NATURAL DIFFERENTIAL OPERATORS AND GRAPH COMPLEXES

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ABSTRACT. We show how the machine invented by S. Merkulov [18, 19, 21] can be used to study and classify natural operators in differential geometry. We also give an interpretation of graph complexes arising in this context in terms of representation theory. As application, we prove several results on classification of natural operators acting on vector fields and connections.

INTRODUCTION

This work started in an attempt to understand S. Merkulov’s idea of “PROP profiles” [18, 21] and see if and how it may be used to investigate natural structures in geometry. It turned out that classifications of these geometric structures in many interesting cases boiled down to calculations of the cohomology of certain graph complexes. More precisely, for a wide class of natural operators, the following principle holds.

Principle. For a given type of natural differential operators, there exists a graph cochain complex $(G^r_*, \delta) = (G^r_0 \xrightarrow{\delta} G^r_1 \xrightarrow{\delta} G^r_2 \xrightarrow{\delta} \cdots)$ such that, in stable ranges,

\[
\{\text{natural operators of a given type}\} \cong H^0(G^r_*, \delta).
\]

Stability means that the dimension of the underlying manifold is bigger than some constant explicitly determined by the type of operators. For example, for multilinear natural operators $TM^{\times d} \to TM$ from the $d$-fold product of the tangent bundle into itself the stability means that $\dim(M) \geq d$. In smaller dimensions, “exotic” operations described in [4] occur.

In all cases we studied, the corresponding graph complex appeared to be acyclic in positive dimensions, so the cohomology describing natural operators was the only nontrivial piece of the cohomology of $(G^r_*, \delta)$. Standard philosophy of strongly homotopy structures [12] suggests that the graph complex $(G^r_*, \delta)$ describes stable strongly homotopy operators of a given type.

Graph complexes arising in the Principle are in fact isomorphic to subspaces of fixed elements in suitable Chevalley-Eilenberg complexes, so, formally speaking, we claim that a certain Chevalley-Eilenberg cohomology is the cohomology of some graph complex. Instances of this phenomenon were systematically used by M. Kontsevich in his seminal paper [9]. The details of operadic graph complexes were then written down by J. Conant [2], J. Conant and K. Vogtmann [3, 4], M. Mulase and M. Penkava [22], M. Penkava [24], and M. Penkava and...
A. Schwarz [23]. What makes the Principle exciting is the miraculous fact that the corresponding graph complexes are of the type studied during the “renaissance of operads” and powerful methods developed in this period culminating in [13, 17, 20] apply.

Another way to view the proposed method is as a formalization of the “abstract tensor calculus” attributed to R. Penrose. When we studied differential geometry in kindergarten, many of us, trying to avoid dozens of indices, drew simple pictures consisting of nodes representing tensors (which resembled little insects) and lines joining legs of these insects symbolizing contraction of indices. We attempt to put this kindergarten approach on a solid footing.

Thus the purpose of this paper is two-fold. The first one is to set up principles of abstract tensor calculus as a useful language for ‘stable’ geometric objects. This will be done in Sections 1–4. The logical continuation should be translating textbooks on differential geometry into this language, because all basic properties of fundamental objects (vector fields, forms, currents, connections and their torsions and curvatures) are of stable nature.

We then show, in Sections 5–7, how results on graph complexes may give explicit classifications of natural operators in stable ranges. As an example we derive from a rather deep result of [14] a characterization of operators on vector fields (Theorem 5.1 and its Corollary 5.3). As another application we prove that all natural operators on linear connections and vector fields, with values in vector fields, are freely generated by compositions of covariant derivatives and Lie brackets, and by traces of these compositions – see Theorems 7.2 and 7.3, and their Corollaries 7.3 and 7.7, in conjunction with Theorems 6.2 and 6.3.

This article is supplemented by [11] in which we explain the relation between invariant tensors and graphs. We believe that [11], which can be read independently, will help to understand the constructions of Sections 3 and 4.

The theory of invariant operators sketched out in this paper leads to directed, not necessarily connected or simply-connected, graphs. A similar theory can be formulated also for symplectic manifolds, where the corresponding graph complexes would be those appearing in the context of anti-modular operads (modular versions of anticyclic operads, see [10, Definition 5.20]). Something close to a symplectic version of our theory has in fact already been worked out in [27].

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1. Natural operators

Informally, a natural differential operator is a recipe that constructs from a geometric object another one, in a natural fashion, and which is locally a function of coordinates and their derivatives.

1.1. Example. Let $M$ be a $n$-dimensional smooth manifold. The classical Lie bracket $X, Y \mapsto [X, Y]$ is a natural operation that constructs from two vector fields on $M$ a third one. Given
a local coordinate system \((x^1, \ldots, x^n)\) on \(M\), the vector fields \(X\) and \(Y\) are locally expressions \(X = \sum_{1 \leq i \leq n} X^i \partial/\partial x^i\), \(Y = \sum_{1 \leq i \leq n} Y^i \partial/\partial x^i\), where \(X^i, Y^i\) are smooth functions on \(M\). If we define \(X^i_j := \partial X^i/\partial x^j\) and \(Y^i_j := \partial Y^i/\partial x^j\), \(1 \leq i, j \leq n\), then the Lie bracket is locally given by the formula \([X, Y] = \sum_{1 \leq i, j \leq n} (X^j Y^i_j - Y^j X^i_j) \partial/\partial x^i\).

In the rest of the paper, we use Einstein’s convention assuming summations over repeated indices. In this context, indices \(i, j, k, \ldots\) will always be natural numbers between 1 and the dimension of the underlying manifold, which will typically be denoted \(n\).

1.2. Example. The covariant derivative \((\Gamma, X, Y) \mapsto \nabla_X Y\) is a natural operator that constructs from a linear connection \(\Gamma\) and vector fields \(X\) and \(Y\), a vector field \(\nabla_X Y\). In local coordinates,

\[
\nabla_X Y = (\Gamma^i_{jk} X^j Y^k + X^j Y^i_j) \frac{\partial}{\partial x^i},
\]

where \(\Gamma^i_{jk}\) are Christoffel symbols.

Natural operations can be composed into more complicated ones. Examples of ‘composed’ operations are the torsion \(T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]\) and the curvature \(R(X, Y)Z := \nabla_{[X,Y]} Z - [\nabla_X, \nabla_Y] Z\) of the linear connection \(\Gamma\).

1.3. Example. Let \(X\) be a vector field and \(\omega\) a 1-form on \(M\). Denote by \(\omega(X) \in C^\infty(M)\) the evaluation of the form \(\omega\) on \(X\). Then \((X, \omega) \mapsto \exp(\omega(X))\) defines a natural differential operator with values in smooth functions. Clearly, the exponential can be replaced by an arbitrary smooth function \(\varphi : \mathbb{R} \to \mathbb{R}\), giving rise to a natural operator \(\mathcal{O}_\varphi(X, \omega) := \varphi(\omega(X))\).

1.4. Example. ‘Randomly’ generated local formulas need not lead to natural operators. As we will see later, neither \(O_1(X, Y) = X^j Y^i \partial/\partial x^2\) nor \(O_2(X, Y) = X^j Y^i_j \partial/\partial x^i\) behaves properly under coordinate changes, so they do not give rise to vector-field valued natural operators.

We may summarize the above examples by saying that a natural differential operator is a recipe given locally as a smooth function in coordinates and their derivatives, such that the local formula is invariant under coordinate changes. After this motivation, we give precise definitions of geometric objects and operators between them. Our exposition follows [7], see also [3].

Denote by \(\text{Man}_n\) the category of \(n\)-dimensional manifolds and open embeddings. Let \(\text{Fib}_n\) be the category of smooth fiber bundles over \(n\)-dimensional manifolds with morphisms differentiable maps covering morphisms of their bases in \(\text{Man}_n\).

1.5. Definition. A natural bundle is a functor \(\mathcal{B} : \text{Man}_n \to \text{Fib}_n\) such that for each \(M \in \text{Man}_n\), \(\mathcal{B}(M)\) is a bundle over \(M\). Moreover, \(\mathcal{B}(M')\) is the restriction of \(\mathcal{B}(M)\) for each open submanifold \(M' \subset M\), the map \(\mathcal{B}(M') \to \mathcal{B}(M)\) induced by \(M' \hookrightarrow M\) being the inclusion \(\mathcal{B}(M') \hookrightarrow \mathcal{B}(M)\).

Let us recall a structure theorem for natural bundles due to Kruka, Palais and Terng [10, 23, 26]. For each \(s \geq 1\) we denote by \(\text{GL}^{(s)}_n\) the group of \(s\)-jets of local diffeomorphisms \(\mathbb{R}^n \to \mathbb{R}^n\) at 0, so that \(\text{GL}^{(1)}_n\) is the ordinary general linear group \(\text{GL}_n\) of linear invertible maps \(A : \mathbb{R}^n \to \mathbb{R}^n\). Let \(F^{(s)}_r(M)\) be the bundle of \(s\)-jets of frames on \(M\) whose fiber over \(z \in M\) consist of \(s\)-jets of local diffeomorphisms of neighborhoods of \(0 \in \mathbb{R}^n\) with neighborhoods of \(z \in M\). It is clear that \(F^{(s)}_r(M)\) is a principal \(\text{GL}^{(s)}_n\)-bundle and \(F^{(1)}_r(M)\) the ordinary \(\text{GL}_n\)-bundle of frames \(Fr(M)\).
1.6. **Theorem** (Krupka, Palais, Terng). For each natural bundle \( \mathcal{B} \), there exists \( l \geq 1 \) and a manifold \( \mathcal{B} \) with a smooth \( GL_n^{(l)} \)-action such that there is a functorial isomorphism

\[
\mathcal{B}(M) \cong Fr^{(l)}(M) \times_{GL_n^{(l)}} \mathcal{B} := (Fr^{(l)}(M) \times \mathcal{B})/GL_n^{(l)}.
\]

Conversely, each smooth \( GL_n^{(l)} \)-manifold \( \mathcal{B} \) induces, via (2), a natural bundle \( \mathcal{B} \). We will call \( \mathcal{B} \) the fiber of the natural bundle \( \mathcal{B} \). If the action of \( GL_n^{(l)} \) on \( \mathcal{B} \) does not reduce to an action of the quotient \( GL_n^{(l-1)} \) we say that \( \mathcal{B} \) has order \( l \).

1.7. **Example.** Vector fields are sections of the tangent bundle \( T(M) \). The fiber of this bundle is \( \mathbb{R}^n \), with the standard action of \( GL_n \). The description \( T(M) \cong Fr(M) \times_{GL_n} \mathbb{R}^n \) is classical.

1.8. **Example.** De Rham \( m \)-forms are sections of the bundle \( \Omega^m(M) \) whose fiber is the space of anti-symmetric \( m \)-linear maps \( \text{Lin}(\wedge^m(\mathbb{R}^n), \mathbb{R}) \), with the obvious induced \( GL_n \)-action. The presentation \( \Omega^m(M) \cong Fr(M) \times_{GL_n} \text{Lin}(\wedge^m(\mathbb{R}^n), \mathbb{R}) \) is also classical. A particular case is \( \Omega^0(M) \cong Fr(M) \times_{GL_n} \mathbb{R} \cong M \times \mathbb{R} \), the bundle whose sections are smooth functions. We will denote this natural bundle by \( \mathbb{R} \), believing there will be no confusion with the symbol for the reals.

1.9. **Example.** Linear connections are sections of the bundle of connections \( \text{Con}(M) \) [8, Section 17.7] which we recall below. Let us first describe the group \( GL_n^{(2)} \). Its elements are expressions of the form \( A = A_1 + A_2 \), where \( A_1 : \mathbb{R}^n \to \mathbb{R}^n \) is a linear invertible map and \( A_2 \) is a linear map from the symmetric product \( \mathbb{R}^n \circ \mathbb{R}^n \) to \( \mathbb{R}^n \). The multiplication in \( GL_n^{(2)} \) is given by

\[
(A_1 + A_2)(B_1 + B_2) := A_1(B_1) + A_1(B_2) + A_2(B_1, B_1).
\]

The unit of \( GL_n^{(2)} \) is \( id_{\mathbb{R}^n} + 0 \) and the inverse is given by the formula

\[
(A_1 + A_2)^{-1} = A_1^{-1} - A_1^{-1}(A_2(A_1^{-1}, A_1^{-1})).
\]

Let \( \mathcal{C} \) be the space of linear maps \( \text{Lin}(\mathbb{R}^n \circ \mathbb{R}^n, \mathbb{R}^n) \), with the left action of \( GL_n^{(2)} \) given as

\[
(Af)(u \otimes v) := A_1f(A_1^{-1}(u), A_1^{-1}(v)) - A_2(A_1^{-1}(u), A_1^{-1}(v)),
\]

for \( f \in \text{Lin}(\mathbb{R}^n \circ \mathbb{R}^n, \mathbb{R}^n) \), \( A = A_1 + A_2 \in GL_n^{(2)} \) and \( u, v \in \mathbb{R}^n \). The bundle of connections is then the order 2 natural bundle represented as \( \text{Con}(M) := Fr^{(2)}(M) \times_{GL_n^{(2)}} \mathcal{C} \). Observe that, while the action of \( GL_n^{(2)} \) on the vector space \( \mathcal{C} \) is not linear, the restricted action of \( GL_n \subset GL_n^{(2)} \) on \( \mathcal{C} \) is the standard action of the general linear group on the space of bilinear maps.

For \( k \geq 0 \) we denote by \( \mathcal{B}^{(k)} \) the bundle of \( k \)-jets of local sections of the natural bundle \( \mathcal{B} \) so that \( \mathcal{B}^{(0)} = \mathcal{B} \). If \( \mathcal{B} \) is represented as in (2), then \( \mathcal{B}^{(k)}(M) \cong Fr^{(k+1)}(M) \times_{GL_n^{(k+1)}} \mathcal{B}^{(k)} \), where \( \mathcal{B}^{(k)} \) is the space of \( k \)-jets of local diffeomorphisms \( \mathbb{R}^n \to \mathbb{R} \) defined in a neighborhood of \( 0 \in \mathbb{R}^n \).

1.10. **Definition.** Let \( \mathcal{F} \) and \( \mathcal{G} \) be natural bundles. A (finite order) **natural differential operator** \( \mathcal{D} : \mathcal{F} \to \mathcal{G} \) is a natural transformation (denoted by the same symbol) \( \mathcal{D} : \mathcal{F}^{(k)} \to \mathcal{G} \), for some \( k \geq 1 \). We denote the space of all natural differential operators \( \mathcal{F} \to \mathcal{G} \) by \( \text{Nat}(\mathcal{F}, \mathcal{G}) \).

If \( \mathcal{F} \) and \( \mathcal{G} \) are natural bundles of order \( \leq l \), with fibers \( \mathcal{F} \) and \( \mathcal{G} \), respectively, then each natural operator in Definition 1.10 is induced by an \( GL_n^{(k+l)} \)-equivariant map \( O : \mathcal{F}^{(k+l)} \to \mathcal{G} \), for some \( k \geq 0 \). Conversely, such an equivariant map induces an operator \( \mathcal{D} : \mathcal{F} \to \mathcal{G} \). This means
that the study of natural operators is reduced to the study of equivariant maps. The procedure described above is therefore called the IT reduction (from invariant-theoretic).

From this moment on, we impose the following assumptions on natural bundles $\mathcal{F}$, $\mathcal{G}$ an operators $\mathcal{O} : \mathcal{F} \to \mathcal{G}$ between them.

**A1** The fibers $\mathcal{F}$ and $\mathcal{G}$ of the bundles $\mathcal{F}$ and $\mathcal{G}$ are vector spaces and the restricted actions of $\text{GL}_n \subset \text{GL}_n^{(l)}$ on $\mathcal{F}$ and $\mathcal{G}$ are rational linear representations.

