LEFSCHETZ PROPERTIES OF JACOBIAN ALGEBRAS AND SINGULARITIES OF HYPERPLANE SECTIONS

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Abstract. Let $V : f = 0$ be a smooth hypersurface of degree $d \geq 3$ in the complex projective space $\mathbb{P}^n$, $n \geq 3$. Let $M(f)$ be the associated Jacobian algebra and $H : \ell = 0$ be a hyperplane in $\mathbb{P}^n$. We related the Lefschetz type properties of the map $\ell : M(f)_k \to M(f)_{k+1}$ induced by the multiplication by $\ell$ to the singularities of the hyperplane section $V \cap H$. We also show that a generic hypersurface $V \subset \mathbb{P}^n$ has at least one section $V \cap H$ containing $n$ ordinary double points in general position. Equivalently, the dual hypersurface $V^\vee$ has at least one normal crossing singularity of multiplicity $n$.

1. Introduction

Let $S = \mathbb{C}[x_0, \ldots, x_n]$ be the graded polynomial ring in $n+1$ variables with complex coefficients, with $n \geq 2$. Let $f \in S_d$ be a homogeneous polynomial such that the hypersurface $V(f) : f = 0$ in the projective space $\mathbb{P}^n$ is smooth. Then the Jacobian algebra $M(f) = S/J(f)$ is a standard graded Artinian Gorenstein algebra with socle degree $T = (n+1)(d-2)$, where we set

$$f_j = \frac{\partial f}{\partial x_j}$$

for $j = 0, \ldots, n$ and $J(f) = (f_0, \ldots, f_n) \subset S$ is the Jacobian ideal of $f$. In fact, the hypersurface $V(f)$ is smooth if and only if $M(f)$ is an Artinian algebra.

Definition 1.1. Let $M = \bigoplus_{i=0}^TM_i$ be an Artinian graded $\mathbb{C}$–algebra with $M_T \neq 0$.

1. The algebra $M$ is said to have the Weak Lefschetz Property in degree $i$, for $i < T$, for short WLP$_i$, if there exists an element $L \in M_1$ such that the multiplication map $L : M_i \to M_{i+1}$ is of maximal rank. We say that the algebra $M$ has WLP if it has WLP$_i$ for all $0 \leq i \leq T-1$.

2. We say that $M$ has the Strong Lefschetz Property in degree $k < T/2$, for short SLP$_k$, if there is $L \in M_1$ such that the linear map $LT^{-2k} : M_k \to M_{T-k}$ is an isomorphism. We say that the algebra $M$ has SLP if it has SLP$_k$ in degree $k$ for all $k < T/2$.

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Example 1.2. This example is due to Stanley [17] and Watanabe [19] and it is considered to be the starting point of the research area of Lefschetz properties for graded algebras. Consider the standard graded Artinian Gorenstein algebra

$$M = \mathbb{C}[x_0, \ldots, x_n] / (x_0^{a_0}, \ldots, x_n^{a_n})$$

with integers \(a_i > 0\), for all \(i = 0, \ldots, n\). Then \(M\) has SLP.

In particular, for a Fermat type polynomial

$$f = x_0^d + \ldots + x_n^d,$$

the Jacobian algebra \(M(f)\) has SLP. By obvious semi-continuity properties, it follows that the Jacobian algebra \(M(f)\) has SLP for a generic polynomial \(f \in S_d\). However, it seems that we have no control on the meaning of the word 'generic' in this claim. In fact, one has the following.

Conjecture 1.3. The Jacobian algebra \(M(f)\) has SLP for any polynomial \(f \in S_d\) such that \(V(f)\) is smooth.

Remark 1.4. We list here the known partial results related to this conjecture.

(1) When \(n = 2\), for any smooth curve \(V(f)\), the associated Jacobian algebra \(M(f)\) has WLP, as follows from the more general results in [14]. Moreover, when \(d = 2d'\) is even, then the multiplication by the square of a generic linear form \(\ell \in S_1\) induces an isomorphism

$$\ell^2 : M(f)_{3d'-4} \to M(f)_{3d'-2}.$$

In particular, when \(d = 4\), the Jacobian algebra \(M(f)\) has the SLP, see [8].

