A parallel decoder for good quantum LDPC codes

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We introduce a parallel decoding algorithm for recently discovered families of asymptotically good quantum low-density parity-check codes. This algorithm provably corrects arbitrary errors of weight linear in the code length, with a logarithmic number of steps. This decoder applies directly to the family of quantum Tanner codes, and serves as a subroutine for expander lifted product codes. Along the way, we exploit recently established bounds on the robustness of random tensor codes to give a tight bound on the minimum distance of quantum Tanner codes.

Quantum low-density parity-check (LDPC) codes hold the promise of drastically reducing the required overhead for fault-tolerant quantum computing compared to approaches based on the surface code [1–5]. Rather surprisingly, this advantage was already established for the class of hypergraph product codes [6] that predates the recent series of breakthroughs [7–9] culminating with the discovery of asymptotically good quantum LDPC codes [10, 11], that is, quantum codes of length \(n\) with a constant encoding rate and a minimum distance \(d\) linear in \(n\). These new codes should significantly improve the performance of quantum computers that can implement long-range gates between arbitrary qubits, for instance ion-based [12], photonic [13], or Rydberg-based [14] architectures. To take full advantage of these capabilities, it is crucial to devise efficient decoding algorithms so as to keep the errors under control during an execution of a quantum algorithm. This in particular requires highly parallelizable decoders that run in logarithmic time, since new errors keep accumulating while the classical decoder tries to identify errors that have occurred earlier. We present such an algorithm for the family of quantum Tanner codes [11]. The same decoder also serves as the main subroutine for the decoding of the expander lifted product codes of Panteleev and Kalachev [10].

There are two interesting settings for decoding algorithms, depending on whether the errors are arbitrary (or adversarial) or whether they follow a simple stochastic model (such as independent and identically distributed errors or local stochastic errors [1]). Before the recent breakthroughs yielding codes with a linear minimum distance, the distinction was crucial because experimentally relevant errors have a weight linear in the code length (since each qubit suffers an error with constant probability) and could only be dealt with by making some assumptions about their distribution. This is problematic in the context of fault tolerance because correlations between errors are essentially impossible to track down. While several decoders perform reasonably well against random noise, even with a noisy syndrome extraction [15–21], they cannot handle much more than \(\sqrt{n}\) errors in an adversarial setting [22, 23].

We will first review the construction of quantum Tanner codes and then present our logarithmic-time decoding algorithm. It is strongly inspired by a sequential, linear-time decoder analyzed in [24], and is an adaptation of the small-set-flip decoder initially designed for hypergraph product codes [22], which was also recently applied to other good quantum LDPC codes [25–27].

Quantum Tanner codes.— These codes are a generalization of classical Tanner codes [28, 29]. They are obtained by enforcing local linear constraints (corresponding to the dual of a small tensor code of constant size) of constant weight on qubits associated with the 2-faces (squares) of a square complex [8, 30]. The square complex that we use appears in the work of Panteleev and Kalachev [10] and can also be thought of as a quadruplicate version of the left-right Cayley complex of Dinur et al. [30]. It is an incidence structure between a set \(V\) of vertices, two sets of edges \(E_A\) and \(E_B\), that we will refer to as \(A\)-edges and \(B\)-edges, and a set \(Q\) of squares. The vertex-set \(V\) is partitioned into four subsets \(V = V_{00} \cup V_{01} \cup V_{10} \cup V_{11}\), corresponding to four copies of a fixed group \(G\), that is, \(V_{ij} = G \times \{i, j\}\). We also have two self-inverse subsets \(A = A^{-1}\) and \(B = B^{-1}\) of the group \(G\) and assume for simplicity that \(A\) and \(B\) are of the same cardinality \(\Delta\). For \(i \in \{0, 1\}\), two vertices \(v = (g, i0) \in V_{0i}\) and \(v' = (g', i1) \in V_{1i}\) are related by an \(A\)-edge if \(g' = ag\) for some \(a \in A\). Similarly, for \(j \in \{0, 1\}\), vertices \(v = (g, 0j)\) and \(v' = (g', 1j)\) are related by a \(B\)-edge if \(g' = gb\) for some \(b \in B\). The sets \(E_A\) and \(E_B\) make up the set of \(A\)-edges and \(B\)-edges respectively and define two graphs \(\mathcal{G}_A = (V, E_A), \mathcal{G}_B = (V, E_B)\), each of which consists of two disjoint copies of the double cover of a Cayley graph over the group \(G\) (with generator set \(A\) for \(\mathcal{G}_A\), and \(B\) for \(\mathcal{G}_B\)). Next, the set \(Q\) of squares is defined as the set of 4-subsets of vertices of the form \(\{(g, 00), (ag, 01), (aqb, 11), (gb, 10)\}\), with the four vertices belonging to distinct copies of \(G\).

If we restrict the vertex set to \(V_0 := V_{00} \cup V_{11}\), every square is now incident to only two vertices: one in \(V_{00}\) and one in \(V_{11}\). The set of squares can then be seen as a set of edges on \(V_0\), and it therefore defines a bipartite graph that we denote by \(\mathcal{G}_0 = (V_0, Q)\). Similarly, the

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restriction to the vertices of $V_1 := V_{00} \cup V_{10}$ defines the graph $G_{11}$, which is an exact replica of $G_{01}$, both graphs are defined over two copies of the group $G$, with $g, g' \in G$ being related by an edge whenever $g' = agb$ for some $a \in A, b \in B$. For any vertex $v$, we denote by $Q(v)$ the $Q$-neighbourhood of $v$ defined as the set of squares incident to $v$. The $Q$-neighbourhood $Q(v)$ has cardinality $\Delta^2$ and is isomorphic to the product set $A \times B$; the situation is illustrated on Fig. 1.

![Figure 1. The four local views $Q(v)$ that contain the square \{(g,00),(ag,01),(agb,11),(gb,10)\} depicted in red and blue. The views of two vertices connected by an $A$-edge (resp. a $B$-edge) share a row depicted in red (resp. a column in blue). The labeling is chosen to ensure that a given square, such as the one in red and blue, is indexed similarly, by $(a, b)$ here, in the four local views. The $\sigma_x$-type generators are codewords of $C_A \otimes C_B$ in the local views of $V_{00} \cup V_{11}$; the $\sigma_y$-type generators are codewords of $C_A^\perp \otimes C_B^\perp$ in the local views of $V_{01} \cup V_{10}$. They automatically commute since their support can only intersect on a shared row or column (as depicted), and the orthogonality of the local codes ensure that they commute on this row or column.](image)

A classical Tanner code on a $\Delta$-regular graph $G = (V, E)$ is the set of words of $\mathbb{F}_2^{256}$ such that on the edge neighbourhood of every vertex $v \in V$, we see a codeword of a small code $C$ of length $\Delta$ [28]. We denote the resulting code by $\text{Tan}(G, C) \subset \mathbb{F}_2^{256}$. Expander codes, obtained by combining good codes $\tilde{C}$ with an expander graph $G$, gave the first explicit family of good classical LDPC codes with a constant encoding rate, a linear minimum distance and an efficient decoder [29]

Quantum Tanner codes are quantum CSS codes formed by two classical Tanner codes $C_0$ and $C_1$ with support on the set $Q$ of squares of a square complex as above. The CSS construction [31, 32] requires both codes to satisfy the orthogonality condition $C_0^\perp \subset C_1$. To this end, we define local codes on the space $\mathbb{F}_2^{A \times B}$ that we may think of as the space of matrices whose rows (columns) are indexed by $A$ (by $B$). If $C_A \subset \mathbb{F}_2^A$ and $C_B \subset \mathbb{F}_2^B$ are two linear codes, we define the tensor (or product) code $C_A \otimes C_B$ as the space of matrices $x$ such that for every $b \in B$ the column vector $(x_{ab})_{a \in A}$ belongs to $C_A$ and for every $a \in A$ the row vector $(x_{ab})_{b \in B}$ belongs to $C_B$. Recalling that the dual $C^\perp$ of a code $C \subset \mathbb{F}_2^d$ is the set of words orthogonal to all words in $C$, we define $C_0$ and $C_1$ to be the classical Tanner codes $C_0 := \text{Tan}(G_{01}^\perp, (C_A \otimes C_B)^\perp)$ and $C_1 := \text{Tan}(G_{11}^\perp, (C_A^\perp \otimes C_B^\perp)^\perp)$, with bits associated to each square of $Q$ and local constraints enforced at the vertices of $V_0$ and $V_1$, respectively. To check the orthogonality condition between the two codes, it is convenient to look at their generators (or parity-checks). We define a $\sigma_x$-generator for $C_0$ (resp. a $\sigma_y$-generator for $C_1$) as a vector of $\mathbb{F}_2^Q$ whose support lies entirely in the $Q$-neighbourhood $Q(v)$ of $V_0$ (resp. $V_1$), and which is equal to a codeword of $C_A \otimes C_B$ (resp. $C_A^\perp \otimes C_B^\perp$) on $Q(v)$. The Tanner code $C_0$ (resp. $C_1$) is defined as the set of vectors orthogonal to all $\sigma_x$-generators (resp. $\sigma_y$-generators). The commutation between both types of generators follows from the fact that if a $\sigma_x$-generator on $v_0 \in V_0$ and a $\sigma_y$-generator on $v_1 \in V_1$ have intersecting supports, then $v_0$ and $v_1$ must be neighbours in the left-right Cayley complex and their local views must intersect on either a column or a row, on which the two generators equal codewords of $C_A$ and $C_A^\perp$, or of $C_B$ and $C_B^\perp$ (see Fig. 1). Since $\Delta$ is chosen constant with respect to $n$, we see that the generators of the code have constant weight and that each qubit only appears in a constant number of generators: the resulting quantum Tanner code is therefore a quantum LDPC code by definition. Choosing $C_A$ and $C_B$ of rates $\rho$ and $1 - \rho$, so that both $C_0$ and $C_1$ have rate $\rho(1 - \rho)$, yields a quantum code with encoding rate $k/n \geq (1 - 2\rho)^2$, as can be seen by counting the number of generators of the code. Here, $k$ is the number of logical qubits.

