Zero Duality Gap in View of Abstract Convexity

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October 21, 2019

Abstract

We extend conditions for zero duality gap to the context of nonconvex and nonsmooth optimization. We use tools provided by the theory of abstract convexity. Recall that every convex lower semicontinuous function is the upper envelope of a set of affine functions. Hence, in the classical convex analysis, affine functions act as “building blocks” or a family “simpler functions”. The abstract convexity approach replaces this family of simpler functions (the linear and affine functions in classical convex analysis), by a different set of functions. These sets are called abstract linear, and abstract affine sets, and denoted by $\mathcal{L}$, and $\mathcal{H}$, respectively. Mimicking the classical setting, abstract convex functions are the upper envelopes of elements of $\mathcal{H}$. By using this approach, we establish new characterizations of zero duality gap under no assumptions on the topology of $\mathcal{L}$. Moreover, under a mild assumption on the set $\mathcal{L}$ (namely, assuming that $\mathcal{L}$ has the weak$^*$-topology), we establish and extend several fundamental results of convex analysis. In particular, we prove that the zero duality gap property can be stated in terms of an inclusion involving $\varepsilon$-subdifferentials.

The weak$^*$ topology $C(\mathcal{L}, X)$ is exploited to obtain the sum rule of abstract $\varepsilon$-subdifferential.

The Banach–Alaoglu–Bourbaki theorem is extended to the abstract linear functional space $\mathcal{L}$. The latter extends a fact recently established by Borwein, Burachik and Yao in the conventional convex case.

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1 Introduction

The theory of abstract convexity, also called convexity without linearity, is a powerful tool that allows us to extend many facts from classical convex analysis to more general frameworks. It has been the focus of active research for the last fifty years because of its many applications in functional analysis, approximation theory, and nonconvex analysis. Nevertheless, just like convex analysis, the development of abstract convexity has been mainly motivated by applications to optimization.

The works [3–6, 18, 20, 21, 24] use abstract convexity for applications to nonconvex optimization. A deep study on abstract convexity can be found in the seminal book of Alex Rubinov [19], see also the monograph of Ivan Sing [22]. Abstract convexity reflects one of the fundamental concepts of convex analysis, which is the fact that every lower semicontinuous convex function $f$ is the upper envelope of affine functions. More precisely, at every point $x$, we have

$$f(x) = \sup \{h(x) : h \text{ is an affine function, } h \leq f\}.$$  \hspace{1cm} (1)

Most results in convex analysis are consequences of two important aspects of (1): (i) the “supremum” operation, and (ii) the set over which this supremum is taken. Results that depend on aspect (ii) are likely to depend on specific properties of linear/affine functions. How to distinguish which facts from convex analysis follow from the “upper envelope” operation, and which follow from the particular structure of the set of linear/affine functions? Abstract convexity is the fundamental tool that addresses this crucial question: it retains the “upper envelope” operation in aspect (i) of (1), but changes the set of functions over which the supremum is taken. These sets of functions are called abstract linear and they naturally induce the abstract affine sets. Since aspect (i) of (1) is retained, global properties of convex analysis may be preserved even when dealing with nonconvex models. Thus, this approach is sometimes called a “non-affine global support function technique” (see for example [3, 12, 19]).

Many tools from convex analysis, such as subgradients and $\varepsilon$-subgradients, have their “abstract” counterparts, obtained again by using abstract linear functions. For instance, the abstract subgradient of an abstract convex function $f$ at a point $x$ collects all the supporting abstract linear functions which are minorants of (i.e., their graphs stay below) $f$, and coincide with $f$ at $x$. This extends the concept of the convex subgradient and provides a valuable tool for studying certain nonconvex optimization problems (see [3, 12, 18, 19]).

Another example is the Fenchel-Moreau conjugate $f^*$ of a function $f$. The definition of $f^*$ uses the set of linear functions, and abstract convexity allows to produce “abstract” types of conjugates.

In [1], zero duality gap is shown to be equivalent to (a) certain properties involving $\varepsilon$-subgradients and (b) other facts involving conjugate functions. One of the aims of the present work is to extend these results to the context of abstract convexity. Additionally, we supplement the sum rule for abstract subdifferentials, improving the result in [12]. To the best of our knowledge, with the exception of [3], this manuscript is the very first attempt to consider explicitly the weak $^\ast$ topology (sequential pointwise convergence topology) on the abstract linear sets to deduce calculus rules of subdifferentials. The fundamental result on the weak $^\ast$ closeness of the dual unit ball of dual spaces $X^\ast$ is extended to general spaces of abstract linear functions $L$ equipped with the weak $^\ast$ topology $C(L, X)$.

The structure of the present paper is as follows. Section 2 recalls some preliminary definitions and facts used throughout the paper. We briefly introduce and study abstract linear functional space, and abstract convexity notions; some results are new in the theory of abstract convexity. In Section 3, we provide some properties that are tantamount to the equality between the conjugate of the sum of some functions and the infimal convolution of the conjugates of those functions, which ensures the zero duality gap. We pose no topological assumptions on the primal space nor on the space of linear functions. A comparison with its forerunner, [1] Theorem 3.2] for convex programming, is established. The necessary
and sufficient characterization for zero duality gap is provided, which is new even in the standard convex analysis. In Section 4, we equip the abstract linear functional space with the weak* topology to extend some classical convex subdifferentials calculus in the framework of abstract convexity. Some of the facts in convex analysis cannot be extended to abstract convexity without imposing additional assumptions. Here, we assume that the epigraphs of conjugate functions admit the weak* subdifferentials. In the next section, we define the abstract set of linear functions.

2 Preliminaries on Abstract Convexity

When mentioning a convex function, or convex set without any further explanation, we mean the standard convexity of convex analysis. We use the notation $\mathbb{R}_{+\infty} := \mathbb{R} \cup \{+\infty\}$ and $\mathbb{R}_{\pm\infty} := \mathbb{R} \cup \{\pm \infty\}$. Given a function $f : X \to \mathbb{R}_{\pm\infty}$, its domain is the set $\text{dom } f := \{x \in X : f(x) < \infty\}$, its epigraph is the set $\text{epi } f := \{(x, \lambda) : \lambda \geq f(x), x \in \text{dom } f\}$. Throughout, $X$ is a nonempty set, which can be thought of as a primal space. Note that we do not assume any algebraic or topological structure on $X$. Let $\mathcal{F} := \{f : X \to \mathbb{R} : f \text{ is a function}\} = \mathbb{R}_X$, i.e., $\mathcal{F}$ is the set of all functions acting from $X$ to $\mathbb{R}$. The addition operator in $\mathcal{F}$ is the conventional one, i.e., $(f_1 + f_2)(x) := f_1(x) + f_2(x)$ for all $x \in X$. As mentioned above, the linear functions and its vertical shifts (which are the affine functions) are at the core of convex analysis. They have a crucial role in the definitions of conjugate functions and $\varepsilon$-subdifferentials. In the next section, we define the abstract set $\mathcal{L}$ of linear functions. We make minimal assumptions on $\mathcal{L}$ which trivially hold in the classical convex case. In Proposition 3 below, we show that some classical features of the conjugate functions and $\varepsilon$-subdifferentials, are still true for this general set $\mathcal{L}$ of linear functions.

2.1 Abstract Linear Space

Definition 1. Let $X$ be a nonempty set. A space of abstract linear functions, denoted by $\mathcal{L}$, is a subset of $\mathcal{F} = \mathbb{R}_X$ that satisfies the following properties.

(a) $\mathcal{L}$ is closed with respect to the addition operator i.e. $f_1, f_2 \in \mathcal{L} \implies f_1 + f_2 \in \mathcal{L}$.

(b) For every $l \in \mathcal{L}$ and $m \in \mathbb{N}$, there exist $l_1, \ldots, l_m \in \mathcal{L}$ such that

$$l = l_1 + \ldots + l_m.$$  \hfill (2) \label{con1}

Remark 2. Assume that $0 \in \mathcal{L}$ and $\mathcal{L}$ verifies Definition 1(a). Then $\mathcal{L}$ automatically verifies property (b) in Definition 1.

Throughout, we will assume $\mathcal{L}$ possesses properties (a) and (b).

Using the set $\mathcal{L}$, we state next the abstract counterparts of infimal convolution, Fenchel conjugate function, and $\varepsilon$-subdifferential.

Definition 3. (i) Let $X$ be a nonempty set equipped with an addition operation. Take $m$ functions $\psi_1, \ldots, \psi_m : X \to \mathbb{R}_{+\infty}$. The infimal convolution of the functions $\psi_1, \ldots, \psi_m$ is the function $\psi_1 \Box \ldots \Box \psi_m : X \to \mathbb{R}_{+\infty}$ defined by

$$\psi_1 \Box \ldots \Box \psi_m(x) := \inf_{x_1 + \ldots + x_m = x} \left\{ \psi_1(x_1) + \ldots + \psi_m(x_m) \right\},$$  \hfill (3) \label{D3.1.1}

with the convention that infimum over an empty set is $+\infty$.

(ii) Let $X$ and $\mathcal{L}$ be as in Definition 1 and $f : X \to \mathbb{R}_{+\infty}$ be any function. The Fenchel conjugate of $f$ is the function $f^* : \mathcal{L} \to \mathbb{R}_{+\infty}$ defined as

$$f^*(l) := \sup_{x \in X} \{l(x) - f(x)\}.$$  \hfill (4) \label{D3.1.2}
Proposition 4. Given a set $X$, a number $\varepsilon \geq 0$, and a function $f : X \to \mathbb{R}_{+\infty}$, we define the $\varepsilon$-subdifferential point-to-set mapping $\partial_{\varepsilon} f : X \rightrightarrows \mathcal{L}$ at a point $x \in \text{dom } f$ as

$$\partial_{\varepsilon} f(x) := \{ l \in \mathcal{L} : f(y) - f(x) - (l(y) - l(x)) + \varepsilon \geq 0 \text{ for all } y \in X \}.$$  \hfill \{D3.1.3\}

We prove next some properties of the concepts defined in (i)–(iii).

**Proposition 4.** Given a set $X$ and a space of abstract linear functionals $\mathcal{L}$, $m \geq 2$, the following statements hold.

(i) For all $x \in X$ and all $\varepsilon \geq 0$, we have

$$l \in \partial_{\varepsilon} f(x) \iff f^*(l) + f(x) \leq l(x) + \varepsilon.$$ \hfill \{P3.2.i\}

(ii) The following inequality holds

$$\left( \sum_{i=1}^{m} f_i \right)^* \leq f_1^* \cdots f_m^* \text{ in } \mathcal{L}.$$ \hfill \{P3.2.1\}

(iii) Let $l \in \mathcal{L}$ and $f : X \to \mathbb{R}_{+\infty}$. Then, $l \in \text{dom } f^*$ if and only if for all $\varepsilon > 0$, there is an $x \in X$ such that $l \in \partial_{\varepsilon} f(x)$.

(iv) For any $\varepsilon > 0$, we always have $\bigcap_{\eta > 0} \partial_{\varepsilon + \eta} f(x) = \partial_{\varepsilon} f(x)$.

(v) For any $m$ functions $f_1, \ldots, f_m : X \to \mathbb{R}_{+\infty}$, any $x \in X$, and any $\varepsilon \geq 0$, we have the following inclusion

$$\bigcap_{\eta > 0} \bigcup_{\varepsilon_1 + \ldots + \varepsilon_m = \varepsilon + \eta, \varepsilon_i \geq 0} \bigcup_{\sum \partial_{\varepsilon_i} f_i(x) \subset \partial_{\varepsilon} \left( \sum_{i=1}^{m} f_i \right)(x).} \hfill \{P3.2.v\}$$

**Proof.** (i) See [19] Proposition 7.10].

(ii) Take $l \in \mathcal{L}$. We consider two cases.

**Case 1.** Assume that $l \notin \text{dom } \left( \sum_{i=1}^{m} f_i \right)^*$. We have

$$\sup_{x \in X} \left\{ l(x) - \sum_{i=1}^{m} f_i(x) \right\} = +\infty.$$

Take any additive decomposition of $l$, i.e., take any finite collection $l_1, \ldots, l_m$ such that $l_1 + \ldots + l_m = l$. We have

$$\sum_{i=1}^{m} f_i^*(l_i) \geq \sup_{x \in X} \left\{ \sum_{i=1}^{m} (l_i(x) - f_i(x)) \right\} = +\infty.$$

Taking infimum over all possible additive decompositions of $l$, we deduce that

$$f_1^* \cdots f_m^*(l) = +\infty.$$

**Case 2.** Assume that $l \in \text{dom } \left( \sum_{i=1}^{m} f_i \right)^*$. Take an arbitrary additive decomposition of $l$, i.e., take any finite collection $l_1, \ldots, l_m$ such that $l_1 + \ldots + l_m = l$. By definition of conjugate function, we have that for every $\varepsilon > 0$, there is an $x \in X$ such that

$$\left( \sum_{i=1}^{m} f_i \right)^*(l) \leq l(x) - \sum_{i=1}^{m} f_i(x) + \varepsilon \leq \sum_{i=1}^{m} l_i(x) - f_i(x) + \varepsilon \leq \sum_{i=1}^{m} f_i^*(l_i) + \varepsilon,$$
where we used the definition of \(t_1, \ldots, t_m\) as additive decomposition of \(l\) in the equality and the definition of conjugate function in the last inequality. Since the additive decomposition is arbitrary, the expression above yields
\[
\left(\sum_{i=1}^{m} f_i\right)^* (l) \leq \inf_{l_1 + \ldots + l_m = l} \left\{ f_1^* (l_1) + \ldots + f_m^* (l_m) \right\} + \epsilon = f_1^* \square \ldots \square f_m^* (l) + \epsilon.
\]

Since the inequality holds for all \(l \in \mathcal{L}\) and \(\epsilon > 0\), we obtain (7).

