The weighted difference substitutions and Nonnegativity
Decision of Forms*

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Abstract In this paper, we study the weighted difference substitutions from geometrical views. First, we
give the geometric meanings of the weighted difference substitutions, and introduce the concept of conver-
gence of the sequence of substitution sets. Then it is proven that the sequence of the successive weighted
difference substitution sets is convergent. Based on the convergence of the sequence of the successive
weighted difference sets, a new, simpler method to prove that if the form $F$ is positive definite on $\mathbb{T}_n$,
then the sequence of sets $\{\text{SDS}^m(F)\}_{m=1}^\infty$ is positively terminating is presented, which is different from
the one given in [11]. That is, we can decide the nonnegativity of a positive definite form by successively
running the weighted difference substitutions finite times. Finally, an algorithm for deciding an indefinite
form with a counter-example is obtained, and some examples are listed by using the obtained algorithm.

Key words Nonnegativity decision of forms; The weighted difference substitutions; Barycentric subdi-
vision

1 Introduction

Theories and methods of nonnegative polynomials have been widely used in robust control,
non-linear control and non-convex optimization \cite{1,2,3}, etc. Some famous research works on
nonnegativity decision of polynomials without cell-decomposition were given by Pólya’s Theorem
\cite{4,5} and papers \cite{6,7}.

Few years ago, Yang \cite{8,9,10} introduced a heuristic method for nonnegativity decision of
polynomials, which is now called Successive Difference Substitution (SDS). It has been applied
to prove a great many polynomial inequalities with more variables and higher degrees. Yang
recommended further studies on SDS and put forward some open problems.

A valuable progress on the topic was made by Yao \cite{11}. He investigated a weighted difference
substitution

\[ \begin{cases} 
  x_1 = t_1 + \frac{t_2}{2} + \cdots + \frac{t_n}{n}, \\
  x_2 = \frac{t_2}{2} + \cdots + \frac{t_n}{n}, \\
  \vdots \\
  x_n = \frac{t_n}{n},
\end{cases} \tag{1} \]

instead of the original difference substitution

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Given the form $F \in \mathbb{R}[x_1, x_2, \ldots, x_n]$, when $[a_1], [a_2], \ldots, [a_m]$ traverse all the permutations of $1, 2, \ldots, n$ respectively, we define the set

$$\text{SDS}^{(m)}(F) = \bigcup_{[a_m]} \cdots \bigcup_{[a_2]} \bigcup_{[a_1]} F(B_{[a_1]}B_{[a_2]} \cdots B_{[a_m]}X^{T_r}),$$

which is called the $m$-times successive weighted difference substitution set of the form $F$. 

and proved that, for a form (namely, a homogeneous polynomial) which is positive definite on $\mathbb{R}_+^n$, the corresponding sequence of SDS sets is positively terminating, where $\mathbb{R}_+^n = \{(x_1, x_2, \ldots, x_n)| x_i \geq 0, i = 1, 2, \ldots, n\}$. That is, we can decide the nonnegativity of a positive definite form by successively running SDS finite times.

This paper is organized as follows. Section 2 introduces some preliminary notions of the weighted difference substitutions. Section 3 provides a new perspective to study the weighted difference substitutions, gives the geometric meanings of them, and proves that the sequence of the successive weighted difference substitution sets is convergent. A new, simpler method to prove that if the form $F$ is positive definite on $\mathbb{R}_+^n$, then the sequence of sets $\{\text{SDS}^{(m)}(F)\}_{m=1}^\infty$ is positively terminating is presented in Section 4, and an algorithm for deciding an indefinite form with a counter-example in Section 5. By using the obtained algorithm, several examples are listed in Section 6.

### 2 Preliminary notions

We first introduce some notations and definitions according to [11] (with some differences). Consider $W_n \in \mathbb{R}^{n \times n}$, where

$$W_n = \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ 0 & \frac{1}{2} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{n} \end{bmatrix}.$$  

Let $[k_1 k_2 \cdots k_n]$ be a permutation of $1, 2, \cdots, n$. $P_{[k_1 k_2 \cdots k_n]} = [a(i, j)]$ is an $n \times n$ matrix for which $a(1, k_1) = 1, a(2, k_2) = 1, \ldots, a(n, k_n) = 1$, and $0$ in all other positions (Permutation matrix).

