Gravito-electromagnetic perturbations and QNMs of regular black holes

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Abstract

In the framework of Einstein’s gravity coupled to nonlinear electromagnetic fields, we study gravito-electromagnetic perturbations of magnetic regular black holes (BHs). The master equations of perturbations are obtained through Chandrasekhar’s formulation, from which it can be seen, different from the electric counterparts, for magnetic BHs gravitational perturbations with odd-parity coupled only to the electromagnetic perturbations with even-parity. We solve the master equations numerically and obtain quasinormal modes (QNMs) for three typical regular BHs. Results show that QNMs of distinct regular BHs differ significantly, and they differ from that of the Reissner–Nordström BH as well. Indications of these results on the stability of these regular BHs are discussed in detail.

Keywords: quasinormal modes, gravitational wave physics, black hole physics

(Some figures may appear in colour only in the online journal)

1. Introduction

In the past few years, significant astronomical discoveries on black holes (BHs) have been made, including the first-ever observations of gravitational waves (GWs) from coalescence of binary BHs [1, 2] and the first images of the M87* and SgrA* shadows [3, 4], confirming
the existence of BHs in the Universe overwhelmingly and opening the new era of astronomical detection. Because of the long range nature of gravity, GW channel has incomparable advantages in astronomical observations providing us a novel detection method beyond the traditional optical method. With the continuous improvement of detection ability of GWs, we are now able to test general relativity (GR) in the strong gravity regime with unprecedented precision by examining phenomena around BHs.

According to GR, BH will be the inevitable end state of the collapse of very massive stars [5–10]. Penrose and Hawking proved that, under the strong energy condition, singularity will appear inevitably during the collapse [11, 12]. However, it is widely believed that singularity is not physical and should be resolved by, for example, taking into account the quantum effects of either matter or gravitational field. For the electromagnetic field, one effective way to include its quantum effects is to consider its nonlinear extensions. Actually, there has been a long history on the study of nonlinear electromagnetic field theories. Early in 1930’s, Heisenberg and Euler showed that higher order corrections to Maxwell theory arose when considering vacuum polarization [13], and Born and Infeld constructed an electromagnetic field theory tempting to obtain a finite self-energy of the electron [14]. Subsequent researches found that Born–Infeld (BI) electromagnetic theory also arises in the low energy effective theory of string theory [15, 16]. Various BH solutions of gravity coupled to nonlinear electromagnetic fields were found and studied extensively in past years [17–25]. More interestingly, regular BH solutions with no essential singularity are indeed found to exist in this framework. Bardeen first attempted to construct BHs free of singularity [26]. Later, Ayón-Beato and García successfully constructed exact regular BH solutions of Einstein equations [27, 28]. More recently, Fan and Wang constructed a new class of general regular BH solutions [29]. The nonlinear electromagnetic theory they constructed reduces to several well known electromagnetic theories that admit Bardeen or Hayward BH in some limit [30–32]. Even if it is temporarily not known whether the nonlinear theory given in [29] can be derived from some more fundamental theory, the nonlinear theory as an effective theory of electrodynamics is worth to be studied itself, so we would like to study the regular BHs given in [29] in the following. We intend to focus on the magnetic regular BH, since magnetic fields are ubiquitous in nature and permeate our Universe at various scales and amplitudes [33, 34]. BHs in magnetic backgrounds are also supported by recent observations [35–38]. Thus, it is interesting and important to study the effects of ambient magnetic fields on the BH dynamics. However, fully understanding the interaction between BHs and magnetic fields is a rather complicated and still open problem in general. The model we consider in this paper can be seen as a simpler and effective way to describe such kind of interaction. Then it is natural to ask the following question: can the regular BHs be distinguished from the standard ones by GW signals?

It is now well known that, in the late-time (the so-called ringdown stage) of the coalescence of binary BHs (or BH–neutron star, binary neutron stars), GW signal emitted is a superposition of series of sinusoidal damping modes. These modes, called QNMs, have characteristic frequencies determined uniquely by parameters of the final formed BH, namely its mass, angular momentum and electromagnetic charge (if present). Also, QNMs depends on the specific theory underlying the BH. Thus it provides us a powerful tool to distinguish different BHs and theories by detecting QNMs in GWs. Amounts of effort have been devoted on studying QNMs of various BHs in variety of theories in past years [39–42], see also reviews [43–46]. Although limited by detection accuracy, data analysis to read off QNMs from the detected GW signals are ongoing and some progresses have been made [47–54]. It is expected that the next generation of detectors can give us more conclusive information about QNMs of BHs. Study of QNMs of regular BHs have also attracted lots of attention [55–72], and results show that regular structure of BH origin may have considerable influences on QNMs. However, previous
studies mainly involve gravitational or matter perturbations separately or test fields, investigations of gravitational and matter perturbations simultaneously in a consistent way are few [73–76]. So, in this paper, we aim to study gravitational and electromagnetic perturbations of regular BHs simultaneously. By applying Chandrasekhar’s formulation [77], we obtain the general master perturbation equations which could be applied to any static and magnetic BHs. We solve the coupled Schrödinger-like master equations and calculate fundamental QNMs of three types of regular BHs given in [29] by numerical method. We hope the results obtained in this paper may help to distinguish distinct types of BHs with GW signals.

