TORUS FIBRATIONS AND LOCALIZATION OF INDEX II
- LOCAL INDEX FOR ACYCLIC COMPATIBLE SYSTEM -

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Abstract. We give a framework of localization for the index of a Dirac-type operator on an open manifold. Suppose the open manifold has a compact subset whose complement is covered by a family of finitely many open subsets, each of which has a structure of the total space of a torus bundle. Under some compatibility and acyclicity conditions we show that the index of the Dirac-type operator is localized on the compact set.

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1. Introduction

This paper is the second of the series concerning a localization of the index of elliptic operators.

For a linear elliptic operator on a closed manifold, its Fredholm index is sometimes determined by the information on a specific subset under appropriate geometric conditions. Such a phenomenon is called localization of index. A typical example is Hopf’s theorem identifying the index of the de Rham operator with the number of zeros of a vector field counted with sign and multiplicity. In this case the geometric condition is given by the vector field, and the index is localized around the zeros of the vector field. Another typical example is Atiyah-Segal’s Lefschetz formula for the equivariant index for a torus action, when the geometric condition is given by the torus action. The index is localized around its fixed point set, and the localization is understood in terms of an algebraic localization of equivariant K-group. In particular when the manifold is symplectic and the elliptic operator is a Dirac-type operator, the localization is extensively investigated using the relation between the algebraic localizations in equivariant K-theory and that in equivariant ordinary cohomology theory.

In the previous paper [4], we investigated the case of a closed symplectic manifold equipped with a prequantizing line bundle and the structure of a Lagrangian fibration, and described a localization of the index of a Dirac-type operator, twisted by the prequantizing line bundle, on the subset consisting of Bohr-Sommerfeld fibers and singular fibers. A novel feature of our method is that we do not use a global group action but use only the structure of a torus bundle on an open subset of the manifold.

In the present paper we generalize our method to deal with the case when we do not have a global torus bundle on the open subset, but we just have the structure of a torus bundle on a neighborhood of each points, which gives a family of torus bundles satisfying some compatibility condition. The various torus bundles may have tori of various dimensions as their fibers. This generalization enables us to describe the localization phenomenon more precisely. Even for the case in the previous paper, we could replace the subset on which the index is localized with a smaller subset. A typical example of our generalization is the localization for the Riemann-Roch number of toric manifolds, for which we would need an orbifold version of our formulation. Moreover we can deal with some prequantized singular Lagrangian fibration without global toric action (Section 6). In our subsequent paper we will use the localization to give an approach to V. Guillemin and S. Sternberg’s conjecture concerning “quantization commutes with reduction” in the case of torus actions. Though our motivating example is the Riemann-Roch number of a symplectic manifold, the localization of the index is formulated for more general cases. In fact we first establish a general framework to formulate the index of elliptic operators on complete manifolds (Section 3). This section is independent of the other sections and the framework might itself be interesting.

The mechanism of our localization is explained as a version of Witten’s deformation, where the potential term itself is a first order differential operator. Our geometric input...
data is a family of torus bundles. Roughly speaking we deform the operator like an adiabatic limit shrinking the various fiber directions at the same time in a compatible manner. The potential term corresponds to some average of the de Rham operators along the various fiber directions.

Formally our localization is formulated as a property for the index of the elliptic operator on an open manifold: let $D$ be an elliptic operator on a (possibly non-compact) manifold $X$, and $V$ is an open subset of $X$ whose complement $X \setminus V$ is compact. Suppose $V$ has a certain geometric structure $s$, by which we can modify $D$ to construct a Fredholm operator. The index of the Fredholm operator depends on the data $(X, V, s, D)$. Suppose the index satisfies the following properties. Firstly the index is deformation invariant. Secondly if $X'$ is an open subset of $X$ containing $X \setminus V$, and hence $X' \setminus V$ is compact. Let $D'$ be the restriction of $D$ on $X'$. We assume that the structure $s$ has its restriction $s'$ on $V' = X' \cap V$. Then we have the index of the Fredholm operator constructed from the data $(X', V', s', D')$. The required excision property is the equality between the two indices. We will construct Fredholm operators which satisfy the above type of excision property. The structure $s$ on $V$ is not extended on the whole $X$. In this sense $X \setminus V$ is regarded as the singular locus of the structure. The index is localized on the singular locus $X \setminus V$, and we call it the local index of the data $(X, V, s, D)$. When $X \setminus V$ is of the form of the disjoint union of finitely many compact subsets, the localized index is equal to the sum of the contributions from the compact subsets.

Our first main result is the construction of the local index when the structure $s$ is the strongly acyclic compatible system defined in Section 2. Our second main result is a few basic properties of the local index, in particular a product formula of the local index.

The organization of this paper is as follows. In Section 2 we define the notion of a strongly acyclic compatible system, which we use as the geometric structure $s$ in the above explanation. In Section 3 we give a formulation of the index of elliptic operators on complete Riemannian manifolds. This formulation is a generalization of the one given in Section 5 of [8]. This section is independent of the other sections. In Section 4 we define the index of elliptic operators using the framework of Section 3 under the assumption that a strongly acyclic compatible system is given on an end of the base manifold. In Section 5 we show a product formula for the index defined in Section 4. In Section 6 we give an example of our formulation using some 4-dimensional Lagrangian fibration with singular fibers. In Appendix we give proofs of the technical lemmas used in the main part.

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2. Compatible fibration and acyclic compatible system

In this paper we work in smooth category. In particular, every manifold is a smooth manifold and every fiber bundle is a smooth fiber bundle.

2.1. Compatible fibration. Let $M$ be a manifold.

Definition 2.1. A compatible fibration on $M$ is a collection of data

- $\{\pi_{\alpha}: V_{\alpha} \to U_{\alpha} \mid \alpha \in A\}$,
- $\{\pi_{\alpha\beta}: V_{\alpha} \cap V_{\beta} \to U_{\alpha\beta} \mid \alpha, \beta \in A \text{ such that } V_{\alpha} \cap V_{\beta} \neq \emptyset \text{ and } \alpha \neq \beta\}$, and
We often denote a compatible fibration on $M$ by \( \{p^a_{\alpha \beta}: \alpha \beta \to \pi_a(V_a \cap V_\beta) \mid \alpha, \beta \in A \text{ such that } V_a \cap V_\beta \neq \emptyset \text{ and } \alpha \neq \beta \} \) satisfying the following properties.

(1) \( \{V_a\}_{a \in A} \) is an open covering of \( M \).
(2) For each \( \alpha \in A \), \( U_\alpha \) is a manifold and \( \pi_{\alpha}: V_\alpha \to U_\alpha \) is a fiber bundle whose fiber is a closed manifold.
(3) For each \( \alpha \) and \( \beta \), we have
\[
V_\alpha \cap V_\beta = \pi^{-1}_\alpha(\pi_\alpha(V_\alpha \cap V_\beta)) = \pi^{-1}_\beta(\pi_\beta(V_\alpha \cap V_\beta)).
\]
(4) For each \( \alpha, \beta \in A \) satisfying \( V_\alpha \cap V_\beta \neq \emptyset \) and \( \alpha \neq \beta \), \( U_{\alpha \beta} \) is a manifold, \( \pi_{\alpha \beta}: V_\alpha \cap V_\beta \to U_{\alpha \beta} \) is a fiber bundle with fiber a closed manifold, \( p^a_{\alpha \beta}: U_{\alpha \beta} \to \pi_a(V_\alpha \cap V_\beta) \) is a fiber bundle with fiber a closed manifold such that the following diagram commutes

\[
\begin{array}{ccc}
V_\alpha \cap V_\beta & \xrightarrow{\pi_{\alpha \beta}} & \pi_\beta(V_\alpha \cap V_\beta) \\
\pi_\alpha|_{V_\alpha \cap V_\beta} & \downarrow & \pi_{\alpha \beta}|_{V_\alpha \cap V_\beta} \\
U_{\alpha \beta} & \xrightarrow{p^a_{\alpha \beta}} & \pi_{\alpha}(V_\alpha \cap V_\beta) \subseteq U_\beta
\end{array}
\]

Here we assume \( U_{\alpha \beta} = U_{\beta \alpha}, \pi_{\alpha \beta} = \pi_{\beta \alpha}, \pi_{\alpha \beta} = \pi_{\beta \alpha} \), and \( p^a_{\alpha \beta} = p^a_{\beta \alpha} \).

(5) For each \( b \in U_{\alpha \beta} \) we have
\[
\pi^{-1}_{\alpha \beta}(b) = \pi^{-1}_\alpha(p^a_{\alpha \beta}(b)) \cap \pi^{-1}_\beta(p^a_{\beta \alpha}(b)).
\]

We often denote a compatible fibration on \( M \) by \( \{\pi_a\}_{a \in A} \) for simplicity.

Let \( \{\pi_a\}_{a \in A} \) be a compatible fibration on \( M \).

**Definition 2.2.** For \( \alpha \in A \) and \( x \in V_\alpha \), we define \( A(\alpha; x) \) as follows
\[
A(\alpha; x) := \{\beta \in A \mid x \in V_\alpha \cap V_\beta, p^\beta_{\alpha \beta}: U_{\alpha \beta} \to \pi_\beta(V_\alpha \cap V_\beta) \text{ is a diffeomorphism} \}.
\]

**Remark 2.3.** We have \( \pi^{-1}_\beta(V_\alpha \cap V_\beta) \subseteq \pi^{-1}_\alpha(V_\alpha \cap V_\beta) \) for \( \beta \in A(\alpha, x) \).

**Definition 2.4.** An open covering \( \{V'_a\}_{a \in A} \) of \( M \) is said to be *admissible* if it satisfies the following conditions
(1) \( V'_a \subseteq V_a \) for each \( \alpha \in A \) and
(2) \( \pi^{-1}_\beta(V'_a \cap V'_\beta) = V'_a \cap V'_\beta \) for each \( \alpha \) and \( \beta \).

Later we will fix an admissible open covering satisfying some good property. (See Assumption 2.12 (2).)

**Proposition 2.5.** Let \( \{V'_a\}_{a \in A} \) be an admissible open covering of \( M \). Then, the collection of data
\[
\begin{align*}
&\{\pi_a|_{V'_a}: V'_a \to U'_a := \pi_a(V'_a) \mid \alpha \in A\} \\
&\{\pi_{\alpha \beta}|_{V'_a \cap V'_\beta}: V'_a \cap V'_\beta \to U'_{\alpha \beta} := \pi_{\alpha \beta}(V'_a \cap V'_\beta) \mid \alpha, \beta \in A \text{ such that } V'_a \cap V'_\beta \neq \emptyset \text{ and } \alpha \neq \beta\}, \text{ and} \\
&\{p^a_{\alpha \beta}|_{U'_{\alpha \beta}}: U'_{\alpha \beta} \to \pi_a(V'_a \cap V'_\beta) \mid \alpha, \beta \in A \text{ such that } V'_a \cap V'_\beta \neq \emptyset \text{ and } \alpha \neq \beta\}
\end{align*}
\]
is a compatible fibration on \( M \).
Suppose that there is an admissible open covering \( \{V'_a\}_{a \in A} \) of \( M \). We take and fix it.

