ON QUASI-PRÜFER AND UM⁻domains

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Abstract. In this note we show that an integral domain \( D \) of finite \( w \)-dimension is a quasi-Prüfer domain if and only if each overring of \( D \) is a \( w \)-Jaffard domain. Similar characterizations of quasi-Prüfer domains are given by replacing \( w \)-Jaffard domain by \( w \)-stably strong \( S \)-domain, and \( w \)-strong \( S \)-domain. We also give new characterizations of UM⁻domains.

1. Introduction

The quasi-Prüfer notion was introduced in [2] for rings (not necessarily domains). As in [9], we say that an integral domain \( D \) is a quasi-Prüfer domain if for each prime ideal \( P \) of \( D \), if \( Q \) is a prime ideal of \( D[X] \) with \( Q \subseteq P[X] \), then \( Q = (Q \cap D)[X] \). It is well known that an integral domain is a Prüfer domain if and only if it is integrally closed and quasi-Prüfer [11, Theorem 19.15]. There are several different equivalent conditions for quasi-Prüfer domains (c.f. [9, 2, 3]).

On the other hand as a \( t \)-analogue, an integral domain \( D \) is called a UM⁻domain [12], if every upper to zero in \( D[X] \) is a maximal \( t \)-ideal and has been studied by several authors (see [8], [6], and [18]). UM⁻domains are closely related to quasi-Prüfer domains in the sense that a domain \( D \) is a UM⁻domain if and only if \( D_P \) is a quasi-Prüfer domain for each \( t \)-prime ideal \( P \) of \( D \) [8, Theorem 1.5]. And the other relation is the characterization of quasi-Prüfer domains due to Fontana, Gabelli and Houston [8, Corollary 3.11]: a domain \( D \) is a quasi-Prüfer domain if and only if each overring of \( D \) is a UM⁻domain.

In [16] we defined and studied the \( w \)-Jaffard domains and proved that all strong Mori domains (domains that satisfy the ACC on \( w \)-ideals) and all UM⁻domains of finite \( w \)-dimension, are \( w \)-Jaffard domains. In [17] we defined and studied a subclass of \( w \)-Jaffard domains, namely the \( w \)-stably strong \( S \)-domains and showed how this notion permit studies of UM⁻domains in the spirit of earlier works on quasi-Prüfer domains. The aim of this paper is to prove that, for a domain \( D \) with some condition on \( w \)-dim(\( D \)), the following statements are equivalent, which gives new descriptions of quasi-Prüfer domains; a result reminiscent of the well-known result of Ayache, Cahen and Echi [2] (see also [9, Theorem 6.7.8]).

1. Each overring of \( D \) is a \( w \)-stably strong \( S \)-domain.
2. Each overring of \( D \) is a \( w \)-strong \( S \)-domain.
3. Each overring of \( D \) is a \( w \)-Jaffard domain.
4. Each overring of \( D \) is a UM⁻domain.
5. \( D \) is a quasi-Prüfer domain.

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Throughout, the letter $D$ denotes an integral domain with quotient field $K$ and $F(D)$ denotes the set of nonzero fractional ideals. Let $f(D)$ be the set of all nonzero finitely generated fractional ideals of $D$. Let $\ast$ be a star operation on the domain $D$. For every $A \in F(D)$, put $A^{\ast} := \bigcup F^\ast$, where the union is taken over all $F \in f(D)$ with $F \subseteq A$. It is easy to see that $\ast_{f}$ is a star operation on $D$. A star operation $\ast$ is called of finite character if $\ast_{f} = \ast$. We say that a nonzero ideal $I$ of $D$ is a $\ast$-ideal of $D$, if $I^\ast = I$; a $\ast$-prime, if $I$ is a prime $\ast$-ideal of $D$. It has become standard to say that a star operation $\ast$ is stable if $(A \cap B)^\ast = A^\ast \cap B^\ast$ for all $A, B \in F(D)$. Given a star operation $\ast$ on an integral domain $D$ it is possible to construct a star operation $\ast$ which is stable and of finite character defined as follows: for each $A \in F(D)$,

$$A^\ast := \{x \in K| xJ \subseteq A, \text{ for some } J \subseteq D, J \in f(D), J^\ast = D\}.$$ 

