GEOMETRY OF NONDEGENERATE POLYNOMIALS: MOTIVIC NEARBY CYCLES AND COHOMOLOGY OF CONTACT LOCI

LÊ QUY THUONG AND NGUYEN TAT THANG

Abstract. We study polynomials with complex coefficients which are nondegenerate in two senses, one of Kouchnirenko and the other with respect to its Newton polyhedron, through data on contact loci and motivic nearby cycles. Introducing an explicit description of these quantities we can answer in part to questions concerning the motivic nearby cycles of restriction functions and the integral identity conjecture in the context of Newton nondegenerate polynomials. Furthermore, in the nondegeneracy in the sense of Kouchnirenko, we give calculations on cohomology groups of the contact loci.

1. Introduction

Let $f$ be a nondegenerate $\mathbb{C}$-polynomial in the sense of Kouchnirenko (cf. Section 3.1) vanishing at the origin $O$ of $\mathbb{C}^d$. The problem of computing the motivic Milnor fiber $S_{f,O}$ in terms of the Newton polyhedron $\Gamma$ of $f$ was firstly mentioned by Guibert in 2002 (cf. [4]). Recently, Steenbrink and Bultot-Nicaise obtain solutions in terms of toric geometry ([15]), or of log smooth models ([2]). Their formula for $S_{f,O}$ allows to compute the Hodge spectrum of the singularity of $f$ at $O$ by means of the additivity of the Hodge spectrum operator. In this article, we will show that the formula also provides a way to explore the following problem for Newton nondegenerate polynomials.

Problem 1. Let $f$ be in $\mathbb{C}[x_1,\ldots,x_d]$ with $f(O) = 0$, and let $H$ be a hyperplane in $\mathbb{C}^d$. What is the relation between $S_{f,O}$ and $S_{f|_{H},O}$?

The question concerns a motivic analogue of a monodromy relation of a complex singularity and its restriction to a generic hyperplane studied early in [8]. For $n \in \mathbb{N}^*$, the $n$-iterated contact locus $X_{n,O}(f)$ admits a decomposition into $\mu_n$-invariant $\mathbb{C}$-subvarieties $X_{J,a}^{(n)}$ along $a \in \mathbb{N}_{\geq 0}$ and $J \subseteq [d] := \{1,\ldots,d\}$. The nondegeneracy of $f$ allows to describe $X_{J,a}^{(n)}$ via $\Gamma$, as in Theorem 3.1 which is the key step to compute the motivic zeta function $Z_{f,O}(T)$ and the motivic Milnor fiber $S_{f,O}$, which yields Theorem 3.2. For every face $\gamma$ of $\Gamma$, let $J_\gamma$ be the unique subset of $[d]$ such that $\gamma$ is contained in the hyperplanes $x_j = 0$ for all $j \notin J_\gamma$ and not contained in the other coordinate hyperplanes, and let $X_\gamma(0)$ (resp. $X_\gamma(1)$) be the $\mathbb{C}$-subvariety of $\mathbb{C}^d$ defined by the face function $f_\gamma$ (resp. $f_\gamma - 1$). Let $K$ be the set of all compact faces of $\Gamma$.

Theorem (see Theorem 3.2). Let $f$ be nondegenerate in the sense of Kouchnirenko such that $f(O) = 0$. Then the identity $S_{f,O} = \sum_{\gamma \in K} (-1)^{|J_\gamma|+1-\dim(\gamma)} ([X_\gamma(1)] - [X_\gamma(0)])$, holds true in the monodromic Grothendieck ring of $\mathbb{C}$-varieties endowed with $\mu$-action.

2000 Mathematics Subject Classification. Primary 14B05, 14B07, 14J17, 32S05, 32S30, 32S55.

Key words and phrases. arc spaces, contact loci, motivic zeta function, motivic Milnor fiber, motivic nearby cycles, Newton polyhedron, nondegeneracy, sheaf cohomology with compact support.
We choose the hyperplane defined by $x_d = 0$ to be $H$ in Problem 1 and consider for any $n \geq m$ in $\mathbb{N}$ the so-called $(n, m)$-iterated contact locus $X_{n,m,O}(f, x_d)$ of the pair $(f, x_d)$. It is a $\mu_n$-invariant $\mathbb{C}$-subvariety of $X_{n,O}(f)$. Then we show in this article that the formal series
\[ Z^\Delta_{f,x_d,O}(T) := \sum_{n \geq m \geq 1} [X_{n,m,O}(f, x_d)] \mathbb{L}^{-(n+m)d} T^n \]
is rational and it can be described via data of $\Gamma$. We put $S^\Delta_{f,x_d,O} := \lim_{T \to \infty} Z^\Delta_{f,x_d,O}(T)$. Using the description of $S^\Delta_{f,x_d,O}$ together with Theorem 3.2, a solution to Problem 1 for the nondegeneracy in the sense of Kouchnirenko can be realized as in the following theorem.

**Theorem** (part of Theorem 3.4). With $f$ as previous, the identity $S_{f,O} = S_{f,\mu,O} + S^\Delta_{f,x_d,O}$ holds in the monodromic Grothendieck ring of $\mathbb{C}$-varieties with $\mu$-action.

We also obtain a similar result on the motivic nearby cycles (Theorem 3.4). An important consequence of Theorems 3.1, 3.2 and 3.4 is a very elementary proof of the integral identity
\[ \gamma \wedge \gamma = \int \gamma \wedge \gamma \]
for Newton nondegenerate polynomials (Corollary 3.5). Here, we are interested in a smaller problem on cohomology groups of $X_{n,O}(f)$ (compare with [1, Theorem 1.1]).

**Problem 2.** Let $f$ be a polynomial over $\mathbb{C}$ vanishing at the origin $O$. Compute the cohomology groups with compact support $H^n_{\text{BM}}(X_{n,O}(f), \mathbb{C})$.

We devote Section 4 to study this problem for nondegenerate singularities in the sense of Kouchnirenko not only using sheaf cohomology with compact support but also the Borel-Moore homology $H^\text{BM}_{s}$. Write $X_{n,O}(f) = \bigcup_{(J,a) \in \mathcal{P}_n} X^{(n)}_{J,a}$, where $\mathcal{P}_n$ is defined right after (4.11). Let $\eta : \mathcal{P}_n \to \mathbb{Z}$ be the function defined by $\eta(J,a) = \dim_{\mathbb{C}} X^{(n)}_{J,a}$. We prove the following results:

**Theorem** (Theorems 4.3, 4.5). For $f$ as in Problem 2 and nondegenerate in the sense of Kouchnirenko, there exist spectral sequences
\[ E^2_{\mu,p,q} := \bigoplus_{\eta(J,a)=p} H^\text{BM}_{p+q}(X^{(n)}_{J,a}) \Rightarrow H^\text{BM}_{p+q}(X_{n,O}(f)), \]
\[ E^1_{\check{}p,q} := \bigoplus_{\eta(J,a)=p} H^\check{}_{p+q}(X^{(n)}_{J,a}, F) \Rightarrow H^\check{}_{p+q}(X_{n,O}(f), F), \]
for any sheaf of abelian groups $F$ on $X_{n,O}(f)$.

In particular, by applying the second spectral sequence with $F$ being a constant sheaf, we obtain a spectral sequence converging to the compact support cohomology groups of contact loci with complex coefficient whose first page is a direct sum of (singular) homology of $X_\gamma(0)$ and $X_\gamma(1)$ (see Corollary 4.5).

2. Preliminaries

2.1. **Monodromic Grothendieck ring of varieties.** Let $S$ be an algebraic $\mathbb{C}$-variety. Let $\text{Var}_S$ be the category of $S$-varieties, with objects being morphisms of algebraic $\mathbb{C}$-varieties $X \to S$ and a morphism in $\text{Var}_S$ from $X \to S$ to $Y \to S$ being a morphism of algebraic $\mathbb{C}$-varieties $X \to Y$ commuting with $X \to S$ and $Y \to S$. Denote by $\mu$ the limit of the projective system $\mu_{nm} \to \mu_n$ given by $x \mapsto x^m$, with $\mu_n = \text{Spec} \mathbb{C}[\xi]/(\xi^n - 1)$ the group scheme over $\mathbb{C}$ of $n$th roots of unity. An action of $\mu$ on a variety $X$ is an action of a group $\mu_n$ on $X$, and...
the action is good if every orbit is contained in an affine open subset of \( X \). By definition, an action of \( \hat{\mu} \) on an affine Zariski bundle \( X \to B \) is affine if it is a lifting of a good action on \( B \) and its restriction to all fibers is affine.

