ASYMPTOTIC STABILITY OF WAVE EQUATIONS COUPLED BY VELOCITIES

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Abstract. This paper is devoted to study the asymptotic stability of wave equations with constant coefficients coupled by velocities. By using Riesz basis approach, multiplier method and frequency domain approach respectively, we find the sufficient and necessary condition, that the coefficients satisfy, leading to the exponential stability of the system. In addition, we give the optimal decay rate in one dimensional case.

1. Introduction and main results. In this paper, we consider the long time behavior of the solution to a system of wave equations with constant coefficients coupled by velocities. In particular, we want to study what kind of conditions, that the coefficients satisfy, lead to the exponential stability of the system.

In the case of scalar wave equation, there are numerous results on asymptotic stability or stabilization with internal or boundary damping. Cox and Zuazua [7] studied, by Fourier analysis, the energy decay of

\[
\begin{cases}
  u_{tt} - u_{xx} + a(x)u_t = 0 & \text{in } (0,1) \times (0, +\infty), \\
  u(0, t) = u(1, t) = 0 & \text{in } (0, +\infty), \\
  (u, u_t)(0) = (u^0, u^1) & \text{in } (0,1)
\end{cases}
\]  

with indefinite damping. As a corollary, the exponential decay of the energy and its optimal rate were shown in [7] when \( a > 0 \) in \((0,1)\). Using Multiplier method, Alabau-Boussouira [3, 4] and Alabau et al. [2] proved the indirect stabilization of wave systems coupled by displacements, for instance,

\[
\begin{cases}
  u_{tt} - \Delta u + a(x)u_t + b(x)v = 0 & \text{in } \Omega \times (0, +\infty), \\
  v_{tt} - \Delta v - b(x)u = 0 & \text{in } \Omega \times (0, +\infty), \\
  u = v = 0 & \text{on } \Gamma \times (0, +\infty), \\
  (u, u_t)(0) = (u^0, u^1), \ (v, v_t)(0) = (v^0, v^1) & \text{in } \Omega
\end{cases}
\]  

That is, the damping is acted only in one equation and the total energy of the whole system decays polynomially due to the coupling effect. Using the criteria of

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polynomial decay in [16], Liu and Rao proved in [17], by frequency domain approach, the polynomial stability of a partially damped wave system with weak coupling by displacements and multiple propagation speeds. Liu and Rao [18] also proved the indirect stabilization, by Riesz basis approach, with optimal polynomial decay rate for a wave system which is coupled by displacements in one dimensional case.

Recently, Alabau-Boussouira, Wang and Yu [5] obtained the indirect stabilization, combining multiplier method and weighted energy techniques, for the following damped wave system with variable coefficients coupled by velocities

\[
\begin{cases}
    u_{tt} - \Delta u + \rho(x, u_t) + b(x)v_t = 0 \quad &\text{in } \Omega \times (0, +\infty), \\
    v_{tt} - \Delta v - b(x)u_t = 0 \quad &\text{in } \Omega \times (0, +\infty), \\
    u = v = 0 \quad &\text{on } \Gamma \times (0, +\infty), \\
    (u, u_t)(0) = (u^0, u^1), \quad (v, v_t)(0) = (v^0, v^1) \quad &\text{in } \Omega
\end{cases}
\]

The decay speed is shown to change corresponding to the various properties of the nonlinear damping \(\rho(x, u_t)\), especially, if \(\rho(x, u_t) = \alpha u_t \quad (\alpha > 0)\), \(b(x) = b > 0\), the total energy decays exponentially. This phenomenon indicates that the velocity coupling has different impact compared to the displacement coupling. Being regarded as perturbation, the coupling through displacements is compact, while the coupling through velocities is bounded. On the other hand, the controllability (asymptotic stability) of coupled wave systems with general coefficients is closely related to the synchronization (asymptotic synchronization), according to the pioneer results on synchronization by Li and Rao [14, 15] (see also [10]). For the above reasons, we focus on the question: What kind of coefficients can lead to the exponential stability of the general coupled wave system by velocities?

Suppose that \(\Omega\) is a bounded open set in \(\mathbb{R}^d\) with \(C^2\) boundary \(\Gamma = \partial \Omega\). Consider the following general wave system coupled by velocities

\[
\begin{cases}
    u_{tt} - \Delta u + \alpha u_t + \beta v_t = 0 \quad &\text{in } \Omega \times (0, +\infty), \\
    v_{tt} - \Delta v + \gamma u_t + \eta v_t = 0 \quad &\text{in } \Omega \times (0, +\infty), \\
    u = v = 0 \quad &\text{on } \Gamma \times (0, +\infty), \\
    (u, u_t)(0) = (u^0, u^1), \quad (v, v_t)(0) = (v^0, v^1) \quad &\text{in } \Omega
\end{cases}
\]

where \(\alpha, \beta, \gamma, \eta \in \mathbb{R}\) are constants.

**Definition 1.1.** System (4) is said to be exponential stable if there exist constants \(M > 0\) and \(\omega > 0\) such that

\[E(t) \leq Me^{-\omega t}E(0)\]

where the (total) energy is defined by

\[E(t) = \frac{1}{2} \int_\Omega |u_t|^2 + |\nabla u|^2 + |v_t|^2 + |\nabla v|^2 dx\]

and \(\omega\) is the corresponding decay rate.

The task of this paper is to find the conditions that the coefficients matrix

\[
\begin{pmatrix}
    \alpha & \beta \\
    \gamma & \eta
\end{pmatrix}
\]

should satisfy such that System (4) is exponential stable. Our main result is the following theorem:
Theorem 1.2. System (4) is exponential stable if and only if
\[
\begin{cases}
\alpha + \beta > 0, \\
\alpha \eta - \beta \gamma > 0
\end{cases}
\] (7)
i.e., the two eigenvalues of the coefficient matrix \( \begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix} \) both have positive real part.

Remark 1. By Proposition 2, the equivalent form of Theorem 1.2, one can see that the condition (7) means that the two components of the solution are essentially both damped.

The organization of the paper is as follows: In Section 2, we give the well-posedness of the system (4) and reduce the original general problem equivalently into the same problem in two canonical forms (see Proposition 2). Then, we prove Proposition 2 in Section 3 by three different approaches successively, that is, Riesz basis approach, multiplier method, frequency domain approach. Finally, some useful extension and remarks are provided in Section 4.

2. Preliminaries. For the simplicity of statements, let \( \mathcal{L}^2 = \mathcal{L}^2(\Omega), \mathcal{H}_0^1 = \mathcal{H}_0^1(\Omega), \mathcal{H}^2 = \mathcal{H}^2(\Omega), \) and \( \mathcal{H} = \mathcal{H}_0^1 \times \mathcal{L}^2 \times \mathcal{H}_0^1 \times \mathcal{L}^2. \) Then \( \mathcal{H} \) is a complex Hilbert space equipped with the inner product
\[
\langle U, \tilde{U} \rangle_{\mathcal{H}} = \int_{\Omega} \nabla u \cdot \nabla \tilde{u} + \beta \gamma + \nabla v \cdot \nabla \tilde{v} + \alpha \eta \, dx.
\] (8)

Let \( U = (u, y, v, z) \) and let \( U_0 = (u_0, v_0, v_1). \) System (4) can be rewritten as the Cauchy Problem in abstract form:
\[
\frac{dU}{dt} = AU, \quad U(0) = U_0.
\] (9)
where \( A : D(A) \rightarrow \mathcal{H} \) is defined by
\[
AU = (y, \Delta u - \alpha y - \beta z, z, \Delta v - \gamma y - \eta z)
\] (10)
with \( D(A) = (\mathcal{H}^2 \cap \mathcal{H}_0^1) \times \mathcal{H}_0^1 \times (\mathcal{H}^2 \cap \mathcal{H}_0^1) \times \mathcal{H}_0^1. \) Obviously, \( A \) can be regarded as a bounded perturbation to the standard wave operator, namely, \( A = A_0 + A_1, \) where \( A_0U = (y, \Delta u, z, \Delta v) \) with \( D(A_0) = D(A) \) and \( A_1U = (0, -\alpha y - \beta z, 0, -\gamma y - \eta z) \) with \( D(A_1) = \mathcal{H}. \)

By classical Hille-Yoshida Theorem and perturbation theory (see [20, page 76, Theorem 1.1]), we have

Proposition 1. \( A \) generates a \( C_0 \)-semigroup \( \{S(t) = e^{tA}\}_{t \geq 0} \) on \( \mathcal{H}. \) If \( U_0 = (u_0, u_1, v_0, v_1) \in \mathcal{H}, \) System (4) has a unique solution \( U(t) = S(t)U_0 \in C^0([0, +\infty); \mathcal{H}). \) If \( U_0 \in D(A), \) System (4) has a unique solution \( U(t) = S(t)U_0 \in C^0([0, +\infty); D(A)) \cap C^1([0, +\infty); \mathcal{H}). \) Moreover, \( \|U(t)\|_{\mathcal{H}}^2 = 2E(t) \) for all \( t \geq 0. \)

For the sake of convenience of statement, we use real Schur decomposition for the coefficient matrix so that we can reduce the original general problem into the same problem in two canonical forms.

