A Dynamic Near-Optimal Algorithm for Online Linear Programming

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Abstract

We consider the online linear programming problem where the constraint matrix is revealed column by column along with the objective function. We provide a $1-o(1)$ competitive algorithm for this surprisingly general class of online problems under the assumption of random order of arrival and some mild conditions on the right-hand-side input. Our learning-based algorithm works by dynamically updating a threshold price vector at geometric time intervals, the price learned from the previous steps is used to determine the decision for the current step. Our result provides a common near-optimal solution to a wide range of online problems including online routing and packing, online combinatorial auction, online adwords matching, many secretary problems, and various resource allocation and revenue management problems. Apart from online problems, the algorithm can also be applied for fast solution of large linear programs by sampling the columns of constraint matrix.

1 Introduction

Online optimization is attracting an increasingly wide attention in computer science, operations research, and management science communities because of its wide applications to electronic markets and dynamic resource allocation problems. In many practical problems, data does not reveal itself at the beginning, but rather comes in an online fashion. For example, in the online combinatorial auction problem, consumers arrive sequentially requesting a subset of goods, each offering a certain bid price for their demand. On observing a request, the seller needs to make an irrevocable decision for that consumer with the overall objective of maximizing the revenue or other social welfare while respecting the inventory constraints. Similarly, in the online routing problem [4], the central organizer receives demands for subsets of edges in a network from the users in a sequential order, each with a certain utility and bid price for his demand. The organizer needs to allocate the network capacity online to those bidders to maximize social welfare. A similar format also appears in knapsack secretary problems [2], online keyword matching problem [7,9], online packing problems [4], and various other online resource allocation problems.

In all these examples mentioned above, the problem takes the format of online linear programming. In an online linear programming problem, the constraint matrix is revealed column by column with the corresponding coefficient in the objective function. After observing the

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1In fact, most of the problems mentioned above are integer programs. While we discuss our ideas and results in terms of linear relaxation of these programs, our results will naturally extend to integer programs. See Section 4.2 for the detailed discussion.
input received so far, the online algorithm must make the current decision without observing the future data. To be precise, consider the linear program

\[
\begin{align*}
\text{maximize} & \quad \sum_{j=1}^{n} \pi_j x_j \\
\text{subject to} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\
& \quad 0 \leq x_j \leq 1, \quad j = 1, \ldots, n
\end{align*}
\]

(1)

where \(\forall j, \pi_j \geq 0, \ a_j = (a_{ij})_{i=1}^{m} \in [0, 1]^m\), and \(b = (b_i)_{i=1}^{m} \in \mathbb{R}^m\). In the corresponding online linear programming problem, at time \(t\), the coefficients \((\pi_t, a_t)\) are revealed, and the algorithm must make a decision \(x_t\). Given the previous \(t-1\) decisions \(x_1, \ldots, x_{t-1}\), and input \((\pi_t, a_t)_{j=1}^{t}\) until time \(t\), the \(t^{th}\) decision is to select \(x_t\) such that

\[
\begin{align*}
\sum_{j=1}^{t} a_{ij} x_j & \leq b_i, \quad i = 1, \ldots, m \\
0 & \leq x_j \leq 1
\end{align*}
\]

(2)

The goal of the online algorithm is to choose \(x_t\)'s such that the objective function \(\sum_{t=1}^{n} \pi_t x_t\) is maximized. Traditionally, people analyze an algorithm base on the worst-case input. However, this leads to very pessimistic bounds for the above online problem: no online algorithm can achieve better than \(O(1/n)\) approximation of the optimal offline solution [2]. Therefore, in this paper, we make the following enabling assumptions:

**Assumption 1.** The columns \(a_j\) (with the objective coefficient \(\pi_j\)) arrive in a random order, i.e., the set of columns can be adversarially picked at the start. However, after they are chosen, \((a_1, a_2, \ldots, a_n)\) and \((a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)})\) have same chance to happen for all permutation \(\sigma\).

**Assumption 2.** We know the total number of columns \(n\) a priori.

The first assumption says that we consider the average behavior of the online algorithm over random permutations. This assumption is reasonable in practical problems, since the order of columns usually appears to be independent of the content of the columns. It is also a standard assumption in many existing literature on online problems, most notably in the classical secretary problems [2].

The second assumption seems restrictive, however is necessary. In [7], the authors show that if Assumption 2 doesn’t hold, then the worst case competitive ratio is bounded away from 1. Note that all the existing algorithms for secretary problems make this assumption [2]. In our algorithm, we will use this quantity \(n\) to decide the length of history used to learn the threshold prices. Of course, we can relax the assumption to the knowledge of \(n\) within at most \(1 \pm \epsilon\) multiplicative error, without affecting the results. Moreover, for certain revenue management and inventory control applications detailed below, the knowledge of \(n\) can be replaced by the knowledge of the length of time horizon under certain arrival assumptions, which is usually available in practice.

In this paper, we present an almost optimal online solution for [2] under above assumptions. We also extend our results to the following more general online linear optimization problems. Consider a sequence of \(n\) non-negative vectors \(f_1, f_2, \ldots, f_n \in \mathbb{R}^k\), \(mn\) non-negative vectors \(g_{i1}, g_{i2}, \ldots, g_{in} \in [0, 1]^k\), \(i = 1, \ldots, m\), and \(k\)-dimensional simplex \(K = \{x \in \mathbb{R}^k : x^T e \leq 1\}\). In this problem, each time we make a \(k\)-dimensional decision \(x_t \in \mathbb{R}^k\), satisfying:

\[
\begin{align*}
\sum_{j=1}^{t} g_{ij} x_j & \leq b_i, \quad i = 1, \ldots, m \\
x_t & \in K \\
x_t & \geq 0, x_t \in \mathbb{R}^k
\end{align*}
\]

(3)

where decision vector \(x_t\) must be chosen only using the knowledge up to time \(t\). The objective is to maximize \(\sum_{t=1}^{n} f_t^T x_t\) over the whole time horizon.
1.1 Our results

In the following, let OPT denote the optimal objective value for the offline problem (1).

**Definition 1.** An online algorithm A is $c$-competitive in random permutation model if the expected value of the online solution obtained by using A is at least a $c$ factor of the optimal offline solution. That is,

$$E_{\sigma} \left[ \sum_{t=1}^{n} \pi_t x_t(\sigma, A) \right] \geq c \text{OPT}$$

where the expectation is taken over uniformly random permutations $\sigma$ of 1, \ldots, $n$, and $x_t(\sigma, A)$ is the $t^{th}$ decision made by algorithm A when the inputs arrive in the order $\sigma$.

Our algorithm is based on the observation that the optimal solution $x^*$ for the offline linear program is almost entirely determined by the optimal dual solution $p^* \in \mathbb{R}^m$ corresponding to the $m$ inequality constraints. The optimal dual solution acts as a threshold price so that $x^*_i > 0$ only if $\pi_i \geq p^T a_i$. Our online algorithm works by learning a threshold price vector from the input received so far. The price vector then determines the decision for the next step. However, instead of computing a new price vector at every step, the algorithm initially waits until $n\epsilon$ steps, and then computes a new price vector every time the history doubles. That is, at time steps $n\epsilon, 2n\epsilon, 4n\epsilon, \ldots$ and so on. We show that our algorithm is $1 - O(\epsilon)$-competitive in random permutation model under certain conditions on the input. Our main result is stated as follows:

**Theorem 1.** For any $\epsilon > 0$, our online algorithm is $1 - O(\epsilon)$-competitive for the online linear program (2) in random permutation model, for all inputs such that

$$B = \min_i b_i \geq \Omega \left( \frac{m \log \left( \frac{n}{\epsilon} \right)}{\epsilon^2} \right)$$

We also give the following alternative conditions for the theorem to hold:

**Corollary 1.** Theorem 1 still holds if condition (7) is replaced by

$$B \geq \Omega \left( \frac{(m \lambda + m^2) \log(1/\epsilon)}{\epsilon^2} \right)$$

where $\lambda = \log \log(\frac{\pi_{max}}{\pi_{min}})$, $\pi_{max} = \min_{j=1,\ldots,n} \pi_j$, $\pi_{min} = \min_{j=1,\ldots,n} \pi_j$.

