Indigenous bundles with nilpotent $p$-curvature

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Abstract

We study indigenous bundles in characteristic $p > 0$ with nilpotent $p$-curvature, and show that they correspond to so-called deformation data. Using this equivalence, we translate the existence problem for deformation data into the existence of polynomial solutions of certain differential equations with additional properties. As in application, we show that $\mathbb{P}^1$ minus four points is hyperbolically ordinary (in the sense of Mochizuki [13]). We also give a concrete application to existence of deformation data with fixed local invariants.

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1 Introduction

The main goal of this paper is to set up an equivalence between indigenous bundles with nilpotent $p$-curvature and deformation data. Deformation data arise in the theory of stable reduction of Galois covers of curves. An aspect of this theory which has not yet been treated in a satisfactory way is the existence of deformation data with given local invariants. Though deformation data are in principal easier objects than indigenous bundles, the existence problem seems more approachable via indigenous bundles. This is especially true for indigenous bundles on $\mathbb{P}^1$. Here we give a concrete interpretation of an indigenous bundle in terms of a solution of an ordinary differential equation in characteristic $p$ with additional properties. The existence problem of deformation data becomes then a variant of Dwork’s accessory parameter problem ([6]).

Let $X$ be a smooth projective curve over an algebraically closed field $k$ of characteristic $p > 0$. A deformation datum $(Z, \omega)$ consist of a cyclic cover $Z \to X$ of order dividing $p - 1$ together with a differential form $\omega$ on $Z$, satisfying certain additional properties (Definition 4.1). Associated to a deformation datum is a set of local invariants called the signature.

Let $f_K : Y_K \to X_K$ be a $G$-Galois cover defined over a field $K$ of characteristic zero. Suppose that $p$ strictly divides the order of $G$, and that $f_K$ has bad reduction at some place $\wp$ of $K$ with residue characteristic $p$. Let $\bar{f}$ denote the stable reduction of $f_K$ at $\wp$ (see [17] for a precise definition). Then to $\bar{f}$ one may associate a set of deformation data; they describe the inseparable part $\bar{f}$ of $f_K$. A particularly interesting case is when $f_K$ is a Belyi map, i.e. when
$X_K = \mathbb{P}_K^1$ and $f_K$ is branched at $\{0, 1, \infty\}$. Here one has a good description of the stable reduction $\tilde{f}$, see [17], [15]. In particular, the inseparable part of $\tilde{f}$ is described by one deformation datum on $X := \mathbb{P}_k^1$, where $k = \mathbb{F}_p$. Deformation data coming from the stable reduction of Belyi maps are called special; they are characterized by a numerical condition on the signature.

The correspondence between Belyi maps with bad reduction and special deformation data is intriguing and, at present, quite mysterious. The only case that is fully understood is worked out in [4]. Suppose $f_K : Y_X \to X_K = \mathbb{P}_K^1$ is a Belyi map which is Galois and whose Galois group is contained in the linear group $\text{GL}_2(\mathbb{F}_p)$. If $f_K$ has bad reduction at $\wp$, then the corresponding special deformation datum is hypergeometric, i.e. it corresponds to a polynomial solution of a certain hypergeometric differential equation. Furthermore, the correspondence between the group theoretic description of the cover $f_K$ and the parameters of this hypergeometric equation is totally explicit. As an application, one gets an if and only if criterion for a Belyi map with Galois group contained in $\text{GL}_2(\mathbb{F}_p)$ to have good reduction at $p$.

Deformation data show up naturally in other situations, too. One example is the lifting problem, i.e. the question of whether a curve with given automorphism group in characteristic $p$ lifts to characteristic 0. See e.g. [5]. Here, as in other situations where deformation data play a role, the difficult problem is to show the existence of deformation data with a given signature. One of the motivations behind the present paper is to translate this existence problem into the existence of a different kind of object which is often more tractable.

In §4 we associate to every deformation datum $(Z, \omega)$ over $X$ the equivalence class of a certain flat vector bundle of rank two $(E, \nabla)$ on $X$, with at most regular singularities. (The equivalence class corresponds to the associated projective bundle $\mathbb{P}(E)$, together with its induced connection.) We show that $(E, \nabla)$ is an indigenous bundle in the sense of Mochizuki, see [13], [14]. Moreover, $(E, \nabla)$ has nilpotent and nonvanishing $p$-curvature. Conversely, we show that every indigenous bundle with nonvanishing and nilpotent $p$-curvature comes from a deformation datum $(Z, \omega)$. A technical part is to relate the invariants on both sides of the equivalence. We show that the signature of a deformation datum can be expressed in terms of the order of the zeros of the $p$-curvature of the corresponding indigenous bundle. The correspondence between indigenous bundles with nilpotent $p$-curvature and deformation data is implicit in Mochizuki’s work. Our construction is more direct and explicit, and is inspired by work of Ihara [10].

The last part of the paper concerns the existence of deformation data on $X = \mathbb{P}_k^1$ with given signature. In this case, there is a classical correspondence between indigenous bundles and second order differential equations

$$L(u) = u'' + p_1 u' + p_2 u = 0$$

with at most regular singularities. Indigenous bundles with nilpotent $p$-curvature correspond to differential equations which admit a polynomial solution. See §5.2 for a more precise statement. Using the results of the first part of the paper, we
translate the existence problem for deformation data into a variant of Dwork’s
accessory parameter problem ([6]). We use this description to discuss the exis-
tence of special deformation data with four singularities in §5.3.

As an application of the description of indigenous bundles on $\mathbb{P}^1$ in terms
of solutions of a differential equation, we show in §6 that a projective line minus
four points is hyperbolically ordinary ([13]). This complements a result of
Mochizuki which states that every generic marked curve is hyperbolically ordi-
nary. In terms of deformation data our result implies that for every $\lambda \in k-\{0,1\}$
there exists a deformation datum $(Z, \omega)$ with singularities $\{0,1,\infty,\lambda\}$ and sig-
nature $\sigma = (0,0,0,0)$.

We also briefly discuss the analogous question for other signatures $\sigma$. In
§6.3 we give a numerical condition on $\sigma$ which determines the dimension of the space
of deformation data on $\mathbb{P}^1$ with four singular points and signature $\sigma$. It is either
zero or one, and both cases occur. In particular, for special deformation data
it is zero. It therefore seems hard to prove the existence of special deformation
data by using Mochizuki’s theory, which is best suited to prove existence results
on generic curves. A different approach to construct special deformation data,
which also uses the language of indigenous bundles, is developed in [2].

2 Indigenous bundles

2.1 The following notation will be fixed throughout this paper. Let $k$ be an
algebraically closed field and $X$ a smooth projective connected curve of genus
$g$ over $k$. We fix $r \geq 0$ pairwise distinct closed points $x_1, \ldots, x_r \in X$, which
we call marked points. We assume that $2g - 2 + r > 0$. We denote by $\Omega^\log_{X/k} =
\Omega_X/k(\sum x_i)$ the sheaf of differential 1-forms on $X$ with at most simple poles in
the marked points and $\tau^\log_{X/k} \cong (\Omega^\log_{X/k})^{-1}$ its dual, i.e. the sheaf of vector fields
on $X$ with at least simple zeros in the marked points.

A flat vector bundle is a vector bundle $E$ on $X$ together with a connection
$\nabla : E \rightarrow E \otimes \Omega^\log_{X/k}$. This means that the connection has regular singularities
at the marked points (see e.g. [11], §11). Two flat vector bundles $E$ and $E'$ are
called equivalent if there exists a flat line bundle $L$ and a horizontal isomorphism
$E' \cong E \otimes L$. Notation: $E \sim E'$.

2.2 Let $(E, \nabla)$ be a flat vector bundle on $X$ of rank two. For $i = 1, \ldots, r$,
we define the monodromy operator $\mu_i$ as an endomorphism of the fiber $E|_{x_i}$ of
$E$ at $x_i$, as follows. Let $t$ be a local parameter at $x_i$. Then $\nabla(t\partial/\partial t)$ defines
a $k$-linear endomorphism of the stalk $E_{x_i}$ of $E$ at $x_i$ which fixes the submodule
$m_{x_i} \cdot E_{x_i}$. Here $m_{x_i}$ denotes the maximal ideal of the local ring $\mathcal{O}_{X,x_i}$. Therefore,$\nabla(t\partial/\partial t)$ induces a $k$-linear endomorphism $\mu_i$ of the fiber $E|_{x_i} = E_{x_i}/m_{x_i} \cdot E_{x_i}$.
One checks easily that $\mu_i$ does not depend on the choice of the parameter $t$.

Let $\alpha_i, \beta_i$ be the two eigenvalues of $\mu_i$. We call $\alpha_i, \beta_i$ the local exponents of
∇ at $x_i$. We distinguish two cases. If $\mu_i$ is not semisimple, then $\alpha_i = \beta_i$ and

$$
\mu_i \sim \begin{pmatrix} \alpha_i & 1 \\ 0 & \alpha_i \end{pmatrix}.
$$

If this is the case then we say that $\nabla$ has *logarithmic monodromy* at $x_i$. If $\mu_i$ is semisimple then

$$
\mu_i \sim \begin{pmatrix} \alpha_i & 0 \\ 0 & \beta_i \end{pmatrix},
$$

and we say that $\nabla$ has *toric monodromy* at $x_i$.

### 2.3 A filtration on a flat vector bundle $(\mathcal{E}, \nabla)$ of rank two consists of a line subbundle $F^1 \mathcal{E} \subset \mathcal{E}$ such that $\text{Gr}^1 \mathcal{E} := \mathcal{E}/F^1 \mathcal{E}$ is also a line bundle. For such a filtration, the connection $\nabla$ induces a Kodaira–Spencer map

$$
\kappa : F^1 \mathcal{E} \longrightarrow \text{Gr}^1 \mathcal{E} \otimes \Omega^{\log}_{X/k}.
$$

If it seems more convenient, we regard $\kappa$ as a morphism

$$
\kappa : \tau^{\log}_{X/k} \longrightarrow (F^1 \mathcal{E})^{-1} \otimes \text{Gr}^1 \mathcal{E}.
$$

Note that, written in either way, $\kappa$ is $\mathcal{O}_X$-linear.

**Definition 2.1** An *indigenous bundle* on $X$ is a flat vector bundle $(\mathcal{E}, \nabla)$ of rank two which satisfies the following conditions:

(i) There exists a filtration $F^1 \mathcal{E} \subset \mathcal{E}$ whose associated Kodaira–Spencer map is an isomorphism.

(ii) The connection $\nabla$ has nontrivial monodromy at every marked point (i.e. $\mu_i \neq 0$ for all $i$).

A filtration $F^1 \mathcal{E} \subset \mathcal{E}$ as in (i) is called a *Hodge filtration*.

**Proposition 2.2** Let $(\mathcal{E}, \nabla)$ be an indigenous bundle.

(i) The Hodge filtration for $(\mathcal{E}, \nabla)$ is unique.

