ON THE VORTEX-POINT CHARGE COMPOSITE: CLASSICAL ORBITS AND QUANTUM BOUND STATES

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Abstract
The possibility of composite systems arising out of a point charge interacting with a Nielsen-Olesen vortex in 2+1-dimensions is investigated. It is shown that classical bounded orbits are possible for certain ranges of parameters. Long lived metastable states are shown to exist, in a semi-classical approach, from the study of the effective potential. Loss of self-adjointness of the Hamiltonian and its subsequent self-adjoint extension in some cases leads to bound states.
1 Introduction

2+1-Dimensional physics has come of age in the last couple of decades. Specifically with the advent of anyons [1], (excitations having arbitrary spin and statistics), and its posited presence ranging from diverse condensed matter systems such as high $T_c$ superconductivity, fractional quantum Hall effect to more exotic scenarios in the high energy regime such as processes in the presence of cosmic strings etc., 2+1-dimensions has ceased to be just a laboratory for testing ideas conceived for the "physical" 3+1-dimensions. The present paper deals with studying the energy spectrum of a quantum (charged) particle in the presence of a Nielsen-Olesen magnetic vortex [2], in 2+1-dimensions.

The motivation is twofold. The physical existence of magnetic vortex lines in type II superconductors [3] (in an external magnetic field) makes the study of a vortex-particle system interesting. Also according to the present lore and as envisaged by Wilczek [1], a possible realisation of anyons in nature is some sort of a composite object, consisting of a magnetic flux tube (of the "fictitious" Chern Simons gauge field) attached to a bose or a fermi particle. Albeit the crucial difference in nature of the two magnetic fields, the existence of (quantum) bound states, (as we demonstrate here), should lend credibility to the hypothetical anyon structure.

Let us elaborate a little on our system, our modes of analysis as well as the results that we have obtained. The system is that of a non-relativistic point charge in the presence of a vortex, the latter being the Nielsen-Olesen vortex solution in the Abelian Higgs Model (AHM) [2, 4]. We restrict ourselves to minimal gauge invariant coupling between the point charge and the gauge field.

We start with a thorough discussion on the classical aspect of the problem, which turns out to be quite tricky. We show that classical bounded trajectories [5] of the particle are allowed for certain ranges of the parameters of the model. This analysis is important since it hints at the possible existence of quantum bound states, which brings us to the Schrodinger equation problem of the charge in presence of a classical vortex potential. One would be too optimistic to conclude that upon quantization, the closed orbits would correspond to bound states. Apparantly this is not the case. Essentially this is because the Hamiltonian in question being positive definite, negative energy bound states can not appear. Still we will show that the effective potential energy profile is such that it allows metastable states of quite long life time. Previous works [6] in a similar vein are those of a point particle in the presence of a ’t Hooft-Polyakov monopole in 3+1-dimensions.

However, bound states in the strict sense do exist, for certain values of effective angular momentum at least. This is related to the loss of self adjointness of the Hamiltonian, due to the singular vortex potential, and its subsequent self adjoint extension, [7, 8], which allows only the above bound states. An early work in this connection is [9].

The paper is organised as follows: in section 2, we introduce the vortex potential and the detailed classical dynamics. In section 3, we discuss the metastable quantum bound states. Section 4 deals with the problem of self adjointness and the ensuing true bound states. We conclude the paper in section 5.

2 Classical dynamics

The Nielsen-Olesen vortex solutions [2] have provided a deep connection between a relativistic field theory, (the AHM), and the system of type II superconductors. If the
superconducting material is invariant along the z-direction, the system is essentially a two-dimensional one, in the \( x- y \)-plane \[4\]. In the former case vortices appear in the two space dimensional slice of the magnetic flux lines extended along the third direction. Hence one can study the vortex solutions in the 2+1-dimensional AHM.

