A deformation of affine Hecke algebra and integrable stochastic particle system

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Abstract

We introduce a deformation of the affine Hecke algebra of type $GL$ which describes the commutation relations of the divided difference operators found by Lascoux and Schützenberger and the multiplication operators. Making use of its representation we construct an integrable stochastic particle system. It is a generalization of the $q$-Boson system due to Sasamoto and Wadati. We also construct eigenfunctions of its generator using the propagation operator. As a result we get the same eigenfunctions for the $(q, \mu, \nu)$-Boson process obtained by Povolotsky.

Keywords: affine Hecke algebra, integrable systems, stochastic processes

1. Introduction

In this article we introduce a deformation of the affine Hecke algebra of type $GL$ and construct an integrable stochastic particle system making use of its representation.

In a previous paper [11] we constructed a discrete analogue of the non-ideal Bose gas with delta-potential interactions on a circle, which we call the periodic delta Bose gas for short. While a discretization of the periodic delta Bose gas and its generalization was studied by van Diejen [3, 4] from the viewpoint of the theory of Macdonald’s spherical functions, our discretization is motivated by the desire to understand an algebraic structure of integrable stochastic models.

The discrete model constructed in [11] contains two parameters. Specializing the parameters suitably and taking the limit as the system size goes to infinity, its Hamiltonian $H$ becomes the time evolution operator for the joint moment of the integrable stochastic system called the O’Connell-Yor semi-discrete directed polymer [8]. We can construct eigenfunctions of $H$ using the propagation operator $G$, which sends an eigenfunction of (a half of) the discrete Laplacian to that of $H$. To define $G$ we generalize the construction by van Diejen and
Emsiz [5] of the integral-reflection operators due to Yang [12] and Gutkin [6]. Then the discrete integral-reflection operators determine a representation of the affine Hecke algebra of type GL.

Recently the author found that the Hamiltonian $H$ is related to another stochastic system more closely. The operator $H$ acts on the space of functions on the orthogonal lattice $\mathbb{Z}^k$, where $k$ is the number of particles, and leaves the space of symmetric functions invariant. Identify the space of symmetric functions with the space of functions on the fundamental chamber $W^k = \{(m_1, ..., m_k) \in \mathbb{Z}^k | m_1 \geq \cdots \geq m_k\}$. We assign to each element $(m_1, ..., m_k)$ of $W^k$ the configuration of $k$ bosonic particles on $\mathcal{Z}$ such that the particles are on the sites $m_1, ..., m_k$. Then, by specializing the two parameters of $H$ in another way and adding a constant, we obtain the transition rate matrix of the $q$-Boson system introduced by Sasamoto and Wadati [10].

In this paper we generalize the above construction of an integrable stochastic particle system. Our ingredient is a deformation of the affine Hecke algebra of type GL. In [7] Lascoux and Schützenberger characterize the difference operators acting on polynomials which satisfy the braid relations. The operators contain four parameters and at a spacial point they turn into the Demazure-Lustzig operators which give a polynomial representation of the affine Hecke algebra. Our deformed algebra arises from the commutation relations between the difference operators due to Lascoux and Schützenberger and the multiplication operators. By definition it has a polynomial representation. Making use of it we can construct the discrete integral-reflection operators with more parameters and define the propagation operator $G$ as before. One of the main results of this article is construction of the discrete Hamiltonian $H$ satisfying the commutation relation $\Delta G = G H$, where $\Delta$ is the discrete Laplacian, in this generalized setting.

Our operator $H$ also leaves the space of symmetric functions invariant, and by specializing the four parameters suitably we obtain a transition rate matrix of a continuous time Markov chain on $W^k$. The resulting model is described as follows. It is a stochastic particle system on the one-dimensional lattice $\mathcal{Z}$ controlled by two parameters $s$ and $q$. The particles can occupy the same site simultaneously. Some particles may move from site $i$ to $i-1$ independently for each $i \in \mathcal{Z}$. The rate at which $r$ particles move to the left from a cluster with $c$ particles is given by

$$s^{r-1} \frac{r-1}{[r]} \prod_{p=0}^{r-1} \frac{[c-p]}{1 + s[c - 1 - p]} \quad (c \geq r \geq 1),$$

where $[n] := (1 - q^n)/(1 - q)$ is the $q$-integer. In the case of $s = 0$, the rate is equal to zero unless $r = 1$ and hence only one particle may move with the rate proportional to $1 - q^r$. Thus we recover the $q$-Boson system.

Using the propagation operator $G$ we can construct symmetric eigenfunctions of the operator $H$ by means of the Bethe ansatz method, which we call the Bethe wave functions. After the specialization of the parameters we obtain the eigenfunctions of the transition rate matrix. They are parameterized by a tuple $z = (z_1, ..., z_k)$ of distinct constants and are given by

$$\sum_{\sigma \in S_k} \prod_{1 \leq i < j \leq k} \frac{q^{z_{\sigma(i)} - z_{\sigma(j)}}}{z_{\sigma(i)} - z_{\sigma(j)}} \prod_{i=1}^{k} \left( \frac{1 - \nu z_{\sigma(i)}}{1 - z_{\sigma(i)}} \right)^{m_i} \left( (m_1, ..., m_k) \in W^k \right),$$

where $\nu := s/(1 - q + s)$. Here we note that they are equal to the eigenfunctions of the generator of the $(q, \mu, \nu)$-Boson process introduced by Povolotsky [9] (see also [2]).
The paper is organized as follows. In section 2 we define the deformation of the affine Hecke algebra and introduce its representations which are the origin of the propagation operator. In section 3 we define the discrete Hamiltonian \( H \) and the propagation operator \( G \), and prove the commutation relation \( HG = GΔ \). Using the operator \( G \) we construct the Bethe wave functions. In section 4 we describe the construction of the stochastic particle system arising from the Hamiltonian \( H \). We prove some polynomial identities which we use to rewrite the operator \( H \) in appendix.

2. A deformation of affine Hecke algebra and its representation

2.1. Preliminaries

Throughout this paper we fix an integer \( k \geq 2 \). Let \( V := \bigoplus_{i=1}^k \mathbb{R} v_i \) be the \( k \)-dimensional Euclidean space with an orthonormal basis \( \{ v_i \}_{i=1}^k \), and \( V^\ast \) the linear dual of \( V \). We let \( \{ e_i \}_{i=1}^k \) denote the dual basis of \( V^\ast \) corresponding to \( \{ v_i \}_{i=1}^k \). Set \( a_{ij} := e_i - e_j \) for \( i, j = 1, \ldots, k \). The subset \( R := \{ a_{ij} \mid i \neq j \} \) of \( V^\ast \) forms the root system of type \( A_{k-1} \) with the simple roots \( a_i := a_{i, i+1} \) (1 \( \leq i < k \)). Denote by \( R^\pm \) the set of the associated positive and negative roots. For \( v \in V \), set

\[
I(v) := \left\{ a \in R^+ \mid a(v) < 0 \right\}.
\]

The Weyl group \( W \) of type \( A_{k-1} \) is generated by the orthogonal reflections \( s_i : V \to V \) (1 \( \leq i < k \)) defined by \( s_i(v) := v - a_i(v)a_i^\vee \), where \( a_i^\vee := v_i - v_{i+1} \) is the simple coroot. Denote the length of \( w \in W \) by \( \ell(w) \).

For any \( v \in V \), the orbit \( Wv \) intersects the closure of the fundamental chamber

\[
\mathcal{C}_+ := \left\{ v \in V \mid a_i(v) \geq 0 \ (i = 1, \ldots, k-1) \right\}
\]

at one point. Take a shortest element \( w \in W \) such that \( wv \in \mathcal{C}_+ \). Then \( I(v) = R^+ \cap w^{-1}R^- \) and hence the shortest element is uniquely determined. Denote it by \( w_v \).

