Quantum mechanical evolution operator in the presence of a scalar linear potential: discussion on the evolved state, phase shift generator and tunneling

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Abstract
We discuss the form of the wave-function of a state subjected to a scalar linear potential, focusing on quantum tunneling. We analyze the phases acquired by the evolved state and show that some are of a pure quantum mechanical origin. We propose a simple experimental scenario to measure one of these phases. We apply the evolution equations to re-analyze the Stern and Gerlach experiment and to demonstrate how to manipulate spin by employing constant electric fields.

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1. Introduction

Scalar linear potentials are widely used in physics, as they can be generated by homogeneous irrotational fields, such as electrostatic or gravitational fields, which are typical in physical problems. Moreover, linear potentials are of general use because they can approximate more sophisticated potentials for sufficiently small distances. Scalar linear potentials are also considered for quantum tunneling (e.g., the Sauter potential [1]). Being able to rigorously evolve a quantum mechanical state subjected to a linear potential is therefore of fundamental and pedagogical importance.

The exact evolution of quantum systems subjected to linear and quadratic potentials has recently attracted interest. The quantum propagator in the presence of a linear potential has been studied in several works [2–5]. The evolution operator related to the most generic time-dependent quadratic potential (with linear terms included) has also been analyzed in the literature using quantum invariants [6, 7]. Here, we derive the evolution operator in the presence of a scalar time-independent linear potential using the Zassenhaus formula [8]. Our simple approach allows for several physical considerations, which are given in the article.

In section 2, we show that the wave-function of a state that has evolved in the presence of a linear potential is given, up to a phase, by the free evolved wave-function (i.e., evolved with no potential) whose argument is shifted by a certain quantity that depends on the potential. In section 3, we consider a Gaussian wave-packet subjected to a linear potential. We study the problem of quantum tunneling, or quantum diffusion, which is the phenomenon where a microscopic object (typically a particle or an atom) can penetrate a potential barrier whose height is larger than the object’s kinetic energy [9, 10]. Because such phenomena are forbidden by the classical laws of mechanics, it is often referred to as a peculiar characteristic of quantum mechanics. In section 4, we discuss the form of the evolved state and the phases it acquires. Some of these phases are shown to stem
from the non-commutativity of momentum and position operators in quantum mechanics. A simple experimental scenario aimed at measuring one of those phases is proposed. In section 5, as a pedagogic application of our evolution equations, we rigorously re-analyze the example of the Stern and Gerlach (SG) experiment. In section 6, we show how to manipulate spin of charged particles by using constant electric fields, instead of the more commonly used magnetic fields. Finally, a summary is given in section 7.

2. Evolution operator and evolved wave-functions

We consider a potential of the form $V_0 x$, where $V_0$ is an arbitrary constant or any operator that commutes with momentum and position operators. Without restriction of generality, we consider only one dimension, which is the $x$ direction. The Hamiltonian may be thus written as

$$\hat{H} = \frac{\hat{p}^2}{2m} + V_0 \hat{x}, \quad (1)$$

where $\hat{p}$ is the linear momentum operator along the $x$ direction and $m$ is the particle mass. Because the Hamiltonian is time-independent, from equation (1) we may obtain the correspondent evolution operator from an initial time $t_i$ to $t$ [11]:

$$\hat{U}(t, t_i) = e^{\frac{\hbar}{2m} \hat{p} V_0 \hat{x} (t-t_i)}, \quad (2)$$

where $\hbar$ denotes the reduced Planck constant. However, because the argument of the exponential in the preceding equation is made of non-commutative operators, its application to ket states is non-trivial. To rewrite equation (2) in a more manageable way, we use the Zassenhaus formula [8]:

$$e^{\hat{A} + \hat{B}} = e^{\hat{A}} e^{\hat{B}} \prod_{n=1}^{\infty} e^{\hat{C}_n}, \quad (3)$$

where

$$\hat{C}_2 = \frac{1}{2} \left[ [\hat{B}, \hat{A}], \hat{B} \right] + \frac{1}{6} \left[ [\hat{B}, \hat{A}], \hat{A} \right].$$

