SHAPOVALOV DETERMINANT FOR LOOP SUPERALGEBRAS

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Abstract. Let a given finite dimensional simple Lie superalgebra $g$ possess an even invariant non-degenerate supersymmetric bilinear form. We show how to recover the quadratic Casimir element for the Kac–Moody superalgebra related to the loop superalgebra with values in $g$ from the quadratic Casimir element for $g$. Our main tool here is an explicit Wick normal form of the even quadratic Casimir operator for the Kac–Moody superalgebra associated with $g$; this Wick normal form is of independent interest.

If $g$ possesses an odd invariant supersymmetric bilinear form we compute the cubic Casimir element.

In addition to the cases of Lie superalgebras $g(A)$ with Cartan matrix $A$ for which the answer was known, we consider the Poisson Lie superalgebra $\mathfrak{po}(0|n)$ and the related Kac–Moody superalgebra.

1. Introduction

1.1. Motivations: physics. For a very lucid detailed exposition of these motivations in the non-super case, see the text-book by P. Di Francesco, P. Mathieu and D. Sénéchal [DMS].

Observe that, in various applications, the simple algebra which initiated the study is often less interesting than certain its “relatives” (its nontrivial central extension, algebra of derivations and the result of iterations of these constructions). In what follows, having the simple object at the center of attention, we keep its relatives in view.

1) Stringy algebras. In a seminal paper [BPZ], A. Belavin, A. Polyakov and A. Zamolodchikov observed that the infinite number of generators of the conformal group in the two-dimensional case generate the Ward identities for correlation functions, and these differential equations (Ward identities) completely specify the behavior of the correlation functions. The components of the stress-energy operator in the conformal field theory form, together with the central charge, the Virasoro algebra; this reduces the study of the conformal theory to the study of (irreducible) highest weight representations of the Virasoro algebra.

V. Dotsenko and V. Fateev [DF] explicitly constructed a large class of conformal theories, so-called minimal models. In their study, the complete description of (irreducible) unitarizable highest-weight representations of a real form of the Virasoro algebra is vital.
2) Loop algebras. Bosonization of free fermions with spin and the internal symmetry group $G$ provide with an example of a nontrivial conformal theory based on the Kac–Moody algebra $\hat{\mathfrak{g}}^{(1)}$. The components of the stress-energy operator for these theories, built out of quadratic forms of fermion current operators for $G^{(1)}$, satisfy the relations for the Virasoro algebra with central charge $C = \frac{\dim \mathfrak{g}}{2} + c_{\mathfrak{g}}$, where $k$ is the value of the central charge of $\hat{\mathfrak{g}}^{(1)}$ and $c_{\mathfrak{g}}$ is the value of the (quadratic) Casimir operator of $\mathfrak{g}$ in the adjoint representation, cf. [GO]. The Hamiltonian of the WZW model can also be built with quadratic forms of current operators and thus also represents a nontrivial conformal field theory. The differential equations for the multi-point correlation functions of the WZW primary fields are the Knizhnik-Zamolodchikov equations introduced in [KZ].

In all these cases the description of the irreducibles is performed by means of the Shapovalov determinant; in what follows we recall its definition and ways to compute it.

3) Super versions. For (relatives of) simple stringy superalgebras, the Shapovalov determinant was computed, except for a number of cases. (A recent paper [KW] claims to solve all the cases, but has various omissions.) In this paper, we consider the Kac-Moody superalgebras; for the stringy Lie superalgebras, see [GLL].

1.2. Means: mathematics. The Casimir elements. From the above we already see that Casimir elements (the elements of the center of the universal enveloping algebra or its completion) are very important. In many problems, it suffices to know only the quadratic Casimir, but we have to have it explicitly.

The Shapovalov determinant is a useful tool for verifying if certain representations of Lie algebras and Lie superalgebras $\mathfrak{g}$ with vacuum vector (either highest or lowest weight one), namely, the Verma modules, are irreducible or not, and even for constructing certain irreducible modules with vacuum vector.

To define the Shapovalov determinant, the algebra $\mathfrak{g}$ must be rather “symmetric”, i.e.,

$$\mathfrak{g}$$ must possess a Cartan subalgebra (i.e., a maximal nilpotent subalgebra coinciding with its normalizer) $\mathfrak{t}$ whose even part is commutative and diagonalizes $\mathfrak{g}$, and such that the weight 0 eigenspace of $\mathfrak{g}$ (relative to $\mathfrak{t}_0$) coincides with $\mathfrak{t}$;

$\mathfrak{g}$ must possess an involutive (or super-involutive if we respect the Sign Rule, see below) anti-automorphism $\sigma$ which interchanges the root vectors (with respect to $\mathfrak{t}_0$) of opposite sign.

Recall that an anti-automorphism of a given Lie superalgebra $\mathfrak{g}$ is a linear map $\sigma: \mathfrak{g} \rightarrow \mathfrak{g}$ that for every $x, y \in \mathfrak{g}$, satisfies

$$\sigma([x, y]) = \begin{cases} (-1)^{p(x)p(y)}[\sigma(y), \sigma(x)] & \text{if we respect the Sign Rule,} \\ [\sigma(y), \sigma(x)] & \text{if we ignore the Sign Rule.} \end{cases}$$

(An endomorphism $\sigma$ is said to be involutive if $\sigma^2 = \id$ and super-involutive if $\sigma^2 = (-1)^{p(x)} \id$. Since the Shapovalov determinant is defined up to a scalar factor, and the Sign

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1The formal definition, see [K], is a bit complicated: $\hat{\mathfrak{g}}^{(1)}$ is a certain subalgebra of the Lie algebra of derivations of the central extension $\tilde{\mathfrak{g}}^{(1)}$ of the loop algebra $\mathfrak{g}^{(1)}$ of maps $S^1 \rightarrow \mathfrak{g}$, where $\mathfrak{g} = \text{Lie}(G)$ is a simple finite dimensional complex Lie algebra, cf. [19].
Rule only affects its sign, we may choose a more convenient definition. In computations it is usually more convenient to ignore the Sign Rule.)

Additionally, the following finiteness condition should hold:

\begin{enumerate}
\item the root spaces of $\mathfrak{g}$ should be finite dimensional;
\item the number of partitions of any positive weight in $U(\mathfrak{g}^+)$, see \cite{9}, into positive roots of $\mathfrak{g}$ should be finite.
\end{enumerate}

Nothing else is needed to define the Shapovalov determinant, but to compute it is much easier in presence of the even quadratic Casimir element $C_2$ (of the center of $U(\mathfrak{g})$ or its completion $\hat{U}(\mathfrak{g})$) or (which is not quite the same if $\dim \mathfrak{g} = \infty$, but suffices for our purposes) in presence of the non-degenerate EVEN bilinear form $B$ on $\mathfrak{g}$. In presence of such an element $C_2$, the Shapovalov determinant is a product of linear terms; for various cases where this statement is proved, see \cite{KK, GL1, Go2, Sh}.

1.3. Cases where an even quadratic Casimir exists. Kac and Kazhdan \cite{KK} computed the Shapovalov determinant for any Lie algebra with symmetrizable Cartan matrix (with arbitrary (complex) entries); their technique is literally applicable to Lie superalgebras with symmetrizable Cartan matrix, as mentioned in \cite{GL1} and expressed in detail in \cite{Go2}.

Moreover, Kac and Kazhdan used the Shapovalov determinant to describe the Jantzen filtration of the Verma modules over the Lie algebras they considered.

