STABLE FINITE ELEMENT PAIR FOR STOKES PROBLEM AND DISCRETE STOKES COMPLEX ON QUADRILATERAL GRIDS

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ABSTRACT. In this paper, we first construct a nonconforming finite element pair for incompressible Stokes problem on quadrilateral grids, and then construct a discrete Stokes complex associated with that finite element pair. The finite element spaces involved consist of piecewise polynomials only, and the divergence-free condition is imposed in a primal formulation. Combined with some existing results, these constructions can be generated onto grids that consist of both triangular and quadrilateral cells.

1. INTRODUCTION

The Stokes problem is an important model problem in applied sciences, which can be used to describe the motion of an incompressible fluid. In this paper, we study the stable finite element method for the two dimensional stationary incompressible Stokes problem of velocity-pressure type. In this context, a stable finite element method of the model problem implies a pair of finite element spaces that are consistent approximations to the Hilbert spaces $(H^1(\Omega))^2$ and $L^2(\Omega)$, respectively, and satisfy the two stability conditions which are, with the detailed technical description given in the following section,

**SC 1:** the coercive condition and inf-sup condition hold uniformly;

**SC 2:** the divergence-free constraint is imposed in a primal formulation.

The first condition falls into the classical theory of Stokes problem, as it provides a necessary and sufficient condition for the well-posedness of the discrete problem; see, e.g., [7,13,30]. The second condition, sometimes known as mass conservation, is also desirable in many applications. Though it is not yet fully revealed how methods that enforce divergence-free would be superior to those that do not, satisfying the property can decouple the pressure error from the velocity error, and can avoid possible instabilities.

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that can arise from violation of mass conservation \[6, 22, 40, 41\]. Various approaches have been utilised to develop methods in regard to the conditions both SC 1 and SC 2, including the discontinuous Galerkin methods \[16, 19–21\], the isogeometric methods \[24, 25\], the least square finite element methods \[9, 12, 18, 33\], and finite element methods with enhanced stabilisation \[8, 11, 15, 17, 45\]. In this paper, we focus ourselves on the traditional finite element methods, and will not mention other approaches too much.

A few finite element pairs have been reported to satisfy the conditions both SC 1 and SC 2. As a natural idea, the conforming \(P^2_k - P_{k-1}\) pairs were constructed for \(k \geq 2\). They are proved to satisfy SC 1 and SC 2 on special types of uniform or quasi-uniform triangulations (Scott-Vogelius \[50, 51\] for \(k \geq 4\), Arnold-Qin \[4\] for \(k = 2\), and Qin \[48\] for \(k = 3\)). By adding extra smoothness to the finite element functions on the vertices other than the corners of the domain, Falk-Neilan \[27\] designed a special family of \(P^2_k - P_{k-1}\) pairs for \(k \geq 4\) that are shown to satisfy SC 1 and SC 2 on general grids without the so-called singular corner vertices. On general triangular grids, Crouzeix-Raviart \[23\] constructed a nonconforming \(P^2_1 - P_0\) element which satisfies both SC 1 and SC 2 in a nonconforming way. A similar nonconforming \(P^2_2 - P_1\) element was constructed by Fortin-Soulie \[29\]. Another natural idea is, for a given velocity space, using its divergence space as the pressure space, and/or reversely, for a given pressure space, looking for a velocity space so that divergence is a surjection. This idea succeeds where nodal basis functions can be constructed, such as the \(Q_{k+1,k} \times Q_{k,k+1} - Q_k\) pair on rectangular grid (see Zhang \[62\] for \(k \geq 2\) and Huang-Zhang \[35\] for \(k = 1\)), and the Mardal-Tai-Winther pair \[42\] on triangular grids, which uses a space of vector functions rather than a tensor product of two scalar function spaces to approximate the velocity field. Kouhia-Stenberg \[50\] also used different function spaces for different components of the velocity and constructs a stable linear element method. Xie-Xu-Xue \[61\] made a way to add divergence-free basis functions onto \(H(\text{div})\)-conforming finite element space to generate the velocity function space, and generate and survey several stable pairs in a unified way. Guzmán-Neilan \[31\] constructed the velocity spaces by adding rational divergence-free functions to \(H(\text{div})\)-conforming functions, and obtained conforming stable pairs. Beside these examples, there have been many finite element pairs that satisfy the condition SC 1, while satisfy the divergence-free condition SC 2 in a dual formulation; for these pairs, see \[10, 14, 30, 49\] and the references therein.
Principally, the condition SC 2 decouples the computations of pressure and velocity, and it would bring convenience once we can present a precise description of the divergence-free velocity subspace. Mathematically, a divergence-free function can be the curl of some other function [30]. This is relevant to the fundamental observation that the incompressible velocity field admits a stream function, and this gives a natural connection between the incompressible Stokes problem and the biharmonic equation. This property can be described in the framework of the Stokes complex originally introduced by [42,60]. There have been various complexes to describe different physical and mathematical observations [2,3]. A powerful tool to design, analyse, understand, and moreover, to apply the stable finite element pair for the Stokes problem is then to reproduce discrete analogous of the Stokes complex where the Sobolev spaces are replaced by corresponding finite element spaces. The existence of such a structure like the discrete Stokes complex makes the connection between the model problems revealed at discrete level, and wider scope of methods and applications of the model problems can be expected. This structure is an intrinsic connection between the finite element pairs, and some of the existing pairs that satisfy both SC 1 and SC 2 have been shown to be associated with specific discrete Stokes complexes. On triangular grids, for instance, discrete Stokes complexes have been established associated with the conforming $P^2_k - P_{k-1}$ element, with([31]) or without([50]) extra smoothness on vertices, and the nonconforming $P^2_1 - P_0$ element pair [26], respectively. Different discrete Stokes complexes were also introduced in [42] and [31], respectively. While when quadrilateral grids are considered, few discrete Stokes complexes are known.

As the quadrilateral grids are widely used where the problem geometry is of quadrilateral nature, in this paper, we study the stable finite element method for Stokes problem that satisfy conditions both SC 1 and SC 2 on quadrilateral grids. Specifically, as it is seen among the existing methods that the nonconforming methodology would in general admit higher flexibility, we develop a nonconforming finite element pair that satisfies both SC 1 and SC 2 in a nonconforming way, and then construct a discrete Stokes complex associated with this element pair. After carry out the discussion on quadrilateral grids, we will then carry out the discussion on grids consisting of both triangular and rectangular cells to obtain parallel analogue results.

On the quadrilateral grids, we use an average continuous piecewise incomplete quadratic polynomial space for the velocity field. The velocity space is the same as the
one used for solving the Poisson equation in Lin-Tobiska-Zhou [38] on rectangular grid, while the nodal parameters of the velocity space had been used by Han [32] with different shape function space on rectangle grids. The same velocity space was also used in Shi-Zhang [52] for rectangular grids to shape a finite element pair for the Stokes problem, where the pressure is approximated by piecewise constant, and the condition $\text{SC 2}$ holds in a dual formulation. Shi-Zhang’s element also relies on a bilinear mapping between the cell and a reference rectangle when forming the shape functions on quadrilateral cells. In this present paper, applying the idea in [47], we define the finite element functions on quadrilateral cells directly, and they are all piecewise polynomials. We use discontinuous piecewise linear polynomial space for the pressure, and both the conditions $\text{SC 1}$ and $\text{SC 2}$ are satisfied.

A discrete Stokes complex is then constructed based on the newly established finite element pair as we prove that the divergence-free part of the discrete velocity space is piecewisely the curl of a quadrilateral Morley element space. The Morley element was originally constructed on triangular grids to solve the fourth order problem with piecewise quadratic polynomials [44], and generalized to arbitrary dimension by Wang-Xu [58], and to elliptic problems of arbitrary order by Wang-Xu [59]. Using the same nodal parameters as the Morley element, Wang-Shi-Xu [57] constructs a rectangle Morley element, which is generalised by Park-Sheen [47] to general convex quadrilateral grids. The Morley element was proved to be associated with a discrete Stokes complex on triangles together with the nonconforming $P_2^1-P_0$ element [26,28]. In this present paper, we show that a discrete Stokes complex connects the quadrilateral Morley element and the newly-developed Stokes element pair on quadrilateral grids.

Because of the similarity of the nodal parameters of the finite elements established on triangular and quadrilateral cells, which implies the same continuity of finite element functions on triangular and quadrilateral triangulations, it is then natural to combine these finite elements together to form discretisation schemes for the biharmonic problem and the Stokes problem, respectively, on a mixed grid involving both triangular and quadrilateral cells. Since the nodal interpolations are defined locally, this combination is straightforward, and finally a same discrete Stokes complex is also established on the mixed grid.

The rest of the paper is as follows. In Section 2 we introduce some preliminaries including the model problems and general finite element discretisation. In Section 3 we introduce a stable finite element pair on quadrilateral grids, and construct a discrete
Stokes complex. In Section 4, we carry out the discussion on a mixed grid. Finally, conclusions are given in Section 5.