$$\hat{C}_3 = \frac{1}{8} \left( \left[ \left[ \left[ \hat{B}, \hat{A} \right], \hat{B} \right], \hat{B} \right] + \left[ [B, A], A, B \right] \right) + \frac{1}{24} \left[ [\hat{B}, \hat{A}], \hat{A} \right], \hat{A}].$$

$$\ldots \ldots$$

By choosing $\hat{A} = -\frac{i}{\hbar} V_0 \hat{x} (t - t_i)$, and $\hat{B} = -\frac{\hat{p}^2}{2m} (t - t_i)$ and by using $[\hat{p}^2, x] = \hat{p} [\hat{p}, x] + [\hat{p}, x] \hat{p} = -2i\hbar \hat{p}$, we readily obtain

$$\hat{C}_2 = -\frac{i}{\hbar} \frac{\hbar}{2m} (t-t_i) \hat{x}, \quad \hat{C}_3 = -\frac{i}{\hbar} \frac{\hbar}{2m} (t-t_i)^2, \quad \hat{C}_{i>4} = 0.$$ 

Finally, by using the fact that $e^{\hat{B}}$ commutes with $e^{\hat{C}_2}$ and $e^{\hat{C}_3}$ commutes with anything, we may write the (exact) evolution operator as

$$\hat{U}(t, t_i) = e^{\frac{i}{\hbar} \frac{\hbar}{2m} (t-t_i) V_0 \hat{x} (t-t_i)} e^{\frac{i}{\hbar} \frac{\hbar}{2m} (t-t_i)^2 \hat{U}_0 (t, t_i)}, \quad (5)$$

where

$$\hat{U}_0(t, t_i) = e^{\frac{i}{\hbar} \frac{p^2}{2m} (t-t_i)}. \quad (6)$$

is the evolution operator in the free case, i.e., if there was no potential. Alternatively, by choosing $\hat{B} = -\frac{\hat{p}^2}{2m} \hat{x} (t - t_i)$ and $\hat{A} = -\frac{\hat{p}^2}{2m} (t - t_i)$ one may analogously derive

$$\hat{U}(t, t_i) = \hat{U}_0(t, t_i) e^{\frac{i}{\hbar} \frac{\hbar}{2m} (t-t_i)^2} e^{\frac{i}{\hbar} \frac{\hbar}{2m} (t-t_i) \hat{V}_0} e^{-\frac{i}{\hbar} \frac{\hbar}{2m} (t-t_i)^2} \hat{U}_0(t, t_i), \quad (7)$$

Equation (5) coincides with equation (2.6) in [3].

We notice that, for times much shorter than the characteristic time of interaction between potential and particle, we can neglect the terms $\sim (t-t_i)^2$ and $\sim (t-t_i)^3$ in the exponents of equations (7) and (5) in favor of the linear terms $\sim (t-t_i)$. The form thus obtained for the evolution operator $\hat{U}(t, t_i)$ would be equal to the one obtainable from equation (2) by considering as if the kinetic energy operator $\hat{p}^2/2m$ and the potential energy operator $V_0 \hat{x}$ commuted. This means that, for short interaction times, the kinetic and the potential energy operators may be considered to approximately commute. This result is expected, as it is a basic step for the path integral formulation of quantum mechanics [12].

Next, we apply equation (5) to an arbitrary initial state $|\alpha(t_i)\rangle$, defined at time $t_i$, to obtain the ket state at time $t$. Multiplying by $\langle x |$ from the left, we obtain the wavefunction of the state at time $t$ in the spatial representation:

$$\Psi(x, t) = \langle x | \alpha(t) \rangle = \langle x | \hat{U}(t, t_i) | \alpha(t_i) \rangle$$

$$= e^{\frac{i}{\hbar} \frac{\hbar}{2m} (t-t_i)^2} e^{\frac{i}{\hbar} \frac{\hbar}{2m} (t-t_i) V_0} \langle x | e^{\frac{i}{\hbar} \frac{\hbar}{2m} V_0 (t-t_i)^2} | \alpha(t_i) \rangle \quad (8)$$