Observe that the lower bound in the condition on $B$ depends on $\log(1/\epsilon)/\epsilon^2$. We may emphasize that this dependence on $\epsilon$ is near-optimal. In [10], the author proves that $k \geq 1/\epsilon^2$ is necessary to get $1 - O(\epsilon)$ competitive ratio in the $k$-secretary problem, which is a special case of our problem with $m = 1, B = k$ and $a_t = 1$ for all $t$.

We also extend our results to the more general online linear programs as introduced in (3):

**Theorem 2.** For any $\epsilon > 0$, our algorithm is $1 - O(\epsilon)$-competitive for the general online linear program (3) in random permutation model, for all inputs such that:

$$B = \min_i b_i \geq \Omega \left( \frac{m \log \left( \frac{nk}{\epsilon} \right)}{\epsilon^2} \right).$$

where

**Remark 1.** Our condition to hold the main result is independent of the size of OPT or objective coefficients, and our result is also independent of any possible distribution of input data. If the largest entry of constraint coefficients does not equal to 1, then our both theorems hold if the condition (4) or (6) is replaced by:

$$b_i \geq \Omega \left( \frac{\bar{a}_i m \log \left( \frac{nk}{\epsilon} \right)}{\epsilon^2} \right), \forall i,$$

where $\bar{a}_i = \max_j \{a_{ij}\}$ or $\bar{a}_i = \max_j \{\|g_{ij}\|_{\infty}\}$. Note that this bound is proportional only to $\log(n)$ so that it is way below to satisfy everyone’s demand.
It is apparent that our generalized problem formulation should cover a wide range of online decision making problems. In the next section, we discuss the related work and some of the applications of our model. As one can see, indeed our result improves the competitive ratios for many online problems studied in the literature.

1.2 Related work

Online decision making has been a topic of wide recent interest in the computer science, operations research, and management science communities. Various special cases of the general problem presented in this paper have been studied extensively in the literature, especially in the context of online auctions and secretary problems. Babaioff et al. [2] provide a comprehensive survey of existing results on the secretary problems. In particular, constant factor competitive ratios have been proven for $k$-secretary and knapsack secretary problems under random permutation model. Further, for many of these problems, a constant competitive ratio is known to be optimal if no additional conditions on input are assumed. Therefore, there has been recent interest in searching for online algorithms whose competitive ratio approaches 1 as the input parameters become large. The first result of this kind appears in [10], where a $1 - O(1/\sqrt{k})$-competitive algorithm is presented for $k$-secretary problem under random permutation model. More recently, Devanur et al. [7] presented a $1 - O(\epsilon)$-competitive algorithm for the online adwords matching problem under assumption of certain lower bounds on OPT in terms of $\epsilon$ and other input parameters. In [7], the authors raise several open questions including the possibility of such near-optimal algorithms for a more general class of online problems. In our work, we give an affirmative answer to this questions by showing a $1 - O(\epsilon)$-competitive algorithm for a large class of online linear programs under a weaker lower bound condition.

A common element in the techniques used in existing work on secretary problems [2] (with the exception of Kleinberg [10]), online combinatorial auction problems [1], and adwords matching problem [7], has been one-time learning of threshold price(s) from first $n\epsilon$ of customers, which is then used to determine the decision for remaining customers. Our algorithm is based on the same learning idea. However, in practice one would expect some benefit from dynamically updating the prices as more and more information is revealed. Then, a question would be: how often and when to update them? Our dynamic price update builds upon this intuition, and we demonstrate that it’s better to update the prices at geometric time intervals—not too soon and not too late. In particular, we present an improvement from a factor of $1/\epsilon^3$ to $1/\epsilon^2$ in the lower bound requirement by using dynamic price updating instead of one-time learning.

In our analysis, we apply many standard techniques from PAC-learning, in particular, concentration bounds and covering arguments. These techniques were also heavily used in [3] and [7]. In [3], price learned from one half of bidders is used for the other half to get an incentive compatible mechanism for combinatorial auctions. Their approach is closely related to the idea of one-time learning of price in online auctions, however, their goal is offline revenue maximization and an unlimited supply is assumed. And [7], as discussed above, considers a special case of our problem. Part of our analyses is inspired by some ideas used there, as will be pointed out in the text.

1.3 Specific Applications

In the following, we show some of the applications of our algorithm. It is worthy noting that for many of the problems we discuss below, our algorithm is the first near-optimal algorithm under the distribution-free model.
1.3.1 Online routing problems

The most direct application of our online algorithm is the online routing problem \([4]\). In this problem, there are \(m\) edges in a network, each edge \(i\) has a bounded capacity \(b_i\). There are \(n\) customers arriving online, each with a request of certain path \(a_t \in \{0,1\}^m\), where \(a_{it} = 1\), if the path of request \(t\) contains edge \(i\), and a utility \(\pi_t\) for his request. The offline problem for the decision maker is given by the following integer program:

\[
\begin{align*}
\text{maximize} & \quad \sum_{t=1}^{n} \pi_t x_t \\
\text{subject to} & \quad \sum_{t=1}^{n} a_{it} x_t \leq b_i, \quad i = 1, \ldots, m \\
& \quad x_t \in \{0,1\}
\end{align*}
\]

By Theorem 1 and its natural extension to integer programs as will be discussed in Section 4.2, our algorithm gives a \(1 - O(\epsilon)\) competitive solution to this problem in the random permutation model as long as the edge capacity is reasonably large. Earlier, a best of \(\log(m)\) competitive algorithm was known for this problem under worst case input model [4].

1.3.2 Online single-minded combinatorial auctions

In this problem, there are \(m\) goods, \(b_i\) units of each good \(i\) are available. There are \(n\) bidders arriving online, each with a bundle of items \(a_t \in \{0,1\}^m\) that he desires to buy, and a limit price \(\pi_t\) for his bundle. The offline problem of maximizing social utility is same as the routing problem formulation given in (7). Due to use of a threshold price mechanism, where threshold price for \(i\)th bidder is computed from the input of previous bidders, it is easy to show that our \(1 - O(\epsilon)\) competitive online mechanism also supports incentive compatibility and voluntary participation. Also one can easily transform this model to revenue maximization. A \(\log(m)\)-competitive algorithm for this problem in random permutation setting can be found in recent work [1].

1.3.3 The online adwords problems

The online adwords allocation problem is essentially the online matching problem. In this problem, there are \(n\) queries arriving online. And, there are \(m\) bidders each with a daily budget \(b_i\), and bid \(\pi_{ij}\) on query \(j\). For \(j\)th query, the decision vector \(x_j\) is an \(m\)-dimensional vector, where \(x_{ij} \in \{0,1\}\) indicates whether the \(j\)th query is allocated to the \(i\)th bidder. Also, since every query can be allocated to at most one bidder, we have the constraint \(x_j^T e \leq 1\). Therefore, the corresponding offline problem can be stated as:

\[
\begin{align*}
\text{maximize} & \quad \sum_{j=1}^{n} \pi_j^T x_j \\
\text{subject to} & \quad \sum_{j=1}^{n} \pi_{ij} x_{ij} \leq b_i, \quad \forall i \\
& \quad x_j^T e \leq 1 \\
& \quad x_j \in \{0,1\}^m
\end{align*}
\]

The linear relaxation of above problem is a special case of the general linear optimization problem \([3]\) with \(f_j = \pi_j, g_{ij} = \pi_{ij} e_i\), where \(e_i\) is the \(i\)th unit vector of all zeros except 1 for the \(i\)th entry. By Theorem 2 (and remarks below the theorem), and extension to integer programs discussed in Section 4.2, our algorithm will give a \(1 - O(\epsilon)\) approximation for this problem given the lower bound condition that for all \(i\), \(\frac{b_i}{\pi_i^{\text{max}}} \geq \frac{m \log(mn/\epsilon)}{\epsilon^2}\), where \(\pi_i^{\text{max}} = \max_j \{\pi_{ij}\}\) is the largest bid by bidder \(i\) among all queries.

