THE LOG PRODUCT FORMULA IN QUANTUM

K-THEORY

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Abstract. We prove a formula expressing the K-theoretic log Gromov-Witten invariants of a product of log smooth varieties $V \times W$ in terms of the invariants of $V$ and $W$. The proof requires introducing log virtual fundamental classes in K-theory and verifying their various functorial properties. We introduce a log version of K-theory and prove the formula there as well.

0. Introduction

The present paper proves a log product formula for quantum K-theory, a K-theoretic version of Gromov-Witten theory. We refer to the book [Ogu18] for background on log geometry, [Her19] for the basics of log normal cones and the log product formula, and [Lee04] for quantum K-theory and K-theoretic virtual classes without log structure.

Let $V, W$ be log smooth quasiprojective log schemes. Write

$$ Q := \overline{M}_{g,n}^! (V) \times_{\overline{M}_{g,n}} \overline{M}_{g,n}^! (W), $$

where $\overline{M}_{g,n}^!(X)$ is the stack of log stable maps to $X$, and $\times^f$ the fs fiber product, or fiber product in the category of fs log schemes [Ogu18, Corollary III.2.1.6]. We have maps

$$ \overline{M}_{g,n}^!(V \times W) \xrightarrow{h} Q \xrightarrow{\Delta^!} \overline{M}_{g,n}^!(V) \times \overline{M}_{g,n}^!(W). $$

The stack $Q$ can acquire a K-theoretic log virtual fundamental class $[Q]^\ell_{\text{vir}}$ – otherwise known as a log virtual structure sheaf $O_Q^{\text{vir}}$ – in two ways: by pulling back that of $\overline{M}_{g,n}^!(V) \times \overline{M}_{g,n}^!(W)$ or pushing forward that of $\overline{M}_{g,n}^!(V \times W)$. The log product formula asserts these are equal:

**Theorem 0.1** (=4.3). The classes

$$ h_* [\overline{M}_{g,n}^!(V \times W)/\mathcal{M}_{g,n}]^{\ell_{\text{vir}}} = \Delta^! [\overline{M}_{g,n}^!(V) \times \overline{M}_{g,n}^!(W)/\mathcal{M}_{g,n} \times \mathcal{M}_{g,n}]^{\ell_{\text{vir}}} $$

are equal in $K_0(Q)$ as well as $K_1(Q)$. Here $K_0(Q) := K(\text{Coh}(Q))$ is the Grothendieck group of coherent sheaves on the lisse-ét site; $K_1(Q)$ is an inverse limit of $K_0$-theories of log blowups of $Q$ defined in Section 3. $\Delta^!$ is the log Gysin map defined in Definition 1.2.

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Writing
\[ ev : \overline{\mathcal{M}}_{g,n}(V) \to V^n, \quad ev : \overline{\mathcal{M}}_{g,n}(W) \to W^n \]
for the evaluation maps, we get an equality between log quantum \( K \)-invariants:

**Corollary 0.2.** If \( \alpha \in K_\circ(V)^{\otimes n}, \beta \in K_\circ(W)^{\otimes n} \), the following invariants are equal
\[
\chi \left( \mathcal{O}_{\overline{\mathcal{M}}_{g,n}(V \times W)}^{\text{evir}} \otimes ev^*(\alpha) \otimes ev^*(\beta) \right) = \chi \left( \Delta^!(\mathcal{O}_{\overline{\mathcal{M}}_{g,n}(V)}^{\text{evir}} \otimes ev^* \alpha \times ev^* \beta) \right).
\]

This follows from Theorem 0.1 and the projection formula. Setting the log structures to be trivial, we obtain the ordinary (non-log) versions of the above results.

**Corollary 0.3.** The “ordinary” (non-log) versions of Theorem 0.1 and Corollary 0.2 also hold.

Without log structure, the log virtual class and Gysin map becomes the ordinary virtual class and Gysin map \( \Delta^! \) in \([Lee04]\).

M. Kontsevich and Yu. Manin formulated the product formula in Gromov–Witten theory in terms of cohomological field theories and proved it in genus zero \([KM96]\). K. Behrend \([Beh99]\) generalized it to all genera. Extensions to log Gromov–Witten theory in Chow Groups were obtained by F. Qu and Y.-P. Lee \([LQ18]\), D. Ranganathan \([Ran19]\), and L. Herr \([Her19]\). Each builds on the proof of Behrend; the present article further adapts Herr’s proof to \( K_\circ \)-theory.

Log \( K_1 \)-theory \( K_1(Q) \) introduced in Section 3 can be traced back to \([IKNU20]\) and \([Bar18]\). Operations in \( K_1 \)-theory require comparing the inverse systems of log blowups of the source and target. Section 3 undertakes this comparison and will be of use in the forthcoming \([HMPW21]\) and an ongoing project of the current authors.

One key ingredient in the formulation of the log product formula is the log Gysin map \( \Delta^! \) in \( K_\circ \)-theory in Section 1. This may be refined to \( K_1 \)-theory in certain situations. The log normal cone and log virtual fundamental class are defined in \( K \)-theory, parallel to the constructions denoted simply \( \Delta^! \) etc. in \([Her19]\) for Chow groups. We stress that the product formula would not be true with the ordinary Gysin map \( \Delta^! \).

The main new technical complication absent for Chow groups comes from the pushforward operation in \( K_\circ \)-theory (Section 2). For example, the naïve translation of Costello’s formula in Chow \([Cos06]\) Theorem 5.0.1] to \( K \)-theory is false for pure degrees \( d > 1 \). Nevertheless, we prove a log version of Costello’s formula in \( K_\circ \)-theory for pure degree one (cf. \([Her19]\) Theorem 4.1, \([HW21]\)).
Theorem 0.4 (\(=2.7\)). Consider an fs pullback square
\[
\begin{array}{ccc}
X' & \xrightarrow{p} & X \\
\downarrow f' & \ell & \downarrow f \\
Y' & \xrightarrow{q} & Y
\end{array}
\]
where \(Y, Y'\) are log smooth, \(q\) proper birational and \(f\) defines a perfect obstruction theory. Endow \(f'\) with the pullback perfect obstruction theory. Under additional technical assumptions specified in Theorem 2.7, we have
\[
p_*[X'/Y']^\text{vir} = [X/Y]^\text{vir} \in K_\circ(X).
\]

The same is later shown in \(K_\dagger\)-theory. Along the way, Hironaka’s pushforward theorem [Hir64, Corollary 2 pg 153], as conjectured by A. Grothendieck, is strengthened to log smooth stacks. We are not aware of a previous proof even for ordinary stacks. Section 4 applies these tools to obtain the log product formula.

To our knowledge, the ordinary product formula in Corollary 0.2 is also new. The log version is more potent, as one can see with toric varieties. Any pair of toric varieties of the same dimension are related by a roof of log blowups. Log virtual fundamental classes are invariant under log blowups as in [AW18, Her19 Theorem 3.10] (for Chow), and Proposition 2.5 below (for \(K\)-theory). Hence all log Gromov-Witten invariants of a fixed dimension with suitable insertions are the same. To compute them, one can take \((\mathbb{P}^1)^n\). The log product formula reduces this to the case of \(\mathbb{P}^1\), whose log Gromov-Witten invariants are related to double ramification cycles. This plan of attack was shown to us independently by Jonathan Wise and Dhruv Ranganathan, and the main obstacle is the complexity of the operations \(\Delta^1\).

The product formula is one of a handful of general tools in the Gromov-Witten repertoire. Versions of the degeneration and localization formulas have been obtained in log Gromov-Witten theory in Chow groups and appear within reach for \(K\)-theory. The present article is a proof of concept, introducing the log Gysin maps and pushforward identities for this ongoing program.

Throughout, we work with fs (fine and saturated) log structures. The primary reason one needs log virtual fundamental classes instead of ordinary ones is that Diagram (1) in Section 4 is not cartesian in the category of ordinary schemes. The fs pullback often differs from the scheme-theoretic pullback, even on underlying schemes.

0.1. Conventions and Definitions. We only consider fs (fine and saturated) log structures. We use the notation \(\mathcal{L}, \mathcal{L}Y\) to refer to Olsson’s stacks \(\text{For}, \text{For} Y\) in [Ols03] with \(T\)-points:
\[
\mathcal{L}(T) := \{\text{a log structure } M_T \text{ on } T.\}
\]
\[ \mathcal{L}X(T) := \begin{cases} 
\text{a log structure } M_T \text{ on } T \text{ and} \\
\text{a map of log algebraic stacks } T \to X 
\end{cases}. \]

All stacks are assumed Artin algebraic, locally of finite type over \( \mathbb{C} \), quasiseparated in the sense of [Sta18, 04YW], and locally noetherian. Those stacks that aren’t quasicoompact, e.g., \( \mathcal{L}Y, \mathcal{M}_{g,n} \), are quickly replaced by suitable quasicompact open substacks to avoid imposing “quasicompact support” conditions on our \( K \)-theories. These are not algebraic stacks in the sense of [LMB99] because they may not adhere to their stricter notion of “quasiseparatedness.” By “log algebraic stack,” we mean a stack \( X \) as above with a map \( X \to \mathcal{L} \).

The notation \( \mathfrak{r}, \times, C^\ell \) denotes the fs pullbacks and log normal cone, instead of the ordinary scheme-theoretic pullbacks or normal cone \( \mathfrak{r}, \times, C \). We use \( \mathfrak{r}, \times, C^\star \) if it so happens that the two coincide. These distinctions are subtle but important, as the difference between pullbacks of schemes and log schemes is the reason for the log normal cone.