where we defined $|\alpha(t)\rangle = \hat{U}_0(t, t_i) |\alpha(t_i)\rangle$, which is the state evolved by the free evolution operator equation (6). By using $\exp(-i\hat{p}\Delta x/\hbar) |x\rangle = |x + \Delta x\rangle$, which follows from the definition of momentum operator as the generator of spatial translations [11], we finally obtain

$$\Psi(x, t) = e^{\frac{i}{\hbar} \frac{\hbar}{2m} (t-t_i)^2} e^{\frac{i}{\hbar} \frac{\hbar}{2m} V_0 (t-t_i)^2} \Psi_0 \left( x + \frac{V_0}{2m} (t-t_i)^2, t \right). \quad (9)$$

It is important to notice that, in the preceding equation, the free evolved wave-function $\Psi_0$ can be of any form. In the special case that $\Psi_0$ is a plane-wave, $\Psi$ will then be a solution of the time-dependent Schrödinger equation related to the Hamiltonian in equation (1) [13]. The modulus squared of the wave-function (i.e., the probability density) simply satisfies

$$|\Psi(x, t)|^2 = \left| \Psi_0 \left( x + \frac{V_0}{2m} (t-t_i)^2, t \right) \right|^2.$$ 

We may obtain analogous relations in the momentum
representation:

\[ \tilde{\Psi}(p, t) = \langle p | \alpha(t) \rangle = \langle p | \hat{U}(t, t_i) | \alpha(t_i) \rangle \]

\[ = \exp \left( \frac{i}{\hbar} \int_{t_i}^{t} \hat{V}_0(p + V_0(t - t_i), t) \mathrm{d}t \right) \tilde{\Psi}_0(p + V_0(t - t_i), t). \]

Equations (9) and (11) relate the wave-functions (in the spatial and linear momentum representations) of a general state evolved in the presence of a linear potential with the wave-functions of the same state evolved without the linear potential. The evolved wave-function is given, up to a phase, by the free evolved wave-function with its argument evolved following the classical equation of motion in the presence of the opposite potential. For example, in the case of spatial representation, the argument is evolved following \( x \rightarrow x + \frac{\hbar}{\sigma \Delta} (t - t_i)^2 \), whereas the classical evolution of the position \( x \rightarrow x = \frac{\hbar}{\sigma \Delta} (t - t_i)^2 \). Although this might initially seem counterintuitive, in section 3, we show how it is not.

3. Evolution of a Gaussian wave-packet

Let us consider, at initial time \( t_i \), a Gaussian wave-packet with momentum mean value \( p_0 \), spatial mean value \( x_0 \), and standard deviation (or width) \( \sigma \), which represents a realistic state in standard experiments:

\[ \Psi^G(x, t) = \frac{1}{\pi^{1/4} \sigma^{1/2}} \exp \left( \frac{i}{\hbar} \int_{x-i\sigma^2}^{x+i\sigma^2} \frac{-i(x-x_0)^2}{2\sigma^2} \right) \equiv \langle x | G \rangle. \]  

Equation (14) may be also read in the following way: when the whole wave-packet is subjected to a linear potential, the (modulus squared of the) wave-packet follows the classical kinematics evolution. Now, let us consider a Gaussian wave-packet that hits a potential barrier whose initial part can be approximated as linear. This initial, approximately linear part of the potential shall be called \( D \). The maximum value of the potential in \( D \) is taken to be higher than the initial average kinetic energy of the wave-packet, this latter being \( \langle E_{kin} \rangle = p_0^2/(2m) \). For convenience, this situation is depicted in figure 1 and showed in the supplementary animation available online (in the animation, the evolution of the wave-packet in the free case is displayed for comparison). Let us further suppose that the difference between the furthest point of \( D \) and the first classical turning point \( a \) is much bigger than half wave-packet spatial width as given at the time \( t_o \) (i.e. \( D' \gg \frac{\sigma(x-a)}{2} \)). Here, \( t_o = p_0/V_0 \) is the time when the wave-packet mean value is at the turning point \( a \). In other words, we suppose that the DeBroglie wave-length of the wave-packet is small compared to the characteristic distance over which the first derivative of the potential varies appreciably. In these chosen settings, the whole wave-packet will be subjected to the same linear potential up to the turning point \( a \). The evolution given by equation (14) then dictates that the wave-packet will be wholly pushed backwards, as one would expect from the classical point of view. Consequently, no tunneling will be permitted to the wave-packet. Therefore, in order to have any chance for quantum tunneling, the wave-packet spatial width must be somewhat larger or comparable to \( D'(\sigma \geq D') \) so that the evolution given by equation (14) may not be applicable.

