Abstract
The field of Circuit Complexity utilises careful analysis of Boolean Circuit Functions in order to extract meaningful information about a range of complexity classes. In particular, the complexity class \( P/\text{Poly} \) has played a central role in much of the historical attempts to tackle the problem of whether solution and verification are equivalent i.e. the central \( P \) versus \( NP \) problem. Whilst circuits can potentially be easier to analyse than Turing Machines due to their non-uniform nature of computation (program size is allowed to depend on the input size), it is notoriously hard to establish lower bounds for them. In this report, we will touch upon several results published by Hastad, Sipser and Razborov that will highlight a dynamic interplay between circuit complexity and many of the central ideas of modern-day complexity theory, and in particular the central importance of Hastad’s Switching Lemma.

Keywords: Natural Proofs, Circuit Complexity, Lower Bounds, \( P \)-\( \text{Poly} \), Uniform and Non-uniform Complexity, Hastad’s Switching Lemma, Restriction and Polynomial Methods, Literature Review

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1 Introduction and Context

1.1 Historical Context
Circuit complexity is a branch of Computational Complexity theory in which Boolean functions are classified according to the size or depth of the Boolean Circuits that compute them. Where the circuit-size complexity of a Boolean function \( f \) is the minimal size of any circuit computing \( f \). The circuit-depth complexity of a Boolean function \( f \) is the minimal depth
of any circuit computing $f$. Boolean functions and the corresponding complexity behind these functions have been under thorough study since the publication of Claude Shannon’s seminal paper in the year of 1949, which demonstrated that almost all Boolean Functions on $n$ variables require circuits of size $O(2^n/n)$.

Our motivations for studying circuit complexity are multi-faceted. Circuit complexity is a reasonable measure for the difficulty measure for functions that take only finitely many inputs i.e. through the lens of circuit complexity, we are able to deduce very meaningful conclusions about finite-input problems - we have more control over the study of program size of algorithms on finite bit instances than if we were simply analyzing the same problem through the lens of Turing Computation. There are also close connections between circuit complexity, de-randomisation and by extension the complexity class BPP.

The best known lower bound on the circuit size for a problem in NP is currently $4.5n - o(n)$, [AB07] this is not very strong and much lower than expected, yet we are still stuck at this stage. For the Polynomial Hierarchy, better lower bounds are known - for example for every $k > 0$, some language in the Polynomial Hierarchy require circuits of size $\Omega(n^k)$.

Whilst proving circuit lower bounds is notoriously difficult, super-polynomial lower bounds have been proved under certain restrictions on the family of circuits used. The first function for which such bounds were obtained was the parity function, which computes the sum of input bits modulo 2. The proof that PARITY was not contained in the complexity class $AC^0$ was first established independently by Ajtai in 1982 and by Furst, Saxe and Sipser in 1984, after which in 1987 Hastad proved the even stronger result that any family of constant-depth circuits computing the parity function require exponential size, where he applied his famous and powerful switching lemma, utilising the key idea of random restrictions, which Roosman, Servedio and Tan extended to the idea of random Projections to prove an average-case depth hierarchy theorem for Boolean Circuits [RST15].

1.2 Setup and Definitions

Definition 1. $t$-CNF is an AND of clauses of width at most $t$. A clause of width $t$ is an OR of $t$ literals. For example, $x_1 \lor \bar{x}_2 \lor x_4$ is a clause of width 3, and $(x_1 \lor x_2) \land (\bar{x}_1 \lor \bar{x}_2)$ is a 2-CNF and another example of a 3-CNF:

$$(A \lor \neg B \lor \neg C) \land (\neg D \lor E \lor F)$$

Definition 2. $s$-DNF is an OR of disjuncts of width at most $s$, where a disjunct of width $s$ is an AND of $s$ literals, with an example below:

$$(A \land \neg B \land \neg C) \lor (\neg D \land E \land F)$$

Definition 3. A restriction $\rho$ is a mapping from $\{x_1, \ldots, x_n\}$ to $\{0, 1\}$. A random restriction $\rho \in R(p,q)$, $0 \leq p, q \leq 1$ is a random restriction such that:

$$P_p[\rho(x_i) = \ast] = p$$

$$P_p[\rho(x_i) = 0] = (1 - p)q$$

$$P_p[\rho(x_i) = 1] = (1 - p)(1 - q)$$

independently, for each $i$.

Definition 4. $AC^k$ is the class of functions computable by polynomial size and $O((\log n)^k)$ depth circuits over infinitely many AND and infinitely many OR and NOT gates

Definition 5. $NC^k$ is the class of functions computable by polynomial size and $O((\log n)^k)$ depth circuits over 2-AND and 2-OR and NOT gates

In particular we know that $AC^k \subseteq NC^{k+1} \subseteq AC^{k+1}$
Definition 6. The complexity class $P/Poly$ is defined as the set of languages recognised by a polynomial-time Turing Machine with polynomial-bounded advice\footnote{We say that a Turing Machine has access to an advice family $\{a_n\}_{n \geq 0}$ where each $a_n$ is a string if while computing on an input of size $n$, the machine is allowed to examine $a_n$. The advice is said to have polynomial size if there is a $c \geq 0$ such that $|a_n| \leq n^c$} function and its relation with circuit complexity is encoded in the fact that:

$$P_{/\text{poly}} = \cup_{c \in \mathbb{N}} \text{SIZE}(n^c)$$

Definition 7.

1.3 Shannon’s Proof

We present Shannon’s beautiful non-constructive proof below.

