Modules over the Noncommutative Torus and Elliptic Curves

Francesco D’Andrea\textsuperscript{1}, Gaetano Fiore\textsuperscript{1,2} and Davide Franco\textsuperscript{1}

\textsuperscript{1} Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, Via Cintia, 80126 Napoli.
\textsuperscript{2} I.N.F.N., Sezione di Napoli, Complesso MSA, Via Cintia, 80126 Napoli

Abstract

Using the Weil-Brezin-Zak transform of solid state physics, we describe line bundles over elliptic curves in terms of Weyl operators. We then discuss the connection with finitely-generated projective modules over the algebra $A_\theta$ of the noncommutative torus. We show that such $A_\theta$-modules have a natural interpretation as Moyal deformations of vector bundles over an elliptic curve $E_\tau$, under the condition that the deformation parameter $\theta$ and the modular parameter $\tau$ satisfy a non-trivial relation. We then conclude with some remarks about formal deformations of vector bundles on the torus and twists based on the Lie algebra of the 3-dimensional Heisenberg group.

1 Introduction

In the study of the ergodic action of a topological group $G$ on a compact space $M$, the orbit space can be efficiently described by using the crossed product $C^\ast$-algebra $C(M) \rtimes G$. The most celebrated example is the irrational rotation algebra $A_\theta$, arising from the action of $\mathbb{Z}$ on $\mathbb{R}/\mathbb{Z}$ generated by an irrational translation $x \mapsto x + \theta$. This is a strict deformation quantization of $C(T^2)$ [17], hence the name “noncommutative torus”, and is one of several examples of quantization of manifolds carrying an action of $T^2$ or more generally $\mathbb{R}^2$ [18].

Finitely-generated and projective $A_\theta$-modules have been classified by Connes and Rieffel in [4, 16]. From a geometric point of view, it is natural to wonder whether they can be obtained as deformations of vector bundles over the torus $T^2$. An obstruction in applying the standard quantization technique for the action of $T^2$ resp. $\mathbb{R}^2$ [18] is given by the fact that the only finitely generated projective $C(T^2)$-modules carrying an action of $\mathbb{R}^2$ are the free modules. More precisely, sections of a non-trivial line bundle $L \to T^2$, when realized as quasi-periodic functions in $C^\infty(\mathbb{R}^2)$, are not stable under the generators $\partial_x, \partial_y$ of the action of $\mathbb{R}^2$ by translation.

One possible solution is to replace the ordinary derivatives with “covariant” derivatives, cf. equation (3.5). These generate a projective representation of $\mathbb{R}^2$ on sections of line bundles, that is an proper representation of the 3-dimensional Heisenberg group $H_3$. One might then use a suitable Drinfeld twist based on $U(h_3)[[\hbar]]$, with $h_3$ the Lie algebra of $H_3$, to deform vector bundles over $T^2$ into modules for the noncommutative torus. A discussion about this strategy is in §5.
In this paper we follow a different approach and realize finitely generated projective $A_\theta$-modules as Moyal deformations of vector bundles over an elliptic curve $E_\tau$: it is indeed possible to use the group $\mathbb{R}^2$ instead of the Heisenberg group, provided the modular parameter $\tau$ is chosen appropriately. For a line bundle with degree $p$, in order for this construction to work the modular parameter and the deformation parameter $\tau$ must satisfy the constraint $\tau - \frac{p\theta}{\tau} i \in \mathbb{Z} + i\mathbb{Z}$.

Line bundles over tori $\mathbb{T}^n$ have a nice physical interpretation as their $L^2$-sections form Hilbert spaces describing (electrically) charged quantum particles on $\mathbb{T}^n$ in the presence of a nonzero magnetic field, see [11] for a physically-minded presentation.

The paper is organized as follows. In §2, we recall definition and properties of the noncommutative torus and its modules. We will be as self-contained as possible, with the purpose of both fixing notations and facilitating the reader. In §3, using the Weil-Brezin-Zak transform (3.6), we give a description of the module of sections of a line bundles over an elliptic curve in terms of Weyl operators, much in the spirit of §2.2, cf. Prop. 3.3. In §4, we show how to obtain modules isomorphic to the ones in §2.2 from a deformation construction that makes use of Moyal star product (2.7); in particular, every module of §2.2 can be obtained in this way. In Prop. 2.2 we explain how to obtain a Hermitian structure similar to the Hermitian structure (3.10) used in [4, 16] from the canonical Hermitian structure of $C^\infty(\mathbb{R}^2)$. Finally, we conclude in §5 with some comments about formal deformations of vector bundles on the torus and twists based on the Lie algebra of the 3-dimensional Heisenberg group.

2 The noncommutative torus

In this section, we collect some basic facts about the noncommutative torus, the description of its algebra using Weyl operators, and recall the construction of finitely generated projective modules, that in our framework replace vector bundles.

2.1 The abstract $C^*$-algebra and Weyl operators

Let $0 \leq \theta < 1$. We denote by $A_\theta$ the universal unital $C^*$-algebra generated by two unitary operators $U$ and $V$ with commutation relation

$$UV = e^{2\pi i\theta} VU.$$ 

If $\theta = 0$, $A_0 \simeq C(\mathbb{T}^2)$ is isomorphic to the $C^*$-algebra of continuous functions on a 2-torus, with standard operations and sup norm. Rather than $A_\theta$, we are interested in the subset $A_\theta^\infty \subset A_\theta$, whose elements are series

$$\sum_{m,n \in \mathbb{Z}} a_{m,n} U^m V^n,$$

where $\{a_{m,n}\}_{m,n \in \mathbb{Z}}$ is a sequence for which the norm

$$p_k(a)^2 = \sup_{m,n \in \mathbb{Z}} (1 + m^2 + n^2)^k |a_{m,n}|^2.$$
is finite for all \( k \in \mathbb{N} \). We refer to such a sequence as “rapid decreasing”, and to the map \((m, n) \mapsto a_{m,n}\) as Schwartz function on \(\mathbb{Z}^2\). The collection of all sequences satisfying the condition above will be denoted by \(\mathcal{S}(\mathbb{Z}^2)\). The set \(A_\theta^\infty\) with the family of norms \(p_k\) is a Fréchet pre \(C^*\)-algebra (cf. e.g. [6, 21] and references therein).

A concrete description of this algebra can be obtained by introducing Weyl operators. Let \(a, b \in \mathbb{R}\). The Weyl operator \(W(a, b)\) is the unitary operators on \(L^2(\mathbb{R})\) defined by

\[
\{ W(a, b)\psi \}(t) = e^{-\pi i ab e^{2\pi i bt} \psi(t-a)}, \quad \psi \in L^2(\mathbb{R}).
\]  

Since

\[
W(a, b)W(c, d) = e^{-\pi i (ad-bc)} W(a+c, b+d),
\]  

the linear span of all Weyl operators is a unital \(*\)-algebra. A faithful unital \(*\)-representation \(\pi\) of \(A_\theta^\infty\) is given on generators by \(\pi(U) := W(1,0)\) and \(\pi(V) := W(0,-\theta)\).

The difference between the rational and irrational case can be understood by considering the associated von Neumann algebra \(N_\theta = \{ W(m, n\theta) : m, n \in \mathbb{Z} \}''\), closure of \(\pi(A_\theta^\infty)\). If \(\theta = p/q\) with \(p, q\) coprime, it can be shown that the center is the subalgebra generated by the operators \(W(q,0)\) and \(W(0,p)\), isomorphic to \(L^\infty(\mathbb{T}^2)\). On the other hand, if \(\theta\) is irrational, the center is trivial and \(N_\theta\) is a factor (in fact, a type \(\Pi_1\) factor).