(2) When \(n = 3\), if \(V(f)\) be any smooth cubic surface in \(\mathbb{P}^3\), then \(M(f)\) has the SLP, see [8]. Moreover, for any smooth surface \(V(f)\), the SLP holds in degree 1, see for instance [1, Theorem B].

(3) For \(n = 4\), if \(V(f)\) be any smooth cubic 3-fold in \(\mathbb{P}^4\), then \(M(f)\) has the SLP, [1, Theorem C].

(4) For arbitrary dimension \(n\) and degree \(d\), one knows that \(M(f)\) has the WLP in degree \(\leq d - 2\), see [16]. The SLP also holds for obvious reasons. Indeed, if \(\ell^T \in J(f)\) for all \(\ell \in S_1\), then the hessian polynomial \(\text{hess}(f) \notin J(f)\) would have no Waring decomposition.

Remark 1.5. Using the duality of the Jacobian algebra \(M(f)\), in order to prove the WLP for \(M(f)\), it is enough to prove the WLP for \(i < T/2\).

In this note, we fix a smooth hypersurface \(V(f) : f = 0\) and a hyperplane \(H : \ell = 0\) in \(\mathbb{P}^n\), with \(n, d \geq 3\), and consider the hyperplane section \(V(f, \ell) = V(f) \cap H\). Under the assumption that \(V(f, \ell)\) is singular, we investigate the injectivity of the multiplication map \(\ell : M(f)_k \to M(f)_{k+1}\) for \(k < T/2\).

Remark 1.6. Note that the hyperplane section \(V(f, \ell)\) has only isolated singularities and, conversely, any hypersurface \(W \subset H\) with only isolated singularities may occur as a section \(V(f, \ell) = V(f) \cap H\) for a certain smooth hypersurface \(V(f)\), see [6, Proposition (11.6)].
Our first main result is stated in terms of some numerical invariants of the hyperplane section \( V(f, \ell) \), which we define now. Clearly \( V(f, \ell) \) is a hypersurface in \( H = \mathbb{P}^{n-1} \). If we choose a system of coordinates \( y = (y_1, \ldots, y_n) \) on \( H \), then \( V(f, \ell) \) given by an equation \( g = 0 \), hence it has a Jacobian ideal \( J(g) \) in the polynomial ring \( R = \mathbb{C}[y_1, \ldots, y_n] \). Let \( I(g) \) be the saturation of the Jacobian ideal \( J(g) \) with respect to the maximal ideal \( (y_1, \ldots, y_n) \) and let \( s(g) \) be the initial degree of the graded ideal \( I(g) \), namely
\[
(1.1) \quad s(g) = \min\{j \in \mathbb{N} : I(g)_j \neq 0\} \leq d - 1.
\]
Let \( g_j \) denote the partial derivative of \( g \) with respect to \( y_j \) and consider the graded \( R \)-module \( \text{Syz}(g) \) of first order syzygies of \( g_1, \ldots, g_n \), namely
\[
(1.2) \quad \text{Syz}(g) = \{a = (a_1, \ldots, a_n) \in R^n : a_1g_1 + \ldots + a_ng_n = 0\}.
\]
Let \( r(g) \) be the initial degree of the graded module \( \text{Syz}(g) \), namely
\[
(1.3) \quad r(g) = \min\{j \in \mathbb{N} : \text{Syz}(g)_j \neq 0\} \leq d - 1.
\]
It is clear that both invariants \( r(g) \) and \( s(g) \) do not depend on the choice of the linear coordinates \( y \) on \( H = \mathbb{P}^{n-1} \). Note that \( V(f, \ell) \) singular implies \( s(g) > 0 \). On the other hand, \( r(g) = 0 \) if and only if \( V(f, \ell) \) is a cone, i.e. one can choose the coordinates \( y \) such that \( g_1 = 0 \), hence \( g \) does not depend on one variable. With this notation, our first main result is the following improvement of the second author result recalled in Remark 1.4 (4).

**Theorem 1.7.** The multiplication map \( \ell : M(f)_k \to M(f)_{k+1} \) is injective for any
\[
k \leq \min\{d - 3 + r(g), d - 3 + s(g)\}.
\]

Our second main result is the following.

