Semilinear elliptic equations with measure data and quasi-regular Dirichlet forms

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Abstract

We study equations of the form \(-Lu = f(x,u) + \mu\), where the operator \(L\) is associated with a quasi-regular possibly non-symmetric Dirichlet form, \(f\) satisfies the monotonicity condition and mild integrability conditions, and \(\mu\) is a bounded smooth measure. We prove general results on existence, uniqueness and regularity of probabilistic solutions, which are expressed in terms of solutions to backward stochastic differential equations. Applications include equations with non-symmetric divergence form operators, with gradient perturbations of some pseudodifferential operators and equations with Ornstein-Uhlenbeck type operators in Hilbert spaces.

1 Introduction

Let \(E\) be a metrizable Lusin space, \(m\) be a positive \(\sigma\)-finite measure on \(\mathcal{B}(E)\) and let \((\mathcal{E}, D(\mathcal{E}))\) be a quasi-regular possibly non-symmetric Dirichlet form on \(L^2(E; m)\). In the present paper we study existence, uniqueness and regularity of solutions of semilinear equations of the form

\[-Lu = f(x,u) + \mu.\]  \hfill (1.1)

Here \(f : E \times \mathbb{R} \to \mathbb{R}\) is a measurable function, \(\mu\) is a smooth signed measure on \(\mathcal{B}(E)\) with respect to the capacity determined by \(\mathcal{E}\) and \(L\) is the operator associated with the form \(\mathcal{E}\), i.e. the unique linear operator on \(L^2(E; m)\) such that

\[D(L) \subset D(\mathcal{E}), \quad (-Lu, v) = \mathcal{E}(u,v), \quad u \in D(L), \ v \in D(\mathcal{E}).\]  \hfill (1.2)

We assume that \(f\) satisfies the monotonicity condition and mild integrability conditions (even weaker than the integrability conditions considered earlier in [1]). As for \(\mu\) we assume that it belongs to the class

\[\mathcal{R} = \{\mu : |\mu| \text{ is smooth and } \hat{G}\phi \cdot \mu \in \mathcal{M}_{0,b} \text{ for some } \phi \in L^1(E; m) \text{ such that } \phi > 0 \text{ m-a.e.}\},\]

where \(|\mu|\) denotes the variation of \(\mu\), \(\mathcal{M}_{0,b}\) is a space of all finite smooth signed measures and \(\hat{G}\) is the co-potential operator associated with \(\mathcal{E}\). In the important case where \(\mathcal{E}\) is transient the class \(\mathcal{R}\) includes \(\mathcal{M}_{0,b}\) but it may happen that \(\mathcal{R}\) also includes some Radon measures of infinite total variation.

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The paper continuous research begun in our paper [12] in which equations of the form (1.1) with $L$ associated with symmetric regular Dirichlet form are studied. The main motivation for writing this paper is to extend results of [12] to encompass equations with non-symmetric operators and equations in infinite dimensions.

As in [12] by a solution of (1.1) we mean a quasi-continuous function $u : E \to \mathbb{R}$ satisfying for quasi-every $x \in E$ the nonlinear Feynman-Kac formula

$$u(x) = E_x \int_0^\zeta f(X_t, u(X_t)) \, dt + E_x \int_0^\zeta dA^\mu_t,$$

(1.3)

where $X = (X, P_x)$ is a Markov process with life-time $\zeta$ associated with the form $\mathcal{E}$, $E_x$ is the expectation with respect to $P_x$ and $A^\mu$ is the additive functional of $X$ corresponding to $\mu$ in the Revuz sense. We show that in the case where $\mathcal{E}$ is transient the solution may be defined in purely analytic way similar to the Stampacchia’s way of defining solutions by duality. Namely, a solution of (1.1) can be defined equivalently as a quasi-continuous function $u$ such that $|\langle \nu, u \rangle| = |\int_E u \, d\nu| < \infty$ for every $\nu$ in the set $S_{00}^{(0)}$ of smooth measures of 0-order energy integral such that $\|U_\nu\|_\infty < \infty$ and

$$\langle \nu, u \rangle = (f(\cdot, u), \bar{U}_\nu) + \langle \mu, \widetilde{\bar{U}_\nu} \rangle, \quad \nu \in S_{00}^{(0)},$$

where $(\cdot, \cdot)$ is the usual scalar product in $L^2(E; m)$, $\bar{U}_\nu$ is the 0-order co-potential of $\nu$ and $\widetilde{\bar{U}_\nu}$ denotes its quasi-continuous version. In the paper we work exclusively with probabilistic definition (1.3) because in our opinion it is simpler and more natural than the definition by duality, and what is even more important, allows us to use directly powerful methods of the theory of Dirichlet forms and Markov processes.

In the paper in Section 3 we prove existence and uniqueness of probabilistic solutions of (3.3) and then in Section 4 we study additional regularity properties of the solutions. Our main result says that under mild assumptions on $f$, $f(\cdot, u) \in L^1(E; m)$ and for every $k > 0$ the truncation $T_k u \equiv (-k) \vee u \wedge k$ belongs to the extended Dirichlet space $\mathcal{F}_e$ of $\mathcal{E}$. Moreover,

$$\mathcal{E}(T_k u, T_k u) \leq k(\|f(\cdot, 0)\|_{L^1(E; m)} + 2\|\mu\|_{TV}),$$

(1.4)

where $\|\mu\|_{TV}$ stands for the total variation norm of $\mu$.

The results of Sections 3 and 4 rely on [12]. Some of the proofs given there more or less closely parallels those of symmetric regular Dirichlet forms. There are, however, sometimes subtle adjustments necessary to fit argument to each new situation. For instance this pertains to the proof of (1.4) and the proofs of some auxiliary but important results on smooth measures and their associated additive functionals. These auxiliary results are proved by using the so-called transfer method. Let us also mention that in Section 2 we prove that if $(\mathcal{E}, D(\mathcal{E}))$ is transient then each smooth bounded measure $\mu$ on $\mathcal{B}(E)$ admits decomposition of the form

$$\mu = \nu + f \cdot m,$$

where $\nu$ is a difference of two measures of 0-order potential and $f \in L^1(E; m)$. This characterization of bounded smooth measures is new even for symmetric regular forms. It generalizes considerably the decomposition proved in [2] for the form associated with the Laplace operator on $D \subset \mathbb{R}^d$. 

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Perhaps the most important part of the paper is Section 5 in which some applications of general results of Sections 2–4 are indicated. We decided to describe in some detail four quite different examples. In the first one we consider equation (1.1) with \(L\) being a non-symmetric divergence form operator that is operator associated with local non-symmetric regular form. In the second example \(L\) is a "divergent free" gradient perturbation of symmetric nonlocal operator on \(\mathbb{R}^d\) whose model example is the \(\alpha\)-laplacian. In that case \(L\) corresponds to a non-symmetric non-local regular form. Then we consider a symmetric non-local operator on some finely open subset \(D \subset \mathbb{R}^d\), which is associated with a symmetric but in general non-regular form. In the last example we consider the Ornstein-Uhlenbeck operator in Hilbert space that is operator associated with a local non-regular form. In each case we formulate specific theorem on existence, uniqueness and regularity of solutions. To our knowledge all these results are new. Finally, at the end of the section we briefly discuss the possibility of other applications of our general results.

2 Preliminaries

Unless otherwise stated, in all the paper we assume that \(E\) is a metrizable Lusin space, i.e. a metrizable space which is the image of a Polish space under a continuous bijective mapping. We adjoin an extra point \(\partial\) to \(E\) as an isolated point. We define the Borel \(\sigma\)-algebra on \(E_\partial \equiv E \cup \{\partial\}\) by putting \(\mathcal{B}(E_\partial) = \mathcal{B}(E) \cup \{B \cup \{\partial\} : B \in \mathcal{B}(E)\}\). We make the convention that any function \(f : E \to \mathbb{R}\) is extended to \(E_\partial\) by putting \(f(\partial) = 0\).

Throughout the paper \(m\) is a \(\sigma\)-finite positive measure on \(\mathcal{B}(E)\). We extend it to \(\mathcal{B}(E_\partial)\) by putting \(m(\{\partial\}) = 0\).

2.1 Quasi-regular Dirichlet forms

**Definition.** A pair \((\mathcal{E}, D(\mathcal{E}))\), where \(D(\mathcal{E})\) is a dense linear subspace of \(L^2(E;m)\) and \(\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E})\) is a bilinear form is called a coercive closed form on \(L^2(E;m)\) if

(a) \((\tilde{\mathcal{E}}, D(\mathcal{E}))\) is positive definite (i.e. \(\tilde{\mathcal{E}}(u,u) \geq 0\) for \(u \in D(\mathcal{E})\)) and closed, where \(\tilde{\mathcal{E}}(u,v) = \frac{1}{2}(\mathcal{E}(u,v) + \mathcal{E}(v,u))\),

(b) \((\mathcal{E}, D(\mathcal{E}))\) satisfies the weak sector condition, i.e. there is \(K > 0\) such that

\[
|\mathcal{E}_1(u,v)| \leq K \mathcal{E}_1(u,u)^{1/2}\mathcal{E}_1(v,v)^{1/2}, \quad u,v \in D(\mathcal{E}).
\]

Here and henceforth, \(\mathcal{E}_\alpha(u,u) = \mathcal{E}(u,u) + \alpha(u,u)\) for \(\alpha \geq 0\). Note that if \((\mathcal{E}, D(\mathcal{E}))\) is a coercive closed form then \(D(\mathcal{E})\) when equipped with the inner product \(\tilde{\mathcal{E}}_1(u,v)\) is a Hilbert space. If

\[
|\mathcal{E}(u,v)| \leq K \mathcal{E}(u,u)^{1/2}\mathcal{E}(v,v)^{1/2}, \quad u,v \in D(\mathcal{E})
\]

for some \(K > 0\) then we say that \((\mathcal{E}, D(\mathcal{E}))\) satisfies the strong sector condition.

By [16, Theorem 1.2.8] every coercive closed form on \(L^2(E;m)\) determines uniquely strongly continuous contraction resolvents \((G_\alpha)_{\alpha > 0}\), \((\hat{G}_\alpha)_{\alpha > 0}\) on \(L^2(E;m)\) such that \(G_\alpha(L^2(E;m)) \subset D(\mathcal{E})\), \(\hat{G}_\alpha(L^2(E;m)) \subset D(\mathcal{E})\) and

\[
\mathcal{E}_\alpha(G_\alpha f,u) = (f,u) = \mathcal{E}_\alpha(u,\hat{G}_\alpha f), \quad f \in L^2(E;m), u \in D(\mathcal{E}), \alpha > 0
\]
By \((T_t)_{t \geq 0}\) (resp. \((\tilde{T}_t)_{t \geq 0}\)) we will denote the strongly continuous contraction semigroup on \(L^2(E; m)\) corresponding to \((G_\alpha)_{\alpha \geq 0}\) (resp. \((\tilde{G}_\alpha)_{\alpha \geq 0}\)). Note that \(T_t, G_\alpha\) and \(\tilde{T}_t, \tilde{G}_\alpha\) can be extended to a semigroup and resolvent on \(L^1(E; m)\) (see [17] Section 1.1).

Define
\[
D(L) = \{ u \in D(\mathcal{E}) : v \mapsto \mathcal{E}(u, v) \text{ is continuous w.r.t. } (-\cdot, \cdot)^{1/2} \text{ on } D(\mathcal{E}) \},
\]
(2.2)
where \((-\cdot, \cdot)\) is the inner product in \(L^2(E; m)\). For \(u \in D(L)\) let \(Lu\) denote the unique element in \(L^2(E; m)\) such that
\[
(-Lu, v) = \mathcal{E}(u, v), \quad v \in D(\mathcal{E}).
\]
(2.3)

By [16], Theorem I.2.16, \(L\) is the generator of \((G_\alpha)_{\alpha \geq 0}\) (and \((T_t)_{t \geq 0}\)) and in fact the generator \(L\) of \((G_\alpha)_{\alpha \geq 0}\) can be characterized as the unique operator on \(L^2(E; m)\) such that (1.2) is satisfied.

**Definition.** A coercive closed form \((\mathcal{E}, D(\mathcal{E}))\) on \(L^2(E; m)\) is called a semi-Dirichlet form if it has the following contraction property: for every \(u \in D(\mathcal{E})\), \(u^+ \wedge 1 \in D(\mathcal{E})\) and
\[
\mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0.
\]

If, in addition, \(\mathcal{E}(u - u^+ \wedge 1, u + u^+ \wedge 1) \geq 0\), then \((\mathcal{E}, D(\mathcal{E}))\) is called a Dirichlet form.

Let \((\mathcal{E}, D(\mathcal{E}))\) be a semi-Dirichlet form on \(L^2(E; m)\). For a closed subset \(F \subset E\) we set
\[
D(\mathcal{E})_F = \{ u \in D(\mathcal{E}) : u = 0 \text{ m-a.e. on } E \setminus F \}.
\]

**Definition.** (a) An increasing sequence \(\{F_k\}_{k \geq 1}\) of closed subsets of \(E\) is called an \(\mathcal{E}\)-nest if \(\bigcup_{k=1}^\infty D(\mathcal{E})_F\) is \(\tilde{\mathcal{E}}^{1/2}\)-dense in \(D(\mathcal{E})\).

(b) A subset \(N \subset E\) is called \(\mathcal{E}\)-exceptional if \(N \subset \bigcap_{k=1}^\infty F_k^c\) for some \(\mathcal{E}\)-nest \(\{F_k\}_{k \in \mathbb{N}}\).

In what follows we say that a property of points in \(E\) holds \(\mathcal{E}\)-quasi-everywhere (\(\mathcal{E}\)-q.e. for short) if the property holds outside some \(\mathcal{E}\)-exceptional set.