Shannon (1949) There is a constant $c$ such that every $n$-ary Boolean function has Circuit Complexity at most $2^n/cn$

Proof. Fix an $n$-ary function $F : \{0,1\}^n \rightarrow \{0,1\}$. Let $x = (x_1, x_2, \cdots, x_n)$ denote the input vector for some integer $k$ to be specified later and define $y = (x_1, \cdots, sx_k)$ and $z = (x_{k+1}, \cdots, x_n)$.

We begin by constructing a $2^k \times 2^{n-k}$ truth table for $F$, where each row is specified by a possible value of $y$ and each column by a possible value of $z$. Suppose that there are only $t$ different column vectors in this matrix. The inequality $t \leq 2^{n-k}$ is obvious; less obvious but still trivial is the inequality $t \leq 2^s$. Let $F_i(z) = 1$ if column $z$ has the the $i$-th pattern and 0 otherwise. Similarly, let $G_i(y)$ be the function specified by the $i$-th column vector. Then we can write:

$$F(x) = F(y, z) = \cup_{i=1}^t G_i(y) \land F_i(z)$$

Suppose for the moment that all the 1 in this table are restricted to $s$ of the $2^k$ rows. This immediately implies that $t \leq 2^s$. Under this assumption, we can build a circuit for $F$ as follows. The inputs are connected to two binary-to-positional converters $B_k$ and $B_{n-k}$ which requires $2^k + 2^{n-k}$ gates. For each $i$ between 1 and $t$, we add two trees of OR gates connected by a single AND to compute $F_i(y) \land G_i(z)$ for each $i$ this requires $2^k + 2^{n-k} + 1$ gates. Finally, we connect all these with another tree of at most $2^s$ OR gates. The total number of gates used in this special case is at most $2(2^k + 2^{n-k}) + 2^s + 1$.

Now back to the general case. We can write any $n$-ary function $F$ as a dis-jection $2^k/s$ different $n$-ary functions, each of which has a truth table with at most $s$ non-zero rows. To construct a circuit for $F$, we build a circuit for each of its $s$-row components, sharing a single pair of binary-to-positional converters. The total gate count for this construction is at most:

$$2^k + 2^{n-k} + \frac{2^k}{s} (2^k + 2^{n-k} + 2^s + 1) = 2^k + 2^{n-k} + \frac{2^n + 2^k(2^s + s^k + 1)}{s}$$

Finally, we are free to choose the parameters $k$ and $s$ to minimize this gate count. If we take $k = 2\log n$ and $s = n - 2\log n$, then the total gate count is at most:

$$n^2 + \frac{2n}{n^2} + \frac{2^n + n^2(2^n/n^2 + n^2 + 1)}{n - 2\log n} = n^2 + \frac{2^n}{n^2} + \frac{2^{n+1} + n^4 + n^2}{n - 2\log n} = O(\frac{2^n}{n})$$

1.4 Karl-Lipton Theorem

The Karl-Lipton Theorem justifies why circuit lower bounds is a viable way to attack the $P$ versus $NP$ problem.
Karl-Lipton Theorem: \( \text{PH} \neq \sum_2 \Rightarrow \text{NP} \subseteq \text{P}/\text{poly} \)

Proof. Suppose \( \text{NP} \subseteq \text{P}/\text{poly} \). We will show that this implies \( \text{PH} \subseteq \sum_2 \). Note that \( \text{PH} \subseteq \sum_2 \) and \( \prod_2 \subseteq \sum_2 \) are equivalent; if the latter were true then by swapping quantifier order, we have that \( \sum_2 = \sum_2 \), collapsing the Polynomial Hierarchy.

By assumption that \( \text{NP} \subseteq \text{P}/\text{poly} \) there exists a polynomially sized circuit family \( \{C_n\} \), which decides SAT. This implies that there exists a polynomial size circuit family \( \{C_{*n}\} \) which, given a boolean formula \( \phi \), finds a satisfying assignment for it.

We construct \( \{C_{*n}\} \) in the following way: we choose a free variable and substitute 1 or 0 for it, then run the decider. If the decider says that one of these boolean formulas has a satisfying assignment, repeat this process on another free variable. When there are no more free variables, we will have substituted a satisfying assignment into \( \phi \).

Let \( L \in \prod_2 \). Then \( x \in L \iff \forall u \exists v \text{ s.t. } \phi(x, u, v) = 0 \)

Then \( \exists C \forall u \phi(x, u, C \ast (x, u)) = 0 \). That is, there exists a circuit which outputs an unsatisfying assignment for \( \phi \) for any \( u \). Then \( L \in \sum_2 \) and the theorem is proven. \( \square \)

2 Hastad’s Switching Lemma [AB07]

Arguments using restrictions (partial assignments to input variables) to simplify unbounded fan-in Boolean circuits have been quite successful for obtaining lower bounds on circuit size and depth, oracles to separate complexity classes, lower bounds on time, processors and memory of PRAMs as well as on the complexity of proofs in bounded-depth proof systems.

Ultimately the key intuition behind Hastad’s switching lemma is to show that an AND of small ORs can be written as an OR of small ANDs if an appropriate restriction is applied. In particular, we can reduce the depth of formulas by 1 at the expense of reducing the number of input variables.

The core to their effectiveness is that they simplify the formulas without completely trivialising the functions that are being computed. Therefore, there is a certain element of creativity attributed to this choice of restrictions, since one chooses a family of restrictions that is tailored to the function being computed and argues that some member of the family has the desired properties. As long as the random restriction is likely to kill a clause, the switching lemma should work.

