Correlated transport in driven atomtronic circuits

Enrico Compagno,1 Guillaume Quesnel,1 Anna Minguzzi,2 Luigi Amico,3,4,5 and Denis Feinberg*1

1Univ. Grenoble-Alpes, CNRS, Institut Néel, 38000 Grenoble, France
2Univ. Grenoble-Alpes, CNRS, LPMMC, 38000 Grenoble, France
3Dipartimento di Fisica e Astronomia 'Ettore Majorana', Via S. Sofia 64, 95127 Catania, Italy
4Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117543, Singapore
5LANEF Chaire d’excellence, Univ. Grenoble-Alpes & CNRS, F-38000 Grenoble, France

(Dated: December 24, 2019)

The quest to develop quantum technology of practical value requires accessing to correlated quantum many-body states. However, how to dynamically trigger suitable quantum effects for applications is still challenging. Here, we propose two new schemes of correlated quantum transport to generate pair and multi-pair boson correlations in neutral matter-wave circuits. Our system is made of a triple-well arranged in a ring-shape configuration, and with a large symmetrical offset among the wells. With the first scheme, working with repulsive static interactions, we demonstrate how a single nonlocal boson pair coherent transfer can be achieved. In the second scheme, we combine strong offsets and a periodic drive of the interaction. In this case, depending on the system parameters, we obtain either multi-pair boson transfer and generalized NOON states of the W type or coherent destruction of tunneling. Our results provide key insights for the preparation and manipulation of many-body quantum correlations in atomtronics devices made of interacting Bose-Einstein condensates.

Introduction and summary of results. Entanglement and interference are key notions that quantum technology aims to harness to both answer fundamental questions in basic science and to fabricate quantum devices with enhanced performances superseding the existing ones [1]. To this end, ultra-cold atom systems plays an important role as they have excellent coherence properties and can be controlled with high precision and flexibility of the operating conditions [2]. Nowadays, cold atoms can be locally addressed on micrometer spatial scales, employing a variety of approaches like painting techniques [3], light sculpting by means of spatial light-modulators or digital micro-mirror devices [4]. In addition, thanks to the latest achievements in the field it is possible to control the spatial features of the atomic confinement on time scales much shorter than atom’s dynamics time scales [3, 5–11]. Atomtronics takes up such progress to realize circuits of cold-atoms guided with laser light beams or magnetic means. Such platform allows both to carry out basic research in many-body physics and to achieve new concepts of devices of practical value [12, 13]. In particular, mesoscopic physics can be studied with a new twist. Different problems have been considered so far. Persistent currents in ring-shaped condensates and the atomic counterpart of SQUID devices have been studied both in the classical and quantum regimes [14–21]. New features of quantum transport in simple circuitual elements have been evidenced in coherent bosonic systems [22–27]. Several schemes have been proposed to study coherence transfer in hybrid solid state quantum devices - cold atom systems [28–34].

In this work, we address correlated transport in atomtronic circuits as a mean to widen the class of states that can be prepared and manipulated in quantum networks. To this end, we study the dynamics of \( N \) neutral bosons trapped in a strongly tilted triple-well ring potential at zero temperature (Fig. 1). A charged matter-wave counterpart of such a system has been studied in mesoscopic physics, where coupled superconductors have been demonstrated to sustain nontrivial generalization of the Josephson effect involving a coherent dc-transport of multi-Cooper pairs [35–39]. Such systems are interesting also as quantum simulators in that they can provide a realization of tunable topological materials [40].

We show here that charged matter-wave and our neutral matter-wave-based system involve rather different phenomena. In particular, we demonstrate that the interplay between interaction and the strong offset is the source of two- and higher order bosonic correlations. In addition, the atomtronic platform grants us the access to physical regimes that are hard, if not impossible, to be accessed with the standard quantum electronic implementations. In particular, one can rely on Feshbach resonances to tune the atom-atom interaction in a wide range of values as well as a function of time. Our physical sys-

![FIG. 1. Scheme of the model. A Bose-Einstein condensate is trapped in a triple-well ring configuration. \( J \) is the tunnelling rate, \( \mu, \mu_2 \) and \( -\mu \) are respectively the local chemical potential bias in sites 1, 2 and 3. The red arrows illustrate the correlated transport process where pairs of bosons move coherently from site 2 towards 1 and 3, and the blue arrows represent pair motion triggered by the periodic drive.](image-url)
inter-particle interaction and \( \mu \) the hopping strength and the static part of the on-site inter-
ber operator. The parameters potential, ie the well o

The two above processes are exploited respectively in the two following schemes. We first consider static repulsive on-
site interactions. Here, the strong offset among the wells makes the single-particle and the correlated nonlocal pair transfers well separated in the dynamics. By suitably choosing the well offset \( \mu_2 \) we generate nearly perfect superpositions of two states that differ by one nonlocal boson pair hopping as in Eq. 2 with \( N_T = 2 \) (Fig. 2). Remarkably, we find that it can occur only for non-zero interparticle interactions, contrarily to the case of quantum electronics in which the correlated pair transport can be achieved also in the non-interacting regime.

In the second scheme, we set \( \mu_2 = 0 \) and modulate the on-site interaction periodically in time around zero [41–45]. We especially focus on a resonance between the drive and the Josephson atom pair oscillations due to the offset, i.e. \( \omega = 2 \mu / \hbar \). We find that such a resonant regime triggers higher-order correlated transport: the drive promotes sequential coherent transfers of boson pairs between wells, while the offset, enhancing or suppressing the particle multi-occupancy, establishes non-trivial correlations in the transport. The exact dynamics is complemented by a high-frequency Floquet analysis, yielding an excellent agreement.

