Predictability of the imitative learning trajectories

Paulo R. A. Campos
Departamento de Física, Universidade Federal de Pernambuco, 52171-900 Recife, PE, Brazil
José F. Fontanari
Instituto de Física de São Carlos, Universidade de São Paulo,
Caixa Postal 369, 13560-970 São Carlos, São Paulo, Brazil

The fitness landscape metaphor plays a central role on the modeling of optimizing principles in many research fields, ranging from evolutionary biology, where it was first introduced, to management research. Here we consider the ensemble of trajectories of the imitative learning search, in which agents exchange information on their fitness and imitate the fittest agent in the population aiming at reaching the global maximum of the fitness landscape. We assess the degree to which the starting and ending points determine the learning trajectories using two measures, namely, the predictability that yields the probability that two randomly chosen trajectories are the same, and the mean path divergence that gauges the dissimilarity between two learning trajectories. We find that the predictability is greater in rugged landscapes than in smooth ones. The mean path divergence, however, is strongly affected by the search parameters — population size and imitation propensity — that obliterate the influence of the underlying landscape. The imitative search becomes more deterministic, in the sense that there are fewer distinct learning trajectories and those trajectories are more similar to each other, with increasing population size and imitation propensity. In addition, we find that the roughness, which measures deviation from additivity of the fitness function, of the learning trajectories is always greater than the roughness estimated over the entire fitness landscape.

I. INTRODUCTION

Search is fundamental for human decision-making because the available choices usually result from discovery through search activities [1]. Hence, given the central role played by imitation on the construction of human culture [2, 3] (or, more restrictedly, of the community of knowledge [4]), which was neatly summarized by the phrase “Imitative learning acts like a synapse, allowing information to leap the gap from one creature to another” [5], it has been suggested that agent-based models of the imitative learning search could reproduce some features of problem-solving strategy and performance of groups [6, 7]. In fact, in the case that the agents are too propense to imitate their more successful peers or that the group is too large, the resulting disastrous performance of the imitative learning search, as compared with the baseline situation where the agents work independently of each other, is reminiscent of the classic Groupthink phenomenon of social psychology that occurs when everyone in a group starts thinking alike [8]. Otherwise, if the imitation propensity and the group size are properly set, then the imitative search greatly improves the group performance, as expected [6, 7].

The standard theoretical framework to study search heuristics is the fitness landscape metaphor first introduced in the context of evolutionary biology to assess the consequences of the genotype-fitness map [9]. The basic idea is that points in a multidimensional space representing all possible gene combinations (genotypes) compose the domain of a real valued function that represents the fitness of each genotype. Since the expected effect of natural selection is the increase of fitness, evolution can be viewed as a hill-climbing or adaptive walk in the space of genotypes. The effectiveness of natural selection would thus be strongly dependent on the ruggedness of the fitness landscape [10].

Within the fitness landscape framework we can, in principle, represent the entire evolutionary history of a population by a trajectory on the landscape, and so the analysis of the ensemble of the evolutionary trajectories can offer an approach to the (open) issue whether evolution is predictable or not. In other words, if “the tape of evolution is replayed” should we expect a completely different outcome? (See [11] and [12] for different viewpoints on this matter.) Of course, the issue here is not whether the population dynamics is deterministic or stochastic, but whether the constraints imposed by the underlying fitness landscape are strong enough to drive the (stochastic) evolution towards the same evolutionary paths.

The study of evolutionary predictability through the analysis of the ensemble of evolutionary trajectories resulted in the proposal of (at least) two measures aiming at quantifying the diversity and divergence of trajectories that begin at a same point in genotype space and end also at a same, distinct point, usually the global maximum of the fitness landscape [13, 15]. The first measure is the predictability that yields the probability that two evolutionary trajectories picked at random from the ensemble of trajectories are the same. The second measure is the mean path divergence that gauges the dissimilarity between two randomly chosen trajectories in the ensemble of trajectories. Those measures indicate that adaptive walks in rugged landscapes are more deterministic and hence more predictable than walks in smooth land-
scapes. However, due mainly to the technical difficulty of defining and following evolutionary trajectories using standard evolutionary algorithms, there are no systematic studies of the effects of different degrees of ruggedness on the statistical properties of the ensemble of trajectories.

Here we consider the ensemble of learning trajectories of a well-mixed population of agents that search for the unique global maximum of NK-fitness landscapes \[16\] using imitative learning. In this context, there is little ambiguity in the definition of a learning trajectory, which is the ordered sequence of states assumed by the model agent, i.e., the fittest agent in the population, who can influence all the other agents through the imitation procedure. We note that the NK model is the canonical modeling approach used in management research to study decision making in complex scenarios where the decision variables interact in determining the value (fitness) of their combinations \[18–21\]. In addition, use of the NK model allows the tuning of the ruggedness of the landscape and hence the study of the effects of the topography of the fitness landscape on the repeatability of the imitative learning trajectories in independent searches.