(iii) Let \(\epsilon > 0\). Due to the definition of \(f^*: f^*(l) = \sup_{x \in X} \{ l(x) - f(x) \} \), then for all \(\epsilon > 0\) we can find an \(x \in X\) such that \(l(x) - f(x) \geq f^*(l) - \epsilon\), or \(l(x) + \epsilon \geq f^*(l) + f(x)\). By (ii), the latter is equivalent to having \(l \in \partial f(x)\).

(iv) Fix \(\epsilon > 0\). It is clear from the definition that \(\partial \epsilon f(x) \subset \partial_{\epsilon + \eta} f(x)\) for every \(\eta > 0\). Hence we deduce that \(\partial \epsilon f(x) \subset \bigcap_{\eta > 0} \partial_{\epsilon + \eta} f(x)\). For the opposite inclusion, fix \(\eta > 0\) and take \(l \in \bigcap_{\eta > 0} \partial_{\epsilon + \eta} f(x)\). By (ii), the latter is equivalent to having \(l(x) + f(x) + f^*(l) \leq \eta + \epsilon\) for all \(\eta > 0\). Thus, \(l(x) + f(x) + f^*(l) \leq \epsilon\). Using (ii) again, we deduce that \(l \in \partial f(x)\).

(v) Take \(\eta > 0\). We claim that (v) is true if the inclusion
\[
\sum_{i=1}^{m} \partial_{\epsilon_i} f_i(x) \subset \partial_{\epsilon + \eta} \left(\sum_{i=1}^{m} f_i\right)(x), \tag{9} \{P3.2.vP1\}
\]
is true for every \(\eta > 0\) and every additive decomposition \(\epsilon_1, \ldots, \epsilon_m\) of \(\epsilon + \eta\). Indeed, consider \(m\) non-negative numbers \(\epsilon_1, \ldots, \epsilon_m\) such that \(\epsilon_1 + \ldots + \epsilon_m = \epsilon + \eta\) and assume that (9) holds. Since the right hand side of (9) does not depend on the choice of \(\epsilon_1, \ldots, \epsilon_m\), we have that
\[
\bigcup_{\epsilon_i \geq 0} \epsilon_1 + \ldots + \epsilon_m = \epsilon + \eta \quad \sum_{i=1}^{m} \partial_{\epsilon_i} f_i(x) \subset \partial_{\epsilon + \eta} \left(\sum_{i=1}^{m} f_i\right)(x).
\]

Now (v) will follow by taking intersection for all \(\eta > 0\) in both sides of the expression above and then using (iv). Therefore, our claim is true and we proceed to establish (9) for every \(\eta > 0\) and every additive decomposition \(\epsilon_1, \ldots, \epsilon_m\) of \(\epsilon + \eta\).

If \(\sum_{i=1}^{m} \partial_{\epsilon_i} f_i(x) = \emptyset\), then inclusion (9) trivially holds. So assume that \(\sum_{i=1}^{m} \partial_{\epsilon_i} f_i(x) \neq \emptyset\) and take \(l \in \sum_{i=1}^{m} \partial_{\epsilon_i} f_i(x)\). Then there are \(l_i \in \partial_{\epsilon_i} f_i(x)\) (\(i = 1, \ldots, m\)) such that \(l = l_1 + \ldots + l_m\). Using (ii) we can write
\[
f^*_i(l_i) + f_i(x) \leq l_i(x) + \epsilon_i \quad \text{for all } i = 1, \ldots, m.
\]

Add up the inequalities above, and use the fact that \(\epsilon_1 + \ldots + \epsilon_m = \epsilon + \eta\), to obtain
\[
\sum_{i=1}^{m} f^*_i(l_i) + \sum_{i=1}^{m} f_i(x) \leq \sum_{i=1}^{m} l_i(x) + \epsilon + \eta = l(x) + \epsilon + \eta,
\]
where we used the definition of \(l\) in the rightmost equality. Using (i), we have
\[
\sum_{i=1}^{m} f^*_i(l_i) \geq \left(\sum_{i=1}^{m} f_i\right)^* (l),
\]
which combined with the inequality above yields
\[
\left(\sum_{i=1}^{m} f_i\right)^* (l) + \left(\sum_{i=1}^{m} f_i\right) (x) \leq l(x) + \epsilon + \eta \iff l \in \partial_{\epsilon + \eta} \left(\sum_{i=1}^{m} f_i\right)(x),
\]
where we used (ii). This establishes (9), and the proof of (v) is complete. \(\square\)

**Remark 5.** Statements in Proposition 3 are well-known facts in classical convex analysis which can be found in many contexts of convex analysis.
2.2 Abstract Convex Functions and Abstract Convex Sets

We start this subsection by defining the abstract affine functions, which are, as in the standard convex analysis, the vertical shifts of the abstract linear functions.\[\{D4.1\}\]

**Definition 6.** Let \(X\) and \(L\) be as in Definition 1. The space of abstract affine functions is defined as \(H := \{l + c : l \in L, c \in \mathbb{R}\}\).

**Remark 7.** The space of affine function \(H\) can be defined independently as an arbitrary set which is closed with respect to the vertical shift operator.

Equipped with the set \(H\), we can now extend the classical notions of convex function and convex set to our abstract framework.\[\{D4.2\}\]

**Definition 8.** Let \(X, L\) and \(H\) be as in Definition 6. We have the following definitions.

(i) Given any function \(f : X \rightarrow \mathbb{R}_{+\infty}\), the set \(\text{supp} f := \{h \in H : h(x) \leq f(x), \forall x \in X\} \subset H\).\[\{\text{suppf}\}\]

Equality holds in (12) for all \(x \in X\) if and only if \(f\) is \(H\)-convex.

(ii) If the function \(f \in H\)-convex, then for all \(x \in \text{dom} f\) and \(\varepsilon > 0\), we have \(\partial f(x) \neq \emptyset\). Conversely, if for all \(x \in X\), and \(\varepsilon > 0\), we always have \(\partial f(x) \neq \emptyset\), then the function \(f\) is \(H\)-convex.
(iii) (Fenchel–Moreau) For all \( x \in X \), we always have
\[
 f^{**}(x) \leq f(x). \tag{13} \]

Equality holds in \( \text{(13)} \) for all \( x \in X \) if and only if \( f \) is \( \mathcal{H} \)-convex.

**Proof.**

(i) Inequality \( \text{(12)} \) holds trivially by the definition of \( \text{supp} f \). When the equality holds in \( \text{(12)} \) for all \( x \in X \), then \( f \) is \( \mathcal{H} \)-convex by Definition \( \text{(8)} \)(ii). Conversely, assume that \( f \) is \( \mathcal{H} \)-convex. By definition, there exists a set \( H \subset \mathcal{H} \) such that
\[
f(x) = \sup_{h \in H} h(x). \]
It is easy to see from the definitions that \( H \subset \text{supp} f \). Since \( f(x) = \sup_{h \in H} h(x) \leq \sup_{h \in \text{supp} f} h(x) \leq f(x) \), then \( \sup_{h \in \text{supp} f} h(x) = f(x) \), for all \( x \in X \). This proves that equality holds in \( \text{(12)} \). The proof of (i) is complete.

(ii) Assuming the function \( f \) is \( \mathcal{H} \)-convex, by (i), we have \( \sup_{h \in \text{supp} f} h(x) = f(x) \). Then for all \( x \in \text{dom} f \) and \( \epsilon > 0 \), there is an affine function \( l + c \in \text{supp} f \) with \( l \in \mathcal{L} \), \( c \in \mathbb{R} \) such that
\[
l(x) + c + \epsilon \geq f(x), \quad \text{and} \quad f(y) \geq l(y) + c, \forall y \in X.
\]
Consequently,
\[
f(y) - f(x) \geq (l(y) - l(x)) - \epsilon, \forall y \in X,
\]
which implies \( l \in \partial_c f(x) \). Conversely, assume that for all \( x \in X \) and \( \epsilon > 0 \), we always have \( \partial_c f(x) \neq \emptyset \). We will prove that
\[
f(x) = \sup_{h \in \text{supp} f} h(x).
\]
By (i), we always have
\[
f(x) \geq \sup_{h \in \text{supp} f} h(x). \tag{14} \]

Take \( \epsilon > 0 \). By our assumption, there is a linear function \( l \in \partial_c f(x) \), or, equivalently,
\[
f(y) \geq l(y) - l(x) - \epsilon + f(x), \quad \forall y \in X.
\]
Then the affine function \( v^*(.) := l(.) - l(x) - \epsilon + f(x) \) belongs to the set \( \text{supp} f \) and \( v^*(x) = l(x) - \epsilon + f(x) - l(x) = f(x) - \epsilon \). Hence,
\[
f(x) - \epsilon = v^*(x) \leq \sup_{h \in \text{supp} f} h(x),
\]
where the inequality holds because \( v^* \in \text{supp} f \). Since the above inequality holds for all \( \epsilon > 0 \), inequality \( \text{(14)} \) now implies that \( f(x) = \sup_{h \in \text{supp} f} h(x) \). Using now the last statement in (i), we conclude \( f \) is a \( \mathcal{H} \)-convex function.

(iii) See \[1]{\text{Theorem 7.1}}. \]

**Remark 11.**

(i) Inequality \( \text{(12)} \) holds for \( f : X \to \mathbb{R}_{+\infty} \) even when \( \text{supp} f = \emptyset \). Indeed, in this case we have \( \sup_{h \in \text{supp} f} h(x) = -\infty < f(x) \). In this situation, however, \( f \) is not an \( \mathcal{H} \)-convex function. Consequently, if \( f \) is \( \mathcal{H} \)-convex, we must have \( \text{supp} f \neq \emptyset \).

(ii) Proposition \( \text{(11)} \)(ii) is not an “if and only if” statement. The first implication holds for all \( \mathcal{H} \)-convex function, whereas the converse implication needs \( \text{dom} f = X \).

Next proposition provides some properties of \( \mathcal{L} \)-convex sets and \( \mathcal{H} \)-convex sets used in the next sections.

**Proposition 12.** Let \( X, \mathcal{L} \) and \( \mathcal{H} \) be as in Definition \( \text{(8)} \) and \( C \subset \mathcal{L} \). The following assertions hold.
(i) The set $C$ is $\mathcal{L}$-convex if and only if there is an $\mathcal{L}$-convex function $f : X \to \mathbb{R}$ such that

$$C = \text{suppf}. $$

In this case, $\text{suppf} \subset \mathcal{L}$, i.e. $\text{suppf} := \{l \in \mathcal{L} : l(x) \leq f(x), \forall x \in X\}.$

(ii) The set $C \subset \mathcal{L}$ is $\mathcal{L}$-convex if and only if there is a $\mathcal{L}$-convex function $f : X \to \mathbb{R}_{+\infty}$ such that

$$C = S_{f^*}^{\leq}(0) := \{l \in \mathcal{L} : f^*(l) \leq 0\}. $$

(iii) Suppose in this part that $\mathcal{L}$ has linear structure (i.e. closed with respect to addition and multiplication by a scalar). If the set $C \subset \mathcal{L}$ is $\mathcal{L}$-convex, then $C$ is closed for convex combinations of its elements. Namely, for all $l_1, l_2 \in C$ and $\alpha \in [0,1]$, we have $\alpha l_1 + (1 - \alpha) l_2 \in C$.

**Proof.**  (i) See [19, Lemma 1.1].

(ii) Assume that $C \subset \mathcal{L}$ is a nonempty $\mathcal{L}$-convex set. We need to find a function $f$ such that $C = S_{f^*}^{\leq}(0)$. Define $f := \sup_{l \in C} l$. By definition, $f$ is $\mathcal{L}$-convex. We need to show that $C = S_{f^*}^{\leq}(0) = \{l \in \mathcal{L} : f^*(l) \leq 0\}$. Indeed, if there is an element $l_0 \notin C$, then by the definition of $\mathcal{L}$-convex set, there is an $x \in X$ such that

$$\begin{equation}
\begin{aligned}
& l_0(x) > \sup_{l \in C} l(x) = f(x),
& \tag{15} \{\text{P4.2P1}\}
\end{aligned}
\end{equation}$$

where we used the definition of $f$ in the equality. This implies that

$$f^*(l_0) = \sup_{x \in X} \{l_0(x) - f(x)\} > 0. $$

Thus, $l_0 \notin S_{f^*}^{\leq}(0)$. This shows that $S_{f^*}^{\leq}(0) \subset C$. For proving the opposite inclusion, take $l_0 \notin S_{f^*}^{\leq}(0)$, then there is an $x \in X$ satisfying (15). This implies $l_0 \notin C$. Altogether, $S_{f^*}^{\leq}(0) = C$. Conversely, assume that there is $f : X \to \mathbb{R}_{+\infty}$ an $\mathcal{L}$-convex function such that $C = S_{f^*}^{\leq}(0) = \{l \in \mathcal{L} : f^*(l) \leq 0\}$. We will show that $C$ is an $\mathcal{L}$-convex set. Indeed, for all $l_0 \notin S_{f^*}^{\leq}(0)$, we have $\sup_{x \in X} \{l_0(x) - f(x)\} > 0$. Then, there is an $x \in X$ such that $l_0(x) - f(x) > 0$, which is (15). By definition, this implies that $C$ is $\mathcal{L}$-convex.

