Galilean symmetry in generalized abelian Schrödinger-Higgs models with and without gauge field interaction

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Abstract

We consider a generalization of nonrelativistic Schrödinger-Higgs Lagrangian by introducing a nonstandard kinetic term. We show that this model is Galilean invariant, we construct the conserved charges associated to the symmetries and realize the algebra of the Galilean group. In addition, we study the model in the presence of a gauge field. We also show that the gauged model is Galilean invariant.

Keywords: Galilean symmetry, Gauge theories, Chern-Simons gauge theory, Symmetries in theory of fields and particles.

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1 Introduction

The case where the space-time group is the Galilean Group is particularly relevant as many condensed matter and quantum systems display explicitly this symmetry. A feature of massive non-relativistic quantum systems is that Galilean boosts only act up-to phase, so that only the 1-parameter central extension of the Galilean group acts unitarily [1]. Later, Bargmann [2] showed that, in 3 or more space dimensions, the Galilean group only admits a 1-parameter central extension, identified with the physical mass. Nevertheless, in Ref. [3], Levy Leblond showed that owing to the abelian nature of planar rotations, the Galilean group in the plane admits a second extension. More recently, a completed study of the Galilean invariance inherent to the bosonic sector was developed in [4, 5, 6, 7] and [8, 9, 10]. Also, the supersymmetric extension of the
Galilean group in 2 + 1 dimension was analyzed in [11], [12, 13] and [14, 15].

In the recent years, theories with nonstandard kinetic term, named \(k\)-field models, have received much attention. The \(k\)-field models are mainly in connection with effective cosmological models [16, 17, 18, 19, 20, 21] as well as the tachyon matter [22], and the ghost condensates [23, 24, 25, 26, 27]. The strong gravitational waves [28] and dark matter [29], are also examples of non-canonical fields in cosmology. Also, topological structure of these models was analyzed [30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43], showing that the \(k\)-theories can support topological soliton solutions both in models of matter as in gauged models.

In this paper we are interested in studying the nonrelativistic Higgs model with a generalized dynamics. This nonstandard dynamics is introduced by a function \(\omega\), which depend on the Higgs field. In particular we will propose \(\omega = \rho^n\) where \(\rho = \phi^\dagger \phi\), being \(\phi\) the Higgs field. We will show that the model is Galilean invariant and we will construct the conserved charges associated with this invariance. We, also, show that our model realize the algebra of the Galilean group. For this purpose, we rewrite the theory in terms of a new field \(\psi\), which is related to the original Higgs field by \(\psi = \phi^{n+1}\). Thus, it is not difficult to show that our model model satisfies the Galilean algebra.

Finally, we analyze a generalized gauge model. In particular, we will concentrate on a generalization of the Jackiw-Pi model [6, 7]. We, also, show that this model is Galilean invariant, realizing the algebra of the group.

2 The model and its symmetries

Let us start by considering the (2 + 1)-dimensional Schrödinger model governed by the action,

\[
S = \int d^3x \left( i\phi^\dagger \partial_0 \phi - \frac{1}{2m} |\partial_i \phi|^2 + \lambda |\phi|^2 \right),
\]

(1)

Here, \(\phi(x)\) is a complex scalar field and \(\lambda\) is a strength coupling constant. Also, the metric tensor is \(g^{\mu\nu} = (1, -1, -1)\).

It is well known that the model (11) presents Galilean invariance [4]. This means that action (11) is invariant under time and space translation, rotations, Galilean boost and the \(U(1)\) symmetry. More precisely, the Schrödinger model remains invariant under the following transformations:

1. Time translation:

\[
\delta t = a,
\]

(2)

where \(a\) is a constant. The infinitesimal time-translation on the field is

\[
\delta \phi = a \partial_0 \phi
\]

(3)
2. Space translation:

$$\delta \mathbf{r} = \mathbf{a},$$  \hspace{1cm} (4)

where $\mathbf{a}$ is a constant 2-vector. The infinitesimal translation of the field is

$$\delta \phi = a_i \partial_i \phi$$  \hspace{1cm} (5)

3. The angular momentum is obtained in a similar way by considering an infinitesimal rotation

$$\delta \mathbf{r} = \theta \mathbf{z} \times \mathbf{r},$$  \hspace{1cm} (6)

where $\theta$ is the rotation angle and the infinitesimal field transformation reads

$$\delta \phi = \theta \mathbf{r} \times \partial \phi$$  \hspace{1cm} (7)

4. Under a Galilean boost

$$\delta \mathbf{r} = \mathbf{v} t,$$  \hspace{1cm} (8)

the field transform as

$$\delta \phi = i m v_i r_i \phi - t v_i \partial_i \phi$$  \hspace{1cm} (9)

5. The Galilean invariance is completed with the inclusion of $U(1)$ symmetry

$$\delta \phi = i \alpha \phi$$  \hspace{1cm} (10)

According to the Noether’s theorem, if under a variation of the field $\delta \phi$, the variation of the Lagrangian density is a surface term, $\delta L = \partial_\mu X^\mu$, then exist a conserved charge associated with such variation of the fields. For our symmetries, these conserved charges are

