Is the Equivalence Principle violated by Generalized Uncertainty Principles and Holography in a brane-world?

Fabio Scardigli\textsuperscript{1,\dagger} and Roberto Casadio\textsuperscript{2,\dagger}

\textsuperscript{1}Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan
\textsuperscript{2}Dipartimento di Fisica, Università di Bologna and I.N.F.N., Sezione di Bologna, via Irnerio 46, 40126 Bologna, Italy

It has been recently debated whether a class of generalized uncertainty principles that include gravitational sources of error are compatible with the holographic principle in models with extra spatial dimensions. We had in fact shown elsewhere that the holographic scaling is lost when more than four space-time dimensions are present. However, we shall show here that the validity of the holographic counting can be maintained also in models with extra spatial dimensions, but at the intriguing price that the equivalence principle for a point-like source be violated and the inertial mass differ from the gravitational mass in a specific non-trivial way.

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I. INTRODUCTION

The topic of generalized uncertainty principles (GUP) is a rather old one. Recently, it has been revived with the addition of gravitational contributions which provide a minimum length of the order of the Planck scale (for a review and examples, see Ref. \cite{1}). An attempt in this direction was taken by Ng and van Dam \cite{2} who suggested to include an error due to the space-time curvature induced by the measuring device, the latter being described, along the lines of Wigner’s 1958 paper \cite{3}, as a system made of a clock, a photon detector and a photon gun, with total mass $m$ and diameter $d = 2a$ (spherical symmetry is assumed for simplicity). A given length $l$ is then measured by timing the photon travel from the gun to a suitably placed (ideally weightless) mirror back. Photons are also supposed to be emitted in spherical waves, in order to avoid recoil and back-reaction effects on the clock’s position. This leads to a GUP which yields the remarkable consequence of suggesting that four-dimensional space-time actually contains (gravitational) degrees of freedom which scale in agreement with the holographic principle \cite{4}.

However, if one tries to extend this result to models with extra spatial dimensions \cite{5,6}, the latter property becomes questionable. It was in fact shown in Ref. \cite{7} that a straightforward extension does not work. Afterward, different authors suggested to modify the Ng and van Dam’s GUP by including yet another possible source of error \cite{8} or by using a black hole for the measuring device \cite{9}. Both attempts seem to recover the holographic counting. However, they also require the detector to satisfy rather specific (indeed peculiar) properties: either it needs to be a black hole, or its size needs to scale in a very specific way with the mass (see Ref. \cite{10} for more details).

Before we proceed, let us recall why it is sensible to place on the same footing a “fundamental” uncertainty principle such as Heisenberg’s and an (apparently) phenomenological gravitational source of error. On general grounds, one understands that in Einstein’s general relativity space-time is a dynamical concept and its quantum description must involve uncertainty. Constructions such as that of Ref. \cite{2} make it clear that the two sources of uncertainty are closely related: the photon shot by the gun moves in a Schwarzschild metric with ADM mass equal to $m$ minus the photon energy $E$ \cite{15}. Since we are timing the photon’s travel, the time-energy uncertainty relation \cite{20} implies that $E$ has an uncertainty $\Delta E \sim \hbar/\Delta t_{em}$ if $\Delta t_{em}$ is the uncertainty in the time of emission. Correspondingly, we cannot determine with infinite accuracy the length of the photon optical path, say from $r_0 > r_g$ to $r > r_0$ (in the detector’s frame), with $r_g(m) = 2G_N m/c^2$, but can just find the lower and upper bounds

$$\int_{r_0}^{r} \frac{d\rho}{1 - \frac{\rho}{r_0}} \equiv c \Delta t_{\text{max}} \gtrsim c \Delta t \gtrsim c \Delta t_{\text{min}} \equiv \int_{r_0}^{r} \frac{d\rho}{1 - \frac{\rho}{r_0}} \ , \quad (1)$$

where $r_\pm = r_g(m - E \pm \Delta E)$. Ref. \cite{2} then suggests to add to other sources of errors the uncertainty in the length of the optical path as the difference

$$\delta l_C \sim c (\Delta t_{\text{max}} - \Delta t_{\text{min}}) \ . \quad (2)$$