**Definition 2.6.** A subset \( C \) of \( M \) is said to be admissible (for \( \{V'_a\}_{a \in A} \)) if \( V'_a \cap C = \pi^{-1}_a(\pi_a(V'_a \cap C)) \) for each \( a \in A \).

**Example 2.7.** Each \( V_a \) (resp. \( V'_a \)) is admissible open subset.

We can restrict a compatible fibration to an admissible open subset.

**Proposition 2.8.** Let \( C \) be a submanifold of \( M \). Suppose \( C \) is admissible. Then the data \( \{\pi'_a|_{C \cap V'_a}: C \cap V'_a \to \pi'_a(C \cap V'_a)\}_{a \in A} \) is a compatible fibration on \( C \).

**Proof.** It is sufficient to show the following three equalities

1. \( \pi^{-1}_a \pi'_a(C \cap V'_a \cap V''_b) = C \cap V'_a \cap V''_b \),
2. \( \pi^{-1}_a \pi'_a (C \cap V'_a \cap V''_b) = C \cap V'_a \cap V''_b \), and
3. \( p^{-1}_a \pi'_a(C \cap V'_a \cap V''_b) = C \cap V'_a \cap V''_b \).

First let us show the right facing inclusion \( \subset \) for 1. For each \( z \in \pi^{-1}_a(\pi'_a(C \cap V'_a \cap V''_b)) \) there exists \( x \in \cap V'_a \cap V''_b \) such that \( \pi'_a(z) = \pi'_a(x) \). Then \( z \in \pi^{-1}_a(\pi'_a(C \cap V'_a \cap V''_b)) \). We show that \( \pi^{-1}_a(z) \subset \pi^{-1}_a(\pi'_a(C \cap V'_a \cap V''_b)) \). Next we show the right facing inclusion \( \subset \) for 2. For each \( z \in \pi^{-1}_a \pi'_a(C \cap V'_a \cap V''_b) \) there exists \( x \in \cap V'_a \cap V''_b \) such that \( \pi'_a(z) = \pi'_a(x) \). Then \( z \in \pi^{-1}_a \pi'_a(C \cap V'_a \cap V''_b) \). Finally we show the right facing inclusion \( \subset \) for 3. For each \( z \in p^{-1}_a \pi'_a(C \cap V'_a \cap V''_b) \) there exists \( x \in \cap V'_a \cap V''_b \) such that \( p^{-1}_a \pi'_a(z) = \pi'_a(x) \). Then \( z \in \pi^{-1}_a \pi'_a(C \cap V'_a \cap V''_b) \).

This shows \( \pi^{-1}_a(z) \subset C \). Hence \( z \in \pi^{-1}_a(\pi'_a(C \cap V'_a \cap V''_b)) \).

**Remark 2.9.** For an admissible submanifold \( C \) of \( V \), the restriction \( \{\pi'_a|_{C \cap V'_a}: C \cap V'_a \to \pi'_a(C \cap V'_a)\}_{a \in A} \) does not satisfy the second and the third conditions in Assumption 2.12 in general.

**Definition 2.10.** Let \( f: M \to \mathbb{R} \) be a function. If \( f \) is constant along fibers of \( \pi'_a|_{V'_a} \) for all \( a \in A \), then we call \( f \) an admissible function (for \( \{V'_a\}_{a \in A} \)).

**Definition 2.11.** Suppose that there is an admissible open covering \( \{V'_a\}_{a \in A} \) and fix it. If a linear map \( I: C^\infty(M) \to C^\infty(M) \) satisfies the following properties, then we call \( I \) an averaging operation for \( \{V'_a\}_{a \in A} \).

1. \( I(f) \) is an admissible function for \( \{V'_a\} \) for all \( f \in C^\infty(M) \).
2. If \( f \) is a constant function, then \( I(f) \) is also a constant function with the same value of \( f \).
3. If \( f \) is a non-negative function, then \( I(f) \) is so.
4. There exists an open covering \( \{V''_a\}_{a \in A} \) of \( M \) which satisfies the following properties.
   • For all \( a \in A \) we have \( V''_a \subset V'_a \).
   • For all \( a \in A \) we have \( \pi^{-1}_a(V''_a) = V''_a \).
   • For all \( f \in C^\infty(V) \) and \( x \in M \) there exists some \( a \in A \) such that \( x \in V''_a \) and
     \[ \min_{y \in \pi^{-1}_a(\pi_a(x))} f(y) \leq I(f)(x) \leq \max_{y \in \pi^{-1}_a(\pi_a(x))} f(y). \]
(5) Let \( f : M \to \mathbb{R} \) be a smooth function. If \( \text{supp} f \) is contained in \( V'_\alpha \) for some \( \alpha \in A \), then \( \text{supp} I(f) \) is also contained in \( V'_\alpha \).

In this paper we impose the following technical assumptions for a compatible fibration
\( \{ \pi_\alpha : V_\alpha \to U_\alpha \}_{\alpha \in A} \).

**Assumption 2.12.**

1. The index set \( A \) is a finite set.
2. There is an admissible open covering \( \{ V'_\alpha \}_{\alpha \in A} \) of \( M \) such that \( \overline{V'_\alpha} \subset V_\alpha \), and we fix it. Hereafter we say *admissible* (resp. an averaging operation) instead of admissible (resp. an averaging operation) for \( \{ V'_\alpha \} \).
3. There is an averaging operation \( I : C^\infty(M) \to C^\infty(M) \) whose definition is given in Definition 2.14.

Using the averaging operation we can construct an admissible partition of unity as in the following.

**Lemma 2.13 (Existence of admissible partition of unity).** Let \( M \) be a manifold with a compatible fibration \( \{ \pi_\alpha \} \). There is a smooth partition of unity \( \{ \rho_\alpha^2 \} \) of the open covering \( M = \bigcup_\alpha V'_\alpha \) such that each \( \rho_\alpha \) is admissible.

**Proof.** Take any partition of unity \( \{ \phi_\alpha \} \) of \( M = \bigcup_\alpha V'_\alpha \). Applying the averaging operation we have a family of admissible functions \( \{ I(\phi_\alpha) \} \). Note that it is another partition of unity of \( \{ V'_\alpha \} \) because of the Property 2,3 and 5 of the averaging operation. We put
\[
\rho_\alpha := I(\phi_\alpha) / \sqrt{\sum_\beta I(\phi_\beta)^2}.
\]
Then \( \{ \rho_\alpha^2 \} \) is a required admissible partition of unity. \( \square \)

We give a sufficient condition for Assumption 2.12.

**Definition 2.14 (Good compatible fibration).** If a compatible fibration \( \{ \pi_\alpha : V_\alpha \to U_\alpha \} \) over \( M \) satisfies 1 and 2 in Assumption 2.12 together with the following 5', then we call \( \{ \pi_\alpha : V_\alpha \to U_\alpha \} \) a good compatible fibration.

5’. If \( V_\alpha \cap V_\beta \neq \emptyset \), then we have \( \alpha \in A(\beta; x) \) or \( \beta \in A(\alpha; x) \) for all \( x \in V_\alpha \cap V_\beta \).

We show the following proposition in Appendix A.

**Proposition 2.15.** If \( \{ \pi_\alpha \}_{\alpha \in A} \) is a good compatible fibration on \( M \), then there exists an open covering \( \{ V'_\alpha \}_{\alpha \in A} \) of \( M \) and a linear map \( I : C^\infty(M) \to C^\infty(M) \) such that \( I \) satisfies all conditions in Definition 2.12 for \( \{ V'_\alpha \}_{\alpha \in A} \).

We also give a sufficient condition for Assumption 2.12 (2) in Appendix B.

Now we define an appropriate notion of Riemannian metric and connection for a compatible fibration. For the projection \( \pi \) of a fiber bundle over a manifold we denote by \( T[\pi] \) the vector bundle consisting of tangent vectors along fibers of \( \pi \). We first note that there exist following four types of short exact sequences for \( i = \alpha, \beta \)

1. \[
0 \to T[\pi_\alpha] \to T V_\alpha \to \pi_\alpha^* T V_\alpha \to 0,
\]
2. \[
0 \to T[\pi_\alpha \beta] \to T (V_\alpha \cap V_\beta) \to \pi_\alpha^* \pi_\beta^* T V_\alpha \cap V_\beta \to 0,
\]
3. \[
0 \to T[\pi_\alpha \beta] \vert_{V_\alpha \cap V_\beta} \to T[\pi_\alpha] \vert_{V_\alpha \cap V_\beta} \to \pi_\alpha^* \pi_\beta^* T[\pi_\alpha \beta] \to 0,
\]
4. \[
0 \to T[\pi_\alpha \beta] \vert_{V_\alpha \cap V_\beta} \to T V_\alpha \cap V_\beta \to \pi_\beta^* T[\pi_\alpha] \vert_{V_\alpha \cap V_\beta} \to 0.
\]

**Definition 2.16.** Let \( E^0, E^1 \) and \( E^2 \) be smooth vector bundles with metrics. A short exact sequence \( 0 \to E^0 \to E^1 \to E^2 \to 0 \) is orthogonally split if the isomorphism \( E^1 \cong
$E^0 \oplus E^2$ defined by the orthogonal splitting with respect to the metric on $E^1$ is isometric with respect to the metrics on $E^0$, $E^1$ and $E^2$.

**Definition 2.17.** A compatible Riemannian metric of a compatible fibration is a collection of metrics on the vector bundles $T[\pi_1], TU_1, \pi_{\alpha\beta}, TU_{\alpha\beta}$ such that the exact sequences (3) and (4) are orthogonally split with respect to these metrics.

From the definition, we have a canonical isometric isomorphism

$$\begin{align*}
(\pi_\alpha) &= \pi_{\alpha\beta}^* T \implies (\pi_\beta) \\
(\pi_{\alpha\beta}^* T) &= (\pi_\alpha) \oplus (\pi_\beta)
\end{align*}$$

over $V_{\alpha} \cap V_{\beta}$.