1. The Hamiltonian, associated to the time translation:

$$H = \int d^2 x \left( \frac{1}{2m} |\partial_t \phi|^2 + \lambda |\phi|^2 \right)$$  \hspace{1cm} (11)
2. The linear momentum, generated by space translation

\[ P_i = \frac{i}{2} \int d^2x (\phi^\dagger \partial_i \phi - \partial_i \phi^\dagger \phi) \]  
(12)

3. The angular momentum, obtained from the variation under rotations

\[ J = \int d^2x \left( - P_1 x_2 + P_2 x_1 \right) \]  
(13)

where,

\[ P_i = \frac{i}{2} \left( \phi^\dagger \partial_i \phi - \partial_i \phi^\dagger \phi \right) \]  
(14)

4. The conserved charge

\[ G_i = \int d^2x \left( P_i t - m x_i \rho \right) \]  
(15)

is generated by Galilean boost. Here, \( \rho = \phi^\dagger \phi \)

5. The mass operator

\[ M = mN = m \int d^2x \rho \]  
(16)

which arises from \( U(1) \) symmetry.

The algebra of the Galilean group may be realized by using the Poisson brackets for functions of the matter fields, which are defined from the symplectic structure of the Lagrangian at fixed time to be

\[ \{ F, G \}_PB = i \int d^2x \left( \frac{\delta F}{\delta \phi^\dagger (r)} \frac{\delta G}{\delta \phi(r)} - \frac{\delta F}{\delta \phi(r)} \frac{\delta G}{\delta \phi^\dagger (r)} \right) \]  
(17)

In the particular case in which \( F = \phi \) and \( G = \phi^\dagger \) we have

\[ [\phi(x), \phi(x')^\dagger] = -i \delta^2(x - x') \]  
(18)

Using the Poisson bracket relations the above conserved charges can be shown to realize the algebra of the Galilean group

\[ [P_i, P_j] = [P_i, H] = [J, H] = [G_i, G_j] = 0 \]
\[ [J, P_i] = \epsilon^{ij} P_j \]
\[ [J, G_i] = \epsilon^{ij} G_j \]
\[ [P_i, G_j] = \delta^{ij} mN \]
\[ [H, G_i] = P_i \]  
(19)
In this section we are interested in exploring a generalization of the model \( \mathcal{M} \). Following the same idea of the works cited in Ref.\([30, 31, 32, 33, 40, 41]\), we modify the model \( \mathcal{M} \) by changing both the canonical kinetic term of the scalar field and the potential term, so that the proposed model is described by the action

\[
S = \int d^3 x \ \omega(\rho) \mathcal{L}_{NR} = \int d^3 x \ \omega(\rho) \left( i \phi^\dagger \partial_0 \phi - \frac{1}{2m} |\partial_i \phi|^2 + \lambda |\phi|^2 \right) = S_1 + S_2 + S_3 ,
\]

where

\[
S_1 = \int d^3 x \ \omega(\rho) i \phi^\dagger \partial_0 \phi
\]

\[
S_2 = - \int d^3 x \ \omega(\rho) \frac{1}{2m} |\partial_i \phi|^2
\]

\[
S_3 = \int d^3 x \ \omega(\rho) \lambda |\phi|^2
\]

(21)

Here, the function \( \omega(\rho) \) is a function of the complex scalar field \( \phi \). In particular, we chose \( \omega(\rho) \) to be a dielectric function of the form

\[
\omega(\rho) = \rho^n = (\phi^\dagger \phi)^n ,
\]

(22)

where \( n \) is a positive integer.

In the next we will calculate the variation of the action \( \mathcal{M} \) under time and space translation, angular rotation, Galilean boost and U(1) transformation. We begin to calculate the variation of the action \( \mathcal{M} \) under time and space translation

\[
\delta \phi = a \partial_0 \phi
\]

(23)

\[
\delta \phi = a_i \partial_i \phi
\]

(24)

The variation \( \delta \phi \) implies

\[
\delta \omega(\rho) = \delta \rho^n = n \rho^{(n-1)} \delta \rho = an \rho^{(n-1)} (\partial_0 \phi^\dagger \phi + \phi^\dagger \partial_0 \phi) = a \partial_0 \rho^n = a \partial_0 \omega(\rho)
\]

(25)

So that,

\[
\delta S_1 = \int d^3 x \left[ i \delta \omega(\rho) \phi^\dagger \partial_0 \phi + i \omega(\rho) \delta (\phi^\dagger \partial_0 \phi) \right] = \int d^3 x \left[ i a \partial_0 \omega(\rho) \phi^\dagger \partial_0 \phi + i \omega(\rho) \delta (\phi^\dagger \partial_0 \phi) \right]
\]

\[
= \int d^3 x \left[ i a \partial_0 \omega(\rho) \phi^\dagger \partial_0 \phi + i \omega(\rho) (a \partial_0 \phi^\dagger \partial_0 \phi + a \phi^\dagger \partial_0^2 \phi) \right]
\]

