Chvátal’s Conjecture and Correlation Inequalities

Ehud Friedgut∗, Jeff Kahn†, Gil Kalai‡, and Nathan Keller§

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Abstract

Chvátal’s conjecture in extremal combinatorics asserts that for any decreasing family \( F \) of subsets of a finite set \( S \), there is a largest intersecting subfamily of \( F \) consisting of all members of \( F \) that include a particular \( x \in S \). In this paper we reformulate the conjecture in terms of influences of variables on Boolean functions and correlation inequalities, and study special cases and variants using tools from discrete Fourier analysis.

1 Introduction

Definitions. A family \( \mathcal{G} \) of subsets of \( [n] = \{1, 2, \ldots, n\} \) is intersecting if \( A \cap B \neq \emptyset \) for any \( A, B \in \mathcal{G} \), and increasing if \( A \supset B \in \mathcal{G} \) implies \( A \in \mathcal{G} \) (and similarly for decreasing).

One of the seminal results (maybe the seminal result) of extremal combinatorics is the Erdős-Ko-Rado theorem [7], which says that, for \( k \leq n/2 \), the maximum size of an intersecting subfamily of the family \( \mathcal{F} \) of all \( k \)-subsets of \( [n] \) is \( \binom{n-1}{k-1} \), the number of \( k \)-sets containing some fixed \( x \in [n] \). Given this, it is natural to ask whether something similar holds for other \( \mathcal{F} \)’s. A celebrated 1972 conjecture of Chvátal [4] says that this is true for every decreasing \( \mathcal{F} \):

Conjecture 1.1 (Chvátal’s Conjecture). For any decreasing \( \mathcal{F} \subseteq 2^{[n]} \), some largest intersecting subfamily has the form \( \{A \in \mathcal{F} : x \in A\} \).

Of course it is no longer the case that any \( x \) will suffice, and the difficulty of identifying a suitable \( x \) is a central reason for the conjecture’s intractability. Chvátal’s Conjecture has been the subject of many papers (and surely far more effort than this published record indicates), but progress to date has been limited, dealing mostly with either very special cases or variants.

In this paper we suggest an analytic approach. We show that Chvátal’s Conjecture can be restated in terms of influences (defined below) and correlation inequalities, providing an opening for use of tools from discrete Fourier analysis.

*Faculty of Mathematics and Computer Science, The Weizmann Institute of Science, Rehovot, Israel. Research supported in part by ISF grant 0398246, and Minerva grant 712023. E-mail: ehudf@math.huji.ac.il
†Department of Mathematics, Rutgers University, Piscataway NJ 08854 USA. Partially supported by NSF grants DMS1201337 and DMS1501962, and BSF grant 2014290. E-mail: jkahn@math.rutgers.edu
‡Einstein Institute of Mathematics, Hebrew University, Jerusalem, Israel. Partially supported by ERC advanced grant 320924, NSF grant DMS1300120, and BSF grant 2014290. E-mail: kalai@math.huji.ac.il
§Department of Mathematics, Bar Ilan University, Ramat Gan, Israel. Partially supported by ISF grant 402/13, BSF grant 2014290, and by the Alon Fellowship. E-mail: nathan.keller27@gmail.com

1A list of more than 20 papers directly related to the conjecture appears at the website: http://users.encs.concordia.ca/chvatal/conjecture.html

2Sean Eberhard independently considered similar relations, motivating a MathOverflow question [5].
of a stronger form of Chvátal’s conjecture suggested by the second author about 25 years ago (see Section 3).

We first recall a few definitions. In what follows, we identify subsets of \([n]\) with elements of the discrete cube \(\Omega = \{0, 1\}^n\) in the natural way and write \(\mu\) for uniform measure on \(\Omega\). The correlation between \(A, B \subseteq \Omega\) is \(\text{Cor}(A, B) = \mu(A \cap B) - \mu(A)\mu(B)\). More generally for \(f, g : \Omega \to \mathbb{R}\) we use \(\text{Cor}(f, g) = \mathbb{E}_\mu[fg] - \mathbb{E}_\mu[f]\mathbb{E}_\mu[g]\) (so \(\text{Cor}(\chi_A, \chi_B) = \text{Cor}(A, B)\)), where we use \(\chi\) for indicator; of course \(\text{Cor}(f, g)\) is just the covariance of \(f\) and \(g\).

A family \(F\) is said to be antipodal if \(|F \cap \{A, A^c\}| = 1\) for each \(A \subseteq \Omega\) (with \(A^c\) the complement of \(A\)).

The influence of the \(k\)th variable on \(A \subseteq \Omega\) is
\[
I_k(A) = 2\mu(\{x \in A | x \oplus e_k \notin A\}),
\]
where \(x \oplus e_k\) is gotten from \(x\) by replacing \(x_k\) by \(1 - x_k\). The total influence of \(A\) is \(I(A) = \sum_{k=1}^n I_k(A)\) and we write \(I_{\min}(A)\) for \(\min_{1 \leq k \leq n} I_k(A)\).

Recall that Harris’ seminal correlation inequality [12] says that \(\text{Cor}(A, B) \geq 0\), for increasing \(A, B\). Michel Talagrand [20] initiated the study of: “How much are increasing sets positively correlated?”, and this question will be a central theme for us as well. As we will see, Chvátal’s Conjecture can also be formulated as a correlation inequality, viz.

**Conjecture 1.2.** For any increasing \(A\) and increasing antipodal \(B\) (both \(\subseteq \Omega\)),
\[
\text{Cor}(A, B) \geq \frac{1}{4} I_{\min}(A).
\]

The equivalence is shown in Section 2. We will also be interested in a weaker but more general possibility:

**Conjecture 1.3.** For any increasing \(A, B \subseteq \Omega\)
\[
\text{Cor}(A, B) \geq c I_{\min}(A)\mu(B)(1 - \mu(B)).
\]

for some fixed (positive) \(c\).

As we will see in Section 7, (2) is not true with \(c > \ln 2\), even if \(B\) is balanced (i.e., \(\mu(B) = 1/2\)); in particular, the antipodality in Conjecture 1.2 cannot be replaced by the weaker assumption that \(B\) is balanced. On the other hand, Kahn’s strong form of Chvátal’s conjecture (Conjecture 3.4 below) implies that (2) does hold with \(c = 1/2\), and the possibility that this relaxation loses only a constant factor seems to us one of the more interesting aspects of the present discussion.

