Relating boundary and interior solutions of the cohomological equation for cocycles by isometries of negatively curved spaces. The Liñeic case

Alexis Moraga and Mario Ponce

Faculty of Mathematics, Pontificia Universidad Católica de Chile, Av. Vicuna Mackenna 4860, Macul, Chile

E-mail: ajmoraga@mat.uc.cl and mponcea@mat.uc.cl

Received 20 April 2021, revised 13 January 2022
Accepted for publication 26 January 2022
Published 16 February 2022

Abstract

We consider the reducibility problem of cocycles by isometries of Gromov hyperbolic metric spaces in the Liñeic setting. We show that provided that the boundary cocycle (that acts on a compact space) is reducible in a suitable Hölder class, then the original cocycle by isometries (that acts on an unbounded space) is also reducible.

Keywords: Livsic problem, cocycles by isometries, skew-product dynamics

Mathematics Subject Classification numbers: 37H05, 37H15, 37A20.

1. Introduction

The study of dynamical properties of a cocycle is substantially simplified when we can reduce it to a cocycle that takes values in a smaller group. More precisely, given a homeomorphism (dynamical system) \( T : \Omega \to \Omega \) over a compact metric space \( \Omega \) and a (topological) group \( G \), we consider a continuous \( G \)-valued cocycle \( A : \Omega \to G \) over the base dynamics \( T \). We are interested in the dynamical behavior of the cocycle, that is, we seek to understand the properties of the dynamical product \( A^n(\omega) := A(T^{n-1}\omega) \cdot A(T^{n-2}\omega) \cdots A(\omega) \). Such kind of cocycles (and its dynamical properties) appear in many situations in dynamical systems and other branches of mathematics. For instance, when \( T \) is a diffeomorphism of a smooth (parallelizable) \( n \)-manifold, the cocycle given by the derivative \( A(\omega) := DT(\omega) \in GL(n, \mathbb{R}) \) provides a great deal
of information about the dynamics of $T$. In fact, this is one of the main approaches to the theory of smooth dynamical systems. The source of examples for the study of these cocycles is vast, including the Schrödinger equation. In that case, the underlying group is $SL(2, \mathbb{R})$ and the (base) dynamical system $T$ corresponds to a minimal linear translation on a torus (see [2, 9]). The Kontsevitch–Zorich cocycle is a keystone for studying the Teichmüller flow over translation surfaces (see [26]).

To reduce a cocycle consists in finding a continuous function $B : \Omega \to G$ such that the conjugated cocycle $B(T\omega)^{-1} \cdot A(\omega) \cdot B(\omega)$ takes values in a small subgroup of $G$. In the case of the trivial subgroup $\{e_G\}$, where $e_G$ is the neutral element, we look for the existence of a function $B : \Omega \to G$ such that

$$B(T\omega) \cdot B(\omega)^{-1} = A(\omega).$$

We call this equation the cohomological equation. In that case we say that $A$ is a coboundary or that $A$ is reducible.

There exists satisfactory results regarding the reducibility of the cocycle, whenever the base map $T$ is elliptic, hyperbolic or a partially hyperbolic dynamical system. For instance, in the minimal case, a lot of work had been assembled under the theory of Gottschalk and Hedlund (see [11, 17, 20]), and the KAM theory (see [2]). In the hyperbolic case, the Livšic theory has produced a great amount of results (see [6, 7, 10, 13, 14, 17, 19, 21, 23, 24]). Recently the solution to the cohomological equation in the Abelian case has been addressed by some authors as a central problem in the partially hyperbolic dynamical system case (see [16, 25]).

Many of the above examples are linear cocycles, that is, the elements of $G$ act as linear maps of suitable vector bundles over the base space $\Omega$. A typical approach allows to consider the elements of the general linear group $GL(d, \mathbb{R})$ as isometries of the set of positives matrices $\text{Pos}(d, \mathbb{R})$ endowed with a suitable metric (resulting into a non-positive curved space, see [15]). With this in mind, we will be interested in cocycles taking values on the group $\text{Isom}(\mathcal{H})$ of isometries of a metric space $\mathcal{H}$ of negative curvature (to be defined in a precise sense). In the context of a minimal base dynamics, in [5] the authors show that a bounded continuous cocycle by isometries of a non-positively curved complete metric space is always reducible, thus showing a general version of the Gottschalk–Hedlund theorem.

In this work we will be interested in the Livšic setting, that is, when $T$ is a hyperbolic homeomorphism (to be defined in a precise sense, see section 3). These maps have (among other features) a dense set of periodic orbits. Hence, it is interesting to study the reducibility of the cocycle restricted to a periodic orbit. Let $\omega$ be such that there exists $n \in \mathbb{N}$ verifying $T^n\omega = \omega$. We obtain a direct obstruction to the existence of a (at least formal) solution of the cohomological equation

$$\prod_{i=0}^{n-1} A(T^i\omega) = \prod_{i=0}^{n-1} B(T^{i+1}\omega) \cdot B(T^i\omega)^{-1} = B(T^n\omega) \cdot B(\omega)^{-1} = e_G.$$

The Livšic problem consists in determining whether the condition (2) is not only necessary but also sufficient for $A$ being a coboundary. This terminology originates in the seminal work of Livšic [19], who proved that this is the case whenever $G$ is Abelian, $A$ is Hölder-continuous and $T$ is a topologically transitive hyperbolic diffeomorphism. Since then, many extensions of this classical result have been proposed. Perhaps the most relevant is Kalinin’s recent version for $G = GL(d, \mathbb{C})$.

**Theorem of Kalinin**, see [13]. Let $T$ be a topologically transitive hyperbolic homeomorphism of a compact metric space $\Omega$. Let $A : \Omega \to GL(d, \mathbb{C})$ be an $\alpha$-Hölder function for which
the condition (2) holds. Then there exists an $\alpha$-Hölder function $B : \Omega \to GL(d, \mathbb{C})$ such that for all $\omega \in \Omega$, $A(\omega) = B(T\omega) \cdot B(\omega)^{-1}$.

This outstanding result is strongly based on the linear action of the matrices. In the context of cocycles that take values on the group of diffeomorphisms of a manifold, the answer to the Livšic problem is unclear. In the case of diffeomorphisms of a closed manifold (compact and without boundary), the recent result by Avila, Kocsard and Liu [1] is the most important step in the theory, extending the previous result by Kocsard and Potrie [18] on circle diffeomorphisms.

**Theorem of Avila, Kocsard and Liu**, see [1]. Let $T$ be a topologically transitive hyperbolic homeomorphism of a compact metric space $\Omega$. Let $A : \Omega \to \text{Diff}^r(M)$ be an $\alpha$-Hölder cocycle, taking values on the group of diffeomorphisms of class $C^r$, $r > 1$, of a smooth closed manifold $M$. If the condition (2) holds then there exists a Hölder continuous map $B : \Omega \to \text{Diff}^r(M)$ such that $B(T\omega) \cdot B(\omega)^{-1} = A(\omega)$.

Our work focuses on the study of the cohomological equation for cocycles of isometries of negatively curved metric spaces in the Livšic setting. The approach we propose is to take advantage of the fact that negatively curved spaces, even though they are non compact, have a natural compactification as they are completed with a boundary at infinity. This boundary has a natural topology, that turns it into a compact space. For example, when we deal with a sectional-negatively curved Riemannian manifold, the boundary at infinity is just a codimension one sphere (see [8]). In general, every isometry extends to a homeomorphism of this boundary. In the context of cocycles by isometries, this produces a new cocycle by homeomorphisms of the boundary at infinity, for which the reducibility question can be formulated (compare with the theorem by Avila, Kocsard and Liu above). The main goal of this work is to relate the reducibility of the cocycle by isometries with the reducibility of the cocycle by homeomorphisms induced on the boundary at infinity.

While the reducibility at the boundary at infinity is a more or less direct consequence of the reducibility of the original cocycle, the other direction is more interesting and complicated. The main result of this article (see theorem 5.3 for the exact statement) says that a continuous cocycle by isometries of a uniquely visible Gromov hyperbolic space over a hyperbolic base dynamics is reducible when the induced boundary cocycle is reducible in certain Hölder class.

