Noise-Tolerant Optomechanical Entanglement via Synthetic Magnetism

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Entanglement of light and multiple vibrations is a key resource for multichannel quantum information processing and memory. However, entanglement generation is generally suppressed, or even fully destroyed, by the dark-mode (DM) effect induced by the coupling of multiple degenerate or near-degenerate vibrational modes to a common optical mode. Here we propose how to generate optomechanical entanglement via DM breaking induced by synthetic magnetism. We find that at nonzero temperature, light and vibrations are separable in the DM-unbreaking regime but entangled in the DM-breaking regime. Remarkably, the threshold thermal phonon number for preserving entanglement in our simulations has been observed to be up to 3 orders of magnitude stronger than that in the DM-unbreaking regime. The application of the DM-breaking mechanism to optomechanical networks can make noise-tolerant entanglement networks feasible. These results are quite general and can initiate advances in quantum resources with immunity against both dark modes and thermal noise.

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Introduction.—Quantum entanglement [1], allowing for inseparable quantum correlations shared by distant parties, is a crucial resource for modern quantum technologies, including quantum metrology, communication, and computation [2]. So far, efficient entanglement of photons with atoms [3–9], trapped ions [10,11], quantum dots [12], and superconducting qubits [13–15] has been demonstrated in both microscopic- and macroscopic-scale devices [16,17]. These entangled states have been used to connect remote long-term memory nodes in distributed quantum networks [18–21].

The cavity optomechanical system is an elegant candidate for implementing quantum information carriers and memory [22–25]. Owing to the remarkable progress in ground-state cooling [26–29] and single-phonon manipulation [30–33], it has become an efficient platform for achieving quantum entanglement between two bosonic modes [34–48]. In particular, macroscopic quantum entanglement involving two massive oscillators has recently been observed in optomechanical platforms [49–52]. Practically, the applicability of modern quantum technologies in optomechanical networks ultimately requires quantum entanglement of light and many vibrations [53–56]. Realization of large-scale photon-phonon entanglement, however, remains an outstanding challenge due to the suppression from the dark-mode effect [57–67] induced by the coupling of an optical mode to multiple degenerate or near-degenerate vibrational modes [61–67], which could be used as multiple nodes in quantum networks [18].

In this Letter, we propose to generate light-vibration entanglement by breaking the dark mode via synthetic magnetism, and reveal its counterintuitive robustness to thermal noise. By introducing a loop-coupled structure, formed by light-vibration couplings and phase-dependent phonon-hopping interactions, a synthetic gauge field is induced and it breaks dark modes. Note that the realization of a reconfigurable synthetic gauge field has recently been reported in phase-dependent loop-coupled optomechanical platforms [68–76]. We find that in the dark-mode-unbreaking (DMU) regime, light-vibration entanglement is destroyed by thermal noise concealed in the dark modes; while in the dark-mode-breaking (DMB) regime, optomechanical entanglement is generated via synthetic magnetism. Surprisingly, in the DMB regime, the threshold phonon number for preserving entanglement has been observed to be up to 3 orders of magnitude stronger than that in the DMU regime. Our Letter describes a general mechanism, and it can provide the means to engineer and protect fragile quantum resources from thermal noises and dark modes, and pave a way towards noise-tolerant quantum networks [18,56].

System and dark-mode control.—We consider a loop-coupled optomechanical system consisting of an optical mode and two vibrational modes [see Fig. 1(a)]. A driving field, with frequency \( \omega_\text{L} \) and amplitude \( |\Omega| = \sqrt{2xP_\text{L}/(\hbar\omega_\text{L})} \) (with laser power \( P_\text{L} \) and cavity-field decay rate \( \kappa \)), is applied to the cavity field. In a rotating...
frame defined by the unitary transformation operator \( \exp(-i\omega L c^\dagger c t) \), the system Hamiltonian reads (\( \hbar = 1 \))

