A Stable Pfaffian State in Bilayer Graphene

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Here we show that the Pfaffian state proposed for the \( \frac{5}{2} \) fractional quantum Hall states in conventional two-dimensional electron systems can be readily realized in a bilayer graphene at one of the Landau levels. The properties and stability of the Pfaffian state at this special Landau level strongly depend on the magnetic field strength. The graphene system shows a transition from the incompressible to a compressible state with increasing magnetic field. At a finite magnetic field of \( \sim 10 \) Tesla, the Pfaffian state in bilayer graphene becomes more stable than its counterpart in conventional electron systems.

Ever since the discovery of the quantum Hall effect at the Landau level filling factor \( \nu = \frac{1}{2} \), the first even-denominator state observed in a single-layer system, it has been very aptly characterized as an “enigma” [2]. It was clear at the outset that this state must be different from the fractional quantum Hall effect (FQHE) in predominantly odd-denominator filling fractions [3, 4]. Understanding this enigmatic state has been a major challenge in all these years [5]. At this half-filled first excited Landau level [9], a novel state described by a pair wave function involving a Pfaffian [7, 8] (or anti-Pfaffian [9]) has been the strongest candidate. More intriguing are the elementary charged excitations at this ground state that have a charge \( e^* = e/4 \) and obey ‘non-abelian’ statistics [13]. Recent observation of the charge \( e^* = e/4 \) quasiparticles at \( \nu = \frac{5}{2} \) quantum Hall state [12] has brought the issue to the fore [13]. It has been suggested that these non-abelian quasiparticles, besides carrying the signatures of Majorana fermions [14] in this system, might even be useful for quantum information storage and processing in an intrinsically fault-tolerant manner [15].

Electrons in another recently discovered two-dimensional system, graphene [16], display a range of truly remarkable behavior [17]. The dynamics of electrons in a single sheet of graphene, a hexagonal honeycombed lattice of carbon atoms is that of massless Dirac fermions with linear dispersion, chiral eigenstates, valley degeneracy, and unusual Landau levels in an external magnetic field [17]. Theoretical studies of FQHE in monolayer [18] and bilayer graphene [19] were reported earlier by us. Recent experimental observations of the \( \nu = \frac{1}{3} \) FQHE in monolayer graphene [20] have provided a glimpse of the role highly correlated electrons play in graphene. Given the acute interest in studying the properties of the \( \nu = \frac{5}{2} \) state in conventional two-dimensional electron gas (2DEG), a natural question to ask is how does this state manifests itself in graphene.

For the conventional (nonrelativistic) 2DEG the incompressible state at \( \nu = \frac{5}{2} \) has been studied numerically for a finite number of electrons [15]. A relatively good (but not 100%) overlap with the Pfaffian state has been found. The overlap of the exact wave function of the finite-size systems with the Pfaffian state can be improved by varying the inter-electron potential. For example, by increasing the thickness of the two-dimensional layer [21], one can improve the overlap with the Pfaffian state and increase the excitation gap of the corresponding incompressible state. The interaction properties of a two-dimensional system are determined by the Haldane pseudopotentials [22], which are the energies of two electrons with relative angular momentum \( m \). The pseudopotentials at the \( n \)-th Landau level are of the form

\[
V^{(n)}_m = \int_0^\infty \frac{q dq}{2\pi} V(q) [F_n(q)]^2 L_m(q^2)e^{-q^2},
\]

where \( L_m(x) \) are the Laguerre polynomials, \( V(q) = 2\pi e^2/(\kappa q_0 q) \) is the Coulomb interaction potential in the momentum space, \( \kappa \) is the dielectric constant, \( \epsilon_0 = (\hbar/cB)^{\frac{1}{2}} \) is the magnetic length, and \( F_n(q) \) are the form factors of the \( n \)-th Landau level.

