Bethe Ansatz calculation of the spectral gap of the asymmetric exclusion process

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Abstract

We present a new derivation of the spectral gap of the totally asymmetric exclusion process on a half-filled ring of size $L$ by using the Bethe Ansatz. We show that, in the large $L$ limit, the Bethe equations reduce to a simple transcendental equation involving the polylogarithm, a classical special function. By solving that equation, the gap and the dynamical exponent are readily obtained. Our method can be extended to a system with an arbitrary density of particles.

Keywords: ASEP, Bethe Ansatz, Dynamical Exponent, Spectral Gap.

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1 Introduction

The asymmetric simple exclusion process (ASEP) is a model of driven diffusive particles on a lattice with hard-core exclusion (for general review see Spohn 1991). The ASEP appears as a minimal building block in a large variety of models for hopping conductivity (Richards 1977), polymer reptation (Widom et al. 1991), traffic flow (Schreckenberg and Wolf 1998) or
surface growth (Krug 1997). In particular, the ASEP in one dimension is a
discrete version of the Kardar-Parisi-Zhang (KPZ) equation (Halpin-Healy
and Zhang 1995). In biophysics, the ASEP has been used to describe the
diffusion of macromolecules through narrow vessels (Levitt 1973) and the
kinetics of protein synthesis on RNA (MacDonald and Gibbs 1969); more
recently, a mapping between sequence-alignment and the exclusion process
has been proposed (Bundschuh 2002). From a theoretical point of view, the
ASEP plays the role of a paradigm in non-equilibrium statistical mechanics:
it displays a variety of features such as boundary induced phase transitions
(Krug 1991), spontaneous symmetry breaking in one dimension (Evans et al.
1995) and dynamical phase separation (Evans et al. 1998).

Exact results for the ASEP in one dimension have been derived using
two complementary approaches (for a review see Derrida 1998). The Matrix
Ansatz (Derrida et al. 1993) allows to calculate steady state properties such
as invariant measures (Speer 1993), current fluctuations in the stationary
state and large deviation functionals (Derrida et al. 2003). The Bethe Ansatz
(Dhar 1987) provides spectral information about the evolution (Markov) op-
erator (Gwa and Spohn 1992; Schütz 1993; Kim 1995) which can be used
to derive large deviation functions (Derrida and Lebowitz 1998; Derrida and
Appert 1999; Derrida and Evans 1999). The exact relation between these two
techniques is still a matter of investigation (Alcaraz et al. 1994; Stinchcombe
and Schütz 1995; Alcaraz and Lazo 2003).

The relaxation time to the stationary state for the ASEP on a lattice of
size $L$ scales typically as a power law, $L^z$, $z$ being the ASEP dynamical ex-
ponent. The calculation of $z$ for a one dimensional system by Bethe Ansatz
is an important exact result that was first announced by Dhar, who found
$z = 3/2$ (Dhar 1987). The spectral gap (and thus $z$) was subsequently calcu-
lated for the half-filling case by Gwa and Spohn (1992) and for an arbitrary
density by Kim (1995) who mapped the ASEP into a non-Hermitian XXZ
Heisenberg spin chain. The one-dimensional ASEP belongs to the KPZ uni-
versality class and therefore the KPZ dynamical exponent in one dimension
is equal to $3/2$; this result was previously deduced from Galilean invariance
and renormalization group arguments (for a review see Krug 1997).

In this work, we present a new method of calculating the spectral gap of
the totally asymmetric exclusion process (TASEP) starting from the Bethe
Ansatz equations. Our method, based on an analytic continuation formula,
circumvents the technical difficulties involved in the derivation of Gwa and
Spohn (1992) and renders the calculation of the ASEP dynamical exponent
much more concise and transparent. Besides, our technique can be readily
extended to the arbitrary density case and allows us to calculate the spectral
gap for the asymmetric exclusion process with a tagged particle.
The outline of this paper is as follows. In section 2, we recall the definition of the TASEP, present the Bethe Ansatz equations without deriving them and summarize their analysis. In section 3, we present our original calculation of the spectral gap in the half-filling case. Concluding remarks and generalizations of our method are given in section 4.

