Sharp non asymptotic oracle inequalities for non parametric computerized tomography model *

Fourdrinier, D. † and Pergamenschikov S.M.‡

November 22, 2018

Abstract

We consider non parametric estimation problem for stochastic tomography regression model, i.e. we consider the estimation problem of function of multivariate variables (image) observed through its Radon transformation calculated with the random errors. For this problem we develop a new adaptive model selection method. By making use the Galchouk and Pergamenschikov approach we construct the model selection procedure for which we show a sharp non asymptotic oracle inequality for the both usual and robust quadratic risks, i.e. we show that the proposed procedure is optimal in the oracle inequalities sense.

AMS 2010 subject classifications: 62C10, 62C20.
Key words and phrases: Radon transform, Fourier transform, Inverse Fourier transform.

* This work was done under the RSF grant 17-11-01049 (National Research Tomsk State University).
† LITIS, EA 4108, Université de Rouen, France and International Laboratory SSP & QF, National Research Tomsk State University, Russia
‡ Laboratory of Mathematics LMRS, University of Rouen, France and International Laboratory SSP & QF, National Research Tomsk State University, Russia e-mail: Serge.Pergamenchtchikov@univ-rouen.fr
1 Introduction

In this paper we consider the multivariate regression model proposed in [18] for computerized tomography problems, i.e. we consider the following regression model

\[ y(\nu, \varsigma) = R(S)(\nu, \varsigma) + \xi(\nu, \varsigma) \]  

(1.1)

where \( R(\cdot) \) is the random transformation, \( S \) is a \( \mathbb{R}^d \rightarrow \mathbb{R} \) function from \( L_2(\mathbb{R}^d) \) such that \( S(x) = 0 \) for \( |x| \geq x^* \) for some fixed \( x^* > 0 \). The vector \( \nu \in \mathbb{R}^d \) with \( |\nu| = 1 \), \( \varsigma \in \mathbb{R} \) and \( \xi(\nu, \varsigma) \) is the noise variable. Our aim, in this paper, is to estimate the function \( S \) based on the vector of observations

\[ y_l = R(S)(\nu_l, \varsigma_l) + \xi_l, \quad 1 \leq l \leq n, \]  

(1.2)

where \( (\xi_l)_{1 \leq l \leq n} \) is i.i.d. sequence with the unknown distribution function \( p \) under which

\[ E_p \xi_1 = 0, \quad E_p \xi_1^2 = \sigma_p \quad \text{and} \quad E_p \xi_1^4 < \infty. \]  

(1.3)

We assume that the distribution \( Q \) belongs to some distribution family \( Q^p_n \) which will be specified below.

In this case we use the robust estimation approach proposed in [4, 13, 14] for the nonparametric estimation. According to this approach we have to construct an estimator \( \hat{S}_n \) (any function of \( (y_l)_{0 \leq l \leq n} \)) for \( S \) to minimize the robust risk defined as

\[ R^*_n(\hat{S}_n, S) = \sup_{p \in P^p_n} R_p(\hat{S}_n, S), \]  

(1.4)

where \( R_p(\cdot, \cdot) \) is the usual quadratic risk of the form

\[ R_p(\hat{S}_n, S) := E_p \| \hat{S}_n - S \|^2 \quad \text{and} \quad \|S\|^2 = \int_{[-x^*, x^*]^d} |S(x)|^2 \, dx. \]  

(1.5)

It is clear that if we don’t know the distribution of the observation one needs to find an estimator which will be optimal for all possible observation distributions. Moreover in this paper we consider the estimation problem in the adaptive setting, i.e. when the regularity of \( S \) is unknown. To this end we use the adaptive method based on the model selection approach. The interest to such statistical procedures is explained by the fact that they provide adaptive solutions for the nonparametric estimation through oracle inequalities which give the non-asymptotic upper bound for the quadratic risk including the minimal risk over chosen family of estimators. It should be noted that for the first time the model selection methods were proposed by Akaike [1] and Mallows [17] for parametric models. Then, these methods had been developed for the nonparametric estimation and the oracle inequalities for
the quadratic risks was obtained by Barron, Birgé and Massart [2], by Fourdrinier and Pergamenshchikov [3] for the regression models in discrete time and [12] in continuous time. Unfortunately, the oracle inequalities obtained in these papers can not provide the efficient estimation in the adaptive setting, since the upper bounds in these inequalities have some fixed coefficients in the main terms which are more than one. To obtain the efficiency property for estimation procedures one has to obtain the sharp oracle inequalities, i.e. in which the factor at the principal term on the right-hand side of the inequality is close to unity. The first result on sharp inequalities is most likely due to Kneip [8] who studied a Gaussian regression model in the discrete time. It will be observed that the derivation of oracle inequalities usually rests upon the fact that the initial model, by applying the Fourier transformation, can be reduced to the Gaussian independent observations. However, such transformation is possible only for Gaussian models with independent homogeneous observations or for inhomogeneous ones with known correlation characteristics. For the general non Gaussian observations one needs to use the approach proposed by Galchouk and Pergamenshchikov [5, 6] for the heteroscedastic regression models in discrete time and developed then by Konev and Pergamenshchikov in [10, 11, 13, 14] for semimartingale models in continuous time. In general the model selection is an adaptive rule \( \hat{\lambda} \) which choses an estimator \( S^* = \hat{S}_\lambda \) from an estimate family \( (\hat{S}_\lambda)_{\lambda \in \Lambda} \). The goal of this selection is to prove the following nonasymptotic oracle inequality: for any sufficient small \( \delta > 0 \) and any observation duration \( n \geq 1 \)

\[
R_p(S^*, S) \leq (1 + \delta) \min_{\lambda \in \Lambda} R_p(\hat{S}_\lambda, S) + \delta^{-1} B_n, \tag{1.6}
\]

where the rest term \( B_n \) is sufficiently small with respect to the minimax convergence rate. Such oracle inequalities are called sharp, since the coefficient in the main term \( 1 + \delta \) is close to one for sufficiently small \( \delta > 0 \). The rest of the paper is organized as follows. In Section 2 we state the main conditions for the model (1.1) and we construct the model selection procedures. In Section 3 we give the main results on the oracle inequalities. In Section 4 we study the main properties of the model (1.1). In Section 5 we prove all results. Appendix 6 contains all technical and auxiliary proofs.

## 2 Model selection

We assume that the noise distribution \( p \) belong to the probability family \( \mathcal{P}_n \) is defined as

\[
\varsigma_* \leq \sigma_p \leq \varsigma^* \quad \text{and} \quad \mathbb{E}_p \varsigma_1^4 \leq \varsigma_1^*, \tag{2.1}
\]
where the unknown bounds \( 0 < \varsigma \leq \varsigma^* \) and \( \varsigma^* \) are functions of \( x_s \), i.e. \( \varsigma_s = \varsigma_s(n) \), \( \varsigma^* = \varsigma^*(n) \) and \( \varsigma^*_1 = \varsigma^*_1(n) \), such that for any \( \bar{\epsilon} > 0 \),

\[
\lim_{n \to \infty} n^{\bar{\epsilon}} \varsigma_s(n) = +\infty \quad \text{and} \quad \lim_{n \to \infty} \frac{\varsigma_s(n) + \varsigma^*_1(n)}{n^{\bar{\epsilon}}} = 0. \tag{2.2}
\]

First, we define the trigonometric basis in \( L^2[−x_s, x_s] \) as

\[
\varphi_k(\varsigma) = \frac{1}{\sqrt{2x_s}} e^{ik\varsigma \pi/x_s} \quad \text{and} \quad i = \sqrt{-1}. \tag{2.3}
\]

For any \( x = (x_1, \ldots, x_d) \in [−x_s, x_s]^d \) we can represent the function \( S \) in \( L^2[−x_s, x_s] \) as

\[
S(x) = \sum_{j=(j_1,\ldots,j_d) \in \mathbb{Z}^d} \theta_j \Phi_j(x), \tag{2.4}
\]

where

\[
\Phi_j(x) = \prod_{l=1}^d \varphi_{j_l}(x_l) \quad \text{and} \quad \theta_j = \int_{[−x_s, x_s]^d} S(z) \Phi_j(z) \, dz.
\]

