Gauge theories of gravity: the nonlinear framework

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Nonlinear realizations of spacetime groups are presented as a versatile mathematical tool providing a common foundation for quite different formulations of gauge theories of gravity. We apply nonlinear realizations in particular to both the Poincaré and the affine group in order to develop Poincaré gauge theory (PGT) and metric-affine gravity (MAG) respectively. Regarding PGT, two alternative nonlinear treatments of the Poincaré group are developed, one of them being suitable to deal with the Lagrangian and the other one with the Hamiltonian version of the same gauge theory. We argue that our Hamiltonian approach to PGT is closely related to Ashtekar’s approach to gravity. On the other hand, a brief survey on MAG clarifies the role played by the metric–affine metric tensor as a Goldsone field. All gravitational quantities in fact—the metric as much as the coframes and connections— are shown to acquire a simple gauge–theoretical interpretation in the nonlinear framework.

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I. INTRODUCTION

In the search for the unification of forces, different alternatives to Einstein’s original General Relativity have been proposed \[1\] \[2\] \[3\] \[4\] \[5\] \[6\] \[7\] \[8\] \[9\] \[10\] \[11\], based on the extension of the gauge principle to spacetime groups. The analogy existing between gauge theories of gravity and the Yang–Mills theories supporting the standard model allows to assimilate gravitation to the remaining forces through the characterization of all interactions—gravity included—as mediated by gauge potentials only. In what follows we will focus our attention on Hehl’s Poincaré gauge theory (PGT) as much as on metric-affine gravity (MAG), see \[12\] \[13\] \[14\] \[15\] \[16\], both formalisms presenting grand adaptability in dealing with a diversity of spacetime actions. We claim their value as a suitable support for the unification of different theoretical points of view on gravitational forces. Indeed, a main result of the present work is to show the close relationship between the Hamiltonian version of PGT and the Ashtekar approach.

The cornerstone of our treatment consists of nonlinear realizations (NLR’s), a mathematical method \[17\] \[18\] \[19\] \[20\] \[21\] \[22\] \[23\] \[24\] \[25\] argued by us to provide a universal foundation for gauge theories of different groups \[26\] \[27\] \[28\] \[29\] \[30\]. The usefulness of NLR’s in the context of gravitational theories becomes apparent mainly when translations are contained in the spacetime gauge group as a subgroup. As a result of the nonlinear approach, the translational connections transform into covector-valued 1-forms suitable to be identified as coframes, so that the dynamical gauge theory becomes indistinguishable from spacetime geometry.

For a gauge group \(G\) to be realized nonlinearly, an auxiliary subgroup \(H \subset G\) is required to be chosen in addition. The freedom in selecting the latter provides the nonlinear method with a considerable flexibility. Indeed, a single theory, say the gauge theory of the Poincaré group \(G\), manifests itself in quite different forms depending on the subgroup \(H\) chosen. As we will see, the NLR of PGT with \(H = \text{Lorentz}\) reveals to be suitable to be taken as the basis for a Lagrangian approach, whereas the one with \(H = \text{SO}(3)\) is especially adapted to a Hamiltonian treatment. In the case of MAG, being \(G\) the affine group, we also consider two different nonlinear approaches, corresponding to the choices of the subgroups \(H = \text{GL}(4,R)\) and \(H = \text{Lorentz}\) respectively. The relation between both NLR’s will be exploited to explain the gauge theoretical origin of the ten degrees of freedom of the metric, as much as their Goldstone nature allowing to rearrange them into redefined fields.

The present work is organized as follows. In Section II we briefly review the mathematical foundations of NLR’s in terms of composite fiber bundles. Section III is devoted to what we call the standard approach to PGT, yielding explicitly Lorentz covariant coframes and spin connections. The resulting formalism is used to build an Einstein–Cartan Lagrangian description of gravity. Then in Section IV we pay attention to the Hamiltonian approach to PGT. Several technical aspects are discussed, such as a Poincaré invariant foliation of spacetime and a general Hamiltonian formalism adapted to exterior calculus. The Hamiltonian dynamics of the Einstein–Cartan action, expressed in terms of real PGT connection variables, is shown to be consistent both with the Lagrangian treatment of Section III and with an alternative approach of the Ashtekar type which we also develop in some detail. Finally in Section V we apply the nonlinear approach to the affine group to derive metric-affine gravity.

II. FOUNDATIONS OF NONLINEAR GAUGE THEORIES

Principal bundles \(P(M,G)\) describe the structure of ordinary gauge theories of internal Lie groups \(G\). This scheme does not hold for nonlinear gauge theories, based on the interplay between the gauge group \(G\) and a subgroup \(H \subset G\). In \[31\] we invoked composite fiber bundles as the suitable topological background underlying nonlinear realizations of local symmetries.

Roughly speaking, a composite bundle is a principal bundle \(P(M,G)\) whose \(G\)-diffeomorphic fibers are regarded themselves as bundles whose structure group \(H\) is a subgroup of the structure group \(G\) of the total bundle. Actually, composite bundles can be built provided a subgroup \(H \subset G\) exists, whose right action on elements \(g \in G\) induces a complete partition of the group manifold \(G\) into mutually disjoint orbits \(gH\). By projecting each of these equivalence classes to a single element (that is to a left coset) of the quotient space \(G/H\), the group manifold \(G\) becomes organized as a bundle \(G/G/H\) with the orbits \(gH\) (diffeomorphic to the subgroup \(H\)) as local fibers and with \(G/H\) as the base space. When attached to points of an auxiliary base manifold \(M\), the local bundles \(G(G/H\), \(H)\) constitute the fibers of a composite bundle.

Locally, composite bundles are diffeomorphic to \(M \times G/H \times H\), so that by singling out either the base space \(M\) or the manifold \(\Sigma \simeq M \times G/H\), two mutually related bundle structures can be recognized in the total bundle space \(P\). On the one hand, the usual bundle structure survives with total \(G\)-diffeomorphic fibers projecting to the bundle base space \(M\). On the other hand, \(P\) can also be regarded as consisting of \(H\)-diffeomorphic fiber branches attached to the manifold \(\Sigma\), the latter playing the role of an intermediate base space. The alternative projections \(\pi_{PM} : P \to M\) and \(\pi_{PB} : P \to \Sigma\) become related to each other by defining an additional mapping \(\pi_{EM} : \Sigma \to M\) such that the ordinary total projection decomposes into two partial projections as \(\pi_{PM} = \pi_{EM} \circ \pi_{PB}\). Correspondingly, the local sections
$s_{MP} : M \to P$ decompose as

$$s_{MP} = s_{\Sigma P} \circ s_{M\Sigma} .$$  \hfill (1)

Let us express the sections introduced in (1) in terms of zero sections –the one associated to $s_{MP}$ denoted as $\sigma_{MP}$ and so on–, that is

$$s_{MP} = R_{\tilde{g}} \circ \sigma_{MP} , \quad \tilde{g} \in G ,$$  \hfill (2)

$$s_{\Sigma P} = R_{a} \circ \sigma_{\Sigma P} , \quad a \in H ,$$  \hfill (3)

and

$$s_{M\Sigma} = R_{b} \circ \sigma_{M\Sigma} , \quad b \in G/H .$$  \hfill (4)

As compatibility conditions, they have to satisfy $\tilde{g} = b \cdot a$ and $\sigma_{MP} = R_{b^{-1}} \circ \sigma_{\Sigma P} \circ R_{b} \circ \sigma_{M\Sigma}$ (or alternatively $\sigma_{\Sigma P} \circ R_{b} = R_{b} \circ \sigma_{\Sigma P}$). For later convenience we introduce the composite section

$$\sigma_{\xi} : M \to \Sigma \to P$$

defined from the total and zero sections in (3) and (4) as

$$\sigma_{\xi}(x) := \sigma_{\Sigma P} \circ s_{M\Sigma}(x) = R_{b} \circ \sigma_{MP}(x) .$$  \hfill (5)

(The $\xi$ in $\sigma_{\xi}(x)$ stands for the parameters labelling the elements $b \in G/H$ displayed as $R_{b}$ in the r.h.s. of (5).) The reason for introducing (5) is its usefulness for expressing the main results on the nonlinear approach, deduced in [31] and summarized below.

By comparing two bundle elements, both of the form (3), differing from each other by the left action $L_{g}$ of elements $g \in G$, in [31] we found the nonlinear transformation law

$$L_{g} \circ \sigma_{\xi}(x) = R_{h} \circ \sigma_{\xi}'(x) ,$$  \hfill (6)

being $R_{h}$ the right action of a certain element $h \in H$ of the subgroup. For practical reasons, in [30] we transformed (6) into the more manageable formula

$$g \cdot b = b' \cdot h ,$$  \hfill (7)

where $g \in G$, $h \in H$, and $b, b' \in G/H$. Notice that (7) reproduces the original form of the nonlinear law as given in [17]. On the other hand, for infinitesimal $g$ and $h = e^{\mu} \approx 1 + \mu$, a gauge transformation is induced by (6) on fields $\psi$ of any representation space of $H$, namely

$$\delta\psi(\sigma_{\xi}(x)) = \rho(\mu)\psi(\sigma_{\xi}(x)) ,$$  \hfill (8)

where $\rho(\mu)$ denotes the suitable representation of the $H$-algebra, see [17] [31].

Since we are interested in building the covariant derivatives of the fields $\psi$ transforming nonlinearly as (8), in [30] we compared the ordinary linear connection, resulting from pulling back the connection 1-form $\omega$ by means of $\sigma_{MP}$, that is

$$A_{M} = \sigma_{MP}^{*}\omega ,$$  \hfill (9)

with the connection characteristic for the nonlinear approach, defined as the pullback of $\omega$ by means of $\sigma_{\xi}$, namely

$$\Gamma_{M} = \sigma_{\xi}^{*}\omega ,$$  \hfill (10)

the difference between $\sigma_{\xi}$ and $\sigma_{MP}$ being displayed in (5). One finds [30] that the nonlinear connection (10) can be expressed in terms of the linear one (9) as

$$\Gamma_{M} = b^{-1}(d + A_{M})b .$$  \hfill (11)

Its gauge transformations, induced by the nonlinear group action (6), are found to be

$$\delta\Gamma_{M} = -(d\mu + [\Gamma_{M}, \mu]) ,$$  \hfill (12)

with $\mu$ the same $H$-algebra-valued parameters as in (5). Being $\Gamma_{M}$ valued on the Lie algebra of the whole group $G$, from [12] one reads out that only those of its components defined on the $H$-algebra behave as true connections.
transforming inhomogeneously, while its components with values on the remaining algebra elements of \(G/H\) transform as \(H\)-tensors. As a result of (8) and (12), covariant differentials defined as
\[
D\psi := (d + \rho(\Gamma_M))\psi, \tag{13}
\]
transform as
\[
\delta D\psi = \rho(\mu)D\psi. \tag{14}
\]
Finally, let us make use of the covariant differential operator
\[
D := d + \Gamma_M, \tag{15}
\]
as read out from (13) without specifying any particular representation, to obtain the field strength as
\[
F := D \wedge D = d\Gamma_M + \Gamma_M \wedge \Gamma_M. \tag{16}
\]
In view of (12) we find (16) to transform as
\[
\delta F = [\mu, F]. \tag{17}
\]
The relevance of the nonlinear approach for the foundation of gauge theories of gravity becomes evident in the following sections, where we apply it to the local treatment of different spacetime groups.

III. STANDARD NONLINEAR POINCARÉ GAUGE THEORY

A. Coframes and Lorentz connections

As a first application of the general formalism established in previous section, let us take the group \(G\) to be the Poincaré group in order to show how its nonlinear local approach gives rise to the Poincaré gauge theory of gravity (PGT). Diverse nonlinear realizations are possible depending on the choice of the auxiliary subgroup \(H \subset G\). In this section we take \(H\) to be the Lorentz group, yielding an explicitly Lorentz covariant four–dimensional formalism which provides the geometrical basis for a Lagrangian approach (developed by us in the language of exterior calculus). Later in Section IV we present the version of PGT resulting from taking \(H\) to be \(SO(3)\), suitable to deal with the \(3 + 1\) decomposition underlying PGT Hamiltonian dynamics (to be treated also in exterior calculus, with differential forms playing the role of dynamical variables).

Our starting point is the fundamental transformation law (6) of nonlinear realizations. After rewriting it in the simplified form (7), for \(G = \text{Poincaré}\) and \(H = \text{Lorentz}\) we parametrize the infinitesimal Poincaré group element \(g \in G\) and the infinitesimal Lorentz group element \(h \in H\) respectively as
\[
g = e^{i\epsilon^\alpha P_\alpha}e^{i\beta_{\alpha\beta}L_{\alpha\beta}} \approx 1 + i\left(\epsilon^\alpha P_\alpha + \beta^{\alpha\beta}L_{\alpha\beta}\right), \tag{18}
\]
and
\[
h = e^{i\mu^{\alpha\beta}L_{\alpha\beta}} \approx 1 + i\mu^{\alpha\beta}L_{\alpha\beta}, \tag{19}
\]
where \(L_{\alpha\beta}\) are the Lorentz generators and \(P_\mu\) the translational ones. As read out from (7), the left action of (18) on elements
\[
b = e^{-i\xi^\alpha P_\alpha} \tag{20}
\]
of the coset space \(G/H\), being (20) identical with elements of the group of translations labelled by the finite parameters \(\xi^\alpha\), induces a right action of (18) on
\[
b' = e^{-i(\xi^\alpha + \delta\xi^\alpha)P_\alpha}. \tag{21}
\]
Replacing (18)–(21) into (7) and taking into account the commutation relations of the Poincaré algebra
\[
[P_\alpha, P_\beta] = 0, \quad [L_{\alpha\beta}, P_\mu] = i\epsilon_{[\alpha}P_{\beta]\mu}, \\
[L_{\alpha\beta}, L_{\mu\nu}] = i\left(\delta_{[\alpha}L_{\nu]\beta] - \epsilon_{\alpha[\mu}L_{\mu]\nu]_{\beta]}\right), \tag{22}
\]
and
with \( o_{\alpha \beta} \) as the the Minkowski metric
\[
o_{\alpha \beta} := \text{diag}( - + + + ), \tag{23}
\]
a simple computation with the help of the Hausdorff-Campbell formula, see Appendix B, yields the value of \( \mu^{\alpha \beta} \) in (19) as much as the variation of the translational coset parameters, namely
\[
\mu^{\alpha \beta} = \beta^{\alpha \beta}, \quad \delta \xi^\alpha = - \beta^{\alpha \beta} \xi^\beta - \epsilon^\alpha, \tag{24}
\]
showing \( \xi^\alpha \) to transform exactly as Minkowskian coordinates.

