A New Renormalization Scheme of Fermion Field in Standard Model

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In order to obtain proper wave-function renormalization constants for unstable fermion and consist with Breit-Wigner formula in the resonant region, We have assumed an extension of the LSZ reduction formula for unstable fermion and adopted on-shell mass renormalization scheme in the framework of without field renormalization. The comparison of gauge dependence of physical amplitude between on-shell mass renormalization and complex-pole mass renormalization has been discussed. After obtaining the fermion wave-function renormalization constants, we extend them to two matrices in order to include the contributions of off-diagonal two-point diagrams at fermion external legs for the convenience of calculations of S-matrix elements.

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I. INTRODUCTION

We are interested in the LSZ reduction formula [1] in the case of unstable particle, for the purpose of obtaining a proper wave function renormalization constant(wrc.) for unstable fermion. In this process we have demanded the form of the unstable particle’s propagator in the resonant region resembles the relativistic Breit-Wigner formula. Therefore the on-shell mass renormalization scheme is a suitable choice compared with the complex-pole mass renormalization scheme [2], as we will discuss in section 2. We also discard the field renormalization constants which have been proven at question in the case of having fermions mixing [3]. The layout of this article is as follows. First we discuss the problem arising from the introduction of field renormalization constants in the case of having fermions mixing. Next we introduce the extended LSZ reduction formula for unstable fermion and obtain the proper fermion wrc.. Next we introduce some notation in order to write precisely, especially for the imaginary part of Feynman diagrams. For unstable particle the imaginary part of their self-energy functions should be considered in their wrc. [3]. Firstly let us introduce some notation in order to write them down. The bare fermion field is renormalized as follows

\[ \Psi_0 = Z^{*L}_i \Psi_i, \quad \bar{\Psi}_0 = \bar{\Psi} Z^{*R}_i, \]  

(1)

with the field renormalization constants \( Z^{*L}_i \) and \( Z^{*R}_i \) [3]

\[ Z^{*L}_i = Z^{*L}_i \gamma_L + Z^{*R}_i \gamma_R, \quad \bar{Z}^{*L}_i = \bar{Z}^{*L}_i \gamma_R + \bar{Z}^{*R}_i \gamma_L. \]  

(2)

with \( \gamma_L \) and \( \gamma_R \) the left- and right-handed projection operator. Within one-loop accuracy we have \( Z^{*L}_i = 1 + \frac{1}{2} \delta Z \) and \( \bar{Z}^{*L}_i = 1 + \frac{1}{2} \delta \bar{Z} \). Thus the inverse fermion propagator is as follows (the letters \( i, j \) denote the family indices)

\[ -iS^{-1}_{ij}(p) = (p - m_i) \delta_{ij} + \hat{\Sigma}_{ij}(p), \]  

(3)

with the one-loop renormalized self-energy

\[ \hat{\Sigma}_{ij}(p) = \hat{\Sigma}^L_{ij}(p^2) \gamma_L + \hat{\Sigma}^R_{ij}(p^2) \gamma_R + \Sigma^S_{ij}(p^2)(m_i \gamma_L + m_j \gamma_R) + \frac{1}{2} \delta \hat{Z}_{ij}(p - m_j) + \frac{1}{2} (p - m_i) \delta \hat{Z}_{ij} - \delta m_i \delta_{ij}. \]  

(4)

If we introduce the field renormalization conditions that the fermion two-point functions are diagonal when the external-line momentums are on their mass shell,

\[ S_{ij}^{-1}(p) u_j(p) |_{p^2 = m_j^2} = 0 \]

\[ \bar{u}_i(p) S_{ij}^{-1}(p') |_{p'^2 = m_i^2} = 0 \quad \text{if} \quad i \neq j. \]  

(5)
with $\bar{u}_i$ and $u_j$ the Dirac spinors, we will encounter a bad thing that there is no solution for the above equations if we keep the "pseudo-hermiticity" relationship between $\bar{Z}^\dagger$ and $Z^\dagger$ [3]

$$\bar{Z}^\dagger = \gamma^0 Z^\dagger \gamma^0$$

This is because of the existence of the imaginary part of the fermion two-point functions, which makes $\Sigma^L_{ij}$, $\Sigma^R_{ij}$ and $\Sigma^S_{ij}$ un-Hermitian [4]. As we know the relationship of Eq.(6) comes from the hermiticity of Lagrangian, which guarantees the S-matrix is unitary, thus cannot be broken.

A method to solve this problem is to introduce two sets of field renormalization constants, one set for the external-line particles, the other set for the inner-line particles in a Feynman diagram [3]. But it brings some complexity and cannot explain why we couldn’t find a consistent field renormalization constant for all particles.

II. WAVE-FUNCTION RENORMALIZATION CONSTANS

Another method to solve the above problem is to abandon field renormalization, since particle fields don’t belong to physical parameters thus don’t need to be renormalized. One effect of this method is that the calculations of Feynman diagrams becomes simpler than before. The other effect will be seen in section 3.

Since there aren’t field renormalization constants, we need to find a way to determine the fermion wrc. We demand the propagator of unstable particle has the following form when the momentum is on mass shell

$$\sim \frac{1}{p^2 - m^2 + i m \Gamma(p^2)}$$

where $m \Gamma(p^2)$ is a real number which represents the imaginary part of the self-energy function times the particle’s wrc. This form resembles the relativistic Breit-wigner formula, thus will be a reasonable assumptions. Under this assumption only the on-shell mass renormalization scheme is suitable, the other scheme, complex-pole mass renormalization scheme, cannot satisfy Eq.(7). We can see this point in the boson case.