So, taking into account the equality (A.4) we can represent the coefficient \( \theta_j \) as

\[
\theta_j = \frac{1}{\sqrt{2x_s}} \int_{−x_s}^{x_s} R(S(\nu_j, \varsigma)) e^{i\beta_j \varsigma} \, d\varsigma,
\]

where \( \beta_j = |j| \pi/x_s \) and \( \nu_j = j/|j| \) for \( |j| > 0 \). Taking into account here the property (A.6) we obtain that for any \( L_j \geq x_s \) the Fourier coefficient can be rewritten as

\[
\theta_j = \frac{1}{\sqrt{2x_s}} \int_{−L_j}^{L_j} R(S(\nu_j, \varsigma)) e^{i\beta_j \varsigma} \, d\varsigma, \tag{2.5}
\]

To calculate this coefficient we use the approximation

\[
a_j = \frac{1}{\sqrt{2x_s}} \sum_{l=1}^q R(S(\nu_j, s_l)) \psi_{j,l} \quad \text{and} \quad \psi_{j,l} = \int_{s_{j,l-1}}^{s_{j,l}} e^{i\beta_j \varsigma} \, d\varsigma, \tag{2.6}
\]

where \( (s_l)_{1 \leq l \leq q} \) is the uniform partition of the interval \([−L_j, L_j]\), i.e.

\[
s_{j,l} = −L_j + \frac{2L_j}{q} l.
\]
The number of points \( q = q_n \) will be chosen later. Using this approximation we estimate \( \theta_j \) as

\[
\hat{\theta}_j = \sum_{l=1}^{q} y_{j,l} \psi_{k,l} \quad \text{and} \quad y_{j,l} = y(\nu_j, s_{j,l}) .
\]  

(2.7)

Therefore, it can be represented as

\[
\hat{\theta}_j = \theta_j + \zeta_j \quad \text{and} \quad \zeta_j = b_j + \frac{1}{\sqrt{q}} \eta_j ,
\]  

(2.8)

where \( b_j = a_j - \theta_j \).

\[
\eta_j = \sqrt{q} \sum_{l=1}^{q} \xi_{j,l} \psi_{j,l} \quad \text{and} \quad \xi_{j,l} = \xi(\nu_j, s_{j,l}) .
\]  

(2.9)

Note that as it shown in Proposition 4.1 the second moment of this random variable is given as

\[
E_p |\eta_j|^2 = \sigma_p \overline{\varpi}_j ,
\]  

(2.10)

where

\[
\overline{\varpi}_j = 4L_j^2 \frac{\sin^2 \beta_j}{\beta_j^2} \quad \text{and} \quad \beta_j = \pi (1 + [1/\nu^*_j]) \nu^*_j \frac{|j|}{q} .
\]  

(2.11)

To obtain the uncorrelated property for the random variables \( \eta_j \) we set \( L_j \) as

\[
L_j = L(\nu_j) = (1 + [1/\nu^*_j]) \nu^*_j x_*,
\]  

(2.11)

where the coefficient \( \nu^*_j \) is the absolute value of the first nonzero component of the vector \( \nu_j = (\nu_{j,1}, \ldots, \nu_{j,d})' \), i.e. \( \nu^*_j = \min\{|\nu_{j,l}| > 0 , 1 \leq l \leq d\} \). Taking into account that in this case \( L_j \geq x_* \) we obtain that

\[
\overline{\varpi}_* = \inf_{|j| \leq q} \overline{\varpi}_j^2 > 0 .
\]  

(2.12)

So, using the estimators (2.7) we will estimate the function \( S \). The idea is the following, first we replace the infinite sum in (2.4) by the finite sum over the set

\[
\mathcal{S}_n = \{-m_n, \ldots, m_n\}^d ,
\]  

(2.13)

where the integer \( m_n \geq 1 \) will be specify below. Then, according to the Pinsker weighted least square method we will replace the Fourier coefficients in (2.4) with its estimators (2.7) multiplied by some coefficient \( 0 \leq \lambda(j) \leq 1 \), i.e.
\[ S_\lambda(x) = \sum_{j \in S_n} \lambda(j) \hat{\theta}_j \Phi_j(x), \quad x \in [-x_*, x_+]^d, \tag{2.14} \]

the weight vector \( \lambda = (\lambda(j))_{j \in S_n} \) belongs to some finite set \( \Lambda \) from \([0, 1]^r \) and \( r_n = (2m_n + 1)^d \). We set

\[ \tilde{i} = \text{card}(\Lambda) \quad \text{and} \quad |\Lambda_*|_* = \max_{\lambda \in \Lambda} \left( \tilde{L}(\lambda) + \tilde{L}_1(\lambda) \right), \tag{2.15} \]

where \( \tilde{L}(\lambda) = \sum_{j \in \mathbb{Z}^d} \lambda(j) \omega_j \) and \( \tilde{L}_1(\lambda) = \sum_{j \in \mathbb{Z}^d} 1_{\{\lambda \neq 0\}} \).

Now we need to write a cost function to choose a weight \( \lambda \in \Lambda \). Of course, it is obvious, that the best way is to minimize the cost function which is equal to the empirical squared error

\[ \text{Err}_n(\lambda) = \|S_\lambda - S\|^2 = \int_{[-x_*, x_+]^d} |\hat{S}_\lambda(x) - S(x)|^2 \, dx, \]

which in our case is equal to

\[ \text{Err}_n(\lambda) = \sum_{j \in \mathbb{Z}^d} \lambda^2(j) |\hat{\theta}_j|^2 - 2 \sum_{j \in \mathbb{Z}^d} \lambda(j) \text{Re} \hat{\theta}_j \tilde{\theta}_j + \|S\|^2, \tag{2.16} \]

where \( \lambda(j) = 0 \) for \( j \in \mathbb{Z}^d \setminus S_n \). Since the coefficients \( \theta_j \) are unknown, we need to replace the term \( \hat{\theta}_j \tilde{\theta}_j \) by an estimate which we choose as

\[ \tilde{\theta}_j = |\hat{\theta}_j|^2 - \frac{\tilde{\sigma}}{\tilde{q}} \omega_j. \tag{2.17} \]

Here \( \tilde{\sigma} \) is an estimate for \( \sigma_p \) which is given in (2.18). If the variance \( \sigma_p \) is known we set \( \tilde{\sigma} = \sigma_p \), otherwise, we can choose as

\[ \tilde{\sigma} = \frac{1}{\tilde{q}} \sum_{j \in T_n} |\hat{\theta}_j|^2 \quad \text{and} \quad \tilde{q} = \frac{\sum_{j \in T_n} \omega_j}{\tilde{q}}, \tag{2.18} \]

where \( T_n = \{[\sqrt{m_n}] + 1, \ldots, m_n\}^d \) and the coefficients \( \omega_j \) are defined in (2.17).

Moreover, for this substitution to the empirical squared error one needs to pay a penalty. Finally, we define the cost function in the following way

\[ J_n(\lambda) = \sum_{j \in \mathbb{Z}^d} \lambda^2(j) |\hat{\theta}_j|^2 - 2 \sum_{j \in \mathbb{Z}^d} \lambda(j) \tilde{\theta}_j + \delta \tilde{P}_n(\lambda), \tag{2.19} \]
where $\rho$ is some positive penalty coefficient which will be chosen later and the penalty term $\hat{P}_n(\lambda)$ we choose as

$$\hat{P}_n(\lambda) = \frac{\hat{\sigma}}{q} \hat{L}(\lambda^2), \quad (2.20)$$

where $\lambda^2 = (\lambda^2(j))_{j \in \mathbb{Z}^d}$. In the case when the $\sigma_p$ is known we set

$$P_n(\lambda) = \frac{\sigma_p}{q} \hat{L}(\lambda^2). \quad (2.21)$$

Now, we define the model selection procedure as

$$\hat{S}_* = \hat{S}_\hat{\lambda} \quad \text{and} \quad \hat{\lambda} = \arg\min_{\lambda \in \Lambda} J_n(\lambda). \quad (2.22)$$

We recall that the set $\Lambda$ is finite so $\hat{\lambda}$ exists. In the case when $\hat{\lambda}$ is not unique, we take one of them.