Let us conclude this section by recalling the deformation point of view. Let \(x, y \in \mathbb{R}^2\) and \(u, v\) be the functions

\[
u(x, y) := e^{2\pi i x}, \quad \upsilon(x, y) := e^{2\pi i y}.
\]  

They are generators of the unital \(C^*\)-algebra \(C(\mathbb{T}^2)\) (any periodic continuous function is a uniform limit of trigonometric polynomials). By standard Fourier analysis, we know that any \(f \in C^\infty(\mathbb{T}^2)\) can be written as

\[
f = \sum_{m,n \in \mathbb{Z}} a_{m,n} u^m v^n,
\]  

and this series converges (uniformly) to a smooth function \(f\) if and only if \(\{a_{m,n}\} \in \mathcal{S}(\mathbb{Z}^2)\).

There is an associative product \(\ast_\theta\) on \(C^\infty(\mathbb{T}^2)\) defined on monomials by

\[(u^j v^k) \ast_\theta (u^m v^n) = \sigma((j,k),(m,n)) u^{j+m} v^{k+n}, \quad \text{for all } j,k,m,n \in \mathbb{Z},
\]  

where \(\sigma : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{C}^*\) is a 2-cocycle in the group cohomology complex of \(\mathbb{Z}^2\), given by:

\[
\sigma((j,k),(m,n)) := e^{i \pi \theta (jn-km)}.
\]

One can then interpret the algebra \((C^\infty(\mathbb{T}^2), \ast_\theta)\) as a cocycle quantization of \(\mathcal{S}(\mathbb{Z}^2)\) with convolution product. The \(C^*\)-completion is the twisted group \(C^*\)-algebra \(C^* (\mathbb{Z}^2, \sigma)\). Note that since \((f \ast_\theta g) \ast = g \ast_\theta f \ast\), we get a \(*\)-algebra with undeformed involution.

A unital \(*\)-algebra isomorphism \(T_\theta : (C^\infty(\mathbb{T}^2), \ast_\theta) \to A_\theta^\infty\) is given on \(f\) as in (2.4) by

\[
T_\theta(f) := \sum_{m,n \in \mathbb{Z}} a_{m,n} e^{-\pi i m \theta} u^m v^n,
\]  

where the phase factors are chosen to have \(T_\theta(f)^* = T_\theta(f^*)\) for all \(f\).
The product (2.5) can be extended to a larger class of functions as explained in [18]. A (well-defined) associative product on the Schwartz space $S(\mathbb{R}^2)$ is given by:

\[
(f *_\theta g)(z) = \frac{4}{\partial z} \int_{\mathbb{C} \times \mathbb{C}} f(z + \xi)g(z + \eta) e^{\frac{4\pi i}{\theta} \text{Im}(\xi \eta)} \, d\xi \, d\eta ,
\]

where in complex coordinates $z = x + iy$ and $dz = dx \, dy$. We refer to (2.7) as the “Moyal product”. This product is extended by duality to tempered distributions, and this allows to define an associative product, by restriction, on several interesting function spaces. A relevant example is the space $B(\mathbb{R}^2) \subset C^\infty(\mathbb{R}^2)$ of smooth functions that are bounded together with all their derivatives. It is a theorem [13, Prop. 2.23] that $B(\mathbb{R}^2)$ with the product above (and suitable seminorms) is a unital Fréchet pre-$C^\ast$-algebra. If $f, g$ are periodic, we recover the product (2.5), cf. [13, Eq. (2.12)], thus justifying using the same symbol for (2.5) and (2.7).

2.2 Heisenberg modules

Finitely generated projective $A_\theta^\infty$-modules were constructed in [4], and then completely classified in [16]. By Serre-Swan theorem they provide a noncommutative analogue of complex smooth vector bundles. The stable range theorem tells us that, on a base space $X$ of real dimension 2, any two complex vector bundles that are stably equivalent are isomorphic, i.e. the map $\text{Vect}(X) \to K^0(X)$ is injective. As a consequence, since $K^0(T^2) \simeq \mathbb{Z} \oplus \mathbb{Z}$, finitely generated projective $C^\infty(T^2)$-modules (complex vector bundles on $T^2$) are classified by two integers, the rank $r$ and the first Chern number $d$, or “degree”. The tensor product of vector bundles gives $K^0(T^2)$ the structure of a unital ring, isomorphic to $\mathbb{Z}[t]/(t^2)$ via the map $(r, d) \mapsto r + td$ [1].

Something similar holds for $A_\theta^\infty$, $\theta \neq 0$. It is proved in [16] that $K_0(A_\theta^\infty)$ has the cancellation property, meaning that the map from equivalence classes of finitely generated projective modules to the $K_0$-group is injective, and since $K_0(A_\theta^\infty) \simeq \mathbb{Z} \oplus \mathbb{Z}$, these are also classified by two integers $p, s$. The explicit definition of the corresponding modules, here denoted $E_{p, s}$, is recalled below,\(^1\) and the aim of this paper will be to describe them as deformations of vector bundles in a suitable sense.

**Definition 2.1** ([4, 16]). Let $p, s \in \mathbb{Z}$ and $p \geq 1$. As a vector space, $E_{p, s} := S(\mathbb{R}) \otimes \mathbb{C}^p$ with $S(\mathbb{R})$ the set of Schwartz functions on $\mathbb{R}$. The right $A_\theta^\infty$-module structure is given by

\[
\psi \triangleleft U = \left\{ W(\frac{\pi}{p} + \theta, 0) \otimes (S^s)^s \right\} \psi , \quad \psi \triangleleft V = \left\{ W(0, 1) \otimes C \right\} \psi ,
\]

where $C, S \in M_p(\mathbb{C})$ are the clock and shift operators:

\[
C = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & e^{2\pi i/p} & 0 & \ldots & 0 \\
0 & 0 & e^{4\pi i/p} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & e^{2(p-1)\pi i/p}
\end{pmatrix} , \quad S = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 1 \\
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0
\end{pmatrix} , \quad (2.8)
\]

and we adopt the convention that $C = S = 1$ for $p = 1$.

\(^1\)Here we use the notations of [21, §7.3], except for a replacement $\theta \to -\theta$. 

4
If \( \theta \) is irrational, any finitely generated projective right \( A_\theta \)-module is isomorphic either to a free module \((A_\theta^\infty)_p\) or to a module \( \mathcal{E}_{p,s} \), with \( p \) and \( s \) coprime \((p > 0 \text{ and } s \neq 0) \) or \( p = 1 \) and \( s = 0 \) [16, 7]. There is also a pre-Hilbert module structure on \( \mathcal{E}_{p,s} \) recalled below.

**Definition 2.2 ([4, 16])**. An \( A_\infty^\theta \)-valued Hermitian structure on \( \mathcal{E}_{p,s} \) is given by

\[
\langle \psi, \varphi \rangle = \sum_{m,n \in \mathbb{Z}} U^m V^n \int_{-\infty}^{\infty} (\psi \triangleleft U^m V^n | \varphi) t dt .
\]

where \( (\cdot) : \mathcal{E}_{p,s} \times \mathcal{E}_{p,s} \to \mathcal{S}(\mathbb{R}) \) is the canonical Hermitian structure of \( \mathcal{E}_{p,s} = \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^p \) as a right pre-Hilbert \( \mathcal{S}(\mathbb{R}) \)-module, given by

\[
(\psi | \varphi)_t := \sum_{\tau=1}^{p} \overline{\psi}(t) \varphi(\tau) ,
\]

for all \( \psi = (\psi_1, \ldots, \psi_p) \) and \( \varphi = (\varphi_1, \ldots, \varphi_p) \in \mathcal{E}_{p,s} \).