Within the framework of the Haldane pseudopotentials it is convenient to study the finite-size system numerically in the spherical geometry. The size of the sphere and the number of single-particle states are determined by the parameter \( S \), where \( 2S \) is the number of magnetic fluxes through the sphere in units of the flux quanta. The single-electron states are characterized by the angular momentum \( S \) and its \( z \) component \( S_z \). For the many-electron system the corresponding states are classified by the total angular momentum \( L \) and its \( z \) component [23].

For a system with \( N \) electrons the \( \nu = \frac{1}{2} \) Pfaffian state is realized at \( 2S = 2N - 3 \). Here the filling factor \( \nu = \frac{1}{2} \) is defined as the filling factor of a given Landau level. In spherical geometry the \( \nu = \frac{1}{2} \) Pfaffian state is the exact ground state only for a very special type of three-particle interaction [8] when the three-particle interaction potential is non-zero only if the total angular momentum of three particles is \( 3S - 3 \). For any two-particle interaction the \( \nu = \frac{1}{2} \) Pfaffian state is not an exact eigenstate, which makes it impossible to continuously connect the Pfaffian state to any exact eigenstate of the two-particle Hamiltonian. By varying the interaction function, i.e.,
the pseudopotentials, the close proximity to the Pfaffian function with an overlap of 99% can be achieved. The \( \nu = \frac{1}{2} \) Pfaffian state is most sensitive to the lowest pseudopotentials, \( V_1, V_3, \) and \( V_5. \)

For a single graphene layer the Landau level wave functions are mixtures of those for Landau levels of nonrelativistic systems; for example the first Landau level in graphene can be expressed in terms of zero and the first Landau wave functions of the nonrelativistic system \([18]\). As a result the form-factor in a single graphene layer takes the form \( F_n(q) = [L_n(q^2/2) + L_{n-1}(q^2/2)]/2^\pi \) for \( n \geq 1 \) and \( F_{n=0} = L_0(q^2/2) \) \([18]\). Numerical analysis of finite-size systems in a spherical geometry with up to 14 electrons shows that the largest excitation gap around 0.02 \( e^2/(\hbar c_0) \) occurs at the \( n = 2 \) graphene Landau level \([24]\). Although the excitation gap at a finite size system in this case is comparable to the \( \nu = \frac{1}{2} \) nonrelativistic system the overlap of the ground state with the Pfaffian state is less than 0.5 at all Landau levels \([24]\). This fact shows that a single graphene layer does not have stable incompressible \( \nu = \frac{1}{2} \) Pfaffian states. Modification of the interaction potential can improve the formation of the \( \nu = \frac{1}{2} \) incompressible Pfaffian state. One such modification can be achieved in a bilayer graphene.

We show here that bilayer graphene can indeed improve the stability of the \( \nu = \frac{1}{2} \) Pfaffian state in graphene. Namely, one of the bilayer Landau levels (for a given valley) has a stable \( \nu = \frac{1}{2} \) Pfaffian state, the properties of which can be controlled by a magnetic field. The maximum overlap of the finite system ground state with the corresponding Pfaffian state occurs at finite values of the magnetic field. The \( \nu = \frac{1}{2} \) incompressible state of a bilayer graphene is more stable than the corresponding state in a conventional two-dimensional system.

We consider a bilayer graphene which consists of two coupled graphene layers with the Bernal stacking arrangement. Each graphene layer has two sublattices, say, \( A \) and \( B \). For the Bernal stacking arrangement, the coupling is mainly between the atoms of sublattice \( A \) of the lower layer and atoms of sublattice \( B' \) of the upper layer. For one projection of spin, e.g., \( +\frac{1}{2} \), the state of the bilayer graphene can be expressed in terms of the four-component spinor \( (\psi_A, \psi_B, \psi_B', \psi_A')^T \) for valley \( K \) and \( (\psi_B', \psi_A', \psi_A, \psi_B)^T \) for valley \( K' \). The subindices \( A, B \) and \( A', B' \) correspond to lower and upper layers respectively. The properties of bilayer graphene can be controlled by a bias voltage, \( \Delta U \), which is the potential difference between the upper and lower layers. The Hamiltonian of the biased bilayer system in a perpendicular magnetic field has the form \([25]\):

\[
H = \xi \begin{pmatrix}
\Delta U/2 & v_F v_\pi & \xi t & 0 \\
v_F v_\pi & \Delta U/2 & 0 & 0 \\
\xi t & 0 & -\Delta U/2 & v_F v_\pi \\
0 & 0 & v_F v_\pi & -\Delta U/2
\end{pmatrix},
\]