**Theorem 1.8.** For any dimension \( n \geq 3 \) and degree \( d \geq 3 \), a generic hypersurface \( V(f) \subset \mathbb{P}^n \) of degree \( d \) has at least one nodal hyperplane section \( V(f, \ell) = V(f) \cap H \), which has exactly \( n \) singularities in general position.

**Remark 1.9.** (i) The fact that a generic hypersurface has all the tangency in general position was established in [2]. This means that for a generic, smooth hypersurface \( V \subset \mathbb{P}^n \) and any hyperplane \( H \subset \mathbb{P}^n \), the singularities of the section \( V \cap H \) are points in general position in \( H \), i.e. the corresponding vectors in the vector space associated to \( H \) are linearly independent. In particular, there are at most \( n \) singularities in any such section \( V \cap H \) when \( V \) is generic.

(ii) The fact that a surface \( S \subset \mathbb{P}^3 \) of degree \( d \geq 3 \) admits tritangent planes \( H \) it is well known, and there are formulas for the number of these planes in terms of the degree \( d \), see for instance [18, Section (8.3)]. The fact that, for \( S \) generic of degree \( d \geq 5 \), the singularities of \( S \cap H \) are exactly 3 nodes follows from [20, Proposition 3]. For a generic hypersurface \( V \subset \mathbb{P}^n \), with \( n \geq 4 \) and \( d = \deg V \geq n + 2 \), it follows from [20, Proposition 4] that all the singularities of a hyperplane section \( V \cap H \) are double points (not necessarily \( A_1 \) singularities) in number at most \( n \).
In section 2 we prove Theorem 1.7. In the rest of the paper, we show that Theorem 1.7 can give the WLP (in some degrees or in all degrees) for some generic classes of hypersurfaces, with a precise meaning of the word ‘generic’, as well as for many new non generic classes of hypersurfaces. In section 3 we apply this result to surfaces $S$ in $\mathbb{P}^3$. The surfaces of degree 4 are considered in Example 3.1 and Example 3.5. The surfaces having a section $C = S \cap H$ which is a nodal curve and such that all irreducible components of $C$ are rational (resp. $C$ is a free or a nearly free curve) are considered in Proposition 3.2 (resp. Proposition 3.4). In section 4, we consider higher dimensional hypersurfaces $V$, having a nodal hyperplane section $Y = V \cap H$ with many singularities, e.g. $Y$ is a Kummer surface in Proposition 4.1, respectively a Chebyshev hypersurface in Proposition 4.2. In section 5 we prove first Theorem 1.8, and then we restate this result as a property of the dual hypersurface, see Theorem 5.2. Finally, application of Theorem 1.8 to WLP of Jacobian algebras are given in Corollary 5.3 and Corollary 5.4.

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2. The proof of Theorem 1.7

Without loss of generality, we may take $\ell = x_0$ and $y_j = x_j$ for all $j = 1, \ldots, n$. Then one may write

$$f(x_0, y) = g(y) + x_0 h(y) + x_0^2 p_2(y) + \ldots + x_0^d p_d(y),$$

where $p_j \in R_{d-j}$. Assume that $\ell : M(f)_k \to M(f)_{k+1}$ is not injective. Then there is a homogeneous polynomial $q \in S_k$ such that $q \notin J(f)_k$ and $x_0 q \in J(f)_{k+1}$.

It follows that one has a relation

$$x_0 q = b_0 f_0 + \ldots + b_n f_n,$$

with homogeneous polynomials $b_j \in S_{k+2-d}$, not all divisible by $x_0$. If we set $x_0 = 0$, this yields

$$(2.1) \quad c_0 h + c_1 g_1 + \ldots + c_n g_n = 0,$$

where not all the polynomials $c_j(y) = b_j(0, y) \in R_{k+2-d}$ are zero. There are two cases to discuss.