(26)

By integration by parts, the last term of this integral, we have,

\[
\delta S_1 = 0
\]

(27)

The variation of \( \mathcal{M}_2 \) under \( \delta \phi \) leads to

\[
\delta S_2 = - \frac{a}{2m} \int d^3 x \left[ (\partial_i \partial_0 \phi^\dagger \partial_i \phi + \partial_i \phi^\dagger \partial_i \partial_0 \phi) \omega(\rho) + |\partial_i \phi|^2 \partial_0 \omega(\rho) \right]
\]

(28)
Then, integrating by parts the first term of this integral, we immediately arrive to
\[ \delta S_2 = 0 \] (29)

Finally, it easy to check that,
\[ \delta S_3 = \lambda \int d^3 x \ [\delta \omega(\rho) \rho + \omega(\rho) \delta \rho] = a \lambda \int d^3 x \ \partial_0 [\omega(\rho) \rho] = a \lambda \int d^3 x \ \partial_0 \rho^{(n+1)} = 0 \] (30)

where we have supposed
\[ \lim_{t,x \to \infty} \phi = 0 \] (31)

Thus, the model is invariant under time translation. Space translation, involves
\[ \delta \omega(\rho) = \delta \rho^n = n \rho^{(n-1)} \delta \rho = a_i n \rho^{(n-1)} (\partial_i \phi^\dagger \phi + \phi^\dagger \partial_i \phi) = a_i \partial_i \rho^n = a_i \partial_i \omega(\rho) \] (32)

Then we have,
\[ \delta S_1 = \int d^3 x \ [i a_i \partial_i \omega(\rho) \phi^\dagger \partial_0 \phi - i a_i \omega(\rho) (\partial_i \phi^\dagger \partial_0 \phi + \phi^\dagger \partial_0 \partial_i \phi)] \] (33)

Integrating by parts the last term of this integral we get
\[ \delta S_1 = 0 \] (34)

The variation wit respect to \( S_2 \) is
\[ \delta S_2 = -\frac{1}{2m} \int d^3 x \ [\left(a_i \partial_i^2 \phi^\dagger \partial_0 \phi + a_i \partial_i \phi^\dagger \partial_0^2 \phi\right) \omega(\rho) + |\partial_i \phi|^2 a_i \partial_i \omega(\rho)] \] (35)

It can be easily seen that integrating by parts the first term, the variation becomes zero.

For \( S_3 \), we have
\[ \delta S_3 = \lambda \int d^3 x \ [\delta \omega(\rho) \rho + \omega(\rho) \delta \rho] = a_i \lambda \int d^3 x \ \partial_i [\omega(\rho) \rho] = a_i \lambda \int d^3 x \ \partial_i \rho^{(n+1)} = 0 \] (36)

The model is also invariant under rotations. Indeed, we have from [7]
\[ \delta \phi = x_1 \partial_2 \phi - x_2 \partial_1 \phi \] (37)

Such that,
\[ \delta \omega(\rho) = n \rho^{(n-1)} \left[x_1 (\partial_2 \phi^\dagger \phi + \phi^\dagger \partial_2 \phi) - x_2 (\partial_1 \phi^\dagger \phi + \phi^\dagger \partial_1 \phi)\right] = n \rho^{(n-1)} (x_1 \partial_2 \omega(\rho) - x_2 \partial_1 \omega(\rho)) = x_1 \partial_2 \omega(\rho) - x_2 \partial_1 \omega(\rho) \] (38)

\[ \delta S_1 = i \int d^3 x \ [\ [x_1 \partial_2 \omega(\rho) - x_2 \partial_1 \omega(\rho)] \phi^\dagger \partial_0 \phi + \omega(\rho) \partial_0 \phi [x_1 \partial_2 \phi^\dagger - x_2 \partial_1 \phi^\dagger] + \omega(\rho) \phi^\dagger \partial_0 [x_1 \partial_2 \phi - x_2 \partial_1 \phi] \] (39)
Thus, the equation (43) is reduced to

\[ \int d^3x \ w(\rho) \phi^\dagger \partial_0 [x_1 \partial_2 \phi - x_2 \partial_1 \phi] = - \int d^3x \ \left[ [x_1 \partial_2 \omega(\rho) - x_2 \partial_1 \omega(\rho)] \phi^\dagger \partial_0 \phi \\
+ \ w(\rho) \partial_0 \phi [x_1 \partial_2 \phi^\dagger - x_2 \partial_1 \phi^\dagger] \right] \]

which implies

\[ \delta S_1 = 0 \] (41)

Variation with respect to \( S_2 \), requires a bit more attention. By using (47) and (48) we have

\[ \delta S_2 = - \frac{1}{2m} \int d^3x \ \left[ \partial_i (x_1 \partial_2 \phi^\dagger - x_2 \partial_1 \phi^\dagger) \partial_i \phi \omega(\rho) + \ w(\rho) \partial_i \phi^\dagger \partial_i (x_1 \partial_2 \phi - x_2 \partial_1 \phi) \\
+ \ |\partial_i \phi|^2 [x_1 \partial_2 \omega(\rho) - x_2 \partial_1 \omega(\rho)] \right] \]