We will make use of the following proposition.

Proposition 2.1. Suppose that \( v, v' \in V \) satisfy \( I(v) \subseteq I(v') \). Then \( w_{v'} = w_{w_{v'}w_v} \) and \( \ell(w_{v'}) = \ell(w_{w_{v'}w_v}) + \ell(w_v) \).

2.2. A deformation of affine Hecke algebra

Let us define a deformation of the affine Hecke algebra of type \( GL_k \).

Definition 2.2. Let \( \alpha, \beta, \gamma, \delta \) be complex constants and set

\[
g := 1 + \beta \gamma - \alpha \delta.
\]

We define the algebra \( A_k \) to be the unital associative \( \mathbb{C} \)-algebra with the generators \( X^\pm_i (1 \leq i \leq k) \) and \( T^j_i (1 \leq i < k) \) satisfying the following relations:
When $\beta = \gamma = 0$, the algebra $A_k$ is isomorphic to the affine Hecke algebra of type $GL_k$.

We will use the property that any symmetric polynomial in $X_1, \ldots, X_k$ commutes with $T_i$ ($1 \leq i < k$) in $A_k$.

Set

$$L := \bigoplus_{i=1}^{k} \mathbb{Z} v_i$$

and denote by $F(L)$ the vector space of $\mathbb{C}$-valued functions on $L$. The Weyl group acts on $F(L)$ by $(w f)(x) := f(w^{-1} x)$. Set

$$F(L)^W := \{ f \in F(L) \mid w f = f \text{ for all } w \in W \}.$$

Now we introduce a right action of $A_k$ on the group algebra $C[L]$ due to Lascoux and Schützenberger [7]. In the following we identify $C[L]$ with the Laurent polynomial ring $C[e^{\pm v_1}, \ldots, e^{\pm v_k}]$.

**Proposition 2.3.** [7] Define the $C$-linear operators $\tilde{X}_i$ ($1 \leq i \leq k$) and $\tilde{T}_i$ ($1 \leq i < k$) acting on $C[L]$ from the right by

$$P \tilde{X}_i := e^{-v_i} P, \quad P \tilde{T}_i := P \cdot s_i + \left( a e^{v_i} + \beta \right) \left( e^{v_i} + \delta \right) \left( P - P \cdot s_i \right),$$

where \( \cdot \) stands for the right action of the Weyl group defined by $e^x, w := e^{w^{-1} x} (x \in L, w \in W)$. Then the assignment $X_i \mapsto \tilde{X}_i$ and $T_i \mapsto \tilde{T}_i$ extends uniquely to a right representation of the algebra $A_k$ on $C[L]$.

Consider the non-degenerate bilinear pairing $C[L] \times F(L) \to \mathbb{C}$ defined by $(e^x, f) := f(x)$ for $x \in L$ and $f \in F(L)$. We define the $C$-linear operators $\tilde{X}_i$ ($1 \leq i \leq k$) and $\tilde{T}_i$ ($1 \leq i < k$) acting on $F(L)$ by

$$P \tilde{X}_i f = (P, \tilde{X}_i f), \quad P \tilde{T}_i f = (P, \tilde{T}_i f).$$

From proposition 2.3 they give a left action of $A_k$ on $F(L)$:

**Proposition 2.4.** The assignment $X_i \mapsto \tilde{X}_i$ and $T_i \mapsto \tilde{T}_i$ extends uniquely to a left representation of the algebra $A_k$ on $F(L)$. The action is explicitly given as follows:

$$\left( \tilde{X}_i f \right)(x) = f(x - v_i).$$
If \( a_i(x) > 0 \) then

\[
(\tilde{H} f)(x) = a\delta f(x) + (1 + \beta\gamma)f(s_i x) + \alpha\gamma \sum_{j=1}^{a_i(x)} f(s_i x + ja_i^\vee) + \nu_{i+1} \\
+ (\alpha\delta + \beta\gamma) \sum_{j=1}^{a_i(x)-1} f(s_i x + ja_i^\vee) + \beta\delta \sum_{j=0}^{a_i(x)-1} f(s_i x + ja_i^\vee - \nu_{i+1}).
\]

When \( a_i(x) = 0 \), we have \((\tilde{H} f)(x) = f(x)\). If \( a_i(x) < 0 \) then

\[
(\tilde{H} f)(x) = -\beta\gamma f(x) + (1 - \alpha\delta)f(s_i x) - \alpha\gamma \sum_{j=0}^{-a_i(x)-1} f(s_i x - ja_i^\vee) + \nu_{i+1} \\
- (\alpha\delta + \beta\gamma) \sum_{j=1}^{-a_i(x)-1} f(s_i x - ja_i^\vee) - \beta\delta \sum_{j=1}^{-a_i(x)} f(s_i x - ja_i^\vee - \nu_{i+1}).
\]

We will often use the fact that \((\tilde{H} f)(x) = 0\) for any \( f \in F(L) \) if \( a_i(x) = 0 \).

3. Discrete Hamiltonian and propagation operator

3.1. Discrete Hamiltonian

Hereafter we assume that

\[
1 + \beta\gamma [n] \neq 0
\]

for any positive integer \( n \), where

\[
[n] := \frac{1 - q^n}{1 - q}
\]

is the \( q \)-integer.

We define the functions \( d_i^\pm (1 \leq i \leq k) \) and \( \delta_{j_1, j_2, \ldots, j_r} (1 \leq j_1 < j_2 < \cdots < j_r \leq k) \) on \( L \) by

\[
d_i^+(x) := \# \{ p \mid i < p \leq k, \ a_p(x) = 0 \}, \\
d_i^-(x) := \# \{ p \mid 1 \leq p < i, \ a_p(x) = 0 \}. \tag{3.1}
\]

and

\[
\delta_{j_1, j_2, \ldots, j_r}(x) := \begin{cases} 
1 & (e_{j_1}(x) = \cdots = e_{j_r}(x)), \\
0 & \text{(otherwise)}.
\end{cases}
\]

If \( r = 1 \) we set \( \delta_j \equiv 1 \) by definition.
Now we define the discrete Hamiltonian \( H \) by

\[
H_d := -\alpha \gamma \sum_{j=1}^{k} \left[ d_j^+ \right] + \beta \gamma \sum_{r=1}^{k} (-\beta \delta)^{-1}[r-1]! q^{-r(r-1)/2} \times \sum_{1 \leq i_1 < \cdots < i_l \leq k} q^{\sum_{r=1}^{l} d_{i_r}^+ d_{i_r}^- - p} \prod_{p=0}^{k-1} \prod_{p=0}^{l-1} (1 + \beta \gamma [d_{i_r}^+ + d_{i_r}^- - p]) \prod_{p=1}^{r} \bar{X}_{i_p},
\]

where \([n]! := \prod_{a=1}^{n} [a] (n > 0)\) and \([0]! := 1\). Note that the index \( j_i \) in the factor \(1 + \beta \gamma [d_{i_r}^+ + d_{i_r}^- - p]\) may be replaced with any \( j_p \) because \( d_{i_r}^+(x) + d_{i_r}^-(x) = d_{j_p}^+(x) + d_{j_p}^-(x)\) if \( \alpha \gamma (x) = 0\).

