An overview of the balanced excited random walk

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Abstract The balanced excited random walk, introduced by Benjamini, Kozma and Schapira in 2011, is defined as a discrete time stochastic process in $\mathbb{Z}^d$, depending on two integer parameters $1 \leq d_1, d_2 \leq d$, which whenever it is at a site $x \in \mathbb{Z}^d$ at time $n$, it jumps to $x \pm e_i$ with uniform probability, where $e_1, \ldots, e_d$ are the canonical vectors, for $1 \leq i \leq d_1$, if the site $x$ was visited for the first time at time $n$, while it jumps to $x \pm e_i$ with uniform probability, for $1 + d - d_2 \leq i \leq d$, if the site $x$ was already visited before time $n$. Here we give an overview of this model when $d_1 + d_2 = d$ and introduce and study the cases when $d_1 + d_2 > d$. In particular, we prove that for all the cases $d \geq 5$ and most cases $d = 4$, the balanced excited random walk is transient.

1 Introduction

We consider an extended version of the balanced excited random walk introduced by Benjamini, Kozma and Schapira in [1]. The balanced excited random walk is defined in any dimension $d \geq 2$, and depends on two integers $d_1, d_2 \in \{1, \ldots, d\}$. For each $1 \leq i \leq d$, let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the canonical vector whose $i$-th coordinate is 1, while all other coordinates are 0. We define the process $(S_n : n \geq 0)$,
called the \textit{balanced excited random walk} on $\mathbb{Z}^d$ as a mixture of two simple random walks, with the initial condition $S_0 = 0$: if at time $n$, $S_n$ visits a site for the first time, with probability $1/(2d_1)$, at time $n + 1$ it performs a simple random walk step using one of the first $d_1$ coordinates, so that for all $1 \leq i \leq d_1$,

$$
\mathbb{P}[S_{n+1} - S_n = e_i | \mathcal{F}_n, S_n \neq S_i \text{ for all } 1 \leq j < n] = \frac{1}{2d_1},
$$

where $\mathcal{F}_n$ is the $\sigma$-algebra generated by $S_0, \ldots, S_n$; on the other hand, if at time $n$, $S_n$ visits a site it has previously visited, at time $n + 1$ it performs a simple random walk using one the last $d_2$ coordinates, so that for all $d - d_2 + 1 \leq i \leq d$,

$$
\mathbb{P}[S_{n+1} - S_n = e_i | \mathcal{F}_n, S_n = S_j \text{ for some } 1 \leq j < n] = \frac{1}{2d_2}.
$$

We call this process $S$ the $M_d(d_1, d_2)$-random walk. In \cite{1}, this random walk was considered in the case when $d_1 + d_2 = d$, which we call the \textit{non-overlapping} case. Here we will focus on the \textit{overlapping} case corresponding to $d_1 + d_2 > d$.

We say that the $M_d(d_1, d_2)$-random walk is transient if any site is visited only finitely many times, while we say that it is recurrent if it visits every site infinitely often. Since a random walk $M_d(d_1, d_2)$ is not Markovian, in principle could be neither transient nor recurrent.

For the non-overlapping case, in 2011 in \cite{1} it was shown that the $M_4(2, 2)$-random walk is transient, while in 2016, Peres, Schapira and Sousi in \cite{7}, showed that the $M_3(1, 2)$-random walk is transient, but the transience of $M_3(2, 1)$-random walk is still an open question.

The main result of this article is the following theorem concerned with the \textit{overlapping} case.

\textbf{Theorem 1.} For every $(d, d_1, d_2)$ with $d \geq 4$, $1 \leq d_1, d_2 \leq d$, $d_1 + d_2 > d$ and $(d, d_1, d_2) \neq (4, 3, 2)$, the $M_d(d_1, d_2)$-random walk is transient.

Theorem\cite{1} has a simple proof for $d \geq 7$, for all admissible values of $d_1$ and $d_2$. Let $r := d_1 + d_2 - d$. Note that if $r \geq 3$ then the walk is transient, since its restriction to the $r$ overlapping coordinates is at least a 3-dimensional simple symmetric random walk with geometrically bounded holding times. We will argue in the next two paragraphs that the walk is also transient if $d_1 - r \geq 3$ or if $d_2 - r \geq 3$. Assuming for the moment that each of the three inequalities $r \geq 3$, $d_1 - r \geq 3$ or $d_2 - r \geq 3$ implies transience, note that if none of them holds we have that $d = d_1 + d_2 - r \leq 6$. We conclude that for $d \geq 7$ the walk is transient for all admissible values of $d_1$ and $d_2$.

