Stability and disturbance attenuation for a switched Markov jump linear system

Collin C. Lutz*, Student Member, IEEE, and Daniel J. Stilwell, Member, IEEE
Bradley Department of Electrical & Computer Engineering
302 Whittemore (0111)
Virginia Tech
Blacksburg, VA 24061
Tel: (540) 231-3204, Fax: (540) 231-3362, Email: {collin,stilwell} at vt.edu

Abstract—We address a class of Markov jump linear systems that are characterized by the underlying Markov process being time-inhomogeneous with a priori unknown transition probabilities. Necessary and sufficient conditions for uniform stochastic stability and uniform stochastic disturbance attenuation are reported. In both cases, conditions are expressed as a set of finite-dimensional linear matrix inequalities that can be solved efficiently.

I. INTRODUCTION

Discrete-Time Markov jump linear system is a stochastic discrete-time linear time-varying system where the time-variation of system matrices is determined by a realization of a Markov chain. The Markov chain may be time-homogeneous (characterized by constant transition probabilities) or time-inhomogeneous (characterized by time-varying transition probabilities), and we slightly abuse terminology by referring to a time-(in)homogeneous Markov jump linear system. We address a switched Markov jump linear system, which is simply a time-inhomogeneous Markov jump linear system where the underlying Markov chain is characterized by an a priori unknown sequence of transition probability matrices that assume one of finitely-many values at each time instant; an a priori unknown switching sequence parameterizes the transition probability matrices at each time instant. As a special case, our analysis also applies to time-inhomogeneous Markov jump linear systems with known transition probabilities that vary in a finite set. In the existing literature, stability (resp. disturbance attenuation) of a time-inhomogeneous Markov jump linear system is equivalent to an infinite-dimensional Lyapunov (resp. storage function) criterion that in general lacks a practical technique for solving. For stochastically stable and contractive systems, we show the existence of Lyapunov and storage functions with finite dependence on the future. This observation leads to necessary and sufficient conditions for uniform stochastic stability and uniform stochastic disturbance attenuation for a switched Markov jump linear system, expressed as a set of finite-dimensional linear matrix inequalities (LMIs), which can be solved efficiently using well-known techniques.

The Markov jump linear system abstraction finds application in many areas including economics [1], fault-tolerant control [2], energy-aware control [3], and networked control [4, 5]. Despite the prevalence of Markov jump linear systems, little attention has been paid to the case when the Markov chain transition probabilities are time-varying, and almost no attention has been paid to the case when the Markov chain transition probabilities are time-varying and a priori unknown.

*Corresponding author

This research was made with Government support under and awarded by DoD, Air Force Office of Scientific Research, National Defense Science and Engineering Graduate (NDSEG) Fellowship, 32 CFR 168a.

Time-varying Markov chain transition probabilities may arise in a variety of situations. Consider, for example, a control system where the plant and controller are connected via a wireless communications network subject to random network delays and/or packet loss (see, e.g., [6]). Network delays and packet loss probabilities are influenced by many factors, including ambient noise, distance between wireless nodes, the presence of other wireless communication nodes on the same network, and sources of interference on the same frequency band [7]. Thus, network delay and packet loss probabilities may vary with time due to, e.g., solar activity, mobile network nodes, evolving network topology, or adversarial disruption (jamming). In some of these scenarios, the time-varying Markov chain transition probabilities may be known in advance, while in other scenarios, the time-variation may be a priori unknown.

Time-homogeneous Markov jump linear systems have been studied quite extensively in the literature. Ji and Chizeck [8] study various second moment stability concepts and the almost sure asymptotic stability for the time-homogeneous case. Costa and Fragoso [9] provide coupled linear matrix inequality conditions equivalent to mean square stability, and Ji and Chizeck [10] characterize the jump linear quadratic Gaussian optimal control problem. More recently, Seiler and Sengupta [11] consider the $H_{\infty}$ control problem and provide a stochastic bounded real lemma that can be used when the Markov chain is time-homogeneous. Lee and Dullerud provide necessary and sufficient conditions for a time-homogeneous Markov jump linear system to be almost surely uniformly exponentially stable [12] and almost surely uniformly strictly contractive [13].

Time-inhomogeneous Markov jump linear systems have received less attention. For the case when the time-varying Markov chain transition probabilities are known, Krtolica et al. [14] provide necessary and sufficient condition for mean square stability in the form of an infinite set of coupled matrix equations, and Aberkane [15] states a similar stability result in terms of an infinite set of LMIs. Fang and Loparo [16] reduce the infinite set of matrix equations in [14] to a finite set when the transition probabilities of the Markov chain are periodic. Aberkane [15] provides a necessary and sufficient condition for stochastic disturbance attenuation in the form of an infinite set of LMIs, which reduces to a finite set when the transition probabilities of the Markov chain are periodic.

For the case when the Markov chain transition probabilities are time-varying and a priori unknown, only sufficient conditions for uniform stochastic stability and uniform stochastic disturbance attenuation have been provided in the literature. Bolzern et al. [17] examine a continuous-time Markov jump linear system with time-varying a priori unknown Markov process transition rates (see, e.g., [18, Sec. 11.4.2]) and provide a sufficient condition for uniform stochastic stability subject to a dwell-time constraint. Lutz and Stilwell [3] examine a particular class of time-inhomogeneous Markov jump

...
linear systems with a priori unknown transition probabilities, provide sufficient conditions for uniform mean square stability and uniform stochastic disturbance attenuation, and present a sufficient condition for uniform stochastic stability subject to an average dwell-time constraint.

Notation
The positive and nonnegative integers are represented by \( \mathbb{N} \) and \( \mathbb{N}_0 \), respectively. The standard Euclidean vector norm and corresponding induced matrix norm are both denoted by \( ||.|| \).

Definition 2: The Markov jump linear system \((G, P, p(0))\) is exponentially mean square stable if there exist \( c \geq 1 \) and \( 0 \leq \lambda < 1 \) such that \( \mathbb{E} [\Phi^T(k, j) \Phi(k, j)] \theta(j) = I \) \( \leq \lambda^{k-j} I \) for all \( i \in \mathcal{N} \) and for all \( k, j \in \mathbb{N}_0 \) such that \( k \geq j \geq 0 \).

At least two notions of disturbance attenuation for Markov jump linear systems have been examined in the literature \([11, 13]\). We find that mean square attenuation best suits our approach.

Definition 3: The Markov jump linear system \((G, P, p(0))\) is mean square strictly contractive if there exists \( \gamma \in (0, 1) \) such that whenever \( x(0) = 0 \), \( ||z||_2 \leq \gamma ||w||_2 \) for all \( w \in \ell^2 \).

The following matrix-valued function will appear extensively in characterizations of disturbance attenuation.

Definition 4: Let \( G \) be given. For \( i \in \mathcal{N} \) and \( X, Y \in \mathbb{S}_n \), define

\[
B(i, X, Y) = \begin{bmatrix} A(i) & B(i) \\ C(i) & D(i) \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(i) & B(i) \\ C(i) & D(i) \end{bmatrix} - \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix}
\]

Exponential mean square stability of the time-inhomogeneous Markov jump linear system \((G, P, p(0))\) with known transition probabilities may be characterized by a stochastic Lyapunov criterion.

Proposition 5 (Thm. 2 of \([14]\), Prop. 1 of \([15]\)): The time-inhomogeneous Markov jump linear system \((G, P, p(0))\) is exponentially mean square stable if and only if there exist \( \eta, \rho > 0 \) and a function \( X : \mathcal{N} \times \mathbb{N}_0 \rightarrow \mathbb{S}_n^+ \) such that \( \eta I \leq X(i, k) \leq \rho I \) and

\[
\lambda^T \left( \begin{array}{c} x(i, k+1) - X(i, k) \end{array} \right) \leq -\nu \|y\|^2
\]

for all \( i \in \mathcal{N}, k \in \mathbb{N}_0 \), and \( y \in \mathbb{R}^m \). Of course, the utility of Proposition 5 for assessing stability of a given system is limited due to the infinite number of matrices being prohibitively difficult to compute in practice.

Disturbance attenuation for a time-inhomogeneous Markov jump linear system can also be characterized by an infinite set of LMIs.

