One loop calculation in lattice QCD with domain-wall quarks

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Abstract

One loop corrections to the domain-wall quark propagator are calculated in massless QCD. It is shown that no additive counter term to the current quark mass is generated in this theory, and the wave function renormalization factor of the massless quark is explicitly evaluated. We also show that an analysis with a simple mean-field approximation can explain properties of the massless quark in numerical simulations of QCD with domain-wall quarks.

11.15Ha, 11.30Rd, 12.38Bx, 12.38.Gc
I. INTRODUCTION

The formulation of the lattice fermion in QCD with the chiral symmetry is one of the most fascinating problems theoretically and practically. Although both Wilson and Kogut-Susskind (KS) fermion formulations have been popularly used for the lattice QCD simulations, some disadvantages remain in these formulations: In the Wilson fermion formulation the quark mass has additive quantum correction and the chiral limit is reached only by the fine tuning of the mass parameter. As a general rule we have to take continuum limit tuning the mass appropriately in order to simulate massless QCD. In the KS fermion formulation the number of flavors is restricted and the original flavor symmetry is broken explicitly to some residual one.

The domain-wall fermion formulation, which was originally proposed to define lattice chiral gauge theories [1], has been applied to the lattice QCD [2]. This formulation is expected to have great advantage over the previous two formulations: An advantage over the KS fermion is that the number of flavor is not fixed. This is manifest from its definition. The other advantage over the Wilson fermion is that mass renormalization is multiplicative ($m_{\text{eff}} = Z_m m_{\text{tree}}$). In other words, if a massless mode exists at the tree level it is stable against the quantum correction. This property is not a trivial one, but only an intuitive discussion on it has been given so far [2]. On the other hand, the recent numerical simulation suggests that the stability of the zero mode holds even non-perturbatively [3]. Therefore an analytical understanding of the domain-wall QCD is now needed. The aim of this paper is to confirm the stability of the massless mode by the lattice perturbation theory and to give explicitly the wave function renormalization of the quark field.

This paper is organized as follows. In section 2 we will give basic tools for the perturbative calculation with the domain-wall fermion. It is enough to present only fermion propagator because other Feynman rules of gauge interaction and gauge propagator are exactly identical to that of the ordinary Wilson fermion. In section 3 we calculate one loop corrections to the fermion propagator. Section 4 is the main part of this paper, where we discuss the renormalization of the zero mode or massless quark field. We take the diagonal basis of the mass matrix of the domain-wall fermion and see that the zero mode is stable against the one loop correction. The wave function renormalization factor of the massless quark field is also given explicitly. Section 5 is devoted to the mean field analysis. We show that properties of the zero mode observed in the numerical simulation [3] are well explained in this approximation. In section 6 we give our conclusion and discussion. In appendices some derivations of formulae used in the text are presented.

In this paper we set the lattice spacing $a = 1$ and take the $SU(N_c)$ gauge group with the gauge coupling constant $g$ and the second Casimir $C_2 = \frac{N^2 - 1}{2N_c}$. We set $N_c = 3$ in the numerical calculations.

II. PERTURBATION THEORY WITH DOMAIN-WALL FERMION
A. Action

We adopt the domain-wall fermion of Shamir type \[2\] to describe massless quarks. The domain-wall fermion is a variant of the Wilson fermion with sufficiently many flavors and special form of the mass matrix. Although it is also interpreted as a five-dimensional Wilson fermion \[1\], we prefer to treat it as the multi-flavor system \[4\].

In this point of view only difference from the Wilson fermion action is the fermion bilinear term. If we separate the QCD action for lattice perturbation theory into fermion and gauge parts,

\[
S = S_{\text{fermion}} + S_{\text{gauge}} + S_{\text{GF}} + S_{\text{FP}} + S_{\text{measure}},
\]

the lattice gauge action \(S_{\text{gauge}}\), the gauge fixing and the FP-ghost term \(S_{\text{GF}} + S_{\text{FP}}\), the invariant measure term \(S_{\text{measure}}\) and the gauge-fermion interaction terms in \(S_{\text{fermion}}\) are exactly same as those in the ordinary Wilson fermion perturbation theory \([5,6]\) with many flavors.

The domain-wall fermion action \(S_{\text{fermion}}\) is written as

\[
S_{\text{fermion}} = \sum_{n,m} \overline{\psi}_{m,s} (\gamma_{\mu} D_{\mu})_{m,n} \psi_{n,s} + \overline{\psi}_{m,s} W_{m,n}^{s,t} P_{+} \psi_{n,t} + \overline{\psi}_{m,s} W_{m,n}^{-s,t} P_{-} \psi_{n,t}
\]

\[+ m_q \overline{\psi}_{m,s} (\delta_{m,n} \delta_{s,N_s} \delta_{t,1} P_{+} + \delta_{m,n} \delta_{s,1} \delta_{t,N_s} P_{-}) \psi_{n,t}
\]

(2)

where \(m, n\) is four dimensional space index, \(s, t = 1, \cdots, N_s\) is the flavor index. Here the Dirac operator is given by

\[
(\gamma_{\mu} D_{\mu})_{n,m} = \sum_{\mu} \frac{1}{2} \gamma_{\mu} \left( U_{n,\mu} \delta_{n+\mu,m} - U_{m,\mu}^{\dagger} \delta_{n-\mu,m} \right)
\]

(3)

and mass matrix \(W_{s,t}^{\pm}\) is defined as

\[
W_{n,m:s,t}^{\pm} = \delta_{s\pm 1,t} \delta_{n,m} - W_{n,m} \delta_{s,t}
\]

(4)

where

\[
W_{n,m} = (1 - M) \delta_{n,m} + \frac{r}{2} \sum_{\mu} \left( U_{n,\mu} \delta_{n+\mu,m} + U_{m,\mu}^{\dagger} \delta_{n-\mu,m} - 2 \delta_{n,m} \right)
\]

(5)

is a sum of the Dirac mass term and the Wilson term, which contain gauge fields at this stage, and \(r\) is the Wilson parameter, which we set \(r = -1\). The parameter \(m_q\) is the current quarks mass, but in this paper we only treat the massless QCD taking \(m_q = 0\). \(P_{\pm}\) is a projection operator defined by

\[
P_{\pm} = \frac{1 \pm \gamma_5}{2}
\]

(6)

In our domain-wall fermion action \([2]\) we have Dirac mass \(M\) besides the current quark mass \(m_q\). Here we have to notice that \(M\) is not the physical quark mass but it is rather an unphysical mass of the cutoff order \((1/a)\) like Wilson term. As will be mentioned later \(M\) has an important role as a parameter of the theory: choosing a suitable value for \(M\) we have a massless fermion mode for the vanishing current quark mass \((m_q = 0)\).
In order to see the massless fermion mode it is more convenient to be in the momentum representation and pull out the bilinear term. The fermion action in the momentum space is written as

$$S_{\text{fermion}} = \int \frac{d^4 p}{(2\pi)^4} \bar{\psi}(-p) \left[ \sum_{\mu} i\gamma_\mu \sin p_\mu + W^+(p)_{s,t} P_+ + W^-(p)_{s,t} P_- \right] \psi(p) t + S_{\text{int.}} \quad (7)$$

where the mass matrix has the following form,

- $W^+(p)_{s,t} = \delta_{s+1,t} - W(p)\delta_{s,t}$

- $= \left( \begin{array}{cccc} -W(p) & 1 & & \\ \vdots & -W(p) & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ -W(p) & & & \end{array} \right) \quad (8)$

- $W^-(p)_{s,t} = \delta_{s-1,t} - W(p)\delta_{s,t}$

- $= \left( \begin{array}{cccc} -W(p) & & & \\ 1 & -W(p) & & \\ & \ddots & \ddots & \\ & & 1 & -W(p) \end{array} \right) \quad (9)$

- $W(p) = 1 - M - r \sum_\mu (1 - \cos p_\mu). \quad (10)$

The gauge interaction term $S_{\text{int.}}$ is identical to that of the Wilson fermion perturbation theory with $N_s$ flavors.

As will be discussed in appendix C, in spite of the presence of the Dirac mass $M$ this fermion system has one massless fermion mode and $N_s - 1$ excited modes with the mass of cut-off order by virtue of this mass matrix form, provided that $|W(p \sim 0)| < 1$ is satisfied by a suitable choice of the Dirac mass $M$. Here we take the momentum region $p_\mu \sim 0$ to see the zero mode with physical momenta. At the momentum $p_\mu \sim \pi$, where the doubler emerges in the naive fermion formulation, the parameter condition is not satisfied ($|W(p \sim \pi)| > 1$), so that all $N_s$ fermion modes have mass of the cut-off order.

**B. Fermion Propagator**

In the next section we will calculate the one loop correction to the fermion propagator. In this subsection we set up the lattice Feynman rules for domain-wall fermion with vanishing current quark mass ($m_q = 0$).

