Forbidden induced subposets in the grid

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Abstract

In this short paper, we prove the following generalization of a result of Methuku and Pálvölgyi. Let $P$ be a poset, then there exists a constant $C_P$ with the following property. Let $k$ and $n$ be arbitrary positive integers such that $n$ is at least the dimension of $P$, and let $w$ be the size of the largest antichain of the grid $[k]^n$ endowed with the usual pointwise ordering. If $S$ is a subset of $[k]^n$ not containing an induced copy of $P$, then $|S| \leq C_P w$.

1 Introduction

The Boolean lattice $2^n$ is the power set of $[n] = \{1, ..., n\}$ ordered by inclusion. If $P$ and $Q$ are posets, a subset $P'$ of $Q$ is a copy of $P$ if the subposet of $Q$ induced on $P'$ is isomorphic to $P$. The following theorem, originally conjectured by Katona, and Lu and Milans [13] was proved by Methuku and Pálvölgyi [16].

Theorem 1. (Methuku, Pálvölgyi [16]) Let $P$ be a poset. Then there exists a constant $C = C(P)$ with the following property. If $S \subseteq 2^n$ such that $S$ does not contain a copy of $P$, then $|S| \leq C_{\binom{n}{\lfloor n/2 \rfloor}}$.

The aim of this paper is to give a slightly different proof of this theorem, which extends from the Boolean lattice to arbitrary grids as well. A grid of size $k_1 \times ... \times k_n$ is the cartesian product $G = [k_1] \times ... \times [k_n]$ endowed with the pointwise ordering, that is, if $(x_1, ..., x_n), (y_1, ..., y_n) \in G$, then $(x_1, ..., x_n) \leq_G (y_1, ..., y_n)$ if $x_i \leq y_i$ for $i \in [n]$. If $k_1 = ... = k_n = k$, we shall write $[k]^n$ instead of $[k_1] \times ... \times [k_n]$.

Before we state our main theorem, let us introduce a few definitions. The dimension (Dushnik-Miller dimension) of a poset $P$, denoted by $\dim P$, is the smallest positive integer $d$ such that $P$ is the intersection of $d$ linear orders. Formally, $\dim P$ is the smallest positive integer $d$ for which there exist $d$ bijections $L_1, ..., L_d : P \to |P|$ such that for every $p, q \in P$, $p \leq_P q$ iff $L_i(p) \leq L_i(q)$ holds for every $i \in [d]$.

Also, the width of a poset $P$ is the size of the largest antichain in $P$ and is denoted by $w(P)$. Let us mention that by a result of Hiraguchi [9], we have $\dim P \leq \min\{|P|/2, w(P)\}$.

The following theorem is the main result of this manuscript.

Theorem 2. Let $P$ be a poset. Then there exists a constant $C_P$ with the following property. Let $k$ and $n$ be positive integers satisfying $n \geq \dim P$, and let $w$ be the width of $[k]^n$. If $S \subseteq [k]^n$ such that $S$ does not contain a copy of $P$, then $|S| \leq C_P w$.

In case $k = 2$, we have $w = \binom{n}{\lfloor n/2 \rfloor}$ by the well known theorem of Sperner [17], so our main theorem is indeed a strengthening of Theorem 1.

Forbidden (weak) subposet and induced subposet problems are extensively studied in the Boolean lattice $2^n$, for recent developments see [2, 3, 8, 13], for example. However, there are not many such results when $2^n$ is replaced with some grid $[k]^n$. A well known result of Erdős [4] is that if $S \subseteq 2^n$ does not contain a chain of size $l$, then $|S|$ is at most the sum of the $l-1$ largest binomial coefficients of order $n$. In fact, $|S| \leq (l-1)\binom{n}{\lfloor n/2 \rfloor}$. This result easily generalizes to $[k]^n$ as well, that is, if $S \subseteq [k]^n$ does no contain a chain of size $l$, then $|S| \leq (l-1)w([k]^n)$. See Section 5 for a short explanation.

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The proof of Theorem 2 is quite short, although it builds on two other results, both of which requires its own introduction. The first of these results, presented in Section 2, is a high dimensional generalization of the celebrated theorem of Marcus and Tardos [14] on matrix patterns. Let us note that the connection between high dimensional permutation matrices and partial orders has been already explored in [16] in the proof of Theorem 1. The new ingredient in the proof of Theorem 2 is the application of special chain decompositions of the grid. This topic shall be introduced in Section 3. The proof of the main theorem is presented in Section 4. Finally, in Section 5, we discuss a possible extension of Theorem 2 concerning the so called Lubell function.

