Gauging $\mathcal{N}=4$ supersymmetric mechanics II: 
$(1,4,3)$ models from the $(4,4,0)$ ones

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Abstract

Exploiting the gauging procedure developed by us in hep-th/0605211, we study the relationships between the models of $\mathcal{N}=4$ mechanics based on the off-shell multiplets $(4,4,0)$ and $(1,4,3)$. We make use of the off-shell $\mathcal{N}=4$, $d=1$ harmonic superspace approach as most adequate for treating this circle of problems. We show that the most general sigma-model type superfield action of the multiplet $(1,4,3)$ can be obtained in a few non-equivalent ways from the $(4,4,0)$ actions invariant under certain three-parameter symmetries, through gauging the latter by the appropriate non-propagating gauge multiplets. We discuss in detail the gauging of both the Pauli-Gürsey $SU(2)$ symmetry and the abelian three-generator shift symmetry. We reveal the $(4,4,0)$ origin of the known mechanisms of generating potential terms for the multiplet $(1,4,3)$, as well as of its superconformal properties. A new description of this multiplet in terms of unconstrained harmonic analytic gauge superfield is proposed. It suggests, in particular, a novel mechanism of generating the $(1,4,3)$ potential terms via coupling to the fermionic off-shell $\mathcal{N}=4$ multiplet $(0,4,4)$.

PACS: 11.30.Pb, 11.15.-q, 11.10.Kk, 03.65.-w
Keywords: Supersymmetry, gauging, isometry, superfield
1 Introduction

An extended $d=1$ supersymmetry possesses some notable features which are not shared by its higher-dimensional counterparts. In view of the distinguished role of supersymmetric quantum mechanics, both as the appropriate simplified “laboratory” for studying various aspects of supersymmetric quantum field theories and as a theory providing superextensions of some intrinsically one-dimensional systems, it is of importance to fully understand these specific features of the $d=1$ supersymmetry and their dynamical manifestations in the corresponding models of supersymmetric mechanics. One of these peculiarities is the so-called “1D automorphic duality” [1]-[3] which relates off-shell $d=1$ supermultiplets with the same number of physical fermions and different divisions of the set of bosonic fields into physical and auxiliary components (see also [4, 5, 6]). The procedure generating the multiplets with a greater number of auxiliary fields from those with the lesser number and the procedure inverse to it can be referred to as the “reduction” and “oxidation”, respectively [7]. The non-linear versions of this duality were considered in [8]-[10]. Using these dualities, the relations between different supersymmetric mechanics models can be studied. In particular, it was argued in [10] that various models of $\mathcal{N}=4$ supersymmetric mechanics based on the off-shell multiplets with 4 physical fermions can be obtained, through the reduction procedure, from the models based on the “root” [12] off-shell multiplet with the field content $(4, 4, 0)$ [1, 11, 8, 9, 12](hereafter, the abbreviating $(n_1, n_2 = n_1, n_3)$ stands for the off-shell multiplet with $n_1$ physical bosons, $n_2$ physical fermions and $n_3$ auxiliary bosonic fields).

In most of the above studies, the reduction procedure was accomplished at the component level and “by hands”: basically by treating the time derivative of some initial physical bosonic field as a new auxiliary field. Recently, we proposed a superfield version of this procedure ensuring the manifest off-shell supersymmetry at all steps [7]. The process of reduction was shown to amount to gauging some isometries of the superfield actions of the multiplet with the maximal number of the physical bosonic fields ($(1, 1, 0)$, $(2, 2, 0)$ and $(4, 4, 0)$ in the $\mathcal{N}=1$, $\mathcal{N}=2$ and $\mathcal{N}=4$ cases) by a “topological” gauge multiplet. The characteristic property of the latter (specific just for the $d=1$ case) is that its only surviving component in the Wess-Zumino gauge is the bosonic “gauge field”. The residual gauge freedom is always realized in such a way that it can be used to kill one or few original bosonic fields. After fully fixing the gauge freedom in this way, one is left with the new off-shell multiplet in which the place of “killed” bosonic fields is occupied by the former gauge fields which possess no kinetic terms and so are auxiliary. Thus the supersymmetric mechanics of $d=1$ supermultiplets with one or another numbers of auxiliary fields naturally arises upon fixing a gauge in a coupled system of the “extreme” multiplet having no auxiliary fields at all and a “topological” supermultiplet which gauges one or another isometry realized on the extreme multiplet. Besides choosing the Wess-Zumino gauge (which basically corresponds to the component consideration of refs. [1]-[6], [10]) one is free to choose another, manifestly supersymmetric gauge in which the whole reduction procedure can be performed in terms of superfields. This possibility to accomplish the reduction in a manifestly supersymmetric superfield fashion is one of the merits of the gauging approach. It can be regarded as an efficient tool of deducing off-shell superfield descriptions of $d=1$ supersymmetric mechanics systems, starting from the system associated with the basic (“root”) multiplet. In most cases, the potential terms of the resulting multiplet are generated by Fayet-Iliopoulos terms of the gauge superfield and/or the gauge-covariantized Wess-Zumino-
like terms of the “root” multiplet. The inverse “oxidation” procedure amounts to constraining the gauge multiplet to be “pure gauge” and so eliminating it altogether.

In [7] we concentrated on the $\mathcal{N}=4$ mechanics and the superfield reduction from the $\mathcal{N}=4$ root multiplet $(4,4,0)$ to the multiplet $(3,4,1)$ (both its linear [13, 14, 15] and non-linear [8, 9] versions). As a by-product we constructed a new non-linear superfield version of the multiplet $(4,4,0)$ in $\mathcal{N}=4, d=1$ harmonic superspace (see also [11], [16]-[18]). Some further examples of our gauging procedure leading to the $\mathcal{N}=4$ multiplets $(0,4,4)$ and $(1,4,3)$ were also considered. The last case corresponds to gauging non-abelian $SU(2)$ symmetry realized on the multiplet $(4,4,0)$.

In [7] we limited our study to a particular case of such gauging, with the free action of the multiplet $(4,4,0)$ as the input. In the present paper we consider the most general situation. We start from the most general $SU(2)$ invariant $(4,4,0)$ action in the harmonic $\mathcal{N}=4, d=1$ superspace and show that its $SU(2)$ gauging generates the generic sigma-model type superfield action of the multiplet $(1,4,3)$ [19, 20]. Also, we show that the latter can be equally reproduced by gauging some other three-parameter isometries admitting a realization on the multiplet $(4,4,0)$. As such one can choose three commuting shift isometries. We discuss the $(4,4,0)$ origin of various mechanisms of generating potential terms of the multiplet $(1,4,3)$ and show how the superconformally invariant actions of the latter can be reproduced within the gauging procedure from the superconformal $(4,4,0)$ actions. As a by-product we find out a new description of the multiplet $(1,4,3)$ in terms of unconstrained harmonic analytic gauge prepotential. This description suggests a new mechanism of generating potential terms of the multiplet $(1,4,3)$ via coupling it to the off-shell fermionic $\mathcal{N}=4$ multiplet $(0,4,4)$. Also, using this formulation, off-shell couplings of the multiplet $(1,4,3)$ to the $(3,4,1)$ one and some extra $(4,4,0)$ multiplets can be easily constructed. Finally, we discuss the description of the “mirror” $(1,4,3)$ multiplet (with a different $SU(2)$ assignment of the component fields) in the $\mathcal{N}=4, d=1$ harmonic superspace and present a simple coupling of the mirror multiplet to the initial $(1,4,3)$ multiplet.

2 $\mathcal{N}=4, d=1$ harmonic superspace

2.1 Basics

Because the $\mathcal{N}=4, d=1$ harmonic superspace (HSS) plays the central role in our construction, we start by recollecting the basics of this approach following refs. [21, 22] and [8, 7].

The $\mathcal{N}=4, d=1$ superspace is defined as the following coordinate set

\[ z = (t, \theta^i, \bar{\theta}^i), \quad \bar{\theta}^i = (\bar{\theta}_i). \] (2.1)

The covariant spinor derivatives are defined as

\[ D^i = \frac{\partial}{\partial \theta^i} + i \bar{\theta}^i \partial_t, \quad \bar{D}_i = \frac{\partial}{\partial \bar{\theta}^i} + i \theta_i \partial_t = - (D^i), \]

\[ \{D^i, \bar{D}_j\} = 2i \delta^i_j \partial_t, \quad \{D^i, \bar{D}^j\} = \{\bar{D}_i, \bar{D}_j\} = 0. \] (2.2)

The $\mathcal{N}=4, d=1$ harmonic superspace (HSS) in the central basis is the following extension of (2.1)

\[ (z, u) = (t, \theta^i, \bar{\theta}^i, u_\pm^i). \] (2.3)
The analytic subspace of HSS is defined as
\[ \tilde{\text{u}}_i^\pm = (u^\pm) \] (2.4)
and is closed under the \( N=4 \) supersymmetry
\[ \delta t_A = -2i \left( \epsilon^i u_i^- \bar{\theta}^+ - \bar{\epsilon}^i u_i^± \theta^+ \right), \quad \delta \theta^± = \epsilon^i u_i^±, \quad \delta \bar{\theta}^± = \bar{\epsilon}^i u_i^±, \quad \delta u_i^± = 0, \] (2.7)
and is real with respect to the generalized conjugation \(- [21]\)
\[ \tilde{t}_A = t_A, \quad \tilde{\theta}^± = \bar{\theta}^±, \quad \tilde{\bar{\theta}}^± = -\bar{\theta}^±, \quad \tilde{u}_i^± = u_i^±, \quad \tilde{\bar{u}}_i^± = -u_i^±. \] (2.8)
In the central basis \( (z, u^±) \), the harmonic derivatives and the harmonic projections of spinor derivatives are defined by
\[ D^± = \partial^{±} = u_i^± \frac{\partial}{\partial u_i}, \quad D^\pm = u_i^± D^i, \quad \bar{D}^± = u_i^± \bar{D}^i. \] (2.9)
In the analytic basis, the same spinor and harmonic derivatives read
\[
D^+ = \frac{\partial}{\partial \theta^+}, \quad \bar{D}^+ = -\frac{\partial}{\partial \bar{\theta}^+}, \quad D^- = -\frac{\partial}{\partial \theta^-} + 2i \theta^- \partial_{t_A}, \quad \bar{D}^- = \frac{\partial}{\partial \bar{\theta}^-} + 2i \bar{\theta}^- \partial_{t_A},
\]
\[
D^{++} = \partial^{++} - 2i \theta^+ \bar{\theta}^+ \partial_{t_A} + \theta^+ \frac{\partial}{\partial \theta^+} + \bar{\theta}^+ \frac{\partial}{\partial \bar{\theta}^+},
\]
\[
D^{--} = \partial^{--} - 2i \theta^- \bar{\theta}^- \partial_{t_A} + \theta^- \frac{\partial}{\partial \theta^-} + \bar{\theta}^- \frac{\partial}{\partial \bar{\theta}^-}.
\] (2.10)
They satisfy the following non-zero (anti)commutation relations
\[
[D^{±}_i, D^{±}_j] = D^{±}_k, \quad [D^{±}_i, D^{±}_j] = D^{±}_k, \quad \{D^+, D^-\} = -\{D^-, D^+\} = 2i \partial_{t_A},
\]
\[
[D^{++}_i, D^{--}_j] = D^0, \quad [D^0, D^{±}_i] = \pm 2 D^{±}_i,
\]
\[
D^0 = u_i^± \frac{\partial}{\partial u_i} - u_i^- \frac{\partial}{\partial u_i} + \theta^+ \frac{\partial}{\partial \theta^+} + \bar{\theta}^+ \frac{\partial}{\partial \bar{\theta}^+} - \theta^- \frac{\partial}{\partial \theta^-} - \bar{\theta}^- \frac{\partial}{\partial \bar{\theta}^-}.
\] (2.11)
The derivatives \( D^+, \bar{D}^+ \) are short in the analytic basis, whence it follows that one can define analytic \( \mathcal{N}=4 \) superfields \( \Phi^{(q)}(\zeta, u) \)
\[ D^+ \Phi^{(q)} = \bar{D}^+ \Phi^{(q)} = 0 \quad \Rightarrow \quad \Phi^{(q)} = \Phi^{(q)}(\zeta, u), \] (2.13)
where \( q \) is the external harmonic \( U(1) \) charge. This Grassmann harmonic analyticity is preserved by the harmonic derivative \( D^{++} \); when applied to \( \Phi^{(q)}(\zeta, u) \), this derivative yields an analytic \( \mathcal{N}=4, d=1 \) superfield of charge \( (q + 2) \).
The measures of integration over the full HSS and its analytic subspace are given, respectively, by

\[
\mu_H = du dt^4 \theta = du dt_A (D^- \bar{D}^-)(D^+ \bar{D}^+) = \mu_A^{(-2)}(D^+ \bar{D}^+),
\]
\[
\mu_H^{(-2)} = du d\zeta^{(-2)} = du dt_A d\theta^+ d\bar{\theta}^+ = du dt_A (D^- \bar{D}^-).
\] (2.14)

The integration over the harmonic two-sphere \(S^2 = \{u_i^+, u_k^--\}\) is normalized so that

\[
\int du 1 = 1,
\] (2.15)

and the integral of any other irreducible monomial of the harmonics is vanishing \([21, 22]\).