In the supplementary animation available online, we chose the mean position, the spatial width and the velocity to be \( 0 \) m, 0.2 cm, and 1 m s\(^{-1}\), respectively, at time \( t = 0 \). The strength of the potential is chosen to allow the wave-packet to travel for a length of 0.3 meters before reaching the classical turning point. At the classical turning point, the spatial width will have grown 10% (see animation). For the best comparison, we adopted the same length unit of the animation for the

\[ \text{This is a less stringent approximation than the assumption for the validity of that Wentzel–Kramers–Brillouin (WKB) approach. Within the WKB approach, which is widely used in quantum tunneling problems, the DeBroglie wave-length of the packet is considered to be small compared to the characteristic distance over which the potential varies appreciably} \]
Based on the preceding considerations, we argue that quantum tunneling reflection and transmission coefficients should directly depend on the spatial width of the wave-packet. However, although there are studies on tunneling with Gaussian wave-packets (see [15] and references therein), to the best of our knowledge, no direct relation between tunneling coefficients and spatial width of the wave-packet has been suggested. In fact, transmission coefficients in quantum tunneling are not normally given as dependent on the wave-packet spatial width, but rather as solely dependent on the energy of the particle (E) and barrier thickness. For instance, within the Wentzel–Kramers–Brillouin approximation, which is the most widely used approach for solving tunneling problems, the transmission coefficient is given by $T(E) = e^{-2\eta_i/\left(1 + e^{-2\eta_i/4}\right)^2}$, where

$$\eta_i = \int_a^b \sqrt{\frac{2m}{\hbar^2}}(V(x) - E) \, dx > 0,$$

and $a$, $b$ are the classical turning points (i.e., $L = b - a$ is the classically forbidden region) [9]. An experimental assessment of the dependence of tunneling coefficients on the spatial width would thus be desirable.

The direct dependence of tunneling coefficients on the spatial width of the wave-packet could have applications in many areas of science: quantum tunneling could be enhanced or suppressed by controlling the spatial width of the state, instead of controlling the energy of the state or the environment surrounding it [16, 17].

Unfortunately, we cannot find an explicit expression here for the transmission and reflection coefficients for a realistic potential barrier with the present quantum mechanical formalism. This is because the Zassenhaus formula does not converge for potentials of order higher than linear, and because a realistic potential barrier cannot be represented by a linear function. Nonetheless, any potential barrier can be approximated to linear for short distances. Based on this, our claim that the tunneling coefficients should depend on the spatial width of the wave-packet holds. In view of the fact that the wave-packet spatial mean value ($x_0$) follows the classical equation of motion, our conjecture is that the fraction of the wave-packet beyond the potential barrier at the classical turning point plays leading role in determining the transmission coefficient in quantum tunneling. Given wave-packets with the same linear momentum mean value, wave-packets with larger spatial width will tunnel more efficiently. This can be validated by preparing Gaussian wave-packets and delaying the arrival of some to the potential barrier. The spatial width $\sigma_D(\Delta t)$ of the wave-packets increases during the free evolution. Thus, the delayed wave-packets will have larger spatial width in respect to the non-delayed wave-packets.

4. Discussion on the phases: phase shift generator

Here, we discuss the expression for the evolution operator in equation (5) more extensively.