Let us now specify the weight coefficients $\lambda = (\lambda(j))_{j \in S_n}$. Consider, for some fixed $0 < \varepsilon < 1$, a numerical grid of the form

$$\mathcal{A} = \{1, \ldots, k^*\} \times \{\varepsilon, 2\varepsilon, \ldots, \lfloor 1/\varepsilon \rfloor \varepsilon\}, \quad (2.23)$$

where $[a]$ means the integer part of the number $a$. We assume that both parameters $k^* \geq 1$ and $\varepsilon$ are functions of $x_*$, i.e. $k^* = k^*(n)$ and $\varepsilon = \varepsilon(n)$, such that

$$\begin{cases}
\lim_{n \to \infty} k^*(n) = +\infty, & \lim_{n \to \infty} \frac{k^*(n)}{\ln n} = 0, \\
\lim_{n \to \infty} \varepsilon(n) = 0 & \text{and} \quad \lim_{n \to \infty} n^\delta \varepsilon(n) = +\infty
\end{cases} \quad (2.24)$$

for any $\delta > 0$. One can take, for example, for $n \geq 2$

$$\varepsilon(n) = \frac{1}{\ln n} \quad \text{and} \quad k^*(n) = k^*_0 + \sqrt{\ln n}, \quad (2.25)$$

where $k^*_0 \geq 0$ is some fixed constant. For each $\alpha = (\beta, 1) \in \mathcal{A}$, we introduce the weight sequence

$$\lambda_\alpha = (\lambda_\alpha(j))_{j \in S_n} \quad \text{and} \quad \lambda_\alpha(j) = \prod_{i=1}^d \hat{\lambda}_\alpha(j_i) \quad (2.26)$$

with the elements

$$\hat{\lambda}_\alpha(t) = 1_{\{1 \leq t < t_*\}} + \left(1 - \left(\frac{t}{\omega_\alpha}\right)^\beta\right) 1_{\{t_* \leq t \leq \omega_\alpha\}},$$
where $t_\ast = 1 + [\ln \nu_n], \omega_\alpha = (\tilde{d}_\beta \nu_n)^{1/(2\beta + d)},$

$$\tilde{d}_\beta = \frac{(\beta + 1)(2\beta + 1)}{\pi^{2\beta}}$$

and $\nu_n = \mathbf{m}_n/\zeta^\ast.$

and the threshold $\zeta^\ast$ is introduced in (2.1). Now we define the set $\Lambda$ as

$$\Lambda = \{\lambda_\alpha, \alpha \in A\}.$$ \hfill (2.27)

It will be noted that in this case the cardinal of the set $\Lambda$ is

$$i_n = k^\ast[1/\varepsilon^2].$$ \hfill (2.28)

Moreover, taking into account that $\tilde{d}_\beta < 1$ for $\beta \geq 1$ we obtain for the set (2.27)

$$|\Lambda| \leq \sup_{\alpha \in A} (\omega_\alpha)^d \leq (\nu_n/\varepsilon)^{d/(2+d)}.$$ \hfill (2.29)

Remark 2.1. Note that the form (2.26) for the weight coefficients was proposed by Pinsker in [22] for the efficient estimation in the nonadaptive case, i.e. when the regularity parameters of the function $S$ are known. In the adaptive case these weight coefficients are used in [13, 14] to show the asymptotic efficiency for model selection procedures.

3 Main results

Now we formulate all non asymptotic oracle inequalities. Before, let us first introduce the following auxiliary function which is used to describe the rest terms in the oracle inequalities.

$$\Psi_p = \left( 1 + \sigma_p + \frac{1}{\sigma_p} \right) E_p \xi^4 i.$$ \hfill (3.1)

First, we obtain the oracle inequality for the risk (1.5).

Theorem 3.1. There exists some constant $\nu > 0$ such that for any $0 < \delta < 1/8,$ any $q \geq 2d\mathbf{m}_n + 2$ and any $p \in \mathcal{P}_n$ the estimator of $S$ given in (2.22) satisfies the following oracle inequality

$$\mathcal{R}_p(\tilde{S}_\ast, S) \leq \frac{(1 + 2\delta)}{1 - 4\delta} \min_{\lambda \in \Lambda} \mathcal{R}_p(\tilde{S}_\lambda, S)$$

$$+ \nu \Psi_p + |\Lambda|^d \left( E_p |\tilde{\sigma} - \sigma_p| + \tilde{b}/q \right) \frac{1}{q^\delta},$$ \hfill (3.2)

where $\tilde{b} = q^2 \sup_{j \in \mathbb{Z}^d} |b_j|^2.$
Note that, if $\sigma_p$ is known we obtain the following results.

**Corollary 3.2.** There exists some constant $\bar{\nu} > 0$ such that for any $0 < \delta < 1/8$, any $q \geq 2d_mn + 2$ and any $p \in P_n$ the estimator of $S$ given in (2.22) satisfies the following oracle inequality

$$R_p(\hat{S}, S) \leq \frac{(1 + 2\delta)}{1 - 4\delta} \min_{\lambda \in \Lambda} R_p(\hat{S}_\lambda, S) + \bar{\nu} \frac{\Psi_p + |\Lambda| \bar{b}/q}{q^\delta}. \quad (3.3)$$

Now we study the estimate (2.18).

**Proposition 3.3.** Assume that the partial derivative $\partial^d / \partial x_1 \ldots \partial x_d S$ is continuous and $\bar{r}$ be the Lipschitz constant for $S$. Then, there exists some positive $\bar{\nu} > 0$ such that for any $m_n \geq 4$, $q_* \geq 1$ and $m_n^d \leq q \leq q_* m_n^d$, $m_n^d \leq q \leq q_* m_n^d$,

$$E_p |\hat{\sigma} - \sigma_p| \leq \bar{\nu} q_* \left( \frac{1 + \bar{r}^2 + \bar{r}_d(S)(1 + E_p v_1)}{m_n^{d/2}} \right), \quad (3.4)$$

where and $\bar{r}_d(S) = \|\partial^d / \partial x_1 \ldots \partial x_d S\|^2$.

Theorem 3.1 and Proposition 3.3 implies the following result.

**Theorem 3.4.** Assume that the partial derivative $\partial^d / \partial x_1 \ldots \partial x_d S$ is continuous and $\bar{r}$ be the Lipschitz constant for $S$. Then, there exists some positive $\bar{\nu} > 0$ such that for any $m_n \geq 4$, $q_* \geq 1$, $m_n^d \leq q \leq q_* m_n^d$, $|\Lambda| \leq m_n^{d/2}$ and any $p \in P_n$ the estimator of $S$ given in (2.22) satisfies the following oracle inequality

$$R_p(\hat{S}, S) \leq \frac{(1 + 2\delta)}{1 - 4\delta} \min_{\lambda \in \Lambda} R_p(\hat{S}_\lambda, S)$$

$$+ \bar{\nu} q_* \frac{\Psi_p (1 + \bar{r}^2 + \bar{r}_d(S))}{q^\delta}. \quad (3.5)$$

The next result presents the non-asymptotic oracle inequality for the robust risk (1.4) for the model selection procedure (2.22), considered with the coefficients (2.26). Using the definition of the probability family $P_n$ in (2.1) and the function (3.1) we can obtain directly the following result.

**Theorem 3.5.** Assume that the partial derivative $\partial^d / \partial x_1 \ldots \partial x_d S$ is continuous and $\bar{r}$ be the Lipschitz constant for $S$. Then, there exists some positive $\bar{\nu} > 0$ such
that for any $m_n \geq 4$, $q_\ast \geq 1$, $m_n^d \leq q \leq q_\ast m_n^d$, the estimator of $S$ given in (2.22) satisfies the following oracle inequality

$$R_n^* (\hat{S}_\ast, S) \leq \frac{(1 + 2\delta)}{1 - 4\delta} \min_{\lambda \in \Lambda} R_n^* (\hat{S}_\lambda, S)$$

$$+ \tilde{v}_q \frac{\Psi_n^* (1 + \tilde{r}_2 + \tilde{r}_d(S))}{q^\delta} \leq (1 + 2\delta) \left( \frac{1}{1 - 4\delta} \min_{\lambda \in \Lambda} R_n^* (\hat{S}_\lambda, S) + \tilde{v}_q \frac{\Psi_n^* (1 + \tilde{r}_2 + \tilde{r}_d(S))}{q^\delta} \right),$$

(3.6)

where the coefficient $\Psi_n^* > 0$ is such that for any $\tilde{\delta} > 0$,

$$\lim_{n \to \infty} \frac{\Psi_n^*}{n^{\tilde{\delta}}} = 0.$$  

(3.7)

Remark 3.1. Note that the principal term in the right-hand side of the inequality (3.6) is best in the class of estimators $(\hat{S}_\lambda, \lambda \in \Lambda)$. Inequalities of such type are called the sharp non-asymptotic oracle inequalities. The inequality is sharp in the sense that the coefficient of the principal term may be chosen as close to 1 as desired.

