Fractional quantization of charge and spin in topological quantum pumps

Pasquale Marra and Roberta Citro
1CNR-SPIN, I-84084 Fisciano (Salerno), Italy
2Department of Physics “E. R. Caianiello”, University of Salerno, I-84084 Fisciano (Salerno), Italy

Topological quantum pumps are topologically equivalent to the quantum Hall state: In these systems, the charge pumped during each pumping cycle is quantized and coincides with the Chern invariant. However, differently from quantum Hall insulators, quantum pumps can exhibit novel phenomena such as the fractional quantization of the charge transport, as a consequence of their distinctive symmetries in parameter space. Here, we report the analogous fractional quantization of the spin transport in a topological spin pump realized in a one-dimensional lattice via a periodically modulated Zeeman field. In the proposed model, which is a spinfull generalization of the Harper-Hofstadter model, the amount of spin current pumped during well-defined fractions of the pumping cycle is quantized as fractions of the spin Chern number. This fractional quantization of spin is topological, and is a direct consequence of the additional symmetries ensuing from the commensuration of the periodic field with the underlying lattice.

I. INTRODUCTION

Topological insulators are characterized by the presence of gapless edge modes which are topologically protected by a nontrivial edge topology invariant. In particular, quantum Hall insulators exhibit chiral edge modes and a quantized Hall conductance, which coincides with the Chern invariant, i.e., an integer capturing the global topological properties of the system. On the other hand, the recently discovered quantum spin Hall insulators exhibit spin-polarized helical edge modes which are protected by a time-reversal symmetric topological invariant or by a spin Chern number. The quantum spin Hall state is equivalent to two copies of the charge Hall state cloned into two decoupled spin channels, with spin up and down electrons moving in opposite directions. As such, the total charge Hall conductance vanishes but the net spin Hall conductance remains finite.

As shown by Thouless, a one-dimensional system with a periodically-modulated field can realize a topological quantum pump. This system can exhibit a topologically non-trivial state which is equivalent to a two-dimensional quantum Hall insulator. In this state, the charge pumped during each pumping cycle is quantized and coincides with the Chern invariant: The quantized charge is topological, being analogous to the transverse conductance of the quantum Hall state. As a further generalization, the topological spin pumping and spin Chern pumping has been theoretically proposed. These are indeed the one-dimensional analogous of a spin quantum Hall state, in view of the fact that the spin pumped is finite despite a vanishing net charge pumped during each cycle. Yet, topological spin pumps remain difficult to realize in condensed matter settings, despite the numerous proposals. On the other hand, ultracold atomic gases in optical lattices have recently proven as an ideal platform of topological charge and spin pumps in optical lattices. On top of that, topological quantum pumps and, in general, low-dimensional systems with periodically-modulated fields, can exhibit novel and characteristic physical phenomena which have no analogue in quantum Hall insulators, such as the presence of fractional edge charges, and the fractional quantization of the charge transport, which is a direct consequence of their peculiar symmetries in parameter space. Indeed, in the presence of a periodically modulated field, the charge pumped at well-defined fractions of the pumping period is quantized as integer fractions of the Chern number, as shown in Ref.

In this work we describe a novel but related physical phenomenon, i.e., the fractional quantization of the spin transport in a spin quantum pump. This system can be realized by a one-dimensional lattice in the presence of a spatially and periodically modulated Zeeman field, with a periodicity which is commensurate with the underlying lattice, and which is modulated adiabatically in time. This non-interacting quantum spin pump exhibits a fractional quantization of the amount of spin pumped at well-defined fractions of the pumping cycle. In particular, we show that the fractional quantization of the spin transport can be probed by measuring the variations of the spin center of masses of the electronic gas. We therefore discuss the experimental implementation of this system in condensed matter systems as well as in ultracold atomic gases in optical lattices, which have been proven as an ideal platform to realize exotic states of matter.

II. FRACTIONAL QUANTIZATION OF CHARGE AND SPIN TRANSPORT

Quantum pumps are defined as insulating systems which can transfer a finite amount of charge or spin as a result of the adiabatic and periodic evolution in time of a driving field. Let us consider the case where the adiabatic evolution of the system is described by a slow and continuous change of a parameter \( \varphi \), and assume that the system is periodic in such parameter with period \( 2\pi \). If the Fermi level lays in any of the intraband gaps of the spectrum, i.e., any energy gap which remains open for any choice of the control parameter \( \varphi \), and
the up and down spin channels are completely decoupled, the charge \( Q \) and the total spin \( S \) pumped during an adiabatic cycle \( \varphi_0 \rightarrow \varphi_0 + 2\pi \) corresponds with the charge and the spin Chern numbers and are given by

\[
Q = C_\uparrow + C_\downarrow, \\
S = C_\uparrow - C_\downarrow,
\]

where

\[
C_\tau = \frac{1}{2\pi} \int_{\varphi_0}^{\varphi_0 + 2\pi} d\varphi \sum_{i} \int_{BZ} dk \Omega_{i\tau}(k, \varphi). 
\]