As is discussed in the previous subsection, the domain-wall fermion action is almost same as that of the ordinary Wilson fermion’s one with $N_s$ flavors. The peculiar feature of the domain-wall fermion is the form of the fermion propagator, which is given by

$$S_P(p)_{s,t} = \left[ i\gamma_\mu \bar{p}_\mu + W^+(p) P_+ + W^-(p) P_- \right]_{s,t}^{-1} \quad (11)$$

where $\bar{p}_\mu = \sin(p_\mu)$. The explicit form is written as
\[
S_F(p)_{s,t} = \left[ (-i\gamma_\mu p_\mu + W^-) G_R(s, t) P_+ + (-i\gamma_\mu p_\mu + W^+) G_L(s, t) P_- \right]_{st}
\]
where
\[
G_R(s, t) \equiv \left( \frac{1}{p^2 + W^+ W^-} \right)_{st}
= G^0(s - t) + A_{++} e^{\alpha(s+t)} + A_{+-} e^{\alpha(s-t)} + A_{-+} e^{-\alpha(s+t)} + A_{--} e^{-\alpha(s-t)}
\] (13)
\[
G^0(s - t) = A \left( e^{\alpha(N_s -|s-t|)} + e^{-\alpha(N_s -|s-t|)} \right)
\] (14)
\[
A_{++} = \frac{A}{e^{\alpha N_s (1 - We^\alpha)} - e^{-\alpha N_s (1 - We^{-\alpha})}} \left( (1 - We^{-\alpha})(e^{-2\alpha N_s} - 1) \right)
\] (15)
\[
A_{+-} = \frac{A}{e^{\alpha N_s (1 - We^\alpha)} - e^{-\alpha N_s (1 - We^{-\alpha})}} \left( We^\alpha (e^{-\alpha} - e^{-\alpha N_s}) \right)
\] (16)
and
\[
G_L(s, t) \equiv \left( \frac{1}{p^2 + W^+ W^-} \right)_{st}
= G^0(s - t) + B_{++} e^{\alpha(s+t)} + B_{+-} e^{\alpha(s-t)} + B_{-+} e^{-\alpha(s+t)} + B_{--} e^{-\alpha(s-t)}
\] (17)
\[
B_{++} = \frac{A}{e^{\alpha N_s (1 - We^\alpha)} - e^{-\alpha N_s (1 - We^{-\alpha})}} \left( e^{-\alpha} - We^{-\alpha} (e^{-2\alpha N_s} - 1) \right)
\] (18)
\[
B_{+-} = \frac{A}{e^{\alpha N_s (1 - We^\alpha)} - e^{-\alpha N_s (1 - We^{-\alpha})}} \left( We^\alpha (e^{-\alpha} - e^{-\alpha N_s}) \right)
\] (19)
Here \(\alpha\) and \(A\) are defined as
\[
cosh \alpha \equiv \frac{1 + W^2 + p^2}{2W(p)}
\] (20)
\[
sinh \alpha = \frac{1}{2W \sqrt{1 - W^2}} + \sum \frac{\sin^2 p_\mu + (\sum \sin^2 p_\mu)^2}{2 \sinh \alpha}
\] (21)
\[
A \equiv \frac{1}{2W \sinh \alpha \, 2 \sinh(\alpha N_s)}
\] (22)
Note that the argument \(p\) of \(W\) and \(\alpha\) is suppressed throughout this paper unless necessary. Since this fermion propagator is invariant under \(\alpha \to -\alpha\), we take the \(\alpha > 0\) without loss of generality. \(G_R\) and \(G_L\) are also symmetric in \((s, t)\). See appendix B for the derivation. In the one-loop calculation we use the above propagator in the \(N_s \to \infty\) limit.

III. ONE LOOP CALCULATION

A. Diagrams

In this section we calculate the one loop correction to the fermion propagator, which is given by two contributions \(\Sigma^{\text{tadpole}}(p) + \Sigma^{\text{half-circle}}(p)\), from diagrams in Fig. [4].
The 1PI fermion 2-point vertex function is given by

\[ V_{1\text{-loop}}^{(2)}(p)_{s,t} = [i\gamma_\mu \sin p_\mu + W^+(p)P_+ + W^-(p)P_- - \Sigma(p)]_{s,t}, \]  

(23)

with

\[ \Sigma(p) = \Sigma^{\text{tadpole}}(p) + \Sigma^{\text{half-circle}}(p). \]  

(24)

In order to investigate the massless mode of \( \Gamma_{1\text{-loop}}^{(2)}(p)_{s,t} \) in the \( p_\mu \to 0 \) limit, we need only the first few terms in the \( p_\mu \) expansion. Since the only dimensionful quantity is the external momentum \( p_\mu \) in our calculation, the higher order terms in the \( p_\mu \) expansion are also higher order in \( a \).

**B. Contribution from Tadpole Diagram**

The contribution from the tadpole diagram is written as

\[ \Sigma^{\text{tadpole}} = \frac{1}{2} g^2 C_2 \sum_\mu (i\gamma_\mu \sin p_\mu - r \cos p_\mu) \int_{-\pi}^{\pi} \frac{d^4l}{(2\pi)^4} \frac{1}{\sin^2 \frac{l}{2}} \delta_{s,t} \]  

\[ = g^2 C_2 T \left( \frac{1}{2} i\phi + 2 \right) \delta_{s,t} + \mathcal{O}(a) \]  

(25)

(26)

where \( T \) is the tadpole loop integral

\[ T = \int_{-\pi}^{\pi} \frac{d^4l}{(2\pi)^4} \frac{1}{\sin^2 \frac{l}{2}} = 0.154612. \]  

(27)

The first term in Eq. (26) is finite, and the second term linearly diverges in the limit \( a \to 0 \). We see that \( \Sigma^{\text{tadpole}} \) is diagonal in flavor space, and its effect is to modify the mass parameter \( M \to \tilde{M} = M - 2g^2 C_2 T \).

**C. Contribution from Half Circle Diagram**

The contribution from the half circle diagram in the Feynman gauge

\[ \Sigma_{s,t}^{\text{half-circle}} = \int_{-\pi}^{\pi} \frac{d^4l}{(2\pi)^4} \sum_\mu (-i g T^a) \left\{ \gamma_\mu \cos \frac{1}{2}(l_\mu + p_\mu) - ir \sin \frac{1}{2}(l_\mu + p_\mu) \right\} \times S_F(l)_{s,t} \times (-i g T^a) \left\{ \gamma_\mu \cos \frac{1}{2}(l_\mu + p_\mu) - ir \sin \frac{1}{2}(l_\mu + p_\mu) \right\} \times \frac{1}{(p - l)^2} \]  

(28)

cannot be calculated analytically because of its complicated dependence on the flavor indices \( s, t \) in the fermion propagator. Here \( \hat{p}_\mu \) is defined as \( \hat{p}_\mu = 2 \sin(p_\mu/2) \).

It is easily seen that the loop integral of (28) has infra-red divergence. As is in the ordinary lattice perturbation theory the infra-red divergence can be written in an analytic form. To do this we separate \( \Sigma_{s,t}^{\text{half-circle}} \) as follows:
\[ \Sigma_{s,t}^{\text{half-circle}}(p) = \Sigma_{s,t}^{\text{lat.}}(p) + \Sigma_{s,t}^{\text{cont.}}(p) \]  

(29)

where

\[ \Sigma_{s,t}^{\text{lat.}}(p) = \Sigma_{s,t}^{\text{half-circle}}(p) - \Sigma_{s,t}^{\text{cont.}}(p) \]  

(30)

and \( \Sigma_{s,t}^{\text{cont.}}(p) \) is introduced to extract the infra-red divergence:

\[ \Sigma_{s,t}^{\text{cont.}}(p) = 2g^2C_2 \int \frac{d^4l}{(2\pi)^4} \frac{-if(C_+P_+ + C_-P_-)_{s,t}}{l^2(p-l)^2} \theta(\pi^2 - l^2) \]

with \((C_+),_s,t = (1-w_0^2)w_0^{s+t-2}, (C_-),_s,t = (1-w_0^2)w_0^{2N_t-s-t}, \) and \(w_0 = W(0)\). In order to have zero modes with the physical momentum, \(w_0\) should be in the region \(w_0^2 \leq 1\). This leads to the condition of \(M\) that \(0 \leq M \leq 2\). Since \(\Sigma_{s,t}^{\text{lat.}}(p)\) is infra-red finite in \(p \to 0\) limit, we can evaluate it in the \(p\) expansion:

\[ \Sigma_{s,t}^{\text{lat.}}(p) = \Sigma_{s,t}^{\text{lat.}}(0) + p_\mu \frac{\partial \Sigma_{s,t}^{\text{lat.}}}{\partial p_\mu}(0) + O(a). \]

(31)

The logarithmically divergent part \(\Sigma_{s,t}^{\text{cont.}}(p)\) can be calculated analytically, while a linearly divergent and finite terms (the first and the second terms in eq.\((31)\)) have to be evaluated by numerical integrations of loop momenta. After a little algebra we have

\[ \Sigma_{s,t}^{\text{half-circle}} = -\left[ i\mathcal{P}(I_{s,t}^+P_+ + I_{s,t}^-P_-) + M_{s,t}^+P_+ + M_{s,t}^-P_- \right] \]

(32)

where \(I^\pm\) and \(M^\pm\) are given by

\[ I_{s,t}^\pm = I_{s,t}^\pm(s, t) + I_{s,t}^{\text{finite}}(s, t) \]