This paper is organized as follows. In section 2, we obtain the second-order master perturbation equations, and rewrite them in the standard form through separating variables and introducing proper new functions. In section 3, we apply the master perturbation equations to obtain fundamental QNMs for three types of regular BHs by numerical method, and discuss their physical indications. The last section is devoted to summary and discussions.

2. Gravito-electromagnetic perturbations

In this section, we perturb the field equations of Einstein gravity coupled to nonlinear electromagnetic field theory, and to obtain second-order perturbation equations. Through separating variables and defining new functions we successfully rewrite the perturbation equations in the standard Schrödinger-type form at last.

2.1. Field equations

Einstein gravity coupled to nonlinear electromagnetic field is described by the action [29]

\[ I = \frac{1}{16\pi} \int d^4x \sqrt{-g} (R - \mathcal{L}(F)) , \]

where \( d^4x \equiv dt dx dy dz \) denotes the volume element in four-dimensional spacetime, \( g \) under the square root is the metric determinant, \( d^4x \) and \( \sqrt{-g} \) constitute the invariant volume element under diffeomorphism. The Ricci scalar \( R \) and \( \mathcal{L}(F) \) denotes the gravitational and electromagnetic sector of the full theory respectively. \( \mathcal{L}(F) \) is given by

\[ \mathcal{L}(F) = \frac{4\mu}{\alpha} \left( \frac{\alpha F}{1 + (\alpha F)^2} \right)^{1/2} , \]

where \( \mu > 0 \) is a dimensionless constant which determines the weak field behaviors of the nonlinear electromagnetic theory, \( \nu \) is another dimensionless parameter whose different values correspond to different particular electromagnetic theory, \( \alpha > 0 \) has the dimension of length squared and it is related to the magnetic charge \( Q_m \) through equation (6) below. \( F \) is defined as \( F \equiv F_{\rho\sigma} F^{\rho\sigma} \) with \( F_{\rho\sigma} \) being the field strength tensor. Standard variation process leads to the equations of motion (EOMs) for the metric

\[ R_{\rho\sigma} - \frac{1}{2} R g_{\rho\sigma} = 2 \left( \mathcal{L}_F F_{\rho\sigma}^2 - \frac{1}{4} g_{\rho\sigma} \mathcal{L} \right) , \]

and the electromagnetic fields

\[ \nabla_{\rho} (\mathcal{L}_F F^{\rho\sigma}) = 0 , \]

respectively, where \( R_{\rho\sigma} \) is Ricci tensor, \( \mathcal{L}_F \) denotes \( d\mathcal{L}(F) / dF \), and \( \nabla_{\mu} \) denotes covariant derivative. A general class of magnetic regular BH solutions were found in [29], the metric and vector potential are given by
where $Q_m$ is the magnetic charge and $q$ is an integration constant which is related to $Q_m$ through equation (6). The parameter $\mu$ are required to be $\mu \geq 3$ in order that the BHs are free of curvature singularities at $r = 0$. $\nu = 1, 2, 3$ corresponds to new Maxwellian, Bardeen and Hayward BHs respectively. Thermodynamic quantities of the regular BHs are given by

$$M_{\text{ADM}} = M, \quad S = \pi r_0^2, \quad Q_m = \frac{q^2}{\sqrt{2} r_0},$$

$$T = \frac{1}{4\pi r_0} \left( 1 - 2\mu \alpha^{-1} q^4 \mu^{-1} (r_0 + q)^{-\mu-1} \right),$$

$$\Psi = -\frac{q}{\sqrt{2} r_0} \left( 3 - (\mu - 3) \frac{q}{r_0} \left( 1 + \frac{q}{r_0} \right)^{-\mu-1} - 3 \right),$$

$$\Pi = \frac{q^3}{4\alpha^2} \left( 1 + (\mu + 1) \frac{q}{r_0} \left( 1 + \frac{q}{r_0} \right)^{-\mu-1} - 1 \right),$$

where $M_{\text{ADM}}$ is the Arnowitt–Deser–Misner (ADM) mass, $S$ is the entropy, $T$ is Hawking temperature, $\Psi$ is the magnetic potential conjugate to $Q_m$, $\Pi$ is conjugate to $\alpha$, and $r_0$ is the radius of the outmost horizon. It is straightforward to check that the first law of thermodynamics

$$dM_{\text{ADM}} = TdS + \Psi dQ_m + \Pi d\alpha$$

and the Smarr formula [78]

