Entanglement Monogamy of Tripartite Quantum States

Chang-shui Yu and He-shan Song
School of Physics and Optoelectronic Technology,
Dalian University of Technology, Dalian 116024, P. R. China
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An interesting monogamy equation with the form of Pythagorean theorem is found for \(2 \otimes 2 \otimes n\)-dimensional pure states, which reveals the relation among bipartite concurrence, concurrence of assistance, and genuine tripartite entanglement. At the same time, a genuine tripartite entanglement monotone as a generalization of 3-tangle is naturally obtained for \((2 \otimes 2 \otimes n)\)-dimensional pure states in terms of a distinct idea. For mixed states, the monogamy equation is reduced to a monogamy inequality. Both results for tripartite quantum states can be employed to multipartite quantum states.

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I. INTRODUCTION

Entanglement is an essential feature of quantum mechanics, which distinguishes quantum from classical world. A key property of entanglement as well as one of the fundamental differences between quantum entanglement and classical correlations is the degree of sharing among many parties—Unlike classical correlations, quantum entanglement is monogamous [1-3], i.e., the degree to which either of two parties can be entangled with anything else seems to be constrained by the entanglement that may exist between the two quantum parties. For the systems of three qubits, a kind of monogamy of bipartite quantum entanglement measured by concurrence [4] was described by Coffman-Kundu-Wootters (CKW) inequality [1]. The generalization to the case of multiple qubits was conjectured by CKW and has been proven recently by Osborne et al [5]. The monogamy inequality dual to CKW inequality based on concurrence of assistance (CoA) [6] was presented for tripartite systems of qubits by Gour et al [7] and the generalized one for multiple qubits was proven in Ref. [8]. In this paper, we find a new and very interesting monogamy equation for \((2 \otimes 2 \otimes n)\)-dimensional (or multiple qubits) quantum pure states which relates the bipartite concurrence, CoA and genuine tripartite entanglement.

In fact, CKW inequality and the dual one correspond to a residual quantity, respectively. It is only for tripartite pure states of qubits that so far the two residual quantities have been shown to be the same and have clear physical meanings. From Ref. [1] and [6], one can learn that it just corresponds to 3-tangle [1]. One of the distinguished advantages of our monogamy equation will be found that the residual quantity has clear physical meanings not only for \((2 \otimes 2 \otimes n)\)-dimensional quantum pure state but also for a general multipartite pure state including a pair of qubits.

Recently it has been realized that entanglement is a useful physical resource for various kinds of quantum information processing [9-12]. Based on the different physics of implementation, there are usually three alternative ways [7] to producing entanglement. The specially important way for quantum communication is the reduction of a multipartite entangled state to an entangled state with fewer parties, which is called "assisted entanglement" quantified by entanglement of assistance (EoA) [13]. An important application of EoA is for tripartite quantum entangled state to maximize the entanglement of two parties (qubits) denoted by Alice and Bob with the assistance of the third party (qudit) named Charlie who is only allowed to do local operations. However, because EoA is not an entanglement monotone [14], one would prefer to the remarkable entanglement monotone—concurrence of assistance (CoA) where concurrence is employed to quantify the entanglement between Alice and Bob. In this process of entanglement preparation, Charlie only makes local operations and classical communications in order to increase the entanglement shared by Alice and Bob. In this way, it is impossible to produce new entanglement. There must exist some trade-off between the increment of entanglement shared by Alice and Bob induced by Charlie and quantum correlations with other parties. Then what are those?

The question is answered in this paper by our interesting monogamy equation. From the equation, one can find that the increment of entanglement shared by Alice and Bob just corresponds to the degree of genuine tripartite entanglement (3-way entanglement) of \((2 \otimes 2 \otimes n)\)-dimensional quantum pure state and is analytically calculable. Hence, the increment naturally characterizes the genuine tripartite entanglement, which is shown to be an entanglement monotone and can be considered as an interesting generalization of 3-tangle in terms of a new idea. In addition, the monogamy equation is reduced to a monogamy inequality for mixed states. The results are also suitable for multipartite quantum states. This paper is organized as follows. We first introduce our interesting monogamy equation for pure states; then for mixed states, we reduce this monogamy equation for pure state to a monogamy inequality; Next we point out these re-
sults are suitable for multipartite quantum states; The conclusion is drawn finally.

