Strengthened PT–symmetry with $P \neq P^\dagger$

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Abstract

In Quantum Mechanics working with non-Hermitian PT–symmetric Hamiltonians (i.e., with an indefinite metric $P$ in Hilbert space) we propose to relax the usual constraint $P = P^\dagger$. We show that this merely induces certain “hidden” symmetries responsible, say, for the degeneracy of levels. Using a triplet of the coupled square wells for illustration we show that the bound states may remain stable in a large domain of couplings.

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1 \(\mathcal{PT}\)–symmetric Quantum Mechanics and its two alternative scenarios

\(\mathcal{PT}\)–symmetric Quantum Mechanics of C. Bender et al [1] studies non-Hermitian Hamiltonians \(H \neq H^\dagger\) with the peculiar property

\[
\mathcal{PT} H = H \mathcal{PT}.
\] (1)

In the original and simplest one-dimensional version of the theory [2] the symbol \(\mathcal{T}\) denotes the complex conjugation, i.e., an antilinear involution with the property \(\mathcal{T} i \partial_t \mathcal{T} = -i \partial_t\) interpreted as time reversal. The Hamiltonians themselves are assumed symmetric so that we may replace \(H \rightarrow h = h^T\) and deduce that \(\mathcal{T} h \mathcal{T} = h^* \equiv h^\dagger\), \(\mathcal{T} h \mathcal{PT} = h^\dagger \mathcal{P}^*\) and

\[
h^\dagger = \mathcal{P}^{-1} h \mathcal{P} = \mathcal{P}^* h (\mathcal{P}^*)^{-1}.
\] (2)

The choice of the parity \(\mathcal{P}\) with the properties \(\mathcal{P} = \mathcal{P}^* = \mathcal{P}^{-1} = \mathcal{P}^\dagger = \mathcal{P}^{\dagger}\) in ref. [2] inspired A. Mostafazadeh [3] who recommended a transition from the physics-inspired symmetry (1) to its mathematically equivalent representation in the form of the \(\mathcal{P}\)–pseudo-Hermiticity requirement

\[
H^\dagger = \mathcal{P} H \mathcal{P}^{-1}.
\] (3)

He emphasized that eq. (3) may be read as an isospectrality property where one might work with asymmetric Hamiltonians and with the “generalized parity” operators which need not be involutive at all, \(\mathcal{P} \rightarrow \mathcal{P} \neq \mathcal{P}^{-1}\).

In such a perspective the boldface symbol \(\mathcal{P}\) may represent an arbitrary auxiliary operator. The involutive character of the Hermitian conjugation in eq. (3) implies that

\[
H = (H^\dagger)^\dagger = (\mathcal{P}^T)^{-1} H^\dagger \mathcal{P}^\dagger = (\mathcal{P}^T)^{-1} \mathcal{P} H \mathcal{P}^{-1} \mathcal{P}^\dagger
\] (4)

which gives us the two alternative possibilities;

[a] a self-adjoint pseudo-metric \(\mathcal{P} = \mathcal{P}^\dagger\) is chosen, or

[b] non-Hermitian operators \(\mathcal{P} \neq \mathcal{P}^\dagger\) are admitted.

In the light of the current literature the “trivial symmetry” scenario [a] seems to be “the only useful” option where \(\mathcal{P}\) becomes a pseudo-metric (often called “indefinite metric”) in the physical Hilbert space of states \(\mathcal{H}\).
We believe that the non-Hermitian alternative \([b]\) may prove equally interesting. Indeed, there is no real reason for ignoring the class of operators \(S = P^{-1} P^\dagger \neq I\) which represent a nontrivial symmetry (4) of the Hamiltonian. In what follows we intend to support such a point of view by an illustrative construction. For this purpose we shall pick up one of the most elementary coupled-channel symmetries \(S \neq I\) derived from the non-involutive and non-Hermitian toy operator

\[
P = \begin{pmatrix} 0 & 0 & \mathcal{P} \\ \mathcal{P} & 0 & 0 \\ 0 & \mathcal{P} & 0 \end{pmatrix} = (\mathcal{P}^\dagger)^{-1} = P^{-2}. \tag{5}\]

Provided that its sub-operator \(\mathcal{P}\) remains defined as the parity, \(\mathcal{P}\psi(x) = \psi(-x)\), it cannot be interpreted as a metric because its eigenvalues are complex.