Crucial for the analysis of quantum Tanner codes is the robustness of the small tensor codes. Very recently, [26, 33] showed that if $C_A$ and $C_B$ are chosen randomly with some fixed rate, then their tensor product is $\kappa$-robust for some constant $\kappa$ independent of the code length $\Delta$. This means that for any dual tensor codeword $x \in (C_A^\perp \otimes C_B^\perp)^\perp$, there exist $c \in C_A \otimes \mathbb{F}_2^r, r \in \mathbb{F}_2^A \otimes C_B$ such that $x = c + r$ and

$$|x| \geq \kappa \Delta(||c|| + ||r||),$$

where $| \cdot |$ denotes the Hamming weight and $\| \cdot \|$ counts the number of nonzero columns (resp. rows) of $c$ (resp. $r$). Note that the distance of the random codes $C_A, C_B$ and their dual will be at least $\delta \Delta$ for some $\delta > 0$ with overwhelming probability. We show in Appendix B that these two facts imply a linear lower bound for the distance of the quantum Tanner code.

**Theorem 1.** For a constant $\Delta$ large enough, the quantum Tanner codes described in the construction above with a Ramanujan left-right Cayley complex have a linear minimum distance:

$$d_{\text{min}} \geq \kappa^2 \delta^2 256 \Delta.$$
The dependency in $\Delta$ of this bound is tight since there exist logical errors of weight $\leq n/\Delta$ [11]. The bound is sharper than that of [11], and its proof is somewhat simpler.

We note that quantum Tanner codes have found recent applications outside of coding theory: they can fool optimisation algorithms exploiting the sum-of-squares hierarchy [34] and are instrumental in the recent proof of the NLT's theorem in quantum complexity theory [35, 36].

Decoding quantum Tanner codes. — We focus here on decoding $\sigma_2$-type errors, which are detected by $\sigma_2$-generators. This is without loss of generality for a CSS code. The general strategy outlined in [24] consists in defining a mismatch vector associated with the error $e \in \mathbb{F}_2^Q$, that summarises how the local decoders associated to the local code around each vertex may disagree about the error, and then try to locally modify this mismatch in order to reduce its weight. It is natural to see the error $e$ as a collection of local views on the vertices of $V_1$: abusing notation slightly, we can write $e = \{ e_v \}_{v \in V_1}$, with local views $e_v$ restricted to $Q(v)$. For each vertex $v \in V_1$, one can compute (in parallel if needed) a local error $\varepsilon_v$ with support on $Q(v)$ of minimal Hamming weight yielding the same local syndrome as $e_v$. This gives a decomposition of the local views of the error $e_v = \varepsilon_v + c_v + r_v$, with $c_v \in C_A \otimes \mathbb{F}_2^Q$, $r_v = \mathbb{F}_2^Q \otimes C_B$. Note that the $k$-robustness property will apply to any such $c_v + r_v$. The issue is that the local views $\{ \varepsilon_v \}_{v \in V_1}$ are in general not consistent and do not define a global error candidate. We measure this inconsistency by defining the mismatch vector:

$$Z := \sum_{v \in V_1} \varepsilon_v \in \mathbb{F}_2^Q.$$  

If it is equal to zero, it means that each square/qubit is affected the same value for the two views it belong to, and the decoder is able to define a global error. Otherwise, the support of $Z$ corresponds to the set of squares for which the local views disagree. Noting that $\sum_{v \in V_1} e_v = 0$ since each square appears twice in this sum, we can rewrite the mismatch (2) as

$$Z = \sum_{v \in V_1} r_v + c_v = C_0 + R_0 + C_1 + R_1,$$

with $C_i := \sum_{v \in V_1} c_v$, and $R_j := \sum_{v \in V_1} r_v$, where we denote $i := 1 - i, j := 1 - j$. This convenient representation of $Z$ highlights the symmetry of the local representations of $C_j + R_0$ on vertices of $V_{ij}$, and one may look for local modifications of $Z$ by adding to it a dual tensor codeword on the $Q$-neighbourhood of any of the four types of vertices.

The main subroutine of the decoding algorithm consists in finding a valid decomposition $Z = C_0 + R_0 + C_1 + R_1$. The simplest approach to this is a sequential decoder that proceeds in a greedy fashion: simply look for a vertex $v$, and local codewords $c_v, r_v$ such that flipping $c_v + r_v$ (replacing $Z$ by $Z + c_v + r_v$) decreases the Hamming weight of $Z$. Our main technical result exploits the expansion properties of the square complex together with the robustness of the local codes to establish the existence of some vertex $v \in V$ and local dual tensor codeword $c_v + r_v$ that decreases the weight of $Z$ almost optimally. Instrumental in this analysis is the notion of the norm of a representation of a mismatch vector $Z$ which can be expressed in many different ways as $Z = C_0 + R_0 + C_1 + R_1$. The “column” vector $C_i$ is expressed as a sum of local vectors with disjoint supports, each of which is a codeword of $C_A$ supported by a column common to two $Q$-neighbourhoods of different types of vertices ($V_i$ and $V_j$). The row vectors $R_j$ admit similar decompositions into $C_B$ components. We will write $\|C_i\| (\|R_j\|)$ to denote the number of non-zero $C_A$ codewords ($C_B$ codewords) in the decomposition of $C_i$ ($R_j$). We will say that a decomposition $Z = C_0 + R_0 + C_1 + R_1$ is minimal if it minimises its norm, namely the value $\|C_0\| + \|R_0\| + \|C_1\| + \|R_1\|$. Finally, we shall say that a vertex $v \in V_{ij}$ is active for a decomposition $(C_0, R_0, C_1, R_1)$, if $C_j + R_i$ is non-zero on $Q(v)$.

**Theorem 2.** Fix $\varepsilon \in (0, 1)$. If the sets of active vertices $S_{ij}$ for a minimum decomposition of a mismatch vector $Z$ satisfy

$$|S_{ij}| \leq \frac{1}{24}\delta^2\varepsilon^3|V_00|$$

then there exists some vertex $v$ and some codeword $x_v$ of the dual tensor code such that

$$|Z| - |Z + x_v| \geq (1 - \varepsilon)|x_v|.$$

Provided the initial error is not too large, we show that the condition on the number of active vertices in Theorem 2 keeps being satisfied, so that one can keep on iterating the procedure, finding a local codeword to flip at each step, and that it will eventually give a decomposition $Z = C_0 + R_0 + C_1 + R_1$ of the mismatch. This decomposition gives a correct guess $\hat{e} = \sum_{v \in V_{00}} \varepsilon_v + C_0 + R_0$ for the error, that differs from $e$ by a sum of generators. This naturally yields a sequential decoder, with parameter $\varepsilon$, described formally in the appendix.

It is easy to see that the pre-processing (defining the mismatch) and the post-processing (computing $\hat{e}$ from the decomposition of $Z$) can be performed in parallel. To get a fully parallel decoder, one simply needs a parallel procedure for decomposing $Z$. This is obtained by looking for vertices that would be candidates for the sequential decoder of parameter $2\mu \in (2\varepsilon, 1)$, i.e. finding some $x_v$ such that $|Z| - |Z + x_v| \geq (1 - 2\mu)|x_v|$. In order to apply several sequential iterations in parallel, the decoder needs the different $c_v + r_v$’s that it will identify to have disjoint supports. This is achieved if one restricts the set of candidate vertices $v$ to a single set $V_{ij}$; so the decoder chooses its 4 parallel options, each of which consists of applying simultaneously all sequential iterations for all the candidate vertices it has identified in a given $V_{ij}$. It then applies the option that maximises the
decrease of $|\hat{Z}|$. The pseudo-code of all the algorithms is presented in Appendix C.1.

Theorem 3. Fix $\varepsilon \in (0, 1/2)$, and $\mu \in (\varepsilon, 1/2)$. If the Hamming weight $|e|$ is less than

$$\frac{1}{2^{12}} \min \left( \frac{\varepsilon^3}{16}, \kappa \right) \kappa^2 \psi^2 n \Delta,$$

then the parallel decoder returns a valid correction in logarithmic time.

The parameter $\mu$ determines the number of guaranteed parallel steps of the decoder. Its optimal value is $\mu = \sqrt{\varepsilon/2}$, which guarantees at most $\log(4/\varepsilon)/\log(1/c)$ parallel steps, where $c = \varepsilon^{-1} - (1 - \sqrt{2\varepsilon})^2/4$. The value of $\varepsilon$ can be tuned to what one wants to achieve. A larger $\varepsilon$ will increase the weight of correctable errors but increases the number of required parallel steps, while a smaller $\varepsilon$ decreases the number of parallel steps but only decodes smaller weight errors.

The main technical result needed to prove that the parallel decoder converges after a logarithmic number of iterations is that there is a linear number of vertices $v$ that can be updated at each step to decrease the weight of $Z$. The idea to prove their existence consists in running the sequential algorithm virtually and then establishing that many of the updates corresponding to the sequential decoder than in fact be performed at the same time. It is detailed in the proof of Theorem 14 in Appendix C.2.

Remark: we have not tried to optimise the power of 2 numerical constants in Theorems 1, 2, 3 for the sake of readability.

Decoding lifted product codes.—We remark that the same decoding algorithm can be applied to the lifted product codes of [10]. Indeed, their decoding can be reduced (with a parallel procedure) to that of quantum Tanner codes as explained in [24]. Furthermore, the decoder also works for hypergraph products of two classical Tanner codes, albeit with smaller correction capabilities since these codes have a distance scaling like $\sqrt{n}$.

