A GENERALIZATION OF THE DAVIS-JANUSZKIEWICZ
CONSTRUCTION AND APPLICATIONS TO TORIC MANIFOLDS
AND ITERATED POLYHEDRAL PRODUCTS

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Abstract. The fundamental Davis-Januszkiewicz construction of toric manifolds is
reinterpretated in order to allow for generalization. Applications involve the simplicial
wedge $J$-construction and Ayzenberg's recent identities arising from composed simpli-
cial complexes.

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1. INTRODUCTION

The topological approach to non-singular toric varieties requires two ingredients:

1. a simple polytope $P^n$ of dimension $n$ having a set $\mathcal{F}$ of $m$ facets and
2. a characteristic function $\lambda: \mathcal{F} \to \mathbb{Z}^n$ which assigns an integer vector to each facet of the simple polytope $P^n$.

The latter can be considered as an $(n \times m)$-matrix $\lambda: \mathbb{Z}^m \to \mathbb{Z}^n$ with integer entries and columns indexed by the facets of $P^n$. A regularity condition, which ensures the smoothness of the toric manifold, requires all $n \times n$ minors of $\lambda$ corresponding to the vertices of $P^n$ to be $+1$ or $-1$.

Associated to the pair $(P^n, \lambda)$, Davis and Januszkiewicz [6], constructed two spaces:

$$L = T^m \times P^n / \sim$$

and a toric manifold

$$M^{2n} = T^n \times P^n / \sim \lambda .$$

The properties of the spaces $L$ have been studied extensively via an alternative general construction developed by Buchstaber and Panov [5], who gave them the name “moment-angle complexes”

$$L = T^m \times P^n / \sim \cong Z(K_P; (D^2, S^1)).$$

In the notation used here, $K_P$ represents the simplicial complex dual to the boundary of a simple polytope $P^n$. From this point of view, the toric manifold $M^{2n}$ is recovered as the quotient $Z(K_P; (D^2, S^1))/\ker \lambda$.

The main results presented in authors’ earlier work [4], arise from a construction on a simplicial complex $K_P$ having $m$ vertices. For each sequence $J = (j_1, j_2, \ldots, j_m)$ of positive integers, a new simplicial complex $K_P(J)$ is constructed,

$$K_P \rightsquigarrow K_P(J).$$

Also, associated to $P^n$ is another simple polytope $P(J)$ and $K_P(J) = K_{P(J)}$. Everything fits together in such a way that, from the toric manifold $(P^n, \lambda, M^{2n})$, it is possible to construct another toric manifold $(P(J), \lambda(J), M(J))$. In the context of moment-angle complexes and polyhedral products ([3]), it is shown in [4] that there is a diffeomorphism of orbit spaces

$$Z(K_P; (D^{2j}, S^{2j-1}))/\ker \lambda \to Z(K_P(J); (D^2, S^1))/\ker \lambda(J)$$

which defines $M(J)$. Here, the left hand side uses the notation of a polyhedral product, (generalized moment-angle complex), associated to the family of pairs

$$(D^{2j}, S^{2j-1}) = \{(D^{2j_i}, S^{2j_i-1})\}_{i=1}^m.$$
These ideas are elaborated upon in Section 5. The right hand side involves an ordinary moment-angle complex and fits in with the formalism of Davis–Januszkiewicz and Buchstaber–Panov. A natural question arises about the left hand side which involves the combinatorics of the smaller simplicial complex $K_P$: where is this visible in the standard Davis–Januszkiewicz construction of toric manifolds? One of the primary goals here is to answer this question.

The details of the Davis–Januszkiewicz construction are reviewed in Section 2 and the modifications necessary to generalize the construction are described in Section 3. The modified construction is interpreted in terms of polyhedral products in Section 4 and the answer to the question posed above is presented in Section 5. Additional applications involving the “composed complex” constructions of A. Ayzenberg [1], are discussed in Sections 6 and 7.

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2. A REVIEW OF THE DAVIS-JANUSZKIEWICZ CONSTRUCTION

A toric manifold $M^{2n}$ is a manifold covered by local charts $\mathbb{C}^n$, each with the standard action of a real $n$-dimensional torus $T^n$, compatible in such a way that the quotient $M^{2n}/T^n$ has the structure of a simple polytope $P^n$. Here, “simple” means that $P^n$ has the property that at each vertex, exactly $n$ facets intersect. Under the $T^n$ action, each copy of $\mathbb{C}^n$ must project to an $\mathbb{R}^n_+$ neighborhood of a vertex of $P^n$. The fundamental construction of Davis and Januszkiewicz [6, Section 1.5] is described briefly below. It realizes all toric manifolds and, in particular, all smooth projective toric varieties. Let

$$\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$$

denote the set of facets of $P^n$. The fact that $P^n$ is simple implies that every codimension-$l$ face $F$ can be written uniquely as

$$F = F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_l}$$

where the $F_{i_j}$ are the facets containing $F$. Let

$$\lambda : \mathcal{F} \longrightarrow \mathbb{Z}^n$$

(2.1)
be a function into an \( n \)-dimensional integer lattice satisfying the \textit{regularity} condition that whenever \( F = F_1 \cap F_2 \cap \cdots \cap F_l \) then \( \{ \lambda(F_i), \lambda(F_{i2}), \ldots, \lambda(F_{il}) \} \) span an \( l \)-dimensional submodule of \( \mathbb{Z}^n \) which is a direct summand. Such a map is called a \textit{characteristic function} associated to \( P^n \). Next, regarding \( \mathbb{R}^n \) as the Lie algebra of \( T^n \), the map \( \lambda \) is used to associate to each codimension-\( l \) face \( F \) of \( P^n \) a rank-\( l \) subgroup \( G_F \subset T^n \). Specifically, writing \( \lambda(F_{i1}) = (\lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{ni}) \) gives