A supplementary reason for the present use of the non-Hermitian \(P\) of eq. (5) emerges once we return back to the symmetric Hamiltonians \(h = h^T\) in eq. (2) where any “early generalization” \(\mathcal{P} \to \mathcal{R}\) of the parity would lead immediately to an alternative explicit constraint

\[
\mathcal{R} \mathcal{R}^* h = h \mathcal{R} \mathcal{R}^*. \tag{6}\]

We see that the “alternative generalized parities” \(\mathcal{R}\) would have to be unitary whenever one assumes that they are not asymmetric. One should keep in mind that the latter consistency constraint definitely differs from its predecessor eq. (4). Of course, it can again be interpreted as imposing an additional symmetry upon the Hamiltonian. Thus, one feels that eq. (3) with self-adjoint \(\mathcal{P}\) need not necessarily offer the only productive way towards a generalization.

### 2 Non-Hermitian triplet of coupled square wells

Equation (5) and the non-Hermitian Hamiltonian of the triple-channel form

\[
H = \begin{pmatrix} -\frac{d^2}{dx^2} + D_a(x) & U_a(x) & V_a(x) \\ U_b(x) & -\frac{d^2}{dx^2} + D_b(x) & W_b(x) \\ V_b(x) & W_a(x) & -\frac{d^2}{dx^2} + D_c(x) \end{pmatrix} \tag{7}\]

will be assumed inter-related by our present modification

\[
H^\dagger = P H P^{-1}, \quad P \neq P^\dagger \tag{8}\]
of the $P$—pseudo-Hermiticity condition (3) re-written in the form adapted to the less common scenario [b]. Our choice of the interaction potentials will be dictated by the exact solvability requirement in a way inspired by the simplicity of the various single-channel square-well models [4] – [6] in scenario [a].

No innovations will occur in the real, Hermitian part of the present potentials,

$$
\text{Re} D_{a,b,c}(x) = \text{Re} U_{a,b}(x) = \text{Re} V_{a,b}(x) = \text{Re} W_{a,b}(x) = 0, \quad x \in (-1, 1),
$$

$$
\text{Re} D_{a,b,c}(x) = \text{Re} U_{a,b}(x) = \text{Re} V_{a,b}(x) = \text{Re} W_{a,b}(x) = \infty, \quad x \notin (-1, 1).
$$

In the same spirit, the imaginary potentials in Hamiltonian (7) will be postulated piecewise constant. Their specification

$$
\text{Im} U_{a,b}(x) = \text{Im} V_{a,b}(x) = \text{Im} W_{a,b}(x) = Y > 0, \quad x \in (-1, 0),
$$

$$
\text{Im} U_{a,b}(x) = \text{Im} V_{a,b}(x) = \text{Im} W_{a,b}(x) = -Y, \quad x \in (0, 1),
$$

$$
\text{Im} D_{a,b,c}(x) = Z, \quad x \in (-1, 0), \quad \text{Im} D_{a,b,c}(x) = -Z, \quad x \in (0, 1)
$$

containing two free real coupling constants results from the pseudo-Hermiticity condition (8) and from our choice of the generalized parity (5). This defines the family of the coupled-channel Schrödinger equations

$$
H \begin{pmatrix}
\varphi_a(x) \\
\varphi_b(x) \\
\varphi_c(x)
\end{pmatrix} = E \begin{pmatrix}
\varphi_a(x) \\
\varphi_b(x) \\
\varphi_c(x)
\end{pmatrix}
$$

where the energies $E$ may be assumed real, for the small couplings $Y$ and $Z$ at least [7]. As long as we put our system in a box (9), the elementary boundary condition

$$
\begin{pmatrix}
\varphi_a(x) \\
\varphi_b(x) \\
\varphi_c(x)
\end{pmatrix} \bigg|_{x=\pm 1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

determines all the bound states of the model.

3 Solutions and the degeneracy of their spectrum

The obvious ansatz for the general solution

$$
\varphi_{a,b,c}(x) = \begin{cases}
C_L^{(a,b,c)} \sin \kappa_L(x + 1), & x \in (-1, 0), \\
C_R^{(a,b,c)} \sin \kappa_R(-x + 1), & x \in (0, 1)
\end{cases}
$$
The resulting three real conditions have to specify the two unknown real parameters matching of all the three wave functions and necessitates only the appropriate boundary conditions and necessitates only the appropriate single complex equation.

\[
C_L^{(a,b,c)} \sin \kappa_L = C_R^{(a,b,c)} \sin \kappa_R, \quad \kappa_L C_L^{(a,b,c)} \cos \kappa_L = -\kappa_R C_R^{(a,b,c)} \cos \kappa_R.
\]