The Grothendieck group \( K^0_\hat{\mu}(\text{Var}_S) \) is defined to be an abelian group generated by symbols \([X \to S], X \) endowed with a good \( \hat{\mu} \)-action and \( X \to S \) in \( \text{Var}_S \), such that:

i) \([X \to S] = [Y \to S]\) if \( X \) and \( Y \) are \( \hat{\mu} \)-equivariant \( S \)-isomorphic;

ii) \([X \to S] = [Y \to S] + [X \setminus Y \to S]\) if \( Y \) is a \( \hat{\mu} \)-invariant closed subvariety in \( X \); and

iii) \([X \times A^1_C, \sigma] = [X \times \mathbb{A}^1_C, \sigma']\) if \( \sigma \) and \( \sigma' \) are liftings of the same \( \hat{\mu} \)-action on \( X \) to \( X \times \mathbb{A}^1_C \).

There is a natural ring structure on \( K^0_\hat{\mu}(\text{Var}_S) \) in which the product is induced by the fiber product over \( S \). The unit \( 1_S \) for the product is the class of the identity morphism \( S \to S \) with \( S \) endowed with trivial \( \hat{\mu} \)-action. Denote by \( \mathbb{L} \) (or \( \mathbb{L}_S \) the class of the trivial line bundle \( S \times \mathbb{A}^1 \to S \), and define the localized ring \( \mathcal{M}_\hat{\mu}^0 \) to be \( K^0_\hat{\mu}(\text{Var}_S)[\mathbb{L}^{-1}] \).

Let \( f : S \to S' \) be a morphism of algebraic \( C \)-variety. Then we have two important morphisms associated to \( f \), which are the ring homomorphism \( f^* : \mathcal{M}_\hat{\mu}^0 \to \mathcal{M}_\hat{\mu}^0 \) induced from the fiber product (the pullback morphism) and the \( \mathcal{M}_C \)-linear homomorphism \( f_* : \mathcal{M}_\hat{\mu}^0 \to \mathcal{M}_\hat{\mu}^0 \), defined by the composition with \( f \) (the push-forward morphism). When \( S' \) is \( \text{Spec} \mathbb{C} \), one usually writes \( f_* \) instead of \( f^* \).

### 2.2. Rational series and limit.

Let \( A \) be either \( \mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}] \) or \( \mathcal{M}_\hat{\mu} \) as a ring. Let \( A[[T]]_{sr} \) be the \( A \)-submodule of \( A[[T]] \) generated by 1 and by finite products of elements of the form \( \mathbb{L}^{aT^b} \) with \( (a, b) \in \mathbb{Z} \times \mathbb{N}_{>0} \). Each element of \( A[[T]]_{sr} \) is called a rational series. By \( \mathbb{R} \), there is a unique \( A \)-linear morphism \( \lim_{T \to \infty} : A[[T]]_{sr} \to A \) which sends \( \mathbb{L}^{aT^b} \) to \( -1 \).

During this article, we denote \([d] := \{1, \ldots, d\}, d \in \mathbb{N}^* \). For \( J \) contained in \([d] \), we denote by \( R^J_\geq \) the set of \((a_j)_{j \in J} \) with \( a_j \in \mathbb{R}_{\geq 0} \) for all \( j \in J \), and by \( R^J_0 \) the subset of \( R^J_\geq \) consisting of \((a_j)_{j \in J} \) with \( a_j > 0 \) for all \( j \in J \). Similarly, one can define the sets \( Z^J_\geq, Z^J_0 \) and \( N^J_\geq \). Let \( \sigma \) be a rational polyhedral convex cone in \( R^J_\geq \) and let \( \overrightarrow{\sigma} \) denote its closure in \( R^J_\geq \) with \( J \) a finite set. Let \( \ell \) and \( \ell' \) be two integer linear forms on \( Z^J \) positive on \( \overrightarrow{\sigma} \setminus \{(0, \ldots, 0)\} \). Lemma 2.1.5 in \( \mathbb{R} \) tells us that if \( \sigma \) is open in its linear span and \( \overrightarrow{\sigma} \) is generated by part of a \( \mathbb{Z} \)-basis of the \( \mathbb{Z} \)-module \( Z^J \), then the series

\[
S_{\sigma, \ell, \ell'}(T) := \sum_{a \in \sigma \cap N^J_\geq} \mathbb{L}^{-\ell(a)} T^{\ell(a)}
\]

is in \( \mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}]/[[T]]_{sr} \) and \( \lim_{T \to \infty} S_{\sigma, \ell, \ell'}(T) = (-1)^{\dim(\sigma)} \).

### 2.3. Motivic nearby cycles of regular functions.

For any \( C \)-variety \( X \), let \( \mathcal{L}_n(X) \) be the space of \( n \)-jets on \( X \), and \( \mathcal{L}(X) \) the arc space on \( X \), which is the limit of the projective system of spaces \( \mathcal{L}_n(X) \) and canonical morphisms \( \mathcal{L}_m(X) \to \mathcal{L}_n(X) \) for \( m \geq n \). The group \( \hat{\mu} \) acts on \( \mathcal{L}_n(X) \) via \( \mu_n \) in such a natural way that \( \xi \cdot \varphi(t) = \varphi(\xi t) \) for every \( \xi \in \mu_n \).

From now on, we assume that the \( C \)-variety \( X \) is smooth and of pure dimension \( d \). Consider a regular function \( f : X \to \mathbb{A}^1_C \), with the zero locus \( X_0 \). For \( n \geq 1 \) one defines the \( n \)-iterated contact locus of \( f \) as follows

\[
\mathcal{X}_n(f) = \{ \varphi \in \mathcal{L}_n(X) \mid f(\varphi) = t^n \mod t^{n+1} \}.
\]

Clearly, this variety is invariant by the \( \hat{\mu} \)-action on \( \mathcal{L}_n(X) \) and admits a morphism to \( X_0 \) given by \( \varphi(t) \mapsto \varphi(0) \), which defines an element \([\mathcal{X}_n(f)] := [\mathcal{X}_n(f) \to X_0] \) in \( \mathcal{M}^\mu_{X_0} \). We consider Denef-Loeser’s motivic zeta function \( Z_f(T) = \sum_{n \geq 1} [\mathcal{X}_n(f)] \mathbb{L}^{-ndT^n} \). They prove in \( \mathbb{R} \) that
Z_f(T) is in \( \mathcal{M}^\mathbb{R}_{X_0}[[T]]_{\text{sr}} \), and call the limit \( S_f := -\lim_{T \to \infty} Z_f(T) \) in \( \mathcal{M}^\mathbb{R}_{X_0} \) the **motivic nearby cycles of** \( f \). If \( x \) is a closed point of \( X_0 \), the \( \mathbb{C} \)-variety

\[
\mathcal{X}_{n,x}(f) = \left\{ \varphi \in \mathcal{L}_n(X) \mid f(\varphi) = t^n \mod t^{n+1}, \varphi(0) = x \right\},
\]

is also invariant by the \( \hat{\mu} \)-action on \( \mathcal{L}_n(X) \), called the **iterated contact locus of** \( f \) at \( x \). It is also proved that the zeta function \( Z_{f,x}(T) = \sum_{n \geq 1} [\mathcal{X}_{n,x}(f)] L^{-nd}T^n \) is in \( \mathcal{M}^\mathbb{C}[[T]]_{\text{sr}} \). The limit \( S_{f,x} = -\lim_{T \to \infty} Z_{f,x}(T) \) is called the **motivic Milnor fiber of** \( f \) at \( x \). Obviously, if \( \ell \) is the inclusion of \( \{x\} \) in \( X_0 \), then \( S_{f,x} = \ell^*S_f \) in \( \mathcal{M}^\mathbb{R}_{X_0} \).

We now modify slightly the motivic zeta functions of several functions in [4] and [6]. For a pair of regular functions \((f, g)\) on \( X \), we denote by \( X_0 := X_0(f, g) \) their common zero locus. For \( n \geq m \in \mathbb{N}^* \), we define

\[
\mathcal{X}_{n,m}(f, g) := \left\{ \varphi \in \mathcal{L}_n(X) \mid f(\varphi) = t^n \mod t^{n+1}, \text{ord}_t g(\varphi) = m \right\}.
\]

We can check that \( \mathcal{X}_{n,m}(f, g) \) is invariant under the natural \( \mu_n \)-action on \( \mathcal{L}_n(X) \), and that there is an obvious morphism of \( \mathbb{C} \)-varieties \( \mathcal{X}_{n,m}(f, g) \to X_0 \); from which we obtain the class \( [\mathcal{X}_{n,m}(f, g)] \) of that morphism in \( \mathcal{M}^\mathbb{R}_{X_0} \). Consider the series

\[
Z_{f,g}(T) := \sum_{n \geq m \geq 1} [\mathcal{X}_{n,m}(f, g)] L^{-nd}T^n
\]

in \( \mathcal{M}^\mathbb{R}_{X_0}[[T]] \). For any closed point \( x \in X_0 \), we can define \( Z_{f,g,x}^\Delta(T) \) as above with \( \mathcal{X}_{n,m}(f, g) \) replaced by its \( \mu_n \)-invariant subvariety \( \mathcal{X}_{n,m,x}(f, g) := \{ \varphi \in \mathcal{X}_{n,m}(f, g) \mid \varphi(0) = x \} \). The rationality of the series \( Z_{f,g}(T) \) and \( Z_{f,g,x}^\Delta(T) \) are stated in [4 Théorème 4.1.2] and [6 Section 2.9], up to the isomorphism of rings \( \mathcal{M}^\mathbb{C}_{X_0} \cong \mathcal{M}^\mathbb{C}_{X_0 \times \mathbb{C}^m} \) (see [5 Proposition 2.6]), where Guibert-Loeser-Merle’s result is done in the framework \( \mathcal{M}^\mathbb{C}_{X_0 \times \mathbb{C}^m} \). Put \( S_{f,g} := -\lim_{T \to \infty} Z_{f,g}(T) \) and \( S_{f,g,x}^\Delta := -\lim_{T \to \infty} Z_{f,g,x}^\Delta(T) \).