Discussion and open questions.—Rapid improvements of decoding algorithms over the last decade have led to a surge of interest for LDPC as a path towards hardware-efficient fault-tolerant quantum computing. The unexpected discovery of good quantum LDPC codes with parameters close to optimal will likely impact this research direction in major ways. The parallel decoding algorithm presented here is a first step towards this goal. Of course, a great number of open questions remains. While our proof requires codes of large size, the decoder is well defined for any quantum Tanner code or lifted product code, and it will be interesting to investigate its performance against random noise. A practical decoder should be robust to errors in the syndrome extraction process. We believe this is the case here, and that an analysis along the lines of [3] could be extended to the present decoder. It also does not suffice to correct errors in order to perform a computation, and one must be able to apply logical gates in a fault-tolerant manner [37]. At the moment, a complete understanding of the logical operators for good LDPC codes is still lacking. Finally, and crucially, we will need examples of good codes of reasonably small size if such codes are to be implemented in real devices.

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In this Appendix, we first review in Section A some technical preliminaries on graph expansion and robustness of dual tensor codes. We establish the lower bound on the minimum distance in Section B. Finally, in Section C, we describe the various decoders (sequential and parallel) and analyse them in detail.

Appendix A: Technical preliminaries

1. Graph expansion

In this section, we recall some useful facts about graph expansion.

Let \( G = (V, E) \) be a graph. Graphs will be undirected but may have multiple edges. For \( S, T \subset V \), let \( E(S, T) \) denote the multiset of edges with one endpoint in \( S \) and one endpoint in \( T \). Let \( \mathcal{G} \) be a connected \( \Delta \)-regular graph on \( n \) vertices, and let \( \Delta = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \) be the eigenvalues of the adjacency matrix of \( \mathcal{G} \). For \( n \geq 3 \), we define \( \lambda(\mathcal{G}) := \max\{|\lambda_i|, \lambda_i \neq \pm \Delta\} \). The graph \( \mathcal{G} \) is said to be Ramanujan if \( \lambda(\mathcal{G}) \leq 2\sqrt{\Delta - 1} \).

We recall the following version of the expander mixing lemma (see e.g. [38]) for bipartite graphs.

**Lemma 4 (Expander mixing lemma).** Let \( G \) be a connected \( \Delta \)-regular bipartite graph on the vertex set \( V_0 \cup V_1 \). For any pair of sets \( S \subset V_0, T \subset V_1 \), it holds that

\[
|E(S, T)| \leq \frac{\Delta}{|V_0|} |S||T| + \lambda(\mathcal{G})\sqrt{|S||T|}.
\]

Below \( V = V_0 \cup V_1 \) stands for the vertex set of a left-right Cayley Complex. Recall that \( V_0 = V_{00} \cup V_{11} \) and \( V_1 = V_{01} \cup V_{10} \) make up four copies of a group \( G \), and that we have defined the graphs \( \mathcal{G}_A \) and \( \mathcal{G}_B \) on \( V \), where \( \mathcal{G}_A \) consists of two copies of the double cover of the left Cayley graph \( \text{Cay}(G, A) \), and \( \mathcal{G}_B \) of two copies of the right Cayley graph \( \text{Cay}(G, B) \). We choose \( \text{Cay}(G, A) \) and \( \text{Cay}(G, B) \) to be non-bipartite, so that their double covers are connected.

Let us introduce another graph \( \mathcal{G}^\square \) in addition to \( \mathcal{G}_A \) and \( \mathcal{G}_B \) that shares the vertex set \( V \) and exists on the left-right Cayley complex. It puts an edge between all pairs of vertices of the form \( \{(g, ij), (agb, \overline{ij})\} \), \( g \in G, a \in A, b \in B, i, j = 0, 1 \). The graph \( \mathcal{G}^\square \) is therefore made up of two connected components, on \( V_0 \) and \( V_1 \), that we denote by \( \mathcal{G}_0^\square \) and \( \mathcal{G}_1^\square \). We note that \( \mathcal{G}^\square \) is regular of degree \( \Delta^2 \), and may have multiple edges.

If \( \text{Cay}(G, A) \) and \( \text{Cay}(G, B) \) are Ramanujan, then \( \mathcal{G}^\square \) inherits their expansion properties. Specifically:

**Lemma 5.** Assume that \( \text{Cay}(G, A) \) and \( \text{Cay}(G, B) \) are Ramanujan graphs, then

\[
\lambda(\mathcal{G}_0^\square) \leq 4\Delta, \quad \lambda(\mathcal{G}_1^\square) \leq 4\Delta.
\]

See [11] for a proof. In the bounds above, one could even replace \( \Delta \) by \( \Delta - 1 \), but we keep \( \Delta \) since it leads to simpler expressions.

2. Robustness of (dual) tensor codes

Besides the expansion properties of the left-right Cayley complex, the other required property to obtain good quantum LDPC codes is the robustness of the local dual tensor codes [11, 33]. In words, it states that any low-weight codeword of the dual tensor code \( C_A \otimes \mathbb{F}_2^A + \mathbb{F}_2^A \otimes C_B \) can be obtained as a sum \( c_v + r_v \) where the Hamming weights of \( c_v \) and \( r_v \) are not much larger than that of \( c_v + r_v \). Very recently, better bounds on the robustness of dual tensor codes obtained from two random codes were obtained in [25, 26, 33]. In particular, [26, 33] prove essentially tight bounds. We recall the main result from [33], where robustness is called product expansion and is defined as follows: for two linear codes \( C_A, C_B \subset \mathbb{F}_2^A \), the dual tensor code \( C_A \otimes \mathbb{F}_2^A + \mathbb{F}_2^A \otimes C_B \) is said to be \( \kappa \)-product expanding, if any codeword \( x \in C_A \otimes \mathbb{F}_2^A + \mathbb{F}_2^A \otimes C_B \) can be written as \( x = c + r \) with \( c \in C_A \otimes \mathbb{F}_2^A, r \in \mathbb{F}_2^A \otimes C_B \) and

\[
|c + r| \geq \kappa \Delta(|c||r|),
\]

where \(|c| (|r|)\) denotes the number of columns (rows) involved in the support of \( c \) (\( r \)).

**Theorem 6 ([33]).** For every \( \rho_A, \rho_B \in (0, 1) \), the dual tensor code \( C_A \otimes \mathbb{F}_2^A + \mathbb{F}_2^A \otimes C_B \) obtained from a uniformly random pair of linear codes \( (C_A, C_B) \), of lengths \( \Delta \) and of codimensions \( \lfloor \rho_A \Delta \rfloor \) and \( \lfloor \rho_B \Delta \rfloor \) respectively, is \( \kappa \)-product-expanding with high probability as \( \Delta \to \infty \) with

\[
\kappa = \frac{1}{2} \min \left( \frac{1}{4} \frac{H_2^{-1} \left( \frac{\rho_A}{8} \right)}{H_2^{-1} \left( \frac{\rho_B}{8} \right)} \right),
\]

where \( H_2(x) \) is the binary entropy function.
Here $H_2^{-1}$ is the inverse of the binary entropy function given by

$$H_2(x) := -x \log_2 x - (1 - x) \log_2(1 - x).$$

**Remark:** for our application we have $\rho_B = 1 - \rho_A$. Therefore, Theorem 6 implies that both dual tensor codes $C_A \otimes F^A_2 + F^A_2 \otimes C_B$ and $C_A^+ \otimes F^A_2 + F^A_2 \otimes C_B^+$ are with high probability $\kappa$-product-expanding for the same value $\kappa$.

**Appendix B: Minimum distance**

Let $x$ be a codeword of $\mathcal{C}_1$ not in $\mathcal{C}_0^+$. We will derive a lower bound on the weight of $x$, that will also be valid for the weight of a codeword of $\mathcal{C}_0$ not in $\mathcal{C}_1^+$, and therefore bound from below the minimum distance of the code.

The codeword $x$ is a Tanner codeword on the graph $G^0_1$, with vertex set $V_1 = V_{01} \cup V_{10}$ and the dual tensor code $C_A \otimes F^A_2 + F^A_2 \otimes C_B$ as inner code. The set $Q$ of coordinates is partitioned into $Q$-neighbourhoods $Q(v)$ of vertices $v$ of $V_{01}$, on which $x$ reduces to a dual-tensor codeword $x_v = c_v + r_v$, with $c_v$ and $r_v$ being $C_A \otimes F^A_2$ codewords and $F^A_2 \otimes C_B$ codewords respectively. We regroup the “column” vectors and “row” vectors and write $C_1 = \sum_{v \in V_{01}} c_v$ and $R_0 = \sum_{v \in V_{10}} r_v$. Similarly, using the partition of $Q$ into $Q(v)$s for $v \in V_{10}$, we define $C_0 = \sum_{v \in V_{10}} c_v$ and $R_1 = \sum_{v \in V_{10}} r_v$.

Let us denote by $\|C_i\| = \sum_{v \in V_{10}} \|c_v\|$ the total number of non-zero column vectors in $C_A$ intervening in the local views of $x$, for the partition over $Q(v), v \in V_{01}$. We write $I = 1 - i$ and $J = 1 - j$ to lighten notation. Similarly, we denote by $\|R_i\|$ the quantity $\|R_i\| = \sum_{v \in V_{01}} \|r_v\|$. Note that there are several possible decompositions of a local view $x_v$ of $x$ as $x_v = c_v + r_v$. If for every $v \in V_1$ we choose one that minimises $\|c_v\| + \|r_v\|$, that is the total number of $C_A$ and $C_B$ codewords in the decomposition of $x_v$, we will obtain a minimal representation $(C_0, R_0, C_1, R_1)$ of $x$ that minimises the quantity $\|C_0\| + \|R_0\| + \|C_1\| + \|R_1\|$. We call the latter quantity the norm of $x$, and denote it by