The mass definition in complex-pole scheme is based on the complex-valued position of the propagator’s pole [2]:

$$\bar{s} = M^2_0 + A(\bar{s}) \ .$$

with $A(p^2)$ the self-energy function of boson. Writing $\bar{s} = m_2^2 - i m_2 \Gamma_2$, the mass and width of the unstable particle may be identified with $m_2$ and $\Gamma_2$, respectively. Given $m_2$ and $\Gamma_2$, there are two other definitions:

$$m_1 = \sqrt{m_2^2 + \Gamma_2^2}, \Gamma_1 = \frac{m_1}{m_2} \Gamma_2,$$

$$\bar{s} \rightarrow \frac{(1 - A'(m^2)^{-1})}{p^2 - m^2 + B}$$

The form of the propagator in resonant region is

$$1 \quad \sim \quad \frac{1}{p^2 - M_0^2 - A(p^2)}$$

with $B = (m^2 - M_0^2 - A(m^2))/(1 - A'(m^2))$. Through simple calculations we find all of these three definitions of mass cannot make $B$ a pure imaginary number. So the complex-pole mass renormalization scheme isn’t suitable for the assumption of Eq.(7).

Because the unstable fermion’s propagator near resonant region must contain Dirac matrices, Eq.(7) isn’t enough. We further assume it has the following form:

$$\frac{i}{\tilde{p} - m_i + \Sigma_{ii}(\tilde{p})} p^2 \rightarrow m_i^2 = \frac{i Z_i^\dagger(\tilde{p} + m_i + i x) \tilde{Z}_i^\dagger}{p^2 - m_i^2 + i m_i \Gamma_i(p^2)} .$$

where $Z_i^\dagger$ and $\tilde{Z}_i^\dagger$ are the fermion wrc., $x$ is a quantity which has nothing to do with our present calculations. The LSZ reduction formula which relates the Green functions to S-matrix elements may have such form:
\[ \int dx_1^i dx_2^j e^{ip_1 x_1} e^{ip_2 x_2} < \Omega | T \bar{\psi}(x_1) \psi(x_2) \cdots | \Omega > \]

\[ P_1^2 \rightarrow m_1^2 - \frac{Z_1^b(p_1 + p_2 + m_1)}{Z_1^b(p_1 + p_2 + m_1 + 1)} < \bar{\psi}(p_1) \cdots | S | \cdots | \psi(p_2) > \]

here we have assumed the Heisenberg field \( \bar{\psi}(x_1) \) can be transformed into the out-state \( < \bar{\psi}(p_1) > \) with an additional coefficient \( \frac{Z_1^b(p_1 + m_1)}{Z_1^b(p_1 + m_1 + 1)} \) and the Heisenberg field \( \psi(x_2) \) can be transformed into the in-state \( | \psi(p_2) > \) with an additional coefficient \( \frac{Z_1^b(p_2 + m_2)}{Z_1^b(p_2 + m_2 + 1)} \). According to Eq.(9), we can sum up the corrections to each external legs in a Green function and analyse the Green function into the amputated part times the corresponding propagators. That is to say

\[ \int dx_1^i dx_2^j e^{ip_1 x_1} e^{ip_2 x_2} < \Omega | T \bar{\psi}(x_1) \psi(x_2) \cdots | \Omega > \]

\[ P_1^2 \rightarrow m_1^2 - \frac{Z_1^b(p_1 + m_1)}{Z_1^b(p_1 + m_1 + 1)} \bar{\psi}(p_1) | S | \cdots | \psi(p_2) > \]

where the subscript \( \text{amp} \) refers to the amputated Green function. Thus the LSZ reduction formula with two fermions at external legs will be:

\[ < p_1 \cdots p_n | S | k_1 \cdots k_m > = Z_{\frac{a + m - 1}{2}} \bar{u}(p_1) Z_{-\frac{1}{2}} Z_{f} \bar{u}(k_1). \]

with \( Z_{\frac{a + m - 1}{2}} \) and \( Z_{-\frac{1}{2}} \) the fermion wrc. and \( Z \) the boson wrc.. The fermion spinors \( \bar{u}(p_1) \) and \( u(k_1) \) have been added to the right hand side of the above equation, as are demanded.

In actual calculations, the fermion propagator needs to be transformed into the form of Eq.(9). To start with, we introduce the renormalized fermion self-energy functions as follows

\[ \hat{\Sigma}_{ii} = \hat{\gamma} (\Sigma_{ii}^L(p^2) \gamma_L + \Sigma_{ii}^R(p^2) \gamma_R) + \Sigma_{ii}^{SL}(p^2) \gamma_L + \Sigma_{ii}^{SR}(p^2) \gamma_R - \delta m_i. \]

Although the self-energy functions contain only the contributions of one-particle-irreducible (1PI) diagrams in usual sense, it’s not denied in principle to include the contributions of reducible diagrams in the self-energy functions. In the present scheme the non-diagonal two-point functions aren’t zero when external momentum is on shell, since there aren’t the conditions of Eqs.(5). Thus the diagonal self-energy functions will contain the contributions of the reducible diagrams:

where the disc is the 1PI diagram. The condition \( k, l, \cdots, m \neq i \) avoids the reduplication in actual calculations. Now we write the inverse fermion propagator as follows

\[ -i S_{ii}^{-1}(\hat{\gamma}) = \hat{\gamma} - m_i + \hat{\Sigma}_{ii}(\hat{\gamma}) = \hat{\gamma} (a \gamma_L + b \gamma_R) + c \gamma_L + d \gamma_R \]

with

\[ a = 1 + \Sigma_{ii}^L(p^2), \quad b = 1 + \Sigma_{ii}^R(p^2), \quad c = \Sigma_{ii}^{SL}(p^2) - m_i - \delta m_i, \quad d = \Sigma_{ii}^{SR}(p^2) - m_i - \delta m_i. \]

and therefore after some algebra [3]

\[ -i S_{ii}^{-1}(\hat{\gamma}) = \frac{\hat{\gamma}(a \gamma_L + b \gamma_R) - c \gamma_L - d \gamma_R}{p^2 ab - cd} \]