We find that the properties of the ensemble of learning trajectories are determined by the interaction between the search dynamics and the underlying fitness landscape. In agreement with the results for evolutionary trajectories, we find that the learning trajectories are more deterministic, in the sense that there are fewer and more similar paths joining the initial and end points of the search, for rugged landscapes than for smooth (additive) ones. However, increasing the ruggedness of the landscape, say by increasing the parameter \(K\) of the NK model, does not necessarily make the search more deterministic. The distance and the difference in fitness between the local maxima are probably influential factors to their attractivity, which greatly impact on the properties of the learning trajectories. The increase of either the population size or the imitation propensity makes the search more deterministic since both parameters enhance the attractivity of the local maxima, thus forcing the trajectories to pass through them. The existence of few escape routes from the local maxima results in more predictable and less divergent learning trajectories.

Another interesting aspect of the study of trajectories, which is not directly related to the predictability issue, is the possibility of assessing the ruggedness, defined as the deviation from additivity \[22\], experienced by the search on a specific learning trajectory. This analysis shows that changing the parameters of the model, i.e., the group size and the imitation propensity, changes the characteristics of the trajectories of the imitative search and that those trajectories exhibit a roughness much greater than the average roughness of the landscape.

The rest of this paper is organized as follows. In Section \[II\] we offer an outline of the NK model of rugged fitness landscapes \[16\] and present two measures of ruggedness used to characterize those landscapes. The rules used by the agents to explore the state space of the NK-landscape in their searches for the global maximum are explained in Section \[III\] together with the definition of the (purged) learning trajectories. The measures of predictability and similarity of the learning trajectories are then introduced in Section \[IV\]. In Section \[V\] we present and analyze the results of our simulations, emphasizing the effect of the ruggedness of the landscape on the learning trajectories. Finally, Section \[VI\] is reserved to our concluding remarks.

II. NK MODEL AND ROUGHNESS

The NK model \[16\] is the choice computational implementation of fitness landscapes that has been extensively used to study optimization problems in population genetics, developmental biology and protein folding \[17\]. In fact, the repute of the NK model goes way beyond the (theoretical) biology realm, as that model is considered a paradigm for problem representation in management research \[18–21\], since it allows the manipulation of the difficulty of the problems and challenges posed to individuals and companies. The NK model of rugged fitness landscapes is defined in the space of binary strings of length \(N\) and so this parameter determines the size of the state space, namely, \(2^N\). The other parameter, \(K = 0, \ldots, N - 1\), determines the range of the epistatic interactions among the bits of the binary string and influences strongly the number of local maxima on the landscape.

More pointedly, the state space of the NK landscape consists of the \(2^N\) distinct binary strings of length \(N\), which we denote by \(x = (x_1, x_2, \ldots, x_N)\) with \(x_i = 0, 1\). To each string \(x\) we associate a fitness value \(F(x)\) that is given by an average of the contributions from each component \(i\) of the string, i.e.,

\[
F(x) = \frac{1}{N} \sum_{i=1}^{N} f_i(x),
\]

where \(f_i\) is the contribution of component \(i\) to the fitness of string \(x\). The local fitness \(f_i\) depends on the state \(x_i\) as well as on the states of \(K\) distinct randomly chosen components, i.e., \(f_i = f_i(x_1, x_2, \ldots, x_K)\), with \(i_1 \neq i_2, \ldots, \neq i_K \neq i\). Hence \(K\) measures the degree of interaction (epistasis) among the components of the strings. Here we assume the \(f_i\)s are \(N\) distinct real-valued functions on \(\{0, 1\}^{K+1}\) and, as usual, we assign to each \(f_i\) a uniformly distributed random number in the unit interval \([0, 1]\). Because of the randomness of \(f_i\), we can guarantee that \(F \in (0, 1)\) has a unique global maximum.

For \(K = 0\), the NK-fitness landscape exhibits a single maximum, which is easily determined by picking for each component \(i\) the state \(x_i = 0\) if \(f_i(0) > f_i(1)\) or the state \(x_i = 1\), otherwise. In time, a string is a maximum if its fitness is greater than the fitness of all its \(N\) neighboring strings (i.e., strings that differ from it at
a single component). For \( K = N - 1 \), the fitness values of neighboring configurations are uncorrelated and so the NK model reduces to the Random Energy model \[23, 24\]. This (uncorrelated) landscape has on the average \( 2^N / (N + 1) \) maxima with respect to single bit flips \[16\].

