New integrable 1D models of superconductivity

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Abstract

In this paper we find new integrable one-dimensional lattice models
of electrons. We classify all such nearest-neighbour integrable models
with $\mathfrak{su}(2) \times \mathfrak{su}(2)$ symmetry following the procedure first introduced
in [1]. We find 12 R-matrices of difference form, some of which can
be related to known models such as the XXX spin chain and the free
Hubbard model, and some are new models. In addition, integrable
generalizations of the Hubbard model are found by keeping the kinetic
term of the Hamiltonian and adding all terms which preserve fermion
number. We find that most of the new models can not be diagonalized
using the standard nested Bethe Ansatz.
1 Introduction

It is important to study strongly correlated electrons to understand physical phenomena such as superconductivity. The prototypical example of a model in which this is possible is the Hubbard model \([2,3]\) which is a basic model of electrons in the conduction band of a solid. To each site of the solid, we associate a four-dimensional Hilbert space. The site can be either vacant, occupied by a single electron with spin up or down, or by a pair of electrons. The Hubbard model Hamiltonian, \(H^{(Hub)}\), written in terms of oscillator algebras, is then given by

\[
H^{(Hub)} = \sum_i \sum_{\alpha=\uparrow,\downarrow} (c^\dagger_{\alpha,i} c_{\alpha,i+1} + c^\dagger_{\alpha,i+1} c_{\alpha,i}) + u n_{\uparrow,i} n_{\downarrow,i}.
\] (1.1)

The kinetic part describes a hopping term that allows electrons to move to neighboring sites whereas the potential term measures the number of electron pairs on each site and \(u\) sets the overall scale.

In the one-dimensional case, it was found that the Hubbard model is integrable \([4]\) which means that there is an underlying \(R\)-matrix, i.e. a solution of the Yang-Baxter equation

\[
R_{12}(u,v)R_{13}(u,w)R_{23}(v,w) = R_{23}(v,w)R_{13}(u,w)R_{12}(u,v)
\] (1.2)

which generates an infinite family of conserved charges which commute with the Hubbard Hamiltonian. Furthermore, the \(R\)-matrix satisfies the regularity condition, \(R_{12}(u,u) = P_{12}\), where \(P_{12}\) is the permutation operator.

It is an interesting question whether there are other integrable models that describe similar physical systems as the Hubbard model. Recently a new approach \([1]\) has been put forward to classify solutions of the Yang-Baxter equation of difference form meaning the \(R\)-matrix satisfies

\[
R(u,v) = R(u-v).
\] (1.3)

The full set of such models is very large. A priori, the Hamiltonian has 256 free parameters and solving coupled polynomial system of equations is a challenging task. However in the present setting, the problem becomes more tractable if we impose some further restrictions on our Hamiltonian. The set of models we will consider share some features with the Hubbard model and have a reduced set of free parameters and we will consider two classes of such models. We will first consider models which have \(\text{su}(2) \times \text{su}(2)\) symmetry and then models whose kinetic part is given by the kinetic part of the Hubbard model. However, the Hubbard model itself will not appear as one of our solutions as its \(R\)-matrix is in fact not of difference form but nevertheless in this way we can construct new integrable models that share many properties with it.

As already mentioned, our first class of models exhibits \(\text{su}(2) \times \text{su}(2)\) symmetry. In the Hubbard model the \(\text{su}_c(2) \times \text{su}_\eta(2)\) algebra is realized as a charge and spin symmetry \([5]\). This symmetry can actually be extended to an algebra called centrally extended \(\text{su}(2|2)\) see \([6]\), which is the symmetry algebra which plays a crucial role in the AdS/CFT correspondence \([7]\).
For this class of models we recover the familiar spin chains whose underlying symmetry algebra contains $\mathfrak{su}(2) \times \mathfrak{su}(2)$, such as the $\mathfrak{su}(4), \mathfrak{su}(2|2), \mathfrak{sp}(4)$ and the $\mathfrak{so}(4)$ spin chains (more detailed in appendix C). In the $\mathfrak{so}(4)$ case, however, we find that the Hamiltonian admits an extra parameter $C$ (see (4.8)) which is not present for regular $\mathfrak{so}(n)$ spin chains. We have checked that the spectrum depends non-trivially on this parameter and it corresponds to the decomposition of $\mathfrak{so}(4) = \mathfrak{su}(2) \times \mathfrak{su}(2)$.

Apart from these well-known spin chains, we find several new models that seem to have interesting physical properties. In particular we find three models in which only electron pairs can propagate. The fermionic degrees of freedom seem to freeze out, but they affect the spectrum non-trivially. The standard Bethe Ansatz approach breaks down for these models and we have not been able to find an alternative way to compute the spectrum, however promising approaches are proposed in discussion. We performed a study of the spectrum for small spin chain lengths and found a very non-trivial structure.

For our second class of models, we consider deformations of the free Hubbard model. We keep the kinetic part of the Hubbard model Hamiltonian and we add an arbitrary potential and a possible new hopping term for electron pairs. We allow for the most general deformation which preserves electron number so that we still have a physical interpretation of our model. It turns out that we find four integrable models. Three of those are simple combinations of lower dimensional integrable spin chains in which the electrons with spin up and down decouple. However, we find one new model which has two free parameters which has a very non-trivial Hamiltonian. In particular, it contains a term which flips the spins of electrons, mixing $|\uparrow\uparrow\rangle$ with $|\downarrow\downarrow\rangle$ just as in the XYZ spin chain. As a consequence, this new model has potentially very interesting physics. It is integrable, but due to the fact that it contains some XYZ type-terms, the standard coordinate Bethe Ansatz can not be applied.

This paper is organized as follows. In the first section we will recapitulate the method from [1] that we will use and discuss our conventions. In the next two sections we discuss $\mathfrak{su}(2) \times \mathfrak{su}(2)$ symmetric models. After this we give the classification of the second class of integrable models. We end with a discussion and conclusions.

2 Set-up and method

We employ the method from [1] to classify one-dimensional integrable models of electrons. We will consider the set-up similar to that of the Hubbard model, which means that the local Hilbert space is four-dimensional. Each lattice site can be empty, occupied by an electron with spin up or down, or by a pair of electrons. This means we will find $16 \times 16$ solutions of the Yang-Baxter equation. A full classification of such models is currently not feasible, but if we impose some symmetry conditions on our Hamiltonian, new solutions of the Yang-Baxter equation can be found.

2.1 Hamiltonian

The main idea of [1] is to consider a general Hamiltonian $\mathcal{H}$, also denoted $Q_2$,

$$
\mathcal{H} = Q_2 = \sum_n \mathcal{H}_{n,n+1},
$$

(2.1)
where the Hamiltonian density $\mathcal{H}$ is a $16 \times 16$ matrix. Then we use the so-called boost operator $[8]$

$$B[Q_2] := \sum_{n=-\infty}^{\infty} n\mathcal{H}_{n,n+1}. \quad (2.2)$$

to generate the higher conserved charges $Q_i$ that are present in integrable systems. More precisely, the boost operator $[8]$ can be used to generate recursively all conserved charges $Q_r$ in the following way

$$Q_{r+1} \sim [B[Q_2], Q_r]. \quad (2.3)$$

By imposing that $0 = [Q_2, Q_3] = [Q_3, Q_4] = \ldots$ we derive a set of coupled polynomial equations on the coefficients of $\mathcal{H}$, which we then solve. For the models we consider in this paper it turns out that imposing $[Q_2, Q_3] = 0$ is a sufficient condition. Indeed, for the Hamiltonians corresponding to solutions of $[Q_2, Q_3] = 0$, we can subsequently solve the Yang-Baxter equation. More precisely, we assume that we can expand the $R$-matrix as

$$R = P + P\mathcal{H}u + P\mathcal{H}^2u^2 + \sum_{n\geq 3} R^{(n)}u^n. \quad (2.4)$$

For each of the Hamiltonians that we find, we are able to find a corresponding $R$-matrix, which proves the integrability of the underlying model.

### 2.2 Identifications

As outlined in [1], finding solutions to the Yang–Baxter equation in this way leads to a large redundancy in solutions. In particular, some solutions can be related to each other by simple transformations and we will identify solutions which can be related in this way. The transformations under which we identify solutions are:

**Normalization** We can clearly multiply any solution of the Yang-Baxter equation by a scalar function.

**Reparametrisation** The $R$-matrix will depend on a number of free parameters. In particular, one is free to choose reparametrisations, thus some solutions that we find can be related by a redefinition of the parameters and clearly do not define a different integrable model.

**Basis transformation** Any local basis transformation $V : \mathbb{C}^4 \to \mathbb{C}^4$ can be applied to the $R$-matrix

$$R \mapsto R^V = (V \otimes V)R(V^{-1} \otimes V^{-1}). \quad (2.5)$$

to define a different $R$-matrix which satisfies the Yang-Baxter equation.

**Twists** If $[R, V \otimes V] = [R, W \otimes W] = 0$ then we can define a twisted model.

$$R \mapsto R^{V,W} = (V \otimes W)R(W^{-1} \otimes V^{-1}). \quad (2.6)$$

Notice that a twist can affect the symmetry properties of the $R$-matrix since $V$ or $W$ need not commute with the symmetry generators.
**Discrete transformations**  It is straightforward to check that if \( R(u) \) is a solution of the Yang-Baxter equation then \( PR(u)P \) and \( R(u)^T \) are solutions as well. This means that transposition and permutation are further discrete transformations that map an integrable Hamiltonian to a different integrable Hamiltonian.

### 2.3 Graded models

To any solution \( R \) of the Yang-Baxter equation, one can associate a different solution \( R^f \) in which the underlying vector space is graded, see for instance [9]. In our case, we consider electrons on a 1D lattice and hence, we would like to introduce a 2\|2 graded vector space as our Hilbert space. In particular, we denote the basis of our vector space as

\[
\{ |\phi_1\rangle, |\phi_2\rangle, |\psi_1\rangle, |\psi_2\rangle \}.
\] (2.7)

We will take \( |\phi_1\rangle, |\phi_2\rangle \) to be even and \( |\psi_1\rangle, |\psi_2\rangle \) to be odd basis vectors respectively. An efficient way to relate the graded and the non-graded models is

\[
\mathcal{H}^f = D^r \mathcal{H} D^r, \quad R^f(u) = D^r R(u) D^r;
\]

with

\[
D^l = \text{Diag}(1, 1, 1, 1, 1, 1, 1, 1, -1, -1, 1, -1, -1, 1), \quad D^r = \text{Diag}(1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1).
\] (2.8)

This map follows from the general considerations in Appendix [A].

### 2.4 Oscillators

Alternatively, the graded Hilbert space can be nicely described by the introduction of two sets of fermionic oscillators \( c_\alpha, c_\alpha^\dagger \) with \( \alpha = \uparrow, \downarrow \) satisfying the usual anti-commutation relations

\[
\{ c_\alpha^\dagger, c_\beta \} = \delta_{\alpha\beta}, \quad \{ c^\dagger, c \} = 1.
\] (2.9)

We also introduce the number operators \( n_\alpha = c_\alpha^\dagger c_\alpha \). These oscillators then generate local four-dimensional vector space \( \mathbb{C}^4 \) by creating fermions with spin up or down by acting on vacuum. We define

\[
|\phi_1\rangle = |0\rangle, \quad |\phi_2\rangle = c_\uparrow^\dagger c_\downarrow^\dagger |0\rangle, \quad |\psi_1\rangle = c_\uparrow^\dagger |0\rangle, \quad |\psi_2\rangle = c_\downarrow^\dagger |0\rangle.
\] (2.10)

This identification offers a clear map from the oscillator algebra to graded \( 4 \times 4 \) matrices. Finally, we can consider numerous copies of the oscillators, acting on each site of the spin chain, i.e. \( c_{\alpha,i} \) acts on the \( i \)th site.