\[
\mathcal{H}_l = \Delta c^\dagger c + \sum_{j=1,2} (\omega_j d_j^\dagger d_j + g_j c^\dagger c (d_j + d_j^\dagger)) + (\Omega c + \Omega^* c^\dagger) + \mathcal{H}_p,
\]

\[
\mathcal{H}_p = \chi(e^{i\theta} d_1^\dagger d_2 + e^{-i\theta} d_2^\dagger d_1),
\]

where \( c^\dagger (c) \) and \( d_j^\dagger (d_j) \) are the creation (annihilation) operators of the cavity-field mode (with resonance frequency \( \omega_c \)) and the \( j \)th vibrational mode (with resonance frequency \( \omega_j \)), respectively. The \( g_j \) terms describe optomechanical interactions between the cavity-field mode and the two vibrational modes, with \( g_j \) being the single-photon optomechanical-coupling strength. The \( \Omega \) term denotes the cavity-field driving with detuning \( \Delta = \omega_c - \omega_1 \). The \( \mathcal{H}_p \) term depicts the phase-dependent phonon-hopping interaction (with coupling strength \( \chi \) and modulation phase \( \Theta \)), which is introduced to create synthetic gauge fields and control the dark-mode effect [77].

To demonstrate the dark-mode effect, we expand the operators \( \sigma \in \{c, d_j, c^\dagger, d_j^\dagger\} \) as a sum of their steady-state average values and fluctuations, i.e., \( \sigma = \langle \sigma \rangle_{ss} + \delta \sigma \). Then we obtain the linearized Hamiltonian in the rotating-wave approximation (RWA) as

\[
\mathcal{H}_{\text{RWA}} = \Delta \delta c^\dagger c + \sum_{j=1,2} (\omega_j \delta d_j^\dagger d_j + g_j (\delta c d_j^\dagger + \text{H.c.})) + \chi(e^{i\theta} \delta d_1^\dagger d_2 + \text{H.c.}),
\]

where \( \delta \) is the normalized driving detuning [77] and \( g_j (\langle c \rangle_{ss}) \) are the linearized optomechanical-coupling strengths. Here \( \langle c \rangle_{ss} = -i\Omega^*/(\kappa + i\Delta) \) is assumed to be real by choosing a proper \( \Omega \). Note that the RWA of the light-vibration interaction is performed only in the demonstration of the dark-mode effect and its breaking. In the derivation of entanglement measures and the numerical simulations, we consider both the beamsplitter-type and two-mode-squeezing-type optomechanical interactions.

When the synthetic magnetism is absent (i.e., \( \chi = 0 \)) and \( \omega_1 = \omega_2 \), the system possesses two hybrid mechanical modes: bright (\( D_+ \)) and dark (\( D_- \)) modes defined by

\[
D_\pm = (G_{1(2)} \delta d_1 \pm G_{2(1)} \delta d_2)/\sqrt{G_1^2 + G_2^2},
\]

which satisfy the bosonic commutation relation \([D_\pm, D_\mp^\dagger] = 1\). The dark mode \( D_- \), which decouples from the system and destroys all quantum resources, can be broken by employing the synthetic magnetism (i.e., \( \chi \neq 0 \) and \( \Theta \neq 0 \)). To clarify this, we introduce two superposition-vibrational modes associated with the synthetic magnetism:

\[
\delta D_\pm = F \delta d_1(\mp) + e^{i\pi/4} K \delta d_2(\pm),
\]

which satisfy the bosonic commutation relation \([\delta D_\pm, \delta D_\mp^\dagger] = 1\). Here \( F = |\delta \omega_0|/\sqrt{(\delta \omega_0)^2 + \chi^2} \) and \( K = \chi F/\delta \omega_0 \), with \( \delta \omega_0 = \tilde{\omega}_0 - \omega_1 \) and the redefined resonance frequencies

\[
\tilde{\omega}_0 = (\omega_1 + \omega_2) \pm \sqrt{\omega_1^2 + \chi^2}/2.
\]