Using the equality

\[
\sum_{j=1}^{k} [d_j^+] = \sum_{j=1}^{k} q^{\delta j^+} d_j^+,
\]

we see that when \( \beta = 0 \) it holds that

\[
H = \sum_{j=1}^{k} q^{\delta j^+} (\bar{X}_j - \alpha \gamma d_j^+).
\]

This operator is introduced in [11] as a discrete analogue of the Hamiltonian of the delta Bose gas under periodic boundary condition.\(^1\)

For convenience we write down the action of \( H \) more explicitly. For a non-empty subset \( J = \{j_1, \ldots, j_m\} (j_1 < \cdots < j_m) \) of \( \{1, 2, \ldots, k\} \), we define the operator \( H_J \) acting on \( F(L) \) by

\[
H_J := -\alpha \gamma \sum_{d=1}^{m-1} \frac{[d]}{1 + \beta \gamma [d]} + \sum_{r=1}^{m} (-\beta \delta)^{-1}[r-1]! q^{-r(r-1)/2} \prod_{p=0}^{m-1} (1 + \beta \gamma [m - 1 - p]) e_r \left( \bar{X}_{j_1}, q\bar{X}_{j_1}, \ldots, q^{m-1}\bar{X}_{j_m} \right),
\]

(3.2)

where \( e_r \) is the elementary symmetric polynomial of degree \( r \). Then the value \((Hf)(x)\) is written as follows.

**Lemma 3.1.** For \( x \in L \), decompose the set \( \{1, 2, \ldots, k\} \) into a direct sum \( \sqcup_{n=1}^{N} J_n^x \) so that \( i \) and \( j \) belong to the same subset \( J_n^x \) if and only if \( \alpha \gamma (x) = 0 \). Then for any \( f \in F(L) \) it holds that

\[
(Hf)(x) = \sum_{n=1}^{N} (H_{J_n^x} f)(x).
\]

(3.3)

From the expression (3.3) we see that

\(^1\) The parameters \( \alpha \) and \( \beta \) in [11] are equal to \( \alpha \gamma \) and \( q = 1 - a\delta \), respectively.
Proposition 3.2. \( H(F(L)^W) \subset F(L)^W \).

Proof. Suppose that \( f \in F(L)^W \). It suffices to show that \( (Hf)(s,x) = Hf(x) \) for any \( x \in L \) and \( 1 \leq i < k \). If \( a_i(x) = 0 \) it is trivial. Let us consider the case of \( a_i(x) \neq 0 \). Denote the transposition \((i, i + 1) \in S_k \) by \( \tau \). Consider the decomposition \( \{1, 2, \ldots, k\} = \bigcup_{k} J^x_n \) given in lemma 3.1 with \( x \) replaced by \( s, x \). Then it holds that \( J^x_n = \tau(J^x_n) \). Note that \( (i, i + 1) \not\in \tau(J^x_n) \) for any \( n \) because \( a_i(x) \neq 0 \). For \( 1 \leq j_1 < \cdots < j_m \leq k \) satisfying \( (i, i + 1) \not\in \tau(J^x_n) \) and \( 0 \leq r \leq m \), it holds that
\[
(e_r(e_{i}^{\tau}, q^x_{i}, q^x_{i+1}, \ldots, q^{m-1}x_{i+1}))^{(s, x)} = (e_r(e_{i}^{\tau}, q^x_{i}, q^x_{i+1}, \ldots, q^{m-1}x_{i+1}))^{(s, x)}
\]
because \( f \in F(L)^W \). Therefore \( (Hf)(s, x) = (Hf)(x) \). From lemma 3.1 we find that \( (Hf)(s, x) = (Hf)(x) \).

For later use we rewrite the operator \( H_J \) using the two equalities below. See appendix for the proof.

Lemma 3.3. Let \( m \) be a positive integer and \( z_1, \ldots, z_m \) commutative indeterminates. Then the following equality holds.
\[
\sum_{r=1}^{m} \frac{(-\beta \delta)^{r-1}[r-1]! q^{-r(r-1)/2}}{\prod_{p=0}^{r-1} (1 + \beta \gamma [m - 1 - p])} e_r(z_1, qz_2, \ldots, q^{m-1}z_m) = \frac{1}{\beta} \sum_{r=1}^{m} \frac{(-\delta)^{r-1}[r-1]!}{\prod_{p=1}^{r} (1 + \beta \gamma[p - 1])} \sum_{1 \leq b_1 < \cdots < b_r \leq m} q^{b_r-m} \sum_{p=1}^{r} \left\{ \prod_{i=1}^{r} (\alpha + \beta q^{p-1}z_{b_i}) - \alpha^r \right\}.
\]

Lemma 3.4. Let \( m \) be a positive integer. Then the following equality holds.
\[
\frac{1}{\beta} \sum_{r=1}^{m} \frac{(-\delta)^{r-1}[r-1]!}{\prod_{p=1}^{r} (1 + \beta \gamma[p - 1])} \alpha^r \sum_{1 \leq b_1 < \cdots < b_r \leq m} q^{b_r-m} \sum_{p=1}^{r} \left\{ \prod_{i=1}^{r} (\alpha + \beta q^{p-1}z_{b_i}) - \alpha^r \right\} = \frac{1}{\beta} \sum_{d=0}^{m-1} \frac{1}{1 + \beta \gamma[d]}. \]

Lemmas 3.3 and 3.4 imply the following formula.

Proposition 3.5. Let \( J = \{j_1, \ldots, j_m\} \) \((j_1 < \cdots < j_m)\) be a non-empty subset of \( \{1, 2, \ldots, k\} \). Then it holds that
\[ H_j = -\frac{\alpha}{\beta} m \]
\[ + \frac{1}{\beta} \sum_{r=1}^{m} (-\beta^{r-1} [r - 1]! \prod_{p=1}^{r} (1 + \beta^{r-1} [p - 1]) \sum_{1 \leq b_1 < \cdots < b_r < m} \left\{ \sum_{q=1}^{m} \left\{ b_p - m \right\} \prod_{p=1}^{r} \left( \alpha + \beta^{p-1} \bar{X}_{b_p} \right) \right\} \times \]

3.2. Propagation operator

Let \( w \) be an element of the Weyl group \( W \) and \( w = s_{i_1} \cdots s_{i_r} \in W \) a reduced expression. Then we set \( \mathcal{T}_w := \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_r} \). It does not depend on the choice of the reduced expression of \( w \).

**Definition 3.6.** We define the propagation operator \( G : F(L) \to F(L) \) by
\[
G(f)(x) := (\mathcal{T}_w f)(w, x).
\]

Hereafter, for \( x \in L \), we denote by \( \sigma_x \) the element of the symmetric group \( S_k \) determined by
\[
\sigma_x(v_j) = v_{\sigma(j)} \quad (1 \leq i \leq k).
\]

Then \( \epsilon_i(x) = \epsilon_{\sigma(i)}(w, x) \) for \( 1 \leq i \leq k \).

In the rest of this subsection we prove the following proposition.

**Proposition 3.7.** Suppose that \( 1 \leq t_1 < \cdots < t_r \leq k \) and that \( x \in L \) satisfies \( \epsilon_{t_1}(x) = \cdots = \epsilon_{t_r}(x) \). Then for any \( f \in F(X) \) it holds that
\[
\left( \mathcal{T}_{i_{t_1}} \cdots \mathcal{T}_{i_{t_r}} G(f) \right)(x)
\]
\[
= \left( \mathcal{T}_{i_{t_1}} \cdots \mathcal{T}_{i_{t_r}} \mathcal{T}_{\sigma(t_1)} \cdots \mathcal{T}_{\sigma(t_r)} (w, x) \right)(w, x),
\]
where \( l = \sigma(t_1) + d^+_n(x) \).

First we note that the functions \( d^\pm_i \) have the following properties.