**Definition 2.1.** $n \times n$ square matrix $B_{[k_1 k_2 \cdots k_n]}$ is defined as follows:

$$B_{[k_1 k_2 \cdots k_n]} = P_{[k_1 k_2 \cdots k_n]}W_n.$$ 

And the set that consists of all $B_{[k_1 k_2 \cdots k_n]}$ is denoted by $PW_n$ (there are $n!$ elements in $PW_n$) and called the weighted difference substitution matrix set. Accordingly, the set of linear transformations

$$\{X^{T_r} = B_{[\alpha]}T_{[\alpha]}^{T_r}|B_{[\alpha]} \in PW_n\},$$

is called the weighted difference substitution set, which consists of $n!$ substitutions, where $X, T \in \mathbb{R}_+^n$, and $X^{T_r}, T^{T_r}$ are respectively the transposes of $X, T$.

**Definition 2.2.** The set of linear transformations

$$\{X^{T_r} = B_{[\alpha_1]}B_{[\alpha_2]} \cdots B_{[\alpha_m]}|T^{T_r}|B_{[\alpha_m]} \in PW_n\},$$

is called the $m$-times successive weighted difference substitution set, which consists of $(n!)^m$ substitutions.

**Definition 2.3.** Given the form $F \in \mathbb{R}[x_1, x_2, \ldots, x_n]$, when $[a_1], [a_2], \ldots, [a_m]$ traverse all the permutations of $1, 2, \ldots, n$ respectively, we define the set

$$\text{SDS}^{(m)}(F) = \bigcup_{[a_m]} \cdots \bigcup_{[a_2]} \bigcup_{[a_1]} F(B_{[a_1]}B_{[a_2]} \cdots B_{[a_m]}X^{T_r}),$$

which is called the $m$-times successive weighted difference substitution set of the form $F$. 

\[\begin{aligned} 
    x_1 &= t_1 + t_2 + \cdots + t_n, \\
    x_2 &= t_2 + \cdots + t_n, \\
    \vdots \\
    x_n &= t_n,
\end{aligned}\]  

(2)
Definition 2.4. We define the sequence of sets \( \{\text{SDS}^{(m)}(F)\}_{m=1}^{\infty} \) as follows

\[
\{\text{SDS}(F)^{(m)}\}_{m=1}^{\infty} = \text{SDS}(F), \text{SDS}^{(2)}(F), \cdots.
\]

It’s time to define the termination of \( \{\text{SDS}^{(m)}(F)\}_{m=1}^{\infty} \), which is directly related to the positive semi-definite property of the form \( F \).

Let \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}^n \), and let \( |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \). Then we write a form \( F \) with degree \( d \) as

\[
F = \sum_{|\alpha|=d} c_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.
\]

Definition 2.5. The form \( F \) is called trivially positive if the coefficients \( c_\alpha \) of every term \( x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \) in \( F \) are nonnegative. If \( F(1, 1, \cdots, 1) < 0 \), then \( F \) is called trivially negative.

Definition 2.6. If the form \( F(X) \geq 0 \) for all \( X \in \mathbb{R}_+^n \), then \( F \) is called positive semi-definite; If \( F(X) > 0 \) for all \( X \neq 0 \in \mathbb{R}_+^n \), \( F \) is called positive definite on \( \mathbb{R}_+^n \); If there are \( X \) and \( Y \in \mathbb{R}_+^n \) such that \( F(X) > 0 \) and \( F(Y) < 0 \), then \( F \) is called indefinite on \( \mathbb{R}_+^n \).

Lemma 2.1. Given the form \( F \) on \( \mathbb{R}_+^n \), if the form \( F \) is trivially positive, then \( F \) is positive semi-definite; If the form \( F \) is trivially negative, then \( F \) isn’t positive semi-definite on \( \mathbb{R}_+^n \).