The linearized Hamiltonian becomes

\[
\mathcal{H}_{\text{RWA}} = \Delta \delta c^\dagger c + \sum_{i=1}^{\pm} (\tilde{\omega}_i \delta d_i^\dagger d_i + (\tilde{G}_i \tilde{\delta} \delta c^\dagger + \text{H.c.})),
\]

where the effective coupling strengths are \( \tilde{G}_\pm = F G_{1(2)}(\mp) + e^{i\pi/4} K G_{2(1)}(\pm) \). In Fig. 1(c), we show \( \tilde{G}_- \) versus \( \Theta \) when \( \omega_1 = \omega_2 \) and \( G_1 = G_2 \). We see that only when \( \Theta = n\pi \) (i.e., the DMU regime), either \( \tilde{D}_+ \) [for an odd \( n \), \( \tilde{G}_- = 0 \) (blue disks)] or \( \tilde{D}_- \) [for an even \( n \), \( \tilde{G}_+ = 0 \) (red disks)] becomes a dark mode. Tuning \( \Theta \neq n\pi \) (for an integer \( n \), the DMU regime) leads to a counterintuitive coupling of the dark mode to the optical mode, which indicates dark-mode breaking. Physically, a reconfigurable synthetic gauge field is realized by modulating \( \Theta \), which results in a flexible switch between the DMB and DMU regimes.

**Langevin equations and their solutions.**—By defining the optical and mechanical quadratures \( \delta X_\nu = (\delta \alpha^\dagger + \delta \alpha)/\sqrt{2} \) and \( \delta Y_\nu = i(\delta \alpha^\dagger - \delta \alpha)/\sqrt{2} \), and the corresponding Hermitian input-noise operators \( X^\nu_\nu = (\alpha^\dagger_\nu + \alpha_\nu)/\sqrt{2} \) and \( Y^\nu_\nu = i(\alpha^\dagger_\nu - \alpha_\nu)/\sqrt{2} \), we obtain the linearized Langevin equations as \( \dot{\mathbf{u}}(t) = \mathbf{A} \mathbf{u}(t) + \mathbf{N}(t) \), where

- \( \mathbf{A} \)
- \( \mathbf{N}(t) \)

FIG. 1. (a) Loop-coupled optomechanical system consisting of two vibrational modes \( d_{j=1,2} \) (with decay rates \( \gamma_j \)) coupled to a common optical mode \( c \) (with decay rate \( \kappa \)) via optomechanical interactions (with strengths \( g_j \)). The two vibrations are coupled to each other through a phase-dependent phonon-exchange coupling (\( \chi \) and \( \Theta \)). (b) An optomechanical network consisting of an optical mode coupled to \( N \) vibrational modes. (c) Effective coupling strengths \( \tilde{G}_\pm \) versus \( \Theta \). By tuning \( \Theta \neq n\pi \) for an integer \( n \), the dark mode can be broken (\( \tilde{G}_\pm \neq 0 \)). The solid disks denote the DMU regime, and the remaining areas correspond to the DMB regime. Here we choose \( \omega_m \) as the frequency scale and set \( \omega_1/\omega_m = 1 \) and \( G_j/\omega_m = \chi/\omega_m = 0.1 \).
we introduce the fluctuation operator vector \( \mathbf{u}(t) = [\delta X_{d_1}, \delta Y_{d_1}, \delta X_{d_2}, \delta Y_{d_1}, \delta X_{d_2}, \delta Y_{d_2}]^T \), the noise operator vector \( \mathbf{N}(t) = \sqrt{2} [\sqrt{\tau_1} X^\text{in}_{d_1}, \sqrt{\tau_1} Y^\text{in}_{d_1}, \sqrt{\tau_2} X^\text{in}_{d_2}, \sqrt{\tau_2} Y^\text{in}_{d_2}, \sqrt{\kappa} X^\text{in}_{c}, \sqrt{\kappa} Y^\text{in}_{c}]^T \), and the coefficient matrix