\[
G_F = \{ (e^{2\pi i (\lambda_{1i} t_1 + \lambda_{12} t_2 + \cdots + \lambda_{1i} t_i)}, \ldots, e^{2\pi i (\lambda_{ni} t_1 + \lambda_{n2} t_2 + \cdots + \lambda_{ni} t_l)}) \in T^n \}
\]

where \( t_i \in \mathbb{R}, i = 1, 2, \ldots, l \). Finally, let \( p \in P^n \) and \( F(p) \) be the unique face with \( p \) in its relative interior. Define an equivalence relation \( \sim \lambda \) on \( T^n \times P^n \) by \((g, p) \sim \lambda (h, q)\) if and only if \( p = q \) and \( g^{-1}h \in G_{F(p)} \cong T^l \). Then

\[
(2.2) \quad M^{2n} \cong M^{2n}(\lambda) = T^n \times P^n / \sim \lambda
\]

is a smooth, closed, connected, \( 2n \)-dimensional manifold with \( T^n \) action induced by left translation [6, page 423]. A projection \( \pi : M^{2n} \to P^n \) onto the polytope is induced from the projection \( T^n \times P^n \to P^n \).

\textit{Remark.} In the cases when \( M^{2n} \) is a projective non-singular toric variety, \( P^n \) and \( \lambda \) encode topologically the information in the defining fan, [5, Chapter 5].

Let \( K_P \) denote the simplicial complex dual to the boundary of simple polytope \( P^n \) having \( m \) facets. Recall that the duality here is in the sense that the facets of \( P^n \) correspond to the vertices of \( K_P \). A set of vertices in \( K_P \) is a simplex if and only if the corresponding facets in \( P^n \) all intersect. Davis and Januszkiewicz constructed a second space in [6], which came to be known as a \textit{moment-angle manifold}, by

\[
(2.3) \quad \mathcal{Z} \cong T^m \times P^n / \sim
\]

where here \( \sim \) does not involve the characteristic \( \lambda \) but the combinatorics of the simplicial complex \( K_P \) only. Here also, the circles in \( T^m \) are indexed by the facets of \( P^n \). The equivalence relation \( \sim \) is defined by analogy with that of (2.2). Specifically, \( \lambda \) in (2.1) is replaced by

\[
(2.4) \quad \theta : \mathcal{F} \to \mathbb{Z}^m
\]

where \( \theta(F_i) = e_i \in \mathbb{Z}^m \).
Constructions (2.3) and (2.2) are related by a quotient map given by the free action of \( \ker \lambda \) on \( \mathbb{Z} \)

\[
T^m \times P^n / \sim \longrightarrow (T^m \times P^n / \sim) / \ker \lambda \cong T^n \times P^n / \sim \lambda
\]
as described in [5, Section 6.1].

3. Modifying the equivalence relations

As above, let \( P^n \) be simple polytope. The construction of (2.3) is generalized easily by first replacing each of the circles in \( T^m \) by spaces \( X_1, X_2, \ldots, X_m \), indexed by the facets of \( P^n \).

**Construction 3.1.** Define an equivalence relation \( \sim_1 \) on the Cartesian product

\[
X_1 \times X_2 \times \cdots \times X_m \times P^n
\]
as follows:

\[
(x_1, x_2, \ldots, x_m, p) \sim_1 (y_1, y_2, \ldots, y_m, q)
\]
if and only if:
(a) \( p = q \) and
(b) when \( p \) is in the relative interior of the face \( F(p) = F_{j_1} \cap F_{j_2} \cap \cdots F_{j_k} \) given as the intersection of the \( k \) facets which are complementary to \( \{F_{i_1}, F_{i_2}, \ldots, F_{i_{m-k}}\} \), then \( x_{i_s} = y_{i_s} \) for all \( s \in \{1, 2, \ldots, m - k\} \).

Equivalence classes of points in \( (X_1 \times X_2 \times \cdots \times X_m) \times P^n / \sim_1 \) are denoted by the symbol \( [(x_1, x_2, \ldots, x_m, p)]_1 \).

Suppose now that \( S^1 \) acts freely on the spaces \( X_1, X_2, \ldots, X_m \), giving an action of \( T^m \) on \( X_1 \times X_2 \times \cdots \times X_m \) in the obvious way. Recall that the function \( \theta \) of (2.4) indexes the “coordinate” circles in \( T^m \) by the facets of \( P^n \). Also, each space \( X_i \) is associated with the facet \( F_i \), so, an intersection of \( k \) facets in \( P^n \) determines a projection \( T^m \longrightarrow T^{m-k} \) and, by this projection, \( T^m \) acts on the product \( X_{i_1} \times X_{i_2} \times \cdots \times X_{i_{m-k}} \).

Next, let \( \lambda \) be a characteristic map specified for the polytope \( P^n \). Then

\[
\ker \lambda \cong T^{m-n} \subset T^m.
\]

For \( k \leq n \), there is the induced action of \( \ker \lambda \subset T^m \) on the product

\[
X_{i_1} \times X_{i_2} \times \cdots \times X_{i_{m-k}}
\]
and a projection
\begin{equation}
\pi_{i_1,i_2,\ldots, i_{m-k}} : X_1 \times X_2 \times \cdots \times X_m/\ker \lambda \longrightarrow X_{i_1} \times X_{i_2} \times \cdots \times X_{i_{m-k}}/\ker \lambda
\end{equation}
corresponding to each intersection of \( k \) facets.

Equivalence classes of points in \( X_1 \times X_2 \times \cdots \times X_m/\ker \lambda \) are denoted by the symbol \([x_1, x_2, \ldots, x_m]_\lambda\). The next Construction generalizes that of (2.2).