### 2.2 Multiplet \(q^{+a}\) and its symmetries

In this paper we shall deal with the multiplet \((4,4,0)\) which is described by a doublet analytic superfield \(q^{+a}(\zeta,u)\) of charge 1 satisfying the harmonic constraint \(^2\)

\[
D^{++} q^{+a} = 0 \Rightarrow q^{+a}(\zeta,u) = f^{ia}(t)u_i^+ + \theta^+ \chi^a(t) + \bar{\theta}^+ \bar{\chi}^a(t) + 2i\theta^+ \bar{\theta}^+ \partial_t f^{ia}(t)u_i^-.
\] (2.16)

It satisfies the pseudoreality condition (see (2.8))

\[
\tilde{q}^{+a} = -q^{+a} \Rightarrow \overline{(f^{ia})} = \epsilon_{ab} \epsilon_{ik} f^{kb}, \ (\chi^a) = \bar{\chi}_a.
\] (2.17)

The Grassmann analyticity conditions together with the harmonic constraint (2.16) imply that in the central basis

\[
q^{+a} = q^a(t,\theta,\bar{\theta})u_i^+, \ D^{(i} q^{k)a} = D^{(i} q^{k)a} = 0.
\] (2.18)

We may write a general off-shell action for the \((4,4,0)\) multiplet as

\[
S_q = \int du dt d^4 \theta L(q^{+a}, q^{-a}, u^\pm), \ q^{-a} \equiv D^{-} q^{+a}.
\] (2.19)

After solving the constraint (2.16) in the central basis of HSS, the superfield \(q^{\pm a}\) may be written in this basis as \(q^{\pm a} = u_i^\pm q^{ia}(t,\theta,\bar{\theta})\), \(D^{(i} q^{k)a} = D^{(i} q^{k)a} = 0\). We then use the notation

\[
L(q^{ia}) = \int du L(q^{+a}, q^{-a}, u^\pm), \ S_q = \int dt d^4 \theta L(q^{ia}).
\] (2.20)

The free action is given by

\[
S_q^{\text{free}} = -\frac{1}{4} \int du dt d^4 \theta (q^{+a} q^{-a}) = \frac{i}{2} \int du d\zeta^{(-2)} (q^{+a} \partial_t q^{a+}).
\] (2.21)

The action (2.19) produces a sigma-model type action in components, with two time derivatives on the bosonic fields and one derivative on the fermions. One can also construct an invariant which in components yields a Wess-Zumino type action, with one time derivative on the

\(^2\)For brevity, in what follows we frequently omit the index "\(A\)" of \(t_A\).
bosonic fields (plus Yukawa-type fermionic terms). It is given by the following general integral over the analytic subspace

\[ S_W^Z = \int dud\zeta (-2) L^{+2}(q^{+a}, u^\pm). \] (2.22)

The free action (2.21) and constraint (2.16) exhibit a number of symmetries. Some of them can be extended to the interaction case, leading to certain restrictions on the form of the general action (2.19). In terms of the component fields, these symmetries become isometries of the target bosonic metric. We shall list here all symmetries of this sort, having in mind to gauge some of them in the next Sections. We will be interested only in those symmetries which commute with $\mathcal{N}=4$ supersymmetry and so can be gauged without passing to the local supersymmetry [7].

**Rotational isometries.** The free action (2.21) and constraint (2.16) are manifestly invariant under the so-called Pauli-Gürsey $SU(2)$ group

\[ \delta_{su(2)} q^{+a} = \lambda_{b}^{a} q^{+b} , \quad \lambda_{a}^{a} = 0 , \quad \overline{(\lambda_{a}^{b})} = -\lambda_{a}^{b}. \] (2.23)

An arbitrary one-parameter subgroup of this group is singled out as

\[ \delta_{1} q^{+a} = \lambda_{1} c_{b}^{a} q^{+b} \equiv \lambda_{1} \mathcal{I}_{1}(c) q^{+a} , \quad \mathcal{I}_{1}(c) = c_{a}^{b} q^{+b} \frac{\partial}{\partial q^{+a}} , \] (2.24)

where $c^{(ab)}$ is an isotriplet of constants

\[ c_{a}^{a} = 0 , \quad \overline{(c^{ab})} = c_{ab}. \] (2.25)

One more isometry of this type is the target space dilatations

\[ \delta_{2} q^{+a} = \lambda_{2} q^{+a} \equiv \lambda_{2} \mathcal{I}_{2} q^{+a} , \quad \mathcal{I}_{2} = q^{+a} \frac{\partial}{\partial q^{+a}} . \] (2.26)

The constraint (2.16) is obviously covariant under these rescalings, while the action (2.21) is not. The simplest invariant action is of the sigma model type [7]. The transformations (2.26) and (2.23) commute with each other.

**Shift isometries.** The action (2.21) (up to boundary terms) and constraint (2.16) are also invariant under the abelian shifts

(a) $\delta_{3} q^{+a} = \lambda_{3} u^{+a} \equiv \lambda_{3} \mathcal{I}_{3} q^{+a} , \quad \mathcal{I}_{3} = u^{+a} \frac{\partial}{\partial q^{+a}} ;$  
(b) $\delta_{4} q^{+a} = \tilde{\lambda}_{b}^{a} u^{+b} . \] (2.27)

An arbitrary one-parameter subgroup of (2.27b) is singled out as

\[ \delta_{5} q^{+a} = \lambda_{5} b_{d}^{(a)} u^{+d} \equiv \lambda_{5} \mathcal{I}_{5} (b) q^{+a} , \quad \mathcal{I}_{5} (b) = b_{d}^{(a)} u^{+d} \frac{\partial}{\partial q^{+a}} , \] (2.28)

where $b^{(ab)}$ is some real constant isotriplet (in general, it is different from $c^{(ab)}$). In what follows, without loss of generality, we normalize all these isotriplets as in [7]

\[ c^{2} = c_{ab} c_{ab} = 2 , \quad b^{2} = b_{ab} b_{ab} = 2 . \] (2.29)
The rotational and shift isometries, being realized on the same $q^{+a}$, do not commute with each other. Their closure is an extension of $SU(2)_{PG}$ by some solvable subgroups. Below we list some subgroups with two and three generators from this closure.

**Commuting subsets**

\[
G_I = \{ \mathcal{I}_1(c), \mathcal{I}_2 \}, \quad G_{II} = \{ \mathcal{I}_3, \mathcal{I}_5(b) \}, \quad G_{III} = \{ \mathcal{I}_5(c), \mathcal{I}_5(b) \}, \quad \quad (2.30)
\]

\[
[\mathcal{I}_1(c), \mathcal{I}_2] = [\mathcal{I}_3, \mathcal{I}_5(b)] = 0.
\]

**Non-commuting subsets**

\[
G_{IV} = \{ \mathcal{I}_2, \mathcal{I}_3 \}, \quad G_V = \{ \mathcal{I}_2, \mathcal{I}_5(b) \}, \quad [\mathcal{I}_3, \mathcal{I}_2] = \mathcal{I}_3, \quad [\mathcal{I}_2, \mathcal{I}_5(b)] = \mathcal{I}_5(b), \quad (2.31)
\]

\[
[\mathcal{I}_1(c), \mathcal{I}_3] = -\mathcal{I}_5(c), \quad [\mathcal{I}_1(c), \mathcal{I}_5(c)] = \mathcal{I}_3, \quad \quad (2.32)
\]

\[
G_{VI} = \{ \mathcal{I}_1(c), \mathcal{I}_3, \mathcal{I}_5(c) \}, \quad [\mathcal{I}_1(c), \mathcal{I}_3] = -\mathcal{I}_5(c), \quad [\mathcal{I}_1(c), \mathcal{I}_5(c)] = \mathcal{I}_3, \quad (2.33)
\]

\[
G_{VII} = \{ \mathcal{I}_1(c), \mathcal{I}_5(b), \mathcal{I}_5(d) \}, \quad (c \cdot b) = (c \cdot d) = (b \cdot d) = 0, \quad d^{ab} \equiv c^b_j b^{fb}, \quad (2.34)
\]

\[
[\mathcal{I}_1(c), \mathcal{I}_5(b)] = -\mathcal{I}_5(d), \quad [\mathcal{I}_1(c), \mathcal{I}_5(d)] = \mathcal{I}_5(b), \quad (2.35)
\]

\[
G_{VIII} = \{ \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_5(c) \}, \quad [\mathcal{I}_2, \mathcal{I}_3] = -\mathcal{I}_3, \quad [\mathcal{I}_2, \mathcal{I}_5(c)] = -\mathcal{I}_5(c), \quad (2.36)
\]

The subclasses of the general $q$-actions (2.19) and (2.22) revealing invariance under various subgroups listed above will be defined in the next Sections, as far as necessary. In fact, some of the above subgroups coincide up to a redefinition of $q^{+a}$ by some constant matrix (preserving the constraint (2.16) and the free action (2.21)). The full list of such isomorphisms is

\[
G_{II} \sim G_{III}, \quad G_{IV} \sim G_V, \quad G_{VI} \sim G_{VII}, \quad G_{VIII} \sim G_{IX}.
\]

For further use, we also present the full structure of the closure of the rotational and shift isometries. Denoting, in the proper basis, the generators of $SU(2)_{PG}$ (2.23) by $T_M$ ($M = 1, 2, 3$), the generator of target dilatations (2.26) by $T$ and the generators of the singlet and triplet shifts in (2.27) by $F$ and $F_M$, we have

\[
[T_M, T_N] = i \varepsilon_{MKN} T_K, \quad [T_M, F_N] = \frac{i}{2} \varepsilon_{MKN} F_K - i \delta_{MN} F, \quad [T_M, F] = \frac{i}{4} F_M, \quad (2.37)
\]

\[
[F_M, F_N] = [F_M, F] = 0, \quad [T, T_M] = 0, \quad [T, F_M] = F_M, \quad [T, F] = F, \quad (2.38)
\]

This algebra generates a subgroup of the Weyl group in 4-dimensional target Euclidean space, with $F$ and $F_M$ forming the 4-translation operator and $SU(2)_{PG}$ being one of two $SU(2)$ factors of the rotation group $SO(4) \sim SU(2) \times SU(2)$.

