Level of student’ algebraic thinking

S Y Maudy*, D Suryadi and E Mulyana
Department of Mathematics Education, Universitas Pendidikan Indonesia, Jl. Dr. Setiabudhi No. 229, Bandung 40154, Indonesia

*septianiyugnimaudy@student.upi.edu

Abstract. The diversity of students algebraic thinking, has its own levels, which is important for the teachers and the students to notice. This research aimed to describe the level of secondary-school students’ algebraic thinking through the assignment of an algebra task to seventh to twelfth graders followed by interview, this research analyzed to what extent their level of algebraic thinking along with underlying experiences. The analysis results indicated that there found levels 0 to 4 from these various grades. Interestingly, the learning of algebra that the student experienced at school did not guarantee their capability to apply algebraic thinking.

1. Introduction
Algebra is one of the critical aspects in mathematics that presents many benefits to life. Algebra is the gateway to higher mathematics [1,2,3]. Algebra exploration is identical with the existence of variables that present a challenge for students. Mathematical interest decreased during adolescence in general and in algebra classes in particular [4,5,6]. Algebraic failure rates continue to be high, especially among low-ability students [6,7]. Yet without the knowledge of algebra, people will not be able to understand many of the ideas discussed in the fields of chemistry, physics, earth sciences, economics, business, psychology, and many other fields [8].

In this case, algebra has many similarities with reading, writing, and arithmetic. More specifically, the following algebraic characteristics are so important that some of the many things cannot be done at all or are not that easy without algebra. Those characteristics include: algebra is the language of generalization, algebra is the language of relations between numbers, and algebra is the language for solving various types of numerical problems [8].

Algebraic thinking can be interpreted as an approach to quantitative situations that emphasize the general relational aspect with tools in the form of a symbol of letters [9]. This is similar to that suggested by Swafford and Langrall, algebraic thinking as the ability to operate on an unknown quantity as if quantity were known [10]. Algebraic thinking occurs by starting with a person’s sensitivity about something or objects that cannot be determined exactly (something unknown, variable, and parameter), then proceed with the analysis of the object and the last one is to model that the object already analyzed in symbol [11,12].

The use of generalizations can be considered as algebraic thinking [13,14,15]. Algebraic thinking is apparent when through the conjecture and argumentation in generalizing data and mathematical relationships expressed in formal language improvement [16]. Rafdord also examined the generalization of the pattern of sequences presented in the form of drawings and identified some generalization strategies [17]. Every student who started school has demonstrated the ability to generalize and abstract
certain case which is the root of algebra [18]. The ability to generalize is an important aspect of algebraic thinking, so this area of algebraic research gets more attention on the mathematics education community [19].

It goes on to say that one type of algebraic thinking is functional thinking, which occurs when a child performs a generalization that links two sets of objects [14]. On the other hand, Carpenter states that relational thinking involves expression and equalization on the overall being compared as a process that is completed step by step [20]. This relational thinking in elementary school is not only used in integer operations but can be developed into fractional operations. Fractions are not too difficult when presented by developing the students’ potential for relational thinking [21]. One focus of this relational thinking is to transform the fractions into a topic understood by the child who supports and reinforces the basics of what has been done to rationalize the number of counts, fractions and operations. This is also in line with Hackenberg & Lee’s research, the fractional knowledge of student influences the algebraic thinking of students in writing algebraic equations [22].

To develop student learning in mathematics, especially algebra, it is important to understand the development of students’ thinking [23]. One means by which teachers can dig up the emergence of algebraic thinking models is by using problem solving. This is in line with Herbert and Brown’s study of algebraic thinking in solving a problem [24]. The problems involving concepts in algebra are given to the student, then the student is asked to solve. Problem solving plays an important role for algebra development [25].

Based on some research on algebraic thinking [17,23], one of the problems that can be used to explore algebraic thinking is the problem associated with the pattern of numbers. It is also in line with the researches by Ake, et al. and Godino, et al. in Western Europe that identifies levels of algebra in mathematical activities in primary schools called proto-algebra [26,27]. One of the problems given was the problem of number pattern. Godino, et al. expands with 3 new levels in high school [27]. In short, the definition of a level is based on dichotomous differences, among others: the existence of intensive algebraic objects (i.e., entities of general or indefinite characters), transformations (operations) applied to these objects based on the application of structural properties, and the type of language used.

How the teachers can help the students if they do not know how far the students’ algebraic thinking. Each student has different characteristics and different ways of thinking. Vygotsky explains that, various stored mental processes can be generated through the learning process and can be operated when one interacts with an adult or collaborates with a fellow friend [28]. When solving problems through self-study, individuals can develop their abilities called actual development. In addition, there is potential development that is a development that occurs as a result of interaction with teachers or other students who have more ability. The distance between actual development and potential development is called zone of proximal development. Learning occurs when students work or learn to handle tasks that have not been studied but those tasks are still within ZPD, which is slightly above the current student development. So it is necessary to determine the current students’ algebraic thinking level to help them achieve their potential development.

