A Strongly Consistent Sparse $k$-means Clustering with Direct $l_1$ Penalization on Variable Weights

Saptarshi Chakraborty, Swagatam Das
Indian Statistical Institute, Kolkata, India

Abstract—We propose the Lasso Weighted $k$-means ($LW$-$k$-means) algorithm as a simple yet efficient sparse clustering procedure for high-dimensional data where the number of features ($p$) can be much larger compared to the number of observations ($n$). In the $LW$-$k$-means algorithm, we introduce a lasso-based penalty term, directly on the feature weights to incorporate feature selection in the framework of sparse clustering. $LW$-$k$-means does not make any distributional assumption of the given dataset and thus, induces a non-parametric method for feature selection. We also analytically investigate the convergence of the underlying optimization procedure in $LW$-$k$-means and establish the strong consistency of our algorithm. $LW$-$k$-means is tested on several real-life and synthetic datasets and through detailed experimental analysis, we find that the performance of the method is highly competitive against some state-of-the-art procedures for clustering and feature selection, not only in terms of clustering accuracy but also with respect to computational time.

Index Terms—Clustering, Unsupervised Learning, Feature Selection, Feature Weighting, Consistency.

I. INTRODUCTION

CLUSTERING is one of the major steps in exploratory data mining and statistical data analysis. It refers to the task of distributing a collection of patterns or data points into more than one non-empty groups or clusters in such a manner that the patterns belonging to the same group may be more identical to each other than those from the other groups [1], [2]. The patterns are usually represented by a vector of variables or observations that are also commonly known as features in the pattern recognition community. The notion of a cluster, as well as the number of clusters in a particular data set, can be ambiguous and subjective. However, most of the popular clustering techniques comply with the human conception of clusters and capture a dense patch of points in the feature space as a cluster. Center-based partitional clustering algorithms identify each cluster in terms of a single point called a centroid or a cluster center, which may or may not be a member of the given dataset. $k$-means [3], [4] is arguably the most popular clustering algorithm in this category. This algorithm separates the data points into $k$ disjoint clusters ($k$ is to be specified beforehand, though) by locally minimizing the total intra-cluster spread i.e. the sum of squares of the distances from each point to the candidate centroids. Obviously, the

algorithm starts with a set of randomly initialized candidate centroids from the feature space of the data and attempts to refine them towards the best representatives of each cluster over the iterations by using a local heuristic procedure. $k$-means may be viewed as a special case of the more general model-based clustering [5], [6], [7] where the set of $k$ centroids can be considered as a model from which the data is generated. Generating a data point in this model consists of first selecting a centroid at random and then adding some noise. For a Gaussian distribution of the noise, this procedure will result into hyper-spherical clusters usually.

With the advancement of sensors and hardware technology, it has now become very easy to acquire a vast amount of real data described over several variables or features, thus giving rise to high-dimensional data. For example, images can contain billions of pixels, text and web documents can have several thousand words, microarray datasets can consist of expression levels of thousands of genes. Curse of dimensionality [8] is a term often coined to describe some fundamental problems associated with the high-dimensional data where the number of features $p$ far exceeds the number of observations $n$ ($p \gg n$). With the increase of dimensions, the difference between the distances of the nearest and furthest neighbors of a point fades out, thus making the notion of clusters almost meaningless [9]. In addition to the problem above, many researchers also concur on the fact that especially for high dimensional data, the meaningful clusters may be present only in subspaces formed with a specific subset of the features available [10], [11], [12], [13]. Different features can exhibit different degrees of relevance to the underlying groups in a practical data with a high possibility. Generally, the machine learning algorithms employ various strategies to select or discard a number of features to deal with this situation. Using all the available features for cluster analysis (and in general for any pattern recognition task) can make the final clustering solutions less accurate when a considerable number of features are not relevant to some clusters [14]. To add to the difficulty further, the problem of selection of an optimal feature subset with respect to some criteria is known to be NP-hard [15]. Also, even the degree of contribution of the relevant features can vary differently to the task of demarcating various groups in the data. Feature weighting is often thought of as a generalization of the widely used feature selection procedures [16], [17], [10], [18]. An implicit assumption of the feature selection methods is that all the selected features are equally relevant to the learning task in hand, whereas, feature weighting algorithms do not make such assumption as each of the selected features
may have a different degree of relevance to a cluster in the data. To our knowledge, Synthesized Clustering (SYNCLUS) [19] is the first $k$-means extension to allow feature weights. SYNCLUS partitions the available features into a number of groups and uses different weights for these groups during a conventional $k$-means clustering process. The convex $k$-means algorithm [17] is an interesting approach to feature weighting by integrating multiple, heterogeneous feature spaces into the $k$-means framework. Another extension of $k$-Means to support feature weights was introduced in [14]. Huang et al. [20], introduced the celebrated Weighted $k$-means algorithm ($WK$-means) which introduces a new step for updating the feature weights derived from the current partition. $WK$-means was later extended to support fuzzy clustering [21] and cluster-dependent weights [22]. Entropy Weighted $k$-means [23], improved $k$-prototypes [24], Minkowski Weighted $k$-means [13], Feature Weight Self-Adjustment $k$-Means [10], Feature Group Weighted $k$-means ($FG-k$-means) [12] are among the notable works in this area. A detailed account of these algorithms and their extensions can be found in [18].

Traditional approaches for feature selection can be broadly categorized into filter and wrapper-based approaches [25], [26]. Filter methods use some kind of proxy measure (just for example, mutual information, Pearson product-moment correlation coefficient, Relief-based algorithms etc.) to score the selected feature subset during the pre-processing phase of the data. On the other hand, the wrapper approaches employ a predictive learning model to evaluate the candidate feature subsets. Although wrapper methods tend to be more accurate than those following a filter-based approach [25], nevertheless, they incur high computational costs due to the need of executing both a feature selection module and a clustering module several times on the possible feature subsets.

Real-world datasets can come with a large number of noise variables, i.e., variables that do not change from cluster to cluster, also implying that the natural groups occurring in the data differ with respect to a small number of variables. Just as an example, only a small fraction of genes (relevant features) contribute to the occurrence of a certain biological activity, while the others in a large fraction, can be irrelevant (noisy features). A good clustering method is expected to identify the relevant features, thus avoiding the derogatory effect of the noisy and irrelevant ones. It is not hard to see that if an algorithm can impose positive weights on the relevant features while assigning exactly zero weights on the noisy ones, the negative influence from the latter class of features can be nullified. Sparse clustering methods closely follow such intuition and aim at partitioning the observations by using only an adaptively selected subset of the available features.

A. Relation to Prior Works

Introducing sparsity in clustering is a well studied field of unsupervised learning. Friedman and Meulman [27] proposed a sparse clustering procedure, called Clustering Objects on Subsets of Attributes (COSA), which in its simplified form, allows different feature weights within a cluster and closely relate to a weighted form of the $k$-means algorithm. Witten and Tibshirani [28] observed that COSA hardly results in a truly sparse clustering since, for a positive value of the tuning parameter involved, all the weights retain non-zero value. As a betterment, they proposed the sparse $k$-means algorithm by using the $l_1$ and $l_2$ penalization to incorporate feature selection. The $l_1$ penalty on the weights result in sparsity (making weights of some of the (irrelevant) features 0) for a small value of a parameter which is tuned by using the Gap Statistic [29]. On the other hand, the $l_2$ penalty is equally important as it causes more than one components of the weight vector to retain non-zero value. Despite its effectiveness, the statistical properties of the sparse $k$-means algorithm including its consistency are yet to be investigated. Unlike the fields of sparse classification and regression, only a few notable extensions on sparse $k$-means emerged subsequently. A regularized version of sparse $k$ means for clustering high dimensional data was proposed in [30], where the authors also established its asymptotic consistency. Arias-Castro and Pu [31] proposed a simple hill climbing approach to optimize the clustering objective in the framework of the sparse $k$ means algorithm.

A very competitive approach for high dimensional clustering, different from the framework of sparse clustering was taken in [32] based on the so-called Influential Feature-based Principal Component Analysis aided with a Higher Criticality based Thresholding (IF-PCA-HCT). This method first selects a small fraction of features with the largest Kolmogorov-Smirnov (KS) scores and then determines the first $k-1$ left singular vectors of the post-selection normalized data matrix. Subsequently, it estimates the clusters by using a classical $k$-means algorithm on these singular vectors. According to [32], the only parameter that needs to be tuned in IF-PCA-HCT is the threshold for the feature selection step. The authors recommended a data-driven rule to set the threshold on the basis of the notion of Higher Criticism (HC) that uses the order statistics of the feature $z$-scores [33].

Another similar approach known as the IF-PCA algorithm was proposed by Jin et al. [34]. For a threshold $t$ this method clusters the dataset by using the classical PCA to all features whose $l_2$ norm is larger that $t$. Pan and Shen [35] proposed the Penalized model-based clustering. This method proposes an EM algorithm to obtain feature selection. Although this method is quite effective, it assumes the likelihood of the data, which can lead to erroneous results if the assumed likelihood is not well suited for the data. This is also the case for IF-HCT-PCA[32] and IF-PCA [34] as they both assume a Gaussian mixture model for the data. As it can be seen from Section III that the proposed method does not suffer from this drawback. In contrast to the Sparse $k$-means algorithm [28] which uses only $l_1$ and $l_2$ terms in the objective function, our proposed method uses only an $l_1$ penalization and also a $\beta$ exponent in the weight terms, which can lead to more efficient feature selection as seen in Section VI-F. In addition, no obvious relation between the Sparse $k$-means and $LW\cdot k$-means is apparent.

Some theoretical works on sparse clustering can be found in [36], [34], [37]. A minimax theory for high-dimensional Gaussian mixture models was proposed by Azizyan et al.
where the authors derived some precise information theoretic bounds on the clustering accuracy and sample complexity of learning a mixture of two isotropic Gaussians in high dimensions under small mean separation. The minimax complexity of learning a mixture of two isotropic Gaussians in low dimensions under small mean separation was derived in [36]. The strong consistency of the Reduced k-Means (RKM) algorithm [38] the under i.i.d sampling was recently established by Terada [39]. Following the methods of [40] and [39], the strong consistency of the factorial k-means algorithm [41] was also proved in [42]. Gallegos and Ritter [43] extended the Pollard’s proof of strong consistency [40] for an affine invariant k-parameters clustering algorithm. Nikulin [44] presented proof for the strong consistency of the divisive information-theoretic feature clustering model in probabilistic space with Kullback-Leibler (KL) divergence. Recently, the strong consistency of the Weighted k-means algorithm for nearmetric spaces under i.i.d. sampling was proved by Chakraborty and Das [45]. The proof of strong consistency presented in this paper is slightly trickier than the aforementioned papers as we need to choose $\alpha$ and $\lambda$ suitably such that the sum of the weights are bounded almost surely and at least one weight is bounded away from 0 almost surely.

In Table I, we highlight some of the works in this field along with their important aspects in terms of feature weighting, feature selection, model assumptions and proof of consistency of the algorithms and try to put our proposed algorithm in the context.

### B. Summary of Our Contributions

We propose a simple sparse clustering framework based on the feature-weighted $k$ means algorithm, where a Lasso penalty is imposed directly on the feature weights and a closed form solution can be reached for updating the weights. The proposed algorithm, which we will refer to as Lasso Weighted $k$ means ($LW$-$k$-means), does not require the assumption of normality of the irrelevant features as required for the IF-HCT-PCA algorithm [32]. We formulate the $LW$-$k$-means as an optimization procedure on an objective function and derive a block coordinate descent type algorithm [46] to optimize the objective function in section IV. We also prove that the proposed algorithm converges after a finite number of iteration in Theorem IV.6. We establish the strong consistency of the proposed $LW$-$k$-means algorithm in Theorem V.4. Conditions ensuring almost sure convergence of the estimator of $LW$-$k$-means with unboundedly increasing sample size are investigated in section V-A. With a detailed experimental analysis, we demonstrate the competitiveness of the proposed algorithm against the baseline $k$-means and $WK$-means algorithms along with the state-of-the-art sparse $k$-means and IF-HCT-PCA algorithms by using several synthetic as well as challenging real-world datasets with a large number of attributes. Through our experimental results, we observe that not only the $LW$-$k$-means outperforms the other state-of-the-art algorithms, but it does so with considerably less computational time. In section VII, we report a simulation study to get an idea about the distribution of the obtained feature weights. The outcomes of the study show that $LW$-$k$-means perfectly identifies the irrelevant features in certain datasets which may deceive some of the state-of-the-art clustering algorithms.