2. Preliminaries

2.1. Stokes problem and the Stokes complex. Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain, and $\Gamma = \partial \Omega$ be the boundary, with $n$ the outward unit normal vector. We consider the incompressible Stokes problem with homogeneous boundary condition:

\[
\begin{aligned}
-\nu \Delta u + \nabla p &= f & \text{in } \Omega, \\
\nabla \cdot u &= 0 & \text{in } \Omega, \\
u \cdot u &= 0 & \text{on } \partial \Omega.
\end{aligned}
\]

Here $\nu$ is the kinematic viscosity, $u$, $p$, and $f$ denote the velocity, the pressure, and the external body force, respectively, and $\Delta$ and $\nabla$ are the Laplacian and gradient operators, respectively. For simplicity, we set $\nu = 1$ in the rest of the paper.

Denote by $H^1(\Omega)$, $H^1_0(\Omega)$, $H^2(\Omega)$, and $H^2_0(\Omega)$ the standard Sobolev spaces as usual, and $L^2_0(\Omega) := \{w \in L^2(\Omega) : \int_\Omega w dx = 0\}$. The variational form of (1) is to find $(u, p) \in (H^1_0(\Omega))^2 \times L^2_0(\Omega)$, such that

\[
\begin{aligned}
(\nabla u, \nabla v) + (\nabla \cdot v, p) &= (f, v) & \forall v \in (H^1_0(\Omega))^2, \\
(\nabla \cdot u, q) &= 0 & \forall q \in L^2_0(\Omega).
\end{aligned}
\]

Here $(\nabla u, \nabla v) = \int_\Omega \sum_{i,j=1}^2 (\nabla u)_{ij}(\nabla v)_{ij} dx$, and $(f, v) = \int_\Omega \sum_{i=1}^2 f_i v_i dx$. The well-posedness of (2) is guaranteed by the facts [30]:

\[
(\nabla v, \nabla v) \geq C_1 ||v||^2_{L^2(\Omega)} \forall v \in (H^1_0(\Omega))^2, \quad \text{and} \quad \inf_{0 \neq q \in L^2_0(\Omega)} \sup_{0 \neq v \in (H^1_0(\Omega))^2} \frac{(\nabla \cdot v, q)}{||v||_{L^2(\Omega)} ||q||_{L^2(\Omega)}} \geq C_2,
\]

with $C_1$ and $C_2$ two positive constants dependent on the domain only.

The biharmonic problem associated with the Stokes problem (1) is:

\[
\begin{aligned}
\Delta^2 \varphi &= F \in H^{-2}(\Omega), & \text{in } \Omega, \\
\partial \varphi / \partial n &= 0, & \text{on } \partial \Omega.
\end{aligned}
\]

The variational problem is to find $\varphi \in H^2_0(\Omega)$, such that

\[
(\nabla^2 \varphi, \nabla^2 \psi) = F(\psi), \ \forall \psi \in H^2_0(\Omega).
\]

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Here \( (\nabla^2 \phi, \nabla^2 \psi) = \int_{\Omega} \sum_{i,j=1}^2 (\nabla^2 \phi)_i (\nabla^2 \psi)_j \, \mathrm{d}x \). Denote by \( \text{curl} \) the curl operator on a scalar function, which is the rotation of the gradient operator \( \nabla \). Define for vector functions the operators \( \text{div} = \nabla \cdot \) and \( \text{curl} = \text{curl} \cdot \). We have the basic relation below.

**Lemma 1.** [30] Let \( \Omega \) be simply connected.

1. \( \text{curl} H^2_0(\Omega) = \{ v \in (H^1_0(\Omega))^2 : \text{div} v = 0 \}, \) and \( \text{div}(H^1_0(\Omega))^2 = L^2_0(\Omega) \).
2. Let \( (u, p) \) be the solution of (2), and \( \varphi \) be the solution of (4), with \( F(\psi) = (f, \text{curl} \psi) \) for \( \psi \in H^2_0(\Omega) \). Then \( u = \text{curl} \varphi \).

We can rewrite Lemma 1 in the form of the Stokes complex [27, 42, 60] which reads

\[
0 \rightarrow H^2_0(\Omega) \xrightarrow{\text{curl}} (H^1_0(\Omega))^2 \xrightarrow{\text{div}} L^2_0(\Omega) \rightarrow 0.
\]

In this sequence, the composition of two consecutive mappings is zero, and the range of each map is the null space of the succeeding map in a simply connected domain.

2.2. **Finite element method for Stoke problem.** When the Sobolev spaces \( (H^1_0(\Omega))^2 \) and \( L^2_0(\Omega) \) are replaced by some finite element spaces \( V_{h0} \) and \( \hat{W}_h \), conforming or nonconforming, we have the finite element problem: find \( (u_h, p_h) \in V_{h0} \times \hat{W}_h \), such that

\[
\begin{cases}
(\nabla_h u_h, \nabla_h v_h) + (\text{div}_h v_h, p_h) = (f, v_h) & \forall v_h \in V_{h0}, \\
(\text{div}_h u_h, q_h) = 0 & \forall q_h \in \hat{W}_h.
\end{cases}
\]

Here \( \nabla_h \) and \( \text{div}_h \) are in the piecewise sense for nonconforming \( V_{h0} \).

We define two stable conditions for the finite element scheme. The subscript “\( h \)” in the norms implies the dependence of the triangulation.

**SC 1:** There exist two positive constants \( \gamma_1 \) and \( \gamma_2 \), such that

\[
\inf_{v_h \in Z_h, v_h \neq 0} \frac{(\nabla_h v_h, \nabla_h v_h)}{||v_h||^2_{1,h}} := \gamma_h^1 > \gamma_1 \text{ on } Z_h := \{ v_h \in V_{h0} : (\text{div}_h v_h, q_h) = 0, \forall q_h \in \hat{W}_h \}
\]

and

\[
\inf_{0 \neq q_h \in \hat{W}_h} \sup_{0 \neq v_h \in V_{h0}} \frac{(\text{div}_h v_h, q_h)}{||v_h||_{1,h} ||q_h||_0} := \gamma_h^2 > \gamma_2.
\]

**SC 2:** \( Z_h = \{ v_h \in V_{h0} : \text{div}_h v_h = 0 \} \).

Evidently, a necessary condition that both of the two conditions hold is that the pressure space is the divergence space of the velocity space, in a conforming or nonconforming way.

We have the convergence result for the finite element problem.
Lemma 2. ([1],[14]) Let the stable condition SC 1 hold. Then the discrete problem ([11]) has a unique solution. Moreover, let \((u, p)\) and \((u_h, p_h)\) be the solution of (2) and (11), respectively, then there exists a constant \(C\) depending only on \(\gamma_1\) and \(\gamma_2\), such that

\[
\|u - u_h\|_{1,h} + \|p - p_h\|_{0,\Omega} \leq C \left( \inf_{v_h \in V_{h,0}} \|u - v_h\|_{1,h} + \inf_{q_h \in W_h} \|p - q_h\|_0 \right. \\
+ \left. \sup_{0 \neq v_h \in V_h} \frac{(\nabla_h u, \nabla_h v_h) + (\text{div}_h v_h, p) - (f, v_h)}{\|v\|_{1,h}} \right).
\]

The last term is the consistency error, which vanishes when \(V_{h,0} \subset (H^1_0(\Omega))^2\).

3. Stable finite element pair and discrete Stokes complex on quadrilateral grids

3.1. Quadrilateral triangulation.

3.1.1. Geometry of convex quadrilateral grid. Let \(Q\) be a convex quadrilateral with \(a_i\) the vertices and \(e_i\) the edges, \(i = 1 : 4\). See Figure 1 for an illustration. Let \(m_i\) be the mid-point of \(e_i\), then the quadrilateral \(\square m_1 m_2 m_3 m_4\) is a parallelogram ([46]). The cross point of \(m_1 m_3\) and \(m_2 m_4\), which is labelled as \(O\), is the midpoint of both \(m_1 m_3\) and \(m_2 m_4\).

Denote \(r = \overrightarrow{Om_3}\) and \(s = \overrightarrow{Om_4}\). Then the coordinates of the vertices in the coordinate system \(rOs\) are \(a_1(-1-\alpha,-1-\beta), a_2(-1+\alpha,-1+\beta), a_3(-1+\alpha,1+\beta)\) and \(a_4(1+\alpha,1+\beta)\) for some \(\alpha, \beta\). Since \(Q\) is convex, \(|\alpha| + |\beta| < 1\) ([47]). Without loss of generality, we assume \(\alpha > 0, \beta > 0\) and \(r \times s > 0\).

Define the shape regularity indicator of the of the cell \(Q\) by \(R_Q := \max\{|r|,|s|,|r \times s|\}\). Evidently \(R_Q \geq 1\), and \(R_Q = 1\) if and only if \(Q\) is a square. A given family of quadrilateral triangulations \(\{Q_h\}\) of \(\Omega\) is said to be regular, if all the shape regularity indicators of the cells of all the triangulations are uniformly bounded.

Define two linear functions \(\xi\) and \(\eta\) by \(\xi(ar + bs) = b\) and \(\eta(ar + bs) = a\). The two functions play the same role on quadrilateral as that of barycentric coordinate on triangles.