**Lemma 3.8.** (1) For \( x \in L, 1 \leq i \leq k \) and \( 1 \leq j < k \), it holds that
\[
d^\pm_i(x) \quad (j \neq i - 1, i),
\]
\[
d^\pm_i(s_j x) = \begin{cases} 
  d^\pm_{j-1}(x) - \theta(a_{j-1}(x) = 0) & (j = i - 1), \\
  d^\pm_{j+1}(x) + \theta(a_{j}(x) = 0) & (j = i),
\end{cases}
\]
where \( \theta(P) = 1 \) or 0 if \( P \) is true or false, respectively.

(2) For any \( x \in L \) and \( 1 \leq i \leq k \), it holds that \( d^\pm_i(x) = d^\pm_i(\sigma_i x) \).
Proof. The proof of (1) is straightforward. Let \(w_i = s_{i_1} \cdots s_{i_l}\) be a reduced expression. Then \(a_{i_j}(s_{i_1}, \ldots, s_{i_l}) \neq 0\) for all \(1 \leq p \leq l\), and hence \(d_i^\pm(x) = d_{a_{i_j}}^\pm(w_i)\).

Lemma 3.9. Suppose that \(1 \leq t_1 < \cdots < t_r \leq k\) and that \(x \in L\) satisfies \(e_i(x) = \cdots = e_r(x)\).

(1) The values \(\sigma_i(t_p) + d_i^+(x)\) and \(\sigma_i(t_p) - d_i^-(x)\) are independent of \(p = 1, 2, \ldots, r\).

(2) Set \(l^\pm = \sigma_i(t_1) = d_i^\pm(x)\). Then \(a_{i_{t_r-1}}(w_i) > 0\), \(a_{i_{t_r-1}}(w_i) = 0 \quad (l^+ \leq i < l^+ - l)\), \(a_{i_{t_r-1}}(w_i) = 0\) and \(l^+ - l \leq \sigma_i(t_r) < \cdots < \sigma_i(t_1) \leq l^+\).

Proof. Since \(e_{\sigma_i(t_1)}(w_i) = \cdots = e_{\sigma_i(t_r)}(w_i)\) and \(w_i, x \in \mathcal{H}\), there exist two integers \(l^\pm\) such that \(1 \leq l^+ \leq k\), \(a_{i_{t_r-1}}(w_i) > 0\), \(a_{i_{t_r-1}}(w_i) > 0\) and \(l^+ - l \leq \sigma_i(t_r) \leq l^+\) for all \(1 \leq p \leq r\). Then we have

\[
d_i^\pm(x) = d_{\sigma_i(t_p)}^\pm(w_i) = (l^+ - \sigma_i(t_p)) \quad (1 \leq p \leq r).
\]

Therefore \(\sigma_i(t_p) \pm d_i^\pm(x)\) is equal to \(l^\pm\) for all \(1 \leq p \leq r\). From the definition of \(d_i^\pm(x)\) we have \(d_i^+(x) > \cdots > d_i^-(x)\). Hence it holds that \(\sigma_i(t_1) < \cdots < \sigma_i(t_r)\) because \(l^+ = d_i^+(x) + \sigma_i(t_p)\) is independent of \(p\).

The following lemma is the key to the proof of proposition 3.7.

Lemma 3.10. Suppose that \(1 \leq t_1 < \cdots < t_r \leq k\) and that \(x \in L\) satisfies \(e_i(x) = \cdots = e_i(x)\). Set \(y = x - \sum_{p=1}^r a_{i_p} = \sigma_i(t_1) + d_i^+(x), m = \sigma_i(t_1) - d_i^-(y)\) and

\[
u_1 := \left(s_{r-r+1}, \ldots, s_{r-f(t_1)}\right) \left(s_{r-r+1}, \ldots, s_{f(t_r)}\right) \left(s_{r-r+1}, \ldots, s_{f(t_r)}\right) \left(s_{r-r+1}, \ldots, s_{f(t_r)}\right),
\]

\[
u_2 := \left(s_{m-r-1}, \ldots, s_{f(t_r)}\right) \left(s_{m-r-1}, \ldots, s_{f(t_r)}\right) \left(s_{m-r-1}, \ldots, s_{f(t_r)}\right) \left(s_{m-r-1}, \ldots, s_{f(t_r)}\right).
\]

Then \(\nu_1 w_1 = \nu_2 w_1\) and \(\ell(\nu_1 w_1) = \ell(\nu_2 w_1) = \ell(\nu_1) + \ell(\nu_2)\) (Note that the right hand sides of (3.4) are reduced expressions of \(u_1\) and \(u_2\)).

Proof. Set \(z = x - \sum_{p=1}^r a_{i_p} / 2\). Since \(\sum_{p=1}^r a_{i_p} / 2 < 1\) for any \(a \in \mathcal{R}^+, I(x)\) and \(I(y)\) are contained in \(I(z)\). Hence proposition 2.1 implies that \(w_i = w_i w_i = w_i w_i = \ell(\nu_1 z) + \ell(\nu_2 z) + \ell(\nu_1)\). Thus it suffices to show that \(w_i = w_i\) and \(w_i = z = \nu_2 = \nu_1\). Here we prove that \(w_i = \nu_1\) for \(w_i = \nu_2\) is similar.

Let us write down the set \(I(w_i)\). It consists of all the positive roots \(a \in \mathcal{R}^+\) such that \(a(w_i) = a(w_i) - \sum_{p=1}^r a_{i_p} / 2 < 0\). Since \(w_i \in \mathcal{H}\) it is equivalent to requiring that \(a(w_i) = 0\) and there exists \(p\) such that \(a(w_i) = 1\) and \(a(w_i) = 0\) if \(q \neq p\). Therefore, from lemma 3.9, we find that

\[
I(w_i) = \bigcup_{p=1}^r \left\{ a_{i_p} | \sigma_i(t_p) < i \leq l, i \neq \sigma_i(t_{p+1}), \ldots, \sigma_i(t_r) \right\}.
\]

It is equal to \(\mathcal{R}^+ \cap u_{i_{t_r-1}}^{-1} (\mathcal{R}^+)\), and hence \(w_i = \nu_1\).

Now let us prove proposition 3.7. We use the notation of lemma 3.10. Since \(a_{i_p}(w_i) = 0\) for \(m \leq i < \sigma_i(t_r)\) and \(a_{i_p}(w_i) = 0\) for \(\sigma_i(t_1) < i < l\), it holds that
\[ w_i y = u_2 w_i y = u_1 w_i \left( x - \sum_{p=1}^{r} v_{p} \right) = u_1 \left( w_i x - \sum_{p=1}^{r} v_{p} \right) = w_i x - \sum_{j=1}^{r} v_{j} \]

and \((T^{-1}_{u_2} g)(w_i y) = g(w_i y)\) for any \( g \in F(L) \). Moreover \( T_{u_2} = T_{u_1} T_{u_1} \). Therefore

\[
(\tilde{X}_i \cdots \tilde{X}_i G(f))(x) = (\tilde{T}_{u_2} f)(w_i y) = (\tilde{T}_{u_1} \tilde{T}_{u_1} f) \left( w_i x - \sum_{j=1}^{r} v_{j} \right) = (\tilde{X}_{i-r+1} \cdots \tilde{X}_i \tilde{T}_{u_1} \tilde{T}_{u_1}(f))(w_i x).
\]

This completes the proof of proposition 3.7.

### 3.3. Commutation relation of \( H \) and \( G \)

In this subsection we prove the following theorem.

**Theorem 3.11.** It holds that \( HG = G(\sum_{i=1}^{k} \tilde{X}_i) \). Therefore, if \( f \in F(L) \) is an eigenfunction of the difference operator \( \sum_{i=1}^{k} \tilde{X}_i \), then \( G(f) \) is an eigenfunction of the discrete Hamiltonian \( H \) with the same eigenvalue.

