VARIATIONS OF LANDAU’S THEOREM FOR \(p\)-REGULAR AND \(p\)-SINGULAR CONJUGACY CLASSES

ALEXANDER MORETÓ AND HUNG NGOC NGUYEN

Dedicated to Professor Nguyễn H. V. Hùng on the occasion of his sixtieth birthday.

Abstract. The well-known Landau’s theorem states that, for any positive integer \(k\), there are finitely many isomorphism classes of finite groups with exactly \(k\) (conjugacy) classes. We study variations of this theorem for \(p\)-regular classes as well as \(p\)-singular classes. We prove several results showing that the structure of a finite group is strongly restricted by the number of \(p\)-regular classes or the number of \(p\)-singular classes of the group. In particular, if \(G\) is a finite group with \(O_p(G) = 1\) then \(|G/F(G)|_{p'}\) is bounded in terms of the number of \(p\)-regular classes of \(G\). However, it is not possible to prove that there are finitely many groups with no nontrivial normal \(p\)-subgroup and \(k\) \(p\)-regular classes without solving some extremely difficult number-theoretic problems (for instance, we would need to show that the number of Fermat primes is finite).

1. Introduction and statement of results

The well-known Landau’s theorem [21] states that, for any positive integer \(k\), there are finitely many isomorphism classes of finite groups with exactly \(k\) conjugacy classes or \(k\) (ordinary) irreducible characters. This theorem is indeed an immediate consequence of the number-theoretic fact proved by Landau himself in the same paper that the equation \(x_1^{-1} + x_2^{-1} + \cdots + x_k^{-1} = 1\) has only finitely many positive integer solutions. Using the classification of finite simple groups, L. Héthelyi and B. Külshammer [13] improved this theorem by replacing all conjugacy classes by only conjugacy classes of elements of prime power order. A block version of Landau’s theorem was proposed by R. Brauer in [3] – given a positive integer \(k\), are there only finitely many isomorphism classes of groups which can occur as defect groups of blocks of finite groups with exactly \(k\) irreducible characters? This was answered affirmatively by Külshammer in [18] for solvable groups and then in [19] for \(p\)-solvable groups. As far...
as we know, Brauer’s question is still open in general and indeed, a positive answer to this question follows if the Alperin-McKay conjecture is correct, as pointed out by B. Külshammer and G. R. Robinson [20].

Let $G$ be a finite group. Landau’s theorem can be restated in the following form: the order of $G$ is bounded in terms of the class number of $G$. That is, if $k$ is the number of classes of $G$, then $|G| < f(k)$ for some (increasing) real-valued function $f$ defined on the set of positive integers. We remark that the number of classes of $G$ is equal to the sum of the multiplicities of class sizes as well as the sum of the multiplicities of character degrees of $G$. A. Moretó [24] and D. A. Craven [6] have strengthened Landau’s theorem by showing that the order of a finite group is bounded in terms of the largest multiplicity of its character degrees. The dual statement for conjugacy classes was proved only for solvable groups by A. Jaikin-Zapirain [15] and seems very difficult to prove for arbitrary groups, see [20].

In this paper, we study other variations of Landau’s theorem. Let $p$ be a prime. Instead of considering all conjugacy classes, we consider only $p$-regular classes which are classes of elements of order not divisible by $p$, or only $p$-singular classes which are classes of elements of order divisible by $p$.

We write $k_{p'}(G)$ to denote the number of $p$-regular classes of $G$. Of course, $k_{p'}(G)$ does not say anything about $O_p(G)$, so it makes sense to assume that $O_p(G) = 1$. The question is: can we bound $|G|$ in terms of $k_{p'}(G)$? (Or, equivalently, we might have asked whether $|G/O_p(G)|$ is bounded in terms of $k_{p'}(G)$, without assuming that $O_p(G) = 1$.) It is easy to see that obtaining an affirmative answer to this question is very hard. Let $q = 2^n + 1$ be a Fermat prime. Consider the Frobenius group $F_n := C_q times C_{q-1}$ and let $F(G)$ denote the Fitting subgroup of $G$. We see that $k_{p'}(F_n) = 2$ and $|F_n/F(F_n)| = q - 1$. This example shows that if $|G/F(G)|$ is bounded in terms of $k_{p'}(G)$ then there would be finitely many Fermat primes. Therefore, it does not even seem feasible to bound $|G/F(G)|$. Of course, one could ask for a counterexample to our question that does not depend on the existence on infinitely many Fermat primes but, as we will discuss in Section 9, our question seems equivalent to some open (and extremely complicated) questions on prime numbers.

Let $O_\infty(G)$ denote the largest normal solvable subgroup of $G$, which is also referred to as the solvable radical of $G$. Our first result is the following.

**Theorem 1.1.** Let $p$ be a fixed prime and $G$ a finite group with $O_p(G) = 1$. If $k = k_{p'}(G)$ denotes the number of $p$-regular classes of $G$, then $|G/O_\infty(G)|$ is $k$-bounded and $|O_\infty(G)/F(G)|$ is metabelian by $k$-bounded.

The conclusion that $|O_\infty(G)/F(G)|$ is metabelian by $k$-bounded can be stated in other words, namely there exists $F(G) \subseteq N \leq O_\infty(G)$ such that $N/F(G)$ is metabelian and $|O_\infty(G)/N|$ is bounded in terms of $k$. Two remarks regarding Theorems 1.1 are in order. Firstly, although it does not say anything about $F(G)$, one easily sees that the nilpotency class of $F(G)$ and even the number of chief factors
of $G$ up to $F(G)$ is at most $k_{p'}(G)$. Therefore, the number of $p$-regular classes of $G$ somehow also controls the structure of $F(G)$. Secondly, we will see in Section 9 that there is not much room for improvement in Theorem 1.1.

Though the problem of bounding $|G/F(G)|$ is out of reach due to some difficult number-theoretic questions, we are able to prove that the $p'$-part of $|G/F(G)|$ is indeed bounded by the number of $p$-regular classes of $G$.

**Theorem 1.2.** Let $p$ be a fixed prime and $G$ a finite group with $O_p(G) = 1$. Then $|G/F(G)|_{p'}$ is bounded in terms of $k_{p'}(G)$.

The classification of groups $G$ with $O_p(G) = 1$ according to $k_{p'}(G)$ has been considered in several papers. First, with a block-theoretic motivation, Y. Ninomiya and T. Wada classified in [30] the groups with $k_{p'}(G) = 2$. Ninomiya then obtained the classification when $k_{p'}(G) = 3$ in [27, 28, 29] and G. Tiedt classified the solvable groups with $k_{p'}(G) = 4$ for $p$ odd in [36]. We have classified the groups with trivial solvable radical and $k_{p'}(G) = 4$.

**Theorem 1.3.** Let $p$ be a prime and $G$ a finite group with trivial solvable radical. Then $k_{p'}(G) = 4$ if and only if one of the following holds:

(i) $p = 2$ and $G \cong A_5$, $\text{PSL}_2(7)$, $A_6 \cdot 2_1$, $A_6 \cdot 2_2$, $\text{PSU}_3(3) \cdot 2$, $\text{PSL}_3(4) \cdot 2_1$, $\text{PSL}_3(4) \cdot 2^2$, or $2F_4(2)' \cdot 2$.

(ii) $p = 3$ and $G \cong A_5$.

(iii) $p = 7$ and $G \cong \text{PSL}_2(7)$.

We conjecture that these are all the nonsolvable groups with four $p$-regular classes and $O_p(G) = 1$ but, at this time, we have not been able to prove this.

Motivated by results on $p$-regular classes, we have considered the analogous question for $p$-singular classes. So let $k_p(G)$ denote the number of conjugacy classes of $p$-singular elements in $G$. We have obtained the following.