To the best of our knowledge, for \( \theta \neq 0 \) there is no analogue of the ring of vector bundles. The analogue of the tensor product of vector bundles would be the tensor product of bimodules over the algebra itself, but while in the commutative case every module is a bimodule, this is not true for a general noncommutative algebra.

It is known, for example, that \( A_\theta \) has non-trivial Morita auto-equivalence bimodules if and only if \( \theta \) is a real quadratic irrationality, i.e. \( \mathbb{Q}(\theta) \) is a real quadratic field [14]. The strong analogy with the theory of complex multiplication of elliptic curves, suggests that noncommutative tori may play a role in number theory similar to the one played by elliptic curves. This idea is the starting point of Manin real multiplication program [14].

Constructing an analogue of the ring of vector bundles is the typical problem that could be addressed, at least formally, with the use of a Drinfeld twist (cf. §5 for a discussion).

### 3 Vector bundles over elliptic curves

In this section we recall some basic facts about elliptic curves, in particular about non-holomorphic factors of automorphy, and establish a correspondence between modules of sections of line bundles and Heisenberg modules.

#### 3.1 Smooth and holomorphic line bundles over elliptic curves

Let us identify \((x, y) \in \mathbb{R}^2 \) with \( z = x + iy \in \mathbb{C} \). Fix \( \tau \in \mathbb{C} \) with \( \text{Im}(\tau) > 0 \), let \( \Lambda := \mathbb{Z} + \tau \mathbb{Z} \) and \( E_\tau = \mathbb{C}/\Lambda \) the corresponding elliptic curve with modular parameter \( \tau \). We use the symbol \( T^2 \) when \( \tau = i \). Equivalently (Jacobi uniformization) \( E_\tau \simeq \mathbb{C}^*/q^{2\mathbb{Z}} \) where

\[
q = e^{\pi i r} ,
\]

the biholomorphism being given by the exponential map \( z \mapsto e^{2\pi iz} \).

The algebra \( C^\infty(E_\tau) \) can be identified with the subalgebra of \( C^\infty(\mathbb{C}) \) made of \( \Lambda \)-invariant functions:

\[
C^\infty(E_\tau) = \{ f \in C^\infty(\mathbb{C}) : f(z + m + n\tau) = f(z) \forall m, n \in \mathbb{Z} \} .
\]
The space $C^\infty(C)$ has a natural structure of $C^\infty(E_\tau)$-module. Let $\alpha: \Lambda \times C \to C^*$ be a smooth function satisfying

$$\alpha(\lambda + \lambda', z) = \alpha(\lambda, z + \lambda') \alpha(\lambda', z),$$

where $z \in C$, and $\lambda, \lambda' \in \Lambda$. In the notations of [2], this is an element of $Z^1(\Lambda, H^0(\mathcal{O}_E))$, where $V = C$. Consider the corresponding set of smooth quasi-periodic functions:

$$\Gamma_{\alpha} := \left\{ f \in C^\infty(C) : f(z + \lambda) = \alpha(\lambda, z) f(z) \, \forall \, \lambda \in \Lambda, z \in C \right\}. \quad (3.1)$$

Then $\Gamma_{\alpha}$ is stable under multiplication by $C^\infty(E_\tau)$, and hence a $C^\infty(E_\tau)$-submodule of $C^\infty(C)$. It is projective and finitely-generated, since its elements can be thought as smooth sections of a line bundle on $E_\tau$ (Appell-Humbert theorem).

Holomorphic elements of $\Gamma_{\alpha}$ are the so-called theta functions for the factor $\alpha$: they form a finite-dimensional vector space (while $\Gamma_{\alpha}$ is finitely generated, but not finite-dimensional). Elements of $\Gamma_{\alpha}$ are called differentiable theta functions [2, pag. 57].

The basic example is the Jacobi theta function [15]:

$$\vartheta(z; q) = \sum_{n \in \mathbb{Z}} q^{n^2} e^{2\pi i n z}, \quad q = e^{\pi i \tau}. \quad (3.2)$$

Hence the factor of automorphy is

$$\alpha(\lambda, z) = q^{-n^2} e^{-2\pi i n z}, \quad z = x + iy, \lambda = m + n \tau. \quad (3.3)$$

This corresponds to a holomorphic line bundle with degree 1. If we forget about the holomorphic structure, this is the unique smooth line bundle with degree 1 modulo isomorphisms. Let $\tau = \omega_x + i \omega_y$, with $\omega_x, \omega_y \in \mathbb{R}$ and $\omega_y > 0$. The isomorphism of $C^\infty(E_\tau)$-modules $f \mapsto e^{-\pi y^2/\omega y} f$ maps $\Gamma_{\alpha}$ into $\Gamma_{\beta}$, with

$$\beta(\lambda, z) = e^{-\pi \omega_x n^2} e^{-2\pi i n x}, \quad z = x + iy, \lambda = m + n \tau, \quad (3.3)$$

a unitary factor of automorphy. Note that a $C^\infty(E_\tau)$-module isomorphism corresponds (is dual to) an isomorphism of smooth vector bundles, but not necessarily of holomorphic vector bundles. In particular, $\Gamma_{\beta}$ are not (smooth) sections of an holomorphic vector bundle, and none of its elements is holomorphic (cf. §3.3).

The general smooth line bundle with degree $p$ can be obtained from the factor of automorphy $\beta^p$.

### 3.2 Sections of line bundles and Heisenberg modules

Let us denote by $\mathcal{F}_{\tau,p}$ the $C^\infty(E_\tau)$-module (3.1) associated to the factor of automorphy $\beta^p$, with $p \in \mathbb{Z}$ and $\beta$ as in (3.3). So, $f \in C^\infty(C)$ belongs to $\mathcal{F}_{\tau,p}$ iff:

$$f(z + 1) = f(z) \quad \text{and} \quad f(z + \tau) = e^{-\pi i p (\omega_x + 2x)} f(z) \quad \forall \, z = x + iy \in C. \quad (3.4)$$
Every $f \in \mathcal{F}_{\tau,p}$ is a bounded function, since from (3.4) it follows that $|f|$ is a continuous periodic function. Two maps $\nabla_1, \nabla_2 : \mathcal{F}_{\tau,p} \to \mathcal{F}_{\tau,p}$ are given by:

$$\nabla_1 f(z) = (\partial_x + \frac{2\pi ip}{\omega_p} y) f(z), \quad \nabla_2 f(z) = \partial_y f(z).$$

(3.5)

One can explicitly check that for any $f$ satisfying (3.4), $\nabla_1 f$ and $\nabla_2 f$ satisfy (3.4) too.

In the next proposition, we study a transform very close to the Weil-Brezin-Zak transform used in solid state physics [12, §1.10] (see also the end of section 2 in [11]). In the following, $[n] := n \mod p$.

**Proposition 3.1.** Every $f \in \mathcal{F}_{\tau,p}$ is of the form

$$f(z) = \sum_{n \in \mathbb{Z}} e^{2\pi inx} e^{\frac{\pi in^2 \omega_p}{p}} f_{[n]}(y + n \frac{\omega_p}{p}),$$

(3.6)

for some Schwartz functions $f_{[1]}, \ldots, f_{[p]} \in \mathcal{S}(\mathbb{R})$, uniquely determined by $f$ and given by:

$$f_{[n]}(y) = \int_0^1 e^{-2\pi inx} e^{\frac{\pi in^2 \omega_p}{p}} f(z) - \frac{p}{\omega_p} \pi i dx \quad \forall n = 1, \ldots, p.$$ 

(3.7)

We denote by $\varphi_{\tau,p} : \mathcal{F}_{\tau,p} \to \mathcal{H}_{\tau,p} := \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^p$ the bijection associating to $f$ the vector valued function $\vec{f} = (f_{[1]}, \ldots, f_{[p]})^t$.