Lower bounds on the correlations of increasing families in terms of influences were obtained by Talagrand [20] (as already mentioned; see Theorem 4.1 below) and by Keller, Mossel, and Sen [17] (Theorem 4.5). In Section 4 we combine these results with results about influences of an individual family (due to Kahn, Kalai, and Linial [15], and Talagrand [19]), to prove Conjecture 1.3 under some (fairly strong) additional hypotheses. We also prove, for general increasing families \(A, B\),
\[
\text{Cor}(A, B) \geq c I_{\min}(A)\mu(B)(1 - \mu(B))/\sqrt{\log(1/I_{\min}(A))}.
\]

These results may be thought of as illustrating connections with existing Fourier technology. Inequality (3), while weak compared to what we are after, may serve as a useful benchmark for future research. In Section 8 we rely on [16] and, perhaps surprisingly, show that Conjecture 1.2 is true in some average sense.
In Sections 5 and 6 we propose and study strengthenings of Harris’ inequality that would imply Conjectures 1.2 and 1.3. One possibility is the following consequence of Kahn’s conjecture. (Here we use $\hat{C}(S)$ for the Fourier coefficient $\hat{\chi}_C(S)$; Fourier definitions are recalled in the next section.)

**Conjecture 1.4.** For any increasing $A, B \subseteq \Omega$,

$$\text{Cor}(A, B) \geq c \sum_{k=1}^{n} I_k(A) \sum_{\{S \mid \hat{B}(S)^2 : S \ni k\}} \frac{1}{|S|},$$

for some universal $c$. If $B$ is antipodal this is true with $c = 1$.

Notice that this gives a lower bound for $\text{Cor}(A, B)$ in terms of a weighted sum of the influences of $A$, with the sum of weights in the antipodal case equal to $1/4$.

2 Reformulation and preliminaries

This section gives the easy equivalence of the two versions of Chvátal’s Conjecture stated in the introduction and some additional background and comments.

2.1 Reformulation

**Proposition 2.1.** For a decreasing $F \subseteq \Omega$ the following statements are equivalent.

(a) There is a $k \in [n]$ for which $\max\{|B| : B \subseteq F \text{ maximal intersecting}\} = |\{A \in F : k \in A\}|$.

(b) There is a $k \in [n]$ such that the maximum correlation of $F$ with a maximal intersecting $B$ is attained by $B = \{A \in \Omega : k \in A\}$.

(c) For any increasing, antipodal $B \subseteq \Omega$, $\text{Cor}(F, B) \leq -\frac{1}{4}(I_{\min}(F))$.

Of course (a) is Conjecture 1.1 while (c) is the same as Conjecture 1.2 since, for any $A, B$, $\text{Cor}(A, B) = -\text{Cor}(A^c, B)$ (more generally, $\text{Cor}(1 - f, g) = \text{Cor}(1, g) - \text{Cor}(f, g) = -\text{Cor}(f, g)$ for any $f, g$).

**Proof.** It is obvious (and standard) that the maximum in (a) is the same as

$$\max\{|F \cap B| : B \text{ maximal intersecting}\},$$

and that each maximal intersecting $B$ has measure $1/2$. (It is easy to see—and was observed e.g. in [3]—that $F$ is maximal intersecting if and only if it is increasing and antipodal.) Thus for maximal intersecting $B$ we have

$$|F \cap B| = 2^n(\text{Cor}(F, B) + \mu(F))/2,$$

which implies the equivalence of (a) and (b) (since we maximize the left side of [3] by maximizing $\text{Cor}(F, B)$). For the equivalence of (c) we just observe that for $B$ as in (b) (sometimes called a “dictatorship”) we have

$$\text{Cor}(F, B) = -\mu(\{A \in F : A \cup \{x\} \notin F\})/2 = -I_k(F)/4.$$
2.2 Harper and Fourier-Walsh

Harper’s classic edge-isoperimetric inequality [11] says (though not originally in this language) that for all \( A \subseteq \Omega \),
\[
I(A) \geq 2\mu(A) \log_2(1/\mu(A)).
\] (6)

In particular \( I(A) \geq 1 \) for balanced \( A \).

\textit{Definition.} For \( f : \Omega \to \mathbb{R} \), the \textit{Fourier-Walsh expansion} of \( f \) is the (unique) representation
\[
f = \sum \{ \alpha_S u_S : S \subseteq [n] \},
\]
where \( u_S(T) = (-1)^{|S \cap T|} \) for \( T \subseteq [n] \). The (Fourier) coefficients \( \alpha_S \) are also denoted \( \hat{f}(S) \).

Since \( \{ u_S \} \) is an orthonormal basis for the space of functions \( f : \Omega \to \mathbb{R} \) (relative to the usual inner product \( \langle \cdot, \cdot \rangle \) with respect to uniform measure), the representation is indeed unique, with \( \hat{f}(S) = \langle f, u_S \rangle \), and we have Parseval’s identity:
\[
\langle f, g \rangle = \sum \hat{f}(S) \hat{g}(S) \quad \forall f, g.
\] (7)

Thus (since \( \mu(f) = \hat{f}(\emptyset) \)),
\[
\text{Cor}(f, g) = \sum_{S \neq \emptyset} \hat{f}(S) \hat{g}(S).
\] (8)

As we have already done above, we will sometimes use \( \hat{C}(S) \) for \( \hat{\chi}_C(S) \). It is standard (see e.g. [15]) that for any \( A \subseteq \Omega \) and \( i \in [n] \), \( I_i(A) = 4 \sum \{ \hat{A}(S)^2 : S \ni i \} \). If \( A \) is decreasing then also \( I_i(A) = 2\hat{A}([i]) \) and if \( A \) is increasing then \( I_i(A) = -2\hat{A}([i]) \).

2.3 The dream relation

The following observation is simple but crucial for our line of thought (cf. the aforementioned Theorems 4.1 and 4.5).

\textbf{Proposition 2.2.} Let \( A, B \) be increasing events with \( \mu(B) = t \). If
\[
\text{Cor}(A, B) \geq \frac{1}{4} \sum I_k(A) I_k(B),
\] (9)
then
\[
\text{Cor}(A, B) \geq \frac{1}{2} I_{\min}(A) t \log_2(1/t).
\] (10)

In particular, if \( B \) is balanced then \( \text{Cor}(A, B) \geq \frac{1}{4} I_{\min}(A) \).