The first $\left(-\frac{i V_0}{\hbar}\Delta t/2 \left( t - t_f \right)^2\right)$ and third $\left(\frac{i V_0}{\hbar}\Delta t/2 \left( t - t_f \right)^2\right)$ phases that multiply the free evolution operator from the left in equation (5) stem directly from the non-commutativity between momentum and position operators, and are thus purely quantum mechanical corrections. On the other hand, the second phase $\left(-\frac{i}{\hbar}\Delta t/2 \left( t - t_f \right)^2\right)$ and free evolution operator ($\hat{U}_f(t, t_f)$) represent the evolution given by the potential and the kinetic energy gained by the traveling particle, respectively. If position and momentum operators commuted, then only these latter terms would be present.

It is somewhat interesting that the phase $\frac{i V_0}{\hbar}\Delta t/2 \left( t - t_f \right)^2$ is directly responsible in equation (9) for the shift in the argument of the free evolved wave-function in the spatial representation, such shift being $V_0/(2m)(t - t_f)^2$. In fact, that shift, together with the shift $p_0\Delta t/m$ given by the free evolution, gives rise to the classical motion of the spatial mean value of the state (see previous section). Therefore, the non-commutativity of momentum and position operators is effectively responsible for the non-relativistic classical motion of the spatial mean value of the wave-packet: $x_0 \rightarrow x_0 + p_0\Delta t/m - V_0\Delta t^2/2m$. We may furthermore notice that the corresponding shift in the momentum representation, which is responsible for the classical motion of the linear momentum mean value, is directly given by the phase containing the linear potential, $\left(-\frac{i}{\hbar}\Delta t/2 \left( t - t_f \right)^2\right)$. Therefore, we may consider the term $\frac{i V_0}{\hbar}\Delta t/2 \left( t - t_f \right)^2$ as a linear potential in the momentum space. In other words, the term $\frac{i V_0}{\hbar}\Delta t$ may be

![Figure 1. A Gaussian wave-packet (blue curve) hits a potential barrier (black curve) whose initial part (D) can be approximated as linear. The potential barrier is in units of the initial average value for the kinetic energy of the wave-packet ($<E_{\text{kin}}^\text{in}> = \frac{p_0^2}{2M}$). The red solid vertical bars denote the classical turning points (a, b). The length $D'$ is chosen to be much larger than half wave-packet spatial width as given at the time $t_0 = \frac{a + b}{2}$. Here, $t_0 = p_0/V_0$ is the time when the wave-packet mean value is at the turning point a. For these settings, as a consequence of the evolution given by equation (14), the wave-packet will be wholly pushed backwards and no tunneling will be permitted.](image-url)
considered the dual of the potential $V_{i}$. The former is generated by the presence of the latter because the latter does not commute with the free Hamiltonian. The presence of both potentials gives symmetry to the evolution of the state in spatial and momentum representations and ensures that, in both representations, the mean value is evolved following the non-relativistic classical motion.

Conversely, the phase $-\frac{i}{\hbar} \mathbf{p}_{i} (t - t_{0})^{3}$ does not play any role in determining the classical evolution of the state. Such a phase originates from the second (and last) expansion term of the Zassenhaus formula, and is therefore a higher order correction with respect to other terms. Indeed, this phase would be the only one missing if we replaced, in the free plane wave $e^{\frac{i}{\hbar} \mathbf{p}_{i} ^{\text{free}} (t - t_{0})}$, the classical transformations $x \to x - \frac{\hbar}{2m} (t - t_{0})^{2}$ and $p \to p - \frac{\hbar}{2m} (t - t_{0})$, as one would do as a first attempt to guess the wave-function of a state subjected to a linear potential. An experiment aimed at ascertaining the existence of this phase would thus be a useful test for quantum mechanics. To this aim, here we sketch a simple experimental scenario that permits such a measurement.