The first two terms of this integral may be rewritten as,

\[ \int d^3x \ \left[ \partial_i (x_1 \partial_2 \phi^\dagger - x_2 \partial_1 \phi^\dagger) \partial_i \phi \omega(\rho) + \ w(\rho) \partial_i \phi^\dagger \partial_i (x_1 \partial_2 \phi - x_2 \partial_1 \phi) \right] \\
= \int d^3x \ \omega(\rho) \ x_1 \left[ \partial_1 \partial_2 \phi^\dagger \partial_1 \phi + \partial_2 \phi^\dagger \partial_2 \phi + \partial_1 \phi^\dagger \partial_2 \phi + \partial_2 \phi^\dagger \partial_1 \phi \right] \\
- \int d^3x \ \omega(\rho) \ x_2 \left[ \partial_1 \partial_2 \phi^\dagger \partial_1 \phi + \partial_1 \partial_2 \phi^\dagger \partial_1 \phi + \partial_1 \partial_2 \phi^\dagger \partial_2 \phi + \partial_2 \partial_1 \phi \partial_2 \phi \right] \]

By integration by parts, we may rewrite the first two terms of each of these last two integrals. That is,

\[ \int d^3x \ \omega(\rho) \ x_1 \partial_1 \partial_2 \phi^\dagger \partial_1 \phi = - \int d^3x \ x_1 \partial_1 \phi^\dagger (\omega(\rho) \partial_1 \partial_2 \phi + \partial_1 \phi \partial_2 \omega(\rho)) \]

\[ \int d^3x \ \omega(\rho) \ x_1 \partial_2 \phi^\dagger \partial_2 \phi = - \int d^3x \ x_1 \partial_2 \phi^\dagger (\omega(\rho) \partial_2 \phi + \partial_2 \phi \partial_2 \omega(\rho)) \]

\[ - \int d^3x \ \omega(\rho) \ x_2 \partial_1 \phi^\dagger \partial_1 \phi = \int d^3x \ x_2 \partial_1 \phi^\dagger (\omega(\rho) \partial_1 \phi + \partial_1 \phi \partial_1 \omega(\rho)) \]

\[ - \int d^3x \ \omega(\rho) \ x_2 \partial_2 \phi^\dagger \partial_2 \phi = \int d^3x \ x_2 \partial_2 \phi^\dagger (\omega(\rho) \partial_2 \phi + \partial_2 \phi \partial_1 \omega(\rho)) \]

Thus, the equation (43) is reduced to

\[ \int d^3x \ \left[ \partial_i (x_1 \partial_2 \phi^\dagger - x_2 \partial_1 \phi^\dagger) \partial_i \phi \omega(\rho) + \ w(\rho) \partial_i \phi^\dagger \partial_i (x_1 \partial_2 \phi - x_2 \partial_1 \phi) \right] \\
= - \int d^3x \ |\partial_i \phi|^2 \left( - x_1 \partial_2 \omega(\rho) + x_2 \partial_1 \omega(\rho) \right) \]

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By introducing (48) in (42), we immediately arrive to
\[ \delta S_2 = 0 \] (49)
The invariance under rotation of the model is completed by the variation of \( S_3 \),
\[ \delta S_3 = \lambda \int d^3x [x_1 \partial_2 \omega(\rho) - x_2 \partial_1 \omega(\rho)] \rho + \omega(\rho) [x_1 \partial_2 \rho - x_2 \partial_1 \rho] \] (50)
Taking into account that \( \omega(\rho) = \rho^n \), we can rewrite this integral as
\[ \delta S_3 = \lambda \int d^3x [x_1 \rho^n \partial_2 \rho - x_2 n \rho^n \partial_1 \rho + x_1 + x_1 \rho^n \partial_2 \rho - x_2 \rho^n \partial_1 \rho] \]
Let us concentrate on the Galilean boost,
\[ \delta \phi = (imv_i x_i - tv_i \partial_i) \phi \] (52)
Under this transformation the variation of \( \omega(\rho) \) is
\[ \delta \omega(\rho) = n \rho^{n+1} \delta \rho = -ntv_i \rho^n \partial_i \rho = -tv_i \partial_i \omega(\rho) \] (53)
Thus, we have for \( S_1 \) the following variation,
\[ \delta S_1 = i \int d^3x \left[ -tv_i \partial_i (x_1 \partial_2 \omega(\rho) \phi) - \omega(\rho) tv_i \partial_i \phi \partial_0 \phi - \omega(\rho) \phi^\dagger tv_i \partial_i \partial_0 \phi \right] \] (54)
Integrating by parts the last term, it is easy to check
\[ \delta S_1 = 0 \] (55)
The variation of \( S_2 \) may be evaluated by using (52) and (53), so that
\[ \delta S_2 = -\frac{1}{2m} \int d^3x \left[ -tv_i \partial_i (x_1 \partial_2 \omega(\rho)) \phi^\dagger \phi + (tv_i \partial_i^2 \phi^\dagger \phi - tv_i \partial_i \phi^\dagger \partial_0 \phi - \omega(\rho) \phi^\dagger tv_i \partial_i \partial_0 \phi) \right] \] (56)
which vanish after integrating by parts the last term of this integral.
The \( S_3 \) is also invariant under Galilean boost. Indeed we have,
\[ \delta S_3 = \lambda \int d^3x \left[ -tv_i \rho \partial_i \omega(\rho) - tv_i \omega(\rho) \partial_i \phi^\dagger \phi + tv_i \omega(\rho) \phi^\dagger \partial_i \phi \right] \] (57)
where the last term may be integrated by parts, arriving to
\[ \delta S_3 = 0 \] (58)
Finally, the \( U(1) \) invariance of (20) is automatically satisfied, since \( \omega(\rho) \) and \( \mathcal{L}_{NR} \) are \( U(1) \) invariant, and then
\[ \delta S = \int d^3x \left( \delta \omega(\rho) \mathcal{L}_{NR} + \omega(\rho) \delta \mathcal{L}_{NR} \right) = 0 \] (59)
3 The conserved charges

