INVARIANT MEASURES FOR RANDOM EXPANDING ON AVERAGE SAUSSOL MAPS

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In this paper, we investigate the existence of random absolutely continuous invariant measures (ACIP) for random expanding on average Saussol maps in higher dimensions. This is done by the establishment of a random Lasota-Yorke inequality for the transfer operators on the space of bounded oscillation. We prove that the number of ergodic skew product ACIPs is finite and provide an upper bound for the number of these ergodic ACIPs. This work can be seen as a generalization of the work in [3] on admissible random Jabłoński maps to a more general class of higher dimensional random maps.

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1. Introduction

In this paper, we study a class of random expanding on average multi-dimensional maps where randomness means, at each iteration of the discrete process, one of a given family of maps is chosen and applied to produce the next stage of the dynamics. The randomness is governed by an external ergodic, invertible, probability preserving system $\sigma : \Omega \rightarrow \Omega$ (quenched setting) where $(\Omega, P)$ is a probability space, including but not restricted to the IID case, where maps are chosen according to a stationary process, see [1]. In real life applications, the relevance of random dynamical systems is clear due to the fact that systems are influenced by external factors or noise.

The long term behaviour of these random maps is unpredictable. Therefore, one attempts to understand their statistical properties through random absolutely continuous invariant measures (ACIPs), which are relevant to describe physically observable events. Generally, there is no measure that is invariant under all the maps at the same time. In this paper, we investigate the existence and bound the number of ergodic skew product ACIPs for the so-called random Saussol maps.

The statistical properties of multi-dimensional piecewise expanding maps are significantly more complicated to analyse than those of one dimensional maps. This is not simply because of technical difficulties, indeed there are intrinsic obstacles. For example, there exist two-dimensional piecewise expanding and $C^2$ maps with singular ergodic properties [27], and examples with no or infinitely many absolutely continuous invariant measures [7].
In [22], Saussol studied the statistical properties and existence of absolutely continuous invariant measures with respect to the Lebesgue measure for a general class of multi-dimensional piecewise expanding maps with singularities. The author establishes a spectral gap in the transfer operators associated to Saussol maps, defined in [22]. The assumptions on these maps naturally appear for maps with discontinuities on some wild sets. The author proved the existence of a finite number ACIPs. Moreover, a key property of the function space involved in [22] gives a constructive upper bound on the number of ergodic ACIPs.

In the literature, Saussol maps were studied by several researchers, among many, we mention [1, 19, 25]. In [19], Hu and Vaienti treated a class of nonsingular transformations with indifferent fixed points without the assumption of any Markov property. They adapted Saussol’s strategy to prove a Lasota-Yorke inequality and obtained the existence of ACIPs that can be finite or infinite. The random IID case of Saussol maps was covered by Aimino, Nicol and Vaienti in [1]. Their work is restricted to the case where the partitions associated to the maps under consideration are finite. Consequently, they were able to use the sufficient condition given in Lemma 2.1 in [22] instead of the hypothesis described in (PE5) given in Section 2 in [22]. They assumed the so-called random covering property to show that their systems are mixing. In [25], Thomine linked his work and proved similar results to Saussol [22] but using Sobolev spaces.

In a more recent work, in [10], Dragičević, Froyland, González-Tokman and Vaienti defined admissible transfer operator cocycles. They presented two classes of examples, one and higher dimensional piecewise expanding maps. In the higher dimensional case, Saussol maps were used over finite partitions with uniform constants. The same authors, in [11], proved a fiberwise almost sure invariance principle for a large class of random dynamical systems. They also provided explicit examples of random dynamical systems, including Saussol maps, and proved the existence of a unique random ACIP. In [23], Tanzi, Pereira and van Strien studied random compositions of small perturbations of dynamical systems modeled by Saussol maps. Their main focus was to study when the compositions of perturbations of a given map result in statistical behaviour close to that of the map itself. Particularly, they proved that the evolution of sufficiently regular mass distributions under the random perturbations stays close to the mass distribution that is invariant under the perturbed map.

This work can be seen as a generalization of the work in [3] on random Jabłoński maps where each component of the map only depends on its corresponding variable. In this paper, we include maps such that the components are allowed to depend on all or some of the variables. In [3], the authors studied the quenched setting of random Jabłoński maps. They proved that the skew product associated to this random dynamical system admits a finite number of ergodic ACIPs. Moreover, two different upper bounds on the number of ergodic skew product ACIPs were provided. Those bounds were motivated by the works of Buzzi [5] on random Lasota-Yorke
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maps in one dimension and Góra, Boyarsky and Proppe [18] on higher dimensional
determined. These bounds used the fact that Jabłoński maps are defined on rectangular
partitions and the maps preserve rectangles. Such bounds are not direct to adapt in
our setting since we do not make any assumption on the shape of partition and we
do not assume any type of geometric preservation under the maps. The bound we
develop in (6.1) is based on an analytical observation that is nonnegative functions
in the space of bounded oscillation are strictly positive on nontrivial balls inside
their support.

In Theorem 2 in [18], Góra, Boyarsky and Proppe obtained a bound on the
number of ergodic ACIPs for piecewise $C^2$ Jabłoński transformations which are suf-
ficiently expansive. In Remark 2.3 in [21], Liverani gave a sufficient condition for the
uniqueness of ACIP using dense orbits. However, in spite of the fact that Batayneh
and González-Tokman [3], Buzzi [5] and Araujo-Solano [2] provided bounds on the
number of ACIPs for random compositions of certain classes of one and multi-
dimensional maps, studying and, particularly, bounding the number of ergodic
ACIPs is still largely open problem. This relates to questions about multiplicity
of Lyapunov exponents in multiplicative ergodic theory.

The plan of the paper is the following: in Section 2, we provide background
materials regarding the function space involved and Oseledets splittings for random
dynamical systems. Section 3 is devoted for defining random Saussol maps. In Sec-
tion 4, we develop a random Lasota-Yorke Inequality and prove the quasi-compact
property. Section 5 provides existence results of random invariant and skew product
ACIPs. In Section 6, we provide an upper bound on the number of ergodic skew
product ACIPs for random Saussol maps. An example is also provided.

2. Background

2.1. The space of bounded oscillation

There are several function spaces to use when studying transfer operators induced
by higher dimensional expanding maps. One of them is the space of functions of
bounded variation in higher dimension, see [17]. Another alternative is fractional
Sobolev spaces, as done in [25]. In this paper, the analysis of the transfer operators
of random Saussol maps requires us to use a function space called the space of
bounded oscillation. This space was first introduced by Keller [20] in one dimension,
developed by Blank [4] and used by Saussol [22] and successively by Buzzi [5] and
Tsujii [26]. Other references where this space was used to provide densities of ACIPs
are [8, 19]. In the rest of the paper, $N > 1$.

