Fast Rotating Relativistic Stars: Spectra and Stability without Approximation

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The oscillations and instabilities of relativistic stars are studied by taking into account, for the first time, the contribution of a dynamic space-time. The study is based on the linearised version of Einstein’s equations and via this approach the oscillation frequencies, the damping and growth times as well as the critical values for the onset of the secular (CFS) instability are presented. The ultimate universal relations for asteroseismology are derived which can lead to relations involving the moment of inertia and Love numbers in an effort to uniquely constrain the equation of state via all possible observables. The results are important for all stages of neutron star’s life but especially to nascent or post-merger cases.

Keywords: Wave generation and sources, Relativistic stars: Rotation, Stability, Oscillations, Equations of state, Asteroseismology

Introduction.—Oscillations and instabilities of neutron stars were always considered among the promising sources for gravitational waves. This was the reason that attracted the interest of Kip Thorne and collaborators [1,2] already from the mid 60s, while the instabilities associated with spinning neutron stars, discovered and analysed by Chandrasekhar [3] with Friedman & Schutz (CFS) [1,3], are still promising sources. Rotational (CFS) instabilities can be excited in core collapse scenarios [3] and they can be potential sources of gravitational waves in the late post-merger phase [7], i.e., when the final object survives for periods longer than a few tenths of seconds. Furthermore, the CFS-instability of the r-mode can be active in low-mass X-ray binaries (LMXBs) [8] and even affect the evolution of single neutron stars as proposed recently [9]

The different oscillation patterns are characterised by their restoring force, e.g., p(pressure)-modes, g(gravity)-modes, r- or i(Coriolis)-modes, s(shear)-modes or w(wave)-modes. The f-mode is the fundamental mode of the p-mode sequence and it is the oscillation most likely to be excited in violent processes such as neutron star mergers or neutron star formation by supernova core-collapse. The f-mode is associated with major density variations and thus can potentially be an emitter of copious amounts of gravitational radiation. The emission of gravitational waves is the primary reason for the mode’s rapid damping at least for newly born neutron stars.

The efforts to associate the patterns of oscillations with the fundamental parameters of the stars, i.e., mass, radius or equation of state (EoS henceforth) was initiated in the mid 90s and continued for almost two decades (gravitational wave asteroseismology) [10,16].

By now, very robust empirical relations have been derived, connecting the observables, i.e., frequency, damping time, moment of inertia with the fundamental parameters of the non-rotating stars. For example, relations of the form \( \sigma_0 \approx \alpha + \beta (M_0/R_0)^{1/2} \) or \( M_\sigma = F(M_0^3/I) \) [10] could provide the average density or the moment of inertia of the neutron star if the frequency, \( \sigma_0 \), of the f-mode is known.

In the era of gravitational wave astronomy, the various oscillation patterns (traced already in numerical simulations, e.g., [7,15]), if observed, can provide a wealth of information about the emitting sources and their effects can leave their imprints both in the gravitational but also in the electromagnetic spectrum.

We indicatively refer to some recent work relating the f-modes to the Love numbers [19,21] and the post-merger SGRBs [22].

In these early works the rotation of the neutron stars was not taken into account and this was their main drawback. In nature, neutron stars always have some degree of rotation which can reach extreme values. In fact, the most relevant scenarios from the point of view of gravitational wave detectability of oscillation modes are likely to involve rapidly rotating stars. Unfortunately, the aforementioned empirical relations cannot be trivially extended to rotating stars. Rotation splits the spectra in a similar fashion as the Zeeman splitting of the spectral lines due to the presence of magnetic fields. In the stars the splitting leads to perturbations propagating in the direction of rotation (co-rotating) and perturbations travelling backwards (counter-rotating). The oscillation frequency, as observed by an observer at infinity, will either increase or decrease depending on the propagation direction of the waves. For slow rotation there will be a shift of the form \( \sigma \approx \sigma_0 \pm \kappa \Omega + O(\Omega^2) \) where \( \sigma_0 \) is the mode frequency of the non-rotating model, \( m \) the angular harmonic index (\( \sigma/m \) is the pattern velocity) and \( \Omega \) the rotation...
rate of the star. If the spin of the star exceeds a critical value, which depends on the EoS, and its radius and mass, i.e. when the pattern velocity of the backwards moving mode becomes smaller than Ω, then the star is becoming unstable to the emission of gravitational radiation. This is the aforementioned CFS-instability [3–5]. This instability is generic (independent of the degree of rotation) for the r-modes [23–24] while it can be excited only for relatively high spin values (Ω ≳ 0.8ΩK) for the f-modes. An extensive discussion can be found in [25, 26].