That’s to say
\[
\frac{i}{\not{p} - m_i + \Sigma_{ii}(\not{p})} = \frac{i(m_i + \delta m_i + \not{p} \gamma_L (1 + \Sigma_{ii}^L) + \not{p} \gamma_R (1 + \Sigma_{ii}^R) - \not{p} \gamma_L \Sigma_{ii}^{S,L} - \not{p} \gamma_R \Sigma_{ii}^{S,R})}{p^2(1 + \Sigma_{ii}^L)(1 + \Sigma_{ii}^R) - (m_i + \delta m_i - \Sigma_{ii}^{S,L})(m_i + \delta m_i - \Sigma_{ii}^{S,R})}.
\] (18)

Adopting the on-shell mass renormalization scheme\(^1\), the mass renormalization condition is

\[
\text{Re}[m_i^2(1 + \Sigma_{ii}^L(m_i^2))(1 + \Sigma_{ii}^R(m_i^2)) - (m_i + \delta m_i - \Sigma_{ii}^{S,L}(m_i^2))(m_i + \delta m_i - \Sigma_{ii}^{S,R}(m_i^2))] = 0
\] (19)

Expanding the denominator of Eq.(18) at \(p^2 = m_i^2\) according to Eq.(9), we obtain

\[
\frac{i}{\not{p} - m_i + \Sigma_{ii}(\not{p})} \rightarrow \frac{i(m_i + \delta m_i + \not{p} \gamma_L (1 + \Sigma_{ii}^L) + \not{p} \gamma_R (1 + \Sigma_{ii}^R) - \not{p} \gamma_L \Sigma_{ii}^{S,L} - \not{p} \gamma_R \Sigma_{ii}^{S,R})}{(p^2 - m_i^2 + im_i \Gamma_i(p^2))A}
\] (20)

with

\[
A = \text{Re}[1 + \Sigma_{ii}^L + \Sigma_{ii}^R + \Sigma_{ii}^L \Sigma_{ii}^R + m_i^2 \frac{\partial}{\partial p^2} (\Sigma_{ii}^L + \Sigma_{ii}^R + \Sigma_{ii}^L \Sigma_{ii}^R) + \frac{\partial \Sigma_{ii}^{S,R}}{\partial p^2}(m_i + \delta m_i - \Sigma_{ii}^{S,R}) + \frac{\partial \Sigma_{ii}^{S,L}}{\partial p^2}(m_i + \delta m_i - \Sigma_{ii}^{S,L})]
\] (21)

\[
m_i \Gamma_i(p^2) = \frac{1}{A} Im[p^2(\Sigma_{ii}^L(p^2) + \Sigma_{ii}^R(p^2) + \Sigma_{ii}^L(p^2) \Sigma_{ii}^R(p^2)) + (m_i + \delta m_i)(\Sigma_{ii}^{S,L}(p^2) + \Sigma_{ii}^{S,R}(p^2)) - \Sigma_{ii}^{S,L}(p^2) \Sigma_{ii}^{S,R}(p^2)]
\] (22)

Noted we have discarded the regular terms in Eq.(20).

From Eq.(9) and (20) we obtain

\[
Z_i^L \frac{Z_i^L}{\not{p}} = (1 + \Sigma_{ii}^L(m_i^2))/A, \quad Z_i^R \frac{Z_i^R}{\not{p}} = (1 + \Sigma_{ii}^R(m_i^2))/A, \quad Z_i^L \frac{Z_i^R}{\not{p}} = (m_i + \delta m_i - \Sigma_{ii}^{S,L}(m_i^2))/(A(m_i + ix)), \quad Z_i^R \frac{Z_i^L}{\not{p}} = (m_i + \delta m_i - \Sigma_{ii}^{S,R}(m_i^2))/(A(m_i + ix)).
\] (23)

In order to solve these equations, we firstly obtain a equation about \(x\)

\[
(m_i + ix)^2(1 + \Sigma_{ii}^L(m_i^2))(1 + \Sigma_{ii}^R(m_i^2)) = (m_i + \delta m_i - \Sigma_{ii}^{S,L}(m_i^2))(m_i + \delta m_i - \Sigma_{ii}^{S,R}(m_i^2)).
\] (24)

Replacing \(m_i + ix\) in Eqs.(23) by this relationship, we obtain

\[
Z_i^L \frac{Z_i^L}{\not{p}} = \frac{1 + \Sigma_{ii}^R(m_i^2)}{A}, \quad Z_i^R \frac{Z_i^R}{\not{p}} = \frac{1 + \Sigma_{ii}^L(m_i^2)}{A},
\]

\[
Z_i^L \frac{Z_i^R}{\not{p}} = \frac{(m_i + \delta m_i - \Sigma_{ii}^{S,R}(m_i^2))(1 + \Sigma_{ii}^R(m_i^2))}{(m_i + \delta m_i - \Sigma_{ii}^{S,L}(m_i^2))(1 + \Sigma_{ii}^L(m_i^2))}, \quad Z_i^R \frac{Z_i^L}{\not{p}} = \frac{(m_i + \delta m_i - \Sigma_{ii}^{S,L}(m_i^2))(1 + \Sigma_{ii}^L(m_i^2))}{(m_i + \delta m_i - \Sigma_{ii}^{S,R}(m_i^2))(1 + \Sigma_{ii}^R(m_i^2))}
\] (25)