Definition 2.1. For a Borel subset $C$ of $\mathbb{R}^N$ and $f \in L^1(\mathbb{R}^N)$, we define the oscillation of $f$ on $C$ by

$$\text{osc}(f, C) := E \sup_C f - E \inf_C f,$$

where $E \sup_C f$ is the essential supremum of $f$ on $C$ and $E \inf_C f$ is the essential
infinum of $f$ on $C$.

For $x \in \mathbb{R}^N$ and $\varepsilon > 0$, denote the open ball of radius $\varepsilon$ centered at $x$ by $B_\varepsilon(x)$. The mapping $x \mapsto \text{osc}(f, B_\varepsilon(x))$ is lower semi-continuous by Proposition 3.1 in \cite{22} and hence measurable, one then can define the $\alpha$-seminorm of $f$ (or the $\alpha$-oscillation of $f$).

**Definition 2.2.** Let $0 < \alpha \leq 1$ and $\tilde{\varepsilon}_0 > 0$ be real numbers and $f \in L^1(\mathbb{R}^N)$. The $\alpha$-seminorm of $f$ (or the $\alpha$-oscillation of $f$) is defined as

$$|f|_{\alpha} := \sup_{\varepsilon \leq \tilde{\varepsilon}_0} \int \varepsilon^{-\alpha} \text{osc}(f, B_\varepsilon(x)) \, dx.$$ (2.2)

**Definition 2.3.** Let $f \in L^1(C)$. We identify $f$ with its extension by zero to $\mathbb{R}^N$. If $|f|_{\alpha}$ is a finite, then $f$ is said to be of bounded oscillation on $C$. The set of all such maps is denoted by $V_\alpha$. For $f \in V_\alpha$, the norm of $f$ is defined by $\|f\| := \|f\|_{L^1} + |f|_{\alpha}$.

The space $V_\alpha$ is called the space of bounded oscillation (the Quasi-Hölder space in \cite{22}). $V_\alpha$ is a Banach space and compactly embedded in $L^1(C)$, see \cite{10}. Note that while the norm depends on $\tilde{\varepsilon}_0$, the space $V_\alpha$ does not, and two choices of $\tilde{\varepsilon}_0$ give rise to two equivalent norms. By Lemma 1 in \cite{8}, the space $V_\alpha$ is a proper subset of the space of bounded variation $(BV(C), \|\cdot\|_{BV})$ in the sense of Definition 1.1 in \cite{14}, indeed $\|f\|_{BV} \leq 2N^{\frac{\alpha}{2}} \|f\|_1$. A crucial property of this space is, in higher dimensions, functions of $V_\alpha$ are bounded \cite{22} but functions of $BV(C)$ are not bounded in general.

The oscillation defined in Equation (2.1) satisfies some properties listed in Proposition 3.2 in \cite{22}. We use these properties when we develop the Lasota-Yorke inequality in Theorem 4.1. We recall this proposition as well as Lemma 3.1 from \cite{22} which gives one of the key properties of nonnegative functions of the space of bounded oscillation $V_\alpha$.

**Proposition 2.1** (\cite{22}). Let $f, f_i, g \in L^\infty(\mathbb{R}^N)$, $g$ be positive function, $a, b, c > 0$ and $K$ be Borel subset of $\mathbb{R}^N$. The oscillation has the following properties: (i) $\text{osc} \left( \sum_i f_i, B_a(\cdot) \right) \leq \sum_i \text{osc}(f_i, B_a(\cdot))$. (ii) $\text{osc}(f1_{K \cap B_a(\cdot)}) \leq \text{osc}(f, K \cap B_a(\cdot))1_K(\cdot) + 2 \left( E \sup_{K \cap B_a(\cdot)} |f| \right)1_{B_a(K) \cap B_a(K^c)(\cdot)}$, where $B_a(K) := \{x : d(x, K) < a\}$ and $d$ is the Euclidean metric and $K^c$ is the complement of $K$. (iii) $\text{osc}(fg, K) \leq \text{osc}(f, K)E \sup_K + \text{osc}(g, K)E \inf_K |f|$. (iv) If $a + b \leq c$, then for all $x \in \mathbb{R}^N$ we have

$$E \sup_{B_a(x)} f \leq \frac{1}{m(B_a(x))} \int_{B_a(x)} \left( f(z) + \text{osc}(f, B_c(z)) \right) \, dz.$$  

**Lemma 2.1** (\cite{22}). For every positive $h \in V_{\alpha}$, $h \not\equiv 0$, there exists a ball on which the infimum of $h$ is strictly positive. The radius $\varepsilon$ of the ball can be taken as

$$\varepsilon = \min(\tilde{\varepsilon}_0, \left( \frac{\int hdm}{|h|_{\alpha}} \right)^{\frac{1}{\alpha}}),$$ (2.3)
where $\varepsilon_0$ as of Definition (2.2).

2.2. Random dynamical systems and Oseledets splittings

**Definition 2.4.** A random dynamical system is a tuple $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{X}, \mathcal{L})$, where the base $\sigma$ is an invertible measure-preserving transformation of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $(\mathcal{X}, || \cdot ||)$ is a Banach space and $\mathcal{L} : \Omega \rightarrow L(\mathcal{X}, \mathcal{X})$ is a family of bounded linear maps of $\mathcal{X}$, called the generator.

A key regularity notion in this work is the concept of $\mathbb{P}$-continuity which was first introduced by Thieullen in [24]. We will apply this in Corollary 5.1.

**Definition 2.5.** For a topological space $\Omega$, equipped with a Borel probability $\mathbb{P}$, a mapping $L$ from $\Omega$ to a topological space $Y$ is said to be $\mathbb{P}$-continuous if $\Omega$ can be expressed as a countable union of Borel sets such that the restriction of $L$ to each of them is continuous.

For convenience, we let $L_\omega := L(\omega)$ be the transfer operator defined in (3.9). A random dynamical system defines a cocycle, given by

\[(k, \omega) \mapsto L_\omega^{(k)} := L_{\sigma^{-1} \omega} \circ \cdots \circ L_\omega \circ L_\omega. \quad (2.4)\]

Multiplicative ergodic theorems deal with random dynamical systems $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{X}, \mathcal{L})$. They give rise to Oseledets splittings or decompositions of $\mathcal{X}$ that depend on $\omega$. We apply the Oseledets splitting theorem for $\mathbb{P}$-continuous random dynamical systems [14] to show that the random invariant densities $h_\omega$ given in Theorem 5.1 belong to the first Oseledets subspace. In Corollary 5.1, we use the finite dimensionality of the first Oseledets subspace to show that the number of ergodic ACIPs is finite.