The first triplet merely defines, say, constants \(C\), and at \(x = x\), the ratio of both of these equations eliminates all these constants and leads to the single complex equation

\[
\kappa_L \cot \kappa_L = -\kappa_R \cot \kappa_R. \tag{14}
\]

Under this constraint our final quantization condition will result from the insertion of the ansatz (13) in the Schrödinger eq. (11) at \(x \in (-1, 0), \),

\[
\begin{pmatrix}
\kappa_L^2 + i Z & i Y & i Y \\
i Y & \kappa_L^2 + i Z & i Y \\
i Y & i Y & \kappa_L^2 + i Z
\end{pmatrix}
\begin{pmatrix}
C_L^{(a)} \\
C_L^{(b)} \\
C_L^{(c)}
\end{pmatrix}
= E
\begin{pmatrix}
C_L^{(a)} \\
C_L^{(b)} \\
C_L^{(c)}
\end{pmatrix}, \tag{15}
\]

and at \(x \in (0, 1), \)

\[
\begin{pmatrix}
\kappa_R^2 - i Z & -i Y & -i Y \\
-i Y & \kappa_R^2 - i Z & -i Y \\
-i Y & -i Y & \kappa_R^2 - i Z
\end{pmatrix}
\begin{pmatrix}
C_R^{(a)} \\
C_R^{(b)} \\
C_R^{(c)}
\end{pmatrix}
= E
\begin{pmatrix}
C_R^{(a)} \\
C_R^{(b)} \\
C_R^{(c)}
\end{pmatrix}. \tag{16}
\]

As long as the energies are assumed real, this indicates that we may put \(\kappa_R = s + it = \kappa_L^*\) with, say, positive \(s > 0\) and any real \(t \in (-\infty, \infty). \) In this notation the complex eq. (14) degenerates to the real implicit formula

\[
2s \sin 2s + 2t \sinh 2t = 0 \tag{17}
\]

which first occurred in ref. [8] and which has thoroughly been studied in ref. [9]. In our present model, eq. (17) has to be combined with the complex secular equation

\[
\det \begin{pmatrix}
(s + it)^2 - i Z - E & -i Y \\
-i Y & (s + it)^2 - i Z - E & -i Y \\
-i Y & -i Y & (s + it)^2 - i Z - E
\end{pmatrix} = 0. \tag{18}
\]

The resulting three real conditions have to specify the two unknown real parameters \(s = s_n, t = t_n\) and the energy \(E = E_n, n = 0, 1, \ldots\). Once we set \(E = s^2 - t^2 - \alpha\)
(with a real $\alpha$) and $Z = 2st + \beta$ (with a real $\beta$) we may re-write eq. (18) as a pair of the real polynomial equations

$$\begin{align*}
\alpha^3 - 3\alpha (\beta^2 - Y^2) &= 0, \\
\beta^3 - 3\beta (\alpha^2 + Y^2) + 2Y^2 &= 0.
\end{align*}$$

(19)

In the preliminary test we shall assume that $\alpha = \alpha_{\text{(tentative)}} \neq 0$. From the first row we get $\alpha_{\text{(tentative)}}^2 = 3(\beta^2 - Y^2)$ which simplifies the second row to the solvable cubic equation with the three roots,

$$\beta_{1\text{(tentative)}} = Y, \quad \beta_{2\text{(tentative)}} = \beta_{3\text{(tentative)}} = -\frac{1}{2}Y, \quad \alpha_{\text{(tentative)}} \neq 0.$$ 

Their insertion in the definition of $\alpha$ gives the respective solutions

$$\alpha_1 = 0, \quad \alpha_{2,3} = \pm \frac{3i}{2}Y$$

all of which contradict our assumptions. We are forced to conclude that we always have the vanishing $\alpha = 0$,

$$E = s^2 - t^2.$$ 

(20)

At $\alpha = 0$ the secular equation (19) leads to the unique triplet of roots

$$\beta_1 = -2Y, \quad \beta_2 = \beta_3 = Y, \quad \alpha = 0.$$ 

(21)

Their respective insertion in eqs. (15) and/or (16) gives the unnormalized eigenvector

$$\left(C_{(1)}^{(a)}, C_{(1)}^{(b)}, C_{(1)}^{(c)}\right) \sim (1, 1, 1)$$

(22)

plus the two other, due to the degeneracy, non-unique vectors available, say, in an orthogonalized representation

$$\left(C_{(2)}^{(a)}, C_{(2)}^{(b)}, C_{(2)}^{(c)}\right) \sim (1, -1, 0), \quad \left(C_{(3)}^{(a)}, C_{(3)}^{(b)}, C_{(3)}^{(c)}\right) \sim (1, 1, -2)$$

(23)

which, incidentally, coincides with the Jacobi-coordinate recipe for the three equal-mass particles [10].