The history of switching lemmas traces back to Furst, Sax and Sipser, but in this project we will be looking at the most powerful of these switching lemmas, which is attributed to Hastad.
2.1 Statement of the Lemma

Theorem 1. Let $f$ be some Boolean Function which can be written as some $t$-CNF. Then, for any integer $s \geq 1$, any $p \in [0,1]$ we have that:

$$P_{\rho \in R(p,1/2)}[f|_{\rho} \text{is not } s-DNF] \leq (5pt)^s$$

Here $f_{\rho}$ is not $s$-DNF means $f|_{\rho}$ can not be written as $s$-DNF

The argument of Hastad’s switching lemma involves the probabilistic method, in particular we argue that the probability that a restriction from the family fails to have the desired properties is strictly less than 1. For example, we can consider an OR of small ANDs and each term in turn. Essentially a term that is falsified by a restriction does not contribute any variables to the AND of small ORs and for each term that is not falsified it is more likely that the term is satisfied than that any variable is contributed in the AND of small ORs.

Note that one very important feature of this lemma is that the probability on the right hand side of the inequality $(5pt)^s$ does not depend on the number of variables. The notion of the random restriction is actually key to Hastad’s Switching Lemma. Indeed if we consider the threshold function $Th^x_n(k)$, which is 1 if and only if:

$$x_1 + x_2 + \cdots + x_n \geq k$$

We can readily see that $Th^x_n(k)$ can be written as $k$-DNF and $(n-k)$-CNF but not as a $(n-k-1)$-CNF. Indeed if $k$ is fixed and $n$ goes to infinity, this example shows, in general, that $t$-CNF can not be written as an $s$-DNF, where $s = s(t)$ only depends on $t$.

2.2 Depth 2 Circuit Computing Parity

2.3 Lower-bound on PARITY

Theorem: For any $d \geq 2$, $\text{PAR}_n$ can not be computed by depth-$d$ size $2^{cn^{1/(d-1)}}$ circuit when $n$ is sufficiently large, where $c$ is a reasonable constant, say $1/11$

Proof. Assume for sake of contradiction that $\text{PAR}_n$ can be computed by some depth $d$ size $S$ circuit $C(x)$, where

$$S < 2^{cn^{1/(d-1)}}$$

and the constant $c$ will be determined later. Without loss of generality, assume that our circuit $C(x)$ is already in standard form (depth $d$ size $S$ circuit can be transformed to standard form of depth $d$ size $dS$, for which we only need to make the constant $c$ slightly larger, that is, $c + \epsilon$ for any $\epsilon > 0$.

Now view $C(x)$ as a depth $d + 1$ circuit by adding a dummy layer consist of AND or OR gates of fan-in 1 to the bottom, alternating each layer. The motivation is to apply switching
lemma for \( t = 1 \). Indeed applying the switching lemma to this circuit with parameter \( t = 1 \), \( s = (1 + \delta)c_H \log S \) and \( p = 1/(2c_H) \) where \( \delta > 0 \) is an arbitrarily small constant, and \( c_H \) denotes the constant 5 in Hastad’s Switching Lemma. That is, we will apply random restriction \( \rho \in R(1/(2c_H), 1/2) \). For each gate in the second bottom layer, which is 1-CNF or 1-DNF with probability at least:

\[
1 - (c_H p t)^s = 1 - \left( \frac{1}{2} \right)^{(1+\delta)c_H \log S} \geq 1 - S^{1-\delta}
\]

that gate can be written as \( s \)-DNF. If all the bottom gates can be switched, we get a depth \( d \) circuit of bottom fan-in \( s = (1 + \delta)c_H \log S \).

Now apply the random restriction \( \rho_i \in R(p, 1/2) \) where:

\[
p = 1/(2c_H S) = 1/(2(1 + \delta)c_H \log S)
\]

for \( i = 1, 2, \cdots, d - 2 \). For \( \rho_i \), for each gate in the second bottom layer, by switching lemma, with probability at least:

\[
1 - (c_H p t)^{2} \geq 1 - S^{1-\delta}
\]

this \( t \)-CNF can be converted to an \( s \)-DNF where \( t = s = (1 + \delta) \log S \). For each \( \rho_i \), the depth of the circuit is supposed to be reduced by 1. Finally, it remains to count the number of times we have applied the switching lemma, which is at most \( S \), the total number of gates in the original circuit. Apply a union bound, with probability \( \geq 1 - S^{-\delta} \to 1 \), circuit \( C(x)|_{\rho} \) can be converted to a depth 2 size \( S \) circuit where:

\[
\rho = \rho_{d-2} \circ \cdots \circ \rho_0 \in R(1/(2c_H 2(1 + \delta) \log S)^{d-2}), 1/2)
\]

Now that we have proved that \( C(x)|_{\rho} \) can be written as a depth 2 size \( S \) circuit with high probability, we observe that parity is still a parity function or its negation after applying any restriction in possibly less number of variables, let us count the number of free variables after applying \( p \). The expected number of free variables is:

\[
\frac{n}{2c_H (2c_H (1 + \delta) \log S)^{d-2}}
\]

By the Chernoff bound, we claim:

\[
m = |\rho^{-1}(*)| > \frac{n}{2c_H (2c_H (1 + 2\delta) \log S)^{d-2}}
\]

with high probability. It is not difficult to prove that any depth 2 circuit computing \( \text{PAR}_m \) should have bottom fan-in \( m \), which implies that:

\[
(1 + \delta) \log S \geq m \geq n/(2c_H (2c_H (1 + 3\delta) \log S)^{d-2})
\]

