Extended Edge States in Finite Hall Systems

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Abstract

We study edge states of a random Schrödinger operator for an electron submitted to a magnetic field in a finite macroscopic two dimensional system of linear dimensions equal to $L$. The $y$ direction is $L$-periodic and in the $x$ direction the electron is confined by two smoothly increasing parallel boundary potentials. We prove that, with large probability, for an energy range in the first spectral gap of the bulk Hamiltonian, the spectrum of the full Hamiltonian consists only on two sets of eigenenergies whose eigenfunctions have average velocities which are strictly positive/negative, uniformly with respect to the size of the system. Our result gives a well defined meaning to the notion of edge states for a finite cylinder with two boundaries, and extends previous studies on systems with only one boundary.
1 Introduction

In this paper we investigate spectral properties of random Hamiltonians describing the dynamics of a spinless quantum particle on a cylinder of circumference $L$ and confined along the cylinder axis by two boundaries separated by the distance $L$. The particle is subject to an external homogeneous magnetic field and a weak random potential. A precise statement of the model is given in section 2. The physical interest of the model comes from the integral quantum Hall effect occurring in disordered two dimensional electronic systems subject to a uniform magnetic field, for example, in the interface of a heterojunction $[\text{KDP}]$, $[\text{PG}]$. In his treatment of this effect Halperin $[\text{H}]$ pointed out the fundamental role played by edge states carrying boundary diamagnetic currents, and it is therefore important to understand the spectral properties of finite but macroscopic quantum Hall samples with boundaries. A short review of the spectral properties of finite quantum Hall systems can be found in $[\text{FM2}]$.

The study of random magnetic Hamiltonians with boundaries is recent and, before we address the case of a (finite) cylinder, we wish to briefly discuss a few existing results. The case of a semi-infinite plane with one planar boundary, modeled by a smooth confining potential $U$ or a Dirichlet condition at $x = 0$, is satisfactorily understood. In this case it is proven that the spectrum of the Hamiltonian $H^c_\omega = H_L + U + V_\omega$, $H_L$ being the Landau Hamiltonian for a uniform magnetic field $B$ and $V_\omega$ an Anderson-type random potential, has absolutely continuous components inside the complement of Landau bands, for $\|V_\omega\|_\infty \ll B$ ($[\text{FGW}]$, $[\text{IBP}]$ and $[\text{MMP}]$). The proof of this statement is essentially based on Mourre theory with conjugate operator $y$. The positivity of $i[H^c_\omega, y]$ in suitable spectral subspaces of $H^c_\omega$ leads to the absolutely continuous nature of the spectrum. Since this commutator is equal to the velocity $v_y$ this means that states in the corresponding spectral subspaces propagate in the $y$–direction along the edge with positive velocity.
For the case of a strip with two boundaries, separated by a distance $L$, few results are known. For a general (random) potential we expect that there is no absolutely continuous component in the spectrum, because the impurities may induce a tunnelling (or backscattering) between the two boundaries and thus propagating edge states along each boundary cannot persist for an infinite time. In [CHS] the authors have shown that such states survive, for a finite time related to the quantum tunnelling time between the two edges. In [EJK] instead of a strip of size $L$, the authors consider a parabolic channel. They show that if the perturbation $V$ is periodic, or if $V$ is small enough and decays fast enough in the $y$–direction, then the absolutely continuous spectrum survives in certain intervals, but their analysis does not cover true Anderson like potentials.

In this work we address the case of a macroscopic finite systems with two confining walls separated by a distance $L$ along the $x$–direction and with the $y$–direction of length $L$ made periodic (i.e. the geometry is that of a cylinder). The left (resp. right) walls are modeled by a smooth confining potential $U_L$ (resp. $U_r$) separated by a distance $L$, and the bulk between them contains impurities modeled by a random Anderson-like potential $V_\omega$. In this case, although the spectrum consist of discrete isolated eigenvalues, we show that there is a well defined notion of edge states associated to each boundary.

Let us explain our main result expressed in Theorem 1. We show that, with large probability, the spectrum of the random Hamiltonian

$$H_\omega = H_L + V_\omega + U_L + U_r$$

in an energy interval $\Delta \subset \left( \frac{1}{2}B + \|V_\omega\|_\infty, \frac{3}{2}B - \|V_\omega\|_\infty \right)$ consists in the union of two sets $\Sigma_L$ and $\Sigma_r$, which are small perturbations of the spectra $\sigma(H_L + U_L + V_\ell)$ and $\sigma(H_L + U_r + V_r)$, of the two single-boundary random Hamiltonians (see Section 2 for their precise definition). As in [FM1], the eigenvalues in $\Sigma_L$ and $\Sigma_r$ are characterised by their average velocity along the periodic direction $J_E = (\psi_E, v_y \psi_E)$: the eigenfunctions corresponding to the eigenvalues in $\Sigma_L$ (resp. $\Sigma_r$)
have a uniformly, negative (resp. positive) velocity, with respect to \( L \). These are the so-called edge states and from the constructions in the proofs it is possible to see that the eigenvalues in \( \Sigma_\ell \) (resp. \( \Sigma_r \)) correspond to eigenfunctions localised in the \( x^- \) direction near the left (resp. right) boundary.

Although our analysis is presented for a sample of size \( L \times L \) the same results can be straightforwardly extended to all geometries where the two boundaries are separated by any distance \( D \) at least \( O(\ln L) \) (assuming the length of the periodic direction is fixed to \( L \)). For distances \( D = O(1) \) our analysis does not hold, a fact which is consistent with [CHS]. In fact, we expect that by using the results in the present paper one could prove that a wave packet localised on the left boundary and with appropriate energy, will propagate along the left boundary up to a finite tunneling time and then, backscatter and propagate along the right boundary and so forth. The tunneling time is set by \( V_\omega \) and the distance \( D \) between the two boundaries. Thus if \( D = O(1) \) with respect to \( L \), this tunneling time is also \( O(1) \), and always remains much smaller than \( O(L) \) which is the time needed for a ballistic flight around the whole periodic direction \( y \).

The paper is organised as follows. In section 2 we present the precise definition of the model and state the main Theorem. Section 3 is concerned with the main mathematical tools used in our analysis: a Wegner estimate and a decoupling scheme of the cylinder into two semi-infinite ones. The proof of the main theorem is then completed in section 4. Some useful estimates and more technical material are collected in the appendices.

## 2 The Model and Main Result

We study the spectral properties of the family of random Hamiltonians

\[
H_\omega = H_L + U_\ell + U_r + V_\omega, \quad \omega \in \Omega_A
\]

(2.1)

acting in the Hilbert space \( L^2(\mathbb{R} \times [-\frac{L}{2}, \frac{L}{2}]) \) with periodic boundary conditions along \( y \): \( \psi(x, -\frac{L}{2}) = \psi(x, \frac{L}{2}) \). We choose the Landau gauge in which the kinetic
part has the form \( H_L = \frac{1}{2} p_x^2 + \frac{1}{2} (p_y - Bx)^2 \) with spectrum given by the Landau levels: \( \sigma(H_L) = \{(n + \frac{1}{2})B; n \in \mathbb{N}\} \). The potentials \( U_\ell \) and \( U_r \) representing the confinement along the \( x \)-direction at \( x = \pm \frac{L}{2} \) are independent of \( y \) and are supposed strictly monotonic, twice differentiable and satisfy

\[
c_1|x + \frac{L}{2}|^{m_1} \leq U_\ell(x) \leq c_2|x + \frac{L}{2}|^{m_2} \quad \text{for } x \leq -\frac{L}{2} \tag{2.2}
\]
\[
c_1|x - \frac{L}{2}|^{m_1} \leq U_r(x) \leq c_2|x - \frac{L}{2}|^{m_2} \quad \text{for } x \geq \frac{L}{2} \tag{2.3}
\]

for some constants \( 0 < c_1 < c_2, \ 2 \leq m_1 < m_2 < \infty \) and \( U_\ell(x) = 0 \) for \( x \geq -\frac{L}{2} \), \( U_r(x) = 0 \) for \( x \leq \frac{L}{2} \). The random potential \( V_\omega \) is given by the sum of local perturbations located at the sites of a finite lattice \( \Lambda = \{(n, m) \in \mathbb{Z}^2; n \in [\frac{-L}{2}, \frac{L}{2}], m \in [\frac{-L}{2}, \frac{L}{2}]\} \). Let \( V \geq 0, \ V \in C^2, \ \|V\|_\infty \leq V_0, \ supp\ V \subset B(0, \frac{1}{4}) \) (the open ball centered at \((0, 0)\) of radius \( \frac{1}{4} \)) and \( X_{n,m}(\omega) \) i.i.d. random variables with common bounded density \( h \in C^2([-1, 1]) \) representing the random strength of each local perturbation. Then \( V_\omega \) has the form

\[
V_\omega(x,y) = \sum_{(n,m) \in \Lambda} X_{n,m}(\omega) V(x-n,y-m) \tag{2.4}
\]

We denote by \( \mathbb{P}_\Lambda \) the product measure defined on the set of all possible realizations \( \Omega_\Lambda = [-1, 1]^\Lambda \). Clearly for each realization \( \omega \in \Omega_\Lambda \) we have \( \|V_\omega\| \leq V_0 \) and we suppose \( V_0 \ll B \).