4 Properties of the regression model (1.1)

First we study the property of the random variables (2.9)

**Proposition 4.1.** For any vectors $j$ and $k$ from $\mathbb{Z}^d$

$$E_p \eta_j \bar{\eta}_k = \sigma_p \varpi_j 1_{(j=k)},$$

(4.1)

where $\varpi_j$ is defined (2.10).

**Proof.** Note that, in view of (2.9) we obtain that

$$E_p \eta_j \bar{\eta}_k = q \sum_{l,l_1=1}^q \psi_{j,l} \psi_{k,l_1} E_p \xi_{j,l} \xi_{k,l_1}.$$

It is clear that, if $\nu_j \neq \nu_k$, then $E_p \xi_{j,l} \xi_{k,l_1} = 0$ and, therefore, $E_p \eta_j \bar{\eta}_k = 0$. Let now $\nu_j = \nu_k$, but $j \neq k$. In this case we obtain that

$$E_p \eta_j \bar{\eta}_k = q \sigma_p \sum_{l=1}^q \psi_{j,l} \bar{\psi}_{k,l}.$$

(4.2)

The functions $\psi_{j,l}$ can be represented as

$$\psi_{j,l} = e^{i\beta_j} s_{j,l} \Upsilon_j(\Delta_j),$$

(4.3)
where \( \Upsilon_j(z) = \int_0^z e^{-i\beta_j x} \, dx \) and \( \Delta_j = 2L_j/q \). Note now, that in view of (2.11) we get that \( L_j = L_k \) for \( \nu_j = \nu_k \). So, in this case for \( j \neq k \)

\[
\sum_{l=1}^{q} \tilde{\psi}_{j,l} \tilde{\psi}_{k,l} = \Upsilon_j(\Delta_j) \Upsilon_k(\Delta_j) \sum_{l=1}^{q} e^{i(\beta_j - \beta_k) s_{j,l}}
\]

\[
= \Upsilon_j(\Delta_j) \Upsilon_k(\Delta_j) Q_j \frac{1 - Q^q_j}{1 - Q_j} e^{-i(\beta_j - \beta_k)L_j},
\]

where \( Q_j = e^{i(\beta_j - \beta_k)\Delta_j} \). Taking into account the definition of \( \nu_j^* \) in (2.11), we obtain that for \( \nu_j = \nu_k \) the difference

\[
l^* = (|j| - |k|)\nu_j^* = |j|\nu_j^* - |k|\nu_k^* \in \mathbb{Z}.
\]

Therefore, taking into account again the definition of \( L_j \) in (2.11), we obtain that

\[
Q^q_j = e^{i(\beta_j - \beta_k)\Delta_j} = e^{2\pi l^*} = 1,
\]

i.e.

\[
\sum_{l=1}^{q} \tilde{\psi}_{j,l} \tilde{\psi}_{k,l} = q |\Upsilon_j(\Delta_j)|^2 \chi_{\{k=j\}} \tag{4.4}
\]

Therefore,

\[
E|\eta_j|^2 = \sigma_p \overline{\omega}_j \quad \text{and} \quad \overline{\omega}_j = q^2 |\Upsilon_j(\Delta_j)|^2.
\]

This implies (4.1). Hence Proposition 4.1.

**Proposition 4.2.** Let \( z = (z_j)_{j \in \mathbb{Z}^d} \) be a family of non random complex numbers. Then

\[
E_p \left| \sum_{j \in \mathbb{Z}^d} z_j \eta_j \right|^2 \leq \overline{\omega}_p |z|^2, \tag{4.5}
\]

where \( \overline{\omega}_p = 16\sigma_p x_* \) and \( |z|^2 = \sum_{j \in \mathbb{Z}^d} |z_j|^2 \).

Now for any non random family of real numbers \( x = (x_j)_{j \in \mathbb{Z}^d} \) with \( |x| < \infty \) we set

\[
U(x) = \sum_{j \in \mathbb{Z}^d} x_j \overline{\eta}_j, \tag{4.6}
\]

where \( \overline{\eta}_j = |\eta_j|^2 - \sigma_p \overline{\omega}_j \).

**Proposition 4.3.** The function (4.6) admits the following upper bound

\[
\sup_{1 \leq \#(x) \leq q} \frac{E_p |U(x)|^2}{|x|^2} \leq c_* \, E_p \xi_1^4, \tag{4.7}
\]

where \( \#(x) = \sum_j 1_{\{x_j \neq 0\}} \) and \( c_* = 5x_*^2 2^9 \).
Proof. First, note that the random variable $\tilde{\eta}_j$ can be represented as

$$\tilde{\eta}_j = q \sum_{l=1}^q \xi_{j,l} |\psi_{j,l}|^2 + 2q \sum_{l=2}^q v_{j,l} \xi_{j,l},$$

where $\tilde{\xi}_{j,l} = \xi_{j,l}^2 - \sigma_p$ and $v_{j,l} = \psi_{j,l} \sum_{t=1}^{l-1} \psi_{j,t} \xi_{j,t}$. Moreover, in view of the equality (4.3) we obtain that

$$\tilde{\eta}_j = q |\Upsilon_j(\Delta_j)|^2 \sum_{l=1}^q \tilde{\xi}_{j,l} + 2q \sum_{l=2}^q v_{j,l} \xi_{j,l}.$$

Therefore, setting

$$U_{1,l}(x) = \sum_j x_j |\Upsilon_j(\Delta_j)|^2 \tilde{\xi}_{j,l} \quad \text{and} \quad U_{2,l}(x) = \sum_j x_j v_{j,l} \xi_{j,l},$$

we can represent the function (4.6) as

$$U(x) = qU_1(x) + 2qU_2(x), \quad (4.8)$$

where $U_1(x) = \sum_{l=1}^q U_{1,l}(x)$ and $U_2(x) = \sum_{l=1}^q U_{2,l}(x)$. Taking into account that the random variables $(\xi_{j,l})_{1 \leq l \leq q}$ are independent with $E\tilde{\xi}_{j,l} = 0$, we obtain that

$$E_p U_1^2(x) = \sum_{l=1}^q E_p U_{1,l}^2(x) \quad \text{and} \quad E_p |U_2(x)|^2 = \sum_{l=1}^q E_p |U_{2,l}(x)|^2.$$

Now, using here that $|\Upsilon_j(\Delta_j)| \leq \Delta_j$, we obtain through the Cauchy-Bunyakovsky-Schwarz inequality that

$$E_p U_{1,l}^2(x) \leq \Delta_j^4 |x|^2 \#(x) E_p \xi_{1,l}^4 \leq \frac{2^8 x^2 E_p \xi_{1,l}^4}{q^4} |x|^2 \#(x).$$

Therefore, for $\#(x) \leq q$ we obtain that

$$E_p U_1^2(x) \leq 2^8 x^2 E_p \xi_{1,l}^4 |x|^2 q^{-2}.$$

Moreover, to estimate the last term in (4.8) note that $U_{2,l}(x)$ can be rewritten as

$$U_{2,l}(x) = \sum_{t=1}^{l-1} \tau_{t,l} \quad \text{and} \quad \tau_{t,l} = \sum_j x_j \psi_{j,t} \tilde{\psi}_{j,t} \xi_{j,t} \xi_{j,l}.$$
It is easy to see that for any 1 ≤ t, s ≤ l − 1
\[ E_p \tau_{t,l} \bar{\tau}_{s,l} = \sum_{j,k} x_j x_k \bar{\psi}_{j,l} \psi_{k,l} \bar{\psi}_{j,s} \psi_{k,s} E_p \xi_{j,s} \xi_{k,s} E_p \xi_{j,l} \xi_{k,l}, \]
i.e. \( E_p \tau_{t,l} \bar{\tau}_{s,l} = 0 \) for \( t \neq s \) and
\[ E_p |\tau_{t,l}|^2 = \sigma_p^4 \sum_{j,k} x_j x_k \bar{\psi}_{j,l} \psi_{k,l} \bar{\psi}_{j,t} \psi_{k,t} 1_{\{\nu_j = \nu_k\}}. \]
Therefore,
\[ E_p |U_{2,l}(x)|^2 = \sum_{t=1}^{l-1} E_p |\tau_{t,l}|^2 \leq \sum_{t=1}^{q} E_p |\tau_{t,l}|^2 \]
\[ = \sigma_p^4 \sum_{j,k} x_j x_k 1_{\{\nu_j = \nu_k\}} \bar{\psi}_{j,l} \psi_{k,l} \sum_{t=1}^{q} \bar{\psi}_{j,t} \psi_{k,t}. \]
Taking into account here the property (4.4), we obtain that
\[ E_p |U_{2,l}(x)|^2 \leq q \sigma_p^4 \sum_k x_k^2 |\gamma_k(D_k)|^4 \leq 2^8 \xi_s^2 \sigma_p^4 q^{-3} |x|^2 \]
and, therefore,
\[ E_p |U_2(x)|^2 \leq 2^{8} \xi_s^2 \sigma_p^4 q^{-2} \leq 2^{8} \xi_s^2 E_p \xi_1^4 q^{-2}. \]
From here it follows (4.7). Hence Proposition 4.3.

\[ \Box \]

5 Proofs

We will prove here most of the results of this paper.