A simple and popular measure of the ruggedness of a landscape is the fraction of maxima \( \eta_{\text{max}} \), i.e., the number of strings with no fitter neighbors divided by the total number of strings in the landscape \[16\]. Here we will also consider an alternative measure that has been used to model empirically adaptive landscapes in the protein evolution literature \[10, 13, 15, 22\]: the roughness \( \rho \) that measures the deviation from additivity of a landscape. Since \( \rho \) is measured over a particular trajectory towards the global maximum, it is a valuable tool to assess the nature of the search on rugged landscapes. In the following we offer a brief sketch of this important measure \[22\].

Without loss of generality, we can assume that the fitness \( F(x) \) of a string \( x \) is given by the sum of an additive term \( F(x) \) and a non-additive term \( \Omega(x) \), i.e.,

\[
F(x) = F(x) + \Omega(x). \tag{2}
\]

Ideally, the non-additive term \( \Omega \) should be a small random component because, in practice, protein fitness landscapes are nearly additive close to the maximum \[10\], but here we impose no condition on \( \Omega \) since \( F(x) \) is an NK landscape fully specified by eq. (1). The additive term is given by

\[
F(x) = F_{\text{max}} + \sum_{j=1}^{N} w_j(x) \tag{3}
\]

where \( F_{\text{max}} = F(x_{\text{max}}) \) is the fitness value of the unique global maximum \( x_{\text{max}} \) of the landscape and

\[
w_j(x) = \begin{cases} 0 & \text{if } x_j = x_{j,\text{max}} \\ v_j & \text{if } x_j \neq x_{j,\text{max}} \end{cases} \tag{4}
\]

Hence, once \( F_{\text{max}} \) is known, we need only to specify the \( N \) unknowns \( v_j \) to determine the values of the additive fitness \( F(x) \) for all the \( 2^N \) strings. This can be done by considering the \( N \) neighbors of the global maximum, \( x_{j,\text{max}} = (x_{1,\text{max}}, \ldots, x_{j,\text{max}}, \ldots, x_{N,\text{max}}) \) for which we can rewrite eq. (3) as

\[
F(x_{j,\text{max}}) = F_{\text{max}} + v_j. \tag{5}
\]

Now, since the NK landscape defined in eq. (1) is clearly additive for \( K = 0 \), we must require that \( F(x) = F(x) \) in this case. This condition is fulfilled provided we define

\[
v_j = F(x_{j,\text{max}}) - F_{\text{max}} \tag{6}
\]

for \( j = 1, \ldots, N \). So the fitness of the additive landscape \( F(x) \) is guaranteed to coincide with the fitness of a general NK landscape \( F(x) \) at the global maximum and at its neighbors. In addition, in the case \( K = 0 \) we have \( F(x) = F(x) \) for all strings, since \( w_j(x_j) = [f_j(x_j) - f_j(x_{j,\text{max}})] / N \).

Let us consider a trajectory on an NK landscape \( F(x) \) of length \( T \) that is comprised of the sequences \( x^1, x^2, \ldots, x^T = x_{\text{max}} \). The roughness \( \rho \) of the landscape measured along this trajectory is defined as \[22\]

\[
\rho = \left[ \frac{1}{T} \sum_{a=1}^{T} [F(x^a) - F(x^a)]^2 \right]^{1/2}, \tag{7}
\]

which depends on the particular trajectory followed by the imitative search on its way towards the global maximum of the landscape. The global roughness of a landscape \( \hat{\rho} \) is defined by summing over all the state space in eq. (7) and setting \( T \) to the size of that space, i.e., \( 2^N \).

In this paper we consider single realizations of NK-fitness landscapes with \( N = 12 \) and \( K = 0, 2, 4, \) and 11. Table I shows the fraction of maxima \( \eta_{\text{max}} \) and the global roughness \( \hat{\rho} \) of those landscapes, which accord to our expectation that those ruggedness measures increase with the parameter \( K \). We note that for \( K = 0 \) we have \( \eta_{\text{max}} = 1/2 \) (there is a single maximum) and \( \hat{\rho} = 0 \) (the landscape is additive). As already pointed out, for \( K = 11 \) the expected fraction of maxima is \( 1/(N + 1) = 1/13 \approx 0.077 \), a result derived by averaging over all possible landscapes with \( N = 12 \) and \( K = 11 \) \[16\]. Our single realization of this ensemble of landscapes exhibits the fraction \( \eta_{\text{max}} = 0.071 \), which indicates it is a representative instance of that ensemble.