**Proof.** Since $f$ is a smooth periodic function of $x$, then $f(z) = \sum_{n \in \mathbb{Z}} e^{2\pi inx} \tilde{f}_n(y)$, for some functions $\tilde{f}_n$ of $y$. The condition

$$f(z + \tau) = \sum_{n \in \mathbb{Z}} e^{2\pi in(x + \omega_p)} \tilde{f}_n(y + \omega_p) = e^{-\pi ip(\omega_x + 2x)} f(z) = \sum_{n \in \mathbb{Z}} e^{-\pi ip \omega_x} e^{2\pi i(n-p)x} \tilde{f}_n(y)$$

gives the recursive relation $\tilde{f}_{n+p}(y) = e^{\pi i(2n+p)\omega_x} \tilde{f}_n(y + \omega_y)$. If we write $n = j + kp$ with $k \in \mathbb{Z}$ and $0 < j < p$, then previous equation has solution

$$\tilde{f}_{j+kp}(y) = e^{\pi i(2jk + k^2)p} \omega_x \tilde{f}_j(y + k \omega_y).$$

Called $f_{[j]}(y) = e^{-\pi i \frac{j^2}{p} \omega_x} \tilde{f}_j(y - \frac{j}{p} \omega_y)$, for $j = 0, \ldots, p-1$, after the replacement $j + kp \to n$ we get (3.6). Using Fourier inversion formula we get (3.7). We must now prove that $f$ is smooth if and only if $f_{[1]}, \ldots, f_{[p]}$ are Schwartz functions.

To simplify the notations, let us give the proof for $\tau = i$ and $p = 1$, but the generalization to arbitrary $p$ and $\tau$ is straightforward. So, what we want to prove is that:

**Lemma 3.2.** $\sum_{n \in \mathbb{Z}} e^{2\pi inx} \tilde{f}(y+n)$ converges to a function $f \in C^\infty(\mathbb{R}^2) \iff \tilde{f} \in \mathcal{S}(\mathbb{R})$.

Let $g_n(z) = e^{2\pi inx} \tilde{f}(y+n)$. By Weierstrass M-test, $\sum_n g_n(z)$ converges (uniformly) to a $C^\infty$-function if for all $j, k$ there exists a sequence of non-negative numbers $C_{n}^{j,k}$ which is summable in $n$ and such that:

$$|\partial_x^j \partial_y^k g_n(z)| \leq C_{n}^{j,k}.$$ 

Fix an $y_0 > 0$. For all $|y| \leq y_0$ we have $|n| = |y + n - y| \leq |y + n| + |y| \leq |y + n| + y_0$, and:

$$|\partial_x^j \partial_y^k g_n(z)| = (2\pi |n|)^j |\partial_y^k \tilde{f}(y+n)| \leq (2\pi)^j n^{-2} \sum_{i=0}^{j+2} \binom{j+2}{i} y_0^{j+2-i}(y+n)^i \partial_y^i \tilde{f}(y+n).$$
for all \( n \neq 0 \). If \( \tilde{f} \) is Schwartz, \( \| \tilde{f} \|_{l,k} := \sup_y |y^j \partial_y^k \tilde{f}(y)| \) is finite for all \( l, k \), and

\[
|\partial_y^j \partial_y^k g_n(z)| \leq (2\pi)^j n^{-2} \sum_{l=0}^{j+2} (l+2) y_0^{j+2-l} ||\tilde{f}||_{l,k} = K_k n^{-2}.
\]

If we set \( C_n = K y_0^{-2} \) for \( n \neq 0 \), this series is summable. Thus \( \sum_n g_n(z) \) is uniformly convergent (together with its derivatives) to a \( C^\infty \)-function on any strip \( \mathbb{R} \times [-y_0, y_0] \subset \mathbb{R}^2 \), and then on the whole \( \mathbb{R}^2 \). This proves the “\( \Leftarrow \)” part.

Assume now the convergence of the series to a \( C^\infty \)-function \( f \). By standard Fourier analysis, \( \tilde{f}(y + n) = \int_0^1 e^{-2\pi i n y} f(x) dx \) and, integrating by parts,

\[
(y + n)^j \partial_y^k \tilde{f}(y + n) = \left( \frac{-i}{2\pi} \right)^j \int_0^1 e^{-2\pi i n y} \nabla_1^j \nabla_2^k f(z) dx.
\]

Recall that \( \nabla_1^j \nabla_2^k f \in \mathcal{F}_{\tau,p} \) is a bounded function. Let \( C_{j,k} \) be its sup norm. Then \( ||\tilde{f}||_{j,k} \leq (2\pi)^{-j} C_{j,k} \), proving that \( \tilde{f} \) is a Schwartz function.

\( \mathcal{F}_{\tau,p} \) is a right \( C^\infty(E_\tau) \)-module. Given two elements \( f, g \), it follows from (3.4) that the product \( f^*g \) is a periodic function, hence an element of \( C^\infty(E_\tau) \). The Hermitian structure \( (f,g) \mapsto f^*g \) turns \( \mathcal{F}_{\tau,p} \) into a pre-Hilbert module.

Note that \( C^\infty(E_\tau) \) is generated by the two unitaries

\[
u_{\tau}(x,y) := e^{2\pi i (x - \frac{m}{\omega} y)} , \quad \nu_{\tau}(x,y) := e^{2\pi i \frac{p}{\omega} y} , \quad (3.8)
\]

that reduces to the basic ones generating \( C(T^2) \) when \( \tau = i \). This because the diffeomorphism

\[
z \mapsto z' = (x - \frac{m}{\omega} y) + i \frac{1}{\omega} y
\]

transforms \( \Lambda \) into \( Z + i Z \), and \( E_\tau \) into the torus \( T^2 \).

**Proposition 3.3.** \( H_{\tau,p} \) is a right pre-Hilbert \( C^\infty(E_\tau) \)-module, isomorphic to \( \mathcal{F}_{\tau,p} \) via the map \( \varphi_{\tau,p} \), if we define the module structure (using Weyl operators) as

\[
f \triangleleft u_{\tau} := \{ W(\frac{\omega}{p}, -\frac{x}{\omega}) \otimes S \} f , \quad f \triangleleft v_{\tau} := \{ W(0, \frac{1}{\omega}) \otimes C^* \} f , \quad (3.9)
\]

were \( C, S \in M_p(\mathbb{C}) \) are the clock and shift operators in (2.8).

The pullback with \( \varphi_{\tau,p} \) of the canonical Hermitian structure of \( \mathcal{F}_{\tau,p} \) is the Hermitian structure on \( H_{\tau,p} \) given by

\[
\langle f, g \rangle = \frac{1}{\omega y} \sum_{m,n \in \mathbb{Z}} u_{\tau}^m v_{\tau}^n \int_{-\infty}^{\infty} (f \triangleleft u_{\tau}^m v_{\tau}^{-n})_t dt . \quad (3.10)
\]

where \( (\cdot) : H_{\tau,p} \times H_{\tau,p} \to S(\mathbb{R}) = S(\mathbb{R}) \otimes \mathbb{C}^p \) as a right pre-Hilbert \( S(\mathbb{R}) \)-module, given by (2.10).