5.1 Proof of Theorem 3.1

First, note that from (2.16) - (2.19) it follows that
\[ \text{Err}_n(\lambda) = J_n(\lambda) + 2 \sum_{j \in \mathbb{Z}^d} \lambda(j) \bar{\theta}_j + ||S||^2 - \delta \bar{P}_n(\lambda), \quad (5.1) \]
where \( \tilde{\theta}_j = \tilde{\theta}_j - \Re \tilde{\theta}_j \tilde{\eta}_j \). Using the definition of \( \tilde{\theta}_j \) in (2.17) we obtain that
\[
\tilde{\theta}_j = \Re \tilde{\theta}_j \zeta_j + |\zeta_j|^2 - \frac{\tilde{\sigma}}{\tilde{\sigma}} \Re \tilde{\theta}_j \zeta_j = \Re \tilde{\theta}_j \zeta_j + |b_j|^2
\]
\[
+ \frac{2}{\sqrt{q}} \Re \tilde{b}_j \eta_j + \frac{1}{q} \eta_j + \frac{\sigma_p - \sigma}{q} \Re \tilde{\theta}_j ,
\]
where \( \tilde{\eta}_j \) is defined in (4.6). Now we set
\[
M(\lambda) = \sum_{j \in \mathbb{Z}} \lambda(j) \Re \tilde{\theta}_j \zeta_j , \quad D_1(\lambda) = \frac{2}{\sqrt{q}} \sum_{j \in \mathbb{Z}} \lambda(j) \Re \tilde{b}_j \eta_j ,
\]
\[
D_2(\lambda) = \sum_{j \in \mathbb{Z}} \lambda(j)|b_j|^2 \quad \text{and} \quad D(\lambda) = D_1(\lambda) + D_2(\lambda) . \tag{5.2}
\]
Using these functions, we can rewrite (5.1) as
\[
\text{Err}_n(\lambda) = J_n(\lambda) + 2 \frac{\sigma_p - \sigma}{q} \tilde{L}(\lambda) + 2M(\lambda) + 2D(\lambda)
\]
\[
+ 2 \sqrt{P_n(\lambda)} \zeta(\lambda) U(e(\lambda)) \frac{\tilde{\sigma} \sigma_p \sqrt{q}}{\sqrt{q}} + ||S||^2 - \delta P_n(\lambda) , \tag{5.3}
\]
where \( e(\lambda) = \lambda/|\lambda|, \tilde{\sigma}(\lambda) = |\lambda|/\sqrt{\tilde{L}(\lambda^2)} \) and the function \( \tilde{L}(\cdot) \) is defined in (2.15). Let now \( \lambda_0 = (\lambda_0(j))_{1 \leq j \leq n} \) be a fixed sequence in \( \Lambda \), \( \hat{\lambda} \) be as in (2.22) and \( \mu_0 = \hat{\lambda} - \lambda_0 \). Substituting \( \lambda_0 \) and \( \hat{\lambda} \) in Equation (5.3), we obtain
\[
\text{Err}_n(\hat{\lambda}) - \text{Err}_n(\lambda_0) = J(\hat{\lambda}) - J(\lambda_0) + 2 \frac{\sigma_p - \sigma}{q} \tilde{L}(\mu_0) + 2M(\mu_0) + 2D(\mu_0)
\]
\[
+ 2 \sqrt{P_n(\lambda_0)} \zeta(\tilde{\lambda}) U(e) \frac{\tilde{\sigma} \sigma_p \sqrt{q}}{\sqrt{q}} - 2 \sqrt{P_n(\lambda_0)} \zeta(\lambda_0) U(e_0) \frac{\tilde{\sigma} \sigma_p \sqrt{q}}{\sqrt{q}}
\]
\[
- \delta \hat{P}_n(\hat{\lambda}) + \delta \hat{P}_n(\lambda_0) , \tag{5.4}
\]
where \( \hat{e} = e(\hat{\lambda}) \) and \( e_0 = e(\lambda_0) \). Note that, by (2.15),
\[
|L(\mu_0)| \leq \tilde{L}(\hat{\lambda}) + L(\lambda) \leq 2|\Lambda|^4 .
\]
Using the inequality
\[
2|ab| \leq \delta a^2 + \delta^{-1} b^2 \tag{5.5}
\]
and taking into account that $P_n(\lambda) > 0$ we obtain that for any $\lambda \in \Lambda$ and any $0 < \tilde{\delta} \leq \delta$

$$2\sqrt{P_n(\lambda)}\frac{\tilde{\varrho}(\lambda)|U(e(\lambda))|}{\sqrt{|q_p|}} \leq \delta P_n(\lambda) + \frac{\tilde{\varrho}^* U^*}{\delta |q_p|},$$

where $\tilde{\varrho}^* = \max_{\lambda \in \Lambda} \tilde{\varrho}^2(\lambda)$ and $U^* = \sup_{\lambda \in \Lambda} U^2((e(\lambda)))$. Note here that for any $\lambda \in \Lambda$

$$|\hat{P}_n(\lambda) - P_n(\lambda)| \leq \frac{\hat{\sigma} - \sigma_p}{q} \hat{L}(\lambda^2) \leq \frac{|\hat{\sigma} - \sigma_p|}{q} |\Lambda|_k.$$ 

So, taking into account that $J(\hat{\lambda}) \leq J(\lambda_0)$, we get for any $0 < \tilde{\delta} \leq \delta < 1$ that

$$\text{Err}_n(\lambda) \leq \text{Err}_n(\lambda_0) + 6 \frac{|\hat{\sigma} - \sigma_p| |\Lambda|_k + 2M(\mu_0) + 2D(\mu_0)}{q} + \frac{2\tilde{\varrho}^* U^*}{\delta |q_p|} + (\tilde{\delta} - \delta) P_n(\hat{\lambda}) + 2\delta P_n(\lambda_0). \quad (5.6)$$

To estimate the third term in the right side of this inequality we represent for any $\mu \in \Lambda_1 = \Lambda - \lambda_0 = \{\lambda - \lambda_0, \lambda \in \Lambda\}$ as

$$M(\mu) = M_1(\mu) + M_2(\mu), \quad (5.7)$$

where

$$M_1(\mu) = \frac{1}{\sqrt{|q|}} \text{Re} \sum_{j \in \mathbb{Z}^d} \mu(j) \tilde{\theta}_j \eta_j \quad \text{and} \quad M_2(\mu) = \sum_{j \in \mathbb{Z}^d} \mu(j) \text{Re} \tilde{\theta}_j b_j$$

Moreover, for any family $v = (v(j))_{j \in \mathbb{Z}^d}$ for which $|v|^2 = \sum_{j \in \mathbb{Z}^d} v^2(j) < \infty$ we set

$$S_v(x) = \sum_{j \in \mathbb{Z}^d} v(j) \theta_j \Phi_j(x). \quad (5.8)$$

Using Proposition 4.2 we obtain that

$$E_p |M_1(\mu)|^2 \leq \frac{\tilde{\varrho}^*}{q} \sum_{j \in \mathbb{Z}^d} \mu^2(j) |\theta_j|^2 = \tilde{\varrho}^* \frac{\|S_{\mu}\|^2}{q}. \quad (5.9)$$

