Whitney algebras and Grassmann’s regressive products

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Abstract

Geometric products on tensor powers $\Lambda(V)^{\otimes m}$ of an exterior algebra and on Whitney algebras [13] provide a rigorous version of Grassmann’s regressive products of 1844 [17]. We study geometric products and their relations with other classical operators on exterior algebras, such as the Hodge $*$–operators and the join and meet products in Cayley-Grassmann algebras [2, 30]. We establish encodings of tensor powers $\Lambda(V)^{\otimes m}$ and of Whitney algebras $W^m(M)$ in terms of letterplace algebras and of their geometric products in terms of divided powers of polarization operators. We use these encodings to provide simple proofs of the Crapo and Schmitt exchange relations in Whitney algebras and of two typical classes of identities in Cayley-Grassmann algebras.

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1 Introduction

In 1986, I. Stewart wrote: “The late nineteenth century witnessed many attempts to develop an algebra of n—dimensional space, by analogy with the representation of the plane by complex numbers. Prominent among them was Hermann Grassmann’s “Die Lineale Ausdehnungslehre” (The Calculus of Extension), published in 1844. Grassmann had the misfortune to write in a discursive, philosophical and obscure style at the time when axiomatic presentation was becoming de riguer in the mathematical world” [30].

As definitively recognized in recent times [28, 29, 26], H. G. Grassmann was indeed the inventor of linear and multilinear algebra, as well as of “geometry” in arbitrary finite dimension ([17], 1844).
Grassmann’s basic idea was to build up a formal system that allows geometric entities to be manipulated in an intrinsic (invariant) way, that is, by making no appeal to a reference system. In Grassmann’s vision, this kind of approach should put together the synthetic and the analytic approaches to geometry in arbitrary finite dimension. As Grassmann wrote in 1877:

“Extension theory forms the abstract basis of the theory of space (geometry), that is it is the pure mathematical science, stripped of all geometric content, whose special application is that theory. […] The purpose of this method of calculation in geometry is to unify the synthetic and analytic methods, that is, to transplant the advantages of each into the soil of the other, so that any construction is accompanied by an elementary analytic operation and conversely. ([17], p. 283 and 285)”

Grassmann’s elementary analytic operations were essentially of two kinds: the outer (progressive) product (the one that is nowadays called the “wedge product” in the exterior algebra) and the regressive product. It is worth claiming that this second kind of product was not a single operation but a family of (unary) operations.

Grassmann himself realized that his general approach was not only obscure but also misleading for the Mathematicians of his time. He published in 1862 a second and simplified version of his *opus magnum* of 1844. In this version, he made the deliberate choice of restricting the definition of regressive product to the special case of those he called real regressive products (see subsection 3.2, for details). None the less, the impact of Grassmann’s work upon the mathematical community of the second half of the XIX century remained almost irrelevant.

There were, to be sure, two bright exceptions: W. K. Clifford and G. Peano, whose ideas laid the foundations of the two main algebraic theories that, at present, fulfill Grassmann’s program in the more efficient way.

In his celebrated paper of 1878 [11], W. K. Clifford introduced what are nowadays called Clifford algebras. About one century later D. Hestenes and his school captured (see, e.g. [20, 21]) the various geometric meanings of Clifford algebras in their full generality by introducing the notion of Geometric Clifford algebra; these algebras are Clifford algebras in which a distinguished element is chosen, the integral. For lack of space, here we don’t speak about this important point of view. We refer the reader to the recent book by H. Li [21].

In his book of 1888 [25], G. Peano made another crucial step into an apparently different direction: he realized that the 1862 regressive product can be defined (at least in dimension 3) in a transparent and intrinsic way by fixing a bracket - a non-degenerate, alternating multilinear form - on the ground vector space. Peano’s ideas were at the origin of the modern notion of Cayley-Grassmann algebra (CG-algebras, for short), developed by G.-C. Rota and his school (see, e.g. [2]). In the last decades, a considerable amount of work has been done dealing with Cayley-Grassmann algebras (see, e.g. [21],

1 At the time of Grassmann, the terms “space” and “geometry” were used to mean just dimensions one, two and three.
Chapters 2, 3 and the Bibliography). These algebras nowadays have important applications both in Mathematics and in Computer Science (Invariant Theory, Geometric theorem proving and Computer vision, to name but a few).

In this paper, we provide a comparative discussion of three classes of algebras that sprang out from the ideas of Grassmann and Peano, and some of their main features and typical applications. These algebras are the Cayley-Grassmann algebra of a Peano space, endowed with the join and meet products; the tensor powers $\Lambda(V)^{\otimes m}$ of the exterior algebra of a vector space $V$, endowed with geometric products, that provide a formalism closer to Grassmann’s original approach; the Whitney algebras $W^m(M)$ of a matroid $M$, endowed with geometric products, at first motivated by the study of representability of matroids. The study of tensor powers of exterior algebras and of Whitney algebras endowed with geometric products is a recent subject ([23, 24, 13, 12, 3]).

The first main concern of the present paper is the discussion of geometric products in the tensor powers $\Lambda(V)^{\otimes m}$ of the exterior algebra of a finite dimensional vector space $V$. Geometric products provide a rigorous version of Grassmann’s regressive products, both in the real and in the formal cases ([17], §125). It is worth noticing that Grassmann failed in finding a geometric meaning of formal geometric products $^2$; in Section 5, we exhibit such a geometric meaning, by showing that geometric products lead to a concise treatment of (generalized) Hodge $*$—operators.

The second main concern of our paper is to establish the encoding of the algebras $\Lambda(V)^{\otimes m}$ and $W^m(M)$ and of their geometric products by means of quotients of skew-symmetric letterplace algebras $\text{Skew}[L|m]$ [6, 3] and place polarization operators (Section 7). Skew-symmetric letterplace algebras are algebras with Straightening law, and the action of place polarization operators implements a Lie action of the general linear Lie algebra $\mathfrak{gl}(m, \mathbb{Z})$ (Section 6). This encoding (foreshadowed by G.-C. Rota) allows the treatment of identities to be simplified.

On the one hand, we have that all the identities among linear combinations of iterated “geometric products” in the algebra $\Lambda(V)^{\otimes m}$ are consequences of the Superalgebraic Straightening law of Grosshans, Rota and Stein [18]. This fact may be regarded as an analog of the Second Fundamental Theorem of Invariant Theory. The application to Whitney algebras is even more relevant. The Straightening law provides a system of $\mathbb{Z}$—linear generators of the ideals that define the algebras $W^m(M)$ (Subsection 7.3). Thus, Whitney algebras are still algebras with Straightening law, and we have an algorithm for the solution of the word problem (for example, see the proof of the exchange relations of Crapo and Schmitt, Section 8). The iterative application of the Straightening laws gives rise to an explosive increment of computational complexity; this is the reason why a systematic investigation of special identities is called for.

On the other hand, identities that hold for geometric products may also be derived from identities that hold in the enveloping algebra $\mathfrak{U}(\mathfrak{gl}(m, \mathbb{Z}))$. Thus, the basic ideas borrowed from the method Capelli virtual variables (see, e.g. [10], [5], [9], [27]) can be applied to manipulations of geometric products (see, e.g. Sections 9 and 10).

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$^2$ In 1877, Grassmann wrote: “In the 1862 Ausdehnungslehre the concept of the formal regressive product is abandoned as sterile, thus simplifying the whole subject.” [17], p.200
2 The algebras: generalities

1. CG – algebras. These algebras are the classical exterior algebras endowed with the join product (the traditional wedge product) and the meet product. As the join product of two extensors that represent subspaces $U$ and $V$ in general position yields an extensor that represents the space direct sum of $U$ and $V$, the meet product represents the intersection space of $U$ and $V$.

In the language of CG-algebras, projective geometry statements and constructions can be expressed as (invariant) identities and equations.

Example 1. (The Desargues Theorem)

In the projective plane $P[K^3]$, consider two triangles $123$ and $1'2'3'$. Consider the three lines $11'$, $22'$ and $33'$. In the language of CG-algebras, we have: the lines $11'$, $22'$ and $33'$ are concurrent if and only if

$$(1 \vee 1') \land (2 \vee 2') \land (3 \vee 3')$$

equals ZERO.

Consider the three points $12 \cap 1'2'$, $13 \cap 1'3'$, $23 \cap 2'3'$. These points are collinear if and only if the bracket

$$[(1 \vee 2) \land (1' \vee 2'), (1 \vee 3) \land (1' \vee 3'), (2 \vee 3) \land (2' \vee 3')]$$

equals ZERO.

In the CG-algebra, we have the identity (see, subsection... below)

$$[(1 \vee 2) \land (1' \vee 2'), (1 \vee 3) \land (1' \vee 3'), (2 \vee 3) \land (2' \vee 3')] = -[1, 2, 3][1', 2', 3'](1 \vee 1') \land (2 \vee 2') \land (3 \land 3').$$

Thus, the preceding identity implies the following geometric statement:

The three points

$$12 \cap 1'2', 13 \cap 1'3', 23 \cap 2'3'$$

are collinear if and only if the three lines

$$11', 22', 33'$$

are concurrent.

It turns out that there is even a third class of unary operations, the Hodge star operators, which is the vector space analog of complementation in Boolean algebra. The Hodge $*$-operators are invariantly defined with respect to the orthogonal group, and formalize Grassmann’s “Erganzung”; these operators implement the duality between meet and join.

The notion of CG-algebra endowed with a Hodge $*$-operator is equivalent to that of Geometric algebra in the sense of Hestenes [4].
2. **Tensor powers of the exterior algebra of a vector space.** (see [23, 24, 13, 15, 12]) These algebras are endowed with a product induced by the wedge product, as well as with a family of linear operators, called geometric products, that yield a rigorous formulation of Grassmann’s 1844 regressive products. In particular, given two extensors that represent subspaces $U$ and $V$, not necessarily in general position, there is a special geometric product that yields the tensor product of two extensors that represent the space sum of $U$ and $V$ and the intersection space of $U$ and $V$, respectively.

3. **Whitney algebras of matroids** (see [13, 12, 3]). This class of algebras provides a generalization of the algebras mentioned in the preceding point, as well as of the White bracket ring [33]. Roughly speaking, Whitney algebras can be regarded as the generalization of the tensor powers of exterior algebras towards algebras associated to systems of points subject to relations of abstract dependence, in the sense of H. Whitney [34]. Again, in Whitney algebras the join product and the geometric products are defined.

## 3 The algebras: basic constructions

We provide explicit constructions of the algebras mentioned above, and a description of their main features. First of all, we fix some terminology and notation.

Let $V$ be an $n$-dimensional vector space over a field $\mathbb{K}$, and let $\Lambda(V)$ be its exterior algebra. An element $A$ of $\Lambda(V)$ that can be written as a product of vectors is called an extensor. To each representation of a given extensor $A$ as a product of vectors there corresponds a basis of one and the same dimensional subspace $\overline{A}$ of $V$; the number of vectors in a representation of $A$ is the step of $A$, and equals the dimension of $\overline{A}$. An extensors $A$ divides an extensor $B$, in the usual sense, if and only if the subspace $\overline{A}$ is contained in the subspace $\overline{B}$. The subspaces of $V$ ordered by set inclusion form a lattice in which, for any two subspaces $U_1$, $U_2$, the greatest lower bound $U_1 \sqcap U_2$ is the subspace set-intersection of $U_1$ and $U_2$, and the least upper bound $U_1 \sqcup U_2$ is the subspace sum of $U_1$ and $U_2$. This lattice is modular, thus for any two subspaces $U_1$, $U_2$ there is a canonical isomorphism between the intervals $[U_1 \sqcap U_2, U_1]$ and $[U_2, U_1 \sqcup U_2]$.