Definition 2.7. Given a form \( F \) on \( \mathbb{R}_+^n \), if there is a positive integer \( k \) such that every element of the set \( \text{SDS}^{(k)}(F) \) is trivially positive, the sequence of sets \( \{\text{SDS}^{(m)}(F)\}^\infty_{m=1} \) is called positively terminating; If there is a positive integer \( k \) and a form \( G \) such that \( G \in \text{SDS}^{(k)}(F) \) and \( G \) is trivially negative, the sequence of sets \( \{\text{SDS}^{(m)}(F)\}^\infty_{m=1} \) is called negatively terminating; The sequence of sets \( \{\text{SDS}^{(m)}(F)\}^\infty_{m=1} \) is neither positively terminating nor negatively terminating, then it is called not terminating.

By Definition 2.7, it’s easy to get the following lemma.

Lemma 2.2. Given the form \( F \) on \( \mathbb{R}_+^n \), if the sequence of sets \( \{\text{SDS}^{(m)}(F)\}^\infty_{m=1} \) is positively terminating, then \( F \) is positive semi-definite on \( \mathbb{R}_+^n \); If the sequence of sets \( \{\text{SDS}^{(m)}(F)\}^\infty_{m=1} \) is negatively terminating, then \( F \) isn’t positive semi-definite on \( \mathbb{R}_+^n \).

Next, we will give the definition of the normalized substitution.

Definition 2.8. Let \( V = [v_{ij}] \) be an \( n \times n \) matrix. If \( \sum_{i=1}^{n} v_{ij} = 1, j = 1, 2, \cdots, n \), \( V \) is called a normalized matrix. And the corresponding substitution

\[
X^{\text{Tr}} = VT^{\text{Tr}},
\]

is called a normalized substitution.

Lemma 2.3. Let \( U = [u_{ij}] = V_1 V_2 \cdots V_k \). If \( V_1, V_2, \cdots, V_k \) are normalized matrices, then \( U \) is a normalized matrix, that is,

\[
\sum_{i=1}^{n} u_{ij} = 1 (j = 1, 2, \cdots, n).
\]

And let \( (x_1, x_2, \cdots, x_n)^{\text{Tr}} = U(t_1, t_2, \cdots, t_n)^{\text{Tr}} \), then \( \sum_{i=1}^{n} x_i = 1 \) iff \( \sum_{i=1}^{n} t_i = 1 \).

The proof of Lemma 2.3 is very straightforward and is omitted. The \((n - 1)\)-dimensional simplex is defined as follows

\[
\Delta_n = \left\{ (x_1, x_2, \cdots, x_n) \mid \sum_{i=1}^{n} x_i = 1, (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n \right\}.
\]
Definition 2.9. If the form $F(X) \geq 0$ for all $X \in \mathbb{T}_n$, then $F$ is called positive semi-definite on $\mathbb{T}_n$; If $F(X) > 0$ for all $X \in \mathbb{T}_n$, then $F$ is called positive definite on $\mathbb{T}_n$; If there are $X$ and $Y \in \mathbb{T}_n$ such that $F(X) > 0$ and $F(Y) < 0$, then $F$ is called indefinite on $\mathbb{T}_n$.

Obviously, we have the following conclusion.

Lemma 2.4. The form $F$ is positive semi-definite (positive definite, indefinite) on $\mathbb{T}_n$ iff $F$ is positive semi-definite (positive definite, indefinite) on $\mathbb{R}^n$.

According to Lemma 2.4, for brevity, we suppose that the form $F$ is defined on $\mathbb{T}_n$ in the remainder of this paper.

3 Convergence of the sequence of successive weighted difference substitution sets.

In this section, we’ll consider the weighted difference substitutions from geometrical views.

Let $X = (x_1, x_2, \cdots, x_n) \in \mathbb{T}_n$. Consider the weighted difference substitution

$$X^T = W_n T^T,$$

where $W_n$ is denoted by $(\mathbf{3})$.