(c) An \(\mathcal{E}\)-q.e. defined function \(u\) is called \(\mathcal{E}\)-quasi-continuous if there exists a nest \(\{F_k\}_{k \in \mathbb{N}}\) such that \(f \in C(\{F_k\})\), where
\[
C(\{F_k\}) = \{ f : A \to \mathbb{R} : \bigcup_{k=1}^\infty F_k \subset A \subset E, f|_{F_k} \text{ is continuous for every } k \in \mathbb{N} \}.
\]

The notions of \(\mathcal{E}\)-nest and \(\mathcal{E}\)-exceptional set can be characterized by certain capacities relative to \((\mathcal{E}, D(\mathcal{E}))\). To formulate this characterization let us fix \(\varphi \in L^2(E; m)\) such that \(0 < \varphi \leq 1\) m-a.e. and for open \(U \subset E\) set
\[
\text{Cap}_{\varphi}(U) = \inf\{ \mathcal{E}_1(u, u) : u \in D(\mathcal{E}), u \geq \tilde{G}_1 \varphi, \text{ m-a.e. on } U \},
\]
where \(\{\tilde{G}_\alpha\}\) is the resolvent associated with \((\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))\). For arbitrary \(A \subset E\) we set
\[
\text{Cap}_{\varphi}(A) = \inf\{ \text{Cap}_{\varphi}(U) : A \subset U \subset E, U \text{ open} \}.
\]
(2.4)

Then by [16] Theorem III.2.11] an increasing sequence \(\{F_k\}_{k \geq 1}\) of closed subsets of \(E\) is an \(\mathcal{E}\)-nest iff \(\lim_{k \to \infty} \text{Cap}_{\varphi}(E \setminus F_k) = 0\), and secondly, \(N \subset E\) is \(\mathcal{E}\)-exceptional iff \(\text{Cap}_{\varphi}(N) = 0\). Notice that from the above it follows in particular that the capacities \(\text{Cap}_{\varphi}\) defined for different \(\varphi \in L^2(E; m)\) such that \(0 < \varphi \leq 1\) m-a.e. are equivalent to each other.
**Definition.** A semi-Dirichlet form \((\mathcal{E}, D(\mathcal{E}))\) on \(L^2(E; m)\) is called quasi-regular if

(a) There exists an \(\mathcal{E}\)-nest consisting of compacts sets.

(b) There exists an \(\mathcal{E}^{1/2}\)-dense subset of \(D(\mathcal{E})\) whose elements have \(\mathcal{E}\)-quasi-continuous \(m\)-versions.

(c) There exists an \(\mathcal{E}\)-exceptional set \(N \subset E\) and \(\{u_n\}_{n \in \mathbb{N}} \subset D(\mathcal{E})\) such that each \(u_n\) has an \(\mathcal{E}\)-quasi-continuous \(m\)-version \(\tilde{u}_n\) and the family \(\{\tilde{u}_n\}_{n \in \mathbb{N}}\) separates the points of \(E \setminus N\).

The notion of quasi-regular semi-Dirichlet form includes the notion of regular semi-Dirichlet form, because from [16, Section IV.4(a)] it follows that if \(E\) is a locally compact separable metric space and \(m\) is a positive Radon measure on \(\mathcal{B}(E)\) then any regular semi-Dirichlet form on \(L^2(E; m)\) is quasi-regular. Important examples of quasi-regular semi-Dirichlet forms which are not regular are given for instance in [16, 18, 19]. One of such examples is considered in Section 5.

**Definition.** A Dirichlet form \((\mathcal{E}, D(\mathcal{E}))\) is called transient if there is an \(m\)-a.e. strictly positive and bounded \(g \in L^1(E; m)\) such that

\[
\int_E |u|g \, dm \leq \mathcal{E}(u, u)^{1/2}, \quad u \in D(\mathcal{E}).
\]  

(2.5)

Notice that transience of a Dirichlet form depends only on its symmetric part. It is known (see [10, Corollary 3.5.34]) that \((\mathcal{E}, D(\mathcal{E}))\) is transient iff the corresponding sub-Markovian semigroup \((T_t)_{t \geq 0}\) is transient, i.e. for all \(u \in L^1(E; m)\) such that \(u \geq 0\) \(m\)-a.e.,

\[
\lim_{N \to \infty} \int_0^N T_t u \, dt < \infty, \quad m\text{-a.e.}
\]

**Definition.** Let \((\mathcal{E}, D(\mathcal{E}))\) be a Dirichlet form. The extended Dirichlet space \(\mathcal{F}_e\) associated with the symmetric Dirichlet form \((\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))\) is the family of measurable functions \(u : E \to \mathbb{R}\) such that \(|u| < \infty\) \(m\)-a.e. and there exists an \(\tilde{\mathcal{E}}\)-Cauchy sequence \(\{u_n\} \subset D(\tilde{\mathcal{E}})\) such that \(u_n \to u\) \(m\)-a.e. The sequence \(\{u_n\}\) is called an approximating sequence for \(u \in \mathcal{F}_e\).

For a Dirichlet form \((\mathcal{E}, D(\mathcal{E}))\) and \(u \in \mathcal{F}_e\) we set \(\mathcal{E}(u, u) = \lim_{n \to \infty} \mathcal{E}(u_n, u_n)\), where \(\{u_n\}\) is an approximating sequence for \(u\) (see [7, Theorem 1.5.2]). If moreover \(\mathcal{E}\) satisfies the strong sector condition then we may extend \(\mathcal{E}\) to \(\tilde{\mathcal{F}}_e\) by putting \(\mathcal{E}(u, v) = \lim_{n \to \infty} \mathcal{E}(u_n, v_n)\) with approximating sequences \(\{u_n\}\) and \(\{v_n\}\) for \(u \in \mathcal{F}_e\) and \(v \in \tilde{\mathcal{F}}_e\), respectively (it is easily seen that \(\mathcal{E}(u, v)\) is independent of the choice of the approximating sequences). Observe that this extension satisfies the strong sector condition, i.e. (2.1) holds true for all \(u, v \in \tilde{\mathcal{F}}_e\).

If \((\mathcal{E}, D(\mathcal{E}))\) is transient then by [7, Lemma 1.5.5], \((\mathcal{F}_e, \tilde{\mathcal{E}})\) is a Hilbert space. Note also that if \((\mathcal{E}, D(\mathcal{E}))\) is a quasi-regular Dirichlet form then by [16, Proposition IV.3.3] each element \(u \in D(\mathcal{E})\) admits a quasi-continuous \(m\)-version denoted by \(\tilde{u}\), and that \(\tilde{u}\) is \(\mathcal{E}\)-q.e. unique for every \(u \in D(\mathcal{E})\). If moreover \((\mathcal{E}, D(\mathcal{E}))\) is transient then the last statement holds true for \(D(\mathcal{E})\) replaced by \(\mathcal{F}_e\) (see Remark [2.2]).

In the remainder of this section we assume that \((\mathcal{E}, D(\mathcal{E}))\) is a quasi-regular Dirichlet form on \(L^2(E; m)\).
2.2 Markov processes associated with Dirichlet forms

By [16, Theorem IV.3.5] there exists an $m$-tight special standard Markov process $X = (\Omega, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, \zeta, (P_x)_{x \in \text{EV}(\partial)})$ with state space $E$, life-time $\zeta$ and cemetery state $\partial$ (see, e.g., [14] or [16, Section IV.1] for precise definitions) which is properly associated with $(\mathcal{E}, D(\mathcal{E}))$. Let $(p_t)_{t \geq 0}$ be the transition semigroup of $X$ defined as

$$p_t f(x) = E_x f(X_t), \quad x \in E, \quad t \geq 0, \quad f \in \mathcal{B}^+(E).$$

The statement that $X$ is properly associated with $(\mathcal{E}, D(\mathcal{E}))$ means that $p_t f$ is a quasi-continuous $m$-version of $T_t f$ for every $t > 0$ and $f \in \mathcal{B}_b \cap L^2(E; m)$ (and hence for every $t > 0$ and $f \in L^2(E; m)$ by [16, Exercise IV.2.9]). Equivalently, by [16, Proposition IV.2.8], the proper association means that $R_\alpha f$ is an $\mathcal{E}$-quasi-continuous $m$-version of $G_\alpha f$ for every $\alpha > 0$ and $f \in \mathcal{B}_b \cap L^2(E; m)$, where $(R_\alpha)_{\alpha > 0}$ is the resolvent of $X$, i.e.

$$R_\alpha f(x) = E_x \int_0^\infty e^{-\alpha t} f(X_t) \, dt, \quad x \in E, \quad \alpha > 0, \quad f \in \mathcal{B}^+(E).$$

By [16, Theorem IV.6.4] the process $X$ is uniquely determined by $(\mathcal{E}, D(\mathcal{E}))$ in the sense that if $X'$ is another process with state space $E$ properly associated with $(\mathcal{E}, D(\mathcal{E}))$ then $X$ and $X'$ are $m$-equivalent, i.e. there is $S \in \mathcal{B}(E)$ such that $m(E \setminus S) = 0$, $S$ is both $X$-invariant and $X'$-invariant, and $p_t f(x) = p'_t f(x)$ for all $x \in S$, $f \in \mathcal{B}_b(E)$, $t > 0$, where $(p'_t)_{t > 0}$ is the transition semigroup of $X'$.

2.3 Smooth measures

Definition. (a) A positive measure $\mu$ on $\mathcal{B}(E)$ is said to be $\mathcal{E}$-smooth ($\mu \in S$ in notation) if $\mu(B) = 0$ for all $\mathcal{E}$-exceptional sets $B \in \mathcal{B}(E)$ and there exists an $\mathcal{E}$-nest $\{F_k\}_{k \in \mathbb{N}}$ of compact sets such that $\mu(F_k) < \infty$ for $k \in \mathbb{N}$.

(b) $\mu \in S$ is said to be of finite energy integral ($\mu \in S_0$ in notation) if there is $c > 0$ such that

$$\int_E |\tilde{v}(x)| \, \mu(dx) \leq c \mathcal{E}_1(v, v)^{1/2}, \quad v \in D(\mathcal{E}). \quad (2.6)$$

(c) Assume additionally that $(\mathcal{E}, D(\mathcal{E}))$ is transient. Then $\mu \in S$ is said to be of finite $0$-order energy integral ($\mu \in S_0^{(0)}$ in notation) if there is $c > 0$ such that

$$\int_E |\tilde{v}(x)| \, \mu(dx) \leq c \mathcal{E}(v, v)^{1/2}, \quad v \in \mathcal{F}_e.$$

If $(\mathcal{E}, D(\mathcal{E}))$ is regular and $E$ is a locally compact separable metric space then the notion of smooth measures defined above coincides with that in [7]. Moreover, if $\mu$ is a positive Radon measure on $E$ such that (2.6) is satisfied for all $v \in C_0(E) \cap D(\mathcal{E})$ then $\mu$ charges no $\mathcal{E}$-exceptional set (see [15, Remark A.2]) and hence $\mu \in S_0$.

By [14, Proposition 2.18(ii)] (or [16, Proposition III.3.6]) the reference measure $m$ is $\mathcal{E}$-smooth. Therefore if $f \in L^1(E; m)$ then $\mu = f \cdot m$ is bounded and smooth. A general result on the structure of bounded smooth measures will be stated in Theorem 2.4.
Let \( \mu \in S_0 \) and \( \alpha > 0 \). Then from the Lax-Milgram theorem (see, e.g., [3] Theorem 2.7.41) it follows that there exist unique elements \( U_\alpha \mu, \hat{U}_\alpha \mu \in D(\mathcal{E}) \) such that

\[
\mathcal{E}_\alpha(U_\alpha \mu, v) = \int_E \tilde{v}(x) \mu(dx) = \mathcal{E}_\alpha(v, \hat{U}_\alpha \mu), \quad v \in D(\mathcal{E}).
\]

Similarly, if \((\mathcal{E}, D(\mathcal{E}))\) satisfies the strong sector condition and \( \mu \in S_0^{(0)} \) then from the Lax-Milgram theorem applied to the Hilbert space \((U, \mathcal{L})\), it follows that there exist unique elements \( U\mu, \hat{U}\mu \in \mathcal{F}_e \) such that

\[
\mathcal{E}(U\mu, v) = \int_E \tilde{v}(x) \mu(dx) = \mathcal{E}(v, \hat{U}\mu), \quad v \in \mathcal{F}_e.
\]

Let \( \mathcal{M}_{0,b} \) denote the subset of \( S_0 \) consisting of all measures \( \mu \) such that \( \|\mu\|_{TV} \leq \infty \), where \( \|\mu\|_{TV} \) denotes the total variation of \( \mu \), and let \( \mathcal{M}_{0,b}^+ \) denote the subset of \( \mathcal{M}_{0,b} \) consisting of all positive measures. A useful characterization of the class \( \mathcal{M}_{0,b} \) provides Theorem [2.3] below. It generalizes [3] Theorem 2.1. To prove the theorem we will need the following lemma.

**Lemma 2.1.** Assume that \((\mathcal{E}, D(\mathcal{E}))\) is transient. If \( \mu \in S \) then there exists a nest \( \{F_n\} \) such that \( 1_{F_n} \cdot \mu \in S_0^{(0)} \) for each \( n \in \mathbb{N} \).

