A BIGROUPOID’S TOPOLOGY

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Abstract. The fundamental bigroupoid of a space captures its homotopy 2-type. When the space is semilocally 2-connected, we can lift the construction to a bigroupoid internal to the category of spaces, such that the invariants of the topological bigroupoid corresponding to the path components and first two homotopy groups are discrete. In fact more is true, in that the topologised fundamental bigroupoid is locally trivial in a way analogous to the case of topological groupoids.

1. Introduction

One of the standard examples of a non-trivial topological groupoid is the fundamental groupoid $\Pi_1(X)$ of a space $X$ which is locally well-behaved. In particular, the existence of this topology (which has to be compatible with composition among other things) is equivalent to the existence of a universal covering space.

Now there are higher analogues of the fundamental groupoid of a space, and indeed the celebrated Homotopy Hypothesis is that spaces and higher groupoids amount to the same thing. The easiest higher groupoid associated to a space is the fundamental bigroupoid $\Pi_2(X)$, which captures not only the path-components, fundamental group and the second homotopy group, but also the first Postnikov invariant (hence the action of $\pi_1(X)$ on $\pi_2(X)$).

It is natural then to consider putting a topology on higher homotopy groupoids in a way analogous to the case of $\Pi_1(X)$. Clearly some assumptions about the local properties of the space are necessary, and indeed we find that a 2-dimensional analogue of semilocal simple connectedness is sufficient. This condition is also necessary if one asks that $\pi_1(\Pi_2(X))$, $i = 0, 1, 2$ are discrete.

Extending this result further up the ladder of groupoids needs to take a different approach, because weak 3-groupoids are quite complicated and after that the explicit, algebraic definitions are no longer useful. One could consider however other models for higher groupoids, such as operadic definitions of weak $n$-groupoids. The analogue of the results in this paper would be that under suitable local connectivity assumptions, the algebras for the operads involved in the definitions would be topological, i.e. in the category of spaces rather than in the category of sets.

The paper essentially falls into two parts: a review of the topology of mapping spaces, with a particular focus on constructing bases for the topology which are sensitive to local connectivity properties. This allows us to translate topological properties from a space to its path and loop spaces. The rough statement is that local connectivity goes down by 1. The second half applies the calculations in the first half to describe the topology on $\Pi_2(X)$, and show it is a topological bigroupoid. An appendix reviews the definition of a bigroupoid.
2. Mapping space topology

First, we recast some facts about the compact-open topology on the path space $X^I$ into a slightly different form. Recall the definition of an open neighbourhood basis for a topology.

**Definition 1.** Let $S$ be a set and for each $s \in S$ let $\{N_s(\lambda)\}_{\lambda \in \Lambda_s}$ be a collection of subsets of $S$. The collection $\{\{N_s(\lambda)\}_{\lambda \in \Lambda_s} | s \in S\}$ is said to be a basis of open neighbourhoods, or open neighbourhood basis, for a topology on $S$ if

1. For all $\lambda \in \Lambda_s$, $s \in N_s(\lambda)$
2. For all pairs $\lambda, \nu \in \Lambda_s$, there is a $\nu \in \Lambda_s$ such that $N_s(\nu) \subset N_s(\lambda) \cap N_s(\mu)$
3. For all $\lambda \in \Lambda_s$ and all $s' \in N_s(\lambda)$, $N_s(\lambda) = N_{s'}(\mu)$ for some $\mu \in \Lambda_s$.

The sets $N_s(\lambda)$ are called basic open neighbourhoods.

There is then a topology $T$ on $S$ where the open sets are defined to be those sets that contain a basic open neighbourhood of each of their points. In this case, we can talk about an open neighbourhood basis for the topological space $(S, T)$.

**Example 2.** Consider the space $\mathbb{R}^n$ (with the usual topology). The sets $(v, C)$ where $C \ni v$ is a convex open subset of $\mathbb{R}^n$, form an open neighbourhood basis.

A non-example of an open neighbourhood basis for $\mathbb{R}^n$ (perhaps to the detriment of the nomenclature) is the collection of open neighbourhoods $(v, B(v, r))$ with $B(v, r)$ an open ball of radius $r$ centred at $v$.

It is sometimes very useful to know when a subset of the basic open neighbourhoods also forms an open neighbourhood basis. First note that the sets $\{N_s(\lambda)\}_{\lambda \in \Lambda_s}$ are partially ordered by inclusions. The following lemma is an easy exercise.

**Lemma 3.** If $\{\{N_s(\lambda)\}_{\lambda \in \Lambda_s} | s \in S\}$ is an open neighbourhood basis for a topology $T$, and

$$\{N_s(\lambda)\}_{\lambda \in \Lambda_s} \subset \{N_s(\lambda)\}_{\lambda \in \Lambda_s}$$

is a cofinal subset for each $s \in S$ ($\lambda_0 \in \Lambda_s \subset \Lambda_s$) such that $\{\{N_s(\lambda_0)\}_{\lambda \in \Lambda_s} | s \in S\}$ is an open neighbourhood basis for a topology $T'$, then $T = T'$.

The open neighbourhood basis $\{\{N_s(\lambda_0)\}_{\lambda \in \Lambda_s} | s \in S\}$ is said to be finer than the open neighbourhood basis $\{\{N_s(\lambda)\}_{\lambda \in \Lambda_s} | s \in S\}.$

**Example 4.** Following on from the previous example, there is a finer basis consisting of the basic open neighbourhoods $(v, B)$ where $B \ni v$ is an open ball in $\mathbb{R}^n$. The reader is encouraged to follow the simple exercise of verifying the conditions of the definition and lemma for these two examples, as they are a simple analogue of the flow of ideas in the next few definitions and lemmata.

We also will need the definition of a topological groupoid, as a means of concisely specifying properties of certain open covers.

**Definition 5.** A topological groupoid is a groupoid such that the sets of objects and arrows are topological spaces and the source, target, unit, multiplication and inversion maps are continuous. Functors between topological groupoids are always assumed to be continuous.

We will only use two examples, both arising from a common construction. Recall first that a space gives a topological groupoid with arrow space equal to the object space and all maps (source, target etc.) the identity. Any map of spaces gives a functor between the associated topological groupoids.

**Example 6.** Let $X$ be a space and let $U = \bigsqcup_\alpha U_\alpha$ be some collection of open sets of $X$. There is an obvious map $j: U \to X$. There is a groupoid $C(U)$ called
Čech groupoid with object space $U$ and arrow space $U \times_M U$. Source and target are projection on the two factors, the unit map is the diagonal and multiplication $U \times X U \times X U \to U \times X U$ is projection on first and last factors.

Recall that a partition of the unit interval $I = [0, 1]$ is a finite, strictly increasing list of elements $\{t_1, \ldots, t_n\}$ of $I$.

Example 7. Given a partition $\{t_1, \ldots, t_n\}$, it defines a closed cover

$$[0, t_1] \bigcap \ldots \bigcap [t_n, 1] \to I.$$ 

Analogous to the Čech groupoid, we define a partition groupoid $\mathbf{p}$ with object space $[0, t_1] \bigcap \ldots \bigcap [t_n, 1]$ and arrow space the fibred product of this space with itself (over $I$).

We point out that there are canonical functors $j: \check{C}(U) \to X$ and $\mathbf{p} \to I$. A functor $\mathbf{p} \to \check{C}(U)$ consists of a sequence of $n + 1$ paths $[t_i, t_{i+1}] \to U_i$ in open sets $U_i$ appearing in $U$, such that the endpoint of the $i^{th}$ path coincides with the starting point of the $(i + 1)^{st}$ path in the intersection $U_i \cap U_{i+1} \subset X$. Lastly, we call a functor $\mathbf{p}' \to \mathbf{p}$ commuting with the maps to $I$ a refinement.

Now let $\gamma: I \to X$ be a path, $\mathbf{p}$ a partition groupoid given by $\{t_1, \ldots, t_n\}$ and $U = \bigcap_{i=0}^n U_i$, a finite collection of open sets of $X$ such that the indicated lift (a functor) exists

$$\begin{array}{ccc}
\mathbf{p} & \xrightarrow{\gamma} & \check{C}(U) \\
\downarrow \gamma & & \downarrow \gamma \\
I & \to & X
\end{array}$$

with $\widehat{\gamma}(\{t_i, t_{i+1}\}) \subset U_i$ (as usual we let $t_0 = 0$ to ensure this makes sense). If this lift exists, we say $\gamma[\mathbf{p}]$ lifts through $\check{C}(U)$, $\gamma[\mathbf{p}]$ denoting the composition $\mathbf{p} \to X$.

Lemma 8. Given a set $N_\gamma(\mathbf{p}, U) \subset X^I$ as described above, and any other path $\eta \in N_\gamma(\mathbf{p}, U)$, we have the equality

$$N_\gamma(\mathbf{p}, U) = N_\eta(\mathbf{p}, U).$$

Proof. If $\eta' \in N_\eta(\mathbf{p}, U)$, then $\eta'[\mathbf{p}]$ lifts through $\check{C}(U)$. But this is precisely the definition of elements in $N_\gamma(\mathbf{p}, U)$. By symmetry we see that these two basic open neighbourhoods are equal. \hfill \Box

In the following sequence of definitions of open neighbourhood bases we shall prove after each one that the sets do indeed form an open neighbourhood basis.

Definition 9. If $X$ is a space, the compact-open topology on the set $C(I, X)$ of paths in $X$ has as basic open neighbourhoods the sets

$$N_\gamma(\mathbf{p}, U) = \{\eta: I \to X \mid \eta[\mathbf{p}] \text{ lifts through } \check{C}(U)\}$$

where $U$ is some finite collection of open sets such that $\gamma[\mathbf{p}]$ lifts through $\check{C}(U)$. The set of paths with this topology will be denoted $X^I$.

Proof. (That these sets form an open neighbourhood basis) The conditions (1) and (3) from definition \ref{def:open neighbourhood basis} are manifest, the latter using lemma \ref{lem:open neighbourhood basis}. For the condition (2), let $N_\gamma(\mathbf{p}, U)$ and $N_\eta(\mathbf{q}, U')$ be basic open neighbourhoods. Consider, for fixed $\gamma \in C(I, X)$, the assignment

$$(\mathbf{p}, U) \mapsto N_\gamma(\mathbf{p}, U).$$
If \( p \) and \( U \) don’t satisfy the conditions in the definition of \( N_\gamma(p,U) \), then put \( \nu(p,U) = \emptyset \), the empty subset of \( C(I,X) \). This gives us a map
\[
\nu: \{ (p,U) \} \to \mathcal{P}(\mathcal{C}(I,X))
\]
to the power set of \( \mathcal{C}(I,X) \), which we claim is not injective (away from \( \emptyset \), where it is obviously not injective).