**Definition 2.6.** Let $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{X}, \mathcal{L})$ be a random dynamical system. An Oseledets splitting for $\mathcal{R}$ consists of a sequence of isolated (exceptional) Lyapunov exponents

\[\infty > \lambda^* = \lambda_1 > \lambda_2 > ... > \lambda_l > \kappa^* \geq -\infty,\]

where the index $l \geq 1$ is allowed to be finite or countably infinite, and a family of $\omega$-dependent splittings,

\[\mathcal{X} = Y_1(\omega) \oplus ... \oplus Y_l(\omega) \oplus V(\omega), \quad (2.5)\]

where for $j = 1, ..., l$, $d_j := \dim(Y_j(\omega)) < \infty$ and $V(\omega) \in \mathcal{G}(\mathcal{X})$, where $\mathcal{G}(\mathcal{X})$ is the Grassmannian of $\mathcal{X}$. For all $j = 1, ..., l$ and $\mathbb{P}$-a.e. $\omega \in \Omega$ we have

\[L_\omega Y_j(\omega) = Y_j(\sigma\omega), \quad L_\omega V(\omega) \subseteq V(\sigma\omega),\]
and
\[
\lim_{s \to \infty} \frac{1}{s} \log \left\| \mathcal{L}_\omega^{(s)} y \right\| = \lambda_j, \forall y \in Y_j(\omega) \setminus \{0\}, \tag{2.6}
\]
\[
\lim_{s \to \infty} \frac{1}{s} \log \left\| \mathcal{L}_\omega^{(s)} v \right\| \leq \kappa^*, \forall v \in V(\omega).
\]

3. Random Saussol maps

**Definition 3.1.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \(\sigma : \Omega \to \Omega\) an invertible, ergodic and \(\mathbb{P}\)-preserving transformation. Let \(C\) be a compact subset of \(\mathbb{R}^N\), with \(\text{clos}(\text{int}(C)) \neq \emptyset\). A random Saussol map \(T\) over \(\sigma\) is a map \(T : \Omega \to \{T_\omega\}_{\omega \in \Omega}\), where \(T_\omega := T(\omega) : C \to C\) such that there exists an at most countable family of disjoint open sets \(U_i \subset C\) and \(V_i\) where \(\text{clos}(U_i) \subset V_i\) for all \(i\) in an indexing set \(I\), and maps
\[
T_{\omega,i} : V_i \to \mathbb{R}^N,
\]
satisfying for some \(0 < \alpha \leq 1\) and small enough \(\varepsilon_0 > 0\): (PE1) There exists \(S > 0\), such that for \(\mathbb{P}\text{-a.e.} \omega \in \Omega\), \(0 < s(\omega) < S\) and for all \(i \in I\) and \(u, v \in T_\omega V_i\) such that \(d(u, v) \leq \varepsilon_0\), we have
\[
d(T_{\omega,i}^{-1} u, T_{\omega,i}^{-1} v) \leq s(\omega)d(u, v), \tag{3.1}
\]
and
\[
\int_{\Omega} \log(s(\omega))d\mathbb{P}(\omega) < 0. \tag{3.2}
\]
(PE2) For \(\mathbb{P}\text{-a.e.} \omega \in \Omega\) and all \(i\), \(T_\omega|_{U_i} = T_{\omega,i}|_{U_i}\) and \(T_{\omega,i}(V_i) \supset B_{\varepsilon_0}(T_{\omega,i}(U_i))\). (PE3) For \(\mathbb{P}\text{-a.e.} \omega \in \Omega\) and all \(i\), \(T_{\omega,i} \in C^1(V_i)\) and \(T_{\omega,i}\) is injective and \(T_{\omega,i}^{-1} \in C^1(T_{\omega,i}V_i)\). Moreover, the determinant is uniformly Hölder: for \(\mathbb{P}\text{-a.e.} \omega \in \Omega\) and all \(i,\varepsilon \leq \varepsilon_0, z \in T_{\omega,i}V_i\) and \(x, y \in B_{\varepsilon}(z) \cap T_{\omega,i}V_i\), we have
\[
|\det DT_{\omega,i}^{-1} x - \det DT_{\omega,i}^{-1} y| \leq c|\det DT_{\omega,i}^{-1} z|\varepsilon^\alpha, \tag{3.3}
\]
for some \(c > 0\). (PE4) For \(\mathbb{P}\text{-a.e.} \omega \in \Omega\), \(m(C \setminus \bigcup_{i \in I} U_i) = 0\). (PE5) For \(\mathbb{P}\text{-a.e.} \omega \in \Omega\) and \(\varepsilon > 0\), let
\[
G_{\omega,\varepsilon_0}(\varepsilon) := \sup_{x \in C} G_{\omega,\varepsilon_0}(x, \varepsilon), \tag{3.4}
\]
where
\[
G_{\omega,\varepsilon_0}(x, \varepsilon) := \sum_{i \in I} \frac{m\left(T_{\omega,i}^{-1} B_\varepsilon(\partial T_\omega U_i) \cap B_{s(\omega)\varepsilon_0}(x)\right)}{m(B_{s(\omega)\varepsilon_0}(x))}. \tag{3.5}
\]

\(^a\)The sets \(U_i\) and \(V_i\) may also depend on \(\omega\). However, we do not make this dependence explicit, unless it becomes relevant for the discussion.
For $\mathbb{P}$-a.e. $\omega \in \Omega$, define $\zeta_{\epsilon_0}(\omega)$ by

$$
\zeta_{\epsilon_0}(\omega) := s(\omega)^\alpha + 2 \sup_{\epsilon \leq \epsilon_0} \frac{G_{\omega,\epsilon}(\epsilon)}{\epsilon^\alpha}(S_{\epsilon_0})^\alpha,
$$

and assume that

$$
\int_{\Omega} \log(\zeta_{\epsilon_0}(\omega)) d\mathbb{P}(\omega) < 0.
$$

In addition, we assume the mapping $\omega \mapsto \mathcal{L}_\omega$ is $\mathbb{P}$-continuous. For $k \in \mathbb{N}$, the $k$-fold composition $T^{(k)}_\omega$ is defined as

$$
T^{(k)}_\omega = T_{\sigma^{k-1}\omega} \circ \ldots \circ T_{\sigma\omega} \circ T_\omega.
$$

For simplicity, we sometimes refer to the range of $T$, that is $\{T_\omega\}_{\omega \in \Omega}$, as the random Saussol map. A random Saussol map gives rise to a random dynamical system, where $\mathcal{X} = V_\alpha$ and $\mathcal{L}_\omega = \mathcal{L}_{T_\omega}$ is the transfer operator defined by

$$
\mathcal{L}_\omega f = \sum_{i \in I} (g_\omega f) \circ T_{\omega,i}^{-1} 1_{T_\omega U_i},
$$

where

$$
g_\omega := \frac{1}{|\det DT_\omega|}.
$$

This can be seen as the expanding on average random version of the deterministic maps studied by Saussol in [22].