\textit{Proof.} Combining (9) and Harper’s inequality gives
\[
\text{Cor}(A, B) \geq \frac{1}{4} \sum I_k(A) I_k(B) \geq \frac{1}{4} I_{\min}(A) I(B) \geq \frac{1}{2} I_{\min}(A) t \log_2(1/t).
\]

\hfill \Box

Again, Proposition 2.2 is mainly motivational; neither (9) (the “dream relation”) nor its consequence (10) is true in general. In this paper, we consider weaker statements of similar type.
3 The conjectures of Kleitman and Kahn

In this section we describe an earlier analytic approach to Chvátal’s Conjecture suggested by the second author in an unpublished manuscript in the early 90s [14]. This built on a strengthening of Chvátal’s Conjecture proposed by Kleitman [18] in 1979.

**Definition 3.1.** Let \( f, g : \Omega \rightarrow \mathbb{R}^+ \). We say that \( f \) flows to \( g \) if there exists \( v : \Omega \times \Omega \rightarrow \mathbb{R}^+ \) such that:

1. For any \( A \in \Omega \), we have \( \sum_B v(A, B) = f(A) \).
2. For any \( B \in \Omega \), we have \( \sum_A v(A, B) = g(B) \).
3. If \( A \not\subseteq B \), then \( v(A, B) = 0 \).

Equivalently (via max-flow min-cut), \( f \) flows to \( g \) if \( \sum_A f(A) = \sum_A g(A) \) and \( f(F) \geq g(F) \) for every decreasing family \( F \) (where \( f(F) = \sum_{A \in F} f(A) \)).

**Notation.** For a “principal” family \( F = F_i = \{ A : i \in A \} \), we set \( \chi_F = \chi_i \) (recalling that \( \chi_F \) is the indicator of \( F \)).

The following strengthening of Chvátal’s conjecture was proposed by Kleitman [18].

**Conjecture 3.2.** For any maximal (w.r.t. inclusion) intersecting \( F \subseteq \Omega \), there is a convex combination \( \sum_{i=1}^n \lambda_i \chi_i \) of \( \chi_1, \ldots, \chi_n \) that flows to \( \chi_F \).

Fishburn [8] observed that this is equivalent to a “functional” form of Chvátal’s conjecture, viz.

**Conjecture 3.3.** For any nonincreasing \( g : \Omega \rightarrow \mathbb{R}^+ \), \( g(F) \) is maximized over intersecting families \( F \) by some \( F_i \).

Of course Chvátal’s Conjecture is just Conjecture 3.3 for \( \{0,1\} \)-valued \( g \).

The suggestion of [14] is a particular set of \( \lambda_i \)'s for Kleitman’s conjecture; these are most easily described in terms of the Fourier-Walsh coefficients.

For \( f : \Omega \rightarrow \mathbb{R} \), set \( f^*(x) = \max(f(x), 0)^2 \) (thus \( f^*(x) = f(x)^2 \) if \( f(x) \) is nonnegative and \( f^*(x) = 0 \) otherwise). We call \( f \) antipodal if \( f(A^c) = -f(A) \) for any \( A \subseteq [n] \). In particular, if \( F \) is an antipodal family, then \( f = 2 \cdot 1_F - 1 \) is an antipodal function.

**Conjecture 3.4** ([14]). For any nondecreasing, antipodal \( f : \Omega \rightarrow \mathbb{R} \), if

\[
\lambda_i = \lambda_i(f) = \sum \{ \hat{f}(S)^2 : \max(S) = i \} \quad 1 \leq i \leq n,
\]

then \( \tilde{f} = \sum_{i=1}^n \lambda_i \chi_i \) flows to \( f^* \).

Note that for an antipodal \( f \), \( \hat{f}(\emptyset) = 0 \), so [7] gives

\[
\sum \lambda_i(f) = \sum_{S \neq \emptyset} \hat{f}^2(S) = \langle f, f \rangle - \hat{f}^2(\emptyset) = 2^{-n} \sum f^2(T) = 2^{-(n-1)} \sum f^*(T). \quad (11)
\]

Thus \( \sum \hat{f}(T) = 2^{n-1} \sum \lambda_i = \sum f^*(T) \), a prerequisite for the conclusion of Conjecture 3.4. In particular, when \( f = 2 \cdot 1_F - 1 \) with \( F \) maximal intersecting, the \( \lambda_i \)'s are convex coefficients, and in this case Conjecture 3.4 strengthens Kleitman’s Conjecture 3.2 by specifying the \( \lambda_i \)'s.

As noted following Definition 3.1 Conjecture 3.4 is equivalent to
Conjecture 3.5. If \( f : \Omega \to \mathbb{R} \) is nondecreasing, antipodal and \( I \subseteq \Omega \) is decreasing, then (with \( \tilde{f} \) as above)
\[
\sum_{A \in I} \tilde{f}(A) \geq \sum_{A \in I} f^*(A).
\] (12)

As noted in [14], the following, superficially more general, version of Conjecture 3.4 is again equivalent.

Conjecture 3.6. Let \( f : \Omega \to \mathbb{R} \) be non-decreasing and antipodal. For each \( S \subseteq [n] \), let \( \lambda_S : [n] \to \mathbb{R}^+ \) be some function satisfying
\[
\sum_{i=1}^n \lambda_S(i) = \hat{f}(S)^2 \quad \text{and} \quad \lambda_S(i) = 0 \quad \forall i \notin S,
\]
and set \( \lambda_i = \sum_{i \in S} \lambda_S(i) \). Then \( \sum_{i=1}^n \lambda_i \chi_i \) flows to \( f^* \).

Conjecture 3.4 is the special case gotten by setting \( \lambda_S(i) = \hat{f}(S)^2 \chi_{\{i=\max S\}} \). Another natural choice is
\[
\lambda_S(i) = |S|^{-1} \hat{f}(S)^2 \chi_{\{i \in S\}}.
\] (13)

Conjecture 3.6 with this choice is weaker than Conjecture 3.4, but of course still sufficient for Conjecture 3.2.

As observed in [14], Conjecture 3.4 (or Conjecture 3.5) also implies a natural extension of Chvátal’s Conjecture to general (not necessarily maximal) increasing, intersecting families:

Conjecture 3.7. For any increasing, intersecting \( F \subseteq \Omega \) and decreasing \( I \subseteq \Omega \), there is an \( i \) such that
\[
\frac{|F_i \cap I|}{2^{n-1}} \geq \frac{|F \cap I|}{|F|}.
\]

Some further discussion of Conjecture 3.5 and variants, and in particular, of some surprising cases in which the conjecture is tight, is provided in [14]. Correlation reformulations of the conjectures of [14] are given in Section 4.