**Theorem 1.4.** Let $p$ be a fixed prime and $G$ a finite group whose non-abelian composition factors have order divisible by $p$ and $O_{p'}(G) = 1$. Assume that $G$ does not contain $\text{PSL}_2(p^f)$ with $f > F$ as a composition factor. Assume furthermore that

(i) if $p = 2$ then the Suzuki groups $Sz(2^f)$ with $f > F$ are not composition factors of $G$; and

(ii) if $p = 3$ then the Ree groups $2^G_2(3^f)$ with $f > F$ are not composition factors of $G$.

Then $|G/O_\infty(G)|$ is $(F, k_p(G))$-bounded and $|O_\infty(G)/F(G)|$ is metabelian by $(F, k_p(G))$-bounded.

The next result is an analog of Theorem 1.2.

**Theorem 1.5.** Assume the hypotheses of Theorem 1.4. Then $|G/F(G)|_p$ is bounded in terms of $F$ and $k_p(G)$. 
All the assumptions on the simple groups $\text{PSL}_2(p^f)$, $\text{Sz}(2^f)$, $\text{^2G}_2(3^f)$ are necessary. The simple linear group $\text{PSL}_2(2^f)$ with $f \geq 1$ has exactly $2(p^{2f} - 1)$ $p$-singular elements and two classes of $p$-singular elements. The Ree group $\text{^2G}_2(3^f)$ with $f \geq 3$ odd has exactly eight classes of $3$-singular elements (three classes of order 3 elements, three of order 9, and two of order 6), see [37]. Finally, we remark that the Suzuki groups $\text{Sz}(2^f)$ with $f \geq 3$ odd are the only simple groups whose orders are not divisible by 3.

One could ask whether we can weaken the hypothesis by considering only classes of $p$-elements. Indeed, our proof shows that if $G$ is solvable and $O_{p'}(G) = 1$ then $G$ has a normal subgroup $N$ such that $N/F(G)$ is metabelian and $|G/N|$ is bounded in terms of the number of classes of $p$-elements. Also, in [25] it was proved that the derived length of a Sylow $p$-subgroup of a finite group is bounded above by (a linear function of) the number of classes of $p$-elements. However, our results fail if we replace “$p$-singular classes” by “classes of $p$-elements”. For instance, the linear group $\text{PSL}_3(3^a)$ has three classes of 3-elements for every $a \in \mathbb{Z}^+$ (see [34]) and one could find counterexamples with any family of simple groups of Lie type.

Since the order of every non-abelian simple group is even, the following is an immediate consequence of Theorems 1.4 and 1.5.

**Corollary 1.6.** Let $G$ be a finite group with $O_2(G) = 1$ and assume that $G$ does not have composition factors isomorphic to $\text{PSL}_2(2^f)$ or $\text{Sz}(2^f)$ with $f > F$. Then

(i) $|G/O_\infty(G)|$ and $|G/F(G)|_2$ are both $(F, k_2(G))$-bounded; and
(ii) $|O_\infty(G)/F(G)|$ is metabelian by $(F, k_2(G))$-bounded.

Our proofs of Theorems 1.1 and 1.4 are divided into two parts. First we prove that $|G/O_\infty(G)|$ is $k_2(G)$-bounded (respectively, $(F, k_2(G))$-bounded) and then we prove our results in full. These theorems are of a qualitative nature only. One could obtain explicit bounds by following the proofs, but these would be far from best possible. For instance, following the proof of Theorem 1.1 we obtain that if $O_\infty(G) = 1$ and $k_2(G) = 4$ then $|G| \leq (11!^4/16)!$ while from Theorem 1.3 we see that indeed $|G| \leq 35,942,400$.

The outline of the paper is as follows. In the next section, we prove some preliminary lemmas which will be needed later in the proofs of the main results. In Section 3, we present some lower bounds for the number of $p$-regular classes of finite simple groups of Lie type. These bounds are crucial in the proof of the first part of Theorem 1.1 in Section 4. Similarly, we state and prove lower bounds for the number of $p$-singular classes of simple groups that we need for Theorem 1.4 in Section 5 and complete the proof of the first part of Theorem 1.4 in Section 6. Next, we prove the
full versions of Theorems 1.1 and 1.4 in Section 7. Proofs of Theorems 1.2 and 1.5 have the same flavor and are carried out in Section 8. We discuss the difficulties of improving on these theorems and some open questions in Section 9. Finally we prove Theorem 1.3 in Section 10.

Remark. When this paper was in the referee process, we learned that part of Theorem 1.1 was already obtained by D. S. Passman in [31]. Passman raised the following problem: Let $G$ have precisely $n$ irreducible representations over an algebraically closed field of characteristic $p > 0$. How much of the structure of $G$ can be bounded by a function of $n$, possibly depending upon $p$? He then proved that $|G/O_p'(O_\infty(G))|$ is bounded in terms of $n$, where $O_p'(O_\infty(G))$ is the smallest normal subgroup of $O_\infty(G)$ such that $O_\infty(G)/O_p'(O_\infty(G))$ is a $p'$-group (see [31, Corollary 3.5]). This relies on a result due to R. M. Guralnick [31, Theorem 3.4] which is similar to Proposition 3.4 of this paper but was proved by a different method. The first part of Theorem 1.1 follows from Passman’s result while the second part and Theorem 1.2 partly solve Passman’s problem.

2. Preliminaries

In this section we prove some easy but useful lemmas. The first two of them are well-known, but for the reader’s convenience we include their short proofs.

Lemma 2.1. Let $G$ be a finite group and $N \trianglelefteq G$. Then $k_{p'}(G/N) \leq k_{p'}(G)$ and $k_p(G/N) \leq k_p(G)$.

Proof. The first part of the lemma is obvious as $k_{p'}(G)$ is exactly the number of irreducible $p'$-Brauer characters of $G$ and every $p'$-Brauer character of $G/N$ can be viewed as a $p'$-Brauer character of $G$. So it remains to prove the second part of the lemma. Let $k = k_p(G/N)$ and let $\{g_1N, g_2N, ..., g_kN\}$ be a collection of representatives for $p$-singular classes of $G/N$. Since $o(gN)$ divides $|G|$ for every $g \in G$, we see that $g_1, g_2, ..., g_k$ are $p$-singular. It is obvious that $g_1, g_2, ..., g_k$ are in different classes of $G$ and hence the lemma follows. □

Lemma 2.2. Let $G$ be a finite group and $N \trianglelefteq G$. Then $k_{p'}(N) \leq |G:N|k_{p'}(G)$ and $k_p(N) \leq |G:N|k_p(G)$.

Proof. Consider the conjugation action of $G$ on the set of $p$-singular classes of $N$. Clearly every orbit of this action has size at most $|G:N|$. Therefore the number of $p$-singular classes of $G$ inside $N$ is at least $k_p(N)/|G:N|$ and it follows that $k_p(N) \leq |G:N|k_p(G)$. The other inequality is proved similarly. □

The following lemma is a consequence of a result of S. M. Seager [33].

Lemma 2.3. Let $G$ be a solvable group and $V$ a faithful irreducible $G$-module with $|V| = r^a$, where $r$ is a prime power. Assume that the number of $G$-orbits in $V$ is $k$. Then one of the following holds:
(1) $|G|$ is $k$-bounded.
(2) $G$ is isomorphic to a subgroup of the wreath product of the affine semilinear
    group $\Gamma(r^a)$ and a symmetric group $S_l$ for some $l$ that is $k$-bounded and $V =
    GF(r^a)^l$.

Proof. This follows immediately from Theorem 2.1 of [17], which is a reformulation
of Theorem 1 of [33]. □

Corollary 2.4. Assume the hypotheses of Lemma 2.3. Then $G$ is metabelian by
$k$-bounded.

Proof. This follows instantly from Lemma 2.3. □

We end this section with a number-theoretic lemma that will be needed in the
proof of Theorem 1.2.