Lemma 2.1. [9, page 79, Theorem 2.3.1] For any \( B = \begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix} \in \mathbb{R}^{2 \times 2}, \) there exists real orthogonal matrix \( P \in \mathbb{R}^{2 \times 2} \) such that \( P^TBP = \tilde{B} \) where \( \tilde{B} \) is in one of the following two canonical forms:
\[\tilde{B} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad (a, b \in \mathbb{R}, b \neq 0)\]
\[\tilde{B} = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \quad (a, b, c \in \mathbb{R}).\]

Moreover, (7) is equivalent to \(a > 0\) in Case i) and to \(a > 0\) and \(c > 0\) in Case ii).

**Lemma 2.2.** Let \(P \in \mathbb{R}^{2 \times 2}\) be a real orthogonal matrix and let \((u, v)\) be solution of (4), then \((\tilde{u}, \tilde{v}) = (u, v)P\) is solution of
\[
\begin{aligned}
\tilde{u}_{tt} - \Delta \tilde{u} + \tilde{\alpha} \tilde{u}_t + \tilde{\beta} \tilde{v}_t &= 0 \quad \text{in} \quad \Omega \times (0, +\infty), \\
\tilde{v}_{tt} - \Delta \tilde{v} + \tilde{\gamma} \tilde{u}_t + \tilde{\eta} \tilde{v}_t &= 0 \quad \text{in} \quad \Omega \times (0, +\infty), \\
\tilde{u} &= \tilde{v} = 0 \quad \text{on} \quad \Gamma \times (0, +\infty), \\
(\tilde{u}, \tilde{v})(0) &= (u^0, v^0)P, \quad (\tilde{u}_t, \tilde{v}_t)(0) = (u^1, v^1)P \quad \text{in} \quad \Omega
\end{aligned}
\]

where \(\tilde{B} = \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{\eta} \end{pmatrix} = P^TBP\) and the energy of (11) is equivalent to that of (4).

Thanks to Lemmas 2.1-2.2, Theorem 1.2 is equivalent to the following Proposition

**Proposition 2.** System (4) is exponential stable if and only if
\[i) \ a > 0 \quad \text{when} \quad B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad (a, b \in \mathbb{R}, b \neq 0);\]
\[ii) \ a > 0 \quad \text{and} \quad c > 0 \quad \text{when} \quad B = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \quad (a, b, c \in \mathbb{R}).\]

3. **Examples.**

4. **Proof of Proposition 2.**

4.1. **Riesz basis approach.** In this subsection, we adopt Riesz basis approach to prove Proposition 2 in one dimensional case, that is \(\Omega = (0, \pi) \subset \mathbb{R}\). The key ingredient is to prove the Riesz basis property and to analyze the spectrum of \(A\). Additionally, we obtain the optimal and explicit decay rate in this situation, see Corollary 1.

In Case i), System (4) can be rewritten as the Cauchy Problem (9) in one dimensional case, where \(A : \mathcal{D}(A) \rightarrow \mathcal{H}\) is defined by
\[
AU = (y, u_{xx} - ay - bz, v_{xx} + by - az) \quad (12)
\]

with \(\mathcal{D}(A) = (\mathcal{H}^2 \cap \mathcal{H}_0^1) \times \mathcal{H}_0^1 \times (\mathcal{H}^2 \cap \mathcal{H}_0^1) \times \mathcal{H}_0^1\).

Let \(\lambda\) be the eigenvalue of \(A\) and \(E = (u, y, v, z) \in \mathcal{D}(A)\) be its eigenvector:
\[
(\lambda I - A)E = 0
\]

It is equivalent to
\[
\begin{aligned}
\lambda u - y &= 0 \quad \text{in} \quad \mathcal{H}_0^1 \\
(\lambda + a)y - u_{xx} + bz &= 0 \quad \text{in} \quad \mathcal{L}^2 \\
\lambda v - z &= 0 \quad \text{in} \quad \mathcal{H}_0^1 \\
(a + \lambda)z - v_{xx} - by &= 0 \quad \text{in} \quad \mathcal{L}^2
\end{aligned}
\]

or further
\[
\begin{aligned}
-u_{xx} + (\lambda + a)\lambda u + b\lambda v &= 0 \quad \text{in} \quad \mathcal{L}^2 \\
-v_{xx} + (\lambda + a)\lambda v - b\lambda u &= 0 \quad \text{in} \quad \mathcal{L}^2 \\
u(0) = u(\pi) = v(0) = v(\pi) &= 0
\end{aligned}
\]
Since $\lambda_1, \lambda_2 \in \mathbb{R}$ are four zeroes of the algebraic equation

$$\nu^4 - 2(\lambda + a)\lambda \nu^2 + [(\lambda + a)^2 + b^2] \lambda^2 = 0$$

and $A_i (i = 1, \cdots, 4)$ are constants to be determined. Then it follows that

$$u(x) = \frac{\sin nx}{n}, \quad \nu = \pm i n \quad (n \in \mathbb{Z}^+)$$

and consequently

$$v(x) = \frac{\lambda^2 + a\lambda + n^2}{-b\lambda} u(x), \quad y(x) = \lambda u(x), \quad z(x) = \lambda v(x)$$

where the eigenvalue $\lambda$ satisfies the characteristic equation

$$\lambda^4 + 2a\lambda^3 + (2n^2 + a^2 + b^2)\lambda^2 + 2an^2\lambda + n^4 = 0$$

or equivalently

$$(\lambda^2 + (a - ib)\lambda + n^2)(\lambda^2 + (a + ib)\lambda + n^2) = 0$$

Consequently, there are two classes of eigenvalues:

$$\lambda_{1,n}^\pm = (\pm X_n - \frac{a}{2}) - i(\pm Y_n - \frac{b}{2}), \quad \lambda_{2,n}^\pm = (\pm X_n - \frac{a}{2}) + i(\pm Y_n - \frac{b}{2})$$

where

$$X_n = \sqrt{\frac{(a^2 - b^2 - 4n^2)^2 + 4a^2b^2 + a^2 - b^2 - 4n^2}{2}}, \quad Y_n = \frac{ab}{X_n}$$

yielding that $X_n$ is a decreasing function of $n \in \mathbb{Z}^+$. It follows that

$$\max_{n \in \mathbb{Z}^+} \{\Re(\lambda_{1,n}^\pm), \Re(\lambda_{2,n}^\pm)\} = \Re(\lambda_{1,1}^+) = \Re(\lambda_{2,1}^+) = X_1 - \frac{a}{2}$$

$$\lambda_{1,n}^\pm \sim \pm in, \quad \lambda_{2,n}^\pm \sim \pm in, \quad \text{as } n \to +\infty$$

The corresponding eigenvectors are

$$E_{1,n}^+ = \sin nx \left( \frac{1}{n}, \frac{\lambda_{1,n}^\pm}{n}, -\frac{i\lambda_{1,n}^\pm}{n} \right), \quad E_{2,n}^+ = \sin nx \left( \frac{1}{n}, \frac{\lambda_{2,n}^\pm}{n}, \frac{i\lambda_{2,n}^\pm}{n} \right)$$

If $a \leq 0$, there exists an eigenvalue with nonnegative real part according to (15)-(16). Then by choosing the corresponding eigenvector as the initial data, it is easy to conclude that System (4) is unstable. If $a > 0$, we have

**Proposition 3.** For any given $a > 0, 0 \neq b \in \mathbb{R}$, \{(E_{1,n}^+, E_{1,n}^-, E_{2,n}^+, E_{2,n}^-)\}_{n \in \mathbb{Z}^+} forms a Riesz basis of the Hilbert space $\mathcal{H}$.