In both schemes we observe that the regimes of interest are dominated by extremely long time scales, featuring a near degeneracy in the static or dynamical energy spectrum [41, 43] [46]. For suitable choices of the interaction strength, Coherent Destruction of Tunnelling (CDT) [41, 43, 47, 48] can occur. Alternatively, interesting entangled states can be dynamically generated. Two-mode NOON [49] and \( W \)-type states, i.e. entangled superpositions of states \(| N, 0, 0 \rangle, |0, N, 0 \rangle, |0, 0, N \rangle \) with even \( N \) (see Fig. 3) can emerge for a wide range of parameters of the system. Finally, we discuss how to detect experimentally the correlated transport in cold atoms settings. We show that second-order momentum correlations, as obtained after a time-of-flight expansion, can distinguish between coherent and localized states achieved in the transfer (Fig. 4).

The model. For a deep ring lattice, we describe the system by means of the three-mode Bose-Hubbard Hamiltonian subjected to a periodic modulation of the interaction strength:

\[
\begin{align*}
H(i) = -J & \sum_{i=1}^{3} (a^\dagger_i a_{i+1} + H.c.) + \\
& \left( \frac{U_0}{2} + \frac{U_1}{2} \sin(\omega t) \right) \sum_{i=1}^{3} \hat{n}_i (\hat{n}_i - 1) - \sum_{i=1}^{3} \mu_i \hat{n}_i,
\end{align*}
\]  

(1)

where \( a_i^\dagger \) create a boson in site \( i \) and \( \hat{n}_i = a_i^\dagger a_i \) is the number operator. The parameters \( J \) and \( U_0 \) quantify respectively the hopping strength and the static part of the on-site interaction, \( U_1 \) is the amplitude of the periodic modulation of the inter-particle interaction and \( \mu_i \) describes the local chemical potential, ie the well offset of the site \( i \). Our aim is to inves-
tigate on how a correlated pair transport of ultra-cold bosonic atoms can be triggered by a suitable choice of \( \mu_i, U_1 \) and \( \omega \).

Coherent transfer of nonlocal boson pairs. We first consider static repulsive interactions, i.e \( U_1 = 0 \) in Eq. (1). We are interested in a two-atom transition from one site, say 2, to the two others 1 and 3, and choose for this purpose \( \mu_1 = -\mu_3 = \mu \). Specifically, starting from the initial state \(| \psi_i \rangle = | n_{1i}, n_{2i}, n_{3i} \rangle \), the correlated transport of \( m \) pairs brings the system coher-
tently through the states \(| \psi_f \rangle = | n_{1i} + m, n_{2i} - 2m, n_{3i} + m \rangle \). \( N_T \) being the total number of pairs, an entangled target state is a superposition of such states:

\[
| \psi_T \rangle = \sum_{m=1}^{N_T} c_m | n_{1i} + m, n_{2i} - 2m, n_{3i} + m \rangle.
\]

(2)

The Hilbert space described by the states \(| n_{1i}, n_{2i}, n_{3i} \rangle \), can be split into subspaces with fixed difference \( n_1 - n_3 \), that contain the target state (2). Correlated processes are favoured by enforcing energy separation of these subspaces, i.e. we set \( \mu \gg J, U_0, \mu_2 \). We use both exact diagonalization and second-order perturbation theory in \( J/\mu \) to analyze how a pair of bosons is transferred with high fidelity within one subspace.

Two-atom and higher-order virtual processes allow to iden-
tify an effective model for single pair transfer:

\[
H_{\text{eff, hop}} = J_{\text{eff}} \left[ (a_i^\dagger)^2 a_1 a_3 + H.c. \right],
\]

where the effective hopping strength \( J_{\text{eff}}^\mu \) can be obtained via quasi-degenerate perturbation theory [50]:

\[
J_{\text{eff}}^\mu = -\frac{J^2 U_0}{\mu^2} \mathcal{F}(U_0, \mu, \mu_2)
\]

(4)

(see Ref. 46 for the full effective model and the exact form of the function \( \mathcal{F} \)). Eq. (4) clearly shows that pair correlated transport requires a non-zero value of the inter-particle interaction \( U_0 \). Otherwise, the boson pair transport is suppressed by destructive interference of virtual paths [46] such as:

\[
\begin{align*}
|0, N, 0 \rangle & \rightarrow |1, N - 1, 1 \rangle \rightarrow |1, N - 2, 1 \rangle \\
|0, N, 0 \rangle & \rightarrow |0, N - 1, 1 \rangle \rightarrow |1, N - 2, 1 \rangle
\end{align*}
\]

(5)

This makes the correlated transport in bosonic systems strikingly different from the process taking place in quantum electronics where a quartet mode (made of pairs of Cooper pairs) exists due to the electron exchange interaction [35–39]. We note that in the aforementioned regime in which correlated transport can take place, the interaction \( U_0 \) must to be large compared to the transfer rates Eq. (4). As a result, coherent combinations involving a large number of states in Eq. (2) cannot be achieved with the present static protocol. Single pair correlated transfer, though, is obtained with high fidelity by fine tuning the potential \( \mu_2 \) to compensate the total interaction energy. It connects states \(| \psi_0 \rangle = | n_{1i}, n_{2i}, n_{3i} \rangle, | \psi_f \rangle = | n_{1i} + 1, n_{2i} - 2, n_{3i} + 1 \rangle \), well-separated from the rest of the spectrum. High fidelity transfer can hold for any number of atoms \( N = 2N_T \) for large enough \( \mu \), leading to a nearly perfect
Correlated transport with periodic driving. To go beyond single-pair transfer we drive the interaction strength as $U(t) = U_1 \sin(\omega t)$, choosing $U_0 = 0$ in Eq. (1) and $\omega \gg J$ (see [51, 52]). A relevant parameter for the dynamics is $K_0 = U_1/\omega$. Given the values of the relative offsets between the bosonic islands (1, 2, 3), one encounters two different Josephson frequencies in a ring geometry: $\omega_1 = \mu$ for single boson transitions between 1 (or 3) and 2, and $\omega = 2\mu$ for single boson transitions between 1 and 3, also coinciding with the energy for boson pairs hopping between 1 (or 3) and 2. Therefore coherent pair correlations are expected to be formed for resonant driving frequencies $\omega = 2\mu$.