### C. A Motivating Example

Before proceeding further, we take a motivating example to illustrate the efficacy of the $LW$-$k$-means procedure (detailed in Section IV) w.r.t the other peer clustering algorithms by considering a sample toy dataset. In Fig. 1a, we show the scatter plot of a synthetic dataset $data1$ (the dataset is available at https://github.com/SaptarshiC98/lwk-means). It is clear that only the $x$-variable contains the cluster structure of the data while the $y$-variable does not. We run five algorithms ($k$-means, $WK$-means, sparse $k$-means, IF-HCT-PCA, and $LW$-$k$-means) on the dataset independently 20 times and report the average CER (Classification Error Rate: proportional to instances misclassified over the whole set of instances) in Table II. We also note the average feature weights for each algorithm. From Table II, we see that only the $LW$-$k$-means assigns a zero feature weight to feature $y$ and also that it achieves an average CER of 0. The presence of an elongated cluster (colored in black in Fig. II) affects the clustering procedure of all the algorithms except $LW$-$k$-means. This elongated cluster, which is non-identically distributed in comparison to the other clusters, increases the Within Sum of Squares (WSS) of the $y$ values, thus increasing its weight. It can be easily seen that for this toy example, the other peer algorithms erroneously detect the $y$ feature to be important for clustering and thus leads to

### Table I: Some Well Known Algorithms on Feature Weighting and Feature Selection

| Algorithm/Reference | Feature Weighting | Feature Selection | Model Assumptions | Consistency Proof |
|---------------------|-------------------|-------------------|-------------------|-------------------|
| $k$-means [5]       | ✓                 | ✗                 | ✗                 | ✓                 |
| $W$-$k$-means [20]  | ✓                 | ✗                 | ✗                 | ✓                 |
| Pan and Shen [35]   | ✗                 | ✓                 | ✓                 | ✓                 |
| Sparse-$k$-means [28]| ✓                 | ✓                 | ✓                 | ✓                 |
| IF-HCT-PCA [32]     | ✓                 | ✓                 | ✓                 | ✓                 |
| IF-PCA [34]         | ✓                 | ✓                 | ✓                 | ✓                 |
| LW-$k$-means (The Proposed Method) | ✓ | ✓ | ✓ | ✓ |

### Table II: Comparison between $LW$-$k$-means and IF-HCT-PCA

| Algorithm         | Feature Weights | Average CER |
|-------------------|-----------------|-------------|
| $k$-means          | $x$             | 1           |
| $WK$-means         | $0.5657$        | $0.4343$    |
| IF-HCT-PCA         | 1               | 1           |
| Sparse $k$-means   | $0.9446$        | $0.3281$    |
| LW-$k$-means       | $0.7587$        | 0           |
inaccurate clustering. This phenomenon is illustrated in Fig. II.

II. BACKGROUND

A. Some Preliminary Concepts

In this section we will discuss briefly about the notion of consistency of an estimator. Before we begin, let us recall the definition of convergence in probability and almost surely.

Definition II.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A sequence of random variables $\{X_n\}_{n \geq 1}$ is said to converge almost surely (a.s. $[P]$) to a random variable $X$ (in the same probability space), written as

$$X_n \overset{a.s.}{\rightarrow} X$$

if $P(\{\omega \in \Omega : X_n(\omega) \rightarrow X(\omega)\}) = 1$.

Definition II.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A sequence of random variables $\{X_n\}_{n \geq 1}$ is said to converge in probability to a random variable $X$ (in the same probability space), written as

$$X_n \overset{P}{\rightarrow} X$$

if $\forall \epsilon > 0$, $\lim_{n \to \infty} P(|X_n - X| > \epsilon) = 0$.

B. The Setup and Notations

Before we start, we discuss the meaning of some symbols used throughout the paper in Table III.

| Symbol | Meaning |
|--------|---------|
| $\mathbb{R}$ | The set of all real numbers |
| $\mathbb{R}_+$ | The set of all non-negative real numbers |
| $\mathbb{R}^p_k$ | $A \subset \mathbb{R}^p | A$ contains $k$ or fewer points $|$ |
| $\mathbb{N}$ | The set of all natural numbers |
| $\mathcal{S}$ | The set $\{2n : n \in \mathbb{N}\}$ |
| $\mathcal{U}$ | The cluster assignment matrix |
| $\mathcal{Z}$ | The centroid matrix whose rows denote the centroids |
| $\mathbf{W}$ | Vector of all the feature weights |
| $\mathcal{N}(\mu, \sigma^2)$ | Normal distribution with mean $\mu$ and variance $\sigma^2$ |
| $\mathcal{U}[a, b]$ | Uniform distribution on the interval $(a, b)$ |
| $\chi^2_d$ | $\chi^2$ distribution with $d$ degrees of freedom |
| $\mathbf{A}^\top$ | Transpose of the matrix $\mathbf{A}$ |
| $\mathbf{1}$ | Vector $(1, \ldots, 1)^\top$ of length $n$ |
| i.i.d | Independent and Identically Distributed |
| i.o. | Infinitely Often |
| a.s. | Almost Surely |
| CER | Classification Error Rate |

Let $\mathcal{X} = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{R}^p$ be a set of $n$ data points which needs to be partitioned into $k$ disjoint and non-empty clusters. Let us also impose $2 \leq k \leq n$ and assume that $k$ is known. Let us now recall the definition of a consistent and
strongly consistent estimator.

**Definition II.3.** An estimator $T_\theta = T_n(x_1, \ldots, x_n)$ is said to be consistent for a parameter $\theta$ if $T_n \xrightarrow{p} \theta$.

**Definition II.4.** An estimator $T_\theta = T_n(x_1, \ldots, x_n)$ is said to be strongly consistent for a parameter $\theta$ if $T_n \xrightarrow{a.s.} \theta$.

A detailed exposure on consistency can be found in [47].

**C. $k$-means Algorithm**

The conventional $k$-means clustering problem can be formally stated as a minimization of the following objective function:

$$k\text{-means}(U, Z) = \sum_{i=1}^{n} \sum_{j=1}^{k} \sum_{l=1}^{p} u_{i,j} d(x_{i,l}, z_{j,l}),$$  \hspace{1cm} (1)

where $U$ is an $n \times k$ cluster assignment matrix (also called partition matrix), $u_{i,j}$ is binary and $u_{i,j} = 1$ means data point $x_i$ belongs to cluster $C_j$, $Z = [z_1', z_2', \ldots, z_k']'$ is a matrix, whose rows represent the $k$ cluster centers, and $d(\cdot, \cdot)$ is the distance metric of choice to measure the dissimilarity between two data points. For the widely popular squared Euclidean distance, $d(x_{i,l}, z_{j,l}) = (x_{i,l} - z_{j,l})^2$. Local minimization of the $k$-means objective function is, most commonly carried out by using a two-step alternating optimization procedure, called the Lloyd’s heuristic and recently a performance guarantee of the method in well clusterable situations was established in [48].

**D. WK- means Algorithm**

In the well-known Weighted $k$-means ($W\text{-}k$-means) algorithm by Huang et al. [20], the feature weights are also updated along with the cluster centers and the partition matrix within a $k$-means framework. In [20], the authors modified the objective function of $k$-means in the following way to achieve an automated learning of the feature weights:

$$W\text{-k}\text{-means}(U, Z, W) = \sum_{i=1}^{n} \sum_{j=1}^{k} \sum_{l=1}^{p} u_{i,j} w_i^\beta d(x_{i,l}, z_{j,l}),$$  \hspace{1cm} (2)

where $W = [w_1, w_2, \ldots, w_p]$ is the vector of weights for the $p$ variables, $\sum_{l=1}^{k} w_l = 1$, and $\beta$ is the exponent of the weights. Huang et al. [20] formulated an alternative optimization based procedure to minimize the objective function with respect to $U$, $Z$ and $W$. The additional step introduced in the $k$-means loop to update the weights use the following closed form upgrade rule: $w_l = \frac{\sum_{i=1}^{n} p^\beta u_{i,j} d(x_{i,l}, z_{j,l})}{\sum_{i=1}^{n} p^\beta}$, where

$$D_l = \sum_{i=1}^{n} \sum_{j=1}^{k} u_{i,j} d(x_{i,l}, z_{j,l}).$$

**E. Sparse $k$-means Algorithm**

Witten and Tibshirani [28] proposed the sparse $k$-means clustering algorithm for feature selection during clustering of high-dimensional data. The sparse $k$-means objective function can be formalized in the following way:

$$P_{\text{Sparse } k\text{-means}}(U, W) = \sum_{i=1}^{n} \left( w_l \sum_{j=1}^{k} \sum_{l=1}^{p} d(x_{i,l}, z_{j,l}) \right) - \sum_{j=1}^{n} \sum_{l=1}^{k} u_{i,j} w_{l} d(x_{i,l}, z_{j,l}).$$  \hspace{1cm} (3)

This objective function is optimized w.r.t. $U$ and $W$ subject to the constraints,

$$||W||_2^2 \leq 1, \quad ||W||_1 \leq s \quad \text{and} \quad w_j \geq 0 \quad \forall j \in \{1, \ldots, p\}.$$
however small it may be, to the irrelevant features but will never assign a zero to it. Thus, $WK$-means fails to perform feature selection, where some features need to be completely discarded.

Let us try to overcome this difficulty by adding a penalty term. If we add a penalty term $\frac{1}{n} \frac{\lambda}{p^2} \sum_{i=1}^{p} D_i |w_i|$, that will equally penalize all the $w_i$’s regardless of whether the feature is distinguishing or not. Instead of doing that we use the penalty term $\frac{1}{n} \frac{\lambda}{p^2} \sum_{i=1}^{p} D_i |w_i|$, which will punish those $w_i$’s for which $D_i$’s are larger. Here $p^2$ is just a normalizing constant. Thus if we use this penalty term, the objective function becomes

$$P(W) = \frac{1}{n} \sum_{i=1}^{p} D_i w_i^\beta + \frac{1}{n} \frac{\lambda}{p^2} \sum_{i=1}^{p} D_i |w_i| = \sum_{i=1}^{p} (w_i^\beta + \frac{\lambda}{p^2} |w_i|) D_i.$$  \hfill (7)

Apart from having the objective function 7, we do not want that the sum of the weights should deviate too much from 1. Thus we substract a penalty term $\alpha(\sum_{i=1}^{p} w_i - 1)$, where, $\alpha > 0$ and the objective function becomes,

$$P(W) = \frac{1}{n} \sum_{i=1}^{p} (w_i^\beta + \frac{\lambda}{p^2} |w_i|) D_i - \alpha \sum_{i=1}^{p} w_i + \alpha.$$  \hfill (8)

Since, $\alpha > 0$ is a constant, minimizing 8 is same as minimizing

$$P(W) = \frac{1}{n} \sum_{i=1}^{p} (w_i^\beta + \frac{\lambda}{p^2} |w_i|) D_i - \alpha \sum_{i=1}^{p} w_i$$  \hfill (9)

w.r.t $W$. Since $\lambda$ is a constant, we change the objective function to 4. In Fig. 2b, we show the contour plot of the objective function 4 with $p = 2$, $\beta = 2$, $D_1 = 1$, $D_2 = 3$, $\lambda = 1$, $\alpha = 0.66$. Clearly the minimization of the objective function occurs on the $x$-axis. Hence the $LW$-$k$-means algorithm can set some of the feature weights to 0 and thus can perform feature selection. The $w_i$’s term provides an additional degree of non-linearity to the $LW$-$k$-means objective function. Also notice that for $\beta = 2$, though sparse $k$-means objective function (3) and the $LW$-$k$-means objective function uses the same term, they are not similar at all. The optimal value for the weight for a given set of cluster centroids for sparse $k$-means algorithm does not have a closed form expression but for $LW$-$k$-means, we can find a closed form expression (section IV), which can be used for hypothesis testing purposes for model based clustering.

In addition, we note the difference between the Regularized $k$-means [49] and $LW$-$k$-means. The former uses a penalization on the centroids of each cluster for feature selection but the latter uses the whole dataset for the same purpose. Since a cluster centroid determined by the underlying $k$-means procedure may not be the actual representative of a whole cluster, using penalization only on the cluster centroids may lead to improper feature selection due to grater loss of information about the naturally occurring groups in the data.

### IV. THE LASSO WEIGHTED $k$-MEANS ALGORITHM AND ITS CONVERGENCE

We can minimize 4 by solving the following three minimization problems.

- **Problem P$_1$:** Fix $Z = Z_0$, $W = W_0$, minimize $P(U, Z_0, W_0)$ w.r.t $U$ subject to the constraints 5a and 5b.
- **Problem P$_2$:** Fix $U = U_0$, $W = W_0$, minimize $P(U_0, Z, W_0)$ w.r.t $Z$.
- **Problem P$_3$:** Fix $Z = Z_0$, $U = U_0$, minimize $P(U_0, Z_0, W)$ w.r.t $W$.

It is easily seen that Problem P$_1$ can be solved by assigning

$$u_{i,j} = \begin{cases} 1 & \text{if } \sum_{l=1}^{p} (w_i^\beta + \frac{\lambda}{p^2} |w_i|) D_l D_j^0 \leq \sum_{l=1}^{p} (w_i^\beta + \frac{\lambda}{p^2} |w_i|) D_l, 1 \leq t \leq k, \\ \text{otherwise.} \end{cases}$$

Problem P$_2$ can also be easily solved by assigning,

$$z_{i,j} = \frac{\sum_{l=1}^{p} u_{i,l} d(x_{i,l}, z_{j,l})}{\sum_{l=1}^{p} u_{i,l}}.$$  

Let, $D_l = \sum_{i=1}^{n} \sum_{j=1}^{k} u_{i,j} d(x_{i,l}, z_{j,l})$. Hence Problem P$_3$ can be stated in the following way. Let $D_l^0$ denote the value of $D_l$ at $Z = Z_0$ and $U = U_0$. We note that the objective function can now be written as

$$P(W) = \frac{1}{n} \sum_{i=1}^{p} (w_i^\beta + \frac{\lambda}{p^2} |w_i|) D_l^0 - \alpha \sum_{i=1}^{p} w_i.$$  \hfill (10)

Now, for solving Problem P$_3$, we note the following.

**Theorem IV.1.** The objective function $P(W)$ in 10 is convex in $W$.

**Proof.** See Appendix A-A. \Box

Now let us solve Problem P$_3$ for the case $p = 1$. For this, we construct an equivalent problem as follows.