2 Forbidden matrix patterns

In this section, we state the theorem of Klazar and Marcus [11] on forbidden matrix patterns. For this, let us introduce a few definitions.

A $d$-dimensional $0-1$ matrix is called $d$-pattern. The weight of a $d$-pattern $M$ is the number of 1’s in $M$ and is denoted by $\omega(M)$. A $d$-pattern $M$ of size $m_1 \times \ldots \times m_d$ contains a $d$-pattern $A$ of size $a_1 \times \ldots \times a_d$, if there exist indices $i_{x,y}$ for $x \in [d], y \in [a_x]$ such that $1 \leq i_{x,1} < i_{x,2} < \ldots < i_{x,a_x} \leq m_x$, and $M(i_{1,y_1}, \ldots, i_{d,y_d}) = 1$ if $A(y_1, \ldots, y_d) = 1$. In other words, $M$ contains $A$ if $M$ has an $a_1 \times \ldots \times a_d$ sized submatrix $M'$, where each 1 of $A$ corresponds to a 1 of $M'$. Say that $M$ avoids $A$, if $M$ does not contain $A$.

A $k \times \ldots \times k$ sized $d$-pattern $A$ is a permutation pattern if each axis-parallel hyperplane of $A$ contains at most one 1 entry.

The following is the main theorem of this section, proved by Marcus and Tardos [14] in the case $d = 2$, and extended by Klazar and Marcus [11] for arbitrary $d$.

**Theorem 3.** (Klazar, Marcus [11]) Let $A$ be a $d$-dimensional permutation pattern. There exists a constant $c_A$ such that for every positive integer $m$, every $m \times \ldots \times m$ sized $d$-pattern $M$ that avoids $A$ satisfies

$$\omega(M) \leq c_A m^{d-1}.$$ 

We shall use the following simple corollary of this theorem.

**Corollary 4.** Let $P$ be a $d$-dimensional poset. There exists a constant $c_P$ such that for every positive integer $m$, if a set $S \subset [m]^d$ does not contain a copy of $P$, then

$$|S| \leq c_P m^{d-1}.$$ 

**Proof.** Let $a = |P|$ and let $L_1, \ldots, L_d : P \to [a]$ be $d$ linear orders, whose intersection is $P$. Define the $a \times \ldots \times a$ sized $d$-pattern $A$ such that

$$A(i_1, \ldots, i_d) = \begin{cases} 
1 & \text{if } (i_1, \ldots, i_d) = (L_1(p), \ldots, L_d(p)) \text{ for some } p \in P, \\
0 & \text{otherwise.}
\end{cases}$$

Clearly, $A$ is a permutation pattern. We show that $c_P = c_A$ suffices, where $c_A$ is the constant defined in Theorem 3. Let $M$ be the $m \times \ldots \times m$ sized $d$-pattern, where $M(j_1, \ldots, j_d) = 1$ if $L(j_1, \ldots, j_d) \in S$. It is easy to see that $\omega(M) = |S|$, and as $S$ does not contain a copy of $P$, $M$ avoids $A$. Hence, $|S| \leq c_P m^{d-1}$. 

The optimal order of the constant $c_A$ in Theorem 3 is also studied. The best general bounds are due to Geneson and Tian [7], who proved that $c_A$ can be chosen to be at most $2^{O_d(k)}$, where $k = \omega(A)$. Also, for each $k, d$ they showed the existence of a $k \times \ldots \times k$ sized $d$-dimensional permutation pattern $B$ such that $c_B$ needs to be at least $2^{\Omega_d(k^{1/d})}$, extending a result of Fox [6]. A similar construction shows that $c_P$ is also at least $2^{\Omega(|P|^{1/2})}$ for certain 2-dimensional posets $P$. This already shows that if the constant $C_P$ in Theorem 2 truly exists, then it has to be exponential in $|P|$ for certain posets $P$. On the other hand, it is not clear whether there exist posets $P$ such that the order of the optimal constant $C(P)$ in Theorem 1 is also exponential in $|P|$.
3 Decomposition into long chains

The following conjecture was proposed by Füredi [6].