4 Chvátal’s Conjecture and off-the-shelf correlation inequalities

We have already mentioned the fundamental inequality of Harris [12], asserting positive (i.e., nonnegative) correlation of any two increasing subsets of \( \Omega \). (There are also some well-known extensions, in particular the “FKG Inequality” of [9] and the “Four Functions Theorem” of [11].) In 1996, Talagrand [20] proved a lower bound on the correlation in Harris’ Inequality in terms of influences. In 2012, Keller, Mossel, and Sen [17] proved an alternative lower bound (incomparable with Talagrand’s). As Conjecture 1.2 (our reformulation of Chvátal’s Conjecture) again asks for a lower bound on correlation of increasing families in terms of influences, it is natural to hope that lower bounds along the lines of [17][20] may help in proving it. Here we review the above bounds and see what they have to say about Conjecture 1.2.

We assume from now on (as we clearly may) that \( \min(A) \) is positive. Note that in what follows “Chvátal’s Conjecture” usually refers to the form in Conjecture 1.2.
4.1 Talagrand’s inequality

In [20], Talagrand proved the following correlation inequality.

**Theorem 4.1.** For any increasing \( A, B \subseteq \Omega \),

\[
\text{Cor}(A, B) \geq c\varphi \left( \sum I_k(A)I_k(B) \right),
\]

where \( \varphi(x) = x/\log(e/x) \), and \( c \) is a universal constant.

(Here and below sums indexed by \( k \) run over \( k \in [n] \).)

Combined with Harper’s inequality (6), Theorem 4.1 yields a weak version of Chvátal’s Conjecture:

**Corollary 4.2.** For \( A, B \subseteq \Omega \) with \( A \) increasing and \( B \) increasing and antipodal,

\[
\text{Cor}(A, B) \geq c\varphi(I_{\min}(A)),
\]

where \( \varphi(x) \) is as in Theorem 4.1 and \( c \) is a universal constant.

**Proof.** From (6) we have

\[
\sum I_k(A)I_k(B) \geq I_{\min}(A)\sum I_k(B) \geq I_{\min}(A),
\]

which, since \( \varphi \) is increasing, gives Corollary 4.2 via Theorem 4.1. \( \square \)

Let us stress that Corollary 4.2 is much weaker than Chvátal’s Conjecture, since \( I_{\min}(A) \) is always \( O(n^{-1/2}) \) (it is largest when \( A \) is “majority”), and is often much smaller. The following proposition says we can do better if we impose some (restrictive but not unnatural) assumptions; here we need to recall the “KKL Theorem” of [15]:

**Theorem 4.3.** There is a fixed \( c > 0 \) such that for any \( A \subseteq \Omega \), there is a \( k \in [n] \) with

\[
I_k(A) \geq c\mu(A)(1 - \mu(A))(\log_2 n)/n.
\]

**Definition.** \( A \subseteq \Omega \) is regular if \( I_i(A) = I_j(A) \ \forall i, j \) (for an increasing \( A \), this means that the sets \( \{ S \in A : i \in S \} \) are all of the same size). Of particular interest here are the weakly symmetric families, those invariant under transitive subgroups of \( \mathfrak{S}_n \).

**Proposition 4.4.** For each \( a > 0 \) there is a \( c = c(a) > 0 \) such that if \( A \subseteq \Omega \) is increasing with \( \mu(A) \in (n^{-a}, 1 - n^{-a}) \) and \( B \subseteq \Omega \) is increasing, balanced and regular, then

\[
\text{Cor}(A, B) > cI_{\min}(A).
\]

Note that the assumption that \( B \) is regular holds in the examples of Section 7 that give the strongest constraint we know on the \( c \) in Conjecture 1.3.

**Proof.** (We use \( c', c'' \ldots \) for positive constants depending on \( a \).) The assertion is the same for \( A^c \) as for \( A \) (since \( \text{Cor}(A, B) = \text{Cor}(A^c, B^c) \) and complementation doesn’t affect influences), so we may assume \( \mu(A) \leq 1/2 \). Theorem 4.3 and our assumptions on \( B \) give \( I_k(B) > c' \log n/n \ \forall k \), implying

\[
\sum I_k(A)I_k(B) \geq c'(\log n/n)\sum I_k(A) \geq c' \log n \cdot I_{\min}(A).
\]
On the other hand, since $\mu(A) \in (n^{-\alpha}, 1/2]$, (9) gives
\[
c'(\log n/n)I(A) \geq 2c'(\log n/n)\mu(A) \log_2(1/\mu(A)) > c''an^{-(a+1)} \log^2 n,
\]
whence
\[
\log(c/\sum I_k(A)I_k(\mathcal{B})) < c'' \log n. \tag{17}
\]
From (16) and (17) we have $\varphi(\sum I_k(A)I_k(\mathcal{B})) > cI_{\min}(A)$, so (15) is given by Theorem 4.1. \hfill \square

Remarks. 1. Of course the above proof supports replacement of $I_{\min}(A)$ in (15) by the average, $I(A)/n$, of the $I_k(A)$'s. As pointed out to us by Alex Samorodnitsky, when $\mathcal{B}$ is “majority” (the “fully symmetric” case), $\text{Cor}(\mathcal{A}, \mathcal{B}) \geq I(A)/(4n)$ for any increasing $\mathcal{A}$; this follows from the fact that $\mathcal{A}$ contains at least as many sets of size $k$ as of size $n-k$ for any $k > n/2$, and is exact when $\mathcal{A}$ is $\{1\}$ or $\Omega \setminus \{0\}$.

2. As shown in [20], Theorem 4.1 is sharp (up to the value of $c$). This is also demonstrated by the examples of Section 7. Still, one may wonder whether it can be improved when one of the two sets is antipodal.

4.2 The inequality of Keller, Mossel and Sen

The following relative of Theorem 4.1 is from [17].

Theorem 4.5. There is a fixed $c > 0$ such that for increasing $\mathcal{A}, \mathcal{B} \subseteq \Omega$,
\[
\text{Cor}(\mathcal{A}, \mathcal{B}) \geq c \sum \psi(I_k(A))\psi(I_k(\mathcal{B})), \tag{18}
\]
where $\psi(x) = x/\sqrt{\log(e/x)}$.

Like Theorem 4.1, this gives a weak version of Chvátal’s conjecture; here we replace (9) by a theorem of Talagrand [19] that sharpens the KKL Theorem:

Theorem 4.6. For $\mathcal{B} \subseteq \Omega$ increasing, $\sum \varphi(I_k(\mathcal{B})) > c\mu(\mathcal{B})(1 - \mu(\mathcal{B}))$ (where $c$ is a positive constant).