We construct an effective time-averaged Hamiltonian following the method proposed by Dalibard and Goldman [53, 54] (see also Ref. 55). The derivation involves elimination of the second and third terms in Eq. (1) by a canonical transformation. The dressed 1 ↔ 3 transitions can be directly averaged on time while the 1 ↔ 2 and 3 ↔ 2 transitions require a Floquet expansion of order $1/\omega$ (see Ref. 46 for the detailed derivation), yielding:

$$H_{\text{eff}} = -J \left[ J_{11}(K_0, h_1 - h_3 - 1) a_1^\dagger a_3 + H.c. \right] + \frac{J^2}{\mu} \left[ L(K_0, h_1, h_2) (a_1^\dagger a_2)^2 + L(K_0, h_2, h_3) (a_2^\dagger a_3)^2 + M(K_0, h_1, h_2, h_3) a_1^\dagger a_2^\dagger a_2 a_3 + H.c. \right] + \frac{J^2}{\mu} \left[ H_2(h_1 + 1) P(K_0, h_1, h_2) - h_1(h_2 + 1) P(K_0, h_2, h_1) + h_2(h_3 + 1) P(K_0, h_2, h_3) - h_3(h_2 + 1) P(K_0, h_3, h_2) \right]$$

where $L, M, N, P$ being infinite series of Bessel functions [46].

Eq. (6) shows that atom pair transfers $2 \leftrightarrow 1$ and $2 \leftrightarrow 3$ as well as $2 \leftrightarrow (1, 3)$ are dynamically generated. Fig. 3 shows the time dynamics of the transfer fidelity between states $|0, N, 0\rangle$, $|N, 0, 0\rangle$ as a function of the interaction and its maximum on a large time interval. The agreement between the exact dynamics obtained from Eq. (1) and the one resulting from the effective Hamiltonian Eq. (6) is remarkable.

For specific values of $K_0$, the fidelity $F_{N,0,0}$ displays marked peaks. Optimal transfer requires a fine tuning of the interaction parameter, especially for large (even) $N$. The system achieves a coherent transfer of $N$ bosons from island 2 either to island 1 or to island 3, at the same $K_0$ value but at different times. Moreover, for $K_0 \gg 1$, the asymptotic expansion of the Bessel functions gives $L \approx \frac{\sin(2K_0)}{2K_0}$ [46]. Therefore the transfer of pairs from the islands 2 to 1 or 3 is suppressed for $K_0 = n\pi/2, n \in \mathbb{N}$. The tomography of the wavefunction shows that in this case the system is trapped in the initial state $|0, N, 0\rangle$, by a mechanism similar to CDT [47]. This is corroborated by the fidelity map of Fig. 3.

More remarkably, we find that the coherent superposition of the states $|N, 0, 0\rangle$, $|0, N, 0\rangle$, $|0, 0, N\rangle$ can be dynamically created for a wider range of system’s parameters. To illustrate the above statement, we represent the state of the system as

$$|\Psi_S(t)\rangle = |\Psi_W(t)\rangle + |\Psi_{W,3}(t)\rangle$$

$$|\Psi_W(t)\rangle = a_1(t) |0, N, 0\rangle + a_2(t) |0, 0, N\rangle + a_3(t) |0, 0, N\rangle$$

with $|\Psi_{W,3}(t)\rangle$ an orthogonal vector to $|\Psi_W(t)\rangle$, and compute the fidelity on $|\Psi_S(t)\rangle$.

$$F_S(t) = |a_1(t)|^2 + |a_2(t)|^2 + |a_3(t)|^2$$

For all the explored values of $N$ (between 2 and 10), we find that $F_S(t)$ displays nearly periodic maxima that are close to 1 (see Fig. 3 and Ref. 46). In particular, in specific intervals of $K_0$ a three-component superposition $W$-like state $\Psi_W$ with comparable amplitudes $|a_1| \approx |a_2| \approx |a_3|$ is generated with high fidelity. We observed that two-component NOON states can also be obtained, with $|a_1| \approx |a_2| \gg |a_3|$. Comparing to Ref. 43 with no offset and a non-resonant drive, here the strong offset and the resonant drive promote very efficient pair correlations and full coherent transfer of $N/2$ pairs.

Readout. The diagnostic of the system relies on the time-of-flight technique: in our dynamical scheme, the condensate

![Graph showing fidelity over time](image-url)
is prepared, driven and then released at the time $t = \bar{t}$. The long time density in real space can be accessed by the Fourier transform at the releasing time $\bar{t}$ [56, 57].

$$\hat{n}(k) = \sum_{i,j} e^{-ik(R_i-R_j)}a_i^\dagger a_j, \quad n(k, \bar{t}) = \langle \Psi(\bar{t}) | \hat{n}(k) | \Psi(\bar{t}) \rangle$$

Despite the seemingly featureless $n(k)$ (Fig. 4a), pair and many-boson pair transfers can be probed through the correlations in $\hat{n}(k)$:

$$\sigma(k, k', \bar{t}) = \langle \Psi(\bar{t}) | \hat{n}(k) \hat{n}(k') | \Psi(\bar{t}) \rangle - n(k, \bar{t}) n(k', \bar{t})$$

A perfect $|0N0\rangle$ state (Fig. 4c) is characterized by a pattern in $\sigma(k, k')$ with $k = -k'$, due to coherent virtual single-atom transitions towards one of the unoccupied sites [46]. Rotated but similar patterns are obtained for states $|N, 0, 0\rangle$, $|0, 0, N\rangle$, with $N = 8$ and $K_0 = 2.59$, $\mu = 17$ (in $J$ units).

