UNIQUENESS FOR $L_p$-VISCOSITY SOLUTIONS FOR
UNIFORMLY PARABOLIC ISAACS EQUATIONS WITH
MEASURABLE LOWER ORDER TERMS

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(Communicated by Hongjie Dong)

Abstract. In this article we present several results concerning uniqueness of
$C$-viscosity and $L_p$-viscosity solutions for fully nonlinear parabolic equations.
In case of the Isaacs equations we allow lower order terms to have just measur-
able bounded coefficients. Higher-order coefficients are assumed to be Hölder
continuous in $x$ with exponent slightly less than $1/2$. This case is treated by
using stability of maximal and minimal $L_p$-viscosity solutions.

1. Introduction. For a real-valued measurable function $H(u, t, x),
\[ u = (u', u'') \in \mathbb{R}^{d+1}, \quad u'' \in \mathbb{S}, \quad (t, x) \in \mathbb{R}^{d+1}, \]
where $\mathbb{S}$ is the set of symmetric $d \times d$ matrices, and sufficiently regular functions
$v(t, x)$ we set
\[ H[v](t, x) = H(v(t, x), Du(t, x), D^2v(t, x), t, x), \]
and we will be dealing with the parabolic equations
\[ \partial_t v(t, x) + H[v](t, x) = 0 \quad (1.1) \]
in subsets of $[0, T) \times \mathbb{R}^d$, where $T \in (0, \infty)$ is fixed. Above
\[ \mathbb{R}^d = \{ x = (x_1, ..., x_d) : x_1, ..., x_d \in \mathbb{R} \}, \]
\[ \partial_t = \frac{\partial}{\partial t}, \quad D^2 u = (D_{ij} u), \quad Du = (D_i u), \quad D_i = \frac{\partial}{\partial x^i}, \quad D_{ij} = D_i D_j. \]
If $R \in (0, \infty)$ and $(t, x) \in \mathbb{R}^{d+1}$, then
\[ B_R = \{ x \in \mathbb{R}^d : |x| < R \}, \quad B_R(x) = x + B_R, \]
\[ C_R = [0, R^2) \times B_R, \quad C_R(t, x) = (t, x) + C_R. \]
We also take a bounded domain $\Omega \subset \mathbb{R}^d$ of class $C^{1,1}$ and set
\[ \Pi = [0, T) \times \Omega, \quad \partial' \Pi = \bar{\Pi} \setminus (\{0\} \times \Omega) \]

Remark 1.1. We assumed that $\Omega \in C^{1,1}$ just to be able to refer to the results
available at this moment, but actually much less is needed for our Theorems 2.1,
4.1, 4.2, 4.3, 4.4, and 5.1 to hold. For instance the exterior cone condition would
suffice.

2000 Mathematics Subject Classification. 35K55, 35B65.
Key words and phrases. Fully nonlinear equations, viscosity solutions, Isaacs equations.
We will be dealing with viscosity solutions of (1.1) in $\Pi$. The following definition is taken from [3] and has the same spirit as in [1].

**Definition 1.1.** For each choice of “regularity” class $R = C$ or $R = L_p$ we say that $u$ is an $R$-viscosity subsolution of (1.1) in $\Pi$ provided that $u$ is continuous in $\Pi$ and, for any $C_r(t_0, x_0) \subset \Pi$ and any function $\phi$, that is continuous in $C_r(t_0, x_0)$ and whose generalized derivatives satisfy $\partial_t \phi, D\phi, D^2 \phi \in R(C_r(t_0, x_0))$, and is such that $u - \phi$ attains its maximum over $C_r(t_0, x_0)$ at $(t_0, x_0)$, we have

$$\lim_{t \to 0} \sup_{C_r(t_0, x_0)} [\partial_t \phi(t, x) + H(u(t, x), D\phi(t, x), D^2 \phi(t, x), t, x)] \geq 0. \quad (1.2)$$

In a natural way one defines $R$-viscosity supersolutions and calls a function an $R$-viscosity solution if it is an $R$-viscosity supersolution and an $R$-viscosity subsolution.

Note that $C_r(t_0, x_0)$ contains $\{(t, x) : t = t_0, |x - x_0| < r\}$, which is part of its boundary. Therefore, the conditions like $D^2 \phi \in C(C_r(t_0, x_0))$ mean that the second-order derivatives of $\phi$ are continuous up to this part of the boundary.

In Section 2 we discuss uniqueness of $C$-viscosity solutions for general equations when $H$ is Lipschitz continuous with respect to $u$. The result we obtain is crucial for proving uniqueness of $L_p$-viscosity solutions in Section 5 for the Isaacs equations with measurable lower order terms. The proof of the main result in Section 2 hinges on Lemma 2.3, whose rather long proof is given in Section 3. Section 4 concentrates on the extremal $L_p$-viscosity solutions, their existence and stability. Precisely the stability of $L_p$-viscosity minimal and maximal solutions is used in Section 5.

2. Uniqueness of $C$-viscosity solutions of parabolic equations. Fix some constants $\delta \in (0, 1], K_0 \in (0, \infty)$, and set

$$S_\delta = \{ a \in S : d^{-1} |\lambda|^2 \geq a^{ij} \lambda^i \lambda^j \geq \delta |\lambda|^2 \quad \forall \lambda \in \mathbb{R}^d \}.$$ 

In the following assumption there is $\gamma = \gamma(d, \delta) \in (1/4, 1/2)$ which we specify in the following way. In Lemma 6.3 of [12]

$$\kappa(d, \delta, p) \in (1, 2)$$

(close to 1) is defined. We take $\delta = \delta(d, \delta) \in (0, \delta)$ (close to 0) from Theorem 4.1 of [12] and set

$$\kappa = \kappa(d, \delta, d + 3), \quad \gamma = \frac{7 - 3\kappa}{12 - 4\kappa} \in (1/4, 1/2).$$

**Assumption 2.1.** (i) The function $H(u, t, x)$ is a continuous function of $(u, t, x)$ and is Lipschitz continuous with respect to $u$ with Lipschitz constant $K_0$.

(ii) At all points of differentiability of $H$ with respect to $u''$ we have $D_{u''} H \in S_\delta$.

(iii) For all values of the arguments we have

$$|H(u, t, x) - H(u, t, y)| \leq K_0 |x - y|^\gamma |u''| + (|u'| + 1) \omega(|x - y|), \quad (2.1)$$

where $\omega(\tau), \tau \geq 0$, is a continuous functions vanishing at the origin.

(iv) sup $\{|H(0, t, x)| : (t, x) \in \mathbb{R}^{d+1}\} = : H < \infty$.

(iv) We are given a $g \in C(\partial^2 \Pi)$.

**Theorem 2.1.** Under the above assumptions there exists a unique $v \in C(\Pi)$ which is a $C$-viscosity solution of (1.1) in $\Pi$ with boundary condition $v = g$ on $\partial^2 \Pi$. Furthermore, there exists a constant $N \in (0, \infty)$ such that for any $\rho > 0$

$$||v||_{C^\gamma(\Pi^\rho)} \leq N \rho^{-\kappa}, \quad (2.2)$$

where $\Pi^\rho = [0, T - \rho^2] \times \Omega^\rho, \Omega^\rho = \{ x : \rho_1(x) > \rho \}, \rho_1(x) = \text{dist}(x, \Omega^\rho)$. 

Remark 2.1. The assumptions of Theorem 2.1 are almost identical to the assumptions made in the elliptic case in [16], that, to the best of our knowledge, provides the most general result to date concerning the uniqueness of $C$-viscosity solutions of class $C^{0,1}$ for the uniformly elliptic case (see Remark 3.1 there). Our Theorem 2.1 is a parabolic counterpart of Trudinger’s result from [16]. Note that unlike [16] we do not prove the comparison principle and do not require a priori our solutions to be of class $C^{0,1}$.

In the parabolic case the uniqueness of $L_p$-viscosity solutions is proved in Lemma 6.2 of [3] when $H$ is independent of $(t,x)$. In the case of the Isaacs equations, under the assumptions on the coefficients guaranteeing that our assumptions are satisfied as well, the statement about the uniqueness of $C$-viscosity solutions is found in Theorem 9.3 of [3]. However, this statement is not provided with a proof with the excuse that its proof is similar to the one known in the elliptic case.

One of the features of our proof is that it also allows one to establish an algebraic rate of convergence of numerical approximations (see [11]).

The proof of Theorem 2.1 is based on a few auxiliary results. Denote

$$[u'] = (u'_1, \ldots, u'_d).$$

Remark 2.2. It is easy to see that if $v(t,x)$ is a $C$-viscosity subsolution of (1.1) in $\Pi$, then, for any constant $c$, the functions $w(t,x) := e^{ct}v(t,x)$ is a $C$-viscosity subsolution of

$$\partial_t w + H^c[w] = 0$$

in $\Pi$, where

$$H^c(u, t, x) := e^{ct}H(e^{-ct}u, t, x) - cu'_0.$$

The function $H^c(u, t, x)$ has the same Lipschitz constant with respect to $(|u'|, u'')$ and its derivative with respect to $u'_0$, wherever it exists, is

$$D_{u'_0} H(e^{-ct}u, t, x) - c \leq K_0 - c.$$

If we take $c = K_0 + 1$ and redefine $H^c$ for $t \notin [0,T]$ as its value at the closest end point of $[0,T]$, then $H^c$ will satisfy all assumptions of Theorem 2.1 with $2K_0 + 1$ in place of $K_0$ and additionally satisfy $D_{u'_0} H \leq -1$.

That is why without loss of generality we suppose that not only Assumption 2.1 is satisfied but also for all values of arguments

$$D_{u'_0} H \leq -1$$

(2.3)

wherever the left-hand side exists.

Below we suppose that the above assumptions are satisfied, take the convex positive homogeneous of degree one function $P(u'')$ on $S$ from Theorem 4.1, and set $P[u](t,x) = P(D^2u(t,x))$. Recall that $\kappa$ is introduced before Assumption 2.1.

Lemma 2.2. (i) For any $K \geq 1$ each of the equations

$$\partial_t u_K + \max (H[u_K], P[u_K] - K) = 0,$$

$$\partial_t u_{-K} + \min (H[u_{-K}], -P[-u_{-K}] + K) = 0,$$

in $\Pi$ (a.e.) with boundary condition $u_{\pm K} = g$ on $\partial'\Pi$ has a unique solution $u_{\pm K} \in W^{1,2}_{p,loc}(\Pi^p) \cap C(\bar{\Pi})$ for any $p \geq 1$ and $\rho > 0$.

(ii) We have $u_{-K} \leq u_K$ and, as $K$ increases, $u_{-K}$ increase and $u_K$ decrease.

(iii) The family $\{u_{-K}, u_K : K \geq 1\}$ is equicontinuous in $\bar{\Pi}$. 

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(iv) There exists a constant $N \in (0, \infty)$ such that for any $\rho > 0$ and $K \geq 1$

$$\|u_K, u_{-K}\|_{C^2(\Pi^\rho)} \leq N \rho^{-\kappa}. \quad (2.6)$$

The existence part in assertion (i) for the sign $+$ follows from Theorem 4.1 which holds under more general assumptions than the ones imposed here. For the sign $-$ it suffices to replace $H(u, t, x)$ with $-H(-u, t, x)$. Uniqueness and assertion (ii) are direct consequences of the maximum principle.

Assertion (iii) for $u_K$ follows from the linear theory and the observation that

$$\max \{H(u', 0, t, x), -K\} \leq |H(u', 0, t, x)| \leq K_0 |u'| + \bar{H}.$$ 

Indeed, for any $K$, there exist $\mathbb{S}_3$-valued $a$, $\mathbb{R}^d$-valued $b$, and real-valued $c \geq 0$ and $f$ such that

$$\partial_t u_K + a^{ij} D_{ij} u_K + b^i D_i u_K - cu_K + f = 0$$

and $|b| \leq K_0$, $c \leq K_0$, $|f| \leq \bar{H}$. For $u_{-K}$ the argument is similar. Assertion (iv) follows from Theorem 2.1 of [12].