Corollary 4.7. There is a fixed $c > 0$ such that for any increasing $\mathcal{A}, \mathcal{B} \subseteq \Omega$,
\[
\text{Cor}(\mathcal{A}, \mathcal{B}) > c\psi(I_{\min}(\mathcal{A}))\mu(\mathcal{B})(1 - \mu(\mathcal{B})).
\]

Proof. Theorem 4.5, the monotonicity of $\psi$ and Theorem 4.6 give
\[
\text{Cor}(\mathcal{A}, \mathcal{B}) > c\psi(I_{\min}(\mathcal{A})) \sum \psi(I_k(\mathcal{B})) > c\psi(I_{\min}(\mathcal{A}))\mu(\mathcal{B})(1 - \mu(\mathcal{B})) \tag{19}
\]
(where the second inequality uses the fact that $\psi(x) \geq \varphi(x)$ for $x \in [0, 1]$). \hfill \square

Corollary 4.7 misses the bound of Conjecture 1.2 by a factor like $\sqrt{\log(1/I_{\min}(\mathcal{A}))}$, which improves the $\log(1/I_{\min}(\mathcal{A}))$ of Corollary 4.2 but is still weak. Of course something is lost in the second inequality of (19), but we don’t see how to exploit this in general. (For regular $\mathcal{B}$, Theorem 4.5 does support a different derivation of Proposition 4.4.) It is tempting to try to replace the bound in (15) by $c \sum \psi_\alpha(I_k(A))\psi_{1-\alpha}(I_k(\mathcal{B}))$, where $\psi_\alpha(x) = x/(\log(e/x))^{\alpha}$ (e.g. $\psi_0$ is the identity, $\psi_{1/2} = \psi$ and $\psi_1 = \varphi$). If true for $\alpha = 0$, this would give Conjecture 1.2 to within a constant factor via the argument of Corollary 4.7 (since it replaces the middle expression in (19) by $cI_{\min}(\mathcal{A}) \sum \varphi(I_k(\mathcal{B}))$); but in fact it is not true for any $\alpha \neq 1/2$ (e.g. for $\alpha < 1/2$ let $\mathcal{B}$ be “majority” and $\mathcal{A} = \{x \in \Omega : \sum x_i > s\}$, with $s$ chosen so that $\mu(\mathcal{A}) = \exp[-\Omega(n)]$.  

8
5 Alternative correlation inequalities

Here we consider a few alternative correlation inequalities. Some of these (if correct) would imply Chvátal’s conjecture, while others may serve as first steps in the direction of the conjecture. Proofs are given in Section 4.

5.1 Reformulations and consequences of Kahn’s conjecture

We begin with a pair of inequalities that reformulate Conjecture 3.5 and the special case of Conjecture 3.6 suggested in [13] for $f$ of the form $2 \cdot 1_S - 1$ (equivalently, for $\{\pm 1\}$-valued $f$). Recall $\hat{C} = \hat{\chi}_c$.

Conjecture 5.1. For increasing $A \subseteq \Omega$ and maximal intersecting $B \subseteq \Omega$,

(a) $\text{Cor}(A, B) \geq \sum_{i=1}^{n} I_i(A) \sum \{\hat{B}(S)^2 : \max(S) = i\}$,

(b) $\text{Cor}(A, B) \geq \sum_{i=1}^{n} I_i(A) \sum \{\hat{B}(S)^2 : S \ni i\}$.

Each of these inequalities has the form $\text{Cor}(A, B) \geq \sum w_i I_i(A)$, with $\sum w_i = \sum \{\hat{B}(S)^2 : S \neq \emptyset\} = \mu(B)(1 - \mu(B)) = 1/4$; thus either implies $\text{Cor}(A, B) \geq \frac{1}{4} I_{\min}(A)$. For comparison with Section 4, note that in each case, $w_i \leq \sum \{\hat{B}(S)^2 : S \ni i\} = \frac{1}{4} I_i(B)$. It is easy to see that (a) is strongest when we index with $I_1(A) \leq \cdots \leq I_n(A)$, and that (a) implies (b) (by averaging over orderings). On the other hand, for Chvátal’s conjecture it is enough to establish the weakest version of (a), in which we take $I_1(A) \geq \cdots \geq I_n(A)$.

The following generalization of Chvátal’s conjecture avoids the antipodality condition.

Conjecture 5.2. For increasing $A, B \subseteq \Omega$,

$$\text{Cor}(A, B) + \text{Cor}(A, B') \geq 2 \mu(B)(1 - \mu(B)) I_{\min}(A).$$

Note that if $B$ is antipodal, then $B' = B$ and $\mu(B)(1 - \mu(B)) = 1/4$; thus Conjecture 5.2 contains Conjecture 1.2.

Proposition 5.3. If Conjecture 3.2 is true then for increasing $A, B \subseteq \Omega$,

$$\text{Cor}(A, B) \geq \frac{1}{2} \sum I_i(A) \sum \{\hat{B}(S)^2 : \max(S) = i\}. \quad (20)$$

Inequality (20) implies Conjecture 1.3 with $c = 1/2$ (a consequence of Conjecture 3.4 mentioned in Section 4), so also Conjecture 5.2 without the 2.

5.2 A symmetric version of Conjecture 5.1(b)

For $i \in [n]$, define $\Delta_i : \mathbb{R}^\Omega \rightarrow \mathbb{R}^\Omega$ by $\Delta_i f(x) = f(x) - f(x \oplus e_i)$. It is easy to see that for any $S \subseteq [n]$, $\Delta_i f(S) = \chi_{\{i \in S\}} f(S)$. We will use $\Delta_i(A)$ for $\Delta_i(1_A)$.

For $g : \Omega \rightarrow \mathbb{R}$ with $\mathbb{E}[g] = 0$ and $\alpha \in [0, 1]$, set $M_\alpha(g) = \sum_S \frac{f_i(S)^2}{|S|^\alpha}$.