Let \( \sigma \) be a sharp fs monoid. Write
\[
\mathbb{A}_\sigma := \text{Spec } \mathbb{C}[\sigma],
\]
\[
\mathfrak{A}_\sigma := [\mathbb{A}_\sigma/\mathbb{A}_\sigma^{gp}],
\]
so that
\[
\text{Hom}(X, \mathfrak{A}_\sigma) = \text{Hom}(\sigma, \Gamma(M_X)).
\]

An Artin cone is an algebraic stack of the form \( \mathbb{A}_\sigma \) with its natural log structure. An Artin fan is a log algebraic stack with a representable strict étale cover by Artin cones. Note the association \( \sigma \mapsto \mathfrak{A}_\sigma \) is contravariant here, as opposed to the covariant conventions in [ACWM17], [AW18].

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1. Log $K_0$-theory

We amalgamate [Her19], [Qu18], [Man08] to develop the basic properties of log virtual fundamental classes in $K_0$-theory.

A locally noetherian algebraic stack $X$ has a lis-ét site $X_{lis-ét}$ consisting of smooth $X$-schemes $T \to X$ with $T$ affine and étale $X$-maps $T' \to T$ between them. We define the categories $QCoh(X), Coh(X)$ of quasi-coherent and coherent sheaves to be cartesian sections of the stack of sheaves of modules, with or without finite generation hypotheses [Ols07]. Define $K_0(X) := K(Coh(X))$ to be the group generated by the coherent sheaves on $X$ modulo relations for exact sequences as in [Qu18, 1.3.2], [HK19, 3A]. This group was denoted $G_{naive}^0(X)$ and shown to coincide with the Thomason-Trobaugh definition of $G$-theory under the assumption $D_{qc}(X)$ is compactly generated in [HK19, Lemma 5.5].

To use excision sequences, we need our stacks to be quasicompact and quasiseparated [Sta18].

Remark 1.1. There are two notions of quasiseparated for a stack $X$ in the literature: the Stacks Project requires the diagonal $\Delta_X$ to be quasicompact and quasiseparated, while [Ols03] and [LMB99] require that the diagonal $\Delta_X$ be quasicompact and separated. We adhere to the weaker notion [Sta18, 04YW], and [Ols03, Theorem 3.2] shows that $L S \to S$ quasiseparated in our sense despite [Ols03, Remark 3.17] pointing out that $L S \to S$ is not quasiseparated in the stronger sense.

Given a morphism $X \to \mathcal{L}Y$, we can find a quasicompact open subset $U \subseteq \mathcal{L}Y$ through which the morphism factors: $X \to U \subseteq \mathcal{L}Y$. Put it another way, any morphism $X \to Y$ with $X$ quasicompact factors as $X \to U \to Y$ with $U \to Y$ quasiseparated and log étale, $U$ quasicompact, and $X \to U$ strict.

Definition 1.2. Assume $X, Y$ are quasiseparated and $X$ is quasicompact. The log (intrinsic) normal cone of a DM type map $f : X \to Y$ is defined by

$$C_{X/Y}^\ell := C_{X/\mathcal{L}Y}.$$

A log perfect obstruction theory for $f$ is an embedding $C_{X/Y}^\ell \subseteq E$ into a vector bundle over $X$. One similarly has a log deformation to the normal cone $\tilde{M}_{X/Y}^\ell = \tilde{M}_{X/\mathcal{L}Y}$ [Qu18]. We think of each as $X$-stacks, writing $C^\ell_f |_T$ for the pullback $C^\ell_f \times_X T$ along a map $T \to X$.

Pick a factorization $X \to U \to Y$ with $X \to U$ strict, $U \to Y$ quasiseparated and log étale as in Remark 1.1. Given a log perfect obstruction theory, define the log Gysin map as

$$f^\dagger : K_0(\mathcal{L}Y) \to K_0(U) \to K_0(X),$$
the composite of restriction to $U$ and the ordinary Gysin map for $X \to U$ using the obstruction theory $C_{X/U} \simeq C_X^\ell \subseteq E$. The log virtual fundamental class $[X/Y]^{\ell \text{vir}}$ is $f^! [\mathcal{O}_{\mathcal{L}Y}]$, if $[\mathcal{O}_{\mathcal{L}Y}]$ is defined. This operation and notation may similarly be extended to the $K_o$-theory of log stacks over $Y$.

**Remark 1.3.** The map $f^! : K_o(\mathcal{L}Y) \to K_o(X)$ does not depend on the choice of $U \subseteq \mathcal{L}Y$. The map $\mathcal{L}Y \to Y$ is locally of finite presentation [Ols03, Theorem 1.1], so $\mathcal{L}Y$ is locally noetherian [Sta18, 01T6] and the map $i : U \subseteq \mathcal{L}Y$ is quasicompact [Sta18, 01OX]. The restriction $K_o(\mathcal{L}Y) \to K_o(U)$ is consequently well-defined because pullback preserves coherence [Ols07, Lemma 6.5].

The ordinary Gysin map $g^!$ for $g : X \to U$ is defined by the usual commutative diagram [Qu18 (0.2)]

$$K_o(C_{X/U}) \xrightarrow{i_*} K_o(M_{X/U}) \xrightarrow{j^*} K_o(U \times \mathbb{A}^1) \xrightarrow{\pi_*} K_o(C_{X/U}).$$

Then $g^! = i^! \circ (j^*)^{-1} \circ \pi^*$. One composes with the inclusion and intersection with the zero section $K_o(C_{X/U}) \to K_o(E) \to K_o(X)$ to get a class in the $K_o$-theory of $X$.

The above serves as a proxy for the analogous diagram with $\mathcal{L}Y$ in place of $U$ because we aren’t aware of a proof of excision for non-quasicompact stacks. Remark that $C_{X/U} \simeq C_X^\ell \simeq C_X^\ell \times_Y U$, $M_{X/U} \simeq M_{X/Y}|_U$ because $U \subseteq \mathcal{L}Y$ is open.

Examine the case of $f^![\mathcal{O}_V]$, for $V$ a log smooth $U$-stack. Take the fs pullback

$$W \xrightarrow{\ell^\vee} V \xrightarrow{\ell} X \xrightarrow{i} U.$$

An inclusion $\tilde{M}_W/V \subseteq \tilde{M}_{X/Y}|_W$ results, and witness $j^* (\mathcal{O}_{\tilde{M}_W/V}) = \mathcal{O}_{\tilde{M}_{X/Y}|_W}$.

Thus $g^![\mathcal{O}_V] = i^![\mathcal{O}_{\tilde{M}_W/V}] = [\mathcal{O}_{C_{X/Y}|_W}]$, and virtual fundamental classes are again fundamental classes of log normal cones.

If $Y$ is log smooth, $[X/Y]^{\ell \text{vir}}$ is independent of $Y$ but not of the obstruction theory $E$. By this, we mean that a composite $X \to Y' \to Y$ with obstruction theories $E$ for $X \to Y$ and $E'$ for $X \to Y'$ coming from two-term perfect complexes $\mathcal{E}^\bullet, \mathcal{E}'^\bullet$ with a compatibility datum

$$L_{Y'/Y}|_X \to \mathcal{E}^\bullet \to \mathcal{E}'^\bullet \to$$

will produce the same virtual fundamental class $[X/Y]^{\ell \text{vir}} = [X/Y']^{\ell \text{vir}}$. The proof is as in [Her19 Theorem 3.12]. We omit the proof but use the notation $[X]^{\ell \text{vir}}$ for $[X/Y]^{\ell \text{vir}}$ if $Y$ is log smooth.
Remark 1.4 (Artin fans). We review a construction generalizing the “skeleton” or cone over the dualizing complex of an s.n.c. divisor \( D \subseteq X \).

The fs log stack \( \mathcal{A}_P \) is like “\( \text{Spec} P \)” for monoids:

\[
\text{Hom}_{fs}(X, \mathcal{A}_P) = \text{Hom}_{\text{mon}}(P, \Gamma(M_X)).
\]

Surprisingly, the stack \( \mathcal{A}_P \) represents a strict-\( \acute{e} \text{tale} \) sheaf among fs log schemes. Put another way, \( \mathcal{A}_P \to \mathcal{L} \) represents an \( \acute{e} \text{tale} \) sheaf on \( \mathcal{L} \)-schemes.

Any fine, finite type log scheme \( X \) has a locally closed stratification \( X = \bigsqcup X_i \) such that \( M_X|_{X_i} \) is locally constant. Suppose \( M_X|_{X_i} \) were actually constant, and write \( M_{X_i} = \Gamma(X_i, M_X|_{X_i}) \). There are generization maps if \( X_i \subseteq X_j \):

\[
M_{X_i} \to M_{X_j}.
\]

If \( I \) is the category of strata with generizations \( X_i \subseteq X_j \) as morphisms, this defines a functor

\[
I \to (\text{Mon})^{\text{fine}} \to \mathcal{L}_{\acute{e}t}; \quad i \mapsto M_{X_i} \mapsto \mathcal{A}_{M_{X_i}}.
\]

Taking the colimit as \( \acute{e} \text{tale} \) sheaves on \( \mathcal{L} \) gives the Artin fan of \( X \) [ACWM17]:

\[
\text{colim}_I \mathcal{A}_{M_{X_i}} \to \mathcal{L}.
\]

If \( M_{X_i} \) are locally constant but not constant, they entail monodromy representations of the strata \( \pi_1(X_i) \). Cover \( X \) instead by strict \( \acute{e} \text{tale} \) maps from schemes with no monodromy. Define \( \mathcal{A}_X \) to be the colimit of the Artin fans of this cover, again as sheaves over \( \mathcal{L} \).