(ii) Fix an index $i \in \{1, \ldots, r\}$ and suppose that $\nabla$ has toric monodromy at $x_i$, with exponents $\alpha_i, \beta_i$. Then $\alpha_i \neq \beta_i$.

(iii) Any equivalent bundle $\mathcal{E}' \sim \mathcal{E}$ is indigenous as well.

Note that Part (i) of the proposition implies that it makes sense to speak about the Hodge filtration of an indigenous bundle.
Proof: Define the height of a filtration $F^1 \mathcal{E} \subset \mathcal{E}$ as the integer

$$\frac{1}{2}(\deg \text{Gr}^1 \mathcal{E} - \deg F^1 \mathcal{E}).$$

By assumption, a Hodge filtration has height $-g + 1 - r/2 < 0$. On the other hand, it is easy to see that two filtrations with negative height are equal. This proves (i). To prove (ii), suppose that $\alpha_i = \beta_i$. Then $\mu_i$ fixes the line $F^1 \mathcal{E}|_{x_i} \subset \mathcal{E}|_{x_i}$. But this would mean that the Kodaira–Spencer map $\kappa$ vanishes at $x_i$, which gives a contradiction. The proof of (iii) is easy and left to the reader. □

Remark 2.3 In [13], indigenous bundles are required to have logarithmic monodromy at the marked points and to have trivial determinant. What we call here an indigenous bundle is a flat vector bundle whose associated projective bundle is torally indigenous, in the sense of [14], §I.4.

3 Indigenous bundles in characteristic $p$

3.1 From now on, we assume that our base field $k$ has odd, positive characteristic $p$. In this section, we develop the theory of indigenous bundles in characteristic $p > 2$. All definitions are essentially due to Mochizuki ([13], [14]).

We set $T := \left(\tau_{\log}^{X/k}\right)^{\otimes p}$. This is a line bundle on $X$ of degree $-p(2g-2+r) < 0$. We endow $T$ with the unique connection $\nabla_T : T \to T \otimes \Omega_{\log}^{X/k}$ such that the subsheaf $T^h$ of horizontal sections consists precisely of the ‘$p$-th powers’, i.e. of sections of the form $D \otimes p$, where $D$ is a section of $\tau_{\log}^{X/k}$ ([11], Theorem 5.1).

Let $(\mathcal{E}, \nabla)$ be a flat vector bundle on $X$. The $p$-curvature of $(\mathcal{E}, \nabla)$ is an $\mathcal{O}_X$-linear morphism

$$\Psi_\mathcal{E} : T \to \text{End}_{\mathcal{O}_X}(\mathcal{E}),$$

defined as follows. Let $D$ be a rational section of $\tau_{\log}^{X/k}$. We regard $D$ as a derivation of the function field $K = k(X)$. Then $D^p := D \circ \cdots \circ D$ is again a derivation of $K$ and $\nabla(D)$ and $\nabla(D^p)$ are $k$-linear endomorphisms of the $K$-vector space $E := \mathcal{E} \otimes_{\mathcal{O}_X} K$. We define

$$\Psi_\mathcal{E}(D^{\otimes p}) := \nabla(D^p) - \nabla(D^p).$$

This is a $K$-linear endomorphism of $E$. One shows that the rule $D^{\otimes p} \mapsto \Psi(D^{\otimes p})$ descents to the desired $\mathcal{O}_X$-linear map $\Psi_\mathcal{E}$ ([11], 5.0.1).

It is important to notice that the $p$-curvature is horizontal in the sense that it commutes with the canonical connections on $T$ and $\text{End}_{\mathcal{O}_X}(\mathcal{E})$. Indeed, by the definitions of these connections, the claim that $\Psi_\mathcal{E}$ is horizontal is equivalent to the fact that the endomorphisms $\Psi_\mathcal{E}(D^{\otimes p})$ and $\nabla(D)$ of $E$ commute. This is easy to check, see also [11], 5.2.2.

Definition 3.1 An indigenous bundle $(\mathcal{E}, \nabla)$ on $X$ is called

(i) active if $\Psi_\mathcal{E} \neq 0$,
(ii) *nilpotent* if the image of $\Psi_E$ consists of nilpotent endomorphisms.

Let us, from now on, assume that $(E, \nabla)$ is active and nilpotent. We are only interested in the equivalence class of $E$ (i.e. in the associated projective bundle $\mathbb{P}(E)$). Therefore we may replace $E$ by any equivalent bundle $E'$ which is also active and nilpotent.

**Definition 3.2** We say that $(E, \nabla)$ is *normalized* if there exists a horizontal and surjective homomorphism $\gamma : E \to O_X$.

Normalized indigenous bundles correspond to Mochizuki’s FL-bundles ([13], §2.1).

We claim that for any indigenous bundle $(E, \nabla)$ which is active and nilpotent there is an equivalent bundle $(E', \nabla')$ which is normalized. Moreover, $(E', \nabla')$ is unique, up to isomorphism.

Let $M \subset E$ be the kernel of $\Psi_E$, i.e. the maximal subbundle on which $\Psi_E$ is zero. Our assumption implies that $M$ and the quotient $L := E/M$ are line bundles. Moreover, it follows from the fact that $\Psi_E$ is horizontal that $M$ is invariant under the connection $\nabla$. In other words, we obtain a short exact sequence of flat vector bundles

\[
0 \to M \to E \to L \to 0.
\]

The $p$-curvature of the induced connections on $M$ and $L$ is zero.

We set $E' := E \otimes L^{-1}$. Since the $p$-curvature on $L$ is zero, $L^{-1}$ admits a nonzero rational horizontal section. Using this fact, it is easy to see that the $p$-curvature of $E'$ is equal to the $p$-curvature of $E$ (here we identify $\text{End}(E)$ and $\text{End}(E')$ in the obvious way). In particular, $E'$ is active and nilpotent. By construction, $O_X$ is a quotient of $E'$. Hence $E'$ is normalized. The uniqueness of $E'$ follows from the fact that $M$ is the maximal saturated line subbundle of $E$ which is invariant under $\nabla$.

**3.2** From now on, we assume that $E$ is normalized. Moreover, we fix a surjective and horizontal morphism $\gamma : E \to O_X$. Note that $\gamma$ is unique up to multiplication by an element of $k^\times$. By definition, $M$ is the kernel of $\gamma$.

**Definition 3.3** A point $x \in X$ where $\Psi_E$ vanishes is called a *spike*. We write $n_x := \text{ord}_x(\Psi_E)$ for the order of vanishing of $\Psi_E$ at $x$ and say that $x$ is a *spike of order* $n_x$.

We may regard the $p$-curvature of $E$ as a nonzero horizontal homomorphism

$$
\Psi_E : T \to \text{Hom}_{O_X}(L, M) = M \otimes L^{-1} \simeq M,
$$

since we assumed that $M$ is normalized. Let $S = \sum_x n_x \cdot x$ be the divisor of zeros of $\Psi_E$ (with support in the spikes). Then $\Psi_E$ induces an isomorphism $T(S) \cong M$. We let $\delta : T(S) \hookrightarrow E$ denote the composition of the last isomorphism.
multiplied by $-1$ with the canonical injection $M \hookrightarrow \mathcal{E}$. From now on, we identify $\mathcal{T}(S)$ with a subbundle of $\mathcal{E}$ via the injection $\delta$. Since $\mathcal{E}$ is normalized, the exact sequence (1) becomes

$$0 \to \mathcal{T}(S) \to \mathcal{E} \xrightarrow{\gamma} \mathcal{O}_X \to 0.$$ 

It follows that the $p$-curvature of $\mathcal{E}$, considered as $\mathcal{O}_X$-linear morphism $\mathcal{T} \to \text{End}_{\mathcal{O}_X}(\mathcal{E})$, is given by the formula

$$\Psi_{\mathcal{E}}(D^\otimes p)(e) = -\gamma(e) \cdot D^\otimes p.$$ 

Here $e$ is a section of $\mathcal{E}$ and $D$ a vector field.

Let $e_1$ be a rational section of the line subbundle $F^1\mathcal{E}$. We set $u := \gamma(e_1) \in \mathcal{O}_{X,x}$ and $e_0 := u^{-1}e_1$. By definition, $e_0$ is a rational section of $F^1\mathcal{E}$ with $\gamma(e_0) = 1$. Note that these two properties characterize $e_0$ uniquely. In particular, $e_0$ is independent of the choice of $e_1$. (However, $e_0$ does depend on the choice of $\gamma$.) We call $e_0$ the canonical section of $\mathcal{E}$.

**Lemma 3.4** The canonical section $e_0$ has at most simple poles and no zeros. The marked points $x_i$ are regular points of $e_0$.

**Proof:** Let $x \in X$ be a closed point and $t$ a local parameter at $x$. We set $D := t\partial/\partial t$ or $D := \partial/\partial t$, depending on whether $x$ is marked or not. We may assume that the section $e_1$ is a generator of $F^1\mathcal{E}$ at $x$. Then $u = \gamma(e_1)$ is an element of the local ring $\mathcal{O}_{X,x}$. We have to show that $u$ has at most simple zeros, and is invertible if $x$ is a marked point.

Let $e_2 := \nabla(D)(e_1)$. Since the Kodaira–Spencer morphism is nonzero at $x$, $(e_1, e_2)$ is an $\mathcal{O}_{X,x}$-basis for the stalk of $\mathcal{E}$ at $x$. Moreover, we have $\gamma(e_2) = D(u)$. Let $e_3$ be an element of $\mathcal{E}_x$ with $\gamma(e_3) = 1$ and write $e_3 = q_1e_1 + q_2e_2$, with $q_1, q_2 \in \mathcal{O}_{X,x}$. Applying $\gamma$ to this equality we get

$$q_1u + q_2D(u) = 1,$$

and hence

$$\min(\text{ord}_x u, \text{ord}_x D(u)) = 0.$$ 

Suppose that $x$ is not marked. Then $D = \partial/\partial t$ and $\text{ord}_x D(u) = \text{ord}_x u - 1$. We conclude that $\text{ord}_x(u) \in \{0, 1\}$. On the other hand, if $x = x_i$ is a marked point, then $D = t\partial/\partial t$ and $\text{ord}_x D(u) = \text{ord}_x u$. Therefore, $\text{ord}_x u = 0$. □

**Definition 3.5** The points where the canonical section $e_0$ has a simple pole are called the supersingular points of $(\mathcal{E}, \nabla)$.

3.3 Let $(\mathcal{E}, \nabla)$ be an active, nilpotent indigenous bundle which we assume to be normalized. We now compute the $p$-curvature in the marked points. Let $t$ be a local parameter at the marked point $x_i$, and set $D_i := t\partial/\partial t$. It is easy to see that $D_i^p = D_i$. Therefore,

$$\Psi_{x_i} := \Psi_{\mathcal{E}}(D_i^\otimes p)|_{x_i} = \mu_i^p - \mu_i.$$
Since the subbundle $\text{Ker}(\gamma) \subset \mathcal{E}$ is invariant under $\nabla$, the monodromy operator $\mu_i$ fixes the line $\text{Ker}(\gamma)|_{x_i} \subset \mathcal{E}|_{x_i}$. Therefore, $\text{Ker}(\gamma)$ has regular singularities at $x_i$. It follows that the local exponents of $(\mathcal{E}, \nabla)$ at $x_i$ are $(\alpha_i, 0)$, where $\alpha_i$ is the local exponent of $(\text{Ker}(\gamma), \nabla)$.