### 3. Motivic nearby cycles of a nondegenerate polynomial and applications

#### 3.1. Newton polyhedron of a polynomial

During this article, we use the symbol \([d]\) for the set \( \{1, \ldots, d\} \), for \( d \) in \( \mathbb{N}^* \). Let \( x = (x_1, \ldots, x_d) \) be a set of \( d \) variables, and let \( f(x) = \sum_{a \in [d]} c_a x^a \) be in \( \mathbb{C}[x] \) with \( f(O) = 0 \), where \( O \) is the origin of \( \mathbb{C}^d \). Let \( \Gamma \) be the Newton polyhedron of \( f \), i.e., the convex hull of the set \( \bigcup_{c_a \neq 0} (\alpha + \mathbb{R}^d_{\geq 0}) \) in \( \mathbb{R}^d_{\geq 0} \). Let \( F \), resp. \( K \), denote the set of all the faces, resp. the compact faces, of \( \Gamma \). For every face \( \gamma \) of \( \Gamma \) (not necessarily compact, the case \( \gamma = \Gamma \) included), we define by \( f_\gamma(x) = \sum_{a \in \gamma} c_a x^a \) the **face function** of \( f \) with respect to \( \gamma \). The polynomial \( f \) is called **nondegenerate on the face** \( \gamma \in F \) if \( f_\gamma \) has no singular point in \( \mathbb{C}^d_{m,\mathbb{C}} \). We say that \( f \) is **nondegenerate in the sense of Kouchnirenko** if it is nondegenerate on every compact face \( \gamma \in K \). If \( f \) is nondegenerate on every face of \( \Gamma \) (including non-compact faces, and \( \Gamma \) itself), we say that \( f \) is **nondegenerate in the sense of Newton polyhedron** or simply **Newton nondegenerate**. Consider the function \( \ell = \ell_\Gamma : \mathbb{R}^d_{\geq 0} \to \mathbb{R} \) which sends \( a \) in \( \mathbb{R}^d_{\geq 0} \) to \( \inf_{\alpha \in \Gamma} \langle a, b \rangle \), where \( \langle \cdot, \cdot \rangle \) is the standard inner product in \( \mathbb{R}^d \). For \( a \in \mathbb{R}^d_{\geq 0} \), we denote by \( \gamma_a \) the face of \( \Gamma \) to which the restriction of the function \( \langle a, \cdot \rangle \) gets its minimum, i.e., \( b \in \Gamma \) is in \( \gamma_a \) if and only if \( \langle a, b \rangle = \ell(a) \). Note that \( \gamma_a \) is a compact face if and only if \( a \) is in \( \mathbb{R}^d_{\geq 0} \). Moreover, \( \gamma_a = \Gamma \) when \( a = (0, \ldots, 0) \) in \( \mathbb{R}^d \), and \( \gamma_a \) is a proper face of \( \Gamma \) otherwise. For every proper face \( \gamma \) of \( \Gamma \), we define

\[
\sigma_\gamma := \sigma_{[d],\gamma} := \{ a \in \mathbb{R}^d_{\geq 0} \mid \gamma = \gamma_a \}.
\]
It is clear that $\sigma_\gamma$ is a cone of dimension $d - \dim(\gamma)$.

For any $J \subseteq [d]$, put $A_J^J := \text{Spec } (\mathbb{C}[(x_i)_{i \in J}])$ and $f^J := f|_{A_J^J}$. If $f$ is nondegenerate in the sense of Kouchnirenko (resp. Newton nondegenerate) then $f^J$ is also nondegenerate in the sense of Kouchnirenko (resp. Newton polyhedron). If $\gamma$ is a proper face of $\Gamma(f^J)$, we denote by $\sigma_{J,\gamma}$ the cone \{ $a \in \mathbb{R}_{\geq 0}^J \mid \gamma = \gamma_a$ \}, which has the dimension $|J| - \dim(\gamma)$.

### 3.2. Contact loci.

Let $f(x_1, \ldots, x_d)$ be as above. For $n \in \mathbb{N}^*$, $k \in \mathbb{N}$ and $J \subseteq [d]$, we denote by $\Delta_{J}^{(n,k)}$ the set of $a \in \{0, \ldots, n\}^J$ such that $\ell_j(a) + k = n$, where $\ell_j$ stands for $\ell_{\Gamma(J^J)}$. For $a \in \Delta_{J}^{(n,k)}$, put

$$X_{J,a}^{(n)} := \{ \varphi \in X_n(f) \mid \text{ord}_j x_j(\varphi) = a_j \forall j \in J, \ x_i(\varphi) \equiv 0 \forall i \not\in J \}.$$

This subvariety of $X_n(f)$ is invariant by the $\mu_n$-action given by $\xi \cdot \varphi(t) = \varphi(\xi t)$, and it defines an element $[X_{J,a}^{(n)}] := [X_{J,a}^{(n)}] \rightarrow X_0$ in $K_0(\text{Var}_{\mathbb{C}})$, where the structure map is given by $\varphi \mapsto \varphi(0)$. Let $P_n$ be the index set consisting of all such pairs $(J,a)$ such that

$$X_n(f) = \bigcup_{(J,a) \in P_n} X_{J,a}^{(n)}.$$  

Note that for every face $\gamma$ of $\Gamma$ (including $\Gamma$ itself), there exists a unique set $J_\gamma \subseteq [d]$ such that $\gamma$ is contained in the hyperplanes $x_j = 0$ for all $j \not\in J_\gamma$ and not contained in other coordinate hyperplanes. By this, the index set $P_n$ in (3.1) is nothing else than the set of pairs $(J_\gamma,a)$ such that $\gamma \in F$ and $a \in \bigcup_{k \in \mathbb{N}} (\sigma_{J_\gamma,\gamma} \cap \Delta_{J_\gamma}^{(n,k)})$.

In particular, if $f$ is Newton nondegenerate, the $\mathbb{C}$-subvariety $X_J(0)$ of $\mathbb{G}^J_{m,\mathbb{C}}$ defined by $f^J(x)$ is smooth for every $J \subseteq [d]$. Thus, we have a $\hat{\mu}$-equivariant Zariski locally trivial fibration $X_{J,a}^{(n)}(0,\ldots,0) \rightarrow X_J(0)$ with fiber $\mathbb{A}_{\mathbb{C}}^{(|J|-1)n}$ (proving this statement is part of the proof of Theorem 3.1 below).

For every face $\gamma \in F$, let us consider the $\mathbb{C}$-varieties

$$X_\gamma(1) := \{ x \in \mathbb{G}^{J_\gamma}_{m,\mathbb{C}} \mid f_\gamma(x) = 1 \} \quad \text{and} \quad X_\gamma(0) := \{ x \in \mathbb{G}^{J_\gamma}_{m,\mathbb{C}} \mid f_\gamma(x) = 0 \}.$$

We always consider the trivial action of $\hat{\mu}$ on the variety $X_\gamma(0)$. Let $a$ be in the relative interior rel.int. $\sigma_\gamma$ of the dual cone $\sigma_\gamma$ of $\gamma$, then $\gamma = \gamma_a$. If $\gamma_a$ is compact, the variety $X_\gamma(1)$ admits a natural $\mu_{\ell_{J_\gamma}(a)}$-action as follows

$$e^{2\pi ir/\ell_{J_\gamma}(a)} \cdot (x_j)_{j \in J_\gamma} := (e^{2\pi ir a_j/\ell_{J_\gamma}(a)} x_j)_{j \in J_\gamma},$$

for $r \in [\ell_{J_\gamma}(a)]$. Let $s = s_J$ denote the sum function: $s(a) = \sum_{j \in J} a_j$ for $a = (a_j)_{j \in J} \in \mathbb{R}^J$.