\[
A = \begin{pmatrix}
-\gamma_1 & \omega_1 & \chi_+ & \chi_- & 0 & 0 \\
-\omega_1 & -\gamma_1 & -\chi_- & \chi_+ & -2G_1 & 0 \\
-\chi_+ & \chi_- & -\gamma_2 & \omega_2 & 0 & 0 \\
-\chi_- & -\chi_+ & -\omega_2 & -\gamma_2 & -2G_2 & 0 \\
0 & 0 & 0 & 0 & -\kappa & \Delta \\
-2G_1 & 0 & -2G_2 & 0 & -\Delta & -\kappa
\end{pmatrix},
\]

(4)

with \( \chi_+ = \chi \sin \theta \) and \( \chi_- = \chi \cos \theta \). The formal solution of the Langevin equation is given by \( \mathbf{u}(t) = \mathbf{M}(t) \mathbf{u}(0) + \int_0^t \mathbf{M}(t-s) \mathbf{N}(s) ds \), where \( \mathbf{M}(t) = \exp(At) \). Note that the parameters used in our simulations satisfy the stability conditions derived from the Routh-Hurwitz criterion [78]. The steady-state properties of the system can be inferred based on the steady-state covariance matrix \( \mathbf{V} \), which is defined by the matrix elements \( \mathbf{V}_{kl} = \langle [u_k(\infty), u_l(\infty)] + [\langle u_k(\infty) u_l(\infty) \rangle]/2 \rangle \) for \( k, l = 1 \rightarrow 6 \). Under the stability conditions, the covariance matrix \( \mathbf{V} \) fulfills the Lyapunov equation \( \mathbf{AV} + \mathbf{VA}^T = -\mathbf{Q} \) [34], where \( \mathbf{Q} = \text{diag}\{\gamma_1(2\bar{n}_1 + 1), \gamma_1(2\bar{n}_2 + 1), \gamma_2(2\bar{n}_2 + 1), \gamma_2(2\bar{n}_1 + 1), \kappa, \kappa \} \).

Generating bipartite entanglement and full tripartite inseparability via DMB.—The logarithmic negativity \( E_{N',j} \) and the minimum residual contangle \( E^{[\text{res}]}_r \), which can be used to quantify bipartite entanglement and full tripartite inseparability [79–83], are, respectively, defined as

\[
E_{N',j} \equiv \max[0, -\ln(2\zeta^-_j)],
\]

(5a)

\[
E^{[\text{res}]}_r \equiv \min_{(r,s,t)} |E^{(\text{res})}_r - E^{[\text{res}]}_s - E^{[\text{res}]}_t|.
\]

(5b)

Here \( \zeta^-_j \equiv 2^{-1/2}(\Sigma \mathbf{V}_j^T - [\Sigma \mathbf{V}_j^2/4 \det \mathbf{V}_j^{1/2}]^{1/2}) \), with \( \Sigma \mathbf{V}_j = \text{det} \mathbf{A}_j + \text{det} \mathbf{B} - 2 \text{det} \mathbf{C}_j \), is the smallest eigenvalue of the partial transpose of the reduced correlation matrix \( \mathbf{V}_j = (A_{ij}^j B_{ij}) \), which is obtained by removing in \( \mathbf{V} \) the rows and columns of the uninteresting mode [77]. In Eq. (5b), \( r, s, t \in \{d_1, d_2, c\} \) denote all the permutations of the three mode indices, \( E^{(\text{res})}_r \) or \( E^{[\text{res}]}_r \) is the contangle of subsystems of \( r \) and \( st \) (or \( t \) and it is defined as the squared logarithmic negativity [77,80]. The residual contangle satisfies the monogamy of quantum entanglement \( E^{(\text{res})}_r \geq E^{[\text{res}]}_s + E^{[\text{res}]}_t \), which is based on the Coffman-Kundu-Wootters monogamy inequality [82]. \( E_{N',j} > 0 \) and \( E^{[\text{res}]}_r > 0 \) mean, respectively, the emergence of bipartite optimale mechanical entanglement and full tripartite inseparability. The full inseparability is an important quantum resource and it is a necessary (but insufficient) condition for the presence of genuine multipartite entanglement [53,83–85]. In particular, this inseparability has recently been detected in experiments [53,85].