Hereafter we set

\[ \tilde{V}_i := \alpha + \beta \tilde{X}_i \quad (1 \leq i \leq k). \]

**Lemma 3.12.** Suppose that \( 1 \leq i < k \) and that \( x \in L \) satisfies \( a_i(x) = 0 \). Then for any \( g \in F(L) \) and \( p \geq 1 \) it holds that

\[
\left( q^{p-1} + \delta[p-1] \tilde{V}_i \right)(\tilde{T}_i + \beta(\gamma + \delta \tilde{X}_{i+1}))(g)(x) = \left( q^p + \delta[p] \tilde{V}_{i+1} \right)(g)(x).
\]

**Proof.** For simplicity we write \( R \equiv P_2 \) if two operators \( R, P_2 \) acting on \( F(L) \) satisfy \((R(g))(x) = (P_2(g))(x)\). Since \( a_i(x) = 0 \) it holds that \( \tilde{T}_i Q \equiv Q \) for any operator \( Q \) acting on \( F(L) \). Using \( \beta(\gamma + \delta \tilde{X}_{i+1}) = q - 1 + \delta \tilde{V}_{i+1} \) we obtain

\[
\left( q^{p-1} + \delta[p-1] \tilde{V}_i \right)(\tilde{T}_i + \beta(\gamma + \delta \tilde{X}_{i+1})) \equiv q^{p-1}(q + \delta \tilde{V}_{i+1} + \delta[p-1] \tilde{V}_i \tilde{T}_i + \delta[p-1] \tilde{V}_i \tilde{V}_{i+1} (q - 1 + \delta \tilde{V}_{i+1}).
\]

Since \( \tilde{V}_i \tilde{V}_i = \tilde{V}_i \tilde{V}_{i+1} = (q - 1 + \delta \tilde{V}_{i+1}) \), it is equivalent to

\[ q^{p-1}(q + \delta \tilde{V}_{i+1}) + \delta[p-1] \tilde{V}_{i+1} = q^p + \delta[p] \tilde{V}_{i+1}. \]

This completes the proof. \( \square \)
Lemma 3.13. Suppose that $r, l$ and $\nu_1, \ldots, \nu_r$ are positive integers satisfying $1 \leq \nu_1 < \cdots < \nu_r \leq l \leq k$. For a subset $I = \{i_1, \ldots, i_d\} (i_1 < \cdots < i_d)$ of $\{1, 2, \ldots, r\}$, set

$$c(I) := (\beta/q \alpha)^d q^{\sum_{p=r+1}^d i_p}, \quad \tilde{Q}_I := \tilde{X}_{i_{d+1}} \cdots \tilde{X}_i \left( \tilde{T}_{i_{d-1}} \cdots \tilde{T}_{i_1} \right) \cdots \left( \tilde{T}_{i_3} \cdots \tilde{T}_{i_1} \right).$$

For $1 \leq p \leq r$ define the operator $\tilde{S}_p : F(L) \to F(L)$ by

$$\tilde{S}_p := \sum_{I \subset \{p+1, \ldots, r\}} c(I) \prod_{i=p}^{l-1} \left( q^{p-1} + \delta[p-1] \tilde{V}_i \right) \cdot \tilde{Q}_I. \quad (3.5)$$

If $x \in L$ satisfies $a_i(x) = 0$ for $\nu_i \leq i < l$, then it holds that

$$\left( \tilde{S}_p(g) \right)(x) = (1 + \beta \gamma [p-1]) \left( \tilde{S}_{p+1} \tilde{V}_p \prod_{i=p}^{l-1} \left( q^{p-1} + \delta[p] \tilde{V}_i \right) \right)(x)$$

for $1 \leq p \leq r$ and $g \in F(L)$, where $\nu_{r+1} = l + 1$ and $\tilde{S}_{r+1}$ is the identity operator.

**Proof.** We use the same symbol $\equiv$ defined in the proof of lemma 3.12. Decompose the sum $(\tilde{S}_p(g))(x)$ into the two parts with $p \in I$ and $p \notin I$. We set $J = I \setminus \{p\}$ and $J = I$ for the first part and the second, respectively. Each term in the sum of the second part is invariant under the action of $\tilde{T}_{i_{l-1}} \cdots \tilde{T}_{i_p}$, because $a_i(x) = 0$ for $\nu_p \leq i < l - U$. Since $\tilde{T}_i (\nu_p \leq i < l - U)$ commutes with any symmetric polynomial in $\nu_{l-1} \cdots \nu_1$, we have

$$\tilde{S}_p \equiv \sum_{J \subset \{p+1, \ldots, r\}} c(J) \prod_{i=p}^{l-1} \left( q^{p-1} + \delta[p-1] \tilde{V}_i \right) \times \left\{ \beta q^{p-1} \tilde{X}_{l-1} + \alpha \left( q^{p-1} + \delta[p-1] \tilde{V}_{l-1} \right) \right\} \left( \tilde{T}_{l-1} \cdots \tilde{T}_p \right) \tilde{Q}_J.$$

It holds that

$$\beta q^{p-1} \tilde{X}_{l-1} + \alpha \left( q^{p-1} + \delta[p-1] \tilde{V}_{l-1} \right) = (1 + \beta \gamma [p-1]) \tilde{V}_{l-1}$$

and

$$\tilde{V}_{l-1} \cdots \tilde{V}_{l-1} \cdots \tilde{V}_{p} = \prod_{i_1 < \cdots < i_p} \left( \tilde{T}_i + \beta \left( \gamma + \delta \tilde{X}_{i+1} \right) \right) \tilde{V}_{i_p},$$

where $\prod_{i_1 < \cdots < i_p} A_i := A_{m-1} A_{m-2} \cdots A_m$ is an ordered product. Now use lemma 3.12 repeatedly. Since $\prod_{i_1 < \cdots < i_p} (q^{p} + \delta[p] \tilde{V}_i)$ and $\tilde{V}_{i_p}$ commute with $\tilde{Q}_J$ for any $J \subset \{p+1, \ldots, r\}$, we obtained the desired equality. \[\square\]

**Proposition 3.14.** Suppose that $1 \leq \nu_1 < \cdots < \nu_r \leq l$ and that $x \in L$ satisfies $\epsilon_i(x) = \cdots = \epsilon_i(1)$. Then for any $f \in F(L)$ it holds that
\[
\left( \prod_{p=1}^{r} \left( \alpha + \beta q^{p-1} \hat{X}_{p} \right) G(f) \right)(x)
= \prod_{p=1}^{r} \left( 1 + \beta \eta [p - 1] \right)
\times \left( \prod_{p=1}^{r} \left( \hat{V}_{\sigma_{e}(p)} \prod_{\alpha_{e}(p) < \sigma_{e}(p+1)} q^{p + \delta[p] V_{\sigma_{e}(p+1)}} \right) \hat{T}_{\sigma_{e}}(f) \right)(w_{i}x),
\]

where \( \sigma_{e}(t_{i+1}) = \sigma_{e}(t_{i}) + d_{n}^{+}(x) + 1 \).

