Algebras of reduced $E$-Fountain semigroups and the generalized right ample identity II

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Abstract

We study the generalized right ample identity which was introduced by the author in a previous paper. Let $S$ be a reduced $E$-Fountain semigroup which satisfies the congruence condition. We prove that it satisfies the generalized right ample identity if and only if every element of $S$ induces a homomorphism of left $S$ actions between certain classes of generalized Green’s relations. In this case, we interpret the associated category as a discrete form of a Peirce decomposition of the semigroup algebra. We also give some natural examples of semigroups satisfying this identity.

1 Introduction

Let $S$ be a semigroup and let $k$ be a field. It is of interest to study the semigroup algebra $kS$. In many cases we can associate with $S$ a category $C$ such that the semigroup algebra $kS$ is isomorphic to the category algebra $kC$. In [24] such a result is obtained for a certain class of $E$-Fountain semigroups and in this context the generalized right ample identity is introduced. Given a subset of idempotents $E$ of $S$ we can define two equivalence relations $\tilde{L}_E$ and $\tilde{R}_E$ on $S$. We say that $a\tilde{L}_Eb$ ($a\tilde{R}_Eb$) if $a$ and $b$ have the same set of right (respectively, left) identities from $E$. The semigroup $S$ is called reduced $E$-Fountain if every $\tilde{L}_E$ and $\tilde{R}_E$-class contains a (unique) idempotent from $E$ and $ef = e \iff fe = e$ for every $e, f \in E$. If in addition $\tilde{L}_E$ and $\tilde{R}_E$ are right and left congruences respectively then we say that $S$ satisfies the congruence condition and we can associate a certain category $C(S)$ with the semigroup $S$ (for full details see [13]). A reduced $E$-Fountain semigroup which satisfies the congruence condition is also called a DRC-semigroup in the literature and this class is under current active research (see [12, 30]). Let $S$ be a reduced $E$-Fountain semigroup

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which satisfies the congruence condition. For every \( a \in S \) we denote by \( a^+ \) (\( a^* \)) the minimal left (respectively, right) identity of \( a \) from \( E \). We say that \( S \) satisfies the generalized right ample identity if 
\[
\left( e (a (eaf)^*)^* \right)^+ = (a (eaf)^*)^+
\]
holds for every \( a \in S \) and \( e, f \in E \). Under some mild conditions on \( S \), it is proved in [24] that if the generalized right ample identity holds then \( kS \) is isomorphic to the algebra \( kC \) of its associated category. This is a generalization of several results ([10, 16, 17, 20, 21, 25]) that were useful in the study of algebras of various classes of semigroups such as inverse semigroups ([26] and [27, Part IV]) and monoids of partial functions ([13, 22, 23]) - see also [9, 29] for different but related approaches. This motivates consideration of the generalized right ample identity. However, one has to admit that this identity seems unnatural and [24] suggests only one motivating example - the Catalan monoid. The goal of this paper is to obtain a deeper understanding of the generalized right ample identity. We will show that it has a more natural description and we will explore additional examples of semigroups which satisfy it.

The semigroup \( S \) acts partially on the left of every \( \mathcal{L}_E \)-class. For every \( \alpha \in S \), right multiplication induces a function from the \( \mathcal{L}_E \)-class of \( \alpha^+ \) to the \( \mathcal{L}_E \)-class \( \alpha^* \). In Section 3.1 we show that this function is a homomorphism of partial \( S \) actions if and only if the generalized right ample identity holds. In this case we can identify the morphisms of the associated category \( C \) with all such homomorphisms. In Section 3.2 we assume that \( S \) is finite and turn to consider \( kS \) modules. Under a certain condition on \( S \), we show that if the generalized right ample condition holds then the left modules formed by linear combinations of elements of an \( \mathcal{L}_E \)-class are projective modules of \( kS \), thus extending a result from [15]. In this case the linear category formed from \( C \) is a Peirce decomposition of \( kS \). Therefore we can think of \( C \) as a “discrete” form of a Peirce decomposition. In section 4 we give two additional examples of semigroups which satisfy the generalized right ample identity. Linear operators on an Hilbert space and order preserving functions with a fixed point. We study in greater detail the later one. In particular, we prove that its algebra is semisimple and isomorphic to the algebra of the (inverse) semigroup of all order-preserving partial permutations.

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2 Preliminaries

2.1 Semigroups

Let $S$ be a semigroup and let $S^1 = S \cup \{1\}$ be the monoid formed by adjoining a formal unit element. Recall that Green’s preorders $\leq_R$, $\leq_L$ and $\leq_J$ are defined by:

\begin{align*}
  a \leq_R b & \iff aS^1 \subseteq bS^1 \\
  a \leq_L b & \iff S^1a \subseteq S^1b \\
  a \leq_J b & \iff S^1aS^1 \subseteq S^1bS^1
\end{align*}

The associated Green’s equivalence relations on $S$ are denoted by $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{J}$. Recall also that $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$. It is well known that $\mathcal{L}$ ($\mathcal{R}$) is a right congruence (respectively, left congruence). We can define a partial order on the set of idempotents $E(S)$ according to

\[ e \leq f \iff ef = e = fe. \]

We assume familiarity with additional basic notions of semigroup theory that can be found in standard textbooks such as [11].

Let $E \subseteq E(S)$ be a subset of idempotents. We define two equivalence relations $\tilde{\mathcal{L}}_E$ and $\tilde{\mathcal{R}}_E$ on $S$ by

\begin{align*}
  a \tilde{\mathcal{L}}_E b & \iff (\forall e \in E \ a e = a \iff b e = b) \\
  a \tilde{\mathcal{R}}_E b & \iff (\forall e \in E \ e a = a \iff e b = b).
\end{align*}

These relations are one type of “generalized” Green’s relations (see [3]). Note that $\mathcal{L} \subseteq \tilde{\mathcal{L}}_E$ and $\mathcal{R} \subseteq \tilde{\mathcal{R}}_E$.

**Definition 2.1.** The semigroup $S$ is called $E$-Fountain if every $\tilde{\mathcal{L}}_E$-class contains an idempotent from $E$ and every $\tilde{\mathcal{R}}_E$-class contains an idempotent from $E$.

We remark that this property is also called “$E$-semiabundant” in the literature.

**Definition 2.2.** An $E$-Fountain semigroup $S$ is called reduced if

\[ ef = e \iff fe = e \]

for every $e, f \in E$.

A reduced $E$-Fountain semigroup is called a “DR semigroup” in [28]. In such a semigroup, every $\tilde{\mathcal{L}}_E$-class contains a unique idempotent from $E$. The unique idempotent in the $\tilde{\mathcal{L}}_E$-class of $a \in S$ is
denoted $a^*$. The idempotent $a^*$ is the minimal right identity of $a$ from the set $E$ (with respect to the natural partial order on idempotents). In other words, if $e \in E$ satisfies $ae = a$ then $ea^* = a^*e = a^*$. Dually, every $\tilde{R}_E$-class contains a unique idempotent. The unique idempotent in the $\tilde{R}_E$-class of $a$ which is also its minimal left identity from the set $E$ is denoted $a^+$. See [28] for proofs and additional details.

**Definition 2.3.** Let $S$ be a reduced $E$-Fountain semigroup. We say that $S$ satisfies the **congruence condition** if $\tilde{L}_E$ is a right congruence and $\tilde{R}_E$ is a left congruence.

It is well known that $S$ satisfies the congruence condition if and only if the identities $(ab)^* = (a^*b)^*$ and $(ab)^+ = (ab^+)^+$ hold - see [3] Lemma 4.1. In this case we can define a category $C(S)$ in the following way. The objects are in one-to-one correspondence with the set $E$. The morphisms are in one to one correspondence with elements of $S$. For every $a \in S$ the associated morphism $C(a)$ has domain $a^*$ and range $a^+$. If the range of $C(a)$ is the domain of $C(b)$ (that is, if $b^* = a^+$) the composition $C(b) \cdot C(a)$ is defined to be $C(ba)$. The assumption $b^* = a^+$ implies that $(ba)^+ = (ba^+)^+ = (bb^*)^+ = b^+$ and likewise $(ba)^* = a^*$ so this is indeed a category - see [13] for additional details.