As teachers, we should be able to anticipate the diversity of student characteristics. As early as possible teachers should be sensitive to where the students’ algebraic thinking level are, so it can help them to bridge each level well. The students can increase their level of thinking if teachers know their students’ positions and assign algebra tasks slightly above the actual abilities of each student and the task is a tiered problem. Every student is different, thus our job is not to equate their diversity, but to try to facilitate each of those differences. The students together with the teachers and friends can try to bridge to the next level. So, the students can increase their algebraic thinking level meaningfully. Based on the description above, in this study, the authors are interested to conduct a research entitled, “Level of Students’ Algebraic Thinking”.


2. Method
This study used qualitative methods with the principle of a priori, in which certain theories are used to reveal the phenomena that occur. The qualitative approach does not change the process or conditions that exist in the field. Qualitative research describes the data in the field without any manipulation done, other than that form of data presented is a form of narrative or description of the analysis.

In the early stages of the study, the authors created algebra tasks prepared in the form of a description problem. It was because the test-shaped description can measure the mental processes of students in putting ideas into the answer. Algebra tasks are based on algebraic literature review. After that algebra task was tested on 220 students grade VII to grade XI. After the first data were analyzed, the researchers selected 10 students to be interviewed. To address the research questions, we conducted a clinical interview study [29]. Student participated in 45-minute, task-based interview: an algebra interview [30].

3. Results and discussion
Onto-Semiotic provides theoretical tools that can help to characterize algebraic thinking in terms of object types and processes involved in the practice of mathematics [31]. Given the intrinsic mathematical practice of algebra can be based on the existence of several types of objects and processes, which are usually considered in the literature as algebra.

Onto-semiotic aims to overcome the problem of meaning and representation by describe the properties of explicit mathematical objects. The theoretical framework used is anthropology, semiotics, and sociocultural [32]. Anthropology is the study of everything about man and his culture, whereas semiotics is the element of language used to represent the practice of mathematics. Duval defines semiotics as a set of interdependent symbols and in accordance with clearly identifiable principles [33]. While social culture is an element that influences the previous elements, namely ontology, anthropology, and semiotics.

The theoretical frameworks used in OSA are (1) the epistemological model of mathematics in anthropological and sociocultural grounds, i.e. the knowledge webs formed in an institutional environment that are strongly influenced by the individual himself and his or her own social and cultural background, (2) the cognitive model in the basis of semiotics formed from the pragmatic nature, meaning the knowledge webs formed from personal objects influenced by the pragmatic or practical nature of a person, and (3) the instructional model corresponding logically to the things has been mentioned above, the intention is matters relating to the interaction that occurs between teachers with students in the classroom or interaction between students and their environment.

Based on the description above, it can be said that the Onto-Semiotic approach is very concerned of the internal and external factors of each individual, also consider the relationship of cognitive domains with social culture. There are six main objects of mathematics involved in the practice of mathematics that are interconnected with each other and then form the cognitive arrangements. The six objects are namely languages, situations, procedures, arguments, propositions, and concepts. Context (in the form of problems) serves to enhance and contextualize thinking activities. Languages (terms, symbols, numbers, images etc.) represent other mathematical objects and serve as tools for action. The argument serves to justify or validate a statement and to explain both deductive and inductive procedures and propositions. Procedures (in the form of mathematical operations, algorithms, and settlement techniques) and propositions (in the form of theorems, properties and others) have the same function of connecting concepts. While the concept obtained through definitions, descriptions, and others.

Different types of objects and algebraic processes can be expressed in different languages, preferably alphanumeric at higher levels of algebra. However, elementary school students may also use other means of expression to represent algebraic objects and processes [34].
Based on some of the things described above, the researchers make a first algebra problem as follows. Kirana has a YouTube account and regularly uploads videos of which numbers increase every month. In the first month, she uploaded a video. In the second month, the video that has been uploaded was 3. In the third and fourth month there are 6 and 10 videos respectively.

a. How many videos have been uploaded in the 7th and 8th months? (Work with at least 3 ways)
b. How many videos have been uploaded in the nth month? (Work with at least 3 ways)

The researchers make the second algebra problem as follows. Joni and Mina have a YouTube account. Each 1 addition of Joni’s subscriber, Mina’s subscriber adds 5. If their numbers of subscriber are 318, how many each subscriber Joni and Mina has? (Work with at least 3 ways).

From the description above, we can see that the students’ algebraic thinking from the existence of algebraic objects and cognitive configuration that occur when working on algebraic problems. In this study, there are four levels of algebraic thinking in the practice of mathematics in high school students from grade VII to XI. Next section will be described each level and some examples of student cases.

3.1. Level 0

Level 0 mathematic practice is in which that does not include algebraic features is described so that students can develop algebraic thinking in primary school with the help of teachers. This issue has not been completed in the literature on early algebra [36]. In this study, it also occurs in high school students. Several rules are proposed for setting algebraic level 0 to the practice of mathematics, i.e. the involvement of large objects, expressed by natural language, numerical, iconic or gestural [26].