### III. IMITATIVE LEARNING

We consider a well-mixed population of \( M \) agents or binary strings of length \( N = 12 \) that explore the state space of an NK-fitness landscape aiming at reaching the global maximum \( x_{\text{max}} = (x_{1,\text{max}}, x_{2,\text{max}}, \ldots, x_{N,\text{max}}) \). Initially, all strings are identical (isogenic population) and set to the antipode of \( x_{\text{max}} \), i.e., \( (1 - x_{1,\text{max}}, 1 - x_{2,\text{max}}, \ldots, 1 - x_{N,\text{max}}) \). The imitative search strategy is based on the presumption that it may be advantageous for an agent to copy or imitate the agent with the highest fitness in the population, the so-called model agent \[25, 26\]. In this paper we use the terms agent and string interchangeably.

| \( K \) | \( \eta_{\text{max}} \) | \( \hat{\rho} \) |
| --- | --- | --- |
| 0 | 0.00024 | 0 |
| 2 | 0.00366 | 0.295 |
| 4 | 0.01147 | 0.6917 |
| 11 | 0.07129 | 1.8612 |

Table I. Fraction of maxima \( \eta_{\text{max}} \) and roughness \( \hat{\rho} \) of the NK-fitness landscapes with \( N = 12 \) used in the computational experiments.
More pointedly, we implement the synchronous or parallel update of the $M$ agents as follows. At time $t$ we first determine the model agent and then we let each agent to choose between two actions. The first action, which happens with probability $1-p$, consists of simply flipping a bit at random. The second action, which happens with probability $p$, is the imitation of the model agent. The model and the target agents are compared and the different bits are singled out. Then the target agent selects at random one of the distinct bits and flips it so that this bit is now the same in both agents. As expected, imitation results in the increase of the similarity between the target and the model agents, which may not necessarily lead to an increase of the fitness of the target agent if the landscape is not additive. In the case the target agent is identical to the model agent, which happens not only in our initial isogenic population setup but also during the search since the imitation procedure reduces the diversity of the strings, the target agent flips a bit at random with probability one.

The parameter $p \in [0,1]$ is the imitation propensity, which is the same for all agents (see [27] for the relaxation of this assumption). The case $p = 0$ corresponds to the baseline situation in which the agents explore the state space independently of each other. The case $p = 1$ corresponds to the situation where only the model string explores the state space through random bit flips; the other strings simply follow the model, thus making the population almost isogenic.

We note that during the increment from $t$ to $t+1$ all $M$ agents are updated either by flipping a bit at random or by copying a bit of the model string. The search ends when one of the agents finds the global maximum and we denote by $t^*$ the time when this happens. Clearly, at time $t = t^*$ the model string is the global maximum $x^{\text{max}}$. We stress that the imitative search always reaches the global maximum. This contrasts with the adaptive walks [16], where only bit flips that increase the fitness of the strings are allowed, which may permanently get stuck in a local maximum of the landscape. For those walks it is important to study the influence of the topography of the landscape on the accessibility of the global maximum [25], whereas for the imitative search the relevant issue is determining the time needed to find the global maximum.

In spite of apparent similarities, the imitative learning search is markedly different from the evolutionary algorithms (see [29, 30]). In fact, the exploration of the state space through the flipping of randomly chosen bits is similar to the mutation operator of those algorithms, with the caveat that in evolutionary algorithms mutation is an error of the reproduction process, whereas in the imitative search flipping a bit and imitation are mutually exclusive processes. The analogy between the imitation and the crossover processes is more flimsy since the model agent is a mandatory parent in all mates but it contributes a single gene (i.e., a single bit) to the offspring which then replaces the other parent, namely, the target agent. Since the contributed gene is not random - it must be absent in the target agent - the genetic analogy is clearly inappropriate and so the imitative learning search stands on its own as a search strategy.

The trajectory of a particular imitative search is the ordered sequence of the model strings that begins at the antipode of $x^{\text{max}}$ and ends at $x^{\text{max}}$. Thus an unpurged trajectory has $t^*$, not necessarily distinct, strings. However, the trajectories we consider here are purged of loops and so they contain typically much fewer than $t^*$ sequences. Explicitly, if a string appears as a model string more than once in a search, thus forming a loop structure, the trajectory is redefined and the loop is removed. In this way no string can appear more than once in the resulting purged learning trajectory. Therefore, for our purposes, a learning trajectory is described by a directed graph with no loop structures and where each node in the graph denotes a model string.