$$\|x\| := \|C_0\| + \|R_0\| + \|C_1\| + \|R_1\|.$$ 

Since $x = \sum_{v \in V_{01}} x_v = C_1 + R_0 = \sum_{v \in V_{10}} x_v = C_0 + R_1$, we have $C_0 + R_0 + C_1 + R_1 = 0$, and therefore $C_0 + R_0 = C_1 + R_1$. We make the remark that the local views of the vector $x^0 : = C_0 + R_0 = C_1 + R_1$ at vertices of $V_0$ are also dual tensor codewords in $C_A \otimes F^B_2 + F^A_2 \otimes C_B$ (which should not be confused with the local views of $x$ on vertices of $V_0$). In other words the vector $x^0$ is a codeword of the Tanner code on the graph $G^0_0$ and the same inner code $C_A \otimes F^B_2 + F^A_2 \otimes C_B$. Now, if $(C_0, R_0, C_1, R_1)$ is a minimal representation for $x$, it may not necessarily be a minimal representation for $x^0$. However, if we consider a local view $x^0_v$ at $v \in V_{00}$, and if its decomposition $x^0_v = c_v + r_v$ is not minimal, where $r_v$ and $c_v$ are the local views at $v$ of $R_0$ and $C_0$, then we may replace the local decomposition by a minimal one, which will equal $x^0_v = (c_v + t_v) + (r_v + t_v)$, where $t_v$ is some tensor codeword in $(C_A \otimes F^B_2) \cap (F^A_2 \otimes C_B)$. This has the effect of changing $C_1 + R_0$ and $C_0 + R_1$ to $C_1 + R_0 + t_v$ and $C_0 + R_1 + t_v$, and of reducing the sum $\|C_0\| + \|R_0\| + \|C_1\| + \|R_1\|$. We observe that in this case $x^0$ is unchanged and $x$ is changed to another codeword of $\mathcal{C}_1$, which stays in the same class $x + \mathcal{C}_0^+$, since the tensor codeword $t_v$ is a generator. The conclusion is that we can keep proceeding in this way until no local modification is possible, at which point we will have replaced $x$ by a codeword of the same class modulo $\mathcal{C}_0^+$, and $(C_0, R_0, C_1, R_1)$ will be a minimal representation of both $x$ and of $x^0$.

We have shown in particular:

**Lemma 7.** If $x \in \mathcal{C}_1 \setminus \mathcal{C}_0^+$, and $\|x\|$ is the minimum norm in the coset $x + \mathcal{C}_0^+$, then a minimal representation for $x$ is also a minimal representation for $x^0$.

We will prove the following lower bound on the norm of a codeword of $\mathcal{C}_1 \setminus \mathcal{C}_0^+$.

**Lemma 8.** For any $x \in \mathcal{C}_1 \setminus \mathcal{C}_0^+$, we have

$$\|x\| \geq \frac{\kappa \delta^2 n}{512 \Delta^2}.$$ 

From Lemma 8 we easily deduce a lower bound on the quantum minimum distance.

**Theorem 9.** The minimum distance of the quantum Tanner code satisfies

$$d_{\text{min}} \geq \frac{\kappa^2 \delta^2 n}{256 \Delta}.$$
Proof. Let $x$ be a codeword of $x \in C_1 \setminus C_0^\perp$, and let $(C_0, R_0, C_1, R_1)$ be a minimal representation for $x$. We have
\[ x = C_1 + R_0 = \sum_{v \in V_{10}} (c_v + r_v). \]
Since the local vectors in this sum have disjoint supports, we have
\[ |x| = \sum_{v \in V_{10}} |c_v + r_v| \geq \sum_{v \in V_{10}} \kappa \Delta (\|c_v\| + \|r_v\|) \]
applying robustness of the local codes and minimality of the representation. Hence $|x| \geq \kappa \Delta (\|C_0\| + \|R_0\|)$, and similarly $|x| \geq \kappa \Delta (\|C_1\| + \|R_1\|)$ by summing over $V_{10}$. Therefore,
\[ |x| \geq \kappa \Delta \frac{1}{2} (\|C_1\| + \|R_0\| + \|C_0\| + \|R_1\|) = \kappa \Delta \frac{1}{2} |x| \]
which proves the lower bound for non-trivial codewords of $C_1$. The same lower bound holds for non-trivial codewords of $C_0^\perp$ by symmetry. \qed

It remains to prove Lemma 8. We may suppose that $x$ achieves the minimum of $\|x\|$ in $x + C_0^\perp$, so that Lemma 7 holds. In the following, $(C_0, R_0, C_1, R_1)$ is a minimal representation of $x$.

Let $S_{ij}$ denote the set of vertices $v$ of $V_{ij}$ for which $C_j + R_i$ is non-zero on $Q(v)$. Let us also set $S_0 = S_{00} \cup S_{11}$, and $S_1 = S_{01} \cup S_{10}$. Let us call a vertex $v$ of $V_{ij}$ exceptional, if $\|c_v\| + \|r_v\|$ is at least $\alpha \Delta$ with $\alpha = \delta^2/256$, where $c_v$ and $r_v$ are the restrictions to $Q(v)$ of $C_j$ and $R_i$ respectively.

Ordinary vertices of $S_{ij}$ are defined as non-exceptional. We remark that any non-zero column $C_A$-vector of $C_j$ (or any row vector of $R_i$) in the $Q$-neighbourhood of any ordinary vertex of $S_{ij}$ has, by definition of ordinary, at most $2\delta \frac{\delta}{2}$ non-zero coordinates in common with $R_i$, which we bound from above by $\frac{1}{2} \delta \Delta$ for the sake of readability.

The following lemma states that whenever $S_{ij}$ is small enough, the number of exceptional vertices is a small fraction of $|S_{ij}|$, of order $O(1/\Delta^2)$, hence the terminology.

Lemma 10. Let $i = 0, 1$. Under the hypothesis $|S_{ij}| \leq \frac{\alpha \delta}{2} |V_{00}|$, we have
\[ |S_{00}^e| \leq \frac{64}{\Delta^2 \alpha^2 \kappa^2} |S_0| \quad \text{and} \quad |S_{11}^e| \leq \frac{64}{\Delta^2 \alpha^2 \kappa^2} |S_1|. \]

Proof. We prove the upper bound for $S_{00}^e$, the other cases being similar. Viewing $C_0 + R_0$ as a subgraph of $G_0^\perp$, we have that vertices of $S_{00}^e$ have degree at least $\kappa \Delta \alpha \Delta$ (applying robustness, which we may do by minimality of $\|C_0\| + \|R_0\| + \|C_1\| + \|R_1\|$), and the Expander mixing Lemma in $G_0^\perp$ gives
\[ |S_{00}^e| \alpha \kappa \Delta^2 \leq |E(S_{00}^e, S_{11})| \leq \Delta^2 \frac{|S_{00}^e| |S_{11}|}{|V_{00}|} + 4 \Delta \sqrt{|S_{00}^e||S_{11}|}. \]
Upper bounding $|S_{11}|/|V_{00}|$ by $\alpha \kappa/2$, we obtain
\[ \frac{1}{2} |S_{00}^e| \Delta^2 \alpha \kappa \leq 4 \Delta \sqrt{|S_{00}^e||S_{11}|} \]
and finally
\[ |S_{00}^e| \leq \frac{64}{\Delta^2 \alpha^2 \kappa^2} |S_{11}| \leq \frac{64}{\Delta^2 \alpha^2 \kappa^2} |S_0|. \]

Let $v$ be a vertex of $V_{ij}$ and consider a column on its local view. Suppose this column supports a non-zero $C_A$-codeword that is part of $C_j$. Recall that this column is shared by the local view of a neighbouring vertex $w$ in $V_{ij}$. Below we consider the set $T$ of vertices of $V_{ij}$ whose local views see at least one non-zero $C_A$-codeword of $C_j$ that is shared by the local view of an ordinary vertex $w \in V_{ij}$. Let $v$ be a vertex of $V_{ij}$ and consider a column on its local view. Suppose this column supports a non-zero $C_A$-codeword that is part of $C_j$. Recall that this column is shared by the local view of a neighbouring vertex $w$ in $V_{ij}$. Below we consider the set $T$ of vertices of $V_{ij}$ whose local views see at least one non-zero $C_A$-codeword of $C_j$ that is shared by the local view of an ordinary vertex $w \in V_{ij}$.

Lemma 11. If we have $|S_{11}| \leq \delta |V_{00}|/4$, then
\[ |T| \leq \frac{64}{\delta^2 \Delta} |S_{11}|. \]
Furthermore, exactly the same result holds if $T$ is defined as the subset of vertices of $S_{11}$ on whose $Q$-neighbourhood $R_T$ displays a non-zero row codeword of $C_B$ that is shared by the local view of an ordinary vertex $w \in S_{11}$.
Proof. We deal with the case when \( i = j = 0 \) and \( T \) is defined as the set of vertices of \( V_{00} \) whose \( Q \)-neighbourhoods share a non-zero column vector with an ordinary vertex of \( V_{10} \). The other cases will hold by symmetry. If we keep only the two sets of vertices \( V_{00} \) and \( V_{01} \), then every square of the square complex becomes incident to two vertices (instead of four), and we obtain a multigraph. In this multigraph, every row of a local view appears in two local views, one of a vertex \( v \in V_{00} \), and the other in a neighbouring vertex of \( V_{01} \). If we collapse this row to a single “square” (which has now become an edge) by identifying them, the bipartite graph over \( V_{00} \cup V_{01} \) that we obtain in this way is exactly the double cover of the Cayley graph \( \text{Cay}(G, A) \). Now the codeword \( x \) induces a subgraph of this graph, which we obtain by putting an edge in the subgraph whenever the row view it originates from is non-zero in \( x \).

By construction, every vertex of \( T \) in this subgraph has degree at least \( \delta \Delta / 2 \) (actually degree at least \( (1 - \frac{\delta}{256}) \delta \Delta \), as discussed above), and its edges fall into \( S_{01} \). Hence,

\[
|T| \frac{\delta \Delta}{2} \leq |E(S_{01}, T)|.
\]

Applying the expander mixing Lemma in the double cover of \( \text{Cay}(G, A) \) we obtain,

\[
|E(S_{01}, T)| \leq \Delta \frac{|S_{01}|}{|V_{01}|} |T| + 2\sqrt{\Delta} \sqrt{|S_{01}| |T|}.
\]

Finally, applying the hypothesis \( |S_{01}| \leq \delta |V_{01}| / 4 \) we get

\[
|T| \frac{\delta \Delta}{4} \leq 2\sqrt{\Delta} \sqrt{|S_{01}| |T|}
\]

and the result follows. We remark that when \( T \) is defined in \( V_{1i} \) as opposed to \( V_{ii} \), it is the codeword \( x^0 = R_0 + C_0 \) that is considered rather than \( x \).