**Theorem 3.1.** Assume that $f \in \mathbb{C}[x]$ is nondegenerate on a face $\gamma \in F$. If $a \in \sigma_{J_\gamma,\gamma} \cap \Delta_{J_\gamma}^{(n,0)}$ (hence $n = \ell_{J_\gamma}(a)$), there is naturally a $\mu_n$-equivariant isomorphism of $\mathbb{C}$-varieties

$$\tau : X_{J_\gamma,a}^{(n)} \rightarrow X_\gamma(1) \times \mathbb{A}_{\mathbb{C}}^{[J_\gamma] \ell_{J_\gamma}(a) - s(a)}.$$

If $k \in \mathbb{N}^*$ and $a \in \sigma_{J_\gamma,\gamma} \cap \Delta_{J_\gamma}^{(n,k)}$, there is a Zariski locally trivial fibration

$$\pi : X_{J_\gamma,a}^{(n)} \rightarrow X_\gamma(0)$$

with fiber $\mathbb{A}_{\mathbb{C}}^{[J_\gamma] \ell_{J_\gamma}(a) + k - s(a) - k}$. 

As a consequence, the identities \( [X_{J, a}^{(n)}] = [X_{(1)}^{(n)}] L_{J} \mid_{J_{\gamma}^{(a)} - s(a)} \) for \( a \in \sigma_{J, \gamma} \cap \Delta_{J_{\gamma}}^{(n, 0)} \), and \( [X_{J, a}^{(n)}] = [X_{\gamma(0)}^{(n)}] L_{J} \mid_{J_{\gamma}^{(a) + k} - s(a) - k} \) for \( k \in \mathbb{N}^* \) and \( a \in \sigma_{J, \gamma} \cap \Delta_{J_{\gamma}}^{(n, k)} \) hold in \( \mathcal{M}_{C}^{a} \).

**Proof.** It suffices to prove the theorem for \( J_{\gamma} = [d] \). Let \( a = (a_{1}, \ldots, a_{d}) \) be in \( \sigma_{J, \gamma} \cap \Delta_{J_{\gamma}}^{(n, 0)} \), hence \( n = \ell(a) \) and \( \gamma = \gamma_{a} \). Every element \( \varphi \) in \( X_{\gamma, a}^{(n)} \) has the form \((\sum_{j=a_{1}}^{a_{d}} b_{ij} t^{j}) \) with \( b_{ia} \neq 0 \) for \( 1 \leq i \leq d \). The coefficient of \( t^{k} \) in \( f(\varphi(t)) \) is nothing but \( f_{\gamma_{a}}(b_{1a_{1}}, \ldots, b_{da_{d}}) \), thus \( (b_{1a_{1}}, \ldots, b_{da_{d}}) \) is in \( X_{\gamma_{a}}(1) \). We deduce that \( X_{\gamma, a}^{(n)} \) is \( \mu_{(a)} \)-equivariant isomorphic to \( X_{\gamma_{a}}(1) \times_{C} \mathbb{A}_{C}^{\ell(a) - s(a)} \) (where the group \( \mu_{(a)} \) acts trivially on \( \mathbb{A}_{C}^{\ell(a) - s(a)} \)) via the map
\[
\theta: \varphi(t) \mapsto \left((b_{ia})_{1 \leq i \leq d}, (b_{ij})_{1 \leq i \leq d, a_{i} < j \leq \ell(a)}\right).
\]
Indeed, for every \( \xi \) in \( \mu_{(a)} \), the element \( \varphi(\xi t) \) is sent to \( ((\xi b_{ia})_{1 \leq i \leq d}, (b_{ij})_{1 \leq i \leq d, a_{i} < j \leq \ell(a)}) \) which equals \( \xi \cdot ((b_{ia})_{1 \leq i \leq d}, (b_{ij})_{1 \leq i \leq d, a_{i} < j \leq \ell(a)}) \). Thus \( \theta \) is a \( \mu_{(a)} \)-equivariant isomorphism.

Now we prove the second statement. Let \( a \) be in \( \sigma_{J, \gamma} \cap \Delta_{J_{\gamma}}^{(n, k)} \) for \( k \in \mathbb{N}^* \), hence \( n = \ell(a) + k \) and \( \gamma = \gamma_{a} \). For \( \varphi \) in \( X_{\gamma, a}^{(n)} \), putting
\[
\widetilde{\varphi} := (t^{-a_{1}} x_{1}(\varphi), \ldots, t^{-a_{d}} x_{d}(\varphi)),
\]
we get
\[
f(\varphi) = t^{\ell(a)} f_{\gamma_{a}}(\varphi) + \sum_{k \geq 1} t^{\ell(a) + k} \sum_{(a, a) = \ell(a) + k} c_{\alpha} \varphi^{\alpha}.
\]
Defining
\[
\widetilde{f}(\widetilde{\varphi}, t) := f_{\gamma_{a}}(\varphi) + \sum_{k \geq 1} t^{k} \sum_{(a, a) = \ell(a) + k} c_{\alpha} \varphi^{\alpha},
\]
we obtain a function
\[
\widetilde{f}: \mathcal{L}_{\ell(a) + k + 1 - a_{1}}(\mathbb{A}_{C}^{1}) \times_{C} \cdots \times_{C} \mathcal{L}_{\ell(a) + k + 1 - a_{d}}(\mathbb{A}_{C}^{1}) \times_{C} \mathbb{A}_{C}^{1} \to \mathbb{A}_{C}^{1}
\]
given by \( \widetilde{f}(\widetilde{\varphi}, t_{0}) := \widetilde{f}(\widetilde{\varphi}(t_{0}), t_{0}) \). It thus follows from \( 3.3 \) that \( \varphi \) is in \( X_{\gamma, a}^{(n)} \) if and only if \( \widetilde{f}(\widetilde{\varphi}, t) = t^{k} \mod t^{k+1} \). Putting \( \widetilde{f}_{\gamma_{a}} = \sum_{j=0}^{\ell(a) - a_{i} + k} b_{ij} t^{j} \) for \( 1 \leq i \leq d \), the latter means that
\[
\begin{align*}
f_{\gamma_{a}}(b_{10}, \ldots, b_{d0}) &= 0 & \text{with } b_{0i} \neq 0 & \text{for } 1 \leq i \leq d, \\
q_{j}(b_{1j}, \ldots, b_{d_{0}j}) + p_{j}((b_{i}t^{j})'_{i}, j') &= 0 & \text{for } 1 \leq j \leq k - 1, \\
q_{k}(b_{1k}, \ldots, b_{d_{0}k}) + p_{k}((b_{i}t^{j})'_{i}, j') &= 1,
\end{align*}
\]
where \( p_{j} \), for \( 1 \leq j \leq k \), are polynomials in variables \( b'_{i} \) with \( i' \leq d \) and \( j' < j \), and
\[
q_{j}(b_{1j}, \ldots, b_{d_{0}j}) = \sum_{i=1}^{d_{0}} \frac{\partial f_{\gamma_{a}}}{\partial x_{i}}(b_{10}, \ldots, b_{d_{0}0}, 0, \ldots, 0) b_{ij}.
\]
Note that the function \( f \) does not depend on \( x_{i} \) for all \( i > d_{0} \).

We consider the morphism \( \pi: X_{\gamma, a}^{(\ell(a) + k)} \to X_{\gamma_{a}}(0) \) which sends the \( \varphi \) described previously to \( (b_{10}, \ldots, b_{d_{0}}) \). Since \( \mu \) acts trivially on \( X_{\gamma_{a}}(0) \), we only need to prove that \( \pi \) is a locally trivial fibration with fiber \( \mathbb{A}_{C}^{\ell(a) - s(a) - k} \). For every \( 1 \leq i \leq d_{0} \), we put
\[
(3.4)
U_{i} := \left\{ (x_{1}, \ldots, x_{d}) \in X_{\gamma_{a}}(0) \mid \frac{\partial f_{\gamma_{a}}}{\partial x_{i}}(x_{1}, \ldots, x_{d}) \neq 0 \right\}.
\]
The nondegeneracy of $f$ on the face $\gamma = \gamma_a$ gives us an open covering $\{U_1, \ldots, U_d\}$ of $X_\gamma(0)$. We construct trivializations of $\pi$ as follows

$$\pi^{-1}(U_i) \xrightarrow{\Phi_{U_i}} U_i \times_C \mathbb{A}^d\mathbb{C}$$

where $e = \sum_{i=1}^d (\ell(a) - a_i + k) - k$ and we identify $\mathbb{A}^d\mathbb{C}$ with the subvariety of $\mathbb{A}^d\mathbb{C}$ defined by the equations $b_{ij} = 0$ for $1 \leq j \leq k - 1$ and $b_{ik} = 1$ in the coordinate system $(\tilde{b}_{ij})$, and for $\varphi$ as previous,

$$\Phi_{U_i}(\varphi) = (\tilde{b}_{10}, \ldots, \tilde{b}_{d0}), (\tilde{b}_{ij})_{1 \leq i \leq d, 1 \leq j \leq \ell(a) - a_i + k),$$

