Routing Permutations on Spectral Expanders via Matchings

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Abstract
We consider the following matching-based routing problem. Initially, each vertex \( v \) of a connected graph \( G \) is occupied by a pebble which has a unique destination \( \pi(v) \). In each round the pebbles across the edges of a selected matching in \( G \) are swapped, and the goal is to route each pebble to its destination vertex in as few rounds as possible. We show that if \( G \) is a sufficiently strong \( d \)-regular spectral expander then any permutation \( \pi \) can be achieved in \( O(\log n) \) rounds. This is optimal for constant \( d \) and resolves a problem of Alon et al. (SIAM J Discret Math 7:516–530, 1994).

Keywords Routing · Expander Graphs · Non-blocking Networks

Mathematics Subject Classification 05C48 · 05C85

1 Introduction
The following routing problem was introduced by Alon et al. [2]. Given a graph \( G \), initially each vertex \( v \) is occupied by a pebble \( p_v \) (both vertices and pebbles are labelled). For each pebble \( p_v \), we are also given its unique destination \( \pi(v) \), that is, \( \pi \) is a permutation on \( V(G) \). One round of routing consists of selecting a matching in \( G \) and swapping pebbles along each edge of such a matching. For example, if a pebble \( p \) currently sits on a vertex \( v \), a pebble \( p' \) on a vertex \( b \), and the edge \( \{a, b\} \) is part of the selected matching, then we move the pebble \( p \) to the vertex \( b \) and the pebble \( p' \) to the vertex \( a \). We denote by \( rt(G, \pi) \) the smallest number of rounds needed to achieve a permutation \( \pi \), and \( rt(G) = \max_{\pi} rt(G, \pi) \). Classes of graph for which \( rt(G) \) has been studied include complete (bipartite) graphs, paths, cycles, trees, hypercubes, and expanders [2, 10, 15, 18]. Note that the described routing is fairly restrictive: not only that at most one pebble can travel through an edge (in each direction) in each round,
but every vertex sends and receives at most one pebble. Most other routing schemes do not put such a hard constraint on the load on vertices, but rather try to optimize load on edges [4, 13].

In this note we are interested in the case where \( G \) is a spectral expander. We say that a graph \( G \) is an \((n, d, \lambda)\)-graph if it is a \( d \)-regular graph with \( n \) vertices and \( \lambda(G) \leq \lambda \), where \( \lambda(G) \) denotes the second largest absolute value of eigenvalues of its adjacency matrix. A fundamental result in spectral graph theory, the Expander Mixing Lemma (e.g. see [9, Lemma 2.5]), demonstrates the importance of \( \lambda(G) \): For any \( S, T \subseteq V(G) \) we have

\[
|e_G(S, T) - |S||T|d/n| \leq \lambda(G) \sqrt{|S||T|},
\]

where \( e_G(S, T) \) counts the number of edges with one endpoint in \( S \) and the other in \( T \) (in the case \( S \cap T \neq \emptyset \), every edge with both endpoints in \( S \cap T \) is counted twice). The largest absolute value of eigenvalues of a \( d \)-regular graph \( G \) is always \( d \), thus \( \lambda(G) \leq d \), and the celebrated Alon–Boppana bound (see [9, Theorem 2.7]) states \( \lambda(G) \geq 2\sqrt{d-1} - o_n(1) \). Note that already for \( \lambda(G) = o(d) \), the inequality (1) implies strong edge and vertex-expansion properties of the graph \( G \). For a thorough introduction to \((n, d, \lambda)\)-graphs, expander graphs, and pseudo-random graphs in general, we refer the reader to [9, 12].

It was shown in [2] that if \( G \) is an \((n, d, \lambda)\)-graph, for any \( \lambda < d \), then

\[
rt(G) = O \left( \frac{d^2}{(d-\lambda)^2 \log^2 n} \right).
\]

We improve this bound under a mild assumption on \( \lambda \).

