SHORTCUT METHOD OF SOLUTION OF GEODESIC EQUATIONS FOR SCHWARZSCHILD BLACK HOLE

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Abstract
It is shown how the use of the Kerr-Schild coordinate system can greatly simplify the formulation of the geodesic equation of the Schwarzschild solution. An application of this formulation to the numerical computation of the aspect of a non-rotating black hole is presented. The generalization to the case of the Kerr solution is presented too.

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1 Introduction.

Since the publication of the Schwarzschild static solution \[1\], it has been well known that the geodesic equation in this space-time can be solved analytically. The purpose of this note is to show that, by the mean of the Eddington coordinate system, the equation of motion of test particles in the Schwarzschild space-time can be greatly simplified so as to provide a more efficient method of solution, using ideas suggested by analysis of the less simple case of the Kerr rotating black hole solution.

The forms of the spherical Schwarzschild metric as expressed in term of the usual Schwarzschild coordinate system and in term of the less usual Eddington coordinate system (which is interpretable as a limiting case of the Kerr-Schild coordinate system for a rotating black hole) are presented in § 2. Taking advantage of the obvious constants of motion which appear in the Schwarzschild space-time, we write down the geodesic equation in term of the Eddington coordinate system as second order equations with respect to some affine parameter.

These results will be useful for many problems for which geodesic motion is concerned. In particular, they can greatly simplify the numerical integration of the test particles trajectory. As an application, we present in § 3 the apparent shape of a non-rotating Black Hole surrounded by an accretion disc as seen by an observer who is travelling towards and ultimately through the event horizon.

2 The Schwarzschild metric and the geodesic equation.

Using the standard Schwarzschild coordinate system \((t, r, \theta, \varphi)\), the line element of the metric takes the form

\[
ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{1}{(1 - \frac{2M}{r})}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \tag{1}
\]

in units such that \(c = G = 1\), where \(M\) is the mass of the Black Hole.

The first integrals of the equations of motion are well known to be expressible as

\[
\dot{t} = \frac{E}{(1 - \frac{2M}{r})}, \quad \dot{r} = \frac{E^2 - (r^2 - 2Mr)(\mu^2 r^2 + K)}{r^2 \sin^2 \theta}, \quad \dot{\theta} = \frac{K - \Phi}{r^2 \sin^2 \theta}, \quad \dot{\varphi} = \frac{\Phi}{r^2 \sin^2 \theta} \tag{2}
\]

where a dot denotes differentiation with respect to some affine parameter, \(\tau\) say, and where the constants of motion \(\mu^2\), \(E\) and \(\Phi\) are respectively the rest-mass, the energy and the angular momentum about the axis \(\sin \theta = 0\) of the test-particle, \(K\) being the Carter’s fourth constant of motion \[3\] which reduces in this simple spherical case to the square of the total angular momentum of the orbiting particle.

Introducing the Eddington coordinate system \((T, x, y, z)\), i.e. the Kerr-Schild coordinate system for the limiting case of a non rotating black hole, where \((x, y, z)\) are “cartesian-like” spatial coordinates and where \(T\) is a retarded time, which are related to the Schwarzschild coordinates by means of
\begin{align*}
x &= r \sin \theta \cos \varphi; \quad y = r \sin \theta \sin \varphi; \quad z = r \cos \theta \\
\text{and} \quad T &= u - r \quad ; \quad du = dt + \frac{1}{1 - \frac{2M}{r}} dr ,
\end{align*}

the metric takes the (Kerr-Schild) form

\[ ds^2 = -dT^2 + dx^2 + dy^2 + dz^2 + \frac{2M}{r^3} (xdx + ydy + zdz + rdT)^2 \]

where

\[ r^2 = x^2 + y^2 + z^2 . \]

At this stage, it is to be noticed that the above coordinate transformation (first published by Eddington in 1924 [4]) solves the “Schwarzschild singularity” \((r = 2M)\) problem in the sense of being regular across the event horizon \(r = 2M\).

In terms of this system, it is straightforward to show that the geodesic equations reduce to

\begin{align*}
\ddot{T} &= \frac{2MK}{r^3 (E - \dot{r})} + E \\
\ddot{x}^i &= -3KM \frac{x^i}{r^3} - \mu^2 M \frac{x^i}{r^3}
\end{align*}

where \(x^i := (x, y, z)\).