Lemma 2.5. Let $p$ be a fixed prime and $r > 1$ be an integer. (Note that $r$
do not need to be fixed.) Let $n_p$ and $n_{p'}$ respectively denote the $p$-part and $p'$-part
of a positive integer $n$. Then

$$\frac{(r^a - 1)_{p'}}{a \cdot a_p} \to \infty \text{ as } a \to \infty.$$ 

Proof. Let $a = p^c b$ with $p \nmid b$. Since $\gcd(r^{pc} - 1, r^b - 1) = r - 1$ and $r^a - 1$ is divisible
by both $r^{pc} - 1$ and $r^b - 1$, we have

$$(r^a - 1)_{p'} \geq \frac{(r^{pc} - 1)_{p'}(r^b - 1)_{p'}}{(r - 1)_{p'}}.$$ 

It follows that

$$\frac{(r^a - 1)_{p'}}{a \cdot a_p} \geq \frac{(r^{pc} - 1)_{p'}}{p^{2c}} \cdot \frac{(r^b - 1)_{p'}}{b(r - 1)_{p'}}.$$ 

By Feit’s theorem [7, 32] on large Zsigmondy primes, for each positive integer $N$,
there exists a Zsigmondy prime $q$ such that $(r^b - 1)_q > bN + 1$ for all but finitely
many pairs of integers $(b, r)$ with $b > 2$ and $r > 1$. We note that when $b$ is large
enough then $q \neq p$. Therefore

$$(r^b - 1)_{p'} > (bN + 1)(r - 1)_{p'},$$

which implies that

$$\frac{(r^b - 1)_{p'}}{b(r - 1)_{p'}} \to \infty \text{ as } b \to \infty.$$ 

It remains to prove that

$$\frac{(r^{pc} - 1)_{p'}}{p^{2c}} \to \infty \text{ as } c \to \infty.$$
First we assume that $p \nmid (r - 1)$. Since $r^{p^c} \equiv r \pmod{p}$, we have $p \nmid (r^{p^c} - 1)$ so that $(r^{p^c} - 1)p' = (r^{p^c} - 1)$ and therefore we should be done in this case. So let us consider the case $p \mid (r - 1)$. Using [22, Lemma 8], we see that $(r^{p^c} - 1)p = p^c(r - 1)p$ if $p > 2$ or $(r - 1)p > p$ and $(r^{p^c} - 1)p = p^c(r + 1)p$ otherwise. In any case, we always have $(r^{p^c} - 1)p \leq p^c(r + 1)$. It follows that

\[
\frac{(r^{p^c} - 1)p'}{p^{2c}} \geq \frac{r^{p^c} - 1}{p^{3c}(r + 1)}.
\]

It is now easy to see that $(r^{p^c} - 1)p' / p^{2c} \to \infty$ as $c \to \infty$, as desired. \(\square\)

3. \textit{p-Regular classes of the simple groups of Lie type}

In this section, we prove Theorem 1.1 for simple groups of Lie type. The next three lemmas provide a lower bound for the number of $p$-regular classes of a simple group of Lie type. These bounds are probably not the best possible but they are enough for our purpose. We basically make use of a result of L. Babai, P. P. Pálfy, and J. Saxl [2] on the proportion of $p$-regular elements in finite simple groups (this has been improved in [1]) and a recent result of J. Fulman and R. M. Guralnick [8] on minimum centralizer sizes in finite classical groups.

**Lemma 3.1.** Let $G = \text{PSL}_n(q)$ or $\text{PSU}_n(q)$ be simple. For any prime $p$, we have

\[
k_{p'}(G) > \frac{q^{n-1}}{6n^3}.
\]

**Proof.** A lower bound for the smallest centralizer size in finite classical groups is given in [8, §6]. In particular, the centralizer size of an element of $\text{GL}_n(q)$ is at least

\[
\frac{q^{n-1}(q - 1)}{e(1 + \log_q(n + 1))}.
\]

Therefore, for every $x \in \text{PSL}_n(q)$, we have

\[
|C_{\text{PSL}_n(q)}(x)| \geq \frac{q^{n-1}(q - 1)}{e(1 + \log_q(n + 1))(q - 1)(q - 1, n)} = \frac{q^{n-1}}{e(1 + \log_q(n + 1))(q - 1, n)}.
\]

On the other hand, by [2] Theorem 1.1, if $G$ is a simple classical group acting naturally on a projective space of dimension $m - 1$, then the proportion of $p$-regular elements of $G$ is at least $1/(2m)$. Therefore the number of $p$-regular elements in $\text{PSL}_n(q)$ is at least

\[
\frac{1}{2n}|\text{PSL}_n(q)|.
\]
We thus obtain a lower bound on the number of classes of $p$-regular elements in $\text{PSL}_n(q)$:

$$k_p(\text{PSL}_n(q)) \geq \frac{q^{n-1}}{2ne(1 + \log_q(n + 1))(q - 1, n)},$$

and the lemma follows for the linear groups.

The arguments for the unitary groups follow similarly by using the fact that the centralizer size of an element of $\text{GU}_n(q)$ is at least $q^n(\frac{1 - 1/q^2}{e(2 + \log_q(n + 1))})^{1/2}$.

**Lemma 3.2.** Let $G = \text{PSp}_{2n}(q)$, $\Omega_{2n+1}(q)$, or $\text{PO}_{2n}^\pm(q)$ be simple. For any prime $p$, we have

$$k_p(G) > \frac{q^n}{120n^2}.$$ 

**Proof.** The proof uses the same ideas as the proof of the previous lemma. We present here a proof for the case $G = \text{PO}_{2n}^\pm(q)$ only.

By [8, Theorem 6.13], the centralizer size of an element of $\text{SO}_{2n}^\pm(q)$ is at least

$$q^n \left(\frac{1 - 1/q}{2e(\log_q(4n + 4))}\right)^{1/2}.$$ 

Therefore, for every $x \in \text{PO}_{2n}^\pm(q)$ we have

$$|C_{\text{PO}_{2n}^\pm(q)}(x)| \geq \frac{q^n}{4} \left(\frac{1 - 1/q}{2e(\log_q(4n + 4))}\right)^{1/2}.$$ 

As the number of $p$-regular elements in $\text{PO}_{2n}^\pm(q)$ is at least $|\text{PO}_{2n}^\pm(q)|/(4n)$, it follows that

$$k_p(\text{PO}_{2n}^\pm(q)) \geq \frac{q^n}{16n} \left(\frac{1 - 1/q}{2e(\log_q(4n + 4))}\right)^{1/2},$$

and the lemma follows for the simple orthogonal groups in even dimension. \hfill \Box

**Lemma 3.3.** Let $G_r(q)$ be a finite simple exceptional group of Lie type of rank $r$ defined over a field of $q$ elements. Then, for any prime $p$,

$$k_p(G_r(q)) > cq^r,$$

where $c$ is a universal constant not depending on $G_r(q)$ and $p$.

**Proof.** By [8] Theorem 6.15], for every $x \in G_r(q)$, we have

$$|C_{G_r(q)}(x)| \geq \frac{q^r}{A(\min\{q, r\})(1 + \log_q r)} \geq \frac{q^r}{32A},$$

where $A$ is an absolute constant not depending on the group $G_r(q)$. For a prime $p$, it is shown in [2] Theorem 1.1] that the proportion of $p$-regular elements in a simple
exceptional group of Lie type is greater than $1/15$. We deduce that

$$k_{p'}(G_r(q)) > \frac{1}{15} \cdot \frac{q^r}{32A} = \frac{q^r}{480A}.$$ 

The lemma now follows with $c := 1/(480A)$. □

Lemmas 3.1, 3.2, and 3.3 imply that a simple group of Lie type has many classes of $p$-regular elements when its order is large. In order to prove Theorem 1.1, we need a bit more than that.