### 3 Hubbard type models

The most general nearest-neightbour Hamiltonian where the local Hilbert space is four-dimensional has 256 components. Fully classifying all integrable solutions is currently not feasible, but we can restrict to a proper subset of physically interesting Hamiltonians with
a smaller amount of free parameters. We would like to restrict to models which exhibit
spin and charge $\mathfrak{su}(2)$ symmetry, similar to the Hubbard model \[3\]. It turns out that
there are two non-trivial four-dimensional representations of $\mathfrak{su}(2) \times \mathfrak{su}(2)$, see Appendix \[B\]. In this section we will consider the case in which the representation can be written as
direct sum which is the case of the Hubbard model.

3.1 $\mathfrak{su}(2) \times \mathfrak{su}(2)$ symmetry

We consider the four-dimensional representation $\rho_{2\mathfrak{su}(2)}$ of $\mathfrak{su}(2) \times \mathfrak{su}(2)$ in which both
$\mathfrak{su}(2)$'s are represented two-dimensionally, see Appendix \[B\]

\[
\rho_{2\mathfrak{su}(2)}(t^L_i \times t^R_j) = \begin{pmatrix}
\rho_2(t^L_i) & 0 \\
0 & \rho_2(t^R_j)
\end{pmatrix}.
\]

(3.1)

For any $A \in \mathfrak{su}(2) \times \mathfrak{su}(2)$, we then demand that

\[
[\mathcal{H}_{12}, \rho_{2\mathfrak{su}(2)}(A) \otimes 1 + 1 \otimes \rho_{2\mathfrak{su}(2)}(A)] = 0.
\]

(3.2)

Examples of models that have this symmetry are the AdS$_5 \times S^5$ superstring, the Hubbard
model and the $\mathfrak{su}(4)$ Heisenberg spin chain. However, only the last model has an $R$-matrix
which is of different form.

Hamiltonian  It is straightforward to show that an $\mathfrak{su}(2) \times \mathfrak{su}(2)$ invariant Hamiltonian
in the above sense takes the form

\[
\mathcal{H}|\phi_\alpha \phi_\beta \rangle = A|\phi_\alpha \phi_\beta \rangle + B|\phi_\beta \phi_\alpha \rangle + C \epsilon_{abc} \epsilon_{a\beta} |\psi_\alpha \psi_\beta \rangle
\]

(3.3)

\[
\mathcal{H}|\phi_\alpha \psi_\beta \rangle = G|\phi_\alpha \psi_\beta \rangle + H|\psi_\beta \phi_\alpha \rangle
\]

(3.4)

\[
\mathcal{H}|\psi_\alpha \phi_\beta \rangle = K|\psi_\alpha \phi_\beta \rangle + L \phi_\alpha \psi_\beta \rangle
\]

(3.5)

\[
\mathcal{H}|\psi_\alpha \psi_\beta \rangle = D|\psi_\alpha \psi_\beta \rangle + E|\psi_\beta \psi_\alpha \rangle + F \epsilon_{abc} \epsilon_{a\beta} |\phi_\alpha \phi_\beta \rangle
\]

(3.6)

Here $\phi_{1,2}$ and $\psi_{1,2}$ span the two independent $\mathfrak{su}(2)$ fundamental representations. Explicitly
in matrix form, the Hamiltonian density is given by

\[
\mathcal{H} = \left(\begin{array}{cccccccccccccccc}
A + B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A + B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A
\end{array}\right). \]

(3.7)

Graded vs. non-Graded  Both the Hubbard model and the AdS$_5 \times S^5$ string model
are graded models. This means that $\phi_\alpha$ are even basis vectors while $\psi_\alpha$ are odd basis
vectors. There is a one-to-one map between the graded and non-graded integrable models
which is given by

\[
(A, B, C, D, E, F, G, H, K, L) \leftrightarrow (A, B, -C, D, E, -F, G, -H, K, -L).
\]

(3.8)

This map can be obtained by following the procedure in appendix \[A\].
Oscillator representation  We can define our $\mathfrak{su}(2) \times \mathfrak{su}(2)$ representation $\rho_{osc}$ in the oscillator language

$$\rho_{osc}(t^L_1) = \frac{1}{2} (c_1^\dagger c_1^\dagger + c_1^\dagger c_1) \quad \rho_{osc}(t^L_2) = \frac{i}{2} (c_1^\dagger c_2^\dagger - c_2^\dagger c_1) \quad \rho_{osc}(t^L_3) = \frac{i}{2} (n_\uparrow + n_\downarrow - 1),$$  

$$(3.9)$$

$$\rho_{osc}(t^R_1) = \frac{1}{2} (c_1^\dagger c_1 + c_1^\dagger c_1^\dagger) \quad \rho_{osc}(t^R_2) = \frac{i}{2} (c_1^\dagger c_1^\dagger - c_1^\dagger c_1) \quad \rho_{osc}(t^R_3) = -\frac{i}{2} (n_\downarrow - n_\uparrow).$$

$$(3.10)$$

It is straightforward to check from the defining anti-commutation relations of the oscillators (2.9) that these operators satisfy the $\mathfrak{su}(2)$ defining relations.

The most general two-site operator which commutes with both of the above $\mathfrak{su}(2)$ oscillator representations again has 10 free parameters $C_i$ and is given by

$$\mathcal{H}_{12} = \sum_{\alpha \neq \beta} \left[ (c_{\alpha,1}^\dagger c_{\alpha,2} + c_{\alpha,1} c_{\alpha,2}^\dagger) (C_1 + C_2(n_{\beta,1} - n_{\beta,2})^2) + (c_{\alpha,1}^\dagger c_{\alpha,2} - c_{\alpha,1} c_{\alpha,2}^\dagger) (C_3(n_{\beta,1} - \frac{1}{2}) + C_4(n_{\beta,2} - \frac{1}{2})) \right]$$

$$+ (c_{\tau,1}^\dagger c_{\tau,2}^\dagger c_{\tau,2} c_{\tau,2}^\dagger + c_{\tau,1} c_{\tau,1}^\dagger c_{\tau,1} c_{\tau,2}^\dagger) C_5 + (c_{\tau,1}^\dagger c_{\tau,1} c_{\tau,2}^\dagger c_{\tau,2} c_{\tau,2}^\dagger + c_{\tau,1}^\dagger c_{\tau,1} c_{\tau,2} c_{\tau,2}^\dagger) C_6$$

$$+ C_7 (n_{\tau,1} - \frac{1}{2})(n_{\tau,2} - \frac{1}{2}) + C_8(n_{\tau,1} - \frac{1}{2})(n_{\tau,2} - \frac{1}{2})$$

$$+ C_9(n_{\tau,1} - n_{\tau,1})^2(n_{\tau,2} - n_{\tau,2})^2 + (C_5 - C_6)(n_{\tau,1}^2 + n_{\tau,2}^2 - 1) (n_{\tau,1}^2 - n_{\tau,1})(n_{\tau,2}^2 - n_{\tau,2})$$

$$+ \frac{1}{2} C_5((n_{\tau,1} - n_{\tau,2})^2 + (n_{\tau,1} - n_{\tau,2})^2) + C_0,$$  

$$(3.11)$$

where

$$C_0 = \frac{1}{2}(B + G + K), \quad C_1 = \frac{1}{2}(L - H), \quad C_2 = \frac{1}{2}(C - F + H - L),$$

$$C_3 = \frac{1}{2}(H + L - C - F), \quad C_4 = \frac{1}{2}(C + F + H + L), \quad C_5 = -B, \quad C_6 = E,$$

$$C_7 = 2A + B - 2K, \quad C_8 = 2A + B - 2G, \quad C_9 = A + B + D + E - G - K.$$  

$$(3.12)$$

### 3.2 Solutions

Following the steps in [1], we take a Hamiltonian of the form (3.7) and compute the corresponding density $Q_3$ for the next conserved charge. Next, we impose that $[Q_2, Q_3] = 0$ and find a set of coupled cubic polynomial equations. Solving this set of equations leads to 45 solutions, which, after identifying solutions according to the transformations discussed in Section 2.2, results in 12 independent solutions which are listed in Table 1. Each of these models is integrable and we will present the corresponding R-matrices in the next section.
Table 1: All non-graded integrable spin chains with charge and spin $su(2)$ symmetry.

### 3.3 R-matrices

As a result of the $su(2) \times su(2)$ symmetry, all of the $R$-matrices corresponding to the Hamiltonians listed above can be expressed as

$$
R_{12}(u) = 
\begin{pmatrix}
    r_1 + r_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & r_1 & 0 & 0 & r_2 & 0 & 0 & 0 & 0 & 0 & 0 & -r_8 & 0 & 0 & r_8 \\
    0 & 0 & r_4 & 0 & 0 & 0 & 0 & r_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & r_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & r_2 & 0 & 0 & r_1 & 0 & 0 & 0 & 0 & 0 & 0 & r_8 & 0 & 0 & r_8 \\
    0 & 0 & 0 & 0 & 0 & 0 & r_1 + r_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & r_4 & 0 & 0 & 0 & r_{10} & 0 & 0 & r_{10} \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & r_7 & 0 & 0 & 0 & 0 & 0 & r_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & r_7 & 0 & 0 & r_3 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & r_7 & 0 & 0 & r_3 & 0 & 0 & 0 & 0 \\
    0 & r_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_5 + r_9 & 0 & 0 \\
\end{pmatrix}
$$

(3.13)

where we have omitted the $u$-dependence on the functions $r_j(u)$ in order to avoid overly bulky expressions. In order to find the $R$-matrix explicitly we must solve the YBE. To this end, we recall that we can express the $R$-matrix in terms of the Hamiltonian density $H$ as

$$
R(u) = P + PHu + PH^2 \frac{u^2}{2} + O(u^3)
$$

(3.14)

Knowing this expansion greatly simplifies solving the YBE. Indeed, already at second order in $u$ it can become apparent that certain entries in $R$ may be equal, or related by a sign change or overall factor. This allows one to consider a reduced ansatz for $R$ where these identifications are introduced. One then attempts to solve the reduced system of functional equations. Specifically, we consider the YBE (1.3) and differentiate wrt $v$ and evaluate the result at $v = 0$. Consistency with (3.14) then places initial conditions on the $r_j$ and their derivatives and we can subsequently solve the resulting ODEs.
For each of the 12 models listed above we will simply list the corresponding non-zero functions $r_j(u)$ only. All the R-matrices presented below are regular (i.e. satisfy $R(0) = P$) and satisfy crossing unitarity $R_{12}(u)R_{21}(-u) = 1$.

