EXISTENCE OF ATTRACTORS, HOMOCLINIC TANGENCIES AND SINGULAR-HYPERBOLICITY FOR FLOWS

A. ARBIETO, A. ROJAS, B. SANTIAGO

ABSTRACT. We prove that every $C^1$ generic three-dimensional flow has either infinitely many sinks, or, finitely many hyperbolic or singular-hyperbolic attractors whose basins form a full Lebesgue measure set. We also prove in the orientable case that the set of accumulation points of the sinks of a $C^1$ generic three-dimensional flow has no dominated splitting with respect to the linear Poincaré flow. As a corollary we obtain that every three-dimensional flow can be $C^1$ approximated by flows with homoclinic tangencies or by singular-Axiom A flows. These results extend [3], [6], [20] and solve a conjecture in [17].

1. INTRODUCTION

Araujo’s Theorem [3] asserts that a $C^1$ generic surface diffeomorphism has either infinitely many sinks (i.e. attracting periodic orbits), or, finitely many hyperbolic attractors whose basins form a full Lebesgue measure set. In the recent paper [4] the authors were able to extend this result from surface diffeomorphisms to three-dimensional flows without singularities. More precisely, they proved that a $C^1$ generic three-dimensional flow without singularities either has infinitely many sinks, or, finitely many hyperbolic attractors whose basins form a full Lebesgue measure set. The present paper goes beyond and extend [4] to the singular case. Indeed, we prove that every $C^1$ generic three-dimensional flow has either infinitely many sinks, or, finitely many hyperbolic or singular-hyperbolic attractors whose basins form a full Lebesgue measure set. The arguments used in the proof will imply in the orientable case that the set of accumulation points of the sinks of a $C^1$ generic three-dimensional flow has no dominated splitting with respect to the linear Poincaré flow. From this we obtain that every three-dimensional flow can be $C^1$ approximated by flows with homoclinic tangencies or by singular-Axiom A flows. This last result extends [6], [20] and solves a conjecture in [17]. Let us state our results in a precise way.

By a three-dimensional flow we mean a $C^1$ vector fields on compact connected boundaryless manifolds $M$ of dimension 3. The corresponding space equipped with the $C^1$ vector field topology will be denoted by $\mathcal{X}^1(M)$. The flow of $X \in \mathcal{X}^1(M)$ is denoted by $X_t$, $t \in \mathbb{R}$. A subset of $\mathcal{X}^1(M)$ is residual if it is a countable intersection of open and dense subsets. We say that a $C^1$ generic three-dimensional flow satisfies a certain property $P$ if there is a residual subset $\mathcal{R}$ of $\mathcal{X}^1(M)$ such that $P$ holds for every element of $\mathcal{R}$. The closure operation is denoted by $\text{Cl}(\cdot)$.

2010 Mathematics Subject Classification. Primary: 37D20; Secondary: 37C70.
Key words and phrases. singular-hyperbolic Attractor, Sink, Three-dimensional flow.
Partially supported by CNPq, FAPERJ and PRONEX/DS from Brazil.
By a critical point of $X$ we mean a point $x$ which is either periodic (i.e. there is a minimal $t_x > 0$ satisfying $X_{t_x}(x) = x$) or singular (i.e. $X(x) = 0$). The eigenvalues of a critical point $x$ are defined respectively as those of the linear automorphism $DX_{t_x}(x) : T_xM \to T_xM$ not corresponding to $X(x)$, or, those of $DX(x)$. A critical point is a sink if its eigenvalues are less than 1 in modulus (periodic case) or with negative real part (singular case). A source will be a sink for the time reversed flow $-X$. Denote by Sink$(X)$ and Source$(X)$ the set of sinks and sources of $X$ respectively.

Given a point $x$ we define the omega-limit set,

$$\omega(x) = \left\{ y \in M : y = \lim_{t_k \to \infty} X_{t_k}(x) \text{ for some sequence } t_k \to \infty \right\}.$$  

(when necessary we shall write $\omega_X(x)$ to indicate the dependence on $X$.) We call a subset $\Lambda \subset M$ invariant if $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$; and transitive if there is $x \in \Lambda$ such that $\Lambda = \omega(x)$. The basin of any subset $\Lambda \subset M$ is defined by

$$W^s(\Lambda) = \{ y \in M : \omega(y) \subset \Lambda \}.$$  

(Sometimes we write $W^s_X(\Lambda)$ to indicate dependence on $X$). An attractor is a transitive set $A$ exhibiting a neighborhood $U$ such that

$$A = \bigcap_{t \geq 0} X_t(U).$$  

A compact invariant set $\Lambda$ is hyperbolic if there are a continuous $DX_\cdot$-invariant tangent bundle decomposition $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda \oplus E^u_\Lambda$ over $\Lambda$ and positive numbers $K, \lambda$ such that $E^s_\Lambda$ is generated by $X(x)$,

$$\|DX_t(x)/E^s_x\| \leq Ke^{-\lambda t} \quad \text{and} \quad \|DX_{-t}(x)/E^u_{X_t(x)}\| \leq K^{-1}e^{\lambda t}, \quad \forall (x, t) \in \Lambda \times \mathbb{R}^+.$$  

On the other hand, a dominated splitting $E \oplus F$ for $X$ over an invariant set $I$ is a continuous tangent bundle $DX_\cdot$-invariant splitting $T_I M = E_I \oplus F_I$ for which there are positive constants $K, \lambda$ satisfying

$$\|DX_t(x)/E_x\| \cdot \|DX_{-t}(X_t(x))/F_{X_t(x)}\| \leq Ke^{-\lambda t}, \quad \forall (x, t) \in I \times \mathbb{R}^+.$$  

In this case we say that the dominating subbundle $E_I$ is contracting if

$$\|DX_t(x)/E_x\| \leq Ke^{-\lambda t}, \quad \forall (x, t) \in I \times \mathbb{R}^+.$$  

The central subbundle $F_I$ is said to be volume expanding if

$$|\det DX_t(x)/F_x|^{-1} \leq Ke^{-\lambda t}, \quad \forall (x, t) \in I \times \mathbb{R}^+.$$  

A compact invariant set is partially hyperbolic if it has a dominated splitting with contracting dominating direction. We say that a partially hyperbolic set is singular-hyperbolic for $X$ if its singularities are all hyperbolic and its central subbundle is volume expanding. A hyperbolic (resp. singular-hyperbolic) attractor for $X$ is an attractor which is simultaneously a hyperbolic (resp. singular-hyperbolic) set for $X$.

With these definitions we can state our first result.

**Theorem A.** A $C^1$ generic three-dimensional flow has either infinitely many sinks, or, finitely many hyperbolic or singular-hyperbolic attractors whose basins form a full Lebesgue measure set.
The method of the proof of the above result (based on [18]) will imply the following result for three-dimensional flows on orientable manifolds. Denote by \( \text{Sing}(X) \) the set of singularities of \( X \). Given \( \Lambda \subset M \) we denote \( \Lambda^* = \Lambda \setminus \text{Sing}(X) \).

We define the vector bundle \( N^X \) over \( M^* \) whose fiber at \( x \in M^* \) is the orthogonal complement of \( X(x) \) in \( T_x \mathcal{M} \). Denoting the projection \( \pi_x : T_x \mathcal{M} \to N^X_x \) we define the Linear Poincaré flow (LPF) \( P_t^X : N^X \to N^X \) by \( P_t^X(x) = \pi_{X(\pi_x(x))} \circ DX_1(t) \) for \( t \in \mathbb{R} \). An invariant set \( \Lambda \) of \( X \) has a LPF-dominated splitting if \( \Lambda^* \neq \emptyset \) and there exist a continuous tangent bundle decomposition \( N^X_{\Lambda^*} = N^u_{\Lambda^*} \oplus N^s_{\Lambda^*} \) with \( \dim N^s_{\Lambda^*} = \dim N^u_{\Lambda^*} = 1 \) (\( \forall x \in \Lambda^* \)) and \( T > 0 \) such that

\[
\left\| P_t^X(x)/N^s_{\Lambda^*} \right\| = \left\| P_t^X(X_T(x))/N^u_{X_T(x)} \right\| \leq \frac{1}{2}, \quad \forall x \in \Lambda^*.
\]

**Theorem B.** If \( X \) is a \( C^1 \) generic three-dimensional flow of an orientable manifold, then neither \( \text{Cl}(\text{Sink}(X)) \setminus \text{Sink}(X) \) nor \( \text{Cl}(\text{Source}(X)) \setminus \text{Source}(X) \) have LPF-dominated splitting.