**Proposition 3.4.** Let $G$ be a simple group of Lie type and let $k^*_p(G)$ denote the number of $\text{Aut}(G)$-classes of $p$-regular elements in $G$. Then we have

$$k^*_p(G) \to \infty \text{ as } |G| \to \infty.$$ 

**Proof.** We remark that $k^*_p(G) \geq k_{p'}(G)/|\text{Out}(G)|$ where $\text{Out}(G)$ is the outer automorphism group of $G$. First we assume that $G = G_r(q)$ is a finite simple exceptional group of Lie type of rank $r$ defined over a field of $q = \ell^f$ elements, where $\ell$ is prime. Then it is known that $|\text{Out}(G)| \leq 6f$ (see [11] for instance). It follows by Lemma 3.3 that

$$k^*_p(G) > \frac{cq^r}{6f}.$$ 

It is now easy to see that $k^*_p(G) \to \infty$ as $q \to \infty$.

The simple classical groups are treated similarly. We will prove only the case of linear groups as an example. So assume that $G = \text{PSL}_n(q)$ where $q = \ell^f$. Then $|\text{Out}(G)| = (2, q - 1)f$ when $n = 2$ and $|\text{Out}(G)| = 2(n, q - 1)f$ when $n > 2$. In particular, $|\text{Out}(G)| \leq 2nf$. Therefore, by Lemma 3.4 we have

$$k^*_p(G) > \frac{q^{n-1}}{12fn^4}.$$ 

Again, one can easily show that $q^{n-1}/(12fn^4) \to \infty$ as either $q \to \infty$ or $n \to \infty$. The proof is complete. □

4. **First part of Theorem 1.1**

The goal of this section is to prove the following result.

**Theorem 4.1.** Let $G$ be a finite group. Then $|G/\text{O}_\infty(G)|$ is $k_{p'}(G)$-bounded.

**Proof.** By Lemma 2.1, we have $k_{p'}(G/\text{O}_\infty(G)) \leq k_{p'}(G)$. Therefore, in order to prove the theorem, it suffices to assume that $\text{O}_\infty(G)$ is trivial. We now need to show that $|G|$ is bounded in terms of $k_{p'}(G)$.

Let $\text{Soc}(G)$ denote the socle of $G$, which in this case coincides with the generalized Fitting subgroup of $G$. Since the generalized Fitting subgroup contains its own centralizer, we have $C_G(\text{Soc}(G)) \subseteq \text{Soc}(G)$. It follows that $G$ is embedded in $\text{Aut}(\text{Soc}(G))$, the automorphism group of $\text{Soc}(G)$. As $|\text{Aut}(\text{Soc}(G))|$ is bounded in
terms of \( \text{Soc}(G) \), it is enough to bound \( \text{Soc}(G) \). Since \( G \) has no nontrivial solvable normal subgroup, \( \text{Soc}(G) \) is isomorphic to a direct product of non-abelian simple groups, say

\[
\text{Soc}(G) \cong S_1 \times S_2 \times \cdots \times S_t.
\]

First, we will show that \( t \) is at most \( k_{p'}(G) \). For each \( 1 \leq i \leq t \), choose a \( p \)-regular element \( x_i \) in \( S_i \). Consider the elements

\[
g_i = (x_1, x_2, \cdots, x_i, 1, \cdots, 1) \in \text{Soc}(G).
\]

As \( \text{o}(g_i) = \text{lcm}(\text{o}(x_1), \text{o}(x_2), \cdots, \text{o}(x_i)) \), these \( t \) elements of \( \text{Soc}(G) \) are all \( p \)-regular. Moreover, since \( G \) permutes the direct factors \( S_1, S_2, \ldots, S_t \) of \( \text{Soc}(G) \), the elements \( g_1, g_2, \ldots, g_t \) belong to different conjugacy classes of \( G \). Hence, \( G \) has at least \( t \) classes of \( p \)-regular elements so that \( t \leq k_{p'}(G) \).

Next, we will show that each \( |S_i| \) is bounded in terms of \( k_{p'}(G) \) for every \( i \). Since there are finitely many sporadic simple groups, we are left with two cases:

(i) Suppose that \( S_i = A_n \) is an alternating group. We will show that \( n \) is bounded in terms of \( k_{p'}(G) \). Assume, to the contrary, that \( n \) can be arbitrarily large. We choose an \( n \) large enough such that the number of primes smaller than \( n \) is at least \( k_{p'}(G) + 2 \). In that case, \( |A_n| = n!/2 \) is divisible by at least \( k_{p'}(G) + 1 \) primes different from \( p \). For each such prime \( q \), there is at least one \( G \)-conjugacy class of \( q \)-elements. Therefore, we deduce that \( G \) would have more than \( k_{p'}(G) \) classes of \( p \)-regular elements, a contradiction.

(ii) Suppose that \( S_i \) is a simple group of Lie type. Assume that \( |S_i| \) can be arbitrarily large. Then, by Theorem 3.4, the number of \( \text{Aut}(S_i) \)-classes of \( p \)-regular elements in \( S_i \), denoted by \( k_{p'}^*(S_i) \), would be arbitrarily large as well. However, if \( x \) and \( y \) lie in different \( \text{Aut}(S_i) \)-classes of \( S_i \), then \((1, \ldots, 1, x, 1, \ldots, 1)\) and \((1, \ldots, 1, y, 1, \ldots, 1)\) lie in different \( G \)-classes, and we therefore deduce that the number of \( p \)-regular classes of \( G \) would be arbitrarily large. This contradiction completes the proof. \( \square \)

5. \( p \)-Singular classes of simple groups

In this section, we prove an analogue of Proposition 3.4 for \( p \)-singular conjugacy classes. The following will be needed in the proof of the first part of Theorem 1.4.

**Proposition 5.1.** Let \( p \) be a prime and \( S \) a non-abelian simple group with \( |S| \) divisible by \( p \). Assume that \( S \neq \text{PSL}_2(p^f) \). Assume furthermore that \( S \neq \text{Sz}(2^f) \) if \( p = 2 \) and \( S \neq 2G_2(3^f) \) if \( p = 3 \). Let \( k_p^*(S) \) denote the number of \( \text{Aut}(S) \)-classes of \( p \)-singular elements inside \( S \). Then we have

\[
k_p^*(S) \to \infty \text{ as } |S| \to \infty.
\]

**Proof.** First we consider the alternating groups of degree at least 5. Since the number of prime divisors of \( |A_n| \) tends to infinity as \( n \) tends to infinity, it is easy to see that \( k_p(A_n) \to \infty \) as \( n \to \infty \). Moreover, as \( k_2^*(A_n) \geq k_2(A_n)/2 \), it follows that \( k_2^*(A_n) \to \infty \)
as \( n \to \infty \) and the proposition follows for the alternating groups. We therefore can assume from now on that \( G \) is a simple group of Lie type. Let \( h(S) \) denote the Coxeter number of the associated Weyl group of \( S \). Following [14], we denote by \( \mu(S) \) the probability that an element of \( S \) is \( p \)-singular. In other words, \( \mu(S) \) is the proportion of \( p \)-singular elements in \( S \).