\textit{Case $d_2 - r \geq 3$.} We claim that with probability 1 the fraction of times when the random walk uses the last $d_2 - r$ coordinates is asymptotically bounded from below by a positive constant and therefore, the random walk is transient. To see this note that whenever the walk makes 3 consecutive steps, the probability that in at least one of these steps it visits a previously visited (old) site is bounded away from 0. Indeed, if in two consecutive steps the walk visits two previously unvisited (new)
sites then with probability $1/(2d_1)$ it backtracks in the next step and, thus, visits an old site.

Case $d_1 - r \geq 3$. We claim that the number of times the random walk uses the first $d_1 - r$ coordinates goes to infinity as $n \to \infty$, which is enough to prove transience. Denote by $r_n$ the number of points in the range of the walk at time $n$. We will show that $r_n \to \infty$ as $n \to \infty$ a.s. For $k \geq 1$, let $n_k = \inf\{n \geq 0 : r_n = k\}$. We will argue that if $n_k < \infty$ then with probability one $n_{k+1} < \infty$. Note that $r_{n_k} = k$, $S_{n_k}$ is a new site, and there are $k - 1$ other sites in the range. Let $n_k < \infty$ and $A_1$ be the event that in the next $k$ steps the walk jumps only in positive coordinate directions. On $A_1$, at times $n_k + 1, n_k + 2, \ldots, n_k + k$ the walk visits $k$ distinct sites of $\mathbb{Z}^d - \{S_{n_k}\}$. Among these sites there are at most $k - 1$ old sites. Therefore, on the event $A_1 \cap \{n_k < \infty\}$ the walk will necessarily visit a new site and $n_{k+1} \leq n_k + k < \infty$. Note that the probability of $A_1$ (given $n_k < \infty$) is $2^{-k}$. If $A_1$ does not occur, then we consider the next $k$ steps and define $A_2$ to be the event that in these next $k$ steps the walk jumps only in the positive coordinate directions, and so on. Since, conditional on $n_k < \infty$, the events $A_1, A_2, \ldots$ are independent and each has probability $2^{-k}$, we conclude that $n_{k+1} < \infty$ with probability one.

Therefore, to complete the proof of Theorem 1 we have only to consider the cases $d = 4, 5, 6$. It will be shown below that the cases $d = 5, 6$ and several cases in $d = 4$, can be derived in an elementary way sometimes using the trace condition of [6]. In a less straightforward way the cases $M_d(2, 4)$ and $M_d(4, 2)$ can be treated through the methods of [11]. The case $M_d(2, 3)$ which is more involved, can be treated through a modification of methods developed by Peres, Schapira and Sousi [7] for the $M_1(1, 2)$-random walk through good controls on martingale increments by sequences of geometric i.i.d. random variables. It is not clear how the above mentioned methods could be applied to the $M_d(3, 2)$-random walk to settle down the transience-recurrence question for it, so this case remains open.

In Section 2 we will give a quick review of the main results that have been previously obtained for the non-overlapping case of the balanced excited random walk. In Section 3 we will prove Theorem 1. In Section 3.1 we will introduce the trace condition of [6], which will be used to prove the cases $d = 5, 6$ and several cases in dimension $d = 4$. In Section 3.2 we will prove the transience of the random walks $M_d(2, 4)$ and $M_d(4, 2)$. While in Section 3.3 we will consider the proof of the transience of the $M_d(2, 3)$-random walk.