**Proof.** Since $a > 0$ and $b \neq 0$, the system (4) has no multiple eigenvalues, thus \{(E_{1,n}^+, E_{1,n}^-, E_{2,n}^+, E_{2,n}^-)\}_{n \in \mathbb{Z}^+} are linearly independent. Note that

$$e_{1,n}^\pm = \sin nx \left( \frac{1}{n}, \pm i, \frac{1}{n}, \mp 1 \right), \quad e_{2,n}^\pm = \sin nx \left( \frac{1}{n}, \pm i, \frac{1}{n}, \mp 1 \right) \quad (n \in \mathbb{Z}^+)$$
forms a Riesz basis of $\mathcal{H}$ and
\[
\langle e_{i,n}^+, e_{j,m}^+ \rangle_{\mathcal{H}} = \langle e_{i,n}^-, e_{j,m}^- \rangle_{\mathcal{H}} = \langle e_{i,n}^+, e_{j,m}^- \rangle_{\mathcal{H}} = 0, \quad \forall n, m \in \mathbb{Z}^+, n \neq m, i, j = 1, 2
\]
\[
\langle e_{1,n}, e_{2,m} \rangle_{\mathcal{H}} = \langle e_{1,n}, e_{2,m} \rangle_{\mathcal{H}} = \langle e_{1,n}^+, e_{2,m}^+ \rangle_{\mathcal{H}} = 0, \quad \forall n \in \mathbb{Z}^+
\]
Then using the asymptotic expansion of the eigenvalues (18), we get
\[
\|e_{i,n}^+ - E_{i,n}^+\|_H^2 + \|e_{i,n}^- - E_{i,n}^-\|_H^2 = O\left(\frac{1}{n^2}\right) \quad i = 1, 2
\]
and thus
\[
\sum_{n \in \mathbb{Z}^+} \sum_{i=1,2} (\|e_{i,n}^+ - E_{i,n}^+\|_H^2 + \|e_{i,n}^- - E_{i,n}^-\|_H^2) < +\infty
\]
By Lemma A.1, \(\{E_{i,n}^+, E_{i,n}^-, E_{2,n}^+, E_{2,n}^-, E_{1,n}^+, E_{1,n}^-, E_{0,n}^+, E_{0,n}^-, E_{0,n}^+, E_{0,n}^-\}\) forms a Riesz basis of $\mathcal{H}$. \(\square\)

For any given $U_0 \in \mathcal{H}$, by Proposition 3, there exist \(\{\alpha_{i,n}^+, \alpha_{1,n}^-, \alpha_{2,n}^+, \alpha_{2,n}^-\}\) \(n \in \mathbb{Z}^+ \subset \mathbb{C}^4\) such that
\[
U_0 = \sum_{n \in \mathbb{Z}^+} \sum_{i=1,2} (\alpha_{i,n}^+ E_{i,n}^+ + \alpha_{i,n}^- E_{i,n}^-)
\]
and
\[
E(0) = 2\|U_0\|_H^2 = \sum_{n \in \mathbb{Z}^+} \sum_{i=1,2} (|\alpha_{i,n}^+|^2 + |\alpha_{i,n}^-|^2) < +\infty
\]
Moreover, the solution of (9) writes
\[
U(t) = \sum_{n \in \mathbb{Z}^+} \sum_{i=1,2} (\alpha_{i,n}^+ e^{\lambda_{i,n}^+ t} E_{i,n}^+ + \alpha_{i,n}^- e^{\lambda_{i,n}^- t} E_{i,n}^-)
\]
In Case ii), consider the eigenvalue and eigenfunction: \((\lambda I - A)E = 0, \ E \in \mathcal{D}(A)\), namely
\[
(y, u_{xx} - ay, z, v_{xx} - by - cz) = \lambda (u, y, v, z) \in \mathcal{H}
\]
Then the eigenvalue satisfies
\[
\lambda^4 + (a + c)\lambda^3 + (2n^2 + ac)\lambda^2 + (a + c)n^2\lambda + n^4 = 0
\]
or equivalently,
\[
(\lambda^2 + a\lambda + n^2)(\lambda^2 + c\lambda + n^2) = 0
\]
thus
\[
\lambda_{i,n}^\pm = \begin{cases} 
-\frac{a \pm \sqrt{a^2 - 4n^2}}{2}, & \text{if } n \leq \frac{a}{2}, \\
-\frac{a \pm \sqrt{4n^2 - a^2}}{2}, & \text{if } n > \frac{a}{2}
\end{cases}
\]
\[
\lambda_{i,n}^\pm = \begin{cases} 
-\frac{c \pm \sqrt{c^2 - 4n^2}}{2}, & \text{if } n \leq \frac{c}{2}, \\
-\frac{c \pm \sqrt{4n^2 - c^2}}{2}, & \text{if } n > \frac{c}{2}
\end{cases}
\]
Clearly
\[
\max_{n \in \mathbb{Z}^+} \Re(\lambda_{i,n}^\pm) = \Re(\lambda_{i,1}^+) \quad i = 1, 2
\]
\[
\lambda_{1,n}^\pm \sim \pm in, \quad \lambda_{2,n}^\pm \sim \pm in, \quad \text{as } n \to +\infty
\]
Therefore all the eigenvalues \(\{\lambda_{i,n}^+, \lambda_{i,n}^-\}\) \(n \in \mathbb{Z}^+\) have negative real parts if and only if $a > 0$ and $c > 0$. Certainly if $a \leq 0$ or $c \leq 0$, System (4) is not asymptotically stable.
When $a > 0$ and $c > 0$, we would like to prove that the eigenvectors as well as root-vectors form a Riesz basis of $\mathcal{H}$. For this purpose, we discuss the different situations that multiple eigenvalues may appear with various values of $a$ and $c$.

**Case 1.** $a \not\in 2\mathbb{Z}^+$, $c \not\in 2\mathbb{Z}^+$ and $a \neq c$. In this case, $\lambda_{1,n}^+, \lambda_{1,n}^-, \lambda_{2,n}^+, \lambda_{2,n}^-$ are distinct and system (4) is decoupled. The eigenvectors corresponding to $\lambda_{1,n}^+, \lambda_{2,n}^-$ ($n \in \mathbb{Z}^+$) are

$$E_{1,n}^\pm = \sin n x \left( \frac{1}{n}, \frac{\lambda_{1,n}^\pm}{n}, \frac{b}{(a-c)n}, \frac{\lambda_{1,n}^\pm b}{(a-c)n} \right), \quad E_{2,n}^\pm = \sin n x \left( 0, 0, \frac{1}{n}, \frac{\lambda_{2,n}^\pm}{n} \right)$$

Comparing $\{E_{1,n}^\pm, E_{2,n}^\pm\}$ with a Riesz basis of $\mathcal{H}$:

$$e_{1,n}^\pm = \sin n x \left( \frac{1}{n}, \pm i, \frac{b}{(a-c)n}, \frac{\pm i b}{a-c} \right), \quad e_{2,n}^\pm = \sin n x \left( 0, 0, \frac{1}{n}, \pm i \right)$$

yields by (24) and Lemma A.1 that $\{E_{1,n}^+, E_{1,n}^-, E_{2,n}^+, E_{2,n}^-\}_{n \in \mathbb{Z}^+}$ forms a Riesz basis of $\mathcal{H}$. Therefore, every initial data $U_0 \in \mathcal{H}$ can be expanded as (19) with (20) and the corresponding solution is given by (21).