3.1.2. Triangulations and grids. Let \(Q_h\) be a regular triangulation of domain \(\Omega\), with the cells being convex quadrilaterals; i.e., \(\Omega = \bigcup_{Q \in Q_h}Q\). Let \(\mathcal{N}_h\) denote the set of all the vertices, \(\mathcal{N}_h = \mathcal{N}_h^i \cup \mathcal{N}_h^b\), with \(\mathcal{N}_h^i\) and \(\mathcal{N}_h^b\) consisting of the interior vertices and the boundary vertices, respectively. Similarly, let \(\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b\) denote the set of all the
edges, with \( E^i_h \) and \( E^b_h \) consisting of the interior edges and the boundary edges, respectively. For an edge \( e \), \( n_e \) is a unit vector normal to \( e \), and \( \tau_e \) is a unit tangential vector of \( e \) such that \( n_e \times \tau_e > 0 \). On the edge \( e \), we use \( \llbracket \cdot \rrbracket_e \) for the jump across \( e \).

Denote by \( \mathfrak{N} \) the number of cells of the triangulation; denote by \( \mathfrak{X}, \mathfrak{X}_I, \mathfrak{X}_B \) and \( \mathfrak{X}_C \) the number of vertices, internal vertices, boundary vertices, and corner vertices, respectively; and denote by \( \mathfrak{E}, \mathfrak{E}_I \) and \( \mathfrak{E}_B \) the number of edges, internal edges, and boundary edges, respectively. Euler’s formula states that \( \mathfrak{N} + \mathfrak{X} = \mathfrak{E} + 1 \).

3.2. An incomplete quadratic finite element on quadrilateral grid.

3.2.1. A finite element on convex quadrilaterals. The quadrilateral finite element presented below coincides with the one given by Lin-Tobiska-Zhou \[38\] on rectangle \( Q \), and we call it the quadrilateral Lin-Tobiska-Zhou(QLTZ) element.

The QLTZ element is defined by \((Q, P^\text{QLTZ}_Q, D^\text{QLTZ}_Q)\) with

1. \( Q \) is a convex quadrilateral;
2. \( P^\text{QLTZ}_Q = P_1(Q) + \text{span}\{\xi^2, \eta^2\} \);
3. the components of \( D^\text{QLTZ}_Q = (d^\text{QLTZ}_Q)_{i=0:4} \) for any \( v \in H^1(Q) \) are:

\[
    d^\text{QLTZ}_0(v) = \int_Q v \, dx, \quad \text{and} \quad d^\text{QLTZ}_i(v) = \int_{e_i} v \, ds, \quad e_i \text{ the edges of } T, \ i = 1 : 4.
\]
The element defined above is unisolvent. Indeed, define

\[
\begin{align*}
\phi_0 &= -3(3\xi^2 + 3\eta^2 - 2\alpha\xi - 2\beta\eta - (4 + \alpha^2 + \beta^2))/(2\alpha^2 + 2\beta^2 + 6), \\
\phi_1 &= -\frac{3}{4}\xi^2 + \frac{\beta - 1}{2}\eta + \frac{3 + \beta^2}{4} - \frac{\beta^2 - \beta + 3}{6}\phi_0, \\
\phi_2 &= -\frac{3}{4}\eta^2 + \frac{\alpha - 1}{2}\xi + \frac{3 + \alpha^2}{4} - \frac{\alpha^2 - \alpha + 3}{6}\phi_0, \\
\phi_3 &= -\frac{3}{4}\xi^2 + \frac{\beta + 1}{2}\eta + \frac{3 + \beta^2}{4} - \frac{\beta^2 + \beta + 3}{6}\phi_0, \\
\phi_4 &= -\frac{3}{4}\eta^2 + \frac{\alpha + 1}{2}\xi + \frac{3 + \alpha^2}{4} - \frac{\alpha^2 + \alpha + 3}{6}\phi_0,
\end{align*}
\]

then \(d_i^{QLTZ}(\phi_j) = \delta_{ij}, i, j = 0 : 4\).

Define the interpolation \(\Pi_Q^{QLTZ} : H^1(Q) \to P_Q^{QLTZ}\) by \(\Pi_Q^{QLTZ}w = \sum_{i=1}^4 \int_{E_i} w\, ds_i + \int_Q w\, dx\phi_0\). Then \(\Pi_Q^{QLTZ}\) is well-defined, and \(\Pi_Q^{QLTZ}w = w\), if \(w \in P_Q^{QLTZ}\).

Let \(w = (w_1, w_2)^T \in (H^1(Q))^2\). We define the interpolator \(\Pi_Q^{QLTZ} : (H^1(Q))^2 \to (p_Q^{QLTZ})^2\) by two steps:

**Step 1:** Construct \(w^1 = (w_1^1, w_2^1)^T\) by \(w_i^1 = \sum_{j=1}^4 \int_{E_j} w_i\, ds_j\) for \(i = 1, 2\).

**Step 2:** Find \(w^2 = (c_1\phi_0, c_2\phi_0)^T\) such that

\[
\int_Q \text{div} w^2 q dx = \int_Q \text{div}(w - w^1) q dx, \quad \forall q \in P_1(Q).
\]

Then define

\[
\Pi_Q^{QLTZ}w := w^1 + w^2.
\]

*Lemma 3.* The interpolator \(\Pi_Q^{QLTZ}\) is well-defined. Moreover, \(\Pi_Q^{QLTZ}w = w\), if \(w \in (p_Q^{QLTZ})^2\), and \(\int_Q \text{div}\Pi_Q^{QLTZ}w q dx = \int_Q \text{div}w q dx, \quad \forall q \in P_1(Q)\).

*Proof.* To show the well-definedness of \(\Pi_Q^{QLTZ}\), we only have to show that the problem \(\textbf{w}\) is well-posed. By the definition of \(w^1\), we obtain that \(\int_{E_i} w - w^1\, ds = 0\) for \(i = 1 : 4\). Thus by the property of \(\phi_0\), \(\int_Q \text{div}(c_1\phi_0, c_2\phi_0)^T dx = 0 = \int_Q \text{div}(w - w^1) dx\) for any \(c_1, c_2\).
Therefore, we only have to show that the equation (8) admits a unique solution pair \((c_1, c_2)\) for \(q\) substituted by \(\xi\) and \(\eta\). The coefficient matrix of the left hand side of (8) is then

\[
\begin{bmatrix}
\int_Q \partial_x \phi_0 \xi dx & \int_Q \partial_y \phi_0 \xi dx \\
\int_Q \partial_x \phi_0 \eta dx & \int_Q \partial_y \phi_0 \eta dx
\end{bmatrix}
= \begin{bmatrix}
\int_Q \partial_x \phi_0 \xi dx & \int_Q \partial_y \phi_0 \xi dx \\
\int_Q \partial_x \phi_0 \eta dx & \int_Q \partial_y \phi_0 \eta dx
\end{bmatrix}
\begin{bmatrix}
r_x & s_x \\
r_y & s_y
\end{bmatrix}^{-1},
\]

and we only have to check the determinant of the coefficient matrix.

Technically, we construct two tables (Tables 1 and 2) about the evaluation of some functions firstly. In particular, Table 1 is used in generating Table 2. For example, we calculate \(\int_Q (\xi^2 \eta) dx = \frac{1}{2} \int_Q (\xi^2 \eta)s \cdot n ds = \frac{1}{2} \sum_{i=1}^4 \int_{e_i} (\xi^2 \eta)s \times \tau_i ds = \frac{4}{3} \alpha \beta r \times s\). Here, we have noted that \(\partial_x \xi = \partial_y \eta = 0\) and \(\partial_y \xi = \partial_x \eta = 1\).

| \(\text{function}(u)\) | 1 \(\xi\) \(\eta\) \(\xi^2\) \(\eta^2\) \(\xi^3\) \(\eta^3\) \(\xi^2 \eta\) |
|-----------------|---|---|---|---|---|---|---|
| \(\int_{e_1} u ds\) | 1 0 -1 (1 - \(\beta\))^2 \(\frac{1}{3}\) \(1 + \frac{\alpha^2}{3}\) 0 \(-1 - \alpha^2\) \(-\frac{(1 - \beta)^2}{3}\) |
| \(\int_{e_2} u ds\) | 1 -1 0 1 + \(\beta^2\) \(\frac{1}{3}\) \(1 - \beta^2\) 0 \(2(1 - \alpha)\beta\) |
| \(\int_{e_3} u ds\) | 1 0 1 \(\frac{(1 + \beta)^2}{3}\) \(1 + \frac{\alpha^2}{3}\) 0 \(1 + \alpha^2\) \(\frac{(1 + \beta)^2}{3}\) |
| \(\int_{e_4} u ds\) | 1 1 0 \(\frac{\beta^2}{3}\) \(\frac{(1 + \alpha)^2}{3}\) \(1 + \beta^2\) 0 \(2(1 + \alpha)\beta\) |

Table 1. Boundary average of some functions.

| \(\text{function}(u)\) | 1 \(\xi\) \(\eta\) \(\xi^2\) \(\eta^2\) \(\xi^3\) \(\eta^3\) \(\xi \eta\) |
|-----------------|---|---|---|---|---|---|---|
| \(\int_Q u dx\) | 4r \times s \(\frac{2\alpha}{3}\)r \times s \(\frac{2\beta}{3}\)r \times s \(\frac{1}{3}(1 + \beta^2)\)r \times s \(\frac{2}{3}(1 + \alpha^2)\)r \times s \(\frac{2}{3}\alpha \beta r \times s\) |

Table 2. Domain average of some functions.

