Pure-state density matrix that competently describes classical chaos

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We work with reference to a well-known semiclassical model, in which quantum degrees of freedom interact with classical ones. We show that, in the classical limit, it is possible to represent classical results (e.g., classical chaos) by means of a pure-state density matrix.

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INTRODUCTION

The classical-quantum transition and the classical limit are certainly frontier issues that constitute an important physics topic [1–3]. On the other hand, the use of semiclassical systems to describe problems in physics has a long historical heritage [4–6]. A particularly important case is to be highlighted, in which quantum effects in one of the two components of a composite system are negligible in comparison to those in the other. Regarding this scenario as classical simplifies the description and provides deep insight into the combined system dynamics [7–11]. This methodology is widely used for the interaction of matter with a field. In this effort we will look at these matters through a well-known semi-quantum model [14–16]. This model has been investigated in great detail from a purely dynamic viewpoint [13–15] and also using statistical quantifiers derived from Information Theory (IT) [16–18]. For this model and in [20], a suitable density matrix was found for describing the system’s route on its way to the classical limit. Rather exhaustive numerical results were presented.

The purpose of this work is to analytically determine what happens with the above mentioned pure-state density matrices in the exact classical limit. Same interesting insight will ensue.

MODEL

We will consider a Hamiltonian $\hat{H}$ containing classical degrees of freedom (DOF) interacting with strictly quantum DOFs. The dynamical equations for the quantum operators will be the canonical ones [14,15], i.e., any operator $\hat{O}$ evolves (in the Heisenberg picture) as

$$\frac{d\hat{O}}{dt} = -i\hbar \{H, \hat{O}\}. \quad (1)$$

The concomitant evolution equation for its mean value $\langle O \rangle \equiv \text{Tr} [\rho \hat{O}(t)]$ will be $\frac{d\langle O \rangle}{dt} = -i\hbar \langle [H, \hat{O}] \rangle$, where the average is taken with respect to a proper quantum density operator $\rho$. Additionally, the classical variables will obey classical Hamiltonian equations of motion, where the generator is the mean value of the Hamiltonian, i.e.,

$$\frac{dA}{dt} = -\frac{\partial \langle H \rangle}{\partial A}, \quad (2a)$$
$$\frac{dP_A}{dt} = -\frac{\partial \langle H \rangle}{\partial P_A}. \quad (2b)$$

The above equations constitute an autonomous set of coupled first-order ordinary differential equations (ODE). Solving it allows for a dynamical description in which no quantum rules are violated, i.e., the commutation relations are trivially conserved for all times. $A$ plays the role of a time-dependent parameter for the quantum system, and the initial conditions are determined by a proper quantum density operator $\rho$.

We consider now a system representing the zero-th mode contribution of a strong external field to the production of charged meson pairs [12,13], whose Hamiltonian is

$$\hat{H} = \frac{1}{2} \left( \frac{\hat{p}^2}{m_q} + \frac{P_A^2}{m_{cl}} + m_q \omega^2 \hat{x}^2 \right). \quad (3)$$

where $\hat{x}$ and $\hat{p}$ are quantum operators, while $A$ and $P_A$ are classical canonical conjugate variables. The term $\omega^2 = \omega_q^2 + \epsilon^2 A^2$ is an interaction one introducing non-linearity in our problem, with $\omega_q$ a frequency. $m_q$ and $m_{cl}$ are quantum and classical masses, respectively. The Hamiltonian [3] is a particular case of a family of semi-classical ones, quadratic in $\hat{x}$ and $\hat{p}$, without linear terms (see below). This family has as a time-invariant a quantity $I$ that relates to the Uncertainty Principle [13] as

$$I = \langle \hat{x}^2 \rangle \langle \hat{p}^2 \rangle - \frac{(\langle \hat{L} \rangle)^2}{4} \geq \frac{\hbar^2}{4}. \quad (4)$$