If $\mathcal{E}$ has a logarithmic singularity at $x_i$ with local exponents $(\alpha_i, \beta_i) = (0, 0)$ then

$$\Psi_{x_i} \sim \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$ 

In particular, $\Psi_{x_i} \neq 0$, so $x_i$ is not a spike.

Similarly, if $\mathcal{E}$ has a toric singularity at $x_i$, with local exponents $(\alpha_i, \beta_i) = (\alpha_i, 0)$, then

$$\Psi_{x_i} \sim \begin{pmatrix} \alpha_i^p - \alpha_i & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Since $(\mathcal{E}, \nabla)$ is nilpotent, we have that $\Psi_{x_i}^2 = 0$. This implies that $\alpha_i \in \mathbb{F}_p^\times$. In particular, $x_i$ is a spike.

**Proposition 3.6** Suppose that $(\mathcal{E}, \nabla)$ is active, nilpotent, and normalized. Let $x \in X$ be a closed point.

(i) If $x = x_i$ is a marked point and $\nabla$ has logarithmic monodromy at $x$, then $n_x = 0$.

(ii) If $x = x_i$ is a marked point and $\nabla$ has toric monodromy at $x$, with exponents $(\alpha_i, 0)$, then $x$ is a spike. The order $n_x$ of this spike satisfies

$$n_x \equiv -\alpha_i \neq 0 \pmod{p}.$$ 

(iii) If $x$ is not a marked point then $n_x \equiv 0 \pmod{p}$.

(iv) If $x$ is supersingular then $n_x = 0$.

**Proof:** Suppose first that $x = x_i$. If $\nabla$ has logarithmic monodromy at $x$, then $n_{x_i} = 0$ by the discussion preceding Proposition 3.6. This proves (i). Suppose that $\nabla$ has toric monodromy at $x_i$. It is well known that the order of vanishing of a horizontal section of a flat line bundle in a regular singular point is congruent to minus the local exponent in that point. Since $\alpha_i$ is the local exponent of $\mathcal{M}$ at $x_i$, we have that

$$n_{x_i} \equiv -\alpha_i \pmod{p}.$$ 

The claim $n_{x_i} \neq 0$ follows now from Proposition 2.2 (ii). This proves (ii). The proof of (iii) is similar.

Let $x$ be a supersingular point, and let $t$ be a local parameter at $x$. Put $D = \partial / \partial t$. We have seen that we may regard $\Psi_{\mathcal{E}}(D^p)$ as homomorphism of $\mathcal{O}_X = \mathcal{L} \to \mathcal{M}$. Let $e_0$ be the canonical section. Then $\gamma(e_0)$ is a section of $\mathcal{O}_X$ which is invertible at $x$. It follows from (2) that $\Psi_{\mathcal{E}}(D^p)e_0 = -D^p$. This shows that $n_x = 0$. 

$\blacksquare$
4 Relation with deformation data

4.1 We start this section by recalling the notion of a deformation datum. These objects arise in the study of bad reduction of Galois covers of curves, see §4.2 for a quick review. We refer to [16] and [17] for a more thorough discussion.

Definition 4.1 A deformation datum on $X$ is a pair $(Z, \omega)$, where $Z \rightarrow X$ is a finite, at most tamely ramified Galois cover $\pi: Z \rightarrow X$ of smooth connected curves, and $\omega$ is a rational section of $\Omega_{Z/k}$. In addition, we require that the following holds.

(i) Let $H$ be the Galois group of $Z \rightarrow X$. For each $\sigma \in H$ we have $\sigma^* \omega = \chi(\sigma) \cdot \omega$, where $\chi: H \rightarrow \mathbb{F}_p^*$ is an injective character.

(ii) The differential $\omega$ is logarithmic, i.e. of the form $du/u$.

Two deformation data $(Z, \omega)$ and $(Z', \omega')$ are called equivalent if there exists an isomorphism of $X$-schemes $\varphi: Z \rightarrow Z'$ and an element $\epsilon \in \mathbb{F}_p^*$ such that $\varphi^* \omega' = \epsilon \cdot \omega$. For each closed point $x \in X$ we define the following invariants:

\[ m_x := |H_z|, \quad h_x := \text{ord}_z \omega + 1, \quad \sigma_x := h_x/m_x. \]

Here $z \in Z$ is any point above $x$ and $H_z \subset H$ denotes the stabilizer of $z$.

Definition 4.2 Let $(Z, \omega)$ be a deformation datum on $X$.

(i) A point $x \in X$ is said to be a supersingular point for $(Z, \omega)$ if $\sigma_x = (p + 1)/(p - 1)$.

(ii) A point $x \in X$ is said to be singular if it is not a supersingular point and $\sigma_x \equiv 1 \pmod{p}$.

The condition of Definition 4.2 (ii) makes sense by Lemma 4.3 (i) below. Lemma 4.3 (iii) implies that there are only finitely many supersingular and singular points. We refer to Example 4.5 for a motivation of the terminology 'supersingular'.

Lemma 4.3 Let $(Z, \omega)$ be a deformation datum.

(i) For all $x \in X$, $m_x$ divides $p - 1$. Moreover, $h_x$ and $m_x$ are relatively prime.

(ii) If $h_x \neq 0$ then $\gcd(p, h_x) = 1$.

(iii) For all but finitely many points $x \in X$ we have $\sigma_x = 1$.

(iv) We have

\[ \sum_{x \in X} (\sigma_x - 1) = 2g - 2. \]
Proof: The statement that $m_x$ divides $p - 1$ follows immediately from Definition 4.1 (i). The statement that $\gcd(h_x, m_x) = 1$ follows from the assumption that $\chi$ is injective. Compare to the proof of [16], Prop. 2.5. Part (ii) is proved in [8], Cor. 1.8.b. Note that $h_x$ is denoted by $-m$ in that paper. Part (iii) is obvious. Part (iv) follows from the Riemann–Roch Theorem. □

Definition 4.4 An **signature** is given by a finite set $M$ and a map $$\sigma : M \to \frac{1}{p-1} \cdot \mathbb{Z}, \quad x \mapsto \sigma_x$$ such that $\sigma_x \geq 0$ and $\sigma_x \neq 1, \frac{p+1}{p-1}$ for all $x \in M$ and such that the number $$d := \frac{p-1}{2} \left( 2g - 2 - \sum_{x \in M} (\sigma_x - 1) \right)$$ is a positive integer. The **singularities** of $\sigma$ are the elements $x \in M$ with $\sigma_x \neq 1 \pmod{p}$.

Given a deformation datum $(Z, \omega)$ on $X$, the invariants $\sigma_x$ defined in (3) give rise to a signature $\sigma$ (where $M$ is the set of points $x \in X$ with $\sigma_x \neq 1, \frac{p+1}{p-1}$). It follows from Lemma 4.3 (iv) that the number $d$ defined above is the number of supersingular points.

We shall always enumerate the singularities of a deformation datum $(Z, \omega)$ as $x_1, \ldots, x_r$ and write $\sigma_i := \sigma_{x_i}$ for $i = 1, \ldots, r$. We say that the deformation datum $(Z, \omega)$ is **trivial** if $2g - 2 + r = 0$. In the rest of this paper, we exclude trivial deformation data.

4.2 We quickly recall how deformation data come up in the theory of reduction of Galois covers. For more details, see e.g. [17]. Let $R$ be a complete discrete valuation ring with fraction field $K$ of characteristic zero and residue field $k = \overline{k}$ of characteristic $p > 0$. Let $f_K : Y_K \to X_K$ be a $G$-Galois cover defined over $K$. We suppose, for simplicity, that $p$ strictly divides the order of $G$ and that the curve $X_K$ has good reduction, i.e. extends to a smooth $R$-curve $X_R$, with special fiber $X := X_R \otimes k$. Let $v$ denote the discrete valuation of the function field of $X_K$ corresponding to the generic point of $X$. It extends the valuation on $K$ and has residue field $k(X)$. Let $A$ denote the completion of the valuation ring of $v$. Choose a valuation of the function field of $Y_K$ extending the valuation $v$ via the map $f_K$, and let $C/A$ denote the resulting extension of valuation rings. After replacing the ground field $K$ by some finite extension, we may suppose that the ramification index of $C/A$ is one.

Suppose that the extension $C/A$ is ramified and let $B/A$ be the maximal unramified subextension. The residue field of $B$ is the function field $k(Z')$ of a Galois cover $Z' \to X$ of order prime to $p$. It is shown in [15] that the extension $C/B$ is a torsor under a finite flat group scheme over $R$. The special fiber of this group scheme is isomorphic to either $\mu_p$ or $\alpha_p$. In particular, the extension of
residue fields of $C/B$ is inseparable of degree $p$ and has the structure of a torsor under $\mu_p$ or $\alpha_p$. It is a classical fact that such a torsor gives rise to a differential form $\omega' \in \Omega_{k(Z')}^k$. If the group scheme is $\mu_p$ (which we shall assume from now on) then $\omega'$ is logarithmic. Namely, in this case the extension $C/B$ is defined by a Kummer equation $y^p = h$, and the image of $h \in B$ in the residue field $k(Z')$ is not a $p$th power. Then $\omega' := dh/h$. One easily checks that the Galois group of $Z'/X$ acts on $\omega$ via a character $\chi: H \to F_p^\times$. We see that the deformation datum $(Z', \omega')$ has all the properties required for a deformation datum, except that $\chi$ may not be injective. However, $\omega'$ descends to a logarithmic differential form $\omega$ on the quotient $Z := Z'/\ker(\chi)$, and we obtain a deformation datum $(Z, \omega)$.

The deformation datum $(Z, \omega)$ contains much valuable information on the stable reduction of $Y_K$. The situation is particularly nice if the cover $f_K: Y_K \to X_K$ is a Belyi map, i.e. if $X_K = \mathbb{P}^1_K$, and $f_K$ is branched precisely over the three points $0, 1, \infty$. If $f_K$ has bad reduction then the above construction does always yield a deformation datum $(Z, \omega)$ (i.e. the $\alpha_p$-case does not occur). Furthermore, the deformation datum $(Z, \omega)$ is special (in the sense of Definition 5.8 below) and almost completely determines the structure of the stable reduction of the curve $Y_K$. See [17] for details.

Example 4.5 The Belyi map

$$\lambda: X(2p) \to X(2) \simeq \mathbb{P}^1$$

between the modular curves of level $2p$ and $2$ is a Galois cover branched at three points, with Galois group $G = \text{PSL}_2(p)$. This cover has a natural model over the field $\mathbb{Q}(\zeta_p)$ which has bad reduction to characteristic $p$. As explained above, the reduction of $X(2p) \to X(2)$ gives rise to a deformation datum $(Z, \omega)$.