The technique of Kac and Kazhdan can be applied even in the absence of symmetrizable Cartan matrix; its presence only makes computation of the needed values of the even quadratic Casimir easier. So it is reasonable to look around for Lie (super)algebras without symmetrizable Cartan matrix but with a non-degenerate even bilinear form. Grozman and Leites considered all such simple Lie superalgebras among $\mathbb{Z}$-graded of polynomial growth and their relatives (for the same reasons that Kac–Moody algebras are “better” than simple loop algebras, the finite dimensional Poisson Lie superalgebras $\text{poi}(0|n)$ are “better” than their simple relatives $\mathfrak{h}'(0|n)$, where $\mathfrak{h}(0|n) = \text{poi}(0|n)/\text{center}$ and $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$): One stringy superalgebra, $t^\mathbb{Z}(1|6)$ (physicists call it $N = 6$ Neveu-Schwarz superalgebra), finite dimensional Poisson superalgebras $\text{poi}(0|2n)$ and Kac–Moody algebras associated with $\text{poi}(0|2n)$. More precisely, \cite{GL1} contains computations of the quadratic Casimirs $C_2$ and description of the irreducible Verma modules over $t^\mathbb{Z}(1|6)$ and $\text{poi}(0|2n)$.

Gorelik and Serganova \cite{GS1} wrote a sequel to \cite{GL1} having produced a more explicit expression of the Shapovalov determinant than that of \cite{GL1}; their rather nontrivial result is an explicit description of Jantzen’s filtration for the Verma modules over $\text{poi}(0|2n)$.

There are many (perhaps, undescrivably many) examples of filtered Lie (super)algebras of polynomial growth (i.e., such that the associated graded Lie (super)algebras are of polynomial growth) and possessing a $C_2$, cf. \cite{GL0, Ko}. The Shapovalov determinant is computed only for one (the simplest) of such algebras, namely, for $\mathfrak{gl}(\lambda)$, where $\lambda \in \mathbb{C}$, and only for the simplest types of Verma modules (\cite{Sh}).

1.4. Cases where no even quadratic Casimirs exist. 1) There is an odd non-degenerate bilinear form. Such are the queer Lie superalgebra $\mathfrak{q}(n)$ — a non-trivial super analog of $\mathfrak{gl}(n)$, the Poisson superalgebras $\text{poi}(0|2n + 1)$, and the Kac-Moody superalgebras associated with them.

\footnote{The hat over $U$ means that the elements of $\hat{U}(\mathfrak{g})$ can be infinite sums of elements of $U(\mathfrak{g})$.}
In the 1990s, Grozman and Leites conjectured that since $q(n)$ and $\mathfrak{poi}(0|2n+1)$ are super analogs of $\mathfrak{gl}(n)$, it is possible to compute their Shapovalov determinant (which is not easy even to define in these and similar cases), but made a mistake computing it and decided that the terms it factorizes into can be of any degree.

Gorelik skilfully used the anti-center and suggested an elegant proof of the fact that, for $q(n)$, the Shapovalov determinant factorizes in the product of linear polynomials, see [Go1].

Among the simple $\mathbb{Z}$-graded Lie superalgebras of polynomial growth that possess a non-degenerate invariant bilinear odd form there is only one stringy Lie superalgebra (we will consider it in [GLL]) as well as $q(n)$, $\mathfrak{poi}(0|2n+1)$, and Kac-Moody algebras associated with them.

2) **There are no non-degenerate bilinear forms.** Such are most of the stringy Lie superalgebras (see [GLL]), and Lie superalgebras $q(2)(n)$ with non-symmetrizable Cartan matrices recently considered in [GS2].

1.5. **Our results and open problems.** Thus, there remained to consider, first of all, the Lie (super)algebras with properties (1), (3). The Kac–Moody (super)algebras associated with the loop (super)algebras with values in finite dimensional “symmetric” Lie (super)algebras are among such algebras, so here we explicitly describe the Wick normal form of the Casimir operator for the Kac–Moody superalgebras $\hat{\mathfrak{g}}^{(1)}$ (as well as “twisted” Kac–Moody superalgebras $\hat{\mathfrak{g}}^{(r)}$ ) in terms of the the Casimir operator for a finite dimensional simple Lie superalgebra $\mathfrak{g}$. This implies a description of the Shapovalov determinant for $\hat{\mathfrak{g}}^{(1)}$ previously known only for Kac–Moody (super)algebras with Cartan matrix. We also conjecture a description of the Shapovalov determinant for Kac–Moody (super)algebra related with $\mathfrak{poi}(0|2n)$ for $n > 2$.

We will show that if $\mathfrak{g}$ possesses an odd invariant non-degenerate symmetric bilinear form, then $U(\mathfrak{g})$ (or $\hat{U}(\mathfrak{g})$) contains a cubic central element. If $\dim \mathfrak{g} < \infty$, this implies, as a rule, that the Shapovalov determinant factorizes in a product of factors of degree $\leq 2$. We also conjecture a description of the Shapovalov determinant for $\mathfrak{poi}(0|2n + 1)$.

2. **Background: Lie superalgebras**

For a detailed background, see [GLS]. In what follows the ground field is $\mathbb{C}$.

2.1. **The Poisson superalgebra.** Let $G(m)$ be the Grassmann superalgebra generated by $\theta_1, \ldots, \theta_m$. The Poisson Lie superalgebra (do not confuse with the Poisson-Lie (super)algebra) $\mathfrak{poi}(0|m)$ has the same superspace as $G(m)$ and the (Poisson) bracket is given by

$$\{f, g\}_{P.b.} = (-1)^{|f||g|} \sum_{j \leq m} \frac{\partial f}{\partial \theta_j} \frac{\partial g}{\partial \theta_j}$$

for any $f, g \in \mathbb{C}[\theta]$. It is often more convenient to re-denote the $\theta$s and set (over $\mathbb{C}$, over $\mathbb{R}$ such a transformation is impossible so the brackets (4) and (5) are not equivalent over $\mathbb{R}$)

$${\xi}_j = \frac{1}{\sqrt{2}}(\theta_j - \sqrt{-1}\theta_{+j}),$$

$${\eta}_j = \frac{1}{\sqrt{2}}(\theta_j + \sqrt{-1}\theta_{+j})$$

for $j \leq r = \left\lceil \frac{m}{2} \right\rceil$,

$$\theta = \theta_{2r+1}$$

if $m$ is odd.
and accordingly modify the bracket (if \( m = 2r \), there is no term with \( \theta \)):

\[
\{ f, g \}_{PB} = (-1)^p(f) \left( \sum_{j \leq m} \left( \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial y_j} + \frac{\partial f}{\partial y_j} \frac{\partial g}{\partial x_j} \right) + \frac{\partial f}{\partial \theta} \frac{\partial g}{\partial \theta} \right).
\]

The quotient of \( \mathfrak{po}(0|n) \) modulo center is the Lie superalgebra \( \mathfrak{h}(0|n) \) of hamiltonian vector fields generated by functions:

\[
H_{f} := (-1)^p(f) \left( \sum_{j \leq m} \left( \frac{\partial f}{\partial x_j} \frac{\partial}{\partial y_j} + \frac{\partial f}{\partial y_j} \frac{\partial}{\partial x_j} \right) + \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta} \right).
\]

2.2. The integral. The still conventional notation \( d^n \theta := d\theta_1 \ldots d\theta_n \) for the volume element in the Berezin integral is totally wrong as was clear already in 1966 from the explicit form of the Berezinian of the Jacobi matrix of the coordinate change. A reasonable notation with compulsory indication of the coordinates is \( \text{vol}(\theta) \); for more motivations, see [Del, GLS].