(33)

\[ I_{s,t}^\pm(s, t) = \frac{1}{16\pi^2}g^2C_2(C_\pm),_{s,t} \left( \ln(\pi^2) + \frac{1}{2} - \ln p^2 \right) \]

(34)

\[ I_{s,t}^{\text{finite}}(s, t) = g^2C_2 \int \frac{d^4l}{(2\pi)^4} \frac{1}{l^2} \left[ \frac{1}{8} \sum_\mu \left( \cos l_\mu(W^-G_R + W^+G_L)(s, t) + \sin^2 l_\mu(G_L + G_R)(s, t) \right) \right. \]

\[ + \sum_\mu \frac{\sin^2 l_\mu}{4(l^2)^2} \left( (W^-G_R + W^+G_L)(s, t) + 2(\sum_\nu \cos^2 l_\nu/2 - 2\cos^2 l_\mu/2)G_L(s, t) \right. \]

\[ \left. + \sum_\nu \sin^2 l_\nu/2G_R(s, t) \right] - g^2C_2(C_+),_{s,t} \int \frac{d^4l}{(2\pi)^4} \frac{1}{(l^2)^2} \theta(\pi^2 - l^2) \]

(35)

\[ I_{s,t}^{\text{finite}}(s, t) = g^2C_2 \int \frac{d^4l}{(2\pi)^4} \frac{1}{l^2} \left[ \frac{1}{8} \sum_\mu \left( \cos l_\mu(W^-G_R + W^+G_L)(s, t) + \sin^2 l_\mu(G_L + G_R)(s, t) \right) \right. \]

\[ + \sum_\mu \frac{\sin^2 l_\mu}{4(l^2)^2} \left( (W^-G_R + W^+G_L)(s, t) + 2(\sum_\nu \cos^2 l_\nu/2 - 2\cos^2 l_\mu/2)G_R(s, t) \right. \]

\[ \left. + \sum_\nu \sin^2 l_\nu/2G_L(s, t) \right] - g^2C_2(C_-),_{s,t} \int \frac{d^4l}{(2\pi)^4} \frac{1}{(l^2)^2} \theta(\pi^2 - l^2) \]

(36)

\[ M_{s,t}^\pm = g^2C_2 \int \frac{d^4l}{(2\pi)^4} \frac{1}{l^2} \sum_\mu \left[ \cos^2 l_\mu/2(W^+G_L)(s, t) - \sin^2 l_\mu/2(W^-G_R)(s, t) \right. \]

(37)
\begin{equation}
+ \frac{1}{2} \sin^2 \mu (G_L + G_R)(s, t) \right]
\end{equation}

\begin{equation}
M_{s,t} = g^2 C_2 \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2} \sum_{\mu} \left[ \cos^2 \mu / 2(W^+ G_R)(s, t) - \sin^2 \mu / 2(W^+ G_L)(s, t) + \frac{1}{2} \sin^2 \mu (G_L + G_R)(s, t) \right]
\end{equation}

By the dimensional counting $I^{\pm}$ has $\ln a^2$ divergence and constant terms in $a$, and $M^{\pm}$ has $1/a$ linear divergence when lattice spacing $a$ is introduced explicitly. Although $M^{\pm}$ may have $\ln a^2$ divergence naively, it is canceled by the algebraic relation

\begin{equation}
(W^+(p = 0))_{s,t}(1 - M)^t = 0
\end{equation}

\begin{equation}
(W^-(p = 0))_{s,t}(1 - M)^{-t} = 0.
\end{equation}

The logarithmic divergence $\ln a^2$ in $I^{\pm}$ is given analytically. As we can see from the form of $(C_+)_{s,t} = (1 - w_0^2)w_0^{s+t-2}$, $I_{\log}^{+}$ is localized in the boundary $(s, t) = (1, 1)$. This is because the logarithmic divergence comes from the effect of massless fermion mode which is localized in the boundary. The other one $I_{\log}^{-}$ is also localized in the other boundary $(s, t) = (N_s, N_s)$.

The finite terms and linearly divergent terms should be calculated by repeating the numerical integration $O(N_s^2)$ times. However, as can be seen in the next section, such huge number of integrations can be avoided for the wave-function renormalization of the quark field. On the other-hand, since the structures of $I^{\pm}_{\text{finite}}$ and $M^{\pm}$ are useful to understand the domain-wall QCD more deeply, we will give them in the separated paper.

**IV. RENORMALIZATION OF QUARKS FIELD**

The result obtained in the previous section is summarized in the following form of the effective action for 2 point function with the scale $p^2 = (\mu a)^2$ at 1-loop level:

\begin{equation}
\Gamma^{(2)} = \overline{\psi}(-p)_s \left[ i\gamma_{\mu} p_{\mu} \left( Z^+ P_+ + Z^- P_- \right) + \overline{W}^+ P_+ + \overline{W}^- P_- \right]_{s,t} \psi(p)_t
\end{equation}

where

\begin{equation}
Z^{\pm} = 1 + g^2 C_2 (I_{\text{tad}} + I_{\log}^{\pm} + I_{\text{finite}}^{\pm})
\end{equation}

\begin{equation}
\overline{W}^{\pm} = W^{\pm}(0) + g^2 C_2 (M_{\text{tad}} + M^{\pm})
\end{equation}

with

\begin{equation}
I_{\text{tad}}(s, t) = -\frac{1}{2} T \delta_{s,t} = -0.077306 \delta_{s,t}
\end{equation}

\begin{equation}
M_{\text{tad}}(s, t) = -2 T \delta_{s,t} = -0.309224 \delta_{s,t}.
\end{equation}

The expressions for $I_{\log}^{\pm}$, $I_{\text{finite}}^{\pm}$ and $M^{\pm}$ are given in the previous section. In this section we consider the renormalization of zero modes, which is interpolated by the quark field: $q(p) = P_+ \psi(p)_1 + P_- \psi(p)_{N_s}$. Here we only present the results and give the detail of derivations in appendix C.
A. Diagonalization of mass matrix and stability of zero modes

For the renormalization of zero modes, it is better to use new basis, \( \psi^d(p) \) which diagonalize mass matrices \( \overline{W}^\pm \). This basis are given by the relation that

\[
\psi^d_s(p) = U_{s,t}P_+\psi(p) + V_{s,t}P_-\psi(p),
\]

where unitary matrices \( U \) and \( V \) satisfy

\[
\begin{align*}
[U\overline{W}^+U^\dagger]_{s,t} &= M^2_s\delta_{s,t} \\
[V\overline{W}^+V^\dagger]_{s,t} &= M^2_s\delta_{s,t}.
\end{align*}
\]

In our notation the mass eigen-value squared \( M^2_s \) is arranged in such a way that \( M^2_N = 0 \), and we can take \( U \) and \( V \) real matrices without loss of generality.

We calculate \( U \) and \( V \) at 1-loop level:

\[
U = (1 + g^2U_0)U_0, \quad V = (1 + g^2V_1)V_0,
\]

where tree level matrices \( U_0 \) and \( V_0 \) are analytically obtained in the large \( N_s \) limit as follows:

\[
[U_0]_{s,t} = \begin{cases} 
(2/N_s)^{1/2}\sin\alpha_s(N_s+1-t) & s \neq N_s \\
(1 - w_0^2)^{1/2}w_0^{(t-1)} & s = N_s
\end{cases}
\]

and \([V_0]_{s,t} = [U_0]_{s,N_s+1-t} \), where \( w_0 = 1 - \overline{M} \) with \( \overline{M} = M + 4(u-1) \). Here \( u = 1 \) for the naive perturbation theory, while \( u = 1 - g^2C_2T/2 \) for the tadpole improved perturbation theory. Hereafter we will call both cases “the tree level” and will not distinguish the two cases unless necessary. If we expand the mass eigen-value squared as \( (M^2)_s = (M^2_0)_s + g^2(M^2_1)_s \), the tree level one is related to the phase factor \( \alpha_s \) such that \( 2w_0\cos\alpha_s = 1 + w_0^2 - (M^2_0)_s \). This phase factor, which also satisfies \( \sin\alpha_sN_s = w_0\sin\alpha_s(N_s+1) \), is explicitly given as \( \alpha_s = \pi s/N_s \) in the large \( N_s \) limit. It is also shown that \( U_0 \) and \( V_0 \) diagonalize \( W^\pm \) itself such that

\[
[V_0W^+U_0^\dagger]_{s,t} = [U_0W^-V_0^\dagger]_{s,t} = (M_0)_s\delta_{s,t}.
\]

We now consider \( \overline{W}^\pm \), which is denoted as \( \overline{W}^\pm = W^\pm + g^2W^1_1 \), where \( (W^\pm_s)_{s,t} = W^\pm(0) = \delta_{t,s+1} - w_0 \) and \( g^2(W_1^1)_{s,t} = g^2C_2(4M^2 + M_{\text{tad}})_{s,t} + 4(1-u)\delta_{s,t} \). To diagonalize \( \overline{W}^+ \cdot \overline{W}^\mp \) at 1-loop order, \( U_1 \) and \( V_1 \) should satisfy

\[
\begin{align*}
(U_1)_{s,t}(M^2_0)_t + (M^2_0)(U^\dagger_1)_{s,t} \\
+ (U_0W^1_0V^\dagger_0 \cdot V_0W^+_0U^\dagger_0)_{s,t} + (U_0W^-_0V^\dagger_0 \cdot V_0W^1_0U^\dagger_0)_{s,t} = (M^2_1)_s\delta_{s,t},
\end{align*}
\]

\[
\begin{align*}
(V_1)_{s,t}(M^2_0)_t + (M^2_0)(V^\dagger_1)_{s,t} \\
+ (V_0W^1_0U^\dagger_0 \cdot U_0W^-_0V^\dagger_0)_{s,t} + (V_0W^+_0U^\dagger_0 \cdot U_0W^-_0V^\dagger_0)_{s,t} = (M^2_1)_s\delta_{s,t}.
\end{align*}
\]