Conclusions— We have considered the minimal system to study the correlated dynamics in bosonic networks: three coupled mesoscopic dots of interacting particles. The correlated transfer results from non-trivial interference effects arising from the combination between suitable offset potentials and driving of local potential. The dynamics of such a strongly driven nonlinear system is dictated by the specific energy level configuration that enhances or suppresses the particle transfer between the dots: nearly time-periodic recurrences between localized and highly entangled states of NOON or W type is clearly observed. The diagnostic of the system states is carried out through the analysis of the momentum distribution that, in a cold atoms setting, corresponds to time of flight images.

Our study provides a new route to engineer complex correlations in quantum networks. The physical system we studied is within the current experimental capabilities in the atomtronic field. It would be interesting to study systems of more coupled dots to engineer multipartite entangled states whose the class of generalized W states provides an example.

Acknowledgments— E. C. thanks S. Bose and L. Banchi for interesting discussions and suggestions. A. M. and D. F. thank H. Perrin for useful insights on the experimental feasibility of the setup. We acknowledge the ANR SuperRing project ANR-15-CE30-0012-02 and the Grenoble LANEF framework ANR-10-LABX-51-01 are acknowledged for their support with mutualized infrastructure.

*Corresponding author.
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Supplemental Material

UNDRIVEN CASE : DERIVATION OF THE EFFECTIVE MODEL FOR CORRELATED PAIR TRANSPORT

To better understand how a correlated pair hopping can be triggered, we consider the transfer of a single pair from the state $|\psi_i\rangle = |n_1, n_2, n_3\rangle$ to the state $|\psi_f\rangle = |n_1 + 1, n_2 - 2, n_3 + 1\rangle$. For a strong offset and no driving ($\mu \gg J, U_0, U_1 = 0$), Eq. 1 of the article can be written as $H = H_0 + V$, where

$$H_0 = \frac{U_0}{2} \sum_{j=1}^{3} n_j(n_j - 1) - \sum_{j=1}^{3} \mu_j n_j ,$$

$$V = -J \sum_{j=1}^{3} (a_j a_j^{\dagger} + H.c.) .$$

Following a second order effective hopping process, the intermediate virtual states are respectively $|\psi_{i1}\rangle = |n_1 + 1, n_2 - 1, n_3\rangle$ and $|\psi_{i2}\rangle = |n_1, n_2 - 1, n_3 + 1\rangle$. By mean of a quasi-degenerate perturbation theory approach, the amplitude probability of the process is given by the sum over the contributions of all intermediate paths. Using Eq. (12) we find that the effective hopping in Eq. 4 of the main paper is given by

$$J_{\text{eff}}^{\mu} = \frac{J^2}{2} \sum_{l,j} \left( \frac{1}{\Delta E_{il}} + \frac{1}{\Delta E_{lj}} \right) = -\frac{J^2 U_0}{\mu^2} F(U_0, \mu, \mu_2)$$

$$= \left( \frac{J^2 U_0 (\mu^2 + \mu_2^2 + 2U_0^2) + 3\mu_2 U_0}{(-\mu + \mu_2 + U_0)(-\mu + \mu_2 + 2U_0)(\mu + \mu_2 + U_0)(\mu + \mu_2 + 2U_0)} \right) ,$$

where we define $\Delta E_{ij} = E_{ij} - E_0 = \langle \psi_i | H_0 | \psi_j \rangle - \langle \psi_i | H_0 | \psi_0 \rangle$ and $|j\rangle = |n_1 + 1, n_2 - 2, n_3 + 1\rangle$, $|l\rangle \in \{ |n_1 + 1, n_2 - 1, n_3\rangle, |n_1, n_2 - 1, n_3 + 1\rangle \}$. From Eq. (13) for $\mu_2 = 0$ and small interaction, we find that $J_{\text{eff}}^{\mu} \approx -J^2 U_0/\mu^2$. Thus, the second order hopping probability is zero for non-interacting particles $U_0 = 0$, due to a destructive interference between the virtual paths in Eq. (13) (Fig. 5).

One can calculate in the same way the local energy terms up to the second order

$$\langle \psi_i | H_{\text{eff}} | \psi_i \rangle \equiv E_{\text{eff},i}^{\mu}$$

$$\langle \psi_f | H_{\text{eff}} | \psi_f \rangle \equiv E_{\text{eff},f}^{\mu}$$

Next order terms are also relevant, defining

$$|l\rangle , |l'\rangle \in \{ |n_1 + 1, n_2 - 1, n_3\rangle, |n_1, n_2 - 1, n_3 + 1\rangle, |n_1 + 2, n_2 - 2, n_3\rangle, |n_1, n_2 - 2, n_3 + 2\rangle \} ,$$

we evaluate the third order hopping term as, for $\mu_2 = 0$ and small $U_0$,

$$J_{\text{eff}}^{\mu} = -\frac{J^3}{2} \sum_{l,l'} \left( \frac{1}{\Delta E_{ll} \Delta E_{ll'}} + \frac{1}{\Delta E_{ll'} \Delta E_{ll'}} \right)$$

$$\approx -\frac{J^3}{\mu^3} (n_1 - n_3) U_0 + \mu_2 .$$

Optimization of the resonance between two states differing by one nonlocal atom pair happens when two levels of the spectrum encounter an anticrossing (Fig. 6). In the vicinity of the anticrossing, these states are little different from states $|0, N, 0\rangle$ and $|1, N - 2, 1\rangle$. This generates a Rabi oscillation with nearly perfect fidelity on extremely long time scales.
FIG. 5. High order hoppings processes between three coupled bosonic islands. Fock states $|n_1, n_2, n_3\rangle$ (black dots) are grouped accordingly to the energy $\mu(n_1 - n_3)$ (horizontal lines). The diagram shows how Fock states can be connected via second order (red) and third order paths (blue and orange).