**Model 1** For this model, we begin by considering the case where all parameters are generic. In this case it is convenient to introduce the parameter $\eta$ and the function $g(u)$ defined by

$$d = \frac{(a + c)^2 \csc^2(\eta)}{4b}, \quad g(u) = \arccot(\tan(\eta)) - \frac{1}{2} u(a + c) \cot(\eta). \quad (3.15)$$

Then we have

$$r_2 = r_6 = 1, \quad r_3 = \frac{a + c}{b} (\cot(\eta) \cot(g(u)) - 1), \quad r_7 = e^{\frac{1}{2} u(a-c)} \csc(g(u)), \quad r_{10} = e^{-u(a-c)} r_7 \quad (3.16)$$

Clearly there are two degenerate cases of the above parameters, namely when $b = 0$ or $a + c = 0$. Hence we must treat these two cases separately. For $b = 0$ we have

$$r_2 = r_6 = 1, \quad r_3 = \frac{a + c}{a + c} - 1, \quad r_7 = e^{au}, \quad r_{10} = e^{cu} \quad (3.17)$$

whereas for $a + c = 0$ we have

$$r_2 = r_6 = 1, \quad r_3 = \frac{4}{b} r_4 = \sqrt{\frac{2}{b}} \tan \left( \sqrt{\frac{b}{d}} u \right), \quad r_7 = e^{au} \csc \left( \sqrt{b d} \right), \quad r_{10} = e^{-2au} r_7 \quad (3.18)$$

**Model 2** This model also has the degenerate cases $b = 0$ and $a + c = 0$, which must be treated separately. For the case when all parameters are generic it is again useful to introduce a parameter $\eta$ and functions $g(u)$ and $h(u)$ defined by

$$d = \frac{(a + c)^2}{4b} \sech^2(\eta), \quad g(u) = \csch\left( \eta - \frac{1}{2} u(a + c) \tanh(\eta) \right), \quad (3.19)$$

$$h(u) = \sinh\left( \frac{1}{2} u(a + c) \tanh(\eta) \right).$$

We then have

$$r_2 = 1, \quad r_3 = \frac{2h}{a + c} \frac{g(u) h(u) \sech(\eta)}{g(h(u) \sech(\eta))}, \quad r_6 = g(u) \sinh \left( \frac{1}{2} u(a + c) \tanh(\eta) + \eta \right), \quad r_7 = e^{\frac{1}{2} u(a-c)} g(u) \sinh(\eta), \quad r_{10} = e^{\frac{1}{2} u(c-a)} g(u) \sinh(\eta) \quad (3.20)$$

When $a + c = 0$ we have that $d = 0$ and we obtain the same degenerate model as obtained from model 1. When $b = 0$ then we obtain

$$r_2 = 1, \quad r_3 = \frac{d}{a + c} \left( e^{u(a+c)} - 1 \right), \quad r_7 = e^{au}, \quad r_{10} = e^{cu} \quad (3.21)$$
Model 3
\begin{align*}
  r_2 &= 1, \quad r_7 = e^{bu} \\
  r_6 &= e^{au}, \quad r_{10} = e^{cu}
\end{align*}

(3.22)

Model 4
\begin{align*}
  r_1 &= -u \rho \ r_2 \\
  r_2 &= (1 - u \rho)^{-1} \\
  r_3 &= -e^{\phi} \ r_1 \\
  r_6 &= 1 \\
  r_7 &= e^{u(a-\rho)} \ r_2 \\
  r_{10} &= e^{u(\rho-a)} \ r_2
\end{align*}

(3.23)

Model 5
\begin{align*}
  r_1 &= r_5 = -u \rho \ r_2 \\
  r_2 &= r_6 = (1 - u \rho)^{-1} \\
  r_3 &= -e^{\phi} \ r_1 \\
  r_7 &= e^{u(a-\rho)} \ r_6 \\
  r_{10} &= e^{u(\rho-a)} \ r_6
\end{align*}

(3.24)

Model 6
\begin{align*}
  r_2 &= 1 \\
  r_6 &= (1 - u \rho)^{-1} \\
  r_5 &= u \rho \ r_6 \\
  r_3 &= e^{\phi} \ r_5 \\
  r_7 &= e^{u(a-\rho)} \ r_2 \\
  r_{10} &= e^{u(\rho-a)} \ r_2
\end{align*}

(3.25)

Model 7
\begin{align*}
  r_1 &= -r_5 = -u \rho \ r_2 \\
  r_2 &= r_6 = (1 - u \rho)^{-1} \\
  r_7 &= e^{u(a-\rho)} r_2 \\
  r_{10} &= e^{u(\rho-a)} r_2
\end{align*}

(3.26)

Model 8
\begin{align*}
  r_1 &= -r_5 = -\tan(u \rho) \\
  r_2 &= 1 - r_1 \\
  r_5 &= 1 + r_1 \\
  r_6 &= 1 + r_1 \\
  r_7 &= r_{10} = 1 \\
  r_8 &= e^{\phi} \ r_1 \\
  r_9 &= -e^{\phi} \ r_1
\end{align*}

(3.27)

Model 9
\begin{align*}
  r_1 &= r_5 \\
  r_2 &= r_6 = 1 - r_1 \\
  r_7 &= r_{10} = 1 \\
  r_8 &= -e^{\phi} \ r_1 \\
  r_9 &= -e^{\phi} \ r_1
\end{align*}

(3.28)

Model 10
\begin{align*}
  r_1 &= r_5 = 2 \left[ e^{\frac{3u}{4}} - 1 \right] \\
  r_2 &= r_6 = \frac{3mu}{e^{\frac{3u}{4}} - 4} + 2 \\
  r_7 &= r_{10} = e^{\frac{1}{4}(3mu)} \\
  r_8 &= e^{-2\phi} r_9 = r_8 = -\frac{1}{2} e^{\frac{3mu}{4} + \phi} \ r_1
\end{align*}

(3.29)
Model 11
\[ r_1 = r_5 = \rho u(3\rho u - 4)\ f(u) \quad r_7 = r_{10} = -2(3\rho u - 2)^{-1} \]
\[ r_2 = r_6 = 4(1 - \rho u)\ f(u) \quad e^{2\phi}r_9 = r_8 = 2\rho u e^{\phi}\ f(u) \quad (3.30) \]
\[ r_3 = r_4 = -\frac{3}{2}u\rho\ r_7 \quad f(u)^{-1} = (\rho u - 2)(3\rho u - 2) \]

Model 12
\[ r_1 = r_5 = r_4^2 \quad r_2 = r_6 = r_7^2 \]
\[ r_3 = -r_4 = -\tanh(u\rho) \quad r_7 = r_{10} = \text{sech}(u\rho) \quad (3.31) \]
\[ r_8 = e^{\phi}r_4 r_7 \quad r_9 = -e^{-\phi}r_4 r_7 \]

3.4 Interpretation of models

In this section we discuss the models that we have listed in Table [1]. We will relate some of them to known models and discuss some of the properties of the new models.

Model 1  Model 1 corresponds to a quadruple embedding of an XXZ-type spin chain with Hamiltonian
\[
\mathcal{H}^{(XXZ)} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & a & b & 0 \\
0 & d & c & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \quad (3.32)
\]
One can show that the spectrum of Model 1 corresponds to the spectrum of \( \mathcal{H}^{(XXZ)} \) where each eigenspace has an extra degeneracy factor of 4.

Model 2  Similarly, Model 2 is a staggered-type XXZ model with Hamiltonian
\[
\mathcal{H}^{(XXZ')} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & a & b & 0 \\
0 & d & c & 0 \\
0 & 0 & 0 & a + c
\end{pmatrix}. \quad (3.33)
\]

Model 3  The Hamiltonian for Model 3 is diagonal and hence it is trivially integrable.

The Hamiltonian for models 4, 5, 6 and 7 all depend on a parameter \( a \) and can be written as follows
\[
\mathcal{H} = \mathcal{H}_0 + \alpha \otimes 1 - 1 \otimes \alpha \quad (3.34)
\]
with
\[
\alpha = \text{Diag}(1, 1, 1 - a, 1 - a) \quad (3.35)
\]
and \( \mathcal{H}_0 \) is the Hamiltonian for \( a = 0 \). Notice that the term \( \alpha \otimes 1 - 1 \otimes \alpha \) does not affect the spectrum for closed spin chains. The \( a \) dependence can be recovered by applying a simple transformation to the \( R \)-matrix. Define
\[
U(u) = \text{Diag}(1, 1, e^{au}, e^{au}). \quad (3.36)
\]
Then
\[ R_{12}(u) = (U(u) \otimes 1)R_{12}(u)|_{u=0}(U(-u) \otimes 1). \] (3.37)

Moreover, the Hamiltonian for models 4, 5, 6 and 7 also all depend on \( \phi \), which corresponds to a particular twist
\[ \mathcal{H}(\phi) = (G_1 \otimes 1) \mathcal{H}(\phi = 0) (G_1^{-1} \otimes 1), \] (3.38)
with
\[ G_1 = \text{Diag}(1, 1, e^{-\phi}, e^{-\phi}). \] (3.39)

Keeping this in mind, we can set \( a, \phi \) to some convenient values in order to compare with known models in the literature, since the general \( a, \phi \) dependence can be easily restored.

**Model 4** After setting \( a = -1, \phi = i\pi \) and \( \rho = -1 \), we find that Model 4 seems to be a modified version of the \( \mathfrak{su}(4) \) spin chain. After applying a simple basis transform which sends the basis vectors \( E^i \mapsto E^{5-i} \), we can write
\[ \mathcal{H}^{(4)} = 1 - P + \epsilon_{ij} \epsilon^{kl} E^i_k \otimes E^j_l = 1 - P + \sum_{i,j=1,2} (E^i_i \otimes E^j_j - E^i_j \otimes E^j_i), \] (3.40)
where the sum runs over the \( i, j, k, l = 1, 2 \). We denote the standard \( 4 \times 4 \) matrix unities by \( E^i_i \). The last term can actually be interpreted as the Hamiltonian of the XXX spin chain restricted to the first two basis vectors (3.40), because in general the identity and permutation operators can be expressed as \( 1 = E^i_i \) and \( P = E^j_j \otimes E^j_j \). Thus we can write
\[ \mathcal{H}^{(4)} = 1 - P + 1_{2} - P_{2}, \] (3.41)
where \( 1_{2} \) and \( P_{2} \) are the identity and permutation operator on a two-dimensional subspace generated by \( E^{1,2} \).

**Model 5** Model 5 corresponds to the twisted \( \mathfrak{su}(4) \) spin chain. More specifically, if we set \( a = -1, \phi = i\pi \) and \( \rho = -1 \) we recover
\[ \mathcal{H}^{(4)} \mapsto 1 - P, \] (3.42)
which indeed is the \( \mathfrak{su}(4) \) spin chain Hamiltonian.

**Model 6** Models 4 and 6 are related to each other by a grading. More precisely, we find for \( a = 1, \phi = i\pi \) and \( \rho = 1 \)
\[ \mathcal{H}^{(6)} = 1 - P^{f} + \epsilon_{ij} \epsilon^{kl} E^i_k \otimes E^j_l, \] (3.43)
where \( P^{f} \) is now the graded permutation.

**Model 7** Model 7 corresponds to the twisted \( \mathfrak{su}(2|2) \) spin chain. We find that we recover
\[ \mathcal{H}^{(4)} \mapsto 1 - P^{f}, \] (3.44)
upon setting \( a = 1, \phi = i\pi \) and \( \rho = 1 \).

For models 8, 9, 10, 11 and 12 the parameter \( \phi \) corresponds to a rescaling of certain basis elements.
\[
\mathcal{H}(\phi) = (G_2 \otimes G_2) \mathcal{H}(\phi = 0) (G_2^{-1} \otimes G_2^{-1}).
\] (3.45)

where
\[
G_2 = \text{Diag}(1, 1, e^{-\phi}, 1).
\] (3.46)

Hence \(\phi\) will not affect the spectrum and can be accounted for by a simple local basis transformation.