**Construction 3.2.** Define an equivalence relation \( \sim_2 \) on the Cartesian product
\[
(X_1 \times X_2 \times \cdots \times X_m/\ker \lambda) \times \mathbb{P}^n
\]
as follows:
\[
([x_1, x_2, \ldots, x_m]_\lambda, p) \sim_2 ([y_1, y_2, \ldots, y_m]_\lambda, q)
\]
if and only if:

(i) \( p = q \)

(ii) when \( p \) is in the relative interior of the face \( F(p) = F_{j_1} \cap F_{j_2} \cap \cdots \cap F_{j_k} \) given as the intersection of the \( k \) facets which are complementary to \( \{F_{i_1}, F_{i_2}, \ldots, F_{i_{m-k}}\} \), then
\[
\pi_{i_1,i_2,\ldots,i_{m-k}}([x_1, x_2, \ldots, x_m]_\lambda) = \pi_{i_1,i_2,\ldots,i_{m-k}}([y_1, y_2, \ldots, y_m]_\lambda).
\]

Equivalence classes of points in \( (X_1 \times X_2 \times \cdots \times X_m/\ker \lambda) \times \mathbb{P}^n/\sim_2 \) are denoted by the symbol \([([x_1, x_2, \ldots, x_m]_\lambda, p])_2\).

Construction 3.2 can be reinterpreted as follows. The group \( \ker \lambda \) acts on the space \((X_1 \times X_2 \times \cdots \times X_m) \times \mathbb{P}^n/\sim_1 \) by
\[
t \cdot [(x_1, x_2, \ldots, x_m, p)]_1 = [t \cdot (x_1, x_2, \ldots, x_m, p)]_1.
\]

Property (b) in Construction 3.1 ensures that the action is well defined. The next lemma, the analogue of (2.5), follows naturally.

**Lemma 3.3.** There is a homeomorphism
\[
h : (X_1 \times X_2 \times \cdots \times X_m/\ker \lambda) \times \mathbb{P}^n/\sim_2 \longrightarrow ((X_1 \times X_2 \times \cdots \times X_m) \times \mathbb{P}^n/\sim_1)/\ker \lambda
\]
given by
\[
h(([x_1, x_2, \ldots, x_m]_\lambda, p)]_2) = [([x_1, x_2, \ldots, x_m, p)]_1)_\lambda.
\]
Proof. To see that $h$ is well defined, suppose
$$\left[[[x_1, x_2, \ldots, x_m]_\lambda, p]\right]_2 = \left[[[y_1, y_2, \ldots, y_m]_\lambda, p]\right]_2$$
with $p$ is in the relative interior of the face $F(p) = F_{j_1} \cap F_{j_2} \cap \cdots \cap F_{j_k}$ given as the intersection of the $k$ facets which are complementary to $\{F_{i_1}, F_{i_2}, \ldots, F_{i_{m-k}}\}$, as in Construction 3.2. Then, $\pi_{i_1,i_2,\ldots,i_{m-k}}([x_1, x_2, \ldots, x_m]) = \pi_{i_1,i_2,\ldots,i_{m-k}}([y_1, y_2, \ldots, y_m])$ and hence,
$$t \cdot (x_{i_1}, x_{i_2}, \ldots, x_{i_{m-k}}) = (y_{i_1}, y_{i_2}, \ldots, y_{i_{m-k}})$$
for some $t \in \ker \lambda$. It follows now from Construction 3.1 that
$$\left[[[x_1, x_2, \ldots, x_m]_\lambda, p]\right]_2 = \left[[[y_1, y_2, \ldots, y_m]_\lambda, p]\right]_2$$
as required.

To check that $h$ is an injection, suppose that
$$h\left([[x_1, x_2, \ldots, x_m]_\lambda, p]\right)_2 = h\left([[y_1, y_2, \ldots, y_m]_\lambda, p]\right)_2.$$
Then $t \in \ker \lambda$ exists so that
$$t \cdot ([x_1, x_2, \ldots, x_m, p])_1 = [t \cdot (x_1, x_2, \ldots, x_m, p)]_1 = [(y_1, y_2, \ldots, y_m, p)]_1.$$ 
Next, write $t \cdot (x_1, x_2, \ldots, x_m) = (u_1, u_2, \ldots, u_m)$. Since $p$ is in the relative interior of the face $F(p) = F_{j_1} \cap F_{j_2} \cap \cdots \cap F_{j_k}$, it follows that
$$u_{i_s} = y_{i_s} \quad \text{for all} \quad s \in \{1, 2, \ldots, m - k\}$$
corresponding to the complementary facets. It follows that
$$\pi_{i_1,i_2,\ldots,i_{m-k}}([x_1, x_2, \ldots, x_m]_\lambda) = \pi_{i_1,i_2,\ldots,i_{m-k}}([y_1, y_2, \ldots, y_m]_\lambda)$$
and hence
$$\left[[[x_1, x_2, \ldots, x_m]_\lambda, p]\right]_2 = \left[[[y_1, y_2, \ldots, y_m]_\lambda, p]\right]_2.$$
It is easy now to see that $h$ is a homeomorphism. \hfill $\Box$

4. Interpreting the generalizations in the Buchstaber–Panov formalism

Construction 2.3 can be analyzed locally. In the neighbourhood of a vertex $v_i$, a simple polytope $P^n$ looks like $\mathbb{R}^n_+$. 
The polytope can be given a *cubical* structure as in [5, Construction 5.8 and Lemma 6.6]. The cube $I^n$, *anchored* by the vertex $v_i$, sits inside the copy of $\mathbb{R}_+^n$ obtained by deleting all faces of $P^n$ which do not contain $v_i$.