II. MONOGAMY EQUATION FOR PURE STATES

Given a tripartite \((2 \otimes 2 \otimes n)\)-dimensional quantum pure state \(|\Psi\rangle_{ABC}\) shared by three parties Alice, Bob and Charlie, where Charlie’s aim is to maximize the entanglement shared by Alice and Bob by local measurements on Charlie’s particle C, the reduced density matrix by tracing over party C can be given by \(\rho_{AB} = Tr_C (|\Psi\rangle_{ABC} \langle \Psi|)\). Let \(E = \{p_i, |\varphi_i^{AB}\rangle\}\) is any a decomposition of \(\rho_{AB}\) such that

\[
\rho_{AB} = \sum_i p_i |\varphi_i^{AB}\rangle \langle \varphi_i^{AB}|,
\]

then CoA is defined \([5,6]\) by

\[
C_a (|\Psi\rangle_{ABC}) = \max_E \sum_i p_i C (|\varphi_i^{AB}\rangle) = C_a (\rho_{AB}) = tr \sqrt{\rho_{AB} \rho_{AB} \sqrt{\rho_{AB}}}
\]

\[
= \sum_{i=1}^n \lambda_i,
\]

where \(\rho_{AB} = (\sigma_y \otimes \sigma_y) \rho_{AB}^* (\sigma_y \otimes \sigma_y), \sigma_y\) is Pauli matrix and

\[
C (\rho_{AB}) = \max \{0, \lambda_1 - \sum_{i>1} \lambda_i\}
\]

is the concurrence of the reduced density matrix \(\rho_{AB}\) with \(\lambda_i\) being the square roots of the eigenvalues of \(\rho_{AB} \rho_{AB}^*\) in decreasing order. With the definitions of CoA and concurrence, we can obtain the following theorem.

**Theorem 1:** For a \((2 \otimes 2 \otimes n)\)-dimensional quantum pure state \(|\Psi\rangle_{ABC}\),

\[
C_a^2 (\rho_{AB}) = C^2 (\rho_{AB}) + \tau^2 (\rho_{AB}),
\]

where \(\tau (\rho_{AB}) = \tau (|\Psi\rangle_{ABC})\) is the genuine tripartite entanglement measure for \(|\Psi\rangle_{ABC}\).

It is very interesting that eq. (6) has an elegant form that is analogous to Pythagorean theorem if one considers CoA as the length of the hypotenuse of a right-angled triangle and considers bipartite concurrence and genuine tripartite entanglement as the lengths of the other two sides of the triangle. Note that the lengths of all the sides are allowed to be zero. The illustration of the relation is shown in Fig. 1 (See the left triangle).

**Proof.** According to the definition of CoA and concurrence, it is obvious that

\[
C_a^2 (\rho_{AB}) - C^2 (\rho_{AB}) \geq 0.
\]

Then the remaining is to prove that \(\tau (\rho_{AB})\) is an entanglement monotone and characterizes the genuine tripartite entanglement of \(|\Psi\rangle_{ABC}\). Next, we first prove that \(\tau (\rho_{AB})\) does not increase under a general tripartite local operation and classical communication (LOCC) denoted by \(\mathcal{M}_k\) where subscript \(k\) labels different outcomes. We first assume that Alice and Bob perform quantum operations \(M_{Akj}\) and \(M_{Bkj}\) on their qubits respectively, where \(\sum_{k,j} M_{Akj} M_{Bkj} \leq I_A\) and \(\sum_{k,j} M_{Akj}^\dagger M_{Bkj} \leq I_B\) are the most general local operations given in terms of the Kraus operator \([15]\) with \(I_A\) and \(I_B\) being the identity operators in Alice’s and Bob’s systems. After local operations, the average CoA can be written as

\[
\frac{1}{\kappa} \sum_{kk'} P_{kk'} \tau (\mathcal{M}_{kk'} (\rho_{AB}))
\]

\[
= \sum_{kk'} P_{kk'} \sqrt{C_a^2 (\mathcal{M}_{kk'} (\rho_{AB})) - C^2 (\mathcal{M}_{kk'} (\rho_{AB}))}
\]

\[
\leq \left\{ \sum_{kk'} P_{kk'} C (\mathcal{M}_{kk'} (\rho_{AB})) \right\}^{1/2}
\]

\[
= \sum_{kk'} \left[ \sum_{jj'} \det (M_{Akj}) \det (M_{Bkj}) \right] \left| \sqrt{C_a^2 (\rho_{AB}) - C^2 (\rho_{AB})} \right|
\]

\[
\leq \sqrt{C_a^2 (\rho_{AB}) - C^2 (\rho_{AB})} = \tau (\rho_{AB}),
\]

where

\[
\mathcal{M}_{kk'} (\rho_{AB}) = \sum_{jj'} (M_{Akj} \otimes M_{Bkj}^\dagger \otimes I_C) / P_{kk'},
\]

and \(P_{kk'} = tr \mathcal{M}_{kk'} (|\Psi\rangle_{ABC} \langle \Psi|)\). Here the first inequality follows from Cauchy-Schwarz inequality:

\[
\sum x_i y_i \leq \left( \sum x_i^2 \right)^{1/2} \left( \sum y_i^2 \right)^{1/2},
\]

the second inequality follows from the geometric-arithmetic inequality

\[
\sum_{kk'} |\det (M_{xkk})| \leq \frac{1}{2} \sum_{kk'} tr M_{xkk}^\dagger M_{xkk} \leq 1, x = A, B,
\]

and the second equation is derived from the fact \([4,16]\) that

\[
C_a (M_{Akj} \rho_{AB} M_{Akj}^\dagger) = |\det (M_{Akj})| C_a (\rho_{AB}),
\]

\[
C (M_{Akj} \rho_{AB} M_{Akj}^\dagger) = |\det (M_{Akj})| C (\rho_{AB}).
\]
and the analogous relations for $M_{Bk^j}$. Eq. (8) shows that $\tau (\rho_{AB})$ does not increase under Alice's and Bob's local operations.

Next we prove that $\tau (\rho_{AB})$ does not increase under Charlie's local operations either. Suppose $\rho_{AB} = \lambda \rho_{1}^{AB} + (1 - \lambda) \rho_{2}^{AB}$, $\lambda \in [0, 1]$, then

$$\lambda \tau (\rho_{1}^{AB}) + (1 - \lambda) \tau (\rho_{2}^{AB})$$

$$= \lambda \sqrt{C_{a}^{2} (\rho_{1}^{AB}) - C^{2} (\rho_{1}^{AB})} + (1 - \lambda) \sqrt{C_{a}^{2} (\rho_{2}^{AB}) - C^{2} (\rho_{2}^{AB})}$$

$$\leq \left\{ \left[ \lambda C_{a} (\rho_{1}^{AB}) + (1 - \lambda) C_{a} (\rho_{1}^{AB}) \right]^{2} - \left[ \lambda C (\rho_{2}^{AB}) + (1 - \lambda) C (\rho_{2}^{AB}) \right]^{2} \right\}^{1/2}$$

$$\leq \sqrt{C_{a}^{2} (\rho_{AB}) - C^{2} (\rho_{AB})} = \tau (\rho_{AB}) ,$$

where the first inequality follows from the Cauchy-Schwarz inequality (10) and the second inequality follows from the definitions of $C_{a} (\rho_{AB})$ and $C (\rho_{AB})$. Eq. (14) shows that $\tau (\rho_{AB})$ is a concave function of $\rho_{AB}$, which proves that $\tau (\rho_{AB})$ does not increase under Charlie's local operations following the same procedure (or Theorem 3) in Ref. [17]. All above show that $\tau (\rho_{AB})$ is an entanglement monotone.

Now we prove that $\tau (\rho_{AB})$ characterizes genuine tripartite entanglement. Based on eq. (4) and eq. (5), it is obvious that

$$\tau (\rho_{AB}) = \left\{ \begin{array}{ll}
\sum_{i=1}^{4} \lambda_{i}, & \sum_{i=2}^{4} \lambda_{i} \\
2 \sqrt{\sum_{i=2}^{4} \lambda_{i}}, & \sum_{i=2}^{4} \lambda_{i} > \sum_{i=2}^{4} \lambda_{i},
\end{array} \right.$$

which is an explicit formulation. Ref. [18] has given a special quantity named "entanglement semi-monotone" that characterizes the genuine tripartite entanglement.

One can find that it requires the same conditions as the quantity introduced in Ref. [18] for $\tau (\rho_{AB})$ to reach zero, which shows that $\tau (\rho_{AB})$ characterizes the genuine tripartite entanglement. The proof is completed. $\square$

In general, multipartite entanglement is quantified in terms of different classifications [19-21]. However, $\tau (\rho_{AB})$ quantifies genuine tripartite entanglement in a new way, i.e., we consider the entanglement of GHZ-state class as the minimal unit [22] in terms of tensor treatment [23] and summarize all the genuine tripartite inseparability without further classifications. It is an interesting generalization of 3-tangle. Theorem 1 shows a very clear physical meaning, i.e. the increment of entanglement between Alice and Bob induced by Charlie is just the genuine tripartite entanglement among them. The meaning can especially easily be understood for tripartite quantum state of qubits. In this case, $\tau (\rho_{AB}) = 2\sqrt{\lambda_{1}\lambda_{2}}$. Two most obvious examples are GHZ state and W state. The entanglement of reduced density matrix of GHZ state is zero, hence Theorem 1 shows that the CoA of GHZ state all comes from the three-way entanglement and equals to 1 (the value of 3-tangle). On the contrary, the W state has no three-way entanglement (only two-way entanglement) [24], hence its CoA is only equal to the concurrence ($\frac{\lambda}{2}$) of two parties. That is to say, for W state, Charlie can not provide any help to increase the entanglement between Alice and Bob.