The hypotheses “log smooth and equidimensional” for log virtual classes in [Her19] were redundant: Remark 1.5. We claim connected, quasicompact log smooth stacks \( X \) are irreducible. In particular, general quasicompact log smooth stacks are equidimensional and have a fundamental class. This makes the extra hypotheses of “equidimensional” or “irreducible” on connected log smooth stacks in [Her19] redundant.

The Artin fan \( \mathcal{A}_X \) of \( X \) is quasicompact because \( X \) is, so [ACWM17, Theorem 4.6.2] supplies a subdivision \( F \to \mathcal{A}_X \) with a strict map \( F \to \mathcal{A}_N^k \) that is necessarily \( \acute{e} \text{tale} \). Denote by \( \widetilde{X} := \widetilde{F} \times^{\mathcal{A}_X} X \) the induced log blowup of \( X \), which has a smooth map \( \widetilde{X} \to \mathcal{A}_N^k \) to a smooth stack \( \mathcal{A}_N^k \) and hence
is smooth. This argument extends [Niz06, Theorem 5.10] to quasicompact log algebraic stacks.

Log blowups are surjective with geometrically connected fibers [Nak17, Proposition 2.6]. A closed (or open), surjective map to a connected topological space with connected fibers has connected source. Hence \( \tilde{X} \) is connected and smooth, which implies irreducible. The same holds for its image \( X \). Furthermore, \( \mathcal{L}X \) is irreducible because \( X \subseteq \mathcal{L}X \) is a dense open. It results that \( [\mathcal{O}_{LX}] \) is well defined.

**Remark 1.6.** Consider a short exact sequence

\[
E \to C \to D
\]

of cone stacks over noetherian \( X \). The pullback is an isomorphism:

\[
f^* : K_o(D) \simeq K_o(C).
\]

The proof of [HK19, Theorem 5.7] shows \( G^{naive}(D) \simeq G^{naive}(C) \) for any vector bundle-torsor, and \( K_o(\cdot) = G^{naive}(\cdot) \). For example, the intersection with the zero section used to define \( f^\dagger \) above is an isomorphism.

In view of [Kha20, Proposition 3.3, Corollary 3.4], this is an application of [Kha20, Theorem 3.16] for perfect stacks \( X \).

**Remark 1.7.** For any irreducible, finite type scheme \( X \), \( K_o(X) \) is generated by \( [\mathcal{O}_V] \) for \( V \subset X \) subschemes as a module over \( K_o(\text{Spec } \mathbb{C}) \). The proof uses noetherian induction to reduce to the affine case and applies the Jordan-H"older filtration.

This observation asserts that the case of \( f^\dagger[\mathcal{O}_V] \) discussed in Remark 1.3 determines \( f^\dagger \). We aren’t aware of a proof for algebraic stacks.

**Remark 1.8.** Consider an fs pullback square of qcqs stacks

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{q} & Y,
\end{array}
\]

where \( f \) and \( q \) are endowed with log perfect obstruction theories \( C_f^\ell \subseteq E \), \( C_q^\ell \subseteq F \). The two composite Gysin maps are equal

\[
f'^\dagger q^\dagger = p^\dagger f^\dagger : K_o(\mathcal{L}Y) \to K_o(X').
\]

This is seen by replacing \( \mathcal{L}Y, \mathcal{L}Y', \mathcal{L}X \) with quasicompact open substacks \( U_Y, U_Y', U_X \) containing the images of \( X' \) and fitting into a commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & U_X \\
\downarrow & & \downarrow \rho \\
U_Y & \xrightarrow{\gamma} & U_X
\end{array}
\]

\[
\begin{array}{ccc}
U_Y' & \xrightarrow{\delta} & U_X \\
\downarrow & & \downarrow \\
U_Y & \xrightarrow{\varepsilon} & U_Y
\end{array}
\]
applying \cite{Qu18} Proposition 2.5 to the pullback square, and \cite{Qu18} Proposition 2.11 to $X' \to U_x \times_{U_Y} U_Y \to U_x$, $X' \to U_x \times_{U_Y} U_Y \to U_Y$. 

**Remark 1.9.** If $Y$ isn’t log smooth but $V \to Y$ is a map from such a log stack, one can pull back the square and the log perfect obstruction theories to $V$. This explains the utility of log perfect obstruction theories for non-log smooth $Y$.

### 2. Pushforward Theorems of Hironaka and Costello

Unlike the previous section, these pushforward theorems are not simple applications of \cite{Qu18} to maps $X \to \mathcal{L}Y$. Even for a proper birational morphism $q : Y' \to Y$ between log smooth stacks, one might not have $Rq_* \mathcal{O}_{\mathcal{L}Y'} = \mathcal{O}_{\mathcal{L}Y}$. Indeed, a log blowup $Y' \to Y$ results in an open embedding $\mathcal{L}Y' \subseteq \mathcal{L}Y$. Nevertheless, analogues of the pushforward theorems of Hironaka and Costello remain true. We also verify the “birational invariance” property of \cite{AW18} generalized in \cite[Theorem 3.10]{Her19}.

A DM type map $f : X \to Y$ of algebraic stacks is **birational** [HH09, Definition A.1] if there is a dense open substack $V \subseteq Y$ whose preimage $f^{-1}V \subseteq X$ is dense and on which $f$ restricts to an isomorphism. This definition coincides with the stacks project if $f$ is locally of finite presentation \cite[0BAC]{Sta18}. Birationality is smooth-local on the target and satisfies the “3 for 2” property:

If $X \xrightarrow{f} Y \xrightarrow{g} Z$ have composite $h = g \circ f$ and two of $f, g, h$ are birational, so is the third.

Recall Hironaka’s pushforward theorem: a proper birational map $p : X \to Y$ of locally finite presentation with $X, Y$ smooth or rational singularities satisfies $Rp_* \mathcal{O}_X = \mathcal{O}_Y$ \cite[Theorem 1.1]{CR15}, \cite[Theorem 5.10]{KM98}, \cite[Corollary 2 pg 153]{Hir64}. The reduction to rational singularities merely resolves the singularities of each and then resolves the map between them in such a way that $Rp_* \mathcal{O}_X = \mathcal{O}_Y$.

**Lemma 2.1.** Let $p : \tilde{X} \to X$ be a log blowup of an fs log smooth algebraic stack $X$ over $\mathbb{C}$. Then

$$R_p \mathcal{O}_{\tilde{X}} = \mathcal{O}_X.$$  

*Proof.* The statement is local in $X$, so assume $X$ is a scheme with smooth global chart $X \to \mathbb{A}_P$ and $\tilde{X}$ is globally pulled back from a toric blowup $s : \tilde{F} \to \mathbb{A}_P$ of toric varieties. This reduces to the map $s : \tilde{F} \to \mathbb{A}_P$ itself, which satisfies $Rs_* \mathcal{O}_{\tilde{F}} = \mathcal{O}_{\mathbb{A}_P}$ because $\tilde{F}$ and $\mathbb{A}_P$ have rational singularities and $s$ is proper birational. \hfill $\square$

**Remark 2.2.** A separated DM stack has finite inertia stacks, hence a coarse moduli space:

A proper unramified map $q$ is finite: \cite[02V5]{Sta18} shows $q$ is locally quasifinite, \cite[01TJ]{Sta18} or \cite[01TD]{Sta18} that $q$ is quasifinite, and \cite{Sta18}...
finally gives that \( q \) is finite. A separated DM stack is defined to have proper unramified diagonal, which is then finite.

**Proposition 2.3** ("Hironaka’s Pushforward Theorem"). Consider a proper birational morphism \( p : X \to Y \) of DM type and locally finite presentation between log smooth algebraic stacks over \( \mathbb{C} \). Then

\[
Rp_*\mathcal{O}_X = \mathcal{O}_Y.
\]

**Proof.** The statement is smooth-local in \( Y \), so assume \( Y \) is an affine scheme. Take log blowups of \( X, Y \) which are smooth [Niz06 Theorem 5.10] and apply Lemma 2.1 to reduce to the case where \( X, Y \) have smooth underlying scheme. This implies \( X \) is a separated DM stack, and we must now prove Hironaka’s pushforward theorem in this case.

The coarse moduli space \( \pi : X \to \overline{X} \) exists, is a finite map, and is compatible with flat base change [Con05]. Then \( R^i\pi_*\mathcal{O}_X = 0 \) by finiteness [Sta18, 03QP] and \( \pi_*\mathcal{O}_X = \mathcal{O}_{\overline{X}} \) by the universal property of \( \pi \) applied to maps to \( \mathbb{A}^1 \) as discussed after [Con05, Theorem 1.1].

The coarse space \( \overline{X} \) has at worst finite quotient singularities [AV02 Lemma 2.2.3], which are rational singularities by [Kov00] or [Bou87]. Choose a proper birational map \( \tilde{t} : \tilde{X} \to \overline{X} \) with smooth source and \( R\tilde{t}_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\overline{X}} \). Then \( \tilde{X} \to Y \) is proper birational between smooth schemes, so Hironaka’s original pushforward theorem applies.

\[ \square \]

The interested reader can make sense of “log rational singularities” and generalize the proposition.