Locally, $T^m \times P^n$ is

\begin{equation}
T^m \times I^n \cong (S^1 \times I)^n \times (S^1)^{m-n}
\end{equation}

Recall that all the circles in $T^m$ are indexed by the facets of the polytope so here, *the order of the factors is confused*. The factors $S^1$ which are paired with a copy of $I$ are those corresponding to the facets of $P^n$ which meet at $v_i$. The effect of the equivalence relation $\sim$ in $T^m \times I^n / \sim$, (2.3), is to convert every $S^1 \times I$ on the right hand side of (4.1) into a disc by collapsing $S^1 \times \{0\}$ to a point. So

\begin{equation}
T^m \times I / \sim \cong (D^2)^n \times (S^1)^{m-n}.
\end{equation}

The vertices of $P^n$ correspond the maximal simplices of the simplicial complex $K_P$ so, assembling the blocks (4.2) gives the moment-angle manifold

\[ Z = T^m \times P^n / \sim \cong Z(K_P; (D^2, S^1)). \]

As described in [6] and [5], the map $\lambda$ determines a subtorus $\ker \lambda = T^{m-n} \subset T^m$ and a commutative diagram of quotient maps

\begin{equation}
\begin{array}{ccc}
T^m \times P^n / \sim & \longrightarrow & T^n \times P^n / \sim \lambda \\
\downarrow \cong & & \downarrow \cong \\
Z(K_P; (D^2, S^1)) & \longrightarrow & Z(K_P; (D^2, S^1))/\ker \lambda.
\end{array}
\end{equation}

The construction above is generalized easily by replacing each of the circles in $T^m$ by spaces $X_1, X_2, \ldots, X_m$ indexed by the facets of $P^n$. Again, locally in the neighbourhood...
of a vertex $v_i$, at which facets $F_{i_1}, F_{i_2}, \ldots, F_{i_n}$ meet, $X_1 \times X_2 \times \cdots \times X_m \times P^n$ is

$$X_1 \times X_2 \times \cdots \times X_m \times I^n$$

$$= (X_{i_1} \times X_{i_2} \times \cdots \times X_{i_n} \times I^n) \times X_{i_{n+1}} \times X_{i_{n+2}} \times \cdots \times X_{i_m}$$

$$= (X_{i_1} \times I) \times (X_{i_2} \times I) \times \cdots \times (X_{i_n} \times I) \times X_{i_{n+1}} \times X_{i_{n+2}} \times \cdots \times X_{i_m}$$

As everything in the Cartesian product is indexed by the facets of the polytope, the order of the factors here has been shuffled naturally. Finally, the equivalence relation $\sim_1$ on

$$(X_1 \times X_2 \times \cdots \times X_m) \times I^n$$

converts every $X_{i_k} \times I$ into $CX_{i_k}$, in a natural way. So,

$$(X_1 \times X_2 \times \cdots \times X_m) \times I^n / \sim_1$$

$$\cong CX_{i_1} \times CX_{i_2} \times \cdots \times CX_{i_n} \times (X_{i_{n+1}} \times X_{i_{n+2}} \times \cdots \times X_{i_m})$$

Assembling over all the vertices of $P^n$ along the common intersection determined by the cubical structure on $P^n$, gives the polyhedral product

$$Z(K_P; (CX_1, X_1)) \subseteq CX_1 \times CX_2 \times \cdots \times CX_m.$$

just as in the case $X_i = S^1$ for standard moment-angle complexes, [5, Section 6.1].

A choice of cubical structure for the simple polytope $P^n$ allows a choice of homeomorphism

$$(4.4) \quad \alpha: X_1 \times X_2 \times \cdots \times X_m \times P^n / \sim_1 \longrightarrow Z(K_P; (CX, X)).$$

The fact that $S^1$ acts (freely) on $X_i$ is now used again. $X_i$. The action extends to $CX_i$ by preserving the cone parameter. Consequently, there is an action of $T^m$ on $CX_1 \times CX_2 \times \cdots \times CX_m$ which extends to

$$Z(K_P; (CX, X)) \subseteq CX_1 \times CX_2 \times \cdots \times CX_m.$$
The next theorem is the analogue of (4.3).

**Theorem 4.1.** The diagram following commutes

\[
\begin{array}{ccc}
X_1 \times X_2 \times \cdots \times X_m \times P^n / \sim_1 & \xrightarrow{\beta} & (X_1 \times X_2 \times \cdots \times X_m / \ker \lambda) \times P^n / \sim_2 \\
\downarrow \cong & & \downarrow \cong \\
Z(K_P; (CX, X)) & \xrightarrow{\delta} & Z(K_P; (CX, X)) / \ker \lambda
\end{array}
\]

where the map \( \beta \) is the composite of the quotient map

\[
((X_1 \times X_2 \times \cdots \times X_m) \times P^n / \sim_1) \rightarrow ((X_1 \times X_2 \times \cdots \times X_m) \times P^n / \sim_1) / \ker \lambda
\]

with the map \( h^{-1} \) of Lemma 3.3 and \( \gamma \) is the composite \( \overline{\alpha} \circ h^{-1} \).

**Proof.** The homeomorphism \( h \) can be used to replace the space in the top right hand corner with the space \((X_1 \times X_2 \times \cdots \times X_m) \times P^n / \sim_1\) / \( \ker \lambda \). This gives a new commutative diagram by the equivariance of the homeomorphism \( \alpha \). The maps \( \beta \) and \( \delta \) are defined in terms of the map \( h^{-1} \) and so the diagram commutes as given. \( \square \)

The next remark confirms that the original Davis-Januszkiewicz constructions are preserved.

**Remark 4.2.** For the case \( X_i = S^1 \) and \( S^1 \) acting on itself in the usual way, Constructions 3.1 and 3.2 agree with those of (2.3) and (2.2).