$$M_{\text{ADM}} = 2TS + \Psi Q_m + 2\Pi \alpha,$$  

are satisfied. To understand the Smarr formula, let us perform the dimensional analysis. From the action (1) we know $\mathcal{L}(\mathcal{F})$ should have the dimension (length)$^{-2}$ in order to keep the action to be dimensionless. Since the parameter $\mu$ is dimensionless, the parameter $\alpha$ should have the dimension (length)$^{\mu}$ taking into count the expression of $\mathcal{L}(\mathcal{F})$ in equation (2). In spacetime with general dimensions $D$, the mass $M$ has dimension (length)$^{D-3}$, thus in four dimensions, $M$ has dimension (length)$^1$. Entropy $S$ has dimension (length)$^2$ since $S$ is proportional to the horizon area. From equation (5) we know, $q$ has the dimension (length)$^1$, thus the magnetic charge has dimension (length)$^1$ considering $Q_m = q^2/\sqrt{2} \alpha$. Assuming the mass $M = M(S, Q_m, \alpha)$ has scaling dimension $d_M$ and each extensive variable has a scaling dimension denoted by $d_S, d_{Q_m}, d_\alpha$ respectively, then we have

$$L^{d_M} = M(L^{d_S}, L^{d_{Q_m}}, Q_m, L^{d_\alpha}).$$

According to Euler’s theorem, rescaling of the extensive variables yield [79, 80]

$$d_M = d_S \left( \frac{\partial M}{\partial S} \right) + d_{Q_m} \left( \frac{\partial M}{\partial Q_m} \right) + d_\alpha \left( \frac{\partial M}{\partial \alpha} \right).$$

From the dimensional analysis above we know $d_M = 1, d_S = 2, d_{Q_m} = 1, d_\alpha = 2$, this together with the Euler’s theorem result in the Smarr formula (8) as expected. To study gravito-electromagnetic perturbations of the background BH spacetime, we adopt Chandrasekhar’s procedure [77]. First, we choose a metric with sufficient generality to describes a non-stationary and axisymmetric spacetime

$$ds^2 = -e^{2\gamma} (dt)^2 + e^{2\psi} (d\varphi - q_3 dx^2 - q_3 dx^3 - \sigma dt)^2 + e^{2\mu_1} (dx^2)^2 + e^{2\mu_3} (dx^3)^2.$$
Note that the metric (11) involves seven quantities $\gamma, \psi, \mu_2, \mu_3, \omega, q_2, q_3$, which are functions of $t, x^2$ and $x^3$. While Einstein’s equation contains only six independent equations, thus among the seven functions there are only six independent ones. It can be shown that the functions are related through

$$\left(\sigma, q_2, 0\right), \left(\sigma, q_3, 0\right), \left(q_2, 0, q_3\right), \left(q_3, 0, q_2\right) = 0. \tag{12}$$

The form of metric (11) can also be used to describe a class of non-stationary and non-axisymmetric spacetime by setting $g^{\mu\nu}(t, \varphi, x^2, x^3) = g^{\mu\nu}(t, x^2, x^3) h(\varphi)$, where the seven quantities above are functions of all four variables $t, \varphi, x^2, x^3$. Anyway, the metric (11) is of sufficient generality for the questions we discussed in this paper.

Following Chandrasekhar’s procedure, for the gravitational sector of the field equations, we apply Cartan’s equations of structure to calculate Riemann and Ricci tensors associated to the metric (11), while for the matter sector we apply tetrad formalism to work out the relevant contributions to EOMs. In order to apply Cartan’s formulation, we take the following basis of one-forms

$$\omega^0 = e^\gamma dt, \quad \omega^1 = e^\psi (dq_2 - q_3 dx^3 - q_2 dx^2 - \sigma dt), \quad \omega^2 = e^{\mu_2} dx^2, \quad \omega^3 = e^{\mu_3} dx^3. \tag{13}$$

In this paper, we use Latin letters for the tetrad indices and Greek letters for the tensor indices. From Cartan’s first equation of structure

$$d\omega^j + \omega^j \wedge \omega^i = 0, \quad (14)$$

one is able to calculate the connection one-forms $\omega^j$, where $\wedge$ is wedge product. And from Cartan’s second equation of structure

$$\frac{1}{2} R^{\ell}_{\,jk}\omega^k \wedge \omega^j = \Omega^\ell_{\,j} = d\omega^j + \omega^j \wedge \omega^j, \tag{15}$$

the Riemann tensor can be worked out. The explicit expressions of spin connection and Riemann tensor can be found in [77], so we do not list here. It’s straightforward to obtain Ricci tensor from Riemann tensor.

To apply the tetrad formalism, we choose the basis of the orthonormal frame as

$$e_{\mu 0} = (-e^\gamma, 0, 0, 0),$$
$$e_{\mu 1} = (-\sigma e^\psi, e^\psi, -q_2 e^\psi, -q_3 e^\psi),$$
$$e_{\mu 2} = (0, 0, e^{\mu_2}, 0),$$
$$e_{\mu 3} = (0, 0, 0, e^{\mu_3}). \tag{16}$$

The corresponding contravariant vectors are given by

$$e^{\mu 0} = (e^{-\gamma}, \sigma e^{-\gamma}, 0, 0),$$
$$e^{\mu 1} = (0, e^{-\psi}, 0, 0),$$
$$e^{\mu 2} = (0, q_2 e^{-\mu_2}, e^{-\mu_2}, 0),$$
$$e^{\mu 3} = (0, q_3 e^{-\mu_3}, 0, e^{-\mu_3}). \tag{17}$$

It’s easy to see that $e_{\mu a} e_{\nu}^a = \eta_{ab} = \text{diag}(-1, 1, 1, 1)$, thus the chosen frame is indeed a locally inertial frame.