For the sake of clarity, we will use upper indices s and u on the coefficients of our models to distinguish between the stable (co-rotating) and the potentially unstable (counter-rotating) branch of the f-mode.

Throughout the article, we employ units in which c = G = M⊙ = 1.

Mathematical Formulation.—We are going to work with the Einstein equations along with the law for the conservation of energy-momentum,

\[ G_{\mu\nu} = 8\pi T_{\mu\nu} \quad \text{and} \quad \nabla_\mu T^{\mu\nu} = 0. \]  

(1)

We restrict ourselves to the study of the dynamics of small perturbations around an equilibrium configuration which allows us to linearise equations (1).

We assume an axisymmetric background described in isotropic coordinates of the form

\[ ds^2 = -e^{2\nu} dt^2 + e^{2\psi} dr^2 + r^2 d\theta^2 + \epsilon e^{2\psi} \sin^2 \theta (d\phi - \omega dt)^2 \]

and generate the equilibrium configurations by means of the nsa-code [27]. The only two non-vanishing components of the 4-velocity are linked via the star’s rotation rate, \( u^\phi = \Omega u^t \).

We model the neutron star to be a perfect fluid without viscosity for which the corresponding energy-momentum tensor takes the form

\[ T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + p g^{\mu\nu}, \]

(3)

where ε is the energy density and p is the pressure. An EoS links the energy density and the pressure to each other; in this Letter, we will—for the purposes of simplicity and comparability—utilise commonly used polytropic EoSs as described in [28]. As in previous studies, we consider sequences of neutron stars along which we keep the central energy density constant, with rotation rates up to their respective mass-shedding limit; we choose different polytropic indices \( N \in [0.68, 1] \). Beyond the polytropic EoS, we employ piecewise-polytropic approximations [29] to three tabulated EoSs, namely APR, SLy, and WFF1, for the scrutiny of more realistic neutron star models, for which we also generate rotational sequences of fixed baryon mass. Our non-rotating configurations have gravitational masses \( M \in [1.3, 2.1]M_\odot \).

In deriving the perturbation equations, we opt for the Lorenz gauge

\[ \nabla^\nu h_{\mu\nu} = 0, \]

(4)

where we denote with \( h_{\mu\nu} \) the trace-reversed metric perturbations. The perturbed Einstein tensor then takes the form

\[ -2\delta G_{\mu\nu} = \Box h_{\mu\nu} + 2\mathcal{R}^\alpha\beta\gamma\delta h_{\alpha\beta} - \mathcal{R}^\alpha\gamma h_{\beta\nu}h_{\alpha\delta} + Rh_{\mu\nu} - g_{\mu\nu}R_{\alpha\beta}h^{\alpha\beta}. \]

(5)

The advantage of this formulation is that the evolution equations for the metric perturbations will take the form of ten coupled wave-like equations while the mixing of temporal and spatial derivatives is avoided, in contrast to the form of the most commonly used gauge choices or the ones without any gauge choice [30].

Our description for the fluid perturbations is inspired by the very convenient formulation that has been developed and successfully used in [31–33] for studies within the Cowling approximation. The system of equations describing the hydrodynamics is essentially the same as in the cited literature, except that the equations gain numerous terms accounting for the space-time perturbations.

Results.—A full list of the results for various equations of state and different sequences of fixed central energy density and fixed baryon mass will be presented in a forthcoming extended article along with the evolution equations used in this effort. Here we will provide some highlighted results in order to demonstrate the existence of asteroseismological relations of various types and we lay out the way that one can make use of these relations in analysing the gravitational wave signals.