Let us start this section by considering the Noether theorem. Let \( \{\theta_c\} = \{\phi, \phi^{\dagger}, \psi, \psi^{\dagger}\} \) the set of the fields of the our system, where \( c \) runs from 1 to 4. The theorem establish that if under a variation of the fields \( \delta\theta_c \), the variation of the Lagrangian density is a surface term, \( \delta\mathcal{L} = \partial_\mu X^\mu \), then exist a conserved current associated with such variation of the fields. The Noether current, assuming the summation convention over the index \( c \), is

\[
j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu \theta_c)} \delta\theta_c - X^\mu
\]

(60)

Note that, from the Noether theorem, the only restriction for \( X^\mu \) is that \( \partial_\mu X^\mu \) be a surface term.

Here we are dealing with

\[
\mathcal{L} = \omega(\rho)\mathcal{L}_{NR} = \omega(\rho) \left( i\phi^{\dagger}\partial_0\phi - \frac{1}{2m}|\partial_i\phi|^2 + \lambda|\phi|^2 \right)
\]

(61)

Such that,

\[
\frac{\partial\mathcal{L}}{\partial(\partial_0 \phi)} = i\phi^{\dagger}\omega(\rho)
\]

(62)

For the special case of the time translation we choose,

\[
X^0 = \mathcal{L}, \quad X^i = 0,
\]

(63)

\[
\{\delta\theta_c\} = \{\partial_0 \phi, \partial_0 \phi^{\dagger}\}
\]

(64)

Thus, we have

\[
H = \int d^2x \ j_0 dx^2 = \int d^2x \ i\phi^{\dagger}\partial_0\phi\omega(\rho) - \mathcal{L} = \int d^2x \ \left( \frac{1}{2m}|\partial_i\phi|^2 - \lambda|\phi|^2 \right)\omega(\rho),
\]

(65)

which is the Hamiltonian of the model [20].

The conserved charge associated to space-translation is obtained as follows,

\[
\frac{\partial\mathcal{L}}{\partial(\partial_0 \phi)} \delta\phi = i\phi^{\dagger}\omega(\rho)\partial_i\phi
\]

(66)

\[
X^0 = 0, \quad X^i = 0,
\]

(67)

Then,

\[
P_i = i \int d^2x \ \phi^{\dagger}\omega(\rho)\partial_i\phi = \frac{i}{2} \int d^2x \left( \phi^{\dagger}\omega(\rho)\partial_i\phi - \partial_i\phi^{\dagger}\omega(\rho)\phi - \rho\partial_i\omega(\rho) \right)
\]

(68)
Using that $\omega(\rho) = \rho^n$, it is easy to check that last term of this integral is zero, indeed we have,

\[
\int d^2 x \, \rho \partial_i \omega(\rho) = \int d^2 x \, \rho \partial_i \rho^n = \int d^2 x \, n \rho^n \partial_i \rho = \frac{n}{n+1} \int d^2 x \, \partial_i \rho^{n+1} = 0 \tag{69}
\]

So, we get

\[
P_i = \frac{i}{2} \int d^2 x \left( \phi^\dagger \partial_i \phi - \partial_i \phi^\dagger \phi \right) \omega(\rho) \tag{70}
\]

This is the conserved charge associated to space-translations, which differs from the usual nonrelativistic $P_i$ in the fact that here we have the function $\omega(\rho)$ multiplying the term $\phi^\dagger \partial_i \phi - \partial_i \phi^\dagger \phi$.