**Case 2.** $a \in 2\mathbb{Z}^+$ or $c \in 2\mathbb{Z}^+$ and $a \neq c$. There are three sub-cases.

**Case 2.1.** $a = 2m \neq c = 2k$, $m, k \in \mathbb{Z}^+$. System (4) has two multiple eigenvalues: $\lambda_{1,m}^+ = -\frac{a}{2} = -m$ and $\lambda_{2,k}^+ = -\frac{c}{2} = -k$. The dimension of the eigen space corresponding to $\lambda_{1,m}^+$ or $\lambda_{2,k}^+$ is one. The corresponding eigen function corresponding to $\lambda_{1,m}^+$ is

$$E_{1,m}^+ = \sin mx \left( \frac{1}{m}, -1, \frac{b}{(a-c)m}, \frac{-b}{a-c} \right)$$

Solving $(\mathcal{A} - \lambda_{1,m}^+ i) E_{1,m}^- = E_{1,m}^+$, we obtain the root vector

$$E_{1,m}^- = \frac{\sin mx}{2m} \left( \frac{1}{m}, 1, \frac{b}{(a-c)m}, \frac{b}{a-c} \right)$$

which is linearly independent of $E_{1,m}^+$. Similarly, we calculate the eigenvector and root vector of $\lambda_{2,k}^+$.

By Lemma A.1, $\{E_{1,n}^+, E_{1,n}^-, E_{2,n}^+, E_{2,n}^-\}_{n \in \mathbb{Z}^+}$ forms a Riesz basis of $\mathcal{H}$. For every $U_0 \in \mathcal{H}$ given by (19) with (20), the solution of (9) is

$$U(t) = \sum_{n \in \mathbb{Z}^+} \sum_{i=1}^2 \left( \alpha_{i,n}^+ e^{\lambda_{i,n}^+ t} E_{i,n}^+ + \alpha_{i,n}^- e^{\lambda_{i,n}^- t} E_{i,n}^- \right) + \alpha_{1,m}^- t e^{\lambda_{1,m}^- t} E_{1,m}^+ + \alpha_{2,k}^- t e^{\lambda_{2,k}^- t} E_{2,k}^+$$

(25)

**Case 2.2.** $a = 2m, m \in \mathbb{Z}^+$ and $c \not\in 2\mathbb{Z}^+$. $\mathcal{A}$ has one multiple eigenvalue $\lambda_{1,m}^+ = -\frac{a}{2} = -m$. For every $U_0 \in \mathcal{H}$ given by (19) with (20), the solution of (9) is

$$U(t) = \sum_{n \in \mathbb{Z}^+} \sum_{i=1}^2 \left( \alpha_{i,n}^+ e^{\lambda_{i,n}^+ t} E_{i,n}^+ + \alpha_{i,n}^- e^{\lambda_{i,n}^- t} E_{i,n}^- \right) + \alpha_{1,m}^- t e^{\lambda_{1,m}^- t} E_{1,m}^+$$

(26)

**Case 2.3.** $c = 2k, k \in \mathbb{Z}^+$ and $a \not\in 2\mathbb{Z}^+$. $\mathcal{A}$ has one multiple eigenvalue $\lambda_{2,k}^+ = -\frac{c}{2} = -k$. Similar to Case 2.2, for every $U_0 \in \mathcal{H}$ given by (19) with (20), the
Case 4. The solution of (9) is

\[ U(t) = \sum_{n \in \mathbb{Z}^+} \sum_{i=1,2} \left( a_{i,n} e^{\lambda_{i,n}^+ t} E_{i,n}^+ + a_{i,n} e^{\lambda_{i,n}^- t} E_{i,n}^- \right) + a_{2,0} e^{\lambda_{2,0}^+ t} E_{2,0}^+ \]  

(27)

Case 3. \( a = c \notin 2\mathbb{Z}^+ \). The two classes of eigenvalues coincide \( \lambda^+_{1,n} = \lambda^+_{2,n} \), \( \lambda^-_{1,n} = \lambda^-_{2,n} \), and the dimension of eigenspace varies with respect to \( b \).

Case 3.1. \( b = 0 \). System (4) is decoupled. The dimension of eigenspace of \( \lambda^+_{1,n} \) is two. The eigenvectors are

\[ E_{1,n}^+ = \sin nx \left( \frac{1}{n}, \lambda^+_{1,n} \right), \quad E_{2,n}^+ = \sin nx \left( 0, 0, \frac{1}{n}, \lambda^+_{1,n} \right) \]

which forms a Riesz basis of \( \mathcal{H} \). For every \( U_0 \in \mathcal{H} \) given by (19) with (20), the solution of (9) is still (21).

Case 3.2. \( b \neq 0 \). The dimension of eigenspace of \( \lambda^+_{1,n}, \lambda^-_{1,n} \) is one. The eigenvector and root vector are

\[ E_{1,n}^+ = \sin nx \left( 0, 0, \frac{1}{n}, \lambda^+_{1,n} \right), \quad E_{2,n}^+ = -\sin nx \left( \frac{a + 2\lambda^+_{1,n}}{b \lambda^+_{1,n}}, \frac{a + 2\lambda^+_{1,n}}{b}, -\frac{1}{2} \lambda^+_{1,n}, -\frac{1}{2} \right) \]

which forms a Riesz basis of \( \mathcal{H} \). For every \( U_0 \in \mathcal{H} \) given by (19) with (20), the solution of (9) is

\[ U(t) = \sum_{n \in \mathbb{Z}^+} \left( \sum_{i=1,2} a_{i,n} e^{\lambda_{i,n}^+ t} E_{i,n}^+ + a_{i,n} e^{\lambda_{i,n}^- t} E_{i,n}^- \right) + a_{2,0} e^{\lambda_{1,n}^+ t} E_{1,n}^+ + a_{2,0} e^{\lambda_{1,n}^- t} E_{1,n}^- \]  

(28)

Case 4. \( a = c = 2m, m \in \mathbb{Z}^+ \). Then \( \lambda^+_{1,n} = \lambda^+_{2,n}, \lambda^-_{1,n} = \lambda^-_{2,n} \) for all \( n \in \mathbb{Z}^+ \).

Case 4.1. \( b = 0 \). System (4) is decoupled. For \( n \neq m \), the algebraic degree of the eigenvalue \( \lambda^+_{1,n} \) is two. The dimension of eigenspace of \( \lambda^+_{1,n} \) is two and the eigenvectors are

\[ E_{1,n}^+ = \sin nx \left( \frac{1}{n}, \lambda^+_{1,n} \right), \quad E_{2,n}^+ = \sin nx \left( 0, 0, \frac{1}{n}, \lambda^+_{2,n} \right) \]

For \( n = m \), the algebraic degree of the eigenvalue \( \lambda^+_{1,m} \) is four. Actually \( \lambda^+_{1,m} = \lambda^+_{2,m} = -m \), and the dimension of its eigenspace is two. The eigenvectors and root eigenvectors are

\[ E_{1,m}^+ = \sin m \left( \frac{1}{m}, -1, 0, 0 \right), \quad E_{2,m}^+ = \sin m \left( 0, 0, \frac{1}{m}, -1 \right) \]

\[ E_{1,m}^- = \sin m \left( \frac{1}{2m}, 1, 0, 0 \right), \quad E_{2,m}^- = \sin m \left( 0, 0, \frac{1}{2m}, 1 \right) \]

For every \( U_0 \in \mathcal{H} \) given by (19) with (20), the solution of (9) is

\[ U(t) = \sum_{n \in \mathbb{Z}^+} \sum_{i=1,2} \left( a_{i,n} e^{\lambda_{i,n}^+ t} E_{i,n}^+ + a_{i,n} e^{\lambda_{i,n}^- t} E_{i,n}^- \right) + a_{2,m} e^{\lambda_{1,m}^+ t} E_{1,m}^+ + a_{2,m} e^{\lambda_{1,m}^- t} E_{1,m}^- \]  

(29)

Case 4.2. \( b \neq 0 \). For \( n \neq m \), the eigenvalues \( \lambda^+_{1,n} \) are the same as in Case 3.2. While for \( n = m \), the dimension of the eigenspace of the eigenvalue \( \lambda^+_{1,m} = \lambda^+_{2,m} = -m \) is
one. We could solve successively the eigenvalue, first order, second order and the third order root vector:

\[ E_{1,m}^+ = \sin \frac{m \pi}{2m} (0, 0, \frac{1}{m}, -1), \quad E_{1,m}^- = \sin \frac{m \pi}{2m} (0, 0, \frac{1}{m}, 1), \]
\[ E_{2,m}^+ = \frac{\sin m \pi}{m} \left( \frac{1}{b m}, -\frac{1}{b}, \frac{1}{2m^2}, 0 \right), \quad E_{2,m}^- = \frac{\sin m \pi}{m^2} \left( \frac{3}{2bm}, -\frac{1}{2b}, \frac{1}{2m^2}, 0 \right). \]