Let us start by introducing the vortex solutions of the AHM \[2, 4\]. The \((c = 1)\) Lagrangian is

\[
L_{AHM} = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} + \frac{1}{2} (D_\mu \phi)^* (D^\mu \phi) - \frac{\lambda}{4} (|\phi|^2 - F^2)^2, \tag{1}
\]

where \( F^{\mu \nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \), \( D_\mu \phi = \partial_\mu \phi - ie A_\mu \phi \). The problem has a cylindrical symmetry and the asymptotic solution, (i.e., the vortex), in polar \((r, \theta)\) coordinates is

\[
A_r = A_\theta = 0, \quad A_\theta (r) \approx \frac{n}{e r} + \frac{\alpha e^{-eFr}}{\sqrt{r}}, \quad r \to \infty, \tag{2}
\]

\[
\phi = e^{in\theta} (F + \beta e^{-\sqrt{\lambda}Fr}). \tag{3}
\]

We will be confined to the \( n = 1 \) or single vortex line or 'flux-tube' sector, since numerical work has indicated it to be stable. \( \alpha \) and \( \beta \) are constants and \( \alpha \) is of the order of \( \sqrt{\frac{e}{eF}} \).

In this work we only consider the particle to be interacting with the gauge potential \( A^\mu \). Obviously, other interactions such as Yukawa or vector couplings with \( \phi \) can be studied as well. Henceforth we consider the gauge field to be external. Clearly this particular solution of \( A^\mu \) in (2) generates no electric field. The magnetic field is

\[
B = \frac{1}{e} 2\pi \delta^{(3)}(\vec{r}) + \alpha \left( \frac{1}{2r} - eF \right) e^{-eF r} \frac{1}{\sqrt{r}} \tag{4}
\]

The matter Lagrangian with \( U(1) \) gauge invariant coupling is

\[
L_{\text{matter}} = \frac{1}{2} M \dot{r}^2 + \frac{1}{2} M r^2 \dot{\theta}^2 + j^\mu A_\mu, \tag{5}
\]

where \( j^\mu \) is the conserved particle current. We denote time and space derivatives of \( O \) by \( \dot{O} \) and \( O' \) respectively. Writing explicitly the potential, we obtain

\[
L_{\text{matter}} = \frac{1}{2} M \dot{r}^2 + \frac{1}{2} M r^2 \dot{\theta}^2 + gr \dot{\theta} \left( \frac{1}{e r} + \frac{\alpha e^{-eF r}}{\sqrt{r}} \right), \tag{6}
\]

Note that the term \( gr \dot{\theta} \left( \frac{1}{e r} \right) = \frac{2}{e} \dot{\theta} \) is a total time derivative and does not influence the classical equations of motion. It is dropped in the present classical analysis. The Lorentz equation for the particle is

\[
M \ddot{X}^i = g e^{ij} \dot{X}^j B, \tag{7}
\]

where \( M \) and \( g \) are mass and charge of the particle and \( B \) is without the \( \delta \)-function term. In polar coordinates we have

\[
\ddot{X} = \hat{r} \ddot{r} + r \dddot{\theta} \hat{\theta}, \tag{8}
\]

\[
\ddot{X} = (\dddot{r} - r \dddot{\theta}) \hat{r} + (r \dddot{\theta} + 2 \dddot{\theta}) \hat{\theta}, \tag{9}
\]

and the Lorentz equation in component form is

\[
M (\dddot{r} - r \dddot{\theta}^2) = gr \dot{\theta} B, \quad M (r \dddot{\theta} + 2 \dddot{\theta}) = -g \dddot{r} B. \tag{10}
\]
Note that \( \frac{2}{e} \) has the dimension of \( h \). In the quantised version the particle charge \( g \) becomes related to \( e \) by \( g = k(eh) \) where \( k \) is some number. The \( \theta \) equation following from (6) is given by

\[
Mr^2\ddot{\theta} + \alpha g \sqrt{r} e^{-eFr} = L,
\]

where \( L \) is a constant. Thus a generalised angular momentum is conserved. Borrowing from the classical terminology, this is one of the integrals of motion, the other being the conserved energy, since a static background interaction is chosen. The \( r \)-equation provides the energy, when we substitute \( \dot{\theta} \) from (11)

\[
Mr^2\ddot{r} = -\frac{d}{dr}\left[\frac{1}{2M}(L - \alpha g \sqrt{r} e^{-eFr})^2\right],
\]

and the conserved energy \( E \) is

\[
E = \frac{1}{2} M\dot{r}^2 + \frac{1}{2M} \left( \frac{L - \alpha g \sqrt{r} e^{-eFr}}{r} \right)^2 = \frac{1}{2} M\dot{r}^2 + \frac{1}{2M} V(r).
\]

This is just the sum of kinetic and effective potential energy.