Let $p$ be an odd prime and $k$ an algebraic closure of the prime field $\mathbb{F}_p$. The Hasse polynomial is defined as

$$u := \sum_{i=1}^{(p-1)/2} \left( \binom{p-1}{i} \right) \lambda^i \in k[\lambda].$$

It has the property that the elliptic curve

$$E_\lambda: \quad y^2 = x(x-1)(x-\lambda)$$

defined over $k$ is supersingular if and only if $u(\lambda) = 0$. It is shown in [4] that the deformation datum $(Z, \omega)$ associated to the cover $X(2p) \to X(2)$ is given by

$$(4) \quad \omega := \frac{z \, d\lambda}{\lambda(\lambda-1)},$$

and where $Z$ is the cover of $X = \mathbb{P}^1_\lambda$ defined generically by the equation $z^{(p-1)/2} = u$. The crucial fact used in the proof is that the Hasse polynomial $u$ is a solution to the Gauss hypergeometric differential equation

$$\lambda(\lambda-1) u'' + (2\lambda - 1) u' + \frac{1}{4} u = 0.$$
In the rest of this section we give a correspondence between deformation data and indigenous bundles in characteristic $p$. In Example 4.5 above, the indigenous bundle corresponding to the deformation datum $(Z, \omega)$ is essentially equivalent to the Gauss hypergeometric differential equation. The construction we shall give may seem very ad hoc. To fully appreciate it, one should have a look at Mochizuki’s theory of $p$-adic uniformization of ordinary hyperbolic curves [13], [14], and the work of Ihara on congruence relations [10].

4.3 We choose a rational function $t$ on $X$ with $dt \neq 0$, and set $D := \partial/\partial t$.

**Lemma 4.6** Let $Z \to X$ be a cyclic cover given by an equation $z^{p-1} = v^{-1}$, where $v \in k(X)$. Let

\[ \omega = z \, dt. \]

(i) The differential form $\omega$ on $Z$ is logarithmic if and only if

\[ D^{p-1}v = -1. \]

(ii) Suppose that $(Z, \omega)$ is a deformation datum, and let $t$ be a local parameter at $x \in X$. Then

\[ \operatorname{ord}_x(v) = (p-1)(1 - \sigma_x). \]

**Proof:** It is a classical fact that a differential form $\omega := z \, dt$ is logarithmic if and only if

\[ D^{p-1}z = -z^p. \]

For an outline of the proof see [7], Exercise 9.6. This exercise is stated only in the case that the genus of $X$ is zero, but one may easily extend the proof to our situation by considering $z$ as in element of $k((t))$. We deduce that

\[ D^{p-1}v = D^{p-1} \frac{z}{z^{2p}} = \frac{1}{z^p} D^{p-1}z = -1. \]

This proves (i). Part (ii) follows immediately from the definition of the signature (3).

For the rest of this subsection, we fix a deformation datum $(Z, \omega)$ on $X$. It is clear that we may write $\omega = z \, dt$, where $z$ is a rational function on $Z$ satisfying an equation $z^{p-1} = v^{-1}$ with $v \in k(X)$. Set

\[ \phi_0 := \omega^{\otimes(1-p)} = v \cdot D^{\otimes(p-1)}. \]

Note that $\phi_0$ depends only on and determines the equivalence class of the deformation datum $(Z, \omega)$. For each point $x \in X$ we define

\[ n_x := \begin{cases} 0 & \text{if } x \text{ is supersingular}, \\ (p-1)\sigma_x & \text{if } x = x_i \text{ is singular}, \\ (p-1)(\sigma_x - 1) & \text{otherwise.} \end{cases} \]

Note that $n_x$ is a nonnegative integer and is zero for all but finitely many points $x$. Moreover, $n_x \equiv 0 \pmod{p}$ if $x$ is not singular. Put $S := \sum_x n_x \cdot x$. 

12
Lemma 4.7  The tensor \( \phi_0 \), considered as a rational section of the line bundle 
\( (\Omega^{\log}_{X/k}) \otimes (1-p)(S) \), has no zeros. The poles of \( \phi_0 \) are precisely the supersingular
points. All poles of \( \phi_0 \) are double poles.

**Proof:** Lemma 4.6 (ii) together with the definition (5) of \( n_x \) implies that

\[
\text{ord}_x \nu = \begin{cases} 
-2 & \text{if } x \text{ is supersingular,} \\
p - 1 - n_x & \text{if } x \text{ is singular,} \\
-n_x & \text{otherwise.}
\end{cases}
\]

It follows immediately that \( \phi_0 \), considered as a rational section of the line bundle 
\( (\Omega^{\log}_{X/k}) \otimes (1-p)(S) \), does not have zeros and has poles of order 2 in the supersin-
gular points and no poles elsewhere. \( \square \)

We now associate to \((Z, \omega)\), an indigenous bundle \((\mathcal{E}, \nabla)\) on \((X; x_i)\) which is
active, nilpotent and normalized. Let \( U \subset X \) denote the complement of the set
of supersingular points. On \( U \), we define a flat vector bundle \( \mathcal{E}|_U \) of rank two
as follows:

\[
\mathcal{E}|_U := T(S)|_U \oplus \mathcal{O}_U \cdot e_0.
\]

Let \( t = \partial/\partial t \) and \( \phi_0 = v \cdot D^{(p-1)} \) be as above. The connection \( \nabla \) is defined
by the matrix equation

\[
\nabla(D)(\mathcal{E}) = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \cdot \mathcal{E}
\]

with respect to the generic basis \( \mathcal{E} = (D^{(p)}, e_0) \) of \( \mathcal{E}|_U \). It is clear that the
definition of \( (\mathcal{E}|_U, \nabla) \) depends only on \( \phi_0 \) and not on the choice of the function
\( t \). We claim that \( \nabla \) has regular singularities in the singular points and no
singularities elsewhere. Namely, let \( x \in U \) and assume that \( t \) is a local parameter
at \( x \). Set \( D_x := \partial/\partial t \) if \( x \) is not a singularity of the deformation datum and
\( D_x := t\partial/\partial t \) otherwise. Then

\[
\nabla(D_x)(\mathcal{E}) = A \cdot \mathcal{E}
\]

where the matrix \( A \) has coefficients which are regular in \( x \). It follows from
Lemma 4.6 (i) that the \( p \)-curvature is given by the rule

\[
\Psi_\xi(D^{(p)})(\nu) = D^{p-1}(v) \cdot \gamma|_U(\nu) \cdot D^{(p)} = -\gamma|_U(\nu) \cdot D^{(p)}.
\]

Here \( \gamma|_U : \mathcal{E}|_U \to \mathcal{O}_U \) is the projection onto the second factor (the coefficient of
\( e_0 \)) and \( \nu \) is an arbitrary section of \( \mathcal{E}|_U \).

**Proposition 4.8**  (i) The bundle \((\mathcal{E}|_U, \nabla)\) extends to an indigenous bundle
\((\mathcal{E}, \nabla)\) on \( X \). This extension is unique if we require that the section \( e_0 \)
lies in the Hodge filtration \( F^1 \mathcal{E} \) and that the projection \( \gamma|_U \) extends to an
epimorphism \( \gamma : \mathcal{E} \to \mathcal{O}_X \).
(ii) The bundle \((\mathcal{E}, \nabla)\) is active, nilpotent, and normalized.

(iii) The supersingular points (Definition 3.5) are exactly the supersingular points for \((Z, \omega)\) (Definition 4.2). The marked points are the singularities of \((Z, \omega)\) (Definition 4.2).

(iv) The order \(n_x\) of a spike is defined by (5).

**Proof:** First we extend the bundle \(\mathcal{E}|_U\) to the supersingular points in such a way that the connection \(\nabla\) is regular in these points. This is a purely local problem. Fix a supersingular point \(x\) for \((Z, \omega)\). We may suppose that the function \(t\) introduced above is a local parameter for \(x\). We have

\[
D_p - 1 (v - t^{p-1}) = 0.
\]

Therefore, there exists a rational function \(w\) on \(X\) such that

\[
D(w) = v - t^{p-1}.
\]

Since \(v\) has a double pole at \(x\), we may assume that \(w\) has a simple pole. Set \(e_1 := e_0 - w \cdot D^{\otimes p}\). We extend the bundle \(\mathcal{E}|_U\) to the point \(x\) in such a way that the pair \((e'_0, e_1)\) is an \(\mathcal{O}_{X,x}\)-basis of the stalk \(\mathcal{E}_x\). In terms of this basis, the connection \(\nabla\) is given by the matrix equation

\[
\nabla(D)(e'_0) = \begin{pmatrix} 0 & t^{p-1} \\ 0 & 0 \end{pmatrix} \cdot e_1.
\]

We see that \(\nabla\) is regular in \(x\). It is also clear that \(\gamma\) extends to an epimorphism in the point \(x\). This finishes the definition of the flat bundle \((\mathcal{E}, \nabla)\).

We now show that \((\mathcal{E}, \nabla)\) is indigenous. We define the subbundle \(F^1\mathcal{E}\) as the maximal subbundle of rank one which has \(e_0\) as rational section. On \(U\), the Kodaira–Spencer map

\[
\kappa : F^1\mathcal{E}|_U = \mathcal{O}_U \cdot e_0 \longrightarrow (\mathcal{E}/F^1\mathcal{E})|_U \otimes \Omega^{\text{reg}}_{U/k} = T(S)|_U \otimes \Omega^{\text{reg}}_{U/k}
\]

sends \(e_0\) to \(\phi_0\). Since \(\phi_0\) has no zeros on \(U\) (Lemma 4.7 (i)), \(\kappa\) is an isomorphism on \(U\). Let \(x\) be a supersingular point. Using the notation of the preceding paragraph, the stalk \(F^1\mathcal{E}_x\) is generated by \(e'_0 := w^{-1}e_0\). Also, the pair \((e'_0, e_1)\) is a basis for the stalk \(\mathcal{E}_x\). A short computation gives:

\[
\nabla(D)(e'_0) = w^{-1}t^{p-1} \cdot e'_0 - w^{-2}v \cdot e_1.
\]

Since \(v\) has a pole of order 2 and \(w\) a simple pole in \(x\), the coefficient of \(e_1\) in (7) has order 0 in \(x\). This means that the Kodaira–Spencer map does not vanish in \(x\). We have shown that \((\mathcal{E}, \nabla)\) is indigenous.

It is also clear that the extension we have defined is unique, under the conditions that it is indigenous with \(e_0\) lying in the Hodge filtration and such that \(\gamma|_U\) extends to an epimorphism \(\gamma : \mathcal{E} \rightarrow \mathcal{O}_X\). This concludes the proof of (i).