For every \( U_0 \in \mathcal{H} \) given by (20) with (20), the solution of (9) is

\[ U(t) = \sum_{n \in \mathbb{Z}^+} \sum_{i=1,2} \left( \alpha_{i,n}^+ e^{\lambda_{i,n}^+ t} E_{i,n}^+ + \alpha_{i,n}^- e^{\lambda_{i,n}^- t} E_{i,n}^- \right) \]
\[ + \sum_{n \neq m} \left( \alpha_{2,m}^+ e^{\lambda_{2,m}^+ t} E_{2,m}^+ + \alpha_{2,m}^- e^{\lambda_{2,m}^- t} E_{2,m}^- \right) \]
\[ + \alpha_{2,m}^+ e^{\lambda_{2,m}^+ t} (t E_{2,m}^- + \frac{t^2}{2} E_{1,m}^- + \frac{t^3}{3!} E_{1,m}^+) \]
\[ + \alpha_{2,m}^- e^{\lambda_{2,m}^- t} (t E_{1,m}^- + \frac{t^2}{2} E_{1,m}^-) \]
\[ + \alpha_{2,m}^- e^{\lambda_{2,m}^- t} E_{1,m}^+ \]  

(30)

Above all, we easily conclude by (20),(21), (25),(26),(27),(28),(29),(30) that

\[ E(t) = 2 \| U(t) \|_{\mathcal{H}}^2 \leq M E(0) e^{-\omega t}, \quad \forall t \in [0, +\infty) \]  

(31)

for some positive constants \( M, \omega \) independent of the initial data. This is the end of Proof of Proposition 2.

From the above proof of Proposition 2, we get the corollary concerning the decay rate.

**Corollary 1.** i) If \( B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \) \((a > 0, b \in \mathbb{R}, b \neq 0)\); then one has (31) with the optimal decay rate \( \omega = -2 \Re(\lambda_{1,1}^+) = -2 \Re(\lambda_{2,1}^-) \). ii) If \( B = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \) \((a > 0, b \in \mathbb{R}, c > 0)\), then one has

\[ E(t) = 2 \| U(t) \|_{\mathcal{H}}^2 \leq M E(0) p e^{-\omega t}, \quad \forall t \in [0, +\infty). \]  

(32)

with the optimal decay rate \( \omega = -2 \max \{ \Re(\lambda_{1,1}^+), \Re(\lambda_{2,1}^-) \} \) and \( p \)

\[ p = \begin{cases} 1, & \text{if } a = c = 2 \text{ and } b = 0 \text{ or } a = 2 \neq c \text{ or } a \neq 2 = c; \\ 3, & \text{if } a = c = 2 \text{ and } b \neq 0; \\ 0, & \text{else.} \end{cases} \]  

(33)

4.2. **Multiplier method.** In this subsection, we apply the multiplier method to establish the decay estimates of the total energy. The key ingredient is to use an integral inequality (see Lemma A.2) which leads to the exponential decay of the energy. In this subsection, the initial data are assumed to be all real functions.

In Case 1), System (4) is reduced to

\[
\begin{cases}
  u_{tt} - \Delta u + au_t + bv_t = 0 & \text{in } \Omega \times (0, +\infty), \\
  v_{tt} - \Delta v - bu_t + av_t = 0 & \text{in } \Omega \times (0, +\infty), \\
  u = v = 0 & \text{on } \Gamma \times (0, +\infty), \\
  (u, u_t)(0) = (u^0, v^0), \quad (v, v_t)(0) = (v^0, v^1) & \text{in } \Omega.
\end{cases}
\]  

(34)
Using the multiplier $u_t$ to $u$-equation and $v_t$ to $v$-equation, we obtain

$$\int_{\Omega} u_t(u_{ttt} - \Delta u + au_t + bv_t) + v_t(v_{ttt} - \Delta v + av_t - bv_t)dx = 0$$

Integrating by parts and using the definition of the total energy $E(t)$, we get

$$\frac{dE(t)}{dt} = -a \int_{\Omega} u_t^2 + v_t^2 dx \leq 0 \quad (35)$$

which indicates that the energy decays if and only if $a > 0$. Then, we can follow the proof of [12, page 114, Theorem 8.13] to conclude by LaSalle Invariance Principle [13] that System 11 is asymptotically stable if $a > 0$.

Using the multiplier $u$ to $u$-equation and $v$ to $v$-equation, we obtain for $0 \leq S \leq T$ that

$$\int_{S}^{T} \int_{\Omega} u(u_{ttt} - \Delta u + au_t + bv_t) + v(v_{ttt} - \Delta v - bu_t + av_t)dxdt = 0$$

and after integration by parts,

$$\int_{S}^{T} \int_{\Omega} |\nabla u|^2 + |\nabla v|^2 dxdt = \int_{S}^{T} \int_{\Omega} u_t^2 + v_t^2 dxdt - b \int_{S}^{T} \int_{\Omega} u_t - v_t dxdt$$

Thus

$$\int_{S}^{T} \int_{\Omega} |\nabla u|^2 + |\nabla v|^2 dxdt \leq (1 + C\varepsilon) \int_{S}^{T} \int_{\Omega} u_t^2 + v_t^2 dxdt + \varepsilon \int_{S}^{T} \int_{\Omega} u_t^2 + v_t^2 dxdt$$

$$- \int_{S}^{T} \int_{\Omega} u_t + v_t + \frac{a}{2}(u^2 + v^2) dxdt$$

$$\leq (1 + C\varepsilon) \int_{S}^{T} \int_{\Omega} u_t^2 + v_t^2 dxdt$$

$$+ C\varepsilon \int_{S}^{T} \int_{\Omega} |\nabla u|^2 + |\nabla v|^2 dxdt$$

$$+ C(E(T) + E(S))$$

Noting (35) implies for $a > 0$

$$\int_{S}^{T} \int_{\Omega} u_t^2 + v_t^2 dxdt = \frac{1}{a}(E(S) - E(T)) \leq \frac{1}{a}E(S)$$

Choosing $\varepsilon > 0$ small we get

$$\int_{S}^{T} \int_{\Omega} |\nabla u|^2 + |\nabla v|^2 dxdt \leq CE(S)$$
and further
\[ \int_S^T E(s)ds \leq CE(S), \quad \forall 0 \leq S \leq T. \]

Thanks to Lemma A.2, we conclude that \( E(t) \) decays to 0 exponentially.

In Case ii), System (4) is reduced to
\[
\begin{align*}
\begin{cases}
\dot{u} - \Delta u + au_t &= 0 & \text{in } \Omega \times (0, +\infty), \\
\dot{v} - \Delta v + bu_t + cv_t &= 0 & \text{in } \Omega \times (0, +\infty), \\
u &= 0 & \text{on } \Gamma \times (0, +\infty), \\
(u, u_t)(0) = (u^0, u^1), & (v, v_t)(0) = (v^0, v^1) & \text{in } \Omega
\end{cases}
\end{align*}
\]

(36)

It is easy to see that if \( a \leq 0 \), then the energy of \( u \) does not decay; while if \( c \leq 0 \), by taking \( u \equiv 0 \), then the energy of \( v \) does not decay. Hence, the total energy \( E(t) \) of (36) does not decay to 0 if \( a \leq 0 \) or \( c \leq 0 \). It remains to prove that if \( a > 0 \) and \( c > 0 \), \( E(t) \) decays to 0 exponentially.