To proceed further, we must ascertain first that the potential energy function \( V(r) \) in (13) does have a profile which is able to sustain bounded classical motion of the particle. Clearly \( V(r) \) is positive infinity at \( r = 0 \) and goes to zero at large \( r \). We want to argue that between \( r = 0 \) and \( r = \infty \), there is one minimum and one maximum, which will give the well. If we express the potential energy \( V(r) \) as \( V(r) = [T(r)]^2 \), then

\[
V(r)' = 2T(r)T'(r)', \quad V(r)'' = 2(T')^2 + 2TT''.
\]

Clearly \( V' = 0 \) has two roots, \( T = 0 \) and \( T' = 0 \). It is easy to see that the root \( T = 0 \) constitutes the minimum. To show that under some restrictions, the other root \( T' = 0 \) does represent a maximum, \( V'' = 2TT'' \) has to be negative. We evaluate \( V'' \) explicitly and use \( T' = 0 \) to express it in the following form

\[
V'' |_{T'=0} = -\frac{2(\alpha g)^2 e^{-2eFr}}{r^2}(\frac{1}{2} - eFr)(\frac{1}{2} - (\frac{1}{2} - eFr)^2).
\]

Hence the allowed range of \( r \) is such that either \( \frac{1}{2} > eFr > 0 \) or \( eFr > \frac{5}{4} \).

Thus a potential well is formed with the minimum at \( r_1 \) obeying

\[
L = \alpha g \sqrt{r_1} e^{-eFr_1},
\]

and the maximum at \( r_2 \) obeying

\[
L = \alpha g \sqrt{r_2} e^{-eFr_2}(\frac{1}{2} + eFr_2),
\]

provided \( r_2 > r_1 \) and \( r_1 \) is within the allowed range. Obviously the minimum value of \( V(r) \) is zero, the expression being positive definite.

Let us now turn to numerical results. Since the equations for the minimum and maximum, (16), (17), involve transcendental functions, we have given graphical solutions in Fig. 1 with the potential well structure in Fig. 2, for the particular solution chosen in Fig. 1. From Fig. 1 one can see that there is a lower bound for the gradient of the straight line (II), below which there is no solution for \( r_1 \). However, for this extreme value of the parameter, solution for \( r_2 \) is such that \( r_2 < r_1 \). On the other hand, there are solutions for
with \( r_2 > r_1 \), which is required for well formation, such that \( r_1 \) is at least less than \( \frac{1}{eF} \), that is the relevant dimensionless variable \( eFr_1 < 1 \). In Fig. 1, (II) and (III) give the values \( eFr_1 = 0.08 \) and \( eFr_2 = 0.21 \). In Fig. 2, we have plotted the dimensionless variables \( V(r)/\frac{LeF_2^2}{2M} \) vs. \( eFr \). The well is formed within the range of the vortex, where the exponential damping can compete with the \( \frac{L^2}{2Mr^2} \) centrifugal term. But in this region the asymptotic solution may not be too reliable.

We now give a rough idea of how the system might look close to the origin. One can have solutions of \( A^\mu \) and \( \phi \) field equations, (obtained from (1)), such that

\[
A_0 = A_r = 0, \quad A_\theta = \alpha eF^2 r + \beta e^3 F^4 r^3, \quad \phi = CeF^2 r + p e^3 F^4 r^3,
\]

with \( \alpha = -\frac{\Lambda}{6}, \quad \beta = \frac{\Lambda^2}{24(1+4\Lambda)}, \quad C^2 = -\frac{\Lambda^2}{3(1+4\Lambda)}, \quad p = -\frac{\Lambda C}{12}, \quad \Lambda = \frac{1}{e^2} \). The field equations are satisfied to \( O((eFr)^n) \), \( n > 4 \). Comparing with [2], we find that near the origin, \( \phi \) goes to zero correctly. The magnetic field is

\[
B = \frac{2\Lambda F}{e} + O(r^2).
\]

(18)

B will behave properly near origin if the \( O(r^2) \) term opposes the constant value, which is true for \( \frac{1}{e^2} > \frac{1}{4} \).