Because we haven’t introduced the fermion field renormalization constants, which contain Dirac matrices thus may change the Lorentz structure of the fermion’s self-energy functions, the self-energy functions in our scheme will keep the Lorentz invariant structure:

\[
\tilde{\Sigma}_{ij}(\not{p}) = \not{p}(\Sigma_{ij}^L(p^2) \gamma_L + \Sigma_{ij}^R(p^2) \gamma_R) + \Sigma_{ij}^S(p^2)(m_{0,i} \gamma_L + m_{0,j} \gamma_R) - \delta m_i \delta_{ij}.
\] (26)

where \(m_{0,i}\) and \(m_{0,j}\) the bare fermion masses. It is obvious that

\[
\Sigma_{ij}^{S,R} = \Sigma_{ij}^{S,L} \equiv m_i \Sigma_{ij}^S
\] (27)

Therefore Eqs.(25) can be simplified as follows

\(^1\)There are many articles discussing the gauge dependence of the on-shell mass renormalization scheme \([5]\), we will discuss this problem in section 4
Because the four equations aren’t independent, we need to find an additional condition. From the strong symmetry between $\bar{Z}$ and $Z$ manifested in the above equations, a reasonable assumption is to set $\bar{Z}^L_i = Z^L_i$, $\bar{Z}^R_i = Z^R_i$, which is also the case of stable fermion. Therefore the final results are

\[
\begin{align*}
Z^L_i &= Z^L_i = (1 + \Sigma^L_{ii}(m^2_i))/A_i, \\
Z^R_i &= Z^R_i = (1 + \Sigma^R_{ii}(m^2_i))/A_i.
\end{align*}
\] (28)

and

\[
d\frac{m_i}{m_i} = \frac{\sqrt{Re[(1 + \Sigma^L_{ii}(m^2_i))(1 + \Sigma^R_{ii}(m^2_i))]}}{1 + Im[\Sigma^S_{ii}(m^2_i)]^2 + Re[\Sigma^S_{ii}(m^2_i)] - 1}.
\] (30)

It is very easy to give the explicit one-loop level results. The wrc., mass counterterm and decay rate are listed below:

\[
\begin{align*}
\bar{Z}^L_i &= Z^L_i = 1 - Re[\Sigma^L_{ii}(m^2_i)] + m_i^2 \frac{\partial}{\partial p^2} (\Sigma^L_{ii}(p^2) + \Sigma^R_{ii}(p^2) + 2\Sigma^S_{ii}(p^2))|_{p^2=m^2_i} + iIm[\Sigma^R_{ii}(m^2_i)]. \\
\bar{Z}^R_i &= Z^R_i = 1 - Re[\Sigma^L_{ii}(m^2_i)] + m_i^2 \frac{\partial}{\partial p^2} (\Sigma^L_{ii}(p^2) + \Sigma^R_{ii}(p^2) + 2\Sigma^S_{ii}(p^2))|_{p^2=m^2_i} + iIm[\Sigma^L_{ii}(m^2_i)].
\end{align*}
\] (31)

\[
\begin{align*}
\delta m_i &= \frac{m_i}{2} Re[\Sigma^L_{ii}(m^2_i) + \Sigma^R_{ii}(m^2_i) + 2\Sigma^S_{ii}(m^2_i)] \\
m_i \Gamma_i &= m_i^2 Im[\Sigma^L_{ii}(m^2_i) + \Sigma^R_{ii}(m^2_i) + 2\Sigma^S_{ii}(m^2_i)].
\end{align*}
\] (32) (33)

Another meaningful result concerning $x$ (which appears in Eq.(9)) is:

\[
x = -\frac{\Gamma_i}{2}
\] (34)

Until now we haven’t discussed the Optical Theorem. It is a very complex problem, so we want to leave it for the future work. Here we only point out that Eq.(33) is really the fermion decay rate at one-loop level [3].

III. COMPARISON BETWEEN DIFFERENT RENORMIALIZATION SCHEMES

Adopting which mass renormalization scheme and whether introducing field renormalization constants determine the difference of the renormalization schemes. Here we will list three other schemes and compare their results.

Firstly we consider the renormalization scheme which adopts the complex-pole mass renormalization scheme and doesn’t introduce field renormalization constants. In this case Eq.(9) isn’t satisfied, but we can assume $\Gamma_i$ is a complex number and independent of the propagator’s momentum. Thus all of the results associated with fermion wrc. in previous section keep unchanged, except for removing the operator $Re$ from $A$, because in this case it is no sense to treat the real part and the imaginary part of the denominator of the fermion propagator differently. Therefore we obtain

\[
\begin{align*}
\bar{Z}^L_i &= Z^L_i = (1 + \Sigma^R_{ii}(m^2_i))/A_1 \\
\bar{Z}^R_i &= Z^R_i = (1 + \Sigma^L_{ii}(m^2_i))/A_1
\end{align*}
\] (35)

with

\[
A_1 = 1 + \Sigma^L_{ii} + \Sigma^R_{ii} + \Sigma^L_{ii}\Sigma^R_{ii} + m_i^2 \frac{\partial}{\partial p^2} (\Sigma^L_{ii} + \Sigma^R_{ii} + \Sigma^L_{ii}\Sigma^R_{ii}) + \frac{\partial \Sigma^S_{ii}}{\partial p^2} (m_i + \delta m_i - \Sigma^S_{ii}) + \frac{\partial \Sigma^S_{ii}}{\partial p^2} (m_i + \delta m_i - \Sigma^S_{ii})
\] (36)

On the other hand, $\delta m_i$ and $\Gamma_i$ are different from the previous results very much. But at one-loop level they are the same as Eq.(32) and (33).