For future use we collect some properties of three simpler random Hamiltonians. Let us first consider the pure single-boundary Hamiltonians

\[
H^0_\alpha = H_L + U_\alpha \quad \alpha = \ell, r . \tag{2.5}
\]

From translation invariance along \( y \) we deduce that for \( L = +\infty \) the spectrum consists of analytic and monotone decreasing (resp. increasing) branches \( \varepsilon_n^\ell(k) \) (resp. \( \varepsilon_n^r(k) \)) where \( k \in \mathbb{R} \) is the wave number associated to \( p_y \). One has \( \lim_{k \to +\infty} \varepsilon_n^\ell(k) = \lim_{k \to -\infty} \varepsilon_n^r(k) = (n + \frac{1}{2})B \) and \( \lim_{k \to -\infty} \varepsilon_n^\ell(k) = \lim_{k \to +\infty} \varepsilon_n^r(k) = +\infty \). Because of periodic boundary conditions along \( y \) the quantum number \( k \) takes discrete values \( \frac{2\pi m}{L} \), \( m \in \mathbb{Z} \). For \( L \) finite the spectrum consists of discrete eigenvalues
\( E_{n,m}^\alpha = \varepsilon_n^\alpha \left( \frac{2\pi m}{L} \right) \) on the spectral branches. Moreover we have

\[
|E_{0,m+1}^\alpha - E_{0,m}^\alpha| \geq \frac{C_0}{L} \quad \alpha = \ell, r
\]  

(2.6)

for each \( m \) such that \( E_{0,m}^\alpha \in \Delta_\varepsilon = \left( \frac{1}{2}B + V_0 + \varepsilon, \frac{3}{2}B - V_0 - \varepsilon \right) \), where \( C_0 > 0 \) is independent of \( m \) and depends only on the spectral branch \( \varepsilon_0^\alpha \). We will suppose that the following hypothesis is fulfilled

**Hypothesis 1.** There exists \( L_0 \) and \( d_0 > 0 \) such that for all \( L > L_0 \)

\[
\text{dist} \left( \sigma(H_\ell^0) \cap \Delta_\varepsilon, \sigma(H_r^0) \cap \Delta_\varepsilon \right) \geq \frac{d_0}{L}.
\]  

(2.7)

In order to fulfill this hypothesis one must take non-symmetric boundary potentials \( U_\ell \) and \( U_r \). We expect that in fact our result still holds for \( U_\ell(x) = U_r(-x) \) because physically the random potential \( V_\omega \) removes with high probability any degeneracy, but in order to control this case one should improve the Wegner estimate in Section 3. In Appendix C we give an example for a situation where this hypothesis is satisfied.

We will make use of the random single-boundary Hamiltonians

\[
H_\alpha = H_L + U_\alpha + V_\omega^\alpha
\]  

(2.8)

where \( V_\omega^\alpha = V_\omega|_{\Lambda_\alpha} \) with \( \Lambda_r = \{(n,m) \in \mathbb{Z}^2; n \in \left[ \frac{L}{2} - \frac{3D}{4}, \frac{L}{2} \right], m \in \left[ -\frac{L}{2}, \frac{L}{2} \right] \} \)
and \( \Lambda_\ell = \{(n,m) \in \mathbb{Z}^2; n \in \left[ -\frac{L}{2}, -\frac{L}{2} + \frac{3D}{4} \right], m \in \left[ -\frac{L}{2}, \frac{L}{2} \right] \} \), where \( D = \sqrt{L} \).

Since the perturbation has compact support and the essential spectrum of \( H_0^\alpha \) is given by the Landau levels, the spectrum of \( H_\alpha \) is discrete with the Landau levels as only accumulation points. We denote it by \( \sigma(H_\alpha) = \{ E_\kappa^\alpha : \kappa \in \mathbb{N} \} \). One can prove \( \square \) that, for each \( \omega \in \Omega_{\Lambda_\alpha} = [-1, 1]^{\Lambda_\alpha} \) (the restriction of the configurations \( \omega \) to the sublattice \( \Lambda_\alpha \)) and for each \( \kappa \) such that \( E_\kappa^\alpha \in \Delta_\varepsilon \), the distance between two consecutive eigenvalues satisfies

\[
|E_{\kappa+1}^\alpha - E_{\kappa}^\alpha| \geq \frac{C}{L} \quad \alpha = \ell, r
\]  

(2.9)

where \( C > 0 \) is uniform in \( \kappa, \omega \) and \( L \). Moreover for each \( E_\kappa^\ell \in \Delta \) (resp. \( E_\kappa^r \in \Delta \)) the average velocity associated to the corresponding eigenfunctions is strictly
negative (resp. positive) uniformly in $L$ (see Appendix B)

$$|J_{E_\alpha}| \geq C' > 0 \quad \alpha = \ell, r.$$ (2.10)

Finally we remark that the Hamiltonian $H_L + V_\omega|_{\tilde{\Lambda}}$ ($\tilde{\Lambda} \subset \Lambda$) has a point spectrum contained in Landau bands

$$\sigma(H_L + V_\omega|_{\tilde{\Lambda}}) \subset \bigcup_{n \geq 0} \left[ (n + \frac{1}{2})B - V_0, (n + \frac{1}{2})B + V_0 \right].$$ (2.11)

When $\tilde{\Lambda}$ is given by

$$\Lambda_b \equiv \tilde{\Lambda} = \{(n, m) \in \mathbb{Z}^2; n \in [-\frac{L}{2} + (\frac{D}{4} - 1), \frac{L}{2} - (\frac{D}{4} - 1)], m \in [-\frac{L}{2}, \frac{L}{2}]\}$$

we call the Hamiltonian $H_L + V_\omega|_{\Lambda_b}$ the bulk Hamiltonian and we denote it by $H_b$. All the Hamiltonians considered so far are densely defined self-adjoint operators.

We now state the main result of this paper.

**Theorem 1.** Let $V_0$ small enough, fix $\varepsilon > 0$ and let $0 < \delta < \frac{B}{2} - V_0 - \varepsilon$. Suppose that (H1) hold. Then there exists $\mu > 0$, $\tilde{L}$ such that if $L > \tilde{L}$ one can find a set $\hat{\Omega} \subset \Omega_\Lambda$ of realizations of the random potential $V_\omega$ with $\mathbb{P}_\Lambda(\hat{\Omega}) \geq 1 - L^{-\nu}$ ($\nu \gg 1$) such that for all $\omega \in \hat{\Omega}$ the spectrum of $H_\omega$ in $\Delta = (B - \delta, B + \delta)$ is the union of two sets $\Sigma_\ell$ and $\Sigma_r$ with the following properties:

a) $\mathcal{E}_\alpha^\alpha \in \Sigma_\alpha$ ($\alpha = \ell, r$) are a small perturbation of $E_\alpha^\alpha \in \sigma(H_\alpha) \cap \Delta$ with

$$|\mathcal{E}_\alpha^\alpha - E_\alpha^\alpha| \leq e^{-\mu \sqrt{B} \sqrt{L}}.$$ (2.12)

b) For $\mathcal{E}_\alpha^\alpha \in \Sigma_\alpha$ the average velocity $J_{\mathcal{E}_\alpha^\alpha}$ of the associated eigenstate satisfies

$$|J_{\mathcal{E}_\alpha^\alpha} - J_{E_\alpha^\alpha}| \leq e^{-\mu \sqrt{B} \sqrt{L}}.$$ (2.13)

That is the eigenfunctions associated to the eigenvalues (of $H_\omega$) in $\Delta$ have an $O(1)$ velocity.

The main tools for the proof of Theorem 1 are developed in section 3. Basically they consist in a Wegner estimate for the random Hamiltonians $H_\alpha$ ($\alpha = \ell, r$) and
a decoupling scheme that links the resolvent of the full Hamiltonian $H_\omega$ with those of $H_\ell$, $H_r$ and $H_b$. In section 4 we prove two propositions that lead to parts a) and b) of Theorem 1. Finally in appendix A we prove some technical results, in appendix B we prove (2.10) and in appendix C we discuss the Hypothesis 1.

Let $x, x' \in \mathbb{R} \times [-\frac{L}{2}, \frac{L}{2}]$, then one can check that

$$|x - x'|_s \equiv \inf_{n \in \mathbb{Z}} \sqrt{(x - x')^2 + (y - y' - nL)^2}$$

has the properties of a distance on $\mathbb{R} \times S_L$ and that it is related to the Euclidian distance $|x - x'| \equiv \sqrt{(x - x')^2 + (y - y')^2}$ by

$$|x - x'|_s \leq |x - x'|.$$

The interest of $| \cdot |_s$ is that, since we are working with a cylindrical geometry all decay estimates are naturally expressed in terms of this distance.

## 3 Wegner Estimates and Decoupling Scheme

We first give a Wegner estimate for the Hamiltonians $H_\alpha$ ($\alpha = \ell, r$). Denote by $P_{0,m}^\alpha$ the projector of $H_\alpha^0$ onto the eigenvalue $E_{0,m}^\alpha$ and by $P_\alpha(I)$ the projector of $H_\alpha$ on an interval $I$. Let $I_m = (E_{0,m-1}^\alpha + \delta_0, E_{0,m}^\alpha - \delta_0)$ and $\Delta_\alpha = \bigcup_{m_0 \leq m \leq m_1} I_m$, for some $-\infty \ll m_0 < m_1 \ll \infty$ and $\delta_0 \ll \frac{C_0}{L}$. The local potentials $V(x-n, y-m)$ will also be denoted by $V_i$, $i = (m,n) \in \Lambda$.