Now, Lemma 10 will tell us that if \( ||x|| \) is too small, there can only be very few exceptional vertices. But what Lemma 11 then tells us, is that the non-zero column codewords in the neighbourhoods of ordinary vertices must cluster in a limited number of \( Q \)-neighbourhoods for vertices of a neighbouring type, yielding too many exceptional vertices and contradicting Lemma 10. We now make this argument formal.

Proof of Lemma 8. Suppose \( x \in c_1 \setminus c_2^1 \) satisfies \( ||x|| < \frac{a^2 \kappa}{32 \Delta} \). We may suppose that \( ||x|| \) achieves its minimum value inside the coset \( x + c_2^0 \), and that \( (C_0, R_0, C_1, R_1) \) is a minimal representation for \( x \). Since we have \( |S_{ij}| \leq ||C_j|| + ||R_i|| \leq ||C_0|| + ||R_0|| + ||C_1|| + ||R_1|| = ||x|| \), we have that \( |S_{ij}| \leq \frac{a^2}{32} |V_{00}| = \frac{1}{2} \kappa |V_{00}| \). Therefore Lemma 10 holds, and so does Lemma 11 since clearly \( \frac{1}{2} \kappa \alpha \leq \delta / 4 \).

Without loss of generality let us suppose \( |S_1| \geq |S_0| \) (otherwise invert their roles in the argument below) and \( |S_{10}| \geq |S_{01}| \) (otherwise invert their roles).

Because there are so few exceptional vertices in \( S_{10} \) (by Lemma 10), the number of ordinary vertices of \( S_{10} \) is almost equal to \( |S_{10}| \) namely it is at least \( a |S_{10}| \) for any constant \( a < 1 \) and \( \Delta \) large enough.

We obtain that either the number of ordinary rows or the number of ordinary columns in (the \( Q \)-neighbourhoods of vertices of) \( S_{10} \) is at least \( a |S_{10}| / 2 \). Suppose without loss of generality* that the number of ordinary columns is at least \( a |S_{10}| / 2 \). Applying Lemma 11, they must cluster among the \( Q \)-neighbourhoods of a set \( T \subset S_{00} \) of size at most \( \frac{a^2}{32} |S_{01}| \leq \frac{a^2}{32} |S_{10}| \) (since we have supposed \( |S_{01}| \leq |S_{10}| \)). Therefore, the average number of non-zero columns of \( C_0 \) on the \( Q \)-neighbourhoods of the vertices of this set \( T \) is at least \( a |S_{10}| / 2 (\frac{a^2}{32} |S_{10}|)^{-1} = a \frac{2}{32} \), which is close to twice the minimum norm of a local view of \( R_i + C_j \) for an exceptional vertex. This shows that a constant proportion of vertices of \( T \) must be exceptional vertices of \( S_{00} \). Now since the \( Q \)-neighbourhood of a vertex can host at most \( \Delta \) columns, we also have \( |T| \geq \frac{a}{2} |S_{10}| \geq \frac{a}{256} |S_1| \) (since \( |S_{10}| \geq |S_{01}| \)), and since \( |S_1| \geq |S_0| \), we have \( |T| \geq \frac{a}{32} |S_0| \). Therefore, for large enough \( \Delta \), we have a contradiction with Lemma 10 which limits the number of exceptional vertices of \( S_{00} \) to not more than \( O(1/\Delta^2) |S_0| \).

* If it is the number of ordinary rows that exceeds \( a |S_{10}| / 2 \), then \( T \) is defined inside \( S_{11} \) instead of \( S_{00} \) and the rest of the argument is unchanged.

Appendix C: Analysis of the decoder

In this section, we present in detail the various decoders: the general decoding procedure including the computation of the mismatch and the post-processing is described in Algorithm 1. We give two procedures for finding a mismatch decomposition of the form \( Z = C_0 + R_0 + C_1 + R_1 \), a sequential procedure described in Algorithm 2 and a parallel one described in Algorithm 3. The parallel mismatch decomposition requires calls to single parallel steps that are described in Algorithm 4.
1. Description of the decoders

Both the sequential decoder and the parallel decoder follow the blueprint of Algorithm 1. They use a pre-processing phase which consists of computing a mismatch vector $Z$. Specifically, on the $Q$-neighbourhood of every vertex of $V_0$ (if one is trying to correct a $\sigma_z$-error), or of $V_1$ (if one is trying to correct a $\sigma_x$-error), the decoder computes a local estimation $\hat{\varepsilon}_v$ of the original error vector $\varepsilon$, and then sums all these $\hat{\varepsilon}_v$’s to make up the mismatch vector $Z$. Note that this common preprocessing phase can be parallelised straightforwardly when needed.

The core of the decoding algorithm is then to uncover a decomposition of the mismatch vector of the form $Z = \hat{C}_0 + \hat{R}_0 + \hat{C}_1 + \hat{R}_1$, where $\hat{C}_i$ is a sum of local (column) codewords of $C_A$ whose coordinates are indexed by a column of a $Q$-neighbourhood $Q(v)$ for $v \in V_{ii}$, and similarly $\hat{R}_i$ is a sum of local row codewords of $C_B$ whose coordinates are indexed by a row in some $Q(v)$, $v \in V_{ii}$. Note that the individual local column codewords of $\hat{C}_i$ also appear in the $Q$-neighbourhoods of vertices of $V_{i\cdot}$, and the row codewords of $\hat{R}_i$ appear likewise in $Q$-neighbourhoods of $v \in V_{\cdot i}$. The mismatch decomposition procedures differ significantly for the sequential decoder and the parallel decoder.

Finally, both decoders use a common postprocessing phase once they have decomposed the mismatch vector $Z$. If one is decoding a $\sigma_z$-error, the decoders output their estimation of the error which is computed as

$$\hat{e} = \sum_{v \in V_{00}} \varepsilon_v + \hat{C}_0 + \hat{R}_0.$$ 

We remark that we also have

$$\hat{e} = \sum_{v \in V_{11}} \varepsilon_v + \hat{C}_1 + \hat{R}_1$$

since the sum of those two quantities is zero by construction and by definition of the mismatch. Similarly, if one is decoding a $\sigma_x$-error, the decoders compute $\hat{e} = \sum_{v \in V_{11}} \varepsilon_v + \hat{C}_0 + \hat{R}_1 = \sum_{v \in V_{11}} \varepsilon_v + \hat{C}_1 + \hat{R}_0$. Since these summations can be done locally on the components of the partition of $Q$ into $Q$-neighbourhoods of the relevant $V_{ij}$, the postprocessing is achieved naturally by means of a parallel computation.

It remains to describe the mismatch decomposition procedures. The sequential mismatch decomposition procedure is given in Algorithm 2. The sequential procedure is rather natural: it consists of initializing a variable $\hat{Z}$ at $Z$, and iteratively looking for a vertex $v \in V$ and a dual tensor codeword $x_v$ supported by $Q(v)$, such that adding $x_v$ to $\hat{Z}$ decreases its weight by a sufficient amount. The decoder then decomposes $x_v$ as $x_v = c_v + r_v$ with $\|c_v\| + \|r_v\|$ minimum, and increments $\hat{C}_j$ by $c_v$ and $\hat{R}_i$ by $r_v$, where $i, j$ are such that $v \in V_{ij}$. Iterations continue until $\hat{Z} = 0$.

The parallel decomposition procedure simultaneously looks for vertices that would be candidates for the sequential decoder. It then wants to apply several sequential iterations in parallel: however, it needs the different $x_v$’s that it will identify to have disjoint supports. This is achieved if one restricts the set of candidate vertices $v$ to a single set $V_{ij}$: so the decoder compares its 4 parallel options, each of which consists of applying simultaneously all sequential iterations for all the candidate vertices it has identified in a given $V_{ij}$. It then applies the option that maximises the decrease of $|\hat{Z}|$. This is summarised in Algorithm 3, which consists of successive applications of single-step parallel procedures described below in Algorithm 4.

2. General proof strategy

Both the sequential decoder and the parallel decoder look for a decomposition of the mismatch vector $Z$ as a sum, over a subset of vertices $v$, of local dual tensor codewords $c_v + r_v$ supported by $Q$-neighbourhoods $Q(v)$. The mismatch decomposition procedure is the core of the decoding algorithm, both in the sequential and in the parallel case.

The sequential mismatch decomposition procedure is parameterized by a constant $\varepsilon \in (0,1)$ and proceeds in the natural way: it looks for a vertex $v \in V$ together with a local dual tensor codeword $x_v = c_v + r_v$ (with $c_v \in C_A \otimes \mathbb{F}_2^R$, $r_v \in \mathbb{F}_2^A \otimes C_B$) such that flipping $x_v$ decreases the Hamming weight of the mismatch by at least $(1-\varepsilon)|x_v|$, in other words, such that $|\hat{Z} + x_v| \leq |\hat{Z}| - (1-\varepsilon)|x_v|$, where $\hat{Z}$ is the current value of the mismatch, initiated at the original mismatch vector $Z$. It then proceeds by updating the current mismatch value to $\hat{Z} = \hat{Z} + x_v$, and continues in this way until $\hat{Z} = 0$, at which point it outputs the sum of the updates $x_v$, which equals $Z$. Theorem 12 below states that the required local codeword $x_v$ always exists for a given mismatch $\hat{Z}$, provided a minimum decomposition of $\hat{Z}$ has sufficiently few active vertices. We now explain what this means.