By $(\mathbf{4})$, if $X = A_1 = (1, 0, \ldots, 0)$, then $T = (1, 0, \ldots, 0)$; If $X = A_2 = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, then $T = (0, 1, \ldots, 0) \cdots$. If $X = A_n = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, then $T = (0, 0, \ldots, 1)$. Moreover, $A_k$ is the barycenter of the (k-1)-dimensional proper face of $\mathbb{T}_n$ which contains the points $A_1, A_2, \cdots, A_k$ for $k = 1, 2, \cdots, n$. According to Lemma $(\mathbf{2.3})$ for all $(x_1, x_2, \cdots, x_n) \in \mathbb{T}_n$, we have $t_1 + t_2 + \cdots + t_n = 1$. Therefore, $A_1 A_2 \cdots A_n$ is a subsimplex of the first barycentric subdivision of $\mathbb{T}_n$, satisfying $W_n = [A_1^T, A_2^T, \cdots, A_n^T]$. And it is indicated that the weighted difference substitution $(\mathbf{4})$ and the subsimplex $A_1 A_2 \cdots A_n$ correspond to each other.

Analogously, the other $n! - 1$ weighted difference substitutions correspond to the other $n! - 1$ sub simplices of the first barycentric subdivision of $\mathbb{T}_n$.

Hence, from geometrical views, the weighted difference substitution set corresponds to the first barycentric subdivision of $\mathbb{T}_n$, that is, there is a one-to-one correspondence between the weighted difference substitutions and the sub simplices of the first barycentric subdivision of $\mathbb{T}_n$.

![Figure 1: Barycentric subdivision](image)

For instance, when $n = 3$, the following six weighted difference substitution matrices correspond to sub simplices labeled as 1-6 in Fig. 1, respectively,

$$
\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, 
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, 
\begin{bmatrix}
0 & 0 & 1 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}, 
\begin{bmatrix}
0 & 0 & 1 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}, 
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}, 
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}.
$$

By Lemma $(\mathbf{2.3})$ and Definition $(\mathbf{2.2})$, we have that the $m$-times successive weighted difference substitution set corresponds to the $m$-th barycentric subdivision of $\mathbb{T}_n$.

Next, we’ll introduce the concept of convergence of the sequence of substitution sets.
Definition 3.1. Let $\sigma$ be a subsimplex of $T_n$, the maximum distance between vertices of $\sigma$ is called the diameter of $\sigma$.

Definition 3.2. Let $K_0 = T_n$, and $K_{i+1}$ be the subdivision of $K_i$ for $i = 1, 2, \ldots$. If for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that all the diameters of the subsimplexes of $K_N$ are less than $\varepsilon$, the subdivision sequence $\{K_i\}_{i=1}^{\infty}$ is called convergent.

Definition 3.3. Suppose that the subdivision scheme through which $K_{i-1}$ is subdivided into $K_i$ corresponds to the substitution set $L_i$ for $i = 1, 2, \ldots$. If the sequence $\{L_i\}_{i=1}^{\infty}$ is convergent, the sequence of substitution sets $\{L_i\}_{i=1}^{\infty}$ is called convergent. If $L_i = L$, $i = 1, 2, \ldots$, briefly, we say the sequence of the successive $L$-substitution sets is convergent.

It's time to consider the convergence of the sequence of the successive weighted difference substitution sets.

Lemma 3.1. \[12, 13\] Let $K$ be a complex. If $K_N$ is the $k$-th barycentric subdivision of $K$, then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that all the diameters of the subsimplexes of $K_N$ are less than $\varepsilon$.

By Lemma 3.1, we have the following theorem, which plays important roles in the proofs of Theorem 3.1 and Theorem 5.3 in the next sections.

Theorem 3.1. The barycentric subdivision sequence $\{K_m\}_{m=1}^{\infty}$ of $T_n$ and the corresponding sequence of the successive weighted difference substitution sets $\{\text{SDS}^{(m)}(F)\}_{m=1}^{\infty}$ are convergent.