On \( \mathfrak{po}(0|n) \), or rather on the space of generating functions, the integral — equal to the coefficient of the highest term in Taylor series expansion in \( \theta \) — determines a non-degenerate invariant supersymmetric bilinear form, of the same parity as \( n \), given by

\[
(f | g) := \int fg \text{ vol}(\theta).
\]

2.3. Cartan subalgebras, maximal tori, roots and coroots. In [BNO, PS], it is shown that the Cartan subalgebras of simple and certain non-simple (like \( \mathfrak{po}, \mathfrak{q} \) and their subquotients) finite dimensional Lie superalgebras are conjugate by inner automorphisms. We always fix a Cartan subalgebra \( \mathfrak{t} \) (for example, for \( \mathfrak{po}(0|2n) \), we take \( \mathfrak{t} = \mathbb{C}[\xi_1 \eta_1, \ldots, \xi_n \eta_n] \)). For any \( \alpha \in \mathfrak{t}_0^\ast \), set

\[
\mathfrak{g}_\alpha := \{ x \in \mathfrak{g} \ | \ \text{ad}_h - \alpha(h))^N(x) = 0 \ \text{for a sufficiently large } N \text{ and every } h \in \mathfrak{t}_0 \}.
\]

Then, as is not difficult to see,

\[
\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{t}_0^\ast} \mathfrak{g}_\alpha \text{ and } \mathfrak{t} \subset \mathfrak{g}_0.
\]

2.3.1. Remark. In what follows we only consider the algebras for which \( \mathfrak{t} = \mathfrak{g}_0 \).

Denote by \( R \) the set of non-zero functionals \( \alpha \in \mathfrak{t}_0^\ast \) for which \( \dim \mathfrak{g}_\alpha \neq 0 \); this \( R \) is called the set of roots of \( \mathfrak{g} \). For the algebras we consider,

there exists an \( H \in \mathfrak{t} \) such that \( \alpha(H) \in \mathbb{R} \) for all \( \alpha \in R \) and \( \alpha(H) \neq 0 \) for \( \alpha 
eq 0 \).

This allows us to split the roots into positive and negative ones by setting

\[
R^+ := \{ \alpha \in R \ | \ \alpha(H) > 0 \}, \quad R^- := \{ \alpha \in R \ | \ \alpha(H) < 0 \};
\]

\[
\mathfrak{g}^\pm = \bigoplus_{\alpha \in \mathfrak{t}^\ast} \mathfrak{g}_\alpha.
\]

If \( \mathfrak{t} = \mathfrak{t}_0 \) and commutative, we can identify \( U(\mathfrak{t}) \) with \( S(\mathfrak{t}) \); let \( HC \) be the Harish-Chandra projection, i.e., the projection on the first direct summand in the following decomposition:

\[
U(\mathfrak{g}) \simeq U(\mathfrak{t}) \bigoplus (\mathfrak{g}^- U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{g}^+) \longrightarrow S(\mathfrak{t}) \simeq U(\mathfrak{t}).
\]
2.4. The Shapovalov determinant. 1) It is purely even and commutative. For a fixed \( \lambda \in t^* \), set \( M^\lambda = U(g)/I \), where \( I \) is the left ideal generated by \( g^+ \) and the elements
\[
\lambda - \lambda(h) \quad \text{for any } h \in t.
\]
The \( g \)-module \( M^\lambda \) is called the Verma module with highest weight \( \lambda \). Obviously, as spaces, \( M^\lambda \cong U(g^-)m^\lambda \), where \( m^\lambda \) is the vacuum vector; and the \( U(g^-) \)-action on \( m^\lambda \) is faithful, i.e., without kernel. The anti-automorphism \( \sigma \), see (2), can be (uniquely) extended to an anti-automorphism of \( U(g) \):
\[
\sigma(x_1 \otimes \ldots \otimes x_k) = \begin{cases} (-1)^{p(x_i)p(x_j)} \sigma(x_k) \otimes \ldots \otimes \sigma(x_1) & \text{minding the Sign Rule;} \\
\sigma(x_k) \otimes \ldots \otimes \sigma(x_1), & \text{ignoring the Sign Rule.}
\end{cases}
\]
On \( U(g^-) \), we define the following bilinear form with values in \( U(t) = S(t) \):
\[
(Xm \mid Ym) = HC(a(X)Y).
\]
Obviously, the distinct weight subspaces of \( U(g^-) \) are orthogonal with respect to this inner product \( (\cdot \mid \cdot) \) defined by (12), and \( M^\lambda \) is irreducible if and only if \( (\cdot \mid \cdot) \) is non-degenerate. Each determinant \( Sh_\mu \) of the Gram matrix of the restriction of the form \( (\cdot \mid \cdot) \) on the subspace \( U(g^-)(\chi) \) of weight \( -\chi \) is referred to as the Shapovalov determinant. Since \( S(t) \) is commutative algebra, these determinants are well-defined (as polynomials in \( t \)); if we do not fix a basis of \( U(g^-)(\chi) \) (as we do in what follows), they are determined up to a scalar factor.

On \( M^\lambda \), define the following bilinear form (with values in \( \mathbb{C} \)):
\[
(Xm^\lambda \mid Ym^\lambda)_\chi = (X|Y)(\chi) \equiv HC(\sigma(X)Y)(\chi) \quad \text{for any } X, Y \in U(g^-).
\]
This definition is natural in the sense that for any \( X, X', Y, Y' \in U(g) \) (not only of \( U(g^-) \)), if \( Xm^\lambda = X'm^\lambda \) and \( Ym^\lambda = Y'm^\lambda \), then
\[
HC(\sigma(X)Y)(\lambda) = HC(\sigma(X')Y')(\lambda).
\]
Obviously
1) the elements of distinct weights of \( M^\lambda \) are orthogonal to each other relative \( (\cdot \mid \cdot)_\chi \);
2) the restriction of the bilinear form \( (\cdot \mid \cdot)_\lambda \) onto the subspace \( M^\lambda(\lambda - \chi) \) of elements of weight \( \lambda - \chi \) is degenerate if and only if \( Sh_\chi(\lambda) = 0 \).
2.4.1. Statement. 1) Any non-trivial submodule of \( M^\lambda \) contains a non-zero element \( v \) whose weight is distinct from \( \lambda \) and \( g^+ v = 0 \) (such an element \( v \) is said to be a singular vector of \( M^\lambda \)).
2) If \( v \in M^\lambda \) is a singular vector, then \( U(g^-)v \) — is a non-trivial submodule of \( M^\lambda \).
3) \( x \in M^\lambda \) is an isotropic element of the form \( (\cdot \mid \cdot)_\chi \) if and only if \( x \) can be represented as \( Av \), where \( A \in U(g^-) \), and \( v \) is a singular vector of \( M^\lambda \).
4) If \( v \in M^\lambda \) is a singular vector of weight \( \mu \) and \( C \) is a central element of \( U(g) \), then
\[
HC(C)(\lambda) = HC(C)(\mu).
\]
Thanks to this statement to describe all irreducible Verma modules is the same as to compute all Shapovalov determinants. Moreover, we see that Casimir elements help to compute Shapovalov determinants.

2) \( t_1 \neq 0 \). (This case is considered in more detail separately.) Among finite dimensional simple Lie superalgebras, this only happens with \( psq(n) \) and \( b'(0|2n-1) \). M. Gorelik [Go1] considered the case of \( psq(n) \); that of \( b'(0|2n-1) \) might, in theory, be obtained from her
results by contraction, but in applications an explicit formula is needed instead of a small talk. In what follows we offer a conjectural formula for the Shapovalov determinant for $\text{poi}(0|2n + 1)$; conjecturally the answer for $\mathfrak{h}'(0|2n + 1)$ for $n > 1$ is analogous.

3. CASIMIR OPERATORS ON LIE SUPERALGEBRAS: THE CASE OF AN EVEN INVARIANT FORM

In this section, let $\mathfrak{g}$ be a finite dimensional Lie superalgebra, and $(\cdot | \cdot)$ an invariant even supersymmetric non-degenerate bilinear form on $\mathfrak{g}$. Let $\{ e_i \}_{i=1}^d$ be a basis of $\mathfrak{g}$. We set $a_{ij} = (e_i | e_j)$ and $(b_{ij}) = (a_{ij})^{-1}$ the inverse matrix.