Now, the \( L_{\text{matter}} \) and the energy are

\[
L_{\text{matter}} = \frac{1}{2} M \dot{r}^2 + \frac{1}{2} M r^2 \dot{\theta}^2 + gr \dot{\theta}(\alpha eF^2 r + O(r^3)),
\]

(19)

\[
E = \frac{1}{2} M \dot{r}^2 + \frac{1}{2Mr^2}(L - \alpha geF^2 r^2)^2.
\]

(20)

Apart from a constant shift of \( \frac{L_{\text{matter}}^2}{2M} \) this potential energy function is just that of a two dimensional harmonic oscillator, with the minimum value of zero at \( r_1 = \sqrt{L/(\alpha geF^2)} \). Hence we have that an approximate solution of the gauge field, valid at short distance, provides in the leading order, a two dimensional circular oscillator potential, which indeed has stable classical orbits. These two limiting results, that is gauge fields at large and small \( r \), do indicate that the system in question is classically stable.

3 Quantum dynamics: metastable states

Let us notice at the outset that the Hamiltonian considered by us is of the form

\[
H = \frac{1}{2M} [\dot{p} - g\vec{A}(r)]^2,
\]

(21)

where in our case, \( \vec{A}(r) \) is the vortex potential [2]. However, without going into details, for a purely magnetic field, \( H \) is a non-negative, hermitian operator. We consider the Hilbert space to be composed of wave functions which are square integrable on the plane and regular at the origin. This means that the energy spectrum is positive and purely continuous, contrary to the nature of bound states. We will discuss later in more detail the Schrodinger equation analysis which also points to the same conclusion as above. For the present, arguing from a physical point of view, a potential well which is everywhere positive and goes to zero asymptotically can not produce a bound state, (for special cases contrary to this conclusion, see Simon in [7]), simply because the particle can tunnel out
and live outside the well. This example clearly shows that closed classical trajectories, which are present here, are in no way a sufficiency condition for quantum bound states. After the above compelling arguments against the existence of bound states in the strict sense, (that is states having negative energy), let us look at the potential profile in Fig. 2 more carefully. We immedietly notice that although the well is quite sharp, (since the terms responsible are exponential), the subsequent fall off towards zero is very gradual, (thanks to the inverse power law). Obviously this will reduce the tunnelling amplitude to a large extent and so the states having positive energies that are well inside the ”well” can not live outside the well due to the greater potential energy. Thus the well quite efficiently traps the particle states and these are termed by us as metastable states or resonances. Even though the square integrable eigen functions are disallowed, in the spirit of semiclassical quantization, we generate harmonic oscillator states about the potential minimum. This will give the spectrum of metastable states.

It is now straightforward to perform the oscillator approximation near $r_1$. While considering the asymptotic solutions, we write $r = r_1 + x$ in $V(r)$ of (13) and keep upto $O(x^2)$ terms and get

$$V(r_1 + x) = \frac{1}{2M(r_1 + x)^2}[L - \alpha g \sqrt{(r_1 + x)e^{-eF(r_1 + x)}}]^2,$$  \hspace{1cm} (22)

$$\approx \frac{1}{2Mr_1^2}[\alpha g \sqrt{r_1}e^{-eFr_1}(\frac{1}{2r_1} - eF)^2]x^2.$$  \hspace{1cm} (23)

Identifying this energy as $\frac{1}{2}kx^2$ we find

$$w = \sqrt{\frac{k}{M}} = \frac{1}{M} \frac{\alpha ge^{-eFr_1}}{\sqrt{r_1}} \left| \frac{1}{2r_1} - eF \right| = \frac{gB(r_1)}{M}.$$  \hspace{1cm} (24)

Note that this frequency corresponds precisely to the Larmor frequency for a charged particle in a magnetic field.