$I$ describes the deviation of the semiquantum system from the classical one given by $I = 0$. The quantity $\hat{L}$ is defined as $\hat{L} = \hat{x} \hat{p} + \hat{p} \hat{x}$. To investigate the classical limit one needs also to consider the classical counterpart of the Hamiltonian [3], in which all variables are classical. In this case $\hat{L}$ is replaced by $L = 2xp$. We analyze in this work the limit $I \to 0$. A well known ODE-theorem establishes uniqueness and a continuous dependence of the ODE-solutions on the initial conditions, if a condition called the Lipschitz one is fulfilled [21]. If the ODE solutions remain bounded as time grows towards infinity, the condition is always satisfied.
Consider semiquantum systems (SS) governed by operators that close a partial Lie algebra with the Hamiltonian. These SS’ dynamics will be ruled by closed systems of equations (CSE), involving also the classical variables. These CSE will depend in continuous fashion on the initial conditions. For instance, this happens with the set \( \hat{x}^2, \hat{p}^2, \hat{L} \) for quadratic (in \( \hat{x} \) and \( \hat{p} \)) Hamiltonians \[13\]. This fact guarantees the existence of the limit \( I \to 0 \) \[13\]. If the Hamiltonian includes lineal terms in \( \hat{x} \) and \( \hat{p} \), \( I \) is no longer a constant of the motion. In this case one uses of \[I\] uses of \( I \) by \( \langle \hat{O} \rangle = \frac{1}{\sqrt{\text{det} |L|}} \text{Tr} \left[ \hat{L}^{-1} \hat{O} \right] \), which is a time-invariant property can be encountered in the celebrated article \[23\], we have \[ \text{det} |L| = \text{det} |\text{det} |L| \hat{O} \left[ \hat{O} \right] \text{det} |L| = 1 \]. Consider semiquantum systems (SS) governed by operators that close a partial Lie algebra with the Hamiltonian \[13\]. These SS’ dynamics will be ruled by closed systems of equations (CSE), involving also the classical variables. These CSE will depend in continuous fashion on the initial conditions. For instance, this happens with the set \( \hat{x}^2, \hat{p}^2, \hat{L} \) for quadratic (in \( \hat{x} \) and \( \hat{p} \)) Hamiltonians \[13\]. This fact guarantees the existence of the limit \( I \to 0 \) \[13\]. If the Hamiltonian includes lineal terms in \( \hat{x} \) and \( \hat{p} \), \( I \) is no longer a constant of the motion. In this case one uses of \( I \) uses of \( I \) by (5). On the other hand, the statistical operator must evolve in time from \[5\] according to the Liouville-von Neumann equation

\[
\frac{i\hbar}{\partial t}(\hat{\rho}) = [\hat{H}, \hat{\rho}] .
\]

As the operators \( \hat{O}_i \) close a partial Lie algebra with respect to the Hamiltonian \( \hat{H} \) \[22, 23\], we have

\[
[\hat{H}(t), \hat{O}_i] = i\hbar \sum_{j=1}^{3} g_{ji}(t)\hat{O}_j , \quad i = 0, 1, \ldots, 3 ,
\]

the statistical operator depends on the time \( t \) according to \[23\]

\[
\hat{\rho}(t) = \exp - \left( \lambda_0 \hat{I} + \lambda_1(t)\hat{x}^2 + \lambda_2(t)\hat{p}^2 + \lambda_3(t)\hat{L} \right).
\]

provided that the Lagrange multipliers \( \lambda_j(t) \) verify the set of differential equations \[23\]

\[
\frac{d\lambda_i}{dt}(t) = \sum_{j=1}^{3} g_{ij}\lambda_j(t) , \quad i = 1, 2, 3 ,
\]

with \( \lambda_j(0) = \lambda_j \) from \[1\]. The demonstration of this property can be encountered in the celebrated article \[23\] and is based on the uniqueness of the solutions of the Liouville Equation and the MaxEnt principle, together with the conservation of the Entropy

\[
S(\hat{\rho}) = -\text{Tr} \left[ \hat{\rho} \ln \hat{\rho} \right] = \lambda_0 + \sum_{i=1}^{3} \lambda_i \langle \hat{O}_i \rangle ,
\]

(Boltzmann’s constant is set equal to unity), which is maximized by the statistical operator \[11\]. From now on we will use the fact that \( \lambda_j(t) = \lambda_j \) to simplify the notation. In this way, Eqs. \[4\]–\[5\] are valid for all \( t \). Additionally, once \( \hat{\rho}(t) \) is obtained, we can determine (in the Schrödinger picture), the temporal evolution of the EV of any operator \( \hat{O} \)

\[
\langle \hat{O} \rangle(t) = \text{Tr}[\hat{\rho}(t)\hat{O}].
\]