2 Bethe Ansatz for the TASEP

2.1 The TASEP model

We consider the totally asymmetric simple exclusion process on a periodic one-dimensional lattice with \( L \) sites (sites \( i \) and \( L + i \) are identical). In this model, the total number \( n \) of particles is conserved. Each lattice site \( i \) (1 \( \leq \) \( i \) \( \leq \) \( L \)) is either occupied by one particle or is empty (exclusion rule). Stochastic dynamical rules govern the evolution of the system: a particle on a site \( i \) at time \( t \) jumps, in the interval between times \( t \) and \( t + dt \), with probability \( dt \) to the neighbouring site \( i + 1 \) if this site is empty. The total number of configurations for \( n \) particles on a ring with \( L \) sites is given by \( \Omega = L!/\left[ n!(L-n)! \right] \). In the stationary state, all configurations have the same probability \( 1/\Omega \) (Derrida 1998).

A configuration can be characterized by the positions of the \( n \) particles on the ring, \((x_1, x_2, \ldots, x_n)\) with \( 1 \leq x_1 < x_2 < \ldots < x_n \leq L \). We call \( \psi_t(x_1, \ldots, x_n) \) the probability of this configuration at time \( t \); the probability distribution \( \psi_t \) of the system at time \( t \) is thus a \( \Omega \)-dimensional vector. As the ASEP is a continuous-time Markov (i.e., memoryless) process, the time evolution of \( \psi_t \) is determined by the master equation

\[
\frac{d\psi_t}{dt} = M\psi_t, \tag{1}
\]

where the transition rate \( \Omega \times \Omega \) matrix \( M \) is the Markov matrix. A right eigenvector \( \psi \) is associated with the eigenvalue \( E \) of \( M \) if

\[
M\psi = E\psi. \tag{2}
\]

The Markov matrix \( M \) is a real non-symmetric matrix and, therefore, its eigenvalues (and eigenvectors) are either real numbers or complex conjugate pairs. The spectrum of \( M \) contains the eigenvalue \( E = 0 \) and the associated right eigenvector is the stationary state \( \psi(x_1, \ldots, x_n) = 1/\Omega \). Because the dynamics is ergodic (i.e., \( M \) is an irreducible and aperiodic Markov matrix), the Perron-Frobenius theorem implies that all eigenvalues \( E \) except 0 have
a strictly negative real part; the relaxation time of the corresponding eigenmode is $\tau = -1/\text{Re}(E)$. (The imaginary part of $E$ gives rise to an oscillatory behaviour).

In this paper, we shall calculate the gap $E_1$, i.e., the non-zero eigenvalue of $M$ with largest real part. The eigenmode associated with $E_1$ has thus the longest relaxation time that scales as $L^z$, $z$ being the dynamical exponent.

### 2.2 The Bethe equations

Writing $M$ explicitly, the eigenvalue equation (2) becomes

$$E\psi(x_1, \ldots, x_n) = \sum_{i} \left[ \psi(x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_n) - \psi(x_1, \ldots, x_n) \right],$$

where the sum runs over the indexes $i$ such that $x_{i-1} < x_i - 1$, i.e., such that the corresponding jump is allowed. The Bethe Ansatz assumes that the eigenvectors $\psi$ can be written in the form

$$\psi(x_1, \ldots, x_n) = \sum_{\sigma \in \Sigma_n} A_{\sigma} z_{\sigma(1)}^{x_1} z_{\sigma(2)}^{x_2} \cdots z_{\sigma(n)}^{x_n},$$

where $\Sigma_n$ is the group of the $n!$ permutations of $n$ indexes. The coefficients $\{A_{\sigma}\}$ and the fugacities $\{z_1, \ldots, z_n\}$ are complex numbers to be determined. The eigenvalue $E$ associated with an eigenvector of the form (4) is given by

$$E = -n + \sum_{i=1}^{n} 1/z_i.$$

Using matching conditions at the boundary surfaces $x_{i-1} = x_i - 1$ and the periodicity of the lattice, it can be shown that a vector $\psi$ of the type (4) is an eigenvector of $M$ if the fugacities $\{z_1, \ldots, z_n\}$ satisfy the Bethe equations

$$(z_i - 1)^n z_i^{-L} = -\prod_{j=1}^{n} (1 - z_j) \quad \text{for} \quad i = 1, \ldots, n.$$  

The procedure for deriving these equations has been thoroughly explained in Halpin-Healy and Zhang (1995) and Derrida (1998).