**Conjecture 5.** For every positive integer $n$, the Boolean lattice $2^{[n]}$ can be partitioned into $(\binom{n}{\lfloor n/2 \rfloor})$ chains such that the size of each chain is $l$ or $l + 1$, where $l = \lfloor 2^n / (\binom{n}{\lfloor n/2 \rfloor}) \rfloor$.

If this conjecture is true, then there exists a partition of $2^{[n]}$ into chains of size $(\sqrt{\pi/2} + o(1))\sqrt{n}$. While Conjecture 5 is still open, the author of this paper [18] proved that $2^{[n]}$ can be partitioned into $(\binom{n}{\lfloor n/2 \rfloor})$ chains such that the size of each chain in the partition is between $0.8\sqrt{n}$ and $25\sqrt{n}$. This result is deduced from a more general one, which we shall state after introducing some further terminology.

In what comes, we shall define the notion of unimodal normalized matching poset, which itself requires a few preliminary definitions. A poset $Q$ is graded if there exists a partition of its elements into subsets $A_0, A_1, \ldots, A_n$ such that $A_0$ is the set of minimal elements, and whenever $x \in A_i$ and $y \in Q$ such that $x < y$ with no $u \in Q$ satisfying $x < u < y$, then $y \in A_{i+1}$. If there exists such a partition, then it is unique and $A_0, A_1, \ldots, A_n$ are the levels of $Q$.

A graded poset $Q$ with levels $A_0, \ldots, A_n$ is unimodal if $|A_0|, \ldots, |A_n|$ is a unimodal sequence, that is, there exists $m \in \{0, \ldots, n\}$ such that $|A_0| \leq \ldots \leq |A_m|$ and $|A_m| \geq |A_{m+1}| \geq \ldots \geq |A_n|$. Also, $Q$ is rank-symmetric, if $|A_i| = |A_{n-i}|$ for $i = 0, \ldots, n$.

A graded poset $Q$ with levels $A_0, \ldots, A_n$ is a normalized matching poset, if for every $0 \leq i, j \leq n$ and $X \subset A_i$, we have

$$\frac{|X|}{|A_i|} \leq \frac{\Gamma(X)}{|A_j|},$$

where $\Gamma(X)$ is the set of elements in $A_j$ which are comparable with an element of $X$. Also, $Q$ has the LYM-property, if every antichain $S \subset Q$ satisfies $\sum_{i=0}^{n} |S \cap A_i|/|A_i| \leq 1$. By a result of Kleitman [12], $Q$ is a normalized matching poset if and only if it has the LYM property.

It is easy to show that the Boolean lattice $2^{[n]}$ is a rank-symmetric, unimodal normalized matching poset. The following extension of Conjecture 5 was proposed by Hsu, Logan and Shahriari [10].

**Conjecture 6.** Let $Q$ be a rank-symmetric, unimodal normalized matching poset of width $w$. Then $Q$ can be partitioned into $w$ chains, each chain in the partition having size $l$ or $l + 1$, where $l = |Q| / w$.

Naturally, this conjecture is open as well. The author of this paper [18] proved the following result concerning Conjecture 6.

**Theorem 7.** (Tomon [18]) Let $Q$ be a unimodal normalized matching poset of width $w$. Then there exists a chain partition of $Q$ into $w$ chains such that the size of each chain in the partition is at least $\frac{Q}{2w} - \frac{1}{2}$. Also, there exists a chain partition of $Q$ into $w$ chains such that the size of each chain in the partition is at most $\frac{2Q}{w} + 5$.

Let us remark that the poset $Q$ in Theorem 7 need not be rank-symmetric. We shall use the following simple corollary of this theorem.

**Corollary 8.** Let $k, n$ be positive integers and let $w$ be the width of the grid $[k]^n$. Then $[k]^n$ can be partitioned into chains such that the size of each chain $C$ in the partition satisfies

$$\frac{k^n}{4w} \leq |C| \leq \frac{k^n}{w}.$$

**Proof.** Note that $[k]^n$ is graded with levels

$$A_i = \{(a_1, \ldots, a_n) \in [k]^n : a_1 + \ldots + a_n = n + i\}$$

for $i = 0, \ldots, kn - n$. It is well known that $[k]^n$ is a rank-symmetric, unimodal normalized matching poset, see p.60-63 in the book of Anderson [1], for example. Hence, by Theorem 7, $[k]^n$ can be partitioned into chains such that the size of each chain is at least $\max\{k^n/(2w - 1/2), 1\} \geq k^n/4w$. If the size of a chain $C$ in this partition is larger than $k^n/w$, then cut $C$ into smaller pieces such that the size of each piece is at least $k^n/4w$, and at most $k^n/w$. The resulting chain partition suffices.