Case 1. $c_0 \neq 0$. Then note that the singular set of the hypersurface $V(f)$ on the hyperplane $x_0 = 0$ is given by the solutions of the system

$$h = g_1 = \ldots = g_n = 0.$$

Since $V(f)$ is smooth and $V(f, \ell)$ is assumed to be singular, it follows that $h$ does not vanish on the singular set of the section $V(f, \ell)$, which is given by the solutions of the system

$$g_1 = \ldots = g_n = 0.$$

This observation and the equation (2.1) imply that $c_0 \in I(g)$ and hence $k + 2 - d = \deg c_0 \geq s(g)$. In other words, we have $k \geq d - 2 + s(g)$. 

Case 2. $c_0 = 0$. Then the equation (2.1) becomes a non-zero homogeneous element $\rho \in \text{Syz}(g)$. It follows that $k + 2 - d = \deg \rho \geq r(g)$. In other words, we have $k \geq d - 2 + r(g)$.

This ends the proof of Theorem 1.7.

3. Surfaces of degree $d \geq 4$ in $\mathbb{P}^3$

In this case the plane section $V(f, \ell)$ is a plane curve of degree $d$, with isolated singularities. We discuss several possibilities.

Example 3.1. Let $V(f)$ be a smooth surface of degree 4 in $\mathbb{P}^3$ which contains no line, and let $H : \ell = 0$ be a plane such that the plane curve $V(f, \ell) = V(f) \cap H$ has

1. either at least 3 singular points with total Tjurina number $\tau(V(f, \ell))$ at most 5, or
2. a $D_4$-singularity (i.e. an ordinary triple point) and a node $A_1$.

Then clearly $s(g) \geq 2$, since any element in $I(g)$ vanishes at the singular points and $V(f, \ell)$ has no line components. Using the lower bound on $\tau(V(f, \ell))$ given in [13], we see that $r(g) \geq 2$. It follows from Theorem 1.7 that the multiplication map $\ell : M(f)_3 \to M(f)_4$ is injective. In this case $T = 8$ hence we conclude that the WLP holds for $M(f)$ when the quartic surface $V(f)$ satisfies the above conditions, in view of Remark 1.5. It is known that for any generic quartic surface $V$ in $\mathbb{P}^3$ there are plane sections $V \cap H$ which are rational curves with 3 nodes, see [5, Theorem 1.2], hence the above conditions hold for a generic quartic surface. Other quartic surfaces are discussed in Example 3.5.

Proposition 3.2. If the smooth surface $V(f) : f = 0$ admits a plane section $V(f, \ell)$ which is a nodal curve and all irreducible components of $V(f, \ell)$ are rational curves, then the corresponding Jacobian algebra $M(f)$ has the WLP.

Proof. Using [11, Theorem 4.1], it follows that $r(g) \geq d - 2$. On the other hand, we have $s(g) \geq d - 2$, using [4, Theorem 3.2]. It follows from Theorem 1.7 that the multiplication map $\ell : M(f)_3 \to M(f)_4$ is injective for $k < 2d - 4 = T/2$. We conclude as in Example 3.1 using Remark 1.5. □

Remark 3.3. We have already noticed in Example 3.1 that, for $d = 4$, any generic quartic surface in $\mathbb{P}^3$ admits a section which is a rational nodal curve, see [5, Theorem 1.2]. On the other hand, for $d \geq 5$, a generic degree $d$ surface in $\mathbb{P}^3$ does not admit a section which is a rational nodal curve. More precisely, in this case any irreducible component $C$ of a section $V(f, \ell)$ of a generic surface has geometric genus satisfying the inequality

\[ g(C) > \frac{d(d - 3)}{2} - 3, \]

see [20, Theorem 1]. In the same paper, Xu shows that the list of singularities on the section $V(f, \ell)$ is one of the following, see [20, Proposition 3].

1. $A_1, 2A_1, 3A_1$;
2. $A_2, A_1A_2$.
In other words, only list of singularities with total Tjurina number \( \tau(V(f, \ell)) \leq 3 \) may occur for a generic surface of degree \( d \geq 5 \). For a similar result, see also [2].

**Proposition 3.4.** If the smooth surface \( V(f) : f = 0 \) admits a plane section \( V(f, \ell) : g = 0 \) which is a free (resp. nearly free) curve with exponents \( (d_1, d_2) \) with \( d_1 \leq d_2 \), then the multiplication map \( \ell : M(f)_k \to M(f)_{k+1} \) is injective for any \( k \leq d + d_1 - 3 \).

