Probing the cosmological singularity with a particle

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Abstract

We examine the transition of a particle across the singularity of the compactified Milne (CM) space. Quantization of the phase space of a particle and testing the quantum stability of its dynamics are consistent to one another. One type of transition of a quantum particle is described by a quantum state that is continuous at the singularity. It indicates the existence of a deterministic link between the propagation of a particle before and after crossing the singularity. Regularization of the CM space leads to the dynamics similar to the dynamics in the de Sitter space. The CM space is a promising model to describe the cosmological singularity deserving further investigation by making use of strings and membranes.

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1. Introduction

Presently available cosmological data indicate that known forms of energy and matter comprise only 4% of the makeup of the Universe. The remaining 96% is unknown, called ‘dark’, but its existence is needed to explain the evolution of the Universe [1, 2]. The dark matter, DM, contributes 22% of the mean density. It is introduced to explain the observed dynamics of galaxies and clusters of galaxies. The dark energy, DE, comprises 74% of the density and is responsible for the observed accelerating expansion. These data mean that we know almost nothing about the dominant components of the Universe!

Understanding the nature and the abundance of the DE and DM within the standard model of cosmology has difficulties [3, 4]. These difficulties have led many physicists to seek anthropic explanations which, unfortunately, have little predictive power. An alternative model has been proposed by Steinhardt and Turok (ST) [5–7]. It is based on the idea of a cyclic evolution, CE, of the Universe. The ST model has been inspired by string/M theories [8]. In its simplest version it assumes that the spacetime can be modelled by the higher dimensional compactified Milne, CM, space. The attraction of the ST model is that it potentially provides...
a complete scenario of the evolution of the universe, one in which the DE and DM play a key role in both the past and the future. The ST model requires DE for its consistency, whereas in the standard model, DE is introduced in a totally ad hoc manner. Demerits of the ST model are extensively discussed in [9]. Response to the criticisms of [9] can be found in [3].

The mathematical structure and self-consistency of the ST model has yet not been fully tested and understood. Such task presents a serious mathematical challenge. It is the subject of our research programme.

The CE model has in each of its cycles a quantum phase including the cosmological singularity, CS. The CS plays a key role because it joins each two consecutive classical phases. Understanding the nature of the CS has primary importance for the CE model. Each CS consists of contraction and expansion phases. A physically correct model of the CS, within the framework of string/M theory, should be able to describe propagation of a p-brane, i.e. an elementary object like a particle, string and membrane, from the pre-singularity to post-singularity epoch. This is the most elementary and fundamental criterion that should be satisfied. It presents a new criterion for testing the CE model. Hitherto, most research has focused on the evolution of scalar perturbations through the CS.

Successful quantization of the dynamics of p-brane will mean that the CM space is a promising candidate to model the evolution of the Universe at the cosmological singularity. Thus, it could be further used in advanced numerical calculations to explain the data of observational cosmology. Failure in quantization may mean that the CS should be modelled by a spacetime more sophisticated than the CM space.

Preliminary insight into the problem has already been achieved by studying classical and quantum dynamics of a test particle in the two-dimensional CM space [10]. The present paper is a continuation of [10] and it addresses the two issues: the Cauchy problem at the CS and the stability problem in the propagation of a particle across the CS. Both issues concern the nature of the CS.

In section 2 we define and make comparison of the two models of the universe: the CM space and the regularized CM space. The classical dynamics of a particle in both spaces is presented in section 3. The quantization of the phase space of a particle is carried out in section 4. In section 5 we examine the stability problem of particle’s dynamics both at classical and quantum levels. We summarize our results, conclude and suggest next steps in section 6.

2. Spacetimes

2.1. The CM space

For completeness, we recall the definition of the CM space used in [10]. It can be specified by the following isometric embedding of the 2D CM space into the 3D Minkowski space

\[
y^0(t, \theta) = t \sqrt{1 + r^2}, \quad y^1(t, \theta) = rt \sin(\theta/r), \quad y^2(t, \theta) = rt \cos(\theta/r),
\]

(1)

where \((t, \theta) \in \mathbb{R}^1 \times S^1\) and \(0 < r \in \mathbb{R}^1\) is a constant labelling compactifications. One has

\[
\frac{r^2}{1 + r^2} (y^0)^2 - (y^1)^2 - (y^2)^2 = 0.
\]

(2)

Equation (2) presents two cones with a common vertex at \((y^0, y^1, y^2) = (0, 0, 0)\). The induced metric on (2) reads

\[
dx^2 = -dt^2 + t^2 d\theta^2.
\]

(3)

Generalization of the 2D CM space to the ND spacetime has the form

\[
dx^2 = -dt^2 + dx^k dx_k + t^2 d\theta^2,
\]

(4)
where \( t, x^k \in \mathbb{R}^1, \theta \in S^1 (k = 1, \ldots, N - 2) \). One term in the metric (4) disappears/appears at \( t = 0 \), thus the CM space may be used to model the big-crunch/big-bang type singularity [8]. In what follows we restrict our considerations to the 2D CM space. Later, we make comments concerning generalizations.

It is clear that the CM space is locally isometric to the Minkowski space at each point except the vertex \( t = 0 \). The CM space is not a manifold, but an orbifold due to this vertex. The Riemann tensor components vanish for \( t \neq 0 \) and cannot be defined at \( t = 0 \), since one dimension disappears/appears there. There is a spacelike singularity at \( t = 0 \) of removable type because any timelike geodesic with \( t < 0 \) can be extended to some timelike geodesic with \( t > 0 \). However, such an extension cannot be unique due to the Cauchy problem for the geodesic equation at the vertex (compact dimension shrinks away and reappears at \( t = 0 \)).

### 2.2. The RCM space

Since trajectory of a test particle coincides (by definition) with timelike geodesic, there is no obstacle for the test particle to reach and leave the CS. However, the Cauchy problem for a geodesic equation at the CS is not well defined. As the result, a test particle ‘does not know where to go’ at the singularity. Thus, the singularity acts as ‘generator’ of uncertainty in the propagation of a test particle from the pre-singularity to post-singularity era. In the present paper we propose to solve this problem by replacement of a test particle by a physical one.