**Model 8**  This model has the interesting feature that fermionic degrees of freedom freeze out. Only pairs of electrons can propagate along the chain. Although the Hamiltonian for \(L = 2\) has all the eigenvalues equal to zero, the same does not occur for more sites. Actually, the model is highly non trivial for more sites, as one can see from the tables 2, 3 and 4.

The degeneracies for this model for \(L = 3, 4, 5\) respectively are the following
\[
\begin{align*}
L = 3 : & \{2, 2, 4, 52\}, \\
L = 4 : & \{1, 1, 1, 1, 14, 14, 224\}, \\
L = 5 : & \{2, 2, 2, 2, 4, 4, 4, 4, 4, 10, 10, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 20, 20, 20, 20, 24, 24, 724\}.
\end{align*}
\] (3.47)

Unfortunately this model does not seem physical since its Hamiltonian is not Hermitian. An Ansatz for the spectrum of this model for up to 1 excitation for any number of sites is presented in section 6.

**Model 9**  This model is similar to Model 8. Just like in Model 8, Model 9 also allows only pairs of electrons move in the chain. The Hamiltonian for this model, for \(\rho = 1\) and \(\phi = 0\), is however Hermitian. This model has a non trivial spectrum and could be very interesting physically.

Its degeneracies are given by
\[
\begin{align*}
L = 3 : & \{8, 4, 52\}, \\
L = 4 : & \{1, 15, 16, 30, 194\} \\
L = 5 : & \{4, 4, 8, 8, 8, 8, 20, 20, 20, 20, 24, 24, 724\}.
\end{align*}
\] (3.48)

An Ansatz for the spectrum of this model with 1-excitation for any number of sites is worked out in the section 6.

**Model 10**  By adding a factor of \(\frac{3\rho}{4}\) in the Hamiltonian for model 10, we see that it looks like models 8 and 9 in the sense that \(G = H = K = L = 0\). Again only pairs of electrons can move, and for \(\rho = 1\) and \(\phi = 0\) the spectrum is all real.

The degeneracies for \(L = 3, 4, 5\) are given by
\[
\begin{align*}
L = 3 : & \{4, 8, 16, 36\}, \\
L = 4 : & \{1, 1, 1, 1, 1, 1, 1, 6, 6, 8, 8, 14, 16, 16, 32, 44, 100\}, \\
L = 5 : & \{4, 4, 8, 8, 8, 8, 8, 8, 8, 8, 12, 16, 16, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 64, 240, 300\}.
\end{align*}
\] (3.49)

\(^1\) Notice we can choose the values of \(\rho\) and \(\phi\) because they can be introduced by twists and basis transformations.
Model 11  Model 11 corresponds to the \( \mathfrak{sp}(4) \) spin chain, \([11][14]\)

\[
R^{\mathfrak{sp}(4)} = u1 + P - \frac{u}{u+3}(-1)^{p(i)+p(k)}E_j^i \otimes E_{5-j}^5.
\]

(3.50)

It can be shown that

\[
\mathcal{H}^{(11)} = \frac{3\rho_2}{2} I - (U \otimes U)\mathcal{H}^{(\mathfrak{sp}(4))}(U^{-1} \otimes U^{-1}), \quad U = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & e^{\phi/2} \\
0 & 0 & 1 & 0 \\
e^{-\phi/2} & 0 & 0 & 0
\end{pmatrix}.
\]

(3.51)

The Bethe Ansatz for this known model has been worked out in for example in \([14][15]\).

Model 12  Model 12 corresponds to the free Hubbard model, i.e. just the kinetic term. In order to see this, we need to consider a twist \((2.6)\) with \(V = \text{diag}(1, -1, i, i)\) and \(W = 1\). On the level of the Hamiltonian we find

\[
\mathcal{H}_{12}^{(V)} = -iV_1 \mathcal{H}V_1^{-1}
\]

(3.52)

This relation is needed to make contact with the regular Hubbard model, because the charge \(\mathfrak{su}_c(2)\) is twisted. Moreover, we need to make the model graded. We also find that we can put \(\phi = 0\) by using a basis transformation, so that

\[
\mathcal{H}^{(H_{ub})} \sim (U \otimes U)D^y(V_1 \mathcal{H}_{12}^{(V)}V_1^{-1})D^y(U^{-1} \otimes U^{-1}),
\]

(3.53)

with

\[
U = \begin{pmatrix}
0 & e^{-\frac{\phi}{2}} & 0 & 0 \\
e^{-\frac{\phi}{2}} & 0 & 0 & 0 \\
0 & 0 & i & 0 \\
0 & 0 & 0 & i
\end{pmatrix}.
\]

(3.54)

4  Other \(\mathfrak{su}(2) \times \mathfrak{su}(2)\) invariant models

We notice that the \(\mathfrak{so}(4) \sim \mathfrak{su}(2) \times \mathfrak{su}(2)\) spin chain is not in our list of spin and charge \(\mathfrak{su}(2)\) invariant models listed in Table 1. This is due to the fact that there is one further non-trivial four-dimensional representation, \(\rho_{2\oplus 2}\), of \(\mathfrak{su}(2) \times \mathfrak{su}(2)\) which gives rise to the \(\mathfrak{so}(4)\) spin chain, see Appendix B. This representation is given by

\[
\rho_{2\oplus 2}(t_i^L) = 1 \otimes \rho_2(t_i), \quad \rho_{2\oplus 2}(t_i^R) = \rho_2(t_i) \otimes 1,
\]

(4.1)

where \(t^L \times t^R \in \mathfrak{su}(2) \times \mathfrak{su}(2)\). It is straightforward to check that the invariant Hamiltonian under the \(\rho_{2\oplus 2}\) representation takes the general form

\[
\mathcal{H} = A + BP + CK + D \epsilon_{ijk} E_j^i \otimes E_k^i,
\]

(4.2)

where \(P\) is the permutation operator and \(K = E_j^i \otimes E_j^i\) is the so-called trace operator. The matrices \(E_j^i\) are the standard \(4 \times 4\) matrix unities.
Following the steps from \[1\], we only find two new integrable Hamiltonians
\[
\mathcal{H}^{(13)} = A - BP + BK + C\epsilon_{ijkl}E_k^i \otimes E_l^j, \\
\mathcal{H}^{(14)} = AK,
\]
and the corresponding R-matrices are given by
\[
R^{(13)} = \frac{(1 + Au)u}{1 - Bu} \left[ (u (B^2 - C^2) - B) + \frac{1 - Bu}{u} P + BK + C\epsilon_{ijkl}E_k^i \otimes E_l^j \right], \tag{4.5}
\]
\[
R^{(14)} = (1 + Au) \left[ \sqrt{3}\coth(\sqrt{3}Bu) - 2 \right] P + \frac{1}{\sqrt{3}\coth(\sqrt{3}Bu) - 1} K. \tag{4.6}
\]
For \(C = 0\), Model 13 corresponds to the usual \(\mathfrak{so}(n)\) (Appendix \[C\]) spin chain for \(n = 4\) \[16\]. The presence of \(C\) corresponds to the fact that exactly in the four-dimensional case there is an extra invariant contraction, where all indices are contracted with the Levi-Civita symbol. The spectrum depends non-trivially on \(C\) and it appears due to the isomorphism \(\mathfrak{so}(4) \sim \mathfrak{su}(2) \times \mathfrak{su}(2)\). Indeed each \(\mathfrak{su}(2)\) subalgebra comes with its own quadratic Casimir. For the usual XXX, spin chain, the Hamiltonian can be written as
\[
\mathcal{H}^{(XXX)} = \sum_i \sigma_i \otimes \sigma_i, \tag{4.7}
\]
where \(\sigma^i\) are the Pauli matrices. In this case we see that this decomposition directly generalizes
\[
\mathcal{H}^{(13)} = A + 2 \sum_i \left[ (B + C)\rho_{\pm2}(t_i^L) \otimes \rho_{\pm2}(t_i^L) + (B - C)\rho_{\pm2}(t_i^R) \otimes \rho_{\pm2}(t_i^R) \right]. \tag{4.8}
\]
In other words, the \(\mathfrak{so}(4)\) spin chain can be written as the sum of two independent XXX spin chains and the spectrum is simply the sum of the energies of the XXX spin chains with the relevant coefficients, see also \[13,14\].

## 5 Generalized Hubbard models

We noticed that Model 12 corresponds to the free Hubbard model. We can use this model as a starting point to see if there are any potentials or interaction terms that can be added to this kinetic term while preserving integrability. In this way we would find new integrable Hubbard like deformations. We know that these new models cannot be \(\mathfrak{su}(2) \times \mathfrak{su}(2)\) invariant. We would like to only consider models which we can interpret as a model of electrons moving on a one-dimensional lattice or conduction band. To this end, we will only include terms which preserve fermion number.

Let \(\mathcal{K}_{Hub} \) denote the kinetic term of the Hubbard model, \(i.e.\)
\[
\mathcal{K}_{Hub} = \sum_{\alpha = \uparrow, \downarrow} (c_{\alpha,1}^\dagger c_{\alpha,2} + c_{\alpha,2}^\dagger c_{\alpha,1}). \tag{5.1}
\]
We add other kinetic/hopping terms which act on two electrons simultaneously. We consider a term which describes the hopping of a pair of electrons \(\mathcal{K}_{pair}\) and a term which flips the spins of the electrons on neighboring sites \(\mathcal{K}_{flip}\)
\[
\mathcal{K}_{pair} = A_1 c_{\uparrow,1}^\dagger c_{\uparrow,2}^\dagger c_{\downarrow,2} c_{\downarrow,1} + A_2 c_{\uparrow,2}^\dagger c_{\downarrow,2}^\dagger c_{\uparrow,1} c_{\downarrow,1}, \tag{5.2}
\]
\[
\mathcal{K}_{flip} = A_3 c_{\uparrow,1}^\dagger c_{\downarrow,2} c_{\downarrow,1} c_{\uparrow,2} + A_4 c_{\downarrow,1}^\dagger c_{\uparrow,2} c_{\downarrow,1} c_{\uparrow,2} + A_5 c_{\uparrow,1}^\dagger c_{\downarrow,2}^\dagger c_{\uparrow,1} c_{\downarrow,2} + A_6 c_{\downarrow,1}^\dagger c_{\downarrow,2}^\dagger c_{\uparrow,1} c_{\downarrow,2}. \tag{5.3}
\]
The $K_{flip}$ term violates spin conservation as it contains terms which sends $|\uparrow\uparrow\rangle \rightarrow |\downarrow\downarrow\rangle$ and $|\downarrow\downarrow\rangle \rightarrow |\uparrow\uparrow\rangle$. We finally consider the most general potential term written in terms of number operators

$$V = B_1 + B_2 n_{\uparrow,1} + B_3 n_{\downarrow,1} + B_4 n_{\uparrow,1}n_{\downarrow,1} + B_5 n_{\uparrow,2} + B_6 n_{\uparrow,1}n_{\uparrow,2} + B_7 n_{\downarrow,1}n_{\uparrow,2} + B_8 n_{\uparrow,1}n_{\downarrow,1}n_{\uparrow,2} + B_9 n_{\downarrow,2} + B_{10} n_{\uparrow,1}n_{\downarrow,2} + B_{11} n_{\downarrow,1}n_{\downarrow,2} + B_{12} n_{\uparrow,1}n_{\downarrow,1}n_{\downarrow,2} + B_{13} n_{\uparrow,2}n_{\downarrow,2} + B_{14} n_{\uparrow,1}n_{\uparrow,2}n_{\downarrow,2} + B_{15} n_{\downarrow,1}n_{\uparrow,2}n_{\downarrow,2} + B_{16} n_{\uparrow,1}n_{\downarrow,1}n_{\uparrow,2}n_{\downarrow,2}. \quad (5.4)$$

The total Hamiltonian whose integrability we will investigate is

$$\mathcal{H} = \mathcal{K}_{Hub} + \mathcal{K}_{pair} + \mathcal{K}_{flip} + V, \quad (5.5)$$

which has 22 free parameters. It is the most general Hamiltonian which preserves the number of electrons and whose single electron hopping term is given by the standard kinetic term $\mathcal{K}_{Hub}$.