(iii) Suppose $C$ be a $\mathcal{L}$-convex set. Take $l_1, l_2 \in C$ and $\alpha \in [0,1]$. We will show that $\alpha l_1 + (1 - \alpha) l_2 \in C$. Let $f := \sup_{l \in C} l$ which is $\mathcal{L}$-convex by construction. By (i), we have $C \subset \text{suppf}$. Hence, for all $x \in X$ we have

$$l_1(x) \leq f(x), \ 	ext{and} \ l_2(x) \leq f(x). $$

Then, it is clear that $\alpha l_1(x) + (1 - \alpha) l_2(x) \leq f(x)$. This implies $\alpha l_1 + (1 - \alpha) l_2 \in \text{suppf} = C$. \hfill \Box

Remark 13. The converse of Proposition [12] (iii) is not true. In the example provided in Section 5, the characterisations of $\mathcal{L}$-convex sets and $\mathcal{H}$-convex sets respectively in Propositions [13] and [14] show that not all the convex sets are $\mathcal{L}$-convex or $\mathcal{H}$-convex.

**Lemma 14.** [13, Lemma 6.1] Let $\mathcal{H}$ be a set of continuous functions defined on a metric space $X$ with the following properties:

(i) $\mathcal{H}$ is a conic set i.e. for all $\lambda > 0$, if $h \in \mathcal{H}$, then $\lambda h \in \mathcal{H}$.

(ii) For each triplet $(\varepsilon, z, U)$, where $\varepsilon > 0$, $z \in X$, and $U$ is a neighborhood of $z$, there exists a function $h \in \mathcal{H}$ such that

$$h(z) > 1 - \varepsilon, \ h(x) \leq 1 \ 	ext{for all} \ x \in U, \ h(x) \leq 0 \ 	ext{for all} \ x \notin U, \ 	ag{16} \{\text{Rub.1}\}$$

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then if a lower semicontinuous function \( f \) is strictly minored by \( \mathcal{H} \) (i.e. there is a \( \tilde{h} \in \mathcal{H} \) such that \( f \geq \tilde{h} \) and \( h + \tilde{h} \in \mathcal{H} \) for all \( h \in \mathcal{H} \), then \( f \) is \( \mathcal{H} \)-convex.

We provide some nontrivial examples for which the abstract linear functional space \( \mathcal{L} \) is not a conventional one.

**Example 15.** [19 Example 7.6] Let \( X \) be a Banach space, \( \mathcal{L} := \{a \|x - x_0\| : a \in \mathbb{R}, x_0 \in X\} \) a set of abstract linear functions, \( \mathcal{H} := \{a \|x - x_0\| + b : a, b \in \mathbb{R}, x_0 \in X\} \) a set of abstract affine functions. There is no linear structure on \( \mathcal{L} \) as it is not closed with respect to the addition operator.

Observe that \( \mathcal{H} \) is a conic set (see Lemma 14), and for each triplet \((\varepsilon, x_0, U)\) where \( \varepsilon > 0 \), \( x_0 \in X \), and \( U \) is a neighbourhood of \( x_0 \), there exists an abstract affine function \( h \in \mathcal{H} \) such that

\[
h(x_0) > 1 - \varepsilon, \quad h(x) \leq 1 \text{ for all } x \in U, \quad h(x) \leq 0 \text{ for all } x \notin U.
\]

Indeed, for any open neighbourhood \( U \) of \( x_0 \), there is a \( r > 0 \) that \( B_r(x_0) \subset U \). Choose \( a := -1/r \) and \( b = 1 \), we have \( h(x) := \frac{\varepsilon}{r} \|x - x_0\| + 1 \). It is easy to see that \( \max_{x \in X} h(x) = h(x_0) = 1 \), and \( h(x) \leq 0 \) for all \( x \notin B_r(x_0) \).

As a consequence, by Lemma 14 the set of all \( \mathcal{H} \)-convex functions contains the set of all lower semicontinuous functions minored by \( \mathcal{H} \). Furthermore, for all bounded below lower semicontinuous functions \( f \), \( \inf_X f \in \mathbb{R} \), we can always find an abstract affine function \( h(x) := -\|x - x_0\| + \inf_X f \), with \( x_0 \in X \), bounding the function \( f \) from below. \(\Box\)

### 3 Conditions for Zero Duality Gap

Let \( X \) and \( \mathcal{L} \) be as in Definition 1. Given \( m \) functions \( f_1, \ldots, f_m : X \to \mathbb{R}_+ \) \((m \geq 2)\), consider the minimization problem

\[
P := \inf \left( \sum_{i=1}^{m} f_i(x_i) \right),
\]

\(s.t. \quad x_1, \ldots, x_m \in X.\)

The dual problem of \((P)\) is given as follows:

\[
d := \sup \left( \sum_{i=1}^{m} -f_i^*(l_i) \right),
\]

\(s.t. \quad l_1, \ldots, l_m \in \mathcal{L}.\)

Problem \((P)\) is a very general minimization problem in which \( X \) is a general nonempty set, and there is no assumption on the convexity of the functions \( f_1, \ldots, f_m \).

Denote by \( v(P) \), \( v(D) \), the optimal values of \((P)\) and \((D)\), respectively. We say that a **zero duality gap holds** for problems \((P)\) and \((D)\) if \( v(P) = v(D) \).

The following characterizes the zero duality gap property for \((P)\) and \((D)\), using the infimal convolution of the conjugate functions \( f_i^* \).

\[
P = \inf \left( \sum_{i=1}^{m} f_i(x_i) \right) = - \left( \sum_{i=1}^{m} f_i \right)^*(0); \quad (P1) \quad (P1)
\]

\[
d = \sup \left( \sum_{i=1}^{m} -f_i^*(l_i) \right) = -(f_1^* \boxplus \ldots \boxplus f_m^*)(0). \quad (D1) \quad (D1)
\]

Thus, the zero duality gap is equivalent to
Theorem 16 below extends [1, Theorem 3.2] to our general framework. It characterizes the condition
\[
\left( \sum_{i=1}^{m} f_i \right)^* (0) = (f_1^* \square \ldots \square f_m^*)(0),
\]  
which clearly guarantees (17).  

**Theorem 16.** Let \( f_1, \ldots, f_m : X \to \mathbb{R}_{+}^{\infty} \) be such that \( \bigcap_{i=1}^{m} \text{dom} f_i \neq \emptyset \). The following statements are equivalent.

(i) There is a \( K > 0 \) such that, for any \( x \in X \) and any \( \varepsilon > 0 \),
\[
\partial_\varepsilon \left( \sum_{i=1}^{m} f_i \right)(x) \subseteq \sum_{i=1}^{m} \partial_{K\varepsilon} f_i(x).
\]  

(ii) \( \left( \sum_{i=1}^{m} f_i \right)^* = f_1^* \square \ldots \square f_m^* \) in \( \mathcal{L} \).

(iii) For every \( x \in \bigcap_{i=1}^{m} \text{dom} f_i \) and any \( \varepsilon \geq 0 \),
\[
\partial_\varepsilon \left( \sum_{i=1}^{m} f_i \right)(x) \supseteq \bigcap_{\eta > 0} \left[ \bigcup_{\varepsilon_i \geq 0, \sum_{i=1}^{m} \varepsilon_i = \varepsilon + \eta} \sum_{i=1}^{m} \partial_{\varepsilon_i} f_i(x) \right].
\]  

**Proof.** (i) ⇒ (ii) Let \( l \in \mathcal{L} \). By Proposition 3(ii), we have
\[
\left( \sum_{i=1}^{m} f_i \right)^*(l) \leq f_1^* \square \ldots \square f_m^*(l).
\]
Let is show the opposite inequality. It is enough to consider the case in which \( l \in \text{dom} \left( \sum_{i=1}^{m} f_i \right)^* \). By (iii) in Proposition 3 for any \( \varepsilon > 0 \), there is a \( x \in X \) that \( l \in \partial_\varepsilon \left( \sum_{i=1}^{m} f_i \right)(x) \). Using now (15), \( l \in \sum_{i=1}^{m} \partial_{\varepsilon_i} f_i(x) \), and there are \( \varepsilon_i \in \partial_{\varepsilon_i} f_i(x) \) \((i = 1, \ldots, m)\) such that \( l = l_1 + \ldots + l_m \). The following inequalities hold by Proposition 3(ii)
\[
f_i^*(l_i) + f_i(x) \leq l_i(x) + K\varepsilon, \quad \forall i = 1, \ldots, m.
\]
Adding up the inequalities above and using the definition of infimal convolution, we can write
\[
(f_1^* \square \ldots \square f_m^*)(l) \leq \sum_{i=1}^{m} f_i^*(l_i) \leq -\sum_{i=1}^{m} f_i(x) + \sum_{i=1}^{m} l_i(x) + mK\varepsilon
\]
\[
= -\sum_{i=1}^{m} f_i(x) + l(x) + mK\varepsilon \leq \left( \sum_{i=1}^{m} f_i \right)^*(l) + mK\varepsilon.
\]
Since the inequality above holds for all \( \varepsilon > 0 \), we deduce that \((f_1^* \square \ldots \square f_m^*)(l) \leq \left( \sum_{i=1}^{m} f_i \right)^*(l) \). Using (15), we obtain (ii). The proof of (i) ⇒ (ii) is complete.

(ii) ⇒ (iii). Let \( \varepsilon \) be a non-negative number, \( x \in \bigcap_{i=1}^{m} \text{dom} f_i \). The inclusion
\[
\partial_\varepsilon \left( \sum_{i=1}^{m} f_i \right)(x) \supseteq \bigcap_{\eta > 0} \left[ \bigcup_{\varepsilon_i \geq 0, \sum_{i=1}^{m} \varepsilon_i = \varepsilon + \eta} \sum_{i=1}^{m} \partial_{\varepsilon_i} f_i(x) \right]
\]
is shown in Proposition 3(v). Let us show the opposite inclusion.
Take \( l \in \partial_e \left( \sum_{i=1}^m f_i \right)(x) \). By Proposition 4 (i), we have that
\[
\left( \sum_{i=1}^m f_i \right)(x) + \left( \sum_{i=1}^m f_i \right)^*(l) \leq l(x) + \varepsilon. \tag{20} \{T3.3.P1\}
\]

Combine assumption (ii) with (20) to obtain, for all \( \eta > 0 \),
\[
(f^*_i \circ \ldots \circ f^*_m)(l) + \eta \leq l(x) - \sum_{i=1}^m f_i(x) + \varepsilon + \eta. \tag{21} \{T3.3.P2\}
\]

Inequality (21) shows that \((f^*_i \circ \ldots \circ f^*_m)(l) < +\infty\). Using the definition of infimal convolution (Definition 3 (i)), there exist \( l_1, \ldots, l_m \in \mathcal{L} \) such that \( l = l_1 + \ldots + l_m \) and
\[
f^*_i(l_1) + \ldots + f^*_m(l_m) \leq (f^*_i \circ \ldots \circ f^*_m)(l) + \eta.
\]

Combine the above inequality with (21) to deduce \( f^*_i(l_1) + \ldots + f^*_m(l_m) \leq l(x) - \sum_{i=1}^m f_i(x) + \varepsilon + \eta \). Equivalently,
\[
\sum_{i=1}^m (f_i(x) + f^*_i(l_i) - l_i(x)) \leq \varepsilon + \eta. \tag{22} \{T3.3.P3\}
\]

Set \( \gamma_i := f_i(x) + f^*_i(l_i) - l_i(x) \) \((i = 1, \ldots, m)\). We have that \( \gamma_i \geq 0 \) by definition of conjugate function. Moreover, from Proposition 4 (i) we have that \( l_i \in \partial \gamma_i f_i(x) \) for all \( i = 1, \ldots, m \). Due to (22), we obtain
\[
\sum_{i=1}^m \gamma_i = \sum_{i=1}^m (f_i(x) + f^*_i(l_i) - l_i(x)) \leq \varepsilon + \eta.
\]

Choose \( \varepsilon_i := \gamma_i + \varepsilon + \eta - \sum_{j=1}^m \gamma_j \) \((i = 1, \ldots, m)\). From the inequality above, we have that \( \varepsilon_i \geq \gamma_i \). Hence, \( l_i \in \partial \varepsilon_i f_i(x) \subset \partial \gamma_i f_i(x) \) for all \( i = 1, \ldots, m \), and \( \sum_{i=1}^m \varepsilon_i = \varepsilon + \eta \). Altogether,
\[
l \in \bigcup_{\varepsilon_1 + \ldots + \varepsilon_m = \varepsilon + \eta} \bigcup_{\sum_{i=1}^m \partial \varepsilon_i f_i(x), \ \forall \eta > 0, \ \varepsilon_i \geq 0}
\]

which yields (iii). The proof of (ii) \( \Rightarrow \) (iii) is complete.