As an application we obtain a solution for Conjecture 1.3 in [17]. A periodic point \( x \) of \( X \) is a saddle if it has eigenvalues of modulus less and bigger than 1 simultaneously. Denote by \( \text{PSaddle}(X) \) the set of periodic saddles of \( X \). As is well known [13], through any \( x \in \text{PSaddle}(X) \) it passes a pair of invariant manifolds, the so-called strong stable and unstable manifolds \( W^{ss}(x) \) and \( W^{uu}(x) \), tangent to \( x \) to the eigenspaces corresponding to the eigenvalue of modulus less and bigger than 1 respectively. Saturating these manifolds with the flow we obtain the stable and unstable manifolds \( W^s(x) \) and \( W^u(x) \) respectively. A homoclinic point associated to \( x \) is a point \( q \) where these last manifolds meet. We say that \( q \) is a transverse homoclinic point if \( T_q W^s(x) \cap T_q W^u(x) \) is the one-dimensional subspace generated by \( X(q) \) and a homoclinic tangency otherwise.

We define the nonwandering set \( \Omega(X) \) as the set of points \( p \) such that for every \( T > 0 \) and every neighborhood \( U \) of \( p \) there is \( t > T \) satisfying \( X_t(U) \cap U \neq \emptyset \).

Following [17], we say that \( X \) is singular-Axiom A if there is a finite disjoint union

\[
\Omega(X) = \Lambda_1 \cup \cdots \cup \Lambda_r,
\]

where each \( \Lambda_i \) for \( 1 \leq i \leq r \) is a transitive hyperbolic set (if \( \Lambda_i \cap \text{Sing}(X) = \emptyset \)) or a singular-hyperbolic attractor for either \( X \) or \( -X \) (otherwise).

With these definitions we can state the following corollary.

**Corollary 1.1.** Every three-dimensional flow can be \( C^1 \) approximated by a flow exhibiting a homoclinic tangency or by a singular-Axiom A flow.

**Proof.** Passing to a finite covering if necessary we can assume that \( M \) is orientable. Let \( R(M) \) denote the set of three-dimensional flows which cannot be \( C^1 \) approximated by ones with homoclinic tangencies. As is well-known [9], \( \text{Cl}(\text{PSaddle}(X)) \) has a LPF-dominated splitting for every \( C^1 \) generic \( X \in R(M) \). Furthermore,

\[
(\text{Cl}(\text{Sink}(X)) \setminus \text{Sink}(X)) \cup (\text{Cl}(\text{Source}(X)) \setminus \text{Source}(X)) \subset \text{Cl}(\text{PSaddle}(X))
\]

Combining this inclusion with Theorem B we obtain \( \text{Cl}(\text{Sink}(X)) \setminus \text{Sink}(X) = \text{Cl}(\text{Source}(X)) \setminus \text{Source}(X) = \emptyset \), and so, \( \text{Sink}(X) \cup \text{Source}(X) \) consists of finitely many orbits, for every \( C^1 \)-generic \( X \in R(M) \). Now we obtain that \( X \) is singular-Axiom A by Theorem A in [18].
2. Proof of theorems A and B

Let $X$ be a three-dimensional flow. Denote by $\text{Crit}(X)$ the set of critical points.

Recall that a periodic point saddle if it has eigenvalues of modulus less and bigger than 1 simultaneously. Analogously for singularities by just replace 1 by 0 and the eigenvalues by their corresponding real parts. Denote by $\text{Sink}(X)$ and $\text{Saddle}(X)$ the set of sinks and saddles of $X$ respectively.

A critical point $x$ is dissipative if the product of its eigenvalues (in the periodic case) or the divergence $\text{div}X(x)$ (in the singular case) is less than 1 (resp. 0). Denote by $\text{Crit}^d(X)$ the set of dissipative critical points. We define the dissipative region by $\text{Dis}(X) = \text{Cl}(\text{Crit}^d(X))$.

For every subset $\Lambda \subset M$ we define the weak basin by

$$W^s_w(\Lambda) = \{ x \in M : \omega(x) \cap \Lambda \neq \emptyset \}.$$  

(This is often called weak region of attraction [7].) With these notations we obtain the following result. Its proof is similar to the corresponding one in [4]:

**Theorem 2.1.** There is a residual subset $R_6$ of three-dimensional flows $X$ for which $W^s_w(\text{Dis}(X))$ has full Lebesgue measure.

The homoclinic class associated to $x \in \text{PSaddle}(X)$ is the closure of the set of transverse homoclinic points $q$ associated to $x$. A homoclinic class of $X$ is the homoclinic class associated to some saddle of $X$.

Given a homoclinic class $H = H_X(p)$ of a three-dimensional flow $X$ we denote by $H_Y = H_Y(p_Y)$ the continuation of $H$, where $p_Y$ is the analytic continuation of $p$ for $Y$ close to $X$ (c.f. [19]).

The following lemma was also proved in [4]. In its statement $\text{Leb}$ denotes the normalized Lebesgue measure of $M$.

**Lemma 2.2.** There is a residual subset $R_{12}$ of three-dimensional flows $X$ such that for every hyperbolic homoclinic class $H$ there are an open neighborhood $O_{X,H}$ of $f$ and a residual subset $R_{X,H}$ of $O_{X,H}$ such that the following properties are equivalent:

1. $\text{Leb}(W^s_w(H_Y)) = 0$ for every $Y \in R_{X,H}$.
2. $H$ is not an attractor.

We say that a compact invariant set $\Lambda$ of a three-dimensional flow $X$ has a spectral decomposition if there is a disjoint decomposition

$$\Lambda = \bigcup_{i=1}^r H_i$$

into finitely many disjoint homoclinic classes $H_i$, $1 \leq i \leq r$, each one being either hyperbolic (if $H_i \cap \text{Sing}(X) = \emptyset$) or a singular-hyperbolic attractor for either $X$ or $-X$ (otherwise).

Now we prove the following result which is similar to one in [4] (we include its proof for the sake of completeness). In its statement $\text{PSaddle}^d(X)$ denotes the set of periodic dissipative saddles of a three-dimensional flow $X$.

**Theorem 2.3.** There is a residual subset $R_{11}$ of three-dimensional flows $Y$ such that if $\text{Cl}(\text{PSaddle}^d(Y))$ has a spectral decomposition, then the following properties are equivalent for every homoclinic $H$ associated to a dissipative periodic saddle:

1. $\text{Leb}(W^s_w(H)) > 0$. 

(b) $H$ is either hyperbolic attractor or a singular-hyperbolic attractor for $Y$.

Proof. Let $R_{12}$ be as in Lemma 2.2. Define the map $S : X^1(M) \to 2^c_c$ by $S(X) = \text{Cl}(\text{PSaddle}_d(X))$. This map is clearly lower-semicontinuous, and so, upper semicontinuous in a residual subset $\mathcal{N}$ (for the corresponding definitions see [14], [15]).

By Lemma 2.4 there is a residual subset $L$ of three-dimensional flows $X$ for which every singular-hyperbolic attractor with singularities of either $X$ or $-X$ has zero Lebesgue measure.