**Theorem IV.2.** Suppose $w \in \mathbb{R}$, $D > 0$, $\alpha \geq 0$, $\lambda \geq 0$, $\beta \in \mathbb{S}$ be scalars. Consider the following single-dimensional optimization problem $P_1^*$:

$$\min_{w} \frac{1}{n} w^\beta D - \alpha w + \frac{\lambda}{np^2} |w| D.$$  \hfill (11)

Let $w^*_1$ be a solution to 11. Consider another single-dimensional optimization problem $P_2^*$:

$$\min_{w} \frac{1}{n} w^\beta D - \alpha w + \frac{\lambda}{np^2} tD,$$  \hfill (12)
subject to
\[ t - w \geq 0, \quad (13) \]
\[ t + w \geq 0. \quad (14) \]
Suppose \((w^*, t^*)\) be a solution of problem \(P_2^*\). Then, \(w^* = w_1^*\).

Proof. See Appendix A-B. □

Before we solve problem \(P_2^*\), we consider the following definition.

**Definition IV.1.** For scalars \(x\) and \(y\) ≥ 0, the function \(S(\cdot, \cdot)\) is defined as,
\[ S(x, y) = \begin{cases} x - y, & \text{if } x > y, \\ x + y, & \text{if } x < -y, \\ 0, & \text{otherwise.} \end{cases} \]

We now solve problem \(P_2^*\) of Theorem IV.2 by using Theorem IV.3.

**Theorem IV.3.** Consider the 1-D optimization problem \(P_2^*\) of Theorem IV.2. Let \(D > 0\) and \((w^*, t^*)\) be a solution to problem \(P_2^*\). Then a solution to this problem exists, is unique and is given by,
\[ w^* = \left[ \frac{1}{\beta} S\left(\frac{n\alpha}{D}, \frac{\lambda}{p^2}\right) \right]^{\frac{1}{p-1}}. \]

Proof. See Appendix A-C. □

In Theorem IV.2, we showed the equivalence of problems \(P_1^*\) and \(P_2^*\) and in Theorem IV.3, we solved problem \(P_2^*\). Hence combining the results of Theorems IV.2 and IV.3, we have the following theorem.

**Theorem IV.4.** Suppose \(w \in \mathbb{R}, D > 0, \alpha \geq 0, \lambda \geq 0, \beta \in S\) be scalars. Consider the following single-dimensional optimization problem,
\[ \min_w \frac{1}{n} w^\beta D - \alpha w + \frac{\lambda}{np^2} |w| D. \quad (15) \]
Then a solution to this problem exists, is unique and is given by,
\[ w^* = \left[ \frac{1}{\beta} S\left(\frac{n\alpha}{D}, \frac{\lambda}{p^2}\right) \right]^{\frac{1}{p-1}}. \]

Proof. The result follows trivially from Theorems IV.2 and IV.3. □

We are now ready to prove Theorem IV.5, which essentially gives us the solution to Problem \(P_2\).

**Theorem IV.5.** Let \(\lambda \geq 0, \alpha \geq 0, D_i > 0\) for all \(d \in \{1, \ldots, p\}\) be scalars. Also let, \(\beta \in S\) and \(p \in \mathbb{N}\). If \(W \in \mathbb{R}^p\), then solution to the problem
\[ \min_{W \in \mathbb{R}^p} P(W) = \frac{1}{n} \sum_{i=1}^{p} (w_i^\beta + \frac{\lambda}{p^2} |w_i|) D_i - \alpha \sum_{i=1}^{p} w_i \]
exists, is unique and is given by,
\[ w_i^t = \left[ \frac{1}{\beta} S\left(\frac{n\alpha}{D_i}, \frac{\lambda}{p^2}\right) \right]^{\frac{1}{p-1}} \quad \forall i \in \{1, \ldots, p\}. \]

Proof. See Appendix A-D. □

Algorithm 1 gives a formal description of the LW-k-means algorithm.

**Algorithm 1: The LW-k-means Algorithm**

**Data:** \(X, k, \frac{1}{\sqrt{p}}, \epsilon\)

**Result:** \(U, Z, W\)

**initialization:** Randomly pick \(k\) datapoints \(x_1, \ldots, x_k\) from \(\{x_1, \ldots, x_n\}\).

Set \(Z = [x_1, \ldots, x_k]^T\), \(W = \frac{1}{p}, \ldots, \frac{1}{p}\), \(P_1 = 0\)

\(P_2 = \) A very large value

while \(|P_1 - P_2| > \epsilon\) do

\[ P_1 = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{k} \sum_{l=1}^{p} (w_i^\beta + \frac{\lambda}{p^2} |w_i|) u_{i,j} d(x_{i,j}, z_{j,l}) - \alpha \sum_{i=1}^{p} w_i \]

Update \(Z\) by

\[ z_{i,j} = \frac{\sum_{i=1}^{n} u_{i,j} d(x_{i,j}, z_{j,l})}{\sum_{i=1}^{n} u_{i,l}} \]

Update \(W\) by

\[ w_i = \begin{cases} 0, & \text{if } D_i = 0, \\ \left[ \frac{1}{\beta} S\left(\frac{n\alpha}{D_i}, \frac{\lambda}{p^2}\right) \right]^{\frac{1}{p-1}}, & \text{otherwise,} \end{cases} \]

where \(D_i = \sum_{j=1}^{k} \sum_{l=1}^{p} u_{i,j} d(x_{i,j}, z_{j,l}) \)

Update \(U\) by

\[ u_{i,j} = \begin{cases} 1, & \text{if } \sum_{i=1}^{p} (w_i^\beta + \frac{\lambda}{p^2} |w_i|) d(x_{i,j}, z_{j,l}) \\ \leq \sum_{i=1}^{p} (w_i^\beta + \frac{\lambda}{p^2} |w_i|) d(x_{i,j}, z_{j,l}), \quad 1 \leq t \leq k \end{cases} \]

\[ 0, \quad \text{otherwise.} \]

\[ P_2 = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{k} \sum_{l=1}^{p} (w_i^\beta + \frac{\lambda}{p^2} |w_i|) u_{i,j} d(x_{i,j}, z_{j,l}) - \alpha \sum_{i=1}^{p} w_i \]
end

We now prove the convergence of the iterative steps in the LW-k-means algorithm. This result is proved in the following theorem. The proof of convergence of the LW-k-means can be directly derived from [46]. We only state the result in Theorem IV.6. The proof of this result is given in Appendix A-E.

**Theorem IV.6.** The LW-k-means algorithm converges after a finite number of iterations.

**V. STRONG CONSISTENCY OF THE LW-K-MEANS ALGORITHM**

In this section, we will prove a strong consistency result pertaining to the LW-k-means algorithm. Our proof of strong consistency result is slightly trickier than that of Pollard [40] in the sense that we have to deal with the weight terms which may not be bounded. We first prove the existence of an \(\alpha\) which depends on the datasets itself (Theorem V.1), such that \(\exists \lambda_0\) for which, we can find an \(l\) such that \(w_i^\beta > C(P) > 0 \forall 0 < \lambda < \lambda_0\) (Theorem V.3). In Theorem V.4, we prove the main result pertaining to the strong
consistency of proposed algorithm. Throughout this section, we will assume that \(d(x, y) = (x - y)^2\), i.e. the distance used is the squared Euclidean distance. We will also assume that the underlying distribution has a finite second moment.

### A. The Strong Consistency Theorem

In this section we prove the strong consistency of the proposed method for the following setup. Let \(X_1, \ldots, X_n\) be independent random variables with a common distribution \(P\) on \(\mathbb{R}^p\).

**Remark 1.** \(P\) can be thought of a mixture distribution in the context of clustering but this assumption is not necessary for the proof.

Let \(P_n\) denote the empirical measure based on \(X_1, \ldots, X_n\). For each measure \(Q\) on \(\mathbb{R}^p\), each \(W \in \mathbb{R}^p\) and each finite subset \(A\) of \(\mathbb{R}^p\), define

\[
\Phi(W, A, Q) := \int \min_{a \in A} \sum_{i=1}^p \left( w_i + \frac{\lambda}{p^2} |w_i| \right) |x_i - a_i|^2 Q(dx)
\]

and

\[
m_k(Q) := \inf \{ \Phi(W, A, Q) | A \in \mathbb{R}^p, W \in \mathbb{R}^p \}.
\]

Here \(\alpha(Q)\) is a functional. \(\alpha(Q)\) and \(\lambda\) are chosen as in Theorems V.1 and V.3. For a given \(k\), let \(A_n, W_n\) denote the optimal sample clusters and weights respectively, i.e. \(\Phi(W_n, A_n, P_n) = m_k(P_n)\). The optimal population cluster centroids and weights are denoted by \(\bar{A}(k)\) and \(\bar{W}(k)\) respectively and they satisfy the relation, \(\Phi(\bar{W}(k), \bar{A}(k), P) = m_k(P)\). Our aim is to show \(A_n \xrightarrow{a.s.} \bar{A}(k)\) and \(W_n \xrightarrow{a.s.} \bar{W}(k)\).

**Theorem V.1.** There exists at least one functional \(\alpha(\cdot)\) such that \(\alpha(P_n) \xrightarrow{a.s.} \alpha(P)\).

**Proof.** See appendix A-F.

**Remark 2.** One can choose \(\alpha\) as follows.

- Run the \(k\)-means algorithm on the entire dataset. Let \(U\) and \(Z\) be the corresponding cluster assignment matrix and the set of centroids respectively.
- Choose \(\alpha_n(P_n) = \frac{1}{\sum_{i=1}^p \left[ \frac{n}{p^2} \right]} \beta^{-r}\).

**Remark 3.** Note that if one chooses \(\alpha\) to be constant, then \(\sum_{i=1}^p w_i = \bar{w} \) will be bounded above by \(\sqrt{\frac{p}{\beta}} \sum_{i=1}^p \frac{n}{p^2} \frac{1}{\bar{w}^{-r}}\), which converges almost surely to a constant by [40]. In what follows, we only require the \(w_i\) terms to be almost surely bounded by a positive constant. That requirement is also satisfied if we choose any positive constant \(\alpha > 0\).

**Theorem V.2.** Let \(U^*_n\) and \(D^*_t\) have the same meaning as in the proof of Theorem V.1. Let \(x_1 = \int x_i P_n(dx)\) denote the mean of the \(j\)th feature and \(D_1 = \sum_{i=1}^n d(x_i, x_1)\). Then \(\exists d' \in \{1, \ldots, p\}\) such that \(D^*_t \leq D_{d'}\).

**Proof.** We prove the theorem using contradiction. Assuming the contrary, suppose, \(D^*_t > D_t \forall d \in \{1, \ldots, p\}\). Then,

\[
m_k(P_n) = \frac{1}{n} \sum_{i=1}^p (w_i^{(n)})^2 + \frac{\lambda}{p^2} |w_i| D_1 - \alpha \sum_{i=1}^p w_i^{(n)}
\]

\[
> \frac{1}{n} \sum_{i=1}^p (w_i^{(n)})^2 + \frac{\lambda}{p^2} |w_i| D_1 - \alpha \sum_{i=1}^p w_i^{(n)},
\]

which is a contradiction since \(m_k(Q) := \inf \{ \Phi(W, A, Q) | A \in \mathbb{R}^p, W \in \mathbb{R}^p \}\).

**Remark 4.** The following theorem illustrates that if \(\lambda\) is chosen inside the range \((0, \lambda_0)\), at least one feature weight is bounded below by a positive constant almost surely. This positive constant depends only on the underlying distribution and is thus denoted by \(C(P)\). Also note that \(\lambda_0\) depends on the underlying distribution of the datapoints.

**Theorem V.3.** There exists a constant \(\lambda_0 > 0 \) and \(d' \in \{1, \ldots, p\}\) such that \(\forall 0 < \lambda < \lambda_0, w_{d'} \geq c(P) > 0\) almost surely.

**Proof.** Let \(x_1 = \int x_i P_n(dx)\) denote the mean of the \(j\)th feature. Let \(D_1 = \sum_{i=1}^n d(x_i, x_1)\). \(D_t\) and \(D_{d'}\) have the same meaning as in the proof of Theorem V.1. Choose \(\alpha(P_n) \xrightarrow{a.s.} \alpha(P)\) as in Theorem V.2. Thus, \(D_{d'}^* \leq D_{d'}\). By the assumption of finite second moment, \(\frac{1}{2} \left( \frac{\lambda}{p^2} \right) \xrightarrow{a.s.} \sigma_{d'}^2\), where, \(\sigma_{d'}^2 = \mathbb{E}\left[ \left( E[(X - E(X))(X - E(X))']_d' \right) d'd' \right]\) is the population variance of the \(d'\)-th feature. Here \(X\) is any random variable having distribution \(P\). Again, \(\alpha(P_n) \xrightarrow{a.s.} \alpha(P)\) (by Theorem V.1). Since, \(S_{x,y}(x, y)\) is a continuous function in \(x, w_{d'}\),

\[
\frac{1}{\beta^2} S \left( \frac{\alpha(P_n)}{\sigma_{d'}^2} \frac{\lambda}{p^2} \right) \xrightarrow{a.s.} \frac{1}{\beta^2} S \left( \frac{\alpha(P)}{\sigma_{d'}^2} \frac{\lambda}{p^2} \right) \xrightarrow{a.s.} \frac{1}{\beta^2} S \left( \frac{\alpha(P)}{\sigma_{d'}^2} \frac{\lambda}{p^2} \right).
\]

We can choose \(\lambda_0 = \frac{1}{4} \frac{\sigma_{d'}^2}{\sigma_{d'}^2}\) and \(C(P) = \frac{1}{2} \left[ \frac{1}{\beta^2} S \left( \frac{\alpha(P)}{\sigma_{d'}^2} \frac{\lambda}{p^2} \right) \right]^{\frac{1}{\beta^2}}\). Thus \(C(P) > 0 \forall 0 < \lambda < \lambda_0\).

We are now ready to prove the main result of this section, i.e. the consistency theorem. The theorem essentially implies that if \(\alpha\) and \(\lambda\) are suitably chosen, the set of optimal cluster centroids and the optimal weights tends to the population optima in an almost sure sense.

