SAMPLING FORMULAS FOR ONE-PARAMETER GROUPS OF OPERATORS IN BANACH SPACES

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Abstract. We extend some results about sampling of entire functions of exponential type to Banach spaces. By using generator $D$ of one-parameter group $e^{tD}$ of isometries of a Banach space $E$ we introduce Bernstein subspaces $\mathcal{B}_\sigma(D)$, $\sigma > 0$, of vectors $f$ in $E$ for which trajectories $e^{tD}f$ are abstract-valued functions of exponential type which are bounded on the real line. This property allows to reduce sampling problems for $e^{tD}f$ with $f \in \mathcal{B}_\sigma(D)$ to known sampling results for regular functions of exponential type $\sigma$.

1. Introduction

The goal of the paper is to extend some theorems about sampling of entire functions of exponential type to Banach spaces. Our framework starts with considering a generator $D$ of one-parameter strongly continuous group of operators $e^{tD}$ in a Banach space $E$. The operator $D$ is used to define analogs of Bernstein subspaces $\mathcal{B}_\sigma(D)$. The main property of vectors $f$ in $\mathcal{B}_\sigma(D)$ is that corresponding trajectories $e^{tD}f$ are abstract-valued functions of exponential type which are bounded on the real line. This fact allows to apply known sampling theorems to every function of the form $\langle e^{tD}f, g^* \rangle \in \mathcal{B}_\infty^\infty(\mathbb{R})$, $g^* \in E^*$, or of the form $t^{-1}\langle e^{D}f - f, g^* \rangle \in \mathcal{B}_2^2(\mathbb{R})$, where $\mathcal{B}_\infty^\infty(\mathbb{R})$, $\mathcal{B}_2^2(\mathbb{R})$ are classical Bernstein spaces on $\mathbb{R}$.

In remark 2.2 we demonstrate that the assumption that $D$ generates a group of operators is somewhat essential if one wants to have non-trivial Bernstein spaces. In section 2 we give a few different descriptions of these spaces and one of them explores recent results in [7] which extend classical Boas formulas [4], [3] for entire functions of exponential type. Note, that in turn, Boas formulas are generalizations of Riesz formulas [19], [18] for trigonometric polynomials.

Section 3 contains two sampling-type formulas which explore regularly spaced samples and section 4 is about two results in the spirit of irregular sampling. Section 5 contains an application to inverse Cauchy problem for abstract Schrödinger equation.

Note, that if $e^{tD}$, $t \in \mathbb{R}$, is a group of operators in a Banach space $E$ then any trajectory $e^{tD}f$, $f \in E$, is completely determined by any (single) sample $e^{\tau D}f$, because for any $t \in \mathbb{R}$

$$e^{tD}f = e^{(t-\tau)D}(e^{\tau D}f).$$

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Our results in sections 3 and 4 have, however, a different nature. They represent a trajectory $e^{tD}f$ as a “linear combination” of a countable number of its samples. Such kind results can be useful when the entire group of operators $e^{tD}$ is unknown and only samples $e^{t_kD}f$ of a trajectory $e^{tD}f$ are given.

It seems to be very interesting that not matter how complicated one-parameter group can be (think, for example, about a Schrödinger operator $D = -\Delta + V(x)$ and the corresponding group $e^{itD}$ in $L_2(\mathbb{R}^d)$) the formulas (3.1), (3.2), (4.1), (4.2), (4.4) are universal in the sense that they contain the same coefficients and the same sets of sampling points.

It was my discussions with Paul Butzer and Gerhard Schmeisser during Sampta 2013 in Jacobs University in Bremen of their beautiful work with Rudolf Stens [7], that stimulated my interest in the topic of the present paper. I am very grateful to them for this.

2. Bernstein vectors in Banach spaces

We assume that $D$ is a generator of one-parameter group of isometries $e^{tD}$ in a Banach space $E$ with the norm $\| \cdot \|$ (for precise definitions see [5], [9]). The notations $D^k$ will be used for the domain of $D^k$, and notation $D^\infty$ for $\bigcap_{k\in\mathbb{N}} D^k$.

**Definition 2.1.** The subspace of exponential vectors $E_\sigma(D)$, $\sigma \geq 0$, is defined as a set of all vectors $f$ in $D^\infty$ for which there exists a constant $C(f, \sigma) > 0$ such that

$$\| D^k f \| \leq C(f, \sigma)\sigma^k, \quad k \in \mathbb{N}. \quad (2.1)$$

Note, that every $E_\sigma(D)$ is clearly a linear subspace of $E$. What is really important is the fact that union of all $E_\sigma(D)$ is dense in $E$ (Corollary 2.2).

**Remark 2.2.** It is worth to stress that if $D$ generates a strongly continuous bounded semigroup then the set $\bigcup_{\sigma \geq 0} E_\sigma(D)$ may not be dense in $E$.

