Real-normalized Whitham hierarchies and the WDVV equations

Anton Dzhamay
Department of Mathematics
Columbia University
New York, NY 10027
e-mail: ad@math.columbia.edu
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Abstract

In this paper we present a construction of a new class of explicit solutions to the WDVV (or associativity) equations. Our construction is based on a relationship between the WDVV equations and Whitham (or modulation) equations. Whitham equations appear in the perturbation theory of exact algebro-geometric solutions of soliton equations and are defined on the moduli space of algebraic curves with some extra algebro-geometric data. It was first observed by Krichever that for curves of genus zero the $\tau$-function of a “universal” Whitham hierarchy gives a solution to the WDVV equations. This construction was later extended by Dubrovin and Krichever to algebraic curves of higher genus. Such extension depends on the choice of a normalization for the corresponding Whitham differentials. Traditionally only complex normalization (or the normalization w.r.t. $\alpha$-cycles) was considered. In this paper we generalize the above construction to the real-normalized case.

Introduction

In the beginning of the 90’s, while studying deformations of 2D topological field theories (TFT), E. Witten, R. Dijkgraaf, E. Verlinde, and H. Verlinde ([DVV91b, DVV91a, Wit90]) wrote down the following overdetermined system of non-linear PDEs for a function $F(t) = F(t_0, t_1, \ldots)$:

\[
F_{\alpha\beta\lambda} (F_{0\lambda\mu})^{-1} F_{\mu\gamma\delta} = F_{\delta\beta\lambda} (F_{0\lambda\mu})^{-1} F_{\mu\gamma\alpha},
\]

(WDVV)

where $F_{\alpha} = \frac{\partial F}{\partial t_{\alpha}}$, and the matrix $\eta_{\alpha\beta} = F_{0\alpha\beta}$ is constant and non-degenerate. These equations are now called the WDVV equations. They appear in the following way [Dij98].

One can show that the structure of a 2D TFT is equivalent to a structure of a Frobenius algebra $A$, i.e., a commutative associative algebra with a unit and a symmetric non-degenerate
bilinear form $\eta$ such that $\eta(a \ast b, d) = \eta(a, b \ast d)$. Let $\{\phi_\alpha\}$ be a basis of $A$ ($\phi_\alpha$ correspond to primary fields in TFT) and let $n_{\alpha\beta} = \langle \phi_\alpha, \phi_\beta \rangle = \langle \phi_\alpha \phi_\beta \rangle_0$, $c_{\alpha\beta\gamma} = \eta(\phi_\alpha, \phi_\beta \ast \phi_\gamma) = \langle \phi_\alpha \phi_\beta \phi_\gamma \rangle_0$. Deformations of a TFT structure considered in [DVV91b, DVV91a, Wit90] correspond to potential deformations of a Frobenius algebra, i.e., there should exist a function $F(t)$, called the WDVV potential, such that $n_{\alpha\beta} = \partial_{\alpha\beta} F(t)$ and $c_{\alpha\beta\gamma}(t) = \partial_{\alpha\beta\gamma} F(t)$. Then the WDVV equations are just the associativity conditions for the deformed algebra structure.

It is now clear that the theory of the WDVV equations and its differential-geometric counterpart, the theory of Frobenius manifolds, are related to a whole spectrum of applications ranging from classical differential geometry (Darboux-Egoroff metrics, $n$-orthogonal curvilinear coordinate systems, deformations of singularities) to quantum cohomology, Gromov-Witten invariants, integrable hierarchies, and the Seiberg-Witten equations.

Solutions of the WDVV equations that can be obtained from the theory of Whitham hierarchies correspond to the topological Landau-Ginzburg theories and minimal models. For $A_n$ minimal models, a Frobenius algebra structure is defined on the space of degree $n - 2$ polynomials in $p$ with the help of the superpotential $W(p) = \frac{p^n}{n}$ by

$$u \ast v = u(p)v(p) \mod W'(p),$$

$$\langle uv \rangle = - \text{res}_\infty \frac{u(p)v(p)}{W'(p)} dp.$$ 

A Frobenius algebra structure is then deformed by deforming the superpotential $W$,

$$W(p|a) = \frac{p^n}{n} + a_{n-2} p^{n-2} + \cdots + a_1 p + a_0,$$

where we used the notation $W(p|a)$ to separate the variable $p$ from the deformation parameters $a_{n-2}, \ldots, a_0$. Let $\xi(p)$ be an $n$th root of $W(p|a)$ in the neighborhood of infinity,

$$\xi^n(p) = nW(p) = p^n + \cdots, \quad \xi(p) = p + O(p^{-1}).$$

Then the deformed basis of primary fields $\phi_\alpha$ is given by

$$(1) \quad \phi_\alpha(p|t) = \frac{1}{\alpha + 1} \frac{d\Omega_{\alpha+1}(p|t)}{dp} = p^\alpha + \cdots,$$

where $\Omega_\alpha(p) = [\xi^\alpha(p)]_+$ is a principal (i.e., polynomial) part of the Laurent expansion of $\xi^\alpha$ in the neighborhood of infinity. The flat coordinates $t_\alpha$ on the space of the deformation parameters are given by $t_\alpha = - \text{res}_\infty \xi^{-(\alpha+1)} p dW$, and $\phi_\alpha(p|t) = - \partial_{t_\alpha} W(p|t)$. Then

$$n_{\alpha\beta} = \langle \phi_\alpha, \phi_\beta \rangle = - \text{res}_\infty \frac{\phi_\alpha \phi_\beta}{W'(p)} dp = \delta_{\alpha+\beta,n-2},$$

$$c_{\alpha\beta\gamma}(t) = (\text{coeff}) \sum_{q_\alpha | dW(q_\alpha) = 0} \text{res}_{q_\alpha} \frac{d\Omega_{\alpha+1} d\Omega_{\beta+1} d\Omega_{\gamma+1}}{dp dW}.$$ 

In [DVV91b] it was shown that $c_{\alpha\beta\gamma}(t)$ satisfy the integrability conditions and therefore $c_{\alpha\beta\gamma}(t) = \partial_{\alpha\beta\gamma} F(t)$. The closed expression for this WDVV potential $F(t)$ was identified by
I. Krichever [Kri92] with the logarithm of the \( \tau \)-function of a certain reduction of the genus zero Whitham hierarchy.

Whitham equations (or modulation equations) appear in the theory of perturbations of exact algebro-geometric solutions of soliton equations and describe “slow drift” on the moduli space of algebro-geometric data. These equations can be defined with the help of certain differentials \( d\Omega_i \) on the universal curve, each differential generating a corresponding Whitham flow on the moduli. A large class of solutions of the Whitham hierarchy is given by the so-called algebraic orbits. Such solutions depend only on finitely many parameters and can be constructed as follows. One starts with a finite-dimensional moduli space, called the universal configuration space, that consists of a curve \( \Gamma \), punctures \( P_{\alpha} \), and a pair of Abelian integrals \( E \) and \( Q \). Then one defines special coordinates on this space (Whitham times), picks a leaf defined by fixing some of them, and maps this leaf to the moduli space of algebro-geometric data in such a way that coordinate lines go to Whitham flows. All information about such algebraic orbit can be encoded in a single function \( \tau(t) \) that depends only on the moduli. This function is called the \( \tau \)-function of an algebraic orbit.

In the genus zero case Abelian integrals become polynomials, and if we choose an algebraic orbit with \( Q = p \), then \( E \) can be identified (up to normalization) with the superpotential \( W \), \( \Omega_i \) define a basis of the corresponding Frobenius algebra, Whitham times \( t_i \) give flat coordinates on the orbit, and \( F(t) = \ln \tau(t) \) is a WDVV potential.

This approach can be generalized to moduli spaces of curves of higher genus. One new feature of a higher genus case is a more complicated topology of the moduli space. As a result, the hierarchy has to be extended to include certain (multivalued) differentials \( d\Omega_A \) that generate additional Whitham flows. Another important issue is a choice of the normalization. Namely, in the genus zero case, differentials \( d\Omega_i \) are completely determined by their expansions in the neighborhoods of marked points \( P_{\alpha} \). In the higher genus case, these conditions define \( d\Omega_i \) only up to a holomorphic differential. This ambiguity is fixed by introducing a normalization condition. There are two main choices — real normalization, which is defined by

\[ \Im \left[ \oint_c d\Omega_i \right] = 0 \quad \forall c \in H_1(\Gamma_g, \mathbb{Z}), \]

and complex normalization (or normalization w.r.t. \( a \)-cycles). Complex normalization requires making a choice of a canonical basis \( B \) of cycles in the homology of \( \Gamma_g \) and is then defined by

\[ \oint_{a_k} d\Omega_i = 0 \quad \forall a_k \in B. \]

Whitham equations were originally derived in [Kri88] for real-normalized differentials. However, after the relationship between Whitham equations and WDVV equations was found in [Kri92] and [Dub92], the focus shifted to the complex normalization condition [Kri94]. There were two main reasons for this. First, in the complex-normalized case the \( \tau \)-function is holomorphic, which is important for string theory applications. Second, the derivation of the expression for the \( \tau \)-function relied on the Riemann bilinear identities. Corresponding
identities for real normalized differentials are technically more complicated. However, complex normalization has certain disadvantages. In particular, it is well-defined only on the extended moduli space that incorporates the choice of a canonical basis $B$ into moduli data.