**Organization of the paper.** In section 2 we develop, in detail, the well-known fact that the existence of the solution to the cohomological equation is equivalent to the saturation of the phase space of a certain dynamical system (the skew-product) by invariant sections. In section 3 we revisit the classical Livšic theory in the real case (which serves both, as a context and as a tool in a key step in the proof of the main result). In section 4 we review the geometric properties of the negatively curved spaces that we will consider (uniquely visible Gromov hyperbolic spaces). Section 5 is the core of this work, as it contains the precise statement of the main theorem and the description of the hypothesis, mainly with respect to the Hölder conditions on the Busemann functions. The section ends with the proof of the main result (theorem 5.3).

### 2. Two equivalent problems

This section is well known to specialists. However, we consider that it is relevant to make explicit the details of the relationship between the reducibility of a cocycle and the saturation of the phase space of the related skew-product dynamics by invariant sections. Let us consider a continuous transformation $T : \Omega \to \Omega$, where $(\Omega, d_\Omega)$ is a compact metric space, a complete metric space $(\mathcal{H}, d_\mathcal{H})$ and a function $A : \Omega \to \text{Isom}(\mathcal{H})$, that takes values on the space of isometries of $\mathcal{H}$.
**Definition 2.1.** We say that $A : \Omega \to \text{Isom}(\mathcal{H})$ is continuous for the compact-uniform topology, if given any compact set $K \subset \mathcal{H}$, any $\omega_0 \in \Omega$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for every $h \in K$ we have

$$d_{\Omega}(\omega_0, \omega) < \delta \Rightarrow d_{\mathcal{H}}(A(\omega) \cdot h, A(\omega_0) \cdot h) < \varepsilon.$$

In this case we will say that the pair $(T, A)$ is a continuous cocycle by isometries of the fiber $\mathcal{H}$ over the base space $\Omega$.

### 2.1. The cohomological equation

The main problem we want to address is about the existence of a solution $B$ to the cohomological equation

\[ \begin{cases} B(T\omega) \cdot B(\omega)^{-1} = A(\omega), \\ B : \Omega \to \text{Isom}(\mathcal{H}) \text{ is continuous for the pointwise topology.} \end{cases} \quad (P1) \]

**Definition 2.2.** In the case that the above is verified, we say that the cocycle $(T, A)$ reducible. We will also consider the following related problem

\[ \begin{cases} B(T\omega) \cdot B(\omega)^{-1} = A(\omega), \\ B : \Omega \to \text{Homeo}(\mathcal{H}) \text{ is continuous for the pointwise topology.} \end{cases} \quad (P1') \]

We write $\text{Homeo}(\mathcal{H})$ for the space of homeomorphisms of $\mathcal{H}$ with the metric topology, endowed with the pointwise topology.

**Remark 2.3.** Given $C \in \text{Isom}(\mathcal{H})$ and a solution $B : \Omega \to \text{Isom}(\mathcal{H})$ to the problem $(P1)$, the function $B^* = B \cdot C^{-1}$ also verifies $(P1)$. Hence, without loss of generality, given $\omega_0 \in \Omega$ we can assume $B(\omega_0) = \text{id}_\mathcal{H}$, the identity map of $\mathcal{H}$. An analogous remark also holds for the problem $(P1')$.

**Lemma 2.4.** $\text{Isom}(\mathcal{H})$ is a closed subspace of $\text{Homeo}(\mathcal{H})$ for the pointwise topology.

**Proof.** Let $(C_i) \subset \text{Isom}(\mathcal{H})$ be a sequence that converges to $C \in \text{Homeo}(\mathcal{H})$. Pick any $h, \tilde{h} \in \mathcal{H}$. The pointwise convergence implies that

$$\lim_{i \to \infty} d_{\mathcal{H}}(C_i(h), C_i(\tilde{h})) = d_{\mathcal{H}}(C(h), C(\tilde{h})).$$

Since $d_{\mathcal{H}}(C_i(h), C_i(\tilde{h})) = d_{\mathcal{H}}(h, \tilde{h})$ we conclude that $C \in \text{Isom}(\mathcal{H})$. \qed

**Lemma 2.5.** If $T$ admits a dense orbit then $(P1) \iff (P1')$.

**Proof.** Suppose $B : \Omega \to \text{Homeo}(\mathcal{H})$ solves $(P1')$. Let $\omega_0 \in \Omega$ be such that its $T$-orbit is dense, and assume that $B(\omega_0) = \text{id}_\mathcal{H}$. We have

$$B(T^n \omega_0) = A(T^{n-1} \omega_0) \cdot A(T^{n-2} \omega_0) \cdots A(\omega_0).$$

Hence $B(T^n \omega_0) \in \text{Isom}(\mathcal{H})$ for every $n \in \mathbb{N}$. Given $\omega \in \Omega$, pick a sequence $(n_i)$ such that $T^{n_i} \omega_0 \to \omega$. Continuity of $B$, together with the previous lemma allow to conclude that $B(\omega) = \lim_{i \to \infty} B(T^{n_i} \omega_0)$ belongs to $\text{Isom}(\mathcal{H})$. The converse implication is direct. \qed
2.2. Skew-product dynamical system

Given a cocycle \((T, A)\) we can construct the following dynamics

\[
F : \Omega \times \mathcal{H} \rightarrow \Omega \times \mathcal{H}
\]

\[
(\omega, h) \mapsto (T\omega, A(\omega) \cdot h),
\]

that is called the related skew-product. The simplest, notwithstanding key example, is the case

\[A(\omega) \equiv id_{\mathcal{H}}.\]

The skew-product \(I_T(\omega, h) = (T\omega, h)\) is called the fibered identity over \(T\).

Let \((T, A)\) be a cocycle. Given \(J : \Omega \rightarrow \text{Homeo}(\mathcal{H})\) we construct

\[
J : \Omega \times \mathcal{H} \rightarrow \Omega \times \mathcal{H}
\]

\[
(\omega, h) \mapsto (\omega, J(\omega) \cdot h).
\]

We consider the following problem, written in the notation above:

\[
(P2) \quad \begin{cases}
J^{-1} \circ F \circ J = I_T, \\
J : \Omega \rightarrow \text{Homeo}(\mathcal{H}) \text{ is continuous for the pointwise topology.}
\end{cases}
\]

In that case we say that the skew-product \(F\) is conjugated to the fibered identity \(I_T\) via a fiberwise preserving homeomorphism.

**Lemma 2.6.** \((P1') \iff (P2)\).

**Proof.** The second coordinate of the relation \(J^{-1} \circ F \circ J = I_T\) is \(J(T\omega)^{-1} \cdot A(\omega) \cdot J(\omega) = id_{\mathcal{H}}\).

\[\square\]

2.3. Invariant sections

In the notation above, the dynamics of the fibered identity \(I_T\) is nothing more than the consideration of many copies of the dynamics of \(T\). In fact, \(I_T^p(\omega, h) = (T^p\omega, h)\). In this way, the space \(\Omega \times \mathcal{H}\) is decomposed (saturated) into a disjoint union of \(I_T\)-invariant sections

\[
\Omega \times \mathcal{H} = \bigcup_{h \in \mathcal{H}} \Omega \times \{h\}.
\]

Assuming the existence of a solution for \((P1')\) (and hence for \((P2)\)), we will see that \(F\) also induces a decomposition of \(\Omega \times \mathcal{H}\) into a disjoint union of continuous \(F\)-invariant sections. Let us define more precisely the elements of this claim.

Given \((\omega_0, h_0) \in \Omega \times \mathcal{H}\), we say that the function \(s : \Omega \rightarrow \mathcal{H}\) is a continuous invariant section passing through \((\omega_0, h_0)\) if

(a) \(s(\omega_0) = h_0\),

(b) For every \(\omega \in \Omega\) we have

\[A(\omega) \cdot s(\omega) = s(T\omega).\]

Example. As commented before, if \(A(\omega) \equiv id_{\mathcal{H}}\) for every \(\omega \in \Omega\) then \(F = I_T\) and for every \((\omega_0, h_0)\) the constant section \(s(\omega) := h_0\) is a continuous invariant section that passes through \((\omega_0, h_0)\).
Example. Let \((T, A)\) a continuous cocycle by isometries and such that \((P1')\) has a continuous solution \(B : \Omega \to \text{Homeo}(\mathcal{H})\) (or equivalently \((P2)\) has a solution). We know that \(\mathcal{F}(\omega, h) = (\omega, B(\omega) \cdot h)\) conjugates \(F\) with \(I_T\). Hence, for \((\omega_0, h_0)\) we define

\[
s_{\omega_0, h_0}(\omega) = B(\omega) \cdot B(\omega_0)^{-1} \cdot h_0.
\]

Continuity is direct from pointwise continuity of \(B\). We can verify the conditions:

(a) \(s_{\omega_0, h_0}(\omega_0) = B(\omega_0) \cdot B(\omega_0)^{-1} \cdot h_0 = h_0\).