**Definition 2.18.** Suppose we have a compatible fibration with a compatible Riemannian metric. A compatible connection is a collection of the splittings of the short exact sequences (1) and (2) such that the isomorphism (5) is equal to the composition of the isomorphisms

$$(\pi_{\alpha\beta}^* T) \oplus (\pi_\alpha^* T) \oplus (\pi_\beta)$$

induced from the splittings.

2.2. Acyclic compatible system.

**Definition 2.19.** Suppose we have a compatible fibration $\{\pi_\alpha\}$ on $M$ with a compatible Riemannian metric. A bundle $W$ over $M$ is a compatible Clifford module bundle if we have the following structures.

(1) $W$ has a structure of a $\mathbb{Z}/2$-graded $Cl(T[\pi_\alpha] \oplus \pi_\alpha^* T U_\alpha)$-module bundle over $V_\alpha$.

(2) Over $V_{\alpha} \cap V_{\beta}$, the above module structures on $V_{\alpha}$ and $V_{\beta}$ are compatible with the isomorphism (5).

The next lemma follows immediately from the definitions of compatible metric, compatible connection and compatible Clifford module bundle.

**Lemma 2.20.** Suppose we have a compatible metric and compatible connection. Then we have a well-defined Riemannian metric on $M$. Moreover if we have a compatible Clifford module bundle, then it has a structure of Clifford module with respect to the well-defined Riemannian metric on $M$.

Let $\{\pi_\alpha\}$ be a compatible fibration on $M$ with compatible Riemannian metric and $W \to M$ a compatible Clifford module bundle.

**Definition 2.21** (Compatible system of Dirac-type operators). A compatible system of Dirac-type operators on $(\{\pi_\alpha\}, W)$ is a data $\{D_\alpha\}$ satisfying the following properties.

(1) $D_\alpha : \Gamma(W|_{V_\alpha}) \to \Gamma(W|_{V_\alpha})$ is an order-one formally self-adjoint differential operator of degree one.

(2) $D_\alpha$ contains only the derivatives along fibers of $\pi_\alpha : V_\alpha \to U_\alpha$, i.e. $D_\alpha$ commutes with multiplication of the pull-back of smooth functions on $U_\alpha$.

(3) The principal symbol $\sigma(D_\alpha)$ of $D_\alpha$ is given by $\sigma(D_\alpha) = c \circ \pi_\alpha \circ \iota_\alpha : T^* V_\alpha \to \text{End}(W|_{V_\alpha})$, where $\iota_\alpha : T[\pi_\alpha] \to TV_\alpha$ is the natural inclusion, $p_\alpha : T^*[\pi_\alpha] \to T[\pi_\alpha]$ is the isomorphism induced by the Riemannian metric and $c : T[\pi_\alpha] \to \text{End}(W|_{V_\alpha})$ is the Clifford multiplication.
For $b \in U_\alpha$ and $u \in T_b U_\alpha$, let $\tilde{u} \in \Gamma(\pi^*_\alpha TU_\alpha|_{\pi^{-1}_\alpha(b)})$ be the section naturally induced by $u$. $\tilde{u}$ acts on $W|_{\pi^{-1}_\alpha(b)}$ by the Clifford multiplication $c(\tilde{u})$. Then $D_\alpha$ and $c(\tilde{u})$ anti-commute each other, i.e.,

$$0 = \{D_\alpha, c(\tilde{u})\} := D_\alpha \circ c(\tilde{u}) + c(\tilde{u}) \circ D_\alpha$$

for all $b \in U_\alpha$ and $u \in T_b U_\alpha$.

(5) If $V_\alpha \cap V_\beta \neq \emptyset$, then the anti-commutator $\{D_\alpha, D_\beta\} := D_\alpha \circ D_\beta + D_\beta \circ D_\alpha$ is a differential operator along fibers of $\pi_{\alpha\beta}$ of order at most 2.

The properties 1, 2, and 3 in Definition 2.21 imply that $D_\alpha$ is of Dirac-type when restricted to each fiber of $\pi_\alpha$.

We call a compatible system of Dirac-type operators $\{D_\alpha\}$ a compatible system for short.

**Remark 2.22.** If there is a compatible system of Dirac-type operators, then, the underlying compatible fibration $\{\pi_\alpha\}_{\alpha \in A}$ must satisfies the following condition on each $V_\alpha \cap V_\beta \neq \emptyset$.

- $T[p^\alpha_{\alpha\beta}]$ and $T[p^\beta_{\alpha\beta}]$ are perpendicular to each other with respect to the metric on $TU_{\alpha\beta}$.

**Definition 2.23** (Acyclic compatible system). A compatible system $\{D_\alpha\}_{\alpha \in A}$ is acyclic if for all $\alpha \in A$, $x \in V_\alpha$ and a family of non-negative numbers $\{t_\beta\}_{\beta \in A(\alpha; x)}$ satisfying $t_\beta > 0$ for some $\beta$, the operator $\sum_{\beta \in A(\alpha; x)} t_\beta D_\beta : \Gamma(W|_{\pi^{-1}_\alpha(\pi_\alpha(x))}) \to \Gamma(W|_{\pi^{-1}_\alpha(\pi_\alpha(x))})$ has zero kernel. Note that the above operator is well-defined because of Remark 2.3.

**Definition 2.24** (strongly acyclic compatible system). A compatible system $\{D_\alpha\}$ is strongly acyclic if it satisfies the following conditions.

1. For each $\alpha$ and $b \in U_\alpha$ $D_\alpha|_{\pi^{-1}_\alpha(b)}$ has zero kernel.
2. If $V_\alpha \cap V_\beta \neq \emptyset$, then the anti-commutator $\{D_\alpha, D_\beta\}$ is a non-negative operator over $V_\alpha \cap V_\beta$.

We first note that the following lemma.

**Lemma 2.25.** A strongly acyclic compatible system is acyclic.

**Proof.** If $\{D_\alpha\}$ is strongly acyclic compatible system, then we have

$$\left(D_\alpha + \sum_{\beta \in A(\alpha; x)} t_\beta D_\beta\right)^2 = D_\alpha^2 + \sum_{\beta} t_\beta D_\beta\{D_\alpha, D_\beta\} + \left(\sum_{\beta} \tau_\beta D_\beta\right)^2 \geq D_\alpha^2$$

for any family of non-negative numbers $\{\tau_\beta\}$. Suppose $\left(\sum_{\beta \in A(\alpha; x)} t_\beta D_\beta\right) s = 0$ for $s \in \Gamma(W|_{\pi^{-1}_\alpha(\pi_\alpha(x))})$. Take $\alpha_0 \in A(\alpha; x)$ so that $t_{\alpha_0}$ is not 0. Then we have

$$\left(D_{\alpha_0} + \sum_{\beta \in A(\alpha_0; x) \setminus \{\alpha_0\}} (t_\beta/t_{\alpha_0}) D_\beta\right) s = 0$$

and $s = 0$ by the above inequality and the first condition in Definition 2.24. \qed

**Remark 2.26.** We often need a compatible fibration $\{\pi_\alpha : V_\alpha \to U_\alpha\}_{\alpha \in A}$ such that $U_\alpha$’s are orbifolds even though $M$ and $V_\alpha$’s are smooth. In fact, such a compatible fibration is necessary when we deal with a torus action. In this case, by using a uniformizing chart.
of $U_{\alpha}$, we extend Definition 2.1. Instead of $\{\pi_{\alpha} : V_{\alpha} \to U_{\alpha}\}_{\alpha \in A}$ we consider the following collection of data.

- $\{q_{\alpha} : \widetilde{V}_{\alpha} \to V_{\alpha} \mid \alpha \in A\}$,
- $\{\pi_{\alpha} : \widetilde{V}_{\alpha} \to \tilde{U}_{\alpha} \mid \alpha \in A\}$,
- $\{\mathcal{F}_{\alpha} \text{ on } V_{\alpha} \mid \alpha \in A\}$,
- $\{\pi_{\alpha\beta} : \widetilde{V}_{\alpha} \times_{\mathcal{F}} \widetilde{V}_{\beta} \to \tilde{U}_{\alpha\beta} \mid \alpha, \beta \in A \text{ such that } V_{\alpha} \cap V_{\beta} \neq \emptyset \text{ and } \alpha \neq \beta\}$, and
- $\{p_{\alpha\beta}^{\alpha} : \tilde{U}_{\alpha\beta} \to \pi_{\alpha}(\widetilde{V}_{\alpha} \times_{\mathcal{F}} \widetilde{V}_{\beta}) \mid \alpha, \beta \in A \text{ such that } V_{\alpha} \cap V_{\beta} \neq \emptyset \text{ and } \alpha \neq \beta\}$,

where for each $\alpha, \beta \in A$, $\pi_{\alpha\beta}$, $p_{\alpha\beta}^{\alpha}$ are fiber bundles, $q_{\alpha}$ is a finite covering and $\mathcal{F}_{\alpha}$ is a foliation on $V_{\alpha}$ which satisfy appropriate compatibilities. See [5] for detail. Note that in this extension $U_{\alpha}$’s do not appear anymore. We define the above data to be good by the same manner as Definition 2.1. In the case where $M$ is an orbifold, as a covering $\widetilde{V}_{\alpha} \to V_{\alpha}$ in the above extension we can adopt a uniformizing chart of $V_{\alpha}$ such that $\widetilde{V}_{\alpha}$ is smooth. For an orbifold version of a compatible system, consult [5].

**Example 2.27.** Let $M$ be $\mathbb{R} \times S^1$ with the standard Riemannian metric and $(t, \theta)$ its standard coordinate. We introduce a compatible fibration on $M$ by

$$\pi_{\alpha} : V_{\alpha} := (-\infty, 1) \times S^1 \to U_{\alpha} := (-\infty, 1)$$

$$\pi_{\beta} : V_{\beta} := (-1, \infty) \times S^1 \to U_{\beta} := (-1, \infty).$$

Let $W$ be the trivial rank 2 Hermitian vector bundle $M \times \mathbb{C}^2$ on $M$ with $\mathbb{Z}/2$-grading $W^0 := M \times (\mathbb{C} \times 0)$, $W^1 := M \times (0 \times \mathbb{C})$.

We define the Clifford multiplication of $Cl(TM)$ on $W$ by

$$c(\partial_\theta) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad c(\partial_t) = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$ 

For smooth functions $f_{\alpha} : V_{\alpha} \to \mathbb{R}$, $f_{\beta} : V_{\beta} \to \mathbb{R}$, let $D_{\alpha}$, $D_{\beta}$ be differential operators on $\Gamma(W|_{V_{\alpha}})$, $\Gamma(W|_{V_{\beta}})$ which are defined by

$$D_{\alpha} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_\theta + f_{\alpha}(t, \theta) \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

$$D_{\beta} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_\theta + f_{\beta}(t, \theta) \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$ 

They are order-one formally self-adjoint differential operators of degree-one. Then, it is easy to see that the data $\{D_{\alpha}, D_{\beta}\}$ is an acyclic compatible system if and only if $f_{\alpha}$ and $f_{\beta}$ satisfy the following properties.