The $\ast$-dimension of $D$ is defined as follows:

$$\ast\text{-dim}(D) = \sup\{\hbar(P) \mid P \text{ is a } \ast\text{-prime ideal of } D\}.$$ 

The most widely studied star operations on $D$ have been the identity $d$, and $v$, $t := v_{f}$, and $w := v$ operations, where $A_{v} := (A^{-1})^{-1}$, with $A^{-1} := (D : A) := \{x \in K| xA \subseteq D\}$. 

Let $D$ be a domain and $T$ an overring of $D$. Let $\ast$ and $\ast'$ be star operations on $D$ and $T$, respectively. One says that $T$ is $(\ast, \ast')$-linked to $D$ if $F^\ast = D \Rightarrow (FT)^{\ast'} = T$ for each nonzero finitely generated ideal $F$ of $D$. As in [4] we say that $T$ is $t$-linked to $D$ if $T$ is $(t, t)$-linked to $D$. As in [6] a domain $D$ is called $t$-linkative if each overring of $D$ is $t$-linked to $D$. As a matter of fact $t$-linkative domains are exactly the domains such that the identity operation coincides with the $w$-operation, that is $DW$-domains in the terminology of [15]. 

If $F \subseteq K$ are fields, then tr. deg.$_{F}(K)$ stands for the transcendence degree of $K$ over $F$. If $P$ is a prime ideal of the domain $D$, then we set $k_{p}(P) := D_{p}/PD_{p}$. 

### 2. $w$-JAFFARD DOMAINS 

First we recall a special case of a general construction for semistar operations (see [16]). Let $D$ be an integral domain with quotient field $K$, let $X, Y$ be two indeterminates over $D$ and $\ast$ be a star operation on $D$. Set $D_{1} := D[X]$, $K_{1} := K(X)$ and take the following subset of $\text{Spec}(D_{1})$:

$$\Theta_{1}^{\ast} := \{Q_{1} \in \text{Spec}(D_{1}) \mid Q_{1} \cap D = (0) \text{ or } (Q_{1} \cap D)^{\ast} \subseteq D\}.$$ 

Set $G_{1}^{\ast} := D_{1}[Y]\langle \cup\{Q_{1}[Y]|Q_{1} \in \Theta_{1}^{\ast}\}\rangle$ and:

$$E_{C_{1}}^{\ast} := E[Y]|G_{1}^{\ast} \cap K_{1}, \text{ for all } E \in F(D_{1}).$$ 

It is proved in [16] Theorem 2.1 that the mapping $[*][X] := \diamond E_{1}^{\ast}$: $F(D_{1}) \rightarrow F(D_{1})$, $E \mapsto E^{[X]}$ is a stable star operation of finite character on $D[X]$, i.e., $[*][X] = [*][X]$. It is also proved that $[*][X] = [*][X]$, $d_{D}[X] = d_{D}[X]$. If $X_{1}, \cdots, X_{r}$ are indeterminates over $D$, for $r \geq 2$, we let

$$[*][X_{1}, \cdots, X_{r}] := ([*][X_{1}, \cdots, X_{r}])[X_{r}].$$ 

For an integer $r$, put $[*][r]$ to denote $[*][X_{1}, \cdots, X_{r}]$ and $D[r]$ to denote $D[X_{1}, \cdots, X_{r}]$. 