Here \( C_\tau \) are the Chern numbers separately for the up and down spin channels \( \tau = \uparrow, \downarrow \). They are defined as the sum over all the filled bands of the integrals of the Berry curvature \( \Omega_{i\tau}(k, \varphi) = \partial_k A_{i\tau}(\varphi) - \partial_\varphi A_{i\tau}(k) \), where \( A_{i\tau}(k) = i\langle \tau | \partial_\varphi | \tau \rangle \) is the Berry connection of the eigenstate \( |\tau\rangle \) of the \( i \)th band with spin \( \tau \). Therefore the charge \( Q \) and the spin \( S \) pumped during an adiabatic cycle are quantized, being respectively equal the sum and the difference of the Chern numbers of each spin channels, and do not depend on the initial phase \( \varphi_0 \). Notice that the parameter \( \varphi \) plays the role of an additional synthetic (non-spatial) dimension \([49, 50]\). The one-dimensional physical system is thus embedded in a two-dimensional parameter space, which is thus analogous to an insulating two-dimensional system. Topological charge pumps (i.e., \( Q \neq 0 \)) are therefore analogous to a quantum Hall insulator, whereas topological spin pumps (i.e., \( S \neq 0 \)) are analogous to a quantum spin Hall insulator.

For arbitrary adiabatic transformations with \( \varphi_0 \rightarrow \varphi_0 + \Delta \varphi \) with \( \Delta \varphi \neq 2\pi \) instead, the charge and spin pumped are not quantized and depend in general on the initial phase \( \varphi_0 \). Nevertheless, if the system exhibit an additional unitary periodicity in the control parameter \( \varphi \), the charge and the spin pumped over fractions of the pumping cycle are quantized as fractions of the charge and spin Chern number. To be more specific, consider the case where the Hamiltonian of the system is periodic up to a unitary transformation \( U \) with respect to the control parameter \( \varphi \) with period \( 2\pi/q \), that is

\[
H(k, \varphi + 2\pi/q) = U H(k, \varphi) U^\dagger, 
\]

with \( q \) integer. In this case the Berry curvature, which is gauge invariant, is a periodic function of the control parameter, i.e., \( \Omega_{i\tau}(k, \varphi + 2\pi/q) = \Omega_{i\tau}(k, \varphi) \). This mandates that the integral of the Berry curvature in Eq. (1) restricted to an adiabatic evolution \( \varphi_0 \rightarrow \varphi_0 + 2\pi/q \) is equal to a fraction \( C_\tau/q \) of the Chern number, as shown in Ref. \([46, 48]\). Therefore, the charge and the spin pumped during an adiabatic transformation \( \Delta \varphi = 2\pi m/q \) which is a multiple of the period \( 2\pi/q \) are given by

\[
Q|_{2\pi m/q} = \frac{m}{q} (C_\uparrow + C_\downarrow), \\
S|_{2\pi m/q} = \frac{m}{q} (C_\uparrow - C_\downarrow),
\]

which gives a fractional charge and spin transferred if, e.g., \( 1 < m < q \). Hence, the charge and the spin pumped over a fraction of the pumping cycle is equal to a fraction of the charge and spin pumped over the whole adiabatic cycle. Let us stress that such fractional quantization of the charge and spin transport is not due to the effect of interactions among particles, but it is solely due to the additional unitary symmetry of the Hamiltonian in the parameter space. Hereafter we show an example of a system which exhibits a fractional quantization of the charge and spin transport.

### III. THE MODEL

A possible route to achieve the fractional quantization of the charge and spin transport in a one-dimensional lattice is by the presence of a periodically amplitude-modulated and unidirectional Zeeman field which at each lattice site \( n \) is given by

\[
b(n) = [b \cos(2\pi\alpha n + \varphi) + b_0] \mathbf{z},
\]

where \( 2\pi\alpha \), and \( \varphi \) are respectively the wavevector and the phase-offset of the periodic modulation, while \( b \) and \( b_0 \) are respectively the amplitude of the Zeeman field modulation and the intensity of a superimposed uniform field. This system can be described by the Hamiltonian in momentum space \( \mathcal{H}(\varphi) = \sum_{k} H(k, \varphi) \) with