**Proof.** For a free curve \( V(f, \ell) : g = 0 \) one has \( J(g) = I(g) \) and hence \( s(g) = d - 1 \). By definition one has \( r(g) = d_1 \) and it is known that \( 2d_1 \leq d - 1 \) which implies \( d_1 \leq d - 1 = s(g) \). This proves our claim for a free curve.

For a nearly free curve \( V(f, \ell) : g = 0 \) one has \( s(g) = \min\{d - 1, d + d_1 - 3\} \), see [12, Corollary 2.17]. By definition \( r(g) = d_1 \) and it is known that \( d_1 + d_2 = d \). It follows that \( d_1 \leq d - 1 \) and also \( d_1 \leq d + d_1 - 3 \) since we suppose \( d \geq 3 \). \( \Box \)

In view of the point (4) in Remark 1.4 the above result is useful only when \( d_1 \geq 2 \).

**Example 3.5.** Let \( V(f) \) be a smooth surface of degree 4 in \( \mathbb{P}^3 \) and let \( H : \ell = 0 \) be a plane such that the plane curve \( V(f, \ell) = V(f) \cap H \) has one of the following lists of singularities, not covered by the discussion in Example 3.1.

1. \( 3A_2 \)
2. \( A_2A_4 \)
3. \( A_6 \)
4. \( 6A_1 \)
5. \( A_1A_5 \)

In the first 3 cases the curve \( V(f, \ell) \) is irreducible and rational, while in case (4) the curve \( V(f, \ell) \) is a union of 4 lines in general position. In particular, the case (4) can be treated using also Proposition 3.2. In case (5), the curve \( V(f, \ell) \) is a union of two conics, meeting at 2 points, with intersection multiplicities 1, and respectively 3. All these curves are shown to be nearly free with exponents \( (2, 2) \), see [12, Example 2.13] and [9, Proposition 5.5]. It follows from Proposition 3.4 that the multiplication map \( \ell : M(f)_k \to M(f)_{k+1} \) is injective. In this case \( T = 8 \) hence we conclude that the WLP holds for \( M(f) \) when the quartic surface \( V(f) \) satisfies the above conditions, in view of Remark 1.5.

**Remark 3.6.** All the possibilities of the singularities of a plane section of a smooth quartic surface in \( \mathbb{P}^3 \) are listed in [3].

### 4. Hypersurfaces in \( \mathbb{P}^n \), \( n \geq 4 \), having sections with many nodes

In this section the hyperplane section \( V(f, \ell) \in H = \mathbb{P}^{n-1} \) is assumed to have only nodes, namely ordinary double points, also known as \( A_1 \)-singularities. In this case, it is known that

\[
(4.1) \quad r(g) = d - 1,
\]

when \( d \geq 3 \), see [7]. To get lower bounds on \( s(g) \) we impose a large number of singularities on the hyperplane section \( V(f, \ell) \). We discuss several cases.
Proposition 4.1. If the smooth 3-fold \( V(f) : f = 0 \) in \( \mathbb{P}^4 \) of degree 4 admits a hyperplane section \( V(f, \ell) : g = 0 \) that is a nodal surface, with at least 10 nodes, not all on a quadric, then the corresponding Jacobian algebra \( M(f) \) has the WLP. In particular, this occurs when the hyperplane section \( K = V(f, \ell) : g = 0 \) is a Kummer surface with 16 nodes.

Proof. Using the inequality (4.1), it follows that \( r(g) \geq 3 \). In the first claim, we assume that \( s(g) \geq 3 \). This claim follows now from Theorem 1.7 and Remark 1.5, since \( T = 10 \) in this case.