Let $\ast$ be a star operation on $D$. A valuation overring $V$ of $D$ is called a $\ast$-valuation overring of $D$ provided that $F^\ast \subseteq FV$, for each $F \in f(D)$. Following [16], the $\ast$-valuative dimension of $D$ is defined as:

$$\ast\text{-dim}_{\ast}(D) := \sup\{\dim(V)|V \text{ is } \ast\text{-valuation overring of } D\}.$$
It is shown in [16] Theorem 4.5 that
\[ \dim_v(D) = \sup \{ w\text{-dim}(R) \mid R \text{ is a } (*,t)\text{-linked over } D \}. \]

It is observed in [16] that we have always the inequality \( \dim(D) \leq \dim_v(D) \).
We say that \( D \) is a \(*\)-Jaffard domain, if \( \dim(D) = \dim_v(D) < \infty \). When \( * = d \) the identity operation then \( d\)-Jaffard domain coincides with the classical Jaffard domain (cf. [1]). It is proved in [16], that \( D \) is a \(*\)-Jaffard domain if and only if
\[ *[X_1, \ldots, X_n] \cdot \dim(D[X_1, \ldots, X_n]) = \dim(D) + n, \]
for each positive integer \( n \). In [19] we gave examples to show that the two classes of \( w\)-Jaffard and Jaffard domains are incomparable by constructing a \( w\)-Jaffard domain which is not Jaffard and a Jaffard domain which is not \( w\)-Jaffard.

At this point we are now prepared to state and prove the first main result of this paper.

**Theorem 2.1.** Let \( D \) be an integral domain of finite \( w\)-dimension. Then the following statements are equivalent:

1. Each overring of \( D \) is a \( w\)-Jaffard domain.
2. \( D \) is a quasi-Prüfer domain.

**Proof.** (1) \( \Rightarrow \) (2) Let \( Q \) be a prime ideal of an overring \( T \) of \( D \), and set \( q := Q \cap D \). Let \( \tau : T_Q \to \mathbb{K}(Q) \) be the canonical surjection and let \( \iota : \mathbb{K}(q) \to \mathbb{K}(Q) \) be the canonical embedding. Consider the following pullback diagram:

\[
\begin{array}{ccc}
D(Q) := \tau^{-1}(\mathbb{K}(q)) & = D_q + QT_Q & \longrightarrow \mathbb{K}(q) \\
\downarrow & & \downarrow \\
\tau & & \tau \\
T_Q & \longrightarrow & \mathbb{K}(Q).
\end{array}
\]

Since \( T_Q \) is quasilocal and \( \mathbb{K}(q) \) is a DW-domain, then \( D(Q) \) is a DW-domain by [15] Theorem 3.1(2). Thus the \( w\)-operation coincides with the identity operation \( d \) for \( D(Q) \). Since by the hypothesis \( D(Q) \) is a \( w\)-Jaffard domain we actually have \( D(Q) \) is a Jaffard domain. On the other hand by [1] Proposition 2.5(a) we have
\[ \dim_v(D(Q)) = \dim_v(T_Q) + \text{tr. deg}_{\mathbb{K}(q)}(\mathbb{K}(Q)). \]

In particular \( \text{tr. deg}_{\mathbb{K}(q)}(\mathbb{K}(Q)) \) and \( \dim_v(T_Q) \) are finite numbers. Note that by [7] Proposition 2.1(5) we have \( \dim(D(Q)) = \dim(T_Q) \) and since \( \dim_v(D(Q)) = \dim(D(Q)) \), we obtain that
\[ \dim(T_Q) = \dim_v(T_Q) + \text{tr. deg}_{\mathbb{K}(q)}(\mathbb{K}(Q)). \]

Since \( \dim(T_Q) \leq \dim_v(T_Q) \), then \( \text{tr. deg}_{\mathbb{K}(q)}(\mathbb{K}(Q)) = 0 \). Consequently \( D \) is a residually algebraic domain, and hence is a quasi-Prüfer domain by [3] Corollary 2.8.