The total number of agent updates to find the global maximum is $Mt^*$ and since we expect that $t^*$ will increase with the size $2^N$ of the state space of the NK-fitness landscape, we define the computational cost $C$ of the search as

$$C = Mt^*/2^N$$

so that $C$ is on the order of 1 for the independent search (see [2] for an analytical derivation of the computational cost of the independent search). Whereas the previous studies of the imitative search focused on the dependence of the computational cost on the two parameters of the search strategy, namely, the number of agents $M$ and the imitation propensity $p$ [6, 2], here our focus is on the statistical characterization of the ensemble of learning trajectories. We use two quantitative measures introduced in the study of evolutionary trajectories [13] - the predictability and the path divergence - which we describe in detail in the next section.

### IV. CHARACTERIZATION OF TRAJECTORIES

For each run of the imitative search, we keep track of the entire (unpurged) learning trajectory comprising $t^*$ strings beginning at the antipode of $x^{\text{max}}$ and ending at $x^{\text{max}}$. We then purge the trajectory from the loops and store it in the trajectory ensemble. This procedure is repeated $A = 10^6$ times. Hence, the trajectory ensemble contains $A$ learning trajectories of the imitative search for a fixed realization of the NK-fitness landscape and for fixed parameters $p$ and $M$. Let us denote by $q_o$ one such a trajectory and by $O(q_o)$ its probability of occurrence in the ensemble, which is given by the ratio between the number of times $q_o$ appears in the ensemble and the ensemble size $A$. The predictability of the imitative search is defined as the probability that two independent runs result in the same (purged) learning trajectory [31] (see also [13, 15]), i.e.,

$$P_2 = \sum_{q_o} O^2(q_o)$$
where the sum is over all distinct trajectories in the trajectory ensemble. This simple measure of the repeatability of the trajectories varies from \( P_2 = 1/A \) in the case all trajectories in the ensemble are different to \( P_2 = 1 \) in the case there is a single trajectory joining the initial and end points of the search \([13]\). The inverse of the predictability, \( 1/P_2 \), can be viewed as the effective number of trajectories that contribute to the imitative search.

A drawback of the predictability \( P_2 \) is that very similar, but not identical, trajectories are counted simply as different trajectories and so this quantity offers no clue on the similarity between the different trajectories followed by the imitative search. Since all learning trajectories have the same starting and ending points, we can use the mean path divergence originally introduced in the study of evolutionary paths \([13]\) to assess the trajectories similarity. The key element here is the divergence \( d(q_\alpha, q_\beta) \) between trajectories \( q_\alpha \) and \( q_\beta \), which may exhibit very distinct lengths, that is calculated as follows. For each string in \( q_\alpha \) we measure the Hamming distances to all strings in \( q_\beta \) and keep the shortest distance only. Then \( d(q_\alpha, q_\beta) \) is defined as the average of these shortest Hamming distances over all strings in \( q_\alpha \). Note that \( d(q_\alpha, q_\alpha) = 0 \) and \( d(q_\alpha, q_\beta) \neq d(q_\beta, q_\alpha) \). Finally, the mean path divergence is defined as

\[
\bar{d} = \frac{1}{\alpha} \sum_{q_\alpha} O(q_\alpha) \sum_{q_\beta} O(q_\beta) d(q_\alpha, q_\beta),
\]

which yields the expected divergence of two trajectories drawn at random from the ensemble of trajectories \([13]\).

V. RESULTS

As a measure of the performance of the population in searching for the global maximum of the NK-fitness landscapes, we consider the mean computational cost \( (C) \), which is obtained by averaging the computational cost \( \langle C \rangle \) over \( 10^6 \) distinct searches for each landscape realization. The dependence of this mean cost on the model parameters \( M \) and \( p \) for homogeneous, well-mixed populations has been extensively studied in previous works \([7]\) and so we will discuss it here for the sake of completeness only.

Figure 1 shows \( (C) \) against the imitation propensity \( p \) for populations of fixed size \( M = 50 \) and for landscapes of distinct ruggedness. For the smooth, additive landscape \( (K = 0) \) the cost decreases monotonously with increasing \( p \). This is expected because in this case the fitness of a string is highly correlated to its distance to the global maximum and so it is always beneficial to imitate the model agent. This scenario changes for rugged landscapes due to the presence of local maxima that may trap the entire population if the imitation propensity is too high. The catastrophic performance observed in this case is akin to the Groupthink phenomenon \([8]\), when everyone in a group starts thinking alike, which can occur when people put unlimited faith in a talented leader (the model agent). In this case, there is an optimal value of the imitation propensity that minimizes the cost. For too rugged landscapes (say, \( K \geq 4 \) in Fig. 1), the best strategy for \( M = 50 \) is never imitate the model (i.e., \( p = 0 \), thus allowing the agents to explore the landscape independently of each other.

Figure 2 reveals a curious result: the hardest challenge to the imitative search is not the most rugged landscape \( (K = 11) \) but the moderately rugged landscape \( (K = 4) \). The reason is that the local maxima for \( K = 4 \) are farther apart than for \( K = 11 \) which makes the escape from them much more costly.

