Online Load Balancing on Unrelated Machines with Startup Costs

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Abstract

Motivated by applications in energy-efficient scheduling in data centers, Khuller, Li, and Saha introduced the machine activation problem as a generalization of the classical optimization problems of minimum set cover and minimum makespan scheduling on parallel machines. In this problem, a set of $n$ jobs have to be distributed among a set of $m$ (unrelated) machines, given the processing time of each job on each machine. Additionally, each machine incurs a startup cost if at least one job is assigned to it. The goal is to produce a schedule of minimum total startup cost subject to a constraint $L$ on its makespan. While Khuller et al considered the offline version of this problem, a typical scenario in scheduling is one where jobs arrive online and have to be assigned to a machine immediately on arrival. We give an $(O(\log mn) \log m), O(\log m))$-competitive randomized online algorithm for this problem, i.e. the schedule produced by our algorithm has a makespan of $O(L \log m)$ with high probability, and a total expected startup cost of $O(\log mn \log m)$ times that of an optimal offline schedule with makespan $L$. Our algorithm is almost optimal since it follows from previous results that the two approximation factors cannot be improved to $o(\log m \log n)$ (under standard complexity assumptions) and $o(\log m)$ respectively.

Our algorithms use the online primal dual framework introduced by Alon et al for the online set cover problem, and subsequently developed further by Buchbinder, Naor and co-authors in various papers. To the best of our knowledge, all previous applications of this framework have been to linear programs (LPs) with either packing or covering constraints. One novelty of our application is that we use this framework for a mixed LP that has both covering and packing constraints. We combine the packing constraint with the objective function to design a potential function on the machines that is exponential in the current load of the machine and linear in the cost of the machine. Then, we create a dynamic order of machines based on this potential function and assign larger fractions of the job to machines that appear earlier in this order. This allocation is somewhat unusual in that the increase in load on a machine is inverse in the value of this potential function itself, i.e. inverse exponential in the current load on the machine. Finally, we show that we can round this fractional solution online using a randomized algorithm. We hope that the algorithmic techniques developed in this paper to simultaneously handle packing and covering constraints will be useful for solving other online optimization problems as well.
1 Introduction

In recent times, the emergence and widespread use of large-scale data centers with massive power requirements has elevated the problem of energy-efficient scheduling to one of paramount importance (see e.g. [6]). A natural strategy for achieving energy savings is that of partial shutdown, i.e. only a subset of machines/processors are active at any point of time. This immediately leads to the following scheduling question: which set of machines should be activated to serve a given set of jobs? Note that such a schedule must address twin objectives:

- The total cost (e.g. in terms of energy consumption) of all machines used in the schedule must be small (thereby achieving energy efficiency).
- The sum of processing times of all jobs assigned to any machine must be small (thereby satisfying throughput requirements).

Motivated by this application, Khuller, Li, and Saha [10] introduced the machine activation problem that involves scheduling jobs to machines so as to minimize the cost of the machines used in the schedule, while ensuring that the “load” on any machine is small. Observe that treating each of these objectives individually leads to classical problems in combinatorial optimization, namely minimum set cover and minimum makespan scheduling on parallel machines, that have been extensively studied over the last thirty years. The novelty of the algorithm proposed in [10] for the machine activation problem lies in being able to handle both objectives simultaneously.

More formally, let $M$ be a set of $m$ machines and $J$ be a set of $n$ jobs, where the processing time of job $j$ on machine $i$ is $p_{ij} > 0$. Further, machine $i$ has startup cost $c_i$. A schedule is defined as an assignment $S : J \rightarrow M$ of jobs to machines; we denote the set of jobs assigned to machine $i$ in schedule $S$ by $J_i^{(S)}$. The set of active machines $M_A^{(S)}$ in schedule $S$ are the machines to which at least one job has been assigned, i.e. $M_A^{(S)} = \{ i \in M : J_i^{(S)} \neq \emptyset \}$, and the cost of schedule $S$ is defined as $\sum_{i \in A^{(S)}} c_i$. The load $\ell_i^{(S)}$ on machine $i$ is the sum of processing times of all jobs assigned to machine $i$, i.e. $\ell_i^{(S)} = \sum_{j \in J_i^{(S)}} p_{ij}$, and the makespan $\ell^{(S)}_{\max}$ is the maximum load on a machine, i.e. $\ell^{(S)}_{\max} = \max_{i \in M} \ell_i^{(S)}$. (Often, we will drop the superscript $(S)$ in the above notation if the schedule is clear from the context.) The objective of the machine activation problem is to obtain a schedule of minimum cost, subject to the constraint that its makespan is at most some given value $L$.

In real-life scheduling tasks, the set of jobs is often not known in advance. This has motivated extensive algorithmic research in online scheduling problems, where the set of machines are available offline but the jobs appear online and have to be scheduled to a machine when they arrive. A natural and important question left open in [10] was to obtain an algorithm for the machine activation problem in the online model. Here, the set of machines, their individual startup costs, and the budget on the total startup cost of the machines activated by the schedule are known offline, but the jobs arrive online. The processing time of a job on each machine is also revealed on arrival of the job. The goal is to assign the arriving job to a machine such that the cost of the resulting schedule is minimized subject to the constraint that its makespan is at most $L$. We call this the online machine activation problem.