**Lemma 2.3.** Under the assumptions of this section, for $K \to \infty$ we have $|u_K - u_{-K}| \to 0$ uniformly in $\Omega$.

This lemma is proved in Section 3.

**Proof of Theorem 2.1.** First we prove uniqueness. Introduce $\psi \in C^2(\mathbb{R}^d)$ as a global barrier for $\Omega$, that is, in $\Omega$ we have $\psi \geq 1$ and

$$a^{ij} D_{ij} \psi + b^i D_i \psi \leq -1$$

for any $(a_{ij}) \in \mathbb{S}_3$, $|b_i| \leq K_0$. Such a $\psi$ can be found in the form $\cosh \mu R - \cosh \mu |x|$ for sufficiently large $\mu$ and $R$.

Then we take and fix a radially symmetric with respect to $x$ function $\zeta = \zeta(t, x)$ of class $C^\infty_c(\mathbb{R}^{d+1})$ with support in $(-1, 0) \times B_1$ and unit integral. For $\varepsilon > 0$ we define $\zeta(\varepsilon(t, x) = \varepsilon^{-d-2} \zeta(\varepsilon^{-2} t, \varepsilon^{-1} x)$ and for locally summable $u(t, x)$ introduce

$$u(\varepsilon)(t, x) = u(t, x) * \zeta(\varepsilon(t, x)). \quad (2.7)$$

Let $\Omega_n$, $n = 2, 3, ..., \ldots$ be a sequence of strictly expanding smooth domains whose union is $\Omega$ and set $\Pi_n = [0, T(1 - 1/n)] \times \Omega_n$. Then for any $n_0 = 3, 4, ...$ and all sufficiently small $\varepsilon > 0$

$$\xi_{\varepsilon, n} := \partial_t \psi(\varepsilon) + \max \{H[\psi(\varepsilon)], P[\psi(\varepsilon)] - K\}$$

is well defined in $\Pi_{n_0}$. Since the second-order derivatives with respect to $x$ and the first derivative with respect to $t$ of $u_K$ are in $L^p(\Pi^\rho)$ for any $p$ and $\rho$, we have $\xi_{\varepsilon, K} \to 0$ as $\varepsilon \downarrow 0$ in any $L^p(\Pi_{n_0})$ for any $K$ and any $p > 1$. Furthermore, $\xi_{\varepsilon, K}$ are continuous because $H(u, t, x)$ is continuous. Therefore, there exist smooth functions $\zeta_{\varepsilon, K}$ on $\Pi_{n_0}$ such that

$$-\varepsilon \leq \xi_{\varepsilon, K} + \zeta_{\varepsilon, K} \leq 0$$

in $\Pi_{n_0}$ for all small $\varepsilon > 0$.

Since $\Omega_{n_0}$ is smooth, by Theorem 1.1 of [4] there exists a unique $u_{\varepsilon, K} \in \bigcap_{p>1} W^{1,2}_{p}(\Pi_{n_0})$ satisfying

$$\partial_t u_{\varepsilon, K} + \sup_{a \in \mathbb{S}_3, |b| \leq K_0}_{0 \leq \varepsilon \leq K_0} \left[ a_{ij} D_{ij} u_{\varepsilon, K} + b^i D_i u_{\varepsilon, K} - cu_{\varepsilon, K} \right] = \zeta_{\varepsilon, K}$$
in $\Pi_{n_0}$ (a.e.) and such that $w_{\varepsilon,K} = 0$ on $\partial \Pi_{n_0}$. By the maximum principle such $w_{\varepsilon,K}$ is unique. Then owing to the continuity of $\zeta_{\varepsilon,K}$, for any $\varepsilon$ and $K$, for all sufficiently small $\beta > 0$, we have

$$\partial_t w_{\varepsilon,K}^{(\beta)} + \sup_{a \in S_\ell,|b| \leq K_0} \left[ a_{ij} D_{ij} w_{\varepsilon,K}^{(\beta)} + b_i D_i w_{\varepsilon,K}^{(\beta)} - c w_{\varepsilon,K}^{(\beta)} \right] \leq \zeta_{\varepsilon,K}^{(\beta)} \leq \zeta_{\varepsilon,K} + \varepsilon \quad (2.8)$$

in $\Pi_{n_0-1}$.

For $w_{\varepsilon,K}^{(\beta)}(t,x) := w_{\varepsilon,K}^{(\beta)} + \varepsilon(T-t)$ now, obviously,

$$\partial_t (u_{\varepsilon,K}^{(\beta)} + w_{\varepsilon,K}^{(\beta)}) + \max \left( H[u_{\varepsilon,K}^{(\beta)} + w_{\varepsilon,K}^{(\beta)}], P[u_{\varepsilon,K}^{(\beta)} + w_{\varepsilon,K}^{(\beta)}] - K \right)$$

$$\leq \partial_t u_{\varepsilon,K}^{(\beta)} + \max \left( H[u_{\varepsilon,K}^{(\beta)}], P[u_{\varepsilon,K}^{(\beta)}] - K \right) + \partial_t w_{\varepsilon,K}^{(\beta)}$$

$$+ \sup_{a \in S_\ell,|b| \leq K_0} \left[ a_{ij} D_{ij} w_{\varepsilon,K}^{(\beta)} + b_i D_i w_{\varepsilon,K}^{(\beta)} - c w_{\varepsilon,K}^{(\beta)} \right] \leq \zeta_{\varepsilon,K} + \zeta_{\varepsilon,K} \leq 0 \quad (2.9)$$

in $\Pi_{n_0-1}$. Hence, in $\Pi_{n_0-1}$

$$\partial_t (u_{\varepsilon,K}^{(\beta)} + w_{\varepsilon,K}^{(\beta)}) + H[u_{\varepsilon,K}^{(\beta)} + w_{\varepsilon,K}^{(\beta)}] \leq 0.$$

Since $H[v] - H[v + \gamma \psi] = a^{ij} D_{ij} \psi + b^i D_i \psi - c \psi$ for some $S_{\beta}$-valued $a$, $\mathbb{R}^d$-valued $b$, such that $|b| \leq K_0$, and $c \geq 1$, we have in $\Pi_{n_0-1}$ that

$$\partial_t (u_{\varepsilon,K}^{(\beta)} + w_{\varepsilon,K}^{(\beta)}) + H[u_{\varepsilon,K}^{(\beta)} + w_{\varepsilon,K}^{(\beta)} + \beta \psi] \leq -\beta < 0.$$

This and the definition of $C$-viscosity solutions imply that, if $v$ is a continuous in $\bar{\Pi}$, $C$-viscosity solution of $\partial_t v + H[v] = 0$ with boundary data $g$, then the minimum of $u_{\varepsilon,K}^{(\beta)} + w_{\varepsilon,K}^{(\beta)} + \beta \psi - v$ in $\Pi_{n_0-1}$ is either nonnegative or is attained on the parabolic boundary of $\Pi_{n_0-1}$. The same conclusion holds after letting first $\beta \downarrow 0$ and then $\varepsilon \downarrow 0$. Combining this with the Aleksandrov estimates showing that $w_{\varepsilon,K} \rightarrow 0$ as $\varepsilon \downarrow 0$ uniformly on $\Pi_{n_0}$, we get that in $\Pi_{n_0}$

$$u_{K} - v \geq - \sup_{n \in \Pi_{n_0-1}} |u_{K} - v|,$$

which after letting $n_0 \rightarrow \infty$ and then $K \rightarrow \infty$ yields $v \leq u$, where $u$ is the common limit of $u_{K}$, $u_{K}$, which exists by Lemma 2.3. By comparing $v$ with $u_{K}$, we get $v \geq u$, and hence uniqueness. After that estimate (2.2) follows immediately from (2.6), and the theorem is proved.

3. Proof of Lemma 2.3. To prove Lemma 2.3 we need an auxiliary result. In the following theorem $\Omega$ can be just any bounded domain. Below by $C^{1,2}_{\text{loc}}(\Pi)$ we mean, as usual, the space of functions $u = u(t,x)$ which are continuous in $\Pi$ along with their derivatives $\partial_t u, D_{ij} u, D_i u$. We recall that $\kappa$ is introduced before Assumption 2.1 and fix a constant

$$\tau \in (0,1).$$

**Theorem 3.1.** Let $u, v \in C^{1,2}_{\text{loc}}(\Pi) \cap C(\bar{\Pi})$ be such that for a constant $K \geq 1$

$$\partial_t u + \max \left( H[u], P[u] - K \right) \geq 0 \geq \partial_t v + \min \left( H[v], -P[-v] + K \right) \quad (3.1)$$

in $\Pi$ and $v \geq u$ on $\partial \Pi$. Also assume that, for a constant $M \in [1, \infty)$,

$$\|u, v\|_{C^{0}(\Pi)} \leq M. \quad (3.2)$$


Then there exist a constant $N \in (0, \infty)$, depending only on $\tau$, the diameter of $\Omega$, $d$, $K_0$, $H$, and $\delta$, and a constant $\eta > 0$, depending only on $\tau$, $d$, and $\delta$, such that, if $K \geq N M^\eta$ and
\[ K \geq T^{-1}, \quad \Pi^{2/\sqrt{\kappa}} \neq \emptyset, \] (3.3)
then in $\Pi$
\[ u(t, x) - v(t, x) \leq N K^{-(\kappa-1)/4} + N M \omega(M^{-1/\tau} K^{-1}). \] (3.4)

Remark 3.1. The purpose of introducing $\tau$ is that for $\omega = t^\tau$ estimate (3.4) becomes
\[ u - v \leq N K^{-(\kappa-1)/4} + N K^{-\tau}, \]
which was used in [11] to estimate the rate of convergence of finite-difference approximations for (1.1).

The statement of this theorem is almost identical to that of Theorem 3.1 of [11] although that theorem is about the Isaacs equations and our equations are more general. However, the most part of the proof follows that of Theorem 3.1 of [11] and, as there, we are going to adapt to our situation an argument from Section 5.A of [1]. For that we need a construction and two lemmas. From the start throughout the section we will only concentrate on $K$ satisfying (3.3).

We take and fix a function $\zeta = \zeta(t, x)$ as before (2.7) and use the notation $u^{(\varepsilon)}$ introduced in (2.7). Recall some standard properties of parabolic mollifiers in which no regularity properties of $\Omega$ are required: If $u \in C^\kappa(\Pi)$, then in $\Pi^\varepsilon$
\[ \varepsilon^{-\kappa}|u - u^{(\varepsilon)}| + \varepsilon^{-(\kappa-1)}|Du - Du^{(\varepsilon)}| \leq N \|u\|_{C^\kappa(\Pi)}, \]
\[ |u^{(\varepsilon)}| + |Du^{(\varepsilon)}| + \varepsilon^{2-\kappa}|D^2 u^{(\varepsilon)}| + \varepsilon^{2-\kappa}|\partial_t u^{(\varepsilon)}| \]
\[ + \varepsilon^{3-\kappa}|D^3 u^{(\varepsilon)}| + \varepsilon^{3-\kappa}|D \partial_t u^{(\varepsilon)}| + \varepsilon^{4-\kappa}|\partial_t D^2 u^{(\varepsilon)}| \leq N \|u\|_{C^\kappa(\Pi)}, \] (3.5)
where the constants $N$ depend only on $d$ and $\kappa$.