As \(\phi_0 = -3(3\xi^2 + 6\eta^2 - 2\alpha \xi - 2\beta \eta - (4 + \alpha^2 + \beta^2))/(2\alpha^2 + 2\beta^2 + 6)\), we have

\[
\partial_x \phi_0 = \frac{-9\eta + 3\beta}{\alpha^2 + \beta^2 + 3}, \quad \text{and} \quad \partial_y \phi_0 = \frac{-9\xi + 3\alpha}{\alpha^2 + \beta^2 + 3}.
\]
Then
\[
\begin{align*}
\int_Q \partial_r \phi_0 \xi dx &= \frac{-1}{\alpha^2 + \beta^2 + 3} \int_Q (9\xi \eta - 3\beta \xi) dx = \frac{-8\alpha \beta}{\alpha^2 + \beta^2 + 3} \mathbf{r} \times \mathbf{s}, \\
\int_Q \partial_s \phi_0 \xi dx &= \frac{-1}{\alpha^2 + \beta^2 + 3} \int_Q (9\xi^2 - 3\alpha \xi) dx = \frac{-4\alpha^2 - 12\beta^2 - 12}{\alpha^2 + \beta^2 + 3} \mathbf{r} \times \mathbf{s}, \\
\int_Q \partial_r \phi_0 \eta dx &= \frac{-1}{\alpha^2 + \beta^2 + 3} \int_Q (9\eta^2 - 3\beta \eta) dx = \frac{-12\alpha^2 + 4\beta^2 - 12}{\alpha^2 + \beta^2 + 3} \mathbf{r} \times \mathbf{s}, \\
\int_Q \partial_s \phi_0 \eta dx &= \frac{-1}{\alpha^2 + \beta^2 + 3} \int_Q (9\xi \eta - 3\alpha \eta) dx = \frac{-8\alpha \beta}{\alpha^2 + \beta^2 + 3} \mathbf{r} \times \mathbf{s},
\end{align*}
\]

and
\[
\det \begin{bmatrix}
\int_Q \partial_r \phi_0 \xi dx & \int_Q \partial_s \phi_0 \xi dx \\
\int_Q \partial_r \phi_0 \eta dx & \int_Q \partial_s \phi_0 \eta dx
\end{bmatrix} = \frac{-48\alpha^4 - 48\beta^4 + 96\alpha^2 \beta^2 + 96\alpha^2 + 96\beta^2 + 144}{(\alpha^2 + \beta^2 + 3)^2} (\mathbf{r} \times \mathbf{s})^2.
\]

Therefore, since \(0 < \alpha, \beta < 1\),
\[
\det \begin{bmatrix}
\int_Q \partial_r \phi_0 \xi dx & \int_Q \partial_s \phi_0 \xi dx \\
\int_Q \partial_r \phi_0 \eta dx & \int_Q \partial_s \phi_0 \eta dx
\end{bmatrix} = \det \begin{bmatrix}
\int_Q \partial_r \phi_0 \xi dx & \int_Q \partial_s \phi_0 \xi dx \\
\int_Q \partial_r \phi_0 \eta dx & \int_Q \partial_s \phi_0 \eta dx
\end{bmatrix} \det \begin{bmatrix} \mathbf{r} & \mathbf{s} \end{bmatrix}^{-1}
\]
\[
= \frac{-48\alpha^4 - 48\beta^4 + 96\alpha^2 \beta^2 + 96\alpha^2 + 96\beta^2 + 144}{(\alpha^2 + \beta^2 + 3)^2} \mathbf{r} \times \mathbf{s} > 0.
\]

This proves the well-posedness of (3), and thus the well-definition of \(\Pi^Q_{\text{QLTZ}}\).

The remaining follows from the definition of the interpolation. This finishes the proof.

\[\square\]

3.2.2. A finite element space for \(H^1(\Omega)\). Associated with \(H^1(\Omega)\), define a finite element space \(V^\text{QLTZ}_h\) by
\[
V^\text{QLTZ}_h := \{ w \in L^2(\Omega) : w|_\Omega \in P^\text{QLTZ}_Q, \int_e w \ ds \ \text{is continuous at } e \in \mathcal{E}^e_h \},
\]
and associated with \(H^1_0(\Omega)\), define a finite element space \(V^\text{QLTZ}_{h0}\) by
\[
V^\text{QLTZ}_{h0} := \{ w_h \in V^\text{QLTZ}_h : \int_e w_h \ ds = 0 \ \text{at } e \in \mathcal{E}^e_h \}.
\]
We define the interpolation operator $\Pi_{h}^{QLTZ}: H^{1}(\Omega) \to V_{h}^{QLTZ}$ by

$$\Pi_{h}^{QLTZ}w \in V_{h}^{QLTZ}, \quad (\Pi_{h}^{QLTZ}w)|_{Q} = \Pi_{Q}^{QLTZ}(w|_{Q}), \text{ for } w \in H^{1}(\Omega).$$

The well-definedness of $\Pi_{h}^{QLTZ}$ is evident. Moreover, $\Pi_{h}^{QLTZ}w \in V_{h0}^{QLTZ}$, if $w \in H_{0}^{1}(\Omega)$.

Associated with $(H^{1}(\Omega))^{2}$ (and $(H_{0}^{1}(\Omega))^{2}$), we define the finite element space $V_{h}^{QLTZ} := (V_{h}^{QLTZ})^{2}$ (and $V_{h0}^{QLTZ} := (V_{h0}^{QLTZ})^{2}$, respectively). Define the interpolation operator $\Pi_{h}^{QLTZ} : (H^{1}(\Omega))^{2} \to V_{h}^{QLTZ}$ by

$$\Pi_{h}^{QLTZ}w \in V_{h}^{QLTZ}, \quad (\Pi_{h}^{QLTZ}w)|_{Q} = \Pi_{Q}^{QLTZ}(w|_{Q}), \text{ for } w \in (H^{1}(\Omega))^{2}.$$ 

Again, $\Pi_{h}^{QLTZ}$ is well-defined, and $\Pi_{h}^{QLTZ}w \in V_{h0}^{QLTZ}$, if $w \in (H_{0}^{1}(\Omega))^{2}$.

Evidently, $\Pi_{h}^{QLTZ}w = w_{h}$ for $w_{h} \in V_{h}^{QLTZ}$ and $\Pi_{h}^{QLTZ}w = w_{h}$ for $w_{h} \in V_{h0}^{QLTZ}$. Thus, since $P_{1}(Q) \subset P_{Q}^{QLTZ}$, by standard technique [38, 47, 54, 57], we have the lemma below.

**Lemma 4.** Let $\{Q_{h}\}$ be a regular family of convex quadrilateral triangulations of $\Omega$.

1. There exists a constant $C$, such that it holds for $w \in H^{s}(\Omega)$, $s = 1, 2$, that

$$|w - \Pi_{h}^{QLTZ}w|_{m,h} \leq C h^{s-m}|w|_{s,\Omega}, \quad 0 \leq m \leq s.$$ 

2. There exists a constant $C$, such that it holds for $w \in (H^{s}(\Omega))^{2}$, $s = 1, 2$, that

$$|w - \Pi_{h}^{QLTZ}w|_{m,h} \leq C h^{s-m}|w|_{s,\Omega}, \quad 0 \leq m \leq s.$$ 

3.2.3. **Application to second order elliptic problem.** We consider the variational problem: find $u \in H_{0}^{1}(\Omega)$, such that

$$(\nabla u, \nabla v) = (f, v), \quad \forall \ v \in H_{0}^{1}(\Omega).$$

Then $V_{h}^{QLTZ}$ is a consistent finite element space. The finite element problem is to find $u_{h} \in V_{h0}^{QLTZ}$, such that

$$(\nabla u_{h}, \nabla v_{h}) = (f, v_{h}), \quad \forall \ v_{h} \in V_{h0}^{QLTZ}.$$ 

Since the edge average of $w_{h} \in V_{h0}^{QLTZ}$ is continuous across internal edges, by the standard technique, we have the error estimate below.
Theorem 5. Let the assumptions of Lemma 4 hold. Let \( u \) and \( u_h \) be the solutions of (9) and (10), respectively.

1. If \( u \in H^2(\Omega) \cap H^1_0(\Omega) \), then \( \|u - u_h\|_{1,h} \lesssim h|u|_{2,\Omega} \).
2. If \( \Omega \) is convex and \( f \in L^2(\Omega) \), then \( \|u - u_h\|_{0,\Omega} \lesssim h^2\|f\|_{0,\Omega} \).