More specifically, we will show two different universal relations providing accurate estimates for the f-mode frequency given some bulk parameters of the star and vice versa. First, we observe a universal behaviour of the f-mode frequency, \( \sigma \), as a function of the star’s spinning frequency, \( \Omega \), along a sequence of fixed central energy density models when we normalise both frequencies with the f-mode frequency, \( \sigma_0 \), of the corresponding non-rotating star. Figure 1 displays this behaviour for more than 100 different neutron star models of each the co- and counter-rotating branches of the f-mode; we model the universal behaviour using the quadratic function

\[ \frac{\sigma}{\sigma_0} = 1 + a_1 \left( \frac{\Omega}{\sigma_0} \right) + a_2 \left( \frac{\Omega}{\sigma_0} \right)^2. \]

(6)

The results of the fit are \( a_1 = -1.02 \) and \( a_2 = -1.55 \) for the potentially unstable branch and \( a_1 = 1.22 \) and \( a_2 = -0.63 \) for the stable branch of the f-mode. The
quadratic fit accounts well for the increasing oblateness of the star with the star’s rotation, however, close to the star’s Kepler limit, deviations from this simple model become visible. We point out that our model predicts that the unstable branch of the $f$-mode becomes susceptible to the CFS-instability once the rotation rate of the star exceeds $\Omega \approx 0.54\sigma_0$ (when considering sequences of constant central energy density).

In earlier studies for non-rotating models [11, 34] fitting relations of the form $\sigma_0 = \alpha + \beta(M_0/R_0)^{1/2}$ were derived. Here, $\alpha$ and $\beta$ can be estimated for the EoSs that fulfil the constraints at the time of observation while $M_0$ and $R_0$ correspond to the mass and radius of the non-rotating model. Thus, this relation in combination with equation (6) connects three fundamental parameters of the sequence, i.e., mass and radius of the non-rotating member with the spin of the observed model. Obviously, from a single observation of the $f$-mode frequency one cannot extract these values but can put constraints among the three of them. Any extra observed oscillation frequency, e.g., both co- and counter-rotating frequencies or knowledge of some parameters of the star, e.g., its mass, will place more stringent constraints.

While the above model for the $f$-mode frequency works well, we derived a list of different astroseismological relations connecting the $f$-mode frequency to the rotation rate normalised by the Kepler velocity of the specific sequence, $\Omega/\Omega_K$, the ratio $T/W$, or the rotational deformation of the star $r_p/r_e$, which will be reported in an upcoming article. At this point, we would like to highlight the finding emerging from our simulations that the critical value for the onset of the CFS-instability is $T/W \approx 0.08 \pm 0.01$, in agreement with the limits given in [35] and in contrast to the widely used Newtonian results $T/W = 0.14$.

Another fitting relation which can be easily implemented in solving the inverse problem is incorporating the effective compactness $\eta := \sqrt{M^2/I_{45}} \approx M/R$, where $M$ is the star’s gravitational mass in solar masses and $I_{45}$ its moment of inertia in multiples of $10^{45}$ g cm$^2$, inspired by [10]. We will be guided by the model employed in the Cowling approximation in [36]. The fitting formula is

$$M\sigma = \left(c_1 + c_2\hat{\Omega} + c_3\hat{\Omega}^2\right) + \left(d_1 + d_3\hat{\Omega}^2\right)\eta,$$  

where $\hat{\Omega} := M\Omega$. Using models based on polytropic as well as on realistic EoSs, the resulting coefficients from a least-squares fit for the counter-rotating branch of the $f$-mode are $(c_1, c_2, c_3)^\text{fit} = (-2.1, -1.3, -0.33)$ and $(d_1, d_2, d_3)^\text{fit} = (3.4, 0.0, 0.087)$; for the co-rotating branch, we find the coefficients $(c_1, c_2, c_3)^\text{fit} = (-2.1, 1.5, -0.89)$ and $(d_1, d_2, d_3)^\text{fit} = (3.4, 0.43)$. Qualitatively, our coefficients agree in order of magnitude with those in [36] in the Cowling approximation; comparing the special cases of no rotation, $\hat{\Omega} = 0$, our fitting formula yields roughly 20 – 40% lower frequencies in the fully general relativistic setup, which is in accordance with expectations.