Next, take the constants $\nu, \varepsilon_0 \in (0, 1)$, specified below in Lemma 3.3 and (3.31), respectively, depending only on $d$, $K_0$, $H$, $\delta$, and the diameter of $\Omega$, introduce
\[ \varepsilon = \varepsilon_0 M^{-1/(\kappa-1)} K^{-(1-\gamma)/(2\gamma)}, \]
and consider the function
\[ W(t, x, y) = u(t, x) - u^{(\varepsilon)}(t, x) - [v(t, y) - u^{(\varepsilon)}(t, y)] - \nu K|x - y|^2 \]
for $(t, x), (t, y) \in \Pi^\varepsilon$. Note that $\Pi^\varepsilon \neq \emptyset$ and even $\Pi^{2\varepsilon} \neq \emptyset$ owing to (3.3) and the fact that $1 - \gamma > \gamma$ and $K, M \geq 1$.

We will need the following simple observation.

**Lemma 3.2.** For any $\chi > 0$ there exists $N = N(\chi, d, \delta)$ such that, if $K \geq N$, then
\[ \varepsilon M \leq \chi K^{-(\kappa-1)/4}. \] (3.6)

Proof. Since $\varepsilon_0 < 1, \kappa \leq 2$, and $M \geq 1$, the left-hand side of (3.6) is less than
\[ K^{-(1-\gamma)/(2\gamma)} = K^{-(5-\kappa)/(14-6\kappa)}. \]
One easily checks that $(5 - \kappa)/(14 - 6\kappa) > 1/2 > (\kappa - 1)/4$ for $\kappa \in (1, 2)$ and this proves the lemma. \qed

Denote by $(\bar{t}, \bar{x}, \bar{y})$ a maximum point of $W$ in $[0, T - \varepsilon^2] \times (\Omega^\varepsilon)^2$. Observe that, obviously,
\[ u(\bar{t}, \bar{x}) - u^{(\varepsilon)}(\bar{t}, \bar{x}) - [v(\bar{t}, \bar{y}) - u^{(\varepsilon)}(\bar{t}, \bar{y})] - \nu K|\bar{x} - \bar{y}|^2 \]
\[ \geq u(\bar{t}, \bar{x}) - u^{(\varepsilon)}(\bar{t}, \bar{x}) - [v(\bar{t}, \bar{x}) - u^{(\varepsilon)}(\bar{t}, \bar{x})], \]
which implies that
\[ |\bar{x} - \bar{y}| \leq N M/(\nu K). \] (3.7)
where and below by \( N \) with indices or without them we denote various constants depending only on \( d, K_0, \tilde{H}, \delta, \tau, \) and the diameter of \( \Omega \), unless specifically stated otherwise. By the way, recall that \( \kappa \) and, hence, \( \gamma \) depend only on \( d \) and \( \delta \).

**Lemma 3.3.** There exist a constant \( \nu \in (0,1) \), depending only on \( d, K_0, \tilde{H}, \delta, \) and the diameter of \( \Omega \), and a constant \( N \in [1, \infty) \) such that, if

\[
K \geq N \nu^{-\eta_1} \varepsilon_0^{(\kappa - 2)\eta_1} M^{n(2-\kappa)/(\kappa - 1)},
\]

where \( \eta_1 = 1 - (2 - \kappa)(1 - \gamma)(2\gamma)^{-1} \) (\( > 0 \)), and \( \bar{x}, \bar{y} \in \Omega' \) and \( \bar{t} < T - \varepsilon^2 \), then

(i) we have

\[
2\nu|\bar{x} - \bar{y}| \leq NMK^{-1}\varepsilon^{\kappa - 1} = N\varepsilon_0^{\kappa - 1}K^{-3+\kappa)/(8\gamma)}, \quad |\bar{x} - \bar{y}| \leq \varepsilon/2;
\]

(ii) for any \( \xi, \eta \in \mathbb{R}^d \)

\[
D_{ij}[u - u^{(\varepsilon)}](\bar{t}, \bar{x})\xi^i\xi^j - D_{ij}[v - u^{(\varepsilon)}](\bar{t}, \bar{y})\eta^i\eta^j \leq 2\nu K|\xi - \eta|^2,
\]

\[
\partial_t[u - u^{(\varepsilon)}](\bar{t}, \bar{x}) \leq \partial_t[v - u^{(\varepsilon)}](\bar{t}, \bar{y});
\]

(iii) we have

\[
\partial_t u(\bar{t}, \bar{x}) + H[u](\bar{t}, \bar{x}) \geq 0,
\]

\[
\partial_t v(\bar{t}, \bar{y}) + H[v](\bar{t}, \bar{y}) \leq 0.
\]

**Proof.** The first inequality in (3.9) follows from (3.5) and the fact that the first derivatives of \( W \) with respect to \( x \) vanish at \( \bar{x} \), that is,

\[
D[\tilde{u} - \bar{u}^{(\varepsilon)}](\bar{t}, \bar{x}) = 2\nu K(\bar{x} - \bar{y}).
\]

Also the matrix of second-order derivatives of \( W \) with respect to \( (x, y) \) is nonpositive at \( (\bar{t}, \bar{x}, \bar{y}) \) as well as its (at least one sided if \( \bar{t} = 0 \)) derivative with respect to \( t \), which yields (ii).

By taking \( \eta = 0 \) in (3.10) and using the fact that \( |D^2u^{(\varepsilon)}| \leq NM\varepsilon^{\kappa - 2} \) we see that

\[
D^2u(\bar{t}, \bar{x}) \leq 2\nu K + NM\varepsilon^{\kappa - 2}.
\]

Similarly,

\[
D^2v(\bar{t}, \bar{x}) \geq -N(\nu K + M\varepsilon^{\kappa - 2}),
\]

which yields

\[
P[u](\bar{t}, \bar{x}) \leq N_1(\nu K + M\varepsilon^{\kappa - 2}), \quad -P[-v](\bar{t}, \bar{y}) \geq -N_1(\nu K + M\varepsilon^{\kappa - 2}).
\]

Since \( H(u, t, x) = a^{ij}u_{ij} + H(u', 0, t, x) \) where \( (a^{ij}) \in S_\delta \) \((a^{ij}) \) depends on \( u, t, x \) and \( |H(u', 0, t, x)| \leq K_0|u'| + \tilde{H} \) and \( M \geq 1 \) and \( \varepsilon < 1 \), we also have (by increasing the above \( N_1 \) if necessary)

\[
H[u](\bar{t}, \bar{x}) \leq N_1(\nu K + M\varepsilon^{\kappa - 2}), \quad H[v](\bar{t}, \bar{y}) \geq -N_1(\nu K + M\varepsilon^{\kappa - 2}).
\]

Also it follows from (3.11) and (3.5) that

\[
\partial_t u(\bar{t}, \bar{x}) \leq \partial_t v(\bar{t}, \bar{y}) + N_2 M\varepsilon^{\kappa - 2}.
\]

Now, if \( H[u](\bar{t}, \bar{x}) \leq P[u](\bar{t}, \bar{x}) - K \), then at \( (\bar{t}, \bar{x}) \)

\[
0 \leq \partial_t u + \max(H[u], P[u] - K) \leq \partial_t u + N_1(\nu K + M\varepsilon^{\kappa - 2}) - K,
\]

\[
\partial_t u \geq K - N_1(\nu K + M\varepsilon^{\kappa - 2})
\]

and at \( (\bar{t}, \bar{y}) \)

\[
0 \geq \partial_t v + \min(H[v], -P[-v] + K) \geq K - 2N_1(\nu K + M\varepsilon^{\kappa - 2}) - N_2 M\varepsilon^{\kappa - 2}.
\]
Here \( M\varepsilon^{\kappa-2} \leq \nu K \), which is equivalent to (3.8) with \( N = 1 \). Hence
\[
K \leq 4N_1\nu K + N_2\nu K,
\]
which is impossible if we choose and fix \( \nu \) such that
\[
(4N_1 + N_2)\nu \leq 1/2.
\]
It follows that \( H[u](\bar{t}, \bar{x}) \geq P[u](\bar{t}, \bar{x}) - K \),
\[
\partial_t u(\bar{t}, \bar{x}) + H[u](\bar{t}, \bar{x}) \geq 0,
\]
and this proves (3.12).

Similarly, if \(-P[-v](\bar{t}, \bar{y}) + K \leq H[v](\bar{t}, \bar{y})\), then at \((\bar{t}, \bar{y})\)
\[
0 \geq \partial_t v - N_1(\nu K + M\varepsilon^{\kappa-2}) + K,
\]
and at \((\bar{t}, \bar{x})\) we have \( \partial_t u \leq N_1(\nu K + M\varepsilon^{\kappa-2}) - K + N_2M\varepsilon^{\kappa-2} \)
\[
0 \leq \partial_t u + \max \left( H[u], P[u] - K \right) \leq -K + 2N_1(\nu K + M\varepsilon^{\kappa-2}) + N_2M\varepsilon^{\kappa-2},
\]
which again is impossible with the above choice of \( \nu \) for \( K \) satisfying (3.8). This
yields (3.13).

Moreover, not only \( M\varepsilon^{\kappa-2} \leq \nu K \) for \( K \) satisfying (3.8), but we also have
\( NM\varepsilon^{\kappa-2} \leq \nu K \), where \( N \) is taken from (3.9), if we increase \( N \) in (3.8). This
yields the second inequality in (3.9).

The lemma is proved. \( \square \)

Everywhere below in this section \( \nu \) is the constant from Lemma 3.3.

**Lemma 3.4.** There exist a constant \( N \) such that, for any \( \mu \geq 0 \), if \( K \geq NM\mu \),
where \( \eta_2 = 8/(5 - \kappa) \), and
\[
W(\bar{t}, \bar{x}, \bar{y}) \geq 2K^{-(\kappa-1)/4} + \mu M\omega(M^{-1/\tau}K^{-1}),
\]
then
\[
u(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y}) - \nu K|\bar{x} - \bar{y}|^2 \geq K^{-(\kappa-1)/4} + \mu M\omega(M^{-1/\tau}K^{-1}).
\]
Furthermore, \( \bar{x}, \bar{y} \in \Omega^{2\varepsilon} \) and \( \bar{t} < T - \varepsilon^2 \).

**Proof.** It follows from (3.7) that (recall that \( \nu \) is already fixed)
\[
|u^{(\varepsilon)}(\bar{t}, \bar{x}) - u^{(\varepsilon)}(\bar{t}, \bar{y})| \leq M|\bar{x} - \bar{y}| \leq N_0M^2/K.
\]
Hence we have from (3.16) that
\[
u(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y}) - \nu K|\bar{x} - \bar{y}|^2 \geq 2K^{-(\kappa-1)/4} - N_0M^2/K + \mu M\omega(M^{-1/\tau}K^{-1}),
\]
and (3.17) follows provided that
\[
N_0M^2/K \leq (1/4)K^{-(\kappa-1)/4},
\]
which indeed holds if
\[
K \geq NM\mu.
\]
Now, if \( \bar{x} \) or \( \bar{y} \) is in \( \tilde{\Omega}^c \setminus \Omega^{2\varepsilon} \), then for appropriate \( \tilde{x} \in \partial\Omega \) and \( \tilde{y} \in \partial\Omega \) either
\[
u(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y}) \leq u(\bar{t}, \bar{x}) - u(\bar{t}, \tilde{x}) + v(\bar{t}, \tilde{x}) - v(\bar{t}, \bar{y}) \leq M(4\varepsilon + |\bar{x} - \bar{y}|)
\]
or
\[
u(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y}) \leq u(\bar{t}, \bar{x}) - u(\bar{t}, \bar{y}) + v(\bar{t}, \bar{y}) - v(\bar{t}, \bar{y}) \leq M(4\varepsilon + |\bar{x} - \bar{y}|).
\]
In any case in light of (3.17), (3.7), and (3.19)
\[
4\varepsilon M + N_0M^2/K - \nu K|\bar{x} - \bar{y}|^2 \geq K^{-(\kappa-1)/4},
\]
Owing to Lemma 3.2, inequality (3.21) is impossible if $K \geq N$ or, upon adjusting $N$ in (3.20) appropriately, if (3.20) holds (recall that $M \geq 1$). Below we assume that (3.20) holds after the modification, so that we have $\bar{\lambda}$.