The test and physical particles differ in a number of ways. For instance, physical particle’s own gravitational field effects its motion [11] and may modify the singularity of the CM space. We assume that these effects may be modelled by replacing the CM space by a regularized compactified Milne, RCM space in such a way that the big-crunch/big-bang type singularity of the CM space is replaced by the big-bounce type singularity. In the RCM space the Cauchy problem does not occur because compact space dimension does not contract to a point, but to some ‘small’ value. As the result the propagation of a particle is uniquely defined in the entire spacetime. Particle’s propagation in the RCM space is similar to the corresponding one in the de Sitter space [12, 13].

We define the RCM space by the following embedding into the 3D Minkowski space

\[
y^0(t, \theta) = t \sqrt{1 + r^2}, \quad y^1(t, \theta) = r \sqrt{t^2 + \epsilon^2 \sin(\theta/r)}, \quad y^2(t, \theta) = r \sqrt{t^2 + \epsilon^2 \cos(\theta/r)},
\]

(5)

and we have the relation

\[
\frac{r^2}{1 + r^2} (y^0)^2 - (y^1)^2 - (y^2)^2 = -\epsilon^2 r^2.
\]

(6)

The induced metric on the RCM space reads

\[
d\hat{s}^2 = \left( \frac{1 + r^2 \epsilon^2}{r^2 + \epsilon^2} \right) dt^2 + (t^2 + \epsilon^2) d\theta^2,
\]

(7)

where \( \epsilon \in \mathbb{R} \) is a small number. It is clear that now the space dimension \( \theta \) does not shrink to zero at \( t = 0 \). The scalar curvature has the form

\[
R_\epsilon = \frac{2 \epsilon^2 (1 + r^2)}{(1 + r^2 + \epsilon^2)^2}
\]

(8)

and the Einstein tensor corresponding to the metric (7) is zero, thus (7) defines some vacuum solution to the 2D Einstein equation.

It is evident that at \( t \neq 0 \) we have

\[
\lim_{\epsilon \to 0} d\hat{s}_\epsilon^2 = -dr^2 + t^2 \ d\theta^2 \quad \text{and} \quad \lim_{\epsilon \to 0} R_\epsilon = 0.
\]

(9)

It is obvious that (6) turns into (2) as \( \epsilon \to 0 \).
Figure 1. Embeddings of RCM and CM spaces.

Figure 1 presents the RCM and CM spaces embedded into the 3D Minkowski space. We can see that the big-crunch/big-bang singularity of the CM space is represented in the RCM by the big-bounce type singularity.

3. Classical dynamics

An action integral, \( A \), describing a relativistic test particle of mass \( m \) in a gravitational field \( g_{kl} \), \((k, l = 0, 1)\) may be defined by

\[
A = \int d\tau \, L(\tau), \quad L(\tau) := \frac{m}{2} \left( \frac{\dot{x}^k \dot{x}^l}{e} g_{kl} - e \right),
\]

where \( \dot{x}^k := dx^k/d\tau \) is an evolution parameter, \( e(\tau) \) denotes the ‘einbein’ on the worldline, \( x^0 \) and \( x^1 \) are time and space coordinates, respectively. In case of the CM and RCM spaces the Lagrangian \( L_\epsilon \) reads

\[
L_\epsilon(\tau) = \frac{m^2}{2e} \left( \frac{\dot{t}^2 + \epsilon^2}{t^2 + \epsilon^2} \dot{\theta}^2 - \left( 1 + \frac{r^2 \epsilon^2}{t^2 + \epsilon^2} \right) \dot{t}^2 - e^2 \right),
\]

where \( \epsilon = 0 \) corresponds to the CM space. The action (10) is invariant under reparametrization with respect to \( \tau \). This gauge symmetry leads to the constraint

\[
\Phi_\epsilon := p_t p_t g^{\epsilon\epsilon} + m^2 = \frac{p_t^2}{(t^2 + \epsilon^2)} - \frac{p_\theta^2}{1 + \frac{r^2 \epsilon^2}{t^2 + \epsilon^2}} + m^2 = 0,
\]

where \( p_t := \partial L_\epsilon / \partial \dot{t} \) and \( p_\theta := \partial L_\epsilon / \partial \dot{\theta} \) are canonical momenta, and where \( g^{\epsilon\epsilon} \) denotes an inverse of the metric \( g_{\epsilon\epsilon} \) defined by the line element (7) (case \( \epsilon = 0 \) corresponds to the CM space).

Variational principle applied to (10) gives equations of motion of a particle

\[
\frac{d}{d\tau} p_\theta = 0, \quad \frac{d}{d\tau} p_t - \frac{\partial L}{\partial \dot{t}} = 0, \quad \frac{\partial L}{\partial e} = 0.
\]

Since during evolution of the system \( p_\theta \) is conserved, due to (13), we can analyse the behaviour of \( p_t \) by making use of the constraint (12). In case of the CM space (\( \epsilon = 0 \)), for \( p_\theta \neq 0 \) there must be \( p_t \rightarrow \infty \) as \( t \rightarrow 0 \). This problem cannot be avoided by different choice of coordinates\(^1\). It is connected with the vanishing/appearance of the space dimension \( \theta \) at

\(^1\) The system of coordinates we use, \((t, \theta) \in \mathbb{R}^1 \times S^1\), is natural for the spacetimes with the topologies presented in figure 1.
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$t = 0$. Another interpretation of this problem is that different geodesics cross each other with the relative speed reaching the speed of light as they approach the singularity at $t = 0$.

The dynamics of a physical particle in the RCM space ($\epsilon \neq 0$) does not suffer from such a problem, since for $p_\theta \neq 0$ the momentum component $p_t$ does not need to ‘blow up’ to satisfy (12).

3.1. Geodesics in CM and RCM spaces

It was found in [10] an analytic general solution to (13), for $\epsilon = 0$, in the form

$$\theta(t) = \theta_0 - \sinh^{-1}\left(\frac{p_0}{mt}\right),$$

(14)

where $(p_\theta, \theta_0) \in \mathbb{R}^1 \times S^1$. It is clear that geodesics (14) ‘blow up’ at $t = 0$, which is visualized in figure 2.

