Response of the QED(2) Vacuum to a Quench:
Long-term Oscillations of the Electric Field and the 
Pair Creation Rate

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Abstract. We consider – within QED(2) – the backreaction to the Schwinger 
pair creation in a time dependent, spatially homogeneous electric field. Our focus 
is the depletion of the external field as a quench and the subsequent long-
term evolution of the resulting electric field. Our numerical solutions of the 
self consistent, fully backreacted dynamical equations exhibit a self-sustaining 
oscillation of both the electric field and the pair number depending on the coupling 
strength.
1. Introduction

The Sauter-Schwinger effect is one of the most important examples of strong-field QED phenomena. It refers to the creation of electron-positron pairs by a spatially homogeneous electric field – the decay of the vacuum [1][2] (cf. [3] for a recent review). A common picture is that virtual, entangled pairs constituting the vacuum are disrupted by the external electric field and lifted on the mass shell, thus loosing their entanglement in the long-time evolution. The rate at which pairs are created by an electric field of strength \( E \) is \( \propto \exp(-\pi E_c/E) \). This rate is exceedingly small for macroscopic installations, since the Sauter-Schwinger (critical) field strength \( E_c = m^2/e = 1.3 \times 10^{18} \text{ V/m} \) is so large, while field strengths presently achievable in the lab are of the order of \( E \approx 0.01E_c \), which results in a huge suppression factor of \( e^{-300} \). (\( m \) and \( e \) the electron/positron mass and charge; we employ natural units with \( c = \hbar = 1 \).)

Nonetheless one hopes upcoming high-intensity optical laser installations can provide the avenue towards the necessary fields. For lasers, the assumption of a constant electric field is not very realistic and a natural generalization is to let it be time dependent. This is called the dynamical Schwinger effect. The special case of a periodic field is dealt with in [4]. For ideas to boost the pair creation rate by superposing a strong, slowly varying field with a weak but fast field, see [5][6][7]. Other setups, not necessarily using lasers, are the field in the vicinity of a super-heavy atomic nucleus [10][14], or a superposed XFEL beam [15][16]. For a survey of these effects, see [17]. Furthermore, ideas have also been put forward to forgo the direct detection of the produced fermions and instead focus on secondary photon signatures [18][21].

These investigations have one thing in common: They suppose the electric (or more generally electro-magnetic) field has no dynamics of its own and is thus unaffected by the created pairs. However, physical intuition suggests the electrons and positrons will produce a current which will in turn generate a counter-acting electro-magnetic field that gets added to the original one. One calls this the backreaction of the fermions on the Maxwell field. Because the Schwing pair production rate is already strongly suppressed for \( E < E_c \), the study of this further diminishing effect was postponed in favor of searching for amplification for non-perturbative effects. But since the backreaction is of principal interest in its own right in illuminating the non-perturbative character of the Schwinger effect, we reconsider it in this paper in the context of 1 + 1 QED (QED(2), also called the massive Schwinger model [22]). Backreactions were considered within QED(2) in [23][25].

A self consistent description of the backreaction is thus needed. This was first accomplished e.g. in [26][29]. In the present contribution we build on theirs and extend it to investigate the long term evolution of the electric field and to investigate how the backreaction affects the created pairs. Put in a nutshell, we consider the response of the QED(2) vacuum to a quench caused by an external electric field.

2. Quantum kinetic equations with backreaction

We use the framework of the quantum kinetic equations, and our derivation follows [30][31]. Incorporating the backreaction is done similarly to [29]. Our starting point is the Dirac equation in 1 + 1 dimensions, \((i\partial - eA + m)\Psi = 0\). The gamma matrices are chosen as \(\gamma^0 = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})\) and \(\gamma^1 = (\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix})\). Since our background field is assumed to be spatially homogeneous, an ansatz for \(\Psi\) via a Fourier transform, \(\Psi(t, x) = \int \frac{d^3p}{(2\pi)^3} \psi(t, p)e^{ipx}\) reduces the Dirac equation to a Schrödinger form

\[
\begin{align*}
\dot{\psi}(t, p) &= h(t, p)\psi(t, p), \\
h(t, p) &= \left(-p + eA(t) \begin{smallmatrix} m \\ m - p - eA(t) \end{smallmatrix}\right).
\end{align*}
\]

The Hamiltonian \(h\) has two time dependent eigenvectors \(U, V\) to the eigenvalues \(\pm\Omega(t, p) = \sqrt{m^2 + (p - eA(t))^2}\). Two linearly independent solutions to the Dirac equations are obtained via the ansatz

\[
\begin{align*}
u(t, p) &= \alpha(t, p)U(t, p) + \beta(t, p)V(t, -p), \\
v(t, -p) &= -\beta^*(t, p)U(t, p) + \alpha^*(t, p)V(t, -p)
\end{align*}
\]

with \(|\alpha|^2 + |\beta|^2 = 1\), resulting in equations for \(\alpha\) and \(\beta\):

\[
\dot{\alpha} = -i\Omega\alpha + \frac{eEm}{2\Omega^2}\beta, \quad \dot{\beta} = -\frac{eEm}{2\Omega^2}\alpha + i\Omega\beta.
\]

To get to observable quantities, we pass over to second quantization by promoting \(\psi\) to an operator on Fock
space. Since we have two bases \((u, v \text{ and } U, V)\) at our disposal, we can expand \(\psi\) in both:

\[
\psi(t, p) = c(p)u(t, p) + d^\dagger(-p)v(t, -p)
= C(t, p)U(t, p) + D^\dagger(t, -p)V(t, -p).
\]  

(4)

The relation between \(c, d\) and \(C, D\) follows from \([2]\) as

\[
\begin{align*}
C(t, p) &= \alpha(t, p)c(p) - \beta^*(t, p)d^\dagger(-p), \\
D^\dagger(t, -p) &= \beta(t, p)c(p) + \alpha^*(t, p)d^\dagger(-p),
\end{align*}
\]

(5)

which is called the Bogoliubov transform. Both sets of operators are fermionic creation/annihilation operators. The vacuum \(|0\rangle\) is annihilated by \(c, d\), and the number of produced pairs is \(\langle 0|C^\dagger C|0\rangle = \langle 0|D^\dagger D|0\rangle = |\beta|^2\). This lets us define the total pair number

\[
n(t) = \int \frac{dp}{2\pi} |\beta(t, p)|^2.
\]

(6)

The second-quantized Hamiltonian is \(H(t) = \int \frac{dp}{2\pi} \psi^\dagger(t, p)H\psi(t, p)\) and when we normal order it in terms of \(C\) and \(D\) (indicated by \(\bullet\ldots\bullet\)) we get

\[
\begin{align*}
\bullet H \bullet &= \int \frac{dp}{2\pi} \Omega[C^\dagger C + D^\dagger D].
\end{align*}
\]

(7)

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\]

(7)

Its expectation value, the energy of the vacuum, is \(\langle 0|\bullet H \bullet |0\rangle = V \times 2 \int \frac{dp}{2\pi} |\beta|^2 \Omega\). The factor \(V = 2\pi\delta(0)\) is the volume of the system and divergent because of the homogeneity.

The particles produced by the electric field will induce a current \(j^\mu(t, x) = e\bar{\Psi}(t, x)\gamma^\mu\Psi(t, x)\). We denote by its mean field part the normal ordered expectation value \(\bar{j}^\mu = \langle 0|\bar{j}^\mu|0\rangle\) which will be constant in \(x\), again due to the homogeneity. It evaluates to \(\bar{j}^0 = 0\) and

\[
\bar{j}^1 = 2e \int \frac{dp}{2\pi} \left[|\beta|^2 \frac{p - eA}{\Omega} - \text{Re}(\alpha^*\beta) \frac{m}{\Omega}\right].
\]

(8)

Note that without normal ordering w.r.t. \(C, D, j^0 \neq 0\), which is unphysical since the external field cannot create a net charge. Also normal ordering w.r.t. \(c, d\) would yield \(\bar{j}^0 = \bar{j}^1 = 0\). This is also unphysical as the particles would not create a (spatial) current.

To this (internal) mean field current we add an arbitrary external current \(j^0_{\text{ext}} = 0, j^1_{\text{ext}} = -E_{\text{ext}}\) which generates the external electric field (that is the quench) and plug both into Maxwell’s equation:

\[
\begin{align*}
\dot{E} &= \dot{E}_{\text{ext}} - 2e \int \frac{dp}{2\pi} \left[|\beta|^2 \frac{p - eA}{\Omega} - \text{Re}(\alpha^*\beta) \frac{m}{\Omega}\right], \\
\dot{A} &= -E.
\end{align*}
\]

(9)

An alternative way of arriving at this equation, pursued in \([29]\), is to start with the total energy density of the system

\[
\epsilon = \frac{E^2}{2} + 2 \int \frac{dp}{2\pi} |\beta|^2 \Omega - \int dt \dot{E}_{\text{ext}} E
\]

(10)

and setting \(\dot{\epsilon} = 0\). The last term is the work the external current must do to counteract the electric field. We use the energy density to check the accuracy of the numerics.

Whichever way one chooses, the equations \([3]\) and \([9]\) together with the initial conditions

\[
\alpha(t_0, p) = 1, \quad \beta(t_0, p) = 0, \quad A(t_0) = 0,
\]

\[
E(t_0) = E_{\text{ext}}(t_0) = 0
\]

(11)

form a well defined system of coupled ordinary differential equations that we are going to evaluate. Note that without the integral incorporating the backreaction in \([9]\), the different momentum modes would be decoupled.

In \(1 + 1\) dimensions, the coupling strength has dimension \([e] = 1\). The fine structure constant is then defined as \(\alpha = e^2/4\pi\Omega m^2\) and we give its value when specifying the strength of the backreaction.

3. Schwinger pair production for various pulse shapes

We will employ two different quenches caused by external electric fields.

3.1. Sauter pulse

The first is the so called Sauter pulse

\[
E_{\text{ext}}(t) = \frac{E_0}{\cosh^2(t/\tau)}.
\]

(12)

Note the initial condition \(E_{\text{ext}}(t_0) = 0\) can only be approximately fulfilled, but to arbitrary precision by choosing \(-t_0\) sufficiently large.

In figure \([1a]\) we show the time evolution of the electric field, determined by \([9]\), with the Sauter pulse \([12]\) as external field \(E_{\text{ext}}\). The first spike is \(E\) closely following \(E_{\text{ext}}\). After the latter has faded away, \(E\) starts to settle into a superposition of oscillations. These have already been noted in \([28, 29]\), and were also found in \([32]\) using different methods, where the Maxwell field was calculated using statistical averages, and in \([33]\), using matrix product states. In \([34]\), a similar effect was found without a driving external field, which the authors call plasmons in QED vacuum and attribute to the vacuum charge polarization. The inset in figure \([1a]\) shows a zoom to the peak of the electric field around \(t = 0\). Increasing the coupling strength screens the electric field more, resulting in a lower net maximum; i.e. the electric field is depleted.
3.2. Flat-top $C^\infty$ pulse

The second pulse shape we employ is a $C^\infty$ pulse with the following properties:

$$E_{\text{ext}}(t) = \begin{cases} 
0, & t \leq 0, \\
E_0, & t_r \leq t \leq t_e + t_f, \\
0, & t \geq 2t_0 + t_f,
\end{cases}$$

and monotonously increasing/decreasing where not specified. Its precise construction can be found in the appendix. In contrast to the Sauter pulse, it has two time scales, the ramping time $t_r$ over which the electric field is switched on and off, and the flat top time $t_f$ over which it is constant. The case $t_f \to \infty$ captures the plain Schwinger effect, but with the additional backreaction. The backreaction again causes some depletion, as evidenced in figure 2(a): The electric fields grows to almost the value of the external field, enters some transient oscillations, and then drops, synchronized with the drop of the external field. Subsequently the field seemingly displays a similar eternal wobbling as in the long-time regime of the Sauter quench, see figure 2(c). The pair number’s oscillations (figure 2(b)) are also synchronized with the field oscillations.

The depletion can be more clearly seen in figure 3 where we plot the electric field for a longer lasting $C^\infty$ pulse and for varying $\alpha$. The sudden change in the external field lifts the total field but also induces transient oscillations. Once the external field is constant, the total field starts declining with a slope that grows as $\alpha$ does, while the oscillations tend to zero (but observe that they take longer to do so for larger coupling strengths). The switching-off of the external field makes the total field swing in the opposite direction after which it enters the long-term oscillating state exhibited in figure 2(c). Note that per (10) the external field does not add energy to the system for $E_{\text{ext}} = \text{const}$. Thus in the flat-top section the energy just gets shifted from the electric field to the created fermions.

4. Summary

In summary we consider the impact of the backreaction on Schwinger type pair creation. The produced pairs screen the external field and facilitate its net depletion. Viewing the external field as a quench to the vacuum, it is interesting to see the vacuum response as a wobbling of the number of created pairs in phase with the long-term oscillations of the induced electric field. For the selected examples and within the considered time

§ Rather a modified version, where the electric field does not extend to the infinite past, but gets turned on smoothly at some time.
The external electric field is plotted in black dashed. The authors gratefully acknowledge inspiring discussions with R. Schützhold, H. Gies, R. Alkofer, D. B. Blaschke and C. Greiner. Many thanks go to S. Smolyansky and A. Panferov for previous common work on the plain Schwinger process. The fruitful collaboration with R. Sauerbrey and T. E. Cowan within the HIBEF project promoted the present investigation.

Appendix A. Construction of the $C^{\infty}$ pulse

To construct the pulse shape for the electric field \[ E^{\text{ext}}(t) = \frac{E_0 \alpha}{m} \left( -t + \frac{f}{f_t} - \frac{f - t}{f_r} \right) \]

this function is $C^{\infty}$ but not analytic, since $r^{(n)}(0) = 0$. Using it, define $s(x) = r(x)/[r(x) + r(1 - x)]$. Observe $s(x \leq 0) = 0$ and $s(x \geq 1) = 1$. This lets us define

First define

\[
 r(x) = \begin{cases} 
 0, & x \leq 0, \\
 e^{-\frac{1}{x}}, & x > 0. 
\end{cases} \tag{A.1}
\]

which has all the properties we claimed for $E^{\text{ext}}$ in \[13\].

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