(b) \(A(\omega) \cdot s_{\omega_0, h_0}(\omega) = A(\omega) \cdot B(\omega) \cdot B(\omega_0)^{-1} \cdot h_0 = B(T_\omega) \cdot B(\omega_0)^{-1} \cdot h_0 = s_{\omega_0, h_0}(T_\omega)\).

This example, and its converse, are the key of this work, since relate the solution of the cohomological equation with the existence of continuous invariant sections for the induced skew-product, that passes through every point of the phase space. Indeed, we will consider the following problem.

Given a continuous cocycle by isometries \((T, A)\), and the corresponding skew-product dynamics \(F(\omega, h) = (T_\omega, A(\omega) \cdot h)\), we consider the following problem

\[
(P3) \begin{cases}
\text{For every } (\omega_0, h_0) \text{ there exists a continuous invariant section } s_{\omega_0, h_0} \\
\text{that passes through } (\omega_0, h_0).
\end{cases}
\]

For every \(\omega_0, \omega \in \Omega\) the map \(h_0 \mapsto s_{\omega_0, h_0}(\omega)\) is continuous.

**Definition 2.7.** In the case that the above is verified, we say that the cocycle \((T, A)\) has a phase space saturated by invariant sections.

**Lemma 2.8.** Let \((T, A)\) be a continuous cocycle by isometries, such that \(T\) admits a dense orbit. If \(s, \tilde{s}\) are two continuous invariant sections that pass through \((\omega_0, h_0) \in \Omega \times \mathcal{H}\) then \(s = \tilde{s}\).

**Proof.** Assume there exists \(\omega_1 \in \Omega\) such that \(\tilde{s}(\omega_1) \neq s(\omega_1)\). Let \(\omega_s\) be such that its \(T\)-orbit is dense. Taking \(T^{n_1}\omega_s\), close enough to \(\omega_1\), we can find a positive integer \(n_s \in \mathbb{N}\) such that

\[
d_{H}(\tilde{s}(T^{n_1}\omega_s), s(T^{n_1}\omega_s)) > \frac{1}{2}d_{H}(\tilde{s}(\omega_1), s(\omega_1)) > 0. \tag{3}
\]

Let \((n_j)\) be a sequence of positive integers such that \(T^{n_j+n_{i_j}}\omega_s \to \omega_0\). Since \(s, \tilde{s}\) are continuous and both pass through \((\omega_0, h_0)\), we obtain that

\[
\lim_{j \to \infty} s(T^{n_j+n_{i_j}}\omega_s) = s(\omega_0) = h_0,
\]

\[
\lim_{j \to \infty} \tilde{s}(T^{n_j+n_{i_j}}\omega_s) = \tilde{s}(\omega_0) = h_0.
\]

Then we have

\[
\lim_{j \to \infty} d_{H}(\tilde{s}(T^{n_j+n_{i_j}}\omega_s), s(T^{n_j+n_{i_j}}\omega_s)) = 0. \tag{4}
\]
Since both $s, \tilde{s}$ are invariant by $F$ we have
\[ s(T^{n+h}\omega_s) = A^n(T^n\omega_s) \cdot s(T^n\omega_s), \]
\[ \tilde{s}(T^{n+h}\omega_s) = A^n(T^n\omega_s) \cdot s(T^n\omega_s). \]
As $A^n(T^n\omega_s) \in \text{Isom}({\mathcal{H}})$ for every $j$, we obtain
\[ d_H(\tilde{s}(T^{n+h}\omega_s), s(T^{n+h}\omega_s)) = d_H(\tilde{s}(T^n\omega_s), s(T^n\omega_s)), \]
which is incompatible with (3) and (4).

**Remark 2.9.** Note that the problem (P3) can also be raised for general cocycles $A : \Omega \to \mathcal{G}$, where $\mathcal{G}$ is a group that acts by homeomorphisms of a topological space. The formal construction of the skew-product is analogous.

**Proposition 2.10.** Let $(T, A)$ be a continuous cocycle by isometries. If $T$ admits a dense orbit then $(P1) \iff (P3)$.

**Proof.** We only have to show that (P3) implies (P2). Fix $\omega_0 \in \Omega$. For $\omega \in \Omega$ we define the map $B(\omega)$ by
\[ B(\omega) \cdot h = s_{\omega_0,h}(\omega). \]
Let us verify that $B$ fulfill the conditions on (P2). First, we note that $B(\omega_0) \cdot h = s_{\omega_0,h}(\omega_0) = h$ and hence $B(\omega_0) = id_\mathcal{H}$. We also have
\[ B(T\omega) \cdot h = s_{\omega_0,h}(T\omega) \]
\[ = A(\omega) \cdot s_{\omega_0,h}(\omega) \]
\[ = A(\omega) \cdot B(\omega) \cdot h, \]
thus, $B(T\omega) = A(\omega) \cdot B(\omega)$. We still have to verify that $B(\omega) \in \text{Homeo}(\mathcal{H})$. The function $h \mapsto s_{\omega_0,h}(\omega) = B(\omega) \cdot h$ is continuous by (P3). Thanks to the uniqueness of the invariant section, the inverse of $B(\omega)$ can be easily computed as
\[ B(\omega)^{-1} \cdot g = s_{\omega_0,g}(\omega), \]
which is also continuous. The pointwise continuity of $\omega \mapsto B(\omega) \in \text{Homeo}(\mathcal{H})$ comes from the continuity of $\omega \mapsto s_{\omega_0,h}(\omega)$. \hfill \Box

In the previous proof, the choice of $\omega_0$ is arbitrary, giving account of the non-uniqueness of the solution to the cohomological equation.

### 3. Classic Livšic theorem

In this section we revisit the classic Livšic theorem on cohomological equations (see [19]). Following [3], we define a *hyperbolic homeomorphism* as follows. Given a compact metric space $\Omega$, a homeomorphism $T : \Omega \to \Omega$ and an arbitrary point $\omega \in \Omega$, let us consider the *local stable and unstable sets at $\omega$* given by
\[ W^s_\epsilon(\omega) = \{ \tilde{\omega} \in \Omega \mid d_\Omega(T^n\omega, T^n\tilde{\omega}) \leq \epsilon, \ \forall \ n \geq 0 \}; \]
\[ W^u_\epsilon(\omega) = \{ \tilde{\omega} \in \Omega \mid d_\Omega(T^{-n}\omega, T^{-n}\tilde{\omega}) \leq \epsilon, \ \forall \ n \geq 0 \}, \]
where $\epsilon$ is any positive real number.
**Definition 3.1.** The homeomorphism $T$ is said hyperbolic whenever it is bi-Lipschitz (both $T$ and $T^{-1}$ are Lipschitz), transitive and there exists constants $\varepsilon, \delta, C_0, \tau > 0$ and continuous functions $\nu_1, \nu_2 : \Omega \to (-\infty, \infty)$ such that the following properties hold:

(a) $d_{\Omega}(T\omega', T\omega'') \leq \nu_1(\omega)d_{\Omega}(\omega', \omega'')$ for any $\omega', \omega'' \in W^s_\varepsilon(\omega)$ and any $\omega \in \Omega$;

(b) $d_{\Omega}(T^{-1}\omega', T^{-1}\omega'') \leq \nu_2(\omega)d_{\Omega}(\omega', \omega'')$ for any $\omega', \omega'' \in W^s_\varepsilon(\omega)$ and any $\omega \in \Omega$;

(c) $\nu_1(T^{-n}\omega) \ldots \nu_1(\omega) \leq C_0e^{-\tau n}$ for any $\omega \in \Omega$ and any $n \geq 1$;

(d) $\nu_2(T^{-(n-1)}\omega) \ldots \nu_2(\omega) \leq C_0e^{-\tau n}$ for any $\omega \in \Omega$ and any $n \geq 1$.