By the flow-version of the main result in [1], there is a residual subset $\mathcal{R}_7$ of three-dimensional flows $X$ such that for every singular-hyperbolic attractor $C$ for $X$ (resp. $-X$) there are neighborhoods $U_{X,C}$ of $C$, $U_{X,C}$ of $X$ and a residual subset $R_{X,C}^0$ of $\mathcal{U}_{X,C}$ such that for all $Y \in \mathcal{R}_{X,C}^0$, if $Z = Y$ (resp. $Z = -Y$) then

\[(1) \quad C_Y = \bigcap_{t \geq 0} Z_t(U_{X,C}) \text{ is a singular-hyperbolic attractor for } Z.\]

Define $\mathcal{R} = R_{12} \cap \mathcal{N} \cap L \cap \mathcal{R}_7$. Clearly $\mathcal{R}$ is a residual subset of three-dimensional flows. Define

\[
\mathcal{A} = \{ f \in \mathcal{R} : \text{Cl}(\text{PSaddle}_d(X)) \text{ has no spectral decomposition} \}.\]

Fix $X \in \mathcal{R} \setminus \mathcal{A}$. Then, $X \in \mathcal{R}$ and Cl(PSaddle$_d(X))$ has a spectral decomposition

\[
\text{Cl}(\text{PSaddle}_d(X)) = \left( \bigcup_{i=1}^{r_x} H^i \right) \cup \left( \bigcup_{j=1}^{a_x} A^j \right) \cup \left( \bigcup_{k=1}^{b_x} R^k \right)
\]

into hyperbolic homoclinic classes $H_i$ ($1 \leq i \leq r_x$), singular-hyperbolic attractors $A^j$ for $X$ ($1 \leq j \leq a_x$), and singular-hyperbolic attractors $R^k$ for $-X$ ($1 \leq k \leq b_x$).

As $X \in \mathcal{R}_{12} \cap \mathcal{R}_7$, we can consider for each $1 \leq i \leq r_x$, $1 \leq j \leq a_x$, and $1 \leq k \leq b_x$ the neighborhoods $O_{X,H_i}$, $U_{X,A^j}$ and $U_{X,R^k}$ of $X$ as well as their residual subsets $\mathcal{R}_{X,H_i}$, $\mathcal{R}_{X,A^j}^0$ and $\mathcal{R}_{X,R^k}^0$ given by Lemma 2.2 and 1 respectively.

Define

\[
\mathcal{O}_X = \left( \bigcup_{i=1}^{r_x} O_{X,H_i} \right) \cap \left( \bigcap_{j=1}^{a_x} U_{X,A^j} \right) \cap \left( \bigcap_{k=1}^{b_x} U_{X,R^k} \right)
\]

and

\[
\mathcal{R}_X = \left( \bigcup_{i=1}^{r_x} \mathcal{R}_{X,H_i} \right) \cap \left( \bigcap_{j=1}^{a_x} \mathcal{R}_{X,A^j}^0 \right) \cap \left( \bigcap_{k=1}^{b_x} \mathcal{R}_{X,R^k}^0 \right).
\]

Clearly $\mathcal{R}_X$ is residual in $\mathcal{O}_X$.

From the proof of Lemma 2.2 in [1], we obtain for each $1 \leq i \leq r_x$ a compact neighborhood $U_{X,i}$ of $H^i$ such that

\[(2) \quad H^i_Y = \bigcap_{t \in \mathcal{R}} Y_t(U_{X,i}) \quad \text{is hyperbolic and equivalent to } H^i, \quad \forall Y \in \mathcal{O}_{Y,H^i}.
\]

As $X \in \mathcal{N}$, $S$ is upper semicontinuous at $X$ so we can further assume that

\[
\text{Cl}(\text{PSaddle}_d(Y)) \subset \left( \bigcup_{i=1}^{r_x} U_{X,i} \right) \cup \left( \bigcup_{j=1}^{a_x} U_{X,A^j} \right) \cup \left( \bigcup_{k=1}^{b_x} U_{X,R^k} \right), \quad \forall Y \in \mathcal{O}_X.
\]
It follows that

\[ \text{Cl}(\text{PSaddle}_{d_i}(Y)) = \left( \bigcup_{i=1}^{r_X} H_i^{\gamma} \right) \cup \left( \bigcup_{j=1}^{a_X} A_j^{\gamma} \right) \cup \left( \bigcup_{k=1}^{b_X} R_k^{\gamma} \right), \quad \forall Y \in \mathcal{R}_X. \]

Next we take a sequence \( X' \in \mathcal{R} \setminus \mathcal{A} \) which is dense in \( \mathcal{R} \setminus \mathcal{A} \).
Replacing \( \mathcal{O}_{X'} \) by \( \mathcal{O}_{X',i} \) where

\[ \mathcal{O}_{X,0} = \mathcal{O}_{X,0} \quad \text{and} \quad \mathcal{O}_{X,i} = \mathcal{O}_{X,i} \setminus \left( \bigcup_{i=0}^{i-1} \mathcal{O}_{X,i+1} \right), \quad \text{for } i \geq 1, \]

we can assume that the collection \( \{ \mathcal{O}_{X,i} : i \in \mathbb{N} \} \) is pairwise disjoint.

Define

\[ \mathcal{O}_{12} = \bigcup_{i \in \mathbb{N}} \mathcal{O}_{X,i} \quad \text{and} \quad \mathcal{R}'_{12} = \bigcup_{i \in \mathbb{N}} \mathcal{R}_{X,i}. \]

We claim that \( \mathcal{R}'_{12} \) is residual in \( \mathcal{O}_{12} \).
Indeed, for all \( i \in \mathbb{N} \) write \( \mathcal{R}_{X,i} = \bigcap_{n \in \mathbb{N}} \mathcal{O}_{X,n}^n \), where \( \mathcal{O}_{X,n}^n \) is open-dense in \( \mathcal{O}_{X,i} \), for every \( n \in \mathbb{N} \). Since \( \{ \mathcal{O}_{X,i} : i \in \mathbb{N} \} \) is pairwise disjoint, we obtain

\[ \bigcap_{n \in \mathbb{N}} \bigcup_{i \in \mathbb{N}} \mathcal{O}_{X,n}^n \subset \bigcup_{n \in \mathbb{N}} \bigcap_{i \in \mathbb{N}} \mathcal{O}_{X,n}^n = \bigcup_{i \in \mathbb{N}} \mathcal{R}_{X,i} = \mathcal{R}'_{12}. \]

As \( \bigcup_{i \in \mathbb{N}} \mathcal{O}_{X,n}^n \) is open-dense in \( \mathcal{O}_{12} \), \( \forall n \in \mathbb{N} \), we obtain the claim.

Finally we define

\[ \mathcal{R}_{11} = \mathcal{A} \cup \mathcal{R}'_{12}. \]

Since \( \mathcal{R} \) is a residual subset of three-dimensional flows, we conclude as in Proposition 2.6 of \[10\] that \( \mathcal{R}_{11} \) is also a residual subset of three-dimensional flows.

Take \( Y \in \mathcal{R}_{11} \) such that \( \text{Cl}(\text{PSaddle}_{d_i}(Y)) \) has a spectral decomposition and let \( H \) be a homoclinic class associated to a dissipative saddle of \( Y \). Then, \( H \subset \text{Cl}(\text{PSaddle}_{d_i}(Y)) \) by Birkhoff-Smale’s Theorem \[12\]. Since \( \text{Cl}(\text{PSaddle}_{d_i}(Y)) \) has spectral decomposition, we have \( Y \notin \mathcal{A} \) so \( Y \in \mathcal{R}'_{12} \) thus \( Y \in \mathcal{R}_X \) for some \( X \in \mathcal{R} \setminus \mathcal{A} \). As \( Y \in \mathcal{R}_X \), \[8\] implies \( H = H_i^\gamma \) for some \( 1 \leq i \leq r_X \) or \( H = A_i^\gamma \) for some \( 1 \leq j \leq a_X \) or \( H = R_k^\gamma \) for some \( 1 \leq k \leq b_X \).

Now, suppose that \( \text{Leb}(W^\psi_Y(H)) > 0 \). Since \( Y \in \mathcal{R}_X \), we have \( Y \in \mathcal{R}_X^0 \) for all \( 1 \leq k \leq b_X \). As \( X \in \mathcal{L} \) and \( W^\psi_Y(R_k^\gamma) \subset R_k^\gamma \) for every \( 1 \leq k \leq b_X \), we conclude by Lemma \[24\] that \( H \neq R_k^\gamma \) for every \( 1 \leq k \leq b_X \).