The fitting formula (7) has the advantage that it does not rely on specifically defined sequences of neutron stars, along which a particular property is held constant. For example, equation (6) depends on the $f$-mode frequency, $\sigma_0$, of the (in a very particular fashion) corresponding non-rotating configuration, which may not even exist in some cases (e.g. supramassive neutron stars supported by fast rotation); the latter model, equation (7), is satisfied with bulk properties of the star of which we want to know the oscillation frequency and vice versa. Another advantage of this formulation (7) is that, as it is proven in [36], a similar formula can be derived for higher multiples, i.e., $\ell = 3$ and $\ell = 4$. This is something that has already been observed in our simulations but we do not have enough data yet in order to provide a close formula. Fitting formula (7) can be useful in imposing further constraints on the parameters of the post-merger objects since it combines the mass and spin of the resulting object with the $f$-mode frequency and, via $\eta$, the moment of inertia, $I$, or the compactness, $M/R$. Thus, the latter two can be further constrained by an observation of an $f$-mode signal, as mass and potentially spin can be extracted from the pre-merger and early post-merger analysis of the signal. The situation becomes more attractive if both co- and counter-rotating modes are observed since only the mass of the post-merger object will be needed to constrain its parameters. This will be an independent yet complementary constraint in the estimation of the radius in addition to those based on the Love numbers.

The fluid modes are subject to damping or growth...
due to the presence of a dynamic space-time (for ease of language, when we speak about damping times in the following, we will nonetheless mean both damping and growth times). Due to the long damping times compared to our evolution times, a direct determination of the damping time from the time signal is numerically unfeasible. Instead, we compute estimates for the damping time of a particular mode by employing the quadrupole formula in its empirically improved variant denoted as R/RQF, from [39].

Fitting a cubic polynomial through the data points in a least-squares fashion yields the universal relation:

\[ \frac{\tau_{\text{R}}}{\tau_{\text{H}}} = \left( \frac{\sigma_{\text{n}}}{\sigma_{0}} \right)^4 \left( \frac{1}{0.6 + 0.03 \left( \frac{\sigma_{\text{n}}}{\sigma_{0}} \right) + 0.4 \left( \frac{\sigma_{\text{n}}}{\sigma_{0}} \right)^2} \right)^4, \]

where \( \text{sgn}(x) \) is the signum function.

This formula is relevant for the counter-rotating and potentially unstable modes which are the most relevant for the gravitational wave emission; especially when the growth time of the instability is quite short as it happens for supramassive neutron stars which can be the end product of the merger [4].

Compared to results in the Cowling approximation, our fitting function deviates only little, supporting the position that the most significant change in the damping time when accounting for a dynamic space-time is due to the alteration of the mode frequency. The observed deviations can be accounted to a modification of the eigenfunctions. In contrast, the actual damping times differ significantly. It was stated explicitly in [10] that the damping times calculated in the Cowling approximation were deviating by a factor 3 from the fully relativistic results in the non-rotating limit. Here, our deviations are less than 25%.

Conclusions and Outlook.—In this Letter we report the first successful extraction of frequencies as well as damping times of the \( |m| = 2 \) f-mode of general relativistic, rapidly rotating neutron stars without the commonly used slow-rotation or Cowling approximation. This Letter, together with forthcoming results, concludes a long-standing open problem, building upon the effort from numerous studies throughout the past five decades. Our code, which is evolving the linearised Einstein equations, enables us to calculate the spectrum of a neutron star at comparably low computational expense and high grid resolution.

We provide different universal relations for the frequency as well as the damping time of the \( |m| = 2 \) f-mode. These universal relations will be an essential piece in the asteroseismological toolkit once the third generation GW observatories will be able to pick up the ring-down and fluid ringing signal following the merger of a binary neutron star system; they allow to solve the inverse problem, leading to accurate measurements of mass and radius of the post-merger object. We also report the discovery of an accurate estimate for the onset of the CFS-instability when the f-mode frequency of the non-rotating member of the family is known and verified the critical value of \( T/W \) for the onset of the instability.

The present results serve as first proof for the availability of computational technology which will allow us to further investigate the spectrum and stability of rapidly rotating neutron stars in full General Relativity and to provide further fruitful universal relations for frequencies and damping times of different modes in the near future. A natural extension of our work will be the investigation of other classes of modes (i.e. higher multipole f-modes, low p-modes and g-modes as well as u-modes) which may be excited in different astrophysical processes; beyond that we are going to extend our code to account for differentially rotating neutron stars with “hot EoS” which are especially relevant for nascent neutron stars or post-merger configurations in the immediate aftermath of the merger.

While the majority of the presented results are based on polytropic models we included results from observationally justified EoS, the list of which will be extended in the future. Rather than expecting major changes in the universal relations, we expect minor modifications at most while, however, painting an astrophysically comprehensive picture.

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