(i) First, we assume that the characteristic of the underlying field of \( S \) is different from \( p \). By [14, Theorem 5.1], we have 
\[
\mu(S) \geq \frac{1}{2ph(S)} \left( 1 - \frac{1}{p} \right)
\]
except when \( p = 3 \) and \( S = PSL_3(q) \) with \( (q - 1)_3 = 3 \) or \( S = PSU_3(q) \) with \( (q + 1)_3 = 3 \), in which case \( \mu(S) = 1/9 \). (Here \( x_3 \) denotes the 3-part of an integer \( x \).) So in any case, we always have
\[
\mu(S) \geq \frac{p - 1}{2ph(S)} > \frac{1}{4h(S)}
\]

The arguments for simple classical groups are fairly similar. We present here a proof for the linear groups. So assume that \( S \cong PSL_n(q) \). As mentioned in the proof of Lemma 3.1 for \( x \in PSL_n(q) \), we have
\[
|C_{PSL_n(q)}(x)| \geq \frac{q^{n-1}}{e(1 + \log_q(n + 1))(q - 1, n)}.
\]
Therefore,
\[
k_p(PSL_n(q)) \geq \frac{q^{n-1}}{4e(1 + \log_q(n + 1))(q - 1, n)h(PSL_n(q))} = \frac{q^{n-1}}{4en(1 + \log_q(n + 1))(q - 1, n)}.
\]
Let \( q = \ell^f \) where \( \ell \) is a prime (unequal to \( p \)). Since \( |\text{Out}(PSL_n(q))| \leq 2f(n, q - 1) \), we deduce that
\[
k_2^*(PSL_n(q)) \geq \frac{q^{n-1}}{8efn(1 + \log_q(n + 1))(q - 1, n)^2}.
\]
It is now easy to see that \( k_2^*(PSL_n(q)) \to \infty \) as \( |PSL_n(q)| \to \infty \).

Now we are left with the exceptional groups. Since \( h(S) \leq 30 \) for every simple exceptional group \( S \), we have \( \mu(S) \geq 1/(120) \). Assume that \( S \) is defined over a field of \( q \) elements with \( r \) the rank of the ambient algebraic group. By [8, Theorem 6.15], for every \( x \in S \), we have
\[
|C_S(x)| \geq \frac{q^r}{A(\min\{q, r\})(1 + \log_q r)} \geq \frac{q^r}{32A},
\]
where \( A \) is an absolute constant. It follows that
\[
k_p(S) \geq \frac{q^r}{120 \cdot 32A} = \frac{q^r}{3840A}.
\]
Thus
\[ k_p^*(S) \geq \frac{q^r}{3840A \cdot 6^f}, \]
where \( q = \ell^f \) for a prime \( \ell \neq p \). Again, we see that \( k_p^*(S) \to \infty \) as \( q \to \infty \).

(ii) Next, we assume that the underlying field of \( S \) has characteristic \( p \). By [14, Theorem 10.1], we have
\[ \mu(S) \geq \frac{2}{5q}. \]
(We note that the bound given in [14, Theorem 10.1] is \( \frac{2}{5q} - \delta \) where \( \delta = 1 \) unless \( S \) is a Suzuki or Ree group, in which case \( \delta = 2 \). In that paper, the authors use notation \( 2B_2(q^2), 2G_2(q^2), 2F_4(q^2) \) to denote the Suzuki and Ree groups. Here we think it is more convenient to write \( 2B_2(q), 2G_2(q), 2F_4(q) \).)

If \( S \) is a simple classical group different from \( \text{PSL}_2(pf) \), by using the lower bound for the centralizer size as in (i), we can also deduce that \( k_p^*(S) \to \infty \) as \( |S| \to \infty \). The same thing is true for exceptional simple groups as long as the rank \( r \) associated to \( S \) is greater than 1 since
\[ \frac{2}{5q} \cdot \frac{q^r}{32A \cdot 6^f} \to \infty \text{ as } q \to \infty, \]
if \( r > 1 \). Therefore, if \( S \) is a simple group of exceptional Lie type in characteristic \( p \) with \( S \neq \text{Sz}(2^f) \) when \( p = 2 \) and \( S \neq 2G_2(3^f) \) when \( p = 3 \), then
\[ k_p^*(S) \to \infty \text{ as } |S| \to \infty. \]
The proof is complete. \( \square \)

6. First part of Theorem 1.4

We are ready to prove the first part of Theorem 1.4, which we restate below for the reader’s convenience. The ideas are quite similar to those in Theorem 4.1.

**Theorem 6.1.** Let \( p \) be a prime and \( G \) a finite group whose non-abelian composition factors have order divisible by \( p \). Assume that \( G \) does not contain \( \text{PSL}_2(pf) \) with \( f > F \) as a composition factor. Assume furthermore that

(i) if \( p = 2 \) then the Suzuki groups \( \text{Sz}(2^f) \) with \( f > F \) are not composition factors of \( G \); and

(ii) if \( p = 3 \) then the Ree groups \( 2G_2(3^f) \) with \( f > F \) are not composition factors of \( G \).

Then \( |G/O_\infty(G)| \) is \( (F, k_p(G)) \)-bounded.

**Proof.** By Lemma 2.1 it suffices to assume that \( O_\infty(G) \) is trivial. Recall that \( G \) does not contain \( \text{PSL}_2(pf) \) with \( f > F \) as a composition factor. When \( p = 2 \) or \( p = 3 \), the Suzuki groups \( \text{Sz}(2^f) \) or respectively the Ree groups \( 2G_2(3^f) \) with \( f > F \) are not composition factors of \( G \). We want to show that \( |G| \) is bounded in terms of \( F \)
and $k_p(G)$. As before, it is enough to bound $|\text{Soc}(G)|$. Since $G$ has no nontrivial solvable normal subgroup, $\text{Soc}(G)$ is isomorphic to a direct product of non-abelian simple groups, say $\text{Soc}(G) \cong S_1 \times S_2 \times \cdots \times S_t$.

First, we will show that $t$ is at most $k_p(G)$. We choose a $p$-singular element $x_1$ in $S_1$ and arbitrary elements $x_2, x_3, \ldots, x_t$ in $S_2, S_3, \ldots, S_t$, respectively. (This can be done since every non-abelian composition factor of $G$ has order divisible by $p$.) Consider the elements

$$g_i = (x_1, x_2, \ldots, x_i, 1, \ldots, 1) \in \text{Soc}(G)$$

where $i = 1, 2, \ldots, t$. As $o(g_i) = \text{lcm}(o(x_1), o(x_2), \ldots, o(x_i))$, these $t$ elements of $\text{Soc}(G)$ are $p$-singular. Moreover, the elements $g_1, g_2, \ldots, g_t$ belong to different conjugacy classes of $G$. We conclude that $G$ has at least $t$ classes of $p$-singular elements so that $t \leq k_p(G)$.

Now it suffices to show that if $S$ is a direct factor of $\text{Soc}(G)$ different from $\text{PSL}_2(p^f)$, and $\text{Sz}(2^f)$ for $p = 2$, and $^2G_2(2^f)$ for $p = 3$, then $|S|$ is bounded in terms of $k_2(G)$. Assume, to the contrary, that $|S|$ can be arbitrarily large. Then, by Proposition 5.1, the number $k_p^*(S_1)$ of $\text{Aut}(S)$-classes of $p$-singular elements in $S$ would be arbitrarily large as well. However, if $x$ and $y$ lie in different $\text{Aut}(S)$-classes of $S_1$, then $(1, \ldots, 1, x, 1, \ldots, 1)$ and $(1, \ldots, 1, y, 1, \ldots, 1)$ lie in different $G$-classes, and we therefore deduce that $k_p(G)$ would be arbitrarily large, as required.

\section{Proofs of Theorems 1.1 and 1.4}

We now can complete the proof of Theorems 1.1 and 1.4. By Theorems 4.1, 6.1 and Lemma 2.2 we may assume in this section that $G$ is solvable.

\textbf{Proof of Theorem 1.1.} Since $O_{p'}(G) = 1$, we have that $F(G)$ is a $p'$-group. Since $F(G/\Phi(G)) = F(G)/\Phi(G)$ and $k_{p'}(G/\Phi(G)) \leq k_p(G)$, we may assume that $\Phi(G) = 1$. By Gashütz’s theorem (see, for instance, Theorem 1.12 of [23]), $H = G/F(G)$ acts faithfully and completely reducibly on $V = F(G)$. Write $V = V_1 \oplus \cdots \oplus V_t$ as a direct sum of irreducible $H$-modules. Now $H$ is isomorphic to a subgroup of the direct product of the groups $G/C_G(V_i)$ for $i = 1, \ldots, t$. By Corollary 2.4 each of these quotient groups is metabelian by $k$-bounded. Also, since $G$ has $k$ classes of $p$-regular elements, we have $t \leq k$. Therefore, the direct product of the groups $G/C_G(V_i)$ for $i = 1, \ldots, t$ is metabelian by $k$-bounded. We deduce that $H$ is metabelian by $k$-bounded. \hfill \Box

The rest of the proof of Theorem 1.4 goes along the same lines. In fact, as we already mentioned in the introduction, we can obtain a bit more.