(2) \( \Rightarrow \) (1) Let \( T \) be an overring of \( D \). We claim that \( T \) is of finite \( w\)-dimension. Since \( D \) is a quasi-Prüfer domain, [16] Theorem 2.4 implies that \( D \) is a \( t\)-linkative and UMt domain. Thus in particular \( T \) is a \( t\)-linked overring of \( D \). Then
\[ w\text{-dim}(T) \leq \sup \{ w\text{-dim}(R) \mid R \text{ is } t\text{-linked over } D \} \]
\[ = w\text{-dim}_{v}(D) = w\text{-dim}(D) < \infty, \]
where the first equality is by [16] Theorem 4.5]. Finally by [17] Corollary 2.6], every UMt domain of finite \( w\)-dimension is a \( w\)-Jaffard domain to deduce that \( T \) is a \( w\)-Jaffard domain. \( \square \)
As an immediate corollary we have:

**Corollary 2.2.** Let $D$ be an integral domain of finite $w$-dimension. Then the following statements are equivalent:

1. Each $t$-linked overring of $D$ is a $w$-Jaffard domain.
2. $D$ is a UM$t$ domain.

**Proof.** (1) $\Rightarrow$ (2) Let $P$ be a $t$-prime ideal of $D$, and $T$ be an overring of $D_P$. Thus $T = T_D \setminus P$ is a $t$-linked overring of $D$ by [5, Proposition 2.9]. Therefore $T$ is a $w$-Jaffard domain by the hypothesis. Consequently $D_P$ is a quasi-Pr"ufer domain by Theorem 2.1. Then $D$ is a UM$t$ domain by [8, Theorem 1.5].

(2) $\Rightarrow$ (1) Let $T$ be a $t$-linked overring of $D$. Then as the proof of Theorem 2.1 we have

$$w\dim(T) \leq \sup\{w\dim(R) \mid R \text{ is } t\text{-linked over } D\}$$

$$= w\dim_u(D) = w\dim(D) < \infty.$$ 

By [17, Corollary 2.6] we get that $T$ is a $w$-Jaffard domain. $\square$

3. $w$-stably strong S-domains

Let $*$ be a star operation on $D$. Following [17] the domain $D$ is called a $*$-strong S-domain, if each pair of adjacent $*$-prime ideals $P_1 \subset P_2$ of $D$, extend to a pair of adjacent $[*X]$-prime ideals $P_1[X] \subset P_2[X]$, of $D[X]$. If for each $n \geq 1$, the polynomial ring $D[n]$ is a $[*n]$-strong S-domain, then $D$ is said to be an $*$-stably strong S-domain. It is observed in [17] that a domain $D$ is $*$-strong S-domain (resp. $*$-stably strong S-domain) if and only if $D_P$ is strong S-domain (resp. stably strong S-domain) for each $*$-prime ideal $P$ of $D$. Thus a strong S-domain (resp. stably strong S-domain) $D$ is $*$-strong S-domain (resp. $*$-stably strong S-domain) for each star operation $*$ on $D$. However, the converse is not true in general; i.e., for some star operation $*$, the domain $D$ might be $*$-strong S-domain (resp. $*$-stably strong S-domain), but $D$ is not strong S-domain (resp. stably strong S-domain). In [14, Example 4.17] Malik and Mott gave an example of a UM$t$ domain (in fact a Krull domain) which is not strong S-domain. But a UM$t$ domain is a $w$-stably strong S-domain (and hence $w$-strong S-domain as well) by [17, Corollary 2.6].

We observe [14, Corollary 2.3] that a finite $w$-dimensional $w$-stably strong S-domain is a $w$-Jaffard domain.

We are now prepared to state and prove the second main result of this paper.

**Theorem 3.1.** Let $D$ be an integral domain of finite $w$-valuative dimension. Then the following statements are equivalent:

1. Each overring of $D$ is a $w$-stably strong S-domain.
2. Each overring of $D$ is a $w$-strong S-domain.
3. Each overring of $D$ is a UM$t$ domain.
4. $D$ is a quasi-Pr"ufer domain.