\[
H(k, \varphi) = c_{k\uparrow}^\dagger (-2t \cos k + b_0 \sigma_z) c_k + \\
\frac{1}{2} e^{i\varphi} c_{k\downarrow}^\dagger b \sigma_z c_{k+2\pi\alpha} + \text{h.c.},
\]

where the Nambu notation has been used with \( c_k^\dagger = [c_{k\uparrow}^\dagger, c_{k\downarrow}^\dagger] \) and \( c_k = [c_{k\uparrow}, c_{k\downarrow}]^\dagger \) the creation and annihilation spinors of states with momentum \( k \) and \( t \) the hopping parameter. Hereafter we assume that the wavevector of the Zeeman field is commensurate to the lattice, i.e., that \( \alpha = p/q \) is a rational number with \( p, q \in \mathbb{Z} \) coprimes. The presence of a finite Zeeman field explicitly breaks time-reversal symmetry in Hamiltonian \( 5 \), which can be regarded as a spinfull generalization of the Harper-Hofstadter Hamiltonian \([49, 50]\). Notice that the periodicity of the Zeeman field is equivalent to a periodic potential acting separately on the two spin channels. Indeed, for unidirectional Zeeman fields and in absence of spin-orbit coupling interactions, the two spin channels are decoupled and Hamiltonian \( 5 \) can be block-diagonized into two independent Harper-Hofstadter Hamiltonians for each spin channel, i.e., \( H(k, \varphi) = H_\uparrow(k, \varphi) + H_\downarrow(k, \varphi) \), where

\[
H_\tau(k, \varphi) = (-2t \cos k + b_0) c_{k\tau}^\dagger c_k + \\
\pm \frac{1}{2} e^{i\varphi} b c_{k+2\pi\alpha} + \text{h.c.},
\]

where the \( \pm \) sign corresponds respectively to the up and down spin channels. Hamiltonian \( 6 \) coincides with the Harper-Hofstadter Hamiltonian \([49, 50]\) realized in a one-dimensional lattice with a periodically-modulated field with wavevector \( 2\pi\alpha \). This Hamiltonian is formally equivalent to the Hamiltonian originally proposed by Hofstadter \([49]\), which describes a two-dimensional electron system in the presence of a strong
magnetic field: The wavevector $\alpha$ corresponds in this case to the magnetic flux per lattice cell in units of the flux quantum $h/e$. Therefore the Chern numbers $C_\tau$ which label each of the $q-1$ intraband gaps of each spin channel separately, are given by the unique integer solution $|C_\tau| < q/2$ of the Diophantine equation\textsuperscript{5,11} $pC_\tau \equiv j_\tau \mod q$, where $j_\tau$ is the gap index with respect to the Hamiltonian $H_0(k, \varphi)$. Notice that the competition between the periodicity of the one-dimensional chain and of the Zeeman field gives rise to a superlattice with spatial periodicity greater than one unit cell. As a consequence, the Brillouin zone and the energy levels are folded into a reduced Brillouin zone $[0, 2\pi/q]$.

Remarkably, the uniform Zeeman term $b_0$ can be used to tune the topological properties of the system and, consequently, the amount of charge and spin pumped during an adiabatic cycle. In fact, since the two spin channels are decoupled, the uniform Zeeman field $b_0$ is proportional to the energy splitting between energy levels with opposite spin and same band index, i.e., $E_{\uparrow i} - E_{\downarrow i} = 2b_0$. Consequently, the relative energy of the $q-1$ intraband gaps of each spin channel can be shifted in energy by tuning the uniform field $b_0$. For instance, by increasing the field, the energies of the two spin channels move respectively upwards and downwards in energy. Hence, the intraband gaps of the two spin channels can overlap in several different ways, giving rise to global gaps which can have different topological invariants $Q$ and $S$ (charge and spin Chern numbers).

Figure 1 shows the topological phase space of a quantum pump with amplitude-modulated Zeeman field with wavevector $2\pi \alpha = 2\pi/3$ as a function of the uniform term $b_0$. The energy bands (dark color) and the global gaps (bright colors) arise from the overlap respectively of the bands and the gaps of the two spin channels. The global gaps are labeled by the charge $Q$ and the spin $S$ pumped during an adiabatic evolution of the phase-offset $\Delta \varphi = 2\pi$. The system exhibits a rich topological phase space, which includes inequivalent states realizing alternatively a pure charge pump (i.e., $Q \neq 0$ and $S = 0$), a spin pump (i.e., $Q = 0$ and $S \neq 0$), or a state which exhibits charge and spin pumping altogether (i.e., $Q \neq 0$ and $S \neq 0$).