### 2.3 “Topological” gauge $\mathcal{N}=4$ superfield

The $\mathcal{N}=4$, $d=1$ “gauge multiplet” is described by a charge 2 unconstrained analytic superfield $V^{++}(\zeta, u)$ the gauge transformation of which in the abelian case reads

\[
\delta V^{++} = D^{++} \Lambda, \quad (2.39)
\]

---

3The second factor is the automorphism $SU(2)$ acting on the doublet indices of harmonics $u^{+i}$, central basis Grassmann variables and the left index of the superfield $q^{a}$ in (2.18). Since it does not commute with $\mathcal{N}=4$ supersymmetry, it cannot be gauged without turning on the whole world-line $\mathcal{N}=4$ supergravity [7].
with $\Lambda(\zeta, u)$ being a charge zero unconstrained analytic superfield parameter. Using this gauge freedom, one can choose the Wess-Zumino gauge, in which the gauge superfield becomes

$$V^{++}(\zeta, u) = 2i(\theta^+\bar{\theta}^+)A(t), \quad \delta A(t) = -\partial_t\Lambda_0(t), \quad \Lambda_0 = \Lambda(\zeta, u)|_{\theta = 0}.$$ (2.40)

We observe that the “gauge” $N=4, d=1$ multiplet locally carries $(0 + 0)$ degrees of freedom and so it is “topological”. Globally the field $A(t)$ can differ from a pure gauge, and this feature allows for its treatment as an auxiliary field in the “unitary” gauges.

As in the $N=2, d=4$ HSS [21, 22], $V^{++}$ gauge-covariantizes the analyticity-preserving harmonic derivative $D^{++}$. Assume that the analytic superfield $\Phi^{(q)}$ is transformed under some abelian gauge isometry as

$$\delta\Lambda \Phi^{(q)} = \Lambda I \Phi^{(q)},$$ (2.41)

where $I$ is the corresponding generator. Then the harmonic derivative $D^{++}$ is covariantized as

$$D^{++}\Phi^{(q)} \implies D^{++}\Phi^{(q)} = (D^{++} - V^{++}I)\Phi^{(q)}.$$ (2.42)

One can also define the second, non-analytic harmonic connection $V^{--}$

$$D^{--} = D^{--} - V^{--}I, \quad \delta V^{--} = D^{--}\Lambda.$$ (2.43)

From the requirement of preserving the algebra of harmonic derivatives (2.11),

$$[D^{++}, D^{--}] = D^0, \quad [D^0, D^{++}] = \pm 2D^{++},$$ (2.44)

the well-known harmonic zero-curvature equation follows

$$D^{++}V^{--} - D^{--}V^{++} = 0.$$ (2.45)

It specifies $V^{--}$ in terms of $V^{++}$. One can also define the covariant spinor derivatives

$$D^- = [D^{--}, D^+] = D^- + (D^+V^{--})I, \quad \bar{D}^- = [D^{--}, \bar{D}^+] = \bar{D}^- + (\bar{D}^+V^{--})I,$$ (2.46)

as well as the covariant time derivative $D_t$:

$$\{D^+, D^-\} = 2iD_t, \quad D_t = \partial_t - \frac{i}{2}(D^+D^+V^{--})I.$$ (2.47)

The vector gauge connection

$$V \equiv D^+\bar{D}^+V^{--}, \quad \delta V = -2i\partial_t\Lambda,$$ (2.48)

is an analytic superfield, $D^+V = \bar{D}^+V = 0$, so $D_t$ preserves the analyticity. In the WZ gauge (2.40)

$$V \implies 2iA(t).$$ (2.49)

We will exploit these relations and their non-abelian generalization in next Sections.
3 Multiplet (1, 4, 3) from gauging $SU(2)_{PG}$

In the full set of rotational and shift symmetries realized on $q^+a$ there are six different subgroups with three generators: $SU(2)_{PG}$ defined in (2.23), the abelian 3-parameter translation set (2.27b) and four solvable subgroups $G_{VI}$ - $G_{IX}$ defined by eqs. (2.33) - (2.36) (modulo isomorphisms (2.37)). In this paper we shall limit our study to the first two options$^4$ and start with gauging $SU(2)_{PG}$. The simplest version of this gauging, with the free $q^+a$ action as the point of departure, was already considered in [7].

3.1 From (4, 4, 0) to (1, 4, 3)

Let us gauge the $SU(2)_{PG}$ symmetry (2.23) by substituting $\lambda^a_{\ b} \rightarrow \Lambda^a_{\ b}(\zeta,u)$,

$$\delta q^+a = \Lambda^a_{\ b} q^+b. \quad (3.1)$$

The constraint (2.16) is covariantized to

$$\nabla^+ q^+a \equiv D^+ q^+a - V^+a b q^+b = 0, \quad (3.2)$$

where the traceless analytic gauge connection $V^+a b$ is transformed as

$$\delta V^+a b = D^+ \Lambda^a_{\ b} + \Lambda^a_{\ c} V^+c b - V^+a c \Lambda^c_{\ b}. \quad (3.3)$$

The analytic superspace form of the free action (2.21) is covariantized by replacing

$$\partial_t q^+a \Rightarrow \nabla t q^+a = \partial_t q^+a - \frac{i}{2} V^a b q^+b, \quad (3.4)$$

where

$$V^a_{\ b} = D^+ \bar{D}^+ V^- - V^- a_{\ b}, \quad (3.5)$$

$$D^+ V^- - V^+ a_{\ c} b + V^- a c V^+ c b = 0, \quad (3.6)$$

$$\delta V^- a_{\ b} = D^- \Lambda^a_{\ b} + \Lambda^a_{\ c} V^- c b - V^- a c \Lambda^c_{\ b}. \quad (3.7)$$

The equivalent form of the action (2.21) in the central basis is covariantized by replacing

$$q^- a = D^- q^+a \Rightarrow \hat{q}^- a \equiv \nabla^- q^+a = D^- q^+a - V^- a q^+b. \quad (3.8)$$

It is straightforward to check the identity of both forms of the covariantized free action.

Using the constraint (3.2) and the harmonic zero curvature condition (3.6), it is easy to check that

$$[\nabla^+, \nabla^-] = D^0, \quad \nabla^+ \hat{q}^- a = q^+a, \quad \nabla^- \hat{q}^- a = 0. \quad (3.9)$$

The subclass of general $q^+$ actions (2.19) enjoying gauge $SU(2)_{PG}$ symmetry is defined as follows

$$S_{\text{gauge}} = \int dt d^4 \theta du L(q^+a \hat{q}^- a, u^\pm). \quad (3.10)$$

$^4$See the concluding Sect. 5 for some comments on the remaining options.
Taking into account the relations (3.2), (3.9), the $SU(2)_{PG}$ invariant
\[ J \equiv q^{+a}q_{a}^{-} \]  
(3.11)
is the only gauge invariant quantity which one can construct. Also, it is easy to show that
\[ D^{++}J = 0, \]  
(3.12)
whence it follows that $J$ does not depend on harmonics in the central basis. Therefore, without loss of generality, one can neglect the harmonic integral in (3.10) together with the dependence on the explicit harmonics in $L(J, u^{\pm})$. Thus, the most general gauge invariant action is obtained, via the replacement (3.8), from the most general globally $SU(2)_{PG}$ invariant action
\[ S_{PG} = \int dt d^{4}\theta \mathcal{L} \left( q^{+a}q_{a}^{-} \right) = \int dt d^{4}\theta \mathcal{L} \left( \frac{1}{2}q^{ia}q_{ia} \right). \]  
(3.13)

As was already shown in [7], the $SU(2)_{PG}$ gauging of the multiplet $q^{ia}$ gives rise to one sort of the $\mathcal{N}=4, d=1$ supermultiplet $(1, 4, 3)$, in such a way that three physical bosonic components of $q^{ia}$ become purely gauge while $V^{++\, (ab)}$ supplies three auxiliary degrees of freedom. In [7] this was demonstrated in the WZ gauge
\[ V^{+a}_{(b)} = 2i\theta^{+}\bar{\theta}^{+}A_{b}^{a}(t), \quad V^{-a}_{(b)} = 2i\theta^{-}\bar{\theta}^{-}A_{b}^{a}(t), \quad V^{a}_{(b)} = D^{+}\bar{D}^{+}V^{-a}_{(b)} = 2iA_{b}^{a}(t), \quad (3.14) \]
\[ \delta_{r}A_{b}^{a} = -\partial_{t}\Lambda_{(0)}^{a}_{b} + \Lambda_{(0)}^{a}_{c}A_{b}^{c} - A_{c}^{d}\Lambda_{(0)}^{c}_{b}, \quad \delta_{r}f^{ia} = \Lambda_{(0)}^{d}_{c}f^{ic}. \]  
(3.15)
where $f^{ia}(t)$ is the first component of $q^{ia}$, $f^{ia}u_{i}^{+} = q^{ia}|_{\theta=0}$. In this gauge, the solution of the covariantized constraint (3.2) is obtained from the solution (2.16) just by the replacement
\[ \partial_{t}f^{ia} \Rightarrow \nabla_{t}f^{ia} = \partial_{t}f^{ia} + A_{b}^{a}f^{ib}. \]  
(3.16)
Splitting $f^{ia}$ as
\[ f^{ia} = \varepsilon^{ia} \frac{1}{\sqrt{2}} f + f^{(ia)}, \]  
(3.17)
and assuming that $f$ has a non-vanishing constant vacuum part, $f = < f > + \ldots, < f > \neq 0$, one observes that the symmetric part in (3.17) can be fully gauged away by the residual $SU(2)$ gauge freedom
\[ f^{ia} \Rightarrow \varepsilon^{ia} \frac{1}{\sqrt{2}} f. \]  
(3.18)
So one ends up with the fields $f(t), \psi^{ia}(t), A^{(ab)}$, which is just the off-shell field content of the multiplet $(1, 4, 3)$. Note that in this gauge the only manifest $SU(2)$ symmetry is the diagonal one in the product $SU(2)_{A} \times SU(2)_{PG}$. It plays the role of automorphism $SU(2)$ group. As usual in WZ gauge, $\mathcal{N}=4$ supersymmetry is not manifest, it should be accompanied by a field-dependent gauge transformation to preserve the WZ gauge and the additional gauge (3.18).

Here we show how to arrive at the multiplet $(1, 4, 3)$ while preserving manifest $\mathcal{N}=4$ supersymmetry.

To this end, we project all doublet $SU(2)_{PG}$ indices on the harmonics $u^{\pm i}$ using the completeness relation (2.4)
\[ q^{+a} = \omega u^{+a} - l^{++}u^{-a}, \quad \omega = q^{+a}u_{a}^{-}, \quad l^{++} = q^{+a}u_{a}^{+}, \]  
(3.19)
\[ V^{\pm (ab)} = u^{+a}u^{+b}V^{\pm (ab)} + u^{-a}u^{-b}V^{\pm (ab)} - 2u^{+(a}u^{-b)}V^{\pm (ab)}, \]
\[ V^{\pm (++)} = V^{\pm (ab)}u_{a}^{+}u_{b}^{+}, \quad V^{\pm (++)} = V^{\pm (ab)}u_{a}^{+}u_{b}^{+}. \]  
(3.20)
In terms of these projections, the transformation laws (3.1), (3.3) and (3.7) read
\[
\begin{align*}
\delta \omega &= -\Lambda^{++} \omega + \Lambda^{-+} l^{++}, \\
\delta l^{++} &= \Lambda^{++} l^{++} - \Lambda^{++} \omega,
\end{align*}
\]
while the constraint (3.2) takes the form
\[
\begin{align*}
(a) \quad D^{++} l^{++} - V^{++(+) +} l^{++} + V^{++(+)} \omega &= 0, \\
(b) \quad D^{++} \omega - l^{++} - V^{++(-)l} + V^{++(+)} \omega &= 0.
\end{align*}
\]
Assuming that \( \omega > 0 \), we observe from the transformation law (3.21) that one can choose the following manifestly \( \mathcal{N}=4 \) supersymmetric gauge
\[
\omega = 1, \quad l^{++} = 0 \quad \Rightarrow \quad q^a = u^a.
\]
In this gauge, the constraints (3.24) imply
\[
V^{++(++)} = V^{++(+)\omega} = 0, \quad V^{++(-)} \equiv \mathcal{V} \neq 0.
\]
The residual gauge freedom is given by
\[
\begin{align*}
\Lambda^{++} &= 0, \quad \Lambda^{+-} = 0, \quad \Lambda^{-+} \neq 0, \\
\delta_r \mathcal{V} &= D^{++} \Lambda^{-}, \\
\delta_r V^{--(++)} &= 0, \quad \delta_r V^{--(+-)} = -\Lambda^{-+} + \Lambda^{-+} V^{--(++)}, \\
\delta_r V^{--(-)} &= D^{--} \Lambda^{-} + 2 \Lambda^{--} V^{--(-)}.\tag{3.29}
\end{align*}
\]
Here \( \Lambda^{-+} = \Lambda^{-+}(\zeta, u) \) is the only unconstrained residual analytic gauge parameter. The harmonic zero-curvature condition (3.6) is rewritten as
\[
\begin{align*}
D^{++} V^{--(++)} &= 0, \\
D^{++} V^{--(+-)} - (1 + \mathcal{V}) V^{--(++)} + \mathcal{V} &= 0, \\
D^{++} V^{--(-)} - D^{--} \mathcal{V} - 2 (1 + \mathcal{V}) V^{--(+-)} &= 0.\tag{3.32}
\end{align*}
\]
These equations determine \( V^{--(++)}, V^{--(+-)} \) and \( V^{--(-)} \) as functions of the analytic gauge potential \( \mathcal{V} \).