□
4 Proof of the main theorem

For the proof of Theorem 2, we need the following estimate on the width of \([k]^n\).

**Lemma 9.** Let \(k, n\) be positive integers such that \(k \geq 2\). Then \(w([k]^n) = \Theta(k^{n-1}/\sqrt{n})\).

**Proof.** We shall use the following bound on the width of grids, which can be found on p.63-68 in [1]. Let \(k_1, \ldots, k_n \geq 2\) be integers and let \(A = \sum_{i=1}^{n} (k_i^2 - 1)\). Then the width of the grid \([k_1] \times \ldots \times [k_n]\) is \(\Theta(k_1 \ldots k_n / \sqrt{n})\).

Setting \(k_1 = \ldots = k_n = k\), we get the desired result. \(\Box\)

In the proof of our main theorem, we shall exploit that \([k]^n\) is the cartesian product of posets. For \(i \in [n]\), let \(P_i = (X_i, <_i)\) be posets, then \(P_1 \times \ldots \times P_n = (X_1 \times \ldots \times X_n, \leq)\) is the cartesian product of \(P_1, \ldots, P_n\), where \((p_1, \ldots, p_n) \leq (q_1, \ldots, q_n)\), if \(p_i \leq q_i\) for \(i \in [n]\). It is easy to see that if \(n = n_1 + \ldots + n_d\), then the grid \([k]^n\) is isomorphic to the cartesian product \([k]^{n_1} \times \ldots \times [k]^{n_d}\).

With a slight abuse of notation, we shall identify the cartesian product with \([k]^n\) itself.

Now we are ready to prove Theorem 2.

**Proof of Theorem 2.** If \(k = 1\), then \([k]^n\) has only one element, so suppose that \(k \geq 2\). Let \(d = \dim P\) and let \(c_P\) be the constant defined in Corollary 4. Write \(n = n_1 + \ldots + n_d\), where \(n_i \in \{\lceil n/d \rceil, \lfloor n/d \rfloor\}\). As \(n \geq d\), we have \(n_1, \ldots, n_d \geq 1\). Let \(w_i\) be the width of \([k]^{n_i}\) and let \(l_i = k^{n_i}/4w_i\). By Lemma 9, we have

\[
w_i = \Theta \left( \frac{k^{n_i-1}}{\sqrt{n/d}} \right) \quad \text{and} \quad l_i = \Theta(k^{n_i}/\sqrt{n/d}).
\]

By Corollary 8, for \(i \in [d]\), there exist a partition of \([k]^{n_i}\) into chains \(C_{i,1}, \ldots, C_{i,s_i}\), such that \(l_i \leq |C_{i,j}| \leq 4l_i\) for \(j \in [s_i]\). These chain partitions yield a partition of \([k]^n = [k]^{n_1} \times \ldots \times [k]^{n_d}\) into the collection of cartesian products \(G_{j_1, \ldots, j_d} = C_{1,j_1} \times \ldots \times C_{d,j_d}\), where \((j_1, \ldots, j_d) \in [s_1] \times \ldots \times [s_d]\).

Note that the poset \(G_{j_1, \ldots, j_d}\) is isomorphic to the \(d\)-dimensional grid \([|C_{1,j_1}|] \times \ldots \times [|C_{d,j_d}|]\). Let \(m = \max_{i \in [d]} l_i\), then \(m = \Theta(k^{n_i}/\sqrt{n/d})\), \(|G_{j_1, \ldots, j_d}| = \Theta(1)^d n^d\), and \(G_{j_1, \ldots, j_d}\) is isomorphic to a subset of the grid \([4m]^d\). Hence, as \(G_{j_1, \ldots, j_d} \cap S\) does not contain an induced copy of \(P\), we have that

\[
|G_{j_1, \ldots, j_d} \cap S| \leq c_P (4m)^{d-1} = O(1)^d c_P \frac{|G_{j_1, \ldots, j_d}|}{m}.
\]