**Proof.** Let \((\mathcal{E}^\#, D(\mathcal{E}^\#))\) denote the regular extension of \((\mathcal{E}, D(\mathcal{E}))\) specified by [16] Theorem VI.1.2] and let \( i : E \to E^\# \) denote the inclusion map. Then \((\mathcal{E}^\#, D(\mathcal{E}^\#))\) is transient and by Lemma IV.4.5 and Corollary VI.1.4 in [16], \( \mu^\# = \mu \circ i^{-1} \) is a smooth measure on \( \mathcal{B}(E^\#) \). Therefore by the 0-order version of [7] Theorem 2.2.4] (see remark following [7] Corollary 2.2.2]) there exists an \( \mathcal{E}^\# \)-nest \( \{\mathcal{F}_k\} \) on \( E^\# \) such that

\[
\mu_k^\# = 1_{F_k} \cdot \mu^\# \in S_0^{(0)}(\mathcal{E}^\#), \quad k \geq 1.
\]

Let \( \{E_k\} \) be an \( \mathcal{E} \)-nest of \([16] \) Theorem VI.1.2] and let \( F_k' = F_k \cap E_k, k \in \mathbb{N} \). By [16] Corollary VI.1.4, \( \{F_k'\} \) is an \( \mathcal{E} \)-nest on \( E \). Put \( \mu_k = 1_{F_k'} \cdot \mu \). We are going to show that \( \mu_k \in S_0^{(0)} \), i.e. for any nonnegative \( u \in \mathcal{F}_e \),

\[
\langle \mu_k, \tilde{u} \rangle \leq c\mathcal{E}(u, u)^{1/2}
\]

for some \( c > 0 \). To this end, let us consider an approximating sequence \( \{u_n\} \) for \( u \) and extend \( u, u_n \) to functions on \( E^\# \) by putting \( u(x) = u_n(x) = 0 \) for \( x \in E^\# \setminus E \). Then

\[
\mathcal{E}^\#(u_n - u_l, u_n - u_l) = \mathcal{E}(u_n - u_l, u_n - u_l) \quad \text{for} \ n, l \in \mathbb{N}, \ \text{so} \ \{u_n^\#\} \ \text{is} \ \mathcal{E}^\# \text{-Cauchy sequence. Moreover, since} \ m^\#(E^\# \setminus E) = 0, \ u_n^\# \to u^\# \ \text{m^\#-a.e. Consequently,} \ \{u_n^\#\} \ \text{is an} \ \mathcal{E}^\# \text{-approximating sequence for} \ u^\#. \ \text{It follows that} \ u^\# \ \text{belongs to the extended} \ \text{space} \ \mathcal{F}_e^\# \ \text{for} \ \mathcal{E}^\# \ \text{and}
\]

\[
\mathcal{E}(u, u) = \mathcal{E}^\#(u^\#, u^\#).
\]

Since \( \tilde{u}^\# \mid_E \) is an \( m \)-version of \( u \) and by [16] Corollary VI.1.4] the function \( \tilde{\mu}^\# \mid_E \) is \( \mathcal{E} \)-quasi-continuous, \( \tilde{u} = u^\# \mid_E \mathcal{E}-\text{q.e. From this and the fact that} \ \mu_k^\# = \mu_k \ \text{on} \ E \ \text{it follows that}
\]

\[
\langle \mu_k, \tilde{u} \rangle = \langle \mu_k, \tilde{u}^\# \mid_E \rangle = \langle \mu_k^\#, \tilde{u}^\# \rangle \leq c\mathcal{E}^\#(u^\#, u^\#)^{1/2},
\]

which gives (2.7). \( \square \)
Remark 2.2. Note that the argument following (2.7) show that each \( u \in \mathcal{F}_e \) admits an \( \mathcal{E} \)-quasi-continuous modification.

In what follows given \( \nu \in S^{(0)}_0 - S^{(0)}_0 \) we denote by \( T_\nu \) the bounded linear operator on \( \mathcal{F}_e \) defined as

\[
T_\nu(u) = \langle \nu, \tilde{u} \rangle = \tilde{\mathcal{E}}(\tilde{U} \nu, u), \quad u \in \mathcal{F}_e,
\]

where \( \tilde{U} \nu \) is the potential of \( \nu \) associated with the form \( \tilde{\mathcal{E}} \) and \( \tilde{u} \) is an \( \tilde{\mathcal{E}} \)-quasi-continuous \( m \)-version of \( u \).

**Theorem 2.3.** Assume that \( (\mathcal{E}, D(\mathcal{E})) \) is transient. If \( \mu \in \mathcal{M}_{0,b} \) then there exist \( \nu \in S^{(0)}_0 - S^{(0)}_0 \) and \( f \in L^1(E; m) \) such that

\[
\mu = \nu + f \cdot m,
\]

i.e. for every bounded \( u \in \mathcal{F}_e \),

\[
\langle \mu, \tilde{u} \rangle = \langle \nu, \tilde{u} \rangle + \langle u, f \rangle = \tilde{\mathcal{E}}(\tilde{U} \nu, u) + \langle f, u \rangle.
\]

**Proof.** Without loss of generality we may assume that \( \mu \) is positive. By Lemma 2.1 there exists a nest \( \{F_n\} \) such that \( 1_{F_n} \cdot \mu \in S^{(0)}_0 \). Clearly \( \mu_n = 1_{F_{n+1}} \cdot 1_{F_n} \cdot \mu \in S^{(0)}_0 \) and, since \( (\bigcup_{n=1}^{\infty} F_n)^c \) is exceptional and \( \mu \) is smooth, \( \mu = \sum_{n=1}^{\infty} \mu_n \). For \( \alpha > 0 \) set \( \mu^\alpha_n = \alpha \tilde{R}_\alpha \mu_n \cdot m \), where \( \{\tilde{R}_\alpha\}_{\alpha > 0} \) is the resolvent of the process associated with \( (\tilde{\mathcal{E}}, D(\mathcal{E})) \). Then \( \mu^\alpha_n \in S \) since \( \alpha \tilde{R}_\alpha \mu_n \in D(\mathcal{E}) \) and \( m \in S \). By [15, Theorem A.8],

\[
\tilde{R}_\alpha \mu_n(x) = E_x \int_0^{\infty} e^{-\alpha t} dA_t^\mu_n
\]

for q.e. \( x \in E \). Hence, by [15, Theorem A.8(iv)], \( \langle \mu^\alpha_n, u \rangle = \alpha \langle u, \tilde{R}_\alpha \mu_n \rangle = \alpha \langle \mu_n, \tilde{R}_\alpha u \rangle \) for every nonnegative Borel measurable \( u \). From this one can deduce that

\[
\langle \mu^\alpha_n, u \rangle = \langle \mu_n, \alpha \tilde{R}_\alpha u \rangle \quad (2.8)
\]

for \( u \in \mathcal{F} \). Let \( u \in \mathcal{F}_e \) and let \( \{u_k\} \subset D(\mathcal{E}) \) be an approximating sequence for \( u \). Then

\[
\langle \mu^\alpha_n, u_k \rangle = \tilde{\mathcal{E}}(\tilde{U} \mu^\alpha_n, u_k) \leq \mathcal{E}(\tilde{U} \mu_n, \tilde{U} \mu^\alpha_n)^{1/2} \mathcal{E}(u_k, u_k)^{1/2} \quad (2.9)
\]

for every \( k \in \mathbb{N} \), because by (2.8),

\[
\langle \mu^\alpha_n, u_k \rangle = \tilde{\mathcal{E}}(\tilde{U} \mu_n, \alpha \tilde{R}_\alpha u_k) \leq \mathcal{E}(\tilde{U} \mu_n, \tilde{U} \mu_n)^{1/2} \mathcal{E}(\alpha \tilde{R}_\alpha u_k, \alpha \tilde{R}_\alpha u_k)^{1/2}
\]

\[
\leq \mathcal{E}(\tilde{U} \mu_n, \tilde{U} \mu_n)^{1/2} \mathcal{E}(u_k, u_k)^{1/2}.
\]

Letting \( k \to \infty \) in (2.9) we get

\[
\langle \mu^\alpha_n, u \rangle \leq \mathcal{E}(\tilde{U} \mu_n, \tilde{U} \mu_n)^{1/2} \mathcal{E}(u, u)^{1/2}.
\]

Thus \( \mu^\alpha_n \in S^{(0)}_0 \) and for every \( n \in \mathbb{N} \),

\[
\sup_{\alpha > 0} \mathcal{E}(\tilde{U} \mu^\alpha_n, \tilde{U} \mu^\alpha_n)^{1/2} \leq \sup_{\alpha > 0} \|T_{\mu^\alpha_n}\| \leq \mathcal{E}(\tilde{U} \mu_n, \tilde{U} \mu_n)^{1/2} < \infty,
\]

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where \( \|T_{\mu_n}\| \) stands for the operator norm of \( T_{\mu_n} \). By the above and the Banach-Saks theorem, for every \( n \in \mathbb{N} \) we can choose by the diagonal method a sequence \( \{\alpha_l\} \) such that the sequence \( \{\hat{U}_\gamma \mu_n\} \), where

\[
\gamma_k(\mu_n) = f_k(\mu_n) \cdot m,
\]

\[
f_k(\mu_n) = \frac{1}{k} \sum_{l=1}^{k} \alpha_l \hat{R}_\alpha \mu_n,
\]

is \( \hat{E} \)-convergent to some \( g \in \mathcal{F}_e \) as \( k \to \infty \). Equivalently, \( \|T_{\gamma_k(\mu_n)} - T\| \to 0 \) as \( k \to \infty \), where \( T(u) = \hat{E}(g, u) \) for \( u \in \mathcal{F}_e \). On the other hand, by [16 Theorem I.2.13], for any \( u \in \mathcal{F} \),

\[
T_{\mu_n}(u) = \hat{E}(\hat{U}_\mu, \hat{R}_\alpha u) \to \hat{E}(\hat{U}_\mu, u) = T_{\mu_n}(u)
\]
as \( \alpha \to \infty \). It follows that in fact \( T = T_{\mu_n} \). We can therefore find a subsequence \( \{k_n\} \) such that

\[
\|T_{\gamma_{k_n}(\mu_n)} - T_{\mu_n}\| \leq 2^{-n}
\]

(2.10)

for every \( n \in \mathbb{N} \). Set

\[
\nu = \sum_{n=1}^{\infty} (\mu_n - \gamma_{k_n}(\mu_n)), \quad f = \sum_{n=1}^{\infty} f_{k_n}(\mu_n).
\]

Then

\[
\mu = \sum_{n=1}^{\infty} \mu_n = \nu + f \cdot m.
\]

Since \( m \) is \( \sigma \)-finite, there exists a sequence \( \{U_l\} \) of Borel subsets of \( E \) such that

\[
\bigcup_{l=1}^{\infty} U_l = E, \quad U_l \subset U_{l+1} \quad \text{and} \quad m(U_l) < \infty, \quad l \in \mathbb{N}.
\]

Since \( \|\alpha \hat{R}_\alpha\| \leq 1 \), for every \( l \in \mathbb{N} \) we have

\[
(f, 1_{U_l}) \leq \sum_{n=1}^{\infty} (f_{k_n}(\mu_n), 1_{U_l}) \leq \sum_{n=1}^{\infty} (\mu_n, 1_{U_l}) \leq \sum_{n=1}^{\infty} \|\mu_n\|_{TV} = \|\mu\|_{TV}.
\]

Hence \( \|f\|_{L^1(E; m)} < \infty \) by the monotone convergence theorem. In particular, \( \nu = \mu - f \cdot m \in S \). That \( \nu \in S_0^{(0)} \) now follows from (2.10). \( \square \)

Remark 2.4. (i) Let \( D \subset \mathbb{R}^d \) be a bounded domain. Consider the classical form

\[
\mathbb{D}(u, v) = \frac{1}{2} \int_D \langle \nabla u, \nabla v \rangle_{\mathbb{R}^d} \, dx, \quad u, v \in D(\mathcal{E}) = H_0^1(D).
\]

It is known that \( (\mathbb{D}, H_0^1(D)) \) is a transient regular Dirichlet form on \( L^2(D; dx) \) (see [7 Example 1.5.1]). If \( \mu \in \mathcal{M}_b^+ \), then \( \mu \in S \) if \( \mu \) charges no set of Newtonian capacity zero. By Poincaré’s inequality, the norms determined by \( \mathbb{D} \) and \( \mathbb{D}_1 \) are equivalent (of course, the norm determined by \( \mathbb{D}_1 \) is the usual norm in the Sobolev space \( H_0^1(D) \)). As a consequence, \( \mathcal{F}_1 = H_0^1(D) \) and \( S_0 = S_0^{(0)} \). It follows in particular that if \( \nu \in S_0^0 \) and \( \nu \in S_0^{(0)} - S_0^{(0)} \) then \( T_\nu \in H^{-1}(D) \). Therefore the decomposition of Theorem \( \text{[2.3]} \) reduces to the decomposition proved in [2 Theorem 2.1] (with \( p = 2 \)).

(ii) In Theorem \( \text{[2.3]} \) if \( u \in \mathcal{F} \), then \( \langle \nu, \tilde{u} \rangle = \mathcal{E}_1(U_1 \nu, u) = \mathcal{E}_1(u, \hat{U}_1 \nu) \). If \( \mathcal{E} \) satisfies the strong sector condition then \( \langle \nu, \tilde{u} \rangle = \mathcal{E}(U \nu, u) = \mathcal{E}(u, \hat{U} \nu) \) for \( u \in \mathcal{F}_e \).

(iii) From the proof of Theorem \( \text{[2.3]} \) it follow that if \( \mu \in \mathcal{M}_{b,0}^+ \) then the \( L^1 \) part \( f \) of its decomposition is positive. In [2 Remark 2.3] an example is given to show that in general this is not true for \( \nu \).
Lemma 2.5. Assume that \((\mathcal{E}, D(\mathcal{E}))\) is transient and satisfies the strong sector condition. If \(\mu \in \mathcal{S}_0^{(0)}\) then \(\{U_\alpha \mu\}\) is weakly \(\mathcal{E}\)-convergent to \(U_\mu\) as \(\alpha \downarrow 0\).

Proof. Let \(v \in \mathcal{F}_e\) and let \(\{v_k\} \subset D(\mathcal{E})\) be an approximating sequence for \(v\).

\[
\mathcal{E}(U_\mu - U_\alpha \mu, v_k) = \alpha(U_\alpha \mu, v_k), \quad \mathcal{E}(G_0 U_\alpha \mu, v_k) = (U_\alpha \mu, v_k).
\]

Hence \(\mathcal{E}(U_\mu - U_\alpha \mu, v_k) = \mathcal{E}(\alpha G_0 U_\alpha \mu, v_k)\). Letting \(k \to \infty\) yields \(\mathcal{E}(U_\mu - U_\alpha \mu, v) = \mathcal{E}(\alpha G_0 U_\alpha \mu, v)\). Consequently, \(U_\mu - U_\alpha \mu = \alpha G_0 U_\alpha \mu\). In the same manner we can see that \(\tilde{U}_\mu - \tilde{U}_\alpha \mu = \alpha G_0 \tilde{U}_\alpha \mu\). Hence

\[
\mathcal{E}(U_\mu - U_\alpha \mu, \tilde{U}_\mu - \tilde{U}_\alpha \mu) = \alpha^2 \mathcal{E}(G_0 U_\alpha \mu, \tilde{G}_0 \tilde{U}_\alpha \mu) = \alpha^2 (G_0 U_\alpha \mu, \tilde{U}_\alpha \mu) \geq 0.
\]

On the other hand,

\[
\mathcal{E}(U_\mu - U_\alpha \mu, \tilde{U}_\mu - \tilde{U}_\alpha \mu) = \mathcal{E}(U_\mu, \tilde{U}_\mu - \tilde{U}_\alpha \mu) + \mathcal{E}(U_\alpha \mu, \tilde{U}_\mu) - \alpha(U_\alpha \mu, \tilde{U}_\alpha \mu)
\]

\[
\leq \langle \mu, \tilde{U}_\mu \rangle - \langle \mu, \tilde{U}_\alpha \mu \rangle.
\]

Since \(\langle \mu, \tilde{U}_\alpha \mu \rangle = \mathcal{E}(\alpha U_\alpha \mu, \tilde{U}_\alpha \mu) = \langle \mu, \tilde{U}_\alpha \mu \rangle\), it follows from the above that

\[
\mathcal{E}(U_\alpha \mu, U_\alpha \mu) + \alpha(U_\alpha \mu, U_\alpha \mu) = \mathcal{E}(U_\alpha \mu, U_\alpha \mu) = \langle \mu, \tilde{U}_\alpha \mu \rangle \leq \langle \mu, \tilde{U}_\alpha \mu \rangle
\]

for \(\alpha > 0\). Hence \(\{U_\alpha \mu\}_{\alpha > 0}\) is \(\bar{\mathcal{E}}\)-bounded and for each \(k \in \mathbb{N}\), \(\alpha(U_\alpha \mu, v_k) \to 0\) as \(\alpha \downarrow 0\).