Let $S$ be a reduced $E$-Fountain semigroup which satisfies the congruence condition. If $E$ is a subband, it is proved in [24, Lemma 3.9] that $E$ is commutative. In this case, $S$ is called an $E$-**Ehresmann** semigroup, see [3] [13] for more facts and examples.

Recall that a semigroup $S$ is called **inverse** if for every $a \in S$ there exists a unique element denoted $a^{-1}$ such that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$ (this is in fact a very special case of an $E$-Ehresmann semigroup where $E = E(S)$). One of the most important examples of inverse semigroups is the symmetric inverse monoid $\mathcal{I}S_X$ which consists of all partial injective transformations (also called partial permutations) on the set $X$.

Denote by $\mathcal{PT}_X$ the monoid of partial functions on a set $X$. A partial action of $S$ on $X$ is a semigroup homomorphism $\psi : S \to \mathcal{PT}_X$. Equivalently, we say that $S$ acts partially on $X$ or that $X$ is a partial action of $S$. Another convention is to write $s \cdot x$ instead of $\psi(s)(x)$ for $s \in S$ and $x \in X$. Note that $s \cdot x$ might be undefined. Assume that $S$ acts partially on two sets $X, Y$. Recall that a function $f : X \to Y$ is a homomorphism of partial $S$ actions if for every $x \in X$ and $s \in S$, $s \cdot x$ is defined if and only if $s \cdot f(x)$ is defined and in this case $f(s \cdot x) = s \cdot f(x)$.

### 2.2 Linear categories

Let $\mathcal{C}$ be a small category. We denote by $\mathcal{C}^0$ and $\mathcal{C}^1$ the sets of objects and morphisms of $\mathcal{C}$ respectively. For $e, f \in \mathcal{C}^0$ we write $\mathcal{C}(e, f)$ for the hom-set of all morphisms whose domain is $e$ and
Let $k$ be a field. A **$k$-linear category** is a category $L$ enriched over the category of $k$-vector spaces. This means that every hom-set of $L$ is a $k$-vector space and the composition of morphisms is a bilinear map with respect to the vector space operations. Note that a $k$-algebra is a $k$-linear category with one object. A **functor** of $k$-linear categories is a category functor which is also a linear transformation when restricted to any hom-set. Let $C$ be a category. We can form a $k$-linear category $k[C]$ in the following way. The objects of $k[C]$ and $C$ are identical, and for every two objects $e, f$ the hom-set $k[C](e, f)$ is the $k$-vector space with basis $C(e, f)$. In other words, $k[C](e, f)$ contains all formal linear combinations of morphisms from $C(e, f)$. The composition of morphisms in $k[C]$ is defined naturally in the only way that extends the composition of $C$ and forms a bilinear map.

### 2.3 Algebras and modules

Let $A$ be a finite dimensional and unital $k$-algebra. Recall that an $A$-module $P$ is called **projective** if the functor $\text{Hom}_A(P, -)$ is an exact functor. Equivalently, $P$ is projective if it is a direct summand of $A^k$ (for some $k \in \mathbb{N}$) as an $A$-module. It is well known that $Ae$ is a projective $A$-module for every idempotent $e \in A$. Let $E = \{e_1, \ldots, e_n\}$ be a set of idempotents from $A$. Recall that $E$ is a complete set of orthogonal idempotents if $\sum_{i=1}^n e_i = 1_A$ and $e_i e_j = 0$ if $i \neq j$. It is well known that in this case $A \simeq \bigoplus_{i=1}^n Ae_i$ as left $A$-modules and $A \simeq \bigoplus_{i,j=1}^n e_i Ae_j$ as $k$-vector spaces. This decomposition is called a **Peirce decomposition** of $A$. Fixing a complete set of orthogonal idempotents $E = \{e_1, \ldots, e_n\}$, we can associate with $A$ a linear category $L(A)$ whose objects are the projective modules of the form $Ae_i$ and its morphisms are the $A$-module homomorphisms between them. In other words, the set of morphisms with domain $Ae_i$ and range $Ae_j$ is $\text{Hom}_A(Ae_i, Ae_j)$. For the sake of simplicity, we call $L(A)$ also a Peirce decomposition of $A$ (note that $\dim_k \text{Hom}_A(Ae_i, Ae_j) = \dim_k e_i Ae_j$). It is well known that the category of all $A$-representations is equivalent to the category of all $L(A)$-representations.

We will be mainly interested in this paper in semigroup and category algebras. The **semigroup algebra** $kS$ of a semigroup $S$ is defined in the following way. It is a $k$-vector space with basis the elements of $S$, that is, it consists of all formal linear combinations

$$\{k_1 s_1 + \ldots + k_n s_n \mid k_i \in k, s_i \in S\}.$$

The multiplication in $kS$ is the linear extension of the semigroup multiplication. Note that in general $kS$ might not possess a unit element.
The category algebra $\mathbb{k}C$ of a (small) category $C$ is defined in the following way. It is a $\mathbb{k}$-vector space with the morphisms of $C$ as a basis, that is, it consists of all formal linear combinations $\{k_1m_1 + \ldots + k_nm_n \mid k_i \in \mathbb{k}, m_i \in C^1\}$.

The multiplication in $\mathbb{k}C$ is the linear extension of the following:

$$m' \cdot m = \begin{cases} m'm & \text{if } m'm \text{ is defined} \\ 0 & \text{otherwise.} \end{cases}$$

If $C$ has a finite number of objects then the unit element of $\mathbb{k}C$ is $\sum_{e \in C^0} 1_e$ where $1_e$ is the identity morphism of the object $e$ of $C$.

### 3 The generalized right ample identity

Let $S$ be a reduced $E$-Fountain semigroup which satisfies the congruence condition and let $C(S)$ be its associated category. In [24] the generalized right ample identity is defined by several equivalent definitions, one of them is the following.

**Definition 3.1.** We say that the generalized right ample identity (or generalized right ample condition) holds in $S$ if the identity

$$\left( e (a(ef)^*)^+ \right)^* = (a(ef)^*)^+$$

holds for every $a \in S$ and $e, f \in E$.

If $S$ is an $E$-Ehresmann semigroup (i.e., where $E$ is commutative or even a subband) it is proved in [24] that the generalized right ample identity reduces to the “standard” well studied right ample identity

$$ea = a(ea)^*$$

for every $a \in S$ and $e \in E$. The motivation for defining the generalized right ample identity comes from the following result. We say that a relation $R$ on $S$ is **principally finite** if for every $a \in S$ the set $\{c \in S \mid cRa\}$ is finite.

**Theorem 3.2** ([24] Theorems 4.2 and 4.4). Let $k$ be a field. Define a relation $\leq_i$ on $S$ by the rule that $a \leq_i b \iff a = be$ for a certain $e \in E$. Assume that $\leq_i$ is principally finite so the linear
transformation $\varphi : kS \to kC(S)$ defined on basis elements by

$$\varphi(a) = \sum_{c \leq_l a} C(c)$$

is well defined. The linear transformation $\varphi$ is a homomorphism of $k$-algebras if and only if $S$ satisfies the generalized right ample identity. If in addition $\leq_l$ is contained in a principally finite partial order then $\varphi$ is an isomorphism of $k$-algebras if and only if $S$ satisfies the generalized right ample identity.

As already mentioned, in this paper we seek a deeper understanding of the generalized right ample identity. The first step is to observe that we can somewhat simplify this identity.

**Lemma 3.3.** The semigroup $S$ satisfies the generalized ample identity if and only if it satisfies the identity

$$(e (a(ea)^*)^+)^* = (a(ea)^*)^+$$

for every $e \in E$ and $a \in S$.

**Proof.** If $S$ satisfies the generalized right ample identity, we can substitute $f = a^*$ and obtain the simplified identity. In the other direction, let $e, f \in E$ and $a \in S$ and set $a' = af$. The simplified identity implies

$$(e (a'(ea')^*)^+)^* = (a'(ea')^*)^+$$

hence

$$(e (af (eaf)^*)^+)^* = (af (eaf)^*)^+. \quad (1)$$

Now, $eaf = eaf$ so $(eaf)^* \leq f$ in the partial order of idempotents. Therefore, $(eaf)^* = (eaf)^*$ so (1) implies that

$$(e (a(ef)^*)^+)^* = (a(ef)^*)^+$$

as required. \qed

### 3.1 Green’s relations and partial actions

In this subsections we give a more natural interpretation for the generalized right ample identity.