On the other hand, using for the linear connection (9) of the Poincaré group the notation
\[
A_M = - i \Gamma^\alpha P_\alpha - i \Gamma^{\alpha \beta} L_{\alpha \beta}, \tag{25}
\]
whose components on the Poincaré algebra are the linear translational contribution \( \Gamma^\alpha \) and the Lorentz one \( \Gamma^{\alpha \beta} \) respectively, we find the nonlinear connection (11) to be
\[
\Gamma_M = - i \vartheta^\alpha P_\alpha - i \Gamma^{\alpha \beta} L_{\alpha \beta}, \tag{26}
\]
with the Lorentz connection \( \Gamma^{\alpha \beta} \) unmodified with respect to the linear case (25), but with the translational connection transformed into
\[
\vartheta^\alpha := D \xi^\alpha + \Gamma^{\alpha}_\alpha. \tag{27}
\]
In view of (12), the components of (26) transform respectively as
\[
\delta \vartheta^\alpha = - \vartheta^\beta \beta^{\alpha \beta}, \quad \delta \Gamma^{\alpha \beta} = D \beta^{\alpha \beta}. \tag{28}
\]
The most relevant result is the first equation in (28). According to it, instead of the linear translational connection \( \Gamma^\alpha \) in (26) transforming inhomogeneously as \( \delta \Gamma^\alpha = - \beta^{\alpha \beta} \beta^\beta + D \epsilon^\alpha \), we have at our disposal a nonlinear translational connection 1-form (27) which is Lorentz covector-valued. The latter will be identified from now on as the Lorentz coframe or tetrad. This feature of deductively providing tetrads with the right transformation properties constitutes one of the main achievements of nonlinear realizations. (Compare with the hypotheses needed to build (27) in the context of the linear approach [32].)

B. Gravitational actions and Lagrangian field equations (in the language of exterior calculus)

Let us show that the coframe \( \vartheta^\alpha \) and the Lorentz connection \( \Gamma^{\alpha \beta} \) introduced above are variables suitable to build Poincaré gauge invariant gravitational actions. (Exterior calculus allows to take differential forms as such –rather than their components– as dynamical variables. Actually we obtain the field equations by varying a Lagrangian density 4–form with respect to the 1–forms \( \vartheta^\alpha \) and \( \Gamma^{\alpha \beta} \) respectively, see below and [16].) The field strengths of the coframe and of the Lorentz connection are found by applying the general expression (16) to the nonlinear Poincaré connection (26), yielding
\[
F = - i T^\alpha P_\alpha - i R^{\alpha \beta} L_{\alpha \beta}. \tag{29}
\]
In (29), the torsion
\[
T^\alpha := D \vartheta^\alpha := d \vartheta^\alpha + \Gamma^\alpha_\beta \wedge \vartheta^\beta \tag{30}
\]
coincides with the translational field strength, while the Lorentzian field strength is the curvature
\[
R_{\alpha \beta} := d \Gamma_{\alpha \beta} + \Gamma^\gamma_\beta \wedge \Gamma_{\alpha \gamma}. \tag{31}
\]
Both (30) and (31) are building blocks for gravitational actions. For instance, with the help of (31) besides the elements of the eta basis defined in Appendix A (built from the coframes (27)), one can express the ordinary Einstein-Cartan gravitational Lagrange density 4–form with cosmological term as
\[
L_{EC} = - \frac{1}{2 l^2} R^{\alpha \beta} \wedge \eta_{\alpha \beta} + \frac{\Lambda}{l^2} \eta, \tag{32}
\]
see [10] [33]. More general Lagrangians including contributions quadratic in the irreducible pieces of curvature and torsion (of the form $^{(1)}R^\alpha\beta \wedge \star R_\alpha\beta$, $^{(1)}T^\alpha \wedge \star T_\alpha$) are extensively studied in the literature [34] [33]. For the sake of simplicity, here we only consider the action $S = \int L_{EC}$ built from [22]. The field equations derived by varying [22] with respect to the tetrads $\theta^\alpha$ are

$$\frac{1}{2} \eta_{\alpha\beta\gamma} \wedge R^\beta\gamma - \Lambda \eta_\alpha = 0,$$

and on the other hand, variation on the Lorentz connection $\Gamma^{\alpha\beta}$ yields

$$D\eta_{\alpha\beta} = 0.$$  

(33)

Since $D\eta_{\alpha\beta} = \eta_{\alpha\beta\gamma} \wedge T^\gamma$, from [34] follows the vanishing of torsion, that is

$$T^\alpha = 0,$$

(35)

implying that the Lorentz connection reduces to the (anholonomic) Christoffel connection

$$\Gamma^{\{\} \alpha \beta} := e_\alpha[i] d\theta_\beta - \frac{1}{2} (e_\alpha)[e_\beta] d\theta^\gamma \partial_\gamma.$$  

(36)

By replacing [30] in [33], the latter reduces to the standard Einstein vacuum equations with cosmological constant defined on a Riemannian space. This can be easily checked by translating [33] to the usual Riemannian language of General Relativity involving the holonomic metric $g_{ij} := \alpha_\alpha \epsilon^{\alpha}_e \epsilon^{\beta}_j$ defined from the tetrads $\theta^\alpha = dx^i e_\alpha$ with the Minkowski metric [28]. The anholonomic Christoffel connection [30] transforms into

$$\Gamma^{\{\} \alpha \beta} := -dx^i e_\alpha[i] \left( \partial_\beta g_{ij} - e_{k\beta} e_{k[i]} \right)$$

(37)

when reexpressed in terms of the ordinary holonomic Christoffel symbol $\Gamma^i_{ij} := \frac{1}{2} g^{kl} \left( \partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij} \right)$, while the curvature [31] with [37] reduces to

$$R_{\alpha\beta} = \frac{1}{2} dx^i \wedge dx^j e_{[\alpha} e_{j\beta]} R_{ij}^k,$$

(38)

being $R_{ij}^k := 2 \left( \partial_i \Gamma^k_{lj} + \Gamma^k_{im} \Gamma^m_{lj} \right)$ the ordinary Riemann tensor. By inserting [38] into [33], using the definitions of the Ricci tensor $R_{ij} := R_{ikj}^k$ and of the scalar curvature $R := g^{ij} R_{ij}$ and making use of the holonomic version of the eta basis of Appendix A, being for instance $\eta^j = \eta^\alpha e^{\alpha}_j = \frac{1}{2} \sqrt{g} e^k_{klm} dx^k \wedge dx^l \wedge dx^m$, the Einstein equations [33] take their standard form

$$\frac{1}{2} \eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma} - \Lambda \eta_\alpha = - e^i_\alpha \left( R_{ij} - \frac{1}{2} g_{ij} R + \Lambda g_{ij} \right) \eta^j = 0.$$  

(39)

The fact that the Einstein-Cartan action [32] reproduces the Einstein equations of General Relativity is a test of the validity of the PGT approach. However, the latter is flexible enough to be applied to extended gravitational actions involving quadratic curvature and torsion terms and giving rise to nonvanishing torsion. In general, the use of Poincaré gauge variables introduces both, a different perspective in the interpretation of gravity as mediated by connections only—translational as much as Lorentzian ones—rather than by the base space metric $g_{ij}$, and moreover the possibility of deriving new results which are meaningless in the purely metrical approach. In order to illustrate the latter point, in the next paragraph we show as an example the coupling of gravitational fields (namely $\theta^\alpha$ and $\Gamma^{\alpha\beta}$) to fundamental matter fields.

C. PGT invariant action of Dirac fields

If PGT is to be regarded as a basic theory of gravity, one has to understand its coupling to matter beyond phenomenological matter sources. Accordingly, a PGT invariant Dirac action is to be added to PGT gravitational Lagrangians like [32] or its generalizations. To do so, the first step consists in finding the explicit form of the covariant derivative [17] of Dirac bispinors with the Poincaré nonlinear connection [20]. As shown in [30], a four–dimensional
realization of the Poincaré generators, $P_\mu$ as much as $L_{\alpha\beta}$, can be built from the gamma matrices in the Dirac representation

$$
\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^a = \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},
$$

being $\gamma_5 := i \gamma^0 \gamma^1 \gamma^2 \gamma^3$. Besides the usual spinor generators

$$\rho(L_{\alpha\beta}) = \sigma_{\alpha\beta} := \frac{i}{8} [\gamma_\alpha, \gamma_\beta],$$

we introduce the finite matrix representation of translational generators as

$$\rho(P_\mu) = \pi_\mu := m \gamma_\mu (1 + \gamma_5),$$

where the dimensional constant $m \sim [L]^{-1}$ (in natural units $\hbar = c = 1$) guarantees the same dimensionality for the intrinsic linear momentum associated to (42) as for the orbital linear momentum $-i \partial_\mu$, see [30]. Both (41) and (42) provide a nontrivial finite matrix realization of the Poincaré algebra (22) in spite of the fact that $\pi_\mu \pi_\nu = 0$. The Poincaré covariant derivative (13) of Dirac fields thus reads

$$D\psi = d\psi - i (\Gamma^{\alpha\beta} \sigma_{\alpha\beta} + \vartheta^\mu \pi_\mu) \psi,$$

transforming in accordance with (14) as $\delta D\psi = i \beta^{\alpha\beta} \sigma_{\alpha\beta} D\psi$. The PGT–invariant Dirac Lagrange density 4-form built with the help of (43)–without explicit mass term– reads

$$L_D = \frac{i}{2} (\bar{\psi} * \gamma \wedge D\psi + D\bar{\psi} \wedge * \gamma \psi),$$

where we use the notation of [14], being $\gamma := \vartheta^\mu \gamma_\mu$ and $* \gamma$ its Hodge dual, and as usual $\bar{\psi} := \psi^\dagger \gamma_0$ and

$$D\bar{\psi} := (D\psi)^\dagger \gamma^0 = d\bar{\psi} + i \bar{\psi} (\Gamma^{\alpha\beta} \sigma_{\alpha\beta} + \vartheta^\mu \pi_\mu).$$

The Dirac matter action (44) in the presence of gravity has the peculiarity of including the intrinsic translational contributions required by the nonlinear gauge approach to PGT, as seen from the covariant derivatives (13) and (14). However, it is interesting to notice that these contributions manifest themselves as a mass term to be added to an explicitly Lorentz-invariant (rather than Poincaré-invariant) Dirac action. Indeed, let us separate the translational parts of (13) and (14) as

$$D\psi := \tilde{D}\psi - i \vartheta^\mu \pi_\mu \psi, \quad D\bar{\psi} := \tilde{D}\bar{\psi} + i \bar{\psi} \vartheta^\mu \pi_\mu,$$

where we denote with tildes the translation-independent pieces with the standard form of Lorentz covariant derivatives. By replacing (13) in (14), we realize that the latter transforms into

$$L_D = \frac{i}{2} (\bar{\psi} * \gamma \wedge \tilde{D}\psi + \tilde{D} \bar{\psi} \wedge * \gamma \psi) + * m \bar{\psi} \psi.$$

To get (47) we made use of the fact that $\vartheta^\alpha \wedge * \vartheta_\beta = \delta^\alpha_\beta \eta$, with $\eta = * 1$ as the 4-dimensional volume element, see Appendix A, so that

$$* \gamma \wedge \vartheta^\mu \pi_\mu = - \eta \gamma^\mu \pi_\mu = * m (1 + \gamma_5),$$

and

$$- \vartheta^\mu \pi_\mu \wedge * \gamma = - \eta \pi_\mu \gamma^\mu = * m (1 - \gamma_5).$$

The particular combination of (13) and (14) in the matter action cancels out the $\gamma_5$ contribution, only remaining the background mass term in (47). Thus, the nonlinear PGT approach to the coupling of translations to Dirac fields predicts the latter ones to be massive.
IV. HAMILTONIAN TREATMENT OF PGT

A. Remark on the diversity of equivalent nonlinear approaches

Previous section was devoted to a nonlinear approach to the gauge theory of the Poincaré group —namely the one with auxiliary subgroup \( H = \text{Lorentz} \)— useful to support Lagrangian dynamics of spacetime. However, we recall that given \( G = \text{Poincaré} \), the choice of \( H \subset G \) is not uniquely predetermined. The outline of Section II showed that nonlinear realizations of a given group \( G \) require to fix, in addition to the total symmetry group \( G \) itself, a subgroup \( H \subset G \) enabling the \( G \)–gauge transformations to act on representation fields of \( H \). No breaking of the original \( G \)–symmetry is needed for it to be realized through explicitly \( H \)–symmetric quantities. We are free to select any among the available subgroups \( H \subset G \) in order to construct diverse versions of the gauge theory of one and the same group \( G \). (Notice that the usual gauge theories of internal groups based on linear realizations rest on the particular NLR corresponding to the choice \( H = G \).) Nonlinear gauge approaches to a group \( G \) corresponding to different auxiliary subgroups \( H_1, H_2 \) are equivalent to each other, being possible to relate them by means of gauge–like redefinitions of the fields, as we will show in Section V. Thus descriptions of the local group \( G \) with either \( H_1 \) or \( H_2 \) constitute different realizations of the same gauge theory.