with $\tilde{b}_{ij} = 0$ if $1 \leq j \leq k - 1$, $\tilde{b}_{ik} = 1$, and $\tilde{b}_{ij} = b_{ij}$ otherwise. Furthermore, the inverse map $\Phi_{U_i}^{-1}$ of $\Phi_{U_i}$ is also a regular morphism given explicitly as follows

$$\Phi_{U_i}^{-1}(\tilde{b}_{ij}) = \left( \frac{\ell(a) - a_{i1} + k}{\sum_{j=0}^{\ell(a) - a_{i1} + k} b_{ij} t^{j+a_{i1}}}, \ldots, \frac{\ell(a) - a_{id} + k}{\sum_{j=0}^{\ell(a) - a_{id} + k} b_{lj} t^{j+a_{id}}} \right),$$

where $b_{ij} = \tilde{b}_{ij}$ for either that $l \neq i$ or that $l = i$ and $k < j \leq \ell(a) - a_i + k$, and

$$b_{ij} = -p_{j}(b_{ij'})_{l \leq d, i' < j} - \sum_{l \leq d, i \neq i'} (\partial f_{\gamma_a}/\partial x_i)(\tilde{b}_{10}, \ldots, \tilde{b}_{d0})\tilde{b}_{ij},$$

for $1 \leq j \leq k - 1$, and

$$b_{ik} = \frac{1 - p_{k}(b_{ij'})_{l \leq d, i' < k} - \sum_{l \leq d, i \neq i'} (\partial f_{\gamma_a}/\partial x_i)(\tilde{b}_{10}, \ldots, \tilde{b}_{d0})\tilde{b}_{ik}}{(\partial f_{\gamma_a}/\partial x_i)(\tilde{b}_{10}, \ldots, \tilde{b}_{d0})}.$$

This proves that $\pi$ is a locally trivial fibration with fiber $\mathbb{A}^d\mathbb{C}\mathbb{C}$.

3.3. Motivic nearby cycles. We still use the notation introduced previously. In particular, we denotes by $F$ (resp. $K$) the set of all the faces (resp. compact faces) of $\Gamma$. For $0 \leq m \leq d$, we denote by $F(m)$ the set of $\gamma \in F$ such that if $\gamma = \gamma_a$ then $a_i \geq 1$ for all $i \in J_\gamma \cap [m + 1, d]$. Note that $F(0) = K$.

**Theorem 3.2.** Let $f \in \mathbb{C}[x_1, \ldots, x_d]$, and let $d_1, d_2 \in \mathbb{N}$ such that $d = d_1 + d_2$. The below identities hold in $M_\mathbb{C}^d$.

(i) If $f$ is Newton nondegenerate and if there is a closed immersion $\iota : \mathbb{A}^{d_1}\mathbb{C} \hookrightarrow X_\gamma$, then

$$\int_{\mathbb{A}^{d_1}\mathbb{C}} \iota^* S_f = \sum_{\gamma \in F(d_1)} (-1)^{|J_\gamma| + 1 - \dim(\gamma)} ([X_\gamma(1)] - [X_\gamma(0)]).$$

(ii) If $f$ is nondegenerate in the sense of Kouchnirenko and $f(O) = 0$, then

$$S_{f,O} = \sum_{\gamma \in K} (-1)^{|J_\gamma| + 1 - \dim(\gamma)} ([X_\gamma(1)] - [X_\gamma(0)]).$$
proved in [3], we see that runs over the pairs \((d_i, a)\) in (i) runs for \(f\).

By the decomposition (3.1) and by Theorem 3.1, we have

\[
\int_{\mathbb{A}^d} t^* [X_n(f)] L^{-nd} = \sum_{\gamma \in F(d_1)} \left( \sum_{a \in \sigma_{J_{d_1}}} [X_{\gamma}(1)] L^{[J_{\gamma}|-d|]} [\langle J_{\gamma},(a) \rangle - s(a)] + \sum_{a \in \sigma_{J_{d_1}}} [X_{\gamma}(0)] L^{[J_{\gamma}|-d|]} [\langle J_{\gamma},(a) \rangle + k - s(a) - k] \right),
\]

where the sum \(\sum_{\text{=}a}\) runs over the pairs \((J_{\gamma}, a)\) with \(a \in \sigma_{J_{\gamma}} \cap \Delta_{J_{\gamma}}^{(n,0)}\), and the sum \(\sum_{\langle\rangle a}\) runs over the pairs \((J_{\gamma}, a)\) with \(a \in \sigma_{J_{\gamma}} \cap \Delta_{J_{\gamma}}^{(n,k)}\) for \(k \geq 1\). By the rationality of \(Z_f(T)\) proved in [3], we see that \(\int_{\mathbb{A}^d} t^*\) commutes with the sum \(\sum_{n \geq 1}\) in \(Z_f(T)\), thus

\[
\int_{\mathbb{A}^d} t^* Z_f(T) = \sum_{\gamma \in F(d_1)} \left( [X_{\gamma}(1)] + [X_{\gamma}(0)] \frac{L^{[J_{\gamma}|-d|]}}{1 - L^{[J_{\gamma}|-d|]}} \sum_{a \in \sigma_{J_{d_1}}} L^{-s(a)} (L^{[J_{\gamma}|-d|]} \langle J_{\gamma},(a) \rangle) \right) .
\]

By [4 Lemme 2.1.5.], we have

\[
\lim_{T \to \infty} \sum_{a \in \sigma_{J_{\gamma}} \cap \Delta_{J_{\gamma}}^{(n,0)}} L^{-s(a)} (L^{[J_{\gamma}|-d|]} \langle J_{\gamma},(a) \rangle) = (-1)^{\dim(\sigma_{J_{\gamma}}, \gamma)} = (-1)^{[J_{\gamma}|-\dim(\gamma]},
\]

from which

\[
\int_{\mathbb{A}^d} t^* S_f = -\lim_{T \to \infty} \int_{\mathbb{A}^d} t^* Z_f(T) = \sum_{\gamma \in F(d_1)} (-1)^{[J_{\gamma}]+1-\dim(\gamma)} (\langle X_{\gamma}(1) \rangle - [X_{\gamma}(0)]) .
\]

(ii) The study is local at \(O\), so we only work with all \(a \in \sigma_{J_{\gamma}} \cap \Delta_{J_{\gamma}}^{(n,k)}\) for \(\gamma \in K\), i.e. we only need the condition that \(f\) is nondegenerate in the sense of Kouchnirenko. Then the argument in (i) runs for \(d_1 = 0\), thus (ii) follows.

\(\square\)

**Remark 3.3.** This result revisits Guibert’s work in [4 Section 2.1] for Newton nondegenerate polynomials \(f\) in a more general setting. Indeed, in [4] Guibert requires \(f\) to have the form \(\sum_{\nu \in \mathbb{N}^d} a_\nu x^\nu\), while we do not. Recently, Bultot-Nicaise in [2] Theorems 7.3.2, 7.3.5] provide a new approach to the motivic zeta functions \(Z_f(T)\) and \(Z_{f,O}(T)\) using log smooth models.

We now consider the relation between the motivic nearby cycles of \(f\) and that of a restriction of \(f\). We write \(\tilde{f}\) for \(f^{[d-1]}\), i.e., \(\tilde{f}(x_1, . . . , x_{d-1}) = f(x_1, . . . , x_{d-1}, 0)\), and write \(\tilde{O}\) for the origin of \(\mathbb{C}^{d-1}\). Let \(\tilde{X}_0\) be the zero locus of \(\tilde{f}\), which can be embedded into \(X_0\). The following theorem may be partially considered as a motivic analogue of D. T. Lé’s work on a monodromy relation of a complex singularity and its restriction to a generic hyperplane (see [8]).

**Theorem 3.4.** Let \(f \in \mathbb{C}[x_1, . . . , x_d]\), and let \(d_1, d_2 \in \mathbb{N}\) such that \(d = d_1 + d_2\). The below identities hold in \(\mathcal{M}_{\mathbb{C}}^d\).