Let us consider two electron beams along the $z$ direction, where the electrons are in phase one with another [18]. One of the two beams is accelerated and subsequently decelerated along an axis orthogonal to the beam direction that is taken to be $x$. To apply equations (5)–(7), the acceleration and deceleration must be due to a linear potential. As shown in figure 2, this could be realized, for instance, by a series of three capacitors of lengths $L = \nu \Delta t$, $2L$, and $L$, where $\Delta t$ is the time the electron spends in the first capacitor and $\nu$ is the beam velocity along $z$ (here, $L$ must be much larger than the spatial width of the electron state). For this example, we must consider the three-dimensional generalization of equation (5), where $\hat{p}^{2}$ is replaced by $\hat{p}^{2} = \hat{p}_{x}^{2} + \hat{p}_{y}^{2} + \hat{p}_{z}^{2}$. Because position and momentum operators along different directions commute, such a replacement can be safely made. By applying such generalized evolution operator to the initial electron state $\ket{p_{i}} \simeq \ket{p_{i} = 0, p_{z} = 0, p_{y} = p_{y}}$, the electron state following electrostatic deflection (which lasts for a time $4\Delta t$) can be easily calculated to be $e^{\frac{i}{\hbar} \mathbf{p}_{i} ^{2} (4\Delta t)} \ket{p_{i}} \equiv e^{\frac{i}{\hbar} \mathbf{p}_{i} ^{2} (4\Delta t)} \ket{p_{i}}$.

We have here redefined $\hat{\mathbf{V}}_{i}(4\Delta t) \ket{p_{i}} \equiv \ket{p_{i}}$ because the phase given by the free evolution operator is shared by both beams and therefore is not measurable. Thus, upon passing the capacitors, the beam acquires a phase shift equal to $-\frac{i}{\hbar} \mathbf{p}_{i} ^{2} (4\Delta t)$ with respect to the other (non-deflected) beam. Such phase shift can be measured when the beams are recombined.

In what follows, we shall denote the experimental apparatus sketched in figure 2 as a phase shift generator (PSG).

From the preceding considerations, we see that when different beams are subjected to different accelerations, a phase difference proportional to the potential-difference squared may appear, if the potentials responsible for the accelerations can be approximated to linear. Therefore, equations (5), (7), and figure 2 might be also useful to estimate the loss of coherence in dealing with charged particles.

Phase shifts of quantum states have been very useful in physics and are objects of current research and debate (e.g., the gravitational phase shift [19, 20] and the Gouy phase [21, 22]). Along the same lines, ascertaining the existence of the phase shift generated by the PSG would be useful for testing quantum mechanics and might also have several applications. We shall see in section 6 how such a phase could be used, for example, in quantum information and spintronics.

5. Application to SG experiment

The SG experiment [23] is rightfully considered of fundamental and pedagogical importance for understanding quantum mechanics. The particles injected in the SG apparatus are subjected to a linear potential. The SG apparatus is therefore probably the best example for a clear application of the evolution equations (5)–(11). The following brief analysis is also motivated by the fact that in textbooks the SG apparatus is normally explained with intuitive, semi-classical arguments (e.g., see [11]), whereas in the literature, it is more rigorously explained with an involved quantum mechanical formalism [24].

In the SG experiment, silver atoms are injected in the apparatus [23]. Out of the 47 electrons of the silver atom, only the outermost electron contributes to the atomic spin, if we neglect the nuclear contribution (which is irrelevant to our discussion). Therefore, it is common to consider the spin state of such electron as characterizing the spin state of the whole atomic system. Because no atomic excitations are to be considered, we may disregard any atomic internal structure. The potential for an SG whose magnetic field is along the $x$ direction is $\mathbf{V}_{i} ^{\text{SG}} \simeq -\frac{e}{m} B_{0} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}$, where $e$ and $m_e$ are the
Manipulating spin by employing electric fields

Manipulating spin by means of electric fields is currently the subject of applied research [27–31]. The advantage of controlling spin by using electric rather than magnetic fields is that the former are easier to generate. Moreover, they allow for controlling spins independently one from another, which is a requirement for building quantum computers [32, 33].

Our goal here is to show how to manipulate the spin of charged particles by employing constant electric fields. This will be achieved by combining a spin beam splitter with a PSG apparatus. For our purposes, and for simplicity, we will consider electrons and use the SG apparatus as the spin beam splitter. The end of the present section presents a brief discussion on feasibility and possible extensions.

We prepare an electron with, for example, momentum and spin along $z$. Its state is therefore described by $|p\rangle \otimes |S_z, \pm\rangle$, with $p = (0, 0, p_z)$. We let such an electron sequentially pass through an SG, then through a PSG, and finally through a second SG. While the first SG is set along the $x$ direction, the second SG is set along the $-x$ direction. This thought experiment is sketched in figure 3, panel a).