For \( \kappa > 0 \), we set the equivalent energy
\[
E_\kappa(t) = \frac{1}{2} \int_\Omega \kappa u_t^2 + v_t^2 + \kappa |\nabla u|^2 + |\nabla v|^2 dx
\]
(37)

Using the multiplier \( \kappa u_t \) to \( u \)-equation, \( v_t \) to \( v \)-equation,
\[
\frac{dE_\kappa(t)}{dt} = - \int_\Omega \kappa u_t^2 + bu_t v_t + cv_t^2 dx
\]

Choosing \( \kappa > 0 \) suitably large, namely, \( b^2 - 4\kappa ac < 0 \) then there exists \( \delta > 0 \) such that
\[
\frac{dE_\kappa(t)}{dt} \leq -\delta \int_\Omega \kappa u_t^2 + v_t^2 dx
\]
(38)

Using the multiplier \( \kappa u_t \) to \( u \)-equation, \( v_t \) to \( v \)-equation,
\[
\int_\Omega \kappa u_t + v v_t dx \bigg|_S^T - \int_S^T \int_\Omega \kappa u_t^2 + v_t^2 dxdt + \int_S^T \int_\Omega \kappa |\nabla u|^2 + |\nabla v|^2 dxdt \\
+ \frac{1}{2} \int_\Omega (\kappa a v^2 + c v^2) dx \bigg|_S^T + b \int_S^T \int_\Omega v u_t dxdt = 0
\]

thus,
\[
\int_S^T \int_\Omega \kappa |\nabla u|^2 + |\nabla v|^2 dxdt = \int_S^T \int_\Omega \kappa u_t^2 + v_t^2 dxdt - b \int_S^T \int_\Omega v u_t dxdt \\
- \int_\Omega \kappa u_t + v v_t + \frac{1}{2} (\kappa a v^2 + c v^2) \bigg|_S^T dx
\]

Thanks to Cauchy Inequality and Poincaré Inequality, we get for every \( \varepsilon > 0 \)
\[
\int_S^T \int_\Omega \kappa |\nabla u|^2 + |\nabla v|^2 dxdt \leq \int_S^T \int_\Omega \kappa u_t^2 + v_t^2 dxdt + \int_S^T \int_\Omega \varepsilon v^2 + C_\varepsilon \kappa u_t^2 dxdt \\
- \int_\Omega \kappa u_t + v v_t + \frac{1}{2} (\kappa a v^2 + c v^2) \bigg|_S^T dx \\
\leq (1 + C_\varepsilon) \int_S^T \int_\Omega \kappa u_t^2 + v_t^2 dxdt + C \varepsilon \int_S^T \int_\Omega |\nabla v|^2 dxdt + C(E_\kappa(T) + E_\kappa(S))
\]
Similarly as in Case i), we can prove by (38) and choosing \( \varepsilon > 0 \) small the integral inequality for \( E_{\kappa} \):

\[
\int_{S}^{T} E_{\kappa}(t) dt \leq C E_{\kappa}(S), \quad \forall 0 \leq S \leq T.
\]

Then it follows by Lemma A.2 \( E_{\kappa}(t) \) (or equivalently, \( E(t) \)) decays to 0 exponentially. This ends the proof of Proposition 2.

4.3. Frequency domain approach. In this subsection, we use the frequency domain approach to prove Proposition 2. More precisely, we want to get the exponential stability of the semigroup through the uniform estimate of the resolvent on the imaginary axis by Lemma A.3 [11, 22] (see also [19]).

For \( U_0 = (u^0, v^0, w^0, v^1) \in \mathcal{H} \), the solution of (9) is \( U(t) = S(t)U_0 \) where \( \{S(t)\}_{t \geq 0} \) is the associated \( C_0 \)-semigroup of operator and \( \|S(t)U_0\|_{\mathcal{L}}^2 = 2E(t) \) for all \( t \geq 0 \).

In Case (i), the energy relation (35) yields that the energy decays, or equivalently \( \{S(t)\}_{t \geq 0} \) is a contraction, if and only if \( a > 0 \). Next, we prove Conditions (49) and (50) in Lemma A.3 are satisfied for \( a > 0 \).

We start, by contradiction arguments, to assume that (49) does not hold, i.e., there exists \( \xi \in \mathbb{R} \) and \( U = (u, y, v, z) \in \mathcal{D}(\mathcal{A}) \) with \( \|U\|_{\mathcal{H}} = 1 \) such that \( (i\xi - \mathcal{A})U = 0 \), namely,

\[
\begin{cases}
  i\xi u - y = 0 & \text{in } \mathcal{H}_0^1 \\
  (i\xi + a)y - \Delta u + b z = 0 & \text{in } L^2 \\
  i\xi v - z = 0 & \text{in } \mathcal{H}_0^1 \\
  (i\xi + a)z - \Delta v - by = 0 & \text{in } L^2
\end{cases}
\]

By definition (8), we easily calculate

\[
\Re((i\xi - \mathcal{A})U, U)_{\mathcal{H}} = a(||y||_{L^2}^2 + ||z||_{L^2}^2) = 0
\]  

(39)

Then it follows that \( y = z = 0 \) in \( L^2 \) and thus

\[
\begin{cases}
  i\xi u = 0 & \text{in } L^2 \\
  \Delta u = 0 & \text{in } L^2 \\
  i\xi v = 0 & \text{in } L^2 \\
  \Delta v = 0 & \text{in } L^2
\end{cases}
\]

The theory of elliptic equation with Dirichlet boundary condition implies that \( u = v = 0 \in \mathcal{H}^2 \cap \mathcal{H}_0^1 \). Consequently we have \( U = 0 \in \mathcal{H} \) which contradicts with \( \|U\|_{\mathcal{H}} = 1 \). Therefore (49) is satisfied for \( a > 0 \).

Next we continue to prove that (50) holds for \( a > 0 \). Otherwise, thanks to the continuity of the resolvent \( \Re(i\xi, \mathcal{A}) \) with respect to \( \xi \in \mathbb{R} \), (50) is not valid at \( \infty \), i.e., there exists \( \{U_n = (u_n, y_n, v_n, z_n)\}_{n \in \mathbb{N}^\ast} \subset \mathcal{D}(\mathcal{A}) \) with \( \|U_n\|_{\mathcal{H}} = 1 \) and \( \{\xi_n\}_{n \in \mathbb{N}^\ast} \subset \mathbb{R} \) with \( |\xi_n| \to +\infty \), \( \|U_n\|_{\mathcal{H}} \to 0 \) as \( n \to +\infty \), namely,

\[
\begin{cases}
  i\xi_n u_n - y_n \to 0 & \text{in } \mathcal{H}_0^1 \\
  (i\xi_n + a)y_n - \Delta u_n + b z_n \to 0 & \text{in } L^2 \\
  i\xi_n v_n - z_n \to 0 & \text{in } \mathcal{H}_0^1 \\
  (i\xi_n + a)z_n - \Delta v_n - by_n \to 0 & \text{in } L^2
\end{cases}
\]

Similarly as (39), we have

\[
\Re((i\xi - \mathcal{A})U_n, U_n)_{\mathcal{H}} = a(||y_n||_{L^2}^2 + ||z_n||_{L^2}^2)
\]  

(40)
On the other hand,
\[ |\Re(\langle i\xi_n - A \rangle U_n, U_n)_{\mathcal{H}}| \leq \|\langle i\xi_n - A \rangle U_n\|_{\mathcal{H}} \cdot \|U_n\|_{\mathcal{H}} = \|\langle i\xi_n - A \rangle U_n\|_{\mathcal{H}} \to 0. \]

Then \( z_n \to 0 \in L^2 \), \( y_n \to 0 \in L^2 \), and thus
\[
\begin{align*}
& \langle i\xi_n u_n \rangle \to 0 \quad \text{in} \quad L^2 \\
& \langle \Delta u_n \rangle \to 0 \quad \text{in} \quad L^2 \\
& \langle i\xi_n v_n \rangle \to 0 \quad \text{in} \quad L^2 \\
& \langle \Delta v_n \rangle \to 0 \quad \text{in} \quad L^2 
\end{align*}
\]

Thanks to Nirenberg Inequality [1, page 135, Theorem 5.2], we get
\[ \|\nabla u_n\|_{L^2} \leq C \|\Delta u_n\|_{L^2} \|u_n\|_{L^2} \leq C \|\frac{\Delta u_n}{\xi_n}\|_{L^2} \|\xi_n u_n\|_{L^2} \to 0 \]
and in a same way \( \nabla v_n \to 0 \in L^2 \). Consequently \( \|U_n\|_{\mathcal{H}} \to 0 \) as \( n \to +\infty \). This
contradicts with \( \|U_n\|_{\mathcal{H}} = 1 \) and implies (50) for \( a > 0 \). The exponential decay of
the semigroup of operator \( \{S(t)\}_{t \geq 0} \) associated to (9) is a consequence of Lemma
A.3.