Keeping in mind that \(|\dot{r}| = E\) when any test-particle reaches the event horizon, the above equations show clearly the irreversible nature of the Schwarzschild Black Hole. Moreover, unlike what is obtained in terms of the Schwarzschild coordinates system, for which one has to take care of the axis \(\sin \theta = 0\), of the sign of \(\dot{r}\) and \(\dot{\theta}\) and of the presence of the event horizon, such differential equations can be numerically solved in a straightforward way in order to determine the path of any test-particle.

### 3 Apparent shape of a non-rotating Black Hole.

Accretion disc are currently supposed to play an important role in several astrophysical situations, especially when high-energy phenomena are involved. Because of the astrophysical interest of such objects, several authors have computed the apparent shape of of black hole with thin accretion disc as seen from infinity ([1] [3] [9]). The main purpose of these calculation was to simulate line profiles which are commonly observed from astrophysical sources which are generally interpreted as emission from an accretion disc around a compact object which may be a black hole. Our present purpose is quite different in the sense that, as an application of the previous formulæ, we will present the apparent shape of a black hole surrounded by a thin accretion disc as seen by an observer who is flying near the hole.

We will assume that the disc is a stationary Keplerian one, orbiting in the plane \(z = 0\). Each particle of the disc then follows a circular orbit with angular velocity
\[ \Omega := \frac{d\varphi}{dt} = \sqrt{\frac{M}{r^3}} \] (9)

Following Page and Thorne [10], the flux of radiation from the surface of the disc is given by

\[
F_e = \frac{3M \dot{M}}{8\pi (\rho^2 - 3)\rho^3} \left\{ \frac{\rho - \sqrt{6} + \sqrt{3}}{2} \log \left( \frac{3 - 2\sqrt{2}}{\rho - \sqrt{3}} \right) \right\} \] (10)

where \( \dot{M} \) is the accretion rate and where we have introduced

\[ \rho = \sqrt{\frac{r}{M}}. \] (11)

As shown by Ellis [5], the observed bolometric flux \( F_o \) is given by

\[ F_o = \frac{F_e}{(1 + z)^4} \] (12)

where the redshift factor \((1 + z)\) is given by the ratio of the two scalar products

\[ (1 + z) = \frac{p_\alpha u_\alpha|_{\text{emission}}}{p_\alpha u_\alpha|_{\text{reception}}} \], (13)

\( p, w \) and \( u \) being the four-velocities of the photon, the emitting particle and the observer respectively. This redshift factor consists of a gravitational part due to the gravitational field of the black hole (which is measurable only in the closed vicinity of the hole), a Doppler part due to the rotation of the disc (the dominant one) and a Doppler part due to the motion of the observer.

It is now straightforward to compute how the black hole will look. Assuming that the observer takes a camera with him, the coordinates of each pixel of the plate of the camera allows to compute the components of the four-velocity of the photon that reaches that pixel (recall that the four-velocity of a photon has only two true degrees of freedom). The four velocity of each photon reaching the eyes of the observer can be obtained in terms of an orthonormal parallel-propagated frame \((\lambda_0, \lambda_1, \lambda_2, \lambda_3)\) along the observer’s trajectory as

\[ p^\mu = \lambda^\mu_0 + \cos \vartheta \lambda^\mu_2 + \sin \vartheta \cos \varphi \lambda^\mu_1 + \sin \vartheta \sin \varphi \lambda^\mu_3, \] (14)

where \( \vartheta \) and \( \varphi \) are two angles describing the photographic plate in the rest frame of the observer.

A direct numerical integration of equations (6) (8) in the special case \( \mu^2 = 0 \) for negative values of the affine parameter \( \tau \) gives the history of the photon and then the luminosity of the source of its origin. Finally, application of the correction factor \((1 + z)^4\) as given by equations (12) and (13) gives the brightness of this fiducial pixel.

The figures represent eight simulated photographs obtained by this procedure. These “snapshots” have been computed at successive steps during the flight of an observer who is on free fall along a parabolic orbit \((\mu^2 = 1, E = 1, K = 12)\) of the Schwarzschild space-time (see figure 1). On the first snapshot, the observer is above the disc at a distant \( r = 1280M \). The observer then crosses the disc (snapshot # 4, \( r = 39M \)) and, finally, goes into the black hole (snapshot #8, \( r = 0.7M \)). During this trip, the observer directs his camera in the direction of his motion except on the last snapshot for which the observer directs his camera toward the exterior of the hole.