**A2** The action of $\text{GL}_n^{(l)}$ on the fiber $\mathcal{G}$ of $\mathcal{G}$ is linear, and

**A3** we consider only polynomial differential operators for which the induced map of the fibers $O : \mathcal{F}^{(k)} \to \mathcal{G}$ is a polynomial map.

Notice that we do not require the action of the full group $\text{GL}_n^{(l)}$ on the fiber of $\mathcal{F}$ to be linear. Assumption A2 is needed for the cohomology in Theorem 2.2 in Section 2 to be well-defined, assumptions A1 and A3 are necessary to relate this cohomology to a graph complex.

Polynomiality A3 rules out operators as $\mathcal{O}_\varphi$ from Example 1.3. There is probably no systematic way how to study operators of this type – imagine that $\varphi$ is an arbitrary, not even real analytic, smooth function. Clearly most if not all “natural” natural operators considered in differential geometry are polynomial, so assumption A3 seems to be justified. As argued in [8, Section 24] and as we will also see later in Remarks 1.2 and 1.3, in some situations the operators possess a certain homogeneity which automatically implies polynomiality.

1.11. **Example.** Given natural bundles $\mathcal{B}'$ and $\mathcal{B}''$ with fibers $\mathcal{B}'$, resp. $\mathcal{B}''$, there is an obviously defined natural bundle $\mathcal{B}' \times \mathcal{B}''$ with fiber $\mathcal{B}' \times \mathcal{B}''$. With this notation, the Lie bracket is a natural operator $[-,-] : T \times T \to T$ and the covariant derivative an operator $\nabla : \text{Con} \times T \times T \to T$, where $T$ is the tangent space functor and Con the bundle of connections recalled in Example 1.9. The corresponding equivariant maps of fibers can be easily read off from local formulas given in Examples 1.1 and 1.2.

1.12. **Example.** The operator $\mathcal{O}_\varphi : T \times \Omega^1 \to C^\infty$ from Example 1.3 is induced by the $\text{GL}_n$-equivariant map $O_\varphi : \mathbb{R}^n \times \mathbb{R}^{n*} \to \mathbb{R}$ given by $o_\varphi(v,\alpha) := \varphi(\alpha(v))$. Clearly, $\mathcal{O}_\varphi$ satisfies A3 if and only if $\varphi : \mathbb{R} \to \mathbb{R}$ is a polynomial.

2. **Natural operators and cohomology**

We start this section by a brief recollection of two classical constructions. For a Lie algebra $\mathfrak{h}$ and a $\mathfrak{h}$-module $W$, the Chevalley-Eilenberg cohomology $H^*(\mathfrak{h},W)$ of $\mathfrak{h}$ with coefficients in $W$ is the cohomology of the cochain complex $(C^*(\mathfrak{h},W),\delta_{CE})$ defined by $C^m(\mathfrak{h},W) := \text{Lin}(\wedge^m \mathfrak{h},W)$, $m \geq 0$, with $\delta_{CE}$ the sum $\delta_{CE} = \delta_1 + \delta_2$, where

\begin{align}
(4) \quad (\delta_1 f)(h_1,\ldots,h_{m+1}) &:= \sum_{1 \leq i \leq m+1} (-1)^{i+1} \cdot h_i f(h_1,\ldots,\hat{h}_i,\ldots,h_{m+1}) \quad \text{and} \\
(5) \quad (\delta_2 f)(h_1,\ldots,h_{m+1}) &:= \sum_{1 \leq i<j \leq m+1} (-1)^{i+j} \cdot f([h_i,h_j],h_1,\ldots,\hat{h}_i,\ldots,\hat{h}_j,\ldots,h_{m+1}),
\end{align}

for $f \in C^m(\mathfrak{h},W)$, $h_1,\ldots,h_{m+1} \in \mathfrak{h}$ and $\hat{\cdot}$ denoting the omission. If $m = 0$, the summation in the right hand side of (5) runs over the empty set, so we put $(\delta_2 f)(h) := 0$ for $f \in C^0(\mathfrak{h},W)$. 
The second notion we need to recall is the semidirect product of groups. Assume that $G$ and $H$ are Lie groups, with $G$ acting on $H$ by homomorphisms. One then defines the semidirect product $G \rtimes H$ as the Cartesian product $G \times H$ with the multiplication

$$(g_1, h_1)(g_2, h_2) := (g_1g_2, g_2^{-1}(h_1)h_2), \quad g_1, g_2 \in G, \ h_1, h_2 \in H.$$ 

Both $G$ and $H$ are subgroups of $G \rtimes H$ and their union $G \cup H$ generates $G \rtimes H$. Let us close this introductory part by formulating a proposition that ties the above two constructions together.

If $W$ is a left $G \rtimes H$-module, the inclusion $H \subset G \rtimes H$ induces a left $H$-action on $W$ which, in turn, induces an infinitesimal action of $\mathfrak{h}$ on $W$. One may therefore consider the cochain complex $(C^*(\mathfrak{h}, W), \delta_{CE})$. Since $G$ acts by homomorphisms, the unit of $H$ is $G$-fixed, so there is an induced action of $G$ on the Lie algebra $\mathfrak{h}$ of $H$. The group $G$ acts also on $W$, via the inclusion $G \subset G \rtimes H$. These two actions give rise, in the usual way, to an action of $G$ on $C^*(\mathfrak{h}, W)$. Let us denote $C^*_G(\mathfrak{h}, W)$ the subspace of $G$-fixed elements of $C^*(\mathfrak{h}, W)$. We have the following:

2.1. **Proposition.** The subspace of fixed elements $C^*_G(\mathfrak{h}, W) \subset C^*(\mathfrak{h}, W)$ is $\delta_{CE}$-closed, so the cohomology $H^*_G(\mathfrak{h}, W) := H^*(C^*_G(\mathfrak{h}, W), \delta_{CE})$ is defined. For $H$ connected, there is an isomorphism

$$(6) \quad H^0_G(\mathfrak{h}, W) \cong W^G \rtimes H,$$

where $W^G \rtimes H$ denotes, as usual, the space of $G \rtimes H$-fixed elements in $W$.

**Proof.** We leave a direct verification of the $\delta_{CE}$-closeness of $C^*_G(\mathfrak{h}, W)$ as a simple exercise to the reader. It is equally easy to see that $H^0_G(\mathfrak{h}, W)$ consists of elements of $W$ which are simultaneously $G$-fixed and $\mathfrak{h}$-invariant. If $H$ is connected, the exponential map is an epimorphism, thus $\mathfrak{h}$-invariant elements in $W$ are precisely those which are $H$-fixed. This, along with the fact that $G \cup H$ generates $G \rtimes H$, gives (6). \qed

In Section 1 we recalled that natural differential operators $\mathcal{O} \in \mathfrak{Nat}(\mathcal{F}, \mathcal{G})$ between natural bundles of order $\leq l$ with fibers $\mathcal{F}$ resp. $\mathcal{G}$, correspond to $GL_n^{(k+l)}$-equivariant maps $O : \mathcal{F}^{(k)} \to \mathcal{G}$ with some $k \geq 0$. This can be expressed by the isomorphism:

$$(7) \quad \mathfrak{Nat}(\mathcal{F}, \mathcal{G}) \cong \bigcup_{k \geq 0} \text{Map}_{GL_n^{(k+l)}}(\mathcal{F}^{(k)}, \mathcal{G}),$$

where $\text{Map}_{GL_n^{(k+l)}}(\mathcal{F}^{(k)}, \mathcal{G})$ is the space of polynomial $GL_n^{(k+l)}$-equivariant maps $\mathcal{F}^{(k)} \to \mathcal{G}$ – see assumption A3 on page 3. The space $\text{Map}(\mathcal{F}^{(k)}, \mathcal{G})$ of all polynomial maps has the standard $GL_n^{(k+l)}$-action induced from the actions on $\mathcal{F}^{(k)}$ and $\mathcal{G}$.

The space of equivariant maps is the fixed subspace $\text{Map}_{GL_n^{(k+l)}}(\mathcal{F}^{(k)}, \mathcal{G}) = \text{Map}(\mathcal{F}^{(k)}, \mathcal{G})^{GL_n^{(k+l)}}$. Let us see how Proposition 2.1 describes these spaces. The crucial observation is that $GL_n^{(s)}$ is, for each $s \geq 1$, a semidirect product [8, Section 13]. If $(\mathbb{R}^n)^{\otimes r}$ denotes the $r$th symmetric power of $\mathbb{R}^n$, $r \geq 1$, then elements of $GL_n^{(s)}$ are expressions $A = A_1 + A_2 + A_3 + \cdots + A_s$, $A_i \in \text{Lin}((\mathbb{R}^n)^{\otimes i}, \mathbb{R}^n)$, $1 \leq i \leq s$, such that $A_1 : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. It is a simple exercise to write formulas for the product and inverse; for $s = 2$ it was done in Example 1.3.

The space $\text{Lin}((\mathbb{R}^n)^{\otimes i}, \mathbb{R}^n)$ is canonically isomorphic to the space $\text{Sym}((\mathbb{R}^n)^{\otimes i}, \mathbb{R}^n)$ of symmetric multilinear maps and we will identify these two spaces in the sequel. Denote by $\text{NGL}_n^{(s)} = \{A = A_1 + A_2 + A_3 + \cdots + A_s \in \text{GL}_n^{(s)}; \ A_1 = \text{id}\}$ the pronipotent radical of $\text{GL}_n^{(s)}$. Then $\text{GL}_n^{(s)}$
is the semidirect product $GL_n^{(s)} = GL_n \rtimes NGL_n^{(s)}$, with $GL_n$ acting on $NGL_n^{(s)}$ by adjunction. Denote finally $ngl_n^{(s)}$ the Lie algebra of $NGL_n^{(s)}$,

\begin{equation}
ngl_n^{(s)} = \{ a = a_2 + a_3 + \cdots + a_s; \ a_i \in Sym \left( (\mathbb{R}^n)^{\otimes i}, \mathbb{R}^n \right), \ 2 \leq i \leq s \}.
\end{equation}

Assume that the action of $GL_n^{(l)}$ on the fiber $\mathcal{G}$ of $\mathcal{B}$ is linear. Then $\text{Map}(\mathcal{F}^{(k)}, \mathcal{G})$ is a linear representation of $GL_n^{(k+1)}$ and Proposition 2.1 applied to $G = GL_n$, $H = NGL_n^{(k+1)}$ and $W = \text{Map}(\mathcal{F}^{(k)}, \mathcal{G})$ gives

\begin{equation}
\text{Map}_{GL_n^{(k+1)}}(\mathcal{F}^{(k)}, \mathcal{G}) \cong H^0_{GL_n}(ngl_n^{(k+1)}, \text{Map}(\mathcal{F}^{(k)}, \mathcal{G})).
\end{equation}

For each $k \geq 0$, the inclusion $\text{Map}(\mathcal{F}^{(k)}, \mathcal{G}) \hookrightarrow \text{Map}(\mathcal{F}^{(k+1)}, \mathcal{G})$ together with the projection $ngl_n^{(k+1)} \rightarrow ngl_n^{(k+1)}$ induces a $GL_n$-invariant inclusion

\begin{equation}
C^*(ngl_n^{(k+1)}, \text{Map}(\mathcal{F}^{(k)}, \mathcal{G})) \hookrightarrow C^*(ngl_n^{(k+1)}, \text{Map}(\mathcal{F}^{(k+1)}, \mathcal{G}))
\end{equation}

which commutes with the differentials. Let us denote

\begin{equation}
C^*(ngl_n^{(\infty)}, \text{Map}(\mathcal{F}^{(\infty)}, \mathcal{G})) := \bigcup_{k \geq 0} \bigcup_{l \geq 1} C^*(ngl_n^{(k+1)}, \text{Map}(\mathcal{F}^{(k)}, \mathcal{G}))
\end{equation}

and $C_{GL_n}(ngl_n^{(\infty)}, \text{Map}(\mathcal{F}^{(\infty)}, \mathcal{G}))$ the $GL_n$-stable subspace of $C^*(ngl_n^{(\infty)}, \text{Map}(\mathcal{F}^{(\infty)}, \mathcal{G}))$. Let

\begin{equation}
H^*_{GL_n}(ngl_n^{(\infty)}, \text{Map}(\mathcal{F}^{(\infty)}, \mathcal{G})) := H^*(C^*_{GL_n}(ngl_n^{(\infty)}, \text{Map}(\mathcal{F}^{(\infty)}, \mathcal{G})), \delta_{CE}).
\end{equation}

Then (7) together with (3) and the fact that cohomology commutes with direct limits implies:

2.2. Theorem. Let $\mathcal{F}$ and $\mathcal{B}$ be natural bundles with fibers $\mathcal{F}$ resp. $\mathcal{G}$ of orders $\leq l$. Suppose that the action of $GL_n^{(l)}$ on $\mathcal{G}$ is linear. Then, under the above notation

\begin{equation}
\text{Nat}(\mathcal{F}, \mathcal{G}) \cong H^0_{GL_n}(ngl_n^{(\infty)}, \text{Map}(\mathcal{F}^{(\infty)}, \mathcal{G})).
\end{equation}

In the following sections we show that, in many interesting cases, the cohomology in the right hand side of (11) is the cohomology of a certain graph complex.

3. Natural operators and graphs

We are going to describe natural differential operators by certain spaces spanned by graphs. Roughly speaking, graphs, viewed as contraction schemes for indices, will encode elementary $GL_n$-invariant tensors in (10). Our approach is based on a translation of the Invariant Tensor Theorem into the graph language explained in (11).

Suppose that $\mathcal{B}$ is a natural bundle satisfying A1 on page 3, so that the induced action of $GL_n \subset GL_n^{(l)}$ on the fiber $\mathcal{B}$ is rational linear. According to standard facts of the representation theory of $GL_n$ recalled, for instance, in [7, § 1.4], an equivalent assumptions is that, as a $GL_n$-module, $\mathcal{B}$ is the direct sum of $GL_n$-modules

\begin{equation}
\mathcal{B} = \bigoplus_{i \leq j \leq b} \mathcal{B}_i,
\end{equation}

\begin{equation}
\mathcal{B}_i
\end{equation}
where $\mathcal{B}_i$ is, for each $1 \leq i \leq b$, either the space $\text{Lin}(\mathbb{R}^{n \otimes q_i}, \mathbb{R}^{n \otimes p_i})$ for some $p_i, q_i \geq 0$, with the standard $\text{GL}_n$-action, or a subspace of this space consisting of maps whose inputs and/or outputs have a specific symmetry, which can for example be expressed by a Young diagram.

In other words, $\mathcal{B}_i$ are spaces of multilinear maps whose coordinates are tensors $T^{a_1,\ldots,a_p}_{b_1,\ldots,b_q}_i$ with $q_i$ input indices and $p_i$ output indices, which may or may not enjoy some kind of symmetry. We will graphically represent these tensors as corollas with $q_i$-inputs and $p_i$ outputs:

\[
\begin{array}{c}
\text{inputs} \quad \text{outputs} \\
\end{array}
\]  

Instead of $\bullet$ we may sometime use different symbols for the node, such as $\nabla$, $\blacksquare$, $\circ$, &c.

### 3.1. Example.

The fiber of the tangent bundle $T$ is $\mathbb{R}^n = \text{Lin}(\mathbb{R}^{n \otimes 0}, \mathbb{R}^{n \otimes 1})$, so one has in (12) $b = 1$, $p_1 = 1$, $q_0 = 0$. Elements of the fiber are tensors $X^a$ symbolized by $\bullet$. The fiber $\mathcal{C}$ of the connection bundle $\text{Con}$ (see Example 1.9) is $\text{Lin}(\mathbb{R}^{n \otimes 2}, \mathbb{R}^{n \otimes 1})$, therefore $b = 1$, $p_1 = 1$ and $q_1 = 2$. Elements of $\mathcal{C}$ are $\text{GL}_n$-tensors (Christoffel symbols) $\Gamma^a_{bc}$ represented by

\[
\begin{array}{c}
\text{inputs} \\
\end{array}
\]

An example with an anti-symmetry is the bundle $\Omega^n$ of de Rham $m$-forms, $m \geq 0$. Its fiber is the space $\text{Lin}(\bigwedge^m \mathbb{R}^n, \mathbb{R}^{n \otimes 0}) = \text{Lin}(\bigwedge^m \mathbb{R}^n, \mathbb{R})$ of anti-symmetric tensors $\omega_{b_1,\ldots,b_m}$.