This implies that \( S \geq 2^{n((d-1)/(2 + 3\delta)c_H)} \). Therefore, as long as constant \( c < 1/(2c_H) \), our theorem holds.

\[\square\]

### 2.4 Matching Upper Bound on Parity

Now we present an equivalent upper bound up to a constant factor for PARITY. Indeed for depth 3, we observe that:

\[
\text{PAR}_n(x_1, \cdots, x_n) = \text{PAR}_m(\text{PAR}_m(y_1), \cdots, \text{PAR}_m(y_m))
\]

where \( m = \sqrt{n} \) and \( y_1 = (x_1, \cdots, x_m), y_2 = (x_{m+1}, \cdots, x_{2m}) \) etc. For the \( \text{PAR}_m \) on the outside, write it as \( m \)-CNF of size \( 2^{m-1} + 1 \) and for the \( \text{PAR}_m \) inside, write it as \( m \)-DNF of size \( 2^{m-1} + 1 \) which is a depth 4 circuit of size:

\[
(1 + m)(2^{m-1} + 1) \leq n2^{\sqrt{n}}
\]

To make it depth 3, simply merge two layers in the middle, both of which are OR gates and the size will not increase. Using the same argument we can prove:
Theorem 2. For any \( d \geq 2 \), \( PAR_n \) can be computed by depth \( d \) circuit of size at most 
\[
n^2 n^{1/(d-1)}
\]

Now if the depth \( d \geq \log n \) it turns out that there exists linear size circuits computing \( PAR_n \).

Theorem 3. \( PAR_n \) can be computed by depth \( \lceil \log n \rceil \) circuit of size \( 2n - 1 \)

Proof. Build a complete binary tree with \( n \) leaves, corresponding to variables \( x_1, \ldots, x_n \); all non-leaf nodes are XOR = PAR\(_2\) gates. Since the depth of the binary tree is \( d = \lceil \log n \rceil \), the number of non-leaf nodes is:
\[
(n - 2^{d-1}) + 2^{d-2} + 2^{d-3} + \cdots + 1 = n - 1
\]

For each XOR gate, we need 2 gates to implement (recall that XOR can be either written as 2-CNF or 2-DNF, and apply the gates merging technique again) except for the top gate (3 gates are needed). The total number of gates is \( 2(n - 1) + 1 = 2n - 1 \)

\( \square \)

2.5 Proof of the Switching Lemma

Let the DNF \( F = T_1 \lor T_2 \lor \cdots \lor T_m \). We restrict variables in two stages.

Stage 1: Restrict variable with probability \( \sqrt{p} \), \( f \to f|_{\rho_1} \):

\[
x_i = \begin{cases} 
0 \text{ with probability } \frac{1 - \sqrt{p}}{2} \\
x_i \text{ with probability } \sqrt{p} \\
1 \text{ with probability } \frac{1 - \sqrt{p}}{2}
\end{cases}
\]

In the first case, we consider terms with fan-in \( \geq 4 \log S \) we have that:
\[
P[\text{Any Term with fan-in } \geq 4 \log S \neq 0] \leq \left( \frac{1 - \sqrt{p}}{2} \right)^{4 \log S} \leq \left( \frac{2}{3} \right)^{4 \log S} \leq \frac{1}{S^3}
\]

Therefore, we have that:
\[
P[\exists \text{Term with fan-in } \geq 4 \log S \text{ doesn’t become } 0] \leq \frac{1}{S^3}
\]

In the second case, we consider terms with fan-in \( \leq 4 \log S \):
\[
P[T_i \text{ depends on } c_0 \text{ variables}] \leq (4 \log S)^{c_0} \left( \sqrt{p} \right)^n \leq \frac{1}{S^3}
\]

Therefore it must be the case that:
\[
P[\exists \text{term with fan-in } \leq 4 \log S, \text{ depends on } \geq c_0 \text{ variables}] \leq \frac{1}{S^3}
\]

where \( c_0 \) is a constant depending on \( c_1 \), so now the DNF is also a \( c_0 \)-DNF.

Stage 2: Restrict variables in \( f|_{\rho_2} \) with probability \( \sqrt{p} \to f|_{\rho_1 \cup \rho_2} \):

\[
x_i = \begin{cases} 
0 \text{ with probability } \frac{1 - \sqrt{p}}{2} \\
x_i \text{ with probability } \sqrt{p} \\
1 \text{ with probability } \frac{1 - \sqrt{p}}{2}
\end{cases}
\]

In the first case, there are many disjoint terms \( T_1, \ldots, T_l \) with \( l \geq 3^{c_0} 4 \log S \) so that:
\[
P[T_i = 1] \geq \left( \frac{1}{3} \right)^{c_0}
\]
\[ P[T_i \neq 1] \leq (1 - \frac{1}{3})^{n^c_0} \]
\[ P[\exists T_i = 1] \geq 1 - ((1 - \frac{1}{3})^{n^c_0})^t = 1 - 2^{-1/3^{n^c_0}} \geq 1 - \frac{1}{S^2} \]

In the second case, the maximum number of disjoint \( T_i \)'s is \( \leq 4 \log S \).

Now if we select disjoint \( T_i \)'s greedily, we find that there exists a set \( H \) with \( 4c_03^{n^c_0} \log S \) variables such that \( \forall i \) we have that \( H \cap T_i \neq \emptyset \)

To finish off the proof, we first restrict variables of \( H \), and we can find a constant \( b \) such that the number of unset variables in \( H \leq b \). We subsequently restrict variables not in \( H \), and use induction on \( c_0 \).