Next we study the renormalization scheme which adopts the complex-pole mass renormalization scheme and has introduced field renormalization constants. We still assume $\Gamma_i$ is a complex number and independent of the propagator’s momentum. Because the field renormalization constants have been introduced, the fermion self-energy functions will contain their contributions. We can write down the fermion self-energy functions in such form:
On the other hand the unit residue renormalization condition amounts to requiring the application of the LSZ reduction formula Eq.(12) at two-point Green functions. The new self-energy function $\Sigma'$ is called "pure" self-energy function by us. It is easy to obtain from the above equation that

$$
\begin{align*}
\Sigma^L_i &= Z^L_{ii} \Sigma^L_{ii}, \\
\Sigma^S,L_i &= Z^L_{ii} \Sigma^S,L_{ii}, \\
\Sigma^R_i &= Z^R_{ii} \Sigma^R_{ii}, \\
\Sigma^S,R_i &= Z^R_{ii} \Sigma^S,R_{ii}.
\end{align*}
$$

(38)

We can write down the inverse fermion propagator in the form of Eq.(15) with

$$
a = \tilde{Z}^L_{ii} Z^L_{ii} + \Sigma^L_{ii} (p^2),
b = \tilde{Z}^R_{ii} Z^R_{ii} + \Sigma^R_{ii} (p^2),
c = \Sigma^S,L_{ii} (p^2) - (m_i + \delta m_i) \tilde{Z}^R_{ii} Z^R_{ii},
d = \Sigma^S,R_{ii} (p^2) - (m_i + \delta m_i) \tilde{Z}^L_{ii} Z^L_{ii}.
$$

(39)

According to Eq.(17), the condition that in the limit $p^2 \to m_i^2$ the chiral structure in the numerator must be canceled out leads to

$$
\tilde{Z}^L_{ii} Z^L_{ii} + \Sigma^L_{ii} (m_i^2) = \tilde{Z}^R_{ii} Z^R_{ii} + \Sigma^R_{ii} (m_i^2),
\Sigma^S,L_{ii} (m_i^2) - (m_i + \delta m_i) \tilde{Z}^R_{ii} Z^R_{ii} = \Sigma^S,R_{ii} (m_i^2) - (m_i + \delta m_i) \tilde{Z}^L_{ii} Z^L_{ii}.
$$

(40)

On the other hand the unit residue renormalization condition amounts to requiring

$$
a = ab + m_i^2 a \frac{\partial}{\partial p^2} b + m_i^2 b \frac{\partial}{\partial p^2} a - c \frac{\partial}{\partial p^2} d - d \frac{\partial}{\partial p^2} c.
$$

(41)

Using Eq.(39) and Eqs.(38), we have

$$
1 = \tilde{Z}^L_{ii} Z^L_{ii} [1 + \tilde{Z}^L_{ii} + m_i^2 \frac{\partial}{\partial p^2} \tilde{Z}^L_{ii} + \Sigma^S,L_{ii} \frac{\partial}{\partial p^2} (\tilde{Z}^L_{ii} + \frac{\partial}{\partial p^2} \tilde{Z}^L_{ii}) + \frac{\partial}{\partial p^2} (m_i + \delta m_i - \tilde{Z}^S,L_{ii}) + \frac{\partial}{\partial p^2} \tilde{Z}^S,L_{ii} + \frac{\partial}{\partial p^2} \Sigma^S,L_{ii} + \frac{\partial}{\partial p^2} \Sigma^S,R_{ii})]
$$

(42)

Like the case in section 2 we need an additional condition to determine the field renormalization constants. If we adopt the same assumption as the ones in the previous section, $\tilde{Z}^L_{ii} = Z^L_{ii}$, $\tilde{Z}^R_{ii} = Z^R_{ii}$, we will obtain

$$
\tilde{Z}^L_{ii} = Z^L_{ii} = (1 + \Sigma^R_{ii} (m_i^2))/A_2,
\tilde{Z}^R_{ii} = Z^R_{ii} = (1 + \Sigma^L_{ii} (m_i^2))/A_2.
$$

(43)

with

$$
A_2 = 1 + \Sigma^L_{ii} + \Sigma^R_{ii} + \Sigma^S,L_{ii} \frac{\partial}{\partial p^2} (\tilde{Z}^L_{ii} + \frac{\partial}{\partial p^2} \tilde{Z}^L_{ii}) + \frac{\partial}{\partial p^2} (m_i + \delta m_i - \tilde{Z}^S,L_{ii}) + \frac{\partial}{\partial p^2} \tilde{Z}^S,L_{ii} + \frac{\partial}{\partial p^2} \Sigma^S,L_{ii} + \frac{\partial}{\partial p^2} \Sigma^S,R_{ii}).
$$

(44)

$A_2$ is the same as $A_1$, except for replacing the self-energy functions with "pure" self-energy functions. But at one-loop level they are the same, so the one-loop results of Eqs.(43) are

$$
\tilde{Z}^L_{ii} = Z^L_{ii} = 1 - \Sigma^L_{ii} (m_i^2) - m_i^2 \frac{\partial}{\partial p^2} (\Sigma^L_{ii} (p^2) + \Sigma^R_{ii} (p^2) + 2 \Sigma^S_{ii} (p^2)) |_{p^2=m_i^2},
\tilde{Z}^R_{ii} = Z^R_{ii} = 1 - \Sigma^R_{ii} (m_i^2) - m_i^2 \frac{\partial}{\partial p^2} (\Sigma^L_{ii} (p^2) + \Sigma^R_{ii} (p^2) + 2 \Sigma^S_{ii} (p^2)) |_{p^2=m_i^2}.
$$

(45)