In the above definition, we ensure that $C$ and the $U_i$’s do not need to be connected and no control on the angles between smooth elements of the partition like the ones in [17, 9]. The family of the $U_i$’s does not need to be finite.

**Remark 3.1.** For the random Saussol map $T = \{T_\omega\}_{\omega \in \Omega}$, the skew product map $F$ on $\Omega \times C$ which encodes the whole dynamics of the system is given by

$$
F(\omega, x) = (\sigma \omega, T_\omega(x)).
$$

4. Lasota-Yorke inequality and quasi-compactness

In this section, we establish a one step Lasota-Yorke inequality on the space of bounded oscillation $V_\alpha$. This inequality is used to show the quasi-compactness property in Corollary 4.1.

**Definition 4.1.** Let $A : \mathcal{X} \rightarrow$ be a bounded linear map. The index of compactness norm of $A$ is

$$
\|A\|_{ic(\mathcal{X})} = \inf\{r > 0 : A(B_{\mathcal{X}}) \text{ can be covered by finitely many balls of radius } r\},
$$

where $B_{\mathcal{X}}$ is the unit ball in $\mathcal{X}$. 

For Proposition 2.1 (ii), we get

\[ \lambda(\omega) = \lim_{k \to \infty} \frac{1}{k} \log \left\| \mathcal{L}_\omega^{(k)} \right\|, \]

and the index of compactness \( K(\omega) \) is

\[ K(\omega) = \lim_{k \to \infty} \frac{1}{k} \log \left\| \mathcal{L}_\omega^{(k)} \right\|_{\mathcal{L}(\mathcal{X})}, \]

whenever these limits exist.

We recall the following remark from [16].

**Remark 4.1.** In Definition 4.2 if \( \sigma \) is ergodic, then \( \lambda \) and \( K \) are constant \( \mathbb{P} \)-almost everywhere. We denote these constants by \( \lambda^* \) and \( K^* \). By definition, we have that \( K^* \leq \lambda^* \). The finiteness of \( \lambda^* \) is implied by the assumption that \( \int_{\Omega} \log^+ \| \mathcal{L}_\omega \| d\mathbb{P}(\omega) < \infty \).

**Definition 4.3.** A random dynamical system \( \mathcal{R} \) with an ergodic base \( \sigma \) is called quasi-compact if \( K^* < \lambda^* \).

**Theorem 4.1.** Let \( T = \{ T_\omega \}_{\omega \in \Omega} \) be a random Saussol map. If \( \varepsilon_0 \) in Definition 2.2 is small enough, then there are positive measurable functions \( \eta, D : \Omega \to \mathbb{R}^+ \) such that \( \int_{\Omega} \log(\eta(\omega)) d\mathbb{P}(\omega) < 0 \) and

\[ \| \mathcal{L}_\omega f \|_\alpha \leq \eta(\omega) \| f \|_\alpha + D(\omega) \| f \|_{L^1}, \]

for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \) and \( f \in V_\alpha \).

**Proof.** Let \( x = (x_1, \ldots, x_n) \in C, \omega \in \Omega \), let \( \alpha \in (0, 1] \) and \( \varepsilon_0 > 0 \) be small enough. By Proposition 2.2 and 2.1 (i), we have

\[ \text{osc}(\mathcal{L}_\omega f, B_\varepsilon(x)) \leq \sum_{i \in I} \text{osc} \left( (g_{\omega, f} \circ (T_{\omega, i}^{-1})^{-1}1_{T_{\omega, i}U_i}, B_\varepsilon(x)) \right). \]

By Proposition 2.1 (ii), we get

\[ \text{osc}(\mathcal{L}_\omega f, B_\varepsilon(x)) \leq \sum_{i \in I} \left( \text{osc}((g_{\omega, f} \circ T_{\omega, i}^{-1}, T_{\omega}U_i \cap B_\varepsilon(x))1_{T_{\omega}U_i}(x) \right) + 2 \left( E \sup_{T_{\omega}U_i \cap B_\varepsilon(x)} |(g_{\omega, f} \circ T_{\omega, i}^{-1})1_{B_\varepsilon(\partial T_{\omega}U_i)}(x) \right) \right).

\[ \leq \sum_{i \in I} \left( \text{osc}(g_{\omega, f}1_{U_i \cap (T_{\omega})^{-1}B_\varepsilon(x)}1_{T_{\omega}U_i}(x) \right) + 2 \left( E \sup_{U_i \cap (T_{\omega})^{-1}B_\varepsilon(x)} |g_{\omega, f}|1_{B_\varepsilon(\partial T_{\omega}U_i)}(x) \right) \right). \] (4.1)

Let

\[ R_{\omega, i}^{(1)}(x) := \text{osc} \left( g_{\omega, f}, U_i \cap (T_{\omega})^{-1}B_\varepsilon(x) \right). \]

For \( x \in T_{\omega}U_i \), let \( y_{\omega, i} := T_{\omega, i}^{-1}x \). Then, by (PE1), we have

\[ R_{\omega, i}^{(1)}(x) \leq \text{osc} \left( g_{\omega, f}, U_i \cap B_{\lambda}(y_{\omega, i}) \right). \]
By Proposition \ref{proposition:bound}, (iii), for almost all \(x \in T_{\omega}U_{i}\), we have
\[
R^{(1)}_{\omega,i}(x) \leq \text{osc}(f, B_{s(\omega)})(y_{\omega,i}) E \sup_{U_{i} \cap B_{s(\omega)}(y_{\omega,i})} g_{\omega} + \text{osc}(g_{\omega}, B_{s(\omega)})(y_{\omega,i}) E \inf_{U_{i} \cap B_{s(\omega)}(y_{\omega,i})} |f|.
\]

Applying (PE3), we have
\[
R^{(1)}_{\omega,i}(x) \leq (1 + cs(\omega)^{\alpha} \varepsilon^{\alpha}) \text{osc}(f, B_{s(\omega)})(y_{\omega,i}) g_{\omega}(y_{\omega,i}) + |f|(y_{\omega,i})g_{\omega}(y_{\omega,i})cs(\omega)^{\alpha} \varepsilon^{\alpha}.
\]

Hence, the first term in (4.1) can be estimated as
\[
\sum_{i \in I} R^{(1)}_{\omega,i} 1_{T_{\omega}U_{i}} \leq (1 + cs(\omega)^{\alpha} \varepsilon^{\alpha}) \mathcal{L}_{\omega}(\text{osc}(f, B_{s(\omega)})(\cdot)) + cs(\omega)^{\alpha} \varepsilon^{\alpha} \mathcal{L}_{\omega}(|f|).
\]