**Corollary 2.4.** In the setting of Proposition 2.3, pushforward identifies fundamental classes:

\[
p_*[\mathcal{O}_X] = [\mathcal{O}_Y] \in K_0(Y).
\]

Hironaka’s pushforward theorem details when fundamental classes are closed under pushforward. What about log virtual fundamental classes?

**Proposition 2.5.** Let \( X \to F \) be a strict morphism of DM type to an Artin fan and \( \tilde{F} \to F \) a proper, birational, DM-type morphism. Write \( \tilde{X} \) for the pullback

\[
\begin{array}{ccc}
\tilde{X} & \longrightarrow & \tilde{F} \\
\downarrow \phi & & \downarrow \\
X & \longrightarrow & F.
\end{array}
\]

This setup allows \( \tilde{X} \to X \) to be a log blowup, a root stack, or composites of such.

Suppose \( f : X \to Y \) is a morphism to a log smooth algebraic stack equipped with a log perfect obstruction theory \( C^\ell_{X/Y} \subseteq E \) and equip \( \tilde{X} \to Y \)
with the induced log perfect obstruction theory. Giving $\tilde{X}$ the induced log perfect obstruction theory, we have

$$p_*[\tilde{X}/Y]^{\text{vir}} = [X/Y]^{\text{vir}} \in K_0(X)$$

**Proof.** Apply compatibility of pushforward and the Gysin map [Qu18, Proposition 2.4] and Corollary 2.4 to the strict pullback square on Artin fans:

$$p_*[\mathcal{O}_{\tilde{X}/\tilde{F}}] = [\mathcal{O}_{C_{X/F}}] \in K_0(X).$$

Note that $C_{\tilde{X}/\tilde{F}} = C^\ell_{\tilde{X}}$, and the same for $X/F$. Consider the map of short exact sequences of cone stacks

$$
\begin{array}{c}
T^\ell_{Y|\tilde{X}} \longrightarrow C^\ell_{X/Y \times \tilde{F}} \xrightarrow{\tilde{r}} C^\ell_{\tilde{X}} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
T^\ell_{Y|X} \longrightarrow C^\ell_{X/Y \times F} \xrightarrow{r} C^\ell_{X}
\end{array}
$$

from [Her19, Proposition 2.5]. Remark that $C^\ell_{\tilde{X}/\tilde{F}} \simeq C^\ell_{X/Y}$, etc. After pulling back the bottom row to $\tilde{X}$, the leftmost vertical map $T^\ell_{Y|\tilde{X}} \to T^\ell_{Y|X}$ becomes an isomorphism; thus the right square is a pullback. Remark 1.6 attests $r^*, \tilde{r}^*$ are isomorphisms, so the commutative square

$$
\begin{array}{c}
K_0(C^\ell_{\tilde{X}/Y}) \xrightarrow{\tilde{t}^*} K_0(C^\ell_{X/Y}) \\
\tilde{r}^* \sim r^* \sim t^*
\end{array}
$$

comprising from compatibility of pullback and pushforward and the equalities $r^*[\mathcal{O}_{C^\ell_{X}}] = [\mathcal{O}_{C^\ell_{X/Y}}]$ etc. imply

$$\tilde{t}^*[\mathcal{O}_{C^\ell_{\tilde{X}/Y}}] = [\mathcal{O}_{C^\ell_{X/Y}}].$$

Composing with the inclusions into the obstruction theories and the Gysin map of the zero section of the obstruction theories, we get our result.

□

The rest of this section concerns Costello’s formula, which will be half of our proof of the log product formula.

**Construction 2.6.** Suppose $f : X \to Y$ is a DM type map between quasicompact log algebraic stacks and $Y$ is log smooth. Quasicompact log algebraic stacks have smooth-locally connected log strata, so [ACWM17, Proposition 3.2.1, Proposition 3.3.2] produces an Artin fan $Y \to \mathcal{A}_Y$ and a
relative Artin fan $\mathcal{A}_{X/Y}$ for the pair, both quasicompact:

$$
\begin{array}{ccc}
X & \rightarrow & W \\
\downarrow & & \downarrow \\
Y & \rightarrow & \mathcal{A}_{X/Y} \\
\downarrow & & \downarrow \\
& & \mathcal{A}_Y \\
& & \downarrow \\
& & \mathcal{L}.
\end{array}
$$

The map $\mathcal{A}_{X/Y} \rightarrow \mathcal{A}_Y$ is log étale, $W$ is log smooth, and $X \rightarrow W$ is strict. All the maps are DM type because Olsson showed $\mathcal{L}^1 \rightarrow \mathcal{L}$ is and the maps $\mathcal{A}_Y \rightarrow \mathcal{L}$, $\mathcal{A}_{X/Y} \rightarrow \mathcal{L}^1$ are representable by construction. The Artin fan of a fine log algebraic stack is locally noetherian. The map $Y \rightarrow \mathcal{A}_Y$ is smooth and thus noetherian; the same argument shows $W$ is noetherian.

Costello’s original pushforward formula [Cos06, Theorem 5.0.1] is incorrect as stated; see [HW21]. We prove a $K_0$-theoretic version of Costello’s corrected pushforward formula in pure degree one:

**Theorem 2.7** ("Costello’s pushforward formula in $K_0$-theory"). Consider an fs pullback square of DM type maps between locally noetherian, locally finite type log algebraic stacks over $\mathbb{C}$:

$$
\begin{array}{ccc}
X' & \rightarrow & X \\
\downarrow & & \downarrow \\
Y' & \rightarrow & Y \\
\end{array}
$$

Suppose $f$ has a log perfect obstruction theory $C^\ell_{X/Y} \subseteq E$ and endow $f'$ with the induced log perfect obstruction theory $C^\ell_{X'/Y'} \subseteq C^\ell_{X/Y}|_{X'} \subseteq E|_{X'}$. Assume $Y', Y$ are log smooth, $X$ is quasicompact, and $q$ is proper birational. Then

$$p_* [X'/Y']^{\text{vir}} = [X/Y]^{\text{vir}} \in K_0(X).$$

**Proof.** The proof is a global version of the argument in [Her19, Theorem 4.1]. Construction 2.6 furnishes us with a factorization $X \rightarrow W \rightarrow Y$ with $X \rightarrow W$ strict and $W \rightarrow Y$ log étale. Take the fs pullback:

$$
\begin{array}{ccc}
X' & \rightarrow & X \\
\downarrow & & \downarrow \\
W' & \rightarrow & W \\
\downarrow & & \downarrow \\
Y' & \rightarrow & Y.
\end{array}
$$

**Claim:** $s : W' \rightarrow W$ is proper birational.

The schematic fiber product $W \times^\text{sch}_{Y'} Y \rightarrow W$ is proper and $W' \rightarrow W \times^\text{sch}_{Y} Y'$ is finite [Ogu18, Proposition III.2.1.5].
Birationality of $s$ is smooth-local in $W$ and $Y$, so assume both are affine schemes. Find a commutative square

\[
\begin{array}{ccc}
\tilde{W} & \rightarrow & W \\
\downarrow & & \downarrow \\
\tilde{Y} & \rightarrow & Y
\end{array}
\]

with horizontal maps log blowups such that $\tilde{W} \rightarrow \tilde{Y}$ is integral [Kat99, Theorem]. Take fs pullbacks $\tilde{W}' := W' \times^\ell_W \tilde{W}, \tilde{Y}' := Y' \times^\ell_Y \tilde{Y}$ to obtain

\[
\begin{array}{ccc}
\tilde{W}' & \rightarrow & \tilde{W} \\
\downarrow & & \downarrow \\
\tilde{Y}' & \rightarrow & \tilde{Y}
\end{array}
\]

The 3 for 2 property of birationality and [Niz06, Proposition 4.3] reduce us to showing $\tilde{s}$ is birational and ensure that $\tilde{Y}' \rightarrow \tilde{Y}$ is. We conclude by observing the log étale and integral $\tilde{W} \rightarrow \tilde{Y}$ is flat [Ogu18, Theorem IV.4.3.5(1)].

Corollary 2.4 shows $s_*[\mathcal{O}_W] = [\mathcal{O}_W]$. Remark $C^\ell_{X/Y} \simeq C_{X/W}$, so we have an ordinary obstruction theory for $X \rightarrow W$. Commutativity of Gysin maps and pushforward [Qu18, Proposition 2.4] gives

$$p_*[X'/W']^{vir} = [X/W]^{vir}$$

and this translates to the statement above via the identifications $C^\ell_{X/Y} \simeq C_{X/W} \subseteq E$, etc.

\[\square\]

**Remark 2.8** (Due to G. Martin). One might wonder whether the analogous equality

$$p_*[X'/Y']^{\ell vir} = d \cdot [X/Y]^{\ell vir} \in K_0(X)$$

holds for proper maps $q : Y' \rightarrow Y$ that are of pure degree $d$ instead of birational. This is false even for $f = id_Y$ in ordinary $K$ Theory.