### 3.1 Cayley-Grassmann algebras

Let $\mathbb{K}$ be a field. Let $V$ be a vector space over $\mathbb{K}$, $\text{dim}(V) = n$, endowed with a bracket $[,]$ (a non-degenerate alternating $n$-multilinear form). The pair $(V,[,])$ is called a Peano space. The CG-algebra of $(V,[,])$ is the exterior algebra $\Lambda(V)$ endowed with two associative products:

1. the usual wedge product, called the join and denoted by the symbol $\lor$;
2. the meet, denoted by the symbol $\land$. This product is defined in the following way.

Let $A$ and $B$ be extensors of steps $a$ and $b$, respectively, with $a + b \geq n$, then:

$$A \land B = \sum_{(A)(n-b,a+b-n)} [A_{(1)}B]A_{(2)} = \sum_{(B)(a+b-n,n-a)} [AB_{(2)}]B_{(1)}.$$
If \( a + b < n \), then \( A \land B = 0 \) by definition.

(Here we use the Sweedler notation for coproduct slices in bialgebras \([31,1]\): for an extensor \( C \) of step \( c \),
\[
\Delta(C)_{(c-h,h)} = \sum_{(C)_{(c-h,h)}} C(1) \otimes C(2),
\]
where \( \text{step}(C(1)) = c - h \) and \( \text{step}(C(2)) = h \).

The fact that there are two completely different ways of computing the meet of two extensors gives a great suppleness and power to the CG-algebra of a Peano space.

The geometric meaning of the join product \( \lor \) (the wedge in the standard language for exterior algebras) is well-known. Let \( A \) and \( B \) be extensors that represent subspaces \( \mathcal{A} \) and \( \mathcal{B} \), respectively. If \( \mathcal{A} \lor \mathcal{B} = (0) \), then \( \mathcal{A} \land \mathcal{B} = \mathcal{A} \land \mathcal{B} \); otherwise, \( A \lor B \) is zero. Not unexpectedly, the geometric meaning of the meet product \( \land \) is the “dual” of the geometric meaning of the join product. If \( \mathcal{A} \land \mathcal{B} = V \), then \( A \lor B = A \lor B \); otherwise, \( A \land B \) is zero.

**Example 2.** (The intersection of two lines in \( \mathbb{P}^2(\mathbb{K}) \)) Let \( (V, [\cdot]) \) be a Peano space over the field \( \mathbb{K} \), \( \dim(V) = 3 \). Let \( p_1, p_2, q_1, q_2 \) be four mutually non-proportional vectors in \( V \) that span \( V \). The 2-extensors \( p_1 \lor p_2, q_1 \lor q_2 \) represent two different lines in \( \mathbb{P}^2(\mathbb{K}) \). The vector \( (p_1 \lor p_2) \land (q_1 \lor q_2) = [p_1, q_1, q_2]p_2 - [p_2, q_1, q_2]p_1 = -[p_1, p_2, q_1]q_2 + [p_1, p_2, q_2]q_1 \) represents the intersection point of the two lines.

### 3.2 Tensor powers of exterior algebras

Let \( \mathbb{K} \) be a field. Let \( V \) be a vector space over \( \mathbb{K} \), \( \dim(V) = n \). Let \( \Lambda(V) \otimes \Lambda(V) \) be the tensor square of the exterior algebra \( \Lambda(V) \), in the category of \( \mathbb{Z}_2 \)-graded \( \mathbb{K} \)-algebras. The product in this algebra is given by

\[
(A_1 \otimes A_2)(B_1 \otimes B_2) = (-1)^{a_2b_1} A_1 B_1 \otimes A_2 B_2
\]
for every extensors \( A_i, B_j \) of steps \( a_i, b_i \) in \( \Lambda(V) \). This algebra is endowed with two families of linear operators, called geometric products:

- ‘rising’ geometric products:

\[
\phi^{(h)}_{21} : \Lambda(V) \otimes \Lambda(V) \to \Lambda(V) \otimes \Lambda(V), \quad h \in \mathbb{Z}^+
\]

defined by setting, for \( A \) and \( B \) extensors of steps \( a \) and \( b \), respectively,

\[
\phi^{(h)}_{21}(A \otimes B) = \sum_{(A)_{(a-h,h)}} A_{(1)} \otimes A_{(2)} B.
\]
• 'lowering' geometric products:
\[ \phi_{12}^{(h)} : \Lambda(V) \otimes \Lambda(V) \to \Lambda(V) \otimes \Lambda(V), \quad h \in \mathbb{Z}^+ \]
defined by setting, for \( A \) and \( B \) extensors of steps \( a \) and \( b \), respectively,
\[ \phi_{12}^{(h)} (A \otimes B) = \sum_{(B)_{(h,b-h)}} AB_{(1)} \otimes B_{(2)}. \]

The Propositions below exploit the geometric meaning of the geometric products \( \phi_{12}^{(h)} \), \( h \) any positive integer. In the language of Grassmann, Proposition 1 deals with real regressive products ([17], §125). The proof of Proposition 1 is nowadays a simple exercise of multilinear algebra; nonetheless, it is worth noticing that this Proposition and its proof are quite close to Grassmann’s way of manipulating extensors ([17], §126, §§130-132).

Proposition 1. Let \( A \) and \( B \) be extensors, and denote by \( \overline{A}, \overline{B} \) the corresponding subspaces of \( V \). In the modular lattice of subspaces of \( V \), consider the isomorphic intervals \([\overline{A} \sim \overline{B}, \overline{A}] \cong [\overline{B}, \overline{A} \sim \overline{B}]\), and denote by \( p \) their common dimension.

- For \( h = p \), we have
  \[ \phi_{21}^{(h)} (A \otimes B) = C \otimes D, \]
  where \( C = \overline{A} \sim \overline{B} \) and \( D = \overline{A} \sim \overline{B}; \)
- For \( h > p \), we have
  \[ \phi_{21}^{(h)} (A \otimes B) = 0. \]

The operators \( \phi_{12}^{(h)} \) have analogous geometric meanings.

Proof. Let \( C \) and \( A' \) be extensors such that \( \overline{C} = \overline{A} \sim \overline{B} \) and \( CA' = A \), step\((A') = p \) (here we use juxtaposition to mean the exterior product of extensors). Let \( a, c \) denote the steps of \( A \) and \( C \), respectively. By definition, and since the exterior algebra is a bialgebra, we have
\[ \phi_{21}^{(h)} (A \otimes B) = \sum_{(a-h,h)} (CA')_{(1)} \otimes (CA')_{(2)} B = \]
\[ \sum_{h_1 + h_2 = h} \left( \sum_{(C)_{(c-h_1, h_1)}} C_{(1)} \otimes C_{(2)} \sum_{(A')_{(p-h_2, h_2)}} (A')_{(1)} \otimes (A')_{(2)} \right) (1 \otimes B) = \]
\[ \sum_{(A')_{(p-h, h)}} C(A')_{(1)} \otimes (A')_{(2)} B. \] (†)

- if \( h > p \), the sum (†) has no summand, hence reduces to zero;

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\(^3\) For the convenience of the reader, we recall that the terms *magnitude*, *system*, nearest covering system and common system of the Ausdehnungslehre corresponds to *extensor*, *subspace*, *sum* and *intersection* of subspaces.
• if \( h = p \), formula (\( \dagger \)) simplifies to \( C \otimes A'B \), and it turns out that \( \overline{A'B} = \overline{A} \sim \overline{B} \).

Example 3. (The intersection of two coplanar lines in \( \mathbb{P}^n(\mathbb{K}) \)) Let \( V \) be a vector space over the field \( \mathbb{K} \), \( \dim(V) = n + 1 \geq 4 \). Let \( p_1, p_2, q_1, q_2 \) be four mutually nonproportional vectors in \( V \) that span a 3-dimensional subspace of \( V \). The 2-extensors \( p_1p_2, q_1q_2 \) represent two different coplanar lines in \( \mathbb{P}^n(\mathbb{K}) \). The problem of finding their intersection point cannot be treated by the meet in the CG-algebra of a Peano space \( (V, [\cdot, \cdot]) \), since \( (p_1p_2) \wedge (q_1q_2) = 0 \). However, the problem can be solved in the context of the algebra \( \Lambda(V) \otimes \Lambda(V) \) endowed with geometric products. There are two extensors \( u \) and \( v \) such that

\[
\phi_{21}^{(1)}(p_1p_2 \otimes q_1q_2) = p_1 \otimes p_2q_1q_2 - p_2 \otimes p_1q_1q_2 = u \otimes v
\]

where

\[
\overline{u} = \overline{p_1p_2} – \overline{q_1q_2} \quad \text{and} \quad \overline{v} = \overline{p_1p_2} – \overline{q_1q_2}.
\]

These extensors can be described in function of the vectors \( p \)'s and \( q \)'s as follows. Without loss of generality, we can assume that the extensor \( p_2q_1q_2 \) represents the plane \( \overline{p_1p_2} – \overline{q_1q_2} \). Notice that there exists a scalar \( \lambda \in \mathbb{K} \) such that \( p_1q_1q_2 = \lambda p_2q_1q_2 \). Now, we have

\[
\phi_{21}^{(1)}(p_1p_2 \otimes q_1q_2) = p_1 \otimes p_2q_1q_2 - p_2 \otimes p_1q_1q_2 = p_1 \otimes p_2q_1q_2 - \lambda p_2 \otimes p_2q_1q_2 = (p_1 - \lambda p_2) \otimes p_2q_1q_2.
\]

Thus, we can take \( u = p_1 - \lambda p_2 \) and \( v = p_2q_1q_2 \). The vector \( p_1 - \lambda p_2 \) represents the intersection point of the coplanar lines \( \overline{p_1p_2}, \overline{q_1q_2} \) in \( \mathbb{P}^n(\mathbb{K}) \).

In the language of Grassmann, Proposition 2 will deal with formal regressive products (\cite{7}, §125). Grassmann used the term formal since he didn’t found any geometric meaning for them. As a matter of fact, Proposition 2 exhibits such a geometric meaning, that will turn out to be closely related to the notion of Hodge **-operator**.

Proposition 2. Let \( A \) and \( B \) be extensors, and denote by \( \overline{A}, \overline{B} \) the corresponding subspaces of \( V \). In the modular lattice of subspaces of \( V \), consider the isomorphic intervals \( [\overline{A} \sim \overline{B}, \overline{A}] \cong [\overline{B}, \overline{A} \sim \overline{B}] \), and denote by \( p \) their common dimension. For \( 0 \leq h \leq p \), we have:

• the left span of \( \phi_{21}^{(h)}(A \otimes B) \) equals the linear span of all the extensors \( H \) that represent subspaces \( \overline{H} \) of codimension \( h \) in the interval \( [\overline{A} \sim \overline{B}, \overline{A}] \);

• the right span of \( \phi_{21}^{(h)}(A \otimes B) \) equals the linear span of all the extensors \( K \) that represent subspaces \( \overline{K} \) of dimension \( h \) in the interval \( [\overline{B}, \overline{A} \sim \overline{B}] \).

The operators \( \phi_{12}^{(h)} \) have analogous geometric meanings.
Proof. Let $C$ and $A'$ be extensors such that $C = \overline{A} \setminus \overline{B}$ and $CA' = A$, $\text{step}(A') = p$. We have

$$\phi_{21}^{(h)}(A \otimes B) = \sum_{(A')_{(p-h,h)}} C(A'_{(1)}) \otimes (A'_{(2)})B. \quad (\dagger)$$

Assume that

$$\Delta(A') = \sum_{(A')_{(p-h,h)}} (A'_{(1)}) \otimes (A'_{(2)})$$

is a minimal representation of the tensor $\Delta(A')$; thus it has exactly $\binom{p}{h}$ terms, the set of the extensors $(A'_{(1)})$ is linearly independent, the set of the extensors $(A'_{(2)})$ is linearly independent.