(e) If $d_{\Omega}(\omega, \omega') < \varepsilon$, then $W^s_\varepsilon(\omega') \cap W^s_\varepsilon(\omega')$ consists of exactly one point, which depends continuously on $(\omega, \omega')$.

Important examples of maps satisfying these properties are hyperbolic diffeomorphisms of compact manifolds.

Given a continuous function $\psi : \Omega \to \mathbb{R}$, we consider the real valued cohomological equation

$$u(T\omega) - u(\omega) = \psi(\omega).$$

Relating the function $u$ with the real translation $t \mapsto t + u(\omega)$, the relation (2) gives raise to the following obstruction to the existence of solutions to (5)

$$(\text{PPO})_{\mathbb{R}} \iff \begin{align*}
\omega &\in \Omega \text{ and } n \in \mathbb{N} \text{ such that } T^n\omega = \omega \\
\sum_{j=0}^{n-1} \psi(T^j\omega) &= 0.
\end{align*}$$

**Theorem 3.2** (see [Livšic [19]]). Let $T : \Omega \to \Omega$ be a hyperbolic homeomorphism defined on a compact metric space $\Omega$. Given a Hölder function $\psi : \Omega \to \mathbb{R}$ verifying the (PPO)$_{\mathbb{R}}$ condition, there exists a Hölder solution $u : \Omega \to \mathbb{R}$ to the cohomological equation (5).

**Remark 3.3.** Given two continuous solutions $u_1, u_2$ to (5), the difference $\Delta u = u_1 - u_2$ verifies $\Delta u(T\omega) = \Delta u(\omega)$ for every $\omega \in \Omega$. Since $T$ admits a dense orbit we conclude that $\Delta u$ is constant. In other words, any two continuous solutions to (5) differ by a constant.

4. Gromov hyperbolic spaces and Busemann functions

This section collects some geometric elements that we use in the setting of our main result (theorem 5.3). The main reference is the almost comprehensive book by Bridson and Häfliger [4], chapter III.

4.1. Gromov hyperbolic spaces

Given a proper metric space $(\mathcal{H}, d_{\mathcal{H}})$, a geodesic is an isometric map $\gamma : [a, b] \subset \mathbb{R} \to \mathcal{H}$. The space $\mathcal{H}$ is called a geodesic space if every pair of points can be joined by a geodesic; if this geodesic is unique we call $\mathcal{H}$ a uniquely geodesic space. A geodesic triangle (or simply a triangle) is a set consisting of the union of three geodesics that join three points $x, y, z$ between each other. Points are called vertices and the geodesic that join two vertices, $x, y$, denoted by $[x, y]$, is called a side of the triangle.
Definition 4.1. Let $\delta > 0$. A geodesic triangle $\triangle ABC$ is called $\delta$-slim if each of its sides is contained in a $\delta$ neighborhood of the union of the other two sides. That is, for every point $p \in [A, B]$ there exists $q \in [B, C] \cup [A, C]$ such that $d_{\mathcal{H}}(p, q) \leq \delta$. A uniquely geodesic space $\mathcal{H}$ is called Gromov hyperbolic if there exists $\delta > 0$ so that every triangle is $\delta$-slim.

Examples of Gromov hyperbolic spaces are real trees (non directed graphs in which two vertices are joined by exactly one path). The classic Poincaré hyperbolic space $\mathbb{H}^2$ is also Gromov hyperbolic, and, as a consequence, CAT($\kappa$) spaces are Gromov hyperbolic, for $\kappa < 0$. A great deal of interesting examples of Gromov hyperbolic spaces were given by Gromov in [12].

Definition 4.2. We say that two geodesic rays $\gamma_1, \gamma_2 : [0, \infty] \to \mathcal{H}$, are equivalent if there exists a real number $C > 0$ such that $\sup_{t \geq 0} d_{\mathcal{H}}(\gamma_1(t), \gamma_2(t)) \leq C$. The Gromov boundary at infinity $\partial \mathcal{H}$ is defined as the equivalence classes among all geodesic rays of the space. Given a geodesic $\gamma : [-\infty, \infty] \to \mathcal{H}$, we write $\gamma(+\infty) = [\gamma]$ and $\gamma(-\infty) = [t \mapsto \gamma(-t)]$, and we call them the ends of $\gamma$.

Proposition 4.3 [see [4] III.3.2.]. Let $\mathcal{H}$ be a Gromov hyperbolic space. The boundary $\partial \mathcal{H}$ is visible, that is, given $x \in \mathcal{H}$ and a boundary point $\alpha \in \partial \mathcal{H}$, there exists a unique geodesic ray $\gamma : [0, \infty] \to \mathcal{H}$ such that $\gamma(+\infty) = \alpha$ and $\gamma(0) = x$. Moreover, given two boundary points $\alpha, \beta \in \partial \mathcal{H}$ there exists a geodesic $\gamma : [-\infty, \infty] \to \mathcal{H}$ such that $\gamma(+\infty) = \alpha$ and $\gamma(-\infty) = \beta$.

Definition 4.4. A Gromov hyperbolic space is said a uniquely visible Gromov hyperbolic space when the geodesic joining every pair of boundary points is unique.

A generalized geodesic ray is a geodesic ray $\gamma : [0, \infty] \to \mathcal{H}$ or a path that is a geodesic until a certain point and then it is constant. The next definition provides a topology for $\partial \mathcal{H}$. From now on we will assume that $\mathcal{H}$ is a uniquely visible Gromov hyperbolic space.

Definition 4.5. Let $p \in \mathcal{H}$. We say that a sequence $(x_n) \subset \mathcal{H}$ converges to a point $x \in \mathcal{H} \cup \partial \mathcal{H}$, if there exist generalized geodesic rays $(\gamma_n)$ such that $\gamma_n(0) = p$ and $\lim_{n \to \infty} \gamma_n(x_n) = x$, and verifying that every subsequence of $(\gamma_n)$ contains a sub subsequence that converges, in the compact-uniform topology, to a generalized ray $\gamma$ with $\gamma(+\infty) = x$.

Given an isometry $A \in \text{Isom}(\mathcal{H})$ and a geodesic ray $\gamma : [0, \infty] \to \mathcal{H}$, the image $A \cdot \gamma$ is a geodesic ray and hence the action of $A$ on $\mathcal{H}$ can be extended to an action $A^* : \partial \mathcal{H} \to \partial \mathcal{H}$ by considering the equivalence classes $A^* \cdot [\gamma] = [A \cdot \gamma]$. Since $A$ is an isometry, $A^*$ is well defined.

The proofs of the following set of lemmas can be found in [4] III.3.2.

Lemma 4.6. Given two isometries $A, B$ we have $(A \circ B)^* = A^* \circ B^*$.

Lemma 4.7. $A^* = id_{\partial \mathcal{H}}$ if and only if $A = id_{\mathcal{H}}$.

Lemma 4.8. If $A \in \text{Isom}(\mathcal{H})$ then $A^* : \partial \mathcal{H} \to \partial \mathcal{H}$ is a homeomorphism.

Lemma 4.9. Let $\alpha_0, \beta_0 \in \partial \mathcal{H}$ and the geodesic $\gamma_0 : [-\infty, \infty] \to \mathcal{H}$ such that $\gamma_0(-\infty) = \beta_0 \in \mathcal{H}$ and $\gamma_0(+\infty) = \alpha_0$. Consider two sequences $(\alpha_n, \beta_n)_{n \in \mathbb{N}}$ in $\partial \mathcal{H}$ such that $\beta_n \to \beta$ and $\alpha_n \to \alpha$. Define $\gamma_n$ as the complete geodesic given by proposition 4.3. such that $\gamma_n(-\infty) = \beta_n$ and $\gamma_n(+\infty) = \alpha_n$. For any $r > 0, s \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that for $n > N$

$$\gamma_n \cap B(\gamma_0(s), r) \neq \emptyset.$$. 
Lemma 4.10. Let \( h \in \mathcal{H} \) and \( r > 0 \) small enough. Then for every pair of points \( x, y \) that belong to the ball \( B(h, r) \), the geodesic segment \( \gamma_{x,y} \) is contained in \( B(h, r) \). That is, the ball \( B(h, r) \) is convex.