Our knowledge of the roots (21) enables us to eliminate one of the real parameters (say, $t$) as lying on one of the two different hyperbolic curves,

$$t = t(\sigma)(s) = \frac{1}{2s} Z_{\text{eff}}(\sigma), \quad Z_{\text{eff}}(1) = Z + 2Y, \quad Z_{\text{eff}}(2, 3) = Z - Y.$$ 

(24)
Our construction of bound states is completed. In terms of the parameters $s$ and $t$ they are determined by formulae (13), (20), (22) and (23). The parameters themselves must be fixed by the pair of eqs. (24) and (17). In a way described more thoroughly in ref. [8], all the physical roots of our secular eq. (17) may be re-parametrized by the formula

$$s = s_n = \frac{(n + 1)\pi}{2} + (-1)^n \varepsilon_n, \quad n = 0, 1, \ldots$$

where the new unknown quantities $\varepsilon_n$ remain small not only in the vicinity of the well known Hermitian case where both the coupling constants $Y$ and $Z$ remain sufficiently small but also at all the sufficiently large $n \geq n_0$. Thus, one may calculate them perturbatively in both these regimes [11].

4 The determination of the domain where the energies remain real

In the $s - t$ plane we may visualize all the roots $(s_n, t_n)$ as intersections of the two hyperbolic curves (24) with all the family of the $(t \to -t)$–symmetric ovals representing the complete graphical solution of our second secular implicit constraint (17) (cf. [8]). As long as $Y > 0$, all the present energies $E_n$ remain real if and only if

$$Y - Z_{\text{crit}} \leq Z \leq Z_{\text{crit}} - 2Y. \quad (26)$$

We may recollect the commentary in ref. [9] and conclude that $Z_{\text{crit}} \approx 4.475$ at the present choice of the units $\hbar = 2m = 1$. In the $(Y, Z)$ plane the condition (26) is satisfied inside a fairly large triangle with (approximate) vertices $(0, 4.4753)$, $(0, -4.4753)$ and $(2.9835, -1.4918)$ (cf. Figure 1).

The critical parameter $Z_{\text{crit}}$ remains the same for several different square-well systems. It determines the boundary of the physical domain in the single-channel square well as well as in all its classical [11] and supersymmetric [12] partners. Still, only its four-digit estimate has been published up to now [9]. Moreover, even that improvement of the original three-digit estimate of ref. [8] by one digit did not seem easy. This apparently discouraged, undeservedly, any other attempts. For example, an interesting alternative approach of ref. [13] using a discretization of the coordinates has only been studied in the lowest possible approximation giving just a schematic initial estimate $4\sqrt{2} \approx 5.66$ of $Z_{\text{crit}}$. In fact, a more complicated problem
with complex energies has been solved during the most successful numerical attempt in ref. [9]. For this reason, let us show now that a systematic improvement of the precision of $Z_{\text{crit}}$ can be made feasible at a reasonable computational cost.

Firstly let us summarize the situation where one locates, graphically [8], the first two single-channel non-Hermitian square-well energies $E_{0,1} = s_{0,1}^2 - t_{0,1}^2$ as related to the neighboring intersections $(s_{0,1}, t_{0,1})$ of the implicitly defined $Z$–independent curve $s = s^{(a)}(t)$ [with $2s^{(a)}(t) \sin[2s^{(a)}(t)] = -2t \sinh 2t$ from eq. (17) above] with the $Z$–dependent but much more elementary hyperbolic curve $s^{(b)}(t) = Z/(2t)$ of eq. (24). In such a graphical setting it was clarified that the maximal $Z = Z_{\text{crit}}$ at which both the energies $E_{0,1}$ remain real is the point at which they [as well as the neighboring intersections $(s_{0,1}, t_{0,1})$] merge. Thus, the value of $Z = Z_{\text{crit}}$ is defined as a parameter of confluence at which $E_0$ precisely coincides with $E_1$.