In this paper we develop a real-normalized version of the above approach. In this case, Whitham hierarchy can be defined on the usual moduli space of curves with some extra algebro-geometric data. First, we define a real leaf in the universal configuration space, introduce Whitham coordinates on this leaf and map it into the moduli space in such a way that the resulting differentials are real-normalized. Then we prove the real-normalized version of the Riemann bilinear identities (generalized to multivalued differentials). Using these identities we find the formula for the $\tau$-function of an algebraic orbit and prove the following theorem for $F(T) = \ln \tau(T)$.

**Theorem.** The third derivatives of $F(T)$ are given by the following formula

$$\partial_{TA TB TC} F(T) = \Re \left[ \sum \text{res}_{q_s} \frac{d\Omega_A d\Omega_B d\Omega_C}{dEdQ} \right]$$

This theorem then implies that if we consider a special class of algebraic orbits by choosing $dQ$ to be a real-normalized differential with a pole of order two at a puncture $P_1$, then Whitham flows define potential deformations of a Frobenius algebra structure with observables corresponding to $\frac{d\Omega_A}{dQ}$, and we have the following theorem.

**Theorem.** The logarithm of a $\tau$-function of a (reduced) real-normalized genus $g$ Whitham hierarchy is a solution to the WDVV equation.

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**Whitham equations**

Whitham equations first appeared in the theory of perturbations of exact algebro-geometric solutions of soliton equations. Such solutions are defined by linear flows on the Jacobian $\text{Jac}(\Gamma_g)$ of an auxiliary algebraic curve $\Gamma_g$ and are expressed in terms of theta functions. Perturbing these solutions by the so-called non-linear WKB (or Whitham averaging [Whi74]) method ([FFM80, DM82, DN83]) results in a “slow drift” on the moduli $\mathcal{M}$ of algebro-geometric data. Equations describing this drift are called Whitham equations.

Although Whitham equations are equations on the moduli, they can be conveniently written with the help of certain Abelian differentials $d\Omega_A(P|I)$, $P \in \Gamma_g$ defined on the universal curve $\mathcal{N}^1_g$,

$$\Gamma_g \longrightarrow \mathcal{N}^1_g \longrightarrow \mathcal{M}.$$
Each of the differentials $d\Omega_A$ is coupled with a corresponding Whitham time $T_A$ defining $A^{th}$ Whitham flow on $\mathcal{M}$. Then, after making a special choice of connection on $\mathcal{N}_g$, Whitham equations can be written in the following implicit form,

$$\partial_A d\Omega_B = \partial_B d\Omega_A.$$  

This form of Whitham equations was first observed by Flashka, Forest, and McLaughlin [FFM80] for the KdV equation and hyperelliptic spectral curves. It was later justified by Krichever [Kri88] in a more general setting of $(2 + 1)$ equations and general spectral curves.

**Whitham hierarchies**

It turns out that one can construct a whole hierarchy of Whitham equations. Following Krichever, we use algebro-geometric approach to construct Whitham hierarchies (Hamiltonian approach to the theory of Whitham equations was developed in [DN83]).

We begin with a local definition. Let $T = \{T_A\}_{A \in \mathcal{A}}$ be a collection of (real or complex) times, indexed by some set $\mathcal{A}$, let $z \in D \subset \mathbb{C}$, and let $\{\Omega_A(z|T)\}_{A \in \mathcal{A}}$ be a collection of functions, meromorphic in $z$, each $\Omega_A$ is coupled with the corresponding time $T_A$. The functions $\Omega_A(z|T)$ should be thought of as pull-backs via Whitham times of Abelian integrals $\Omega_A = \int d\Omega_A$ from a leaf spanned by Whitham flows in $\mathcal{M}$, with $z$ corresponding to a local coordinate along the fiber. Define a 1-form $\omega$ by

$$\omega = \sum_A \Omega_A(z|T) dT_A.$$

Then, by definition, *Whitham hierarchy* is given by the generating equation

$$\delta \omega \wedge \delta \omega = 0,$$

where

$$\delta \omega = \sum_A \partial_B \Omega_A dT_B \wedge dT_A + \frac{d\Omega_A}{dz} dz \wedge dT_A.$$

This is equivalent to the following two equations,

$$\sum_{\{A,B,C\}} \varepsilon^{\{A,B,C\}} \left( \frac{d\Omega_A}{dz} \right) (\partial_B \Omega_C) = 0, \quad \sum_{\{A,B,C,D\}} \varepsilon^{\{A,B,C,D\}} (\partial_A \Omega_B)(\partial_C \Omega_D) = 0,$$

where we sum over all possible permutations of a fixed collection of indexes and $\varepsilon$ is a sign of a permutation.

Usually we have one marked index $A_0 \in \mathcal{A}$ with

$$X = T_{A_0}, \quad p(z|T) = \Omega_{A_0}(z|T).$$
Then, as long as \( \frac{dp}{d\bar{z}} \neq 0 \), we can make a change of coordinates from \((z, T)\) to \((p, T)\). In these coordinates, equation (3) written for \(A_0, A, B\) takes the form

\[
\partial_A \Omega_B(p|T) - \partial_B \Omega_A(p|T) + \{\Omega_A, \Omega_B\}(p|T) = 0,
\]

where

\[
\{\Omega_A, \Omega_B\}(p|T) = \partial_X \Omega_A \frac{d\Omega_B}{dp} - \partial_X \Omega_B \frac{d\Omega_A}{dp}
\]

is the usual Poisson bracket. Equations in this form are called zero-curvature equations. They can be interpreted as a compatibility conditions for the following Hamiltonian system,

\[
\partial_A E(p|T) = \{E, \Omega_A\}(p|T).
\]

Then, if zero-curvature equations are satisfied, there exists (locally) a solution \(E = E(p|T)\) of this system and, as long as \(\frac{dE}{dp} \neq 0\), we can again change coordinates from \((p, T)\) to \((E, T)\). In these coordinates, the above system takes the form

\[
\partial_A p(E|T) = \partial_X \Omega_A(E|T)
\]

and its compatibility conditions can be written as

\[
\partial_A \Omega_B(E|T) = \partial_B \Omega_A(E|T).
\]

Therefore, there exists (locally) a function \(S = S(E|T)\) such that \(\Omega_A(E|T) = \partial_A S(E|T)\). This function \(S(E|T)\) is called a prepotential of the Whitham hierarchy.

**Algebraic orbits**

One can use the above formalism to construct certain exact solutions of Whitham equations. These solutions depend on finitely many parameters and are obtained as follows [Kri94]. First we construct Whitham flows on finite-dimensional submanifolds, called algebraic orbits, of \(\mathcal{M}\) (note that \(\mathcal{M}\) is usually infinite-dimensional) together with Whitham equations that they satisfy. Then these equations are extended to the whole \(\mathcal{M}\). By making a correct choice of algebraic data, one can obtain usual Whitham equations of the soliton theory.

We illustrate this idea in the special case of the so-called dispersionless Lax equations. In this case we take algebraic curve of genus zero, \(\Gamma \simeq \mathbb{C}P^1\), with a single puncture \(P_1 = \infty\). The moduli space is equivalent to a moduli space of local coordinates around \(P_1\),

\[
\mathcal{M} = \hat{\mathcal{M}}_{0,1} = \{\Gamma \simeq \mathbb{C}P^1; P_1 = \infty; z(P)\} \simeq \{\nu_s, s = 1, \ldots\},
\]

where

\[
z^{-1}(p) = \mathcal{V}(p) = p + \nu_1 p^{-1} + \nu_2 p^{-2} + \cdots,
\]
and $p$ is the standard coordinate on $\mathbb{C}$. Note that in this case $\Omega_1 = p$, i.e., the marked index is $i = 1$. Then, by definition, the dispersionless Lax hierarchy (or dispersionless KP hierarchy) is a set of evolution equations on $\nu_s = \nu_s(T)$:

$$\partial_i \xi(p|T) = \{\xi, \xi_i^+\} = \{\xi, \Omega_i\}, \quad \text{where } \partial_i = \frac{\partial}{\partial T_i}. $$

This hierarchy can be thought of as a quasi-classical limit of the usual KP hierarchy, and algebraic orbits correspond to $n^{th}$ order reductions of the KP hierarchy, i.e., $n$KdV hierarchies. Namely, let

$$E(p) = p^n + u_{n-2}p^{n-2} + \cdots + u_0$$

and define $\xi(p)$ by the condition $\xi(p)^n = E(p)$. Then $\nu_s = \nu_s(u_0, \ldots, u_{n-2})$ and we obtain a finite-dimensional leaf in $\mathcal{M}$. Corresponding evolution equations on $u_i = u_i(T)$ can be written as

$$\partial_i E(p|T) = \{E, (E^\xi)_+\} \quad \text{in } (p, T) \text{ coordinates,}$$

$$\partial_i p(E|T) = \partial_X \Omega_i(E|T) \quad \text{in } (E, T) \text{ coordinates.}$$

Then the solution $E = E(p|T)$ (i.e., $u_i = u_i(T)$) can be obtained as follows [Kri88, Kri92].