**Proof.** From (3.6) and (3.8) we get

\[
v_{\tau}(z)f(z) = e^{2\pi i \frac{p}{\omega} y} \sum_{n \in \mathbb{Z}} e^{2\pi i n x} e^{\pi i n^2 \frac{p}{\omega}} f_{[n]}(y + n\frac{\omega}{p})
\]
\[ f_{[n]}(y) = e^{-2\pi in/p} e^{2\pi i \frac{1}{p} y} f_{[n]}(y + n \frac{\omega_y}{p}) \]

where

\[ f'_{[n]}(y) = e^{-2\pi in/p} e^{2\pi i \frac{1}{p} y} f_{[n]}(y). \]

If we call \( f = \varphi_{\tau,p}(f) = (f_{[1]}, f_{[2]}, \ldots, f_{[p]})^t \), then one can check that \( f' = \varphi_{\tau,p}(v_{\tau}f) = (f'_{[1]}, f'_{[2]}, \ldots, f'_{[p]})^t = \{W(0, 1 \frac{\omega_y}{p}) \otimes C^* \} f \), that is the second formula in (3.9). Similarly,

\[
\begin{align*}
\tilde{u}_\tau(z) f(z) & = e^{2\pi i (x - \frac{\omega_y}{p} z)} \sum_{n \in \mathbb{Z}} e^{2\pi i n x} e^{\frac{2\pi i n}{p} \omega_y} f_{[n]}(y + n \frac{\omega_y}{p}) \\
& = \sum_{n \in \mathbb{Z}} e^{2\pi i (n+1) x} e^{\pi i (n+1)^2 \frac{\omega_y}{p}} e^{-2\pi i \frac{\omega_y}{p} (y + n \frac{\omega_y}{p})} f_{[n]}(y + n \frac{\omega_y}{p}) \\
& = \sum_{n' \in \mathbb{Z}} e^{2\pi i n' x} e^{\pi i n'^2 \frac{\omega_y}{p}} e^{-2\pi i \frac{\omega_y}{p} (y + n' \frac{\omega_y}{p})} f'_{[n'-1]}(y - \frac{\omega_y}{p} + n' \frac{\omega_y}{p}),
\end{align*}
\]

where \( n' = n + 1 \). If we call

\[ f''_{[n]}(y) = e^{\pi i \frac{\omega_y}{p} y} e^{-2\pi i \frac{\omega_y}{p} y} f_{[n-1]}(y - \frac{\omega_y}{p}), \]

then

\[
\begin{align*}
\tilde{u}_\tau(z) f(z) & = \sum_{n \in \mathbb{Z}} e^{2\pi i n x} e^{\frac{2\pi i n}{p} \omega_y} f''_{[n]}(y + n \frac{\omega_y}{p}).
\end{align*}
\]

One can check that

\[ f''_{[n]} = W\left(\frac{\omega_y}{p}, -\frac{\omega_y}{\omega_y}\right) f_{[n-1]}, \]

proving the first equation in (3.9).

It remains compute \( f^*g \). Using (3.6) again we get:

\[
(f^*g)(z) = \sum_{m,n \in \mathbb{Z}} e^{2\pi i (n-m) x} e^{\pi i (n^2-m^2) \frac{\omega_y}{p}} f^*_{[m]}(y + m \frac{\omega_y}{p}) g_{[n]}(y + n \frac{\omega_y}{p}) \\
= \sum_{k,m \in \mathbb{Z}} e^{2\pi i k x} e^{\pi i (k^2+2km) \frac{\omega_y}{p}} f^*_{[m]}(y + m \frac{\omega_y}{p}) g_{[m+k]}(y + (m+k) \frac{\omega_y}{p}),
\]

where we called \( n = m + k \). This can be rewritten as follows. Using the identity

\[ F(y) = \frac{1}{\omega_y} \sum_{n \in \mathbb{Z}} e^{2\pi i n \frac{1}{\omega_y} t} \int_0^{\omega_y} e^{-2\pi i n \frac{1}{\omega_y} t} F(t) dt \]

we get

\[ f^*g = \frac{1}{\omega_y} \sum_{k,m,n \in \mathbb{Z}} u_{\tau}^k v_{\tau}^m e^{\pi i (k^2+2km) \frac{\omega_y}{p}} \int_0^{\omega_y} e^{2\pi i \frac{1}{\omega_y} (k \omega_y - n)} f^*_{[m]}(t+m \frac{\omega_y}{p}) g_{[m+k]}(t+(m+k) \frac{\omega_y}{p}) dt. \]

Let \( f = (f_{[0]}, f_{[1]}, \ldots, f_{[p-1]})^t \) as above, and \( g = (g_{[0]}, g_{[1]}, \ldots, g_{[p-1]})^t \). Note that

\[
\begin{align*}
\tilde{g} & = u_{\tau}^{-k} v_{\tau}^{-n} \left\{ W\left(\frac{\omega_y}{p}, -\frac{\omega_y}{\omega_y}\right) \otimes S \right\}^{-k} \left\{ W(0, 1 \frac{1}{\omega_y}) \otimes C^* \right\}^{-n} g \\
& = \left\{ W\left(\frac{k \omega_y}{p}, \frac{k \omega_y}{\omega_y}\right) W\left(0, -\frac{n \omega_y}{\omega_y}\right) \right\}^{-k} \otimes S^{-k} C^n g.
\end{align*}
\]
we get
\[ f \] hence there exists an open set (the support of \( f \))
\[ \Gamma \] for all \( \alpha \).

**Proof.**
Let \( \bar{\alpha} \) and with projection sending the class of \((z, w)\) to the class of \(z\) in \(E_\tau\). The set of smooth sections of this line bundle can be identified with the set (3.1). It is easy to characterize which \( L_\alpha \) have (non-zero) holomorphic sections.

**Proposition 3.4.** \( \Gamma_\alpha \) contains non-zero holomorphic functions only if there exists an open set where all the functions \( z \mapsto \alpha(\lambda, z) \) are holomorphic (for all \( \lambda \in \Lambda \)).

**Proof.**
Let \( \bar{\partial} = \frac{1}{i}(\partial_x + i\partial_y) \) and suppose \( f \in \Gamma_\alpha \) is holomorphic and not identically zero. By deriving the relation \( f(z + \lambda) = \alpha(\lambda, z)f(z) \) one gets
\[ \bar{\partial}f = 0, \]
hence there exists an open set (the support of \( f \)), independent of \( \lambda \), where \( \bar{\partial}\alpha(\lambda, z) = 0 \) for all \( \lambda \in \Lambda \).

As a corollary, the unitary factor of automorphy \( \beta_p \), with \( \beta \) as in (3.3), gives a module \( \Gamma_{\beta_p} \) with no non-zero holomorphic elements, for any \( p \in \mathbb{Z} \setminus \{0\} \). The \( C^\infty(E_\tau) \)-module map \( f \mapsto e^{i\pi y^2/\omega_y}f \) sends \( \Gamma_{\beta_p} \) into the module \( \Gamma_{\alpha_p} \), with \( \alpha \) as in (3.2). It is well-known, and easy to check explicitly (see Prop. 3.5 below), that \( \Gamma_{\alpha_p} \) has non-zero holomorphic elements, proving that the line bundles \( L_{\alpha_p} \) and \( L_{\beta_p} \) are isomorphic as smooth line bundles, but not as holomorphic line bundles. Similarly to (3.6), we have:
Proposition 3.5. Every \( f \in \Gamma_{\alpha p} \) is of the form
\[
f(z) = e^{\pi p y^2/\omega_y} \sum_{n \in \mathbb{Z}} e^{2\pi i n x} e^{\pi i n^2 \omega_x/\omega_y} f_{[n]}(y + n \omega_y^p),
\]
for some Schwartz functions \( f_{[1]}, \ldots, f_{[p]} \in \mathcal{S}(\mathbb{R}) \).

As a consequence,

Proposition 3.6. Let \( q = e^{\pi i \tau} \). Every holomorphic function \( f \in \Gamma_{\alpha p} \) is of the form
\[
f(z) = \sum_{n \in \mathbb{Z}} c_{[n]} q^{-n^2/p} e^{2\pi i n z},
\]
where \( c_{[n]} \) are complex numbers. The space of holomorphic elements has dimension \( |p| \).