Proposition 6 (Thm. 1 of \([15]\)): Assume \((G, P, p(0))\) is weakly controllable and \( p(k) > 0 \) for all \( k \in \mathcal{N} \) and \( k \in \mathbb{N}_0 \). The time-inhomogeneous Markov jump linear system \((G, P, p(0))\) is exponentially mean square stable and mean square strictly contractive if and only if there exist \( \eta, \rho, \nu > 0 \) and a function \( X : \mathcal{N} \times \mathbb{N}_0 \rightarrow \mathbb{S}_n^+ \) such that \( \eta I \leq X(i, k) \leq \rho I \) and \( B(i, X(i, k+1)) - X(i, k) \leq -\nu \|y\|^2 \) for all \( i \in \mathcal{N}, k \in \mathbb{N}_0 \), and \( y \in \mathbb{R}^m \). Of course, the utility of Proposition 6 for assessing disturbance attenuation properties of a given system is limited due to the infinite number of matrices being prohibitively difficult to compute in practice.

III. SWITCHED MARKOV JUMP LINEAR SYSTEM

In this section, we examine a time-inhomogeneous Markov jump linear system with a priori unknown time-varying transition probabilities. We assume that the sequence \( P \) of transition probability matrices is not known in advance but takes values in some finite set of matrices \( \{\Pi(1), \ldots, \Pi(J)\} \) where \( \Pi(s) \in \mathcal{T}_N \) for \( s \in \mathcal{J} = \{1, \ldots, J\} \).
Given a priori unknown sequence $\psi : \mathbb{N} \to \mathcal{J}$. The notation $\pi_{ij}(\psi(k))$ denotes the $ij$-th element of matrix $\Pi(\psi(k))$.

A switched Markov jump linear system, denoted $(\mathcal{G}, \Pi, \Psi, p(0))$, is defined to be the collection of Markov jump linear systems $\{ (\mathcal{G}, \Pi \circ \psi, p(0)) : \psi \in \Psi \}$ where $\Psi$ is the application-specific set of all sequences that may occur. Depending on the application, $\Psi$ could be the set of all sequences in $\mathcal{J}$. Alternatively, some applications may disallow certain sequences from occurring due to problem-specific information available. Each member of the switched Markov jump linear system is driven by a different time-inhomogeneous Markov chain with transition probabilities given by $\Pi(\psi(k))$, $k \in \mathbb{N}$ for some sequence $\psi \in \Psi$. The switched modifier here is used to draw analogy to deterministic switched systems (see, e.g., [23]), and we often refer to $\psi$ as a switching sequence. For $M \in \mathbb{N}$ and $k \in \mathbb{N}_0$, we define $\psi_M(k) = (\psi(k+1), \psi(k+2), \ldots, \psi(k+M))$. Additionally, the set of all sequences of length $M$ that occur in $\Psi$ is denoted $\Psi_M = \{ \psi_M(t) : \psi \in \Psi, t \in \mathbb{N}_0 \}$ and is a subset of $\mathcal{J}^M$.

### A. Stability

Since we now address time-inhomogeneous Markov jump linear systems where the sequence of transition probability matrices is not known a priori, we modify the definition of stability so that it applies uniformly over all possible sequences of transition probability matrices.

**Definition 7:** The switched Markov jump linear system $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable if there exist $c \geq 1$ and $0 < \lambda < 1$ such that $E[\Phi_i(k, j)\Phi(k, j) | \theta(k) = i] \leq c\lambda^{k-j}I$ for all $i \in \mathcal{N}$, all $j$, $k \in \mathbb{N}_0$ such that $k \geq j \geq 0$, and all $\psi \in \Psi$.

Uniformity in Definition 7 refers to the uniform decay rate for all $\psi \in \Psi$. Thus, uniform exponential mean square stability ensures that each individual Markov jump linear system in the family $(\mathcal{G}, \Pi, \Psi, p(0))$ is exponentially mean square stable, and all members share a common uniform decay rate.

The goal in this section is to establish a necessary and sufficient condition for uniform stability that is more tractable than an infinite set of LMs. It is well-known that any stable discrete-time linear time-varying system admits a time-varying quadratic Lyapunov function; it is less well-known that the usual construction (e.g., see [20] Thm. 23.3) can be modified so that at each time instant, the Lyapunov function depends on only a finite number of the past system parameter matrices [12, Lem. 4]. Inspired by this fact, the following lemma constructs a stochastic Lyapunov function for a stable time-inhomogeneous Markov jump linear system that depends on only a finite number of the future transition probability matrices.

**Lemma 8:** Suppose system $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable with stability constant $c$ and decay rate $\lambda$ in Definition 7. Let $M = \max\{ \log(1/\epsilon^2) - 2, 0 \}$ so that $c\lambda^{M+2} < 1$. Then for each $\psi \in \Psi$, $V_\psi(i, k) := \sum_{j=k}^{M+k-1} E[\Phi_i(j, k)\Phi(j, k) | \theta(k) = i]$ satisfies

$$
\begin{align*}
\eta I & \leq V_\psi(i, k) \leq \rho I \quad \text{(2a)} \\
A^T(i)\tilde{Y}_\psi(i, k+1)A(i) - V_\psi(i, k) & \leq -\nu I \quad \text{(2b)}
\end{align*}
$$

for all $i \in \mathcal{N}$ and all $k \in \mathbb{N}_0$ where $\tilde{Y}_\psi(i, k+1) = \sum_{j=1}^{N} \pi_{ij}(\psi(k+1))\tilde{Y}_\psi(j, k+1)$, $\eta = 1$, $\rho = c/(1 - \lambda)$, and $\nu = 1 - c\lambda^{M+2}$.

**Proof:** The inequalities in (2a) follow readily from Definition 7 and the definition of $V_\psi$. For convenience, define $\Gamma(j, k) = \Phi_i(j, k)\Phi(j, k)$. Then

$$
A^T(i)\tilde{Y}_\psi(i, k+1)A(i) = \sum_{j=k+1}^{k+M+2} E\left[A^T(\theta(k))\Gamma(j, k+1)A(\theta(k)) | \theta(k) = i \right] = Y_\psi(i, k) - I + E\left[\Gamma(k+M+2, k) | \theta(k) = i \right],
$$

where (3) follows by plugging in the definition of $\tilde{Y}_\psi(i, k+1)$. The order of summation, and recognizing an iterated expectation. Definition 7 and equation (4) yield (2b) with $\nu > 0$ by the hypothesis on $M$.

**Remark 9:** Note that $\sum_{j=k}^{k+M+2} \Phi_i(j, k)\Phi(j, k)$ from Lemma 8 is a function of the random variables $\theta(k), \ldots, \theta(k + M)$. The joint probability distribution $P(\theta(k+1) = i_1, \ldots, \theta(k + M) = i_M | \theta(k) = i_0) = \frac{1}{M+1} \sum_{i=1}^{M+1} \pi_{i_0-i}\psi(k+l)$ is required to compute the expectation in the definition of $Y_\psi(i, k)$ in Lemma 8. Since the joint probability distribution is determined by the conditional value of $\theta(k)$ and the value of $\psi_M(k), Y_\psi(i, k)$ may be computed with knowledge of only $i$ and $\psi_M(k)$. Since $\mathcal{N}$ and $\mathcal{J}$ are finite sets, $\cup_{i \in \mathcal{N}} \{ Y_\psi(i, k) \}$ is a finite set of matrices with no more than $NJ^M$ elements. Fix $\psi \in \Psi$ arbitrarily. For $i \in \mathcal{N}$, $k \in \mathbb{N}_0$, and $y \in \mathbb{R}^n$, define $V_\psi(i, k, y) := y^T Y_\psi(i, k) y$. By (2), $V_\psi(i, k, y)$ is a quartic stochastic Lyapunov function for system $(\mathcal{G}, \Pi \circ \psi, p(0))$. Thus, uniform stability of the family $(\mathcal{G}, \Pi, \Psi, p(0))$ guarantees the existence of a finite set of matrices that may be used to construct a time-varying quadratic stochastic Lyapunov function for any member of the family.

The next theorem, inspired by [12] Thm. 9, provides a necessary and sufficient condition, expressed as a set of finite-dimensional LMIs, for uniform exponential mean square stability of a switched Markov jump linear system.

**Theorem 10:** The switched Markov jump linear system $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable if and only if there exist $M \in \mathbb{N}_0$ and a function $X : \mathcal{N} \times \Psi_M \to \mathbb{S}_+^n$ such that

$$
A^T(i) \sum_{j=1}^{N} \pi_{ij} X(j, r_2, \ldots, r_{M+1}) A(i) - X(i, r_1, \ldots, r_M) < 0
$$

for any $(r_1, \ldots, r_{M+1}) \in \Psi_{M+1}$ and $i \in \mathcal{N}$.