In Case (ii), it is easy to see, as in Section 3.2, that if \( a \leq 0 \) or \( c \leq 0 \), system (36)
is not asymptotically stable. It remains to prove that (36) is exponentially stable if
\( a > 0 \) and \( c > 0 \).

We introduce an equivalent inner product of (8),
\[ \langle U, \tilde{U} \rangle_{\mathcal{H}_\kappa} = \int_\Omega \kappa \nabla u \cdot \nabla \tilde{u} + \kappa \tilde{y} \cdot \tilde{y} + \nabla \tilde{v} \cdot \nabla \tilde{v} + \pi \tilde{z} \, dx, \]
(41)

where \( \kappa > 0 \). Clearly, \( \mathcal{H} \) is a Hilbert space under \( \langle \cdot, \cdot \rangle_{\mathcal{H}_\kappa} \) and is denoted by \( \mathcal{H}_\kappa \)
with the corresponding norm \( \| \cdot \|_{\mathcal{H}_\kappa} \). For \( U_0 = (w^0, u^1, v^1) \in \mathcal{H}_\kappa \), the solution
of (9) is \( U(t) = S(t)U_0 \) where \( \{S(t)\}_{t \geq 0} \) is the associated \( C_0 \)-semigroup of operator
and \( \|S(t)U_0\|_{\mathcal{H}_\kappa}^2 = 2E_k(t) \) for all \( t \geq 0 \). By Lemma A.3, it suffices to prove that
Conditions (49) and (50) hold for \( a > 0 \) and \( c > 0 \).

We start, by contradiction arguments, to assume that (49) does not hold, i.e.,
there exists \( \xi \in \mathbb{R} \) and \( U = (u, v, y, z) \in \mathcal{D}(A) \) with \( \|U\|_{\mathcal{H}_\kappa} = 1 \) such that
\( (i\xi - A)U = 0 \), namely,
\[
\begin{align*}
& (i\xi - y) = 0 \quad \text{in} \quad \mathcal{H}_0^1 \\
& (i\xi + a)y - \Delta u = 0 \quad \text{in} \quad L^2 \\
& i\xi v - z = 0 \quad \text{in} \quad \mathcal{H}_0^1 \\
& (i\xi + c)z - \Delta v + by = 0 \quad \text{in} \quad L^2 
\end{align*}
\]

We compute
\[ \Re(\langle i\xi - A \rangle U, U)_{\mathcal{H}_\kappa} = a\kappa \|y\|_{L^2}^2 + c\|z\|_{L^2}^2 + b\Re(y, z)_{L^2} = 0. \]
(42)
Taking \( \kappa > 0 \) suitably large, namely, \( b^2 - 4\kappa ac < 0 \) then there exists \( \delta > 0 \) such that
\[ 0 = \Re(\langle i\xi - A \rangle U, U)_{\mathcal{H}_\kappa} \geq \delta (\|y\|_{L^2}^2 + \|z\|_{L^2}^2). \]
(43)
Then \( y = z = 0 \in \mathcal{L}^2 \) and thus
\[
\begin{aligned}
&u = 0 \quad \text{in} \quad \mathcal{L}^2 \\
&\Delta u = 0 \quad \text{in} \quad \mathcal{L}^2 \\
&v = 0 \quad \text{in} \quad \mathcal{L}^2 \\
&\nabla v = 0 \quad \text{in} \quad \mathcal{L}^2 
\end{aligned}
\]

Obviously the theory of elliptic equation with Dirichlet boundary condition implies that \( u = v = 0 \in \mathcal{H}^2 \cap \mathcal{H}^1_c \). Consequently we have \( U = 0 \in \mathcal{H}_n \) which contradicts with \( \|U\|_{\mathcal{H}_n} = 1 \) and yields that (49) is satisfied for \( a > 0 \) and \( c > 0 \).

Next we continue to prove that (50) holds for \( a > 0 \). Otherwise, thanks to the continuity of the resolvent \( \mathcal{R}(i\xi, A) \) with respect to \( \xi \in \mathbb{R} \), (50) is not valid at \( \infty \), i.e., there exists \( \{U_n = (u_n, y_n, v_n, z_n)\}_{n \in \mathbb{Z}^+} \subset \mathcal{D}(A) \) with \( \|U_n\|_{\mathcal{H}_n} = 1 \) and \( \{\xi_n\}_{n \in \mathbb{Z}^+} \subset \mathbb{R} \) with \( |\xi_n| \to +\infty \), \( \|i\xi_n - A\|_{\mathcal{H}_n} \to 0 \) as \( n \to +\infty \), namely,
\[
\begin{aligned}
&i\xi_n u_n - y_n \to 0 \quad \text{in} \quad \mathcal{H}_0^1 \\
&(i\xi_n + a)y_n - \Delta u_n \to 0 \quad \text{in} \quad \mathcal{L}^2 \\
&i\xi_n v_n - z_n \to 0 \quad \text{in} \quad \mathcal{H}_0^1 \\
&(i\xi_n + c)z_n - \Delta v_n + by_n \to 0 \quad \text{in} \quad \mathcal{L}^2 
\end{aligned}
\]

Let \( \kappa > \frac{b^2}{4ac} \). We get similarly as (42) and (43) that
\[
\Re((i\xi_n - A)U_n, U_n)_{\mathcal{H}_n} = \alpha \kappa \|y_n\|_{\mathcal{L}^2}^2 + c\|z_n\|_{\mathcal{L}^2}^2 + b\Re(y_n, z_n)_{\mathcal{L}^2} \geq \delta(\|y_n\|_{\mathcal{L}^2}^2 + \|z_n\|_{\mathcal{L}^2}^2) \tag{44}
\]
for some \( \delta > 0 \). On the other hand,
\[
|\Re((i\xi_n - A)U_n, U_n)_{\mathcal{H}_n}| \leq \|i\xi_n - A\|_{\mathcal{H}_n} \cdot \|V_n\|_{\mathcal{H}_n} = \|i\xi_n - A\|_{\mathcal{H}_n} \to 0.
\]
Hence we have \( y_n \to 0 \in \mathcal{L}^2, z_n \to 0 \in \mathcal{L}^2 \) and thus
\[
\begin{aligned}
&i\xi_n u_n \to 0 \quad \text{in} \quad \mathcal{L}^2 \\
&\Delta u_n \to 0 \quad \text{in} \quad \mathcal{L}^2 \\
&i\xi_n v_n \to 0 \quad \text{in} \quad \mathcal{L}^2 \\
&\Delta v_n \to 0 \quad \text{in} \quad \mathcal{L}^2 
\end{aligned}
\]

Thanks to Nirenberg Inequality [1, page 135, Theorem 5.2], we get
\[
\|\nabla u_n\|_{\mathcal{L}^2} \leq C\|\Delta u_n\|_{\mathcal{L}^2}^{\frac{1}{2}}\|u_n\|_{\mathcal{L}^2}^{\frac{1}{2}} \leq C\|\Delta u_n\|_{\mathcal{H}_n}^{\frac{1}{2}}\|\xi_n u_n\|_{\mathcal{L}^2}^{\frac{1}{2}} \to 0
\]
and in a same way \( \nabla v_n \to 0 \in \mathcal{L}^2 \). Consequently \( \|U_n\|_{\mathcal{H}_n} \to 0 \) as \( n \to +\infty \). This contradicts with \( \|U_n\|_{\mathcal{H}_n} = 1 \) and implies (50) for \( a > 0 \) and \( c > 0 \). The exponential decay of the semigroup \( \{S(t)\}_{t \geq 0} \) associated to (9) is again a consequence of Lemma A.3.

5. Extension and remarks. In this section, we give several remarks as extension of the main results. They can be proved by the above-mentioned classical approaches without essential difficulties, thus we omit the proof.
Remark 2. If the boundary conditions in (4) are replaced by mixed types of boundary conditions, including Neumann and Robin conditions, we can adopt the above three approaches to conclude that (7) is still the sufficient and necessary condition of exponential stability of the system. For instance,
\[
\begin{align*}
  u_{tt} - \Delta u + \alpha u_t + \beta v_t &= 0 & \text{in } \Omega \times (0, +\infty), \\
v_{tt} - \Delta v + \gamma u_t + \eta v_t &= 0 & \text{in } \Omega \times (0, +\infty), \\
v &= 0 & \text{on } \Gamma_0 \times (0, +\infty), \\
\frac{\partial u}{\partial \nu} + \sigma u - \mu v &= 0 & \text{on } \Gamma_1 \times (0, +\infty), \\
(u, u_t)(0) &= (u^0, u^1), & (v, v_t)(0) &= (v^0, v^1) & \text{in } \Omega
\end{align*}
\]
where \( \Gamma_0 \cup \Gamma_1 = \partial \Omega, \Gamma_0 \cap \Gamma_1 = \emptyset \) and \( \sigma \geq 0, \nu \geq 0 \).