(i) Suppose that \(f\) is Newton nondegenerate and that \(\mathbb{A}^{d_1}_{\mathbb{C}}\) is embedded in \(\tilde{X}_0 \subseteq X_0\). Denote by \(\iota\) the inclusion of \(\mathbb{A}^{d_1}_{\mathbb{C}}\) in both \(X_0\) and \(\tilde{X}_0\). Then

\[
\int_{\mathbb{A}^d} \iota^* S_f = \int_{\mathbb{A}^{d_1}} \iota^* S_{\tilde{f}} + \int_{\mathbb{A}^{d_2}} \iota^* S_{f,x_0} .
\]

(ii) If \(f\) is nondegenerate in the sense of Kouchnirenko and \(f(O) = 0\), then

\[
S_{f,O} = S_{\tilde{f},\tilde{O}} + S_{f,x_0} .
\]
Proof. As in the proof of Theorem 3.2, the proof method for (ii) is the same as for (i) but with \( d_1 = 0 \). So, we are only going to prove (i). By the definition of \((n, m)\)-iterated contact loci, we have

\[
\mathcal{X}_{n,m,O}(f, xd) = \bigcup_{(J_\gamma, a) \in P_{n,m}} \mathcal{X}_{J_\gamma,a}^{(n)}.
\]

We deduce from Section 2.3 and the method in the proof of Theorem 3.2 that \( \int_{\mathbb{A}_C^d} i^* Z_{f,x}^T(T) \) is equal in \( \mathcal{M}_{C}^\mu[[T]] \) to

\[
\sum_{\gamma \in F(d_1)} \sum_{a \in \sigma_{J_\gamma,\gamma}} \left[ \mathcal{X}_{J_\gamma,a}^{(\ell_\gamma(a))} \right] \sum_{\ell_\gamma(a) \geq a_d \geq 1} \mathbb{L}^{-d \ell_\gamma(a)} \mathbb{T}^{\ell_\gamma(a)}
\]

plus

\[
\sum_{\gamma \in F(d_1)} \sum_{a \in \sigma_{J_\gamma,\gamma}} \left[ \mathcal{X}_{J_\gamma,a}^{(\ell_\gamma(a)+k)} \right] \sum_{\ell_\gamma(a)+k \geq a_d \geq 1, k \geq 1} \mathbb{L}^{-d \ell_\gamma(a)+k} \mathbb{T}^{\ell_\gamma(a)+k}.
\]

We apply Theorem 3.1 to \( \gamma \in F(d_1) \) and \( a \in \sigma_{J_\gamma,\gamma} \). If \( d \in J_\gamma \), then \( \ell_\gamma(a) + k \geq a_d \geq 1 \) automatically for any \( k \in \mathbb{N} \). If \( d \notin J_\gamma \), then the inequalities \( \ell_\gamma(a) + k \geq a_d \geq 1 \) is in the situation of \( \mathcal{N} \), Lemma 2.10], in which the corresponding series has the limit zero. Thus

\[
\int_{\mathbb{A}_C^d} i^* S_{f,x}^T = \int_{\mathbb{A}_C^d} i^* S_{f,x}^T = \sum_{\gamma \in F(d_1), d \in J_\gamma} (-1)^{|J_\gamma|+1-\dim(\gamma)} \left( [X_\gamma(1)] - [X_\gamma(0)] \right),
\]

from which, by Theorem 3.2

\[
\int_{\mathbb{A}_C^d} i^* S_f = \sum_{\gamma \in F(d_1), d \notin J_\gamma} (-1)^{|J_\gamma|+1-\dim(\gamma)} \left( [X_\gamma(1)] - [X_\gamma(0)] \right) + \int_{\mathbb{A}_C^d} i^* S_{f,x}^T.
\]

The condition \( d \notin J_\gamma \) means that \( J_\gamma \subseteq [d-1] \), hence the first sum \( \sum_{\gamma \in K, d \notin J_\gamma} \) in the above decomposition of \( S_{f,O} \) is nothing but \( S_{f,O} \), again by Theorem 3.2.

To state and prove the below corollary, we denote by \( O_l \) the origin of the affine space \( \mathbb{C}^d_l \) for \( 1 \leq l \leq 3 \).

**Corollary 3.5** (Integral identity conjecture). Let \((x, y, z)\) be the standard coordinates of the affine space \( \mathbb{C}^d_l = \mathbb{C}^d_l \times \mathbb{C}^{d_2} \times \mathbb{C}^{d_3} \). Let \( f \) be a Newton nondegenerate polynomial in \( \mathbb{C}[x, y, z] \) such that \( f(0) = 0 \) and \( f(\lambda x, \lambda^{-1} y, z) = f(x, y, z) \) for all \( \lambda \) in \( \mathbb{C}^* \). Then the integral identity \( \int_{\mathbb{A}_C^d} i^* S_f = \mathbb{L}^{d_1} S_{f,0 \times \mathbb{A}_C^d} \) holds in \( \mathcal{M}_{C}^\mu \), with \( i \) the inclusion of \( \mathbb{A}_C^d \) in \( X_0 \).

**Proof.** Put \( f_0 := f \); and for \( 1 \leq j \leq d_2 \), let \( f_j \) be the restriction of \( f_{j-1} \) to \( y_j^{-1}(0) \). For abuse of notation we use \( i \) commonly for the inclusions of \( \mathbb{A}_C^d \) in the varieties \( f_j^{-1}(0) \). By applying Theorem 3.4 (several times for part (i)) we have

\[
\int_{\mathbb{A}_C^d} i^* S_f = \int_{\mathbb{A}_C^d} i^* S_{f_0} + \sum_{j=1}^{d_2} \int_{\mathbb{A}_C^d} i^* S_{f_{j-1},y_j}.
\]

The hypothesis on \( f \) implies that \( f(x, 0, z) = f(0, 0, z) \), hence

\[
\int_{\mathbb{A}_C^d} i^* S_{f_0} = \mathbb{L}^{d_1} S_{f_0 \times \mathbb{A}_C^d}.
\]
It thus remains to prove that $\int_{K^d_{C}} t^*S_{f,j-1,y} = 0$ in $\mathcal{M}_{C}^{d}$, for every $1 \leq j \leq d_{2}$, and it suffices to check this with $j = 1$. Using the proof of Theorem 3.4 we have

\begin{equation}
(3.5) \quad \int_{K^d_{C}} t^*S_{f,y} = \sum_{\gamma \in F(d_{1}), d_{1} + 1 \in J_{\gamma}} (-1)^{|J_{\gamma}| + 1 - \dim(\gamma)} ([X_{\gamma}(1)] - [X_{\gamma}(0)]).
\end{equation}

For all $\gamma \in F(d_{1})$ with $J_{\gamma}$ containing $d_{1} + 1$, write $J_{\gamma} = J_{1} \sqcup J_{2} \sqcup J_{3}$ with $J_{1} \subseteq [d_{1}], J_{2} \subseteq [d_{1} + [d_{2}]]$, and $J_{3} \subseteq [d_{1} + d_{2} + [d_{3}]]$. Observe that $J_{2}$ is nonempty because $d_{1} + 1 \in J_{2}$, it thus implies from the hypothesis on $f$ that $J_{1}$ is also nonempty. Write $\gamma = \tilde{\gamma} + \mathbb{R}_{\geq 0}$ with $\tilde{\gamma}$ a compact face, then $I \subseteq [d_{1}], J_{\gamma} = J_{\tilde{\gamma}}$ and $\dim(\gamma) = |I| + \dim(\tilde{\gamma})$. Since $f(\lambda x, \lambda^{-1} y, z) = f(x, y, z)$ for all $\lambda \in \mathbb{C}^{*}$, we can write $f$ in the form

$$f(x, y, z) = \sum_{s(\alpha) = s(\beta)} a_{\alpha, \beta, \gamma} x^{\alpha} y^{\beta} z^{\gamma}.$$ 

Since the hyperplane $\{ s(\alpha) = s(\beta) \}$ in $\mathbb{R}_{d_{0}}^{d}$ goes through the origin, its intersection with $\partial \Gamma$ has codimension at least 2. This together with $J_{1} \neq \emptyset$ implies that $I \neq \emptyset$. Using Lemmas 3.2, 3.3, 4.7 in [9] we can rewrite (3.5) as follows

$$\int_{K^d_{C}} t^*S_{f,y} = \sum_{\tilde{\gamma} \in K} (-1)^{|J_{\tilde{\gamma}}| + 1 - \dim(\tilde{\gamma})} ([X_{\tilde{\gamma}}(1)] - [X_{\tilde{\gamma}}(0)]) \sum_{I \subseteq M_{\tilde{\gamma}}} (-1)^{|I|},$$

in which there exists for each $\tilde{\gamma} \in K$ a unique $M_{\tilde{\gamma}} \neq \emptyset$ such that the previous identity holds. Since $\sum_{I \subseteq M_{\tilde{\gamma}}} (-1)^{|I|} = 0$, we get $\int_{K^d_{C}} t^*S_{f,y} = 0$ in $\mathcal{M}_{C}^{d}$. The corollary has been proved. \(\Box\)

**Remark 3.6.** In fact, the integral identity conjecture states for formal series. It plays a crucial role in Kontsevich-Soibelman’s theory of motivic Donaldson-Thomas invariants for noncommutative Calabi-Yau threefolds (see [7]). We refer to [9], [10], [14], [12] and [11] for proofs of different versions of the conjecture.