Earlier a \(1 - O(\epsilon)\) algorithm was provided for this problem by [7]. We may point out that we have eliminated the condition on \(\text{OPT}\) obtained by [7] and only have a condition on \(b_i\) which may be checkable before the execution, a property which is not provided by the condition of former type. Further, our lower bounds on \(B\) are weaker by an \(\epsilon\) factor to the one obtained.
in [7]. Later, we show that this improvement is a result of dynamically learning the price at geometric intervals, instead of one-time learning in [7]. Richer models incorporating other aspects of sponsored search such as multiple slots, can be formulated by redefining $f_j, g_{ij}, K$ to obtain similar results.

### 1.3.4 Yield management problems

Online yield management problem is to allocate perishable resources to demands in order to increase the revenue by best online matching the resource capacity and demand in a given time horizon $T$ [8][11]. It has wide applications including airline booking, hotel reservation, ticket selling, media and the internet resource allocation problems. In this problem, there are many types of product $j$, $j = 1, 2, ..., J$, and several resources $b_i$, $i = 1, ..., m$. To sell a unit demand of product $j$, it requires to consume $t_{ij}$ unit of resource $i$ for all $i$. Buyers demanding each type of product come in a stationary Poisson process and each offers a price $\pi$ for his or her unit product demand. The objective of the seller is to maximize his or her revenue in the time horizon $T$ while respecting given resource constraints. The offline problem is exactly the same as the online routing problem:

$$\begin{align*}
\text{maximize} & \quad \sum_t \pi_t x_t \\
\text{subject to} & \quad \sum_t a_t x_t \leq b_t, \\
& \quad x_t \in \{0, 1\}, \quad \forall t,
\end{align*}$$

(8)

where $a_t$ is a type of product $t_j$, $j = 1, ..., J$. In this problem, we may not know the exact number $n$ of the total buyers in advance. But since the buyers arrive according to a stationary Poisson process, they can be viewed as randomly ordered and we can use the bids in the first $\epsilon T$ time period to learn the optimal dual price and apply to the remaining time horizon. Given $T$ large enough, the number of buyers in $\epsilon T$ time will be approximately $\epsilon n$ and therefore our algorithm will give a near-optimal solution to this problem, even though the arrival rate of buyers for each product is not known to the seller.

### 1.3.5 Inventory control problems with replenishment

This problem is similar to the multi-period yield management problem discussed in the previous subsection. The sellers have $m$ items to sell and each time a bidder comes and requests a certain bundle of items $a_j$ and offers a price $\pi_j$. In this problem, we have periodic selling period. In each period, the seller has to choose an inventory $b$ at the beginning, and then allocate the demand of the buyers during this period. Each unit of $b_i$ costs capital $c_i$ and the total investment $\sum_i b_i c_i$ in each period is limited by budget $C$. There are many periods in the whole time horizon and the objective is to maximize the total revenue in the whole time horizon. The offline problem of each period, for all bidders arrive in that period, is as follows:

$$\begin{align*}
\text{maximize} & \quad x^T b \\
\text{subject to} & \quad \sum_t \pi_t x_t \\
& \quad \sum_t a_t x_t \leq b, \\
& \quad \sum_{i=1}^m c_i b_i \leq C, \\
& \quad 0 \leq x_t \leq 1 \quad \forall t
\end{align*}$$

(9)

Note that given $b$, the problem for one period is exactly as we discussed before. Given the bids come in a random permutation over the total time horizon, our analysis will show that the itemized demands $b$ for the previous period will be approximately the same for the rest periods, and online pricing the bids for the rest of periods based on the itemized dual prices learnt from the previous period would give a revenue that is close to the optimal revenue of the offline problem over the whole time horizon.
The rest of the paper is organized as follows. In Section 2 and 3, we present our online algorithm and prove that it achieves $1 - O(\epsilon)$ competitive ratio under mild conditions on the input. To keep the discussion simple and clear, we start in Section 2 with a simpler one-time learning algorithm. While the analysis for this simpler algorithm will be useful to demonstrate our proof techniques, the results obtained in this setting are weaker than those obtained by our dynamic price update algorithm, which is discussed in Section 3. In Section 4, we present the extension to multidimensional online linear programs and the applicability of our model to solving large static linear programs. Then we conclude our paper.

2 One-time learning algorithm

For the linear program (1), we use $p \in \mathbb{R}^m$ to denote the dual variable associated with the first set of constraints $\sum_{t} a_t x_t \leq b$. Let $\hat{p}$ denote the optimal dual solution to the following partial linear program defined only on the input until time $s = \lceil n\epsilon \rceil$:

$$\begin{align*}
\maximize & \sum_{t=1}^{s} \pi_t x_t \\
\text{subject to} & \sum_{t=1}^{s} a_{it} x_t \leq (1 - \epsilon) \frac{\pi_b}{n} b_i, \quad i = 1, \ldots, m \\
& 0 \leq x_t \leq 1, \quad t = 1, \ldots, s
\end{align*}$$

(10)

Also, for any given dual price vector $p$, define the allocation rule $x_t(p)$ as:

$$x_t(p) = \begin{cases} 0 & \text{if } \pi_t \leq p^T a_t \\ 1 & \text{if } \pi_t > p^T a_t \end{cases}$$

(11)

Our one-time learning algorithm can now be stated as follows:

Algorithm 1 One-time Learning Algorithm (OLA)

1. Initialize $s = \lceil n\epsilon \rceil$, $x_t = 0$, for all $t \leq s$. And $\hat{p}$ is defined as above.
2. Repeat for $t = s + 1, s + 2, \ldots$
   (a) If $a_{it} x_t(\hat{p}) \leq b_i - \sum_{j=1}^{t-1} a_{ij} x_j$, set $x_t = x_t(\hat{p})$; otherwise, set $x_t = 0$. Output $x_t$.

This algorithm learns a dual price vector using the first $\lceil n\epsilon \rceil$ arrivals. Then, at each time $t > \lceil n\epsilon \rceil$, it uses this price vector to decide the current allocation, and executes this decision as long as it doesn’t violate any of the constraints. An attractive feature of this algorithm is that it requires to solve a linear program only once, and the linear program it solves is significantly smaller, defined only on $\lceil n\epsilon \rceil$ variables. In the next subsection, we prove the following proposition regarding the competitive ratio of this algorithm, which relies on a stronger condition than Theorem 1:

Proposition 1. For any $\epsilon > 0$, the one-time learning algorithm is $1 - 6\epsilon$ competitive for online linear programming, if

$$B = \min_i b_i \geq \frac{6m \log(n/\epsilon)}{\epsilon^3}$$

(12)

2.1 Competitive ratio analysis

Observe that the one-time learning algorithm waits until time $s = \lceil n\epsilon \rceil$, and then sets the solution at time $t$ as $x_t(\hat{p})$, unless there is a constraint violation. To prove the competitive ratio of this algorithm, we first prove that with high probability, $x_t(\hat{p})$ satisfies all the constraints of the linear program. Then, we show that the expected value $\sum_t \pi_t x_t(\hat{p})$ is close to the optimal offline objective value. For simplicity of the discussion, we assume $s = n\epsilon$ in the following.
To start with, we observe that if $\mathbf{p}^*$ is the optimal dual solution to (1), then \( \{x_t(\mathbf{p}^*)\} \) is close to the primal optimal solution $\mathbf{x}^*$. That is, learning the dual price is sufficient to determine the primal solution. We make the following simplifying assumption:

**Assumption 3.** The inputs of this problem are in general position, namely for any price vector $\mathbf{p}$, there can be at most $m$ columns such that $\mathbf{p}^T \mathbf{a}_t = \pi_t$.