**Theorem V.4.** Suppose that \(\int \|x\|^2 P(dx) < \infty\) and for each \(j = 1, \ldots, k\), there is a unique set \(\bar{A}(j)\) and a unique \(W_j \in \mathbb{R}^p\) such that \(\Phi(W_j, \bar{A}(j), P) = m_k(P)\) and \(\alpha\) and \(\lambda\) are chosen according to Theorem V.1 and V.3 respectively. Then \(A_n \xrightarrow{a.s.} \bar{A}(k)\) and \(W_n \xrightarrow{a.s.} \bar{W}(k)\). Moreover, \(\Phi(W_n, A_n, P_n) \xrightarrow{a.s.} \Phi(W(k), \bar{A}(k), P)\).

**Proof.** We will prove the theorem using the following steps.
Step 1. There exists \( M > 0 \) such that \( B(M) \) contains at least one point of \( A_n \) almost surely, i.e. there exists \( M > 0 \) such that

\[
P((\cup_{n=1}^{\infty} \cap_{m=n}^{\infty} \{ \omega : A_m(\omega) \cap B(M) \neq \emptyset \})) = 1.
\]

Proof of Step 1. Let \( r > 0 \) be such that \( B(r) \) has a positive \( P \)-measure. By our assumptions, \( \Phi(W_n, A_n, P_n) \leq \Phi(1_n, A_0, P_n) \) for any set \( A_0 \) containing at most \( k \) points. Choose \( A_0 = \{ 0 \} \). Then,

\[
\Phi(1_n, A_0, P_n) = \left( 1 + \frac{\lambda}{p^2} \right) \int \| x \|^2 P_n(dx) - a(P)(P_n).
\]

Thus,

\[
\Phi(1_n, A_0, P_n) \overset{a.s.}{\to} \left( 1 + \frac{\lambda}{p^2} \right) \int \| x \|^2 P(dx) - a(P).
\]

Let \( \Omega' = \{ \omega \in \Omega : \forall n \in \mathbb{N} \exists m \geq n \text{ s.t. } A_m \cap B(M) = \emptyset \} \). By the Axiom of Choice [50], for any \( \omega \in \Omega' \), there exists a sequence \( \{ n_k \}_{k \in \mathbb{N}} \) such that \( n_i < n_j \) for \( i < j \) and \( A_{n_k} \cap B_M(x_0) = \emptyset \). Now, for this sequence,

\[
\limsup_h \Phi(W_{n_h}, A_{n_h}, P_{n_h}) \geq \lim_h \left[ C(P)^\beta + \frac{\lambda}{p^2} C(P) \right] (M - r)^2 P_n(B(r)) - \alpha(P) \geq \liminf_h (\alpha(P)) \sum_{i=1}^p w_i^{(n_k)}.
\]

Thus,

\[
\limsup_h \Phi(W_{n_h}, A_{n_h}, P_{n_h}) \geq \left[ C(P)^\beta + \frac{\lambda}{p^2} C(P) \right] (M - r)^2 P(B(r)) - \alpha(P) \overset{a.s.}{\to} \left( 1 + \frac{\lambda}{p^2} \right) \int \| x \|^2 P(dx) - a(P).
\]

This would make \( \Phi(W_{n_h}, A_{n_h}, P_{n_h}) > \Phi(1_n, A_0, P_n) \) i.o., which is a contradiction.

Step 2. For \( n \) large enough, \( B(5M) \) contains all points of \( A_n \) almost surely, i.e.

\[
P((\cup_{n=1}^{\infty} \cap_{m=n}^{\infty} \{ \omega : A_m(\omega) \subset B(5M) \})) = 1.
\]

Proof of Step 2. We use induction for the proof of this step. We have seen from Step 1, the conclusions of this claim is valid. We assume this claim is valid for optimal allocation of \( 1, \ldots, k-1 \) cluster centroids.

Suppose \( A_k \) contains at least one point outside \( B(5M) \). Now if we delete this cluster centroid, at worst, the center \( a_1 \), which is known to lie inside \( B(M) \) might have to accept points that were previously assigned to cluster centroids outside \( B(5M) \). These sample points must have been at a distance at least \( 2M \) from the origin, otherwise, they would have been closer to the centroid \( a_1 \), than to any other centroid outside \( B(5M) \). Hence, the extra contribution to \( \Phi(\cdot, \cdot, P_n) \), due to deleting the centroids outside \( B(5M) \) is atmost

\[
\int \| x \|^2 \geq 2M \sum_{l=1}^p (w_l^{(n)})^2 \| x_l - a_l \|^2 P_n(dx) - \alpha(P_n) \sum_{l=1}^p w_l^{(n)} \\
\leq (1 + \frac{\lambda}{p^2}) \int \| x \|^2 \geq 2M \sum_{l=1}^p (x_l - a_l) \| x_l \|^2 P_n(dx) - \alpha(P_n) \sum_{l=1}^p w_l^{(n)} \\
\leq 2(1 + \frac{\lambda}{p^2}) \int \| x \|^2 \geq 2M \| x \|^2 P_n(dx) \\
\leq 4\left( 1 + \frac{\lambda}{p^2} \right) \int \| x \|^2 P_n(dx) \overset{a.s.}{\to} 0.
\]

Let \( A_n^* \) be obtained by deleting the centroids outside \( B(5M) \) of \( A_n \). Since \( A_n^* \) has almost \( k-1 \) points, we have \( \Phi(W_n, A_n^*, P_n) \geq \Phi(W_n, B_n, P_n) \), where \( W_n \) and \( B_n \) denote the optimal set of weights and optimal set of cluster centroids for \( k-1 \) centers respectively. Let \( \Omega'' = \{ \omega \in \Omega : \forall n \in \mathbb{N} \exists m \geq n, A_m(\omega) \not\subset B(5M) \} \). Now by Axiom of Choice, for any \( \omega \in \Omega'' \), there exists a sequence \( \{ n_k \}_{k \in \mathbb{N}} \) such that \( n_i < n_j \) for \( i < j \) and \( A_{n_k} \not\subset B(5M) \).

\[
m_{k-1}(P) \leq \liminf_h \Phi(W_{n_h}, A_{n_h^*}, P_{n_h}) \text{ a.s.} \\
\leq \limsup_h \Phi(W_{n_h}, A_{n_h^*}, P_{n_h}) \\
+ 4\left( 1 + \frac{\lambda}{p^2} \right) \int \| x \|^2 P_n(dx) \\
\leq \liminf_h \Phi(W, A, P_h) + 4\left( 1 + \frac{\lambda}{p^2} \right) \int \| x \|^2 P(dx),
\]

for any \( A \) having \( k \) or fewer points and for any \( W \in \mathbb{R}^P \). Choose \( A = \bar{A}(k) \) and \( W = W(k) \). Choose \( \epsilon > 0 \) such that \( m_k(P) + \epsilon < m_{k-1}(P) \). Choose \( M \) large enough such that \( 4\left( 1 + \frac{\lambda}{p^2} \right) \int \| x \|^2 P(dx) < \epsilon \). Thus, the last bound of Eqn 17 is less than \( \Phi(W(k), \bar{A}(k), P) + \epsilon = m_k(P) + \epsilon > m_{k-1}(P) \), which is a contradiction.

Hence, for \( n \) large enough, it suffices to search for \( A_n \) among the class of sets, \( \xi_k := \{ A \subset B(5M) | A \text{ contains } k \text{ or fewer points} \} \). For the final requirement on \( M \), we assume that \( M \) is large enough so that \( \xi_k \) contains \( \bar{A}(k) \). Under the topology induced by the Hausdorff metric, \( \xi_k \) is compact. Let \( \Gamma_k = [0, b] \times \ldots [0, b] \) (\( p \) times), where \( b \) is such that \( b > 1 \) and \( W(k) < b \) \( \forall A \in \{ 1, \ldots, p \} \). As proved in Theorem V.7, the map \( (W, A) \to \Phi(W, A, P) \) is continuous on \( (\Gamma_k \times \xi_k, \bar{N}) \). The function \( \Phi(\cdot, \cdot, P) \) has the property that given any neighbourhood \( N \) of \( \Phi(k, \bar{A}(k), P) \) (depending on \( k \)) \( \Phi(W, A, P) \geq \Phi(W(k), \bar{A}(k), P) + \eta \), for every \( (W, A) \in \Gamma_k \times \xi_k \).

Now by uniform SLLN (Theorem V.6), we have,

\[
\sup_{(W, A) \in \Gamma_k \times \xi_k} | \Phi(W, A, P_n) - \Phi(W, A, P) | \overset{a.s.}{\to} 0.
\]

We need to show that \( (W_n, A_n) \text{ eventually lies inside } N \). It is enough to show that \( \Phi(W_n, A_n, P_n) < \Phi(W(k), \bar{A}(k), P) + \eta \), eventually. This follows from

\[
\Phi(W_n, A_n, P_n) \leq \Phi(W(k), \bar{A}(k), P_n),
\]
\[ \Phi(W_n, A_n, P_n) - \Phi(W_n, A_n, P) \xrightarrow{a.s.} 0, \]

and

\[ \Phi(\bar{W}(k), \bar{A}(k), P_n) - \Phi(\bar{W}(k), \bar{A}(k), P) \xrightarrow{a.s.} 0. \]

Similarly for \( n \) large enough,

\[ \Phi(W_n, A_n, P_n) = \inf_{\{\Phi(W, A, P)\mid W \in \Gamma_k, A \in \xi_k\}} \xrightarrow{a.s.} \inf_{\{\Phi(W, A, P)\mid W \in \Gamma_k, A \in \xi_k\}} \Phi(W, A, P) = m_k(P). \]

\[ \square \]

### B. Uniform SLLN and continuity of \( \Phi(\cdot, \cdot, P) \)

In this section, we prove a uniform SLLN for the function \( \Phi(\cdot, \cdot, P) \) in Theorem V.5.

**Theorem V.5.** Let \( \mathcal{G} \) denote the family of all \( P \)-integrable functions of the form \( g_{W,A}(x) := \min_{x \in A} \sum_{l=1}^{p} (w_l^\beta + \frac{\lambda}{p^2}|w_l|)(x_l - a_l + \delta_l)^2 \), where \( A \in \xi_k \) and \( W \in \Gamma_k \). Then \( \sup_{g \in \mathcal{G}} \int g dP - \int g dP \xrightarrow{a.s.} 0. \)

**Proof.** It is enough to show that for every \( \epsilon > 0 \), \( \exists \) a finite class of functions \( \mathcal{G}_\epsilon \), such that for each \( g \in \mathcal{G} \), there exists functions \( \hat{g}, \tilde{g} \in \mathcal{G}_\epsilon \) such that \( \int (\hat{g} - \tilde{g}) dP < \epsilon. \)

Let \( D_{\delta_1} \) be a finite subset of \( B(5M) \) such that every point of \( B(5M) \) lies within a \( \delta_1 \) distance of at least one point of \( D_{\delta_1} \). Also let \( D_{\delta_2} \) be a finite subset of \( \Gamma_k \) such that every point of \( \Gamma_k \) is within a \( \delta_2 \) distance of at least one point of \( D_{\delta_2} \). \( \delta_1 \) and \( \delta_2 \) will be chosen later. Let, \( \xi_k, \delta_k = \{ A \in \xi_k \mid A \subset D_{\delta_1} \} \) and \( \Gamma_k, \delta_k = \{ W \in \Gamma_k \mid W \subset D_{\delta_2} \} \). Take \( \mathcal{G}_\epsilon \) to be the class of functions of the form

\[ \min_{A \in A'} \sum_{l=1}^{p} ((w_l + \delta_2)^2 + \frac{\lambda}{p^2}|w_l + \delta_2|)(x_l - a_l + \delta_1)^2, \]

where \( A' \) ranges over \( \xi_k, \delta_1 \) and \( W \) ranges over \( \Gamma_k, \delta_2 \).