For symmetry rotation the conserved current is

\[
j^0 = i \phi^\dagger (x_1 \partial_2 \phi - x_2 \partial_1 \phi) \omega(\rho) \tag{71}
\]

After integration by parts it is not difficult to arrive to

\[
J = \int d^2 x \, j^0 = i \int d^2 x \left( -\frac{1}{2} \phi^\dagger \partial_1 \phi + \frac{1}{2} \phi \partial_1 \phi^\dagger \right) x_2 \omega(\rho) + \left( -\frac{1}{2} \phi^\dagger \partial_2 \phi + \frac{1}{2} \phi \partial_2 \phi^\dagger \right) x_1 \omega(\rho) + \frac{1}{2} \rho x_2 \partial_1 \omega(\rho) - \frac{1}{2} \rho x_1 \partial_2 \omega(\rho) \tag{72}
\]

Again, if we take into account that $\omega(\rho) = \rho^n$, the last two terms of this formula may be eliminated. Thus, the expression (72) is reduced to

\[
J = \int d^2 x \left( -P_1 x_2 + P_2 x_1 \right), \tag{73}
\]

which is the usual expression of the Angular momentum.

The Galilean boost lead us to

\[
\frac{\partial L}{\partial (\partial_0 \phi)} \delta \phi = i \phi^\dagger \omega(\rho)(i m v_i x_i \phi - t v_i \partial_i \phi) \tag{74}
\]

Therefore,

\[
G = \int d^2 x \left( -m v_i x_i \rho \omega(\rho) + i t v_i \phi^\dagger \partial_i \phi \omega(\rho) \right) = \int d^2 x \left( -m v_i x_i \rho \omega(\rho) + P_i t v_i \right) \tag{75}
\]

Let us, now, examine conserved charge associated to $U(1)$ symmetry. The conserved current is

\[
j_0 = \frac{\partial L}{\partial (\partial_0 \phi)} \delta \phi = i \phi^\dagger \omega(\rho) i \alpha \phi = -\alpha \rho^{n+1} \tag{76}
\]

So that the conserved charge is

\[
N = -\alpha \int d^2 x \, \rho^{n+1} \tag{77}
\]

which is a generalization of the usual mass operator (16).
4 The Galilean Algebra

In this section we shall study the algebra of the generators associated to the symmetry transformations studied in section (2). We have seen in section (2) that the algebra of the Galilean group is realized by the Poisson bracket (17). Also, the Poisson bracket (17) implies the commutation relation (18), which is the fundamental relation to construct the algebra (19). The commutator (18), is the usual commutator between the fundamental field of the theory and its canonical conjugate, which is usually defined as

\[ \pi = \frac{\partial L}{\partial (\partial_0 \phi)} = i\phi^\dagger \]  

(78)

So that,

\[ [\phi(x), \pi(x')] = \delta^2(x - x') \]  

(79)

However, this mechanism entails problems when we try to apply it to our model. The problem lies in the fact that, here, \( \pi \) is not \( i\phi^\dagger \). Indeed,

\[ \pi = \frac{\partial L}{\partial (\partial_0 \phi)} = i\phi^\dagger \omega(\rho) = i(\phi^\dagger)^{n+1}\phi^n \]  

(80)

Thus, the definition (17) of the Poisson bracket does not apply. For this reason, we should rewrite the model in terms of new fields, such that we can recover a commutation relation, similar to (18). For this purpose, we focus on the action (20). We can rewrite this action in the following way,

\[
S = \int d^3x \left( i\phi^\dagger \partial_0 \phi - \frac{1}{2m} |\partial_i \phi|^2 + \lambda \rho \right) \omega(\rho) = \int d^2x \left( i\phi^\dagger \partial_0 \rho^n - \frac{1}{2m} |\partial_i \phi|^2 \rho^n + \lambda \rho^{n+1} \right) \\
= \int d^3x \left( i(\phi^{n+1})^\dagger \phi^n \partial_0 \phi - \frac{1}{2m} \phi^n \partial_i \phi (\phi^n)^\dagger \partial_i \phi^\dagger + \lambda (\phi^{n+1})^\dagger \phi^{n+1} \right) \\
= \int d^3x \left( \frac{i}{n+1} (\phi^{n+1})^\dagger \partial_0 \phi^{n+1} - \frac{1}{2m(n+1)^2} \partial_i (\phi^{n+1}) \partial_i (\phi^{n+1})^\dagger + \lambda (\phi^{n+1})^\dagger \phi^{n+1} \right) \tag{81}
\]

From (81), it is natural to define new fields, such that

\[ \psi = \phi^{n+1}, \quad \psi^\dagger = (\phi^\dagger)^{n+1}, \]  

(82)

Thus, the action (81) is rewritten as

\[
S = \int d^2x \left( \frac{i}{n+1} \psi^\dagger \partial_0 \psi - \frac{1}{2m(n+1)^2} \partial_i (\psi) \partial_i (\psi)^\dagger + \lambda \psi^\dagger \psi^\dagger \psi \right) \tag{83}
\]

Comparing this action with the nonrelativistic action (1), we see immediately that both are very similar. Then, the canonical conjugate field is

\[ \pi = \frac{\partial L}{\partial (\partial_0 \psi)} = \frac{i}{n+1} \psi^\dagger, \]  

(84)
and we can define the Poisson bracket, in terms of the new fields, following the definition (17),

\[
\{F,G\}_{PB} = i \int d^2x \left( \frac{\delta F}{\delta \psi^\dagger} \frac{\delta G}{\delta \psi} - \frac{\delta F}{\delta \psi} \frac{\delta G}{\delta \psi^\dagger} \right)
\]