**Integrable solutions** Following our procedure, we find four integrable solutions that have an $R$-matrix of difference form. These models do not include the usual Hubbard model since that model has an $R$-matrix that can not be written in difference form. We find that there are no integrable models of this type which have a non-zero pair hopping term $\mathcal{K}_{pair}$.

First, there are three independent models that only have a non-trivial $V$

$$\mathcal{H}^{(15)} = \mathcal{K}_{Hub} + a_1(n_{\uparrow,1} - n_{\downarrow,1})^2 + a_2(n_{\uparrow,1} - n_{\uparrow,2}) + a_3(n_{\downarrow,1} - n_{\downarrow,2})^2 + a_4(n_{\downarrow,1} - n_{\downarrow,2}) \quad (5.6)$$

$$\mathcal{H}^{(16)} = \mathcal{K}_{Hub} + a_1(n_{\uparrow,1} - n_{\downarrow,1})^2 + a_2(n_{\uparrow,1} - n_{\uparrow,2}) + a_3(n_{\downarrow,1} + n_{\downarrow,2}) + a_4(n_{\downarrow,1} - n_{\downarrow,2}) \quad (5.7)$$

$$\mathcal{H}^{(17)} = \mathcal{K}_{Hub} + a_1(n_{\uparrow,1} + n_{\downarrow,2})^2 + a_2(n_{\uparrow,1} - n_{\uparrow,2}) + a_3(n_{\downarrow,1} + n_{\downarrow,2}) + a_4(n_{\downarrow,1} - n_{\downarrow,2}) \quad (5.8)$$

These models separate and the Hamiltonians can be written as

$$\mathcal{H} = \mathcal{H}_{\uparrow} + \mathcal{H}_{\downarrow}. \quad (5.9)$$

Hence they are simply a direct sum of two two-dimensional integrable systems.

Secondly, we find a Hamiltonian that has a non-trivial spin flip interaction $\mathcal{K}_{flip}$ as well as a potential part

$$\mathcal{H}^{(18)} = \mathcal{K}_{Hub} + a\left(c_{\uparrow,1}^\dagger c_{\downarrow,2}^\dagger c_{\downarrow,1} c_{\downarrow,2} + c_{\downarrow,1}^\dagger c_{\downarrow,2}^\dagger c_{\downarrow,1} c_{\downarrow,2} + c_{\uparrow,1}^\dagger c_{\downarrow,2}^\dagger c_{\downarrow,1} c_{\downarrow,2} + c_{\downarrow,1}^\dagger c_{\downarrow,2}^\dagger c_{\downarrow,1} c_{\downarrow,2}\right) + (2a - b)(n_{\uparrow,1} + n_{\downarrow,1}) + b(n_{\uparrow,2} + n_{\downarrow,2}) - a(n_{\uparrow,1} + n_{\downarrow,1})(n_{\uparrow,2} + n_{\downarrow,2}). \quad (5.10)$$

Notice that this model does not preserve spin orientation and consequently is a type of XYZ deformation of the Hubbard potential. This model clearly does not separate as Models 15-17 did and to our knowledge is a new model of electrons on a one-dimensional lattice. The model has two free parameters and could have very interesting limits, spectral reductions and phase diagram. Since spin is not conserved by this model, the conventional Bethe Ansatz approach is not applicable. It would be a very interesting problem to find the spectrum of this model and to study its physical properties or quantum algebraic formalism derived model.
**R-matrices** The $R$-matrix for the XYZ-type Hubbard model corresponding to $\mathcal{H}^{(18)}$ is given by

$$
R_{12}^{(18)}(u) = f(u)
\begin{pmatrix}
    r_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & r_2 & 0 & 0 & r_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & r_3 & 0 & 0 & 0 & 0 & 0 & 0 & r_7 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & r_4 & 0 & 0 & 0 & 0 & 0 & r_7 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & r_5 & 0 & 0 & r_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_7 & 0 & r_7 \\
    0 & 0 & 0 & r_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & r_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & r_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & r_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{r_8}{r_3} & 0 & 0 & 0 & 0 & 0 & 0 & r_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_7 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_7 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_7 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_7 & 0 & 0 & 0 \\
\end{pmatrix}
$$

where we have the following functions

$$
\begin{align*}
    r_1 &= \cos(\theta + u \cos(\theta)) \\
    r_2 &= \frac{r_3^2}{r_1} \\
    r_3 &= \sin(u \cos(\theta)) \\
    r_4 &= \cos(\theta) \cos(u \cos(\theta)) \\
    r_5 &= -\cos(\theta) \tan(\theta + u \cos(\theta)) r_3 \\
    r_6 &= \frac{r_7^2}{r_1} \\
    r_7 &= \cos(\theta) e^{u(a_2 + \sin(\theta))} \\
    r_8 &= \frac{r_7}{r_3} \\
    r_9 &= \frac{s_7}{r_1} r_3 \\
    r_{10} &= \sin(\theta) r_3 \\
    r_{11} &= \frac{1}{4} (\cos(2\theta) - \cos(2u \cos(\theta)) + \cos(2(\theta + u \cos(\theta))) + 3) r_1^{-1} \\
    r_{12} &= \frac{s_7^2}{r_1} \\
\end{align*}
$$

and have defined $s_7(u) = r_7(-u)$ and $f(u) = (2a_2 u + 1) \sec(\theta + u \cos(\theta))$. The $R$-matrix satisfies the braided unitarity condition

$$
R_{12}(u)R_{21}(-u) = 1 - 4a_2^2 u^2.
$$

### 6 Towards the spectrum for Models 8, 9 and 10

Models 8 and 9 have $G = H = K = L = 0$, which means that one electron can not move by itself in the chain, only moving when in pairs, as can be seen in equations (3.3)-(3.6). Notice that Model 10 can be brought to this form by adding to its Hamiltonian a term proportional to identity

$$
\mathcal{H}^{(10)} = \mathcal{H}^{(10)} + \frac{3}{4} \rho 1.
$$

By doing this, one obtains

$$
A = D = \frac{7}{4} \rho, \quad B = E = -1, \quad C = \frac{e^{-\phi}}{2} \quad \text{and} \quad F = \frac{e^{\phi}}{2}.
$$

Now we can deal with these three models at the same time. From now one in this paper, every time we mention model 10 we will be referring to (6.1).

Because one electron alone can not move in the spin chain, but a pair of electrons can, we will refer to $\psi_1$ as $\frac{1}{2}$-excitation, and to a pair of $\psi$’s or one $\phi_2$ as 1-excitation.
6.1 Vacuum

Let us define the vacuum as

\[ |\Lambda_0\rangle \equiv |\phi_1\phi_1\ldots\phi_1\rangle. \]  

(6.3)

According to (3.3) \( \mathcal{H}_{12} |\phi_1\phi_1\rangle = (A + B) |\phi_1\phi_1\rangle \) so for the reference state, a periodic spin chain of \( L \) sites has eigenvalue

\[ A_0 = L(A + B). \]  

(6.4)

Notice that for model 8 and 9 is \( A = 0 \) because \( A = -B \). For the modified model 10, this not happens.

6.2 \( \frac{1}{2} \)-excitation

So, the eigenstates with half excitation are

\[ |\Lambda_{\frac{1}{2}}\rangle = |\psi_{\alpha,j}\rangle \equiv |\phi_1\ldots\phi_1\psi_{\alpha}\phi_1\phi_1\ldots\phi_1\rangle \]  

(6.5)

where the \( \psi_{\alpha} \) is in the \( j \)-th site of the spin chain. The corresponding eigenvalues are given by

\[ A_{\frac{1}{2}} = (L - 2)(A + B) \]  

(6.6)

with degeneracy \( d = 2L \). The reason for the degeneracy is that there are \( L \) different positions to put the \( \psi_{\alpha} \), and two possible values for \( \alpha \).

6.3 1-excitation

For one excitation, as already mentioned, we can have one \( \phi_2 \) or two \( \psi_{\alpha} \)'s. There are \( 2L^2 - L \) eigenvalues (and corresponding eigenvectors) with 1-excitation. Let us see how to construct them.

**Case 1** Let us start by considering the case with the two \( \psi \)'s being separated by one or more \( \phi_1 \)'s, i.e.

\[ |\phi_1\ldots\phi_1\psi_{\alpha}\phi_1\phi_1\ldots\phi_1\rangle \]  

(6.7)

with \( k > j + 1 \).

In this case the only contribution comes from the action of the Hamiltonian in pairs of \( \phi_1 \)'s. The states constructed in this way are already eigenstates of the Hamiltonian and have eigenvalue equal to \( (L - 4)(A + B) \) for \( L > 3 \). For \( L = 3 \) there is no way of the eigenstate (6.7) exist because of the periodicity of the spin chain.

**Case 2** Another case considered separately is when the two \( \psi \)'s are together but they are equal to each other, i.e.

\[ |\phi_1\ldots\phi_1\psi_{\alpha}\psi_{\alpha}\phi_1\phi_1\ldots\phi_1\rangle \]  

(6.8)

Such state will be an eigenstate of the Hamiltonian with eigenvalue \( (L - 3)(A + B) + (D + E) \).
Case 3  The most general state with one excitation (excluding case 1 and case 2) can be written as

$$|A_1\rangle \equiv \sum_j c_j |j\rangle + \sum_j h_{\alpha\beta,j} |\psi_{\alpha,j}\psi_{\beta,j+1}\rangle$$  \hspace{1cm} (6.9)$$

where $\alpha \neq \beta$, sum in repeated greek indices is assumed, and

$$|\psi_{\alpha L}\psi_{\beta L+1}\rangle = |\psi_{\beta,1}\psi_{\alpha L}\rangle,$$  \hspace{1cm} (6.10)$$

$$c_0 = c_L, \quad c_{i+1} = c_1,$$  \hspace{1cm} (6.11)$$

$$h_{\alpha\beta,0} = h_{\alpha\beta,L}, \quad h_{\alpha\beta,L+1} = h_{\alpha\beta,1},$$  \hspace{1cm} (6.12)$$

and

$$|j\rangle \equiv |\phi_1...\phi_j\rangle,$$  \hspace{1cm} (6.13)$$

$$|\psi_{\alpha,j}\psi_{\beta,j+1}\rangle \equiv |\phi_1...\phi_j\psi_{\alpha+j}\psi_{\beta,j+1}\rangle.$$  \hspace{1cm} (6.14)$$

Now let us see for which values of $c_i$ and $h_{\alpha\beta,j}$ the state $|A_1\rangle$ will be an eigenstate of the Hamiltonian.