The aim of this paper is to take the opposite perspective with respect to some previous works and to show that the GUP of Refs. \cite{2,7} and the holographic principle can be both kept valid consistently. However, we shall then show that this leads to another principle being violated, namely the detector’s inertial mass and gravitational mass must differ in models with extra spatial dimensions. We shall write explicitly the fundamental constants $c$, $\hbar$ and Newton’s constant $G_N$ or, alternatively, the Planck length $\ell_P = (G_N \hbar/c^3)^{1/2}$ or mass $M_p = \hbar/2c\ell_P$ [respectively replaced by $G_{(n+4)}$, $\hbar_{(n+4)}$, and $\ell_{(n+4)}$].

\textsuperscript{\dagger}Electronic address: fabio@yukawa.kyoto-u.ac.jp
\textsuperscript{\dagger}Electronic address: casadio@bo.infn.it
\( \ell_{(4+n)} = (G_{(4+n)} \hbar/c^3)^{1/4} \) and \( M_{(4+n)} = \hbar/2c \ell_{(4+n)} \) in 4 + \( n \) dimensions.

II. GRAVITATIONAL GUP

Suppose we wish to measure a distance \( l \) with the detector described in the Introduction. If \( \Delta x \) is the initial uncertainty in the position of the clock, after the time \( T = 2l/c \) taken by the photon to return to the detector, the uncertainty in the actual length of the segment \( l \) becomes

\[
\Delta x_{\text{tot}} = \Delta x + T \Delta v = \Delta x + \frac{\hbar T}{2m \Delta x},
\]

where \( \Delta v \) is the uncertainty in the detector’s velocity from Heisenberg’s principle. Upon minimizing \( \Delta x_{\text{tot}} \) with respect to \( \Delta x \) we obtain Wigner’s quantum mechanical error [2]

\[
\delta l_{\text{QM}} \simeq 2 \left( \frac{\hbar l}{mc} \right)^{1/2},
\]

which we seem to be able to reduce as much as we want by choosing \( m \) very large. But gravity now gets in the way as mentioned before.

An important remark is that the measuring device cannot be a black hole, or it could not serve as a photon gun [12],

\[
a > r_g \quad \Rightarrow \quad \delta l_{\text{QM}}^2 \gtrsim 8l_p^2 \frac{l}{a}.
\]

Besides, we need now to include in the computation the gravitational error of Eq. [2] with \( r = a + l > r_g = a \gg r_g \). We consider for \( \ell_{(4+n)} \) the lower and upper bounds allowed by total energy conservation, corresponding to the two limiting cases \( E - \Delta E = 0 \) and \( E + \Delta E = m \), respectively. This yields, for distances \( l \gtrsim a \),

\[
\delta l_C \simeq r_g \log \left( \frac{a + l}{a} \right) \gtrsim r_g \log 2 \approx \frac{r_g}{2}.
\]

Note that \( \delta l_C \) increases with increasing detector’s mass and the total error becomes

\[
\delta l_{\text{tot}} = \delta l_{\text{QM}} + \delta l_C \simeq 2 \left( \frac{\hbar l}{mc} \right)^{1/2} + \frac{G_N m}{c^2}.
\]

For a given \( l \), this error can only be minimized with respect to the mass of the clock, which yields

\[
(\delta l_{\text{tot}})_{\text{min}} \simeq 3 \left( \frac{\hbar^2 l}{p^2} \right)^{1/3},
\]

for \( m = 2 M_p (l/p)^{1/3} \). The global uncertainty on \( l \) therefore contains precisely the term proportional to \( l^{1/3} \) required by the holography [21].