2 Overview of the balanced excited random walk

The balanced excited random walk was introduced in its non-overlapping version by Benjamini, Kozma and Schapira in [11]. A precursor of the balanced excited random walk, is the excited random walk, introduced by Benjamini and Wilson in 2003 [2], which is defined in terms of a parameter $0 < p < 1$ as follows: the random walk $(X_n : n \geq 0)$ has the state space $\mathbb{Z}^d$ starting at $X_0 = 0$; whenever the random walk visits a site for the first time, it jumps with probability $(1 + p)/2d$ in direction
$e_1$, probability $(1 - p)/2d$ in direction $-e_1$ and with probability $1/2d$ in the other directions; whenever the random walk visits a site which it already visited previously it jumps with uniform probability in directions $\pm e_i, 1 \leq i \leq d$. Benjamini and Wilson proved in [2] that the model is transient for $d > 1$. A central limit theorem and a law of large numbers for $d > 1$ was proven in [3] and [4]. A general review of the model can be found in [5]. Often the methods used to prove transience, the law of large numbers and the central limit theorem for the excited random walk, are based on the ballisticity of the model (the fact that the velocity is non-zero), through the use of regeneration times. This means that most of these methods are not well suited to study the balanced excited random walk, which is not ballistic. For the moment, a few results have been obtained for the balanced excited random walk, where basically for each case a different technique has been developed. The first result was obtained by Benjamini, Kozma and Schapira in [1] for the $M_4(2, 2)$ case is the following theorem.

**Theorem 2 (Benjamini, Kozma and Schapira, 2011).** The $M_4(2, 2)$-random walk is transient.

The proof of Theorem 2 is based on obtaining good enough estimates for the probability that a 2-dimensional random walk returns to its starting point in a time interval $[n/c(\log n)^2, cn]$, for some constant $c > 0$, and on the range of the random walk. This then allows to decouple using independence the first 2 coordinates from the last 2 ones. In this article, we will apply this method to derive the transience in the $M_4(4, 2)$ and $M_4(2, 4)$ cases of Theorem 1.

In 2016, Peres, Sousi and Schapira in [7], considered the case $M_3(1, 2)$ proving the following result.

**Theorem 3 (Peres, Schapira and Sousi, 2016).** The $M_3(1, 2)$-random walk is transient.

The proof of Theorem 3 is based on obtaining good enough estimates for the probability that a 2-dimensional random walk returns to its starting point in a time interval $[n/c(\log n)^2, cn]$, for some constant $c > 0$, and on the range of the random walk. This then allows to decouple using independence the first 2 coordinates from the last 2 ones. In this article, we will apply this method to derive the transience in the $M_4(4, 2)$ and $M_4(2, 4)$ cases of Theorem 1.

**3 Proof of Theorem 1**

We will divide the proof of Theorem 1 in three steps. With the exception of the cases $M_4(1, 4), M_4(4, 1), M_4(2, 4), M_4(4, 2)$ and $M_4(2, 3)$, we will use an important result
of Peres, Popov and Sousi \cite{6}. The cases \( M_4(1,4) \) and \( M_4(4,1) \) will be derived as those in dimension \( d \geq 7 \). For the cases \( M_4(2,4) \) and \( M_4(4,2) \) we will show how the argument of \cite{1} can be adapted. And the case \( M_4(2,3) \) is handled as in \cite{7}.

3.1 The trace condition

Here we will recall the so called trace condition of \cite{6} which is a general condition under which a generalized version of the balanced random walk is transient, and see how it can be used to prove Theorem 1 for the cases different from \( M_4(3,4) \), \( M_4(4,3) \), \( M_4(3,3) \) and \( M_4(4,4) \).

Given \( d \geq 1 \) and \( m \geq 1 \), consider probability measures \( \mu_1, \ldots, \mu_m \) on \( \mathbb{R}^d \) and for each \( 1 \leq i \leq m \), let \( (\xi_{n}^{i} : n \geq 1) \) be an i.i.d. sequence of random variables distributed according to \( \mu_i \). We say that a stochastic process \( (\ell_{k} : k \geq 0) \) is an adapted rule with respect to a filtration \( (\mathcal{F}_n : n \geq 0) \) of the process, if for each \( k \geq 0 \), \( \ell_{k} \) is \( \mathcal{F}_k \)-measurable. We now define the random walk \( (X_n : n \geq 0) \) generated by the probability measures \( \mu_1, \ldots, \mu_m \) and the adapted rule \( \ell \) by

\[
X_{n+1} = X_n + \xi_{n+1}^{\ell_{n+1}}, \quad \text{for} \quad n \geq 0.
\]

Let \( \mu \) be a measure on \( \mathbb{R}^d \). \( \mu \) is called of mean 0 if \( \int x \, d\mu = 0 \). The measure \( \mu \) is said to have \( \beta \) moments if for any random variable \( Z \) distributed according to \( \mu \), \( ||Z|| \) has moment of order \( \beta \). The covariance matrix of \( \mu \), \( \text{Var}(\mu) \), is defined as the covariance of \( Z \).