Let \( p \) be given by \{ \( t_1, \ldots, t_n \) \}, and for a refinement \( p' \to p \) let \( m_i \) be the number of regions of \( p' \) that are mapped to \([t_i,t_{i+1}] \subset p\). Then given \( U = \bigsqcup_{i=1}^n U_i \) such that \( \nu(p,U) \) is not empty, define
\[
U_m = \prod_{i=1}^n m_i U_i,
\]
whereupon the path \( \gamma[p'] \) lifts through \( \check{C}(U_{m}) \). In fact we have the equality
\[
N_\gamma(p',U_m) = N_\gamma(p,U),
\]
as a simple pasting argument shows, and hence \( \nu \) is not injective. Thus if we are given a common refinement \( pq \) and sets \( N_\gamma(p,U) \), \( N_\gamma(q,U') \) we can find \( U_m \) and \( U'_m \) such that
\[
N_\gamma(p,U) = N_\gamma(pq,U_m) \quad \text{and} \quad N_\gamma(q,U') = N_\gamma(pq,U'_m).
\]
In this case the number of open sets making up \( U_m \) and \( U'_m \) are the same, so they can be paired off as \( (U_m)_i \cap (U'_m)_i \), unlike the open sets comprising \( U \) and \( U' \).

Then, considering \( N_\gamma(p,U) \cap N_\gamma(q,U') = N_\gamma(pq,U_m) \cap N_\gamma(pq,U'_m) \), define \( V_i = (U_m)_i \cap (U'_m)_i \) for all \( i \), and \( V = \bigsqcup V_i \). There are obvious functors \( \check{C}(V) \to \check{C}(U_m) \) and \( \check{C}(V) \to \check{C}(U'_m) \).

Since \( \gamma[pq] \) lifts through both \( \check{C}(U_m) \) and \( \check{C}(U'_m) \), it can be seen to lift through \( \check{C}(V) \). We can thus consider the set \( N_\gamma(pq,V) \). Any path \( \eta \) in \( X \) such that \( \eta[pq] \) lifts through \( \check{C}(V) \) also lifts through \( \check{C}(U_m) \) and \( \check{C}(U'_m) \), so \( \eta \in N_\gamma(pq,U) \cap N_\gamma(pq,U') \).

Thus
\[
N_\gamma(pq,V) \subset N_\gamma(pq,U) \cap N_\gamma(pq,U') = N_\gamma(p,U) \cap N_\gamma(q,U')
\]
as needed. \( \square \)

**Remark 10.** Ordinarily, the compact-open topology on a mapping space is defined using a subsbasis, but \( I \) is compact, and the given basic open neighbourhoods are cofinal in those given by finite intersections of subbasic neighbourhoods, and so define the same topology.

When the finite collection \( U \) of open sets is replaced by a finite collection of basic open neighbourhoods we find that this still defines an open neighbourhood basis for the compact-open topology.

**Lemma 11.** The sets
\[
N_\gamma(p,W) = \{ \eta: I \to X \mid \eta[p] \text{ lifts through } \check{C}(W) \},
\]
where \( W \) is a finite collection of basic open neighbourhoods of \( X \) such that \( \gamma[p] \) lifts through \( \check{C}(W) \), is a basis of open neighbourhoods for \( X^I \).

**Proof.** The proof that this is indeed a basis of neighbourhoods and is a basis of neighbourhoods for \( X^I \) will proceed in tandem. Clearly basic open neighbourhoods of this sort are also basic open neighbourhoods of the sort given in definition \( [7] \). As with the treatment of the first basis for compact-open topology, conditions (1) and (3) in definition \( [6] \) are easily seen to hold, again using lemma \( [9] \). To show that condition (2) holds, we define the set \( N_\gamma(p,V) \subset N_\gamma(p,W) \cap N_\gamma(q,W') \) as
in the previous proof. This is a basic open neighbourhood for the compact-open topology as in definition [3]. Now if we show that any such basic open neighbourhood contains a basic open neighbourhood \( N_\gamma(p, W) \) as defined in the lemma, we have both shown that sets of this form comprise an open neighbourhood basis, and that they are cofinal in basic open neighbourhoods of the form \( N_\gamma(p, U) \).

Consider then a basic open neighbourhood \( N_\gamma(p, U) \) as in definition [8]. The open sets \( U_i \) in the collection \( U \) are a union of basic open neighbourhoods, \( U_i = \bigcup_{\alpha \in J_i} W_\eta^\alpha \). Pull the cover

\[
\prod_{i=0}^n \prod_{\alpha \in J_i} W_\eta^\alpha \to X
\]

back along \( \gamma \) and choose a finite subcover \( \prod_{i=0}^n \prod_{\alpha = 1}^{k_i} \gamma^\ast W_i^\alpha \). Denote by \( W = \prod_{i,\alpha} W_i^\alpha \) the corresponding collection of \( k_0 + k_1 + \ldots + k_n \) basic open neighbourhoods of \( X \). This clearly covers the image of \( \gamma \). Choose a refinement \( p' \to p \) such that \( \gamma[p'] \) lifts through \( \hat{C}(W) \).

If \( \eta \in N_\gamma(p, W) \), \( \eta[p'] \) lifts through \( \hat{C}(W) \) and hence through \( \hat{C}(U) \). To show that \( \eta \in N_\gamma(p, U) \) we just need to show that \( \eta[p'] \to \hat{C}(U) \) factors through \( p \):

Let \((t_i^-, t_i^+)\) be an arrow in \( p' \) which maps to an identity arrow in \( p \). We need to show that \( (t_i^-, t_i^+) \) is mapped to an identity arrow in \( \hat{C}(U) \), which would imply the diagonal arrow in the above diagram factors through \( p \).

Let \( \hat{C}(W)_i \to U_i \) be the pullback of the map \( (+) \) along \( \text{disc}(U_i) \to \hat{C}(U) \). If \([t_i, t_{i+1}]\) is a region of \( p \) and \( p'(i) = [t_i, t_{i+1}] \times_{p'} p' \), then \( p'(i) \to \hat{C}(U) \) lands in \( \text{disc}(U_i) \) and so descends to \([t_i, t_{i+1}]\). Repeating this argument for each \( i \) gives the required result. We then apply lemma [3] and so the sets \( N_\gamma(p, W) \) form an open neighbourhood basis for the compact-open topology. \( \square \)

We shall define special open sets \( N_\gamma(p, W) \) which are just basic open neighbourhoods \( N_\gamma(p, W) \) where \( W = \prod_{i=0}^n W_i \) such that \( W_{2i+1} \subset W_{2i} \cap W_{2i+2} \) for \( i = 0, \ldots, n - 1 \).

**Lemma 12.** For every basic open neighbourhood \( N_\gamma(p, W) \) there is an open neighbourhood \( N_\gamma(p, W^*) \subset N_\gamma(p, W) \).

**Proof.** If \( W = \prod_{i=0}^n W_i \) and \( p \) is given by \( \{t_1, \ldots, t_n\} \), define \( W^*_i = W_i \) for each \( i = 0, \ldots, n \), and choose a basic open neighbourhood \( W_{2i+1}^* \subset W_i \cap W_{i+1} = W_{2i}^* \cap W_{2i+2} \) of \( \gamma(t_i) \). Let \( W^* := \prod_{i=0}^n W_i^* \). Then for \( i = 1, \ldots, n \), choose an \( \varepsilon_i > 0 \) such that \( \gamma([t_i, t_i + \varepsilon_i]) \subset W_{2i+1}^* \) and \( t_i + \varepsilon_i < t_{i+1} \). The figure gives a schematic picture of this construction for \( n = 3 \):
Let \( p' \) be given by \( \{ t_1, t_1 + \varepsilon_1, \ldots, t_n, t_n + \varepsilon_n \} \). Then \( \gamma[p'] \) lifts through \( \check{C}(W^*) \), so we can consider the basic open neighbourhood \( N^{*}_{\gamma}(p', W^*) \). Applying the argument from the end of the proof of lemma 11 we can see that any element \( \eta \) of \( N^{*}_{\gamma}(p', W^*) \) is such that \( \eta[p'] \), which lifts through \( \check{C}(W^*) \) and hence \( \check{C}(W) \), descends to a function \( \eta[p'] \to \check{C}(W) \), and so is an element of \( N^{*}_{\gamma}(p, W) \).

Given a pair of basic open neighbourhoods \( W_i, W_{i+1} \) as per the definition of \( N^{*}_{\gamma}(p, W) \), we know that either \( W_i \cap W_{i+1} = W_i \) or \( W_i \cap W_{i+1} = W_{i+1} \). Thus each intersection \( W_i \cap W_{i+1} \) for \( i = 0, \ldots, n - 1 \) is a basic open neighbourhood.

**Proposition 13.** The open sets \( N^{*}_{\gamma}(p, W) \) form an open neighbourhood basis for the compact-open topology on \( X^I \).

**Proof.** As in the previous two proofs, the sets \( N^{*}_{\gamma}(p, W) \) easily satisfy conditions (1) and (3) of definition 1. The intersection \( N^{*}_{\gamma}(p, W) \cap N^{*}_{\gamma}(p', W') \) contains an open set of the form \( N_{\gamma}(p, U) \), and by lemma 11 it contains an open set \( N_{\gamma}(p, W'_w) \). Using lemma 12 there is a subset of \( N_{\gamma}(p, W'_w) \) of the form \( N^{*}_{\gamma}(p, W''_w) \). Thus we see that the given open sets satisfies condition (2) of definition 1 and are cofinal in the basic open neighbourhoods from lemma 11. Hence they form an open neighbourhood basis for \( X^I \).

In light of this result we can use any of these open neighbourhood bases when dealing with the compact-open topology. We can then transfer the topological properties of \( X \) described in terms of basic open neighbourhoods to the topological properties of \( X^I \), and various subspaces, described in terms of basic open neighbourhoods.

**Definition 14.** Let \( n \) be a positive integer. A space \( X \) is called **semilocally \( n \)-connected** if it has a basis of \( (n - 1) \)-connected open neighbourhoods \( N_X \) such that \( \pi_n(N_X) \to \pi_n(X) \) is the trivial map (for any choice of basepoint). We say a space is **semilocally \( 0 \)-connected** if for any basic neighbourhood \( N_X \) and any two points \( x, y \in N_X \), there is a path from \( x \) to \( y \) in \( X \).