Lemma 4.11. Let \( \alpha \in \partial \mathcal{H} \). For every \( h \in \mathcal{H} \) let \( \gamma_{h} : [0, \infty) \to \mathcal{H} \) be the unique geodesic ray joining \( h \) with \( \alpha \). Extend \( \gamma_{h} \) to a complete geodesic and denote by \( \beta_{h} \) the point \( \gamma_{h}(-\infty) \in \partial \mathcal{H} \). The map \( h \mapsto \beta_{h} \) is continuous.

4.2. Busemann functions

Let \( (\mathcal{H}, d_{\mathcal{H}}) \) be a Gromov hyperbolic space. Given \( \alpha \in \partial \mathcal{H} \) and a point \( p \in \mathcal{H} \), we define the Busemann function \( b_{p,\alpha} : \mathcal{H} \to \mathbb{R} \), in the direction \( \alpha \) and with base point \( p \) as

\[
b_{p,\alpha}(h) = \lim_{n \to \infty} d_{\mathcal{H}}(x_{n}, h) - d_{\mathcal{H}}(x_{n}, p),
\]

where \( (x_{n}) \subset \mathcal{H} \) is any sequence such that \( x_{n} \to \alpha \). The convergence and independence on \( (x_{n}) \) relies on the triangle inequality and the hyperbolicity of \( \mathcal{H} \).

Definition 4.12. Given any real number \( r \in \mathbb{R} \), the level set \( b_{p,\alpha}^{-1}(r) \) is called an horosphere centered at \( \alpha \).

Lemma 4.13. Let \( \gamma : \mathbb{R} \to \mathcal{H} \) be a geodesic. If \( \gamma(+\infty) = \alpha \) and \( b_{p,\alpha}(\gamma(0)) = 0 \) then \( b_{p,\alpha}(\gamma(s)) = b_{p,\alpha}(\gamma(0)) + s \) for every \( r, s \in \mathbb{R} \).

Proof. Note that \( x_{n} = \gamma(n) \) verifies \( x_{n} \to \alpha \). We have

\[
d_{\mathcal{H}}(\gamma(n), \gamma(s)) - d_{\mathcal{H}}(\gamma(n), p) = n - s - (d_{\mathcal{H}}(\gamma(n), \gamma(0)) + d_{\mathcal{H}}(\gamma(n), \gamma(0)))
\]

\[
= -s + (d_{\mathcal{H}}(\gamma(n), \gamma(0)) - d_{\mathcal{H}}(\gamma(n), p)).
\]

Taking \( n \to \infty \) above we obtain

\[
b_{p,\alpha}(\gamma(s)) = -s - b_{p,\alpha}(\gamma(0))
\]

\[
= -s. \square
\]

Lemma 4.14. Let \( A \in \text{Isom}(\mathcal{H}) \), \( \alpha \in \partial \mathcal{H} \) and \( p \in \mathcal{H} \). The following holds

\[
b_{p,A \cdot \alpha}(A \cdot p) = -b_{p,\alpha}(A^{-1} \cdot p) = b_{A^{-1} p, \alpha}(p).
\]

Proof. Let \( (x_{n}) \subset \mathcal{H} \) such that \( x_{n} \to \alpha \). Hence \( A \cdot x_{n} \to A^{\ast} \cdot \alpha \). Then

\[
b_{p,A \cdot \alpha}(A \cdot p) = \lim_{n \to \infty} d_{\mathcal{H}}(A \cdot x_{n}, A \cdot p) - d_{\mathcal{H}}(A \cdot x_{n}, p)
\]

\[
= \lim_{n \to \infty} - (d_{\mathcal{H}}(x_{n}, A^{-1} \cdot p) - d_{\mathcal{H}}(x_{n}, p))
\]

\[
= -b_{p,\alpha}(A^{-1} \cdot p). \square
\]

Lemma 4.15. If \( \alpha \in \partial \mathcal{H} \) and \( p \in \mathcal{H} \) then for every \( h, g \in \mathcal{H} \) one has

\[
|b_{p,\alpha}(h) - b_{p,\alpha}(g)| \leq d_{\mathcal{H}}(h, g).
\]
The geodesic space $H$ isometrically embedded in the boundary $\partial H$ has a continuous action by isometries such that there exists a pointwise continuous solution

$$\tilde{d}_H(x_n, h) - d_H(x_n, p) = \tilde{d}_H(x_n, h) + d_H(x_n, p).$$

We conclude taking $n \to \infty$. \hfill \Box

**Lemma 4.16.** Given $p, p_1, p_2, h \in H$ and $\alpha \in \partial H$ we have

$$b_{p_1, \alpha}(h) = b_{p_2, \alpha}(h) + b_{p_1, \alpha}(p_2).$$

**Proof.** Given $x_n \to \alpha$ one knows

$$d_H(x_n, h) - d_H(x_n, p_1) = d_H(x_n, h) - d_H(x_n, p_2) + d_H(x_n, p_2) - d_H(x_n, p_1).$$

We conclude taking $n \to \infty$. \hfill \Box

**5. Boundary solutions and interior solutions**

**Definition 5.1.** Let $(T, A)$ be a continuous cocycle by isometries of a Gromov hyperbolic space $H$. By considering the boundary action $A^*: \Omega \to \text{Homeo}(\partial H)$ we can induce a cocycle by homeomorphisms of the boundary $\partial H$, defined as $A^*(\omega) = (A(\omega))^*$. We call $(T, A')$ the boundary cocycle induced by $(T, A)$.

**Lemma 5.2.** If $(T, A)$ is continuous then $(T, A')$ is continuous for the pointwise topology.

**Proof.** Let $\omega_n \to \omega \in \Omega$, $\alpha \in \partial H$ and a geodesic $\gamma$ such that $\gamma(+\infty) = \alpha$. Since $A$ is continuous for the compact-uniform topology, the geodesics $A(\omega_n) \cdot \gamma$ converge to the geodesic $A(\omega) \cdot \gamma$ in the compact-uniform topology. This implies that $A^*(\omega_n) \cdot \alpha \to A^*(\omega) \cdot \alpha$. \hfill \Box

This lemma allows to fit the boundary cocycle $(T, A')$ into the framework of the problem (P1'), that is, to look for a pointwise continuous solution $B: \Omega \to \text{Homeo}(\partial H)$ to the cohomological equation $B(T(\omega) \cdot B(\omega)^{-1} = A^*(\omega)$.

This work is entirely devoted to relate the solution of the cohomological equations for the cocycle of isometries $(T, A)$ (so called interior solution) with the solution of the cohomological equation for the boundary cocycle $(T, A')$ (so called boundary solution).

Obtaining solutions to the problems ((P1), (P2) or (P3)) posed for the boundary cocycle (boundary solutions) from the existence of solutions for the cocycle by isometries (interior solutions) can be addressed in several ways. For instance, if $(T, A)$ is a continuous cocycle by isometries such that there exists a pointwise continuous solution $B: \Omega \to \text{Isom}(H)$ of the cohomological equation $B(T(\omega) \cdot B(\omega)^{-1} = A(\omega)$, then lemma 4.6. gives

$$B^*(T(\omega) \cdot B^*(\omega)^{-1} = A^*(\omega).$$

However, pointwise continuity of $B^*$ cannot be assured in all situations. The existence of solutions to the problem (P3) for the cocycle $(T, A)$ gives a solution to (P3) for the boundary cocycle $(T, A^*)$ in the following way. Assume $(T, A)$ has solutions to (P3), that is, through every point $(\omega_0, h_0)$ a continuous $A$-invariant section $s_{\omega_0, h_0}(\omega)$ passes, and such that for fixed $h(\omega_0, \omega)$ the map $h \mapsto s_{\omega_0, h}(\omega)$ is continuous. Fix $p \in H$ and any pair $(\omega_0, \alpha_0) \in \Omega \times \partial H$. Let $\gamma_0: \mathbb{R}^+ \to H$ be the geodesic ray that connects $p$ to $\alpha_0$ and let $\overline{p} = \gamma_0(1)$. Let $\gamma_0: \mathbb{R}^+ \to H$ be the geodesic ray that starts at $s_{\omega_0, p}(\omega)$ and passes through $s_{\omega_0, \alpha}(\omega)$. Define $s_{\omega_0, \alpha}(\omega) = \gamma_0(+\infty)$. It is easy to show that $s_{\omega_0, \alpha}(\omega)$ is a continuous $A^*$-invariant section that passes through $(\omega_0, \alpha_0)$.  