3.1. Statement. The quadratic element

$$\Omega_0 = \sum_{i,j} b_{ij} e_i \otimes e_j \in U(\mathfrak{g}).$$

is a central (Casimir) element of $U(\mathfrak{g})$, in particular,

$$[x, \Omega_0] = 0 \quad \text{for all} \ x \in \mathfrak{g}.$$

Proof. It suffices to prove (15) for each $x \in \{ e_i \}_{i=1}^d$. Let $c_{ij}^k$ be structure constants in this basis. Let $p_i = p(e_i)$ be the parities. We have

$$[e_k, \Omega_0] = \sum_{i,j} b_{ij} ([e_k, e_i] \otimes e_j + (-1)^{p_i p_k} e_i \otimes [e_k, e_j]) =$$

$$\sum_{i,j} b_{ij} \sum_l (c_{kl}^i e_l \otimes e_j + (-1)^{p_i p_k} c_{kj}^l e_i \otimes e_l) =$$

$$\sum_{i,j} \sum_l (b_{ij} c_{kl}^j + (-1)^{p_i p_k} b_{il} c_{kl}^j) e_i \otimes e_j.$$

So, to prove the statement, it suffices to show that

$$\sum_l (b_{ij} c_{kl}^j + (-1)^{p_i p_k} b_{il} c_{kl}^j) = 0 \quad \text{for all} \ i, j, k \in \overline{1,d}.$$

The invariance of the form $(\cdot | \cdot)$ implies that

$$0 = ([e_p, e_k] | e_r) - (e_p | [e_k, e_r]) = \sum_s (c_{pk}^s a_{sr} - c_{ks}^r a_{ps}) \quad \text{for all} \ k, p, r \in \overline{1,d}.$$

Let us multiply this equality by $b_{ip} b_{rj}$ and sum over $p$ and $r$. Since $(a_{ij})(b_{ij}) = 1_n$, we get

$$0 = \sum_p c_{pk}^j b_{ip} - \sum_r c_{kr}^i b_{rj} = \sum_l (c_{kl}^j b_{il} - c_{kl}^i b_{lj}) = -\sum_l ((-1)^{p_i p_k} c_{kl}^j b_{il} + c_{kl}^i b_{lj}).$$

Since $(\cdot | \cdot)$ is even, the supermatrix $(b_{ij})$ is also even, and the first term in the last sum can be non-zero only if $p_i = p_l$, so this equality is equivalent to (17). □

Given a loop superalgebra $\mathfrak{g}^{(1)} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, define the Kac–Moody Lie superalgebra

$$\hat{\mathfrak{g}}^{(1)} = \text{Span}(\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}], u, z),$$

where $u = t^{\frac{d}{dt}}$ and $z$ are even,
with the following bracket:

$$ [t^m x, t^n y] = t^{m+n} [x, y] + m \delta_{m,-n} (x \mid y) z \quad \text{for all } x, y \in \mathfrak{g}, \quad m, n \in \mathbb{Z}; $$

(19)  

$$ [z, X] = 0 \quad \text{for all } X \in \hat{\mathfrak{g}}^{(1)}; $$

$$ [u, t^n x] = nt^n x \quad \text{for all } x \in \mathfrak{g}, \quad n \in \mathbb{Z}. $$

Set

$$ \Omega' = \sum_{n=-\infty}^{\infty} \sum_{i,j} b_{ij} t^n e_i \otimes t^{-n} e_j \in \hat{U}(\hat{\mathfrak{g}}^{(1)}). $$

3.2. Statement. $[X, 2u \otimes z + \Omega] = 0$ for any $X \in \hat{\mathfrak{g}}^{(1)}$.

**Proof.** It is easy to check this for $X = u, z$. Now let $X = t^m e_k$. Let us compute $[X, \Omega']$ up to terms with $z$: we get

$$ \sum_{n=-\infty}^{\infty} \sum_{i,j} b_{ij} (t^{m+n} [e_k, e_i] \otimes t^{-n} e_j + (-1)^{p_ip_k} t^n e_i \otimes t^{m-n} [e_k, e_j]) = $$

$$ \sum_{n=-\infty}^{\infty} \sum_{i,j} b_{ij} (t^{m+n} [e_k, e_i] \otimes t^{-n} e_j + (-1)^{p_i p_k} t^{m+n} e_i \otimes t^{-n} [e_k, e_j]) = $$

$$ \sum_{n=-\infty}^{\infty} \sum_{i,j} b_{ij} (t^{m+n} \otimes t^{-n}) \cdot [e_k, \Omega_0] = 0, $$

where the term in quotation marks is understood as follows: having represented $[e_k, \Omega_0]$ as the sum of quadratic elements, we multiply, in each of them, the first factor by $t^{m+n}$ and the second one by $t^{-n}$.

Now, let us compute the terms with $z$ in $[X, \Omega']$. We get these terms only from the terms in the sum with $n = \pm m$, so we get

$$ \sum_{i,j} b_{ij} (m e_k \mid e_i) z \otimes t^m e_j + (-1)^{p_i p_k} m t^n e_i \otimes (e_k \mid e_j) z = $$

(21)  

$$ m \sum_{i,j} b_{ij} (a_{ki} z \otimes t^m e_j + (-1)^{p_i p_k} t^m e_i \otimes a_{kj} z). $$

Since $b_{ij} \neq 0$ only for $p_i = p_j$, and $(a_{ij})(b_{ij}) = 1_d$, it follows that (21) is equal to

$$ m \left( \sum_j \delta_{kj} z \otimes t^m e_j + \sum_i \delta_{ik} t^m e_i \otimes z \right) = m (z \otimes X + X \otimes z) = 2m X \otimes Z. $$

Since $[X, 2u \otimes z] = -2m X \otimes z$, we see that $[X, 2u \otimes z + \Omega'] = 0$. \qed

Now we want to express the quadratic central element of $\hat{U}(\hat{\mathfrak{g}}^{(1)})$ in the Wick normal form, i.e., so that, in every tensor product, the first factor has a non-positive degree with respect to $u$, and the second one has a non-negative degree. The Wick normal form of an operator $\Omega$ will be denoted by $: \Omega :$. Set

$$ \Omega^{pm} = \sum_{n=1}^{\infty} \sum_{i,j} b_{ij} t^{-n} e_i \otimes t^n e_j. $$
Let us compute \([X, \Omega_0 + 2\Omega^m + 2u \otimes z]\). The bracket obviously vanishes for \(X = u\) and \(z\); so let \(X = t^n x\). A computation similar to the previous one shows that \([X, 2u \otimes z]\) cancels with monomials from \([X, \Omega_0 + 2\Omega^m]\) containing \(z\), so we only need to compute \([X, \Omega_0 + 2\Omega^m]\) up to elements with \(z\). If \(X \in \mathfrak{g}\), then \(X\) commutes with \(\Omega_0\) (as it was shown before) and, similarly, with every term in the sum over \(n\) in \(\Omega^m\). Consider the case \(X = t^n x\), where \(m > 0\). From \([X, \Omega^m]\) we get

\[
\sum_{n=1}^{\infty} \sum_{i,j} b_{ij}(t^{m-n}[e_k, e_i] \otimes t^n e_j) = (-1)^{p_{ij}} t^{-n} e_i \otimes t^{m+n}[e_k, e_j] = \sum_{n=m+1}^{\infty} \sum_{i,j} (-1)^{p_{ij}} t^{m-n} e_i \otimes t^n[e_k, e_j] = \sum_{n=m+1}^{\infty} \sum_{i,j} t^{m-n} \otimes t^n \cdot [e_k, \Omega_0] = \sum_{n=1}^{\infty} \sum_{i,j} b_{ij} t^{m-n}[e_k, e_i] \otimes t^n e_j.
\]