Given a matrix \( A \), we call \( \lambda_{\text{max}}(A) \) its maximal eigenvalue and \( A' \) its transpose. In \cite{6}, the following result was proven.

**Theorem 4 (Peres, Popov and Sousi, 2013).** Let \( \mu_1, \ldots, \mu_m \) be measures in \( \mathbb{R}^d \), \( d \geq 3 \), with zero mean and \( 2 + \beta \) moments, for some \( \beta > 0 \). Assume that there is a matrix \( A \) such that the trace condition is satisfied:

\[
\text{tr}(A \, \text{Var}(\mu_i) \, A') > 2\lambda_{\text{max}}(A \, \text{Var}(\mu_i) \, A')
\]

for all \( 1 \leq i \leq m \). Then any random walk \( X \) generated by these measures and any adapted rule is transient.

It follows from Theorem \cite{4} that whenever \( d_1 \geq 3 \) and \( d_2 \geq 3 \), the trace condition is satisfied, with \( A = I \), for the two corresponding matrices associated to the first \( d_1 \) and last \( d_2 \) dimensions, and hence the \( M_4(d_1,d_2) \)-random walk is transient. Hence, by the discussion right after the statement of Theorem \cite{4} in Section \cite{4} we see that the only cases which are not covered by Theorem \cite{4} correspond to

\[
d_1 - r \leq 2, \quad r \leq 2 \quad \text{and} \quad d_2 - r \leq 2,
\]

and
\[
\min\{d_1, d_2\} \leq 2. 
\]

But (1) implies that \(\max\{d_1, d_2\} \leq 2 + r\). Thus,
\[
d_1 + d_2 = \max\{d_1, d_2\} + \min\{d_1, d_2\} \leq 4 + r,
\]
so that \(d = d_1 + d_2 - r \leq 4\). This proves the transience for all the cases when \(d \geq 5\).

Now note that in dimension \(d = 4\) the random walks \(M_4(3, 3), M_4(3, 4), M_4(4, 3)\) and \(M_4(4, 4)\) satisfy \(d_1 \geq 3\) and \(d_2 \geq 3\), so that the trace condition of [6] is satisfied.

Finally, that the random walks \(M_4(1, 4)\) and \(M_4(4, 1)\) satisfy \(r \geq 3, d_1 - r \geq 3\) or \(d_2 - r \geq 3\), so that they are also transient.

### 3.2 The random walks \(M_4(2, 4)\) and \(M_4(4, 2)\)

Consider the \(M_4(4, 2)\)-random walk and call \(r_n\) the cardinality of its range at time \(n\). Let us use the notation \(S = (X, Y)\) for the \(M_4(4, 2)\)-random walk, where \(X\) are the first two components and \(Y\) the last two ones. We will also call \(r_n^{(1)}\) the number of times up to time \(n\) that the random walk jumped using the \(X\) coordinates while it was at a site that it visited for the first time and \(r_n^{(2)} := r_n - r_n^{(1)}\). In analogy with Lemma 1 of [1], we have the following result.

**Lemma 1.** For any \(M > 0\) and each \(i = 1, 2\), there exists a constant \(C > 0\) such that
\[
P[n/(C \log n)^2 \leq r_n^{(i)} \leq 99n/100] = 1 - o\left(n^{-M}\right). \tag{2}
\]

**Proof.** First note that in analogy to the proof Lemma 1 of [1], we have that
\[
P[n/(C \log n)^2 \leq r_n \leq 99n/100] = 1 - o\left(n^{-M}\right).
\]

Since each time the random walk is at a newly visited site with probability \(1/2\) it jumps using the \(X\) random walk and with probability \(1/2\) the \(Y\) random walk, by standard large deviation estimates, we deduce (3).

Now note that
\[
\{(X_k, Y_k) : k \geq 1\} = \{(U_1(r_k^{(1)}), U_2(r_k^{(2)}) + V(k - r_{k-1})) : k \geq 1\}, \tag{3}
\]
where \(U_1, U_2\) and \(V\) are three independent simple random walks in \(\mathbb{Z}^2\). It follows from the identity (3) and Lemma 1 used to bound the components \(r_n^{(1)}\) and \(r_n^{(2)}\) of the range of the walk, that
\[
P[0 \in \{S_n, \ldots, S_{2n}\}] \leq P[0 \in \{U(n/(C \log n)^2), \ldots, U(2n)\}]
\times P[0 \in \{W(n/(C \log n)^2), \ldots, W(2n)\} + o(n^{-M})], \tag{4}
\]
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where \( U \) and \( W \) are simple symmetric random walks on \( \mathbb{Z}^2 \). At this point, we recall Lemma 2 of [1].