1. $f_{\alpha}(t, \theta) \notin \mathbb{Z}$ for any $(t, \theta) \in V_{\alpha}$. The same property also holds for $f_{\beta}$.
2. $\frac{t_{\alpha}f_{\alpha}(t, \theta) + t_{\beta}f_{\beta}(t, \theta)}{t_{\alpha} + t_{\beta}} \notin \mathbb{Z}$ for any $(t, \theta) \in V_{\alpha} \cap V_{\beta}$ and any non-negative real numbers $t_{\alpha}$, $t_{\beta}$ which satisfy $t_{\alpha} + t_{\beta} \neq 0$.

**Example 2.28.** For non-negative integers $m$ and $n$ satisfying $n \leq m$ let $M$ be $\mathbb{R}^{2m-n} \times T^n$, where we regard $T^n$ as $\mathbb{R}^n/(2\pi\mathbb{Z})^n$. Let $A$ be an ordered set, $\{V_{\alpha}^i\}_{\alpha \in A}$ a finite open covering of $\mathbb{R}^{2m-n}$, and $\{R_{\alpha}\}_{\alpha \in A}$ a family of subspaces of $\mathbb{R}^n$ spanned by rational vectors. We assume that if $\alpha < \beta$, then $R_{\alpha} \subset R_{\beta}$. We put $V_{\alpha} := V_{\alpha}^i \times T^n$ and $T_{\alpha} := R_{\alpha}/R_{\alpha} \cap (2\pi\mathbb{Z})^n$.

Define $U_{\alpha}$ to be $V_{\alpha}^i \times T^n/T_{\alpha}$ and $\pi_{\alpha} : V_{\alpha} \to U_{\alpha}$ to be the natural projection. Then these data define a good compatible fibration on $M$. 

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TORUS FIBRATIONS AND LOCALIZATION OF INDEX II 9
Let \((g, J)\) be the pair of the Riemannian metric and the almost complex structure on \(M\) which is defined by
\[
g_x \left( \sum_{i=1}^{2m-n} a_i^1 \partial_y^i + \sum_{i=1}^{n} b_i^1 \partial_y^i, \sum_{j=1}^{2m-n} a_j^2 \partial_y^j + \sum_{j=1}^{n} b_j^2 \partial_y^j \right) = \sum_{i=1}^{2m-n} a_i^1 a_i^2 + \sum_{i=1}^{n} b_i^1 b_i^2,
\]
for \(x = (y, \theta) \in M\). Note that since \(g\) is invariant under \(J\), \((g, J)\) defines the Hermitian metric on \(M\) by
\[
h_x(u, v) = g_x(u, v) + \sqrt{-1}g_x(u, Jv)
\]
for \(u, v \in T_xM\). By using the horizontal lift \(\pi^*_aTU_\alpha \rightarrow TV_\alpha\) with respect to \(g\), it is obvious that \(\{\pi_\alpha\}\) is equipped with a compatible Riemannian metric and a compatible connection.

Next we define a compatible Clifford module bundle \(W\) and a strongly acyclic compatible system \(\{D_\alpha\}_{\alpha \in \mathcal{A}}\) in the following way. Take a Hermitian line bundle \((L, \nabla^L)\) with Hermitian connection on \(M\) whose restriction to \(\pi^{-1}_\alpha(b)\) is a flat connection for each \(\alpha \in \mathcal{A}\) and \(b \in U_\alpha\). We assume the following condition.

\((\ast)\) For all \(\alpha \in \mathcal{A}\) and \(b \in U_\alpha\) the restriction \(\nabla^L|_{\pi^{-1}_\alpha(b)}\) is not trivially flat connection, i.e., its holonomy representation is non-trivial.

We define a Hermitian vector bundle \(W\) on \(M\) by
\[
W := \wedge^\bullet_C TM_C \otimes \mathcal{L}_a,
\]
where \(TM_C\) is the tangent bundle regarded as a complex vector bundle with \(J\). A Clifford module structure \(c: Cl(TM) \rightarrow \text{End}(W)\) is defined by
\[
c(u)(\varphi) = u \wedge \varphi - u_* \varphi
\]
for \(u \in TM\), \(\varphi \in W\), where \(\wedge\) is the interior product with respect to \(h\), namely,
\[
v_\perp(v_1 \wedge v_2 \wedge \cdots \wedge v_k) := \sum_{i=1}^k (-1)^{i-1}h(v_i, v)v_1 \wedge v_i \wedge \cdots \wedge v_k,
\]
for \(v_1, \ldots, v_k \in TM\).

Let \(\nabla^L TM_C\) be the Hermitian connection on \(\wedge^\bullet_C TM_C\) which is induced from the Levi-Civita connection on \(TM\) with respect to \(g\). Two connections \(\nabla^{\wedge^\bullet C TM_C}\) and \(\nabla^L\) define the Hermitian connection on \(W\) by
\[
\nabla := \nabla^{\wedge^\bullet C TM_C} \otimes \text{id} + \text{id} \otimes \nabla^L.
\]
Then we define \(D_\alpha\) by
\[
D_\alpha := c \circ \iota_\alpha^* \circ \nabla|_{V_\alpha} : \Gamma (W|_{V_\alpha}) \xrightarrow{\nabla|_{V_\alpha}} \Gamma (T^*V_\alpha \otimes W|_{V_\alpha})
\xrightarrow{\iota_\alpha^*} \Gamma (T^*[\pi_\alpha] \otimes W|_{V_\alpha})
\xrightarrow{\iota_\alpha} \Gamma (W|_{V_\alpha}),
\]
where \(\iota_\alpha: T[\pi_\alpha] \rightarrow TV_\alpha\) is the natural inclusion.

By the construction it is obvious that \(\{D_\alpha\}\) satisfies the condition 1, 2, and 3 in Definition 2.21. The condition 4 in Definition 2.21 follows from the fact that \(g\) restricted to
each fiber of \( \pi_\alpha \) is flat. We can show that for each \( \alpha \) and \( b \in U_\alpha \) the kernel of \( D_\alpha|_{\pi_\alpha^{-1}(b)} \) vanishes. It follows from Property (*) and the following lemma.

**Lemma 2.29.** Let \((E, \nabla^E) \to T\) be a flat Hermitian line bundle on an n-torus. If the degree zero cohomology \( H^0(T; E) \) with local system \((E, \nabla^E)\) vanishes, then all cohomologies \( H^\bullet(T; E) \) vanish.

**Proof.** Fix a flat Riemannian metric on \( T \). Let \((x_1, \ldots, x_n)\) be a coordinate on \( T \) such that \( \{\partial/\partial x_i\} \) forms a parallel orthonormal frame of the tangent bundle. Let \( \nabla^T \) be the Levi-Civita connection of the flat metric. Then the induced connection \( \nabla^W := \nabla^E \otimes \text{id} + \text{id} \otimes \nabla^T \) on \( W := E \otimes \wedge^k T^* T \) is a flat connection and \((W, \nabla^W)\) is a Dirac bundle in the sense of [10]. The general Bochner identity for Dirac bundles (Theorem 8.2 in [10]) implies that any harmonic section in \( \Gamma(W) \) is parallel. If \( \omega = \sum_I s_I dx_I \in \Gamma(W) \) is a harmonic section of degree \( k \), where \( I \) runs multi-indices \( I = (i_1 < \cdots < i_k) \) and \( dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k} \), then we have \( 0 = \nabla^W \omega = \sum_I (\nabla^E s_I) dx_I \), and hence, \( \nabla^E s_I = 0 \). Since \( H^0(M; E) = 0 \) we have \( s_I = 0 \) and \( \omega = 0 \).

These facts and Proposition 2.34, which will be shown in the next subsection, imply that \( \{D_\alpha\} \) is a strongly acyclic compatible system.

2.3. **Example from torus action.** Suppose an \( n \)-dimensional torus \( G \) acts on a smooth compact manifold \( \hat{M} \) smoothly and let \( M \) be a \( G \)-invariant open subset of \( \hat{M} \). Let \( A \) be the set of all the subgroups of \( G \) which appear as stabilizers

\[
G_x := \{g \in G | gx = x\}
\]

at some points \( x \in M \). Note that \( A \) is a finite set and has a partial order with respect to the inclusion.

The following lemma is useful to construct a good compatible fibration satisfying a convenient property for some cases with torus actions. We give a proof in Appendix C.

**Lemma 2.30** (Existence of a good open covering). There exists an open covering \( \{V_H\}_{H \in A} \) of \( M \) parameterized by \( A \) satisfying the following properties.

1. Each \( V_H \) is \( G \)-invariant.
2. For each \( x \in V_H \) we have \( G_x \subset H \).
3. If \( V_H \cap V_{H'} \neq \emptyset \), then we have \( H \subset H' \) or \( H \supset H' \).

Moreover if for each \( x \in M \) there is an open neighborhood \( V_x \) of \( x \) such that \( V_{g_x} = gV_x \) for all \( g \in G \), then \( \{V_H\}_{H \in A} \) satisfies \( V_H \subset \bigcup_{G_x = H} V_x \).

**Proposition 2.31.** In the above case, \( M \) has a structure of a good compatible fibration.

**Proof.** We endow \( G \) with a rational flat Riemannian metric. Precisely speaking we take a Euclidean metric on the Lie algebra of \( G \) such that the intersection of the integral lattice and the lattice generated by some orthonormal basis has rank \( n \). We extend it to the whole \( G \).

For a subgroup \( H \) of \( G \) let \( H^\perp \) be the orthogonal complement of \( H \) defined as the image of the orthogonal complement of the Lie algebra of \( H \) by the exponential map. Since the metric is rational \( H^\perp \) is well-defined as a compact subgroup of \( G \) and it has only finitely many intersection points \( H \cap H^\perp \).

Let \( \{V_H\}_{H \in A} \) be the open covering of \( M \) in Lemma 2.30. For each \( H \in A \) we define \( U_H \) to be \( V_H/H^\perp \) and \( \pi_H : V_H \to U_H \) to be the natural projection. Then the data
\{\pi_H : V_H \to U_H \mid H \in A\} define the good compatible fibration because of the property of the good covering. \qed

**Remark 2.32.** In the above proof, since the intersection \(H \cap H^\perp\) has finitely many points the quotient space \(U_H\) may have orbifold singularities. For such a case we can construct a compatible fibration in the sense of Remark 2.26 as follows.