The obvious solution $z_1 = \ldots = z_n = 1$ provides the stationary distribution with eigenvalue 0. More generally, given a solution $\{z_1, \ldots, z_n\}$ of Eq. (6), the corresponding eigenvalue $E$ is obtained from Eq. (5); moreover, the coefficients $\{A_{\sigma}\}$ and thus the eigenvector $\psi$ are uniquely determined. In order to have a complete basis of eigenvectors, $\Omega$ independent solutions of the Bethe equations (6) are needed.
Following Gwa and Spohn (1992), we introduce $Z_i = 2/z_i - 1$. The equations (4) and (5) then become, respectively,

$$2E = -n + \sum_{j=1}^{n} Z_j,$$

(7)

and

$$(1 - Z_i)^n (1 + Z_i)^{L-n} = -2^L \prod_{j=1}^{n} \frac{Z_j - 1}{Z_j + 1} \quad \text{for} \quad i = 1, \ldots, n.$$  

(8)

We notice that the left hand side of Eq. (8) is a polynomial in $Z_i$ whereas the right hand side (r.h.s.) is independent of the index $i$.

The analysis of the Bethe equations is simplified if only half-filled models are considered, that is, if $L = 2n$ (Gwa and Spohn 1992). Equation (8) then reduces to

$$(1 - Z_i^2)^n = -4^n \prod_{j=1}^{n} \frac{Z_j - 1}{Z_j + 1} \quad \text{for} \quad i = 1, \ldots, n.$$  

(9)

The half-filling restriction does not affect the physical behaviour of the ASEP: in the large $L$ limit, models with arbitrary density, $\rho = n/L \in ]0, 1[$, belong to the same universality class. However, systems with vanishingly small density of particles ($\rho \to 0$) or holes ($\rho \to 1$) exhibit a different behaviour and will not be discussed here.

2.3 Analysis of Bethe equations

Taking advantage of the fact that the r.h.s. of Eq. (9) is independent of the index $i$, the Bethe equations can be reformulated as follows. Consider the polynomial equation

$$(1 - Z^2)^n = Y,$$  

(10)

where $Y$ is a given complex number. Writing

$$Y = -e^{u\pi},$$  

(11)

$u$ being a complex number with $-1 \leq \text{Im}(u) < 1$, we obtain the $n$-th roots of $Y$

$$y_m = e^{(u+i)\pi/n} e^{(m-1)2i\pi/n} \quad \text{for} \quad m = 1, \ldots, n.$$  

(12)

The $y_m$’s are evenly spaced on a circle of center 0 and radius $|Y|^{1/n}$ and are labeled counter-clockwise $0 \leq \text{Arg}(y_1) < \text{Arg}(y_2) < \ldots < \text{Arg}(y_n) < 2\pi$. Thus, the $2n$ solutions ($Z_1, \ldots, Z_{2n}$) of Eq. (10) are

$$Z_m = (1 - y_m)^{1/2}; \quad Z_{m+n} = -Z_m \quad \text{with} \quad m = 1, \ldots, n.$$  

(13)
The branch cut of the function \( z^{1/2} \) is, as usual, the real semi-axis \((-\infty, 0]\), i.e., for \( m = 1, \ldots, n \), the argument of \( Z_m \) belongs to \([-\pi/2, \pi/2]\). We explain in Appendix \[3\] that each \( Z_m \) is an analytic function of \( Y \) in the complex plane with a branch cut along \([0, +\infty)\) and that the locus of the \( Z_m \)'s is a remarkable curve called a Cassini oval.

We now choose \( n \) different roots \((Z_{c(j)})_{j=1,n}\) among \((Z_1, \ldots, Z_{2n})\), such that the choice function \( c : \{1, \ldots, n\} \to \{1, \ldots, 2n\} \) satisfies
\[
1 \leq c(1) < \ldots < c(n) \leq 2n .
\] (14)

There are precisely \( \Omega \) such choice functions, \( \Omega = (2n)!/n!^2 \) being the size of the Markov matrix. Finally, we define
\[
A_c(Y) = -4^n \prod_{j=1}^{n} \frac{Z_{c(j)} - 1}{Z_{c(j)} + 1} ,
\] (15)

where \( A_c \) is a function of \( Y \) and of the choice function \( c \). The Bethe equations \([3]\) are equivalent to the self-consistency equation
\[
A_c(Y) = Y .
\] (16)

Given the choice function \( c \) and a root \( Y \) of this equation, the \( Z_{c(j)} \)'s are determined from Eq. \((10)\) and the corresponding eigenvalue \( E_c \) is obtained from Eq. \((7)\).