Thus in the supersymmetric gauge (3.25) we are left with the analytic gauge superfield \( \mathcal{V}(\zeta, u), \delta_r \mathcal{V} = D^{++} \Lambda^{-} \), as the basic object encompassing the whole field content of the system consisting of the \((4, 4, 0)\) multiplet and gauge \( SU(2)_{PG} \) superfield. The general action (3.10) takes the simple form
\[
\begin{align*}
S_{\text{gauge}} &= \int dt d^4 \theta \mathcal{L}(J), \\
J &= q^a \nabla^{--(+)q^a} + 1 - V^{--(++)}, \quad D^{++} J = 0.\tag{3.34}
\end{align*}
\]
where we took into account eq. (3.30). Using the harmonic independence of \(J\) in the central basis, it is easy to find from (3.31) the expression of \(J\) in terms of \(V\):

\[
J = \frac{1}{1 + \mathcal{W}}, \quad \mathcal{W}(t, \theta^i, \bar{\theta}_k) \equiv \int du \mathcal{V}\left(t - 2i\theta^i\bar{\theta}^k u^+_i u^-_k, \theta^i u^+_i, \bar{\theta}^k u^+_k, u^-_i \right). \tag{3.35}
\]

To reveal the field content carried by \(\mathcal{V}\), we should fully exploit the residual infinite-dimensional gauge freedom (3.28). The full WZ form of \(\mathcal{V}\) is easily found to be

\[
\mathcal{V}(\zeta, u) = v_0(t_A) + \theta^+ \psi(t_A) u^-_i + \bar{\theta}^+ \bar{\psi}(t_A) u^+_i + 3\theta^i \bar{\theta}^+ A^{(ik)}(t_A) u^-_i u^+_k, \tag{3.36}
\]

without any further residual gauge freedom, \(\Lambda^- = 0\). Thus we end up with the off-shell \(\mathcal{N}=4\) supermultiplet (1, 4, 3) in the new formulation in terms of the analytic gauge prepotential \(\mathcal{V}(\zeta, u)\). The off-shell transformation properties of the component fields in (3.36) can be found from the transformation law

\[
\delta \mathcal{V} = (\epsilon^i Q_i + \bar{\epsilon}_i Q^i) \mathcal{V} + D^{++} \Lambda^-_{\text{comp}}, \tag{3.37}
\]

where the first part is induced by the \(N = 4\) supertranslations (2.7) \((Q_i = -u^+_i \frac{\partial}{\partial \theta^i} + \ldots, Q^i = u^+_i \frac{\partial}{\partial \bar{\theta}^i} + \ldots)\), while the second part is the compensating gauge transformation needed to preserve the WZ gauge (3.36):

\[
\Lambda^-_{\text{comp}} = \frac{1}{2} \left( \epsilon^{(i} \bar{\psi}^{k)} + \bar{\epsilon}^{(i} \psi^{k)} \right) u^-_i u^+_k + \left( \theta^+ \bar{\epsilon}^i - \bar{\theta}^+ \epsilon^i \right) A^{(ik)}(t_A) u^-_i u^+_k u^-_i. \tag{3.38}
\]

The meaning of the \(\mathcal{N}=4\) superfield \(\mathcal{W}\) defined in (3.35) can be also easily understood. First of all, by construction it is invariant under the gauge transformations (3.28), so one can always choose WZ form (3.36) for \(\mathcal{V}\) in (3.35), i.e. the field content of \(\mathcal{W}\) is just (1, 4, 3). Secondly, using the analyticity of \(\mathcal{V}\), \(D^+ \mathcal{V} = \bar{D}^+ \mathcal{V} = 0\), and the completeness relation (2.4), it is easy to show that

\[
[D^i, \bar{D}_i] \mathcal{W} = -2 \int du \left( D^- D^+ + \bar{D}^- \bar{D}^+ \right) \mathcal{V} = 0. \tag{3.39}
\]

These are just the constraints which define the (1, 4, 3) multiplet in the ordinary \(\mathcal{N}=4\) superspace [19, 20].

Thus we have shown that the most general sigma-model type action of the \(\mathcal{N}=4\) multiplet (1, 4, 3) can be reproduced from the most general SU(2)\(_{\text{PG}}\) invariant action of the multiplet (4, 4, 0) by gauging the SU(2)\(_{\text{PG}}\) symmetry using a “topological” gauge \(\mathcal{N}=4\) supermultiplet.

For further use, let us note that, before imposing any gauge-fixing condition, one can use the constraints (3.24) to covariantly express the gauge superfields \(V^{++(+-)}\) and \(V^{++(++)}\) in terms of \(V^{++(--)}\), \(\omega\) and \(l^{++}\)

\[
V^{++(+-)} = \frac{1}{\omega} \left\{ l^{++} \left[ 1 + V^{++(--)} \right] - D^{++} \omega \right\}, \quad V^{++(++)} = \frac{1}{\omega^2} \left\{ (l^{++})^2 \left[ 1 + V^{++(--)} \right] - D^{++}(l^{++} \omega) \right\}. \tag{3.40}
\]

Taking into account that the analytic superfields \((\omega - 1)\) and \(l^{++}\), in view of their inhomogeneous transformation laws (3.21), can be treated as Goldstone superfields related to the
“spontaneous breaking” of local $SU(2)_{PG}$ symmetry down to its abelian subgroup with the
analytic parameter $\Lambda^{--}$, eqs. (3.40) supplies a nice example of the inverse Higgs phenomenon
[23]. This phenomenon, in particular, provides a possibility to covariantly express gauge fields
associated with the coset generators of the given nonlinearly realized symmetry in terms of
the Goldstone fields and gauge fields belonging to the linear stability subgroup. Using the
transformation laws (3.21) and (3.22), it is easy to check that the above “composite” gauge
superfields $V^{++}$ and $V^{++}$ possess the correct gauge transformation properties. In the
“unitary” gauge $\omega = 1, l^+ = 0$ these superfields vanish as it should be. Also, in accord with
the general reasoning of ref. [23], one can construct a new gauge connection

$$\tilde{V}^{++} = \frac{1}{\omega^2} [(1 - \omega^2) + V^{++}]$$

(3.41)

which has the “would-be abelian” $SU(2)_{PG}$ gauge transformation law

$$\delta \tilde{V}^{++} = D^{++} \left( \frac{1}{\omega^2} \Lambda^{--} \right).$$

(3.42)

We have chosen $\tilde{V}^{++}$ in such a way that it is equal to $V^{++}$ in the unitary gauge. In
what follows, this gauge connection will be used to construct an invariant Fayet-Iliopoulos (FI)
term for the multiplet $(1, 4, 3)$.

3.2 Superconformal properties

Now we turn to discussing the superconformal properties of the new description of the multiplet
$(1, 4, 3)$ and some immediate applications thereof. We shall need to know the realization of the
most general $N=4, d = 1$ superconformal group $D(2, 1; \alpha)$ in the analytic basis of $N=4$
superspace [8]. Since the whole set of the $D(2, 1; \alpha)$ transformations is contained in the closure
of Poincaré supersymmetry (2.7) and conformal supersymmetry, it is sufficient to explicitly give
only the transformations of the latter:

$$\delta' \theta^+ = \eta^+ t_A + 2i(1+\alpha)\eta^- (\theta^+ \bar{\theta}^+), \quad \delta' \theta^- = \eta^- t_A + 2i(1+\alpha)\eta^+ (\theta^- \bar{\theta}^-),$$

(3.43)

$$\delta' u^+_i = \Lambda^+ u^+_i, \quad \delta' u^-_i = 0,$$

$$\Lambda^+ = -2i\alpha (\eta^+ \bar{\theta}^+ - \eta^- \bar{\theta}^-), \quad (D^{++})^2 \Lambda^+ = 0,$$

(3.44)

$$\delta' \theta^- = \eta^- t_A + 2i\eta^- [(1+\alpha)\theta^+ \bar{\theta}^- + \theta^- \bar{\theta}^+] + 2i\alpha \eta^- \theta^- \bar{\theta}^- - 2i(1+\alpha)\eta^+ \theta^+ \bar{\theta}^-,$$

(3.45)

$$\delta' t_A = -2i t_A (\eta^- \bar{\theta}^- - \eta^+ \bar{\theta}^+), \quad \delta' \bar{\theta}^- = \bar{\delta} \bar{\theta}^-.$$  

(3.46)

Here, $\eta^\pm = \eta^i u_i^\pm$, $\bar{\eta}^\pm = \bar{\eta}^i u_i^\pm$ and $\eta^i, \bar{\eta}^i$ are the corresponding Grassmann parameters and the
involution $\bar{}$ is defined in (2.8). We also need the transformation properties of the harmonic
derivatives, the $(4, 4, 0)$ superfield $q^{+a}$, gauge potentials $V^{\pm \pm}$ and the measures of integration
over the full and analytic harmonic superspaces (2.14)

$$\delta' D^{++} = -\Lambda^{++} D^0, \quad \delta' D^{--} = -(D^{--} \Lambda^{++}) D^{--}, \quad \delta' D^0 = 0,$$

(3.47)

$$\delta' q^{+a} = \Lambda^{+a} q^{+a}, \quad \delta' V^{++} = 0, \quad \delta' V^{--} = -(D^{--} \Lambda^{++}) V^{--},$$

(3.48)

$$\delta' \mu_A^{(2)} = \left( \partial_A \delta' t_A + \partial^{--} \Lambda^{++} - \partial_{\theta^+} \delta' \theta^+ - \partial_{\bar{\theta}^+} \delta' \bar{\theta}^+ \right) \mu_A^{(2)} = 0,$$

(3.49)
\[ \delta' \mu_H = \left( \partial_A \delta' t_A + \partial^- \Lambda^{++}_{sc} + \partial_{\bar{\theta}} \delta' \theta^+ - \partial_{\theta} \delta' \bar{\theta}^+ - \partial_{\theta} \delta' \bar{\theta}^- - \partial_{\bar{\theta}} \delta' \theta^- \right) \mu_H \]

\[ = 2i \left[ (1 - \alpha)(\eta^- \bar{\theta}^+ - \bar{\eta}^+ \theta^-) - (1 + \alpha)(\eta^+ \bar{\theta}^- - \bar{\eta}^- \theta^+) \right] \mu_H. \tag{3.50} \]

The integration measures are evidently invariant under the \( \mathcal{N}=4 \) Poincaré supersymmetry (2.7).

Note that for the special values of the parameter \( \alpha \),

\[ (a) \quad \alpha = -1, \quad (b) \quad \alpha = 0, \tag{3.51} \]

the supergroup \( D(2,1;\alpha) \) is reduced to two different \( PSU(1,1|2) \) supergroups, such that one of the two commuting \( R \)-symmetry subgroups \( SU(2) \subset D(2,1;\alpha) \) is identified with \( SU(2) \subset \) \( PSU(1,1|2) \), while the other decouples and acts as outer automorphisms of this \( PSU(1,1|2) \). In particular, as follows from (3.44), the \( PSU(1,1|2) \) supergroup corresponding to the choice \( \alpha = 0 \) does not affect harmonic variables at all.