**Our Contributions.** Our main contribution is a randomized online algorithm for the machine activation problem with a bicriteria competitive ratio of $(O(\log (mn)) \log m), O(\log m))$: suppose an offline optimal schedule for an instance of our problem has cost $B$ and makespan at most $L$; then our online algorithm produces a schedule of expected cost $O(B \log (mn) \log m)$ and makespan $O(L \log m)$ with high probability.
Theorem 1. There is a randomized online algorithm for the machine activation problem that has a bicriteria competitive ratio of $(O(\log(nm) \log m), O(\log m))$.

In the minimum set cover problem, we are given a collection of subsets defined on a universe of elements. The goal is to select a minimum cost sub-collection such that every element of the universe is in at least one selected subset. If the elements appear online and the current selection of subsets must cover every element that has appeared thus far, a lower bound of $\Omega(\log m \log n)$ is known \footnote{The dual was not used explicitly in Alon et al’s original analysis. In fact, we will also not use the dual explicitly even though our algorithm can also be analyzed via the dual.} for $m$ sets and $n$ elements, under standard complexity assumptions. Since the set cover problem is a special case of the machine activation problem where the limit $L$ on the makespan of the schedule is $\infty$, the competitive ratio in the cost of the schedule for any online algorithm for the machine activation problem must be $\Omega(\log m \log n)$.

On the other hand, the (minimum makespan) scheduling problem for unrelated parallel machines is defined as that of distributing $n$ jobs among $m$ machines so as to minimize the makespan of the schedule (machines do not have cost). It can be shown using standard techniques that an online algorithm that produces a schedule of makespan at most $\alpha L$ for the online machine activation problem can be used to obtain an $O(\alpha)$-competitive algorithm for the online scheduling problem on unrelated parallel machines. It is well-known \footnote{The dual was not used explicitly in Alon et al’s original analysis. In fact, we will also not use the dual explicitly even though our algorithm can also be analyzed via the dual.} that the competitive ratio of any algorithm for the latter problem is $\Omega(\log m)$; therefore, this lower bound also holds for the online machine activation problem.

Theorem 2. No algorithm for the online machine activation problem can have a competitive ratio of $\omega(\log m)$ in the makespan. Further, under standard complexity assumptions, no algorithm for the online machine activation problem can have an approximation factor of $\omega(\log m \log n)$ in the cost of the schedule.

Our Techniques. Our algorithm draws inspiration from the techniques used to solve the online versions of the set cover problem and the scheduling problem for unrelated parallel machines. So, let us first summarize the key ideas involved in these two algorithms. For the latter problem, Aspnes et al \footnote{The dual was not used explicitly in Alon et al’s original analysis. In fact, we will also not use the dual explicitly even though our algorithm can also be analyzed via the dual.} gave an elegant solution based on the following exponential potential function: if the current load on machine $i$ is $\ell_i$, then its potential is $a^{\ell_i}$ for some constant $a$. The algorithm assigns the arriving job to a machine that suffers the minimum increase of potential, i.e. to machine $i = \arg \min_{i \in M}(a^{\ell_i + p_j} - a^{\ell_i})$. Observe that if all processing times are scaled down sufficiently and $a$ is small enough, then $a^{\ell_i + p_j} - a^{\ell_i} \approx a^{\ell_i}(a - 1)p_j$, i.e. the increase in potential function is linear in the processing time but exponential in the current load on the machine. Therefore, the algorithm favors lightly loaded machines in preference to those offering low processing times.

It was shown in \footnote{The dual was not used explicitly in Alon et al’s original analysis. In fact, we will also not use the dual explicitly even though our algorithm can also be analyzed via the dual.} that this strong “bias” for lightly loaded machines ensures that the sum of the potentials of all machines is at most $O(m)$ times that in an optimal offline schedule, thereby leading to a competitive ratio of $O(\log m)$ on the makespan of the schedule.

For the online set cover problem, Alon et al \footnote{The dual was not used explicitly in Alon et al’s original analysis. In fact, we will also not use the dual explicitly even though our algorithm can also be analyzed via the dual.} introduced a two-phase online primal dual framework that works as follows. In the first phase, the goal is to obtain a feasible fractional solution to the online instance of the problem. In each step of this phase, in response to a new constraint (i.e. a new element in the online set cover problem) that arrives online, the fractional solution is updated to preserve feasibility while ensuring that the cost incurred can be accounted for by a suitably updated dual solution. In the second phase, the fractional solution is rounded online to obtain an integer solution. It is important to note that while the two phases are presented sequentially for clarity, the algorithm needs to operate both phases (the fractional updates followed by the rounding) in response to response to the arrival of a new constraint. These two phases aim to notionally distinguish between the information-theoretic aspect of the online problem and
the computational aspect of rounding a fractional solution. In fact, Alon et al showed for \( m \) sets and \( n \) elements, this two-phase framework can be used to design an algorithm that has a competitive ratio of \( O(\log m \log n) \), where the \( O(\log m) \) factor arises in the first phase due to information-theoretic limitations of the online algorithm, and the \( O(\log n) \) factor arises in the second phase due to computational limitations encapsulated by the integrality gap of the linear program (LP) for set cover.