Note that in this type of semiclassical problem, the \( g_{ij} \) of Eqs. \[10\] and \[12\] depend on the classical variables \( A \) and \( P_A \). We use equation \[13\] (with \( \hat{O} = \hat{H} \)) in order to obtain \( \langle \hat{H} \rangle \) and thus describe, via Eqs. \[2\], the temporal evolution of \( A \) and \( P_A \). The idea is then to regard the set of equations \[12\], together with the equations \[2\], as a single autonomous first-order system. Note that the classical equations in turn depend on the mean values. In this case the presence of the term \( \langle \hat{x}^2 \rangle \) in the equation for \( P_A \) introduces an additional non-linearity (as such a term is a function of the multipliers) through

\[
\langle \hat{x}^2 \rangle(t) = \text{Tr}[\hat{\rho}(t)\hat{x}^2],
\]

but we will presently see that this non-linearity can be easily handled.
SOME CONVENIENT MATHEMATICAL RESULTS

It is necessary to calculate \( \lambda_0 \) to relate the initial values of the multipliers and their respective EV’s, using Eq. \( \text{[7]} \). We begin by performing a change of representation, made by recourse to the unitary transformation \( \text{[20]} \)

\[
\begin{align*}
\dot{X} &= \frac{\sqrt{2}}{2} \left( \frac{\lambda_2}{\lambda_1} \right)^{1/4} \left( \frac{\lambda_T}{\lambda_V} \right)^{1/4} X + \left( \frac{\lambda_V}{\lambda_T} \right)^{1/4} \dot{P} \text{ (16a)} \\
\dot{P} &= \frac{\sqrt{2}}{2} \left( \frac{\lambda_1}{\lambda_2} \right)^{1/4} \left( \frac{\lambda_T}{\lambda_V} \right)^{1/4} X + \left( \frac{\lambda_V}{\lambda_T} \right)^{1/4} \dot{P} \text{ (16b)}
\end{align*}
\]

where \( \lambda_V = \sqrt{\lambda_1 \lambda_2} + \lambda_3 \) and \( \lambda_T = \sqrt{\lambda_1 \lambda_2} - \lambda_3 \). For reasons of convergence, \( \lambda_1, \lambda_2, \) and \( \lambda_1 \lambda_2 - \lambda_3^2 \) must be positive. Then, \( \lambda_V \) and \( \lambda_T \) become positive too and \( I_{\lambda} \) in \( \text{[18]} \) is well defined. Of course, the transformation \( \text{[10]} \) preserves commutation relations. Thus, \( I \) is also preserved. These new operators are not dimensionless ones [they are expressed in units of the square root of an action and do not depend on \( \hbar \), which is a convenient fact at the time of going over to the classical limit]. Further, \( X \) and \( P \), via the \( \lambda \)'s that appear as coefficients in their definition, are explicitly time-dependent and contain all the relevant information regarding the classical degrees of freedom. Now \( \rho(t) \) becomes \( \text{[20]} \)

\[
\dot{\rho}(t) = \exp(-\lambda_0 t) \exp \left[ -I_{\lambda} \left( X^2 + P^2 \right) \right].
\]

The quantity \( I_{\lambda} \) defined as

\[
I_{\lambda} = \left( \lambda_1 \lambda_2 - \lambda_3^2 \right)^{1/2},
\]

a constant of the motion \( \text{[21]} \). This invariant is the equivalent of the one in Eq. \( \text{[4]} \), expressed in terms of the \( \lambda \)'s. Despite the characteristics assigned to \( X \) and \( P \), the operator \( X^2 + P^2 \) has a discrete spectrum, one resembling that of the Harmonic Oscillator, because the commutation relations are preserved for all time. After a little algebra, it is easy to see from \( \text{[3]} \) that

\[
\lambda_0 = -\ln \left[ \exp(h I_{\lambda}) - \exp(-h I_{\lambda}) \right],
\]

and using Eq. \( \text{[7]} \) (or Eq. \( \text{[11]} \)), the particular EV’s can be cast in the fashion \( \text{[20]} \)

\[
\begin{align*}
\langle \dot{X}^2 \rangle &= \frac{T(I_{\lambda})}{I_{\lambda}} \lambda_2, \quad \text{(20a)} \\
\langle \dot{P}^2 \rangle &= \frac{T(I_{\lambda})}{I_{\lambda}} \lambda_1, \quad \text{(20b)} \\
\langle \dot{L} \rangle &= -2 \frac{T(I_{\lambda})}{I_{\lambda}} \lambda_3, \quad \text{(20c)}
\end{align*}
\]

with \( T(I_{\lambda}) \) given by \( \text{[20]} \)

\[
T(I_{\lambda}) = \frac{\hbar}{2} \left( \frac{\exp(2 \hbar I_{\lambda}) + 1}{\exp(2 \hbar I_{\lambda}) - 1} \right).
\]