For $\epsilon \neq 0$, equations (13) read

$$m(t^2 + \epsilon^2)\dot{\theta} = p_\theta = \text{const}, \quad \epsilon^2 = \left(1 + \frac{r^2 \epsilon^2}{t^2 + \epsilon^2}\right)\dot{t}^2 - (t^2 + \epsilon^2)\dot{\theta}^2$$

(15)

and

$$\left(1 + \frac{r^2 \epsilon^2}{t^2 + \epsilon^2}\right)\dot{t} - \left(1 + \frac{r^2 \epsilon^2}{t^2 + \epsilon^2}\right)\left(\frac{\dot{\epsilon}}{\epsilon}\right)\dot{\theta} - \frac{r^2 \epsilon^2}{(t^2 + \epsilon^2)^2} \dot{\theta} \dot{t} - \dot{\theta}^2 \dot{t} = 0.$$  

(16)

From (15) and (16) we get

$$\left(\frac{d\dot{\theta}}{d\tau}\right)^2 = \frac{p_\theta^2 (1 + \frac{r^2 \epsilon^2}{t^2 + \epsilon^2})}{m^2(t^2 + \epsilon^2)^2 + p_\theta^2(t^2 + \epsilon^2)},$$

(17)

where $p_\theta \in \mathbb{R}^1$. General solution to (17) reads

$$\theta(t) = \theta_0 + p_\theta \int_{-\infty}^{t} d\tau \sqrt{\frac{1 + \frac{r^2 \epsilon^2}{t^2 + \epsilon^2}}{m^2(t^2 + \epsilon^2)^2 + p_\theta^2(t^2 + \epsilon^2)}},$$

(18)

where $\theta_0 \in S^1$. The integral in (18) cannot be calculated analytically. Numerical solution of (17) is presented in figure 2.

Figure 2 shows that a geodesic in the RCM space is bounded and continuous in the neighbourhood of the singularity. In contrary, a geodesic in the CM space (drawing by making use of (14)) blows up as $t \to \pm 0$.

3.2. Phase space and basic observables

We define a phase space to be the set of independent parameters (variables) defining all particle geodesics. Thus the phase space, $\Gamma$, for (18) reads

$$\Gamma := \{(\sigma, p_\sigma) \mid \sigma \in \mathbb{R}^1 \text{ mod } 2\pi r, p_\sigma \in \mathbb{R}^1\} = S^1 \times \mathbb{R}^1.$$  

(19)
The Cauchy problem at the singularity results from the vanishing/appearance of the space dimension $\theta$ at $t = 0$. It is fairly probable that any simple regularization of the singularity of the CM space that prevents such collapse will lead to the cylindrical phase space (19).

In [10] we have analysed four types of propagations of a particle in the CM space. Now we can see that the regularization prefers the propagation in the CM space of the de Sitter type (see, section 3.4 of [10]), because only in this case the phase space topology has the form (19).

Now, let us identify the basic canonical functions on the phase space, i.e. observables that can be used to define any composite observable of the underlying classical system. In case a phase space includes a variable with non-trivial topology, i.e. different from $\mathbb{R}^1$, it is a serious problem. However, it has been solved in two (equivalent) ways not long ago. In what follows we use the method used in the group theoretical quantization (see [14] and references therein).

In the next section we explain relation with another method.

A natural choice [14] of the basic functions on (19) is

$$
S := \sin(\sigma/r), \quad C := \cos(\sigma/r), \quad P := rp_\sigma.
$$

The basic observables $S$ and $C$ are smooth single-valued functions on $S^1$ (contrary to $\sigma$). The observables (20) satisfy the Euclidean algebra $e(2)$ on $\Gamma$

$$
\{S, C\} = 0, \quad \{P, S\} = C, \quad \{P, C\} = -S,
$$

where

$$
\{\cdot, \cdot\} := \frac{\partial}{\partial p_\sigma} \frac{\partial}{\partial \sigma} - \frac{\partial}{\partial \sigma} \frac{\partial}{\partial p_\sigma}.
$$

It is shown in [14] that the Euclidean group $E(2)$ can be used as the canonical group [15] of the phase space $\Gamma$.

4. Quantization of phase space

By quantization we mean finding an irreducible unitary representation of the symmetry group of the phase space of the underlying classical system.

The group $E(2)$ has the following irreducible unitary representation [14]:

$$
[U(\alpha)\psi](\beta) := \psi[(\beta - \alpha) \text{ mod } 2\pi], \quad \text{for rotations } z \rightarrow e^{iaz},
$$

$$
[U(t)\psi](\beta) := [\exp -i(a \cos \beta + b \sin \beta)]\psi(\beta), \quad \text{for translations } z \rightarrow z + t,
$$

where $z = |z| e^{i\beta}, \ t = a + b$ i, and where $\psi \in L^2(S^1)$.

Making use of the Stone theorem, we can find an (essentially) self-adjoint representation of the algebra (21). One has

$$
[\hat{C}, \hat{S}] = 0, \quad [\hat{P}, \hat{S}] = -i\hat{C}, \quad [\hat{P}, \hat{C}] = i\hat{S},
$$

where

$$
\hat{P}\psi(\beta) := -i\frac{\partial}{\partial \beta}\psi(\beta), \quad \hat{S}\psi(\beta) := \sin \beta \psi(\beta), \quad \hat{C}\psi(\beta) := \cos \beta \psi(\beta).
$$

The domain, $\Omega_\lambda$, of operators $\hat{P}, \hat{S}, \hat{C}$ reads

$$
\Omega_\lambda := \{\psi \in L^2(S^1)|\psi \in C^\infty[0, 2\pi], \psi^{(n)}(2\pi) = e^{i\lambda}\psi^{(n)}(0), \ n = 0, 1, 2, \ldots\},
$$

where $0 \leq \lambda < 2\pi$ labels various representations of $e(2)$ algebra. The space $\Omega_\lambda$ is dense in $L^2(S^1)$ so the unbounded operator $\hat{P}$ is well defined. As the operators $\hat{S}$ and $\hat{C}$ are bounded on the entire $L^2(S^1)$, the space $\Omega_\lambda$ is a common invariant domain for all operators and their products.
In [10] we have found that the representation of the algebra specific to the case considered there in section 3.4, has the form

\[ [\hat{\alpha}, \hat{U}] = \hat{U}, \]