Minimize \( \sum_{i \in M} c_i x_i \) subject to

\[
\sum_{j \in J} p_{ij} y_{ij} \leq x_i \quad \forall i \in M \quad (1)
\]

\[
y_{ij} \leq x_i \quad \forall i \in M, j \in J \quad (2)
\]

\[
\sum_{i \in M} y_{ij} \geq 1 \quad \forall j \in J \quad (3)
\]

\[
x_i \in \{0,1\} \quad \forall i \in M \quad (4)
\]

\[
y_{ij} \in \{0,1\} \quad \forall i \in M, j \in J \quad (5)
\]

Figure 1: The integer scheduling linear program (or ISLP). In the fractional scheduling linear program (or FSLP), Eqns. (4) and (5) are relaxed to \( 0 \leq x_i \leq 1 \) for all machines \( i \in M \), and \( 0 \leq y_{ij} \leq 1 \) for all machines \( i \in M \) and jobs \( j \in J \), respectively.

This two-phase primal dual framework has since been extensively used for various online problems (see [7] for a survey); our algorithm also uses this framework. However, to the best of our knowledge, whereas all previous applications of the framework have been to LPs that have exclusively covering or packing constraints, we apply the framework to a mixed LP. Consider the integer LP formulation of our problem given in Fig. 1 (we call this the integer scheduling LP or ISLP). The variable \( x_i \) is 1 iff machine \( i \) is active, and \( y_{ij} \) is 1 iff job \( j \) is assigned to machine \( i \). In the fractional relaxation (which we call the fractional scheduling LP or FSLP), these variables are constrained to be in the range \([0,1]\) instead of Eqns. (4) and (5). Note that we have both covering (Eqn. (3)) and packing (Eqn. (1)) constraints in the ISLP/FSLP.

We interpret the online set cover algorithm as one that maintains a bound on a potential function that is linear in the cost of the active machines, whereas the online scheduling algorithm translates its objective of minimizing makespan into maintaining a bound on the value of a potential function that is exponential in the load on each machine. We design a potential function that combines both objectives: it is linear in the startup cost and exponential in the load on a machine. Our fractional algorithm preferentially assigns (fractions of) jobs to machines in a way that leads to a small increase in this potential. It should be noted that even if we use this potential function, we cannot afford to simply assign the entire job to the machine that would suffer the minimum increase in potential—such a greedy strategy can be easily shown to fail even for the special case of the online set cover problem. Instead, our algorithm creates a dynamic list of machines in increasing order of its change in potential if the current job were assigned to it, and then assigns larger fractions of the job to machines that appear earlier in the order. This allocation is also somewhat unusual in that the increase in load on a machine is inverse in the value of this potential function itself, i.e. inverse exponential in the current load on the machine. For some of the machines in the order, this might also involve increasing the fraction to which the machine is active, i.e. increasing the value of \( x_i \), if the assignment violates Eqn. (1) or (2).

The analysis of the fractional algorithm involves proving a bound on the value of the cumulative potential function over all the machines. Depending on the behavior of a fixed optimal offline solution (which is unknown to the algorithm and is used only for analysis), we classify jobs into three different categories, and prove a bound on the total increase in potential due to jobs in each individual category using three different techniques:
The increase of potential for jobs in category 1 is charged globally to the offline optimal solution using a primal dual argument (without introducing the dual solution explicitly) similar in spirit to that used by Alon et al in [1] for the online set cover problem.

We give a bound on the increase in potential for each individual job in category 2 by showing that the ratio of increase in potential to the fraction of job assigned is bounded.

We give a global bound on the increase of potential for jobs in category 3 by using a recursive argument which is similar in spirit to the one used in the online scheduling algorithm for unrelated machines by Aspnes et al [2].

The novelty of our analysis lies in being able to seamlessly combine, and non-trivially extend, the disparate techniques from [1] and [2]. Ultimately, we show that the potential of the fractional schedule produced by our algorithm is $O(m \log m)$, where the cost and makespan of the offline optimal solution are respectively $\Omega(m)$ and 1 by a suitable initial scaling. We hope that the algorithmic techniques developed in this paper to simultaneously handle packing and covering constraints will be useful for solving other online optimization problems that can be expressed as mixed LPs as well.