Using the fact that \((U_1,V_1)_{s,t} = -(U_1,V_1)_{t,s}\) implied by the unitarity and the reality, and \( V_0W^+_0U^\dagger_0 \) and \( U_0W^-_0V^\dagger_0 \) are diagonal, we can easily solve the above equation as

\[
(U_1)_{s,t} = \frac{(M_0)_{t}(\overline{W}^1_{1})_{s,t} + (M_0)_{s}(\overline{W}^-_{1})_{t,s}}{(M_0^2)_s - (M_0^2)_t}
\]

\[
(V_1)_{s,t} = \frac{(\overline{W}^-_{1})_{s,t}(M_0)_t + (\overline{W}^1_{1})_{t,s}(M_0)_s}{(M_0^2)_s - (M_0^2)_t}
\]
for \( s \neq t \), and

\[
(M^2_s)_s = 2(\overline{W}_1)_{s,s}(M_0)_s, \quad (U_1)_{s,s} = (V_1)_{s,s} = 0,
\]

where \( \overline{W}_1 = V_0W_1U_0^\dagger \). The mass eigen-value squared \( M^2_s = (M_0^2)_s + g^2(M^2_I)_s \) obtained above leads to the mass eigen-value \( M_s \) itself: \( M_s = (M_0)_s + g^2(\overline{W}_1)_{s,s} \). Note that \( M_{N_s} = 0 \) since \( (M_0)_{N_s} = 0 \) and \( (\overline{W}_1)_{N_s,N_s} = 0 \) in the large \( N_s \) limit as is shown in appendix C. This result explicitly demonstrates the stability of the zero modes against 1-loop corrections in domain-wall QCD. As in the case at the tree level, it is shown that

\[
(V\overline{W}^+U^\dagger)_{s,t} = (U\overline{W}^-V^\dagger)_{s,t} = M_s\delta_{s,t} + O(g^4).
\]

B. Wave function renormalization for quark fields

After diagonalization of the mass matrix, the effective action for the zero mode field \( \psi^d(p)_{N_s} = \chi_0(p) \) becomes

\[
\bar{\chi}_0(-p) \left[ i\gamma_\mu P_\mu \left( \overline{Z}_+ P_+ + \overline{Z}_- P_- \right) \right] \chi_0(p)
\]

where

\[
\overline{Z}_\pm = 1 - g^2C_2 \frac{T}{2} + \frac{g^2C_2}{16\pi^2} \left( \log \pi^2 + \frac{1}{2} - \log(\mu a)^2 \right) + g^2(I^d_{\pm,N_s,N_s})
\]

with \( I^d_+ = C_2(U_0I_{\infty}^+U_0^\dagger) \) and \( I^d_- = C_2(V_0I_{\infty}^-V_0^\dagger) \). Since the interpolating quark field \( q(p) \) is expressed as \( q(p) = (U_{N_s}P_+ + V_{N_s,N_s}P_-)\chi_0(p) \), and \( \langle \chi_0(p)\bar{\chi}_0(-p) \rangle \)

\[
\langle q(p)\bar{q}(-p) \rangle = \left[ \frac{U_{N_s}^2}{Z_+} P_+ + \frac{V_{N_s,N_s}^2}{Z_-} P_- \right] \left[ i\gamma_\mu P_\mu \right] \frac{-1}{p^2},
\]

Therefore, the renormalized quark field \( Q(p) \), which satisfies \( \langle Q(p)\bar{Q}(-p) \rangle = \frac{-i\gamma_\mu P_\mu}{p^2} \), is given by \( Q(p) = [(Z^+_F)^{1/2}P_+ + (Z^-_F)^{1/2}P_-]q(p) \) with \( Z^+_F = \overline{Z}_+ \frac{U_{N_s}^2}{U_{N_s,N_s}^2} \) and \( Z^-_F = \overline{Z}_- \frac{V_{N_s,N_s}^2}{U_{N_s,N_s}^2} \). Since an explicit evaluation shows that \( (I^d_{\pm,N_s,N_s}) = (I^d_{N_s,N_s}) \equiv I^d \), thus \( \overline{Z}_+ = \overline{Z}_- \equiv \overline{Z} \), and \( (U_{N_s})^2 = (V_{N_s,N_s})^2 = 1 - w_0^2 \), we finally obtain \( Z^+_F = \overline{Z} \equiv Z_F = \frac{\overline{Z}}{1 - w_0^2} \) where

\[
\overline{Z} = 1 - g^2C_2 \frac{T}{2} + \frac{g^2}{16\pi^2}C_2(\log \pi^2 + \frac{1}{2} - \log(\mu a)^2) + g^2I^d.
\]

Here one unknown constant \( I^d \) is given by
\[ I^d = C_2 \int \frac{d^4l}{(2\pi)^4} \left\{ \frac{1}{8l^2} \sum_{\mu} \left[ \sin^2 l_{\mu}(\bar{G}_R + \bar{G}_L) + 2 \cos l_{\mu}(w_0 - W(l))\bar{G}_R \right] \\
+ \frac{\sin^2 l_{\mu}}{2(l^2)^2} \left[ (w_0 - W(l))\bar{G}_R + (\sum_{\nu} \cos^2 l_{\nu}/2 - 2 \cos^2 l_{\mu}/2)\bar{G}_L + (\sum_{\nu} \sin^2 l_{\nu}/2)\bar{G}_R \right] \\
- \frac{1}{(l^2)^2} \theta(\pi^2 - l^2) \right\} \] (57)

where

\[ \bar{G}_L = A \left[ \bar{G} - \frac{e^{\alpha} - W}{e^{-\alpha} - W} \frac{1}{(e^{\alpha} - w_0)^2} \right] \]

\[ \bar{G}_R = A \left[ \bar{G} - \frac{1}{(e^{\alpha} - w_0)^2} \right] \]

with

\[ A = \frac{1 - w_0^2}{2W \sinh \alpha} \]

\[ \bar{G} = \frac{\sinh \alpha_0 - \sinh \alpha}{2w_0 \sinh \alpha_0 (\cosh \alpha_0 - \cosh \alpha)} \]

and \( e^{-\alpha_0} = w_0 \). The numerical value of \( I^d \) is given in Table I at several values of \( \tilde{M} \), together with the total 1-loop renormalization factor \( Z_1 \) ( \( \tilde{Z} \equiv 1 + g^2 Z_1 \) ) at \( \mu a = 1 \) and the ratio of the non-tadpole contribution \( (Z_1)_{\text{non-tad}} \equiv I^d + \frac{C_2}{16\pi^2} (\log \pi^2 + 0.5) = I^d + 0.02355 \) to the total one. Note also that the tadpole contribution gives \( Z_{\text{tad}} \equiv -C_2 T/2 = -0.1031 \). From this table, we see that \( I^d \) is small and depends on \( \tilde{M} \) very weakly: The value \( I^d = -0.01945 \) at \( \tilde{M} = 0.05 \) monotonically increases (decreases in the absolute value) to \( I^d = -0.01222 \) at \( \tilde{M} = 0.95 \). Furthermore the non-tadpole contribution \( Z_{\text{non-tad}} \) is relatively small: 4% at \( \tilde{M} = 0.05 \) and 12% at \( \tilde{M} = 0.95 \), so that the tadpole contribution becomes dominant at all \( \tilde{M} \). This justifies the use of the tree-level result with the tadpole improvement. Since \( Z_1 \simeq 0.1 \), the one-loop correction to the \( Z \) factor is about 10% at \( g^2 \sim 1.0 \).