**Proof.** Proposition 3.7 implies that the left hand side is equal to \((\hat{S}_{i} \hat{T}_{\sigma_{e}}(f))(w_{i}x)\), where \( \hat{S}_{i} \) is the operator defined by (3.5) with \( \nu_{p} = \sigma_{e}(t_{p}) (1 \leq p \leq r) \) and \( l = \sigma_{e}(t_{1}) + d_{n}^{+}(x) \). Since we have \( a_{i}(w_{i}x) = 0 \) for \( \sigma_{e}(t_{i}) \leq i < \sigma_{e}(t_{i}) + d_{n}^{+}(x) \) because of lemma 3.9, we can apply lemma 3.13 repeatedly and get the above formula. \( \square \)

Now we are ready to prove theorem 3.11. Fix \( x \in L \) and decompose \( \{1, 2, \ldots, k\} = \bigcup_{n=1}^{N} J_{n}^{+} \) as described in lemma 3.1. Then each set \( \sigma_{e}(J_{n}^{+}) \) is an interval of successive integers. Take one component \( J_{n}^{+} \) and set \( l^{+} = \min \sigma_{e}(J_{n}^{+}) \) and \( l^{+} = \max \sigma_{e}(J_{n}^{+}) \). From proposition 3.5 and proposition 3.14, we see that

\[
(H_{J_{n}^{+}} G(f))(x) = -\frac{\alpha}{\beta} m \hat{T}_{\sigma_{e}}(f)(w_{i}x)
+ \frac{1}{\beta} \sum_{r=1}^{m} (-\delta)^{-1}[r - 1]! \sum_{l^{+} \leq l_{1} < \cdots < l_{m} \leq l^{+}} q^{\sum_{r=1}^{m} (l_{r} - l^{+})}
\times \left( \prod_{p=1}^{r} \left( \hat{V}_{\nu_{p}} \prod_{\nu_{p} < \nu_{p+1}} q^{p + \delta[p] V_{\nu_{p+1}}} \right) \hat{T}_{\sigma_{e}}(f) \right)(w_{i}x),
\]

where \( \nu_{r+1} = l^{+} + 1 \). Now use the polynomial identity

\[
\sum_{r=1}^{m} (-\delta)^{-1}[r - 1]! \sum_{l^{+} \leq l_{1} < \cdots < l_{m} \leq l^{+}} q^{\sum_{r=1}^{m} (l_{r} - l^{+})}
\times \prod_{p=1}^{r} \left( z_{c_{p}} \prod_{c_{p} < c_{p+1}} q^{p + \delta[p] z_{c_{p+1}}} \right) = \sum_{i=1}^{m} z_{i},
\]

where \( z_{1}, \ldots, z_{m} \) are commutative indeterminates and \( c_{r+1} = m + 1 \). Finally we find that

\[
(H_{J_{n}^{+}} G(f))(x) = \left( \sum_{j \in J_{n}^{+}} \hat{X}_{\sigma_{e}(j)} \hat{T}_{\sigma_{e}}(f) \right)(w_{i}x).
\]

From (3.3) we have

\[
(HG(f))(x) = \left( \sum_{n=1}^{N} \sum_{j \in J_{n}^{+}} \hat{X}_{\sigma_{e}(j)} \hat{T}_{\sigma_{e}}(f) \right)(w_{i}x) = \left( \sum_{j=1}^{l} \hat{X}_{\sigma_{e}(j)} \hat{T}_{\sigma_{e}}(f) \right)(w_{i}x).
\]

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Since \( \sum_{j=1}^{k} \hat{X}_j \) commutes with \( \hat{T}_i \) (1 \( \leq \) \( k \)), it is equal to
\[
\left( \hat{T}_i, \sum_{j=1}^{k} \hat{X}_j (f) \right) (w, x) = G \left( \sum_{j=1}^{k} \hat{X}_j f \right) (x).
\]
This completes the proof of theorem 3.11.

### 3.4. Bethe wave functions

Using theorem 3.11 we can construct symmetric eigenfunctions of \( H \), which we call the Bethe wave functions. Set
\[
L_+ := L \cap \mathcal{T}_+.
\]

**Proposition 3.15.** For a tuple \( p = (p_1, \ldots, p_k) \) of distinct complex parameters, define the function \( \Phi_p \in F(L)^W \) by
\[
\Phi_p \left|_{L_+} \right. = \sum_{\sigma \in \mathfrak{S}_k} \prod_{1 \leq \sigma(\ell) \leq \sigma(k)} \left( 1 + \frac{\alpha + \beta \sigma(\ell)}{\gamma \sigma(\ell) - \delta \sigma(\ell)} \right) \prod_{j=1}^{k} p_{\sigma(j)}. \tag{3.6}
\]
Then \( \Phi_p \) is an eigenfunction of the discrete Hamiltonian \( H \) with eigenvalue \( \sum_{j=1}^{k} p_j \).

**Proof.** Denote by \( h_p \) the function defined by the right hand side of (3.6) on the whole lattice \( L \). For \( \lambda \in V^\text{st} \) define the function \( e^\lambda \in F(L) \) by \( e^\lambda(x) := e^{\lambda(x)} \). Then it holds that
\[
\hat{T}_i e^\lambda = \left( s_i + \frac{\alpha + \beta e^{\lambda(v)}}{\lambda(v) - \delta e^{\lambda(v+1)}} \right) \left( s_i - 1 \right) e^\lambda
\]
for \( 1 \leq i \leq k \). It implies that \( \hat{T}_i h_p = h_p \) for any \( 1 \leq i \leq k \). Hence we have
\[
G \left( h_p \right) (x) = \left( \hat{T}_i h_p \right) (w, x) = h_p (w, x) = \Phi_p (w, x) = \Phi_p (x),
\]
that is \( G(h_p) = \Phi_p \). Since \( h_p \) is an eigenfunction of \( \sum_{j=1}^{k} \hat{X}_j \) with eigenvalue \( \sum_{j=1}^{k} p_j \), it holds that \( H \Phi_p = \left( \sum_{j=1}^{k} p_j \right) \Phi_p \) because of theorem 3.11.

### 4. Construction of integrable stochastic particle system

Hereafter we identify the space of symmetric functions \( F(L)^W \) with the space of functions on \( L_+ \). Denote it by \( F(L_+) \). A linear operator \( Q \) on \( F(L_+) \) is said to be stochastic if it is given in the form \( (Qf)(x) = \sum_{y \neq x} c(y, x) f(y) - f(x) \) where \( c(y, x) \geq 0 \).

A stochastic operator on \( F(L_+) \) determines a stochastic one-dimensional particle system with continuous time as follows. Denote by \( S_k \) the set of configurations of \( k \) bosonic particles on the one-dimensional lattice \( \mathbb{Z} \). For \( x = \sum_{j=1}^{k} m_j v_j \), denote by \( \nu(x) \) the configuration of \( k \) particles on \( \mathbb{Z} \) such that the particles are on the sites \( m_1, \ldots, m_k \). For example, if \( k = 6 \) and \( x = 3 v_1 + 3 v_2 + 3 v_3 + v_4 - 2 v_5 - 2 v_6 \), \( \nu(x) \) is the configuration where three particles are located on the site 3, one particle on the site 1 and two particles on the site -2. Then the map \( \nu : L_+ \to S_k \) is bijection. We identify \( F(L_+) \) and the set of functions on \( S_k \) through the map \( \nu \).
Then the stochastic operator $Q$ is regarded as the backward generator of the stochastic process on $S_k$ with continuous time, where $c(y, x)$ gives the rate at which the state $\nu(x)$ changes to $\nu(y)$.

Now we give a sufficient condition for $H|_{F(L)^W} = H|_{F(L_+)}$ to be stochastic up to constant.

**Proposition 4.1.** Let $\lambda$ be a constant. The operator $\tilde{H} := (H + \lambda)|_{F(L_+)}$ is stochastic only if $\lambda = -k$ and $(\alpha + \beta)(\gamma + \delta) = 0$.

To prove proposition 4.1, we introduce the cluster coordinate of a point in $L_+$ following [1]. For $x \in L_+$ we determine a set of positive integers $c(M)$ and $c_i(1 \leq i \leq M)$ by the property that $\sum_{i=1}^{M} c_i = k$, $c_i(x) > c_{i+1}(x) > \cdots > c_{i+c_i(x)}(x)$, and $c_j(x) = c_{i+c_i(x)}(x)$ if $c_1 + \cdots + c_{j-1} < j \leq c_1 + \cdots + c_i$. We call the tuple $(c_1, \ldots, c_M)$ the *cluster coordinate* of $x \in L_+$. It describes the number of particles in each cluster in the configuration $\nu(x)$.