**Theorem.** Let

$$S(p|T) = \sum_i T_i\Omega_i(p|u) = \sum_i T_i \xi^i + O(\xi^{-1}). \quad (5)$$

Require that

$$\frac{dS}{dp}(q_s) = 0 \text{ for all } q_s \text{ such that } \frac{dE}{dp}(q_s) = 0. \quad (6)$$

Equation (6) is equivalent to a collection of equations $F_k(u, T) = 0$ that implicitly define $u_i = u_i(T)$.

Then $u(T)$ is a solution of the dispersionless Lax equations.

The theorem follows from the fact that $S$ defined above is a global prepotential, i.e.,

$$\partial_i S(E|T) = \Omega_i(E|T) = \begin{cases} \xi^i + O(\xi^{-1}) \text{ near } P_1 \\ \text{has no other poles} \end{cases}. $$

From the definition of $S$ it is clear that the first condition is satisfied. The role of the condition (6) is to ensure that $\partial_i S(E|T)$ is holomorphic on $\Gamma - P_1$.

Note that the condition (6) can be also written in the form $dS = QdE$ for some polynomial $Q(p|T)$. Conversely, we can recover $T^i_i$ from $S$ and $\xi$ by the formula

$$T^i_i = -\frac{1}{i} \text{res}_{P_1} \xi^i dS.$$
This observation motivates the following alternative approach. Consider the so-called universal configuration space
\[ \mathcal{M}_0(n, m) = \{ \Gamma = \mathbb{C}P^1; P_1 = \infty; [z]_n; E, Q \} \simeq \{ u_0, \ldots, u_{n-2}; b_0, \ldots, b_m \}, \]
where
\[ E(p) = p^n + u_{n-2}p^{n-2} + \cdots + u_0, \quad Q(p) = b_mp^m + \cdots + b_0, \]
\([z]_n\) is an \(n\)-jet of a local coordinate near \(P_1\) and we require that \(E = z^{-n} + O(z)\) near \(P_1\).

By definition, put
\[ dS = Q \, dE, \quad T_i = -\frac{1}{i} \text{res}_{P_1} z^i dS. \]
Then
\[ \partial_i dS(E|T) = d\Omega_i(E|T), \]
condition (7) is equivalent to a collection of equations \(T_k = T_k(u, b)\), which can be inverted, \(u_i = u_i(T)\), \(b_j = b_j(T)\), and therefore we obtain a map from \(\mathcal{M}_{0,1}\) to an algebraic orbit in \(\hat{\mathcal{M}}_0\).

\(\tau\)-functions and solutions to the WDVV equations

For each algebraic orbit constructed above corresponds a so-called \(\tau\)-function. By definition,
\[ \ln \tau(T) = F(T), \]
where
\[ F(T) = \frac{1}{2} \text{res}_\infty \left( \sum_i T_i \mathfrak{v}^i \right) dS, \]
and \(dS\) is a (differential of) the prepotential of the corresponding algebraic orbit. Then we have the following theorem (Krichever [Kri92]).

**Theorem.** The derivatives of \(F(T)\) are given by the following formulas:
\[ \partial_i F(T) = \text{res}_\infty \mathfrak{v}^i dS \]
\[ \partial_{ij} F(T) = \text{res}_\infty \mathfrak{v}^j d\Omega_i = \text{res}_\infty \mathfrak{v}^j d\Omega_i \]
\[ \partial_{ijk} F(T) = \sum_{q \in dE(q_\lambda) = 0} \text{res}_{q_\lambda} \frac{d\Omega_i d\Omega_j d\Omega_k}{dQ dE}, \]
where \(Q = \frac{dS}{dE}\).

**Corollary.** If we choose
\[ Q(p|T) = p, \quad E(p|T_1, \ldots, T_{n+1}) = p^n + u_{n-2}(T)p^{n-2} + \cdots + u_0(T), \]
then \(F(T)\) is a WDVV potential for \(A_{n-1}\) Landau-Ginzburg model defined by a superpotential
\[ W(p|t_0, \ldots, t_{n-2}) = \frac{1}{n} E(p|t_0, \frac{t_1}{2}, \ldots, \frac{t_{n-2}}{n}, 0, \frac{1}{n+1}). \]
Higher genus case

The above approach can be generalized to algebraic curves of higher genus. This question was considered by Dubrovin [Dub92] for Hurwitz spaces and by Krichever [Kri94] in general. In the higher genus case, the “universal” moduli space

\[ \hat{\mathcal{M}}_{g,N} = \{ \Gamma_g; P_\alpha; \mathfrak{t}_\alpha^{-1}(P) = z_\alpha(P) \} \]

consists of the following algebro-geometric data:

- smooth algebraic curve \( \Gamma_g \) of genus \( g \),
- collection of marked points \( P_\alpha \in \Gamma_g \), \( \alpha = 1, \ldots, N \),
- local coordinates \( z_\alpha(P) \) in the neighborhoods of \( P_\alpha \), \( z_\alpha(P_\alpha) = 0 \).

For simplicity, we concentrate exclusively on a single puncture case, \( N = 1 \).

To construct a realization of a Whitham hierarchy on \( \hat{\mathcal{M}}_{g,1} \), one can proceed as follows (see [Kri94]). Instead of polynomials \( \Omega_i \) one has to consider Abelian differentials of the second kind \( d\Omega_i \) on \( \Gamma_g \) with prescribed behavior near the puncture,

\[ d\Omega_i = d(\mathfrak{t}^i + O(\mathfrak{t}^{-1})) \quad \text{near } P_1. \]  

However, condition (9) specifies \( d\Omega_i \) only up to a holomorphic differential. To define \( d\Omega_i \) uniquely, one has to impose certain normalization conditions. There are two choices — real normalization and complex normalization (or normalization w.r.t. a-cycles). Real normalization is defined by the condition

\[ \Im \left[ \oint_c d\Omega_i \right] = 0 \quad \forall c \in H_1(\Gamma_g, \mathbb{Z}). \]  

To define complex normalization, one has to first choose a canonical homology basis for \( \Gamma_g \),

\[ \mathcal{B} = \{ a_1, \ldots, a_g; b_1, \ldots, b_g \mid a_i, b_j \in H_1(\Gamma_g, \mathbb{Z}), a_i \cdot a_j = b_i \cdot b_j = 0, a_i \cdot b_j = \delta_{ij} \}. \]

As a result, it is necessary to consider the extended moduli space

\[ \hat{\mathcal{M}}_{g,1}^\ast = \{ \Gamma_g; P_1; z(P); \mathcal{B} \}, \]

which is a covering of \( \hat{\mathcal{M}}_{g,1} \). The differentials \( d\Omega_i \) are then uniquely normalized by the condition

\[ \oint_{a_k} d\Omega_i = 0 \quad \forall a_k \in \mathcal{B}. \]

As before, the marked index is \( i = 1 \) and \( p(P) = \int P \, d\Omega_1 \).
Universal configuration space

In order to construct solutions corresponding to algebraic orbits, it is convenient to use the universal configuration space approach. Following [KP97, KP98], define the universal configuration space for Whitham hierarchies,

\[ \mathcal{M}_g(n, m) = \{ \Gamma_g; P_1; [z]_n; E, Q \}, \]

to be the moduli space of the following data:

- genus \( g \) Riemann surface \( \Gamma_g \),
- marked point \( P_1 \in \Gamma_g \),
- an \( n \)-jet \([z]_n\) of local coordinates \( z(P) \) in the neighborhoods of \( P_1 \), \( z(P_1) = 0 \),
- an Abelian integral \( E \) with pole of order \( n \) at \( P_1 \) such that
  \[ dE \sim d(z^{-n} + O(z)) \quad \text{near } P_1, \quad z \in [z]_n, \]
- an Abelian integral \( Q \) with pole of order \( m \) at \( P_1 \).

More precisely, by Abelian integrals we mean pairs \( E = (dE, P_0^E) \), \( Q = (dQ, P_0^Q) \), where \( dE \), \( dQ \) are meromorphic differentials of the second kind on \( \Gamma_g \), holomorphic on \( \Gamma_g - P_1 \), and with poles of order \( n + 1 \) and \( m + 1 \) at \( P_1 \), and

\[ E(P) = \int_{P_0^E}^P dE, \quad Q(P) = \int_{P_0^Q}^P dQ. \]

Alternatively, one can choose a local coordinate \( z \in [z]_n \) and define \( E \) and \( Q \) to be pairs \( E = (dE, cE) \), \( Q = (dQ, c_Q) \), where near \( P_1 \),

\[ E \sim z^{-n} + cE + O(z), \quad Q \sim c_m^Q z^{-m} + \cdots + c_1^Q z^{-1} + c_Q + O(z). \]

Note that after \( E \) is chosen, there is a preferred local coordinate \( z_* \in [z]_n \) defined by

\[ E = z_*^{-n} \quad \text{near } P_1. \]

The moduli space \( \mathcal{M}_g(n, m) \) is a complex manifold of dimension \( 5g + n + m \) with at most orbifold singularities. We also need to consider smaller \((4g + n - 1)\)-dimensional moduli space

\[ \mathcal{M}_g(n) = \{ \Gamma_g; P_1; [z]_n; E \}, \]

as well as the corresponding extended moduli spaces \( \mathcal{M}_g^*(n, m) \) and \( \mathcal{M}_g^*(n) \).
Construction of algebraic orbits