1644
The next is the main result in this work, showing that in the context of the Lišíc results, somehow the converse of the above relations holds. The definitions of the required $\text{à la }$ Hölder conditions are described in section 5.2. Recall definitions 2.2 and 2.7 for reducibility and saturation by invariant sections, respectively. The proof is developed along the following sections.

**Theorem 5.3.** Let $T: \Omega \to \Omega$ be a hyperbolic homeomorphism (in the sense of section 3) defined on a compact metric space $\Omega$. Let $(T,A)$ be a continuous cocycle by isometries of a uniquely visible Gromov hyperbolic space $H$. If $(T,A)$ satisfies

- $(T,A)$ is $\tau$-Hölder for some $\tau > 0$;
- The boundary cocycle $(T,A^*)$ has a phase space that is saturated by invariant sections, and these sections are of class $\tau$-Hölder–Busemann;

then $(T,A)$ is reducible.

5.1. Periodic point obstruction

As discussed during the introduction, we see that a necessary condition on $(T,A)$ in order to admit a solution to the cohomological equation (problem $(P1')$ or problem $(P1)$) is the following periodic point obstruction $(PPO)$.

$$
\begin{align*}
\omega \in \Omega \text{ and } n \in \mathbb{N} \text{ such that } T^n\omega &= \omega \\
A^n(\omega) &= \text{id}_H.
\end{align*}
$$

The PPO condition on $(T,A)$ can be rewritten in the following way in terms of the skew-product $F$.

$$
\begin{align*}
\omega \in \Omega \text{ and } n \in \mathbb{N} \text{ such that } T^n\omega &= \omega \\
F^n(\omega,h) &= (\omega, h) \forall h \in H.
\end{align*}
$$

**Lemma 5.4.** If the boundary cocycle $(T,A^*)$ has a phase space saturated by invariant sections then the cocycle by isometries $(T,A)$ verifies the $(PPO)$ condition.

**Proof.** Let $\omega \in \Omega$ be such that $T^n\omega = \omega$. Let $\alpha \in \partial H$ and $s^\omega_{\alpha,\omega}$ be the invariant section that passes through $\alpha$ at $\omega$. Since the section is invariant, we see that $(A^*)^n(\omega) \cdot \alpha = s^\omega_{\alpha,\omega}(T^n\omega) = s^\omega_{\alpha,\omega}(\omega) = \alpha$, that is, $(A^*)^n(\omega) = \text{id}_{\partial H}$. Lemmas 4.6 and 4.7 allow to conclude that $A^n(\omega) = \text{id}_H$. \(\square\)

5.2. Hölder conditions

In this section we introduce the suitable Hölder conditions that we use in the statement of our main result.

**Definition 5.7.** We say that a cocycle by isometries $(T,A)$ is $\tau$-Hölder, $\tau > 0$, if for every bounded set $K \subset H$ there exists $C_K > 0$ such that for every $\omega_1, \omega_2 \in \Omega$, $h \in K$ one has

$$
d_H(A(\omega_1) \cdot h, A(\omega_2) \cdot h) \leq C_K d_H(\omega_1, \omega_2)^\tau.
$$
Lemma 5.8. Let \((T, A)\) be a \(\tau\)-Hölder cocycle by isometries and \(p \in \mathcal{H}\). There exists \(C_p > 0\) such that for every \(\omega_1, \omega_2 \in \Omega\)

\[
d_H(A(\omega_1)^{-1} \cdot p, A(\omega_2)^{-1} \cdot p) \leq C_p d_\Omega(\omega_1, \omega_2)^\tau.
\]

Proof. Since \(A(\omega_1)\) is an isometry, for every \(h\) we have

\[
d_h(A(\omega_1) \cdot h, A(\omega_2) \cdot h) = d_h(h, A(\omega_1)^{-1} \cdot A(\omega_2) \cdot h).
\]

Notice that the set \(K_p = \{A^{-1}(\omega) \cdot p \mid \omega \in \Omega\}\) is bounded (since \(d_h(A(\omega)^{-1} \cdot p, p) = d_h(p, A(\omega) \cdot p)\), and hence we can take a uniform constant \(C_p = C_{K_p}\) such that for every \(h = A(\omega_2)^{-1} \cdot p \in K_p\) we have the desired inequality. \(\square\)

Definition 5.9. We say that a section \(\alpha : \Omega \to \partial \mathcal{H}\) is \(\tau\)-Hölder–Busemann, \(\tau > 0\), if for every bounded set \(K \subset \mathcal{H}\) there exists \(D_K > 0\) such that for every \(\omega_1, \omega_2 \in \Omega, h \in K\) one has

\[
|b_{p,\alpha(\omega_1)}(h) - b_{p,\alpha(\omega_2)}(h)| \leq D_K d_\Omega(\omega_1, \omega_2)^\tau.
\]

Lemma 5.10. Let \(\alpha : \Omega \to \partial \mathcal{H}\) be a \(\tau\)-Hölder–Busemann section and \(p \in \mathcal{H}\). There exists \(D_p > 0\) such that for every \(\omega_1, \omega_2 \in \Omega\)

\[
|b_{p,\alpha(\omega_1)}(A(\omega_1)^{-1} \cdot p) - b_{p,\alpha(\omega_2)}(A(\omega_2)^{-1} \cdot p)| \leq D_p d_\Omega(\omega_1, \omega_2)^\tau.
\]

5.3. The induced cocycle in the space of horospheres

In general, for a boundary point \(\alpha \in \partial \mathcal{H}\) and an isometry \(A\) of \(\mathcal{H}\), the action of \(A\) takes a horosphere \(B\), centered at \(\alpha\), into the horosphere \(A \cdot B\), centered at \(A^* \cdot \alpha\). We are going to take advantage of this fact to produce an auxiliary cocycle that will be crucial in our proof.

Suppose that there exists a section \(\alpha : \Omega \to \partial \mathcal{H}\) that is continuous and invariant for the action of \(A^*\) on the boundary \(\partial \mathcal{H}\), that is \(A^*(\omega) \cdot \alpha(\omega) = \alpha(T_\omega)\) for every \(\omega \in \Omega\). Roughly speaking, this invariant section in the boundary, together with the considerations of the previous paragraph, allows us to see that the skew-product \(\tilde{F}\), induced by the cocycle by isometries \((T, A)\), acts as a skew-product on a fibered space that consist of \(\Omega\) in the base and, for \(\omega \in \Omega\), the fiber consists in the disjoint union of horospheres centered at \(\alpha(\omega)\). More precisely,

\[
\tilde{F}(\omega, B) = (T_\omega, A(\omega) \cdot B),
\]

where \(B\) is a horosphere centered at \(\alpha(\omega)\). Notice that \(A(\omega) \cdot B\) is a horosphere centered at \(A^*(\omega) \cdot \alpha(\omega) = \alpha(T_\omega)\). Since horospheres are objects of codimension one, more precisely, they are level sets of Busemann functions, the action of \(\tilde{F}\) can be seen as an action on the product \(\Omega \times \mathbb{R}\). In the next paragraphs we develop in details this construction, taking special care of the parameterization on the set of horospheres that performs this idea in a rigorous way.
From here on we will assume that \( \mathcal{H} \) is a uniquely visible Gromov hyperbolic space. Fix a base point \( p \in \mathcal{H} \). For every \( \omega \in \Omega \) we consider the unique geodesic \( \eta_\omega : \mathbb{R} \to \mathcal{H} \) such that \( \eta_\omega(0) = p \) and \( \eta_\omega(+\infty) = \alpha(\omega) \). We will identify the horosphere \( b_{\rho_0(\omega)}^{-1}(r) \) with the (unique) point in the intersection \( \eta_\omega \cap b_{\rho_0(\omega)}^{-1}(r) \). Thanks to lemma 4.13 we know that this point corresponds to \( \eta_\omega(-r) \). We define then the following skew-product map that collects the ideas of the previous discussion about the action of \( F \) in the set of horospheres.