**Proof:** Suppose there exist $M$ and $X$ such that (5) holds. Let $\psi \in \Psi$ be arbitrary. Define $V_\psi(i, k) := X(i, \psi_M(k))$. Since $\mathcal{N} \times \Psi_{M+1} \subset \mathcal{N} \times \mathcal{J}^M + 1$ is a finite set, inequality (5) holds uniformly, and we can find $\eta, \rho, \nu > 0$ such that $\eta I \leq V_\psi(i, k) \leq \rho I$ and $A^T(i)Y_\psi(i, k+1)A(i) - Y_\psi(i, k) \leq -\nu I$ for all $i \in \mathcal{N}$, $k \in \mathbb{N}_0$, and $\psi \in \Psi$. Thus, $y^T Y_\psi(i, k) y$ is a stochastic Lyapunov function for the single system $(\mathcal{G}, \Pi \circ \psi, p(0))$ and guarantees exponential mean square stability by Proposition 3 with $\epsilon = \rho/\eta$ and $\lambda = 1 - \nu/\rho$. Since $\psi$ was arbitrary and the same $c$ and $\lambda$ work for any $\psi \in \Psi$, $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable.

Conversely, assume that $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable with stability constant $c$ and decay rate $\lambda$. Fix $M \in \mathbb{N}_0$ such that $c\lambda^{M+2} < 1$. Let $(i, r_1, \ldots, r_{M+1}) \in \mathcal{N} \times \Psi_{M+1}$ be arbitrary. By definition of $\Psi_{M+1}$, there exist $\psi \in \Psi$ and $t \in \mathbb{N}_0$ such that $\psi_M(t) = (r_1, \ldots, r_{M+1})$. Construct $Y_\psi$ as in Lemma 8 and recall from Remark 9 that $Y_\psi(i, t)$ depends only on $(i, \psi_M(t))$. Thus, define $X(i, r_1, \ldots, r_M) := Y_\psi(i, t)$ and define $X(i, r_2, \ldots, r_{M+1}) := Y_\psi(i, t+1)$. One recovers every inequality in (5) from (2).

**Remark 11:** For any $M \in \mathbb{N}_0$, $\text{Im } X$ in Theorem 10 is finite. Thus, for each $M \in \mathbb{N}_0$, the number of LMIs specified in (5) is finite. The stability of a switched Markov jump linear system may be investigated using an iterative algorithm. First, set $M = 0$ and check...
if the LMIs in (5) are feasible. If not, increment $M$ and repeat. If the switched Markov jump linear system is stable, Theorem 10 says that this algorithm will stop in a finite amount of time with some finite value of $M$. A conservative estimate for $M$ is based on the uniform decay rate of the switched Markov jump linear system (see Lemma 8).

Remark 12: Theorem 10 provides a practical approach for investigating the stability of a single time-inhomogeneous Markov jump linear system with known transition probability matrices that vary in a finite set (let $\mathcal{F}$ be the set containing a single sequence).

Remark 13: Consider the case when $J = 1$ and $\Psi = \{1, 1, \ldots\}$. The switched Markov jump linear system $G(\Pi, \Psi, p(0))$ reduces to a single time-homogeneous Markov jump linear system $G(\Pi \circ \psi_1, p(0))$ where $\psi_1 \equiv 1$. For any $M$, the set $\Psi_M$ contains only a single element $(1, \ldots, 1)$, and the set $\mathcal{N} \times \Psi_M$ contains only $N$ elements. For $i \in \mathcal{N}$, define $Z(i) := X(i, 1, \ldots, 1)$ where $X$ is as in Theorem 10. Then (5) reduces to $A_i^T(\bar{\pi}_i)^T Z(j) = A_i - Z(i) < 0$, which is the well-known stability criterion (see Thm. 2.1) for time-homogeneous Markov jump linear systems; this well-known result is a corollary of Theorem 10.

B. Disturbance Attenuation

We now address disturbance attenuation for a time-inhomogeneous Markov jump linear system where the sequence of transition probability matrices is not known a priori. Accordingly, we modify the definition of disturbance attenuation so that it applies uniformly over all possible sequences of transition probability matrices.

Definition 14: The switched Markov jump linear system $G(\Pi, \Psi, p(0))$ is uniformly mean square strictly contractive if there exists $\gamma \in (0, 1)$ such that whenever $x(0) = 0$, $||x||_{2, e} \leq \gamma ||u||_{2, e}$ for all $w \in \mathcal{F}$ and all $\psi \in \Psi$.

The goal of this section is to establish a KYP-like result for switched Markov jump linear systems in terms of finite-dimensional LMIs similar to Theorem 10. The main result can be found in Theorem 25. The necessity of the LMIs is the difficult part of the proof and hinges on the existence of the matrix-valued functions in Lemma 15. Like Lemma 8, at any time instant each matrix-valued function depends only on a finite number of the future transition probability matrices.

Lemma 15: Assume $p_k(\psi) > 0$ for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in \mathbb{N}_0$. If $G(\Pi, \Psi, p(0))$ is uniformly exponentially mean square stable and uniformly mean square strictly contractive then there exist $\eta, \rho, \nu > 0$ and $M \in \mathbb{N}_0$ such that for each $\psi \in \Psi$, there exists $Y_\psi : \mathcal{N} \times \mathbb{N}_0 \to \mathbb{S}^+_\nu$, such that $Y_\psi(i, k) \in \mathbb{S}^+_{\nu}$ only on $i$ and $\psi_M(k)$ and satisfies

$$\eta I \leq Y_\psi(i, k) \leq \rho I \quad (6a)$$

$$B(i, \tilde{Y}_\psi(i, k + 1), Y_\psi(i, k)) \leq -\nu I \quad (6b)$$

for all $i \in \mathcal{N}$ and all $k \in \mathbb{N}_0$ where $\tilde{Y}_\psi(i, k + 1) = \sum_{j = 1}^N \pi_{ij}(\psi(k + 1)) Y_\psi(j, k + 1)$.

The construction of the functions $Y_\psi, \psi \in \Psi$ requires the intermediate results contained in this section up to and including Lemma 25 and the proof of Lemma 15 follows directly from Lemma 25. For the moment, suppose that functions $Y_\psi, \psi \in \Psi$ have been found that satisfy Lemma 15 and define $V_\psi(i, k, y) := y^T Y_\psi(i, k) y$ for $i \in \mathcal{N}, k \in \mathbb{N}_0$, and $y \in \mathbb{R}^n$. Then $V_\psi$ is a quadratic stochastic storage function for system $G(\Pi, \Psi, p(0))$ that at each time instant depends only on $i$ and $\psi_M(k)$. Since $\mathcal{N} \times \mathcal{F}$ are finite sets, $\cup_{\psi \in \Psi} \Im Y_\psi$ is a finite set of matrices with no more than $NJ_M$ elements. Thus, uniform stability and contractiveness of the switched Markov jump linear system $G(\Pi, \Psi, p(0))$ guarantees the existence of a finite set of matrices that may be used to construct a time-varying quadratic stochastic storage function for any individual Markov jump linear system in the family.

Riccati difference equations defined in terms of the following operators play a key role in the construction of the functions $Y_\psi, \psi \in \Psi$ in Lemma 15.

Definition 16: Let $\mathcal{G}$ be given. For $i \in \mathcal{N}$ and $X \in \mathcal{S}_\nu$, define

$$L(i, X) = A_i^T(i) X A_i(i) + C_i^T(i) C(i)$$

$$R(i, X) = B_i^T(i) X A_i(i) + D_i^T(i) C(i)$$

$$W(i, X) = I - B_i^T(i) X B_i(i) - D_i^T(i) D(i)$$

$$\mathcal{M}(i, X) = \begin{bmatrix} L(i, X) & R(i, X) \\ R(i, X) & -W(i, X) \end{bmatrix}$$

For $i \in \mathcal{N}$ let $\mathcal{X}_i = \{X \in \mathcal{S}_\nu : W(i, X) \text{ invertible}\}$. For $i \in \mathcal{N}$ and $X \in \mathcal{X}_i$, define

$$S(i, X) = L(i, X) + R(i, X) W^{-1}(i, X) R(i, X).$$

Given a modified set of matrices $\{A(i), B(i), C(i), D(i) : i \in \mathcal{N}\}$, let $L(i, X)$, $R(i, X)$, $W(i, X)$, and $S(i, X)$ be defined as above but with $C(i)$ in place of $C(i)$ and $D(i)$ in place of $D(i)$.