Hence the metastable energy spectrum consists of the levels

$$E_n = (n + \frac{1}{2})\hbar w.$$  \hspace{1cm} (25)

(The anharmonic corrections can be included as well).

Near the minimum, the energy of the system in oscillator approximation becomes

$$E = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} kx^2 = \frac{1}{2M} p_x^2 + \frac{1}{2} kx^2.$$  \hspace{1cm} (26)

Alternatively the Bohr-Sommerfield condition can be used to obtain the spectrum of (26), which is an equation of ellipse in phase space. Thus $\int p_x dx$ is the area of the ellipse, which is a multiple of $\hbar$. One can also compute the phase integral $\int_a^b p_x dr$ where $a$ and $b$ are the classical turning points obtained by solving (20) when $M\dot{r} = 0$. However near the potential minimum,

$$\int_a^b p_x dr = \int_a^b \sqrt{2ME} dr = \sqrt{2ME}, \ (b - a) \approx x.$$  \hspace{1cm} (27)

All these will give rise to a spectrum similar as that of (25).
In case of the solutions valid near the origin, once again expanding near \( r_1 \) and writing \( V(r_1 + x) = \frac{1}{2} x^2 \), we find

\[
w = \sqrt{\frac{k}{M}} = \frac{gB}{M}
\]

with \( B \) computed in (18). This is also of the same form as (24). Here we can give the exact energy levels or wave functions for the point particle.

Indeed, one can do a detailed Bohr-Sommerfield quantisation, by introducing radial and angular quantum numbers for the \( r \) and \( \theta \) phase space integrals. The standard way is to replace \( \frac{d}{dt} \) by \( \frac{d}{d\theta} \) (using the \( \theta \) equation of (11)) in the \( r \) equation obtained from (6), in order to obtain an equation for the orbit [5]. The energy \( E \) appears as a parameter. Using the orbit equation and the definition of the quantum numbers, the energy is expressed in terms of the quantum numbers. However, due to the complicated nature of the equations of motion, we have adopted a more pedestrian approach to obtain the metastable state energy spectrum in a harmonic oscillator approximation about the potential minimum.

4 Quantum dynamics: true bound states

Let us start this section with some preliminaries, which are distinct for 2+1-dimensions from the conventional 3+1-dimensions, only in some technical details. For a generic central potential \( V(r) \), the complete set of eigenfunctions are of the form \( \psi(r, \theta) = f(r) \exp(il\theta) \), with \( l \) an integer for singlevaluedness, where \( f(r) \) satisfied the radial equation

\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{l^2}{r^2} + \frac{2M}{\hbar^2} (E - V(r)) \right] f(r) = 0 \tag{29}
\]

with \( f(0) = 0 \) [10]. In our particular case, with the explicit form of \( V(r) \) from (13), we have to solve the radial equation

\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \left( \frac{L}{r} + \frac{\alpha g}{e\sqrt{r}} e^{-eF r} \right)^2 - k^2 \right] f_L(r) = 0 \tag{30}
\]

where \( L = l - \frac{g}{e} \), \( leZ \) and \( k^2 = 2ME \geq 0 \) with \( f_L(0) = 0 \). Note that here the total time derivative term in (6) has not been dropped, so that the full expression of \( B \) in (4) is being considered.

One should not confuse the addition of \( \frac{g}{e} \) in the angular momentum (which came from the \( \frac{\alpha g}{e} \theta \) term mentioned above in the Lagrangian), with the ageold recipe of changing particle statistics in the quantum version. In external field problem angular momentum is quantised in integers even though the expression differs from the canonical one [11]. The effect of total derivative terms in quantum mechanics is briefly discussed in [1].