Lemma 3.5. Let

$$\beta \in [0, \delta/2].$$

Then for any $\alpha \geq -4\nu$ we have

$$\beta \alpha - \frac{\alpha^2}{\alpha + 4\nu} \leq 2(\nu/\delta)^2.$$

Proof of Theorem 3.1. Fix a (large) constant $\mu > 0$ to be specified later as a constant, depending only on $d$, $K_0$, $\bar{H}$, $\delta$, $\tau$, and the diameter of $\Omega$, recall that $\nu$ is found in Lemma 3.3 and first assume that

$$W(t, x, y) \leq 2K^{-(\kappa-1)/4} + \mu M\omega(M^{-1/\tau} K^{-1})$$

for $(t, x), (t, y) \in \Pi^\circ$. Observe that for any point $(t, x) \in \Pi$ one can find a point $(s, y) \in \Pi^\circ$ with $|x - y| \leq \varepsilon$ and $|t - s| \leq \varepsilon^2$ and then

$$u(t, x) - v(t, x) \leq u(s, y) - v(s, y) + NM\varepsilon \leq W(s, y, y) + NM\varepsilon$$

$$\leq 2K^{-(\kappa-1)/4} + NM\varepsilon + \mu M\omega(M^{-1/\tau} K^{-1}).$$

In that case, as follows from Lemma 3.2, (3.4) holds for $K$ satisfying (3.20) with any $N \geq 1$ in (3.20).

It is clear now that, to prove the theorem, it suffices to find $\mu$ such that the inequality (3.16) is impossible if $K \geq NM^\eta$ with $N$ and $\eta$ as in the statement of the theorem and at least not smaller than those in (3.20). Of course, we will argue by contradiction and suppose that (3.16) holds, despite (3.20) is valid and (3.8) is satisfied with $\nu$ fixed in Lemma 3.3 and $\varepsilon_0 \in (0, 1)$, which is yet to be specified.

Then (3.17) holds, and, in particular,

$$u(\bar{t}, x) \geq v(\bar{t}, y).$$

(3.22)

Also by Lemma 3.4 the points $\bar{x}, \bar{y}$ are in $\Omega^\circ$ (even in $\Omega^{2\varepsilon}$) and $\bar{t} < T - \varepsilon^2$, so that we can use the conclusions of Lemma 3.3.

Denote

$$ A = D^2[u - u(\varepsilon)](\bar{t}, x), \quad B = D^2[v - u(\varepsilon)](\bar{t}, y)$$

and interpret matrices as linear operators in a usual way and constants as operators of multiplications by these constants. Observe that (3.10) implies that the operator $B + 2\nu K$ is nonnegative and, hence, $B + 4\nu K$ is strictly positive. Then for

$$\eta = 4\nu K(B + 4\nu K)^{-1}\xi, \quad \eta - \xi = -B(B + 4\nu K)^{-1}\xi$$

inequality (3.10) yields

$$\langle A\xi, \xi \rangle \leq \langle B4\nu K(B + 4\nu K)^{-1}\xi, 4\nu K(B + 4\nu K)^{-1}\xi \rangle + 4\nu K\|B(B + 4\nu K)^{-1}\xi\|^2$$

$$= \langle 4\nu KB(B + 4\nu K)^{-1}\xi, \xi \rangle.$$
Hence $A \leq 4\nu KB(B + 4\nu K)^{-1}$ and
\[
D^2 u(\bar{t}, \bar{x}) \leq D^2 u^{(e)}(\bar{t}, \bar{x}) + 4\nu KB(B + 4\nu K)^{-1} - B^2(B + 4\nu K)^{-1} + D^2 u^{(e)}(\bar{t}, \bar{x}) - D^2 u^{(e)}(\bar{t}, \bar{y}) - D^2 u^{(e)}(\bar{t}, \bar{y}).
\]

We now use (3.9) to get that $|\bar{x} - \bar{y}| \leq NMK^{-1}\varepsilon^{\kappa-1}$, and in light of (3.5) that $|D^2 u^{(e)}(\bar{t}, \bar{x}) - D^2 u^{(e)}(\bar{t}, \bar{y})| \leq N\varepsilon^{\kappa-3} |\bar{x} - \bar{y}| \leq NC$.

where
\[
C = M^2 K^{-1}\varepsilon^{2\kappa-4}.
\]

Also (3.11) reads
\[
\partial_t u(\bar{t}, \bar{x}) \leq \partial_t v(\bar{t}, \bar{y}) + \partial_t u^{(e)}(\bar{t}, \bar{x}) - \partial_t u^{(e)}(\bar{t}, \bar{y})
\]
and as is easy to see
\[
|\partial_t u^{(e)}(\bar{t}, \bar{x}) - \partial_t u^{(e)}(\bar{t}, \bar{y})| \leq NC.
\]

Thus far, we have
\[
D^2 u(\bar{t}, \bar{x}) \leq D^2 v(\bar{t}, \bar{y}) - B^2(B + 4\nu K)^{-1} + NC,
\]
\[
\partial_t u(\bar{t}, \bar{x}) \leq \partial_t v(\bar{t}, \bar{y}) + NC.
\]

Next,
\[
D_t u(\bar{t}, \bar{x}) = 2\nu K (\bar{x}^i - \bar{y}^i) + D_t u^{(e)}(\bar{t}, \bar{x}),
\]
\[
D_t v(\bar{t}, \bar{y}) = 2\nu K (\bar{x}^i - \bar{y}^i) + D_t u^{(e)}(\bar{t}, \bar{y}),
\]
\[
D_t u(\bar{t}, \bar{x}) - D_t v(\bar{t}, \bar{y}) = D_t u^{(e)}(\bar{t}, \bar{x}) - D_t u^{(e)}(\bar{t}, \bar{y}),
\]
where
\[
|D_t u^{(e)}(\bar{t}, \bar{x}) - D_t u^{(e)}(\bar{t}, \bar{y})| \leq N\varepsilon^{\kappa-2} |\bar{x} - \bar{y}| \leq NC
\]
and, therefore,
\[
|D(u(\bar{t}, \bar{x}) - D v(\bar{t}, \bar{y})| \leq NC.
\]

It follows from (3.23), (3.24), (3.12), and (3.22) by Assumption 2.1 and the fact that $D_{\bar{u}_i} H \leq -1$ (see (2.3)) that
\[
0 \leq \partial_t u(\bar{t}, \bar{x}) + H(u(\bar{t}, \bar{x}), D u(\bar{t}, \bar{x}), D^2 v(\bar{t}, \bar{y}) - B^2(B + 4\nu K)^{-1} + NC, \bar{t}, \bar{x})
\]
\[
\leq \partial_t v(\bar{t}, \bar{y}) + NC + H(v(\bar{t}, \bar{y}), D v(\bar{t}, \bar{y}), D^2 v(\bar{t}, \bar{y}) + NC, \bar{t}, \bar{x})
\]
\[
- \left[u(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y})\right] = \partial_t B^2(B + 4\nu K)^{-1}
\]
\[
\leq \partial_t v(\bar{t}, \bar{y}) + NC + H(v(\bar{t}, \bar{y}), D v(\bar{t}, \bar{y}), D^2 v(\bar{t}, \bar{y}), \bar{t}, \bar{x})
\]
\[
- \left[u(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y})\right] - \partial_t B^2(B + 4\nu K)^{-1} + NC.
\]

This along with (3.13) and Assumption 2.1 (iii) yields (recall that $M \geq 1$)
\[
u(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y}) \leq K_0 |\bar{x} - \bar{y}|^p |D^2 v(\bar{t}, \bar{y})| + NM \varepsilon^{\kappa} (\bar{x} - \bar{y})
\]
\[
- \partial_t B^2(B + 4\nu K)^{-1} + NC.
\]

Upon combining this with (3.17) we arrive at
\[
2K^{-\kappa-1/4} + \mu M \varepsilon^{\kappa} (\bar{x} - \bar{y})^p + N_1 M \varepsilon^{\kappa} (\bar{x} - \bar{y}) - \partial_t B^2(B + 4\nu K)^{-1}.
\]

Now we choose $\mu = N_1$ and observe that (see (3.9))
\[
\bar{x} - \bar{y} \leq N_2 K^{-3+\kappa}/(8\gamma) \leq M^{-1/\tau} K^{-1}
\]
and this brings the proof of the theorem to an end.

Here, owing to (3.5) and (3.9),

\[ |x - \gamma| |D^2v(t, \bar{y})| \leq N_{\varepsilon_0}^\gamma K^{-(3+\kappa)/8} (|B| + |D^2u(z)(t, \bar{y})|) \]

\[ \leq N_{\varepsilon_0}^\gamma K^{-(3+\kappa)/8} |B| + N_4 M^{1/(\kappa-1)} \varepsilon_0^{(\kappa-1)\gamma + \kappa - 2} K^{\theta_2(\kappa-1)/4}, \]

where \( \theta_2 = (7\kappa - 19)/(14 - 6\kappa) \), and as is easy to see \( \theta_2 \leq -3/2 \) for \( \kappa \in [1, 2] \), so that

\[ N_4 M^{1/(\kappa-1)} \varepsilon_0^{(\kappa-1)\gamma + \kappa - 2} K^{\theta_2(\kappa-1)/4} \leq (1/4) K^{-(\kappa-1)/4} \]

if, for instance,

\[ K \geq (4N_4 M^{1/(\kappa-1)} \varepsilon_0^{(\kappa-1)\gamma + \kappa - 2})^{8/(\kappa-1)}. \] (3.28)

In what concerns the last term in (3.30), note that

\[ N_3 C = N_3 M^{2/(\kappa-1)} \varepsilon_0^{2\kappa - 4} K^{\theta_3(\kappa-1)/4}, \]

where \( \theta_3 = (4\kappa - 12)/(7 - 3\kappa) \leq -2 \) for \( \kappa \in [1, 2] \), so that

\[ N_3 C \leq (1/4) K^{-(\kappa-1)/4} \]

if

\[ K \geq (4N_3 M^{2/(\kappa-1)} \varepsilon_0^{2\kappa - 4})^{4/(\kappa-1)}. \] (3.29)

We conclude that, for \( K \) satisfying (3.28) and (3.29), relation (3.27) yields

\[ K^{-(\kappa-1)/4} \leq N_{\varepsilon_0}^\gamma K^{-(3+\kappa)/8} |B| - \delta tr B^2 (B + 4\nu K)^{-1}. \] (3.30)

Next, observe that, by Lemma 3.5 applied after we diagonalize \( B \) and set \( B = K \alpha \) implies that the right-hand side of (3.30) is

\[ K \left[ N_{\varepsilon_0}^\gamma K^{-(3+\kappa)/8} |\alpha| - \delta tr \alpha^2 (\alpha + 4\nu)^{-1} \right] \leq N_5 \varepsilon_0^{2(\kappa-1)\gamma} K^{-(\kappa-1)/4}, \]

where the inequality holds owing to Cauchy’s inequality. We can certainly assume that \( N_5 \geq 1 \) and then we can choose \( \varepsilon_0 \in (0, 1) \) so that

\[ N_5 \varepsilon_0^{2(\kappa-1)\gamma} = 1/2, \] (3.31)

which along with (3.30) leads to the desired contradiction:

\[ K^{-(\kappa-1)/4} \leq (1/2) K^{-(\kappa-1)/4}. \]

With so specified \( \varepsilon_0 \) we rewrite condition (3.8) (with \( \nu \) fixed in Lemma 3.3), conditions (3.20), (3.26), (3.28), and (3.29) as

\[ K \geq N M^{(2-\kappa)/(\kappa-1)}, \quad K \geq N M^{\theta_2}, \quad K \geq N M^{1/(\theta_1 \tau)}, \quad K \geq N M^{8/(\kappa-1)^2}. \]