For each \(x \in M\) let \(G_x\) be the orbit through \(x\). Let \(\{V_H\}_{H \in A}\) be the open covering of \(M\) in Lemma 2.30. For each \(H \in A\) and \(x \in V_H\) we take a slice \(U_{H,x} \subset V_H\) at \(x\) such that \(U_{H,x}\) is diffeomorphic to an open disc with dimension \(\dim M - \dim G_x\) and \(U_{H,x}\) intersects transversely with \(G_x\) at only one point \(x\). Let \(V_{H,x}\) be a union of all orbits which intersects with \(U_{H,x}\). \(V_{H,x}\) is a tubular neighborhood of \(G_x\). We put \(\tilde{V}_{H,x} := G_x^\perp \times U_{H,x}\) and denote by \(q_{H,x} : \tilde{V}_{H,x} \to V_{H,x}\) the map which is defined by the action. \(q_{H,x}\) is a finite covering. Since \(x \in V_H\), by the second property of a good open covering, we have \(G_x \subset H\). Then, by definition, we have \(H^\perp \subset G_x^\perp\). In particular, \(H^\perp\) acts freely on \(\tilde{V}_{H,x}\) by multiplication to the first factor. We define \(\tilde{U}_{H,x}\) to be the quotient space of the \(H^\perp\)-action on \(\tilde{V}_{H,x}\) and \(\pi_{H,x} : \tilde{U}_{H,x} \to \tilde{U}_H\) to be the natural projection.

Since \(M\) is a \(G\)-invariant open subset of a compact \(G\)-manifold for each \(H \in A\) we can choose finitely many points \(A(H) \subset M\) such that \(\{V_{H,a}\}_{a \in A(H)}\) covers \(V_H\). Then, \(\{q_{H,a}, \pi_{H,a} \mid H \in A, a \in A(H)\}\) is a good compatible fibration in the sense of Remark 2.26.

**Remark 2.33.** We will show in Section 6 that there is an example that has a good compatible fibration, but does not have a global torus action.

### 2.3.1. Family of flat torus bundles

Let \(\{\pi_\alpha\}_{\alpha \in A}\) be a compatible fibration on \(V\) with a compatible Riemannian metric and \(W\) a compatible Clifford module bundle on \(V\). We show the following.

**Proposition 2.34.** Suppose that an acyclic compatible system \(\{\pi_\alpha, W, D_\alpha\}\) satisfies the following four conditions.

- \(\pi_\alpha : V_\alpha \to U_\alpha\) is a good compatible fibration.
- \(\pi_\alpha : V_\alpha \to U_\alpha\) is a flat torus bundle for all \(\alpha\).
- There is a Clifford connection \(\nabla\) on \(W\) such that the restriction of \(\nabla\) on each fiber of \(\pi_\alpha\) is a flat connection for all \(\alpha\).
- \(D_\alpha\) is the Dirac operator along fibers of \(\pi_\alpha\) defined by \(\nabla|_{V_\alpha}\) for all \(\alpha \in A\).

Then for all \(\alpha, \beta \in A\) such that \(p^\alpha_\beta : U_{\alpha \beta} \to \pi_\beta(V_\alpha \cap V_\beta)\) is a diffeomorphism, the anti-commutator \(D_\alpha \circ D_\beta + D_\beta \circ D_\alpha\) is a non-negative operator along fibers of \(\pi_\beta\). In particular if \(\ker D_\alpha = 0\) for all \(\alpha \in A\), then \(\{\pi_\alpha, W, D_\alpha\}\) is strongly acyclic.

It suffices to show the non-negativity of the anti-commutator along fibers. We consider the following model.

- \(E\) : Euclidean space
- \(\Gamma\) : maximal lattice of \(E\)
- \(F := E/\Gamma\) : flat torus
- \(W \to F\) : \(Cl(TF)\)-module bundle
- \(\nabla : \Gamma(W) \to \Gamma(TF \otimes W)\) : flat Clifford connection of \(W\)
- \(c : TF \otimes W \to W\) : Clifford action of \(TF\)
- \(A\) : finite set
- \(\{E_\alpha\}_{\alpha \in A}\) : family of subspaces of \(E\)
• \{p_\alpha\} : family of orthogonal projections to \{E_\alpha\}
• We assume \(p_\alpha p_\beta = p_\beta p_\alpha\) for all \(\alpha, \beta \in A\).

Using the metric we have the identification \(TF = T^*F = F \times E\). For a symmetric endomorphism \(S : E \to E\) let \(\hat{S} : F \times E \to F \times E\) be the induced bundle map on the (co)tangent bundle. We define a differential operator \(D_S\) by the composition

\[
D_S := c \circ \hat{S} \circ \nabla : \Gamma(W) \to \Gamma(W).
\]

Since \(S\) is symmetric \(D_S\) is a self-adjoint operator. Let \(S_1\) and \(S_2\) be symmetric endomorphisms which commute each other. By the direct computation using the orthonormal basis of \(E\) consisting of simultaneously eigenvectors of \(S_1\) and \(S_2\) we have the following equality

\[
D_{S_1} \circ D_{S_2} + D_{S_2} \circ D_{S_1} = 2\nabla^* \circ \hat{S}_1 \circ \hat{S}_2 \circ \nabla,
\]

where \(\nabla^* : \Gamma(TF \otimes W) \to \Gamma(W)\) is the adjoint operator of \(\nabla\).

In particular, by putting \(S_1 := p_\alpha\) and \(S_2 := p_\beta\) we have the following.

**Lemma 2.35.** \(D_\alpha D_\beta + D_\beta D_\alpha = 2D_{\alpha \beta}^2\), where \(D_{\alpha \beta}\) is the self-adjoint operator \(c \circ \hat{p}_{\alpha \beta} \circ \nabla\) defined by the projection \(p_{\alpha \beta}\) to the intersection \(E_\alpha \cap E_\beta\).

**Proof of Proposition 2.34.** Since \(\{\pi_\alpha\}\) is a good compatible fibration we have a family of tori at each point on \(V\) which comes from a family of subspaces whose projections commute each other. Then the claim follows from Lemma 2.35.

Note that a product of good compatible fibrations is not a good compatible fibration. To formulate a product formula, we will need the following Proposition 2.36 which is a generalization of Proposition 2.34. We prepare a notation.

• Suppose \(\pi_\alpha : V_\alpha \to U_\alpha\) is a torus bundle with structure group \(GL_n(\mathbb{Z}) \times \mathbb{R}^n/\mathbb{Z}^n\).
• For each \(x \in V_\alpha\) and \(v \in T[\pi_\alpha]_x\), we have a vector field on the fiber \(\pi_\alpha^{-1}\pi_\alpha(x)\) which is parallel on the torus and is extending \(v\).
• Using a local trivialization of the torus bundle \(\pi_\alpha : V_\alpha \to U_\alpha\) on a neighborhood of the fiber \(\pi_\alpha^{-1}\pi_\alpha(x)\), we extend the vector field to a vector field \(X_\alpha(v)\) on the neighborhood.

**Proposition 2.36.** Suppose that an acyclic compatible system \(\{\pi_\alpha, W, D_\alpha\}\) satisfies the following three conditions.

• For \(x \in V_\alpha \cap V_\beta, v_\alpha \in T[\pi_\alpha]_x\) and \(v_\beta \in T[\pi_\beta]_x, we have \([X_\alpha(v_\alpha), X_\beta(v_\beta)] = 0\). Moreover, if \(v_\alpha = v_\beta\), then, \(X_\alpha(v_\alpha)\) and \(X_\beta(v_\beta)\) coincide on a neighborhood of \(x \in M\).
• There is a Clifford connection \(\nabla\) on \(W\) such that the restriction of \(\nabla\) on each fiber of \(\pi_\alpha\) is a flat connection for all \(\alpha\).
• \(D_\alpha\) is the Dirac operator along fibers of \(\pi_\alpha\) defined by \(\nabla|_{V_\alpha}\) for all \(\alpha \in A\).

Then for all \(\alpha, \beta \in A\) such that \(p_{\alpha \beta}^\beta : U_{\alpha \beta} \to \pi_\beta(V_\alpha \cap V_\beta)\) is a diffeomorphism, the anti-commutator \(D_\alpha \circ D_\beta + D_\beta \circ D_\alpha\) is a non-negative operator along fibers of \(\pi_\beta\). In particular if \(\ker D_\alpha = 0\) for all \(\alpha \in A\), then \(\{\pi_\alpha, W, D_\alpha\}\) is strongly acyclic.

**Proof.** We can use the same local model used in the proof of Proposition 2.34. □
2.3.2. Symplectic manifold with a torus action. Let \((M, \omega)\) be a \(2m\)-dimensional symplectic manifold equipped with a Hamiltonian action of an \(n\)-dimensional torus \(G\). In this case each orbit is an affine isotropic torus in \(M\). Suppose that there is a \(G\)-equivariant prequantizing line bundle \((L, \nabla)\) on \((M, \omega)\), i.e., \(L\) is a Hermitian line bundle over \(M\) with a Hermitian connection \(\nabla\) whose curvature form is equal to \(-2\pi \sqrt{-1}\omega\), and all these data are \(G\)-equivariant. Since an orbit is isotropic the restriction of \((L, \nabla)\) on each orbit is a flat line bundle. Let \(A = \{H\}\) be the set of all subgroups of \(G\) which appear as a stabilizer subgroup of \(G\)-action at some point in \(M\). We assume that \(A\) is a finite set. We show the following.

**Proposition 2.37.** Suppose that for each \(x \in M\) there exists an open set \(V_x\) such that for all \(y \in V_x\) the restriction of \(L\) on \(G_x^+\)-orbit \(G_x^+ \cdot y\) has no nontrivial parallel sections. Then \(M\) is equipped with a strongly acyclic compatible system parametrized by \(A\).

**Remark 2.38.** The assumption of Proposition 2.37 implies that there is no global non-zero parallel section on each \(G\)-orbit. However the converse does not hold in general. On the other hand in the case of Hamiltonian torus action, the moment map image of a \(G\)-orbit has a global non-zero parallel section on it is a lattice point in the dual of the Lie algebra. In this case it may be expected that there would exist a strongly acyclic compatible system on the complement of the inverse image of lattice points. If it were the case, we would have a localization of the Riemann-Roch number as a sum of contributions from lattice points. Proposition 2.37, however, does not realize this expectation. Actually the complement of the inverse image of the lattice points does not satisfy the assumption of Proposition 2.37 in general. We would need to delete neighborhoods of some extra points from the complement to make it satisfy the assumption. In fact it is not hard to check that the contributions from the extra points to the Riemann-Roch number turn out to be zero, for instance, in the case of toric action, which implies that the localization to the lattice points is practically realized.