Suppose that \(\{U_\alpha \mu\}\) converges \(\bar{\mathcal{E}}\)-weakly to some \(f \in \mathcal{F}_e\) as \(\alpha \downarrow 0\). Since

\[
\mathcal{E}(U_\alpha \mu, v_k) = \langle \mu, \check{v}_k \rangle - \alpha(U_\alpha \mu, v_k),
\]

letting \(\alpha \downarrow 0\) shows that \(\mathcal{E}(f, v_k) = \langle \mu, \check{v} \rangle = \mathcal{E}(U_\mu, v_k)\). Letting \(k \to \infty\) we get \(\mathcal{E}(f, v) = \mathcal{E}(U_\mu, v)\) for \(v \in \mathcal{F}_e\). Thus \(f = U_\mu\), and the proof is complete.

2.4 Smooth measures and additive functionals

Let \(\mathcal{X}\) be the Markov process properly associated with \((\mathcal{E}, D(\mathcal{E}))\). By [16] Theorem VI.2.4 there is a one-to-one correspondence between \(\mathcal{E}\)-smooth measures \(\mu\) on \(\mathcal{B}(E)\) and positive continuous additive functionals (PCAFs) \(A\) of \(\mathcal{X}\). It is given by the relation

\[
\lim_{t \downarrow 0} \frac{1}{t} E_0 \int_0^t f(X_s) \, dA_s = \int_E f \, d\mu, \quad f \in \mathcal{B}^+(E).
\] (2.11)

In what follows the additive functional corresponding to \(\mu\) in the sense of (2.11) will be denoted by \(A^\mu\). In the important case where \(\mu = f \cdot m\) for some \(f \in L^1(E; m)\) the additive functional \(A^\mu\) is given by

\[
A_t^\mu = \int_0^t f(X_s) \, dt, \quad t \geq 0.
\]

The following lemma generalizes [12] Lemma 4.3,
Lemma 2.6. If $A$ is a PCAF of $X$ such that $E_xA_\zeta < \infty$ for m-a.e. $x \in E$ then $u : E \to \overline{\mathbb{R}}$ defined as

$$u(x) = E_xA_\zeta, \quad x \in E$$

is $\mathcal{E}$-quasi-continuous. In particular, $u$ is $\mathcal{E}$-q.e. finite.

Proof. Let $(\mathcal{E}^#, (D(\mathcal{E}^#)))$ denote the regular extension of $(\mathcal{E}, D(\mathcal{E}))$ specified by [16, Theorem VI.1.2]. By [16, Theorem VI.1.2], $X$ can be trivially extended to a Hunt process $X^#$ on $E^#$ properly associated with the form $(\mathcal{E}^#, (D(\mathcal{E}^#)))$. Let us extend $A$ to a PCAF of $X^#$ by putting

$$A^#_t(\omega) = A_t(\omega), \quad t \geq 0, \quad \omega \in \Omega, \quad A^#_0(\omega) = 0, \quad t \geq 0, \quad \omega \in E^# \setminus E. \quad (2.12)$$

By the assumption and the fact that $m^#(E^# \setminus E) = 0$,

$$E^#xA^#_\zeta < \infty, \quad m^#-a.e.$$

Therefore, by [12, Lemma 4.3], the function $u^#(x) = E^#_xA^#_\zeta$ is $\mathcal{E}^#$-quasi-continuous on $E^#$. By [16, Corollary VI.1.4], $u^#_{|E}$ is $\mathcal{E}$-quasi-continuous on $E$, which proves the first part of the lemma since $u^#_{|E}(x) = E_xA_\zeta, \quad x \in E$. The second part is immediate from the definition of quasi-continuity. \hfill \Box

Lemma 2.7. Assume that $(\mathcal{E}, D(\mathcal{E}))$ is transient and satisfies the strong sector condition. If $\mu \in S_0^{(0)}$ then $u$ defined as

$$u(x) = E_xA^\mu_\zeta, \quad x \in E$$

is a quasi-continuous version of $U_\mu$.

Proof. By [15, Proposition A.7], for every $\alpha > 0$ the function $R_\alpha \mu$ defined by $R_\alpha \mu(x) = E_x \int_0^\infty e^{-\alpha t}dA^\mu_t, \quad x \in E$, is a quasi-continuous version of $U_\alpha \mu$. Therefore by Lemma 2.5 and the Banach-Saks theorem there exists sequences $\alpha_n \downarrow 0$ and $\{n_k\}$ such that the Cesàro mean sequence $\{w_n = (1/n) \sum_{k=1}^n u_{n_k}\}$, where $u_n = R_{\alpha_n} \mu$, is $\bar{\mathcal{E}}$-convergent to $U_\mu$. On the other hand, by the monotone convergence theorem, $u_n(x) \to u(x)$ for $x \in E$, and hence $w_n(x) \to u(x)$ for $x \in E$. Consequently, $\{w_n\}$ is an approximating sequence for $u$. Therefore $u \in \mathcal{F}_e$ and

$$\bar{\mathcal{E}}(u - U_\mu, u - U_\mu)^{1/2} \leq \lim_{n \to \infty} (\bar{\mathcal{E}}(u-w_n, u-w_n)^{1/2} + \bar{\mathcal{E}}(U_\mu - w_n, U_\mu - w_n)^{1/2}) = 0.$$

Since $(\bar{\mathcal{E}}, \mathcal{F}_e)$ is a Hilbert space, it follows that $u$ is an $m$-version of $U_\mu$. To show that $u$ is quasi-continuous, let us first note that by [16, Proposition III.3.3] there is a nest $\{F_k\}$ such that $\{u_n\} \subset C(\{F_k\})$. Since $\mathcal{E}$ is quasi-regular, there exists an $\mathcal{E}$-nest $\{E_k\}$ consisting of compact sets. Write $F'_k = F_k \cap E_k$. Then $\{F'_k\}$ is an $\mathcal{E}$-nest consisting of compact sets and $\{u_n\} \subset C(\{F'_k\})$. Since $u_n|F'_k \not\succ u|F'_k$ as $n \to \infty$ for each $k \in \mathbb{N}$, applying Dini’s theorem shows that $u \in C(\{F'_k\})$, which is our claim. \hfill \Box

Let $S_0^{(0)}$ denote the subset of $S_0^{(0)}$ consisting of all measures $\nu$ such that $\nu(E) < \infty$ and $\|U\nu\|_\infty < \infty$. 

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Lemma 2.8. Assume that \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) is transient and satisfies the strong sector condition. If \(\mu \in S, \nu \in S^{(0)}_0\) then for any nonnegative Borel function \(f\),

\[
E_\nu \int_0^\zeta f(X_t) \, dA^\mu_t = \langle f \cdot \mu, \tilde{U} \nu \rangle.
\]  
(2.13)

Proof. By Lemma 2.7 there exist a nest \(\{F_n\}\) such that \(1_{F_n} |f| : |\mu| \in S^{(0)}_0\). By [15, Theorem A.8], for every \(\alpha > 0\) the function \(x \mapsto E_x \int_0^\zeta e^{-\alpha t}1_{F_n} f(X_t) \, dA^\mu_t\) is a quasi-continuous version of \(U_\alpha(1_{F_n} f \cdot \mu)\). Hence

\[
E_\nu \int_0^\zeta e^{-\alpha t}1_{F_n} f(X_t) \, dA^\mu_t = \langle U_\alpha(1_{F_n} f \cdot \mu), \nu \rangle = \langle 1_{F_n} f \cdot \mu, \tilde{U} \nu \rangle.
\]  
(2.14)

Letting \(\alpha \downarrow 0\) and applying the monotone convergence theorem the left-hand side of (2.14) and Lemma 2.5 the right-hand side of (2.14) we obtain

\[
E_\nu \int_0^\zeta 1_{F_n} f(X_t) \, dA^\mu_t = \langle 1_{F_n} f \cdot \mu, \tilde{U} \nu \rangle.
\]  
(2.15)

Letting \(n \to \infty\) in (2.15) yields (2.15) with \(F_n\) replaced by \(\bigcup_{n=1}^\infty F_n\), which implies (2.13) because \((\bigcup_{n=1}^\infty F_n)^c\) is an exceptional set. \(\square\)

Lemma 2.9. Assume that \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) is transient, \(\mu_1 \in S, \mu_2 \in \mathcal{M}_{0,b}^+\). If

\[
E_x \int_0^\zeta dA^\mu_1(t) \leq E_x \int_0^\zeta dA^\mu_2(t)
\]

for \(m\)-a.e. \(x \in E\) then \(\|\mu_1\|_{TV} \leq \|\mu_2\|_{TV}\).

Proof. Let \((\mathcal{E}^\#, \mathcal{D}(\mathcal{E}^\#))\), \(\mu^\#\) be defined as in the proof of Lemma 2.6 and let \((A^\mu)^\#\) be defined by (2.12) with \(A\) replaced by \(A^\mu\). It is an elementary check that \((A^\mu)^\# = A^\mu^\#\). By the assumptions and the fact that \(m^\#(E^\# \setminus E) = 0\),

\[
E_x^\# \int_0^\zeta dA^\mu_1^\#(t) \leq E_x^\# \int_0^\zeta dA^\mu_2^\#(t)
\]

for \(m\)-a.e. \(x \in E^\#\). Clearly \(\mu_2^\# \in \mathcal{M}_{0,b}(E^\#)\). Therefore \(\|\mu_1^\#\|_{TV} \leq \|\mu_2^\#\|_{TV}\) by [12, Lemma 5.4], and hence \(\|\mu_1\|_{TV} \leq \|\mu_2\|_{TV}\). \(\square\)

The following lemma is probably known, but we do not have a reference.

Lemma 2.10. Assume that \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) is transient and satisfies the strong sector condition. Let \(B \in \mathcal{B}(E)\). If \(\nu(B) = 0\) for every \(\nu \in S^{(0)}_0\) then \(B\) is \(\mathcal{E}\)-exceptional.

Proof. Let \((\mathcal{E}^\#, \mathcal{D}(\mathcal{E}^\#))\) be the regular extension of \((\hat{\mathcal{E}}, \mathcal{D}(\hat{\mathcal{E}}))\) specified in [16, Theorem VI.1.2]. Let \(\nu^\#\) be a smooth measure on \(\mathcal{B}(\mathcal{E}^\#)\) and let \(\nu = \nu^\#|_{\mathcal{B}(E)}\). If \(A \in \mathcal{B}(E)\) is \(\mathcal{E}\)-exceptional then by [16, Corollary VI.1.4], \(A\) is \(\mathcal{E}^\#\)-exceptional, and hence \(\nu(A) = \nu^\#(A) = 0\). Moreover, if \(\{F_k\}\) is a nest in \(E^\#\) such that \(\nu^\#(F_k) < \infty\) for \(k \in \mathbb{N}\) and \(\{E_k\}\) is a nest in \(E\) of [16, Theorem VI.1.2] then \(\{F_k \cap E_k\}\) is an \(\mathcal{E}\)-nest of compact
sets in $E$ such that $\nu(F_k \cap E_k) < \infty$, $k \in \mathbb{N}$. Thus $\nu$ is a smooth measure on $B(E)$. If moreover $\nu^# \in S^{(0)}_0(E^#)$ then $\nu(E) < \infty$ and for $\eta \in D(\mathcal{E})$,

$$\langle \nu, \tilde{\eta} \rangle = \langle \nu^#(E^#), \eta \rangle \leq c\mathcal{E}^{1/2}(\eta, \eta) = c\mathcal{E}(\eta, \eta)^{1/2}.$$  

From this in the same manner as in the proof of Lemma 2.1 one can deduce that $\langle \nu, \tilde{\eta} \rangle \leq c\mathcal{E}(\eta, \eta)^{1/2}$ for $\eta \in F_e$, i.e. that $\nu \in S^{(0)}_0$. From Lemma 2.7 and the fact that $A^{\nu^#} = (A^\nu)^#$ it follows now that $U\nu^#_{|E}$ is an $m$-version of $U\nu$. Therefore $\|U\nu\|_\infty < \infty$, which proves that $\nu \in S^{(0)}_0$. From this and the assumption it follows that $\nu^#(B) = \nu(B) = 0$ for every $\nu^# \in S^{(0)}_0(E^#)$. Therefore from the 0-order version of [7 Theorem 2.2.3] we conclude that $\text{Cap}^{#}_{1,1}(B) = 0$, where $\text{Cap}^#$ denotes the capacity relative to $(\mathcal{E}^#, D(\mathcal{E}^#))$. Hence $\text{Cap}^{#}_{\phi}(B) = 0$ by [16 Proposition VI.1.5], and consequently $\text{Cap}_{\phi}(B) = 0$ by [16 Corollary VI.1.4] ($\text{Cap}_{\phi}$ is defined by (2.4)). By remark following (2.4) this implies that $B$ is $\mathcal{E}$-exceptional.  

## 3 Probabilistic solutions

In this section we assume that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form on $L^2(E; m)$. We will need the following assumptions:

(A1) $f : E \times \mathbb{R} \to \mathbb{R}$ is measurable and $y \mapsto f(x, y)$ is continuous for every $x \in E$,

(A2) $(f(x, y_1) - f(x, y_2))(y_1 - y_2) \leq 0$ for every $y_1, y_2 \in \mathbb{R}$ and $x \in E$,

(A3) $f(\cdot, y) \in L^1(E; m)$ for every $y \in \mathbb{R}$,

(A4) $f(\cdot, 0) \in L^1(E; m)$, $\mu \in \mathcal{M}_{0,b}$

and

(A3$^*$) for every $y \in \mathbb{R}$ the function $f(\cdot, y)$ is quasi-$L^1$ with respect to $(\mathcal{E}, D(\mathcal{E}))$, i.e. $t \mapsto f(X_t, y)$ belongs to $L^1_{\text{loc}}((\mathbb{R}^+) \times \mathbb{R})$ for q.e. $x \in E$,

(A4$^*$) $E_x \int_0^\zeta |f(X_t, 0)| dt < \infty$, $E_x \int_0^\zeta d|A_t|^\mu_t < \infty$ for $m$-a.e. $x \in E$.