4. **Cohomology groups of contact loci of nondegenerate singularities**

As before, let $f$ be in $\mathbb{C}[x_{1}, \ldots, x_{d}]$ which vanishes at $O$. In this section, we always assume that $f$ is nondegenerate in the sense of Kouchnirenko (say for short that $f$ is nondegenerate).

4.1. **Borel-Moore homology groups of contact loci.** Consider the following decomposition of $\chi_{n,O}(f)$, which is the local version of (3.1),

\begin{equation}
(4.1) \quad \chi_{n,O}(f) = \bigcup_{(J,a) \in \overline{P}_{n}} \chi_{J,a}^{(n)},
\end{equation}

where $\overline{P}_{n}$ is the set of all the pairs $(J, a)$ such that $\gamma \in K$ and $a \in \bigcup_{k \in \mathbb{N}} \left( \sigma_{J_{\gamma}, \gamma} \cap \Delta_{J_{\gamma}}^{(n,k)} \right)$. Remark that, as $\gamma \in K$, we have $\sigma_{J_{\gamma}, \gamma} \cap \Delta_{J_{\gamma}}^{(n,k)} = (\mathbb{N}^{*})_{J_{\gamma}} \cap \sigma_{J_{\gamma}, \gamma} \cap \Delta_{J_{\gamma}}^{(n,k)}$. We consider an ordering in $\overline{P}_{n}$ as follows: for $(J, a)$ and $(J', a')$ in $\overline{P}_{n}$, $(J', a') \leq (J, a)$ if and only if $J' \subseteq J$ and $a_{i} \leq a'_{i}$ for all $i \in J'$, where $a = (a_{i})_{i \in J}$ and $a' = (a'_{i})_{i \in J'}$. The following lemma is straightforward.

**Lemma 4.1.** Let $n$ be in $\mathbb{N}^{*}$. For all $(J, a)$ and $(J', a')$ in $\overline{P}_{n}$, the following are equivalent:

(i) $(J', a') \leq (J, a)$,

(ii) $\chi_{J', a'}^{(n)} \subseteq \chi_{J,a}^{(n)}$,

(iii) $\chi_{J', a'}^{(n)} \cap \chi_{J,a}^{(n)} \neq \emptyset$,

the closure taken in the usual topology. Consequently, $\overline{\chi_{J,a}^{(n)}} = \bigcup_{(J', a') \leq (J, a)} \chi_{J', a'}^{(n)}$ for $(J, a) \in \overline{P}_{n}$. 


Consider the function \( \eta : \tilde{\mathcal{P}}_n \to \mathbb{Z} \) given by \( \eta(J, a) = \dim_{\mathbb{C}} \mathcal{X}^{(n)}_{J, a} \), for every \( n \in \mathbb{N}^* \). Put
\[
S_p := \bigcup_{(J, a) \in \tilde{\mathcal{P}}_n, \eta(J, a) \leq p} \mathcal{X}^{(n)}_{J, a},
\]
for \( p \in \mathbb{N} \). The below is a property of \( \eta \) and \( S_p \)'s.

**Lemma 4.2.** Let \( n \) be in \( \mathbb{N}^* \).

(i) If \((J', a') \leq (J, a)\) in \( \tilde{\mathcal{P}}_n \), then \( \eta(J', a') \leq \eta(J, a) \).

(ii) For all \( p \in \mathbb{N} \), \( S_p \) are closed and \( S_p \subseteq S_{p+1} \). As a consequence, there is a filtration of \( \mathcal{X}_{n, O}(f) \) by closed subspaces:
\[
\mathcal{X}_{n, O}(f) = S_{d_0} \supseteq S_{d_0-1} \supseteq \cdots \supseteq S_{-1} = \emptyset,
\]
where \( d_0 \) denotes the \( \mathbb{C} \)-dimension of \( \mathcal{X}_{n, O}(f) \).

**Proof.** The first statement (i) is trivial. To prove (ii) we take the closure of \( S_p \); then using Lemma 4.1 we get
\[
\overline{S}_p = \bigcup_{\eta(J, a) \leq p} \overline{\mathcal{X}^{(n)}_{J, a}} = \bigcup_{\eta(J, a) \leq p} \bigcup_{(J', a') \leq (J, a)} \mathcal{X}^{(n)}_{J', a'}.
\]
This decomposition combined with (i) implies that \( \overline{S}_p \subseteq S_p \), which proves that \( S_p \) is a closed subspace. The remaining statements of (ii) are trivial. \( \square \)

A main result of this section is the following theorem. To express the result, we work with the Borel-Moore homology \( H^\text{BM}_p \).

**Theorem 4.3.** Let \( f \in \mathbb{C}[x_1, \ldots, x_d] \) be nondegenerate, \( n \in \mathbb{N}^* \) and \( f(O) = 0 \). Then there is a spectral sequence
\[
E^1_{p, q} := \bigoplus_{(J, a) \in \tilde{\mathcal{P}}_n, \eta(J, a) = p} H^\text{BM}_{p+q}(\mathcal{X}^{(n)}_{J, a}) \Rightarrow H^\text{BM}_{p+q}(\mathcal{X}_{n, O}(f)).
\]

**Proof.** We have the following the Gysin exact sequence
\[
\cdots \rightarrow H^\text{BM}_{p+q}(S_{p-1}) \rightarrow H^\text{BM}_{p+q}(S_p) \rightarrow H^\text{BM}_{p+q}(S_p \setminus S_{p-1}) \rightarrow H^\text{BM}_{p+q-1}(S_{p-1}) \rightarrow \cdots.
\]
Put
\[
A_{p, q} := H^\text{BM}_{p+q}(S_p), \quad E_{p, q} := H^\text{BM}_{p+q}(S_p \setminus S_{p-1}).
\]
Then we have the bigraded \( \mathbb{Z} \)-modules \( A := \bigoplus_{p, q} A_{p, q} \) and \( E := \bigoplus_{p, q} E_{p, q} \). The previous exact sequence induces the exact couple \( (A, E; h, i, j) \), where \( h : A \to A \) is induced from the inclusions \( S_m \subseteq S_{m+1}, i : A \to E \) and \( j : E \to A \) are induced from the above exact sequence. Since the filtration in Lemma 4.2 (ii) is finite, that exact couple gives us the following spectral sequence
\[
E^1_{p, q} := E_{p, q} = H^\text{BM}_{p+q}(S_p \setminus S_{p-1}) \Rightarrow H^\text{BM}_{p+q}(\mathcal{X}_{n, O}(f)).
\]

On the other hand, we have
\[
S_p \setminus S_{p-1} = \bigcup_{\eta(J, a) = p} \overline{\mathcal{X}^{(n)}_{J, a}}.
\]
One claims that for two different pairs \((J, a), (J', a')\) in \( \tilde{\mathcal{P}}_n \) which \( \eta(J, a) = \eta(J', a') = p \) then
\[
\overline{\mathcal{X}^{(n)}_{J, a}} \cap \overline{\mathcal{X}^{(n)}_{J', a'}} = \emptyset \quad \text{and} \quad \overline{\mathcal{X}^{(n)}_{J', a'}} \cap \overline{\mathcal{X}^{(n)}_{J, a}} = \emptyset.
\]
Indeed, if otherwise, suppose that
\[ \chi^{(n)}_{J',a'} \cap \overline{\chi^{(n)}_{J,a}} \neq \emptyset. \]

By Lemma 4.1 we obtain that \( \chi^{(n)}_{J',a'} \subseteq \overline{\chi^{(n)}_{J,a}} \), but \( \chi^{(n)}_{J',a'} \) and \( \chi^{(n)}_{J,a} \) are two disjoint smooth manifolds, then \( \eta(J',a') < \eta(J,a) \). This is a contradiction.

Therefore, in the set \( S_p \setminus S_{p-1} \) with the induced topology, each set \( \chi^{(n)}_{J,a} \) which \( \eta(J,a) = p \) is open, hence, is also closed. This implies that
\[
H^{BM}_{p+q}(S_p \setminus S_{p-1}) = \bigoplus_{(J,a) \in \mathcal{P}_n, \eta(J,a) = p} H^{BM}_{p+q}(\chi^{(n)}_{J,a}).
\]

The theorem is then proved. \( \square \)

**Corollary 4.4.** With the hypothesis as in Theorem 4.3, there is an isomorphism of groups
\[
H^{BM}_{2d_0}(X_{n,O}(f)) \cong \mathbb{Z}^s,
\]
where \( s \) is the number of connected components of \( X_{n,O}(f) \) which have the same complex dimension \( d_0 \) as \( X_{n,O}(f) \).

### 4.2. Sheaf cohomology groups of contact loci.

In this subsection, we are going to prove the following theorem.