In our case, with positive \( eF \), for large \( r \), (that is away from the origin), the exponential terms die out quickly leaving only the inverse power law falloff, which controls the large \( r \) behaviour. Now, if the Hilbert space is that of regular square integrable wave functions on the plane, then the spectrum is purely continuous with \( k \geq 0 \). In fact the eigenfunctions asymptotically behave as

\[
\psi_L(kr) \approx C_l J_L(kr) \tag{31}
\]

where \( C_l \) is a constant [10]. The above analysis rigorously shows that bound states in general are not possible.
However, one must be more careful in imposing boundary conditions [12] on the wave functions due to the presence of the $\delta$-function term in the magnetic field. This points towards a contact interaction which viciates the self adjointness of the Hamiltonian [7] at the origin. The remedy is to look for self adjoint extensions (if present) of the Hamiltonian and this can lead to bound states [8] for some restricted set of parameters.

Generally we tend to overlook the difference between Hermitian and self adjoint operators in quantum physics, but there is a difference in the structure of their respective domains, which crucially governs the dynamics in many cases of physical interest. For unbounded operators, (that do not have bounded expectation values, such as energy, momentum, angular momentum and position), one has to specify the domain of the operator as well as its action on the domain. Without getting involved in too many technical details, an operator $T$ is called Hermitian if $T \subset T^*$, that is $D(T) \subset D(T^*)$ and $T\phi = T^*\phi$ for all $\phi \in D(T)$. Here the adjoint $T^*$ is defined as $(T\psi,\phi) = (\psi,T^*\phi)$ for all $\psi,\phi \in D(T)$ with $D(T)$ denoting the domain of $T$. $T$ is self-adjoint if $T = T^*$ that is iff $T$ is Hermitian and $D(T) = D(T^*)$. Only the self-adjoint operators can be exponentiated to give one parameter unitary groups which governs the dynamics in quantum mechanics.

The basic criterion of self-adjointness is the following: Suppose $T$ is a self-adjoint operator and a $\phi$ exists such that $T^*\phi = i\phi$. Then also $T\phi = i\phi$ and

$$-i(\phi,\phi) = (i\phi,\phi) = (T\phi,\phi) = (\phi,T^*\phi) = (\phi,T\phi) = i(\phi,\phi)$$

which implies $\phi = 0$ and this is true for $T^*\phi = -i\phi$ as well. Hence for self-adjointness, it must be ensured that $\text{ker}(T^* \pm i) = 0$ or $D(T^*) = D(T)$. However, if $T$ is not self-adjoint on a domain, but has $n_\pm$ independent solutions $T^*\phi_i = \pm i\phi_i$ for some $\phi_i \in D(T^*)$ then, only if $n_+ = n_- = n$, one is allowed to make an extension of $T$ to $\tau$, where $D(T) \subset D(\tau)$ and $\tau\phi = T\phi$ for all $\phi \in D(T)$), such that $\tau$ is self-adjoint. Basically one modifies the domain of $T$ to $\tau$ such that $D(\tau) = D(T^*)$ and this also ensures that $\text{ker}(\tau \pm i) = 0$. The vectors $\phi_i$ generate the deficiency subspace and $n_\pm$ are referred as the deficiency indices. Note that for simple differential operators, as in the present case, $T^*$ is the same as $T$, but acting on $D(T^*)$, which in general is different from $D(T)$.

Our task is now to express the Hamiltonian in a suitable form $T$ and find independent, normalisable solutions of

$$T\phi_{\pm}^i = \pm i\phi_{\pm}^i, \quad (32)$$

Since we are looking at regions close to the origin, the exponential term is dropped, compared to more singular centrifugal term. Replacing $f(r)$ and $kr$ by $\frac{u(r)}{\sqrt{r}}$ and $\rho$ in (29), we obtain

$$\frac{d^2u_p}{d\rho^2} + \left(1 - \frac{p(p+1)}{\rho^2}\right)u_p = 0, \quad (33)$$

where $p(p+1) = -(\frac{1}{r} - L^2)$. In general the solutions of the Schroedinger equation as well as its first order partial derivatives will be continuous, uniform and bounded functions over all space, including $\rho = 0$. Here we also have $f(0) = 0$. This induces the following natural boundary conditions on $u_p$ as well;

$$u_p(0) = \frac{du_p(0)}{d\rho} = 0. \quad (34)$$

Let us see somewhat heuristically why the singularity problem at $r = 0$ affects only some of the $p$ states. The general solution of (33) near the origin is