Let us note that in our previous paper [12] devoted to equations of the form (1.1) we followed [11] in assuming that $f$ satisfies (A1), (A2), (A4) and the following condition: for every $r > 0$, $F_r \in L^1(E; m)$, where $F_r(x) = \sup_{|y| \leq r} |f(x, y)|$. Obviously (A3) is weaker than the last condition. Likewise, our condition (A3$^*$) is weaker than the corresponding condition in [12].

Let us define the co-potential operator as

$$\hat{G}\phi = \lim_{n \to \infty} \hat{G}_{1/n}\phi, \quad \phi \in L^1(E; m), \ \phi \geq 0$$

and for given $\mu \in S$ set

$$R\mu(x) = E_x \int_0^\zeta dA^\mu_t, \quad x \in E.$$ 

**Lemma 3.1.** If $(\mathcal{E}, D(\mathcal{E}))$ is transient then for any $\mu \in S$ and $\phi \in L^2(E; m)$,

$$\langle R\mu, \phi \rangle = \langle \mu, \hat{G}\phi \rangle.$$  

(3.1)
Proof. By Lemma 2.1 there is a nest \( \{F_n\} \) such that \( 1_{F_n} \cdot \mu \in S_0^{(0)} \) for each \( n \in \mathbb{N} \). Let

\[
R_\alpha(1_{F_n} \cdot \mu)(x) = E_x \int_0^\zeta e^{-\alpha t} 1_{F_n}(X_t) \, dA_t^\mu, \quad \alpha > 0, \quad x \in E.
\]

Since \( R_\alpha(1_{F_n} \cdot \mu) \) is an \( m \)-version of \( U_\alpha(1_{F_n} \cdot \mu) \),

\[
\langle 1_{F_n} \cdot \mu, \tilde{U}_\alpha \phi \rangle = \mathcal{E}_\alpha(U_\alpha(1_{F_n} \cdot \mu), \tilde{U}_\alpha \varphi) = \mathcal{E}_\alpha(R_\alpha(1_{F_n} \cdot \mu), \tilde{U}_\alpha \varphi).
\]

Hence

\[
\langle 1_{F_n} \cdot \mu, \tilde{G}_\alpha \phi \rangle = \mathcal{E}_\alpha(R_\alpha(1_{F_n} \cdot \mu), \phi).
\]

(3.2)

Letting \( \alpha \downarrow 0 \) and then \( n \to \infty \) in (3.2) gives (3.1). \( \square \)

Let

\[
\mathcal{R} = \{ \mu : |\mu| \in S \text{ and } \tilde{G}_\alpha \mu \in \mathcal{M}_{0,b} \text{ for some } \phi \in L^1(E; m) \text{ such that } \phi > 0 \text{ m-a.e.} \}.
\]

If \( \mu \) is smooth and \( R|\mu| < \infty \text{ m-a.e.} \) then from (3.1) and the fact that \( m \) is \( \sigma \)-finite it follows that \( \mu \in \mathcal{R} \). Furthermore, if \( \mu \in \mathcal{R} \) then by (3.1), \( R|\mu| < \infty \text{ m-a.e.} \). Thus \( \mathcal{R} \) can be equivalently defined as

\[
\mathcal{R} = \{ \mu : \mu \text{ is smooth, } R|\mu| < \infty, \text{ m-a.e.} \}.
\]

It follows in particular that (A4*) is satisfied iff \( f(\cdot,0) \cdot m \in \mathcal{R} \) and \( \mu \in \mathcal{R} \).

**Proposition 3.2.** If \( (\mathcal{E},D(\mathcal{E})) \) is transient then \( \mathcal{M}_{0,b} \subset \mathcal{R} \).

**Proof.** Follows from [17] Corollary 1.3.6] applied to the dual form \( (\tilde{\mathcal{E}},D(\mathcal{E})) \). \( \square \)

In general the inclusion in Proposition 3.2 is strict. To see this let us consider the form \( (\mathcal{D},H_1^0(D)) \) of Remark 2.4(i). If \( d \geq 3 \) and \( D \subset \mathbb{R}^d \) is an open bounded set with smooth boundary then \( R1 \) is a continuous strictly positive function such that \( R1(x) \approx \delta(x) \) for \( x \in D \), where \( \delta(x) = \text{dist}(x, \partial D) \) (for the last property see [13] Proposition 4.9]). Since \( R1 \) is an \( m \)-version of \( G1 = \tilde{G}1 \), it follows that \( L^1(D; \delta(x) \, dx) \in \mathcal{R} \), so \( \mathcal{R} \) contains positive Radon measures of infinite total variation. Elliptic and parabolic equations with right-hand side in \( L^q(D; \delta(x) \, dx) \) \( (q \geq 1) \) space are studied for instance in [5].

**Remark 3.3.** Assume that \( (\mathcal{E},D(\mathcal{E})) \) is transient. Then by Lemma 2.6 and Proposition 3.2 (A3) implies (A3*) and (A4) implies (A4*).

### 3.1 BSDEs

Let \( \mathbb{X} = (\Omega, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, \zeta, (P_x)_{x \in E \cup \{\partial\}}) \) be an \( m \)-tight special standard Markov process properly associated with \( (\mathcal{E},D(\mathcal{E})) \). We will need the following classes of processes defined on \( \Omega \).

\( \mathcal{D} \) is the space of all \( (\mathcal{F}_t) \)-progressively measurable càdlàg processes and \( \mathcal{D}^q(P_x) \), \( q > 0 \), is the subspace of \( \mathcal{D} \) consisting of all processes \( Y \) such that \( E_x \sup_{t \geq 0} |Y_t|^q < \infty \).
\[ \mathcal{M}(P_x) \text{ (resp. } \mathcal{M}_{loc}(P_x) \text{) is the space of all càdlàg ((\mathcal{F}_t), P_x)\text{-martingales (resp. local martingales) } M \text{ such that } M_0 = 0. \mathcal{M}^q(P_x), q > 0, \text{ is the subspace of } \mathcal{M}(P_x) \text{ consisting of all martingales } M \text{ such that } E([M]_\infty)^{p/2} < \infty. \]

We will say that a càdlàg (\mathcal{F}_t)-adapted process \( Y \) is of Doob’s class (D) under \( P_x \) if the collection \( \{Y_\tau, \tau \text{ is a finite valued (} \mathcal{F}_t\text{-stopping time}\} \) is uniformly integrable under \( P_x \).

**Definition.** Let \( f : E \times \mathbb{R} \to \mathbb{R} \) be a measurable function and let \( A^\mu \) be a CAF of \( \mathbb{X} \) corresponding to some \( \mu \in \mathcal{S} \). We say that under the measure \( P_x \) a pair \( (Y^x, M^x) \) is a solution of the backward stochastic differential equation with terminal time \( \zeta \) and coefficient \( f + dA^\mu \) (BSDE\( _x^\zeta(f + dA^\mu) \) for short) if

\[
(Y^x) \in \mathcal{D}, \quad Y_{t \wedge \zeta} \to 0 \quad \text{P}_x\text{-a.s. as } t \to \infty,
\]

\[
\text{exists at most one solution } (Y^x, M^x) \text{ of BSDE}_x^\zeta(f + dA^\mu) \text{ for short) if}
\]

\[
(a) \quad Y^x \in \mathcal{D}, \quad Y_{t \wedge \zeta} \to 0 \quad \text{P}_x\text{-a.s. as } t \to \infty,
\]

\[
\text{is of class (D) under } P_x \text{ and } M^x \in \mathcal{M}_{loc}(P_x),
\]

\[
(b) \quad \text{For every } T > 0, \quad [0, T] \ni t \mapsto f(X_t, Y^x_t) \in L^1(0, T) \text{ and }
\]

\[
Y^x_t = Y^x_{T \wedge \zeta} + \int_{t \wedge \zeta}^{T \wedge \zeta} f(X_s, Y^x_s) \, ds + \int_{t \wedge \zeta}^{T \wedge \zeta} dA^\mu_s - \int_{t \wedge \zeta}^{T \wedge \zeta} dM^x_s, \quad t \in [0, T], \quad \text{P}_x\text{-a.s.}
\]

In [12] existence and uniqueness results for more general (not necessarily Markov type) backward equations are proved. From these results the following theorem follows.

**Theorem 3.4.** Let \( f : E \times \mathbb{R} \to \mathbb{R} \) be a measurable function and \( \mu \in \mathcal{S} \).

(i) If \( f \) satisfies (A2) then for every \( x \in E \) there exists at most one solution \( (Y^x, M^x) \) of BSDE\( _x^\zeta(f + dA^\mu) \).

(ii) If \( f \) satisfies (A1), (A2), (A3*) and \( E_x(\int_0^\zeta |f(X_t, 0)| \, dt) < \infty \) for some \( \mu \in \mathcal{S} \) then there exists a solution \( (Y^x, M^x) \) of BSDE\( _x^\zeta(f + dA^\mu) \) such that \( (Y^x, M^x) \in \mathcal{D}(P_x) \otimes \mathcal{M}(P_x) \) for \( q \in (0, 1) \) and \( M^x \) is a uniformly integrable martingale.

**Proof.** The uniqueness part is a direct consequence of [12, Corollary 3.2]. The existence part follows from [12, Theorem 3.4].

### 3.2 Existence and uniqueness of probabilistic solutions

Let \( (L, D(L)) \) be the operator defined by Eq. (2.1) (or (1.2)).

**Definition.** Let \( \mu \in \mathcal{S} \). We say that an \( \mathcal{E}\)-quasi-continuous function \( u : E \to \mathbb{R} \) is a probabilistic solution of the equation

\[
- Lu = f_u + \mu,
\]

where \( f_u(x) = f(x, u(x)) \) for \( x \in E \), if \( E_x(\int_0^\zeta |f_u(X_t)| \, dt) < \infty \) and

\[
u(x) = E_x \int_0^\zeta f_u(X_t) \, dt + E_x \int_0^\zeta dA^\mu_t
\]

for q.e. \( x \in E \).
In what follows we say that a function $u : E \to \mathbb{R}$ is of class (FD) if the process $t \mapsto u(X_t)$ is of class (D) under the measure $P_x$ for q.e. $x \in E$. Similarly, we say that $u \in \mathcal{F} \mathcal{D}^q$ if the process $t \mapsto u(X_t)$ belongs to $\mathcal{D}^q$ under $P_x$ for q.e. $x \in E$.

**Theorem 3.5.** Assume (A1), (A2), (A3\textsuperscript{*}), (A4\textsuperscript{*}). Then there exists a unique probabilistic solution $u$ of (3.3). Actually, $u$ is of class (FD) and $u \in \mathcal{F} \mathcal{D}^q$ for $q \in (0, 1)$. Moreover, for q.e. $x \in E$ there exists a unique solution $(Y^x, M^x)$ of BSDE\textsubscript{f}((f + dA\textsuperscript{u})). In fact,

$$u(X_t) = Y^x_t, \quad t \geq 0, \quad P_x\text{-a.s.}$$

**Proof.** From Lemma 2.6 it follows that under (A4\textsuperscript{*}) the second assumption in part (ii) of Theorem 3.4 is satisfied for q.e. $x \in E$. To prove the theorem it suffices now to use Theorem 3.4 and repeat step by step arguments from the proof of [12, Theorem 4.7].

Let us note that from Proposition 3.2 and [12, Lemma 2.3] it follows that under the assumptions of Theorem 3.5

$$E_x \int_0^\zeta |f_u(X_t)| dt \leq E_x \left( \int_0^\zeta |f(X_t, 0)| dt + \int_0^\zeta d|A^\mu|_t \right)$$

for m-a.e. $x \in E$, where $u$ is a probabilistic solution of (3.3).

### 3.3 Probabilistic solutions vs. solutions in the sense of duality

Assume that $(\mathcal{E}, D(\mathcal{E}))$ is transient and satisfies the strong sector condition. Let $\mathcal{A}$ denote the space of all $\mathcal{E}$-quasi-continuous functions $u : E \to \mathbb{R}$ such that $u \in L^1(E; \nu)$ for every $\nu \in S_0^{(0)}$. Following [12] we adopt the following definition.

**Definition.** Let $\mu \in \mathcal{M}_{0,b}$. We say that $u : E \to \mathbb{R}$ is a solution of (3.3) in the sense of duality if $u \in \mathcal{A}$, $f_u \in L^1(E; m)$ and

$$\langle \nu, u \rangle = \langle f_u, \tilde{\nu} \rangle + \langle \mu, \tilde{\nu} \rangle, \quad \nu \in S_0^{(0)}.$$  

(3.6)

Note that by the very definition of $S_0^{(0)}$, if $\nu \in S_0^{(0)}$ and $u \in \mathcal{F}_e$ then $\tilde{u} \in L^1(E; \nu)$. As a consequence, if $u \in \mathcal{F}_e$ then $\tilde{u} \in \mathcal{A}$.

**Proposition 3.6.** Assume that $(\mathcal{E}, D(\mathcal{E}))$ is transient, satisfies the strong sector condition and that $\mu \in \mathcal{M}_{0,b}$. If $u$ is $\mathcal{E}$-quasi-continuous and $f_u \in L^1(E; m)$, then $u$ is a probabilistic solution of (3.3) iff it is a solution of (3.3) in the sense of duality.