Consider the $k$-algebra

$$A = k[x_1, x_2, x_3]/(x_1^3 + x_2^3 + x_3^3)$$

and its spectrum $Z = \text{Spec } A$. Let $X$ be the minimal resolution of $Z$ given by blowing up the ideal $(x_1, x_2, x_3)$:

$$X := \text{Proj} \left( A[y_1, y_2, y_3]/(x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_2y_3 - x_2y_2) \right).$$

The composite $q : X \rightarrow Y = \mathbb{A}^2 = \text{Spec } k[x_1, x_2]$ of the blowup and the natural projection is of pure degree 3. Now compute $R^3q_*(\mathcal{O}_X) = H^3(\mathcal{O}_X)$ using the Čech complex of the open cover $U_i = (y_i \neq 0) \subset X, i = 1, 2, 3$:

$$0 \rightarrow C^0(U, \mathcal{O}_X) \rightarrow C^1(U, \mathcal{O}_X) \rightarrow C^2(U, \mathcal{O}_X) \rightarrow 0,$$
with cohomology

\[ H^0(X, \mathcal{O}_X) = A \cdot (1 \oplus 1 \oplus 1), \]
\[ H^1(X, \mathcal{O}_X) = A \cdot \left( \frac{y_1^2}{y_2 y_3} \oplus \frac{-y_2^2}{y_1 y_3} \oplus \frac{y_3^2}{y_1 y_2} \right), \]
\[ H^2(X, \mathcal{O}_X) = 0. \]

Conclude that

\[ Rq_* \mathcal{O}_X = [\tilde{A}] - [\tilde{A}] + 0 = 0 \neq 3 \cdot [\mathcal{O}_Y]. \]

3. Log $K_1$-theory

This section emerged from conversations with Sam Molcho and Jonathan Wise.

Recall that a morphism $\tilde{X} \to X$ of log algebraic stacks is called a log blowup if, strict étale locally on $X$, it is the fs pullback of a subdivision of a toric variety at a coherent monoidal ideal.

One can equivalently ask that $X \to \mathbb{A}_P$ be strict or $X$ be an atomic neighborhood by localizing further, or pull back instead from a subdivision of toric stacks $\mathcal{A}_\Sigma \to \mathcal{A}_P$. Write $\text{Sub}_X$ for the category of log blowups $\tilde{X} \to X$ of a log algebraic stack $X$, for example if $X$ is an Artin fan. Log blowups are monomorphisms among fs log stacks and fs fiber products preserve log blowups, so $\text{Sub}_X$ is a cofiltered preorder. All $X$-morphisms $\tilde{X}_1 \to \tilde{X}_2$ between log blowups of $X$ are themselves log blowups.

Recall that a functor $p : I \to J$ is initial if the comma category

\[ (p/j) := \{ i \to i' | p(i) \to p(i') \text{ lies over } j \} \]

is nonempty and connected for each $j \in J$. This is the dual to concepts variously called “final” and “cofinal” in the literature and has nothing to do with initial objects in functor categories. If $p$ is initial and $f : J \to C$ any functor, the natural map

\[ \lim_J f \to \lim_I f \circ p \]

is an isomorphism. A subcategory is an initial system if the inclusion functor is initial.

A map $f : X \to Y$ of log algebraic stacks induces a functor $f^* : \text{Sub}_Y \to \text{Sub}_X$ sending $\tilde{Y} \to Y$ to $\tilde{Y} \times_X X$. If $f$ is itself a log blowup, the map $f_1 : \text{Sub}_X \to \text{Sub}_Y$ sending $\tilde{X} \to X$ to the composite $\tilde{X} \to Y$ is a section of $f^*$ and each is initial. The key technical observation of this section is that $f^*$ is initial for $f$ strict.

**Definition 3.1.** Define *log $K_1$-theory* of a log algebraic stack $X$

\[ K_1(X) := \lim_{\tilde{X} \to X} K_0(\tilde{X}) \]
as the inverse limit under pushforwards along log blowups of $X$. It has natural maps $K_1(X) \to K_0(X)$ and $K_1(X) \to K_0(\tilde{X})$ for any log blowup $\tilde{X} \to X$.

**Remark 3.2.** Given a proper morphism $p : X \to Y$, compose the natural map on limits with the levelwise pushforward

$$\lim_{\tilde{X} \in \text{Sub}_X} K_0(\tilde{X}) \to \lim_{\tilde{Y} \in \text{Sub}_Y} K_0(\tilde{Y} \times_Y X) \to \lim_{\tilde{Y} \in \text{Sub}_Y} K_0(\tilde{Y})$$

to get a morphism $p_* : K_1(X) \to K_1(Y)$. If $\text{Sub}_Y \to \text{Sub}_X$ is initial, $f$ has a log perfect obstruction theory, and levelwise Gysin Maps are compatible with pushforwards, we will also have Gysin maps $f^\dagger : K_1(Y) \to K_1(X)$. A similar definition $\lim_{\tilde{X} \in \text{Sub}_X} A_\ast(\tilde{X})$ for Chow groups was made in [HPS19].

Log $K_1$-theory can likewise be defined as the colimit under pullbacks of log blowups as in [IKNU20], [Bar18] using $K_0$-theory. One could instead take Gysin maps as the transition morphisms, provided you require the total spaces to be smooth and use the canonical obstruction theory of an l.c.i. There are natural variants taking (co)limits over log blowups as well as root stacks.

**Remark 3.3.** Lemma 2.1 ensures that the class $([\mathcal{O}_{\tilde{X}}])_{\tilde{X} \in \text{Sub}_X}$ is well-defined in $K_1(X)$ for log smooth $X$ over $\mathbb{C}$. Note this class pushes forward to the ordinary fundamental class in $K_0(X)$.

**Definition 3.4.** Suppose $f : X \to Y$ is equipped with a log perfect obstruction theory and $Y$ is log smooth. Define the log virtual fundamental class in $K_1$-theory to be the sequence

$$([\tilde{X}/Y]^{\text{vir}})_{\tilde{X} \in \text{Sub}_X} \in K_1(X).$$

Proposition 2.5 verifies this sequence lies in the limit $K_1(X)$. One can similarly define $f^\dagger[\mathcal{O}_V]$ for log smooth stacks $V \to Y$ but we don’t define this operation in full generality on $K_1$.

**Lemma 3.5.** Suppose $E \to F$ is a strict map of quasicompact Artin fans and $\tilde{E} \to E$ a subdivision. There exists a subdivision $\tilde{F} \to F$ such that the pullback refines $\tilde{E}$:

$$\tilde{F} \times_F E$$

$\tilde{E} \to E.$

**Proof.** Reduce to the case $F = \mathcal{A}_N^k$ using [ACWM17, Theorem 4.6.2] and $E = \mathcal{A}_\sigma$ a single Artin cone. The strict map $\mathcal{A}_\sigma \to \mathcal{A}_N^k$ is necessarily an isomorphism onto a subcone. Then [SST18, Proposition 4.6] exhibits an initial system of subdivisions of $\mathcal{A}_N^k$. □
All log blowups are refined by pulling back subdivisions of the Artin fan \( \mathcal{A}_X \) of \( X \) if \( X \) is quasicompact. This is because one can assume \( X \) is atomic and the strict map \( X \to \mathcal{A}_P \) induces an isomorphism \( P \cong \Gamma(\mathcal{M}_X) \). We stop short of showing all log blowups are pulled back from the Artin fan because doing so would require gluing the above local subdivisions, but Lemma 3.5 refines each local subdivision of a cone of \( \mathcal{A}_X \) by a global subdivision.

**Lemma 3.6.** Suppose \( f : X \to Y \) is a strict map of quasicompact log algebraic stacks. The functor \( f^* : \text{Sub}_Y \to \text{Sub}_X \) is initial.

**Proof.** Consider a log blowup \( \tilde{X} \to X \) that we wish to refine. Assume it is pulled back from a subdivision of \( \mathcal{A}_X \) because these form an initial system as remarked above. The induced map \( \mathcal{A}_X \to \mathcal{A}_Y \) is strict with quasicompact source and target and Lemma 3.5 concludes.

\[ \square \]

**Proposition 3.7.** Suppose a quasicompact DM type morphism \( f : X \to Y \) of log algebraic stacks with \( Y \) quasicompact induces epimorphisms

\[ \mathcal{M}_{Y,f(\mathfrak{m})} \to \mathcal{M}_{X,\mathfrak{m}} \]

on stalks at geometric points \( \mathfrak{m} \to X \). There is a log blowup \( \tilde{Y} \to Y \) such that the fs pullback

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\
\downarrow{p} & & \downarrow{q} \\
X & \xrightarrow{f} & Y
\end{array}
\]

has \( \tilde{f} \) strict.

**Proof.** *Case:* First suppose \( X, Y \) are schemes.

Locally on \( Y \), one chooses a log blowup making \( X \to Y \) \( \mathbb{Q} \)-integral [Ogu18, Theorem III.2.6.7]. We can refine these local log blowups by a global one using quasicompactness of \( Y \) and assume \( f : X \to Y \) is \( \mathbb{Q} \)-integral, in particular exact [IKN05, Remark pg. 58]. The stalks of the characteristic monoids for a log blowup are epimorphisms, so the property of epimorphic stalks is preserved.