In the present section we are going to develop a nonlinear local realization of the Poincaré group having as auxiliary subgroup \( H = \text{SO}(3) \) instead of the Lorentz group considered previously. The resulting \( \text{SO}(3) \)–covariant formalism reveals to be useful for the Hamiltonian treatment of the PGT approach to gravity. As before, we characterize dynamical variables by means of differential forms. An exterior calculus formulation of Hamiltonian dynamics, fit to gauge theories, is briefly outlined in the following. It is mainly based on a proposal by Wallner [35], suitably adapted to the present nonlinear approach to PGT with \( H = \text{SO}(3) \).

B. Poincaré invariant foliation of spacetime

Nonlinear realizations of \( G = \text{Poincaré} \) with \( H = \text{SO}(3) \), as derived immediately from the general formalism on NLR’s of Section II, are displayed in Appendix C. The quantities introduced there will be invoked in what follows in the order needed for our purposes.

First of all, observe that instead of the four–dimensional representation [27] of the tetrad transforming as a Lorentz covector as shown in [28], in the nonlinear approach of Appendix C we find the coframe \( \hat{\vartheta}^a \) splitted through definition [17] into an \( \text{SO}(3) \) singlet \( \hat{\vartheta}^0 \) plus an \( \text{SO}(3) \) covector \( \hat{\vartheta}^a \), whose explicit gauge transformations are given by [22] and [23] respectively. The invariance [22] of the time component \( \hat{\vartheta}^0 \) suggests to perform an invariant foliation of spacetime into spatial slices as follows.

From the 1–form basis [17], we define the dual vector basis \( \epsilon_a \) such that \( \epsilon_a \epsilon^a = \delta_a^a \). Starting from the relation \( [\epsilon_a, \epsilon_b] = (\epsilon_a \epsilon^a \epsilon^b \epsilon_b) \epsilon_c \), holding in the 4–dimensional space, the necessary and sufficient condition for a foliation into 3–dimensional hypersurfaces normal to \( \epsilon_a \) to exist, according to the Frobenius’ theorem, is that the spatial restriction of the former formula yields \( [\hat{\epsilon}_a, \hat{\epsilon}_b] = (\hat{\epsilon}_a \hat{\epsilon}^a \hat{\epsilon}^b \hat{\epsilon}^b) \hat{\epsilon}_c \), not involving \( \epsilon_a \) in the r.h.s., or equivalently

\[
\hat{\vartheta}^0 \wedge d \hat{\vartheta}^a = 0.
\] (50)

Notice that, according to [22], the foliation condition (50) is Poincaré invariant. The general solution of (50) reads

\[
\hat{\vartheta}^0 = u^0 \, d \tau.
\] (51)

In view of (51), let us define \( u^\alpha := \partial_{\tau} [\hat{\vartheta}^0 \epsilon^\alpha] \) such that \( \partial_{\tau} = u^\alpha \epsilon_a - u^0 \hat{\epsilon}_a + u^a \hat{\epsilon}_a \), so that

\[
\epsilon_a = \frac{1}{u^0} (\partial_{\tau} - u^a \hat{\epsilon}_a),
\] (52)

satisfying \( \hat{\epsilon}_a \epsilon^a = 1 \). In what follows, we take (52) as the invariant timelike vector field defining the foliation direction of spacetime. Accordingly, it becomes possible to perform the decomposition of any p–form \( \alpha \) [36] into a longitudinal and a transversal part with respect to (52) as

\[
\alpha = \hat{\vartheta}^0 \wedge \alpha_\perp + \alpha_\parallel,
\] (53)

with respective definitions

\[
\alpha_\perp := \hat{\epsilon}_a \epsilon^\alpha, \quad \alpha_\parallel := \hat{\epsilon}_a \left( \hat{\vartheta}^0 \wedge \epsilon^\alpha \right).
\] (54)
Correspondingly, the foliation of the Hodge dual (53) reads

\[ *\alpha = (-1)^p \hat{\partial}^0 \wedge \#_a \alpha - \#_a \alpha, \tag{55} \]

where the asterisc * stands for the Hodge dual in four dimensions while # represents its three–dimensional restriction, see [37], [38], [36]. On the other hand, the exterior differential of any p–form decomposes as

\[ d\alpha = \hat{\partial}^0 \wedge \left( l_{\hat{e}_0} \alpha - \frac{1}{u^0} d (u^0 \alpha) \right) + d\alpha, \tag{56} \]

where we introduced the Lie derivative with respect to \( \hat{e}_0 \) defined as

\[ l_{\hat{e}_0} \alpha := d (\hat{e}_0 \alpha) + (\hat{e}_0 \alpha) , \tag{57} \]

respectively. On the other hand, being the Lagrangian density a 4–form, its transversal part vanishes, decomposing simply as

\[ L = \hat{\partial}^0 \wedge L_{\perp}. \tag{60} \]

From the Lagrangian normal part \( L_{\perp} \) in (60) we define the momenta

\[ \#_{\pi^{\alpha}} := \frac{\partial L_{\perp}}{\partial (l_{\hat{e}_0} A)} , \quad \#_{\pi^{\hat{\alpha}}} := \frac{\partial L_{\perp}}{\partial (l_{\hat{e}_0} \hat{\alpha})} , \tag{61} \]

and with their help we define the Hamiltonian 3–form

\[ \mathcal{H} := u^0 \left( l_{\hat{e}_0} A_{\perp} \#_{\pi^{\perp}} + l_{\hat{e}_0} \hat{\alpha} \wedge \#_{\pi^{\perp}} - L_{\perp} \right) . \tag{62} \]

In view of (61), from (62) we reconstruct the Lagrangian density (60) by multiplying by \( d\tau \), getting

\[ L = d\tau \wedge \left( u^0 l_{\hat{e}_0} A_{\perp} \#_{\pi^{\perp}} + u^0 l_{\hat{e}_0} \hat{\alpha} \wedge \#_{\pi^{\perp}} - \mathcal{H} \right) . \tag{63} \]
Variations of $\mathcal{H}$, with $\mathcal{H}$ taken to be a functional of the gauge potentials and their momenta, yield the field equations

$$u^0 \iota_0 A_\perp = \frac{\delta \mathcal{H}}{\delta \pi^{A_\perp}}, \quad u^0 \iota_0 A = \frac{\delta \mathcal{H}}{\delta \pi^A}, \quad u^0 \iota_0 \pi^{A_\perp} = -\frac{\delta \mathcal{H}}{\delta A_\perp}, \quad u^0 \iota_0 \pi^A = -\frac{\delta \mathcal{H}}{\delta A}. \quad (64)$$

(As a technical detail, we follow the convention of putting the variations of the generalized coordinates to the left and those of their conjugate momenta to the right.) On the other hand, the Lie derivative of an arbitrary $p$–form defined on the 3–space slices and being a functional of the dynamical variables can be expanded as

$$l_{\iota_0} \omega = \iota_0 A_\perp \frac{\delta \omega}{\delta A_\perp} + \iota_0 A \wedge \frac{\delta \omega}{\delta A} + \frac{\delta \omega}{\delta \pi^{A_\perp}} \iota_0 \pi^{A_\perp} + \frac{\delta \omega}{\delta \pi^A} \iota_0 \pi^A. \quad (65)$$

We define generalized Poisson brackets representing the time evolution of differential forms as the expressions resulting from substituting the field equations (64) into (65), that is

$$u^0 \iota_0 \omega = \{ \omega, \mathcal{H} \} := \frac{\delta \mathcal{H}}{\delta \pi^{A_\perp}} \frac{\delta \omega}{\delta A_\perp} - \frac{\delta \omega}{\delta \pi^A} \frac{\delta \mathcal{H}}{\delta A} + \frac{\delta \omega}{\delta \pi^{A_\perp}} \frac{\delta \mathcal{H}}{\delta A_\perp} \wedge \frac{\delta \omega}{\delta \pi^A} \wedge \frac{\delta \mathcal{H}}{\delta A}. \quad (66)$$

Eq.(66) is a particular case of the more general definition

$$\{ \alpha(x), \beta(y) \} := \int \left[ \frac{\partial \beta(y)}{\partial \pi^i(z)} \wedge \frac{\partial \alpha(x)}{\partial Q^i(z)} - \frac{\partial \alpha(x)}{\partial \pi^i(z)} \wedge \frac{\partial \beta(y)}{\partial Q^i(z)} \right] \wedge \eta(z) \quad (67)$$

of Poisson brackets for dynamical variables characterized by differential forms [55], where the arbitrary forms $\alpha$ and $\beta$ are functionals of the canonically conjugate variables concisely denoted as $Q^i, \# \Pi_i$. From (67) we check that the fundamental Poisson brackets satisfy

$$\{ Q^i(x), Q^j(y) \} = 0, \quad \{ \# \Pi_i(x), \# \Pi_j(y) \} = 0, \quad (68)$$

$$\{ Q^i(x), \# \Pi_j(y) \} = \delta^i_j \delta^3(x - y), \quad (69)$$

as expected. Poisson brackets (66) provide the formal instrument needed to calculate the time evolution of dynamical variables from the Hamiltonian 3–form (62). Let us mention a few theorems concerning them, useful for practical calculations. From definition (66) follows the antisymmetry condition

$$\{ \omega, \mathcal{H} \} = -\{ \mathcal{H}, \omega \}. \quad (70)$$

In view of the chain rule of the Lie derivative, that is $\iota_0 (\sigma \wedge \omega) = \iota_0 \sigma \wedge \omega + \sigma \wedge \iota_0 \omega$, we deduce the distributive property

$$\{ \sigma \wedge \omega, \mathcal{H} \} = \{ \sigma, \mathcal{H} \} \wedge \omega + \sigma \wedge \{ \omega, \mathcal{H} \}. \quad (71)$$

From the normal part of the identity $d \wedge d \alpha \equiv 0$, namely $\iota_0 d \alpha - \frac{1}{u} d(u^0 \iota_0 \alpha) \equiv 0$, it follows

$$\{ \omega, \mathcal{H} \} - d\{ \omega, \mathcal{H} \} = 0, \quad (72)$$

generalizable to any form defined on the transversal 3–spaces. With these theorems at hand, we are ready to attack the Hamiltonian dynamics of a PGT gravitational system.

D. Hamiltonian constraints of PGT

As in Section III, we consider the Einstein–Cartan Lagrangian 4–form [82], to which we are going to apply the Hamiltonian formalism outlined previously, completed by taking into account the fact that PGT–gravity constitutes a constrained system [32], [40], [41], [42], [43]. We make use of the nonlinear version of PGT of Appendices C and D. Due to the formal identity of definitions (C17), (C18), (C19) with gauge transformations, PGT invariants can be alternatively expressed in terms of quantities corresponding to different NLR’s, so that the PGT invariant action [92]...
can be rewritten in terms of the nonlinear variables $\hat{\phi}^a =: X^a$, $\hat{\Gamma}^{ab} =: \epsilon^{ac} A^c$ defined in \([C18], [C19]\). The new form we get for \([D2]\) reads

$$L = -\frac{1}{2l^2} \tilde{R}_{\alpha\beta} \wedge \hat{\eta}^{\alpha\beta} + \frac{\Lambda}{l^2} \hat{\eta} = -\frac{1}{l^2} \left[ D X^a \wedge \pi_a + \hat{\eta}^b \wedge (\hat{\vartheta}_a \wedge R^a - \Lambda \eta) \right]. \quad (73)$$

In \([D3]\), the components of the four–dimensional nonlinear Lorentz curvature relate to the corresponding $SO(3)$ quantities as shown in \([C31]\), while the $SO(3)$ eta–basis elements are given in \([D12]\). In order to calculate the momenta as defined in \([D4]\), we have to find out the normal part of the Lagrangian \([D3]\). Making use of the decompositions \([D2], [D10]\) one gets

$$L_\perp = -\frac{1}{l^2} \left\{ \left[ L e_i X^a - \frac{1}{u^0} D (u^0 X^a_\perp) \right] \wedge \pi_a + \vartheta^a \wedge R_a - \Lambda \eta \right\}. \quad (74)$$

(In \([D4]\) and from now on we simplify the notation by suppressing the hat over the $SO(3)$–valued coframes and basis vectors.) The only nonvanishing momentum obtained from \([D4]\) is

$$\# \pi^a X = \frac{\partial L_\perp}{\partial \left( u^0 X^a_\perp \right)} = -\frac{1}{l^2} \pi_a, \quad (75)$$

while the remaining ones $\# \pi^a u^0$, $\# \pi^a \vartheta$, $\# \pi^a A^a$, $\# \pi^a X^a$, $\# \pi^a X^a_\perp$ equal zero. All of them together with \([D4]\) constitute the set of primary constraints.

The total Hamiltonian 3–form of a constrained system is built as follows. Starting from the canonical Hamiltonian \([D2]\) —adapted in our case to the variables $u^0$, $\vartheta^a$, $A^a_\perp$, $A^a_\parallel$, $X^a_\perp$, $X^a$, $X^a_\perp$, we rewrite it, whenever possible, in terms of covariant expressions, and then we replace the factors multiplying the primary constraints by Lagrange multipliers $\beta^i$. So we get

$$\mathcal{H} = u^0 \left\{ \frac{1}{l^2} \left( X^a_\perp D \pi_a + \vartheta_a \wedge R^a - \Lambda \eta \right) \right. \right.$$

$$\left. \left. - A^a_\parallel \left[ D \pi^a_\perp + \pi_{ab} \epsilon \left( X^b \pi^a_\parallel + X^b \pi^a_\perp + \vartheta^b \wedge \pi^a_\perp \right) \right] \right. \right.$$

$$\left. + \beta^0 \# \pi^a u^0 + \beta^a \# \pi^a \vartheta + \beta^a \# \pi^a A^a + \beta^a \# \pi^a X^a + \beta^a \# \pi^a X^a_\perp \right. \right.$$

$$\left. + \frac{1}{l^2} \left( \frac{1}{l^2} \pi_a + \frac{1}{l^2} \vartheta_a \right) \right\}. \quad (76)$$

From \([D6]\), the time evolution of any dynamical variable is calculable in principle with the help of Poisson brackets of the form \([D6], [D9]\), suitably generalized to the whole set of conjugate variables $u^0$, $\# \pi^a u^0$, $\vartheta^a$, $\# \pi^a \vartheta$, $A^a_\parallel$, $\# \pi^a A^a$, $A^a_\parallel$, $\# \pi^a X^a$, $\# \pi^a X^a_\perp$.