**Proposition 1.** Let $V_0$ sufficiently small with respect to $B$, $E \in \Delta_\alpha \cap \Delta_\epsilon$ and $I = [E - \delta, E + \delta] \subset I_m$. Then

$$\mathbb{P}_{\Lambda_\alpha} \{ \text{dist}(\sigma(H_\alpha), E) < \delta \} \leq \| h \|_{\infty} \delta \text{dist}(I, E_{0,m}^\alpha)^{-2} V_0^2 L^4$$

where $E_{0,m}^\alpha$ is the closest eigenvalue of $\sigma(H_\alpha^0)$ to the interval $I$.

**Proof.** We first observe that $V_1^{1/2} P_{0,m}^\alpha V_1^{1/2}$ is trace class. Indeed, using $\| AB \|_i \leq \| A \|_i \| B \|_i$ ($i = 1, 2$) and $\| AB \|_1 \leq \| A \|_2 \| B \|_2$ we get $\| V_1^{1/2} P_{0,m}^\alpha V_1^{1/2} \|_1 \leq \| V_1^{1/2} P_{0,m}^\alpha \|_2 \| P_{0,m}^\alpha V_1^{1/2} \|_2 \leq V_0 \| P_{0,m}^\alpha \|_2^2 \leq V_0$. 

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We have $E \in \Delta_\alpha \cap \Delta_\delta$, and $I = [E - \tilde{\delta}, E + \tilde{\delta}]$ for $\tilde{\delta}$ small enough (we require that $I \subset \Delta_\alpha \cap \Delta_\delta$). By the Chebyshev inequality we have

$$\mathbb{P}_{\Lambda_\alpha} \{ \text{dist}(\sigma(H_\alpha), E) < \delta \} = \mathbb{P}_{\Lambda_\alpha} \{ \text{Tr} \ P_\alpha(I) \geq 1 \} \leq \mathbb{E}_{\Lambda_\alpha} \{ \text{Tr} \ P_\alpha(I) \}$$

(3.2)

where $\mathbb{E}_{\Lambda_\alpha}$ is the expectation with respect to the random variables in $\Lambda_\alpha$.

We first give an estimate on $\text{Tr} \ P_\alpha(I)$. Let $E_{0,m}^\alpha$ the closest eigenvalue of $\sigma(H_\alpha^0)$ to $I$ and $m_i$ $(i = 0, 1)$ s.t. $\text{dist}(E_{0,m_i}^\alpha, E_{0,m_i}^\alpha) = O(B)$. Let also $P_\alpha^\alpha = \sum_{m>m_1} P_{0,m}^\alpha$ and $P_\alpha^\alpha = \sum_{m<m_0} P_{0,m}^\alpha$.

Using $P_\alpha^\alpha(H_\alpha^0 - E)P_\alpha^\alpha \geq 0$ and $P_\alpha^\alpha R_\alpha^0(E)P_\alpha^\alpha \leq \text{dist}(E_{0,m_1+1}, E)^{-1} P_\alpha^\alpha$ we can write

$$P_\alpha(I)P_\alpha^\alpha P_\alpha(I) = P_\alpha(I)P_\alpha^\alpha(H_\alpha^0 - E)^{1/2} R_\alpha^0(E)(H_\alpha^0 - E)^{1/2} P_\alpha^\alpha P_\alpha(I) \leq \text{dist}(E_{0,m_1+1}, E)^{-1} [P_\alpha(I)(H_\alpha - E)P_\alpha^\alpha P_\alpha(I) - P_\alpha(I)V_0 P_\alpha P_\alpha(I)]$$

(3.3)

and thus

$$\|P_\alpha(I)P_\alpha^\alpha P_\alpha(I)\| \leq \text{dist}(E_{0,m_1+1}, E)^{-1} \left( \frac{|I|}{2} + V_0 \right) \leq \frac{1}{4}$$

(3.4)

if, as we can suppose, $V_0$ is sufficiently small ($\text{dist}(E_{0,m_1+1}, E)^{-1} V_0 = O(\frac{1}{B})$). In a similar way we get

$$\|P_\alpha(I)P_\alpha^\alpha P_\alpha(I)\| \leq \text{dist}(E_{0,m_0-1}, E)^{-1} \left( \frac{|I|}{2} + V_0 \right) \leq \frac{1}{4}.$$  

(3.5)

Now

$$\text{Tr} \ P_\alpha(I)P_\alpha^\alpha = \text{Tr} \ P_\alpha(I)P_\alpha^\alpha P_\alpha(I) \leq \|P_\alpha(I)P_\alpha^\alpha P_\alpha(I)\| \text{Tr} \ P_\alpha(I)$$

(3.6)

and similarly for $\text{Tr} \ P_\alpha(I)P_\alpha^\alpha$. Therefore, using $1 = P_\alpha^\alpha + P_\alpha^\alpha + \sum_{m_0 \leq m \leq m_1} P_{0,m}^\alpha$, together with (3.4) and (3.5) we obtain

$$\text{Tr} \ P_\alpha(I) \leq 2 \sum_{m_0 \leq m \leq m_1} \text{Tr} \ P_\alpha(I) P_{0,m}^\alpha P_\alpha(I).$$

(3.7)

Since

$$\text{dist}(I, E_{0,m}^\alpha)^2 P_\alpha(I)^2 \leq (P_\alpha(I)(H_\alpha - E_{0,m}^\alpha)P_\alpha(I))^2$$

(3.8)
and \( \text{dist}(I, E_{0,m}^\alpha)^{-1} \leq \text{dist}(I, E_{0,m}^\alpha)^{-1} \) for all \( m_0 \leq m \leq m_1 \), it follows that

\[
\text{Tr} P_{0,m}^\alpha P_\alpha(I)P_{0,m}^\alpha \leq \text{dist}(I, E_{0,m}^\alpha)^{-2} \times \\
\times \text{Tr}(P_{0,k}^\alpha P_\alpha(I)(H_\alpha - E_{0,m}^\alpha)P_\alpha(I)(H_\alpha - E_{0,m}^\alpha)P_{0,m}^\alpha(\) \)
\]

\[
= \text{dist}(I, E_{0,m}^\alpha)^{-2} \text{Tr}(P_{0,m}^\alpha V_\omega^\alpha P_\alpha(I)V_\omega^\alpha P_{0,m}^\alpha) . \quad (3.9)
\]

Thus, taking the expectation value in \( [3.7] \) and using that there are \( O(L) \) \( m \)'s between \( m_0 \) and \( m_1 \), we get

\[
\mathbb{E}_{\Lambda_{\alpha}} \{ \text{Tr} P_\alpha(I) \} \leq 2 \cdot O(L) \cdot \text{dist}(I, E_{0,m}^\alpha)^{-2} \sup_{m_0 \leq m \leq m_1} \mathbb{E}_{\Lambda_{\alpha}} \{ \text{Tr}(P_{0,m}^\alpha V_\omega^\alpha P_\alpha(I)V_\omega^\alpha P_{0,m}^\alpha) \} .
\]

\[
(3.10)
\]

It remains to estimate the expectation value in the right hand side of \( (3.10) \). Here we follows a method of Combes and Hislop [CH]. Writing \( V_\omega^\alpha = \sum_{i \in \Lambda_{\alpha}} X_i(\omega)V_i \)

\[
\text{Tr} P_{0,m}^\alpha V_\omega^\alpha P_\alpha(I)V_\omega^\alpha P_{0,m}^\alpha = \sum_{i,j \in \Lambda_{\alpha}^2} X_i(\omega)X_j(\omega) \text{Tr} P_{0,m}^\alpha V_i P_\alpha(I)V_j P_{0,m}^\alpha \quad (3.11)
\]

\[
= \sum_{i,j \in \Lambda_{\alpha}^2} X_i(\omega)X_j(\omega) \text{Tr} V_j^{1/2} P_{0,m}^\alpha V_i^{1/2} V_j^{1/2} V_i^{1/2} P_\alpha(I)V_j^{1/2} .
\]

Since \( V_j^{1/2} P_{0,m}^\alpha V_i^{1/2} \) is trace class we can introduce the singular value decomposition

\[
V_j^{1/2} P_{0,m}^\alpha V_i^{1/2} = \sum_{n=0}^{\infty} \mu_n(u_n, \cdot)v_n \quad (3.12)
\]

where \( \sum_{n=0}^{\infty} \mu_n = \| V_j^{1/2} P_{0,m}^\alpha V_i^{1/2} \|_1 \). Then

\[
\text{Tr} V_j^{1/2} P_{0,k}^\alpha V_i^{1/2} V_j^{1/2} P_\alpha(I)V_j^{1/2} = \sum_{n=0}^{\infty} \mu_n(u_n, V_i^{1/2} P_\alpha(I)V_j^{1/2} v_n
\]

\[
\leq \sum_{n=0}^{\infty} \mu_n(v_n, V_j^{1/2} P_\alpha(I)V_j^{1/2} v_n)^{1/2}(u_n, V_i^{1/2} P_\alpha(I)V_i^{1/2} u_n)^{1/2}
\]

\[
\leq \frac{1}{2} \sum_{n=0}^{\infty} \mu_n \left\{ (v_n, V_j^{1/2} P_\alpha(I)V_j^{1/2} v_n) + (u_n, V_i^{1/2} P_\alpha(I)V_i^{1/2} u_n) \right\} . \quad (3.13)
\]