From this point onwards, \( \lesssim, \gtrsim, \) and \( \equiv \) respectively denote \( \leq, \geq, \) and equal up to a constant. The hidden constants depend on the domain. And, when triangulation is involved, they also depend on the shape-regularity of the triangulation, but they do not depend on \( h \) or any other mesh parameter.

3.3. A stable finite element pair for Stokes problem. For Stokes problem, \( V_h^{QLTZ} \) provides a consistent finite element space for the velocity field; we in lack of a space \( W_h^Q \) for the pressure. To satisfy SC 1 and SC 2 at the same time, we need \( \text{div}_h V_h^{QLTZ} = W_h^Q \).

Here we use the space of piecewise linear polynomials for the pressure field. Define \( W_h^Q := \{ q_h \in L^2(\Omega) : q_h|_Q \in P_1(Q), \forall Q \in T_h \} \) and \( \tilde{W}_h^Q = W_h^Q \cap L^2(\Omega) \). The finite element problem is to find \( (u_h, p_h) \in V_h^{QLTZ} \times \tilde{W}_h^Q \), such that

\[
\begin{cases}
(\nabla_h u_h, \nabla_h v_h) + (\text{div}_h v_h, p_h) = (f, v_h) \\
(\text{div}_h u_h, q_h) = 0
\end{cases} \quad \forall v_h \in V_h^{QLTZ}, \quad \forall q_h \in \tilde{W}_h^Q.
\]

Lemma 6. The inf-sup condition holds for \( V_h^{QLTZ} \times \tilde{W}_h^Q \) that

\[
\inf_{q_h \in \tilde{W}_h^Q} \sup_{v_h \in V_h^{QLTZ}} \frac{(\text{div}_h v_h, q_h)}{\|v_h\|_{1,h} \|q_h\|_{0,\Omega}} \geq C(\text{independent of } h).
\]

Proof. Given \( q_h \in \tilde{W}_h^Q \subset L^2_0(\Omega) \), there exists \( w \in (H^1_0(\Omega))^2 \), such that \( q_h = \text{div} w \), and \( \|w\|_{1,\Omega} \leq C_1 \|\text{div} w\|_{0,\Omega} \). Here \( C_1 \) is a generic constant depending on the domain only. Define \( w_h := \Pi_h^{QLTZ} w \), and then \( \int_Q \text{div} w_q dx = \int_Q \text{div} w dx \) for \( q \in P_1(Q) \). Since \( \text{div}(w_h) \in P_1(Q) \) and \( (\text{div} w)_{|Q} = q_h|_Q \in P_1(Q) \), this implies \( \text{div}(w_h) = (\text{div} w)|_Q \) and further \( \text{div}_h w_h = \text{div} w \). Therefore,

\[
\sup_{v_h \in V_h^{QLTZ}} \frac{(\text{div}_h v_h, q_h)}{\|v_h\|_{1,h} \|q_h\|_{0,\Omega}} \geq \frac{(\text{div}_h w_h, q_h)}{\|w_h\|_{1,h} \|q_h\|_{0,\Omega}} \geq C' \frac{(\text{div} w, q_h)}{\|w\|_{1,\Omega} \|q_h\|_{0,\Omega}} \geq C.
\]

The last second inequality follows from Lemma 4. This finishes the proof. \( \square \)
Remark 7. By the proof of Lemma 6, \( \text{div}_h V^\text{QLTZ}_{h_0} = \hat{W}_h^Q \). Simultaneously, \( \text{curl}_h V^\text{QLTZ}_{h_0} = \hat{W}_h^Q \).

Again, since the edge averages of \( w_h \in V^\text{QLTZ}_{h_0} \) are continuous across internal edges, by the standard technique, the theorem below follows from Lemmas 2 and 6.

Theorem 8. Let \((u, p)\) and \((u_h, p_h)\) be the solutions of (2) and (11), respectively. Then 
\[
\text{div}_h u_h = 0.
\]
Moreover,
\[
\begin{aligned}
(1) & \quad \text{If } u \in (H^2(\Omega) \cap H^1_0(\Omega))^2 \text{ and } p \in H^1(\Omega) \cap L^2_0(\Omega), \text{ then } \\
& \quad \|u - u_h\|_{1,h} + \|p - p_h\|_{0,\Omega} \leq h(|u|_{2,\Omega} + |p|_{1,\Omega}); \\
(2) & \quad \text{If } \Omega \text{ is convex, then } \|u - u_h\|_{0,\Omega} \leq h^2\|f\|_{0,\Omega}.
\end{aligned}
\]

3.4. Discrete Stokes complex.

3.4.1. A Morley element on convex quadrilateral grid. This quadrilateral Morley element is given by Park-Sheen [47].

The quadrilateral Morley element is defined by \((Q, P^M_Q, D^M_Q)\) with
\[
\begin{aligned}
(1) & \quad Q \text{ is a convex quadrilateral;} \\
(2) & \quad P^M_Q = P_2(Q) + \text{span}\{\xi^3, \eta^3\}; \\
(3) & \quad \text{the components of } D^M_Q = \{d^M_i, d^M_{i+4}\}_{i=1:4} \text{ for any } v \in H^2(Q) \text{ are:}
\end{aligned}
\]
\[
d^M_i(v) = v(a_i), \quad a_i \text{ the vertices of } T; \quad d^M_{i+4}(v) = \int_{e_i} \partial_{n_{e_i}} v \, ds, \quad e_i \text{ the edges of } T.
\]

Given a regular convex quadrilateral triangulation of \( \Omega \), define the Morley element space \( M^Q_h \) as
\[
M^Q_h := \{w_h \in L^2(\Omega) : w_h|_Q \in P^M_Q, \ w_h(a) \text{ is continuous at } a \in N_h, \ \int_e \partial_{n_e} w_h \, ds \ \text{ is continuous across } e \in \mathcal{E}_h^i\}.
\]

And, associated with \( H^2_0(\Omega) \), define
\[
M^Q_{h_0} := \{w_h \in M_h : w_h(a) \text{ vanishes at } a \in N^b_h, \ \int_e \partial_{n_e} w_h \, ds \ \text{ vanishes at } e \in \mathcal{E}_h^b\}.
\]
The Morley element provides consistent approximation of fourth-order problems on quadrilateral grids. Let us consider the model problem: find \( u \in H^3_0(\Omega) \), such that
\[
(\nabla^2 u, \nabla^2 v) = (f, v) \quad \forall \ v \in H^2_0(\Omega).
\]
The finite element problem is to find \( u_h \in M^Q_{h0} \), such that
\[
(\nabla_h^2 u_h, \nabla_h^2 v_h) = (f, v_h) \quad \forall \ v_h \in M^Q_{h0}.
\]

**Lemma 9.** Let \( u \) and \( u_h \) be the solution of (13) and (14), respectively.

1. Assume \( u \in H^3(\Omega) \cap H^2_0(\Omega) \), then \( |u - u_h|_{2,h} \lesssim h(|u|_{3,\Omega} + \|f\|_{0,\Omega}) \).
2. If further \( \Omega \) is a convex polygon, then \( |u - u_h|_{1,h} \lesssim h^2(|u|_{3,\Omega} + \|f\|_{0,\Omega}) \).

**3.4.2. A discrete Stokes complex.** The lemma below plays a fundamental role in the construction of the discrete Stokes complex.

**Lemma 10.** \( \text{curl}_h M^Q_{h0} = \tilde{V}^{\text{QLTZ}}_{h0} := \{ w_h \in V^{\text{QLTZ}}_{h0} : \text{div}_h w_h = 0 \} \).

**Proof.** It is obvious that \( \text{curl}_h M^Q_{h0} \subset \tilde{V}^{\text{QLTZ}}_{h0} \). To prove the other direction, we only have to show that the dimension of the two spaces are the same. By Remark [7]
\[
\dim(\tilde{V}^{\text{QLTZ}}_{h0}) = \dim(V^{\text{QLTZ}}_{h0}) - \dim(\text{div}_h V^{\text{QLTZ}}_{h0}) = \dim(V^{\text{QLTZ}}_{h0}) - \dim(\hat{W}^Q_h)
\]
\[
= 2(\tilde{\gamma} + \tilde{\epsilon}_I) - (3\tilde{\gamma} - 1) = \tilde{\epsilon}_I + \chi_I = \dim(M^Q_{h0}) = \dim(\text{curl}_h M^Q_{h0}).
\]
This finishes the proof. \( \square \)

Define \( \Pi^{QM}_h : H^2(\Omega) \to M^Q_h \) by \( \Pi^{QM}_h \varphi \in M^Q_h \) such that
\[
\Pi^{QM}_h \varphi(a) = \varphi(a), \ \forall \ a \in N_h, \ \text{and} \ \int_e \partial_n \Pi^{QM}_h \varphi \ ds = \int_e \partial_n \varphi \ ds, \ \forall \ e \in E_h, \ \varphi \in H^2(\Omega).
\]
Define \( \Pi^0_h \) the \( L^2 \)-projection to \( W^Q_h \). Summing all discussions above, we obtain a main result of the paper as below.