Primary constraints are required to be stable. That is, their respective evolutions in time are enforced to vanish, giving rise to four secondary constraints plus two conditions on the Lagrange multipliers as follows. On the one hand, the evolution equations

$$u^0 L_{\vartheta_a} \# \pi^a u^0 = -\frac{1}{l^2} \left( \varphi^{(0)} + X^a_\perp \varphi_a^{(3)} \right) + A^a_\parallel \varphi_a^{(1)} \right\}, \quad (77)$$

$$u^0 L_e^\alpha \# \pi^a \vartheta = u^0 \varphi_a^{(1)} \right\}, \quad (78)$$

$$u^0 L_{\vartheta_a} \# \pi^a \vartheta = -\frac{1}{l^2} \varphi_a^{(2)} \right\}, \quad (79)$$

$$u^0 L_{\vartheta_a} \# \pi^a X^a_\perp = -\frac{u^0}{l^2} \varphi_a^{(3)} \right\}, \quad (80)$$

when put equal to zero, yield the secondary constraints

$$\varphi^{(0)} := \vartheta_a \wedge R^a - \Lambda \eta \right\}, \quad (81)$$

$$\varphi_a^{(1)} := D \pi^a_\perp + \epsilon_{abc} \left( X^b \pi^a_\parallel + X^b \pi^a_\perp + \vartheta^b \wedge \pi^a_\perp \right) \right\}, \quad (82)$$

$$\varphi_a^{(2)} := D (u^0 \vartheta_a) + u^0 X^b \vartheta_b \wedge \vartheta_a \right\}, \quad (83)$$

$$\varphi_a^{(3)} := D \pi_a \right\}, \quad (84)$$

$$\varphi_a^{(0)} := \vartheta_a \wedge R^a - \Lambda \eta \right\}, \quad (81)$$

$$\varphi_a^{(1)} := D \pi^a_\perp + \epsilon_{abc} \left( X^b \pi^a_\parallel + X^b \pi^a_\perp + \vartheta^b \wedge \pi^a_\perp \right) \right\}, \quad (82)$$

$$\varphi_a^{(2)} := D (u^0 \vartheta_a) + u^0 X^b \vartheta_b \wedge \vartheta_a \right\}, \quad (83)$$

$$\varphi_a^{(3)} := D \pi_a \right\}, \quad (84)$$
while on the other hand the vanishing of the time evolution of the remaining primary constraints, namely

\[ u^0 L_{(a)} \left( \pi_a^0 + \frac{1}{L^2} \eta_a \right) = -\frac{1}{L^2} \eta_{ab} \wedge \left( \beta_b^a + u^0 X^a_b \right), \]  

(85)

\[ u^0 L_{(a)} \pi_a^0 = -\frac{1}{L^2} \left\{ \left[ \beta_b^a - D \left( u^0 X^a_b \right) \right] \wedge \eta_{ab} + u^0 \left( \Lambda_{ab} - \Lambda \eta_{ab} \right) \right\}, \]  

(86)

fixes conditions on the Lagrange multipliers \( \beta_b^a \) and \( \beta_a^a \) respectively. By solving (85) and (86) equaled to zero we get

\[ \beta_b^a = -u^0 X^a_b, \]  

(87)

as much as

\[ \beta_a^a = D \left( X^a_b + u^0 \right) \left[ \pi_a^0 - e^a \right] \left( \partial_b \wedge \pi^b \right) - \frac{1}{2} \vartheta a \# \left( \partial_b \wedge \pi^b \right) \right] + \left( X^a_b \right) \right] \right]. \]  

(88)

The simplification in (88) follows from taking \( \varphi^{(0)} \) in (81) into account. (The symbol \( \approx \) indicates that the equation holds weakly, that is in the subspace of the phase space where all constraints hold.)

Let us deduce several consequences of the secondary constraints (82)–(84). Having in mind also the primary ones, the constraint (83) reduces weakly to \( \varphi_a^{(1)} \approx -\frac{1}{2} \vartheta a \wedge \vartheta b \wedge X^b \approx 0 \) implying \( \vartheta a \wedge X^a \approx 0 \). On the other hand, the vanishing of (83) implies \( \varphi_a^{(3)} := D \eta_a = \pi_{ab} \wedge D \vartheta^b \approx 0 \). Replacing here \( D \vartheta^b \approx -\left( \log u^0 + X^a \vartheta a \right) \wedge \vartheta^b \) as deduced from the constraint (83), one gets \( \left( \log u^0 + X^a \vartheta a \right) \wedge \pi_{ar} \approx 0 \), proving that \( \log u^0 + X^a \vartheta a \approx 0 \) is a constraint by itself, so that \( D \vartheta^a \) being proportional to it, also vanishes weakly. In summary, (82)–(84) with the help of the primary constraints give rise to

\[ \log u^0 + X^a \vartheta a \approx 0, \quad \vartheta a \wedge X^a \approx 0, \quad D \vartheta^a \approx 0, \]  

(89)

whose geometrical meaning as the vanishing of several torsion pieces becomes clear by comparison with (D1), (D2).

On the other hand, (88) can be further simplified. In view of the definition of \( \pi^a \) in (D10), we put \( \vartheta a \wedge \pi^a = \vartheta a \wedge \pi^a + \frac{1}{2} \left[ \epsilon_a \epsilon_b \right] \left( \vartheta d \wedge X^d \right) \epsilon^a b c \pi^c \), where the last term vanishes according to the second equation in (89), and so does the first term of the r.h.s. since the covariant differential of the third equation in (89) yields \( D D \vartheta^a = \pi_{ab} \wedge F^b = \vartheta^a \wedge \vartheta b \wedge \pi^b \approx 0 \), so that \( \vartheta a \wedge \pi^a \approx 0 \). Thus we conclude

\[ \vartheta a \wedge \pi^a \approx 0. \]  

(90)

Replacing (90) in (88), the latter reduces to its ultimate form

\[ \beta_a^a \approx D \left( X^a_b + u^0 \right) \pi^a, \]  

(91)

According to the general treatment of constrained systems (39)–(41), we furthermore have to require the secondary constraints (82)–(84) to be stable in time. The time evolution of (82) is found to automatically satisfy

\[ u^0 L_{(a)} \varphi_a^{(1)} \approx 0, \]  

(92)

by simply taking into account the already known constraints. However, new conditions on the Lagrange multipliers are necessary to guarantee the stability of (83) and (84). For the latter we find

\[ u^0 L_{(a)} \varphi_a^{(3)} \approx \eta_{ab} \wedge \left( D \beta_b^a - \eta_{ab} \wedge \pi^a \right), \]  

(93)

implying, when enforced to vanish,

\[ \beta_a^a = \pi_{ab} - \vartheta^a \left( \vartheta d \wedge \pi^d \right) - \frac{1}{2} \vartheta^a \left( \vartheta b \wedge \pi^b \right), \]  

(94)

which in view of (87) with (88) reduces to

\[ \beta_a^a \approx u^0 \left[ e^a b c X^b X^c - \left( X^a_b + e^a \right) \left( \vartheta b \wedge \pi^b \right) \right]. \]  

(95)
On the other hand, time evolution of (83) is calculated to be
\[
u^0 L_{e_a} \varphi_a^{(2)} \approx -u^0 \vartheta_a \wedge \left[ \frac{d}{u^0} \left( \frac{\beta_\alpha}{u^0} a + \beta^{b}_{X_{ab}} \vartheta_b + X^b \beta^\alpha_b \right) \right],
\]
whose vanishing ensures the stability \( L_{e_a} \left( d \log u^0 + X^a \vartheta_a \right) \approx 0 \) of the first equation in (99). Finally we require the stability of the constraint (97). Making use of (91) and (93) we find
\[
u^0 L_{e_a} \varphi(0) \approx -\frac{1}{u^0 \nu^2} \left[ u^0 \left( \vartheta_a \wedge \beta_\alpha a - u^0 X^a \vartheta_{ab} \wedge X^b \right) \right].
\]
The differentiated quantity in (97) can be calculated in view of (95) with (89), yielding
\[
\vartheta_a \wedge \beta_\alpha a - u^0 X^a \vartheta_{ab} \wedge X^b \approx u^0 \# D X^a + u^0 X^a \# \left[ e_{ab} \left( \vartheta_b \wedge X^b \right) \right] \approx u^0 \# D X^a,
\]
so that (97) transforms into
\[
u^0 L_{e_a} \varphi(0) \approx -\frac{1}{u^0 \nu^2} \left[ (u^0)^2 \varphi^{(4)} \right],
\]
where we defined
\[
\varphi^{(4)} := \vartheta_a \wedge \# D X^a.
\]
The vanishing of (99) is the stability condition of (91), so that in principle (99) should be taken as a new constraint. However, one can check that for (99) to be stable, \( \varphi^{(4)} \) as defined in (100) must vanish, so that (100) itself, rather than the less restrictive condition (99), is to be considered as the new constraint.

In view of the vanishing of (100), the Lagrange multiplier (95) reduces to
\[
\beta_\alpha a \approx u^0 \left( \epsilon^{abc} X^b X^c - \# D X^a \right),
\]
and finally one can prove that the constraint (100) is stable, thus completing our search for the constraints (and for the solved Lagrange multipliers) of the theory.

E. Comparison to the Lagrangian approach to PGT

The meaning of the gravitational equations obtained in the context of the Hamiltonian approach to PGT becomes clarified by comparing them with the ordinary PGT Lagrangian equations (83), (85) derived from the same action \( S_0 \). Besides the constraints of previous subsection, we have to consider the evolution in time of the spatial triad \( \vartheta^a \), of the \( SO(3) \) connection \( A^a \) and of the nonlinear boost connection \( X^a \). As we know, their Lie derivatives along the time direction \( e_0 \) are found from (13) with the help of the Poisson brackets (66). Taking into account the previously calculated values of the Lagrange multipliers (84), (91) and (101), and the definitions of covariantized Lie derivatives in (122), (99) and (105), we get
\[
u^0 L_{e_0} \vartheta^a = \beta^a_\alpha \approx -u^0 \bar{X}^a \Rightarrow L_{e_0} \vartheta^a + \bar{X}^a \approx 0,
\]
\[
u^0 F^a_{\perp} = \beta^a_\alpha \approx u^0 \left( \epsilon^{abc} X^b X^c - \# D X^a \right) \Rightarrow \bar{R}^a_{\perp} \approx -\# (D X^a),
\]
\[
u^0 L_{e_0} \bar{X}^a = \beta^a_\alpha \approx \bar{D} \left( u^0 \bar{X}^a \right) + u^0 \# \bar{R}^a \Rightarrow L_{e_0} \bar{X}^a - \frac{1}{u^0 \nu^2} \bar{D} \left( u^0 \bar{X}^a \right) \approx \# \bar{R}^a.
\]
The evolution equations (102) plus the set of constraints found above summarize the PGT Hamiltonian dynamics we want to compare with the Lagrangian field equations (83), (85). In order to do so, we decompose the latter ones into their longitudinal and transversal parts making use of the results of Appendix D. Let us begin with the Lagrangian result (35) of vanishing torsion, which in view of (111), (112), takes the form
\[
0 = \bar{T}^0 = -\tilde{\vartheta}^0 \wedge \left( \frac{d}{u^0} \log u^0 + X^a \tilde{\vartheta}_a \right) + \dot{\vartheta}_a \wedge \bar{X}^a,
\]
\[
0 = \bar{T}^a = \tilde{\vartheta}^0 \wedge \left( L_{e_0} \tilde{\vartheta}^a + \bar{X}^a \right) + \bar{D} \tilde{\vartheta}^a,
\]
where we reintroduced the hat notation of Appendices C and D in order to avoid confusions. It is easy to check that the four equations contained in \(103\), \(106\) coincide with the Hamiltonian constraints \(81\) together with the evolution equation \(102\). The Hamiltonian equations involved acquire in this way an explicit geometrical meaning.

On the other hand, the Einstein equations \(33\) decompose as the time component

\[
0 = \frac{1}{2} \dot{\eta}_{ab} \wedge \dot{R}^{b\gamma} - \Lambda \dot{\eta}_a = -\dot{\vartheta}^0 \wedge \left( \dot{\vartheta}_a \wedge R^a_{\perp} \right) + \left( \dot{\vartheta}_a \wedge \mathcal{R}^a - \Lambda \eta_a \right),
\]

and the spatial components

\[
0 = \frac{1}{2} \dot{\eta}_{ab} \wedge \dot{R}^{b\gamma} - \Lambda \dot{\eta}_a = -\dot{\vartheta}^0 \wedge \left( \left[ L_{\eta_{ab}} X^b - \frac{1}{u^2} D \left( u^0 X^b \right) \right] \wedge \eta_{ab} + \mathcal{R}_{ab} - \Lambda \eta_{ab} \right) - \tau_{ab} \wedge D X^b,
\]

respectively. Regarding \(107\), notice that the transversal part is the constraint \(81\), while the vanishing of the longitudinal part follows from \(103\) with the constraint \(100\). Furthermore, \(106\) also yields the vanishing of the transversal part of \(108\) since trivially \(0 \approx \dot{\vartheta}_a \wedge \varphi^{(a)} = \dot{\vartheta}_a \wedge \varphi^b \wedge D X^b = \tau_{ab} \wedge D X^b\), where we made use of the Hodge dual relations of Appendix A, suitably adapted to three–dimensional space. The vanishing of the longitudinal part of \(108\) follows from performing the exterior product of \(102\) by \(\tau_{ab}\). The coincidence of the resulting expression with the one in \(103\) can be easily proved as follows. We start with the three–dimensional-adapted identity \(\# \mathcal{R}_b \wedge \eta_a = \dot{\vartheta}_a \wedge \mathcal{R}_b\), see Appendix A, and contract it with \(\dot{\vartheta}_b\). Then, by invoking the constraints \(81\) and \(90\) together with the identity \(#(\dot{\vartheta}_a) = (\dot{\vartheta}_b)\), see Appendix A, we get \(\# \mathcal{R}_b \wedge \eta_{ab} \approx -(\mathcal{R}_a - \Lambda \eta_a)\). Thus we were able to deduce the complete set of Lagrangian equations \(105\)–\(108\) from the Hamiltonian approach to PGT.