FIG. 6. (Top) Anticrossing in the center of the energy spectrum, as a function of the offset parameter $\mu_2$; (Bottom) Rabi-like oscillations of the fidelity at the anticrossing, with $\mu = 20, U = 0.9, \mu_2 = 4.003$. Here $N = 6$

FLOQUET EXPANSION AND THE FULL EFFECTIVE MODEL.

We consider a near-resonant three-mode model ($\omega = 2\mu$) with a strong offset between site 1 and 3.

\[
H = -J \sum_{i=1}^{3} (a_i a_{i+1}^\dagger + H.c.) + \\
+ \frac{U_1}{2} \sin(\omega\tau) \sum_{i=1}^{3} \hat{n}_i(\hat{n}_i - 1) - \sum_{i=1}^{3} \mu_i \hat{n}_i ,
\]

We apply the analysis carried out by Goldman et al.[1] (see also Ref. 2) to obtain an effective model in the strongly driven regime. Since $U_1 = K_0 \omega$ where $K_0 \sim 1$ and $\mu = 2\omega$, two terms in the limits of $\omega \to \infty$ diverges. They can be eliminated away
by a unitary transformation in the system. We first rewrite the Hamiltonian in a reference frame rotating with the driving term, as

$$\tilde{H} = \mathcal{R}^\dagger(t)H(t)\mathcal{R}(t) - i\mathcal{R}^\dagger(t)\partial_t\mathcal{R}(t)$$ (18)

with

$$\mathcal{R}(t) = -\exp\left(i\int_0^t \mathcal{O}(\tau)\,d\tau\right)$$ (19)
$$\mathcal{O}(t) = \mu(\mathring{n}_1 - \mathring{n}_3) + \frac{U_1}{2}\sin(\omega t)\sum_j \mathring{n}_j(\mathring{n}_j - 1)$$ (20)

The Hamiltonian takes the form:

$$\tilde{H}(t) = -J\left(a_2^\dagger\exp\left[iK_0(\mathring{n}_2 - \mathring{n}_1)\sin\omega t - i\frac{\omega}{2}t\right]a_1 + a_2^\dagger\exp\left[iK_0(\mathring{n}_2 - \mathring{n}_3)\sin\omega t + i\frac{\omega}{2}t\right]a_3\right)$$
$$+ a_3^\dagger\exp\left[iK_0(\mathring{n}_1 - \mathring{n}_3)\sin\omega t + i\omega t\right]a_1 + H.c.$$.

$$= \tilde{H}^{(21)}(t) + \tilde{H}^{(23)}(t) + \tilde{H}^{(13)}(t)$$ (21)

Following the high-frequency development, the effective Hamiltonian at order zero is given by:

$$H^{(0)}_{eff} = \frac{1}{T}\int_0^T \tilde{H}(t)\,dt$$ (22)

The term $\tilde{H}^{(13)}(t)$ obviously has a nonzero time average on a period $T = \frac{2\pi}{\omega}$, yielding the following term corresponding to transitions between sites 1 and 3:

$$H^{(13)}_{eff} = -J a_1^\dagger J_1[K_0(\mathring{n}_2 - \mathring{n}_3)]a_3 + H.c.$$ (23)

On the other hand, due to the half-frequency factor in the transitions from site 2 to sites 1 and 3, the corresponding terms have a zero average on the period $2T$, which is the actual period of the transformed Hamiltonian $\tilde{H}$. Then one needs to perform a Floquet expansion to first order in $\frac{J}{\omega}$, yielding terms of order $\frac{J^2}{\omega^2}$:

$$H^{(21)}_{eff} + H^{(23)}_{eff} = \sum_{p>0} \frac{1}{p\mu} \left[\tilde{H}_{12}^p + \tilde{H}_{23}^p + \tilde{H}_{12}^p + \tilde{H}_{23}^p\right]$$ (24)

where $\tilde{H}_{ij}^p$ denotes the $p$-th harmonic of $\tilde{H}_{ij}$.

The calculation of the harmonics and of the commutators is straightforward and yields:

$$H_{eff} = -J\left[J_1[K_0(\mathring{n}_1 - \mathring{n}_3 + 1)]a_1^\dagger a_3 + H.c.\right]$$
$$+ \frac{J^2}{\mu} \left[\mathcal{L}(K_0, \mathring{n}_1, 2)\left(a_1^\dagger a_2^\dagger a_2 a_1\right)^2 + \mathcal{L}(K_0, \mathring{n}_2, 3)\left(a_3^\dagger a_3\right)^2\right.$$
$$+ \mathcal{M}(K_0, \mathring{n}_1, 2, 3)\left(a_1^\dagger a_1^\dagger a_3^\dagger a_3\right)^2 + \mathcal{N}(K_0, \mathring{n}_1, 2, 3) a_1^\dagger a_3^\dagger a_3 + H.c.]$$
$$+ \frac{J^2}{\mu} \left[\mathring{n}_2(\mathring{n}_2 + 1)\mathcal{P}(K_0, \mathring{n}_2, 1) - \mathring{n}_1(\mathring{n}_2 + 1)\mathcal{P}(K_0, \mathring{n}_1, 1) + \mathring{n}_2(\mathring{n}_3 + 1)\mathcal{P}(K_0, \mathring{n}_2, 3) - \mathring{n}_3(\mathring{n}_2 + 1)\mathcal{P}(K_0, \mathring{n}_3, 2)\right]$$