Now Razborov provided a simpler version of this proof which is presented in [AB07]:

**Proof.** We will present a more simplified proof of the Switching Lemma due to Razborov.

Let \( R_t \) denote the set of all restrictions to \( t \) variables, where \( t \geq n/2 \). Then:

\[ |R_t| = \binom{n}{t} 2^t \]

The set of bad restrictions \( \rho \) - those for which \( D(f|\rho) > s \) is greater than \( s \) - is a subset of these. To show that this subset is small, we give a one-to-one mapping from it to the Cartesian product of three sets: \( R_{t+s} \), the set of restrictions to \( (t+s) \) variables, a set \( \text{code}(k, s) \) of size \( k^{O(s)} \) and the set \( \{0,1\}^s \) (The set \( \text{code}(k, s) \) is explained below). This Cartesian product has size \( \binom{n}{t+s} 2^{t+s} k^{O(s)} 2^s \). Thus the probability of picking a bad restriction is bounded by:

\[ \frac{\binom{n}{t+s} 2^{t+s} k^{O(s)} 2^s}{\binom{n}{t} 2^t} \]

Intuitively, this ratio is small because \( k, s \) are to be thought of as constant and \( t > n/2 \), and therefore:

\[ \left( \frac{n}{t} \right)^2 \gg \left( \frac{n}{t+s} \right)^{2^{t+s}} \]

Formally, by using the correct constants as well as the approximation \( \binom{n}{t} \sim (n/e)^t \) we can upper-bound the ratio as follows:

\[ \frac{7(n-t)k}{2n} \]

Therefore to prove the Switching Lemma it suffices to describe the one-to-one mapping mentioned above. This uses the notion of a canonical decision tree for \( f \). We take the \( k \)-DNF circuit for \( f \) and order its terms (i.e. the \( \land \) gates in Figure 1) arbitrarily and within each term we order the variables. The canonical decision tree queries all the variables in the first term in order, then all the variables in the second term, and so on until the function value is determined.

Suppose that restriction \( \rho \) is bad, that is, \( D(f|\rho) > s \). The canonical decision tree for \( f|\rho \) is defined in the same way as for \( f \), using the same order for terms and variables. Since the decision tree complexity of \( f|\rho \) is at least \( s \), there is a path of length at least \( s \) from the root to a leaf. This path defines a partial assignment to the input variables; denote it by \( \pi \). The rough intuition is that the one-to-one mapping takes \( \rho \) to itself plus \( \pi \).

let us reason about restriction \( \rho \). None of the \( \land \) gates outputs 1 under \( \rho \), otherwise \( f|\rho \) would be determined and would not require a decision tree. Some terms output 0, but not all, since that would also fix the overall output. Imagine walking down the path \( \pi \). Let \( t_1 \) be the first term that is not set to zero under \( \rho \). Then \( \pi \) must query all the unixed variables in \( t_1 \). Denote the part of path \( \pi \) that deals with \( t_1 \) by \( \pi_1 \); that is, \( \pi_1 \) is an assignment to the variables of \( t_1 \). Since \( f \) is not determined even after \( s \) steps in \( \pi \), we conclude that \( \pi_1 \) sets \( t_1 \) to zero.

Let \( t_2 \) be the next term not yet set to zero by \( \rho \) and \( \pi_1 \); again, the path must set \( t_2 \) to zero. Let \( \pi_2 \) denote the assignment to the variables of \( t_2 \) along the path. Then \( \pi_2 \) also sets
to zero. This process continues until we have dealt with \(m\) terms when \(\pi\) has reached depth \(s\). Each of these terms was not set by \(\rho\) and, except perhaps for \(\pi_m\), is set to zero after \(s\) queries in \(\pi\). The disjoint union of these \(\pi_i\) terms contains assignments for at least \(s\) variables that were unfixed in \(\rho\).

Our mapping will map \(\rho\) to:

\[
(\rho \cup \sigma_1 \cup \cdots \cup \sigma_m, c, z)
\]

where the term \(\sigma_i\) is the unique set of assignments that makes \(t_i\) true, \(c = c_1 c_2 \cdots c_m\) is in code\((k, s)\) and \(z \in \{0, 1\}^s\). In defining this mapping we are crucially relying on the fact that there is only one way to make a term true, namely to set all its literals to 1.

To show that the mapping is one-to-one, we show how to invert it uniquely. This is harder than it looks since \(a\ priori\) there is no way to identify \(\rho\) from \(\rho \cup \sigma_1 \cup \cdots \cup \sigma_m\). The main idea is that the information in \(c\) and \(z\) allows us to extract \(\rho\) from the union.

Suppose that we are given the assignment \(\rho \cup \sigma_1 \cdots \cup \sigma_m\). We can plug this assignment into \(f\) and then infer which term serves as \(t_1\). It is the first one to be set true. The first \(k\) bits of \(c\), say \(c_1\), are an indicator string showing which variables in \(t_1\) are set by \(\sigma_1\). We can reconstruct \(\pi_1\) from \(\sigma_1\) using the string \(z\), which indicates which of the \(s\) bits fixed in the decision tree differ between the \(\pi\) assignments and the \(\sigma\) assignments.

Having reconstructed \(\pi_1\), we can work out which term is \(t_2\): it is the first true term under the restriction \(\rho \cup \pi_1 \cup \sigma_2 \cdots \cup \sigma_m\). The next \(k\) bits of \(c\), denoted \(c_2\), give us \(\sigma_2\) and continue this process until we have processed all \(m\) terms and figured out what \(\sigma_1, \cdots, \sigma_m\) are. Thus we have figured out \(\rho\), so the mapping is one-to-one.