An application of the spectral averaging theorem (see [CH]) shows that

\[
\mathbb{E}_{\Lambda_{\alpha}} \{ (v_n, V_j^{1/2} P_\alpha(I)V_j^{1/2} v_n) \} \leq \| h \|_\infty 2\delta \quad (3.14)
\]
as well as for the term with $j$ replacing $i$ and $v_n$ replacing $u_n$. Combining (3.10), (3.13), (3.14) and (3.11) we get

$$
E_{\lambda_n}{\{\text{Tr} P_{\alpha}(I)\}} \leq 4 \cdot O(L) \cdot \|h\|_{\infty} \bar{\delta} \text{dist}(I, E_{0,m}^{\alpha})^{-2} V_0^2 \sum_{i,j \in \Lambda_0^{\alpha}} \|V_{i/2}^{1/2} P_{0,m}^{\alpha} V_{j/2}^{1/2}\|_1
$$

\leq 4 \cdot O(L) \cdot \|h\|_{\infty} \bar{\delta} \text{dist}(I, E_{0,m}^{\alpha})^{-2} V_0^2 |\Lambda_0^{\alpha}|^2.

(3.15)

We now turn to the decoupling scheme. By a decoupling formula [BG], [BCD] the resolvent $R(z) = (z - H_\omega)^{-1}$ can be expressed, up to a small term, as the sum of $R_\alpha(z) = (z - H_\alpha)^{-1}$ ($\alpha = \ell, r$) and $R_b(z) = (z - H_b)^{-1}$. We set $D = \sqrt{L}$ and introduce the characteristic functions

$$
\tilde{J}_\ell(x) = \chi_{[-\infty, -\frac{L}{2} + \frac{3D}{4}]}(x) \quad \tilde{J}_b(x) = \chi_{[-\frac{L}{2} + \frac{3D}{4}, \frac{L}{2} - \frac{D}{4}]}(x)
$$

$$
\tilde{J}_r(x) = \chi_{[\frac{L}{2} - \frac{D}{4}, +\infty]}(x).
$$

(3.16)

We will also use three bounded $C^\infty(\mathbb{R})$ functions $|J_i(x)| \leq 1$, $i \in \mathcal{I} \equiv \{\ell, b, r\}$, with bounded first and second derivatives $\sup_x |\partial_x^n J_i(x)| \leq 2$, $n = 1, 2$, and such that

$$
J_\ell(x) = \begin{cases} 1 & \text{if } x \leq -\frac{L}{2} + \frac{3D}{4} \\ 0 & \text{if } x \geq -\frac{L}{2} + \frac{3D}{4} + 1 \end{cases}
J_b(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{L}{2} - \frac{D}{4} \\ 0 & \text{if } |x| \geq \frac{L}{2} - \frac{D}{4} + 1 \end{cases}
$$

$$
J_r(x) = \begin{cases} 1 & \text{if } x \geq \frac{L}{2} - \frac{3D}{4} \\ 0 & \text{if } x \leq \frac{L}{2} - \frac{3D}{4} - 1 \end{cases}.
$$

(3.17)

For $i \in \mathcal{I}$ we have $H_\omega J_i = H_i J_i$ and the decoupling formula is [BG]

$$
R(z) = \left(\sum_{i \in \mathcal{I}} J_i R_i(z) J_i\right) (1 - \mathcal{K}(z))^{-1}
$$

(3.18)

where

$$
\mathcal{K}(z) = \sum_{i \in \mathcal{I}} K_i(z) = \frac{1}{2} [p_x^2, J_i(z) R_i(z) J_i].
$$

(3.19)

The main result of this part is a lemma about $\|\mathcal{K}(z)\|$ for $z$ such that $\text{dist}(z, \sigma(H_\alpha)) \geq e^{-\bar{\mu} \sqrt{B} \sqrt{\Lambda}}$, for a suitable $\bar{\mu} > 0$ and $\text{dist}(z, \sigma(H_b)) \geq \varepsilon$. 

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Figure 1: The system of decoupling functions $J_i$ ($i \in \mathcal{I}$).

**Proposition 2.** Let $\varepsilon > 0$, and $z \in \Delta_\varepsilon$ such that $\text{dist}(z, \sigma(H_\ell) \cup \sigma(H_r)) \geq e^{-\bar{\mu}\sqrt{B\sqrt{L}}}$ with $\bar{\mu} < \frac{1}{192}$. Then for $L$ large enough there exists $C(B,V_0,\varepsilon) > 0$ and $\tilde{\gamma} > 0$ independent of $L$ such that

$$\|K(z)\| \leq C(B,V_0,\varepsilon)e^{-\tilde{\gamma}\sqrt{B\sqrt{L}}}.$$  \hspace{1cm} (3.20)

**Proof.** Computing the commutator in the definition of $K_i(z)$ we have

$$K_i(z) = -\frac{1}{2}(\partial_x^2 J_i) R_b(z) \tilde{J}_i - (\partial_x J_i) \partial_x R_b(z) \tilde{J}_i.$$  \hspace{1cm} (3.21)

Then

$$\|K_b(z)\| \leq \frac{1}{2} \|(\partial_x^2 J_b) R_b(z) \tilde{J}_b\| + \|\partial_x J_b\| \partial_x R_b(z) \tilde{J}_b\|$$  \hspace{1cm} (3.22)

$$\|K_\alpha(z)\| \leq \frac{1}{2} \|(\partial_x^2 J_\alpha) R_b^b(z) \tilde{J}_\alpha\| + \frac{1}{2} \|(\partial_x^2 J_\alpha) R_\alpha^b(z) U_\alpha\| \text{dist}(z, \sigma(H_\alpha))^{-1}$$  \hspace{1cm} (3.23)

$$+ \|\partial_x J_\alpha\| \partial_x R_\alpha(z) \tilde{J}_\alpha\| + \|\partial_x J_\alpha\| \partial_x R_\alpha^b(z) U_\alpha\| \text{dist}(z, \sigma(H_\alpha))^{-1}$$

where for the the second term we used the second resolvent identity and where $R_\alpha^b(z) = (z - [H_L + V_\alpha])^{-1}$.

We have to estimate norms of the form $\|f \partial_x^a \tilde{R}(z) g\|$ ($a = 0, 1$) where here $\tilde{R}(z)$ is $R_b(z)$ or $R_\alpha^b(z)$, $f = \partial_x^m J_i$ and $g = \tilde{J}_i$ or $g = U_\alpha$.

Using the second resolvent formula we develop $\tilde{R}(z)$ in its Neumann series, denote
\(V_\omega|_{\tilde{\Lambda}} \equiv W \) (\(\tilde{\Lambda} = \Lambda_b \) or \(\Lambda_n\))

\[
\tilde{R}(z) = \sum_{n=0}^{\infty} R_0(z)[WR_0(z)]^n
\]

(3.24)

where \(R_0(z) = (z - H_L)^{-1}\). The norm convergence is ensured since we are in a spectral gap, indeed

\[
\|WR_0(z)\| \leq V_0 \text{dist}(z, \sigma(H_L))^{-1} \leq \frac{V_0}{V_0 + \varepsilon} < 1.
\]

(3.25)

Therefore

\[
\|f \partial^2_x \tilde{R}(z)g\| \leq \sum_{n=1}^{\infty} \|f \partial^2_x R_0(z)[WR_0(z)]^ng\|
\]

(3.26)

and we have to control the operator norms \(\|f \partial^2_x R_0(z)[WR_0(z)]^ng\|\).

