Specific heat and effects of strong pairing fluctuations in a superfluid Fermi atom gas in the BCS-BEC crossover region

Pieter van Wyk, Daisuke Inotani, Yoji Ohashi
Department of Physics, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan

Abstract. We theoretically investigate the specific heat at constant volume $C_V$ in the BCS(Bardeen-Cooper-Schrieffer)-BEC(Bose-Einstein-condensation)-crossover regime of an ultracold Fermi gas, below the superfluid phase transition temperature $T_c$. Within the strong-coupling framework developed by Nozières and Schmitt-Rink, we show that the temperature dependence of $C_V$ drastically changes as one passes through the crossover region, and is sensitive to strong fluctuations in the Cooper channel near the unitarity limit. We also compare our results to a recent experiment on a $^6$Li unitary Fermi gas. Since fluctuation effects are a crucial key in the BCS-BEC-crossover phenomenon, our results would be helpful in considering how the fermionic BCS superfluid changes into BEC with increasing the interaction strength, from the viewpoint of specific heat.

1. Introduction

Superfluid Fermi gases have recently attracted much theoretical and experimental attention as a highly controllable many-body quantum system[1, 2, 3, 4, 5, 6, 7, 8]. Using a tunable pairing interaction associated with a Feshbach resonance[6], we have realised the so-called BCS (Bardeen-Cooper-Schrieffer)-BEC (Bose-Einstein-condensation) crossover phenomenon, where we can study a Fermi superfluid and a Bose superfluid in a unified manner. In the BCS-BEC crossover region, the system is dominated by strong superfluid fluctuations, so that fluctuation corrections to physical properties in this regime have extensively been discussed[9, 10, 11]. Recently various thermodynamic quantities have become accessible in cold Fermi gas physics[12, 13, 14, 15], allowing us to directly compare strong-coupling theoretical results with experimental data observed in the BCS-BEC crossover region[10, 16, 17, 18].

The specific heat at constant volume $C_V$, which has recently become experimentally accessible in cold Fermi gas physics[18], was shown to be sensitive to pairing fluctuations above the phase transition temperature $T_c$ in the cases of an isotropic s-wave pairing interaction[19], as well as an anisotropic p-wave [20, 21, 22] one. Using this strong point, Refs.[19] and [20] succeeded in identifying the region where pairing fluctuations dominate over normal-state properties, in the phase diagram of a Fermi gas in terms of the temperature and the s-wave and p-wave interaction strength, respectively. Thus, it is an interesting topic to see whether or not this thermodynamic quantity is also useful to detect pairing (superfluid) fluctuations below $T_c$. In this regard, we briefly note that a similar study has been done for the spin susceptibility in the BCS-BEC crossover regime of a superfluid Fermi gas[23].

In this paper, we theoretically investigate the specific heat at constant volume $C_V$ in the superfluid Fermi gas. We show that inclusion of superfluid fluctuations within the strong-coupling theory developed by Nozières and Schmitt-Rink (NSR)[24] extended to the superfluid phase bellow $T_c$ [25] naturally gives...
\[ \Omega_{\text{NSR}} = -U + \ldots \]

\[ \tau_i, \tau_j \]

**Figure 1.** Feynman diagrams describing superfluid corrections \( \Omega_{\text{NSR}} \) to the thermodynamic potential \( \Omega \).

\(-U < 0\) is an attractive interaction. \( \hat{\Pi}_{ij} \) is the \( ij \) component of the particle-particle correlation function in Eq.(5).

A \( T^3 \) behavior of \( C_V \) at low temperatures in the unitarity limit, coming from collective excitations of a Goldstone mode with gapless linear dispersion. We point out that this temperature dependence is qualitatively different from the well-known \( T^2 \) behavior in an ideal Bose gas expected in the strong coupling BEC limit, as well as the exponential suppression in the mean-field BCS result (where effects of single-particle excitations are only taken into account), so that \( C_V \) may be used to identify the region where phase fluctuations of the superfluid order parameter dominate over low-energy properties of a superfluid Fermi gas in the BCS-BEC crossover region. Throughout this paper we take \( \hbar = k_B = 1 \), and the system volume \( V \) is taken to be unity, for simplicity.

2. Formalism

We consider a two component uniform superfluid Fermi gas, described by the ordinary BCS Hamiltonian given in the Nambu representation

\[ H = \sum_p \Psi_p^\dagger \begin{bmatrix} \varepsilon_p \tau_3 - \Delta \tau_1 \end{bmatrix} \Psi_p - U \sum_q \rho_+ (q) \rho_- (-q), \]  

where

\[ \Psi_p = \begin{bmatrix} c_{p,\uparrow} \\ c_{p,\downarrow} \end{bmatrix}, \]

is the Nambu field acting on particle hole space.