Proof. From (3.12) we get
\[
2 \bar{\partial} f(z) = i e^{\pi p y^2/\omega_y} \sum_{n \in \mathbb{Z}} e^{2\pi i n x} e^{\pi i n^2 \omega_x/\omega_y} (\frac{2\pi p}{\omega_y} y + 2\pi n + \partial_y) f_{[n]}(y + n \omega_y^p).
\]
Thus \( \bar{\partial} f = 0 \) if and only if
\[
(2\pi p y + \partial_y) f_{[n]}(y) = 0
\]
for all \( y \in \mathbb{R} \). The general solution is \( f_{[n]}(y) = c_{[n]} e^{-\pi p y^2/\omega_y} \), hence the thesis. \( \blacksquare \)

3.4 Connections and local trivialization

A connection \( \nabla : \Gamma_{\beta p} \to \Gamma_{\beta p} \otimes \Omega^1 \) is given by \( \nabla f = (\nabla_1 f) dx + (\nabla_2 f) dy \), where \( \nabla_1 \) and \( \nabla_2 \) are given in (3.5). One can explicitly check that the Leibniz rule is satisfied (hence, the above formulas define indeed a connection). The corresponding connection 1-form is \( \omega = \frac{2\pi p}{\omega_y} y dx \) (living on the covering \( C \) of \( E_\tau \)) and the curvature is
\[
\Omega = d\omega = -\frac{2\pi i p}{\omega_y} dx \wedge dy.
\]
Integrating over a fundamental domain we get \(-\frac{2\pi i p}{\omega_y}\) times the area of the parallelogram with vertices 0, 1, \( \tau \) and 1 + \( \tau \) (that is equal to \( \omega_y \)). Thus
\[
\int_{E_\tau} \frac{i}{2\pi} \Omega = p,
\]
proving that the Chern number of the corresponding line bundle is \( p \).

It is an interesting exercise to do a doublecheck using a local trivialization, since it gives us as a byproduct an explicit formula for a projection representing the \( K \)-theory class of the module \( \Gamma_{\beta p} \). From Serre-Swan theorem, if \( g_{jk} : U_j \cap U_k \to \mathbb{C}^* \) are transition functions of the line bundle relative to a (finite) open cover \( \{U_j\}_{j=1}^n \) of \( E_\tau \), and \( \psi_j \in C(E_\tau) \) are such that \( \{\vert \psi_j \vert \}^2 \) is a partition of unity subordinated to the cover, then an idempotent matrix is given by (no summation implied):
\[
P = (\bar{\psi}_j g_{jk} \psi_k).
\]
The matrix elements of $P$ are global continuous functions on $E_{\tau}$, even if $g_{jk}$ in general are not. The cocycle condition for the transition functions guarantees that $P^2 = P$, and one can prove that the module of sections of the line bundle is isomorphic to the finitely generated projective module associated to $P$ [19]. Choosing smooth transition functions and partition of unity, one gets a smooth idempotent. If $g_{jk}$ have values in $U(1)$ rather than $\mathbb{C}^*$, the idempotent is a projection.

Since here the complex structure is irrelevant, to simplify the discussion let us set $\tau = i$. The fundamental domain is then a square, and we can identify $T^2$ with $[0, 1] \times (0, 1)/\sim$, where the equivalence relation is $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$ for all $x, y$. The line bundle with factor of automorphy $\beta$ can be described as follows. The total space is $[0, 1] \times [0, 1] \times \mathbb{C}/\sim$, where $(0, y, w) \sim (1, y, w)$ and $(x, 0, w) \sim (x, 1, e^{-2\pi ip}w)$ for all $x, y, w$. We choose two charts $U_1 = \{0 < y < 1\}$ and $U_2 = \{y \neq \frac{1}{2}\}$ on $T^2$. Two local sections $s_1$ and $s_2$ are defined as follows:

$$s_1 : U_1 \to \mathbb{C}, \quad s_1(x, y) = 1$$

$$s_2 : U_2 \to \mathbb{C}, \quad s_2(x, y) = \begin{cases} 1 & \text{if } 0 \leq y < \frac{1}{2}, \\ e^{2\pi ip} & \text{if } \frac{1}{2} < y \leq 1. \end{cases}$$

Since $s_i(0, y) = s_i(1, y)$, for $i = 1, 2$, and $s_2(x, 0) = e^{-2\pi ip}s_2(x, 1)$, the sections are well defined. On $U_1 \cap U_2$, $s_1(x, y) = g(x, y)s_2(x, y)$ where the transition function is

$$g(x, y) = \begin{cases} 1 & \text{if } 0 < y < \frac{1}{2}, \\ e^{2\pi ip} & \text{if } \frac{1}{2} < y < 1. \end{cases}$$

For any smooth choice of $\{\psi_1, \psi_2\}$, if we call $\Psi = (\psi_1, g\psi_2)^t$, the corresponding projection is $P = \Psi \Psi^t$, with row-by-column multiplication understood. The Grassmannian connection has connection 1-form

$$\Psi^t d\Psi = \overline{\psi}_1 d\psi_1 + \overline{\psi}_2 d\psi_2 + |\psi_2|^2 g^*dg - \frac{1}{2} d(|\psi_1|^2 + |\psi_2|^2) + |\psi_2|^2 g^*dg = \frac{1}{2} d1 + |\psi_2|^2 g^*dg = |\psi_2|^2 g^*dg .$$

An explicit computation gives

$$\Psi^t d\Psi = |\psi_2|^2 \begin{cases} 0 & \text{if } 0 < y \leq \frac{1}{2}, \\ 2\pi ipdx & \text{if } \frac{1}{2} \leq y < 1. \end{cases}$$

Note that this is only well-defined on the chart $U_1$ (is zero for $y = \frac{1}{2}$). The curvature is

$$\Omega = \begin{cases} 0 & \text{if } 0 < y \leq \frac{1}{2}, \\ -2\pi ip dx \wedge d|\psi_2|^2 & \text{if } \frac{1}{2} \leq y < 1. \end{cases}$$

Since $\psi_j$ vanishes outside $U_j$ and $\psi_1(y)^2 + \psi_2(y)^2 = 1$, then $\psi_2(\frac{1}{2}) = 0$ and from $\psi_1(1) = 0$ we get $\psi_2(1)^2 = 1$. Then:

$$\int_{T^2} \frac{i}{2\pi} \Omega = p\int_0^1 dx \int_{1/2}^1 d|\psi_2|^2 = p |\psi_2(1)|^2 - p |\psi_2(\frac{1}{2})|^2 = p ,$$

confirming that the degree is $p$. 

12
4 Vector bundles on the noncommutative torus

We now come back to the pre $C^*$-algebra $A^\theta_\infty$. As one can easily check, $C^\infty(\mathbb{R}^2)$ is a $A^\infty_\theta$-module, with module structure given on generators by:

\[
(U \triangleright f)(x,y) = e^{2\pi i x}f(x,y + \frac{1}{2}\theta), \quad (f \triangleright U)(x,y) = e^{2\pi i y}f(x,y - \frac{1}{2}\theta), \quad (V \triangleright f)(x,y) = e^{2\pi i y}f(x - \frac{1}{2}\theta, y), \quad (f \triangleright V)(x,y) = e^{2\pi i y}f(x + \frac{1}{2}\theta, y).
\]

(4.1)

Let $J$ be the antilinear involutive map:

\[
(Jf)(x,y) = \overline{f(-x,-y)}.
\]

As one can check on generators, conjugation by $J$ sends the algebra $A^\infty_\theta$ into its commutant, and in particular transforms the left action into the right action and vice versa.