Indeed, consider a strongly continuous bounded semigroup $T(t)$ in $L_2(0, \infty)$ defined for every $f \in L_2(0, \infty)$ as $T(t)f(x) = f(x-t)$, if $x \geq t$ and $T(t)f(x) = 0$, if $0 \leq x < t$. Inequality (2.1) implies that if $f \in E_\sigma(D)$ then for any $g \in L_2(0, \infty)$ the function $(T(t)f) g$ is analytic in $t$. Thus if $g$ has compact support then $(T(t)f) g$ is zero for all $t$ which implies that $f$ is zero. In other words in this case every space $E_\sigma(D)$ is trivial.

**Definition 2.3.** The Bernstein subspace $B_\sigma(D)$, $\sigma \geq 0$, is defined as a set of all vectors $f$ in $E$ which belong to $D^\infty$ and for which

$$\| D^k f \| \leq \sigma^k \| f \|, \quad k \in \mathbb{N}. \quad (2.2)$$

**Lemma 2.4** ([11]). Let $D$ be a generator of an one parameter group of operators $e^{tD}$ in a Banach space $E$ and $\| e^{tD}f \| = \| f \|$. If for some $f \in E$ there exists an $\sigma > 0$ such that the quantity

$$\sup_{k \in \mathbb{N}} \| D^k f \| \sigma^{-k} = R(f, \sigma)$$

is finite, then $R(f, \sigma) \leq \| f \|$.

**Proof.** By assumption $\| D^r f \| \leq R(f, \sigma)\sigma^r, r \in \mathbb{N}$. Now for any complex number $z$ we have
It implies that for any functional \( h^* \in E^* \) the scalar function \( \langle e^{zD}f, h^* \rangle \) is an entire function of exponential type \( \sigma \) which is bounded on the real axis \( \mathbb{R} \) by the constant \( \|h^*\|\|f\| \). An application of the Bernstein inequality gives

\[
\|e^{zD}f, h^*\|_{C(\mathbb{R})} = \| (d/dt)^k \langle e^{zD}f, h^* \rangle \|_{C(\mathbb{R})} \leq \|h^*\|\|f\|.
\]

The last one gives for \( t = 0 \)

\[
|\langle D^k f, h^* \rangle| \leq \sigma^k\|h^*\|\|f\|.
\]

Choosing \( h^* \) such that \( \|h^*\| = 1 \) and \( \langle D^k f, h^* \rangle = \|D^k f\| \) we obtain the inequality \( \|D^k f\| \leq \sigma^k\|f\| , k \in \mathbb{N} \), which gives

\[
R(f, \sigma) = \sup_{k \in \mathbb{N}} (\sigma^{-k}\|D^k f\|) \leq \|f\|.
\]

Lemma is proved. \( \square \)

**Theorem 2.5 (17).** Let \( D \) be a generator of one-parameter group of operators \( e^{tD} \) in a Banach space \( E \) and \( \|e^{tD}f\| = \|f\| \). Then for every \( \sigma \geq 0 \)

\[
\mathcal{B}_\sigma(D) = \mathcal{E}_\sigma(D), \ \sigma \geq 0,
\]

**Proof.** The inclusion \( \mathcal{B}_\sigma(D) \subseteq \mathcal{E}_\sigma(D), \ \sigma \geq 0 \), is obvious. The opposite inclusion follows from the previous Lemma. \( \square \)

Motivated by results in [17] we introduce the following bounded operators

\[
(2.3) \quad \mathcal{B}^{(2m-1)}_D f = \left( \frac{\sigma}{\pi} \right)^{2m-1} \sum_{k \in \mathbb{Z}} (-1)^{k+1} A_{m,k} e^{\pi(k-1/2)D} f, \quad f \in E, \ \sigma > 0, \ m \in \mathbb{N},
\]

\[
(2.4) \quad \mathcal{B}^{(2m)}_D f = \left( \frac{\sigma}{\pi} \right)^{2m} \sum_{k \in \mathbb{Z}} (-1)^{k+1} B_{m,k} e^{\pi k D} f, \quad f \in E, \ \sigma > 0, \ m \in \mathbb{N},
\]

where

\[
A_{m,k} = (-1)^{k+1} \text{sinc}(2m-1) \left( \frac{1}{2} - k \right) = \frac{(2m-1)!}{\pi(k-\frac{1}{2})^{2m}} \sum_{j=0}^{m-1} \frac{(-1)^j}{(2j)!} \left( \pi(k-\frac{1}{2})^2 \right)^{2j}, \ m \in \mathbb{N},
\]

for \( k \in \mathbb{Z} \) and

\[
(2.5) \quad B_{m,k} = (-1)^{k+1} \text{sinc}(2m)(-k) = \frac{(2m)!}{\pi k^{2m+1}} \sum_{j=0}^{m-1} \frac{(-1)^j (\pi k)^{2j+1}}{(2j+1)!}, \ m \in \mathbb{N}, \ k \in \mathbb{Z} \setminus 0,
\]