For small values of \( n \), the above described procedure allows us to compute numerical solutions of the Bethe equations. From our numerical observations, we conjecture that for each choice function \( c \) (among the \( \Omega \) possible choice functions), the self-consistency Eq. \((16)\) has a unique solution \( Y \) that yields one eigenvector \( \psi_c \) and one eigenvalue \( E_c \). This suggests that the Bethe equations yield a complete basis of eigenvectors for the ASEP.

Let us first consider the choice function \( c(j) = j \), i.e., the selected \( Z_j \)'s are the \( n \) solutions of Eq. \((10)\) with largest real parts. As this choice plays an important role in the following analysis, we define
\[
A_0(Y) = -4^n \prod_{j=1}^{n} \frac{Z_j - 1}{Z_j + 1} ,
\] (17)
\[
2E_0 = -n + \sum_{j=1}^{n} Z_j .
\] (18)

We emphasize that \( E_0 \) is an implicit function of \( Y \). The equation \( A_0(Y) = Y \) has the solution \( Y = 0 \) that yields \( Z_j = 1 \) for all \( j \) and provides the stationary distribution (or ground state) with eigenvalue 0.
Numerical observations (Gwa and Spohn 1992) indicate that the first excited eigenvalue \( E_1 \) corresponds to the choice \( c(j) = j \) for \( j = 1, \ldots, n - 1 \) and \( c(n) = n + 1 \); i.e., the first excited state is obtained from the ground state by the excitation \( (n \to n + 1) \). Writing \( A_1(Y) \) and \( E_1 \) for the functions \( A_c \) and \( E_c \) corresponding to this choice function, we have, from Eqs. (15, 17 and 18),

\[
A_1(Y) = A_0(Y) \left( \frac{Z_1 - 1}{Z_1 + 1} \right) \frac{Z_n - 1}{Z_n + 1},
\]

\[
2E_1 = 2E_0 - (Z_1 + Z_n),
\]

where we have used \( Z_{n+1} = -Z_1 \). The excitation \( (1 \to 2n) \), that is, \( c(j) = j + 1 \) for \( j = 1, \ldots, n - 1 \) and \( c(n) = 2n \), also leads to Eqs. (19) and (20) and thus to the same eigenvalue \( E_1 \). The first excited state has therefore a degeneracy of order 2.

Consequently, in order to find the expression for the gap \( E_1 \), we must solve the self-consistency equation

\[
A_1(Y) = Y,
\]

calculate the \( Z_j \)'s for \( j = 1, \ldots, n \), and finally deduce \( E_1 \) from Eq. (20).

In the above discussion, we closely followed Gwa and Spohn (1992) to present the Bethe Ansatz equations for the TASEP. We shall now solve these equations and calculate the gap by a radically different and simpler method.

3 Calculation of the gap

Let us define \( F(Y) \) as

\[
A_0(Y) = Y \exp(F(Y)).
\]

In Appendix A we have derived the following identities, valid for \( |Y| \leq 1 \),

\[
F(Y) = \sum_{k=1}^{\infty} \frac{w_{kn}}{k} Y^k,
\]

\[
-4E_0 = \sum_{k=1}^{\infty} \frac{w_{kn-1}}{k} Y^k,
\]

where the \( w_k \)'s are given by

\[
w_k = \frac{(2k - 1)!!}{(2k)!!} = \frac{(2k)!}{(k! 2^k)^2}.
\]
From the Stirling formula, the leading order behaviour of \( w_k \) for \( k \to \infty \), is given by
\[
w_k \sim \frac{1}{\sqrt{\pi k}}. \tag{26}\]

From the power series (23 and 24) we deduce that \( A_0(Y) \) and \( E_0 \) are analytic functions of the complex variable \( Y \) inside the unit circle. This property is not obvious \textit{a priori}: the functions \( A_0(Y) \) and \( E_0 \), defined in Eqs. (17) and (18), respectively, depend implicitly on \( Y \) via the \( Z_m \)'s that involve a branch cut along \([0, +\infty)\). Indeed, for a generic choice function \( c(j) \), \( A_c(Y) \) and \( E_c \) are analytic only in the complex \( Y \) plane with a cut along \([0, +\infty)\) and, therefore, are not analytic in the neighbourhood of \( Y = 0 \). This special property of \( A_0(Y) \) and \( E_0 \) is obtained by an explicit calculation in Appendix A and from geometrical considerations in Appendix B.