If \( H = A_i^\gamma \) for some \( 1 \leq j \leq a_X \) then \( H \) is an attractor and we are done. Otherwise, \( H = H_i^\gamma \) for some \( 1 \leq i \leq r_X \). As \( Y \in \mathcal{R}_X \), we have \( Y \in \mathcal{R}_X^H \) and, since \( f \in \mathcal{R}_{12} \), we conclude from Lemma \[22\] that \( H_i^\gamma \) is an attractor. But by \[24\] we have that \( H_i^\gamma \) and \( H^\gamma \) are equivalent, so, \( H_i^\gamma \) is an attractor too and we are done. \[\square\]

We shall need the following lemma which was essentially proved in \[5\].

**Lemma 2.4.** There is a residual subset \( \mathcal{L} \) of three-dimensional flows \( X \) for which every singular-hyperbolic attractor with singularities of either \( X \) or \( -X \) has zero Lebesgue measure.
Proof. As in [5], for any open set $U$ and any three-dimensional vector field $Y$, let $\Lambda_Y(U) = \bigcap_{t \in \mathbb{R}} Y_t(U)$ be the maximal invariant set of $Y$ in $U$. Define $\mathcal{U}(U)$ as the set of flows $Y$ such that $\Lambda_Y(U)$ is a singular-hyperbolic set with singularities of $Y$. It follows that $\mathcal{U}(U)$ is open in $\mathcal{X}^1(M)$.

Now define $\mathcal{U}(U)_n$ as the set of $Y \in \mathcal{U}(U)$ such that $\text{Leb}(\Lambda_Y(U)) < 1/n$. It was proved in [5] that $\mathcal{U}(U)_n$ is open and dense in $\mathcal{U}(U)$.

Define $\mathcal{R}(U)_n = \mathcal{U}(U)_n \cup (\mathcal{X}^1(M) \setminus \overline{\mathcal{U}(U)})$ which is open and dense set in $\mathcal{X}^1(M)$. Let $\{U_m\}$ be a countable basis of the topology, and $\{O_m\}$ be the set of finite unions of such $U_m$’s. Define

$$\mathcal{L} = \bigcap_m \bigcap_n \mathcal{R}(O_m)_n.$$ 

This is clearly a residual subset of three-dimensional flows. We can assume without loss of generality that $\mathcal{L}$ is symmetric, i.e., $X \in \mathcal{L}$ if and only if $-X \in \mathcal{L}$. Take $X \in \mathcal{L}$. Let $\Lambda = \Lambda_X(O_m)$. Then $X \in \mathcal{U}(O_m)$ and so $X \in \mathcal{U}(O_m)_n$ for every $n$ thus $\text{Leb}(\Lambda) = 0$. Analogously, since $\mathcal{L}$ is symmetric, we obtain that $\text{Leb}(\Lambda) = 0$ for every singular-hyperbolic attractor with singularities of $-X$.

In the sequel we obtain the following key result representing the new ingredient with respect to [4]. Its proof will use the methods in [18]. In its statement card(Sink($X$)) denotes the cardinality of the set of different orbits of a three-dimensional flow $X$ contained in Sink($X$).

**Theorem 2.5.** There is a residual subset $Q$ of three-dimensional flows $X$ such that if $\text{card}(\text{Sink}(X)) < \infty$, then $\overline{\text{Cl}(\text{PSaddle}_d(X))}$ has a spectral decomposition.

**Proof.** First we state some useful notations.

Given a three-dimensional flow $Y$ and a point $p$ we denote by $O_Y(p) = \{Y_t(p) : t \in \mathbb{R}\}$ the $Y$-orbit of $p$. If $p \in \text{PSaddle}_d(Y)$ we denote by $E^s_{p,Y}$ and $E^u_{p,Y}$ the eigenspaces corresponding to the eigenvalues of modulus less and bigger than 1 respectively.

Denote by $\lambda(p,Y)$ and $\mu(p,Y)$ the eigenvalues of $p$ satisfying

$$|\lambda(p,Y)| < 1 < |\mu(p,Y)|.$$

Define the index of a singularity $\sigma$ as the number $\text{Ind}(\sigma)$ of eigenvalues with negative real part.

We say that a singularity $\sigma$ of $Y$ is Lorenz-like for $Y$ if its eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are real and satisfy $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$ (up to some order). It follows in particular that $\sigma$ is hyperbolic (i.e. without eigenvalues of zero real part) of index 2. Furthermore, the invariant manifold theory [18] implies the existence of stable and unstable manifolds $W^s_{\sigma,Y}(\sigma)$, $W^u_{\sigma,Y}(\sigma)$ tangent at $\sigma$ to the eigenspaces $\{\lambda_2, \lambda_3\}$ and $\lambda_1$ respectively. There is an additional invariant manifold $W^{ss}_{\sigma,Y}(\sigma)$, the strong stable manifold, contained in $W^s_{\sigma,Y}(\sigma)$ and tangent at $\sigma$ to the eigenspace corresponding to $\lambda_1$. We shall denote by $E^s_{\sigma,Y}$ and $E^u_{\sigma,Y}$ the eigenspaces associated to the set of eigenvalues $\lambda_2$ and $\{\lambda_3, \lambda_1\}$ respectively.

Let $S(M)$ be the set of three-dimensional flows $X$ with $\text{card}(\text{Sink}(X)) < \infty$ such that

$$\text{card}(\text{Sink}(Y)) = \text{card}(\text{Sink}(X)), \quad \text{for every } X \text{ close to } Y.$$

Every $X \in S(M)$ satisfies the following properties:
There is a LPF-dominated splitting over \( PSaddle_0^d(X) \setminus Sing(X) \), where \( PSaddle_0^d(X) \) denotes the set of points \( x \) for which there are sequences \( Y_k \to X \) and \( x_k \in PSaddle_d(X_k) \) such that \( x_k \to x \) (c.f. [21]).

There are a neighborhood \( \mathcal{U}_X \), \( 0 < \lambda < 1 \) and \( \alpha > 0 \) such that if \( (p, Y) \in PSaddle_d(Y) \times \mathcal{U}_X \), then

(a) \( |\lambda(p, Y)| < \lambda^{t_{p,Y}} \).
(b) \( |\mu(p, Y)| > \lambda^{-t_{p,Y}} \).

Indeed, the first property follows from the proof of Proposition 5.3 in [4] and the second from the proof of Theorem 3.6 in [13] (see also the proof of lemmas 7.2 and 7.3 in [4]).

In addition to this we also have the existence of a residual subset of three-dimensional flows \( \mathcal{R}_\mathcal{R} \) such that every \( X \in S(M) \cap \mathcal{R}_\mathcal{R} \) satisfies that:

- Every \( \sigma \in Sing(X) \cap \text{Cl}(PSaddle_d(X)) \) with \( Ind(\sigma) = 2 \) is Lorenz-like for \( X \) and satisfies \( \text{Cl}(PSaddle_d(X)) \cap W^{ss,X}(\sigma) = \{\sigma\} \).
- Every \( \sigma \in Sing(X) \cap \text{Cl}(PSaddle_d(X)) \) with \( Ind(\sigma) = 1 \) is Lorenz-like for \( -X \) and satisfies \( \text{Cl}(PSaddle_d(X)) \cap W^{uu,X}(\sigma) = \{\sigma\} \), where \( W^{uu,X}(\sigma) = W^{ss,-X}(\sigma) \).

Indeed, as in the remark after Lemma 2.13 in [8], there is a residual subset \( \mathcal{R}_\mathcal{R} \) of three-dimensional flows \( X \) such that every \( \sigma \in Sing(X) \) accumulated by periodic orbits is Lorenz-like for either \( X \) or \( -X \) depending on whether \( \sigma \) has three real eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \) satisfying either \( \lambda_2 < \lambda_3 < 0 < \lambda_1 \) or \( \lambda_2 < 0 < \lambda_3 < \lambda_1 \) (up to some order).