In the second phase of the algorithm, we use an online randomized rounding scheme to obtain an integer solution. To ensure that the expected cost of the integer schedule is bounded by that of the fractional schedule, each machine $i$ is activated with probability proportional to $x_i$. This is implemented online using standard techniques. The more challenging aspect of the rounding is the actual scheduling of jobs to active machines. The natural approach would be to assign job $j$ to machine $i$ with probability $y_{ij}$. Translated to conditional probabilities, this implies that job $j$ should be assigned to machine $i$ with probability $z_{ij} = y_{ij}/x_{ij}$ if machine $i$ is active, where $x_{ij}$ is the value of $x_i$ at the end of the update to the fractional solution for job $j$. This immediately implies that the expected load on a machine in the integer schedule is at most that in the fractional schedule. However, our goal is to obtain a bound on the makespan of the integer solution; in fact, since the events of jobs being assigned to a fixed machine are positively correlated, a bound on the expected load does not immediately yield a concentration bound on the load. Instead, we show that even if job $j$ were to be assigned to machine $i$ unconditionally with probability $z_{ij}$, the expected load on machine $i$ given by $\sum_{j \in J} z_{ij}$ would be small. This overcomes the problem of positive correlation mentioned above since the events are no longer conditioned on machine $i$ being active. We now derive concentration bounds on the load on a machine, which translates to a bound on the makespan of the integer schedule thereby proving Theorem 1.

Previous Work. Many variants of the machine scheduling (or load balancing) problem have been extensively studied in the literature. Perhaps the most celebrated result in scheduling theory is a 2-approximation for the offline minimum makespan scheduling problem for unrelated machines due to Lenstra, Shmoys and Tardos [12], which was later simplified by Shmoys and Tardos [16]. Rather surprisingly, this algorithm continues to offer the best competitive ratio for this problem (and even for several natural special cases such as the restricted assignment problem) even after more than two decades of research. In the online setting, Graham [8, 9] showed that the natural greedy heuristic achieves a competitive ratio of $2 - 1/m$ for $m$ identical machines. The competitive ratio of this problem has been subsequently improved in a series of results (see e.g. [5] and subsequent improvements). For the more general restricted assignment problem where the processing time of each job $j$ on any machine is either some value $p_j$ or $\infty$, an online algorithm having competitive ratio $O(\log m)$ was designed by Azar, Naor and Rom [4]. This algorithm was later generalized to the unrelated machines scenario by Aspnes et al [2] with the same competitive ratio. Various other models
and objectives have been considered for the load balancing problem; for a comprehensive survey, see [3] and [15]. In particular, the machine activation problem was introduced by Khuller et al in [10], where they gave an $O(2(1 + 1/\epsilon)(1 + \ln(n/OPT)), 2 + \epsilon)$-approximation algorithm for any $\epsilon > 0$. Recently, this result was extended by Khuller and Li [13] to a more general set of cost functions.

2 The Fractional Algorithm

\[
\text{Minimize } \sum_{i \in M} c_i x_i \quad \text{subject to}
\]

\[
\sum_{j \in J} p_{ij} y_{ij} \leq 6x_i L 
\]

\[
y_{ij} \leq 2x_i \quad \forall \ i \in M, \ j \in J
\]

\[
\sum_{i \in M} y_{ij} \geq 1 \quad \forall \ j \in J
\]

\[
0 \leq x_i \leq 1 \quad \forall i \in M
\]

\[
0 \leq y_{ij} \leq 1 \quad \forall i \in M, \ j \in J
\]

Figure 2: The relaxed fractional scheduling linear program (or RFSLP). Eqn 6 is enforced only for partially active machines $i$. (Note that for inactive machines, Eqs. 6 and 7 are identical.)

In this section, we will describe the online updates to the fractional solution to maintain feasibility for FSLP on receiving a new job $j$. This involves updating the values of $y_{ij}$ (denoting the fraction of job $j$ assigned to machine $i$) so as to satisfy Eqn. 5 and corresponding updates to the values of $x_i$ if Eqns. 1 or 2 is violated. In fact, we relax the constraints in FSLP in two ways. Let machine $i$ be said to be inactive, partially active or fully active depending on whether $x_i = 0$, $0 < x_i < 1$ or $x_i = 1$ respectively. First, for technical reasons, we relax Eqns. 1 and 2 to Eqns. 6 and 7 respectively (see Fig. 2). Further, we enforce Eqn. 1 only if $x_i < 1$, i.e. if machine $i$ is not fully active. The load on a fully active machine will be bounded separately in the analysis. We call this the relaxed fractional scheduling LP or RFSLP.

Before describing these updates, let us set up some conventions that we will use throughout the paper. We divide all processing times by $L$ at the outset; this allows us to assume that the makespan of the optimal solution is 1. We also assume that we know the value $\alpha$ of the optimal offline (integer) solution, i.e. the minimum startup cost of an offline assignment of jobs to machines that has makespan at most 1. This is also without loss of generality because we can guess the value of the optimal solution, doubling our guess whenever the current algorithmic solution exceeds the cost bounds that we are going to prove (thereby implying that our current guess is too small). In the following discussion, it is sufficient to know the value of $\alpha$ up to a multiplicative factor of 2, but for simplicity of presentation, we will assume that we know it exactly.

Our algorithm uses the value of $n$, which is the total number of jobs that arrive online. If this value is not known offline, each job estimates $n$ by assuming that it is the last job. We can show that using such estimates for $n$ incurs a small additive factor of $O(\log \log n)$ in the makespan, and an additive factor of $O(\log n \log(mn))$ in the cost of the schedule. However, for simplicity, the rest of the paper assumes that the value of $n$ is known offline.