Then the sum $(\dagger)$ has $\binom{p}{h}$ terms; each of the extensors $C(A'_{(1)})$ represents a subspace of codimension $h$ in the interval $[\overline{A} \setminus \overline{B}, \overline{A}]$, and the set of these extensors is linearly independent; each of the extensors $(A'_{(2)})B$ represents a subspace of dimension $h$ in the interval $[\overline{B}, \overline{A} \setminus \overline{B}]$, and the set of these extensors is linearly independent.

The left span of $\phi_{21}^{(h)}(A \otimes B)$ is then the space spanned by the extensors $C(A'_{(1)})$, which turns out to be the space spanned by all the extensors $H$ that represent subspaces $\overline{H}$ of codimension $h$ in the interval $[\overline{A} \setminus \overline{B}, \overline{A}]$.

The right span of $\phi_{21}^{(h)}(A \otimes B)$ is then the space spanned by the extensors $(A'_{(2)})B$, which turns out to be the space spanned by all the extensors $K$ that represent subspaces $\overline{K}$ of dimension $h$ in the interval $[\overline{B}, \overline{A} \setminus \overline{B}]$.

Remark 1. We mention that geometric products provide a simple characterization of the inclusion relation between subspaces. Given two extensors $A$ and $B$,

$$\overline{A} \subseteq \overline{B} \text{ if and only if } \phi_{21}^{(1)}(A \otimes B) = 0.$$  

Example 4. (Two non-coplanar lines in $\mathbb{P}^n(\mathbb{K})$, $n \geq 3$) Let $V$ be a vector space over the field $\mathbb{K}$, $\dim(V)n + 1 \geq 4$. Let $p_1, p_2, q_1, q_2$ be four vectors in $V$ that span a 4-dimensional subspace of $V$. The 2-extensors $p_1p_2, q_1q_2$ represent two non-coplanar lines in $\mathbb{P}^n(\mathbb{K})$. The tensor

$$\phi_{21}^{(1)}(p_1p_2 \otimes q_1q_2) = p_1 \otimes p_2q_1q_2 - p_2 \otimes p_1q_1q_2$$

is not a decomposable tensor. This representation is a minimal one.

The left span of $\phi_{21}^{(1)}(p_1p_2 \otimes q_1q_2)$ is the vector space spanned by the set of vectors $\{p_1, p_2\}$, that equals the vector space spanned by the vectors that represent the points in the projective line $\overline{p_1p_2}$.

The right span of $\phi_{21}^{(1)}(p_1p_2 \otimes q_1q_2)$ is the vector space spanned by the set of 3-extensors $\{p_2q_1q_2, -p_1q_1q_2\}$, that equals the vector space spanned by the 3-extensors that represent the planes that contain the projective line $\overline{p_1q_2}$, and are contained in the projective subspace $\overline{p_1p_2q_2}$.

More generally, let $\Lambda(V)^{\otimes m}$ be the $m$-th tensor power of the exterior algebra $\Lambda(V)$, in the category of $\mathbb{Z}_2$-graded $\mathbb{K}$-algebras. On this algebra we have a family of geometric products

$$\phi_{ij}^{(h)} : \Lambda(V)^{\otimes m} \to \Lambda(V)^{\otimes m},$$
for every \( m = 2, 3, \ldots \) and \( 1 \leq i, j \leq m, i \neq j \), defined as follows.

Let \( i < j \); for any \( A_1, \ldots, A_m \) extensors of steps \( a_1, \ldots, a_m \). The rising geometric products are given by

\[
\phi^{(h)}_{ji} (A_1 \otimes \cdots \otimes A_i \otimes \cdots \otimes A_j \otimes \cdots \otimes A_m) = (-1)^{h(a_{i+1}+\cdots+a_{j-1})} \sum_{(A_i)_{(a_i-h,h)}} A_1 \otimes \cdots \otimes (A_i)_{(1)} \otimes \cdots \otimes (A_i)_{(2)} A_j \otimes \cdots \otimes A_m.
\]

The lowering geometric products are given by

\[
\phi^{(h)}_{ij} (A_1 \otimes \cdots \otimes A_i \otimes \cdots \otimes A_j \otimes \cdots \otimes A_m) = (-1)^{h(a_{i+1}+\cdots+a_{j-1})} \sum_{(A_i)_{(h,a_j-h)}} A_1 \otimes \cdots \otimes A_i (A_j)_{(1)} \otimes \cdots \otimes (A_j)_{(2)} \otimes \cdots \otimes A_m.
\]

### 3.3 Whitney algebras

Let \( M = M(S) \) be a matroid of rank \( n \) over a set \( S \). Let \( \text{Skew}(S) \) be the free skew-symmetric algebra over \( \mathbb{Z} \) on the set \( S \), and let \( \text{Skew}(S)^{\otimes m} \) be the \( m \)-th tensor power algebra of the \( \mathbb{Z} \)-algebra \( \text{Skew}(S) \), in the category of \( \mathbb{Z}_2 \)-graded algebras.

For each dependent subset \( \{v_1, \ldots, v_p\} \) in \( M = M(S) \), we consider the monomial \( w = v_1 \cdots v_p \) in \( \text{Skew}(S) \) and all its possible slices in \( \text{Skew}(S)^{\otimes m} \):

\[
\Delta_{(i_1, \ldots, i_p)}(w) = \sum_{(w)_{(i_1, \ldots, i_p)}} w_{(1)} \otimes \cdots \otimes w_{(m)},
\]

where \( i_1 + \cdots + i_m = p \). Let \( I(M) \) be the (bilateral) ideal of \( \text{Skew}(S)^{\otimes m} \) generated by all these slices.

The \( m \)-th Whitney algebra of \( M \) is the quotient algebra

\[
W^m(M) = \text{Skew}(S)^{\otimes m}/I(M).
\]

Notice that in \( W^1(M) \) words corresponding to independent subsets of \( S \) that span the same flat are not necessarily equal up to a scalar. For every tensor \( u_1 \otimes u_2 \otimes \cdots \otimes u_m \) in \( \text{Skew}(S)^{\otimes m} \), its image in \( W^m(M) \) is denoted by \( u_1 \circ u_2 \circ \cdots \circ u_m \).

In the tensor power algebra \( \text{Skew}(V)^{\otimes m} \), geometric products are defined in strict analogy with those defined in \( \Lambda(V)^{\otimes m} \), and are still denoted by \( \phi^{(h)}_{ij} \), for every \( m = 2, 3, \ldots \) and \( 1 \leq i, j \leq m, i \neq j \).

Indeed, geometric products \( \phi^{(h)}_{ij} \) are well-defined in the Whitney algebra \( W^m(M) \). In the original approach of Crapo and Schmitt [13], this is a non-trivial fact founded on the theory of lax Hopf algebras. In Section 7, we show that this fact directly follows from the letterplace encoding of Whitney algebras in combination with the definition of geometric products in terms of divided powers of place polarization operators.
4 Relations among the algebras

4.1 From tensor powers of exterior algebras to CG-Algebras

Let $V$ be a vector space, $dim(V) = n$. In the exterior algebra $\Lambda(V)$ we fix an extensor $E$ of step $n$. $E$ is usually called the integral. The choice of the integral induces the choice of a bracket on $V$, by setting:

$$[x_1, \ldots, x_n]E = x_1 \cdots x_n, \quad \text{for every } x_1, \ldots, x_n \in V.$$

(Here we use juxtaposition to denote the wedge product).

Thus, the vector space $V$ becomes a Peano space $(V, [\ ]).$ The meet product in the CG-algebra of $(V, [\ ])$ is recovered from the geometric product in $\Lambda(V)^\otimes 2$ by means of the following identity. Let $A$ and $B$ be extensors of steps $a$ and $b$, respectively. We have

$$\phi^{(n-a)}_{12} (A \otimes B) = \sum_{(B)_{n-a,a+b-n}} AB(1) \otimes B(2) = \sum_{(B)_{n-a,a+b-n}} E \otimes [AB(1)]B(2) = (-1)^{(a+b-n)(n-a)} E \otimes (A \wedge B)$$

Clearly, $\phi^{(n-b)}_{21} (A \otimes B) = 1 \otimes (A \lor B)$.

Similarly, we have

$$\phi^{(a)}_{12} (A \otimes B) = \sum_{(A)_{a+b-n,n-b}} A(1) \otimes A(2) B = \sum_{(A)_{a+b-n,n-b}} A(1)[A(2)B] \otimes E = (-1)^{(a+b-n)(n-b)} (A \wedge B) \otimes E$$

Clearly, $\phi^{(b)}_{12} (A \otimes B) = (A \lor B) \otimes 1$.

4.2 From Whitney algebras to tensor powers of exterior algebras.

Let $M = M(S)$ be a matroid on a finite set $S$, and $V$ a vector space over some field $\mathbb{K}$. A representation of $M$ in $V$ is a mapping $g : S \rightarrow V$ such that a set $A \subseteq S$ is independent in $M$ iff $g$ is one-to-one on $A$ and the set $g(A)$ is linearly independent in $V$. A matroid that admits a representation is said to be a representable matroid. In this case, for every $m \in \mathbb{Z}^+$, there is exactly one ring morphism $\hat{g}^m : W^m(M) \rightarrow \Lambda^\otimes m(V)$ such that

$$\hat{g}^m(1 \circ \cdots \circ x \circ 1 \circ \cdots \circ 1) = 1 \otimes \cdots \otimes g(x) \otimes 1 \otimes \cdots \otimes 1$$

for all $x$ in $S$. Furthermore we have:

$$\hat{g}^m \circ \phi^{(h)}_{ij} = \phi^{(h)}_{ij} \circ \hat{g}^m.$$
for every \( h \in \mathbb{Z}^+ \), and every \( 1 \leq i, j \leq m \), with \( i \neq j \).

A matroid \( M \) is representable if and only if in each Whitney algebra \( W^m(M) \), \( m = 1, 2, \ldots \) of \( M \) each product \( w_1 \circ w_2 \circ \ldots \circ w_m \) of words \( w_i \) corresponding to independent subsets of \( S \) is not zero (\cite{13} Corollary 8.9, p. 256).

5 The Hodge operators

5.1 The Hodge \( \ast \) operators

Let \( V \) be a vector space, and let \( (f_1, \ldots, f_n) \) be an ordered basis of \( V \). To each subset \( I = \{i_1, \ldots, i_k\} \) of the set \( \{1, \ldots, n\} \), where \( 1 \leq i_1 < \cdots < i_k \leq n \), corresponds a canonical extensor \( f_I = f_{i_1} \lor \cdots \lor f_{i_k} \); for \( I = \emptyset \) we have \( f_I = 1 \), and for \( I = \{1, \ldots, n\} \) we set \( f_I = F \). When there is no danger of misunderstanding, we will identify a subset with the corresponding canonical extensor, and instead of \( f_I \) we will write \( I \).

The Hodge star operator associated to the basis \( (f_1, \ldots, f_n) \) is the linear operator \( \ast : \Lambda(V) \to \Lambda(V) \) whose values on the canonical extensors are given by

\[
\ast A = \epsilon_A A',
\]

where \( A' \) is the complementary subset of \( A \) in \( \{1, \ldots, n\} \), and \( \epsilon_A \) is the sign implicitly defined by \( F = \epsilon_A AA' \). We have \( \ast 1 = F \) and \( \ast F = 1 \); notice that

\[
\ast (\ast A) = (-1)^{k(n-k)} A, \quad k = \text{step}(A).
\]

Remark 2. The Hodge \( \ast \) operators are defined by choosing a linearly ordered basis of the vector space \( V \). A natural question arises: under which conditions two Hodge \( \ast \) operators coincide? The answer is a classical result.