To estimate this function for a random family $\mu = (\mu(j))_{j \in \mathbb{Z}^d}$ we set

$$Z^* = \sup_{x \in \Lambda_1} \frac{q|M_1(x)|^2}{\|S_x\|^2}. \quad (5.10)$$
So, through the inequality (5.5), we get

\[ 2|M_1(\mu)| \leq \delta\|S_\mu\|^2 + \frac{Z^*}{q\delta}. \]

It is clear that the last term here can be estimated as

\[ E_p Z^* \leq \sum_{x \in \Lambda_1} \frac{qE_p|M_1(x)|^2}{\|S_x\|^2} \leq \sum_{x \in \Lambda_1} \tilde{\omega}_p = \tilde{\omega}_p \hat{i}, \quad (5.11) \]

where \( \hat{i} = \text{card}(\Lambda) \). Using again the inequality (5.5) we obtain that for any \( x \in \Lambda_1 \)

\[ 2|M_2(x)| \leq \delta\|S_x\|^2 + \frac{1}{\delta} \sum_{j \in \mathbb{Z}^d} |x(j)||b_j|^2 \leq \delta\|S_x\|^2 + \frac{2|\Lambda|b_*}{\delta}, \quad (5.12) \]

where \( b_* = \sup_{j \in \mathbb{Z}^d} |b_j|^2 \). Thus,

\[ 2|M(\mu)| \leq 2\delta\|S_\mu\|^2 + \frac{Z^*}{q\delta} + \frac{2|\Lambda|b_*}{\delta}. \quad (5.13) \]

Moreover, note that, for any \( x \in \Lambda_1 \),

\[ \|S_x\|^2 - \|\tilde{S}_x\|^2 = \sum_{j \in \mathbb{Z}^d} x^2(j)(|\theta_j|^2 - |\tilde{\theta}_j|^2) \]

\[ \leq -2M_1(x^2) - 2M_2(x^2), \quad (5.14) \]

where \( x^2 = (x^2(j))_{j \in \mathbb{Z}^d} \). Taking into account that, for any \( x \in \Lambda_1 \) the components \( |x(j)| \leq 1 \), we can estimate this term as in (5.9), i.e.,

\[ \mathbb{E}_p |M_1(x^2)|^2 \leq \tilde{\omega}_p \|S_x\|^2 / q. \]

Similarly to the previous reasoning we set

\[ Z^*_1 = \sup_{x \in \Lambda_1} \frac{q|M_1(x^2)|^2}{\|S_x\|^2} \]

and we get

\[ \mathbb{E}_p Z^*_1 \leq \tilde{\omega}_p \hat{i}. \quad (5.15) \]

Using the same arguments as in (5.13), we can derive

\[ 2|M_1(x^2)| \leq \delta\|S_x\|^2 + \frac{Z^*}{q\delta}. \]
Similarly to (5.12) we can estimate
\[ 2|M_2(x^2)| \leq \delta \| S_x \|^2 + \frac{2|\Lambda|_* b_*}{\delta}. \]

From here and (5.14) we get
\[ \| S_x \|^2 \leq \| \hat{S}_x \|^2 + \frac{Z^*_1 + Z^*_1}{q\delta(1 - 2\delta)} + \frac{2|\Lambda|_* b_*}{1 - 2\delta}. \]

for any \( 0 < \delta < 1 \). Using this bound in (5.13) yields
\[ 2|M(x)| \leq \frac{2\delta\| \hat{S}_x \|^2}{1 - 2\delta} + \frac{Z^*_1 + Z^*_1}{q\delta(1 - 2\delta)} + \frac{2|\Lambda|_* b_*}{1 - 2\delta}. \]

Taking into account that
\[ \| \hat{S}_{\mu_0} \|^2 = \| \hat{S}_\lambda - \hat{S}_{\lambda_0} \|^2 = \| (\hat{S}_\lambda - S) - (\hat{S}_{\lambda_0} - S) \|^2 \leq 2(Err_n(\hat{\lambda}) + Err_n(\lambda_0)), \]

we obtain
\[ 2|M(\mu_0)| \leq \frac{2\delta(Err_n(\hat{\lambda}) + Err_n(\lambda_0))}{1 - 2\delta} + \frac{Z^*_1 + Z^*_1}{q\delta(1 - 2\delta)} + \frac{2|\Lambda|_* b_*}{1 - 2\delta}. \]

Let us estimate now the term \( D(\mu_0) \). Using the inequality (5.5) we obtain that for any \( \lambda \in \Lambda \) and \( 0 < \tilde{\delta} < \delta < 1 \)
\[ |D_1(\lambda)| \leq \frac{\tilde{\delta}}{q} \sum_{j \in \mathbb{Z}^d} \lambda^2(j) |\eta_j|^2 + \frac{1}{\delta} \sum_{j \in \mathbb{Z}^d} 1_{\{\lambda(j) \neq 0\}} |b_j|^2 \]
\[ \leq \tilde{\delta} P_n(\lambda) + \frac{\tilde{\delta}}{q} U(\lambda^2) + \frac{|\Lambda|_* b_*}{\delta}. \]

Taking into account here that
\[ \frac{|U(\lambda^2)|}{q} \leq \frac{|\lambda|}{q} \frac{|U(e(\lambda^2))|}{q} = \sqrt{P_n(\lambda)} \frac{|U(e(\lambda^2))|}{\sqrt{\sigma_p q}} \]
\[ \leq \delta P_n(\lambda) + \frac{\tilde{\varepsilon}^* U^*_1}{\sigma_p q \delta}, \]

where \( U^*_1 = \sup_{\lambda \in \Lambda} U^2((e(\lambda^2))). \). This implies that for any \( \lambda \in \Lambda \)
\[ |D_1(\lambda)| \leq 2\delta P_n(\lambda) + \frac{\tilde{\varepsilon}^* U^*_1}{\sigma_p q} + \frac{|\Lambda|_* b_*}{\delta}. \]
and, therefore,
\[ |D_1(\mu_0)| \leq 2\delta P_n(\lambda) + 2\delta P_n(\lambda_0) + \frac{2\tilde{z} \star U_1^*}{\sigma_p q} + \frac{2|\Lambda|_* b_*}{\delta}. \]

Moreover, similarly to the upper bound (5.12) we get
\[ |D_2(\mu_0)| \leq \max_{\lambda \in \Lambda} D_2(\lambda) + D_2(\lambda_0) \leq 2|\Lambda|_* b_* . \]

Finally, we obtain that
\[ 2|D(\mu_0)| \leq 4\delta P_n(\lambda) + 4\delta P_n(\lambda_0) + \frac{4\tilde{z} \star U_1^*}{\sigma_p q} + \frac{8|\Lambda|_* b_*}{\delta}. \quad (5.17) \]

So, using the upper bound (5.6), we obtain that
\[ \text{Err}_n(\lambda) \leq \text{Err}_n(\lambda_0) + \frac{2\delta(\text{Err}_n(\lambda) + \text{Err}_n(\lambda_0))}{1 - 2\delta} + \frac{6|\tilde{\sigma} - \sigma_p|}{q} |\Lambda|_* + \frac{Z^* + Z^1_*}{q\delta(1 - 2\delta)} + \frac{8|\Lambda|_* b_*}{(1 - 2\delta)\delta} \]
\[ + \frac{8\tilde{z} \star (U^* + U_1^*)}{\delta \sigma_p q} + (5\delta - \delta) \text{Err}_n(\lambda) + 6\delta P_n(\lambda_0). \]

So, choosing \( \delta = \delta/5 \), we obtain that
\[ \text{Err}_n(\lambda) \leq \frac{\text{Err}_n(\lambda_0)}{1 - 4\delta} + 6 \frac{|\tilde{\sigma} - \sigma_p|}{q(1 - 4\delta)} |\Lambda|_* + \frac{Z^* + Z^1_*}{q\delta(1 - 4\delta)} + \frac{40|\Lambda|_* b_*}{(1 - 4\delta)\delta} \]
\[ + \frac{40\tilde{z} \star (U^* + U_1^*)}{\delta \sigma_p q} + \frac{6\delta}{1 - 4\delta} P_n(\lambda_0). \]