(28)

with

\[ \hat{\alpha} \phi(\beta) := -i \frac{d}{d\beta} \phi(\beta), \quad \hat{U} \phi(\beta) := e^{i\beta} \phi(\beta), \]

(29)

where \( 0 \leq \beta < 2\pi \) and \( \phi \in \Omega_\lambda \). However, both representations, (26) and (29), are in fact the same owing to

\[ e^{i\beta} = \cos \beta + i \sin \beta, \quad [\cos \beta, \sin \beta] = 0. \]

(30)

The space \( \Omega_\lambda \), where \( 0 \leq \lambda < 2\pi \), may be spanned by the set of orthonormal eigenfunctions of the operator \( \hat{\alpha} \),

\[ f_{k,\lambda}(\beta) := (2\pi)^{-1/2} \exp i\beta (k + \lambda/2\pi), \quad k = 0, \pm 1, \pm 2, \ldots. \]

(31)

However, the functions (31) are continuous on \( S^1 \) only in the case when \( \lambda = 0 \). Thus, the requirement of the continuity removes the ambiguity of quantization.

## 5. Stability of system

To examine the stability problem of our system, we use the Hamiltonian formulation of the dynamics of a particle.

### 5.1. Classical level

By stability of the dynamics of a classical particle we mean such an evolution of a particle that can be described by the canonical variables which are bounded and continuous functions.

Direct application of the results of [34] gives the following expression for the extended Hamiltonian [16, 17], \( H_\epsilon \), of a particle

\[ H_\epsilon = \frac{1}{2} C_\epsilon \Phi_1 \epsilon, \]

(32)

where \( C_\epsilon \) is an arbitrary function of an evolution parameter \( \tau \), and where \( \Phi_1 \epsilon \) is the first-class constraint defined by (12). The equations of motion for canonical variables \((t, \theta; p_t, p_\theta)\) read

\[
\frac{dt}{d\tau} = \{t, H_\epsilon\} = -C_\epsilon(\tau) \frac{p_t(t^2 + \epsilon^2)}{t^2 + \epsilon^2 + r^2 \epsilon^2},
\]

(33)

\[
\frac{d\theta}{d\tau} = \{\theta, H_\epsilon\} = C_\epsilon(\tau) \frac{p_\theta}{t^2 + \epsilon^2},
\]

(34)

\[
\frac{dp_t}{d\tau} = \{p_t, H_\epsilon\} = C_\epsilon(\tau) \left( \frac{tp_\theta^2}{(t^2 + \epsilon^2)^2} + \frac{tp_\theta^2 \epsilon^2 r^2}{(t^2 + \epsilon^2 + r^2 \epsilon^2)^2} \right),
\]

(35)

\[
\frac{dp_\theta}{d\tau} = \{p_\theta, H_\epsilon\} = 0,
\]

(36)

where

\[
\{\cdot, \cdot\} = \frac{\partial \cdot}{\partial t} \frac{\partial \cdot}{\partial p_t} - \frac{\partial \cdot}{\partial p_t} \frac{\partial \cdot}{\partial t} + \frac{\partial \cdot}{\partial \theta} \frac{\partial \cdot}{\partial p_\theta} - \frac{\partial \cdot}{\partial p_\theta} \frac{\partial \cdot}{\partial \theta}.
\]

To solve (33)–(36), we use the gauge \( \tau = t \). In this gauge (33) leads to

\[ C_\epsilon(t) = -\frac{t^2 + \epsilon^2 + r^2 \epsilon^2}{p_t(t^2 + \epsilon^2)}. \]

(37)
Insertion of (37) into (35) and taking into account (12) gives

\[ \frac{d}{dt} p_t^2 = -\frac{2t p_t^2 r^2 \epsilon^2}{(t^2 + \epsilon^2)(t^2 + \epsilon^2 + r^2 \epsilon^2)} - \frac{2t p_0^2 (t^2 + \epsilon^2 + r^2 \epsilon^2)}{(t^2 + \epsilon^2)^3}. \]  

(38)

Solution to (38) reads

\[ p_t^2 = c_1 \left( \frac{c_1 p_0^2}{t^2 + \epsilon^2} + c_2 \right) \frac{t^2 + \epsilon^2 + r^2 \epsilon^2}{t^2 + \epsilon^2}, \]  

(39)

where \( p_0 \) does not depend on time due to (36). This result is consistent with the constraint (12) if we put \( c_1 = 1 \) and \( c_2 = m^2 \). Insertion of (39) into (37) yields an explicit expression for \( C_\epsilon(t) \). Next, insertion of so obtained \( C_\epsilon(t) \) into (34) gives (17) with the solution (18). Thus, we have found complete solutions to equations (33)–(36).

It results from the functional form of solutions that for \( \epsilon \neq 0 \) the propagation of canonical variables is regular, i.e. has no singularities for any value of time. One may also verify that the constraint equation (12), with \( p_t \) determined by (39), is satisfied for each value of time either. Since \( C_\epsilon(t) \), determined by (37) is bounded, the Hamiltonian \( H_\epsilon \) defined by (32) is weakly zero due to the constraint (12).

Now, let us analyse the case \( \epsilon = 0 \), which corresponds to the evolution of a particle in the CM space. The solutions of (33)–(36), in the gauge \( \tau = t \), are the following:

\[ C_0(t) = -1/p_t, \]  

(40)

\[ p_t^2 = p_0^2 / t^2 + m^2 \]  

(41)

\[ \theta(t) = -\sinh^{-1}(p_0/m t) + \text{const}, \]  

(42)

(where \( p_0 = \text{const} \), in agreement with (14). The constraint equation (12) reads

\[ \Phi_0 = p_0^2 / t^2 - p_t^2 + m^2 = 0. \]  

(43)

It results from (41) and (42) that only for \( p_0 = 0 \) the propagation of a particle in the CM space is regular for any value of time. The Hamiltonian (32) is also regular and is weakly equal to zero. Quite different situation occurs in the case \( p_0 \neq 0 \). Equations (41)–(43) and the Hamiltonian are singular at \( t = 0 \). Thus the dynamics of a particle is unstable.

