Note on parity factors of regular graphs

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Abstract

In this paper, we obtain a sufficient condition for the existence of parity factors in a regular graph in terms of edge-connectivity. Moreover, we also show that our condition is sharp.

1 Preliminaries

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices of a graph $G$ is called the order of $G$ and is denoted by $n$. On the other hand, the number of edges of $G$ is called the size of $G$ and is denoted by $e$. For a vertex $v$ of graph $G$, the number of edges of $G$ incident with $v$ is called the degree of $v$ in $G$ and is denoted by $d_G(v)$. For two subsets $S, T \subseteq V(G)$, let $e_G(S, T)$ denote the number of edges of $G$ joining $S$ to $T$.

Let therefore $g, f : V \to \mathbb{Z}^+$ such that $g(v) \leq f(v)$ and $g(v) \equiv f(v) \pmod 2$ for every $v \in V$. Then a spanning subgraph $F$ of $G$ is called a $(g, f)$-parity-factor, if $g(v) \leq d_F(v) \leq f(v)$ and $d_F(v) \equiv f(v) \pmod 2$ for all $v \in V$. Let $a, b$ be two integers such that $1 \leq a \leq b$ and $a \equiv b \pmod 2$. If $g(v) \equiv a$ and $f(v) \equiv b$ for all $v \in V(G)$, then a $(g, f)$-parity-factor is called $(a, b)$-parity factor. When $a = 1$, $(a, b)$-parity factor is called $(1, n)$-odd factor.

For a general graph $G$ and an integer $k$, a spanning subgraph $F$ such that

$$d_F(x) = k \quad \text{for all } x \in V(G)$$

is called a $k$-factor. In fact, a $k$-factor is also a $(k, k)$-parity factor.

Now let us recall one of the most classic results due to Petersen.

**Theorem 1.1 (Petersen [8])** Let $r$ and $k$ be integers such that $1 \leq k \leq r$. Every $2r$-regular graph has a $2k$-factor.

By the edge-connectivity, Gallai [4] proved the following result.

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*This work was supported by the National Natural Science Foundation of China (No. 11101329)
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Theorem 1.2 (Gallai \cite{4}) Let \( r \) and \( k \) be integers such that \( 1 \leq k < r \), and \( G \) an \( m \)-edge-connected \( r \)-regular graph, where \( m \geq 1 \). If one of the following conditions holds, then \( G \) has a \( k \)-factor:

(i) \( r \) is even, \( k \) is odd, \( |G| \) is even, and \( \frac{r}{m} \leq k \leq r(1 - \frac{1}{m}) \);
(ii) \( r \) is odd, \( k \) is even and \( 2 \leq k \leq r(1 - \frac{1}{m}) \);
(iii) \( r \) and \( k \) are both odd and \( \frac{r}{m} \leq k \).

Bollobás, Saito and Wormald \cite{2} improved above result.

Theorem 1.3 (Bollobás, Saito and Wormald ) Let \( r \) and \( k \) be integers such that \( 1 \leq k < r \), and \( G \) be an \( m \)-edge-connected \( r \)-regular graph, where \( m \geq 1 \) is a positive integer. Let \( m^* \in \{m, m + 1\} \) such that \( m^* \equiv 1 \pmod{2} \). If one of the following conditions holds, then \( G \) has a \( k \)-factor:

(i) \( r \) is odd, \( k \) is even and \( 2 \leq k \leq r(1 - \frac{1}{m^*}) \);
(ii) \( r \) and \( k \) are both odd and \( \frac{r}{m^*} \leq k \).

In this paper, we extend Gallai as well as Bollobás, Saito and Wormald result to \((a, b)\)-parity factor. The main tool in our proofs is the famous theorem of Lovász (see \cite{7}).

Theorem 1.4 (Lovász \cite{7}) \( G \) has a \((g, f)\)-parity factor if and only if for all disjoint subsets \( S \) and \( T \) of \( V(G) \),

\[
\delta(S, T) = f(S) + \sum_{x \in T} d_G(x) - g(T) - e_G(S, T) - \tau \geq 0,
\]

where \( \tau \) denotes the number of components \( C \), called \( f \)-odd components of \( G - (S \cup T) \) such that \( e_G(V(C), T) + f(V(C)) \equiv 1 \pmod{2} \). Moreover, \( \delta(S, T) \equiv f(V(G)) \pmod{2} \).

2 Main Theorem

Theorem 2.1 Let \( a, b \) and \( r \) be integers such that \( 1 \leq a \leq b < r \) and \( a \equiv b \pmod{2} \). Let \( G \) be a \( m \)-edge-connected \( r \)-regular graph with \( n \) vertices. If one of the following conditions holds, then \( G \) has a \((a, b)\)-parity factor.

(i) \( r \) is even, \( a, b \) are odd, \( |G| \) is even, \( \frac{r}{m} \leq b \) and \( a \leq r(1 - \frac{1}{m}) \);
(ii) \( r \) is odd, \( a, b \) are even and \( a \leq r(1 - \frac{1}{m}) \);
(iii) \( r, a, b \) are odd and \( \frac{r}{m^*} \leq b \).
Proof. By Theorem 1.3 (ii) and (iii) are followed. Now we prove (i). Let \( \theta_1 = \frac{a}{r} \) and \( \theta_2 = \frac{b}{r} \). Then \( 0 < \theta_1 \leq \theta_2 < 1 \). Suppose that \( G \) contains no \( (a, b) \)-parity factors. By Theorem 1.4, there exist two disjoint subsets \( S \) and \( T \) of \( V(G) \) such that \( S \cup T \neq \emptyset \), and
\[
-2 \geq \delta(S, T) = b|S| + \sum_{x \in T} d_G(x) - a|T| - e_G(S, T) - \tau,
\]
where \( \tau \) is the number of \( a \)-odd (i.e. \( b \)-odd) components \( C \) of \( G - (S \cup T) \). Let \( C_1, \ldots, C_\tau \) denote \( a \)-odd components of \( G - S - T \) and \( D = C_1 \cup \cdots \cup C_\tau \).