\textbf{Theorem 7.1.} Let $G$ be a solvable group with $O_{p'}(G) = 1$. Then there exists a normal subgroup $N$ containing $F(G)$ such that $N/F(G)$ is metabelian and $|G/N|$ is bounded in terms of the number of classes of $p$-elements of $G$. 
Proof. Let $k$ be the number of conjugacy classes of $p$-elements of $G$. Since $O_p(G) = 1$, the Fitting subgroup $F(G)$ is a $p$-group. Since $F(G/F(G)) = F(G)/F(G)$ and the number of classes of $p$-elements of $G/F(G)$ is at most the number of classes of $p$-elements of $G$, we may assume that $F(G) = 1$. By Gaschütz’s theorem, $H = G/F(G)$ acts faithfully and completely reducibly on $V = F(G)$. As in the proof above, we deduce that $H$ is metabelian by $k$-bounded.

8. Proofs of Theorems 1.2 and 1.5

We will prove Theorems 1.2 and 1.5 in this section. Since their proofs are partly similar, we will skip some details in the proof of Theorem 1.5.

Proof of Theorem 1.2. In light of Theorem 1.1 and Lemma 2.2 we may assume that $G$ is solvable. We keep some of the notation in the proof of Theorem 1.1. As before, we may assume that $Φ(G) = 1$. Then Gaschütz’s theorem implies that $H = G/F(G)$ acts faithfully and completely reducibly on $V = F(G)$. Write

$$V = V_1 \oplus \cdots \oplus V_t$$

as a direct sum of irreducible $H$-modules. We have that $H$ is isomorphic to a subgroup of the direct product of the groups $G/C_G(V_i)$ for $i = 1, \ldots, t$. Since $G$ has $k$ classes of $p$-regular elements, we have $t \leq k$. We write $G_i := G/C_G(V_i)$ and notice that it suffices to bound $|G_i|_{p'}$ for each $i = 1, \ldots, t$. We note that the number of orbits of $G_i$ on $V_i$ is at most $k$. Therefore, by Lemma 2.3, we may assume that $G_i$ is isomorphic to a subgroup of the wreath product of $Γ(r^α)$ and $S_t$, for some $l$ that is $k$-bounded. Also,

$$V_i = W_{i1} \oplus \cdots \oplus W_{it},$$

where $W_{ij} ≅ GF(r^α)$ for every $1 \leq i \leq t$ and $1 \leq j \leq l$.

With a slight abuse of notation, we view $G_i$ as a subgroup of $Γ(r^α) \wr S_t$. In order to bound $|G_i|_{p'}$ it suffices to bound $|G_i \cap Γ(r^α)|_{p'}$. We write

$$Γ(r^α)^l = Γ_1 \times \cdots × Γ_t.$$  

For each $1 \leq j \leq l$, we have $Γ_j = H_j B_j$ where $H_j$ is the Galois group $Gal(GF(r^α)/GF(r))$ and $B_j = GF(r^α)^*$. In order to bound $|G_i \cap Γ(r^α)|_{p'}$ it suffices to bound $|α_j(G_i \cap Γ(r^α))|_{p'}$ for each $1 \leq j \leq l$.

Claim 1: $|α_j(G_i \cap Γ(r^α)) : α_j(G_i \cap B_1 \times \cdots \times B_l)|_{p'}$ is $k$-bounded.

Proof: Since $G_i$ is a quotient of $G$ and $k = k_{p'}(G)$, it follows that $G_i$ has at most $k$ $p$-regular classes. The index of $G_i \cap Γ(r^α)$ in $G_i$ is $k$-bounded, so the number of
$p$-regular classes of $G_i \cap \Gamma(r^a)^l$ is $k$-bounded. Therefore, the number of $p$-regular classes of $\alpha_j(G_i \cap \Gamma(r^a)^l)$ is $k$-bounded.

Now, we have that

$$(G_i \cap \Gamma(r^a)^l)/(G_i \cap B_1 \times \cdots \times B_i)$$

is abelian and since

$$\alpha_j(G_i \cap \Gamma(r^a)^l)/\alpha_j(G_i \cap B_1 \times \cdots \times B_i)$$

is isomorphic to a quotient of this group, it is also abelian. By the previous paragraph, the $p'$-part of its order is $k$-bounded and the claim is proved.

**Claim 2:** $|\alpha_j(G_i \cap B_1 \times \cdots \times B_i)|_{p'}/a$ is $k$-bounded.

Proof: We observe that $|\alpha_j(G_i \cap \Gamma(r^a)^l) : \alpha_j(G_i \cap B_1 \times \cdots \times B_i)|$ divides $a$. Also, the group $\alpha_j(G_i \cap B_1 \times \cdots \times B_i)$ is abelian. Therefore,

$$|\alpha_j(G_i \cap B_1 \times \cdots \times B_i)|_{p'} = k_{p'}(\alpha_j(G_i \cap B_1 \times \cdots \times B_i))$$

$$\leq k_{p'}(\alpha_j(G_i \cap \Gamma(r^a)^l)) \frac{|\alpha_j(G_i \cap \Gamma(r^a)^l)|}{|\alpha_j(G_i \cap B_1 \times \cdots \times B_i)|}$$

$$\leq a k_{p'}(\alpha_j(G_i \cap \Gamma(r^a)^l)).$$

As argued above, the number of $p$-regular classes of $\alpha_j(G_i \cap \Gamma(r^a)^l)$ is $k$-bounded and thus the claim follows.

Recall that the number of orbits of $G_i$ on $V_i$ is at most $k$. Therefore the number of orbits of $G_i \cap \Gamma(r^a)^l$ on $V_i$ is at most $k|G_i : G_i \cap \Gamma(r^a)^l|$, which is a $k$-bounded quantity. We deduce that the number of orbits of $\alpha_j(G_i \cap \Gamma(r^a)^l)$ on $W_{ij}$ is $k$-bounded for every $j$. Therefore,

$$(r^a - 1)/|\alpha_j(G_i \cap \Gamma(r^a)^l)|$$

is $k$-bounded.

In other words, we have

$$\frac{(r^a - 1)_{p'}}{|\alpha_j(G_i \cap \Gamma(r^a)^l)|_{p'}} \cdot \frac{(r^a - 1)_p}{|\alpha_j(G_i \cap \Gamma(r^a)^l)|_p}$$

is $k$-bounded.

Since $\alpha_j(G_i \cap \Gamma(r^a)^l)$ is a subgroup of $\Gamma$ whose order is $a(r^a - 1)$, it follows that

$$|\alpha_j(G_i \cap \Gamma(r^a)^l)|_p \leq a_p(r^a - 1)_p.$$

We therefore find that

$$\frac{(r^a - 1)_{p'}}{a_p \cdot |\alpha_j(G_i \cap \Gamma(r^a)^l)|_p}$$

is $k$-bounded.

Combining this with the claims above, we have

$$\frac{(r^a - 1)_{p'}}{a \cdot a_p}$$

is $k$-bounded.
Proof of Theorem 1.5. By Theorem 6.1 and Lemma 2.2 we may assume that $\Phi$ is solvable. Let $k := k_p(G)$ be the number of $p$-singular classes of $G$. As before, we may assume that $\Phi(G) = 1$. Then Gaschütz’s theorem implies that $G/F(G)$ acts faithfully and completely reducibly on $F(G)$, which is a $p$-group since $O_{p'}(G) = 1$.