It is easy to check superconformal covariance of the constraints (2.16), (3.2) and the harmonic zero-curvature conditions (2.45), (3.6) at any \( \alpha \). The quantity \( J \) defined in (3.34) is transformed as

\[ \delta' J = \left( 2\Lambda_{sc} - D^- \Lambda^{++}_{sc} \right) J \equiv \Lambda_0 J, \quad D^{++} \Lambda_0 = 0. \tag{3.52} \]

The superconformal properties of the basic quantities in the gauge (3.25) can be easily found from the condition of preserving this gauge (which fixes the relevant compensating gauge transformations) and the transformation property of the harmonic measure \( du \) in the central basis

\[ \delta' du = du \left( \eta^+ D^- \Lambda^{++}_{sc} + \eta^- D^- \Lambda^{++}_{sc} \right). \tag{3.53} \]

We obtain

\[ \delta' \mathcal{V} = -2\Lambda_{sc} (1 + \mathcal{V}), \quad \delta' \mathcal{V}^{--} = -\Lambda_0 \left[ 1 - \mathcal{V}^{--} \right], \quad \delta' \mathcal{W} = -\Lambda_0 \left( 1 + \mathcal{W} \right). \tag{3.54} \]

Taking into account that in this gauge \( J = 1 - \mathcal{V}^{--} \), we see that (3.54) agrees with (3.52).~

Also, it is convenient to define \( \mathcal{U} = 1 + \mathcal{W} \), so that

\[ J = \frac{1}{\mathcal{U}}, \quad \delta' J = \Lambda_0 J, \quad \delta' \mathcal{U} = -\Lambda_0 \mathcal{U}. \tag{3.55} \]

The object \( \mathcal{U} \) satisfies the same constraints (3.39) as \( \mathcal{W} \), but has simpler transformation properties.

Let us recall the form of superconformally invariant actions of the multiplet \((1,4,3)\). Within the above gauging procedure, for all \( \alpha \) except the special value \( \alpha = 0 \), they are uniquely defined by the corresponding actions of the multiplet \((4,4,0)\). The latter are formulated in terms of \( q^+ a^- \sim q^{-a} \). To obtain the invariant action of thee multiplet \((1,4,3)\), one just should make the replacement \( q^+ a^- \to J = q^+ a^- \) in the corresponding \((4,4,0)\) action. For \( \alpha \neq 0, -1 \) such subclass of the general sigma-model action (2.20) is given by

\[ S_{(sc)}^\alpha = \gamma \int dt d^4 \theta \frac{J}{U} = \gamma \int dt d^4 \theta U^{-\frac{1}{2}}, \tag{3.56} \]

where \( \gamma \) is a normalization constant. In particular, the free \( q^+ \) action (2.21) is invariant under the supergroup \( D(2,1;\alpha = 1) \sim OSp(4^*|2) \), and the associated superconformal \((1,4,3)\) action is just

\[ S_{(sc)}^{\alpha = 1} = \gamma \int dt d^4 \theta J = \gamma \int dt d^4 \theta U^{-1}. \tag{3.57} \]
This action contains a non-trivial self-interaction, despite the fact that it was obtained by gauging the free $q^+a$ action. It is the only one which admits a local representation as an integral over the analytic superspace. This is just the example we have considered in [7]. On the other hand, the choice of $\alpha = -1/2$ yields the free $(1, 4, 3)$ action

$$S_{(sc)}^{\alpha=-1/2} = \gamma \int dtd^4\theta U^2,$$

(3.58)

though it is obtained by gauging a non-trivial sigma-model $q^+$ action, with the superfield Lagrangian $L \sim (q^a q_a)^{-2}$ in (2.20).5

The cases of $\alpha = -1$ and $\alpha = 0$ at which $D(2,1;\alpha)$ degenerates into the $PSU(1,1|2)$ supergroups (times outer $SU(2)$ automorphisms) require a separate consideration.

At $\alpha = -1$ the action (3.56) is still invariant, but now the Lagrangian is just $U = 1 + \mathcal{W}$ and the action identically vanishes as a consequence of the constraints (3.39). The meaningful action in this case reads [19, 15]

$$S_{(sc)}^{\alpha=-1} \sim \int dtd^4\theta \log U,$$

(3.59)

which is invariant under $PSU(1,1|2)$ (and the extra $SU(2)$ automorphisms), up to a total derivative in the integrand.

In the case of $\alpha = 0$ the situation is even more subtle. As seen from (3.44), in this case the harmonic variables are inert under the corresponding $PSU(1,1|2)$ 6 and the superfields $V, U$ are transformed with zero conformal weight. On the other hand, the integration measure $\mu_H$ is still not invariant,

$$\delta'\mu_H = 2i (\bar{\eta}^i \theta_i - \eta^k \bar{\theta}_k) \mu_H.$$  

(3.60)

The only way to construct the invariant action in this case is to modify the transformation law of the analytic prepotential $V$ under the conformal supersymmetry:

$$\delta'_{mod} V = 4i (\bar{\eta}^- \theta^+ - \eta^- \bar{\theta}^+)$$

(3.61)

(actually, the coefficient in the r.h.s. can be an arbitrary non-zero real number; it was chosen in this way just for further convenience, using the freedom of rescaling $V$). Respectively, the superfield $U$ is now transformed as

$$\delta'_{mod} U = -2i (\bar{\eta}^i \theta_i - \eta^k \bar{\theta}_k).$$

(3.62)

The extra terms in some other $PSU(1,1|2)$ transformations can be found by taking the Lie brackets of (3.61), (3.62) with Poincaré supersymmetry (in fact, when acting on $V$, the symmetry $PSU(1,1|2)$ closes modulo some particular gauge transformation). The invariant action for the case $\alpha = 0$ is then as follows

$$S_{(sc)}^{\alpha=0} \sim \int dtd^4\theta e^{U}.$$  

(3.63)

The modified $\alpha = 0$ superconformal transformation (3.61) cannot be obtained from any modification of the $\alpha = 0$ transformations of $q^+a$ and/or $V^{++(ab)}$ before fixing the unitary

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5In ref. [10] there is an erroneous statement that all superconformally invariant actions of the reduced multiplets are generated from the free $q^+a$ action.

6They are still transformed under the extra automorphisms $SU(2)$ acting on the indices $i, k$. 

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gauge with respect to the local $SU(2)_{PG}$. This becomes possible within the alternative gauging of the $(4, 4, 0)$ system considered in the next Section.

Let us now discuss the issue of superconformally invariant potentials of the multiplet $(1, 4, 3)$ in the description through the analytic prepotential $\mathcal{V}$. It is easy to see that there exist no $SU(2)_{PG}$ invariant WZ-type $q^{+a}$ Lagrangians among those in (2.22). Thus it seems impossible to generate the potential terms for the multiplet $(1, 4, 3)$ by $SU(2)_{PG}$ gauging of any $q^{+a}$ action. Nevertheless, such terms can be constructed with the help of the gauge superfield $V^{++(ab)}$. Prior to imposing any $SU(2)_{PG}$ gauge, one can define the gauge invariant FI term

$$S_{FI} = \int dud\zeta (-2) c^{+2} \tilde{V}^{++(-)} = \int dud\zeta (-2) c^{+2} \mathcal{V} \quad \text{where} \quad c^{+2} = c^{ik} u_i^+ u_k^+ , \quad [c] = cm^{-1},$$

(3.64)

where $\tilde{V}^{++(-)}$ was defined in (3.41), (3.42). This action is invariant under (3.42) and (3.28) thanks to the condition $D^{++} c^{++} = 0$, and in (3.64) we made use of the property that $\tilde{V}^{++(-)} = \mathcal{V}$ in the unitary gauge $\omega = 1, l^{++} = 0$. In the WZ gauge (2.40) the component Lagrangian following from (3.64) is $\propto c^{ik} A_{ik}$. When (3.64) is added to some non-trivial action (3.33), eliminating the auxiliary field $A_{ik}$ gives rise to a non-trivial potential of the physical bosonic field $v_0$ (plus the appropriate Yukawa-type fermionic terms) [19, 20].

Inspecting the superconformal properties of (3.64), one can check that it is superconformally $(PSU(1, 1|2))$ invariant only at $\alpha = 0$ (the shift (3.61) is linear in the analytic coordinates $\theta^+, \bar{\theta}^+$ and so does not affect (3.64)). Thus one possibility to construct the superconformally invariant action of the multiplet $(1, 4, 3)$ with a non-trivial scalar potential is to sum up (3.64) with (3.63),

$$\tilde{S}_{(sc)}^{\alpha=0} = \gamma \int dt d^4 \theta \bar{e} \mathcal{U} + \int dud\zeta (-2) c^{+2} \mathcal{V}.$$

(3.65)

After elimination of $A_{ik}$ there comes out the conformal potential $\sim e^{-(1+|v_0|)}$ with the strength $\sim c^2 = c^{ik} \tilde{c}_{ik}$ (see below). The extra automorphisms $SU(2)$ acting on the indices $i, k$ is obviously broken down to some $U(1)$ due to the presence of the constant triplet $c^{ik}$ in (3.64), (3.65).

The only alternative mechanism of generating superconformally invariant potential term for the multiplet $(1, 4, 3)$ which was known to date [19] requires $\alpha = -1$. Though (3.64) is not superconformally invariant in this case and so cannot be used, one can modify the constraints (3.39) by inserting two arbitrary constants (one complex and one real) in their r.h.s. They form a constant isotriplet with respect to the second $R$-symmetry $SU(2)$ subgroup (the one which provides outer automorphisms of $PSU(1, 1|2)$ corresponding to the choice $\alpha = -1$). Exploiting this broken $SU(2)$ symmetry, one can choose the frame where

$$(D)^2 \bar{\mathcal{U}} = (\bar{D})^2 \tilde{\mathcal{U}} = 0, \quad [D^i, \bar{D}_i] \bar{\mathcal{U}} = f, \quad f = f^*.$$

(3.66)

With $\bar{\mathcal{U}}$ transforming as in (3.55), at $\alpha = -1$ the expressions in the l.h.s. of these constraints can be checked to be scalars of zero conformal weight, so one can equate them to some non-zero constants without contradiction with the superconformal $PSU(1, 1|2)$ symmetry. The substitution of $\bar{\mathcal{U}}$ into (3.59) for $\mathcal{U}$ once again yields the conformal potential for $v_0$ in the component action (with strength $\sim f^2$).

It is interesting to see how the modified constraints emerge within the analytic prepotential description of the multiplet $(1, 4, 3)$. The superfield $\bar{\mathcal{U}}$ is related to $\mathcal{U} = 1 + f du \mathcal{V}$ in the following way

$$\tilde{\mathcal{U}} = \mathcal{U} + \frac{1}{2} f \tilde{\theta} \bar{\theta}_i = \mathcal{U} + \frac{1}{2} f (\tilde{\theta}^+ \theta^- - \bar{\theta}^- \bar{\theta}^+) .$$

(3.67)
This superfield has precisely the same transformation properties with respect to \( PSU(1,1|2) \) corresponding to \( \alpha = -1 \) as the superfield \( U \), i.e. \( \delta U = -\Lambda_0^{(\alpha=-1)} U \), provided that the analytic prepotential \( \mathcal{V} \) possesses the following modified transformation rules with respect to the Poincaré and conformal supersymmetries
\[
\tilde{\delta} \mathcal{V} = -2\Lambda_{sc}^{(\alpha=-1)}(1 + \mathcal{V}) + f \left[ (\epsilon^- + t_A \eta^-) \bar{\theta}^+ + (\bar{\epsilon}^- + t_A \bar{\eta}^-) \theta^+ \right]. \tag{3.68}
\]

It is easy to check that the closure of the modified transformations coincides with the original one (i.e. for \( f = 0 \)) modulo some special gauge transformation of the form (3.28). The latter does not make contribution to \( \mathcal{W} = \int du \mathcal{V} \) owing to the \( u \)-integral. Note that the second term in (3.67) cannot be re-absorbed into \( \mathcal{V} \) because it contains non-analytic Grassmann coordinates.