Let us start by seeing how the Hamiltonians act on the state (6.9). Following (3.3)-(3.6) we obtain

$$\mathbb{H}|j\rangle = (L-2)(A+B)|j\rangle + 2A|j\rangle + B|j-1\rangle + B|j+1\rangle +$$

$$+ CE^\alpha\beta \left(|\psi_{\alpha,j-1}\psi_{\beta,j}\rangle - |\psi_{\alpha,j}\psi_{\beta,j+1}\rangle\right),$$  \hspace{1cm} (6.15)$$

and

$$\mathbb{H}|\psi_{\alpha,j}\psi_{\beta,j+1}\rangle = (L-3)(A+B)|\psi_{\alpha,j}\psi_{\beta,j+1}\rangle +$$

$$+ D|\psi_{\alpha,j}\psi_{\beta,j+1}\rangle + E|\psi_{\beta,j}\psi_{\alpha,j+1}\rangle + F_e^\alpha\beta \langle j+1 | - | j\rangle).$$  \hspace{1cm} (6.16)$$

Now we can write the action of the Hamiltonian in (6.9) as

$$\mathbb{H}|A_1\rangle = \sum_j c_j (L-2)(A+B) + 2Ac_j + Bc_{j+1} + Bc_{j-1} +$$

$$+ h_{\alpha\beta,j-1}e^{\alpha\beta}\mathbb{F} - h_{\alpha\beta,j}e^{\alpha\beta}\mathbb{F} |j\rangle +$$

$$+ \sum_{j=1}^L \left[(L-3)(A+B)h_{\alpha\beta,j} + c_{j+1}e^{\alpha\beta}C - c_je^{\alpha\beta}C +

+ Dh_{\alpha\beta,j} + Eh_{\beta\alpha,j}\right] |\psi_{\alpha,j}\psi_{\beta,j+1}\rangle.$$  \hspace{1cm} (6.17)$$

To find this formula we used the fact that the sum in $j$ is periodic to relabel the coefficients and let everything in terms of $|j\rangle$ and $|\psi_{\alpha,j}\psi_{\beta,j+1}\rangle$.

But $\mathbb{H}|A_1\rangle$ is also equal to

$$\mathbb{H}|A_1\rangle = \sum_j c_j A_1 |j\rangle + \sum_j h_{\alpha\beta,j} A_1 |\psi_{\alpha,j}\psi_{\beta,j+1}\rangle.$$  \hspace{1cm} (6.18)$$

By comparing (6.17) with (6.18) we obtain the two following conditions

$$c_j (L-2)(A+B) + 2Ac_j + Bc_{j+1} + Bc_{j-1} + h_{\alpha\beta,j-1}e^{\alpha\beta}\mathbb{F} - h_{\alpha\beta,j}e^{\alpha\beta}\mathbb{F} = A_1 c_j$$  \hspace{1cm} (6.19)$$

(L-3)(A+B)h_{\alpha\beta,j} + c_{j+1}e^{\alpha\beta}C - c_je^{\alpha\beta}C + Dh_{\alpha\beta,j} + Eh_{\beta\alpha,j} = A_1 h_{\alpha\beta,j}$$  \hspace{1cm} (6.20)$$
Multiplying the equation (6.19) by \( h_\gamma \delta, k \) and then using the equation (6.20) we obtain

\[
(A + B)c_j h_\gamma \delta, k + (2A - D)c_j h_\gamma \delta, k + B(c_{j+1} + c_{j-1})h_\gamma \delta, k - E c_j \epsilon_\gamma \delta C = 0.
\]

(6.21)

Remember \( j, k = 1, \ldots, L \), \( \alpha, \beta, \gamma, \delta = 1, 2 \) and sum in repeated greek indices is assumed.

The equations (6.21) are the conditions \( c_j \) and \( h_{\alpha \beta, j} \) have to satisfy in order to \(|A_1\rangle\) be eigenstates of the Hamiltonian and \( A_1 \) be its eigenvalues.

The procedure is then the following, one solves the equations (6.21) and substitute the \( c_j \) and \( h_{\alpha \beta, j} \) found, in the equations (6.19) and (6.20) to obtain the eigenvalues, and substitute them on (6.9) to obtain the eigenvectors.

All the equations presented in this subsection are valid for 1-excitation for any \( L > 2 \). For \( L = 3, 4, 5 \) we solved equations (6.19), (6.20) and (6.21) algebraically (using Mathematica) and we obtained exactly the eigenvalues for 1-excitation presented in the Tables (2)-(10).

### 6.4 \( \frac{3}{2} \)-excitations and more

For more than 1-excitation we were unable to find a general formula for the eigenvalues and eigenvectors. For \( L = 3, 4, 5 \) we computed the spectrum making use of the reduced Hamiltonians as presented in section D. It would be very interesting to compute ABA for this models since at least model 9 and 10 seem to have very interesting physical properties.

Remember that there is a symmetry \( p \leftrightarrow (L - p) \) with \( p \) being the number of excitations. Therefore, since we know the eigenvalues for vacuum, \( \frac{1}{2} \)-excitation and 1-excitation, we automatically have the ones for \( p = L, p = L - \frac{1}{2} \) and \( p = L - 1 \).

### 7 Discussion

In this paper we classified integral spin chains which can be identified with electrons in a conduction band. This means that we consider a four-dimensional local Hilbert space where each site can be empty, contain a single electron or an electron pair. We then use the recently proposed method of [1] to classify all integrable models that have additional spin and charge symmetry and whose \( R \)-matrix is of difference form. We recover all known models that exhibit this symmetry, such as the \( \text{su}(4) \) and \( \text{sp}(4) \) spin chain, but in addition we find several new models.

The models that we were not able to identify are Models 4, 6, 8, 9, 10, 14 and 18. Models 4 and 6 are a slightly modified version of the \( \text{su}(4) \) and \( \text{su}(2|2) \) spin chain. They have rational \( R \)-matrices which hints at an underlying Yangian symmetry. It would be an interesting question to see if these models can be generalized to \( \text{su}(m) \times \text{su}(n) \). More interesting seem to be models 8, 9 and 10, in which the fermionic degrees of freedom seem to freeze out and the only dynamical degrees of freedom correspond to electron pairs. There are many interesting research directions that can now be pursued.

First and foremost, it would be interesting to do a full classification of with less symmetric models. For instance, only assuming that spin and charge are preserved, there are 35 free parameters. One quickly finds that there will be several thousand solutions, which should contain many new and interesting integrable models. So far we have not been able to find all solutions. Many of these solutions will naturally be related by
basis transformations, twist and reparameterizations. It is very technically challenging to perform this identification.

Secondly, it is important to find the spectrum of the new models that we have found. The new models do not seem to be solvable by means of the standard coordinate (nested) Bethe Ansatz. Using the algebraic Bethe Ansatz approach might be a way to derive the spectrum. At the very least it would be interesting to work out the RTT relations and find the quantum algebras that underly these models.

After this, it would be interesting to study the physical properties of the new models. It would be particularly worthwhile to consider the thermodynamic or continuum limit. Indeed, Model 18 actually depends on two coupling constants and might have a non-trivial phase diagram. Moreover, it would be very interesting to investigate if there are two-dimensional field theories whose scattering matrix would correspond to the $R$-matrices of our new models.

Finally, there are some further open questions regarding our approach to finding integrable systems of different sizes. In this paper, we again confirm that $[Q_2, Q_3] = 0$ is a sufficient condition for these models. It would be very interesting to see an integrable model for which $[Q_3, Q_4] = 0$ would impose new constraints. We can also apply our method to look at other types of models. For instance we could consider models in higher dimensions and look at generalized Hubbard models of the type [17] or consider three-dimensional models and compare with a recent paper where a set of these solutions where recently classified [18].

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A Grading

In this appendix we work out the map between a non-graded $R$-matrix and its graded counterpart. The non-graded $R$-matrix is defined as

$$R_{jj+1}(u) = R_{\beta\delta}^{\alpha\gamma}(u) E_{j\alpha}^{\beta} E_{j+1\gamma}^{\delta}, \quad (A.1)$$

with

$$E_{j\alpha}^{\beta} = 1^\otimes(j-1) \otimes E_{\alpha}^{\beta} \otimes 1^\otimes(L-j), \quad (A.2)$$

where $(E_{\alpha}^{\beta})_i^j = \delta_i^\alpha \delta_j^\beta$ and $\alpha, \beta = 1, ..., 4$.

The graded $R$-matrix is then defined as

$$R_{jj+1}^I(u) = (-1)^{p(\gamma) + p(\alpha)(p(\beta) + p(\gamma))} R_{\gamma\delta}^{\alpha\beta}(u) e_{j\alpha}^{\gamma} e_{j+1\beta}^{\delta}, \quad (A.3)$$

where

$$e_{j\alpha}^{\beta} = (-1)^{p(\alpha) + p(\beta)} \sum_{k=1}^{L} p(\gamma_k) 1^\otimes(j-1) \otimes E_{\alpha}^{\beta} \otimes E_{\gamma_{j+1}}^{\gamma_k} \otimes ... \otimes E_{\gamma_{L}}^{\gamma_k}. \quad (A.4)$$

We are considering a 2/2 graded vector space as our Hilbert space therefore we assume $p(1) = p(2) = 0$ and $p(3) = p(4) = 1$. Summation over repeated Greek indices is assumed.

The graded Permutation operator is defined as
\[ P_{jj+1}^f = (-1)^{p(\beta)} e_j^{\alpha} e_{j+1}^{\beta}. \]  

The graded \( R \)-matrix \( R_{jk}^f \) satisfies the following YBE

\[ R_{12}^f(u - v) R_{13}^f(u) R_{23}^f(v) = R_{23}^f(v) R_{13}^f(u) R_{12}^f(u - v), \]  

regularity

\[ R_{jk}^f(0) = P_{jk}^f, \]  

and unitarity

\[ R_{jk}^f(u) R_{kj}^f(-u) \propto 1. \]  

Notice that \( R_{13}^f(u) = P_{12}^f R_{23}^f(u) P_{12}^f \) and \( P_{13}^f = P_{12}^f P_{23}^f P_{12}^f. \)

Using the graded \( R \)-matrix \( (A.3) \) we are able to construct the Hamiltonian

\[ H_{jj+1}^f = \partial_u \left( P_{jj+1}^f R_{jj+1}^f(u) \right) \bigg|_{u=0}. \]  

By comparing them with the non-graded Hamiltonians we find the map in \( (3.8) \).

## B 4D representations of \( \mathfrak{su}(2) \times \mathfrak{su}(2) \)

We are interested in models that have \( \mathfrak{su}(2) \times \mathfrak{su}(2) \) symmetry. Since we only consider a four dimensional local Hilbert space, we need to classify all four-dimensional representations of this semi-simple Lie algebra. Clearly any representation of \( \mathfrak{su}(2) \times \mathfrak{su}(2) \) automatically induces a four-dimensional representation on both \( \mathfrak{su}(2) \) factors.

Let us first focus on the first copy of \( \mathfrak{su}(2) \). There are five possible four-dimensional representations of \( \mathfrak{su}(2) \). Most of them are reducible and hence they decompose into irreducible representations. In particular, we find the following decompositions \( 1 \oplus 1 \oplus 1 \oplus 1, 2 \oplus 1 \oplus 1, 2 \oplus 2, 3 \oplus 1, 4 \). We are only interested in non-trivial representations, hence we will not consider \( 1 \oplus 1 \oplus 1 \oplus 1 \). Fixing this four-dimensional representation, we then consider three general \( 4 \times 4 \) matrices and impose that they form an \( \mathfrak{su}(2) \) algebra and commute with the representation of the first \( \mathfrak{su}(2) \) factor. This is enough, up to some similarity transformations, to fix the representation of the second \( \mathfrak{su}(2) \) factor.