Since \( M \geq 1 \), for \( \eta' \) defined as the sum of the above powers of \( M \) and \( N' \) defined as the sum of the above \( N \)'s, the inequality (3.16) is impossible for \( K \geq N' M^{\eta'} \) and this brings the proof of the theorem to an end. \( \square \)
Proof of Lemma 2.3. Let $\Omega_n$, $n = 2, 3, \ldots$, be a sequence of strictly expanding smooth domains whose union is $\Omega$ and set $\Pi_n = [0, T(1-1/n)] \times \Omega_n$. Then for any $n_0 = 3, 4, \ldots$ and all sufficiently small $\varepsilon > 0$

$$
\xi_{\varepsilon,K} = \partial_t u_{\varepsilon,K}^{(\varepsilon)} + \max \left( H\left[u_{\varepsilon,K}^{(\varepsilon)}\right], P[u_{\varepsilon,K}^{(\varepsilon)}] - K\right),
$$

$$
\xi_{\varepsilon,-K} = \partial_t u_{\varepsilon,K}^{(\varepsilon)} + \min \left( H\left[u_{\varepsilon,K}^{(\varepsilon)}\right], -P[-u_{\varepsilon,K}^{(\varepsilon)}] + K\right),
$$

are well defined in $\Pi_{n_0}$. Since the second-order derivatives with respect to $x$ and the first derivative with respect to $t$ of $u_{\pm,K}$ are in $L_p(\Pi^p)$ for any $p$ and $p$, we have $\xi_{\varepsilon, \pm K} \to 0$ as $\varepsilon \downarrow 0$ in any $L_p(\Pi_{n_0})$ for any $K$ and any $p > 1$. Furthermore, $\xi_{\varepsilon, \pm K}$ are continuous because $H(u,t,x)$ is continuous. Therefore, there exist smooth functions $\zeta_{\varepsilon,K}$ on $\Pi_{n_0}$ such that

$$
-\varepsilon \leq \zeta_{\varepsilon,K} - \min \left( \zeta_{\varepsilon,K}, -\zeta_{\varepsilon,-K}\right) \leq 0
$$
in $\Pi_{n_0}$ for all small $\varepsilon > 0$.

Since $\Omega_{n_0}$ is smooth, by Theorem 1.1 of [4] there exists a unique $w_{\varepsilon,K} \in \bigcap_{p > 1} W^{1,2}_p(\Pi_{n_0})$ satisfying

$$
\partial_t w_{\varepsilon,K} + \sup_{a \in \mathbb{R}, \|b\| \leq K_0} \left[ a_{ij} D_{ij} w_{\varepsilon,K} + b_{i} D_i w_{\varepsilon,K} - cw_{\varepsilon,K} \right] = \zeta_{\varepsilon,K}
$$
in $\Pi_{n_0}$ (a.e.) and such that $w_{\varepsilon,K} = 0$ on $\partial'\Pi_{n_0}$. By the maximum principle such $w_{\varepsilon,K}$ is unique. Then owing to the continuity of $\zeta_{\varepsilon,K}$, for any $\varepsilon$ and $K$ for all sufficiently small $\beta > 0$, we have

$$
\partial_t w_{\varepsilon,K}^{(\beta)} + \sup_{a \in \mathbb{R}, \|b\| \leq K_0} \left[ a_{ij} D_{ij} w_{\varepsilon,K}^{(\beta)} + b_{i} D_i w_{\varepsilon,K}^{(\beta)} - cw_{\varepsilon,K}^{(\beta)} \right] \leq \zeta_{\varepsilon,K} \leq \zeta_{\varepsilon,K} + \varepsilon
$$
in $\Pi_{n_0-1}$.

For $w_{\varepsilon,K}^{(\beta)}(t,x) := w_{\varepsilon,K}^{(\beta)} + \varepsilon(T-t)$ now similarly to (2.9)

$$
\partial_t \left( u_{\varepsilon,K}^{(\varepsilon)} - w_{\varepsilon,K}^{(\beta)} \right) + \max \left( H\left[u_{\varepsilon,K}^{(\varepsilon)} - w_{\varepsilon,K}^{(\beta)}\right], P[u_{\varepsilon,K}^{(\varepsilon)} - w_{\varepsilon,K}^{(\beta)}] - K\right)
$$

$$
\geq \partial_t u_{\varepsilon,K}^{(\varepsilon)} + \max \left( H\left[u_{\varepsilon,K}^{(\varepsilon)}\right], P[u_{\varepsilon,K}^{(\varepsilon)}] - K\right) + \varepsilon
$$

$$
- \sup_{a \in \mathbb{R}, \|b\| \leq K_0} \left[ a_{ij} D_{ij} u_{\varepsilon,K}^{(\varepsilon)} + b_{i} D_i u_{\varepsilon,K}^{(\varepsilon)} - cw_{\varepsilon,K}^{(\beta)} \right] \geq \xi_{\varepsilon,K} - \zeta_{\varepsilon,K} \geq 0,
$$

$$
\partial_t (u_{\varepsilon,K}^{(\varepsilon)} + w_{\varepsilon,K}^{(\beta)}) + \min \left( H\left[u_{\varepsilon,K}^{(\varepsilon)} + w_{\varepsilon,K}^{(\beta)}\right], -P[-u_{\varepsilon,K}^{(\varepsilon)}] - w_{\varepsilon,K}^{(\beta)} + K\right)
$$

$$
\leq \partial_t u_{\varepsilon,K}^{(\varepsilon)} + \min \left( H\left[u_{\varepsilon,K}^{(\varepsilon)}\right], -P[-u_{\varepsilon,K}^{(\varepsilon)}] + K\right) + \partial_t w_{\varepsilon,K}^{(\beta)}
$$

$$
+ \sup_{a \in \mathbb{R}, \|b\| \leq K_0} \left[ a_{ij} D_{ij} w_{\varepsilon,K}^{(\beta)} + b_{i} D_i w_{\varepsilon,K}^{(\beta)} - cw_{\varepsilon,K}^{(\beta)} \right] \leq \zeta_{\varepsilon,-K} + \zeta_{\varepsilon,K} \leq 0
$$
in $\Pi_{n_0-1}$.

After setting

$$
\mu_{\varepsilon,K}^{(\beta)} = \sup_{\partial'\Pi_{n_0-1}} \left( u_{\varepsilon,K}^{(\varepsilon)} - u_{\varepsilon,K}^{(\varepsilon)} - 2w_{\varepsilon,K}^{(\beta)} \right)
$$

we conclude by Theorem 3.1 applied to $u_{\varepsilon,K}^{(\varepsilon)} - w_{\varepsilon,K}^{(\beta)}$ and $u_{\varepsilon,K}^{(\varepsilon)} + w_{\varepsilon,K}^{(\beta)} + \mu_{\varepsilon,K}^{(\beta)}$ in place of $u$ and $v$, respectively, that there exist a constant $N \in (0, \infty)$, depending only on $\tau$, the diameter of $\Omega$, $d$, $K_0$, $H$, and $\delta$, and a constant $\eta > 0$, depending only on $\tau$, $d$, and $\delta$, such that, if $K \geq N\left(M_{\varepsilon,K}^{(\beta)}\right)^\eta$, then

$$
u_{\varepsilon,K}^{(\varepsilon)} - u_{\varepsilon,K}^{(\varepsilon)} \leq \mu_{\varepsilon,K}^{(\beta)} + 2w_{\varepsilon,K}^{(\beta)} + NK^{-(\kappa-1)/4} + N M_{\varepsilon,K}^{(\beta)} \omega\left(\left(M_{\varepsilon,K}^{(\beta)}\right)^{-1/\tau} K^{-1}\right)$$
in $\Pi_{n_0 - 1}$, where $M_{\varepsilon,K}^\beta \geq 1$ is any number satisfying
\[ M_{\varepsilon,K}^\beta \geq \|u_K^{(\varepsilon)} - w_{\varepsilon,K}^{\beta} - u_{\varepsilon,K}^{(\varepsilon)} + w_{\varepsilon,K}^{\beta} + \mu_{\varepsilon,K}^{\beta}\|_{C^r(\Pi_{n_0 - 1})}. \]

By letting $\beta \downarrow 0$ we obviously obtain that, if $K \geq N(M_{\varepsilon,K})^0$, then
\[ u_K^{(\varepsilon)} - u_{-K}^{(\varepsilon)} \leq \mu_{\varepsilon,K} + 2w_{\varepsilon,K} + NK^{-(\kappa-1)/4} + N\mu_{\varepsilon,K}\omega(M_{\varepsilon,K}^{-1})^{-1} \]
in $\Pi_{n_0 - 1}$, where
\[ \mu_{\varepsilon,K} = \sup_{\partial\Pi_{n_0 - 1}} (u_K^{(\varepsilon)} - u_{-K}^{(\varepsilon)} - 2w_{\varepsilon,K})^+. \]
and $M_{\varepsilon,K} \geq 1$ is any number satisfying
\[ M_{\varepsilon,K} \geq \|u_K^{(\varepsilon)} - w_{\varepsilon,K}^{(\varepsilon)} + u_{\varepsilon,K}^{(\varepsilon)} + \mu_{\varepsilon,K}\|_{C^r(\Pi_{n_0 - 1})}. \]

First we discuss what is happening as $\varepsilon \downarrow 0$. By Theorem 1.1 of [4] we obtain $w_{\varepsilon,K} \to 0$ in $W^{1,2}_{p}(\Pi_{n_0})$ for any $p > 1$, which by embedding theorems implies that $w_{\varepsilon,K} \to 0$ in $C^\kappa(\Pi_{n_0})$. Obviously, the constants $\mu_{\varepsilon,K}$ converge in $C^\kappa(\Pi_{n_0 - 1})$ to
\[ \sup_{\partial\Pi_{n_0 - 1}} (u_K - u_{-K})^+. \]
Now Lemma 2.2 (iv), applied in $\Pi_{n_0 - 1}$, implies that for sufficiently small $\varepsilon$ one can take $N\varepsilon^{-\kappa}(n_0)$ as $M_{\varepsilon,K}$, where $\varepsilon(n_0)$ is the distance between the boundaries of $\Omega_{n_0}$ and $\Omega_{n_0 - 1}$ and $N$ is independent of $K$ and $\varepsilon(n_0)$. Thus, for sufficiently small $\varepsilon$, if $K \geq N\varepsilon^{-\kappa}(n_0)$, then
\[ u_K^{(\varepsilon)} - u_{-K}^{(\varepsilon)} \leq \mu_{\varepsilon,K} + 2w_{\varepsilon,K} + NK^{-(\kappa-1)/4} + N\varepsilon^{-\kappa}(n_0)\omega(N\varepsilon^{\kappa/\tau}(n_0)K^{-1}) \]
in $\Pi_{n_0 - 1}$, which after letting $\varepsilon \downarrow 0$ yields
\[ u_K - u_{-K} \leq NK^{-(\kappa-1)/4} + N\varepsilon^{-\kappa}(n_0)\omega(N\varepsilon^{\kappa/\tau}(n_0)K^{-1}) + \sup_{\partial\Pi_{n_0 - 1}} (u_K - u_{-K})^+ \]
in $\Pi_{n_0 - 1}$. Hence in $\Pi$
\[ u_K - u_{-K} \leq NK^{-(\kappa-1)/4} + N\varepsilon^{-\kappa}(n_0)\omega(N\varepsilon^{\kappa/\tau}(n_0)K^{-1}) + \sup_{\Pi_{n_0 - 1}} (u_K - u_{-K})^+ \]
\[ \leq NK^{-(\kappa-1)/4} + N\varepsilon^{-\kappa}(n_0)\omega(N\varepsilon^{\kappa/\tau}(n_0)K^{-1}) + \xi(n_0) \]
where $\xi(n_0) \to 0$ as $n_0 \to \infty$ by Lemma 2.2 (iii). This obviously proves the lemma because, as is noted in Lemma 2.2 (ii), we have $u_{-K} \leq u_K$.}

4. Existence of maximal and minimal $L_p$-viscosity solutions. Fix constants, $K_0, T \in (0, \infty)$, $p > d + 2$, $\delta \in (0, 1]$, and fix a nonnegative $\tilde{H} \in L_p(\mathbb{R}^{d+1})$.