Let \( F = (f_1, \ldots, f_n) \) and \( F' = (f'_1, \ldots, f'_n) \) be two ordered bases of \( V \), and let \( \ast \) and \( \ast' \) be the Hodge operators associated to them, respectively. The operators \( \ast \) and \( \ast' \) coincide if and only if the transition matrix from \( F \) to \( F' \) is an orthogonal matrix (see. e.g, \cite{2}, Theorem 6.2).

We recall that the Hodge \( \ast \) operators implement the duality between join and meet (\cite{2}, Theorem 6.3). More specifically, in the exterior algebra \( \Lambda(V) \) we fix an integral \( E \), and, consequently, a bracket \( [\ ] \). Thus, the vector space \( V \) becomes a Peano space \( (V, [\ ]) \), and we can consistently consider its CG-algebra. Then we have the identities

\[
[F] \ast (A \lor B) = (\ast A) \land (\ast B),
\]

\[
[F]^{-1} \ast (A \land B) = (\ast A) \lor (\ast B).
\]

Now, assume that \( F = (f_1, \ldots, f_n) \) is a unimodular basis of \( V \), that is \( [f_1, \ldots, f_n] = 1 \), or, equivalently, \( F = f_1 \lor \cdots \lor f_n = E \). Then the preceding identities specialize to

\[
\ast (A \lor B) = (\ast A) \land (\ast B),
\]

\[
\ast (A \land B) = (\ast A) \lor (\ast B).
\]

The preceding results generalize to a result on geometric products.
**Theorem 1.** Let $*$ be the Hodge star operator associated to a basis of the vector space $V$. We have the identity

$$(\ast \otimes \ast) \circ_{21} (A \otimes B) = (-1)^{h(a+b-n)} \circ_{12} (\ast \otimes \ast) (A \otimes B),$$

where $A$ and $B$ are extensors of steps $a$ and $b$, respectively.

**Proof.** On the one hand, we have

$$(\ast \otimes \ast) \circ_{21} (A \otimes B) = (\ast \otimes \ast) \sum_H \zeta_H' \zeta_H'' (A \cap H') \otimes (B \cup H),$$

where $H$ runs through the $h-$subsets of $A \cap B'$ and

$$A = \zeta_H'(A \cap H') H, \quad HB = \zeta_H''(B \cup H).$$

On the other hand, we have

$$\circ_{12} (\ast \otimes \ast) (A \otimes B) = \circ_{12} (\epsilon_A \epsilon_B A' \otimes B')$$

$$= \sum_H \epsilon_A \epsilon_B \eta_H' \eta_H'' (A' \cup H) \otimes (B' \cap H'),$$

where $H$ runs through the $h-$subsets of $B' \cap A$ and

$$B' = \eta_H'(B' \cap H'), \quad A' H = \eta_H''(A' \cup H).$$

A rather tedious sign computation gives

$$\zeta_H' \zeta_H'' \epsilon_B \epsilon_H A \cap H' = (-1)^{h(a+b-n)} \epsilon_A \epsilon_B \eta_H' \eta_H''.$$

\[ \Box \]

In the last part of this subsection we will describe the way to specialize the preceding Theorem in order to obtain the main result that relates Hodge operators to join and meet in CG-algebras.

Let $V$ be a vector space, $\dim(V) = n$. In the exterior algebra $\Lambda(V)$ we fix an integral $E$, and, consequently, a bracket $[ ]$. Thus, the vector space $V$ becomes a Peano space $(V, [ ])$, and we can consistently consider its CG-algebra.

In the previous Theorem, by setting $h = a$, the step of $A$, we have that the left hand side becomes

$$(\ast \otimes \ast) \circ_{21}^{(a)} (A \otimes B) = (\ast \otimes \ast) (1 \otimes (A \vee B)) = F \otimes \ast (A \vee B) = [F]E \otimes \ast (A \vee B),$$

the right hand side becomes

$$\circ_{12}^{(a)} (\ast \otimes \ast) (A \otimes B) = \circ_{12}^{(a)} ((\ast A) \otimes (\ast B)) = (-1)^{a(a+b-n)} E \otimes ((\ast A) \land (\ast B)),$$
thus we get
\[ [F] \ast (A \lor B) = (\ast A) \land (\ast B). \]

In the previous Theorem, by setting \( h = n - b \), the costep of \( B \), we have that the left hand side becomes
\[
(* \otimes *) \diamond_{21}^{(n-b)} (A \otimes B) = (-1)^{(a+b-n)(n-b)} (\ast (A \land B) \otimes E)
\]
\[ = (-1)^{(a+b-n)(n-b)} [F]^{-1} (\ast (A \land B) \otimes 1), \]

the right hand side becomes
\[
\diamond_{12}^{(n-b)} (\ast \otimes \ast) (A \otimes B) = \diamond_{12}^{(n-b)} ((\ast A) \otimes (\ast B)) = ((\ast A) \lor (\ast B)) \otimes 1,
\]

thus we get
\[ [F]^{-1} \ast (A \land B) = (\ast A) \lor (\ast B). \]

**Remark 3.** From Remark 1, it follows that the Hodge \( \ast - \) operator induces an antitone mapping on the lattice of subspaces. Indeed, by setting in the previous Theorem \( h = 1 \), we have
\[
(* \otimes *) \diamond_{21}^{(1)} (A \otimes B) = (-1)^{(a+b-n)(1)} \diamond_{12}^{(1)} (\ast \otimes \ast) (A \otimes B).
\]
The left hand side equals zero if and only if \( \overline{A} \subseteq B \); the right hand side equals zero if and only if \( \overline{\ast A} \supseteq \overline{\ast B} \). Thus,
\[
\overline{A} \subseteq B \text{ if and only if } \overline{\ast A} \supseteq \overline{\ast B}.
\]

### 5.2 The generalized Hodge operators

To each tensor \( t \) in the tensor product of two vector spaces, there are associated two spaces \( L_t, R_t \), its left and right spans, a class of isomorphism \( L_t \rightarrow R_t \), and a canonical nonsingular bilinear map \( L_t \times R_t \rightarrow \mathbb{K} \). We will exploit these constructions and invariants for the geometric product of two extensors in \( \Lambda(V) \), which is a tensor in \( \Lambda(V) \otimes \Lambda(V) \).

Let \( (f_1, \ldots, f_n) \) be a linearly ordered basis of a vector space \( V \), set \( F = f_1 \lor \cdots \lor f_n \), and let \([ \ ]\) be the corresponding bracket. Consider the tensor
\[
\left( \sum_{h=0}^{n} \diamond_{21}^{(h)} (F \otimes 1) \right) = \sum_{(F)} F_{(1)} \otimes F_{(2)},
\]

and notice that: the left span and the right span are \( \Lambda(V) \); for any minimal representation in which \( F_{(1)} \) and \( F_{(2)} \) are subwords of \( F \), the associated linear map \( \Lambda(V) \rightarrow \Lambda(V) \) is the Hodge \( \ast - \) operator associated to the basis \( (f_1, \ldots, f_n) \); the canonical bilinear mapping \( \Lambda(V) \times \Lambda(V) \rightarrow \mathbb{K} \) is the cap-product, defined by setting \( X \times Y \mapsto [XY] \), for any \( X, Y \) extensors of complementary steps, and \( X \times Y \mapsto 0 \) otherwise.

In the following, we will see how these facts generalize to any geometric product. Consider two extensors \( A, B \). Let \( A = CA' \) be a factorization of \( A \) as the product of two extensors, where \( C \) is an extensor such that \( C = A \rightharpoonup B \), and set \( D = A'B \), so that \( \overline{D} = \overline{A} \rightharpoonup \overline{B} \). Denote by \( p \) the step of \( A' \).
Consider the tensor
\[
\left( \sum_{h=0}^{n} \odot_{21}^{(h)} \right) (A \otimes B) = \sum_{(A')} CA'_{(1)} \otimes A'_{(2)} B
\]
where
\[
\Delta(A') = \sum_{(A')} A'_{(1)} \otimes A'_{(2)}.
\]
Choose a minimal representation of \(\Delta(A')\), by representing the extensor \(A'\) as an exterior product \(a_1 \vee \cdots \vee a_p, \ a_i \in V\), and writing it as
\[
\Delta(A') = \sum_{i=0}^{r} \epsilon_i A'_{i,(1)} \otimes A'_{i,(2)},
\]
where \(A'_{i,(1)}\) and \(A'_{i,(2)}\) are increasing subwords of the word \(a_1 \vee \cdots \vee a_p\) (in the preceding coproduct expression, we explicitly write the sign of the summands; it is a slight modification of the Sweedler notation that will turn to be useful in the subsequent part of this paragraph. The symbol \(\epsilon_i\) represents a sign, whose meaning should be obvious).

The representation
\[
\left( \sum_{h=0}^{n} \odot_{21}^{(h)} \right) (A \otimes B) = \sum_{i=1}^{r} \epsilon_i CA'_{i,(1)} \otimes A'_{i,(2)} B \quad (\dagger)
\]
is independent, thus minimal. Thus we have:

1. The left span is \(L = \left\langle CA'_{i,(1)}; \ i = 1, \ldots, r \right\rangle\) which is the linear span of all the extensors that represent subspaces in the interval \([A \sim B, A]\).

The right span is \(R = \left\langle \epsilon_i A'_{i,(2)} B; \ i = 1, \ldots, r \right\rangle\), which is the linear span of all the extensors that represent subspaces in the interval \([B, A \sim B]\).

2. The generalized Hodge \(\star_{\dagger}-\)operator associated to the minimal representation (\(\dagger\)) is defined to be the linear operator
\[
\star_{\dagger} : L \rightarrow R, \quad \star_{\dagger} : CA'_{i,(1)} \mapsto \epsilon_i A'_{i,(2)} B, \quad i = 1, \ldots, r.
\]

Consider the factorization
\[
\star_{\dagger} = \varphi \circ \tilde{\star}.
\]
where
\[
\tilde{\star} : L \rightarrow L, \quad \tilde{\star} : CA'_{i,(1)} \mapsto \epsilon_i CA'_{i,(2)}, \quad i = 1, \ldots, r,
\]
\[
\varphi : L \rightarrow R, \quad \varphi : CH' \mapsto H'B.
\]

Notice that: the linear map \(\tilde{\star}\) is antitone, since it can be identified with the Hodge \(\star-\)operator
\[
\star : \Lambda(A') \rightarrow \Lambda(A'), \quad \star : A'_{i,(1)} \mapsto \epsilon_i A'_{i,(2)};\]
the linear map \(\varphi\) is monotone. Thus the generalized Hodge \(\star_{\dagger}-\)operator \(\star_{\dagger} = \varphi \circ \tilde{\star}\) is antitone.
3. The canonical pairing $\beta : L \times R \to K$ is defined by

$$\beta (C A'_i (1), \epsilon_j A'_j (2) B) = \delta_{ij}, \quad i, j = 1, \ldots, r.$$ 

We have also the following intrinsic characterization of $\beta$.

Let $X$ be an extensor in $L$, let $Y$ be an extensor in $R$, and denote by $k$ the relative step of $X$ with respect to $C$. Then

$$\Diamond^{(k)}_{21} (X \otimes Y) = \beta (X, Y) C \otimes D.$$ 

Indeed, for $X = C A'_i (1)$ and $Y = \epsilon_j A'_j (2) B$, we have

$$\Diamond^{(k)}_{21} (C A'_i (1) \otimes \epsilon_j A'_j (2) B) = \epsilon_j C \otimes A'_i (1) A'_j (2) B = \delta_{ij} C \otimes D.$$

6 Letterplace algebras and polarization operators

The algebras $\Lambda (V)^{\otimes m}$ and $W^m (M)$ and their geometric products admit natural and efficient descriptions in terms of (skew-symmetric) Letterplace algebras and place polarization operators (see, e.g. [5]).