Finally, we define the set code\((k, s)\): this is the set of all sequences of \(k\)-bit binary strings in which each string has at least one 1 bit and the total number of 1 bits is at most \(s\). It can be shown by induction on \(s\) that:

\[
|\text{code}(k, s)| \leq \left(\frac{k}{\ln 2}\right)^s
\]

2.6 Proof that PARITY \(\not\in AC^0\)

We will prove this theorem in two steps, the first of which we will show that for any given \(AC_0\) circuit, there is a low degree polynomial that approximates the circuit, and the second of which we will show that parity cannot be approximated by a low degree polynomial.

**Lemma:** Every function \(f : \mathbb{F}_p^n \rightarrow \mathbb{F}\) is computed by a unique polynomial of degree at most \(p - 1\) in each variable.

**Proof.** Given any \(a \in \mathbb{F}_p^n\), consider the polynomial:

\[
1_a = \prod_{i=1}^{n} \prod_{z \in \mathbb{F}_p, z \neq a_i} \frac{(X_i - z_i)}{(a_i - z_i)}
\]

We have that:

\[
1_a(b) = \begin{cases} 
1 & \text{if } a = b \\
0 & \text{else}
\end{cases}
\]

Furthermore, each variable has degree at most \(p - 1\) in each variable. Now given any function \(f\), we can represent \(f\) using the polynomial:

\[
f(X_1, \cdots, X_n) = \sum_{a \in \mathbb{F}_p^n} f(a) \cdot 1_a
\]

To prove that this polynomial is unique, note that the space of polynomials whose degree is at most \(p - 1\) in each variable is spanned by monomials where the degree in each of the variables is at most \(p - 1\), so it is a space of dimension \(p^n\) (i.e. there are \(p^n\) monomials).
Similarly, the space of functions $f$ is also of dimension $p^n$. Therefore the correspondence must be one to one.

Suppose that we are given a circuit $C \in \text{AC}_0$, we build an approximating polynomial gate by gate. The input gates are relatively straightforward, $x_i$ is a good approximation to the $i$-th input. Similarly, the negation of $f_i$ is the same as the polynomial $1 - f_i$.

The difficult case is a function like $f_1 \land f_2 \land \cdots \land f_t$ which can be computed by a single gate in the circuit. The naive approach would be to use the polynomial $\prod_{i=1}^t f_i$. However, this gives a polynomial whose degree may be as large as the fan-in of the gate, which is too large for our purposes.

We will use the following trick, let $S \subset [t]$ be a completely random set, and consider the function $\sum_{i \in S} f_i$. Then we have the following claim.

**Claim:** If there is some $j$ such that $f_j \neq 0$, then $P_{S}[\sum_{i \in S} f_i = 0] \leq 1/2$

**Proof.** Observe that for every set $T \subseteq [n] - \{j\}$, it cannot be that both:

$$\sum_{i \in T} f_i = 0$$

and:

$$f_j + \sum_{i \in T} f_i = 0$$

Therefore at most half of the sets can give a non-zero sum. \qed

Now note that squaring turns non-zero values into 1, so let us select independent uniformly random sets $S_1, \ldots, S_t \subseteq [t]$ and use the approximation:

$$g = 1 - \prod_{k=1}^t (1 - (\sum_{i \in S_k} f_i)^2)$$

**Claim:** If each $f_i$ has degree at most $r$, then $g$ has degree at most $2tr$ and:

$$P[g \neq f_1 \lor f_2 \lor \cdots \lor f_t] \leq 2^{-t}$$

Overall, if the circuit is of depth $h$, and has $s$ gates, then this process produces a polynomial whose degree is at most $(2l)^h$ that agrees with the circuit on any fixed input except with probability $s2^{-l}$ by the union bound. Therefore in expectation, the polynomial we produce will compute the correct value on a $1 - s2^{-l}$ fraction of all inputs.

Setting $l = \log^2 n$ we obtain a polynomial of degree $\text{polylog}(n)$ that agrees with the circuit on all but one percent of the inputs.

Now it remains to prove the following theorem.

**Theorem:** Let $f$ be any polynomial over $\mathbb{F}_2$ in $n$ variables whose degree is $d$. Then $f$ can compute the parity on at most $1/2 + O(d/\sqrt{n})$ fraction of all inputs.

**Proof.** Consider the polynomial:

$$g(Y_1, \ldots, Y_n) = f(Y_1 - 1, Y_2 - 1, \ldots, Y_n - 1) + 1$$

The key point is that when $Y_1, \ldots, Y_n \in \{1, -1\}$, if $f$ computes the parity of $n$ bits, then $g$ computes the product $\prod Y_i$. Therefore, we have found a degree $d$ polynomial that can compute the same quantity as the product of $n$ variables. We shall show that this computation cannot work on a large fraction of inputs using a counting argument.

Let $T \subseteq \{1, -1\}^n$ denote the set of inputs for which $g(y) = \prod y_i$. To complete the proof, it will suffice to show that $T$ consists of at most $1/2 + O(d/\sqrt{n})$ fraction of all strings.
Consider the set of all functions $q : T \rightarrow \mathbb{F}_3$. This is a space dimension $|T|$. We shall show how to compute every such function using a low degree polynomial.