Also according to the setting in Section 1 we take a bounded domain $\Omega \subset \mathbb{R}^d$ of class $C^{1,1}$ and set $\Pi = [0, T) \times \Omega$.

Assumption 4.1. (i) The function $H$ is a nonincreasing function of $u'_0$, is continuous with respect to $u'_0$, uniformly with respect to other variables $[u']$, $(t,x) \in \mathbb{R}^{d+1}$, $u'' \in \mathbb{S}$, is measurable with respect to $(t,x)$ for any $u$, and is Lipschitz continuous in $[u']$ with Lipschitz constant independent of $u'_0, u''$, $(t,x)$.

(ii) For any $u'/(t,x) \in \mathbb{R}^{d+1}$, the function $H(u,t,x)$ is Lipschitz continuous with respect to $u''$ and at all points of differentiability of $H(u,t,x)$ with respect to $u''$, we have $D_{u''} H \in \mathbb{S}_\delta$. 


(iii) for all \( u', (t, x) \in \mathbb{R}^{d+1} \)
\[
|H(u', 0, t, x)| \leq K_0|u'| + \bar{H}(t, x).
\]

**Assumption 4.2.** We are given a function \( g \in C(\partial \Pi) \).

We are going to use the following local version of Theorem 1.14 of [13], proved there for \( g \in W^{1,2}_p(\mathbb{R}^{d+1}) \) with the solution in global rather than local spaces \( W^{1,2}_{p,\text{loc}} \). This local version is easier to prove because no boundary estimates are needed and we will provide the proof elsewhere.

**Theorem 4.1.** There exists a convex positive homogeneous of degree one function \( P(u') \) such that at all points of its differentiability \( D_{u'}P \in \mathbb{S}_\delta \), where \( \delta = \delta(d, \delta) \in (0, \delta) \), and for \( P[u] = P(D^2u) \) and any \( K > 0 \) there exists \( v \in W^{1,2}_p(\Pi^0) \cap C(\bar{\Pi}) \), for any \( p > 0 \), such that \( v = g \) on \( \partial \Omega \) and the equation
\[
\partial_tv + \max(H[v], P[v] - K) = 0,
\]
holds (a.e.) in \( \Pi \).

By the maximum principle the solutions \( v = v_K \) are unique and decrease as \( K \to \infty \).

**Theorem 4.2.** Under the above assumptions, as \( K \to \infty \), \( v_K \) converges uniformly on \( \Pi \) to a continuous function \( v \) which is an \( L_{d+1}\)-viscosity solutions of (1.1) with boundary condition \( v = g \) on \( \partial \Omega \). Furthermore, \( v \) is the maximal \( L_{d+1}\)-viscosity subsolution of (1.1) of class \( C(\bar{\Pi}) \) with given boundary condition.

**Remark 4.1.** To obtain an \( L_{d+1}\)-viscosity solution which is a minimal \( L_{d+1}\)-viscosity supersolution, it suffices to consider
\[
\partial_tv + \min(H[v], -P[-v] + K) = 0,
\]
which reduces to (4.1) if we replace \( v \) with \( -v \) and \( H(u, t, x) \) with \( -H(-u, t, x) \).

This yields the following result.

**Theorem 4.3.** Let \( v_{-K} \in W^{1,2}_{p,\text{loc}}(\Pi) \cap C(\bar{\Pi}) \) denote a unique solution of (4.2) (a.e.) in \( \Pi \) with boundary data \( v_{-K} = g \) on \( \partial \Omega \). Then, as \( K \to \infty \), \( v_{-K} \) converges uniformly on \( \Pi \) to a continuous function \( w \) which is an \( L_{d+1}\)-viscosity solutions of (1.1) with boundary condition \( w = g \) on \( \partial \Omega \). Furthermore, \( w \) is the minimal \( L_{d+1}\)-viscosity supersolution of (1.1) of class \( C(\bar{\Pi}) \) with given boundary condition.

**Remark 4.2.** The existence of extremal \( C\)-viscosity solutions is proved in the elliptic and parabolic cases in [2] when \( H \) is a continuous function. Our function \( H(u, t, x) \) is just measurable in \( (t, x) \) and we are dealing with \( L_{d+1}\)-viscosity solutions.

Also note that the existence of the extremal \( L_p\)-viscosity solution for the elliptic case was proved in [5] and [6] with no continuity assumption on \( H \) with respect to \( x \). We provide a method which in principle allows one to find it.

Here is a stability result for the extremal \( L_{d+1}\)-viscosity solutions. In the following assumption there are two objects: \( \kappa_1 = \kappa(d, \delta) \in (1, 2) \) (close to 1), and \( \theta = \theta(\kappa, d, \delta) \in (0, 1] \) (close to 0), \( \kappa \in (1, \kappa_1) \). The values of \( \kappa_1 \) and \( \theta \) are specified in the proof of Lemma 5.3 of [10].

**Assumption 4.3.** We have a representation
\[
H(u, t, x) = F(u'', t, x) + G(u, t, x).
\]
\(\text{(i) The functions } F \text{ and } G \text{ are measurable functions of their arguments.}\)

\(\text{(ii) For all values of the arguments }\)
\[|G(u, t, x)| \leq K_0|u'| + H(t, x).\]

\(\text{(iii) The function } F \text{ is positive homogeneous of degree one with respect to } u''\), is Lipschitz continuous with respect to \( u'' \), and at all points of differentiability of \( F \) with respect to \( u'' \) we have \( D_{u''}F \in \mathbb{S}_3 \).

\(\text{(iv) For any } R \in (0, R_0], \ (t, x) \in \mathbb{R}^{d+1}, \text{ and } u'' \in \mathbb{S} \text{ with } |u''| = 1 \ (|u'| := (\text{tr } u''u'')^{1/2}), \text{ we have }\)
\[
\theta_{R,t,x} := \int_{C_R(t,x)} |F(u'', s, y) - \bar{F}_{R,x}(u'', s)| \, ds \, dy \leq \theta,
\]
\[\text{where}\]
\[
\bar{F}_{R,x}(u'', s) = \int_{B_R(x)} F(u'', s, y) \, dy.
\]

Take \( \kappa_1 = \kappa_1(d, \delta) \in (1, 2) \) and \( \theta = \theta(\kappa, d, \delta) \in (0, 1] \) that are introduced before Assumption 4.3. Then fix a \( \kappa \) satisfying
\[1 < \kappa < \left[2 - (d + 2)/p\right] \wedge \kappa_1.\]

**Theorem 4.4.** Let \( H_n \), \( n = 0, 1, \ldots, \) satisfy Assumptions 4.1 and 4.3 with \( R_0 \in (0, 1] \) independent of \( n \), with \( K_0 \) and \( \bar{H} \) from the beginning of the section, \( \theta(\kappa, d, \delta)/2 \) in place of \( \theta \) (specified above) and have, perhaps, different Lipschitz constants with respect to \( |u'| \) for different \( n \). Suppose that \( H_0 \) is Lipschitz continuous in \( u \) with a constant independent of \( (t, x) \). Let \( v_n, n = 0, 1, \ldots, \) be the maximal \( L_{d+1} \)-viscosity solutions of class \( C(\Pi) \) of \( \partial_t v_n + H_n[v_n] = 0 \) in \( \Pi \) with boundary condition \( v_n = g_n \) on \( \partial^\prime \Pi \), where \( g_n \in C(\Pi) \) and \( g_n \rightarrow g_0 \) in \( C(\Pi) \) as \( n \rightarrow \infty \).

Assume that for any \( M > 0 \)
\[
\Delta_{n,M}(t, x) := \sup_{|u| \leq M} \left| H_n(u, t, x) - H_0(u, t, x) \right| \to 0 \quad (4.3)
\]
in \( L_{d+1}(\Pi) \) as \( n \rightarrow \infty \). Also assume that for all values of the arguments and \( n \)
\[
\left| H_n(u, t, x) - H_0(u, t, x) \right| \leq \bar{H}(t, x)(1 + |u'|). \quad (4.4)
\]

Then \( v_n \rightarrow v_0 \) in \( C(\Pi) \) as \( n \rightarrow \infty \). The same holds true if \( v_n \) are minimal \( L_{d+1} \)-viscosity solutions of class \( C(\Pi) \).

**Proof.** According to Theorem 4.2, it suffices to show that
\[
\sup_{K \geq 1} \left( v_{n,K} - v_{0,K} \right) \to 0, \quad (4.5)
\]
in \( C(\Pi) \), where \( v_{n,K} \) are the solutions of
\[
\partial_t v_{n,K} + \max \left( H_n[v_{n,K}], P[v_{n,K}] - K \right) = 0 \quad \text{in } \Pi \ (\text{a.e.}) \text{ with boundary condition } v_{n,K} = g_n \text{ on } \partial^\prime \Pi.
\]

Observe that
\[
\left| \max \left( H_n[v_{n,K}], P[v_{n,K}] - K \right) - \max \left( F_n[v_{n,K}], P[v_{n,K}] - K \right) \right| \leq |G_n[v_{n,K}]| = |b^j D_{ij} v_{n,K} + c v_{n,K} + \tau \bar{H}|
\]
for certain measurable \( \mathbb{R}^d \)-valued \( b \), and measurable real-valued \( \tau \) and \( c \), such that \( |b|, |c| \leq K_0 \) and \( |\tau| \leq 1 \). Also \( \max \left( F_n[0], P[0] - K \right) = 0 \). Therefore, by the mean-value theorem we have
\[
\partial_t v_{n,K} + a^{ij} D_{ij} v_{n,K} + b^j D_{i} v_{n,K} + c v_{n,K} + \tau \bar{H} = 0 \quad (4.6)
\]
(a.e.) in Π for certain measurable $S_3$-valued $(a_{ij})$, and perhaps different measurable $\mathbb{R}^d$-valued $b$, and measurable real-valued $\tau$ and $c$, such that $|b|, |c| \leq K_0$, and $|\tau| \leq 1$. By the parabolic Aleksandrov estimates (4.6) implies that $|v_{n,K}|$ are uniformly bounded in $\Pi$ and by the linear theory of parabolic equations we conclude that the family $\{v_{n,K} : K \geq 1, n \geq 0\}$ is precompact in $C(\bar{\Pi})$.

Next, fix a $\rho > 0$ such that $\Pi^\rho \neq \emptyset$ and observe that, as we know from [4], [8], there is a number $\gamma = \gamma(d, \delta, K_0) \in (0, 1)$ such that there is a constant $N$, depending only on $\rho, d, \delta$, and $K_0$, such that for any cylinder $C_\rho(t_0, x_0) \subset \Pi$ we have due to (4.6) that

$$\int_{C_\rho(t_0, x_0)} |D^2v_{n,K}|^\gamma \, dx \, dt \leq N \sup_{\Pi} |v_{n,K}| + N \left(\int_{\Pi} |cv_{n,K} + \tau \tilde{H}|^{d+1} \, dx \, dt \right)^{\gamma/(d+1)}$$

for all $n, K$. Here the right-hand side is dominated by a constant independent of $n, K$, and it follows, by Chebyshev’s inequality that there is a constant $N$ (perhaps depending on $\rho$) for which

$$|\Pi^\rho \cap \{ |D^2v_{n,K}| \geq M \}| \leq NM^{-\gamma}$$

for all $n \geq 0, K \geq 1, M > 0$.