III. MONOGAMY INEQUALITY FOR MIXED STATES

For a given mixed state $\rho_{ABC}$, CoA can be extended to mixed states in terms of convex roof construction [15], i.e.,

$$C_{a} (\rho_{ABC}) = \min \sum_{i} p_{i} C_{a} (|\psi_{i}^{ABC}\rangle),$$

where the minimum is taken over all decompositions $\{p_{i}, |\psi_{i}^{ABC}\rangle\}$ of $\rho_{ABC}$. Thus we have the following theorem.

Theorem 2. - For a $(2 \otimes 2 \otimes n)$- dimensional mixed state $\rho_{ABC}$,

$$C_{a}^{2} (\rho_{ABC}) \geq C^{2} (\rho_{AB}) + \tau^{2} (\rho_{ABC}),$$

where $\tau (\rho_{ABC})$ is the genuine tripartite entanglement measure for mixed states by extending $\tau (\cdot)$ of pure states in terms of convex roof construction and $\rho_{AB} = tr_{C} \rho_{ABC}$.

Analogous to Theorem 1, one can easily find that the relation of Theorem 2 corresponds to an obtuse-angled triangle after a simple algebra, where CoA corresponds to the length of the side opposite to the obtuse angle. See the right triangle in Fig.1 for the illustration.

Proof. Suppose $\{p_{k}, |\psi_{k}^{ABC}\rangle\}$ is the optimal decomposition in the sense of
\[ \tau(\rho_{ABC}) = \sum_{k} p_{k} \tau(|\psi^{k}\rangle_{ABC}) = \sum_{k} p_{k} \tau(\sigma^{k}_{AB}) , \quad (18) \]

where \( \sigma^{k}_{AB} = tr_{C} [ |\psi^{k}\rangle_{ABC} \langle \psi^{k}|] \). According to Theorem 1, we have

\[
\tau(\rho_{ABC}) = \sum_{k} p_{k} \sqrt{C_{a}^{2}(\sigma^{k}_{AB}) - C^{2}(\sigma^{k}_{AB})} \\
\leq \sqrt{\left[ \sum_{k} p_{k} C_{a}(\sigma^{k}_{AB}) \right]^{2} - \left[ \sum_{k} p_{k} C(\sigma^{k}_{AB}) \right]^{2}} \\
\leq \sqrt{C_{a}^{2}(\rho_{ABC}) - C^{2}(\rho_{ABC})}, \quad (19)
\]

where the first inequality follows from Cauchy-Schwarz inequality (10) and the second inequality holds based on the definitions of \( C_{a}(\rho_{AB}) \) and \( C(\rho_{AB}) \). Eq. (19) finishes the proof. \( \square \)

IV. MONOGAMY FOR Multipartite QUANTUM STATES

Any a given \( N \)-partite quantum state can always be considered as a \((2 \otimes 2 \otimes X)\)-dimensional tripartite quantum states with \( X \) denoting the total dimension of \( N - 2 \) subsystems so long as the state includes at least two qubits, hence both the two theorems hold in these cases. However, it is especially worthy of being noted that the two qubits must be owned by Alice and Bob respectively and the other \( N - 2 \) subsystems should be at Charlie's hand and be considered as a whole. Charlie is allowed to perform any nonlocal operation on the \( N - 2 \) subsystems. In addition, there may be different groupings [25] of a multipartite quantum state especially for multipartite quantum states of qubits, hence there exist many analogous monogamy equations (for pure states) or monogamy inequalities (for mixed states) for the same quantum state. For pure states, every monogamy equation will lead to a genuine \((2 \otimes 2 \otimes X)\)-dimensional tripartite entanglement monotone that quantifies the genuine tripartite entanglement of the tripartite state generated by the corresponding grouping.

V. CONCLUSION AND DISCUSSION

We have presented an interesting monogamy equation with elegant form for \((2 \otimes 2 \otimes n)\)-dimensional quantum pure states, which, for the first time, reveals the relation among bipartite concurrence, CoA and genuine tripartite entanglement. The equation naturally leads to a genuine tripartite entanglement measure for \((2 \otimes 2 \otimes n)\)-dimensional tripartite quantum pure states, which quantifies tripartite entanglement in terms of a new idea. The monogamy equation can be reduced to a monogamy inequality for mixed states. Both the results for tripartite quantum states are also suitable for multipartite quantum states. We hope that the current results can shed new light on not only the monogamy of entanglement but also the quantification of multipartite entanglement.

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tripartite entanglement (three-way entanglement). Thus entanglement of GHZ-state class can be understood as the minimal unit that quantifies genuine tripartite inseparability.

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