**Theorem 11.** The discrete Stokes complex holds as below:
\[
0 \longrightarrow M^Q_{h0} \xrightarrow{\text{curl}_h} V^{\text{QLTZ}}_{h0} \xrightarrow{\text{div}_h} \hat{W}^Q_h \longrightarrow 0.
\]
Moreover,
\[
\text{curl}_h \Pi^{QM}_h = \Pi^{\text{QLTZ}}_h \text{curl} \text{ on } H^2_0(\Omega), \text{ and } \text{div}_h \Pi^{\text{QLTZ}}_h = \Pi^0_h \text{div} \text{ on } (H^1_0(\Omega))^2.
\]
Proof. The discrete Stokes complex follows from Lemma 10 and Remark 7. We only have to prove the commutativity (16).

Given \( \varphi \in H^2_0(\Omega) \), by the definition of \( \Pi^Q_h \), we have for \( e \in E_h \) that

\[
\int_e \nabla \times \Pi^Q_h \varphi \cdot n_e \, ds = \int_e \nabla \times \Pi^Q_h \varphi \cdot \tau_e \, ds \tau_e
\]

\[
= \int_e \partial_{\tau_e} \Pi^Q_h \varphi \, ds \, n_e + \int_e \partial_{n_e} \Pi^Q_h \varphi \, ds \, \tau_e = (\Pi^Q_h \varphi(e_L) - \Pi^Q_h \varphi(e_R)) n_e + \int_e \partial_{n_e} \Pi^Q_h \varphi \, ds \tau_e
\]

\[
= (\varphi(e_L) - \varphi(e_R)) n_e + \int_e \partial_{n_e} \varphi \, ds \tau_e = \int_e \nabla \varphi \, ds .
\]

Note that \( \text{div}_h \nabla \times \Pi^Q_h \varphi = 0 = \text{div}_h \nabla_{\text{QLTZ}} \varphi \). Therefore, \( \nabla \times \Pi^Q_h \varphi = \nabla_{\text{QLTZ}} \varphi \).

Similarly we can prove for \( v \in (H^1_0(\Omega))^2 \) that \( \text{div}_h \nabla_{\text{QLTZ}} v = \text{div}_h v \). This finishes the proof.

Theorem 11 can also be written as this exact sequence and commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^2_0(\Omega) & \xrightarrow{\nabla} & (H^1_0(\Omega))^2 & \xrightarrow{\text{div}} & L^2_0(\Omega) & \longrightarrow & 0 \\
\downarrow \Pi^Q_h & & \downarrow \Pi^Q_{\text{QLTZ}} & & \downarrow \Pi^0_h & & 0 \\
0 & \longrightarrow & M^0_{h0} & \xrightarrow{\nabla} & V^Q_{h0} & \xrightarrow{\text{div}_h} & \hat{V}^Q_h & \longrightarrow & 0.
\end{array}
\]

4. Finite elements and discrete Stokes complex on a mixed grid

In this section, we generalise the results in Section 3 from quadrilateral grids to the mixed grid that consists of both triangular and quadrilateral cells. The technical issues are the same as that in Section 3 and we list the main results and omit the details.

4.1. Mixed triangulation with triangular and quadrilateral cells. Let \( T_h \) be a shape-regular triangulation of domain \( \Omega \), with the cells being triangles or convex quadrilaterals. Again, let \( N_h \) denote the set of all the vertices, \( N_h = N^i_h \cup N^b_h \), with \( N^i_h \) and \( N^b_h \) consisting of the interior vertices and the boundary vertices, respectively. Similarly, let \( E_h = E^i_h \cup E^b_h \) denote the set of all the edges, with \( E^i_h \) and \( E^b_h \) consisting of the interior edges and the boundary edges, respectively. For an edge \( e \), \( n_e \) is a unit vector normal to \( e \), and \( \tau_e \) is a unit tangential vector of \( e \) such that \( n_e \times \tau_e > 0 \). On the edge \( e \), we use \( \| \cdot \|_e \) for the jump across \( e \).
Again, denote by $F$ the number of cells of the triangulation, denote by $X$, $X_I$, $X_B$ and $X_C$ the number of vertices, internal vertices, boundary vertices, and corner vertices, respectively, and denote by $E$, $E_I$ and $E_B$ the number of edges, internal edges, and boundary edges, respectively. Euler’s formula states that $F + X = E + 1$. In the remaining of this section, we use $Q$ to denote a quadrilateral cell, and $T$ for a triangular cell. This will not bring ambiguity according to the context. Denote by $\#Q$ and $\#T$ the number of quadrilateral and triangular cells, respectively.

4.2. **Finite element spaces on a mixed grid.** Associated with the triangulation, we define several finite element spaces for the stream function, the velocity and the pressure, respectively. Associated with the stream function, define

$$M_{h}^{\text{mix}} := \{ w_h \in L^2(\Omega) : w_h|_Q \in P^M_Q, w_h|_T \in P_2(T), $$

$$w_h(a) \text{ is continuous at } a \in N_h, \quad \int_e \partial_n w_h \, ds \text{ is continuous on } e \in E^{h}_i \}.$$  

And, associated with $H^2_0(\Omega)$, define

$$M_{h0}^{\text{mix}} := \{ w_h \in M_h : w_h(a) = 0 \text{ at } a \in N^b_h, \quad \int_e \partial_n w_h \, ds = 0 \text{ at } e \in E^b_h \}.$$ 

Define the interpolation $\Pi^{M, \text{mix}}_h : H^2(\Omega) \rightarrow M_{h}^{\text{mix}}$ by $\Pi^{M, \text{mix}}_h \varphi \in M_{h}^{\text{mix}}$, by

$$\Pi^{M, \text{mix}}_h \varphi(a) = \varphi(a), \quad \forall \ a \in N_h, \quad \text{and} \quad \int_e \partial_n \Pi^{M, \text{mix}}_h \varphi \, ds = \int_e \partial_n \varphi \, ds, \quad \forall \ e \in E_h,$$

for $\varphi \in H^2(\Omega)$. Then $\Pi^{M, \text{mix}}_h$ is well-defined, and $\Pi^{M, \text{mix}}_h H^2_0(\Omega) = M_{h0}^{\text{mix}}$.

Define associated with $H^1(\Omega)$

$$V_{h}^{\text{mix}} := \{ w \in L^2(\Omega) : w|_Q \in P_Q, w|_T \in P_1(T), \quad \int_e w \, ds \text{ is continuous on } e \in E^{h}_i \},$$

and associated with $H^1_0(\Omega)$,

$$V_{h0}^{\text{mix}} := \{ w_h \in V_h : \int_e w_h \, ds = 0, \text{ for } e \in E^b_h \}.$$
Define $V_{h}^{\text{mix}} = (V_{h}^{\text{mix}})^{2}$ and $V_{h0}^{\text{mix}} = (V_{h0}^{\text{mix}})^{2}$. Define $\Pi_{h}^{V_{h}^{\text{mix}}} : (H^{1}(\Omega))^{2} \to V_{h}^{\text{mix}}$ by $\Pi_{h}^{V_{h}^{\text{mix}}} w \in V_{h}^{\text{mix}}$ such that

$$\int_{e} \Pi_{h}^{V_{h}^{\text{mix}}} w \, ds = \int_{e} w \, ds, \quad \forall \, e \in \mathcal{E}_{h}, \quad \text{and} \quad (\Pi_{h}^{V_{h}^{\text{mix}}} w)|_{\mathcal{Q}} = \Pi_{\mathcal{Q}}^{\text{QLTZ}} w$$

for $w \in (H^{1}(\Omega))^{2}$. Then $\Pi_{h}^{V_{h}^{\text{mix}}}$ is well-defined, and $\Pi_{h}^{V_{h}^{\text{mix}}}(H^{1}_{0}(\Omega))^{2} = V_{h0}^{\text{mix}}$.

Define associated with $L^{2}(\Omega)$

$$W_{h}^{\text{mix}} := \{ q_{h} \in L^{2}(\Omega) : q_{h}|_{\mathcal{Q}} \in P_{1}(\mathcal{Q}), \quad q_{h}|_{T} \in P_{0}(T) \},$$

and associated with $L^{2}_{0}(\Omega)$, $\tilde{W}_{h}^{\text{mix}} := W_{h}^{\text{mix}} \cap L^{2}_{0}(\Omega)$. Define $\Pi_{h}^{0,\text{mix}}$ the $L^{2}$-projection to $W_{h}^{\text{mix}}$. Then $\Pi_{h}^{0,\text{mix}} L^{2}_{0}(\Omega) = \tilde{W}_{h}^{\text{mix}}$.

Note that the finite element functions in the spaces $(M_{h}^{\text{mix}}, V_{h}^{\text{mix}} \times W_{h}^{\text{mix}})$ coincide with the original Morley element functions and the nonconforming $P_{1}^{2} - P_{0}$ element functions if restricted on triangular cells, and with quadrilateral Morley element functions and the newly-developed Stokes element functions if restricted on quadrilateral cells. The operators $\Pi_{h}^{M,\text{mix}}$, $\Pi_{h}^{V,\text{mix}}$ and $\Pi_{h}^{0,\text{mix}}$ are all defined cell by cell. Roughly speaking, they are “sum direct” of the interpolations with respect to the finite elements defined on individual cells.