For convenience we introduce the tensor

$$P_{ab} = e^{\mu a} e^{\nu b} \mathcal{L}_F F_{\mu \nu}, \tag{18}$$

with which EOM of the electromagnetic fields (4) can now be rewritten as

$$\eta^{\mu a} P_{am|m} = 0, \tag{19}$$
where $P_{\alpha\beta\gamma\delta} \equiv e^{\mu}_\alpha e^{\nu}_\beta e^{\rho}_\gamma \nabla_{\rho} P_{\mu\nu}$. It is easy to note that the Bianchi identity

$$F_{[abc]} = 0$$

(20)

for the tetrad component of strength tensor $F_{ab} = e^{a}_\mu e^{b}_\nu F_{\mu\nu}$. is still satisfied, where the square bracket denotes the three indices $a, b, c$ to be totally anti-symmetric. After some lengthy but straightforward calculations and considering the metric (11), equations (19) and (20) reduce to the following two sets of equations

\[
\begin{align*}
(e^{\phi+\mu_2+\mu_3} F_{12} )_3 + (e^{\phi+\mu_3} F_{31} )_2 &= 0, \\
(e^{\phi+\gamma} F_{01} )_2 + (e^{\phi+\mu_2} F_{12} )_0 &= 0, \\
(e^{\phi+\gamma} F_{01} )_3 + (e^{\phi+\mu_3} F_{12} )_0 &= 0, \\
(e^{\mu_2+\mu_3} P_{01} )_0 + (e^{\gamma+\mu_3} P_{12} )_2 + (e^{\gamma+\mu_2} P_{13} )_3 &= e^{\phi+\mu_3} P_{02} Q_{02} + e^{\phi+\mu_2} P_{03} Q_{03} - e^{\phi+\gamma} P_{23} Q_{23}
\end{align*}
\]

and

\[
\begin{align*}
(e^{\phi+\mu_2} P_{02} )_2 + (e^{\phi+\mu_3} P_{03} )_3 &= 0, \\
(e^{\phi+\mu_2} P_{03} )_0 - (e^{\phi+\gamma} P_{23} )_2 &= 0, \\
(e^{\phi+\gamma} P_{02} )_0 + (e^{\phi+\mu_2} P_{23} )_3 &= 0, \\
(e^{\gamma+\mu_2} F_{02} )_3 - (e^{\gamma+\mu_3} F_{03} )_2 + (e^{\mu_2+\mu_3} F_{23} )_0 &= e^{\phi+\gamma} F_{01} Q_{23} + e^{\phi+\mu_2} F_{12} Q_{03} - e^{\phi+\mu_3} F_{13} Q_{02}.
\end{align*}
\]

(22)

Where the notations $f_A \equiv f_{\lambda} + q_{\lambda} f_1$, $Q_{AB} \equiv q_{AB} - q_{BA}$ are defined for convenience, and the subscript $i$ denotes taking partial derivative with respect to $x^i$. Note that the above equations are not independent, the first equation in each set is just the integrable condition of the following two equations. Note also that the set of equation (21) involve only quantities which reverse their signs under the transformation $\varphi \rightarrow -\varphi$, while the set of equation (22) involve only quantities that keep invariant under $\varphi \rightarrow -\varphi$. The quantities in (21) and (22) are called to be odd (or axial) and even (or polar) respectively.

### 2.2. First order perturbation equations for metric and electromagnetic fields

Now we consider perturbations of the field equations. In the above, we have noted that the EOMs of electromagnetic field can be splitted into two sets with just odd quantities (21) or just even quantities (22) respectively. The metric perturbations can also be divided into two sets, the perturbations $\sigma, q_2, q_3$ reverse their signs while the perturbations $\delta \gamma, \delta \psi, \delta \mu_2, \delta \mu_3$ keep invariant to keep the metric (11) invariant under the transformation $\varphi \rightarrow -\varphi$, these two kinds of metric perturbations are also called to be odd and even respectively. From the metric (11) it is obvious to see that the odd metric perturbations induce a dragging of inertial frame and impart a rotation of the BH, while the even metric perturbations impart no such rotation.