(iii) \( \Rightarrow \) (i) We will show that (i) holds for \( K = 2 \). By (10), for all \( x \in \bigcap_{i=1}^m \text{dom } f_i(x) \) and \( \varepsilon > 0 \) we have
\[
\partial \varepsilon \left( \sum_{i=1}^m f_i \right)(x) = \bigcap_{\eta > 0} \left[ \bigcup_{\varepsilon_1 + \ldots + \varepsilon_m = \varepsilon + \eta} \bigcup_{\varepsilon_i \geq 0} \sum_{i=1}^m \partial \varepsilon_i f_i(x) \right] \subseteq \bigcap_{\eta > 0} \sum_{i=1}^m \partial \varepsilon f_i(x) \subseteq \bigcup_{\sum_{i=1}^m \partial \varepsilon f_i(x)} \bigcup_{i=1}^m \partial \varepsilon f_i(x),
\]

where we used the fact that \( \partial \varepsilon_i f_i(x) \subset \partial \varepsilon f_i(x) \) for all \( i = 1, \ldots, m \) in the first inclusion. The last inclusion is obtained by choosing \( \eta = \varepsilon \). With \( x \notin \bigcap_{i=1}^m \text{dom } f_i(x) \), we have \( \partial \varepsilon \left( \sum_{i=1}^m f_i \right)(x) = \emptyset \). Therefore, we have shown that (10) holds for all \( x \in X \). The proof of (iii) \( \Rightarrow \) (i) is complete.

\( \square \)
As mentioned above, Theorem 3.2 is an extension from the classical convex case to the framework of abstract convexity of the main result in [1, Theorem 3.2]. More precisely, the authors consider the case when \( \mathcal{L} = X^* \), the classical dual space of all continuous linear functions, and derive the constraint qualifications for zero duality gap of a convex optimization problem. We quote their main result for the convenience of comparison.

**Theorem 17.** [1, Theorem 3.2] Let \( X \) be a normed vector space, \( X^* \) its conjugate space with weak* topology, \( m \in \mathbb{N} \), and \( f_i : X \to \mathbb{R}_+^\infty \) be proper lower semicontinuous convex functions where \( i \in \{1, \ldots, m\} \). Then the following four conditions are equivalent

(i) There exists \( K > 0 \) such that for every \( x \in \bigcap_{i=1}^m \text{dom } f_i \), and every \( \varepsilon > 0 \),

\[
\text{cl} \left[ \sum_{i=1}^m \partial_i f_i(x) \right] \subset \bigcap_{i=1}^m \partial_{K\varepsilon} f_i(x).
\]  \( \{\text{T2.1}\} \)

(ii) \( (\sum_{i=1}^m f_i)^* = f_1^* \bigcap \ldots \bigcap f_m^* \) in \( X^* \).

(iii) \( f_1^* \bigcap \ldots \bigcap f_m^* \) is weak* lower semicontinuous.

(iv) For every \( x \in X \) and \( \varepsilon > 0 \),

\[
\partial_\varepsilon (f_1 + \ldots + f_m)(x) = \bigcap_{\eta > 0} \left[ \bigcup_{\varepsilon_i \geq 0, \sum_{i=1}^m \varepsilon_i = \varepsilon + \eta} (\partial_{\varepsilon_1} f_1(x) + \ldots + \partial_{\varepsilon_m} f_m(x)) \right].
\]  \( \{\text{T2.2}\} \)

In the standard linear functional space \( X^* \), condition (18) and condition (23) are equivalent (both being equivalent to statement (ii) in Theorem 16 and Theorem 17). We will show in the next proposition that (23) \( \Leftrightarrow \) (18) holds in any abstract linear functional space \( \mathcal{L} \) as long as the topology in \( \mathcal{L} \) possesses property (25) stated below. Recall the following known result for the conventional linear functional space \( X^* \) equipped with the weak* topology (see [25, Corollary 2.6.7]): for all lower semicontinuous convex functions \( f_1, \ldots, f_m \), with \( m \geq 2 \), \( x \in X \) and \( \varepsilon \geq 0 \), we have

\[
\partial_\varepsilon \left( \sum_{i=1}^m f_i \right)(x) = \bigcap_{\eta > 0} \left[ \bigcup_{\varepsilon_i \geq 0, \sum_{i=1}^m \varepsilon_i = \varepsilon + \eta} \left( \sum_{i=1}^m \partial_{\varepsilon_i} f_i(x) \right) \right].
\]  \( \{\text{F2}\} \)

This calculus rule is the key ingredient in the proof of the implication (i) \( \Rightarrow \) (ii) in [1, Theorem 3.2].

**Proposition 18.** Given a set \( X \), a space of abstract linear functionals \( \mathcal{L} \), to functions \( f_1, \ldots, f_m : X \to \mathbb{R}_+^\infty \) \( (m > 1) \), and \( x \in X \), for any topology defined on \( \mathcal{L} \), if the equality (23) holds, then the two conditions (18) and (23) are equivalent.

**Proof.** Observe that for any positive numbers \( \eta, \varepsilon \) with \( 0 < \eta \leq \varepsilon \), we have

\[
\bigcup_{\varepsilon_1 + \ldots + \varepsilon_m = \varepsilon + \eta, \varepsilon_i \geq 0} \sum_{i=1}^m \partial_{\varepsilon_i} f_i(x) \subset \sum_{i=1}^m \partial_{\varepsilon_i + \eta} f_i(x) \subset \sum_{i=1}^m \partial_{2\varepsilon_i} f_i(x).
\]  \( \{\text{P3.4.P1}\} \)

Assume (23) holds with \( K > 0 \). Using (25), (26), and (29), we obtain (18) as follows:

\[
\partial_\varepsilon \left( \sum_{i=1}^m f_i \right)(x) = \bigcap_{\eta > 0} \left[ \bigcup_{\varepsilon_i \geq 0, \varepsilon_1 + \ldots + \varepsilon_m = \varepsilon + \eta} \left( \sum_{i=1}^m \partial_{\varepsilon_i} f_i(x) \right) \right] \subset \bigcap_{\eta > 0} \left( \sum_{i=1}^m \partial_{2\varepsilon_i} f_i(x) \right) \subset \bigcap_{\eta > 0} \left( \sum_{i=1}^m \partial_{K\varepsilon} f_i(x) \right).
\]
Conversely, assume the condition \[18\] holds with \(K > 0\). Take in \[24\] the intersection of all \(\eta > 0\) in the left hand side. Use also \[24\] and \[18\], to deduce the following inclusions

\[
\text{cl} \left( \sum_{i=1}^{m} \partial_{i} f_{i}(x) \right) \subset \bigcap_{\eta > 0} \text{cl} \left( \bigcup_{\epsilon_{i} \geq 0, \epsilon_{1} + \ldots + \epsilon_{m} = m\epsilon/2 + \eta} \sum_{i=1}^{m} \partial_{i} f_{i}(x) \right)
\]

which gives \[24\] for \(K = \frac{mK}{2}\). \(\square\)

Remark 19. (i) From the proof above we see that, if \[24\] holds for \(K > 0\), then \[18\] holds for \(2K\). On the other hand, if \[18\] holds for \(K > 0\), then \[24\] holds for \(mK/2\).

(ii) When \(0 \in \mathcal{L}\), part (ii) in Theorem \[18\] below (or in Theorem \[17\] above) ensures the zero duality gap. In our Theorem \[16\] we drop the assumption that all the functions \(f_{1}, \ldots, f_{m}\) are lower semicontinuous convex as in Theorem \[17\] and refine the core argument in the proof in \[11\]. However, without the convexity, condition \[18\] might not be easily satisfied. When the functions \(f_{i}\) (\(i = 1, \ldots, m\)) are not convex, there will exist some \(x \in X\) and \(\epsilon > 0\) such that \(\partial_{i} f_{i}(x) = \emptyset\) (see Proposition \[15\] (ii)). In this situation, condition \[18\] will not hold.

The zero duality gap property is equivalent to equality \[17\], which is clearly less restrictive than condition (ii) in Theorem \[16\]. In the next theorem, we relax condition \[18\] as well as item (ii) in Theorem \[16\] to obtain a necessary and sufficient condition for the zero duality gap property. This characterization is new even in classical convex analysis.

**Theorem 20.** Suppose \(X\) is a set, \(\mathcal{L}\) is a space of abstract linear functionals with \(0 \in \mathcal{L}\), and \(f_{i} : X \to \mathbb{R}_{+\infty}\) are functions with \(\bigcap_{i=1}^{m} \text{dom} f_{i} \neq \emptyset\) where \(i \in \{1, 2, \ldots, m\}\) (\(m > 1\)). Then, the following conditions are equivalent.

(i) For all \(\epsilon > 0\), there exists a \(x \in X\) such that

\[
\partial_{i} f_{i}(x) + \ldots + \partial_{i} f_{m}(x) \ni 0.
\]

(ii) \(\big(\sum_{i=1}^{m} f_{i}\big)^{*}(0) = f_{1}^{\star} \square \ldots \square f_{m}^{\star}(0) < +\infty\).

**Proof**

(i) \(\Rightarrow\) (ii). Assume that statement (i) holds; that is for all \(\epsilon > 0\) there exists \(x \in X\) such that \(0 \in \partial_{i} \big(\sum_{i=1}^{m} f_{i}\big)(x)\). This allows us to use Proposition \[4\] (iii) and deduce that \(0 \in \text{dom} \big(\sum_{i=1}^{m} f_{i}\big)^{*}\), or, equivalently, that \(\big(\sum_{i=1}^{m} f_{i}\big)^{*}(0) < +\infty\). Due to Proposition \[4\] (ii), we only need to show that \(\big(\sum_{i=1}^{m} f_{i}\big)^{*}(0) \geq f_{1}^{\star} \square \ldots \square f_{m}^{\star}(0)\). By \[27\], there are \(l_{i} \in \partial_{i} f_{i}(x)\) (\(i = 1, \ldots, m\)) such that \(0 = l_{1} + \ldots + l_{m}\) and

\[
f_{i}^{\star}(l_{i}) + f_{i}(x) \leq l_{i}(x) + \epsilon, \quad \forall i = 1, \ldots, m.
\]

This implies

\[
\sum_{i=1}^{m} f_{i}^{\star}(l_{i}) + \sum_{i=1}^{m} f_{i}(x) \leq m\epsilon.
\]

We then have the following inequalities

\[
f_{1}^{\star} \square \ldots \square f_{m}^{\star}(0) \leq \sum_{i=1}^{m} f_{i}^{\star}(l_{i}) \leq m\epsilon - \sum_{i=1}^{m} f_{i}(x) \leq m\epsilon + \bigg(\sum_{i=1}^{m} f_{i}\bigg)^{*}(0),
\]

where we used the definition of conjugate function in the rightmost inequality. Letting \(\epsilon \downarrow 0\), we obtain (ii).

\[\{\text{Ab_con_R1}\}\]

\[\{\text{T3.5}\}\]

\[\{\text{T3.5.1}\}\]
(ii) ⇒ (i). Using the inequality in (ii), we have that \( 0 \in \text{dom } (\sum_{i=1}^{m} f_i)^* \). Take \( \varepsilon > 0 \).

By Proposition \( \text{T3.5.P1} \) (iii) we can find a \( x \in X \) such that

\[
0 \in \partial_k \left( \sum_{i=1}^{m} f_i \right) (x).
\]

Using the equality in (i) and Proposition \( \text{T3.5.P1} \) (i), we have

\[
f_i^* \bigcap \ldots \bigcap f_m^* (0) = \left( \sum_{i=1}^{m} f_i \right)^* \leq \varepsilon - \left( \sum_{i=1}^{m} f_i \right) (x).
\] \( \text{T3.5.P1} \)

By Definition \( \text{T3.5.P1} \) (i), there are \( l_1, \ldots, l_m \in \mathcal{L} \) such that \( l_1 + \ldots + l_m = 0 \) and

\[
f_i^* (l_i) + \ldots + f_m^* (l_m) \leq f_i^* \bigcap \ldots \bigcap f_m^* (0) - \varepsilon / 2.
\] \( \text{T5.5.P2} \)

Thus, we have the following estimation

\[
f_i^* (l_i) + \ldots + f_m^* (l_m) \leq f_i^* \bigcap \ldots \bigcap f_m^* (0) - \varepsilon / 2 \leq \varepsilon / 2 - \left( \sum_{i=1}^{m} f_i \right) (x).
\]

We derive

\[
\varepsilon / 2 \geq \sum_{i=1}^{m} (f_i^* (l_i) + f_i (x) - l_i(x)).
\] \( \text{T5.5.P3} \)

And we also have \( f_i (x) + f_i^* (l_i) - l_i(x) \geq 0 \) for all \( i = 1, \ldots, m \), then

\[
-(f_i^* (l_i) + f_i (x) - l_i(x)) + \varepsilon / 2 \geq \sum_{j=1, j \neq i}^{m} (f_j^* (l_j) + f_j (x) - l_j(x)) \geq 0, \quad \forall i = 1, \ldots, m.
\]

Using the inequality above, we can write

\[
f_i^* (l_i) + f_i (x) \leq l_i(x) + \varepsilon / 2, \quad \forall i = 1, \ldots, m,
\]

which by Proposition \( \text{T3.5.P1} \) (i) yields \( l_i \in \partial_{\varepsilon / 2} f_i(x) \) for all \( i = 1, \ldots, m \). Since \( \partial_{\varepsilon / 2} f_i(x) \subset \partial f_i(x) \) we deduce that

\[
0 = l_1 + \ldots + l_m \in \sum_{i=1}^{m} \partial f_i(x).
\]

Since \( n \in \mathbb{N} \) is arbitrary, we conclude that

\[
0 \in \bigcap_{n \in \mathbb{N}} \sum_{i=1}^{m} \partial f_i(x),
\]

which is (i).