We display in Figs. 2(a) and 2(b) \( E_{N',j} \) and \( E^{[\text{res}]}_r \) versus the driving detuning \( \Delta \), in both the DMU and DMB regimes. This demonstrates that light and vibrations are separable in the DMU regime (\( E_{N',j} = 0 \) and \( E^{[\text{res}]}_r = 0 \), see the lower horizontal solid lines), but entangled and full inseparability in the DMB regime (\( E^{[\text{res}]}_r = 0.14(0.12) \) and \( E^{[\text{res}]}_r = 0.013 \), see the upper dashed curves and symbols). In the DMU regime, thermal phonons concealed in the dark mode cannot be extracted by the optomechanical cooling channel, and then quantum entanglement is completely destroyed by the residual thermal noise [61,66]. In the DMB regime, a large entanglement is achieved around the red-sideband resonance (\( \Delta \approx \omega_m \)), corresponding to the optimal cooling. In addition, we have confirmed that in the blue-detuning regime, the introduced synthetic magnetism (i.e., the DMB mechanism) can enhance opto-mechanical entanglement [77]. These results indicate that
entanglement and full tripartite inseparability completely related to a strong quantum interference between two and Figs. 2(e) and 2(f)]. We find that in the DMU regime, there ¯n and protect fragile quantum resources against dark modes, DMB entanglement provides a feasible way to create optomechanical entanglement [77].

constructive and destructive interferences caused by the entanglement generation. Out of this window, both the of the dark-mode effect dominates the enhancement of the synthetic magnetism may slightly degrade the entangle-
cavity-field decay rate, because the decay rates of the valley is determined by the spectral resolution in addition, we see from Fig. 2(e) that when ω1 ≠ ω2, the synthetic magnetism may slightly degrade the entangle-
entanglement is fully destroyed (E_N,j = 0) by the thermal noise in the dark mode, and it is independent of κ (see lower horizontal solid lines). However, in the DMB regime, light-vibration entanglement is generated owing to dark-mode breaking, and E_N,j exist only in the resolved-sideband regime κ/ω_m < 1 (see upper dashed curves). The maximal entanglement is located at κ ≈ 0.2ω_m, which is consistent with the typical deep-resolved-sideband conditions [26–29]. Electromechanical systems are excellent candidates for experimental demonstrations of noise-tolerant entanglement, because good cavities have been reported in this platform [77].

Entangled optomechanical networks.—We generalize the DMB approach for optomechanical networks where an optical mode couples to N ≥ 3 vibrational modes via the optomechanical interactions H_{opc} = \sum_{j=1}^{N} g j e^{j} c(d_j + d_j^{+}), and the nearest-neighbor vibrational modes are coupled through the phase-dependent phonon-exchange couplings H_{pec} = \sum_{j=1}^{N-1} \chi_j (e^{j} d_j d_{j+1} + H.c.) [see Fig. 1(b)]. We have confirmed that these phases are governed by the term \sum_{j=1}^{N} \Theta_j (j \in [2,N]) [77], and hence we assume \Theta_j = \pi and \Theta_{j \in [2,N-1]} = 0 in our simulations.