In this paper, for the magnetic regular BHs we take the combination of the odd gravitational perturbations with even electromagnetic perturbations following [74] (but in different formalism), i.e. we study the perturbed Einstein equations with odd parity caused by $\sigma, q_2, q_3$ combined with the perturbed electromagnetic equations with even parity (22), calculations show that only in this way the master equations of perturbations can be worked out. This is different from the case discussed in [75] and [77], where for the electrically charged BHs, the master equations are given by odd gravitational perturbations coupled to odd electromagnetic perturbations.
We perturb the Einstein equations
\[ R_{ab} = -2 \left( \mathcal{L}_F F_{am} F_{b}^m - \frac{1}{4} \eta_{ab} \mathcal{L}(F) \right), \]
and obtain
\[ \delta R_{ab} = -4 \mathcal{L}_F \delta F_{pq} F^{pq} F_{am} F_{b}^m - 2 \mathcal{L}_F \delta F_{am} F_{b}^m - 2 \mathcal{L}_F \delta F_{am} F_{b}^m + \eta_{ab} \mathcal{L}_F \delta F_{pq} F^{pq}. \]  \( \tag{24} \)
To leading order of the perturbations, we have
\[ \delta R_{12} = -2 \left( \mathcal{L}_F F_{13} \delta F_{23} \right), \quad \delta R_{13} = 0. \]  \( \tag{25} \)
The components of the strength tensor are given by
\[ F_{03} = \sigma e^{-\gamma} \frac{Q_m}{r} \sin \theta, \quad F_{13} = \frac{Q_m}{r^2}, \quad F_{23} = q_2 e^{-\gamma} \frac{Q_m}{r} \sin \theta \]  \( \tag{26} \)
with all other components vanishing. Note that the only non-vanishing component of the background strength tensor is \( F_{13} \), the components \( F_{03}, F_{23} \) and all other components are small perturbations, which can be denoted as \( \delta F_{03}, \delta F_{23}, \) etc.

The linear perturbation version of the set of equation (22) are given by
\[ (e^{\phi+\mu} \delta P_{03}),_0 - (e^{\phi+\gamma} \delta P_{23}),_2 = 0, \]  \( \tag{27} \)
\[ (e^{\phi+\mu} \delta P_{02}),_0 + (e^{\phi+\gamma} \delta P_{23}),_2 = 0, \]  \( \tag{28} \)
\[ (e^{\gamma+\mu} \delta F_{02}),_3 - (e^{\gamma+\gamma} \delta F_{03}),_2 + (e^{\mu+\gamma} \delta F_{23}),_0 = -e^{\phi+\mu} F_{13} Q_{02}. \]  \( \tag{29} \)
For the BH background (5), taking derivative of equation (29) with respect to \( x^i \) (or \( t \)) and combining equations (27) and (28) give rise to
\[ \left[ \frac{1}{L_r} \frac{1}{r^2 \sin \theta} \left( r \sin \theta e^{\gamma} \delta P_{23} \right) \right],_3 + \left[ \frac{e^{\gamma}}{L_r} \left( r e^{\gamma} \delta P_{23} \right) \right],_2 - \left( \frac{r e^{-\gamma}}{L_r} \delta P_{23} \right),_{0,0} = -r^2 \sin \theta F_{13} (q_{2,0,0} - \sigma_{2,0}). \]  \( \tag{30} \)
Introducing the function \( Q(r, \theta, t) \equiv r^2 e^{\gamma} Q_{23} \sin^3 \theta \), the perturbed Einstein equation (25) can be written as
\[ \frac{\partial Q}{\partial \theta} - r^4 (q_{2,0} - \sigma_{2,0}) \sin^3 \theta = 4 r^3 e^{\gamma} \sin^2 \theta F_{13} \delta P_{33}, \]  \( \tag{31} \)
\[ \frac{\partial Q}{\partial r} + r^2 e^{-2\gamma} (q_{3,0} - \sigma_{3,0}) \sin^3 \theta = 0. \]  \( \tag{32} \)
Supposing the time dependence of the perturbation to be \( Q(r, \theta, t) = \tilde{Q}(r, \theta) e^{i \omega t} \), introducing two new functions \( \Delta \equiv r^2 e^{\gamma} \) and \( B(r, \theta) \equiv \delta P_{33} \sin \theta \) for convenience, and eliminating \( \sigma \) from the above two equations, we have
\[ r^4 \frac{\partial}{\partial r} \left( \frac{\Delta \partial \tilde{Q}}{r^4 \partial r} \right) + \sin^3 \theta \frac{\partial}{\partial \theta} \left( \frac{1}{\sin ^3 \theta} \frac{\partial \tilde{Q}}{\partial \theta} \right) + \omega^2 r^4 \frac{\Delta}{\Delta} \tilde{Q} = 4 r^3 \sin^3 \theta F_{13} e^{\gamma} \frac{\partial}{\partial \theta} \left( \frac{B}{\sin ^3 \theta} \right). \]  \( \tag{33} \)
Combining equations (30) and (31) and eliminating \( (q_{2,0} - \sigma_{2,0}) \), we have
\[ \sin \theta \left[ \frac{1}{L_r} \frac{1}{r^2 \sin \theta} \left( r e^{\gamma} B, \theta \right) \right] + \left[ \frac{e^{\gamma}}{L_r} \left( r e^{\gamma} B, \gamma \right) \right] = -r^2 e^{-\gamma} \frac{\partial Q}{\partial \theta} + 4 r e^{\gamma} B F_{13}^2. \]  \( \tag{34} \)
Now, we separate the variables \( r \) and \( \theta \) in equations (33) and (34). Suppose the functions \( \tilde{Q}(r, \theta) \) and \( B(r, \theta) \) can be written in the form