We will show in the next theorem that if the inclusion \( \text{T3.5.P1} \) holds for a fixed \( x \in X \) for all \( \varepsilon > 0 \), or equivalently

\[
\bigcap_{\varepsilon > 0} (\partial f_1(x) + \ldots + \partial f_m(x)) \ni 0,
\] \( \text{T3.6.1} \)

then it characterizes a stronger property.
Theorem 21. Suppose $X$ is a set, $\mathcal{L}$ is a space of abstract linear functionals with $0 \in \mathcal{L}$, and $f_i : X \rightarrow \mathbb{R}_{+\infty}$ are functions with $\bigcap_{i=1}^{m} \text{dom} f_i \neq \emptyset$ where $i \in \{1, 2, \ldots, m\} \ (m \geq 1)$. Then the following conditions are equivalent.

(i) There exists a $x \in X$ such that $31$ holds.

(ii) $\left(\sum_{i=1}^{m} f_i^*(0) \right) = f_1^* \sqcup \ldots \sqcup f_m^*(0) < +\infty$. Additionally, the supremum (as in (ii)) is attained at $x \in X$.

Proof. $\ (i) \Rightarrow (ii)$. Suppose there is an $x \in X$ such that $31$ holds. By Theorem 20 we deduce the first statement in part (ii). Let us show the second statement in (ii). By $31$, for all $\varepsilon > 0$, there are $l_1, \ldots, l_m \in \mathcal{L}$ such that $l_1 + \ldots + l_m = 0$, and $l_i \in \partial_i f_i(x)$ for all $i = 1, \ldots, m$. We have

$$f_i^*(l_i) + f_i(x) \leq l_i(x) + \varepsilon, \quad \text{for all } i = 1, \ldots, m,$$

which implies

$$\sum_{i=1}^{m} f_i^*(l_i) + \sum_{i=1}^{m} f_i(x) \leq m\varepsilon.$$

Then, we have

$$f_1^* \sqcup \ldots \sqcup f_m^*(0) \leq \sum_{i=1}^{m} f_i^*(l_i) \leq -\left(\sum_{i=1}^{m} f_i^*(0)\right) + \left(\sum_{i=1}^{m} f_i^*(0)\right) \leq \sum_{i=1}^{m} f_i^*(0) + m\varepsilon.$$

Let $\varepsilon \downarrow 0$, we have equalities in the expression above. Hence, $f_1^* \sqcup \ldots \sqcup f_m^*(0) = \left(\sum_{i=1}^{m} f_i^*(0)\right)$ and

$$-\left(\sum_{i=1}^{m} f_i^*(0)\right) = \left(\sum_{i=1}^{m} f_i^*(0)\right),$$

this implies that the supremum in the expression of the conjugate function is attained at $x$. This establishes the second statement in (ii).

(ii) $\Rightarrow$ (i). Suppose there is an $x \in X$ such that

$$f_1^* \sqcup \ldots \sqcup f_m^*(0) = \left(\sum_{i=1}^{m} f_i^*(0)\right) \ (32) \ \{T3.6.P1\}$$

For every $\varepsilon > 0$, there are $l_1, \ldots, l_m \in \mathcal{L}$ such that $l_1 + \ldots + l_m = 0$ and

$$f_1^*(l_1) + \ldots + f_m^*(l_m) \leq f_1^* \sqcup \ldots \sqcup f_m^*(0) + \varepsilon \ \{T3.6.P2\} \ (33)$$

Combining $32$, and $33$, we obtain

$$f_1^*(l_1) + \ldots + f_m^*(l_m) \leq -\left(\sum_{i=1}^{m} f_i^*(0)\right) + \varepsilon \ \{T3.6.P3\} \ (34)$$

Since $f_1^*(l_j) + f_j(x) - l_j(x) \geq 0$ for all $j = 1, \ldots, m$, we have

$$f_1^*(l_j) + f_j(x) - l_j(x) \leq f_1^*(l_j) + f_j(x) - l_j(x) + \sum_{j=1, j \neq i}^{m} [f_j^*(l_j) + f_j(x) - l_j(x)]$$

$$\leq \sum_{j=1}^{m} (f_j^*(l_j)) + \sum_{j=1}^{m} f_j(x) - \sum_{j=1}^{m} l_j(x)$$

$$\leq \sum_{j=1}^{m} f_j(x) + \varepsilon, \quad \forall i = 1, \ldots, m.$$
Remark 22. The proof (ii) $\Rightarrow$ (i) in Theorem P4.11 is analogous to that of Theorem P4.20.

4 Abstract Convexity with Weak* Topology

In this section, we consider specifically the weak* topology $\mathcal{C}(\mathcal{L}, X)$ (see [2, Section 3.3]) on the abstract linear functional space $\mathcal{L}$. We expand some fundamental results of standard convex analysis in the framework of abstract convexity. Condition (23) is fully extended into the abstract convexity framework using means of the weak* topology.

4.1 Weak* Topology

Recall that $\mathcal{F}$ is the set of all real functions defined on $X$. On $\mathcal{F}$, the weak* topology $\mathcal{C}(\mathcal{F}, X)$ is the weakest topology that makes all the functions $x : \mathcal{F} \to \mathbb{R}, f \mapsto f(x)$ continuous. Equivalently, for any sequence $(f_n) \subset \mathcal{F}, f_n \to f \in \mathcal{F}$ if and only if $f_n(x) \to f(x)$ for all $x \in X$ (pointwise sequential convergence topology) (see [2, Proposition 3.1]). Recall that $\tau(\mathcal{F}, \mathcal{C}(\mathcal{F}, X))$ is a Hausdorff space (the proof of this fact is similar to the one in [2, Proposition 3.1]).

In the context of [2], the weak* topology is studied for the dual space $X^\ast$ of a normed linear vector space $X$. In order to study general abstract linear functional spaces $\mathcal{L}$, here, we generalize several fundamental results in [2] to the set of function $\mathcal{L}$ which satisfy certain conditions as below. Note that in our analysis, there is no topology required for $X$.

We define the scalar multiplication on $\mathcal{L}$ as usual: $(\alpha l)(x) = \alpha(l(x))$ for all $x \in X$. We will assume that $\mathcal{L}$ is a weak* closed subset of $\mathcal{F}$. More precisely, $\mathcal{L}$ satisfies the following conditions:

- A- closed with respect to addition and multiplication by a scalar: if $l, h \in \mathcal{L}, \lambda, \beta \in \mathbb{R}$, then $\lambda l + \beta h \in \mathcal{L}$.
- B- $\mathcal{L}$ is weak*-closed in $\mathcal{F}$.

It is clear that when condition (A) is satisfied, $\mathcal{L}$ contains the element $0_{\mathcal{F}} (0_{\mathcal{F}}(x) := 0$ for all $x \in X$).

From now on, without any further explanation, we consider the weak* topology inherited by $\mathcal{L}$. Observe that the evaluation functions $x : \mathcal{L} \to \mathbb{R}, l \mapsto l(x), x \in X$ are linear in the conventional sense. Thus, the functions $|x| : \mathcal{L} \to \mathbb{R}, |x|(l) = |l(x)|$ are seminorms. In the next proposition, we show that the space $\mathcal{L}$ is a locally convex topological vector space.

Proposition 23. Let $X$ be a set, $\mathcal{L}$ a space of abstract linear functionals equipped with the weak* topology, $l_0 \in \mathcal{L}$, $\varepsilon > 0$, and a set of vectors $\{x_1, \ldots, x_n\}$ in $X$. Then the set

$$V(x_1, \ldots, x_n|\varepsilon) = \{l \in \mathcal{L} : |l(x_i) - l_0(x_i)| < \varepsilon, \quad \forall i = 1, \ldots, n\}$$

is a neighbourhood of $l_0$ in $\mathcal{L}$. Moreover, the collection of all $V(x_1, \ldots, x_n|\varepsilon), n \in \mathbb{N}, x_1, \ldots, x_n \in X$ and $\varepsilon > 0$ forms a basis of neighbourhoods of $l_0$ in $\mathcal{L}$.

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Thus

(i) We show that \( L \)

Since the functions \( x_i \) \((i = 1, \ldots, n)\) are continuous, we have \( V_i \) \((i = 1, \ldots, n)\) are open. Thus \( V = \bigcap_{i=1}^{n} V_i \) is open, or \( V \) is a neighbourhood of 0.

On the other hand, by the definition of the weak* topology, the basis of neighborhoods of 0 is a collection of all finite intersections of sets of the form \( x^{-1}(U) \), where \( x \in X \) and \( U \) is a neighborhood of \( l_0(x) \) in \( \mathbb{R} \). Thus, the assertion holds.

\[ \square \]

**Theorem 24.** Suppose \( X \) is a set, and \( \mathcal{L} \) is a space of abstract linear functionals equipped with the weak* topology. Assume further that conditions (A), (B) hold, then \( \mathcal{L} \) is a locally convex topological vector space.

**Proof.** Note that all the functions \( |x| : \mathcal{L} \to \mathbb{R} \), \( |x|:l := |l(x)| \) with \( x \in X \), are semi-norms, and by Proposition E4.3 the weak* topology in \( \mathcal{L} \) is the coarsest topology that guarantees the continuity of \( |x| \), with \( x \in X \). Hence, \( \mathcal{L} \) is a locally convex topological vector space. Calling that a vector space is called locally convex topological vector space along with a family of semi-norms \( |x| \) \((x \in X)\) if and only if its topology is the coarsest topology for which all the semi-norms \( |x| \) \((x \in X)\) are continuous.

\[ \square \]

**Example 25.** [12 Example 2.1] Let \( \mathcal{L} \) be a set of functions defined on the Euclidean space \( \mathbb{R}^n \), comprising all the functions \( 1 := \sum_{i=0}^{n} a_i h_i \in \mathcal{L} \) where \( a_i \in \mathbb{R}, i = 1, \ldots, n \) and \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \)

\[ h_0(x) = \|x\|^2, \ h_1(x) = x_1, \ldots, h_n(x) = x_n. \]

The space \( \mathcal{L} \) possesses the following properties.

**Proposition 26.** (i) \( \mathcal{L} \) is closed in \( \mathcal{F} \) with respect to weak* topology.

(ii) If the sequence (or net) \((h_j)_{j \in I}, h_i := a_0^j h_0 + \ldots + a_n^j h_n \to h = a_0^0 h_0 + \ldots + a_n^0 h_n, \)

then \( a_i^j \to a_i^0 \) for all \( j = 0, 1, \ldots, n \).

(iii) The space \( \mathcal{L} \) equipped with weak* topology is homeomorphic to \( \mathbb{R}^{n+1} \) with the standard Euclidean norm.

**Proof.** We show that \( \mathcal{L} \) is closed in \( \mathcal{F} \). Let \((l_i)_{i \in I} \) be a net taken in \( \mathcal{L} \) such that \( l_i \to l \in \mathcal{F} \) with \( l_i := a_0^i h_0 + \ldots + a_n^i h_n \). We have \( l_i(1, 0, \ldots, 0) + l_i(-1, 0, \ldots, 0) = 2a_0^i \) \((l_0(x), 0) = 2a_0^0, \)

or \( a_i^0 \to a_i \). At the same time, \( l_i(x) - a_0^i \|x\|^2 \to l(x) - a_0^0 \|x\|^2 \) for all \( x \in X \) and \( l_i - a_0^0 \|x\|^2 \) is linear function for all \( i \in I \)

(by definition of \( \mathcal{L} \)). Thus, \( l - a_0^0 \|x\|^2 \) is also a linear function. Hence, \( l \in \mathcal{L} \).

(ii) It is shown in (i) that if \( l_i \to l \), then \( a_0^i \to a_0 \) and \( (l_i - a_0^i \|x\|^2) \to (l - a_0^0 \|x\|^2) \). That means \( a_j^i \to a_j \) for all \( j = 0, 1, \ldots, n \).

(iii) Consider the bijective mapping \( \varphi : \mathcal{L} \to \mathbb{R}^n \) with \( \varphi(l) := (a_0, a_1, \ldots, a_n) \), where \( l = \sum_{i=0}^{n} a_i h_i \).

In the view of Proposition E4.3 (ii), we have \( \varphi \) is continuous. On the other hand, we also have if \( (a_0^i, a_1^i, \ldots, a_n^i)_{i \in I} \to (a_0^0, a_1^0, \ldots, a_n^0) \), then \( (\varphi^{-1}(a_0^i, a_1^i, \ldots, a_n^i))_{i \in I} = \sum_{j=0}^{n} a_j^i h_j \to \sum_{j=0}^{n} a_j^0 h_j = \varphi^{-1}(a_0^0, a_1^0, \ldots, a_n^0) \). Hence, \( \varphi^{-1} \) is continuous.

\[ \square \]

We can treat \( \mathcal{L} \) as the vector space \( \mathbb{R}^{n+1} \) with the basis \( h_0, h_1, \ldots, h_n \). The weak* topology is equivalent to the Euclidean norm in \( \mathbb{R}^{n+1} \).

In the conventional dual space \( X^* \) of \( X \), the compactness of the unit ball \( \mathbb{B}^* \) is of utmost importance. Here we establish a generalization of the Banach-Alaoglu-Bourbaki (see Corollary E4.3) to real functional space \( \mathcal{L} \).