$$u_p \approx A\rho^{p+1} + B\rho^{-p}. \quad (35)$$
For $p$ different from 0 or -1, to maintain regularity at $\rho = 0$, either $A$ or $B$ has to vanish and $u_p(0) = 0$. Thus the singularity problem does not matter for these states. But for $p$ equal to 0 or -1, $u_0$ or $u_{-1}$ can be finite but non-zero at the origin. For them the imposition of the boundary condition $u(0) = 0$ clashes with self-adjoint property of the Hamiltonian. This problem and the subsequent solution is elaborated below.

Let us factorise (33)

$$\left(\frac{\partial}{\partial \rho} - \frac{p + 1}{\rho}\right) u_p = v_p, \quad \left(\frac{\partial}{\partial \rho} + \frac{p + 1}{\rho}\right) v_p = -u_p.$$  \hspace{1cm} (36)

or

$$\left(\frac{\partial}{\partial \rho} + \frac{p}{\rho}\right) u_p = v_p, \quad \left(\frac{\partial}{\partial \rho} - \frac{p}{\rho}\right) v_p = -u_p.$$  \hspace{1cm} (37)

For $p$ taking values 0 or 1, the pairs of equations become identical and reduce to

$$T \begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{\partial}{\partial \rho} \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} v \\ u \end{pmatrix}.$$  \hspace{1cm} (38)

The deficiency subspace is generated by a pair of normalised vectors (i.e. $n_+ = n_- = 1$)

$$T \chi_\pm = \pm i\chi_\pm, \quad \chi_\pm = e^{-r} \begin{pmatrix} -1 \\ \pm i \end{pmatrix}. \hspace{1cm} (39)$$

Now the extension of $T$ is $\tau$ where $T = \tau$ with domain

$$D(\tau) = \{ \psi + \beta \chi_+ + \gamma \beta \chi_- | \psi \epsilon D(T), \beta \epsilon C\},$$  \hspace{1cm} (40)

and the operation of $\tau$ on $D(\tau)$ is

$$\tau(\psi + \beta \chi_+ + \gamma \beta \chi_-) = T\psi + i\beta \chi_+ - i\gamma \beta \chi_-,$$  \hspace{1cm} (41)

where $| \gamma |$ is an isometry that maps $\chi_+ \rightarrow \chi_-$. We parametrise $\gamma = e^{i\alpha}$ and there are different extensions for different $\alpha$. In this extended domain the deficiency indices vanish,

$$n_\pm(\tau_{\alpha}) = 0.$$  \hspace{1cm} (42)

This is the self-adjoint extension explained before. The vital role played by the contact term in the magnetic field in (4), which gave rise to the $\frac{2}{e}$ term in $L$ becomes manifest only now. Note that in the expression $p(p + 1) = -(\frac{l}{4} - (l - \frac{2}{e})^2)$, since $l$ is an integer, the LHS can vanish, (i.e., $p = 0, -1$), only if the $\frac{2}{e}$ term cancels $\frac{1}{4}$ in the RHS.

From a slightly different angle, let us now see how the boundary conditions clash with the self-adjointness of the Hamiltonian at the origin. Indeed, this is probably an easier way to understand the connection between deficiency subspaces and the bound state spectrum in question. From the definition of the self adjoint operator, it follows that the Hamiltonian is self adjoint if

$$\langle \phi, T\psi \rangle - \langle T\phi, \psi \rangle = 0$$

with $\phi = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \epsilon D(T^*)$ and $\phi = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \epsilon D(T)$. Using the natural boundary conditions $u_2(0) = v_2(0) = 0$ and $u_2(\infty) = v_2(\infty) = 0$, (since these specify $D(T)$), the above condition reduces to

$$v_1^*(0)u_2(0) - u_1^*(0)v_2(0) = 0.$$  \hspace{1cm} (43)
Thus the above equation vanishes, independent of any boundary condition on \( \phi \), which is in the domain of \( T^* \). Thus \( D(T^*) > D(T) \) since it consists of similar vectors as in \( D(T) \), but without the boundary condition. This makes \( T \) not self-adjoint in this domain. The cure is to moderate the too strong boundary condition on \( D(T) \) by eg.,