Unfortunately, in \( 4 + n \) dimensions this does not seem to work. When \( a + l \) is shorter than the size \( L \) of the extra dimensions, one can use the \( 4 + n \)-dimensional Schwarzschild metric [13]

\[
ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu = -F(r) \, c^2 \, dt^2 + F(r)^{-1} \, dr^2 + r^2 \, d\Omega_{n+2}^2, \quad (9)
\]

where Greek indices run from 0 to 3 + \( n \) (Latin indices will denote spatial coordinates) with

\[
F(r) = 1 - C/r^{1+n}, \quad (10a)
\]

and

\[
C = \frac{16 \pi G_{4+n} m}{(2 + n) A_{2+n} c^2}, \quad (10b)
\]

\( A_{2+n} \) being the surface area of the unit \( (2 + n) \)-sphere. Upon repeating analogous steps, one then finds [7]

\[
(\delta l_{\text{tot}})_{\text{min}} \sim \left( a^{-n} \ell_{(4+n)}^2 l \right)^{1/3}.
\]

The above expression, even in the rather ideal case \( a \sim \ell_{(4+n)} \), yields the following scaling for the number of degrees of freedom in a volume \( V \) of size \( l \),

\[
\mathcal{N}(V) = \left( \frac{l}{(\delta l_{\text{tot}})_{\text{min}}} \right)^{3+n} \sim \left( \frac{l}{\ell_{(4+n)}} \right)^{2(1+\frac{2}{n})}, \quad (12)
\]

and the holographic counting holds in four-dimensions \((n = 0)\) but is lost when \( n > 0 \).

III. GUP, HOLOGRAPHY AND THE EQUIVALENCE PRINCIPLE

Let us now point out that, beside the GUP proposed by Ng and van Dam, the result in Eq. [11] deeply relies on the use of the black hole metric [9] and its dependence on the mass \( m \). In particular, the expression for the parameter \( C \) is obtained by taking the weak field limit [13] in which the metric can be written as \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), with \( |h_{\mu\nu}| \ll 1 \) in the asymptotic region far from any source. The linearized metric \( h_{\mu\nu} \), in the harmonic gauge, obeys the Poisson equation

\[
\nabla^2 h_{\mu\nu} = -\frac{16 \pi G_{4+n}}{c^4} \bar{T}_{\mu\nu}, \quad (13)
\]

with a source \( \bar{T}_{\mu\nu} \) related to the stress-energy tensor by

\[
\bar{T}_{\mu\nu} = \left( T_{\mu\nu} - \frac{1}{2+n} \eta_{\mu\nu} T \right). \quad (14)
\]

The condition that the system be non-relativistic means that time derivatives can be considered much smaller than spatial derivatives, so that the components of the stress energy tensor can be ordered as \( |T_{00}| \gg |T_{ij}| \gg
A solution to Eq. (13) is then given by

\[ h_{\mu \nu}(x) = \frac{16 \pi G_{4+n}}{(1 + n) A_{2+n} c^4} \int_{|x|\sim |y|} \frac{\bar{T}_00(y)}{|x - y|^{2+n}} d^{3+n}y \]

\[ \simeq \frac{16 \pi G_{4+n}}{(1 + n) A_{2+n} c^4 \pi^{1+n}} \int T_\mu \nu d^{3+n}y + \frac{16 \pi G_{4+n}}{A_{2+n} c^4} \frac{x^k}{r^{3+n}} \int y^k \bar{T}_{\mu \nu} d^{3+n}y + \ldots, \]

where the approximate equality is obtained by expanding for \( r = |x| \gg |y| \). Myers and Perry define the 4 + n-dimensional ADM mass \( m \) as

\[ \int \bar{T}_{00} d^{3+n}x = m c^2, \]

so that one obtains the natural generalization of the Newtonian potential to 4 + n dimensions,

\[ h_{00} \simeq \frac{16 \pi G_{4+n}}{(2 + n) A_{2+n} c^2} \frac{m}{r^{1+n}} = \frac{C}{r^{1+n}}, \]

One can now wonder if the metric defined by Eqs. (9)-(10) can be modified in such a way that the holographic principle be fulfilled also in 4 + n dimensions, thus suitably changing the counting of degrees of freedom given in Eq. (12). In other words, we shall assume the holographic principle as a constraint to fix the form of the 4 + n-dimensional black hole metric. Of course, this new metric must still satisfy the 4 + n-dimensional Einstein equations (13), which is a very strong constraint and it seems therefore sensible to change the original metric as little as possible. On the other hand, we note that the Myers-Perry solution exhibits a complete 3 + n-dimensional spherical symmetry, which means that it ignores the weight of the brane \( \bar{T}_{\mu \nu} \). All things considered, the deformation of the metric (9)-(10) which we shall use consists in allowing for a departure from a linear relation between the inertial mass and the gravitational ADM mass of the form

