A NOTE CONCERNING A PROPERTY OF SYMPLECTIC MATRICES

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Abstract. This note provides a counterexample to a proposition stated in [1] regarding the neighborhood of certain 4 × 4 symplectic matrices.

1. Introduction. We denote by $I_n$ the $n \times n$ identity matrix, by $J$ the standard $4 \times 4$ symplectic matrix, i.e.

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \in \mathbb{R}^{4 \times 4},$$

and by $\text{Sp}^0(\mathbb{R}^4) = \{ S \in \mathbb{R}^{4 \times 4} : S^TJS = J \}$ the corresponding symplectic group, which shall be equipped with some norm. Furthermore a matrix $S \in \text{Sp}^0(\mathbb{R}^4)$ is called elliptic if the spectrum $\sigma(S)$ is contained in $S^1 \setminus \{ \pm 1 \}$.

In section 2 we will present a continuous family of symplectic matrices contradicting the following statement:

**Proposition 1** (Prop. 2.1 of [1]). Assume that $P$ is a matrix satisfying

$$P \in \text{Sp}^0(\mathbb{R}^4), \quad P \neq I_4, \quad \dim \ker(P - I_4) \neq 2, \quad \sigma(P) = \{ 1 \}. \quad (1)$$

Then there exists a neighborhood $\mathcal{U} \subset \text{Sp}^0(\mathbb{R}^4)$ of $P$ such that a matrix $S \in \mathcal{U}$ is elliptic if and only if the following conditions hold

$$\det(S - I_4) > 0 \quad \text{and} \quad \text{tr} S < 4. \quad (2)$$

This proposition has been used in the proof of Theorem 1.1 of [1] to obtain a spectral stability result for periodic solutions of a perturbed Kepler problem. It has not been used for the instability result contained in the same theorem.

2. A family of symplectic matrices. For $\varepsilon \geq 0$ we define the $2 \times 2$ matrices

$$A_\varepsilon = \begin{pmatrix} 1 & -\varepsilon \\ \varepsilon & 1 \end{pmatrix}, \quad B_\varepsilon = \begin{pmatrix} 1 & -\varepsilon \\ 0 & 0 \end{pmatrix}, \quad C_\varepsilon = \frac{1}{1 + \varepsilon^2} \begin{pmatrix} 1 & -\varepsilon \\ \varepsilon & 1 \end{pmatrix},$$

as well as the $4 \times 4$ matrix

$$P_\varepsilon = \begin{pmatrix} A_\varepsilon & B_\varepsilon \\ 0 & C_\varepsilon \end{pmatrix}.$$
Clearly \([0, \infty) \ni \varepsilon \mapsto P_\varepsilon \in \mathbb{R}^{4\times 4}\) is continuous. Moreover \(P_\varepsilon \in \text{Sp}(\mathbb{R}^4)\), since
\[
P_\varepsilon^T J P_\varepsilon = \begin{pmatrix} 0 & A_\varepsilon^T C_\varepsilon \\ -C_\varepsilon^T A_\varepsilon & B_\varepsilon^T C_\varepsilon - C_\varepsilon^T B_\varepsilon \end{pmatrix} = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} = J.
\]
It follows that
\[
P_0 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]
satisfies condition (1). Next we show that for positive \(\varepsilon\) the matrix \(P_\varepsilon\) satisfies condition (2) and is not elliptic. We have
\[
\text{tr}(P_\varepsilon) = 2 + \frac{2}{1+\varepsilon^2} < 4.
\]
The characteristic polynomial \(\chi_\varepsilon\) of \(P_\varepsilon\) is given by
\[
\chi_\varepsilon(\lambda) = ((1 - \lambda)^2 + \varepsilon^2)^2 \left(\frac{1}{1+\varepsilon^2} - \lambda\right)^2 + \frac{\varepsilon^2}{(1+\varepsilon^2)^2},
\]
so especially
\[
\det(P_\varepsilon - I_4) = \chi_\varepsilon(1) = \frac{\varepsilon^4}{1+\varepsilon^2} > 0
\]
and we also see that the spectrum
\[
\sigma(P_\varepsilon) = \{1 \pm i\varepsilon, (1 \pm i\varepsilon)^{-1}\}
\]
is not contained in \(S^1\).

Thus there exists no neighborhood of \(P_0\), on which condition (2) implies ellipticity.

3. Invariant Lagrangian splittings. A Lagrangian splitting of \(\mathbb{R}^4\) is a decomposition \(\mathbb{R}^4 = U \oplus V\) into two-dimensional subspaces satisfying \(u_1^T J u_2 = 0\), \(v_1^T J v_2 = 0\) for all \(u_1, u_2 \in U\), \(v_1, v_2 \in V\).

For \(\varepsilon > 0\) the planes \(U = \text{span}\{e_1, e_2\}\), \(V = \text{span}\{v_1^\varepsilon, v_2^\varepsilon\}\), where
\[
e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_1^\varepsilon = \begin{pmatrix} 1 + \varepsilon^2 \\ -\varepsilon(1 + \varepsilon^2) \\ -2\varepsilon^2 \\ \varepsilon^3 \end{pmatrix}, \quad v_2^\varepsilon = \begin{pmatrix} 0 \\ 1 + \varepsilon^2 \\ -\varepsilon^3 \\ -2\varepsilon^2 \end{pmatrix},
\]
form a Lagrangian splitting of \(\mathbb{R}^4\). Moreover, since
\[
P_\varepsilon e_1 = e_1 + \varepsilon e_2, \quad P_\varepsilon e_2 = -\varepsilon e_1 + e_2,
\]
\[
P_\varepsilon v_1^\varepsilon = \frac{1}{1+\varepsilon^2} v_1^\varepsilon + \frac{\varepsilon}{1+\varepsilon^2} v_2^\varepsilon, \quad P_\varepsilon v_2^\varepsilon = -\frac{\varepsilon}{1+\varepsilon^2} v_1^\varepsilon + \frac{1}{1+\varepsilon^2} v_2^\varepsilon,
\]
the splitting \(\mathbb{R}^4 = U \oplus V\) is invariant under \(P_\varepsilon\).

On the other hand in the limiting case \(\varepsilon = 0\) the map \(P_0\) does not admit an invariant Lagrangian splitting: Indeed let \(U_0 \oplus V_0\) be a splitting of \(\mathbb{R}^4\) into \(P_0\)-invariant planes. We can assume that \(U_0\) (otherwise \(V_0\)) contains a vector of the form \(u_1 = a e_1 + b e_2 + e_3 + c e_4\) with \(a, b, c \in \mathbb{R}\). By the invariance also \(u_2 = P_0 u_1 = (a + 1) e_1 + b e_2 + e_3 + c e_4 \in U_0\). So \(u_1^T J u_2 = -1\) implies that the splitting is not Lagrangian.

This elaboration shows that the family \((P_\varepsilon)_{\varepsilon \in [0, \infty)}\) contradicts also a lemma on which the proof of Proposition 1 is based:
Lemma 3.1 (Lem. 2.5 of [1]). Let \( \{S_n\} \) be a sequence of matrices in \( \text{Sp}(\mathbb{R}^4) \) converging to \( S \). In addition assume that for each \( n \geq 0 \) there exists a splitting of \( \mathbb{R}^4 \) by Lagrangian planes that are invariant under \( S_n \). Then there exists another splitting by Lagrangian planes that are invariant under \( S \).

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REFERENCES

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