Given \( A = \{a_1, \ldots, a_k\} \in \xi_k \), there exists \( A^0 = \{a_{i_1}^{(0)}, \ldots, a_{i_k}^{(0)}\} \in \xi_k, \delta_1 \), such that \( H(A, A') < \delta_1 \) (choose \( a_{i_l}^{(0)} \) such that \( |a_l - a_{i_l}^{(0)}| < \delta_1 \)). Also note that given \( W \in \Gamma_k \), there exists \( W^{(0)} \in \Gamma_k, \delta_2 \). For given \( g_{W,A} \in \mathcal{G} \), take,

\[ \hat{g}_{W,A} := \min_{A \in A^0} \sum_{l=1}^{p} ((\max_{l=1}^{p} |w_l^{(0)} - \delta_0, 0|)^2 + \frac{\lambda}{p^2}|\max_{l=1}^{p} |w_l^{(0)} - \delta_0, 0|\max_{l=1}^{p} (x_l - a_l - \delta_1, 0)|^2 \]

and

\[ \tilde{g}_{W,A} := \min_{A \in A^0} \sum_{l=1}^{p} ((\max_{l=1}^{p} |w_l^{(0)} + \delta_0, 0|)^2 + \frac{\lambda}{p^2}|\max_{l=1}^{p} |w_l^{(0)} + \delta_0, 0|\max_{l=1}^{p} (x_l - a_l + \delta_1, 0)|^2 \]

Clearly, \( \hat{g}_{W,A} \leq g_{W,A} \leq \tilde{g}_{W,A} \). Now by taking \( R > 5M \), we have

\[ \int (\hat{g}_{W,A} - \tilde{g}_{W,A}) dP(dx) \]

\[ \leq \sum_{l=1}^{k} \int\left( \sum_{l=1}^{p} ((\max_{l=1}^{p} |w_l^{(0)} + \delta_0, 0|)^2 + \frac{\lambda}{p^2}|\max_{l=1}^{p} |w_l^{(0)} + \delta_0, 0|\max_{l=1}^{p} (x_l - a_l + \delta_1, 0)|^2 \]

\[ + \frac{\lambda}{p^2}|\max_{l=1}^{p} |w_l^{(0)} + \delta_0, 0|\max_{l=1}^{p} (x_l - a_l - \delta_1, 0)|^2 \right) P(dx) \]

\[ \leq \kappa \sup_{\lambda > \gamma M} \sup_{\lambda < 5 \gamma M} \sup_{\lambda < L} \left( (\max_{l=1}^{p} |w_l^{(0)} + \delta_0, 0|)^2 + \frac{\lambda}{p^2}|\max_{l=1}^{p} |w_l^{(0)} + \delta_0, 0|\max_{l=1}^{p} (x_l - a_l + \delta_1, 0)|^2 \right) \]

\[ \int (\hat{g}_{W,A} - \tilde{g}_{W,A}) dP(dx) \]

The second term can be made smaller than \( \epsilon/2 \) if \( R \) is made large enough. Now appealing to the uniform continuity of the function \( (\max_{l=1}^{p} |w_l^{(0)} + \delta_0, 0|)^2 + \frac{\lambda}{p^2}|\max_{l=1}^{p} |w_l^{(0)} + \delta_0, 0|\max_{l=1}^{p} (x_l - a_l + \delta_1, 0)|^2 \) on bounded sets, we can find \( \delta_1 \) and \( \delta_2 \) small enough such that the first term is less than \( \epsilon/2 \). Hence the result. \( \square \)

**Theorem V.6.** Let \( \mathcal{G} \) denote the family of all \( P \)-integrable functions of the form \( g_{W,A}(x) := \min_{x \in A} \sum_{l=1}^{p} (w_l^\beta + \frac{\lambda}{p^2}|w_l|)(x_l - a_l - \delta_0)^2 \), where \( A \in \xi_k \) and \( W \in \Gamma_k \). Let \( g_{W,A,P_n}(x) = \min_{x \in A} \sum_{l=1}^{p} (w_l^\beta + \frac{\lambda}{p^2}|w_l|)(x_l - a_l - \delta_0)^2 - \alpha(P_n) \sum_{l=1}^{p} w_l \). Then the following holds:

1. \( \int g_{W,A,P_n}(x) dP_n(x) = \Phi(W, A, P_n) \).
2. \( \sup_{W,A} \int g_{W,A,P_n} P_n - \int g_{W,A,P} P_n \xrightarrow{a.s.} 0. \)

**Proof.** Part (1) follows trivially. We only prove part (2).

\[ \left| \int g_{W,A,P_n} P_n - \int g_{W,A,P} P \right| \]

\[ \leq \left| \int g_{W,A,P_n} P_n - \int g_{W,A,P} P + \sum_{l=1}^{p} w_l |\alpha(P_n) - \alpha(P)| \right| \]

\[ \leq \left| \int g_{W,A,P_n} P_n - \int g_{W,A,P} P + b|\alpha(P_n) - \alpha(P)| \right| \]

Hence,

\[ \sup_{W,A} \int g_{W,A,P_n} P_n - \int g_{W,A,P} P \]

\[ \leq \sup_{W,A} \int g_{W,A,P_n} P_n - \int g_{W,A,P} P + \sup_{W,A} b|\alpha(P_n) - \alpha(P)| \]

\[ = \sup_{W,A} \int g_{W,A,P_n} P_n - \int g_{W,A,P} P + b|\alpha(P_n) - \alpha(P)| \xrightarrow{a.s.} 0. \]

The last almost sure convergence of Eqn 20 is true since the first term converges to 0 a.s. (Theorem V.5) and the second term converges to 0 a.s. (Theorem V.1). \( \square \)

Before proceeding any further let us first define two function classes.
The first term can be made smaller than \( \varepsilon \). Fix \( \Gamma \).

**Lemma V.1.** The family of functions \( \mathcal{F}_1 \) is equicontinuous.

**Proof.** See Appendix A-G.

**Lemma V.2.** The family of functions \( \mathcal{F}_2 \) is equicontinuous.

**Proof.** See Appendix A-H.

Before we state the next theorem, note that, the map \( (W, A) \rightarrow \Phi(W, A, P) \) is from \( \Gamma_k \times \xi_k \rightarrow \mathbb{R} \). \( \Gamma_k \times \xi_k \) is a metric space with the metric

\[
d_0((W_1, A_1), (W_2, A_2)) := \|W_1 - W_2\|_2^2 + H(A_1, A_2),
\]

where, \( H(\cdot, \cdot) \) is the Hausdorff metric.

**Theorem V.7.** The map \( (W, A) \rightarrow \Phi(W, A, P) \) is continuous on \( \Gamma_k \times \xi_k \).

**Proof.** Fix \((W_0, A_0) \in \Gamma_k \times \xi_k\). From triangle inequality, we get,

\[
|\Phi(W, A, P) - \Phi(W_0, A_0, P)| \leq |\Phi(W, A, P) - \Phi(W, A_0, P)| + |\Phi(W, A_0, P) - \Phi(W_0, A_0, P)|.
\]

The first term can be made smaller than \( \epsilon/2 \) if \( A \) is chosen close enough to \( A_0 \) (in Hausdorff sense). This follows from Lemma V.1. The second term can also be made smaller than \( \epsilon/2 \) if \( W \) is chosen close enough to \( W_0 \) (in Euclidean sense). This follows from Lemma V.2. Hence the result.

**VI. EXPERIMENTAL RESULTS**

In this section, we present the experimental results on various real-life and synthetic datasets. All the experiments were undertaken on an HP laptop with Intel(R) Core(TM) i3-5010U 2.10 GHz processor, 4GB RAM, 64-bit Windows 8.1 operating system. The datasets and codes used in the experiments are publicly available from https://github.com/SaptarshiC98/lwk-means.

**A. Regularization Paths**

In this section we discuss the concept of regularization paths in the context of \( LW-k \)-means. The term regularization path was first introduced in the context of lasso [52]. We introduce two new concepts called mean regularization path and median regularization path in the context of \( LW-k \)-means. Suppose we have a sequence \( \{\lambda_i\}_{i=1}^n \) of length \( n \) of \( \lambda \) values. After setting \( \lambda = \lambda_i \), we run the \( LW-k \)-means algorithm \( t \) times (say). Hence, we have a set of \( t \) weights, \( W_1, \ldots, W_t \). Hence we can take the estimates of the average weight to be the mean of these \( t \) vectors. Let this estimate be \( W^*_i \). Thus, for each value \( \lambda_i \), we get the mean weights \( W^*_i \). This sequence of \( W^*_i \)’s, \( \{W^*_i\}_{i=1}^t \) is defined to be the mean regularization path. Similarly one can define the median regularization path by taking the median of the weights instead of the mean.

**B. Case Studies in Microarray Datasets**

A typical microarray dataset has several thousands of genes and fewer than 100 samples. We use the Leukemia and Lymphoma datasets to illustrate the effectiveness of the \( LW-k \)-means algorithm. We do not include \( k \)-means and IF-HCT-PCA in the following examples since both the algorithms do not perform feature weighting.

1) Example 1: The Leukemia dataset consists of 3571 gene expressions and 72 samples. The dataset was collected by Golub et al. [53]. We run the \( LW-k \)-means algorithm 100 times for each value of \( \lambda \) and note the average value of the different feature weights. We also note the average CER for different \( \lambda \) values.

In Fig. 3, we show the regularization paths for the Leukemia dataset. In Fig. 5, we plot the average misclassification error rate for the same dataset. It is evident from Fig. 5, that as we decrease \( \lambda \) the average CER drops down abruptly around \( \lambda = 0.52 \times 10^{-3} \). From Fig. 3, we observe that only few features are selected on an average, 10 for \( \lambda = 0.6 \times 10^{-3} \) when \( \lambda > 0.5 \times 10^{-3} \). Possibly these features do not completely reveal the cluster structure of the dataset. As \( \lambda \) is decreased, the CER remains more or less stable. We also run the \( WK \)-means and sparse \( k \)-means algorithms 100 times (we performed the experiment 100 times to get a more consistent view of the feature weight) on the Leukemia dataset and compute the median of the weights for different features. In Fig. 4a and 4b, we plot these feature weights against the corresponding features for \( WK \)-means and sparse \( k \)-means respectively. It can be easily seen that \( WK \)-means and sparse \( k \)-means do not assign zero weight to all the features. In Fig. 4c, we plot corresponding average (median) feature weights assigned by the \( LW-k \)-means algorithm. It can be easily observed that \( LW-k \)-means assigns zero feature weights to many of the features.

2) Example 2: The Lymphoma dataset consists of 4026 gene expressions and 62 samples. The dataset was collected by Alizadeh et al. [54]. We run the \( LW-k \)-means algorithm 100 times for each value of \( \lambda \) and note both the mean and median values of the different feature weights. We also note the both the mean and median CER’s for different \( \lambda \) values.
mean CER curve is less smooth than the median CER curve. We also see from Fig. 8, that the median regularization path is smoother relative to the mean regularization paths and in Fig. 8, we plot the average (both mean and median) CER for different values of $\lambda$.

In Fig. 6, we plot the average (both mean and median) regularization paths and in Fig. 8, we plot the average (both mean and median) CER for different values of $\lambda$. We observe that the median regularization path is smoother relative to the mean regularization path. We also see from Fig. 8, that the mean CER curve is less smooth than the median CER curve.

The non-smooth mean regularization paths indicate a few cases where due to a bad initialization, the solutions got stuck at a local minimum instead of the global minima of the objective function. During our experiments we observed that there were a few times when we got a bad initialization for cluster centroids, thus adversely affecting the mean regularization path and mean CER. On the other hand, the median is more robust against outliers and thus the corresponding regularization paths and CER are smoother compared to those corresponding to the mean. From Fig. 8b, we observe that there is a sudden drop in the misclassification error rate around $\lambda = 6.2 \times 10^{-4}$ and it remains stable when $\lambda$ is further decreased. This might be due to the fact that when $\lambda$ is high, no features are selected and as $\lambda$ is decreased to around $\lambda = 6.2 \times 10^{-4}$, the relevant features are selected. Also note that these features have higher weights than other features, even when $\lambda$ is quite small. The above facts indicate that indeed the $LW$-k-means detects the features which contain the cluster structure of the data.

We also run the $WK$-means and sparse $k$-means algorithms 100 times on the Lymphoma dataset and compute the median of the weights for different features. In figures 7a and 7b, we plot these feature weights against the corresponding features for $WK$-means and sparse $k$-means respectively. It can be easily seen that $WK$-means and sparse $k$-means do not assign zero feature weights and thus in effect do not perform a feature selection. In Fig. 7c, we plot the corresponding average (median) feature weights assigned by the $LW$-$k$-means algorithm. It is easily observed that $LW$-k-means assigns zero feature weights to many of the features.

C. Choice of $\lambda$

Let us illustrate with the example of the synthetic toy1 dataset (generated by us) which has 10 features of which
only the first 4 are the distinguishing ones. The dataset is available from https://github.com/SaptarshiC98/lwk-means. We take different values of $\lambda$ and iterate the $LW$-$k$-means algorithm 20 times and take the average value of the weights assigned to different features by the algorithm. Fig. 9 shows the average value of the feature weights for different values of $\lambda$. This figure is similar to the regularization paths for the lasso [52].

Here the key observation is that as $\lambda$ increases, the weights decrease on an average and eventually becomes 0. From Fig. 9, it is evident that the $LW$-$k$-means correctly identifies that the first 4 features are important for revealing the cluster structure for the dataset. Here, an appropriate guess for $\lambda$ might be any value between 0.1 and 0.5. It is clear from this toy example, if the dataset has a proper cluster structure, after a threshold, increasing $\lambda$ slightly does not reduce the number of feature selected.

D. Experimental Results on Real-life Datasets

1) Description of the Datasets: The datasets are collected from the Arizona State University (ASU) Repository (http://featureselection.asu.edu/datasets.php), Keel Repository [55], and The UCI Machine Learning Repository [56]. In Table IV, a summary description of the datasets is provided. The COIL2, ORL2, YALE2 datasets are constructed by taking the first 144, 20 and 22 instances from the COIL20, ORL, and Yale image datasets respectively. The Breast Cancer and Lung Cancer datasets were analyzed and grouped into two classes in [57]. A description of all the genomic datasets can be found in [32].
TABLE IV: Description of the Real-life Datasets

| Dataname      | Source               | k | n  | p    |
|---------------|----------------------|---|----|------|
| Brain Cancer  | Pomeroy [32]         | 5 | 42 | 5597 |
| Leukemia      | Gordon et al. [58]   | 2 | 72 | 3571 |
| Lung Cancer   | Bhattacharjee et al. [59] | 2 | 203| 12,600|
| Lymphoma      | Alizadeh et al. [54] | 3 | 62 | 4026 |
| SuCancer      | Su et al. [32]       | 2 | 174| 7909 |
| Wine          | Keel                 | 3 | 178| 13   |
| COIL2         | ASU                  | 5 | 360| 1024 |
| ORL2          | ASU                  | 2 | 20 | 1024 |
| YALE2         | ASU                  | 2 | 22 | 1024 |
| ALLAML        | ASU                  | 2 | 72 | 7129 |
| Appendicitis  | Keel                 | 2 | 106| 7    |
| WDBC          | Keel                 | 2 | 569| 30   |
| GLIOMA        | ASU                  | 4 | 50 | 4434 |

2) Performance Index: For comparing the performance of various algorithms on the same dataset, we use the Classification Error Rate (CER) [60] between two partitions $T_1$ and $T_2$ of the patterns as the cluster validation index. This index measures the mismatch between two partitions of a given set of patterns with a value 0 representing no mismatch and a value 1 representing complete mismatch.