The assumption is not necessarily true for all inputs. However, one can always randomly perturb $\pi_t$ by arbitrarily small amount $\eta$ through adding a random variable $\xi$ taking uniform distribution on interval $[0, \eta]$. In this way, with probability 1, no $\mathbf{p}$ can satisfy $m + 1$ equations simultaneously among $\mathbf{p}^T \mathbf{a}_t = \pi_t$, and the effect of this perturbation on the objective can be made arbitrarily small.

Given this assumption, we can use complementary conditions of linear program (1) to observe that:

**Lemma 1.** $x_t(\mathbf{p}^*) \leq x_t^*$, and under Assumption 3 $x_t^*$ and $x_t(\mathbf{p}^*)$ differ on at most $m$ values of $t$.

However, in the online algorithm, we use the price $\hat{\mathbf{p}}$ learned from first few inputs, instead of the optimal dual price. The remaining discussion attempts to show that the learned price will be sufficiently accurate for our purpose. Note that the random order assumption can be interpreted to mean that the first $s$ inputs are a uniform random sample without replacement of size $s$ from the $n$ inputs. Let $S$ denote this sample set of size $s$, and $N$ denote the complete set of size $n$. Consider the sample linear program (10) defined on the sample set $S$ with right hand side set as $(1 - \epsilon)\mathbf{b}$. Then, $\hat{\mathbf{p}}$ was constructed as the optimal dual price of the sample linear program, which we refer to as the sample dual price. We prove the following lemma:

**Lemma 2.** The primal solution constructed using sample dual price is a feasible solution to the linear program (1) with high probability. More precisely, with probability $1 - \epsilon$,

$$
\sum_{t=1}^{n} a_{it} x_t(\hat{\mathbf{p}}) \leq b_i, \quad \forall i = 1, \ldots, m
$$

given $B \geq \frac{6m \log(n/\epsilon)}{\epsilon^2}$.

**Proof.** The proof will proceed as follows: Consider any fixed price $\mathbf{p}$. We say a random sample $S$ is “bad” for this $\mathbf{p}$ if and only if $\mathbf{p} = \hat{\mathbf{p}}(S)$, but $\sum_{t=1}^{n} a_{it} x_t(\mathbf{p}) > b_i$ for some $i$. First, we show that the probability of bad samples is small for every fixed $\mathbf{p}$. Then, we take union bound over all “distinct” prices to prove that with high probability the learned price $\hat{\mathbf{p}}$ will be such that $\sum_{t=1}^{n} a_{it} x_t(\hat{\mathbf{p}}) \leq b_i$ for all $i$.

To start with, we fix $\mathbf{p}$ and $i$. Define $Y_i = a_{it} x_t(\mathbf{p})$. If $\mathbf{p}$ is an optimal dual solution for sample linear program on $S$, then by the complementary conditions, we have:

$$
\sum_{t \in S} Y_i = \sum_{t \in S} a_{it} x_t(\mathbf{p}) \leq (1 - \epsilon) b_i
$$

(13)

Therefore, the probability of bad samples is bounded by:

$$
P(\sum_{t \in S} Y_i \leq (1 - \epsilon) b_i, \sum_{t \in N} Y_i \geq b_i) \leq P(\| \sum_{t \in S} Y_i - \epsilon \sum_{t \in N} Y_i \| \geq \epsilon^2 b_i) \leq 2 \exp\left(\frac{-\epsilon^2 b_i^2}{2 \epsilon^2}\right) \leq \delta
$$

where $\delta = \frac{\epsilon^2 b_i^2}{2 \epsilon^2}$. The last step follows from Hoeffding-Bernstein’s Inequality (Lemma 10 in appendix A), and the assumption made on $B$.

Next, we take a union bound over all “distinct” $\mathbf{p}$’s. We call two price vectors $\mathbf{p}$ and $\mathbf{q}$ distinct if and only if they result in distinct solutions, i.e., $\{x_t(\mathbf{p})\} \neq \{x_t(\mathbf{q})\}$. Note that we

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\[2\] This technique for resolving ties was also used in [7].
only need to consider distinct prices, since otherwise all the \( Y_t \)'s are exactly the same. Thus, each distinct \( p \) is characterized by a unique separation of \( n \) points \( \{ \{ a_t, \pi_t \} \}_{t=1}^{n} \) in \( m \)-dimensional space by a hyperplane. By results from computational geometry \[12\], the total number of such distinct prices is at most \( n^m \). Taking union bound over \( n^m \) distinct prices, and \( i = 1, \ldots, m \), we get the desired result.

Above we showed that with high probability, \( x_t(\hat{p}) \) is a feasible solution. In the following, we show that it actually is a near-optimal solution if we include the objective value in the learning part.

**Lemma 3.** The primal solution constructed using sample dual price is a near-optimal solution to the linear program \[1\] with high probability. More precisely, with probability \( 1 - \epsilon \),

\[
\sum_{t \in N} \pi_t x_t(\hat{p}) \geq (1 - 3\epsilon)OPT
\]

given \( B \geq \frac{6m \log(n/\epsilon)}{\epsilon^3} \).

**Proof.** The proof of this lemma is based on two observations. First, \( \{ x_t(\hat{p}) \}_{t=1}^{n} \) and \( \hat{p} \) satisfies all the complementarity conditions, and hence is an optimal primal-dual solution of the following linear program

\[
\text{maximize} \quad \sum_{t \in N} \pi_t x_t \\
\text{subject to} \quad \sum_{t \in N} a_{it} x_t \leq \hat{b}_i \quad i = 1, \ldots, m \\
0 \leq x_j \leq 1 \quad j = 1, \ldots, n
\]

where \( \hat{b}_i = \sum_{t \in N} a_{it} x_t(\hat{p}) \) if \( \hat{p}_i > 0 \), and \( \hat{b}_i = b_i \), if \( \hat{p}_i = 0 \).