\[
(2.6) \quad B_{m,k} = (-1)^{k+1} \text{sinc}(2m)(-k) = \frac{(2m)!}{\pi k^{2m+1}} \sum_{j=0}^{m-1} \frac{(-1)^j (\pi k)^{2j+1}}{(2j+1)!}, \ m \in \mathbb{N}, \ k \in \mathbb{Z} \setminus 0,
\]
and
\[
B_{m,0} = (-1)^{m+1} \frac{\pi^{2m}}{2m+1}, \quad m \in \mathbb{N}.
\]
Both series converge in \(E\) due to the following formulas (see [7])
\[
(\sigma \pi)^{2m-1} \sum_{k \in \mathbb{Z}} |A_{m,k}| = \sigma^{2m-1},
\]
(\sigma \pi)^{2m} \sum_{k \in \mathbb{Z}} |B_{m,k}| = \sigma^{2m}.
Since \(\|e^{tD}f\| = \|f\|\) it implies that
\[
\|B_{\frac{2m-1}{m}D}(\sigma)\| \leq \sigma^{2m-1}\|f\|, \quad \|B_{\frac{2m}{m}D}(\sigma)f\| \leq \sigma^{2m}\|f\|, \quad f \in E.
\]
For the following theorem see [11], [12], [16], [17].

**Theorem 2.6.** If \(D\) generates a one-parameter strongly continuous bounded group of operators \(e^{tD}\) in a Banach space \(E\) then the following conditions are equivalent:
1. \(f\) belongs to \(B_\sigma(D)\).
2. The abstract-valued function \(e^{tD}f\) is entire abstract-valued function of exponential type \(\sigma\) which is bounded on the real line.
3. For every functional \(g^* \in E^*\) the function \(\langle e^{tD}f, g^* \rangle\) is entire function of exponential type \(\sigma\) which is bounded on the real line.
4. The following Boas-type interpolation formulas hold true for \(r \in \mathbb{N}\)
\[
D^r f = B_{\frac{r}{D}}(\sigma)f, \quad f \in B_\sigma(D).
\]

**Corollary 2.1.** Every \(B_\sigma(D)\) is a closed linear subspace of \(E\).

Let’s introduce the operator \(\Delta^m s f = (I - e^{sD})^m f, \quad m \in \mathbb{N}\), and the modulus of continuity [5]
\[
\Omega_m(f, s) = \sup_{|\tau| \leq s} \|\Delta^m \tau f\|.
\]
The following theorem is proved in [14], [16], [17].

**Theorem 2.7.** There exists a constant \(C > 0\) such that for all \(\sigma > 0\) and all \(f \in D^k\)
\[
\inf_{g \in B_\sigma(D)} \|f - g\| \leq C \sigma^{-k} \Omega_m(D^k f, \sigma^{-1}), \quad 0 \leq k \leq m.
\]

**Corollary 2.2.** The set \(\bigcup_{\sigma \geq 0} B_\sigma(D)\) is dense in \(E\).

**Definition 2.8.** For a given \(f \in E\) the notation \(\sigma_f\) will be used for the smallest finite real number (if any) for which
\[
\|D^k f\| \leq \sigma_f^k \|f\|, \quad k \in \mathbb{N}.
\]
If there is no such finite number we assume that \(\sigma_f = \infty\).

Now we are going to prove another characterization of Bernstein spaces. In the case of Hilbert spaces corresponding result was proved in [13], [15].

**Theorem 2.9.** Let \(f \in E\) belongs to a space \(B_\sigma(D)\), for some \(0 < \sigma < \infty\). Then the following limit exists
\[
d_f = \lim_{k \to \infty} \|D^k f\|^{1/k}
\]
and \( d_f = \sigma_f \).

Conversely, if \( f \in D^\infty \) and \( d_f, f = \lim_{k \to \infty} \|D^k f\|^{1/k} \), exists and is finite, then \( f \in B_{d_f}(D) \) and \( d_f = \sigma_f \).

Let us introduce the Favard constants (see [1], Ch. V) which are defined as

\[
K_j = \frac{4}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^{(j+1)}}{(2r+1)^{j+1}}, \quad j, \ r \in \mathbb{N}.
\]

It is known [1], Ch. V, that the sequence of all Favard constants with even indices is strictly increasing and belongs to the interval \([1, 4/\pi]\) and the sequence of all Favard constants with odd indices is strictly decreasing and belongs to the interval \((\pi/4, \pi/2]\), i.e.,

\[
(2.13) \quad K_{2j} \in [1, 4/\pi), \ K_{2j+1} \in (\pi/4, \pi/2].
\]

We will need the following generalization of the classical Kolmogorov inequality. It is worth noting that the inequality was first proved by Kolmogorov for \( L^\infty(\mathbb{R}) \) and later extended to \( L^p(\mathbb{R}) \) for \( 1 \leq p < \infty \) by Stein [22] and that is why it is known as the Stein-Kolmogorov inequality.