As a consequence of the bulk-edge correspondence, nontrivial gaps are characterized by the presence of edge states at the boundary between the system and, e.g., a trivial insulating phase. Since the spin channels are decoupled, these edge states are spin-polarized, i.e., they correspond to states with a well-defined spin. Figure 2 shows the bulk energy bands and the edge states at the two boundaries of the system for each spin channel as a function of the phase-offset $\varphi$. For each spin channel as a function of the phase-offset $\varphi$, the intraband gaps can overlap in several different ways, giving rise to global gaps which can have different topological invariants $Q$ and $S$. The intraband gaps are all topologically invariant $Q$ and $S$.
channel the number of edge states at each boundary is equal to the corresponding Chern number $C_\tau$. The two panels correspond to different values of the uniform Zeeman term $b_0$. The two intraband gaps in Fig. 2(a) with $C_\tau = C_{\downarrow} = \pm 1$ correspond to the pure charge pumping state $Q = \pm 2$ in Fig. [1]. The central intraband gap in Fig. 2(b) with $C_\tau = -C_{\downarrow} = 1$ correspond instead to the pure spin pumping state $S = 2$, whereas the other four intraband gaps with $C_\tau = \pm 1$ and $C_{\downarrow} = \pm 1$ correspond to states with $Q \neq 0$ and $S \neq 0$ (both charge and spin are pumped) shown in Fig. [1].

We notice that a similar system[14,52] can be realized in a one-dimensional lattice with a spin-dependent potential given by

$$v_\tau(n) = v \cos(2\pi a n \pm \varphi), \tag{7}$$

where the $\pm$ sign is for the up and down spin $\tau = \uparrow, \downarrow$ respectively. Such a system can be also block-diagonalized into two copies of Harper-Hofstadter Hamiltonians respectively for each spin channel. Differently from the case of a periodically modulated Zeeman field of Eq. (4), this system does not break the time-reversal symmetry and consequently the charge Chern number is zero[14], being $C_\tau = -C_{\downarrow}$. Therefore, the charge pumped over an adiabatic cycle vanishes $Q = 0$, whereas the spin pumped is nonzero $S \neq 0$ in the nontrivial phases where $C_\tau = -C_{\downarrow} = \pm 1$.

IV. FRACTIONAL QUANTIZATION OF SPIN TRANSPORT

As discussed in Sec. II, the charge and spin are fractionally quantized when Eq. (2) is satisfied, i.e., if the Hamiltonian is periodic in the phase-offset $\varphi$ up to unitary transformations. In particular, such unitary transformations correspond to a lattice translations. It is easy to show from Eq. (6) that $H(k, \varphi + m2\pi a) = T(m)H(k, \varphi)T(-m)$ where $T(m)$ is the translation operator which translates the lattice by $m$ sites.

From this follows that[40] the Hamiltonian $H(k, \varphi)$ is periodic in the phase-offset $\varphi$ with period $\Delta \varphi = 2\pi/q$ up to a lattice translation, i.e.,

$$H(k, \varphi + m2\pi/q) = T(C_m)H(k, \varphi)T(-C_m) \tag{8}$$

where $T(C_m)$ translates the lattice by $C_m$ sites, with $C_m$ the integer solution of the Diophantine equation $pC_m \equiv m \mod q$. In other words, changes of the phase-offset $\Delta \varphi = m2\pi/q$ (i.e., integer multiple of $2\pi/q$) are equivalent to a discrete lattice translation. An analogous periodicity up to unitary transformations can be derived in the case of a one-dimensional discrete chain with a spin-dependent potential described by Eq. (7).

Let us now verify numerically the fractional quantization of the spin transport, by calculating directly the spin pumped as the integral of the Berry curvature as a function of the variation of the phase-offset. Figure 3(a) shows the spin pumped during an adiabatic evolution of the system $\varphi_0 \rightarrow \varphi_0 + \Delta \varphi$ with $2\pi a = 2\pi/3$ in the nontrivial state with $S = 2$ and $Q = 0$, for different values of the initial phase-offset $\varphi_0$. One can see that the amount of spin transferred is quantized as fractions of the total spin Chern number for $\Delta \varphi = 2\pi m/q$ according to Eq. (4), independently from the initial value of the phase-offset $\varphi_0$. The whole adiabatic cycle $\Delta \varphi = 2\pi$ corresponds to a pumped spin equal to the total spin Chern number $S = 2$. We notice that the transferred spin is not quantized for $\Delta \varphi \neq 2\pi m/q$. The system exhibit also a fractional quantization of the charge pumped in the nontrivial phases with $Q \neq 0$.