**Proof.** Let $u$ be a solution of (3.3) in the sense of duality. Let us denote by $w(x)$ the right-hand side of (3.4) if it is finite and put $w(x) = 0$ otherwise. By Proposition 3.2 $w$ is finite m-a.e., and hence, by Lemma 2.6 $w$ is quasi-continuous. By Lemma 2.8 $w \in \mathcal{A}$ and $\langle \nu, w \rangle$ is equal to the right-hand side of (3.6). Thus $\langle \nu, u \rangle = \langle \nu, w \rangle$ for $\nu \in S_0^{(0)}$. Lemma 2.10 now shows that $u = \nu \mathcal{E}$-q.e. since $u, \nu$ are $\mathcal{E}$-quasi-continuous. Conversely, assume that $u$ is a probabilistic solution of (3.3). Then again by Lemma 2.8 $u \in \mathcal{A}$ and $u$ satisfies (3.6). \[\Box\]

**Proposition 3.7.** Assume that $(\mathcal{E}, D(\mathcal{E}))$ is transient and (A4) is satisfied.
(i) If \( u \) is a probabilistic solution of (3.3) then \( f_u \in L^1(E; m) \) and
\[
\|f_u\|_{L^1(E; m)} \leq \|f(\cdot, 0)\|_{L^1(E; m)} + \|\mu\|_{TV}. \tag{3.7}
\]

(ii) If moreover \( (E, D(E)) \) satisfies the strong sector condition then \( u \) is a probabilistic solution of (3.3) iff it is a solution of (3.3) in the sense of duality.

**Proof.** Assertion (i) follows from (3.5) and Lemma 2.9 whereas (ii) follows from (i) and Proposition 3.6.

\[\square\]

# 4 Regularity of probabilistic solutions

For \( k > 0 \) and \( u : E \to \mathbb{R} \) set
\[
T_k u(x) = (-k) \vee u(x) \wedge k, \quad x \in E.
\]

**Lemma 4.1.** Assume that \( (E, D(E)) \) is a quasi-regular transient Dirichlet form. Then for every \( k > 0 \),
\[
E(T_k u, T_k u) \leq E(u, T_k u) \tag{4.1}
\]
for all \( u \in D(E) \). If moreover \( (E, D(E)) \) satisfies the strong sector condition then (4.1) holds for all \( u \in F_e \).

**Proof.** Let \( u \in D(E) \). We may and will assume that \( u \) is quasi-continuous. Let \( (\hat{E}, D(\hat{E})) \) denote the form dual to \( E \), i.e. \( \hat{E}(u, v) = E(u, v), u, v \in D(E) \). Since \( \hat{E} \) is a semi-Dirichlet form, it follows from [16, Theorem 1.4.4] that \( u \wedge k \in D(\hat{E}) \) and
\[
\hat{E}(u \wedge k, u - u \wedge k) \geq 0 \tag{4.2}
\]
for every \( u \in D(E) \) and \( k \geq 0 \). Moreover, since \( (-k) \vee u = -[(-u) \wedge k] \in D(E) \), we conclude that \( T_k u = ((-k) \vee u) \wedge k \in D(E) \). We have
\[
\hat{E}(T_k u, u - T_k u) = \hat{E}(T_k u, u - (-k) \vee u) + \hat{E}(T_k u, (-k) \vee u - T_k u)
\]
\[
\geq \hat{E}(T_k u, u - ((-k) \vee u)),
\]
because by (4.2),
\[
\hat{E}(T_k u, (-k) \vee u - T_k u) = \hat{E}((((-k) \vee u) \wedge k, (-k) \vee u - ((-k) \vee u) \wedge k) \geq 0.
\]
Since
\[
\hat{E}(T_k u, u - ((-k) \vee u)) = \hat{E}(u^+ \wedge k - u^- \wedge k, -u^- + u^- \wedge k)
\]
\[
\geq \hat{E}(u^+ \wedge k, -u^- + u^- \wedge k)
\]
\[
= \hat{E}(u^+ \wedge k, -(u + k)^-),
\]
it follows from the above that
\[
\hat{E}(T_k u, u - T_k u) \geq \hat{E}(u^+ \wedge k, -(u + k)^-). \tag{4.3}
\]
We know that \( u^+ \wedge k, v = (u + k)^- = u^- - (u^- \wedge k) \in D(E) \). Since \( \{x \in E : u(x) \leq -k\} \) is a quasi-support of \( v \) and \( u^+ \wedge k \) equals zero on the quasi-open set \( \{x \in E : u(x) < -k/2\} \)
containing \( \{ x \in E : u(x) \leq -k \} \), it follows from [8, Theorem 4.1] that there is a \( \sigma \)-finite positive measure \( J \) on \( E \times E \backslash d \) (d stands for the diagonal) and a \( \sigma \)-finite positive measure \( K \) on \( E \) such that

\[
\hat{E}(u^+ \land k, (u + k)^-) = 2 \int_{E \times E \backslash d} ((u^+ \land k)(y) - (u^+ \land k)(x))(u(y) + k)^- J(dx, dy)
\]

\[
+ \int_E (u^+ \land k)(x)(u(x) + k)^- K(dx).
\]

Hence

\[
\hat{E}(u^+ \land k, (u + k)^-) = -2 \int_{E \times E \backslash d} (u^+ \land k)(x)(u(y) + k)^- J(dx, dy) \leq 0.
\]

This and (4.3) give \( \hat{E}(T_k u, u - T_k u) \geq 0 \), which is equivalent to (4.1). Now assume that \( E \) satisfies (2.1) and \( u \in F_e \). Let us consider an approximating sequence \( \{ u_n \} \subset D(E) \) for \( u \). By [7, Theorem 1.5.3], \( T_k u_n \in F_e \) and \( \hat{E}(T_k u_n, T_k u_n) \leq \hat{E}(u_n, u_n) \) for each \( n \in \mathbb{N} \). Since \( \{ u_n \} \) is \( \hat{E} \)-convergent, it follows that \( \sup_{n \geq 1} \hat{E}(T_k u_n, T_k u_n) < \infty \). Since \( (F_e, \hat{E}) \) is a Hilbert space, applying the Banach-Saks theorem we can find a subsequence \( \{ n_i \} \) such that the Cesàro mean sequence \( \{ w_N = (1/N) \sum_{i=1}^{N} T_k(u_{n_i}) \} \) is \( \hat{E} \)-convergent to some \( w \in F_e \). Since \( \hat{E} \) is transient, there is an \( m \)-a.e. strictly positive and bounded \( g \in L^1(E; m) \) such that

\[
\int_E |w_N - v|^g dm \leq E(w_N - w, w_N - w)^{1/2} \rightarrow 0.
\]

On the other hand, since \( u_n \rightarrow u \) \( m \)-a.e., applying the Lebesgue dominated convergence theorems shows that \( \int_E |w_N - T_k u|^g dm \rightarrow 0 \). Consequently, \( w = T_k u \) and \( \{ T_k u_n \} \) converges \( \hat{E} \)-weakly to \( T_k u \). From this and the first part of the proof it follows that

\[
E(T_k u, T_k u) \leq \liminf_{n \rightarrow \infty} E(T_k u_n, T_k u_n) \leq \liminf_{n \rightarrow \infty} E(u_n, T_k u_n).
\]

(4.4)

Moreover, using (2.1) and the facts that \( \{ u_n \} \) is \( \hat{E} \)-convergent to \( u \) and \( \{ T_k u_n \} \) is \( \hat{E} \)-weakly convergent to \( T_k u \) we conclude the last limit in (4.4) equals \( E(u, T_k u) \), which completes the proof of the second assertion of the lemma.

Theorem 4.2. Assume that \( (E, D(E)) \) is a quasi-regular transient Dirichlet form and \( \mu \in M_{0,b} \). Then if \( u \) is a solution of (3.3) and \( f_u \in L^1(E; m) \) then \( T_k u \in F_e \) for every \( k > 0 \). Moreover, for every \( k > 0 \),

\[
E(T_k u, T_k u) \leq k(\| f_u \|_{L^1(E; m)} + \| \mu \|_{TV}).
\]

(4.5)

Proof. By Lemma (2.1) there exists a nest \( \{ F_n \} \) such that \( 1_{F_n} (|f_u| \cdot m + 1_{F_n} \cdot |\mu|) \in S_0^{(0)} \). For \( \alpha > 0 \) set

\[
u^\alpha_n(x) = Ex \int_0^\infty e^{-\alpha t} 1_{F_n} f_u(X_t) dt + Ex \int_0^\infty e^{-\alpha t} 1_{F_n} (X_t) dA^\mu_t, \quad x \in E
\]

and \( \mu_n = 1_{F_n} f_u \cdot m + 1_{F_n} \cdot \mu \). By [15, Theorem A.8],

\[
u^\alpha_n(x) = \hat{U}_{\alpha \mu_n}(x).
\]
for q.e. \( x \in E \). Hence \( u_n^\alpha \in F \) and \( T_k u_n^\alpha \in D(\mathcal{E}) \) since every normal contraction operates on \((\mathcal{E}, D(\mathcal{E}))\). Therefore

\[
\mathcal{E}_\alpha(u_n^\alpha, T_k u_n^\alpha) = \mathcal{E}_\alpha(U_n \mu_n, T_k u_n^\alpha) = \int_E \tilde{T}_k u_n^\alpha \, d\mu_n \leq k(\|f_u\|_{L^1(E;m)} + \|\mu\|_{TV}).
\]

By Lemma 2.1 applied to the form \( \mathcal{E}_\alpha \),

\[
\mathcal{E}_\alpha(T_k u_n^\alpha, T_k u_n^\alpha) \leq \mathcal{E}_\alpha(u_n^\alpha, T_k u_n^\alpha).
\]

Consequently,

\[
\mathcal{E}(T_k u_n^\alpha, T_k u_n^\alpha) \leq k(\|f_u\|_{L^1(E;m)} + \|\mu\|_{TV}) - N.
\]

By the Banach-Saks theorem we can choose a sequence \( \{\alpha_l\} \) such that \( \alpha_l \downarrow 0 \) as \( l \to \infty \) and the sequence \( \{w_N = (1/N) \sum_{l=1}^N T_k u_n^\alpha\} \) is \( \mathcal{E} \)-convergent. Moreover, from Lemma 2.7 one can deduce that \( u_n^\alpha(x) \to u_n(x) \) as \( \alpha \downarrow 0 \) for q.e. \( x \in E \). Hence \( T_k u_n^\alpha \to T_k u_n \) m.a.e. and consequently, \( w_n \to T_k u_n \) m.a.e. Thus \( \{w_N\} \) is an approximating sequence for \( T_k u_n \). By (4.6), \( \mathcal{E}(w_N, w_N) \leq k(\|f_u\|_{L^1(E;m)} + \|\mu\|_{TV}) \) for every \( N \in \mathbb{N} \). Hence

\[
\mathcal{E}(T_k u_n, T_k u_n) = \lim_{N \to \infty} \mathcal{E}(w_N, w_N) \leq k(\|f_u\|_{L^1(E;m)} + \|\mu\|_{TV}).
\]

Since \( u_n \to u \) q.e. we now apply the above arguments again, with \( T_k u_n^\alpha \) replaced by \( T_k u_n \), to obtain (4.5). \( \square \)

**Corollary 4.3.** If \((\mathcal{E}, D(\mathcal{E}))\) is a quasi-regular transient Dirichlet form and \( f, \mu \) satisfy (A1), (A2), (A3*), (A4) then there exists a unique solution \( u \) of (3.3). Moreover, \( u \) is of class \((FD)\), \( u \in FD^q \) for \( q \in (0, 1) \) and (3.7), (1.8) are satisfied.

**Proof.** Follows immediately from Theorem 3.5, Proposition 3.7 and Theorem 4.2. \( \square \)

**Remark 4.4.** Assume that \((\mathcal{E}, D(\mathcal{E}))\) is transient, satisfies the strong sector condition, and that \( \mu \in S_0^{(0)} \). If \( u \) is a solution of (3.3) such that \( f_u \cdot m \in S_0^{(0)} \) then \( u \) is a weak solution of (3.3), i.e. for every \( v \in F_e \),

\[
\mathcal{E}(u, v) = (f_u, v) + \langle \mu, \bar{v} \rangle.
\]

Indeed, by Lemma 2.1 if \( \mu, f_u \cdot m \in S_0^{(0)} \) then \( u \) satisfying (3.4) is a quasi-continuous version of \( U(f_u \cdot m + \mu) \), which implies (4.7). Note that \( f_u \cdot m \in S_0^{(0)} \) for instance if \( f_u \in L^2(E; m) \) and for some \( c > 0 \), \( \mathcal{E}_1(u, u) \leq c \mathcal{E}(u, u) \), \( u \in F_e \), because then \( f_u \cdot m \in S_0 \) and \( S_0 = S_0^{(0)} \).

**Remark 4.5.** (i) Example 5.7 in [12] shows that in general under (A1)–(A4) the solution \( u \) of (3.3) may not be locally integrable.

(ii) Assume (A1), (A2), (A3*) (A4*) and let \( u \) be a probabilistic solution of (3.3) of Theorem 3.5. Then from (3.4), (3.5) it follows that \( |u(x)| \leq R(|\sigma(., 0) \cdot m + 2|\mu| \).

Therefore the condition

\[
(|f(., 0)|, \hat{G}1) + \langle |\mu|, \hat{G}1 \rangle < \infty
\]

is sufficient to guarantee integrability of \( u \). One interesting situation in which (4.8) holds true is given at the end of Section 5.3.
5 Applications

In this section we show by examples how our general results work in practice. Propositions 5.2–5.4 below concerning nonlocal operators and operators in Hilbert spaces are new even in the linear case, i.e. if $f \equiv 0$. To our knowledge Proposition 5.1 concerning nonsymmetric local form is also new.

5.1 Classical nonsymmetric local regular forms

We start with nonsymmetric forms associated with divergence form operators. Let $D$ be an bounded open subset of $\mathbb{R}^d$, $d \geq 3$, and let $m$ be the Lebesgue measure on $D$. Assume that $a : D \to \mathbb{R}^d \otimes \mathbb{R}^d$, $b, d : D \to \mathbb{R}^d$ and $c : D \to \mathbb{R}$ are measurable functions such that

(a) $c - \sum_{i=1}^d \frac{\partial b_i}{\partial x_i} \geq 0$ and $c - \sum_{i=1}^d \frac{\partial b_i}{\partial x_i} \geq 0$ in the sense of Schwartz distributions,

(b) There exist $\lambda > 0$, $M > 0$ such that $\sum_{i,j=1}^d \bar{a}_{ij}\xi_i\xi_j \geq \lambda|\xi|^2$ for all $\xi \in \mathbb{R}^d$ and $|\bar{a}_{ij}| \leq M$ for $i, j = 1, \ldots, d$, where $\bar{a}_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$, $\bar{a}_{ij} = \frac{1}{2}(a_{ij} - a_{ji})$,

(c) $c \in L^{d/2}_{loc}(D; dx)$ and $b_i, d_i \in L^{d}_{loc}(D; dx)$, $b_i - d_i \in L^d(D; dx) \cup L^\infty(D; dx)$ for $i = 1, \ldots, d$.