**5. Application to the construction of infinite families of toric manifolds**

5.1. **The case of odd spheres.** The first application is to the infinite families of toric manifolds constructed in [4] and summarized briefly below.

Let \( K \) be a simplicial complex of dimension \( n - 1 \) on vertices \( \{v_1, v_2, \ldots, v_m\} \). Given a sequence of positive integers \( J = (j_1, j_2, \ldots, j_m) \), define a new simplicial complex \( K(J) \) on new vertices

\[
\{v_{11}, \ldots, v_{1j_1}, v_{21}, \ldots, v_{2j_2}, \ldots, v_{m1}, \ldots, v_{mj_m}\},
\]

with the property that

\[
\{v_{i11}, \ldots, v_{i1j_{i1}}, \ldots, v_{ik1}, \ldots, v_{ikj_{ik}}\}
\]
is a minimal non-face of $K(J)$ if and only if $\{v_{i_1}, \ldots, v_{i_k}\}$ is a minimal non-face of $K$. Moreover, all minimal non-faces of $K(J)$ have this form. A result of Provan and Billera, [8, page 578], ensures that if $K = K_P$ is dual to the boundary of a simple polytope $P$, then $K(J)$ is dual to the boundary of another simple polytope $P(J)$. That is

$$K_P(J) = K_{P(J)}.$$  

It is the case also that the polytope $P(J)$ can be constructed directly from $P$ in a straightforward way.

Let $(P^n, \lambda, M^{2n})$ specify a toric manifold as in (2.2). From this, it is possible to construct another toric manifold $(P(J), \lambda(J), M(J))$ where the numbers $m$ and $n$, from Section 2, transform as follows.

$$\begin{bmatrix} m & n & m - n \end{bmatrix} \sim \begin{bmatrix} d(J) = j_1 + j_2 + \cdots + j_m \\ d(J) - m + n \\ m - n \end{bmatrix}$$

In terms of the original characteristic map $\lambda$, the matrix specifying the characteristic map $\lambda(J)$ is given in Figure 1 below. In that figure, $I_{j_i - 1}$ represents the identity matrix of size $j_i - 1$. It is clear from the form of the matrix $\lambda(J)$ and the definition of $K(J)$, that the Davis-Januszkiewicz cohomology calculation, [6, Theorem 4.14], expresses the integral cohomology of $M(J)$ in terms of the sequence $J$, the original matrix $\lambda$ and the combinatorics of $K$. 

Figure 1. The matrix \( \lambda(J) \)
Also in [4] is an interpretation of this construction of the toric manifolds $M(J)$ in terms of generalized moment-angle complexes. To see this, consider the family of CW pairs

$$(D^{2j}, S^{2j-1}) = \{(D^{2j_i}, S^{2j_i})\}_{i=1}^{m}$$

and the associated generalized moment-angle complex $Z(K_P; (D^{2j}, S^{2j-1}))$. There is an inclusion of tori $T^m \rightarrow T^{d(J)}$ which includes the $i$th circle in $T^m$ by the diagonal

$$S^1 \rightarrow (S^1)^{j_i}.$$ 

This gives an action of $T^m$ on $Z(K_P(J); (D^{2j}, S^{2j-1}))$. Also, via a choice of diffeomorphism $D^{2j_i} \cong (D^2)^{j_i}$, there is an action of $T^{d(J)}$ on the moment-angle complex $Z(K_P; (D^{2j}, S^{2j-1}))$. With this understood, there are $T^m$ and $T^{d(J)}$-equivariant diffeomorphisms

$$(5.4) \quad Z(K_P; (D^{2j}, S^{2j-1})) \rightarrow Z(K_P(J); (D^2, S^1))$$

from which arises a diffeomorphism of orbit spaces

$$Z(K_P; (D^{2j}, S^{2j-1}))/\ker \lambda \rightarrow Z(K_P(J); (D^2, S^1))/\ker \lambda(J)$$

which defines $M(J)$. (Here, $\ker \lambda$ and $\ker \lambda(J)$ are isomorphic subgroups of $T^{d(J)}$.)

The appearance of the toric manifold $M(J)$ as the right hand side of (5.5) is perplexing because that space is not reflected in either the fundamental construction (2.2) or in diagram (4.3). The matter is resolved by the diagram of Theorem 4.1 where the space appears in the bottom right of the diagram with $X_i = S^{2j_i-1}$, and $CX_i = D^{2j_i}$; the group $S^1$ acts freely on $S^{2j_i-1}$ in the usual way. This observation is formalized in the next theorem.

**Theorem 5.1.** The toric manifolds $M(J)$, defined by either the original Davis-Januszkiewicz construction (2.2) or equivalently, by the quotients (5.5), are examples of Construction 3.2 as follows

$$M(J) = \left( T^{d(J)-m+n} \times P(J) \right)/\sim_{\lambda(J)} \cong \left( S^{2j_1-1} \times S^{2j_2-1} \times \cdots \times S^{2j_m-1}/\ker \lambda \right) \times P^n/\sim_2 .$$

**Proof.** This result follows immediately from the right hand side of diagram (4.5). \hfill \Box

Notice here that the advantage of the right hand side is the use of the (generalized) Davis-Januszkiewicz construction with the polytope $P^n$, which is in general much smaller than $P(J)$ and has simpler combinatorics.

**Remark 5.2.** Notice that the part of $\lambda(J)$ in Figure 1 reproduced below,
is essentially the \((j_2 - 1) \times j_2\) matrix

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 & -1 \\
0 & 1 & \cdots & 0 & -1 \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
0 & 0 & \cdots & 1 & -1
\end{bmatrix}
\]

(5.6)

which is the characteristic matrix for the diagonal \(S^1\) action on \(S^{2j_2-1}\). Indeed, the \(i\)th “block row” of \(\lambda(J)\) is the characteristic matrix for the diagonal \(S^1\) action on \(S^{2j_i-1}\). This particular connection to odd spheres becomes evident in the light of Construction 3.2 but was not obvious at the time that [4] was written. This observation becomes relevant in the next section.