Now, take \( X \in S(M) \cap \mathcal{R}_\mathcal{R} \). Since \( X \in S(M) \), we have that \( PSaddle_0^d(X) \setminus Sing(X) \) has a LPF-dominated splitting and then \( \text{Cl}(PSaddle_d(X)) \setminus Sing(X) \) also does because \( \text{Cl}(PSaddle_d(X)) \subset PSaddle_0^d(X) \). Therefore, if \( \sigma \in Sing(X) \cap \text{Cl}(PSaddle_d(X)) \), Proposition 2.4 in [10] implies that \( \sigma \) has three different real eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \) satisfying \( \lambda_2 < \lambda_3 < 0 < \lambda_1 \) (up to some order). Since \( X \in \mathcal{R}_\mathcal{R} \), we conclude that \( \sigma \) is Lorenz-like for \( X \). To prove \( \text{Cl}(PSaddle_d(X)) \cap W^{ss,X}(\sigma) = \{\sigma\} \) we assume by contradiction that this is not the case. Then, there is \( x \in (\text{Cl}(PSaddle_d(X)) \cap W^{ss,X}(\sigma)) \setminus \{\sigma\} \). Choose sequences \( x_n \in \text{Cl}(PSaddle_d(X)) \) and \( t_n \to \infty \) such that \( x_n \to x \) and \( X_{t_n}(x_n) \to y \) for some \( y \in W^{uu,X}(\sigma) \setminus \{\sigma\} \). Let \( N_{x,X}^s \oplus N_{x,X}^u \) denote the LPF-dominated splitting of \( \text{Cl}(PSaddle_d(X)) \setminus Sing(X) \). We have \( N_{x,X}^s = N_x \cap W^{s,X}(\sigma) \) by Proposition 2.2 in [10] and so \( N_{x_n}^s \) tends to be tangent to \( W^{s,X}(\sigma) \) as \( n \to \infty \). On the other hand, Proposition 2.4 in [10] says that \( N_{y,X}^u \) is almost parallel to \( E_{\sigma,X}^{s,X} \). Therefore, the directions \( N_{x_n,X}^{s,X} \) tends to have positive angle with \( E_{\sigma,X}^{s,X} \). But using that \( \lambda_2 < \lambda_3 \) we can see that \( N_{x_n,X}^{s,X} = P_{-t_n}(X_{t_n}(x_n))N_{x_{t_n,X}}^{s,X} \) tends to be transversal to \( W^{s,X}(\sigma) \) nearby \( x \). As this is a contradiction, we obtain the result. The second property can be proved analogously.

On the other hand, there is another residual subset \( \mathcal{Q}_1 \) of three-dimensional flows for which every compact invariant set without singularities but with a LPF-dominated splitting is hyperbolic.

Indeed, by Lemma 3.1 in [8] we have that there is a residual subset \( \mathcal{Q}_1 \) of three-dimensional flows for which every transitive set without singularities but with a LPF-dominated splitting is hyperbolic. Fix \( X \in \mathcal{Q}_1 \) and a compact invariant set
without singularities but with a LPF-dominated splitting $N^X_A = N^X_A \ominus N^u_X$. Suppose by contradiction that $\Lambda$ is not hyperbolic. Then, by Zorn’s Lemma, there is a minimally nonhyperbolic set $\Lambda_0 \subset \Lambda$ (c.f. p.983 in [20]). Assume for a while that $\Lambda_0$ is not transitive. Then, $\omega(x)$ and $\alpha(x) = \omega_{-X}(x)$ are proper subsets of $\Lambda_0$, $\forall x \in \Lambda_0$. Therefore, both sets are hyperbolic and then we have
\[ \lim_{t \to \infty} \|P^X_t(x)/N^X_x\| = \lim_{t \to \infty} \|P^{-X}_{-t}(x)/N^u_x\| = 0, \forall x \in \Lambda_0, \]
which easily implies that $\Lambda_0$ is hyperbolic. Since this is a contradiction, we conclude that $\Lambda_0$ is transitive. As $X \in Q_1$ and $\Lambda_0$ has a LPF-dominated splitting (by restriction), we conclude that $\Lambda_0$ is hyperbolic, a contradiction once more proving the result.

Next we recall that a compact invariant set $\Lambda$ of a flow $X$ is Lyapunov stable for $X$ if for every neighborhood $U$ of $\Lambda$ there is a neighborhood $V \subset U$ of $\Lambda$ such that $X_t(V) \subset U$, for all $t \geq 0$.

It follows from [9, 17] that there is a residual subset $\mathcal{D}$ of three-dimensional flows $X$ such that if $\sigma \in \text{Sing}(X) \cap \text{Cl}(\text{PSaddle}_d(X))$ and $\text{Ind}(\sigma) = 2$, then $\text{Cl}(W^u(\sigma))$ is a Lyapunov stable set for $X$ with dense singular unstable branches contained in $\text{Cl}(\text{PSaddle}_d(X))$. Analogously, if $\text{Ind}(\sigma) = 1$, then $\text{Cl}(W^s(\sigma))$ is a Lyapunov stable set for $-X$ with dense singular stable branches contained in $\text{Cl}(\text{PSaddle}_d(X))$.

From these properties we derive easily that every $X \in S(M) \cap \mathcal{R}_7 \cap \mathcal{D}$ and every $\sigma \in \text{Sing}(X) \cap \text{Cl}(\text{PSaddle}_d(X))$ satisfies one of the following alternatives:

(c) If $\text{Ind}(\sigma) = 2$, then every $\sigma' \in \text{Sing}(X) \cap \text{Cl}(W^u(\sigma))$ is Lorenz-like for $X$.

(d) If $\text{Ind}(\sigma) = 1$, then every $\sigma' \in \text{Sing}(X) \cap \text{Cl}(W^s(\sigma))$ is Lorenz-like for $-X$.

Given a three-dimensional flow $Y$ we define
\[ E^c_{u,Y} = E^u_{p,Y} \oplus E^Y_p, \quad \forall p \in \text{PSaddle}_d(Y). \]

We claim that there is a residual subset of three-dimensional flows $\mathcal{R}_{15}$ such that for every $X \in S(M) \cap \mathcal{R}_{15}$ and every $\sigma \in \text{Sing}(X) \cap \text{Cl}(\text{PSaddle}_d(X))$ there are neighborhoods $V_X$ of $X$, $U_\sigma$ of $\sigma$ and $\beta_\sigma > 0$ such that if $Y \in V_X$ and $x \in \text{PSaddle}_d(Y)$ satisfies $O_Y(x) \cap U_\sigma \neq \emptyset$, then

(4) $\angle(E^u_x, E^c_{u,Y}) > \beta_\sigma$, if $\text{Ind}(\sigma) = 2$ and

(5) $\angle(E^s_x, E^c_{u,Y}) > \beta_\sigma$, if $\text{Ind}(\sigma) = 1$.

(This step corresponds to Theorem 3.7 in [13].)

Indeed, we just take $\mathcal{R}_{15} = Q_1 \cap \mathcal{D} \cap \mathcal{R}_7 \cap \mathcal{I}$ where $\mathcal{I}$ is the set of upper semi-continuity points of the the map $\phi : X \mapsto \text{Cl}(\text{PSaddle}_d(X))$.

To prove (4) it suffices to show the following assertions, corresponding to propositions 4.1 and 4.2 of [13] respectively, for any $X \in S(M) \cap \mathcal{R}_{15}$ and $\sigma \in \text{Sing}(X) \cap \text{Cl}(\text{PSaddle}_d(X))$ with $\text{Ind}(\sigma) = 2$ ($B_\delta(\cdot)$ denotes the $\delta$-ball operation):

At 1. Given $\epsilon > 0$ there are a neighborhood $V_{X,\sigma}$ of $X$ and $\delta > 0$ such that for all $Y \in V_{X,\sigma}$, if $p \in \text{PSaddle}_d(Y) \cap B_\delta(\sigma_Y)$ then

(a) $\angle(E^u_p, E^{ss}_p) < \epsilon$.