\[
V : \Omega \times \mathbb{R} \to \Omega \times \mathbb{R},
\]

\[
(\omega, t) \mapsto (T\omega, -b_{\rho_0(T\omega)}(A(\omega) \cdot \eta_\omega(t)))
\]

**Lemma 5.11.** The skew-product \( V \) is a continuous cocycle by translations of the real line taking the form

\[
V(\omega, t) = (T\omega, t + \phi(\omega)),
\]

where

\[
\phi(\omega) = -b_{\rho_0(T\omega)}(A(\omega) \cdot p).
\]

**Proof.** Since \( A(\omega) \) is an isometry, the curve \( A(\omega) \cdot \eta_\omega \) is a geodesic, and the invariance of \( \alpha \) under \( A^\ast \) implies

\[
(A(\omega) \cdot \eta_\omega)(+\infty) = \alpha(T\omega).
\]

Lemma 4.13 implies

\[
-b_{\rho_0(T\omega)}(A(\omega) \cdot \eta_\omega(t)) = t - b_{\rho_0(T\omega)}(A(\omega) \cdot \eta_\omega(0)) = t - b_{\rho_0(T\omega)}(A(\omega) \cdot p).
\]

With the aim of formalizing the idea that the action induced by \( F \) on the set of horospheres centered on \( \alpha \) is given precisely by \( V \), we define the function \( P(\omega, h) = (\omega, -b_{\rho_0(\omega)}(h)) \), which assigns to each point \( h \in \mathcal{H} \) the parameter corresponding to the horosphere centered on \( \alpha(\omega) \) that contains \( h \).

**Lemma 5.12.** Using the preceding notation, we have

\[
P \circ F = V \circ P.
\]

**Proof.** Since \( A \) is an isometry and \( A^\ast(\omega) \cdot \alpha(\omega) = \alpha(T\omega) \), for every \( h \in \mathcal{H} \) we have

\[
b_{A(\omega) \circ \rho_0(T\omega)}(A(\omega) \cdot h) = b_{\rho_0(\omega)}(h).
\]

Moreover, lemma 4.16 implies

\[
b_{A(\omega) \circ \rho_0(T\omega)}(A(\omega) \cdot h) = b_{\rho_0(T\omega)}(A(\omega) \cdot h) - b_{\rho_0(T\omega)}(A(\omega) \cdot p).
\]

Equating the right sides of equations (8) and (9), we obtain

\[
-b_{\rho_0(T\omega)}(A(\omega) \cdot h) = -b_{\rho_0(\omega)}(h) - b_{\rho_0(T\omega)}(A(\omega) \cdot p),
\]

which is the second coordinate of the equality \( P \circ F = V \circ P \).
Proof. Let \( \omega_1, \omega_2 \in \Omega \). Recall that \( \alpha(T\omega_1) = A^*(\omega_1) \cdot \alpha(\omega_1) \) and \( \alpha(T\omega_2) = A^*(\omega_2) \cdot \alpha(\omega_2) \). Hence, lemma 4.14 implies that
\[
\phi(\omega_2) - \phi(\omega_1) = b_{\rho(\omega_1)}(A(\omega_1) \cdot p) - b_{\rho(\omega_2)}(A(\omega_2) \cdot p) \\
= b_{\rho(\omega_2)}(A^{-1}(\omega_1) \cdot p) - b_{\rho(\omega_1)}(A^{-1}(\omega_2) \cdot p) \\
+ b_{\rho(\omega_2)}(A^{-1}(\omega_2) \cdot p) - b_{\rho(\omega_1)}(A^{-1}(\omega_1) \cdot p).
\]
Combining lemmas 4.15, 5.8 and 5.10, we obtain
\[
|\phi(\omega_2) - \phi(\omega_1)| \leq |b_{\rho(\omega_2)}(A^{-1}(\omega_2) \cdot p) - b_{\rho(\omega_1)}(A^{-1}(\omega_1) \cdot p)| \\
+ |b_{\rho(\omega_2)}(A^{-1}(\omega_1) \cdot p) - b_{\rho(\omega_1)}(A^{-1}(\omega_1) \cdot p)| \\
\leq d_\Omega(A^{-1}(\omega_2) \cdot p, A^{-1}(\omega_1) \cdot p) + Cd_\Omega(\omega_1, \omega_2)^2 \\
\leq 2Cd_\Omega(\omega_1, \omega_2)^2.
\]
\[\square\]

Lemma 5.14. If \((T, \Omega)\) verifies the (PPO) condition then \(\phi\) verifies the \((\text{PPO})_{\mathbb{R}}\) condition.

Proof. Let \( \bar{\omega} \in \Omega \) and \( \bar{n} \in \mathbb{N} \) such that \( T^{\bar{n}}\bar{\omega} = \bar{\omega} \). Notice that (7) implies \( P \circ F^\bar{n} = V^\bar{n} \circ P \). The (PPO') condition reads \( F^\bar{n}(\bar{\omega}, p) = (\bar{\omega}, p) \), which gives
\[
P(\bar{\omega}, p) = V^\bar{n}(P(\bar{\omega}, p)) \\
= V^\bar{n}(\bar{\omega}, 0) \\
= \left( \bar{\omega}, \sum_{j=0}^{\bar{n}-1} \phi(T^j\bar{\omega}) \right).
\]
Since \( P(\bar{\omega}, p) = (\bar{\omega}, 0) \) we obtain \( \sum_{j=0}^{\bar{n}-1} \phi(T^j\bar{\omega}) = 0 \).
\[\square\]

5.4. Proof of the theorem 5.3

According to proposition 2.10 we only need to show that \((T, A)\) has a phase space saturated by invariant sections. The strategy is as follows. Given any point \((\omega_0, h_0) \in \Omega \times \mathcal{H}\), a direct application of the classic Liñśsic theorem to the cocycle \( V \) will provide us with a continuous section of horospheres that is invariant under \( A \) and such that \( h_0 \) is contained in the horosphere corresponding to \( \omega_0 \). This is an invariant section of co-dimension one objects. In order to obtain an invariant section with exactly one point on each fiber, we are going to create a complementary invariant section of one-dimensional objects (geodesics), so that the geodesic corresponding to \( \omega_0 \) contains \( h_0 \). The intersection of these two invariant sections will provide us with the desired invariant section taking values on \( \mathcal{H} \). Rigorous details are given below.

Let \((\omega_0, h_0) \in \Omega \times \mathcal{H}\). Take \( \alpha_0 \in \partial\mathcal{H} \). Let \( \alpha : \Omega \to \partial\mathcal{H} \) be the \( \tau\)-Hölder–Busemann section that is invariant under \( A^* \) and such that \( \alpha(\omega_0) = \alpha_0 \). Under the hypotheses of the theorem 5.3, lemmas 5.4, 5.13 and 5.14 allow us to apply the classic Liñśsic theorem to the real valued
cocycle \((T, V)\) (constructed using the current invariant section \(\alpha\)). This gives us a \(\tau\)-Hölder solution \(u : \Omega \to \mathbb{R}\) to the cohomological equation

\[
u(T \omega) - u(\omega) = \phi(\omega),
\]

where \(\phi(\omega)\) is defined at (6). We can also choose \(u\) such that \(u(\omega_0) = -b_{p,\alpha(\omega_0)}(h_0)\).

**Lemma 5.15.** There exists a section of horospheres \(\omega \mapsto B_\omega\) such that each \(B_\omega\) is centered at \(\alpha(\omega)\) and is invariant under the action of \(F\), that is, \(A(\omega) \cdot B_\omega = B_T(\omega)\). Moreover \(h_0 \in B_{\omega_0}\).