Observe that the reciprocal derivation is not possible. Indeed, the longitudinal part of \(107\), that is \(\dot{\vartheta}_a \wedge \varphi^{(a)} \approx 0\), is obtained as the trace of \(106\) provided \(100\) vanishes. However, equation \(108\) itself has not a Lagrangian equivalent. Something similar can be said about \(103\) and the longitudinal part of \(108\). It is precisely the presence of \(103\) and \(104\) that makes it possible to put the Hamiltonian Einstein equations together into a very simple \(SO(3)\)–covariant formulation on four–dimensional spacetime. Taking into account \(103\) and \(106\), both \(103\) and the Hodge dual of \(103\) rearrange into

\[
D X^a - \ast \mathcal{R}^a \approx 0,
\]

while the trace \(\dot{\vartheta}_a \wedge \mathcal{R}_a \approx 0\) of \(108\) besides the constraint \(81\) are summarized by the four–dimensional formula

\[
\dot{\vartheta}_a \wedge \mathcal{R}_a - \Lambda \dot{\eta}_a \approx 0,
\]

being \(\dot{\eta}_a := \dot{\vartheta}_a \wedge \eta =: \tau_a\), see \(107\). Equations \(109\), \(110\) with the additional conditions \(103\), \(106\) of vanishing torsion constitute the condensed form of the Hamiltonian PGT equations derived from the ordinary Einstein–Cartan action \(32\) in vacuum.

F. Relation to Ashtekar variables

We are interested in calling attention on the close relationship in which the variables \(C17\), \(C19\) introduced by us in the context of Hamiltonian PGT stand to the Ashtekar variables, see \(10\) \(11\) \(44\) \(36\) \(45\). To make the link apparent, notice that the Lagrangian dynamics of the Einstein–Cartan action \(32\) is not modified by adding to it a term as

\[
L = L_{EC} - \beta \frac{1}{2 \ell^2} \dot{\mathcal{R}}^{\alpha\beta} \wedge \dot{\mathcal{R}}_{\alpha\beta} = -\frac{1}{2 \ell^2} \left( \dot{\mathcal{R}}^{\alpha\beta} + \beta \dot{\mathcal{R}}^{*\alpha\beta} \right) \wedge \dot{\mathcal{R}}_{\alpha\beta} + \Lambda \dot{\eta},
\]

the latter being proportional (with arbitrary constant coefficient \(\beta\)) to the Lie dual of the curvature, that is to

\[
\dot{\mathcal{R}}^{\alpha\beta} := \frac{1}{2} \dot{\mathcal{R}}_{\alpha\beta}^{\mu
 \nu} \dot{R}_{\mu\nu},
\]

(not to be confused with the Hodge dual considered in Appendix A). Indeed, as compared with the Lagrangian approach of Subsection III.B, the contributions of the additional term in \(111\) to the field equation analogous to \(33\) still imply zero torsion \(33\), while the modification of the Einstein equation \(33\) as deduced from \(111\) is enlarged by a term proportional to \(\dot{R}_{\mu\nu} \wedge \dot{\vartheta}_\nu \equiv -\dot{D} \dot{T}_\mu\), which also vanishes for vanishing torsion. So the Lagrangian dynamics
derived from (111) is indistinguishable from the one obtained from (12). We will show that also the Hamiltonian
equations coincide with those of the standard Einstein–Cartan case.

Let us start by reexpressing (111) in terms of the PGT variables (C17)–(C19) making use of (C31), so that the
main term in (111) becomes
\[ \frac{1}{2} \left( \hat{R}_{a\beta} + \beta \hat{R}^*_{a\beta} \right) \wedge \hat{\eta}^{\alpha\beta} = \left( \vartheta^0 \wedge \vartheta^a + \beta \eta^{0a} \right) \wedge \mathcal{R}_a + \left( \beta \vartheta^0 \wedge \vartheta^a - \eta^{0a} \right) \wedge D X_a. \] (113)

Then we combine the PGT connection fields (C18), (C19), namely the SO(3) connections \( A^a \) and the nonlinear boost
connections \( X^a \), into a modified SO(3) connection
\[ \hat{A}^a := A^a + \beta X^a, \] (114)
which we claim to be a variable of the Ashtekar type, as will be justified by the following development. For later
convenience, in (114) the constant \( \beta \) is chosen to be the same as in (111), without further determining its value [46],
non even prejudging for the moment about its real or complex character. The SO(3) field strength built from (116)
reads
\[ \hat{F}^a := d \hat{A}^a + \frac{1}{2} \epsilon^{abc} \hat{A}^b \wedge \hat{A}^c = \mathcal{R}^a + \beta D X^a + (\beta^2 + 1) \frac{1}{2} \epsilon^{abc} X^b \wedge X^c, \] (115)
compare with (C29), (C30). Replacing (115) in (113) (as much as the covariant derivative \( DX^a \) in (115), defined as
(C29), by \( \hat{D}X^a \) in terms of (114)), we get
\[ \frac{1}{2} \left( \hat{R}_{a\beta} + \beta \hat{R}^*_{a\beta} \right) \wedge \hat{\eta}^{\alpha\beta} = \left( \vartheta^0 \wedge \vartheta^a + \beta \eta^{0a} \right) \wedge \hat{F}_a - (\beta^2 + 1) \left[ \eta^{0a} \wedge \hat{D} X_a + (\vartheta^0 \wedge \vartheta^a - \beta \eta^{0a}) \wedge \frac{1}{2} \epsilon^{abc} X^b \wedge X^c \right]. \] (116)
We are free to maintain the value of \( \beta \) arbitrary or even to choose it to be real despite the Lorentzian signature we
are dealing with [46], but it is obvious that a major simplification of (116) follows from taking \( \beta^2 = -1 \). In particular
we fix \( \beta = i \), so that (111) becomes an action of the Jacobson–Smolin type [44], namely
\[ L = -\frac{1}{12} \left( \hat{R}_{a\beta} + \hat{\eta}^{\alpha\beta} \right) + \Lambda \hat{\eta} = -\frac{1}{2} \left( \vartheta^0 \wedge \vartheta^a + i \eta^{0a} \right) \wedge \hat{F}_a + \frac{\Lambda}{12} \hat{\eta}, \] (117)
with the anti–self–dual curvature defined from the curvature and its Lie dual (12) as
\[ (-) \hat{R}_{a\beta} := \frac{1}{2} \left( \hat{R}_{a\beta} + i \hat{R}^*_{a\beta} \right), \] (118)
thus satisfying \((-) \hat{R}^*_{a\beta} = -i (-) \hat{R}_{a\beta} \). The components of the corresponding anti–self–dual connection
\[ (-) \hat{\Gamma}_{a\beta} := \frac{1}{2} \left( \hat{\Gamma}_{a\beta} + i \hat{\Gamma}^*_{a\beta} \right), \quad \text{with} \quad \hat{\Gamma}^*_{a\beta} := \frac{1}{2} \hat{\eta}^{\alpha\beta}_{\mu\nu} \hat{\Gamma}_{\mu\nu}, \] (119)
relate to the complex Ashtekar connection (113) with \( \beta = i \) as
\[ \hat{A}^a := A^a + i X^a = \epsilon^{abc} (-) \hat{\Gamma}^*_{bc} = 2i (-) \hat{\Gamma}^0a, \] (120)
while the field strength (115) reduces to
\[ \hat{F}^a = \mathcal{R}^a + i D X^a. \] (121)

For what follows, it is convenient to rewrite (117) making use of the identity \( \vartheta^0 \wedge \vartheta^a \wedge \hat{F}_a \equiv \hat{F}_a \wedge \eta^{0a} \), see Appendix A, together with the notation of (D12), as the complex Lagrangian
\[ L = \frac{1}{12} \left[ \left( i \hat{F}^a - * \hat{F}^a \right) \wedge \hat{\pi}_a + \Lambda \vartheta^0 \wedge \hat{\pi} \right], \] (122)
to which we proceed to apply the Hamiltonian treatment developed previously. The normal part of (122) reads
\[ L_\perp = \frac{1}{12} \left[ \left( i \hat{F}^a_\perp - \hat{\pi}^a \right) \wedge \hat{\pi}_a + \Lambda \hat{\pi} \right], \] (123)
where we used definitions
\[ \tilde{F}_\perp := l_0 \tilde{\mathbf{A}} - \frac{1}{u_0} \tilde{D} \left( u^0 \tilde{\mathbf{A}} \right), \quad \tilde{F}^a := \frac{d}{u_0} \tilde{\mathbf{A}}^a + \frac{i}{2} \epsilon^{abc} \tilde{\mathbf{A}}^b \wedge \tilde{\mathbf{A}}^c, \]
(124)

analogous to those in \[129\]. All momenta calculated from \[125\] result to be primary constraints, being the only nonvanishing one
\[ \# \pi^A_a := \frac{\partial L_\perp}{\partial \left( l_0 \tilde{\mathbf{A}}^a \right)} = \frac{i}{l^2} \eta_a, \]
(125)

compare with \[75\], whereas the remaining ones \# \pi^u_a, \# \pi^0_a, \# \pi^{A_+}_a \) are all equal to zero. Proceeding as in Subsection \textbf{IV. D}, we build the total Hamiltonian 3–form
\[ \mathcal{H} = u^0 \left[ \frac{1}{l^2} \left( \partial_a \wedge \tilde{F}^a - \Lambda \eta \right) \right. - \tilde{A}^a \left( \tilde{D} \# \pi^A_a + \epsilon_{abc} \varphi^b \wedge \# \pi^c_a \right) \]
\[ + \beta_{u^0} \# \pi^0_a + \beta^a \wedge \# \pi^0_a + \beta^a \# A^A_+ + \beta^a \# \pi^{A_+}_a \wedge \# \pi^A_a \right) \]
(126)

Time evolution of any dynamical variable is calculable with the help of the Poisson brackets \[66\] adapted to the conjugate variables \( u^0, \# \pi^0_a; \partial^a, \# \pi^0_a; A^A_+, \# \pi^{A_+}_a; \tilde{A}_\perp, \# \pi^{A_+}_a \). Repeating the steps of Subsection \textbf{IV. D}, the stability conditions of the primary constraints yield on the one hand the secondary constraints
\[ \varphi^{(0)} := \partial_a \wedge \tilde{F}^a - \Lambda \eta, \]
(127)
\[ \varphi^{(1)} := \tilde{D} \# \pi^A_a + \epsilon_{abc} \varphi^b \wedge \# \pi^c_a, \]
(128)
and on the other hand the conditions on the Lagrange multipliers
\[ i \eta_{ab} \wedge \beta^b_0 - \tilde{D} (u^0 \partial_a) = 0, \]
(129)
\[ i \eta_{ab} \wedge \beta^b_0 + u^0 \left( \tilde{F}_a - \Lambda \eta_a \right) = 0. \]
(130)

From \[125\] follows
\[ \beta^0_0 = -i \left( \epsilon^a \eta \# \tilde{D} (u^0 \partial_a) \right) \varphi^b + \frac{i}{2} \varphi^a \# \left( \tilde{D} (u^0 \partial_a) \wedge \varphi^b \right), \]
(131)
and from \[126\] with \[127\]
\[ \beta^a_0 \approx i u^0 \left( \# \tilde{F}^a - \epsilon^a \eta \right) \left( \varphi^b \wedge \# \tilde{F}^b \right). \]
(132)

The constraint \[128\] is stable. Instead, \[127\] requires the additional stability condition
\[ \beta^0_0 \wedge \left( \tilde{F}_a - \Lambda \eta_a \right) + \partial_a \wedge \tilde{D} \beta^0_a \approx 0, \]
(133)

which, by making use of \[125\], \[126\] and \[128\], transforms into
\[ \tilde{D} \left( u^0 \right)^2 \left( \varphi_a \wedge \# \tilde{F}^a \right) \approx 0. \]
(134)

The stability of \[131\] requires
\[ \varphi_a \wedge \# \tilde{F}^a \approx 0, \]
(135)

constituting a new constraint, replacing the –less restrictive– previously found \[129\]. Substitution of \[131\] in \[132\] yields
\[ \beta^0_0 \approx i u^0 \# \tilde{F}^a. \]
(136)
Our search for constraints finishes by checking that (133) is stable. Let us at this point argue in favor of the equivalence between the Ashtekar and the Hamiltonian PGT approach to gravity. The proof requires first to support the strict Ashtekar character of the present treatment; we achieve it by showing that, although somewhat hidden by the exterior calculus notation, the already obtained constraints satisfied by the complex variables (120) coincide with the Ashtekar constraints. The second step consists in demonstrating that the complex approach in terms of (120) – that is, in terms of variables of the Ashtekar type built from PGT quantities – constitutes an alternative formulation of the real Hamiltonian approach to PGT as presented in IV. D, E. Actually, we are going to show that, by decomposing the dynamical equations of the Ashtekar kind into their real and imaginary parts, they reproduce the Hamiltonian PGT equations.