**Lemma 2 (Benjamini, Kozma and Schapira, 2011).** Let \( U \) be a simple random walk on \( \mathbb{Z}^2 \) and let \( t \in [n/(\log n)^3, 2n] \). Then

\[
P[0 \in \{U(t), \ldots, U(2n)\}] = O\left(\frac{\log \log n}{\log n}\right).
\]

Combining inequality (4) with Lemma 2 we conclude that there is a constant \( C > 0 \) such that for any \( n > 1 \) (see Proposition 1 of [1])

\[
P[0 \in \{S_n, \ldots, S_{2n}\}] \leq C\left(\frac{\log \log n}{\log n}\right)^2.
\]

Hence,

\[
\sum_{k=0}^{\infty} P[0 \in \{S_{2k}, \ldots, S_{2k+1}\}] < \infty,
\]

and the transience of the \( M_4(4, 2) \)-random walk follows form Borel-Cantelli. A similar argument can be used to prove the transience of the \( M_4(2, 4) \)-random walk.

### 3.3 The \( M_4(2, 3) \)-random walk

Here we will follow the method developed by Peres, Schapira and Sousi in [7]. We first state Proposition 2.1 of [7].

**Proposition 1 (Peres, Schapira and Sousi, 2016).** Let \( \rho > 0 \) and \( C_1, C_2 > 0 \). Let \( M \) be a martingale with quadratic variation \( V \) and assume that \( (G_k : k \geq 0) \) is a sequence of i.i.d. geometric random variables with mean \( C_1 \) such that for all \( k \geq 0 \),

\[
|M_{k+1} - M_k| \leq C_2 G_k.
\]

For all \( n \geq 1 \) and \( 1 \leq k \leq \log_2(n) \) let \( t_k := n - \frac{k}{2} \) and

\[
A_k := \left\{ V_{k+1} - V_k \geq \rho \frac{t_{k+1} - t_k}{(\log n)^{2a}} \right\}.
\]

Suppose that for some \( N \geq 1 \) and \( 1 \leq k_1 < \cdots < k_N < \log_2(n)/2 \) one has that

\[
P\left( \cap_{i=1}^{N} A_{k_i} \right) = 1.
\]

Then, there exists constant \( c > 0 \) and a positive integer \( n_0 \) such that for all \( a \in (0, 1) \) and \( n \geq n_0 \) one has that
\[ \mathbb{P}(M_n = 0) \leq \exp\left(-cN/(\log n)^{\alpha}\right). \]

**Remark 1.** Proposition [1] is slightly modified with respect to Proposition 2.1 of [7] since we have allowed the mean \( C_1 \) of the geometric random variables to be arbitrary and the bound (5) to have an arbitrary constant \( C_2 \).

Let us now note that the \( M_d(2,3) \)-random walk \((S_n : n \geq 0)\) can be defined as follows. Suppose \((\zeta_n : n \geq 1)\) is a sequence of i.i.d. random variables taking each of the values \((0, \pm 1, 0, 0), (0, 0, \pm 1, 0)\) and \((0, 0, 0, \pm 1)\) with probability \(1/6\), while \((\xi_n : n \geq 1)\) is a sequence of i.i.d. random variables (independent from the previous sequence) taking each of the values \((0, \pm 1, 0,0)\) and \((\pm 1, 0,0,0)\) with probability \(1/4\). Define now recursively, \(S_0 = 0\), and

\[ S_{n+1} = S_n + \Delta_{n+1} \]

where the step is

\[ \Delta_{n+1} = \begin{cases} \xi_{r_n}, & \text{if } r_n = r_{n-1} + 1 \\ \xi_{\Delta_{n+1} - r_n}, & \text{if } r_n = r_{n-1} \end{cases} \]

and \( r_n = \#\{S_0, \ldots, S_n\} \) as before is the cardinality of the range of the random walk at time \( n \) (note that formally \( r_{-1} = 0 \)).