It is curious that the prepotential realization of the \((1,4,3)\) multiplet suggests one more mechanism of generating conformal potential for the bosonic field \( v_0 \). Again, it only works for the \( PSU(1,1|2) \) case with \( \alpha = -1 \). It is non-minimal, because it uses a superconformal coupling to the extra off-shell multiplet \((0,4,4)\). The latter contains no physical bosons at all and comprises 4 fermionic fields and 4 bosonic auxiliary fields. It is described by the fermionic analog of \( q^{+a} \), the superfield \( \Psi^{+m} \), \( (\Psi^{+m}) = \Psi^+_m \), subjected to the constraint [8]
\[
D^{++}\Psi^{+m} = 0 \Rightarrow \Psi^{+m} = \psi^{im}u_i^+ + \theta^+a^m + \bar{\theta}^+\bar{a}^m + 2i\theta^+\bar{\theta}^+\partial_i\psi^{im}u_i^- . \tag{3.69}
\]

With respect to the doublet index \( m \) \((m = 1,2)\), it is transformed by some extra \( SU(2) \) which commutes with \( N=4 \) and so is an analog of \( SU(2)_{PG} \) (it does not necessarily coincide with the latter). The requirement of superconformal covariance of the constraint (3.69) uniquely fixes the superconformal \( D(2,1;\alpha) \) transformation rule of \( \Psi^{+m} \), for any \( \alpha \), as follows
\[
\delta'\Psi^{+m} = \Lambda_{sc} \Psi^{+m} . \tag{3.70}
\]

The free action of \( \Psi^{+m} \),
\[
S^{\psi}_{free} = \int dud\zeta^{(-2)} \Psi^{+m} \Psi^+_m , \tag{3.71}
\]
is obviously not invariant under \( D(2,1;\alpha) \). However, recalling the transformation law (3.54), we observe that this action can be easily promoted to a superconformal invariant by coupling \( \Psi^{+m} \) to the \((1,4,3)\) multiplet
\[
S^{\psi}_{(sc)} = \int dud\zeta^{(-2)}(1 + \mathcal{V}) \Psi^{+m} \Psi^+_m . \tag{3.72}
\]

This action is superconformal at any \( \alpha \), and it also respects the gauge invariance (3.28) as a consequence of the constraint (3.69). However, a simple analysis shows that in components it yields only a bilinear term in the auxiliary fields \( a^m, \bar{a}^m \) and therefore cannot produce a non-trivial potential of the field \( v_0 \). To get such a potential, one needs to add a FI-type term to (3.72)
\[
S^{\psi}_{FI} = \int dud\zeta^{(-2)} \left( \theta^+\xi_m \Psi^{+m} + \bar{\theta}^+\bar{\xi}^m \Psi^+_m \right) , \tag{3.73}
\]
where \( \xi_m, \bar{\xi}^m \) is a constant doublet which breaks the extra \( SU(2) \) acting on the indices \( m \).

Using the transformation properties (3.43), (3.49) and (3.70), as well as the constraint (3.69), it is easy to show that (3.73) at \( \alpha = -1 \) is invariant under both Poincaré and conformal supersymmetries, up to a total derivative in the integrand. After passing to components, this
action produces terms which are linear in the auxiliary fields $a^m, \bar{a}^m$. Eliminating these fields in the total $D(2, 1; \alpha = -1)$ invariant action

$$S_{tot} = S_{(sc)}^{\alpha=1} + S_{(sc)}^{\alpha=-1} + S_{FI}^{\psi}, \quad (3.74)$$

one again reproduces the conformal potential for $v_0$. The price for this is the enlargement of the physical fermionic sector of the model from 4 to 8 fields, still with the presence of only one physical bosonic field. Taking into account that the $N=4$ multiplets $(1, 4, 3)$ and $(0, 4, 4)$ can be joined into the off-shell $N=8$ multiplet $(1, 8, 7)$ [24], one can expect a hidden $N=8$ supersymmetry in this combined system.

Note that the two mechanisms of obtaining superconformally invariant potential terms at $\alpha = -1$ described above cannot coexist since the action (3.72) is not invariant under the modified superconformal transformations (3.68).

### 3.3 Examples of component actions

Let us give a few examples of bosonic component actions. We shall need the form of the bosonic sector of the superfield $J$ and $\mathcal{U} = 1 + \mathcal{W}$ defined in (3.34) and (3.35). Passing to the central basis in the prepotential $\mathcal{V}(\zeta, u)$, we find from (3.35)

$$\mathcal{U} = 1 + \mathcal{W} = (1 + v_0) [\theta^- \hat{\theta}^- A^{++} - (\theta^- \hat{\theta}^+ + \theta^+ \hat{\theta}^-) A^{+-} + \theta^+ \hat{\theta}^+ A^{--}] + \theta^+ \hat{\theta}^+ \theta^- \hat{\theta}^- \partial^2 v_0, \quad (3.75)$$

where all component fields are functions of $t$ and we still used the harmonic projections of the Grassmann coordinates (in fact, the harmonic dependence in (3.75) is fake, which immediately follows from the easily checkable relation $\partial^{++} \mathcal{U} = 0$). The corresponding expression for the superfield $J = \mathcal{U}^{-1}$ is

$$J = \frac{1}{1 + v_0} \left\{ 1 - \frac{1}{1 + v_0} [\theta^- \hat{\theta}^- A^{++} - (\theta^- \hat{\theta}^+ + \theta^+ \hat{\theta}^-) A^{+-} + \theta^+ \hat{\theta}^+ A^{--}] - \frac{1}{1 + v_0} \theta^+ \hat{\theta}^+ \theta^- \hat{\theta}^- [\partial^2 v_0 - \frac{1}{1 + v_0} A^{ik} A_{ik}] \right\}. \quad (3.76)$$

Using these explicit expressions, we find, in particular,

$$S_{(sc)}^{\alpha=1} \Rightarrow S_{(sc)}^{\alpha=1} \sim \int dt [(\partial_t \rho)^2 - \frac{1}{8} \rho^6 (A^{ik} A_{ik})], \quad \rho = (1 + v_0)^{-1/2}, \quad (3.77)$$

$$S_{(sc)}^{\alpha=-1} \Rightarrow S_{(sc)}^{\alpha=-1} \sim \int dt [(\partial_t \rho)^2 - \frac{1}{8} \rho^2 (A^{ik} A_{ik})], \quad \rho = \sqrt{1 + v_0}, \quad (3.78)$$

$$S_{(sc)}^{\alpha=0} \Rightarrow S_{(sc)}^{\alpha=0} \sim \int dt [(\partial_t \rho)^2 - \frac{1}{8} \rho^2 (A^{ik} A_{ik})], \quad \rho = 2 e^{\frac{1}{2}(1 + v_0)^{1/2}}, \quad (3.79)$$

where the superfield actions were defined in (3.57), (3.59) and (3.63). For $\bar{\mathcal{U}} = \mathcal{U} + \frac{1}{2} f (\hat{\theta}^+ \hat{\theta}^- - \hat{\theta}^- \hat{\theta}^+) \theta \bar{\theta}$ there appears the additional (conformal) potential term in (3.78)

$$- \frac{1}{16} \int dt \frac{f^2}{\rho^2}.$$
The FI term (3.64) yields

\[ S_{FI} \Rightarrow S_{FI}^{bos} = i \int dt c^{ik} A_{ik}. \] (3.80)

After eliminating \( A^{ik} \) from the total action (3.65) (with \( \gamma = -1 \) for simplicity), one obtains

\[ \tilde{S}^{\alpha=0}_{(sc)} \Rightarrow \tilde{S}^{\alpha=0}_{bos} = \int dt \left[ (\partial_t \rho)^2 - \frac{2v^2}{\rho^2} \right]. \] (3.81)

The sum of actions (3.72) and (3.73) gives rise to the following bosonic contribution

\[ S_{(sc)}^{\psi} + S_{FI}^{\psi} \Rightarrow \int dt \left[ 2(1 + v_0) a^m \bar{a}_m + (\xi_m a^m - \bar{\xi}^m a_m) \right], \] (3.82)

which, upon eliminating the auxiliary fields \( a^m, \bar{a}_m \), again adds the conformal potential to the action (3.78) in the total action (3.74):

\[ S^{\alpha=-1}_{bos} \Rightarrow S^{\alpha=-1}_{bos} - \frac{1}{2} \int dt \frac{\xi_m \bar{\xi}^m}{\rho^2}. \] (3.83)

### 3.4 “Mirror” (1, 4, 3) multiplet

To close this Section, we make a few comments on the description of the “mirror” (1, 4, 3) multiplet in the considered setting. The basic difference between this multiplet [20] and the one discussed above is that its three auxiliary fields form a triplet with respect to the second (hidden) \( SU(2) \) automorphism group of \( \mathcal{N}=4, d = 1 \) Poincaré superalgebra. They are singlets with respect to the manifest automorphism \( SU(2) \) acting on the doublet indices \( i \) of harmonics and Grassmann coordinates. One could consider an alternative \( \mathcal{N}=4, d = 1 \) harmonic superspace, with just this second \( SU(2) \) being harmonized. In this superspace the “mirror” (1, 4, 3) multiplet is described in the same way as the multiplet we dealt with here, the only difference being in the \( D(2,1;\alpha) \) superconformal properties, such that the special cases \( \alpha = 0 \) and \( \alpha = -1 \) switch with respect to each other (the formal coincidence with the description in the \( \mathcal{N} = 4, d = 1 \) harmonic superspace considered here can be restored by passing to the parameter \( \beta = - (\alpha + 1) \)). On the other hand, in the framework of the harmonic superspace considered here this alternative (1, 4, 3) multiplet is described by a general superfield \( \Omega \) subjected to the following constraints [7]:

\[ (a) \ D^+ \bar{D}^+ \Omega = 0, \quad (b) \ D^{++} \Omega = 0. \] (3.84)

In the analytic basis, these constraints imply

\[ \Omega = \Sigma(\zeta, u) + i \left[ \theta^- \Psi^+ (\zeta, u) + \bar{\theta}^- \bar{\Psi}^+(\zeta, u) \right], \] (3.85)

\[ (a) \ D^{++} \Psi^+ = D^{++} \bar{\Psi}^+ = 0, \quad (b) \ D^{++} \Sigma + i \left( \theta^+ \Psi^+ + \bar{\theta}^+ \bar{\Psi}^+ \right) = 0. \] (3.86)

The general solution of (3.86) is

\[ \Psi^+ = \psi^i u^+_i + \theta^+ s + \bar{\theta}^+ r + 2i \theta^+ \bar{\theta}^+ \partial_t \psi^i u^-_i, \]
\[ \bar{\Psi}^+ = -\bar{\psi}^i u^+_i - \theta^+ s + \bar{\theta}^+ r - 2i \theta^+ \bar{\theta}^+ \partial_t \bar{\psi}^i u^-_i, \] (3.87)
\[ \Sigma = \sigma - i \theta^+ \psi^i u^-_i + i \bar{\theta}^+ \bar{\psi}^i u^-_i. \] (3.88)
where
\[ \text{Re} \sigma = \partial_t \sigma. \quad (3.89) \]
The independent fields \( \sigma(t), \psi^i(t), s(t) \), \( \text{Im} r(t) \) constitute the alternative off-shell \((1,4,3)\) multiplet.

The superconformal properties of the superfields \( \Sigma, \Psi, \bar{\Psi} \) can be easily defined and the relevant actions can be constructed analogously to those presented above. Once again, the superconformally invariant potential terms can be constructed only for \( \alpha = 0 \) and \( \alpha = -1 \). At \( \alpha = 0 \), the expression in the l.h.s. of (3.84a) has the conformal weight zero, so one can consider the more general condition
\[ D^+ \bar{D}^+ \Omega = \bar{c}^{++}, \quad \bar{c}^{++} = \bar{c}^{ik} u^+_i u^+_k, \quad D^{++} \bar{c}^{++} = 0. \quad (3.90) \]

The constants \( \bar{c}^{ik} \) have the dimension of mass and, via the constraint (3.84b), properly modify (3.85) - (3.88). After substitution of the modified superfields into the sigma-model type superconformal action of \( \Omega \), one obtains the conformal potential for \( \sigma_0 \), in the same way as for \( v_0 \) in the case \( \alpha = 0 \). Also, it is easy to couple the two kinds of \((1,4,3)\) multiplets to each other through an interaction similar to (3.72)
\[ S_{I-II} \sim \int dud\zeta (-2) (1 + V) \Psi^+ \bar{\Psi}^+, \quad (3.92) \]
where \( \Psi, \bar{\Psi} \) satisfy the constraints (3.86a) corresponding to the choice (3.84) (i.e. for \( \bar{c}^{ik} = 0 \) in (3.90)). In the future we hope to come back to a more detailed analysis of these models and their possible implications in such long-standing problems as constructing \( \mathcal{N}=4 \) extensions of the Calogero and Calogero-Moser integrable systems [25].