Remark 3. If the damping and the coupling are both acted on the boundary, we can adopt the multiplier method to conclude that (7) is still the sufficient and necessary condition of exponential stability of the following system
\[
\begin{align*}
  u_{tt} - \Delta u &= 0 & \text{in } \Omega \times (0, +\infty), \\
v_{tt} - \Delta v &= 0 & \text{in } \Omega \times (0, +\infty), \\
u &= v = 0 & \text{on } \Gamma_0 \times (0, +\infty), \\
\frac{\partial u}{\partial \nu} + \alpha u_t + \beta v_t &= 0 & \text{on } \Gamma_1 \times (0, +\infty), \\
\frac{\partial v}{\partial \nu} + \gamma u_t + \eta v_t &= 0 & \text{on } \Gamma_1 \times (0, +\infty), \\
(u, u_t)(0) &= (u^0, u^1), & (v, v_t)(0) &= (v^0, v^1) & \text{in } \Omega
\end{align*}
\]
where \( \Gamma_0 \cup \Gamma_1 = \partial \Omega, \Gamma_0 \cap \Gamma_1 = \emptyset \) with some suitable geometric conditions (see [6, 12]). For instance, one can assume that there exists \( x_0 \in \mathbb{R}^n \), such that \( \Gamma_0 = \{ x \in \partial \Omega | (x-x_0) \cdot \nu(x) \leq 0 \} \) and \( \Gamma_1 = \{ x \in \partial \Omega | (x-x_0) \cdot \nu(x) > 0 \} \neq \emptyset \) where \( \nu(x) \) is the unit outer normal filed.

Remark 4. If we consider a wave system with multiple propagation speeds
\[
\begin{align*}
  u_{tt} - \Delta u + \alpha u_t + \beta v_t &= 0 & \text{in } \Omega \times (0, +\infty), \\
v_{tt} - c^2 \Delta v + \gamma u_t + \eta v_t &= 0 & \text{in } \Omega \times (0, +\infty), \\
u &= v = 0 & \text{on } \Gamma \times (0, +\infty), \\
(u, u_t)(0) &= (u^0, u^1), & (v, v_t)(0) &= (v^0, v^1) & \text{in } \Omega
\end{align*}
\]
with \( c \neq 1 \), Proposition 2 still holds for (47) if the coefficient matrix \( B = \begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix} \) or \( B^T \) is of the two canonical forms. As for the general situations, we have the following sufficient conditions for the exponential stability of (47):
\[
\begin{align*}
\alpha &> 0, \\
\eta &> 0, \\
\alpha \eta - \beta \gamma &> 0
\end{align*}
\]
Sketch of proof of Remark 4. We give a sketch of proof by multiplier method. If the coefficients \( B = \begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix} \) or \( B^T \) is of the two canonical forms, we just follow the
proof of Proposition 2. As for the general situations apart from the two canonical cases we set

\[ E_k(t) = \frac{1}{2} \int_\Omega k u_t^2 + k|\nabla u|^2 + v_t^2 + \epsilon^2 |\nabla v|^2 \, dx \]

with \( k > 0 \). Using the multiplier \( ku_t \) to \( u \)-equation, \( v_t \) to \( v \)-equation, we easily arrive at

\[ \frac{dE_k(t)}{dt} = -\int_\Omega k\alpha u_t^2 + (k\beta + \gamma)u_tv_t + \eta v_t^2 \, dx \]

To prove the exponential stability of (47), it suffices (but is not necessary) to prove that there exist \( k > 0 \) and \( \delta > 0 \) such that

\[ k\alpha p^2 + (k\beta + \gamma)pq + \eta q^2 \geq \delta(p^2 + q^2), \quad \forall p, q \in \mathbb{R} \]

which is equivalent to the existence of \( k > 0 \) such that

\[ (k\beta + \gamma)^2 - 4k\alpha \eta < 0 \]

with \( \alpha > 0, \eta > 0 \). Since \( \beta\gamma \neq 0 \) and \( \alpha\eta - \beta\gamma > 0 \), we get, by taking \( k = \frac{2\alpha\eta - \beta\gamma}{\beta^2} > 0 \), that

\[ (k\beta + \gamma)^2 - 4k\alpha \eta = \beta^2 \left( k - \frac{2\alpha\eta - \beta\gamma}{\beta^2} \right)^2 + \frac{4\alpha\eta(\beta\gamma - \alpha\eta)}{\beta^2} = \frac{4\alpha\eta(\beta\gamma - \alpha\eta)}{\beta^2} < 0 \]

It follows that

\[ \frac{dE_k(t)}{dt} = -\delta \int_\Omega u_t^2 + v_t^2 \, dx \]

for some \( \delta > 0 \). Then one can follow the arguments as we did for (36) to prove the integral inequality for \( E_k \):

\[ \int_S^T E_k(t) \, dt \leq CE_k(S), \quad \forall 0 \leq S \leq T. \]

The exponential decay of \( E_k(t) \) and that of \( E(t) \) are the consequence of Lemma A.2.

\[ \square \]

**Remark 5** (Comparison of Three Approaches). The advantage of Riesz basis approach is to give the explicit expression of the solution so that the relation between the exponential stability of the system and the coefficients can be relatively easy to discover. In addition, the optimal decay rate can be obtained through very careful analysis of both the spectrum and the eigenvectors. However, the Riesz basis properties of the eigenvectors are hard to check in higher dimensional case. The multiplier method is rather simple to apply without restrains in space dimension and works for variable coefficients case (even for nonlinear problems), but it requires strong geometric assumptions. The frequency domain approach is also applicable for all space dimension and for the variable coefficients case without much information of the eigenvalues and eigenvectors. Nevertheless, the optimal decay rate can not be obtained in general by multiplier method or by frequency domain approach.

**Appendix A. Appendix.**

**Lemma A.1.** Let \( \{e_n\}_{n \in \mathbb{Z}^+} \) be a Riesz basis on Hilbert space \( \mathcal{H} \). Let \( \{E_n\}_{n \in \mathbb{Z}^+} \) be a basis on \( \mathcal{H} \). If \( \sum_{n \in \mathbb{Z}^+} ||E_n - e_n||^2 < +\infty \), then \( \{E_n\}_{n \in \mathbb{Z}^+} \) is a Riesz basis on \( \mathcal{H} \).

Lemma A.1 is an equivalent form of Bari’s Theorem, see [8, page 317, Theorem 2.3]. The special case that \( \{e_n\}_{n \in \mathbb{Z}^+} \) is an orthogonal Riesz basis of \( \mathcal{H} \) can be found as [21, App. D, Theorem 3].
Lemma A.2. [12, page 103, Theorem 8.1] Let \( E : [0, +\infty) \to [0, +\infty) \) be a non-increasing function. Suppose that there exists \( C > 0 \) such that
\[
\int_S^T E(s) \, ds \leq CE(S), \quad \forall 0 \leq S \leq T,
\]
then
\[
E(t) \leq E(0) e^{-t/C}, \quad \forall t \geq C
\]

Lemma A.3. [19, page 4, Theorem 1.3.2] Let \( \{S(t) = e^tA\}_{t \geq 0} \) be a \( C_0 \)-semigroup of contractions on a Hilbert space \( \mathcal{H} \). Then \( \{S(t)\}_{t \geq 0} \) is exponentially stable if and only if the following conditions are satisfied
\[
\begin{align}
(1) & \quad \mathbb{i}\mathbb{R} = \{i\xi \mid \xi \in \mathbb{R}\} \subset \rho(A); \\
(2) & \quad \limsup_{|\xi| \to +\infty} ||R(i\xi, A)|| < +\infty.
\end{align}
\]

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