**Lemma 2.10.** Let \( f \in D^\infty \). Then, the following inequality holds

\[
(2.14) \quad \|D^k f\|^n \leq C_{k,n} \|D^n f\|^k \|f\|^{n-k}, \quad 0 \leq k \leq n,
\]

where \( C_{k,n} = (K_{n-k})^n / (K_n)^{n-k} \).

**Proof.** Indeed, for any \( h^* \in E^* \) the Kolmogorov inequality [22] applied to the entire function \( \langle e^{tD} f, h^* \rangle \) gives

\[
\left\| \left( \frac{d}{dt} \right)^k \langle e^{tD} f, h^* \rangle \right\|_{C(\mathbb{R}^1)}^n \leq C_{k,n} \left\| \left( \frac{d}{dt} \right)^n \langle e^{tD} f, h^* \rangle \right\|_{C(\mathbb{R}^1)}^k \times
\]

\[
\left\| \langle e^{tD} f, h^* \rangle \right\|_{C(\mathbb{R}^1)}^{n-k}, \quad 0 < k < n,
\]

or

\[
\left\| \langle e^{tD} D^k f, h^* \rangle \right\|_{C(\mathbb{R}^1)}^n \leq C_{k,n} \left\| \langle e^{tD} D^n f, h^* \rangle \right\|_{C(\mathbb{R}^1)}^k \left\| \langle e^{tD} f, h^* \rangle \right\|_{C(\mathbb{R}^1)}^{n-k}.
\]

Applying the Schwartz inequality to the right-hand side, we obtain

\[
\left\| \langle e^{tD} D^k f, h^* \rangle \right\|_{C(\mathbb{R}^1)}^n \leq C_{k,n} \left\| h^* \right\|^k \|D^n f\|^k \|h^*\|^{n-k} \|f\|^{n-k},
\]

which, when \( t = 0 \), yields

\[
\left| \langle D^k f, h^* \rangle \right|^n \leq C_{k,n} \left\| h^* \right\|^n \|D^n f\|^k \|f\|^{n-k}.
\]

By choosing \( h \) such that \( \left| \langle D^k f, h^* \rangle \right| = \|D^k f\| \) and \( \|h^*\| = 1 \) we obtain (2.13). \( \square \)

**Proof of Theorem 2.9**

From Lemma 2.10 we have

\[
\|D^k f\|^n \leq C_{k,n} \|D^n f\|^k \|f\|^{n-k}, \quad 0 \leq k \leq n.
\]

Without loss of generality, let us assume that \( \|f\| = 1 \). Thus,

\[
\|D^k f\|^{1/k} \leq (\pi/2)^{1/k} \|D^n f\|^{1/n}, \quad 0 \leq k \leq n.
\]
Let $k$ be arbitrary but fixed. It follows that

$$\|D^k f\|^{1/k} \leq (\pi/2)^{1/kn} \|D^n f\|^{1/n},$$

for all $n \geq k$, which implies that

$$\|D^k f\|^{1/k} \leq \lim_{n \to \infty} \|D^n f\|^{1/n}.$$ 

But since this inequality is true for all $k > 0$, we obtain that

$$\lim_{k \to \infty} \|D^k f\|^{1/k} \leq \lim_{n \to \infty} \|D^n f\|^{1/n},$$

which proves that $d_f = \lim_{k \to \infty} \|D^k f\|^{1/k}$ exists.

Since $f \in B_\sigma(D)$ the constant $\sigma_f$ is finite and we have

$$\|D^k f\|^{1/k} \leq \sigma_f \|f\|^{1/k},$$

and by taking the limit as $k \to \infty$ we obtain

$$d_f \leq \sigma_f.$$ 

To show that $d_f = \sigma_f$, let us assume that $d_f < \sigma_f$. Therefore, there exist $M > 0$ and $\sigma > 0$ such that $0 < d_f < \sigma < \sigma_f$ and

$$\|D^k f\| \leq M\sigma^k, \quad \text{for all } k > 0,$$

which, in view of Lemma 2.4 implies that $f \in B_\sigma(D)$. Now by repeating the argument in the first part of the proof we obtain $d_f = \sigma_f$, where $\sigma_f = \inf \{\sigma : f \in B_\sigma(D)\}$.

Theorem 2.9 is proved.

Now consider the following abstract Cauchy problem for the operator $D$.

\[(2.15) \quad \frac{du(t)}{dt} = Du(t), \quad u(0) = f,\]

where $u : \mathbb{R} \to E$ is an abstract function with values in $E$. Since solutions of this problem given by the formula $u(t) = e^{tD}f$ we obtain the following result.

**Theorem 2.11.** A vector $f \in E$, belongs to $B_\sigma(D)$ if and only if the solution $u(t)$ of the corresponding Cauchy problem \[(2.15)\] has the following properties:

1) as a function of $t$, it has an analytic extension $u(z), z \in \mathbb{C}$ to the complex plane $\mathbb{C}$ as an entire function;

2) it has exponential type $\sigma$ in the variable $z$, that is

$$\|u(z)\|_E \leq e^{\sigma|z|}\|f\|_E.$$

and it is bounded on the real line.