Further, we deduce from \( \text{[20]} \) that

\[
T(I_{\lambda}) = \sqrt{T},
\]

Now, by recourse to the Eqs. \( \text{[2]}, \text{[12]}, \) and \( \text{[20A]} \), we are in position to write down our dynamical system of equations as a closed one in both multipliers and classical variables. We have \( \text{[20]} \)

\[
\begin{align*}
\frac{d\lambda_1}{dt} &= 2m_q \omega^2 \lambda_3, \quad \text{(23a)} \\
\frac{d\lambda_2}{dt} &= -\frac{2}{m_q} \lambda_3, \quad \text{(23b)} \\
\frac{d\lambda_3}{dt} &= -\frac{1}{m_q} \lambda_1 + m_q \omega^2 \lambda_2, \quad \text{(23c)} \\
\frac{dA}{dt} &= P_A, \quad \text{(23d)} \\
\frac{dP_A}{dt} &= -e^2 m_q A T(I_{\lambda}) \lambda_2. \quad \text{(23e)}
\end{align*}
\]

This system associates a kind of phase-space to the density operator \( \text{[11]} \), determined by classical variables and Lagrange multipliers. The system \( \text{[23]} \) depends in nonlinear fashion upon the classical variable \( A \), via \( \omega^2 \), but the non-linear term \( T(I_{\lambda}) \) in \( \text{[23e]} \) is easily tractable as a function of \( I \), using \( \text{[24]} \). This non-linearity is thus replaced by a dependence upon \( I \) plus the initial conditions. This last dependence emerges via the invariant \( I_{\lambda} \) (which in turn is fixed by \( \dot{\rho}(0) \), i.e. by the initial values of the Lagrange multipliers).

USEFUL PREVIOUS RESULTS

In \( \text{[24]} \), we investigated the dynamics described by the density operator \( \text{[11]} \) as a function of the relative energy \( E_r \), defined as \( E_r = \frac{|E|}{m_q \omega^2} \). The classical limit obtains for \( E_r \to \infty \) (a particular case is \( I \to 0 \), which we will study below).

In \( \text{[24]} \) we also showed that, by augmenting \( E_r \) (for example decreasing \( I \)), the physical system passes through three regions: a quasiclassical one, a transitional one, and a classical one. As \( E_r \) grows, complexity augments and, eventually, chaos emerges. This is a phenomenon of a semi-classical nature, since the classical dynamics-stage has, obviously, not yet been reached. Remark on the coexistence of the Uncertainty Principle with chaos and also on that, having \( \dot{\rho}(t) \), one can know the time dependence of any expectation value via Eq. \( \text{[11]} \).

Also, from Eqs. \( \text{[24]} \) and \( \text{[22]} \), we found in \( \text{[20]} \) that

\[
I_{\lambda} = \frac{1}{2} \frac{\ln \left( T + \frac{\hbar}{2} \right)}{\ln \left( T - \frac{\hbar}{2} \right)}, \quad \text{(24)}
\]

relating \( I_{\lambda} \) to \( I \). Note here that as \( I \) decreases, \( I_{\lambda} \) augments. If \( I \) approaches \( \hbar^2/4 \), then \( I_{\lambda} \to \infty \), since \( X^2 + P^2 \).
approaches the ground state. Even then $I \neq 0$. Thus, we do not reach the classical limit yet. We need to take the limit $\hbar \to 0$ and still $I_\lambda \to \infty$ holds [20].