We may reorganise a decomposition $\hat{Z} = \sum_{v \in V} c_v + r_v$ of $\hat{Z}$ into a sum of local dual tensor codewords as

$$\hat{Z} = \hat{C}_0 + \hat{R}_0 + \hat{C}_1 + \hat{R}_1,$$ (C1)
We may now state Theorem 12. If the sets of active vertices $S_{ij}$ for a minimum decomposition of a mismatch vector $Z$ satisfies

$$|S_{ij}| \leq \frac{1}{212} \delta^2 \varepsilon^3 \kappa |V_{00}|$$

\[\text{(C2)}\]

over all possible decompositions (C1) of $\hat{Z}$. We shall say that a decomposition (C1) of $\hat{Z}$ is minimum if (C2) equals $\|\hat{Z}\|$. Finally, we shall say that for a decomposition (C1), the vertex $v$ in $V_{ij}$ is active, if $\hat{C}_j + \hat{R}_i$ is non-zero on $Q(v)$. We may now state Theorem 12:

**Theorem 12.** Fix $\varepsilon \in (0, 1)$. If the sets of active vertices $S_{ij}$ for a minimum decomposition of a mismatch vector $Z$ satisfies

$$|S_{ij}| \leq \frac{1}{212} \delta^2 \varepsilon^3 \kappa |V_{00}|$$

where $\hat{C}_i = \sum_{v \in V_{ij} \cup V_{ij \setminus V_{ij}} \setminus v} c_v$ and $\hat{R}_i = \sum_{v \in V_{ij} \cup V_{ij \setminus V_{ij}} \setminus v} r_v$. Now we may decompose $\hat{C}_0$ as a sum of of $c_v$’s, for $v$ restricted to $V_{00}$ (as opposed to $v$ ranging over $V_{00} \cup V_{10}$), in which case this decomposition is unique, and we shall denote $\|\hat{C}_0\|$ the sum of all corresponding $\|c_v\|s$, where we recall that $\|c_v\|$ denotes the number of individual column vectors that make up the local codeword in $C_A \otimes F_2^B$. Note that we obtain the same value $\|\hat{C}_0\|$, if we decompose $\hat{C}_0$ as a sum of $c_v$’s, for $v$ ranging in $V_{10}$. The quantities $\|\hat{C}_1\|$, $\|\hat{R}_0\|$, $\|\hat{R}_1\|$ are defined similarly. We now define the norm $\|\hat{Z}\|$ of $\hat{Z}$ to be the minimum value of $\|\hat{C}_0\| + \|\hat{R}_0\| + \|\hat{C}_1\| + \|\hat{R}_1\|$ over all possible decompositions (C1) of $\hat{Z}$. We shall say that a decomposition (C1) of $\hat{Z}$ is minimum if (C2) equals $\|\hat{Z}\|$. Finally, we shall say that for a decomposition (C1), the vertex $v$ in $V_{ij}$ is active, if $\hat{C}_j + \hat{R}_i$ is non-zero on $Q(v)$.
In particular, states that if the weight throughout the sequential decoder twice for the analysis of the parallel decoder. In particular, 

\[ \tilde{C}_0 = 0, \tilde{R}_0 = 0, \tilde{C}_1 = 0, \tilde{R}_1 = 0 \text{ and } \tilde{Z} = Z. \]

while \( \tilde{Z} \neq 0 \) do

1. Set \( \tilde{C}_0 = 0, \tilde{R}_0 = 0, \tilde{C}_1 = 0, \tilde{R}_1 = 0 \) and \( \tilde{Z} = Z \).

2. if \( C = 0 \) and \( R = 0 \) return Failure.

3. \( C_j \leftarrow \tilde{C}_j + \tilde{C} \).

4. \( R_i \leftarrow \tilde{R}_i + \tilde{R} \).

5. \( \tilde{Z} \leftarrow \tilde{Z} + \tilde{C} + \tilde{R} \).

6. return \( (\tilde{C}_0, \tilde{R}_0, \tilde{C}_1, \tilde{R}_1) \).

Theorem 13. Fix \( \varepsilon \in (0, 1) \). If the Hamming weight \( |e| \) is less than

\[ \frac{1}{2^{11}} \min \left( \frac{\varepsilon^3}{16}, \kappa \right) (1 - \varepsilon) \kappa^2 \delta^2 \frac{n}{\Delta}, \]

then the sequential decoder with parameter \( \varepsilon \) returns a valid correction \( \hat{e} \), namely a correction equivalent to \( e \) (differing from \( e \) by an element of the stabilizer group).

We give a detailed proof in Appendix C.4.

We then exploit Theorem 13 on the sequential decoder twice for the analysis of the parallel decoder. In particular, the theorem guarantees that there exists at least one good vertex and local codeword that can be flipped. For the parallel decoder, we require that there are many such vertices, more precisely a number linear in the error weight (or in the Hamming weight of the mismatch). To show this, and exhibit many such good vertices, the first idea is to take a good decomposition of the mismatch, that is a decomposition consisting in a set of \( \{x_v\} \) with almost nonoverlapping supports. One can easily obtain such a set by running (virtually) the sequential decoder with a small parameter \( \varepsilon \) on the initial mismatch. Then expansion in the complex will imply that most of these \( x_v \) have the property that flipping them significantly decreases the weight of the mismatch, say by some \( (1 - 2\mu) |x_v| \), with \( \varepsilon < 2\mu < 1 \). The correctness of the sequential decoder (with parameter \( 2\mu \), this time) then implies the correctness
of the parallel decoder, and provided that we choose $2\mu$ sufficiently larger than $\varepsilon$, we also obtain that the parallel decoder terminates in a logarithmic number of steps.

The parallel decoder will try to find a decomposition of the mismatch $Z$ in the following manner: at each step, it will pick one of the four sets of vertices $V_{ij}$ (chosen so as to maximize the decrease of the Hamming weight $|Z|$) and flip local dual tensor codewords $x_v = c_v + r_v$ in neighbourhoods of vertices $v \in V_{ij}$ if it decreases the Hamming weight of $Z$ by at least $(1 - 2\mu)|x_v|$. Here, $\mu$ will be strictly larger than $\varepsilon$. In order to analyse the parallel decoder, the idea is to run the sequential decoder with parameter $\varepsilon$ and then exploit the corresponding decomposition of the mismatch. We then show that the parallel decoder will recover a decomposition compatible with that given by a sequential decoder with parameter $2\mu > \varepsilon$. This implies both the correctness of the algorithm as well as its logarithmic complexity.

The correctness of the parallel decoding algorithm is a simple corollary since the execution of the parallel decoding algorithm yields a valid execution of the sequential decoding algorithm, by picking an arbitrary order for the local changes made during a parallel step. The more challenging part is to establish that it terminates in a logarithmic number of steps.

**Theorem 14.** Fix $\varepsilon \in (0,1/2)$, and $\mu \in (\varepsilon,1/2)$. If the Hamming weight $|e|$ is less than

$$
\frac{1}{2^{12}} \min \left( \frac{\varepsilon^3}{16}, \kappa^2 \delta^2 \frac{n}{\Delta} \right),
$$

then the parallel decoder with parameter $2\mu$ returns a valid correction in logarithmic time.

**Proof.** Without loss of generality, we assume the error $e$ to be a $\sigma_x$-type error.

We first establish the correctness of the algorithm. At each parallel step, all the local decisions consist in flipping some $x_v = c_v + r_v$ such that $|Z| - |Z + x_v| \geq (1 - 2\mu)|x_v|$ for vertices $v$ restricted to a single set $V_{ij}$. Since the various $x_v$ flipped in parallel are not overlapping, one would obtain the same outcome by flipping them sequentially instead (in an arbitrary order). This corresponds to a valid execution of the sequential decoding algorithm with parameter $2\mu$. The correctness of the parallel decoder then results from Theorem 13.

Let us turn to the complexity of the algorithm. The main challenge consists in showing that there exist many vertices where such a local flip can be applied at each parallel step. In particular, we want to show that the parallel decoder will not simply flip a single local codeword at each step, in which case it could take up to a linear number of steps before obtaining the decomposition of the mismatch. We first consider a execution of the sequential decoder with parameter $\varepsilon$ on the mismatch $Z$. By definition of the sequential decoder, and since the initial error weight satisfies the assumption of Theorem 13, it gives a sequence $(z_1, z_2, \ldots, z_m)$ for local dual tensor codewords $z_k := c_{v_k} + r_{v_k}$ such that $Z = \sum_{i=1}^{m} z_k$ and $|Z + \sum_{k=1}^{\ell-1} z_k| \geq |Z + \sum_{k=1}^{\ell} z_k| + (1 - \varepsilon)|z_k|$, for all $\ell \leq m$. Unravelling the sum immediately shows that:

$$
|Z| \geq (1 - \varepsilon) \sum_{i=1}^{m} |z_k|. \quad (C3)
$$

Our goal is to prove that many of the $z_k$ can be flipped to decrease $|Z|$ by at least $(1 - 2\mu)|z_k|$. To this end, we partition the set of indices $[m]$ into two parts, a good set $G$ and a bad set $B$, defined as

$$
G := \{ k \in [m] : |Z| - |Z + z_k| \geq (1 - 2\mu)|z_k| \},
$$

and $B := [m] \setminus G$. Note that for any $b \in B$, it holds that $|Z \cap z_b| \leq (1 - \mu)|z_b|$. This implies

$$
|Z + \sum_{b \in B} z_b| \geq |Z| - |Z \cap \sum_{b \in B} z_b|
$$

$$
\geq |Z| - \sum_{b \in B} |Z \cap z_b|
$$

$$
\geq |Z| - (1 - \mu) \sum_{b \in B} |z_b|.
$$
Writing \( Z = \sum_{g \in G} z_g + \sum_{b \in B} z_b \) together with (C3), we obtain
\[
| \sum_{g \in G} z_g | \geq (1 - \varepsilon) \left( (\sum_{g \in G} |z_g| + \sum_{b \in B} |z_b|) - (1 - \mu) \sum_{b \in B} |z_b| \right) \\
= (1 - \varepsilon) \sum_{g \in G} |z_g| + (\mu - \varepsilon) \sum_{b \in B} |z_b| \\
= (1 - \varepsilon) \sum_{g \in G} |z_g| - (\mu - \varepsilon) \sum_{g \in G} |z_g| + (\mu - \varepsilon) \sum_{g \in G} |z_g| + (\mu - \varepsilon) \sum_{b \in B} |z_b| \\
\geq (1 - \mu) \sum_{g \in G} |z_g| + (\mu - \varepsilon) \sum_{b \in B} |z_b| \\
\geq (1 - \mu)|\sum_{g \in G} z_g | \geq \frac{1}{4}(1 - \frac{\varepsilon}{\mu})|Z|.
\]

Let us partition the set \( G \) into \( G = \bigcup_{i,j=0}^1 G_{ij} \) with \( g \in G_{ij} \) if the local tensor codeword \( z_g \) lies in the \( Q \)-neighborhood of a vertex of \( V_{ij} \).