3. **Sampling-type formulas for one-parameter groups**

We assume that $D$ generates one-parameter strongly cononuous bounded group of operators $e^{tD}, t \in \mathbb{R}$, in a Banach space $E$. In this section we prove explicit formulas for a trajectory $e^{tD}f$ with $f \in B_\sigma(D)$ in terms of a countable number of equally spaced spaced samples.
Theorem 3.1. If \( f \in B_\sigma(D) \) then the following sampling formulas hold for \( t \in \mathbb{R} \)

\[
e^{tD}f = f + tDf\text{sinc}\left(\frac{\alpha t}{\pi}\right) + t \sum_{k \neq 0} \frac{e^{\frac{ik\pi}{\sigma}Df} - f}{k\pi} \text{sinc}\left(\frac{\alpha t}{\pi} - k\right),
\]

(3.1)

\[
f = e^{tD}f - t(e^{tD}Df)\text{sinc}\left(\frac{\alpha t}{\pi}\right) - t \sum_{k \neq 0} \frac{e^{\frac{(k\pi + t)\pi}{\sigma}Df} - e^{tDf}}{k\pi} \text{sinc}\left(\frac{\alpha t}{\pi} + k\right).
\]

(3.2)

Remark 3.2. It is worth to note that if \( t \neq 0 \), then right-hand side of (3.2) does not contain vector \( f \) and we obtain a "linear combination" of \( f \) in terms of vectors \( e^{\frac{k\pi}{\sigma}Df}, \ k \in \mathbb{Z} \), and \( e^{tD}Df \).

Proof. If \( f \in B_\sigma(D) \) then for any \( g^* \in E^* \) the function \( F(t) = \langle e^{tD}f, g^* \rangle \) belongs to \( B_\sigma^\infty(\mathbb{R}) \).

We consider \( F_1(t) \in B_2^\sigma(\mathbb{R}) \), which is defined as follows. If \( t \neq 0 \) then

\[
F_1(t) = \frac{F(t) - F(0)}{t} = \left\langle \frac{e^{tD}f - f}{t}, g^* \right\rangle,
\]

and if \( t = 0 \) then \( F_1(t) = \frac{d}{dt}F(t)|_{t=0} = \langle Df, g^* \rangle \). We have

\[
F_1(t) = \sum_k F_1\left(\frac{k\pi}{\sigma}\right) \text{sinc}\left(\frac{\alpha t}{\pi} - k\right),
\]

which means that for any \( g^* \in E^* \)

\[
\left\langle \frac{e^{tD}f - f}{t}, g^* \right\rangle = \sum_k \left\langle \frac{e^{\frac{k\pi}{\sigma}Df} - f}{k\pi}, g^* \right\rangle \text{sinc}\left(\frac{\alpha t}{\pi} - k\right).
\]

Since

\[
\text{sinc}^{(n)}(\pi x) = \sum_{j=0}^{n} \frac{(-1)^j j!}{\pi x^{n+1}} \sum_{j=0}^{n} \sin \left(\pi x + \frac{2jx}{2}\right) \frac{(-1)^j (\pi x)^j}{j!}
\]

one has the estimate

\[
|\text{sinc}^{(n)}(\pi x)| \leq \frac{C}{|x|^n}, \quad n = 0, 1, ...
\]

which implies convergence in \( E \) of the series

\[
\sum_k \frac{e^{\frac{k\pi}{\sigma}Df} - f}{k\pi} \text{sinc}\left(\frac{\alpha t}{\pi} - k\right).
\]

It leads to the equality for any \( g^* \in E^* \)

\[
\left\langle \frac{e^{tD}f - f}{t}, g^* \right\rangle = \left\langle \sum_k \frac{e^{\frac{k\pi}{\sigma}Df} - f}{k\pi} \text{sinc}\left(\frac{\alpha t}{\pi} - k\right), g^* \right\rangle, \quad t \neq 0,
\]

and if \( t = 0 \) it gives the identity \( \langle Df, g^* \rangle = \langle Df, g^* \rangle \sum_k \text{sinc} k \). Thus,

(3.3)

\[
\frac{e^{tD}f - f}{t} = \sum_k \frac{e^{\frac{k\pi}{\sigma}Df} - f}{k\pi} \text{sinc}\left(\frac{\alpha t}{\pi} - k\right), \quad t \neq 0.
\]
or for every \( t \in \mathbb{R} \)
\[
e^{tD} f = f + tDf \text{sinc} \left( \frac{\sigma t}{\pi} \right) + t \sum_{k \neq 0} \frac{e^{\frac{kt}{\sigma}}Df - f}{\frac{k}{\pi}} \text{sinc} \left( \frac{\sigma t}{\pi} - k \right).
\]

Thus, (3.1) is proved.