The fractional quantization of the charge pumped corresponds to an analogous fractional quantization of the variation $\Delta \langle r \rangle$ of the center of charge Eq. (40), which is defined in a finite system as $\langle r \rangle = \frac{1}{N} \int_0^L dr \rho(r) r$, where $\rho(r)$ is the local density of states, $N$ the total number of particles, and $L$ the length of the system. In a spinfull system, let us define the total center of charge and the ‘spin dipole moment’ (i.e., the
adiabatic transformation

\[ \Delta \] the variation of the 'spin dipole moment'

\[ \tau \] channels

nal state (\system). In other words, the system comes back to the origin-

ations of the center of charge and of the spin dipole moment

\[ \Delta \] system. By integrating the continuity equation, one obtains that the

\[ \rho \] with

[54x540]

\[ \rho_{\tau}(r) \] being the local density of states for each of the spin

channels \( \tau = \uparrow, \downarrow \). As shown in Ref. [40] for a spinless system,

by integrating the continuity equation, one obtains that the variation of the total center of charge

ing this argument separately to each spin channel, it follows

pumped through the bulk up to an integer number. By apply-

[54x609]

\[ C \] mod 1, \[ C \] mod 1,

where \( \nu = aN/L \) is the number of particles per lattice site (\( a \) is the lattice parameter). Notice that Eq. (10) is the analogous

of Eq. (3) for a finite system.

Figure 3(b) shows the variations of the spin dipole moment

\[ \Delta \] during an adiabatic evolution of the system \( \phi_0 \rightarrow \phi_0 + \Delta \phi \) with \( 2\pi \alpha = 2\pi/3 \) in the

nontrivial state with \( S = 2 \) and \( Q = 0 \), for different values of the initial phase-offset \( \phi_0 \). The spin dipole moment is calculated
directly from Eq. (9) for an isolated and finite system (\( L = 165 \) lattice sites) confined by a box-shaped potential. The variations of the spin dipole moment are quantized as fractions of the total spin Chern number for \( \Delta \phi = 2\pi m/q \) according to Eq. (10) up to an integer, and independently from the initial value of the phase-offset \( \phi_0 \).

Notice that for a whole adiabatic cycle \( \Delta \phi = 2\pi \), the vari-

ations of the center of charge and of the spin dipole moment always vanish, as a consequence of charge conservation in

the system. In other words, the system comes back to the original state (\( \Delta (r_{\uparrow} - r_{\downarrow}) = 0 \)) at the end of the cycle. For this reason, the shift of the center of charge due to the topologi-

cal pumping has to be counterbalanced by an opposite change of the particle density at the edges of the system (cf. Eq. 7 of Ref. [40]). This finite-size effect produces a discontinuous jump of the center of charge in systems with sharp boundary conditions (box-shaped potential), as in Fig. 3(b).

V. EXPERIMENTAL REALIZATION

The experimental implementation of our proposal requires the realization of periodical and unidirectional Zeeman fields with a periodicity comparable with the underlying one-dimensional lattice. Such kind of fields can be induced in nanowires by the presence of a contiguous antiferromagnetic material with a magnetic ordering vector which is commensurate with the lattice parameter of the wire, or via Moiré patterns realized by graphene on a ferromagnetic substrate [53, 54]. Notice that periodically modulated Zeeman fields have been already considered in the context of topological quantum states in many proposals [48, 55–62]. Moreover, this system can be implemented at a mesoscopic scale via artificial one-dimensional superlattices realized using nanolithographic design on ultrathin films [63] or quantum dot solids [64], in the presence of nanomagnets. However, the continuous control of the phase-offset \( \phi \) appears to be difficult to achieve in condensed matter.

On the other hand, topological spin pumps can be realized in optical lattices via ultracold fermionic atoms with two internal Zeeman states acting as pseudospins [47, 65–66]. In these systems, a periodically modulated Zeeman field can be induced by a combination of radio frequency and optical-Raman coupling fields, which simultaneously couple the spin states of an ultracold Bose-Einstein condensate [23, 25]. Alternatively, the spin-dependent potential of Eq. (7) can be realized in optical lattices via two counterpropagating laser beams with linear polarization vectors [14, 52] forming an angle which corresponds to the phase-offset \( \phi \). The fractional quantization of the spin transport can be verified by directly measuring the variation of the spin center of masses of the atomic cloud as a result of pumping by in situ imaging [31, 32].

VI. CONCLUSIONS

In this work we have described a novel property which can be observed in topological quantum pumps, i.e., the fractional quantization of the spin transport. This phenomenon can be observed in the presence of a periodically modulated Zeeman field which is commensurate with the underlying lattice. These systems can be realized in optical lattices of cold atoms, in nanowires in the presence of nanoengineered heterostructures, or in artificial one-dimensional superlattices.