Part (ii) follows from the construction of \((\mathcal{E}, \nabla)\). Part (iv) follows from (6). Let \(x \in X\) be a supersingular point for \((Z, \omega)\) (Definition 4.2). The map \(\gamma|_U\) extends to a map \(\gamma : \mathcal{E} \rightarrow \mathcal{O}_X\), by sending \(e_0\) to 1. This makes \(e_0\) into a canonical section (with respect to \(\gamma\)), as defined in §4.3. By definition of the extension of \(\mathcal{E}\) to \(x\), the section \(e_1\) is invertible at \(x\). This implies that \(e_0\) has a simple
pole at $x$. Therefore $x$ is a supersingular point of the bundle $(\mathcal{E}, \nabla)$. Since $e_0$ is regular on $U$, it follows that the two notions of supersingularity agree.

Equation (5), together with Definition 4.2, implies that $n_x \equiv 0 \mod p$ if $x$ is not supersingular and not a singularity of $(Z, \omega)$. This implies that the marked points of $(\mathcal{E}, \nabla)$ are exactly the singularities of $(Z, \omega)$. \hfill \Box

Remark 4.9 Lemma 4.3 (ii) states that $\sigma_x = h_x/m_x$ with $m_x|(p - 1)$ and $\gcd(h_x, p) = 1$. Note that this is compatible with Proposition 3.6. Namely, if $x = x_i$ is a marked point of $(\mathcal{E}, \nabla)$ then $\sigma_i = n_i/(p - 1)$ where $n_i$ is either 0 or prime to $p$. If $x$ is not a marked point, then $n_x \equiv 0 \mod p$ and $\sigma_x = n_x/(p - 1) + 1 = (n_x + p - 1)/(p - 1)$ and $\gcd(n_x + p - 1, p) = 1$.

4.4 We now reverse the construction of §4.3. Let $(\mathcal{E}, \nabla)$ be an indigenous bundle on $X$ with regular singularities in the marked points $x_1, \ldots, x_r$. Suppose that $(\mathcal{E}, \nabla)$ is active, nilpotent, and normalized (§3.1).

Proposition 4.10 There exists a deformation datum $(Z, \omega)$ with singular points $x_1, \ldots, x_r$ such that the indigenous bundle associated to $(Z, \omega)$ by the construction of §4.3 is isomorphic to $(\mathcal{E}, \nabla)$. Moreover, $(Z, \omega)$ is unique up to equivalence.

Proof: Choose a horizontal and surjective homomorphism $\gamma : \mathcal{E} \to \mathcal{O}_X$ and identify the kernel of $\gamma$ with the line bundle $\mathcal{T}(S)$, where $S = \sum n_x \cdot x$ is the zero divisor of the $p$-curvature. Let $e_0$ be the canonical section of $\mathcal{E}$, corresponding to the choice of $\gamma$ (§3.1). Set

$$\phi_0 := \nabla(e_0).$$

Since $\gamma(e_0) = 1$, by definition, it follows that $\gamma(\nabla(D)e_0) = \nabla(D)\gamma(e_0) = 0$. Therefore $\phi_0$ is a rational section of the bundle $\mathcal{T}(S) \otimes \Omega^{log}\log_{X/k} = (\Omega^{log}_{X/k})^{\otimes(1-p)}(S)$. It is easy to check that $\phi_0$ does not depend on the choice of $\gamma$.

Let $t$ be a rational function on $X$ with $dt \neq 0$ and write $\phi_0 = v(dt) \otimes (1-p)$. Let $Z$ be a connected component of the nonsingular projective curve with generic equation $z^{p-1} = v^{-1}$.

Set $\omega := z \, dt$. One checks that the equivalence class of the pair $(Z, \omega)$ does not depend on the choice of the function $t$ (the notion of equivalence we use is that of Definition 4.1).

We claim that the pair $(Z, \omega)$ is a deformation datum. Indeed, by definition, the pair $(Z, \omega)$ satisfies (i) of Definition 4.1. Therefore it suffices to show that $\omega$ is logarithmic. Let $D := \partial/\partial t$. Then $D^p = 0$. By the definition of $\phi_0$, we have $\nabla(D)(e_0) = v \cdot D^{\otimes p}$. It follows that

$$\Psi_e(D^{\otimes p})(e_0) = \nabla(D)^{p-1}(v \cdot D^{\otimes p}) = D^{p-1}(v) \cdot D^{\otimes p}.$$

Recall that we have chosen $e_0$ such that $\gamma(e_0) = 1$. Therefore, (2) and (8) imply

$$D^{p-1}(v) = -1.$$
Lemma 4.6 (i) implies that $\omega$ is a logarithmic differential form. This proves that $(Z, \omega)$ is a deformation datum.

Let $(E', \nabla')$ be the indigenous bundle associated to the deformation datum $(Z, \omega)$, by the construction of §4.3. It is clear that the restriction of $E$ and of $E'$ to the complement $U$ of the supersingular points are isomorphic. Therefore, it follows from the uniqueness part of Proposition 4.8 (i) that $E$ and $E'$ are isomorphic. This finishes the proof of the lemma.

We can summarize the results of this section as follows.

**Theorem 4.11** The construction of §4.3 establishes a bijection between the set of equivalence classes of indigenous bundles on $X$ which are active and nilpotent and the set of equivalence classes of deformation data. Suppose that the deformation datum $(Z, \omega)$ corresponds to the indigenous bundle $(E, \nabla)$.

Then

(i) the supersingular points of $(Z, \omega)$ agree with the supersingular points of $(E, \nabla)$,

(ii) the singular points of $(Z, \omega)$ are the marked points of $(E, \nabla)$,

(iii) the points $x$ with $\sigma_x \neq 1$ and $\sigma_x \equiv 1 \pmod{p}$ are the unmarked spikes of $(E, \nabla)$

The relation between the order $n_x$ of a spike $x$ of $(E, \nabla)$ and the invariant $\sigma_x$ associated to $(Z, \omega)$ is expressed by (5).

In the situation of Theorem 4.11, we may talk about the signature of the deformation datum $(Z, \omega)$ as the signature of the indigenous bundle $(E, \nabla)$.

**Definition 4.12** Let $(X; \{x_1, \ldots, x_r\})$ be a marked curve over an algebraically closed field $k$ of characteristic $p > 0$. An indigenous bundle $(E, \nabla)$ on $(X; x_i)$ is called admissible if it does not have spikes outside the marked points.

**Corollary 4.13** Let $(E, \nabla)$ be an active and nilpotent indigenous bundle on $X$. Let $r$ the number of marked points and $d$ the number of supersingular points. Suppose that

$$\frac{2d}{p-1} + \sum_{i=1}^{r} \sigma_i = 2g(X) - 2 + r.$$

Then $(E, \nabla)$ is admissible.

**Proof:** This follows immediately from Theorem 4.11 and Lemma 4.3 (iv). In fact, the assumption implies that there are no points $x \in X$ with $\sigma_x \neq 1$ except for the singular and the supersingular points. \qed
5 Explicit construction of deformation data

In this section we suppose that $X = \mathbb{P}_k^1$. We are interested in constructing deformation data via their associated indigenous bundles. It turns out that on $\mathbb{P}^1$ it is more practical to replace indigenous bundles by their associated differential equation.

5.1 Let $t$ denote the standard parameter on $X = \mathbb{P}_k^1$. Set $D := \partial/\partial t$. We write $f'$ instead of $D(f)$. Let $r \geq 3$ and $x_1, \ldots, x_r \in X$ pairwise distinct closed points. We assume that $x_r = \infty$. Set $U_0 := X - \{\infty\} = A^1_k$ and $U := X - \{x_1, \ldots, x_r\}$.

Let $(\mathcal{E}, \nabla)$ be an indigenous bundle on $(X; x_i)$ which is active and nilpotent. We assume that $(\mathcal{E}, \nabla)$ is normalized and choose a horizontal and surjective morphism $\gamma : \mathcal{E} \rightarrow \mathcal{O}_X$ ($\S 3.1$). We let $e_1$ be an everywhere invertible section of $F^1\mathcal{E}$ on $U_0$; it is unique up to multiplication by a constant in $k^\times$. Set $e_2 := \nabla(D)(e_1)$. It follows from Definition 2.1 (i) that $e := (e_1, e_2)$ forms a basis of $\mathcal{E}$ on $U$. Hence we may write $\nabla(D)(e_2) = -p_2 e_1 - p_1 e_2$, or

$$\nabla(D)(e) = \begin{pmatrix} 0 & -p_2 \\ 1 & -p_1 \end{pmatrix} \cdot e,$$

where $p_1, p_2$ are rational functions which are regular on $U$. The differential operator associated to $(\mathcal{E}, \nabla)$ is the second order operator

$$L(u) := u'' + p_1 u' + p_2 u.$$

In $[9]$, Appendix, one can find a more general dictionary between flat bundles on $\mathbb{P}^1$ and ordinary differential operators. In particular, it is shown that the following properties of the operator $L$ follow from the corresponding properties of $(\mathcal{E}, \nabla)$:

- $L$ has regular singularities in the marked points $x_i$,
- the local exponents at $x_i \neq \infty$ are $(\alpha_i, 0)$ (the same as those of $(\mathcal{E}, \nabla)$).

Remark 5.1 For some authors $L(u) = 0$ is the differential equation associated to the dual of the bundle $(\mathcal{E}, \nabla)$, since it describes the horizontal vectors of the flat vector bundle dual to $(\mathcal{E}, \nabla)$, see $[9]$ appendix. The advantage of our convention is that the local exponents of $(\mathcal{E}, \nabla)$ coincide with the notion of local exponents of the differential operator $L$ in the classical sense, see for example $[19]$, $\S 1.2.5$. 

By [9], nilpotence of the $p$-curvature of $(\mathcal{E}, \nabla)$ is equivalent to the assertion that the equation $L(u) = 0$ has sufficiently many solutions in the weak sense. This means that both the equation $L(u) = 0$ and its Wronskian equation $L_W(w) = w' + p_1 w$ have an algebraic solution. The nonvanishing of the $p$-curvature means that the solution space of $L(u) = 0$ has dimension one over $k(t^p)$.

**Proposition 5.2** Let $L$ be the differential operator associated to the active, nilpotent, normalized bundle $(\mathcal{E}, \nabla)$.

(i) The equation $L(u) = 0$ has a polynomial solution $u \in k[t]$ with simple zeros exactly in the supersingular points of $(\mathcal{E}, \nabla)$. It is unique up to multiplication by a constant in $k^\times$.

(ii) The local exponents of $L$ at $x_r = \infty$ are $(-d + \alpha_r, -d)$, where $d$ is the number of supersingular points of $(\mathcal{E}, \nabla)$.

**Proof:** It follows from the choice of $e_1$ that $u := \gamma(e_1)$ is a polynomial that does not vanish at the marked points. A simple computation shows that $L(u) = 0$. This proves the first assertion of (i). The remaining assertions of (i) are clear. See also the proof of Lemma 3.4.