But then

\[
|S| = \sum_{(j_1, \ldots, j_d) \in [s_1] \times \ldots \times [s_d]} |G_{j_1, \ldots, j_d} \cap S| = O(1)^d c_P \sum_{(j_1, \ldots, j_d) \in [s_1] \times \ldots \times [s_d]} \frac{|G_{j_1, \ldots, j_d}|}{m} = O(1)^d c_P \frac{k^n}{m} = O(1)^d c_P \frac{k^{n-1}}{\sqrt{n/d}} = O(1)^d c_P \sqrt{d w}.
\]

Hence, setting \(C_P = C_0^d c_P\) with some large absolute constant \(C_0\), we have \(|S| \leq C_P w\). \(\square\)

5 Lubell function

For a graded poset \(Q\) with levels \(A_0, \ldots, A_n\), define the Lubell mass of the subset \(S \subset Q\) as

\[
L_Q(S) = \sum_{i=0}^{n} \frac{|S \cap A_i|}{|A_i|}.
\]

Note that every level of \(Q\) is an antichain, so we have the trivial inequality

\[
\frac{|S|}{w(Q)} \leq L_Q(S).
\]

The following strengthening of Theorem 1 was conjectured by Lu and Milans [13], and it was proved by Mérouch [15].
Theorem 10. (Méroueh [15]) Let $P$ be a poset. There exists a constant $c(P)$ such that for every positive integer $n$, if $S \subseteq [2^n]$ does not contain a copy of $P$, then

$$L_{[2^n]}(S) \leq c(P).$$

Also, a simple consequence of the LYM property is that if $Q$ is a normalized matching poset and $P$ is a chain, then $L_Q(S) \leq |P| - 1$ holds for every $S \subseteq Q$ not containing a copy of $P$. This is true because $S$ is the union of at most $|P| - 1$ antichains and the Lubell mass of any antichain in $Q$ is at most 1 by the LYM property. In fact, we have $L_{[k]^n}(S) \leq |P| - 1$ for every positive integer $k$ and $n$, and $S \subseteq [k]^n$ not containing a copy of the chain $P$.

Hence, it might be natural to conjecture the following common strengthening of Theorem 2 and Theorem 10.

If $P$ is a poset, there exist a constant $C(P)$ such that for every positive integer $k, n$ satisfying $k \geq 2, n \geq \dim P$, if $S \subseteq [k]^n$ does not contain a copy of $P$, then $L_{[k]^n}(S) \leq C(P)$.

However, it does not take much effort to show that this conjecture is false, even for small posets $P$. An immediate counterexample is when $P$ is an antichain on 2 elements and $S \subseteq [k]^2$ is a maximal chain. In this case, $S$ clearly does not contain a copy of $P$, but $L_{[k]^2}(S) = 1/2 + 2 \sum_{i=1}^{k-1} 1/i = \Theta(\log k)$. Let us present another example which shows that this conjecture cannot be saved even by increasing the lower bound on $n$.

Let $K$ be the poset on three elements $a, b, c$ with the only comparable pair $a < b$.

Claim 11. For every positive integer $k, n$ satisfying $n \geq 2$, there exists $S \subseteq [k]^n$ such that $S$ does not contain a copy of $K$ and

$$L_{[k]^n}(S) \geq \Omega(1)^n \log k.$$ 

Proof. Let $A_0, \ldots, A_{kn-n}$ be the levels of $[k]^n$. Also, let $s = |\log_2 k| - 1$, and for $i = 0, \ldots, s - 1$, let

$$B_i = \{(a_1, \ldots, a_n) \in [k]^n : \forall j \in [d], 2^i \leq a_j < 2^{i+1}\}.$$

Also, let $r_i = \lfloor (3 \cdot 2^i - 1)n/2 \rfloor$ and

$$S_i = \{(a_1, \ldots, a_n) \in B_i : a_1 + \ldots + a_n = r_i\}.$$

Note that $B_i$ is isomorphic to $[2^i]^n$ and $S_i$ is a maximal sized antichain in $B_i$. Set $S = \cup_{\ell=0}^s S_i$.

First of all, we show that $S$ does not contain a copy of $K$. Suppose that $0 \leq i < j \leq s - 1$, then $x < y$ holds for any two elements $x \in B_i$ and $y \in B_j$. Suppose that $\{a, b, c\} \in S$ is a copy of $K$. As $a$ and $c$ are incomparable, we must have that $a, c \in S_i$ for some $i \in \{0, \ldots, s-1\}$. Also, $b$ and $c$ are incomparable, so $b \in S_i$ as well. But then $a$ and $b$ are incomparable as $S_i$ is an antichain, contradiction.