[^1]: The name ‘spin dipole moment’ is used in analogy to the dipole moment of a charge distribution, which is given by \( \langle r_{\uparrow} + r_{\downarrow} \rangle = \langle r_{\uparrow} \rangle + \langle r_{\downarrow} \rangle \) where \( \langle r_{\pm} \rangle \) are respectively the center of mass of the positive and negative charges of the distribution.

[1] M. Z. Hasan and C. L. Kane, Colloquium: Topological insulators, Rev. Mod. Phys. 82, 3045 (2010).
[2] X.-L. Qi and S.-C. Zhang, Topological insulators and superconductors, Rev. Mod. Phys. 83, 1057 (2011).
[3] K. v. Klitzing, G. Dorda, and M. Pepper, New method for high-accuracy determination of the fine-structure constant based on
quantized Hall resistance, Phys. Rev. Lett. 45, 494 (1980).
[4] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Quantized Hall conductance in a two-dimensional periodic potential, Phys. Rev. Lett. 49, 405 (1982).
[5] C. L. Kane and E. J. Mele, Quantum spin Hall effect in graphene, Phys. Rev. Lett. 95, 226801 (2005).
[6] C. L. Kane and E. J. Mele, Z2 topological order and the quantum spin Hall effect, Phys. Rev. Lett. 95, 146802 (2005).
[7] B. A. Bernevig and S.-C. Zhang, Quantum spin Hall effect, Rev. Mod. Phys. 83, 105701 (2011).
[8] N. Sheng, Z. Y. Weng, L. Sheng, and F. D. M. Haldane, Quantum spin-Hall effect and topologically invariant Chern numbers, Phys. Rev. Lett. 97, 036808 (2006).
[9] D. J. Thouless, Quantization of particle transport, Phys. Rev. B 27, 6083 (1983).
[10] Q. Niu and D. J. Thouless, Quantised adiabatic charge transport in the presence of substrate disorder and many-body interaction, J. Phys. A: Math. Gen. 17, 2453 (1984).
[11] L. Fu and C. L. Kane, Time reversal polarization and a Z2 adiabatic spin pump, Phys. Rev. B 74, 195312 (2006).
[12] R. Shindou, Quantum spin pump in S=1/2 antiferromagnetic chains – holonomy of phase operators in sine-Gordon theory, J. Phys. Soc. Jpn. 74, 1214 (2005).
[13] R. Citro, F. Romeo, and N. Andrei, Electrically controlled pumping of spin currents in topological insulators, Phys. Rev. Lett. 108, 013638 (2012).
[14] D. Ferraro, G. Dolcetto, R. Citro, and M. Sassetti, Spin current pumping in helical Luttinger liquids, Phys. Rev. B 87, 245419 (2013).
[15] C. Q. Zhou, Y. F. Zhang, L. Sheng, D. N. Sheng, and D. Y. Xing, Proposal for a topological spin Chern pump, Phys. Rev. B 90, 085133 (2014).
[16] W. Y. Deng, W. Luo, H. Geng, M. N. Chen, L. Sheng, and D. Y. Xing, Non-adiabatic topological spin pumping, New J. Phys. 17, 103018 (2015).
[17] M. N. Chen, L. Sheng, R. Shen, D. N. Sheng, and D. Y. Xing, Spin Chern pumping from the bulk of two-dimensional topological insulators, Phys. Rev. B 91, 125117 (2015).
[18] M. Lewenstein, A. Sanpera, V. Ahufinger, B. Damski, A. Sen, et al., Ultracold atomic gases in optical lattices: mimicking condensed matter physics and beyond, Adv. Phys. 56, 243 (2007).
[19] I. Bloch, J. Dalibard, and W. Zwerger, Many-body physics with ultracold gases, Rev. Mod. Phys. 80, 885 (2008).
[20] J. Dalibard, F. Gerbier, G. Juzeliūnas, and P. Öhberg, Colloquium: Artificial gauge potentials for neutral atoms, Rev. Mod. Phys. 83, 1523 (2011).
[21] M. Aidelsburger, M. Atala, S. Nascimbène, S. Trotzky, Y.-A. Chen, and I. Bloch, Experimental realization of strong effective magnetic fields in an optical lattice, Phys. Rev. Lett. 107, 255301 (2011).
[22] K. Jiménez-García, L. J. LeBlanc, R. A. Williams, M. C. Beeler, A. R. Perry, and I. B. Spielman, Peierls substitution in an engineered lattice potential, Phys. Rev. Lett. 108, 255303 (2012).
[23] J. Struck, C. Olschläger, M. Weinberg, P. Hauke, J. Simonet, A. Eckardt, M. Lewenstein, K. Sengstock, and P. Windpassinger, Tunable gauge potential for neutral and spinless particles in driven optical lattices, Phys. Rev. Lett. 108, 225304 (2012).