For any vector \(\varphi \in L^2(\mathbb{R} \times [-\frac{L}{2}, \frac{L}{2}])\) with \(\|\varphi\| = 1\)

\[
\|f \partial^2_x R_0(z)[WR_0(z)]^ng\varphi\|^2 = \int_{\text{supp } f} |f(x)|^2 |(\partial^2_x R_0(z)[WR_0(z)]^ng\varphi)(x)|^2 \, dx
\]

(3.27)

For the integrand in (3.27) we have

\[
J \equiv |(\partial^2_x R_0(z)[WR_0(z)]^ng\varphi)(x)| \leq \int_{\text{supp } g} \int \cdots \int dx_n \times
\]

(3.28)

\[
\times |\partial^2_x R_0(x, x_1; z)||W(x_1)||R_0(x_1, x_2; z)||\cdots||R_0(x_n, x'; z)||g(x')||\varphi(x')|.
\]

Now, taking out \(\|W\|_\infty\) and using Lemma 1, Appendix A we get

\[
J \leq \left( cB^2 \frac{V_0}{V_0 + \varepsilon} \right)^n \int_{\text{supp } g} \int \cdots \int dx_n e^{-\gamma \sqrt{B} \sum_{i=0}^{n} |x_i - x_{i+1}|} \times
\]

(3.29)

\[
\times |\Phi^1(x - x_1, t)| \cdots |\Phi^0(x_n - x', t)||g(x')||\varphi(x')|
\]

where \(x_0 = x\) and \(x_{n+1} = x'\). Splitting the exponential and making the change of variables \(x - x_1 = -z_1, \ldots, x_{n-1} - x_n = -z_n\) we get (with \(x_n = x_n(\{z_i\}, x)\) and \(A = cB^2 \frac{V_0}{V_0 + \varepsilon}\))

\[
J \leq A^n \sup_{z_1: \ldots : z_n} \left\{ \int_{\text{supp } g} e^{-\frac{2}{3} \gamma \sqrt{B} |x - x'|} |g(x')||\varphi(x')||\Phi^0(|x_n - x'|) e^{-\frac{1}{3} \gamma \sqrt{B} |x_n - x'|} \, dx' \right\} \times
\]

(3.30)

\[
\times \left[ \int_{\mathbb{R}^2} |\Phi^1(|z|)| e^{-\frac{2}{3} \gamma \sqrt{B} |z|} \, dz \right] \left[ \int_{\mathbb{R}^2} |\Phi^0(|z|)| e^{-\frac{1}{3} \gamma \sqrt{B} |z|} \, dz \right]^{n-1}
\]

(3.31)
Splitting the exponential and using the Schwartz inequality we have the estimate

\[ \sup_{z_1, \ldots, z_n} \mathcal{X} \leq \sup_{x' \in \text{supp } g} e^{-\frac{1}{3} \tilde{\gamma} \sqrt{B|x-x'|}} \left\{ \int_{\mathbb{R}^2} |\Phi^0(|w|)|^2 e^{-\frac{2}{3} \tilde{\gamma} \sqrt{B}|w|} \, dw \right\}^{1/2} \times \]

\[ \times \left( \sup_{x' \in \text{supp } g} e^{-\frac{2}{3} \tilde{\gamma} \sqrt{B}|x-x'|} |g(x')|^2 \right)^{1/2} \| \varphi \| . \] (3.32)

Now, since \( U_\alpha \) do not grow too fast (see (2.2), (2.3))
\( (\sup_{x' \in \text{supp } g} e^{-\frac{2}{3} \tilde{\gamma} \sqrt{B}|x-x'|} |g(x')|^2)^{1/2} \) is bounded by a numerical constant.

On the other hand the term \( \int_{\mathbb{R}^2} |\Phi^0(|w|)|^2 e^{-\frac{2}{3} \tilde{\gamma} \sqrt{B}|w|} \, dw \) is bounded by a constant depending only on \( B \).

Moreover the terms \( \mathcal{Y} \) and \( \mathcal{Z} \) are also bounded by a constant depending only on \( B \) and not on \( L \). This leads to

\[ \| f \partial_x^n [R_0(z)] g \varphi \| \leq \| f \|_\infty \tilde{C}(B)(\tilde{C}(B)A)^n e^{-\frac{1}{12} \tilde{\gamma} \sqrt{BD}} \| \varphi \| . \] (3.33)

Therefore, if \( V_0 \) is small enough the series (3.26) converges and

\[ \| f \partial_x^n \tilde{R}(z)g \| \leq \tilde{C}(B, V_0) \sqrt{L} e^{-\frac{1}{12} \tilde{\gamma} \sqrt{BD}} . \] (3.34)

This implies

\[ \| K_b(z) \| \leq \varepsilon^{-1} \sqrt{L} C(B, V_0) e^{-\frac{1}{12} \tilde{\gamma} \sqrt{L}} \] (3.35)

\[ \| K_\alpha(z) \| \leq \sqrt{L} e^{\tilde{\mu} \sqrt{L}} C(B, V_0) e^{-\frac{1}{12} \tilde{\gamma} \sqrt{L}} \quad \alpha = \ell, r \] (3.36)

thus \( \| \mathcal{K}(z) \| \leq C(B, V_0, \varepsilon) e^{-\tilde{\gamma} \sqrt{L}} \) where \( 2\tilde{\gamma} = \tilde{\gamma} \frac{\ell}{12} - \bar{\mu} \). Since \( \tilde{\gamma} = \frac{1}{16} \) in Lemma [1, Appendix A] we must take \( \bar{\mu} < \frac{1}{192} \). \( \square \)

We remark that in the proof above we have proved the following statement (see (3.34)) that will be useful in the next section

\[ \| (1 - \tilde{J}_\alpha) \tilde{R}_b(z)g \| \leq \tilde{C}(B, V_0, \varepsilon) e^{-\tilde{\gamma} \sqrt{L}} . \] (3.37)

where \( g = U_\alpha \) or \( g = \chi_B \) (\( B \subset \mathbb{R} \times [-L, L] \)) with \( \text{dist}(\text{supp } g, \text{supp}(1-\tilde{J}_\alpha)) = O(D) \)

and \( \tilde{R}_b(z) \) a resolvent associated to a generic bulk Hamiltonian \( (H_L + V_\omega|_\Lambda) \).
4 Projector estimates and the proof of Theorem 1

In this section we prove two propositions that lead to Theorem 1. Let $D' = \{\kappa : E^\alpha_\kappa \in \Delta, \alpha = \ell, r\}$, card$(D') = \mathcal{O}(L)$, where $\Delta \subset \Delta_\epsilon$ is given in section 2.

**Proposition 3.** For $L$ large enough, with probability greater then $1-L^{-\nu}$ ($\nu \gg 1$), we have for all $\kappa \in D'$

$$\|P - P_\alpha(E^\alpha_\kappa)\| \leq e^{-\gamma \sqrt{B} \sqrt{L}}$$

(4.1)

where $P_\alpha(E^\alpha_\kappa)$ is the projector associated to $H_\alpha$ onto $E^\alpha_\kappa$ and $P$ is the projector associated to $H_\omega$ onto $\{z \in \mathbb{C} : |z - E^\alpha_\kappa| \leq e^{-\bar{\mu} \sqrt{B} \sqrt{L}}\}$.

**Proof.** (1): Let $E = \{m : E^\alpha_0,m \in \Delta, \alpha = \ell, r\}$, card$(E) = \mathcal{O}(L)$, and let

$$\hat{\Omega}_\ell = \{\omega \in \Omega_{\Lambda_\ell} : \text{dist}(E^\ell_0,m,\sigma(H_\ell)) \geq L^{-\sigma}, \forall m \in E\} ,$$

(4.2)

with $\sigma > 11$, this set has probability

$$\mathbb{P}_{\Lambda_\ell}(\hat{\Omega}_\ell) \geq 1 - L^{-(\sigma-8)}.$$  

(4.3)

Indeed for a fixed $m \in E$, using Proposition 1 and (H1) one gets

$$\mathbb{P}_{\Lambda_\ell}\{\omega \in \Omega_{\Lambda_\ell} : \text{dist}(E^\ell_0,m,\sigma(H_\ell)) \geq L^{-\sigma}, \text{ for one } m \in E\} \geq 1 - C'(h,V_0)L^{-\sigma}L^4(L^{-\sigma} - L^{-3\sigma})^{-2} \geq 1 - C(h,V_0)L^{6-\sigma}.$$  

(4.4)

For a given realisation $\omega_\ell \in \hat{\Omega}_\ell$ let

$$\hat{\Omega}_r(\omega_\ell) = \{\omega \in \Omega_{\Lambda_r} : \text{dist}(E^r_\kappa,\sigma(H_r)) \geq L^{-3\sigma}, \forall \kappa \in D'\} ,$$

(4.5)

this set has probability

$$\mathbb{P}_{\Lambda_r}(\hat{\Omega}_r(\omega_\ell)|\omega_\ell) \geq 1 - L^{-(\sigma-6)}.$$  

(4.6)

uniformly with respect to the realisations of $\hat{\Omega}_\ell$. Indeed

$$\mathbb{P}_{\Lambda_r}\{\omega \in \Omega_{\Lambda_r} : \text{dist}(E^r_\kappa,\sigma(H_r)) \geq L^{-3\sigma}, \text{ for one } \kappa \in D'\} \geq 1 - C'(h,V_0)L^{-3\sigma}L^4(L^{-\sigma} - L^{-3\sigma})^{-2} \geq 1 - C(h,V_0)L^{4-\sigma}.$$  

(4.7)
It follows that the set
\[
\hat{\Omega}(\ell) = \left\{ \omega = (\omega_\ell, \omega_b, \omega_r) \in \Omega : \omega_\ell \in \hat{\Omega}_\ell, \omega_b \in \Omega_b, \omega_r \in \hat{\Omega}_r(\omega_\ell) \right\}
\] (4.8)
\[
\Omega_b = \Omega|_{\Lambda_b \setminus (\Lambda_\ell \cup \Lambda_r)}
\]
has probability
\[
\mathbb{P}_\Lambda(\hat{\Omega}(\ell)) = \mathbb{P}_{\Lambda_b}(\hat{\Omega}_b) \mathbb{E}_{\Lambda_\ell} \left\{ \mathbb{P}_{\Lambda_r}(\hat{\Omega}_r|\omega_\ell) | \omega_\ell \in \hat{\Omega}_\ell \right\}
\]
(4.9)

\[\geq (1 - L^{-(\sigma-6)}) \mathbb{P}_{\Lambda_r}(\hat{\Omega}_r) \geq 1 - L^{-(\sigma-9)}\]

(2): We now work with a given \(\omega \in \hat{\Omega}(\ell)\). Take \(\tilde{\mu} > 0\) as in Proposition 2 and \(L\) large enough such that for all \(\kappa \in \mathcal{D}^{' \Gamma_\kappa} = \left\{ z \in \mathbb{C} : |z - E_\kappa^L| \leq e^{-\tilde{\mu}\sqrt{B}\sqrt{L}} \right\} \cap \sigma(H_r) = \emptyset\), and remark that \(\text{Tr} P_b(\Delta) = 0\) (\(P_b\) the projector associated to \(H_b\)).