Sometimes we will need decorations of nodes. For example, the product bundle $T \times T$ has fiber $\mathbb{R}^n \times \mathbb{R}^n$ generated by tensors $X^a, Y^a$ which will be denoted

\[
\begin{array}{c}
\downarrow 1 \quad \downarrow 2 \quad \text{or} \quad \uparrow X \quad \uparrow Y \\
\end{array}
\]

Let $\mathcal{B}$ be a natural bundle with fiber $\mathcal{B}$ decomposed as in (12). It is easy to see that the fiber $\mathcal{B}^{(k)}$ of the $k$-jet bundle $\mathcal{B}^{(k)}$ decomposes, as a $\text{GL}_n$-module, into $\mathcal{B}^{(k)} = \bigoplus_{1 \leq i \leq b} \mathcal{B}_i^{(k)}$, where

\[
\mathcal{B}_i^{(k)} = \bigoplus_{0 \leq v \leq k} \text{Sym}(\mathbb{R}^{n \otimes v}, \mathbb{R}) \otimes \mathcal{B}_i.
\]

This means that if elements of $\mathcal{B}_i$ are tensors $T^{a_1,\ldots,a_p}_{b_1,\ldots,b_q}_i$, elements of $\mathcal{B}^{(k)}_i$ are tensors $(s_1,\ldots,s_v)T^{a_1,\ldots,a_p}_{b_1,\ldots,b_q}_i$, $v \leq k$, with braces indicating the symmetry in $(s_1,\ldots,s_v)$. In terms of pictures this amounts to adding new symmetric inputs to corollas (13), so elements of $\mathcal{B}^{(k)}_i$ will be symbolized by

\[
\begin{array}{c}
\text{inputs} \quad \text{outputs} \\
\end{array}
\]

### 3.2. Example.

The fiber of the $k$th tangent bundle $T^{(k)}$ is the space of tensors

\[
X^a_{(s_1,\ldots,s_v)} := \frac{\partial^u X^a}{\partial x^{s_1} \cdots \partial x^{s_v}}, \quad v \leq k,
\]

\[
\begin{array}{c}
\text{inputs} \quad \text{outputs} \\
\end{array}
\]
which we draw as

\[(17) \quad \text{, } v \leq k.\]

The fiber of the bundle \(\text{Con}^{(k)}\) is the space of tensors \((s_1, \ldots, s_u)\Gamma^a_{bc} \coloneqq \frac{\partial^a}{\partial x^{s_1} \cdots \partial x^{s_u}}, v \leq k\), depicted as

\[(18) \quad v \text{ inputs } \quad , v \leq k.\]

As follows from (19), \(\text{ngl}^{(k+1)}_n = \bigoplus_{2 \leq u \leq k+l} \text{Sym}(\mathbb{R}^n \otimes^u, \mathbb{R}^n)\). Therefore \(\text{ngl}^{(k+1)}_n\) is the space of symmetric tensors \(\phi^b_{(s_1, \ldots, s_u)}, 2 \leq u \leq k + l\), or in pictures,

\[(19) \quad u \text{ inputs } \quad , 2 \leq u \leq k + l.\]

In what follows, white corollas (19) will always denote elements of \(\text{ngl}^{(k+1)}_n\) for some \(k + l \geq 2\).

In the rest of this section we construct a graded space \(\mathcal{G}^*_\mathcal{F}, \mathcal{G}\) spanned by graphs representing \(\text{GL}_n\)-invariant cochains in \(C^*_{\text{GL}_n}(\text{ngl}^{(\infty)}_n, \text{Map}(\mathcal{F}^{(\infty)}, \mathcal{G}))\). The differentials will be studied in the next section.

Suppose that the natural bundles \(\mathcal{F}\) and \(\mathcal{G}\) satisfy assumption A1 on page 3, and see what can be said about the space \(C^m_{\text{GL}_n}(\text{ngl}^{(k+1)}_n, \text{Map}(\mathcal{F}^{(k)}, \mathcal{G}))\) of \(\text{GL}_n\)-equivariant polynomial maps from \(\text{Map}(\text{ngl}^{(k+1)}_n \times \mathcal{F}^{(k)}, \mathcal{G})\) that are \(m\)-homogeneous and antisymmetric in \(\text{ngl}^{(k+1)}_n\). By the polynomiality assumption A3,

\[(20) \quad C^m_{\text{GL}_n}(\text{ngl}^{(k+1)}_n, \text{Map}(\mathcal{F}^{(k)}, \mathcal{G})) \cong \bigoplus_{t \geq 0} \text{Lin}_{\text{GL}_n} \left( \bigwedge^m \text{ngl}^{(k+1)}_n \otimes \mathcal{F}^{(k)\otimes t}, \mathcal{G} \right),\]

where \(\text{Lin}_{\text{GL}_n}(\cdot, \cdot)\) denotes the space of \(\text{GL}_n\)-equivariant linear maps.

Let us decompose the fibers \(\mathcal{F}\) and \(\mathcal{G}\) of natural bundles \(\mathcal{F}\) and \(\mathcal{G}\) into the direct sum (12),

\(\mathcal{F} = \bigoplus_{1 \leq i \leq f} \mathcal{F}_i\) and \(\mathcal{G} = \bigoplus_{1 \leq i \leq g} \mathcal{G}_i\). By (14), the components of the fiber \(\mathcal{F}^{(k)}\) of the \(k\)-jet bundle \(\mathcal{F}^{(k)}, k \geq 0\), are the direct sums \(\mathcal{F}_i^{(k)} = \bigoplus_{0 \leq v \leq k} \mathcal{F}_i^{[v]}, 1 \leq i \leq f\), with \(\mathcal{F}_i^{[v]} \coloneqq \text{Sym}(\mathbb{R}^n \otimes^v, \mathbb{R}^n) \otimes \mathcal{F}_i\).

Using the above decompositions and description (13) of \(\text{ngl}^{(k+1)}_n\), one can rewrite the right hand side of (20) into

\[(21) \quad \bigoplus_{l \geq 0} \bigoplus_{S(k,l,t)} \text{Lin}_{\text{GL}_n} \left( \bigwedge_{1 \leq i \leq m} \text{Sym}(\mathbb{R}^n \otimes^{u_i}, \mathbb{R}^n) \otimes \bigotimes_{1 \leq s \leq t} \mathcal{G}_i^{[v_{s,i}]}, \mathcal{G}_i \right),\]

where \(S(k,l,t)\) is the set of integers \(u_1, \ldots, u_m, i_1, \ldots, i_t, v_1, \ldots, v_t\) and \(i\) such that

\[2 \leq u_1, \ldots, u_m \leq k + l, \quad 1 \leq i_1, \ldots, i_t \leq f, \quad 0 \leq v_1, \ldots, v_t \leq k \quad \text{and} \quad 1 \leq i \leq g.\]

Let us fix a multiindex \(\omega = (u_1, \ldots, u_m, i_1, \ldots, i_t, v_1, \ldots, v_t, i) \in S(k, l, t)\). By our assumptions, the space \(\mathcal{F}_i^{[v_{s,i}]}\) is, for each \(1 \leq s \leq t\), isomorphic to the space

\(\text{Lin}^{\mathcal{D}_{\mathcal{G}_i}}_{\mathcal{J}_s}(\mathbb{R}^n \otimes^{u_i + q_{i,s}}, \mathbb{R}^n \otimes^{p_{i,s}}) \coloneqq \{ f \in \text{Lin}(\mathbb{R}^n \otimes^{u_i + q_{i,s}}, \mathbb{R}^n \otimes^{p_{i,s}}); \ f s = 0 = tf \text{ for } s \in \mathcal{J}_s, t \in \mathcal{D}_{\mathcal{G}_i}\}.\)
of linear maps having a symmetry specified by subsets $\mathcal{I} \subset k[\Sigma_{v_s + q_{i_s}}], \mathcal{O} \subset k[\Sigma_{p_{i_s}}]$, see also [11, Remark 4.4]. Similarly, $G_i \simeq \text{Lin}^G_{\mathcal{I}}(\mathbb{R}^{n \otimes c}, \mathbb{R}^{n \otimes d})$, for some $c, d \geq 0$ and subsets $\mathcal{I} \subset k[\Sigma_{c}], \mathcal{O} \subset k[\Sigma_{d}]$. The expression

$$
\text{Lin}_{\text{GL}_n} \left( \bigwedge_{1 \leq i \leq m} \text{Sym}(\mathbb{R}^{n \otimes u_i}, \mathbb{R}^n) \otimes \bigotimes_{1 \leq s \leq t} \mathcal{G}_{i_s}^{[v_s]}, G_i \right)
$$

in (21) is therefore isomorphic to

$$
\text{Lin}_{\text{GL}_n} \left( \bigwedge_{1 \leq i \leq m} \text{Sym}(\mathbb{R}^{n \otimes u_i}, \mathbb{R}^n) \otimes \bigotimes_{1 \leq s \leq t} \text{Lin}^G_{\mathcal{J}}(\mathbb{R}^{n \otimes (v_s + q_{i_s})}, \mathbb{R}^{n \otimes p_{i_s}}), \text{Lin}^G_{\mathcal{O}}(\mathbb{R}^{n \otimes c}, \mathbb{R}^{n \otimes d}) \right).
$$

Let us remark that in all applications discussed in this paper, we will always have $p_{i_s} = 1$ for $1 \leq s \leq t$, $c = 0$ and $d = 1$.

Observe that (22) is the space in (24) of [11], with an appropriate choice of the parameters, which in this case is $r := t - m$, and

$$
h_i := u_i \quad \text{for} \quad 1 \leq i \leq m, \quad \text{and} \quad h_i := v_s + q_{i_s}, \quad \mathcal{J}_i := \mathcal{J}_s, \quad \mathcal{O}_i := \mathcal{O}_s \quad \text{for} \quad i = s + m, \quad 1 \leq s \leq t,
$$

therefore the methods developed in [11] apply. We believe that the reader can tolerate a certain incompatibility between the notation used in this paper and the notation of [11] – the alphabet does not have enough letters to avoid notational conflicts.

By Proposition 4.8 and Remark 4.10 of [11], the space (23) is related to the space $\mathcal{G}_{\omega}^m$ spanned by graphs with vertices of three types:

1. **1st type:** $t$ ‘black’ vertices (15) with $p_i := p_{i_s}, q_i := q_{i_s}$ and $v := v_s$, representing tensors in $\mathcal{G}_{j_s}^{[v_s]}, 1 \leq s \leq t$,

2. **2nd type:** one vertex (13) with $p_i := c$ and $q_i := d$ called the anchor, representing tensors in the dual $\mathcal{G}_i^*$ of $G_i$, and

3. **3rd type:** $m$ ‘white’ vertices (19) with $u := u_i$ representing generators of the Lie algebra $\mathfrak{gl}_n^{(k+t)}, 1 \leq i \leq m$.

Our graphs are directed and oriented, where an orientation is, by definition, an equivalence class of linear orders of the set of white vertices, modulo the relation identifying orders that differ by an even number of transpositions. If the orientations of two graphs $G'$ and $G''$ differ by an odd number of transpositions, we put $G' = -G''$ in $\mathcal{G}_{\omega}^m$. This notion of orientation is not the traditional one but resembles orientations in various graph complexes [16, § II.5.5].

The graphs spanning $\mathcal{G}_{\omega}^m$ are not required to be connected, and multiple edges and loops are allowed. The vertices above are Merkulov’s genes [21]. The unique vertex of the 2nd type marks the place where we evaluate the composition along the graph at an element of $\mathcal{G}^*$, which explains the dualization in the definition of this vertex.
Proposition 4.8 of [11] (or its obvious extension mentioned in [11] Remark 4.10), combined
with the isomorphism between (22) and (23), gives an epimorphism

\[ R_{n,\omega}^m : \mathcal{G}_\omega^m \twoheadrightarrow \text{Lin}_{GL_n} \left( \bigwedge_{1 \leq i \leq m} \text{Sym}(\mathbb{R}^{n \otimes u_i}, \mathbb{R}^n) \otimes \bigotimes_{1 \leq s \leq t} \mathcal{F}_{ls}^{[\tau_i]}, \mathcal{G}_t \right) \]

which is, by [11] Proposition 4.9, a monomorphism if \( n + m \geq \) the number of edges of graphs in \( \mathcal{G}_\omega^m \). The central result of this section, Theorem 3.3 below, uses the limit

\[ \mathcal{G}_{\omega,\mathcal{G}}^m := \bigcup_{k \geq 0} \bigcup_{t \geq 0} \bigoplus_{\omega \in S(k,l,t)} \mathcal{G}_\omega^m. \]

The space \( \mathcal{G}_{\omega,\mathcal{G}}^m \) is spanned by graphs with an arbitrary number of the 1st type vertices with an arbitrary \( v \geq 0 \) in (17), one 2nd type vertex representing tensors in \( \mathcal{G}_i^1 \) for \( 1 \leq i \leq g \), and \( m \) 3rd type vertices with an arbitrary \( u \geq 2 \) in (19).

3.3. Theorem. The epimorphisms \( R_{n,\omega}^m \) in (24) assemble, for each \( m \geq 0 \), into a surjection

\[ R_n : \mathcal{G}_{\omega,\mathcal{G}}^m \twoheadrightarrow C_{GL_n}^m \left( \mathfrak{gl}_{m}^{(\infty)}, \text{Map}(\mathcal{F}^{(\infty)}), \mathcal{G} \right). \]

The restriction

\[ R_n^m(e) : \mathcal{G}_{\omega,\mathcal{G}}^m(e) \twoheadrightarrow C_{GL_n}^m \left( \mathfrak{gl}_{m}^{(\infty)}, \text{Map}(\mathcal{F}^{(\infty)}), \mathcal{G} \right) \]

of the map \( R_n^m \) to the subspace \( \mathcal{G}_{\omega,\mathcal{G}}^m(e) \subset \mathcal{G}_{\omega,\mathcal{G}}^m \) spanned by graphs with \( \leq e \) edges, is a monomorphism whenever \( n = \dim(M) \geq e - m \).

Proof. The maps \( R_{n,\omega}^{m} \) of (22) assemble, for each \( k \geq 0 \) and \( l \geq 1 \), into an epimorphism

\[ R_{n,k,l}^m := \bigoplus_{t \geq 0} \bigoplus_{\omega \in S(k,l,t)} R_{n,\omega}^m : \bigoplus_{t \geq 0} \bigoplus_{\omega \in S(k,l,t)} R_{n,\omega}^m \twoheadrightarrow \bigoplus_{t \geq 0} \text{Lin}_{GL_n} \left( \bigwedge_{m} \mathfrak{gl}_{n}^{(k+l)} \otimes \mathcal{F}^{(k)\otimes t}, \mathcal{G} \right). \]

Recalling (10), (22), and the definition (23) of the graph complex \( \mathcal{G}_{\omega,\mathcal{G}}^m \), we conclude that \( R_n^m := \bigoplus_{k \geq 0} \bigoplus_{l \geq 1} R_{n,k,l}^m \) is the desired surjection (26). The second part of the theorem follows from [11], Proposition 4.9] applied to the constituents \( R_{n,\omega}^{m} \) of \( R_n^m \).