Let us now write the position at time \( n \) of the \( M_d(2,3) \) random walk as

\[ S_n = (X_n, Y_n, Z_n, W_n). \]

Define recursively the sequence of stopping times \((\tau_k : k \geq 0)\) by \( \tau_0 = 0 \) and for \( k \geq 1 \),

\[ \tau_k := \inf\{n > \tau_{k-1} : (Z_n, W_n) \neq (Z_{\tau_{k-1}}, W_{\tau_{k-1}})\}. \]

Note that \( r_0 = 1 \) and \( \tau_k < \infty \) a.s. for all \( k \geq 0 \). Furthermore, the process \((U_k : k \geq 0)\) defined by

\[ U_k := (Z_{\tau_k}, W_{\tau_k}), \]

is a simple random walk in dimension \( d = 2 \), and is equal to the simple random walk with steps defined by the last two coordinates of \( \zeta \). Let us now call \( P_U \) the law of \( S \) conditionally on the whole \( U \) process. Note that the first coordinate \( \{X_n : n \geq 0\} \) is an \( \mathcal{F}_n := \sigma\{\Delta_k : k \leq n\} \)-martingale with respect to \( P_U \), since

\[ E_U(X_{n+1} - X_n \mid \mathcal{F}_n) = 1_{r_n=r_{n-1}+1} E(\xi_{r_n} \cdot e_1 \mid \mathcal{F}_n, U), \]

\( U \) is \( \sigma(\zeta_k : k \geq 1) \)-measurable as it is defined only in terms of the sequence \((\zeta_k 1_{\{\pi_{34}(\zeta_k) \neq 0\}})_{k \geq 1}\), \((\pi_{34}) \) being the projection in the 3rd and 4th coordinates, and
by independence. Hence, \(\{M_m : m \geq 0\}\) with \(M_m := X_{t_m}\) is a \(\mathcal{F}_m\)-martingale with respect to \(P_U\), where \(\mathcal{F}_m := \mathcal{F}_{t_m}\). To prove the theorem, it is enough to show that \(\{(M_n, U_n) : n \geq 0\}\) is transient (under \(P\)). Let us call \(r_U(n)\) the cardinality of the range of the random walk \(U\) at time \(n\). For each \(n \geq 0\) and \(k \geq 0\), let
\[
t_k := n - n/2^k
\]
and
\[
\mathcal{K} := \left\{ k \in \{1, \ldots, (\log n)^{3/4}\} : r_U(t_{k+1}) - r_U(t_k) \geq \rho (t_{k+1} - t_k)/\log n \right\}.
\]
We will show that
\[
P(M_n = U_n = 0) = E[P_U(M_n = 0)1\{|\mathcal{K}| \geq \rho (\log n)^{3/4}, U_n = 0\}] + E[P_U(M_n = 0)1\{|\mathcal{K}| < \rho (\log n)^{3/4}, U_n = 0\}],
\]
is summable in \(n\), for \(\rho = \rho_0\) chosen appropriately. At this point, let us recall Proposition 3.4 of \([7]\), which is a statement about simple symmetric random walks.

**Proposition 2 (Peres, Schapira and Sousi, 2016).** For \(k \geq 1\), consider \(t_k\) as defined in \((7)\). Then, for \(\mathcal{K}\) as defined in \((8)\), we have that there exist positive constants \(\alpha, C_3, C_4\) and \(\rho_0\) such that for all \(\rho < \rho_0\),
\[
P(|\mathcal{K}| \leq \rho (\log n)^{3/4} | U_n = 0) \leq C_3 e^{-C_4 (\log n)^\alpha}.
\]
Choosing \(\rho = \rho_0 \leq 1\) small enough, by Proposition 2 we have the following bound for the second term on the right-hand side of \((9)\).
\[
E[P_U(M_n = 0)1\{|\mathcal{K}| < \rho_0 (\log n)^{3/4}, U_n = 0\}] \leq C_5 \frac{1}{n} \exp(-C_6 (\log n)^\alpha),
\]
for some positive \(C_5\) and \(\alpha\), where we have used the fact that \(P(U_n = 0) \leq \frac{C_5}{n}\) for some constant \(C_5 > 0\).