Second, one can show that if \( \hat{p}_i > 0 \), then with probability \( 1 - \epsilon \), \( \hat{b}_i \geq (1 - 3\epsilon)b_i \). To see this, observe that \( \hat{p} \) is optimal dual solution of sample linear program on set \( S \), let \( \hat{x} \) be the optimal primal solution. Then, by complementarity conditions of the linear program, if \( \hat{p}_i > 0 \), then the \( i^{th} \) constraint must be satisfied by equality. That is, \( \sum_{t \in S} a_{it} \hat{x}_t = (1 - \epsilon)b_i \). Then, given the observation made in Lemma \[1\] and that \( B = \min \{ b_i \} \geq \frac{m}{2\epsilon} \), we get:

\[
\sum_{t \in S} a_{it} x_t(\hat{p}) \geq \sum_{t \in S} a_{it} \hat{x}_t - m \geq (1 - 2\epsilon)b_i.
\]

Then, using the Hoeffding-Bernstein’s Inequality, in a manner similar to the proof of Lemma \[2\] we can show that (the proof is given in Appendix A.3) given the lower bound on \( B \), with probability at least \( 1 - \epsilon \):

\[
\hat{b}_i = \sum_{t \in N} a_{it} x_t(\hat{p}) \geq (1 - 3\epsilon)b_i
\]

Lastly, observing that whenever \[18\] holds, given an optimal solution \( x^* \) to \[1\], \( (1 - 3\epsilon)x^* \) will be a feasible solution to \[16\]. Therefore, the optimal value of \[16\] is at least \( (1 - 3\epsilon)OPT \), which is equivalently saying that

\[
\sum_{t=1}^{n} \pi_t x_t(\hat{p}) \geq (1 - 3\epsilon)OPT
\]

Therefore, the objective value for online solution taken over entire period \( \{1, \ldots, n\} \) is near optimal. However, the online algorithm does not make any decision in the learning period \( S \), and only the decisions from period \( \{ s + 1, \ldots, n \} \) contribute to the objective value. The following lemma that relates sample optimal to the optimal value of a linear program will bound the contribution from the learning period:
Lemma 4. Let \( \text{OPT}(S) \) denote the optimal value of the linear program (10) over sample \( S \), and \( \text{OPT}(N) \) denote the optimal value of the offline linear program (1) over \( N \). Then,

\[
\mathbb{E}[\text{OPT}(S)] \leq \epsilon \text{OPT}(N)
\]

Proof. Let \((x^*, p^*, y^*)\) and \((\hat{x}, \hat{p}, \hat{y})\) denote the optimal primal-dual solution of linear program (1) on \( N \), and sample linear program on \( S \), respectively.

\[
(p^*, y^*) = \arg \min_{p, y} \quad b^T p + \sum_{t \in N} y_t \quad \text{s.t.} \quad p^T a_t + y_t \geq \pi_t, \quad t \in N, \quad p, y \geq 0
\]

\[
(\hat{p}, \hat{y}) = \arg \min_{p, y} \quad (1 - \epsilon) b^T p + \sum_{t \in S} y_t \quad \text{s.t.} \quad p^T a_t + y_t \geq \pi_t, \quad t \in S, \quad p, y \geq 0
\]

Since \( S \subseteq N \), \( p^*, y^* \) is a feasible solution to the dual of linear program on \( S \), by weak duality theorem:

\[
\text{OPT}(S) \leq \epsilon b^T p^* + \sum_{t \in S} y^*_t
\]

Therefore,

\[
\mathbb{E}[\text{OPT}(S)] \leq \epsilon b^T p^* + \mathbb{E}[\sum_{t \in S} y^*_t] = \epsilon (b^T p^* + \sum_{t \in N} y^*_t) = \epsilon \text{OPT}(N)
\]

Now, we are ready to prove Proposition 1.

Proof of Proposition 1 Using Lemma 2 and Lemma 3 with probability at least \( 1 - 2\epsilon \), the following event happen:

\[
\sum_{i=1}^{n} a_{it} x_i(\hat{p}) \leq b_i, \quad i = 1, \ldots, m
\]

\[
\sum_{i=1}^{n} \pi_i x_i(\hat{p}) \geq (1 - 3\epsilon) \text{OPT}
\]

That is, the decisions \( x_i(\hat{p}) \) are feasible and the objective value taken over the entire period \( \{1, \ldots, n\} \) is near optimal. Denote this event by \( \mathcal{E} \), where \( \Pr(\mathcal{E}) \geq 1 - 2\epsilon \). We have by Lemma 2, 3 and Lemma 4

\[
\mathbb{E}[\sum_{t \in N \setminus S} \pi_t x_t(\hat{p}) | \mathcal{E}] \geq (1 - 3\epsilon) \text{OPT} - \mathbb{E}[\sum_{t \in S} \pi_t x_t(\hat{p}) | \mathcal{E}] \geq (1 - 3\epsilon) \text{OPT} - \frac{\text{OPT}}{1 - 2\epsilon}
\]

Therefore,

\[
\mathbb{E}[\sum_{t=s+1}^{n} \pi_t x_t(\hat{p})] \geq \mathbb{E}[\sum_{t \in N \setminus S} \pi_t x_t(\hat{p}) | \mathcal{E}] \cdot \Pr(\mathcal{E}) \geq (1 - 6\epsilon) \text{OPT}
\]

3 Dynamic price update algorithm

The algorithm discussed in the previous section uses the first \( n\epsilon \) inputs to learn the price, and then applies it in the remaining time horizon. While this one-time learning algorithm has its own merits, in particular, requires solving only a small linear problem defined on \( n\epsilon \) variables, the lower bound required on \( B \) is stronger than that claimed in Theorem 1 by an \( \epsilon \) factor.
In this section, we discuss an improved dynamic price update algorithm that will achieve the result in Theorem 1. Instead of computing the price only once, the algorithm will update the price every time the history doubles, that is, it will learn a new price at time periods \( t = n\epsilon, 2n\epsilon, 4n\epsilon, \ldots \). To be precise, let \( \hat{p}_\ell \) denote the optimal dual solution for the following partial linear program defined only on the input until time \( \ell \):

\[
\begin{align*}
\text{maximize} & \quad \sum_{t=1}^{\ell} \pi_t x_t \\
\text{subject to} & \quad \sum_{t=1}^{\ell} a_{it} x_t \leq (1 - h_\ell) \frac{\ell}{n} b_i, \quad i = 1, \ldots, m \\
& \quad 0 \leq x_t \leq 1, \quad t = 1, \ldots, \ell
\end{align*}
\]

where the set of numbers \( h_\ell \) are defined as follows:

\[
h_\ell = \epsilon \sqrt{\frac{\ell}{n}} \quad \forall \ell
\]

Also, for any given dual price vector \( \hat{p}_\ell \), define the allocation rule \( x_t(\hat{p}_\ell) \) as earlier in (11). Then, our dynamic price update algorithm can be stated as follows:

**Algorithm 2 Dynamic Pricing Algorithm (DPA)**

1. Initialize \( t_0 = \lceil n\epsilon \rceil, x_t = \hat{x}_t = 0 \), for all \( t \leq t_0 \).
2. Repeat for \( t = t_0 + 1, t_0 + 2, \ldots \)
   (a) Set \( \hat{x}_t = x_t(\hat{p}_\ell) \), where \( \ell = \lceil 2^r n\epsilon \rceil \) for largest \( r \) such that \( \ell < t \).
   (b) If \( a_{it} \hat{x}_t \leq b_i - \sum_{j=1}^{t-1} a_{ij} \hat{x}_j \), set \( x_t = \hat{x}_t \); otherwise, set \( x_t = 0 \). Output \( x_t \).

Note that we update the price \( \lceil \log_2 (1/\epsilon) \rceil \) times during the whole time horizon. Thus, the algorithm requires more computation, but as we show next it requires a weaker lower bound on \( B \) for proving the same competitive ratio. The intuition behind this improvement is as follows. Note that initially, at \( \ell = n\epsilon, h_\ell = \sqrt{\epsilon} > \epsilon \). Thus, more slack is available, and so the large deviation argument for constraint satisfaction (as in Lemma 2) requires a weaker condition on \( B \). The numbers \( h_\ell \) decrease as \( \ell \) increases. However, for large values of \( \ell \), sample size is larger, making the weaker condition on \( B \) sufficient for our purpose. Also, \( h_\ell \) decrease rapidly enough, so that the overall loss on the objective value is not significant. The careful choice of numbers \( h_\ell \) will play a key role in proving our results.