**Theorem 1.1** There exists \( d_0, C \in \mathbb{N} \) such that if \( G \) is an \((n, d, \lambda)\)-graph with \( d \in [d_0, n-1] \) and \( \lambda < d/72 \), then

\[
rt(G) \leq \frac{C \log n}{\log(d/\lambda)}.
\]

The special case of Theorem 1.1 where \( d \geq n^\alpha \) and \( \lambda = K \sqrt{d} \), for some constants \( \alpha, K > 0 \) (that is, \( G \) is a very dense graph with, asymptotically, the strongest possible spectral expansion properties) was recently obtained by Horn and Purcilly [10]. In the case where \( d \) is a constant, Theorem 1.1 gives \( rt(G) = O(\log n) \) which answers a problem raised by Alon et al. [2]. In general, Theorem 1.1 is easily seen to be optimal for \( \lambda < d^{1-\varepsilon} \), for any constant \( \varepsilon > 0 \), as in this case the obtained bound \( rt(G) = O(\log_d(n)) \) asymptotically matches the diameter of \( G \), which is clearly a lower bound on \( rt(G) \).

Throughout the proof we omit the use of floors and ceilings. Constants can be adjusted such that all inequalities hold with a sufficient margin to compensate for this.
2 Nonblocking Generalised Matchings

In this section we state some notions and results from the theory of wide-sense nonblocking networks, the main machinery underlying our proof of Theorem 1.1.

Definition 2.1 Given \(d, t \in \mathbb{N}\), we say that a bipartite graph \(G = (A \cup B, E)\) is \((d, t)\)-nonblocking if there exists a family \(S\) of subsets of \(E\), called the safe states, such that the following holds:

1. \(\emptyset \in S\),
2. If \(E' \subseteq E'\) and \(E' \in S\) then \(E'' \in S\), and
3. Given \(E' \in S\) of size \(|E'| < t\) and a vertex \(v \in A\) with \(\deg_{E'}(v) < d\) (that is, \(v\) is incident to less than \(d\) edges in \(E'\)), there exists an edge \(e = (v, w) \in E \setminus E'\) such that \(E' \cup \{e\} \in S\) and \(w\) is not incident to any edge in \(E'\).

In other words, a bipartite graph is \((d, t)\)-nonblocking if one can dynamically extend and shrink a star-matching, with the only restriction that at every point each star has at most \(d\) rays and their total size is at most \(t\).

The following result is due to Feldman et al. [6, Proposition 1]. It is proven in the same way as the more known result, at least within the combinatorics community, of Friedman and Pippenger [7] on embeddings of trees in expanders.

Lemma 2.2 Let \(G = (A \cup B, E)\) be a bipartite graph and \(d, a, d \in \mathbb{N}\). If for every \(X \subseteq A\) of size \(1 \leq |X| \leq 2a\) there are at least \(2d|X|\) vertices in \(B\) adjacent to some vertex in \(X\), then \(G\) is \((d/a, n/24)\)-nonblocking.

The following lemma verifies the condition of Lemma 2.2 for bipartite subgraphs of spectral expanders.

Lemma 2.3 Let \(G\) be an \((n, d, \lambda)\)-graph with \(\lambda < d/72\), and let \(V(G) = A \cup B\) be a partition of the vertex set. If every vertex \(v \in A\) has at least \(d/3\) neighbours in \(B\), then the bipartite graph \(G[A, B]\) is \((d/\lambda, n/24)\)-nonblocking.

Proof We verify the conditions of Lemma 2.2 with \(a = \lambda n/(24d)\). Consider some \(X \subseteq A\) of size \(|X| \leq 2a \leq \lambda n/(12d)\). Suppose, towards a contradiction, that the set \(Y \subseteq B\) of vertices adjacent to some vertex in \(X\) is of size \(|Y| < 2(d/\lambda)|X|\). On the one hand, by the assumption of the lemma we have

\[e(X, Y) \geq |X|d/3.\]

On the other hand, from (1) together with the upper bound on \(|Y|, |X|\) and \(\lambda\), we have

\[e(X, Y) \leq d|X||Y|/n + \lambda \sqrt{|X||Y|} < 2d^2|X|^2/(\lambda n) + |X|\sqrt{2\lambda d} \leq |X|d/3,
\]

thus we reach a contradiction. \(\square\)
3 Proof of Theorem 1.1

Definition 3.1 Given $k \in \mathbb{N}$, we say that a family of paths $\mathcal{P}$ in a graph $G$ is $k$-matching-switchable if the following holds:

- Each path $P \in \mathcal{P}$ is of the same odd length $\ell = 2k + 1$ (that is, each path has $\ell$ edges),
- All endpoints are distinct, and
- For each $z \in \{1, \ldots, k\}$, the set of edges $E_z(\mathcal{P})$, consisting of the $z$-th and $(\ell + 1 - z)$-th edge in each path in $\mathcal{P}$, forms a matching.

The main property of a $k$-matching-switchable family $\mathcal{P}$, already used by Alon et al. [2], is that if we perform $2k + 1$ rounds of routing where in round $z \leq k$ we swap pebbles across $E_z(\mathcal{P})$, in round $k + 1$ across edges corresponding to the $(k + 1)$-st edge from each path in $\mathcal{P}$ (note that this is indeed a matching), and in round $z > k + 1$ across $E_{2k+2-z}(\mathcal{P})$, then in the end all the pebbles other than the endpoints of paths in $\mathcal{P}$ remain where they were before, and the pebbles corresponding to endpoints of each path $P \in \mathcal{P}$ are swapped.

Proof of Theorem 1.1 It is implicit in [2, Theorem 2] that every permutation can be represented as a composition of two permutations of order 2. Thus it suffices to prove Theorem 1.1 assuming $\pi$ is of order 2, that is, $\pi^2$ is the identity permutation.

Using the probabilistic method, we first show that there exists a partition $V(G) = V_1 \cup V_2$ such that, for each vertex $v \in V(G)$, we have:

(i) $v, \pi(v) \in V_i$ for some $i \in \{1, 2\}$, and
(ii) $v$ has at least $d/3$ neighbours in both $V_1$ and $V_2$.

For each cycle in $\pi$ (which is of length either 1 or 2 by the assumption on the order of $\pi$) toss an independent (fair) coin to decide whether to put the vertices of the cycle into $V_1$ or $V_2$. Let $E_v$ denote the event that the vertex $v$ has fewer than $d/3$ neighbours in either $V_1$ or $V_2$. By the Chernoff–Hoeffding inequalities and union-bound, $E_v$ happens with probability at most $\exp(-\Theta(d))$. Two events $E_v$ and $E_w$ are independent if $v$ and $w$ are at distance at least 3, thus each $E_v$ is dependent on at most $d^2$ other events. Therefore, by the Lovász Local Lemma (see [3, Corollary 5.1.2]) there is a non-zero probability that none of the events happens, assuming $d \geq d_0$ is sufficiently large, implying that a desired partition exists.

Let $D = d/\lambda$. Suppose we are given a subset $W \subseteq V_i$ of size $|W| \leq \varepsilon n$, for $\varepsilon = 1/72$, such that if $v \in W$ and $\pi(v) \neq v$ then $\pi(v) \notin W$. We show that there exists a family of $k$-matching-routable paths $\mathcal{P}_W = \{P_v\}_{v \in W}$, for $k = \log_D(n)$, where each $P_v \in \mathcal{P}_W$ connects $v$ and $\pi(v)$. As observed earlier, these paths define a routing scheme which swaps the endpoints of paths in $\mathcal{P}_W$ while leaving everything else intact. By greedily taking a new set $W \subseteq V_i$ ($|W| \leq \varepsilon n$) of vertices which are not yet routed and using the corresponding family $\mathcal{P}_W$ to swap the endpoints, every pebble reaches its destination after at most $(2k + 1) \cdot 2[1/\varepsilon]$ rounds, giving the desired bound.