Exactness entails a pullback

\[
\begin{array}{ccc}
\mathcal{M}_{Y,f(\mathfrak{m})} & \xrightarrow{\mathcal{M}_{Y,f(\mathfrak{m})}} & \mathcal{M}_{X,\mathfrak{m}} \\
\downarrow & & \downarrow \\
\mathcal{M}^{op}_{Y,f(\mathfrak{m})} & \xrightarrow{\mathcal{M}^{op}_{Y,f(\mathfrak{m})}} & \mathcal{M}^{op}_{X,\mathfrak{m}}
\end{array}
\]

with horizontal arrows both epimorphisms. Epimorphisms of groups are surjections [La20a], so the top horizontal arrow is a surjection. Exactness ensures it is also an injection, hence an isomorphism.
We now reduce the general statement for log stacks to the above case. Cover $Y$ by a strict smooth map $V \to Y$ from a quasicompact scheme and choose a blowup $\tilde{V} \to V$ pulled back from $\mathcal{X}_V$ such that $\tilde{V} \times_V X \to \tilde{V}$ is strict. Find a log blowup $\tilde{Y} \to Y$ refining $\tilde{V} \to V$ using Lemma 3.6 to conclude.

\textbf{Corollary 3.8.} If $f : X \to Y$ satisfies the hypotheses of Proposition 3.7, the functor $f^* : \text{Sub}_Y \to \text{Sub}_X$ is initial.

\textit{Proof.} Use notation as in Proposition 3.7. The composite $p_! f^* q^* = p_! p^* f^* : \text{Sub}_Y \to \text{Sub}_X$ is initial as a composite of functors coming from log blowups and the strict $\tilde{f}$ as in Lemma 3.6. Hence $f^*$ is initial.

\textbf{Definition 3.9.} If $f : X \to Y$ satisfies the hypotheses of Proposition 3.7 and furthermore is equipped with a log perfect obstruction theory $C_\ell^f \subseteq E$, we can define $f^\dagger$ on $K^\dagger$-theory as the composite:

$$K^\dagger(X) := \lim_{\tilde{Y} \in \text{Sub}_Y} K_0(\tilde{Y}) \xrightarrow{\tilde{f}} \lim_{\tilde{Y} \in \text{Sub}_Y} K_0(\tilde{Y} \times_\ell X) \simeq \lim_{\tilde{X} \in \text{Sub}_X} K_0(\tilde{X}) =: K^\dagger(X).$$

The first map is induced from the levelwise Gysin maps on $\tilde{f} : \tilde{Y} \times_\ell X \to \tilde{Y}$, while the isomorphism comes from Corollary 3.8 showing $f^* : \text{Sub}_Y \to \text{Sub}_X$ is initial.

\textbf{Proposition 3.10.} Consider an fs pullback square

$$\begin{array}{ccc}
X' & \xrightarrow{\tilde{f}} & X \\
\downarrow^f & & \downarrow^f \\
Y' & \xrightarrow{q} & Y
\end{array}$$

of DM type maps between log algebraic stacks with $Y', Y$ log smooth. Endow $f$ with a log perfect obstruction theory $C_\ell^f \subseteq E$ and equip $f'$ with the induced log perfect obstruction theory $C_\ell^{f'} \subseteq C_\ell^f|_{X'} \subseteq E|_{X'}$.

1. If $q$ is proper birational and $X$ is quasicompact, then $p_* [X'/Y']^{\ell\text{vir}} = [X/Y]^{\ell\text{vir}} \in K^\dagger(X)$.

2. Suppose $q : Y' \to Y$ satisfies the hypotheses of Proposition 3.7. Endow $q$ with a log perfect obstruction theory $C_\ell^q \subseteq F$. Then $q^\dagger [X/Y]^{\ell\text{vir}} = [X'/Y']^{\ell\text{vir}} \in K^\dagger(X')$.

\textit{Proof.} The map $t : K^\dagger(X') \to \lim_{\tilde{X} \in \text{Sub}_X} K_0(\tilde{X} \times_\ell X')$ sends the sequence $[X'/Y']^{\ell\text{vir}} := ([\tilde{X}'/Y']^{\ell\text{vir}})_{\tilde{X}' \in \text{Sub}_{X'}}$, to the sequence $([\tilde{X} \times_\ell X'/Y']^{\ell\text{vir}})_{\tilde{X} \in \text{Sub}_X}$. Costello’s pushforward Theorem 2.7 shows the
levelwise pushforward sends this sequence to the \((\widetilde{X}/Y)^{\ell_{vir}})_{X \in \text{Sub}_X} =: [X/Y]^{\ell_{vir}} \in K_{1}(X)

Similarly, one argues the levelwise Gysin maps send \((\widetilde{X}/Y)^{\ell_{vir}})_{X \in \text{Sub}_X} to \((\widetilde{X} \times_X X'/Y')^{\ell_{vir}})_{X \in \text{Sub}_X} by applying Remark \[8\] and then \(t^{-1}\) sends this to \((\widetilde{X}'/Y')^{\ell_{vir}})_{X' \in \text{Sub}_{X'}} =: [X'/Y']^{\ell_{vir}}.

\[\square\]

\textbf{Remark 3.11.} Write \(s : K_{1}(X) \rightarrow K_{0}(X)\), etc. for the natural projections. Under the assumptions of Proposition \[3.10\]

\(sq^{\dagger}[X/Y]^{\ell_{vir}} = [X'/Y']^{\ell_{vir}} = q^{\dagger}s[X/Y]^{\ell_{vir}},\)

\(sp^*[X'/Y']^{\ell_{vir}} = [X/Y]^{\ell_{vir}} = p^*s[X'/Y']^{\ell_{vir}}.\)

In this sense, the operations \(q^{\dagger}, p^{*}\) defined variously on \(K_{1}\)-theory or \(K_{0}\)-theory are compatible.

4. The Log Product Formula

Let \(V, W\) be log smooth quasiprojective schemes throughout this section. Write \(\mathcal{M}_{g,n} \subseteq \mathcal{M}_{g,n}\) for the open substack of stable curves in the moduli space of all genus-\(g\), \(n\)-marked nodal curves. Equip each with the divisorial log structure from the singular locus, an s.n.c. divisor. The stack \(\mathcal{M}_{g,n}^{\dagger}(V)\) of \textit{log stable maps} sends an fs log scheme \(T\) to diagrams of fs log schemes

\[
\begin{array}{ccc}
C & \rightarrow & V \\
\downarrow & & \\
T & & \\
\end{array}
\]

where \(C\) is an fs log smooth curve over \(T\) with genus \(g\) and \(n\)-markings such that the underlying diagram of schemes is a stable map of curves. We emphasize that \(\mathcal{M}_{g,n}^{\dagger}(V)\) denotes the stack \(\mathcal{M}(V)\) of Gross-Siebert \[11\] parametrizing \textit{log} stable maps and not the ordinary space of stable maps.

The stack \(\mathcal{D}\) consists of diagrams \((C' \leftarrow C \rightarrow C'')\) of genus \(g\), \(n\)-pointed prestable curves over \(T\) such that \(C \rightarrow C' \times C''\) is stable. Equivalently, no unstable rational component of \(C\) is contracted under both projections.

We have an fs pullback square:

\[
\begin{array}{cccc}
\mathcal{M}_{g,n}^{\dagger}(V \times W) & \rightarrow_{h} & Q & \rightarrow_{\ell} \mathcal{M}_{g,n}^{\dagger}(V) \times \mathcal{M}_{g,n}^{\dagger}(W) \\
\downarrow^{c} & & \downarrow^{b} & \\
\mathcal{D} & \rightarrow_{\nu} & Q' & \rightarrow_{\ell} \mathcal{M}_{g,n} \times \mathcal{M}_{g,n} \\
\downarrow & & \downarrow & \\
\mathcal{M}_{g,n} & \rightarrow_{\Delta} & \mathcal{M}_{g,n} \times \mathcal{M}_{g,n} & \\
\end{array}
\]

(1)
One lemma and the study of this diagram in [Her19] suffice to achieve the log product formula.

**Lemma 4.1.** Provided \( q \) is log flat and integral in the fs pullback square

\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & Y,
\end{array}
\]

the natural inclusion \( C^\ell_f \subseteq C^\ell_f|_{X'} \) is an isomorphism.

**Proof.** The claim is smooth-local in \( C^\ell_f \), hence in \( X, Y \) by Lemmas 2.15, 2.16 of [Her19]. Assume \( X, Y \) are affine schemes equipped with a global chart by Artin Cones:

\[
\begin{array}{ccc}
X & \longrightarrow & \mathcal{A}_\sigma \\
\downarrow & & \downarrow \\
Y & \longrightarrow & \mathcal{A}_\tau.
\end{array}
\]

Write \( W := Y \times_{\mathcal{A}_\tau} \mathcal{A}_\sigma \) as in Construction 2.6 and take the fs pullback of the factorization:

\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow & & \downarrow \\
W' & \longrightarrow & W \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & Y.
\end{array}
\]

Then \( X \to W \) is strict, \( W \to Y \) is log étale, and we have a map of short exact sequences of cone stacks

\[
\begin{array}{ccc}
T^\ell_{W'/Y'}|_{X'} & \longrightarrow & C^\bullet |_{X'/W'} \\
\downarrow & & \downarrow \\
T^\ell_{W/Y}|_{X} & \longrightarrow & C^\bullet |_{X/W}.
\end{array}
\]

Log flatness of \( q \) ensures the leftmost arrow becomes an isomorphism after pullback to \( X' \) [Ols05 1.1 (iv)], so the right square of cones is cartesian. It suffices to show the map of ordinary normal cones \( C^\bullet |_{X'/W'} \to C^\bullet |_{X/W}|_{X'} \) is an isomorphism. Note \( W' \to W \) is log flat and integral, hence it is flat [Ogu18, Theorem IV.4.3.5(1)] and the result is standard.

\[\square\]

**Remark 4.2.** The map \( \mathcal{D} \to \mathcal{M}_{g,n} \) sending a trio \( (C' \leftarrow C \rightarrow C'') \) of partial stabilizations to their source \( C \) is log étale [Her19, Remark 5.9]. Not only does this equate the normal cones \( C^\bullet |_{\mathcal{M}_{g,n}(V \times W)/\mathcal{D}} \cong C^\bullet |_{\mathcal{M}_{g,n}(V \times W)/\mathcal{M}_{g,n}} \), but [Her19 Lemma 5.10] identifies the natural obstruction theory on \( c \) with
the one pulled back from $a$. We similarly equip $b$ with the pulled back obstruction theory from $a$.