We turn now to the study of the learning trajectories using the two measures introduced in Section IV.
shows the predictability $P_2$ for the learning trajectories that resulted in the mean computational costs exhibited in Fig. 1. We recall that since the size of the ensemble of trajectories is $A = 10^6$ our results are not statistically reliable in the regime that $P_2 \approx 10^{-6}$. Nevertheless, the results of Fig. 2 are bewildering since they show that even if the computational costs are practically indistinguishable for the different landscapes, as in the case of small $p$, the ensembles of learning trajectories are completely different. In particular, for the smooth ($K = 0$) or the nearly smooth ($K = 2$) landscapes, for which the local maxima have little effect on the search (see Fig. 1), the learning trajectories are practically unpredictable: each run follows a distinct learning trajectory. For the more rugged landscapes, however, the local maxima seem to act as mandatory rest areas of the imitative search resulting in a substantial increase of the predictability. This effect is enhanced by the increase of the imitation propensity, as expected, making the dynamics almost deterministic for the landscape with $K = 4$.

In contrast to the computational cost (Fig. 1), the ensemble of trajectories shows a strong dependence on the ruggedness of the landscape even in the case $p = 0$ (independent search). This is so because the learning trajectories are the trajectories of the model strings, i.e., the strings of highest fitness at a given time, and so the evolution of those strings must necessarily be influenced by the fitness landscape. In this case, the attractivity of the local maxima is due to the effect of reverse bit flippings.

As pointed out in Section IV, the mean path divergence $d$ offers a view of the similarity of the learning trajectories. This quantity is shown in Fig. 3 for the same trajectories considered in the analysis of the predictability. The monotonic decreasing of $d$ with increasing $p$ is expected since the overlap among trajectories must increase due to the boosted influence of the local maxima. We note that there is no direct relation between the predictability $P_2$ and the path divergence $d$. For instance, consider a situation where the search can follow two trajectories only and that those, say equally probable, trajectories have negligible overlap. This hypothetical scenario would produce highly divergent and highly predictable trajectories. Thus, although there are fewer distinct trajectories in the ensemble for $K = 11$ than in the ensemble for $K = 0$ (see Fig. 2), the distinct trajectories for $K = 11$ are much more divergent than those for $K = 0$. In this line, the comparison between the results for $K = 4$ and $K = 0$ for small $p$ indicates that the few distinct trajectories that compose the ensemble of trajectories for $K = 4$ are as divergent as the multitude of trajectories of the ensemble for $K = 0$. However, for high $p$ the ensemble of trajectories for $K = 4$ exhibits the hallmarks of a deterministic dynamics, namely, high predictability and low divergence.

The previous analysis considered a fixed population size $M = 50$ and focused on the influence of the imitation propensity $p$ on the properties of the ensemble of learning trajectories. Regarding the predictability of stochastic dynamic processes, the study of the influence of the population size $M$ is important as one usually expects that the dynamics becomes more deterministic as the population increases. Figure 4 summarizes our findings for a fixed NK-fitness landscape with $N = 12$ and $K = 4$. Both measures $P_2$ and $d$ indicate that the imitative search becomes more deterministic, in the sense that there are fewer distinct learning trajectories and those trajectories are more similar to each other, as $M$ increases. The reason for this is the strengthening of the attractivity of the local maxima as evinced by the increase of the computational cost. In fact, if the search is forced to visit the regions around those maxima, the resulting trajectories must exhibit a high overlap. Large populations increase the attractivity of the local maxima simply because they allow the existence of several copies of the model agent and this makes it very hard for the imitative search to explore other regions of the state space through the flipping of bits at random, since the extra copies attract the updated model agent back to the local maximum. Hence what makes the dynamics more deterministic with increasing $M$ is not the decrease of the size of fluctuations around the deterministic trajectory, but the mandatory passage through the local maxima of the landscape and the few routes of escape from them.

Finally, Fig. 5 reveals the power of the trajectory-dependent roughness $\rho$ defined in eq. (7) to explain the nature of the imitative learning search. Actually, since $\rho$ is defined for a particular trajectory, this figure presents the average of $\rho$ over the $A = 10^6$ trajectories in the ensemble of learning trajectories. The results show that the imitative search seems to experience a more rugged landscape as $p$ or $M$ increases, although the landscape is fixed for the data shown in the figure. It is interesting that the roughness of the learning trajectories are always greater than the global roughness $\bar{\rho}$ of the landscape. As
the ending points of the dynamic trajectories. Although the original motivation to study ensembles of evolutionary trajectories was to offer an answer to the ultimate (and still open) question whether evolution is deterministic and hence predictable or stochastic and hence unpredictable [13-15], the measures for the predictability and the divergence of trajectories introduced in those studies could also be fruitfully applied to study dynamical systems outside the biological realm.