\[
\tilde{Q}(r, \theta) = \tilde{Q}(r) C_{l+1}^{-3/2}(\theta),
\]

\[
B(r, \theta) = B(r) \sin \theta \frac{dC_{l+1}^{-3/2}(\theta)}{d\theta} = 3B(r)C_{l+1}^{-1/2}(\theta),
\]

where \( C_{l+1}^{-3/2}(\theta) \), \( C_{l+1}^{-1/2}(\theta) \) are Gegenbauer functions, and the relation \( \frac{1}{\sin \theta} \frac{dC_n^\rho}{d\theta} = -2\rho C_n^{\rho+1} \) has been used in (36), then equation (33) becomes

\[
r^4 \frac{d}{dr} \left( \frac{\Delta}{r^2} \frac{d\tilde{Q}}{dr} \right) C_{l+1}^{-3/2}(\theta) + \sin^2 \theta \frac{d}{d\theta} \left( \frac{1}{\sin^2 \theta} \frac{dC_{l+1}^{-3/2}(\theta)}{d\theta} \right) \tilde{Q}(r)
+ \omega^2 \frac{r^d}{\Delta} \tilde{Q} C_{l+1}^{-3/2}(\theta) = 4r^3 \sin^3 \theta F_{13} e^\gamma \frac{d}{d\theta} \left( \frac{1}{\sin^2 \theta} \frac{dC_{l+1}^{-3/2}(\theta)}{d\theta} \right) F_{13}.
\]

Recalling the equation

\[
\left[ \frac{d}{d\theta} \sin^{2\rho} \theta \frac{d}{d\theta} + n(n + 2\rho) \sin^{2\rho} \theta \right] C_n^\rho(\theta) = 0
\]

satisfied by the Gegenbauer function, equation (37) can be simplified to

\[
r^4 \frac{d}{dr} \left( \frac{\Delta}{r^2} \frac{d\tilde{Q}}{dr} \right) -(l+2)(l-1)\tilde{Q}(r) + \omega^2 \frac{r^d}{\Delta} \tilde{Q} + 4r^3 \sin^3 \theta F_{13} e^\gamma (l+2)(l-1)B(r) = 0.
\]

Substituting equation (36) into equation (34), and repeating the procedure to obtain equation (39), then equation (34) reduces to

\[
\frac{l(l+1)}{r L_F} B(r) + \frac{d}{dr} \left[ \frac{e^{\gamma} \frac{d}{dr} (\text{re}^\gamma B(r))}{L_F} \right] + \omega^2 r^2 \frac{e^{-\gamma} B(r)}{L_F} = \frac{F_{13}}{r^2} \tilde{Q}(r) + 4re^\gamma F_{13} B(r).
\]

Up to now, we have succeeded in separating variables of the master equations.

Next we need to write the master equations (39) and (40) into the standard form. First we introduce the tortoise coordinate through \( dr_* = \frac{r}{L_F} dr \), and then we define two new functions \( H_1(r) \) and \( H_2(r) \)

\[
\tilde{Q}(r) = rH_1(r), \quad \text{re}^\gamma B(r) = \sqrt{L_F} H_2(r),
\]

these efforts enable us to rewrite equations (39) and (40) into the standard Schrödinger-type form finally

\[
\left( \frac{d^2}{dr_*^2} + \omega^2 \right) H_1 + \left( \frac{\Delta \Delta'}{r^2} - \frac{4\Delta^2}{r^2} - (l+2)(l-1) \frac{\Delta}{r^2} \right) H_1
+ 4(l+2)(l-1)F_{13} \sqrt{L_F} \frac{\Delta}{r^2} H_2 = 0,
\]

\[
\left( \frac{d^2}{dr_*^2} + \omega^2 \right) H_2 + \left( -l(l+1) \frac{\Delta}{r^2} - \frac{2\Delta^2}{r^2} \sqrt{L_F} \frac{\Delta'}{L_F} + \frac{\Delta^2}{r^2} (\sqrt{L_F} \frac{\Delta'}{L_F})' \right) H_2
+ \frac{\Delta \Delta'}{\sqrt{L_F}} (\sqrt{L_F} \frac{\Delta'}{L_F})' + \frac{\Delta^2}{r^2} (\sqrt{L_F} \frac{\Delta'}{L_F})' - 4\Delta^2 (\sqrt{L_F} \frac{\Delta'}{L_F})' L_F F_{13} \right) H_2 + \frac{\Delta}{r^2} F_{13} \sqrt{L_F} H_1 = 0.
\]
condition and outgoing wave condition respectively, that is necessary. Precisely, at the horizon and at infinity, the solution should take ingoing wave can be read out from equations (42).