We define the virtual cost of job $j$ on machine $i$ as

\[
\eta_i(j) = \begin{cases} 
  c_i a_i^{x_i-1} p_{ij}, & \text{if machine } i \text{ is fully active, i.e. } x_i = 1 \\
  c_i p_{ij}, & \text{otherwise}
\end{cases}
\]
where \(a\) is a constant that we will fix later. (Recall that \(\ell_i\) represents the load on machine \(i\), i.e. \(\ell_i = \sum_{j \in J} p_{ij} y_{ij}\).) Let \(M(j)\) denote an ordering of machines in non-decreasing order of virtual cost \(\eta_i(j)\) for job \(j\). Let \(P(j)\) denote the maximal prefix of \(M(j)\) such that \(\sum_{i \in P(j)} x_i < 1\). (Note that \(P(j)\) may be empty.) If \(P(j) \neq M(j)\), then \(k(j)\) denotes the first machine in \(M(j)\) that is not in \(P(j)\); \(k(j)\) is undefined if \(P(j) = M(j)\).

If \(x_i\) is increased to \(x_i + \Delta x_i\) for a partially active machine \(i\), then we say that the effective capacity created by this increase for job \(j\) is \(\min(2x_i, 6\Delta x_i / p_{ij})\). Note that the effective capacity created by an increase in \(x_i\) is a feasible increase in the value of \(y_{ij}\) independent of the current load on machine \(i\).

### 2.1 The Algorithm

The algorithm has two phases—an offline pre-processing phase, and an online phase that (fractionally) schedules the arriving jobs.

**Pre-processing.** We multiply the startup cost of every machine by \(\alpha / m\), and discard machines with startup cost greater than \(m\) (after the scaling) at the outset. Further, for every machine \(i\) whose startup cost is at most 1, we increase its cost to 1, and initialize \(x_i\) to 1. For all other machines with \(1 < c_i \leq m\), we initialize \(x_i\) to \(1 / m\). At the end of the pre-processing phase, we have the following properties:

- The cost of an optimal solution is between \(m\) and \(2m\).
- The cost of every machine is between 1 and \(m\).
- Every machine whose cost is 1 is fully active; all other machines have \(x_i = 1 / m\).

**Online Algorithm.** Suppose job \(j\) arrives online. We increase \(y_{ij}\)'s using the following rules repeatedly until Eqn. 8 is satisfied.

- **Type A:** \(k(j)\) is undefined (i.e. \(P(j) = M(j)\)) or \(x_{k(j)} < 1\) (i.e. machine \(k(j)\) is not fully active). We increase \(x_i\) to \(\min(x_i(1 + 1/c_i n), 1)\) for each machine \(i \in P(j)\) (and also for \(i = k(j)\) if it is defined), and correspondingly increase \(y_{ij}\) by the effective capacity created in each machine.

- **Type B:** \(x_{k(j)} = 1\) (i.e. machine \(k(j)\) is fully active). We increase \(x_i\) to \(\min(x_i(1 + 1/c_i n), 1)\) for each machine \(i \in P(j)\), and correspondingly increase \(y_{ij}\) by the effective capacity created in each machine. Further, we increase \(y_{k(j)j}\) by \(6 / \eta_{k(j)}(j) n\).

### 2.2 Analysis

Our goal is to show the following bounds on the makespan and cost of the fractional schedule.

**Lemma 1.** The fractional schedule produced by the online algorithm satisfies \(\sum_{j \in J} y_{ij} p_{ij} = O(\log m)\) for each machine \(i\), and \(\sum_{i \in M} c_i x_i = O(m \log m)\).

We introduce a potential function \(\phi_i\) for machine \(i\) defined as

\[
\phi_i = \begin{cases} 
  c_i a^{\ell_i - 1}, & \text{if machine } i \text{ is fully active, i.e. } x_i = 1 \\
  c_i x_i, & \text{otherwise.}
\end{cases}
\]

The cumulative potential function \(\phi = \sum_{i \in M} \phi_i\). Our goal will be to show that \(\phi = O(m \log m)\). This will immediately imply Lemma 1.
We prove the bound on $\phi$ in three steps. First, we bound the increase of $\phi$ in the pre-processing phase; next, we bound the increase of $\phi$ in each algorithmic step (of either type A or type B); and finally, we bound the total number of algorithmic steps.

**Pre-processing.** The next lemma bounds the increase of $\phi$ in the pre-processing phase.

**Lemma 2.** At the end of the pre-processing phase, $\phi \leq m$.

**Proof.** The startup cost of each machine that is fully active after pre-processing is 1; on the other, every partially active machine $i$ has $c_i \leq m$ and $x_i = 1/m$ after pre-processing. \hfill $\square$

**Single Algorithmic Step.** Now, we bound the increase in $\phi$ due to a single algorithmic step of either type.

**Lemma 3.** The increase in potential in a single algorithmic step of type A is at most $2/n$.

**Proof.** The total increase in potential in an algorithmic step of type A (due to increase in the value of $x_i$ for machines $i \in P(j) \cup \{k(j)\}$) is at most $\sum_{i \in P(j) \cup \{k(j)\}} c_i(x_i/c_i)n = (\sum_{i \in P(j) \cup \{k(j)\}} x_i)/n < 2/n$. \hfill $\square$

**Lemma 4.** For any constant $1 < a < 13/12$, the increase in potential in a single algorithmic step of type B is at most $2/n$.