Accordingly, in this paper we have analyzed the learning trajectories of the imitative search [9] on NK-fitness landscapes using the predictability measure $P_2$, which yields the probability that two independent runs of the search follow the same trajectory on the state space [31], and the path divergence $d$, which measures the dissimilarity of the learning trajectories [13]. Regarding the use of these measures, a great advantage of the imitative search over the traditional algorithms used to model the evolutionary dynamics is the ease to define a learning trajectory, which is the ordered, loop-free sequence of model strings that ends at the (unique) global maximum of the landscape and begins at its antipode. Since in the imitative search the model strings play an important role in guiding the population towards the global maximum, it is natural to use them to compose the learning trajectories. In the evolutionary algorithms, however, there is no such a natural choice: the fittest string at a given generation is more likely to contribute offspring to the next generation but does not have a global guiding role as the model string of the imitative learning, and the consensus string is an artificial construct that plays no role in the population dynamics.

We find that the measures $P_2$ and $d$ depend not only on the underlying fitness landscape but also on the way the population explores the landscape, which may enhance some features of its topography as, for instance, the attractiveness of the local maxima. This effect makes it difficult to draw general conclusions on the influence of the topography on the learning trajectories. Nevertheless, our results for the predictability of the trajectories (see Fig. 3) accord with the empirical findings that the landscape ruggedness caused by magnitude and sign epistasis constrains the pathways of protein evolution without shutting off pathways to the maximum [10]. This is exactly what we observe when we compare the predictability for the additive ($K = 0$) landscape with the predictability for rugged landscapes. This result is expected since a rugged landscape exhibits a multitude of minima that are actively avoided by both the imitative and the evolutionary searches. Avoiding these minima as well as the regions of low fitness around them, restricts the learning trajectories and hence increases the predictability. However, the predictability exhibits a non-monotonic dependence on the ruggedness of the NK-fitness landscapes: although the trajectories on a rugged landscape are more predictable than on a smooth landscape, increasing the ruggedness of the landscape does not necessarily increase the predictability of the trajectories.

VI. DISCUSSION

The study of the statistical properties of the ensembles of trajectories offers an alternative perspective to assess the behavior of agent-based models, whose analysis has mostly focused on the length and on the nature of

![Graph](image1)

**Fig. 4.** Influence of the population size $M$ on (a) the mean computational cost $\langle C \rangle$, (b) the predictability $P_2$ and (c) the mean path divergence $d$, for the imitation propensities $p = 0.1, 0.3, 0.5$ as indicated. The parameters of the landscape are $N = 12$ and $K = 4$.

![Graph](image2)

**Fig. 5.** Mean roughness $\langle \rho \rangle$ of the learning trajectories as function of (a) the imitation propensity $p$ for $M = 50$ and (b) the population size $M$ for $p = 0.1, 0.3, 0.5$ as indicated. The dashed lines show the global roughness of the landscape, $\hat{\rho} = 0.6917$. The parameters of the landscape are $N = 12$ and $K = 4$.

expected, for large $M$ or $p$ the roughness saturates to a value that depends only on the landscape and that characterizes the trajectory of maximum roughness on it.
The high predictability and low divergence trajectories observed when the population size $M$ or the imitation propensity $p$ are large correlate very well with the regions of high computational cost, which indicates that these trajectories were trapped in local maxima, the escape of which was very costly to the imitative search. We note that the reason the local maxima makes the search more deterministic is not only that they attract the trajectories but that there are only a few effective escape routes away from them. This perspective allows us to understand why the computational cost and the predictability are higher for the landscape with $K = 4$ than for the uncorrelated landscape with $K = 11$ (see Figs. 1 and 2): although the latter landscape exhibits more local maxima, there are many effective and distinct escape paths from them. The mean path divergence (see Fig. 3) accords with and adds to this scenario by showing that the escape paths are more divergent for the uncorrelated landscape.

In addition to the analysis of the predictability and divergence measures, the knowledge of the learning trajectories allows us to examine the influence of the model parameters on the way the imitative search experiences the landscape. This is possible due to the introduction of the trajectory-dependent roughness $\rho$, defined in eq. (7), that reveals quite explicitly that the imitative search follows qualitatively distinct trajectories as the control parameters $M$ and $p$ change. As pointed out, large values of these parameters increase the attractiveness of the local maxima, which results in an increase of the roughness of the learning trajectories. In fact, since $\rho$ measures essentially the deviation from additivity [22] and the local maxima are the hallmarks of a non-additive landscape, a trajectory that visits a large number of them (or their vicinities) is guaranteed to maximize the roughness.