If in (3.1) we replace \( f \) by \( (e^{\tau D} f) \) for a \( \tau \in \mathbb{R} \) we will have
\[
e^{tD} (e^{\tau D} f) = e^{\tau D} f + tD(e^{\tau D} f) \text{sinc} \left( \frac{\sigma t}{\pi} \right) + t \sum_{k \neq 0} \frac{e^{\frac{kt}{\sigma}}D(e^{\tau D} f) - (e^{\tau D} f)}{\frac{k}{\pi}} \text{sinc} \left( \frac{\sigma t}{\pi} - k \right).
\]

For \( t = -\tau \) we obtain the next formula which holds for any \( \tau \in \mathbb{R} \), \( f \in B_\sigma(D) \),
\[
(3.5) \quad f = e^{\tau D} f - \tau \sum_{k \neq 0} \frac{e^{\frac{kt}{\sigma}(\tau + \frac{D}{\pi})}f - e^{\frac{kt}{\sigma}D}f}{\frac{k}{\pi}} \text{sinc} \left( \frac{\sigma t}{\pi} + k \right) - \tau D(e^{\tau D} f) \text{sinc} \left( \frac{\sigma t}{\pi} \right),
\]
which is the formula (3.2). Theorem is proved.

The next Theorem is a generalization of what is known as Valiron-Tschakaloff sampling/interpolation formula [6].

**Theorem 3.3.** For \( f \in B_\sigma(D) \), \( \sigma > 0 \), we have for all \( z \in \mathbb{C} \)
\[
e^{zD} f = z \text{sinc} \left( \frac{\sigma Z}{\pi} \right) Df + \sum_{k \neq 0} \frac{\sigma Z}{k}\pi \text{sinc} \left( \frac{\sigma Z}{\pi} - k \right) e^{\frac{k}{\pi}Df}
\]

**Proof.** If \( F \in B_{\sigma}(D) \), \( \sigma > 0 \), then for all \( z \in \mathbb{C} \) the following Valiron-Tschakaloff sampling/interpolation formula holds [6]
\[
(3.7) \quad F(z) = z \text{sinc} \left( \frac{\sigma Z}{\pi} \right) F'(0) + \text{sinc} \left( \frac{\sigma Z}{\pi} \right) F(0) + \sum_{k \neq 0} \frac{\sigma Z}{k}\pi \text{sinc} \left( \frac{\sigma Z}{\pi} - k \right) F \left( \frac{k}{\sigma} \right)
\]

For \( f \in B_{\sigma}(D) \), \( \sigma > 0 \) and \( g^* \in E^* \) we have \( F(z) = \langle e^{zD} f, g^* \rangle \in B_{\sigma}(D) \) and thus
\[
\langle e^{zD} f, g^* \rangle = z \text{sinc} \left( \frac{\sigma Z}{\pi} \right) \langle Df, g^* \rangle + \text{sinc} \left( \frac{\sigma Z}{\pi} \right) \langle f, g^* \rangle + \sum_{k \neq 0} \frac{\sigma Z}{k}\pi \text{sinc} \left( \frac{\sigma Z}{\pi} - k \right) \langle e^{\frac{k}{\pi}Df}, g^* \rangle.
\]

Since the series
\[
\sum_{k \neq 0} \frac{\sigma Z}{k}\pi \text{sinc} \left( \frac{\sigma Z}{\pi} - k \right) e^{\frac{k}{\pi}Df}
\]
converges in \( E \) for every fixed \( z \) we obtain the formula (3.6).

**Theorem 3.4.** If \( f \in B_{\sigma}(D) \) then the following sampling formula holds for \( t \in \mathbb{R} \) and \( n \in \mathbb{N} \)
\[
e^{tD} D^n f = \sum_{k} \frac{e^{\frac{kt}{\sigma}}Df - f}{\frac{k}{\pi}} \left\{ n \text{sinc}^{(n-1)} \left( \frac{\sigma t}{\pi} - k \right) + \frac{\sigma t}{\pi} \text{sinc}^{(n)} \left( \frac{\sigma t}{\pi} - k \right) \right\}.
\]
In particular, for \( n \in \mathbb{N} \) one has
\[
D^n f = Q^n_D(\sigma)f,
\]
where the bounded operator \( Q^n_D(\sigma) \) is given by the formula
\[
Q^n_D(\sigma)f = n \sum_k \frac{e^{\frac{k\pi D}{\sigma}}f - f}{k\pi} \left[ \text{sinc}^{(n-1)}(-k) + \text{sinc}^n(-k) \right].
\]