\[ \int \bar{T}_{00} d^{3+n}x = M_{(4+n)} c^2 \left( \frac{m}{M_{(4+n)}} \right)^{\gamma(n)}, \]

where \( \gamma \) is a (yet) unspecified function of \( n \). Although this ansatz is not the only one that can in principle be conceived, it really is one of the simplest possible, as Eq. (13) means that the gravitational mass \( M \) and inertial mass \( m \) of the source (the detector) are related by

\[ M = M_{(4+n)} \left( \frac{m}{M_{(4+n)}} \right)^{\gamma(n)}. \]

The equivalence principle would thus be violated for any function \( \gamma \neq 1 \).

Eq. (19) yields a total error in length measurements given by

\[ \delta l_{\text{tot}} = \delta l_{\text{QM}} + \delta l_{\text{C}} \simeq \frac{J}{\sqrt{m}} + K m^\gamma, \]

where

\[ J = 2 \left( \frac{\hbar l}{c} \right)^{1/2}, \quad K = \frac{2^n - 1}{n 2^n a^n} \frac{16 \pi G_{4+n} M_{(4+n)}^{(1-\gamma)}}{(2 + n) A_{2+n} c^2} . \]

Upon minimizing \( \delta l_{\text{tot}} \) with respect to \( m \), one obtains

\[ (\delta l_{\text{tot}})_{\text{min}} \sim \frac{1}{\gamma + 1}. \]

Hence, if we require that holography holds, namely \( (\delta l_{\text{tot}})_{\text{min}} \sim (l)^{1/2} \), we must also have

\[ \frac{\gamma}{2} + 1 = \frac{1}{3 + n}. \]

Therefore, the equivalence principle must be violated at distances shorter than the size \( L \) of the extra dimensions, as well as Newton’s law is modified in 4 + n dimensions (i.e., \( F \sim 1/\nu^{2+n} \)).

**IV. YET ANOTHER VIEW**

For the sake of completeness and in order to further support our results, we report and comment hereafter on a more recent proposal of Ng’s. In Ref. [16], he describes a different argument to reconcile GUP with holography. He says: “...To see this, let the clock be a light-clock consisting of a spherical cavity of diameter \( d = 2a \), surrounded by a mirror wall of mass \( m \), between which bounces a beam of light. For the uncertainty in distance measurement not to be greater than \( \delta l \), the clock must tick off time fast enough that \( d/c \lesssim \delta l/c \). But \( d \), the size of the clock, must be larger than the Schwarzschild radius \( r_S \equiv 2G N m/c^2 \) of the mirror, for otherwise one cannot read the time registered on the clock. From these two requirements, it follows that

\[ \delta l \gtrsim G N m/c^2. \]

Thus general relativity alone would suggest using a light clock to do the measurement.”

On combining the last inequality with Wigner’s bound in Eq. (4), Ng readily obtains the expression in Eq. (7) and then, minimizing with respect to \( m \), the result (13) for the minimum total error. This works flawlessly in four dimensions, and, again in Ref. [16], Ng applies the same kind of argument also to space-times with 4 + n dimensions. He considers the clock as completely immersed in the extra dimensions, that is \( 0 < d < L \), so that the metric (4) can be used. The condition that the clock be not a black hole,

\[ a > r_S = C^{1+n}, \]

where \( C = \frac{1}{\nu^{2+n}} \).
and the fact that the error must be larger than the size of the clock, \( \delta l \gtrsim d \), then imply that

\[
\delta l \gtrsim C^{\frac{1}{n+2}} .
\]

(27)

This means (including Wigner’s quantum error and omitting unimportant numerical factors) that

\[
\delta l \gtrsim \left( \frac{\hbar l}{m c} \right)^{1/2} + C^{\frac{1}{n+2}} .
\]

(28)

Since, from Eq. (10), \( C \sim m \), after extremizing for \( m \), Ng obtains the total error

\[
\delta l_{\text{tot}} \sim \left( \ell_{(4+n)} \right)^{\frac{1}{n+2}} ,
\]

(29)

so that the holographic scaling is satisfied also in \( 4 + n \) dimensions. Of course, this result is in sharp contrast with the one obtained in Ref. [1].