**V. MEAN FIELD ANALYSIS AT FINITE \( N_s \)**

As seen in the previous sections, due to the presence of off-diagonal terms in the extra dimension, analysis of the 1-loop correction to domain-wall quarks becomes too complicated to be easily applied to results of the numerical simulations, which should be performed on finite \( N_s \). In this section we adopt an approximated but simpler method to analyze the effect of 1-loop corrections. We call the method the mean field (MF) analysis since the link variable \( U_{n,\mu} \) in the fermion action is simply replaced by the mean field \( u \) which is independent on \( n \) and \( \mu \). After this replacement the fermion propagator can be explicitly calculated and result is identical to the one given in Appendix B with the replacement such that \( x \rightarrow ux \) and \( \cos p_{\mu} \rightarrow u \cos p_{\mu} \). In perturbation theory this is equivalent to the tree level analysis with the tadpole improvement, which has been shown in the previous section to give about 90% of the wave function renormalization factor at 1-loop level.
Since we are interested in the zero mode at $s = 1$, we set $s = t = 1$ in the propagator. In this case the zero mode appears in $B_+ e^{-2\alpha} G_L$, which is given at non-zero $m_q$ by

$$B_+ e^{-2\alpha} = \frac{(1 - We^{-\alpha})(1 - m_q^2)}{2W \sinh(\alpha)F}$$  \hfill (58)

where

$$F = We^{\alpha} - 1 + m_q^2(1 - We^{-\alpha}) - 4m_q \cdot W \cdot \sinh(\alpha)e^{-\alpha N_s} + e^{-2\alpha N_s}(1 - We^{-\alpha} + m_q^2(We^{\alpha} - 1)).$$  \hfill (59)

In the small momentum limit, this leads to

$$\lim_{r^2 \to 0} B_+ e^{-2\alpha} = \frac{Z^{-1}}{p^2 + m_F^2} \times \frac{(1 - w_0^2)^2 + p^2 u w_0^2}{(1 - w_0^2)^2 + p^2 u (1 + w_0^2)}$$  \hfill (60)

where $Z^{-1} = \frac{1 - m_q^2}{Au}$, $m_F^2 = \frac{B}{Au}$ and $w_0 = 1 - M + 4(1 - u) = 1 - \tilde{M}$ with

$$A = \frac{1}{1 - w_0^2} \left[ 1 + m_q^2 w_0^2 - w_0(1 - w_0^2) + m_q w_0 \tilde{N}_s \right]$$

$$\times \left\{ 2N_s(1 - w_0^2) - 1 - w_0^2 + 2w_0(1 - w_0^2) - N_s(1 - w_0^2)^2/w_0 \right\}$$

$$+ w_0^{2N_s} \left\{ w_0^2 + m_q^2 - 2N_s(1 - w_0^2) - w_0(1 - w_0^2) + N_s(1 - w_0^2)^2/w_0 \right\}$$

$$B = (1 - w_0^2) \left[ m_q^2 - 2m_q w_0 N_s + w_0^{2N_s} \right].$$

Since the pole in the second factor is in general larger than the physical pole in the first factor, we neglect the second factor in the latter analysis.

Now we use the above formula to understand the behavior of the zero mode observed in ref. [3]. For the value of $u$ there are several choices. The tadpole diagram alone gives

$$u = 1 - g^2 C_2 T/2 = 1 - 0.10307 g^2 \approx \exp[-0.10307 g^2],$$

where we may take the bare coupling $2N_s/\beta$ or the renormalized coupling $g_{MS}^2(\pi/a)$ for $g^2$ in the above formula. Alternatively we may also use the “observed” value of $u$: $u = P^{1/4}$ where $P$ is the average value of the plaquette normalized to unity. We adopt the latter one in our analysis. The configurations in ref. [3] generated at $\beta = 5.7$ and $m_q a = 0.01$ by the dynamical Kogut-Susskind quark action give $P = 0.5772$, which leads to $u = 0.872$. In ref. [3] two remarkable features are found for the zero mode: no zero mode is observed for $N_s = 4$ and the zero mode is observed at $M = 1.7$ but not at $M \leq 1.0$ for $N_s = 10$. To explain these we calculate $m_F$ as a function of $M$ for both $N_s = 4$ and 10 at $m_q = 0, 0.01, 0.02, 0.03$, and plot the results in Fig. 4, where solid lines are for $N_s = 4$ and dashed lines for $N_s = 10$. Four lines for each $N_s$ correspond to $m_q = 0, 0.01, 0.02, 0.03$ from below to above around $M = 1.5$. The result tells us the followings. The allowed range for the light fermion is very narrow for $N_s = 4$ (roughly $1.4 < M < 1.6$). This may be a reason why the light state could not be found in the simulation [3]. Note that the allowed range for the zero mode is $0.512 < M < 2.512$ in the $N_s \to \infty$ limit. Although the allowed range becomes
larger for \( N_s = 10 \) (1.1 < \( M < 1.9 \)), no light state appears at \( M \leq 1.0 \), as observed in the simulation. Furthermore the order of the fermion mass \( m_F \) is reversed to the order of the current quark mass \( m_q \) at \( M \leq 1.0 \): \( m_F \) is largest at \( m_q = 0 \). The plot also supports the fact that the zero mode is observed at \( M = 1.7 \) in the simulation.

As seen in the above the behavior of the numerical simulation is understandable by the MF analysis, which can supply useful informations on the tuning of parameters in numerical simulations such as \( N_s, M \) or \( m_q \) before-hand. For example we may take \( N_s = 4 \) for the simulations, which reduces the cost of both CPU time and memory a lot, if \( M \) is appropriately chosen ( \( M \simeq 1.5 \) for \( U = 0.872 \) ).

VI. CONCLUSION AND DISCUSSION

In this paper we calculate one-loop correction to the fermion propagator in the massless lattice QCD formulated via domain-wall fermions. We show that the zero mode is stable against the one-loop correction: no additative counter term to the quark mass is generated in the large \( N_s \) limit. This property is very different from and superior to the ordinary Wilson fermion formulation. We explicitly calculate the wave-function renormalization factor for the massless quarks and show that the tadpole contribution becomes dominant at all \( \tilde{M} \). We also adopt the mean-field analysis to this model, demonstrating that it can qualitatively explain data obtained in the numerical simulation \[3\].

Although our results strongly indicate that the domain-wall QCD can avoid the fine tuning problem of the quark mass, the mechanism which gives the zero mode in this formulation has not been fully understood yet. Since our proof for the stability of the zero mode contains an explicit calculation at 1-loop (: (\( \tilde{W}_1 \))\(_{N_s,N_s} = 0 \)), it can not be easily carried over to higher orders. The result of numerical simulation \[3\] suggests that the zero mode is also stable against the non-perturbative dynamics. There may be a yet unknown symmetry which ensures the existence of zero mode in the large \( N_s \) limit. To find such a symmetry will be important for our understanding of the formulation.

In this paper only the wave-function renormalization factor is explicitly evaluated. Based on the method developed in this paper, it is possible to calculate more complicated quantities such as renormalization factors for the quark mass, currents and 4-fermi operators, which are necessary to get the continuum physics from numerical simulations. The results of this paper also suggest that the smeared quark operator \( q^{\text{smea r}} = \sum_s (w_0^s P_+ \psi_s + w_0^{N_s-s} P_- \psi_s) \) may give better signals than \( q = P_+ \psi_1 + P_- \psi_{N_s} \) does, since it has a larger overlap to zero modes.

After this work has been completed, there appears a new paper \[7\], in which the stability of the zero mode is generally considered.

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The domain-wall fermion propagator is already given by the Eq. (12). which is given by the present purpose. One of them is the fermion interaction vertex with one gluon field, the number of interactions in the lattice perturbation theory, only two of them are needed for

\[ S_{\text{meas}} = -\frac{1}{2} \sum_n \sum_{\mu} \text{tr} \ln \left( \frac{1 - \cos \left( gA_\mu^\alpha(n + \frac{1}{2}\hat{\mu}) \text{ad}(T^\alpha) \right)}{(gA_\mu^\alpha(n + \frac{1}{2}\hat{\mu}) \text{ad}(T^\alpha))^2} \right)_{ab} \]  

(64)

where \( g \) is the coupling of the \( SU(N_c) \) gauge, \( \beta = 2N_c/g^2. \) \( \alpha \) is the gauge parameter. The actions \( S_{FP} \) and \( S_{\text{meas}} \) is not needed in our calculation at one loop level.

The momentum representation of gauge part is

\[ S_{\text{gauge}} + S_{\text{GF}} = \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} A_\mu^\alpha(-p) \left[ \hat{p}^2 \delta_{\mu \nu} - (1 - \frac{1}{\alpha})\hat{p}_{\mu} \hat{p}_{\nu} \right] A_\nu^\alpha(p) \]  

(65)

here \( + \cdots \) denotes the gluon self interactions which do not come into play in our calculation.

The fermion-gauge interaction terms in the momentum representation is

\[ S_{\text{int.}} = \sum_{n=1}^{\infty} \int \frac{d^4 l_1}{(2\pi)^4} \cdot \frac{d^4 l_2}{(2\pi)^4} \cdot \delta^4(k + p + l_1 + \cdots + l_n) \]  

\[ \times \frac{i^n}{n!} g^n A_\mu^a(l_1) \cdots A_\mu^a(l_n) \bar{\psi}(k) T^{a_1} \cdots T^{a_n} \]  

\[ \times \left[ \frac{\gamma_\mu}{2} \left( e^{\frac{i}{2\alpha}(p_{\mu} - k_{\mu})} - (-) \right) e^{-\frac{i}{\alpha}(p_{\mu} - k_{\mu})} \right] \psi(p)_s. \]  

(66)

The domain-wall fermion propagator is already given by the Eq. (12).