Then by [16, Proposition II.2.11] the form $(\mathcal{E}, C_0^\infty(D))$, where

$$\mathcal{E}(u, v) = \int_D \langle a\nabla u, \nabla v \rangle_{\mathbb{R}^d} \, dx + \int_D \langle (b, \nabla u)_{\mathbb{R}^d} v + (d, \nabla v)_{\mathbb{R}^d} u \rangle \, dx + \int_D c u v \, dx,$$

is closable and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a regular Dirichlet form on $L^2(D; dx)$. By (a) and (b),

$$\mathcal{E}(u, u) \geq \int_D \langle a\nabla u, \nabla u \rangle_{\mathbb{R}^d} \, dx = \int_D \langle \bar{a}\nabla u, \nabla u \rangle_{\mathbb{R}^d} \, dx \geq \lambda^{-1} \int_D \langle \nabla u, \nabla u \rangle_{\mathbb{R}^d} \, dx \quad (5.1)$$

for $u \in H_0^1(D)$, and hence, by Poincaré’s inequality, there is $C_1 > 0$ such that

$$\mathcal{E}(u, u) \geq C_1(u, u) \quad (5.2)$$

for $u \in H_0^1(D)$. Consequently, $(\mathcal{E}, D(\mathcal{E}))$ satisfies the strong sector condition. From the calculations made on pp. 50–51 in [16] it follows that there exists $C_2 > 0$ depending on $\lambda$ and the coefficients $a, b, c, d$ such that

$$\mathcal{E}(u, u) \leq C_2 \mathbb{D}_1(u, u), \quad (5.3)$$

where $\mathbb{D}$ is defined in Remark 2.2. By (5.1)–(5.3), $D(\mathcal{E}) = H_0^1(D)$. From this, (5.1) and the fact that $(\mathbb{D}, H_0^1(D))$ is transient it follows that $(\mathcal{E}, D(\mathcal{E}))$ is transient as well.

The operator corresponding to $(\mathcal{E}, D(\mathcal{E}))$ in the sense of [12] has the form

$$Lu = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial u}{\partial x_j}) - \sum_{i=1}^d b_i \frac{\partial u}{\partial x_i} + \sum_{i=1}^d \frac{\partial}{\partial x_i} (d_i u) - cu.$$
Also note that by (5.1), (5.3) and Poincaré’s inequality the set $S^{(0)}_0$ determined by the form $\mathcal{E}$ coincides with the set $S^{(0)}_0$ determined by $\mathbb{D}$. Hence, if $\mu \in \mathcal{M}_{0,b}$ then $\mu = \nu + f$ for some $f \in L^1(D;dx)$ and $\nu \in \mathcal{M}_{0,b}$ such that $T_\nu \in H^{-1}(D)$. From this and the well known characterization of $H^{-1}(D)$ it follows that if $\mu \in \mathcal{M}_{0,b}$ then

$$\mu = f^0 - \text{div} F \quad (5.4)$$

for some $f^0 \in L^1(D;dx)$ and $F = (F^1, \ldots, F^d)$ such that $F^i \in L^2(D;dx)$, $i = 1, \ldots, d$.

From the above considerations and Corollary 4.3 we obtain the following proposition.

**Proposition 5.1.** Let $D \subset \mathbb{R}^d$, $d \geq 3$, be a bounded domain and let $a, b, c, d$ satisfy (a)–(c). If $f, \mu$ satisfy (A1)–(A4) then there exists a unique probabilistic solution of the problem

$$-Lu = f_u + \mu \quad \text{on } D, \quad u|_{\partial D} = 0.$$

Moreover, $f_u \in L^1(D;dx)$, $T_k u \in H^0_1(D)$ for every $k > 0$ and (3.7), (4.5) hold true.

### 5.2 Gradient perturbations of nonlocal symmetric regular forms on $\mathbb{R}^d$

The following example of a nonlocal nonsymmetric regular Dirichlet form is borrowed from [9].

Let $\psi: \mathbb{R}^d \to \mathbb{R}$ be a continuous negative definite function, i.e. $\psi(0) \geq 0$ and $\xi \mapsto e^{-t\psi(\xi)}$ is positive definite for $t \geq 0$, and for $s \in \mathbb{R}$ let $H^{\psi,s}$ denote the Hilbert space

$$H^{\psi,s} = \{ u \in L^2(\mathbb{R}^d;dx) : \|u\|_{\psi,s} < \infty \}$$

where

$$\|u\|_{\psi,s}^2 = \int_{\mathbb{R}^d} (1 + \psi(\xi))^s |\hat{u}(\xi)|^2 d\xi$$

and $\hat{u}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} u(x) dx$, $\xi \in \mathbb{R}^d$. If $\psi(\xi) = |\xi|^2$ then $H^{\psi,s}$ coincides with the usual fractional Sobolev space $H^s$. Basic properties of the spaces $H^{\psi,s}$ are to be found in [9, Section 3.10].

Given $\psi$ as above and $b = (b_1, \ldots, b_d): \mathbb{R}^d \to \mathbb{R}^d$ such that $b_i \in C^1_b(\mathbb{R}^d)$ for $i = 1, \ldots, d$ let us define forms $\Psi, \mathcal{B}$ by putting

$$\Psi(u, v) = \int_{\mathbb{R}^d} \psi(\xi) \hat{u}(\xi) \hat{v}(\xi) d\xi, \quad u, v \in H^{\psi,1}$$

and

$$\mathcal{B}(u, v) = \int_{\mathbb{R}^d} \langle b_i, \nabla u \rangle_{\mathbb{R}^d} v dx, \quad u, v \in C^\infty_0(\mathbb{R}^d). \quad (5.5)$$

Consider the following assumptions on $\psi, b$:

(a) $1/\psi$ is locally integrable on $\mathbb{R}^d$,

(b) There exist $\alpha \in (1, 2]$, $\kappa > 0$, $R > 0$ such that $\psi(\xi) \geq \kappa |\xi|^\alpha$ if $|\xi| > R$,

(c) $b_i \in C^1_b(\mathbb{R}^d)$ for $i = 1, \ldots, d$ and $\text{div} b = 0$. 

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It is known (see, e.g., [7, Example 1.4.1]) that \((\Psi, H^{\psi,1})\) is a symmetric regular Dirichlet form on \(L^2(\mathbb{R}^d; dx)\). By [7, Example 1.5.2] it is transient iff (a) is satisfied. By [9, Corollary 4.7.35] there exists \(c > 0\) (depending on \(b\)) such that \(|B(u, v)| \leq c\|u\|_{H^{1/2}}\|v\|_{H^{1/2}}\). Hence, if (b) is satisfied then \(H^{\psi,1} \subset H^{1/2}\) and

\[
|B(u, v)| \leq C\|u\|_{\psi,1}\|v\|_{\psi,1}
\]

for some \(C > 0\). Since \(C_0^\infty(\mathbb{R}^d)\) is dense in \(H^{\psi,1}\) (see [9, Theorem 3.10.3]), it follows that under (b) we may extend (5.5) to a continuous bilinear form \((B, H^{\psi,1})\). If moreover (c) is satisfied, then by the integration by parts formula,

\[
B(u, u) = -\frac{1}{2} \int_{\mathbb{R}^d} u^2 \text{div} b \, dx = 0
\]

for \(u \in C_0^\infty(\mathbb{R}^d)\) and hence for all \(u \in H^{\psi,1}\). Using the integration by parts formula one can also check (see [9, Example 4.7.36]) that if \(\text{div} b = 0\) then \((B, H^{\psi,1})\) has the contraction properties required in the definition of a Dirichlet form and hence is a Dirichlet form. Finally, let us consider the form

\[
E(u, v) = \Psi(u, v) + B(u, v), \quad u, v \in H^{\psi,1}.
\]

From the properties of \(\Psi, B\) mentioned above it follows that if (a)–(c) are satisfied then \((E, H^{\psi,1})\) is a regular transient Dirichlet form on \(L^2(\mathbb{R}^d; dx)\) and the extended Dirichlet space for \(E\) coincides with the extended Dirichlet space \(H^{\psi,1}_e\) for \(\Psi\). The latter can be characterized for \(\psi\) of the form \(\psi(\xi) = c|\xi|^\alpha\) for some \(\alpha \in (0, 2]\), \(c > 0\) (see [7, Example 1.5.2] or [10, Example 3.5.55]). From that characterization it follows that if \(\psi\) satisfies (b) and \(\alpha < d\) (i.e. (a) is satisfied) then \(H^{\psi,1}_e \hookrightarrow L^p(\mathbb{R}^d)\) with \(p = 2d/(d - \alpha)\) and \(\|u\|_{L^p(\mathbb{R}^d; dx)} \leq C\Psi(u, u)^{1/2}\) for \(u \in H^{\psi,1}\) (see [10, Corollary 3.5.60]).

The operator associated with \(\Psi\) is a pseudodifferential operator \(\psi(\nabla)\) which for \(u \in C_0^\infty(\mathbb{R}^d)\) has the form

\[
\psi(\nabla)u(x) = \int_{\mathbb{R}^d} e^{i(x, \xi)} \psi(\xi) \hat{u}(\xi) \, d\xi, \quad x \in \mathbb{R}^d.
\]

**Proposition 5.2.** Assume (A1)–(A4) and (a)–(c). Then there exists a unique probabilistic solution of the equation

\[-\psi(\nabla)u - (b, \nabla u) = f_u + \mu.\]

Moreover, \(f_u \in L^1(\mathbb{R}^d; dx), T_k u \in H^{\psi,1}_e\) for every \(k > 0\) and (3.7), (4.5) hold true.

Proposition 5.2 holds true for operators corresponding to (5.8) with \(\Psi\) replaced by arbitrary symmetric regular Dirichlet form with domain \(H^{\psi,1}\). For examples of such forms see Examples 4.7.30 and 4.7.31 in [9] and Remark 2.6.8 and Theorem 2.6.10 in [10].

### 5.3 Nonlocal symmetric forms on \(D \subset \mathbb{R}^d\)

Let \(\psi\) be a continuous negative definite function satisfying conditions (a), (b) of Subsection 5.2 and let \(D \subset \mathbb{R}^d\) be a nearly Borel measurable set finely open with respect
to the process associated with the form $\Psi$. Set $L^2_D(\mathbb{R}^d; dx) = \{ u \in L^2(\mathbb{R}^d; dx) : u = 0 \text{ a.e. on } D^c \}$ and
$$H^\psi_{D,1} = \{ u \in H^\psi_{D,1} : \tilde{u} = 0 \text{ q.e. on } D^c \}.$$ Then by [3, Theorem 3.3.8] the form $(\Psi, H^\psi_{D,1})$ is a quasi-regular Dirichlet form on $L^2_D(\mathbb{R}^d; dx)$. If $\alpha < d$ then it is transient by [7, Theorem 3.3.8]. Therefore Corollary 4.3 leads to the following proposition.

**Proposition 5.3.** Let assumptions of Proposition 5.2 hold and let $D$ be a nearly Borel finely open subset of $\mathbb{R}^d$ with $d > \alpha$. Then there exists a unique probabilistic solution of the problem
$$-\psi(\nabla)u = f_u + \mu \quad \text{in } D, \quad u = 0 \quad \text{in } D^c. \quad (5.9)$$ Moreover, $f_u \in L^1(D; dx)$, $T_k u \in H^\psi_{D,e}$ for every $k > 0$ and [3.7], [4.5] hold true.

Let us remark that if $D$ is bounded then $H^\psi_{D,e} = H^\psi_{D,1}$, because $H^\psi_{D,1} \hookrightarrow L^2(D; dx)$ in that case. If $D$ is open and has smooth boundary then as in [11] we may define the space $H_0^\psi_{D,1}(D)$ as follows. Given $u \in C^\infty_0(D)$ we extend it to $\mathbb{R}^d$ by setting $u = 0$ on $D^c$. We then obtain a function $u \in C^\infty_0(\mathbb{R}^d)$ with support in $D$. Consequently, we may regard $C^\infty_0(D)$ as a subspace of $H^\psi_{D,1}$ and therefore define $H_0^\psi_{D,1}(D)$ as the closure of $C^\infty_0(D)$ in $H^\psi_{D,1}$. By [7, Theorem 4.4.3], $C^\infty_0(D)$ is a special standard core of $(\Psi, H^\psi_{D,1})$, and hence, by [7, Lemma 2.3.4], $H^\psi_{D,1} = H_0^\psi_{D,1}(D)$.

Assume that $d \geq 3$ and $D \subset \mathbb{R}^d$ is a bounded open set with a $C^1$-boundary. Let us consider the form $(\Psi, H^\psi_{D,1})$ with $\psi(\xi) = c|\xi|^\alpha$ for some $c > 0, \alpha \in (0, 2]$. By [13, Proposition 4.9] there exist constants $0 < c_1 < c_2$ depending only on $d, \alpha, D$ such that
$$c_1 \delta^{\alpha/2}(x) \leq R_1(x) \leq c_2 \delta^{\alpha/2}(x), \quad x \in D,$$ where $\delta(x) = \text{dist}(x, \partial D)$. From this, Theorem 3.5 and Remark 4.5 and it follows that if $f$ satisfies (A1), (A2), (A3$^*$) and $f(\cdot, 0) \in L^1(D; \delta^{\alpha/2}(x) dx), \int_D \delta^{\alpha/2}(x)|\mu|(dx) < \infty$ then the probabilistic solution $u$ of (3.3) belongs to $L^1(D; dx)$.

### 5.4 Dirichlet forms on infinite dimensional state space

Let $H$ be a separable real Hilbert space and let $A, Q$ be linear operators on $H$. Assume that

(a) $A : D(A) \subset H \rightarrow H$ generates a strongly continuous semigroup $\{ e^{-tA} \}$ in $H$ such that $\| e^{tA} \| \leq M e^{-\omega t}$, $t \geq 0$, for some $M > 0, \omega > 0$,

(b) $Q$ is bounded, $Q = Q^* > 0$ and $\sup_{t>0} \text{Tr} Q_t < \infty$, where $Q_t = \int_0^t e^{sA}Q e^{sA^*} ds$,

(c) $Q_\infty(H) \subset D(A)$, where $Q_\infty = \int_0^\infty e^{tA}Q e^{tA^*} dt$.