The two perturbation equations (42) and (43) are coupled and can be cast in a compact form as

\[
\left( \frac{d^2}{dr^2} + \omega^2 - V(r) \right) Y = 0, 
\]

where \( Y = \left( \begin{array}{c} H_1 \\ H_2 \end{array} \right) \), \( V(r) \) is a 2 \times 2 matrix, whose matrix elements which are functions of \( r \) can be read out from equations (42) and (43). To obtain QNMs, physical boundary conditions are necessary. Precisely, at the horizon and at infinity, the solution should take ingoing wave condition and outgoing wave condition respectively, that is

\[
Y_i \sim b_i e^{-i\omega r}, \quad r \to r_+, \\
Y_i \sim b_i e^{i\omega r}, \quad r \to \infty, 
\]

where \( Y_i = H_1, H_2 \) for \( i = 1, 2 \) respectively, and \( b_1, B_2 \) are coefficients.

With the above boundary conditions, it is now an eigenvalue problem to obtain quasinormal frequency \( \omega \) from the perturbation equation (44), which can be solved by applying the matrix-valued numerical integration method [82]. The basic idea of [82], is to directly integrate the
system through shooting method from the horizon outward to infinity by imposing the boundary conditions (45). Numerical results of fundamental modes, which are expected to dominate versus $q$. For more visual comparison, we also plot the numerical data in figures 1 and 2. For both the electromagnetic and gravitational perturbations, we all plot three cases with $\ell = 2,3,4$ respectively. For every $\ell$ we give two plots which describe $\Re \omega$ versus $q/M$ and $\Im \omega$ versus $q/M$ respectively. Gravito-electromagnetic QNMs of RN BH carrying magnetic charge in GR are also shown for comparison. From the tables and figures, following silent properties of QNMs can be observed:

| $\ell$ | $q/M$ | $l=2$ | $l=3$ | $l=4$ |
|-------|-------|-------|-------|-------|
|       | $M_{S_{1}}$ | $M_{S_{2}}$ | $M_{S_{1}}$ | $M_{S_{2}}$ | $M_{S_{1}}$ | $M_{S_{2}}$ |
| 0     | 0.457596 | 0.373672 | 0.656899 | 0.599444 | 0.853095 | 0.809178 |
|       | 0.0950045 | 0.0889623i | 0.0956164i | 0.0927032i | 0.0958601i | 0.0941641i |
| 0.1   | 0.468975 | 0.373045 | 0.666858 | 0.599029 | 0.861766 | 0.809553 |
|       | 0.0966194 | 0.0885194i | 0.0957702i | 0.0918889i | 0.0970081i | 0.0937172i |
| 0.2   | 0.474839 | 0.37312 | 0.67422 | 0.598863 | 0.871742 | 0.809297 |
|       | 0.0965579 | 0.0882093i | 0.0954732i | 0.0915312i | 0.0967447i | 0.0932967i |
| 0.3   | 0.48484 | 0.373449 | 0.686297 | 0.599205 | 0.887226 | 0.810104 |
|       | 0.0962131 | 0.0876668i | 0.095169i | 0.0909092i | 0.0954353i | 0.0925481i |
| 0.4   | 0.499273 | 0.374301 | 0.703836 | 0.600697 | 0.907249 | 0.812951 |
|       | 0.0953048 | 0.0868343i | 0.0952043i | 0.0899732i | 0.09402i | 0.0914434i |
| 0.5   | 0.518875 | 0.376017 | 0.729105 | 0.604038 | 0.935047 | 0.818769 |
|       | 0.0937642 | 0.085577i | 0.0943042i | 0.0885838i | 0.0943714i | 0.0898618i |
| 0.6   | 0.546067 | 0.37901 | 0.761723 | 0.610073 | 0.974356 | 0.828703 |
|       | 0.0914386 | 0.0835757i | 0.0906819i | 0.0863319i | 0.0903898i | 0.087355i |

| $\ell$ | $q/M$ | $l=2$ | $l=3$ | $l=4$ |
|-------|-------|-------|-------|-------|
|       | $M_{S_{1}}$ | $M_{S_{2}}$ | $M_{S_{1}}$ | $M_{S_{2}}$ | $M_{S_{1}}$ | $M_{S_{2}}$ |
| 0     | 0.457596 | 0.373672 | 0.656899 | 0.599444 | 0.853095 | 0.809178 |
|       | 0.0950045 | 0.0889623i | 0.0956164i | 0.0927032i | 0.0958601i | 0.0941641i |
| 0.2   | 0.484066 | 0.373029 | 0.676035 | 0.599096 | 0.868435 | 0.809700 |
|       | 0.0971561 | 0.088577i | 0.0958228i | 0.0919502i | 0.0969739i | 0.0937901i |
| 0.4   | 0.487026 | 0.372949 | 0.679654 | 0.598688 | 0.873348 | 0.809999 |
|       | 0.0967646 | 0.0882681i | 0.0954973i | 0.0915619i | 0.0964386i | 0.0933831i |
| 0.6   | 0.495228 | 0.372795 | 0.689621 | 0.597882 | 0.885956 | 0.807840 |
|       | 0.095471 | 0.0873902i | 0.0946848i | 0.0904792i | 0.0948336i | 0.0922095i |
| 0.8   | 0.512083 | 0.372704 | 0.710699 | 0.597269 | 0.909488 | 0.807622 |
|       | 0.0921845 | 0.0854734i | 0.0926479i | 0.0881798i | 0.0917749i | 0.0896405i |
| 0.9   | 0.525768 | 0.372739 | 0.727753 | 0.597416 | 0.928251 | 0.808613 |
|       | 0.0891379 | 0.0834854i | 0.0896736i | 0.0852594i | 0.089680i | 0.0874720i |
| 0.95  | 0.534733 | 0.372759 | 0.738341 | 0.597685 | 0.940533 | 0.809554 |
|       | 0.086849 | 0.0827797i | 0.0872509i | 0.0849986i | 0.0877607i | 0.086444i |
Figure 1. Fundamental modes of the electromagnetic branch $M_3$. From left to right the plots correspond to $\ell = 2, 3, 4$ respectively. For each $\ell$, the upper plot describes $\text{Re}\omega$ versus $q/M$ while the lower plot describes $\text{Im}\omega$ versus $q/M$.