### 4 Gauging shift isometries

The multiplet \((1,4,3)\) can equally be reproduced by gauging three mutually commuting shift isometries \(2.27b\).

After promoting \( \lambda_{ab} \) in \((2.27b)\) to \( \tilde{\lambda}_{ab}(\zeta, u) \),
\[ \delta q^{+a} = \tilde{\lambda}_{ab} u^{+b}, \quad \tilde{\lambda}^a_a = 0, \quad (4.1) \]
the constraint \((2.16)\) should be covariantized by introducing three abelian analytic gauge connections \( V^{++(ab)} \)
\[ D^{++} q^{+a} + V^{++(ab)} u^+_b = 0, \quad \delta V^{++(ab)} = D^{++} \tilde{\lambda}^{(ab)}. \quad (4.2) \]
Like in the \( SU(2)_{PG} \) case, one can introduce non-analytic gauge connection \( V^{--(ab)} \),
\[ D^{++} V^{--(ab)} - D^{--} V^{++(ab)} = 0, \quad \delta V^{--(ab)} = D^{--} \tilde{\lambda}^{(ab)}. \quad (4.3) \]
and define the non-analytic superfield \( \hat{q}^{-a} \) as

\[
\hat{q}^{-a} = \nabla^{--}q^{-a} = D^{--}q^{-a} + V^{--(ab)}u_b^+ , \quad \delta \hat{q}^{-a} = \tilde{\Lambda}^a u^b ;
\]

\[
\nabla^{++} \hat{q}^{-a} = D^{++} \hat{q}^{-a} + V^{++(ab)}u_b^- = q^{++a} , \quad \nabla^{--} \hat{q}^{-a} = D^{--} \hat{q}^{-a} + V^{--(ab)}u_b^- = 0.
\] (4.4)

Note that \( \delta \nabla^{--} \hat{q}^{-a} = 0 \), and the last relation in (4.5) follows from the equation \( \nabla^{++} \nabla^{--} \hat{q}^{-a} = D^{++} \nabla^{--} \hat{q}^{-a} = 0 \), which can be easily proved using the constraint in (4.2) and the harmonic flatness condition in (4.3).

Passing to the harmonic projections \( \omega, l^{++} \) and \( V^{++(\pm \pm)}, V^{++(\mp \mp)} \) by the same relations as in the previous Section, we observe that

\[
\delta \omega = -\tilde{\Lambda}^{+-} , \quad \delta l^{++} = -\tilde{\Lambda}^{++} ,
\] (4.6)

where the involved analytic parameters are the proper projections of \( \tilde{\Lambda}^{(ab)} \). Eqs. (4.6) suggest the choice of the manifestly supersymmetric unitary gauge in this case as

\[
\omega = l^{++} = 0.
\] (4.7)

In this gauge, as follows from (4.2),

\[
V^{++(+ +)} = V^{++(+ -)} = 0
\] (4.8)

and the only residual gauge symmetry is

\[
\delta V^{++(--)} = D^{++} \tilde{\Lambda}^{--} , \quad \tilde{\Lambda}^{--} = \tilde{\Lambda}^{--}(\zeta, u).
\] (4.9)

The harmonic flatness condition is reduced to the set

\[
D^{++} V^{--(++)} = 0 , \quad V^{--(++)} = 0 , \quad V^{--(+-)} = 0 , \quad V^{--(--)} = 0.
\] (4.10)

which is just the abelian version of (3.30) - (3.32). We see that, like in the case of \( SU(2)_{PG} \) gauging, the basic object encoding the irreducible field content in the unitary gauge is the analytic superfield \( \mathcal{V} = V^{++(--)} \) with the gauge transformation law \( \delta \mathcal{V} = D^{++} \tilde{\Lambda}^{--} \) which allows one to choose the WZ gauge (3.36) with the \((\mathbf{1}, \mathbf{4}, \mathbf{3})\) off-shell field content. From eqs. (4.10), (4.11) we deduce the expression for \( V^{--(++)} \) in terms of \( \mathcal{V} \) which coincides with the one given in eq. (3.35)

\[
V^{--(++)} \equiv \mathcal{W} = \int d\mathcal{V}.
\] (4.13)

The remaining projections \( V^{--(+-)} \) and \( V^{--(--)} \) can be also expressed through \( \mathcal{V} \) from (4.10) - (4.12); like in the \( SU(2)_{PG} \) case, they seem to be of no need for constructing the corresponding invariant actions. Indeed, the superfield \( V^{--(++)} = \mathcal{W} \) in eq. (4.13) by construction satisfies the constraints (3.39) defining the off-shell multiplet \((\mathbf{1}, \mathbf{4}, \mathbf{3})\) in the ordinary \( \mathcal{N}=4 \) superspace. The most general sigma-model type off-shell action of this multiplet is given by an expression similar to (3.33)

\[
S = \int dt d^4\theta \mathcal{L}(\mathcal{W}).
\] (4.14)
It clearly has the same degree of generality as (3.33) in view of the relation $J = 1/(1 + W)$. However, these actions are obtained by gauging different subclasses of the general action of the superfield $(4, 4, 0)$. While in the previous case one should proceed from the $SU(2)_{PG}$ invariant subclass, which corresponds to the restriction to the Lagrangians (3.13) depending on the only $SU(2)_{PG}$ invariant structure $J = q^+q_\alpha$, in the case considered here we need to start from the subclass possessing an invariance under the shifts (2.27b). It is easy to construct the appropriate unique invariant combination of the superfields $q^+a$ and $q^-a = D^--q^+$:

$$I_0 = q^--u^+_a - q^+u^-_a, \quad D^{++}I_0 = 0.$$

The sigma-model $q^+a$ actions invariant under (2.27b) are then constructed as

$$S_{\text{shift}} = \int \mu H L(I_0, u) = \int dt d^4\theta L'(I_0),$$

where the second form of the action is achievable due to the property that $I_0$ does not depend on harmonics. Passing to the gauge invariant actions is then accomplished by covariantizing the constraint (2.16) as in (4.2) and making the substitution $q^-a \Rightarrow \tilde{q}^-a$ in (4.15) and (4.16):

$$I_0 \Rightarrow I = (D^{--}q^+)u^+ - q^+u^-_a, \quad D^{++}I = 0,$$

$$S_{\text{shift}} \Rightarrow S_{\text{shift}}^{\text{loc}} = \int dt d^4\theta L(I).$$

The unitary gauge (4.7) implies $q^+a = 0$, so the invariant $I$ is reduced just to $W$

$$I = V^{--(++)} = W.$$

Like in the previous case, there exist no WZ-type $q^+a$ actions (2.40) invariant under (2.27b), so the only way of generating potential terms of the eventual $(1, 4, 3)$ multiplet from some gauge invariant actions of the system of superfields $q^+a$ and $V^{++(ab)}$ is the FI term of the gauge superfield. Due to the abelian structure of the gauge group, such term is given, before any gauge-fixing, by

$$S_{\text{shift}}^{\text{FI}} = \int d\zeta^{(-2)} c_{(ab)} V^{++(ab)}, \quad \overline{c_{(ab)}} = c_{(ab)}.$$

Using the constraint (4.2), one can replace $V^{++(++)}$ and $V^{++(+-)}$ by their inverse Higgs expressions

$$V^{++(++)} = -D^{++}l^{++}, \quad V^{++(+-)} = l^{++} - D^{++}\omega.$$

Then, up to a total harmonic derivative, (4.20) can be rewritten as

$$S_{\text{shift}}^{\text{FI}} = \int d\zeta^{(-2)} c^{++} (V - 2\omega).$$

It is still gauge invariant up to a total derivative in the Lagrangian. In the unitary gauge (4.7) it coincides with (3.64) of the $SU(2)_{PG}$ case. The object

$$\tilde{V} = V - 2\omega, \quad \delta\tilde{V} = D^{++}\tilde{\Lambda}^{--}$$

is the abelian analog of the modified gauge connection (3.41).

Despite the formal coincidence of the final outputs in the manifestly supersymmetric unitary gauge in both cases, there is one important difference related to the superconformal invariance.
In the SU(2)$_{PG}$ case, the gauge covariantization preserves the superconformal $D(2, 1; \alpha)$ covariance of the original constraint (2.16). Also, the invariant $J$ has nice superconformal properties both before and after performing the SU(2)$_{PG}$ gauging. As a result, for any $\alpha \neq 0$ there is a one-to-one correspondence between the superconformally invariant sigma-model type actions of $q^{+a}$ and those of the multiplet $(1, 4, 3)$ emerging as a particular gauge of the original $q^{+a}, V^{++(ab)}$ system. In the case of the gauging of three shift isometries, the gauge-covariantized constraint (4.2) breaks the original superconformal invariance for any $\alpha$ except $\alpha = 0$. Also, the superconformal transformations, at any $\alpha \neq 0$, do not take the invariant $I$ into itself, as opposed to the SU(2)$_{PG}$ invariant $J$. On the other hand, staying in the unitary gauge with $I$ as the only object accommodating the irreducible $(1, 4, 3)$ field content, one can forget about the precise $(4, 4, 0)$ origin of this multiplet and construct from $I$ any actions of the multiplet $(1, 4, 3)$, including the superconformally invariant ones described in the previous Section. The property that the same $(1, 4, 3)$ actions can be obtained by gauging two non-equivalent global symmetries realized on the multiplet $(4, 4, 0)$ is in fact one more manifestation of the non-uniqueness of the “oxidizing” procedure which is inverse to gauging. Indeed, given a sigma-model type $(1, 4, 3)$ action, it can be “oxidized” either to the $(4, 4, 0)$ action (3.13) or to (4.16). Only the first oxidation inherits the superconformal invariance (at $\alpha \neq 0$): starting from a superconformally invariant $(1, 4, 3)$ action one arrives at the $(4, 4, 0)$ action which also respects the same superconformal invariance. The second version of the oxidizing procedure generically lacks superconformal covariance.

As an example, let us discuss the covariantization of the free $q^{+a}$ action (2.21) within the alternative gauging under consideration. Like in the previous case, we shall deal with the full superspace form of this action. While in the SU(2)$_{PG}$ case the gauging is accomplished just by the replacement $D^- q^{+a} \Rightarrow \nabla^- q^{+a}$, it is not so in the rigid case, just because even in the rigid case the action (2.21) is invariant under (2.27b) up to a total derivative in the Lagrangian. The gauge-invariant (once again, up to a total harmonic derivative) superfield Lagrangian in this case proves to be as follows

$$L_{\text{gauge}}^{\text{free}} = q^{+a} D^- q^+_a - 2V^{--(ab)} u^+_a q^+_b + 2V^{--(ab)} V^{++(c)} u^+_a u^-_c; \quad (4.24)$$

In the unitary gauge $q^{+a} = 0$, it is simplified to

$$L_{\text{gauge}}^{\text{free}} = 2V^{--(ab)} V^{++(c)} u^+_a u^-_c; \quad (4.25)$$

It is curious that the Lagrangian (4.24) coincides, modulo a total harmonic derivative, with the square of the gauge invariant quantity $I$ defined in (4.17)

$$L_{\text{gauge}}^{\text{free}} = I^2 = [V^{--(++)}]^2; \quad (4.26)$$

where the second equality is valid in the unitary gauge (recall (4.19)). To prove the equivalence of (4.24) and (4.26), it is sufficient to compare their gauge-fixed forms. The r.h.s. in (4.25) can be rewritten as

$$V^{--(++)} V^{++(--)} - V^{--(--)} V^{++(++)} = V^{--(++)} V^{++(--)}; \quad (4.27)$$

where we used the property (4.8) which is valid in the unitary gauge. Then we represent one of two $V^{--(++)}$ in (4.26) as $V^{--(++)} = V^{--(ab)} D^{++} u^+_a u^-_b$, integrate by parts with respect to $D^{++}$, use the relations (4.3), (4.10) and once again (4.8) to reduce $[V^{--(++)}]^2$ just to the form (4.27).
We see that in the shift case the gauging of the free $\mathbf{(4,4,0)}$ action yields the free action of the multiplet $\mathbf{(1,4,3)}$. This should be contrasted with the $SU(2)_{PG}$ gauging which produces from the free $q^{+\alpha}$ action the $\mathbf{(1,4,3)}$ action (3.57) involving a non-trivial self-interaction [7]. This simple example illustrates the non-compatibility of the shift gauging with superconformal invariance at $\alpha \neq 0$: the superconformal symmetry leaving invariant the free $q^{+\alpha}$ action is $D(2,1;\alpha = 1)$, while the free action of the multiplet $\mathbf{(1,4,3)}$ is invariant under $D(2,1;\alpha = -1/2)$. No such an inconsistency takes place in the case of the $SU(2)_{PG}$ gauging.