A simple and important example of $A, Q$ satisfying (a)–(c) is $Q = I$ and a self-adjoint operator $A$ such that $\langle Ax, x \rangle_H \leq -\omega|x|^2_H$, $x \in D(A)$, for some $\omega > 0$ and $A^{-1}$ is of trace class. In this example $Q_\infty = -\frac{1}{2} A^{-1}$. Other examples are to be found for instance in [6].
By (a) the operators $Q_t$, $Q_\infty$ are well defined and by (b), $Q_\infty$ is of trace class. Let $\gamma$ denote the Gaussian measure on $H$ with mean 0 and covariance operator $Q_\infty$. We consider the form

$$E(u,v) = -\int_H \langle \nabla u, AQ_\infty \nabla v \rangle d\gamma, \quad u, v \in FC^\infty_b.$$  

(5.10)

Here $FC^\infty_b$ is the space of finitely based smooth bounded functions, i.e.

$$FC^\infty_b = \{ u : H \to \mathbb{R} : u(x) = f(\langle x, e_1 \rangle, \ldots, \langle x, e_m \rangle), m \in \mathbb{N}, f \in C^\infty_b(\mathbb{R}^m) \}$$

for some orthonormal basis $\{e_k\}$ of $H$ consisting of eigenvectors of $Q_\infty$ and $\nabla$ is the $H$-gradient defined for $u \in FC^\infty_b$ as the unique element of $H$ such that $\langle \nabla u(x), h \rangle_H = \frac{du}{dh}(x)$ for $x \in H$ (the last derivative is the Fréchet derivative in the direction $h$, i.e. $\frac{du}{dh}(x) = \frac{d}{dh}u(x + sh)|_{s=0}$). Under (a)–(c) the form $(E, FC^\infty_b)$ is closable and its closure, which we will denote by $(\mathcal{E}, FC^\infty_b)$, is a coercive closed form on $L^2(H;\gamma)$ (see Theorem 2.2, Remark 2.3 and Lemma 3.3 in [6]). Using the product rule for $\nabla$ on $FC^\infty_b$ one can check in the same way as in [16, Section II.2(d)] (see also [16, Section II.3(c)]) that it has the Dirichlet property. Finally, by results of [16, Section IV.4], it is quasi-regular.

By [6, Theorem 3.6] the semigroup $\{P_t\}$ on $L^2(H;\gamma)$ associated with $(\mathcal{E}, W^{1,2}_Q(H))$ is the Ornstein-Uhlenbeck semigroup of the form

$$P_tf(x) = \int_H f(y)N(e^{tA}x, Q_t)(dy), \quad x \in H,$$

where $N(e^{tA}x, Q_t)$ is the gaussian measure on $H$ with mean $e^{tA}x$ and covariance operator $Q_t$. The generator of $\{P_t\}$ has the form

$$Lu = \frac{1}{2} \text{Tr}(Q\Delta u) - \langle x, A^* \nabla u \rangle_H.$$  

It is worth noting that if (a)–(c) are satisfied and $Q^{-1}$ is bounded then a bounded measure $\mu$ on $H$ is smooth iff it admits a decomposition similar to (5.4). Indeed, if (a)–(c) are satisfied and $\mu \in \mathcal{M}_{0,b}$ then by Remark 2.4 there is $\nu \in S_0$ and $f \in L^1(H;\gamma)$ such that $\langle \mu, \tilde{u} \rangle = \mathcal{E}_1(U_t \nu, u) + \int_H fu \, d\gamma$ for every $u \in W^{1,2}_Q(H)$. By Lapunov’s equation (see [6, Lemma 3.3]), $\mathcal{E}(U_t \nu, u) = \frac{1}{2} \int_H \langle Q\nabla U_t \nu, \nabla u \rangle_H \, d\gamma$. Hence

$$\langle \mu, \tilde{u} \rangle = \frac{1}{2} \int_H \langle F, \nabla u \rangle_H \, d\gamma + \int_H f^0 u \, d\gamma$$

with $F = Q\nabla U_t \nu \in L^2(H;\gamma)$, $f^0 = U_t \nu + f \in L^1(H;\gamma)$. On the other hand, if $F \in C^\infty_b(H;H)$ has finite divergence with respect to $\gamma$ (see [3, Section 11.1] for the precise definition) then by [3, Lemma 11.1.9],

$$\int_H \langle F, \nabla u \rangle_H \, d\gamma = -\int_H (\text{div}_\gamma F)u \, d\gamma,$$

(5.11)

where $\text{div}_\gamma F(x) = \text{div} F(x) - \langle Q_\infty^{-1} x, F(x) \rangle_H$, $x \in H$. This makes it legitimate to write $\mu$ in the form

$$\mu = f^0 - \text{div}_\gamma F.$$  

(5.12)
Conversely, if $\mu \in M_b$ and $\mu$ is of the form \(5.12\) for some $f^0 \in L^1(H; \gamma)$, $F \in L^2(H; \gamma)$ then $\text{div}_\gamma F \in M_b$ and if, $Q^{-1}$ is bounded, then by \(5.11\) and Lapunov’s equation,

$$|\langle \text{div}_\gamma F, u \rangle| \leq \|Q^{-1/2}F\|_{L^2(H;\gamma)}\|Q^{1/2}\nabla u\|_{L^2(H;\gamma)} \leq C\tilde{\mathcal{E}}(u, u)^{1/2}$$

for all bounded $u \in W^{1,2}_Q(H)$. That $\text{div}_\gamma F$ is smooth now follows from \([7, \text{Lemma 2.2.3}]\) applied to the form $\tilde{\mathcal{E}}$.

Since for every $\lambda > 0$ the form $(\mathcal{E}_\lambda, W^{1,2}_Q(D))$ is transient, from the above remarks and Corollary \(4.3\) we obtain the following proposition.

**Proposition 5.4.** Assume \((A1)-(A4)\) and \((a)-(c)\). Then for every $\lambda > 0$ there exists a unique probabilistic solution to the equation

$$-Lu + \lambda u = f_u + \mu.$$ 

Moreover, $f_u \in L^1(H;\gamma)$, $T_k u \in W^{1,2}_Q(D)$ for every $k > 0$ and \(5.7, \ 1.5\) hold true.

For generalizations of forms \(5.10\) to operators $Q$ depending on $x$ or more general measures on topological vector spaces then gaussian measures on Hilbert spaces we refer the reader to \([16, \text{Section II.3}], \ [18]\) and to the references therein).

### 5.5 Concluding remarks

In this subsection we briefly outline how general results on transformation of Dirichlet forms can by applied to obtain other interesting examples of quasi-regular forms.

(i) (Perturbation of Dirichlet forms) Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-Dirichlet form and let $\nu \in S$. Set

$$\mathcal{E}^\nu(u, v) = \mathcal{E}(u, v) + \int_E \tilde{u}\tilde{v} \, d\nu, \quad u, v \in D(\mathcal{E}^\nu),$$

where $D(\mathcal{E}^\nu) = D(\mathcal{E}) \cap L^2(E; \nu)$. By \([19, \text{Proposition 2.3}]\), $(\mathcal{E}^\nu, D(\mathcal{E}^\nu))$ is a quasi-regular Dirichlet form on $L^2(E; m)$. In our context an important example of $\nu$ is $\nu(dx) = V(x) m(dx)$ for some nonnegative $V \in L^1(E; m) \cap L^\infty(E; m)$. In this case $D(\mathcal{E}^\nu) \equiv D(\mathcal{E}^\nu) = D(\mathcal{E})$. Moreover, $(\mathcal{E}^\nu, D(\mathcal{E}^\nu))$ satisfies the strong sector condition if $(\mathcal{E}, D(\mathcal{E}))$ satisfies it and from \((2.5)\) it follows immediately that $(\mathcal{E}^\nu, D(\mathcal{E}^\nu))$ is transient if $(\mathcal{E}, D(\mathcal{E}))$ is transient or $V$ is $m$-a.e. strictly positive. Therefore Propositions \(5.1\) and \(5.4\) hold true for operators $L$ replaced by $L - V$ (In Proposition \(5.4\) we can take $\lambda \geq 0$ if $V$ is $m \equiv \mu$-a.e. strictly positive), and Proposition \(5.3\) holds for $\psi(\nabla)$ replaced by $V$. Note that the perturbed regular form may become non-regular. For instance, in \([10, \text{Section II.2(e)}]\) one can find an example of $V$ such that the classical form $(\mathcal{D}, H^1(\mathbb{R}^d))$ (see Remark \(2.4\)) perturbed by $V$ is not regular.

(ii) (Superposition of closed forms) For $k = 1, 2$ let $(\mathcal{E}^{(k)}, D^{(k)})$ be a closable symmetric bilinear form on $L^2(E; m)$. Set

$$\mathcal{E}(u, v) = \mathcal{E}^{(1)}(u, v) + \mathcal{E}^{(2)}(u, v), \quad u, v \in D,$$

where $D = \{u \in D^{(1)} \cap D^{(2)} : \mathcal{E}^{(1)}(u, u) + \mathcal{E}^{(2)}(u, u) < \infty\}$. By \([16, \text{Proposition I.3.7}]\) the form $(\mathcal{E}, D)$ is closable on $L^2(E; m)$. We may use this property and examples considered in Section \(5.1\)–\(5.4\) to construct new quasi-regular Dirichlet forms. To illustrate how
this work in practice, following [16, Remark II.3.16] we consider the form \((E, FC_0^\infty)\) of Section 5.1 and a symmetric finite positive measure on \((H \times H, B(H) \otimes B(H))\) such that the form
\[
J(u, v) = \int_H \int_H (u(x) - u(y))(v(x) - v(y)) J(dx\,dy), \quad u, v \in FC_0^\infty
\]
is closable. Then the form \((E + J, FC_0^\infty)\) is closable and its closure is a symmetric quasi-regular Dirichlet form. Thus we have constructed an infinite-dimensional (and so non-regular) Dirichlet form which is nonlocal. For the operator corresponding to that form one can formulate an analogue of Proposition 5.4.

General results on superposition of closed form are to be found in [7, Section 3.1] and [16, Proposition I.3.7].

(iii) (Parts of forms) Let \((E, D(E))\) be a symmetric regular Dirichlet form on \(L^2(E; m)\) and let \(D \subset E\) be a nearly Borel measurable finely open with respect to the process \(X\) associated with \((E, D(E))\). Set \(L^2_D(E; m) = \{u \in L^2(E; m) : u = 0\text{ m.a.e. on } D^c \}\) and \(F_D = \{u \in D(E) : \tilde{u} = 0\text{ q.e. on } D^c \}\). By [3, Theorem 3.3.8] the form \((E, F_D)\) on \(L^2_D(E; m)\), called the part of \((E, D(E))\) on \(D\), is a quasi-regular Dirichlet form (if \(D\) is open then it is regular). We can use this result to get solutions of Dirichlet problems of the form (5.9) with \(\psi(\nabla)\) replaced by arbitrary operator associated with a symmetric regular Dirichlet form.

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References

[1] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre and J.L. Vazquez, An \(L^1\)-theory of existence and uniqueness of solutions of nonlinear elliptic equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 22 (1995), 241–273.

[2] L. Boccardo, T. Gallouët and L. Orsina, Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data, Ann. Inst. H. Poincaré Anal. Non Linéaire 13 (1996), 539–551.

[3] Z.-Q. Chen and M. Fukushima, Symmetric markov processes, time change, and boundary theory, Princeton University Press, Princeton, 2012.

[4] G. Da Prato and J. Zabczyk, Second Order Partial Differential Equations in Hilbert Spaces, Cambridge University Press, Cambridge 2002.

[5] M. Fila, Ph. Souplet, and F.B. Weissler, Linear and nonlinear heat equations in \(L^p\) spaces and universal bounds for global solutions, Math. Ann. 320 (2001), 87–113.

[6] M. Fuhrman, Analyticity of transition semigroups and closability of bilinear forms in Hilbert spaces, Studia Math. 115 (1995), 53–71.

[7] M. Fukushima, Y. Oshima and M. Takeda, Dirichlet Forms and Symmetric Markov Processes, Walter de Gruyter, Berlin, 1994.
[8] Ze-Chun Hu, Zhi-Ming Ma and Wei Sun, Extensions of Lévy-Khintchine formula and Beurling-Deny formula in semi-Dirichlet forms setting, J. Funct. Anal. 239 (2006), 179–213.

[9] N. Jacob, Pseudo-Differential Operators and Markov Processes. Vol. I: Fourier Analysis and Semigroups, Imperial College Press, London, 2001.

[10] N. Jacob, Pseudo-Differential Operators and Markov Processes. Vol. II: Generators and Their Potential Theory, Imperial College Press, London, 2002.

[11] N. Jacob and V. Moroz, On the semilinear Dirichlet problem for a class of nonlocal operators generating Dirichlet forms, Progr. Nonlinear Differential Equations Appl. 40 (2000), 191–204.

[12] T. Klimsiak and A. Rozkosz, Dirichlet forms and semilinear elliptic equations with measure data, J. Funct. Anal. 265 (2013), 890–925.

[13] T. Kulczycki, Properties of Green function of symmetric stable processes, Probab. Math. Statist. 17 (1997), 339–364.

[14] Z.-M. Ma, L. Overbeck and M. Röckner, Markov processes associated with semi-Dirichlet forms, Osaka J. Math. 32 (1995), 97–117.

[15] Li Ma, Zhi-Ming Ma and Wei Sun, Fukushima’s decomposition for diffusions associated with semi-Dirichlet forms, Stoch. Dyn. 12 (2012), 1250003, 31 pp.

[16] Z.-M. Ma and M. Röckner, Introduction to the Theory of (Non–Symmetric) Dirichlet Forms, Springer–Verlag, Berlin, 1992.

[17] Y. Oshima, Semi-Dirichlet Forms and Markov Processes. Walter de Gruyter, Berlin, 2013.

[18] M. Röckner, Lp-analysis of finite and infinite dimensional diffusion operators, in: Stochastic PDE’s and Kolmogorov Equations in Infinite Dimensions (Cetraro, Italy 1998), G. Da Prato (ed.), Lecture Notes in Math. 1715 Springer, Berlin, 1999, 65–116.

[19] M. Röckner and B. Schmuland, Quasi-regular Dirichlet forms: examples and counterexamples, Canad. J. Math. 47 (1995), 165–200.

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