To finish with preparations, set

$$H_{n,K} = \max(H_n, P - K), \quad F_{n,K} = \max(F_n, P),$$

$G_{n,K} = H_{n,K} - F_{n,K}$, where $F_n$ is taken from Assumption 4.3 written for $H_n$. Then $F_{n,K}$ and $G_{n,K}$ satisfy Assumption 4.3 (i) and (ii) with the same $K_0$ and $\tilde{H}$. Assumption 4.3 (iii) also is satisfied with $\delta$ in place of $\delta$. Finally, easy manipulations, using the fact that in the assumptions of the theorem we suppose that Assumption 4.3 (iv) is satisfied for $F_n$ with $\theta(\kappa, d, \delta)/2$ in place of $\theta$, show that Assumption 4.3 (iv) is satisfied for $F_{n,K}$ with $\theta(\kappa, d, \delta)$.

Thus, all the assumptions of Theorem 2.1 of [10] are satisfied apart from $g \in W^{1,2}_\infty(\mathbb{R}^{d+1})$ and $\Omega \in C^2$. Standard approximation techniques show that these assumptions can be replaced with the current ones.

Now since $v_{n,K}$ is a classical solution of (4.1), we obtain from that theorem, for any small $\rho > 0$, the estimates of the $C^{1+\gamma}(\Pi^\rho)$-norms of $v_{n,K}$ uniform with respect to $n$ and $K$. Therefore, by interpolation theorems we get

$$\sup_{\Pi^\rho} |Dv_{n,K}| \leq N,$$  \hspace{1cm} (4.8)

where and below by $N$ we denote various constants independent of $K$ and $n$, perhaps depending on $\rho$.

Now set $w_{n,K} = v_{0,K} - v_{n,K}$, and observe that

$$0 = \partial_t w_{n,K} + I_1 + I_2 + I_3,$$

where

$$I_1 = \max(H_0[v_{0,K}], P[v_{0,K}] - K) - \max(H_0(v_{0,K}, Dv_{0,K}, D^2v_{n,K}), P[v_{n,K}] - K) = a_{ij}D_{ij}w_n,$$

$$I_2 = \max(H_0(v_{0,K}, Dv_{n,K}, D^2v_{n,K}), P[v_{n,K}] - K) - \max(H_0(v_{n,K}, Dv_{n,K}, D^2v_{n,K}), P[v_{n,K}] - K),$$

$$I_3 = \max(H_0[v_{n,K}], P[v_{n,K}] - K)$$
\[ - \max \left( H_n[v_{n,K}], P[v_{n,K}] - K \right), \]
and \((a^{ij})\) is an \(S^2\)-valued function. By assumption
\[
|I_2| \leq N \left( |w_{n,K}| + |Dw_{n,K}| \right).
\]
Therefore,
\[
0 = \partial_t w_{n,K} + a^{ij}D_{ij}w_{n,K} + b^i D_i w_{n,K} + cw_{n,K} + I_3,
\]
where \(b\) and \(c\) are bounded uniformly with respect to \(n,K\).

Upon observing that by assumption and \((4.8)\) in \(\Pi^p\) for any \(M > 0\) we have
\[
|I_3| \leq \bar{H}N|D^2w_{n,K}| + \Delta_n, M + N I_{D^2v_{n,K}|\leq M}
\]
and using the parabolic Aleksandrov estimates in \(\Pi^p\) we conclude that there exists a constant \(N\) such that for all \(n,K,M\) in \(\Pi\)
\[
|v_{0,K} - v_{n,K}| \leq N\|H|_{D^2v_{n,K}|\geq M}\|_{L^{d+1}(\Pi^p)} + N\|\Delta_n, M + N\|_{L^{d+1}(\Pi^p)} + \sup_{\Pi^p|\Pi^p}|v_{0,K} - v_{n,K}|. \tag{4.9}
\]

Here
\[
\sup_{n,K}\|H|_{D^2v_{n,K}|\geq M}\|_{L^{d+1}(\Pi^p)} \to 0
\]
as \(M \to \infty\), since \(\bar{H} \in L^{d+1}(\Pi)\) and \((4.7)\) holds. Therefore, by first taking the sup's with respect to \(K \geq 1\) in \((4.9)\), then sending \(n \to \infty\), using assumption \((4.3)\), and then sending \(M \to \infty\), we infer from \((4.9)\) that, for any small \(p > 0\)
\[
\limsup_{n \to \infty} \sup_{\Pi^p}|v_{0,K} - v_{n,K}| \leq \limsup_{n \to \infty} \sup_{\Pi^p}|v_{0,K} - v_{n,K}|.
\]

After that it only remains to set \(p \downarrow 0\) and use the equicontinuity of \(v_{n,K}\) and the fact that \(g_n \to g_0\) uniformly in \(\partial \Pi\). The theorem is proved. \(\square\)

**Remark 4.3.** It follows from the above proof that \(\bar{H}\) in \((4.4)\) can be replaced with \(H_n\), provided that the family \(|H_n|^{d+1}\) is uniformly integrable over \(\Pi\).

An obvious consequence of this theorem is the stability of uniqueness.

**Corollary 4.5.** Suppose that for any \(n = 1,2,...\) there is only one \(L^{d+1}\)-viscosity solutions of class \(C(\Pi)\) of \(\partial_t v_n + H_n[v_n] = 0\) in \(\Pi\) with boundary condition \(v_n = g_n\) on \(\partial \Pi\). Then the same holds for \(n = 0\).

Coming back to Theorem 4.2, observe that, as we have mentioned above, by the maximum principle \(v_K\) decreases as \(K\) increases. The precompactness of \(\{v_K, K \geq 1\}\) in \(C(\Pi)\) is proved in the same way as in the above proof after \((4.6)\) using the fact that
\[
\max \left( H_n[v_K], P[v_K] - K \right) = \max \left( H_n(v_K, Dv_K, 0, 0), P[0] - K \right) = a^{ij}D_{ij}v_K,
\]
where \((a^{ij})\) is \(S^2\)-valued and
\[
\max \left( H_n(v_K, Dv_K, 0, 0), P[0] - K \right) = \max \left( H_n(v_K, Dv_K, 0, 0), K \right) = b^i D_i v_{n,K} + cv_{n,K} + \tau \bar{H}.
\]

It follows that \(v_K\) converges uniformly on \(\bar{\Pi}\) as \(K \to \infty\) to a function \(v \in C(\bar{\Pi})\). To prove that \(v\) is an \(L^{d+1}\)-viscosity solution we need the following.
Lemma 4.6. There is a constant $N$ depending only on $d$, $\delta$, and the Lipschitz constant of $H$ with respect to $[u'] = (u_1', \ldots, u_d')$ such that for any $r \in (0, 1]$ and $C_r(t, x)$ satisfying $C_r(t, x) \subset \Pi$ and $\phi \in W^{1,2}_{d+1}(C_r(t, x))$ we have on $C_r(t, x)$ that

\begin{align*}
v &\leq \phi + N r^{d/(d+1)} \left\| (\partial_t \phi + H[\phi])^+ \right\|_{L^{d+1}(C_r(t,x))} + \max_{\partial^C_r(t,x)} (v - \phi)^+. \quad (4.10) \\
v &\geq \phi - N r^{d/(d+1)} \left\| (\partial_t \phi + H[\phi])^- \right\|_{L^{d+1}(C_r(t,x))} - \max_{\partial^C_r(t,x)} (v - \phi)^-. \quad (4.11)
\end{align*}

Proof. Observe that in $C_r(t, x)$ (a.e.)

\begin{align*}
- \partial_t \phi - \max_h (H[\phi], P[\phi] - K) \\
= - \partial_t \phi - \max_h (H[\phi], P[\phi] - K) + \partial_t v_K + \max_h (H[v_K], P[v_K] - K)
= & \partial_t v_K + a_{ij} D_{ij} (v_K - \phi) + b_i D_i (v_K - \phi) - c (v_K - \phi),
\end{align*}

where $a = (a_{ij})$ is an $S_{\delta}$-valued function, $b = (b_i)$ is bounded by the Lipschitz constant of $H$ with respect to $[u']$, and $c \geq 0$. It follows by the parabolic Aleksandrov estimates that

\begin{align*}
v_K &\leq \phi + \max_{\partial^C_r(t,x)} (v_K - \phi)^+ \\
&\quad + N r^{d/(d+1)} \left\| (\partial_t \phi + \max_h (H[\phi], P[\phi] - K))^+ \right\|_{L^{d+1}(C_r(t,x))}, \quad (4.12)
\end{align*}

where the constant $N$ is of the type described in the statement of the present lemma. We obtain (4.10) from (4.12) by letting $K \to \infty$. In the same way (4.11) is established. The lemma is proved.

Proof of Theorem 4.2. First we prove that $v$ is an $L_{d+1}$-viscosity solution. Let $(t_0, x_0) \in \Pi$ and $\phi \in W^{1,2}_{d+1,\text{loc}}(\Pi)$ be such that $v - \phi$ attains a local maximum at $(t_0, x_0)$. Then for $\varepsilon > 0$ and all small $r > 0$ for

$$\phi_{\varepsilon,r}(t, x) = \phi(t, x) - \phi(t_0, x_0) + v(t_0, x_0) + \varepsilon \left( |x - x_0|^2 + t - t_0 - r^2 \right)$$

we have that

$$\max_{\partial^C_r(t_0, x_0)} (v - \phi_{\varepsilon,r})^+ = 0.$$

Hence, by Lemma 4.6

$$\varepsilon r^2 = (v - \phi_{\varepsilon,r})(t_0, x_0) \leq N r^{d/(d+1)} \left\| (\partial_t \phi_{\varepsilon,r} + H[\phi_{\varepsilon,r}])^+ \right\|_{L^{d+1}(C_r(t_0, x_0))},$$

$$\varepsilon^{d+1} \leq N r^{-d/(d+2)} \left\| (\partial_t \phi_{\varepsilon,r} + H[\phi_{\varepsilon,r}])^+ \right\|^{d+1}_{L^{d+1}(C_r(t_0, x_0))}.$$ 

By letting $r \downarrow 0$ and using the continuity of $H(u, t, x)$ in $u'$, which is assumed to be uniform with respect to other variables and also using the continuity of $\phi$ (embedding theorems) and $v$, we obtain

$$N \lim_{r \downarrow 0} \text{ess sup}_{C_r(t_0, x_0)} \left( \partial_t \phi_{\varepsilon}(t, x) + H(v(t, x), D\phi_{\varepsilon}(t, x), D^2\phi_{\varepsilon}(t, x)) \right) \geq \varepsilon. \quad (4.13)$$

where $\phi_{\varepsilon} = \phi + \varepsilon (|x - x_0|^2 + t - t_0)$. Finally, observe that $H(u, t, x)$ is Lipschitz continuous with respect to $[u']$, $u''$ with Lipschitz constant independent of $u_0'$, $u''$ by assumption. Then letting $\varepsilon \downarrow 0$ in (4.13) proves that $v$ is an $L_{d+1}$-viscosity subsolution. The fact that it is also an $L_{d+1}$-viscosity supersolution is proved similarly on the basis of (4.11).

Finally, we prove that $v$ is the maximal continuous $L_{d+1}$-viscosity subsolution. Let $w$ be an $L_{d+1}$-viscosity subsolution of (1.1) of class $C(\Pi)$ with boundary data

\begin{align*}
v &\leq \phi + N r^{d/(d+1)} \left\| (\partial_t \phi + H[\phi])^+ \right\|_{L^{d+1}(C_r(t,x))} + \max_{\partial^C_r(t,x)} (v - \phi)^+. \quad (4.10) \\
v &\geq \phi - N r^{d/(d+1)} \left\| (\partial_t \phi + H[\phi])^- \right\|_{L^{d+1}(C_r(t,x))} - \max_{\partial^C_r(t,x)} (v - \phi)^-. \quad (4.11)
\end{align*}
g. To prove that \( v \geq w \), it suffices to show that for any \( \varepsilon > 0 \) and \( K > 1 \) we have \( u_K + \varepsilon(T-t) \geq w \) in \( \overline{\Pi} \).