with the definitions:
\[ L(K_0, \hbar, \hbar') = \sum_{m \geq 0} \frac{1}{(2m + 1)} \left( J_{m+1}[K_0(\hbar - \hbar' - 1)]J_{-m}[K_0(\hbar - \hbar' - 3)] - J_{-m}[K_0(\hbar - \hbar' - 1)]J_{m+1}[K_0(\hbar - \hbar' - 3)] \right) \]

\[ M(K_0, \hbar_i=1,2,3) = \sum_{m \geq 0} \frac{1}{(2m + 1)} \left( J_{m}[K_0(\hbar_2 - \hbar_i - 1)]J_{-m}[K_0(\hbar_2 - \hbar_i - 3)] - J_{-m}[K_0(\hbar_2 - \hbar_i - 1)]J_{m}[K_0(\hbar_2 - \hbar_i - 3)] \right) \]

\[ N(K_0, \hbar_i=1,2,3) = \sum_{m \geq 0} \frac{1}{(2m + 1)} \left( \left[ \left( J_{m+1}[K_0(\hbar_1 - \hbar_2 - 1)]J_{-m}[K_0(\hbar_1 - \hbar_2 - 3)] - J_{-m}[K_0(\hbar_1 - \hbar_2)]J_{m+1}[K_0(\hbar_1 - \hbar_2 - 1)] \right)(\hbar_2 + 1) \right] \]

\[ \mathcal{P}(K_0, \hbar, \hbar') = \sum_{m \geq 0} \frac{1}{(2m + 1)} \left( J_{m}[K_0(\hbar - \hbar' - 1)]J_{m+1}[K_0(\hbar - \hbar' - 1)] \right)^2 \]

This effective Hamiltonian can be further simplified in the limit of large interaction \( K_0 \gg 1 \), where the asymptotic expansion holds:

\[ J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{n\pi}{2} - \frac{\pi}{4} \right) \]  \hspace{1cm} (27)

yielding after a few steps:

\[ H_{\text{eff}}^{\text{Asympt}} = -J \sqrt{\frac{2}{\pi K_0(\hbar_1 - \hbar_3) - 1}} \sin \left( K_0(\hbar_1 - \hbar_3 - 1) - \frac{\pi}{4} \right) a_i^+ a_3 \]

\[ + \frac{J^2}{\mu} \frac{1}{sinc(2K_0)} \left[ \frac{1}{\sqrt{(\hbar_1 - \hbar_2 - 1)(\hbar_1 - \hbar_2 - 3)}} \left( a_i^+ a_2 \right)^2 + \frac{1}{\sqrt{(\hbar_2 - \hbar_3 - 1)(\hbar_2 - \hbar_3 - 3)}} \left( a_i^+ a_2 \right)^2 \right] \]

\[ + \frac{J^2}{2K_0\mu} \left[ \frac{\cos(K_0(\hbar_1 - \hbar_3 - 1))}{\sqrt{(\hbar_1 - \hbar_2 - 1)(\hbar_3 - \hbar_2 - 2)}} - \frac{\cos(K_0(\hbar_1 - \hbar_3 + 1))}{\sqrt{(\hbar_1 - \hbar_2 - 2)(\hbar_3 - \hbar_2 - 1)}} \right] \left( a_i^+ a_3 \right)^2 \]

\[ + \frac{J^2}{2\mu} \left( sinc(2K_0(\hbar_2 - \hbar_1 - 1)) \hat{n}_1 (\hat{n}_2 + 1) - sinc(2K_0(\hbar_2 - \hbar_1 + 1)) \hat{n}_2 (\hat{n}_1 + 1) \right) \]

\[ + sinc(2K_0(\hbar_2 - \hbar_1)) \hat{n}_2 (\hat{n}_3 + 1) - sinc(2K_0(\hbar_3 - \hbar_1 + 1)) \hat{n}_3 (\hat{n}_2 + 1) \right] + H.c. \]  \hspace{1cm} (31)

where \( \text{sinc}(x) = \frac{\sin(x)}{x} \). Inspection of the successive terms helps to understand several features of the exact numerical solution. First, the term (30) vanishes when the wells 1, 3 have the same occupation. Second, terms (29) vanish when \( K_0 = \frac{n\pi}{2} \), therefore Coherent Destruction of Tunneling (CDT) is obtained starting from state \( |0, N, 0\rangle \) around those \( K_0 \) values. Despite the approximation contained in the asymptotic expansion (28-31), this explains well the map found in the main article (Figure 3b).

**ADDITIONAL DATA IN THE DRIVEN CASE**

We present here some wavefunction trajectories for higher (and even) atom numbers. For \( N = 6, 8, 10 \) one finds the same trends as those presented in the main article: tuning the interaction \( K_0 \) allows to periodically achieve a high-fidelity superposition of states \( |0N\rangle, |N00\rangle, |00N\rangle \), in particular W-type states (Figure 7). Notice the nearly periodical appearance of such states, with a very long period (several \( 100J^{-1} \)) related to anticrossings in the Floquet pseudo-energy spectrum.