**Proof.** For each \(\omega \in \Omega\) define the horosphere (centered at \(\alpha(\omega)\)) as

\[
B_\omega = b_{p,\alpha(\omega)}^{-1}(-u(\omega)).
\]

Let us compute

\[
V \circ P(\omega, B_\omega) = V(\omega, -b_{p,\alpha(\omega)}(b_{p,\alpha(\omega)}^{-1}(-u(\omega))))
\]

\[
= V(\omega, u(\omega))
\]

\[
= (T \omega, u(\omega) + \phi(\omega))
\]

\[
= (T \omega, u(T \omega))
\]

\[
= P(T \omega, B_T(\omega)).
\]

Using that \(V \circ P = P \circ F\), we obtain \(P(T \omega, B_T(\omega)) = P(T \omega, A(\omega) \cdot B_\omega)\). Both \(B_T(\omega)\) and \(A(\omega) \cdot B_\omega\) are horospheres centered at \(\alpha(T \omega)\), hence they coincide. Since \(b_{p,\alpha(\omega)}(h_0) = -u(\omega_0)\) we have \(h_0 \in B_{\omega_0}\). \(\square\)

**Lemma 5.16.** There exists a section of geodesics \(\omega \mapsto \gamma_\omega\) such that each \(\gamma_\omega\) is invariant under the action of \(F\), that is, \(A(\omega) \cdot \gamma_\omega = \gamma_T(\omega)\). Moreover \(h_0 \in \gamma_{\omega_0}\).

**Proof.** Let \(\gamma_{\omega_0} : \mathbb{R} \to \mathcal{H}\) be a geodesic such that \(\gamma_{\omega_0}(u(\omega_0)) = h_0\) and \(\gamma_{\omega_0}(+\infty) = \alpha(\omega_0) \in \partial \mathcal{H}\). Notice that \(b_{p,\alpha(\omega_0)}(\gamma_{\omega_0}(0)) = 0\). Since \((T, A^*)\) has a phase space that is saturated by invariant sections, there exists a continuous section \(\beta : \Omega \to \mathcal{H}\), that is \(A^*\)-invariant and such that \(\beta(\omega_0) = \gamma_{\omega_0}(+\infty)\). For every \(\omega \in \Omega\) define the geodesic \(\gamma_\omega : \mathbb{R} \to \mathcal{H}\) to be the unique geodesic such that

\[
\gamma_\omega(-\infty) = \beta(\omega), \quad \gamma_\omega(0) = b_{p,\alpha(\omega_0)}^{-1}(0) \quad \text{and} \quad \gamma_\omega(+\infty) = \alpha(\omega).
\]

\(\square\)

**Lemma 5.17.** For every \(\omega \in \Omega\) the intersection \(\gamma_\omega \cap B_\omega\) consists exactly of the point \(\gamma_\omega(u(\omega))\).

**Proof.** Lemma 4.13 implies that \(b_{p,\alpha(\omega)}(\gamma_\omega(s)) = -s\) for every \(s \in \mathbb{R}\). Hence, \(\gamma_\omega(s) \in B_\omega\) if and only if \(s = u(\omega)\). \(\square\)

**Lemma 5.18.** The section \(s_{\omega_0, h_0} : \Omega \to \mathcal{H}\) defined as \(s_{\omega_0, h_0}(\omega) = \gamma_\omega(u(\omega))\) is a continuous \(F\)-invariant section, and \(s_{\omega_0, h_0}(\omega_0) = h_0\).

**Proof.** Invariance under the action of \(F\) is obtained by combining lemmas 5.15–5.17. The construction of \(\gamma_{\omega_0}\) gives \(\gamma_{\omega_0}(u(\omega_0)) = h_0 = s_{\omega_0, h_0}(\omega_0)\). Since \(\omega \mapsto u(\omega)\) is continuous then it suffices to show that \(\omega \mapsto \gamma_\omega(0)\) is continuous. Recall that \(\gamma_\omega(0)\) is the unique point on \(\gamma_\omega\) such that \(b_{p,\alpha(\omega)}(\gamma_\omega(0)) = 0\). Let \(\omega \in \Omega, \varepsilon > 0\).
By hypothesis, for fixed $h$ the Busemann functions $\omega \mapsto b_{p_0(\omega)}(h)$ are uniformly Hölder, for $h$ in a bounded set. Hence, for $\omega$ close enough to $\overline{\omega}$ we have

\[ b_{p_0(\omega)}(\frac{B(\gamma_\omega(-\varepsilon),\varepsilon/2)}{2}) > b_{p_0(\omega)}(\frac{\gamma_\omega(-\varepsilon)}{2})/2 = \varepsilon/2 > 0, \]
\[ b_{p_0(\omega)}(\frac{B(\gamma_\omega(\varepsilon),\varepsilon/2)}{2}) < b_{p_0(\omega)}(\frac{\gamma_\omega(\varepsilon)}{2})/2 = -\varepsilon/2 < 0. \]

For $\omega$ close enough to $\overline{\omega}$, the geodesic $\gamma_\omega$ traverses through the small balls $B(\gamma_\omega(\varepsilon),\varepsilon/2)$ and $B(\gamma_\omega(-\varepsilon),\varepsilon/2)$. Hence, using the convexity of balls, the function

\[ t \mapsto b_{p_0(\omega)}(\gamma_\omega(t)) \]

reaches its unique zero inside the ball $B(\gamma_\omega(0),\varepsilon)$. □

**Lemma 5.19.** For every $\omega_0, \omega \in \Omega$ the map $h \mapsto s_{\omega_0,h}(\omega)$ is continuous.

**Proof.** Denote by $u_0$ the solution of (10) that verifies $u_0(\omega_0) = -b_{p_0(\omega_0)}(h)$. From remark 3.3 we know that for distinct $h, \overline{h} \in \mathcal{H}$, the solutions $u_h, u_{\overline{h}}$ to (10) differ by a constant, that is

\[ u_{\overline{h}} = u_0 + \left( u_{\overline{h}}(\omega_0) - u_0(\omega_0) \right), \]
\[ = u_0 + \left( b_{p_0(\omega_0)}(\overline{h}) - b_{p_0(\omega_0)}(h) \right). \]

This, and lemma 4.15 yield

\[ |u_{\overline{h}}(\omega) - u_{\overline{h}}(\omega)| \leq d_H(h, \overline{h}). \] (11)

The remaining part of the proof relies just in geometric arguments, whose continuity closely follows the uniqueness of geodesics with endpoints in $H \cup \partial H$. Let us recall how the construction of $s_{\omega_0,h}(\omega)$ depends on $h$. We construct a geodesic $\gamma_{\omega_0} := \gamma_{\omega_0,h}$ that joins $h \in H$ to $\alpha(\omega_0) \in \partial H$. We denote by $\beta_{\omega,h}$ the point $\gamma_{\omega,h}(\omega)$. Lemma 4.11 says that $h \mapsto \beta_{\omega,h}$ is continuous. Afterwards, we consider the continuous $\Lambda^*$-invariant section $\beta_h : \Omega \rightarrow \partial H$ that passes through $\beta_{\omega,h}$. This section depends (pointwise on $\omega$) continuously on $h$. For each $\omega \in \Omega$ we consider the geodesic $\gamma_{\omega,h}$ that joins $\beta_{\omega,h}$ with $\alpha(\omega)$. Continuity (with respect to $h$) of $\gamma_{\omega,h}(u_0(\omega))$ follows.

Summarizing, the proof of theorem 5.3 is complete, since in lemma 5.18 we have constructed a continuous invariant section that passes through any point $(\omega_0, h_0)$. Lemma 5.19 provides the technical condition of continuity that allows us to reconstruct the solutions to the cohomological equation in the class $\text{Homeo}(\mathcal{H})$ (as detailed in proposition 2.10).

5.5. Final comments and perspectives

It is natural to wonder if the reducibility criterion for isometry cocycles presented in this article has any chance of being applied in specific cases. The first direction has to do with the hypotheses about the Hölder–Busemann conditions on the boundary solutions. It is not difficult to verify that in the case of strong hyperbolic spaces, whose boundary at infinity can be metrized with the help of the Gromov product (see [22]), the hypotheses about the regularity of the invariant sections at the boundary can be replaced by a classical Hölder condition. In another direction, the Avila, Kocsard and Liu theorem [1] allows to expect possible applications to the Livšic problem for isometry cocycles that extend smoothly enough to the boundary at infinity. This because the lemmas 4.6 and 4.7 directly imply that the PPO condition in the isometry cocycle implies the PPO condition in the boundary cocycle.
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