We now follow the notation in the proof of Theorem 1.2. In particular, we define $V_i, G_i, W_{ij}, \alpha_j$, and $\Gamma = H_jB_j$ with $1 \leq i \leq t$ and $1 \leq j \leq l$ as before. In order to bound $|G/F(G)|_p$, it suffices to bound $|\alpha_j(G_i \cap \Gamma(r^a)^l)|_p$ for each $i$ and $j$.

As in the proof of Theorem 1.2, we can argue that

$$k_p(\alpha_j(G_i \cap \Gamma(r^a)^l))$$

and

$$|\alpha_j(G_i \cap \Gamma(r^a)^l) : \alpha_j(G_i \cap B_1 \times \cdots \times B_l)|_p$$

are both $k$-bounded. Furthermore, since

$$|\alpha_j(G_i \cap \Gamma(r^a)^l) : \alpha_j(G_i \cap B_1 \times \cdots \times B_l)|_p$$

divides $a$, we have

$$k_p(\alpha_j(G_i \cap B_1 \times \cdots \times B_l)) \leq a k_p(\alpha_j(G_i \cap \Gamma(r^a)^l)).$$

Since the group $\alpha_j(G_i \cap B_1 \times \cdots \times B_l)$ is abelian, we see that

$$k_p(\alpha_j(G_i \cap B_1 \times \cdots \times B_l)) = |\alpha_j(G_i \cap B_1 \times \cdots \times B_l)|_p (|\alpha_j(G_i \cap B_1 \times \cdots \times B_l)|_p - 1).$$

As $k_p(\alpha_j(G_i \cap \Gamma(r^a)^l))$ is $k$-bounded, it follows that

$$\frac{1}{a} |\alpha_j(G_i \cap B_1 \times \cdots \times B_l)|_p (|\alpha_j(G_i \cap B_1 \times \cdots \times B_l)|_p - 1)$$

is $k$-bounded and we are done. So we may assume that

$$|\alpha_j(G_i \cap B_1 \times \cdots \times B_l)|_p \geq 2$$

so that

$$|\alpha_j(G_i \cap B_1 \times \cdots \times B_l)|_p - 1 \geq \frac{1}{2} |\alpha_j(G_i \cap B_1 \times \cdots \times B_l)|_p.$$

We then deduce that

$$\frac{|\alpha_j(G_i \cap B_1 \times \cdots \times B_l)|}{a}$$

is $k$-bounded.

Similar to the proof of Theorem 1.2, we know that

$$(r^a - 1)/|\alpha_j(G_i \cap \Gamma(r^a)^l)|$$

is $k$-bounded.
Using the conclusion of the previous paragraph and the fact that
\[ |\alpha_j(G_i \cap \Gamma(r^a)) : \alpha_j(G_i \cap B_1 \times \cdots \times B_l)| \] divides \( a \),
we deduce that
\[ \frac{r^a - 1}{a^2} \] is \( k \)-bounded. Therefore, the integer \( a \) must be \( k \)-bounded, which in turn implies that
\[ |\alpha_j(G_i \cap B_1 \times \cdots \times B_l)|_p \] is \( k \)-bounded.
As mentioned above that
\[ |\alpha_j(G_i \cap \Gamma(r^a)) : \alpha_j(G_i \cap B_1 \times \cdots \times B_l)|_p \] is \( k \)-bounded,
we conclude that
\[ |\alpha_j(G_i \cap \Gamma(r^a))|_p \] is \( k \)-bounded, as required. \( \square \)

9. Examples and open questions

There does not seem to be much room for improvement in Theorem 1.1. For instance, let \( V_{l,q} = GF(q^{2^l}) \) where \( q \) is any Fermat prime and \( l \) any positive integer, and \( H_{l,q} = C_{q-1} lP \), where \( P \) is a Sylow 2-subgroup of \( S_{2^l} \). Take \( G_{l,q} := H_{l,q} V_{l,q} \). We see that the number of 2-regular classes of \( G \) is \( 2^{l+1} \) (and a complete set of representatives of these classes is \( (1, \ldots, 1), (a, 1, \ldots, 1), (a, a, 1, \ldots, 1), \ldots, (a, \ldots, a) \)). If we fix \( l \) then all the groups \( G_{l,q} \) have the same number of 2-regular classes. It turns out that the order of the groups \( G_{l,q} \) is bounded if and only if there are finitely many Fermat primes. On the other hand, if we fix \( q \) and take \( l \) large enough then \( G_{l,q} \) has arbitrarily large derived length. Therefore, there is no hope to prove that if \( G \) is any finite group with \( O_p(G) = 1 \) and \( k = k'_p(G) \) then either \( G \) is metabelian or \( |G/F(G)| \) is \( k \)-bounded, for instance (unless we prove that there are finitely many Fermat primes first).

We have already seen that if we want to bound the order of a group \( G \) with \( O_p(G) = 1 \) in terms of the number of \( p \)-regular classes we need to bound the number of Fermat primes. In fact, looking at the classification of groups with few \( p \)-regular classes we see the following.

**Theorem 9.1.** We have the following.

(i) The cardinality of the set of groups with two \( p \)-regular classes and no nontrivial normal \( p \)-group (for some prime \( p \)) is bounded if and only if there are finitely many Fermat and Mersenne primes.

(ii) The cardinality of the set of groups with three \( p \)-regular classes and no nontrivial normal \( p \)-group (for some prime \( p \)) is bounded if and only if there are finitely many Fermat primes, finitely many primes of the form \( 2r^n + 1 \) where \( r \) is prime, and finitely many 3-powers of the form \( 2r^n + 1 \) where \( r \) is prime.
(iii) The cardinality of the set of groups with four $p$-regular classes and no nontrivial normal $p$-group for some odd prime $p$ is bounded if and only if there are finitely many Mersenne primes and finitely many prime powers of the form $4r^n + 1$, where $r$ is prime.

Proof. Part (i) follows from the classification of groups with $k_p'(G) = 2$ in [30]. Part (ii) follows from the classification of groups with $k_p'(G) = 3$ in [27, 28, 29]. Part (iii) follows from [30], noting that in the third family of groups that appear in the statement of Main Theorem it should say that $p = 2^n - 1$ is a Mersenne prime. □

We remark that, in fact, using the groups that appear in Ninomiya’s and Tiedt’s theorems it is easy to build up groups similar to the groups $G_{l,q}$ in the first paragraph of this section for odd primes.

In view of these results and the proof of Theorem 1.1 we suspect that it should be possible to prove that the cardinality of the set of groups $G$ with $k$ $p$-regular classes and $O_p(G) = 1$ is $k$-bounded if and only if there are finitely many primes of certain forms. Our proof basically shows that it suffices to study actions of affine linear groups on modules and see what happens when there are few orbits.

As mentioned in the introduction, Landau’s theorem was strengthened by Craven and Moretó by showing that the order of a finite group $G$ is bounded in terms of the largest multiplicity of irreducible character degrees of $G$. We think that Theorem 1.1 can also be strengthened in this direction.

**Question 9.2.** Let $p$ be a fixed prime and $G$ a finite group with $O_p(G) = 1$. Let $m$ be the largest multiplicity of $p$-Brauer character degrees or the largest multiplicity of $p$-regular class sizes of $G$. Is it true that $|G/O_\infty(G)|$ is $m$-bounded and $O_\infty(G)/F(G)$ is metabelian by $m$-bounded?

Another direction to generalise Theorems 1.1 and 1.2 is to go from a single prime $p$ to a set of primes $\pi$.

**Question 9.3.** Let $\pi$ be a fixed finite set of primes of cardinality at most 2 and $G$ a finite group. Is it true that $|G/O_\infty(G)|$ is bounded in terms of the number of conjugacy classes of $\pi$-regular elements of $G$?