We have
\[
\sum_{g \in G} z_g = \sum_{i,j=0}^1 \sum_{g \in G_{ij}} z_g.
\]

We obtain that there exists \((i, j) \in \{0, 1\}^2 \) such that
\[
| \sum_{g \in G_{ij}} z_g | \geq \frac{1}{4}(1 - \frac{\varepsilon}{\mu})|Z|.
\]

Since the corresponding \( z_g \) have disjoint supports, we get by definition of \( G \) that
\[
|Z + \sum_{g \in G_{ij}} z_g| \leq |Z| - (1 - 2\mu) \sum_{g \in G_{ij}} z_g \leq \left(1 - \frac{1}{4}(1 - 2\mu)(1 - \frac{\varepsilon}{\mu})\right)|Z|.
\]

Choosing \( \mu > \varepsilon \) implies that the decoder terminates in at most \( \frac{\log|Z|}{\log(1/\varepsilon)} \) parallel steps, with \( c = 1 - \frac{1}{4}(1 - 2\mu)(1 - \frac{\varepsilon}{\mu}) \).

The maximum value of \( 1/c \) is achieved for \( \mu = \sqrt{\varepsilon/2} \) which gives \( c = 1 - (1 - \sqrt{2\varepsilon})^2/4 \).

\[
3. \ \text{Proof of Theorem 12}
\]

The proof strategy follows the same footsteps as the proof of the minimum distance. When studying the minimum distance, we analysed the set \( C_j + R_i \), which coincided with \( C_7 + R_7 \). For the decoding, we start by identifying the mismatch \( Z \) associated with the error and take a minimum decomposition \( C_0 + R_0 + C_1 + R_1 \). The relevant set is now the support of \( (C_j + R_i) \cap (C_7 \cap R_7) \), which is smaller than the set considered when analysing the minimum distance. However, under the assumption that the sequential decoder is stalled, this set cannot be too small, and essentially the same techniques as before will allow us to arrive at a contradiction.

\[
a. \ \text{A stalled sequential decoder, Exceptional vertices, ordinary rows and columns}
\]

We consider a \( \sigma_x \)-type error \( e \) and define its associated mismatch \( Z \). We work with a minimal decomposition of \( Z \):
\[
Z = C_0 + R_0 + C_1 + R_1,
\]

meaning that the quantity \( \|C_0\| + \|R_0\| + \|C_1\| + \|R_1\| \) is minimal. To each vertex \( v \in V \), this decomposition associates codewords \( c_v \in C_A \otimes F_2^B \) and \( r_v \in F_2^A \otimes C_B \). We say that a vertex \( v \in V_{ij} \) is an active vertex if \( c_v + r_v \neq 0 \), i.e. if \( C_j + R_i \) is non-zero on \( Q(v) \), and we denote by \( S_{ij} \) the sets of active vertices in \( V_{ij} \).
The sequential decoder with parameter $\varepsilon$ searches for some vertex $v \in V$ and a dual tensor codeword $x_v \in (C_A^r \otimes C_B^r)^\perp = C_A \otimes F_B^r + F_A^r \otimes C_B$ such that flipping $x_v$ decrease the Hamming weight of $Z$ by at least $(1-\varepsilon)|x_v|$. To prove Theorem 12 we will assume there is no such vertex and work towards a contradiction.

We will follow the blueprint of Appendix B, and define exceptional and ordinary vertices of $S_{ij}$ as before, namely a vertex $v \in S_{ij}$ is said to be exceptional, if the local dual tensor codeword $x_v = c_v + r_v$, equal to the restriction of $C_j + R_i$ to $Q(v)$, satisfies $\|c_v\| + \|r_v\| > \alpha \Delta$. Here we will take $\alpha = \frac{1}{27} \delta^2 \varepsilon^2$. The set of exceptional vertices of $S_{ij}$ is denoted by $S_{ij}^e$ and non-exceptional vertices are called ordinary. Let us furthermore call an ordinary column (row) of $v \in S_{ij}$ a column (row) of the $Q$-neighbourhood $Q(v)$ on which $C_j$ ($R_i$) is non-zero, and for a vertex $v$ that is ordinary. Note that when talking about ordinary columns (or rows) it is important to specify for which vertex $v$ or $v'$ of $ij$ since this column appears in two different local views for two different vertices, and may be ordinary for one vertex and not for the other.

**Lemma 15.** Assume that the sequential decoder of parameter $\varepsilon$ is stalled. For all $v \in S_{ij}$, and all dual tensor codewords $x_v$, components of $C_j + R_i$, we have

$$|x_v \cap (C_j + R_i)| \geq \frac{\varepsilon}{2}|x_v|.$$

Furthermore, let $y_v$ be the subvector of $C_j$ supported by some ordinary column for $v \in S_{ij}$. Then

$$|y_v \cap (C_j + R_i)| \geq \frac{\varepsilon}{4}|y_v|.$$

**Proof.** Note that for any two binary vectors $x, z$, identifying them with their supports we have that $|z| - |z + x| \leq (1 - \varepsilon)|x|$ is equivalent to $2|z \cap x| \leq (2 - \varepsilon)|x|$, since $|z + x| = |z| - 2|z \cap x| + |x|$. Since the decoder is stalled, we have

$$|Z| - |Z + x_v| \leq (1 - \varepsilon)|x_v|$$

which therefore gives

$$2|Z \cap x_v| \leq (2 - \varepsilon)|x_v|.$$

Note that $x_v \subset C_j + R_i$ and therefore $Z \cap x_v = x_v + ((C_j + R_i) \cap x_v)$ and $|Z \cap x_v| = |x_v| - |(C_j + R_i) \cap x_v|$. Combining this with the previous inequality proves the first claim of the Lemma.

To prove the second claim of the Lemma we argue that, since $y_v$ is an ordinary column vector, that there is some $y' \subset y_v$ such that $|y'| \leq \alpha \Delta$ and $y_v + y' \subset C_j + R_i$. Specifically, $y'$ is supported by the intersection of $y_v$ and $R_i$. From our choice of $\alpha$ we clearly have $\alpha \Delta \leq \frac{\delta}{4} \varepsilon \Delta$, so that $|y'| \leq \frac{\varepsilon}{4}|y_v|$. We have

$$2|Z \cap y_v| \leq (2 - \varepsilon)|y_v|,$$

otherwise we could decode at vertex $v$ by flipping $y_v$. Since $y_v + y' \subset y_v$, we can write $2|Z \cap (y_v + y')| \leq (2 - \varepsilon)|y_v|$, and since $y_v + y' \subset C_j + R_i$, we have $|Z \cap (y_v + y')| = |y_v + y'| - |(C_j + R_i) \cap (y_v + y')|$, whence

$$2|y_v| - 2|y'| - (2 - \varepsilon)|y_v| \leq 2|(C_j + R_i) \cap (y_v + y')| \leq 2|(C_j + R_i) \cap y_v|$$

which proves the claim, since $|y'| \leq \frac{\varepsilon}{4}|y_v|$. □

Recall that minimality of the representation $(C_0, R_0, C_1, R_1)$ and the robustness property (Lemma 6) implies that

$$|c_v + r_v| \geq \kappa (\|c_v\| + \|r_v\|) \Delta$$

whenever $c_v + r_v$ is the local representation of $C_j + R_i$ at $v \in S_{ij}$. In particular, an exceptional vertex is such that $|c_v + r_v| \geq \alpha \kappa \Delta^2$.

**b. Exceptional vertices are rare**

**Lemma 16.** Let $i = 0, 1$. Under the hypothesis $|S_{ij}| \leq \frac{\alpha \varepsilon}{4} |V_0|$, we have

$$|S_{ij}^e| \leq \frac{2^8}{\alpha^2 \kappa^2 \varepsilon^2} \frac{1}{\Delta^2} |S_0| \quad \text{and} \quad |S_{ij}^c| \leq \frac{2^8}{\alpha^2 \kappa^2 \varepsilon^2} \frac{1}{\Delta^2} |S_1|.$$
Proof. The proof follows closely that of Lemma 10. We prove the upper bound for $S_{00}$, the other cases being similar.

Viewing $(C_0 + R_0) \cap (C_1 + R_1)$ as a subgraph of $S_0^2$, we have that vertices of $S_{00}$ have degree at least $\kappa \varepsilon \Delta^2/2$, by Lemma 15 and robustness.

By the Expander mixing Lemma in $G_0^2$, we therefore have
\[
\frac{1}{2} \alpha \kappa \varepsilon \Delta^2 |S_{00}^e| \leq |E(S_{00}^e, S_{11})| \leq \Delta \sqrt{|S_{00}^e||S_{11}|} + 4\Delta \sqrt{|S_{00}^e||S_{11}|}.
\]
Writing $|S_{11}|/|V_{00}| \leq \frac{256\varepsilon}{\Delta}$, we get
\[
\frac{1}{4} \alpha \kappa \varepsilon \Delta^2 |S_{00}^e|^{1/2} \leq 4\Delta |S_{11}|^{1/2}.
\]
Because there are so few exceptional vertices in $S$, the number of ordinary columns is at least $\frac{256\varepsilon}{\Delta}$. Applying Lemma 17, they must cluster among the $Q$-neighbourhoods of $C_B$ that is shared by the local view of an ordinary vertex $w \in S_T$.