Now, let us estimate the Lubell mass of $S$. As $S_i$ is a maximal sized antichain in $B_i$, we have $|S_i| = \Theta(2^{(n-1)} / \sqrt{n})$ by Lemma 9. But $S_i$ is contained in the level $A_{r_i-n}$, which satisfies

$$|A_{r_i-n}| \leq \binom{r_i}{n-1} < \left(\frac{er_i}{n-1}\right)^{n-1} \leq O(1)^n \cdot 2^{(n-1)}.$$

Hence, we have

$$L_{[k]^n}(S) = \sum_{i=0}^{s-1} \frac{|S_i|}{|A_{r_i-n}|} \geq \Omega(1)^n s = \Omega(1)^n \log k.$$

On the other hand, it is not hard to show that with $n$ fixed, the Lubell mass of a set $S \subseteq [k]^n$ not containing a copy of $P$ is bounded by some multiple of $\log k$.

Claim 12. Let $P$ be a poset and let $n \geq \dim P$ be an integer. There exist constants $C(P)$ and $\alpha_n$ such that for every positive integer $k$, if $S \subseteq [k]^n$ does not contain a copy of $P$, then

$$L_{[k]^n}(S) \leq C(P)\alpha_n \log k.$$
Proof. Again, let $A_0,...,A_{kn-n}$ be the levels of $[k]^n$. Let

$$Q^- = \{(a_1,...,a_n) \in [k]^n : a_1 + ... + a_n \leq (k+1)n/2\},$$

that is $Q^-$ is the lower half of $[k]^n$, and let $Q^+ = [k]^n \setminus Q^-$ be the upper half of $[k]^n$. We prove that the Lubell mass of $S^+ = S \cap Q^+$ is bounded by $C_P \beta_n \log k$, where $C_P$ is the constant defined in Theorem 2 and $\beta_n$ is some function of $n$. Then, a symmetric argument shows that the Lubell mass of $S^- = S \cap Q^-$ is also bounded by $C_P \beta_n \log k$, finishing our proof.

Let $s = \lfloor \log_2(k+1)n/2 \rfloor$, and for $i = 0,...,s-1$, let

$$C_i = \{(a_1,...,a_n) \in Q^- : 2^i \leq a_1 + ... + a_n < 2^{i+1}\}. $$

Also, let $S_i = C_i \cap S^-$. Clearly, $C_i \subset [2^{i+1}]^n$, so by Theorem 2 and Lemma 9, we have

$$|S_i| \leq O(C_P 2^{(i+1)(n-1)/\sqrt{n}}).$$

Also, $C_i$ is the union of the levels $A_{2^i-n},...,A_{r-n}$, where $r = \min\{2^{i+1} - 1, (k+1)n/2\}$, which satisfy

$$|A_{2^i-n}| \leq ... \leq |A_{r-n}|.$$  

The size of $A_{2^i-n}$ is at least $(2^i / 2n)^{n-1}$, as for any choice $a_1,...,a_{n-1} \in [2^i / n, 2^i / n + 2^i / n^2]$, there exists a unique $a_n$ such that $(a_1,...,a_n) \in A_{2^i-n}$. Hence, we have

$$L_{[k]^n}(S) = \sum_{j=2^i-n}^{r-n} \frac{|S_i \cap A_j|}{|A_j|} \leq \frac{(2n)^{n-1}|S_i|}{2^{(n-1)}} \leq O(C_P (2n^2)^{n-1}),$$

which gives

$$L_{[k]^n}(S^-) = \sum_{i=0}^{s-1} L_{[k]^n}(S_i) = O(C_P (2n^2)^{n-1} s) = O(C_P (2n^2)^{n-1} \log k).$$

\[ \square \]

In the light of these results, we believe that the following generalization of Theorem 10 might be true.

Conjecture 13. Let $P$ be a poset. There exists a constant $C(P)$ such that the following holds. For every positive integer $k$ and $n$ satisfying $n \geq \dim P$, if $S \subset [k]^n$ does not contain a copy of $P$, then

$$L_{[k]^n}(S) \leq C(P) \log k.$$

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