6.1 Skew-symmetric Letterplace algebras and place polarization operators.

First of all, take a set of symbols $L$ and the set $m = \{1, 2, \ldots, m\}$ of the first $m$ positive integers; the set $[L|m] = \{(x|i); x \in L, i \in m\}$ is called the set of letterplace variables. The elements of $L$ are called letters, the elements of $m$ are called places.

The skew-symmetric Letterplace algebra $Skew [L|m]$ is the free associative skew-symmetric (unitary) $\mathbb{Z}$–algebra generated by the set $[L|m]$.

Given $h, k \in m$, the place polarization operator from $h$ to $k$ is the unique derivation $D_{kh}$ of $Skew [L|m]$ such that

$$D_{kh} (x|i) = \delta_{hi} (x|k),$$

for every $x \in L, i \in m$. In particular, the action of a polarization on a monomial is given by

$$D_{kh} ((x_{i_1}|j_1)(x_{i_2}|j_2) \cdots (x_{i_r}|j_n)) =$$

$$= \sum \delta_{h_j} (x_{i_1}|j_1) \cdots (x_{i_{r-1}}|j_{r-1})(x_{i_r}|k)(x_{i_{r+1}}|j_{r+1}) \cdots (x_{i_n}|j_n).$$

The place polarization operators satisfy the commutation relations

$$D_{kh} D_{ji} - D_{ji} D_{kh} = \delta_{hj} D_{ki} - \delta_{ki} D_{jh},$$

thus they implement a representation of the general linear Lie algebra $gl(m, \mathbb{Z})$ on $Skew [L|m]$. 
6.2 Biproducts in skew-symmetric letterplace algebras and divided powers of polarization operators

We provide a brief description of the divided power notation for the biproducts

\[(w|i_1^{(q_1)}i_2^{(q_2)}\ldots i_p^{(q_p)}), \quad w \text{ a word on } L, \quad i_1, \ldots, i_p \in m,\]

in the skew-symmetric letterplace algebra \(Skew[L|m]\). In the following, we will denote the length of a word \(w \in Mon[L]\) by the symbol \(|w|\).

- Let \(p, q \in \mathbb{Z}^+, x_1, x_2, \ldots, x_p \in L, i \in m\). Then
  \[(x_1x_2\cdots x_p|i^{(q)}) = (x_1|i)(x_2|i)\cdots(x_p|i),\]
  if \(p = q\), and is zero otherwise. We also use the convention that
  \[(w|i^{(0)}) = 1, \quad 1 \text{ the unity of the algebra},\]
  if \(w\) is the empty word, and equals zero otherwise.

- (Laplace expansion) Let \(w\) be a word on the alphabet \(L\). Let \(q_1, \ldots, q_m\) be nonnegative integers. Then
  \[(w|i_1^{(q_1)}i_2^{(q_2)}\ldots i_p^{(q_p)}) = \sum_{(w)(q_1, \ldots, q_p)} (w_{(1)}|i_1^{(q_1)})(w_{(2)}|i_2^{(q_2)})\cdots(w_{(p)}|i_p^{(q_p)}),\]
  if \(|w| = q_1 + \cdots + q_p\), and is zero otherwise.

The biproducts are skew-symmetric in the letters:

\[\cdots x_i \cdots x_j \cdots|i_1^{(q_1)} \cdots i_p^{(q_p)} = -(\cdots x_j \cdots x_i \cdots|i_1^{(q_1)} \cdots i_p^{(q_p)}),\]

and symmetric in the places:

\[(x_1 \cdots x_t|\cdots s^{(q_s)}\cdots)t^{(q_t)} \cdots = (x_1 \cdots x_t|\cdots t^{(q_t)}\cdots s^{(q_s)} \cdots);\]

thus, in particular, each biproduct can be written in the form

\[(w|1^{(q_1)}2^{(q_2)}\cdots m^{(q_m)}).\]

**Remark 4.** The biproducts satisfy the anticommutation relations

\[(w|i_1^{(p_1)}\cdots s^{(p_s)}) (w'|j_1^{(q_1)}\cdots j_t^{(q_t)}) = (-1)^{pq} (w'|j_1^{(q_1)}\cdots j_t^{(q_t)}) (w|i_1^{(p_1)}\cdots s^{(p_s)}),\]

where \(p = \sum p_i\) and \(q = \sum q_i\).

Now we consider the action of place polarization operators on biproducts.

**Proposition 3.** For each nonnegative integer \(h\) and \(1 \leq i < j \leq m\), we have

\[D_{ji}^h(w|\cdots i^{(q_i)}\cdots j^{(q_j)}\cdots) = \begin{cases} \frac{(q_i+h)!}{q_i!} (w|\cdots i^{(q_i-h)}\cdots j^{(q_j+h)}\cdots) \\ 0 \end{cases}\]

according to \(h \leq q_i\), or \(h > q_i\). An analogous result holds for \(i > j\).
Let \( i, j \in m \), with \( i \neq j \). Given a nonnegative integer \( h \), the \( h \)-th divided power \( D_{ji}^{(h)} \) of the place polarization \( D_{ji}^{(h)} \) is, by definition, the operator

\[ D_{ji}^{(h)} = \frac{D_{ji}^{(h)}}{h!}. \]

We claim that the next corollary implies that the action of a divided power of a place polarization operator is well-defined on the algebra \( \text{Skew}[L|m] \).

**Corollary 1.** For each natural integer \( h \) and \( 1 \leq i < j \leq m \), we have

\[ D_{ji}^{(h)}(w|i(q_i)\ldots j(q_j)\ldots) = \begin{cases} (q_j + h)(q_i - h)\ldots j(q_j)\ldots & \text{if } h \leq q_i, \text{ or } h > q_i. \\ 0 & \text{if } h \leq q_i, \text{ or } h > q_i. \end{cases} \]

according to \( h \leq q_i \), or \( h > q_i \). In particular,

\[ D_{ji}^{(h)}(w|[q]) = \delta_{ik}(w|i(q_i - h)j(q_j)), \]

An analogous result holds for \( i > j \).

### 6.3 The Straightening Law

In the context of the skew-symmetric Letterplace algebra \( \text{Skew}[L|m] \), the supersymmetric Straightening Law of Grosshans, Rota and Stein [18] can be stated as follows.

**Theorem 2.** Let \( u, v, w \) be words on \( L \); let \( p_1, \ldots, p_m \) and \( q_1, \ldots, q_m \), be \( m \)-tuples of nonnegative integers, and set \( \sum p_i = p \) and \( \sum q_i = q \). Then

\[
\sum_{(v)} (uv_1)|1(p_1)\ldots m(p_m))(v_2w|1(q_1)\ldots m(q_m)) = (-1)^{|u||v|} \times \\
\sum_{(u),r} (-1)^{|u_2|} c_r(vu_1)|1(p_1+r_1)\ldots m(p_m+r_m))(u_2w|1(q_1-r_1)\ldots m(q_m-r_m)),
\]

where \( c_r = \prod_{i} \binom{p_i+r_i}{r_i} \). The sum in the left-hand side is extended to all the slices of the word \( v \); the sum in the right-hand side is extended to all the slices of the word \( u \), and to all the nonnegative integers \( r_i \) such that \( 0 \leq r_i \leq q_i \).

**Corollary 2.** Let \( u, v \) be words on \( L \); let \( 1 \leq i \neq j \leq m \), and \( p, q \) nonnegative integers. Then

\[
\sum_{(v)} (uv_1)i(p)v_2j(q) = (-1)^{|u||v|} \sum_{(u),r} (-1)^{|u_2|} (vu_1)i(p)r)(u_2j(q-r)).
\]

The sum in the left-hand side is extended to all the slices of the word \( v \); the sum in the right-hand side is extended to all the slices of the word \( u \), and to all the nonnegative integers \( r \) such that \( 0 \leq r \leq q \).
7 Letterplace encodings

7.1 The letterplace encoding of the algebra $Skew[L]^{\otimes m}$

Given a set $L$, consider the free associative skew-symmetric (unitary) $\mathbb{Z}$–algebra $Skew[L]$ generated by the set $L$. On each $\mathbb{Z}$–algebra tensor power $Skew[L]^{\otimes m}$, $m \in \mathbb{Z}^+$, we may consider the “formal” geometric products $\diamond_{ji}^{(h)}$, defined and denoted in the same way as in Subsection 3.2. There is a $\mathbb{Z}$-algebra isomorphism

$$\Phi : Skew[L|m] \rightarrow Skew[L]^{\otimes m},$$

$$(x|i) \mapsto 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1,$$

(where $x \in L$ is in the $i$–th fold in the tensor product).

**Proposition 4.** The geometric products on $Skew[L]^{\otimes m}$ correspond, via the isomorphism $\Phi$, to the divided powers of place-polarizations on $Skew[L|m]$:

$$\diamond_{ji}^{(h)} = \Phi \circ D_{ji}^{(h)} \circ \Phi^{-1}, \quad i \neq j, \quad h \in \mathbb{Z}^+.$$

**Proof.** We consider the case of raising geometric products, i.e. the case $i < j$. We will prove that the operators $\diamond_{ji}^{(h)} \circ \Phi$ and $\Phi \circ D_{ji}^{(h)}$ have the same action on the monomials of the type

$$(A_1|l^{(q_1)}) \cdots (A_i|l^{(q_i)}) \cdots (A_j|j^{(q_j)}) \cdots (A_m|m^{(q_m)}).$$

On the one hand, we have

$$\diamond_{ji}^{(h)} \left( \Phi \left( \cdots (A_i|l^{(q_i)}) \cdots (A_j|j^{(q_j)}) \cdots \right) \right) =$$

$$\diamond_{ji}^{(h)} \left( \cdots \otimes A_i \otimes \cdots \otimes A_j \otimes \cdots \right) =$$

$$= (-1)^{h(q_{i+1} + \cdots + q_{j-1})} \sum_{(A_i)_{(q_i-h,h)}} \cdots \otimes (A_1)_{(1)} \otimes \cdots \otimes (A_i)_{(2)} A_j \otimes \cdots.$$

On the other hand, we have

$$\Phi \left( D_{ji}^{(h)} \left( \cdots (A_i|l^{(q_i)}) \cdots (A_j|j^{(q_j)}) \cdots \right) \right) =$$

$$\Phi \left( \cdots (A_i|l^{(q_i-h)}j^{(h)}) \cdots (A_j|j^{(q_j)}) \cdots \right) =$$

$$\Phi \left( \sum_{(A_i)_{(q_i-h,h)}} \cdots ((A_i)_{(1)}|l^{(q_i-h)}) ((A_i)_{(2)}|j^{(h)}) \cdots (A_j|j^{(q_j)}) \cdots \right) =$$

$$(-1)^{h(q_{i+1} + \cdots + q_{j-1})} \Phi \left( \sum_{(A_i)} \cdots ((A_i)_{(1)}|l^{(q_i-h)}) \cdots ((A_i)_{(2)} A_j|j^{(q_j+h)}) \cdots \right) =$$

$$= (-1)^{h(q_{i+1} + \cdots + q_{j-1})} \sum_{(A_i)} \cdots \otimes (A_1)_{(1)} \otimes \cdots \otimes (A_i)_{(2)} A_j \otimes \cdots.$$
In particular, we have that each geometric product $\diamond_{ji}^{(1)}$ correspond to a polarization $D_{ij}$, for $i \neq j$; this leads us to define the $\diamond_{ji}^{(1)}$ as the correspondent of the polarization $D_{ii}$.

We write each geometric product $\diamond_{ji}^{(1)}$ briefly as $\diamond_{ji}$, and call it a simple geometric product, for $1 \leq i, j \leq m$. The encoding of tensor powers of free skew-symmetric algebras and geometric products with skew-symmetric letterplace algebras and polarization operators leads to the following characterization and properties of simple geometric products.