Using Eqs. (19), (22) and (23), the self-consistency equation (21), that determines the gap, reduces to
\[
\sum_{k=1}^{\infty} \frac{w_k}{k} Y^k = \ln \frac{1 - Z_1}{1 + Z_1} + \ln \frac{1 - Z_n}{1 + Z_n}. \tag{27}\]

From Eq. (20) and Eq. (21), we have
\[
-4E_1 = \sum_{k=1}^{\infty} \frac{w_{kn-1}}{k} Y^k + 2Z_1 + 2Z_n. \tag{28}\]

Combining Eq. (27) and Eq. (28), we obtain
\[
-4E_1 = \sum_{k=1}^{\infty} \frac{w_{kn}}{k(2kn - 1)} Y^k + \left(2Z_1 + \ln \frac{1 - Z_1}{1 + Z_1}\right) + \left(2Z_n + \ln \frac{1 - Z_n}{1 + Z_n}\right), \tag{29}\]

where we have used \((2kn)w_{kn} = (2kn - 1)w_{kn-1}\).

Thus, to find the gap for a half-filled system with \( n \) particles, we must solve Eq. (27) for \( Y \) and substitute the result in Eq. (29). We emphasize that the power series in these equations represent, inside the unit disk, analytic functions of \( Y \) that are defined in the whole complex plane with a cut along \([1, +\infty)\).

We now consider the thermodynamic limit, \( n \to \infty \). We obtain, at leading order, from Eq. (26)
\[
F(Y) = \sum_{k=1}^{\infty} \frac{w_k}{k} Y^k \to \frac{1}{\sqrt{\pi n}} \text{Li}_{3/2}(Y), \tag{30}\]
\[
\sum_{k=1}^{\infty} \frac{w_{kn}}{k(2kn - 1)} Y^k \to \frac{1}{2\sqrt{\pi n^3}} \text{Li}_{5/2}(Y), \tag{31}\]
where we have used the polylogarithm function of index $s$

$$\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}. \quad (32)$$

By virtue of the integral representation

$$\text{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - z} \, dt, \quad (33)$$

the function $\text{Li}_s$ can be extended by analytic continuation to the whole complex plane with a branch cut along the real semi-axis $[1, +\infty)$. In the large $n$ limit, we deduce from Eqs. (11, 12 and 13) that

$$Z_1 = (1 - y_1)^{1/2} = \sqrt{\frac{\pi}{n}} (-u - i)^{1/2} + O\left(\frac{1}{n^{3/2}}\right), \quad (34)$$

$$Z_n = (1 - y_n)^{1/2} = \sqrt{\frac{\pi}{n}} (-u + i)^{1/2} + O\left(\frac{1}{n^{3/2}}\right), \quad (35)$$

where we have supposed that $Y = -e^{u\pi}$ remains finite when $n \to \infty$. Using these expressions and the expansion $\ln \frac{1-Z}{1+Z} = -2Z - \frac{2}{3}Z^3 + O(Z^5)$, Eq. (27) reduces to

$$\text{Li}_{3/2}(-e^{u\pi}) = -2\pi \left[(-u + i)^{1/2} + (-u - i)^{1/2}\right], \quad (36)$$

and the gap (29), at the leading order, is given by

$$E_1 = \frac{1}{n^{3/2}} \left\{ \frac{-1}{8\sqrt{\pi}} \text{Li}_{5/2}(-e^{u\pi}) + \frac{\pi^{3/2}}{6} \left[(-u + i)^{3/2} + (-u - i)^{3/2}\right] \right\}. \quad (37)$$

[Notice that the r.h.s. of Eq. (36) and of Eq. (37) are real when $u$ is real.] With the help of the Maple software, we find a unique solution of Eq. (36) in the strip $-1 \leq \text{Im}(u) < 1$ that is real and is given by

$$u = 1.119 068 802 804 474 \ldots \quad (38)$$

Inserting this value of $u$ in Eq. (37) yields the large $n$ (or large $L$) behaviour of the gap

$$E_1 = -\frac{2.301 345 960 455 050 \ldots}{n^{3/2}} = -\frac{6.509 189 337 976 136 \ldots}{L^{3/2}}. \quad (39)$$

This is precisely the result obtained by Gwa and Spohn (1992). This gap scales as $L^{-3/2}$ and is real for the TASEP in the half-filling case.
4 Summary and discussion