Write $(\gamma_1, \gamma_2)$ for the local exponents of $L$ at $\infty$. An easy computation relating the local exponents of $L$ with those of $(\mathcal{E}, \nabla)$ shows that $\gamma_1 - \gamma_2 = \alpha_r$ (up to renumbering). The Fuchs’ Relation (see [19], §1.2.6) states that

\begin{equation}
\tag{10}
r - 2 = \gamma_1 + \gamma_2 + \sum_{i \neq r} \alpha_i = 2\gamma_2 + \sum_{i=1}^{r} \alpha_i \in \mathbb{F}_p.
\end{equation}

Let $n_x$ denote the spike order of $(\mathcal{E}, \nabla)$ at a point $x \in X$ and $\sigma_x$ the invariant attached to $x$ and the deformation datum corresponding to $(\mathcal{E}, \nabla)$. Recall that

\[ \frac{n_x}{p - 1} = \begin{cases} 
\sigma_x & \text{if } x = x_i, \\
0 & \text{if } x \text{ is supersingular,} \\
\sigma_x - 1 & \text{otherwise.}
\end{cases} \]

We have seen in §3.3 that $n_x \equiv -\alpha_i \mod p$ if $x = x_i$ and is congruent to 0 otherwise. Moreover, $\sigma_x - 1 = 2/(p - 1)$ if $x$ is supersingular. Therefore Lemma 4.3 (iv) implies that

\[ r - 2 = \sum_{i=1}^{r} \sigma_i + \sum_{x \neq x_i} (\sigma_x - 1) \equiv -2d + \sum_{i=1}^{r} \alpha_i \pmod{p}. \]

Together with (10), this shows that $\gamma_2 = -d$ and $\gamma_1 = -d + \alpha_r$. \qed
The goal of this section is to give a converse to Proposition 5.2. We fix distinct points $x_1, \ldots, x_r = \infty$ in $\mathbb{P}^1_k$ and a signature $\sigma$ with $r$ singularities. Let $d$ be the positive integer of Definition 4.4 (iii). We shall give a necessary and sufficient condition on a differential operator $L$ to be associated to an indigenous bundle with singularities $x_i$ and signature $\sigma$, as in §5.1.

Let $0 \leq \alpha_i < p$ be such that $\sigma_i \equiv \alpha_i \mod p$. Put

$$Q = \prod_{i=1}^{r-1} (t-x_i)^{1-\alpha_i}.$$  

A general differential operator $L$ of degree two with regular singularities at marked points $x_1, \ldots, x_r$ and local exponents $(\alpha_i, 0)$ at $x_i \neq \infty$ and $(-d+\alpha_r, -d)$ at $x_r = \infty$ can be written as $L = (\partial/\partial t)^2 + p_1(\partial/\partial t) + p_2$, where

$$p_1 = \sum_{i=1}^{r-1} -\alpha_i + 1, \quad p_2 = \frac{d(d-\alpha_r)t^{r-3} - \beta_{r-4}t^{r-4} - \cdots - \beta_0}{\prod_{i=1}^{r-1}(t-x_i)}$$

for constants $\beta_0, \ldots, \beta_{r-4}$. The constants $\beta_j$ are called the accessory parameters.

Note that $Q'/Q = p_1$ by (11). Before stating the existence result, we prove an easy lemma.

Lemma 5.3 Let $(\mathcal{E}, \nabla)$ be an active and nilpotent indigenous bundle of signature $\sigma$. Let $e_1, e_2$ be as in §5.1 and $u$ as in Proposition 5.2. The horizontal sections of $(\mathcal{E}, \nabla)$ are

$$Qw^p(-u'e_1 + ue_2),$$

for an arbitrary element $w \in k(t)$.

Proof: The assumption that $(\mathcal{E}, \nabla)$ is active implies that the solution space is 1-dimensional over $k(t)^p$. Therefore it suffices to show that $L(Q(-u'e_1 + ue_2)) = 0$. This follows by direct verification.

Proposition 5.4 Let $L = (\partial/\partial t)^2 + p_1(\partial/\partial t) + p_2$ be a second order differential operator with regular singularities in $x_i$ and local exponents $(\alpha_i, 0)$ (resp. $(-d+\alpha_r, -d)$ in $x_i$ for $i \neq r$ (resp. $x_r$). Suppose that $L$ has a polynomial solution $u$ of degree $d$ such that

(i) $u$ does not have zeros at the marked points,

(ii) we have

$$\text{ord}_x \frac{1}{Qu^2} = \begin{cases} 
0 & \text{if } x \text{ is supersingular}, \\
(p-1)\sigma_i - \alpha_i & \text{if } x = x_i, \\
(p-1)(\sigma_x - 1) & \text{otherwise}.
\end{cases}$$

19
Then $L$ is associated to an indigenous bundle $(E, \nabla)$ which is active, nilpotent and normalized. Moreover, the signature of $(E, \nabla)$ is $\sigma$. The supersingular points of $(E, \nabla)$ are exactly the zeros of $u$.

**Proof:** We define an indigenous bundle $(E, \nabla)$ corresponding to $u$ as follows. We define a function $w$ by

$$D_{p}^{-1} \left( \frac{1}{Qu^2} \right) = -w.$$ 

Set $\tilde{Q} := w^{-p}Q$. Then

$$D_{p}^{-1} \left( \frac{1}{Qu^2} \right) = -1.$$ 

On $U_0 = A^1$ we let $E$ be the trivial bundle with basis $e_1, e_2$ and with the connection $\nabla$ defined by (9).

A straightforward computation using the Fuchs’ relation (10) and expression for the local exponents at $\infty$ shows that we may take the pair $(e_1, \nabla(t\partial/\partial t)(e_1))$ as a basis of $E$ in a neighborhood of $x = \infty$, compare to [19], §1.2.6. This defines the bundle $(E, \nabla)$ on $X = \mathbb{P}_k^1$. We define the filtration $F^1E \subset E$ as the maximal line subbundle which contains $e_1$ as a rational section. It follows immediately that the Kodaira–Spencer map is an isomorphism.

Let $M$ be the kernel of the $p$-curvature of $E$. We leave it to the reader to check that the image of $e_1/u$ in $L := E/M$ is invertible on $U_0$. One computes that

$$\nabla(D) \frac{e_1}{u} = \frac{1}{Qu^2} \eta,$$

where $\eta = \tilde{Q}(-u'e_1 + ue_2)$ is a horizontal section of $E$ (Lemma 5.3). Therefore

$$\Psi_E(D^{\otimes p}) \frac{e_1}{u} = D_{p}^{-1} \left( \frac{1}{Qu^2} \right) \eta = -\eta,$$

by (12). Using the notation of (2), the $p$-curvature considered as $\mathcal{O}_X$-linear morphism $T \to \text{End}_{\mathcal{O}_X}(E)$ is given by

$$\Psi_E(D^{\otimes p}) \frac{e_1}{u} = -D^{\otimes p}.$$ 

Therefore, the bundle $(E, \nabla)$ is indigenous, active, nilpotent and normalized. Moreover, $e_0 := e_1/u$ is the canonical section of the bundle $(E, \nabla)$, as defined in §3.2.

It follows from (13) that the order of zero of $\Psi_E$ is

$$n_x = \begin{cases} 
\alpha_i + \text{ord}_x D_{p}^{-1} \left( \frac{1}{Qu^2} \right) & \text{if } x = x_i \text{ is a marked point}, \\
\text{ord}_x D_{p}^{-1} \left( \frac{1}{Qu^2} \right) & \text{otherwise}.
\end{cases}$$

This shows that the signature of $(E, \nabla)$ is as stated in the proposition. \hfill $\square$
Dwork’s accessory parameter problem is to find values for \((\beta_j)\) such that the operator \(L\) has nilpotent curvature. We are interested in those \(L\) which satisfy the stronger conditions of Proposition 5.4 above. Namely, the \(p\)-curvature of \(L\) should be nilpotent and nonvanishing and \(L(u) = 0\) should have a polynomial solution of a given degree \(d\) which does not vanish at the marked points.

**Definition 5.5** A solution of the strong accessory parameter problem with signature \((\sigma_i)\) and supersingular degree \(d\) is a tuple \((x_i; \beta_j) \in k^r \times k^{r-3}\) such that the associated operator \(L\) has a polynomial solution \(u\) of degree \(d\) which satisfies the conditions of Proposition 5.4.

Let \((x_i; \beta_j)\) be a solution of the strong accessory parameter problem. Let \(L\) be the differential operator and \((E, \nabla)\) the (normalized) indigenous bundle corresponding to the solution. Let \(u\) be the polynomial solution of \(L(u) = 0\) of degree \(d\). According to Theorem 4.11, \((E, \nabla)\) gives rise to a deformation datum \((Z, \omega)\). The following proposition shows that we can write down \((Z, \omega)\) very explicitly in terms of the solution \(u\).

Let \(x_{r+1}, \ldots, x_s\) be the set of spikes which are not marked points. As usual, we write \(n_i = n_{x_i}\) for the order of the \(p\)-curvature \(\Psi_E\) at \(x_i\). Let \(a_i\) be the unique integer such that \(0 \leq a_i < p-1\) and \(a_i \equiv n_i \mod p-1\). We may define a positive integer \(\nu_i\) by

\[
\nu_i = \frac{n_i - a_i}{p-1}
\]

if \(1 \leq i \leq r\) and \( \nu_i = 1 + \frac{n_i - a_i}{p-1}\) otherwise. Recall that \(n_i \equiv -a_i \mod p\) if \(1 \leq i \leq r\) and \(n_i \equiv 0 \mod p\) otherwise. Therefore \(\alpha_i \equiv \nu_i - a_i \mod p\) if \(1 \leq i \leq r\).

**Proposition 5.6** There is a constant \(\epsilon \in k^\times\) such that the deformation datum is \((Z, \omega)\), where \(Z \rightarrow X\) is the tamely ramified cyclic cover with generic equation

\[
z^{p-1} = \prod_{i \neq r} (t - x_i)^{a_i} u^2
\]

and

\[
\omega = \frac{\epsilon z dt}{\prod_{i \neq r} (t - x_i)^{-\nu_i +1}}.
\]

**Proof:** At every point \(x \in X\) we have the invariant \(\sigma_x = h_x/m_x\) attached to the deformation datum \((Z, \omega)\). We have seen in §4 that \(\sigma_x = (p+1)/(p-1)\) if and only if \(x\) is a supersingular point. By Proposition 5.2, this means that \(x\) is a zero of the polynomial \(u\). At all other points \(x = x_i\) we have

\[
\sigma_i := \sigma_{x_i} = \nu_i + \frac{a_i}{p-1}
\]

with \(a_i\) and \(\nu_i\) as defined before the statement of the proposition. It is now easy to see that the cyclic cover \(Z \rightarrow X\) can be identified with the cover defined by (14). Furthermore, the differential in (15) has the same divisor as \(\omega\) and is therefore equal to \(\omega\) for a suitable constant \(\epsilon\). The proposition is proved. \(\square\)
In [2], Proposition 3.2.2 a direct proof of this proposition is given which does not use the comparison with indigenous bundles. A key point is to show that $1/Qu^2$ does not have residues in the supersingular points. This follows in our set-up from Proposition 3.6 (iv).