3) Computational Protocols: The following computational protocols were followed during the experiment.

Algorithms under consideration: The $LW$-$k$-means algorithm, $WK$-$k$-means algorithm [20], the IF-HCT-PCA algorithm [32] and the sparse $k$-means algorithm [28].

We set the value of $\beta$ to 4 for both the $LW$-$k$-means and $WK$-$k$-means algorithms throughout the experiments. The value of $\lambda$ was chosen by performing some hand-tuned experiments. To choose the value of $\alpha$, we first run the $k$-means algorithm until convergence. Then use the value of $\alpha = \frac{1}{\sum_{i=1}^{n} \sum_{j=1}^{k} u_{i,j} d(x_{i,j}, z_{j,l})}$. Here $D_1 = \sum_{i=1}^{n} \sum_{j=1}^{k} u_{i,j} d(x_{i,j}, z_{j,l})$. This is the value of the Lagrange multiplier in $WK$-$k$-means algorithm [20].

Performance comparison: For each of the last three algorithms, we start with a set of randomly chosen centroids and iterate until convergence. We run each algorithm independently 20 times on each of the datasets and calculate the CER. We standardized the datasets prior to applying the algorithms for all the five algorithms.

E. Discussions

In this section, we discuss some of the results obtained by using $LW$-$k$-means algorithm for clustering various datasets. In Tables V and VI, we report the mean CER obtained by $LW$-$k$-means, $WK$-$k$-means, $k$-means, IF-HCT-PCA and sparse $k$-means. The values of $\lambda$ for $LW$-$k$-means are also mentioned in both Tables V and VI.

In Table V, we report the mean CER obtained by $LW$-$k$-means, $WK$-$k$-means, $k$-means, IF-HCT-PCA, and sparse $k$-means. It is evident from Table V that the $LW$-$k$-means outperforms three of the state of the art algorithms (except sparse $k$-means) in all the synthetic datasets. Though the $k$-means and $LW$-$k$-means give the same CER for the synthetic datasets, the time taken by sparse $k$-means is much more compared to $LW$-$k$-means. Also for some of the synthetic datasets, sparse $k$-means fails to identify all the relevant feature as discussed in section VI-F.

As revealed from Table VI, the $LW$-$k$-means outperforms the IF-HCT-PCA in 11 of the 13 real-life datasets. In Table VII, we note the average time taken by each of the $LW$-$k$-means, IF-HCT-PCA and sparse $k$-means. Computation of the threshold by Higher Criticism thresholding increases the runtime of the IF-HCT-PCA algorithm. Also the computation of the tuning parameter via the gap statistics increases the runtime of the sparse $k$-means algorithm. We also note the average number of selected features for the three algorithms in Table VII. It is clear from Table VII, $LW$-$k$-means also achieves better results in much lesser time compared to that of IF-HCT-PCA.

From Table VI, it can be seen that the $LW$-$k$-means outperforms the sparse $k$-means in all the 6 microarray datasets. For the other datasets, $LW$-$k$-means and sparse $k$-means give comparable results. Also, it is clear from Table VII, $LW$-$k$-means achieves it in much lesser time compared to sparse $k$-means. Also note from Table VII, the sparse $k$-means gives non-zero weights to all the features except for $YALE2$ and $ORL2$ datasets. Thus, in effect, for all the other datasets, sparse $k$-means does not perform feature selection. It can also be seen that $LW$-$k$-means achieves almost the same level of accuracy using much smaller number of features for the two aforementioned datasets.

F. Discussions on Feature Selection

In this section, we compare the feature selection aspects between $LW$-$k$-means, IF-HCT-PCA and sparse $k$-means algorithms. We only discuss compare the three algorithms for synthetic datasets, since the importance of each feature is known beforehand.

Before we proceed, we define a new concept called the ground truth relevance vector of a dataset. The ground truth relevance vector of a dataset $D$ is defined as, $T^*_k = (t_1, \ldots, t_p)$, where $t_i = 1$ if $i^{th}$ feature is important in revealing the cluster structure of the dataset, $t_i = 0$, otherwise. In general, this vector is not known beforehand. The objective of any feature selection algorithm is to estimate it.

Similarly we define relevance vector of a feature selection algorithm $A$ and a dataset $D$. It is a binary vector assigned by feature selection algorithm $A$ to the dataset $D$ and is defined by, $T^A_k = (t_1, \ldots, t_p)$, where $t_i = 1$ if $i^{th}$ feature is selected by algorithm $A$, $t_i = 0$, otherwise.

For the synthetic datasets, we already know the ground truth relevance vector for these datasets. We use Matthews Correlation Coefficient (MCC) [61] to compare between the ground truth relevance vector and the relevance vector assigned by the algorithms $LW$-$k$-means, IF-HCT-PCA and sparse $k$-means. MCC lies between 1 and +1. A coefficient of +1 represents a perfect agreement between the ground truth and the algorithm with respect to feature selection, 1 indicates total disagreement between the same and 0 denotes no better than random feature selection. The MCC between the ground truth relevance vector and the relevance vector assigned by the algorithms $LW$-$k$-means, IF-HCT-PCA, and sparse $k$-means is shown in Table
### Table V: CER for Synthetic Datasets

| Datasets  | LW-\(k\)-means(\(\lambda\)) | WK-\(k\)-means | \(k\)-means | IF-HCT-PCA | Sparse \(k\)-means |
|-----------|------------------------------|----------------|------------|------------|-----------------|
| s1        | 0 (0.04)                    | 0.0642         | 0.3012     | 0.3333     | 0               |
| s2        | 0 (0.02)                    | 0.1507         | 0.1398     | 0.34       | 0               |
| s3        | 0 (0.02)                    | 0.0401         | 0.0865     | 0.6667     | 0               |
| s4        | 0 (0.007)                   | 0.087          | 0.2000     | 0.3167     | 0               |
| s5        | 0 (0.007)                   | 0.1065         | 0.1172     | 0.3267     | 0               |
| s6        | 0 (0.002)                   | 0.0465         | 0.1537     | 0.3567     | 0               |
| s8        | 0 (0.0005)                  | 0.1272         | 0.0653     | 0.3067     | 0               |
| hd6       | 0 (0.005)                   | 0.2567         | 0.3062     | 0.34       | 0               |
| sim1      | 0 (0.1)                     | 0.0203         | 0.0452     | 0.333      | 0               |
| f1        | 0.0267 (0.0019)             | 0.6158         | 0.6138     | 0.3700     | 0.5938333       |
| f5        | 0.0100 (0.0006)             | 0.6337         | 0.6260     | 0.2767     | 0.5328333       |

### Table VI: CER for Real-life Datasets

| Datasets  | LW-\(k\)-means(\(\lambda\)) | WK-\(k\)-means | \(k\)-means | IF-HCT-PCA | Sparse \(k\)-means |
|-----------|------------------------------|----------------|------------|------------|-----------------|
| Brain     | 0.2381 (0.0005)              | 0.4452         | 0.2865     | 0.2624     | 0.2857          |
| Leukemia  | 0.0278 (0.0005)              | 0.2419         | 0.2789     | 0.0695     | 0.2778          |
| Lung Cancer | 0.2167 (0.000162)           | 0.4672         | 0.4361     | 0.2172     | 0.3300          |
| Lymphoma  | 0.0161 (0.0006)              | 0.3266         | 0.3877     | 0.0657     | 0.2741          |
| SuCancer  | 0.4770 (0.0003)              | 0.4822         | 0.4772     | 0.5000     | 0.4770          |
| Wine      | 0.0506 (0.1)                 | 0.0896         | 0.3047     | 0.1404     | 0.0506          |
| COIL5     | 0.4031 (0.001)               | 0.4365         | 0.4261     | 0.4889     | 0.3639          |
| ORL2      | 0.0500 (0.005)               | 0.1053         | 0.1351     | 0.3015     | 0.0512          |
| YALE2     | 0.1364 (0.002)               | 0.1523         | 0.1364     | 0.4545     | 0.1364          |
| ALLAML    | 0.2500 (0.0002)              | 0.3486         | 0.2562     | 0.2693     | 0.2546          |
| Appendicitis | 0.1981 (0.17)            | 0.3642         | 0.3156     | 0.1509     | 0.1905          |
| WDBC      | 0.0756 (0.0001)              | 0.0758         | 0.0901     | 0.1494     | 0.0810          |
| GLIOMA    | 0.4 (0.0005)                 | 0.424          | 0.442      | 0.6        | 0.4             |

### Table VII: Comparison between LW-\(k\)-means and IF-HCT-PCA

| Datasets  | Number of Selected Features | Time (in seconds) |
|-----------|-----------------------------|-------------------|
| LW-\(k\)-means | IF-HCT-PCA | Sparse \(k\)-means | LW-\(k\)-means | IF-HCT-PCA | Sparse \(k\)-means |
| Brain     | 14                          | 429               | 5597        | 2.407632   | 186.951822       | 324.26 |
| Leukemia  | 28                          | 213               | 3571        | 1.008672   | 48.983883        | 0      |
| Lung Cancer | 148                        | 418               | 12600       | 0.219742   | 273.263245       | 4.49   |
| Lymphoma  | 32                          | 44                | 4026        | 0.156323   | 14.038397        | 43.41  |
| SuCancer  | 7909                        | 6                 | 7909        | 28         | 0.0161           | 0.0465 |
| Wine      | 13                          | 4                 | 13          | 62         | 0.2172           | 94.39  |
| COIL5     | 332.2                       | 441               | 1024        | 0.0267     | 0.1272           | 0.0465 |
| ORL2      | 92                          | 324               | 4358        | 0.156323   | 14.038397        | 43.41  |
| YALE2     | 33                          | 31                | 159         | 0.0267     | 0.1272           | 0.0465 |
| ALLAML    | 357                         | 213               | 7129        | 0.156323   | 14.038397        | 43.41  |
| GLIOMA    | 77                          | 50                | 4358        | 0.156323   | 14.038397        | 43.41  |
| Appendicitis | 5                     | 7                 | 7           | 0.241305   | 0.1152437        | 2.87   |
| WDBC      | 30                          | 13                | 30          | 0.30246    | 0.1152437        | 2.87   |

### VIII. Simulation Study

From Table VIII, it is clear that LW-\(k\)-means correctly identifies all the relevant features and thus leads to an MCC of +1 for each of the synthetic datasets, whereas, IF-HCT-PCA performs no better than a random feature selection. For the sparse \(k\)-means algorithm, it identifies only a subset of the relevant features as important for datasets s2, s3, s4, s5, s6, s7 and correctly identifies all of the features in only datasets s1, hd6, and sim1. Also for datasets f1 and f5, the sparse \(k\)-means algorithm performs no better than random selection of the features.

### A. Example 1

We simulated 50 datasets each of which have 4 clusters consisting of 100 points each. Let \(X_i\) be a random point from the \(i^{th}\) cluster, where \(i \in \{1, 2, 3, 4\}\). Let \(X_i = (X^{(1)}_i, X^{(2)}_i)'\). The dataset is simulated as follows.

- \(X^{(1)}_i\) are i.i.d from \(N(0, 1)\).
- \(X^{(2)}_i\) are i.i.d from \(N(0, 1)\).
- \(X^{(2)}_i\) are i.i.d from \(N(7, 1)\).
• $X_2^{(2)}$ are i.i.d from $\mathcal{N}(2,1)$.
• $X_1^{(3)}$ are i.i.d from $\mathcal{N}(13,1)$.
• $X_2^{(3)}$ are i.i.d from $\mathcal{N}(-2,1)$.
• $X_1^{(4)}$ are i.i.d from $\mathcal{N}(19,1)$.
• $X_2^{(4)}$ are i.i.d from $\text{Unif}(10,10)$.

We run the sparse $k$-means and $LW$-$K$-means algorithms 10 times on each dataset and noted the average of the feature weights. We do this procedure on each of the 50 datasets. In Fig. 10, we plot the histogram for the features $x_1$ and $x_2$. From Fig. 10, it is clear that feature $x_1$ has a cluster structure and feature $x_2$ does not. In Fig. 11, we plot the boxplot of the average weights assigned by the $LW$-$k$-means and sparse $k$-means algorithms to features $x_1$ and $x_2$ for all the 50 datasets. Fig. 11 shows that sparse $k$-means assigns a feature weight of 0.32 to the unimportant feature $x_2$, whereas $LW$-$k$-means assigns $x_2$, zero feature weight and hence is capable of proper feature selection.

![Fig. 10: Histogram of features $x_1$ and $x_2$ of the W2 dataset. Clearly Feature $x_1$ has a cluster structure and feature $x_2$ doesn’t.](image)

Thus each of the datasets has only the first 50 features relevant and the other features irrelevant. For each of these 70 datasets, we run $LW$-$k$-means (with $\lambda = 0.005$) and $WK$-means 40 times and note the average (mean) weights assigned to different features by both the $LW$-$k$-means and $WK$-means algorithms.

In Fig. 12a, we show the boxplot of the average weights assigned by the $WK$-means algorithm to feature 1 to 50 for all the 70 datasets. In Fig. 12b, we show the corresponding boxplot for the $LW$-$k$-means algorithm. Fig. 12 clearly show a lesser variability for the weights assigned by $LW$-$k$-means compared to that of $WK$-means. In Fig. 13, we plot the corresponding boxplot for the rest of the features. For space constraints, we only plotted the boxplots corresponding to features 500 to 550 for the $WK$-means algorithm. From Fig. 13a it is clear that the average weights assigned by $WK$-means for the irrelevant features are somewhat close to zero but not exactly zero. On the other hand, the average weights assigned by $LW$-$k$-means for the irrelevant features are exactly equal to zero as shown in fig. 13b.