From \([X, \Omega_0 + \Omega^m]\) we get

\[
\sum_{n=0}^{\infty} \sum_{i,j} b_{ij}(t^{m-n}[e_k, e_i] \otimes t^n e_j) = \sum_{n=m}^{\infty} \sum_{i,j} (-1)^{p_{ij}} t^{-n} e_i \otimes t^{m+n}[e_k, e_j] = \sum_{n=0}^{\infty} \sum_{i,j} b_{ij}(-1)^{p_{ij}} t^{m-n} e_i \otimes t^n[e_k, e_j] = \sum_{n=m}^{\infty} \sum_{i,j} t^{m+n} \otimes t^{-n} \cdot [e_k, \Omega_0] = \sum_{n=0}^{\infty} \sum_{i,j} b_{ij}(-1)^{p_{ij}} t^{m-n} e_i \otimes t^n[e_k, e_j].
\]

Changing \(n \rightarrow m - n\), \(i \leftrightarrow j\), we get

\[- \sum_{n=1}^{m} \sum_{i,j} b_{ij}(-1)^{p_{ij}} t^{m} e_j \otimes t^{m-n}[e_k, e_i] = - \sum_{n=1}^{m} \sum_{i,j} b_{ij}(-1)^{p_{ij}} t^{m} e_j \otimes t^{m-n}[e_k, e_i].\]

So, for \(m > 0\), we have

\[
[t^m e_k, \Omega_0 + 2\Omega^m + 2u \otimes z] = \sum_{n=1}^{m} \sum_{i,j} b_{ij}(t^{m-n}[e_k, e_i] \otimes t^n e_j - (-1)^{p_{ij}} t^n e_j \otimes t^{m-n}[e_k, e_i]) = \sum_{n=1}^{m} \sum_{i,j} t^m b_{ij}[e_k, e_i], e_j] = m t^n \sum_{i,j} b_{ij}[e_k, e_i], e_j].
\]

We similarly get the same result for \(m < 0\). So we get the following:
3.3. **Statement.** If, on $\mathfrak{g}$, the map

$$A : x \mapsto \sum_{i,j} b_{ij}[[x, e_i], e_j]$$

is equal to $\lambda$ id, where id is the identity operator, then

1) the element

$$\Omega := \Omega_0 + 2\Omega^{nm} + 2u \otimes z + \lambda u$$

is central in $\widehat{U(\mathfrak{g}^{(1)})}$.

2) Both $\Omega_0$ and $\Omega$ can be represented in the Wick normal form.

3) The linear terms of $\Omega_0 :$ and $\Omega :$ differ by $\lambda u$.

3.3.1. **Remark.**

1) Heading 2) holds since $\Omega_0$ is a finite sum, and $2\Omega^{nm} + 2u \otimes z + \lambda u$ is already in the normal form.

2) Conjecture: If $A$ is not scalar, then no non-zero central quadratic element of $\widehat{U(\mathfrak{g}^{(1)})}$, if any such exists, can be expressed in the Wick normal form.

3.4. **Statement.** The map $A$ commutes with the $\mathfrak{g}$-action.

**Proof.** According to (17), we have

$$[e_k, AX] = \sum_{i,j} b_{ij} \left( [[[e_k, x], e_i], e_j] + (-1)^{p_k p(x)} [[x, [e_k, e_i]], e_j] + (-1)^{p_k (p(x) + p_1)} [[x, e_i], [e_k, e_j]] \right) =$$

$$A[e_k, x] + (-1)^{p_k p(x)} \sum_{i,j} b_{ij} \left( [[x, [e_k, e_i]], e_j] + (-1)^{p_k p_1} [[x, e_i], [e_k, e_j]] \right)$$

and

$$\sum_{i,j} b_{ij} \left( [[x, [e_k, e_i]], e_j] + (-1)^{p_k p_1} [[x, e_i], [e_k, e_j]] \right) =$$

$$\sum_{i,j} b_{ij} (c_{kl}^i [[x, e_i], e_j] + (-1)^{p_k p_1} c_{kl}^i [[x, e_i], e_j]) =$$

$$\sum_{i,j} \left( \sum_{l} (b_{ij} c_{kl}^i + (-1)^{p_k p_1} b_{ij} c_{kl}^i) \right) [[x, e_i], e_j] = 0.$$

So we see that

$$[e_k, AX] = A[e_k, x] \quad \text{for all } k \in \overline{1, d} \text{ and } X \in \mathfrak{g}.$$

So we have

3.4.1. **Remark.** From Schur’s lemma we deduce that if $\mathfrak{g}$ is simple, then $A$ is a scalar operator. For example, if $\mathfrak{g} = \mathfrak{sl}(m|n)$ and $(X \mid Y) + \text{tr}(XY)$, then $A = 2(m - n)$. Note that if $\mathfrak{g}$ is a direct sum of simple algebras, then $A$ is a direct sum of the corresponding scalar operators, and hence it may be not a scalar.

3.5. **Statement.** Let $\mathfrak{g} = \mathfrak{poi}(0|2n)$. Then $A = 0$ if $n > 1$ and not a scalar operator if $n = 1$.

**Proof.** If $n = 1$, then direct computation shows that $A$ does not act by 0 on homogeneous elements of degree 1 and 2 whereas $A(1) = 0$ since 1 lies in the center.

Note that if $x, y$ are homogenous polynomials such that $(x \mid y) \neq 0$, then $\deg x + \deg y = 2n$. Thus, for any homogenous polynomial $X$, if $AX \neq 0$, then $\deg AX = \deg X + 2n - 4$. 
If \( n = 2 \), then \( A(\theta_1 \theta_2 \theta_3 \theta_4) = 0 \): This element must be either 0 or a homogenous polynomial of degree 4, and the only (up to a scalar factor) such polynomial — \( \theta_1 \theta_2 \theta_3 \theta_4 \) — does not lie in \( \mathfrak{g}' \). Since any element of the basis can be represented as
\[
\frac{\partial}{\partial \theta_i} \cdot \frac{\partial}{\partial \theta_i} \cdot \theta_1 \theta_2 \theta_3 \theta_4 = \{ \theta_{i_1}, \{ \ldots, \{ \theta_{i_k}, \theta_1 \theta_2 \theta_3 \theta_4 \} \ldots \} \text{ for } k = 0, 1, 2, 3, 4,
\]
where \( \{ \cdot, \cdot \}_{PB} \) is the Poisson bracket and \( A \) commutes with the \( \text{poi}(0|2n) \)-action, we see that \( A = 0 \).

If \( n \geq 3 \), then \( AX = 0 \) for any homogenous polynomial \( X \) of degree \( \geq 5 \). Since by bracketing with a polynomial of degree 5 one can get any polynomial of degree \( \leq 4 \), and since \( A \) commutes with the algebra action, we deduce that \( A = 0 \).

\[ \square \]

### 3.6. Twisted Kac–Moody (super)algebras

Now, let us consider the case where \( \mathfrak{g} \) can be represented, as a linear space, as \( \bar{\mathfrak{g}}_0 \oplus \bar{\mathfrak{g}}_1 \) so that
\[
[\bar{\mathfrak{g}}_0, \bar{\mathfrak{g}}_0] + [\bar{\mathfrak{g}}_1, \bar{\mathfrak{g}}_1] \subset \bar{\mathfrak{g}}_0; \quad [\bar{\mathfrak{g}}_0, \bar{\mathfrak{g}}_1] \subset \bar{\mathfrak{g}}_1; \quad (\bar{\mathfrak{g}}_0 \mid \bar{\mathfrak{g}}_1) = 0.
\]
In other words, the tilde indicates a \( \mathbb{Z}/2 \)-grading on \( \mathfrak{g} \) (\( a \text{ priori} \) having nothing in common with the parity) respected by the (non-degenerate, even) inner product \( (\cdot | \cdot) \).