In particular, if \(F = \psi\) and \(G = \psi^\dagger\) we recover the usual commutation relation between the fundamental field and its canonical conjugate,

\[
\{\psi,\psi^\dagger\}_{PB} = [\psi,\psi^\dagger] = i \int d^2x \left( - \delta^2(x-x') \delta^2(x-x') \right) = -i \delta^2(x-x')
\]

In order to verify that the algebra (19) is also true for our model (20), we rewrite the conserved charges in terms of the fields \(\psi\) and \(\psi^\dagger\),

\[
H = \int d^2x \left( \frac{1}{2m(n+1)^2} \partial_i(\psi) \partial_i(\psi^\dagger) - \lambda \psi^\dagger \psi \right)
\]

\[
P_i = \frac{i}{2(n+1)} \int d^2x \left( \psi^\dagger \partial_i \psi - \partial_i \psi^\dagger \psi \right)
\]

\[
M = \int d^2x \left( - P_1 x_2 + P_2 x_1 \right)
\]

\[
G = \int d^2x \left( P_i t v_i - m v_i x_i \psi^\dagger \psi \right)
\]

\[
N = -\alpha \int d^2x \ \psi^\dagger \psi
\]

where,

\[
P_i = \frac{i}{2(n+1)} \left( \psi^\dagger \partial_i \psi - \partial_i \psi^\dagger \psi \right)
\]

Thus, the conserved charges written in terms of \(\psi\) and \(\psi^\dagger\) as well as the commutation relation between \(\psi\) and \(\psi^\dagger\), lead us to similar context of the nonrelativistic case analyzed in section (2). Therefore, it is easy to understand that the generalized model (20) satisfied the algebra of the Galilean group expressed in (19).

5 Gauged model

Let us consider the model, in which Higgs field is coupled to a gauge field \(A_\mu(x)\),

\[
S = S_A + \int d^3x \ \omega(\rho) \mathcal{L}_{NR} = S_A + \int d^3x \ \omega(\rho) \left( i \phi^\dagger D_0 \phi - \frac{1}{2m} |D_i \phi|^2 + \lambda |\phi|^4 \right),
\]

where the covariant derivative is

\[
D_\mu = \partial_\mu + i e A_\mu \quad (\mu = 0, 1, 2)
\]
and $S_A$ denote the dynamics of the gauge field. In particular we will assume that $S_A$ is a $2 + 1$ dimensional Chern-Simons action, given by,

$$S_{cs} = \frac{\kappa}{4} \int d^3 x e^{\mu \nu \alpha} A_\mu F_{\nu \alpha} = \kappa \int d^3 x (A_0 F_{12} + A_2 \partial_0 A_1)$$

(91)

As we know [6], the nonrelativistic model

$$S_{cs} + S_{NR} = S_{cs} + \int d^3 x \left( i \phi \psi^\dagger \partial_0 \phi - \frac{1}{2m} |D_i \phi|^2 + \lambda |\phi|^4 \right) ,$$

(92)

is Galilean invariant and satisfies the algebra of the formula [19] inherent to the Galilean group. Our task, here, is show that action (89) is also Galilean invariant and satisfies (19). In order to show this, it is convenient to use the mechanism of the previous section. That is, express the relevant quantities in terms of $\psi$ and $\psi^\dagger$ instead of $\phi$ and $\phi^\dagger$. Thus, the action (89) reads as,

$$S = \int d^3 x \left( i \psi^\dagger \left[ \frac{1}{n+1} \partial_0 \psi + i e A_0 \psi \right] ight)
- \frac{1}{2m} \left( \frac{1}{n+1} \partial_i \psi^\dagger \left( \frac{1}{n+1} \partial_i \psi + i e A_i \psi \right) + \lambda (\psi^\dagger \psi)^2 \right) + S_{cs}$$

(93)

In terms of the fields $\psi$ and $\psi^\dagger$,

$$S = \int d^3 x \left( i \psi^\dagger \left[ \frac{1}{n+1} \partial_0 \psi + i e A_0 \psi \right] ight)
- \frac{1}{2m} \left( \frac{1}{n+1} \partial_i \psi^\dagger \left( \frac{1}{n+1} \partial_i \psi + i e A_i \psi \right) + \lambda (\psi^\dagger \psi)^2 \right) + S_{cs}$$

(94)

Let us, now, define the action $S'$, such that $S' = (n + 1) S$,

$$S' = \int d^3 x \left( i \psi^\dagger \left[ \frac{1}{n+1} \partial_0 \psi + i e A_0 \psi \right] ight)
- \frac{1}{2m} \left( \frac{1}{n+1} \partial_i \psi^\dagger \left( \frac{1}{n+1} \partial_i \psi + i e A_i \psi \right) + \lambda (\psi^\dagger \psi)^2 \right) + S_{cs}$$