(1)

where \( t \) is the inter-layer hopping integral, \( \pi_\pm = \pi_x \pm i \pi_y, \)

\( \bar{\pi} = \bar{p} + e\vec{A}/c, \)

\( \bar{p} \) is an electron two-dimensional momentum, \( \vec{A} \) is the vector potential, \( v_F, \approx 10^6 \) m/s is the fermi velocity, and \( \xi = + (K \text{ valley}) \) or \( - (K' \text{ valley}). \)

The discrete eigenstates of the Hamiltonian (1) can be found from the following equation \([25]\):

\[
(\epsilon + \xi \delta)^2 - 2(n+1)(\epsilon - \xi \delta)^2 - 2n = (\epsilon^2 - \delta^2)t^2,
\]

(2)

where \( \delta = \Delta U/2 \) and all energies are expressed in units of \( \hbar v_F/\epsilon_0 \). For a given value of \( n \) there are four bilayer Landau levels which are characterized by the index \( n \) and the energy \( \epsilon \) of the level. The corresponding wave functions can be expressed in terms of \( n, |n-1|, \) and \( n+1 \) conventional Landau wave functions \([25]\). The resulting form factors \( F_n,\epsilon(q) \) were derived in \([18]\). The form factors and the corresponding pseudopotentials allow us to find the energy spectrum of a finite \( N \) electron system in the spherical geometry \([19, 22]\). We
report our calculations for $N = 8, 10,$ and 14 electron systems. To determine the incompressibility of the system we calculated the excitation gap and the overlap of the ground state wave function with the Pfaffian function. We consider only one valley, for example, valley K. The results are similar for $K'$. 

For all but one bilayer Landau levels the overlap of the $\nu = \frac{1}{2}$ ground state with the Pfaffian state is found to be small ($< 0.6$). At the same time there is one special Landau level (for each valley) at which the $\nu = \frac{1}{2}$ ground state is well described by the Pfaffian function. This level corresponds to one of the solutions of Eq. (2) with $n = 0$. At a small bias voltage, $\Delta U,$ the wave function corresponding to this Landau level is the mixture of the conventional Landau level wave functions with indices 0 and 1. The wave functions of this special level have the form of $\langle \phi_0, 0, 0, (t/\sqrt{2})\phi_1 \rangle$, where $\phi_n$ are $n$-th ‘nonrelativistic’ Landau functions and $t$ is in units of $\hbar v_F/\ell_0$. Then the corresponding form factor is $F(q) = (L_0 + (t^2/2)L_1)/(1 + t^2/2)$. At small values of the dimensionless hopping integral, $t(\ell_0/\hbar v_F)$, the interaction within this level is similar to the one at the lowest Landau level of a conventional system, which does not show an incompressible $\nu = \frac{1}{2}$ state. At large values of $t(\ell_0/\hbar v_F)$ the special bilayer Landau level is similar to the $n = 1$ Landau level of the conventional system and shows the $\nu = \frac{1}{2}$ Pfaffian state. By varying the magnetic field, the dimensionless inter-layer hopping integral is changed which modify the interaction within the Landau level and changes the properties of the $\nu = \frac{1}{2}$ state. We present the numerical results only for this special bilayer Landau level.

At the zero bias voltage this special Landau level has zero energy and is degenerate with another level, which has the form $\langle 0, 0, 0, \phi_0 \rangle$. In addition to this accidental degeneracy, each level has two-fold valley degeneracy, which make the zero energy state four-fold degenerate. At a finite bias voltage this degeneracy is completely lifted and the special Landau level of the bilayer can be isolated. In Fig. 1(a) we show the lowest Landau levels of a bilayer at finite bias voltage. Two solid red lines correspond to the special Landau levels of the two val-

leyes. The many-particles properties of these two levels are identical. Therefore, we study the $\nu = \frac{1}{2}$ state for only one valley.