It is natural to ask how much the generalized Green’s relations are similar to the standard ones. We show that a certain property of Green’s relations holds also for the generalized relations if and only if the generalized right ample identity holds.

Let $S$ be a general semigroup and let $x \in S$. Recall that $L_x$ denotes the $L$-class of $x$. 


Lemma 3.4. The semigroup \( S \) acts on the left of the set \( L_x \) by partial functions according to

\[
  s \cdot x = \begin{cases} 
    sx & \text{if } sx \in L_x \\
    \text{undefined} & \text{otherwise}
  \end{cases}
\]

Proof. Note that \( S^1 x = \{ s \in S \mid s \leq_L x \} \) and \( L_x^\perp = \{ s \in S \mid s \leq_L x \land s \notin L_x \} \) are both left ideals of \( S \). Then \( L_x \simeq S^1 x / L_x^\perp \) is a quotient of two left ideals and therefore a partial left \( S \) action.

Let \( \alpha \in S \) and \( e, f \in E(S) \) such that \( e \mathcal{R} \alpha \mathcal{L} f \). Set \( \beta \) to be the inverse of \( \alpha \) such that \( \alpha \beta = e \) and \( \beta \alpha = f \). According to Green’s lemma [11, Lemma 2.2.1] the function \( \rho_\alpha : L_e \to L_f \) defined by \( \rho_\alpha(x) = x\alpha \) is a well defined bijection whose inverse is \( \rho_\beta : L_f \to L_e \).

Lemma 3.5. The function \( \rho_\alpha \) is a homomorphism of partial \( S \) actions.

Proof. Let \( x \in L_e \) and \( s \in S \). If \( sx \in L_e \) and \( sx\alpha \in L_f \) it is clear that

\[
  \rho_\alpha(sx) = sx\alpha = sp_\alpha(x).
\]

It is left to show that

\[
  sx \in L_e \iff sx\alpha \in L_f.
\]

If \( sx \in L_e \) then \( \rho_\alpha(sx) = sx\alpha \in L_f \) because \( \rho_\alpha \) is well defined. In the other direction, if \( sx\alpha \in L_f \) then

\[
  \rho_\beta(sx\alpha) = sx\alpha\beta = sx \in L_e.
\]

We now show that the generalized right ample identity is equivalent to a similar property for the generalized Green’s relation \( \tilde{L}_E \). Fix \( S \) to be a reduced \( E \)-Fountain semigroup which satisfies the congruence condition. We denote the \( \tilde{L}_E \)-class of \( e \in E \) by \( \tilde{L}_E(e) \). It is proved in [15] Lemma 5.1] that for every \( e \in E \), the semigroup \( S \) acts on the left of the set \( \tilde{L}_E(e) \) by partial functions according to

\[
  s \cdot x = \begin{cases} 
    sx & \text{if } sx \in \tilde{L}_E(e) \\
    \text{undefined} & \text{otherwise}
  \end{cases}
\]

for \( s \in S \) and \( x \in \tilde{L}_E(e) \).

For every \( \alpha \in S \) we can define a function \( r_\alpha : \tilde{L}_E(\alpha^+) \to \tilde{L}_E(\alpha^*) \) by \( r_\alpha(x) = x\alpha \). Indeed, if \( x \in \tilde{L}_E(\alpha^+) \) then

\[
  (r_\alpha(x))^* = (x\alpha)^* = (x^*\alpha)^* = (\alpha^+\alpha)^* = \alpha^*
\]
so $r_{\alpha}(x) \in \tilde{L}_E(\alpha^*)$ hence $r_{\alpha}$ is a well-defined function.

In general, $r_{\alpha}$ is not a homomorphism of partial $S$ actions but it is if the generalized right ample identity holds.

**Theorem 3.6.** The semigroup $S$ satisfies the generalized right ample condition if and only if $r_{\alpha}$ is a homomorphism of partial $S$-actions for every $\alpha \in S$.

**Proof.** First assume $S$ satisfies the generalized right ample condition. Let $x \in \tilde{L}_E(\alpha^+)$. If $sx \in \tilde{L}_E(\alpha^+)$ and $sx\alpha \in \tilde{L}_E(\alpha^*)$ it is obvious that

$$r_{\alpha}(sx) = sx\alpha = sr_{\alpha}(x).$$

It is left to show that

$$sx \in \tilde{L}_E(\alpha^+) \iff sx\alpha \in \tilde{L}_E(\alpha^*).$$

If $sx \in \tilde{L}_E(\alpha^+)$ then $(sx)^* = \alpha^+$ and,

$$(sx\alpha)^* = ((sx)^*\alpha)^* = (\alpha^+\alpha)^* = \alpha^*$$

so $sx\alpha \in \tilde{L}_E(\alpha^*)$ as required. In the other direction, assume $sx\alpha \in \tilde{L}_E(\alpha^*)$ so $(sx\alpha)^* = \alpha^*$. Choose $a = \alpha$ and $c = (sx)^*$. The generalized right ample identity implies

$$((sx)^*(\alpha(sx\alpha)^*)^+) = (\alpha((sx)^*\alpha)^*)^+.$$ 

The congruence condition implies $(sx\alpha)^* = ((sx)^*\alpha)^*$ so we obtain

$$((sx)^*(\alpha(sx\alpha)^*)^+) = (\alpha(sx\alpha)^*)^+.$$ 

By assumption, $(sx\alpha)^* = \alpha^*$ so

$$((sx)^*(\alpha\alpha^*)^+) = (\alpha\alpha^*)^+$$

hence

$$((sx)^*\alpha^+) = \alpha^+$$

which implies

$$(sx\alpha^+)^* = \alpha^+$$

by another use of the congruence condition.

Finally, $\alpha^+ = x^*$ so $\alpha^+ = (sx\alpha)^* = (sx)^*$ hence $sx \in \tilde{L}_E(\alpha^*)$ as required. This finishes the “only if” part. For the other direction assume $r_{\alpha}$ is a homomorphism of partial $S$ actions for every $\alpha \in S$, and choose $e \in E$ and $a \in S$. 

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We need to prove
\[(e(a(ea)^*)^*)^* = (a(ea)^*)^*.\]
First note that \(eaa^* = ea\) hence \((ea)^* \leq a^*\). This implies
\[(a(ea)^*)^* = (a^*(ea)^*) = ((ea)^*)^* = (ea)^*.
Now, consider the function
\[r_{a(ea)^*} : \tilde{L}_E((a(ea)^*)^*) \to \tilde{L}_E((a(ea)^*)^*) = \tilde{L}_E((ea)^*).\]
Note that
\[er_{a(ea)^*}((a(ea)^*)^*) = e(a(ea)^*)^* a(ea)^* = ea(ea)^* = ea \in \tilde{L}_E((ea)^*).\]
The fact that \(r_{a(ea)^*}\) is a homomorphism of partial \(S\) actions implies that \(e(a(ea)^*)^* \in \tilde{L}_E((a(ea)^*)^*)\) hence
\[(e(a(ea)^*)^*)^* = (a(ea)^*)^+\]
as required. \(\square\)

For future reference we state also the dual result.

**Corollary 3.7.** The semigroup \(S\) satisfies the generalized left ample identity
\[((ae)^+a)e = (ae)^+a\]
(for every \(a \in S, e \in E\)) if and only if the function \(l_\alpha : \tilde{R}_E(\alpha)^* \to \tilde{R}_E(\alpha^+)\) defined by \(l_\alpha(x) = \alpha x\) is a homomorphism of partial right \(S\)-actions for every \(\alpha \in S\).

It turns out that every homomorphism of partial \(S\) actions between sets of the form \(\tilde{L}_E(e)\) is \(r_\alpha\) for a certain \(\alpha \in S\) as we now show.