Assume the contrary and observe that, since \( u_K + \varepsilon(T-t) \geq w \) on \( \partial \overline{\Pi} \), there is a point \( (t_0, x_0) \in \Pi \) such that

\[
\gamma := u_K(t_0, x_0) + \varepsilon(T-t_0) - w(t_0, x_0) < 0
\]

and \( u_K + \varepsilon(T-t) - w \geq \gamma \) in \( C_p(t_0, x_0) \) for sufficiently small \( \rho > 0 \), so that by definition

\[
-\varepsilon + \lim \sup_{r \downarrow 0} \varepsilon \sup_{r \downarrow 0} C_r(t_0, x_0) = \partial_t u_K(t, x) + H(w(t, x), Du_K(t, x), D^2 u_K(t, x), t, x) \geq 0.
\]

Since \( H \) is a decreasing function with respect to \( u_0' \), in light of (4.14),

\[
-\varepsilon + \lim \sup_{r \downarrow 0} \varepsilon \sup_{r \downarrow 0} C_r(t_0, x_0) = \partial_t u_K(t, x) + H(u_K(t, x), Du_K(t, x), D^2 u_K(t, x), t, x) \geq 0.
\]

This is however impossible since \( \partial_t u_K + H[u_K] \leq 0 \) in \( \Pi \) (a.e.). This contradiction finishes proving the theorem. \( \square \)

5. Uniqueness of \( L_p \)-viscosity solutions for parabolic Isaacs equations. Fix some constants \( \delta \in (0, 1], K_0, T \in (0, \infty), p > d + 2 \). Assume that we are given countable sets \( A \) and \( B \), and, for each \( \alpha \in A \) and \( \beta \in B \), we are given an \( \mathbb{S}_\delta \)-valued function \( a^{\alpha \beta}(t, x) \) on \( \mathbb{R}^{d+1} = \{ (t, x) : t \in \mathbb{R}, x \in \mathbb{R}^d \} \), a real-valued function \( b^{\alpha \beta}(u', t, x) \) on \( \mathbb{R}^{d+1} \times \mathbb{R}^d \), and a real-valued \( \tilde{H}(t, x) \geq 0 \) on \( \mathbb{R}^{d+1} \).

**Assumption 5.1.**

(i) The above introduced functions are measurable.

(ii) The function \( a^{\alpha \beta}(t, x) \) is uniformly continuous with respect to \( (t, x) \) uniformly with respect to \( \alpha, \beta \), and, with \( \gamma \) introduced before Assumption 2.1, for all values of indices and arguments

\[
|a^{\alpha \beta}(t, x) - a^{\alpha \beta}(t, y)| \leq K_0|x - y|^\gamma.
\]

(iii) The function \( b^{\alpha \beta}(u', t, x) \) is nonincreasing with respect to \( u'_0 \), is Lipschitz continuous with respect to \( u' \) with Lipschitz constant \( K_0 \), and, for all values of indices and arguments

\[
|b^{\alpha \beta}(u', t, x)| \leq K_0|u'| + \tilde{H}(t, x).
\]

(iv) We have \( \tilde{H} \in L_p(\mathbb{R}^{d+1}) \).

For

\[
u = (u', u''), \quad u' = (u'_0, u'_1, \ldots, u'_d) \in \mathbb{R}^{d+1}, \quad u'' \in \mathbb{S},
\]

and \( (t, x) \in \mathbb{R}^{d+1} \) introduce

\[
H(u, t, x) = \sup_{\alpha \in A} \inf_{\beta \in B} [a^{\alpha \beta}_{ij}(t, x)u''_{ij} + b^{\alpha \beta}(u', t, x)]
\]

(where as everywhere the summation convention is enforced and the summations are done inside the brackets).

For sufficiently smooth functions \( u = u(t, x) \) define

\[
H[u](t, x) = H(u(t, x), Du(t, x), D^2 u(t, x), t, x).
\]

As is usual in this article we take an open bounded subset \( \Omega \) of \( \mathbb{R}^d \) of class \( C^{1,1} \) and set

\[
\Pi = [0, T) \times \Omega.
\]

Here is the main results of this section.
Theorem 5.1. Under the above assumption for any \( g \in C(\partial \Pi) \) there exists a unique continuous in \( \Pi, L_{d+1} \)-viscosity solution \( g \) of the Isaacs equation

\[
\partial_t u + H[u] = 0
\]

in \( \Pi \) with boundary condition \( u = g \) on \( \partial \Pi \).

Remark 5.1. Under Assumption 5.1 (i), (ii), in case

\[
b^\alpha\beta(u',t,x) = \sum_{i=1}^{d} b_i^{\alpha\beta}(t,x)u'_i - c^{\alpha\beta}(t,x)u'_0 + f^{\alpha\beta}(t,x)
\]

with uniformly continuous (in \( t,x \)) uniformly in \( \alpha, \beta \) and uniformly bounded coefficients and the free terms, the uniqueness of \( C \)-viscosity solutions is stated without proof in Theorem 9.3 in [3]. This case is covered by Theorem 2.1.

For general \( H \), not necessarily related to Isaacs equations, uniqueness is claimed for \( \partial_t u + H(D^2u) = 0 \) in Lemma 4.7 of [17]. It is proved for \( L_p \)-viscosity solutions in Lemma 6.2 of [3] in case \( H \) is independent of \( (t,x) \) with no reference to Wang’s Lemma 4.7 of [17].

In the elliptic case Jensen and Świȩch [5] proved the uniqueness of continuous \( L_p \)-viscosity solutions for Isaacs equations, assuming that \( b^{\alpha\beta}_i, c^{\alpha\beta} \) are bounded, \( \sup_{\alpha,\beta} |f^{\alpha\beta}| \in L_p \) and Assumptions 5.1 (i), (ii) are satisfied. Their proof uses a remarkable Corollary 1.6 of Świȩch [15] of which the parabolic counterpart is given in [3].

An important difference with [5] here is that we consider lower-order terms in a more general form, but in [5] the summability assumption is weaker (some \( p < d \) are allowed).

We prove Theorem 5.1 by using Corollary 4.5 after some preparations.

Fix a nonnegative \( \zeta \in C_0^\infty(\mathbb{R}^{d+1}) \) which integrates to one and for \( n = 1,2,..., \) introduce \( \zeta_n(t,x) = n^{d+1}\zeta(nt, nx) \). Also, for real-valued \( \xi = \xi^{\alpha\beta} \) given on \( A \times B \) and \( u',(t,x) \in \mathbb{R}^{d+1} \) define

\[
\mathcal{H}_0(\xi,u',t,x) = \sup_{\alpha \in A} \inf_{\beta \in B} \left[ \xi^{\alpha\beta} + b^{\alpha\beta}(u',t,x) \right],
\]

\[
\mathcal{H}_n(\xi,u',t,x) = \mathcal{H}_0(\xi,u',t,x) \ast \zeta_n(t,x), \quad n = 1,2,...,
\]

where the convolution is performed with respect to \( (t,x) \). We will also use the notation

\[
|\xi| = \sup_{\alpha \in A} \sup_{\beta \in B} |\xi^{\alpha\beta}|.
\]

Here are a few properties of \( \mathcal{H}_n \).

Lemma 5.2. (i) For each \( n \geq 1 \), the function \( \mathcal{H}_n(\xi,u',t,x) \) is continuous, is Lipschitz continuous in \( u' \) with Lipschitz constant \( K_0 \), and the function \( |\mathcal{H}_n(0,0,t,x)| \) is bounded.

(ii) For each \( n \geq 1 \), the function \( \mathcal{H}_n(\xi,u',t,x) \) is infinitely differentiable with respect to \( x \) (and \( t \)), and there exists a constant \( N \) (depending on \( n \)) such that \( |D_x \mathcal{H}_n| \leq N(1 + |u'|) \) for all values of arguments.

(iii) For all values of the arguments

\[
|\mathcal{H}_n(\xi,u',t,x) - \mathcal{H}_0(\xi,u',t,x)| \leq 2K_0|u'| + \hat{H} + \hat{\hat{H}},
\]

where

\[
\hat{H} = \sup_{n} \hat{H} \ast \zeta_n.
\]
(iv) For each $M > 0$

\[
\delta_{n,M}(t,x) := \sup_{|u'| \leq M} \sup_{|\xi| \leq M} |H_n(\xi, u', t, x) - H_0(\xi, u', t, x)| \to 0 \quad (5.3)
\]

in $L_{d+1}(\Pi)$ as $n \to \infty$.

**Proof.** Assertions (i)-(iii) are quite elementary and their proofs are left to the reader. To prove (iv) set

\[
\delta_n(\xi, u', t, x) = H_n(\xi, u', t, x) - H_0(\xi, u', t, x)
\]

and observe that for any $(\xi, u')$

\[
\lim_{n \to \infty} \|\delta_n(\xi, u', \cdot)\|_{L_{d+1}(\Pi)} \to 0
\]

by the $L_{d+1}$-continuity of $L_{d+1}$-functions. Furthermore, by the Lipschitz continuity of $H_n$ with respect to $(\xi, u')$ uniform with respect to $n, t, x$, for any $\varepsilon > 0$, one can find $m$ and $(\xi_k, u'_k), k = 1, \ldots, m$, such that $|\xi_k|, |u'_k| \leq M$ and any $(\xi, u')$ satisfying $|\xi|, |u'| \leq M$ has a neighbor $(\xi_k, u'_k)$ such that

\[
|H_n(\xi, u', t, x) - H_n(\xi_k, u'_k, t, x)| \leq \varepsilon
\]

in $\Pi$ for any $n \geq 0$. It follows that

\[
\delta_{n,M}(t,x) \leq \max_{k=1,\ldots,m} \left| \delta_n(\xi_k, u'_k, t, x) \right| + \varepsilon.
\]

Then

\[
\lim_{n \to \infty} \|\delta_{n,M}\|_{L_{d+1}(\Pi)} \leq \lim_{n \to \infty} \sum_{k=1}^{m} \|\delta_n(\xi_k, u'_k, \cdot)\|_{L_{d+1}(\Pi)} + N\varepsilon = N\varepsilon,
\]

where $N$ is independent of $\varepsilon$. This certainly proves (iv) and the lemma.

**Proof of Theorem 5.1.** First we check that the assumptions of Theorem 4.4 are satisfied for

\[H_n(u, t, x) = H_n(\xi, u', t, x), \quad \xi^{\alpha\beta} = \alpha_{ij}^{\alpha\beta} u''_{ij}.
\]

Assumptions (4.3), (4.4) are taken care of in Lemma 5.2. Assumption 4.1 is easily seen to be satisfied with $\bar{H}$ and $\hat{H}$ in place of $H$.

In what concerns Assumption 4.3 we set

\[F_n(u', t, x) = F_0(u', t, x) = \sup_{\alpha \in A} \inf_{\beta \in B} \alpha_{ij}^{\alpha\beta}(t, x)u''_{ij}.
\]

Then Assumptions 4.3 (ii), (iii) are obviously satisfied. The remaining assumptions of Theorem 4.4 is Assumption 4.3 (iv), which is satisfied for any $\theta > 0$ if $R_0$ is chosen appropriately in light of Assumption 5.1 (ii).

Thus Theorem 4.4 is applicable. Furthermore, for each $n \geq 1$ the assumptions of Theorem 2.1 are satisfied in light of Lemma 5.2 (ii) and Assumption 5.1 (ii). Also uniqueness of $C$-viscosity solutions implies that of $L_p$-viscosity solutions.

Now, the combination of Theorem 2.1 and Corollary 4.5 (proved under the assumptions of Theorem 4.4) immediately yields the desired result.

**Acknowledgments.** The author is sincerely grateful to the referees for useful comments.
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Received October 2017; revised February 2018.

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