**Proof.** The implication (1) $\Rightarrow$ (2) is trivial, and (3) $\Rightarrow$ (1) holds by [17, Corollary 2.6].
(2) ⇒ (4) Let \( Q \) be a prime ideal of an overring \( T \) of \( D \) and set \( q := Q \cap D \). As in the proof of Theorem 2.1 we have the following pullback diagram:

\[
\begin{array}{ccc}
D(Q) & \longrightarrow & \mathbb{K}(q) \\
\downarrow & & \downarrow \\
T_Q & \tau & \mathbb{K}(Q).
\end{array}
\]

Since \( T_Q \) is quasilocal and \( \mathbb{K}(q) \) is a DW-domain, then \( D(Q) \) is a DW-domain by [15, Theorem 3.1(2)]. Thus the \( w \)-operation coincides with the identity operation \( d \) for \( D(Q) \). Since by the hypothesis \( D(Q) \) is a \( w \)-strong S-domain, we actually have \( D(Q) \) is a strong S-domain. Next we claim that \( D(Q) \) is of finite dimension. Indeed since \( D(Q) \) is a DW-domain it is in fact a \( t \)-linked overring of \( D \). Then

\[
\dim(D(Q)) = w-\dim(D(Q))
\]

\[
\leq \sup \{ w-\dim(R) | R \text{ is } t \text{-linked over } D \}
\]

\[
= w-\dim_w(D) < \infty,
\]

where the second equality is by [16, Theorem 4.5]. On the other hand by [1, Proposition 2.7] we have the inequality below

\[
1 + \dim(T_Q) + \min \{ \text{tr. deg}_{\mathbb{K}(q)}(\mathbb{K}(Q)), 1 \} \leq \dim(D(Q)[X])
\]

\[
= \dim(D(Q)) + 1
\]

\[
= \dim(T_Q) + 1.
\]

The first equality holds since \( D(Q) \) is strong S-domain and [13, Theorem 39], and the second one holds by [7, Proposition 2.1(5)]. Thus \( \text{tr. deg}_{\mathbb{K}(q)}(\mathbb{K}(Q)) = 0 \). Consequently \( D \) is a residually algebraic domain and hence is a quasi-Prüfer domain by [8, Corollary 2.8].

(4) ⇒ (3) Suppose that \( D \) is a quasi-Prüfer domain and let \( T \) be an overring of \( D \). Thus \( T \) is also a quasi-Prüfer domain. Therefore \( T \) is a UMt domain by [8, Theorem 2.4]. □

As an immediate corollary we have:

**Corollary 3.2.** Let \( D \) be an integral domain of finite \( w \)-valuative dimension. Then the following statements are equivalent:

1. Each \( t \)-linked overring of \( D \) is a \( w \)-stably strong S-domain.
2. Each \( t \)-linked overring of \( D \) is a \( w \)-strong S-domain.
3. Each \( t \)-linked overring of \( D \) is a UMt domain.
4. \( D \) is a UMt domain.

**Proof.** The implication (1) ⇒ (2) is trivial.

For (2) ⇒ (4) let \( P \) be a \( t \)-prime ideal of \( D \), and \( T \) be an overring of \( D_P \). Thus \( T = T_{D \setminus P} \) is a \( t \)-linked overring of \( D \) by [5, Proposition 2.9]. Therefore \( T \) is a \( w \)-strong S-domain by the hypothesis. Consequently \( D_P \) is a quasi-Prüfer domain by Theorem 3.1. Then \( D \) is a UMt domain by [8, Theorem 1.5].

(4) ⇒ (3) Suppose \( T \) is a \( t \)-linked overring of \( D \). Then \( T \) is a UMt domain by [8, Theorem 3.1].

(3) ⇒ (1) Is true by [17, Corollary 2.6]. □
Note that the equivalence (3) $\Leftrightarrow$ (4) in Theorem 3.1 (resp. Corollary 3.2) is well known [8, Corollary 3.11] (resp. [4, Theorem 2.6]), but our proof is completely different.

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