5.2. Quantum level

By stability of dynamics of a quantum particle we mean the boundedness from below of its quantum Hamiltonian.

To construct the quantum Hamiltonian of a particle we use the following mapping (see, e.g. [18])

\[ p_k p_l g^{kl} \rightarrow \Box := (-g)^{-1/2} \partial_k \left[ (-g)^{1/2} g^{kl} \partial_l \right], \]  

(44)

where \( g := \det[g_{kl}] \) and \( \partial_k := \partial/\partial x^k \). The Laplace–Beltrami operator, \( \Box \), is invariant under the change of spacetime coordinates and it leads to Hamiltonians that give results consistent with experiments [18], and which has been used in theoretical cosmology (see [34] and references therein).

This way we have also verified, in the gauge \( \tau = t \), the equivalence between our Hamiltonian and the Lagrangian formulations of the dynamics of a particle.

The division of the set of geodesics into regular and unstable depends on the choice of coordinates, but it always includes the unstable ones.
In the case of the CM space the Hamiltonian, for \( t < 0 \) or \( t > 0 \), reads

\[
\hat{H} = \Box + m^2 = \frac{\partial^2}{\partial t^2} + \frac{1}{t} \frac{\partial}{\partial t} - \frac{1}{t^2} \frac{\partial^2}{\partial \theta^2} + m^2. \tag{45}
\]

The operator \( \hat{H} \) was obtained by making use of (44), (3) and (32) in the gauge \( C_\epsilon = 2 \) (for \( \epsilon = 0 \)). In this gauge\(^4\) the Hamiltonian equals the first class constraint (43). Thus the Dirac quantization scheme [16, 17] leads to the equation

\[
\hat{H} \psi(\theta, t) = 0. \tag{46}
\]

The space of solutions to (46) defines the domain of boundedness of \( \hat{H} \) from below (and from above).

Let us find the non-zero solutions of (46). Separating the variables

\[
\psi(\theta, t) = A(\theta)B(t) \tag{47}
\]

leads to the equations

\[
\frac{d^2 A}{d\theta^2} + \rho^2 A = 0, \quad \rho \in \mathbb{R} \tag{48}
\]

and

\[
\frac{d^2 B}{dt^2} + \frac{1}{t} \frac{dB}{dt} + \frac{m^2 t^2 + \rho^2}{t^2} B = 0, \quad t \neq 0, \tag{49}
\]

where \( \rho \) is a constant of separation. Two independent continuous solutions on \( S^1 \) read

\[
A_1(\rho, \theta) = a_1 \cos(\rho \theta), \quad A_2(\rho, \theta) = a_2 \sin(\rho \theta), \quad a_1, a_2 \in \mathbb{R}. \tag{50}
\]

Two independent solutions on \( \mathbb{R} \) (for \( t < 0 \) or \( t > 0 \) have the form [19, 20]

\[
B_1(\rho, t) = b_1 \text{Re} J(i\rho, mt), \quad B_2(\rho, t) = b_2 \text{Re} Y(i\rho, mt), \quad b_1, b_2 \in \mathbb{C}, \tag{51}
\]

where \( \text{Re} J \) and \( \text{Re} Y \) are the real parts of Bessel’s and Neumann’s functions, respectively. Since \( \rho \in \mathbb{R} \), the number of independent solutions is \( 2 \times 2 \times \infty \) (for \( t < 0 \) and \( t > 0 \)).

At this stage we define the scalar product on the space of solutions (50) and (51) as follows:

\[
\langle \psi_1 | \psi_2 \rangle := \int_{\Gamma} d\mu \bar{\psi}_1 \psi_2, \quad d\mu := \sqrt{-g} \, d\theta \, dt = |t| \, d\theta \, dt, \tag{52}
\]

where \( \Gamma := [-T, 0] \times S^1 \) (with \( T > 0 \)) in the pre-singularity epoch, and \( \Gamma := [0, T] \times S^1 \) in the post-singularity epoch. We assume that the CM space can be used to model the universe only during its quantum phase, which lasts the period \([-T, T]\).

Now we construct an orthonormal basis, in the left neighbourhood of the cosmological singularity, out of the solutions (50) and (51). One can verify that the solutions (50) are orthonormal and continuous on \( S^1 \) if \( a_1 = \pi^{-1/2} = a_2 \) and \( \rho \theta = 0, \pm 1, \pm 2, \ldots \). (This set of functions coincides with the basis (31) that spans the subspace \( \Omega_0 \) if we replace \( k \) by \( \rho \).) Some effort is needed to construct the set of orthonormal functions out of \( \text{Re} J(i\rho, mt) \) and \( \text{Re} Y(i\rho, mt) \). First, one may verify that these functions are square-integrable on the interval \([-T, T]\). This is due to the choice of the measure in the scalar product (52), which leads to the boundedness of the corresponding integrands. Second, having normalizable set of four independent functions, for each \( \rho \), we can turn it into an orthonormal set by making use of the Gram–Schmidt procedure (see, e.g. [20]). Our orthonormal and countable set of functions may be used to define the span \( \mathcal{F} \). The completion of \( \mathcal{F} \) in the norm induced by the scalar

\(^4\) In the preceding subsection, concerning the classical dynamics, the choice of gauge was different. But since the theory we use is gauge invariant, the different choice of the gauge does not effect physical results.
product (52) defines the Hilbert spaces \( L^2(\tilde{\Gamma} \times S^1, d\mu) \). It is clear that the same procedure applies to the right neighbourhood of the singularity.

Finally, we can prove that the Hamiltonian (45) is self-adjoint on \( L^2(\tilde{\Gamma} \times S^1, d\mu) \). The proof is immediate if we rewrite (46) in the form
\[
\Box \psi = -m^2 \psi.
\]
It is evident that on the orthonormal basis that we have constructed above the operator \( \Box \) is an identity operator multiplied by a real constant \(-m^2\). The operator \( \Box \) is bounded since
\[
\|\Box\| := \sup_{\|\psi\|=1} \|\Box \psi\| = \sup_{\|\psi\|=1} \|-m^2 \psi\| = m^2 < \infty,
\]
where \( \|\psi\| := \sqrt{\langle \psi | \psi \rangle} \). The operator \( \Box \) is also symmetric, because \( m \) is a real constant. Since \( \Box \) is bounded and symmetric, it is a self-adjoint operator (see, e.g. [22]). Clearly, the self-adjointness of the Hamiltonian (45) results from the self-adjointness of \( \Box \).