**Lemma 3.8.** Let \(S\) be a reduced \(E\)-Fountain semigroup. Let \(e, f \in E\) and assume that \(F : \tilde{L}_E(e) \to \tilde{L}_E(f)\) is a homomorphism of partial left \(S\) actions. Then \(F = r_\alpha\) for \(\alpha \in S\) with \(\alpha^+ = e\) and \(\alpha^* = f\).

**Proof.** Set \(\alpha = F(e)\). It is clear that \(\alpha^+ = f\). To show \(\alpha^* = e\) first observe that
\[\alpha = F(e) = F(ee) = eF(e) = e\alpha\]
so \(\alpha^+ \leq e\). Next,
\[\alpha = \alpha^+ \alpha = \alpha^+ F(e)\]
The fact that $F$ is a homomorphism of partial $S$ actions implies that
\[ \alpha^+ e \in \tilde{L}_E(e) \]
and $\alpha^+ \leq e$ implies $\alpha^+ = \alpha^+ e$. Therefore, $\alpha^+ \in \tilde{L}_E(e)$ so $\alpha^+ = e$ by the uniqueness of idempotents from $E$ in an $\tilde{L}_E$-class. Finally, for every $x \in \tilde{L}_E(e)$ we have $x^* = e$ so
\[ F(x) = F(xe) = xF(e) = x = r_\alpha(x) \]
therefore, $F = r_\alpha$ as required.

As a corollary we obtain a concrete interpretation of the associated category $\mathcal{C}(S)$. Define $\mathcal{D}(S)$ to be a category as follows: The objects of $\mathcal{D}(S)$ are sets of the form $\tilde{L}_E(e)$ for $e \in E$ and the hom-set $\mathcal{D}(S)(e, f)$ is the set of all homomorphisms of partial left $S$ actions $F : \tilde{L}_E(e) \to \tilde{L}_E(f)$.

**Corollary 3.9.** Let $S$ be a reduced $E$-Fountain semigroup which satisfies the congruence condition and the generalized right ample identity. The category $\mathcal{C}(S)^{\text{op}}$ is isomorphic to the category $\mathcal{D}(S)$.

**Proof.** Define $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ on objects by $\mathcal{F}(e) = \tilde{L}_E(e)$ and on morphisms by $\mathcal{F}(C(\alpha)) = r_\alpha$. First observe that $\mathcal{F}(C(e)) = r_e$ is the identity function on $\tilde{L}_E(e)$ for every $e \in E$. Note also that the domain $C(\alpha)$ is $\alpha^*$ and its range is $\alpha^+$ while $r_\alpha : \tilde{L}_E(\alpha^+) \to \tilde{L}_E(\alpha^*)$. In particular, if $\alpha^+ = \beta^*$ for $\alpha, \beta \in S$ then
\[ r_\alpha r_\beta = r_\beta r_\alpha \]
(composing functions right to left) so
\[ \mathcal{F}(C(\beta)C(\alpha)) = \mathcal{F}(C(\beta\alpha)) = r_{\beta\alpha} = r_\alpha r_\beta = \mathcal{F}(C(\alpha))\mathcal{F}(C(\beta)) \]
and $\mathcal{F}$ is indeed a contravariant functor. $\mathcal{F}$ is clearly a bijection on objects. To see that $\mathcal{F}$ is injective on hom-sets, consider two morphisms $C(\alpha), C(\alpha')$ with domain $\alpha^*$ and range $\alpha^+$ such that $r_\alpha = \mathcal{F}(C(\alpha)) = \mathcal{F}(C(\alpha')) = r_{\alpha'}$. Then
\[ \alpha = \alpha^+ \alpha = r_\alpha(\alpha^+) = r_{\alpha'}(\alpha^+) = \alpha^+ = \alpha' \]
hence $C(\alpha) = C(\alpha')$. Finally, Lemma 3.8 shows that $\mathcal{F}$ is onto on hom-sets and therefore it is an isomorphism between $\mathcal{C}(S)^{\text{op}}$ and $\mathcal{D}$. \qed
3.2 The category $\mathcal{C}(S)$ as a discrete Peirce decomposition

As before, fix $S$ to be a reduced $E$-Fountain semigroup which satisfies the congruence condition. Let $k$ be a field. We denote by $k\tilde{L}_E(e)$ the $k$-vector space of all formal linear combinations

$$k\tilde{L}_E(e) = \{k_1x_1 + \ldots + k_nx_n \mid k_i \in k, \ x_i \in \tilde{L}_E(e)\}.$$ 

It has a structure of a $kS$-module according to

$$s \cdot x = \begin{cases} sx & sx \in \tilde{L}_E(e) \\ 0 & sx \notin \tilde{L}_E(e) \end{cases}$$

for $s \in S$ and $x \in \tilde{L}_E(e)$. It is clear that we can extend $r_\alpha$ to a linear transformation

$$r_\alpha : k\tilde{L}_E(\alpha^+) \to k\tilde{L}_E(\alpha^*)$$

which we denote $r_\alpha$ as well for the sake of simplicity. It is routine to verify that $r_\alpha : \tilde{L}_E(\alpha^+) \to \tilde{L}_E(\alpha^*)$ is a homomorphism of left partial $S$ actions if and only if the extended function $r_\alpha : k\tilde{L}_E(\alpha^+) \to k\tilde{L}_E(\alpha^*)$ is a homomorphism of left $kS$-modules. Therefore, Theorem 3.6 implies the following corollary.

**Corollary 3.10.** The semigroup $S$ satisfies the generalized right ample condition if and only if $r_\alpha : k\tilde{L}_E(\alpha^+) \to k\tilde{L}_E(\alpha^*)$ is a $kS$-module homomorphism for every $\alpha \in S$.

Again, we can view the category $\mathcal{C}(S)$ as a category whose objects are left modules of the form $k\tilde{L}_E(e)$ and whose morphisms are certain homomorphisms, but it is not true that every $kS$-module homomorphism $F : k\tilde{L}_E(e) \to k\tilde{L}_E(f)$ is of the form $F = r_\alpha$ for $\alpha^+ = e$ and $\alpha^* = f$. However, we will see later that under certain conditions, every $kS$-module homomorphism is a linear combination of homomorphism of this form.

From now we fix $S$ to be a finite reduced $E$-Fountain semigroup with the congruence condition. Some of the results in the sequel are true also for infinite semigroups with weaker finiteness conditions but for simplicity we prefer to deal only with the finite case. Our goal in this section is to obtain a version of Corollary 3.9 for a linear category of $kS$-modules.

Recall that the relation $\preceq_1$ is defined on $S$ by the rule that $a \preceq_1 b \iff a = be$ for a certain $e \in E$. It is proved in [24] Lemma 3.5 that $a \preceq_1 b \iff a = ba^*$ and therefore if $c \preceq_1 b$ with $c^* = e$ then $c = be$.

From now on we assume that $S$ satisfies the generalized right ample identity and $\preceq_1$ is contained in a partial order so we can use the isomorphism $kS \simeq k\mathcal{C}(S)$ from Theorem 3.2 freely. For instance, we can assume that $kS$ has a unit element because the category algebra $k\mathcal{C}(S)$ is unital.

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Theorem 3.2 allows us to obtain some information on projective modules of $kS$. It is clear that $kC(S) \cdot C(e)$ is a left projective $kC(S)$-module for every $e \in E$ so it can also be viewed as a projective $kS$-module according to

$$s \ast C(m) = \varphi(s) \cdot C(m) = \sum_{t \leq s} C(t) \cdot C(m)$$

where $\varphi$ is as defined in Theorem 3.2.

Moreover, we have a decomposition

$$kS \cong kC(S) \cong \bigoplus_{e \in E} kC(S) \cdot C(e)$$

as $kS$-modules.

The next theorem is a generalization of [15, Proposition 6.6].

**Theorem 3.11.** For every $e \in E$ there is an isomorphism $k\tilde{L}_E(e) \cong kC(S) \cdot C(e)$ of $kS$-modules. In particular, $k\tilde{L}_E(e)$ is projective for every $e \in E$ and $kS \cong \bigoplus_{e \in E} k\tilde{L}_E(e)$ as $kS$-modules.