We show now that \( s(g) \geq 3 \) in the case of a section being a Kummer surface. Let \( X \) be the minimal resolution of \( K \). Denote by \( H' \) the pull-back on \( X \) of a plane section of \( K \) and let \( E_1, \ldots, E_{16} \) be the exceptional divisors. Assume there is a quadric \( Q \) in the hyperplane \( H = \mathbb{P}^3 \), passing through all the 16 nodes of \( K \). Then the pull-back on \( X \) of this quadric \( Q \) gives an effective divisor \( D \) in the linear system

\[
|2H' - \sum_{i=1}^{16} E_i|.
\]

On the other hand, there are 16 planes in \( H \), classically called tropes, each of them tangent to \( K \) along a conic passing through 6 of the singularities of \( K \), see [15, Chapter 1]. Hence the proper transform \( C_j \) of one of these conics satisfies

\[
2C_j = H' - \sum_{i \in I_j} E_i,
\]

with \(|I_j| = 6\). It follows that

\[
C_j \cdot D = \frac{1}{2} \left( H' - \sum_{i \in I_j} E_i \right) \cdot \left( 2H' - \sum_{i=1}^{16} E_i \right) = -2.
\]

Since \( C_j \) is irreducible, this implies that \( D \) contains the curve \( C_j \). Note that \( D \cdot H' = 8, C_j \cdot H' = 2 \) for all \( j = 1, \ldots, 16 \), hand hence \( D \) cannot contain all the conics \( C_j \). This contradiction shows that \( I(g)_2 = 0 \), and hence \( s(g) \geq 3 \).

\[\square\]

There is a family of nodal hypersurfaces \( X : g = 0 \), in any dimension \( n \) and degree \( d \), for which the subtle invariant \( s(g) \) is known. They are the Chebyshev hypersurfaces, see [10] for their definition. The equality

\[
(4.2) \quad s(g) = d - 2,
\]

follows from [10, Proposition 3.1].

Proposition 4.2. If the smooth hypersurface \( V(f) : f = 0 \) in \( \mathbb{P}^n \), \( n \geq 4 \), of degree \( d \geq 3 \) admits a hyperplane section \( V(f, \ell) : g = 0 \) which is a Chebyshev hypersurface, then the multiplication map \( \ell : M(f)_k \rightarrow M(f)_{k+1} \) is injective for any \( k \leq 2d - 5 \).

Proof. It is enough to use Theorem 1.7 and the formulas (4.1) and (4.2).
5. GENERIC HYPER_SURFACES OF DEGREE D IN $\mathbb{P}^n$, WITH D, n ≥ 3, HAVE SECTIONS WITH n NODES IN GENERAL POSITION

We consider first the projective space $\mathbb{P}^{n-1}$ and the subset $Z_n \subset \mathbb{P}^{n-1} = \mathbb{P}(\mathbb{C}^n)$ given by the classes $p_i$ of the canonical basis $e_i$, $i = 1, \ldots, n$ of the vector space $\mathbb{C}^n$.

**Proposition 5.1.** For any degree $d \geq 3$, there is a hypersurface $Y \subset \mathbb{P}^{n-1}$, with $n \geq 3$, of degree $d$ having as singularities $n$ nodes $A_1$, located at the points in $Z_n$.

**Proof.** Let $y = (y_1, \ldots, y_n)$ be the coordinates on $\mathbb{P}^{n-1}$. We consider first the case $d = 3$ and take $Y$ to be the hypersurface $g(y) = 0$, where

$$g(y) = \sum_{1 \leq i < j < k \leq n} y_i y_j y_k.$$  

It is easy to see that $Y$ has an $A_1$-singularity at each point $p_i$, for $i = 1, \ldots, n$. Now we show that there are no other singularities. Note that for the partial derivative $g_i$ of $g$ with respect to $y_i$ we have

$$g_i(y) = \sum_{1 \leq j < k \leq n, j \neq i, k \neq i} y_j y_k.$$  

Assume that $g_i(y) = 0$ for $i = 1, \ldots, n$ and take the sum of all these equations. We get in this way

$$\sum_{1 \leq j < k \leq n} y_j y_k = 0.$$  

Subtracting the equation (5.1) from (5.2) we get

$$y_i \sum_{1 \leq j < n, j \neq i} y_j = 0.$$  

If we assume that $y_{i_1} \neq 0$ and $y_{i_2} \neq 0$ for some indices $1 \leq i_1 < i_2 \leq n$, the equation (5.3) implies that $y_{i_1} = y_{i_2}$. Hence, for any singular point $y^0$ of $Y$, there is an integer $a$ with $1 \leq a \leq n$ such that $a$ coordinates of $y^0$ are equal to 1, and the remaining $n - a$ coordinates are 0. The equation (5.3) implies that only the case $a = 1$ is possible, and hence $y^0$ is one of the points $p_i$. This completes the proof in the case $d = 3$.

