The \((\lambda \Phi^4)_4\) theory on the lattice: effective potential and triviality.

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We compute numerically the effective potential for the \((\lambda \Phi^4)_4\) theory on the lattice. Three different methods were used to determine the critical bare mass for the chosen bare coupling value. Two different methods for obtaining the effective potential were used as a control on the results. We compare our numerical results with three theoretical descriptions. Our latticedata are in quite good agreement with the “Triviality and Spontaneous Symmetry Breaking” picture.

1. Introduction.

The response to an external field is a standard tool to obtain non-perturbative information in quantum field theories. This is even more important in a scalar field theory since the introduction of an external current gives directly the effective potential (up to an integration constant), thus providing important information on the stability of the system.

The conventional interpretation of the triviality in \((\lambda \Phi^4)_4\) theories is based on Renormalization-Group-Improved-Perturbation-Theory (RGIPT) \cite{2,3,4}. This picture predicts a second-order phase transition and a vanishing Higgs mass \(m_h\) in the continuum limit if \(v\), the physical v.e.v, is held fixed. An alternative interpretation \cite{5,6} predicts a phase transition that is very weakly first-order and that is very weakly first-order and that

\[
\begin{align*}
\int \mathrm{d}x \{ & \frac{1}{2} \sum_{\mu} \left[ \Phi(x + \hat{e}_\mu) - \Phi(x) \right]^2 + \\
& \frac{r_0}{2} \Phi^2(x) + \frac{\lambda_0}{4} \Phi^4(x) - J \Phi(x) \} = 0
\end{align*}
\]

where \(x\) stands for a generic lattice site, the lattice fields are expressed in lattice units \((a = 1)\), and \(\lambda_0 > 0\). For SSB the basic quantity is the expectation value of the bare scalar field \(\Phi(x)\) (B=Bare) \(\langle \Phi \rangle_J = \phi_B(J)\), since determining \(\phi_B(J)\) at several \(J\)-values is equivalent to inverting the relation \(J = J(\phi_B) = dV_{\text{eff}} / d\phi_B\) involving the effective potential \(V_{\text{eff}}(\phi_B)\). In this way, starting from the action in Eq. (1), the effective potential of the theory is rigorously defined up to an arbitrary integration constant (usually chosen to fix \(V_{\text{eff}}(0) = 0\)) and is convex downward \cite{7}.

Since we want to simulate the \((\lambda \Phi^4)_4\) lattice theory close to the continuum limit we have to determine the critical value \(r_c\) of the bare mass parameter \(r_0\) at \(J = 0\), which defines the ‘Coleman-Weinberg regime’ where \(m\), the mass gap of the symmetric phases, vanishes. The critical bare mass parameter \(r_c\) can be determined from the susceptibility \(\chi = 1 / \Omega_{\text{latt}} \left[ \langle \Phi^2 \rangle - \langle \Phi \rangle^2 \right]\) where \(\Phi = 1 / \Omega_{\text{latt}} \sum_x \Phi(x)\) is the average field for a given configuration and the brackets stand for the average on the lattice configurations produced in a Monte Carlo run.

One expects that, near the critical region, \(\chi^{-1} \sim \)
modulus logarithmic corrections to the effective field scaling law. One can thus determine $r_c$ by extrapolation to vanishing $1/((2s))$. Strictly speaking, this method is valid only for a second-order phase transition where the phase transition value $r_0 = r_s$ coincides with $r_c$ and such that, at $r_c = r_s$, both $m$ and $m_b$ vanish.

In the case of a very weak first-order phase transition where $|r_c - r_s|/r_c \sim \exp(-1/(2s))$, $s \equiv 3\lambda_0/16\pi^2 \ll 1$, the induced numerical uncertainty should be negligible.

For the Monte Carlo simulation of the lattice field theory described by Eq. (1) we followed the upgrade of the scalar field $\Phi(x)$ (using Metropolis) by the upgrade of the sign of $\Phi(x)$ according to the embedded Ising dynamics [3]. The zero external field spin-flip probability of the Swendsen-Wang algorithm is slightly modified to take into account the external current $J$.

Our data [6] are well described by the simple linear fit $\chi^{-1} = a(r - r_c)$ and do not show evidence of the logarithmic corrections. We evaluate the above quantity both in the symmetric and broken phase. We determined $r_c$ also through the following fit [2] to the generalized magnetization $\langle \Phi \rangle$:

$$\langle \Phi \rangle = \alpha(r_c - r)^{1/2} \ln |r - r_c|^{\beta} + \delta.$$  \hspace{1cm} (2)

Combining the above estimates for $r_c$ we get $r_c = -0.2280(9)$, in perfect agreement with the independent analysis [10] which predicts, for $\lambda_0 = 0.5$ and $L = 16$, $r_c = -0.2279(10)$. Thus we have three independent and consistent evaluations of $r_c$ at $\lambda_0 = 0.5$ on a $16^4$ lattice, that represent a precise input definition of the ’Coleman–Weinberg regime’ with the action Eq. (1) at $J = 0$.