B. Example 2

We simulated 70 datasets each of which have 3 clusters consisting of 100 points each. Let $X_i$ be a random point from the $i^{th}$ cluster, where $i \in \{1,2,3\}$. Let $X_i = (X_1^{(i)}, \ldots, X_{100}^{(i)})$. The datasets are simulated as follows.

• $X_1^{(1)}$ are i.i.d from $\mathcal{N}(0,1) \ \forall j \in \{1, \ldots, 50\}$.
• $X_2^{(1)}$ are i.i.d from $\mathcal{N}(5,1) \ \forall j \in \{1, \ldots, 50\}$.
• $X_1^{(2)}$ are i.i.d from $\mathcal{N}(10,1) \ \forall j \in \{1, \ldots, 50\}$.
• $X_2^{(2)}$ are i.i.d from $\chi^2(5) \ \forall j \in \{51, \ldots, 1000\}$.
• $X_1^{(3)}$ is independent of $X_1^{(h)} \ \forall i, h \in \{1,2,3\}$ and $\forall j, k \in \{1, \ldots, 1000\}$ such that $(i,j) \neq (h,k)$.

VIII. CONCLUSION AND FUTURE WORKS

In this paper, we introduced an alternative sparse $k$-means algorithm based on the Lasso penalization of feature weighting. We derived the expression of the solution to the $LW$-$k$-means objective, theoretically, using KKT conditions of optimality. We also proved the convergence of the proposed algorithm. Since $LW$-$k$-means does not make any distributional assumptions of the given data, it works well even when the irrelevant
features, the 70 datasets. The boxplot shows a lesser variability for the features 51 to 1000 for all the 70 datasets. The boxplot shows that sparse k-means gives a lesser variability for the features 1 to 50 for all the 70 datasets. We validated our claim by performing detailed experiments on 9 synthetic and 13 real-life datasets. We also undertook a simulation study to find out the variability of the feature weights assigned by the LW-k-means and WK-means and found that LW-k-means always assigns zero weight to the irrelevant features for the appropriate value of λ. We also proposed an objective method to choose the value of the tuning parameter α in the algorithm.

Some possible extension of the proposed method might be to extend it to fuzzy clustering, to give a probabilistic interpretation of the feature weights assigned by the proposed algorithm and also to use different divergence measures to enhance the performance of the algorithm. One can also explore the possibility to prove the strong consistency of the proposed algorithm for different divergence measures, prove the local optimality of the obtained partial optimal solutions and also to choose the value of λ in an user independent fashion.

APPENDIX A

PROOFS OF VARIOUS THEOREMS AND LEMMATA OF THE PAPER

A. Proof of Theorem IV.1

Proof. Clearly, \( P(W) = h(W) + g(W) \), where

\[
h(W) = \frac{1}{n} \sum_{l=1}^{p} w_l^\beta D_l^0 - \alpha \sum_{l=1}^{p} w_l
\]

and

\[
g(W) = \frac{\lambda}{np^2} \sum_{l=1}^{p} |w_l| D_l^0.
\]

Now, \( \frac{\partial^2 h}{\partial w^2} = \beta(\beta - 1)w_l^{\beta - 2}D_l \geq 0 \). Hence \( h(W) \) is convex. It is also easy to see that \( g(W) \) is convex. Hence \( P(W) \) being the sum of two convex functions is convex. \( \square \)

B. Proof of Theorem IV.2

Proof. By Theorem IV.1, the objective function in 11 is convex. Let \((w_1, t_1)\) and \((w_2, t_2)\) satisfy constraints 13 and 14. Let \( \gamma \in (0, 1) \), \( t = \gamma t_1 + (1 - \gamma)t_2 \) and \( w = \gamma w_1 + (1 - \gamma)w_2 \). Then, \( t - w = \gamma(t_1 - w_1) + (1 - \gamma)(t_2 - w_2) \geq 0 \) and \( t + w = \gamma(t_1 + w_1) + (1 - \gamma)(t_2 + w_2) \geq 0 \). Hence \((w, t)\) satisfy constraints 13 and 14 and the constraint set of problem \( P_2^* \) is convex. The Hessian of the objective function in 12 is

\[
H(w, t) = \frac{1}{n} \begin{bmatrix} \beta(\beta - 1)w^{\beta - 2}D & 0 \\ 0 & 0 \end{bmatrix}
\]

which is clearly positive semi-definite. Hence the objective function of problem \( P_2^* \) is convex. Thus, any local minimizer of problem \( P_2^* \) is also a global minimizer.

Since \((w^*, t^*)\) is a local (hence global) minimizer of problem \( P_2^* \), for all \((w, t)\) that satisfy Eqn 13 and 14,

\[
\frac{1}{n} w^\beta D - \alpha w^* + \lambda \frac{1}{np^2} t^* \leq \frac{1}{n} w^\beta D - \alpha w + \lambda \frac{1}{np^2} t D. \tag{21}
\]

Taking \( w = w^*_1 \) and \( t = |w^*_1| \) in Eqn 21, we get,

\[
\frac{1}{n} w^\beta D - \alpha w^* + \lambda \frac{1}{np^2} |w|^* D \leq \frac{1}{n} w^*_1 w^\beta D - \alpha w^*_1 + \lambda \frac{1}{np^2} |w^*_1| D. \tag{22}
\]
Again, since \( w^*_1 \) is a solution to problem \( P_1^* \),
\[
\frac{1}{n} w_1^{*\beta} D - \alpha w_1^* + \frac{\lambda}{np^2} |w_1^*| D \leq \frac{1}{n} w_1^{*\beta} D - \alpha w_1^* + \frac{\lambda}{np^2} |w_1^*| D.
\] (23)
Adding Eqn 22 and 23, we get
\[
t^* \leq |w^*|.
\] (24)
Again, from constraints 13 and 14, we get
\[
t^* \geq |w^*|.
\] (25)
Hence, from Eqn 24 and 25, we get
\[
t^* = |w^*|.
\] (26)
Substituting Eqn 26 in Eqn 22, we get
\[
\frac{1}{n} w_1^{*\beta} D - \alpha w_1^* + \frac{\lambda}{np^2} |w_1^*| D = \frac{1}{n} w_1^{*\beta} D - \alpha w_1^* + \frac{\lambda}{np^2} |w_1^*| D.
\] (27)
Hence from Eqn 23 and 27, we get
\[
\frac{1}{n} w_1^{*\beta} D - \alpha w_1^* + \frac{\lambda}{np^2} |w_1^*| D = \frac{1}{n} w_1^{*\beta} D - \alpha w_1^* + \frac{\lambda}{np^2} |w_1^*| D.
\] (28)
Since, Eqn 28 is true for all \( \alpha \geq 0 \) and \( \lambda \geq 0 \), \( w^* = w_1^* \).

**C. Proof of Theorem IV.3**

**Proof.** The Lagrangian for the single-dimensional optimization problem \( P_2^* \) is given by,
\[
\mathcal{L}(w, t, \lambda_1, \lambda_2) = \frac{1}{n} w^\beta D - \alpha w + \frac{\lambda}{np^2} t D - \lambda_1 (t-w) - \lambda_2 (t+w).
\]
The Karush-Kuhn-Tucker (KKT) necessary conditions of optimality for \((w^*, t^*)\) is given by,
\[
\frac{\partial \mathcal{L}}{\partial w} = 0
\implies \frac{1}{n} \beta w_1^{*\beta-1} D = \alpha - \lambda_1 + \lambda_2 \quad (29)
\]
\[
\frac{\partial \mathcal{L}}{\partial t} = 0
\implies \frac{\lambda}{np^2} D = \lambda_1 + \lambda_2 \quad (30)
\]
\[
t-w \geq 0.
\] (31)
\[
t+w \geq 0.
\] (32)
\[
\lambda_1, \lambda_2 \geq 0.
\] (33)
\[
\lambda_1 (t-w) = 0.
\] (34)
\[
\lambda_2 (t+w) = 0.
\] (35)

Now let us consider the following situations:

**Case-1** \( \frac{n\alpha}{D^2} > \frac{\lambda}{p^2} \):
\[
\frac{1}{n} \beta w_1^{*\beta-1} D = \alpha - \frac{\lambda}{np^2} D + 2\lambda_2 > 0 \implies w > 0.
\]
From, Eqn 31, \( t > 0 \). Thus, from Eqn 34, \( \lambda_2 = 0 \). Hence,
\[
\frac{1}{n} \beta w_1^{*\beta-1} D = \alpha - \frac{\lambda}{np^2} D \implies w^* = \left[ \frac{1}{\beta \left( \frac{n\alpha}{D^2} - \frac{\lambda}{p^2} \right)} \right]^{\frac{1}{\beta - 1}}.
\]

**Case-2** \( \frac{n\alpha}{D^2} \leq \frac{\lambda}{p^2} \):
If \( w > 0, (t+w) > 0 \) which implies \( \lambda_2 = 0 \).
\[
\frac{1}{n} \beta w_1^{*\beta-1} D = \alpha - \frac{\lambda}{np^2} D \leq 0 \implies w \leq 0,
\]
which is a contradiction.
Now if \( w < 0, (t-w) > 0 \) which implies \( \lambda_1 = 0 \). From Eqn 29 and 30, it is easily seen that \( \frac{1}{n} \beta w_1^{*\beta-1} D = \alpha + \frac{\lambda}{np^2} D \implies w \geq 0 \), which is again a contradiction. Hence the only possibility is \( w = 0 \). Now, since \( \frac{n\alpha}{D^2} \geq 0 \), from **Case 1** and 2, we conclude that \( w^* = \left[ \frac{1}{\beta \left( \frac{n\alpha}{D^2} - \frac{\lambda}{p^2} \right)} \right]^{\frac{1}{\beta - 1}} \).

**D. Proof of Theorem IV.5**

**Proof.** Now to solve Problem \( P_3 \), note that Problem \( P_3 \) is separable in \( W \) i.e. we can write \( P(W) \) as
\[
P(W) = \sum_{i=1}^{p} P_i(w_i),
\] (35)
where \( P_i(w_i) = \frac{1}{n} w_i^{\beta} D_i^0 - \alpha w_i + \frac{\lambda}{np^2} |w_i| D_i^0 \). Now since Problem \( P_3 \) is separable, it is enough to solve Problem \( P_1' \) \( \forall d \in \{1, \ldots, p\} \) and combine the solutions to solve Problem \( P_3 \). Here Problem \( P_1' \) \( (d \in \{1, \ldots, p\}) \) is given by,
\[
\text{minimize } P_i(w_i) = \frac{1}{n} w_i^{\beta} D_i^0 - \alpha w_i w.r.t \ w_i.
\] (36)
The Theorem follows trivially from Theorem IV.4.

**E. Proof of Theorem IV.6**

**Proof.** Let \( f_m \) be the value of the objective function at the end of the \( m^{th} \) iteration of the algorithm. Since each step of the inner **while** loop of the algorithm decreases the value of the objective function, \( f_t \geq f_{t+1} \forall t \in \mathbb{N} \). Again note that, \( f_t \geq 0 \forall t \in \mathbb{N} \). Hence the sequence \( \{f_m\}_{m=1}^{\infty} \) is a decreasing sequence of reals bounded below by 0. Hence, by monotone convergence theorem, \( \{f_m\}_{m=1}^{\infty} \) converges. Now since \( \{f_m\}_{m=1}^{\infty} \) is convergent hence Cauchy and thus \( \exists N_0 \in \mathbb{N} \) such that if \( n \geq N_0, |f_{n+1} - f_n| < \epsilon \), which is the stopping criterion of the algorithm. Thus, the \( LW^k \)-means algorithm converges in a finite number of iteration.

**F. Proof of Theorem V.1**

**Proof.** Let \( D_i \) denote the minimum value of the \( k \)-means objective function for only the \( d^{th} \) feature of the dataset i.e. \( \{x_{1,1}, \ldots, x_{n,1}\} \). Let \( U_{n}^* \) denote the cluster assignment matrix corresponding to the optimal set of centroids \( A_n = \{a_1, \ldots, a_k\} \). Let \( D_i^* = \sum_{l=1}^{p} \sum_{j=1}^{k} u_{n,l} c(x_{l,1}, z_{j,1}) \). It is easy to see that \( D_i^* \geq D_i \). Hence, \( \frac{1}{D_i^*} \leq \frac{1}{D_i} \). Thus,
\[
\frac{1}{n} \frac{\beta - 1}{D_i} \geq \frac{1}{n} \frac{\beta - 1}{D_i} \geq \alpha_n(P_n).
\]
\[
\left( \sum_{l=1}^{p} \left( \frac{n\alpha}{D_i^*} \right)^{\frac{1}{\beta - 1}} \right)^{\beta - 1} \geq \left( \sum_{l=1}^{p} \left( \frac{n\alpha}{D_i} \right)^{\frac{1}{\beta - 1}} \right)^{\beta - 1} \geq \alpha_n(P_n).
\]
We know that \( w_i^{(n)} = \left[ \frac{1}{\beta \left( \frac{n\alpha}{D_i^*} - \frac{\lambda}{p^2} \right)} \right]^{\frac{1}{\beta - 1}} \).
Thus,
\[
w_i^{(n)} \leq \left[ \frac{1}{\beta \left( \frac{n\alpha}{D_i^*} \right)} \right]^{\frac{1}{\beta - 1}}.
\]
Thus,
\[ \sum_{l=1}^{p} w_l^{(n)} = \sum_{l=1}^{p} \left[ \frac{1}{\beta D_l^2} \right] \leq (n \alpha(P_n)) \frac{1}{\beta} \sum_{l=1}^{p} \left[ \frac{1}{\beta D_l^2} \right] = \alpha(P_n) \frac{1}{\beta} \sum_{l=1}^{p} \left[ \frac{n}{\beta D_l^2} \right] \leq 1. \]

The almost sure convergence of \( \alpha(P_n) \) follows from the strong consistency of the k-means algorithm [40].