Let \( t_i^{L/R} \) denote the first and second set of \( \mathfrak{su}(2) \) generators in \( \mathfrak{su}(2) \times \mathfrak{su}(2) \) respectively. They satisfy

\[ [t_i^{L/R}, t_j^{L/R}] = \epsilon_{ijk} t_k^{L/R}. \]  

Let \( \rho_n(t_i) \) denote the \( n \)-dimensional irreducible representation of the generators \( t_i \) of \( \mathfrak{su}(2) \), then we find the following representations.

**2 \( \oplus \) 1 \( \oplus \) 1** In this case, the two dimensional representation is embedded as a direct sum. More explicitly, the first factor is simply embedded in the upper left \( 2 \times 2 \) block, \( i.e. \)

\[ \rho_{2\oplus 1\oplus 1}(t_i^L) = \rho_2(t_i) \oplus 0 = \begin{pmatrix} \rho_2(t_i) & 0 \\ 0 & 0 \end{pmatrix}. \]  

(B.2)
It is easy to see that this immediately implies that the second representation has to be two-dimensional and embedded in the lower right block

\[ \rho_{2\oplus 1} \oplus 1(t_i^R) = 0 \oplus \rho_2(t_i) = \begin{pmatrix} 0 & 0 \\ 0 & \rho_2(t_i) \end{pmatrix}. \]  (B.3)

These are the usual spin and charge \(\mathfrak{su}(2)\) symmetries of the Hubbard model.

2 \(\oplus\) 2 In this situation, the two-dimensional \(\mathfrak{su}(2)\) representation is embedded diagonally and can be written as

\[ \rho_{2\oplus 2}(t_i^L) = 1 \otimes \rho_2(t_i) = \begin{pmatrix} \rho_2(t_i) & 0 \\ 0 & \rho_2(t_i) \end{pmatrix}. \]  (B.4)

It is easy to check that the only other non-trivial representation of \(\mathfrak{su}(2)\) commuting with this is

\[ \rho_{2\oplus 2}(t_i^R) = \rho_2(t_i) \otimes 1. \]  (B.5)

3 \(\oplus\) 1 and 4 When one of the \(\mathfrak{su}(2)\) representations contains a three- or four-dimensional irreducible component, then it is straightforward to show that there is no non-trivial representation for the second \(\mathfrak{su}(2)\) representation that commutes with it.

C \(\mathfrak{sp}(2n)\) and \(\mathfrak{so}(n)\) spin chains

Here we collect a few basic facts on \(\mathfrak{sp}(2n)\) and \(\mathfrak{so}(n)\) spin chains.

\(\mathfrak{sp}(2n)\) integrable models. For models with compact symplectic \(\mathfrak{sp}(2n)\) (\(C_n\)) symmetry, such as Model 11, one finds an \(R\)-matrix from Yang-Baxter equation as first found in [11]

\[ R^{(\mathfrak{sp})}_{12}(u) = u(u + n + 1) + (u + n + 1)\mathcal{P}_{12} - u\mathcal{R}_{12} \]  (C.1)

where, \(\mathcal{P}_{12}\) is the permutation operator and \(\mathcal{R}_{12}\) acts in \(\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}\), which in block basis \((\mathbb{C}^n \oplus \mathbb{C}^n)\)

\[ \mathcal{R}_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & K_{12} & -K_{12} & 0 \\ 0 & -K_{12} & K_{12} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_{12} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \]  (C.2)

\(K_{12}\) acts in \(\mathbb{C}^n \otimes \mathbb{C}^n\) and \(R^{(\mathfrak{sp})}_{12}(u)\) obeys unitarity and crossing-symmetry. Consequently, the \(\mathfrak{sp}(2n)\) invariant model provides the following Hamiltonian and dispersion relation

\[ \mathcal{H}^{(\mathfrak{sp})}_{2n} = \sum_{k=1}^{N} \left[ \mathcal{P}_{kk+1} - \frac{1}{n+1} K_{kk+1} \right] + N \frac{1}{u_k^2 + \frac{1}{4}} \]  (C.3)

where \(K\) as of above and \(p\) is the excitation number. These models are generalisations of the Heisenberg XXX-model, they constitute analogues of \(\mathfrak{su}(n)\) fundamental magnetic as pointed out in [19,13].
**so(n) models** An Integrable spin with $SO(n)$ symmetry is a one-dimensional lattice of $L$ sites and local Hilbert spaces $\mathbb{C}^n$. Thus each site contains $n$ states, which transform in the standard way

$$L^{ab}|\nu^c⟩ = i \left( \delta_{bc}|\nu^a⟩ - \delta_{ac}|\nu^b⟩ \right)$$

$$[L^{ab}, L^{cd}] = i \left[ \delta_{ad}L^{bc} + \delta_{bc}L^{ad} - (\delta_{ac}L^{bd} + \delta_{bd}L^{ac}) \right]$$  \hspace{1cm} (C.4)

with Lie generators $L^{ab}$ satisfying the so($n$) algebra relations. The tensor product of two fundamental representations decomposes into a direct sum of a singlet, antisymmetric and symmetric components, which conventionally for dimensions of irreducible representations represents

$$n \otimes n = 1 \oplus \frac{n(n-1)}{2} \oplus \frac{(n+2)(n-1)}{2}$$ \hspace{1cm} (C.5)

which is crucial for wave function state description in three channels. From here one can get conventional bilinear-biquadratic Hamiltonian

$$H_{BB}^{so(n)} = \frac{1}{2n} \sum_i \left[ (n-1) + \sum_{a<b} L^{ab}_{i} L^{ab}_{i+1} + \left( \sum_{a<b} L^{ab}_{i} L^{ab}_{i+1} \right)^2 \right]$$ \hspace{1cm} (C.6)

We can also obtain one-parametric extension

$$H_{BB; \theta}^{so(n)} = \sum_i \left[ \cos \theta \sum_{a<b} L^{ab}_{i} L^{ab}_{i+1} + \sin \theta \left( \sum_{a<b} L^{ab}_{i} L^{ab}_{i+1} \right)^2 \right]$$ \hspace{1cm} (C.7)

Important to note that $n$ even or odd requires separate analysis.

**D The spectrum for models 8, 9 and 10**

**D.1 Reduced Hamiltonians**

The size of the Hamiltonian increases exponentially with the number of sites. So it is not easy to directly diagonalize it. Actually, even when the direct diagonalization is possible, is not easy to know from which type of excitation each eigenvalue is coming from, since Mathematica mixes different excitations when computing the eigenvectors.

A better way to do this is to use reduced Hamiltonians. The idea is to construct a set $v = v_1, ..., v_m$ of all the possible vectors with a certain number of excitations. And then if these vectors satisfy

$$v_i^T . v_j = \delta_{ij}$$ \hspace{1cm} (D.1)

with $T$ denoting transposition, we can define the reduced Hamiltonian as

$$\mathbb{H}_{red} = v_i^T . \mathbb{H} . v_j$$ \hspace{1cm} (D.2)

where $i$ and $j$ go from 1 to the total number of vectors for that number of excitations.

Let us see how this works for one excitation, for example. For 1-excitation we can have one $\phi_2$ or two $\phi_1$’s. So, the number of ways to put $\phi_2$ in a spin chain of length $L$ is

$$\# \text{ of Permutations}(|\phi_1 ... \phi_1 \phi_2 \phi_1 ... \phi_1⟩) = \frac{L!}{1!(L-1)!} = L,$$ \hspace{1cm} (D.3)
while the number of ways to put two $\psi$’s is

$$\# \text{ of Permutations} \left( |\phi_1...\phi_1\psi_1...\phi_1\psi_1| \right) = \begin{cases} 2 \left( \frac{L!}{2(L-2)!} \right) = L(L-1) & \text{for } \alpha = \beta \\ \left( \frac{L!}{1!(L-2)!} \right) = L(L-1) & \text{for } \alpha \neq \beta \end{cases}$$ (D.4)

So, for 1-excitation we have a total of $2L^2 - L$ possible vectors, so $i$ and $j$ in equations (D.1) and (D.2) are $i, j = 1, ..., (2L^2 - L)$. Now it is just to use these vectors to construct the reduced Hamiltonian $\mathbb{H}_{\text{red}}$ in equation (D.2). By diagonalizing this Hamiltonian one obtains all the eigenvalues with 1-excitation. One can repeat this procedure until have all the possible excitations for a given number of sites.

Notice, that diagonalize $\mathbb{H}_{\text{red}}$ is a lot easier than diagonalize the full $\mathbb{H}$ because the size of the matrix is much smaller, and it has the advantage of providing the information of which eigenvalues comes from which excitation.

### D.2 Spectrum computed using the reduced Hamiltonians

In this subsection we construct tables with the spectrum of Hamiltonians for $L = 3, 4, 5$ for Models 8, 9 and 10 using the reduced Hamiltonians.

The tables not show all the possible excitations, because the Hamiltonian $\mathbb{H}$ has a symmetry $p \leftrightarrow L - p$ where $p$ is the number of excitations$^2$. It is therefore enough to show half of the excitations. For $L = 3$, for example, the state with 1-excitation ($p = 1$) has the same eigenvalues as the state with 2-excitations ($p = 2$).

In the following, $d$ means degeneracy and $\Lambda$ means eigenvalue.

#### D.2.1 Model 8

For $L=3$, the Hamiltonian has the following eigenvalues

$$\{2(-2), 2(2), 4(-1), 4(1), 52(0)\}$$ (D.5)

meaning that, there are two eigenvalues equal to $-2$, two equal to 2, four eigenvalues equal to $-1$, etc. These eigenvalues are distributed in the spectrum as presented in Table 2. The degeneracies are given by

| Number of excitations | $d(\Lambda)$ |
|-----------------------|--------------|
| 0                     | $\{1(0)\}$  |
| $\frac{1}{2}$         | $\{6(0)\}$  |
| 1                     | $\{1(-2), 2(1), 12(0)\}$ |
| $\frac{3}{2}$        | $\{2(2), 4(-1), 14(0)\}$ |

**Table 2:** Eigenvalues and their corresponding number of excitations for a chain with 3 sites for model 8.

$^2$We checked this claim for models 8, 9 and 10 with $L = 3, 4, 5$ but it is a direct consequence of the $\text{su}(2) \times \text{su}(2)$
For \( L = 4 \), the Hamiltonian has the following eigenvalues

\[
\left\{ 14(-2), 14(2), 1(-2\sqrt{3}), 1(2\sqrt{3}), 1(2i), 1(-2i), 224(0) \right\}
\]  

which are distributed in the spectrum as in Table 3.

| Number of excitations | \( d(A) \) |
|-----------------------|------------|
| 0                     | \{1(0)\}  |
| \( \frac{1}{2} \)     | \{8(0)\}  |
| 1                     | \{1(-2), 1(2), 26(0)\} |
| \( \frac{3}{2} \)     | \{4(-2), 4(2), 48(0)\} |
| 2                     | \{1(2i), 1(-2i), 4(-2), 4(2), 58(0), 1(-2\sqrt{3}), 1(2\sqrt{3})\} |

Table 3: Eigenvalues and their corresponding number of excitations for a chain with 4 sites for model 8.