Without loss of generality, suppose $W \subseteq V_1$. Consider $k = \log_D(n)$ bipartite graphs $G_z = G[A_z, B_z]$, for $z \in \{1, \ldots, k\}$, where $A_z = V_1$ if $z$ is odd and $A_z = V_2$ otherwise, and $B_z = V(G) \setminus A_z$. By Lemma 2.3, each $G_z$ is $(D, n/24)$-nonblocking.
Let \((w_1, \ldots, w_t)\) be an arbitrary ordering of the vertices in \(W\). We show, by induction on \(i \in \{0, \ldots, t\}\), that there exists a family of \(k\)-matching-switchable paths \(\mathcal{P}_i = (P_{w_j})_{j \leq i}\), with each \(P_{w_j}\) connecting \(w_j\) to \(\pi(w_j)\), such that \(E_z(\mathcal{P}_i)\) is a safe state in \(G_z\) for each \(z \in \{1, \ldots, k\}\). Note that the statement vacuously holds for \(i = 0\). Suppose it holds for some \(i < t\). We show that it then also holds for \(i + 1\) using the following procedure to find a new path \(P_{w_{i+1}}\). Let \(S_0 = \{w_{i+1}\}\) and \(S_0 = \{\pi(w_{i+1})\}\), and set \(s_0 = 1\). For \(z = 1, \ldots, k\), iteratively, let \(M_z\) and \(M'_z\) be sets of \(s_z \coloneqq \min\{\varepsilon n, s_{z-1} D\}\) edges in \(G_z\) incident to \(S_{z-1}\) and \(S'_{z-1}\), respectively, such that each vertex in \(B_z\) is incident to at most one edge in \(E_z(\mathcal{P}_i) \cup M_z \cup M'_z\) and \(E_z(\mathcal{P}_i) \cup M_z \cup M'_z\) is a safe state in \(G_z\). These sets can be found by successively applying \((P3)\) (Definition 2.1) with \(v \in S_{z-1} = S_{z-1} \cup S'_{z-1}\), such that we ask for at most \(D\) edges incident to each such vertex and in total for at most \(s_z\) edges incident to each set \(S_{z-1}\) and \(S'_{z-1}\). Property \((P3)\) can indeed be applied as \(|E_z(\mathcal{P}_i)| < |W| \leq \varepsilon n\) and we further ask for at most \(2\varepsilon n\) edges, thus we are always in a safe state with fewer than \(3\varepsilon n \leq n/24\) edges (recall that \(G_z\) is \((D, n/24)\)-nonblocking). Finally, let \(S_z(S'_z)\) be the endpoints of \(M_z\) \((M'_z)\) in \(B_z\) (for the next iteration, recall that \(B_z = A_{z+1}\)). After all \(k\) iterations are done, the choice of \(k\) implies \(|S_k| = |S'_k| = \varepsilon n\), thus by (1) there exists an edge \(e\) between some \(v_k \in S_k\) and \(v'_k \in S'_k\). Now going backwards with \(z = k, \ldots, 1\), let \(m_z \in M_z\) be the unique edge incident to \(v_z\), and \(m'_z \in M'_z\) the unique edge incident to \(v'_z\). Set \(v_{z-1}(v'_{z-1})\) to be the other endpoint of \(m_z\) \((m'_z)\), and proceed to the next iteration. The edges \(m_1, m_2, \ldots, m_k, e, m'_{k-1}, m'_{k-1}, \ldots, m'_1\) then define a path \(P_{w_{i+1}}\) such that \(\mathcal{P}_{i+1}\) satisfies the inductive hypothesis. This finishes the proof. \(\square\)

4 Concluding Remarks

The idea of using nonblocking properties of expanders to reach a large set of vertices, connect two of them, and then remove all unused ones is usually attributed to Daniel Johannsen [11] and is a standard technique today. For other recent applications of this idea, see [5, 8, 14, 16, 17]. The main difference between these applications and the presented one is that we do not aim to find paths which are entirely vertex-disjoint but rather only ‘locally’, that is, they are disjoint with respect to one step of the routing. While this makes the task easier on the one hand, it also allows for more endpoints to be dealt with simultaneously, thus making the task harder on the other hand. Consequently, unlike most of the other results which rely directly on tree embeddings of Friedman and Pippenger [7], the ‘local’ vertex-disjoint property is achieved through a repeated application of a related result of Feldman et al. [6] (Lemma 2.2 in this note), with each step of the routing corresponding to an edge in a distinct copy of a bipartite subgraph of \(G\).