In functional task, which is problem 1, the student wrote the first eight values as a sequence of the number of videos corresponding to each month, along with the criteria for obtaining those values, that was to add up with consecutive number. The student used an extensive object that was expressed in the natural, numerical, iconic language involved. It is where the numeric and visual languages are used to express certain values. Visual languages used are tables, diagrams, and images. The student did not try to generalize the criteria or the initial sequence to the end of the correspondence. The existence of level 0 can be indicated from the student 4’s answer as shown in Figure 1 below.

Figure 1. Student 4’s answer on problem 1.

In a structural task, which is problem 2, an operation is performed on an extensive object. The student wrote the sequence values of the number of Joni and Mina subscriber by summation of consecutive numeral numbers with an inefficient comparison relationship. Joni plus 1, Mina plus 5, continuously until total number of both are 318. It can be seen from the student’s answer in solving problem 2 as shown in Figure 2. In addition, the existence of level 0 can be indicated from the student 4’s answer.
3.2. Level 1
Intensive objects, whose generalities are explicitly identified by natural language, numerical, iconic or gestural, are involved. Symbols that refer to known intensive objects are used, but there is no operation with them [26]. For functional task, on the first eight values, the students wrote a sequence of formations extrapolated to the next value indicating the type of factual generalization [34]. It is limited to a concrete level, but a starting point that can be transformed into another form. The student’s answers indicating level 1 of algebraic thinking can be seen as shown in Figure 3 below which is represented by Student 1’s answer.

In relation to structural tasks and the nature of operations is applied and it is symbolically stated that unknown data can be involved. The comparison relationship between two variables in which Mina and Joni, the comparison 1 to 5 are used and continued by using the nature of the operation (addition, division, multiplication by fraction). Level 1 algebraic thinking can be indicated from Student 1’s answer in resolving problem 2 as shown in Figure 4.
3.3. Level 2
Specificities or variables expressed in symbolic language to refer to intensive objects that known be involved, but related to the spatial or temporal information of the context [26]. In functional tasks, the students found the correct formula to calculate the number of videos declared in alphanumeric language. The students did not operate with variables to obtain a canonical expression form of correspondence criteria. The justification or reason of the formula is based on the reasoning of the expressed contextualized generalization [34]. There is an explicit use of generic elements and the correspondence of the amount of video expressed in contextual form. However, only the use of literal symbols in general expression is not sufficient to recognize the presence of algebraic practices. The existence of level 2 algebraic thinking indicated by the students as in the Student 2’s answer shown in Figure 5 below.

![Figure 5. Student 2’s answer on problem 2.](image)

In a structural task, the student performs a generational activity by forming the equation $ax \pm b = c$. Variables expressed in symbolic language to designate the intensive object involved but only related to spatial or temporal information in the context. It can be seen from the student’s answer in solving problem 2 as shown in Figure 6.

![Figure 6. Student 2’s answer on problem 2.](image)

3.4. Level 3
Intensive objects are produced that are literally-symbolically represented, and operations are performed with them; transformation is made in the form of a symbolic expression that preserves equality. The operation is performed on the unknown to solve the equation of the form $Ax \pm B = Cx \pm D$, and symbolic and decontextual canonical rules for the expression of pattern and function are formulated. The level 3 of algebra is operating with intensive objects that are symbolically represented, and therefore they have contextual connotations. On functional tasks, generic symbolic generations are generated. This can be indicated from Student 3’s answer in solving the problem 1 as shown in Figure 7.
In structural tasks, the student formed symbolic expressions, in the form of \( ax + by = c \) equations and operations performed on the unknown to solve the equation. This can be seen from Student 3’s answer (see Figure 8).

![Figure 7. Student 3’s answer on problem 1.](image)

In this section, the algebraic-level model extends to the high and middle school mathematics activities, recognizing three additional levels of algebra for this stage of education. Using parameters and treatments can be a criterion for determining higher levels of algebra, as they are associated with the presence of family equations and functions. The parameter intervention will be associated with the fourth level of algebra [27]. In this study, there were few students who reached level 4.

### 3.5. Level 4

The use of parameters to express the equation or function indicates a higher level of algebraic thinking, regarding the third level of algebra considered by Ake et al., which are associated with processes operating with unknown or variables [26]. This is the first encounter with parameters and coefficients variables involving parametric function ranges, i.e. functions that assign parameter values for each function or certain equations. Examples of this stage are, for example, related to the problem of linear functions and quadratic equations. In functional tasks, in solving problem 1, there were students who form algebraic expressions that represent problems by using variables and parameters. The existence of level 4 thinks this algebra can be indicated from the following Student 4’s answers (see Figure 9).

![Figure 8. Student 3’s answer on problem 2.](image)
4. Conclusion

Based on the results of the study, there are five levels (level 0 to 4) of algebraic thinking found in high school students. The five levels show the changes of algebraic objects as the students completed functional and structural tasks. Each student has a different way of algebra thinking and the way of thinking has a level. Although the students are in the same class, they have different levels of thinking. This needs to be realized together, so that the students can go through algebra thinking well not just passing but can pass meaningfully.

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