**Proof of Proposition 2.37.** By taking the union along \(G\)-orbits we may assume that each \(V_x\) is \(G\)-invariant. According to Lemma 2.30 and Remark 2.31 we have a good compatible fibration on \(M\) using a good open covering \(\{V_H\}\) parameterized by \(A = \{H\}\). Note that each \(V_H\) is contained in \(\cup_{G_x = H} V_x\). Fix a \(G\)-invariant \(\omega\)-compatible almost complex structure \(J\) on \(M\). Then using the associated \(G\)-invariant metric \(g_J\) we can construct a compatible Riemannian metric on \(M\) as follows. It is sufficient to construct metrics of \(T[\pi_H]\) for each \(H \in A\) and \(T[p_{HK}]\) for a sequence \(H \subset K\), and check the compatibility coming from (2) and (3). The metric on \(T[\pi_H]\) is defined as the restriction of \(g_J\), and the metric on \(TU_H = T(V_H/H^\perp)\) is defined as the quotient metric of \(g_J\). Note that the fiber of \(p_{HK}\) is the quotient of the \(H^\perp\)-orbit by the \(K^\perp\)-orbit. Then the metric on \(T[p_{HK}]\) is defined as the quotient of the restriction of the metric \(g_J\) on \(H^\perp\)-orbit by the \(K^\perp\)-action. It is straightforward to check the compatibilities. We remark that the metric \(g_J\) restricted to each orbit is a flat affine metric because it is \(G\)-invariant.

Let \(W\) be the \(\mathbb{Z}/2\)-graded compatible Clifford module bundle

\[
W = \bigwedge^* TM \otimes L
\]

with Clifford module structure \(c: Cl(TM) \to \text{End}(W)\) defined by (6). We show that \(W\) is a compatible Clifford module bundle with some more additional structures. For \(H \in A\), let \((T[\pi_H] \oplus JT[\pi_H])^\perp\) be the orthogonal complement of \(T[\pi_H] \oplus JT[\pi_H]\) with respect
to \(g_J\). Since \((g_J, J)\) is \(G\)-invariant \(H^+\)-action preserves \((T[\pi_H] \oplus JT[\pi_H])^\perp\) and \(JT[\pi_H]\). Then we define

\[
E_H := (T[\pi_H] \oplus JT[\pi_H])^\perp / H^\perp,
\]

\[
F_H := JT[\pi_H] / H^\perp.
\]

It is obvious that \(TU_H\) has the natural isometry \(TU_H \cong E_H \oplus F_H\).

Since \(g_J\) is invariant under \(J\), \(J\) preserves \((T[\pi_H] \oplus JT[\pi_H])^\perp\). In particular \(T[\pi_H] \oplus JT[\pi_H]\) and \((T[\pi_H] \oplus JT[\pi_H])^\perp\) have the structures of Hermitian vector bundles with respect to the restriction of \((g_J, J)\) to them. Moreover \((g_J, J)\) is \(G\)-invariant it descends to the Hermitian structure on \(E_H\). Then, we define \(W_{1,H}\) and \(W_{2,H}\) by

\[
W_{1,H} := \Lambda^+_C(T[\pi_H] \oplus JT[\pi_H]) C \otimes L,
\]

\[
W_{2,H} := \Lambda^+_C(E_H) C \otimes L,
\]

and define the Clifford module structures

\[
c_{1,H} : Cl(T[\pi_H] \oplus \pi^*_H F_H) \to \text{End} (W_{1,H}) ,
\]

\[
c_{2,H} : Cl (E_H) \to \text{End} (W_{2,H})
\]

by the same formula as in (6). By definition, \(TV_H\) has a decomposition

\[
TV_H = (T[\pi_H] \oplus \pi^*_H F_H) \oplus \pi^*_H E_H
\]

as Hermitian vector bundles. With respect to this decomposition there are the following isomorphisms

\[
\text{Cl}(TM)|_{V_H} \cong \text{Cl}(T[\pi_H] \oplus \pi^*_H F_H) \otimes \text{Cl}(\pi^*_H E_H),
\]

\[
\cong \text{Cl}(T[\pi_H] \oplus \pi^*_H TU_H)
\]

\[
W|_{V_H} \cong W_{1,H} \otimes \pi^*_H W_{2,H}.
\]

Then by the direct calculation one can check \(c = c_{1,H} \otimes c_{2,H}\) under the above identifications.

Now we define a strongly acyclic compatible system \(\{D_H\}\) on \(W\). Let \(\nabla^{T[\pi_H]} : \Gamma(TV_H) \rightarrow \Gamma(T^*[\pi_H] \otimes TV_H)\) be the the family of Levi-Civita connections along fibers of \(\pi_H\), namely,

\[
\nabla^{T[\pi_H]} = \iota^*_H \otimes q_H \circ \nabla^{TM} \circ q_H,
\]

where \(\iota_H : T[\pi_H] \rightarrow TV_H\) is the natural inclusion, \(\nabla^{TM}\) is the Levi-Civita connection on \(TM\) with respect to \(g_J\), and \(q_H : TV_H \rightarrow TV_H\) is the orthogonal projection to \(T[\pi_H]\) with respect to \(g_J\). \(\nabla^{T[\pi_H]}\) induces the family of Hermitian connections on \(\Lambda^+_C TM_C|_{V_H}\) along fibers of \(\pi_H\), which is denoted by \(\nabla^{\Lambda^+_C TM_C|_{V_H}}\). We define the family of Hermitian connections \(\nabla^H\) on \(W|_{V_H}\) along fibers of \(\pi_H\) by

\[
\nabla^H := \nabla^{\Lambda^+_C TM_C|_{V_H}} \otimes \text{id} \otimes \big( \iota^*_H \otimes \text{id} \circ \nabla^L \big) : \Gamma(W|_{V_H}) \rightarrow \Gamma(T^*[\pi_H] \otimes W|_{V_H}).
\]

Then we define \(D_H : \Gamma(W|_{V_H}) \rightarrow \Gamma(W|_{V_H})\) to be the family of de Rham operators along fibers of \(\pi_H\) which is defined by

\[
D_H := c_{1,H} \circ p_H \circ \nabla^H,
\]

where \(p_H : T^*[\pi_H] \rightarrow T[\pi_H]\) is the isomorphism via \(g_J\).

The restriction of \(L\) on each \(H^+\)-orbit has no non-trivial parallel sections because of the property \(V_H \subset \cup_{G_e = H} V_e\) and our assumption. It implies \(\ker D_H = 0\) by Lemma 2.29. Moreover since the collection of data \(\{\pi_H : V_H \rightarrow U_H, W, D_H\}\) satisfies the assumptions in Proposition 2.34 \(\{D_H\}\) is strongly acyclic. \(\Box\)
Remark 2.39. In the above proof we used the isomorphism between $\mathbb{Z}/2$-graded Clifford algebras

$$Cl(T_1 \oplus T_2) \cong Cl(T_1) \otimes Cl(T_2)$$

for two real vector bundles $T_1$ and $T_2$ with inner products. Our convention is as follows. Let $A = A^0 \oplus A^1$ and $B = B^0 \oplus B^1$ be two $\mathbb{Z}/2$-graded algebras. Then we define a structure of a $\mathbb{Z}/2$-graded algebra on the tensor product $A \otimes B$ as follows. The $\mathbb{Z}/2$-grading is defined as the one for vector spaces,

$$A \otimes B = ((A^0 \otimes B^0) \oplus (A^1 \otimes B^1)) \oplus ((A^0 \otimes B^1) \oplus (A^1 \otimes B^0)).$$

The multiplicative structure is defined by

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{\deg b \cdot \deg a'} (aa') \otimes (bb'),$$

where $a, a' \in A^0 \cup A^1$ and $b, b' \in B^0 \cup B^1$. Now let $R_A$ and $R_B$ be $\mathbb{Z}/2$-graded $A$ and $B$ modules respectively. Then we define a structure of a $\mathbb{Z}/2$-graded $A \otimes B$-module on the tensor product $R_A \otimes R_B$ by the following formula.

$$(a \otimes b) \cdot (r_A \otimes r_B) = (-1)^{\deg b \cdot \deg r_A} (ar_A \otimes br_B),$$

where $a \in A^0 \cup A^1$, $b \in B^0 \cup B^1$, $r_A \in R_A^0 \cup R_A^1$ and $r_B \in R_B^0 \cup R_B^1$.

3. An index theory for complete Riemannian manifolds

3.1. Formulation of index on complete manifolds. Suppose $M$ is a complete Riemannian manifold, $W$ is a $\mathbb{Z}/2$-graded Hermitian vector bundle, and $\sigma : TM \to \text{End}(W)$ is a homomorphism such that $\sigma(v)$ is a skew-Hermitian isomorphism of degree-one for each $v \in TM \setminus \{0\}$. Let $D$ be a degree-one formally self-adjoint first-order elliptic differential operator on $W$ with principal symbol $\sigma$. We assume that $\sigma$ and the coefficients of $D$ are smooth. We formulate an index theory on $M$ under the following assumption.

Assumption 3.1. \begin{itemize}
    \item $D$ has finite propagation speed: there exists a positive real number $C_0$ satisfying $|\sigma| \leq C_0$ uniformly on $M$,
    \item There exist a positive real number $\lambda_0 > 0$ and an open subset $V$ of $M$ with its complement $M \setminus V$ compact such that
    $$\lambda_0 ||s||_V^2 \leq ||Ds||_V^2$$
    for any smooth compactly-supported section $s$ of $W$ with support contained in $V$.
\end{itemize}

It is known that the finite propagation speed implies that $D$ is essentially self-adjoint. We will give a direct proof of the following theorem.

Theorem 3.2. $D$ is essentially self-adjoint as an operator on $L^2$-sections of $W$ and its spectrum is discrete in $(-\sqrt{\lambda_0}, \sqrt{\lambda_0})$. \hfill $\Box$

The proof of the first part is given as Lemma 3.11 in Section 3.3. The rest of the statement follows from Proposition 3.13 in Section 3.3.

Definition 3.3. $E_\lambda$ is the vector space of smooth sections $s$ of $W$ such that $s$ is $L^2$-bounded and satisfies $D^2s = \lambda s$.