**Proposition 5.** The simple geometric products on the algebra $\text{Skew}[L]^{\otimes m}$ enjoy the following properties:

- Each $\diamond_{ij}$ is a derivation on the algebra $\text{Skew}[L]^{\otimes m}$; it is the unique derivation such that
  $$\diamond_{ij} (\cdots \otimes 1 \otimes a \otimes 1 \cdots) = \delta_{ih} \cdots \otimes 1 \otimes 1 \cdots \otimes 1 \otimes a \otimes 1 \cdots,$$
  for any $a \in L$. In the left hand side $a$ occurs in the $h$–th fold, in the right hand side $a$ occurs in the $j$–th fold.

- The space spanned by the $\diamond_{ij}$ for $1 \leq i, j \leq m$ is a Lie subalgebra of the Lie algebra of derivations on $\text{Skew}[L]^{\otimes m}$; furthermore, simple geometric products satisfy the commutation relations
  $$\diamond_{ij} \diamond_{hk} - \diamond_{hk} \diamond_{ij} = \delta_{jh} \diamond_{ik} - \delta_{ki} \diamond_{hj}.$$
  Thus, the mapping that sends each $m \times m$ elementary matrix with $1$ in the $(i, j)$–th entry to the simple geometric product $\diamond_{ij}$ induces a Lie representation of the general linear Lie algebra $gl(m, \mathbb{Z})$ in the Lie algebra of derivations of $\text{Skew}[L]^{\otimes m}$.

Notice that the geometric products are divided powers of simple geometric products:

$$\diamond_{ij}^{(h)} = \frac{\diamond_{ij}^{h}}{h!}, \quad i \neq j.$$

### 7.2 The Letterplace encoding of $\Lambda(V)^{\otimes m}$.

Let $\mathbb{K}$ be a field, and $V$ a finite-dimensional $\mathbb{K}$–vector space. For each tensor power $\Lambda(V)^{\otimes m}$ of the exterior algebra of $V$, we have the isomorphisms

$$\Lambda(V)^{\otimes m} \cong \frac{\mathbb{K} \otimes \text{Skew}[V]^{\otimes m}}{I_m(V)} \cong \frac{\mathbb{K} \otimes \text{Skew}[V[m]}{J_m(V)},$$

where: $I_m(V)$ is the bilateral ideal of the algebra $\mathbb{K} \otimes \text{Skew}[V]^{\otimes m}$ generated by the elements of the form $1 \otimes \cdots \otimes v \otimes \cdots 1 - \lambda \cdot 1 \otimes \cdots \otimes v_1 \otimes \cdots 1 - \mu \cdot 1 \otimes \cdots \otimes v_2 \otimes \cdots 1$ for every $v = \lambda v_1 + \mu v_2 \in V$, and every fold of the tensor; $J_m(V)$ is the bilateral ideal of the
algebra $\mathbb{K} \otimes \text{Skew}[V|m]$ generated by the elements of the form $(v|i) - \lambda(v_1|i) - \mu(v_2|i)$ for every $v = \lambda v_1 + \mu v_2 \in V$, and every $1 \leq i \leq m$. Thus we have an isomorphism

$$\Phi_V : \frac{\mathbb{K} \otimes \text{Skew}[V|m]}{J_m(V)} \to \Lambda(V)^{\otimes m}.$$ 

Now, each place polarization on the algebra $\mathbb{K} \otimes \text{Skew}[V|m]$ leaves invariant the ideal $J_m(V)$, thus it induces a place polarization on the quotient algebra $(\mathbb{K} \otimes \text{Skew}[V|m])/J_m(V)$.

From the results of subsection 7.1 we get, in particular:

- The geometric products on $\Lambda(V)^{\otimes m}$ correspond, via the isomorphism $\Phi_V$, to the divided power place polarizations on $(\mathbb{K} \otimes \text{Skew}[V|m])/J_m(V)$:

$$\phi^{(h)}_{ji} = \Phi_V \circ D^{(h)}_{ji} \circ (\Phi_V)^{-1};$$

- the mapping that sends each $m \times m$ elementary matrix with 1 in the $(i,j)$-th entry to the simple geometric product $\phi_{ij}$ induces a Lie representation of the general linear Lie algebra $gl(m, \mathbb{K})$ in the Lie algebra of derivations of $\Lambda(V)^{\otimes m}$.

### 7.3 The Letterplace encoding of $W^m(M)$.

Let $M = M(S)$ be a matroid of rank $n$ over a set $S$. Consider the $m$-th Whitney algebra of $M$,

$$W^m(M) = \text{Skew}(S)^{\otimes m}/I(M),$$

where $I(M)$ is the bilateral ideal of $\text{Skew}(S)^{\otimes m}$ generated by the slices of dependent words on $S$.

Under the isomorphism

$$\Phi : \text{Skew}[S|m] \to \text{Skew}(S)^{\otimes m},$$

the ideal $I(M)$ in $\text{Skew}(S)^{\otimes m}$ corresponds to the ideal $J(M)$ in $\text{Skew}[S|m]$ generated by the biproducts $(w|1^{(p_1)} \cdots m^{(p_m)})$, for every word $w = x_1 x_2 \cdots x_p$ whose corresponding set $\{x_1, x_2, \ldots, x_p\} \subseteq S$ is dependent in $M$. Indeed,

$$\Phi(w|1^{(p_1)} \cdots m^{(p_m)}) = \Phi \sum_{(w|1^{(p_1)} \cdots m^{(p_m)})} (w(1)|1^{(p_1)} \cdots (w(m)|m^{(p_m)})$$

$$= \sum_{(w|1^{(p_1)} \cdots m^{(p_m)})} w(1) \otimes \cdots \otimes w(p).$$

Thus we have an isomorphism

$$\Phi_M : \text{Skew}[S|m]/J(M) \to \text{Skew}(S)^{\otimes m}/I(M) = W^m(M).$$

We refer to the algebra

$$W^m(M) = \text{Skew}[S|m]/J(M)$$

as the letterplace encoding of $W^m(M)$. 
Now, under the isomorphism
\[ \Phi : \text{Skew}[S|m] \rightarrow \text{Skew}(S)^{\otimes m}, \]
the geometric products on \( \text{Skew}(S)^{\otimes m} \) correspond to the divided power polarizations on \( \text{Skew}[S|m] \). The divided power polarizations on \( \text{Skew}[S|m] \) leave invariant the ideal \( J(M) \), so they are well-defined on the quotient algebra \( W^m(M) \). Thus, the geometric products on \( \text{Skew}(S)^{\otimes m} \) leave invariant the ideal \( I(M) \), so they are well-defined on the quotient algebra, the Whitney algebra \( W^m(M) \).

From the results of subsection 7.1 we get, in particular:

- the mapping that sends each \( m \times m \) elementary matrix with 1 in the \((i, j)\)-th entry to the simple geometric product \( \diamond_{ij} \) induces a Lie representation of the general linear Lie algebra \( gl(m, \mathbb{Z}) \) in the Lie algebra of derivations of \( W^m(M) \).

8 The Exchange relations

This subsection is devoted to a discussion of what Crapo and Schmitt called the fundamental exchange relations in the Whitney algebra ([13], p. 218 and Theorem 7.4). These relations, in particular, generalize the two definitions of the meet in the CG-algebra of a Peano space.

The original approach to Whitney algebras makes use of the heavy categorical machinery of “lax Hopf algebras”. The proof of the exchange relations is based on the “Zipper Lemma” ([13], Theorem 5.7, p. 240), which is an identity satisfied by the homogeneous components of coproducts in a free skew-symmetric algebra.

Thanks to the letterplace encoding of the Whitney algebra, the exchange relations turn out to be an almost direct consequence of the Straightening Laws of Grosshans, Rota and Stein [18].

**Proposition 6.** Let \( M(S) \) be a matroid on the set \( S \). Let \( u \) and \( v \) be words of legth \( p \) and \( q \) on \( S \), associated to independent subsets \( A \) and \( B \), respectively. Let \( k \) be the nonnegative integer such that \( \rho(A \cup B) + k = p + q \). In the algebra \( W^2(M) \) we have

\[
\sum_{(v)q \rightarrow k, k} (uv(1)|1^{(p+q-k)}) (v(2)|2^{(k)}) = (-1)^{pq+k} \sum_{(u)p \rightarrow k, k} (vu(1)|1^{(p+q-k)}) (u(2)|2^{(k)}).
\]

**Proof.** From Corollary 2 (Subsect. 6.3), we have

\[
\sum_{(v)q \rightarrow k, k} (uv(1)|1^{(p+q-k)}) (v(2)|2^{(k)}) = (-1)^{pq} \sum_{t=0}^{k} (-1)^{t} \sum_{(u)p \rightarrow t, t} (vu(1)|1^{(p+q-k)}2^{(k-t)}) (u(2)|2^{(t)}).
\]
Now, for each \( t \), to each coproduct slice \((u)_{p-t,t}\) corresponds a word \( vu_{(1)} \) of length \( q + p - t \); from the assumption \( \rho(A \cup B) = p + q - k \) it follows that for \( t < k \) the word \( vu_{(1)} \) corresponds to a dependent subset of \( S \), or contains a repeated symbol; in any case the biproduct \((vu_{(1)}|^{[p+q-k]}2^{(k-t)})\) vanishes for \( t < k \). Thus the right hand side of the above equality simplifies to

\[
(-1)^{pq+k} \sum_{(u)_{p-k,k}} (vu_{(1)}|^{[p+q-k]})(u_{(2)}|^{[2^k]}).
\]

By translating the previous Proposition in the Whitney algebra \( W^2(M) \), and making a trivial sign computation, we get

**Theorem 3.** (The exchange relations of Crapo and Schmitt [13], Thm. 7.4)

Under the same assumptions of the preceding Proposition, in the algebra \( W^2(M) \) we have:

\[
\sum_{(u)_{q-k,k}} uv_{(2)} \circ v_{(1)} = \sum_{(v)_{p-k,k}} u_{(1)}v \circ u_{(2)}.
\]

### 9 Alternative Laws

One of the more traditional and significant themes (on the path traced by Grassmann) in the study Grassmann-Clifford Geometric Calculus consists in the study of invariant identities that describe geometrical statements and constructions (see e.g. [17], Appendix III p. 285ff. (1877), [16], [19], [22], [21]). In order to make effective this approach, one has to develop a systematic work of the identities that hold for meet and join of extensors in a CG-algebra; these identities were called in [2] the alternative laws.

In the following, for any extensors \( A, B \) of steps \( a, b \), we set

\[
A \wedge B := \sum_{(A)_{(a+b-n,n-b)}} A_{(1)}[A_{(2)}B] = (-1)^{(a+b-n)(n-b)} A \wedge B.
\]

The operation \( \wedge \) is associative up to a sign. From now on, each expression involving \( \wedge \) is thought as nested from left to right. Notice that

\[
\diamond_{21}^{(n-b)} (A \otimes B) = (A \wedge B) \otimes E; \quad \diamond_{12}^{(b)} (A \otimes B) = (A \vee B) \otimes 1.
\]

#### 9.1 A permanental identity

From the Lie commutation relations, one gets that the simple geometric products with indexes \( 0, i_1, \ldots, i_r, j'_1, \ldots, j'_r \), where \( 0 \) is distinct from each of the other indexes and each of the \( i' \)s is distinct from each of the \( j' \)s, satisfy the identity

\[
\diamond_{j'_0} \cdots \diamond_{j'_s} \diamond_{0_{i_1}} \cdots \diamond_{0_{i_r}} = \per(\diamond_{j'_0} + \sum_{s=1}^{r} \diamond_{*} \cdots \diamond_{j'_s}) \quad (\dagger)
\]
The operator in formula \((\dag)\) is said to be a *permanental Capelli operator*. The left-hand side is called 'virtual form' of the right hand side; the right-hand side is called an expansion of the left-hand side, and the terms of the sum are the "Capelli queues" of the expansion (see, e.g. [10], [27]).