5.2. A simple illustration. The toric manifold \(\mathbb{C}P^2\) is made usually by the construction (2.2) using a two-simplex as the simple polytope. The diagram below illustrates Construction 3.2 in this case. The ingredients are as follows:

(1) \(P^n = \Delta^1\) a one-simplex. Here \(n = 1\) and \(m = 2\),

(2) \(J = (1, 2)\) so that \(X_1 = S^1\) and \(X_2 = S^3\) with the usual free \(S^1\) action,

(3) the characteristic map \(\lambda: \mathbb{Z}^2 \to \mathbb{Z}\) is given by the matrix \([1, -1]\) and \(\ker\lambda \cong T^1\) sits inside \(T^2\) as \(t \mapsto (t, t^{-1})\).

Remark. Here, \(P(J) = \Delta^1(1, 2) = \Delta^2\) a two-simplex, the usual polytope used to construct \(\mathbb{C}P^2\) as a toric manifold from (2.2).
In the diagram, the symbol $\rightsquigarrow$ represents the projection (3.1) which appears in part (ii) of Construction 3.2. The diagram presents $\mathbb{C}P^2$ as the cone on $S^3$ attached to $\mathbb{C}P^1$ via the Hopf map

$$(S^1 \times S^3)/\ker \lambda \sim S^3/\ker \lambda = \mathbb{C}P^1$$

$$\mathbb{C}P^2 \cong (S^1 \times S^3/\ker \lambda) \times \Delta^1/\sim_2$$

6. Application to iterated polyhedral products

The odd spheres of Section 5 are themselves examples of moment-angle complexes

$$S^{2j_i-1} \cong Z(K_{\Delta^i}; (D^2, S^1))$$

where $K_{\Delta^i}$ is the simplicial complex dual to the boundary of the simplex $\Delta^i$. Every moment-angle complex $Z(K; (D^2, S^1))$ supports a free circle action and so it’s natural to ask about the case $X_i = Z(K_i; (D^2, S^1))$ in Construction 3.2 for a collection
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\{K_1, K_2, \ldots, K_m\} of arbitrary simplicial complexes. In this case, (4.4) becomes

(6.1) \( (X_1 \times X_2 \times \cdots \times X_m) \times P^m / \sim_1 \cong Z(K_P; [CZ(K_i; (D^2, S^1)), Z(K_i; (D^2, S^1))]). \)

6.1. A generalization of the construction \(K(J)\). The problem of finding an analogue of (5.4) and (5.5) now presents itself. Those diffeomorphisms follow from [4, Theorem 7.2] which is a more general result about the behaviour of polyhedral products with respect to “exponentiation” of CW pairs. Recent work by Anton Ayzenberg [1], generalizing this exponentiation construction becomes relevant to understanding the problem further. A brief description of Ayzenberg’s construction, tailored to the context here, follows.

Let \(K\) be a simplicial complex on \(m\) vertices and \(\{K_1, K_2, \ldots, K_m\}\) a collection of \(m\) simplicial complexes on \(j_1, j_2, \ldots, j_m\) vertices respectively. From these ingredients, a new simplicial complex \(K(K_1, K_2, \ldots, K_m)\), on \(j_1 + j_2 + \cdots + j_m\) vertices, is constructed by

(6.2) \( K(K_1, K_2, \ldots, K_m) = \bigcup_{\sigma \in K} V_\sigma \subset \Delta^{j_1-1} \ast \Delta^{j_2-1} \ast \cdots \ast \Delta^{j_m-1} \)

where

\( V_\sigma = B_1 * B_2 * \cdots * B_m \) with \( B_i = \begin{cases} \Delta^{j_i-1} & \text{if } i \in \sigma \\ K_i & \text{if } i \notin \sigma. \end{cases} \)

Remark 6.1. In this language, the construction \(K(J)\) at the beginning of subsection [5.1] is just \(K(\partial \Delta^{j_1-1}, \partial \Delta^{j_2-1}, \ldots, \partial \Delta^{j_m-1})\) where \(\partial \Delta^{j_i-1}\) is the boundary of the \((j_i - 1)\)-simplex.

For \(K = K_P\), the result analogous to (5.4) is the following.

Theorem 6.2. [1] Proposition 5.1

\[ Z(K_P; ([D^2]^{j_i}, Z(K_i; (D^2, S^1)))) = Z(K_P(K_1, K_2, \ldots, K_m); (D^2, S^1)). \]

The next task is to relate these spaces to the right hand side of (6.1). The following proposition addresses this point. As usual, set \(d(J) = j_1 + j_2 + \cdots + j_m\) and, to simplify the notation, set

\( Z(K_i) := Z(K_i; (D^2, S^1)). \)

Proposition 6.3. There is a \(T^{d(J)}\)-equivariant homotopy equivalence of polyhedral products

(6.3) \( Z(K_P; (CZ(K_i), Z(K_i))) \simeq Z(K_P; ([D^2]^{j_i}, Z(K_i))). \)
Proof. Let $K$ be a simplicial complex on $j$ vertices. Then if $K$ is not the $(j - 1)$-simplex $\Delta^{j-1}$, there is simplicial embedding $K \rightarrow \partial\Delta^{j-1}$ into the boundary. This induces an inclusion

$$Z(K; (D^2, S^1)) \rightarrow Z(\partial\Delta^{j-1}; (D^2, S^1)) = \partial((D^2)^j) \cong S^{2j-1}$$

equivariant with respect to the action of the $j$-torus $T^j$. In turn, this extends to an equivariant homotopy equivalence on cones:

$$CZ(K; (D^2, S^1)) \rightarrow CS^{2j-1} \cong D^{2j}$$

where the action of $T^j$ preserves the cone parameter. Next, choose a standard $T^j$–equivariant diffeomorphism $h: D^{2j} \rightarrow (D^2)^j$ to get an equivariant homotopy equivalence of CW pairs

$$(CZ(K; (D^2, S^1)), Z(K; (D^2, S^1))) \xrightarrow{h} ((D^2)^j, Z(K; (D^2, S^1))).$$