(b) $\angle(E^c_{u,Y}, E^{ss}_p) < \epsilon$. 
Given $\delta > 0$ there are a neighborhood $O$ of $X$ and $C > 0$ such that if $Y \in O$ and $p \in \text{PSaddle}_d(Y)$ with $\text{dist}(p, \text{Sing}(X) \cap \text{Cl}(\text{PSaddle}_d(X))) > \delta$, then

$$\angle(E^s_p, Y, E^{cu}_p, Y) > C.$$ 

To prove A1-(a) we proceed as in p. 417 of [9]. By contradiction suppose that it is not true. Then, there are $\gamma > 0$ and sequences $Y^n \to X$, $p_n \in \text{PSaddle}_d(Y^n) \to \sigma$ such that

$$\angle(E^s_{p_n}, Y^n, E^{ss}_{p_n}, Y^n) > \gamma, \quad \forall n \in \mathbb{N}.$$ 

As in [18] we take small cross sections $\Sigma^u_{\delta, \delta'}$ and $\Sigma^u_\delta$ located close to the singularities in $\text{Cl}(W^u(\sigma))$ all of which are Lorenz-like (by (c) above). It turns out that since $p_n \to \sigma$, there are times $t_n \to \infty$ satisfying $q_n = Y^n_{t_n}(p_n) \in \Sigma^u_\delta$. Using the above inequality we obtain

$$\angle(E^s_{q_n}, Y^n, E^{y^n}) \to 0.$$ 

Next consider the first $s_n > 0$ such that

$$\tilde{q}_n = Y^n_{s_n}(q_n) \in \Sigma^u_{\delta, \delta'}.$$ 

We obtain

$$\angle(E^s_{\tilde{q}_n}, Y^n, E^{s,y^n}) \to 0.$$ 

To see why, we assume two cases: either $s_n$ is bounded or not. If it does, then the above limit follows from the corresponding one for $q_n$. If not, we consider a limit point $q$ of the sequence $Y^n_{s_n}(q_n)$ with $s_n \to \infty$. After observing that the $X$-orbit of $q$ cannot accumulate any index 1 singularity we obtain easily that $q \in \Gamma$, where

$$\Gamma = \bigcap_{t \in \mathbb{R}} X_t (\text{Cl}(\text{PSaddle}_d(X)) \setminus B_{T'}(\text{Sing}(X) \cap \text{Cl}(\text{PSaddle}_d(X)))),$$ 

for some $\delta^* > 0$ small. Clearly $\Gamma$ is a compact invariant subset of $X$ contained in $\text{Cl}(\text{PSaddle}_d(X)) \setminus \text{Sing}(X)$. Since $X \in S(M)$, we have that $\Gamma$ has a LPF-dominated splitting, and so, it is hyperbolic because $X \in Q_1$. This allows us to repeat the proof in p. 419 to obtain (9) which, together with (b) above, implies that $\angle(E^s_{q_n}, Y^n, E^{s,y^n})$ is bounded away from zero. But now we consider the first positive time $r_n$ satisfying $\tilde{q}_n = Y^n_{r_n}(q_n) \in \Sigma^u_{\delta}$. We get as in p. 419 in [18] that $\angle(E^s_{\tilde{q}_n}, Y^n, E^{s,y^n}) \to 0$ and, since $\angle(E^s_{\tilde{q}_n}, Y^n, E^{s,y^n})$ is bounded away from 0, we also obtain $\angle(E^{u,y^n}_{\tilde{q}_n}, E^{s,y^n}_{\tilde{q}_n}) \to 0$. All this together yield $\angle(E^{s,y^n}_{\tilde{q}_n}, E^{u,y^n}_{\tilde{q}_n}) \to 0$ which contradicts (b). This contradiction completes the proof of A1-(a). The bound in A1-(b) follows easily from the methods in [10]. This completes the proof of A1. A2 follows exactly as in p. 421 of [18]. Now A1 and A2 imply (1) as in [18]. To prove (2) we only need to repeat the above proof with $-Y$ instead of $Y$. Taking into account the symmetric relations below:

$$\lambda(p, -Y) = \mu^{-1}(p, Y), \quad \mu(p, -Y) = \lambda^{-1}(p, Y), \quad E^s_{p, -Y} = E^{s,Y}_p \quad \text{and} \quad E^{u,Y}_{p, -Y} = E^{u,Y}_p.$$ 

Once we prove (1) and (2) we use them together with (a) and (b), as in the proof of Theorem F in [9], to obtain that for every $X \in R_{15} \cap S(M)$ there is a neighborhood $K_X$, $0 < \rho < 1$, $c > 0$, $\delta > 0$ and $T_0 > 0$ satisfying the following properties for every $Y \in K_X$ and every $x \in \text{PSaddle}_d(Y)$ satisfying $t_{x,Y} > T_0$ and $O_Y(x) \cap B_\delta(\sigma) \neq \emptyset$:
If \( \text{Ind}(\sigma) = 2 \), then
\[
\|DY_T(p)/E_p^{s,Y}\| \cdot \|DY_{-T}(p)/E^{c_{u,Y}}_{-T}(p)\| \leq c\rho^T, \quad \forall T > 0.
\]

If \( \text{Ind}(\sigma) = 1 \), then
\[
\|D(-Y)_T(p)/E_p^{s,-Y}\| \cdot \|D(-Y)_{-T}(p)/E^{c_{u,-Y}}_{-T}(p)\| \leq c\rho^T, \quad \forall T > 0.
\]

Since we can assume that \( X \) is Kupka-Smale (by the Kupka-Smale Theorem [12]), the set of periodic orbits with period \( \leq T_0 \) of \( X \) is finite. If one of these orbits (say \( O \)) do not belong to \( \text{Cl}(\text{Cl}(\text{PSaddle}_d(X))\setminus \{x \in \text{PSaddle}_d(X) : t x < T_0\}) \) then it must happen that \( O \) is isolated in the sense that \( \text{Cl}(\text{PSaddle}_d(X)) \setminus O \) is a closed subset. Therefore, up to a finite number of isolated periodic orbits, we can assume that the set \( \text{PSaddle}_d^p(X) = \{p \in \text{PSaddle}_d(X) : t_p x \geq T_0\} \) is dense in \( \text{Cl}(\text{PSaddle}_d(X)) \).

Then, as in p.400 of [13] we obtain the following properties:

- If \( \text{Ind}(\sigma) = 2 \), then the splitting \( E^{s,X} \oplus E^{c_{u,X}} \) extends to a dominated splitting \( E \oplus F \) for \( X \) over \( \text{Cl}(W^u(\sigma)) \) with \( \text{dim}(E) = 1 \) and \( E^X \subset F \).
- If \( \text{Ind}(\sigma) = 1 \) the splitting \( E^{s,-X} \oplus E^{c_{u,-X}} \) extends to a dominated splitting \( E \oplus F \) for \( -X \) over \( \text{Cl}(W^s(\sigma)) \) with \( \text{dim}(E) = 1 \) and \( E^{-X} \subset F \).

Therefore, we conclude from (c) and (d) above, lemmas 3.2 and 3.4 in [8] and Theorem D in [17] that if \( X \in \mathcal{R}_{15} \cap S(M) \) and \( \sigma \in \text{Sing}(X) \cap \text{Cl}(\text{PSaddle}_d(X)) \), then:

- If \( \text{Ind}(\sigma) = 2 \), then \( \text{Cl}(W^u(\sigma)) \) is a singular-hyperbolic attractor for \( X \).
- If \( \text{Ind}(\sigma) = 1 \), then \( \text{Cl}(W^s(\sigma)) \) is a singular-hyperbolic attractor for \( -X \).

Next, we define \( \phi : \mathbb{R}^1(M) \to 2_M^c \) by \( \phi(X) = \text{Cl}(\text{Sink}(X)) \). This map is clearly lower semicontinuous, and so, upper semicontinuous in a residual subset \( \mathcal{C} \) of \( \mathbb{R}^1(M) \) ([16], [11]). If \( X \in \mathcal{C} \) satisfies \( \text{card}(\text{Sink}(X)) < \infty \), then the upper semicontinuity of \( \phi \) at \( X \) do imply \( X \in S(M) \).

Finally we define
\[
\mathcal{Q} = \mathcal{R}_{15} \cap \mathcal{C}.
\]
Clearly \( \mathcal{Q} \) is a residual subset of three-dimensional flows.