Remark 4.1. The relation between (4.1) and Moyal product (2.7) is the following. The space $B(\mathbb{R}^2)$ of bounded smooth functions with all derivatives bounded is an $A^\infty_\theta$-submodule of $C^\infty(\mathbb{R}^2)$. For $a \in A^\infty_\theta$ and $f \in B(\mathbb{R}^2)$, one can check that

\[
(a \triangleright f)(z+1) = (a \triangleright f)(z), \quad (f \triangleright a)(z+1) = e^{2\pi i \omega_x} e^{-\pi i p\omega_y} (f \triangleright a)(z),
\]

where as usual $z = x + iy$ and $\omega = \omega_x + i\omega_y$, and we used the property (3.4) of $f$. So, the first condition in (3.4) is always satisfied by $f \triangleright a$ and $f \triangleright V$, while the second is satisfied if and only if $\omega_x$ and $\omega_y - \frac{p\theta}{2}$ are integers, that is what we wanted to prove.

Proposition 4.2. The vector space $F_{r,p}$ is a right $A^\infty_\theta$-module if and only if

\[
\tau - \frac{p\theta}{2} i \in \mathbb{Z} + i\mathbb{Z}.
\]

Proof. We are looking for necessary and sufficient conditions on $\tau$ such that, for any $f$ satisfying (3.4), $f \triangleright U$ and $f \triangleright V$ satisfy (3.4) too. If $f$ satisfies (3.4), from (4.1) we get:

\[
(f \triangleright U)(z+1) = (f \triangleright U)(z), \quad (f \triangleright V)(z+1) = e^{2\pi i \omega_x} e^{-\pi i p\omega_y} (f \triangleright U)(z),
\]

where as usual $z = x + iy$ and $\tau = \omega_x + i\omega_y$, and we used the property (3.4) of $f$. So, the first condition in (3.4) is always satisfied by $f \triangleright U$ and $f \triangleright V$, while the second is satisfied if and only if $\omega_x$ and $\omega_y - \frac{p\theta}{2}$ are integers, that is what we wanted to prove.

Now that we established that the vector spaces in Prop. 4.2 are $A^\infty_\theta$-modules, we want to give a description in terms of Weyl operators, in order to compare them with Def. 2.1. In the rest of this section, we assume that

\[
\tau = r + i(s + \frac{p\theta}{2})
\]
for some $r, s \in \mathbb{Z}$. Using the vector space isomorphism $\varphi_{r,p} : \mathcal{F}_{r,p} \to \mathcal{H}_{r,p} := \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^p$ of Prop. 3.1 we transport the right module structure (4.1) to $\mathcal{H}_{r,p}$. Let

$$\mathcal{f} \bullet a = \varphi_{r,p}(\varphi_{r,p}^{-1}(\mathcal{f}) \triangle a)$$

for all $a \in A^\infty_p$ and $\mathcal{f} \in \mathcal{H}_{r,p}$.

**Proposition 4.3.** For all $\mathcal{f} \in \mathcal{H}_{r,p}$,

$$\mathcal{f} \bullet U = e^{\pi i \frac{\theta}{2}} \{ W(\frac{r}{p} + \theta, 0) \otimes (C^*)^r \mathcal{S} \} \mathcal{f}, \quad \mathcal{f} \bullet V = \{ W(0, 1) \otimes (C^*)^s \} \mathcal{f}, \quad (4.2)$$

where $C, S \in M_p(\mathbb{C})$ are the clock and shift operators.

**Proof.** Here $\omega_x = r$ and $\omega_y = s + \frac{\theta}{2}$, with $r, s \in \mathbb{Z}$. From (3.6) and (4.1), for all $\mathcal{f} \in \mathcal{F}_{r,p}$:

$$(f \triangle V)(x, y) = e^{2\pi i x} \sum_{n \in \mathbb{Z}} e^{2\pi i nx} e^{\pi i \theta e^\frac{x^2}{2r}} f_{[n]}(y + n \frac{\omega_y}{p})$$

$$= \sum_{n \in \mathbb{Z}} e^{2\pi i nx} e^{\pi i \theta e^\frac{x^2}{2r}} \{ e^{-2\pi i ns/p} W(0, 1) f_{[n]} \}(y + n \frac{\omega_y}{p}).$$

This proves the second equation in (4.2). Concerning the first one, using (3.6) and (4.1):

$$(f \triangle U)(x, y) = e^{2\pi i x} \sum_{n \in \mathbb{Z}} e^{2\pi i nx} e^{\pi i \theta e^\frac{x^2}{2r}} f_{[n]}(y + n \frac{\omega_y}{p} - \frac{1}{2}\theta)$$

and with the replacement $n \to n - 1$:

$$= \sum_{n \in \mathbb{Z}} e^{2\pi i nx} e^{\pi i (n^2 - 2n + 1) e^\frac{x^2}{2r}} f_{[n-1]}(y + n \frac{\omega_y}{p} - \frac{1}{2}\theta)$$

$$= \sum_{n \in \mathbb{Z}} e^{2\pi i nx} e^{\pi i (n^2 - 2n + 1) e^\frac{x^2}{2r}} \{ W(\frac{r}{p} + \theta, 0) f_{[n-1]} \}(y + n \frac{\omega_y}{p}).$$

This proves the first equation in (4.2). \qed

For $\theta = 0$ and $\tau = i$ (that means $r = 0$, $s = 1$), we recover the module of Prop. 3.3. For arbitrary $\theta$, if $r = 0$ and $s \neq 0$ then the map $T : \mathbb{C}^p \to \mathbb{C}^p$ given by

$$(Tw)_n = w_{-sn \mod p}$$

is invertible (in fact unitary), and $\text{id}_{\mathcal{S}(\mathbb{R})} \otimes T$ is a right $A^\infty_p$-module isomorphism between $\mathcal{H}_{r,p}$ with module structure (4.2) and the module $\mathcal{E}_{p,s}$ of Def. 2.1. If $r = 0$ and $s = 0$ then the module $\mathcal{H}_{r,1}$ coincides with $\mathcal{E}_{1,0}$.

Notice that, modulo an isomorphism, every Heisenberg module of §2.2 is the deformation of a line bundle (no vector bundles of higher rank appear).

To conclude this section, we now generalize the second part of Prop. 3.3 and show how to get a Hermitian structure similar to (2.9) from a canonical (and simpler) Hermitian structure on $\mathcal{F}_{r,p}$. In the classical case, for all $f, g \in \mathcal{F}_{r,p}$ the product $\overline{fg}$ is periodic, and the map $(f, g) \mapsto \overline{fg}$ is a Hermitian structure on $\mathcal{F}_{r,p}$ that transformed with the isomorphism $\varphi_{r,p}$ becomes (3.10). The noncommutative analogue of this fact is the content of next proposition, where the pointwise product $\overline{fg}$ is replaced by Moyal product $\overline{f} \ast_\theta g$. This
can be done for arbitrary Chern number \( p \), but we must assume that \( r = 0 \) and \( s = 1 \), meaning that we are focusing on modules isomorphic to “the rank 1” Heisenberg modules \( \mathcal{E}_{p,1} \) (the reason is that, in the \( y \) direction, \( \mathcal{T} \ast \theta g \) is periodic with period \( s \), so it belongs to \( \mathcal{C}^\infty(\mathbb{T}^2) \) only if \( s = 1 \)).