Conjecture 5.4. There is a fixed $c > 0$ such that for any $A, B \subseteq \Omega$ with $A$ increasing and $B$ increasing and balanced,

$$\text{Cor}(A, B) \geq c \sum_i M_\alpha(\Delta_i(A)) M_{1-\alpha}(\Delta_i(B)) = c \sum_{S,T \neq \emptyset} \frac{|S \cap T| \hat{\chi}(S)^2 \hat{\chi}(T)^2}{|S|^{\alpha} |T|^{1-\alpha}}. \quad (21)$$
Increasing and balanced,

Proposition 5.5. There is a fixed \( c > 0 \) such that for any \( A, B \subseteq \Omega \) with \( A \) increasing and \( B \) increasing and balanced,

\[
\text{Cor}(A, B) \geq c \sum \frac{|S \cap T| \hat{A}(S)^2 \hat{B}(T)^2}{\sqrt{|S||T|}}. \tag{22}
\]

5.3 Diagonal forms of Conjecture 5.1(b)

An immediate consequence of Proposition 5.5 is:

Corollary 5.6. There is a fixed \( c > 0 \) such that for any increasing \( A, B \subseteq \Omega \),

\[
\text{Cor}(A, B) \geq c \sum_{S \neq \emptyset} \hat{A}(S)^2 \hat{B}(S)^2. \tag{23}
\]

Remark. The inequality [23] is a lower bound on the correlation of two increasing functions in terms of the (normalized) \( \ell_2 \)-norm of their convolution. It would be interesting to extend it to other contexts and to find a proof that’s more direct than the one in Section 6.3. A (very) weak consequence of Conjecture 5.1 (see the sentence following Conjecture 5.4) is:

Conjecture 5.7. For increasing \( A \subseteq \Omega \) and maximal intersecting \( B \subseteq \Omega \),

\[
\text{Cor}(A, B) \geq 4 \sum_{S \neq \emptyset} \hat{A}(S)^2 \hat{B}(S)^2. \tag{24}
\]

We expect even more to be true:

Conjecture 5.8. For any increasing \( A, B \subseteq \Omega \),

\[
\text{Cor}(A, B) \geq \sum_{S \neq \emptyset} |S| \hat{A}(S)^2 \hat{B}(S)^2.
\]

5.4 Inequalities involving the total influence

We would like to (perhaps optimistically) suggest the following family of inequalities.

Conjecture 5.9. There is a fixed \( c > 0 \) such that for increasing \( A, B \subseteq \Omega \) and \( \alpha \in [0, 1] \),

\[
\text{Cor}(A, B) \geq c \left( \frac{\text{Var}(A)}{I(A)} \right)^\alpha \left( \frac{\text{Var}(B)}{I(B)} \right)^{1-\alpha} \sum I_i(A) I_i(B) \tag{25}
\]

(where \( \text{Var}(A) = \mu(A)(1 - \mu(A)) \)). The case \( \alpha = 0 \) would imply Conjecture 1.3, giving, for increasing \( A \) and \( B \),

\[
\text{Cor}(A, B) \geq c \text{Var}(B) \sum I_i(A) I_i(B) \geq c \text{Var}(B) I_{\min}(A) \sum I_i(B) = c \mu(B)(1 - \mu(B)) I_{\min}(A).
\]

For \( \alpha = 1/2 \), Conjecture 5.9 is reminiscent of Theorem 4.5 and we are inclined to believe that it is true, at least in this case.

We may also strengthen Conjecture 5.2 to a variant of the case \( \alpha = 0 \) of Conjecture 5.9.

Conjecture 5.10. For increasing \( A, B \subseteq \Omega \),

\[
\text{Cor}(A, B) + \text{Cor}(A, B') \geq 2 \mu(A)(1 - \mu(A)) \sum I_i(A) I_i(B)/I(B). \tag{26}
\]
6 Alternative correlation inequalities: Proofs

6.1 Connection with Conjectures 3.5 and 3.6

Here we show equivalence of Conjecture 5.1(a) and the restriction of Conjecture 3.5 to $f$'s of the form $2\chi_F - 1$ with $F \subseteq \Omega$ antipodal (maximal intersecting). A similar argument shows that (b) is equivalent to Conjecture 3.6 for the same class of $f$'s and $\lambda_S$'s as in [13].

Proof. For $f$ as above we have $f^* = \chi_F$, so the inequality (12) of Conjecture 3.5 becomes

$$\sum_{A \in \mathcal{I}} \tilde{f}(A) \geq \sum_{A \in \mathcal{I}} \chi_F(A) = |F \cap \mathcal{I}|$$

which we may rewrite as

$$\langle \tilde{f}, \chi_F \rangle \geq \mu(F \cap \mathcal{I})$$

(since $\sum_{A \in \mathcal{I}} \tilde{f}(A) = \sum_A \tilde{f}(A)\chi_F(A) = 2^n \langle \tilde{f}, \chi_F \rangle$). Note also that

$$\mu(\tilde{f}) = \sum \lambda_i(f)\mu(\chi_i) = 1/2 \quad (= \mu(F))$$

(using $\mu(\cdot)$ for expectation), since the $\lambda_i$'s are convex coefficients (see following Conjecture 3.4).

Thus (27) is equivalent to Cor($\tilde{f}, \chi_F$) $\geq$ Cor($F, \mathcal{I}$) or, with $J = \mathcal{I}^c$ (see paragraph following Proposition 2.1),

$$\text{Cor}(F, J) \geq \text{Cor}(\tilde{f}, \chi_J).$$

(28)

To evaluate the r.h.s. notice that, with $\lambda_i(f) = \lambda_i$,

$$\tilde{f} = \sum \lambda_i \chi_i = \sum \lambda_i (1 - u_{(i)})/2 = 1/2 - (1/2) \sum \lambda_i u_{(i)}$$

(note $\chi_i = (1 - u_{(i)})/2$)—that is, the Fourier coefficients, $\alpha_S$, of $\tilde{f}$ are given by: $\alpha_{\emptyset} = 1/2$; $\alpha_{(i)} = -\lambda_i/2$; and $\alpha_S = 0$ if $|S| > 1$—and that for increasing $J \subseteq \Omega$, $\hat{\chi}_J\{\{i\}\} = -I_i(J)/2$ (for any $i$). Thus, recalling (8), we have

$$\text{Cor}(\tilde{f}, \chi_J) = -\frac{1}{2} \sum \lambda_i (-\frac{1}{2} I_i(J)) = \frac{1}{4} \sum I_i(J) \sum \{\hat{\chi}_J(S)^2 : \text{max}(S) = i\};$$

so (28) is the same as Conjecture 5.1(a) (with $(A, B) = (J, F)$). \hfill $\Box$

6.2 Proof of Proposition 5.3

Regard $A$ and $B$ as subsets of $2^{[2,n]}$ (with $[2, n] = \{2, 3, \ldots, n\}$), define $A', B' \subseteq 2^{[n]}$ by

$$A' = A \cup \{A \cup \{1\} : A \in A\} \quad \text{and} \quad B' = \{B \cup \{1\} : B \in B\},$$

and set $\mathcal{I} = (A')^c$. Let $f : \{0, 1\}^{n+1} \rightarrow \{-1, 0, 1\}$ be the antipodal function with $f(x) \equiv 1$ on $B'$ and $f(x) \equiv 0$ on $\{B : 1 \in B \notin B'\}$. The argument of Section 6.1 may be repeated essentially verbatim to show that the inequality $\sum_{A \in \mathcal{I}} \tilde{f}(A) \geq \sum_{A \in \mathcal{I}} f^*(A)$ of Conjecture 3.5 is equivalent to

$$\text{Cor}(A', B') \geq \frac{1}{4} \sum I_i(A') \sum \{\hat{f}(S)^2 : \text{max}(S) = i\}. \quad (29)$$

This implies Proposition 5.3 as follows.