In view of Proposition 4.3 we estimate the expectation of the term \( U^* + U_1^* \) as
\[ E_p(U^* + U_1^*) \leq \sum_{\lambda \in \Lambda} \left( E_p U^2(e(\lambda)) + E_p U^2(e(\lambda^2)) \right) \leq 2i c_4 E_p \xi_1^4 . \]

Taking into account that \( 0 < \delta \leq 1/8 \), we get
\[ R(\hat{S}, S) \leq \frac{R(\tilde{S}, \lambda_0, S)}{1 - 4\delta} + \frac{12|\Lambda|_* E_p |\tilde{\sigma} - \sigma_p|}{q} + \frac{4\tilde{z} \star i}{q\delta} + \frac{80|\Lambda|_* b_*}{\delta} \]
\[ + \frac{80\tilde{z} \star i c_4 E_p \xi_1^4}{\delta \sigma_p q} + \frac{2\delta}{(1 - 4\delta)} P_n(\lambda_0) . \]

Using the upper bound for \( P_n(\lambda_0) \) in Lemma A.3, one obtains (3.1), that finishes the proof. \( \square \)
5.2 Proof of Proposition 3.3

We use here the same method as in [10]. First of all note that the definition (2.8) implies that

\[ \hat{\sigma} = \frac{1}{q} \sum_{j \in T_n} |\theta_j|^2 + \frac{2}{q} M_n + \frac{1}{q} \sum_{j \in T_n} |\zeta_j|^2 , \tag{5.18} \]

where \( M_n = \text{Re} \sum_{j \in T_n} \tilde{\theta}_j \zeta_j \). In Lemma A.2 we show that

\[ \sum_{j \in Q_n} |\theta_j|^2 \leq \frac{x^2 d^*/\pi^2 m^{d/2} \tau_d(S)}{2} + \sum_{j \in T_n} |b_j|^2 , \tag{5.19} \]

where \( Q_n = \{ \lfloor \sqrt{m_n} \rfloor + 1, \ldots, d \} \). To estimate the second term in (5.18) we represent it as

\[ M_n = M_{1,n} + M_{2,n} , \]

where

\[ M_{1,n} = \text{Re} \sum_{j \in T_n} \tilde{\theta}_j b_j \text{ and } M_{2,n} = \frac{1}{\sqrt{q}} \text{Re} \sum_{j \in T_n} \tilde{\theta}_j \eta_j . \]

To estimate \( M_{1,n} \) note that

\[ 2|M_{1,n}| \leq \sum_{j \in T_n} |\theta_j|^2 + \sum_{j \in T_n} |b_j|^2 \leq \frac{x^2 d^*/\pi^2 m^{d/2} \tau_d(S)}{2} + \sum_{j \in T_n} |b_j|^2 . \]

To estimate the last term in this inequality we use the coefficient \( \tilde{b} \) defined in (3.1) and the fact that \( q \geq m_n^d \), i.e.

\[ \sum_{j \in T_n} |b_j|^2 \leq \tilde{b} \frac{m_n^d}{q^2} \leq \frac{\tilde{b}}{q} . \tag{5.20} \]

Therefore,

\[ 2|M_{1,n}| \leq \frac{x^2 d^*/\pi^2 m^{d/2} \tau_d(S)}{2} + \frac{\tilde{b}}{q} . \]

Moreover, the term \( M_{2,n} \) can be estimated through Proposition 4.2 as

\[ E_p |M_{2,n}|^2 \leq \frac{x^2_d p}{q} \sum_{j \in T_n} |\theta_j|^2 \leq \frac{x^2_d p \tau_d(S)}{2} \frac{x^2 d^*/\pi^2 m^{d/2} q}{q} , \]
while the absolute value of this term can be estimated as
\[
2 \mathbb{E}_p |M_{2,n}| \leq 2 \frac{\sqrt{\mathbb{E}_p \tilde{\tau}_d(S) x^d_s}}{\pi^d m_n^{d/4} \sqrt{q}} \leq \frac{\tilde{\omega}_p \tilde{\tau}_d(S) x^d_s}{\pi^d m_n^{d/2}} + \frac{1}{q}.
\]
Therefore,
\[
2 \mathbb{E}_p \frac{|M_n|}{q} \leq \frac{(1 + \tilde{\omega}_p) \tilde{\tau}_d(S) x^d_s}{\pi^d m_n^{d/2} - q} + \frac{1 + \bar{b}}{qq}.
\]
We can represent the last term in (5.18) as
\[
\frac{1}{q} \sum_{j \in T_n} |\zeta_j|^2 = \frac{1}{qq} \sum_{j \in T_n} |\eta_j|^2 + \frac{1}{q} \sum_{j \in T_n} |b_j|^2 + 2 \frac{\text{Re} \sum_{j \in T_n} \eta_j \bar{b}_j}{q \sqrt{q}}.
\]
Moreover, using the definition (4.6) we obtain
\[
\frac{1}{q} \sum_{j \in T_n} |\eta_j|^2 = \frac{\sigma_p \sum_{j \in T_n} \omega_j}{q} + \frac{m_n^{d/2} U(\bar{x})}{q} = \frac{\sigma_p q + m_n^{d/2} U(\bar{x})}{q}
\]
with \(\bar{x}_j = 1_{\{j \in T_n\}} / m_n^{d/2}\). Therefore, from Proposition 4.3 it follows that
\[
\mathbb{E}_p \left| \frac{1}{qq} \sum_{j \in T_n} |\eta_j|^2 - \sigma_p \right| \leq \frac{m_n^{d/2} \sqrt{c_s E_p \xi_1^4}}{qq}.
\]
Moreover, using here (2.12), we obtain that for \(m_n \geq 4\)
\[
\mathbb{E}_p \left| \frac{1}{qq} \sum_{j \in T_n} |\eta_j|^2 - \sigma_p \right| \leq \frac{m_n^{d/2} \sqrt{c_s E_p \xi_1^4}}{\sqrt{\omega_s (m_n - \sqrt{m_n})^d}} \leq \frac{2^d \sqrt{c_s E_p \xi_1^4}}{\sqrt{\omega_s} m_n^{d/2}}.
\]
To estimate the last term in (5.21) we use this bound (5.20) and again Proposition 4.2, i.e.
\[
\mathbb{E} \left| \sum_{j \in T_n} \eta_j \bar{b}_j \right|^2 \leq \tilde{\omega}_p \sum_{j \in T_n} |b_j|^2 \leq \frac{\tilde{\omega}_p \tilde{b}}{q}.
\]
Thus,
\[
\mathbb{E}_p \left| \frac{1}{q} \sum_{j \in T_n} |\zeta_j|^2 - \sigma_p \right| \leq \frac{2^d \sqrt{c_s E_p \xi_1^4}}{\sqrt{\omega_s} m_n^{d/2} + \tilde{b} \frac{q}{qq}} + \frac{2 \sqrt{\tilde{\omega}_p \tilde{b}}}{qq} \leq \frac{2^d \sqrt{c_s E_p \xi_1^4}}{\sqrt{\omega_s} m_n^{d/2}} + \frac{2 \tilde{b} + \tilde{\omega}_p}{qq}.
\]
It should be noted also that for $q \leq q_*$ we can estimate $\bar{q}$ from below as

$$\bar{q} = \sum_{j \in T_n} \frac{\omega_j}{q} \geq \frac{\sqrt{\omega_*}}{2^d} \frac{m_n^{d'}}{q} \geq \frac{\sqrt{\omega_*}}{2^d} \frac{1}{q_*} > 0.$$ 

Moreover, using Lemma A.1 we can estimate directly $b_j$ as

$$\sup_{j \in \mathbb{Z}^d} |b_j| \leq \frac{16\hat{r} (2x_*)^d}{(2\pi)^{d/2}} \frac{1}{q}.$$ 

From here we obtain the bound (3.4) and hence the desired result. \qed

**Acknowledgments.** The last author is partially supported by RFBR Grant 16-01-00121, by the Ministry of Education and Science of the Russian Federation in the framework of the research project No 2.3208.2017/4.6, by the Russian Federal Professor program (Project No 1.472.2016/1.4, Ministry of Education and Science of the Russian Federation) and “The Tomsk State University competitiveness improvement programme” Grant 8.1.18.2018.