A general approach for constructing algebraic orbits in the higher genus case is the following. First, choose a leaf \( \mathcal{V}(n, m) \) in \( \mathcal{M}_g^*(n, m) \). The precise definition of such leaf depends on the choice of the normalization. Then introduce special coordinates \( T_A \), called Whitham times, on \( \mathcal{V}(n, m) \), and define the prepotential \( S \) of the leaf by the formula

\[
dS = QdE.
\]

Whitham differentials \( d\Omega_A \) are then obtained from \( dS \) by

\[
\partial_{T_A}dS(E|T) = d\Omega_A(E|T),
\]

where we use a connection on \( \mathcal{N}_g^1 \) given by choosing \( E = \text{const} \) to be horizontal sections. Note that \( dS = \sum_A T_A d\Omega_A \). The leaf \( \mathcal{V}(n, m) \) is then mapped to the corresponding algebraic orbit \( \mathcal{O}(n) \) by the following sequence of maps:

\[
\begin{align*}
\mathcal{V}(n, m) & \quad \longrightarrow \quad \mathcal{V}(n) \quad \longrightarrow \quad \mathcal{O}(n) \\
\cup & \\
\mathcal{V}^\circ(n) & \quad \longrightarrow \quad \mathcal{O}^\circ(n) \\
\{\Gamma_g; P_1; [z]_n; E, Q\} & \quad \longrightarrow \quad \{\Gamma_g; P_1; [z]_n; E\} \quad \longrightarrow \quad \{\Gamma_g; P_1; z^*\},
\end{align*}
\]

where \( \mathcal{V}^\circ(n) \) is a leaf of a subfibration of \( \mathcal{V}(n, m) \) defined by fixing the periods of \( dE \). Then, in the coordinates \( (E, T) \) on the “universal curve” \( \mathcal{N}_g \), over \( \mathcal{O}^\circ(n) \), the differential \( dS(E|T) = Q(E|T)dE \) satisfies the equations \( \partial_A dS(E|T) = d\Omega_A(E|T) \). Therefore, \( dS \) is a prepotential, \( \partial_A^* \Omega_A(E|T) = \partial_{T_A} p(E|T) \), and we obtain solutions to Whitham equations. Note that Whitham coordinates on \( \mathcal{V}(n, m) \) go to Whitham flows on the algebraic orbit.

For each Whitham derivative \( \partial_{T_A} \) we define a dual “integral” operator \( \oint_{T_A} \) in such a way that

\[
\oint_{T_A} \partial_{T_B} dS = \oint_{T_A} d\Omega_B = \oint_{T_B} d\Omega_A = \oint_{T_B} \partial_{T_A} dS \quad \forall T_A, T_B.
\]

The exact definition of such operators \( \oint_{T_A} \) can be rather non-trivial and usually includes integration with some weight over a cycle and, maybe, some correction terms. The above identities can be thought of as a generalization of Riemann bilinear identities for Whitham differentials, and they are proved along the same lines. The \( \tau \)-function of an algebraic orbit is then defined by \( \tau(T) = e^{F(T)} \), and

\[
F(T) = \frac{1}{2} \sum_{T_A} T_A \oint_{T_A} dS.
\]

It encodes all information about the differentials \( d\Omega_A \) and the curve \( \Gamma_g \), and its first and
second derivatives are given by
\[
\partial_T A F(T) = \frac{1}{2} \oint_{T_A} dS + \frac{1}{2} \sum_{T_B} T_B \oint_{T_B} d\Omega_A \\
= \frac{1}{2} \oint_{T_A} dS + \frac{1}{2} \sum_{T_B} T_B \oint_{T_B} d\Omega_B = \oint_{T_A} dS,
\]
\[
\partial_{T_AT_B} F(T) = \oint_{T_A} d\Omega_B = \oint_{T_B} d\Omega_A.
\]

Note that the equality of mixed partials of \( F(T) \) corresponds to equation (14). Third derivatives of \( F(T) \) are then given by the residue type formula.

**Complex-normalized case**

In this case, one chooses
\[
t_k = -\frac{1}{k} \text{res}_{P_1} z^k Q dE, \quad k = 1, \ldots, n + m,
\]
\[
t_{h,k} = \oint_{a_k}^\leftarrow Q dE, \quad t_{E,a_k} = \oint_{a_k} dE, \quad t_{Q,a_k} = \oint_{a_k} dQ, \quad t_{E,b_k} = \oint_{b_k} dE, \quad t_{Q,b_k} = \oint_{b_k} dQ,
\]
as coordinates on \( \mathcal{M}_g(n,m) \). The notation \( a_k^- \) indicates that the integral has to be taken over the right side of the cycle. A complex \((3g+n+m)\)-dimensional \( \mathcal{V}_C(n,m) \subset \mathcal{M}_g^*(n,m) \) is defined by imposing the \( a \)-normalization condition on the Abelian differentials \( dE \) and \( dQ \),
\[
\oint_{a_k} dE = \oint_{a_k} dQ = 0 \quad \forall a_k \in \mathcal{B}.
\]

Note that the projection of \( \mathcal{V}_C(n,m) \) to \( \mathcal{M}_g(n,m) \) is not well-defined. Whitham coordinates on the leaf are then given by
\[
T_k = -\frac{1}{k} \text{res}_{P_1} z^k dS, \quad k = 1, \ldots, n + m,
\]
\[
T_{h,k} = \oint_{a_k} dS, \quad T_{E,k} = -\oint_{b_k} dQ, \quad T_{Q,k} = \oint_{b_k} dE, \quad k = 1, \ldots, g
\]

Corresponding Whitham differentials are divided into the following four groups.

- \( \partial_{T_k} dS(E|T) = d\Omega_k(E|T) \) are the usual complex-normalized Whitham differentials with prescribed behavior near the puncture,
  \[
  d\Omega_k = d(t^k + O(t^{-1})) \text{ near } P_1, \quad \oint_{a_1} d\Omega_k = 0.
  \]

- \( \partial_{T_{h,k}} dS(E|T) = d\Omega_{h,k}(E|T) \) are holomorphic differentials that are dual to the canonical basis of cycles,
  \[
  \oint_{a_i} d\Omega_{h,k} = \delta_{ik}.
  \]
\[
\partial_T E_k dS(E|T) = d\Omega_E k(E|T) \text{ are } \alpha\text{-normalized and holomorphic everywhere on } \Gamma_g \text{ except for the } \alpha\text{-cycles, where they have jumps,}
\]
\[
d\Omega_E k(P_a^+ - P_a^-) = \delta_k dE(P_{a_i}), \quad \oint_{a_i} d\Omega_E k = 0.
\]

\[
\partial_T Q_k dS(E|T) = d\Omega_Q k(E|T) \text{ are } \alpha\text{-normalized and holomorphic everywhere on } \Gamma_g \text{ except for the } \alpha\text{-cycles, where they have jumps,}
\]
\[
d\Omega_Q k(P_a^+ - P_a^-) = \delta_k dQ(P_{a_i}), \quad \oint_{a_i} d\Omega_Q k = 0.
\]

The duals \( \oint_T A \) are then given by
\[
\oint_T d\Omega = \text{res}_P, \quad \oint d\Omega = \frac{-1}{2\pi \sqrt{-1}} \oint d\Omega, \quad \oint d\Omega = \frac{-1}{2\pi \sqrt{-1}} \oint E d\Omega,
\]
where the notation \( \oint_{[b_k]} d\Omega \) indicates that a certain correction terms have to be added to make the integral independent on small cycle deformations. In particular,
\[
\oint_{[b_k]} dS = \oint_{b_k} dS + T_{E,k} E(a_k^+ \cap b_k).
\]

This remark becomes very important in the real-normalized case and is discussed at length in the next section.

The third derivatives of \( F(T) \) are given by the following theorem ([Kri94]).

**Theorem.**
\[
\partial_{TATBTC} F(T) = \sum_{q,s} \text{res}_{q,s} \frac{d\Omega_A d\Omega_B d\Omega_C}{dE dQ}.
\]

**Remark.** The residue type formula for the third derivatives of \( F(T) \) implies that if we consider the reduced hierarchy with \( dQ = dp \), then \( F(T) \) is a solution for the WDVV equation.

**Real-normalized Whitham hierarchies**

**Real leaf and Whitham times** We begin the study of the real-normalized case by introducing the following real-analytic coordinate system on \( \mathcal{M}_g(n,m) \):
\[
t_k^e = \Re \left[ \frac{-1}{k} \text{res}_P z^k Q dE \right], \quad t_k^i = \Im \left[ \frac{-1}{k} \text{res}_P z^k Q dE \right], \quad k = 1, \ldots, n + m,
\]
and
\[ t_{E,a_k}^i = \Re \left[ \oint_{a_k} dE \right], \quad t_{E,b_k}^i = \Im \left[ \oint_{b_k} dE \right], \quad t_{Q,a_k}^i = \Re \left[ \oint_{a_k} dQ \right], \quad t_{Q,b_k}^i = \Im \left[ \oint_{b_k} dQ \right], \]
\[ t_{h,a_k}^i = \Re \left[ \oint_{a_k} QdE \right], \quad t_{h,b_k}^i = \Im \left[ \oint_{b_k} QdE \right], \quad k = 1, \ldots, g. \]

Note that all coordinates except \( t_{h,a_k}^i, t_{h,b_k}^i \) are just the real and imaginary parts of the corresponding complex analytic coordinates. A real leaf \( \mathcal{V}_R(n,m) \subset \mathcal{M}_g(n,m) \) is defined to be a zero set of the coordinates \( t_{E,a_k}^i, t_{E,b_k}^i, t_{Q,a_k}^i, t_{Q,b_k}^i \). Alternatively, \( \mathcal{V}_R(n,m) \) can be defined by the following 4g basis-independent equations,
\[ (16) \quad \Re \left[ \oint_c dE \right] = 0, \quad \Re \left[ \oint_c dQ \right] = 0, \quad \forall c \in H_1(\Gamma_g, \mathbb{Z}). \]

Therefore, \( \mathcal{V}_R(n,m) \) is well-defined as a real-analytic submanifold of both \( \mathcal{M}_g(n,m) \) and \( \mathcal{M}_g(n,m) \).