The stabilization map $\mathcal{M}_{g,n} \to \mathcal{M}_{g,n}$ is integral and log smooth. To see this, use the smooth cover $\coprod_{k} \mathcal{M}_{g,n+k} \to \mathcal{M}_{g,n}$ and recognize the composites $\mathcal{M}_{g,n+k} \to \mathcal{M}_{g,n}$ as iterations of the universal curve $\mathcal{M}_{g,m+1} \to \mathcal{M}_{g,m}$. Give $\Delta$ the canonical log perfect obstruction theory from the isomorphism $C^{\ell}_{\mathcal{M}_{g,n}/\mathcal{M}_{g,n} \times \mathcal{M}_{g,n}} \simeq N^{\ell}_{\mathcal{M}_{g,n}/\mathcal{M}_{g,n} \times \mathcal{M}_{g,n}}$ as in [Her19, Remark 3.7]. Since $s \times s$ is log flat and integral, Lemma 4.1 equates $C^{\ell}_{\mathcal{M}_{g,n}/\mathcal{M}_{g,n} \times \mathcal{M}_{g,n}} \simeq C^{\ell}_{\mathcal{M}_{g,n}/\mathcal{M}_{g,n} \times \mathcal{M}_{g,n}} |_{Q'}$ and the corresponding Gysin maps $\phi^{\dagger} = \Delta^{\dagger}$ are equal.

The log smooth stabilization map also means $Q'$ is log smooth, and we can write $[Q/Q']^{\ell \text{vir}} = [Q]^{\ell \text{vir}}$ as permitted by Remark 1.3.

**Theorem 4.3** (“The log product formula”). The log product formula holds in $K_{0}$-theory as well as $K_{1}$-theory: the classes

$$h_{*}[\mathcal{M}^{\dagger}_{g,n}(V \times W)]^{\ell \text{vir}} = \Delta^{\dagger}[\mathcal{M}^{\dagger}_{g,n}(V) \times \mathcal{M}^{\dagger}_{g,n}(W)]^{\ell \text{vir}}$$

are equal in $K_{0}(Q)$ as well as $K_{1}(Q)$.

**Proof.** We claim both sides of the equality compute $[Q/Q']^{\ell \text{vir}}$ as in [Beh99]. Remark 3.11 shows it suffices to prove the equality in $K_{1}(Q)$.

Note that $\phi$ induces surjective maps

$$\mathcal{M}_{\mathcal{M}_{g,n} \times \mathcal{M}_{g,n}, \phi(x)} \to \mathcal{M}_{Q', x}$$

at geometric points $x \to Q'$, so $q^{\dagger}$ is well-defined on $K_{1}$-theory as in Definition 3.9. Proposition 3.10 refines Costello’s Formula 2.7 and the compatibility we’ll need of Remark 1.8 to $K_{1}$-theory. In particular, $\phi^{\dagger}$ and $\nu_{*}$ both preserve log virtual fundamental classes:

$$\phi^{\dagger}[\mathcal{M}^{\dagger}_{g,n}(V) \times \mathcal{M}^{\dagger}_{g,n}(W)]^{\ell \text{vir}} = [Q/Q']^{\ell \text{vir}}$$

$$\nu_{*}[\mathcal{M}^{\dagger}_{g,n}(V \times W)]^{\ell \text{vir}} = [Q/Q']^{\ell \text{vir}}.$$

Combining these with the equalities $[\mathcal{M}^{\dagger}_{g,n}(V \times W)]^{\ell \text{vir}} = [\mathcal{M}^{\dagger}_{g,n}(V \times W)]^{\ell \text{vir}}$ and $\phi^{\dagger} = \Delta^{\dagger}$ from Remark 4.2 gives the result.

**Remark 4.4.** One may also prove Theorem 4.3 as in [Ran19]. One replaces $\mathcal{M}_{g,n} \times \mathcal{M}_{g,n}$ by $\mathcal{M}_{g,n}(\mathcal{X}_{V}) \times \mathcal{M}_{g,n}(\mathcal{X}_{W})$, makes similar adjustments to $\mathcal{D}$, $Q'$, and takes judicious log blowups to make the analogue of $s \times s$ flat. Using $\mathcal{M}_{g,n}(\mathcal{X}_{V})$, etc. instead of $\mathcal{M}_{g,n}$ makes the maps $a, c$ strict, so their log virtual fundamental classes agree with the ordinary schematic ones. One applies the usual versions of Costello’s formula and compatibility of Gysin maps in squares instead of our log-adapted versions.
5. Relative variants and a Counterexample

We offer a relative variant of Theorem 4.3 as in [LQ18, §2.4]. This variant makes it easier to see the necessity of $\Delta^!$ instead of $\Delta^\circ$ in Theorem 4.3. We end with a counterexample to the version of the relative variant where $\Delta^\circ$ replaces $\Delta^!$, justifying our technology.

Let $V \to S$, $W \to T$ be log smooth, quasiprojective maps. We allow $V, W, S, T$ to be algebraic stacks as long as $V \to S$, $W \to T$ are representable by schemes; the ensuing fibers of $\overline{M}_{g,n}(V/S) \to S$ are stable maps to schemes. Our counterexample uses relative stable maps to $[\mathbb{P}^1/G_m] \to B G_m$, like “rubber” maps but without expansions.

Form the analogue of Diagram (1):

\[
\begin{array}{cccc}
\overline{M}_{g,n}(V \times W/S \times T) & \xrightarrow{h} & Q & \xrightarrow{\ell} & \overline{M}_{g,n}(V/S) \times \overline{M}_{g,n}(W/T) \\
\downarrow_{\nu} & \quad & \downarrow_{\phi} & \quad & \downarrow_{s \times s} \\
\mathcal{D} & \xrightarrow{\ell} & Q' & \xrightarrow{\phi} & M_{g,n} \times M_{g,n} \\
\downarrow & \quad & \downarrow & \quad & \downarrow \\
\overline{M}_{g,n} & \xrightarrow{\Delta} & \overline{M}_{g,n} \times \overline{M}_{g,n}.
\end{array}
\] (2)

**Remark 5.1.** Log stable maps $C \to V$ have more discrete data than simply genus $g$ and marked points $n$. They also have contact orders at the marked points and the curve class $\beta$ of their image.

Let $\Gamma$ contain the genus $g$, the number $n$ of marked points, and the contact orders of a stable map $C \to V \times W$ over $S \times T$. This induces discrete data $\Gamma', \Gamma''$ describing the maps $C' \to V$, $C'' \to W$ obtained by stabilization.

One way to bundle this data together is in the Artin fan $\overline{M}_{g,n}(V/S)$. The map from a log stack $X$ to its Artin fan is geometrically connected by construction, but not necessarily surjective if $X$ is not log smooth. This induces an injection from the connected components of log stable maps to those of its Artin fan, which is more closely related to tropical curves.

Given $\Gamma', \Gamma''$ discrete data for maps to $V, W$, there is at most one $\Gamma$ for maps to $V \times W$ that induces $\Gamma'$ and $\Gamma''$. If $g, n$ differ in $\Gamma'$, $\Gamma''$, there are none.

**Theorem 5.2 (4.3).** Fix discrete data $\Gamma$ for a stable map $C \to V \times W$ over $S \times T$ as in Remark 5.1. The log product formula holds in $K_{\ell}$-theory and $K_{\circ}$-theory:

\[
h_*[\overline{M}_{1}(V \times W/S \times T)]^{\ell \vir} = \Delta^![\overline{M}_{1'}(V/S) \times \overline{M}_{1''}(W/T)]^{\ell \vir}
\]

in $K_1(Q)$ and $K_0(Q)$.

**Proof.** Omitted. For the discrete data, the normal cones themselves $C_{X/Y}$ respect disjoint unions in the source. \(\square\)
5.1. Counterexample to the ordinary log product formula. We claim the “ordinary product formula” with $\Delta^!$ in the place of $\Delta^\dagger$ is false:

$$h'(\overline{\mathcal{M}}_{g,n}(V \times W))^{\text{vir}} \neq \Delta^!(\overline{\mathcal{M}}_{g,n}(V) \times \overline{\mathcal{M}}_{g,n}(W))^{\text{vir}},$$

where $h'$ is the map

$$\overline{\mathcal{M}}_{g,n}(V \times W) \xrightarrow{h} Q \xrightarrow{\pi} \overline{\mathcal{M}}_{g,n}(V) \times_{\overline{\mathcal{M}}_{g,n}(V)} \overline{\mathcal{M}}_{g,n}(W)$$

to the ordinary scheme-theoretic fiber product. This necessitates the introduction of $\Delta^\dagger$ in Section 1.