In Fig. 1(b,c) we show the magnetic field dependence of the overlap of the $\nu = \frac{1}{2}$ ground state with the Pfaffian state and the corresponding excitation gap. At a small magnetic field the dimensionless hopping integral is large and the system becomes similar to the conventional system at the $n = 1$ Landau level. With increasing magnetic field the properties of the system changes non-monotonically and the overlap with the Pfaffian state reaches its maximum at a magnetic field of $\sim 10$ Tesla (and for $t = 400$ meV). The overlap at this point is $\approx 0.92$, which is a big improvement over the nonrelativistic system ($\sim 0.75$). The dimensionless hopping integral at this point is $t(\ell_0/\hbar v_F) \approx 4.89$.

At a large magnetic field the system is close to the $n = 0$ nonrelativistic Landau level, the overlap with the Pfaffian state becomes small and the $\nu = \frac{1}{2}$ state is finally compressible. This dependence on the magnetic field opens up interesting possibilities to investigate the stability and appearance and disappearance of the $\nu = \frac{1}{2}$ Pfaffian state in a single bilayer Landau level. Although the Pfaffian state becomes unstable only at large magnetic fields, this property strongly depends on the value of the hopping integral. At smaller hopping integrals the magnetic field range of stability of the Pfaffian state...
shrinks. For example, at $t = 300$ meV the Pfaffian state is expected to be unstable at $B \sim 40$ Tesla (see Fig. 1). Another parameter which controls the properties of the graphene bilayer is the bias voltage. Although the bias voltage modifies the bilayer wave functions, we found that the overlap of the ground state with the Pfaffian state and the excitation gap have weak dependence on the bias voltage within a broad range of $\Delta U$ (see Fig. 2). The overlap monotonically decreases with increasing $\Delta U$, which suppresses the overlap by only a few percent. The large excitation gap and the large overlap observed for different system sizes are shown in Fig. 3.

The stability and the strength of the Pfaffian state can be also analyzed in terms of the general dependence of the pseudopotentials, $V_m$, on the relative angular momentum, $m$. We characterize the interaction potential of the bilayer graphene by two parameters: $V_1/V_2$ and $V_2/V_3$. These parameters depend on the magnetic field. By varying the magnetic field, this dependence can be shown as a line in the $(V_1/V_2)-(V_2/V_3)$ plane (Fig. 4). That line connects the initial point at $B = 0$ to the final point, corresponding to large magnetic field, $B = \infty$. The $\nu = \frac{1}{2}$ bilayer graphene system at the initial and final points are identical to the conventional systems at the first ($n = 1$) and zero ($n = 0$) Landau levels, respectively. In Ref. [13] the region of the compressible $\nu = \frac{1}{2}$ state and the region of strong overlap with the Pfaffian state were identified (see Fig. 4). With increasing magnetic field, the $\nu = \frac{1}{2}$ bilayer graphene system transforms from a $\nu = \frac{1}{2}$ nonrelativistic state (at small values of $B$) to a more stable incompressible state with large overlap, and finally to a compressible state (at a large magnetic field). For the hopping integral $t = 400$ meV, the transition from the incompressible to a compressible $\nu = \frac{1}{2}$ state occurs at $B \sim 100$ Tesla.

In conclusion, a stable incompressible $\nu = \frac{1}{2}$ Pfaffian state can in fact be observed in a bilayer graphene only at one Landau level. The properties of this state strongly depend on the value of the magnetic field. With an increasing magnetic field, the $\nu = \frac{1}{2}$ state transforms from an incompressible state at a small magnetic field to a compressible state at a large magnetic field. At intermediate values of the magnetic field, $B \sim 10$ Tesla, the $\nu = \frac{1}{2}$ state becomes more stable than the corresponding state in a conventional two-dimensional electron system.

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