In this case, we can define a **twisted Kac–Moody Lie superalgebra**
\begin{equation}
\bar{\mathfrak{g}}^{(2)} = \text{Span}(\bar{\mathfrak{g}}_0 \otimes \mathbb{C}[t^2, t^{-2}], \bar{\mathfrak{g}}_1 \otimes \mathbb{C}[t^2, t^{-2}], u, z),
\end{equation}
where \( u = t^q \frac{\partial}{\partial t} \) and \( z \) are even, with the bracket \( [19] \). For the list of simple twisted Kac-Moody superalgebras, see [FLS].

Let \( A \) still be a scalar operator \( \lambda \). Let \( \{ e_i^0 \}_{i=1}^{d_0} \) be a basis of \( \bar{\mathfrak{g}}_0 \), let \( \{ e_i^1 \}_{i=1}^{d_1} \) be a basis of \( \bar{\mathfrak{g}}_1 \); set
\[
a_{ij}^0 = (e_i^0 | e_j^0); \quad a_{ij}^1 = (e_i^1 | e_j^1), \quad \text{and} \quad (b_{ij}^0) = (a_{ij}^0)^{-1}, (b_{ij}^1) = (a_{ij}^1)^{-1}.
\]

Set
\[
\Omega' = \sum_{i,j} b_{ij}^0 e_i^0 \otimes e_j^0 \in U(\bar{\mathfrak{g}}_0);
\]
\[
\Omega^\pm = \sum_{n=1}^{\infty} \left( \sum_{i,j} b_{ij}^0 t^{-2n} e_i^0 \otimes t^{2n} e_j^0 + \sum_{i,j} b_{ij}^1 t^{-2n+1} e_i^1 \otimes t^{2n-1} e_j^1 \right) \in \hat{U}(\bar{\mathfrak{g}}^{(2)});
\]
\begin{equation}
\Omega = \Omega' + 2\Omega^\pm + 2u \otimes z + \lambda u.
\end{equation}

Note that \( \bar{\mathfrak{g}}_0 \) is a subalgebra of \( \mathfrak{g} \), and \( \Omega' \) can be computed for \( \bar{\mathfrak{g}}_0 \) in the same way as \( \Omega_0 \) was computed for \( \mathfrak{g} \). Then, as in the previous section, we can prove the following

### 3.7. Statement

*The element \( \Omega \) (see (23)) is a central element in \( \hat{U}(\bar{\mathfrak{g}}^{(2)}) \); its linear part is equal to the linear part of \( \Omega' \) plus \( \lambda u \), i.e.,*

*the linear part of \( \Omega \) (not counting \( \lambda u \)) is defined by \( \bar{\mathfrak{g}}_0 \) only.*

Similarly, if the algebra \( \mathfrak{g} \) has a \( \mathbb{Z}/r \)-grading \( \mathfrak{g} = \bigoplus_{s=0}^{r-1} \bar{\mathfrak{g}}_s \), we can construct the algebra
\[
\bar{\mathfrak{g}}^{(r)} = \text{Span}(\bigoplus_{s=0}^{r-1} \bar{\mathfrak{g}}_s \otimes t^s \mathbb{C}[t^r, t^{-r}], u, z)
\]
with the bracket \([19]\). As earlier, set:

\[ e^s_i — the basis elements in \tilde{\mathfrak{g}}_s; \]

\[ a^s_{ij} = (e^s_i | e^s_j); \quad (b^s_{ij}) = (a^s_{ij})^{-1}; \]

\[ \Omega'_0 = \sum_{i,j} b^0_{ij} e^0_i \otimes e^0_j \in U(\tilde{\mathfrak{g}}_0); \]

\[ \Omega^{pm} = \sum_{s=0}^{r-1} \sum_{n=1}^{\infty} \sum_{i,j} b^{s} t^{-2n+s} e^s_i \otimes t^{2n-s} e^s_j \in \hat{U}(\hat{\mathfrak{g}}^{(r)}). \]

If \( A = \lambda I \), then \( \Omega = \Omega'_0 + 2\Omega^{pm} + 2u \otimes z + \lambda u \) is a central element.

For the simple finite dimensional Lie (super)algebras having such a grading with \( r > 2 \) (i.e., \( \mathfrak{sl}(2m+1|2n+1) \) with the automorphism “minus supertransposition\(^{[10]} \), \( \mathfrak{g} = \mathfrak{osp}(4|2; \sqrt{3}) \) and \( \mathfrak{o}(8) \) with the gradings induced by the order 3 automorphisms), \( \text{these gradings are not compatible with the weight ones.} \)

Therefore the Cartan subalgebra of the Lie (super)algebra \( \hat{\mathfrak{g}}^{(r)} \) should be construct not on the base of the Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \), but on the base of the Cartan subalgebra of \( \tilde{\mathfrak{g}}_0 \), which may have nothing in common with \( \mathfrak{h} \). (For example, if \( \mathfrak{g} = \mathfrak{sl}(2m+1|2n+1) \), then \( \tilde{\mathfrak{g}}_0 \simeq \mathfrak{o}(2m+1) \oplus \mathfrak{o}(2n+1) \), and \( \mathfrak{h} \cap \tilde{\mathfrak{g}}_0 = \{0\} \).) Therefore, although the Casimir elements of \( \mathfrak{g} \) and \( \hat{\mathfrak{g}}^{(r)} \) look alike, their Shapovalov determinants look completely different.

4. **Casimir Operators on Lie Superalgebras: The Case of an Odd Form**

In this section, \( \mathfrak{g} \) is a finite dimensional Lie superalgebra on which there is an invariant odd supersymmetric (here: this is the same as just symmetric) non-degenerate bilinear form \((· | ·)\). Let, as above, \( \{e_i\}_{i=1}^d \) be a basis of \( \mathfrak{g} \), and

\[ a_{ij} = (e_i | e_j), \quad \text{and} \quad (b_{ij}) = (a_{ij})^{-1}. \]

4.1. **Statement.** The cubic element

\[ C_3 = \sum (-1)^p c^k_{ij} b_{lm} b_{ji} e_k \otimes e_l \otimes e_m \]

is central in \( U(\mathfrak{g}) \).

Proof is similar as that of Statement 3.2.

4.2. **Remark.** Although the expression of \( C_3 \) looks less symmetric than that of \( \Omega \) from Statement 3.2, it is possible to show that the coefficient

\[ F(k, l, m) := \sum (-1)^p c^k_{ij} b_{lm} b_{ji} \]

of \( e_k \otimes e_l \otimes e_m \) obeys the Sign Rule applied to \( e_k \otimes e_l \otimes e_m \), relative permutation of the indices \( k, l, m \), i.e.,

\[ F(k, l, m) = (-1)^{p_{kh}} F(l, k, m) = (-1)^{p_{pm}} F(k, m, l). \]

Therefore if \( \mathfrak{g} \) is not commutative, then the degree of \( C_3 \) in \( U(\mathfrak{g}) \) is equal to 3 (and not less).

---

\(^{[10]}\)Its order seems to be equal to 4, but for all superdimensions, except for \((2m+1|2n+1)\), its order is equal to 2 modulo the group of inner automorphisms (see \([\text{Sm}]\), where all outer automorphisms of all finite dimensional Lie superalgebras are listed).
Given a Lie superalgebra $\mathfrak{g}$ with an odd invariant form, we can construct the Lie superalgebra $\mathfrak{g}^{(1)} = \text{Span}(\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}], u, z)$ with relations \([19]\), but with an odd $z$.