Finally we consider the renormalization scheme which adopts the on-shell mass renormalization scheme and has introduced field renormalization constants. In this scheme, the self-energy functions can still be written down in the form of Eqs.(37) and the renormalization conditions of Eqs.(40) keep unchanged. The only variation comes from the unit residue condition. In on-shell scheme we should treat the real part and the imaginary part of the denominator of the fermion propagator in different ways. Thus the unit residue condition becomes

$$
a = Re[ab + m_i^2 a \frac{\partial}{\partial p^2} b + m_i^2 b \frac{\partial}{\partial p^2} a - c \frac{\partial}{\partial p^2} d - d \frac{\partial}{\partial p^2} c].
$$

(46)

after some algebra we have
functions Heaviside function. which are different from Eqs.(31) and (45). \[ \xi \] arising from the introduction of field renormalization constants. renormalization. It not only simplifies the calculations of S-matrix elements, but also avoids the possible deviation thus doesn't need to be renormalized. So the best way to renormalize a quantum field theory is to discard field enhancement the complexity of calculating S-matrix elements. As it's very known, "bare" field isn't a physical quantity which not need to be renormalized. The consistent assumption: \[ \bar{\phi}_{L}^{i} = \frac{1}{(1 + \Sigma_{ii}^{L})(1 + \Sigma_{ii}^{R})} \partial_{p^2} \Sigma_{ii}^{R} \]

\[ A_{3} = 1 + \text{Re} \left[ \frac{\partial}{\partial p^2} \Sigma_{ii}^{L} + \frac{\partial}{\partial p^2} \Sigma_{ii}^{R} + \frac{\partial}{\partial p^2} \Sigma_{ii}^{S} \right] \]

which are different from all of the previous results. At one-loop level, Eqs.(48) becomes

\[ \bar{Z}_{ii}^{L} = \frac{Z_{ii}^{L}}{1 \Sigma_{ii}^{L}(m_{i}^{2}) - m_{i}^2 \partial_{p^2} \Sigma_{ii}^{L}(p^2) + \Sigma_{ii}^{R}(p^2) + 2 \Sigma_{ii}^{S}(p^2)/p^2 = m_{i}^{2}} ; \]

\[ \bar{Z}_{ii}^{R} = \frac{Z_{ii}^{R}}{1 - \Sigma_{ii}^{R}(m_{i}^{2}) - m_{i}^2 \partial_{p^2} \Sigma_{ii}^{R}(m_{i}^{2}) + \Sigma_{ii}^{L}(m_{i}^{2}) + 2 \Sigma_{ii}^{S}(m_{i}^{2})/p^2 = m_{i}^{2}} . \]

which are different from Eqs.(31) and (45). Through these comparison we find that whether to introduce field renormalization constants or not may change the fermion wr..(comparing Eqs.(50) with Eqs.(31)). On the other hand, introducing field renormalization constants will enhance the complexity of calculating S-matrix elements. As it's very known, "bare" field isn't a physical quantity thus doesn't need to be renormalized. So the best way to renormalize a quantum field theory is to discard field renormalization. It not only simplifies the calculations of S-matrix elements, but also avoids the possible deviation arising from the introduction of field renormalization constants.

**IV. GAUGE DEPENDENCE OF RENORMALIZATION SCHEMES**

In this section we will discuss the gauge dependence of the renormalization schemes we have mentioned. For convenience we only discuss two typical renormalization schemes. The first scheme is the scheme we have adopted, the second scheme is the scheme which adopts the complex-pole scheme and contains field renormalization constants. At one-loop level, the results of \( \delta m_i \) and \( \Gamma_i \) are the same in these two schemes. The difference comes from the imaginary part of the fermion's wrf., if you compare Eqs.(31) with Eq.(45). So we only need to consider the imaginary part of the fermion self-energy functions in the following calculations.

We have selected two physical processes to test the gauge dependence of the physical amplitude in these two renormalization schemes. One process is W gauge boson decays into a lepton and an anti-neutrino. For our purpose it is sufficient to only consider the dependence of W boson gauge parameter \( \xi_W \). Thus there are only two Feynman diagrams which contribute to the imaginary part of the lepton self-energy functions, as shown in Fig.1

\[ \text{FIG. 1. lepton's self-energy diagrams which have contributions to the } \xi_W \text{-dependent imaginary part of lepton's self-energy functions} \]

Using the cutting rules [6], we can calculate the imaginary part of the lepton's wrf.. In the first scheme we have: \( \text{restrained } \xi_W > 0 \)

\[ I m[Z_{ii}^{L}] = \frac{e^2(m_{i}^{2} - M_{W}^{2} \xi_W)^2 \theta[m_{i}^{2} - M_{W}^{2} \xi_W]}{64 \pi M_{W}^{2} s_{W}^{2} m_{i}^{2}} \]

with \( m_{i}^{2} \) the lepton's mass, \( M_{W} \) the W gauge boson mass, \( s_{W} \) the sine of the weak mixing angle \( \theta_W \) and \( \theta \) the Heaviside function.
In Fig. 2 we show the irreducible diagrams that contribute to the $\xi_W$-dependent imaginary part of the one-loop $W^- \rightarrow e_i \bar{\nu}_i$ amplitude. Using the cutting rules we obtain

$$Im[T_1] = A_L F_L + B_L G_L + A_L Im[Z_{ii}^L] e/(2\sqrt{2}s_W)$$

(52)

with

$$A_L = \bar{u}(p_2)\frac{\gamma_l \nu(p_1)}{M_W} \bar{u}(p_2)\frac{\gamma_l \nu(p_1)}{M_W}.$$  

(53)

and the form factors

$$F_L = \frac{e^3(\xi_W^2 - 3\xi_W^2 + 2\theta[1 - \xi_W])}{48\sqrt{2}\pi s_W} - \frac{e^3(\xi_W^2 - 3\xi_W^2 + 2\theta[1 - \xi_W])}{64\sqrt{2}\pi s_W s_W M_W^2 m_W^2 \xi_W},$$

$$G_L = \frac{e^3(\xi_W - 1)^3 m_{l,i} M_W \theta[1 - \xi_W]}{32\sqrt{2}\pi s_W (M_W^2 - m_W^2)}.$$  