Next we look at the case $d = 4$ and take $Y$ to be the hypersurface $g(y) = 0$, where

$$g(y) = \sum_{1 \leq i < j \leq n} y_i^2 y_j^2.$$  

It is easy to see that $Y$ has an $A_1$-singularity at each point $p_i$, for $i = 1, \ldots, n$. Now we show that there are no other singularities. In this case we have

$$g_i(y) = 2y_i \sum_{1 \leq j \leq n, j \neq i} y_j^2.$$  

If we assume $g_i(y) = 0$ for all $i$, and that $y_{i_1} \neq 0$ and $y_{i_2} \neq 0$ for some indices $1 \leq i_1 < i_2 \leq n$, the equation (5.4) implies that $y_{i_1}^2 = y_{i_2}^2$. Hence, for any singular point $y^0$ of $Y$, there is an integer $a$ with $1 \leq a \leq n$ such that $a$ coordinates of $y^0$
are equal to ±1, and the remaining \( n - a \) coordinates are 0. The equation \((5.3)\) implies that only the case \( a = 1 \) is possible, and hence \( y^0 \) is one of the points \( p_i \). This completes the proof in the case \( d = 4 \).

Finally, to treat the case, let
\[
h_i(y) = y_i^{d-2} \sum_{1 \leq j \leq n, j \neq i} y_j^2.
\]

Note that \( h_i \) has a singularity of type \( A_1 \) at \( p_i \) and vanishes of order \( d - 2 \geq 2 \) at the other points \( p_j \), for \( j \neq i \). Consider the linear system spanned by \( h_1, \ldots, h_n \). It is easy to see that the base locus \( h_1 = \ldots = h_n = 0 \) of this linear system is exactly the set \( Z_n \). It follows, by Bertini’s Theorem, that a generic member \( Y \) of this linear system is smooth except possibly at the points of \( Z_n \). The choice of the \( h_i \) implies that \( Y \) has an \( A_1 \) singularity at each point in \( Z_n \).

Now we give a proof of Theorem 1.8 stated in the Introduction. Using Remark 1.6 and Proposition 5.1 it follows that, any dimension \( n \geq 3 \) and degree \( d \geq 3 \), there are smooth hypersurfaces \( V(f) \subset \mathbb{P}^n \) of degree \( d \) which have at least one nodal hyperplane section \( V(f, \ell) = V(f) \cap H \), with exactly \( n \) singularities in general position. Let \( B = \mathbb{P}(S_d)_0 \) be the set of classes \([f]\) of polynomials \( f \in S_d \) such that \( V(f) : f = 0 \) is a smooth, degree \( d \) hypersurface in \( \mathbb{P}^n \). Let \( \mathcal{A}(n, d) \subset B \) be the subset of such hypersurfaces, which have at least one nodal hyperplane section \( V(f, \ell) = V(f) \cap H \), with exactly \( n \) singularities in general position. We know already that \( \mathcal{A}(n, d) \neq \emptyset \). We have to show that \( \mathcal{A}(n, d) \) is dense in \( B \). It is easy to see that \( \mathcal{A}(n, d) \) is a constructible subset in \( B \). To show that \( \mathcal{A}(n, d) \) is dense in \( B \) it is enough to show that \( \mathcal{A}(n, d) \) contains an open subset of \( B \) in the strong complex topology. Assume that \([f] \in \mathcal{A}(n, d)\) and consider the corresponding dual mapping
\[
\phi_f : V(f) \to (\mathbb{P}^n)^\vee, \quad x \mapsto (f_0(x) : f_1(x) : \ldots : f_n(x)).
\]
Then the dual hypersurface
\[
V(f)^\vee = \phi_f(V(f))
\]
has a normal crossing singularity of multiplicity \( n \) at the point of the dual projective space \((\mathbb{P}^n)^\vee\) corresponding to \( H \), see for instance the equivalences (11.33) in [4]. The set \( B \) is open, hence there are arbitrarily small open neighborhood \( U \) with 
\([f] \in U \subset B \). For a polynomial \( f' \in S_d \), such that 
\([f'] \in U \), the corresponding dual mapping
\[
\phi_{f'} : V(f') \to (\mathbb{P}^n)^\vee,
\]
can be regarded as a small deformation of the dual mapping \( \phi_f \). It follows that the dual variety \( V(f')^\vee \) is a small deformation of the dual variety \( V(f)^\vee \). A normal crossing singularity is stable under small deformations, due to the fact that the analytic multi-germ
\[
\phi_f : (V(f), Z_f) \to ((\mathbb{P}^n)^\vee, H)
\]
is stable, where \( Z_f \) is the set of \( n \) nodes in \( V(f) \cap H \). Hence the dual variety \( V(f')^\vee \)
must have as well a normal crossing singularity of multiplicity \( n \), which corresponds
to a hyperplane section of $V(f')$ with $n$ nodes. This shows that $[f'] \in A(n, d)$ when $U$ is chosen small enough.