At this point the curves $s^{(a)}(t) = \pi - \varepsilon(t)$ and $s^{(b)}(t)$ will osculate at a certain “intersection” point $t_{\text{crit}}$ and “exceptional” energy $E_{\text{crit}}$. We have to guarantee that both our curves and both their tangents coincide,

$$
\varepsilon(t_{\text{crit}}) = \pi - \frac{Z_{\text{crit}}}{2t_{\text{crit}}}, \quad \partial_t \varepsilon(t_{\text{crit}}) = \frac{Z_{\text{crit}}}{2t_{\text{crit}}^2}.
$$

(27)

The derivative is defined in terms of the positive shift function $\varepsilon(t) < \pi/4$,

$$
\partial_t \varepsilon(t) = -\frac{\sinh 2t + 2t \cosh 2t}{2 [\pi - \varepsilon(t)] \cos 2\varepsilon(t) - \sin 2\varepsilon(t)}
$$

but the definition of the function $\varepsilon(t)$ itself is merely implicit,

$$
\sin [2\varepsilon(t)] = \frac{t \sinh 2t}{\pi - \varepsilon(t)}.
$$

Fortunately, it may be re-interpreted as a quickly convergent iterative recipe of a ‘generalized continued-fraction’ type,

$$
\varepsilon_{(\text{new})}(t) = \frac{1}{2} \arcsin \left[ 2 \frac{t \sinh 2t}{\pi - \varepsilon_{(\text{old})}(t)} \right].
$$

(28)

With the initial $\varepsilon_{(\text{lower})}(t) = \pi/4$ and $\varepsilon_{(\text{upper})}(t) = 0$, the $N$–th iteration of eq. (28) represents the desired explicit definition of the respective approximate functions $\varepsilon_{(\text{lower})}(t)$ and $\varepsilon_{(\text{upper})}(t)$. Their knowledge enables us to solve the two coupled algebraic equations (27) for the two unknown quantities $t_{\text{crit}}$ and $Z_{\text{crit}}$ at each particular choice of $N$. The both-sided convergence of this recipe is illustrated in Table 1. For the sake of completeness, its last-row items may be complemented by the corresponding values of $t_{\text{crit}}^{(\text{lower})} = 0.839393459$, $t_{\text{crit}}^{(\text{upper})} = 0.839393461$, $s_{\text{crit}}^{(\text{lower})} = 2.665799044$, $s_{\text{crit}}^{(\text{upper})} = 2.665799069$ and $E_{\text{crit}}^{(\text{lower})} = 6.401903165$ and $E_{\text{crit}}^{(\text{upper})} = 6.401903294$. 
5 A remark on the interpretation of the model

In summary, we felt inspired by several physical applications of scenario [a] which have recently been offered within relativistic quantum field theory [14] and first-quantized relativistic quantum mechanics [15] as well as in quantum cosmology [16] and in the classical magnetohydrodynamics [17]. In these cases one often employs the partitioned and manifestly Hermitian and involutive \( P \), in the latter three contexts at least [18]. In our present letter we complemented these studies by an illustration of several new possibilities emerging within the scenario [b].

In our present coupled-channel model the Hilbert space is partitioned in subspaces. In fact, there is no real novelty in such a procedure of the model-building as the various \( P \)-pseudo-Hermitian coupled-channel operators are known to result from the relativistic Sakata-Taketani equations [19, 20] and from their various higher-spin generalizations and/or non-relativistic analogues [21]. A fresh example of a coupling of channels in non-Hermitian context may be found in our recent Klein-Gordon study of the influence of certain external solvable delta-function interactions [22]. The Hermitian part of our forces was postulated there in the same deep square-well form of eq. (9) and only the pseudo-metric \( P = P^\dagger \) has been chosen Hermitian, of type [a].

Let us emphasize that from the point of view of Quantum Mechanics it is important to know that there exists (at least one) specification of the scalar product which leads to the positive definite physical norm. Unfortunately, its \textit{explicit construction} is usually fairly complicated in practice. In this sense, the exact solvability of the square-well-like models simplifies significantly the \textit{perturbative} construction of the related “physical” metric \( \Theta \neq P \) in Hilbert space [11]. In the other words, our knowledge of \( \Theta \) enables us to assert that all the observables in our models acquire the so called quasi-Hermiticity property [23] and that only in terms of the (by definition, positive-definite and self-adjoint) \( \Theta \) they may be assigned the standard probabilistic interpretation [23] – [28].