3.4. Example. Let us discuss the case \( \mathcal{G} = T \times T \) and \( \mathcal{G} = T \), where \( T \) is the tangent bundle functor. Graphs spanning the vector space \( \mathcal{G}_{T \times T,T} \) have finite number of the 1st type vertices (17)

marking the places where to insert tensors \( X^a_{(s_1,\ldots,s_v)} \) and \( Y^a_{(s_1,\ldots,s_v)} \) of the fiber of \( (T \times T)^{(\infty)} \). The unique vertex \( \bullet \) of the 2nd type is the place to insert a tensor of the fiber \( \mathbb{R}^{n \otimes s} \) of \( T^* \). There of course will also be \( m \) vertices (19) of the 3rd type for generators of \( \mathfrak{gl}_{m}^{(\infty)} \).

Observe that we omitted braces indicating the symmetry because inputs of all vertices are symmetric and no confusion may occur. Let us inspect how \( \mathcal{G}^0_{T \times T,T} \) describes \( GL_n \)-equivariant maps in \( \text{Map}_{GL_n}(\text{Map}(\mathbb{R} \times \mathbb{R}^{n}^{(\infty)}, \mathbb{R}^{n})) = C_{GL_n}^0(\mathfrak{gl}_{m}^{(\infty)}, \text{Map}(\mathbb{R} \times \mathbb{R}^{n}^{(\infty)}, \mathbb{R}^{n})). \) The graph
describes the equivariant map that sends an element \((X^a, X_b^a, Y^a, Y_b^a) \in (\mathbb{R}^n \times \mathbb{R}^n)^{(1)}\) into the element \((X^j Y^a) \in \mathbb{R}^n\). It is precisely the map \(O_2\) considered in Example 1.4. The linear combination

\[
\ast \bullet Y \ast \bullet X - \ast \bullet X \ast \bullet Y
\]

represents the local formula \((X^a, X_b^a, Y^a, Y_b^a) \mapsto (X^j Y^a - Y^j X^a)\) for the Lie bracket \([X, Y]\) of two vector fields. We allow also graphs as

\[
\bullet Y \bullet X ,
\]

which represents the map \((X^a, X_b^a, Y^a, Y_b^a) \mapsto (X^a Y^i)\) involving the trace \(Y^i\) of \(Y\). An example of a degree 1 cochain in \(C^1_{GL_n}(\text{ngl}_n^{(2)}, \text{Map}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n))\) is provided by

\[
\ast \bullet X \bullet Y
\]

which defines the \(GL_n\)-equivariant 1-cochain \((\varphi_{bc}^a, X^a, Y^a) \mapsto (\varphi_{ij}^a X^i Y^j)\).

As explained in [11, Remark 5.2], for degrees \(\geq 2\) our interpretation of graphs involves the antisymmetrization in white vertices. For instance, the graph

\[
\begin{array}{c}
1 \\
2 \\
\bullet X \\
\bullet Y
\end{array}
\]

represents the 2-cochain \((\varphi_{bc}^a, \psi_{bc}^a, X^a, Y^a) \mapsto (\varphi_{ij}^a \psi_{il}^j - \psi_{ij}^a \varphi_{il}^j) X^k Y^l\). The reason why the expected traditional \(\frac{1}{2!}\)-factor is missing is explained in Remark 4.5.

3.5. Example. In this example we express local formulas for the covariant derivative, torsion and curvature in terms of graphs. The covariant derivative is the operator \(\nabla : \text{Con} \times T^{\times 2} \to T\) locally given by the graph

\[
(27)\quad \nabla_{XY}:
\]

which is a graphical form of formula (1). The torsion \(T : \text{Con} \times T^{\times 2} \to T\) is given by

\[
T(X,Y):
\]

and the curvature \(R : \text{Con} \times T^{\times 3} \to T\) as

\[
R(X,Y)Z :
\]
3.6. Example. This example shows that the map $R^n_m$ from Theorem 3.3 need not be a monomorphism below the ‘stable range.’ Consider again the two graphs from Example 3.4:

\[ G_1 := \begin{array}{c}
\bullet \\
X
\end{array} \text{and } G_2 := \begin{array}{c}
\triangleleft X \\
\bullet
\end{array}. \]

The number of edges of both graphs is 2. As we already saw, $G_1$ represents the local formula $\sum_{1 \leq i,j \leq n} X^i \partial Y^j / \partial x^j \partial / \partial x^i$ and $G_2$ the formula $\sum_{1 \leq i,j \leq n} \partial Y^j / \partial x^j X^i \partial / \partial x^i$. For $n = 1$ both formulas give the same result, namely $X \partial Y / \partial x \partial / \partial x$, therefore $R^n_1(G_1) = R^n_1(G_2)$. For $n \geq 2$ one clearly has $R^n_1(G_1) \neq R^n_1(G_2)$.

4. The differential

In this section we express the restriction of the Chevalley-Eilenberg differential onto the subcomplex $C^*_\text{GL}_n(\text{ngl}_n^{(\infty)}; \text{Map}(\mathbb{F}^{(\infty)}, \mathcal{G}))$ of $\text{GL}_n$-equivariant cochains in terms of graph complexes. Let us describe first the bracket in the limit $\text{ngl}_n^{(\infty)} = \bigcup_{s \geq 2} \text{ngl}_n^{(s)}$ of Lie algebras $\text{ngl}_n^{(s)}$ recalled in [3]. If finite sums $a = a_2 + a_3 + a_4 + \cdots$ and $b = b_2 + b_3 + b_4 + \cdots$ are elements of $\text{ngl}_n^{(\infty)}$, $a_u, b_u \in \text{Sym}(\mathbb{R}^n)^{\otimes u}$, $u \geq 2$, then $[a, b] = [a, b]_3 + [a, b]_4 + \cdots$ (no quadratic term) with

\[ [a, b]_u = \sum_{s+t = u+1} \sum_{1 \leq i \leq s} \left( S(a_s \circ_i b_t) - S(b_s \circ_i a_t) \right), \]

where $S(\cdot)$ denotes the symmetrization (see Remark 4.3) of a linear map $\mathbb{R}^{n \otimes u} \rightarrow \mathbb{R}^n$, $a_s \circ_i b_t$ is the insertion of $b_t$ into the $i$th slot of $a_s$ and $b_s \circ_i a_t$ has the similar obvious meaning. For $v_1, \ldots, v_u \in \mathbb{R}^n$ we easily get

\[ [a, b]_u(v_1, \ldots, v_u) = \sum_{s+t = u+1} \frac{s!t!}{u!} \sum_{\sigma} \left\{ a_s(b_t(v_{\sigma(1)}, \ldots, v_{\sigma(t)}), v_{\sigma(t+1)}, \ldots, v_{\sigma(u)}) - b_s(a_t(v_{\sigma(1)}, \ldots, v_{\sigma(t)}), v_{\sigma(t+1)}, \ldots, v_{\sigma(u)}) \right\}, \]

where $\sigma$ runs over all $(t, s-1)$-unshuffles $\sigma$, i.e. permutations $\sigma \in \Sigma_u$ such that $\sigma(1) < \ldots < \sigma(t)$, $\sigma(t+1) < \ldots < \sigma(u)$.

4.1. Remark. In the rest of the paper, we will consider $\text{ngl}_n^{(\infty)}$ with the modified Lie bracket, given by formula (28) without the $\frac{s!t!}{u!}$ -coefficients. Since this modified Lie algebra is isomorphic to the original one, via the isomorphism $a_s \mapsto s! \cdot a_s$, for $a_s \in \text{Sym}(\mathbb{R}^n)^{\otimes s}$, $s \geq 2$, our modification is purely conventional. The advantage of this modified bracket is that the corresponding replacement rule (29) is a linear combination of graphs without fractional coefficients.

To help the reader to appreciate the idea of the differential, we start with an informal definition. A precise formula including signs and orientations is given in (32). At the beginning of Section 4 we decomposed the CE-differential into the sum $\delta_{CE} = \delta_1 + \delta_2$. Let us analyze the action of the second piece $\delta_2$ first. A graph $G$ representing a $\text{GL}_m$-invariant $m$-cochain has $m$ white vertices that mark the places where to insert elements of $\text{ngl}_n^{(\infty)}$. Let us label, for $m \geq 1$, these white vertices by $\ell \in \{1, \ldots, m\}$ and denote the vertex labelled $\ell$ by $w_\ell$. If $m = 0$, there are no white vertices and no labelling is necessary.
The effect of the differential $\delta_2$ on the graph $G$ is, by the definition recalled in (3), the following. For each $\ell \in \{1, \ldots, m\}$ insert to the vertex $w_\ell$ the element $[h_i, h_j]$ and to the remaining white vertices elements $h_1, \ldots, \hat{h}_i, \ldots, \hat{h}_j, \ldots, h_{m+1}$, make the summation over all $1 \leq i < j \leq m+1$ and antisymmetrize in $h_1, \ldots, h_{m+1}$. Denote the resulting $(m+1)$-cochain by $G_\ell$. Then $\delta_2(G) = \varepsilon_1 \cdot G_1 + \cdots + \varepsilon_1 \cdot G_m$, where $\varepsilon_1, \ldots, \varepsilon_m \in \{-1, +1\}$ are appropriate signs. A moment’s reflection reveals that $G_\ell$ is obtained by replacing the vertex $w_\ell$ by:

\[
\begin{array}{c}
\text{u inputs} \\
\begin{array}{c}
\text{t+s=u+1} \\
\text{t}
\end{array}
\end{array}
\]

\[
\sum \left( \begin{array}{c}
\cdots, s \\
\text{ush}
\end{array} \right),
\]

where the braces $(-)_{\text{ush}}$ indicate that the summation over all $(t, s-1)$-unshuffles of the inputs has been performed. This is precisely the formula for the generators of the homological vector field introduced by Merkulov [19, 21]. One also recognizes (29) as the graphical representation of the axioms of $L_\infty$-algebras as given in [12, page 160].

A similar analysis shows that $\delta_1$ acts by replacing each vertex of type 1 or 2 by the pictorial representation of the action of $\mathfrak{gl}_{n}^{(\infty)}$ on tensors corresponding to this vertex. We will show instances of these ‘pictorial presentations’ in the following two examples.

4.2. Example. Consider a symmetric map $\xi : \mathbb{R}^{\otimes \uplus v} \rightarrow \mathbb{R}^n$ representing an element in the fiber of the $k$-th tangent space $T^{(k)}$ with coordinates $X^a_{(s_1, \ldots, s_v)}$ (see (14) of Example 3.2). The action of $a = a_2 + a_3 + a_4 + \cdots \in \mathfrak{gl}_{n}^{(\infty)}$ on $\xi$ is given by $a\xi = (a\xi)_{u+1} + (a\xi)_{u+2} + \cdots$, where

\[
(a\xi)_v = \sum_{s+u=v+1} \left( \sum_{1 \leq i \leq s} S(a_s, \xi) - \sum_{1 \leq i \leq v} S(\xi, a_s) \right).
\]

Removing fractional coefficients by modifying the $\mathfrak{gl}_{n}^{(\infty)}$-action (compare Remark 4.1), one can graphically express the above rule by the following polarization of (29):

\[
\begin{array}{c}
\text{v inputs} \\
\begin{array}{c}
\text{u}
\end{array}
\end{array}
\]

\[
\sum \left( \begin{array}{c}
\cdots, s \\
\text{ush}
\end{array} \right) - \left( \begin{array}{c}
\cdots, s \\
\text{ush}
\end{array} \right).
\]

4.3. Example. Let us write two initial replacement rules for the connection and its derivatives. The first one is the infinitesimal version of (3):

\[
\begin{array}{c}
\text{v inputs} \\
\begin{array}{c}
\text{u}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{u} \\
\begin{array}{c}
\text{t} \\
\text{v}
\end{array}
\end{array}
\end{array}
\]

\[
-(\begin{array}{c}
\begin{array}{c}
\text{u} \\
\text{v}
\end{array}
\end{array}).
\]

The next one is a graphical form of an equation that can be found in [8, Section 17.7] (but notice a different convention for covariant derivatives used in [8]):

\[
\begin{array}{c}
\begin{array}{c}
\text{u}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{s}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{u}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{v}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{u}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{v}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{s}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{u}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{v}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{s}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{u}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{v}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{s}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{u}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{v}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{s}
\end{array}
\end{array}
\]
We are not going to give a general formula. For our purposes, it will be enough to know that it is of the form

\[(31)\]

where \(G_w\) is a linear combination of 2-vertex trees with one vertex (18), with \(v < w\), and one vertex (19) with \(u < w + 2\).

Let us write a formal definition of the graph differential. For each oriented graph \(G \in \mathcal{G}_r^m\) we define \(\delta(G) \in \mathcal{G}_r^{m+1}\) as the sum over the set \(\text{Vert}(G)\) of vertices of \(G\),

\[(32)\]

where \(\delta_v\) is the replacement of the vertex \(v\) determined by the type of \(v\) and geometric data as explained above. The signs \(\varepsilon_v\) and the orientations of the graphs in \(\delta_v(G)\) are determined in the following way.

(i) The operation \(\delta_v\) replaces a 1st or 2nd type vertex \(v\) by a linear combination of graphs containing precisely one white vertex. The orientation of the graphs in \(\delta_v(G)\) is given by the unique linear order such that this new white vertex is the minimal element and the relative order of the remaining white vertices is unchanged. The sign \(\varepsilon_v\) is +1. Symbolically

\[(33)\]

(ii) Let \(v\) be a white vertex. We may assume that, after changing the sign of the graph \(G\) if necessary, \(v\) is the minimal element in an order determining the orientation. The orientation of graphs in \(\delta_v(G)\) is then given by requiring that the lower left white vertex in the right hand side of (29) is the minimal one, the upper right white vertex of (29) is the next one, and that the relative order of the remaining white vertices is unchanged. The sign \(\varepsilon_v\) is again +1. Symbolically,

\[(34)\]

We leave as a simple exercise to derive from the rule (ii) that, if the white vertex \(v\) is the \(i\)th element of a linear order determining the orientation of \(G\), for some \(1 \leq i \leq m\), the orientations of graphs in \(\delta_v(G)\) are symbolically expressed as

\[(34)\]

Let us emphasize that the applications in this paper use only the initial part \(\delta : \mathcal{G}_r^0 \to \mathcal{G}_r^1\) of the differential. Since the graphs spanning \(\mathcal{G}_r^0\) (resp. \(\mathcal{G}_r^1\)) have no white vertices (resp. one white vertex), the orientation issue is trivial and all \(\varepsilon_v\)’s in (32) are +1.

4.4. **Theorem.** The object \(\mathcal{G}_r^*\) is a cochain complex and the maps \(R^m_n\) in (22) assemble into a cochain map \(R_n^* : (\mathcal{G}_r^*, \delta) \to (C^*_{\text{GL}_n} (\mathfrak{gl}_n^{(\infty)}, \text{Map}(\mathcal{F}^{(\infty)}, \mathcal{G})), \delta_{CE})\).
Proof. Using the antisymmetry of \( f \), one can rewrite equations (4) and (5) into

\[
(\delta_1 f)(h_1, \ldots, h_{m+1}) = \frac{1}{m!} \text{Ant}(h_1 f(h_2, \ldots, h_{m+1})) \quad \text{and}
\]

\[
(\delta_2 f)(h_1, \ldots, h_{m+1}) = \frac{1}{2(m-1)!} \text{Ant}(f([h_2, h_1], h_3, \ldots, h_{m+1})) ,
\]

where \( \text{Ant}(\cdot) \) denotes the antisymmetrization, see Remark 4.5 below. If the multilinear map \( f \) itself is an antisymmetrization \( \text{Ant}(\cdot) \) by \( C \)-subobjects, denoted for the purposes of this remark by \( \Sigma_k \), one can rewrite the above displays into

\[
(35) \quad (\delta_1 f)(h_1, \ldots, h_{m+1}) = \text{Ant}(h_1 F(h_2, \ldots, h_{m+1})) \quad \text{and}
\]

\[
(36) \quad (\delta_2 f)(h_1, \ldots, h_{m+1}) = \text{Ant}\left( \sum_{1 \leq i \leq m} (-1)^{i+1} F(h_1, \ldots, h_{i-1}, [h_{i+1}, h_i], h_{i+2}, \ldots, h_{m+1}) \right) .
\]

After this preparation, we prove that \( R_n^m \) is a chain map by verifying that \( (R_n^{m+1} \circ \delta)(G) = (\delta_{CE} \circ R_n^m)(G) \) for each graph \( G \) generating \( \mathfrak{g}_m \). After choosing a linear order of white vertices of \( G \) compatible with its orientation, an appropriate version of the ‘state sum’ (11) of [11] gives a multilinear map \( F \) such that \( R_n^m(G) = \text{Ant}(F) \), see [11, Remark 5.2].