**Theorem 27.** Suppose \( X \) is a nonempty set (possibly linear vector space), \( \mathcal{F} \) is the set of all real functions acting from \( X \) to \( \mathbb{R} \), and \( F : \mathcal{F} \setminus \mathbb{R} \) are any real functions defined on \( X \). Let \( K := \{ f^* \in \mathcal{F} : G(x) \leq f^*(x) \leq F(x), \forall x \in X \} \) be a subset of \( \mathcal{F} \). Then, \( K \) is a weak* compact set.

\[ \square \]
Proof. Recall that the set \( Y := \mathbb{R}^X \) consists of all maps from \( X \) to \( \mathbb{R} \). Consider \( Y \) equipped with the product topology. Let \( \phi : \mathcal{F} \to Y \) be defined as \( \phi(f) := (f(x))_{x \in X} \). It is well known that \( \phi \) is a homeomorphism from \( \mathcal{F} \) to \( \phi(\mathcal{F}) = Y \) (w.r.t. the product topology). We will show that the set \( H := \phi(K) = \{ y \in Y : G(x) \leq y(x) \leq F(x) \forall x \in X \} \) is a compact set. Indeed, we have
\[
H = \prod_{x \in X} [G(x), F(x)],
\]
a product of compact sets. By Tychonoff’s Theorem, \( H \) is compact, and so is \( K \). \( \square \)

**Corollary 28.** Suppose \( X \) is a linear vector space, \( \mathcal{F} \) is the set of all real functions from \( X \) to \( \mathbb{R} \) equipped with the weak* topology, and \( A \) is a weak* closed subset of \( \mathcal{F} \). Then \( A \) is compact if the functions \( F := \sup_{f \in A} f^* \) and \( G := \sup_{f \in A} (-f^*) \) are real functions i.e. \( F(x), G(x) \in \mathbb{R} \) for all \( x \in X \).

Proof. Observe that \( G = -\inf_{f \in A} f^* \). Then we have
\[
A = \{ f^* \in \mathcal{F} : -G(x) \leq f^*(x) \leq F(x), \forall x \in X \} \cap A.
\]
By Theorem 27 and the weak* closedness of \( A \), we have \( A \) is a weak* compact set. \( \square \)

**Corollary 29** (Banach–Alaoglu–Bourbaki). [3, Theorem 3.16] Suppose \( X \) is a normed vector space, and \( X^* \) is its dual space. Then \( B^* := \{ x^* \in L : |\langle x^*, x \rangle | \leq \| x \| \ \forall x \in X \} \) is a weak* compact set.

Proof. The assertion straightforwardly follows from Corollary 28. \( \square \)

### 4.2 Sum Rule Calculus for Subdifferentials

In this subsection, we improve one of the main results [12, Theorem 3.2]. In [12], the authors studied the extended sum rule for abstract convex functions using the additivity property of the epigraph of the conjugate functions. The additivity condition on the sets \( \text{epi} f^* \) and \( \text{epi} g^* \) is stated as follows:
\[
\text{epi} f^* + \text{epi} g^* = \text{epi} (f + g)^*.
\]
However, this additivity condition may not be valid even for \( f, g \) convex functions in the classical sense, because it entails weak*–closedness of the set \( \text{epi} f^* + \text{epi} g^* \). It is well known that sum of closed sets is not closed in general. Condition \( \text{(35)} \) is even stronger than condition \( (f + g)^* = f^* \cup g^* \) (cf. [12, Corollary 5.1]).

Our aim is to sharpen condition \( \text{(35)} \) by the following condition
\[
\text{cl}^* (\text{epi} f^* + \text{epi} g^*) = \text{epi} (f + g)^*.
\]
where \( \text{cl}^* A \) denotes the weak*–closure of the set \( A \). The weak* topology defined on \( \mathcal{L} \times \mathbb{R} \) is the product topology of the topology \( \tau(\mathcal{L}, \mathcal{C}(\mathcal{L}, X)) \) and the standard topology on \( \mathbb{R} \). Since the set on the right-hand side of \( \text{(35)} \) is weak*-closed, the additivity condition \( \text{(35)} \) readily implies \( \text{(36)} \). Moreover, condition \( \text{(36)} \) is more convenient because (as shown in [28] and [12, Corollary 3.1]) it holds for standard convex functions. We will use the less restrictive condition \( \text{(36)} \) in the next section. Namely, it will allow us to apply Theorem 17 for establishing new zero duality gap characterizations.

**Theorem 30.** [12, Theorem 3.2] Let \( X \) be a set and let \( \mathcal{L} \) be a set of abstract linear functions defined on \( X \). Let \( f, g : X \to \mathbb{R}_{+\infty} \) be \( \mathcal{H} \)-convex functions such that \( \text{dom} f \cap \text{dom} g \neq \emptyset \). Then, equality \( \text{(35)} \) holds if and only if for any \( \varepsilon \geq 0 \),
\[
\partial_\varepsilon (f + g)(x) = \bigcup_{\varepsilon_1 \varepsilon_2 = \varepsilon, \varepsilon_1, \varepsilon_2 \geq 0} \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} g(x), \quad x \in \text{dom} f \cap \text{dom} g.
\]

We present in the next theorem our main result of this subsection.
Theorem 31. Let $X$ be a nonempty set, and $\mathcal{L}$ an abstract linear functional space which is weak* closed in $\mathcal{F}$ and $f_1, \ldots, f_m : X \to \mathbb{R}_\infty$ be functions defined on $X$ with $\cap_{i=1}^m \text{dom } f_i \neq \emptyset$. Assume that (38) holds. Then for any number $\varepsilon \geq 0$, we have $\{x \in \cap_{i=1}^m \text{dom } f_i | \text{ condition (35) holds.} \}$.

Proof. Assume that (38) holds. Take $\varepsilon \geq 0$. For all $x \in X$, due to Proposition 4 (ii), we have
\[
\partial_{\varepsilon} \left( \sum_{i=1}^m f_i \right)(x) \supset \bigcap_{\eta > 0} \text{cl}^* \left( \bigcup_{\varepsilon_i, \varepsilon_{i+1}, \ldots, \varepsilon_m = \varepsilon + \eta} \sum_{i=1}^m \partial_{\varepsilon_i} f_i(x) \right).
\]

Now we prove the converse inclusion. Let $l \in \partial_{\varepsilon} \left( \sum_{i=1}^m f_i \right)(x)$. Then, by Proposition 4 (ii), we have
\[
\left( l, l(x) + \varepsilon - \sum_{i=1}^m f_i(x) \right) \in \text{epi} \left( \sum_{i=1}^m f_i \right)^* \cap \text{cl}^* \left( \sum_{i=1}^m \text{epi } f_i^* \right).
\]

There are nets $(l_{1,i}, l_{1,i}, \ldots, (l_{1,i}, l_{1,i}, l_{1,i}), l_{1,i}) \in I$ with $I$ is a directed set such that
\[
(l_{j,i}, \lambda_{j,i}) \in \text{epi } f_j^*, \quad j \in \{1, \ldots, m\}, i \in I,
\]
\[
\sum_{j=1}^m l_{j,i} \xrightarrow{epi} l, \quad \sum_{j=1}^m \lambda_{j,i} \xrightarrow{l} l(x) + \varepsilon - \sum_{i=1}^m f_i(x).
\]

Take $\eta > 0$. By (40), there is $i_0 \in I$ such that for all $i \geq i_0$ we have
\[
\sum_{j=1}^m \lambda_{j,i} \leq \sum_{j=1}^m l_{j,i}(x) + (\varepsilon + \eta) - \sum_{i=1}^m f_i(x)
\]

Thus, $l \in \text{cl}^* \left( \bigcup_{\varepsilon_i, \varepsilon_{i+1}, \ldots, \varepsilon_m = \varepsilon + \eta} \sum_{i=1}^m \partial_{\varepsilon_i} f_i(x) \right)$ for all $\eta > 0$. Hence, we proved (29). \[\square\]

Remark 32. If the condition (29) holds, and additionally $\sum_{i=1}^m \text{epi } f_i^*$ is weak* closed, then condition (38) holds.

Proposition 33. Suppose $X$ is a vector space, $\mathcal{L}$ is its abstract functional space. Then,
(i) for any $x \in X$, the function $x(\cdot) : \mathcal{L} \to \mathbb{R}_\infty$, $l \mapsto l(x)$ is weak* continuous;
(ii) for every function $f : X \to \mathbb{R}_\infty$, its conjugate function $f^*$ is weak* lower semicontinuous.

Proof. (i) Take $l \in \mathcal{L}$. For all $(l_i)_{i \in I} \subset \mathcal{L}$ ($I$ is a directed set) such that $l_i \to l$ w.r.t weak* topology, we have
\[
\lim x(l_i) = \lim l_i(x) = l(x) = x(l).
\]

Thus, $x(\cdot)$ is weak* continuous.
(ii) Take $f : X \to \mathbb{R}_{+\infty}$, and consider $f^*(l) = \sup_{x \in X} \{l(x) - f(x)\}$ for all $l \in \mathcal{L}$. Then, the epigraph of $f^*$ is

\[ \text{epi} f^* := \{(l, \lambda) : f^*(l) \leq \lambda \} = \bigcap_{x \in X} \left\{ (l, \lambda) : (l(x) - f(x)) \leq \lambda \right\} = \bigcap_{x \in X} \text{epi} [x(\cdot) - f(\cdot)] . \]

Since the function $[x(\cdot) + f(x)] : \mathcal{L} \to \mathbb{R}_{+\infty}$ is weak* continuous, the set $\text{epi} [x(\cdot) - f(\cdot)]$ is weak* closed in $\mathcal{L} \times \mathbb{R}$, and so is $\text{epi} f^*$. This implies the lower semicontinuity of the function $f^*$.

\[ \square \]  

\{P4.6\}

**Proposition 34.** Let $X$ be a set, $\mathcal{L}$, $\mathcal{H}$ be spaces of abstract linear functionals and abstract affine functionals, respectively, equipped with the weak* topology. If the set $A \subset \mathcal{H}$ is an $\mathcal{H}$-convex subset, then $A$ is weak* closed in $\mathcal{H}$.

**Proof.** Let $f : X \to \mathbb{R}_{+\infty}$ be a $\mathcal{H}$-convex function such that $\text{supp} f = A$. Consider the set $B := \text{cl}^* A \subset \mathcal{H}$ and the function $g(x) := \sup_{h \in B} h(x)$. We claim that it is sufficient to show that $g \equiv f$. Indeed, if this holds, then for all $h \in B$ we have $h \leq g = f$, and thus $h \in A$, or $A = \text{cl}^* A$. Observe that $g$ is also an $\mathcal{H}$-convex function and by construction $g(x) \geq f(x)$ (due to $B = \text{cl}^* A \supset A$). Thus, we proceed to establish the claim that $g \equiv f$. Take $x \in X$, we need to show that $f(x) \geq g(x)$. Indeed, for all $h \in B$, there exists a net $(h_i)_{i \in I} \subset A$, with $I$ a directed set, such that $h_i \to h$ w.r.t. $w^*$ convergence and

\[ f(x) \geq h_i(x) \to h(x). \]

Taking the supremum in the right hand side for all $h \in B$, we conclude that $f(x) \geq g(x)$. Since this holds for $x$ arbitrary, the proof is complete.

\[ \square \]

4.3 Zero Duality Gap with Weak* Topology

We next present our main theorem of this section.

**Theorem 35.** Let $X$ be a normed vector space, $\mathcal{L}$ a space of abstract linear functionals with weak* topology, $m \in \mathbb{N}$, and $f_i : X \to \mathbb{R}_{+\infty}$ where $i \in \{1, \ldots, m\}$. Then consider the following five conditions

(i) There exists $K > 0$ such that for every $x \in \bigcap_{i=1}^m \text{dom} f_i$, and every $\varepsilon > 0$, \[13\] holds;

(ii) There exists $K > 0$ such that for every $x \in \bigcap_{i=1}^m \text{dom} f_i$, and every $\varepsilon > 0$, \[23\] holds with respect to weak* topology;

(iii) \[ \sum_{i=1}^m f_i^* = f_1^* \square \ldots \square f_m^* \text{ in } X^*; \]

(iv) $f_1^* \square \ldots \square f_m^*$ is weak* lower semicontinuous;

(v) For every $x \in X$ and $\varepsilon \geq 0$, \[23\] holds.

We have (i) $\iff$ (ii) $\iff$ (v) $\iff$ (iv) $\iff$ (ii) .

Additionally, if the sum rule \[23\] (or condition \[13\]) holds, then all five statements are equivalent.

**Proof.** All the implications (except (ii) $\iff$ (iii)) are clear due to Theorem \[10\], Theorem \[17\], Proposition \[18\] and Proposition \[33\].

(i) $\iff$ (iii) $\iff$ (v) is proved in Theorem \[10\].

(ii) $\Rightarrow$ (iv) is trivial based on the fact that $(\sum_{i=1}^m f_i)^*$ is weak* lower semicontinuous (see Proposition \[33\]).

(iv) $\Rightarrow$ (ii). The proof proceeds exactly the same as the proof of (iii) $\Rightarrow$ (i) in \[1\] Theorem 3.2.

If the sum rule \[23\] holds, then by Proposition \[18\] (i) $\iff$ (ii).

\[ \square \]

A similar series of intriguing corollaries as in in \[1\] can be deduced with more or less identical proofs to those in \[1\].
5 Example

In this section, we consider a nontrivial example, where the weak* closed additivity condition holds for any $H$-convex functions. Thus, the important sum rule also holds. Here, we use the notations different to previous sections.