Take \( X \in \mathcal{Q} \) with \( \text{card}(\text{Sink}(X)) < \infty \). Since \( X \in \mathcal{C} \), we obtain \( X \in S(M) \) thus \( X \in \mathcal{R}_{15} \cap S(M) \). Then, if \( \sigma \in \text{Sing}(X) \cap \text{Cl}(\text{PSaddle}_d(X)) \), \( \text{Cl}(W^u(\sigma)) \) is singular-hyperbolic for \( X \) (if \( \text{Ind}(\sigma) = 2 \)) and that \( \text{Cl}(W^s(\sigma)) \) is a singular-hyperbolic attractor for \( -X \) (if \( \text{Ind}(\sigma) = 1 \)).

Now we observe that if \( p \in \text{PSaddle}_d(X) \) then \( H(p) \subset \text{Cl}(\text{Saddle}_d(X)) \) by the Birkhoff-Smale Theorem. From this we obtain
\[
\text{Cl}(\text{PSaddle}_d(X)) = \text{Cl}\left(\bigcup \{H(p) : p \in \text{PSaddle}_d(X)\}\right).
\]

We claim that the family \( \{H(p) : p \in \text{PSaddle}_d(X)\} \) is finite. Otherwise, there is an infinite sequence \( p_k \in \text{PSaddle}_d(X) \) yielding infinitely many distinct homoclinic classes \( H(p_k) \). Consider the closure \( \text{Cl}(\bigcup_k H(p_k)) \), which is a compact invariant set contained in \( \text{Cl}(\text{PSaddle}_d(X)) \). If this closure does not contain any singularity, then it would be a hyperbolic set (this follows because \( \mathcal{R}_{15} \subset \mathcal{Q}_1 \)). Since there are infinitely many distinct homoclinic classes in this closure, we obtain a contradiction proving that \( \text{Cl}(\bigcup_k H(p_k)) \) contains a singularity \( \sigma \in \text{Cl}(\text{PSaddle}_d(X)) \). If \( \text{Ind}(\sigma) = 2 \) then \( \sigma \) lies in \( \text{Cl}(W^u(\sigma)) \) which is an attractor, and so, we can assume that \( H(p_k) \subset \text{Cl}(W^u(\sigma)) \) for every \( k \) thus \( H(p_k) = \text{Cl}(W^u(\sigma)) \) for every \( k \) which is absurd. Analogously for \( \text{Ind}(\sigma) = 1 \) the claim is proved. Combining with (7) we obtain the desired spectral decomposition. 
\( \square \)
Proof of Theorem A. Define $\mathcal{R} = \mathcal{R}_6 \cap \mathcal{R}_{11} \cap \mathcal{Q}$, where $\mathcal{R}_6$, $\mathcal{R}_{11}$ and $\mathcal{Q}$ are the residual subsets given by theorems 2.1, 2.3 and 2.5 respectively. Suppose that $X \in \mathcal{R}$ has no infinitely many sinks. Then, $\text{card}(\text{Sink}(X)) < \infty$. Since $X \in \mathcal{Q}$, we conclude by Theorem 2.5 that $\text{Cl}(\text{PSaddle}_d(X))$ has a spectral decomposition. Since $X \in \mathcal{R}_{11}$, Theorem 2.3 implies that every homoclinic $H$ associated to a dissipative periodic saddle of $X$ with $\text{Leb}(W^s_f(H)) > 0$ is an attractor of $X$. Since $X \in \mathcal{R}_6$, we have that $\text{Leb}(W^u_s(\text{Dis}(X))) = 1$ by Theorem 2.1.

Now, we consider the following decomposition:

$$\text{Dis}(X) = \text{Cl}(\text{Saddle}_d(X) \cap \text{Sing}(X)) \cup \text{Cl}(\text{PSaddle}_d(X)) \cup \text{Sink}(X),$$

valid in the Kupka-Smale case (which is generic). From this we obtain the union

$$W^s_u(\text{Dis}(X)) = \left( \bigcup \{W^s(\sigma) : \sigma \in \text{Saddle}_d(X) \cap \text{Sing}(X) \text{ and } W^u_s(\sigma) = W^s(\sigma) \} \right) \cup \left( \bigcup \{W^s(\sigma) : \sigma \in \text{Saddle}_d(X) \cap \text{Sing}(X) \text{ and } W^u_s(\sigma) \neq W^s(\sigma) \} \right) \cup W^u(\text{Cl}(\text{PSaddle}_d(X))) \cup W^s(\text{Sink}(X)).$$

But it is easy to check that the first element in the right-hand union above has zero Lebesgue measure and, by the Hayashi’s connecting lemma [11], we can assume without loss of generality that every $\sigma \in \text{Saddle}_d(X) \cap \text{Sing}(X)$ satisfying $W^u_s(\sigma) \neq W^s(\sigma)$ lies in $\text{Cl}(\text{PSaddle}_d(X))$. Since $W^u_s(\text{Dis}(X))$ has full Lebesgue measure, we conclude that

$$\text{Leb}(W^u_s(\text{Cl}(\text{PSaddle}_d(X))) \cup W^s(\text{Sink}(X))) = 1.$$ 

Now, we use the spectral decomposition

$$\text{Cl}(\text{PSaddle}_d(X)) = \bigcup_{i=1}^{r} H_i$$

into finitely many disjoint homoclinic classes $H_i$, $1 \leq i \leq r$, each one being either hyperbolic (if $H_i \cap \text{Sing}(X) = \emptyset$) or a singular-hyperbolic attractor for either $X$ or $-X$ (otherwise), yielding

$$\text{Leb} \left( \bigcup_{i=1}^{r} W^s_u(H_i) \cup W^s(\text{Sink}(X)) \right) = 1.$$ 

But the results in Section 3 of [9] imply that each $H_i$ can be written as $H_i = \Lambda^+ \cap \Lambda^-$, where $\Lambda^\pm$ is a Lyapunov stable set for $\pm X$. We conclude from Lemma 2.2 in [9] that $W^s_u(H_i) = W^s(H_i)$ thus

$$\text{Leb} \left( \bigcup_{i=1}^{r} W^s(H_i) \cup W^s(\text{Sink}(X)) \right) = 1.$$ 

Let $1 \leq i_1 \leq \cdots \leq i_d \leq r$ be such that $\text{Leb}(W^s(H_{i_k})) > 0$ for every $1 \leq k \leq d$. As the basin of the remainder homoclinic classes in the collection $H_1, \ldots, H_r$ are negligible, we can remove them from the above union yielding

$$\text{Leb} \left( \bigcup_{k=1}^{d} W^s(H_{i_k}) \cup \left( \bigcup_{j=1}^{i} W^s(s_j) \right) \right) = 1,$$

where the $s_j$’s above correspond to the finitely many orbits of $X$ in $\text{Sink}(X)$. Since $f \in \mathcal{R}_{11}$, we have from Theorem 2.3 that $H_{i_k}$ is an attractor which is either hyperbolic or singular-hyperbolic for $X$, $\forall 1 \leq k \leq d$. From this we obtain the result. □
Proof of Theorem B. Suppose by contradiction that there is a $C^1$ generic three-dimensional flow of an orientable manifold such that $\text{Cl}(\text{Sink}(X)) \setminus \text{Sink}(X)$ has a LPF-dominated splitting. Then, $\text{card}(\text{Sink}(X)) = \infty$ and $X$ has finitely many periodic sinks with nonreal eigenvalues. Since $X$ is $C^1$ generic, we obtain that the number of orbits of sinks with nonreal eigenvalues is locally constant at $X$. From this we can assume without loss of generality that every sink of a nearby flow is periodic with real eigenvalues. Furthermore, we obtain the following alternatives: If $\text{Ind}(\sigma) = 2$, then $\sigma$ is Lorenz-like for $X$ and satisfies
\[(\text{Cl}(\text{Sink}(X)) \setminus \text{Sink}(X)) \cap W^{ss,X}(\sigma) = \{\sigma\},\]
and, if $\text{Ind}(\sigma) = 1$, then $\sigma$ is Lorenz-like for $-X$ and satisfies
\[(\text{Cl}(\text{Sink}(X)) \setminus \text{Sink}(X)) \cap W^{wu,X}(\sigma) = \{\sigma\}.
As before, these alternatives imply the following ones:

1. If $\text{Ind}(\sigma) = 2$, then every $\sigma' \in \text{Sing}(X) \cap \text{Cl}(W^u(\sigma))$ is Lorenz-like for $X$.
2. If $\text{Ind}(\sigma) = 1$, then every $\sigma' \in \text{Sing}(X) \cap \text{Cl}(W^s(\sigma))$ is Lorenz-like for $-X$.