Our first task is to rewrite the constraints (127), (128) and (135) in a language suitable to reveal them as the well known Ashtekar constraints. (When comparing the following results with the standard equations, for instance (4) and (6) of [27], the reader must have in mind the interchanged role of the latin letters in Ashtekar’s notation for indices as compared with ours, being in our case those of the beginning of the alphabet reserved for internal indices, while those of the middle of the alphabet are assigned by us to the general coordinate indices of the underlying four–dimensional manifold.) Let us begin with the constraint (128), transforming with the help of the primary constraints $\pi^a_0$ and (120) into $\varphi^{(1)}_a \approx \frac{1}{2} \hat{D} \eta_a$. Its 3–dimensional Hodge dual manifests itself as the Gauss law

$$g_a := \# (D \eta_a) = \frac{1}{e} \hat{D} (e a_i) \approx 0,$$

with $e$ as the determinant built from the components of the triad $\vartheta^a = dx^b e_a^b$. Similarly, from (135) we get

$$\# (\vartheta_a \wedge \# F^a) = g^b F_{ba} = dx^i F_{in} \approx 0 \Rightarrow \forall i := e a^j \hat{F}_{ij} \approx 0,$$

where one recognizes Ashtekar’s vector constraint. Finally, the Hodge dual of (127) becomes the ordinary scalar constraint, namely

$$S := \# (\vartheta_a \wedge \# \vec{F}^a - \Lambda \eta) = \frac{1}{2} e a^b c \hat{F}_{bc} - \Lambda = \frac{1}{2} e a^b c \bar{e} c^i \hat{F}_{ij} - \Lambda \approx 0.$$

In view of (137)–(139), the full identification of (120) with the Ashtekar variables will be complete once the spin connection, and thus (120) itself, becomes entirely determined by the coframe as a consequence of the vanishing of torsion, as will be shown below.

On the other hand, the announced proof of the exact coincidence between the present results and the PGT ones in IV. D, E requires to reproduce here the dynamical equations (109) and (110) together with the zero torsion conditions (105)–(106). We proceed as follows. From (120) we calculate the evolution equations for $\vec{A}$ to be

$$u^0 l_a \vec{A}^a = \beta^a + \hat{D} (u^0 \vec{A}_\perp).$$

Taking the value (136) into account with the first definition in (124), from (130) we get

$$\vec{F}_\perp = i \# \vec{F}^a.$$

Equation (141) can be rewritten in 4–dimensional notation by recalling

$$\vec{F}^a = \tilde{\vartheta}^0 \wedge \vec{F}^a_\perp + \vec{F}^a, \quad \ast \vec{F}^a = \tilde{\vartheta}^0 \wedge \# \vec{F}^a - \# \vec{F}^a_\perp,$$

according to (53) and (55) respectively. So, in four dimensions, (141) transforms into

$$\ast \vec{F}^a = -i \vec{F}^a,$$

establishing a simple relation between the field strength and its Hodge dual. (By the way, notice that (138) guarantees the automatic fulfillment of the gauge theoretical equation $\hat{D} \ast \vec{F}^a = -i \hat{D} \vec{F}^a \equiv 0$.) Furthermore, (127) and (141) with (135) yield

$$\vartheta_a \wedge \vec{F}^a - \Lambda \eta_a = 0.$$

Let us show that (143) and (144) constitute an alternative way to display the previously found PGT equations (109) and (110) respectively. Indeed, taking (120) into account one checks that (143) is a shorthand for $(DX^a - \ast R^a)$ –
\[ i^* (D X^\alpha - \ast R^\alpha) \approx 0 \], doubly reproducing (109), while (114) can be rewritten as \( \left( \hat{\vartheta}_a \wedge R^a - \Lambda \hat{\eta}_b \right) - i \hat{\vartheta}_a \wedge D X^\alpha \approx 0 \).

The imaginary contribution, reexpressed in terms of the torsion components (C27), (C28), reads \( \hat{\vartheta}_a \wedge D X^\alpha \equiv -d\hat{T}^0 + \hat{T}_a \wedge X^\alpha \), so that provided the torsion vanishes, (144) reproduces (110). If this is the case, the dynamical equations derived from both approaches coincide.

The result of zero torsion follows in fact on the one hand from (128), reduced in view of the primary constraints \# \pi_a^0 \approx 0\) and (129), to \( \varphi_a^{(i)} \approx \frac{i}{2} D \hat{T}_a = \frac{i}{2} \eta_{ab} \wedge D \vartheta^b \approx 0 \), and on the other hand from (129) with the value of \( \beta_0^a \) given by the evolution equation for the triad, namely \( \beta_0^b = u^0 \hat{L}_{a_b} \vartheta^b \). Combining both results into a single four–dimensional expression in order to facilitate calculations, we get

\[
\frac{1}{u^0} \vartheta^0 \wedge \left[ i \eta_{ab} \wedge u^0 \hat{L}_{a_b} \vartheta^b - D \left( u^0 \vartheta_a \right) \right] - i \eta_{ab} \wedge D \vartheta^b = D \left( \vartheta^0 \wedge \vartheta^a + i \eta^{0a} \right) \approx 0 ,
\]

with \( D \) as the \( SO(3) \) covariant derivative built with the complex connection (120). Taking into account the expressions (C27) and (C28) for torsion, we find

\[
0 \approx \hat{D} \left( \vartheta^0 \wedge \vartheta^a + i \eta^{0a} \right) = T^0 \wedge \vartheta^a - \vartheta^0 \wedge T^a + i \eta^{0a} \wedge T^b ,
\]

whose unique solution is the vanishing of the whole torsion \( T^\alpha \).

Observe that zero torsion allows to simplify \( \beta_0^a = -u^0 X^a - i e^a \right \} d u^0 \),

\[ (147) \]

that is, in terms of the real and imaginary parts of (120) separately, rather than in terms of the whole complex connection (120). Compare (147) with (157).

A more relevant consequence of \( T^\alpha \approx 0 \) is that (120) becomes expressible in terms of the torsion free connection \( \hat{\Gamma}^{(i)}_{\mu \nu} \) of the form displayed in (38) as

\[
\hat{A}^a = -i \frac{1}{2} \left[ e^\mu \right] e^\nu \left( \vartheta^0 \wedge \vartheta^a + i \eta^{0a} \right) \] \[ \hat{\Gamma}^{(i)}_{\mu \nu} = \frac{1}{2} \epsilon^{a bc} \hat{\Gamma}^{(i)}_{0 bc} + i \hat{\Gamma}^{(i)}_{a 0} ,
\]

see (C18), (C19). This corresponds the correspondence between Hamiltonian PGT built exclusively in terms of real quantities as developed before, and the present Hamiltonian treatment in terms of complex Ashtekar variables, the latter satisfying the Ashtekar constraints (137)–(139) and being built from coframes as shown in (148). We claim that the identification of the Ashtekar complex connection as the combination (120) of the PGT real fields (C13) and (C19) allows to regard both Hamiltonian approaches to gravity—Ashtekar’s and PGT— as alternative reformulations of each other.

V. METRIC-AFFINE GRAVITY

Finally, let us briefly illustrate the nonlinear techniques when applied to a spacetime group other than the Poincaré group. We consider in particular the affine group giving rise to metric–affine gravity (MAG) (10) (24) (25) (26) (28), which constitutes an open and active research field—see (10) (18) and references therein—proposed as an alternative to more usual descriptions of gravity. In the various nonlinear approaches to PGT studied in previous sections, we remarked the reinterpretation of tetrad as gauge–theoretical quantities, specifically as nonlinear transitive connections. This result remaining valid in the context of the MAG theory to be presented here, we are going to pay further attention to the origin of the degrees of freedom of the MAG–metric, which also turn out to be of gauge–theoretical nature as Goldstone fields. To make this point apparent, we consider two different nonlinear approaches to the affine group \( G = A(4, R) \), corresponding to the choices of the auxiliary subgroup either as the general linear group \( H_1 = GL(4, R) \) or as the homogeneous Lorentz group \( H_2 = SO(3, 1) \), and then we relate them to each other. The reason for doing so is that by simply applying the standard nonlinear gauge procedure to the affine group with \( H_1 = GL(4, R) \), no metric tensor becomes manifest. For the latter to be deduced, the formalism obtained for \( H_1 = GL(4, R) \) has to be compared to the one derived for \( H_2 = SO(3, 1) \), as will be shown immediately.

Let us start with the nonlinear realization of \( G = A(4, R) \) with \( H_1 = GL(4, R) \). Proceeding as usual, we replace in the simplified form (7) of (10) the suitable \( G \) elements

\[
g = e^{i \epsilon^a P_a + i \omega_\alpha^b \Lambda^\alpha \beta} \approx I + i \epsilon^a P_a + i \omega_\alpha^b \Lambda^\alpha \beta ,
\]

(149)
with infinitesimal transformation parameters $\epsilon^\alpha$ and $\omega^\alpha_\beta$, as much as the $H_1$ elements

$$h := e^{i \epsilon^\alpha \Lambda^\alpha_\beta} \approx I + i \epsilon^\alpha \Lambda^\alpha_\beta,$$

(150)

with the also infinitesimal group parameter $\nu^\alpha_\beta$, and

$$\tilde{b} = e^{-i \xi^\alpha P_\alpha}, \quad \tilde{b}' = e^{-i(\xi^\alpha + \delta \xi^\alpha) P_\alpha},$$

(151)

where the finite translational parameters $\xi^\alpha$ label the cosets $\tilde{b} \in G/H_1$. The tildes in (151) are introduced to distinguish the tilded $\tilde{b}$'s from the untilded $b$ in (159) below. Using the Hausdorff–Campbell formula (B1) with the commutation relations of the affine group

$$[P_\alpha, P_\beta] = 0,$$

$$[\Lambda^\alpha_\beta, P_\gamma] = \delta^\alpha_\gamma P_\beta,$$

$$[\Lambda^\alpha_\beta, \Lambda^\gamma_\delta] = \delta^\alpha_\delta \Lambda^\gamma_\beta - \delta^\gamma_\delta \Lambda^\alpha_\beta,$$

(152)

we find the value of $\nu^\alpha_\beta$ in (160) and the variation of the coset parameters $\xi^\alpha$ in (151) to be respectively

$$\nu^\alpha_\beta = \omega^\alpha_\beta,$$

$$\delta \xi^\alpha = -\xi^\beta \omega^\alpha_\beta - \epsilon^\alpha,$$

(153)

compare with the analogous PGT results (23). The nonlinear connection (11) is built in terms of the linear affine connection

$$A_M := -i \Gamma^\alpha_\beta P_\alpha - i \epsilon \Lambda^\alpha_\beta,$$

(154)

whose components, the translational and the $GL(4, R)$ connection, transform respectively as

$$\delta \Gamma^\alpha_\beta = - \Gamma^\alpha_\beta \omega^\alpha_\beta + \epsilon \alpha,$$

$$\delta \Lambda^\alpha_\beta = \epsilon \alpha,$$

(155)

Replacing (151) in (11) we get

$$\tilde{\Gamma}_M := \tilde{b}^{-1}(d + A_M) \tilde{b} = -i \tilde{\phi}^\alpha P_\alpha - i \tilde{\Gamma}^\alpha_\beta \Lambda^\alpha_\beta,$$

(156)

where

$$\tilde{\phi}^\alpha := \epsilon \alpha,$$

$$\tilde{\Gamma}^\alpha_\beta = \Gamma^\alpha_\beta,$$

(157)

As in the case of (151), we denote these objects with a tilde for later convenience. Applying (12), it is trivial to find

$$\delta \tilde{\phi}^\alpha = - \tilde{\phi}^\beta \omega^\alpha_\beta,$$

$$\delta \tilde{\Gamma}^\alpha_\beta = \tilde{\Gamma}^\alpha_\beta,$$

(158)

showing that the coframe $\tilde{\phi}^\alpha$ in (157) transforms as a $GL(4, R)$ covector, in contrast to the linear translational connection, see (153), while $\tilde{\Gamma}^\alpha_\beta$ remains unchanged as a $GL(4, R)$ connection.

So far, no metric tensor is derived from the gauging of the affine group. In order to deductively obtain a metric as a gauge–theoretical quantity, we have to consider a second nonlinear realization of $G = A(4, R)$ with auxiliary subgroup $H_2 = SO(3, 1)$. Being the homogeneous Lorentz group a (pseudo-)orthogonal group, it is equipped with a Cartan-Killing metric, namely the invariant Minkowski metric $\alpha_\beta$. When taken as the auxiliary subgroup of the nonlinear realization, the Lorentz group induces an automatic metrization of the theory. Certainly, as long as we only attend to the realization with $H_2 = SO(3, 1)$, the metric can just be a constant, the Lorentz invariance $\delta \alpha_\beta = 0$ still holding under gauge transformations of the whole affine group since, as a general feature of the nonlinear procedure, the total group $G$ acts formally as its subgroup $H_2$, see (8). Nevertheless, we are going to show how to establish the correspondence to the realization with $H_1 = GL(4, R)$ studied above, in such a way that, by means of redefinitions isomorphic to gauge transformations, ten Goldstone–like degrees of freedom may be either rearranged in the gauge potentials or displayed as a variable metric tensor, depending on the nonlinear realization we consider, either with $H_1 = SO(3, 1)$ or with $H_1 = GL(4, R)$, see (153)–(158) below.

To study the case with $H_2 = SO(3, 1)$, we start by splitting the generators of the general linear group into the sum of symmetric plus antisymmetric (Lorentz) parts as $\Lambda^\alpha_\beta = S^\alpha_\beta + L^\alpha_\beta$. Then we apply the general formula (17) with the particular parametrization

$$g = e^{i \epsilon^\alpha P_\alpha} e^{i \alpha_\beta S^\alpha_\beta} e^{i \beta_\gamma L^\alpha_\beta}, \quad b := e^{-i \xi^\alpha P_\alpha} e^{i h_\alpha S^\alpha_\beta}, \quad h := e^{i \omega^\alpha_\beta L^\alpha_\beta},$$

(159)
where the infinitesimal transformation parameters of the affine group are the translatable ones \( e^\alpha \) and the general linear parameters in (149), decomposed into symmetric plus antisymmetric contributions as \( \omega_{\alpha}^\beta = \alpha_{\alpha}^\beta + \beta_{\alpha}^\beta \), while the infinitesimal nonlinear parameters \( u_{\alpha}^\beta \) correspond to the Lorentz subgroup \( H_2 \). From (10) we find the variations \( \xi^\alpha = \xi^\alpha + \delta \xi^\alpha \) and \( h'^\alpha_\beta = h^\alpha_\beta + \delta h^\alpha_\beta \) of the coset parameters of \( b \) in (159) to be respectively
\[
\delta \xi^\alpha = -\xi^\beta (\alpha^\beta_\alpha + \beta^\beta_\alpha) - e^\alpha, \quad \delta r_{\alpha}^\beta = (\alpha_{\alpha}^\gamma + \beta_{\alpha}^\gamma) r_{\gamma}^\beta - u_{\gamma}^\beta r_{\alpha}^\gamma,
\]
where we made use of the definition
\[
r_{\alpha}^\beta := \left( e^b \right)_{\alpha}^\beta := \delta r_{\alpha}^\beta + h_{\alpha}^\beta + \frac{1}{2!} h_{\alpha}^\gamma h_{\gamma}^\beta + \ldots
\]
(161)

It is easy to check that, contrary to \( \mu_{\alpha}^\beta \) in (24), the nonlinear Lorentz parameters \( u_{\alpha}^\beta \) relevant for nonlinear transformations differ from the linear ones \( \beta_{\alpha}^\beta \). But we do not need to know their explicit form, which can be calculated from the vanishing of the antisymmetric part of the second equation in (160), see (26) (29).