Consider $X := \mathbb{R}$, denote $\phi_a(t) := at^2$ for some $a \in \mathbb{R}$ and $\Psi_{a,b}(t) = at^2 + b$ for some $a,b \in \mathbb{R}$. Let the set $\mathcal{L} := \{\phi_a : a \in \mathbb{R}\}$ be the space of abstract linear functions, and $H = \{\Psi_{a,b} : a,b \in \mathbb{R}\}$ the space of abstract affine functions. The weak* topology on $\mathcal{L}$ and $H$ are isomorphism to the standard Euclidean norm on $\mathbb{R}$ and $\mathbb{R}^2$ respectively.

We next characterize $\mathcal{L}$-convex sets, and $H$-convex sets for this example.

**Proposition 36.** A set $C \subset \mathcal{L}$-convex if and only if
\[ C = \{\phi_a : a \leq A\} \]
with $A$ is a number in $\mathbb{R}$.

*Proof.* The set of the form $C := \{\phi_a : a \leq A\}$, $(A \in \mathbb{R})$ is a $\mathcal{L}$-convex set, since $C$ is the support set of the function $\phi_A$ in $\mathcal{L}$.

Conversely, suppose $C$ is a $\mathcal{L}$-convex set and let $A := \sup\{a : \phi_a \in C\}$. We have $C \subset \{\phi_a : a \leq A\}$, by the definition of $A$. On the other hand, since for every $a \leq A$, $\phi_a(t) \leq \phi_A(t)$ for all $t \in \mathbb{R}$, then $C \supset \{\phi_a : a \leq A\}$. 

**Proposition 37.** A set $C$ is $H$-convex, $C \neq H$ if and only if the following properties hold
(i) $C$ is weak* closed, convex, and upper bounded (i.e. there are numbers $A,B \in \mathbb{R}$ that are upper bounds of $a$ and $b$ where $\Psi_{a,b} \in C$).
(ii) if $\Psi_{a,b} \in C$, then $\Psi_{a',b'} \in C$ for all $a' \leq a, b' \leq b$.

We first recall the standard strict convex separation theorem.

**Theorem 38.** [Theorem 1.7] Let $X$ be a normed vector space, $A \subset X$ and $B \subset X$ be two nonempty convex subsets such that $A \cap B = \emptyset$. Assume that $A$ is closed and $B$ is compact. Then there exists a closed hyperplane that strictly separates $A$ and $B$ i.e. there is $x^* \in X^* \setminus \{0\}$ such that
\[ \sup_{x \in A} \langle x^*, x \rangle < \inf_{y \in B} \langle x^*, y \rangle. \]

*Proof.* Suppose $C$ is a $H$-convex set and let $f := \sup_{h \in C} h$. In views of Proposition 12(iii) and Proposition $\Psi_{a,b} \in C$ is weak* closed and convex. Let
\[ A := \sup\{a : \Psi_{a,b} \in C, \text{for some } b \in \mathbb{R}\}, \quad (41) \]
\[ B := \sup\{b : \Psi_{a,b} \in C, \text{for some } a \in \mathbb{R}\}. \quad (42) \]
We must have $A,B \in \mathbb{R}$, otherwise $A = +\infty$ or $B = +\infty$, then $f \equiv +\infty$, which implies $C \equiv H$, a contradiction to our assumption.

For all $\Psi_{a,b} \in C$, and any numbers $a' \leq a, b' \leq b$, we have $\Psi_{a',b'}(t) \leq \Psi_{a,b}(t) \leq f(t)$ for all $t \in \mathbb{R}$. Thus, $\Psi_{a',b'} \in \text{supp} f = C$.

One way implication is clear.

Conversely, suppose the set $C$ satisfies conditions (i) and (ii). Let $\Psi_{a,b} \notin C$ and $A,B$ are defined as in (11), (12). Due to (i), $A,B \in \mathbb{R}$.

**Case 1.** $\bar{a} > A$.

Observe that $\Psi_{A,B}(t) \geq \sup_{h \in C} h(t)$ for all $t \in \mathbb{R}$. Since $\bar{a} > A$, then for some $t$ with the absolute value $|t|$ sufficiently large, we always have $(\bar{a} - A)t^2 > B - b$. Thus, $\Psi_{a,b}(t) > \sup_{h \in C} h(t)$.

**Case 2.** $\bar{a} \leq A$. Then $\bar{b} > B$. 

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Consider the set $C^2 := \{(a, b) : \Psi_{a, b} \in C\}$ in $\mathbb{R}^2$. Since $C$ is a weak* closed convex set and $\Psi_{a, b} \notin C$, then $C^2$ is a closed convex set in $\mathbb{R}^2$ and $(\bar{a}, \bar{b}) \notin C^2$. By the strict convex separation theorem (Theorem 38), there is $(x^*, y^*) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ such that

$$\bar{a}x^* + \bar{b}y^* > \sup_{(a, b) \in C^2} ax^* + by^*. \quad (43) \\{E5.3.1\}$$

We consider three cases.

(i) If $y^* < 0$, then due to property (ii), for every $\Psi_{a, b} \in C$ we can fix parameter $a$ and let $b \to -\infty$. Then,

$$\sup_{(a, b) \in C^2} ax^* + by^* \to -\infty + \infty,$$

which is a contradiction to (43).

(ii) If $x^* = 0$, then $x^* \neq 0$.

If $x^* > 0$, then due to our assumption we have $Ax^* \geq \bar{a}x^*$, which contradicts to (43). $$(\bar{a}x^*) > \sup_{(a, b) \in C^2}(ax^*) = Ax^*.$$

If $x^* < 0$, then due to property (ii) and with an analogous argument with case (i), we have

$$\sup_{(a, b) \in C^2} ax^* \to -\infty + \infty.$$ Altogether, it yields a contradiction.

(iii) If $y^* > 0$, then we can divide both sides of (43) by $y^*$ to obtain

$$\frac{\bar{a}x^*}{y^*} + \frac{\bar{b}}{y^*} > \sup_{(a, b) \in C^2} \frac{ax^*}{y^*} + b.$$

Arguing as in (ii), we claim that $\frac{x^*}{y^*} \geq 0$. Let $t := \sqrt{\frac{x^*}{y^*}}$ we have

$$\bar{a}t^2 + \bar{b} > \sup_{(a, b) \in C^2} at^2 + b, \quad \text{or} \quad \Psi_{\bar{a}, \bar{b}}(t) > \sup_{h \in C} h(t).$$

Combining case 1 and case 2, we conclude that $C$ is a $H$–convex set.

It suffices to establish that condition (43) is satisfied by proving that if $C_1, C_2$ are $H$–convex sets, then $\text{cl}^*(C_1 + C_2)$ is $H$–convex. This assertion will be explained in the next theorem.

**Theorem 39.** Let $H$ be a space of affine functions (defined in Example 3) and $f_1, f_2$ be $H$–convex functions, we have $(\text{supp} f_1 + \text{supp} f_2)^{w^*}$ is a $H$–convex set. Consequently,

$$\text{cl}^* (\text{epi} f^* + \text{epi} g^*) = \text{epi} (f + g)^*.$$ 

**Proof.** Since the set $\text{cl}^* (\text{supp} f + \text{supp} g)$ satisfies the two conditions in Proposition 87, then it is a $H$–convex set. Since $f + g = \sup_{x \in \text{cl}^* (\text{supp} f + \text{supp} g)} \Psi$, then $f + g$ is a $H$–convex function with $\text{supp}(f + g) = (\text{supp} f + \text{supp} g)^{w^*}$. This implies

$$\text{cl}^* (\text{epi} f^* + \text{epi} g^*) = \text{epi} (f + g)^*.$$ 

Indeed, since $\text{epi} (f + g)^* = \{(\phi_a, b) : \Psi_{a, -b} \in \text{supp}(f + g)\}$, then for all $(\phi_a, b) \in \text{epi} (f + g)^*$, there are nets $(\Psi_{a_i, b_i})_{i \in I}$ and $(\Psi_{a'_i, b'_i})_{i \in I}$ such that

$$\Psi_{a_i, b_i} \in \text{supp} f, \quad \Psi_{a'_i, b'_i} \in \text{supp} g, \quad \forall i \in I,$n

$$\Psi_{a_i, b_i} + \Psi_{a'_i, b'_i} \overset{w^*}{\to} \Psi_{a, -b}. \quad (44) \\{EX1_con\}$$

Condition (44) is equivalent to $(a_i + a'_i, b_i + b'_i) \to (a, -b)$ in the standard Euclidean space $\mathbb{R}^2$. Thus, $a_i + a'_i \to a$ and $b_i + b'_i \to -b$. This implies $(\phi_a, b) \in \text{cl}^* (\text{epi} f^* + \text{epi} g^*)$. 

\[\square\]
We next consider some particular functions defined on $X$

\begin{align*}
f_1(x) &:= x^4 - x^2; \quad \text{(45) \{P2-f_1\}} \\
f_2(x) &:= 1 - 2|x|; \quad \text{(46) \{P2-f_2\}} \\
f_3(x) &:= \begin{cases} 1 - 2|x| & -0.5 \leq x \leq 0.5, \\ 0 & \text{otherwise.} \end{cases} \quad \text{(47)}
\end{align*}

The minimization problem

\[ p := \min (f_1 + f_2 + f_3), \]

has the dual problem

\[ d := \sup (-f_1^* - f_2^* - f_3^*). \]

We will show that the duality gap holds for problems $(p)$ and $(d)$.

**Proposition 40.** The functions $f_1, f_2, f_3$ are $\mathcal{H}$-convex. The support sets of $f_1, f_2, f_3$ are

\begin{align*}
\text{supp } f_1 &= \left\{ \psi_{a,b} : b \leq -\frac{(1+a)^2}{4} \right\} \bigcup \left\{ \psi_{a,b} : a \leq -1, b \leq 0 \right\}, \quad \text{(48) \{E4.1\}} \\
\text{supp } f_2 &= \left\{ \psi_{a,b} : a \leq 0, b \leq \min_{t \in \mathbb{R}} \{1 - 2|t| - at^2\} \right\}, \quad \text{(49) \{E4.2.1\}} \\
\text{supp } f_3 &= \left\{ \psi_{a,b} : a \leq 0, b \leq \min_{t \in [-0.5, 0.5]} \{1 - 2|t| - at^2\} \right\}. \quad \text{(50) \{E4.3.1\}}
\end{align*}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figs/supp.png}
\caption{Figures 1: supp$_{f_1}$, 2: supp$_{f_2}$, 3: supp$_{f_3}$}
\end{figure}

**Proof.** (i) Take $\psi_{a,b} \in \mathcal{H}$ with $b \leq -\frac{(1+a)^2}{4}$. We have

\[ f_1(t) = t^4 - t^2 \geq \psi_{a,b}(t) = at^2 + b \iff \left( t^2 - \frac{1 + a}{2} \right)^2 \geq b + \frac{(1 + a)^2}{4}, \quad \forall t \in \mathbb{R}. \]

Observe that the inequality on the right hand side holds for all $t \in \mathbb{R}$ and $b \leq \frac{(1+a)^2}{4}$.

This implies $\left\{ \psi_{a,b} : b \leq -\frac{(1+a)^2}{4} \right\} \subset \text{supp } f_1$.

Take $\psi_{a,b} \in \mathcal{H}$ with $a \leq -1, b \leq 0$, we also have

\[ f_1(t) = t^4 + (-a - 1)t^2 \geq b, \quad \forall t \in \mathbb{R}, \]

which implies $\left\{ \psi_{a,b} : a \leq -1, b \leq 0 \right\} \subset \text{supp } f_1$.

Altogether, it yields the inclusion

\[ \left\{ \psi_{a,b} : b \leq -\frac{(1+a)^2}{4} \right\} \bigcup \left\{ \psi_{a,b} : a \leq -1, b \leq 0 \right\} \subset \text{supp } f_1. \]

Take $\psi_{a,b} \in \text{supp } f_1$ and assume that

\[ \psi_{a,b} \notin \left\{ \psi_{a,b} : b \leq -\frac{(1+a)^2}{4} \right\} \bigcup \left\{ \psi_{a,b} : a \leq -1, b \leq 0 \right\}. \]
This implies $b + \frac{(1 + a)^2}{4} > 0$ and $\sqrt{b + \frac{(1 + a)^2}{4}} + \frac{a + b}{2} > 0$.

Then, for all $t \in \left( -\sqrt{b + \frac{(1 + a)^2}{4}} + \frac{a + b}{2}, \sqrt{b + \frac{(1 + a)^2}{4}} + \frac{a + b}{2} \right)$, we have

$$t^4 - (a + 1)t^2 - b < 0,$$

which yields $t^4 - t^2 < at^2 + b$, contradicting the assumption $\Psi_{a,b} \in \text{supp}_{f_1}$. Hence,

$$\text{supp}_{f_1} = \left\{ \Psi_{a,b} : b \leq -\frac{(1 + a)^2}{4} \right\} \cup \left\{ \Psi_{a,b} : a \leq -1, b \leq 0 \right\}.$$

It remains to show that

$$f_1 = \sup_{\Psi \in \text{supp}_{f_1}} \Psi.$$

For all $t \in \mathbb{R}$, we choose $a = 2t^2 - 1$, $b = -t^4$. Then, $b = -\frac{(a + 1)^2}{4}$. By (50), $\Psi_{a,b} \in \text{supp}_{f_1}$, and

$$f_1(t) = x^4 - t^2 = at^2 + b = \Psi_{a,b}(t).$$

(ii) The proof of (50) for $f_2$ can be proceeded similar to one in (iii).