The functorial properties of the polyhedral product \cite[Lemma 2.2.1]{7} and an application of (6.5) for each $i = 1, 2, \ldots, m$ completes the proof. □

Remark. In the case that $j = 4$ and $K$ is dual to the boundary of the square, the inclusion (6.4) is

$$Z(K; (D^2, S^1)) = S^3 \times S^3 \rightarrow Z(\partial\Delta^3; (D^2, S^1)) \cong S^7 \quad (\cong S^3 \ast S^3)$$

and the corresponding homotopy equivalence of pairs is

$$((C(S^3 \times S^3), S^3 \times S^3) \rightarrow ((D^2)^4, S^3 \times S^3)),$$ eviariant with respect to the action of $T^4$.

6.2. The case of moment-angle complexes. As before, let $P^n$ be a simple polytope having $m$ facets, equipped with a characteristic function

$$\lambda: \mathcal{F} \rightarrow \mathbb{Z}^n$$

satisfying the regularity condition following (2.1). Regarding

$$\ker \lambda \hookrightarrow T^m \hookrightarrow T^{d(J)}$$

as in (5.3) and the case of odd spheres, there is a natural free action of $\ker \lambda$ on both sides of (6.5) yielding a homotopy equivalence of orbit spaces

$$Z(K_P; (\overline{CZ(K_i)}, Z(K_i))) / \ker \lambda \cong Z(K_P; ((D^2)^{2i}, Z(K_i))) / \ker \lambda.$$ Combining Theorems 4.5, 6.6 and (6.6) gives now the main observation of this section.
Theorem 6.4. For a simple polytope $P^n$, characteristic function $\lambda$ and $X_i = Z(K_i)$, Construction 3.2 corresponds, up to homotopy, to a quotient of a moment-angle complex by a free action of $\ker \lambda$ as follows:

$$\left( X_1 \times X_2 \times \cdots \times X_m / \ker \lambda \right) \times P^n / \sim_2 \cong Z(K_P; (CZ(K_1), Z(K_2))) / \ker \lambda$$

$$\cong Z(K_P; ((D^2)^{j_i}, Z(K_i))) / \ker \lambda$$

$$\cong Z(K_P(K_1, K_2, \ldots, K_m); (D^2, S^1)) / \ker \lambda.$$

It should be noted that in general, the space $Z(K_P(K_1, K_2, \ldots, K_m); (D^2, S^1)) / \ker \lambda$ might not be a manifold because $K_P(K_1, K_2, \ldots, K_m)$ is dual to the boundary of a simple polytope only in the case that all the $K_i$ are boundaries of simplices.

7. Further generalizations

Away from the diagonal circle action, the situation becomes a little more complicated. Ayzenberg’s construction \cite{1}, can be done in the realm of polytopes. In particular, given a simple polytope $P^n$ having $m$ facets and a sequence of simple polytopes \{ $P_1, P_2, \ldots, P_m$ \} where $P_i = P_i^{n_i}$ is a simple polytope of dimension $n_i$ having $j_i$ facets, the construction yields a new polytope

$$P^n \rightarrow P^n(P_1, P_2, \ldots, P_m).$$

Though the new polytope is not simple when the $P_i$ differ from simplices, it does retain some nice properties. A simplicial complex $K_P(P_1, P_2, \ldots, P_m)$ is associated to it as the nerve complex. On the level of simplicial complexes, the construction is written

$$K_P \rightarrow K_P(K_P_1, K_P_2, \ldots, K_P_m)$$

as in (6.2). As expected, it is shown in \cite{1} that

$$K_P(P_1, P_2, \ldots, P_m) = K_P(K_P_1, K_P_2, \ldots, K_P_m).$$

Under this operation, the numbers $m$ and $n$ transform by the analogue of (5.2):

$$\begin{bmatrix}
  m \\
  n \\
  m - n
\end{bmatrix}
\sim
\begin{bmatrix}
  d(J) = j_1 + j_2 + \cdots + j_m \\
  n + N \\
  d(J) - n - N
\end{bmatrix},$$

where $N = n_1 + n_2 + \cdots + n_m$. Notice that (7.1) reduces to (5.2) for the case $P_i = \Delta^{j_i - 1}$. 
7.1. **A full torus action.** For each \( i = 1,2,\ldots,m, \) let

\[
\lambda_i: \mathbb{Z}^{j_i} \longrightarrow \mathbb{Z}^{n_i}
\]

be a characteristic function on \( P_i \). It has ker \( \lambda_i \cong T^{j_i-n_i} \) which acts freely on \( Z(K_{P_i}; (D^2, S^1)) \).