The answer for the above question is negative if the cardinality of $\pi$ is 3 or higher. For instance, a direct product of any number of copies of $A_5$ has no nontrivial $\{2, 3, 5\}$-regular classes but its order can be arbitrarily large. However, as long as the prime 2 is not in $\pi$, we believe that the same conclusion still holds.

**Question 9.4.** Let $\pi$ be a fixed finite set of primes such that $2 \notin \pi$ and $G$ a finite group. Is it true that $|G/O_\infty(G)|$ is bounded in terms of the number of conjugacy classes of $\pi$-regular elements of $G$?
10. Groups with Four $p$-Regular Classes - Theorem 1.3

It would be interesting to classify the nonsolvable groups with $O_p(G) = 1$ and at most five $p$-regular classes. The reason for this is that we expect the number of these groups to be finite. However, even if the number of nonsolvable groups with six $p$-regular classes and $O_p(G) = 1$ were finite, proving this would be out of reach. The reason is that, for instance, the direct products $S_5 \times F_n$, where the $F_n$ are the Frobenius groups defined in the introduction, have six 2-regular classes.

We now start working toward a proof of Theorem 1.3. We will use Ninomiya’s classification of nonsolvable groups with three $p$-regular classes.

**Lemma 10.1** (Ninomiya [27]). Let $G$ be a finite non-solvable group with $O_p(G) = 1$. Then $k_p(G) = 3$ if and only if one of the following holds:

(i) $p = 2$ and $G \cong S_5$, $PSL_2(7) \cdot 2$, $A_6 \cdot 2_3$, or $A_6 \cdot 2^2$.
(ii) $p = 3$ and $G \cong PSL_2(8) \cdot 3$.
(iii) $p = 5$ and $G \cong A_5$.

In the next two results we also consider groups with five $p$-regular classes. This is not necessary for the proof of Theorem 1.3, but we have decided to include it because it does not make the proofs much longer and it would be helpful for the classification on nonsolvable groups with five $p$-regular classes and trivial solvable radical.

**Lemma 10.2.** Let $S$ be a simple group whose order has exactly four prime divisors. Let $G$ be an almost simple group with socle $S$. Then

(i) $G$ has exactly four $p$-regular classes if and only if $p = 2$ and $G \cong PSL_2(3) \cdot 2_1$, $PSL_3(4) \cdot 2^2$, or $2F_4(2') \cdot 2$.
(ii) $G$ has exactly five $p$-regular classes if and only if $p = 2$ and $G \cong S_7$, $M_{11}$, $M_{12} \cdot 2$, $PSL_3(4) \cdot 2^2$, $PSL_3(4) \cdot 2^3$, $2F_4(2')$, $PSL_2(11) \cdot 2$.

**Proof.** It is not known whether the collection of finite simple groups whose order has exactly four prime divisors is finite or not. However, Y. Bugeaud, Z. Cao, and M. Mignotte showed in [4] that if $S$ is such a group, then $S \cong PSL_2(q)$ where $q$ is a prime power satisfying

\[
q(q^2 - 1) = \gcd(2, q - 1)2^{a_1}3^{a_2}r^{a_3}s^{a_4}
\]

with $r$ and $s$ prime numbers such that $3 < r < s$, or $S$ belongs to a finite list of ‘small’ simple groups, see [4, Theorem 1] for the list. Using [5, 16], it is routine to check the list and find all the groups with exactly four or five $p$-regular classes.

So we are left with the case $S \cong PSL_2(q)$ mentioned above. By [8, Theorem 6.4], the centralizer size of an element of $GL_2(q)$ can be estimated by

\[
|C_{GL_2(q)}(g)| \geq \frac{q(q - 1)}{e(1 + \log_q 3)}.
\]
Therefore, the centralizer size of an element of $\text{PSL}_2(q)$ is at least
\[ \frac{q(q - 1)}{e(1 + \log_q 3)(q - 1)(q - 1, 2)} = \frac{q}{e(1 + \log_q 3)(q - 1, 2)}. \]

On the other hand, by [1, Theorem 1.1], the proportion of $p$-regular elements of $\text{PSL}_2(q)$ is greater than or equal to $1/4$. In particular, the number of $p$-regular elements in $\text{PSL}_2(q)$ is at least $(1/4)|\text{PSL}_n(q)|$ and it follows that
\[ k_{p'}(\text{PSL}_2(q)) \geq \frac{q}{4e(1 + \log_q 3)(2,q - 1)}, \]

which in turns implies that
\[ 5 \geq k_{p'}(G) \geq \frac{q}{4ef(1 + \log_q 3)(2,q - 1)^2}, \]

where $q = \ell^f$ for a prime $\ell$. This inequality together with the partial solution of Equation 10.1 in [4] yield
\[ q \in \{2^4, 2^5, 2^7, 3^3, 3^4, 5^2, 7^2, 11, 13, 19, 23, 31, 37, 47, 53, 73, 97\}. \]

Using [16, 9], we have checked that there is no such group with exactly four $p$-regular classes and only one group with exactly five $p$-regular classes and this group is $\text{PSL}_2(11) \cdot 2$. □

**Lemma 10.3.** Let $G$ be a finite group with $O_\infty(G) = 1$. If $k_{p'}(G) \leq 5$, then $G$ is almost simple.

**Proof.** Let $M$ be a minimal normal subgroup of $G$. Then $M$ is a direct product of $n$ copies of a non-abelian simple group, say
\[ M \cong S \times S \times \cdots \times S. \]

First we prove that $n = 1$. Assume the contrary that $n \geq 2$. Since $|S|$ has at least three prime divisors, we can choose two elements $x_1$ and $x_2$ in $S$ of prime order different from $p$ and $o(x_1) \neq o(x_2)$. Consider the following $p$-regular elements:
\[ (x_1, 1, 1, \cdots, 1), (x_1, x_1, 1, \cdots, 1), (x_2, 1, 1, \cdots, 1), (x_2, x_2, 1, \cdots, 1), (x_1, x_2, 1, \cdots, 1). \]

It is clear that these elements belong to different classes of $G$. Therefore, $G$ has at least six classes of $p$-regular elements (including the trivial class), a contradiction. So $n = 1$ or in other words, $M = S$ is simple.

Next we prove that $C_G(S) = 1$. Again assume the contrary that $C_G(S) \neq 1$. Since $C_G(S) \lhd G$ and $O_\infty(G) = 1$, $C_G(S)$ is non-solvable and hence $|C_G(S)|$ has at least three prime divisors. As above, we can choose two elements $y_1$ and $y_2$ in $C_G(S)$ of prime order different from $p$ and $o(y_1) \neq o(y_2)$. Then the elements
\[ 1, x_1, x_2, y_1, y_2, x_1 y_1 \]
belong to different classes of $G$ and again we have a contradiction. Thus we have shown that $C_G(S) = 1$ so that $G \leq \text{Aut}(S)$. So $G$ is almost simple with socle $S$. □

Now we are ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** By Lemma 10.3 we know that $G$ is almost simple with socle $S$. Since $G$ has exactly four $p$-regular classes, the number of prime divisors of $|S|$ is at most 4. First, we assume that the number of prime divisors of $|S|$ is 3. There are only 8 such simple groups (see [12] or [10, p. 12]) and we have

$$S \in \{A_5, \text{PSL}_2(7), A_6, \text{PSL}_2(8), \text{PSL}_2(17), \text{PSL}_3(3), \text{PSU}_3(3), \text{PSU}_4(2)\}.$$  

Using [10], one can easily find all almost simple groups with socle $S$ in this list and exactly four $p$-regular classes.

Now we are left with the case where $|S|$ has exactly four prime divisors. But this is done in Lemma 10.2(i). The proof is complete. □

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**Department d’Àlgebra, Universitat de València, 46100 Burjassot, València, Spain**

**E-mail address:** alexander.moreto@uv.es

**Department of Mathematics, The University of Akron, Akron, Ohio 44325, USA**

**E-mail address:** hungnguyen@uakron.edu