- When $q = 0$, the BH solution (5) reduces to the well-known Schwarzschild BH. And the perturbation equations (42) and (43) reduces to that of gravito-electromagnetic perturbations of Schwarzschild BH in GR. So in this case, QNMs of the three types of regular BHs and RN BH coincide, which confirms the validity of our numerical method.
- When $q \neq 0$, quasinormal frequencies of the three types of regular BHs differ significantly, and they are very different from that of the RN BH. This may give us a way to distinguish different theories and BHs.
Figure 2. Fundamental modes of gravitational branch $M_{S2}$. From left to right the plots correspond to $\ell = 2, 3, 4$ respectively. For each $\ell$, the upper plot describes $\text{Re}\omega$ versus $q/M$ while the lower plot describes $\text{Im}\omega$ versus $q/M$.

- In the parameter space we considered, all the quasinormal frequencies have negative imaginary part $\text{Im}\omega < 0$ suggesting that these BHs are stable under gravito-electromagnetic perturbations.
- For the electromagnetic branch $M_{S1}$, with the increase of the magnetic charge $q$, the real part of quasinormal frequency $\text{Re}\omega$ increases monotonically for all the three types of regular BHs. Meanwhile, the dependence of the imaginary part of quasinormal frequency $\text{Im}\omega$ on $q$
is complicated relying on the value of \( \ell \). For \( \ell = 2 \) mode which is believed to dominate the ringdown stage, with the increase of \( q \), \( \text{Im}\omega \) of \( \nu = 1 \) type decreases monotonically, while that of \( \nu = 2 \) or \( \nu = 3 \) type first decreases then increases. This means that the new Maxwellian BH with \( \nu = 1 \) becomes more stable when \( q \) becomes larger.

- For the gravitational branch \( M_{S2} \), with the increase of the charge \( q \), \( \text{Re}\omega \) of either \( \nu = 1 \) or \( \nu = 2 \) type increases monotonically, while that of \( \nu = 3 \) type is nearly unaffected. Meanwhile, \( \text{Im}\omega \) of either \( \nu = 2 \) or \( \nu = 3 \) increases monotonically, while that of \( \nu = 1 \) depends on \( q \) in a complicated way. This means that Bardeen-type and Hayward-type BHs becomes less stable when \( q \) becomes larger.

- It is more interesting to see the effect of the value of \( \nu \) on \( \text{Im}\omega \). From the figures, one can see that in both branches, for fixed \( q \), QNMs of \( \nu = 1 \) type always have the most negative \( \text{Im}\omega \) meaning that the Maxwellian BH is the most stable among the three types of regular BHs and also more stable than RN BH.

**4. Summary and discussion**

In this work, we studied the gravito-electromagnetic perturbations of magnetic regular BHs. By applying Chandrasekhar’s approach, the master equations can be obtained for gravitational perturbations with odd-parity coupled to the electromagnetic perturbations with even-parity. This is different from the electric BHs case, for which the odd-parity gravitational perturbations coupled to odd-parity electromagnetic perturbations. The master equations can be used to solve out QNMs of the regular BHs. We fixed \( \mu = 3 \) and take \( \nu = 1, 2, \) and 3 which correspond to the new Maxwellian, Bardeen-type and Hayward-type BHs respectively, and calculate the corresponding QNMs by applying numerical method. From the results, one can see that QNMs depend on the magnetic charge \( q \), and QNMs of the three types of regular BHs differ significantly, they are very different from that of magnetic RN BH as well. In the parameter space we considered, all the quasinormal frequencies have negative imaginary part \( \text{Im}\omega \) suggesting that these BHs are all stable under the perturbations. By comparing the three types of regular BHs and the standard magnetic RN BH, it is interesting to see that QNMs of \( \nu = 1 \) type BH always have the most negative \( \text{Im}\omega \) indicating that the Maxwellian BH is most stable. These properties of QNMs may provide a way to distinguish the different theories and BHs with GW signals.

In this work, we focus on the case of spherical regular BHs. It is natural to extend the current study to rotating case which will be more realistic from astrophysical viewpoint [83–88]. We would like to leave this for further investigations.

**Data availability statement**

No new data were created or analyzed in this study.

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