4.3. Conjecture. If $\mathfrak{g}$ is a simple Lie superalgebra, then $\hat{U}(\mathfrak{g}^{(1)})$ possesses a degree 3 central element that can be represented in the Wick normal form.

5. Kac–Moody-type Lie superalgebras based on queer Lie superalgebras

5.1. What to do if the Cartan subalgebra has odd elements. Here we consider Shapovalov determinant for Lie superalgebras $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{t}$ such that $0 < n_\mathfrak{t} = \dim \mathfrak{t}_\mathfrak{t} < \infty$ and $\mathfrak{t}_\mathfrak{t}$ is commutative. Among $\mathbb{Z}$-graded simple Lie superalgebras of polynomial grows (and their relatives) the algebras with these properties are only the Ramond algebras $R(N) = t^H(1|N)$ (see \([GLL]\)), the relatives of $\mathfrak{q}(n)$ considered by Gorelik \([Go1]\), and the algebras whose Shapovalov determinant nobody considered yet: relatives of $\mathfrak{po}(0|2n+1)$ and Kac–Moody algebras associated with $\mathfrak{q}(n)$, $\mathfrak{po}(0|2n+1)$.

The Shapovalov form $(\cdot | \cdot)$ defined in \([12]\) takes values in $U(\mathfrak{t})$. If $\mathfrak{t}$ is purely even, then $U(\mathfrak{t}) = S(\mathfrak{t})$ is a commutative algebra, and the determinant of $(\cdot | \cdot)$ is well-defined up to a scalar factor. If $\mathfrak{t}_\mathfrak{t} \neq 0$, it is not even clear how to define the Shapovalov determinant.

The simplest way to treat this problem is to consider only Verma modules with 1-dimensional vacuum on which $\mathfrak{t}_\mathfrak{t}$ acts trivially. This restriction \([24]\), however, imposes bounds on possible weights of the vacuum: Under such restriction $[\mathfrak{t}_\mathfrak{t}, \mathfrak{t}_\mathfrak{t}]$ also trivially acts on the vacuum. If $[\mathfrak{t}_\mathfrak{t}, \mathfrak{t}_\mathfrak{t}] = \mathfrak{t}_0$, then the only 1-dimensional $\mathfrak{t}$-module is, up to parity change, the trivial one.

M. Gorelik \([Go1]\) writes that J. Bernstein suggested to define the Shapovalov determinant as follows. First of all, note that $U(\mathfrak{t})$ is a Clifford algebra over $S(\mathfrak{t}_0)$. For a 1-dimensional $\mathfrak{t}_0$-module $\mathbb{C}(\lambda)$ of weight $\lambda \in \mathfrak{t}_0^*$, we define the vacuum module to be

\[
U(\mathfrak{t}) \otimes_{U(\mathfrak{t}_0)} \mathbb{C}(\lambda),
\]

Clearly, $\text{sdim} U(\mathfrak{t}) \otimes_{U(\mathfrak{t}_0)} \mathbb{C}(\lambda) = 2^{n_\mathfrak{t} - 1} | 2^{n_{\mathfrak{t} - 1}}$.

By the Poincaré-Birkhoff-Witt theorem, $U(\mathfrak{t})$ is a filtered algebra, the associated graded algebra being $S(\mathfrak{t}_0) \otimes \wedge(\mathfrak{t}_\mathfrak{t})$. Thus, there is a natural map (the composition of the contraction with the Berezin integral) $f : U(\mathfrak{t}) \rightarrow S(\mathfrak{t}_0) \otimes \wedge^{\dim \mathfrak{t}_\mathfrak{t}}(\mathfrak{t}_\mathfrak{t}) \cong S(\mathfrak{t}_0)$ defined up to a scalar factor, but this suffices for us. So if $\mathfrak{t}_\mathfrak{t} \neq 0$, another Shapovalov form, we call it Bernstein’s Shapovalov form (BSh-form, for short) can be defined to be:

\[
B(\cdot | \cdot) = \int (\cdot | \cdot).
\]

This form takes values in the commutative algebra $S(\mathfrak{t}_0)$, so its determinant is well-defined.

Let us consider separately the case where $n_{\mathfrak{t} - 1} = 1$. In this case, $U(\mathfrak{t})$ is a commutative (not super-commutative!) superalgebra. Thus, if we use $U(\mathfrak{t}) \otimes_{U(\mathfrak{t}_0)} \mathbb{C}(\lambda)$ for the vacuum, the usual Shapovalov form has determinant well defined up to a scalar factor. However, this determinant is equal to the one Gorelik defined \([Go1]\) up to some power of a non-zero element of $\mathfrak{t}_\mathfrak{t}$.

5.2. Open problems. 1) We can construct Kac–Moody Lie superalgebras from Lie superalgebras $\mathfrak{q}(n)$ and $\mathfrak{po}(0|2n - 1)$ (and their “relatives”) using the corresponding odd analog of the invariant Killing form (so $z$ is odd).
In \( g = q(n) \) and \( sq(n) \), there is a central element \( E = 1_{n|n} \), so \([t^n E, \widehat{g(1)}] = \text{Span}(t^n E, z)\). For any element \( x \in U(\widehat{g(1)}_{-}) \), we have
\[
\int HC(x \otimes t^n E) = 0
\]
(since \( HC(x \otimes t^n E) \) can not contain the maximal product of odd elements of Cartan subalgebra), i.e., this element lies in the kernel of the BSh-form. If \( g = pq(n) \) (or, again, \( sq(n) \)), then the invariant form is also degenerate. Since \( \int HC(x \otimes t^n X) \) does not contain \( z \) for any element \( X \) from the kernel of the invariant form and any \( x \in U(\widehat{g(1)}_{-}) \), it follows that \( t^n X \) lies in the kernel of the BSh-form.

Thus, in order the BSh-form be non-degenerate in the generic case, we must consider loops with values in \( g = psq(n) \); the same applies to \( poi(0|2n - 1) \): for non-degeneracy of the BSh-form, we must consider loops with values in \( h'(0|2n - 1) \). We were unable to compute the Shapovalov determinant for such loops.

2) The expression of the Shapovalov determinant depends on the system of positive roots; therefore it is vital to know how to pass from one system to another one. For finite dimensional Lie algebras and Kac–Moody algebras the passage is performed by elements of the Weyl group. For Lie superalgebras such a passage is performed by means of reflections first introduced by Skornyakov, and independently by Penkov and Serganova \([PS]\).

Describe all systems of positive roots (at least, an algorithm: how to pass from one system to another one and an explicit form of at least one of them) for \( g = poi \) or \( h' \) (perhaps \( g \) augmented by the grading operator).

6. Explicit Formulas

6.1. Lie superalgebras with a symmetrizable Cartan matrix. Let \( g = g(A, I) \), where \( I \) is a set of indices labeling odd coroots \( h_i \in t \) dual to the simple roots \( \alpha_i \), be a Lie superalgebra with symmetrizable Cartan matrix \( A = (A_{ij}) \). Since \( A \) is symmetrizable, there exists an invertible diagonal matrix \( D = \text{diag}(d_1, \ldots, d_n) \) such that \( B = DA \) is a symmetric matrix.