To bound the first term on the right-hand side of \((9)\), we will use Proposition 1 with \(a = 1/2\) and \(\rho = \rho_0/4\). Let us first show that \((6)\) is satisfied. Indeed, note that for each \(n \geq 0\) when \(U_n\) is at a new site, \(E_U[|M_{n+1} - M_n|^2] \geq 1/2\). Therefore, for all \(k \in \mathcal{K}\), with \(\rho = \rho_0\), one has that
\[
\begin{align*}
V_{t_{k+1}} - V_t = & \sum_{n=t_k+1}^{t_{k+1}} E_U[(M_n - M_{n-1})^2 | \mathcal{F}_{n-1}] \\
\geq & \left( r_U(t_{k+1} - 1) - r_U(t_k - 1) \right)/2 \geq \left( r_U(t_{k+1}) - r_U(t_k) - 1 \right)/2 \\
\geq & \left( \rho_0/4 \right) (t_{k+1} - t_k)/(\log n)^{2\alpha}.
\end{align*}
\]
Hence, on the event $|\mathcal{X}| \geq \rho_0 (\log n)^{3/4}$, we have that there exist $k_1, \ldots, k_N \in \mathcal{X}$ with $N = \lceil \rho_0 (\log n)^{3/4} \rceil$ such that

$$P_U \left( \cap_{i=1}^N A_k \right) = 1.$$  

Let us now show that there is a sequence of i.i.d. random variables $(G_k : k \geq 0)$ such that (5) is satisfied with $C_1 = 24$ and $C_2 = 3$. Indeed, note that

$$|M_{n+1} - M_n| = |X_{\tau_{n+1}} - X_{\tau_n}| \leq \sum_{k=\tau_0}^\infty |X_{(k+1)\wedge \tau_{n+1}} - X_{k\wedge \tau_{n+1}}|. \quad (11)$$

Note that the right-hand side of (11) is the number of steps of $X$ between times $\tau_n$ and $\tau_{n+1}$. Now, at each time $k$ (with $k$ starting at $\tau_n$) that a step in $X$ is made there is a probability of at least $\frac{1}{2^3} \times \frac{2}{3} = \frac{1}{3}$ that the random walk $S$ makes three successive steps at times $k+1$, $k+2$ and $k+3$, such that in one of them a step in $U$ is made and at most two of these steps are of the $U$ random walk: if the random walk is at a site previously visited at time $k$, with probability $2/3$ at time $k+1$ the $U$ random walk will move; if the random walk is at a site which it had never visited before at time $k$, with probability $\frac{1}{2^3} \times \frac{2}{3} = \frac{1}{3}$, there will be $3$ successive steps of $S$ at times $k+1$, $k+2$ and $k+3$, with the first $2$ steps being of the $X$ random walk and the third step of $U$ (we just need to move in the $e_1$ direction using $X$ at time $k+1$, immediately follow it at time $k+2$ by a reverse step in the $-e_1$ direction using $X$ again, and then immediately at time $k + 3$ do a step in $U$). Since this happens independently each $3$ steps in the time scale of $X$ (time increases by one unit whenever $X$ moves), we see that we can bound the martingale increments choosing i.i.d. geometric random variables $(G_k : k \geq 0)$ of parameter $1/24$ in (5) multiplied by 3.

Remark 2. The sequence of i.i.d. geometric random variables constructed above is not the optimal one, in the sense that it is possible to construct other sequences of i.i.d. geometric random variables of parameter larger than $1/24$.

Since now we know that (6) and (5) are satisfied, by Proposition 1, there exist $n_0 \geq 1$ and $C_7 > 0$ such that on the event $|\mathcal{X}| \geq \rho_0 (\log n)^{3/4}$ we have that for $n \geq n_0$,

$$P_U(M_n = 0) \leq e^{-C_7 \rho_0 (\log n)^{3/4}}.$$ 

Hence, for $n \geq n_0$ we have

$$E[P_U(M_n = 0) 1\{|\mathcal{X}| \geq \rho_0 (\log n)^{3/4}, U_n = 0\}] \leq C_3 \frac{1}{n} e^{-C_7 \rho_0 \frac{(\log n)^{3/4}}{\log n}}. \quad (12)$$

Using the bounds (10) and (12) back in (2) gives us that there exist constants $C_8 > 0$, $C_9 > 0$ and some $\beta > 0$, such that

$$P(M_n = U_n = 0) \leq \frac{1}{n} C_8 e^{-C_9 (\log n)^{\beta}}$$
An overview of the balanced excited random walk

By the Borel-Cantelli lemma, we conclude that the process \((M, U)\) is transient, which gives the transience of \(S\).

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