**Proof.** The second statement follows from the first. To prove the required isomorphism, define $\Phi : k\tilde{L}_E(e) \rightarrow kC(S) \cdot C(e)$ on basis elements by

$$\Phi(x) = C(x).$$

Note that $x \in \tilde{L}_E(e) \iff x^* = e$ which means that the domain of $C(x)$ is $e$. Therefore, $\Phi$ is a well defined isomorphism of $k$-vector spaces. It is left to show that for every $s \in S$

$$\Phi(sx) = s \ast \Phi(x).$$

**Case 1.** If $sx \in \tilde{L}_E(e)$ then $(s^*x)^* = (sx)^* = x^*$. Choose $a = x$, $e = s^*$ so the generalized right ample condition implies that

$$(s^* (x (s^*x))^+)^* = (x (s^*x))^+.$$

Plugging $(s^*x)^* = x^*$ we obtain

$$(s^* (xx^*)^+)^* = (xx^*)^+$$

so

$$(s^* x^*)^* = x^+$$
and the congruence condition implies

$$(sx^+)^* = (s^*x^+) = x^+$$

Now

$$s \preceq \Phi(x) = \sum_{t \leq s} C(t) \cdot C(x).$$

The composition $C(t) \cdot C(x)$ is defined if and only if $t^* = x^+$ so $C(t) \cdot C(x) = 0$ if $t \neq sx^+$. On the other hand, we have already established that $(sx^+)^* = x^+$ so

$$s \preceq \Phi(x) = \sum_{t \leq s} C(t) \cdot C(x) = C(sx^+)C(x) = C(sx^+x) = C(sx) = \Phi(sx)$$

as required.

**Case 2.** If $sx \notin \tilde{L}_E(e)$ then $s \cdot x = 0$. This happens when $(sx)^* \neq x^*$. Assume that $s \preceq \Phi(x) \neq 0$. Since $s \preceq \Phi(x) = \sum_{t \leq s} C(t) \cdot C(x)$ there exists $t$ such that $t \leq s$ and $t^* = x^+$. This implies that $t = sx^+$ so $(sx^+)^* = x^+$. However,

$$(sx)^* = (sx^+) = ((sx^+)^*) = (x^+) = x^*$$

which is a contradiction. Therefore,

$$s \preceq \Phi(x) = 0 = \Phi(sx)$$

also in this case.

**Corollary 3.12.** For every $e, f \in E$, the set

$$B = \{ r_\alpha | \alpha \in S, \ a^+ = e, \ a^* = f \}$$

is a basis for the $k$-vector space $\text{Hom}_k \tilde{L}_E(e, \tilde{L}_E(f))$.

**Proof.** Corollary 3.10 implies that $r_\alpha \in \text{Hom}_k(k\tilde{L}_E(e), k\tilde{L}_E(f))$ if $\alpha^+ = e$ and $\alpha^* = f$. Let $B = \{ r_\alpha_1, \ldots, r_\alpha_l \}$. To show that $B$ is linear independent, assume

$$k_1r_{\alpha_1} + \ldots + k_lr_{\alpha_l} = 0$$

but
so in particular
\[ k_1 r_{\alpha_1}(e) + \ldots + k_l r_{\alpha_l}(e) = 0. \]

Since \( r_{\alpha_i}(e) = e \alpha_i = \alpha_i \), we obtain

\[ k_1 \alpha_1 + \ldots + k_l \alpha_l = 0 \]

which implies \( k_1 = \ldots = k_l = 0 \) since \( \{\alpha_1, \ldots, \alpha_l\} \) is a linearly independent set in \( \mathbb{k} \tilde{L}_E(f) \). Now, according to Theorem 3.11 we know that

\[
\dim \mathbb{k} \text{Hom}_{\mathbb{k}S}(\mathbb{k} \tilde{L}_E(e), \mathbb{k} \tilde{L}_E(f)) = \dim \mathbb{k} \text{Hom}_{\mathbb{k}S}(\mathbb{k} \tilde{C}(S) \cdot C(e), \mathbb{k} \tilde{C}(S) \cdot C(f)) = \dim \mathbb{k} C(e) \tilde{C}(S) C(f)
\]

and this is precisely the number of \( \alpha \in S \) such that \( \alpha^+ = e \) and \( \alpha^* = f \). In conclusion, \( \{r_{\alpha} \mid \alpha \in S, \quad \alpha^+ = e, \quad \alpha^* = f\} \) is a linearly independent set whose size is the dimension of \( \text{Hom}_{\mathbb{k}S}(\mathbb{k} \tilde{L}_E(e), \mathbb{k} \tilde{L}_E(f)) \) and therefore it is a basis.

Set \( \mathcal{L}(\mathbb{k}S) \) to be the Peirce decomposition of \( \mathbb{k}S \) whose set of objects are the projective modules of the form \( \mathbb{k} \tilde{L}_E(e) \) for \( e \in E \) and \( \mathcal{L}(\mathbb{k}S)(\mathbb{k} \tilde{L}_E(e), \mathbb{k} \tilde{L}_E(f)) = \text{Hom}_{\mathbb{k}S}(\mathbb{k} \tilde{L}_E(e), \mathbb{k} \tilde{L}_E(f)) \) for every \( e, f \in E \).

**Theorem 3.13.** There is an isomorphism of linear categories \( \mathbb{k}[(\tilde{C}(S))]^{\text{op}} \simeq \mathcal{L}(\mathbb{k}S) \).

**Proof.** Define a functor of \( \mathbb{k} \)-linear categories \( \mathcal{F} : \mathbb{k}[(\tilde{C}(S))] \to \mathcal{L}(\mathbb{k}S) \) in the following way. On objects,

\[ \mathcal{F}(e) = \mathbb{k} \tilde{L}_E(e) \]

and \( \mathcal{F}(C(\alpha)) = r_{\alpha} \) defines \( \mathcal{F} \) on the bases of the hom-sets. The argument in Corollary 3.9 shows that \( \mathcal{F} \) is a contravariant functor of \( \mathbb{k} \)-linear categories. It is clearly a bijection on objects and Corollary 3.12 shows that \( \mathcal{F} \) is an isomorphism of hom-sets.

**Theorem 3.13** shows that up to taking the opposite category, \( \mathbb{k}[(\tilde{C}(S))] \) is a Peirce decomposition of \( \mathbb{k}S \). In fact, we can think of the associated category \( \tilde{C}(S) \) as being a “discrete” Peirce decomposition of the semigroup \( S \) itself.

### 4 Examples

A good way to show that the generalized right ample condition is a natural property is to give some natural examples of semigroups which satisfy it. If the set of idempotents \( E \) is a subband of \( S \), it is
shown in [24] that the generalized right ample identity reduces to the well studied “standard” right ample identity. So we are interested in examples where $E$ is not a subband of $S$. One important example - the Catalan monoid - was already given in [24] (and will be recalled in the sequel as well).

In this section we give two additional examples for such semigroups.

4.1 Linear operators on an Hilbert space

Let $H$ be a Hilbert space and let $S$ be the algebra of all bounded linear operators on $H$ (this is one of the main examples of a Rickart $\ast$-ring and a Baer $\ast$-ring, see [1, 28]). For every closed subspace $U \subseteq H$ we associate the orthogonal projection $P_U \in S$ onto it. Recall that $P_U$ is an idempotent and $\text{im}(P_U) = U = \ker(P_U)^\perp$ (where $\text{im}T$ and $\ker T$ are the image and kernel of the linear operator $T$ and $V^\perp$ is the orthogonal complement of $V$). It is easy to check that $P_U P_V = P_U \iff P_V P_U = P_U \iff U \subseteq V$.

Now, consider $S$ as a multiplicative monoid. This is an example of a Baer $\ast$-semigroup [5]. If we set $E = \{P_U \mid U$ is a closed subspace of $H\}$, it is not difficult to check that for $T, L \in S$

$$T\tilde{L}_E L \iff \ker(T) = \ker(L), \quad T\tilde{R}_E L \iff \text{im}(T) = \text{im}(L).$$

It follows that $S$ is a reduced $E$-Fountain semigroup where $T^* = P_{\ker(T)}^\perp$ and $T^+ = P_{\text{im}(T)}$. Note that in general $P_U P_V \neq P_V P_U$ so $E$ is not a subband of $S$.