**Proof.** The total increase in potential for machines $i \in P(j)$ due to an algorithmic step of type B is at most $\sum_{i \in P(j)} c_i(x_i/c_in) = (\sum_{i \in P(j)} x_i)/n < 2/n$. The other source of increase in potential is the scheduling of a fraction of job $j$ to machine $k(j)$, due of which the load on machine $k(j)$ increases by $6/c_{k(j)}d^{k(j)-1}$. The resulting increase of potential $\phi_{k(j)}$ is

$$\phi_{k(j)}(d^{k(j)-1}+6/c_{k(j)}d^{k(j)-1}n-d^{k(j)-1}) = c_{k(j)}d^{k(j)-1}(6/c_{k(j)}d^{k(j)-1}n-1) < \frac{12(a-1)}{c_{k(j)}d^{k(j)-1}n} < \frac{1}{n}.$$  

The penultimate inequality follows from the property that $(1+x)^{1/y} < e^{x/y} < 1 + 2x/y$, for any $y \geq x > 0$. \hfill $\square$

**Number of Algorithmic Steps.** We classify the algorithmic steps according to a fixed optimal offline (integer) schedule that we call OPT. Suppose OPT assigns job $j$ to machine OPT $(j)$, and let $M_{OPT}$ denote the machines that are active in the optimal offline schedule. The three categories are:

1. $OPT(j) \in P(j)$.
2. $OPT(j) \notin P(j)$ and OPT $(j)$ is partially active.
3. OPT $(j)$ is fully active.

The next lemma bounds the total increase in potential due to algorithmic steps in the first category above.

**Lemma 5.** The total increase in potential due to all algorithmic steps in the first category is $O(m \log m)$.

**Proof.** In any algorithmic step of the first category, the value of $x_{OPT(j)}$ either increases to $x_{OPT(j)}(1 + \frac{1}{c_{OPT(j)}n})$ or to 1. Since $x_i$ is initialized to at least $1/m$ for each machine $i$ in the pre-processing phase, there are at most $m + \sum_{i \in M_{OPT}} c_i n \log m = O(mn \log m)$ algorithmic steps of the first category. The lemma now follows from Lemmas 3 and 4. \hfill $\square$
The next lemma bounds the total increase in potential due to algorithmic steps in the second category.

**Lemma 6.** The total increase in potential due to all algorithmic steps in the second category is \( O(m) \).

**Proof.** Consider the first \( 2c_{OPT(j)}p_{OPT(j)} \) algorithmic steps in the second category for any particular job \( j \). We have two cases:

- **Case 1:** these algorithmic steps contain at least \( c_{OPT(j)}p_{OPT(j)} \) steps of type B, or
- **Case 2:** these algorithmic steps contain at least \( c_{OPT(j)}p_{OPT(j)} \) steps of type A.

In case 1, each algorithmic step of type B creates an effective capacity of \( 6/\eta_{k(j)} \) in machine \( k(j) \). Since in each such algorithmic step, \( \eta_{k(j)} \geq \eta_{OPT(j)} = c_{OPT(j)}p_{OPT(j)} \), the total effective capacity created by these algorithmic steps is at least 1.

In case 2, let \( R(j) \) denote the set \( P(j) \cup \{ k(j) \} \) for the last of these algorithmic steps. (Note that since \( OPT(j) \notin P(j) \), \( k(j) \) is defined.) Further, let \( x_i^{(1)} \) and \( x_i^{(2)} \) respectively denote the value of \( x_i \) for machine \( i \) before the last algorithmic step for job \( j \), and after the last algorithmic step. For each machine \( i \in R(j) \), \( x_i \) has been increased in each of at least \( c_{OPT(j)}p_{OPT(j)} \) algorithmic steps of type A. Thus, for each machine \( i \in R(j) \),

\[
x_i^{(2)} \geq x_i^{(1)} \left( 1 + \frac{1}{c_i n} \right)^{c_{OPT(j)}p_{OPT(j)}} \geq x_i^{(1)} \left( 1 + \frac{1}{c_i n} \right)^{c_{OPT(j)}p_{OPT(j)}} \geq x_i^{(1)} \frac{2c_{OPT(j)}p_{OPT(j)}}{c_i} \geq x_i^{(1)} \frac{2p_{ij}}{p_{ij}} \tag{11}
\]

since \( c_{OPT(j)}p_{OPT(j)} \geq c_i p_{ij} \) for all machines \( i \in R(j) \). The total effective capacity created in these steps is

\[
\frac{6(x_i^{(2)} - x_i^{(1)})}{p_{ij}} \geq \frac{6x_i^{(2)}(1 - 2^{-p_{ij}})}{p_{ij}} \geq \frac{6x_i^{(2)} - 2}{2} > x_i^{(2)}.
\]