The identities we will discuss in the next two subsections turn out to be consequences of different Laplace expansions of the permanental term \(\phi_{j^i_{ki}}\) in formula \((\dag)\).

### 9.2 Two typical alternative laws in CG-algebras

One of the simplest alternative laws is the following (see, e.g. [2], Corollary 7.2):

**Proposition 7.** In the Cayley-Grassmann algebra of a Peano space \((V, [\ ]}\)) we have:

\[
(a_1 \lor \cdots \lor a_r) \land B_{1'} \cdots \land B_{r'} = \sum_{\sigma \in S_r} (-1)^{|\sigma|} (a_{\sigma(1)} \land B_{1'}) \cdots (a_{\sigma(r)} \land B_{r'})
\]

for any vectors \(a_1, \ldots, a_r\) in \(V = \Lambda^1(V)\) and any covectors \(B_{1'}, \ldots, B_{r'}\) in \(\Lambda^{n-1}(V) \cong V^*\).

**Proof.** By specializing the identity \((\dag)\) on the tensor power of exterior algebra \(\Lambda(V)\) with folds indexed \(0, 1, \ldots, r, 1', \ldots, r'\), we have the operator identity

\[
\diamond_{r'0} \cdots \diamond_{1'0} \diamond_{0r} \cdots \diamond_{01} = \sum_{\sigma \in S_r} \diamond_{r'sr} \cdots \diamond_{1's1} + \sum_{s=1}^r \diamond_{ss} \cdots \diamond_{s0}.
\]

By evaluating the expression on the left hand side on the tensor \(1 \otimes a_1 \otimes \cdots \otimes a_r \otimes B_{1'} \otimes \cdots \otimes B_{r'}\), we get

\[
\diamond_{r'0} \cdots \diamond_{1'0} \diamond_{0r} \cdots \diamond_{01} (1 \otimes a_1 \otimes \cdots \otimes a_r \otimes B_{1'} \otimes \cdots \otimes B_{r'})
= \diamond_{r'0} \cdots \diamond_{1'0} ((a_1 \lor \cdots \lor a_r) \otimes 1 \otimes \cdots \otimes 1 \otimes B_{1'} \otimes \cdots \otimes B_{r'})
= (-1)^{r+n+1} ((a_1 \lor \cdots \lor a_r) \land B_{1'} \land \cdots \land B_{r'}) \otimes 1 \otimes \cdots \otimes 1 \otimes E \otimes \cdots \otimes E.
\]

By evaluating the expression on the right hand side on the tensor \(1 \otimes a_1 \otimes \cdots \otimes a_r \otimes B_{1'} \otimes \cdots \otimes B_{r'}\), since the Capelli queues vanish on this tensor, we get

\[
\sum_{\sigma \in S_r} \diamond_{r'sr} \cdots \diamond_{1's1} (1 \otimes a_1 \otimes \cdots \otimes a_r \otimes B_{1'} \otimes \cdots \otimes B_{r'})
= \sum_{\sigma \in S_r} (-1)^{r+n+|\sigma|} (a_{\sigma(1)} \land B_{1'}) \cdots (a_{\sigma(r)} \land B_{r'}) \otimes 1 \otimes \cdots \otimes 1 \otimes E \otimes \cdots \otimes E.
\]

The statement follows by equating the last steps of the two evaluations, and clearing up the signs. \(\square\)

One of the deepest alternative laws is the last one in ([2], Corollary 7.11); it can be viewed as playing a role in the CG-algebra similar to the role of the distributive law for union and intersection of sets in Boolean algebra.
Proposition 8. In the Cayley-Grassmann algebra of a Peano space $(V, [])$, let $C_1, \ldots, C_r$ be extensors of steps $n-q_1 > 0, \ldots, n-q_r > 0$, $q_i > 0$. Set $p = q_1 + \cdots + q_r$. Let $A$ and $B$ be extensors, $\text{step}(A) = s$, $\text{step}(B) = k$, $s + k = p$. Then

$$(A \lor B) \land (C_1 \land \cdots \land C_r) = A \land \sum_{(i_1, \ldots, i_r)} \varepsilon_{i_1, \ldots, i_r} \sum_{(B)} (B(1) \lor C_1) \land \cdots \land (B(r) \lor C_r),$$

where: the first sum is taken over all the $r$-tuples $(i_1, \ldots, i_r) \vdash s$, the second sum is taken over all the $r$-slices of $B$ of type $(q_1 - i_1, \ldots, q_r - i_r) \vdash p - s = k$, and

$$\varepsilon_{i_1, \ldots, i_r} = sg \left( \sum_{r \geq h > k \geq 1} i_h(q_k - i_k) \right).$$

Proof. Let us consider the tensor $1 \otimes A \otimes B \otimes C_1 \otimes \cdots \otimes C_r$ in the tensor space of $r + 3$ copies of the exterior algebra $\Lambda(V)$, where the copies are indexed $0, 1, 2, 3, \ldots, 2 + r$. We have

$$\diamondsuit_{2+r,0}^{(q_r)} \cdots \diamondsuit_{30}^{(q_1)} \diamondsuit_{02}^{(k)} \diamondsuit_{01}^{(s)} (1 \otimes A \otimes B \otimes C_1 \otimes \cdots \otimes C_r)$$

$$= \diamondsuit_{2+r,0}^{(q_r)} \cdots \diamondsuit_{30}^{(q_1)} (AB \otimes 1 \otimes 1 \otimes C_1 \otimes \cdots \otimes C_r)$$

$$= \beta_1 \times (AB \land C_1 \land \cdots \land C_r) \times 1 \otimes 1 \otimes 1 \otimes E \otimes \cdots \otimes E$$

$$= \gamma_1 \beta_1 \times (AB \land C_1 \land C_2 \land \cdots \land C_r) \times 1 \otimes 1 \otimes 1 \otimes E \otimes E \otimes \cdots \otimes E,$$

where

$$\beta_1 = sg \left( n \sum_{r \geq h \geq 1} q_h(h - 1) \right), \quad \gamma_1 = sg \left( \sum_{r \geq h > k \geq 1} q_hq_k \right).$$

Notice that the operator $\diamondsuit_{2+r,0}^{(q_r)} \cdots \diamondsuit_{30}^{(q_1)} \diamondsuit_{02}^{(k)} \diamondsuit_{01}^{(s)}$ is a “normalized” permanental Capelli operator in the sense of formula (†). Thus, by applying a Laplace expansion (of steps $(s, k)$), we get:

$$\diamondsuit_{2+r,0}^{(q_r)} \cdots \diamondsuit_{30}^{(q_1)} \diamondsuit_{02}^{(k)} \diamondsuit_{01}^{(s)} = \diamondsuit_{2+r,0}^{(q_r)} \cdots \diamondsuit_{30}^{(q_1)} \diamondsuit_{01}^{(s)} \diamondsuit_{02}^{(k)}$$

$$= \sum_{(i_1, \ldots, i_r)} \left( \diamondsuit_{2+r,1}^{(i_r)} \cdots \diamondsuit_{31}^{(i_1)} \right) \times \left( \diamondsuit_{2+r,2}^{(q_r-i_r)} \cdots \diamondsuit_{32}^{(q_1-i_1)} \right) + \sum \diamondsuit_{s, \cdots, s} \cdots \diamondsuit_{s,0},$$

where the first sum is taken over all the $r$-tuples $(i_1, \ldots, i_r) \vdash s$, and the Capelli queues $\diamondsuit_{s, \cdots, s} \cdots \diamondsuit_{s,0}$ annihilate the tensor $1 \otimes A \otimes B \otimes C_1 \otimes C_2 \cdots \otimes C_r$. Thus, we have

$$\diamondsuit_{2+r,0}^{(q_r)} \cdots \diamondsuit_{30}^{(q_1)} \diamondsuit_{02}^{(k)} \diamondsuit_{01}^{(s)} (1 \otimes A \otimes B \otimes C_1 \otimes \cdots \otimes C_r)$$

$$= \sum_{(i_1, \ldots, i_r)} \diamondsuit_{2+r,1}^{(i_r)} \cdots \diamondsuit_{31}^{(i_1)} \times \diamondsuit_{2+r,2}^{(q_r-i_r)} \cdots \diamondsuit_{32}^{(q_1-i_1)} (1 \otimes A \otimes B \otimes C_1 \otimes \cdots \otimes C_r)$$

$$= \sum_{(i_1, \ldots, i_r)} \alpha \sum_{(B)} \diamondsuit_{2+r,1}^{(i_r)} \cdots \diamondsuit_{31}^{(i_1)} (1 \otimes A \otimes 1 \otimes B(1)C_1 \otimes \cdots \otimes B(r)C_r)$$

$$= \sum_{(i_1, \ldots, i_r)} \alpha \beta \sum_{(B)} (A \land B(1)C_1 \land \cdots \land B(r)C_r) \times 1 \otimes 1 \otimes 1 \otimes E \otimes \cdots \otimes E$$

$$= \sum_{(i_1, \ldots, i_r)} \alpha \beta \gamma \sum_{(B)} (A \land B(1)C_1 \land \cdots \land B(r))C_r) \times 1 \otimes 1 \otimes 1 \otimes E \otimes \cdots \otimes E$$
where the first sum is taken over all the $r$–tuples $(i_1, \ldots, i_r) \vdash s$, the second sum is taken over all the $r$–slices of $B$ of type $(q_1 - i_1, \ldots, q_r - i_r) \vdash p - s = k$, and

$$
\alpha = sg \left( \sum_{r \geq h > k \geq 1} (q_h - i_h)(n - q_k) \right), \quad \beta_2 = sg \left( n \sum_{r \geq h \geq 1} i_h(h - 1) \right),
$$

$$
\gamma_2 = sg \left( \sum_{r \geq h > k \geq 1} i_h i_k \right).
$$

Thus we have

$$
\gamma_1 \beta_1 \times (AB \wedge C_1 \wedge C_2 \wedge \cdots \wedge C_r) = \sum \alpha \beta_2 \gamma_2 \sum (A \wedge B(1)C_1 \wedge \cdots \wedge B(r)C_r),
$$

and it turns out that

$$
\beta_1 \gamma_1 \alpha \beta_2 \gamma_2 = \varepsilon.
$$

10 The modular law

In the exterior algebra $\Lambda(V)$ of a vector space $V$, let $A, B, C$ be extensors, where $A$ divides $C$. Thus $\overline{A}, \overline{B}, \overline{C}$ are subspaces of $V$, with $\overline{A} \subseteq \overline{C}$, and in the lattice of subspaces of $V$ we have the modular law

$$
(\overline{A} \quad \sim \overline{B}) \quad \sim \overline{C} = \overline{A} \quad \sim (\overline{B} \quad \sim \overline{C}).
$$

It is natural to expect that this law has a counterpart in terms of geometric products on the tensor power $\Lambda(V) \otimes^3$.