The fermion-gluon interaction vertices are given by (66). Although there are infinite number of interactions in the lattice perturbation theory, only two of them are needed for the present purpose. One of them is the fermion interaction vertex with one gluon field, which is given by

\[ V_1(k, p; l, a; \mu) = -ig T^a \left\{ \gamma_\mu \cos \frac{1}{2}(-k_{\mu} + p_{\mu}) - ir \sin \frac{1}{2}(-k_{\mu} + p_{\mu}) \right\}. \]  

(67)

The other is the vertex with two gluon fields, given by

\[ V_2(k, p; l_1, a, l_2, b; \mu) = \frac{1}{2} g^2 \frac{1}{2} \{ T^a, T^b \} \left\{ i\gamma_\mu \sin \frac{1}{2}(-k_{\mu} + p_{\mu}) - r \cos \frac{1}{2}(-k_{\mu} + p_{\mu}) \right\} \delta_{\mu \nu}. \]  

(68)

The gluon propagator is given by

\[ G_{\mu \nu}^{ab}(p) = \frac{1}{p^2} \left[ \delta_{\mu \nu} - (1 - \alpha)\frac{\hat{p}_\mu \hat{p}_\nu}{p^2} \right] \delta_{ab}. \]  

(69)

We set \( \alpha = 1 \) in this paper.
APPENDIX B. DERIVATION OF FREE FERMION PROPAGATOR

In this appendix we derive the free fermion propagator, used in the text. For the later use in perturbative analyses of this model, non-zero current quark mass $m_q$ for finite $N_s$ is considered. See also Refs. [8,4,9]. We also derive the propagator with Majorana mass terms, which becomes important for the lattice definition of the $N = 1$ supersymmetric model via domain-wall fermions [10,11].

A. Propagator with non-zero $m_q$

The free fermion propagator has the following form:

$$S_F(p)_{s,t} = [(-i \gamma_\mu \overline{P}_\mu + W^-)G_R(s,t)P_+ + (-i \gamma_\mu \overline{P}_\mu + W^+)G_L(s,t)P_-]_{st}$$

where

$$G_R(s,t) \equiv \left(\frac{1}{p^2 + W^+W^-}\right)_{st} \quad \text{and} \quad G_L(s,t) \equiv \left(\frac{1}{p^2 + W^-W^+}\right)_{st}$$

with

$$(W^+_{m,s,t}) = (W^+)_{s,t} + m_q \delta_{s,N_s} \delta_{t,1} \quad \text{and} \quad (W^-_{m,s,t}) = (W^-)_{s,t} + m_q \delta_{s,1} \delta_{t,N_s} \quad (70)$$

We first consider $G_R$. The following equation is satisfied for $G_R$:

$$\sum_t \left[(x + W^+W^-)_{s,t} + m_q (W^+_{st} \delta_{t,N_s} + \delta_{s,N_s} W^-_{1t}) + m_q^2 \delta_{s,N_s} \delta_{t,N_s}\right] G_R(t,u) = \delta_{su} \quad (71)$$

with $x = p^2$. Therefore, except $s = N_s$ or 1, this equation is satisfied by

$$G_R(s,t) = G(s,t) + A_{++} e^{\alpha(s+t)} + A_{+-} e^{\alpha(s-t)} + A_{-+} e^{-\alpha(s+t)} + A_{--} e^{-\alpha(s-t)} \quad (72)$$

where

$$G(s,t) = A \left(e^{\alpha(N_s-|s-t|)} + e^{-\alpha(N_s-|s-t|)}\right) \quad (73)$$

becomes a special solution to the equation $(x + W^+W^-)G_R = 1$, with

$$\cosh \alpha \equiv \frac{1 + W^2 + x}{2W(p)} \quad A \equiv \frac{1}{2W \sinh \alpha} \frac{1}{2 \sinh(\alpha N_s)} \quad (74)$$

and other terms are general solutions to the equation $(x + W^+W^-)G_R = 0$. We can fix their coefficients $A_{\pm \pm}$ by a boundary condition at $s = 1$:

$$(x + W^2 + 1)G_R(1,t) - W \cdot G_R(2,t) - W \cdot m_q \cdot G_R(N_s,t) = \delta_{1t} \quad (75)$$

which is simplified to
\[ G_R(0,t) - m_q G_R(N_s,t) = 0, \tag{76} \]

and another boundary condition at \( s = N_s \):
\[ (x + W^2) G_R(N_s,t) - W \cdot G_R(N_s - 1,t) - W \cdot m_q \cdot G_R(1,t) + m_q^2 G_R(N_s,t) = \delta_{N_s,t}, \tag{77} \]

which is reduced to
\[ G_R(N_s,t) - W \cdot G_R(N_s + 1,t) + W \cdot m_q G_R(1,t) - m_q^2 G_R(N_s,t) = 0. \tag{78} \]

Plugging eq. (72) into eqs. (76) and (78) leads to
\[
\begin{pmatrix}
1 - m_q e^{\alpha N_s} \\
1 - m_q e^{-\alpha N_s}
\end{pmatrix}
= -A
\begin{pmatrix}
e^{-\alpha N_s} - m_q \\
1 - W e^{-\alpha} - m_q^2 + W m_q e^{-\alpha(N_s+1)}
\end{pmatrix}
\begin{pmatrix}
A_{++} & A_{+-} \\
A_{-+} & A_{--}
\end{pmatrix}
\begin{pmatrix}
e^{\alpha N_s} - m_q \\
1 - W e^{\alpha} - m_q^2 + W m_q e^{\alpha(N_s+1)}
\end{pmatrix}, \tag{79}
\]

Solving this we obtain
\[
\begin{pmatrix}
A_{++} \\
A_{--}
\end{pmatrix}
= \frac{A}{F}
\begin{pmatrix}
(e^{-2\alpha N_s} - 1)(1 - W e^{-\alpha})(1 - m_q^2) \\
2 W \sinh(\alpha)(1 - 2 m_q \cosh(\alpha N_s) + m_q^2)
\end{pmatrix}
\begin{pmatrix}
A_{++} & A_{+-} \\
A_{-+} & A_{--}
\end{pmatrix}
\begin{pmatrix}
e^{\alpha N_s} - m_q \\
1 - W e^{\alpha} - m_q^2 + W m_q e^{\alpha(N_s+1)}
\end{pmatrix},
\]

where
\[
F = e^{\alpha N_s} [1 - W e^{\alpha} + m_q^2 (W e^{-\alpha} - 1)] + 4 W m_q \sinh(\alpha) + e^{-\alpha N_s} [W e^{-\alpha} - 1 + m_q^2 (1 - W e^{\alpha})]. \tag{80}
\]

Similarly, plugging the general solution for \( G_L \)
\[ G_L(s,t) = G(s,t) + B_{++} e^{\alpha(s+t)} + B_{+-} e^{\alpha(s-t)} + B_{-+} e^{\alpha(-s+t)} + B_{--} e^{\alpha(-s-t)} \tag{81} \]

into the boundary conditions
\[
G_L(N_s + 1,t) - m_q G_L(1,t) = 0 \tag{82}
\]
\[
G_L(1,t) - W \cdot G_L(0,t) + W \cdot m_q G_L(N_s,t) - m_q^2 G_L(1,t) = 0, \tag{83}
\]

we finally obtain
\[
\begin{pmatrix}
B_{++} \\
B_{-+}
\end{pmatrix}
= \frac{A}{F}
\begin{pmatrix}
(e^{-2\alpha N_s} - 1)e^{-\alpha}(e^{-\alpha} - W)(1 - m_q^2) \\
2 W \sinh(\alpha)(1 - 2 m_q \cosh(\alpha N_s) + m_q^2)
\end{pmatrix}
\begin{pmatrix}
B_{++} & B_{+-} \\
B_{-+} & B_{--}
\end{pmatrix}
\begin{pmatrix}
(e^{\alpha N_s} - m_q) \\
1 - W e^{\alpha} - m_q^2 + W m_q e^{\alpha(N_s+1)}
\end{pmatrix},
\]

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B. Propagator with the Majorana mass term at $N_s$

For an application of the free fermion propagator obtained in the domain-wall model, we consider the model with the Majorana mass term on the anti-boundary at $s = N_s$, which has been proposed for a lattice definition of the $N = 1$ super Yang-Mills theory \cite{10,11}. Here we set $m_q = 0$.

A free fermion action of the model with the Majorana mass $m_0$ can be written in the momentum space as

$$S = \frac{1}{2} \bar{\Psi}(-p) D_{s,t}(p) \Psi(p)$$  \hspace{1cm} (84)

where

$$\Psi_s(p) = \left( \psi_s(p), \bar{\psi}_s(p) \right), \quad \bar{\Psi}_s(p) = \left( \bar{\psi}_s(p) \psi_s(p) \right),$$  \hspace{1cm} (85)

and

$$D(p) = T_0(p) + m_0 X = \begin{pmatrix} D_0(p) & 0 \\ 0 & -D_0(-p)^T \end{pmatrix} + m_0 \delta^2 P_+ IP_-$$  \hspace{1cm} (86)

with $(\delta^2)_{s,t} \equiv \delta_{s,N_s} \delta_{N_s,t}$, and

$$P_+ = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix}, \quad P_- = \begin{pmatrix} P_- & 0 \\ 0 & P_+ \end{pmatrix}, \quad I = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & 0 & \sigma_2 \\ \sigma_2 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \end{pmatrix}$$

in terms of $8 \times 8$ matrices. Here

$$D_0(p) = i \gamma_\mu \pi_\mu + W^+ P_+ + W^- P_-,$$  \hspace{1cm} (87)

is an inverse of the massless free fermion propagator in the domain-wall QCD. By expanding $D^{-1}$ in $m_0$ and rearranging it we obtain

$$D^{-1} = \sum_{n=0}^{\infty} \left( -T_0^{-1} m_0 P_+ IP_- \delta^2 \right)^n T_0^{-1}$$

$$= \sum_{n=0}^{\infty} \left( -m_0 \right)^n T_0^{-1} \delta P_+ Z^{-1} IP_- \delta T_0^{-1}$$  \hspace{1cm} (88)

where $Z = IP_- \delta T_0^{-1} \delta P_+$.