Writing $\mu$ and $\mu'$ for uniform measure on $2^{[2,n]}$ and $2^{[n]}$ respectively, we easily see, first, that

$$\mu'(A') = \mu(A), \quad \mu'(B') = \mu(B)/2 \quad \text{and} \quad \mu'(A' \cap B') = \mu(A \cap B)/2,$$

implying

$$\text{Cor}(A', B') = \text{Cor}(A, B)/2,$$
and, second, that \( I_1(A') = 0 \) and \( I_i(A') = I_i(A) \) for \( i \in [2, n] \). Moreover it is easy to see that for \( S \subseteq [2, n] \),
\[
\mathcal{B}(S) = \begin{cases} 
\hat{f}(S) & \text{if } |S| \text{ is even}, \\
-\hat{f}(S \cup \{1\}) & \text{if } |S| \text{ is odd}, 
\end{cases}
\]
which accounts for all of \( \hat{f} \) since antipodality implies \( \hat{f}(T) = 0 \) if \( T \) is even. Finally, combining these observations, we find that \( (29) \) is in fact the same as \( (20) \).

\section{Proof of Proposition 5.5}

We need the following extension of Talagrand's \cite{19} Prop. 2.3.

**Lemma 6.1.** For any \( \alpha \in [0, 1] \) there is a \( c = c(\alpha) \) such that for any \( f : \Omega \to \mathbb{R} \) with \( \mathbb{E}[f] = 0 \),
\[ M_\alpha(f) := \sum_S \frac{\hat{f}(S)^2}{|S|^{\alpha}} \leq c \left( \log \left( \frac{\|f\|_2}{\|f\|_1} \right) \right)^{-\alpha} \|f\|_2. \]

Talagrand proves this with \( \alpha = 1 \) but for more general product measures \( \mu_p \). (Proposition 5.5 below also holds in this greater generality, given natural definitions which we omit.) At any rate, the proof of Lemma 6.1 follows his nearly verbatim and will not be given here.

**Proof of Proposition 5.5** For any \( C \subseteq 2^{[n]} \), \( f = \Delta_i(C) \) satisfies \( f(x) \in \{0, \pm1\} \) for all \( x \), \( \mathbb{E}[f] = 0 \), and \( \|f\|_2^2 = \|f\|_1 = I_i(C) \). Thus, Lemma 6.1 gives (with \( \psi \) as in Section 4.2)
\[ M_{1/2}(f) \leq c'I_i(C) \left( \log(e/\sqrt{I_i(C)}) \right)^{-1/2} \leq c' \psi(I_i(C)). \]
Applying this for each \( i \in [n] \) and \( C \in \{A, B\} \) and using Theorem 4.5 we have
\[ \sum M_{1/2}(\Delta_i(A))M_{1/2}(\Delta_i(B)) \leq (c')^2 \sum \psi(I_i(A))\psi(I_i(B)) \leq c'' \text{Cor}(A, B), \]
completing the proof.

\section{Can the antipodality assumption be removed?}

Here we show that, as mentioned earlier, Conjecture 1.3 fails for \( c > \ln 2 \), even assuming \( \mathcal{B} \) is balanced; in particular we cannot relax the antipodality in Conjecture 1.2 to the requirement that \( \mathcal{B} \) be balanced and increasing. Our example is based on the "tribes" construction of Ben-Or and Linial \cite{2}. (For simplicity we settle for \( \mathcal{B} \) only approximately balanced.)

**Example.** To define the tribes family \( \mathcal{A} \) we consider an equipartition \( [n] = S_1 \cup \cdots \cup S_{n/r} \) (with \( r \) to be specified; for present purposes we assume \( r|n \)), and, now thinking of \( \Omega \) as \( 2^{[n]} \), set
\[ \mathcal{A} = \{ A \subseteq [n] : \exists i, A \supseteq S_i \}. \]
Then \( \mathcal{B} := \mathcal{A}' = \{ B \subseteq [n] : B \cap S_i \neq \emptyset \forall i \} \). These are of course increasing with \( \mu(\mathcal{B}) = 1 - \mu(\mathcal{A}) \) (as for any dual pair). To arrange \( \mu(\mathcal{A}) \sim 1/2 \) we take \( r = \lfloor r(n) \rfloor \), where
\[ r(n) = \log_2 n - \log_2 \log_2 n + \log_2 \log_2 n, \]
for simplicity confining ourselves to \( n \)'s for which \( r|n \) and \( r(n) - r = o(1) \). We then have
\[ \mu(\mathcal{B}) = (1 - 2^{-r})^{n/r} = \exp[-2^{-r} n/r + O(4^{-r} n)] \to 1/2 \]
(since $2^r \sim n/(r \ln 2)$, where as usual $a \sim b$ means $a/b \to 1$).

For the correlation we work with $\mathcal{A}^c$; we have (with a little calculation)

$$
\mu(\mathcal{B}|\mathcal{A}^c) = (1 - (2^r - 1)^{-1})^{n/r} = \frac{2r}{(2^r - 1)^r}^{n/r} = \mu(\mathcal{B}) \left(1 - (2^r - 1)^{-2}\right)^{n/r},
$$

whence

$$
\text{Cor}(\mathcal{A}, \mathcal{B}) = -\text{Cor}(\mathcal{A}^c, \mathcal{B}) = -\mu(\mathcal{A}^c) \mu(\mathcal{B}) \left(1 - (2^r - 1)^{-2}\right)^{n/r} - 1 \sim \mu(\mathcal{A}^c) \mu(\mathcal{B})(4^r)^{-1}n.
$$

On the other hand the influence of $i \in S_j$ on $\mathcal{A}$ is the probability that a uniform subset of $[n]$ contains $S_j \setminus \{i\}$ and contains no $S_\ell$ with $\ell \neq j$; the common value of the $I_k(\mathcal{A})$’s is thus

$$
2^{-r+1} \left(1 - 2^{-r}\right)^{(n/r) - 1} \sim 2^{-r+1} \mu(\mathcal{B})
$$

and we have

$$
\text{Cor}(\mathcal{A}, \mathcal{B})/I_{\min}(\mathcal{A}) \sim 2^{-r+1} \mu(\mathcal{A}^c)(4^r)^{-1}n \sim \frac{n}{4r^2} \sim \frac{\ln 2}{4}.
$$

It is perhaps surprising (or suggestive?) that the above $\mathcal{B}$’s are so different from the families $\mathcal{F}_i$ that provide the lower bound in Conjecture 1.2.