It is not difficult to see that \( R_n^m \) translates the part of the differential \( \delta(G) \) in (32) given by the summation over the 1st and 2nd type vertices into formula (35) for \( \delta_1(f) \) and the part of \( \delta(G) \) given by the summation over the white vertices to formula (36) for \( \delta_2(f) \). This fact is also reflected by the obvious similarity between formulas (35) and (36) for the Chevalley-Eilenberg differential and symbolic formulas (33) and (34) for the graph differential.

The condition \( \delta^2 = 0 \) can be verified directly using the fact that the local replacement rules used in (32) are duals of Lie algebra actions and checking that the orientations were defined in such a way that the signs combine properly. One may, however, proceed also as follows.

Since both the domain and target of the map \( R_n^m \), as well as \( R_n^m \) itself, are defined in terms of “standard representations,” \( R_n^m \) makes sense for an arbitrary natural \( n \). Let \( G \in \mathfrak{g}_m \). By the finitary nature of objects involved, there exists \( e \geq 0 \) such that all graphs that constitute \( \delta^2(G) \in \mathfrak{g}_m^{m+2} \) have \( \leq e \) edges. Choose \( n \geq e - m - 2 \). We already know that \( R_n^m \) commutes with the differentials, therefore \( R_n^{m+2}(\delta^2(G)) = \delta_{CE}(R_n^m(G)) = 0 \). By the second part of Theorem 3.3 this implies that \( \delta^2(G) = 0 \).

4.5. Remark. In this paper, the antisymmetrization of an element \( x \) of some (say) right \( \Sigma_k \)-module, \( k \geq 1 \), is given by the formula \( \text{Ant}(x) := \sum_{\sigma \in \Sigma_k} \text{sgn}(\sigma) \cdot x\sigma \), without the traditional \( \frac{1}{k!} \). This convention is forced by the standard definition of the Lie algebra associated to an associative algebra \( (A, \cdot) \) – the bracket \( [a', a''] := a' \cdot a'' - a'' \cdot a' \), \( a', a'' \in A \), does not involve the \( \frac{1}{2!} \) factor. On the other hand, we define the symmetrization of \( x \) as above by the expected formula \( S(x) := \frac{1}{k!} \sum_{\sigma \in \Sigma_k} x\sigma \).

4.6. Remark. Applications of our theory will often be based on a suitable choice of a subspace of \( \mathfrak{rat}(\mathfrak{g}, \mathfrak{g}) \), together with the corresponding subcomplex of the graph complex \( \mathfrak{g}_m \). These subobjects, denoted for the purposes of this remark by \( \mathfrak{rat}(\mathfrak{g}, \mathfrak{g}) \) and \( \mathfrak{g}_m^{*} = (\mathfrak{g}_m, \mathfrak{g}, \mathfrak{g}) \), will be chosen so that the number of edges of graphs spanning \( \mathfrak{g}_m^{*} \) will be, for each \( m \geq 0 \), bounded by \( C + m \), where \( C \) is a fixed constant.
An example is the subcomplex $\mathcal{G}^r_\bullet(d)$ of the graph complex $\mathcal{G}^r_{T \times d, T}$, introduced in Section 6, that describes $d$-multilinear operators on vector fields. Graphs spanning $\mathcal{G}^r_\bullet(d)$ have precisely $d + m$ edges, so $C = d$ for this subcomplex. Another example is the subcomplex $\mathcal{G}^r_{\mathcal{V}, (d)}$ of $\mathcal{G}^r_{\mathcal{V}, (d, d, T)}$ describing ‘connected’ $d$-multilinear operators used in Section 7. Each degree $m$ graph spanning this subcomplex has at most $2d + m - 1$ edges, i.e. $C = 2d - 1$ in this case. The third example is the complex $\mathcal{G}_\nu^r_{\mathcal{V}, (d)}$ introduced on page 19 describing ‘connected’ operators in $\mathcal{Nat}(\mathcal{C} \otimes T^\otimes d, \mathbb{R})$. For this complex, $C := 2d$.

Let $(\mathcal{G}^r_{\mathcal{V}, \bullet}, \delta), \mathcal{Nat}(\mathcal{F}, \mathcal{G})$ and the constant $C$ be as above. By Theorem 2.2 combined with Theorem 4.4, the restriction $R_n^\ast$ of $R_n$ induces the map
\begin{equation}
H^0(R_n^\ast) : H^0(\mathcal{G}^r_{\mathcal{V}, \bullet}, \delta) \to \mathcal{Nat}(\mathcal{F}, \mathcal{G})
\end{equation}
which is an isomorphism in stable dimensions. By this we mean that the dimension $n$ of the underlying manifold $M$ is $\geq C$. If this happens, then the map $R_n^\ast$ is, by [1], Proposition 4.9, a chain isomorphism, so $H^0(R_n^\ast)$ is an isomorphism, too. If the dimension of $M$ is less than the stable dimension, one cannot say anything about the induced map $H^0(R_n)$, although the chain map $R_n^\ast$ is still a chain epimorphism.

4.7. Example. In this example we prove a baby version of Theorem 5.1. Namely, we show that the only natural bilinear operations on vector fields on manifolds of dimensions $\geq 2$ are scalar multiples of the Lie bracket. It will be convenient to have ready some initial cases of formula (30) for the replacement rule of vertices representing vector fields and their derivatives:
\[
\delta(\bullet) = 0, \quad \delta(\bullet) = \bullet, \quad \delta(\bullet) = -\bullet + \bullet + \bullet, \ldots
\]
It is also clear that $\delta(\circ) = 0$.

Let us denote by $\mathcal{G}^r_{T \otimes T, T} \subset \mathcal{G}^r_{T \times T, T}$ the subcomplex describing bilinear operators. Its degree 0 part $\mathcal{G}^r_{T \otimes T, T}$ is spanned by

\[
\begin{align*}
\bullet \otimes \circ, & \quad \circ \otimes \bullet, \quad \bullet \otimes \bullet, \quad \circ \otimes \circ
\end{align*}
\]
and
\[
\begin{align*}
\bullet \otimes \circ, & \quad \circ \otimes \bullet
\end{align*}
\]
One easily calculates the differential of the leftmost term:
\begin{equation}
\delta\left(\begin{array}{c}
\bullet \\
\circ
\end{array}\right) = \delta(\bullet) + \delta(\circ) + \delta(\bullet) + \delta(\bullet) = 0 \in \mathcal{G}^r_{T \otimes T, T}
\end{equation}
and similarly one gets
\[
\delta\left(\begin{array}{c}
\bullet \\
\circ
\end{array}\right) = \delta(\bullet) + \delta(\circ) + \delta(\bullet) + \delta(\bullet) = 0 \in \mathcal{G}^r_{T \otimes T, T}
\]
The formula for the differential of the remaining two generators of $\mathcal{G}^r_{T \otimes T, T}$ is obtained by interchanging $X \leftrightarrow Y$ in the previous two displays. One clearly has
\[
\delta\left(\begin{array}{c}
\bullet \\
\circ
\end{array}\right) = \delta(\bullet) + \delta(\circ) + \delta(\bullet) + \delta(\bullet) = 0,
\]
because the inputs of white vertices are symmetric. It is easy to verify that the element
\begin{equation}
b := \begin{array}{c}
\bullet \\
\circ
\end{array} \in \mathcal{G}^r_{T \otimes T, T}
\end{equation}
representing the Lie bracket in fact spans all cochains in $\mathfrak{g}_{T \otimes T}^0$. We conclude that $H^0(\mathfrak{g}_{T \otimes T}^*, \delta)$ is one-dimensional, generated by the bracket $[X, Y]$. The complex $\mathfrak{g}_{T \otimes T}^*$ clearly fits into the scheme discussed in Remark 4.6 (with $C = 2$), which proves Theorem 5.1 for $d = 2$.

4.8. Example. We close this section by an example suggested by the referee which will further illuminate the meaning of the graph differential. The graph

(40)

represents the local expression

(41) $$(X^i \frac{\partial}{\partial x^i}, Y^j \frac{\partial}{\partial y^j}) \mapsto X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial y^j}.$$ 

If \{y'\} is a different set of coordinates, then $X$ and $Y$ transforms to $X^i \frac{\partial y'}{\partial x^i} \frac{\partial}{\partial y'}$ and $Y^j \frac{\partial y'}{\partial x^j} \frac{\partial}{\partial y'}$, respectively. Having this transformed $X$ act on the transformed $Y$ gives

$$X^i \frac{\partial y'}{\partial x^i} \frac{\partial}{\partial y'} \left( Y^j \frac{\partial y'}{\partial x^j} \frac{\partial}{\partial y'} \right) \frac{\partial}{\partial y'} = X^i \frac{\partial y^s}{\partial x^i} \frac{\partial Y^j}{\partial y^s} \frac{\partial y^r}{\partial y^r} + X^i \frac{\partial y^s}{\partial x^i} Y^j \frac{\partial}{\partial y^s} \left( \frac{\partial y^r}{\partial x^j} \frac{\partial}{\partial y^r} \right).$$

The first term in the right hand side is equal to the expression in (41) under change-of-coordinates, so the second term represents the extent to which this expression is not invariant. It is equal to $X^i Y^j \frac{\partial^2 y^r}{\partial x^i \partial x^j} \frac{\partial}{\partial y^r}$, which translates directly to the formula (38) for the differential of (40) in the graph complex.

5. Operations on vector fields

In this section we consider differential operators acting on a finite number of vector fields $X, Y, Z, \ldots$ with values in vector fields, that is, operators in $\mathfrak{nat}(T^{\times \infty}, T) := \bigcup_{d \geq 0} \mathfrak{nat}(T^{\times d}, T)$. The first statement of this section is:

5.1. Theorem. Let $M$ be a smooth manifold and $d$ a natural number such that $\dim(M) \geq d$. Then each $d$-multilinear natural operator from vector fields to vector fields is a sum of iterations of the Lie bracket containing each of $d$ variables precisely once, and all relations between these expressions follow from the Jacobi identity and antisymmetry. In particular, there are precisely $(d - 1)!$ linearly independent operators of the above type.

Theorem 5.1 is an obvious consequence of Proposition 5.6 below and the formula for the dimension of the $k$th piece of the operad $\mathfrak{Lie}$ for Lie algebras that can be found for example in [6, Example 3.1.12]. Theorem 5.1 describes multilinear operators and does not cover operators as $\mathfrak{O}(X, Y, Z) := [X, Y] + [X, [X, Z]]$ but can easily be extended to cover also these cases. Since all operators are assumed to be polynomial, they decompose into the sum of their homogeneous parts. For instance, $\mathfrak{O}(X, Y, Z)$ is the sum of the homogeneity-2 part $[X, Y]$ and the homogeneity-3 part $[X, [X, Z]]$.

5.2. Remark. Let us explain the decomposition of operators $\mathfrak{O} \in \mathfrak{nat}(T^{\times \infty}, T)$ into homogeneous parts in more detail. The local formula $O$ for the operator $\mathfrak{O}$ is the sum $O = O_1 + \cdots + O_r$, where $O_d$ is the part of $O$ consisting of terms with precisely $d$ occurrences of the vector field.
variables. The action of the structure group $\text{GL}_n^{(\infty)}$ on the typical fiber of the prolongation of $T^{\times \infty}$ is linear, which is expressed by the manifest linearity of the replacement rule \( [\text{III}] \) in the vector field variable. This implies that the map $O$ is $\text{GL}_n^{(\infty)}$-equivariant if and only if each of its homogeneous components $O_d$ is $\text{GL}_n^{(\infty)}$-equivariant, $1 \leq d \leq r$. Therefore $\mathcal{O} = \mathcal{O}_1 + \cdots + \mathcal{O}_r$, where $\mathcal{O}_d$ is the operator defined by the local formula $O_d$, $1 \leq d \leq r$.

We conclude that to classify operators of the above type, it suffices to classify homogeneous operators. It is a standard fact that each homogeneous operator of degree $d$ is either $d$-multilinear or a sum of operators obtained from $d$-multilinear operators by repeating one or more of their variables. We will call this procedure the depolarization of multilinear operators. Theorem 5.1 therefore implies the following corollary.

5.3. Corollary. Let $M$ be a smooth manifold. Each natural differential operator from vector fields on $M$ to vector fields on $M$ whose all components are of homogeneity $\leq \dim(M)$ is a sum of iterations of the Lie bracket. All relations between these iterations follow from the Jacobi identity and antisymmetry.

In Example 4.7 we studied the graph complex $\mathcal{G}^*(T \otimes T)$ describing bilinear operators. Bearing this example in mind, we introduce $\mathcal{G}^*(d) = \mathcal{G}^*_{T \otimes \otimes T} \subset \mathcal{G}^*_{T \times \otimes T}$, the subcomplex describing $d$-multilinear operators. Its degree $m$ component is spanned by graphs with $d$ vertices of the first type labelled by $X_1, \ldots, X_d$, $m$ white vertices of the third type and one 2nd type vertex that we call the anchor. Observe that $\mathcal{G}^*(d)$ is precisely the graph complex $\mathcal{G}^*_{(b)\otimes (c)}$ of Corollary 5.1 with $b := d$ and $c := 0$. The collection $\mathcal{G}^* = \{\mathcal{G}^*(d)\}_{d \geq 1}$ of degree 0 subspaces admits two types of operations.

(i) For graphs $G' \in \mathcal{G}^*(u)$, $G'' \in \mathcal{G}^*(v)$ and $1 \leq i \leq u$, one has the $\circ_i$-product $G' \circ_i G'' \in \mathcal{G}^*(u + v - 1)$ given by the following straightforward extension of the Chapoton-Livernet vertex insertion \([11, \text{§ 1.5}]\) to non-simply connected graphs. Assume that $X_1', \ldots, X_u'$ are the black vertices of $G'$, $X_1'', \ldots, X_v''$ the black vertices of $G''$ and $\text{In}(X'_i)$ the set of inputs of $X'_i$ in $G'$. Then

\[
G' \circ_i G'' := \sum_{f: \text{In}(X'_i) \rightarrow \{X''_1, \ldots, X''_v\}} G' \circ_i^f G'' \in \mathcal{G}^*(u + v - 1),
\]

where $G' \circ_i^f G'' \in \mathcal{G}^*(u + v - 1)$ is the graph obtained by replacing the vertex $X'_i$ of $G'$ by $G''$ and grafting the inputs of $X'_i$ on black vertices of $G''$ following $f$.

In more detail, one starts by cutting off the anchor $\bullet$ of $G''$ and grafts the resulting free edge on the vertex of $G'$ immediately above $X'_i$. Then one grafts each input edge $e$ of $X'_i$ on the vertex $f(e)$ of $G''$. Finally, one changes the labels $X'_1, \ldots, X'_{i-1}, X''_1, \ldots, X''_v, X'_{i+1}, \ldots, X'_u$ of the black vertices of the graph obtained in this way into $X_1, \ldots, X_{u+v-1}$.

(ii) One has the right action of the symmetric group: for each $G \in \mathcal{G}^*(d)$ and a permutation $\sigma \in \Sigma_d$, one has $G\sigma \in \mathcal{G}^*(d)$ given by permuting the labels $X_1, \ldots, X_d$ of the black vertices of $G$ according to $\sigma$.