4.3. Finite element schemes for boundary value problems. Since the nodal interpolation operators are locally defined, and sufficient weak continuity conditions have been imposed across the interface, these finite elements defined previously provide convergent finite element schemes for the specific boundary value problems.

Fourth order problem. We consider the finite element problem: find $u_{h} \in M_{h0}^{\text{mix}}$, such that

$$\nabla^{2}_{h}u_{h}, \nabla^{2}_{h}v_{h}) = (f, v_{h}) \quad \forall \, v_{h} \in M_{h0}^{\text{mix}}.$$

Since the average of the gradient of the $M_{h0}^{\text{mix}}$ functions at the internal edges are continuous, by standard technique [47, 53, 57], we have the convergence result below.

Lemma 12. Let $u$ and $u_{h}$ be the solution of (13) and (18), respectively.

1. Assume $u \in H^{3}(\Omega) \cap H^{2}_{0}(\Omega)$, then $|u - u_{h}|_{2,h} \leq h(|u|_{3,\Omega} + \|f\|_{0,\Omega})$.

2. If further $\Omega$ is a convex polygon, then $|u - u_{h}|_{1,h} \leq h^{2}(|u|_{3,\Omega} + \|f\|_{0,\Omega})$. 

Stokes problem. Now we consider the finite element problem: find \((u_h, p_h) \in V_{h0}^{\text{mix}} \times W_{h0}^{\text{mix}}\) such that

\[
\begin{aligned}
(\nabla_h u_h, \nabla v_h) + (\text{div}_h v_h, p_h) &= (f, v_h) \quad \forall \ v_h \in V_{h0}^{\text{mix}}, \\
(\text{div}_h u_h, q_h) &= 0 \quad \forall \ q_h \in W_{h0}^{\text{mix}}.
\end{aligned}
\tag{19}
\]

By the same technique as the proof of Lemma 6, we obtain the lemma below.

**Lemma 13.** The inf-sup condition holds for \(V_{h0}^{\text{mix}} \times W_{h0}^{\text{mix}}\) that

\[
\inf_{q_h \in W_{h0}^{\text{mix}}} \sup_{v_h \in V_{h0}^{\text{mix}}} \frac{\langle \text{div}_h v_h, q_h \rangle}{\|v_h\|_{1,h} \|q_h\|_{0,\Omega}} \geq C \text{(independent of } h)\.
\]

Moreover, \(\text{div}_h V_{h0}^{\text{mix}} = W_{h0}^{\text{mix}}\), and \(\text{curl}_h V_{h0}^{\text{mix}} = W_{h0}^{\text{mix}}\).

Again, since the edge average of \(w_h \in V_{h0}^{\text{mix}}\) is continuous across internal edges, by the standard technique, we obtain the estimate below.

**Lemma 14.** Let \((u, p)\) and \((u_h, p_h)\) be the solutions of (2) and (19), respectively. Then \(\text{div}_h u_h = 0\). Moreover,

1. If \(u \in (H^2(\Omega) \cap H^1_0(\Omega))^2\) and \(p \in H^1(\Omega) \cap L^2_0(\Omega)\), then
   \[
   \|u - u_h\|_{1,h} + \|p - p_h\|_{0,\Omega} \leq h(|u|_{2,\Omega} + |p|_{1,\Omega});
   \]
2. If \(\Omega\) is convex, then \(\|u - u_h\|_{0,\Omega} \leq h^2\|f\|_{0,\Omega}\).

4.4. **Discrete Stokes complex on a mixed grid.** Direct calculation leads to that

\[
\dim(V_{h0}^{\text{mix}}) - \dim(W_{h0}^{\text{mix}}) = 2(\#E + \#Q) - (3\#Q + \#T - 1) = \#E + \#\mathcal{T} = \dim(M_{h0}^{\text{mix}}).
\]

Therefore, we can apply the technique of the proof of Lemma 10 and of the proof of Theorem 11 to obtain the theorem below.

**Theorem 15.** The discrete Stokes complex holds as below:

\[
0 \rightarrow M_{h0}^{\text{mix}} \xrightarrow{\text{curl}_h} V_{h0}^{\text{mix}} \xrightarrow{\text{div}_h} W_{h0}^{\text{mix}} \rightarrow 0.
\]

Moreover,

\[
\text{curl}_h \Pi_h^{M,\text{mix}} = \Pi_h^{V,\text{mix}} \text{curl on } H^2_0(\Omega), \quad \text{and} \quad \text{div}_h \Pi_h^{V,\text{mix}} = \Pi_h^{0,\text{mix}} \text{div on } (H^1_0(\Omega))^2.
\]
Again, the theorem can be written as the exact sequence and the commutative diagram below:

\[
0 \rightarrow H^2_0(\Omega) \xrightarrow{\text{curl}} (H^1_0(\Omega))^2 \xrightarrow{\text{div}} L^2_0(\Omega) \rightarrow 0
\]

\[
0 \rightarrow \Pi_{h}^{M,\text{mix}} \xrightarrow{\text{curl}_h} \Pi_{h}^{V,\text{mix}} \xrightarrow{\text{div}_h} \Pi_{h}^{0,\text{mix}} \rightarrow 0.
\]

5. Concluding remarks

In this paper, we construct stable finite element pairs that satisfy the stability conditions both SC 1 and SC 2 on grids that admit triangular and general convex quadrilateral cells, namely, the pair satisfies the inf-sup stability condition, and the restriction of the solution to an element is exactly divergence-free and the scheme can be seen as a mass conservative one. Different from most existing finite element pairs on quadrilateral grids in the literature, the construction of the newly-developed quadrilateral finite element spaces does not rely on a rectangle reference cell, and the finite element spaces thus consist of piecewise polynomials only. Discrete Stokes complexes are constructed associatedly.

As the constraint of divergence-free is imposed piecewisely, this finite element would have potential applications for the parametric related problems [42, 61]. The exact sequence property provides a precise description of the kernel space involved in the Stokes problem, and will also help to design preconditioners and solvers for the resulting linear systems [28, 34, 43, 61]. These will be discussed in future works.

As the finite elements constructed in this present paper fall into the category of the nonconforming type, an issue is that the uniform Korn’s inequality would fail [37]. This issue will be discussed in future works. We also remark here that the piecewise mass conservation property of the finite element pair makes it potentially one fit for the elasticity problem, and we refer to [1] for a relevant discussion.

References

[1] Arnold, D.: On nonconforming linear-constant elements for some variants of the Stokes equations, Istit. Lombardo Accad. Sci. Lett. Rend. A 127, 83–93 (1994)

[2] Arnold, D.N., Falk, R.S., Winther, R.: Differential complexes and stability of finite element methods. I. The de Rham complex. In: Arnold, D.N. et al. (eds.): Compatible spatial discretizations. (The IMA Volumes in Mathematics and its Applications 142) Berlin: Springer (2006), pp. 23–46
[3] Arnold, D.N., Falk, R.S., Winther, R.: Finite element exterior calculus, homological techniques, and applications. Acta Numerica 15, 1–155 (2006)
[4] Arnold, D. N., Qin, J.: Quadratic velocity/linear pressure Stokes elements, in Proceedings of Advances in Computer Methods for Partial Differential Equations VII, R. Vichnevetsky and R. S. Steplemen, eds., AICA, 1992.
[5] Arnold, D. N., Scott, L.R., Vogelius, M.: Regular inversion of the divergence operator with Dirichlet conditions on a polygon, Ann. Sc. Norm. Super Pisa Cl. Sci. (5) 15, 169–192(1988)
[6] Auricchio, F., Beirão da Veiga, L., Lovadina, C., Reali, A.: The importance of the exact satisfaction of the incompressibility constraint in nonlinear elasticity: mixed FEMs versus NURBS-based approximations, Comput. Methods Appl. Mech. Engrg. 199, 314–323(2010)
[7] Babuška, I.: The finite element method with Lagrangian multipliers, Numer. Math. 20, 179–192(1973)
[8] Bejanov, B., Guermond, J.-L., Minev, P. D.: A locally DIV-free projection scheme for incompressible flows based on non-conforming finite elements, Int. J. Numer. Meth. Fluids 49 549–568(2005)
[9] Bochev, P., Lai, J., Olson, L.: A non-conforming least-squares finite element method for incompressible fluid flow problems, Int. J. Numer. Meth. Fluids 72, 375–402(2013)
[10] Boffi, D., Brezzi, F., Fortin, M.: Finite elements for the Stokes problem, in Mixed Finite Elements, Compatibility Conditions, and Applications, C.I.M.E. Summer School, Springer-Verlag, Berlin, 2008.
[11] Boffi, D., Cavallini, N., Gardini, F., Gastaldi, L.: Local mass conservation of Stokes finite elements, J. Sci. Comput. 52, 383–400(2012)
[12] Bolton, P., Thatcher, R.W.: On mass conservation in least-squares methods, Journal of Computational Physics 203, 287–304(2005)
[13] Brezzi, F.: On the existence, uniqueness and approximation of saddle-point problems arising from Lagrange multipliers, R.A.I.R.O. Anal. Numer. R2, 129–151(1974)
[14] Brezzi, F., Fortin, M.: Mixed and Hybrid Finite Element Methods, Springer Ser. Comput. Math., 15, Springer-Verlag, New York, 1991.
[15] Burman, E., Linke, A.: Stabilized finite element schemes for incompressible flow using Scott-Vogelius elements, Applied Numerical Mathematics 58, 1704–1719(2008)
[16] Carrero, J., Cockburn, B., Schötzau, D.: Hybridized globally divergence-free LDG methods. I. The Stokes problem, Math. Comp. 75, 533–563(2006)
[17] Case, M.A., Ervin, V.J., Linke, A., Rebholz, L.G.: A connection between Scott-Vogelius and grad-div stabilized Taylor-Hood FE approximations of the Navier-Stokes equations, SIAM J. Numer. Anal. 49, 1461–1481(2011)
[18] Chang, C., Nelson, J.: Least-Squares finite element method for the Stokes problem with zero residual of mass conservation, SIAM J. Numer. Anal. 34, 480–489(1997)
[19] Cockburn, B., Kanschat, G., Schötzau, D.: A note on discontinuous Galerkin divergence free solutions of the Navier-Stokes equations, J. Sci. Comput. 31, 61–73(2007)
[20] Cockburn, B., Kanschat, G., Schötzau, D.: A locally conservative LDG method for the incompressible Navier-Stokes equations, Math. Comp. 74, 1067–1095(2004)