(95)

where, here, the covariant derivative is defined as

$$D'_\mu \psi = \partial_\mu \psi + i e A_\mu \psi \ , \ (\mu = 0, 1, 2) ,$$

(96)

the $S'_{cs}$ is

$$S'_{cs} = \frac{\kappa_1}{4} \int d^3 x e^{\mu \nu \alpha} A_\mu F_{\nu \alpha} = \kappa_1 \int d^3 x (A_0 F_{12} + A_2 \partial_0 A_1) ,$$

(97)

and the coupling constants $e_1$, $\kappa_1$ and $\lambda_1$ are

$$e_1 = e(n + 1), \quad \kappa_1 = \kappa(n + 1), \quad \lambda_1 = \lambda(n + 1)$$

(98)

Thus, the action (95) is identical the to nonrelativistic Chern-Simons action (92) and therefore satisfied the same field equations, which lead us to following solution [6, 7],

$$A(x, t) = \frac{e_1}{\kappa_1} \nabla \times \int d^2 r' G(r' - r) \psi^\dagger(t, r') \psi(t, r')$$

(99)
where $G(r)$ is the Green’s function for the Laplacian in two dimensions,

$$G(r) = \frac{1}{2\pi \ln |r|} \quad (100)$$

Consider, now, the variation of the fields $\psi$ and $\psi^\dagger$ under time and space translation, rotations, Galilean boost and the $U(1)$ transformation. From reference [6, 7] it is well known that the infinitesimal gauge covariant time-translation on the fields of the action (92) are

$$\delta \phi = aD_0 \phi, \quad \delta A_0 = 0, \quad \delta A_i = aE_i = a(\partial_0 A_i - \partial_i A_0) \quad (101)$$

Under space translation and rotations we have respectively,

$$\delta \phi = a_i D_i \phi, \quad \delta A_0 = a_i E_i, \quad \delta A_i = \epsilon^{ij} a_j B = \epsilon^{ij} a_j (\partial_1 A_2 - \partial_2 A_1) \quad (102)$$

Finally under an infinitesimal Galilean boost the fields transform as,

$$\delta \phi = (imv_i x_i - tv_i D_i) \phi, \quad \delta A_0 = tv_i E_i, \quad \delta A_i = \epsilon^{ij} v_j B t \quad (103)$$

In addition we know that the conserved charges associated to these symmetries are

$$H_{NR} = \int d^2 x \left( \frac{1}{2m} |D_i \phi|^2 - \lambda |\phi|^4 \right), \quad (105)$$

$$P_{NR} = \int d^2 x \, \mathcal{P}_i = \frac{i}{2} \int d^2 x \left( \phi^\dagger D_i \phi - D_i \phi^\dagger \phi \right), \quad (106)$$

$$J_{NR} = \int d^2 x \left( - \mathcal{P}_1 x_2 + \mathcal{P}_2 x_1 \right), \quad (107)$$

$$G_i = \int d^2 x \left( \mathcal{P}_i t - mx_i \rho \right) \quad (108)$$

The Galilean group is completed with the inclusion of the mass operator

$$M = mN = m \int d^2 x \rho \quad (109)$$

The conservation of $M$ arise as a consequence of a $U(1)$ symmetry

$$\delta \phi = i\alpha \phi \quad (110)$$

Thus, from (101), (102), (103), (104) and (110) we may construct the variations of the field $\psi$,

$$\delta t \psi = \delta \phi^{n+1} = (n+1)\phi^n \delta \phi = (n+1)\phi^n aD_0 \phi$$

$$= (n+1)a(\frac{1}{n+1} \delta_{D_0} \phi^{n+1} + i e A_0 \phi^{n+1}) = a(\partial_0 \psi + i e A_0 \psi) = aD_0' \psi \quad (111)$$
\[ \delta_s \psi = (n + 1) \phi^n a_i D_i \phi = a_i (\partial_i \psi + i e_1 A_i \psi) = a_i D'_i \psi \] (112)

\[ \delta_r \psi = \theta (x_1 D'_2 \psi - x_2 D'_1 \psi) \] (113)

\[ \delta_g \psi = (i m (n + 1) v_i x_i - t v_i D'_i) \psi \] (114)

\[ \delta_u(1) \psi = -i \alpha (n + 1) \psi \] (115)

where, the subindex \( t \), \( s \), \( r \), \( g \) and \( u(1) \) indicate the variation with respect to time translation, space translation, rotation, Galilean boost and \( U(1) \) transformation respectively.

Thus, starting from the action (89) we have arrived to the action (95) which is identical to nonrelativistic Chern-Simons action (92). In addition the variations of the field \( \psi \), (111), (112), (113), (114) and (115) are identical to those of the nonrelativistic Chern-Simons model. Therefore, our model (89) admits Galilean invariance. On the other hand it is not difficult to conclude that conserved charges associated to the Galilean invariance of the action (95) will be the same as those related to the Chern-Simons model (92) and therefore realize the algebra of the Galilean group (19).

In summary we have proposed a generalization of the nonrelativistic Schrödinger-Higgs model. We have shown that this generalized model admits Galilean invariance. In addition we show the Galilean invariance of a generalization of the Jackiw-Pi model.

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