5.4. Proposition. The collection $\mathcal{G}^* = \{\mathcal{G}^*(d)\}_{d \geq 1}$ with the above operations is an operad with unit $\bullet \in \mathcal{G}^*(1) \{14\}$. The operad structure of $\mathcal{G}^*$ restricts to $H^0(\mathcal{G}^*, \delta) = \text{Ker}(\delta : \mathcal{G}^* \rightarrow \mathcal{G}^*)$. 
The operad axioms for the operations in (i) and (ii) above are verified directly, compare also [8, § 1.5]. The simplest way to see that the operad structure of $\mathcal{G}_r^*$ restricts to the kernel of $\delta$ is to extend the operations (i) and (ii), in the obvious manner, to the graded collection $\mathcal{G}_r^*$, making $(\mathcal{G}_r^*, \delta)$ a dg-operad. This, in particular, would mean that $\delta$ is a derivation with respect to these extended $\circ_i$-operations, which implies the second part of the proposition. \[
\]

5.5. Example. An instructive example of the vertex insertion can be found in [8, § 1.5]. We present here a simpler one, taken from the proof of [8, Theorem 1.9]. Let $p$ be the graph

Then one has

The above display implies that the associator $Ass(p) := p \circ_1 p - p \circ_2 p$ equals

and is therefore symmetric in $X_2$ and $X_3$. This, by definition, means that $p$ represents a pre-Lie multiplication [8, § 1.1]. We will see that $\mathcal{G}_r^*$ is indeed closely related to the pre-Lie operad $p\mathcal{L}ie$.

Let $\tau \in \Sigma_2$ be the generator. By standard properties of pre-Lie algebras [8, Proposition 1.2], the antisymmetrization $p(\tau - \mathbb{1})$ of the element $p$ from Example 5.5 is a Lie bracket. Observe that $p(\tau - \mathbb{1})$ equals the element $b$ introduced in (33).

5.6. Proposition. The 0th cohomology $H^0(\mathcal{G}_r^*(d), \delta)$ is, for each $d \geq 2$, generated by the Lie bracket $b = p(\tau - \mathbb{1}) \in H^0(\mathcal{G}_r^*(2), \delta)$, by iterating operations (i) and (ii) above. There are no relations between these iterations other than those following from the Jacobi identity and antisymmetry.

A compact formulation of Proposition 5.6 is that the operad $H^0(\mathcal{G}_r^*, \delta) = \{H^0(\mathcal{G}_r^*(d), \delta)\}_{d \geq 1}$ is isomorphic to the operad $\mathcal{L}ie = \{\mathcal{L}ie(d)\}_{d \geq 1}$ for Lie algebras [16, Example II.3.34], via an isomorphism that sends the generator $\beta \in \mathcal{L}ie(2)$ of $\mathcal{L}ie$ into $b \in \mathcal{G}_r^*(2)$. Graphs spanning $\mathcal{G}_r^0(d)$ have $d$ edges which explains the stability condition $\dim(M) \geq d$ in Theorem 5.1. The rest of this section is devoted to a proof of its main result.

Proof of Proposition 5.6. It is clear from formulas (29), (31) and $\delta (\mathbb{1}) = 0$ that the differential preserves connected components of underlying graphs. Therefore, for each $d \geq 1$, $\mathcal{G}_r^*(d)$ is the direct sum $\mathcal{G}_r^*(d) = \bigoplus_{c \geq 1} \mathcal{G}_r^c(d)$, where $\mathcal{G}_r^c(d)$ denotes the subcomplex spanned by graphs with $c$ connected components. In particular, $\mathcal{G}_r^1(0)$ is the subcomplex of connected graphs. It is easy to see that $\mathcal{G}_r^0_1$ is a suboperad of $\mathcal{G}_r^*$.

As the Lie bracket represented by $b \in \mathcal{G}_r^0_1(2)$ is antisymmetric and satisfies the Jacobi identity, the rule $F(\beta) := b$, where $\beta \in \mathcal{L}ie(2)$ is the generator, defines an operad homomorphism $F : \mathcal{L}ie \rightarrow \mathcal{G}_r^0_1$. Since the Lie bracket and its iterations are natural operators, $Im(F) \subset \ker(\delta : \mathcal{G}_r^0_1 \rightarrow \mathcal{G}_r^1)$.

Proposition 5.6 will clearly be established if we prove that
(i) the operad map $F : \mathcal{L}ie \to \mathcal{G}_{\ast 1}^0$ induces an isomorphism $\mathcal{L}ie \cong H^0(\mathcal{G}_{\ast 1}^0, \delta)$, and

(ii) $H^0(\mathcal{G}_{\ast c}^i(d), \delta) = 0$, for each $c \geq 2$, $d \geq 1$.

Part (i) is highly nontrivial, but it in fact has already been proved in [14]. Indeed, the operad $\mathcal{G}_{\ast 1}^0$ is precisely the operad $p\mathcal{L}ie$ describing pre-Lie algebras [1] and $F : \mathcal{L}ie \to \mathcal{G}_{\ast 1}^0$ corresponds, under the identification $\mathcal{G}_{\ast 1}^0 \cong p\mathcal{L}ie$, to the inclusion $i : \mathcal{L}ie \hookrightarrow p\mathcal{L}ie$ induced by the antisymmetrization of the pre-Lie product. The dg operad $r\mathcal{L}ie$ of [14] coincides, in degrees 0 and 1, with the complex $\mathcal{G}_{\ast 1}^0$ and the isomorphism in (i) is isomorphism (2) of [14].

Let us prove (ii). For each $m \geq 0$, $d \geq 1$, consider the span $\mathcal{G}_{\ast c}^m(d)$ of connected graphs with $d$ vertices $X_1, \ldots, X_d$ of type 1, $m$ ‘white’ vertices of type 3 and no vertex of type 2. The direct sum $\mathcal{G}_{\ast c}^m(d) = \bigoplus_{m \geq 0} \mathcal{G}_{\ast c}^m(d)$ is a cochain complex, with the differential defined in the same way as the differential in $\mathcal{G}_{\ast c}^m(d)$ and denoted again by $\delta$. We claim that, for each $c \geq 2$ and $d \geq 1$, there is an isomorphism of cochain complexes

$$\mathcal{G}_{\ast c}^m(d) \cong \bigoplus_{i_1 + \cdots + i_c = d} (\mathcal{G}_{\ast 1}^i(i_1) \otimes (\mathcal{G}_{\ast 2}^i(i_2) \otimes \cdots \otimes (\mathcal{G}_{\ast c}^i(i_c))))$$

where $\otimes$ as usual denotes the symmetric product. To prove this isomorphism, observe that each graph $G \in \mathcal{G}_{\ast c}^m(d)$ decomposes into the disjoint union

$$G = G_1 \sqcup G_2 \sqcup \cdots \sqcup G_c,$$

of its connected components. Precisely one of these components contains the unique type 2 vertex $\bullet$, assume it is $G_1$. Then $G_1 \in \mathcal{G}_{\ast 1}^m(i_1)$ and $G_s \in \mathcal{G}_{\ast c}^m(i_s)$ for $2 \leq s \leq c$, with some $i_1 + \cdots + i_c = d$. Decomposition (13) is clearly unique up to the order of $G_2, \ldots, G_c$ and is preserved by the differential. This proves (12). By Künneth and Mashke’s theorems, (ii) follows from $H^0(\mathcal{G}_{\ast c}^m(d), \delta) = 0$, $d \geq 1$, which is the same as showing that

$$\delta : \mathcal{G}_{\ast c}^m(d) \to \mathcal{G}_{\ast c}^m(d)$$

is a monomorphism for each $d \geq 1$.

Let us inspect the structure of $\mathcal{G}_{\ast c}^m(d)$. It is clear from simple graph combinatorics that each graph in $\mathcal{G}_{\ast c}^m(d)$ has genus 1, therefore it contains a unique wheel. Denote $\mathcal{G}_{\ast c}^m(d, w) \subset \mathcal{G}_{\ast c}^m(d)$ the subspace spanned by graphs that have precisely $w$ vertices (of either type) on the wheel, $w \geq 0$. It is obvious from (29) and (30) that $\delta(\mathcal{G}_{\ast c}^m(d, w)) \subset \mathcal{G}_{\ast c}^{m+1}(d, w) \oplus \mathcal{G}_{\ast c}^{m+1}(d, w+1)$, for $d \geq 1$, $w \geq 0$; see also Figure 1. Let us denote by $\delta^0$ the component of $\delta$ that preserves the number of vertices on the wheel and $\delta^1$ the component that raises it by one. We claim that in order to prove (14), it is enough to verify that

$$\delta^0 : \mathcal{G}_{\ast c}^m(d) \to \mathcal{G}_{\ast c}^m(d)$$

is a monomorphism for each $d \geq 1$.

The spaces $\mathcal{G}_{\ast c}^m(d, p)$ form a bicomplex $(\mathcal{G}_{\ast c}^m(d, p), \delta)$ with $\mathcal{G}_{\ast c}^m(d, p) := \mathcal{G}_{\ast c}^{m,0}(d, p)$ and $\delta$ the sum $\delta^0 + \delta^1$, where $\delta^0 : \mathcal{G}_{\ast c}^m(d) \to \mathcal{G}_{\ast c}^{m+1}(d)$ and $\delta^0 : \mathcal{G}_{\ast c}^m(d) \to \mathcal{G}_{\ast c}^{m+1}(d)$ are defined above. Condition (15) then implies (14) via a standard spectral sequence argument. The only subtlety is that our bicomplex is not a first quadrant one, thus the convergence of the related spectral sequence has to be checked. We therefore decided to prove the implication $(13) \implies (14)$ by the following elementary calculation.

Suppose that (14) does not hold and let $x \in \mathcal{G}_{\ast c}^m(d)$ be such that $\delta(x) = 0$ while $x \neq 0$. There exists a decomposition $x = x_a + x_{a+1} + \cdots + x_{a+s}$ with $x_w \in \mathcal{G}_{\ast c}^m(d, w)$ for $a \leq w \leq a + s$ in
which \( x_a \neq 0 \). Since \( \delta^0(x_a) \) is the component of \( \delta(x) \) in \( \mathcal{G}^{1}_{\cdot \circ}(d,a) \), \( \delta^0(x_a) = 0 \). Then (15) implies \( x_a = 0 \), a contradiction.

Denote by \( \mathcal{G}^1_{\cdot \circ}(d,w) \subset \mathcal{G}^1_{\cdot \circ}(d,w) \) the subspace spanned by graphs with one binary white vertex on the wheel, as in the left graph in Figure 2. Both \( \mathcal{G}^1_{\cdot \circ}(d,w) \) and \( \mathcal{G}^0_{\cdot \circ}(d,w) \) have canonical bases provided by isomorphism classes of graphs, therefore one has a canonical projection \( \pi : \mathcal{G}^1_{\cdot \circ}(d,w) \to \mathcal{G}^0_{\cdot \circ}(d,w) \). In addition to the projection, there is a second map \( r : \mathcal{G}^1_{\cdot \circ}(d,w) \to \mathcal{G}^0_{\cdot \circ}(d,w) \) whose definition is clear from Figure 2.

Let \( G \in \mathcal{G}^0_{\cdot \circ}(d,w) \) be a graph. Observe that \( \mathcal{G}^0_{\cdot \circ}(d,0) = 0 \), we may therefore assume \( w \geq 1 \). Recall that the differential \( \delta(G) \) is the sum (12) of local replacements \( \delta_v(G) \) over \( v \in \text{Vert}(G) \). Let \( \text{Vert}_G \subset \text{Vert}(G) \) be the subset of vertices on the wheel. For \( v \in \text{Vert}_G \), the contribution \( \delta_v(G) \) contains precisely one graph in \( \mathcal{G}^1_{\cdot \circ}(d,w) \) with the binary white vertex – see again Figure 2. Denote this graph \( \delta^0_v(G) \) and define \( \delta^0(G) := \sum_{v \in \text{Vert}_G} \delta^0_v(G) \). It is clear that \( \text{Im}(\delta^0) \subset \mathcal{G}^1_{\cdot \circ}(d,w) \), \( \delta^0 = \pi \circ \delta^0 \) and \( r \circ \delta^0 = w \cdot id \). Combining these facts, we obtain \( r \circ \pi \circ \delta^0 = w \cdot id \), which implies (15) and finishes the proof.

We believe that one can even show that the complex \( (\mathcal{G}^*_{\cdot \circ}(d),\delta) \) used in the above proof is acyclic in all dimensions. Let us close this section by formulating the following interesting consequence of the proof of Proposition 5.6.

5.7. Corollary. In stable dimensions, there are no nontrivial differential operators from vector fields to functions.

Proof. It is clear that \( d \)-multilinear operators from vector fields to functions are described by the graph complex \( \mathcal{G}^*_{\cdot \circ}(d) \) introduced in our proof of Proposition 5.4. Condition (14) implies that there are no nontrivial \( d \)-multilinear operators of this type. The corollary then follows from the standard (de)polarization trick.
6. Structure of the space of natural operators

In Example 1.8 we considered the trivial natural bundle $\mathbb{R}$ whose sections are smooth functions. Let $\mathcal{F}$ be another natural bundle. The space $\text{Nat}(\mathcal{F}, \mathbb{R})$ of natural operators $O: \mathcal{F} \to \mathbb{R}$ with the ‘pointwise’ multiplication is a commutative algebra, with unit $1$ the operator that sends all sections of $\mathcal{F}$ into the constant section $1 \in \mathbb{R}$. This indicates that spaces of natural operators may sometimes have a rich algebraic structure that can be used to simplify their classification.

6.1. Definition. We say that $\mathcal{F}$ is a bundle with connected replacement rules if the replacement rules send a connected graph to a linear combination of connected graphs.

All natural bundles considered in this paper have connected replacement rules, and the author does not know any ‘natural’ natural operator that has not. We will see that the space of natural operators between bundles with connected replacement rules exhibits some freeness property.

Before we formulate the first statement of this type, we introduce the following convention.

The graph complex $\mathcal{G}^r_{\mathcal{F}, \mathbb{R}}$ for operators in $\text{Nat}(\mathcal{F}, \mathbb{R})$ is spanned by graphs with vertices of the 1st type representing tensors in a prolongation of the fiber of $\mathcal{F}$, vertices of the third type and one 2nd type vertex which in this case has no inputs and no outputs. Therefore is an isolated vertex bearing no information and we discard it from the picture. With this convention, graphs spanning $\mathcal{G}^r_{\mathcal{F}, \mathbb{R}}$ have vertices of the 1st and 3rd type only. The disjoint union of graphs spanning $\mathcal{G}^r_{\mathcal{F}, \mathbb{R}}$ translates into the pointwise multiplication of the corresponding operators and the unit $1 \in \text{Nat}(\mathcal{F}, \mathbb{R})$ is represented by the ‘exceptional’ empty graph.

6.2. Theorem. Let $\mathcal{F}$ be a natural bundle with connected replacement rules. Then, in stable dimensions, the commutative unital algebra $\text{Nat}(\mathcal{F}, \mathbb{R})$ is free, generated by the subspace $\text{Nat}_1(\mathcal{F}, \mathbb{R})$ of natural operators represented by connected graphs. In other words, $\text{Nat}(\mathcal{F}, \mathbb{R}) \cong \mathbb{R}[\text{Nat}_1(\mathcal{F}, \mathbb{R})]$, where $\mathbb{R}[-]$ denotes the polynomial algebra functor.

Proof. Each graph spanning $\mathcal{G}^r_{\mathcal{F}, \mathbb{R}}$ decomposes into the disjoint union of its connected components. The differential $\delta$, by assumption, preserves this decomposition which is clearly unique up to the order of components. The proof is finished by recalling that the disjoint union of graphs expresses the pointwise multiplication of operators.

Let $\mathcal{F}, \mathcal{G}$ be natural bundles. The pointwise multiplication makes the space $\text{Nat}(\mathcal{F}, \mathcal{G})$ a unital module over the unital algebra $\text{Nat}(\mathcal{F}, \mathbb{R})$. We prove a structure theorem also for this space.