Before introducing Whitham coordinates on \( \mathcal{V}_R(n,m) \), we have to make the following important observation. In what follows we work with multivalued differentials that usually have the form \( fd\omega \), where \( f \) is an Abelian integral and the differential \( d\omega \) can again be multivalued. These differentials are well-defined only if we cut the surface \( \Gamma_g \). Thus, we always assume that some canonical basis \( \mathcal{B} \) of \( \Gamma_g \) is chosen, and we cut our surface along the representatives \( a_k, b_k \) of cycles in \( \mathcal{B} \). Sometimes these cuts are not enough and we need one extra cut \( \gamma \). Let us introduce the following notation.

**Notation.** By \( \hat{\Gamma}_g \) we denote the normal polygon obtained from \( \Gamma_g \) by cutting along the \( a \) and \( b \) cycles. For each cycle \( a_l \) we denote by \( a_i^+ \) its left and by \( a_i^- \) its right sides. Same notation applies to \( b \)-cycles. The point of intersection of \( a_i^- \) and \( b_i^+ \) cycles is denoted by \( \Phi_l \) and the point of intersection of \( a_i^+ \) and \( b_i^- \) cycles is denoted by \( \Phi_0 \). By \( (\Gamma_g)_{\text{cut}} \) we denote a Riemann surface obtained from \( \Gamma_g \) by making a cut \( \gamma \) from \( \Phi_0 \) to \( P_1 \), and by \( (\hat{\Gamma}_g)_{\text{cut}} \) we denote its normal polygon. Note that topologically \( (\Gamma_g)_{\text{cut}} \simeq \Gamma_g - P_1 \).

The multivalued differentials that we consider are single valued on either \( \hat{\Gamma}_g \) or \( (\hat{\Gamma}_g)_{\text{cut}} \) and, as differentials on \( \Gamma_g \) or \( (\Gamma_g)_{\text{cut}} \), they can have jumps across the \( a \) and \( b \) cycles, and a cut \( \gamma \). Note that for such differentials, the integral over a cycle depends not only on the homology class of the cycle, but also on the side of the cycle and the actual choice of a representative in the homology class. Thus, in order to make our construction independent of a choice of such representative in the homology class, for each differential \( fd\omega \) we introduce a corresponding cocycle \([fd\omega]\) by the following procedure. First, we define \([fd\omega]\) on basic cycles \( a_k, b_k \) by adding certain correction terms to \( \oint fd\omega \), and then extend it to an arbitrary cycle by linearity. We use the following notation.
Notation. For any cycle \( c = \alpha_1 a_1 + \cdots + \beta_g b_g \), by
\[
\oint_c f \omega = [f \omega](c) = \alpha_1 [f \omega](a_1) + \cdots + \beta_g [f \omega](b_g)
\]
we denote the value of \([f \omega]\) on a cycle \( c \). Note that \([f \omega](a_k)\) and \([f \omega](b_k)\) still depend on a side of the cycle. We always choose a right side for \( a \)-cycles and left side for \( b \)-cycles,
\[
\oint_{[a_k]} f \omega := \oint_{[a_k^-]} f \omega, \quad \oint_{[b_k]} f \omega := \oint_{[b_k^+]} f \omega.
\]
However, it is convenient to indicate the side explicitly in the intermediate calculations. For consistency, we use same notation for usual Abelian differentials on \( \Gamma_g \). In addition, for any differential \( df \), we denote by \( f^l = \int_{\Phi^l} df \) the value of the corresponding Abelian integral \( f \) at the point \( \Phi^l \) of intersection of the \( a^-_l \) and \( b^+_l \) cycles. We also put \( f^r = \Re [f(\Phi^l)] \), \( f^i = \Im [f(\Phi^l)] \).

We now define Whitham times on \( V\mathbb{R}(n, m) \) by
\[
T^r_k = \Re \left[ \frac{-1}{k} \text{res}_{P_1} z^k Q dE \right], \quad T^i_k = \Im \left[ \frac{-1}{k} \text{res}_{P_1} z^k Q dE \right], \quad k = 1, \ldots, n + m
\]
and
\[
T^a_{h,k} = \Im \left[ \oint_{[a_k]} Q dE \right], \quad T^b_{h,k} = \Im \left[ \oint_{[b_k^+]} Q dE \right], \quad T^a_{E,k} = -\Im \left[ \oint_{[b_k]} dQ \right],
\]
\[
T^b_{E,k} = \Im \left[ \oint_{[a_k]} dQ \right], \quad T^a_{Q,k} = \Im \left[ \oint_{[a_k]} dE \right], \quad T^b_{Q,k} = -\Im \left[ \oint_{[b_k]} dE \right].
\]
Except for some relabeling, the main change occurs in the definition of \( T^a_{h,k}, T^b_{h,k} \):
\[
T^a_{h,k} = t^i_{h,a_k} - t^i_{Q,a_k} E^r_k, \quad T^b_{h,k} = t^i_{h,b_k} - t^i_{Q,b_k} E^r_k.
\]

Prepotential. The prepotential \( dS = Q dE \) of the real leaf \( V\mathbb{R}(n, m) \) is described by the following proposition.

Proposition 1. The prepotential \( dS = Q dE \) is a meromorphic differential on \( \Gamma_g \), holomorphic on \( \Gamma_g - P_1 \), with pole of order \( n + m + 1 \) at \( P_1 \), and with jumps on \( a \) and \( b \) cycles. Near \( P_1 \),
\[
dS \sim d \left( \sum_{j=1}^{m+n} (T^r_j + \sqrt{-1} T^i_j) \xi^j + O(z) \right) + R^s \frac{dz}{z},
\]
where \( z = \xi^{-1} \) is our preferred local coordinate in the neighborhood of \( P_1 \), and
\[
R^s = \text{res}_{P_1} dS = \frac{1}{2\pi \sqrt{-1}} \sum_{l=1}^{g} (T^a_{E,l} T^b_{Q,l} - T^b_{E,l} T^a_{Q,l}).
\]
The jumps of $dS$ across the $a$ and $b$-cycles come from the jumps of $Q$ and are given by

$$dS(P_{ai}^+ - P_{ai}^-) = \sqrt{-1}T_{E,l}^a dE, \quad dS(P_{bi}^+ - P_{bi}^-) = \sqrt{-1}T_{E,l}^b dE.$$  \hspace{1cm} (18)

The corresponding cocycle $[dS]$ is well-defined only on $\Gamma_g - P_1$, and is given by

$$\oint_{[a_i^+]} dS = \oint_{a_i^-} dS - \sqrt{-1}T_{E,l}^b E_l^- + T_{Q,l}^b T_{E,l}^a E_l^- - \sqrt{-1}T_{E,l}^a E_l^2 / 2,$$

$$\oint_{[b_i^+]} dS = \oint_{b_i^-} dS - \sqrt{-1}T_{Q,l}^a S_l^- + T_{Q,l}^a T_{E,l}^b E_l^- + \sqrt{-1}T_{E,l}^a E_l^2 / 2,$$

$$\oint_{[a_i^-]} SdE = \oint_{a_i^+} SdE - (\omega_S)_i^{a^-} E_l^- + \sqrt{-1}T_{E,l}^b E_l^2 / 2,$$

$$\oint_{[b_i^-]} SdE = \oint_{b_i^+} SdE - (\omega_S)^{b^+} E_l^- - \sqrt{-1}T_{E,l}^a E_l^2 / 2.$$

The proof of this proposition is a direct calculation.

Since $dS$ has a residue at $P_1$, the corresponding integral $S(P) = \int^P dS$ is well-defined only on $(\hat{\Gamma}_g)_{\text{cut}}$. We choose an additive normalization constant in such a way that the regular part $S(P_1)$ vanishes at $P_1$. We can now consider cocycles corresponding to the differentials $EdS$ and $SdE$, which are well-defined on $\Gamma_g - P_1$.