4 Nonnegativity decision of forms

Given a form $F$ on $T_n$. We know that if the sequence of sets $\{\text{SDS}^{(m)}(F)\}_{m=1}^{\infty}$ is negatively terminating, then we can conclude that $F$ is positive semi-definite on $T_n$. Thus there is a natural question, that is, which kind of forms can be solved by the method? Yao [11] proved that, for a positive definite form $F$ on $T_n$, the sequence of sets $\{\text{SDS}^{(m)}(F)\}_{m=1}^{\infty}$ is positively terminating.

In this section, we present a new, simpler method to prove the conclusion, which is based on the convergence of the sequence of the successive weighted difference substitution sets.

Theorem 4.1. Let the form $F$ be positive definite on $T_n$, then the sequence of sets $\{\text{SDS}^{(m)}(F)\}_{m=1}^{\infty}$ is positively terminating.

Proof. We only give the proof for the ternary form with degree $d$, and the multivariate form can be gotten by induction.

Suppose that $F(x_1, x_2, x_3) = \sum_{i+j+k=d} a_{ijk} x_1^i x_2^j x_3^k$. An arbitrary $m$-times successive weighted difference substitution can be written as

\[
\begin{align*}
&x_1 = k_1 t_1 + (k_1 + \alpha_1) t_2 + (k_1 + \beta_1) t_3, \\
x_2 = k_2 t_1 + (k_2 + \alpha_2) t_2 + (k_2 + \beta_2) t_3, \\
x_3 = k_3 t_1 + (k_3 + \alpha_3) t_2 + (k_3 + \beta_3) t_3,
\end{align*}
\]  

(5)

where $\sum_{i=1}^{3} k_i = 1$, $\sum_{i=1}^{3} \alpha_i = 0$ and $\sum_{i=1}^{3} \beta_i = 0$.

Let $t = t_1 + t_2 + t_3$, then (5) becomes

\[
\begin{align*}
x_1 = k_1 t + \alpha_1 t_2 + \beta_1 t_3, \\
x_2 = k_2 t + \alpha_2 t_2 + \beta_2 t_3, \\
x_3 = k_3 t + \alpha_3 t_2 + \beta_3 t_3.
\end{align*}
\]  

(6)
Thus

\[ \Phi(t_1, t_2, t_3) = F(k_1 t + \alpha_1 t_2 + \beta_1 t_3, k_2 t + \alpha_2 t_2 + \beta_2 t_3, k_3 t + \alpha_3 t_2 + \beta_3 t_3) \]

\[ = \sum_{i+j+k=d} a_{ijk} (k_1 t + \alpha_1 t_2 + \beta_1 t_3)^i (k_2 t + \alpha_2 t_2 + \beta_2 t_3)^j (k_3 t + \alpha_3 t_2 + \beta_3 t_3)^k \]

\[ = \sum_{i+j+k=d} a_{ijk} (k_1 t)^i + \sum_{p+q+r=i, p \neq i} \frac{i!}{p!q!r!} k_1^p \alpha_1^p \beta_1^r t^p t_2^q t_3^r. \]

\[ (k_1^2 t^2 + \sum_{p+q+r=i, p \neq i} \frac{j!}{p!q!r!} k_1^p \alpha_1^p \beta_1^r t^p t_2^q t_3^r) (k_1 t)^i + \sum_{p+q+r=k, p \neq k} \frac{k!}{p!q!r!} k_1^p \alpha_1^p \beta_1^r t^p t_2^q t_3^r) \]

\[ = \sum_{i+j+k=d} a_{ijk} k_1^i k_2^j k_3^k d + \sum_{i+j+k=d} \phi_{ijk}(k_1, k_2, k_3, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) t_1^i t_2^j t_3^k \]

\[ = F(k_1, k_2, k_3) t^d + \sum_{i+j+k=d} \phi_{ijk}(k_1, k_2, k_3, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) t_1^i t_2^j t_3^k \]

\[ = \sum_{i+j+k=d} \left( \frac{d!}{i!j!k!} F(k_1, k_2, k_3) + \phi_{ijk}(k_1, k_2, k_3, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \right) t_1^i t_2^j t_3^k \]

where

\[ A_{ijk} = \frac{d!}{i!j!k!} F(k_1, k_2, k_3) + \phi_{ijk}(k_1, k_2, k_3, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3). \]