In view of the widespread use of the fitness (or adaptive) landscape metaphor on so many distinct branches of science [17], the analysis of the ensemble of trajectories using the tools discussed in this paper is likely to greatly impact on our understanding of a vast class of dynamical systems. Here we have shown how this analysis shed light on the intricate interaction between the imitative learning search and the underlying fitness landscape, revealing the crucial role the local maxima play as guideposts on the route towards the global maximum.

ACKNOWLEDGMENTS

P.R.A.C. is supported in part by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) by Grant No. 303497/2014-9 and Fundação de Amparo à Ciência e Tecnologia do Estado de Pernambuco (FACEPE) under Project No. APQ-0464-1.05/15. J.F.F. is supported in part by Grant No. 2017/23288-0, Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) and by Grant No. 305058/2017-7, Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq).

[1] H.A. Simon, Models of Man: Social and Rational (John Wiley & Sons, New York, 1957).
[2] S. Blackmore, The Meme Machine (Oxford University Press, Oxford, 2000).
[3] R. Boyd and P.J. Richerson, The Origin and Evolution of Cultures (Oxford University Press, Oxford, 2005).
[4] S. Sloman and P. Fernbach, The Knowledge Illusion: Why We Never Think Alone (Riverhead Books, New York, 2017).
[5] H. Bloom, Global Brain: The Evolution of Mass Mind from the Big Bang to the 21st Century (Wiley, New York, 2001).
[6] J.F. Fontanari, PLoS ONE 9, e110517 (2014).
[7] J.F. Fontanari, Eur. Phys. J. B 88, 251 (2015).
[8] I.L. Janis, Groupthink: psychological studies of policy decisions and fiascoes (Houghton Mifflin, Boston, 1982).
[9] S. Wright, Proc. Sixth Int. Cong. Genet. 1, 356 (1932).
[10] M. Carneiro and D.L. Hartl, Proc. Natl. Acad. Sci. USA 107, 1747 (2010).
[11] S.J. Gould, Full House: The spread of excellence from Plato to Darwin (Three Rivers Press New York, 1997).
[12] S.C. Morris, Life’s Solution: Inevitable Humans in a Lonely Universe (Cambridge University Press, Cambridge, UK, 2004).
[13] A.E. Lobkovsky, Y.I. Wolf and E.V. Koonin, PLoS Comput. Biol. 7, e1002302 (2011).
[14] A.E. Lobkovsky and E.V. Koonin, Frontiers Genet. 3, 246 (2012).
[15] J.A.G. de Visser and J. Krug, Nat. Rev. Genet. 15, 480 (2014).
[16] S.A. Kauffman and S. Levin, J. Theor. Biol. 128, 11 (1987).
[17] S.A. Kauffman, At Home in the Universe: The Search for Laws of Self-Organization and Complexity (Oxford University Press, New York, 1995).
[18] D.A. Levinthal, Manag. Sci. 43, 934 (1997).
[19] D. Lazer and A. Friedman, Admin. Sci. Quart. 52, 667 (2007).
[20] S. Billsinger, N. Stiegitz and T.R Schumacher, Organ. Sci. 25, 93 (2013).
[21] S.M. Reia, S. Herrmann and J.F. Fontanari, Phys. Rev. E 95, 022305 (2017).
[22] T. Aita, M. Iwakura and Y. Husimi, Protein Eng. 14, 633 (2001).
[23] B. Derrida, Phys. Rev. B 24, 2613 (1981).
[24] D.B. Saakian and J.F. Fontanari, Phys. Rev. E 80, 041903 (2009).
[25] L. Rendell, R. Boyd, D. Cownden, M. Enquist, K. Eriksen, M.W. Feldman, L. Fogarty, S. Ghirlanda, T. Lilliecrap and K.N. Laland, Science 328, 208 (2010).
[26] A.J. King, L. Cheng, S.D. Starke and J.P. Myatt, Biol. Lett. 8, 197 (2012).
[27] J.F. Fontanari, EPL 113, 28009 (2016).
[28] B. Schmiegelt and J. Krug, J. Statist. Phys. 154, 334 (2014).
[29] D.E. Goldberg, *Genetic Algorithms in Search, Optimization, and Machine Learning* (Addison-Wesley, Reading, 1989).

[30] T. Bäck, *Evolutionary Algorithms in Theory and Practice: Evolution Strategies, Evolutionary Programming, Genetic Algorithms* (Oxford University Press, New York, 1996).

[31] S.W. Roy, *PLoS ONE* **4**, e4500 (2009).