We need to introduce two auxiliary Hamiltonians \(H_1\) and \(H_2\) defined as follows:
\[
H_1 = H_L + V_\omega^L|_{\Lambda_1}
\]
(4.10)
\[
H_2 = H_L + V_\omega^L|_{\Lambda_2} + U_\ell
\]
(4.11)
where \(\Lambda_2 = \left\{ (n, m) \in \mathbb{Z}^2 ; n \in \left[ -\frac{L}{2}, -\frac{L}{2} + (\frac{L}{4} - 1) \right], m \in \left[ -\frac{L}{2}, \frac{L}{2} \right] \right\}\), and \(\Lambda_1 = \Lambda_\ell \setminus \Lambda_2\), of course \(H_\ell = H_2 + V_\omega^L|_{\Lambda_1}\).

From the decoupling formula (3.18) we have
\[
R(z) - R_\ell(z) = \left( \sum_{i \in \mathcal{I}} J_i R_i(z) \tilde{J}_i \right) \left( \sum_{n=1}^{\infty} K(z)^n \right) - (1 - J_\ell) R_\ell(z)
\]
\[= J_\ell R_\ell(z)(1 - \tilde{J}_\ell) + J_b R_b(z) \tilde{J}_b + J_r R_r(z) \tilde{J}_r + \cdots \]
(4.12)
integrating over \(\partial \Gamma_\kappa\) and taking the operator norm we get
\[
\|P - P_\ell(E_\kappa^L)\| \leq e^{-\tilde{\mu}\sqrt{B}\sqrt{L}} \left( \sum_{i \in \mathcal{I}} \sup_{z \in \partial \Gamma_\kappa} \|R_i(z)\| \right) \frac{\sup_{z \in \partial \Gamma_\kappa} \|K(z)\|}{1 - \sup_{z \in \partial \Gamma_\kappa} \|K(z)\|} + \|P_\ell(E_\kappa^L)\| + \|J_\ell P_\ell(E_\kappa^L)(1 - \tilde{J}_\ell)\|
\]
\[= a + b + c. \]
(4.13)
For the first term we note that for \(L\) large enough \(e^{-\tilde{\mu}\sqrt{B}\sqrt{L}} \sup_{z \in \partial \Gamma_\kappa} \|R_i(z)\| \leq 1\) \((i \in \mathcal{I})\). Indeed, for \(i = \ell\) we have \(\sup_{z \in \partial \Gamma_\kappa} \|R_\ell(z)\| = e^{\tilde{\mu}\sqrt{B}\sqrt{L}}\) by construction, for \(i = b\) we have \(\sup_{z \in \partial \Gamma_\kappa} \|R_b(z)\| = e^{-1}\), and for \(i = r\) \(\sup_{z \in \partial \Gamma_\kappa} \|R_r(z)\| = \left(L^{-3\sigma} - e^{-\tilde{\mu}\sqrt{B}\sqrt{L}}\right)^{-1}\). Then, applying Proposition 2 we get
\[
a \leq 2C(B, V_0, \varepsilon)e^{-\tilde{\gamma}\sqrt{B}\sqrt{L}}. \]
(4.14)
For the second and third term we first observe that by the second resolvent formula

\[
\frac{P_\ell(E_\kappa^\ell)}{(z - E_\kappa^\ell)} = (z - H_1)^{-1} P_\ell(E_\kappa^\ell) + (z - H_1)^{-1} [V_\omega^\ell|_{\Lambda_2} + U_\ell] \frac{P_\ell(E_\kappa^\ell)}{(z - E_\kappa^\ell)}. \tag{4.15}
\]

and integrating (4.15) along \(\partial \Gamma_\kappa\) we obtain (using \(\sigma(H_1) \cap \Delta_\varepsilon = \emptyset\))

\[
P_\ell(E_\kappa^\ell) = R_1(E_\kappa^\ell)[V_\omega^\ell|_{\Lambda_2} + U_\ell] P_\ell(E_\kappa^\ell) \tag{4.16}
\]

\[
P_\ell(E_\kappa^\ell)[V_\omega^\ell|_{\Lambda_2} + U_\ell] R_1(E_\kappa^\ell). \tag{4.17}
\]

Therefore, using (4.16) for \(b\) and (4.17) for \(c\) we get

\[
b \leq \| (1 - J_\ell) R_1(E_\kappa^\ell)[V_\omega^\ell|_{\Lambda_2} + U_\ell] \| \leq \| (1 - \tilde{J}_\ell) R_1(E_\kappa^\ell)[V_\omega^\ell|_{\Lambda_2} + U_\ell] \| \tag{4.18}
\]

\[
c \leq \| (1 - \tilde{J}_\ell) R_1(E_\kappa^\ell)[V_\omega^\ell|_{\Lambda_2} + U_\ell] \|. \tag{4.19}
\]

Using (3.37) we get

\[
b + c \leq 2 \left( V_0 L^2 \|(1 - \tilde{J}_\ell) R_1(E_\kappa^\ell)[V_\omega^\ell|_{\Lambda_2} \| + \|(1 - \tilde{J}_\ell) R_1(E_\kappa^\ell) U_\ell \| \right)
\]

\[
\leq 2 \tilde{C}(B, V_0, \varepsilon) L^2 e^{-\tilde{\gamma} \sqrt{B} \sqrt{\mathcal{T}}}. \tag{4.20}
\]

Thus

\[
\| P - P_\ell(E_\kappa^\ell) \| \leq e^{-\gamma \sqrt{B} \sqrt{\mathcal{T}}}. \tag{4.21}
\]

By repeating the above proof in a symmetrical way we get for \(\omega\) in a set \(\hat{\Omega}^{(r)}\) similar to \(\hat{\Omega}^{(\ell)}\)

\[
\| P - P_\ell(E_\kappa^\ell) \| \leq e^{-\gamma \sqrt{B} \sqrt{\mathcal{T}}}. \tag{4.22}
\]

Finally we have both (4.21) and (4.22) for \(\omega \in \hat{\Omega} = \hat{\Omega}^{(\ell)} \cap \hat{\Omega}^{(r)}\) with \(\mathbb{P}_\Lambda \geq 1 - L^{-\nu}, \nu = \sigma - 10\). Note that we can take \(\nu' \gg 1\) by taking \(\sigma \gg 11\).

The estimate on the norm difference of the projectors implies that their dimensions are the same and that \(E_\kappa^\alpha \in \sigma(H_\omega)\) is a small perturbation of \(E_\kappa^\alpha\): this gives part \(a)\) of Theorem 1.
Proposition 4. Let $\omega \in \hat{\Omega}$. Then there exists $\hat{\mu} > 0$ such that the velocity associated to each eigenvalue $\mathcal{E}^\alpha_\kappa$ of $H_\omega$ in $\Delta$ satisfies

\[ |J_{E^\alpha} - J_{E^\alpha}| \leq e^{-\hat{\mu}\sqrt{B\sqrt{T}}}. \]  \tag{4.23}

Proof. Let $J_{E^\alpha} = \text{Tr} v_y P(\mathcal{E}^\alpha_\kappa)$ the average velocity associated to the eigenvalue $\mathcal{E}^\alpha_\kappa \in \sigma(H_\omega)$ and $J_{E^\alpha} = \text{Tr} v_y P_\alpha(E^\alpha_\kappa)$ that associated to the eigenvalue $E^\alpha_\kappa$ of $H_\alpha$.