(iii) We prove $f_3$ is a $\mathcal{H}$-convex function, and (50) holds. For all $\Psi_{a,b} \in \mathcal{H}$ with $a \leq 0$, and $b \leq \min_{t \in [-0.5,0.5]} \{1 - 2|t| - at^2\}$, we have

$$b \leq 1 - 2|t| - at^2, \quad \forall t \in [-0.5,0.5].$$

This implies $\Psi_{a,b}(t) = at^2 + b \leq 1 - 2|t| = f_3(t)$ for all $t \in [-0.5,0.5]$.

On the other hand, we also have

$$\Psi_{a,b}(0.5) = \Psi_{a,b}(-0.5) \leq f_3(0.5) = f_3(-0.5) = 0.$$ 

Since $a \leq 0$, the function $\Psi_{a,b}$ increases on $(-\infty,0)$, and decreases on $(0,\infty)$. Hence,

$$\Psi_{a,b}(t) \leq \Psi_{a,b}(-0.5) \leq 0 = f_3(t), \quad \forall t \in (-\infty,-0.5),$$

$$\Psi_{a,b}(t) \leq \Psi_{a,b}(0.5) \leq 0 = f_3(t), \quad \forall t \in (0.5,\infty).$$

Altogether, it yields one implication

$$\text{supp}_{f_3} \supset \left\{ \Psi_{a,b} : a \leq 0, b \leq \min_{t \in [-0.5,0.5]} \{1 - 2|t| - at^2\} \right\}.$$

Conversely, take $\Psi_{a,b} \in \text{supp}_{f_3}$.

We must have $a \leq 0$, since otherwise $\lim_{n \to \infty} \Psi_{a,b}(t) = +\infty$, which contradicts $\Psi_{a,b}(t) \leq f_3(t)$ for all $t \in \mathbb{R}$.

If $b > \min_{t \in [-0.5,0.5]} \{1 - 2|t| - at^2\}$, then there is an $t_0 \in [-0.5,0.5]$ such that

$$b > 1 - 2|t_0| - at_0^2 = \min_{t \in [-0.5,0.5]} \{1 - 2|t| - at^2\}.$$ 

Thus, $\Psi_{a,b}(t_0) = at_0^2 + b > 1 - 2|t_0| = f_3(t_0)$, a contradiction. Hence, $b \leq \min_{t \in [-0.5,0.5]} \{1 - 2|t| - at^2\}$. Thus,

$$\text{supp}_{f_3} \subset \left\{ \Psi_{a,b} : a \leq 0, b \leq \min_{t \in [-0.5,0.5]} \{1 - 2|t| - at^2\} \right\}.$$

The equality (50) holds.

Now, we prove $f_3$ is $\mathcal{H}$-convex by showing that

$$f_3(x) = \sup_{\Psi \in \text{supp}_{f_3}} \Psi(x), \quad x \in \mathbb{R}. \quad (51) \{\text{supp}_{f_3}\}$$

Obviously, $f_3(x) \geq \sup_{\Psi \in \text{supp}_{f_3}} \Psi(x), \forall x \in \mathbb{R}$. We aim to show in the follows that $f_3(x) \leq \sup_{\Psi \in \text{supp}_{f_3}} \Psi(x), \forall x \in \mathbb{R}$. 

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1. For \( x \not\in (-0.5, 0.5), \) consider \( \Psi \equiv 0 \in \text{supp} f_3. \) We have \( f_3(x) = \Psi(x) = 0. \)

1.2 For \( x \in (-0.5, 0.5) \setminus \{0\}, \) consider \( \Psi_{a,b} \in \text{supp} f_3 \) with \( a = \frac{1}{|x|} \leq 0 \) and

\[
b = \min_{t \in (-0.5, 0.5)} \{ 1 - 2 |t| - at^2 \} = \min_{t \in (-0.5, 0.5)} \left\{ 1 - 2 |t| + \frac{1}{|x|} t^2 \right\} = 1 - |x|.
\]

Then, \( \Psi_{a,b}(x) = ax^2 + b = 1 - 2 |x| = f_3(x). \)

1.3 For \( x = 0, \) \( f_3(0) = 1. \) There is a sequence of affine functions \( (\Psi_{a_n,b_n}) \subset \text{supp} f_3, \) with \( a_n = -\frac{1}{n} \) and \( b_n = 1 - \frac{1}{n}, \) \( n \geq 2. \) Observe that \( a_n < 0 \) and

\[
b_n = \min_{t \in (-0.5, 0.5)} \left\{ 1 - 2 |t| - \frac{1}{n} t^2 \right\}, \quad \forall n \geq 2.
\]

Furthermore, we also have

\[
f_3(0) = 1 \geq \sup_{\Psi \in \text{supp} f_3} \Psi(0) \geq \lim_{n \to \infty} \Psi_n(0) = \lim_{n \to \infty} \left( 1 - \frac{1}{n} \right) = 1 = f_3(0).
\]

 Altogether, we have \( H^\infty \) holds, hence, \( f_3 \) is \( H^\infty \)-convex.

\[\Box\]

**Proposition 41.** The subdifferential operators of the functions \( f_1, f_2, f_3 \) are

\[
\partial f_1(x) = \begin{cases} \{ \phi_a : a \leq -1 \} & x = 0, \\ \{ \phi_a : a = 2x^2 - 1 \} & \text{otherwise}; \end{cases} \tag{52} \]

\[
\partial f_2(x) = \begin{cases} \emptyset & x = 0, \\ \{ \phi_{\frac{1}{|x|}} \} & \text{otherwise}; \end{cases} \tag{53} \]

\[
\partial f_3(x) = \begin{cases} \emptyset & x = 0, \\ \{ \phi_{\frac{1}{|x|}} \} & x \in (-0.5, 0.5) \setminus \{0\}, \\ \{ \phi_a : a \in [-2,0] \} & x = \pm0.5, \\ \{0\} & \text{otherwise}. \end{cases} \tag{54} \]

**Proof.** (i) Take \( x \in \mathbb{R}. \) The linear function \( \phi_a \in \partial f_1(x) \) if and only if

\[
\phi_a + (f_1(x) - ax^2) \in \text{supp} f_1.
\]

By \( [15], \) \( f_1(x) - ax^2 \leq \frac{1 + ax^2}{4} \) or \( a \leq -1 \) and \( f_1(x) - ax^2 \leq 0. \) If \( x = 0, \) then \( a \leq -1. \) If \( x \neq 0, \) then \( x^4 - (a + 1)x^2 \leq \frac{1 + ax^2}{4}, \) equivalently \( (x^2 - \frac{1}{2}x)^2 \leq 0, \) hence \( a = 2x^2 - 1. \)

(ii) The arguments to \( [20] \) for \( f_2 \) are similar to \( [21] \) for \( f_3 \) as below.

(iii) Consider the Fenchel conjugate function

\[
f_3^*(\phi_a) = \sup_{x \in \mathbb{R}} \{ \phi_a(x) - f_3(x) \} = \begin{cases} +\infty & a > 0, \\ \frac{4}{a} & a \in [-2,0], \\ -1 - \frac{4}{a} & a < -2. \end{cases}
\]

Then,

\[
\phi_a \in \partial f_3(x) \iff f_3^*(\phi_a) + f_3(x) = \phi_a(x).
\]

We consider three separate cases.

1. \( x = 0, \) \( f_3(x) = 1. \) Then, \( \phi_a \in \partial f_3(0) \iff f_3^*(\phi_a) + 1 = 0. \) There does not exist \( a \in \mathbb{R} \) such that \( f_3^*(\phi_a) + 1 = 0, \) thus \( \partial f_3(0) = \emptyset. \)
2. $x \in (-0.5, -0.5) \setminus \{0\}$, $f_3(x) = 1 - 2|x|$. Then, $\phi_\alpha \in \partial f_3(x) \iff f_3^*(\phi_\alpha) + 1 - 2|x| = \phi_\alpha(x)$. Observe that we must have $a \leq 0$.
If $a \in [-2, 0]$, then $\frac{1}{2} + 1 - 2|x| = \phi_\alpha(x) = ax^2$, or $a = \frac{1}{1 + 2|x|} > 0$, a contradiction.
If $a < -2$, then

$$-1 - \frac{1}{a} + 1 - 2|x| = ax^2,$$ 

equivalently $(ax)^2 + 2a|x| + 1 = 0$.

Hence, $a = \frac{1}{1 + 2|x|}$, and $\phi_\alpha(t) = \frac{1}{1 + 2|x|}$.

3. $x \notin (-0.5, 0.5)$, $f_3(x) = 0$. Then, $\phi_\alpha \in \partial f_3(x) \iff f_3^*(\phi_\alpha) = ax^2$.
If $a < -2$, then $-1 - \frac{1}{a} = ax^2$, or $-a - 1 = (ax)^2 \geq (0.5a)^2$, which yields $(0.5a + 1)^2 \leq 0$, a contradiction.
If $a \in [-2, 0]$, then $\frac{1}{2} = ax^2$, which only holds true when $a = 0$ or $x = \pm 0.5$.
Hence, $a \in [0, -2]$ if $x = \pm 0.5$, and $a = 0$ otherwise.

□

Proposition 42. The zero duality gap of $(p) : f_1 + f_2 + f_3$ and $(d) : f_1^* + f_2^* + f_3^*$ holds.

Proof. Observe that

$$0 \in \partial f_1(1) + \partial f_2(1) + \partial f_3(1), \quad \text{and} \quad 0 \in \partial f_1(-1) + \partial f_2(-1) + \partial f_3(-1).$$

By Theorem [21] $(f_1 + f_2 + f_3)^*(0) = f_1^* \square f_2^* \square f_3^*(0) < +\infty$ and the supremum attained at $\pm 1$. The sum $f_1 + f_2 + f_3$ attains its global minimum at $\pm 1$.

□

References

[1] J. Borwein, R. S. Burachik, and L. Yao. Conditions for zero duality gap in convex programming. J. Non. Con. Anal., 15(1):167–190, 2014.
[2] H. Brezis. Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, USA, 2011.
[3] R. S. Burachik and A. M. Rubinov. On global optimality conditions via separation functions. J. Optim. Theory Appl., 18.
[4] R. S. Burachik and A. M. Rubinov. On abstract convexity and set valued analysis. J. Non. Con. Anal., 9(1), 2008.
[5] M.H. Daryaei and H. Mohebi. Abstract convexity of extended real-valued icr functions. Optimization, 62:835–855, 2013.
[6] A. R. Doagooei and H. Mohebi. Monotonic analysis over ordered topological vector spaces: Iv. J. Glob. Optim., 45.
[7] J. Dutta, J.E. Martínez-Legaz, and A. M. Rubinov. Monotonic analysis over cones: Iii. Convex Anal., 15.
[8] J. Dutta, J.E. Martínez-Legaz, and A. M. Rubinov. Monotonic analysis over cones: I. Optimization, 53:165–177, 2004.
[9] J. Dutta, J.E. Martínez-Legaz, and A. M. Rubinov. Monotonic analysis over cones: Ii. Optimization, 53:529–547, 2004.
[10] A. C. Eberhard and H. Mohebi. Abstract convexity, global optimization and data classification. Set-Valued Anal, 18:79–108, 2010.
[11] A.D. Ioffe. Abstract convexity and non-smooth analysis. Adv. Math. Econ, 3.
[12] J. Jeyakumar, A. M. Rubinov, and W. Zhiyou. Generalized fenchels conjugation formulas and duality for abstract convex functions. J. Optim. Theory Appl., 132:441–458, 2007.

26
[13] J. E. Martnez-Legaz. Quasiconvex duality theory by generalized conjugation methods. *Optimization*, 19:1029–49, 1988.

[14] H. Mohebi and M. Samet. Abstract convexity of topical functions. *J. Glob. Optim.*, 58:365–357, 2014.

[15] A. Nedić, A. Ozdaglar, and A. M. Rubinov. Abstract convexity for nonconvex optimization duality. *Optimization*, 56:655–674, 2007.

[16] A. M. Rubinov. Abstract convexity, global optimization and data classification. *OPSEARCH*, 38(3):247–265, 2001.

[17] A. M. Rubinov and A. Uderzo. On global optimality conditions via separation functions. *J. Optim. Theory Appl.*, 109:345–370, 2001.

[18] A. M. Rubinov and Z. Y. Wu. Optimality conditions in global optimization and their applications. *Math. Program., Ser. B*, 120:101–123, 2009.

[19] A. M. Rubinov. *Abstract Convexity and Global Optimization*. Kluwer Academic Publishers, Dordrecht, 2000.

[20] A. M. Rubinov and Z. Dzalilov. Abstract convexity of positively homogeneous functions. *Journal of Statistics and Management Systems*, 5:1–20, 2002.

[21] A. P. Shveidel. Abstract convex sets with respect to the class of general min-type functions. *Optimization*, 52:571–579, 2003.

[22] I. Sing. *Abstract convex analysis*. Wiley-Interscience, New York, 2006.

[23] A. S. Strekalovsky. Global optimality conditions and exact penalization. *Optim. Lett.*, Nov 2017.

[24] C. Yao and S. Li. Vector topical function, abstract convexity and image space analysis. *J. Optim. Theory Appl.*, 177:717–742, 2018.

[25] C. Zalinescu. *Convex Analysis in General Vector Spaces*. World scientific, 2002.