[24] Y. V. Kartashov, V. V. Konotop, and F. K. Abdullaev, Gap solitons in a spin-orbit-coupled Bose-Einstein condensate, Phys. Rev. Lett. 111, 060402 (2013).
[25] N. Goldman, G. Juzeliūnas, P. Öhberg, and I. B. Spielman, Light-induced gauge fields for ultracold atoms, Rep. Prog. Phys. 77, 126401 (2014).
[26] L.-J. Lang, X. Cai, and S. Chen, Edge states and topological phases in one-dimensional optical superlattices, Phys. Rev. Lett. 108, 220401 (2012).
[27] M. Aidelsburger, M. Atala, M. Lohse, J. T. Barreiro, B. Paredes, and I. Bloch, Realization of the Hofstadter Hamiltonian with ultracold atoms in optical lattices, Phys. Rev. Lett. 111, 185301 (2013).
[28] H. Miyake, G. A. Siviloglou, C. J. Kennedy, W. C. Burton, and W. Ketterle, Realizing the Harper Hamiltonian with laser-assisted tunneling in optical lattices, Phys. Rev. Lett. 111, 185302 (2013).
[29] M. Aidelsburger, M. Lohse, C. Schweizer, M. Atala, J. T. Barreiro, S. Nascimbène, N. R. Cooper, I. Bloch, and N. Goldman, Measuring the Chern number of Hofstadter bands with ultracold bosonic atoms, Nat. Phys. 11, 162 (2015).
[30] M. Lohse, C. Schweizer, O. Zilberberg, M. Aidelsburger, and I. Bloch, A Thouless quantum pump with ultracold bosonic atoms in an optical superlattice, Nat. Phys. 12, 350 (2016).
[31] S. Nakajima, T. Tomita, S. Taie, T. Ichinose, H. Ozawa, L. Wang, M. Troyer, and Y. Takahashi, Topological Thouless pumping of ultracold fermions, Nat. Phys. 12, 296 (2016).
[32] R. Citro, Ultracold atoms: A topological charge pump, Nat. Phys. 12, 285 (2016).
[33] M. Aidelsburger, M. Lohse, C. Schweizer, and I. Bloch, Spin pumping and measurement of spin currents in optical superlattices, Phys. Rev. Lett. 117, 170405 (2016).
[34] P. L. e. S. Lopes, P. Ghaemi, S. Ryu, and T. L. Hughes, Competing adiabatic Thouless pumps in enlarged parameter spaces, Phys. Rev. B 94, 235160 (2016).
[35] T.-S. Zeng, W. Zhu, and D. N. Sheng, Fractional charge pumping of interacting bosons in one-dimensional superlattices, Phys. Rev. B 94, 235139 (2016).
[36] F. Ronetti, M. Carrega, D. Ferraro, J. Rech, J. Jonckheere, T. Martin, and M. Sassetti, Polarized heat current generated by quantum pumping in two-dimensional topological insulators, Phys. Rev. B 95, 115412 (2017).
[37] S. Gangadharaiah, L. Trifunovic, and D. Loss, Localized end states in density modulated quantum wires and rings, Phys. Rev. Lett. 108, 136803 (2012).
[38] J.-H. Park, G. Yang, J. Klinovaja, P. Stano, and D. Loss, Fractional boundary charges in quantum dot arrays with density modulation, Phys. Rev. B 94, 075416 (2016).
[39] P. Marra, R. Citro, and C. Ortiz, Fractional quantization of the topological charge pumping in a one-dimensional superlattice, Phys. Rev. B 91, 125411 (2015).
[40] D.-W. Zhang, Z.-D. Wang, and S.-L. Zhu, Relativistic quantum effects of Dirac particles simulated by ultracold atoms, Front. Phys. 7, 31 (2012).
[41] D.-W. Zhang, L.-B. Shao, Z.-Y. Xue, H. Yan, Z. D. Wang, and S.-L. Zhu, Particle-number fractionalization of a one-dimensional atomic Fermi gas with synthetic spin-orbit coupling, Phys. Rev. A 86, 063616 (2012).
[42] D.-W. Zhang, S.-L. Zhu, and Z. D. Wang, Simulating and exploring Weyl semimetal physics with cold atoms in a two-dimensional optical lattice, Phys. Rev. A 92, 013632 (2015).
[43] D.-W. Zhang, Y. X. Zhao, R.-B. Liu, Z.-Y. Xue, S.-L. Zhu, and Z. D. Wang, Quantum simulation of exotic PT-invariant topological nodal loop bands with ultracold atoms in an optical lattice, Phys. Rev. A 93, 043617 (2016).
[44] D. Xiao, M.-C. Chang, and Q. Niu, Berry phase effects on electronic properties, Rev. Mod. Phys. 82, 1959 (2010).
[46] Y. E. Kraus, Z. Ringel, and O. Zilberberg, *Four-dimensional quantum Hall effect in a two-dimensional quasicrystal*, Phys. Rev. Lett. **111**, 226401 (2013).