We have constructed the two Hilbert spaces: one for the pre-singularity epoch, \( \mathcal{H}^{(-)} \), and another one to describe the post-singularity epoch, \( \mathcal{H}^{(+)} \). Next problem is to ‘glue’ them into a single Hilbert space, \( \mathcal{H} = L^2([-T, T] \times S^1, d\mu) \), that is needed to describe the entire quantum phase. From the mathematical point of view the gluing seems to be problematic because the Cauchy problem for the equation (46) is not well defined at \( t = 0 \), and because we have assumed that \( t \neq 0 \) in the process of separation of variables to get equations (48) and (49). However, arguing based on the physics of the problem enables the gluing. First of all we have already agreed that a classical test particle is able to go across the singularity (see, subsection 2.2). One can also verify that the probability density
\[
P(\theta, t) := \sqrt{-g} |\psi(\theta, t)|^2 = |t| |\psi(\theta, t)|^2
\]
is bounded and continuous in the domain \([-T, T] \times S^1\). Figures 3 and 4 illustrate the behaviour of \( P(\theta, t) \) for two examples of gluing the solutions having \( \rho = 0 \). The cases with \( \rho \neq 0 \) have similar properties. Thus, the assumption that the gluing is possible is justified. However one can glue the two Hilbert spaces in more than one way, as it was done in the quantization of the phase space in our previous paper [10]. In what follows we present two cases, which are radically different.

5.2.1. Deterministic propagation. Among all solutions (51) there is one, corresponding to \( \rho = 0 \), that attracts an attention [19]. It reads
\[
B_1(0, mt) = b_1 \Re J(0, mt), \quad b_1 \in \mathbb{R},
\]

Figure 3. Probability density corresponding to \( \psi(\theta, t) = A_1(0, \theta) \Re J(0, t) \).
and has the following power series expansion close to \( t = 0 \):

\[
B_1(0, x)/b_1 = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \mathcal{O}[x^8].
\]  

(57)

It is visualized in figure 5(a). The solution (56) is smooth at the singularity, in spite of the fact that (49) is singular at \( t = 0 \).

It defines a solution to (46) that does not depend on \( \theta \), since the non-zero solution (50) with \( \rho = 0 \) is just a constant. Thus, it is insensitive to the problem that one cannot choose a common coordinate system for both \( t < 0 \) and \( t > 0 \).

The solution \( B_1 \) (and the trivial solution \( B_0 = 0 \)) can be used to construct a one-dimensional Hilbert space \( \mathcal{H} = L^2([-T, T] \times S^1, d\mu) \). The scalar product is defined by (52) with \( \tilde{\Gamma} \) replaced by \( \Gamma := [-T, T] \times S^1 \). It is obvious that the Hamiltonian is self-adjoint on \( \mathcal{H} \).

The solution (56) is continuous at the singularity. It describes an unambiguous propagation of a quantum particle. Thus, we call it a deterministic propagation. It is similar to the propagation of a particle in the RCM space considered in the next subsection.

Since (49) is a second-order differential equation, it should have two independent solutions. However, the second solution cannot be continuous at \( t = 0 \). One may argue as follows: the solution (56) may be obtained by ignoring the restriction \( t \neq 0 \) and solving (49) with the following initial conditions:

\[
B(0, 0) = 1, \quad dB(0, 0)/dt = 0.
\]  

(58)

Equations (49) and (58) are consistent, because the middle term of the rhs of (49) may be equal to zero due to (58) so the resulting equation would be non-singular at \( t = 0 \). Another initial condition of the form \( B(0, 0) = \text{const} \) and \( dB(0, 0)/dt = 0 \) would be linearly dependent on (58). Thus, it could not lead to the solution which would be continuous at \( t = 0 \) and linearly independent on (56).
This qualitative reasoning can be replaced by a rigorous derivation using the power series expansion method [20]. Applying this method one obtains that near the singularity $t = 0$ the solution to (49) behaves like $t^n$ and that the corresponding indicial equation reads

$$\omega^2 = -\rho^2. \quad (59)$$

Thus, for $\rho \neq 0$ the two solutions behave like $t^{\pm i\rho}$, i.e. are bounded but not continuous (see, equation (51)). For $\rho = 0$ the indicial equation has only one solution $\omega = 0$ which leads to an analytic solution to (49) defined by (56). In such a case, it results from the method of solving the singular linear second-order equations [20], the second solution to (49) may behave like $\ln|t|$. In fact it reads [19]

$$B_2(0, \theta_1) = b_2 \text{Re} Y(0, \theta_1), \quad b_2 \in \mathbb{R}, \quad (60)$$

and is visualized in figure 5(b). It cannot be called a deterministic propagation due to the discontinuity at the singularity $t = 0$.

5.2.2. Indeterministic propagation.

All solutions (51), except (56), are discontinuous at $t = 0$. This property is connected with the singularity of (49) at $t = 0$. It is clear that due to such an obstacle the identification of corresponding solutions on both sides of the singularity is impossible. However there are two natural constructions of a Hilbert space out of $H^(-)$ and $H^(+)$ which one can apply.