[21] Cockburn, B., Kanschat, G., Schötzau, D.: The local discontinuous Galerkin method for linearized incompressible fluid flow: a review, Computers & Fluids 34, 491–506(2005)

[22] Cousins, B.R., Brone, S. L., Linke, A., Rebholz, L.G., Wang, Z.: Efficient Linear Solvers for Incompressible Flow Simulations using Scott-Vogelius Finite Elements, Numer Methods Partial Differential Eq 29, 1217–1237(2013)

[23] Courzeix, M., Raviart, P.-A.: Conforming and non conforming finite element methods for solving the stationary Stokes equations R.A.I.R.O. R3, 33–76(1973)

[24] Evans, J.A., Hughes, T.J.R.: Isogeometric divergence-conforming B-splines for the steady Navier-Stokes equations, Math. Models Methods Appl. Sci. 23, 1421–1478(2013)

[25] Evans, J.A., Hughes, T.J.R.: Isogeometric divergence-conforming B-splines for the unsteady Navier-Stokes equations, J. Comput. Phys. 241, 141–167(2013)

[26] Falk, R., Morley, E.: Equivalence of finite element methods for problems in elasticity, SIAM J. Numer. Anal. 27, 1486–1505(1990)

[27] Falk, R., Neilan, M.: Stokes complexes and the construction of stable finite elements with pointwise mass conservation, SIAM J. NUMER. ANAL. 51, 1308–1326(2013)

[28] Feng, C., Xu, J., Zhang, S.: Optimal solver for Morley element problem for biharmonic equation on shape-regular grids, preprint.

[29] Fortin, M., Soulie, M.: A non-conforming piecewise quadratic finite element on triangles, International journal for numerical methods in engineering, 19, 505–520(1983)

[30] Girault, V, Raviart, P.-A.: Finite Element Methods for the Navier-Stokes Equations, Springer-Verlag, Berlin, 1986.

[31] Guzmán, J., Neilan, M.: Conforming and divergence-free Stokes elements on general triangular meshes, Math. Comp., in press.

[32] Han, H.: Nonconforming Elements In The Mixed Finite Element Method, J. Comp. Math. 2, 223–233(1984)

[33] Heys, J.J., Lee, E., Manteuffel, T.A., McCormick, S.F.: On mash-conserving least-squares methods, SIAM J. SCI. COMPUT. 28, 1675–1693(2006)

[34] Hiptmair, R., Xu, J.: Nodal auxiliary space preconditioning in $H(\text{curl})$ and $H(\text{div})$ spaces, SIAM J. Numer. Anal. 45, 2483–2509(2007)

[35] Huang, Y., Zhang, S., A lowest order divergence-free finite element on rectangular grids, Frontiers of Mathematics in China 6, 253–270(2011)

[36] Kouhia, R., Stenberg, R.: A linear nonconforming finite element method for nearly incompressible elasticity and Stokes flow, Comput. Methods Appl. Mech. Engrg. 124, 195–212(1995)

[37] Knobloch, P., Tobiska, L.: On Korns first inequality for quadrilateral nonconforming finite elements of first order approximation properties, Int. J. Numer. Anal. Modeling 2, 439–458(2005)
[38] Lin, Q., Tobiska, L., Zhou, A.: Superconvergence and extrapolation of non-conforming low order finite elements applied to the Poisson equation, IMA Journal of Numerical Analysis, 25 160–181(2005)

[39] Linke, A.: Divergence-Free Mixed Finite Elements for the Incompressible Navier-Stokes Equation, Ph.D. thesis, University of Erlangen, 2008.

[40] Linke, A.: Collision in a cross-shaped domain: A steady 2d Navier-Stokes example demonstrating the importance of mass conservation in CFD, Comp. Meth. Appl. Mech. Eng. 198, 3278–3286(2009)

[41] Linke, A., Matthies, G., Tobiska, L., Non-nested multi-grid solvers for mixed divergence free scott-vogelius discretizations, Computing, 83, 87–107(2008)

[42] Mardal, K.A., Tai, X.-C., Winther, R.: A robust finite element method for Darcy-Stokes flow, SIAM J. Numer. Anal. 40, 1605–1631(2002)

[43] Mardal, K.A., Schöberl, J., Winther, R.: A Uniform Inf-Sup Condition with Applications to Pre-conditioning, preprint, arXiv:1201.1513 [math.NA], 2012.

[44] Morley, L.S.D.: The triangular equilibrium element in the solution of plate bending problems, Aeronautical Quarterly 19, 149–169(1968)

[45] Olshanskii, M.A., Reusken, A.: Grad-div stabilization for Stokes equations, Math.Comp. 73, 1699–1718(2004)

[46] Park, C., Sheen, D.: P1-nonconforming quadrilateral finite element methods for second-order elliptic problems, SIAM J. Numer. Anal. 41, 624–640(2003)

[47] Park, C., Sheen, D.: A quadrilateral Morley element for biharmonic equations, Numer. Math. 124, 395–413(2013)

[48] Qin, J.: On the Convergence of Some Low Order Mixed Finite Elements for Incompressible Fluids, Ph.D. Thesis, Penn State University, Department of Mathematics, (1994)

[49] Rannacher, R.: Finite Element Methods for the Incompressible Navier-Stokes Equations, in Fundamental Directions in Mathematical Fluid Mechanics, editors Giovanni P. Galdi, John G. Heywood and Rolf Rannacher, Springer, 191–293 (2000).

[50] Scott, L.R., Vogelius, M.: Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials, RAIRO, Modélisation Math. Anal. Numer. 19, 111–143(1985)

[51] Scott, L.R., Vogelius, L.R.: Conforming finite element methods for incompressible and nearly incompressible continua, in Lect. Appl. Math., 22 (1985), pp. 221–244.

[52] Shi, D.-Y., Zhang, Y.-R.: A nonconforming anisotropic finite element approximation with moving grids for Stokes problem, Journal of Computational Mathematics, 24, 561–578(2006)

[53] Shi, Z.C., On the error estimates of Morley element, Numer. Math. Sin. 12, 113–118(1990) (in Chinese)

[54] Shi, Z.C., Wang, M.: Finite element methods, Science Press, Beijing, 2013.

[55] Tai, X.-C., Winther, R.: A discrete de Rham complex with enhanced smoothness, Calcolo 43, 287–306(2006)
[56] Vogelius, M.: A right-inverse for the divergence operator in spaces of piecewise polynomials. Application to the p-version of the finite element method, Numer. Math. 41, 19–37(1983)

[57] Wang, M., Shi, Z.C., Xu, J.: Some n-rectangle nonconforming elements for fourth order elliptic equations, J. Comput. Math. 25, 408–420(2007)

[58] Wang, M., Xu, J.: The Morley element for fourth order elliptic equations in any dimensions, Numer. Math. 103, 155–169(2006)

[59] Wang, M., Xu, J.: Minimal finite element spaces for 2m-th-order partial differential equations in $\mathbb{R}^n$, Math. Comp. 82, 25–43(2013)

[60] Tai, X.-C., Winther, R.: A discrete de Rham complex with enhanced smoothness, Calcolo 43, 287–306(2006)

[61] Xie, X., Xu, J., Xue, G.: Uniformly-stable finite element methods for Darcy-Stokes-Brinkman models, Journal of Computational Mathematics 26, 437–455(2008)

[62] Zhang, S.: A family of $Q_{k+1,k} \times Q_{k,k+1}$ divergence-free finite elements on rectangular grids, SIAM J. Numer. Anal. 47, 2090–2107(2009)

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