Periodic Position Observation of a Particle in a Harmonic Potential

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(Dated: January 8, 2020)

Abstract

We consider a particle in harmonic oscillator potential, whose position is periodically measured with an instrument of finite precision. We show that the distribution of the measured positions tends to a limiting distribution when the number of measurements tends to infinity. We derive the expression for the limiting position distribution and validate it with numerical simulation.

PACS numbers:
I. INTRODUCTION

The operational approach to quantum mechanics \cite{1-3} has, among other things, systematically expanded the notion of ideal projective measurements to include imprecise and unsharp measurements. This has been fruitful for a number of practical \cite{4, 6} as well as foundational problems \cite{7, 8}.

Unsharp measurements have been used to maintain coherence in the presence of noise \cite{4}. Choudhary et al. \cite{5} have suggested their application in the measurement of qubit levels of a trapped ion. The evolution of a superconducting qubit subjected to unsharp measurement has been investigated \cite{6}. Schemes for reliable state estimation with sequential \cite{9} and continuous-time unsharp measurements \cite{10} have been suggested.

A special class of quantum measurements, called quantum nondemolition (QND) measurements have been widely used in monitoring a quantum oscillator \cite{11, 12}. This form of measurement can in principle leave the quantum state undisturbed. This could be useful for extremely high precision measurements such as in certain schemes of gravitational wave detection. The statistical behavior of a quantum oscillator subjected to a sequence of QND measurements has been formally worked out by Matta and Pierro \cite{13}.

Our aim in this work is to characterize the statistical distribution of a sequence of position measurements of a quantum oscillator.

The interpretation of a wave function in the position basis is that the absolute value of its square is the probability density of a position measurement \cite{14}. Since a measurement in quantum mechanics changes the state of the system, the notion of probability density is valid only in the case of an ensemble of identically prepared states. In contrast, in this work, we explore the consequences of periodic measurements on the same quantum system.

The test system under study is the harmonic oscillator because of its ubiquity and analytical ease. The system starts with a given wavefunction. When an ideal position measurement is made, the wavefunction is supposed to collapse to a delta function, whose position would be a random number following the probability distribution given by the wavefunction just before the collapse. Subsequently, the wavefunction would evolve following the Schrödinger equation until the next observation. In order for the above scheme to work in numerical simulation, we need the wavefunction to be smooth. So we consider the state immediately after a collapse to be a narrow Gaussian function, whose width represents the accuracy of
FIG. 1: Evolution of the probability density function with periodic measurement measurement. In fact, it is supposed to collapse to a wavefunction which is a product of the wavefunction before measurement and the Gaussian function representing the measurement process. However, if the measurement process is represented by a Gaussian function that is narrow enough compared to the spread of the particle wavefunction, the post-collapse wavefunction can be aptly represented by the narrow Gaussian alone.

Using this schema, we show that the distribution of position measurements tends to a limiting distribution in the limit of infinite measurements. The limiting distribution is a Gaussian function centered at zero, and with a standard deviation that depends on the frequency of measurement, the width of the Gaussian function following a collapse, and the characteristic parameters of the harmonic oscillator. We obtain the limiting distribution in closed form and highlight some of its features. These results are validated using numerical simulation.

II. PROBLEM STATEMENT AND NUMERICAL RESULTS

A particle is placed in a harmonic oscillator potential

\[ V = \frac{1}{2}m\omega^2x^2 \]  

The particle is initially in a state \( \Psi(x,0) \) and we subject it to periodic measurements at intervals of time \( t_m \).
We assume that the measuring instrument has a finite precision so that the state of system collapses to a narrow Gaussian function centered at \( x_M \) with standard deviation \( \sigma_M \) after each measurement (this allows the wavefunction to be differentiable):

\[
\Psi(x, t_M) \xrightarrow{\text{measurement}} \mathcal{G}(x_M, \sigma_M).
\]

The case of ideal measurements is easily derivable by letting \( \sigma_M \) go to zero.