We demonstrate that when turning off synthetic magnetism (i.e., \chi_j = 0), there exists only a single bright mode B = \sum_{j=1}^{N} d_j e^{j} / \sqrt{N} and (N − 1) dark modes, with the \ellth dark mode expressed as D_{\ell \in [1,N-1]} = \sum_{j=1}^{N} \delta_j d_\ell e^{j (j-[(N+1)/2])/(N-1)} / \sqrt{N}. In the presence of synthetic magnetism (i.e., \chi_j ≠ 0), all the dark modes are broken by tuning \Theta_j ≠ 2\pi n for an integer n [77]. This provides a possibility of switching between the DMB and DMU regimes in optomechanical networks.

We reveal that light and all the vibrations are separable (E_N,j = 0, see lower horizontal solid lines) in the DMU regime, but they are entangled (E_N,j > 0, see upper symbols) in the DMB regime [see Figs. 4(a) and 4(b)]. Larger entanglement for optomechanical networks can be achieved for the red-sideband resonance (Δ ≈ ω_m) and \chi_j/\omega_m ∈ [0.1,0.15] when \Theta_j = \pi [see Figs. 4(a)–4(c)].

FIG. 3. (a) E_N,j=1,2 Versus \hbar/\hbar in the DMU (solid curves) and DMB (dashed curves) regimes. (b) E_N,j versus κ in the DMU and DMU regimes. In the DMU regime, quantum entanglement is fully destroyed (E_N,j = 0) by the thermal noise in the dark mode, and it is independent of κ (see lower horizontal solid lines). However, in the DMB regime, light-vibration entanglement is generated owing to dark-mode breaking, and E_N,j exist only in the resolved-sideband regime κ/ω_m < 1 (see upper dashed curves). The maximal entanglement is located at κ ≈ 0.2ω_m, which is consistent with the typical deep-resolved-sideband conditions [26–29]. Electromechanical systems are excellent candidates for experimental demonstrations of noise-tolerant entanglement, because good cavities have been reported in this platform [77].

Noise-tolerant optomechanical entanglement.—The DMB entanglement provides a feasible way to create and protect fragile quantum resources against dark modes, and can enable the construction of noise-tolerant quantum devices. We can see from Fig. 3(a) that, in the DMU regime, quantum entanglement emerges only when \hbar/\hbar ≤ 1 (see solid curves), while in the DMB regime, it can persist for a threshold value near \hbar/\hbar ≈ 10^3 (see dashed curves), which is 3 orders of magnitude larger than that in the DMU regime. In Fig. 3(b), we plot E_N,j versus κ in both the DMB
Physically, the resulting synthetic gauge fields lead to breaking all the dark modes, and make the light-vibration entanglement networks feasible [see Fig. 4(d)]. This indicates that the entanglement networks, with immunity against dark modes, can be realized by applying the DMB mechanism to optomechanical networks.

**Discussions and conclusions.**—Our scheme can be implemented using either photonic-crystal optomechanical-cavity configurations [71] or electromechanical systems [50,51], where synthetic magnetism can be, respectively, induced by employing two auxiliary-cavity fields or coupling two vibrations to a superconducting charge qubit [77]. We made detailed parameter analyses and numerical simulations by performing a comparison between the experimental parameters and our simulated parameters [77]. For coupled optomechanical networks, a potential experimental challenge concerning the coherent and in-phase pump of the mechanical modes by a common cavity field should be overcome in realistic experimental setups. Note that the entanglement addressed here (between the intracavity-optical and vibrational modes) is different from that between the output-light and mechanical modes [25,86]. The latter entanglement can be analyzed using the input-output relation and the intracavity field-vibration couplings [25,86].

In conclusion, we showed how to achieve both dark-mode-immune and noise-tolerant entanglement via synthetic magnetism. We revealed that both bipartite entanglement and full tripartite inseparability arise from the DMB mechanism, without which they vanish. In particular, our simulations indicated that the threshold phonon number for preserving entanglement could reach 3 orders of magnitude of that in the DMU regime. This study could enable constructing large-scale entanglement networks with dark-mode immunity and noise tolerance.

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