We have used two independent methods to compute the effective potential. Firstly, we ran simulations of the lattice action Eq. (1) for 16 different values of the external source in the range $0.01 \leq |J| \leq 0.70$. In this way, as outlined in Eqs. (2-4), we directly obtain the slope of the effective potential (from which $V_{\text{eff}}$ can be obtained, up to an additive integration constant). We performed our numerical simulations by using for the sign upgrade both the Metropolis and the S-W cluster algorithm.

As a third additional check of our results, we performed a calculation using an alternative approach [11] to $V_{\text{eff}}$ based on the approximate effective potential $U_{\text{eff}}(\phi_B; \Omega)$ defined through

$$\exp\{-U_{\text{eff}}(\phi_B; \Omega)\} = \int [D\Phi] \delta \left( \frac{1}{\Omega} \int d^4 x \Phi(x) - \phi_B \right) \exp -S[\Phi].$$ \hspace{1cm} (3)

In the limit in which the 4-volume $\Omega \rightarrow \infty$, $U_{\text{eff}}$ tends to the exact $V_{\text{eff}}(\phi_B)$. The difference between $U_{\text{eff}}(\phi_B; \Omega)$ and $V_{\text{eff}}(\phi_B)$ gives both a consistency check of our calculations and an indication of the effects due to the finiteness of our lattice.

3. Comparing theory with the lattice data.

We can now compare our three different sets of lattice data with the existing theoretical expectations.

In the case of the picture in which triviality is compatible with SSB [4,6] the predicted form (in the Coleman–Weinberg case, $r_0 = r_c$, where no quadratic term is present in the effective potential) is:

$$J_{\text{triv}}(\phi_B) = \frac{dV_{\text{triv}}}{d\phi_B} = \alpha \phi_B^3 \ln(\phi_B^2) + \gamma \phi_B^3.$$ \hspace{1cm} (4)

where $\alpha$ and $\gamma$ are free parameters. (Their values are approximation-dependent within the class of “triviality-compatible” approximations.)

The RGIPT prediction exists in various slightly different forms in the literature. We have first used the full two-loop calculation of Ford and Jones [12] in the dimensional regularization scheme. The theoretical prediction for $J^{2-\text{loop}}(\phi_B) = dV^{2-\text{loop}}/d\phi_B$ depends on two free parameters: the ’t Hooft scale $\mu$ and the mass parameter $M^2$ of the classical potential.

A different version of the RGIPT prediction, which re-sums various terms, is given in Eq. (242), Sect. 5.4.2, of the textbook [13] by Itzykson and Drouffe (ID), namely

$$J^{1D}(\phi_B) = \frac{A \phi_B}{\ln |\mu|} \frac{(4\pi)^2 \phi_B^3}{18 \ln |\phi_B|},$$ \hspace{1cm} (5)

and again we have two free parameters $A$ and $\mu$.

The results of fitting the lattice data to the three theoretical predictions, are shown in Table 1.

4. Conclusions.

We have performed a numerical experiment to test two basically different and alternative pictures of
Table 1
Results of the fits of $J_{\text{triv}}$, $J_{2\text{-loop}}$, and $J_{\text{ID}}$ to our three different sets of lattice data.

| data          | $J_{\text{triv}}$ | $J_{2\text{-loop}}$ | $J_{\text{ID}}$ |
|---------------|-------------------|---------------------|-----------------|
| Metropolis    | $\alpha = 0.0152(2)$ | $\mu = 8.0304(449)$ | $|\mu| = 2.70(1) \times 10^8$ |
|               | $\gamma = 0.4496(1)$ | $M^2 = -0.0025(1)$  | $|A| = -0.0055(3)$ |
|               | $\chi^2 = \frac{15}{16-2}$ | $\chi^2 = \frac{142}{16-2}$ | $\chi^2 = \frac{116}{16-2}$ |
| Swendsen-Wang | $\alpha = 0.0152(1)$  | $\mu = 8.0128(283)$ | $|\mu| = 2.70(1) \times 10^8$ |
|               | $\gamma = 0.44962(4)$ | $M^2 = -0.00249(7)$ | $|A| = -0.0055(2)$ |
|               | $\chi^2 = \frac{13}{16-2}$  | $\chi^2 = \frac{284}{16-2}$ | $\chi^2 = \frac{223}{16-2}$ |
| constraint eff. pot. | $\alpha = 0.0156(2)$  | $\mu = 7.9883(455)$ | $|\mu| = 2.69(1) \times 10^8$ |
|               | $\gamma = 0.4494(1)$  | $M^2 = -0.0028(1)$ | $|A| = -0.0063(3)$ |
|               | $\chi^2 = \frac{10}{16-2}$  | $\chi^2 = \frac{109}{16-2}$ | $\chi^2 = \frac{85}{16-2}$ |

‘triviality’ in the ‘Coleman-Weinberg-regime’. Our results are striking and confirm that the lattice data cannot be reproduced by all theoretical models. Indeed the results of fitting the conventional theoretical predictions to our lattice data gives totally unacceptable values of the $\chi^2/f$. On the other hand Eq. (4), based on the alternative picture, gives excellent fits to all sets of data. This is an important evidence for the unconventional interpretation of ‘triviality’ of Refs. [4–6].

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