**Lemma 4.1.** $S$ satisfies the congruence condition.

**Proof.** Let $T, L, R \in L(H)$. First assume $T\tilde{L}_E L$ so $\ker(T) = \ker(L)$. We have

$$x \in \ker(TR) \iff R(x) \in \ker T \iff R(x) \in \ker L \iff x \in \ker(LR)$$

so $\ker(TR) = \ker(LR)$ and $\tilde{L}_E$ is a right congruence. If $T\tilde{R}_E L$ then $\text{im}(T) = \text{im}(L)$. In this case

$$x \in \text{im}(RT) \iff \exists y \in \text{im}(T), \quad R(y) = x \iff \exists y \in \text{im}(L), \quad R(y) = x \iff x \in \text{im}(RL)$$

so $\tilde{R}_E$ is a left congruence.

**Lemma 4.2.** $S$ satisfies the generalized right ample identity.

**Proof.** In view of Theorem 3.6 we will show that every $T \in S$ induces a homomorphism of partial $S$-actions $r_T : \tilde{L}_E(T^+) \to \tilde{L}_E(T^+)$. Set $\text{im}(T) = V$ and let $R \in \tilde{L}_E(T^+) = \tilde{L}_E(P_V)$ and $L \in S$. We

---

1This example is part of the ArXiv version of [24] but was omitted from the final paper due to referee’s request.
need to show that \( L \cdot R \) is defined if and only if \( L \cdot r_T(R) = L \cdot (RT) \) is defined. In other words, we need to prove that

\[
L R \in \tilde{\mathcal{L}}_E(T^+) \iff LRT \in \tilde{\mathcal{L}}_E(T^+).
\]

One direction is immediate since \( LR \in \tilde{\mathcal{L}}_E(T^+) \implies (LR)^* = P_V \implies (LRT)^* = ((LR)^*T)^* = (P_V T)^* = T^* \) so \( LRT \in \tilde{\mathcal{L}}_E(T^+) \). For the other direction, assume \( LR \notin \tilde{\mathcal{L}}_E(T^+) \). Note that \( \ker(R) = V^+ \) and \( \ker(LR) \supseteq \ker(R) \) so \( LR \notin \tilde{\mathcal{L}}_E(T^+) \) implies \( \ker(LR) \supseteq \ker(R) \). Choose \( w \in H \) such that \( LR(w) = 0 \) but \( R(w) \neq 0 \). Set \( v = P_V (w) \) and since \( R P_V = R \) we have \( LR(v) = 0 \) and \( R(v) \neq 0 \). Now, since \( v \in V = \text{im}(T) \), we can choose \( u \in S \) such that \( T(u) = v \). This implies \( LRT(u) = 0 \) but \( RT(u) \neq 0 \) so \( \ker(LRT) \supseteq \ker(RT) \) hence \( LRT \notin \tilde{\mathcal{L}}_E(T^+) \) as required.

Remark 4.3. \( S \) satisfies also the generalized left ample identity. For \( T \in S \) we denote by \( T^t \) its adjoint operator (we do not use \( T^* \) here to avoid confusion with the unary operation of the \( E \)-Fountain structure). It is clear that \((P_U)^t = P_U\), \((T^t)^+ = T^+\) and \((T^4)^* = T^*\) so every identity immediately implies its dual in \( S \).

4.2 Order-preserving functions with a fixed point

Let \( \mathcal{T}_n \) be the monoid of all functions \( f : [n] \to [n] \) (where \([n] = \{1, \ldots, n\}\)). A function is called order-preserving if \( i_1 \leq i_2 \) implies \( f(i_1) \leq f(i_2) \). Denote by \( \mathcal{O}_n \) the monoid of all order-preserving functions \( f : [n] \to [n] \). We denote by \( \mathcal{OF}_n \) the monoid of all order-preserving functions \( f : [n] \to [n] \) with the fixed point \( f(n) = n \).

Example 4.4. The monoid \( \mathcal{OF}_3 \) consists of the following 6 elements:

\[
\begin{pmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 1 & 3
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 3 \\
1 & 3 & 3 \\
2 & 3 & 3
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 3 \\
1 & 3 & 3 \\
3 & 3 & 3
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 1 & 3
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
2 & 2 & 3
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
3 & 3 & 3
\end{pmatrix}
\]

Clearly, \( \mathcal{OF}_n \) is a submonoid of \( \mathcal{O}_n \) and note the analogy with the definition of the monoid \( \mathcal{PT}_n \) of all partial functions on an \( n \)-element set. \( \mathcal{PT}_n \) can be identified with the submonoid of \( \mathcal{T}_{n+1} \) which consists of all functions \( f : [n + 1] \to [n + 1] \) such that \( f(n + 1) = n + 1 \).

To the author’s knowledge, the monoid \( \mathcal{OF}_n \) has not been studied before. In this section we will discuss some properties of this monoid with emphasis on its structure as a reduced \( E \)-Fountain semigroup.

Let \( f \in \mathcal{OF}_n \). Recall that the kernel of \( f - \ker(f) \) is the equivalence relation on its domain defined by \( (i_1, i_2) \in \ker(f) \iff f(i_1) = f(i_2) \). Since \( f \) is order-preserving, the kernel classes of \( f \) are intervals - if \( i_1 \leq i_2 \leq i_3 \) and \( (i_1, i_3) \in \ker(f) \) then \( (i_1, i_2) \in \ker(f) \) as well.
Let \( K_1, \ldots, K_{t+1} \) be the kernel classes of \( f \) (where \( 0 \leq l \leq n-1 \)). Let \( y_i = f(K_i) \) and assume that the indices are arranged such that \( y_1 < y_2 < \cdots < y_{l+1} \). Choose \( x_i \) to be the maximal element of \( K_i \). Since \( f(n) = n \) it must be the case that \( y_{l+1} = n \) and \( x_{l+1} = n \). Now, set

\[
X = \{x_1, \ldots, x_i\}, \quad Y = \{y_1, \ldots, y_i\}
\]

so from every \( f \in \mathcal{OF}_n \) we can extract two sets \( X, Y \subseteq [n-1] \) such that \( |X| = |Y| \). In the other direction, from two sets \( X = \{x_1, \ldots, x_i\} \) and \( Y = \{y_1, \ldots, y_i\} \) we can retrieve \( f \) because

\[
f(x) = \begin{cases} 
  y_1 & x \leq x_1 \\
  y_i & x_{i-1} < x \leq x_i \quad (2 \leq i \leq l) \\
  n & x_i < x.
\end{cases}
\]

As a conclusion, we observe that there is a one-to-one correspondence between functions \( f \in \mathcal{OF}_n \) and pairs of sets \( X, Y \subseteq [n-1] \) such that \( |X| = |Y| \). Note that \( Y = \text{im}(f) \setminus \{n\} \) (where \( \text{im}(f) \) is the image of \( f \)) and \( X \) contains the maximal elements of the kernel classes of \( f \) except \( n \). From now on we denote by \( f_{X,Y} \) the function associated with \( X, Y \subseteq [n-1] \). For instance, the functions from Example 4.3 are

\[
f_{\{1,2\},\{1,2\}}, f_{\{1\},\{1\}}, f_{\{1\},\{2\}}, f_{\{2\},\{1\}}, f_{\{2\},\{2\}}, f_{\emptyset,\emptyset}.
\]

**Corollary 4.5.** The number of elements of \( \mathcal{OF}_n \) is

\[
\sum_{k=0}^{n-1} \binom{n-1}{k}^2 = \binom{2n-2}{n-1}.
\]

**Proof.** We sum over all pairs \( X, Y \subseteq [n-1] \) with the same size. \( \square \)

Another observation will be useful. A function \( f : [n] \to [n] \) is called order-increasing if \( i \leq f(i) \) for every \( i \in [n] \). In particular, if \( f \) is order-increasing then \( f(n) = n \). Let \( \mathcal{C}_n \) be the submonoid of \( \mathcal{OF}_n \) which consists of all order-preserving and order-increasing functions \( f : [n] \to [n] \). This monoid is called the Catalan monoid because the size of \( \mathcal{C}_n \) is the \( n \)-th Catalan number \( \left[ \begin{array}{l} 1 \end{array} \right. \) Theorem 14.2.8]. It is well known that \( \mathcal{C}_n \) is a \( J \)-trivial monoid \[27 \] Proposition 17.17]. Given \( X = \{x_1, \ldots, x_l\} \) and \( Y = \{y_1, \ldots, y_l\} \) we say that \( X \leq Y \) if \( x_i \leq y_i \) for \( 1 \leq i \leq l \). It is easy to see that \( f_{X,Y} \in \mathcal{OF}_n \) is order-increasing if and only if \( X \leq Y \) (see also \[18 \]). Therefore,

\[
\mathcal{C}_n = \{f_{X,Y} \in \mathcal{OF}_n \mid X \leq Y\}.
\]

In the rest of this section we consider several properties of the monoid \( \mathcal{OF}_n \) and its algebra.