The larger \( N \), the finer the tuning of \( K_0 \) necessary to obtain high fidelities. This can be ascribed to the increasing size of the Hilbert space and to the search for anticrossings in the Floquet pseudo-energy spectrum.

**READOUT MAPS: DETAILS AND EXPLOITATION.**

The two states \( |\Psi_{w}\rangle \) and \( |\Psi_{pair}\rangle \) that we have shown to be generated in our system are superpositions of states that are connected by at least one pair transfer. Therefore, the usual Time-Of-Flight observable \( n(k) = \langle \hat{n}(k) \rangle \) is flat for both states and
one has to analyze higher-order correlations to probe the presence of these states. We are interested in the following quantity:

$$\sigma_{\Psi}(k, k') = \langle \hat{h}(k)\hat{h}(k') \rangle - \langle \hat{h}(k) \rangle \langle \hat{h}(k') \rangle$$

with

$$\hat{h}(k) = \frac{|w(k)|^2}{N} \sum_{i,j} e^{i(\vec{r}_i - \vec{r}_j) \cdot \vec{k}} b_i^\dagger b_j, \quad w(k) = e^{-\frac{x^2}{4}(32)}$$

$x$ defining a phenomenological broadening. The vectors $\vec{r}_i$ defines the position of the sites representing the three wells. These sites are represented as the three summits of an equilateral triangle of length $a = 1$ (Fig. 8). We choose the origin of the coordinates as the center of this triangle.

$$\sigma_{\Psi_{\text{pair}}}(k, +k) = |w(k)|^4 \left[ \frac{2}{N} + \frac{4 \sqrt{N(N-1)}}{N^2} |\alpha| |\beta| \cos \left( \sqrt{3}k_y + \phi_\alpha - \phi_\beta \right) \right]$$

Static case

Let us now consider $|\Psi_{\text{pair}}\rangle = \alpha|0, N, 0\rangle + \beta|1, N - 2, 1\rangle$. The two states are connected by operators like $b_1^\dagger b_2^\dagger b_3^\dagger b_4$ or $b_2^\dagger b_1^\dagger b_3^\dagger b_4$ so that in the case $k = +k$, the phase acquired is $e^{i(\vec{r}_3 - \vec{r}_4) \cdot \vec{k}} = e^{i \sqrt{3}k_y}$. Finally:

$$\sigma_{\Psi_{\text{pair}}}(k, +k) = |w(k)|^4 \left[ \frac{2}{N} + \frac{4 \sqrt{N(N-1)}}{N^2} |\alpha| |\beta| \cos \left( \sqrt{3}k_y + \phi_\alpha - \phi_\beta \right) \right]$$

FIG. 7. Trajectories and readout maps for (from top to bottom) $N = 4, K_0 = 2.4, \mu = 15; N = 6, K_0 = 2.7, \mu = 19; \text{and} N = 10, K_0 = 1.16, \mu = 11.7$, starting at $t = 0$ from state $|00\rangle$; (left panels) Fidelities for states $|00\rangle, |N00\rangle, |00N\rangle$ and their sum $F_S$; (right panels) two-atom correlation $\sigma(k, -k)$ for a W-type state realized at times indicated by a vertical dotted line on the left panels.

FIG. 8. Geometrical representation of the three-well BEC system.
where \( \phi_\alpha, \phi_\beta \) are the phases of coefficients \( \alpha, \beta. \)

We can see that the resulting pattern is made of stripes modulated along \( k_x \) with a periodicity \( \frac{2\pi}{\sqrt{3}} \) (see Fig. 4b of the paper). Moreover, as \(|0, N, 0\rangle \) and \(|1, N-2, 1\rangle \) are connected by only one pair transfer, \( \sigma_{\Psi_{\text{pair}}}(k, +k) \) also depends on the phase difference between the coefficient \( \alpha \) and \( \beta \). Therefore one can directly probe the coherence of this state by measuring \( \sigma_{\Psi_{\text{pair}}}(k, +k) \).

### Ideal cases

In the three-mode Bose-Hubbard with a strong offset between the sites 1 and 3, we have shown that the modulation of the inter-particle interaction can dynamically trigger W-type superposition states of the form:

\[
|\Psi_W\rangle = \alpha|N00\rangle + \beta|0N0\rangle + \gamma|00N\rangle
\]  

One easily checks that the usual Time-Of-Flight observable \( n(k) = \langle \hat{n}(k) \rangle \) is flat. The same holds for different components of \( |\Psi_W\rangle \) individually and for \( |\Psi_W\rangle \) itself. For a `perfect' state \( |\Psi_W\rangle \), there is only one contribution to \( \sigma_{\Psi_{\text{pair}}}(k, k') \) which leads to a \( k \)-dependence. This contribution is due to virtual processes where a boson hops from a site \( i \) to a site \( j \) and comes back so that the phase acquired is \( (\vec{k} - \vec{k'}) \cdot x_{ij} \). Looking at the case \( k^2 = -\vec{k} \) we find:

![FIG. 9. Readout maps for ideal cases: fully localized (top panels), ideal W- and W-type state, and NOON state (bottom panels).](image)

\[
\Re(\sigma_{\Psi}(k, -k)) = \frac{|w(k)|^4}{N} \left[ (|\alpha|^2 + |\beta|^2) \cos(k_x) + (|\alpha|^2 + |\gamma|^2) \cos(2k_x) + (|\beta|^2 + |\gamma|^2) \cos(k_x - \sqrt{3}k_y) \right]
\]  