Integrating both sides yields
\[
\int \sum_{i \in I} R^{(1)}_{\omega,i} 1_{T_{\omega}U_{i}} \leq (1 + cs(\omega)^{\alpha} \varepsilon^{\alpha}) \int_{\mathbb{R}^N} \text{osc}(f, B_{s(\omega)})(\cdot)) + cs(\omega)^{\alpha} \varepsilon^{\alpha} \int_{\mathbb{R}^N} |f|.
\]

By definition of \(|f|_{\alpha}\) in (2.2) with \(\tilde{\varepsilon}_{0} = S\varepsilon_{0}\), we have
\[
\int \sum_{i \in I} R^{(1)}_{\omega,i} 1_{T_{\omega}U_{i}} \leq (1 + cs(\omega)^{\alpha} \varepsilon^{\alpha})(s(\omega)\varepsilon)^{\alpha} |f|_{\alpha} + c(s(\omega)\varepsilon)^{\alpha} \|f\|_{L^1}.
\]

For the second term in (4.1), let
\[
R^{(2)}_{\omega,i}(x) := \left( \frac{1}{(1 + cs(\omega)^{\alpha} \varepsilon^{\alpha})} \int_{\mathbb{R}^N} |g_{\omega}(y)| \right) 1_{B_{s}(\partial T_{\omega}U_{i})}(x).
\]

If \(x \notin B_{c}(T_{\omega}U_{i})\) then \(R^{(2)}_{\omega,i}(x) = 0\). Using the definition of \(g_{\omega}\), (PE1) and (3.3), we get
\[
R^{(2)}_{\omega,i}(x) \leq \left( \frac{1}{(1 + cs(\omega)^{\alpha} \varepsilon^{\alpha})} \int_{\mathbb{R}^N} |f| \right) |\det(D(T_{\omega}U_{i})^{-1}x| (1 + cs(\omega)^{\alpha} \varepsilon^{\alpha}) 1_{B_{s}(\partial T_{\omega}U_{i})}(x).
\]

Integrating both sides over \(\mathbb{R}^N\) followed by a change of variable \(x = T_{\omega,i}y_{\omega,i}\) gives
\[
\frac{1}{(1 + cs(\omega)^{\alpha} \varepsilon^{\alpha})} \int_{\mathbb{R}^N} R^{(2)}_{\omega,i}(x) dx \leq \int_{\mathbb{R}^N} 1_{B_{s}(\partial T_{\omega}U_{i})}(T_{\omega,i}y_{\omega,i}) \sup_{B_{s}(\partial T_{\omega}U_{i})}(y_{\omega,i}) |f| dy_{\omega,i}.
\]

By Proposition \ref{proposition:bound}, (iv), choosing \(a = s(\omega)\varepsilon\), \(b = (S - s(\omega))\varepsilon_{0}\) and \(c = S\varepsilon_{0}\), we get (4.3) is less than or equal to
\[
\int_{\mathbb{R}^N} \frac{1_{B_{s}(\partial T_{\omega}U_{i})}(y_{\omega,i})}{m(B_{s-s(\omega)}\varepsilon_{0})(y)} dy \int_{B_{s-s(\omega)}\varepsilon_{0}(y)} (|f|(z) + \text{osc}(f, B_{S\varepsilon_{0}}(z))) dz,
\]

which becomes, after changing the order of integration,
\[
\int_{\mathbb{R}^N} (|f|(z) + \text{osc}(f, B_{S\varepsilon_{0}}(z))) dz \int_{\mathbb{R}^N} \frac{1_{(T_{\omega}U_{i})^{-1}B_{s}(\partial T_{\omega}U_{i})}(y) 1_{B_{S-s(\omega)}\varepsilon_{0}(z)}(y)}{m(B_{S-s(\omega)}\varepsilon_{0})(y))} dy.
\]
Finally, since the measure of a ball depends only on its radius, we can replace the second integral by
\[ m\left( T_{\omega,i}^{-1}B_{\varepsilon}(\partial T_{\omega}U_i) \cap B_{(S-s(\omega))\varepsilon_0}(z) \right) / m\left( B_{(S-s(\omega))\varepsilon_0}(z) \right). \]

By the definitions of \( G_{\omega,\varepsilon_0}(\varepsilon) \) in (3.5), we get
\[ \frac{1}{1 + \cos(\omega)\alpha s(\omega)} \sum_i R_{i}^{(2)}(x) dx \leq G_{\omega,\varepsilon_0}(\varepsilon)(S\varepsilon_0)^\alpha \| f \|_\alpha + G_{\omega,\varepsilon_0}(\varepsilon) \| f \|_{L^1}. \] (4.4)

By combining (4.2) and (4.4) into (4.1) and dividing both sides by \( \varepsilon^\alpha \) and taking the supremum over all \( \varepsilon \leq \tilde{\varepsilon}_0 = S\varepsilon_0 \), we get
\[ \| L_\omega f \|_\alpha \leq \eta(\omega) \| f \|_\alpha + D(\omega) \| f \|_{L^1}, \] (4.5)

where
\[ \eta(\omega) = (1 + \cos(\omega)\alpha s(\omega)) \left( s(\omega) + 2 \sup_{\varepsilon \leq \varepsilon_0} \frac{G_{\omega,\varepsilon_0}(\varepsilon)}{\varepsilon^\alpha}(S\varepsilon_0)^\alpha \right), \] (4.6)
\[ D(\omega) = \cos(\omega)\alpha + 2(1 + \cos(\omega)\alpha s(\omega)) \sup_{\varepsilon \leq \varepsilon_0} \frac{G_{\omega,\varepsilon_0}(\varepsilon)}{\varepsilon^\alpha} \]
\[ \leq \cos(\omega)\alpha + (1 + \cos(\omega)\alpha s(\omega)) (\max(S,1)\varepsilon_0)^{-\alpha}. \] (4.7)

Taking \( \varepsilon_0 \) small enough and using (3.7), we have
\[ \int_{\Omega} \log(\eta(\omega)) d\mathbb{P}(\omega) < 0. \]

The following lemma is taken from [16] applied to our setting.