For any $p \in \text{Per}(X)$ we denote by $\lambda(p, X)$ and $\mu(p, X)$ the two eigenvalues of $p$ so that
\[|\lambda(p, X)| \leq |\mu(p, X)|.\]
The corresponding eigenspaces will be denoted by $E_p^{−,X}$ and $E_p^{+,X}$. We have the symmetric relations
\[\lambda(p, -X) = \mu^{-1}(p, X), \mu(p, -X) = \lambda^{-1}(p, X), E_p^{−,X} = E_p^{+,X}, E_p^{+,−X} = E_p^{−,−X}.

We obtain from the fact that the number of sinks with nonreal eigenvalues is locally constant at $X$ that there is a fixed number $0 < \lambda < 1$ and a neighborhood $U_X$ of $X$ satisfying:
\[\begin{align*}
&\text{(a) } \frac{\lambda(p, Y)}{|\mu(p, Y)|} \leq \lambda^{l,p,Y} \text{ and} \\
&\text{(b) } \angle(E_p^{−,X}, E_p^{+,X}) > \alpha, \text{ for every } (p, Y) \in \text{Sink}(Y) \times U_X. \\
\end{align*}\]

Using these properties we obtain as in the proof of Theorem 2.5 that there are neighborhoods $V_X$ of $X$, $U_\sigma$ of $\sigma$ and $\beta_\sigma > 0$ such that if $Y \in V_X$ and $x \in \text{Sink}(Y)$ satisfies $O_Y(x) \cap U_\sigma \neq \emptyset$, then
\[\angle(E_x^{−,Y}, E_x^{cu,Y}) > \beta_\sigma, \quad \text{if } \text{Ind}(\sigma) = 2\]
and
\[\angle(E_x^{−,−Y}, E_x^{cu,−Y}) > \beta_\sigma, \quad \text{if } \text{Ind}(\sigma) = 1.\]

Consequently there are a neighborhood $K_X$ of $X$, $0 < \rho < 1$, $c > 0$, $\delta > 0$ and $T_0 > 0$ satisfying the following properties for every $Y \in K_X$ and every $x \in \text{Cl}(\text{Sink}(Y)) \setminus \text{Sink}(Y)$ satisfying $t_{x,Y} > T_0$ and $O_Y(x) \cap B_\delta(\sigma) \neq \emptyset$:

- If $\text{Ind}(\sigma) = 2$, then
\[\|DY_T(p)/E_p^{−,Y}\| \cdot \|DY_T(p)/E_p^{cu,Y}\| \leq cp^T, \quad \forall T > 0.\]
- If $\text{Ind}(\sigma) = 1$, then
\[\|D(−Y)_T(p)/E_p^{−,Y}\| \cdot \|D(−Y)_T(p)/E_p^{cu,−Y}\| \leq cp^T, \quad \forall T > 0.\]

Using these dominations as before we obtain the following:

- If $\text{Ind}(\sigma) = 2$, then $\text{Cl}(W^u(\sigma))$ is a singular-hyperbolic attractor for $X$.
- If $\text{Ind}(\sigma) = 1$, then $\text{Cl}(W^s(\sigma))$ is a singular-hyperbolic attractor for $-X$. 
Since a singular-hyperbolic attractor for either $X$ or $-X$ cannot be accumulated by sinks we conclude that
\[ \text{Sing}(X) \cap (\text{Cl}(\text{Sink}(X)) \setminus \text{Sink}(X)) = \emptyset. \]
Since there is a LPF-dominated splitting, we conclude that $\text{Cl}(\text{Sink}(X)) \setminus \text{Sink}(X)$ is a hyperbolic set. Since there are only a finite number of orbits of sinks in a neighborhood of a hyperbolic set, we conclude that $\text{card}(\text{Sink}(X)) < \infty$ which is absurd. This concludes the proof. □

References

[1] Abdenur, F., Attractors of generic diffeomorphisms are persistent, Nonlinearity 16 (2003), no. 1, 301–311.
[2] Alves, J.F., Araújo, V., Pacifico, M.J., Pinheiro, V., On the volume of singular-hyperbolic sets, Dyn. Syst. 22 (2007), no. 3, 249–267.
[3] Araújo, A., Existência de atratores hiperbólicos para difeomorfismos de superfícies (Portuguese), Preprint IMPA Série F, No 23/88, 1988.
[4] Arbieto, A., Morales, C.A., Santiago, B., On Araújo’s Theorem for flows, Preprint (2013) arXiv:1307.5796v1 [math.DS] 22 Jul 2013.
[5] Arbieto, A., Obata, D.J., On attractors and their basins, Preprint 2012.
[6] Arroyo, A, Rodríguez Hertz, F., Homoclinic bifurcations and uniform hyperbolicity for three-dimensional flows, Ann. Inst. H. Poincaré Anal. Non Linéaire 20 (2003), no. 5, 805–841.
[7] Bhatia, N.P., Szegö, G.P., Stability theory of dynamical systems, Die Grundlehren der mathematischen Wissenschaften, Band 161 Springer-Verlag, New York-Berlin 1970.
[8] Bonatti, C., Gan, S., Yang, D., Dominated chain recurrent class with singularities, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) (To appear).
[9] Carballo, C.M., Morales, C.A., Pacifico, M.J., Homoclinic classes for generic C^1 vector fields, Ergodic Theory Dynam. Systems 23 (2003), no. 2, 403–415.
[10] Doering, C.I., Persistently transitive vector fields on three-dimensional manifolds, Dynamical systems and bifurcation theory (Rio de Janeiro, 1985), 5989, Pitman Res. Notes Math. Ser., 160, Longman Sci. Tech., Harlow, 1987.
[11] Hayashi, S., Connecting invariant manifolds and the solution of the C1 stability and -stability conjectures for flows, Ann. of Math. (2) 145 (1997), no. 1, 81–137.
[12] Hasselblatt, B., Katok, A., Introduction to the modern theory of dynamical systems (with a supplementary chapter by Katok and Leonardo Mendoza), Encyclopedia of Mathematics and its Applications, 54. Cambridge University Press, Cambridge, 1995.
[13] Hirsch, M., Pugh, C., Shub, M., Invariant manifolds, Lec. Not. in Math. 583 (1977), Springer-Verlag.
[14] Kuratowski, K., Topology. Vol. II, New edition, revised and augmented. Translated from the French by A. Kirkor Academic Press, New York-London; Państwowe Wydawnictwo Naukowe Polish Scientific Publishers, Warsaw 1968.
[15] Kuratowski, K., Topology. Vol. I, New edition, revised and augmented. Translated from the French by J. Jaworowski Academic Press, New York-London; Państwowe Wydawnictwo Naukowe, Warsaw 1966.
[16] Morales, C.A., Another dichotomy for surface diffeomorphisms, Proc. Amer. Math. Soc. 137 (2009), no. 8, 2639–2644.
[17] Morales, C.A., Pacifico, M.J., A dichotomy for three-dimensional vector fields, Ergodic Theory Dynam. Systems 23 (2003), no. 5, 1575–1600.
[18] Morales, C.A., Pacifico, M.J., Pujals, E.R., Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers, Ann. of Math. (2) 160 (2004), no. 2, 375–432.
[19] Palis, J., Takens, F., Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations. Fractal dimensions and infinitely many attractors. Cambridge Studies in Advanced Mathematics, 35. Cambridge University Press, Cambridge, 1993.
[20] Pujals, E.R., Sambarino, M., Homoclinic tangencies and hyperbolicity for surface diffeomorphisms, Ann. of Math. (2) 151 (2000), 961–1023.
[21] Wen, L., On the preperiodic set, Discrete Contin. Dynam. Systems 6 (2000), no. 1, 237–241.
Instituto de Matemática, Universidade Federal do Rio de Janeiro, P. O. Box 68530, 21945-970 Rio de Janeiro, Brazil.

E-mail address: arbieto@im.ufrj.br, bruno_santiago@im.ufrj.br