Note that inequality (5) may be rewritten in the terms of the operators in Definition 16. Expanding the left side of (6b) gives

$$B(i, \tilde{Y}_\psi(i, k + 1), Y_\psi(i, k))$$

$$= \mathcal{M}(i, \tilde{Y}_\psi(i, k + 1)) - \begin{bmatrix} Y_\psi(i, k) & 0 \\ 0 & 0 \end{bmatrix}.$$

(7)

By the Schur complement, (7) is negative definite if and only if $W(i, \tilde{Y}_\psi(i, k + 1)) > 0$ and $Y_\psi(i, k) > S(i, \tilde{Y}_\psi(i, k + 1))$. Using these inequalities as a guide, we shall examine finite-horizon Riccati difference equations defined by the recursive relation and final condition

$$X_\psi(i, k, T) = S(i, X_\psi(i, k, T)) \quad (8a)$$

$$X_\psi(i, T + 1, T) = 0 \quad (8b)$$

where $i \in \mathcal{N}, T \in \mathbb{N}_0$ (the horizon), $0 \leq k \leq T, \psi \in \Psi$, and $X_\psi(i, k, T) = \sum_{j = 1}^N \pi_{ij}(\psi(k + 1)) X_{\psi(j, k + 1, T)}$. For a fixed $\psi \in \Psi$ and $T \in \mathbb{N}_0$, the solution $X_\psi(\cdot, \cdot, T)$ to (5) may be computed iteratively backwards-in-time starting with the final condition. However, we first need to verify that the inverse specified in (8a) is well-defined. The algebraic identity and special input in the next lemma aid in this task.

Lemma 17: Let $k \in \mathbb{N}_0, \theta(k) \in \mathcal{N}, X \in \mathcal{S}_\nu$, and $x(k), z(k), w(k)$ be as in (1). Then

$$z^T(k) z(k) - w^T(k) w(k) + x^T(k + 1) x(k + 1)$$

$$= \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^T \mathcal{M}(\theta(k), X) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}. \quad (9)$$

If $X \in \mathcal{X}_{\theta(k)}$ and $w(k) = W^{-1}(\theta(k), X) R(\theta(k), X) z(k)$ then

$$z^T(k) z(k) - w^T(k) w(k) + x^T(k + 1) x(k + 1)$$

$$= \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^T \mathcal{M}(\theta(k), X) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} = x^T(k) S(\theta(k), X) x(k). \quad (10)$$

Proof: The proof of Lemma 17 follows from simple matrix algebra.

The following lemma establishes that the Riccati recursive relation in (8a) is well-defined when $G(\Pi, \Psi, p(0))$ is uniformly mean square strictly contractive.

Lemma 18: Assume $p_k(\psi) > 0$ for all $\psi \in \Psi, i \in \mathcal{N},$ and $k \in \mathbb{N}_0$. If $G(\Pi, \Psi, p(0))$ is uniformly mean square strictly contractive then there exists $\nu > 0$ such that

$$W(i, \tilde{X}_\psi(i, k + 1, T)) \geq \nu I$$
for all $\psi \in \Psi$, $i \in \mathcal{N}$, $T \in N_0$, and $0 \leq k < T$ where $X_{\psi}$ is defined by the recursive relation and final condition in (8).

**Proof:** Let $x(0) = 0$. Arbitrarily fix $\psi \in \Psi$ and $T \in N_0$. Let $i \in \mathcal{N}$ and consider $w$ such that $w(T) = y(\psi(T) + 1)y$, and $w(k) = 0$ for $k \neq T$, where $y$ is an arbitrary vector, and $\chi(\theta(T) = i)$ is the indicator function of the set $\{\theta(T) = i\} \subset \Omega$. Note that $E[\chi(\theta(T) = i)] = P(\theta(T) = i)$. With $w$ defined above, $x(k) = 0$ for $k \leq T$ and $z(k) = 0$ for $k \leq T - 1$. Definition (14) gives

$$||z||_{2,e}^2 - ||w||_{2,e}^2 \leq -\nu ||w||_{2,e}^2$$

(11)

for any $\psi \in \Psi$ and $w \in L_2$ where $\nu = 1 - \gamma^2$. Then

$$-\nu ||w||_{2,e}^2 = -\nu P\{\theta(T) = i\} y^T y$$

(12)

$$\geq \sum_{k=0}^{\nu} E[|z(k)|^2 - ||w(k)||^2]$$

(13)

$$= E[z(T)z(T) - w(T)w(T)]$$

(14)

$$= E[z(T)z(T) - w(T)w(T)]$$

(15)

where (12) follows from the definition of $w$; (13) follows from (11); (14) follows since $X_{\psi}(i, T + 1, T)$ is 0; and (15) follows from (11) and $x(T) = 0$. Thus, $W(i, X_{\psi}(i, T + 1, T)) \geq \nu I$ since $y$ was an arbitrary vector.

Now fix $0 \leq t \leq T$ and assume $W(i, X_{\psi}(i, k + 1, T)) \geq \nu I$ for $t \leq k \leq T$ and $i \in \mathcal{N}$. Consider $w$ of the form

$$w(k) = \begin{cases} 0 & k \leq t - 2 \\ \chi(\theta(k) = i)y & k = t - 1 \\ W^{-1}(\theta(k), \tilde{X}_{\psi}(\theta(k), k + 1, T)) & : k \geq T + 1 \end{cases}$$

(16)

Then $x(k) = 0$ for $k \leq t - 1$ and $z(k) = 0$ for $k \leq t - 2$. Define $V(k) = x^T(k)X_{\psi}(\theta(k), k + 1, T)x(k)$ and $\tilde{V}(k) = x^T(k)\tilde{X}_{\psi}(\theta(k) - 1, k, T)x(k)$. Then

$$\sum_{k=0}^{T} E[|z(k)|^2 - ||w(k)||^2]$$

(17)

$$= \sum_{k=t-1}^{T} E[|z(k)|^2 - ||w(k)||^2] + \tilde{V}(k) - V(k)$$

(18)

$$= E[-w^T(t-1)W(\theta(t-1), \tilde{X}_{\psi}(\theta(t-1), t, T))w(t-1)]$$

(19)

$$= -P(\theta(t-1) = i) y^T W(i, \tilde{X}_{\psi}(i, t, T)) y$$

(20)

$$= -\nu E[y^T w^T(t-1)w(t-1)]$$

(21)

where (18) follows after recognizing a telescoping sum, realizing $V(t-1) = V(T+1) = 0$, and applying an iterated expectation; (19) follows from Lemma (17) $x(t-1)$ is the recursive relation (8a) and definition of $V(k)$; and (20) follows from (11). Since $y$ was an arbitrary vector, $W(i, \tilde{X}_{\psi}(i, t, T)) \geq \nu I$. The result follows by induction.

**Remark 19:** The input specified in (16) is similar to disturbance inputs constructed in (11) and (15). The techniques used in Lemma (18) show that if

$$w(k) = \begin{cases} \chi(\theta(k), \tilde{X}_{\psi}(\theta(k), k + 1, T)) & : k \leq T - 1 \\ 0 & : k \geq T + 1 \end{cases}$$

(22)

then

$$\sum_{k=0}^{T} E[z^T(k)z(k) - w^T(k)w(k)] = E[x^T(0)X_{\psi}(\theta(0), 0, T)x(0)]$$

(23)

maximizes the quantity in (7) (see (24) Lemma 2.1).

A hypothesis in the statement of Lemma (18) can be expressed as a requirement on the possible sequences $\Pi(\psi(k))$ of stochastic matrices and the initial distribution $p(0)$.

**Proposition 20:** For all $\psi \in \Psi$, any $i \in \mathcal{N}$, and all $k \in N_0$, $P_i(0) > 0$ if and only if for all $\psi \in \Psi$ and all $\kappa \in \kappa$, each column of $\Pi(\psi(k))$ is nonzero, and $P_i(0) > 0$ for all $i \in \mathcal{N}$.

**Proof:** Use induction and the identity $P_{\pi}(\theta(k) = i) = \sum_{i=1}^{N} \pi_i(\psi(k))P_{\theta(k) = i}$.

The following property is key for finding a uniform upper bound on solutions to (8).

**Lemma 21:** Fix $\psi \in \Psi$ and $t \in N_0$. Define $\psi_0$ to be a shifted version of $\psi$ so that $\psi_0(k) = \psi(t + k)$ for $k \in \kappa$, and define $P_i(0) = P(t)$. If $(G, \Pi(\psi_0), p(0))$ is exponentially mean square stable and mean square strictly contractive, then $(G, \Pi \circ \psi_0, p(0))$ is exponentially mean square stable and mean square strictly contractive. Furthermore,

$$X_{\psi}(i, t, T) = X_{\psi_0}(i, 0, T - t)$$

(24)

for $i \in \mathcal{N}$ and $0 \leq t \leq T$ where $X_{\psi}$ and $X_{\psi_0}$ are defined by (5).