The first inequality follows from Eqn. 11 while the second inequality follows from the observation that for any \( z \leq 1 \), we have \( (1 - 2^{-z})/z \geq [1 - (1 - z + z^2/2)]/z = 1 - z/2 \geq 1/2 \). Hence, the effective capacity created by these algorithmic steps is at least \( \sum_{i \in R(j)} x_i^{(2)} \geq 1 \). The lemma now follows from Lemmas 5 and 4 coupled with the fact that \( \sum_{j \in J} c_{OPT(j)}p_{OPT(j)} \leq 2m \). \( \square \)

Finally, we bound the total increase of potential due to algorithmic steps in the third category.

**Lemma 7.** For any \( 1 < a < 13/12 \), the total increase in potential due to all algorithmic steps in the third category is at most \( 2 + (2/3) \sum_{i \in M_A} c_i a^{L_i-1} \), where \( L_i \) is the final load on machine \( i \) in the schedule produced by the algorithm and \( M_A \) is the set of machines that are fully activated by the algorithm.

**Proof.** First, we consider an algorithmic step of type A. In such a step, \( k(j) \) must be defined since \( OPT(j) \notin P(j) \); thus, \( \sum_{i \in P(j) \cup \{ k(j) \}} x_i \geq 1 \). Further, for each machine \( i \in P(j) \cup \{ k(j) \} \), we have \( c_i p_{ij} = \eta_i \leq \eta_{OPT(j)} \). Thus, the fraction of job \( j \) assigned in this algorithmic step is at least

\[
\sum_{i \in P(j) \cup \{ k(j) \}} \min \left( \frac{6x_i}{c_i p_{ij} n}, 2x_i \right) \geq \min \left( \frac{3}{\eta_{OPT(j)}(j)n}, 1 \right),
\]

since \( \sum_{i \in P(j) \cup \{ k(j) \}} x_i \geq 1 \). We first consider the situation where the whole of job \( j \) was assigned in this algorithmic step. In this case, the increase in potential due to job \( j \) is \( \sum_{i \in P(j) \cup \{ k(j) \}} (c_i x_i)/n < 2/n \); cumulatively for all jobs, such increases in potential add up to at most 2. Otherwise, the sum of increase in
over all machines in this algorithmic step is at least $3/\eta_{OPT}(j)n$. Now, consider an algorithmic step of type B. Then the increase in $y_{k(j)}$ is $6/\eta_{k(j)}n \geq 6/\eta_{OPT}(j)n$ since $\eta_{OPT}(j) \geq \eta_{k(j)}$. In either case, the total number of algorithmic steps in the third category for job $j$ is at most $\eta_{OPT}(j)n/3$.

Note that $\eta_{OPT}(j) \leq c_{OPT}(j)a_{OPT}(j) - 1p_{OPT}(j)j$. Therefore, summing over all jobs, the total increase in potential due to algorithmic steps in the third category is bounded by (using Lemmas 3 and 4)

$$2 + 2\sum_{j \in J} c_{OPT}(j)a_{OPT}^{-1}p_{OPT}(j)j \leq 2 + \frac{2}{3}\sum_{i \in M_{OPT} \cap M_A} c_i a_i^{-1} \leq 2 + \frac{2}{3}\sum_{i \in M_A} c_i a_i^{-1},$$

where the penultimate inequality follows from the fact that the makespan of the optimal offline schedule is at most 1.

Finally, we bound the total potential $\phi$; this immediately yields Lemma 8.

**Lemma 8.** The online fractional algorithm produces a schedule that satisfies $\phi = O(m \log m)$.

**Proof.** Lemmas 2, 5, 6 and 7 imply $\phi \leq O(m \log m) + (2/3)\phi$, which proves the lemma.

### 3 The Online Randomized Rounding Procedure

In this section, we give an online randomized rounding scheme for the fractional solution produced by the algorithm in the previous section.

#### 3.1 The Algorithm

For each machine $i$, we select (offline) a number $r_i$ uniformly at random and independently from $[0, 1]$. In response to a new job $j$ arriving online, the algorithm updates the schedule in three steps:

- **Fractional step.** The fractional schedule is updated according to the algorithm described in the previous section.

- **Activation step.** Each inactive machine $i$ that satisfies $r_i \leq 5x_i(j) \ln(mn)$ is activated.

- **Assignment step.** Let $M_A(j)$ be the set of active machines in the integer solution. Let $z_{ij} = y_{ij}/2x_i(j)$ if $x_i(j) < 1/5 \ln(mn)$ and $z_{ij} = y_{ij}$ otherwise. Let $q_{ij}$ be the normalized probability proportional to $z_{ij}$ in a distribution defined on the set of machines $M_A(j)$. We assign job $j$ to machine $i$ with probability $q_{ij}$.