Proof. Because \( F_1 \in B_2^2(\mathbb{R}) \) we have
\[
\left( \frac{d}{dt} \right)^n F_1(t) = \sum_k F_1 \left( \frac{k\pi}{\sigma} \right) \text{sinc}^n \left( \frac{\sigma t - k}{\pi} \right)
\]
and since
\[
\left( \frac{d}{dt} \right)^n F(t) = n \left( \frac{d}{dt} \right)^{n-1} F_1(t) + t \left( \frac{d}{dt} \right)^n F_1(t)
\]
we obtain
\[
\left( \frac{d}{dt} \right)^n F(t) = n \sum_k F_1 \left( \frac{k\pi}{\sigma} \right) \text{sinc}^n \left( \frac{\sigma t - k}{\pi} \right) + \frac{\sigma t}{\pi} \sum_k F_1 \left( \frac{k\pi}{\sigma} \right) \text{sinc}^n \left( \frac{\sigma t - k}{\pi} \right)
\]
Because \( \left( \frac{d}{dt} \right)^n F(t) = \langle D^n e^{tD} f, g^* \rangle \), and
\[
F_1 \left( \frac{k\pi}{\sigma} \right) = \langle e^{\frac{k\pi D}{\sigma}}f - f, g^* \rangle
\]
we obtain that for \( t \in \mathbb{R}, \ n \in \mathbb{N} \),
\[
D^n e^{tD} f = \sum_k e^{\frac{k\pi D}{\sigma}}f - f \left[ n \text{sinc}^{(n-1)} \left( \frac{\sigma t - k}{\pi} \right) + \frac{\sigma t}{\pi} \text{sinc}^n \left( \frac{\sigma t - k}{\pi} \right) \right].
\]
Theorem is proved. \( \square \)

4. Irregular sampling theorems

In \cite{8} the following fact was proved.

Theorem 4.1. Let \( \{t_n\}_{n \in \mathbb{Z}} \) be a sequence of real numbers such that
\[
\sup_{n \in \mathbb{Z}} |t_n - n| < 1/4.
\]
Define the entire function
\[
G(z) = (z - t_0) \prod_{n=1}^{\infty} \left( 1 - \frac{z}{t_n} \right) \left( 1 - \frac{z}{t_{-n}} \right).
\]
Then for all \( f \in B_2^2(\mathbb{R}) \) we have
\[
f(t) = \sum_{n \in \mathbb{Z}} f(t_n) \frac{G(t)}{G(t_n)(t - t_n)}
\]
uniformly on all compact subsets of \( \mathbb{R} \).

An analog of this result for Banach spaces is the following.
Theorem 4.2. If \( D \) generates a one-parameter strongly continuous group \( e^{tD} \) of isometries in a Banach space \( E \). Suppose that assumptions of Theorem 4.1 are satisfied and \( t_0 \neq 0 \). Then for all \( f \in B_2(D), \ g^* \in E^* \) and every \( t \in \mathbb{R} \) the following formulas hold

\[
\langle e^{tD}f, g^* \rangle = \langle f, g^* \rangle + t \sum_{n \in \mathbb{Z}} \left( \frac{\langle e^{tnD}f, g^* \rangle - \langle f, g^* \rangle}{t_n} \right) G(t),
\]

and

\[
\langle f, g^* \rangle = \langle e^{tD}f, g^* \rangle + t \sum_{n \in \mathbb{Z}} \left( \frac{\langle e^{(tn-t)D}f, g^* \rangle - \langle e^{tD}f, g^* \rangle}{t_n} \right) G(-t)
\]

uniformly on all compact subsets of \( \mathbb{R} \).

Remark 4.3. The last formula represents a "measurement" \( \langle f, g^* \rangle \) through "measurements" \( \langle e^{(tn-t)D}f, g^* \rangle \) and \( \langle e^{tD}f, g^* \rangle \) which are different from \( \langle f, g^* \rangle \).

Proof. If \( f \in B_2(D) \) then for any \( g^* \in E^* \) the function \( F(t) = \langle e^{tD}f, g^* \rangle \) belongs to \( B^*_2(\mathbb{R}) \). We consider \( F_1 \in B^*_2(\mathbb{R}) \), which is defined as follows. If \( t \neq 0 \) then

\[
F_1(t) = \frac{F(t) - F(0)}{t} = \left\langle \frac{e^{tD}f - f}{t}, g^* \right\rangle, \ g^* \in E^*,
\]

and if \( t = 0 \) then

\[
F_1(t) = \frac{d}{dt} F(t)|_{t=0} = \langle Df, g^* \rangle.
\]

We have

\[
F_1(t) = \sum_{n \in \mathbb{Z}} F_1(t_n) \frac{G(t)}{G(t_n)(t - t_n)}
\]

or

\[
\left\langle \frac{e^{tD}f - f}{t}, g^* \right\rangle = \sum_{n \in \mathbb{Z}} \left\langle \frac{e^{tnD}f - f}{t_n}, g^* \right\rangle \frac{G(t)}{G(t_n)(t - t_n)}
\]

uniformly on all compact subsets of \( \mathbb{R} \).