**Lemma 4.1** ([16]). Suppose we have the following inequality
\[ \| L_\omega f \|_\alpha \leq A(\omega) \| f \|_\alpha + B(\omega) \| f \|_{L^1}, \]
for all \( f \in V_\alpha \) where \( A(\omega) \) and \( B(\omega) \) are measurable and
\[ \int_{\Omega} \log(A(\omega)) d\mathbb{P}(\omega) < 0. \]
Then there exists a full measure subset \( \Omega_1 \subseteq \Omega \) with the following property
\[ \lim_{k \to \infty} \frac{1}{k} \log \| e_\omega^{(k)} \|_{\mathcal{C}(X)} \leq \int_{\Omega} \log(A(\omega)) d\mathbb{P}(\omega) \] for all \( \bar{\omega} \in \Omega_1 \).

**Corollary 4.1.** Let \( T = \{ T_\omega \}_{\omega \in \Omega} \) be a random Saussol map. Provided \( \varepsilon_0 \) in (2.2) is small enough, the following hold. (i) The random dynamical system generated by \( T \) is quasi-compact and (ii) its maximal Lyapunov exponent \( \lambda^* \) is zero.

**Proof.** [Proof of Corollary 4.1 (i)] By Lemma 4.1 and (4.5), the index of compactness \( K^* \) is bounded above by \( \int_{\Omega} \log(\eta(\omega)) d\mathbb{P}(\omega) < 0. \) Next, we show that \( \lambda^* \geq 0. \)
Since the Perron–Frobenius operator $L^{(k)}_\omega$ is a Markov operator for each $\omega \in \Omega$, then for any density function $f \in V_\alpha$, we have that
\[
\|L^{(k)}_\omega f\|_\alpha \geq \|L^{(k)}_\omega f\|_{L^1} = \|f\|_{L^1} = 1,
\]
which shows that
\[
\lambda^* \geq 0. \tag{4.8}
\]

**Proof.** [Proof of Corollary 4.1 (ii)] To prove $\lambda^* \leq 0$. We have $\|L_\omega\|_{L^1} \leq 1$, it suffices to consider the growth of the $\alpha$-oscillation of the term $L^{(k)}_\omega f$. Applying the arguments in Lemma C.5 in [16] and Proposition 1.4 in [5], the functions $\eta(\omega)$ and $D(\omega)$ can be redefined such that $\xi \eta^{-\alpha}$ is satisfied and $D(\omega)$ is uniformly bounded by $\tilde{D}$ and $\tilde{D} \geq \xi \tilde{\varepsilon}^{-\alpha}$, where $\xi$ is defined the proof of Theorem 5.1. Therefore, we have
\[
\|L_\omega f\|_\alpha \leq \eta(\omega) \|f\|_{L^1} + \tilde{D} \|f\|_{L^1}, \tag{4.9}
\]
and $\int_\Omega \log(\eta(\omega)) d\mathbb{P}(\omega) < 0$. Iterating (4.9), we get a bound on the sequence $(\|L^{(k)}_\omega f\|_\alpha)_{k=1}^\infty$. Hence,
\[
\lim_{k \to \infty} \frac{1}{k} \log \|L^{(k)}_\omega f\|_\alpha \leq 0.
\]
and since this is true for $\mathbb{P}$-a.e. $\omega \in \Omega$, we get $\lambda^* \leq 0$. By (4.8), we have $\lambda^* = 0$.

5. Existence of random invariant and skew product ACIPs, finiteness, and physicality of measures

**Definition 5.1.** Let $T = \{T_\omega\}_{\omega \in \Omega}$ be a random Saussol map. A family $\{\mu_\omega\}_{\omega \in \Omega}$ is called a random invariant measure for $T$ if $\mu_\omega$ is a probability measure on $D$, the map $\omega \mapsto \mu_\omega$ is measurable and
\[
T_\omega \mu_\omega = \mu_{\sigma \omega}, \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega.
\]
A family $\{h_\omega\}_{\omega \in \Omega}$ is called a random invariant density for $T$ if $h_\omega \geq 0$, $h_\omega \in L^1(C)$, $\|h_\omega\|_{L^1} = 1$, the map $\omega \mapsto h_\omega$ is measurable and
\[
L_\omega h_\omega = h_{\sigma \omega}, \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega.
\]

**Theorem 5.1.** Consider a random Saussol map $T$. If $\varepsilon_0$ is small enough, then for each $\omega \in \Omega$ and $k = 1, 2, \ldots$, we define
\[
h^k_\omega = (L_{\sigma^{-1} \omega} \circ \cdots \circ L_{\sigma^{-k} \omega})1,
\]
where $1 \in V_\alpha$ is the constant function and for each $s = 1, 2, \ldots$, we define
\[
H^s_\omega = \frac{1}{s} \sum_{k=1}^s h^k_\omega.
\]
Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$:

(i) the sequence $\{H^s_\omega\}_{s \in \mathbb{N}}$ is relatively compact in $L^1$; and

(ii) the following limit exists,

\[
\lim_{s \to \infty} H^s_\omega =: h_\omega \in L^1.
\]

Moreover, $\{h_\omega\}_{\omega \in \Omega}$ is a random invariant density for $T$.

**Proof.** For $k = 1, 2, \ldots$, and $\mathbb{P}$-a.e. $\omega \in \Omega$, the following holds,

\[
h^k_\omega = L^{(k)}_{\sigma^k} \omega_1.
\]

Applying the hybrid Lasota-Yorke inequality (4.9) to estimate $\|h^k_\omega\|_\alpha$, we get

\[
\|h^k_\omega\|_\alpha = \left\|L^{(k)}_{\sigma^k} 1\right\|_\alpha \leq \|L_{\sigma^k}^{-1} \circ \cdots \circ L_{\sigma^2}^{-1} \circ L_{\sigma^1}\|_\alpha \\
\leq \eta(\sigma^{-1}\omega) \|L_{\sigma^2}^{-1} \circ \cdots \circ L_{\sigma^1}\|_\alpha \\
+ \hat{D} \|L_{\sigma^2}^{-1} \circ \cdots \circ L_{\sigma^1}\|_\alpha \\
\leq \eta(\sigma^{-1}\omega) \eta(\sigma^{-2}\omega) \cdots \eta(\sigma^{-k}\omega) \|1\|_\alpha \\
+ \hat{D} \|L_{\sigma^2}^{-1} \circ \cdots \circ L_{\sigma^1}\|_\alpha.
\]

Since $\|1\|_\alpha = 0$, $\|1\|_{L^1} = m(C)$ and the transfer operator is contractive in $L^1$, we have

\[
\|h^k_\omega\|_\alpha \leq \hat{D} m(C) \left(1 + \eta(\sigma^{-1}\omega) + \eta(\sigma^{-2}\omega) \eta(\sigma^{-1}\omega) + \cdots + \eta(\sigma^{-k}\omega) \eta(\sigma^{-1}\omega)\right) \\
= \hat{D} m(C) \left(1 + \sum_{j=1}^{k} \eta^{(j)}(\sigma^{-j}\omega)\right),
\]

where for each $j \in \mathbb{N}$, $\eta^{(j)}(\sigma^{-j}\omega) := \eta(\sigma^{-1}\omega) \eta(\sigma^{-2}\omega) \cdots \eta(\sigma^{-j}\omega)$. Note that for $j = 1, 2, \ldots$, we have

\[
\frac{1}{j} \log \eta^{(j)}(\sigma^{-j}\omega) = \frac{1}{j} \sum_{i=1}^{j} \log \eta(\sigma^{-i}\omega).
\]

Applying Birkhoff ergodic theorem, we get that the above time average converges as $j \to \infty$ to the space average $\int_{\Omega} \log(\eta(\omega)) d\mathbb{P}(\omega) =: \log(\bar{\xi}) < 0$, for some $0 < \bar{\xi} < 1$.