**Proposition 2.** The cocycles $[EdS]$ and $[SdE]$ are given by

$$\oint_{[a_i^+]} EdS = \oint_{a_i^-} EdS + \sqrt{-1}T_{Q,l}^b S_l^- + T_{Q,l}^b T_{E,l}^a E_l^- - \sqrt{-1}T_{E,l}^b E_l^2 / 2,$$

$$\oint_{[b_i^+]} EdS = \oint_{b_i^-} EdS - \sqrt{-1}T_{Q,l}^a S_l^- + T_{Q,l}^a T_{E,l}^b E_l^- + \sqrt{-1}T_{E,l}^a E_l^2 / 2,$$

$$\oint_{[a_i^-]} SdE = \oint_{a_i^+} SdE - (\omega_S)_i^{a^-} E_l^- + \sqrt{-1}T_{E,l}^b E_l^2 / 2,$$

$$\oint_{[b_i^-]} SdE = \oint_{b_i^+} SdE - (\omega_S)^{b^+} E_l^- - \sqrt{-1}T_{E,l}^a E_l^2 / 2.$$

Moreover, the following “integration by parts” formulas hold:

$$\Re \left[ \oint_{[a_i^-]} SdE \right] = -\Re \left[ \oint_{[a_i^+]} EdS + T_{h,l}^b T_{Q,l}^a \right],$$

$$\Re \left[ \oint_{[b_i^-]} SdE \right] = -\Re \left[ \oint_{[b_i^+]} EdS - T_{h,l}^b T_{Q,l}^a \right].$$

**Whitham differentials** Similarly to the complex-normalized case, Whitham differentials $d\Omega_A(E|T) = \partial_{T_A} dS(E|T)$ can be divided into four groups:

- $d\Omega_k^A$ and $d\Omega_k^b$ are real-normalized meromorphic differentials with prescribed singularities at $P_1$, i.e., exactly the differentials of the real-normalized Whitham hierarchy,

- $d\Omega_{h,k}^a$ and $d\Omega_{h,k}^b$ form a canonical real basis in the space of real-normalized holomorphic differentials that is dual to our basis $\mathcal{B}$,

- $d\Omega_{E,k}^a$ and $d\Omega_{E,k}^b$ are meromorphic differentials with a simple pole at $P_1$ and a $dE$-jump across $a$ and $b$-cycles,
• $d\Omega^a_{Q,k}$ and $d\Omega^b_{Q,k}$ are meromorphic differentials with a simple pole at $P_1$ and a $dQ$-jump across $a$ and $b$-cycles.

In the sequel we restrict to leaves $\mathcal{V}_{Q,k}(n, m)$ of a foliation defined by the level sets of $T^a_{Q,k}$, $T^b_{Q,k}$. Then all the differentials but $d\Omega^a_{Q,k}$ and $d\Omega^b_{Q,k}$ generate flows preserving the foliation. For this reason, we do not describe differentials $d\Omega^a_{Q,k}$, $d\Omega^b_{Q,k}$ in detail. We also need to consider corresponding Abelian differentials, which we always normalize by $(\Omega_A)(P_1) = 0$.

**Proposition 3.** For any Whitham time $T_A$, the differential $d\Omega_A(E|T) = \partial_{T_A}dS(E|T)$ is holomorphic on $\Gamma_g - P_1$.

**Proof.** The only possible extra poles can appear when $E(P)$ does not define a local coordinate, i.e., at the points $q_s$ such that $dE(q_s) = 0$. Assuming for simplicity that $dE$ has a simple pole at $q_s$, let $E_s(T) = E(q_s(T)|T)$, and choose $\xi(E|T) = \sqrt{E - E_s(T)}$ to be a local coordinate near $q_s$. Then we have

$$\partial_{T_A}dS(E|T) = \partial_{T_A}(Q(\xi(E|T)|T))d(\xi^2 + E_s)$$

$$= \frac{dQ - \partial_{T_A}(E_s)}{\sqrt{E - E_s}}2\xi d\xi = -(\partial_{T_A}E_s)dQ(E|T),$$

which is holomorphic at $q_s$. \hfill \Box

Main properties of the Whitham differentials are summarized in the following proposition.

**Proposition 4.**

• Differentials $d\Omega_k^a(E|T)$ and $d\Omega_k^b(E|T)$ are real-normalized,

$$\Im \left[ \oint_C d\Omega_k^a \right] = \Im \left[ \oint_C d\Omega_k^b \right] = 0 \quad \forall c \in H_1(\Gamma_g, \mathbb{Z}),$$

meromorphic on $\Gamma_g$, with a single pole at $P_1$, where

$$d\Omega_k^a \sim d(t^k + O(1)), \quad d\Omega_k^b \sim d(\sqrt{1-t^k} + O(1)) \quad \text{res}_{P_1} \Omega_k^a = \text{res}_{P_1} \Omega_k^b = 0.$$

• Differentials $d\Omega_{h,k}^a(E|T)$ and $d\Omega_{h,k}^b(E|T)$ are holomorphic on $\Gamma_g$ and form a canonical basis in the space of real-normalized holomorphic differentials dual to our homology basis $\mathcal{B}$, i.e.,

$$\Im \left[ \oint_{a_l} d\Omega_{h,k}^a \right] = \delta_{kl}, \quad \Im \left[ \oint_{b_l} d\Omega_{h,k}^a \right] = 0,$$

$$\Im \left[ \oint_{a_l} d\Omega_{h,k}^b \right] = 0, \quad \Im \left[ \oint_{b_l} d\Omega_{h,k}^b \right] = \delta_{kl}$$
• Differentials $d\Omega^a_{E,k}(E|T)$ and $d\Omega^b_{E,k}(E|T)$ are meromorphic on $\Gamma_g$, holomorphic on $\Gamma_g - P_1$, have a simple pole at $P_1$ which is balanced by a single jump across one of the cycles:

\[
\text{res}_{P_1} d\Omega^a_{E,k} = \frac{1}{2\pi \sqrt{-1}} T^a_{Q,k} = \frac{1}{2\pi} \oint_{a_k} dE, \quad d\Omega^a_{E,k}(P_{a_i}^+ - P_{a_i}^-) = \sqrt{-1} \delta_{ki} dE(P_{a_i}),
\]

\[
\text{res}_{P_1} d\Omega^b_{E,k} = \frac{-1}{2\pi \sqrt{-1}} T^a_{Q,k} = \frac{1}{2\pi} \oint_{b_k} dE, \quad d\Omega^b_{E,k}(P_{b_i}^+ - P_{b_i}^-) = 0,
\]

differentials themselves are not real-normalized, but the corresponding cocycles are,

\[
\Im \left[ \oint_{[c]} d\Omega^a_{E,k} \right] = \Im \left[ \oint_{[c]} d\Omega^b_{E,k} \right] = 0.
\]

**Duality and Riemann Bilinear Relations**  In the real-normalized case, the dual “integral” operators $\oint_{T_A}$ are given by the following formulas:

\[
\oint_{T^i_A} d\Omega = \Re \left[ \text{res}_{P_1} \psi^i d\Omega \right], \quad \oint_{T^j_B} d\Omega = \Re \left[ \text{res}_{P_1} \sqrt{-1} \psi^j d\Omega \right],
\]

\[
\oint_{T^i_{E,h,k}} d\Omega = \Re \left[ \frac{1}{2\pi} \oint_{[b_k]} d\Omega \right], \quad \oint_{T^i_{h,k}} d\Omega = \Re \left[ \frac{1}{2\pi} \oint_{[a_k]} d\Omega \right],
\]

\[
\oint_{T^j_{E,h,k}} d\Omega = \Re \left[ \frac{1}{2\pi} \left( \oint_{[b_k]} E d\Omega + \text{Corr}^a_k(d\Omega) \right) \right],
\]

\[
\oint_{T^j_{E,h,k}} d\Omega = \Re \left[ \frac{1}{2\pi} \left( \oint_{[a_k]} E d\Omega + \text{Corr}^b_k(d\Omega) \right) \right],
\]

where the correction terms $\text{Corr}(d\Omega)$ are given by

\[
\text{Corr}^a_k(dS) = T^a_{h,k} T^b_{Q,k} + T^b_{h,k} T^a_{Q,k}, \quad \text{Corr}^b_k(dS) = -T^a_{h,k} T^b_{Q,k} + T^b_{h,k} T^a_{Q,k},
\]

\[
\text{Corr}(d\Omega_A) = \partial_{T_A} \text{Corr}(dS).
\]

We now have to establish relations (14).

**Proposition 5.** Whitham differentials satisfy the identity

\[
(19) \quad \oint_{T_A} d\Omega_B = \oint_{T_B} d\Omega_A
\]

**Proof.** This proposition is proved by a direct calculation. We illustrate it by considering the following cases.

First, let $T_A = T^i_A$ and $T_B = T^j_B$. Then we have to prove the following identity:

\[
\oint_{T^i_A} d\Omega^j_B = \Re \left[ \text{res}_{P_1} \psi^i d\Omega^j \right] = \Re \left[ \text{res}_{P_1} \psi^i d\Omega^j \right] = \oint_{T^j_B} d\Omega^i_A.
\]
But
\[ \text{res}_{P_1} \psi d\Omega^r_j = - \text{res}_{P_1} \psi d\Omega^r_j = - \text{res}_{P_1} \Omega_j^r d\psi - \text{res}_{P_1} \Omega_j^r d(\psi)_- \]
\[ = - \text{res}_{P_1} \Omega_j^r d\psi - \text{res}_{P_1} \psi d(\psi)_- = - \text{res}_{P_1} \Omega_j^r d\psi + \text{res}_{P_1} (\psi d\Omega^r_i). \]

Taking real parts and observing that
\[ \Re \left[ \text{res}_{P_1} \Omega_j^r d\Omega^r_i \right] = \Re \left[ \frac{1}{2\pi i} \oint_{\Gamma_{\psi}} \Omega_j^r d\Omega^r_i \right] = 0, \]
we obtain the desired identity.