(8)

Obviously,

\[ \lim_{(k_1, k_2, k_3) \to (0,0,0,0,0,0)} \phi_{ijk}(k_1, k_2, k_3, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = 0. \]

(9)

And since \( F(x_1, x_2, x_3) \) is positive definite on \( T_n \), there exists \( \varepsilon > 0 \) such that

\[ F(k_1, k_2, k_3) \geq \varepsilon > 0. \]

(10)

On the one hand, the vertex of the subsimplex which corresponds to the successive weighted difference substitution \([5]\) are respectively

\[ (k_1, k_2, k_3), (k_1 + \alpha_1, k_2 + \alpha_2, k_3 + \alpha_3), (k_1 + \beta_1, k_2 + \beta_2, k_3 + \beta_3). \]

By Theorem \([4.1]\), \( \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \) can be sufficiently small when \( m \) is sufficiently large.

On the other hand, for \( F \) is continuous on \( T_n \) and by \([5]-[10]\), we have \( A_{ijk} > 0 \) when \( \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \) are sufficiently small.

Putting together the above two aspects, we have that there exists a sufficiently large integer \( m \) such that \( F \) becomes trivially positive by \([5]\). For the successive weighted difference substitution \([5]\) is arbitrary, the sequence of sets \( \{SDS^m(F)\}_{m=1}^\infty \) is positively terminating.

According to the proof of Theorem \([4.1]\) we obtain the following conclusion.

**Corollary 4.1.** Let the form \( F \) be positive definite on \( T_n \), then by an arbitrary \( m \)-times successive weighted difference substitution, when \( m \) is sufficiently larger, \( F \) can become a nonlacunary trivially positive form.

Theorem \([4.1]\) is somewhat analogous to Pólya’s Theorem. However, many examples show that Pólya’s Theorem seems almost useless to positive semi-definite forms except for few cases, while SDS is demonstrated very helpful to positive semi-definite ones as well.
Decision of indefinite forms

Many problems, such as the inequality disproving, are always transformed into decision of indefinite forms.

Given a form $F$ on $T_n$. Suppose that there exists $X_0 \in T_n$, such that $F(X_0) > 0$. It is well-known to us that if the sequence of sets $\{SDS^{(m)}(F)\}_{m=1}^{\infty}$ is negatively terminating, then $F$ is indefinite on $T_n$. Then it follows a question naturally: for an indefinite form $F$ on $T_n$, is the $\{SDS^{(m)}(F)\}_{m=1}^{\infty}$ negatively terminating? The following theorem answers the question.

**Theorem 5.1.** Let the form $F$ be indefinite on $T_n$, then the sequence of sets $\{SDS^{(m)}(F)\}_{m=1}^{\infty}$ is negatively terminating.

**Proof.** Since the form $F$ is indefinite on $T_n$, there exists $X_0 \in T_n$ such that $F(X_0) < 0$. And $F$ is continuous on $T_n$, so there exists a neighborhood $U(X_0) \subset T_n$ of $X_0$ (If $X_0$ is on the boundary of $T_n$, then we take $U(X_0) \cap T_n$) such that $F(X) < 0$ for all $X \in U(X_0)$. For the barycentric subdivision sequence of $T_n$ is convergent, then there exists a subcomplex $\sigma$ of the $k$-th barycentric subdivision of $T_n$, which corresponds to the $k$-times successive weighted difference substitution

$$X^{Tr} = B_{[1]}B_{[2]}\cdots B_{[k]}T^{Tr}, \ B_{[1]}, B_{[2]}, \cdots, B_{[k]} \in PW_n,$$

satisfying $\sigma \subset U(X_0)$, where $k$ is a sufficiently larger integer, and $PW_n$ is the weighted difference substitution matrix set. Thus, $-F(X)$ is positive definite on $\sigma$. By Theorem 4.1, the sequence of sets $\{SDS^{(m)}(-F)\}_{m=1}^{\infty}$ is positively terminating, so there exists an $l$-times successive weighted difference substitution