The system was simulated by evolving the wavefunction for \( t_M \) seconds and drawing a random sample \( x_M \) every \( t_M \) seconds from the probability density \( |\Psi(x)|^2 \) just before measurement. Just after the measurement it was replaced by a narrow Gaussian of standard deviation \( \sigma_M \) and centered at \( x_M \). The state was then allowed to evolve until the next measurement following the Schrödinger equation (Fig. 1). The process was repeated to obtain the limiting distribution of samples, which we plot in Fig. 2(a). Numerical simulations for various values of \( t_M \) and \( \sigma_M \) revealed that the limiting distribution is always Gaussian. The standard deviation of the samples was found to rapidly converge to a constant value as the number of samples was increased (Fig. 2(b)).

In order to check the dependence of the limiting standard deviation (\( \sigma_\infty \)) on the accuracy of the measuring device, we obtained the results for different values of \( \sigma_M \). The results
FIG. 3: Limiting standard deviation versus (a) the standard deviation of collapsed wavefunction, and (b) the natural frequency of the harmonic oscillator.

are presented in Fig. 3(a). Similarly, we explore the dependence of the limiting standard deviation on the natural frequency of the harmonic oscillator, which we plot in Fig. 3(b). We also found that the limit distribution is independent of the initial wavefunction.

III. DERIVATION OF LIMIT DISTRIBUTION

We now obtain the expression for the limiting distribution and its dependence on various parameters.

The evolution of a Gaussian wave packet in a harmonic potential is a well known result. Let the initial wavepacket be

\[ \Psi(x, 0) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_x}} \exp\left\{ -\frac{(x-x_0)^2}{4\sigma_x^2} \right\} \]  

(2)

where \( x_0 \) is the initial center of the Gaussian wave packet, \( \sigma_x \) is the initial width of wave packet. Then the probability density at time \( t \) is given by

\[ |\Psi(x, t)|^2 = \frac{1}{\sqrt{2\pi}\sigma(t)} \exp\left\{ -\frac{(x-x_0 \cos \omega t)^2}{2\sigma(t)^2} \right\} \]  

(3)
where
\[
\sigma(t) = \frac{\sigma_{gs}^2}{2\sqrt{2}\sigma_{x0}} \left[ 4 \left( \frac{\sigma_{x0}}{\sigma_{gs}} \right)^4 + 1 + \left( 4 \left( \frac{\sigma_{x0}}{\sigma_{gs}} \right)^4 - 1 \right) \cos 2\omega t \right]
\] (4)
and $\sigma_{gs} = \sqrt{\hbar/m\omega}$ is the width of the ground-state eigenfunction.

For the sake of succinctness, we shall refer to a Gaussian in $x$ centered at $\mu$ with standard deviation $\sigma$ as $G(x-\mu,\sigma)$, whereby the time evolution of a Gaussian wavepacket can be expressed as
\[
G(x-x_{0},\sigma_{x0}) \longrightarrow G(x-x_{0}\cos \omega t,\sigma(t))
\] (5)

At $t = 0$ we start with a Gaussian wavepacket $G(x_{0},\sigma_{x0})$ centered at $x = 0$, and width $\sigma_{x0}$. We repeatedly measure the position of the particle after fixed time intervals of $t_{M}$. At each measurement a random value of the position is chosen following the distribution of $|\Psi|^2$ at that time instant.

A measurement collapses the wavefunction. The imprecise instrument is assumed to collapse the wavefunction into a narrow Gaussian wavepacket
\[
\Psi_{i}(x,0) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_{M}^{i}}} \exp \left\{ -\frac{(x-x_{Mi})^2}{4\sigma_{M}^{i}} \right\}
\]
where $x_{Mi}$ is the outcome of the $i^{\text{th}}$ measurement. The next measurement happens after a time $t_{M}$. The probability density for the wavefunction just before the next measurement can be calculated using equation (3)
\[
|\Psi_{i}(x,t_{M})|^2 = \frac{1}{\sqrt{2\pi}\sigma(t_{M})} \exp \left\{ -\frac{(x-x_{Mi}\cos \omega t_{M})^2}{2\sigma^2(t_{M})} \right\}
\]
\[
= G(x-x_{Mi}\cos \omega t_{M},\sigma(t_{M}))
\]
For the first measurement the distribution is
\[
D_{1}(x) = |\Psi(x,t_{M})|^2
\]
\[
= \frac{1}{\sqrt{2\pi}\sigma_{0}(t_{M})} \exp \left\{ -\frac{x^2}{2\sigma_{0}^2(t_{M})} \right\}
\]
\[
= G(x,\sigma_{0}(t_{M}))
\]