Given a deformation datum $(Z, \omega)$, its signature $\sigma$ determines the number $d$ of supersingular points, by Lemma 4 (iv):

\begin{equation}
(16)
\frac{p-1}{2}(s-2 - \sum_{i=1}^{s} \sigma_i).
\end{equation}

The proof of Proposition 5.6 shows that $(Z, \omega)$ is essentially determined by its signature and the position of the singular and the supersingular points. Therefore, in order to construct a deformation datum with given signature, one has to determine the position of the singular points $x_i$ and the polynomial $u$ of degree $d$ (whose zeros are the supersingular points) such that the differential $\omega'$ defined in Proposition 5.6 is logarithmic. Modulo the automorphisms of $X = \mathbb{P}^1$, this conditions gives rise to $2 + d$ equations in $r + d - 3$ variables.

On the other hand, by Proposition 5.6 the deformation datum $(Z, \omega)$ corresponds to a solution of the strong accessory parameter problem. This problem corresponds to $2(r-3)$ equations in $2(r-3)$ variables. So if $d$ is large compared to $r$, this gives a much better method to construct deformation data. This is particularly striking for $r = 3$, as the next example illustrates.

**Example 5.7** Suppose that $r = 3$. We may assume that the marked points are $(x_1, x_2, x_3) = (0, 1, \infty)$. Corollary 4.13 implies that there are no spikes outside the marked points. Equation 16 implies that $\nu_i = 0$ for $i = 1, 2, 3$ and that $a_1 + a_2 + a_3 \leq p - 1$ is even. We have $d = (p-1 - a_1 - a_2 - a_3)/2$. Since the number of accessory parameters is zero, there exists a unique order two differential equation with regular singularities at $0, 1, \infty$ with local exponents $(-a_1, 0), (-a_2, 0), (-a_3 - d, -d)$. This is the hypergeometric equation

\[ t(t-1)u'' + [(A + B + 1)t - C]u' + ABu = 0, \]

where $A := (1 + a_1 + a_2 + a_3)/2$, $B := (1 + a_1 + a_2 - a_3)/2$ and $C := 1 + a_1$. It is proved in [4], Proposition 3.2, that this equation has a polynomial solution $u$ of degree $d$. It follows that (14) and (15) define a deformation datum $(Z, \omega)$ with singular points 0, 1, $\infty$ and signature $(a_i/(p-1))_{i=1,2,3}$.

For $a_1 = a_2 = a_3 = 0$ this is Example 4.5, giving rise to the Hasse polynomial and Gauss’ hypergeometric equation.

**5.3** In the rest of this section, we consider the case of special deformation data. As explained in §4.2, these correspond to the reduction of $G$-Galois covers of $\mathbb{P}^1$ branched at 3 points. Here $G$ is some group whose order is strictly divisible by $p$. Let $(Z, \omega)$ be a deformation datum on $X = \mathbb{P}^1$, with singular points signature $(\sigma_1, \ldots, \sigma_s)$. As in §5.2, we write $\sigma_i = \nu_i + a_i/(p-1)$ with $\nu_i \geq 0$ and $0 \leq a_i < p - 1$. 

22
Definition 5.8 The deformation datum \((Z, \omega)\) is called special if \(\nu_i = 0\) for exactly three indices \(i\) and \(\nu_i = 1\) for the other indices.

Note that if \((Z, \omega)\) is special then \(\sigma_x \not\equiv 1 \mod p\), by Corollary 4.13. In other words, the corresponding indigenous bundle \((E, \nabla)\) is admissible (Definition 4.12).

We consider the case of special deformation data with four singularities. We may assume that the singular points are \(x_4 = 0, x_1 = 1, x_2 = \lambda, x_3 = \infty\), for a variable \(\lambda\), and that \(\nu_1 = \nu_2 = \nu_3 = 0\) and \(\nu_4 = 1\). Let \(p \geq 7\) and \((a_1, a_2, a_3, a_4) = (0, 0, 1, 3)\). Put \(d = (p - 5)/2\).

We start by describing the equations \(F, G \in \mathbb{F}_p[\lambda, \beta]\) whose zero locus corresponds to the set of deformation data with signature \((0, 0, 1/(p-1), (p+2)/(p-1))\), by using the conditions of Proposition 5.4. Set \(u = \sum_{i \geq 0} u_i t^i\), with \(u_0 = 1\). Assuming that \(L(u) = 0\), we find a recursion

\[
\lambda A(i) u_{i+1} = (B(i) + \beta) u_i - C(i) u_{i-1},
\]

with

\[
A(i) = (i + 1)(i + a_4), \quad B(i) = i^2(1 + \lambda) + i(a_2 + a_4 + \lambda(a_1 + a_4)),
\]

\[
C(i) = (i - 1 - d)(i - 1 - d - a_3).
\]

for the coefficients of \(u\). Since \(p - a_4 = p - 3 > d = (p - 5)/2\), we have that \(A(i) \neq 0\), for \(i = 0, \ldots, d\). We let \(F\) be the numerator of \(u(d + 1)\). Since \(C(d + 1) = 0\), it follows that \(F = 0\) implies that \(L\) has a solution \(u\) of degree \(d\).

(In [1] we find an interpretation of this equation as eigenvalue problem.) One easily sees that \(\deg_d(F) = d + 1\).

Let

\[
w = \frac{1}{Qu^2} = \frac{1}{t^3(t-1)(t-\lambda)u^2}.
\]

One computes that

\[
\text{Res}_{t=0} w = u_0^2 + (-2u_1 u_0 + 3u_1^2 - 2u_2 u_0 + u_0^2)\lambda^2 + (u_0^2 - 2u_0 u_1)\lambda.
\]

The recursion (17) for the \(u_i\) implies that

\[
u = 1 - \frac{\beta}{3\lambda} t + \frac{\beta^2 - 4(1 + \lambda)\beta - 6}{24\lambda^2} t^2 + \text{higher order terms}.
\]

We replace \(u\) by \(24\lambda^2 u\) and define \(G\) to be the numerator of the residue at \(t = 0\) of \(w\), i.e.

\[
G = 576\lambda^2 + (480\beta + 576)\lambda + 864 + 60\beta^2 + 480\beta.
\]

The problem is now to show that for every \(p \geq 7\) there exist solutions \((\lambda, \beta) \in \mathbb{F}_p^2\) of \(F = G = 0\) such that the points \((0, 1, \infty, \lambda, \tau_i)\) are pairwise distinct. Here \(\tau_i\) are the zeros of \(u\), i.e. the supersingular points. For \(x \neq 0, 1, \lambda, \infty\), we have that \(\text{ord}_x(u) \equiv 1 \mod p\). Since \(\text{deg}_x(u) < p\), it follows that if the points \((0, 1, \lambda, \infty, \tau_i)\) are not pairwise distinct, then \(\lambda \in \{0, 1, \infty\}\).
Table 1: The number of deformation data with signature $(0,0,\frac{1}{p-1},\frac{p+2}{p-1})$

| $p$ | $d$ | total deg | good deg | $p$ | $d$ | total deg | good deg |
|-----|-----|-----------|----------|-----|-----|-----------|----------|
| 7   | 1   | 12        | 3        | 53  | 29  | 148       | 48       |
| 11  | 3   | 24        | 8        | 59  | 27  | 168       | 56       |
| 13  | 4   | 30        | 10       | 61  | 28  | 174       | 58       |
| 17  | 6   | 33        | 3        | 67  | 31  | 191       | 62       |
| 19  | 7   | 48        | 16       | 71  | 33  | 203       | 67       |
| 23  | 9   | 60        | 18       | 73  | 24  | 150       | 50       |
| 29  | 12  | 78        | 25       | 79  | 37  | 226       | 73       |
| 31  | 13  | 83        | 27       | 83  | 39  | 239       | 79       |
| 41  | 18  | 112       | 36       | 83  | 39  | 239       | 79       |
| 43  | 19  | 118       | 38       | 97  | 46  | 282       | 93       |
| 47  | 21  | 132       | 43       |     |     |           |          |

A solution $(\lambda, \beta) \in \overline{\mathbb{F}}^2_p$ of $F = G = 0$ is called a good solution if $\lambda \not\in \{0, 1\}$. Table 1 gives a list of the degree of the total solution space and the degree of the good solutions as function of $p$. We already remarked that all good solutions have multiplicity one. The progression of the good degree strongly suggests that there are deformation data for all $p \geq 7$.

We finish the paper with another example. Let $p \geq 11$ and $(a_1, a_2, a_3, a_4) = (0, 0, p-8, 3)$. We have that $d = (p-1-a_1-a_2-a_3-a_4)/2 = 2$. We compute equations $F, G \in \overline{\mathbb{F}}_p[\lambda, \beta]$ for the locus of deformation data, as above. One finds that

\[
F = 40\beta\lambda^2 + (80\beta + 14\beta^2 + 360)\lambda + 132\beta + 14\beta^2 + 360 + \beta^3,
\]
\[
G = -4\beta\lambda^3 + (3\beta^2 - 12\beta - 24)\lambda^2 + (12 - 8\beta)\lambda + 12.
\]

As in the previous example, the bad solutions satisfy $\lambda = 0, 1$. Using $p \geq 11$, one shows that there are no trivial solutions in this case. We conclude that for all $p \geq 11$ there exist special deformation data with signature $(0, 0, (p-8)/(p-1), (p+2)/(p-1))$.

It is plausible but unproven that every special deformation datum arises from a Belyi map with bad reduction. For results in this direction see [3]. In this paper one finds a sample of possible applications of existence results for deformation data to Galois theory.
6 Ordinary indigenous bundles

6.1 In this section we discuss Mochizuki’s notion of hyperbolically ordinary indigenous bundles. For a discussion of this notion we refer to [14], Introduction §2.1 (page 72) and Chapter IV.

Let \((g, r)\) be integers such that \(2g - 2 + r \geq 3\). Denote by \(\mathcal{M}_{g,r}/\mathbb{F}_p\) the stack of \(r\)-marked curves \((X;\{x_1, \ldots, x_r\})\) with \(g(X) = g\). For simplicity, we call such curves \((g, r)\)-marked. Let \(N_{g,r}\) be the stack of admissible indigenous bundles \((\mathcal{E}, \nabla)\) on an \((g, r)\)-marked curve \((X; x_i)\). We write \(\pi : N_{g,r} \rightarrow M_{g,r}\) for the natural projection.

**Definition 6.1** Let \((\mathcal{E}, \nabla)\) be an admissible indigenous bundle on a \((g, r)\)-marked curve \((X; x_i)\). We say that \((\mathcal{E}, \nabla)\) is hyperbolically ordinary if \(\pi : N_{g,r} \rightarrow M_{g,r}\) is étale at the point corresponding to \((\mathcal{E}, \nabla)\). We write \(N_{g,r}^{\text{ord}} \subset N_{g,r}\) for the substack of hyperbolically ordinary indigenous bundles. A \((g, r)\)-marked curve \((X; x_i)\) is called hyperbolically ordinary if it lies in the image of \(\pi : N_{g,r}^{\text{ord}} \rightarrow M_{g,r}\).