**Proof:** Consider the Markov jump linear system modulated by a shifted random process

$$[x_i(t + 1), x_j(t + 1)] = \begin{bmatrix} A(\theta(i)) & B(\theta(i)) \\ C(\theta(i)) & D(\theta(i)) \end{bmatrix} [x_i(t), x_j(t)]$$

(25)

where $\theta(i) = \theta(t + k)$ for $k \in N_0$ and $w_i \in L_2$. Note that this system may be denoted by $(G, \Pi \circ \psi_0, p(0))$. Now $(G, \Pi \circ \psi_0, p(0))$ is exponentially mean square stable since $E[\Phi_t^i(k, j, \Phi_t(k, j) | \phi_t(j)) = \Phi_t^i(k, j, \Phi_t(k, j) + t | \phi_t(j))] \leq e^{\gamma(t+k)(t+1)}$ I where $\Phi_t$ is the random state transition matrix for the system in (23).

Now let $x_0(0) = x(0) = 0$, and let $w_i \in L_2$ be arbitrary. Define $w$ such that $w(k) = 0$ when $k < t$, and $w(k) = w_i(k - t)$ when $k \geq t$. Note that $w_i \in L_2$ implies $w \in L_2$, and $\|w\|_{2,e} = \|w_i\|_{2,e}$. Furthermore, $z_0 = 0$ for $k \leq t - 1$, and $x(k) = 0$ for $0 \leq k \leq t$. It is easily shown that $z_0(k) = z(t + k)$ for $k \in N_0$ and $\|z_0\|_{2,e} = \|z\|_{2,e}$. Since $(G, \Pi \circ \psi, p(0))$ is mean square strictly contractive, $\|z_0\|_{2,e} = \|z\|_{2,e} \leq \gamma \|w_i\|_{2,e}$. Since $w_i \in L_2$ was arbitrary, $(G, \Pi \circ \psi_0, p(0))$ is mean square strictly contractive.

Now to prove (24), note that the base case $X_{\psi}(i, T, T) = X_{\psi_0}(i, T - t, T - t)$ holds for all $i \in \mathcal{N}$. For the inductive hypothesis, assume for some $0 \leq k < T - t - 1$ that

$$X_{\psi}(i, T - k, T) = X_{\psi_0}(i, T - t - k, T - t)$$

(26)

for all $i \in \mathcal{N}$. Then $X_{\psi}(i, T - k, T)$ admits the same recursive relation as $X_{\psi_0}(i, T - t - k, T - t)$, and

$$S \left( \sum_{j=1}^{N} \pi_j(\psi(T - k))X_{\psi}(j, T - k, T) \right)$$

(27)

$$= S \left( \sum_{j=1}^{N} \pi_j(\psi(T - t - k))X_{\psi_0}(j, T - t - k, T - t) \right)$$

(28)

where (28) follows from the inductive hypothesis and the fact that $\psi(T - k) = \psi(T - t - k)$. Equation (22) follows by induction.

A uniform upper bound on solutions to (8) is established in the following lemma using Lemma 21 and a technique similar to [25 Sec. B.2.3].

**Lemma 22:** Assume $p_i(k) > 0$ for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in N_0$. If $(G, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable and uniformly mean square strictly contractive then there exists $\rho > 0$ such that

$$0 \leq X_{\psi}(i, k, T) \leq \rho I$$

(29)
for all $i \in \mathcal{N}$, any $T \in \mathbb{N}_0$, all $\psi \in \Psi$, and all $0 \leq k \leq T + 1$ where $X_\psi$ is defined in (35).

**Proof:** Arbitrarily fix $\psi \in \Psi$ and $T \in \mathbb{N}_0$. Define $w$ as in (21).

Then

$$E \left[ x^T(0)X_\psi(\theta(0), 0, T)x(0) \right]$$

$$= \sum_{k=0}^{T} E \left[ x^T(k)z(k) - w^T(k)w(k) \right]$$

$$\leq \|z\|_{2,e}^2 - \|w\|_{2,e}^2$$

where (26) follows from Remark 19. By linearity, $z$ is the zero-input response and $z_{\psi}$ is the zero-state response (e.g., see [20, Ch. 20]). By the Cauchy-Schwarz inequality

$$\|z\|_{2,e}^2 \leq \|z_{\psi}\|_{2,e}^2 + \|z_{\psi}\|_{2,e}^2 + 2 \|z_{\psi}\|_{2,e} \|z_{\psi}\|_{2,e}$$

Since $(\mathcal{G}, \Pi, p(0))$ is uniformly exponentially mean square stable,

$$\|z_{\psi}\|_{2,e}^2 \leq \frac{1}{2}$$

for all $\psi \in \Psi$ where $\delta = \max_{i \in \mathcal{N}} (\lambda_{\max}(C(i)^T C(i))) c/(1 - \lambda)$ and $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue. By Definition 14

$$\|z_{\psi}\|_{2,e}^2 \leq \|z_{\psi}\|_{2,e}^2$$

for all $\psi \in \Psi$ and $w \in \mathcal{C}$ where $\nu = 1 - \gamma^2$. Then

$$\|z\|_{2,e}^2 - \|w\|_{2,e}^2 \leq -\nu \|w\|_{2,e}^2 + 2\sqrt{\nu} E[|x(0)|^2] \|w\|_{2,e}^2$$

$$\leq (\delta + \delta/\nu) E[|x(0)|^2]$$

$$\leq \rho E[|x(0)|^2]$$

for all $\psi \in \Psi$ and all $w \in \mathcal{C}$ where $\rho = \delta + \delta/\nu$; (32) follows from (28), (29), (30), and (31); and, (33) follows by completing the square. Choose any $i \in \mathcal{N}$ and let $x(0) = \chi(\theta(0) = i)y$ where $y$ is an arbitrary vector. Then (27) and (34) imply

$$P(\theta(0) = i) = \chi(\theta(0) = i)^T X_\psi(i, 0, T) y \leq \rho P(\theta(0) = i) \|y\|$$

Since $y$ was arbitrary, the upper bound in (35) holds for $k = 0$. The general case follows from (21).

That $0 \leq X_\psi(i, k, T)$ for all $i \in \mathcal{N}$, $T \in \mathbb{N}_0$, $\psi \in \Psi$, and $0 \leq k \leq T$ can be seen clearly from (21).

We now examine perturbed finite-horizon Riccati difference equations defined by the recursive relation and final condition

$$X_\psi(i, k, T, \epsilon) = S(i, X_\psi(i, k + 1, T, \epsilon)) + \epsilon I$$

$$X_\psi(i, T + 1, T, \epsilon) = 0$$

where $i \in \mathcal{N}$, $T \in \mathbb{N}_0$, $0 \leq k \leq T$, $\psi \in \Psi$, $\epsilon \geq 0$, and $X_\psi(i, k, 1, T, \epsilon) = \sum_{j=1}^{T} \pi_j \psi(k + 1)X_\psi(j, k + 1, T, \epsilon)$ for fixed $\psi \in \Psi$, $T \in \mathbb{N}_0$, and $\epsilon$. The solution $X_\psi(i, \cdot, T, \epsilon)$ to (35) may be computed iteratively backwards-in-time starting with the final condition. An augmented and perturbed system utilized in the following theorem shows that solutions to (35) are uniformly positive definite as well as uniformly bounded.

**Theorem 23:** Assume $p_0(k) > 0$ for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in \mathbb{N}_0$. If $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable and uniformly mean square strictly contractive then there exist $\eta, \rho, \nu > 0$ such that for all $\epsilon \in [0, \eta]$, $\epsilon I \leq W(i, X_\psi(i, k + 1, T, \epsilon))$ and

$$\epsilon I \leq X_\psi(i, k, T, \epsilon) \leq \rho I$$

for all $i \in \mathcal{N}$, $T \in \mathbb{N}_0$, $0 \leq k \leq T$, and $\psi \in \Psi$ where $X_\psi$ is defined by the recursive relation and final condition in (35).