The proof of Lemma 2.2 does not provide an efficient algorithm for finding an edge guaranteed by \((P3)\). However, under somewhat stronger requirements on \(G\), which are satisfied in our case, Aggarwal et al. [1, Theorem 2.2.7] gave a polynomial time algorithm for finding such an edge, which in turn translates into a polynomial time algorithm for finding a routing scheme in Theorem 1.1.
The following problem remains open: is it true that if \( G \) is a \( d \)-regular graph with the Cheeger constant \( h(G) \geq \varepsilon \), for some \( \varepsilon > 0 \), where

\[
h(G) = \min \{ e(X, V(G) \setminus X)/|X| : X \subseteq V(G) \text{ and } |X| \leq |V(G)|/2 \},
\]

then \( \text{rt}(G) = O_{d, \varepsilon}(\log n) \)? The best known bound is \( \text{rt}(G) = O((d/\varepsilon)^4 \log^2 n) \), due to Alon et al. [2].

References

1. Aggarwal, A., Bar-Noy, A., Coppersmith, D., Ramaswami, R., Schieber, B., Sudan, M.: Efficient routing in optical networks. J. ACM 43(6), 973–1001 (1996)
2. Alon, N., Chung, F.R.K., Graham, R.L.: Routing permutations on graphs via matchings. SIAM J. Discret. Math. 7(3), 513–530 (1994)
3. Alon, N., Spencer, J.H.: The probabilistic method. In: Wiley-Interscience Series in Discrete Mathematics and Optimization, 4th edn. Wiley, Hoboken (2016)
4. Broder, A.Z., Frieze, A.M., Upfal, E.: Existence and construction of edge-disjoint paths on expander graphs. SIAM J. Comput. 23(5), 976–989 (1994)
5. Draganić, N., Krivelevich, M., Nenadov, R.: Rolling backwards can move you forward: on embedding problems in sparse expanders. Trans. Am. Math. Soc. 375(7), 5195–5216 (2022)
6. Feldman, P., Friedman, J., Pippenger, N.: Wide-sense nonblocking networks. SIAM J. Discret. Math. 1(2), 158–173 (1988)
7. Friedman, J., Pippenger, N.: Expanding graphs contain all small trees. Combinatorica 7, 71–76 (1987)
8. Glebov, R.: On Hamilton cycles and other spanning structures. PhD thesis, Freie Universität Berlin (2013)
9. Hoory, S., Linial, N., Widgerson, A.: Expander graphs and their applications. Bull. Am. Math. Soc. 43(4), 439–561 (2006)
10. Horn, P., Purcilly, A.: Routing number of dense and expanding graphs. J. Comb. 11(2), 329–350 (2020)
11. Johannsen, D.: Personal communication
12. Krivelevich, M., Sudakov, B.: Pseudo-random graphs. In: More Sets, Graphs and Numbers. A Salute to Vera Sós and András Hajnal, pp. 199-262. Springer, János Bolyai Mathematical Society, Berlin, Budapest (2006)
13. Leighton, F.T., Maggs, B.M., Rao, S.B.: Packet routing and job-shop scheduling in \( O(\text{congestion} + \text{dilation}) \) steps. Combinatorica 14(2), 167–186 (1994)
14. Letzter, S., Pokrovskiy, A., Yepremyan, L.: Size-Ramsey numbers of powers of hypergraph trees and long subdivisions. arXiv Preprint at arXiv:2103.01942 (2021)
15. Li, W.-T., Lu, L., Yang, Y.: Routing numbers of cycles, complete bipartite graphs, and hypercubes. SIAM J. Discret. Math. 24(4), 1482–1494 (2010)
16. Montgomery, R.: Hamiltonicity in random graphs is born resilient. J. Comb. Theory Ser. B 139, 316–341 (2019)
17. Montgomery, R.: Spanning trees in random graphs. Adv. Math. 356, 92 (2019)
18. Zhang, L.: Optimal bounds for matching routing on trees. SIAM J. Discret. Math. 12(1), 64–77 (1999)

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