$$X^{Tr} = B_{[1]}B_{[2]}\cdots B_{[j]}T^{Tr}, \ B_{[1]}, B_{[2]}, \cdots, B_{[j]} \in PW_n,$$

satisfying that $F(B_{[1]}B_{[2]}\cdots B_{[i]}B_{[j]}B_{[j]}B_{[j]}T^{Tr})$ is trivially negative. Therefore, the sequence of sets $\{SDS^{(m)}(F)\}_{m=1}^{\infty}$ is negatively terminating.

By the proving process of Theorem 5.1, we obtain the following algorithm, which is used to decide the nonnegativity of a form or to decide an indefinite form with a counter-example.

**Algorithm (YYS)**

Input: the form $F \in \mathbb{Q}[x_1, x_2, \cdots, x_n]$, where $F$ is positive definite or indefinite on $\mathbb{R}^n_+$.

Output: “The form $F$ is positive semi-definite”, or “$\tilde{X}_0, F(\tilde{X}_0) < 0$”.

step1: Let $F = \{F\}$.

step2: Compute $\bigcup_{F \in F} SDS(F)$,

Let

$$\mathcal{F} = \bigcup_{F \in \mathcal{F}} SDS(F) - \{\text{trivially positive forms in } \bigcup_{F \in \mathcal{F}} SDS(F)\} \triangleq \{F_{[1]}, F_{[2]}, \cdots, F_{[k]}\},$$

where

$$F_{[i]} = F(B_{[i]}X^{Tr}), \ B_{[i]} \in PW_n.$$  

step3: Let $L = [1, 2, \cdots, [k]]$.

step31: If $\mathcal{F}$ is null, then output “the form $F$ is positive semi-definite”, and terminate.

step32: If there is a trivially negative form $F_{[i]} \in \mathcal{F}$, then output

“$\tilde{X}_0 = B_{[L[i]0]}\left[\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right]^{Tr}, \ F(\tilde{X}_0) < 0$”,

and terminate, where

$$B_{[L[i]0]} = B_{[L[i1]1]}B_{[L[i2]2]}B_{[L[i3]3]} \cdots B_{[L[ik]k]}.$$
$L[i]$ is the $i$-th component of $L$, $L[i][j]$ is the $j$-th component of $L[i]$, and $m$ is the total number of the components of $L[i]$.

step33: Else, Compute $\bigcup_{F \in \mathcal{F}} SDS(F)$. Let

$$
\mathcal{F} = \bigcup_{F \in \mathcal{F}} SDS(F) - \{\text{trivially positive forms in } \bigcup_{F \in \mathcal{F}} SDS(F)\}
$$

$$
\Delta = \{F_{[\text{op}(L[1]), 1]}, \cdots, F_{[\text{op}(L[1]), l_1]}, F_{[\text{op}(L[2]), 1]}, \cdots, F_{[\text{op}(L[2]), l_2]},
\cdots, F_{[\text{op}(L[k]), 1]}, \cdots, F_{[\text{op}(L[k]), l_k]}\},
$$

where

$$
F_{[\text{op}(L[i]), j]} = F(B_{[\text{op}(L[i]), j]}X^T),
$$

and $\text{op}(L[i])$ extracts operands from $\text{op}(L[i])$. And let

$$
L = [\text{op}(L[1]), 1], \cdots, [\text{op}(L[1]), l_1], [\text{op}(L[2]), 1], \cdots, [\text{op}(L[2]), l_2],
\cdots, [\text{op}(L[k]), 1], \cdots, [\text{op}(L[k]), l_k],
$$

then go to step3.

By Algorithm YYS, we design a Maple program called YYS, see Appendix. To the program YYS, there are some positive semi-definite forms making the program do not terminate, that is, we can’t decide these positive semi-definite forms by the method.

6 Examples

In this section, we demonstrate the program YYS with some examples.