First we observe that $v_y P(\mathcal{E}^\alpha_\kappa)$ is trace class. Indeed, $v_y P(\mathcal{E}^\alpha_\kappa) = v_y P(\mathcal{E}^\alpha_\kappa) P(\mathcal{E}^\alpha_\kappa)$ with $v_y P(\mathcal{E}^\alpha_\kappa)$ bounded and $\|P(\mathcal{E}^\alpha_\kappa)\|_1 = \text{Tr} P(\mathcal{E}^\alpha_\kappa) = \text{Tr} P_\alpha(E^\alpha_\kappa) = 1$.

\[ \|v_y P(\mathcal{E}^\alpha_\kappa)\|_1^2 \leq \|v_y P(\mathcal{E}^\alpha_\kappa)\|^2 \leq \|P(\mathcal{E}^\alpha_\kappa) v_y^2 P(\mathcal{E}^\alpha_\kappa)\| \] \tag{4.24}

\[ \leq 2\|P(\mathcal{E}^\alpha_\kappa)(H_\omega - V_\omega)P(\mathcal{E}^\alpha_\kappa)\| \leq (3B + 2V_0) \]

To get the second inequality one has simply added positive terms to $v_y^2$. Similarly

\[ \|v_y P_\alpha(E^\alpha_\kappa)\|_1^2 \leq (3B + 2V_0). \] \tag{4.25}

With the help of the identity

\[ P(\mathcal{E}^\alpha_\kappa) - P_\alpha(E^\alpha_\kappa) = [P(\mathcal{E}^\alpha_\kappa) - P_\alpha(E^\alpha_\kappa)]^2 + [P(\mathcal{E}^\alpha_\kappa) - P_\alpha(E^\alpha_\kappa)]P_\alpha(E^\alpha_\kappa) \]

\[ + \ P_\alpha(E^\alpha_\kappa) [P(\mathcal{E}^\alpha_\kappa) - P_\alpha(E^\alpha_\kappa)] \] \tag{4.26}

we get

\[ |J_{E^\alpha} - J_{E^\alpha}| = |\text{Tr} v_y [P(\mathcal{E}^\alpha_\kappa) - P_\alpha(E^\alpha_\kappa)]| \leq |\text{Tr} v_y [P(\mathcal{E}^\alpha_\kappa) - P_\alpha(E^\alpha_\kappa)]|^2 \]

\[ + \ |\text{Tr} v_y [P(\mathcal{E}^\alpha_\kappa) - P_\alpha(E^\alpha_\kappa)]P_\alpha(E^\alpha_\kappa)| \]

\[ + \ |\text{Tr} v_y P_\alpha(E^\alpha_\kappa) [P(\mathcal{E}^\alpha_\kappa) - P_\alpha(E^\alpha_\kappa)]|. \] \tag{4.27}

and then, from (4.24) and (4.25), we get

\[ |J_{E^\alpha} - J_{E^\alpha}| \leq 2(\|v_y P(\mathcal{E}^\alpha_\kappa)\|_1 + \|v_y P_\alpha(E^\alpha_\kappa)\|_1) \|P(\mathcal{E}^\alpha_\kappa) - P_\alpha(E^\alpha_\kappa)\| \] \tag{4.28}

\[ \leq 4(3B + 2V_0)^{1/2}\|P(\mathcal{E}^\alpha_\kappa) - P_\alpha(E^\alpha_\kappa)\|. \]

Combining this last inequality with Proposition 3, we get the result. \(\square\)

From Proposition 4 and the result of Appendix B given in (2.10) we obtain part b) of Theorem 1.
A Estimate of the Green function $R_0(x, x'; z)$

In this appendix we give the necessary decay property of the kernel $R_0(x, x'; z)$ with periodic boundary conditions along $y$. The exact formula for $R_0(x, x'; z)$ can be found in [FM1]. We introduce the following notation

$$\Phi^\alpha(|x - x'|_*)$$

$$= \begin{cases} 
1 + \ln \left( \frac{B}{2} |x - x'|^2 \right), & \alpha = 0 \\
1 + \left[ \ln \left( \frac{B}{2} |x - x'|^2 \right) + (1 + \ln \left( \frac{B}{2} |x - x'|^2 \right)) |x - x'|^{-1} \right], & \alpha = 1.
\end{cases} \quad (A.1)$$

**Lemma 1.** If $|\text{Im} \, z| \leq 1$, $\text{Re} \, z \in \left[\frac{1}{2}, \frac{3}{2} \right] B[ then, for $L$ large enough, there exists $C(z, B)$ positive constant independent of $L$ such that $(\alpha = 0, 1)$

$$|\partial_x^\alpha R_0(x, x'; z)| \leq C'(z, B)e^{-\frac{B}{8}|x-x'|^2} \Phi^\alpha(|x - x'|_*)$$

$$\leq C(z, B)e^{-\gamma \sqrt{B}|x-x'|} \Phi^\alpha(|x - x'|_*) \quad (A.2)$$

where $C(z, B) = cB^2 \text{dist}(z, \sigma(H_L))^{-1}$ with $c$ a numerical positive constant and $\gamma = \frac{1}{16}$.

**Proof.** As in [FM1] we can prove that (for $L$ large enough the logarithmic divergences appear only for $\vert m \vert \leq 1$ and the sum over $\vert m \vert > 1$ converge)

$$|\partial_x^\alpha R_0(x, x'; z)| \leq \frac{C'(z,B)}{3} e^{-\frac{B}{8}|x-x'|^2} + \sum_{|m|\leq 1} |\partial_x^\alpha R_0^\infty(x, y - mL, x'; z)| \quad (A.3)$$

with

$$|\partial_x^\alpha R_0^\infty(x, x'; z)| \quad (A.4)$$

$$\leq \begin{cases} 
\frac{C'(z,B)}{3} e^{-\frac{B}{8}|x-x'|^2} \left\{ 1 + \frac{1}{\mathbb{E}(0,\sqrt{2B-1})}(|x - x'|) \ln \left( \frac{B}{2} |x - x'|^2 \right) \right\}, & \alpha = 0 \\
\frac{C'(z,B)}{3} e^{-\frac{B}{8}|x-x'|^2} \left\{ 1 + \frac{1}{\mathbb{E}(0,\sqrt{2B-1})}(|x - x'|) \ln \left( \frac{B}{2} |x - x'|^2 \right) \right\} + (1 + \ln \left( \frac{B}{2} |x - x'|^2 \right)) |x - x'|^{-1} \right\}, & \alpha = 1.
\end{cases}$$

Now, using $|x - x'|_* \leq |x - x'|$, we can replace the Euclidean distance with the distance $| \cdot |_*$ in all the terms in the RHS of (A.4), since all these functions are decreasing. To obtain the same bound for the terms $|m| \leq 1$ in the sum we just drop the characteristic functions $1_{\mathbb{E}(0,\sqrt{2B-1})}$.

\hfill \Box
B  Average velocity of the eigenstate associated to $E_\kappa^\alpha$

In this appendix we prove following [F] that the eigenstates corresponding to the eigenvalues of $H_\alpha$ ($\alpha = \ell, r$) in an energy interval $\Delta = (B - \delta, B + \delta) \subset \Delta_\epsilon$ have an average velocity that is strictly positive/negative uniformly in $L$, that is, if we have $H_\alpha \psi_\kappa^\alpha = E_\kappa^\alpha \psi_\kappa^\alpha$ then

$$|(\psi_\kappa^\alpha, v_y \psi_\kappa^\alpha)| \geq C' > 0 .$$  \hspace{1cm} (B.1)

¿From the eigenvalue equation we have

$$\| (H_0^\alpha - E_\kappa^\alpha) \psi_\kappa^\alpha \|^2 = \| V_\omega^\alpha \varphi_\kappa \|^2 \leq V_0^2 .$$ \hspace{1cm} (B.2)

We now expand $\psi_\kappa^\alpha$ on the eigenfunctions of $H_0^\alpha$ denoted

$$\{ \varphi_{n,k}^\alpha \}_{n \in \mathbb{N}, k \in \mathbb{Z}}$$

where $\varphi_{n,k}^\alpha$ is the solution on the eigenvalue problem

$$\left[ \frac{1}{2} p_x^2 + \frac{1}{2} (k - B x)^2 + U_\alpha \right] \varphi_{n,k}^\alpha = E_{n,k}^\alpha \varphi_{n,k}^\alpha .$$

$$\psi_\kappa^\alpha(x,y) = \sum_{n=0}^{\infty} \sum_{m \in \mathbb{Z}} \psi_n(m) \varphi_{n,m}(x,y) .$$ \hspace{1cm} (B.3)

and of course

$$\| \psi_\kappa^\alpha \|^2 = \sum_{n=0}^{\infty} \sum_{m \in \mathbb{Z}} | \psi_n(m) |^2 = 1 .$$ \hspace{1cm} (B.4)

¿From (B.3) the equation (B.2) becomes

$$\sum_{n=0}^{\infty} \sum_{m \in \mathbb{Z}} | \psi_n(m) |^2 \left( E_{n,m}^\alpha - E_{\kappa}^\alpha \right)^2 \leq V_0^2$$ \hspace{1cm} (B.5)

thus since each term in the sum is positive we have

$$\sum_{m \in \mathbb{Z}} | \psi_0(m) |^2 \left( E_{0,m}^\alpha - E_{\kappa}^\alpha \right)^2 \leq V_0^2$$ \hspace{1cm} (B.6)

We remark that for $n \geq 1$ one has $|E_{n,m}^\alpha - E_{\kappa}^\alpha| \geq \frac{B}{2} - \delta$, this leads to

$$\| \psi_\kappa \|^2 \equiv \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} | \psi_n(m) |^2 \leq \frac{V_0^2}{(B - \delta)^2} .$$ \hspace{1cm} (B.7)
Let $m^*$ such that $|E_{0,m^*}^\alpha - E_\kappa^\alpha|$ is minimal, and for a fixed $a$ independent of $L$ let $\mathcal{A} = [m^* - a, m^* + a]$. Then from (B.3)

$$V_0^2 \geq \sum_{m \in \mathbb{Z}} |\psi_0(m)|^2 (E_{0,m}^\alpha - E_\kappa^\alpha)^2 \geq \sum_{m \in \mathcal{A}^c} |\psi_0(m)|^2 (E_{0,m}^\alpha - E_\kappa^\alpha)^2$$

$$\geq \inf_{m \in \mathcal{A}^c} (E_{0,m}^\alpha - E_\kappa^\alpha)^2 \sum_{m \in \mathcal{A}^c} |\psi_0(m)|^2$$