\( c_{p,\sigma} \) is the annihilation operator of a Fermi atom with pseudospin \( \sigma = \uparrow, \downarrow \), describing two atomic hyperfine states contributing to Cooper-pairs. \( \xi_p = \varepsilon_p - \mu = \frac{p^2}{2m} - \mu \) is the kinetic energy, measured from the Fermi chemical potential \( \mu \) (where \( m \) is an atomic mass), \( \Delta \) is the superfluid order parameter, and \(-U < 0\) is an attractive contact type s-wave interaction. \( \tau_j (j = 1, 2, 3) \) are the Pauli matrices. In Eq.(1) \( \rho_\pm (q) = \frac{\rho_1 (q) \pm i \rho_2 (q)}{2} \), where \( \rho_j (q) = \sum_p \Psi_p^\dagger \tau_j \Psi_p \) are generalised density operators describing amplitude \( (j = 1) \) and phase \( (j = 2) \) fluctuations of the superfluid order parameter \( \Delta [25, 26] \).

The interaction strength is measured in terms of the inverse \( s \)-wave scattering length \( a_s^{-1} \), which is related to \(-U\) through the renormalization prescription,

\[ \frac{1}{U} = -\frac{m}{4\pi a_s} + \sum_p \frac{1}{2\varepsilon_p}. \]

In the NSR theory, fluctuation corrections \( \equiv \Omega_{\text{NSR}} \) to the thermodynamic potential \( \Omega = \Omega_0 + \Omega_{\text{NSR}} \) are diagrammatically given as Fig.1, where

\[ \Omega_0 = \sum_p \left[ \xi_p - E_p + \frac{\Delta^2}{2\varepsilon_p} + 2T \ln \left[ 1 - n_F (E_p) \right] \right] - \frac{m\Delta^2}{4\pi a_s} \]
is the thermodynamic potential in the mean-field BCS theory. In Eq.(4) \( E_p = \sqrt{\varepsilon_p^2 + \Delta^2} \), and \( n_F(E_p) = [e^{E_p/T} + 1]^{-1} \) is the Fermi distribution function. Summing up the diagrams in Fig.1, we have

\[
\Omega_{\text{NSR}} = \frac{T}{2} \sum_{q, i\nu_n} \ln \left[ \frac{m}{4\pi a_s} - \Pi(q, i\nu_n) + \frac{1}{2\varepsilon_p} \right], \tag{5}
\]

where

\[
\Pi(q, i\nu_n) = \begin{bmatrix} \Pi^{++}(q, i\nu_n) & \Pi^{+-}(q, i\nu_n) \\ \Pi^{-+}(q, i\nu_n) & \Pi^{--}(q, i\nu_n) \end{bmatrix}, \tag{6}
\]

with

\[
\Pi^{++}(q, i\nu_n) = -\frac{1}{4} \sum_{p, s=\pm 1} \tanh \frac{E_s}{2T} \left( \frac{E_s + sE_-}{E_s + sE_- + 2\varepsilon_n^2} \right) \left( 1 + s \frac{\xi_+ - \xi_-}{E_- + E_+} + i\nu_n \left( \frac{\xi_+ + \xi_-}{E_- + E_+} \right) \right),
\]

\[
\Pi^{--}(q, i\nu_n) = \frac{1}{4} \sum_{p, s=\pm 1} \frac{\Delta^2}{(E_s + sE_-)(E_s + sE_- + 2\varepsilon_n^2)} \left( \frac{\tanh \frac{E_s}{2T}}{2T} + s \frac{E_+}{2T} \right), \tag{7}
\]

being the pair correlation function describing superfluid fluctuations (where \( \xi_\pm = \xi_{p\pm p'/2} \) and \( E_\pm = E_{p\pm p'/2} \)).