**PRESENT RESULTS REGARDING THE CLASSICAL LIMIT (CL)**

Our present elaborations begin at this point. We are going to analytically study the limit $I \to 0$ of the density operator [17]. Speaking of a CL entails that both $\hbar$ and $I \to 0$, even if our EVs numerical results are independent of the actual numerical value of $\hbar$. In going to this limit we must always respect the restriction [4]. Two roads are open to us

1. Take first $h \to 0$ and then $I \to 0$. Classical statistics and quantum one are both compatible with [4], for any $\hbar > 0$ (quantum) or for $\hbar = 0$ (classical). In the limit $\hbar \to 0$, the density matrix [17] adopts the form

$$\rho = \frac{I}{Tr[I]}, \quad (25)$$

with $I$ the identity matrix. One has

$$\lim_{\hbar \to 0} I_\lambda = \frac{1}{2 \sqrt{I}}, \quad (26)$$

as a result of

$$\lim_{\hbar \to 0} \hbar I_\lambda = 0, \quad (27)$$

where we employed Eq. [24]. [25] is the maximally mixed density matrix of diagonal elements $1/n, n \in \mathbb{N}$, with $n \to \infty$. Such matrix should arise out of a decoherence process. We can not now take the limit $I \to 0$.

2. Proceed to effect $\lim_{h \to 0} \lim_{I \to \hbar^2/4} I_\lambda$, $\Lambda$ referring here to any of our quantities of interest. This second choice of venue respects the restriction [4] and would constitute the correct way to go. According to [24], we have

$$\lim_{h \to 0} \lim_{I \to \hbar^2/4} I_\lambda = \infty, \quad (28a)$$

$$\lim_{h \to 0} \lim_{I \to \hbar^2/4} \hbar I_\lambda = \infty, \quad (28b)$$

$$\lim_{h \to 0} \lim_{I \to \hbar^2/4} \lambda_i = \infty, \quad i = 0, 1, 2, \quad (28c)$$

$$\lim_{h \to 0} \lim_{I \to \hbar^2/4} |\lambda_3| = \infty. \quad (28d)$$

Note that in the second instance, when $I$ tends to its minimum possible value $\hbar^2/4$, $\rho$ [17] tends to its ground state. Thus, considering the pseudo generalized temperature $1/I_\lambda$, we ascertain that $1/I_\lambda \to 0$. Remark that $I_\lambda$ depends on both the classical variables and the initial conditions for the EVs. Our results holds also for $h \to 0$. Lo and behold, we have found that the classical limit is represented by a pure-state density matrix!.

Looking at the asymptotic behavior of $\lambda_0$ en [19], we see that $\exp(-\lambda_0) \sim \exp(h I_\lambda)$, entailing that the asymptotic eigenvalues of $\rho$ become $\exp[-n h I_\lambda], n = 0, 1, 2, \ldots$. Thus, $\rho$ [17] (or [4]), asymptotically, in its eigen-basis has the associate density matrix $R(t)$

$$R(t) = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (29)$$

This is a rather surprising. Not only the classical features of the semiclassical evolution depicted in Figs. 1 and 2 of [20] are represented by a mixed quantum density matrix, but purely classical results with $I = 0$, are masked by a pure-state density matrix. In the first case, semi-classical Chaos is obtained. In the second, directly classical Chaos.

Any asymptotic mean value will evolve **classically.** EVs ($\hat{X}^m \hat{P}^m$) will be null for all time, thus being trivially classic.

Additionally, not only the mean values of the set $(\hat{x}^2, \hat{p}^2, \hat{L})$ will evolve asymptotically with the classical equations corresponding to the Hamiltonian totally classic, if not surprisingly enough, $(\hat{x}^n \hat{p}^m)(t) = \langle \hat{x}^n \hat{p}^m(t)\rangle$ with initial conditions deducible as powers of $\langle \hat{x}(0)\rangle$ and $\langle \hat{p}(0)\rangle$, with $\langle \hat{x}(0)\rangle = \pm \sqrt{\langle \hat{x}^2(0)\rangle}$ and $\langle \hat{p}(0)\rangle = \pm \sqrt{\langle \hat{p}^2(0)\rangle}$. This follows from [14] and [16] after slight manipulation.

As a proof of the correctness of our results, it is easy to see that $I$ calculated with $\rho(t)$ given by [29] vanishes. Denoting the ground state by $|0\rangle$, we have $<0|\hat{x}^2|0\rangle = <0|\hat{p}^2|0\rangle = \lim_{\hbar \to 0} h/2$ and $<0|\hat{L}|0\rangle = 0$, so that $I = 0$. Moreover, via [19], we obtain an entropy $S = -\lambda_0 - 2 I_{\lambda} \sqrt{T}$, a decreasing monotonic function of $I$, with asymptotic value $S = 0$, as expected for a pure state.