The notion hyperbolically ordinary should not be confused with the notion of an ordinary abelian variety. Since the latter notion plays no role in this paper, we will call “hyperbolically ordinary” ordinary for short.

Example 4.5 implies that \((\mathbb{P}^1_k; \{0, 1, \infty\})\) is ordinary. See also [13], page 1045 and [14], §IV.2.1, page 205. Mochizuki shows that the generic \((g, r)\)-marked curve is ordinary, using the case \((g, r) = (0, 3)\) and a deformation argument ([13], Corollary III.3.8, page 1048). In this section we want to show that \(\pi : N_{0,4}^{\text{ord}} \rightarrow M_{0,4}\) is surjective. In other words, every \((0, 4)\)-marked curve is ordinary. For this we use the translation of indigenous bundles in terms of solutions of differential equations (§5.2). This result could probably also be shown directly using the language of indigenous bundles. Mochizuki shows that \(\pi : N_{g,r} \rightarrow M_{g,r}\) is finite and flat of degree \(p^{3g-3+r}\) ([13], Theorem II.2.3, page 1029f). For \(g = 0\) this already follows from the work of Dwork [6]. For \((g, r) = (0, 4)\) we find therefore that \(\pi\) is separable. This also follows from Lemma 6.3 below.

6.2 Let \((\mathbb{P}^1_k; x_i)\) be a \((0, 4)\)-marked curve. We may choose a suitable coordinate \(t\) on \(\mathbb{P}^1_k\) such that \(x_1 = 0, x_2 = 1, x_3 = \lambda, x_4 = \infty\). This identifies \(M_{0,4}\) with \(\mathbb{P}^1_{\lambda} - \{0, 1, \infty\}\). Our goal is to construct indigenous bundles \((\mathcal{E}, \nabla)\) on \((\mathbb{P}^1_k; x_i)\) with logarithmic monodromy in \(x_i\) for \(i = 1, \ldots, 4\). Proposition 5.4 translates this into the existence of suitable solutions of the differential operator

\[
L = \frac{\partial^2}{\partial t^2} + \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{t-\lambda}\right) \frac{\partial}{\partial t} + \frac{t - \beta}{t(t-1)(t-\lambda)}
\]

(18)
Lemma 6.2  
(i) Let \((\beta, \lambda)\) be such that \(L = L_{\lambda, \beta}\) has a polynomial solution \(u \in k(\lambda, \beta)[t]\). Then \(L\) has a unique monic polynomial solution of degree \(p - 1\).

(ii) For \(L = L_{\lambda, \beta}\) as in (i), the corresponding indigenous bundle is active and admissible (Definition 4.12).

**Proof:** Part (i) is proved in [1], Lemma 1.

Suppose that \(L\) has a polynomial solution \(u \in k(\lambda, \beta)[t]\) of degree \(p - 1\). Fix \(i \in \{1, 2, 3\}\). Since the local exponents of \(L\) at \(t = x_i\) are \((0, 0)\), we have that \(\text{ord}_{x_i}(u) \equiv 0 \mod p\). Since the degree of \(u\) is \(p - 1 < p\), we conclude therefore that \(u\) does not have a zero at \(t = x_i\). Proposition 5.4 implies therefore that \(u\) corresponds to an indigenous bundle \((\xi, \nabla)\).

Suppose that the \(p\)-curvature of \(L\) is zero. Then the differential operator \(L\) has two linearly independent polynomial solutions \(u_1, u_2\) ([9], appendix.) Now Proposition 5.1 of [9] implies that the local exponents \((\gamma_1, \gamma_2)\) of \(L\) at \(t = \infty\) are distinct. But this contradicts our assumptions. We conclude that \(L\) is active. Corollary 4.13 implies that \((\xi, \nabla)\) does not have unmarked spikes. \(\square\)

Beukers’ proof of Lemma 6.2 (i) is a concrete version of [9], Lemma 1, page 174. The proof uses the recursion for the coefficients of a solution \(u\), together with a description of the accessory parameter problem as eigenvalue problem. Beukers shows that \(\pi : N_{0,4} \to M_{0,4}\) is surjective. This also follows from the work of Mochizuki. Unfortunately, Beukers’ description only seems to works for 4 marked points.

Lemma 6.3 There exists an irreducible component \(N_1 \subset N_{0,4}\) such that the restriction \(\pi_1 : N_1 \to M_{0,4}\) isomorphic to \(\mathbb{P}_1^1 \setminus \{0, 1, \infty\}\) of \(\pi\) has degree 1.

**Proof:** Let \(N_1 = \text{Spec}(k(\lambda, 1/(\lambda - 1), \beta)/(\lambda + \beta + 1))\), and write \(\pi_1 : N_1 \to \mathbb{P}_1^1\) for the natural projection. We claim that the differential operator \(L\) has a unique monic solution \(u\) of degree \(p - 1\) over \(N_1\).

Let \(\bar{u} = \sum_{i \geq 0} u_i t^i \in \mathbb{P}_1^1[[t]]\) be a solution of \(L\) with \(u_0 \neq 0\). One easily checks that the coefficients \(u_i\) of \(u\) satisfy a recursion
\[
\lambda(i + 1)^2 u_{i+1} = (1 + \lambda)(i^2 + i + 1)u_i - i^2 u_{i-1}.
\]
Note that \(u := \sum_{i=0}^{p-1} u_i t^i\) is a solution of \(L\) if and only if
\[
(1 + \lambda)u_{p-1} - u_{p-2} = 0.
\]
Moreover, \(u_1, \ldots, u_{p-1}\) are uniquely determined by (17) and \(u_0\).

Put \(v_{p-1-i} := u_i \lambda^i\) for \(i = 0, \ldots, p - 1\). One easily deduces from (19) that the \(v_i\) satisfy the same recursion (19). Since the coefficients \(v_0, \ldots, v_{p-1}\) are uniquely determined by \(u_0\) and the recursion, it follows that
\[
\frac{u_i}{u_0} = \frac{v_i}{v_0} = \frac{u_{p-1-i}}{\lambda^{p-1-i} u_{p-1}}, \quad i = 0, \ldots, p - 1.
\]
Since $u_1 = (1 + \lambda)u_0/\lambda$, we deduce that (20) is satisfied. □

Lemma 6.3 can be understood as follows. The marked curve $(\mathbb{P}^1_k, \{0, 1, \lambda, \infty\})$ has an automorphism $\sigma: t \mapsto \lambda/t$. In Lemma 6.3, we have chosen the accessory parameter $\beta$ so that $\sigma$ is a symmetry of the differential operator $L$.

**Proposition 6.4** The morphism
\[ \pi: N_{0,4}^{\sigma} \rightarrow M_{0,4} \]
is surjective.

**Proof:** Lemma 6.3 implies that $N_{0,4}$ has an irreducible component such that the restriction of $\pi$ to this component has degree 1. We denote this component also by $N_1$. Lemma 6.2 (ii) states that all points of $N_{0,4}$ corresponds to admissible bundles. Therefore it follows from [13], Corollary II.2.16 (page 1043), that $N_1$ is smooth over $\mathbb{P}^1_{\mathbb{P}}$. Hence $N_1$ does not intersect the rest of $N_{0,4}$. This implies that the restriction of $\pi$ to $N_1$ is étale, and $N_1$ is a substack of $N_{0,4}^{\sigma}$. The proposition follows. □

The following corollary immediately follows from the proof of Proposition 6.4. This answers an expectation of Mochizuki ([14], first remark on page 205).

**Corollary 6.5** The stack $N_{0,4}$ is disconnected for all $p \geq 3$.

It is interesting to study what the complement of $N_1$ in $N_{0,4}$ looks like. The following lemma describes this for $p = 5$. It can be easily checked using for example magma. The lemma illustrates that admissible and ordinary are in general two different notions.

**Lemma 6.6** Let $p = 5$. Then $N_{0,4}$ is the disjoint union of two smooth absolutely irreducible components $N_1$ and $N_2$. The restriction of $\pi$ to $N_2$ has degree 4, and is ramified at 12 points of order 2. The branch locus consists of 8 points.

To finish this section, we consider the analogous question in the torally indigenous case. Namely, for $i = 1, 2, 3, 4$ let $0 \leq \sigma_i = a_i/(p-1) < 1$ be rational numbers, and put $\sigma = (\sigma_i)$. We write $N_{0,4}(\sigma)$ for the stack of admissible indigenous bundles of signature $\sigma$. Here the signature of an indigenous bundle is the signature of the corresponding deformation datum. Recall that the signature can be expressed in terms of the order of the spike in $t = x_i$. The stacks $N_{0,4}(\sigma)$ do not occur in the work of Mochizuki, since Mochizuki does not consider the signature as combinatorial invariant. However, $N_{0,4}(\sigma)$ is a substack of $N_{g,r}[a_1 + a_2 + a_3 + a_4]$ of admissible indigenous bundles which are spiked of strength $a_1 + a_2 + a_3 + a_4$ ([14], Introduction §1.2, page 39).

The following proposition is proved in [2], §3.4. It uses the deformation theory of $\mu_p$-torsors ([18]). The proposition illustrates that the situation becomes more complicated for arbitrary signature. Namely, the morphism $N_{0,4}^{\sigma}(\sigma) \rightarrow M_{0,4}$ is not always surjective.
Proposition 6.7 Suppose that $a_1 + a_2 + a_3 + a_4$ is even and nonzero. We define a unique integer $0 \leq d < p - 1$ by the property

$$2d + a_1 + a_2 + a_3 + a_4 \equiv 0 \mod p,$$

and put $\nu = (2d + a_1 + a_2 + a_3 + a_4)/(p - 1)$.

(i) If $\nu = 1$ and $N_{0,4}(\sigma)$ is nonempty, then $\dim N_{0,4}(\sigma) = 0$.

(ii) If $\nu = 2$, then $\dim N_{0,4}(\sigma) = 1$, and $\pi : N_{0,4}^{ord}(\sigma) \to M_{0,4}$ is surjective.

(iii) $\nu = 3$, then $N_{0,4}(\sigma)$ is empty.

Proof: The statement on the dimension of $N_{0,4}(\sigma)$ is proved in [2], Lemma 3.4.2. The surjectivity of $\pi$ in (ii) is proved in [2], Proposition 3.4.3. The nonexistence in case (iii) follows from Lemma 4.3 (iv).

A similar proposition for special deformation data has been proved in [18], Theorem 5.14. Let $\sigma = (\sigma_i)$ be the signature of a special deformation datum. I.e. there are three indices $i$ such that $0 \leq \sigma_i < 1$. For all other $i$ we have that $1 \leq \sigma_i < 2$. Then the space $N_{0,3}(\sigma)$ of special deformation data has dimension 0, if it is nonempty. If $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is as in Example 5.7, then $N_{0,3}(\sigma)$ is exactly one point. Therefore, the corresponding indigenous bundles could also be called ordinary.

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