We denote the standard deviation of this distribution as $\sigma_{0}$ to distinguish it from all subsequent standard deviations. The densities before all subsequent measurements have the same width as they all start from a collapsed state whose standard deviation is identical in all cases. The expected distribution for the second measurement is
\[ D_2(x) = \int_{-\infty}^{\infty} D_1(x_{m1}) |\Psi_1(x, t_{m}|^2 \, dx_{m1} \]
\[ = G \left( x, \sigma(t_{m}) \sqrt{1 + \left( \frac{\sigma_0(t_{m})}{\sigma(t_{m})} \right)^2 \cos^2 \omega t_{m}} \right) \]

The derivation of this result can be found in the Appendix A. Similarly, for the third measurement, the density is

\[ D_3(x) = \int_{-\infty}^{\infty} D_2(x_{m2}) |\Psi_2(x, t_{m}|^2 \, dx_{m2} \]
\[ = G \left( x, \sigma(t_{m}) \sqrt{1 + \cos^2 \omega t_{m} + \left( \frac{\sigma_0(t_{m})}{\sigma(t_{m})} \right)^2 \cos^4 \omega t_{m}} \right) \]

And for the \( n \)th measurement the density is

\[ D_n(x) = G \left( x, \sigma(t_{m}) \sqrt{1 + \cos^2 \omega t_{m} + \cdots + \left( \frac{\sigma_0(t_{m})}{\sigma(t_{m})} \right)^{2 \cos^2(n-1)\omega t_{m}}} \right) \]

The geometric series converges if \( \cos^2 \omega t_{m} < 1 \)

\[ D_\infty(x) = G \left( x, \sigma(t_{m}) \sqrt{\frac{1}{1 - \cos^2 \omega t_{m}}} \right) \]
\[ = G \left( x, \sigma(t_{m}) \frac{1}{\sin \omega t_{m}} \right) \]
\[ = G \left( x, \sigma(t_{m}) \frac{\sin \omega t_{m}}{\sin \omega t_{m}} \right) \]

or,

\[ \sigma_\infty = \left| \frac{\sigma(t_{m})}{\sin \omega t_{m}} \right| \]

The mean of all these densities is then the distribution of samples, one taken from each \( D_i \). This is the mean of the densities \( D_i \). So we have

\[ \sigma_\infty^2 = \sigma(t_{m})^2 + \sigma(t_{m})^2 \cos^2 \omega t_{m} + \cdots + \sigma_0(t_{m})^2 \cos^2(i-1)\omega t_{m} \]

The mean of all these densities is again a Gaussian with mean at zero and variance \( s_\infty^2 \) given by the mean of the individual variances given by equation (8).
\[
\sigma_\infty = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{\infty} \sigma_i^2 = \left\{ \frac{\sigma(t_\infty)}{\sin \omega t_\infty} \right\}^2
\]

The calculation of this result is given in Appendix B.

\[
\therefore s_\infty = \left| \frac{\sigma(t_\infty)}{\sin \omega t_\infty} \right| = \sigma_\infty
\] (9)

We find that \( s_\infty \) is the same as \( \sigma_\infty \). This means that the distribution of measurement outcomes \( x_{m1}, x_{m2}, \ldots, x_{mn} \) itself converges to \( D_\infty \) as \( n \to \infty \).

Substituting equation (4) in equation (9) we get

\[
\sigma_\infty = \left| \frac{\sigma_0^2 \sqrt{4\left(\frac{\sigma_{00}}{\sigma_0}\right)^4 + 1 + 4\left(\frac{\sigma_{00}}{\sigma_0}\right)^4 - 1}}{2\sqrt{2} \sigma_\infty \sin \omega t_\infty} \cos 2\omega t_\infty \right|
\]

In Fig. 3 the analytical result given by equation (9) is plotted with continuous lines while the numerical results are plotted with dots. In both these cases the theoretical results and simulation show good agreement.
IV. ANALYSIS

We found the limit distribution to be a Gaussian centered at zero with standard deviation given by equation (9) which can be simplified to

\[ \sigma_\infty = \sqrt{\sigma_m^2 \cot^2 \omega t_m + \frac{\sigma_0^4}{4\sigma_m^2}} \]