\[ \square \]

As we have seen in the proof above, Theorem 1.8 can be reformulated in the following equivalent way.

**Theorem 5.2.** For any dimension $n \geq 3$ and degree $d \geq 3$, the dual hypersurface $V(f)\vee$ of a generic hypersurface $V(f) \subset \mathbb{P}^n$ of degree $d$ has at least one normal crossing singularity of multiplicity $n$. 

Using the results of this section, we can improve by one, in the case of generic hypersurfaces, the result of the second author, mentioned above in Remark 1.4, (4).

**Corollary 5.3.** If the smooth hypersurface $V(f) : f = 0$ in $\mathbb{P}^n$, $n \geq 3$, of degree $d \geq 3$ admits a nodal hyperplane section $V(f, \ell) = V(f) \cap H$, which has exactly $n$ singularities in general position, then the multiplication map $\ell : M(f)_k \to M(f)_{k+1}$ is injective for any $k \leq d-1$. In particular, this property holds for a generic hypersurface $V(f) : f = 0$ in $\mathbb{P}^n$, $n \geq 3$, of degree $d \geq 3$.

**Proof.** It is enough to use Theorem 1.7, the formulas (4.1) and the obvious fact that $s(g) \geq 2$ in this situation.

\[ \square \]

The following is the special case of the previous result.

**Corollary 5.4.** If the smooth 4-fold $V(f) : f = 0$ in $\mathbb{P}^5$ of degree 3 admits a hyperplane section $V(f, \ell) = V \cap H : g = 0$ that is a nodal 3-fold, with at least 5 nodes, not all on a hyperplane in $H$, then the corresponding Jacobian algebra $M(f)$ has the WLP. In particular, this occurs for a generic 4-fold $V(f) : f = 0$ in $\mathbb{P}^5$ of degree 3 and for a smooth 4-fold $V(f) : f = 0$ in $\mathbb{P}^5$ of degree 3 having a hyperplane section $V(f, \ell) : g = 0$ which is a Segre 3-fold with 10 nodes.

**Proof.** Using the inequality (4.1), it follows that $r(g) \geq 2$. The assumptions in the first claim imply that $s(g) \geq 2$. This claim follows now from Theorem 1.7 and Remark 1.5 since $T = 6$ in this case. The fact that a this situation occurs for a generic 4-fold $V(f) : f = 0$ in $\mathbb{P}^5$ of degree 3 follows from Corollary 5.3.

We treat next the case of the Segre 3-fold section. The Segre 3-fold is unique up-to a projective transformation, and can be given by the equation

\[ X : g = x_1^3 + x_2^3 + x_3^2 + x_4^3 + x_5^3 - (x_1 + x_2 + x_3 + x_4 + x_5)^3 = 0, \]

where $x_1, x_2, x_3, x_4, x_5$ are the coordinates on $H = \mathbb{P}^4$. The 10 nodes are located at $(1 : 1 : 1 : -1 : -1)$ and the other 9 points obtained by permuting the coordinates. Using this description of the singular set of $X$, it follows that $I(g)_1 = 0$, and hence $s(g) \geq 2$. We conclude as for the first claim above.

\[ \square \]

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