Let \( P_{x_0,x_1}X \) be the fibre of \( (ev_0, ev_1): X^I \to X \times X \) over \( (x_0, x_1) \). Notice that the based loop space \( \Omega X \) at a point \( x \) is \( P_{x,x}X \). We shall denote by \( P_{x}X \) the space of paths based at \( x \), i.e. the fibre of \( ev_0: X^I \to X \) at \( x \). The space of free loops \( LX = X^{S^1} \) (given the compact-open topology) can be identified with the inverse image \( (ev_0, ev_1)^{-1}(X) \) of the diagonal \( X \hookrightarrow X \times X \). If there is no confusion, we will usually denote the based loop space simply by \( \Omega X \).
The following theorem is more general than we need, but is of independent interest. Although a more general theorem is stated in [Wad55], the proof is only implied from the slightly weaker case that is proved in loc. cit., namely when $X$ is locally $n$-connected. That proof is intended for an analogous result for the local properties of the mapping space $X^I$ for $P$ any finite polyhedron, and for various subspaces thereof. As a result, the proof has to deal with the fact $P$ is not one-dimensional, and so is necessarily quite complicated.

**Theorem 15.** If a space $X$ is semilocally $n$-connected, $n \geq 1$, the spaces $X^I$, $P_x X$, $P_{x,y} X$ and $\Omega_x X = P_{x,x} X$ are all semilocally $(n-1)$-connected.

**Proof.** First of all, assume that $X$ is semilocally $1$-connected, let $\gamma \in X^I$ and $N^*_\gamma (p, W)$ be a basic neighbourhood where $p$ is given by $\{t_1, \ldots, t_m\}$. Temporarily define $t_0 := 0$ and $t_{m+1} := 1$. Then given two points $\gamma_0, \gamma_1 \in N^*_\gamma (p, W)$, we know that for each $i = 0, \ldots, m+1$, $\gamma_0(t_i), \gamma_1(t_i) \in W_{i-1} \cap W_i$, which is a basic open neighbourhood of $X$. Let $\eta_i$ be a path in $W_{i-1} \cap W_i$ from $\gamma_0(t_i)$ to $\gamma_1(t_i)$ for $i = 1, \ldots, m$, and let $\eta_0$, be a path from $\gamma_0(0)$ to $\gamma_0(0)$ in $W_0$ and $\eta_{m+1}$ be a path from $\gamma_0(1)$ to $\gamma_1(1)$ in $W_0$. The sequence of paths

\begin{equation}
\gamma_0(t_i) \sim \cdots \pi \sim \gamma_1(t_i)
\end{equation}

then defines a loop in $W_i$ for $i = 1, \ldots, m - 1$. As $X$ is semilocally $1$-connected, there is a surface in $X$ of which this loop is the boundary.

These surfaces patch together to form a free homotopy in $X$ between the paths $\gamma_0$ and $\gamma_1$. By adjointness, this defines a path in $X^I$ between the points $\gamma_0$ and $\gamma_1$. Thus $X^I$ is semilocally 0-connected.

If we consider the subspace $P_x X$ (resp. $P_{x,y} X$), then we take the path $\eta_0$ (resp. the paths $\eta_0$ and $\eta_{n+1}$) to be constant. This implies that the path in $X^I$ defined in the previous paragraph lands in $P_x X$ (resp. $P_{x,y} X$), and so those subspaces are likewise semilocally 0-connected.

Now assume that $X$ is semilocally $n$-connected with $n \geq 2$ and that $N^*_\gamma (p, W)$ is a basic open neighbourhood of the point $\gamma$. Consider the $k$-sphere $S^k$ ($k \geq 0$) to be pointed by the 'north pole' $N$. Let $f: S^k \to X$ be a map in $N^*_\gamma (p, W)$ such that at $f(N) = \gamma$. By adjointness, this determines a map $\tilde{f}: S^k \times I \to X$ such
that the restriction $\tilde{f}_{s_i}^{S^k \times [t_i, t_{i+1}]}$ factors through $W_i$ for $i = 0, \ldots, m$. Note that if we further restrict this map to $\tilde{f}_{s_i}^{S^k \times \{t_i\}}$ then for $i = 1, \ldots, m$ it factors through $W_{i-1} \cap W_i$, which is a basic open neighbourhood by the assumption on $W$. We also have maps $\tilde{f}_{s_0}^{S^k \times \{0\}}$ landing in $W_0$, and $\tilde{f}_{s_0}^{S^k \times \{1\}}$ landing in $W_m$. The assumption on $X$ implies that that the basic open neighbourhoods are $(n-1)$-connected, so that for $k = 0, \ldots, m - 1$, there are maps $\eta_i : B^{k+1} \to W_{i-1} \cap W_i$, for $i = 1, \ldots, m$

$\eta_0 : B^{k+1} \to W_0$ and $\eta_{m+1} : B^{k+1} \to W_m$ filling these spheres.

Now for $k = 0, \ldots, n-1$ and $i = 0, \ldots, m - 1$ the maps $\eta_i$ define, together with the cylinders $\tilde{f}_{s_i}^{S^k \times [t_i, t_{i+1}]}$, maps $\xi_i$ from a (space homeomorphic to a) $k+1$-sphere to $W_i$. As $W_i$ is $(n-1)$-connected, for $k = 0, \ldots, n-2$ and $i = 0, \ldots, m$ there is a map $\nu_i : B^{k+1} \times [t_i, t_{i+1}] \to W_i$ filling the sphere. The $m+1$ maps $\nu_i$ paste together to form a homotopy $B^{k+1} \times I \to X$ and a map $B^{k+1} \to N_\gamma^\ast (p, W)$ filling the map from the sphere we started with. Thus the basic open neighbourhood $N_\gamma^\ast (p, W)$ is $(n-2)$-connected.

If we now take $k = n - 1$, then we can find maps $\nu_i : B^n \times [t_i, t_{i+1}] \to X$ filling the sphere. These paste together to give a homotopy $B^n \times I \to X$ and so a map $B^n \to X^I$ filling the sphere $S^{n-1} \to N_\gamma^\ast (p, W) \hookrightarrow X^I$. This implies that the map $\pi_{n-1}(N_\gamma^\ast (p, W), \gamma) \to \pi_{n-1}(X^I)$ is trivial, and so $X^I$ is semilocally $(n-1)$-connected.

If we again consider the subspaces $P_x X$ and $P_{x,y} X$, we can choose the maps $\eta_0$ and $\eta_m$ to be constant (where appropriate) and so this ensures the maps $B^{k+1} \to X^I$ constructed above factor through the relevant subspace. □

As a corollary we get a much simpler proof of another special case of the theorem from [Wad55], namely for the mapping space $X^{Sm}$, or more specifically the set of based maps.

**Corollary 16.** If $X$ is semilocally $n$-connected and $m \leq n$, the space $(X, x)^{(Sm, \eta)}$ of pointed maps is semilocally $(n-m)$-connected.

**Proof.** This is an easy induction on $m$ using theorem 15 using $X^{Sm} = \Omega^m X$, the $m$-fold based loop space. □

We can also discuss the local homotopical properties of the space $LX$, as long as we make one further refinement to the neighbourhood basis it inherits from $X^I$. Let $N_\gamma^\ast (p, W)$ denote an open neighbourhood $N_\gamma^\ast (p, W)$ of $LX$ with $W = \bigsqcup_{i=0}^{2n+1}$ where for $i = 1, \ldots, n - 1$, we have $W_{2i+1} \subset W_{2i} \cap W_{2i+2}$ and $W_{2n+1} \subset W_{2n} \cap W_0$. The proofs of the following lemma and proposition are almost identical to that of lemma 12 and proposition 13, so we omit them.

**Lemma 17.** For every basic open neighbourhood $N_\gamma^\ast (p, W)$ of $LX$, there is a basic open neighbourhood of the form $N_\gamma^\ast (p', W')$ contained in $N_\gamma^\ast (p, W)$.

**Proposition 18.** The sets $N_\gamma^\ast (p, W)$ form an open neighbourhood basis for the compact-open topology on $LX$.

We then have the following analogue of theorem 15.

**Theorem 19.** If the space $X$ be semilocally $n$-connected, the space $LX$ is semilocally $(n-1)$-connected.

**Proof.** Assume that the point $\gamma \in LX$ has a basic neighbourhood $N_\gamma^\ast (p, W)$ where $W = \bigsqcup_{i=0}^{m}$. The proof proceeds along the same lines as that of theorem 15 except we let $\eta_0 = \eta_{m+1} : B^{k+1} \to W_m \cap W_0$. This is enough to ensure that the rest of the proof goes through and that for $k = 0, \ldots, n - 2$ we have
maps $B^{k+1} \to N_0^\omega(p, W)$ expressing the $k$-connectedness of $N_0^\omega(p, W)$, and maps $B^n \to LX$ that give us the result that $\pi_{n-1}(N_0^\omega(p, W)) \to \pi_{n-1}(LX)$ is the trivial homomorphism.

\[ \square \]

**Lemma 22.** The map traversed in the opposite direction is continuous.

**Proof.** Let $\gamma_1, \gamma_2 \in X^I$ be paths in $X$, and $N := N_{\gamma_2, \gamma_1}(p, W)$ a basic open neighbourhood as given by lemma 11. We can assume that $p$ is given by

\[ \{t_1, \ldots, t_n, \frac{1}{2}, t'_1, \ldots, t'_m\}, \]

else we can refine $p$ and alter $W$ so that it does without changing $N$ (as specified in the proof following definition 6). The collection $W$ of basic open neighbourhoods then looks like

\[ W = \prod_{i=0}^n W_i^1 \prod_{j=0}^m W_j^2 =: W^1 \prod W^2. \]

Define the refinement $p' \to p$ by adding an additional two points $\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon$ to the specification of $p$, where $\epsilon$ is small enough that the image of $[\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$ under $\gamma_2 \cdot \gamma$ lands in a basic open neighbourhood $W_{n+1} \subset W_{\frac{1}{2}} \cap W_0^2$. Then defining

\[ W' = \prod_{i=0}^n W_i^1 \prod_{j=0}^m W_{n+1}^1 \prod W_{n+1}^2 \prod_{j=0}^m W_j^2 =: W^1 \prod W^2, \]

we see that $(\gamma_2 \cdot \gamma_1)[p]$ lifts through $W'$. There is then a subset

\[ N' := N_{\gamma_2, \gamma_1}(p', W') \subset N_{\gamma_2, \gamma_1}(p, W) \]

We now set $N_1 = N_{\gamma_1}(p_1, W^1 \prod W_{n+1}^1)$, $N_2 = N_{\gamma_2}(p_2, W_{n+1}^1 \prod W^2)$ where $p_1$ is given by $\{2t_1, \ldots, 2t_n, 1 - 2\epsilon\}$ and $p_2$ is given by $\{2\epsilon, 2t'_1 - 1, \ldots, 2t'_m - 1\}$. Thus $p'$ is the concatenation $p_1 \vee p_2$.

The fibred product $N_2 \times_X N_1$ consists of pairs of paths $\eta_1, \eta_2$ that lift through $W_1$ and $W_2$ resp., whose endpoints match and in particular, $\eta_1(1) = \eta_2(1) = (\eta_2 \cdot \eta_1)(\frac{1}{2}) \in W_1^1 \cap W_0^2$. Thus $\eta_2 \cdot \eta_1[p]$ lifts through $W$, and so the image of $N_2 \times_X N_1$ under concatenation is contained in $N$, so concatenation is continuous.

\[ \square \]

**Lemma 23.** The ‘reverse’ map $X^I \to X^I$ sending a path $\gamma$ to the same path traversed in the opposite direction is continuous.
3. The topological fundamental bigroupoid of a space

One can put a topology on the fundamental groupoid of a space \(X\) if it is semilocally 1-connected. In this section we shall generalise this to the fundamental bigroupoid defined in [Ste00] [HKK01]. It requires local conditions on the free loop space \(LX\), which as we saw in the previous section, can be phrased in terms of the topology of \(X\). We shall also describe the conditions algebraically using the fundamental bigroupoid.

We shall first treat the case of the fundamental groupoid, as though it is long-known we shall need it again in the second part of this section. Assume the space \(X\) is semilocally 1-connected. Since the set of objects of \(\Pi_1(X)\) is the set of paths \(C(I, X)\) in \(X\) quotiented by the equivalence relation ‘homotopic rel endpoints’. Let \(x, y\) be points in \(X\), and without loss of generality we can assume they are in the same path-component. Let \(W_x\) and \(W_y\) be basic open neighbourhoods of \(x\) and \(y\) respectively. Notice that they are path-connected by assumption, and the homomorphisms \(\pi_1(W_x, x)\rightarrow\pi_1(X, x), \pi_1(W_y, y)\rightarrow\pi_1(X, y)\) are trivial.

For \([\gamma]\) a homotopy class of paths from \(x\) to \(y\), we now describe an open neighbourhood basis for \(\Pi_1(X)\). Define the sets \(N_{[\gamma]}(W_x, W_y) = \{[\eta\cdot \gamma \cdot \eta'] \in \Pi_1(X)_1| \eta(0) = \gamma(0), \eta_y(0) = \gamma(1)\}\), where the operation \(-\cdot-\) is the usual concatenation of paths, with the first path on the right and the second on the left. Note that these are homotopy classes in \(X\), as opposed to taking homotopies of paths of the form \(\eta \cdot \gamma \cdot \eta'\).

**Proposition 24.** The sets \(N_{[\gamma]}(W_x, W_y)\) form an open neighbourhood basis for \(\Pi_1(X)_1\).

**Proof.** We have \(\gamma \in N_{[\gamma]}(W_x, W_y)\) by definition, condition (1) from definition \([\gamma]\) holds. If \([\omega] \in N_{[\gamma]}(W_x, W_y)\), then for all \([\omega'] \in N_{[\gamma]}(W_x, W_y)\) we can write
\[
[\omega'] = [\eta_x' \cdot \gamma \cdot \eta_y'] = [\eta_x' \cdot \eta_x \cdot \gamma \cdot \eta_y \cdot \eta_y'] = [\eta_x' \cdot \omega \cdot \eta_y'],
\]
where \([\omega] = [\eta_x \cdot \gamma \cdot \eta_y]\). Thus \(N_{[\omega]}(W_x, W_y) \subset N_{[\gamma]}(W_x, W_y)\). Since \([\gamma] = [\eta_x' \cdot \eta_x \cdot \gamma \cdot \eta_y \cdot \eta_y'] \in N_{[\omega]}(W_x, W_y)\), we can use symmetry to show that \(N_{[\omega]}(W_x, W_y) \subset N_{[\gamma]}(W_x, W_y)\), and condition (3) in definition \([\gamma]\) is satisfied.

To show that condition (2) is satisfied, let \(N_{[\gamma]}(W_x, W_y), N_{[\gamma']} (W_x', W_y')\) be a pair of putative basic neighbourhoods of \([\gamma]\). Let \(W_x' \subset W_x \cap W_y'\) and \(W_y' \subset W_y \cap W_y'\) be basic open neighbourhoods of \(x\) and \(y\). The set \(N_{[\gamma]}(W_x', W_y')\) is then contained in \(N_{[\gamma]}(W_x, W_y) \cap N_{[\gamma]}(W_x', W_y')\).

Although we now have topologies on the sets \(\Pi_1(X)_0\) and \(\Pi_1(X)_1\), we do not know that they form a topological groupoid – composition and other structure maps need to be checked for continuity.

**Proposition 25.** With the topology as described above, \(\Pi_1(X)\) is a topological groupoid for \(X\) a semilocally 1-connected space.

**Proof.** We need to check the continuity of four maps, namely
\[
(s, t) : \Pi_1(X)_1 \rightarrow X \times X,
\]
\[
(-) : \Pi_1(X)_1 \rightarrow \Pi_1(X)_1,
\]
\[
e : X \rightarrow \Pi_1(X)_1,
\]
\[
m : \Pi_1(X)_1 \times_X \Pi_1(X)_1 \rightarrow \Pi_1(X)_1
\]
We shall use the following criterion to check for continuity:

- A map \( f : X \to Y \) between topological spaces is continuous if and only if for every \( x \in X \) and basic open neighbourhood \( N_Y \) of \( f(x) \), there is a basic open neighbourhood \( N_X \) of \( x \) such that \( N_X \subset f^{-1}(N_Y) \) (equivalently, \( f(N_X) \subset N_Y \)).

Let \( \gamma \) be a point in \( \Pi_1(X)_1 \), and set \( (x, y) = (s[\gamma], t[\gamma]) \). The inverse image

\[
(s, t)^{-1}(W_x \times W_y)
\]

contains the basic open neighbourhood \( N[\gamma](W_x, W_y) \), so \((s, t)\) is continuous.

Given the basic open neighbourhood \( N[\gamma](W_x, W_y) \), it is simple to check that

\[
(N[\gamma](W_x, W_y)) = N[\gamma](W_y, W_x),
\]

so \((-\)) is continuous.

For \( x \in X \), consider the basic open neighbourhood \( N[\text{id}_x](W_x, W_x^x) \). The inverse image \( e^{-1}(N[\text{id}_x](W_x, W_x^x)) \) is the intersection \( W_x \cap W_x^x \). There is a basic open neighbourhood \( W_x^\alpha \subset W_x \cap W_x^x \), so \( e \) is continuous.

It now only remains to show that multiplication in \( \Pi_1(X) \) is continuous. For composable arrows \( \gamma_1 : x \to y \) and \( \gamma_2 : y \to z \), let \( N[\gamma_2, \gamma_1](W_x, W_z) \) be a basic open neighbourhood. If \( W_y \) is a basic open neighbourhood of \( y \) the set \( N[\gamma_2](W_y, W_x) \times X \supset\subset N[\gamma_1](W_x, W_y) \) is a basic open neighbourhood of \( (\gamma_2, \gamma_1) \) in \( \Pi_1(X)_1 \times_X \Pi_1(X)_1 \). Arrows in the image of this set under \( m \) look like

\[
[\lambda_1 \cdot \gamma_2 \cdot \lambda_0 \cdot \eta_1 \cdot \gamma_1 \cdot \eta_0],
\]

where \( \lambda_1 \) is a path in \( W_y \), \( \lambda_0 \) and \( \eta_1 \) are paths in \( W_y \) such that \( \lambda_0(\epsilon) = \eta_1(\epsilon + 1) \) for \( \epsilon = 0, 1 \mod 2 \) and \( \eta_0 \) is a path in \( W_y \). Now the composite \( \lambda_0 \cdot \eta_1 \) is a loop in \( W_y \) at \( y \). The arrow \( [\lambda_0 \cdot \eta_1] \) is equal to \( \text{id}_y \) in \( \Pi_1(X) \) by the assumption that \( X \) is semi-locally 1-connected. Thus

\[
[\lambda_1 \cdot \gamma_2 \cdot \lambda_0 \cdot \eta_1 \cdot \gamma_1 \cdot \eta_0] = [\lambda_1 \cdot \gamma_2 \cdot \gamma_1 \cdot \eta_0]
\]

and we have an inclusion

\[
m( N[\gamma_2](W_y, W_x) \times_X N[\gamma_1](W_x, W_y) ) \subset N[\gamma_2, \gamma_1](W_x, W_z).
\]

This implies multiplication is continuous. \( \square \)

Topological groupoids have a notion of equivalence which is weaker than the usual internal equivalence in the sense of having a pair of functors forming an equivalence. We will not go into this, but will point out that sometimes a topological groupoid can be weakly equivalent to a topological groupoid equipped with the discrete topology. We will give a definition which can be shown to be equivalent to the usual definition.

Definition 26. A \textit{weakly discrete groupoid} \( X \) is a topological groupoid such that each hom-space \( X(x, y) = (s, t)^{-1}(x, y) \) is discrete and locally trivial: for each object \( p \in X_0 \) there is an open neighbourhood \( U_p \subset p \) in \( X_0 \) and a lift

\[
\begin{tikzcd}
X_1 \ar{d}{(s, t)} \ar{dd}[swap]{\{p\} \times U_p} \\
X_0 \times X_0
\end{tikzcd}
\]
as indicated.

Proposition 27. For a semi-locally 1-connected space \( X \), the topological groupoid \( \Pi_1(X) \) is weakly discrete.
Proof. We will first show $\Pi_1(X)_1 \to X \times X$ is a covering space. The fact it’s a covering space means that homs are discrete. Only need then to show local triviality, but this follows from the fact path components of $X$ are open. Let $X = \coprod_\alpha X_\alpha$ with each $X_\alpha$ a connected (path-)component. Clearly the fibres over $X_\alpha \times X_\beta$ for $\alpha \neq \beta$ are empty, so we can just consider the restriction of $\Pi_1(X)_1$ to each $X_\alpha \times X_\alpha$, from which it follows we can assume $X$ connected. It is also immediate that the image of $(s, t)$ is open.

Let $(x, y) \in X^2$ and $W_x \times W_y$ be a basic open neighbourhood of $(x, y)$, this means that $W_x, W_y$ are path-connected and the inclusion maps induce zero maps on fundamental groups. Let $N_{[\gamma]}(W_x, W_y)$ be a basic neighbourhood. The restriction $(s, t)|_{N_{[\gamma]}(W_x, W_y)}$ maps surjectively onto $W_x \times W_y$, using the path-connectedness of $W_x$ and $W_y$. Consider now the surjective map $s|_{N_{[\gamma]}(W_x, W_y)} : N_{[\gamma]}(W_x, W_y) \to W_x$.

Assume there are two paths $\eta_1, \eta_2$ in $N_{[\gamma]}(W_x, W_y)$ with source $x' \in W_x$ and target $y' \in W_y$. We know that $[\eta_1] = [\omega_1 \cdot \gamma]$ and $[\eta_2] = [\omega_2 \cdot \gamma]$ and so $\omega_1 - \omega_1$ is a loop in $W_y$ based at $y$. By the assumption on $X$, this loop is null-homotopic in $X$, or in other words, $[\omega_1] = [\omega_2]$ in $\Pi_1(X)_1$, so $[\eta_1] = [\eta_2]$. Using a similar argument for $W_x$, we get the result that $(s, t)|_{N_{[\gamma]}(W_x, W_y)}$ is a bijection. It is easily seen that $(s, t)$ maps basic open neighbourhoods to basic open neighbourhoods, and so is an open map, hence an isomorphism. The sets $N_{[\gamma]}(W_x, W_y), N_{[\gamma']}s(W_x, W_y)$ are disjoint for $[\gamma] \neq [\gamma']$, by arguments from the proof of proposition 24. Since every arrow $x' \to y'$ in $\Pi_1(X)$ for $x' \in W_x$ and $y' \in W_y$ lies in some $N_{[\gamma]}(W_x, W_y)$, we have an isomorphism

$$\Pi_1(X) \times_{X^2} (W_x \times W_y) \cong (W_x \times W_y) \times \Pi_2(X)(x, y)$$

and so $\Pi_1(X) \to X \times X$ is a covering space.

To assist in further proofs of continuity, we give a small lemma.

**Lemma 28.** For a semilocally 1-connected space $X$, the map $[-] : X^I \to \Pi_1(X)_1$ is continuous.

Proof. Let $N_{[\gamma]}(W_x, W_y)$ be a basic open neighbourhood. The inverse image $[-]^{-1}N_{[\gamma]}(W_x, W_y)$ consists of points in the open set $U_{x,y} := (ev_0, ev_1)^{-1}(W_x \times W_y) \subset X^I$ that are connected by a path in $U_{x,y}$ to a point of the form $\eta_x \cdot (\gamma \cdot \eta_y)$. Note that every such point is connected by a path in $U_{x,y}$ to the point $\gamma$ — this can be seen by constructing a free homotopy connecting the path $\eta_x \cdot (\gamma \cdot \eta_y)$ to the path $\gamma$. Now $X^I$ is semilocally 0-connected by theorem 15, we can choose a basic open neighbourhood $N^*_\gamma(p, W)$ with $W = \coprod_{i=0}^m W_i$ such that $W_0 = W_x$ and $W_m = W_y$. Every point $\eta$ in this neighbourhood is connected by a path $\Gamma_0$ in $X^I$ to $\gamma$. Moreover, we can choose this path, as in the proof of theorem 15, to be such that $ev_0 \circ \Gamma_0(t) \in W_x$ and $ev_1 \circ \Gamma_0(t) \in W_y$ for all $t \in I$. Thus the neighbourhood $N^*_\gamma(p, W)$ is a subset of $U_{x,y}$, and $m$ is continuous.

Now if we are given a homotopy $Y \times I \to X$, that is a map $Y \to X^I$, we get a map $Y \to \Pi_1(X)_1$ by composition with $[-]$.

To describe the topological fundamental bigroupoid of a space, we first need to define a topological bigroupoid. The definition of bigroupoid is recalled in the appendix. The full diagrammatic definition of an internal bicategory appears in the original article on bicategories [Ben67]. Since we are only interested in topological bigroupoids—bigroupoids in Top, a concrete category—we can refer to elements of objects with impunity. This means that the pointwise coherence diagrams in the
appendix are still valid, and we do not need to display three-dimensional commuting
diagrams of internal natural transformations.

**Definition 29.** A topological bigroupoid $B$ is a topological groupoid $B_1$ equipped
with a functor $(S, T): B_1 \rightarrow \text{disc}(B_0 \times B_0)$ for a space $B_0$ together with

- functors

$$
\begin{align*}
C &: B_1 \times_{\text{disc}(B_0), T} B_1 \rightarrow B_1 \\
I &: \text{disc}(B_0) \rightarrow B_1
\end{align*}
$$

over $\text{disc}(B_0 \times B_0)$ and a functor

$$
(\_): B_1 \rightarrow B_1
$$

covering the swap map $\text{disc}(B_0 \times B_0) \rightarrow \text{disc}(B_0 \times B_0)$.

- continuous maps

$$(2) \begin{align*}
a &: \text{Obj}(B_1) \times_{S, B_0, T} \text{Obj}(B_1) \times_{S, B_0, T} \text{Obj}(B_1) \rightarrow \text{Mor}(B_1) \\
r &: \text{Obj}(B_1) \rightarrow \text{Mor}(B_1) \\
l &: \text{Obj}(B_1) \rightarrow \text{Mor}(B_1) \\
e &: \text{Obj}(B_1) \rightarrow \text{Mor}(B_1) \\
i &: \text{Obj}(B_1) \rightarrow \text{Mor}(B_1)
\end{align*}$$

which are the component maps of natural isomorphisms

$$
\begin{array}{c}
B_1 \times_{S, \text{disc}(B_0), T} B_1 \xrightarrow{id \times C} B_1 \times_{S, \text{disc}(B_0), T} B_1 \\
\xrightarrow{C \times \text{id}} B_1 \times_{S, \text{disc}(B_0), T} B_1 \xrightarrow{a} B_1 \\
\xrightarrow{C} B_1
\end{array}
$$

$$
\begin{array}{c}
B_1 \times_{S, \text{disc}(B_0), T} \text{disc}(B_0) \xrightarrow{id \times I} B_1 \times_{S, \text{disc}(B_0), T} B_1 \\
\xrightarrow{I \times \text{id}} \text{disc}(B_0) \times_{\text{disc}(B_0), T} B_1 \xrightarrow{l} B_1 \\
\xrightarrow{r} B_1
\end{array}
$$

$$
\begin{array}{c}
B_1 \xrightarrow{(\_), id} B_1 \times_{S, \text{disc}(B_0), T} B_1 \\
\xrightarrow{\text{id}, (\_)} B_1 \times_{S, \text{disc}(B_0), T} B_1 \xrightarrow{(\_), i} B_1 \\
\xrightarrow{S} B_1 \times_{S, \text{disc}(B_0), T} B_1 \xrightarrow{T} \text{disc}(B_0)
\end{array}
$$

These are required to satisfy the usual coherence diagrams as given in definitions
42 and 43 in the appendix.

The full definition of the fundamental bigroupoid $\Pi_2(X)$ can be found in [Ste00]
[HKK01], but it can be described in rough detail as follows: the objects are points of
the space $X$, the arrows are paths $I \rightarrow X$ (not homotopy classes) and the 2-arrows
are homotopy classes of homotopies between paths. The horizontal composition of 2-arrows is by pasting such that source and target paths are concatenated, and vertical composition is pasting of homotopies. Horizontal composition, left and right units and inverses are only coherent rather than strict. We will describe a topological bigroupoid $\Pi^T_2(X)$ lifting $\Pi_2(X)$.

Since the 1-arrows of $\Pi^T_2(X)$ are paths in $X$, we can let the topology on $\Pi^T_2(X)$ be the compact-open topology from the previous section. We also let the topology on the objects of $\Pi^T_2(X)$ be that of $X$. The object components of the functors $S,T$, which are evaluation at 0 and 1 respectively, are clearly continuous, as is the map $X \to X^I$ sending a point to a constant path. All we need to have a candidate for being a topological fundamental bigroupoid is a topology on the set of 2-arrows.

Recall that the composition of 2-tracks $[f], [g]$ along a path (vertical composition) is denoted by $[f + g]$, and the concatenation (horizontal composition) is denoted $[f \cdot g]$. The inverse of a 2-track $[f]$ for this composition is written $-[f] = [-f]$. If $[f]$ is a 2-track with representative $f : I^2 \to X$, let $\llbracket f \rrbracket : I \to X^I$ be the corresponding path.

**Lemma 30.** Let $[h] \in \Pi^T_2(X)_2$ be a 2-track, $U_0, U_1$ basic open neighbourhoods of $s_0[h], t_0[h] \in X$ respectively, and

$$V_0 = N_{s_1[h]}(p_0, \prod_{i=0}^{n_0} W_i^{(0)}) \quad \text{and} \quad V_1 = N_{t_1[h]}(p_1, \prod_{i=0}^{n_1} W_i^{(1)})$$

basic open neighbourhoods in $X^I$. Also assume that

$$U_0 \subset W_0^{(0)} \cap W_1^{(1)}, \quad U_1 \subset W_0^{(0)} \cap W_1^{(1)}.$$

Then the sets

$$\langle [h], U_0, U_1, V_0, V_1 \rangle := \{ [f] \in \Pi^T_2(X)_2 \mid \exists \beta_\epsilon : I \to U_\epsilon (\epsilon = 0, 1) \text{ and } \gamma_{\lambda_\epsilon} : I \to V_\epsilon (\epsilon = 0, 1) \text{ such that } [f] = \llbracket \lambda_0 + (\text{id}_{\beta_0} \cdot (h \cdot \text{id}_{\beta_0})) + \lambda_0 \rrbracket \},$$

form an open neighbourhood basis for $\Pi^T_2(X)_2$.

**Proof.** Algebraically the elements of the basic open neighbourhoods $\langle [h], U_0, U_1, V_0, V_1 \rangle$ look like diagrams

with some hidden bracketing on the whiskering of $h$ by $\beta_0, \beta_1$. Here is a topological viewpoint of the same element of $\langle [h], U_0, U_1, V_0, V_1 \rangle$:
Or more schematically,

\[ \begin{array}{c}
\lambda_0 \\
\hline
U_0 & h & U_1 \\
\lambda_1
\end{array} \]

It is immediate from the definition that \([h] \in \langle [h], U_0, U_1, V_0, V_1 \rangle\). To see that for \([f] \in \langle [h], U_0, U_1, V_0, V_1 \rangle\), we have \([h] \in \langle [f], U_0, U_1, V_0, V_1 \rangle\), we can use the fact \(\Pi^T_2(X)\) is a bigroupoid, and apply the compose/concatenate with the (weak) inverse of everything in sight. We do not display the all structure morphisms (associator etc.), relying on coherence for bicategories. If we have

\[ g_0 \quad \| \quad [f] \quad \| \quad g_1 \]

\[ g_0 \quad \| \quad [h] \quad \| \quad g_1 \]

\[ \beta_0 \quad \| \quad [h] \quad \| \quad \beta_1 \]

\[ t_1 [h] \quad \| \quad \lambda_1 \]

\[ s_1 [h] \quad \| \quad \lambda_0 \]
then

\[
\begin{array}{c}
\bullet \rightarrow \bullet \rightarrow \bullet \\
\beta_0 \hspace{1cm} \beta_0 \hspace{1cm} \beta_0 \\
\downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
\uparrow \downarrow \downarrow \downarrow \\
t_1[h] \hspace{1cm} t_1[h] \hspace{1cm} t_1[h]
\end{array}
\]

and so

\[
\begin{array}{c}
\bullet \rightarrow \bullet \rightarrow \bullet \\
\beta_0 \hspace{1cm} \beta_0 \hspace{1cm} \beta_0 \\
\downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
\uparrow \downarrow \downarrow \downarrow \\
t_1[h] \hspace{1cm} t_1[h] \hspace{1cm} t_1[h]
\end{array}
\]

We thus only need to show that the intersection

\[(3) \langle h, U_0, U_1, V_0, V_1 \rangle \cap \langle h, U'_0, U'_1, V'_0, V'_1 \rangle \]

contains a basic open neighbourhood. Choose basic open neighbourhoods

\[
\begin{align*}
V'_0 & := N_{s_0[h]}(p_0, \prod_{i=0}^{n_0} W_{i}^{(0)}) \subset V_0 \cap V'_0, \\
V''_1 & := N_{t_1[h]}(p_1, \prod_{i=0}^{n_1} W_{i}^{(1)}) \subset V_1 \cap V'_1
\end{align*}
\]

of the points \(s_0[h], t_1[h]\) respectively and basic open neighbourhoods

\[
\begin{align*}
U'_0 & \subset U_0 \cap U'_0 \cap W_0^{(0)} \cap W_0^{(1)}, \\
U''_1 & \subset U_1 \cap U'_1 \cap W_{n_0}^{(0)} \cap W_{n_1}^{(1)}
\end{align*}
\]

of the points \(s_0[h], t_0[h]\) respectively. The four basic open neighbourhoods satisfy

the conditions necessary to make the set

\[\langle h, U''_0, U''_1, V''_0, V''_1 \rangle\]
a basic open neighbourhood. By inspection this is contained in \([\mathcal{B}]\) as required. \(\Box\)

Now recall that the map \((s_1, t_1) : B_2 \to B_1 \times B_1\) for \(B\) a bigroupoid factors through \(B_1 \times B_0 \times B_0 B_1\). In the case of \(\Pi^T(X)\), this gives a function

\[
\Pi^T(X)_2 \to X^I \times X^I \times X^I \simeq LX.
\]

of the underlying sets. If \(L_0X\) denotes the (path) component of the null-homotopic loops, then clearly \(\text{im}(s, t) = L_0X\), which is open and closed in \(LX\) by our assumptions on \(X\).

We also introduce the notation \(p_1 \lor p_2\) for partition groupoids \(p_i\), which is meant to indicate the join and rescaling, covering the same operation on intervals.

**Lemma 31.** With the topology from lemma \([30]\) \((s_1, t_1) : \Pi^T(X)_2 \to L_0X\) is a covering space when \(X\) is semilocally 2-connected.

**Proof.** Recall that when \(X\) is semilocally 2-connected, \(LX\) is semilocally 1-connected, with path-connected basic open neighbourhoods. Let \(\omega\) be a point in \(L_0X\), corresponding to the paths \(\gamma_1, \gamma_2 : I \to X\) from \(x\) to \(y\). Let \(N := N^\omega_\gamma(p, W)\) be a basic open neighbourhood in \(L_0X\) where

\[
W = W_0 \prod_{i=1}^n W_i \prod_{j=n+1}^k W_j = W_0 \prod_{j=n+2} W_j = W_0 \prod_{j=n+2} W_j,
\]

and without loss of generality \(p = p_1 \lor p_2\), such that \(N_{\gamma_1}(p_1, W^1)\) and \(N_{\gamma_2}(p_2, W^2)\) are basic open neighbourhoods. Consider now the pullback

\[
\xymatrix{ N \times_{L_0X} \Pi^T(X)_2 \ar[r] \ar[d] & \Pi^T(X)_2 \ar[d] \\
L_0X \ar[r] & L_0X}
\]

which we want to show is a product \(N \times \Pi^T(X)(\gamma_1, \gamma_2)\). For \([h] \in \Pi^T(X)(\gamma_1, \gamma_2) = (s_1, t_1)^{-1}(\omega)\), define the following basic open neighbourhood:

\[
\langle [h] \rangle := ([h], W_0, W_{n+1}, W_1, W^2)
\]

By definition, the neighbourhoods \(N_{\gamma_1}(p_1, W^1)\) and \(N_{\gamma_2}(p_2, W^2)\) are path-connected, so the map \(\langle [h] \rangle \to N\) is surjective. Using the same arguments as in the proof of proposition \([27]\), it is also surjective and open, hence an isomorphism. We also know that if \([h] \neq [h']\), the neighbourhood \(\langle [h] \rangle\) is disjoint from \(\langle [h'] \rangle\), because if they shared a common point, they would be equal (see the proof of lemma \([30]\)). Every 2-track in \(N \times_{L_0X} \Pi^T(X)_2 \to N\) lies in some \(\langle [h] \rangle\), so there is an isomorphism

\[
N \times_{L_0X} \Pi^T(X)_2 \simeq N \times \Pi^T(X)(\gamma_1, \gamma_2)
\]

and \(\Pi^T(X)_2 \to L_0X\) is a covering space. \(\Box\)

**Remark 32.** Since \(L_0X\) is open and closed in \(LX\), we know that \(\Pi^T(X)_2 \to LX\) is a covering space where the fibres over the complement \(LX - L_0X\) are empty.

This lemma implies that the two composite maps \(s_1, t_1 : \Pi^T(X)_2 \to LX \rightrightarrows X^I\) are continuous. In fact these are the source and target for a topological groupoid

**Lemma 33.** The 2-tracks and paths in a space, with the topologies as above, form a topological groupoid \(\Pi^T(X) : (\Pi^T(X)_2 \rightrightarrows X^I)\).
Proof. We have already seen that the source and target maps are continuous, we only need to show that the unit map $\text{id}_{(-)}\circ (\text{id}_s)$ is continuous. For the unit map, let $\gamma \in X^I$, and $(\text{id}_s) := (\text{id}_s, U_0, U_1, V_0, V_1)$ be a basic open neighbourhood. Define $C := \text{id}_{(-)}((\text{id}_s))$ and consider the image of $C$ under $\text{id}_{(-)}$:

$$\text{id}_{(-)}(C) = \{ \eta \in (\text{id}_s) | \eta = [\lambda_1 + (\text{id}_{\beta_1} \cdot (\text{id}_s \cdot \text{id}_{\beta_0}) + \lambda_0] = \text{id}_\chi \}$$

Then $s_1(\lambda_1) = t_1(\lambda_0) = \beta_1 \cdot (\gamma \cdot \beta_0)$, $t_1(\lambda_1) = s_1(\lambda_0) = \chi$ and $\lambda_0 = -\lambda_1 := \lambda$. As $\lambda_0$ is a path in $V_0$ and $\lambda_1$ a path in $V_1$, we see that $\lambda$ is a path in $V_0 \cap V_1$ which implies $\chi \in V_0 \cap V_1$. If we choose a basic neighbourhood $V_2 \subset V_0 \cap V_1 \subset X^I$ of $\gamma$, then $\text{id}_{(-)}(V_2) \subset (\text{id}_s)$, and so the unit map is continuous.

We now need to show the map

$$+: \Pi^T_2(X)_2 \times_{X^I} \Pi^T_2(X)_2 \to \Pi^T_2(X)_2$$

is continuous. Let $[h_1], [h_2]$ be a pair of composable arrows, and let $(\langle h_2 + h_1 \rangle) := \langle [h_2 + h_1], U_0, U_1, V_1, V_2 \rangle$ be a basic open neighbourhood. Choose a basic open neighbourhood $V_1 = N_X(p, W)$ of $\gamma = s_1[h_2] = t_1[h_1]$ in $X^I$ such that the open neighbourhoods $U_0$ and $U_1$ are the first and last basic open neighbourhoods in the collection $W$. Consider the image

$$\mathcal{I} := +(\langle [h_2], U_0, U_1, V_1, V_2 \rangle \times_{X^I} \langle [h_1], U_0, U_1, V_0, V_1 \rangle).$$

The following diagram is a schematic of what an element in the image looks like:

![Diagram](attachment:image.png)

The thick lines are identified, and the circles are the basic opens $U_0, U_1 \subset X$. Topologically this is a disk with a cylinder $I \times S^1$ glued to it along some $I \times \{\theta\}$. For this 2-track to be an element of our original neighbourhood $\langle [h_2 + h_1] \rangle$ we need to show that the surface that goes ‘under’ the cylinder is homotopic (rel boundary) to the one that goes ‘over’ the cylinder, i.e. that there is a filler for the cylinder. Then a generic 2-track $[f_2 + f_1] \in \mathcal{I}$ is equal to one of the form

$$[\lambda_1 + (\text{id}_{\beta_1} \cdot ([h_2 + h_1] \cdot \text{id}_{\beta_0}) + \lambda_0] \in \langle [h_2 + h_1] \rangle$$

which schematically looks like
The trapezoidal regions in the first picture correspond to paths in $V_1$, which under the identification of the marked edges paste to form a loop in $V_1$. As $X^I$ is semilocally 1-connected, there is a filler for this loop in $X^I$. This implies that there is the homotopy we require, and so $+$ is continuous.

It is clear from the definition of the basic open neighbourhoods of $\Pi^T_2(X)_2$ that

$- (\langle [h], U_0, U_1, V_0, V_1 \rangle) = \langle [-h], U_0, U_1, V_1, V_0 \rangle$

and so $-(-)$ is manifestly continuous.

The maps $ev_0, ev_1 : X^I \to X$ give us a functor $\Pi^T_2(X)_1 \to \text{disc}(X \times X)$ of topological groupoids. We now have all the ingredients for a topological bigroupoid, but first a lemma about pasting open neighbourhoods of paths with matching endpoints.

Let $\gamma_1, \gamma_2 \in X^I$ be paths such that $\gamma_1(1) = \gamma_2(0)$ and let $N_1 := N_{\gamma_1}(p_1, W^1)$, $N_2 := N_{\gamma_2}(p_2, W^2)$ be basic open neighbourhoods. For an open set $U \subset W^1 \cap W^2$ (these being the last open sets in their respective collections), define subsets of $X^I$,

$M_1 := \{ \eta \in N_1 | \eta(1) \in U \}, \quad M_2 := \{ \eta \in N_2 | \eta(0) \in U \}.$

We define the pullback $M_1 \times_X M_2$ as a subset of $X^I \times_X X^I$ where this latter pullback is by the maps $ev_0, ev_1$. The proof of the following lemma should be obvious.

**Lemma 34.** The image of the set $M_1 \times_X M_2$ under concatenation of paths is the basic open neighbourhood

$N_{\gamma_2 \cdot \gamma_1}(p_1 \lor p_2, W^1 \amalg U \amalg W^2).$

We shall denote the image of $M_1 \times_X M_2$ as in the lemma by $N_1 \#_V N_2$.

**Proposition 35.** $\Pi^T_2(X)_2$ is a topological bigroupoid.

**Proof.** We need to show that the identity assigning functor

$\text{disc}(X) \to \Pi^T_2(X)_1,$

the concatenation and reverse functors,

$(-) \cdot (-) : \Pi^T_2(X)_1 \times_{\text{disc}(X)} \Pi^T_2(X)_1 \to \Pi^T_2(X)_1,$

$(-) : \Pi^T_2(X)_1 \to \Pi^T_2(X)_1,$
and the structure maps in (2) are continuous. The first follows from lemma 22, and the continuity of the object components of the second two are just lemmas 21 and 23. On the arrow space, the reverse functor clearly sends basic open neighbourhoods to basic open neighbourhoods,

\[ \langle [h], U_0, U_1, V_0, V_1 \rangle = \langle [h], U_1, U_0, V_0, V_1 \rangle, \]

and so is continuous.

Let \( \langle [h_2 \cdot h_1], U_0, U_1, V_0, V_1 \rangle \) be a basic open neighbourhood in \( \Pi^T_2(X)_2 \), where we have the basic open neighbourhoods

\[ V_0 = N_{s_1[h_2 \cdot h_1]}(p_0, W^0), \quad V_1 = N_{t_1[h_2 \cdot h_1]}(p_1, W^1) \]

in \( X^I \) where

\[ W^0 = \prod_{i=0}^{n} W^0_i, \quad W^1 = \prod_{j=0}^{m} W^1_j, \quad n, m \geq 3. \]

We can assume that \( p_0 = q^0_0 \lor q^0_2 \) and \( p_0 = q^1_1 \lor q^1_2 \). Let the partition groupoids be given by the following data

- \( q^0_0 \): \( \{ t_1, \ldots, t_k \} \)
- \( q^0_2 \): \( \{ t_{k+2}, \ldots, t_n \} \)
- \( q^1_1 \): \( \{ t'_1, \ldots, t'_l \} \)
- \( q^1_2 \): \( \{ t'_{l+2}, \ldots, t'_m \} \).

We now define the neighbourhoods

\[ V^0_1 := N_{s_1[h_1]}(q^0_1 \prod_{i=0}^{k} W^0_i), \]
\[ V^0_2 := N_{s_1[h_2]}(q^0_2 \prod_{i=k+2}^{n} W^0_i), \]
\[ V^1_1 := N_{t_1[h_1]}(q^1_1 \prod_{j=0}^{l} W^1_j), \]
\[ V^1_2 := N_{t_1[h_2]}(q^1_2 \prod_{j=l+2}^{m} W^1_j). \]

Consider the image of the fibred product \( \langle [h_1], U_0, U_1, V^0_1, V^1_1 \rangle \times X \langle [h_2], U_1, U_2, V^0_2, V^1_2 \rangle \) under concatenation, any element of which looks like

\[ \text{Referring to the object space } X^I \text{ of } \Pi^T_2(X)_1 \text{. Likewise, 'arrow components' refer to the arrow space of this groupoid, corresponding to the 2-arrow space of the bigroupoid} \]

\[ \text{Referring to the object space } X^I \text{ of } \Pi^T_2(X)_1 \text{. Likewise, 'arrow components' refer to the arrow space of this groupoid, corresponding to the 2-arrow space of the bigroupoid.} \]
where the two points marked \(\times\) are identified, so the line between them is a circle. Since the open set \(U_1 \subset X\) is 1-connected, there is a filler for this circle, and there is a homotopy between this surface and one of the form

Also, by lemma 34, the surfaces \(\lambda_0 \cdot \lambda_1, \lambda_2 \cdot \lambda_1\) are elements of \(V_0^1 \# U_1 V_2^1\) respectively. Then the image of the open set \(\langle [h_1], U_0, U_1, V_0^1, V_1^1 \rangle \times_X \langle [h_2], U_2, V_1^2, V_2^2 \rangle\) under concatenation is contained in \(\langle [h_2 \cdot h_1], U_0, U_1, V_0, V_1 \rangle\).

The assiduous reader will have already noticed that the following relations hold for the (component maps of) the structure morphisms of \(\Pi^T_2(X)\):

\[
l = r \circ (-), \quad e = - (i \circ (-)).
\]

This means that we only need to check the continuity of \(a\) and two of the other four structure maps.

For the associator \(a: X^I \times_X X^I \times_X X^I \to \Pi^T_2(X)_2\), we take a basic open neighbourhood

\[
\langle a_{\gamma_1, \gamma_2, \gamma_3} \rangle := \langle a_{\gamma_1, \gamma_2, \gamma_3}, U_0, U_1, V_0, V_1 \rangle
\]

and by continuity of concatenation of paths choose a basic open neighbourhood \(N\) of \(\langle \gamma_1, \gamma_2, \gamma_3 \rangle\) in \(X^I \times_X X^I \times_X X^I\) whose image under the composite

\[
X^I \times_X X^I \times_X X^I \xrightarrow{a} \Pi^T_2(X)_2 \xrightarrow{(s_1, t_1)} X^I \times_X X^I \times_X X^I
\]

is contained in \(V_0 \times_X V \times X V^1\). Also let \(U \subset X^I \times_X X^I \times_X X^I\) be a basic open neighbourhood whose image under

\[
X^I \times_X X^I \times_X X^I \xrightarrow{a} \Pi^T_2(X)_2 \xrightarrow{(s_1, t_1)} X^I \times_X X^I \xrightarrow{(s_0, t_0)} X_0 \times X_0
\]
is contained within $U_0 \times U_1$. Then if $N' \subset N \cap U$ is a basic open neighbourhood of $(\gamma_1, \gamma_2, \gamma_3)$, its image under $a$ is contained in $\langle a, \gamma_1, \gamma_2, \gamma_3 \rangle$, so $a$ is continuous.

The continuity of the other structure maps is proved similarly, and left as an exercise for the reader. \hfill \Box

It is expected that for a reasonable definition of a weak equivalence of bicategories internal to $\text{Top}$, the canonical 2-functor $\Pi_2(X) \to \Pi_2^2(X)$, where $\Pi_2(X)$ is equipped with the discrete topology, is such a weak equivalence. In any case, we can define strict 2-functors between topological bigroupoids, and these are the only such morphisms we shall need here.

**Definition 36.** A strict 2-functor $F : B \to B'$ between topological bigroupoids $B, B'$ consists of a continuous map $F_0 : B_0 \to B'_0$ and a functor $F_1 : B_1 \to B'_1$ commuting with $(S, T)$ and the various structure maps from definition 29.

We define the category of topological bigroupoids and continuous strict 2-functors and denote it by $\text{Bigpd}(\text{Top})$. Let $\text{Top}_{s2c}$ denote the full subcategory (of $\text{Top}$) of semilocally 2-connected spaces.

**Theorem 37.** There is a functor

$$\Pi_2^T : \text{Top}_{s2c} \to \text{Bigpd}(\text{Top}),$$

given on objects by the construction described above, which lifts the fundamental bigroupoid functor $\Pi_2$ of Stevenson and Hardie-Kamps-Kieboom.

**Proof.** We only need to check that the strict 2-functor $f_* : \Pi_2^T(X) \to \Pi_2^T(Y)$ induced by a map $f : X \to Y$ in continuous. Recall from [HKK01] that this strict 2-functor is given by $f$ on objects and post composition with $f$ on 1- and 2-arrows. We then just need to check that this is continuous on 2-arrows, as it is obvious that it is continuous on objects and 1-arrows.

Let $\langle (f \circ h) \rangle := \langle f \circ h, U^Y_0, U^Y, V_0, V_1 \rangle$ be a basic open neighbourhood in $\Pi_2^T(Y)_2$, and choose basic open neighbourhoods $W_\epsilon \in f^{-1}(V_\epsilon)$ in $X^I$ for $\epsilon = 0, 1$. If $W_0 = \coprod_{i=0}^n W_i^0$ and $W_1 = \coprod_{i=0}^m W_i^1$, then choose basic open neighbourhoods

$$U^X_0 \subset f^{-1}(U^Y_0) \cap W_0^0 \cap W_1^0, \quad U^X_1 \subset f^{-1}(U^Y_1) \cap W_0^1 \cap W_m^1$$

in $X$. It is then clear that $f_*\langle (h), U^X_0, U^X_1, W_0, W_1 \rangle \subset \langle (f \circ h) \rangle$, and so $f_*$ is a continuous 2-functor. \hfill \Box

Now there is a notion of local triviality of topological bigroupoids analogous to that of ordinary topological groupoids. This requires a subsidiary definition

**Definition 38.** Let $p : X \to M$ be a functor between topological groupoids such that $M$ is a topological space. An anasection is a pair $(V, \sigma)$ where $j : V \to M$ is an open cover of $M$ and $\sigma : \check{C}(V) \to X$ is a functor such that $j = p \circ \sigma$.

We can picture $(V, \sigma)$ as being an $X$-valued Čech cocycle on $M$ satisfying a particular property. Note also that an ordinary section of $p$ (which is essentially just a section of the object component of $p$) is also an anasection.

**Definition 39.** Let $B$ be a topological bigroupoid such that $X = B_0$ is locally path-connected. We say $B$ is locally trivial if the following conditions hold:

(I) The image of $(s_1, t_1) : B_2 \to B_1 \times_{B_0} B_1$ is open and closed, and $B_2 \to \text{im}(s_1, t_1)$ admits local sections.

(II) For every $b, b' \in B_0$ there is an open neighbourhood $U$ of $b'$ such that for all $g \in S^{-1}(b)$ there is an anasection $(V, \sigma)$ such that there is an arrow $g \xrightarrow{\sim} \sigma(v)$ in $S^{-1}(b)$ for some $v \in V$. 


If $B$ satisfies just condition (II) it will be called a submersive bigroupoid\footnote{Compare with the definition of a topological submersion: a map $p: M \to N$ of spaces such that every $m \in M$ there is a local section $s: U \to M$ of $p$ such that $m = s(u)$.}

In fact, composing a local section with the restriction of the inversion functor $B_1 \to B_1$, we get local sections of target fibre $T^{-1}(b)_0 \to B_0$. Given a pair of local sections, one of the source fibre and one of the target fibre, they determine a map to the fibre product $B_1 \times_{\text{disc}(B_0)} B_1$, which can be composed with the horizontal composition functor to give a local section of $(S, T): \text{Obj}(B_1) \to B_0 \times B_0$.

We will not actually use this definition as it stands, because we are only interested in locally trivial bigroupoids that satisfy a stronger version of condition (I):

**Definition 40.** A topological bigroupoid $B$ will be called locally weakly discrete if

(I) The map $(s_1, t_1): B_2 \to B_1 \times_{B_0} B_1$ is a covering space.

Note that condition (I) implies condition (I) from definition \[39\]

This nomenclature is consistent with the usage of the word ‘locally’ in the theory of bicategories, in that condition (I) implies that the groupoid $B(a, b) = (S, T)^{-1}(a, b)$ is locally trivial with discrete hom-spaces, and hence weakly discrete. However, if we merely assume the fibres of $B_1 \to \text{disc}(B_0 \times B_0)$ are weakly discrete, it does not follow that we have a locally weakly discrete bigroupoid as defined above.

Recall that a space is locally contractible if it has a neighbourhood basis of contractible open sets. We shall call a space semilocally contractible if it has a neighbourhood basis such that the inclusion maps are null-homotopic.

**Proposition 41.** If $X$ is semilocally contractible, $\Pi^T_2(X)$ is locally trivial and locally weakly discrete.

**Proof.** We already know that $\Pi^T_2(X)$ is locally weakly discrete, by lemma \[31\] hence we only need to show it is submersive.

Let $x_0$ be any point in $X$ and let $\gamma \in P_{x_0}X$. Let $U$ be a neighbourhood of $x_1 := \gamma(1)$ such that $U \hookrightarrow X$ is null-homotopic. Then the map $P_{x_1}X \to X$ admits a local section $s: U \to P_{x_1}X$, which we claim can be chosen such that $\lambda := s(x_1)$, which is a loop in $X$, is null-homotopic. If this is not the case, compose the section with the map $P_{x_1}X \to P_{x_0}X$ given by preconcatenation with $\lambda$, then the new section sends $x_1$ to $\lambda \cdot \lambda$, which is null-homotopic. We then compose the section $s$ with the map $P_{x_0}X \to P_{x_0}X$ which is preconcatenation with $\gamma$ to get a section $s'$. Since $s'(x_1) = \lambda \cdot \gamma$, which is homotopic to $\gamma$ rel endpoints, we have an anasection

\[
\text{disc}(U) \overset{\epsilon}{\leftarrow} \text{disc}(U) \overset{s'}{\to} S^{-1}(x_0),
\]

such that $\gamma$ is isomorphic to an object in the image of $s'$. Thus $\Pi^T_2(X)$ is a submersive groupoid \[ \square \]

**Appendix A. Definition of a bigroupoid**

The following is adapted from [Lei98] and [HKK01], themselves a distilling of the original source [Ben67]. We shall only be interested in small bicategories and groupoids, that is, those where the data forms a set.

**Definition 42.** A bicategory $\mathbf{B}$ is given by the following data:

- A set $\mathbf{B}_0$ called the 0-cells or objects of $\mathbf{B}$,
• A small category $\mathcal{B}_1$ with a functor

$$(S, T) : \mathcal{B}_1 \to \text{disc}(\mathcal{B}_0) \times \text{disc}(\mathcal{B}_0).$$

The fibre of $(S, T)$ at $(A, B) \in \mathcal{B}_0 \times \mathcal{B}_0$ is denoted $\mathcal{B}(A, B)$ and is called a hom-category. The objects $f, g, \ldots$ of $\mathcal{B}_1$ are called 1-cells, or 1-arrows, and the arrows $\alpha, \beta, \ldots$ of $\mathcal{B}_1$ are called 2-cells, or 2-arrows. The functors $S, T$ are the source and target functors. The composition in $\mathcal{B}_1$ will be denoted $(\alpha, \beta) \mapsto \alpha \cdot \beta$ where the target of $\alpha$ is the source of $\beta$. This is also called vertical composition.

• Functors

$c_{ABC} : \mathcal{B}(B, C) \times \mathcal{B}(A, B) \to \mathcal{B}(A, C)$,

for each $A, B, C \in \mathcal{B}_0$, called horizontal composition, and an element

$I_A : * \to \mathcal{B}(A, A)$.

for each $A \in \mathcal{B}_0$, picking out a 1-cell $A \to A$ called the weak unit of $A$. Horizontal composition is denoted $(w, v) \mapsto w \circ v$ where $T(v) = S(w)$, and $v, w$ are either 1-cells or 2-cells.

• Natural isomorphisms

$\mathcal{B}(C, D) \times \mathcal{B}(B, C) \times \mathcal{B}(A, B) \xrightarrow{id \times c_{ABC}} \mathcal{B}(C, D) \times \mathcal{B}(A, C)$

$\xrightarrow{c_{BCD} \times id} \mathcal{B}(B, D) \times \mathcal{B}(A, B) \xrightarrow{c_{ABD}} \mathcal{B}(A, D)$

given by invertible 2-cells

$a_{hgf} : (h \circ g) \circ f \Rightarrow f \circ (g \circ f),$

for composable $f, g, h \in \text{Obj} \mathcal{B}_1$ and natural isomorphisms

$\mathcal{B}(A, B) \times * \xrightarrow{id \times I_A} \mathcal{B}(A, B)$

$\xrightarrow{r_{AB} \sim} \mathcal{B}(A, B) \xrightarrow{c_{ABB}} \mathcal{B}(A, B)$

$\xrightarrow{I_B \times id \sim} \mathcal{B}(B, B) \times \mathcal{B}(A, B) \xrightarrow{c_{ABB}} \mathcal{B}(A, B)$

given by invertible 2-cells

$r_f : f \circ I_A \Rightarrow f, \quad l_f : I_B \circ f \Rightarrow f.$

where $A = S(F)$ and $B = T(F)$. 
The following diagrams are required to commute:

\[ ((k \circ h) \circ g) \circ f \xrightarrow{a_{khg} \circ \text{id}_f} (k \circ (h \circ g)) \circ f \]

\[ (k \circ h) \circ (g \circ f) \]

\[ k \circ ((h \circ g) \circ f) \]

\[ (g \circ I) \circ f \xrightarrow{a_{gI}} g \circ (I \circ f) \]

\[ g \circ f \]

\[ (g \circ f) \circ (f \circ f) \xrightarrow{\text{id} \circ a_{fI}} f \circ (f \circ f) \]

\[ I \circ f \xrightarrow{\text{id} \circ \text{id}_f} f \circ (f \circ f) \]

If the 2-cells \(a, l, r\) are all identity 2-cells, then the bicategory is called a 2-category, or \textit{strict} 2-category for emphasis.

**Definition 43.** A \textit{bigroupoid} is a bicategory \(B\) such that \(B_1\) is a groupoid, and the following additional data for each \(A, B \in B_0\): 

- **Functors**
  \[ (-) : B(A, B) \to B(B, A) \]

- **Natural isomorphisms**
  \[
  B(A, B) \xrightarrow{(-, \text{id})} B(B, A) \times B(A, B) \quad B(A, B) \xrightarrow{(\text{id}, (-))} B(A, B) \times B(B, A)
  \]

\[
\begin{array}{ccc}
\text{B}(A, B) & \xrightarrow{(\text{-}, \text{id})} & \text{B}(B, A) \times \text{B}(A, B) \\
\downarrow & & \downarrow \\
\text{B}(A, A) & \xrightarrow{c_{AB}} & \text{B}(A, A) \\
\downarrow & & \downarrow \\
A & \xrightarrow{l_A} & \text{B}(A, A)
\end{array}
\quad
\begin{array}{ccc}
\text{B}(A, B) & \xrightarrow{(\text{id}, \text{-})} & \text{B}(A, B) \times \text{B}(B, A) \\
\downarrow & & \downarrow \\
\text{B}(B, B) & \xrightarrow{c_{B,AB}} & \text{B}(B, B) \\
\downarrow & & \downarrow \\
B & \xrightarrow{l_B} & \text{B}(B, B)
\end{array}
\]

The following diagram is required to commute

\[
\begin{array}{ccc}
I \circ f & \xrightarrow{i_f \circ \text{id}_f} & (f \circ f) \circ f \\
\downarrow & & \downarrow \\
f & \xrightarrow{id_f \circ c_f} & f \circ I
\end{array}
\]

\[
\begin{array}{ccc}
I \circ f & \xrightarrow{i_f \circ \text{id}_f} & (f \circ f) \circ f \\
\downarrow & & \downarrow \\
f & \xrightarrow{id_f \circ c_f} & f \circ I
\end{array}
\]
It is a consequence of the other axioms that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{F} \circ I \\ ^{r_f} \downarrow & \xrightarrow{\text{id}_{\mathcal{F}}} & \mathcal{F} \circ (f \circ \mathcal{F}) \\ \mathcal{F} & \longrightarrow & (f \circ \mathcal{F}) \circ \mathcal{F} \\
\mathcal{F} \circ (f \circ \mathcal{F}) & \xrightarrow{\alpha_{\mathcal{F}, \mathcal{F}}} & (f \circ \mathcal{F}) \circ \mathcal{F} \\
& & \mathcal{F} \circ I \\
\mathcal{F} \circ (f \circ \mathcal{F}) & \xrightarrow{e_f \circ \text{id}_{\mathcal{F}}} & \mathcal{F} \circ I \\
\end{array}
\]

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