We obtain a non-dimensional form by dividing throughout by \( \sigma_0 \),

\[ \frac{\sigma_\infty}{\sigma_0} = \sqrt{\left(\frac{\sigma_m}{\sigma_0}\right)^2 \cot^2 2\pi \frac{t_m}{T} + \frac{1}{4} \left(\frac{\sigma_0}{\sigma_m}\right)^2} \]

or,

\[ \varsigma_\infty = \sqrt{\varsigma_m^2 \cot^2 2\pi \tau_m + \frac{1}{4\varsigma_m^2}} \]

where \( \varsigma_\infty = \frac{\sigma_\infty}{\sigma_0}, \varsigma_m = \frac{\sigma_m}{\sigma_0} \) and \( \tau_m = \frac{t_m}{T} \). These substitutions are advantageous because they are dimensionless quantities independent of the length and time-scales of any particular harmonic oscillator.

In Fig. 4(a) we see how \( \varsigma_\infty \) changes when we vary \( \varsigma_m \), for particular values of \( \tau_m \). For \( \varsigma_m \to 0 \), \( \varsigma_\infty \) grows hyperbolically (\( \sim \frac{1}{2\varsigma_m} \)). It has a minimum value at \( \varsigma_m = \sqrt{\tan \frac{2\pi \tau_m}{2}} \). And as \( \varsigma_m \to \infty \), it grows linearly in \( \varsigma_m \) with slope \( |\cot 2\pi \tau_m| \). For \( \tau_m \in [0, \frac{1}{4}] \), the slope of the linear asymptote varies from \( \infty \) to 0. After \( \tau_m = \frac{1}{4} \), the process reverses itself till \( \tau_m = \frac{1}{2} \), after which the pattern repeats periodically.

In Fig. 4(b) we see how \( \varsigma_\infty \) changes with \( \tau_m \). The plots are periodic in \( \tau_m \) with period \( \frac{1}{2} \). The curves have minima at \( \frac{n}{2} - \frac{1}{4}, n \in \mathbb{N} \). As \( \varsigma_m \) increases \( \varsigma_\infty \) gets steeper and the minima tend to zero.

In Fig. 5 the dependence of \( \varsigma_\infty \) on both \( \tau_m \) and \( \varsigma_m \) have been consolidated into a single surface plot.

V. CONCLUSION

We have investigated the statistical distribution of periodic measurements on a single quantum system (in this case a quantum harmonic oscillator). We find that the measurement outcomes follow a Gaussian distribution with mean zero.
An analytical expression for the standard deviation of the limiting distribution was derived and was validated with numerical simulation. The standard deviation of this distribution was found to depend on the accuracy and frequency of measurements, and the natural frequency of the harmonic oscillator. This distribution was found to be independent of the initial wavefunction.

We have shown that there is an optimal accuracy of measurement that minimizes the standard deviation of the limit distribution. We also found that certain measurement intervals minimize the standard deviation of the limit distribution. These results may be useful for localizing a particle at the center of a well with the least uncertainty.
ACKNOWLEDGEMENT

AA acknowledges the financial support from Institute Fellowship of IISER Kolkata. SB acknowledges the J C Bose National Fellowship provided by SERB, Government of India, Grant No. SB/S2/JCB-023/2015.

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Appendix A: Derivation of \( D_2(x) \)

\[
D_2(x) = \int_{-\infty}^{\infty} D_1(x_{\mu_1}) \left| \Psi_1(x, t_{\mu_1}) \right|^2 dx_{\mu_1}
\]

\[
= \int_{-\infty}^{\infty} \mathcal{G}(x_{\mu_1}, \sigma_0(t_{\mu_1})) \mathcal{G}(x - x_{\mu_1} \cos \omega t_{\mu_1}, \sigma(t_{\mu_1})) dx_{\mu_1}
\]

\[
= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_0(t_{\mu_1})} \exp \left\{ -\frac{(x_{\mu_1})^2}{2\sigma_0(t_{\mu_1})^2} \right\} dx_{\mu_1}
\]