6.3. Theorem. Suppose that both $\mathcal{F}$ and $\mathcal{G}$ are bundles with connected replacement rules. Then, in stable dimensions, $\text{Nat}(\mathcal{F}, \mathcal{G})$ is the free $\text{Nat}(\mathcal{F}, \mathbb{R})$-module generated by the subspace $\text{Nat}_1(\mathcal{F}, \mathcal{G})$ of operators represented by connected graphs,

$$\text{Nat}(\mathcal{F}, \mathcal{G}) \cong \text{Nat}_1(\mathcal{F}, \mathcal{G}) \otimes \text{Nat}(\mathcal{F}, \mathbb{R}).$$

Proof. The proof is similar to the proof of Theorem 6.2. The graph complex $\mathcal{G}^r_{\mathcal{F}, \mathcal{G}}$ describing operators in $\text{Nat}(\mathcal{F}, \mathcal{G})$ is spanned by graphs with vertices of the first and third types, and one vertex of the second type. Each such a graph is the disjoint union of its connected components as in $\mathcal{G}^r_{\mathcal{F}, \mathbb{R}}$ and the differential preserves this decomposition. Precisely one of these components
contains the vertex of the third type thus representing an operator in \( \mathfrak{N} \mathfrak{a} \mathfrak{t}_1(\mathfrak{F}, \mathfrak{G}) \). The remaining components describe operators from \( \mathfrak{N} \mathfrak{a} \mathfrak{t}_1(\mathfrak{F}, \mathbb{R}) \) and assemble, via the pointwise multiplication, into an operator in \( \mathfrak{N} \mathfrak{a} \mathfrak{t}(\mathfrak{F}, \mathbb{R}) \).

Theorems [6.2] and [6.3] imply that in order to classify operators in \( \mathfrak{N} \mathfrak{a} \mathfrak{t}(\mathfrak{F}, \mathfrak{G}) \), it is enough to understand the ‘connected’ subspaces \( \mathfrak{N} \mathfrak{a} \mathfrak{t}_1(\mathfrak{F}, \mathbb{R}) \) and \( \mathfrak{N} \mathfrak{a} \mathfrak{t}_1(\mathfrak{F}, \mathfrak{G}) \). We will use this fact in the next section.

6.4. Example. In Section [3] we studied natural operators on vector fields with values in vector fields, that is, operators in \( \mathfrak{N} \mathfrak{a} \mathfrak{t}(T^{\imes n}, T) := \bigcup_{d \geq 0} \mathfrak{N} \mathfrak{a} \mathfrak{t}(T^{\times d}, T) \). We also considered operators with values in functions and proved, in Corollary [5.7], that there are no nontrivial operators of this type in stable dimensions.

This means that \( \mathfrak{N} \mathfrak{a} \mathfrak{t}(T^{\times \infty}, \mathbb{R}) \) is the trivial commutative algebra \( \mathbb{R} \) and \( [\square] \) reduces to the isomorphism \( \mathfrak{N} \mathfrak{a} \mathfrak{t}(T^{\times \infty}, T) \cong \mathfrak{N} \mathfrak{a} \mathfrak{t}_1(T^{\times \infty}, T) \) which says that all operators from vector fields to vector fields live, in stable dimensions, on connected graphs.

7. Operators on connections and vector fields

We will consider operators acting on a linear connection \( \Gamma \) and a finite number of vector fields \( X, Y, Z, \ldots \), with values in vector fields, such as the covariant derivative \( \nabla_X Y \), torsion \( T(X, Y) \) and curvature \( R(X, Y) Z \) recalled in Example [1.2]. By Theorems [7.2] and [7.3], the structure of the space \( \mathfrak{N} \mathfrak{a} \mathfrak{t}(\text{Con} \times T^{\times \infty}, T) := \bigcup_{d \geq 0} \mathfrak{N} \mathfrak{a} \mathfrak{t}(\text{Con} \times T^{\times d}, T) \) of these operators is determined by the ‘connected’ subspaces \( \mathfrak{N} \mathfrak{a} \mathfrak{t}_1(\text{Con} \times T^{\times \infty}, T) \) and \( \mathfrak{N} \mathfrak{a} \mathfrak{t}_1(\text{Con} \times T^{\times \infty}, \mathbb{R}) \). In this section we describe these spaces. The following remark should be compared to Remark [5.2] in Section [3].

7.1. Remark. The local formula \( O \) for a natural differential operator \( \mathcal{D} \) in \( \mathfrak{N} \mathfrak{a} \mathfrak{t}(\text{Con} \times T^{\times \infty}, T) \) or in \( \mathfrak{N} \mathfrak{a} \mathfrak{t}(\text{Con} \times T^{\times \infty}, \mathbb{R}) \) decomposes into \( O = \sum_{a, b \geq 0} O_{a, b} \) (finite sum), where \( O_{a, b} \) is the part of \( O \) containing precisely \( a \) \( \nabla \)-variables and \( b \) vector field variables. For example, the local formula \( [\Pi] \) for the covariant derivative represented by the graph in (27) is the sum \( O_{1,2} + O_{0,2} \), where \( O_{1,2}(X, Y, \Gamma) := \Gamma_j^i X^j Y^k \partial / \partial x^i \) and \( O_{0,2}(X, Y, \Gamma) := X^j Y^i \partial / \partial x^i \).

In contrast to Section [3], here the action of the structure group \( \text{GL}_n^{(\infty)} \) on the typical fiber is linear only in the vector-field variables - the non-linearity in the \( \nabla \)-variables is manifested in the presence of the ‘isolated’ white vertex in the replacement rule (31). Nevertheless, one may still decompose \( \mathcal{D} = \mathcal{D}_1 + \cdots + \mathcal{D}_r \), with \( \mathcal{D}_k \) the operator represented by the local formula \( O_d := \sum_{a \geq 0} O_{a, d}, 1 \leq d \leq r \). Therefore homogeneity and multilinearity in this section always refer to the vector fields variables.

The first half of this section will be devoted to the study of the space \( \mathfrak{N} \mathfrak{a} \mathfrak{t}_1(\text{Con} \times T^{\times \infty}, T) \), the space \( \mathfrak{N} \mathfrak{a} \mathfrak{t}_1(\text{Con} \times T^{\times \infty}, \mathbb{R}) \) will be addressed in the second half of this section. As in Section [3], we start with multilinear operators.

7.2. Theorem. Let \( d \geq 0 \). On smooth manifolds of dimension \( \geq 2d - 1 \), each \( d \)-multilinear operator in \( \mathfrak{N} \mathfrak{a} \mathfrak{t}_1(\text{Con} \times T^{\times d}, T) \) is a linear combination of iterations of the covariant derivative and the Lie bracket which contains each of the vector fields \( X_1, \ldots, X_d \) exactly once. All relations follow from the anticommutativity and the Jacobi identity of the Lie bracket.
If $g_d$ denotes the number of linearly independent operators of this type, the generating function $g(t) = \sum_{d \geq 1} \frac{1}{d!} g_d t^d$ is determined by the functional equation

$$e^{g(t)} (1 - t - g^2(t)) = 1.$$ (47)

Equation (47) can be expanded into inductive formula (53) from which one can calculate some initial values of $g_k$ as $g_1 = 1$, $g_2 = 3$, $g_3 = 26$, &c. Theorem 7.2 will follow from Proposition 7.4 below. The depolarization of Theorem 7.2 is:

7.3. Corollary. On a smooth manifold $M$, each operator from $\mathfrak{N}at_1(\text{Con} \times T^{\times \infty}, T)$ whose all components are of homogeneity $\leq \frac{1}{2}(\dim(M) + 1)$ is a linear combination of compositions of the covariant derivative and the Lie bracket. All relations between these compositions follow from the anticommutativity and the Jacobi identity of the Lie bracket.

The central object will be the subcomplex $\mathcal{G}_r^{\bullet}(d)$ of the graph complex $\mathcal{G}_r^{\bullet(\text{Con} \times T^{\times d}, T, \text{nat})}$ describing ‘connected’ $d$-multilinear operators. Its degree $m$ piece $\mathcal{G}_r^{m}(d)$ is spanned by connected graphs with $d$ vertices (17) labelled by $X_1, \ldots, X_d$, some number of vertices (18) labelled $\nabla$, $m$ white vertices (19) and one vertex $\Box$. It is clear that $\mathcal{G}_r^{\bullet}(d)$ is precisely the subcomplex spanned by connected graphs, of the direct sum $\mathcal{G}_r^{\bullet}(d) := \bigoplus_{c \geq 0} \mathcal{G}^{\bullet(\nabla(c))}$, where $\mathcal{G}^{\bullet(\nabla(c))}$ is the graph complex of [11, Corollary 5.1]. As in Proposition 5.4, one easily sees that the collection $\mathcal{G}_r^{\bullet}(d) := \{\mathcal{G}_r^{\bullet}(d)\}_{d \geq 1}$ forms an operad. It is also not difficult to verify that each graph spanning $\mathcal{G}_r^{m}(d)$ has at most $2d + m - 1$ edges, which explains the stability condition in Theorem 7.2.

Let $\mathcal{P} = \{\mathcal{P}(d)\}_{d \geq 1}$ be the operad describing algebras with two independent operations – a bilinear product $\ast$ satisfying no other conditions and a Lie bracket. Of course, $\mathcal{P}$ is the free product (= the coproduct in the category of operads, see [13, p. 137]) of the free operad $\Gamma(\ast)$ generated by the bilinear operation $\ast$ and the operad $\mathcal{L}ie$ for Lie algebras, $\mathcal{P} = \Gamma(\ast) \ast \mathcal{L}ie$. Recall that we denoted by $\beta \in \mathcal{L}ie(2)$ the generator.

Define the operad homomorphism $F : \mathcal{P} \to \mathcal{G}_r^{\bullet}(1)$ by $F(\beta) := b$ and $F(\ast) := c$, where $b \in \mathcal{G}_r^{0}(1)$ is the graph (28) representing the Lie bracket and $c \in \mathcal{G}_r^{0}(1)$ the graph (27) for the covariant derivative. As in Section 3 we easily see that $F$ is well-defined and that $\text{Im}(F) \subseteq \text{Ker}(\delta : \mathcal{G}_r^{0}(1) \to \mathcal{G}_r^{1}(1))$. Theorem 7.2 clearly follows from

7.4. Proposition. The map $F : \mathcal{P} \to \mathcal{G}_r^{0}(1)$ induces an isomorphism $\mathcal{P} \cong H^0(\mathcal{G}_r^{0}(1), \delta)$. The generating function $p(t) := \sum_{d \geq 1} \frac{1}{d!} \dim(\mathcal{P}(d)) \cdot t^d$ for the operad $\mathcal{P}$ satisfies (44).

Proof. The map $F$ embeds into the following diagram of operads and their homomorphisms:

$$\begin{array}{ccc}
\mathcal{P} = \Gamma(\ast) \ast \mathcal{L}ie & \xrightarrow{F} & \mathcal{G}_r^{0}(1) \\
\downarrow \text{id} \ast \ast & & \downarrow T \\
\Gamma(\ast) \ast p\mathcal{L}ie & \rightarrow & \Gamma(\ast) \ast p\mathcal{L}ie.
\end{array}$$ (48)

Let us define the remaining maps in 48. As in 1, one can show that the operad $\mathcal{G}_r^{0}(1)$ is isomorphic to the operad $\Gamma(\ast) \ast p\mathcal{L}ie$ governing structures consisting of a bilinear multiplication $\ast$ and an independent pre-Lie product $\circ$. The map $A : \mathcal{G}_r^{0}(1) \to \Gamma(\ast) \ast p\mathcal{L}ie$ in 48 is the
isomorphism that sends the graph
\[
X \star Y \in \Gamma(\star)(2)
\]
into \(X \circ Y \in \pi \mathcal{L}ie(2)\) and the graph
\[
X \in \mathfrak{g}_r^0(2)
\]
into \(X \circ Y \in \pi \mathcal{L}ie(2)\). The map \(T : \Gamma(\star) \star \pi \mathcal{L}ie \to \Gamma(\star) \star \pi \mathcal{L}ie\) is the ‘twist’ \(T(X \star Y) := X \star Y - Y \circ X\) and \(T(X \circ Y) := X \circ Y\). It is evident that the composition \(TAF\) coincides with the coproduct \(\eta \star \iota\) of the identity \(\eta : \Gamma(\star) \to \Gamma(\star)\) and the map \(\iota : \pi \mathcal{L}ie \to \pi \mathcal{L}ie\) given by the antisymmetrization of the pre-Lie product \(\iota([X, Y]) := Y \circ X - X \circ Y\), which is an inclusion by \([4, \text{Proposition } 3.1]\). This implies that \(\eta \star \iota\) is a monomorphism, therefore \(F\) is a monomorphism, too.

Now, to prove that \(F\) induces an isomorphism \(\mathcal{P} \cong H^*(\mathfrak{g}_r^0(2), \delta)\), it suffices to show that the dimensions of the spaces \(H^0(\mathfrak{g}_r^0(2), \delta)\) and \(\mathcal{P}(d)\) are the same, for each \(d \geq 1\). Our calculation of the dimension of \(H^0(\mathfrak{g}_r^0(2), \delta)\) will be based on the fact that \((\mathfrak{g}_r^0(2), \delta)\) forms a bicomplex. For integers \(p, q\) denote by \(\mathfrak{g}_r^{p,q}(2)\) the subspace of \(\mathfrak{g}_r^{p+1,q}(2)\) spanned by graphs with precisely \(-p\) \(\nabla\)-vertices. It immediately follows from the replacement rules \([29], [30]\) and \([31]\) that \(\delta = \delta' + \delta''\), where \(\delta'(\mathfrak{g}_r^{p,q}(2)) \subset \mathfrak{g}_r^{p+1,q}(2)\) and \(\delta''(\mathfrak{g}_r^{p,q}(2)) \subset \mathfrak{g}_r^{p,q+1}(2)\). It is also clear from simple graph combinatorics that the bicomplex \((\mathfrak{g}_r^*(2), \delta)\) is bounded by the triangle \(p = 0, p + q = 0\) and \(q = d - 1\), see Figure 3. The horizontal differential \(\delta'\) in \(\mathfrak{g}_r^*(2)\) is easy to describe – it replaces \(\nabla\)-vertices according the rule
\[
(v) \rightarrow \quad - \quad (v + 2), \quad v \geq 0,
\]
and leaves other vertices unchanged.