Using $Z^2 = -x(G_R(p)_{N_s,N_s})^2 P_+$ and suming over $n$, we finally get

$$D^{-1} = T_0^{-1} + \left[ -m_0 T_0^{-1} \delta P_+ IP_- \delta T_0^{-1} + m_0^2 T_0^{-1} \delta P_+ Z IP_- \delta T_0^{-1} \right] \times \frac{1}{1 + m_0^2 x(G_R(p)_{N_s,N_s})^2}.$$  \hspace{1cm} (89)

Explicitly this formula gives, in terms of $2 \times 2$ block notations,
The two eigen-state equations lead to the same equation

\[ D(p)_{11}^{-1} = -D(-p)_{22}^{-1} = \langle \psi(p) \bar{\psi}(-p) \rangle \]

\[ = (-i \gamma_\mu \bar{\psi}_\mu) [Z_+(p)P_+ + Z_-(p)P_-] + M_+(p)P_+ + M_-P_- \]  \hspace{1cm} (90)

where

\[ Z_+(p)_{s,t} = G_R(p)_{s,t} - \frac{m_0^2 x G}{1 + m_0^2 x G^2} G_R(p)_{s,N_s} G_R(p)_{N_s,t} \]

\[ Z_-(p)_{s,t} = G_L(p)_{s,t} + \frac{m_0^2 G}{1 + m_0^2 x G^2} (W^- G_R(p))_{s,N_s} (G_R(p)W^+)_{N_s,t} \]

\[ M_+(p)_{s,t} = (W^- G_R(p))_{s,t} - \frac{m_0^2 x G}{1 + m_0^2 x G^2} (W^- G_R(p))_{s,N_s} G_R(p)_{N_s,t} \]

\[ M_-(p)_{s,t} = (W^+ G_L(p))_{s,t} - \frac{m_0^2 x G}{1 + m_0^2 x G^2} G_R(p)_{s,N_s} (G_R(p)W^+)_{N_s,t} \]

with \( G \equiv G_R(p)_{N_s,N_s} \). Similarly

\[ (D(p)_{12})_{s,t} = \langle \psi(p) \bar{\psi}(-p) t \rangle \]

\[ = \frac{m_0}{1 + m_0^2 x G^2} \left[ x G_R(p)_{s,N_s} G_R(p)_{N_s,t} I_2 P_+ - i \gamma_\mu \bar{\psi}_\mu G_R(p)_{s,N_s} (G_R(p)W^+)_{N_s,t} I_2 P_+ \right. \]

\[ + i \gamma_\mu (W^- G_R(p))_{s,N_s} G_R(p)_{N_s,t} I_2 P_- + (W^- G_R(p))_{s,N_s} (G_R(p)W^+)_{N_s,t} I_2 P_- \] \hspace{1cm} (91)

and

\[ (D(p)_{21})_{s,t} = \langle \bar{\psi}(p) \psi(-p) t \rangle \]

\[ = \frac{m_0}{1 + m_0^2 x G^2} \left[ x G_R(p)_{s,N_s} G_R(p)_{N_s,t} I_2 P_+ + i \gamma_\mu \bar{\psi}_\mu G_R(p)_{s,N_s} (G_R(p)W^+)_{N_s,t} I_2 P_- \right. \]

\[ - i \gamma_\mu (W^- G_R(p))_{s,N_s} G_R(p)_{N_s,t} I_2 P_+ + (W^- G_R(p))_{s,N_s} (G_R(p)W^+)_{N_s,t} I_2 P_- \] \hspace{1cm} (92)

See ref. [11] for an application of this result.

**APPENDIX C. PROPERTIES OF DIAGONALIZATION MATRICES**

In this appendix we derive several properties of diagonalization matrices, \( U \) and \( V \), which are used for the renormalization of quarks fields.

Let us consider the tree level diagonalization of matrices \( (W_0^+ \cdot W_0^+) \). To diagonalize \( (W_0^+ \cdot W_0^+) \), we have to solve the eigen-value problems \( (W_0^+ \cdot W_0^+)_{s,t} \phi_\pm^i(t) = (M_0^2)_{i} \phi_\pm^i(s) \), then \( U_0 \) and \( V_0 \) are given by normalized eigenvectors \( \phi_\pm^i \): \( (U_0)_{s,t} = \phi_+^i(t) \) and \( (V_0)_{s,t} = \phi_-^i(t) \). The two eigen-state equations lead to the same equation

\[ - w_0 \left( \phi_\pm^i(s + 1) + \phi_\pm^i(s - 1) \right) + \left( 1 + w_0^2 - (M_0^2)_{i} \right) \phi_\pm^i(s) = 0 \] \hspace{1cm} (93)

but with different boundary conditions:

\[ - w_0 \phi_+^i(0) + \phi_+(1) = 0, \quad \phi_+^i(N_s + 1) = 0 \] \hspace{1cm} (94)
or

\[- w_0 \phi_i^\dagger (N_s + 1) + \phi_- (N_s) = 0, \quad \phi^\dagger_-(0) = 0. \tag{95}\]

Therefore, once \(\phi^\dagger_+(s)\) is known, the other is easily obtained through \(\phi^\dagger_+(s) = \phi^\dagger_+(N_s + 1 - s)\). Hereafter we consider \(\phi^\dagger_+(s)\) only and drop the suffices + and i.

There are two types of solutions to the eigen-state equation. For \((M_0^2)_i \leq (1 - w_0)^2\) we have a damping solution \(\phi(s) = Ae^{-\alpha s}\) with \(\cosh \alpha = \frac{1 + w_0^2 - (M_0^2)_i}{2w_0}\). The first boundary condition leads to \(e^{-\alpha} = w_0\). This implies \((M_0^2)_i = 0\), and therefore \(\phi(s)\) is nothing but the zero mode solution of the domain-wall QCD. For this solution \(w_0\) should satisfy \(w_0^2 \leq 1\) \((0 \leq M \leq 2)\). The other boundary condition can be satisfied in the large \(N_s\) limit. The normalization constant becomes \(A = (1 - w_0^2)^{1/2}\). Note that there are no other damping solutions which satisfy the first boundary condition.

If the eigen-value is in the region \((1 - w_0)^2 \leq (M_0^2)_i \leq (1 + w_0)^2\), we have an oscillating solution \(\phi(s) = Ae^{i\alpha s} + Be^{-i\alpha s}\) with \(\cos \alpha = \frac{1 + w_0^2 - (M_0^2)_i}{2w_0}\). The two boundary conditions imply

\[
\begin{pmatrix}
  e^{i\alpha} - w_0 & e^{-i\alpha} - w_0 \\
e^{i\alpha(N_s+1)} & e^{-i\alpha(N_s+1)}
\end{pmatrix}
\begin{pmatrix}
  A \\
  B
\end{pmatrix} = 0. \tag{96}
\]

The existence of the non-trivial solution requires \(w_0 \sin \alpha(N_s + 1) = \sin \alpha N_s\), which leads to \(\phi(s) = -Ae^{i\alpha(N_s+1)} \sin \alpha(N_s + 1 - s) \equiv A_0 \sin \alpha(N_s + 1 - s)\). Without loss of generality we can take real \(A_0\), and the normalization condition gives \(A_0 = (2/N_s)^{1/2}(1 + O(1/N_s))\). Setting \(\alpha = a/N_s\) we reduce the equation for \(\alpha\) to \(w_0 \sin \alpha = \sin a\) in the large \(N_s\) limit. The solutions \(a = \pi n\) with integer \(n\) to this equation is translated to \(N_s - 1\) independent solutions: \(\alpha = \pi n/N_s\) with \(n = 1, 2, \cdots, N_s - 1\). (Note that \(0 \leq \alpha_s \leq \pi\) since \(\sin \alpha_s > 0\).)

Therefore, all eigen-values and eigen-vectors are now obtained, giving

\[ [U_0]_{s,t} = \begin{cases} (2/N_s)^{1/2} \sin \alpha_s(N_s + 1 - t) & s \neq N_s \\ \left(1 - w_0^2\right)^{1/2} w_0^{(t-1)} & s = N_s \end{cases} \tag{97} \]

and \([V_0]_{s,t} = [U_0]_{s,N_s+1-t}\).

Next we prove some properties of \(U_0\) and \(V_0\). It is noted that \(U_0\) and \(V_0\) can also diagonalize \(W_0^\pm\) :

\[
\left(V_0 W_0^+ U_0^\dagger\right)_{s,t} = \delta_{s,t} f_s \tag{98}
\]

where

\[ f_s = \begin{cases} w_0 \cos \alpha_s(N_s + 1) - \cos \alpha_s N_s & s \neq N_s \\ 0 & s = N_s \end{cases}. \tag{99} \]

Using the equation for \(\alpha_s\) \((s \neq N_s)\) we can show

\[
f_s^2 = w_0^2 + 1 - 2w_0 \left[\sin \alpha_s N_s \sin \alpha(N_s + 1) + \cos \alpha_s N_s \cos \alpha_s(N_s + 1)\right] \\
= w_0^2 + 1 - 2w_0 \cos \alpha_s = (M_0^2)_s. \tag{100}
\]
This proves \( f_s = (M_0)_s \) for all \( s \).