\[
= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_0(t_{\mu_1})} \exp \left\{ -\frac{(x - x_{\mu_1} \cos \omega t_{\mu_1})^2}{2\sigma(t_{\mu_1})^2} \right\} dx_{\mu_1}
\]

\[
= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma(t_{\mu_1}) \sec \omega t_{\mu_1}} \exp \left\{ -\frac{(x_{\mu_1})^2}{2\sigma_0(t_{\mu_1})^2} \right\} \sec \omega t_{\mu_1} dx_{\mu_1}
\]

\[
= \int_{-\infty}^{\infty} \sec \omega t_{\mu_1} \mathcal{G}(x_{\mu_1}, \sigma_1(t_{\mu_1})) \mathcal{G}(x_{\mu_1} - x \sec \omega t_{\mu_1}, \sigma(t_{\mu_1}) \sec \omega t_{\mu_1}) dx_{\mu_1}
\]

Using the following result about the integral of the product of two Gaussians

\[
\int_{-\infty}^{\infty} \mathcal{G}(x - \mu_1, \sigma_1) \mathcal{G}(x - \mu_2, \sigma_2) dx = \mathcal{G} \left( \mu_1 - \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2} \right)
\]

we have

\[
D_2(x) = \sec \omega t_{\mu_1} \mathcal{G} \left( x \sec \omega t_{\mu_1}, \sqrt{\sigma_0(t_{\mu_1})^2 + \sigma(t_{\mu_1})^2 \sec^2 \omega t_{\mu_1}} \right)
\]

\[
= \sec \omega t_{\mu_1} \exp \left\{ -\frac{x^2 \sec^2 \omega t_{\mu_1}}{2(\sigma(t_{\mu_1})^2 \sec^2 \omega t_{\mu_1} + \sigma_0(t_{\mu_1}))^2} \right\}
\]

\[
= \exp \left\{ -\frac{x^2}{2(\sigma(t_{\mu_1})^2 + \sigma_0(t_{\mu_1}) \cos^2 \omega t_{\mu_1})^2} \right\}
\]

\[
D_2(x) = \mathcal{G} \left( x, \sigma(t_{\mu_1}) \sqrt{1 + \left( \frac{\sigma_0(t_{\mu_1})}{\sigma(t_{\mu_1})} \right)^2 \cos^2 \omega t_{\mu_1}} \right)
\]
Appendix B: Calculating $s_\infty$

\[ s_\infty^2 = \lim_{n \to \infty} \frac{1}{n} \sigma_0(t_m)^2 + \frac{1}{n} \{ \sigma(t_m)^2 + \sigma_0(t_m)\cos^2 \omega t_m \} \]
\[ + \frac{1}{n} \{ \sigma(t_m)^2 + \cos^2 \omega t_m + \cdots + \sigma_0(t_m)\cos^{2(i-1)} \omega t_m \} \]
\[ + \cdots \]
\[ = \lim_{n \to \infty} \left[ \frac{\sigma_0(t_m)^2}{n} \left\{ 1 + \cos^2 \omega t_m + \cos^4 \omega t_m + \cdots \right\} \right. \]
\[ + \frac{\sigma(t_m)^2}{n} \left[ n + (n - 1) \cos^2 \omega t_m + \cdots \right. \]
\[ + \left. \left( n - (i - 1) \right) \cos^{2(i-1)} \omega t_m + \cdots \right] \]
\[ = \lim_{n \to \infty} \frac{\sigma_0(t_m)^2 \cosec^2 \omega t_m}{n} + \frac{\sigma(t_m)^2}{n} \sum_{k=1}^{\infty} \left[ n - (k - 1) \right] (\cos^2 \omega t_m)^{(k-1)} \]

If $\cos^2 \omega t_m < 1$ the second term is a convergent arithmetico-geometric series and we have

\[ s_\infty^2 = \lim_{n \to \infty} \left[ \frac{\sigma_0(t_m)^2 \cosec^2 \omega t_m}{n} + \frac{\sigma(t_m)^2}{n} \left\{ \frac{n}{1 - \cos^2 \omega t_m} - \frac{\cos^2 \omega t_m}{1 - \cos^2 \omega t_m} \right\} \right] \]
\[ \therefore s_\infty^2 = \left\{ \frac{\sigma(t_m)}{\sin \omega t_m} \right\}^2 \]