Theorem 3.2 implies that $E_\lambda$ is zero for $\lambda < 0$, and $E_\lambda$ is finite dimensional for $\lambda < \lambda_0$. Moreover there are only discrete values $\lambda < \lambda_0$ for which $E_\lambda$ is non-zero. Note that the super dimension of $E_\lambda$ is zero for $0 < \lambda < \lambda_0$, and hence the super dimension of $\oplus_{\lambda < \lambda_0} E_\lambda$ is constant for $0 < \lambda_1 < \lambda_0$. 

**Definition 3.4.** \( \text{ind } D \) is the super dimension of \( E_0 \), or the super dimension of \( \oplus_{\lambda < \lambda_1} E_\lambda \) for \( 0 < \lambda_1 < \lambda_0 \).

The index has the following deformation invariance. Let \( \{ D_t \} (|t| < \epsilon) \) be a one-parameter family of degree-one formally self-adjoint first-order elliptic differential operators on \( W \) with principal symbols \( \{ \sigma_t \} \).

**Assumption 3.5.**
- Each \( D_t \) and \( \sigma_t \) satisfy Assumption 3.1 for common \( \lambda_0 \) and \( V \).
- On each compact subset of \( M \) the coefficients of \( D_t \) are \( C^\infty \) convergent to those of \( D_0 \) as \( t \to 0 \).

We do not assume that the propagation speed is uniform with respect to \( t \).

We will show the following theorem in Section 3.4.

**Theorem 3.6.** Under Assumption 3.5 \( \text{ind } D_t \) is constant with respect to \( t \).

**Remark 3.7.** So far we are fixing \( M \) and \( W \). In Section 3.5 we will formulate a deformation for which \( M \) and \( W \) can vary. We give a proof of Theorem 3.6 so that it can be directly generalized to this case. The generalization immediately implies an excision property of index for complete Riemannian manifolds.

### 3.2. Partial integration.

We need two partial integration formulas. In general let \( W \) be a Hermitian vector bundle over a complete Riemannian manifold \( M \), and \( D_\tau : \Gamma(W) \to \Gamma(W) \) be a first order partial differential operator on \( W \) with smooth coefficients whose principal symbol is \( \tau \). We assume that \( D_\tau \) has finite propagation speed, i.e., \( \tau \) is a smooth \( L^\infty \)-bounded section of \( TM \otimes \text{End}(W) \).

**Lemma 3.8.** Let \( s \in \Gamma(W) \) is an \( L^2 \)-bounded section such that \( D_\tau^* D_\tau s \) is also \( L^2 \)-bounded. Then \( D_\tau s \) is also \( L^2 \)-bounded and we have

\[
\int_M (D_\tau^* D_\tau s, s) = \int_M |D_\tau s|^2.
\]

**Lemma 3.9.** Suppose \( s_0 \) and \( s_1 \) are \( L^2 \)-bounded sections of \( W \) such that \( D_\tau s_0 \) and \( D_\tau^* s_1 \) are also \( L^2 \)-bounded. Then we have

\[
\int_M (D_\tau s_0, s_1) = \int_M (s_0, D_\tau^* s_1).
\]

We follow the argument in [8] using a family of cut-off functions:

**Lemma 3.10.** Let \( M \) be a complete Riemannian manifold.

1. There is a smooth proper function \( f : M \to \mathbb{R} \) such that \( |df| \) is bounded and \( f^{-1}((\infty, c]) \) is compact for any \( c \).
2. There is a constant \( C > 0 \) such that for each \( \epsilon > 0 \) and \( a \in \mathbb{R} \), we have a compact supported function \( \rho_{a, \epsilon} : M \to [0, 1] \) which is equal to 1 on \( f^{-1}((\infty, a]) \), and satisfies \( |d\rho_{a, \epsilon}| < C\epsilon \).

A proof of the above lemma is given in [7]. For completeness we give a detailed construction in Appendix [D]. The existence of such a function in 1 of Lemma 3.10 is equivalent to the completeness of \( M \). For more details see [7].

If we choose a Hermitian connection \( \nabla \) on \( W \), we can describe \( D_\tau s = \tau \cdot \nabla s + Bs \), where \( B \) is a smooth section of \( \text{End}(W) \) and \( \cdot \) is the combination of the product \( \text{End}(W) \otimes W \to W \) and the contraction \( T^* M \otimes TM \to \mathbb{R} \). We do not assume that \( B \) is bounded.
Lemma 1.1 B. From the equality
\[ \int_M (D^*_\tau D_\tau s, \rho^2_{a,\epsilon} s) = \int_M (D_\tau s, D_\tau (\rho^2_{a,\epsilon} s)) \]
(7)
\[ = \int_M (D_\tau s, \rho^2_{a,\epsilon} D_\tau s) + \int_M (D_\tau s, 2\rho_{a,\epsilon} \tau (d\rho_{a,\epsilon}) s), \]
there is a constant \( C \) independent of \( s, a, \epsilon \) such that
\[ ||D^*_\tau D_\tau s||_2 ||s||_2 \geq ||\rho_{a,\epsilon} D_\tau s||^2_2 - C\epsilon ||\rho_{a,\epsilon} D_\tau s||_2. \]
It implies that, as \( \epsilon \) increases, \( ||\rho_{a,\epsilon} D_\tau s||_2 \) is bounded, i.e., \( D_\tau s \) is \( L^2 \)-bounded. Using (7) again we have
\[ \int_M (D^*_\tau D_\tau s, s) = ||D_\tau s||^2_2 + I, \quad |I| \leq C\epsilon ||D_\tau s||_2 ||s||_2. \]
Taking \( \epsilon \to 0 \), we obtain the required equality.

When \( s \) is not smooth, take a smooth compactly supported section which approximate \( s \) in \( L^2_2 \)-norm on the support of \( \rho_{a,\epsilon} \). Then we can reduce the argument to the smooth case.

Proof of Lemma 3.8. We first assume that \( s \) is smooth. We have
\[ 0 = \int_M (D_\tau (\rho_{a,\epsilon} s_0), s_1) - \int_M (s_0, D^*_\tau (\rho_{a,\epsilon} s_1)) = \int_M (D_\tau s_0, s_1) - \int_M (s_0, D^*_\tau s_1) + I' \]
with an error term \( I' \) satisfying \( |I'| \leq C\epsilon ||s_0||_2 ||s_1||_2 \), which implies the required equality. When \( s \) is not smooth, we can reduce the argument to the smooth case as in the proof of Lemma 3.8.

Using the cut off function and a standard argument we can also show:

Lemma 3.11. Under Assumption \[3.7\] \( D \) is essentially self-adjoint.

Proof of Lemma 3.11 Suppose \( L^2 \)-sections \( u \) and \( v \) satisfies \( Du = v \) weakly. We show that for any \( \epsilon > 0 \) there is a compactly supported smooth section \( u_\epsilon \) satisfying \( ||u_\epsilon - u||_M \leq 2\epsilon \) and \( ||Du_\epsilon - v||_M \leq 4\epsilon \). Take a compact subset \( K \) such that \( ||u||_{M \setminus K}, ||v||_{M \setminus K} < \epsilon \). Using Lemma 3.10 choose a smooth compactly-supported function \( \rho : M \to [0,1] \) satisfying \( \rho = 1 \) on \( K \) and \( |d\rho| < 1 \). Let \( K' \) be the compact support of \( \rho \). Since \( D \) is elliptic, the weak equality \( Du = v \) and the regularity theorem imply that \( u \) is locally \( L^2_1 \)-bounded and there is a smooth sections \( u' \) satisfying \( ||u' - u||_{K'} < \epsilon \) and \( ||Du' - v||_{K'} < \epsilon \). Then we have
\[ ||\rho u' - u||_M \leq ||\rho(u' - u)||_{K'} + ||(1 - \rho)u||_{M \setminus K} \leq 2\epsilon \]
and
\[ ||D(\rho u') - v||_M = ||d\rho \cdot (u' - u) + d\rho \cdot u + \rho(Du - v) - (1 - \rho)v||_M \]
\[ \leq ||u' - u||_{K' \setminus K} + ||u||_{K' \setminus K} + ||Du' - v||_{K'} + ||v||_{M \setminus K} \]
\[ \leq 4\epsilon. \]
It implies that \( D \) is essentially self-adjoint.
3.3. Min-max principle. In this section we use Assumption 3.1 for a single operator \( D \), and Assumption 3.5 for a one-parameter family \( \{D_t\} \).

**Lemma 3.12.** For any compact subset \( K \) containing \( M \setminus V \) there is a compact set \( K' \) containing \( K \) such that if \( s \) and \( Ds \) are \( L^2 \)-bounded, then we have the estimate

\[
\lambda_0^{1/2} ||s||_{M \setminus K} - 2\lambda_0^{1/2} ||s||_{K' \setminus K} \leq ||Ds||_{M \setminus K}.
\]

Moreover if the coefficients of \( D_t \) are \( C^\infty \)-convergent to those of \( D_0 = D \) on any compact set as \( t \to 0 \), then we can choose \( K' \) so that the above estimate is valid for any \( t \) sufficiently close to 0.

**Proof.** Lemma 3.11 implies that we can assume that \( s \) is smooth and compactly supported without loss of generality. From Lemma 3.10 for any \( \varepsilon > 0 \), there is a compact set \( K' \) containing \( K \) and a smooth non-negative function \( \rho : M \to \mathbb{R} \) such that \( \rho = 1 \) on \( M \setminus K' \), \( \rho = 0 \) on \( K \) and \( |d\rho| \leq \varepsilon \). Then the above estimate follows from the next two inequalities

\[
||D(\rho s)||_M \geq \lambda_0^{1/2} ||\rho s||_V \geq \lambda_0^{1/2} ||s||_{M \setminus K} - \lambda_0^{1/2} ||s||_{K' \setminus K},
\]

\[
||D(\rho s)||_M \leq ||\rho Ds||_M + ||\sigma \cdot ((d\rho) \otimes s)||_M \leq ||Ds||_{M \setminus K} + C(D, K')\varepsilon ||s||_{K' \setminus K},
\]

where \( C(D, K') := \max_K |\sigma| \). The last statement of the lemma follows from the fact that \( C(D_t, K') \) is continuous with respect to \( t \). \( \square \)

**Proposition 3.13.** Suppose \( 0 \leq \lambda_1 < \lambda_0 \). Let \( \{s_i\} \) be a sequence of \( L^2 \)-sections of \( W \) satisfying \( ||s_i||_M = 1 \), and \( \{t_i\} \) is a sequence convergent to 0. Suppose each \( D_{t_i}s_i \) is \( L^2 \)-bounded and satisfies \( ||D_{t_i}s_i||_M^2 \leq \lambda_1 \). Then there is a subsequence \( \{s_{i'}\} \) which is weakly convergent to some non-zero \( s_\infty \neq 0 \) such that \( D_0s_\infty \) is \( L^2 \)-bounded and satisfies

\[
(8) \quad ||D_0s_\infty||_M^2 \leq \lambda_1 ||s_\infty||_M^2.
\]

**Proof.** Take a subsequence \( \{s_{i'}\} \) so that \( \{s_{i'}\} \) and \( \{D_{t_i}s_{i'}\} \) are weakly convergent to some \( s_\infty \) and \( u_\infty \) in \( L^2(M, W) \) respectively. Since \( D_t \) is a smooth family, for each smooth compactly-supported section \( \phi \) the sequence \( D_{t_{i'}} \phi \) is strongly convergent to \( D_0\phi \). The equality \( \int_M (D_{t_{i'}} \phi, s_{i'}) = \int_M (\phi, D_{t_{i'}} s_{i'}) \) implies \( \int_M (D_0\phi, s_\infty) = \int_M (\phi, u_\infty) \), i.e., \( D_0s_\infty = u_\infty \) weakly.

