n-2} F(t, t+1) - p(t+1) C_l(t+1) \quad (7.4)$$

Multiplying (7.3) on both sides with $\lambda_l^{n-2} F(t, t+1)$, then multiplying both sides of (7.4) on both sides with $C_l(t)$ and adding the subsequent equations gives

$$0 = C_l(t) \left( \sum_{j=1}^{n-2} \lambda_l^{j-1} p(t+j+1) F(t+j+1, t+1) - p(t+1) C_l(t+1) \right)$$

$$+ \lambda_l^{n-2} F(t, t+1) \left( \sum_{j=1}^{n-1} \lambda_l^{j-1} p(t+j) F(t+j, t) \right)$$

$$= C_l(t) \left( \sum_{j=2}^{n-1} \lambda_l^{j-1} p(t+j) F(t+j, t+1) - p(t+1) C_l(t+1) \right)$$

$$+ \lambda_l^{n-2} F(t, t+1) \left( \sum_{j=1}^{n-1} \lambda_l^{j-1} p(t+j) F(t+j, t) \right). \quad (7.5)$$

If $C_l(t) = 0$, then by (7.4) there exists a linear combination of

$$p(t+1), \ldots, p(t+n-1)$$

that equals zero. If $C_l(t) \neq 0$, then there exists a linear combination of $p(t+1), \ldots, p(t+n-1)$ that equals zero by (7.5). This means for $n = 3$ that $p(t+1)$ and $p(t+2)$ are linearly dependent and therefore, as we can replace $t$ with $t+1$, that $p(t+1)$, $p(t+2)$ and $p(t+3)$ are linearly dependent, which makes it impossible for $q_1$, $q_2$, $q_3$ to be a choreographic solution. For $n = 4$ it means that $p(t+3)$ lies in the span of $p(t+1)$ and $p(t+2)$ and, again replacing $t$ by $t+1$, that $p(t+4)$ lies in the span of $p(t+2)$ and $p(t+3)$,
meaning that all \( p(t+j), j \in \{1, 2, 3, 4\} \) lie in the span of \( p(t+1) \) and \( p(t+2) \). This can only happen if all the point masses move along a great circle of a sphere for all \( t \), which is not a case we are considering in this paper.

For \( n = 5 \) we have that if (7.2) does not hold for an \( l \neq 0 \), then, as \( \text{Im} \lambda_0^l = 0 \), we have that

\[
0 = \sum_{j=2}^{4} \sin \left( \frac{2\pi l j - 1}{5} (p(t+j) - \sigma(p(t+j) \odot p(t)) p(t)) \right) \frac{(\sigma - \sigma(p(t+j) \odot p(t))^2)^{\frac{3}{2}}}{(\sigma - \sigma(p(t+j) \odot p(t))^2)^{\frac{3}{2}}}.
\]

(7.6)

If \( C_l = 0 \), then by (7.6) we have that \( p(t+4) \) can be written as a linear combination of \( p(t+2) \) and \( p(t+3) \) and by extension all \( p(t+j) \) can be written as a linear combination of \( p(t+2) \) and \( p(t+3) \). If \( C_l \neq 0 \), then taking the imaginary parts on both sides of (7.6) we again get that \( p(t+4) \) can be written as a linear combination of \( p(t+2) \) and \( p(t+3) \) and by extension all \( p(t+j) \) can be written as a linear combination of \( p(t+2) \) and \( p(t+3) \). So again all masses are equal, unless perhaps all point masses lie on a great circle. If \( n = 6 \) and there is an \( l \notin \{0, 3\} \) such that (7.2) does not hold, then

\[
0 = \sum_{\substack{j=2 \atop j \neq 4}}^{5} \sin \left( \frac{2\pi l j - 1}{n} (p(t+j) - \sigma(p(t+j) \odot p(t)) p(t)) \right) \frac{(\sigma - \sigma(p(t+j) \odot p(t))^2)^{\frac{3}{2}}}{(\sigma - \sigma(p(t+j) \odot p(t))^2)^{\frac{3}{2}}}.
\]

(7.7)

at which point we can repeat the argument we used for the case \( n = 5 \). This proves that for \( n < 6 \) for \( \sigma = -1 \) all masses are equal and for \( n = 6 \) there exist at most two different values for the masses. For \( \sigma = 1 \) we have the same two results, provided the point masses do not move along a great circle.

8 Proof of Corollary 1.6

Let \( q_1, ..., q_{n+1} \) be a solution of (1.2), where \( q_1, ..., q_n \) are an equally spaced choreography and \( q_{n+1} = (0, 0, 1)^T \). Let \( \hat{\sigma} = (0, 0, 1)^T \). Then by (1.2) we have for \( k \in \{1, ..., n\} \) that

\[
\frac{\ddot{p}(t+k) + \sigma(p(t+k) \odot \dot{p}(t+k)) p(t+k)}{\sigma - \sigma(p(t+k) \odot p(t+k))^2} = \sum_{j=1}^{n} \frac{m_j(p(t+j) - \sigma(p(t+k) \odot p(t+j)) p(t+k))}{(\sigma - \sigma(p(t+k) \odot p(t+j))^2)^{\frac{3}{2}}}
\]

\[
+ \frac{m_{n+1}(\hat{\sigma} - \sigma(p(t+k) \odot \hat{\sigma}) p(t+k))}{(\sigma - \sigma(p(t+k) \odot \hat{\sigma})^2)^{\frac{3}{2}}},
\]

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which, writing \( s = t + k \), can be rewritten as

\[
\ddot{p}(s) + \sigma(\dot{p}(s) \odot \dot{p}(s))p(s) = \sum_{j=1}^{n-1} \frac{m_{j+k}(p(s + j) - \sigma(p(s) \odot p(s + j))p(s))}{(\sigma - \sigma(p(s) \odot p(s + j))^2)^{\frac{3}{2}}} + \frac{m_{n+1}(\tilde{e} - \sigma(p(s) \odot \tilde{e})p(s))}{(\sigma - \sigma(p(s) \odot \tilde{e})^2)^{\frac{3}{2}}}.
\]

(8.1)

Summing both sides of (8.1) with respect to \( k \) from 1 to \( n \), dividing both sides of the resulting equation by \( n \) and subtracting that equation from (8.1) then gives (2.20), which is the result on which the proof of Theorem 1.5 was based. So for \( n < 6 \) for \( \sigma = -1 \) all masses are equal and for \( n = 6 \) there exist at most two different values for the masses. For \( \sigma = 1 \) we have the same two results, provided the point masses do not move along a great circle.

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