In this work, we have calculated the gap of the TASEP in the limit of a large size system by using the Bethe Ansatz. We first take the large $n$ limit of the Bethe equations inside the unit circle, then perform the analytic continuation of these equations in the whole complex plane with a branch cut along $[1, +\infty)$ and finally solve them. Gwa and Spohn (1992), on the contrary, first represent the analytic continuation of the Bethe equations for a fixed value of $n$ as a $n$-dependent complex integral (thanks to the Euler-Maclaurin formula) and then extract the gap from the large $n$ limit of this integral representation which is rather a delicate operation. We have shown here that the derivation of the TASEP gap is greatly simplified by performing the calculations in the reverse order, that is taking the large $n$ limit first and the analytic continuation afterwards.

We do not claim that it is always true that large $n$ limit and analytic continuation are commuting operations. If the solution $Y$ of Eq. (21), in the large $n$ limit, diverges to $\infty$ or approaches asymptotically the branch cut, reversing the order of operations may not be possible. Fortunately, in our problem, $Y$ remains a bounded negative real number when $n \to \infty$.

We have applied our method to several other problems but, for sake of conciseness, we simply list these additional calculations without giving the details. Our technique provides the subleading corrections to the gap and allows us to calculate the eigenvalue of the highest excited state, of a finite excitation above the ground state, or below the highest excited state. The gap for the totally asymmetric exclusion process with an arbitrary density $\rho$ can also be obtained by elementary means, and we find in agreement with (Kim 1995)

$$E_1(\rho) = 2\sqrt{\rho(1-\rho)} \ E_1(\rho = 1/2) \pm \frac{2i\pi}{L}(2\rho - 1). \quad (40)$$

where $E_1(\rho = 1/2)$ is the gap for the half-filling case given in Eq. (39); we notice that for a density other than one-half the gap has a non-zero imaginary part. We have also studied generalizations of the ASEP by introducing a tagged particle that has the same dynamics as the other particles: the gap then scales as $L^{-5/2}$.

Finally, we emphasize that the formula (33) already appeared in the work of Derrida and Appert (1999): indeed, Eqs. (27) and (28) are similar to those used in their calculation of large deviation functions of the ASEP by Bethe Ansatz.
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A Derivation of Eq. (23) and Eq. (24)

The numbers $w_k$ defined in Eq. (25) are the coefficients of the Taylor series

$$\frac{1}{\sqrt{1-x}} = \sum_{k=0}^{\infty} w_k x^k. \quad (41)$$

By integration, we find

$$\sqrt{1-x} = 1 - \frac{1}{2} \sum_{k=1}^{\infty} \frac{w_{k-1}}{k} x^k. \quad (42)$$

Recalling that $Z_m = \sqrt{1-y_m}$ [see Eqs. (12) and (13)], we obtain

$$\sum_{m=1}^{n} Z_m = n - \frac{1}{2} \sum_{k=1}^{\infty} \frac{w_{k-1}}{k} \sum_{m=1}^{n} y_m. \quad (43)$$

The fact that the $y_m$’s are the $n$-th roots of $Y$ leads to the following relation

$$\sum_{m=1}^{n} y_m^k = \begin{cases} nY^{k/n} & \text{if } k \text{ is a multiple of } n \\ 0 & \text{otherwise}. \end{cases} \quad (44)$$

Inserting this relation in Eq. (43), we obtain

$$\sum_{m=1}^{n} Z_m = n - \frac{1}{2} \sum_{k=1}^{\infty} \frac{w_{nk-1}}{k} Y^k. \quad (45)$$

We thus find, thanks to the crucial identity (44), that $\sum_{m=1}^{n} Z_m$ is analytic in $Y$ inside the unit circle. Finally, substituting Eq. (45) in Eq. (18), we obtain Eq. (24).

The derivation of Eq. (23) follows similar steps. We first notice that the Taylor expansion of the function

$$f(x) = \ln \left( \frac{4}{x} \frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} \right) \quad (46)$$

is given by

$$f(x) = \sum_{k=1}^{\infty} \frac{w_k}{k} x^k. \quad (47)$$
[This follows from \( f(0) = 0 \) and \( f'(x) = \frac{1}{x} \left( \frac{1}{\sqrt{1-x}} - 1 \right) \). Using Eq. (17) and the identity \( \prod_{m=1}^{n} (-y_m) = -Y \), we deduce that

\[
A_0(Y) = Y \prod_{m=1}^{n} \frac{4}{y_m} \frac{1 - Z_m}{1 + Z_m}.
\] (48)

From Eq. (22) and Eqs. (46), (47) and (48), we obtain

\[
F(Y) = \sum_{m=1}^{n} f(y_m) = \sum_{k=1}^{\infty} \frac{w_k}{k} \sum_{m=1}^{n} y_m^k.
\] (49)

This equation reduces to Eq. (23) by virtue of the identity (44).

