On relative OR-complexity of Boolean matrices and their complements *

Igor S. Sergeev†

We construct explicit Boolean square matrices whose rectifier complexity (OR-complexity) differs significantly from the complexity of the complement matrices. This note can be viewed as an addition to the material of [2, §5.6].

Recall that rectifier \((m, n)\)-circuit is an oriented graph with \(n\) vertices labeled as inputs and \(m\) vertices labeled as outputs. Rectifier circuit (OR-circuit) implements a Boolean \(m \times n\) matrix \(A = (A[i, j])\) iff for any \(i\) and \(j\) the value \(A[i, j]\) indicates the existence of an oriented path from \(j\)-th input to \(i\)-th output. Complexity of a circuit is the number of edges in it, circuit depth is the maximal length of an oriented path. See details in [2, 5].

We denote by \(\text{OR}(A)\) the complexity of an edge-minimal circuit implementing a given matrix \(A\); if we speak about circuits of depth \(\leq d\), then the corresponding complexity is denoted by \(\text{OR}_d(A)\).

It was proved in [2] via method [3] the existence of \(n \times n\)-matrices \(A\) satisfying

\[
\frac{\text{OR}(\bar{A})}{\text{OR}(A)} = \Omega(n / \log^3 n).
\]

Note that due to general results [5, 6] on the asymptotic complexity of the class of Boolean matrices the ratio in the question cannot exceed \(\Theta(n / \log n)\).

A \(k\)-rectangle is an all-ones \(k \times k\) matrix. A matrix is \(k\)-free if it does not contain a \(k\)-rectangle as a submatrix.

It was established in [2] the existence of an \(n \times n\) matrix \(A\) simple for depth-2 circuits, \(\text{OR}_2(A) = O(n \log^2 n)\), whose complement matrix \(\bar{A}\) is 2-free and has relatively high weight (the number of ones) \(|\bar{A}| = \Omega(n^{5/4})\). As a consequence of [6], \(\text{OR}(\bar{A}) = \text{OR}_2(\bar{A}) = |\bar{A}|\).

Below, we provide an explicit construction of matrices satisfying similar conditions.

**Theorem 1.** (i) For an explicit Boolean \(n \times n\) matrix \(C\):

\[
\frac{\text{OR}(\bar{C})}{\text{OR}(C)} = n \cdot 2^{-O(\sqrt{\ln n \ln \ln n})}.
\]

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*Research is supported in part by RFBR, grant 14–01–00671a.
†e-mail: isserg@gmail.com
For an explicit Boolean \( n \times n \) matrix \( C \) the following conditions hold: \( \text{OR}(C) = O(n) \), matrix \( \overline{C} \) is 2-free and \( |\overline{C}| = \Omega(n^{4/3}) \).

(Recall that the weight of any 2-free matrix is at most \( n^{3/2} + n \).) The proof of the theorem is based on the following simple combinatorial lemma.

**Lemma 1.** Let the weight of an \( n \times n \) matrix \( A \) be \( |A| \geq 2n^{3/2} \). Then \( A \) contains \( \Omega((|A|/n)^4) \) 2-rectangles.

**Proof.** Say that a row covers a pair \( u \) of two columns, if this row has ones in these columns. If \( a_i \) denotes the number of ones in the \( i \)-th row of \( A \), then the number of pairs of columns covered by the rows of \( A \) is

\[
\sigma = \sum_{i=1}^{n} \left( \frac{a_i}{2} \right) = \frac{1}{2} \sum_{i=1}^{n} a_i^2 - \frac{|A|}{2} \geq \left( \frac{\sum_{i=1}^{n} a_i^2}{2n} \right) - \frac{|A|}{2} = \frac{|A|^2}{2n} - \frac{|A|}{2} \geq \frac{|A|^2}{4n}.
\]

Let \( b_u \) be the number of rows covering the pair \( u \) of columns. Then \( \sum_u b_u = \sigma \). Thus, the number of 2-rectangles in \( A \) is

\[
\sum_u \left( \frac{b_u}{2} \right) = \frac{1}{2} \sum_u b_u^2 - \frac{\sigma}{2} \geq \left( \frac{\sum_u b_u^2}{n(n-1)} \right) - \frac{\sigma}{2} = \frac{\sigma^2}{n(n-1)} - \frac{\sigma}{2} \geq \frac{\sigma^2}{2n^2} = \Omega \left( \left( \frac{|A|}{n} \right)^4 \right).
\]

\( \square \)

Let \( n = \binom{m}{2} \). Given an \( m \times m \) matrix \( A \) construct an \( n \times n \) matrix \( B \) as follows. Label rows and columns of \( B \) by 2-element subsets of \( [m] \). Set \( B[a, b] = 1 \) iff \( a \times b \) forms a 2-rectangle in \( B \).

**Lemma 2.** If \( A \) is \( k \)-free, then \( B \) is \( K \)-free, \( K = \left( \binom{k-1}{2} \right) + 1 \).

**Proof.** Suppose that \( B \) contains a \( K \)-rectangle at the intersection of rows \( s_1, \ldots, s_K \) and columns \( t_1, \ldots, t_K \). Then \( A \) contains a rectangle at the intersection of rows \( \cup s_i \) and columns \( \cup t_i \). But necessarily \( |\cup s_i|, |\cup t_i| \geq k \), contradicting \( k \)-freeness of \( A \). \( \square \)

**Lemma 3.** If \( A \) is \( k \)-free and \( |A| \geq 2m^{3/2} \), then

\[
\text{OR}(B) = \Omega \left( \left( \frac{|A|}{kn} \right)^4 \right),
\]

on the other hand, \( \text{OR}_3(B) = O(n) \).
Proof. By Lemma 1, $|B| = \Omega((|A|/n)^4)$, and Lemma 2 implies that $B$ is $K$-free. Therefore, by the Nechiporuk’s theorem \([6]\)

$$\text{OR}(B) \geq \frac{|B|}{K^2} = \Omega\left(\left(\frac{|A|}{kn}\right)^4\right).$$

We are left to show that the matrix $\bar{B}$ can be implemented by a depth-3 circuit of linear complexity. Take a depth-3 circuit where the nodes on the second and the third layer are numbers $1, \ldots, m$, and there is an edge joining an input or an output $a$ with a node $i$ iff $i \in a$. The edges between the second and the third layers are drawn according to the entries of the matrix $\bar{A}$.

By the construction, the circuit has $O(m^2)$ edges. Indeed, it implements the matrix $\bar{B}$ since there exists a path connecting an input $a$ with an output $b$ iff the submatrix at the intersection of rows $b$ and columns $a$ is not all-zero.

To prove p. (i) of the Theorem take $m \times m$ norm-matrix $A$ \([4]\), which is $\Delta$-free and has $m^2/\Delta$ ones, where $\Delta = 2^{O(\sqrt{\log m \log \log m})}$, under appropriate choice of parameters. Put $C = \bar{B}$.

To prove p. (ii) take 3-free $m \times m$ Brown’s matrix $A$ \([1]\) of weight $\Theta(m^{5/3})$. Put $C = \bar{B}$. \[\square\]

The author is grateful to Stasys Jukna for suggestions improving the presentation.

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