It is also important to note that \( U_0 \) \( (V_0) \) diagonalizes \( I_{\log}^+ \) \( (I_{\log}^-) \) terms, since

\[
\left( U_0 w_0^{s+t-2} U_0^\dagger \right)_{s,t} = \delta_{s,t} \delta_{s,N_s} (1 - w_0^2) \sum_{s,t} w_0^{2(s+t-2)} + O(1/N_s)
\]

\[
= \delta_{s,t} \delta_{s,N_s} (1 - w_0^2)^{-1} + O(1/N_s).
\]  \hspace{1cm} (101)

Now let us consider quantities including \( g^2 \) contributions. We first show that \( U \) and \( V \) diagonalize \( \mathbf{W}^\pm \) at this order:

\[
V \cdot \mathbf{W}^+ \cdot U^\dagger = (1 + g^2 V_1) V_0 (W_0^+ + g^2 W_1^+) U_0^\dagger (1 + g^2 U_1^\dagger)
\]

\[
= V_0 \cdot W_0^+ U_0^\dagger + g^2 \left\{ V_1 V_0 W_0^+ U_0^\dagger + V_0 W_0 U_0^\dagger U_1^\dagger + \mathbf{W}^+ \right\}
\]  \hspace{1cm} (102)

where the coefficient of the \( g^2 \) term is simplified to

\[
(V_1)_{s,t} (M_0)_t + (M_0)_s (U_1^\dagger)_{s,t} + (\mathbf{W}^+)_{s,t} = (\mathbf{W}^+)_{s,t} \left( 1 + \frac{(M_0^2)_t - (M_0^2)_s}{(M_0^2)_s - (M_0^2)_t} \right) = 0
\]  \hspace{1cm} (103)

for \( s \neq t \), and becomes \( (\mathbf{W}^+)_{s,s} \) for \( s = t \). Eq. (102) then becomes

\[
= (M_0 + g^2 \mathbf{W}^+) 1.
\]

It is necessary for the stability of the zero mode to show that \( (\mathbf{W}^+)_{N_s,N_s} = 0 \). This can be proven as follows.

\[
(\mathbf{W}^+)_{N_s,N_s} = (1 - w_0^2) \sum_{s,t} w_0^{N_s-s} (W_1^+)_{s,t} w_0^{t-1}
\]  \hspace{1cm} (104)

where \( W_1^+ \) is composed of the sum \( G_L, G_R, W_0^+ G_L \) and \( W_0^- G_R \) but

\[
\sum_{s,t} w_0^{N_s-s} G(s,t) w_0^{t-1} = O(N_s w_0^{N_s}) \to 0
\]

for all \( G = G_L, G_R, W_0^+ G_L \) and \( W_0^- G_R \) in the large \( N_s \) limit. Therefore eq. (104) vanishes.

For the wave-function renormalization factor we have to know

\[
U_{N_s,1} = V_{N_s,N_s} = (U_0)_{N_s,1} + g^2 \sum_{t \neq N_s} (U_1)_{N_s,t} (U_0)_{t,1}.
\]

Fortunately, since \( (U_0)_{t,1} = (2/N_s)^{1/2} \sin \alpha_t (1 - 1) = 0 \), there is no order \( g^2 \) contribution and it becomes \( U_{N_s,1} = (1 - w_0^2)^{1/2} \).

Finally we would like to evaluate \( I^\pm_k \) from \( I^\pm_{\text{finite}} \). If we define

\[
\langle F(s,t) \rangle_U = \sum_{s,t} (U_0)_{N_s,s} F(s,t) (U_0)_{N_s,t}
\]

and

\[
\langle F(s,t) \rangle_V = \sum_{s,t} (V_0)_{N_s,s} F(s,t) (V_0)_{N_s,t},
\]

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we can show
\[
\langle e^{-\alpha |s-t|} \rangle_{U,V} = (1 - w_0^2) \frac{\sinh \alpha_0 - \sinh \alpha}{2w_0 \sinh \alpha_0 (\cosh \alpha_0 - \cosh \alpha)}
\]
with \( e^{-\alpha_0} = w_0\),
\[
\langle e^{-\alpha(s+t-2)} \rangle_U = \langle e^{-\alpha(2N_s-s-t)} \rangle_V = (1 - w_0^2) \frac{e^{2\alpha}}{(e^\alpha - w_0)^2}
\]
and
\[
\langle e^{-\alpha(s+t-2)} \rangle_V = \langle e^{-\alpha(2N_s-s-t)} \rangle_U = 0.
\]
Using these formula we obtain
\[
\langle G_L(s,t) \rangle_U = \frac{1 - w_0^2}{2W \sinh \alpha} \left[ \frac{\sinh \alpha_0 - \sinh \alpha}{2w_0 \sinh \alpha_0 (\cosh \alpha_0 - \cosh \alpha)} - \frac{e^\alpha - W}{e^\alpha - W (e^\alpha - w_0)^2} \right] = \langle G_R(s,t) \rangle_V \equiv \tilde{G}_L
\]
(105)
\[
\langle G_R(s,t) \rangle_U = \frac{1 - w_0^2}{2W \sinh \alpha} \left[ \frac{\sinh \alpha_0 - \sinh \alpha}{2w_0 \sinh \alpha_0 (\cosh \alpha_0 - \cosh \alpha)} - \frac{1}{(e^\alpha - w_0)^2} \right] = \langle G_L(s,t) \rangle_V \equiv \tilde{G}_R
\]
(106)
and
\[
\langle W_0^+ G_L(s,t) \rangle_{U,V} = \langle W_0^- G_R(s,t) \rangle_{U,V} = (w_0 - W) \tilde{G}_R.
\]
The explicit expression for \( I^\pm_{\text{finite}} \) is reduced to the final result in terms of \( \tilde{G}_{L/R} \): \( I^d_+ = I^d_- \equiv I^d \) where
\[
I^d = C_2 \int \frac{d^4 l}{(2\pi)^4} \left\{ \frac{1}{8l^2} \sum_{\mu} \left[ \sin^2 l_\mu (\tilde{G}_R + \tilde{G}_L) + 2 \cos l_\mu (w_0 - W(l)) \tilde{G}_R \right] \\
+ \sum_{\mu} \frac{\sin^2 l_\mu}{2(l^2)^2} \left[ (w_0 - W(l))\tilde{G}_R + (\sum_{\nu} \cos^2 l_{\nu}/2 - 2 \cos^2 l_\mu/2) \tilde{G}_L + \sum_{\nu} (\sin^2 l_{\nu}/2) \tilde{G}_R \right] \\
- \frac{1}{(l^2)^2} \theta(\pi^2 - l^2) \right\}.
\]
(107)
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FIG. 1. Diagrams which contribute to the one-loop correction to the fermion propagator. Above: Tadpole diagram. Below: Half-circle diagram.

FIG. 2. The fermion mass $m_F$ obtained in the mean-field approximation as a function of $M$ for $N_s=4$ (solid lines) and $N_s=10$ (dashed line), at $m_q=0$, 0.01, 0.02, 0.03 from below to above around $M = 1.5$. 
TABLES

TABLE I. Value of $I^d$ vs $\tilde{M}$, together with $Z_1$ and $(Z_1)_{\text{non-tad}}/Z_1$

| $\tilde{M}$ | $I^d$       | $Z_1$    | $(Z_1)_{\text{non-tad}}/Z_1$ |
|------------|------------|----------|-----------------------------|
| 0.05       | -0.01945(5)| -0.09897 | 0.041                       |
| 0.10       | -0.01871(5)| -0.09822 | 0.049                       |
| 0.15       | -0.01804(5)| -0.09756 | 0.056                       |
| 0.20       | -0.01744(5)| -0.09696 | 0.063                       |
| 0.25       | -0.01688(5)| -0.09640 | 0.069                       |
| 0.30       | -0.01636(5)| -0.09589 | 0.075                       |
| 0.35       | -0.01588(5)| -0.09541 | 0.080                       |
| 0.40       | -0.01544(5)| -0.09496 | 0.085                       |
| 0.45       | -0.01502(5)| -0.09454 | 0.090                       |
| 0.50       | -0.01463(5)| -0.09415 | 0.095                       |
| 0.55       | -0.01426(5)| -0.09378 | 0.099                       |
| 0.60       | -0.01392(5)| -0.09345 | 0.103                       |
| 0.65       | -0.01361(5)| -0.09313 | 0.107                       |
| 0.70       | -0.01332(5)| -0.09284 | 0.110                       |
| 0.75       | -0.01305(5)| -0.09257 | 0.113                       |
| 0.80       | -0.01281(5)| -0.09233 | 0.116                       |
| 0.85       | -0.01259(5)| -0.09211 | 0.119                       |
| 0.90       | -0.01239(5)| -0.09191 | 0.121                       |
| 0.95       | -0.01222(5)| -0.09174 | 0.124                       |