### 3.1 Competitive ratio analysis

The analysis for the dynamic algorithm proceeds in a manner similar to that for the one-time learning algorithm. However, stronger results for the price learned in each period need to be proven here. In this proof, for simplicity of discussion, we assume that \( \epsilon = 2^{-E} \) for some integer \( E \), and that the numbers \( \ell = 2^r n\epsilon \) for \( r = 0, 1, 2, \ldots, E - 1 \) are all integers. Let \( L = \{n\epsilon, 2n\epsilon, \ldots, 2^{E-1}n\epsilon \} \).

Lemma 5 and 6 are in parallel to Lemma 2 and 3 in the previous section, however require a weaker condition on \( B \):

**Lemma 5.** For any \( \epsilon > 0 \), with probability \( 1 - \epsilon \):

\[
\sum_{t=\ell+1}^{2\ell} a_{it} x_t(\hat{p}_\ell) \leq \frac{\ell}{n} b_i, \quad \text{for all } i \in \{1, \ldots, m\}, \ell \in L
\]

given \( B = \min_i b_i \geq \frac{20m \log n}{\epsilon^2} \).
Lemma 6. And let $OPT_{\ell}$ an optimal primal-dual solution for the linear program $d$ that is defined on variables till time $\ell$.

Proof. To start with, we fix $p$ and $\ell$. This time, we say a random order is “bad” for this $p$ if and only if there exists $l \in L$, such that $p = \hat{p}$ but $\sum_{t=1}^{2\ell} a_{it}x_t(\hat{p}^t) > \frac{1}{2}b_l$ for some $i$. Consider the $i$th component $\sum_{t=1}^{2\ell} a_{it}x_t$ for a fixed $i$. For ease of notation, we temporarily omit the subscript $i$. Define $Y_i = a_{it}x_t(p)$. If $p$ is an optimal dual solution for (20), then by the complementarity conditions, we have:

$$\sum_{t=1}^{\ell} Y_t = \sum_{t=1}^{\ell} a_{it}x_t(p) \leq (1 - h_{\ell})b_{\ell}$$

(22)

Therefore, the probability of “bad” permutations for $p$ is bounded by:

$$P(\sum_{t=1}^{\ell} Y_t \leq (1 - h_{\ell})\frac{b_{\ell}}{n}, \sum_{t=1}^{2\ell} Y_t \geq \frac{2b_{\ell}}{n}) \leq P(\sum_{t=1}^{\ell} Y_t \leq (1 - h_{\ell})\frac{b_{\ell}}{n}, \sum_{t=1}^{2\ell} Y_t \geq \frac{2b_{\ell}}{n}) + P(\sum_{t=1}^{\ell} Y_t - \frac{1}{2} \sum_{t=1}^{2\ell} Y_t \geq \frac{h_{\ell}b_{\ell}}{n}, \sum_{t=1}^{2\ell} Y_t \leq \frac{2b_{\ell}}{n})$$

Define $\delta = \frac{\ell}{m+n+R}$. Using Hoeffding-Bernstein’s Inequality (Lemma 10 in appendix, here $R = 2\ell$, $\sigma_R^2 \leq b/n$, and $\Delta_R \leq 1$), we have:

$$P(\sum_{t=1}^{\ell} Y_t \leq (1 - h_{\ell})\frac{b_{\ell}}{n}, \sum_{t=1}^{2\ell} Y_t \geq \frac{2b_{\ell}}{n}) \leq \exp\left(-\frac{\ell b_{\ell}^2}{2m}\right) \leq \frac{\delta}{2}$$

and

$$P(\sum_{t=1}^{\ell} Y_t - \frac{1}{2} \sum_{t=1}^{2\ell} Y_t \geq \frac{h_{\ell}b_{\ell}}{n}, \sum_{t=1}^{2\ell} Y_t \leq \frac{2b_{\ell}}{n}) \leq 2 \exp\left(-\frac{\ell b_{\ell}^2}{2(n+2R)}\right) \leq \frac{\delta}{2}$$

where the last steps hold because $h_{\ell} \leq 1$, and the assumption made on $B$.

Next, we take a union bound over $n^m$ distinct prices, $i = 1, \ldots, m$, and $E$ values of $\ell$, the lemma is proved.

In the following, we will use some notations. Let $LP_s(d)$ denote the partial linear program that is defined on variables till time $s$, i.e. $(x_1, \ldots, x_s)$, with right hand side in the inequality constraints set as $d$. That is,

$$\begin{align*}
\text{maximize} & \quad \sum_{t=1}^{s} \pi_t x_t \\
\text{subject to} & \quad \sum_{t=1}^{\ell} a_{it}x_t \leq d_i \\
& \quad 0 \leq x_t \leq 1 \quad t = 1, \ldots, s
\end{align*}$$

And let $OPT_s(d)$ denote the optimal objective value for $LP_s(d)$.

Lemma 6. With probability at least $1 - \epsilon$, for all $\ell \in L$:

$$\sum_{t=1}^{2\ell} \pi_t x_t(\hat{p}^t) \geq (1 - 2h_{\ell} - \epsilon)OPT_{2\ell}(\frac{2\ell b_{\ell}}{n})$$

given $B = \min_i b_i \geq \frac{20m\log n}{\epsilon^2}$. 

Proof. Let $\hat{b}_i = \sum_{j=1}^{2\ell} a_{ij}x_j(\hat{p}^t)$ for $i$ such that $p_i^t > 0$, and $\hat{b}_i = \frac{2n}{\ell}b_i$, otherwise. Then, note that the solution pair $((x_1(p^t)), \cdots, x_s(p^t))$, satisfies all the complementarity conditions, and therefore is an optimal primal-dual solution for the linear program $LP_{2\ell}(\hat{b})$:

$$\begin{align*}
\text{maximize} & \quad \sum_{t=1}^{2\ell} \pi_t x_t \\
\text{subject to} & \quad \sum_{t=1}^{2\ell} a_{it}x_t \leq \hat{b}_i \\
& \quad 0 \leq x_t \leq 1 \quad t = 1, \ldots, 2\ell
\end{align*}$$

(23)

This means

$$\sum_{t=1}^{2\ell} \pi_t x_t(\hat{p}^t) = OPT_{2\ell}(\hat{b}) \geq \min_i \left(\frac{\hat{b}_i}{\frac{2n}{\ell}b_i}\right)OPT_{2\ell}(\frac{2\ell b_{\ell}}{n})$$

(24)
Now, let us analyze the ratio $\hat{b}_i$. By definition, for $i$ such that $p^\ell_i = 0$, $\hat{b}_i = 2\ell b_i/n$. Otherwise, using techniques similar to the proof of Lemma 5, we can prove that with probability $1 - \epsilon$,

$$\hat{b}_i = \sum_{t=1}^{2\ell} a_{it} x_t(\hat{p}^\ell) \geq (1 - 2\ell - \epsilon) \frac{2\ell}{n} b_i$$

(25)

A detailed proof of inequality (25) appears in appendix B.1. And the lemma follows from (25).