**Example 1.** Show that the following form is positive semi-definite on $\mathbb{R}^4$,

$$
F(x, y, z) = x(x-y)^5 - y(-z-y)^5 - z(x-z)^5.
$$

Utilize the program YYS and execute order YYS($F, [x,y,z]$). The procedure need successively run the weighted difference substitutions 3 times, then outputs: “The form $F$ is positive semi-definite.”

**Example 2.** Show that the following form is indefinite on $\mathbb{R}^4$.

$$
F(x, y, z) = 7x^3 - 12x^2y - 12x^2z + 6xy^2 + 12xyz + 6xz^2 - \frac{9}{10}y^3 - 3y^2z - 3yz^2 - \frac{4}{5}z^3.
$$

Executing order YYS($F, [x,y,z]$), we have a counter-example by successively running the weighted difference substitutions 2 times:

$$
\bar{X}_0 = \left(\frac{37}{108}, \frac{49}{108}, \frac{11}{54}\right)^T, \quad F(\bar{X}_0) < 0.
$$

Obviously, $F(1,0,0) = 7 > 0$, so the form $F$ is indefinite on $\mathbb{R}^4$.

**Example 3.** Let $x \geq 0, y \geq 0, z \geq 0$, and $x + y + z \neq 0$. Try to decide whether the following inequality holds for all $p \in \mathbb{N}$.

$$
\frac{2}{3} \left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right) - \left(\frac{x^p + y^p + z^p}{3}\right)^\frac{1}{p} \geq 0.
$$

Take off denominators of the left polynomial, and denote the new polynomials by $F_1, F_2, \cdots, F_6$ for $p = 1, 2, \cdots, 6$, respectively. Execute order YYS($F_p, [x,y,z]$), $p = 1, 2, \cdots, 6$. For $p = 1, 2, \cdots, 5$, the procedure only need run the weighted difference substitutions 1 time, then outputs: “The form $F$ is positive semi-definite”. For $p = 6$, we have a counter-example by successively running the weighted difference substitutions 5 times:

$$
\bar{X}_0 = \left(\frac{2159}{5832}, \frac{3685}{11664}, \frac{3661}{11664}\right)^T, \quad F_6(\bar{X}_0) < 0.
$$

So the inequality can’t hold for $p = 6$. 


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Appendix. Maple Program YYS

YYS:=proc(poly,var)
local a,b,A,f,i,j,k,m,n,r,s,t,F,G,H,M,W,newvar,st,Var:
  uses combinat, LinearAlgebra:
  F:=[poly,[]]:
  n:=nops(var):
  Var:=convert(var,Vector):
  W:=(n)->Matrix(n,n,(i,j)->'if'(i<=j,1/j,0)):
  b:=permute(n): a:=W(n):
  A:=seq(seq(a[b[i][j]],j=1..n),i=1..n!):
  for i to nops(A) do
    for j to n do
      newvar[i,j]:=op(j,convert(A[i].Var,list)):
    od:
  od:
  r:=100:
  for s to r do
    m:=nops(F):
    f:=[]:
    for k to m do
      G[k]:=[]:
      for i from 1 to nops(A) do
        od:
      od:
    od:
  od:
end:
st := {seq(Var[j] = newvar[i, j], j = 1..n)};
G[k] := [op(G[k]), [expand(subs(st, F[k][1])), [op(F[k][2]), i]]]:
od:
F := [seq(op(G[u]), u = 1..nops(F))]:
for i to nops(F) do
  if max([coeffs(F[i][1])]) < 0 then
    M := IdentityMatrix(n):
    for j from 2 to nops(F[i][2]) do
      M := M . A[F[i][2][j]]:
    od:
    print(convert(M . Vector[column](n, 1/n), list)):
    return("The form is indefinite");
  elif min([coeffs(F[i][1])]) > 0 then
    f := [op(f), i]:
  fi:
  od:
if nops(f) > 0 then
  F := subs({seq(F[f[t]] = NULL, t = 1..nops(f))}, F):
  fi:
if nops(F) = 0 then
  print(s);
  return("The form is positive semi-definite");
  fi:
od:
end proc: