ON THE EXISTENCE OF A DERIVED EQUIVALENCE BETWEEN A KOSZUL ALGEBRA AND ITS YONEDA ALGEBRA

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Abstract. In this paper we focus on the relations between the derived categories of a Koszul algebra and its Yoneda algebra, in particular we want to consider the cases where these categories are triangularly equivalent. We prove that the simply connected Koszul algebras are derived equivalent to their Yoneda algebras. We consider derived discrete Koszul algebras, and we give necessary and sufficient conditions for these Koszul algebras to be derived equivalent to their Yoneda algebras. Finally, we look at the Koszul algebras such that they are derived equivalent to a hereditary algebra. In the case that the hereditary algebra is tame, we characterize when these algebras are derived equivalent to their Yoneda algebras.

We dedicate this work to the memory of Dieter Happel.

In our context, algebras will always be finite dimensional, and of the form $A = kQ/I$, where $Q$ is a quiver and $I$ is an homogeneous ideal with generators in degrees bigger equal to two and $k$ is an algebraically closed field. Most of the cases, the homogeneous ideal $I$ is generated in degree two. The ideal of $kQ$ generated by the paths of length one will be denoted by $J$. Our modules will be finitely generated left modules. In this paper, the derived category of an algebra will be the bounded derived category of finitely generated modules over the algebra. We say that two algebras are derived equivalent if there exists a triangulated equivalence between their derived categories. We will denote by $\mathbb{Z}$ the ring of integers.

We are interested in the study of some aspects of the derived categories of Koszul algebras. In particular, we are looking for the cases where there exists a derived equivalence between a Koszul algebra and its Yoneda algebra. It was proven by Beilinson, Ginzburg and Soergel in [9], under the assumption of $A$ being finite dimensional and $E(A)$ being Noetherian, that there exists a derived equivalence between the derived categories of the finitely generated graded modules over a Koszul algebra and its Yoneda algebra.

It has been shown by Dag Madsen in [23], that it does not exist a triangulated equivalence of unbounded derived categories between a Koszul algebra and its Yoneda algebra. He observed that derived equivalence between the Koszul algebra and its dual may exist only in the case where that Koszul algebra is finite.

2000 Mathematics Subject Classification. Primary 18E30. Secondary 16S37.

The first author was supported by grants from CAPES, the second author was partially supported by grants from the CAPES, CNPq and FAPESP and the third author was supported by grants from CONICET..
dimensional with finite global dimension. We will show that, in particular cases, this derived equivalence exists.

Bautista and Liu, in [7], proved that a finite dimensional algebra $A$ with radical square zero is derived equivalent to a hereditary algebra if and only if its ordinary quiver $Q$ is gradable. Moreover, they showed that in this case, $D^b(A) \cong D^b(kQ^{op})$. Since $kQ/(J^2)$ is the Yoneda algebra of $kQ$, we observe that the desired derived equivalence exists in this case. We note that if the underlying graph of a quiver is a tree, then it is a gradable quiver. We start generalizing that result of Bautista and Liu to other classes of algebras.

An algebra is called a triangular algebra when its ordinary quiver $Q_A$ contains no oriented cycles. We recall that a triangular algebra $A$ is called simply connected if, for every presentation of $A$ given by a pair $(Q_A, I_{\mu})$, we have a null fundamental group, that is, $\pi_1(Q_A, I_{\mu}) = 0$. Now, we are in a position to state our first result.

**Theorem 1:** Let $A$ be a simply connected Koszul algebra with finite global dimension. Then the Koszul algebra $A$ is derived equivalent to its Yoneda algebra $E(A)$.

We recall that, a derived category $D^b(A)$ is said to be discrete if, for every vector $n = (n_i)_{i \in \mathbb{Z}}$ of natural numbers, there are only finitely many isomorphism classes of indecomposable objects in $D^b(A)$ with the homological dimension vector equal to $n$. We refer [24] to the reader. Using the results of Bobinski, Geiss and Skowronski, [10], we are able to characterize when a derived discrete Koszul algebra is derived equivalent to its Yoneda algebra. It follows from [10] that $A$ has a discrete derived category if and only if $A$ is derived equivalent to a hereditary algebra of Dynkin type or $D^b(A) \cong D^b(A(r,n,m))$, where $A(r,n,m)$ is a quotient of the path algebra over the cycle $C(n,m)$ with exactly one source and one sink, where $n$ is the number of clockwise arrows, $m$ is the number of counterclockwise arrows and $r > 1$ is the number of consecutive clockwise quadratic relations from the vertex $(n-r)$ to the vertex zero. For those algebras $A(r,n,m)$, we have the cycle $C(n,m)$ given by the following quiver.

\[
\begin{array}{c}
-1 \leftarrow \ldots \leftarrow \ldots \leftarrow \ldots \leftarrow -m + 1 \\
\rightarrow \\
0 \leftarrow \ldots \leftarrow n - r \leftarrow \ldots \leftarrow 1 \\
\rightarrow \leftarrow \\
-n \leftarrow \ldots \leftarrow (n-r) \leftarrow \ldots \leftarrow 0 \\
\rightarrow \rightarrow \end{array}
\]

We now are able to state our second theorem.

**Theorem 2.** Let $A$ be a finite global dimension Koszul algebra having a discrete derived category. Then $A$ and $E(A)$ are derived equivalent if and only if $A$ is derived equivalent to a hereditary algebra of Dynkin type or $A$ is derived equivalent to $A(r,n,m)$.

Finally, we consider Koszul algebras which are derived equivalent to hereditary algebras, that is, Koszul algebras, piecewise hereditary of the quiver type. In the
case that the hereditary algebra is tame, we are able to characterize when those Koszul algebras are derived equivalent to their Yoneda algebras. We get the following theorem.

\textbf{Theorem 3.} Let $A$ be a Koszul algebra derived equivalent to a hereditary algebra $H = kQ$. Then the following statement holds.

a) If $A$ is simply connected, then $A$ is derived equivalent to its Yoneda algebra $E(A)$, and $Q$ is a tree.

b) If $H$ is of Euclidean type, then $A$ is derived equivalent to its Yoneda algebra $E(A)$ if and only if $A$ is simply connected or $A$ is derived equivalent to an iterated tilted algebra of type $A_s$, for some $s \in \mathbb{Z}$ and $s \geq 1$, over the non-oriented cycle $C(n,n)$.

In Section 1, we begin reviewing some definitions and results on the theory of Koszul algebras, derived categories, tilting theory and gentle algebras. In Section 2, we prove our main result, we show that, simply connected Koszul algebras are derived equivalent to their Yoneda algebras. We dedicated Section 3 to the study of derived discrete Koszul algebras, and we characterize when they are derived equivalent to their Yoneda algebras. In Section 4, we consider Koszul algebras which are derived equivalent to a hereditary algebra $H$. We give necessary and sufficient conditions, in the case that $H$ is tame, for these algebras and their Yoneda algebras to be derived equivalent. We show that in this case, we have to consider Koszul algebras which are derived equivalent to $A_s$ and Koszul algebras derived equivalent to hereditary algebras of tree type.

1. Preliminaries.

Let $A$ denote a positively $\mathbb{Z}$-graded algebra $A = A_0 \oplus A_1 \oplus \ldots$ such that each $A_i$ for $i \geq 0$ is a finitely generated $A_0$-bimodule. We also denote by $A_0$ the semisimple graded $A$-module $A/(A_1 \oplus \ldots)$. The algebra $A$ is called Koszul algebra if $A_0$ admits a graded minimal projective resolution

$$\ldots \rightarrow P^2 \rightarrow P^1 \rightarrow A_0 \rightarrow 0,$$

such that $P^i$ is generated in degree $i$.

We denote $E(A) = \prod_{n \geq 0} \text{Ext}^n_A(A_0, A_0)$ the Yoneda algebra of the graded algebra as above. We recall that the Yoneda algebra of a Koszul algebra is also a Koszul algebra. We recall that a graded algebra $A$ is a \textit{Koszul algebra} if and only if its Yoneda algebra is generated in degree zero and one, that is, the elements in $\text{Ext}^1_A(A_0, A_0)$ generate all higher extension groups under the Yoneda product. It is also known that $E(E(A)) \cong A$ as graded algebra if and only if $A$ is a Koszul algebra.

We also recall that, every Koszul algebra is a quadratic algebra, that is, $A = kQ/I$ where $I$ is generated is generated in degree two. It follows from Koszul duality that $A$ is a finite dimensional Koszul algebra if and only if its Yoneda algebra $E(A)$ has finite global dimension. We refer [14] for further details. As examples of Koszul algebras we have hereditary algebras and monomial quadratic algebras (e.g. gentle
algebras), and quadratic finite dimensional algebras with global dimension 2. We refer Priddy or Green and Martinez-Villa in [22] and [14], for other examples.

Let $A$ be an abelian category and $D(A)$ the derived category of $A$. We will consider $A = \text{mod}-A$ the category of finitely generated modules of a finite dimensional $k$-algebra $A$ where $k$ is a field. The category of finitely generated graded modules of a graded algebra with zero degree morphisms will be denoted by $\text{gr}A$, as usual. Since we assume that $A$ has finite global dimension then the bounded derived category of $A$, denoted by $D^b(A)$, and $K^b(P)$ the bounded homotopy category of the complexes of projectives $A$-modules, are equivalent triangulated categories, [19].

We recall a few definitions and necessary results on the theory of tilting and cotilting modules. Let $Q$ be a finite quiver without oriented cycles and $G$ its underlying graph. Given a sink $v \in G_0$, the quiver $\sigma^+_v(Q)$ has the same underlying graph $G$ and arrows defined in the following way: If $\alpha : u \to v$ is an arrow in $Q$ then $v \to u$ is an arrow in $\sigma^+_v(Q)$. If $\alpha : u \to w$ is an arrow in $Q$ for $w \neq v$ then it remains an arrow in $\sigma^+_v(Q)$. Given a source $v \in G_0$, the quiver $\sigma^-_v(Q)$ is defined dually. Those two operations are called reflections. Let $Q'$ be a finite quivers without oriented cycles, we say that $Q$ can be obtained from $Q'$ by a sequence of reflections, if there exists $v_1, \ldots, v_n \in Q'$ such that $v_i$ is a sink or a source of $\sigma^-_{v_i-1} \cdot \cdot \cdot \sigma^-_{v_1}(Q')$ and $Q = \sigma^+_{v_1} \cdot \cdot \cdot \sigma^+_{v_n}(Q')$. In this case we say that $Q$ and $Q'$ are equivalent and we denoted by $Q \cong Q'$.

We recall the following statement.

**Lemma**([19]) If $Q$ and $Q'$ are finite quivers without oriented cycles, then $Q \cong Q'$ if and only if $D^b(kQ) \cong D^b(kQ')$.

We observe that if $Q \cong Q'$ then the underlying graph $G$ of $Q$ and of $Q'$ are the same. A converse statement is not true, in general. However if $G$ is a tree and $Q, Q'$ are quivers with that same underlying graph $G$ then $Q \cong Q'$.

We say that the $A$-module $T$ is a tilting module provides the following conditions:

1. The projective dimension $\text{pd}_AT \leq 1$.
2. $\text{Ext}^1_A(T, T) = 0$.
3. There is a short exact sequence $0 \to A \to T' \to T'' \to 0$, with $T'$ and $T'' \in \text{add}(T)$.

Dually we can define a cotilting module. We recall the definition of APR-tilting module. Let $S(i)$ be the simple module associated to a sink (or a source), then $T = \tau^{-1}S(i) \oplus \oplus_{j \neq i} P_j \ (T = \oplus_{j \neq i} I_j \oplus \tau S(i))$ is tilting (cotilting) module called the APR-tilting (cotilting) module associated to $i$. Observe that a reflection in a quiver $Q$ at the vertex $i$ corresponds to apply the APR-tilting (or cotilting) module associated with the vertex $i$ and to compute the endomorphism algebra of this tilting (cotilting) module.

We recall that a algebra $A$ is said to be a tilted algebra if $A = \text{End}_H(T)^{op}$, where $H$ is a hereditary algebra and $T$ is a tilting module, and an algebra $B$ is said to be iterated tilted algebra, if there exists a family of algebras $(A_i)_{0 \leq i \leq n}$ and a family of tilting $A_i$-modules $T_i = A_iT$, where $A_0 = A$, $A_{i+1} = \text{End}_{A_i}(T)^{op}$, with $A$ is a path algebra, and $B = A_n$.

It follows by [19] that $A$ is derived equivalent to a hereditary algebra $H = kQ$ if and only if $A$ is an iterated tilted algebra of type $Q$. 
We recall a very important class of algebras that will be useful for our purposes. Following [3], we have the following description of triangular gentle algebras.

**Theorem** ([3]) Let \( A = kQ/I \) where \( Q \) is a connected triangular quiver and \( I \) an admissible ideal. Then \( A \) is a gentle algebra if and only if the following conditions are satisfied:

1. The ideal \( I \) is generated by a set of paths of length two.
2. The number of arrows in \( Q \) with a prescribed source or target is at most two.
3. For any arrow \( \alpha \in Q \), there is at most one arrow \( \beta \) and one arrow \( \gamma \) such that \( \alpha\beta \) and \( \gamma\alpha \) are not in \( I_A \).
4. For any arrow \( \alpha \in Q \), there is at most one arrow \( \eta \) and one arrow \( \zeta \) such that \( \eta\alpha \) and \( \alpha\zeta \) belong to \( I_A \).

It follows from this description that a Koszul algebra \( A \) is a gentle if and only if its Yoneda algebra is also a gentle algebra. We get from [1] that a gentle algebra whose quiver is a tree coincides exactly with the iterated tilted algebra of type \( \tilde{A}_n \).

It follows from [3] that an algebra is iterated tilted from a hereditary algebra of type \( \tilde{A}_n \) if and only if it is gentle and satisfies the following conditions:

1. \( Q_A \) has exactly \( n + 1 \) vertices.
2. \( Q_A \) contains a unique (non-oriented) cycle \( C \).
3. On \( C \) the number of clockwise oriented relations equals the number of counterclockwise oriented relations. (**clock condition**).

This characterization of iterated tilted algebras of type \( \tilde{A}_n \) will be very useful in Section 3.

2. **Simply connected Koszul algebras**

In this section we show that a simply connected Koszul algebra \( A \) and its Yoneda algebra are derived equivalent. It is known that a triangular algebra \( A \) is simply connected if and only if \( A \) admits no proper Galois covering, we refer [2] for further details. The next lemma is a standard result, we give a proof for the sake of completeness.

**Lemma 2.1.** Let \( V = \oplus_{g \in G} V_g \) be a \( G \)-graded finite dimensional vector space. We consider the dual space of \( V \) graded in the following way \( V^* = \oplus_{g \in G} V_g^* \). If \( W \) is a \( G \)-graded subspace of \( V \) then the subspace \( W^\perp \) of \( V^* \) consisting of the maps that vanish in \( W \) is a \( G \)-graded subspace.

**Proof:** We consider for each \( g \) a basis of the nonvanish subspaces \( V_g \) and we add them up to form a basis of \( V \). We observe that \( V_g^* \) has a basis, the dual basis of \( V_g \). Let \( W = \oplus_{g \in G} V_g \cap W \) the \( G \)-graded subspace of \( V \). We take \( f \in W^\perp \) and we write \( f = f_{g_1} + \cdots + f_{g_t} \) with \( f_{g_i} \in V_{g_i}^* \). We will show that each \( f_{g_i} \in W^\perp \).

Let \( w \in W_{g_i} \), then \( f_{g_i}(w) = 0 \) for \( i \neq j \) hence \( f(w) = f_{g_i}(w) = 0 \). Now take any \( w = w_{g_1} + \cdots + w_{g_t} \), with \( w_{g_i} \in W \cap V_{g_i} \). So \( f_{g_i}(w) = f_{g_i}(w_j) = 0 \) since \( f \in W^\perp \) thus \( f_{g_i} \in W^\perp \). \( \blacksquare \)
We shall present below an important step to prove our main result on this section. Let be $G$ a group and $A$ a $G$-graded algebra. One may consider the algebra $A$ as a $G$-graded category over $k$ defined by one category object given by $A$ and morphisms given by the elements of $A$. We denote that category by $A_G$. We consider the smash product category of $A_G$ by $G$ denoted for short by $A\#G$. We refer [12] to definition and related results. We recall that the smash product $A\#G$ has a free $G$-action and $|G|$ objects.

**Lemma 2.2.** Let $A$ be a finite dimensional Koszul $G$-graded algebra with finite global dimension. Then the smash product $A\#G$ is a connected Galois covering of $A$ if and only if $E(A)\#G$ is a connected Galois covering of $E(A)$.

**Proof:** We recall that $A = kQ/I$ with $Q$ a finite quiver and $I$ a $G$-homogeneous graded ideal. We know that $I$ is a $\mathbb{Z}$-graded ideal generated in degree 2, that is, $I = I(2) \oplus I(3) \oplus I(4) \oplus \cdots = < I(2) >$ with $I(2) = kQ(2) \cap I$. It is known also that $E(A) = kQ^{op}/O(I)$ for $Q^{op}$ the opposite quiver of $Q$ and $O(I)$ the orthogonal ideal of $I$. We refer the reader to [14] and [15] for more details. Since $A$ is a $G$-graded algebra, it follows from Lemma 2.1 that $E(A)$ is also a $G$-graded algebra.

We claim that if the smash product $A\#G$ is a connected Galois covering of $A$ then $E(A)\#G$ is also a connected Galois covering of $E(A)$. In fact we shall prove that the smash product $E(A)\#G$ is the ext-algebra of $A\#G$ and our claim will follow from the description of the ext-algebra of a Koszul algebra and the fact that $E(E(A)) = A$ as graded algebra for Koszul algebras.

Let $A\#G = k\mathcal{R}(Q)/\mathcal{R}(I)$ a connected Galois covering of $A = kQ/I$ where $\mathcal{R}(Q)$ is the covering of the quiver $Q$ and $\mathcal{R}(I)$ is the lifting of the ideal $I$, see [16]. Since $E(A) = kQ^{op}/O(I)$ one may consider a Galois covering of $E(A)$ given by the smash product $E(A)\#G = k\mathcal{R}(Q^{op})/\mathcal{R}(O(I))$ in the same sense.

We observe that any $\mathcal{R}(Q) \to Q$ is a covering if and only if $\mathcal{R}(Q)^{op} \to Q^{op}$ is a covering. Moreover $\mathcal{R}(Q^{op}) = [\mathcal{R}(Q)]^{op}$.

According to the description of the Yoneda algebra in [15] and from the remark above we obtain that $E(A\#G) = E(k\mathcal{R}(Q)/\mathcal{R}(I)) = k[\mathcal{R}(Q)]^{op}/O(O(I)) = k\mathcal{R}(Q^{op})/O(O(I))$. Thus it is enough to prove that $O(O(I)) = O(O(I))$. We consider the following diagram:

$$
\begin{array}{ccc}
\mathcal{R}(I) & \leftrightarrow & \mathcal{O}(\mathcal{R}(I)) \\
p & \downarrow & \pi \\
I & \leftrightarrow & \mathcal{O}(I)
\end{array}
$$

Let $\alpha' \in O(\mathcal{R}(I))$. We have that $\alpha'$ is orthogonal to any element in $\mathcal{R}(I)$, one say, $< (\alpha')^{op}, \beta >= 0$ for any $\beta \in \mathcal{R}(I)$.

Let $\alpha \in O(\mathcal{O}(I))$. Then we have $\pi(\alpha) \in O(I)$ hence $< \pi(\alpha), \eta >= 0$ for any $\eta \in I$. It follows that $< (\pi(\alpha))^{op}, p(\beta) >= 0$ for every $\beta \in p^{-1}(\eta)$ with $\eta \in I$ since $\beta \in \mathcal{R}(I)$. We now assume that $< (\alpha)^{op}, \beta >= 0$ for some $\beta \in \mathcal{R}(I)$. Thus $\alpha \notin O(\mathcal{R}(I))$ hence $\pi(\alpha) \notin p(O(\mathcal{R}(I))) = O(I)$. Since $p(\mathcal{R}(O(I))) = p(O(\mathcal{R}(I))) = O(I)$ we conclude that $\mathcal{R}(O(I)) \subset O(\mathcal{R}(I))$ hence they are equal and our result is proved. 

We now give an important remark which follows from [16], [2], [3]. A $G$-weight of a quiver $Q$ in a group $G$ is just a map $w : Q_1 \to G$, where $Q_1$ is the set of arrows of
Any $G$-weight induces in a natural way a grading on the path algebra $kQ$. When the ideal $I$ is a $G$-homogeneous ideal then it induces a grading on $A = kQ/I$ and we say that $A$ has a $G$-grading induced by $w$. In this case, Green in [10] constructed a covering of $A$ which is isomorphic to the smash product $A\#G$, see [17]. The next result is a straightforward consequence of that remark and the results on Lemma 2.1 and Lemma 2.2 given above.

**Proposition 2.3.** Let $A$ be a connected basic triangular algebra. The following conditions are equivalent:

1. $A$ is simply connected
2. Given any group $G$ and any weighted grading on $A$ then $A\#G$ is isomorphic to $|G|$ copies of $A$.
3. Given any non trivial group $G$ and any weighted grading on $A$, then $A\#G$ is disconnected.

**Theorem 2.4.** Let $A$ be a finite dimensional Koszul algebra, with finite global dimension. Then $A$ is a simply connected algebra if and only if its Yoneda algebra is a simply connected algebra.

**Proof:** The proof of the theorem will follow from the Lemma 2.2, the proposition above and the fact that $E(E(A)) \cong A$ as graded algebras for Koszul algebras.

The statement above does not hold in general, even for quadratic algebras. The next example show us a quadratic non-koszul algebra that does not satisfy that statement.

**Example 1:** Let $A$ the quiver algebra given by

![Quiver Diagram]

with the commutative relation $\beta\beta' = \alpha'\gamma$ and the monomial relations $\alpha\beta = 0 = \gamma\delta$. Since $A$ is constricted its homotopy group does not depend on the presentation of $A$, [8], and with our presentation it is trivial. So $A$ is a simply connected algebra. We observe that the simple $S_1$ is a projective dimension 3 non-koszul module hence we have a presentation of $E(A)$ over $k$ as a quiver algebra given by $(Q', O(I))$ where $Q'$ is the quiver $Q^{op}$ adjointed to an arrow from the vertex 6 to the vertex 1 and $O(I)$ is the ortogonal ideal of $I$ hence $E(A)$ is not a simply connected algebra.

We now present the main result of this section.

**Theorem 2.5.** Let $A = kQ/I$ be a simply connected Koszul algebra. Then $A$ and $E(A)$ are derived equivalent.
Proof: It follows from [12] the following equivalence of categories \( \text{mod} - \tilde{A} \cong \text{gr} A \), where \( \tilde{A} \) is a Galois covering of the algebra \( A \). It was shown in [12] that the category \( \text{mod} ZA \) is isomorphic to the smash product category \( A\# Z \). Hence we have \( \text{mod} - \tilde{A} \cong \text{gr} A \cong \text{mod} ZA \cong A\# Z \). Since \( A \) is a simply connected algebra it follows from Proposition 2.3 that \( \text{mod} - \tilde{A} \) is a disconnected product of categories indexed by \( Z \) all isomorphic to \( \text{mod} - A \).

We recall from Lemma 2.2 that \( \text{D}^b(E(A)\# Z) \cong \text{D}^b(E(A)) \), that is, \( \text{D}^b(A\# Z) \cong \text{D}^b(E(A)\# Z) \). Since \( A \) admits no proper Galois covering, it follows that \( \text{D}^b(A) \cong \text{D}^b(E(A)) \) as we claimed.

Corollary 2.6. Let \( A = kQ/I \) a Koszul algebra with finite global dimension. Then

1. If \( Q \) is tree then \( D^b(A) \cong D^b(E(A)) \).
2. If \( A \) is a simply connected iterated tilted algebra from \( kQ \) then \( D^b(A) \cong D^b(E(A)) \) and \( D^b(E(A)) \cong D^b(kQ') \).

Proof: We recall from [18] that monomial quadratic algebras are Koszul algebras. Hence the first item follows straightforward from the Theorem 2.5 since \( Q \) is a tree. The first equivalence in the second item follows from Theorem 2.5 and the second equivalence follows from the hypothesis over \( A \). ■

It follows from Theorem 2.5 that if \( A \) is a gentle algebra whose quiver is a tree then \( A \) and its Yoneda algebra \( E(A) \) are both derived equivalent to an hereditary algebra of type \( A_n \). We notice that the quiver \( Q \) and \( Q' \) presented on the corollary of the Theorem 2.5 are not equivalent in general. The example below will show that statement.

Example 2: Let the algebra \( A \) given by the quiver

\[
\begin{array}{cccc}
1 & \xrightarrow{\alpha} & 2 & \xrightarrow{\beta} & 3 \\
& & \downarrow{\gamma} & & \\
& & 4 & &
\end{array}
\]

with \( \alpha \beta = 0 \). We have \( A \) is an iterated tilted algebra of type \( A_4 \) and \( Q_A \neq A_4 \).

3. Derived discrete Koszul algebras.

In this section we study Koszul algebras having derived discrete categories. An important tool for us will be the characterization given in [9] for these the algebras.

We recall from [9], that \( A \) has discrete derived category if and only if \( A \) is derived equivalent to a hereditary algebra of Dynkin type or \( D^b(A) \cong D^b(A(r, n, m)) \). Moreover the algebras \( A(r, n, m) \) and \( A(s, n', m') \) are derived equivalent if and only if \( r = s, n = n' \) and \( m = m' \), see also [24]. It also follows that if \( A \) has discrete derived category and \( B \) is derived equivalent to \( A \) then \( A \) is of Dynkin type if and only if \( B \) is of Dynkin type.
We recall that $A(r, n, m)$ is a quotient of the path algebra over the cycle $C(n, m)$, as described in the introduction. In order to prove the main result on this section we will need the following lemma.

**Lemma 3.1.** Let $A = A(r, n, m)$ be a discrete Koszul gentle algebra. Then $E(A)$ is a discrete Koszul gentle algebra if and only if $|m - n + r| > 0$. Furthermore, in this case, $A$ and $E(A)$ are derived equivalent if and only if $m = n$.

**Proof:** Let $A = kC(n, m)/I(r)$ where $I(r)$ is generated by $r$ consecutive monomial quadratic relations from the vertex $(n - r)$ to the vertex zero. Therefore $E(A) = kC(m, n)/I(s)$ where $I(s)$ is the ideal generated by $(m + 1)$ consecutive monomial quadratic relations from the vertex zero to the vertex $(- m)$ and $(n - r + 1)$ consecutive monomial quadratic relations from vertex $(n - r)$ to the vertex $(- m)$. We recall [10] to have $E(A)$ derived equivalent to the algebra $A(r', m, n)$ where $r' = |m - n + r|$. Hence $E(A)$ is a discrete Koszul gentle algebra when $r' > 0$, otherwise $E(A)$ will be an iterated tilted algebra of type $\tilde{A}_n$, hence it is not a discrete koszul gentle algebra. We now observe that the algebras $A(r, n, m)$ and $A(s, m, n)$ are derived equivalent if and only if $r = s$ and $m = n$. Since $r' = |m - n + r|$ it follows that $r = r'$ if and only if $m = n$. Hence $A$ and $E(A)$ are derived equivalent if and only if $m = n$.

The next example illustrates the result above.

**Example 3:** Let $A = kQ/I$ where the quiver $Q$ is given by

\[
\begin{array}{ccc}
1 & \overset{\alpha}{\rightarrow} & 2 \\
\downarrow{} & & \downarrow{} \\
3 & & 4
\end{array}
\]

and the ideal $I$ is generated by the relations $\alpha\beta$ and $\beta\gamma$. The algebra $A$ is derived equivalent to the algebra $A(2, 3, 2)$, and $E(A)$ is derived equivalent to $A(1, 3, 2)$, accordingly to the classification in [10], moreover $m - n + r = 1$. We also observe that $A$ is derived equivalent to a hereditary algebra and $E(A)$ is not. This fact shows that, even in the derived discrete case not always exists a derived equivalence between the Koszul algebra and its Yoneda algebra.

We shall now present the main result of this section. We will characterize when there exists a derived equivalence between a derived discrete Koszul algebra and its Yoneda algebra.

**Theorem 3.2.** Let $A$ be a finite global dimension Koszul algebra having a discrete derived category. Then $A$ and $E(A)$ are derived equivalent if and only if $A$ is derived equivalent to a hereditary algebra of Dynkin type or $A$ is derived equivalent to $A(r, n, n)$.

**Proof:** If $A$ is of Dynkin type algebra, that is, $A = kQ/I$ where $Q$ is given by a Dynkin diagram then $A$ is a simply connected algebra and the result follows from Theorem 2.5. We will follow the technics and arguments presented in [10] to complete the proof of our result.
The case when $A = A(r, n, m) = kC(n, m)/I(r)$ for $C(n, m)$ a cycle as defined above and the graded ideal $I(r)$ generated by $r$ consecutive monomial quadratic clockwise relations from the vertex $(n - r)$ to the vertex zero is proved on the Lemma 3.1 given above.

The next case we will consider is given by $A = kC(n, m)/I(r)$ with $I(r)$ generated by $r^+$ monomial quadratic clockwise relations and $r^-$ monomial quadratic counterclockwise relations. It follows from [10] that $A$ is derived equivalent to the discrete algebra $B = kC(m, n)/I(r')$ where $I(r')$ is generated by $r' = |r^+ - r^-|$ monomial consecutive clockwise relations from the vertex $(n - r)$ to zero. On the other hand we have $E(A) = kC(m, n)/I(s)$ where $I(s)$ is the ideal generated by $(m - r^-)$ monomial quadratic clockwise relations and $(n - r^+)$ monomial quadratic counterclockwise relations hence we have from [10] that $E(A)$ is derived equivalent to the discrete algebra $B' = kC(m, n)/I(s')$ where $I(s')$ is generated by $s' = |m - n + r|$ monomial clockwise consecutive relations. It follows that $A$ and $E(A)$ are derived equivalent if and only if $B$ and $B'$ are derived equivalent, and that happens exactly when $r' = s'$ and $m = n$, if and only if $m = n$.

Finally, we may consider the general case where $A$ has exactly one non-oriented cycle and branches connected to that cycle. We observe that $E(A)$ has the same underlying graph. We take $A = kQ/I$ and $E(A) = kQ^{op}/I'$ where $I'$ is the quadratic dual ideal of $I$. We have from [10] that $A$ is derived equivalent to the discrete algebra $B_1 = kQ/I_1$ where $I_1$ is generated by $I$ minus all relations at the branches of $Q$ and the monomial quadratic paths connecting the cycle of $Q$ with the branches belonging to $I$. We follow the construction of $E(A)$ presented in [15] to have $E(A)$ derived equivalent to $B_1 = kQ^{op}/I'_1$ where $I'_1$ is generated by $I'$ minus all relations at the branches of $Q^{op}$ in the same sense given to ideals $I$ and $I_1$ above.

We will follow the same steps presented in [10] to obtain the classification of discrete algebras. One may eliminate the relations from the cycle of $Q$ to its branches and vice-versa using the technics presented in [10]. Hence we will obtain a derived equivalent algebra given by one non-oriented cycle with relations on it and branches leaving or reaching the middle points of that relations on the cycle. On may also obtain all branches toward the cycle. We apply that procedure to $A$ and $E(A)$ at the same time to obtain $A$ and $E(A)$ derived equivalent to $B_2 = kQ'/I_2$ and $B_2' = kQ'^{op}/I'_2$ respectively with $I_2$ and $I'_2$ generated by $I_1$ and $I'_1$ minus relations connecting branches and the cycle of $Q$ and $Q^{op}$ respectively.

The next step will introduce branches inside the cycle. Thus we will obtain $A$ derived equivalent to $B_3 = kC(n, m)/I_3$ where $C(n, m)$ is a non-oriented cycle having $n$ clockwise arrows, $m$ counterclockwise arrows and $I_3$ is generated by $r^+$ monomial quadratic clockwise relations and $r^-$ monomial counterclockwise relations. Moreover $A$ is derived equivalent to $kC(n, m)/I(r)$ where $r = |r^+ - r^-|$. One may apply the same procedure to obtain $E(A)$ derived equivalent to $kC(m, n)/I(s)$ where $s = |m - n + r|$. It follows that $A$ and $E(A)$ are derived equivalent if and only if $kC(n, m)/I(r)$ and $kC(m, n)/I(s)$ are derived equivalent. It is possible exactly when $A(r, n, m)$ and $A(s, m, n)$ are derived equivalent, that is, exactly when $r = s$ and $m = n$. Hence $A$ and $E(A)$ are derived equivalent exactly when $m = n$. ■

We give now two examples related to our previous result.

Example 4 : $(m = n)$Let the algebra $A = kQ/I$ with $Q$ given by
with relations $\beta \gamma = \theta \eta = 0$.

We observe that $A$ is derived equivalent to $A' = kC(4,4)/I$ where the cycle $C(4,4)$ is given by the following quiver

and $I$ is generated by the relations $\alpha \beta = \beta \gamma = \theta \eta = 0$. Hence $A$ is derived equivalent to the algebra $A(1,4,4)$. 

**Example 5:** (2-cycles gentle algebras.) We consider $A = kQ/I$ with $Q$ given by

with relation $\alpha \beta = \beta \alpha = 0$. We observe that its Yoneda algebra is not a finite dimensional algebra.

4. **Koszul algebras derived equivalent to hereditary algebras**

In this section we study Koszul algebras which are derived equivalent to hereditary algebras. In the case that the hereditary algebra is tame, we are are able to characterize when these Koszul algebras are derived equivalent to their Yoneda algebras. We start the section showing that a quadratic algebra whose quiver is a tree is derived equivalent to a hereditary algebra.

**Proposition 4.1.** Let $A = kQ/I$ be a finite dimensional quadratic algebra with $Q$ a tree. Then $A$ is derived equivalent to an hereditary algebra.

**Proof:** One may follows the same procedure as presented in [11] in order to obtain an iterated tilted algebra from $A$ which is an hereditary algebra. For the sake of the reader we will presented the main steps to prove our result.
Let assume that we have a relation \( \rho \) ending on a vertex associated to a projective simple \( A \)-module. One may consider the APR-tilting module associated to that simple \( A \)-module. Then the tilted algebra obtained is a quotient of a path algebra by an ideal in \( \{I - \rho\} \). One may apply that procedure for each relation such that there is not another relation begining on the middle vertex of that relation in order to reduce the ideal of relations of \( A \). Since \( A \) has no cycles we will obtain an hereditary algebra as an iterated tilted algebra from \( A \). Our assertion will follow from [19].

Remark: We would like to observe that the quadratic hypothesis in the former proposition is essential. Since for any \( n \geq 13 \) the algebra whose quiver is the linearly ordered \( A_n \) with the relations generated by the set of all paths of length 3 is not derived equivalent to a hereditary algebra, as it is shown in [20].

Corollary 4.2. Let \( A = kQ/I \) be a Koszul algebra with \( Q \) a tree. Then \( A \) and \( E(A) \) are derived equivalent to the same hereditary algebra.

Proof: Since \( A \) is a Koszul algebra it is a quadratic algebra hence this result will follow from the proposition above and Theorem 2.5.

Remark: We observe that a Koszul algebra \( A = kQ/I \) with \( Q \) a tree is not derived equivalent to \( kQ \), in general. We refer to the reader the example 2 given above to illustrate that fact.

Proposition 4.3. Let \( A \) be Koszul algebra derived equivalent to a hereditary algebra \( H = kQ \). If \( A \) is simply connected then \( Q \) is a tree and \( A \) is derived equivalent to its Yoneda algebra.

Proof: Since \( A \) is a simply connected algebra it follows from Thm. 2.5 that \( E(A) \) is derived equivalent to \( A \). The fact that \( Q \) is a tree follows from [21].

A natural question to investigate, in the above situation, is given by the following question. If the Koszul algebra \( A \) is a tilted algebra of type \( Q \), then its Yoneda algebra \( E(A) \) is also a tilted algebra of type \( Q \). We found a negative answer to that question. The next example exhibit a Koszul algebra tilted from a hereditary algebra \( Q \) whose Yoneda algebra is not a tilted algebra of \( Q \).

Example 6: Let \( H = kQ \) be the path algebra whose quiver is

\[
1 \rightarrow 3 \rightarrow 4 \rightarrow 5
\]

Let \( A \) the quotient of \( kQ \) by the ideal generated by the relation \( \alpha \beta \). We observe that \( A \) is an endomorphism ring of the tilting \( H \)-module \( T = P_1 \oplus P_2 \oplus P_3 \oplus P_5 \oplus \tau^{-1}S_4 \). We have that \( E(H) = kQ^{op}/r^{2}_{op} \), where \( r^{2}_{op} \) is the radical square of \( kQ^{op} \) and \( E(A) \) the quotient of the path algebra whose quiver is

\[
1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \leftarrow 5,
\]

by the ideal generated by the relations \( \theta^{op} \gamma^{op} = \gamma^{op} \theta^{op} \). Hence \( E(A) \) is not a tilted algebra since it has global dimension three.
We now consider Koszul algebras which are derived equivalent to a hereditary algebra of Euclidean type. It follows from [4] and [21] that if $A$ is simply connected and derived equivalent to a hereditary algebra $H = kQ$, then $Q$ is a tree. In particular if $H$ is of Euclidean type and $A$ is not simply connected, then $Q$ is of type $\tilde{A}_n$. We shall study Koszul algebras which are iterated tilted algebra of type $\tilde{A}_n$. We recall that the quiver of these algebras have exactly one non-oriented cycle and their generating ideal satisfy the clock condition on the cycle, as described on the previous section. We also observe that these algebras do not have discrete derived category. We will say that an algebra $A$ is combed when its Gabriel Quiver is a cycle with exactly one sink and one source. It follows that if the Gabriel quiver of some algebra $A$ is given by the non-oriented cycle $C(n,m)$ hence $A$ is combed, (section 3).

The following proposition is an important tool to prove our main result on this section.

**Proposition 4.4.** An algebra $A$ is a monomial quadratic algebra over the cycle $C(n,m)$ with $n = m$ satisfying the clock condition if and only if $E(A)$ satisfies these same properties.

**Proof:** Since $A = kC(n,m)/I$ is a combed quadratic monomial algebra we have that $E(A)$ is also a combed quadratic monomial algebra over the non-oriented cycle $C(n,m)$. We recall that the relations in $E(A)$ are generated in $kQ$ by all monomial quadratic paths which are not in $I$. It follows that $E(A)$ satisfies the clock condition if and only if $A$ satisfies that same condition. \[\Box\]

**Corollary 4.5.** If $A$ is a monomial quadratic algebra with Gabriel quiver given by the non-oriented cycle $C(n,n)$ satisfying the clock condition, then $D^b(A) \cong D^b(E(A))$.

**Proof:** We have from the Proposition 4.4 that $A$ and $E(A)$ are both combed Koszul algebras over the cycle $C(n,n)$ satisfying the clock condition. Furthermore, that cycle has the same orientation for both algebras. It follows from [3] that both are derived equivalent to the same hereditary algebra of type $\tilde{A}_s$ for some suitable $s$. Hence they are derived equivalent. \[\Box\]

The following example shows that the hypothesis on the cycle $C(n,m)$ satisfying $n = m$ can not be dropped on the result above.

**Example 7:** Let the algebra $A = kQ/I$ where $Q$ is given by

\[
\begin{array}{ccc}
1 & \xrightarrow{\alpha} & 2 \xrightarrow{\beta} & 3 \\
\downarrow^{\theta} & & \downarrow^{\gamma} & \\
5 & \xrightarrow{\eta} & 4
\end{array}
\]

and $I$ generated by the relations $\alpha\beta$ and $\eta\theta$. We have that the Yoneda algebra of $A$ has presentation $E(A) = kQ/ \langle \beta\gamma \rangle$. We observe that $E(A)$ has discrete derived category and $A$ is iterated tilted of type $\tilde{A}_4$. 
We will apply the results above to obtain the conditions for a derived equivalence between a Koszul algebras and its Yoneda algebra when the Gabriel quiver of that algebras have a unique non-oriented cycles $C$ satisfying the clock condition. We observe that those algebras are not combed algebras, in general. We will denote $C$ by $C_{(n,m)}$ to identify the non-oriented cycle with $n$ clockwise arrows and $m$ counterclockwise arrows, not necessarily with exactly one sink and one source. We shall consider the case where the underlying quiver of the gentle algebra $A$ is any cycle $C_{(n,m)}$. We present the following result.

**Proposition 4.6.** Let $A$ be Koszul gentle algebra whose Gabriel quiver is exactly one non-oriented cycle $C_{(n,m)}$ satisfying the clock condition. Then we have the following.

1. If $n = m$ then $D^b(A) \cong D^b(E(A))$.
2. If $n \neq m$ then $D^b(E(A)) \cong D^b(A(s,n,m))$, that is, $E(A)$ is derived discrete.

**Proof:** We observe that under the hypothesis over $A$ we have $D^b(A) \cong D^b(A')$ where $A'$ is a monomial quadratic combed algebra with the non-oriented cycle $C(n,m)$ satisfying the clock condition.

We assume that $n = m$. Following the same arguments given by the proof of the Proposition 4.4 one may conclude that $E(A)$ is a monomial quadratic algebra having the same underlying quiver given by the non-oriented cycle $C_{(n,m)}$ satisfying the clock condition. It follows that $E(A)$ is derived equivalent to $A'$ and our proof of the first item is complete.

We now assume that $n \neq m$. We recall that $A = kC_{(n,m)}/I$ with $I$ generated by $r$ monomial quadratic clockwise relations and $r$ monomial quadratic counterclockwise relations. Hence, $E(A) = kC_{(n,m)}/I'$ where $I'$ is the ideal generated by $(m' - r)$ monomial quadratic clockwise relations and $(n' - r)$ monomial quadratic counterclockwise relations, where $m'$ and $n'$ are the number of the paths of length 2 in $C_{(n,m)}$ which are not in $I$, on the clockwise and counterclockwise directions, respectively. We observe that $m' = n'$ if and only if $m = n$. It follows from [10] that $E(A)$ is derived equivalent to the discrete algebra $A(s,n,m)$ where $s = |m' - n'|$.

**Corollary 4.7.** Let $A$ be a Koszul algebra derived equivalent to an iterated tilted algebra of type $A_s$ over the non-oriented cycle $C_{(n,m)}$. Then $A$ is derived equivalent to its Yoneda algebra $E(A)$ is and only if $n = m$.

**Proof:** Since $A$ is derived equivalent to an iterated tilted algebra of type $A_s$, we know from [3] that $A$ is a gentle algebra having the unique non-oriented cycle $C$ on $Q_A$ satisfying the clock condition.

The quiver $Q_A$ of $A$ also can be described as a branch enlargement of a non-oriented cycle $C_{(p,q)}$, for some pair $(p,q)$ where the $l$ branches are iterated tilted algebras of type $A_{r+1}$, or equivalently, gentle algebras of tree type. The arrows $\alpha$ in the quiver $Q_A$ whose link each branch with the cycle $C$ could point to the cycle, or opposite to it. The new relations on the cycle $C_{(n,m)}$ could be a zero relation of type $i\alpha \beta (\beta \alpha)$, with $\beta$ an arrow of the cycle if $\alpha$ point to the cycle (or opposite to the cycle), or $\alpha$ is not involved in any relation with arrows of the cycle if the target point of the arrow $\alpha$ is a middle point of a zero relation involving arrows of the cycle $C$, say ii) $\beta \gamma$, where $\beta$ and $\gamma$ are arrows in that cycle. Thus one may
obtain from $A$ an iterated tilted algebra of type $\hat{A}_s$ for $s = lr + p + q - 1$ given by a quotient of the path algebra over the non-oriented cycle $C_{(n,m)}$ for some pair $(n, m)$ and a monomial quadratic ideal $I$ satisfying the condition presented on section 1.

One may describe the Yoneda algebra of $A$ in the same way. We get that branches on $Q_{E(A)}$ are still gentle algebras of tree type, and relations of type i) become relations of type ii), and conversely relations of type ii) become of type i). We recall that the unique non-oriented cycle of $E(A)$ is the same cycle $C_{(p,q)}$. Thus $E(A)$ is derived equivalent to an iterated tilted algebra of type $\hat{A}_s$, for the same $s$ as we found for $A$ over the non-oriented cycle $C_{(n,m)}$ if and only if that cycle satisfies the clock conditions over the Yoneda algebra. Hence the result follows from the Proposition 4.4 and Proposition 4.6 above, when we have $n = m$.}

We will illustrate the result on the corollary above with the next example. **Example 8:** Let the algebra $A = kQ/I$ with $Q$ the quiver given on Example 4 and $I$ generated by the relations $\beta\gamma = \theta\eta = \eta\psi = 0$. Hence we have $A$ derived equivalent to an iterated tilted algebra of type $\hat{A}_s$ whose quiver is the cycle $C(4,4)$ and relations are given by $\alpha\beta = \beta\gamma = 0 = \theta\eta = \eta\psi$.

Now we may state the main theorem of this section.

**Theorem 4.8.** Let $A$ be a Koszul algebra derived equivalent to a hereditary algebra $H = kQ$. Then the following statements hold.

a) If $A$ is simply connected, then $A$ is derived equivalent to its Yoneda algebra $E(A)$, and $Q$ is a tree.

b) If $H$ is of Euclidean type, then $A$ is derived equivalent to its Yoneda algebra $E(A)$ if and only if $A$ is simply connected or $A$ is derived equivalent to an iterated tilted algebra of type $\hat{A}_s$ over the non-oriented cycle $C_{(n,n)}$.

**Proof:** The item a) follows from 4.3. For the proof of the item b) since $A$ is not simply connected, and $H$ is tame, it follows from [4], and [21] that $A$ is derived equivalent to an iterated tilted of type $\hat{A}_s$. Then the result follows from 4.7.

**Acknowledgments:** We would like to thank E.L.Green for the initial questions and motivation.

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