Choose $\xi$ such that $0 < \bar{\xi} < \xi < 1$ and $\xi > \frac{1}{\bar{\xi}}$. For large enough $j_0(\omega)$, we have that $\eta^{(j)}(\sigma^{-j}\omega) < \xi^j$, for all $j \geq j_0(\omega)$.

Let $\theta(\omega)$ be defined as

\[
\theta(\omega) := \max_{1 \leq j \leq j_0(\omega)} \left(\frac{\eta^{(j)}(\sigma^{-j}\omega)}{\xi^j}, 1\right),
\]

where $\xi > 1$ and $\xi$ is chosen such that $0 < \bar{\xi} < \xi < 1$. For large enough $j_0(\omega)$, we have that $\eta^{(j)}(\sigma^{-j}\omega) < \xi^j$, for all $j \geq j_0(\omega)$.
and hence for all $j$, we have that

$$
\eta^{(j)}(\omega) < \theta(\omega) \xi^j.
$$

Taking the sum over $j$, by (5.2), we get

$$
\|h_k^\omega\|_\alpha \leq \tilde{D}m(C)(1 + \theta(\omega) \sum_{j=1}^{\infty} \xi^j)
$$

$$
\leq \tilde{D}m(C)(1 + \theta(\omega) \sum_{j=1}^{\infty} \xi^j) = \tilde{D}m(C)(1 + \frac{\theta(\omega)}{1-\xi}),
$$

since $0 < \xi < 1$. Let

$$
\Theta(\omega) := \tilde{D}m(C)(1 + \frac{\theta(\omega)}{1-\xi}),
$$

then we have proven that for every $k \in \mathbb{N}$

$$
\|h_k^\omega\|_\alpha \leq \Theta(\omega).
$$

From this inequality, it follows that the sequence of averages $\{\|h_k^\omega\|_\alpha\}_{k \in \mathbb{N}}$ is bounded and hence the sequence $\{H_s^\omega\}_{s \in \mathbb{N}}$ too. Therefore, $\{H_s^\omega\}_{s \in \mathbb{N}}$ is relatively compact in $L^1$ by Lemma A.1 in [21]. This establishes (i). Then, the same argument in the proof of [3], we have $\{H_s^\omega\}_{s \in \mathbb{N}}$ converges in the strong sense to a random invariant density $h_\omega$, as in (5.1). The relative compactness of $V_\alpha$ in $L^1$ implies that $h_\omega \in V_\alpha$. This proves (ii).

**Remark 5.1.** For $\mathbb{P}$-a.e. $\omega \in \Omega$, let $\mu_\omega$ on the fiber $\{\omega\} \times C \subset \Omega \times C$, as

$$
d\mu_\omega = h_\omega, \tag{5.7}
$$

where $h_\omega$ is given by (5.11). Then, $\mu_\omega$ is a random invariant ACIP and the measure $\mu$ defined on $\mathbb{P} \times m$-measurable sets $A \subseteq \Omega \times C$ by

$$
\mu(A) = \int_\Omega \mu_\omega(A) d\mathbb{P}(\omega), \tag{5.8}
$$

is an ACIP for the associated skew product $F$ defined in (3.10).

For the rest of the paper, we assume

$$
\int_\Omega \log^+ \|\mathcal{L}_\omega\|_\alpha d\mathbb{P}(\omega) < \infty.
$$

By Corollary 4.4, random Saussol maps give rise to quasi-compact random dynamical systems with $\lambda_1 = 0$. Therefore, Theorem 17 in [14] implies the following.

**Corollary 5.1.** For $\mathbb{P}$-a.e. $\omega \in \Omega$, the random invariant density $h_\omega$ given in (5.11) belongs to the Oseledets space $Y_1(\omega)$ given in (2.5). Moreover, the number $r$ of ergodic skew product ACIPs $\mu_1, \ldots, \mu_r$ defined in Equation (5.8) is finite and

$$
r \leq d_1 = \dim(Y_1(\omega)). \tag{5.9}
$$
The proof of this corollary is the same as the proof of Corollary 4.6 in [3].

**Definition 5.2.** Consider the tuple \((\Omega, \mathcal{F}, \mathbb{P}, \sigma, T)\) where \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space, \(\sigma: \Omega \to \Omega\) an ergodic and invertible \(\mathbb{P}\)-preserving transformation and \(T = \{T_\omega: \omega \in \Omega\} \subseteq \mathbb{R}^n\). A probability measure \(\nu\) on \(C\) is called physical if for \(\mathbb{P}\)-a.e. \(\omega \in \Omega\), the Lebesgue measure of the random basin \(RB_\omega(\nu)\) of \(\nu\) at \(\omega\) is positive where

\[
RB_\omega(\nu) = \{x \in C : \frac{1}{s} \sum_{k=0}^{s-1} \delta_{T_\omega^k(x)} \to \nu \text{ as } s \to \infty\},
\]

where \(\delta_x\) is the Dirac measure at a point \(x\) and the convergence in (5.10) is in the weak convergence sense.

The next result due to Buzzi applies in our setting.

**Theorem 5.2 ([5]).** Let \(\mu\) be one of the measures \(\mu_i : i = 1, \ldots, r\) given in Corollary 5.1. Then, the marginal measure of \(\mu\) on \(C\), denoted by \(\nu\), is a physical measure on \(C\).

The union of all basins of the of the physical measures \(\nu_i\) coming from the marginals of \(\mu_i\) on \(C\), \(i = 1, \ldots, r\) has full Lebesgue measure, which means Lebesgue almost everywhere, the asymptotic long term behaviour of a full \(\mathbb{P}\)-measure set of random orbits will be described by these physical measures.

6. An upper bound on the number of ergodic skew product ACIPs

**Theorem 6.1.** Assume that \(m(C) \geq 1\). The number \(r\) of ergodic skew product ACIPs defined in Corollary 5.1 satisfies

\[
r \leq \frac{m(C)}{\gamma_N} E \inf_{\omega \in \Omega} \left( \frac{\Theta(\omega)}{\gamma} \right),
\]

where \(\Theta(\omega)\) is defined in (5.5) and \(\gamma_N\) is the volume of the \(N\)-dimensional unit ball.