The next step is to mimic the construction of \( \lambda(J) \) in section [5.1](#). To this end, denote by \( \tilde{\lambda}_i \) the first \( j_i - 1 \) columns of the \((n_i \times j_i)\)-matrix \((\lambda^j_{lk}) \) corresponding to \( \lambda_i \). The last column of \((\lambda^j_{lk}) \) is

\[
\begin{bmatrix}
\lambda^1_{i j_i} \\
\lambda^2_{i j_i} \\
\vdots \\
\lambda^n_{i j_i}
\end{bmatrix}
\]

(7.2)

Entirely by analogy with the case of odd spheres in section [5.1](#) in particular Remark [5.2](#), the matrices \( \lambda, \lambda_1, \lambda_2, \ldots, \lambda_m \) are used to construct an \(((n + N) \times d(J))\)-matrix \( \lambda(J, N) \), shown in Figure 3, which defines a map

\[
\lambda(J, N): \mathbb{Z}^{d(J)} \longrightarrow \mathbb{Z}^{n+N}.
\]

The characteristic matrix corresponding to the diagonal \( S^1 \) action on the odd sphere \( S^{2j_i-1} \) which appears in [5.6](#), is replaced by \( \lambda_i \) which has ker \( \lambda_i \cong T^{j_i-n_i} \) acting freely on \( Z(K_{P_i}; (D^2, S^1)) \). The blocks \( I_{j_i-1} \) in Figure 1 are replaced by \( \tilde{\lambda}_i \) and the last columns of “−1”, by (7.2).
\begin{figure}
\begin{center}
\begin{tabular}{cccccccccc}
\hline
$\lambda_1$ & 0 & $\cdots$ & 0 & $\lambda^1_{1j_1}$ & $\lambda^1_{2j_1}$ & $\ddots$ & $\lambda^1_{n_1j_1}$ \\
0 & $\lambda_2$ & 0 & 0 & 0 & $\lambda^2_{1j_2}$ & $\ddots$ & $\lambda^2_{n_2j_2}$ \\
$\vdots$ & $\vdots$ & $\ddots$ & $\vdots$ & $\ddots$ & $\vdots$ & $\ddots$ \\\n0 & $\vdots$ & $\ddots$ & $\lambda_m$ & $\cdots$ & $\lambda^m_{1j_m}$ & $\lambda^m_{2j_m}$ & $\ddots$ \\
0 & 0 & 0 & 0 & 0 & $\lambda^m_{n_mj_m}$ & $\ddots$ & $\lambda$ \\
\hline
\end{tabular}
\end{center}
\caption{The matrix $\lambda(N, J)$}
\end{figure}
7.2. **The rank of the matrix** $\lambda(J, N)$. The matrix corresponding to $\lambda$ is a characteristic matrix and so can be written in *refined* block form as:

$$\lambda = I_n | S$$

where $I_n$ is the $n \times n$-identity matrix and $S$ is of size $n \times (m - n)$. Similarly, the matrix corresponding to $\lambda_i$ can be written in the block form as

$$I_{n_i} | S_i$$

where $I_{n_i}$ is the $n_i \times n_i$-identity matrix and $S_i$ is of size $n_i \times (j_i - 1 - n_i)$. This observation allows the conclusion that the row rank of $\lambda(J, N)$ is $N + n$ and the next proposition follows.

**Proposition 7.1.** The row rank of the matrix $\lambda(J, N)$ is $N + n$ and so

$$\ker \lambda(J, N) \cong T^{d(J) - N - n}.$$ 

7.3. **A new toric space construction.** The inclusion $\ker \lambda(J, N) \hookrightarrow T^{d(J)}$ gives an action of $\ker \lambda(J, N)$ on the $T^{d(J)}$-equivariantly homotopy equivalent spaces

$$Z(K_P; (CZ(K_{P_1}), \overline{Z(K_{P_1})}))) \cong Z(K_P; (\overline{(D^2)^{j_1}, \overline{Z(K_{P_1})}})).$$

In this context of simple polytopes, Theorem 6.2 gives

$$Z(K_P; (\overline{(D^2)^{j_1}, \overline{Z(K_{P_1})}})) = Z(K_{P(P_1, P_2, ..., P_m)}; (D^2, S^1)).$$

The next theorem follows from the assembly of this information.

**Theorem 7.2.** There is a homotopy equivalence of orbit spaces

$$Z(K_P; (CZ(K_{P_1}), \overline{Z(K_{P_1})}))/\ker \lambda(J, N) \cong Z(K_{P(P_1, P_2, ..., P_m)}; (D^2, S^1))/\ker \lambda(J, N)$$

where $K_{P(P_1, P_2)}$ is the nerve complex of the $d(J)$-facetted, $(n + N)$-dimensional polytope $P(P_1, P_2, ..., P_m)$ and $\ker \lambda(J, N)$ is isomorphic to a torus of dimension $d(J) - (n + N)$.

**Remark.** In a natural way, the form of the matrix $\lambda(J, N)$ indicates that

$$Q = \ker \lambda_1 \oplus \ker \lambda_2 \oplus \cdots \oplus \ker \lambda_m$$

can be considered as a $(d(J) - N)$-dimensional subspace of $Z^{d(J)}$. Work in progress by the authors has, among other things, the goal of characterizing the cases when $\ker \lambda(J, N)$ is an $(d(J) - N - n)$-dimensional subspace of $Q$. In particular, this would imply that the action in Theorem 7.2 is free. There would follow a natural generalization of Construction
in which the free $S^1$ action of $X_i$ is replaced with a free $T^{j_i-n_i}$ action on each $X_i$. In the case above, $X_i = Z(K_{P_i})$ with $\ker \lambda_i \cong T^{j_i-n_i}$ acting freely. The group $\ker \lambda(J,N)$ acts on $Z(K_{P_1}) \times Z(K_{P_2}) \times \cdots \times Z(K_{P_m})$ via the inclusion $\ker \lambda(J,N) \to T^{d(J)}$. So, with very little change in the definitions, the left hand side of the equivalence in Theorem 7.2 would be identified as

$$\left( Z(K_{P_1}) \times Z(K_{P_2}) \times \cdots \times Z(K_{P_m}) / \ker \lambda(J,N) \right) \times P^n / \sim_2.$$ 

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