During its several steps, by using equation (5), we find that the electron state is given by (up to an overall phase):

$$|p\rangle \otimes |S_z, \pm\rangle = \frac{1}{\sqrt{2}} |p\rangle \otimes (|S_z, +\rangle \pm |S_z, -\rangle)$$

where the momentum $p = (+\Delta p, 0, p_z)$ and $p = (-\Delta p, 0, p_z)$, whereas $\Delta p = \frac{e \Delta \phi}{\gamma mc^2} B_0 (t - t_i)$. The probability density of measuring the generic state $|p\rangle \langle p'|$ after an interaction time $(t - t_i)$ is therefore

$$\begin{align*}
\text{Tr} \left[ |p\rangle \langle p'| \hat{\rho}(t_i) \right] &= \frac{1}{2} \delta (p - p'),
\end{align*}$$

which is the well-known SG outcome.

6. Manipulating spin by employing electric fields

Electronic charge and mass, respectively, $\hat{\sigma}$ is the Pauli spin operator along the $x$ direction, and $B_0$ is the strength of the magnetic field. The state of the atoms before entering the SG is completely mixed. Thus, it may not be described by a ket state, but rather it can be described by the following density operator [11, 25]:

$$\hat{\rho}(t_i) = \frac{1}{2} \left( |S_z, +\rangle \langle S_z, +| + |S_z, -\rangle \langle S_z, -| \right).$$

where $p = (0, 0, p_z)$ and $|S_z, \pm\rangle$ are spin-1/2 states along the $x$ direction. Setting $V_0 = \hat{V}_0 \equiv -\left( \frac{e \Delta \phi}{\gamma mc^2} B_0 \right) \hat{\sigma}$, in equation (5) (which is allowed, because $\hat{\sigma}$ commutes with any of the operators $\hat{p}, \hat{\mathcal{p}}, \hat{\chi}, \hat{\eta}, \hat{\gamma}, \hat{\zeta}$), the evolution of the density operator can be easily computed [26]:

$$\hat{\rho}(t) = \hat{U}(t, t_i) \hat{\rho}(t_i) \hat{U}^\dagger(t, t_i)$$

where $p = (\pm \Delta p, 0, p_z)$ and $p = (\Delta p, 0, p_z)$, whereas $\Delta p = \frac{e \Delta \phi}{\gamma mc^2} B_0 (t - t_i)$. The probability density of measuring the generic state $|p\rangle \langle p'|$ after an interaction time $(t - t_i)$ is therefore

$$\begin{align*}
\text{Tr} \left[ |p\rangle \langle p'| \hat{\rho}(t_i) \right] &= \frac{1}{2} \delta (p - p_z),
\end{align*}$$

which is the well-known SG outcome.

Figure 3. (a) A spin flipper built by employing SG and PSG apparatuses. (b) By removing the PSG, the spin is not flipped.
Here, we used the SG apparatus for simplicity, but any spin beam splitter can be used to produce the given results. Spin beam splitters of nanoscopic and mesoscopic dimensions have in fact been realized for electrons (e.g., [35–38]). therefore, our scenario for spin manipulation is feasible with current technology.

A microscopic realization of PSG jointly with a spin beam splitter may be applied in quantum information and spintronics (including atomtronics, if charged atoms are employed) [39–42]. In fact, as shown in this section, this combination works as a gate $|0\rangle + |1\rangle \rightarrow |0\rangle + e^{i\mu} |1\rangle$, where $\mu$ is any desired phase.

7. Summary

In summary, we used the Zassenhaus formula to re-derive the quantum mechanical evolution operator in the presence of a scalar linear potential. We discussed the form of the wave-function of the evolved state, focusing on quantum tunneling. We then analyzed the phases that the evolved state acquires. We proposed an experimental scenario to measure one particular phase given by the non-commutativity of momentum and position operators. We applied the evolution equations to rigorously re-analyze the SG experiment and to demonstrate how to manipulate spin using constant electric fields.

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