Proof. We follow closely the proof of Lemma 11. We deal with the case when $i = j = 0$ and $T$ is defined as the set of vertices of $V_{00}$ whose $Q$-neighbourhoods share a non-zero column vector with an ordinary vertex of $V_{10}$. The other cases will hold by symmetry.

Again we keep only the two sets of vertices $V_{00}$ and $V_{01}$, and collapse rows of local views to single edges, and we look at the graph induced by the squares of $(C_1 + R_0) \cap (C_0 + R_1)$ on the vertex set $T \cup S_0$. What the second claim of Lemma 15 tells us, is that the degree of any vertex of $T$ in this subgraph is at least $\delta \Delta^2$.

As in the proof of Lemma 11, we apply the expander mixing Lemma in the double cover of Cay($G$, $A$) to obtain, and collapse rows of local views to single edges, and we look at the graph induced by the squares of $(C_1 + R_0) \cap (C_0 + R_1)$ on the vertex set $T \cup S_0$. What the second claim of Lemma 15 tells us, is that the degree of any vertex of $T$ in this subgraph is at least $\delta \Delta^2$.

As in the proof of Lemma 11, we apply the expander mixing Lemma in the double cover of Cay($G$, $A$) to obtain,
\[
|T| \frac{\varepsilon \delta \Delta}{4} \leq E(S_0, T) \leq \Delta |S_0||T|/|V_0| + 2\sqrt{\Delta} \sqrt{|S_0||T|}.
\]
Applying the hypothesis $|S_0| \leq \varepsilon |V_0|/8$ we get
\[
|T| \frac{\varepsilon \delta \Delta}{8} \leq 2\sqrt{\Delta} \sqrt{|S_0||T|}
\]
and the result follows.

Proof of Theorem 12. The hypothesis of the theorem translates into $|S_{ij}| \leq \frac{\alpha \kappa \varepsilon}{4}|V_{00}|$. Therefore Lemma 16 holds, and so does Lemma 17, since clearly $\frac{1}{4} \kappa \varepsilon \leq \varepsilon /8$.

Without loss of generality, let us suppose $|S_1| \geq |S_0|$ (otherwise invert their roles in the argument below) and $|S_{10}| \geq |S_{01}|$ (otherwise invert their roles).

Because there are so few exceptional vertices in $S_{10}$ (by Lemma 16), the number of ordinary vertices of $S_{10}$ is almost equal to $|S_{10}|$, namely it is at least $a|S_{10}|$ for any constant $a < 1$ and $\Delta$ large enough.

We obtain that either the number of ordinary rows or the number of ordinary columns in (the $Q$-neighbourhoods of vertices of) $S_{10}$ is at least $a|S_{10}|/2$. Suppose without loss of generality that the number of ordinary columns is at least $a|S_{10}|/2$. Applying Lemma 17, they must cluster among the $Q$-neighbourhoods of a set $T \subset S_{00}$ of size at most $\frac{256\varepsilon}{\Delta} |S_{01}|$ (since we have supposed $|S_{01}| \leq |S_{10}|$). Therefore, the average number of non-zero columns
of $C_0$ on the $Q$-neighbourhoods of the vertices of this set $T$ is at least $\alpha^{\frac{3\gamma^2}{2}\Delta}$, which is close to twice the minimum norm $\alpha\Delta$ of a local view of $R_0 + C_j$ for an exceptional vertex.

Therefore, a constant proportion of vertices of $T$ must be exceptional vertices of $S_{00}$. Now since the $Q$-neighbourhood of a vertex can host at most $\Delta$ columns, we also have $|T| \geq \frac{1}{\Delta^2}|S_{00}| \geq \frac{\alpha}{4\Delta^2}|S_1|$ (since $|S_{00}| \geq |S_0|$), and since $|S_1| \geq |S_0|$ we have $|T| \geq \frac{\alpha}{16}|S_0|$. Therefore, for large enough $\Delta$, we have a contradiction with Lemma 16 which limits the number of exceptional vertices of $S_{00}$ to not more than $O(1/\Delta^2)|S_0|$. \hfill $\square$

4. Proof of Theorem 13

Without loss of generality we may suppose the error $e$ to be a $\sigma_z$-type error.

We need to guarantee that the upper bound on $|S_{ij}|$ required by Theorem 12 is satisfied throughout the decoding procedure, until we reach a zero mismatch $\tilde{Z}$. Recall that $S_{ij}$ is the set of active vertices of $V_j$ in a minimum decomposition of $\tilde{Z}$. We will argue that $|S_{ij}| < \|\tilde{Z}\|$, so we track the evolution of $\|\tilde{Z}\|$ during the mismatch decomposition procedure. During the preprocessing phase, we have that $Z = \sum_{v \in V_1} e_v = \sum_{v \in V_1} x_v$, where $e_v$ is the projection of the error vector $e$ on $Q(v)$ and where $x_v = c_v + r_v$ is the dual tensor codeword that is the difference between $c_v$ and the decoder’s initial evaluation $\varepsilon_v$ of the error. The minimality of $|\varepsilon_v| = |e_v + x_v|$ implies that $|\varepsilon_v| \leq |e_v|$ and therefore that $|x_v| \leq 2|e_v|$. This gives

$$|Z| \leq \sum_{v \in V_1} |x_v| \leq 2 \sum_{v \in V_{10}} |e_v| = 4|e|.$$  \hfill (C4)

Writing $\|x_v\| := |c_v| + \|r_v\|$ and applying robustness to $x_v$, we also have $\kappa \Delta \|x_v\| \leq |x_v| \leq 2|e_v|$, whence

$$\|x_v\| \leq \frac{2}{\kappa \Delta} |e_v|.$$  

Summing over all vertices of $V_1$, we obtain

$$\|Z\| \leq \sum_{v \in V_1} \|x_v\| \leq \sum_{v \in V_{10}} \|x_v\| + \sum_{v \in V_{10}} \|x_v\| \leq \frac{2}{\kappa \Delta} \sum_{v \in V_{10}} |e_v| + \frac{2}{\kappa \Delta} \sum_{v \in V_{10}} |e_v| = \frac{4}{\kappa \Delta} |e|$$

since $|e| = \sum_{v \in V_{10}} |e_v| = \sum_{v \in V_{10}} |e_v|$.

Assume that after the $m$th round of sequential decoding, the decoder has flipped successively $x_{v_1}, x_{v_2}, \ldots, x_{v_m}$. We must have in particular

$$(1 - \varepsilon)(|x_{v_1}| + |x_{v_2}| + \cdots + |x_{v_m}|) \leq |Z|.$$

Now, every time we modify $\tilde{Z}$ by adding some $x_v$ to it, we have that $\|\tilde{Z}\|$ is increased by at most $\|x_v\|$. Robustness implies that $\|x_v\| \leq \frac{1}{\kappa \Delta} |e_v|$ and we may therefore write

$$\|x_{v_1}\| + \|x_{v_2}\| + \cdots + \|x_{v_m}\| \leq \frac{1}{\kappa \Delta (1 - \varepsilon)} |Z|.$$  \hfill (C5)

Using (C4) we therefore obtain that

$$\|\tilde{Z}\| \leq \|Z\| + \|x_{v_1}\| + \|x_{v_2}\| + \cdots + \|x_{v_m}\| \leq \frac{4}{\kappa \Delta} \left(1 + \frac{1}{1 - \varepsilon}\right) |e| \leq \frac{8}{(1 - \varepsilon)\kappa \Delta} |e|$$

since $1 + 1/(1 - \varepsilon) \leq 2/(1 - \varepsilon)$. Therefore, if we impose the condition

$$|e| \leq \frac{\kappa \Delta (1 - \varepsilon)}{8} \frac{1}{2\beta^2 \varepsilon^3} \kappa |V_{00}| = \frac{1}{2\beta^2} \frac{\varepsilon^3}{16} \kappa^2 \delta^2 (1 - \varepsilon) \frac{n}{\Delta}$$

we obtain that $|S_{ij}| \leq \|\tilde{Z}\|$ must always be bounded from above by $\frac{1}{2\beta^2 \varepsilon^3} \kappa |V_{00}|$ throughout the decoding procedure.

We also need to check that the output $\tilde{e}$ of the decoder is correct. This will be the case provided that $|e + \tilde{e}| < d_{\min}(Q)$. Recall that

$$|e + \tilde{e}| = \left| \sum_{v \in V_{10}} (e_v + \varepsilon_v) + \tilde{C}_0 + \tilde{R}_1 \right|$$  \hfill (C6)
and that \( |\sum_{v \in V_0} (e_v + \varepsilon_v)| = |\sum_{v \in V_0} x_v| \leq 2|e| \). To evaluate \( |\hat{C}_0 + \hat{R}_1| \), we make the remark that every local dual tensor codeword \( x_v \) that is used by an iteration of the sequential decoder is decomposed into a minimal representation \( x_v = c_v + r_v \), and contributes at most \( \|x_v\| \) non-zero row and column vectors to \( \hat{C}_0 + \hat{R}_1 \). Therefore,

\[
|\hat{C}_0 + \hat{R}_1| \leq \Delta \sum_v \|x_v\|
\]

where the sum runs over all vertices used by the decoder. Applying (C5), we obtain

\[
|\hat{C}_0 + \hat{R}_1| \leq \frac{1}{\kappa(1 - \varepsilon)} |Z|.
\]

Writing \( |Z| \leq 4|e| \) (from (C4)), we get

\[
|\hat{C}_0 + \hat{R}_1| \leq \frac{4}{\kappa(1 - \varepsilon)} |e|.
\]

Since \( \kappa < 1 \), we may write \( 2|e| \leq \frac{4}{\kappa(1 - \varepsilon)} |e| \), and (C6) now gives us

\[
|e + \hat{e}| \leq \frac{4}{\kappa(1 - \varepsilon)} |e| + |\hat{C}_0 + \hat{R}_1| \leq \frac{8}{\kappa(1 - \varepsilon)} |e|
\]

which is smaller than the minimum distance of the quantum code whenever \( |e| \leq \frac{\xi}{\kappa^2} \delta^2 (1 - \varepsilon) \). This concludes the proof of Theorem 13.