In our present non-relativistic coupled-channel example the transition to the scenario [b] did not influence the methods of the construction of the physical metric so that they need not be discussed separately. Even all of their technical details remain the same for our specific choice of the operator \( P \) since \( P^3 = P \). This is an accidental aspect of our assumptions (5) and (8) which makes the validity of the standard rule (3) an immediate consequence of our assumption (8).
In an opposite direction one may say that our present assumptions concerning the
symmetry of the Hamiltonian are stronger since our $H$ commutes with the product
$S = [P^{-1}]^\dagger P \neq I$. In effect we preserve and complement the “old” eq. (3) by another
assumption. As long as we impose more symmetry, the degeneracy of some levels
occurs. From the practical point of view such a phenomenon is the consequence of
our choice of the complicated $P \neq P^\dagger$. Of course, a wealth of new features of the
spectrum may be expected to emerge from the more systematic study of some less
schematic non-self-adjoint “generalized parities” $P$ in the future.

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Table 1: Numerical determination of the critical coupling

| iteration $N$ | $Z_{\text{crit}}^{(\text{lower})}$ | $Z_{\text{crit}}^{(\text{upper})}$ |
|---------------|-----------------------------------|-----------------------------------|
| 0             | 4.299                             | 4.663                             |
| 2             | 4.4614                            | 4.4857                            |
| 4             | 4.47431                           | 4.47601                           |
| 6             | 4.475239                          | 4.475357                          |
| 8             | 4.47530381                        | 4.4753119                         |
| 10            | 4.475308262                       | 4.475308823                       |
| 12            | 4.475308560                       | 4.475308614                       |

Table captions

Table 1. Numerical determination of the critical coupling

Figure captions

Figure 1. Triangular domain of the allowed couplings $Y$ and $Z$
References

[1] C. M. Bender, S. Boettcher and P. N. Meisinger, J. Math. Phys. 40 (1999) 2201.

[2] C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80 (1998) 4243

[3] A. Mostafazadeh, J. Math. Phys. 43 (2002) 205.

[4] M. Znojil, J. Math. Phys. 45 (2004) 4418.

[5] M. Znojil, J. Math. Phys. 46 (2005) 062109.

[6] H. Bíla, V. Jakubský, M. Znojil, B. Bagchi, S. Mallik and C. Quesne, Czech. J. Phys. 55 (2005) 1075.

[7] H. Langer and C. Tretter, Czech. J. Phys. 54 (2004) 1113.

[8] M. Znojil, Phys. Lett. A. 285 (2001) 7.

[9] M. Znojil and G. Lévai, Mod. Phys. Letters A 16 (2001) 2273.

[10] M. Znojil, J. Phys. A: Math. Gen. 36 (2003) 9929.

[11] A. Mostafazadeh and A. Batal, J. Phys. A: Math. Gen. 37 (2004) 11645.

[12] B. Bagchi, S. Mallik and C. Quesne, Mod. Phys. Lett. A 17 (2002) 1651.

[13] S. Weigert, Czech. J. Phys. 55 (2005) 1183.

[14] C. M. Bender, Czech. J. Phys. 54 (2004) 13.

[15] A. Mostafazadeh, Class. Quantum Grav. 20 (2003) 155.

[16] A. Mostafazadeh, Czech. J. Phys. 54 (2004) 93.

[17] U. Günther and F. Stefani, Czech. J. Phys. 55 (2005) 1099.

[18] M. Znojil, H. Bíla and V. Jakubský, Czech. J. Phys. 54 (2004) 1143.

[19] S. Sakata and M. Taketani, Proc. Phys. Math. Soc. Japan 22 (1940) 757.

[20] H. Feshbach and F. Villars, Rev. Mod. Phys. 30 (1958) 24.

[21] W. I. Fushchych and A. G. Nikitin, Symmetries of equations of quantum mechanics, Allerton Press, New York, 1994.
[22] M. Znojil, Czech. J. Phys. 55 (2005) 1187.

[23] F. G. Scholtz, H. B. Geyer and F. J. W. Hahne, Ann. Phys. (NY) 213 (1992) 74.

[24] M. Znojil, math-ph/0104012 and Rendiconti del Circ. Mat. di Palermo, Ser. II, Suppl. 72 (2004) 211.

[25] R. Kretschmer and L. Szymanowski, quant-ph/0105054 and Phys. Lett. A 325 (2004) 112.

[26] C. M. Bender, D. C. Brody and H. F. Jones, Phys. Rev. Lett. 89 (2002) 0270401.

[27] A. Mostafazadeh, arXiv: quant-ph/0310164.

[28] C. M. Bender, D. C. Brody and H. F. Jones, Phys. Rev. Lett. 92 (2004) 0119902 (erratum).