#### 3.2 Analysis

The next lemma is an immediate consequence of the fact that machine $i$ is active in the integer solution with probability $\min(5x_i \ln(mn), 1)$.

**Lemma 9.** The total startup cost of all machines activated in the integer schedule is $O(m \log m \log(mn))$ in expectation.
Proof. The expected startup cost of machine $i$ in the integer schedule is at most $5c_i x_i \ln(mn)$; the lemma now follows from Lemma 1 and linearity of expectation.

To bound the makespan of the integer solution, we first bound the probabilities $q_{ij}$. (This lemma also shows that with high probability, at least one machine is active for even one job, thereby proving correctness of the algorithm.)

**Lemma 10.** With probability at least $1 - 1/m$, $q_{ij} \leq z_{ij}$ for all machines $i$ and jobs $j$.

Proof. We show that $\sum_{j \in \mathcal{M}_i(j)} z_{ij} \geq 1$ with probability at least $1 - 1/mn$ for any job $j$; the lemma then follows using the union bound over all jobs. We classify machines into $M_1(j)$ and $M_2(j)$ depending on whether or not $x_i(j) \geq 1/5 \ln(mn)$. Every machine $i \in M_1(j)$ is also in $M_A(j)$. Therefore, the contribution of machines in $M_1(j)$ to $\sum_{j \in \mathcal{M}_i(j)} z_{ij}$ is exactly $\sum_{j \in \mathcal{M}_1(j)} y_{ij}$. On the other hand, the contribution of each machine $i \in M_2(j)$ to this sum is $y_{ij} / 2 x_i(j)$ with probability $5 x_i(j) \ln(mn)$, and 0 otherwise. Define a random variable $Z_{ij} = 1$ with probability $5y_{ij} \ln(mn) / 2$, and 0 otherwise. Since $y_{ij} / 2 x_i(j) \leq 1$,

$$
\mathbb{P} \left[ \sum_{j \in M_2(j)} z_{ij} < \sum_{j \in M_1(j)} y_{ij} \right] \leq \mathbb{P} \left[ \sum_{j \in M_2(j)} Z_{ij} < \sum_{j \in M_1(j)} y_{ij} \right] < \frac{1}{mn},$$

by Chernoff bounds (cf. e.g. [14]).

**Lemma 11.** The makespan of the integer schedule is $O(\log m)$ with probability $1 - 2/m$.

Proof. First, we prove that the load on any machine in the integer schedule is $O(\log m + \sum_{j \in \mathcal{J}} y_{ij} p_{ij})$ with probability at least $1 - 1/m^2$ conditioned on the following:

- Lemma 10 holds, i.e. $q_{ij} \leq z_{ij}$ for all machines $i$ and jobs $j$, and
- Machine $i$ is active from the outset in the integer algorithm.

Given that machine $i$ is active from the outset and $q_{ij} \leq z_{ij}$, the load on machine $i$ due to job $j$ is $p_{ij}$ with probability at most $z_{ij}$. Now,

$$\sum_{j \in \mathcal{J}} z_{ij} p_{ij} = \sum_{j \in \mathcal{J} : x_i(j) < 1/5 \ln(mn)} \frac{y_{ij} p_{ij}}{2 x_i(j)} + \sum_{j \in \mathcal{J} : x_i(j) \geq 1/5 \ln(mn)} y_{ij} p_{ij} \leq \sum_{j \in \mathcal{J} : x_i(j) < 1} \frac{y_{ij} p_{ij}}{2 x_i(j)} + \sum_{j \in \mathcal{J}} y_{ij} p_{ij}.$$ 

Let job $j'$ immediately precede job $j$ in the online order; if $j$ is the first job, $x_i(j') = 1/m$. Then,

$$\sum_{j \in \mathcal{J} : x_i(j) < 1} \frac{y_{ij} p_{ij}}{2 x_i(j)} \leq 3 \sum_{j \in \mathcal{J} : x_i(j) < 1} \frac{x_i(j) - x_i(j')} {x_i(j)} \leq 3 \sum_{j \in \mathcal{J} : x_i(j) < 1} \int_{w=x_i(j')}^{x_i(j)} \frac{dw}{w} \leq 3 \int_{1/m}^{1} \frac{1}{w} \leq 3 \ln m.$$ 

The first inequality follows from the fractional algorithm which assigns a fraction $y_{ij} \leq 6(x_i(j) - x_i(j')) / p_{ij}$ of job $j$ to machine $i$. Since $y_{ij} p_{ij} \leq p_{ij} \leq 1$ for all jobs $j$, it follows using Chernoff bounds that the load on machine $i$ is $O(\log m + \sum_{j \in \mathcal{J}} y_{ij} p_{ij})$ with probability at least $1 - 1/m^2$.

Using the union bound over all machines and Lemma 10, we can now claim that the load on machine $i$ is $O(\log m + \sum_{j \in \mathcal{J}} y_{ij} p_{ij})$ (unconditionally) for all machines $i$, with probability at least $1 - 2/m$. The lemma now follows using Lemma 1.

Finally, we note that Lemmas 9 and 11 imply Theorem 1.
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