**Proposition 4.4.** Let \( \tau = i(1 + \frac{1}{2}p\theta) \). Then, for all \( f, g \in \mathcal{F}_{r,p} \):

\[
T_\theta(\mathcal{T} \ast \theta g) = \sum_{m,n \in \mathbb{Z}} V^n U^m \int_{-\infty}^{+\infty} (f \triangleright V^n U^m g) \, dt,
\]

where \( (\cdot)_t \) is canonical inner product in (2.10), \( f = \varphi_{r,p}(f) \) and \( g = \varphi_{r,p}(g) \) as in Prop. 3.1, and \( T_\theta : \mathcal{C}^\infty(\mathbb{T}^2) \to A_\theta^\infty \) is the quantization map (2.6).

**Proof.** For \( \tau = i(s + \frac{1}{2}p\theta) \), equations (3.6) and (2.7) give:

\[
(\mathcal{T} \ast \theta g)(z) = \sum_{m,n \in \mathbb{Z}} 4\frac{1}{\sqrt{z}} u_n^{-m} \int e^{-2\pi i m \xi_1} e^{2\pi i n \eta_1} e^{4\pi i (\eta_1 \xi_2 - n \xi_1)} \times
\]

\[
f_{[m]}^*(y + \xi_2 + m^2 \theta + m^2 \eta) g_{[n]}(y + \eta_2 + n^2 \theta + n^2 \eta) \, d\xi d\eta
\]

\[
= \sum_{m,k \in \mathbb{Z}} u(x, y)^k f_{[m]}^*(y + m^2 \theta - k^2 \theta) g_{[m+k]}(y + (m + k)^2 \theta + k^2 \theta),
\]

where \( \xi = \xi_1 + \xi_2 \), \( \eta = \eta_1 + 2i\eta_2 \) and we called \( n = m + k \). Using (3.11) with \( \omega_y = 1 \):

\[
\mathcal{T} \ast \theta g = \sum_{k,m,n \in \mathbb{Z}} u_k^m v^n \int_0^1 e^{-2\pi i n t} f_{[m]}^*(t + m^2 \theta - k^2 \theta) g_{[m+k]}(t + (m + k)^2 \theta + k^2 \theta) \, dt.
\]

From (4.2):

\[
g \triangleleft U^{-k} V^{-n} = \{W(0,1) \otimes (C^*)^n\}^{-n} \{W(\frac{\pi}{p}, \theta, 0) \otimes S\}^{-k} g
\]

\[
= \{W(0,-n) \otimes C^{ns}\} \{W(-k \frac{\pi}{p} - k\theta, 0) \otimes S^{-k}\} g
\]

\[
= e^{\pi i k t} \{W(-k \frac{\pi}{p} - k\theta, -n) \otimes C^{ns} S^{-k}\} g,
\]

where we used (2.1). The \([m]\)-th component, evaluated at \( t + m^2 \theta - k^2 \theta \), gives

\[
e^{-\pi i k \theta} e^{2\pi i m_k \frac{\theta}{p}} \{W(-k \frac{\pi}{p} - k\theta, -n) g_{[m+k]}\}(t + m^2 \theta - k^2 \theta)
\]

\[
= e^{-\pi i k \theta} e^{-2\pi i k t} g_{[m+k]}(t + (m + k)^2 \theta + k^2 \theta).
\]

Hence for \( s = 1 \):

\[
\mathcal{T} \ast \theta g = \sum_{k,m,n \in \mathbb{Z}} e^{-\pi i k \theta} u_k^m v^n \int_0^1 \{f_{[m]}^*(g \triangleleft U^{-k} V^{-n})_{[m]}\}(t + m^2 \theta - k^2 \theta) \, dt
\]

\[
= \sum_{k,n \in \mathbb{Z}} e^{-\pi i k \theta} u_k^m v^n \int_{-\infty}^{+\infty} (f_{[m]} g \triangleleft U^{-n} V^n) dt,
\]

where \( (\cdot)_t \) is the Hermitian structure in (2.10). Using then (2.6) one finds:

\[
T_\theta(\mathcal{T} \ast \theta g) = \sum_{m,n \in \mathbb{Z}} e^{-2\pi i mn \theta} U^n V^m \int_{-\infty}^{+\infty} (f_{[m]} g \triangleleft U^{-m} V^n) dt.
\]

The observation that \( (f_{[m]} g \triangleleft U^{-m} V^n)_t = (f \triangleright V^n U^m g)_t \) and \( U^n V^m = e^{2\pi i mn \theta} V^n U^m \) concludes the proof. \( \blacksquare \)

15
5 A formal approach: Hopf cocycles and Drinfeld twists

In this section, we conclude with some remarks about formal deformations of vector bundles on the torus using twists based on the universal enveloping algebra of the Heisenberg Lie algebra $h_3$, following the ideas in [10].

At a formal level, a powerful tool to deform algebras together with their modules is

by means of a Drinfeld twist [8, 9] (good reviews are also in [20, 3]). If $B$ is a bialgebra, $A$ a left $B$-algebra module and $M$ an $A \times B$-module, using a cocycle twist based on $B$ we can deform the coproduct of $B$ and get a new bialgebra $B^*$, we can deform the product of $A$ and get a new algebra $A^*$, and deform $M$ into an $A^* \times B^*$-module $M^*$. If $A = C^\infty(\mathbb{T}^2)[[\hbar]]$, with the cocycle twist $F = e^{i\hbar \partial_x \wedge \partial_y}$ based on $U(\mathbb{H}^2)[[\hbar]]$ we can construct a deformation quantization of $A$ that is the formal analogue of the algebra $A_\hbar^\infty$ of the noncommutative torus. The next step would be to deform the modules of sections of line bundles $\mathcal{F}_{i,p} \subset C^\infty(\mathbb{C})$ (here $\tau = i$). The problem is that [11] these submodules are not stable under the generators of translations $\partial_x, \partial_y$ (since the charts of any local trivialization are not invariant under the group of translations) unless $p = 0$, and we cannot use the abelian twist above. In other words, if by contraddiction $\partial_x, \partial_y$ map $\mathcal{F}_{i,p}$ into itself, they can be used to construct a flat connection on the line bundle, meaning that the 1st Chern number must be $p = 0$.

On the other hand the two operators in (3.5) map $\mathcal{F}_{i,p}$ into itself. Called

$$a = \frac{1}{2}(\nabla_1 + i \nabla_2), \quad a^\dagger = \frac{1}{2}(-\nabla_1 + i \nabla_2),$$

and $q$ the operator on $\mathcal{O} := \bigoplus_{p \in \mathbb{Z}} \mathcal{F}_{i,p}$ that is equal to $\pi p$ times the identity on the $p$-th summand, one finds that these three operators satisfy the commutation relations of the Heisenberg Lie algebra $h_3$:

$$[a, a^\dagger] = q, \quad [q, .] = 0.$$

Now $\mathcal{O}$ is a graded algebra, the ring of line bundles on $\mathbb{T}^2$. It is a $U(h_3)$-algebra module, so given any cocycle twist based on $U(h_3)[[\hbar]]$, we can produce a new graded associative algebra $\mathcal{O}_\hbar = \bigoplus_{p \in \mathbb{Z}} (\mathcal{F}_{i,p})_\hbar$, and each $(\mathcal{F}_{i,p})_\hbar$ will be automatically a module for the subalgebra $(\mathcal{F}_{i,0})_\hbar$, deformation of the algebra of functions on the torus itself.

In [10] cocycle twists of abelian type have been used for this scope in higher dimensional tori, where the Lie algebra generated by $q$ and the $\nabla_\alpha$'s admit multi-dimensional Cartan subalgebras. Using instead a twist with non-trivial coassociator [9], associativity of the product of deformed module algebras and of their modules is not guaranteed and must be verified case by case. A detailed study of Heisenberg twists, and their use to study the dynamics of charged particles on the noncommutative torus, is postponed to future works.

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