\section*{G. Proof of Lemma V.1}

\textbf{Proof.} If \( A, B \in \xi_\delta \) such that \( H(A, B) < \delta \), then for each \( b \in B, \exists a(b) \in A \) such that \( |b_i - a_i(b)| < \delta \forall d \in \{1, \ldots, p\} \).

\[ \Phi(W, A, P) - \Phi(W, B, P) \]
\[ = \min a \in A \sum_{l=1}^{p} (w_l^a - w_l^b) |(x_l - a_i)|^2 \]
\[ - \min b \in B \sum_{l=1}^{p} (w_l^b - w_l^a) |(x_l - b_i)|^2 \]
\[ \leq \max b \in B \sum_{l=1}^{p} (w_l^a - w_l^b) |(x_l - a_i)|^2 - |(x_l - a_i)|^2 \]
\[ \leq \min b \in B \sum_{l=1}^{p} (w_l^a - w_l^b) (2x_l + 10M) \delta P(d) \]
\[ = \int_{||x|| \leq R} \sum_{l=1}^{p} w_l^a |x_l|(2x_l + 10M) \delta P(dx) \]
\[ + \int_{||x|| > R} \sum_{l=1}^{p} w_l^a |x_l|(2x_l + 10M) \delta P(dx) \]
\[ \leq \left( \int_{||x|| \leq R} \sum_{l=1}^{p} w_l^a |x_l|(2x_l + 10M) \delta P(dx) \right) \]
\[ + \int_{||x|| > R} \sum_{l=1}^{p} w_l^a |x_l|(2x_l + 10M) \delta P(dx) \]
\[ \leq \left( \int_{||x|| \leq R} \sum_{l=1}^{p} w_l^a |x_l|(2x_l + 10M) \delta P(dx) \right) \]
\[ + \int_{||x|| > R} \sum_{l=1}^{p} w_l^a |x_l| \delta \left( \int_{||x|| > R} (2x_l + 10M) P(dx) \right) \]
\[ = \sum_{l=1}^{p} (w_l^a - w_l^b) |(x_l - a_i)|^2 \]

The second term can be made smaller than \( \epsilon/2 \) if \( R \) is chosen sufficiently large. Appealing to the continuity of the function \( f(W, A, P) \), the first term can be made smaller than \( \epsilon/2 \), if \( \delta \) is chosen sufficiently small. Similarly one can show that, \( \Phi(W', A, P) - \Phi(W, A, P) < \epsilon \). Hence the result. \hfill \qed

\section*{H. Proof of Lemma V.2}

\textbf{Proof.} Let, \( W, W' \in \Gamma_k \) such that \( \| W - W' \| < \delta \). Take \( R > 5M \). Thus,

\[ \Phi(W, A, P) - \Phi(W', A, P) \]
\[ = \int \min a \in A \sum_{l=1}^{p} (w_l^a - w_l^b) |(x_l - a_i)|^2 \]
\[ - \int \min a \in A \sum_{l=1}^{p} (w_l^a - w_l^b) |(x_l - b_i)|^2 \]
\[ \leq \int \sum_{a \in A} \sum_{l=1}^{p} (w_l^a - w_l^b) |(x_l - a_i)|^2 \]
\[ \text{REFERENCES} \]

[1] R. Xu and D. Wunsch, “Survey of clustering algorithms,” IEEE Trans.

[2] K.-C. Wong, “A short survey on data clustering algorithms,” in Soft Computing and Machine Intelligence (ISCMII), 2015 Second Internation-

[3] J. B. MacQueen, “Some methods for classification and analysis of multivariate observations,” vol. 1, pp. 281–297, 1967.

[4] A. K. Jain, “Data clustering: 50 years beyond k-means,” Pattern Recogn.

[5] P. D. McNicholas, “Model-based clustering,” Journal of Classification, vol. 33, no. 3, pp. 331–373, Oct 2016.

[6] C. Fraley and A. E. Raftery, “How many clusters? which clustering method? answers via model-based cluster analysis,” The Computer Journal, vol. 41, pp. 578–588, 1998.

[7] G. J. McLachlan and S. Rathnayake, “On the number of components in a gaussian mixture model,” Wiley Int. Rev. Data Min. and Knowl. Disc., vol. 4, no. 5, pp. 341–355, Sep. 2014.

[8] R. Bellman, “Dynamic programming princeton university press prince-

[9] K. Beyer, J. Goldstein, R. Ramakrishnan, and U. Shaft, “When is nearest neighbor meaningful?” in International conference on database theory. Springer, 1999, pp. 217–235.

[10] C.-Y. Tsai and C.-C. Chiu, “Developing a feature weight self-adjustment mechanism for a k-means clustering algorithm,” Computational statistics & data analysis, vol. 52, no. 10, pp. 4658–4672, 2008.

[11] H. Liu and L. Yu, “Toward integrating feature selection algorithms for classification and clustering,” IEEE Transactions on knowledge and data engineering, vol. 17, no. 4, pp. 491–502, 2005.

[12] X. Chen, Y. Ye, X. Xu, and J. Z. Huang, “A feature group weighting method for subspace clustering of high-dimensional data,” Pattern Recognition, vol. 45, no. 1, pp. 434–446, 2012.

[13] R. C. De Amorim and B. Mirkin, “Minkowski metric, feature weight-

[14] E. Y. Chan, W. K. Ching, M. K. Ng, and J. Z. Huang, “An optimization algorithm for clustering using weighted dissimilarity measures,” Pattern Recognition, vol. 37, no. 5, pp. 943–952, 2004.

[15] A. Blum and R. L. Rivest, “Training a 3-node neural network is np-

[16] D. Wettscberek, D. W. Aha, and T. Mohri, “A review and empirical evaluation of feature weighting methods for a class of lazy learning algorithms,” in Lazy learning. Springer, 1997, pp. 273–314.

[17] D. S. Modha and W. S. Spangler, “Feature weighting in k-means clustering,” Machine learning, vol. 52, no. 3, pp. 217–237, 2003.

[18] R. C. de Amorim, “A survey on feature weighting based k-means algorithms,” Journal of Classification, vol. 33, no. 2, pp. 210–242, 2016.

[19] W. S. DeSarbo, J. D. Carroll, L. A. Clark, and P. E. Green, “Synthesized clustering: A method for amalgamating alternative clustering bases with differential weighting of variables,” Psychometrika, vol. 49, no. 1, pp. 57–78, 1984.
[20] J. Z. Huang, M. K. Ng, H. Rong, and Z. Li, “Automated variable weighting in k-means type clustering,” IEEE Transactions on Pattern Analysis and Machine Intelligence, vol. 27, no. 5, pp. 657–668, 2005.
[21] C. Li and J. Yu, “A novel fuzzy c-means clustering algorithm,” in RSKT. Springer, 2006, pp. 510–515.
[22] J. Z. Huang, J. Xu, M. Ng, and Y. Ye, “Weighting method for feature selection in k-means,” Computational Methods of feature selection, pp. 193–209, 2008.
[23] L. Jing, M. K. Ng, and J. Z. Huang, “An entropy weighting k-means algorithm for subspace clustering of high-dimensional sparse data,” IEEE Transactions on knowledge and data engineering, vol. 19, no. 8, 2007.
[24] J. G. Dy, “Unsupervised feature selection,” Computational methods of feature selection, pp. 19–39, 2008.
[25] T. J. Chich, The axiom of choice. Tata McGraw-Hill Education, 2006.
[26] J. Huang, Z. T. Ke, W. Wang et al., “Phase transitions for high dimensional data and its asymptotic consistency,” Electronic J. Statist., vol. 6, pp. 148–167, 2012. [Online]. Available: https://doi.org/10.1214/12-EJS668
[27] A. A. Alizadeh, M. B. Eisen, R. E. Davis, C. Ma, I. S. Lossos, T. R. Golub, D. K. Slonim, P. Tamayo, C. Huard, M. Gaasenbeek, J. P. Mesirov, H. Coller, M. L. Loh, J. R. Downing, M. A. Caligiuri et al., “Molecular classification of cancer: class discovery and class prediction by gene expression monitoring,” Science, vol. 286, no. 5439, pp. 331–337, 1999.
[28] R. Kohavi and G. H. John, “Wrappers for feature subset selection,” Artificial intelligence, vol. 97, no. 1–2, pp. 273–324, 1997.
[29] J. H. Friedman and J. J. Meulman, “Clustering objects on subsets of attributes (with discussion),” Journal of the Royal Statistical Society: Series B (Statistical Methodology), vol. 66, no. 4, pp. 815–849.
[30] J. J. Ke, W. Wang, and Y. Fang, “Regularized k-means clustering of high-dimensional data and its asymptotic consistency,” Electron. J. Statist., vol. 6, pp. 148–167, 2012. [Online]. Available: http://nikulin15a.html
[31] P. Tseng, “Convergence of a block coordinate descent method for nondifferentiable minimization,” Journal of optimization theory and applications, vol. 109, no. 3, pp. 475–494, 2001.
[32] E. L. Lehmann and G. Casella, Theory of point estimation. Springer Science & Business Media, 2006.
[33] R. Ostrovsky, Y. Rabani, L. J. Schulman, and C. Swamy, “The effectiveness of lloyd-type methods for the k-means problem,” J. ACM, vol. 59, no. 6, pp. 28:1–28:22, Jan. 2013.
[34] T. J. Chich, The axiom of choice. Courier Corporation, 2008.
[35] W. Rudin, Real and complex analysis. Tata McGraw-Hill Education, 2006.
[36] R. Tibshirani, “Regression shrinkage and selection via the lasso,” Journal of the Royal Statistical Society. Series B (Methodological), pp. 267–288, 1996.
[37] T. R. Golub, D. K. Slonim, P. Tamayo, C. Huard, M. Gaasenbeek, J. P. Mesirov, H. Coller, M. L. Loh, J. R. Downing, M. A. Caligiuri et al., “Molecular classification of cancer: class discovery and class prediction by gene expression monitoring,” Science, vol. 286, no. 5439, pp. 331–337, 1999.
[38] A. A. Alizadeh, M. B. Eisen, R. E. Davis, C. Ma, I. S. Lossos, A. Rosenwald, J. C. Boldrick, H. Sabet, T. Tran, Y. Xu et al., “Distinct types of diffuse large b-cell lymphoma identified by gene expression profiling,” Nature, vol. 403, no. 6769, p. 503, 2000.
[39] Y. Terada, J. Alcalá, A. Fernández, J. Luengo, J. Durães, S. García, L. Sánchez, and F. Herrera, “Keel data-mining software tool: Data set repository, integration of algorithms and experimental analysis framework,” Journal of Multiple-Valued Logic and Soft Computing, vol. 17, no. 2-3, pp. 255–287, 2010.
[40] M. Lichman, “UCI machine learning repository,” 2013. [Online]. Available: http://archive.ics.uci.edu/ml
[41] M. R. Yousefi, J. Hua, C. Sima, and E. R. Dougherty, “Reporting bias when using real data sets to analyze classification performance,” Bioinformatics, vol. 26, no. 1, pp. 68–76, 2009.
[42] G. J. Gordon, R. V. Jensen, L.-L. Hsiao, S. R. Gullans, J. E. Blumenstock, S. Ramaswamy, W. G. Richards, D. J. Sugarbaker, and R. Bueno, “Translation of microarray data into clinically relevant cancer diagnostic tests using gene expression ratios in lung cancer and mesothelioma,” Cancer research, vol. 62, no. 17, pp. 4963–4967, 2002.
[43] A. Bhattacharjee, W. G. Richards, J. Staunton, C. Li, S. Monti, P. Vasa, C. Ladd, J. Beheshti, R. Bueno, M. Gillette et al., “Classification of human lung carcinomas by mrna expression profiling reveals distinct adenocarcinoma subclasses,” Proceedings of the National Academy of Sciences, vol. 98, no. 24, pp. 13790–13795, 2001.
[44] J. Friedmann, T. Hastie, and R. Tibshirani, The elements of statistical learning. Springer series in statistics New York, 2001, vol. 1.
[45] B. W. Matthews, “Comparison of the predicted and observed secondary structure of t4 phage lysozyme,” Biochimica et Biophysica Acta (BBA)-Protein Structure, vol. 405, no. 2, pp. 442–451, 1975.

Saptarshi Chakraborty received his B. Stat. degree in Statistics from the Indian Statistical Institute, Kolkata in 2018 and is currently pursuing his M. Stat. degree (in Statistics) at the same institute. He was also a summer exchange student at the Big Data Summer Institute, University of Michigan, USA in 2018, where he worked on the application of Machine Learning algorithms on medical data. His current research interests are Statistical Learning (both supervised and unsupervised), Evolutionary Computing and Visual Cryptography.
Swagatam Das is currently serving as an associate professor at the Electronics and Communication Sciences Unit, Indian Statistical Institute, Kolkata, India. He has published more than 250 research articles in peer-reviewed journals and international conferences. He is the founding coeditor-in-chief of Swarm and Evolutionary Computation, an international journal from Elsevier. Dr. Das has 16,000+ Google Scholar citations and an H-index of 60 till date. He is also the recipient of the 2015 Thomson Reuters Research Excellence India Citation Award as the highest cited researcher from India in Engineering and Computer Science category between 2010 to 2014.