(B.8)

thus

$$\sum_{m \in \mathcal{A}} |\psi_0(m)|^2 \leq V_0^2 \sup_{m \in \mathcal{A}^c} (E_{0,m}^\alpha - E_\kappa^\alpha)^{-2}.$$ (B.9)

From (B.4) and (B.7) we get

$$1 \geq \sum_{m \in \mathbb{Z}} |\psi_0(m)|^2 \geq 1 - \frac{V_0^2}{(\frac{\pi}{2} - \delta)^2}.$$ (B.10)

Combining the last equation and (B.9) we get

$$\sum_{m \in \mathcal{A}} |\psi_0(m)|^2 \geq 1 - V_0^2 \left[\frac{1}{(\frac{\pi}{2} - \delta)^2} + \sup_{m \in \mathcal{A}^c} (E_{0,m}^\alpha - E_\kappa^\alpha)^{-2}\right].$$ (B.11)

Decompose now $\psi_\kappa^\alpha$ as $\psi_\kappa^\alpha = \psi_0 + \psi_*$, then

$$|(\psi_\kappa^\alpha, v_y \psi_\kappa^\alpha)| \geq \left|(\psi_0, v_y \psi_0)\right| - |(\psi_*, v_y \psi_*)| - 2(\psi_*, v_y \psi_0)$$ (B.12)

the first term can be written as

$$\int_{\mathbb{R}} dx \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dy \left\{ \sum_{m' \in \mathbb{Z}} \psi_0^*(m') e^{-i2m'y} \frac{\varphi_0(m')}{\sqrt{L}} \sum_{m \in \mathbb{Z}} \psi_0(m) v_y \frac{\xi m y}{\sqrt{L}} \varphi_0,m(x) \right\}$$

$$= \sum_{m \in \mathbb{Z}} |\psi_0(m)|^2 \int_{\mathbb{R}} dx (k - Bx) |\varphi_0,m(x)|^2$$

$$= \sum_{m \in \mathbb{Z}} |\psi_0(m)|^2 |\partial_k E_0^\alpha(k)|_{k = \frac{2m}{L}}$$ (B.13)

The partial derivative of $E_0^\alpha$ is the average velocity $\partial_k E_0^\alpha(k)|_{k = \frac{2m}{L}} = J_{E_0^\alpha}$, thus

$$|(\psi_0, v_y \psi_0)| \geq \left| \sum_{m \in \mathbb{Z}} |\psi_0(m)|^2 J_{E_0^\alpha} \right|$$

$$\geq |J_{E_0^\alpha}| \left\{ 1 - V_0^2 \left[\frac{1}{(\frac{\pi}{2} - \delta)^2} + \sup_{m \in \mathcal{A}^c} (E_{0,m}^\alpha - E_\kappa^\alpha)^{-2}\right]\right\}$$ (B.14)
for a suitable \( \tilde{m} \in A \), and we have \(|J_{E_{0,m}^\alpha}| > 0\). The second term can be bounded as follows \(|(\psi_*, v_y \psi_*)| \leq ||\psi_*||||v_y \psi_*|| \leq \frac{V_0}{\delta} ||v_y \psi_*|| \) and

\[
||v_y \psi_*||^2 = 2 \left( \psi_*, \frac{1}{2} \left( p_y - Bx \right)^2 \psi_* \right)
\leq 2 \left( \psi_*, \left[ \frac{1}{2} p_x^2 + \frac{1}{2} \left( p_y - Bx \right)^2 + U_\alpha \right] \psi_* \right)
+ 2 \left( \psi_0, \left[ \frac{1}{2} p_x^2 + \frac{1}{2} \left( p_y - Bx \right)^2 + U_\alpha \right] \psi_0 \right) = 2 (\psi_\kappa^\alpha, H_\alpha^0 \psi_\kappa^\alpha)
= 2(\psi_\kappa^\alpha, H_\alpha \psi_\kappa^\alpha) - 2(\psi_\kappa^\alpha, V_\omega \psi_\kappa^\alpha) \leq 2(E_\kappa^\alpha + V_0). \tag{B.15}
\]

This leads to the bound

\[
|(\psi_*, v_y \psi_*)| \leq \frac{V_0}{\frac{1}{2} - \delta} \sqrt{2(E_\kappa^\alpha + V_0)} \tag{B.16}
\]

A similar argument gives the same bound for the third term.

Finally

\[
|(\psi_\kappa^\alpha, v_y \psi_\kappa^\alpha)| \geq |J_{E_{0,m}}| \left\{ 1 - V_0^2 \left[ \frac{1}{(\frac{1}{2} - \delta)^2} + \sup_{m \in A^c} \left( E_{0,m}^\alpha - E_\kappa^\alpha \right)^2 \right] \right\}
- 3 \frac{V_0}{\frac{1}{2} - \delta} \sqrt{2(E_\kappa^\alpha + V_0)} \tag{B.17}
\]

that is strictly positive for a sufficiently small \( V_0 > 0 \) (we can remark that the important condition is \( V_0 \ll B \)).

## C Discussion of hypothesis 1

In this section we indicate a way in which hypothesis \( (H1) \) can be achieved explicitly. We thank F. Bentosela for pointing out this possibility to one of us. We take two symmetric confining walls \( U_\ell(-x) = U_r(x) = U(x) \) and add a magnetic flux tube of intensity \( 0 \leq \Phi \leq 2\pi \) along the cylinder axis. Below we check that the magnetic flux lifts the degeneracy of the levels on the two sides of the sample. In this case the pure edge Hamiltonians are

\[
H_\ell^0[\Phi] = \frac{1}{2} p_x^2 + \frac{1}{2} \left( p_y - Bx + \frac{\Phi}{2} \right)^2 + U(-x) \tag{C.1}
\]

\[
H_r^0[\Phi] = \frac{1}{2} p_x^2 + \frac{1}{2} \left( p_y - Bx + \frac{\Phi}{2} \right)^2 + U(x). \tag{C.2}
\]
The spectra of these Hamiltonians are
\[
\sigma(H^0_\alpha[\Phi]) = \{ E_{n,m}^\alpha(\Phi) : n \in \mathbb{N}, m \in \mathbb{Z} \}. \tag{C.3}
\]
with \( E_{n,m}^\alpha(\Phi) = \epsilon_n^\alpha(\frac{2\pi m}{L} + \frac{\Phi}{L}) \). We consider here only the first spectral branches and note that from the symmetry of the walls, for \( \Phi = 0 \)
\[
\epsilon_0^\ell(-\frac{2\pi}{L}m) = \epsilon_0^r(\frac{2\pi}{L}m) \quad \forall \ m \in \mathbb{Z} \tag{C.4}
\]
We have
\[
\epsilon_0^\ell(-\frac{2\pi}{L}m + \frac{\Phi}{L}) = \epsilon_0^\ell(-\frac{2\pi}{L}m) + \partial_k \epsilon_0^\ell(k_\ell) \frac{\Phi}{L} \tag{C.5}
\]
\[
\epsilon_0^r(\frac{2\pi}{L}m + \frac{\Phi}{L}) = \epsilon_0^r(\frac{2\pi}{L}m) + \partial_k \epsilon_0^r(k_r) \frac{\Phi}{L} \tag{C.6}
\]
for a suitable \( \frac{2\pi}{L}(-m) \leq k_\ell \leq \frac{2\pi}{L}(-m) + \frac{\Phi}{L} \) and \( \frac{2\pi}{L}m \leq k_r \leq \frac{2\pi}{L}m + \frac{\Phi}{L} \). Thus
\[
\left| \epsilon_0^\ell(-\frac{2\pi}{L}m + \frac{\Phi}{L}) - \epsilon_0^r(\frac{2\pi}{L}m + \frac{\Phi}{L}) \right| = \frac{\Phi}{L} \left| \partial_k \epsilon_0^\ell(k_\ell) - \partial_k \epsilon_0^r(k_r) \right| \geq 2\frac{\Phi}{L} \left| \partial_k \epsilon_0^\ell(k_\ell) \right| \geq 2C \frac{\Phi}{L} \tag{C.7}
\]
where \( C > 0 \). A similar argument shows that
\[
\left| \epsilon_0^\ell(-\frac{2\pi(m+1)}{L} + \frac{\Phi}{L}) - \epsilon_0^r(\frac{2\pi m}{L} + \frac{\Phi}{L}) \right| = \frac{\Phi}{L} \left[ \partial_k \epsilon_0^\ell(k_\ell) - \partial_k \epsilon_0^r(k_r) \right] - \frac{2\pi}{L} \partial_k \epsilon_0^\ell(k_\ell) \geq \frac{2\Phi}{L} \left| \partial_k \epsilon_0^\ell(k_\ell) \right| - \frac{2\pi}{L} \left| \partial_k \epsilon_0^r(k_r) \right| \geq 2C \frac{\Phi}{L} \tag{C.8}
\]
Then, by fixing \( \Phi^* \) such that \( 0 < \Phi^* < \pi \) or \( \pi < \Phi^* < 2\pi \) we achieve (2.7).

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