On the root lattice \( \Delta = \text{Span}_\mathbb{Z}(\alpha_1, \ldots, \alpha_n) \), define:

1) a symmetric bilinear form \((\cdot, \cdot)\) by setting
\[
(\alpha_i, \alpha_j) = B_{ij} \quad \text{for any } i, j = 1, \ldots, n;
\]

2) a linear function \( F \) by setting
\[
F(\alpha_i) = \frac{1}{2} B_{ii} \quad \text{for any } i, j = 1, \ldots, n;
\]

\( F \) is, essentially, an analog of the function \((\rho, \cdot)\), where if \( \dim g = \infty \) (the sums are taken multiplicities counted)
\[
\rho := \frac{1}{2} \left( \sum_{\alpha \in R^+_0} \alpha - \sum_{\alpha \in R^+_1} \alpha \right),
\]
where \( R^+_0 \) and \( R^+_1 \) are the subsets of \( R^+ \) consisting of even and odd roots, respectively, meaning that the corresponding root vectors are even (resp. odd).
3) for any $\gamma = \sum k_i \alpha_i \in \Delta$, set

$$h_\gamma = \sum k_i d_i h_i$$

(note that, generally, $h_{\alpha_i} \neq h_i$).

Let $R \subset \Delta$ be a root system and $R^+$ the system of positive roots. Since every root space is either even or odd, every root can be endowed with a parity. Let

$$\mathcal{R}_0^+ = \{\alpha \in R^+ | p(\alpha) = 0, \frac{1}{2} \alpha \notin R\}; \quad \mathcal{R}_1^+ = \{\alpha \in R^+ | p(\alpha) = 1, 2\alpha \notin R\}.$$

Define the **partition function**, also called **Kostant function** and denoted, when Kostant actively worked in representation theory, $K$ in his honor, on the set of weights of $U(\mathfrak{g}_-)$ by the formula

$$K(\mu) = \dim U(\mathfrak{g}_-)(\mu).$$

### 6.2. Theorem ([M] [Go2]).

For the $\mathfrak{g}(A, I)$-module $M^\chi$, we have:

$$\text{Sh}_\chi = \prod_{\alpha \in \mathcal{R}_0^+} \prod_{m \in \mathbb{N}} (h_\alpha + F(\alpha) - \frac{m}{2}(\alpha, \alpha))^K(\chi - m\alpha) \prod_{\alpha \in \mathcal{R}_1^+} \prod_{m=2k+1 \in \mathbb{N}} (h_\alpha + F(\alpha) - \frac{m}{2}(\alpha, \alpha))^K(\chi - m\alpha)$$

### 6.2.1. Remark.

Do not use this expression for $\chi$ which are not weights of $U(\mathfrak{g})$ (for example, for $\chi = k\alpha$, where $\alpha$ is an odd simple root such that $2\alpha$ is not a root and $k > 1$); in such cases the formula may produce a non-scalar (hence, wrong) result.

### 6.3. $\mathfrak{po}(0|2n + 1)$.

Let the indeterminates be $\xi_i, \eta_i, \theta$, where $1 \leq i \leq n$, and let the Poisson bracket be of the form

$$[f, g]_\text{P.b.} = (-1)^{p(f)} \sum_{i=1}^n \left( \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i} \frac{\partial g}{\partial \xi_i} + \frac{\partial f}{\partial \theta} \frac{\partial g}{\partial \theta} \right).$$

For the Cartan subalgebra take $\mathbb{C}[\xi_1 \eta_1, \ldots, \xi_n \eta_n, \theta]$. Let $\epsilon_i$ be the weight of $\xi_i$, where $i = 1, \ldots, n$. For any weight $\gamma = \sum c_i \epsilon_i$, set

$$h_\gamma := \sum c_i \xi_1 \eta_1 \ldots \xi_i \eta_i \ldots \xi_n \eta_n.$$

Set also $h_{\text{max}} := \xi_1 \eta_1 \ldots \xi_n \eta_n$.

### 6.3.1. Conjecture.

Let $\mathfrak{g} = \mathfrak{po}(0|2n + 1)$, where $n > 2$, and let there be selected one of the systems of simple roots containing all the $\epsilon_i$. Then $\text{Sh}_\chi$ factorizes in the product of linear terms of the form $h_\gamma$, where the $\gamma$ are the roots of $\mathfrak{g}$, such that $\chi - \gamma$ positive or zero weights $h_{\text{max}}$.

For $\mathfrak{po}(0|3)$ and $\mathfrak{po}(0|5)$, all systems of positive roots are conjugate, and hence it suffices to compute $\text{Sh}_\chi$ for one system. For $\mathfrak{po}(0|3)$, we select the system consisting of $\epsilon_1$, and for $\mathfrak{po}(0|5)$, the one consisting of $\epsilon_1, \epsilon_2, \epsilon_1 \pm \epsilon_2$.

Set:

- $h_1 = \xi_1 \eta_1$; $h_0 = 1$ (the central element of $\mathfrak{po}(0|3)$) for $\mathfrak{po}(0|3)$;
- $h_{10} = \xi_1 \eta_1$; $h_{01} = \xi_2 \eta_2$; $h_{11} = \xi_1 \eta_1 \xi_2 \eta_2$ for $\mathfrak{po}(0|5)$.
6.3.2. Conjecture. If \( g = \mathfrak{po}(0|3) \), then \( \text{Sh}_{k+1} \) factorizes into the product of linear factors of the form

\[ h_0 \quad \text{and} \quad h_1 - m/2, \quad \text{where} \; m = 1, \ldots, k. \]

6.3.3. Conjecture. If \( g = \mathfrak{po}(0|5) \), then, for \( \chi = k\epsilon_1 + l\epsilon_2 \) (such that \( \chi \) is positive if \( k, k+l \geq 0 \) and at least one of the numbers \( k, l \) is positive), \( \text{Sh}_\chi \) factorizes into the product of linear factors of the form

\[ h_{11}; \]
\[ h_{01}; \quad \text{if} \; k, k + l \geq 1; \]
\[ h_{10}; \quad \text{if} \; k + l \geq 1; \]
\[ h_{01} - h_{10} + m, \quad \text{for} \; m \in \mathbb{Z}, \; 1 \leq m \leq k; \]
\[ h_{01} + h_{10} - m, \quad \text{for} \; m \in \mathbb{Z}, \; 1 \leq m \leq k, (k + l)/2. \]

6.3.4. Remark. In the two latter conjectures “scalar summands” are meant to be the scalar terms of \( U(\mathfrak{po}(0|2n + 1)) \), not the central elements of \( \mathfrak{po}(0|2n + 1) \) itself.

6.4. \( \mathfrak{po}(0|2n)^{(1)} \). Let the indeterminants be \( \xi_i, \eta_i \), where \( 1 \leq i \leq n \), and the bracket in \( \mathfrak{po}(0|2n) \) is of the form:

\[ [f, g]_{\mathfrak{po}} = (-1)^{p(f)} \sum_{i=1}^{n} \left( \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i} \frac{\partial g}{\partial \xi_i} \right). \]

For the Cartan subalgebra we take \( \mathbb{C}[\xi_1 \eta_1, \ldots, \xi_n \eta_n] \oplus \text{Span}(u, z) \). Let \( \epsilon_i \) be the weight of \( \xi_i \), where \( i = 1, \ldots, n \), and \( \epsilon' \) be the weight of \( t \). For any weight \( \gamma = \sum c_i \epsilon_i + c' \epsilon' \), we set

\[ h_\gamma := \sum c_i \xi_1 \eta_1 \cdots \xi_i \eta_i \cdots \xi_n \eta_n + c' z. \]

Set also \( h_{\max} := \xi_1 \eta_1 \cdots \xi_n \eta_n \).

6.4.1. Conjecture. Let \( g = \mathfrak{po}(0|2n)^{(1)} \), where \( n > 2 \), and let there be selected one of the system of positive roots containing all roots \( \epsilon_i, \epsilon' \). Then \( \text{Sh}_\chi \) factorizes into the product of linear factors of the form \( h_{\max} \) and \( h_\gamma \), where the \( \gamma \) are positive roots of \( g \) such that \( \chi - \gamma \) are either positive or zero weights.

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