The degeneracies are given by

\[ \{1, 1, 1, 14, 14, 224\} \]  

For \( L = 5 \), the Hamiltonian has the following eigenvalues

\[
\left\{ 4\left(\frac{1}{2}(3 + \sqrt{5})\right), 4\left(\frac{1}{2}(3 - \sqrt{5})\right), 4\left(\frac{1}{2}(-3 + \sqrt{5})\right), 4\left(\frac{1}{2}(-3 - \sqrt{5})\right), 20\left(\frac{1}{2}(1 + \sqrt{5})\right), 20\left(\frac{1}{2}(1 - \sqrt{5})\right), 20\left(\frac{1}{2}(-1 + \sqrt{5})\right), 20\left(\frac{1}{2}(-1 - \sqrt{5})\right), 2(1 + \sqrt{5}), 2(1 - \sqrt{5}), 2(-1 + \sqrt{5}), 2(-1 - \sqrt{5}), 10(-2), 10(2), 24(-1), 24(1), 724(0), 16(-1, 9563), 16(1, 9563), 16(-1.82709), 16(1.82709), 16(-1.33826), 16(1.33826), 16(-0.209057), 16(0.209057) \right\}
\]  

which are distributed in the spectrum as in Table 4.

The degeneracies are given by

\[ \{2, 2, 2, 2, 4, 4, 4, 4, 4, 10, 10, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16, 20, 20, 20, 20, 20, 24, 24, 724\} \]  

D.2.2 Model 9

For \( L = 3 \), the spectrum is given by the following eigenvalues

\[ \{8(5), 4(2), 52(0)\} \]  

which are distributed as in Table 5.

So degeneracies for this case are given by
| Number of excitations | d (A)                              |
|-----------------------|------------------------------------|
| 0                     | \{1(0)\}                          |
| \(\frac{1}{2}\)      | \{6(0)\}                           |
| 1                     | \{2(5), 1(2), 12(0)\}             |
| \(\frac{3}{2}\)      | \{4(5), 2(2), 14(0)\}            |

Table 4: Eigenvalues and their corresponding number of excitations for a chain with 5 sites for model 8.

| Number of excitations | d (A)                              |
|-----------------------|------------------------------------|
| 0                     | \{1(0)\}                          |
| \(\frac{1}{2}\)      | \{6(0)\}                           |
| 1                     | \{2(5), 1(2), 12(0)\}             |
| \(\frac{3}{2}\)      | \{4(5), 2(2), 14(0)\}            |

Table 5: Eigenvalues and their corresponding number of excitations for a chain with 3 sites for model 9.

\{8, 4, 52\}. \hspace{1cm} (D.12)

For L=4, the spectrum is given by the following eigenvalues

\{1(10), 15(2), 16(6), 30(4), 194(0)\}. \hspace{1cm} (D.13)

These eigenvalues are distributed in the spectrum as in Table 6.

So the degeneracies in this case are

\{1, 15, 16, 30, 194\}. \hspace{1cm} (D.14)

For L=5, the spectrum is given by the following eigenvalues

\\[
\begin{align*}
\{ & 40 \left( \frac{1}{2}(9 + \sqrt{5}) \right), 40 \left( \frac{1}{2}(9 - \sqrt{5}) \right), 4(5 + \sqrt{5}), 4(5 - \sqrt{5}), 32(5), \\
& 20(2), 724(0), 8(10.0507), 8(7.49137), 8(3.8906), 8(3.56732), \\
& 32(5.9563), 32(4.20906), 32(2.66174), 32(2.17291) \}\end{align*}
\] (D.15)
Table 6: Eigenvalues and their corresponding number of excitations for a chain with 4 sites for model 9.

| Number of excitations | d (A) |
|-----------------------|-------|
| 0                    | \{1(0)\} |
| 1/2                  | \{8(0)\} |
| 1                    | \{1(6), 2(4), 1(2), 24(0)\} |
| 3/2                  | \{4(6), 8(4), 4(2), 40(0)\} |
| 2                    | \{1(10), 6(6), 10(4), 5(2), 48(0)\} |

Table 7: Eigenvalues and their corresponding number of excitations for a chain with 5 sites for model 9.

| Number of excitations | d (A) |
|-----------------------|-------|
| 0                    | \{1(0)\} |
| 1/2                  | \{10(0)\} |
| 1                    | \{2 \{1/2(9 + \sqrt{5})\}, 2 \{1/2(9 - \sqrt{5})\}, 1(2), 40(0)\} |
| 3/2                  | \{4 \{1/2(9 + \sqrt{5})\}, 4 \{1/2(9 - \sqrt{5})\}, 4(5), 2(2), 90(0) \}
|                        | \{4(5.95630), 4(4.20906), 4(2.66174), 4(2.17291)\} |
| 2                    | \{8 \{1/2(9 + \sqrt{5})\}, 8 \{1/2(9 - \sqrt{5})\}, 1(5 + \sqrt{5}), 1(5 - \sqrt{5})\}
|                        | \{8(5.95630), 8(4.20906), 2(3.8906), 2(3.56732), 8(2.66174), 8(2.17291)\} |
| 5/2                  | \{12 \{1/2(9 + \sqrt{5})\}, 12 \{1/2(9 - \sqrt{5})\}, 2(5 + \sqrt{5}), 2(5 - \sqrt{5})\}
|                        | \{8(5.95630), 8(4.20906), 4(3.8906), 4(3.56732), 8(2.66174), 8(2.17291)\} |

which are distributed as in Table 7.

The degeneracies in this case are

\{4, 4, 8, 8, 8, 8, 8, 8, 8, 20, 32, 32, 32, 32, 32, 32, 40, 40, 724\}. \hspace{1cm} (D.16)

D.2.3 Model 10

For L=3, the Hamiltonian \( H \) has the following eigenvalues

\[ \left\{ 8 \left( \frac{23}{4} \right), 4 \left( \frac{11}{4} \right), 16 \left( \frac{9}{4} \right), 36 \left( \frac{3}{4} \right) \right\} \]

which are distributed in the spectrum as in Table 8.

The degeneracies for this case are given by

\{4, 8, 16, 36\}. \hspace{1cm} (D.18)
Table 8: Eigenvalues and their corresponding number of excitations for a chain with 3 sites for model 10.

For L=4, the Hamiltonian $H^{(10)}$ has the following eigenvalues

$$
\left\{ 1 \left( \frac{1}{4} (29 + \sqrt{145}) \right), 1 \left( \frac{1}{4} (29 - \sqrt{145}) \right), 1 \left( \frac{1}{4} (21 + \sqrt{33}) \right), 1 \left( \frac{1}{4} (21 - \sqrt{33}) \right), 16 \left( \frac{1}{2} (7 + \sqrt{7}) \right), 16 \left( \frac{1}{2} (7 - \sqrt{7}) \right), 1(9), 6 \left( \frac{15}{2} \right), 8(6), 14 \left( \frac{11}{2} \right), 1(5), 6 \left( \frac{7}{2} \right), 44(3), 8(2), 100 \left( \frac{3}{2} \right), 32(0) \right\}
$$

which are distributed in the spectrum as in Table 9.

Table 9: Eigenvalues and their corresponding number of excitations for a chain with 4 sites for model 10.

The degeneracies are

$$\{1, 1, 1, 1, 1, 6, 6, 8, 8, 14, 16, 16, 32, 44, 100\}.$$  (D.20)

For L=5, the eigenvalues of the Hamiltonian are given by

$$
\left\{ 16 \left( \frac{1}{4} (27 + 2\sqrt{5}) \right), 16 \left( \frac{1}{4} (27 - 2\sqrt{5}) \right), 24 \left( \frac{1}{4} (19 + 2\sqrt{7}) \right), 24 \left( \frac{1}{4} (19 - 2\sqrt{7}) \right), 4 \left( \frac{1}{4} (26 + \sqrt{65}) \right), 4 \left( \frac{1}{4} (26 - \sqrt{65}) \right), 8 \left( \frac{29}{4} \right), 8 \left( \frac{17}{4} \right), 64 \left( \frac{15}{4} \right), 12 \left( \frac{11}{4} \right) \right\}.
$$
and they are distributed in the spectrum as in the Table 10.

| Number of excitations | d (A) |
|------------------------|-------|
| 0                      | \{1 \left( \frac{15}{4} \right) \} |
| \frac{1}{2}            | \{10\left(\frac{5}{2}\right)\} |
| 1                      | \{2 \left( \frac{11}{4} \right), 40 \left( \frac{3}{2} \right), 30 \left( \frac{3}{2} \right), 1 \left( \frac{3}{4} \right), 15 \left( \frac{9}{4} \right), 20 \left( \frac{9}{4} \right), 2 \left( \frac{1}{2} (27 + 2\sqrt{5}) \right), 2 \left( \frac{1}{2} (27 - 2\sqrt{5}) \right) \} |
| 2                      | \{1 \left( \frac{1}{4} (26 + \sqrt{65}) \right), 1 \left( \frac{1}{4} (26 - \sqrt{65}) \right), 2 \left( \frac{1}{4} (27 + 2\sqrt{5}) \right), 2 \left( \frac{1}{4} (27 - 2\sqrt{5}) \right), 6 \left( \frac{1}{4} (19 + 2\sqrt{7}) \right), 6 \left( \frac{1}{4} (19 - 2\sqrt{7}) \right), 2 \left( \frac{20}{3} \right), 1 \left( \frac{40}{3} \right), 10 \left( \frac{14}{3} \right), 3 \left( \frac{14}{3} \right), 60 \left( \frac{14}{3} \right), 40 \left( \frac{14}{3} \right), 2 (10.9277), 2(8.31678), 2(8.2063), 6(6.72073), 6(6.49331), 2(6.45906), 2(6.06519), 2(5.9032), 6(5.52133), 6(4.91174), 2(4.91174), 2(4.42291), 6(4.33029), 6(3.82412), 6(3.14143), 6(2.95068), 6(2.83293), 6(2.7698) \} |
| \frac{5}{2}            | \{2 \left( \frac{1}{4} (26 + \sqrt{65}) \right), 2 \left( \frac{1}{4} (26 - \sqrt{65}) \right), 8 \left( \frac{1}{4} (27 + 2\sqrt{5}) \right), 8 \left( \frac{1}{4} (27 - 2\sqrt{5}) \right), 4 \left( \frac{1}{4} (19 + 2\sqrt{7}) \right), 4 \left( \frac{1}{4} (19 - 2\sqrt{7}) \right), 4 \left( \frac{20}{3} \right), 4 \left( \frac{14}{3} \right), 32 \left( \frac{14}{3} \right), 2 \left( \frac{14}{3} \right), 50 \left( \frac{14}{3} \right), 60 \left( \frac{14}{3} \right), 4(10.9277), 4(8.31678), 4(8.2063), 6(6.72073), 4(6.49331), 4(6.45906), 4(6.06519), 4(5.9032), 4(5.52133), 4(4.91174), 4(4.91174), 4(4.42291), 4(4.33029), 4(3.82412), 4(3.14143), 4(2.95068), 4(2.83293), 4(2.7698) \} |

Table 10: Eigenvalues and their corresponding number of excitations for a chain with 5 sites for model 10.

The degeneracies for this case is then given by

\[ \{4, 4, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 12, 16, 16, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 64, 240, 300 \} \]  

(D.22)

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