**Green’s relations** It is easy to verify that \( f_{Y,Z}f_{X,Y} = f_{X,Z} \) so \( f_{X,Y}f_{Y,X}f_{X,Y} = f_{X,Y} \) and therefore \( \mathcal{OF}_n \) is a regular monoid. As a regular submonoid of \( \mathcal{T}_n \) it is easy to describe its \( \mathcal{R} \) and
\[ \mathcal{L} \text{ relations (see } \text{[11, Proposition 2.4.2 and Exercise 16 on page 63].} \]

**Lemma 4.6.** Let \( f_{X,Y}, f_{Z,W} \in \mathcal{O}F_n \) then

\[
f_{X,Y} R f_{Z,W} \iff \text{im}(f_{X,Y}) = \text{im}(f_{Z,W}) \iff Y = W
\]

\[
f_{X,Y} L f_{Z,W} \iff \ker(f_{X,Y}) = \ker(f_{Z,W}) \iff X = Z.
\]

It follows immediately that \( f_{X,Y} \mathcal{H} f_{Z,W} \iff f_{X,Y} = f_{Z,W} \) so \( \mathcal{O}F_n \) is an \( \mathcal{H} \)-trivial monoid (which is also clear from the fact that \( \mathcal{O}F_n \) is a regular submonoid of \( \mathcal{O}_n \)). In particular, it has only trivial subgroups.

Next, we turn to describe the \( \mathcal{J} \)-classes. Recall that the rank of a function \( f \) is the size of its image. For \( f_{X,Y} \in \mathcal{O}F_n \) we have \( \text{rank}(f_{X,Y}) = |X| + 1 = |Y| + 1 \).

**Lemma 4.7.** Let \( f_{X,Y}, f_{Z,W} \in \mathcal{O}F_n \) then \( f_{X,Y} \mathcal{J} f_{Z,W} \iff \text{rank}(f_{X,Y}) = \text{rank}(f_{Z,W}) \).

**Proof.** Since \( \mathcal{O}F_n \) is a finite monoid we know that \( \mathcal{J} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R} \) (see [11, Proposition 2.1.4]). Now Lemma 4.6 implies that \( f_{X,Y} \mathcal{J} f_{Z,W} \iff f_{X,Y} R f_{Z,Y} L f_{Z,W} \iff |Z| = |Y| \iff \text{rank}(f_{X,Y}) = \text{rank}(f_{Z,W}) \).

\[
\]

In conclusion, we obtain a neat description for the “eggbox” diagrams of \( \mathcal{O}F_n \). The monoid \( \mathcal{O}F_n \) has \( n \) \( \mathcal{J} \)-classes. For every \( 0 \leq k \leq n - 1 \) we can associate a \( \mathcal{J} \)-class \( J_k \) of all functions \( f_{X,Y} \) such that \( |X| = |Y| = k \). Both the \( \mathcal{R} \) and the \( \mathcal{L} \) classes in \( J_k \) are indexed by subsets \( Z \subseteq [n - 1] \). For every \( Z \subseteq [n - 1] \) we have an \( \mathcal{R} \) class

\[
R_Z = \{ f_{X,Z} \mid X \subseteq [n - 1], \ |X| = |Z| \}
\]

and an \( \mathcal{L} \) class

\[
L_Z = \{ f_{Z,Y} \mid Y \subseteq [n - 1], \ |Y| = |Z| \}.
\]

Every \( \mathcal{H} \)-class contains one element. Every \( \mathcal{L} \)-class (\( \mathcal{R} \)-class) in \( J_k \) contains \( \binom{n-1}{k} \) elements. The \( \mathcal{J} \)-class \( J_k \) contains \( \binom{n-1}{k} \) \( \mathcal{L} \)-classes (\( \mathcal{R} \)-classes) and a total of \( \binom{n-1}{k}^2 \) elements.

**Reduced \( E \)-Fountain structure** Set \( E = \{ f_{X,X} \mid X \subseteq [n - 1] \} \subseteq E(\mathcal{O}F_n) \). Note that \( E \) does not contain all the idempotents of \( \mathcal{O}F_n \), but it is precisely the set of idempotents of \( C_n \). According to Lemma 4.6 we know that

\[
f_{Y,Y} R f_{X,Y} L f_{X,X}
\]

\[
19
\]
for every $X, Y \subseteq [n-1]$ with $|X| = |Y|$. Therefore, every $\mathcal{L}$-class and every $\mathcal{R}$-class contains an element of $E$. Since $\mathcal{L} \subseteq \bar{\mathcal{L}}_E$ and $\mathcal{R} \subseteq \bar{\mathcal{R}}_E$, we deduce that every $\bar{\mathcal{L}}_E$-class and every $\bar{\mathcal{R}}_E$-class contains an element of $E$ so $\mathcal{O}F_n$ is an $E$-Fountain semigroup. Now, it is well known (see [2] Lemma 3.6]) that the idempotents of a $J$-trivial semigroup satisfy the property $ef = e \iff fe = e$. Since $E$ is the set of idempotents of the $J$-trivial semigroup $\mathcal{C}_n$ we obtain that $\mathcal{O}F_n$ is in fact a reduced $E$-Fountain semigroup. Note that 

$$(f_{X,Y})^* = f_{X,X}, \quad (f_{X,Y})^+ = f_{Y,Y}$$

and therefore

$$f_{X,Y} \bar{\mathcal{L}}_Ef_{Z,W} \iff X = Z, \quad f_{X,Y} \bar{\mathcal{R}}_Ef_{Z,W} \iff Y = W.$$ 

In particular we have

$$\bar{\mathcal{L}}_E = \mathcal{L}, \quad \bar{\mathcal{R}}_E = \mathcal{R}. \quad (2)$$

The relation $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence [11] Proposition 2.1.2] so by Equality 2 we deduce that $\mathcal{O}F_n$ satisfies the congruence condition and therefore has an associated category $\mathcal{C}(\mathcal{O}F_n)$. The objects of $\mathcal{C}(\mathcal{O}F_n)$ are in one-to one correspondence with sets $X \subseteq [n-1]$. Every $f_{X,Y} \in \mathcal{O}F_n$ correspond to a morphism from $X$ to $Y$ so there exists a (unique) morphism from $X$ to $Y$ if and only if $|X| = |Y|$. In other words, $\mathcal{C}(\mathcal{O}F_n)$ represent the equivalence relation $\sim$ defined on subsets of $[n-1]$ by $X \sim Y \iff |X| = |Y|$. Two objects $X, Y \subseteq [n-1]$ are isomorphic if and only if $|X| = |Y|$ and in this case there is a unique morphism from $X$ to $Y$. In particular, $\mathcal{C}(\mathcal{O}F_n)$ is both a groupoid and a locally trivial category (i.e., a category where the only endomorphisms are the identity morphisms). Finally, Equality 2 and Lemma 3.5 implies that $\mathcal{O}F_n$ satisfies both the right and left generalized ample identities.

Since the Catalan monoid $\mathcal{C}_n$ is a submonoid of $\mathcal{O}F_n$ and $E \subseteq \mathcal{C}_n$ it is also clear that $\mathcal{C}_n$ satisfies the right and left generalized ample identities. This is a much neater proof of this fact than the one given in [24].