We outline a counterexample due to Dhruv Ranganathan [Ran20] and David Holmes, which gives the following:

$$\Delta^!(\overline{\mathcal{M}}_{1,2}(\mathbb{P}^1) \times \overline{\mathcal{M}}_{1,2}(\mathbb{P}^1))^{\text{vir}} \neq \pi_* \Delta^!(\overline{\mathcal{M}}_{1,2}(\mathbb{P}^1) \times \overline{\mathcal{M}}_{1,2}(\mathbb{P}^1))^{\text{vir}}.$$

For notational simplicity, write $\text{DR}_{\Gamma}$ for $\overline{\mathcal{M}}_{1,2}(\mathbb{P}^1/\mathbb{C}^*)/\mathbb{B}_C^*$. Here we equip $\mathbb{P}^1$ with divisorial log structure from 0 and $\infty$ and the moduli space consists of relative stable maps

$$C \xrightarrow{f} [\mathbb{P}^1/\mathbb{C}^*]$$

of degree 2 from genus-one, 2-marked curves with contact order 2 at both 0 and $\infty$.

If no confusion may arise, we denote also $\text{DR}_{\Gamma}$ as its image on $\mathcal{M}_{1,2}$, which has $\mathbb{Z}/2\mathbb{Z}$-action at all points as their stack structures. $\text{DR}_{\Gamma}$ can be
understood as a degree-three multisection from $M_{1,1}$ to $M_{1,2}$ as in Figure 1. We forbid the two marked points from coming together in $\text{DR}_\Gamma$, a component that is sometimes included in double ramification cycles.

**Counterexample in Chow.** We first give the computation in Chow theory. By definition, we have

$$\Delta^!(\text{DR}_\Gamma \times \text{DR}_\Gamma) = \text{DR}_\Gamma \cdot \text{DR}_\Gamma.$$ 

To compute $\Delta^!(\text{DR}_\Gamma \times \text{DR}_\Gamma)$ explicitly, we introduce the following procedure using tropical geometry.

Let $\text{DR}_{\Gamma}^{trop}$ and $\overline{M}_{1,2}^{trop}$ be the Artin fans of $\text{DR}_\Gamma$ and $\overline{M}_{1,2}$ respectively. Think of them as stacky cone complexes with strict maps $\text{DR}_\Gamma \to \text{DR}_{\Gamma}^{trop}$ and $\overline{M}_{1,2} \to \overline{M}_{1,2}^{trop}$. Both $\text{DR}_{\Gamma}^{trop}$ and $\overline{M}_{1,2}^{trop}$ have $\mathbb{Z}/2$ stack structure from interchanging the edges $e_1$ and $e_2$. The map $\text{DR}_{\Gamma}^{trop} \to \overline{M}_{1,2}^{trop}$ is the inclusion of the diagonal in the quadrant spanned by $e_1$ and $e_2$. It does not map cones surjectively onto cones, which corresponds to a non-flat map of toric stacks. Take a subdivision $\overline{M}_{1,2}^{trop}$ of $\overline{M}_{1,2}^{trop}$ by adding the diagonal to the $e_1$-$e_2$
quadrant as in Figure 2. This subdivision induces a log blowup

$$\pi : \overline{M}_{1,2} := \overline{M}_{1,2} \times \overline{M}_{1,2}^{\text{trop}} \to \overline{M}_{1,2}$$

such that the factorization $\text{DR}_{\Gamma}^\dagger \to \overline{M}_{1,2}$ is strict, where $\text{DR}_{\Gamma}^\dagger$ is the strict transformation of $\text{DR}_{\Gamma}$. Let $E = \pi^{-1}(p)$ be the exceptional divisor with $\mathbb{Z}/2\mathbb{Z}$-action at all points as stack structure.

Now we have

$$\Delta^\dagger(\text{DR}_{\Gamma} \times \text{DR}_{\Gamma}) = \text{DR}_{\Gamma}^\dagger \cdot \text{DR}_{\Gamma}^\dagger$$

and hence

$$\pi_* \Delta^\dagger(\text{DR}_{\Gamma} \times \text{DR}_{\Gamma}) = \pi_*(\text{DR}_{\Gamma}^\dagger \cdot \text{DR}_{\Gamma}^\dagger)$$

$$= \pi_*(\pi^\dagger \text{DR}_{\Gamma} - E) \cdot (\pi^\dagger \text{DR}_{\Gamma} - E)$$

$$= \text{DR}_{\Gamma} \cdot \text{DR}_{\Gamma} - \frac{1}{4} \neq \Delta^\dagger(\text{DR}_{\Gamma} \times \text{DR}_{\Gamma}).$$

Note that $\pi^\dagger \text{DR}_{\Gamma} \cdot E = 0$ by projection formula and $E \cdot E = -\frac{1}{4}$. This gives the inequality for Chow theory.

**Counterexample in $K$-theory.** For $K$-theory, note first that

$$\Delta^\dagger[\mathcal{O}_{\text{DR}_{\Gamma}} \times \mathcal{O}_{\text{DR}_{\Gamma}}] := \phi \left( \left[ \mathcal{O}_{\Delta(\overline{M}_{1,2})} \otimes L (\mathcal{O}_{\text{DR}_{\Gamma}} \times \mathcal{O}_{\text{DR}_{\Gamma}}) \right] \right)$$

$$= \lambda_{-1}(N^*_\text{DR}_{\Gamma}/\overline{M}_{1,2}) \cdot \mathcal{O}_{\text{DR}_{\Gamma}},$$

where $\lambda_{-1}(F) := \sum_i (-1)^i \Lambda^i F$ is the alternating sum of exterior power of $F$. We view both $\mathcal{O}_{\Delta(\overline{M}_{1,2})}$ and $\mathcal{O}_{\text{DR}_{\Gamma}} \times \mathcal{O}_{\text{DR}_{\Gamma}}$ as sheaves on $\overline{M}_{1,2} \times \overline{M}_{1,2}$ and $\phi : K_o^\Delta(\overline{M}_{1,2}) (\overline{M}_{1,2} \times \overline{M}_{1,2}) \simeq K_o(\overline{M}_{1,2})$ is a canonical isomorphism, where $K^Z_o(X)$ denote the $K$-theory of $X$ supported on $Z$. The last equality is the excess intersection formula in $K$-theory [FL85, VI. Theorem 1.3]. One can also compute the derived tensor product $\otimes L$ by taking locally free resolution of $\mathcal{O}_{\Delta(\overline{M}_{1,2})}$ and $\mathcal{O}_{\text{DR}_{\Gamma}}$.

The computation of $\Delta^\dagger[\mathcal{O}_{\text{DR}_{\Gamma}} \times \mathcal{O}_{\text{DR}_{\Gamma}}]$ is similar:

$$\Delta^\dagger[\mathcal{O}_{\text{DR}_{\Gamma}} \times \mathcal{O}_{\text{DR}_{\Gamma}}] := \phi \left( \left[ \mathcal{O}_{\Delta(\overline{M}_{1,2})} \otimes L (\mathcal{O}_{\text{DR}_{\Gamma}}^\dagger \times \mathcal{O}_{\text{DR}_{\Gamma}}^\dagger) \right] \right),$$
Therefore, assuming the third equality below, we have
\[
\pi_* \Delta_I^I[\mathcal{O}_{\text{DR}} \times \mathcal{O}_{\text{DR}}] = \pi_* \phi \left( \left( \mathcal{O}_{\Delta(M,1)} \otimes \mathcal{O}_{\text{DR}} \right) \right) = \pi_* \lambda_1 \left( N^*_{\text{DR}/\overline{M_{1,2}}} \right) \cdot \mathcal{O}_{\text{DR}} \\
= -2 \mathcal{O}_p + \lambda_1 \left( N^*_{\text{DR}/\overline{M_{1,2}}} \right) \cdot \mathcal{O}_{\text{DR}} \\
= -2 \mathcal{O}_p + \Delta_I[\mathcal{O}_{\text{DR}} \times \mathcal{O}_{\text{DR}}] \\
\neq \Delta_I[\mathcal{O}_{\text{DR}} \times \mathcal{O}_{\text{DR}}],
\]

exactly as claimed. We are left to show the third equality in (3). Write
\[
\mathcal{O}_{\text{DR}}^I = \pi^* \mathcal{O}_{\text{DR}} - \mathcal{O}_E + \mathcal{O}_{\text{DR}}^I \cap E.
\]
The desired equality follows easily from the following equations
\[
\pi_* (\mathcal{O}_{\text{DR}}^I \cdot \mathcal{O}_E) = \pi_* \mathcal{O}_{\text{DR}}^I \cap E = \mathcal{O}_p; \\
\pi_* (\mathcal{O}_{\text{DR}}^I \cdot \mathcal{O}_{\text{DR}}^I \cap E) = 0; \\
\pi_* (\mathcal{O}_E \cdot \mathcal{O}_{\text{DR}}^I \cap E) = \pi_* (\pi^* \mathcal{O}_p \cdot \mathcal{O}_{\text{DR}}^I \cap E) = \mathcal{O}_p \cdot \mathcal{O}_p = 0; \\
\pi_* (\mathcal{O}_{\text{DR}}^I \cap E \cdot \mathcal{O}_{\text{DR}}^I \cap E) = 0.
\]
The last three equations use the following simple observation. For any locally free sheaf \( V \),
\[
V \cdot \mathcal{O}_{\text{DR}}^I \cap E = \mathcal{O}_{\text{DR}}^{\oplus \text{rk}(V)}
\]
since \( \text{DR}^I \cap E \) is a point.

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