## 6 Appendix

### A.1 Properties of the Radon transformation

First, we recall some basic definitions valid for $\mathbb{R}^d \rightarrow \mathbb{R}$ functions $f$ belonging to the Schwartz space and which can be found, for instance, in [19]. The Fourier transform of $f$ is given by

$$T(f)(z) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ix \cdot z} \, dx \quad (A.1)$$

and its inverse Fourier transform by

$$T^{-1}(f)(\eta) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(y) e^{iy \cdot \eta} \, dy \quad (A.2)$$

where $\cdot$ denotes the inner product in $\mathbb{R}^d$.

For any $\nu$ in the unit sphere $S$ and any $\varsigma \in \mathbb{R}$, the Radon transform of $f$ is defined by

$$R(f)(\nu, \varsigma) = \int_{\nu \perp} f(\varsigma + y) \, dy, \quad (A.3)$$
where $\nu^\perp$ is the subspace orthogonal to $\nu$ in $\mathbb{R}^d$. Setting
\[ A_\nu(f)(\varsigma) = R(f)(\nu, \varsigma), \tag{A.4} \]
it is easily seen that
\[ T(f)(\varsigma \nu) = T \circ A_\nu(f)(\varsigma). \tag{A.5} \]
Taking into account that for $|\varsigma| \geq x^*$ and for $y \in \nu^\perp$ the norm $||\varsigma \nu + y||^2 = ||\varsigma||^2 + ||y||^2 \geq N^2$, we obtain that
\[ R(f)(\nu, \varsigma) = 0 \quad \text{for any} \quad |\varsigma| \geq N. \tag{A.6} \]

The following lemma gives a Lipschitzian property of the Radon transform.

**Lemma A.1.** Let $f$ be a Lipschitzian function from $\mathbb{R}^d$ into $\mathbb{R}$ with Lipschitz constant $\tilde{r}$ and with compact support a centered ball of radius $x^*$.

Then its Radon transform is Lipschitzian with Lipschitz constant $\tilde{r} x^d - 1$; more precisely, for any $\nu \in S$ and any $(s_1, s_2) \in \mathbb{R}^2$,
\[ |R(f)(\nu, s_1) - R(f)(\nu, s_2)| \leq \tilde{r} x^{d-1}_* |s_1 - s_2|. \tag{A.7} \]

**Proof.** Let $\nu \in S$ and $(s_1, s_2) \in \mathbb{R}^2$. By definition of the Radon transform, we have
\[
|R(f)(\nu, s_1) - R(f)(\nu, s_2)| = \left| \int_{\mathbb{R}^{d-1}} (f(\nu s_1 + y) - f(\nu s_2 + y)) \, dy \right|
\]
\[
= \left| \int_{|y| \leq N} (f(\nu s_1 + y) - f(\nu s_2 + y)) \, dy \right|
\]
\[
\leq \int_{|y| \leq N} |f(\nu s_1 + y) - f(\nu s_2 + y)| \, dy
\]
\[
\leq \tilde{r} x^{d-1}_* |s_1 - s_2|.
\]

Hence Lemma A.1. $\square$

**A.2 Property of the Fourier coefficients**

**Lemma A.2.** Assume that the partial derivative $\partial^d / \partial x_1 \ldots \partial x_d$ of $S$ is continuous. Then the equality (5.19) holds true.

**Proof.** Integrating by parts we obtain that
\[
\theta_j = \frac{x_j^d}{i^d \pi^d \prod_{i=1}^d j_i} \int_{[-x_*, x_*]^d} \frac{\partial^d}{\partial x_1 \ldots \partial x_d} S(z) \Phi_j(z) \, dz.
\]
So, applying here the Bunyakovsky - Cauchy - Swarths inequality we obtain the upper bound (5.19). Hence lemma A.2. $\square$
A.3 Property of the penalty term

Lemma A.3. For any $\lambda \in \Lambda$,

$$P_n(\lambda) \leq E_p Err_n(\lambda)$$

where the coefficient $P_n(\lambda)$ was defined in (5.2).

Proof. By the definition of $Err_n(\lambda)$ one has

$$E_p Err_n(\lambda) = \sum_{j \in \mathbb{Z}^d} E_p |(\lambda(j) - 1)\theta_j + \lambda(j) \zeta_j|^2$$

$$= \sum_{j \in \mathbb{Z}^d} E_p \left| (\lambda(j) - 1)\theta_j + \lambda(j) b_j + \lambda(j) \frac{1}{\sqrt{q}} \eta_j \right|^2$$

$$\geq \frac{1}{q} \sum_{j \in \mathbb{Z}^d} \lambda^2(j) E_p |\eta_j|^2 = P_n(\lambda).$$

Hence lemma A.3. $\square$

References

[1] H. Akaike. A new look at the statistical model identification. IEEE Trans. on Automatic Control 19 (1974) 716–723.

[2] Barron A, Birgé L, Massart P (1999) Risk bounds for model selection via penalization. Probab. Theory Relat. Fields 113:301–415

[3] D. Fourdrinier and S. M. Pergamenshchikov. Improved selection model method for the regression with dependent noise. Annals of the Institute of Statistical Mathematics 59, 435–464, 2007.

[4] Galtchouk LI, Pergamenshchikov SM (2006) Asymptotically efficient estimates for non parametric regression models. Statistics and Probability Letters 76(8) : 852 – 860

[5] Galtchouk LI, Pergamenshchikov SM (2009) Sharp non-asymptotic oracle inequalities for nonparametric heteroscedastic regression models. Journal of Nonparametric Statistics 21(1):1 – 16

23
[6] Galtchouk LI, Pergamenshchikov SM (2009) Adaptive asymptotically efficient estimation in heteroscedastic nonparametric regression. Journal of Korean Statistical Society 38(4):305–322

[7] I. A. Ibragimov and R. Z. Khasminskii. Statistical Estimation: Asymptotic Theory. Springer, Berlin-New York, 1981.

[8] Kneip A (1994) Ordered linear smoothers. Annals of Statistics 22:835–866

[9] Konev VV, Pergamenshchikov SM (2003) Sequential estimation of the parameters in a trigonometric regression model with the Gaussian coloured noise. Statistical Inference for Stochastic Processes 6:215–235

[10] Konev VV, Pergamenshchikov SM (2009) Nonparametric estimation in a semimartingale regression model. Part 1. Oracle Inequalities. Journal of Mathematics and Mechanics of Tomsk State University 3:23–41

[11] Konev VV, Pergamenshchikov SM (2009) Nonparametric estimation in a semimartingale regression model. Part 2. Robust asymptotic efficiency. Journal of Mathematics and Mechanics of Tomsk State University 4:31–45

[12] Konev VV, Pergamenshchikov SM (2010) General model selection estimation of a periodic regression with a Gaussian noise. Annals of the Institute of Statistical Mathematics 62:1083–1111

[13] Konev VV, Pergamenshchikov SM (2012) Efficient robust nonparametric estimation in a semimartingale regression model. Ann Inst Henri Poincaré Probab Stat 48(4):1217–1244

[14] Konev VV, Pergamenshchikov SM (2015) Robust model selection for a semimartingale continuous time regression from discrete data. Stochastic processes and their applications 125:294 – 326

[15] Konev VV, Pergamenshchikov SM, Pchelintsev E (2014) Estimation of a regression with the pulse type noise from discrete data. Theory Probab Appl 58(3):442–457

[16] A. P. Korostelev and A. B. Tsybakov Optimal rates of convergence of estimate in the stochastic problem of computerized tomography.- Problems Inform. Transmission 27 (1), 73 – 81, 1991

[17] C. Mallows. Some comments on $C_p$. Technometrics, 15, 661–675, 1973.

[18] F. Natterer. The Mathematics of Computerized Tomography. - Wiley, 1986.
[19] F. Natterer. A Sobolev Space Analysis of Picture Reconstruction, *SIAM, Journal on Applied Mathematics*, **39**, (3), 402 – 411, 1980.

[20] M. Nussbaum. Spline smoothing in regression models and asymptotic efficiency in $L_2$. *Ann. Statist.*, **13**, 984–997, 1985.

[21] A. A. Novikov. On discontinuous martingales. *Theory Probab. Appl.*, **20** (1), 11–26, 1975.

[22] M. S. Pinsker. Optimal filtration of square integrable signals in gaussian white noise. *Problems of transmission information*, **17**, 120–133, 1981.

[23] A. G. Sveshnikov and A. N. Tikhonov *The theory of functions of a complex variable*, Translated from the Russian, English translation, Mir Publishers, 1978