In terms of the cluster coordinate the action of $H$ for $f \in F(L)^W$ is written as follows. Fix $x \in L_+$ and let $(c_1, \ldots, c_M)$ be its cluster coordinate. Then

$$
(Hf)(x) = \sum_{i=1}^{M} \left\{ -\alpha f \sum_{d=1}^{c_i} \frac{[d]}{1 + \beta x} \right\} f(x)
+ \sum_{i=1}^{c_i} (-\beta \delta)^{r-1}[r-1]! q^{-r} \left( \frac{1}{1 + \beta x} \right) e_i \left( 1, q, \ldots, q^{c_i-1} \right)
\times f \left( x - \sum_{p=0}^{r-1} v_{c_i+c_i+p} \right). 
$$

We use this formula in the proof below.

**Proof of proposition 4.1.** If $c_1, \ldots, c_M$ are all equal to one, then $(Hf)(x) = \sum_{j=1}^{M} f(x - v_j)$. Hence $\lambda$ should be equal to $-k$ so that $\tilde{H}$ is stochastic.

In general, set

$$
K_m := -m - \alpha f \sum_{d=1}^{m-1} \frac{[d]}{1 + \beta x} 
+ \sum_{i=1}^{c_i} (-\beta \delta)^{r-1}[r-1]! q^{-r} \left( \frac{1}{1 + \beta x} \right) e_i \left( 1, q, \ldots, q^{m-1} \right).
$$

The operator $\tilde{H}$ is stochastic only if $\sum_{i=1}^{M} K_{c_i} = 0$ for any tuple $(c_1, \ldots, c_M)$ of positive integers such that $\sum_{i=1}^{M} c_i = k$. Since $K_1 = 0$ and $K_2 = -(\alpha + \beta)(\gamma + \delta)/(1 + \beta x)$, we see that $(\alpha + \beta)(\gamma + \delta)$ should be zero.

Moreover, we have the following property.

**Lemma 4.2.** If $(\alpha + \beta)(\gamma + \delta) = 0$, the constant $K_m$ defined by (4.2) is equal to zero for any $m \geq 1$. 

\[ \]
Proof. From lemmas 3.3 and 3.4 it holds that
\[ K_m = -\left(1 + \frac{\beta}{\alpha}\right)m + \frac{1}{\beta} \sum_{r=1}^{m} (-\delta)^{r-1}[r-1]! q^{-(m-1)r} e_r \left(1, q, \ldots, q^{m-1}\right) \prod_{p=1}^{r} \frac{\alpha + \beta q^{p-1}}{1 + \beta r[p-1]} \]

Using this expression we find that
\[ K_m - K_{m-1} = -\frac{(\alpha + \beta)(\gamma + \delta)}{1 + \beta r} \sum_{r=1}^{m-1} (-\delta)^{r-1}[r]! q^{-(m-2)r} e_r \left(1, q, \ldots, q^{m-2}\right) \prod_{p=2}^{r} \frac{\alpha + \beta q^{p-1}}{1 + \beta r[p]} \]
for \( m \geq 2 \). This completes the proof because \( K_1 = 0 \).

Now we define the stochastic operator \( \mathcal{H}(s, q) \) on \( F(L_+) \) by
\[
(\mathcal{H}(s, q)f)(x) = \sum_{i=1}^{M} \sum_{c_1} \sum_{r=1}^{c} \frac{[c - p]}{[r]} \prod_{p=0}^{r-1} \frac{1}{1 + s[c - 1 - p]} \times \left(f \left(x - \sum_{p=0}^{r-1} c_{c+p+\cdots+c_r-p}\right) - f(x)\right),
\]

where \((c_1, \ldots, c_M)\) is the cluster coordinate of \( x \). It determines the stochastic particle system on \( \mathbb{Z} \) described as follows. In continuous time some particles may move from site \( i \) to \( i - 1 \) independently for each \( i \in \mathbb{Z} \). The rate at which \( r \) particles move to the left from a cluster with \( c \) particles is given by
\[
\frac{s^{r-1}}{[r]} \prod_{p=0}^{r-1} \frac{[c - p]}{[c - 1 - p]} \quad (c \geq r \geq 1).
\]
It is non-negative if, for example, \( s \geq 0 \) and \( 0 < q < 1 \).

As a consequence we have the following proposition.

Proposition 4.3. When \( (\alpha + \beta)(\gamma + \delta) = 0 \), it holds that
\[
(\mathcal{H} - k)_{|F(L_+)} = \begin{cases} \mathcal{H}(q^{-(r-1)}u, q^{r-1}) & (\alpha + \beta = 0), \\ \mathcal{H}(\beta r, q) & (\gamma + \delta = 0). \end{cases}
\]

Proof. Use the equality
\[
q^{-r(r-1)/2} e_r \left(1, q, \ldots, q^{m-1}\right) = \prod_{p=0}^{r-1} \frac{[m - p]}{[r]!},
\]
and we obtain the desired formula.

Moreover, using proposition 3.15, we obtain eigenfunctions of \( \mathcal{H}(s, q) \):
Proposition 4.4. Let \( z = (z_1, \ldots, z_k) \) be a tuple of distinct complex parameters, and set
\[
\nu := \frac{s}{1 - q + s}.
\]
(4.3)

Then the function \( \Psi_z^* \) on \( L_k \) defined by
\[
\Psi_z^* := \sum_{\sigma \in \mathcal{S}_k} \prod_{1 \leq i < j \leq k} \frac{q_{\sigma(i)} - z_{\sigma(j)}}{z_{\sigma(i)} - z_{\sigma(j)}} \prod_{i=1}^k \left( \frac{1 - \nu z_{\sigma(i)}}{1 - z_{\sigma(i)}} \right)^{c_i}
\]
satisfies
\[
\mathcal{H}(s, q) \Psi_z^* = (\nu - 1) \sum_{i=1}^k \frac{z_i}{1 - \nu z_i} \Psi_z^*.
\]

Proof. We use proposition 3.15 in the case where \( \delta = -\gamma \). Note that \( q = 1 + \beta \gamma - a\delta = 1 + (\alpha + \beta)\gamma \). Setting \( p_i = (1 - z_i)/(1 + \beta z_i/\alpha) \), we have
\[
1 + \left( \frac{\alpha + \beta p_j}{p_j - p_i} \right) \left( \gamma + \delta p_i \right) = \frac{q z_i - z_j}{z_i - z_j}.
\]

Set \( s = \beta \gamma \). Then \( \beta/\alpha \) is equal to \(-\nu\). Thus we see that \( \Psi_z^* \) is an eigenfunction of \( \mathcal{H}(s, q) = (\mathcal{H} - k)|_{F(L_k)} \) with eigenvalue
\[
\sum_{i=1}^k p_j - k = \sum_{i=1}^k \frac{1 - z_j}{1 - \nu z_i} - k = (\nu - 1) \sum_{i=1}^k \frac{z_i}{1 - \nu z_i}.
\]
This completes the proof. \( \square \)

It should be noted that the function \( \Psi_z^* \) is equal to the eigenfunction for the \((q, \mu, \nu)\)-Boson process constructed by means of the coordinate Bethe ansatz [9].

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Appendix

Here we prove lemmas 3.3 and 3.4. For that purpose we show the following equality.