The superfluid order parameter \( \Delta \) and the Fermi chemical potential \( \mu \) for a given temperature \( T \) are self consistently determined from the coupled particle number equation \( N = N_0 + N_{\text{NSR}} \) (where \( N_0 = -\frac{\partial N_{\text{NSR}}}{\partial \mu} \) \( T \)) and \( N_{\text{NSR}} = -\frac{\partial N_{\text{NSR}}}{\partial \mu} \) with the so-called Thouless criterion[27]

\[
\det \left[ \frac{m}{4\pi a_s} - \Pi(q = 0, i\nu_n = 0) + \frac{1}{2\varepsilon_p} \right] = 0. \tag{8}
\]

The specific heat at constant volume \( C_V \) is then calculated from the relation

\[
C_V = \left( \frac{\partial E}{\partial T} \right)_N, \tag{9}
\]

where the internal energy \( E \) of the system is conveniently obtained from the Legendre transformation

\[
E = \Omega - T \left( \frac{\partial \Omega}{\partial T} \right)_\mu - \mu \left( \frac{\partial \Omega}{\partial \mu} \right)_T. \tag{10}
\]

3. Results

Figure 2 (a) shows the specific heat at constant volume \( C_V \) as a function of temperature in a unitary superfluid Fermi gas. In this figure, we find that \( C_V \) behaves as \( C_V \sim T^3 \) in the low temperature region \( (T/T_c \leq 0.6) \). Since the \( T^3 \)-behaviour of \( C_V \) is characteristic of a system with linear dispersion, this result indicates that the strong-coupling corrections in the NSR theory enables us to include effects of a collective Goldstone mode (having a phonon-like linear dispersion) associated with the broken \( U(1) \) gauge symmetry in the superfluid phase. When the NSR term \( \Omega_{\text{NSR}} \) in Eq.(5) is ignored, the resulting ordinary BCS theory gives the internal energy

\[
E \simeq \Omega_0 - T \left( \frac{\partial \Omega_0}{\partial T} \right)_\mu - \mu \left( \frac{\partial \Omega_0}{\partial \mu} \right)_T
= \sum_p \left[ \xi_p - E_p + \frac{\Delta^2}{2\varepsilon_p} + 2E_p n_F(E_p) \right] - \frac{m\Delta^2}{4\pi a_s} + \mu N_0. \tag{11}
\]
In this paper, we have discussed the specific heat at constant volume \( C_V \) in a superfluid Fermi gas within the NSR theory. We showed that \( C_V \) at unitarity exhibits a \( T^3 \)-behaviour in a wide temperature region below \( T_c \), indicating that excitations are dominated by a collective Goldstone mode with gapless linear 

\[
E \simeq \sum_q \varepsilon_q^B n_B (\varepsilon_q^B) - E_{bi} \frac{1}{2} N, \tag{12}
\]

where \( \varepsilon_q^B = q^2 / 2(2m) \) is the kinetic energy, \( n_B (\varepsilon_q^B) = \left[ e^{\varepsilon_q^B / T} - 1 \right]^{-1} \) is the Bose distribution function, and \( E_{bi} = 1 / \hbar v_F^2 \) is the binding energy of a molecular boson. The specific heat calculated from Eq.(12) is simply given by

\[
C_V = \frac{5}{\pi^2} \left( m^* \right)^{3/2} \int_0^{\infty} dx x^{3/2} e^{-x^{3/2}}. \]

This clearly shows that the specific heat in this regime is proportional to \( T^{3/2} \), which we show in Fig.2 (c).

While one cannot see the \( T^{3/2} \)-behaviour of \( C_V \) in Fig.2 (a) \( C_V \) should continuously change into the BEC result shown in Fig.2 (c). Thus, when we examine how the temperature region where \( C_V \) exhibits the \( T^3 \)-behaviour gradually disappears as one enters the strong-coupling regime, it would give the region where phase fluctuations of the order parameter dominate over the superfluid properties of the system. On the other hand, since the velocity of the Goldstone mode approaches the value \( v_F / \sqrt{3} \) of the Anderson-Bogoliubov mode in the weak-coupling BCS regime (where \( v_F \) is the Fermi velocity), it is unclear whether the \( T^3 \)-behaviour of \( C_V \) seen in the unitarity limit is gradually replaced by the exponential temperature dependence shown in Fig.2 (b), as one decreases the interaction strength, which remains as our future problem.

Figure 3 compares our result with the recent experiment on a \(^6\)Li unitary Fermi gas [18]. We see that the calculated \( C_V \) in the superfluid phase agrees with the observed data, supporting the validity of the present strong-coupling approach to this thermodynamic quantity.

4. Conclusion

In this paper, we have discussed the specific heat at constant volume \( C_V \) in a superfluid Fermi gas within the NSR theory. We showed that \( C_V \) at unitarity exhibits a \( T^3 \)-behaviour in a wide temperature region below \( T_c \), indicating that excitations are dominated by a collective Goldstone mode with gapless linear...
Figure 3. Comparison of the calculated specific heat $C_V$ in NSR theory with the recent experiment on a $^{6}\text{Li}$ Fermi gas.

dispersion. Since this temperature dependence is not obtained in the BEC limit, as well as the mean-field BCS theory, we expect that the $T^3$-behaviour of this thermodynamic quantity would be useful for identifying the region where phase fluctuations of the superfluid order parameter are crucial for system properties in the phase diagram of an ultracold Fermi gas in terms of the temperature and interaction strength.