**Proof:** Consider the augmented switched Markov jump linear system $(\mathcal{G}_i, \Pi, p(0))$ where $\mathcal{G}_i = \{A(i), B(i), C(i), D(i) : i \in \mathcal{N}\}$ where $C(i) = [C(i)^T (i) \sqrt{T}]$, and $D(i) = [D(i)^T (i) \sqrt{T}]$. First note $(\mathcal{G}, \Pi, p(0))$ is uniformly exponentially mean square stable for any $\epsilon$ since $\mathcal{G}_i$ share the same matrices $A(i), i \in \mathcal{N}$. Since $(\mathcal{G}, \Pi, p(0))$ is uniformly mean square strictly contractive, there exists $\eta > 0$ small enough so that for all $\epsilon \in [0, \eta]$ the augmented system $(\mathcal{G}, \Pi, p(\epsilon))$ is uniformly mean square strictly contractive. By Lemma 18 there exists $\nu > 0$ such that for all $\epsilon \in [0, \eta]$, $\nu I \leq W_i(i, X_\psi(i, k + 1, T, \epsilon))$ for all $i \in \mathcal{N}$, $T \in \mathbb{N}_0$, $\psi \in \Psi$, $0 \leq k \leq T$ where $X_\psi(i, T + 1, T, \epsilon) = 0$ and

$$X_\psi(i, k, T, \epsilon) = S(i, X_\psi(i, k + 1, T, \epsilon)) + \epsilon I$$

By Lemma 22 there exists $\rho > 0$ such that for all $\epsilon \in [0, \eta]$, $0 \leq X_\psi(i, k, T, \epsilon) \leq \rho I$ for all $i \in \mathcal{N}$, $T \in \mathbb{N}_0$, $\psi \in \Psi$, and $0 \leq k \leq T + 1$. That $X_\psi(i, k, T, \epsilon) \geq \epsilon I$ for $\epsilon \in \mathcal{N}$ and $0 \leq k \leq T$ follows clearly from (37).

The following lemma allows comparison of the solutions of two Riccati difference equations in (35) with different values for $\epsilon$.

**Lemma 24 (Lem. 2.6 of [13]):** For $i \in \mathcal{N}$ and $X \in \mathcal{X}_i$, define

$$F(i, X) = A(i) + B(i)W^{-1}(i, X)R(i, X).$$

Let $Y \in \mathcal{X}_i$ and $\Delta = X - Y$. Then the following algebraic identities hold.

$$S(i, X) - S(i, Y) \leq F(i, Y) \Delta F(i, Y)^T$$

$$\leq F(i, X) \Delta F(i, Y)^T$$

Before proceeding, the following technical lemma is needed which is similar in nature to [13, Thm. 2.7(a)]. The following lemma examines the random state transition matrix defined by

$$\phi(k, j, T) = F(\theta(k - 1), \tilde{X}_{\psi}(\theta(k - 1), k, T, \epsilon)) \times \cdots \times F(\theta(j), \tilde{X}_{\psi}(\theta(j), j + 1, T, \epsilon))$$

when $k$ and $j$ are such that $0 \leq j < k \leq T$, and $\phi(k, j, T) = I$ when $k = j$. Here, $F(i, X)$ is defined as in (25), and $X_\psi$ is defined in (35) for a stable and contractive system $(\mathcal{G}, \Pi, p(0))$. Note that $\phi$ is only defined for $0 \leq j \leq k \leq T$ and that dependence of $\phi$ on $\psi$ and $\epsilon$ is suppressed. The state transition matrix in (41) arises from the recurrence $x(k + 1) = F(\theta(k), \tilde{X}_{\psi}(\theta(k), k + 1, T, \epsilon))x(k)$, which is only defined for $0 \leq k \leq T$.

**Lemma 25:** Assume $p_0(k) > 0$ for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in \mathbb{N}_0$. If $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable and uniformly mean square strictly contractive then there exists $\eta > 0$ such that for any $\epsilon \in (0, \eta)$ there exist $0 \leq \lambda < 1$ and $c_0 > 0$ such that

$$E \left[ \phi^T(k, j, T) \phi(k, j, T) \right] \leq c_0 \lambda^{k-j} I$$

for all $i \in \mathcal{N}$, $T \in \mathbb{N}_0$, $\psi \in \Psi$, and $0 \leq j \leq k \leq T$ where $\phi$ is defined in (41).

**Proof:** Let $\eta > 0$ be as in Theorem 23 and fix $\epsilon \in (0, \eta)$. Fix $\epsilon_0 \in (\epsilon, \eta)$ and let $\epsilon = \epsilon_0 - \epsilon$. Define $Z_{\psi}(i, k, T, \epsilon) := X_{\psi}(i, k, T, \epsilon_0) - X_{\psi}(i, k, T, \epsilon)$ and define $Z_{\psi}(i, k, T, \epsilon) := $
\[ \sum_{j=1}^{N} \pi_{ij}(\psi(k)) Z_{\psi}(j, k, T, \epsilon) \] for each \( i \in \mathcal{N} \). Let \( T \in \mathbb{N} \) and \( \psi \in \Psi \) be arbitrary, and, for notational convenience, define \( F(i, k) := F(i, \tilde{X}_{\psi}(i, k+1, T, \epsilon)) \) where \( i \in \mathcal{N} \) and \( 0 \leq k \leq T \).

Then
\[
Z_{\psi}(i, k, T, \epsilon) = S(i, \tilde{X}_{\psi}(i, k+1, T, \epsilon_0)) + \epsilon_{0}I - S(i, \tilde{X}_{\psi}(i, k+1, T, \epsilon)) - \epsilon I \geq F^T(i, k) (\tilde{X}_{\psi}(i, k+1, T, \epsilon_0) - \tilde{X}_{\psi}(i, k+1, T, \epsilon)) \times F(i, k) + \epsilon I = F^T(i, k) \tilde{Z}_{\psi}(i, k, T, \epsilon) F(i, k) + \epsilon I
\]

where (43) follows from (35), and (44) follows from (39). Additionally, from (45) and (46)
\[
\epsilon I \leq Z_{\psi}(i, k, T, \epsilon) \leq \tilde{p}I
\]

for all \( \psi \in \Psi \), \( T \in \mathbb{N} \), \( i \in \mathcal{N} \), and all \( 0 \leq k \leq T \) where \( \tilde{p} = \rho - \epsilon \). Using stochastic Lyapunov function arguments as in the proof of (13)[Thm. 2], inequalities (45) and (46) ensure that solutions to \( x(k+1) = F(\theta(k), k)x(k) \) decay exponentially in mean square with \( \lambda_\epsilon = 1 - \epsilon / \tilde{p} \) and \( \epsilon_c = \tilde{p}/\rho \). \( \blacksquare \)

We are now ready to provide an explicit construction for \( Y_{\psi} \) in Lemma 15. The construction in the following lemma ensures that for each \( k \in \mathbb{N} \), \( Y_{\psi}(i, k) \) depends only on \( i \) and \( \psi_M(k) \).

**Lemma 26:** Assume \( p_1(k) > 0 \) for all \( \psi \in \Psi \), \( i \in \mathcal{N} \), and \( k \in \mathbb{N} \). If \( (\mathcal{G}, \Psi, p(0)) \) is uniformly exponentially mean square stable and uniformly mean square contractive then there exist \( \eta, \rho > 0 \) such that for all \( e \in (0, \eta) \), there exist \( M \in \mathbb{N} \) and \( \nu > 0 \) such that \( Y_{\psi}(i, k) := X_{\psi}(i, k, M, \epsilon) \) satisfies
\[
\epsilon I \leq Y_{\psi}(i, k) \leq \rho I \quad (47a)
\]
\[
\|Y_{\psi}(i, k+1) - Y_{\psi}(i, k)\| \leq -\nu I = -\nu M \quad (47b)
\]

for all \( \psi \in \Psi \), \( i \in \mathcal{N} \), and \( k \in \mathbb{N} \) where \( Y_{\psi}(i, k+1) = \sum_{j=1}^{N} \pi_{ij}(\psi(k+1)) Y_{\psi}(j, k+1) \) and \( X_{\psi} \) is defined in (35).