The nonlinear connection corresponding to the choice \( H_2 = SO(3, 1) \) is obtained by replacing (154) into (11) taking into account the decomposition \( \Lambda^\alpha_{\beta} = S^\alpha_{\beta} + L^\alpha_{\beta} \), with \( b \) as given in (159). We get
\[
\Gamma_M := b^{-1}(d + A_M)b = -i \vartheta^\alpha P_{\alpha} - i \Gamma_{\alpha}^\beta\left( S^\alpha_{\beta} + L^\alpha_{\beta} \right),
\]
(162)
with
\[
\vartheta^\alpha := \left( (GL) D \xi^\beta + (T) \right)r^\alpha_\beta, \quad \Gamma_{\alpha}^\beta := \left( r^{-1} \right)^\gamma_\alpha \left[ (GL) \Gamma^\lambda_\gamma r_{\lambda}^\beta - d r_{\gamma}^\beta \right].
\]
(163)

The coframe \( \vartheta^\alpha \) in (163) transforms as a Lorentz covector, that is
\[
\delta \vartheta^\alpha = -\vartheta^\beta u_{\beta}^\alpha,
\]
with \( u_{\beta}^\alpha \) as the nonlinear Lorentz parameters, whereas the linear connection in (163), taken as a whole, behaves as a Lorentz connection
\[
\delta \Gamma_{\alpha}^\beta = D u_{\alpha}^\beta.
\]
(165)

Observe however that, as read out from the r.h.s. of (162), the decomposition into two sectors of the Lie algebra of \( GL(4, R) \) gives rise to a splitting of the linear connection into the sum of a symmetric plus an antisymmetric part. Only the latter, with values on the Lorentz algebra, behaves as a true Lorentz connection, while the symmetric part (that is, the nonmetricity \( Q_{\alpha\beta} := 2 \Gamma_{(\alpha\beta)} \)) is a Lorentz tensor, varying as
\[
\delta Q_{\alpha\beta} = 2 u_{(\alpha}^\gamma Q_{\beta)\gamma}.
\]
(166)

Having completed the nonlinear realization of the affine group with the auxiliary subgroup \( H_2 \), we are ready to establish the correspondence between it and the one with \( H_1 \). The affine objects of the approach with \( H_1 = GL(4, R) \) are displayed in (157), distinguished by tildes, while those of the \( H_2 = SO(3, 1) \) case are written without tildes in (160). By comparing (157) and (160) to each other, we find out that the relation between both kinds of quantities is isomorphic to a finite gauge transformation expressible as
\[
\tilde{\vartheta}^\alpha = \left( GL \right) D \xi^\alpha + \Gamma^\alpha = \vartheta^\beta \left( r^{-1} \right)^\beta_\alpha,
\]
(167)
and
\[
\tilde{\Gamma}_{\alpha}^\beta = \Gamma_{\alpha}^\beta = r_{\alpha}^\gamma \left[ \Gamma_{\gamma}^\lambda \left( r^{-1} \right)^\lambda_\gamma \beta - d \left( r^{-1} \right)^\gamma_\beta \right],
\]
(168)
with the main difference that the matrix \( r_{\alpha}^\beta \) as given by (161) is not a gauge transformation matrix, but consists of coset fields varying as shown in (160). It is precisely this peculiar transformation property of \( r_{\alpha}^\beta \), involving both the linear \( (\omega_{\alpha}^\beta = \alpha_{\alpha}^\beta + \beta_{\alpha}^\beta) \) as much as the nonlinear group parameters \( (u_{\alpha}^\beta) \), that is responsible for the difference between the gauge transformations \( \left( GL \right) versus \left( T \right) \) – of the objects with and without tildes respectively, related to each other by \( r_{\alpha}^\beta \) as displayed in (167) and (168).
In analogy to the latter equations, a correspondence can be established between the Minkowski metric \( o_{\alpha\beta} \), existing in the \( H_2 = SO(3, 1) \) approach as a natural invariant, and a correlated MAG-metric tensor \( \tilde{g}_{\alpha\beta} \) defined in the context of the approach with \( H_1 = GL(4, R) \) as

\[
\tilde{g}_{\alpha\beta} := r_a^{\mu} r_b^{\nu} o_{\mu\nu}.
\] (169)

The MAG-metric tensor (169) plays the role of a Goldstone field \[21\] \[29\]. Actually, the ten degrees of freedom associated to it drop out by inverting the "gauge transformation" (169), together with (167) and (168). In other words, it is possible to absorb the metric variables into redefined gauge potentials by using the nonlinear realization with the Lorentz group \( H_2 = SO(3, 1) \) as the auxiliary subgroup instead of \( H_1 = GL(4, R) \). Accordingly, affine invariants can be alternatively displayed in terms of explicit general linear quantities (with tildes), as in the standard formulation of MAG \[16\], or in terms of explicit Lorentz objects (without tildes) with the metric fixed to be Minkowskian. For instance, the line element can be doubly expressed as

\[
ds^2 = \tilde{g}_{\alpha\beta} \bar{\theta}^{\alpha} \otimes \bar{\theta}^{\beta} = o_{\alpha\beta} \bar{\theta}^{\alpha} \otimes \bar{\theta}^{\beta}.
\] (170)

In accordance with the Goldstone–like nature of the MAG–metric, the field equations obtained by varying affine invariant actions with respect to \( \tilde{g}_{\alpha\beta} \) are known to be redundant \[16\]. However, we won’t enter the study of details concerning MAG–dynamics. The interested reader is referred to the literature, where quite general actions were studied, involving quadratic curvature, torsion and nonmetricity terms, for which a number of exact solutions were found \[49\] \[50\] \[51\] \[52\] \[53\] \[54\] \[55\] \[56\]. Discussions on the problem of the inclusion of matter sources in the MAG scheme can also be found for instance in references \[57\] \[58\] \[59\] \[60\] \[61\] concerning fundamental matter.

VI. CONCLUSIONS

We presented a number of applications of NLR’s to the foundation of different gravitational gauge theories in order to illustrate the variety of fields in which the method reveals to be useful, providing underlying mathematical unity and simplicity. In particular, it is worth to recall once more that the Hamiltonian approach developed in Section IV in terms of PGT connection variables revealed to be dynamically equivalent to a theory of the Ashtekar type. (So that, conversely, the latter can be regarded as a reformulation of the Hamiltonian Poincaré gauge theory built from the Einstein–Cartan action.)

As a general result derived from the different examples studied by us, we want to remark that thanks the NLR’s the description of interactions is achieved exclusively in terms of connections, in accordance with the general gauge–theoretical program. Neither the coframes nor the MAG–metric are to be regarded as separate gravitational potentials of specific nature, but rather as ordinary Yang-Mills objects. Indeed, the coframes are interpreted as a kind of gauge potentials, namely as nonlinear translative connections, while the metric of MAG is found to be a Goldstone field playing no fundamental physical role, since its degrees of freedom can be transferred to redefined gauge potentials. In the limit of vanishing Poincaré connections (corresponding to zero gravitational forces), the tetrads \[24\] reduce to the special relativistic ones \( \bar{\theta}^{\alpha} = d\xi^\alpha \), the fields \( \xi^\alpha \) playing the role of ordinary coordinates—as read out from their transformations \[24\]—so that the Minkowski space of Special relativity can be seen as the residual structure left by the dynamical theory of spacetime when gravitational interactions are switched off.

Matter sources in the context of NLR’s were exemplified by Dirac fields in PGT, whose coupling to translations gives rise to a background fermion mass contribution. Instead, the inclusion of fundamental matter in the context of nonlinear metric–affine gravity remains only partially explored. A further natural extension of the nonlinear method not yet developed consists in its application to mechanisms of spontaneous symmetry breaking—from \( G \) to a residual symmetry \( H \subset G \) in the case of external as much as of internal groups. Let us also hope that, although restricted for the time being to classical aspects of gravity, the nonlinear framework can become an useful tool to deal with quantum aspects of gravitational gauge theories.

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APPENDIX A: ETA BASIS

The objects constituting the eta basis defined in the present Appendix, built as the Hodge duals of exterior products of tetrads [16], are convenient to simplify the notation when dealing with differential forms. In terms of [27] we define the Levi-Civita object (that is, the 0-form element of the eta basis) as

\[ \eta^{\alpha \beta \gamma \delta} := \star (\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta), \]

(A1)

and with the help of it the 1-form element

\[ \eta^{\alpha \beta \gamma} := \star (\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma) = \frac{1}{2!} \eta^{\alpha \beta \gamma \delta} \vartheta^\delta, \]

(A2)

the eta-basis 2-form element

\[ \eta^{\alpha \beta} := \star (\vartheta^\alpha \wedge \vartheta^\beta) = \frac{1}{3!} \eta^{\alpha \beta \gamma \delta} \vartheta^\gamma \wedge \vartheta^\delta, \]

(A3)

the 3-form element (dual of the tetrad)

\[ \eta^{\alpha} := \star \vartheta^\alpha = \frac{1}{3!} \eta^{\alpha \beta \gamma \delta} \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta, \]

(A4)

and the 4-form of the eta-basis, or four–dimensional volume element

\[ \eta := \star 1 = \frac{1}{4!} \eta^{\alpha \beta \gamma \delta} \vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta. \]

(A5)

The exterior product of the coframe \( \vartheta^\mu \) with the elements (A1)–(A4) of the eta basis yields respectively

\[ \vartheta^\mu \wedge \eta_{\alpha \beta \gamma \delta} = -\delta^\mu_\alpha \eta_{\beta \gamma \delta} + \delta^\mu_\beta \eta_{\alpha \beta \gamma} - \delta^\mu_\gamma \eta_{\beta \alpha \delta} + \delta^\mu_\delta \eta_{\gamma \beta \alpha}, \]

(A6)

\[ \vartheta^\mu \wedge \eta_{\alpha \beta \gamma} = \delta^\mu_\alpha \eta_{\beta \gamma} + \delta^\mu_\beta \eta_{\alpha \gamma} + \delta^\mu_\gamma \eta_{\beta \alpha}, \]

(A7)

\[ \vartheta^\mu \wedge \eta_{\alpha \beta} = -\delta^\mu_\alpha \eta_{\beta} + \delta^\mu_\beta \eta_{\alpha}, \]

(A8)

\[ \vartheta^\mu \wedge \eta_{\alpha} = \delta^\mu_\alpha \eta. \]

(A9)

For an arbitrary \( p \)-form \( \alpha \) on four–dimensional space with Lorentzian signature, the double application of the Hodge dual operator reproduces \( \alpha \) itself up to the sign as \( \star \star \alpha = (-1)^{p(4-p)+1} \alpha \). A further relation involving Hodge duality reads \( \star (\alpha \wedge \vartheta^\mu) = e^\mu_\nu \star \alpha, \) while for differential forms \( \alpha, \beta \) of the same degree \( p \), equation \( \star \alpha \wedge \beta = \star \beta \wedge \alpha \) holds. The eta basis (A1)–(A5) and the algebraic relations of the present Appendix are extensively used thorough the whole work.