8 Chvátal’s Conjecture is true “on average”

Another initially plausible inequality suggested by (30), is

$$
\text{Cor}(\mathcal{A}, \mathcal{B}) \geq \frac{1}{2} \mu(\mathcal{B}) \log_2(1/\mu(\mathcal{B}))I_{\min}(\mathcal{A});
$$

The example from the previous section shows that this is false even when $\mathcal{A}, \mathcal{B}$ are balanced. When they are not balanced the failure of (30) is more severe; e.g. when $\mathcal{A} = 2^{[n]} \setminus \{\emptyset\}$, we have $\text{Cor}(\mathcal{A}, \mathcal{B}) = \frac{1}{2} \mu(\mathcal{B})I_{\min}(\mathcal{A})$ for any $\mathcal{B} \neq 2^{[n]}$. But as we will see in this section, (30) does hold on average when $\mathcal{A}$ and $\mathcal{B}$ are drawn from a family of increasing sets of equal measure.

In [16], the fourth author proved the following “average-case” variant of Theorem 4.1.

**Theorem 8.1.** For a family $\mathcal{F}$ of increasing subsets of $\Omega$ and $\mathcal{A}, \mathcal{B}$ drawn uniformly and independently from $\mathcal{F}$,

$$
\mathbb{E}\text{Cor}(\mathcal{A}, \mathcal{B}) \geq \frac{1}{4} \mathbb{E} \sum I_k(\mathcal{A})I_k(\mathcal{B}).
$$

This immediately implies a corresponding variant of Chvátal’s Conjecture, even in the more general setting of Conjecture 1.3:

**Proposition 8.2.** For a family $\mathcal{F}$ of increasing subsets of $\Omega$, each of measure $t \in (0, 1)$, and $\mathcal{A}, \mathcal{B}$ drawn uniformly and independently from $\mathcal{F}$,

$$
\mathbb{E}\text{Cor}(\mathcal{A}, \mathcal{B}) \geq \frac{1}{4} t \log_2(1/t) \mathbb{E}I_{\min}(\mathcal{A}).
$$

In particular when $t = 1/2$,

$$
\mathbb{E}\text{Cor}(\mathcal{A}, \mathcal{B}) \geq \frac{1}{4} \mathbb{E}I_{\min}(\mathcal{A}).
$$

Thus Conjecture 1.2 does hold in an average sense; but note that this is true even with balance in place of antipodality, where we have seen that the conjecture proper is not true. More generally, Proposition 8.2 gives truth on average of the incorrect inequality (30).
Proof. Using Theorem 8.1 for the first inequality and 9 for the last, we have
\[ \mathbb{E} \text{Cor}(A, B) \geq \frac{1}{4} \mathbb{E} \sum I_k(A) I_k(B) \geq \frac{1}{4} \mathbb{E} [I_{\text{min}}(A) I(B)] \geq \frac{1}{4} t \log_2(1/t) \mathbb{E} I_{\text{min}}(A). \]

We next show that Proposition 8.2 can sometimes be strengthened. Here we need another result of Talagrand [20] and Chang [3] (see also [13] for the constant):

**Theorem 8.3.** For increasing \( B \subseteq \Omega \), \( \sum I_k(B)^2 \leq 8 \mu(B)^2 \ln(1/\mu(B)) \).

For \( A \subseteq \Omega \) and \( \gamma > 0 \), write \( s_\gamma(A) \) for the sum of the smallest \( \gamma \log_2(1/\mu(A)) \) influences of \( A \). (So we are now thinking of \( \mu(A) \) as somewhat small. Of course, strictly speaking, we should say \( \gamma \log_2(1/\mu(A)) \in \mathbb{N} \).)

**Proposition 8.4.** For a family \( F \) of increasing subsets of \( \Omega \), each of measure \( t \in (0, 1) \), and \( A, B \) drawn uniformly and independently from \( F \),
\[ \mathbb{E} \text{Cor}(A, B) \geq (4\gamma)^{-1}(2 - 2\sqrt{2\gamma \log_2 e}) t \cdot s_\gamma(A). \]

**Proof.** By Theorem 8.1 it suffices to show that for increasing \( A, B \subseteq \Omega \) with \( \mu(A) = \mu(B) = t \),
\[ \sum I_k(A) I_k(B) \geq \gamma^{-1}(2 - 2\sqrt{2\gamma \log_2 e}) t \cdot s_\gamma(A); \tag{31} \]
this is seen as follows. We may assume that \( I_1(A) \leq \cdots \leq I_n(A) \) and then, since \( B \) appears only on the l.h.s. of (31), that \( I_1(B) \geq \cdots \geq I_n(B) \) (by the “Rearrangement Inequality,” e.g. [10, Theorem 368]).

Set \( q = \gamma \log_2(1/t) \). Using Theorem 8.3 and Cauchy-Schwarz we have
\[ 8t^2 \ln(1/t) \geq \sum_{k \leq q} I_k(B)^2 \geq 1/q \left( \sum_{k \leq q} I_k(B) \right)^2 , \]
implying \( \sum_{k \leq q} I_k(B) \leq t \sqrt{8q \ln(1/t)} = \alpha t \log_2(1/t) \), with \( \alpha = 2\sqrt{2\gamma \log_2 e} \), and, by (9),
\[ \sum_{k > q} I_k(B) \geq (2 - \alpha) t \log_2(1/t). \]

But then
\[ \sum I_k(A) I_k(B) \geq I_q(A) \sum_{k > q} I_k(B) \geq (s_\gamma(A)/q)(2 - \alpha) t \log_2(1/t) = \gamma^{-1}(2 - \alpha) t \cdot s_\gamma(A). \]

\[ \square \]

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