(54)

Thus we have

$$Im[T_1] = A_L \left( - \frac{e^3(\xi_W^2 - 3\xi_W^2 + 2\theta[1 - \xi_W])}{48\sqrt{2}\pi s_W} - \frac{e^3(\xi_W^2 - 3\xi_W^2 + 2\theta[1 - \xi_W])}{64\sqrt{2}\pi s_W s_W M_W^2 m_W^2 \xi_W} \right) +$$

$$B_L \left( \frac{e^3(\xi_W - 1)^3 m_{l,i} M_W \theta[1 - \xi_W]}{32\sqrt{2}\pi s_W (M_W^2 - m_W^2)} \right).$$

(55)

We find that the imaginary part of the one-loop $W^- \rightarrow e_i \bar{\nu}_i$ amplitude is gauge dependent in our scheme. On the other hand, the second scheme gives

$$Im[Z_{ii}^L] = \frac{e^2(m_{l,i}^2 - M_{W}^2 \xi_W)^2 \theta[1 - \xi_W]}{32\pi M_W^2 s_W m_W^2 m_W^2 \xi_W^2 \xi_W^2}$$

(56)

and the imaginary part of the one-loop $W^- \rightarrow e_i \bar{\nu}_i$ amplitude (according to the above results) is

$$Im[T_1] = -A_L \frac{e^3(\xi_W^2 - 3\xi_W^2 + 2\theta[1 - \xi_W])}{48\sqrt{2}\pi s_W} + B_L \frac{e^3(\xi_W - 1)^3 m_{l,i} M_W \theta[1 - \xi_W]}{32\sqrt{2}\pi s_W (M_W^2 - m_W^2)}.$$  

(57)

also a gauge dependent quantity. At one-loop level Eq.(55) and (57) don’t contribute to the modulus square of the physical amplitude. But at two-loop level they will have the contributions: $Im[T_1]^1 Im[T_1]$ or $Im[T_1]^1 Im[T_1]$. At the limit $m_{l,i} \rightarrow 0$ they have the same contributions as follows

$$Im[T_1]^1 Im[T_1]|_{m_{l,i} \rightarrow 0} = Im[T_1]^1 Im[T_1]|_{m_{l,i} \rightarrow 0} = \frac{e^6 M_{W}^2 (\xi_W - 1)^2 (\xi_W^2 - 2\xi_W - 2)^2 \theta[1 - \xi_W]}{32\pi^2 s_W^2}.$$  

(58)
Of course there will be no such gauge-dependent terms if we adopt the unitary gauge. We hope that the contributions from the two-loop diagrams of the process $W^- \rightarrow e_i \bar{v}_j$ may cancel these gauge-dependent terms.

Another process we have studied is $Z$ gauge boson decays into a pair down-type quarks. The calculations are more complex than the previous ones. The imaginary parts of the down-type quark's wrc. are the same in these two schemes in the limit $m_{d,i} \rightarrow 0$ ($m_{d,i}$ is the down-type quark's mass). They are as follows

$$\text{Im}[M(Z \rightarrow d_i \bar{d}_i)]_{m_{d,i} \rightarrow 0} = A_L \left[ -\frac{1}{384 \pi c_w s_w} e^3 \left( 1 - 4 c_w^2 \xi_W \right)^{3/2} \theta[1 - 4 c_w^2 \xi_W] + \frac{1}{192 \pi c_w s_w} e^3 \left( (\xi_W - 1)^2 c_w^4 - 2(\xi_W - 5) c_w^2 + 1 \right) \sqrt{(\xi_W - 1)^2 c_w^2 - 2(\xi_W + 1) c_w^2 + 1} \theta \left[ 1/c_w - \sqrt{\xi_W - 1} \right] \right]$$

$$\text{Im}[M(Z \rightarrow d_i \bar{d}_i)] \text{Im}[M(Z \rightarrow d_i \bar{d}_i)]_{m_{d,i} \rightarrow 0} = -\frac{1}{384 \pi c_w s_w} e^3 M_W^2 \left( 4 c_w^2 \xi_W - 1 \right) \theta[1 - 4 c_w^2 \xi_W] + \frac{1}{192 \pi c_w s_w} e^3 M_W^2 \left( (\xi_W - 1)^2 c_w^4 - 2(\xi_W - 5) c_w^2 + 1 \right) \theta \left[ 1/c_w - \sqrt{\xi_W - 1} \right]$$

The contribution of this quantity to the modulus square of the $Z \rightarrow d_i \bar{d}_i$ amplitude at two-loop level is

$$\text{Im}[M(Z \rightarrow d_i \bar{d}_i)] \text{Im}[M(Z \rightarrow d_i \bar{d}_i)]_{m_{d,i} \rightarrow 0} = -\frac{1}{384 \pi c_w s_w} e^3 M_W^2 \left( 4 c_w^2 \xi_W - 1 \right) \theta[1 - 4 c_w^2 \xi_W] + \frac{1}{192 \pi c_w s_w} e^3 M_W^2 \left( (\xi_W - 1)^2 c_w^4 - 2(\xi_W - 5) c_w^2 + 1 \right) \theta \left[ 1/c_w - \sqrt{\xi_W - 1} \right]$$

There are also gauge-dependent terms. But we don’t know how they can be cancelled in the full modulus square of the $Z \rightarrow d_i \bar{d}_i$ amplitude at two-loop level. This problem will be left for our next work.

Through these discussions we can draw a conclusion that based on the present knowledge we cannot judge which scheme is the better.