By Fact 2, every such function $q$ can be computed by a polynomial. Note that in any such polynomial, since $y_i \in \{-1, 1\}$, we have that $y_i^2 = 1$ so we can assume that each variable has degree at most 1. Now suppose $I \subset [n]$ is a set of size more than $n/2$, then for $y \in T$:

$$\prod_{i \in I} y_i = \left( \prod_{i=1}^n y_i \right) \left( \prod_{i \notin I} y_i \right) = g(y) \left( \prod_{i \notin I} y_i \right)$$

In this way, we can express every monomial of $q$ with low degree terms, and so obtain a polynomial of degree at most $n/2 + d$ that computes $q$.

The space of all such polynomials is spanned by $\sum_{i=0}^{n/2+d} \binom{n}{i}$ monomials. Therefore we get that:

$$|T| \leq \sum_{i=0}^{n/2+d} \binom{n}{i}$$

$$\leq 2^{n/2} + \sum_{i=n/2+1}^{d} \binom{n}{i}$$

$$\leq 2^{n/2} + O(d \cdot 2^n/\sqrt{n}) = 2^n (1/2 + O(d/\sqrt{n}))$$

\[ \square \]

3 Beyond Canonical Hastad [Bea94]

Whilst uniform restrictions consider the variables set equally likely to values 0 or 1, we can also consider restrictions whether the input variables are set to these values with imbalanced probability. This method is used to get circuit lower bounds for the CLIQUE problem.

The key idea is to provide weights to restrictions which reflect the probability of the restriction being chosen as a random member of the set. Indeed suppose that in the context of Hastad’s switching lemma, we wanted to argue that there is a restriction which is strongly biased towards setting bits to 1 and keeps the decision tree height small. The basic switching Lemma does not give us this information because as $n$ gets large it is much more unlikely to get a restriction with even a constant factor bias than the probability of failure.

3.1 Other Restriction Methods

The Stars$_m$ restriction is defined as a map from $\{x_1, \ldots, x_n\} \rightarrow \{0, 1, *\}$ with exactly $m$ stars and behaves similarly to $R_{m/n}$. The corresponding switching lemma is:

$$P[\text{DT}_{\text{depth}}(k - \text{DNF}_{\text{Stars}_m} \geq t)] \leq O((m/n)k)^t$$

Consider the $q$-biased $p$-restriction $R_{p,q}$:

$$R_{p,q}(x_i) = \begin{cases} * \text{ w.p. } p \\ 1 \text{ w.p. } (1-p)q \\ 0 \text{ w.p. } (1-p)(1-q) \end{cases}$$

Now the corresponding switching lemma (for $q \leq 0.5$) is as follows:

$$P[\text{DT}_{\text{depth}}(k - \text{DNF}_{R_{p,q}} \geq t)] \leq O(pk/q)^t$$

This is typically used for average case lower bounds under $q$-biased distributions on $\{0,1\}^n$.

Another switching lemma forms the foundation of the so-called “clique switching lemma” for the random restriction on $\binom{n}{2}$ variables such that the stars are edges of a clique on
a $p$-random set of vertices and the non-stars are set to 1 with probability $q$ and 0 with probability $1 - q$.

The corresponding switching lemma for $q \leq 0.5$ is as follows:

$$PDT_{\text{depth}}(k - \text{DNF}_{\text{Clique}_{p,q}}) \geq t \leq O(pk/q)O(k+t)t$$

In particular, this provides a $n^{\Omega(k/d^2)}$ lower bound for $k$-CLIQUE$_n$. Readers are encouraged to look at [Bea94] for a more in-depth analysis on these biased-restrictions, their applications and motivations.

### 3.2 Consequences of Parity not in $AC^0$

PARITY $\notin AC_0$ is the way to obtain an oracle that separates PSPACE from the Polynomial Hierarchy. Hence, no proof that relativises can be used to separate PSPACE from the Polynomial hierarchy. There are many consequences of this result threaded throughout Complexity Theory literature, where three of the key results are highlighted below.

**Fourier Concentration:** The Fourier expansion of a boolean function is its representation as a low-degree polynomial. Using Hastad’s switching lemma, we can show that the Fourier Expansion of any function in $AC^0$ has all of its larger coefficients concentrated on its low order Fourier coefficients; $AC^0$ functions can be approximated by low degree polynomials.

**Pseudo-Random Generators for $AC^0$:** De-randomization studies the possibility of removing or reducing the amount of randomization used by randomized algorithms while still maintaining their efficiency and correctness. Nisan and Wigderson have proved that the randomized analogues of $AC^0$, $RAC^0$ and $BPAC^0$ can be de-randomized in poly-logarithmic space and quasi-polynomial time.

**$AC^0$-Circuit SAT and $\#$-SAT Algorithms:** Given an $AC^0$ circuit, determine whether there exists an input $x$ which evaluates that circuit to 1. Impagliazzo, Matthew and Paturi have demonstrated that Hastad’s Switching Lemma can be used to provide non-trivial algorithms for $AC^0$-circuit SAT.

### 3.3 Comparison with Polynomial Method

The polynomial method for proving lower bounds on PARITY is similar to the method of random restrictions in the sense that we take advantage of a property which is common to all circuits of small size and constant depth with PARITY does not have. This property is that circuits of small size and constant depth can be represented by low degree polynomials with high probability.

In particular, we use the following fact and two lemmas to prove the result.

**Fact:** Any function $g : \{0, 1\}^n \to \mathbb{R}$ has degree at most $d$ if and only if $\hat{g}_\alpha = 0$ for all $\alpha$ such that $|\alpha| > d$.

**Lemma 1:** For every circuit $C$ of size $S$ and depth $d$, there is a function $g : \{0, 1\}^n \to \mathbb{R}$ of degree $O((\log S)^{2d})$ such that $g$ and $C$ agree on at least $3/4$ fraction of $\{0, 1\}^n$.