If we pick a non-zero \( \tau \) such that \( \tau \neq t_n \) for all \( t_n \) and set \( f \in [4.1] \) to \( e^\tau Df \) then for \( t = -\tau \) we will have the following formula which does not have vector \( f \) on the right-hand side

\[
\langle f, g^* \rangle = \langle e^{\tau D}f, g^* \rangle + \tau \sum_{n \in \mathbb{Z}} \left( \frac{\langle e^{(tn-\tau)D}f, g^* \rangle - \langle e^{\tau D}f, g^* \rangle}{t_n} \right) G(-\tau)
\]

Theorem is proved.

In [21] the following result can be found.

Theorem 4.4. Let \( \{t_n\}_{n \in \mathbb{Z}} \) be a sequence of real numbers such that

\[
\sup_{n \in \mathbb{Z}} |t_n - n| < 1/4.
\]

Define the entire function

\[
G(z) = (z - t_0) \prod_{n=1}^{\infty} \left( 1 - \frac{z}{t_n} \right) \left( 1 - \frac{z}{t_{-n}} \right).
\]
Let \( \delta \) be any positive number such that \( 0 < \delta < \pi \). Then for all \( f \in B_{\pi - \delta}(\mathbb{R}) \) we have

\[
f(z) = \sum_{n \in \mathbb{Z}} f(t_n) \frac{G(z)}{G'(t_n)(z - t_n)}
\]

uniformly on all compact subsets of \( \mathbb{C} \).

From here we obtain the next fact.

**Theorem 4.5.** Suppose that \( D \) generates a one-parameter strongly continuous group \( e^{itD} \) of isometries in a Banach space \( E \). Then in the same notations as in Theorem 4.4 one has that for all \( f \in B_{\pi - \delta}(D) \) and all \( g^* \in E^* \) the following formula holds

(4.4)

\[
\langle e^{zD}f, g^* \rangle = \sum_{n \in \mathbb{Z}} \langle e^{tnD}f, g^* \rangle \frac{G(z)}{G'(t_n)(z - t_n)}
\]

uniformly on all compact subsets of \( \mathbb{C} \).

**Proof.** Proof follows from Theorem 4.4 since for any \( g^* \in E^* \) the function \( \langle e^{zD}f, g^* \rangle \) belongs to \( B_{\pi - \delta}(\mathbb{R}) \). \( \square \)

Note that if in the last formula we will set \( z \) to zero and assume that \( t_0 \neq 0 \) we will have a representation of \( \langle f, g^* \rangle \) in terms of samples \( \langle e^{tnD}f, g^* \rangle \neq \langle f, g^* \rangle \) i.e.

(4.5)

\[
\langle f, g^* \rangle = -\sum_{n \in \mathbb{Z}} \langle e^{tnD}f, g^* \rangle \frac{G(0)}{G'(t_n)t_n}.
\]

5. An application to abstract Schrödinger equation

We now assume that \( E \) is a Hilbert space and \( D \) is a self-adjoint operator. Then \( e^{itD} \) is one-parameter group of isometries of \( E \). By the spectral theory \([2]\), there exist a direct integral of Hilbert spaces \( A = \int A(\lambda)dm(\lambda) \) and a unitary operator \( F_D \) from \( E \) onto \( A \), which transforms the domain \( D_k \) of the operator \( D^k \) onto \( A_k = \{ a \in A| \lambda^k a \in A \} \) with norm

\[
\| a(\lambda) \|_{A_k} = \left( \int_{-\infty}^{\infty} \lambda^{2k} \| a(\lambda) \|_{A(\lambda)}^2 dm(\lambda) \right)^{1/2}
\]

and \( F_D(Df) = \lambda(F_Df) \), \( f \in D_1 \).

In this situation one can prove \([11]\) the following.

**Theorem 5.1.** A vector \( f \in E \) belongs to \( B_\sigma(D) \) if and only if support of \( F_Df \) is in \([-\sigma, \sigma]\).

We consider an abstract Cauchy problem for a self-adjoint operator \( D \) which consists of finding an abstract-valued function \( u : \mathbb{R} \to E \) which satisfies Schrödinger equation and has bandlimited initial condition

(5.1)

\[
\frac{du(t)}{dt} = iDu(t), \quad u(0) = f \in B_\sigma(D)
\]

(see \([3], [9]\) for more details).

In this case formula (4.5) can be treated as a solution to inverse problem associated with (5.1).
Theorem 5.2. If conditions of Theorem 4.5 are satisfied and \( t_0 \neq 0 \) then initial condition \( f \in B_{\pi - \delta}(D), \ 0 < \delta < \pi \) in (5.1) can be reconstructed (in weak sense) from the values of the solution \( u(t_n) \) by using the formula

\[
\langle f, g^* \rangle = -\sum_{n \in \mathbb{Z}} \langle u(t_n), g^* \rangle \frac{G(0)}{G'(t_n)t_n} g^* \in E^*.
\]

Similar results can be formulated by using formulas (3.5) and (1.3).

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