## B Roots of the Bethe Equations and Cassini ovals

The polynomial equation

\[(1 - Z^2)^n = Y\] (50)

where \( Y \) is a fixed complex number, has \( 2n \) solutions, \( (Z_1, \ldots, Z_{2n}) \). The purpose of this appendix is to explain how these solutions can be labeled in a coherent way so that each root \( Z_m(Y) \) is an analytic function of \( Y \).

We first notice that \( y_m \), defined in Eq. (12), is an analytic function of \( Y \) in the complex plane with a branch cut along the real semi-axis \([0, +\infty)\). Nevertheless when \( Y \) crosses \([0, +\infty)\), the functions \( y_1, y_2, \ldots, y_n \) are the analytic continuations (above the axis) of respectively \( y_n, y_1, \ldots, y_{n-1} \): thus the existence of the branch cut along \([0, +\infty)\) is due to the labeling of the roots.

The complex numbers \( (Z_1, \ldots, Z_{2n}) \) belong to the curve defined by

\[|Z - 1| \cdot |Z + 1| = r \quad \text{with} \quad r = |Y|^{1/n}\] (51)

and called a Cassini oval. A Cassini oval is the conformal transformation of the circle of center 1 and radius \( r \) by the function \( z \rightarrow z^{1/2} \). Its shape depends on whether the point 0 is inside or outside the circle, i.e., whether \( r < 1 \) or not. When \( r < 1 \), the curve of equation (51) consists of two ovals around the points \( Z = \pm 1 \). The numbers \( (Z_1, \ldots, Z_n) \) lie on the right oval and \( (Z_{n+1}, \ldots, Z_{2n}) \) on the left oval. For the marginal case, \( r = 1 \), the curve is the lemniscate of Bernoulli, with a multiple point at \( Z = 0 \). When \( r > 1 \), the Cassini oval is a single loop with a peanut shape (when \( r \in ]1, 2[ \)) or an oval shape (\( r \geq 2 \)). See Fig. 11 where the cases \( r = 0.9, 1 \) and 1.1 are drawn. In the large-\( r \) limit, the oval tends to the circle of radius \( \sqrt{r} \).
Figure 1: Labeling the roots $Z_m$ of the equation $(1 - Z^2)^n = Y$. Here $Y = e^{i\phi}r^n$ with $n = 10$, $\phi = \pi/2$, and $r \in \{0.9, 1, 1.1\}$. The continuous curves are the corresponding Cassini ovals (see text). When $r$ is fixed and $\phi$ goes from 0 to $2\pi$, each $Z_m$ slips counter-clockwise along the Cassini ovals. Then, the jump $\phi = 2\pi \to 0$, i.e., $Y$ crosses $[0, +\infty)$, consists of a global shift of the labels $m$ around each continuous curve.

We now discuss the analyticity properties of $Z_m(Y)$: $Z_m$ is an analytic function of $y_m$ with a branch cut along $(-\infty, 0]$; this branch cut is compatible with that of $y_m(Y)$. Consequently $Z_m$ is an analytic function of $Y$ with a branch cut along $[0, +\infty)$. Moreover when $Y$ crosses the real segment $[0, 1]$, the functions $Z_1(Y), Z_2(Y), \ldots, Z_n(Y)$ are the analytic continuations (above the axis) of the functions $Z_n(Y), Z_1(Y), \ldots, Z_{n-1}(Y)$ respectively. See Fig. 1 (A similar property is true for the functions $Z_{n+1}(Y), Z_{n+2}(Y), \ldots, Z_{2n}(Y)$ that belong to the left oval). But when $Y$ crosses $[1, +\infty)$, the functions $Z_1(Y), Z_2(Y), \ldots, Z_{2n}(Y)$ are the analytic continuations (above the axis) of the functions $Z_{2n}(Y), Z_1(Y), \ldots, Z_{2n-1}(Y)$ respectively. Consequently the branch cut of the function $A_0(Y)$ defined in Eq. (17) is $[1, +\infty)$ and not $[0, +\infty)$.

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