(a) Tensor product of Hilbert spaces

The Hilbert space is defined in a standard way [21] as $\mathcal{H} := \mathcal{H}^(-) \otimes \mathcal{H}^($+)$ and it consists of functions of the form

$$f(t_1, \theta_1; t_2, \theta_2) \equiv (f^(-) \otimes f^($+$$))(t_1, \theta_1; t_2, \theta_2) := f^(-)(t_1, \theta_1) f^($+$$)(t_2, \theta_2), \quad (61)$$

where $f^(-) \in \mathcal{H}^(-)$ and $f^($+$$) \in \mathcal{H}^($+$$). The scalar product reads

$$\langle f | g \rangle := \langle f^(-) | g^(-) \rangle \langle f^($+$$) | g^($+$$) \rangle, \quad (62)$$

where

$$\langle f^(-) | g^(-) \rangle := \int _{-T}^{0} dt_1 \int _{0}^{2\pi} d\theta_1 |f^(-)(t_1, \theta_1)g^(-)(t_1, \theta_1)|, \quad (63)$$

and

$$\langle f^($+$$) | g^($+$$) \rangle := \int _{0}^{T} dt_2 \int _{0}^{2\pi} d\theta_2 |f^($+$$)(t_2, \theta_2)g^($+$$)(t_2, \theta_2)|. \quad (64)$$

The action of the Hamiltonian is defined by

$$\hat{H}(f^(-) \otimes f^($+$$)) := (\hat{H} f^(-)) \otimes f^($+$$) + f^(-) \otimes (\hat{H} f^($+$$)). \quad (65)$$

The Hamiltonian is clearly self-adjoint on $\mathcal{H}$.

The quantum system described in this way appears to consist of two independent parts. In fact it describes the same quantum particle but in two subsequent time intervals separated by the singularity at $t = 0$.

(b) Direct sum of Hilbert spaces

Another standard way [21] of defining the Hilbert space is $\mathcal{H} := \mathcal{H}^(-) \bigoplus \mathcal{H}^($+$$$. The scalar product reads

$$\langle f_1 | f_2 \rangle := \langle f_1^(-) | f_2^(-) \rangle + \langle f_1^($+$$) | f_2^($+$$) \rangle. \quad (66)$$
where
\[ f_k := (f_k^{(-)}, f_k^{(+)}), \quad k = 1, 2, \] (67)

and where \( f_k^{(-)} \) and \( f_k^{(+)} \) are two completely independent solutions in the pre-singularity and post-singularity epochs, respectively. (The rhs of (66) is defined by (63) and (64).)

The Hamiltonian action on \( \mathcal{H} \) reads
\[ \mathcal{H} \ni (f^{(-)}, f^{(+)}), \quad \mathcal{H} := (\hat{H} f^{(-)}, \hat{H} f^{(+)}) \in \mathcal{H}. \] (68)

It is obvious that \( \hat{H} \) is self-adjoint on \( \mathcal{H} \).

By the construction, the space \( \mathcal{H}(-) \oplus \mathcal{H}(+) \) includes vectors like \((f^{(-)}, 0)\) and \((0, f^{(+)}),\) which give non-vanishing contribution to (66) (but yield zero in case (62)). The former state describes the annihilation of a particle at \( t = 0 \). The latter corresponds to the creation of a particle at the singularity. These type of states do not describe the propagation of a particle across the singularity. The annihilation/creation of a massive particle would change the background. Such events should be eliminated from our model because we consider a test particle which, by definition, cannot modify the background spacetime. Since \( \mathcal{H}(-) \) and \( \mathcal{H}(+) \), being vector spaces, must include the zero solutions, the Hilbert space \( \mathcal{H}(-) \oplus \mathcal{H}(+) \) cannot model the quantum phase of our system.

### 5.3. Regularization

In the RCM space the quantum Hamiltonian, \( \hat{H}_\epsilon \), for any \( t \in [-T, T] \), reads (we use the gauge \( C_\epsilon = 2 \))
\[
\hat{H}_\epsilon = \frac{\sqrt{t^2 + \epsilon^2 + \epsilon^2 r^2}}{t^2 + \epsilon^2} \frac{\partial^2}{\partial \theta^2} = \frac{t^2 + \epsilon^2}{\sqrt{t^2 + \epsilon^2 + \epsilon^2 r^2}} \frac{\partial^2}{\partial t^2} - \frac{t(t^2 + \epsilon^2 + \epsilon^2 r^2)}{(t^2 + \epsilon^2 + \epsilon^2 r^2)^{3/2}} \frac{\partial}{\partial t} - \sqrt{t^2 + \epsilon^2 + \epsilon^2 r^2} m^2 \tag{69}
\]

Since the Hamiltonian is equal to the first-class constraint, the physical states are solutions to the equation
\[ \hat{H}_\epsilon \Psi = 0. \tag{70} \]

As in the case of \( \epsilon = 0 \), the space of solutions to (70) defines the domain of boundedness of \( \hat{H}_\epsilon \). Substituting \( \Psi(\theta, t) = A(\theta) B(t) \) into (70) yields
\[ \frac{d^2 A}{d \theta^2} + \rho^2 A = 0, \quad \rho \in \mathbb{R}, \tag{71} \]
and
\[ \frac{(t^2 + \epsilon^2)^2}{t^2 + \epsilon^2 + r^2 \epsilon^2} \frac{d^2 B}{d t^2} + \frac{t(t^2 + \epsilon^2)(t^2 + \epsilon^2 + 2\epsilon^2 \rho^2)}{(t^2 + \epsilon^2 + 2r^2 \epsilon^2)^2} \frac{dB}{dt} + m^2 (t^2 + \epsilon^2 + \rho^2) B = 0. \tag{72} \]

Equation (71) looks the same as in the non-regularized case (48) so the two independent solutions on \( S^1_\epsilon \) read
\[ A_1(\rho, \theta) = \pi^{-1/2} \cos(\rho \theta), \quad A_2(\rho, \theta) = \pi^{-1/2} \sin(\rho \theta), \tag{73} \]
where \( r \rho = 0, \pm 1, \pm 2, \ldots \) (orthogonality and continuity conditions).