**The semigroup algebra** Define a partial order $\preceq$ on $\mathcal{O}F_n$ by

$$f_{Z,W} \preceq f_{X,Y} \iff W \subseteq Y \text{ or } (W = Y \text{ and } Z \preceq X).$$

It is easy to verify that this is indeed a partial order (it is the lexicographic order on the cartesian product of $\subseteq$ and $\leq$).

**Lemma 4.8.** $\preceq \subseteq \leq$

**Proof.** Let $f_{X,Y} \in \mathcal{O}F_n$ and $f_{Z,Z} \in E$. We need to show that $f_{X,Y}f_{Z,Z} \preceq f_{X,Y}$. First, note that
\( \text{im}(f_{X,Y}f_{Z,Z}) \subseteq \text{im}(f_{X,Y}) = Y \). If \( \text{im}(f_{X,Y}f_{Z,Z}) \not\subseteq Y \) then we are done. It is left to consider the option that \( \text{im}(f_{X,Y}f_{Z,Z}) = Y \). In other words we assume that \( f_{X,Y}f_{Z,Z} \mathcal{R} f_{X,Y} \). The fact that \( \mathcal{R} \) is a left congruence implies that

\[
f_{X,X}f_{Z,Z} = f_{Y,X}f_{X,Y}f_{Z,Z} \mathcal{R} f_{Y,X}f_{X,Y} = f_{X,X}
\]

so

\[
\text{im}(f_{X,X}f_{Z,Z}) = X.
\]

Therefore, we can write

\[
f_{X,X}f_{Z,Z} = f_{Z',X}
\]

and observe that \( f_{X,X}, f_{Z,Z} \in \mathcal{C}_n \) so \( f_{Z',X} \in \mathcal{C}_n \) as well hence \( Z' \leq X \). Finally,

\[
f_{X,Y}f_{Z,Z} = f_{X,Y}f_{X,X}f_{Z,Z} = f_{X,Y}f_{Z',X} = f_{Z',Y}.
\]

and therefore \( f_{X,Y}f_{Z,Z} \preccurlyeq f_{X,Y} \) as required. \( \square \)

Theorem 3.2 and Lemma 4.8 implies that for every field \( k \) we have an isomorphism of algebras

\[
k \mathcal{O}_F \simeq k \mathcal{C}(\mathcal{O}_F).
\]

This isomorphism allows us to obtain a lot of information on the algebra \( k \mathcal{O}_F \). For instance we have the following immediate corollary:

**Lemma 4.9.** For every field \( k \) the algebra \( k \mathcal{O}_F \) is semisimple.

*Proof.* It is well known that the algebra \( k \mathcal{G} \) of a finite groupoid \( \mathcal{G} \) is semisimple if and only if the order of every endomorphism group is invertible in \( k \). In our case, the endomorphism groups of \( \mathcal{C}(\mathcal{O}_F) \) are trivial so \( k \mathcal{C}(\mathcal{O}_F) \) is semisimple for every field \( k \) and hence also \( k \mathcal{O}_F \). \( \square \)

We remark that not many examples are known for non-inverse semigroups with a semisimple algebra. Another notable example is the semigroup of matrices over a finite field of appropriate characteristic (see [27, Section 5.6]).

**The monoid of order-preserving partial permutations** Recall that the symmetric inverse monoid \( \mathcal{I}_S \) is the monoid of all partial permutations on the set \( [n] \). Our next observation is related to the submonoid \( \mathcal{I}_O \) of all order-preserving partial permutations (also denoted sometimes \( \mathcal{P}_O \) - see [3, 4, 6]). In other words, \( \mathcal{I}_O \) consists of all partial permutations \( \theta : [n] \to [n] \) such that \( \theta(i) < \theta(j) \) if \( i, j \) are in the domain of \( \theta \) and \( i < j \). It is clear that every \( \theta \in \mathcal{I}_O \) is completely
determined by its domain and image. In the other direction, given \( X, Y \subseteq [n] \) such that \(|X| = |Y|\) there exists a unique \( \theta_{X,Y} \in \mathcal{IO}_n \) whose domain is \( X \) and image is \( Y \). Therefore, elements of \( \mathcal{IO}_n \) can also be indexed by pairs of sets \( X, Y \subseteq [n] \) such that \(|X| = |Y|\). It is well known that \( \mathcal{IO}_n \) is an inverse semigroup so in particular, it is a reduced \( E \)-Fountain semigroup for \( E = E(\mathcal{IO}_n) \) which satisfies the congruence condition. Moreover, its idempotents commute and it satisfies the left and right ample conditions. Consider the associated category \( \mathcal{C}(\mathcal{IO}_n) \) (which is in fact an inductive groupoid - see [14, Chapter 4]). The idempotents of \( \mathcal{IO}_n \) are the partial permutations of the form \( \theta_{X,X} \) - where the domain equals the image. In other words these are precisely the partial identities. So objects of \( \mathcal{C}(\mathcal{IO}_n) \) are in one to one correspondence with sets \( X \subseteq [n] \). For every \( \theta \in \mathcal{IO}_n \) there exists a unique morphism \( \mathcal{C}(\theta) \) from the domain of \( \theta \) to its image. It is now easy to deduce that \( \mathcal{C}(\mathcal{IO}_n) \) and \( \mathcal{C}(\mathcal{OF}_{n+1}) \) are isomorphic! According to Theorem 3.2 (which for the case of inverse semigroups is precisely [25, Theorem 4.2]) we know that \( k\mathcal{C}(\mathcal{IO}_n) \simeq k\mathcal{IO}_n \) for every field \( k \). So we end up with the following result.

**Proposition 4.10.** Let \( k \) be a field, there is an isomorphism of algebras

\[
k\mathcal{OF}_{n+1} \simeq k\mathcal{IO}_n.
\]

**Remark 4.11.** Note that we have an explicit description of this isomorphism. An isomorphism \( \varphi : k\mathcal{OF}_{n+1} \to k\mathcal{C}(\mathcal{OF}_{n+1}) \) is given in Theorem 3.2 and according to [25], an isomorphism \( \psi : \mathcal{C}(k\mathcal{IO}_n) \to k\mathcal{IO}_n \) can be defined by

\[
\psi(\mathcal{C}(\theta)) = \sum_{\tau \leq \theta} \mu(\tau, \theta) \tau
\]

where \( \leq \) is the standard partial order on an inverse semigroup and \( \mu \) is its related Möbius function (see [18, Chapter 3]).

As a final observation, we note that a similar isomorphism holds for certain submonoids. Consider the Catalan monoid \( \mathcal{C}_n \subseteq \mathcal{OF}_n \). The objects of \( \mathcal{C}(\mathcal{C}_n) \) are sets \( X \subseteq [n-1] \) and there is a unique morphism from \( X \) to \( Y \) if \((|X| = |Y|) \) and \( X \leq Y \). Now consider the monoid \( \mathcal{IC}_n \subseteq \mathcal{IO}_n \) of all order-preserving and order-increasing partial permutations (see [7, Chapter 14]). In other words, a partial permutation \( \theta \) belongs to \( \mathcal{IC}_n \) if \( \theta \in \mathcal{IO}_n \) and \( i \leq \theta(i) \) for every \( i \) in the domain of \( \theta \). \( \mathcal{IC}_n \) is not an inverse semigroup. However, since \( E(\mathcal{IC}_n) = E(\mathcal{IO}_n) \) it is clear that it is a reduced \( E \)-Fountain semigroup which satisfies both the congruence and the right\left ample conditions. It is also clear that \( \theta_{X,Y} \in \mathcal{IO}_n \) belongs to \( \mathcal{IC}_n \) if and only if \( X \leq Y \). Therefore, \( \mathcal{C}(\mathcal{IC}_n) \) is isomorphic to \( \mathcal{C}(\mathcal{C}_{n+1}) \) and we obtain:
Corollary 4.12. Let \( k \) be a field, there is an isomorphism of algebras

\[ kC_{n+1} \cong kIC_n. \]

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\(^2\)This is an unpublished observation of the authors of [16] (private communication).
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