The argument of Ng can be however criticized in the following way. The key point is again, like in Ref. [8], the relation between the size of the clock \( d \) and its mass \( m \). As we have shown in Ref. [10], the size of the clock can be considered as an error (more precisely, the actual error \( b \) is very likely much smaller than \( d \), i.e., \( b \ll d \)), provided one also considers that, to all practical purposes, there is no universal relation between the size \( d \) of a clock and its mass \( m \). Therefore, although the inequality (20) should in general hold (otherwise the clock would be a black hole), it cannot be used to establish (27) as a universal expression of the error \( \delta l \) in terms of the clock’s mass \( m \). In fact, on following a similar logic, one could then consider as a valid lower bound for the error \( \delta l \) any expression containing the mass of the clock itself, provided it limits from below the size of the clock. For example, one could require that the size \( d \) be larger than the clock’s Compton wavelength, \( d > \hbar/m c \), in order to have a classical clock. This would immediately yield a completely different (and non-holographic) scaling of the total error \( \delta l_{\text{tot}} \). Therefore, the use of the Schwarzschild radius of the clock in the expression (28) as a measure of the gravitational part of the error appears to be completely arbitrary. On the other hand, if one correctly considers the size of the clock as a contribution to the error, but strictly independent of \( m \),

\[
\delta l \gtrsim \left( \frac{\hbar l}{m c} \right)^{1/2} + d ,
\]

(30)

then the holographic scaling in \( 4 + n \) dimensions is not recovered [10].

Finally, we point out that, even adopting a device like that of Refs. [10], a gravitational error originated by the uncertainty in the ADM mass of the Schwarzschild metric of the kind discussed in the Introduction should still be included. The complete and correct final expression of the total error would therefore be

\[
\delta l_{\text{tot}} = \delta l_{\text{QM}} + \delta l_{\text{c}} \simeq \frac{J}{\sqrt{m}} + K m^n + b ,
\]

(31)

where \( b (\ll d) \) does not depend on \( m \). This would again yield the same conclusion following from Eq. (20).

V. CONCLUSIONS

We have shown how a gravitational error originated by the quantum mechanical uncertainty in the ADM mass of a detector inevitably affects any measurements of length. This leads to Ng and van Dam’s GUP, which has the remarkable property of respecting the holographic counting in four dimensions. When extra spatial dimensions are present, the holographic scaling is violated. However, holography can be restored if one instead allows for a violation of the equivalence principle at short distances (below the size of extra dimensions). This violation could in principle be tested (see, e.g., Ref. [17]), and its extent is related to the number of extra dimensions. The connections of the present scenario with other models where the equivalence principle is also violated are worth of further investigation. To this aim, the results reported for example in Refs. [18] seem to be particularly promising. Such results, although sometimes worked out in a stringy oriented scenario (for example D-brane induced gravity) or in the framework of loop quantum gravity, seem anyway to match, at least for the key aspects, the more phenomenological arguments given here.

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[19] This is already a great simplification of the actual setup, but unfortunately no exact solution of the Einstein equations is known which describes this two-body problem.
[20] Let us note that we are using the time-energy uncertainty relation in the standard way, which is usually employed in order to estimate the life-time of meta-stable states or the width of spectral lines (see, e.g., Ref. [11]).
[21] It easy to check that the quantum mechanical bound in Eq. (5) is always smaller than the global uncertainty (8), as long as the clock’s size a is larger than (8) itself. In fact, $2\sqrt{2} \ell_p (l/a)^{1/2} < 3 \left(\ell_p^2 l\right)^{1/3}$ iff $a > (8/9)(\ell_p^2 l)^{1/3}$, a condition easily met in all meaningful cases.
[22] The effect of the brane on black holes has never been computed exactly. There are however both numerical [14] and analytical [15] works which show that it should squeeze the horizon along the extra dimension(s), thus breaking the full spherical symmetry.
[23] Note that gravity does not propagate in less than four dimensions and our results are therefore expected to hold only for $n \geq 0$ [8] (for which $\gamma > 0$ and $M$ is finite).