It is interesting that the only $N=4$ $d=1$ superconformal symmetry which is consistent with the gauging considered here corresponds to the exceptional case $\alpha = 0$ in which $D(2,1;\alpha)$ degenerates into $PSU(1,1|2)$ and an extra $SU(2)$ automorphisms group. Indeed, the covariantized constraint (4.2) is manifestly invariant under the $\alpha = 0$ version of the transformations (3.43) - (3.48). What is even more essential is that the constraint (4.2) is also invariant under the following modified transformation of $q^{+\alpha}$

$$\delta'_{\text{mod}} q^{+\alpha} = 2i\beta(\eta^{a}\bar{\theta}^{+} - \bar{\eta}^{a}\theta^{+}). \quad (4.28)$$

Here, $\beta$ is a constant which can be fixed at any non-zero value by simultaneously rescaling $q^{+\alpha}$, $V^{\pm \pm}$ and the gauge parameters $\Lambda^{(ab)}$. We will choose $\beta = 1$. Note that the possibility of such a modification (missed in [8]) exists already at the rigid level, since the original constraint (2.16) is invariant under such an additional shift. In the unitary gauge, with $\beta = 1$ and taking into account the appropriate compensating gauge transformations, the analytic prepotential $V$ and the superfield $W$ transform as

$$\delta'_{\text{mod}} V = 4i(\bar{\eta}^{-}\theta^{+} - \eta^{-}\bar{\theta}^{+}), \quad \delta'_{\text{mod}} W = 2i(\eta^{i}\bar{\theta}_{i} - \bar{\eta}^{i}\theta_{i}), \quad (4.29)$$

which coincides with the $\alpha = 0$ transformation laws (3.61) and (3.62) of the previous Section. In the $SU(2)_{PG}$ case these transformations cannot be derived from the “first principles”, i.e. prior to imposing any gauge-fixing condition, because the constraint (3.2) is not covariant under (4.28). On the other hand, in the alternative approach where abelian shift symmetries are gauged, this becomes possible since (4.2) is covariant under (4.28). Thus, as regards the superconformal properties, the two different ways of deducing the multiplet $\mathbf{(1,4,3)}$ by gauging three-parameter rigid isometries of the $\mathbf{(4,4,0)}$ multiplet are complementary to each other: the $SU(2)_{PG}$ gauging is compatible with the $D(2,1;\alpha)$ symmetries for all $\alpha \neq 0$, while the second gauging suits for treating the exceptional $\alpha = 0$ case. Note that the gauge invariant quantity $I$ defined in (4.15), (4.17) has the following $\alpha = 0$ transformation properties

$$\delta_{\text{mod}} I = 2i(\eta^{i}\bar{\theta}_{i} - \bar{\eta}^{i}\theta_{i}) \quad (4.30)$$

both in the rigid and local cases, so the superconformally invariant Lagrangian (3.63) of the multiplet $\mathbf{(1,4,3)}$ is obtained via the abelian gauging, $q^{-\alpha} \rightarrow \hat{q}^{-\alpha} = \nabla^{-\alpha} q^{+\alpha}$, of the following particular case of the Lagrangians in (4.16)

$$L_{(sc)}^{\alpha=0} = e^{L_{0}} = e^{-q^{-a}u^{a}_{3}} e^{-q^{+a}u^{a}_{3}} = e^{(q^{a}\xi_{a})}. \quad (4.31)$$

The corresponding $q^{+}$ action is invariant under the $\alpha = 0$ superconformal group $PSU(1,1|2)$. 

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5 Concluding remarks

In this paper, we continued the study of implications of the gauging procedure of ref. [7] in the models of $\mathcal{N}=4$ supersymmetric mechanics. We have shown that the general models associated with the off-shell multiplet $(1, 4, 3)$ can be recovered, in a manifestly supersymmetric superfield form, by gauging certain three-parameter symmetries appearing in special subclasses of the superfield actions of the multiplet $(4, 4, 0)$, thereby confirming the role of the latter as the basic (or “root”) multiplet for constructing various models of $\mathcal{N}=4$ mechanics. We have found a new description of the multiplet $(1, 4, 3)$ in terms of the unconstrained harmonic analytic gauge superfield $V(\zeta, u)$, $\delta V = D^{++}\Lambda^{-+}(\zeta, u)$. Since the multiplets $(4, 4, 0)$, $(3, 4, 1)$ and $(0, 4, 4)$ also admit a natural description as $\mathcal{N}=4$ harmonic analytic superfields [8], we conclude that the $\mathcal{N}=4$, $d=1$ harmonic analytic superspace plays a key role in $\mathcal{N}=4$ mechanics. Actually, the chiral $\mathcal{N}=4$, $d=1$ multiplets $(2, 4, 2)$, both linear [26, 19, 20] and nonlinear [9], also admit an alternative description in terms of $\mathcal{N}=4$ analytic superfields [27]. The new off-shell formulation of the multiplet $(1, 4, 3)$ allowed us to find a new mechanism of generating potential terms for this multiplet and to write simple off-shell couplings of this multiplet to the “mirror” $(1, 4, 3)$ multiplet (which can also be formulated in the $\mathcal{N}=4$, $d=1$ harmonic superspace). Also note that it is easy to couple the $(1, 4, 3)$ multiplet to the off-shell multiplet $(3, 4, 1)$ in the description via the analytic superfield $W(\zeta, u), D^{++}W^{++} = 0$ [8]. The superconformally invariant form of this coupling is given by the following unique analytic superspace integral\(^7\)

$$S_{V-W} \sim \int d\zeta (\zeta^{-2}) (1 + V) W^{++}.$$  \hspace{1cm} (5.1)

It is gauge invariant because of the constraint $D^{++}W^{++} = 0$. In the bosonic sector it yields direct couplings of the physical fields of one multiplet to the auxiliary fields of the other one and can also be used to generate non-trivial scalar potentials in the coupled system of two multiplets after eliminating the auxiliary fields. One can also couple the multiplet $(1, 4, 3)$ to some extra $(4, 4, 0)$ multiplet $Q^a$, $D^{++}Q^a = 0$, via the substitutions $W^{++} \rightarrow Q^a u^+_a$ or $W^{++} \rightarrow Q^a c_{ab} Q^{+b}$ in (5.1) (only the second one preserves the superconformal invariance [8]).

We hope that these findings and new tools will help us to gain further insights into the problem of constructing $\mathcal{N}=4$ extensions of some important bosonic systems, such as the integrable many-component Calogero-type models [25].

In the process of our study we exhibited (in Subsection 2.2) the full set of symmetries inherent to the free superfield action of the multiplet $(4, 4, 0)$. Some of them admit an extension to more general $(4, 4, 0)$ actions (like $SU(2)_{PC}$ (2.23) or its abelian shift analog (2.27b)) while some others do not. In particular, it seems impossible to construct, out of $q^{+a}, q^{-a} = D^{-+}q^{+a}$ and harmonics $u^+_i$, any tensorial invariant of the symmetries associated with the solvable three-generator algebras (2.33) - (2.36). However, even in this case we can get a $(1, 4, 3)$ action with a non-trivial interaction as the result of the appropriate gauging of the free $q^{+a}$ action. Let us end up with an example of such gauging.

For definiteness we choose the symmetry associated with (2.33). Its local version is spanned by the following set of gauge transformations

$$\delta_1 q^{+a} = \Lambda_1 c^a_b q^{+b}, \quad \delta_2 q^{+a} = \Lambda_2 u^{+a}, \quad \delta_3 q^{+a} = \Lambda_3 c^a_b u^{+b}. \hspace{1cm} (5.2)$$

\(^7\)The superconformal invariance can be broken by adding, to the Lagrangian in (5.1), the term $\sim W^{++}$ with an arbitrary coupling constant.
The gauge covariantization of the constraint (2.16) and of $q^{-a} = D^--q^+_a$ can be easily constructed

$$D^{++} q^+_a - V^+_1 c_b q^+ b - V^+_2 u^+_a - V^+_3 c_b u^+_b = 0,$$

(5.3)

$$\nabla^{--} q^{-a} = D^{--} q^{-a} - V^{-1} c_b q^+ b - V^{-2} u^+_a - V^{-3} c_b u^+_b.$$  

(5.4)

Here the gauge potentials are transformed as

$$\delta V^{\pm\pm}_1 = D^{\pm\pm} \Lambda_1, \quad \delta V^{\pm\pm}_2 = D^{\pm\pm} \Lambda_2 - \Lambda_1 V^{\pm\pm}_3 + \Lambda_3 V^{\pm\pm}_1,$$

$$\delta V^{\pm\pm}_3 = D^{\pm\pm} \Lambda_3 + \Lambda_1 V^{\pm\pm}_2 - \Lambda_2 V^{\pm\pm}_1,$$

(5.5)

and satisfy the following zero-curvature conditions

$$D^{++}_1 V^{--}_1 - D^{--}_1 V^{++}_1 = 0,$$

$$D^{++}_2 V^{--}_2 - D^{--}_2 V^{++}_1 + V^{++}_1 V^{--}_2 - V^{++}_3 V^{--}_1 = 0,$$

$$D^{++}_3 V^{--}_2 - D^{--}_3 V^{++}_1 - V^{++}_2 V^{--}_1 + V^{++}_1 V^{--}_1 = 0.$$  

(5.6)

The correct gauge covariantization of the free $q^+_a$ Lagrangian $\sim q^+_a D^{--} q^+_a$ in the present case is given by

$$q^+_a D^{--} q^+_a \Rightarrow q^+_a D^{--} q^+_a - V^{--}_1 (q^+_a c_{ab} q^+_b) - 2V^{--}_2 (q^+_a u^+_a) - 2V^{--}_3 (q^+_a c_{ab} u^+_b) + 2 \left( V^{++}_1 V^{--}_2 - V^{++}_3 V^{--}_3 \right) (u^+_a c_{ab} u^+_b).$$

(5.7)

Under (5.2) this expression transforms into a total harmonic derivative and so is not a tensor. The corresponding action can of course be rewritten as an integral over the analytic superspace. One can also add a FI term

$$\sim \int dud\zeta(-2) V^{++}_1.$$  

(5.8)

After passing to the WZ gauge in (5.3), (5.7) and (5.8) ($V^{\pm\pm}_B = \theta^{\mp}\bar{\theta}^{\pm} A_B$), descending to components, properly fixing the residual 3-parameter gauge freedom and eliminating the auxiliary fields $A_B$, one is left with a non-trivial action of a self-interacting $(1, 4, 3)$ multiplet.

**Acknowledgements**

The work of E.I. was supported in part by the NATO grant PST.GLG.980302, the RFBR grant 06-02-16684, grant INTAS 05-7928 and a grant of Heisenberg-Landau program. He thanks Laboratoire de Physique, ENS Lyon, for the kind hospitality extended to him during the course of this study.

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