Note that
\[
-2 \geq \delta(S, T) = b|S| + \sum_{x \in T} d_G(x) - a|T| - e_G(S, T) - \tau
\]
\[
= b|S| + (r - a)|T| - e_G(S, T) - \tau
\]
\[
= \theta_2 r|S| + (1 - \theta_1)r|T| - e_G(S, T) - \tau
\]
\[
= \theta_2 \sum_{x \in S} d_G(x) + (1 - \theta_1) \sum_{x \in T} d_G(x) - e_G(S, T) - \tau
\]
\[
\geq \theta_2 e_G(S, T) + \sum_{i=1}^{\tau} e_G(S, C_i) + (1 - \theta_1)(e_G(S, T) + \sum_{i=1}^{\tau} e_G(T, C_i)) - e_G(S, T) - \tau
\]
\[
= \sum_{i=1}^{\tau} (\theta_2 e_G(S, C_i) + (1 - \theta_1)e_G(T, C_i) - 1) + (\theta_2 - \theta_1)e_G(S, T)
\]
\[
\geq \sum_{i=1}^{\tau} (\theta_2 e_G(S, C_i) + (1 - \theta_1)e_G(T, C_i) - 1).
\]
Since \( G \) is connected and \( 0 < \theta_1 \leq \theta_2 < 1 \), so \( \theta_2 e_G(S, C_i) + (1 - \theta_1)e_G(T, C_i) > 0 \) for each \( C_i \). Hence we will obtain a contradiction by showing that for every \( C = C_i, 1 \leq i \leq \tau \), we have
\[
\theta_2 e_G(S, C) + (1 - \theta_1)e_G(T, C) \geq 1. \tag{2}
\]
These inequalities together with the previous inequality imply
\[
-2 \geq \delta_G(S, T) \geq \sum_{i=1}^{\tau} (\theta_2 e_G(S, C_i) + (1 - \theta_1)e_G(T, C_i) - 1)
\]
\[
> \sum_{i=1}^{\tau - 2} (\theta_2 e_G(S, C_i) + (1 - \theta_1)e_G(T, C_i) - 1) - 2 \geq -2,
\]
which is impossible. Since \( C \) is a \( a \)-odd component of \( G - (S \cup T) \), we have
\[
a|C| + e_G(T, C) \equiv 1 \pmod{2}. \tag{3}
\]
Moreover, since
\[
r|C| = \sum_{x \in V(C)} d_G(x) = e_G(S \cup T, C) + 2|E(C)|,
\]

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we have
\[ r|C| = e_G(S \cup T, C) \pmod{2}. \] (4)

It is obvious that the two inequalities \( e_G(S, C) \geq 1 \) and \( e_G(T, C) \geq 1 \) imply
\[ \theta_2 e_G(S, C) + (1 - \theta_1) e_G(T, C) \geq \theta_2 + 1 - \theta_1 = 1. \]

Hence we may assume \( e_G(S, C) = 0 \) or \( e_G(T, C) = 0 \).

Firstly, we consider (i). If \( e_G(S, C) = 0 \), then \( e_G(T, C) \geq m \). Since \( a \leq r(1 - \frac{1}{m}) \), then \( \theta_1 \leq 1 - \frac{1}{m} \) and so \( 1 \leq (1 - \theta_1)m \). By substituting \( e_G(T, C) \geq m \) and \( e_G(S, C) = 0 \) into (2), we have
\[ (1 - \theta_1)e_G(T, C) \geq (1 - \theta_1)m \geq 1. \]

If \( e_G(T, C) = 0 \), then \( e_G(S, C) \geq m \). Since \( \frac{a}{m} \leq b \), hence \( \theta_2 m \geq 1 \), and so we obtain
\[ \theta_2 e_G(S, C) \geq \theta_2 m \geq 1. \]

Consequently, condition (i) guarantees (2) holds and thus (i) is true. Consequently the proof is complete. \( \square \)

**Remark:** The edge-connectivity conditions in Theorem 2.1 are sharp.

We give the description for (i). For (ii) and (iii), the constructions are similar but slightly more complicated. Let \( r \geq 2 \) be an even integer, \( a, b \geq 1 \) two odd integers and \( 2 \leq m \leq r - 2 \) an even integer such that \( b < r/m \) or \( r(1 - \frac{1}{m}) < a \). Since \( G \) has a \((a, b)\)-parity factor if and only if \( G \) has a \((r - b, r - a)\)-parity factor, so we can assume \( b < r/m \). Let \( J(r, m) \) be the complete graph \( K_{r+1} \) from which a matching of size \( m/2 \) is deleted. Take \( r \) disjoint copies of \( J(r, m) \). Add \( m \) new vertices and connect each of these vertices to a vertex of degree \( r - 1 \) of \( J(r, m) \). This gives an \( m \)-edge-connected \( r \)-regular graph denoted by \( G \). Let \( S \) denote the set of \( m \) new vertices and \( T = \emptyset \). Let \( \tau \) denote the number of components \( C \), called \( a \)-odd components of \( G - (S \cup T) \) such that \( e_G(V(C), T) + a|C| \equiv 1 \pmod{2} \). Then we have \( \tau = r \), and
\[ \delta(S, T) = b|S| + \sum_{x \in T} d_{G-S}(x) - a|T| - \tau(S, T) = bm - r < 0. \]

So by Theorem 1.4, \( G \) contains no \((a, b)\)-parity factors.

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