V. INTRODUCTION OF OFF-DIAGONAL WAVE-FUNCTION RENORMALIZATION CONSTANTS

In order to simplify the calculations of the corrections of fermion external legs in a Feynman diagram we can change the fermion wrc. to two matrices, which will contain the contributions of the off-diagonal two-point diagrams at fermion external legs. In order to simplify the calculations we need to define two new self-energy functions as follows:

$$i\Sigma_{ij} = \sum_{k} \tilde{g}_{ijk} \exp \left( \sum_{k,l} \Sigma_{ijkl} \right)$$

and the corresponding renormalized self-energy functions (after adding proper counterterms to the above diagrams):

$$\hat{\Sigma}_{ij} = \hat{\Sigma}_{ij} \left( \hat{\Sigma}_{ij} \gamma_L + \Sigma_{ij} \gamma_R \right)$$

We will see in a second that these new self-energy functions have no other meanings except for bringing about great convenience in actual calculations.

Now we introduce the fermion wrc. matrices $Z_{ij}$ and $Z_{ij}^\dagger$ as follows:

$$< p_i | \Sigma | p_j > = \hat{u}(m_i) Z_{ij}^\dagger M_{ij} Z_{ij} u(m_j)$$

$$= \hat{u}(m_i) Z_{ij}^\dagger \delta_{ij} + \hat{u}(m_i) Z_{ij}^\dagger \left( \sum_{i,j} \hat{\Sigma}_{ij} \left( \hat{\Sigma}_{ij} \gamma_{ij} + \Sigma_{ij} \gamma_{ij} \right) \right)$$

with $\delta_{ij} = 1$ when $i = j$ and $M^{\text{amp}}$ the amplitude of completely amputated diagrams (without off-diagonal self-energy diagrams at fermion external legs). Noted we have rarely used the definitions of Eqs.(62) in the second line of Eqs.(63). The last line of Eqs.(63) gives the definitions of fermion wrc. matrices:

$$Z_{ij}^\dagger u(m_j) = \left( \hat{u}(m_i) Z_{ij}^\dagger \right) \left( \hat{\Sigma}_{ij}(\hat{\Sigma}_{ij} \gamma_{ij} + \Sigma_{ij} \gamma_{ij}) \right) u(m_j)$$

$$u(m_i) Z_{ij}^\dagger = \left( \hat{u}(m_i) Z_{ij}^\dagger \right) \left( \hat{\Sigma}_{ij}(\hat{\Sigma}_{ij} \gamma_{ij} + \Sigma_{ij} \gamma_{ij}) \right) u(m_j)$$

(64)
after some algebra we have

\[
Z_{ij}^{L/S} = Z_i^{L/S} \delta_{ij} + \frac{\delta_{ij}}{m_i^2 - m_1^2} \left[ m_i Z_i^{L/S} \Sigma_{ij}^L(m_1^2) + m_i m_j Z_i^{L/S} \Sigma_{ij}^R(m_1^2) + m_j Z_i^{L/S} \bar{\Sigma}_{ij}^L(m_1^2) + m_j Z_i^{L/S} \bar{\Sigma}_{ij}^R(m_1^2) \right],
\]

and

\[
Z_{ij}^{R/S} = Z_j^{R/S} \delta_{ij} + \frac{\delta_{ij}}{m_j^2 - m_1^2} \left[ m_i m_j Z_i^{R/S} \Sigma_{ij}^R(m_1^2) + m_j Z_i^{R/S} \Sigma_{ij}^L(m_1^2) + m_j Z_i^{R/S} \bar{\Sigma}_{ij}^R(m_1^2) + m_j Z_i^{R/S} \bar{\Sigma}_{ij}^L(m_1^2) \right].
\]

At one-loop level these results agree with Eqs.(3.3) and Eqs.(3.4) in Ref. [3].

Using these wrc. matrices we can calculate S-matrix elements without considering any of the corrections of fermion external legs in a Feynman diagram. Noted such matrices are also suitable for gauge bosons which have mixing between each other.

VI. CONCLUSION

In this paper we have attempted to extend LSZ reduction formula for unstable fermion. We adopted the assumption of Eq.(7), which resembles the relativistic Breit-Wigner formula thus can describe the behavior of unstable particle’s propagator in the resonant region. Under this assumption only the on-shell mass renormalization scheme is suitable (see section 2). In consideration of the difficulty of introducing fermion field renormalization constants [3] we have abandoned field renormalization. We assume the unstable fermion’s propagator has the form of Eq.(9) in the limit \( p^2 \to m_1^2 \). Thus we can get the LSZ reduction formula Eq.(12) for unstable fermion and obtain the proper fermion wrc. Eqs.(29).

Next we change our assumption and compare the difference between different renormalization schemes. There are four basic renormalization schemes, corresponding to whether or not to introduce field renormalization constants and to adopt which mass renormalization scheme, on-shell scheme or complex-pole scheme [2]. Through comparison we find that whether to introduce field renormalization constants or not will affect the values of fermion wrc. and demonstrate its difference in the imaginary part of physical amplitude, and so does the choice of mass renormalization scheme (see section 3 and 4). This discrepancy is gauge dependent and can manifest itself at the modulus square of physical amplitude at two-loop level. But we haven’t found strong evidence to judge which scheme is better that the others.

Next we change the fermion wrc. to two matrices, in order to contain the corrections of fermion external legs in a Feynman diagram. The new fermion wrc. matrices, according to Eqs.(63), will simplify the calculations of physical amplitude, because we don’t need to calculate the corrections of off-diagonal self-energy diagrams at fermion external legs. We note that the concept of wrc. Matrix can be extended to bosons provided they have mixing in themselves.

In conclusion, we recommend the renormalization scheme which adopts the on-shell mass renormalization scheme and abandons field renormalization, because this renormalization scheme is very simple and clear. We will study this point further in the future.

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