**Proof.** Given a circuit $C$ of size $S$ and depth $d$, for every gate we pick independently an approximating function $g_i$ with parameter $\epsilon = \frac{1}{15}$ and replace the gate by $g_i$. Then for a given input, the probability that the new function so defined computes $C(x)$ correctly is at least the probability that the results of all the gates are correctly computed, which is at least $\frac{3}{4}$. In particular, there is a function among those generated this way that agrees with $C()$ on at least $3/4$ fraction of inputs. Each $g_i$ has degree at most $O((\log S)^2)$, because the fan-in of each gate is at most $S$, and the degree of the function defined in the construction is at most $O((\log S)^{2d})$.

**Lemma 2:** Let $g : \{0, 1\}^n \to \mathbb{R}$ be a function that agrees with PARITY on at least $3/4$ fraction of $\{0, 1\}^n$. Then the degree of $g$ is $\Omega(\sqrt{n})$.
Proof. Let \( g : \{0,1\}^n \to \mathbb{R} \) be a function of degree at most \( t \) that agrees with \( \text{PARITY} \) on at least \( 3/4 \) fraction of inputs. Let \( G : \{-1,1\}^n \to \mathbb{R} \) be defined as:
\[
G(x) = 1 - 2g(\frac{1}{2} - \frac{1}{2}x_1, \cdots, \frac{1}{2} - \frac{1}{2}x_n)
\]
Now note that \( G \) is still of degree at most \( t \), and \( G \) agrees with the function \( \prod(x_1, \cdots, x_n) = x_1 \cdot x_2 \cdots x_n \) on at least \( 3/4 \) fraction of \( \{-1,1\}^n \).

Define \( A \) to be the set of \( x \in \{-1,1\}^n \) such that \( G(x) = \prod x_i \):
\[
A = \{ x : G(x) = \prod_{i=1}^n x_i \}
\]

Then \( |A| \geq \frac{3}{4} \cdot 2^n \) by our initial assumption. Now consider the set \( F \) of all functions \( f : A \to \mathbb{R} \). These form a vector space of dimension \( |A| \) over the reals. We know that any function \( f \) in this set can be written as:
\[
f(x) = \sum_\alpha f_\alpha \prod_{i \in \alpha} x_i
\]

Over \( A \), \( G(x) = \prod_{i=1}^n x_i \) and therefore for \( x \in A \):
\[
\prod_{i \in \alpha} x_i = G(x) \prod_{i \notin \alpha} x_i
\]

Now by our initial assumption, \( G(x) \) is a polynomial of degree at most \( t \). Therefore, for every \( \alpha \) such that \( |\alpha| \geq \frac{n}{2} \), we can replace \( \prod_{i \notin \alpha} x_i \) by a polynomial of degree less than or equal to \( t + \frac{n}{2} \). Every such function \( f \) which belong to \( F \) can be written as a polynomial of degree at most \( t + \frac{n}{2} \). Hence the set \( \{ \prod_{i \in \alpha} x_i \}_{|\alpha| \leq t+\frac{n}{2}} \) forms a basis for the set \( S \). As there must be at least \( |A| \) such monomials, this implies that:
\[
\sum_{k=0}^{t+\frac{n}{2}} \binom{n}{k} \geq \frac{3}{4} \cdot 2^n
\]

And in particular:
\[
\sum_{k=\frac{n}{4}}^{t+\frac{n}{2}} \binom{n}{k} \geq \frac{1}{4} \cdot 2^n
\]

Now we know from Stirling’s approximation that every binomial coefficient \( \binom{n}{k} \) is at most \( O(2^n/\sqrt{n}) \), and therefore we obtain:
\[
O(\frac{t}{\sqrt{n}} \cdot 2^n) \geq \frac{1}{4} \cdot 2^n
\]

This implies that \( t = \Omega(\sqrt{n}) \). \( \square \)

And the theorem is subsequently proved as follows.

Proof. From Lemma 1, we have that there is a function \( g : \{0,1\}^n \to \mathbb{R} \) that agrees with \( \text{PARITY} \) on a \( 3/4 \) fraction of \( \{0,1\}^n \), whose degree is at most \( O((\log S)^2d) \). From Lemma 2, we can deduce that the degree of \( g \) must be at least \( \Omega((\sqrt{n}) \) so that:
\[
(\log S)^{2d} = \Omega(\sqrt{n})
\]

which is equivalent to:
\[
S = 2^{\Omega(n^{1/4d})}
\]

\( \square \)
In comparison with the polynomial method, the method of random restrictions prove a stronger and tighter lower bound. Furthermore, they use a property of the parity function which is true of other functions. Therefore, it truly is a general method to attack circuit lower bounds.

Polynomial methods can also accomplish this, and further has the advantage that it can be applied to any circuit model in which gates can be approximated by low-degree polynomials. For example, consider the AC0 model in which we have NOT gates, unbounded fan-in AND and OR gates and also MOD3 gates that, given boolean inputs $x_1, \ldots, x_n$ output 1 if and only if $\sum_i x_i \equiv 1 \lor 2 \mod 3$.

The method of random restrictions cannot be applied to such circuits, because a MOD3 gate has a value that remains under-determined as long as at least three variables are not fixed and requires a CNF of size exponential in the number of non-fixed variables. We can prove that every $AC^0^3$ circuit of depth $d$ that computes PARITY must size at least $2^{\Omega(n^{1/(4d)})}$.

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