Let \( T_A = T_{h,k}^a, T_B = T_{h,l}^a. \) Then
\[ \oint_{T_{h,k}^a} d\Omega_{h,l}^a = \Re \left[ \oint_{[b_k^+]_{h,l}} d\Omega_{h,l}^a \right] = \Re \left[ \oint_{[b_k^+]_{h,k}} d\Omega_{h,k}^a \right] = \oint_{T_{h,l}^a} d\Omega_{h,k}^a \]
is just the usual Riemann bilinear identity for a canonical basis of real-normalized holomorphic differentials and it is proved in the regular way:
\[ 0 = \Re \left[ \text{res}_{P_1} \Omega_{h,k}^a d\Omega_{h,l}^a \right] = \Re \left[ \frac{1}{2\pi} \left( - \oint_{b_k^+} d\Omega_{h,k}^a + \oint_{b_k^-} d\Omega_{h,l}^a \right) \right]. \]

The most difficult identities to establish correspond to \( T_A = T_{E,k}^a, T_B = T_{E,s}^a \) and the like, since \( d\Omega_{E,k}^a, d\Omega_{E,s}^b \) have simple poles at \( P_1. \) In this case, let \( C_{\varepsilon} \) be a small circle around \( P_1, \Phi_{\varepsilon} \in C_{\varepsilon}, \) and let \( \gamma_{\varepsilon} \) be a cut from \( \Phi_0 \) to \( \Phi_{\varepsilon}. \) We consider the integral \( \Omega_{E,k}^a d\Omega_{E,s}^a \) along the contour
\[ \sum_{l=1}^g (a_i^+ + b_i^+ - a_i^- - b_i^-) + \gamma_{\varepsilon}^+ - \gamma_{\varepsilon}^- + C_{\varepsilon}. \]

First, we compute that
\[ \Re \left[ \frac{1}{2\pi i} \sum_{l=1}^g \left( \oint_{a_i^+} + \oint_{b_i^+} - \oint_{a_i^-} - \oint_{b_i^-} \right) \Omega_{E,k}^a d\Omega_{E,s}^a \right] \]
\[ = \Re \left[ \frac{1}{2\pi} \left( \oint_{[a_k]} \text{Ed}\Omega_{E,s}^a + \oint_{[a_s]} \Omega_{E,k}^a dE + \delta_{ks} E_k^x T_{Q,k}^b - \left( \oint_{[a_s]} d\Omega_{E,k}^a \right) E_s^x \right) \right] \]
\[ = \Re \left[ \frac{1}{2\pi} \left( \oint_{[a_k]} \text{Ed}\Omega_{E,s}^a + \oint_{[a_s]} \Omega_{E,k}^a dE + T_{Q,k}^b \left( (\Omega_{E,s}^a)^k + \delta_{ks} E_s^x \right) \right) \right]. \]

Note that
\[ (\Omega_{E,s}^a)^k + \delta_{ks} E_k^x = (\Omega_{E,s}^a)_0 + \Im \left[ \int_{\Phi_0} d\Omega_{E,s}^a \right] + \delta_{ks} E_k^x \]
\[ = (\Omega_{E,s}^a)_0 + \Im \left[ \Omega_{E,s}^a (\Phi_0^{-}) \right] = \Im \left[ \Omega_{E,s}^a (\Phi_0^{+}) \right], \]
since the only non-trivial imaginary contribution for periods of $d\Omega_{E,s}^a$ comes from $b$-periods, and in order for $b^+$-contribution not to be canceled by $b^-$-contribution, we need $k = s$, in which case $\Im \left[ \int_{\Phi_0}^{\Phi} d\Omega_{E,s}^a \right] = -\delta_{ks} E_k^s$.

On the cut $\gamma$ we have

$$\left( \int_{\gamma^+} - \int_{\gamma^-} \right) (\Omega_{E,k}^a d\Omega_{E,s}^a) = \int_{\Phi_0^+}^{\Phi^+} T_{Q,k}^b d\Omega_{E,s}^a = T_{Q,k}^b (\Omega_{E,s}^a(\Phi^+_\varepsilon) - \Omega_{E,s}^a(\Phi^+_0)).$$

To evaluate $\int_{\gamma} \Omega_{E,k}^a d\Omega_{E,s}^a$, we rewrite everything in terms of a local coordinate $z$ in the neighborhood of $P_1$:

$$d\Omega_{E,k}^a = \frac{T_{Q,k}^b}{2\pi \sqrt{-1}} \frac{dz}{z} + (\alpha + O(z))dz, \quad \Omega_{E,k}^a = \frac{T_{Q,k}^b}{2\pi \sqrt{-1}} \log(z) + O(z).$$

Then

$$\int_{\Phi_0^+}^{\Phi^+} \Omega_{E,k}^a d\Omega_{E,s}^a = \int_{\Phi_0^+}^{\Phi^+} \frac{T_{Q,k}^b T_{Q,s}^b}{8\pi^2} \frac{\log(z)}{z} dz + O(\varepsilon)$$

$$= \frac{T_{Q,k}^b T_{Q,s}^b}{8\pi^2} \left( \frac{\log(z)}{\Phi^+_\varepsilon} \right) + O(\varepsilon)$$

$$= \frac{T_{Q,k}^b T_{Q,s}^b}{8\pi^2} (4\pi \sqrt{-1} \log(z(\Phi^+_\varepsilon) + 4\pi^2) + O(\varepsilon))$$

$$= -\frac{T_{Q,k}^b}{2} (\Omega_{E,s}^a(\Phi^+_\varepsilon)) + \frac{T_{Q,k}^b T_{Q,s}^b}{2} + O(\varepsilon).$$

Collecting all of the above together, we obtain

$$0 = \Re \left[ \frac{1}{2\pi \sqrt{-1}} \left( \sum_{k=1}^{g} \left( \oint_{[a_1]} E d\Omega_{E,s}^a + \oint_{[a_s]} \Omega_{E,k}^a dE \right) \right) \right]$$

$$= \Re \left[ \frac{1}{2\pi} \left( \oint_{[a_s]} E d\Omega_{E,s}^a + \oint_{[a_s]} \Omega_{E,k}^a dE \right) \right]$$

$$\rightarrow \Re \left[ \frac{1}{2\pi} \left( \oint_{[a_s]} E d\Omega_{E,s}^a + \oint_{[a_s]} \Omega_{E,k}^a dE \right) \right] \quad \text{as } \varepsilon \rightarrow 0,$$

which proves our identity. All other cases are somewhat intermediate in difficulty to the cases considered above and are proved along the same lines.

The third derivatives of $F(T)$ are given by the following theorem.

**Theorem 1.**

$$\partial_{T_A T_B T_C} F(T) = \Re \left[ \sum_{Q_a} \text{Res}_{Q_a} \frac{d\Omega_A d\Omega_B d\Omega_C}{dE dQ} \right]$$
Proof. First we prove the following formula. For any two Whitham differentials $d\Omega_B$, $d\Omega_C$ we have
\[
\Re \left[ \frac{1}{2\pi i} \oint_{\partial \mathcal{I}} (\partial_{T_A} \Omega_B) d\Omega_C \right] = \Re \left[ \operatorname{res}_{P_1} (\partial_{T_A} \Omega_B) d\Omega_C \right]
\]  
(21)
\[+ \Re \left[ \sum_{q_s \mid dE(q_s) = 0} \operatorname{res}_{q_s} \frac{d\Omega_A d\Omega_B d\Omega_C}{dEdQ} \right].
\]
To establish this identity, note that since $d\Omega_C$ is holomorphic outside of $P_1$, the right hand side of our equation is a sum of residues at $P_1$ and at poles of $\partial_{T_A} \Omega_B$. Using $E$ as a coordinate, we see that
\[
\partial_{T_A} \Omega_B(E|T) = \partial_{T_A} \partial_{T_B} S(E|T),
\]
is holomorphic. So the only extra poles can appear at $q_s$ such that $dE(q_s) = 0$. Assuming that $q_s$ is a simple zero of $dE$ and using $\xi(P|T) = \sqrt{E(P) - E_s(T)}$ as a local coordinate near $q_s$, we have
\[
\Omega_B(P|T) = \Omega_B(q_s(T)) + \frac{d\Omega_B}{d\xi}(q_s(T)) \xi(P|T) + O(\xi(P|T)^2)
\]
\[
(\partial_{T_A} \Omega_B)(P|T) = \frac{d\Omega_B}{d\xi}(q_s(T)) - \frac{\partial_{T_A} E_s(T)}{2\xi(P|T)} + O(1)
\]
\[
(\partial_{T_A} Q)(P|T) = \frac{dQ}{d\xi}(q_s(T)) - \frac{\partial_{T_A} E_s(T)}{2\xi(P|T)} + O(1).
\]
At the same time,
\[
(\partial_{T_A} Q)(P|T) = \partial_{T_A} \frac{dS(P|T)}{dE(P)} = \frac{d\Omega_A}{dE}(P|T).
\]
Therefore,
\[
(\partial_{T_A} \Omega_B)(P|T) = \frac{d\Omega_B}{dQ}(q_s(T)) d\frac{d\Omega_A}{dE}(P|T) + O(1)
\]
\[
= \frac{d\Omega_B}{dQ}(q_s(T)) \frac{d\Omega_A}{dE}(P|T) + O(1)
\]
and we have
\[
\operatorname{res}_{q_s}(\partial_{T_A} \Omega_B) d\Omega_C = \operatorname{res}_{q_s} \frac{d\Omega_B}{dQ}(q_s(T)) \frac{d\Omega_A}{dE} d\Omega_C(P|T)
\]
\[
= \operatorname{res}_{q_s} \frac{d\Omega_A d\Omega_B d\Omega_C}{dEdQ}(P|T).
\]
The rest of the proof is a direct calculation, which we illustrate by three different cases.
\[
\partial_{T_A} (\partial_{T_j} \partial_{T_j} F(T)) = \Re \left[ \operatorname{res}_{P_1} \xi'(d(\partial_{T_A} \Omega_j^i)) \right] = \Re \left[ \operatorname{res}_{P_1} \Omega_j^i d(\partial_{T_A} \Omega_j^i) \right]
\]
\[
= -\Re \left[ \operatorname{res}_{P_1} \partial_{T_A} \Omega_j^i d\Omega_j^i \right] = \Re \left[ \sum_{q_s \mid dE(q_s) = 0} \operatorname{res}_{q_s} \frac{d\Omega_A d\Omega_j^i d\Omega_j^i}{dEdQ} \right],
\]
since $\partial_{T_{1}}\Omega_{j}^{i}$ is holomorphic at $P_{1}$ and
\[
\Re\left[\frac{1}{2\pi\sqrt{-1}}\oint_{\hat{\Gamma}g} (\partial_{T_{1}}\Omega_{j}^{i}) d\Omega_{i}^{j}\right] = 0.
\]
\[
\partial_{T_{A}}(\partial_{T_{a}} \partial_{T_{h,k}} F(T)) = \Re\left[\frac{-1}{2\pi\sqrt{-1}} \oint_{[b_{k}]} d(\partial_{T_{A}}\Omega_{a}^{h,l})\right] = \Re\left[\frac{1}{2\pi\sqrt{-1}} \oint_{[a_{k}]} (\partial_{T_{A}}\Omega_{a}^{h,k}) d\Omega_{h,k}^{a}\right]
\]
\[
= \Re\left[ \sum_{q_{s}|dE(q_{s})=0} \text{res}_{q_{s}} \frac{d\Omega_{A}d\Omega_{h,k}^{a}d\Omega_{h,k}^{a}}{dEdQ} \right],
\]
since $\text{res}_{P_{1}}(\partial_{T_{A}}\Omega_{a}^{h,k})d\Omega_{h,k}^{a} = 0$.