Consider the pair of integers $p, q$ with the following geometric meanings:

$$
p = \rho (\overline{A}/(\overline{A} \sim \overline{B})) = \rho ((\overline{A} \sim \overline{B} \sim \overline{C})/(\overline{B} \sim \overline{C})) = \rho ((\overline{A} \sim \overline{B})/\overline{B})
$$

$$
q = \rho ((\overline{B} \sim \overline{C})/\overline{C}) = \rho ((\overline{A} \sim \overline{B})/(\overline{A} \sim \overline{B} \sim \overline{C})) = \rho (\overline{B}/(\overline{B} \sim \overline{C})).
$$

On the one hand, we have

$$
\phi_{32}^{(q)} \phi_{21}^{(p)} (A \otimes B \otimes C) = A_1 \otimes B_1 \otimes C_1,
$$

where

$$
\overline{A}_1 = \overline{A} \sim \overline{B}
$$

$$
\overline{B}_1 = (\overline{A} \sim \overline{B}) \sim \overline{C}
$$

$$
\overline{C}_1 = \overline{A} \sim \overline{B} \sim \overline{C}.
$$

On the other hand, we have

$$
\phi_{21}^{(p)} \phi_{32}^{(q)} (A \otimes B \otimes C) = A_2 \otimes B_2 \otimes C_2,
$$
where
\[
\begin{align*}
A_2 &= A \otimes B \ominus C \\
B_2 &= A \otimes (B \ominus C) \\
C_2 &= B \ominus C.
\end{align*}
\]
Thus, \( A_1 = A_2, \ B_1 = B_2, \ C_1 = C_2, \) and it is natural to expect that \( A_1 \otimes B_1 \otimes C_1 = A_2 \otimes B_2 \otimes C_2, \) i.e.,
\[
\left( \diamondots^{(q)}_{32} \diamondots^{(p)}_{21} (A \otimes B \otimes C) \right) = \diamondots^{(p)}_{21} \diamondots^{(q)}_{32} (A \otimes B \otimes C)
\]
This identity is true, and turns out to be a special case of a more general one. Given any pair of non-negative integers \( s, t, \) if \( A \) divides \( C, \) we have
\[
\left( \diamondots^{(t)}_{32} \diamondots^{(s)}_{21} / t! \right) (A \otimes B \otimes C) = \left( \diamondots^{(t-1)}_{32} / t! \diamondots^{(s-1)}_{21} / s! \right) (A \otimes B \otimes C) = \left( \diamondots^{(t-1)}_{32} / t! \diamondots^{(s-1)}_{21} / s! \right) (A \otimes B \otimes C) = \ldots = \diamondots^{(t)}_{32} / t! (A \otimes B \otimes C),
\]
since, under the assumption that \( A \) divides \( C, \) we have
\[
\diamondots_{31} (A \otimes B \otimes C) = 0.
\]

11 Appendix: Left span, right span, and other invariants of a tensor

In this section, we recall some facts about basic invariants of a tensor in the tensor product of two spaces.

1. Let \( V \) and \( W \) be two vector spaces, and let \( V \otimes W \) their tensor product. Any tensor \( t \in V \otimes W \) has various representations. The left span of a representation
\[
t = \sum_{i=1}^{p} v_i \otimes w_i
\]
of \( t \) is the space \( \langle v_1, \ldots, v_p \rangle \) generated by the vectors of \( V \) occurring in the representation. The left span of \( t \) is the space \( L_t \) intersection of the left spans of all the representations of \( t. \)

If the vectors \( w_1, \ldots, w_p \) of \( W \) occurring in the representation are independent, the representation is called right-independent. Any tensor admits a right-independent representation.

**Proposition 9.** The left span of a right-independent representation of a tensor \( t \) is contained in the left span of any representation of \( t. \) As a consequence, the left span \( L_t \) of \( t \) is the left span of any right-independent representations of \( t. \)
2. Analogous concepts and results are obtained by exchanging left and right.
We use the term “independent representation” instead of “(left and right)-independent representation”. Any tensor admits an independent representation.

**Proposition 10.** The number of summands of an independent representation of a tensor \( t \) is less than or equal to the number of summands in any other representation of \( t \); when equality holds, the latter representation is independent.

Due to the preceding Proposition, an independent representation is also said to be a **minimal representation**.

3. In any left-independent representation of a tensor \( t \),
\[
t = \sum_{i=1}^{p} v_i \otimes w_i
\]
the list \((w_1, \ldots, w_p)\) of the vectors occurring on the right is uniquely determined by the list \((v_1, \ldots, v_p)\) of the vectors occurring on the left. The linear mapping
\[
L_t \to R_t, \quad v_i \mapsto w_i, \quad i = 1, \ldots, p
\]
is an isomorphism.

**Proposition 11.** Let \( \phi : L_t \to R_t \) and \( \phi' : L_t \to R_t \) be the linear isomorphism associated to two independent representations
\[
t = \sum_{i=1}^{p} v_i \otimes w_i = \sum_{i=1}^{p} v'_i \otimes w'_i
\]
of a tensor \( t \). Then \( \phi = \phi' \) if and only if the transition matrix for the basis \((v_1, \ldots, v_p)\) to the basis \((v'_1, \ldots, v'_p)\) of \( L_t \) is an orthogonal matrix.

4. From the previous item, there is no canonical linear mapping from the left span to the right span of a tensor. There is, instead, a canonical bilinear mapping on the product of the left span by the right span.

**Proposition 12.** Let \( t \) be a tensor in \( V \otimes W \), and let \( t = \sum_{i=1}^{p} v_i \otimes w_i \) be an independent representation of \( t \); then, the bilinear mapping
\[
\beta : L_t \times R_t \to \mathbb{K}, \quad \beta(v_i, w_j) = \delta_{ij}
\]
depends only on the tensor \( t \).

**Proof.** A preliminary remark. Notice that
\[
\sum_i (v_i \cdot T) \otimes w_i = \sum_i \left( \sum_h v_h T_{hi} \right) \otimes w_i = \sum_i v_i \otimes \left( \sum_k T_{ik} w_k \right) = \sum_i v_i \otimes (T \cdot w_i).
\]
For every $T$ non singular, we have

$$t = \sum_i (v_i \cdot T) \otimes (T^{-1} w_i) = \sum_i v_i \otimes w_i.$$  

In plain words, for any two minimal representations

$$t = \sum_i v_i \otimes w_i = \sum_i v'_i \otimes w'_i,$$

there is a unique non singular matrix $T$ such that $v'_i = v_i \cdot T$, $w'_i = T^{-1} \cdot w_i$.

Let $\beta$ be the bilinear form associated to the representation $t = \sum_i v_i \otimes w_i$, that is, such that $\beta(v_i, w_j) = \delta_{ij}$.

Now, it is clear that

$$\beta (v_i \cdot T, w_j) = \beta (v_i, T \cdot w_j),$$

for every $T$. This immediately implies

$$\beta (v_i \cdot T, T^{-1} \cdot w_j) = \beta (v_i, w_j),$$

that is, the bilinear form $\beta$ is the same as the bilinear form associated to the representation $t = \sum_i v'_i \otimes w'_i$. \qed

References

[1] Abe, E.: *Hopf algebras*, Cambridge Tracts in Mathematics, 74. Cambridge University Press, Cambridge-New York, (1980)

[2] Barnabei, M., Brini, A., Rota, G.-C.: *On the Exterior Calculus of Invariant Theory*. J. of Algebra 96, 120-160 (1985)

[3] Berget, A.: *Tableaux in the Whitney module of a matroid*. Seminaire Lotharingien de Combinatoire 63, Article B63f, pp. 17 (2010)

[4] Bravi, P., Brini, A.: *Remarks on Invariant Geometric Calculus, Cayley-Grassmann Algebras and Geometric Clifford Algebras*. In: Crapo,H., Senato, D. (eds.) Algebraic Combinatorics and Computer Science, pp. 129–150, Springer, Milano (2001)

[5] Brini, A.: *Combinatorics, Superalgebras, Invariant Theory and Representation Theory*. Seminaire Lotharingien de Combinatoire 55, Article B55g, pp. 117 (2007)

[6] Brini, A.: *Private communication to A. Berget*. (2009)

[7] Brini A., Palareti A., Teolis A.: *Gordan–Capelli series in superalgebras*. Proc. Natl. Acad. Sci. USA 85, 1330–1333 (1988)

[8] Brini, A., Regonati,F., Teolis,A.: *Grassmann geometric calculus, invariant theory and superalgebras*. In: Crapo,H., Senato, D. (eds.) Algebraic Combinatorics and Computer Science, pp. 151–196, Springer, Milano (2001)

[9] Brini A., Teolis A.: *Young-Capelli symmetrizers in superalgebras*. Proc. Natl. Acad. Sci. USA 86, 775–778 (1989)

[10] Brini, A., Regonati, F., Teolis, A.: *The Method of Virtual Variables and Representations of Lie Superalgebras*. In: Ablamowicz, R. (ed.) Clifford Algebras - Applications to Mathematics, Physics, and Engineering, Progress in Mathematical Physics, vol. 34, pp. 245–263, Birkhauser, Boston (2004)
[11] Clifford, W.K.: *Application of Grassmann's Extensive Algebra*. Amer. J. of Mathematics, 350–358 (1878)

[12] Crapo, H.: *An Algebra of Pieces of Space – Hermann Grassmann to Gian Carlo Rota*. In: Damiani, E., D'Antonia, O., Marra, V. and Palombi, F.(eds.) From Combinatorics to Philosophy, pp. 61–90, Springer, (2009)

[13] Crapo, H., Schmitt, W.: *The Whitney algebra of a matroid*. J. Comb. Theory A 91, 215–263 (2000)

[14] Dieudonne, J.: *The tragedy of Grassmann*. Linear and Multilinear Algebra 8, 1-14 (1979/80)

[15] Fauser, B.: *A treatise on quantum Clifford algebra*, Kostanz, Abilitationsschrift, arXiv:math.QA/0202059 (2002)

[16] Forder, H.D.: *The Calculus of Extension*, Chelsea Publishing Co., New York (1960)

[17] Grassmann, H.G.: *Die Lineale Ausdehnungslehre*, Verlag von Otto Wigand, Leipzig (1844). Translated by Kannenberg, L.C.: *The Ausdehnungslehre of 1844 and Other Works*, La Salle: Open Court Publ., Chicago (1995)

[18] Grosshans, F.D., Rota, G.-C., Stein, J.A.: *Invariant theory and Superalgebras*, Amer. Math. Soc., Providence, RI (1987)

[19] Hawrylycz, M.: *Arguesian identities in invariant theory*. Advances in Math. 122, 148 (1996)

[20] Hestenes, D., Sobczyk, G., *Clifford Algebra to Geometric Calculus*, Reidel (1984)

[21] Li, H., *Invariant Algebras and Geometric Reasoning*, World Publishing Co. (2008)

[22] Mainetti, M., Yan, C.H.: *Arguesian identities in linear lattices*. Advances in Math. 144, 5093 (1999)

[23] Mourrain, B.: *Approche effective de la theorie des invariants des groupes classiques*, PhD Thesis, Ecole Polytechnique (1991)

[24] Mourrain, B., Stolfi, N.: *Computational symbolic geometry*. In: White, N. (ed.), Invariant Methods in Discrete and Computational Geometry, pp. 107–139, Reidel, (1995)

[25] Peano, G.: *Calcolo geometrico secondo l’Ausdehnungslehre di H. Grassmann*, Fratelli Bocca Editori, Torino (1888)

[26] Petsche, H.J. (Chairperson): *Grassmann Bicentennial Conference*, Potsdam and Szczecin (2009)

[27] Regonati, F.: *On the Combinatorics of Young-Capelli Symmetrizers*. Seminaire Lotharingien de Combinatoire 62, Article B62d, pp. 36 (2010)

[28] Schubring, G. (Chairperson): *The Grassmann Jubileum*. International Conference on the occasion of the 150th anniversary of the publication of the Ausdehnungslehre, Rugen and Szczecin (1994)

[29] G Schubring (ed.): *Hermann Grassmann (1809-1877) : visionary mathematician, scientist and neohumanist scholar*, Kluwer, Dordrecht (1996)

[30] Stewart, I.: *Hermann Grassmann was right*. Nature 321, 17 (1986)

[31] Sweedler, Moss E.: *Hopf algebras*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York (1969)

[32] Weyl, H.: *The Classical Groups. Their invariants and representations*, Princeton University Press, Princeton, New York (1946)

[33] White, N.L.: *The bracket ring of a combinatorial geometry I*. Trans. of Amer. Math Soc. 202, 79–95 (1975)
[34] Whitney, H.: *On the Abstract Properties of Linear Dependence*. Amer. J. Math. **57**, no. 3, 509–533 (1935)