Acknowledgments

We would like to thank M. Matsumoto, D. Kagamihara, and D. Kharga for useful discussions. This work was supported by KiPAS project in Keio University. YO was also supported by Grant-in-Aid for Scientific research from MEXT and JSPS in Japan (No.15K00178, No.15H00840, No.16K05503).

5. References

[1] C. A. Regal, M. Greiner and D. S. Jin, Phys. Rev. Lett. 92, 040403 (2004).
[2] J. Kinast, S. L. Hemmer, M. E. Gehm, A. Turlapov and J. E. Thomas, Phys. Rev. Lett. 92, 150402 (2004).
[3] M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, C. Chin, J. H. Denschlag and R. Grimm, Phys. Rev. Lett. 92, 203201 (2004).
[4] M. W. Zwierlein, J. R. Abo-Shaeer, A. Schirotzek, C. H. Schunck and W. Ketterle, Nature 435, 1047 (2005).
[5] W. Ketterle and M. W. Zwierlein, arXiv:cond-mat/0801.2500 (2008).
[6] C. Chin, R. Grimm, P. Julienne and E. Tiesinga, Rev. Mod. Phys. 82, 1225 (2010).
[7] W. Zwerger, The BCS-BEC Crossover and the Unitary Fermi Gas (Springer - Verlag, Berlin, 2012).
[8] H. Hu, X. J. Lui and P. D. Drummond, Phys. Rev. A 73, 023617 (2006).
[9] S. Tsuchiya, R. Watanabe, Y. Ohashi, Phys. Rev. A 80, 033613 (2009).
[10] H. Hu, X. Lui and P. Drummond, Phys. Rev. A 77, 061605 (2008).
[11] A. Perali, F. Palestini, P. Pieri, G. C. Strinati, J. T. Stewart, J. P. Gaebler, T. E. Drake and D. S. Jin, Phys. Rev. Lett. 106, 060402 (2011).
[12] J. Kinast, A. Turpalov, J. E. Thomas, Q. Chen, J. Stajic and K. Levin, Science, 307, 1296 (2005).
[13] L. Luo and J. E. Thomas, J. Low. Temp. Phys 154, 1 (2009).
[14] M. Horikoshi, S. Nakajima, M. Ueda and T. Mukaiyama, Science 327, 442 (2010).
[15] S. Nascimbene, N. Navon, K. J. Jiang, F. Chevy and C. Salomon, Nature 463, 1057 (2010).
[16] R. Haussmann, W. Rantner, S. Cerrito and W. Zwerger, Phys. Rev. A 75, 023610 (2007).
[17] A. Perali, F. Palestini, P. Pieri, G. C. Strinati, J. T. Stewart, J. P. Gaebler, T. E. Drake and D. S. Jin, Phys. Rev. Lett. 106, 060402 (2011).
[18] M. J. H. Ku, A. T. Sommer, L. W. Cheuk, M. W. Zwierlein, Science 335, 563 (2012).
[19] P. van Wyk, H. Tajima, R. Hanai and Y. Ohashi, Phys. Rev. A 93, 013621 (2016).
[20] D. Inotani, P. van Wyk and Y. Ohashi, JPSJ 85, 123301 (2016).
[21] D. Inotani, P. van Wyk and Y. Ohashi, JPSJ 86, 024302 (2017).
[22] D. Inotani, P. van Wyk and Y. Ohashi, JPSJ 86, 044301 (2017).
[23] H. Tajima, R. Hanai, and Y. Ohashi, Phys. Rev. A 93, 013610 (2016).
[24] P. Nozières, and S. Schmitt-Rink, J. Low Temp. Phys. 59, 195 (1985).
[25] Y. Ohashi and A. Griffin, Phys. Rev. A 67, 063612 (2003).
[26] N. Fukushima, Y. Ohashi, E. Taylor and A. Griffin, Phys. Rev. A 75, 033609 (2007).
[27] D. Thouless, Ann. Phys. 10, 553 (1960).
[28] J. R. Engelbrecht, M. Randeria and C. A. R. Sáde Melo, Phys. Rev. B 55, 15153 (1997).
[29] R. Haussmann, W. Rantner, S. Cerrito, and W. Zwerger, Phys. Rev. A 75, 023610 (2007).