Next, we prove the following lemma relating the sample optimal to the optimal value of the offline linear program:

**Lemma 7.** For any $\ell$,

$$\mathbb{E}\left[\text{OPT}_\ell(\ell/n)\right] \leq \frac{\ell}{n} \text{OPT}$$

**Proof.** The proof is exactly the same as the proof for Lemma 4.

**Proof of Theorem 1:** Observe that the output of the online solution at time $t \in \{\ell + 1, \ldots, 2\ell\}$ is $x_t(\hat{p}^\ell)$ as long as the constraints are not violated. By Lemma 5 and Lemma 6, with probability at least $1 - 2\epsilon$:

$$\sum_{t=\ell+1}^{2\ell} a_{it} x_t(\hat{p}^\ell) \leq \frac{\ell}{n} b_i,$$

for all $i \in \{1, \ldots, m\}$, $\ell \in L$.

$$\sum_{t=1}^{2\ell} \pi_t x_t(\hat{p}^\ell) \geq (1 - 2\ell - \epsilon) \text{OPT}_{2\ell}(\ell/n)$$

Denote this event by $\mathcal{E}$, where $\Pr(\mathcal{E}) \geq 1 - 2\epsilon$. Given this event, the expected objective value achieved by the online solution:

$$\mathbb{E}\left[\sum_{\ell \in L} \sum_{t=\ell+1}^{2\ell} \pi_t x_t(\hat{p}^\ell) | \mathcal{E}\right]$$

\[\geq \sum_{\ell \in L} \mathbb{E}\left[\sum_{t=1}^{\ell} \pi_t x_t(\hat{p}^\ell) | \mathcal{E}\right] - \sum_{\ell \in L} \mathbb{E}\left[\sum_{t=1}^{\ell} \pi_t x_t(\hat{p}^\ell) | \mathcal{E}\right] \]

\[\geq \sum_{\ell \in L} (1 - 2h_\ell - \epsilon) \mathbb{E}\left[\text{OPT}_{2\ell}(2\ell/n) | \mathcal{E}\right] - \sum_{\ell \in L} \mathbb{E}\left[\text{OPT}_\ell(\ell/n) | \mathcal{E}\right] \]

\[\geq \text{OPT} - \sum_{\ell \in L} 2h_\ell \mathbb{E}\left[\text{OPT}_{2\ell}(2\ell/n) | \mathcal{E}\right] - \epsilon \sum_{\ell \in L} \mathbb{E}\left[\text{OPT}_{2\ell}(2\ell/n) | \mathcal{E}\right] - \mathbb{E}[\text{OPT}_{2\ell}(\ell/n)] \]

\[\geq \text{OPT} - \frac{1}{Pr(\mathcal{E})} \sum_{\ell \in L} 2h_\ell \mathbb{E}\left[\text{OPT}_{2\ell}(2\ell/n) \right] - \frac{\epsilon}{Pr(\mathcal{E})} \sum_{\ell \in L} \mathbb{E}\left[\text{OPT}_{2\ell}(2\ell/n) \right] - \frac{1}{Pr(\mathcal{E})} \mathbb{E}[\text{OPT}_{2\ell}(\ell/n)] \]

\[\geq \text{OPT} - \frac{4}{1 - 2\epsilon} \sum_{\ell \in L} h_\ell \frac{\ell}{n} \text{OPT} - \frac{\epsilon}{1 - 2\epsilon} \sum_{\ell \in L} \frac{\ell}{n} \text{OPT} - \frac{\epsilon}{1 - 2\epsilon} \text{OPT} \]

\[\geq \text{OPT} - \frac{12\epsilon}{1 - 2\epsilon} \text{OPT} \]
where the last inequality follows from the fact that
\[\sum_{\ell \in L} \frac{\ell}{n} = (1 - \epsilon), \text{ and } \sum_{\ell \in L} h_{\ell} \frac{\ell}{n} = \epsilon \sum_{\ell \in L} \sqrt{\frac{\ell}{n}} \leq 2.5\epsilon\]

Therefore,
\[
E[\sum_{\ell \in L} \sum_{I=1}^{2\ell} \pi_{\ell} x_{\ell}(\hat{p}^\ell)] \geq E[\sum_{\ell \in L} \sum_{I=1}^{2\ell} \pi_{\ell} x_{\ell}(\hat{p}^\ell)|\mathcal{E}] \Pr(\mathcal{E}) \geq (1 - 14\epsilon)\text{OPT}
\]

Thus, Theorem 1 is proved.

Remark 2. To remove the dependence on \(\log n\) in the lower bound on \(B\) as in Corollary 1, we use the observation that the number of points \(n\) in the expression \(n^m\) for number of distinct prices can be replaced by number of “distinct points” among \((a_i, \pi_i)_{i=1}^n\) which can reduced to \(O(\log_{1+\epsilon} (\lambda) \log_{1+\epsilon}^m (1/\epsilon))\) by a simple preprocessing of the input introducing a multiplicative error of at most \(1 - \epsilon\). And in this case, the condition on \(B\) is
\[B \geq \frac{20(m\lambda + m^2 \log (1/\epsilon))}{\epsilon^2}\]

4 Extensions and Conclusions

We present a few extensions and implications of our results.

4.1 Online multi-dimensional linear program

We consider the following more general online linear programs with multidimensional decisions \(x_t \in \mathbb{R}^k\) at each step, as defined in Section 1:

\[
\begin{align*}
\text{maximize} & \quad \sum_{i=1}^{n} f_i^T x_t \\
\text{subject to} & \quad \sum_{i=1}^{n} g_i^T x_t \leq b_i, \quad i = 1, \ldots, m \\
& \quad x_t^T e \leq 1, x_t \geq 0 \quad \forall t \\
& \quad x_t \in \mathbb{R}^k
\end{align*}
\]

(27)

Our online algorithm remains essentially the same (as described in Section 3), with \(x_t(p)\) now defined as follows:
\[
x_t(p) = \begin{cases} 
0 & \text{if for all } j, f_{ij} \leq \sum_i p_i g_{ij} \\
e_r & \text{otherwise, where } r \in \arg\max_j (f_{ij} - \sum_i p_i g_{ij}) 
\end{cases}
\]

(28)

Using the complementary conditions of (27), and the lower bound condition on \(B\) as assumed in Theorem 2, we can prove following lemma; the proofs are very similar to the proofs for the one-dimensional case, and will be provided in the appendix.

Lemma 8. \(x_t^*\) and \(x_t(p^*)\) differ in at most \(m\) values of \(t\).

Lemma 9. Let \(p\) and \(q\) are distinct if \(x_t(p) \neq x_t(q)\). Then, there are at most \(n^m k^{2m}\) distinct price vectors.

With the above lemmas, the proof of Theorem 2 will follow exactly as the proof for Theorem 1.
4.2 Online Integer programs

From the definition of \( x_t(p) \) in (11) for linear programs, the algorithm always outputs integer solutions. And, since the competitive ratio analysis will compare the online solution to the optimal solution of the corresponding linear relaxation, the competitive ratio stated in Theorem 1 also holds for the online integer programs. The same observation holds for the general online linear programs introduced in Section 4.1 since it also outputs integer solutions. Our result implies a common belief: when relatively sufficient resource quantities are to be allocated to a large number of small demands, linear programming solutions possess a small gap to integer programming solutions.

4.3 Fast solution of large linear programs by column sampling

Apart from online problems, our algorithm can also be applied for solving (offline) linear programs that are too large to consider all the variables explicitly. Similar to the one-time learning online solution, one could randomly sample a small subset \( n \epsilon \) of variables, and use the dual solution \( \hat{p} \) for this smaller program to set the values of variables \( x_j \) as \( x_j(\hat{p}) \). This approach is very similar to the popular column generation method used for solving large linear programs [6].

Our result provides the first rigorous analysis of the approximation achieved by the approach of reducing the linear program size by randomly selecting a subset of columns.

To conclude, we have provided a \( 1 - o(1) \) competitive algorithm for a general class of online linear programming problems under the assumption of random order of arrival and some mild conditions on the right-hand-side input. These conditions are independent of the optimal objective value, objective function coefficients, or distributions of input data. The application of this algorithm includes various online resource allocation problems which is typically very hard to get a near-optimal bounds in the online context. This is the first near-optimal algorithm for general online optimization problems.