\[
\partial_{T_{A}}(\partial_{T_{E}} \partial_{T_{E}} F(T)) = \Re\left[\frac{-1}{2\pi\sqrt{-1}} \oint_{[a_{k}]} Ed(\partial_{T_{A}}\Omega_{E}^{a,k})\right] = \Re\left[\frac{1}{2\pi\sqrt{-1}} \oint_{[a_{k}]} (\partial_{T_{A}}\Omega_{E}^{a,k}) dE\right]
\]
\[
= \Re\left[ \frac{1}{2\pi\sqrt{-1}} \oint_{\hat{\Gamma}g} (\partial_{T_{A}}\Omega_{E}^{a,k}) d\Omega_{E,k}^{E} \right]
\]
\[
= \Re\left[ \sum_{q_{s}|dE(q_{s})=0} \text{res}_{q_{s}} \frac{d\Omega_{A}d\Omega_{E,k}^{a}d\Omega_{E,k}^{a}}{dEdQ} \right],
\]
since $\text{res}_{P_{1}}(\partial_{T_{A}}\Omega_{E,k}^{a})d\Omega_{E,k}^{a} = 0$ and $(\partial_{T_{A}}\Omega_{E,k}^{a})$ is well-defined on $\hat{\Gamma}_{g}$. \hfill \Box

\section*{Reduction to Frobenius Structures and Solutions to the WDVV equations}

In the real-normalized case we choose the marked variable to be $T_{1}$. It correspond to the direction of unity,

(22) \[
\partial_{T_{1}} = e \sim d\Omega_{1}^{r}.
\]

In addition, we choose

(23) \[
dQ = \sqrt{-1}d\Omega_{1}^{r},
\]

and truncate the hierarchy by considering $T_{i}^{r}, T_{i}^{l}$ only for $i \leq n$. Then we have the following theorem.

\begin{theorem}
Given the above assumptions,

(24) \[
\partial_{T_{1}T_{A}T_{B}} F(T) = \begin{cases}
\frac{i}{n}, & d\Omega_{A} = d\Omega_{i}^{r}, d\Omega_{B} = d\Omega_{j}^{l}, \text{ and } i + j = n \\
\frac{2i}{n}, & d\Omega_{A} = d\Omega_{i}^{r}, d\Omega_{B} = d\Omega_{j}^{l}, \text{ and } i + j = n \\
\frac{1}{2\pi}, & d\Omega_{A} = d\Omega_{E,k}^{a}, d\Omega_{B} = d\Omega_{h,k}^{a} \\
\frac{1}{2\pi}, & d\Omega_{A} = d\Omega_{E,k}^{b}, d\Omega_{B} = d\Omega_{h,k}^{a} \\
0 & \text{otherwise}.
\end{cases}
\]

Therefore, the coordinates are flat and $F(T)$ is a solution to the WDVV equation.
\end{theorem}
Proof. Let us rewrite the formula for the third derivative with \( dQ = \sqrt{-1}d\Omega_1 \):

\[
\partial_{T_1 T_2 T_3} F(T) = \Re \left[ \sum_{q, dE(q),=0} \text{res}_q \frac{d\Omega_A d\Omega_B}{\sqrt{-1}dE} \right] = \Re \left[ \frac{1}{2\pi \sqrt{-1}} \oint_{\partial F_g} \frac{d\Omega_A d\Omega_B}{\sqrt{-1}dE} - \text{res}_{P_1} \frac{d\Omega_A d\Omega_B}{\sqrt{-1}dE} \right].
\]

First let us see what is happening near the puncture \( P_1 \). Since \( dE \sim d(z^{-n} + O(1)) \) near \( P_1 \), \( \frac{dz}{dE} \sim z^{n+1} \) has a zero of order \( n + 1 \) at \( P_1 \). It is clear that for the truncated hierarchy the only differentials that can contribute to the \( \text{res}_{P_1}(*) \)-term are \( d\Omega_1^i \) and \( d\Omega_1 \). Assume that near \( P_1 \),

\[
d\Omega_A \sim d(\varepsilon_A z^{-i} + O(1)), \quad d\Omega_B \sim d(\varepsilon_B z^{-j} + O(1)), \quad \varepsilon = 1 \text{ or } \sqrt{-1}.
\]

Then

\[
d\Omega_A d\Omega_B = (\varepsilon_A (-i)z^{-i-1} + O(1))(\varepsilon_B (-j)z^{-j-1} + O(1))(dz)^2
\]

\[
= \{\varepsilon_A \varepsilon_B (i+j)z^{-(i+j)-2} + O(z^{-i-1}) + O(z^{-j-1}) + O(1)\}(dz)^2
\]

\[
\Re \left[ -\text{res}_{P_1} \frac{d\Omega_A d\Omega_B}{\sqrt{-1}dE} \right] = \Re \left[ \frac{ij}{-n} \sqrt{-1} \varepsilon_A \varepsilon_B \right] \delta_{i+j,n} = \frac{ij}{n} \delta_{i+j,n} \delta_{\varepsilon A \varepsilon B, \sqrt{-1}}.
\]

Now let us consider the boundary term \( \frac{1}{2\pi \sqrt{-1}} \oint_{\partial F_g} \frac{d\Omega_A d\Omega_B}{\sqrt{-1}dE} \). For it to be non-zero, at least one of \( d\Omega_A, d\Omega_B \) should be \( d\Omega_{E,k}^a \) or \( d\Omega_{E,k}^b \). For example, let \( d\Omega_A = d\Omega_{E,k}^a \). Then, if \( d\Omega_B \neq d\Omega_{E,s}^a, d\Omega_{E,s}^b \),

\[
\frac{d\Omega_{E,k}^a}{dE}(P_{a_l}^+ - P_{a_l}^-) = \sqrt{-1} \delta_{kl} \quad \frac{d\Omega_{E,k}^a}{dE}(P_{b_l}^+ - P_{b_l}^-) = \begin{cases} \frac{1}{2\pi}, & d\Omega_B = d\Omega_{E,k}^a \\ 0, & \text{otherwise} \end{cases}
\]

and if \( d\Omega_B = d\Omega_{E,s}^a \) or \( d\Omega_B = d\Omega_{E,s}^b \),

\[
\Re \left[ \frac{1}{2\pi \sqrt{-1}} \oint_{\partial F_g} \frac{d\Omega_{E,k}^a d\Omega_{E,s}^a}{\sqrt{-1}dE} \right] = \Re \left[ \frac{1}{2\pi \sqrt{-1}} \oint_{\partial F_g} \frac{d\Omega_{E,k}^b d\Omega_{E,s}^b}{\sqrt{-1}dE} \right] = 0.
\]

All the remaining cases are similar to the ones considered, and this concludes the proof of the theorem. \( \square \)

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