Equation (72) is non-singular in \([-T, T]\) so for each \( \rho \) it has two independent solutions which are bounded and smooth in the entire interval. One may represent these solutions by a symmetric and an anti-symmetric functions. We do not try to find the analytic solutions to (72). What we really need to know are general properties of them. In what follows we further analyse only numerical solutions to (72) by making use of [19].
Since in the regularized case the solutions are continuous in the entire interval \([-T, T]\), the problem of gluing the solutions (the main problem in case \(\epsilon = 0\)) does not occur at all. Thus, the construction of the Hilbert space, \(\mathcal{H}_n\), by using the space of solutions to (71) and (72) is straightforward. The construction of the basis in \(\mathcal{H}_n\) may be done by analogy to the construction of the basis in \(\mathcal{H}^{(\epsilon)}\) (described in subsection 2.2). The only difference is that now \(t \in [-T, T]\) and instead of (51) we use the solutions to (72). Let us denote them by \(B_i(\rho, mt)\), where \(i = s\) and \(i = a\) stand for symmetric and anti-symmetric solutions, respectively. Figures 6 and 7 present two examples of solutions to (72) for \(\rho = 1\) and the corresponding probability densities. We can see that \(P(\theta, t)\) is a bounded and continuous function on \([-T, T] \times S^1\), as in the case of \(\epsilon = 0\) (cp with figures 3 and 4).

We define the scalar product as follows:

\[
\langle \psi_1 | \psi_2 \rangle := \int_{\Gamma} d\mu \overline{\psi}_1 \psi_2, \quad d\mu := \sqrt{-g} \, d\theta \, dt = \sqrt{t^2 + \epsilon^2 + r^2 \epsilon^2} \, d\theta \, dt, \tag{74}
\]

where \(\Gamma := [-T, T] \times S^1\), and where an explicit form of \(d\mu\) is found by making use of (7).

It is evident that \(\hat{H}_n\) is self-adjoint on \(\mathcal{H}_n\). The main difference between the deterministic case with \(\epsilon = 0\) and the present case \(\epsilon > 0\) is that in the former case the Hilbert space is one dimensional (\(\rho = 0\)), whereas in the latter case it is \(2 \times 2 \times \infty\) dimensional (\(r \rho \in \mathbb{Z}\)).

6. Summary and conclusions

The Cauchy problem at the cosmological singularity of the geodesic equations may be ‘resolved’ by the regularization, which replaces the double conical vertex of the CM space by a space with the vertex of the big-bounce type, i.e. with non-vanishing space dimension.
at the singularity. We have presented a specific example of such regularization of the CM space. Both classical and quantum dynamics of a particle in the regularized CM space are deterministic and stable. We have examined these aspects of the dynamics at the phase space and Hamiltonian levels. The classical and quantum dynamics of a particle in the regularized CM space is similar to the dynamics in the de Sitter space [12, 13]. We are conscious that our regularization of the singularity is rather ad hoc. However, most regularizations of the CM space which protect against the disappearance of the $\theta$-dimension at the vertex would solve the Cauchy problem at the singularity and lead to the dynamics of a particle similar to the one considered in our regularized CM space. However, examination of this problem is beyond the scope of the present paper, but will be considered elsewhere.

The classical dynamics in the CM space is unstable (apart from the one class of geodesics). However, the quantum dynamics is well defined. The Cauchy problem of the geodesics is not an obstacle to the quantization. The examination of the quantum stability has revealed surprising result that in one case a quantum particle propagates deterministically in the sense that it can be described by a quantum state that is continuous at the singularity. This case is very interesting as it says that there can exist deterministic link between the data of the pre-singularity and post-singularity epochs. All other states have discontinuity at the singularity of the CM space, but they can be used successfully to construct a Hilbert space. This way we have proved the stability of the dynamics of a quantum particle.

At the quantum level the stability condition requiring the boundedness from below of the Hamiltonian operator means the imposition of the first-class constraint onto the space of quantum states to get the space of physical quantum states. The resulting equation depends on all spacetime coordinates. In the pre-singularity and post-singularity epochs the CM space is locally isometric to the Minkowski space [10]. Owing to this isometry, the stability condition is in fact the Klein–Gordon, KG, equation. The space of solutions to the KG equation in these two epochs and the corresponding Hilbert space are fortunately non-trivial ones, otherwise our quantum theory of a particle would be empty.

Quantization of the phase space carried out in section 4 (and in our previous paper [10]), corresponds to some extent to the method of quantization in which one first solves constraints at the classical level and then quantize the resulting theory. Quantization, that we call here examination of the stability at the quantum level, is effectively the method in which we impose the constraint, but at the quantum level. The results we have obtained within both methods of quantization are consistent. It means that the quantum theory of a particle in the compactified Milne space does exist. The CM space seems to model the cosmological singularity in a satisfactory way.

It turns out that the timelike geodesics of our CM space may have interpretation in terms of cosmological solutions of some sophisticated higher dimensional field theories [23–25]. We have already discussed some aspects of this connection in our previous paper (see section 5 of [10]). Presently, we can say that in one case (see section 4 of [23] and section 5.2.1 of this paper) this analogy extends to the quantum level: transition of a particle through the cosmological singularity in both models is mathematically well defined. In both cases the operator constraint is used to select quantum physical states. Elaboration of this analogy needs an extension of our results to the Misner space (that consists of the Milne and Rindler spaces) because it is the spacetime used in [23]. Another subtlety is connected with the fact that we carry out analysis in the compactified space, whereas the authors of [23] use the covering space.

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6 Our result should be further confirmed by the examination of the dynamics of a particle in a higher dimensional CM space.
There exists another model to describe the evolution of the universe based on string/M theory. It is called the pre-big-bang model [26]. However, the ST model is more self-consistent and complete.

Another sophisticated model called loop quantum cosmology, LQC, is based on non-perturbative formulation of quantum gravity called the loop quantum gravity, LQG [27, 28]. It is claimed that the CS is resolved in this approach [29]. However, this issue seems to be still open due to the assumptions made in the process of truncating the infinite number degrees of freedom of the LQG to the finite number used in the LQC [30, 31]. This model has problems also in obtaining a unique semi-classical approximations [32], which are required to link the quantum phase with the nearby classical phase in the evolution of the Universe. For response to [32] we recommend [33].

Quantization of dynamics of extended objects in the CM space is our next step. There exist promising results on propagation of a string and membrane [34, 35]. However, these results concern extended objects in the low energy states called the zero-mode states and quantum evolution is approximated by a semi-classical model. Recently, we have quantized the dynamics of a string in the CM space rigorously [36], but our results concern only the zero-mode state of a string. For drawing firm conclusions about the physics of the problem, one should also examine the non-zero modes. Work is in progress.

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