The brightness of the pictures has been computed by means of formulae (10) and (12) while the coloration has been added artificially. The apparent position of distant stars has also been computed.
This helps to show the gravitational lens effect which is more conspicuous on snapshot #2 ($r = 121M$): some distant stars, which are more or less perfectly aligned with the observer–black hole system, appear as pieces of rings in the sky. The opacity of the disc has been extremely minimized in order to make more visible the second (and even third) image of the disc and the distant stars. Moreover, this allows to show both the appearance of the upper and down sides of the disc.

The trajectories of the photons and the link between these trajectories and the apparent shape of the hole as seen by the observer are schematically given in figure 2. Examination of figure 2 helps to understand the apparent shape of the disc. Some of the photons emitted by the disc are strongly deflected by the gravitational field (not the curvature) before reaching the eyes of the observer. Hence, the observer is able to see the upper rear side of the disc which appears like surrounding the top of the black hole (primary image on figure 2). In the same way, the observer can see the down rear side of the disc which seems to surround the bottom of the black hole. Now, some photons whose impact parameter is very close to the capturing one make one (even two) turn(s) around the black hole before reaching the eyes of the observer. This leads to the formation of a second (third) image of the disc.

The numerical scheme used in these calculation is of second order. The integration has been done where respect to the parameter $\lambda$ which is related to the affine parameter by

$$d\lambda = \frac{d\tau}{r^2}$$

and which is well suited to the form of equations (7–8) in the sense that it exploits the fact that, far from the hole, null geodesics are straight lines. The code is fully parallelized. Vectorisation is also possible but less efficient since the number and the kind of operations needed to compute the value of each pixel of the screen vary from a pixel to another one.
Figure 2: schematical representation of the photon trajectories.

4 Conclusion.

We have shown that the use of the Eddington coordinates system in the Schwarzschild space-time facilitates the description of the geodesical motion in that space-time. This formulation is well adapted to direct numerical integration because it avoid the usual troubles tied to the spherical-type coordinates system and the trouble tied to the pseudo-singularity \( r = 2M \).

Such a formulation can be extended to the non-spherical case of the Kerr black hole solution. One can show that, in the Kerr-Schild coordinate system, the equations of motion of a test particle of zero rest-mass can be written in the form:

\[
\Sigma^3 \frac{\ddot{x}}{M} = -4ar \frac{Q}{\Delta} \Sigma \dot{y} \\
+ (\Sigma - 4r^2) \sin \theta \cos \psi \left\{ K - \left( a \frac{Q}{\Delta} \right)^2 \right\} \\
- ar \sin \theta \sin \psi \frac{Q}{\Delta} \left\{ 4(E\Sigma - Q) + (4a^2 - \Sigma)\frac{Q}{\Delta} \right\}
\]

\[
\Sigma^3 \frac{\ddot{y}}{M} = +4ar \frac{Q}{\Delta} \Sigma \ddot{x} \\
+ (\Sigma - 4r^2) \sin \theta \sin \psi \left\{ K - \left( a \frac{Q}{\Delta} \right)^2 \right\}
\]

(16) (17)
\[ +ar \sin \theta \cos \psi \frac{Q}{\Delta} \left\{ 4(E\Sigma - Q) + (4a^2 - \Sigma)\frac{Q}{\Delta} \right\} \]

\[ \Sigma^3 \frac{\ddot{z}}{M} = -K \cos \theta (3r^2 - a^2 \cos^2 \theta) \quad (18) \]

\[ \dot{T} = \frac{2MKr}{\Sigma(E - \Sigma r)} + E \quad (19) \]

where we have introduced the quantities

\[ \Sigma = r^2 + a^2 \cos^2 \theta \quad (20) \]
\[ \mathcal{E} = E(r^2 + a^2)^2 - a\Phi \quad (21) \]
\[ Q = \Sigma \dot{r} + \mathcal{E} \quad (22) \]

These results have been used in the study of profiles and shifts of lines emitted from Keplerian accretion disc around in X-ray binaries containing a neutron star or a black hole \[\text{[6]}\] and to the study of microlensing effects in active galactic nuclei \[\text{[7]}\].

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