APPENDIX B: HAUSDORFF–CAMPBELL FORMULAS

In order to make the present exposition as self contained as possible, we give the well known formulas

\[ e^{-A}Be^A = B - [A, B] + \frac{1}{2!} [A, [A, B]] - \frac{1}{3!} [A, [A, [A, B]]] + ... \]

(B1)

\[ e^{-A}de^A = dA - \frac{1}{2!} [A, dA] + \frac{1}{3!} [A, [A, dA]] - ... \]

(B2)

\[ e^{A+\delta A} = e^A + \delta e^A + O \left((\delta A)^2\right) = e^A \left(1 + e^{-A} \delta e^A\right), \]

(B3)

useful for checking calculations.
APPENDIX C: NONLINEAR REALIZATION OF THE POINCARE GROUP WITH SO(3) AS AUXILIARY SUBGROUP

By decomposing the Lorentz generators $L_{\alpha\beta}$ into boosts $K_a$ and space rotations $S_a$, defined respectively as

$$K_a := 2L_{a0}, \quad S_a := -\epsilon_{abc}L_{bc} \quad (a = 1, 2, 3),$$

the commutation relations transform into

$$[S_a, S_b] = -i\epsilon_{abc}S_c,$$
$$[K_a, K_b] = i\epsilon_{abc}S_c,$$
$$[S_a, K_b] = -i\epsilon_{abc}K_c,$$
$$[S_a, P_0] = 0,$$
$$[S_a, P_b] = -i\epsilon_{abc}P_c,$$
$$[K_a, P_0] = iP_a,$$
$$[K_a, P_b] = i\delta_{ab}P_0,$$
$$[P_a, P_b] = [P_a, P_0] = [P_0, P_b] = 0.$$  \hfill (C1)

In the nonlinear transformation law, we take the infinitesimal Poincaré group elements $g \in G$, and the $SO(3)$ group elements $h \in H$, to be respectively

$$g = e^{i\epsilon^\alpha P_\alpha e^{i\theta^a K_a}} \approx 1 + i (\epsilon^0 P_0 + \epsilon^a P_a + \zeta^a K_a + \theta^a S_a)$$  \hfill (C3)

and

$$h = e^{i\Theta^a S_a} \approx 1 + i \Theta^a S_a,$$  \hfill (C4)

and further we parametrize $b \in G/H$ as

$$b = e^{-i\xi^\alpha P_\alpha e^{i\lambda^a K_a}},$$  \hfill (C5)

being $\xi^\alpha$ and $\lambda^a$ finite coset fields. Eq. with the particular choices (C3)–(C5) yields on the one hand the variation of the translational parameters

$$\delta\xi^0 = -\zeta^a \xi_a - \epsilon^0,$$  \hfill (C6)
$$\delta\xi^a = \epsilon_{abc} \theta^b \xi^c - \zeta^a \xi^0 - \epsilon^a,$$  \hfill (C7)

which, since $\zeta^a := \beta^a$ and $\theta^a := \frac{1}{2}\epsilon_{abc} \beta^b$ as read out from (C3), can be rewritten as

$$\delta\xi^a = -\beta^a \xi^\beta - \epsilon^a,$$  \hfill (C8)

compare with (24), showing that the coset parameters $\xi^\alpha$ associated with the translations behave in fact as coordinates. On the other hand, the variations of the boost parameters in (C5) turn out to be

$$\delta\lambda^a = \epsilon_{abc} \theta^b \lambda^c + \zeta^a |\lambda| \coth |\lambda| + \frac{\lambda^a \lambda^b \zeta^b}{|\lambda|^2} (1 - |\lambda| \coth |\lambda|), \quad |\lambda| := \sqrt{\lambda_a \lambda^a}. \hfill (C9)$$

Instead of dealing with $\lambda^a$, it is preferable to introduce the velocity fields

$$\beta^a := -\frac{\lambda^a}{|\lambda|} \tanh |\lambda|, \quad \gamma := \frac{1}{\sqrt{1 - \beta^2}},$$  \hfill (C10)

varying as

$$\delta\gamma = -\zeta^a (\gamma \beta_a), \quad \delta (\gamma \beta^a) = \epsilon_{abc} \theta^b (\gamma \beta^c) - \zeta^a \gamma,$$  \hfill (C11)

that is, as the components of a Lorentz four–vector $(\gamma, \gamma \beta^a)$, as can be easily checked by comparing (C11), (C12) with (C6), (C7).
Finally, (7) also enforces $\Theta^a$ in (14) to be

$$\Theta^a = \theta^a + \frac{\gamma}{1 + \gamma} \epsilon_{bc} \beta^b \beta^c. \quad (C13)$$

According to (8), $\Theta^a$ is the modified $SO(3)$ gauge parameter throw which the nonlinear action of the Poincaré group takes place as

$$\delta \psi = i \Theta^a \rho (S_a) \psi \quad (C14)$$
on fields $\psi$ of representation spaces of $SO(3)$, being $\rho (S_a)$ the corresponding representation of the $SO(3)$ generators.

Now we turn our attention to the nonlinear gauge fields (11), defined from the ordinary linear Poincaré connection

$$A_M := \frac{(T)}{Lor} \Gamma^\alpha_\mu P_\alpha - i \Gamma^\alpha_\beta L_{\alpha \beta}, \quad (C15)$$

standing $\Gamma^\alpha_\mu$ for the translational and $\Gamma^\alpha_\beta$ for the Lorentz contribution. (Although modified by this additional specification, (C15) is identical with (25).) Making use of (11) we introduce for (11) the notation

$$\Gamma_M = -i \hat{\partial}^\alpha P_\alpha - i \hat{\Gamma}^\alpha_\beta L_{\alpha \beta} = -i \hat{\partial}^0 P_0 - i \hat{\Gamma}^\alpha_\beta L_{\alpha \beta} + i X^\alpha K_\alpha + i A^a S_a, \quad (C16)$$

where a simple application of the Hausdorff-Campbell formulas of Appendix B yields

$$\hat{\partial}^\alpha := \hat{\partial}^\beta b_\beta^\alpha, \quad (C17)$$

$$X^\alpha := \hat{\Gamma}^{\alpha 0} = (b^{-1})^{0\mu} \left( \frac{Lor}{\mu \nu} b^{\mu \nu} - d b_\mu^\alpha \right), \quad (C18)$$

$$A^a := \frac{1}{2} \epsilon_{bc} \hat{\Gamma}^{bc} = \frac{1}{2} \epsilon_{bc} (b^{-1})^{\mu \nu} \left( \frac{Lor}{\mu \nu} b^{c \nu} - d b_\mu^c \right), \quad (C19)$$

expressed with the help of the boost matrix

$$b_0^0 = (b^{-1})^{00} := \gamma, \quad b_0^a = -(b^{-1})^{0a} := -\gamma \beta^a,$$

$$b_a^0 = -(b^{-1})^{a0} := -\gamma \beta_a, \quad b_a^b = (b^{-1})^{ab} := \delta_a^b + (\gamma - 1) \frac{\beta_a \beta^b}{\beta^2}, \quad (C20)$$

built from the fields (10). The Lorentz covectors

$$\vartheta^\alpha := \frac{Lor}{D \xi^\alpha} + \Gamma^\alpha, \quad (C21)$$

in the r.h.s. of (10) are identical with the Lorentz coframes (15) of the nonlinear approach studied above. (The abbreviation Lor over the covariant differentials in (21) indicates that they are constructed with the linear Lorentz connection $\Gamma^\alpha_\beta$ in (15).) Despite the formal analogy of (C17)–(C19) with gauge transformations, in fact the coset parameters $\xi^\alpha$, and thus (10) and (20), are fields of the theory rather than gauge parameters. Consequently, (C17)–(C19) are definitions of new variables whose transformation properties depend on (11), (12). Actually, while $\vartheta^\alpha$ in (21) transforms as a Lorentz covector and $\Gamma^\alpha_\beta$ in (15) as a Lorentz connection, for the quantities defined in (C17)–(C19) we find

$$\delta \hat{\partial}^0 = 0, \quad (C22)$$

$$\delta \hat{\partial}^\alpha = \epsilon_{bc} \Theta^b \hat{\partial}^c, \quad (C23)$$

$$\delta X^\alpha = \epsilon_{bc} \Theta^b X^c, \quad (C24)$$

$$\delta A^a = -D \Theta^a := - \left( d \Theta^a + \epsilon_{bc} A^b \Theta^c \right). \quad (C25)$$

That is, the tetrads become split into an $SO(3)$ singlet –the invariant time component $\hat{\partial}^0$– plus an $SO(3)$ covector –the triad $\hat{\partial}^\alpha$–. The nonlinear boost connection 1–forms $X^\alpha$ also transform as the components of an $SO(3)$ covector. Only the $SO(3)$ connections $A^a$ retain their connection character. The nonlinear field strength (10) built from (16) reads

$$F := d \Gamma_M + \Gamma_M \wedge \Gamma_M = -i \hat{T}^0 P_0 - i \hat{T}^a P_a + i (D X^a) K_a + i R^a S_a, \quad (C26)$$
where we introduce the definition of the torsion
\[
\hat{T}^0 := d \hat{\theta}^0 + \hat{\Gamma}^0_\mu \wedge \hat{\theta}^\mu = d \hat{\theta}^0 + \hat{\alpha}_a \wedge X^a, \tag{C27}
\]
\[
\hat{T}^a := d \hat{\theta}^a + \hat{\Gamma}^a_\mu \wedge \hat{\theta}^\mu = D \hat{\theta}^a + \hat{\theta}^a \wedge X^a, \tag{C28}
\]
the boost curvature
\[
D X^a := d X^a + \epsilon^a_{bc} A^b \wedge X^c, \tag{C29}
\]
and the rotational curvature
\[
\mathcal{R}^a := F^a - \frac{1}{2} \epsilon^a_{bc} X^b \wedge X^c, \quad \text{with} \quad F^a := d A^a + \frac{1}{2} \epsilon^a_{bc} A^b \wedge A^c, \tag{C30}
\]
respectively. It is trivial to check
\[
D X^a = \hat{R}^0 a, \quad \mathcal{R}^a = \frac{1}{2} \epsilon^a_{bc} \hat{R}^{bc}, \tag{C31}
\]
relating (C27), (C30) to the four-dimensional curvature \(\hat{R}_a \beta := d \hat{\Gamma}_a \beta + \hat{\Gamma}_\gamma \beta \wedge \hat{\Gamma}_\alpha \gamma\), with the same form as (C31) but built from the Lorentz connection \(\hat{\Gamma}^\alpha \beta\) in (C16).

**APPENDIX D: FOLIATION OF SEVERAL POINCARE OBJECTS**

In the present Appendix we apply the foliation procedure of Subsection IV. B to the quantities introduced in Appendix C. Regarding the fundamental objects (C17), (C18), notice that trivially the zero component of the tetrad (C17), with the form \(\hat{\theta}^0 = u^0 d \tau\) as in (C31), only includes a longitudinal contribution, whereas \(\hat{\theta}^a = \hat{\alpha} a\) only contains a transversal one. On the other hand, the boost nonlinear connection (C18) and the SO(3) connection (C19) decompose as \(X^a = \hat{\theta}^0 X^a + X^a\) and \(A^a = \hat{\theta}^0 A^a + \hat{\Gamma}^a_{bc} A^b \wedge A^c\) respectively. Furthermore, the decomposition of the torsion components (C27), (C28) takes the form
\[
\hat{T}^0 = -\hat{\theta}^0 \wedge \left( \frac{d \log u^0 + X^a \hat{\alpha}_a}{u^0} \right) + \hat{\alpha}_a \wedge X^a, \tag{D1}
\]
\[
\hat{T}^a = \hat{\theta}^a \wedge \left( L_{\hat{\theta}^0} \hat{\alpha}_a + X^a \right), \tag{D2}
\]
where the covariantized Lie derivative and the transversal part of the covariant differential are respectively defined as
\[
L_{\hat{\theta}^0} \hat{\alpha}_a := \hat{\alpha}_a D \hat{\theta}^a = l_{\hat{\theta}^0} \hat{\alpha}_a + \epsilon^a_{bc} A^b \wedge \hat{\alpha}_c, \quad D \hat{\theta}^a := \hat{d} \hat{\alpha}_a + \epsilon^a_{bc} A^b \wedge \hat{\alpha}_c. \tag{D3}
\]
The boost curvature (C27) splits into longitudinal and transversal parts as
\[
D X^a = \hat{\theta}^0 \wedge \left[ L_{\hat{\theta}^0} X^a - \frac{1}{u^0} D \left( u^0 X^a \right) \right] + D X^a, \tag{D4}
\]
in terms of the covariant derivatives
\[
L_{\hat{\theta}^0} X^a := \hat{\alpha}_a D X^a = l_{\hat{\theta}^0} X^a + \epsilon^a_{bc} A^b X^c, \tag{D5}
\]
\[
D \left( u^0 X^a \right) := \hat{d} \left( u^0 X^a \right) + \epsilon_{bc} A^b \left( u^0 X^c \right), \tag{D6}
\]
\[
D X^a := l_{\hat{\theta}^0} A^a - \frac{1}{u^0} D \left( u^0 A^a \right), \tag{D7}
\]
compare with (59). The SO(3) curvature (C30) and its Hodge dual, see (59), decompose respectively as
\[
\mathcal{R}^a = \hat{\theta}^0 \wedge \mathcal{R}^a + \mathcal{R}^a, \quad * \mathcal{R}^a = \hat{\theta}^0 \wedge * \mathcal{R}^a - * \mathcal{R}^a, \tag{D8}
\]
with definitions
\[
\mathcal{R}^a_\perp := F^a_{\perp} - \epsilon^a_{bc} X^b X^c, \quad F^a := l_{\hat{\theta}^0} A^a - \frac{1}{u^0} D \left( u^0 A^a \right). \tag{D9}
\]
and

\[ R^a := F^a - \frac{1}{2} \epsilon^{bc} X^b \wedge X^c, \quad F^a := i \Delta^a + \frac{1}{2} \epsilon^{bc} \Delta^b \wedge \Delta^c. \]  

In order to complete the set of foliated objects needed in Section IV, we give here the 3+1 decomposition of the 4-dimensional eta basis of Appendix A as

\[ \hat{\eta}_{abc} = - \partial^0 \hat{\eta}_{abc}, \quad \hat{\eta}_{ab} = \hat{\delta}^0 \wedge \hat{\eta}_{ab}, \quad \hat{\eta}_a = - \hat{\delta}^0 \wedge \hat{\eta}_a, \quad \hat{\eta} = \hat{\delta}^0 \wedge \hat{\eta}; \]

where the bar over the etas in (D11) means their restriction to the three-space as

\[ \hat{\eta}_{abc} := \hat{\eta}_{abc} = \epsilon_{abc}, \]
\[ \hat{\eta}_{ab} := \hat{\eta}_{ab} = \epsilon_{abc} \hat{\delta}^c, \]
\[ \hat{\eta}_a := \hat{\eta}_a = \frac{1}{2} \epsilon_{abc} \hat{\delta}^b \wedge \hat{\delta}^c, \]
\[ \hat{\eta} := \hat{\eta}_0 = \frac{1}{3!} \epsilon_{abc} \hat{\delta}^a \wedge \hat{\delta}^b \wedge \hat{\delta}^c. \]

The identification of \( \hat{\eta}_{abc} \) with the group constants \( \epsilon_{abc} \) of \( SO(3) \) in (D12) is possible due to the fact that, being the holonomic \( SO(3) \) metric the Kronecker delta, one has \( \hat{\eta}_{abc} = \# \left( \hat{\delta}_a \wedge \hat{\delta}_b \wedge \hat{\delta}_c \right) = \sqrt{\det(\delta_{mn})} \epsilon_{abc} = \epsilon_{abc} \).

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