**Theorem 4.5.** Let \( f \in \mathbb{C}[x_1, \ldots, x_d] \) be nondegenerate, \( n \in \mathbb{N}^* \) and \( f(O) = 0 \). Let \( \mathcal{F} \) be an arbitrary sheaf of abelian groups on \( X_{n,O}(f) \). Then, there is a spectral sequence
\[
E^{p,q}_1 := \bigoplus_{(J,a) \in \mathcal{P}_n, \eta(J,a) = p} H^p_c(\chi^{(n)}_{J,a}, \mathcal{F}) \implies H^{p+q}(X_{n,O}(f), \mathcal{F}).
\]

**Proof.** We use the notation in Lemma 4.2. For simplicity, we write \( S \) for \( S_{d_0} = X_{n,O}(f) \). For any \( 0 \leq p \leq d_0 \), we put \( S^p := S_p \setminus S_{p-1}, \) which is a \( \mu_n \)-invariant subset of \( S \). Consider the inclusions \( j_p : S^p \hookrightarrow S_p, k_p : S \setminus S^p \hookrightarrow S \) and \( i_p : S_p \hookrightarrow S \). Put \( \mathcal{F}_p := (j_p)_!(i_p)^{-1} \mathcal{F} \) and \( \mathcal{F}^p := (k_p^{-1})(i_p)^{-1} \mathcal{F} \) for every \( p \geq 1 \), with the convention \( \mathcal{F}^0 := \mathcal{F} \). Then we have the exact sequences
\[
0 \to \mathcal{F}^p(\mathcal{F}) \to \mathcal{F}^p(\mathcal{F}) \quad \text{and} \quad 0 \to (i_p)_* \mathcal{F}_p \to \mathcal{F}|_{S_p} \to \mathcal{F}|_{S_{p-1}},
\]
in which by \( \mathcal{F}|_{S_p} \) we mean \((i_p)_*(i_p)^{-1}\mathcal{F}\). Therefore we have the following diagram

\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{F}^p(\mathcal{F}) & \to & \mathcal{F} & \to & \mathcal{F}|_{S_p} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (i_p)_* \mathcal{F}_p & \to & \mathcal{F}|_{S_p} & \to & 0 \\
\end{array}
\]

It implies from the snake lemma that \( \mathcal{F}^p(\mathcal{F})/\mathcal{F}^{p+1}(\mathcal{F}) \cong (i_p)_* \mathcal{F}_p \). Thus there is a filtration of \( \mathcal{F} \) by “skeleta”: \( \mathcal{F} = \mathcal{F}^0(\mathcal{F}) \supseteq \mathcal{F}^1(\mathcal{F}) \supseteq \cdots \). It gives the following spectral sequence of cohomology groups with compact support
\[
E^{p,q}_1(\mathcal{F}, \mathcal{F}) := H^{p+q}(\mathcal{F}, (i_p)_* \mathcal{F}_p) \implies H^{p+q}(\mathcal{F}, \mathcal{F}).
\]
On the other hand, by Theorem 3.1, 
\[ H^m_c(S_p, \mathcal{F}_p) \cong H^m_c(S^\circ_p, \mathcal{F}_p) \] 
for any \( m \) in \( \mathbb{N} \). Also, by the isomorphisms given by the extension by zero sheaf, we have
\[ H^m_c(S_p, \mathcal{F}_p) = H^m_c(S_p, (j_p)!((j_p)^{-1}(i_p)^{-1}\mathcal{F})) \cong H^m_c(S^\circ_p, (j_p)^{-1}(i_p)^{-1}\mathcal{F}). \]

We have that
\[ \mathcal{S}^\circ_p = \bigcup_{\eta(j,a) = p} \mathcal{X}^{(n)}_{J,a}. \]

Then by the reason as in the proof of Theorem 4.3, we get
\[ H^m_c(S^\circ_p, (j_p)^{-1}(i_p)^{-1}\mathcal{F}) = \bigoplus_{(J,a) \in \mathcal{P}_n, \eta(J,a) = p} H^{p+q}(\mathcal{X}^{(n)}_{J,a}, (l,J,a)^{-1}(j_p)^{-1}(i_p)^{-1}\mathcal{F}), \]

where \( l,J,a \) is the inclusion of \( \mathcal{X}^{(n)}_{J,a} \) in \( S^\circ_p \). For simplicity of notation, we write \( H^{p+q}(\mathcal{X}^{(n)}_{J,a}, \mathcal{F}) \) instead of \( H^{p+q}(\mathcal{X}^{(n)}_{J,a}, (l,J,a)^{-1}(j_p)^{-1}(i_p)^{-1}\mathcal{F}) \). The proof is completed.

Now, we consider the spectral sequence (4.2) for constant sheaf. We need some notation, for each \( \gamma \in K, k \in \mathbb{N}, n \in \mathbb{N}^* \) and \( p \in \mathbb{Z} \), we denote by \( D^{(n)}_{\gamma,k,p} \) the set of all \( a \in \sigma_{J_a,\gamma} \cap \Delta^{(n)}_{J_a} \) such that \( d - 1 + |J_a|n - s(a) - k = p \), which is a finite set. Then, for each \( (J,a) \in \mathcal{P}_n \) which \( \eta(J,a) = p \), there exist \( \gamma \in K \) and \( k \in \mathbb{N} \) such that \( J = J_a \) and \( a \in D^{(n)}_{\gamma,k,p} \).

The summands in the spectral sequence (4.2) are described more explicitly in case of constant sheaf as below.

**Lemma 4.6.** Let \( \gamma \in K, n \in \mathbb{N}^* \) and \( p,q \in \mathbb{Z} \). Then, for \( J = J_a \) and \( a \in D^{(n)}_{\gamma,0,p} \) we have
\[ (4.4) \quad H^{p+q}_{c}(\mathcal{X}^{(n)}_{J_a}, \mathbb{C}) \cong H_{p-q}(X_{\gamma}(1), \mathbb{C}). \]

**Proof.** It follows from Theorem 3.1 that \( \mathcal{X}^{(n)}_{J_a} \) is a \( p \)-dimensional complex manifold and is homeomorphic to \( X_{\gamma}(1) \times \mathbb{C}^{|J_a|\ell_{J_a}(a) - s(a)} \). Then, by combining the duality and the Kunneth formula we get the conclusion.

We also have the following description for the cohomology of \( \mathcal{X}^{(n)}_{J_a} \) for \( J = J_a \) and \( a \in D^{(n)}_{\gamma,k,p}, k \in \mathbb{N}^* \).

**Lemma 4.7.** Let \( \gamma \in K, n \in \mathbb{N}^* \) and \( p,q \in \mathbb{Z} \). Then, for \( J = J_a \) and \( a \in D^{(n)}_{\gamma,k,p} \) we have
\[ (4.5) \quad H^{p+q}_{c}(\mathcal{X}^{(n)}_{J_a}, \mathbb{C}) \cong H_{p-q}(X_{\gamma}(0), \mathbb{C}). \]

**Proof.** Since \( \mathcal{X}^{(n)}_{J_a} \) is a \( p \)-dimensional complex manifold, then by duality, we have
\[ H^{p+q}_{c}(\mathcal{X}^{(n)}_{J_a}, \mathbb{C}) \cong H_{p-q}(\mathcal{X}^{(n)}_{J_a}, \mathbb{C}). \]

On the other hand, by Theorem 5.1 \( \mathcal{X}^{(n)}_{J_a} \) is a locally trivial fibration on \( X_{\gamma}(0) \) with fiber \( \mathbb{C}^{|J_a|(|J_a|(a) + k) - s(a) - k} \) which is contractible. Hence, by the spectral sequence for (Serre) fibration, we obtain that \( H_{p-q}(\mathcal{X}^{(n)}_{J_a}, \mathbb{C}) \cong H_{p-q}(X_{\gamma}(0), \mathbb{C}) \). The proof is complete.

We have the following result concerning cohomology of contact loci.
Corollary 4.8. Let \( f \in \mathbb{C}[x_1, \ldots, x_d] \) be nondegenerate, \( n \in \mathbb{N}^* \) and \( f(0) = 0 \). Then, there is a spectral sequence (4.6)
\[
E_1^{p,q} := \bigoplus_{\gamma \in K} \left( H_{p-q}(X_{\gamma}(1), \mathbb{C}[D_{\gamma,0,p}^{(n)}]) \oplus H_{p-q}(X_{\gamma}(0), \mathbb{C}[D_{\gamma,k,p}^{(n)}]) \right) \Rightarrow H_{p+q,c}(X_{n,O}(f), \mathbb{C}).
\]

Proof. Apply Theorem 4.5 for \( F \) to be the constant sheaf on \( X_{n,O}(f) \) associated to the field of complex numbers \( \mathbb{C} \), since the inverse image of constant sheaf is a constant sheaf, the Corollary is a direct consequence of Theorem 4.5 and the above lemmas.

Acknowledgement. The first author is deeply grateful to François Loeser for introducing him to Problem 1. The authors thank Vietnam Institute for Advanced Study in Mathematics (VIASM) and Department of Mathematics - KU Leuven for warm hospitality during their visits.

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