As the states \(|000\rangle, |0N0\rangle\) and \(|00N\rangle\) are not connected by second order hopping, we can write:

\[
\sigma_{\Psi_{\text{pair}}}(k, k') = \sigma_{|000\rangle}(k, k') + \sigma_{|0N0\rangle}(k, k') + \sigma_{|00N\rangle}(k, k')
\]  

with:

\[
\Re[\sigma_{|000\rangle}(k, -k)] = \frac{|w(k)|^4}{N} |\beta|^2 \left( \cos(k_x + \sqrt{3}k_y) + \cos(k_x - \sqrt{3}k_y) \right)
\]  

\[
\Re[\sigma_{|0N0\rangle}(k, -k)] = \frac{|w(k)|^4}{N} |\alpha|^2 \left( \cos(k_x + \sqrt{3}k_y) + \cos(2k_x) \right)
\]  

\[
\Re[\sigma_{|00N\rangle}(k, -k)] = \frac{|w(k)|^4}{N} |\gamma|^2 \left( \cos(k_x - \sqrt{3}k_y) + \cos(2k_x) \right)
\]
Therefore the probabilities in state $W$ can be recovered using the following relations:

$$|α|^2 = -\frac{N}{2|w(\frac{\pi}{2}, -\frac{\sqrt{2}}{2})|^4} \text{Re} \left[ \sigma(k, -k) \right]_{k = \frac{\pi}{2}, -\frac{\sqrt{2}}{2}}$$

$$|β|^2 = -\frac{N}{2|w(\pi, 0)|^4} \text{Re} \left[ \sigma(k, -k) \right]_{k = (\pi, 0)}$$

$$|γ|^2 = -\frac{N}{2|w(\frac{\pi}{2}, +\frac{\sqrt{2}}{2})|^4} \text{Re} \left[ \sigma(k, -k) \right]_{k = \frac{\pi}{2}, +\frac{\sqrt{2}}{2}}$$

$$1 = -\frac{N}{2} \text{Re} \left[ \sigma(k, -k) \right]_{k = (0, 0)}$$

As we can see in Fig. 9, for a symmetric setup the three Fock states that compose the superposition state give the same pattern but with a different orientation. The perfect W state with $α = β = γ$ yields a pattern with an hexagonal symmetry and an asymmetric amplitude with respect to zero. A NOON state (for instance $α = β, γ = 0$) and an asymmetric superposition give a pattern similar to that of the perfect W state but distorted in different directions, depending on the coefficients $α, β$ and $γ$. These characteristics allow to probe the presence of a superposition of the three states $|N, 0, 0\rangle$, $|0, N, 0\rangle$ and $|0, 0, N\rangle$.

Other generated states

Here we show some results for $N=8$ and for superposition states that are generated dynamically: Starting from the state $|0\rangle$, we let it evolve and stop $T = 198$ t.I, when the system is in a good superposition state. At this time, the fidelity for such a state is $0.9$. This means that the wavefunction has also sizeable components over a few other states. As $σ_{Ψ_w}(k, k)$ is flat for a perfect superposition state, this observable can indicate the contributions of the other parasitic states. To see this effect, one can delete all the terms $|p\rangle$ in the wavefunction for which the coefficient $|C_p|^2 < \kappa$ and plot the quantities $σ_{Ψ_w}(k, -k)$ and $σ_{Ψ_w}(k, k)$ for different values of $κ$ (Fig. 10). For $κ = 0.0$, all the states are present, for $κ = 0.1$ there is a small fraction of the parasitic states and for $κ = 0.2$ there are only the three states $|N00\rangle$, $|0N0\rangle$ and $|00N\rangle$.

The parasitic states can also alter the values of $σ_{Ψ_w}(k, -k)$ so that it is more difficult to recover the composition of $|Ψ_w\rangle$ with the help of the analytical formula for the perfect state.

In the case $κ = 0.1$, there are only two parasitic states: $|N - 2, 0, 2\rangle$ and $|2, 0, N - 2\rangle$. The only contributions to $σ_{Ψ_w}(k, k)$ are the terms which transfers 2 particles between 1 and 3, in the same direction. So that the result is an oscillation in the direction $k_x$ (the period is $π$, as we can see in Fig. 10). The two quantities $σ_{Ψ_w}(k, k)$ and $σ_{Ψ_w}(k, -k)$ are useful to probe the superposition and to see how far we are from this superposition.

Let us consider that $|Ψ\rangle = |Ψ_w\rangle + ε(|N - 2, 0, 2\rangle + |2, 0, N - 2\rangle)$ like in the case $κ = 0.1$. At order $ε$, the only contribution is obtained when the operator $b^*_x b_x^* b_{x_1} b_{x_1}$ connects $|N - 2, 0, 2\rangle + |2, 0, N - 2\rangle$ with $|Ψ_w\rangle$.

There are only two terms so that the result is: $σ_{Ψ}(k, k') = σ_{Ψ_w}(k, k') + 2ε \frac{\sqrt{2N(N-1)}}{N^2} (e^{i(\kappa + k)\cdot x_1} + e^{i(k+k')\cdot x_1})$. So in the case $κ = k^2$, we can characterize the parasitic states.

$$σ_{Ψ}(k, +k) = σ_{Ψ_w}(k, +k) + 4|ω(k')|^4 \frac{\sqrt{2N(N-1)}}{N^2} \cos (2k_x) + O(ε^2)$$

[1] N. Goldman and J. Dalibard, Phys. Rev. X, 4, 031027 (2014).
[2] A. Eckardt and E Anisimovas, New J. Phys. 17, 093039 (2015)
FIG. 10. Readout maps for a dynamically generated state, as a function of the filtering parameter $\kappa$ (see text). (Left) $\sigma(k, -k)$ is slightly modified by the parasitic states; (Right) $\sigma(k, k)$ is a direct fingerprint of the parasitic states.