\[
u_2(0) + akv_2(0) = u_2(0) + au'_2(0) = 0, \tag{44}
\]

where \( a \) is an arbitrary real parameter. This is the extension \( \tau \). This ensures identical boundary conditions on \( \psi \) and \( \phi \), and hence the domains \( D(\tau^*) \) and \( D(\tau) \) are identical. \( a \) and \( \alpha \) of (41), (44), are related in the following way,

\[
a = \frac{1}{k} \cot \alpha . \tag{45}
\]

Now we can get the bound state spectrum easily. For either \( p = 0 \) or \( p = -1 \) we have

\[
u' = kv, \quad v' = -ku, \tag{46}
\]

We also have from the extended domain

\[
u(0) + au'(0) = 0. \tag{47}
\]

Using the above equations we get

\[
-\frac{1}{k} v'(0) + akv(0) = 0. \tag{48}
\]

Let us consider some form of \( u \) and \( v \) which vanish at \( r = \infty \).

\[
u = Pe^{-\gamma r}, \quad v = Qe^{-\gamma r}. \tag{49}
\]

\( P, Q \) and \( \gamma \) are constants. Solving the above set of equations we find

\[
k^2 = -\frac{1}{a^2}
\]

, and so the energy is

\[
E = -\frac{1}{2Ma^2}. \tag{50}
\]

Note that we used the two component factorised form from (33) since the (matrix) differential operator in (38) is Hermitian in its domain, although individually \( i \frac{d}{dr} \) is Hermitian. However, since \( \frac{d^2}{dr^2} \) is Hermitian, we can use (33) directly as well. Choosing \( p = 0 \) or -1 with

\[
u = Ae^{-\gamma r}, \quad u(0) + au'(0) = 0
\]

we again obtain \( k^2 = -\frac{1}{a^2} \). The difference in these two formulations is nontrivially manifested in the deficiency index analysis where the domains of these operators are different.

Thus (50) is the cherished expression for the bound state spectrum. These states correspond to the parameter values such that \( L = l - \frac{\alpha}{2} = \frac{1}{2} \) with integer values for \( l \). So far we only know that \( a \) is real. There are an infinite possibility of different extensions for distinct values of \( a \) and still further conditions are required to uniquely specify \( a \). For a zero energy bound state, we require \( a = \infty \) and \( u'(0) = 0 \) but there is no restriction on \( u(0) \).
5 Conclusion

We have considered a system of point charge, interacting with a Nielsen-Olesen vortex of the abelian Higgs model, in 2+1-dimensions. We have shown that classical bounded orbits are allowed, for certain restricted range of parameters.

Next we quantise the particle motion, treating the potential as external. From general arguments it is established that in general negative energy bound states are not present. However, the nature of the effective potential energy allows metastable states of considerable lifetime.

On the other hand, presence of the contact interaction, (due to the vortex), makes the analysis subtle for certain angular momenta. We have shown that at least for some values of the parameters, at the origin the self adjoint property of the Hamiltonian is lost, which is restored by the self adjoint extensions, and in this new domain, bound states appear. There is a one parameter arbitraryness in the bound state energy spectrum which can be removed by introducing further physical input.

Recently we have come across the papers [13] where the self-adjoint extensions of the Dirac Hamiltonian have been studied, in the context of Aharanov-Bohm effect and in the presence of a magnetic vortex.

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Figure Captions

Fig. 1

$I \rightarrow e^x \text{ vs. } x$, $II \rightarrow \alpha x \text{ vs. } x$, $III \rightarrow \alpha \frac{x}{4}(x+1)^2 \text{ vs. } x$, where $x = 2eFr$, $\alpha = \frac{1}{2}\left(\frac{4}{eL}\right)^2 = 7.4$

Fig. 2

$V(x) = \frac{1}{x^2}(1 - \sqrt{2\alpha xe^{-x}})^2$ with $\alpha = 7.4$