Lemma A.1. Let \( m \) be a positive integer and \( x, y, z_1, \ldots, z_m \) commutative indeterminates. For \( 1 \leq s \leq m \), set
\[ I_{m,s}(x, y) = \sum_{r=0}^{m-s} \frac{[r + s - 1]! \ ((q - 1)x - y)^r}{\prod_{a=1}^{s+r+1} (x + [a - 1]y)} \times \sum_{1 \leq b_1, \ldots, b_m} q^{s+s-1} e_s \left( z_{b_1}, qz_{b_2}, \ldots, q^{s+s-1}z_{b_m} \right). \] (A.1)

where \( e_s \) is the elementary symmetric polynomial of degree \( s \). Then it holds that

\[ I_{m,s}(x, y) = \frac{[s - 1]! q^{-(r+1)/2}}{\prod_{a=1}^{s-1} (x + [m - 1 - a]y)} e_s \left( z_1, qz_2, \ldots, q^{m-1}z_m \right). \] (A.2)

**Proof.** First we prove

\[ \sum_{1 \leq b_1, \ldots, b_m} q^{s+s-1} e_s \left( z_{b_1}, qz_{b_2}, \ldots, q^{s+s-1}z_{b_m} \right) = q^{s-1/2} e_s (q^{s+1}, q^{s+2}, \ldots, q^m) e_s (qz_1, q^2z_2, \ldots, q^{m-s}z_m). \] (A.3)

for \( m \geq 1, r \geq 0 \) and \( s \geq 0 \) satisfying \( r + s \leq m \) by induction on \( m \). If \( m = 1 \) it is trivial. Suppose that \( m > 1 \). Since the equality holds trivially when \( r = 0 \) or \( s = 0 \), we assume that \( r > 0 \) and \( s > 0 \). Denote the left hand side by \( K_{m,r,s} \). Using

\[ e_s \left( z_{b_1}, qz_{b_2}, \ldots, q^{s+s-1}z_{b_m} \right) = e_s \left( z_{b_1}, qz_{b_2}, \ldots, q^{s+s-2}z_{b_{m-1}} \right) + q^{s+s-1}z_{b_s} e_{s-1} \left( z_{b_1}, qz_{b_2}, \ldots, q^{s+s-2}z_{b_{m-1}} \right), \]

we see that

\[ K_{m,r,s} = \sum_{b=r+s}^{m} q^b \left( K_{b-1,r-1,s} + q^{s+s-1}z_{b} K_{b-1,r,s-1} \right). \]

From the hypothesis of the induction it is easy to equal

\[ q^{s-1/2} \sum_{b=r+s}^{m} q^b \left\{ e_{s-1} \left( q^{s+1}, \ldots, q^{b-2} \right) e_s \left( qz_1, \ldots, q^{b-1}z_{b-1} \right) + z_b \ e_s \left( q^{s+1}, \ldots, q^b \right) e_{s-1} \left( qz_1, \ldots, q^{b-1}z_{b-1} \right) \right\}. \]

Use

\[ q^b z_b \ e_{s-1} \left( qz_1, \ldots, q^{b-1}z_{b-1} \right) = e_s \left( qz_1, \ldots, q^b z_b \right) - e_s \left( qz_1, \ldots, q^{b-1} z_{b-1} \right) \]

and

\[ q^b e_{s-1} \left( q^{s+1}, \ldots, q^{b-1} \right) = e_s \left( q^{s+1}, \ldots, q^b \right) = e_s \left( q^{s+1}, \ldots, q^{b-1} \right). \] (A.4)

successively. Then we get the right hand side of (A.3).

Now let us prove (A.2). Using (A.3) we see that

\[ I_{m,s}(x, y) = q^{(r+1)/2-s} e_s \left( z_1, qz_2, \ldots, q^{m-1}z_m \right) J_{m,s}(x, y), \]
where $J_{m,s}(x,y)$ is given by

$$J_{m,s}(x,y) := \sum_{r=0}^{m-s} \frac{[r+s-1]!}{\prod_{a=1}^{r+s} (x+a-1)y} e_r(q^{s+1}, q^{s+2}, \ldots, q^n).$$

It suffices to show that

$$J_{m,s}(x,y) = q^{-s^2+ms} \frac{[s-1]!}{\prod_{a=0}^{s-1} (x+[m-1-a]y)}$$

for $1 \leq s \leq m$. From the equality (A.4) with $b$ replaced by $m$ and $x+[n]y = x+y+q[n-1]y$ for $n \geq 1$, we find $J_{m,s}(x,y) = q^s J_{m-1,s}(x+y, qy)$ for $m > s$. Now the desired equality (A.5) can be proved by induction on $m$.

**Proof of lemma 3.3.** We rewrite the right hand side. Expand the product

$$\prod_{a=1}^{r} (a + \beta q^{a-1}x_b) - \alpha' = \sum_{r=1}^{r} a^{r-s} \beta^{r-1} e_r(z_b, qz_b, \ldots, q^{r-1}z_b),$$

and exchange the order of the summation with respect to $r$ and $s$. Using

$$(-\delta)^{r+s+1} \alpha' \beta^{r-1} = (-\beta \delta)^{r-1}(q-1-\beta x)^r,$$

we see that the right hand side is equal to $\sum_{r=1}^{s-1} (-\beta \delta)^{r-1} I_{m,s}(1, \beta y)$, where $I_{m,s}(x,y)$ is defined by (A.1). It is equal to the left hand side because of lemma A.1.

**Proof of lemma 3.4.** Set

$$K_m(x,y) := \sum_{r=1}^{m} \frac{[r-1]! ((q-1)x-y)^{r-1}q^{-mr}}{\prod_{a=1}^{r} (x+[a-1]y)} e_r(q, q^2, \ldots, q^n).$$

Then the left hand side is equal to $\alpha K_m(1, \beta y) / \beta$. Hence it suffices to show that

$$K_m(x,y) = \sum_{a=0}^{m-1} \frac{1}{x+[a]y}.$$ 

In the same way as the proof of (A.5), we find the recurrence relation $K_m(x,y) = 1/x + K_{m-1}(x+y, qy)$ for $m > 1$. Now the equality above can be proved by induction on $m$.

### References

[1] Borodin A, Corwin I, Petrov L and Sasamoto T 2013 Spectral theory for the $q$-Boson particle system arXiv:1308.3475
[2] Corwin I 2014 The $(q, \mu, \nu)$-Boson process and $(q, \mu, \nu)$-TASEP arXiv:1401.3321
[3] van Diejen J F 2004 On the Plancherel formula for the (discrete) Laplacian in a Weyl chamber with repulsive boundary conditions at the walls Ann. Henri Poincare 5 135–68
[4] van Diejen J F 2006 Diagonalization of an integrable discretization of the repulsive delta Bose gas on the circle Commun. Math. Phys. 267 451–76
[5] van Diejen J F and Emsiz E 2012 Unitary representations of affine Hecke algebras related to Macdonald spherical functions J. Algebra 354 189–210
[6] Gutkin E 1982 Integrable systems with delta-potential Duke Math. J. 49 1–21
[7] Lascoux A and Schützenberger M P 1987 Symmetrization operators on polynomial rings *Funct. Anal. Appl.* **21** 77–8

[8] O’Connell N and Yor M 2001 Brownian analogues of Burke’s theorem *Stoch. Process. Appl.* **96** 285–304

[9] Povolotsky A M 2013 On the integrability of zero-range chipping models with factorized steady states *J. Phys. A: Math. Theor.* **46** 465205

[10] Sasamoto T and Wadati M 1998 Exact results for one-dimensional totally asymmetric diffusion models *J. Phys. A: Math. Gen.* **31** 6057–71

[11] Takeyama Y 2014 A discrete analogue of periodic delta Bose gas and affine Hecke algebra *Funkcialaj Ekvacioj* **57** 107–18

[12] Yang C N 1967 Some exact results for the many-body problem in one dimension with repulsive delta-function interaction *Phys. Rev. Lett.* **19** 1312–5