Our dynamic learning-based algorithm works by dynamically updating a threshold price vector at geometric time intervals. This geometric learning frequency may also be of interest to statistical and machine learning communities. It essentially indicates that not only it might be bad to react too slow, but also to react too fast.

There are many remaining questions. Could the condition on the right-hand-side vector be further improved? Could we handle linear programming with both buy and sell customers, that is, some coefficients of \( a_j \) are negative? Could our online and learning algorithm be adapted to approximately solve more dynamic and stochastic programming problems?

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A Supporting lemmas for Section 2

A.1 Hoeffding-Bernstein inequality

By Theorem 2.14.19 in [13]:

**Lemma 10.** [Hoeffding-Bernstein inequality] Let $u_1, u_2, ..., u_r$ be a random sample without replacement from the real numbers $\{c_1, c_2, ..., c_r\}$. Then for every $t > 0$,

$$P\left(\left|\sum_i u_i - r\bar{c}\right| \geq t\right) \leq 2\exp\left(-\frac{t^2}{2r\sigma_R^2 + t\Delta_R}\right)$$

where $\Delta_R = \max_i c_i - \min_i c_i$, $\bar{c} = \frac{1}{r}\sum_i c_i$, and $\sigma_R^2 = \frac{1}{r}\sum_{i=1}^r (c_i - \bar{c})^2$.

A.2 Proof of Lemma 1

Let $x^*, p^*$ be optimal primal-dual solution of the offline problem (1). From KKT conditions of (1), $x_t^* = 1$ if $(\hat{p})^T a_t < \pi_t$ and $x_t^* = 0$ if $(p^*)^T a_t > \pi_t$. Therefore, $x_t(p^*) = x_t^*$ if $(p^*)^T a_t \neq \pi_t$. By assumption 3 there are atmost $m$ values of $t$ such that $(p^*)^T a_t \neq \pi_t$.

A.3 Proof of inequality (18)

We prove that with probability $1 - \epsilon$

$$\hat{b}_i = \sum_{t \in N} a_{it} x_t(\hat{p}) \geq (1 - 3\epsilon)b_i$$

given $\sum_{t \in S} a_{it} x_t(\hat{p}) \geq (1 - 2\epsilon)c b_i$. The proof is very similar to the proof of Lemma 2. Fix a price vector $p$ and $i$. Define a permutation is “bad” for $p$, if both (a) $\sum_{t \in S} a_{it} x_t(p) \geq (1 - 2\epsilon)c b_i$ and (b) $\sum_{t \in N} a_{it} x_t(p) \leq (1 - 3\epsilon)b_i$ hold.

Define $Y_t = a_{it} x_t(p)$. Then, the probability of bad permutations is bounded by:
\[
\Pr(|\sum_{t \in S} Y_t - \epsilon \sum_{t \in N} Y_t| \geq c^2 b_i) \sum_{t \in N} Y_t \leq (1 - 3\epsilon) b_i) \leq 2 \exp \left( -\frac{b_i c^2}{3} \right) \leq \frac{e}{m n^m} \quad (30)
\]

where the last inequality follows from the assumption that \( b_i \geq \frac{6m \log(n/\epsilon)}{\epsilon^2} \). Summing over \( n^m \) distinct prices and \( i = 1, \ldots, m \), we get the desired inequality.

**B Supporting lemmas for Section 3**

**B.1 Proof of inequality (25)**

*Proof.* The proof is very similar to the proof of Lemma 1. Fix a \( p, \ell \) and \( i \in \{1, \ldots, m\} \). Define “bad” permutations for \( p, i, \ell \) as those permutations such that all the following conditions hold:

(a) \( p = \hat{p}^\ell \), that is, \( p \) is the price learned as the optimal dual solution for (20),
(b) \( p_i > 0 \), and
(c) \( \sum_{\ell=1}^{2\ell} a_{it} x_t(p) \leq (1 - 2h_\ell - \epsilon) \frac{2\ell}{n} b_i \).

We will show that the probability of these bad permutations is small.

Define \( Y_t = a_{it} x_t(p) \). If \( p \) is an optimal dual solution for (20), and \( p_i > 0 \), then by the KKT conditions the \( i^{th} \) inequality constraint holds with equality. Therefore, by observation made in Lemma 1, we have:

\[
\sum_{t=1}^{\ell} Y_t = \sum_{t=1}^{\ell} a_{it} x_t(p) \geq (1 - h_\ell) \frac{2\ell}{n} b_i - m \geq (1 - h_\ell - \epsilon) \frac{2\ell}{n} b_i \quad (31)
\]

where the second last inequality follows from assumption \( B = \min_i b_i \geq \frac{m}{2\ell} \), and \( \ell \geq n \epsilon \).

Therefore, the probability of “bad” permutations for \( p, i, \ell \) is bounded by:

\[
P(|\sum_{t=1}^{\ell} Y_t - \frac{1}{2} \sum_{t=1}^{2\ell} Y_t| \geq h_\ell \frac{b_i}{n}| \sum_{t=1}^{\ell} Y_t \leq (1 - 2h_\ell - \epsilon) \frac{2\ell}{n} b_i) \leq 2 \exp \left( -\frac{b_i c^2}{3} \right) \leq \delta
\]

where \( \delta = \frac{e}{m n^m} \). The last inequality follows from the assumption on \( B \). Next, we take a union bound over the \( n^m \) “distinct” \( p^\ell \)'s, \( i = 1, \ldots, m \), and \( E \) values of \( \ell \), we conclude that with probability \( 1 - \epsilon \)

\[
\sum_{t=1}^{2\ell} a_{it} \hat{x}_t(p^\ell) \geq (1 - 2h_\ell - \epsilon) \frac{2\ell}{n} b_i
\]

for all \( i \) such that \( \hat{p}_i > 0 \) and all \( \ell \). \( \Box \)

**C Online multi-dimensional linear program**

**C.1 Proof of Lemma 5**

*Proof.* Using Lagrangian duality, observe that given optimal dual solution \( p^* \), optimal solution \( x^* \) is given by:

\[
\begin{align*}
\text{maximize} & \quad f_t^T x_t - \sum_i p^*_i g_{ij}^T x_t \\
\text{subject to} & \quad x_t^T e \leq 1, x_t \geq 0
\end{align*}
\]

(32)

Therefore, it must be true that if \( x_{t^t} = 1 \), then \( r \in \arg \max_j f_{tj} - (p^*)^T g_{tj} \) and \( f_{tr} - (p^*)^T g_{tr} \geq 0 \)

This means that for \( t \)'s such that \( \max_j f_{tj} - (p^*)^T g_{tj} \) is strictly positive and \( \arg \max_j \) returns a unique solution, \( x_t(p^*) \) and \( x^*_t \) are identical. By random perturbation argument there can be atmost \( m \) values of \( t \) which do not satisfy this condition (for each such \( t \), \( p \) satisfies an equation \( f_{tj} - p^T g_{tj} = f_{i} - p^T g_{i} \) for some \( j, l \), or \( f_{tj} - p^T g_{tj} = 0 \) for some \( j \)). This means \( x^* \) and \( x_t(p^*) \) differ in atmost \( m \) positions. \( \Box \)
C.2 Proof of Lemma 9

Proof. Consider $nk^2$ expressions

\begin{align*}
    f_{ij} - p^T g_{ij} - (f_{il} - p^T g_{il}), & \quad 1 \leq j, l \leq k, j \neq l, 1 \leq t \leq n \\
    f_{ij} - p^T g_{ij}, & \quad 1 \leq j \leq k, 1 \leq t \leq n
\end{align*}

$x_t(p)$ is completely determined once we determine the subset of expressions out of these $nk^2$ expressions that are assigned a non-negative value. By theory of computational geometry, there can be at most $(nk^2)^m$ such distinct assignments.