**Proof:** Let \( \eta, \rho \) be as in Theorem 23 and choose \( e \in (0, \eta) \) so that (47a) is verified automatically. Let \( \lambda_\epsilon \) and \( \epsilon_c \) be as in Lemma 25. Choose \( M \in \mathbb{N} \) such that \( \epsilon_c \lambda_\epsilon^{M+1} < \epsilon / \rho \). Then

\[
S(\tilde{X}_{\psi}(i, k+1)) - Y_{\psi}(i, k) + \epsilon I = S(i, \tilde{X}_{\psi}(i, k+1, k+M, 1, \epsilon)) - S(i, \tilde{X}_{\psi}(i, k+1, k+M, \epsilon)) = F^T(i, \tilde{X}_{\psi}(i, k+1, k+M, 1, \epsilon)) \times \tilde{X}_{\psi}(i, k+1, k+M, 1, \epsilon) - \tilde{X}_{\psi}(i, k+1, k+M, \epsilon)) \times F(i, \tilde{X}_{\psi}(i, k+1, k+M))
\]

where (48) follows from (49). But the middle term in (48) can be written
\[
\sum_{j=1}^{N} \pi_{ij}(\psi(k+1)) \left[ S(j, \tilde{X}_{\psi}(j, k+2, k+M, 1, \epsilon)) \right. - S(j, \tilde{X}_{\psi}(j, k+2, k+M, \epsilon)) \right] = \mathbb{E}\left[ F^T(\theta(k+1), \tilde{X}_{\psi}(\theta(k+1), k+2, k+M+1, \epsilon)) \times (\tilde{X}_{\psi}(\theta(k+1), k+2, k+M+1, \epsilon) - \tilde{X}_{\psi}(\theta(k+1), k+2, k+M, \epsilon)) \times F(\theta(k+1), k+2, k+M, \epsilon) \right] \times \tilde{Z}_{\psi}(\theta(k+1), k+2, k+M, \epsilon) \mid \theta(k) = i \right] \leq 0
\]

where (49) follows from (35a), and (50) results after applying (48) to (49). Proceeding in an iterative fashion,
\[
S(i, \tilde{X}_{\psi}(i, k+1)) - Y_{\psi}(i, k) + \epsilon I = \mathbb{E}\left[ \phi^T(\kappa, M+1, k, M+1, 1) \times (\tilde{X}_{\psi}(\theta(k+M), k+M+1, k+M+1, \epsilon)) \times \tilde{Z}_{\psi}(\theta(k+M), k+M+1, k+M, \epsilon) \mid \theta(k) = i \right]
\]

Note that the middle term in (51) satisfies
\[
\epsilon I \leq \tilde{X}_{\psi}(\theta(k+M), k+M+1, k+M+1, \epsilon) - \tilde{p}I \quad (52)
\]

for all values of \( \theta(k+M) \in \mathcal{N} \). Let \( y \in \mathbb{R}^n \) be arbitrary, and for convenience define \( \phi_1 = \phi(k+M+1, k, k+M+1), \phi_2 = \phi(k+M+1, k, k+M) \), and \( X = \tilde{X}_{\psi}(\theta(k+M), k+M+1, k+M+1, \epsilon) \). Then
\[
\psi^T\left( S(i, \tilde{Y}_{\psi}(i, k+1)) - Y_{\psi}(i, k) + \epsilon I \right) y = \psi^T \mathbb{E}\left[ \phi_1^* X \phi_2 \mid \theta(k) = i \right] y \leq \sqrt{\mathbb{E}\left[ \psi_1^2 X \psi_1 \mid \theta(k) = i \right] \mathbb{E}\left[ \psi_2^2 X \psi_2 \mid \theta(k) = i \right]} \leq \psi^T \mathbb{E}\left[ \psi_1^2 X \psi_1 \mid \theta(k) = i \right] \mathbb{E}\left[ \psi_2^2 X \psi_2 \mid \theta(k) = i \right] \]

where \( \psi_1 = \phi_1 y \) and \( \psi_2 = \phi_2 y \).

By (57), \( S(i, \tilde{Y}_{\psi}(i, k+1)) - Y_{\psi}(i, k) \leq -\nu I \) where \( \nu = \epsilon - \epsilon_0 \). Application of the Schur complement yields (47b). \( \blacksquare \)

**Lemma 26** uses techniques similar to those found in [13 Thm. 2.7(b)] where it is shown that the time-varying version of the KYP inequality associated with a uniformly stable and contractive linear time-varying system admits a solution with finite memory of past parameters.

**Remark 27:** The construction in Lemma 26 ensures that \( Y_{\psi}(i, k) \) may be computed with knowledge of only \( i \) and \( \psi_M(k) \). Indeed, if \( t \neq k \) but \( \psi_M(k) = \psi_M(t) \) then \( Y_{\psi}(i, k) = Y_{\psi}(i, t) \). This claim can be easily established using the recursive relation (35a) and base case \( X_{\psi}(i, k+M+1, k+M, \epsilon) = X_{\psi}(i, t+M+1, t+M, \epsilon) = 0 \).

The following theorem, inspired by [3 Thm. 3.3], provides a necessary and sufficient condition, expressed as a set of finite-dimensional LMIs, for uniform exponential mean square stability and uniform mean square strict contractiveness of a switched Markov jump linear system.

**Theorem 28:** Assume \( p_1(k) > 0 \) for all \( \psi \in \Psi \), \( i \in \mathcal{N} \), and \( k \in \mathbb{N} \). The switched Markov jump linear system \( (\mathcal{G}, \Psi, p(0)) \) is uniformly exponentially mean square stable and uniformly mean square strictly contractive if and only if there exist \( M \in \mathbb{N} \) and a function \( X : \mathcal{N} \times \Psi_M \rightarrow S_M^+ \) such that for any \( (r_1, \ldots, r_{M+1}) \) in \( \Psi_{M+1} \) and \( i \in \mathcal{N} \)

\[
\mathbb{E}\left[ \sum_{j=1}^{N} \pi_{ij}(r_1) X(j, r_2, \ldots, r_{M+1}), X(i, r_1, \ldots, r_M) \right] < 0
\]
Proof: Suppose there exist $M$ and $X$ such that (58) holds. Note that the upper left block of (58) implies (5) so uniform exponential mean square stability of $(G, \Pi, \Psi, P(0))$ follows from Theorem 10. Since $\mathcal{N} \times \Psi_{M+1} \subset \mathcal{N} \times \mathcal{J}^{M+1}$ is a finite set, inequality (58) holds uniformly, and we can find $0 < \nu < 1$ such that $B(i, \sum_{j=1}^{\mathcal{N}} \pi_{ij}(r_1)X(j, r_2, \ldots, r_{M+1}), X(i, r_1, \ldots, r_M)) \leq -\nu I$ for any $(i, r_1, \ldots, r_{M+1}) \in \mathcal{N} \times \Psi_{M+1}$. Let $\psi \in \Psi$ be arbitrary. Define $Y_0(i, k) := X(i, \pi(M)(k))$. Using (7) and (9) to rewrite $B$, it follows that

$$E\left[\|x(k)\|_2^2 + x^T(k) + Y_0(\theta(k+1) + 1)x(k+1)\right] - x^T(k)Y_\psi(\theta(k), k)x(k) \leq (1 - \nu)E\left[\|w(k)\|_2^2\right].$$

(59)

Inequality (59), positive definiteness of $Y_0(i, k)$, and $x(0) = 0$ imply $\sum_{k=0}^{\infty} E\left[\|x(k)\|_2^2\right] \leq (1 - \nu)\sum_{k=0}^{\infty} E\left[\|w(k)\|_2^2\right]$ for all $l \in \mathbb{N}_0$. Since $\psi \in \Psi$ was arbitrary, Definition 14 is satisfied with $\gamma = \sqrt{1 - \nu}$ so $(G, \Pi, \Psi, P(0))$ is uniformly mean square strictly contractive.

Conversely, assume that $(G, \Pi, \Psi, P(0))$ is uniformly exponentially mean square stable and uniformly mean square strictly contractive. Let $\eta, \nu$ be as in Lemma 26. Fix $\epsilon \in (0, \eta)$, and let $M$ and $\nu$ be defined as in Lemma 26. Let $(i, r_1, \ldots, r_{M+1}) \in \mathcal{N} \times \Psi_{M+1}$ be arbitrary. By definition of $\Psi_{M+1}$, there exist $\psi \in \Psi$ and $t \in \mathbb{N}_0$ such that $\psi_{M+1}(t) = (r_1, \ldots, r_{M+1})$. Construct $Y_0$ as in Lemma 26 and recall from Remark 27 that $Y_0(i, t)$ depends only on $(i, \psi_{M}(t))$. Thus, define $X(i, r_1, \ldots, r_{M+1}) := Y_0(i, t)$ and define $X(i, r_2, \ldots, r_{M+1}) := Y_0(i, t + 1)$. One recovers every inequality in (58) from (47).

Remark 29: Theorem 28 provides a practical approach for investigating the contractivity of a single time-inhomogeneous Markov jump linear system with known transition probability matrices that vary in a finite set (let $\Psi$ be the set containing a single sequence).