### 7.5. Remark
At this point we need to make a digression and observe that \((\mathfrak{g}_r^*(2), \delta')\) is a particular case of the following construction. For each collection \((U^*, \vartheta_U) = \{(U^*(s), \vartheta_U)\}_{s \geq 2}\) of right \(\Sigma_s\)-modules \((U^*(s), \vartheta_U)\), one may consider the complex \(\mathfrak{g}_r^*[U](d) = (\mathfrak{g}_r^*[U^*](d), \vartheta_U)\) spanned by connected graphs with \(d\) vertices \([17]\) labelled \(X_1, \ldots, X_d\), one vertex \(\uparrow\) and a finite number of vertices decorated by elements of \(U\). The grading of \(\mathfrak{g}_r^*[U^*](d)\) is induced by the grading of \(U^*\) and the differential \(\vartheta_U\) replaces \(U\)-decorated vertices, one at a time, by their \(\vartheta_U\)-images and leaves other vertices unchanged. It is a standard fact \([17]\) (see also \([13, \text{Theorem } 21]\)
that the assignment $(U^*, \vartheta_U) \mapsto (\gr^s_{1*}[U^*](d), \vartheta)$ is a polynomial, hence exact, functor, so
\begin{equation}
(50)
H^*(\gr^s_{1*}[U^*](d), \vartheta) \cong \gr^s_{1*}[H^*(U, \vartheta_U)](d).
\end{equation}

Let now $(E^*, \vartheta_E) = \{(E^*(s), \vartheta_E)\}_{s \geq 2}$ be such that $E^0(s)$ is spanned by symbols $[\mathbb{H}]$, with $v + 2 = s$, $E^1(s)$ by symbols $[\mathbb{I}]$ with $u = s$, and $E^m(s) = 0$ for $m \geq 2$. The differential $\vartheta_E$ is defined by replacement rule $[\mathbb{L}]$. More formally, $E^0(s) = \text{Ind}_{\Sigma_{s-2}^v}^{\Sigma_{s-2}}(1_{s-2})$ and $E^1(s) = 1_s$, where $1_{s-2}$ (resp. $1_s$) denotes the trivial representation of the symmetric group $\Sigma_{s-2}$ (resp. $\Sigma_s$). The differential $\vartheta_E$ then sends the generator $1 \in 1_{s-2}$ into $-1 \in 1_s$. It is clear that, with this particular choice of the collection $(E^*, \vartheta_E)$,
\begin{equation}
(51)
(\gr^s_{1*}(d), \delta') \cong (\gr^s_{1*}[E^*](d), \vartheta).
\end{equation}

Let us continue with the proof of Proposition 7.4. Equations (50) and (51) in Remark 7.5 imply that
\begin{equation}
(52)
H^*(\gr^s_{1*}(d), \delta') = \gr^s_{1*}[H^*(E, \vartheta_E)](d).
\end{equation}
Since $\vartheta_E : E^0(s) \rightarrow E^1(s)$ is an epimorphism, the collection $H^*(E, \vartheta_E) = \{H^*(E(s), \vartheta_E)\}_{s \geq 2}$ is concentrated in degree 0 and $H^0(E(s), \vartheta_E)$ is the kernel of the map $\vartheta_E : E^0(s) \rightarrow E^1(s)$. We conclude that $\gr^s_{1*}[H^*(E, \vartheta_E)](d)$ is spanned by graphs with $d$ vertices $[\mathbb{J}]$, labelled $X_1, \ldots, X_d$, one vertex $\blacktriangleleft$ and some number of vertices decorated by the collection $H^0(E, \vartheta_E) = \{H^0(E(s), \vartheta_E)\}_{s \geq 2}$.

In particular, the graded space $\gr^s_{1*}[H^*(E, \vartheta_E)](d)$ and hence, by (52), also the horizontal cohomology $H^*(\gr^s_{1*}(d), \delta')$, is concentrated in degree 0. This implies that the first term $(E^0(d), d_1) = (H^0(\gr^s_{1*}(d), \delta'), d_1)$ of the corresponding spectral sequence is supported by the diagonal $p + q = 0$, so this spectral sequence degenerates at this level and
\[ \dim(H^0(\gr^s_{1*}(d), \delta)) = \dim(H^0(\gr^s_{1*}(d), \delta')) = \dim(\gr^s_{1*}[H^0(E, \vartheta_E)](d)). \]

Denote the common value of the dimensions in the above display $g_d$. We claim that the sequence $\{g_d\}_{d \geq 1}$ satisfies the recursion:
\begin{equation}
(53)
\frac{g_{n+1}}{(n+1)!} = \frac{g_n}{n!} + \frac{1}{2} \sum_{i+j=n} \frac{g_i g_j}{i! j!} + \frac{1}{3!} \sum_{i+j+k=n} \frac{g_i g_j g_k}{i! j! k!} + \frac{1}{4!} \sum_{i+j+k+l=n} \frac{g_i g_j g_k g_l}{i! j! k! l!} + \cdots
\end{equation}
\[ + \frac{2(2-1)-1}{2!} \sum_{i+j=n+1} \frac{g_i g_j}{i! j!} + \frac{3(3-1)-1}{3!} \sum_{i+j+k=n+1} \frac{g_i g_j g_k}{i! j! k!} + \cdots. \]
This can be seen as follows. Graphs $G$ spanning $\gr^0_{1*}[H^0(E, \vartheta_E)](d)$ are rooted trees with a distinguished vertex (= root) $\blacktriangleleft$. The vertex of $G$ adjacent to the root might either be a vertex $[\mathbb{M}]$ or a vertex decorated by $H^0(E, \vartheta_E)$. The contribution from trees of the first type is reflected by the first line of (53), in which the coefficients $1, 1/2, 1/3!, \ldots$ equal $\dim(1_s)/s!$, $s \geq 1$, where $1_s$ is the trivial representation of the symmetric group $\Sigma_s$ spanned by the vertex $[\mathbb{M}]$ with $u = s$. The second line of (53) counts contributions from trees of the second type. The coefficients are $\dim(H^0(E(s), \vartheta_E))/s!$, $s \geq 2$. It is simple to assemble (53) into equation (47).

Let us show that the generating function $p(t) := \sum_{d \geq 1} \frac{1}{d!} \dim(\mathcal{P}(d)) \cdot t^d$ for the operad $\mathcal{P}$ also satisfies (47). Since $\mathcal{P}$ is, as the coproduct of quadratic Koszul operads, itself quadratic Koszul, one has the functional equation [4, Theorem 3.3.2]:
\begin{equation}
(54)
q(-p(t)) = -t.
\end{equation}
relating $p$ with the generating function $q(t) := \sum_{d \geq 1} \frac{1}{d!} \dim(Q(d)) \cdot t^d$ of its quadratic dual $Q$.

For convenience of the reader, we make a digression and briefly recall the definition of quadratic operads and their quadratic duals. Details can be found in [16, II.3.2] or in the original source [3]. An operad $A$ is quadratic if it is the quotient $\Gamma(E)/(R)$ of the free operad $\Gamma(E)$ on the right $\Sigma_2$-module $E := A(2)$ of arity-two operations of $A$, modulo the operadic ideal $(R)$ generated by some subspace $R \subset \Gamma(E)(3)$.

Each quadratic operad $A = \Gamma(E)/(R)$ as above has its quadratic dual $A^!$ [16, Definition II.3.37] defined as follows. Let us denote $E^\vee := E^* \otimes \text{sgn}_2$ the linear dual of the right $\Sigma_2$-module $E$ twisted by the signum representation. One then has a natural isomorphism $\Gamma(E^\vee)(3) \cong \Gamma(E)(3)^* \otimes \Sigma_2$ of right $\Sigma_3$-modules. Let $R^\perp \subset \Gamma(E^\vee)(3)$ denote the annihilator of $R$ in $\Gamma(E^\vee)(3) \cong \Gamma(E)(3)^*$. The quadratic dual of $A$ is the quotient $A^! := \Gamma(E^\vee)/(R^\perp)$.

To describe the quadratic dual $Q$ of the operad $P$ introduced on page 25 is an easy task. The operad $Q$ governs algebras $V$ with two bilinear operations, $\cdot$ and $*$, such that $\cdot$ is commutative associative, $*$ is ‘nilpotent’ $(a \ast b) \ast c = a \ast (b \ast c) = 0$, $a, b, c \in V$, and these two operations annihilate each other: $(a \ast b) \ast c = a \ast (b \ast c) = a \ast (b \ast c) = 0$, $a, b, c \in V$. It is immediately obvious that

$$\dim(Q(1)) = 1, \quad \dim(Q(2)) = 3 \quad \text{and} \quad \dim(Q(d)) = \dim(\text{Com}(d)) = 1 \quad \text{for} \quad d \geq 3,$$

where $\text{Com}$ denotes the operad for commutative associative algebras. The generating function for $Q$ therefore equals $q(t) = e^t - 1 + t^2$ and equation (23) gives $e^{-p(t)} - 1 + p(t)^2 = -t$, which is equivalent to (17). We proved that the generating functions $g(t)$ and $p(t)$ satisfy the same functional equation and, by definition, the same initial condition $p(0) = g(0) = 0$, therefore they coincide and $\dim(H^0(\text{Gr}_d^0\bigvee(d), \delta)) = \dim(P(d))$ for each $d \geq 1$.

In the rest of this section we study operators in $\text{Nat}_1(\text{Con} \times T^{\infty}, \mathbb{R})$. Roughly speaking, we prove that all operators in this space are traces in the following sense. Let $\mathcal{O} \in \text{Nat}(\text{Con} \times T^{\infty}, T)$ be an operator acting on vector fields $X_0, X_1, X_2, \ldots$ and a connection $\Gamma$. Suppose that $\mathcal{O}$ is a linear order 0 differential operator in $X_0$. This means that the local formula $O(X_0, X_1, X_2, \ldots, \Gamma) \in \mathbb{R}$ for $\mathcal{O}$ is a linear function of $X_0$ and does not contain derivatives of $X_0$. For such an operator we define $Tr_{X_0}(\mathcal{O}) \in \text{Nat}(\text{Con} \times T^{\infty}, \mathbb{R})$ by the local formula

$$Tr_{X_0}(O)(X_1, X_2, \ldots, \Gamma) := \text{Trace}(O(-, X_1, X_2, \ldots, \Gamma) : \mathbb{R}^n \to \mathbb{R}^n) \in \mathbb{R}.$$ 

It is easy to see that $Tr_{X_0}(\mathcal{O})$ is well defined. Let us formulate a structure theorem for multilinear operators from $\text{Nat}_1(\text{Con} \times T^{\infty}, \mathbb{R})$.

7.6. Theorem. Let $d \geq 0$. On smooth manifolds of dimension $\geq 2d$, each $d$-multilinear operator in $\text{Nat}_1(\text{Con} \times T^d, \mathbb{R})$ is the trace of a $(d+1)$-multilinear operator from $\text{Nat}_1(\text{Con} \times T^{d+1}, T)$.

Theorem 7.6 will follow from Proposition 7.8 below. A depolarized version of Theorem 7.6 is:

7.7. Corollary. On a smooth manifold $M$, each operator from $\text{Nat}_1(\text{Con} \times T^{\infty}, \mathbb{R})$ whose all components are of homogeneity $\leq \frac{1}{2} \dim(M)$ is a trace of an operator from $\text{Nat}_1(\text{Con} \times T^{\infty}, T)$.

Denote by $\text{Gr}_d^0\bigvee(d)$ the graph complex describing operators in $\text{Nat}_1(\text{Con} \times T^{d}, \mathbb{R})$. The degree $m$-component of this complex is spanned by connected graphs with $d$ vertices (17) labelled...
$X_1, \ldots, X_d$, some number of vertices (xFFFF) labelled $\nabla$ and $m$ white vertices (g0v). It is not difficult to see that the number of edges of graphs spanning $\mathcal{G}_{\mathcal{V} \cup}^0(d)$ is $\leq 2d$, which explains the stability assumption in Theorem 7.6.

We will also consider the subcomplex $\mathcal{G}_{\mathcal{V} \cup}^m(d) \subset \mathcal{G}_{\mathcal{V} \cup}^m(d+1)$ of graphs describing operators in $\mathcal{G}_{\mathcal{V} \cup}(Con \times T^{\otimes(d+1)}, T)$ for which the trace is defined. Clearly, the degree $m$ component $\mathcal{G}_{\mathcal{V} \cup}^m(d)$ of this subcomplex is spanned by connected graphs with one vertex $\downarrow$ labelled $X_0$, one vertex $\uparrow$, $d$ vertices (FFFF) labelled $X_1, \ldots, X_d$, a finite number of vertices (FFFF) labelled $\nabla$ and $m$ white vertices (g0v). The trace is represented by the map $Tr : \mathcal{G}_{\mathcal{V} \cup}^m(d) \rightarrow \mathcal{G}_{\mathcal{V} \cup}^m(d)$ that removes the vertices $\downarrow$ and $\uparrow$ and connects the two loose edges created in this way by a directed wheel. It is clear that this map commutes with the differentials. We now establish Theorem 7.6 by proving the following.

7.8. Proposition. The map $Tr : (\mathcal{G}_{\mathcal{V} \cup}^m(d), \delta) \rightarrow (\mathcal{G}_{\mathcal{V} \cup}^m(d), \delta)$ induces an epimorphism of cohomology $H^0(\mathcal{G}_{\mathcal{V} \cup}^m(d), \delta) \rightarrow H^0(\mathcal{G}_{\mathcal{V} \cup}^m(d), \delta)$.

Proof. As in the proof of Proposition 7.4 we observe that both $(\mathcal{G}_{\mathcal{V} \cup}^m(d), \delta)$ and $(\mathcal{G}_{\mathcal{V} \cup}^m(d), \delta)$ are bicomplexes, with $\mathcal{G}_{\mathcal{V} \cup}^{p,q}(d)$ (resp. $\mathcal{G}_{\mathcal{V} \cup}^{p,q}(d)$) spanned by graphs in $\mathcal{G}_{\mathcal{V} \cup}^{p,q}(d)$ (resp. $\mathcal{G}_{\mathcal{V} \cup}^{p,q}(d)$) with precisely $-p$ $\nabla$-vertices. The differential in both complexes decomposes as $\delta = \delta' + \delta''$ where $\delta'$ (the ‘horizontal part’) raises the $p$-degree by one and preserves the $q$-degree, and $\delta''$ (the ‘vertical part’) preserves the $q$-degree and raises the $p$-degree by one.

The map $Tr : (\mathcal{G}_{\mathcal{V} \cup}^m(d), \delta) \rightarrow (\mathcal{G}_{\mathcal{V} \cup}^m(d), \delta)$ obviously preserves the bigradings, therefore it induces the map

$H^0(Tr, \delta') : H^0(\mathcal{G}_{\mathcal{V} \cup}^m(d), \delta') \rightarrow H^0(\mathcal{G}_{\mathcal{V} \cup}^m(d), \delta')$

of the horizontal cohomology. Using the same considerations as in the proof of Proposition 7.4, we identify this map with

$Tr : \mathcal{G}_{\mathcal{V} \cup}^m[\mathcal{H}^*(E, \partial_E)](d) \rightarrow \mathcal{G}_{\mathcal{V} \cup}^m[\mathcal{H}^*(E, \partial_E)](d),$

where $(E^*, \partial_E)$ is the dg-collection introduced in Remark 7.3 and the graph complexes in (56) are defined analogously as the graph complex $\mathcal{G}_{\mathcal{V} \cup}^m[\mathcal{H}^*(E, \partial_E)](d)$ used in the proof of Proposition 7.4.

Let us show that the map in (56) is an epimorphism. Consider a graph $G$ in $\mathcal{G}_{\mathcal{V} \cup}^m[\mathcal{H}^*(E, \partial_E)](d)$ and choose a directed edge $e$ in the (unique) wheel of $G$. Let $\hat{G}$ be the graph in $\mathcal{G}_{\mathcal{V} \cup}^m[\mathcal{H}^*(E, \partial_E)](d)$ obtained by cutting $e$ in the middle and decorating the loose ends thus created by vertices $\downarrow$ and $\uparrow$ as in the following display:

Clearly $Tr(\hat{G}) = G$ which proves that (56) is surjective. So, we have two spectral sequences, $(E^{p,q}_0, d_0)$ and $(F^{p,q}_0, d_0)$, such that

$(E^{p,q}_0, d_0) = (\mathcal{G}_{\mathcal{V} \cup}^{p,q}(d), \delta)$, $(F^{p,q}_0, d_0) = (\mathcal{G}_{\mathcal{V} \cup}^{p,q}(d), \delta),$

and the map $Tr : (E^{p,q}_0, d_0) \rightarrow (F^{p,q}_0, d_0)$ induced by the trace map $Tr : \mathcal{G}_{\mathcal{V} \cup}^m(d) \rightarrow \mathcal{G}_{\mathcal{V} \cup}^m(d)$. The map $Tr : (E^{p,q}_1, d_1) \rightarrow (F^{p,q}_1, d_1)$ of the first levels of the spectral sequences is (FFFF) and we
identified this map with epimorphism (56). It is also clear that the first terms of both spectral
sequences are supported by the diagonal $p + q = 0$, so these spectral sequences degenerate
at this level. A standard argument then implies that the map $H^0(Tr) : H^0(\mathfrak{g}_+^* \mathcal{V}_T, (d), \delta) \rightarrow
H^0(\mathfrak{g}_+^* \mathcal{V}_0, (d), \delta)$ in Proposition 7.8 is an epimorphism. □

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