**The pertinent classical statistical treatment**

Let us think of

$$\rho(x, p, t) = \exp - (\lambda_{0cl} + \lambda_{1cl} x^2 + \lambda_{2cl} p^2 + \lambda_{3cl} L), \quad (30)$$

equivalent to [14]. Here $x^2$, $p^2$ and $L = 2xp$ are simple functions, of course. The mean value of any general $F(x, p, t)$, for all $t$, is given via $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x, p, t) \rho(x, p, t) \, dx \, dp$. Using a transformation equivalent to [19], but for classical variables, we obtain the classical version of [17], with $\lambda_{0cl} = \ln(\pi/I_{\lambda cl})$. 

After some manipulation we are led to
\[ \langle x^2 \rangle = \frac{\sqrt{I_{cl}}}{I_{cl}} \lambda_{2cl}, \quad (31a) \]
\[ \langle p^2 \rangle = \frac{\sqrt{I_{cl}}}{I_{cl}} \lambda_{1cl}, \quad (31b) \]
\[ \langle L \rangle = -\frac{2\sqrt{I_{cl}}}{I_{cl}} \lambda_{3cl}. \quad (31c) \]

Now \( I_{cl} \) is the classical version of \( I \),
\[ I_{cl} = (\lambda_{1cl} \lambda_{2cl} - \lambda_{3cl}^2)^{1/2}, \quad (32) \]
that is also a time-invariant quantity, since the \( \lambda_{cl} \) obey the same system of equations used in the quantum treatment (Eqs. \( 23 \)). The classical version of \( I \) reads \( I_{cl} = \langle x^2 \rangle \langle p^2 \rangle - \langle L \rangle^2/4 \), but it no longer satisfies (4), but obeys instead
\[ I_{cl} \geq 0. \quad (33) \]
Moreover, Eqs. (31) coincide with Eqs. (20), together with (24). However, in this instance the dependence of \( I_{cl} \) with \( I_{cl} \) is not given by (24), since
\[ I_{cl} = \frac{1}{2\sqrt{I_{cl}}}, \quad (34) \]
but will coincide with Eq. (24), as one may expect. These classical results confirm that the limit \( h \to 0 \) is consistent with both classical and quantum statistics. Obviously, to complete the present analysis, the limit \( I_{cl} \to 0 \) (or \( I_{cl} \to \infty \)) is demanded. The probability density function (30) will read
\[ \lim_{I_{cl} \to 0} \rho(x, p, t) = \delta(X)\delta(P), \quad (35) \]
being a Dirac delta function of \( X \) and \( P \), as one should expect. In the limit (35) also \( \langle x^n \rangle = 0 \) at all times and all results with total certainty are obtained via (10). Total certitude is achieved without need for any kind of statistical reasoning.

**CONCLUSIONS**

In this work we have exhaustively investigated the classical limit of a density operator \( \rho \) associated to a well-known non-linear semi-classical system that possesses both classical and quantum interacting degrees of freedom. This \( \rho \) was presented previously in [20], in a context of incomplete prior information. In [20] its authors detected three well delimited and different regions in traversing the road towards the classical limit. These zones were characterized by the parameter \( E_r = \frac{|E|}{n_{eq}^2} \), con \( E_r \to \infty \), with \( E \) the total energy and \( I \) a dynamical invariant intimately linked to the uncertainty principle.

One had a quasiclassical region, a transitional one, and a classical zone. As \( E_r \) grows, complexity augments and, eventually, chaos emerges. This was a phenomenon of a semi-classical nature.

*It is article focused attention specifically on the classical limit per se, not on the road to it as in [24].* A purely analytical treatment was effected, for \( I \to 0 \). Two possible paths were contemplated to perform our study. The first was to research the \( h \to 0 \) calculation. Some difficulties were encountered in such instance, that were discussed in the text.

The second path turned to be both correct and coherent. It consist in taking first \( \lim I \to h^2/4 \), approaching the minimum \( I \) –value that quantum mechanics permits. A posteriori one deals with the limit \( h \to 0 \). In quite a counter-intuitive fashion, we stumbled on an asymptotic density matrix \( R \) corresponding to a pure state [24]. \( R \) adequately describes all facets of our classical features, i.e., those pertaining to a classical Hamiltonian. Any and all \( R \)-mean values behave as classical variables. For example, \( \langle x^n \rangle \rangle = \langle x^n \rangle \rangle (t) = \langle x^n \rangle \rangle (p)^m (t) \). We conclusively showed that \( R \) competently describes classical chaos.

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