[47] A. Celi, P. Massignan, J. Ruseckas, N. Goldman, I. B. Spielman, G. Juzel'linas, and M. Lewenstein, *Synthetic gauge fields in synthetic dimensions*, Phys. Rev. Lett. **112**, 043001 (2014).

[48] P. Marra and M. Cuoco, *Pinning Majorana states to domain walls in amplitude-modulated magnetic textures*, arXiv:1606.08450 (2016).

[49] D. R. Hofstadter, *Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields*, Phys. Rev. **B 14**, 2239 (1976).

[50] P. G. Harper, *Single band motion of conduction electrons in a uniform magnetic field*, Proc. Phys. Soc. A **68**, 874 (1955).

[51] D. Osadchy and J. E. Avron, *Hofstadter butterfly as quantum phase diagram*, J. Math. Phys. **42**, 5665 (2001).

[52] D.-W. Zhang and S. Cao, *Measuring the spin Chern number in time-reversal-invariant Hofstadter optical lattices*, Phys. Lett. A **380**, 3541 (2016).

[53] D. Prezzi, D. Eom, K. T. Rim, H. Zhou, S. Xiao, C. Nuckolls, T. F. Heinz, G. W. Flynn, and M. S. Hybertsen, *Edge structures for nanoscale graphene islands on Co(0001) surfaces*, ACS Nano **8**, 5765 (2014).

[54] L. L. Patera, F. Bianchini, G. Troiano, C. Dri, C. Cepek, M. Persessi, C. Africh, and G. Comelli, *Temperature-driven changes of the graphene edge structure on Ni(111): Substrate vs hydrogen passivation*, Nano Letters **15**, 56 (2015).

[55] J. Klinovaja, P. Stano, and D. Loss, *Transition from fractional to Majorana fermions in Rashba nanowires*, Phys. Rev. Lett. **109**, 236801 (2012).

[56] S. Nadj-Perge, I. K. Drozdov, B. A. Bernevig, and A. Yazdani, *Proposal for realizing Majorana fermions in chains of magnetic atoms on a superconductor*, Phys. Rev. B **88**, 020407 (2013).

[57] B. Braunecker and P. Simon, *Interplay between classical magnetic moments and superconductivity in quantum one-dimensional conductors: toward a self-sustained topological Majorana phase*, Phys. Rev. Lett. **111**, 147202 (2013).

[58] F. Pientka, L. I. Glazman, and F. von Oppen, *Topological superconducting phase in helical Shiba chains*, Phys. Rev. B **88**, 155420 (2013).

[59] J. Klinovaja, P. Stano, A. Yazdani, and D. Loss, *Topological superconductivity and majorana fermions in RKKY systems*, Phys. Rev. Lett. **111**, 186805 (2013).

[60] M. M. Vazifeh and M. Franz, *Self-organized topological state with Majorana fermions*, Phys. Rev. Lett. **111**, 206802 (2013).

[61] Y. Kim, M. Cheng, B. Bauer, R. M. Lutchyn, and S. Das Sarma, *Helical order in one-dimensional magnetic atom chains and possible emergence of Majorana bound states*, Phys. Rev. B **90**, 060401 (2014).

[62] A. Saha, D. Rainis, R. P. Tiwari, and D. Loss, *Quantum charge pumping through fractional fermions in charge density modulated quantum wires and Rashba nanowires*, Phys. Rev. B **90**, 035422 (2014).

[63] A. Tadjine, G. Allan, and C. Delerue, *From lattice Hamiltonians to tunable band structures by lithographic design*, Phys. Rev. B **94**, 075441 (2016).

[64] M. Polini, F. Guinea, M. Lewenstein, H. C. Manoharan, and V. Pellegrini, *Artificial honeycomb lattices for electrons, atoms and photons*, Nat. Nano. **8**, 625 (2013).

[65] N. R. Cooper and A. M. Rey, *Adiabatic control of atomic dressed states for transport and sensing*, Phys. Rev. A **92**, 021401 (2015).

[66] L. Taddia, E. Cornfeld, D. Rossini, L. Mazza, E. Sela, and R. Fazio, *Topological fractional pumping with alkaline-earth-like ultracold atoms*, arXiv:1607.07842 (2016).