**Proof.** For \(\mathbb{P}\)-a.e. \(\omega \in \Omega\), by (5.6), we know that \(\|h_\omega\|_{\gamma} \leq \Theta(\omega)\) and \(\int h_\omega dm = 1\). By Lemma 2.1, for \(\mathbb{P}\)-a.e. \(\omega \in \Omega\), the infimum of \(h_\omega\) is strictly positive on some ball of radius

\[
\min(\varepsilon_0, \left(\frac{1}{\Theta(\omega)}\right)^{\frac{1}{\gamma}}) = \left(\frac{1}{\Theta(\omega)}\right)^{\frac{1}{\gamma}}.
\]

This last step is justified as follows. By (5.5), we have

\[
\Theta(\omega) = Dm(C)(1 + \frac{\theta(\omega)}{1 - \xi}).
\]

By (5.3), we have \(\theta(\omega) \geq 1\) and \(m(C) \geq 1\) by assumption, therefore

\[
\Theta(\omega) \geq \frac{D}{1 - \xi}.
\]
In the proof of Corollary 4.1, $\tilde{D}$ is chosen such that $\tilde{D} \geq \xi \bar{\varepsilon}_0^{-\alpha}$, thus

$$\Theta(\omega) \geq \frac{\xi}{1 - \xi} \bar{\varepsilon}_0^{-\alpha}. $$

Note that $\xi$, introduced in the proof of Theorem 5.1, can be chosen such that $\xi > \frac{1}{2}$ (since otherwise, we can choose $\xi$ to be $1 - \xi$), thus

$$\left( \frac{1}{\Theta(\omega)} \right)^{\frac{1}{\alpha}} < \bar{\varepsilon}_0. $$

Since (6.2) is true for $P$-a.e. $\omega \in \Omega$, it follows that the number $r$ of ergodic skew product ACIPs is bounded above by the essential infimum of maximal number of balls of radius $\left( \frac{1}{\Theta(\omega)} \right)^{\frac{1}{\alpha}}$ contained in $C$, which is bounded by

$$m(C) \frac{E \inf_{\omega \in \Omega} \left( \Theta(\omega) \eta \right)^{\frac{1}{\alpha}}}{\gamma_N}. $$

\[\square\]

**Remark 6.1.** If the upper bound in (6.1) is strictly less than 2, then we get uniqueness of the number of ergodic skew product ACIPs. In such a case, one can use the results given in [12, 13] to investigate quenched limit theorems in this setting.

The next example shows how to verify (PE5) once the partition is finite and the boundaries of the $U_i$'s are piecewise smooth boundaries. This example is motivated from Lemma 2.1 in [22] adapted to our random setting.

**Example 6.1.** In Definition 3.1, suppose that $T$ satisfies (PE1) through (PE4) and $\mathcal{I}$ is finite such that the boundaries of the $U_{\omega,i}$'s are included in piecewise $C^1$ codimension one embedded compact submanifolds. Denote by

$$Y(\omega) := \sup_{x \in \mathbb{R}^N} \sum_{i \in I} \# \{ \text{smooth pieces intersecting } \partial U_{\omega,i} \text{ containing } x \} $$

and

$$\Lambda(\omega) := s(\omega)^\alpha + 4Y(\omega) \frac{\gamma_{N-1}}{\gamma_N} \frac{s(\omega)}{(S - s(\omega))^{-\alpha}}. \quad (6.3)$$

Suppose that there exists $\rho > 0$ such that $\int_{\Omega} \log(\Lambda(\omega)) d\mathbb{P}(\omega) < \rho$, then (PE5) holds. To see this, fix $\omega \in \Omega$, $i \in \mathcal{I}$, $\varepsilon \leq \varepsilon_0$ and $x \in \mathbb{R}^N$. By (PE1), we have

$$T^{-1}_{\omega,i} B_{\varepsilon}(\partial T_{\omega} U_{\omega,i}) \cap B_{(S - s(\omega))\varepsilon_0} (x) \subset B_{s(\omega)\varepsilon}(\partial U_{\omega,i}) \cap B_{(S - s(\omega))\varepsilon_0} (x).$$

By the assumption, we have

$$\partial U_{\omega,i} = \bigcup_{j \in \mathcal{J}_{\omega,i}} \Gamma_{\omega,i,j},$$

where $\mathcal{J}_{\omega,i}$ is a finite indexing set and $\Gamma_{\omega,i,j}$ is a compact $C^1$ embedded submanifold. Therefore, we have

$$T^{-1}_{\omega,i} B_{\varepsilon}(\partial T_{\omega} U_{\omega,i}) \cap B_{(S - s(\omega))\varepsilon_0} (x) \subset \bigcup_{j \in \mathcal{J}_{\omega,i}} B_{s(\omega)\varepsilon}(\Gamma_{\omega,i,j}) \cap B_{(S - s(\omega))\varepsilon_0} (x).$$
Arguing as in the proof of Lemma 2.1 in [22], for small \( \varepsilon \), we get
\[
m\left(B_{s(\omega)}\varepsilon(\Gamma_{\omega,i,j}) \cap B_{S-s(\omega)}\varepsilon_0(x)\right) \leq 2s(\omega)\varepsilon\gamma_{N-1}(S-s(\omega))^{N-1}(1+o(1)),
\]
which implies
\[
G_{\omega,\varepsilon_0}(\varepsilon) \leq 2Y(\omega)\frac{\gamma_{N-1}}{\gamma_N} \frac{s(\omega)\varepsilon}{(S-s(\omega))\varepsilon_0}(1+o(1)). \tag{6.4}
\]
Since the number of the \( \Gamma_{\omega,i,j} \)'s is finite, by taking \( \varepsilon_0 \) small enough, we get
\[
\int_{\Omega} \log(\zeta_{\varepsilon_0}(\omega))d\mathbb{P}(\omega) = \int_{\Omega} \log(s(\omega)^\alpha + 2\sup_{\varepsilon \leq \varepsilon_0} G_{\omega,\varepsilon_0}(\varepsilon)(S\varepsilon_0)^\alpha) d\mathbb{P}(\omega)
\leq \int_{\Omega} \log(s(\omega)^\alpha + 4Y(\omega)\frac{\gamma_{N-1}}{\gamma_N} \frac{s(\omega)}{S} S_\alpha) d\mathbb{P}(\omega)
= \int_{\Omega} \log(\Lambda(\omega))d\mathbb{P}(\omega) < \varrho.
\]

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