Remark 30: Consider the case when $J = 1$ and $\Psi = \{(1, \ldots, 1)\}$. The switched Markov jump linear system $(G, \Pi, \Psi, P(0))$ reduces to a single time-inhomogeneous Markov jump linear system $(G \circ \psi_1, P(0))$ where $\psi_1 \equiv 1$. For any $M$, the set $\Psi_{M} \cup \{1\}$ contains only a single element $(1, \ldots, 1)$, and the set $\mathcal{N} \times \Psi_{M}$ contains only $N$ elements. For $i \in \mathcal{N}$, define $\mathcal{Z}(i) := \{X(i, 1, \ldots, 1)\}$ where $X$ is as in Theorem 28. Then (58) reduces to $B(i, \mathcal{Z}(i)) = 0$ where $\mathcal{Z}(i) = \sum_{j=1}^{\mathcal{N}} \pi_{ij}(1)Z(j)$, which is the same inequality found in the well-known bounded real lemma for time-homogeneous Markov jump linear systems [11] Thm. 2. Theorem 28 however, does not require $(G \circ \psi_1, P(0))$ to be weakly controllable as in (11). Thus, the weak controllability hypothesis of (11) Thm. 2 can be replaced by the weaker (see Proposition 31) hypothesis that $p_1(k) > 0$ for all $i \in \mathcal{N}$ and each column of $P(1)$ is nonzero.

Proposition 31: Let $P(1)$ be a stochastic matrix and $\psi_1 \equiv 1$. If the time-homogeneous Markov jump linear system $(G \circ \psi_1, P(0))$ is weakly controllable and $p_1(0) > 0$ for all $i \in \mathcal{N}$, then $p_1(k) > 0$ for all $i \in \mathcal{N}$ and each column of $P(1)$ is nonzero.

Proof: The contrapositive is proved. Suppose the conclusion of the conditional statement is false. Then by Proposition 20, $P(1)$ has a zero column and/or $p_1(0) = 0$ for some $i \in \mathcal{N}$. If the $j$-th column of $P(1)$ is zero, then $\pi_{ij}(1) = 0$ for all $i \in \mathcal{N}$ and $P \{X(k) = x, \theta(k) = j\} = \sum_{i=1}^{\mathcal{N}} \pi_{ij}(1)P \{\theta(k) = j\} = \pi_{ij}(1)I = 0$ for all $k \geq 1$. Thus, the final state $(x, j)$ has zero probability for all $k \geq 1$ and any input $w(k) \in L^2$ so $(G \circ \psi_1, P(0))$ is not weakly controllable.

IV. EXAMPLES

Example 32: Consider the switched Markov jump linear system $(G, \Pi, \Psi, P(0))$ where

$$A(1) = \begin{bmatrix} 0.08 & 0.15 & 0.30 \\ 0.20 & 0.60 & 0.10 \\ 0.50 & 0.20 & 0.40 \end{bmatrix}, \quad B(1) = \begin{bmatrix} 0.10 & 0.70 \\ 0.50 & 0.80 \\ 0.20 & 0.40 \end{bmatrix},$$

$$C(1) = \begin{bmatrix} 0.18 & 0.03 & 0.01 \\ 0.01 & 0.07 & 0.06 \\ 0.02 & 0.03 & 0.15 \end{bmatrix}, \quad D(1) = \begin{bmatrix} 0.01 & 0 \\ 0.08 & 0.05 \\ 0.0 \\ 0.01 \end{bmatrix},$$

$$A(2) = \begin{bmatrix} -0.06 & 0.40 & 0.70 \\ 0.35 & -0.07 & 0.10 \\ 0.23 & -0.04 & 0.51 \end{bmatrix}, \quad B(2) = \begin{bmatrix} 0.90 & 0.47 \\ 0.54 & 0.28 \end{bmatrix},$$

$$C(2) = \begin{bmatrix} 0.03 & -0.02 & 0.03 \\ 0.07 & 0.09 & 0.10 \\ 0.07 & 0.02 & 0.08 \end{bmatrix}, \quad D(2) = \begin{bmatrix} 0.01 & -0.11 \\ 0.0 \\ 0.05 \end{bmatrix},$$

$$\Pi(1) = \begin{bmatrix} 0.46 & 0.54 \\ 0.40 & 0.60 \end{bmatrix}, \quad \Pi(2) = \begin{bmatrix} 0.01 & 0.99 \\ 0.05 & 0.95 \end{bmatrix}.$$
Markov chain vary with time in an a priori unknown manner. The mean square concepts examined are appropriate when at least one subsystem \((A(i), B(i), C(i), D(i))\) is not stable or not contractive when considered as a linear time-invariant system. Necessary and sufficient conditions for a switched Markov jump linear system to be uniformly exponentially mean square stable and uniformly mean square strictly contractive were developed. The conditions are convex and can be directly applied in practice.

REFERENCES

[1] J. B. do Val and T. Başar, “Receding horizon control of jump linear systems and a macroeconomic policy problem,” J. Econ. Dyn. Control, vol. 23, no. 8, pp. 1099–1131, 1999.

[2] H. J. Chizeck, “Fault tolerant optimal control,” Sc.D. dissertation, Massachusetts Inst. Technol., Cambridge, MA, 1982.

[3] C. C. Lutz and D. J. Stilwell, “Energy-aware control: \(\ell_2\) gain for closed-loop systems implemented with stochastic schedulers,” in Proc. Amer. Control Conf. Washington, DC, USA: IEEE, 2013, pp. 5313–5319.

[4] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, “A survey of recent results in networked control systems,” Proc. IEEE, vol. 95, no. 1, pp. 138–162, 2007.

[5] P. Seiler and R. Sengupta, “An \(H_{\infty}\) approach to networked control,” IEEE Trans. Autom. Control, vol. 50, no. 3, pp. 356–364, 2005.

[6] L. Xiao, A. Hassibi, and J. P. How, “Control with random communication delays via a discrete-time jump system approach,” in Proc. Amer. Control Conf. Chicago, IL, USA: IEEE, 2000, pp. 2199–2204.

[7] N. J. Ploplys, P. A. Kawka, and A. G. Alleyne, “Closed-loop control with switching transition rates: mean square stability with dwell-time,” IEEE Control Syst. Mag., vol. 24, no. 3, pp. 58–71, 2004.

[8] Y. Ji, H. J. Chizeck, X. Feng, and K. A. Loparo, “Stability and control of discrete-time jump linear systems,” Control Theory Adv. Technol., vol. 7, no. 2, pp. 247–270, Jun. 1991.

[9] O. L. Costa and M. D. Fragoso, “Stability results for discrete-time linear systems with Markovian jumping parameters,” J. Math. Anal. Appl., vol. 179, no. 1, pp. 154–178, 1993.

[10] Y. Ji and H. J. Chizeck, “Jump linear quadratic Gaussian control: steady-state solution and testable conditions,” Control Theory Adv. Technol., vol. 6, no. 3, pp. 289–319, Sep. 1990.

[11] P. Seiler and R. Sengupta, “A bounded real lemma for jump systems,” IEEE Trans. Autom. Control, vol. 48, no. 9, pp. 1651–1654, Sep. 2003.

[12] J.-W. Lee and G. E. Dullerud, “Uniform stabilization of discrete-time switched and Markovian jump linear systems,” Automatica, vol. 42, no. 2, pp. 205–218, 2006.

[13] ——, “Optimal disturbance attenuation for discrete-time switched and Markovian jump linear systems,” SIAM J. Control Optim., vol. 45, no. 4, pp. 1329–1358, 2006.

[14] R. Krtolica, Ü. Özgüner, H. Chan, H. Göktas, J. Winkelman, and M. Liubakka, “Stability of linear feedback systems with random communication delays,” Int. J. Control, vol. 59, no. 4, pp. 925–953, 1994.

[15] S. Aberkane, “Bounded real lemma for nonhomogeneous Markovian jump linear systems,” IEEE Trans. Autom. Control, vol. 58, no. 3, pp. 797–801, Mar. 2013.

[16] Y. Fang and K. A. Loparo, “Stochastic stability of jump linear systems,” IEEE Trans. Autom. Control, vol. 47, no. 7, pp. 1204–1208, 2002.

[17] P. Bolzern, P. Colaneri, and G. De Nicolao, “Markov jump linear systems with switching transition rates: mean square stability with dwell-time,” Automatica, vol. 46, no. 6, pp. 1081–1088, 2010.

[18] A. Leon-Garcia, Probability, Statistics, and Random Processes for Electrical Engineering, 3rd ed. Pearson Education, 2008.

[19] K. Hrbacek and T. Jech, Introduction to Set Theory, 3rd ed. Marcel Dekker, 1999.

[20] W. J. Rugh, Linear system theory. Prentice-Hall, 1996.

[21] J. F. Sturm, “Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones,” Optimization Meth. & Soft., vol. 11, no. 1-4, pp. 625–653, 1999.