Since \( \{D_{t_i}s_{i'}\} \) is \( L^2 \)-bounded, Assumption 3.5 and a priori estimate imply that on any compact set \( s_{i'} \) is strongly \( L^2 \)-convergent to \( s_\infty \).

On the other hand for any compact set \( K \) containing \( M \setminus V \) there exists a compact set \( K' \) such that

\[
\lambda_0^{1/2} ||s_{i'}||_{M \setminus K} - 2\lambda_0^{1/2} ||s_{i'}||_{K' \setminus K} \leq ||D_{t_{i'}}s_{i'}||_{M \setminus K} \leq \lambda_1^{1/2}
\]

by Lemma 3.12. If \( s_\infty \) is 0, then we have \( ||s_{i'}||_{K' \setminus K} \) converges to 0, which contradicts to \( ||s_{i'}||_M = 1 \) and \( \lambda_1 < \lambda_0 \).

Suppose the estimate (8) does not hold. Then for any \( \varepsilon > 0 \) and any sufficiently small \( \varepsilon' > 0 \) there exists a compact set \( K \) containing \( M \setminus V \) satisfying \( ||s_\infty||_{M \setminus K} < \varepsilon \) and \( \lambda_1 ||s_{i'}||_K^2 + \varepsilon' < ||D_0s_\infty||_K^2 \). We choose \( \varepsilon \) and \( \varepsilon' \) so that they satisfy \( 8\varepsilon \lambda_0 (1 + 2\varepsilon) < \varepsilon'/2 \). Note that the weak convergence implies \( ||D_0s_\infty||_K^2 \leq \liminf_{t_{i'} \to \infty} ||D_{t_{i'}}s_{i'}||_K^2 \). Since \( s_{i'} \) is strongly \( L^2 \)-convergent to \( s_\infty \) on the compact set \( K \), we have

\[
(9) \quad \lambda_1 ||s_{i'}||_K^2 + \frac{\varepsilon'}{2} < ||D_{t_{i'}}s_{i'}||_K^2
\]
for sufficiently large $i'$. Let $K'$ be the compact set containing $K$ which gives the estimate in Lemma 3.12 for sufficiently small $t$. Since $s_r$ is strongly $L^2$-convergent to $s_\infty$ on the pre-compact set $K' \setminus K$, we have $\|s_r\|_{K \setminus K} < 2\epsilon$ for sufficiently large $i'$. The estimate in Lemma 3.12 implies that we have $\lambda_0^{1/2}\|s_r\|_{M \setminus K} \leq \|Ds_r\|_{M \setminus K} + 4\epsilon\lambda_0^{1/2}$ for sufficiently large $i'$. Taking square, and using $\lambda_1 < \lambda_0$ and $\|Ds_r\|_{M \setminus K} \leq \lambda_1^{1/2}$, we obtain

$$\lambda_1\|s_r\|^2_{M \setminus K} \leq \|Ds_r\|_{M \setminus K}^2 + 8\epsilon\lambda_0(1 + 2\epsilon)$$

Adding with (9) we have $\lambda_1\|s_r\|^2_{M} < \|Ds_r\|_{M}^2$ for sufficiently large $i'$, which contradicts our assumption. \qed

For a single operator $D$ we have

**Proposition 3.14.**

(1) Suppose $\lambda < \lambda' < \lambda_0$.

(a) If $s \in E_\lambda$, then $Ds$ is $L^2$-bounded and $\|Ds\|^2_M = \lambda\|s\|^2_M$.

(b) $E_\lambda$ and $E_{\lambda'}$ are $L^2$-orthogonal to each other.

(2) Suppose $0 \leq \lambda_1 < \lambda_0$.

(a) $\dim \oplus_{\lambda \leq \lambda_1} E_\lambda(D) < \infty$.

(b) Let $R_{\lambda_1}$ be the set of $L^2$-bounded sections $s$ satisfying $\|s\|_M = 1$ and $\|Ds\|^2_M < \lambda_0$ such that $s$ is $L^2$-orthogonal to $\oplus_{\lambda \leq \lambda_1} E_\lambda(D)$. If $R_{\lambda_1}$ is not empty, then the functional $I_{\lambda_1} : R_{\lambda_1} \to [0, \lambda_0)$, $I_{\lambda_1}(s) = \|Ds\|^2_M$ attains its minimum value.

(c) Let $\lambda_2 = \|Ds_0\|^2_M$ be the minimum value of $I_{\lambda_1}$ at a minimum $s_0$. Then we have $\lambda_1 < \lambda_2 < \lambda_0$ and $s_0 \in E_{\lambda_2}$.

**Proof.** The first statement for $\lambda < \lambda' < \lambda_0$ follows from the partial integration formulas Lemma 3.8 and Lemma 3.9.

Suppose $\lambda_1 < \lambda_0$. If $\oplus_{\lambda \leq \lambda_1} E_\lambda(D)$ is not finite dimensional, then we have a sequence $e_i$ in the infinite space with $\|e_i\|_M = 1$ and mutually $L^2$-orthogonal each other. Proposition 3.13 implies that we have a weakly convergent limit for a subsequence with non zero limit, which is a contradiction.

Suppose $s_i$ is a sequence in $R_{\lambda_1}$ such that $I_{\lambda_1}(s_i)$ convergent to the infimum of $I_{\lambda_1}$. Proposition 3.13 implies that we have a weakly convergent limit $s_\infty \neq 0$ for a subsequence such that $s_0 := s_\infty/\|s_\infty\|$ is an element of $R_\lambda$ which attains the infimum. For any compactly-supported smooth section $s'$, let $s''$ be the $L^2$-orthogonal projection of $s'$ to $\oplus_{\lambda \leq \lambda_1} E_\lambda(D)$ and put $s''' = s' - s''$. Since $s_0$ attains the minimum of $I_{\lambda_1}$, the derivative of $(s_0 + ts''')(\|s_0 + ts'''\|_M$ at 0 vanishes, and we obtain

$$\int_M (Ds_0, Ds''') = \lambda_2 \int_M (s_0, s''') .$$

Since $Ds''' = Ds' - Ds''$ and $D^2s''' = D^2s' - D^2s''$ is $L^2$-bounded, Lemma 3.9 implies

$$\int_M (s_0, D^2s''') = \lambda_2 \int_M (s_0, s''').$$

On the other hand we have

$$\int_M (s_0, D^2s') = \lambda_2 \int_M (s_0, s').$$

These relations imply

$$\int_M (s_0, D^2s') = \lambda_2 \int_M (s_0, s'),$$

$$\int_M (s_0, D^2s'') = \lambda_2 \int_M (s_0, s''),$$

$$\int_M (s_0, D^2s''') = \lambda_2 \int_M (s_0, s''').$$
i.e., $D^2 s_0 = \lambda_2 s_0$ weakly. Lemma 3.8 implies that $D s_0$ is $L^2$-bounded and $\|D s_0\|_M^2 = \lambda_2 \|s_0\|_M = \lambda_2$. Since $s_0$ is $L^2$-orthogonal to $\oplus_{\lambda < \lambda_1} E_{\lambda}$, we have $\lambda_1 < \lambda_2$. The regularity theorem implies $s_0$ is smooth and hence $s_0 \in E_{\lambda_2}$.

**Corollary 3.15.** Suppose $\lambda_1 < \lambda_0$. Let $E$ be a $\mathbb{Z}/2$ graded subspace of $L^2(M, W)$ such that such that $D s$ is in $L^2(M, W)$ and $\|D s\|_M^2 \leq \lambda_1 \|s\|_M^2$ for any $s \in E$. Then $E$ is finite dimensional and

$$\dim \oplus_{\lambda \leq \lambda_1} E_{\lambda}(D) \geq \dim E.$$  

Moreover the above inequality holds for each degree of $\mathbb{Z}/2$.

### 3.4. Deformation invariance of index.

For a family $\{D_i\}$ we have:

**Proposition 3.16.** Suppose $\lambda_1 < \lambda_0$. Let $\{t_i\}$ be a sequence convergent to 0 and $E^{(i)}$ be a $\mathbb{Z}/2$ graded subspace of $L^2(M, W)$ such that $D_{t_i} s$ is in $L^2(M, W)$ and $\|D_{t_i} s\|_M^2 \leq \lambda_1 \|s\|_M^2$ for any $i$ and $s \in E_i$. Then each $E^{(i)}$ is finite dimensional and

$$\dim \oplus_{\lambda \leq \lambda_1} E_{\lambda}(D_0) \geq \limsup_{i \to \infty} \dim E^{(i)}.$$  

Moreover the above inequality holds for each degree of $\mathbb{Z}/2$.

**Proof.** Suppose dim $\oplus_{\lambda \leq \lambda_1} E_{\lambda}(D_0) < \dim E^{(i)}$ for a subsequence $\{i'\}$. Let $s_{i'}$ be an element of $E^{(i')}$ with $\|s_{i'}\|_M = 1$ which is $L^2$-orthogonal to $\oplus_{\lambda \leq \lambda_1} E_{\lambda}(D_0)$. Let $s_{i\infty}$ be the $L^2$-bounded section given by Proposition 3.13. Then the weak limit $s_{i\infty}$ is also $L^2$ orthogonal to $\oplus_{\lambda \leq \lambda_1} E_{\lambda}(D_0)$, which contradicts Proposition 3.14.

**Remark 3.17.** In the above proof the choice of $s_{i'}$ can be generalized as follows: Fix an $L^2$-orthonormal basis $e_1, e_2, \ldots, e_N$ of the finite dimensional space $\oplus_{\lambda \leq \lambda_1} E_{\lambda}(D_0)$. Fix any sequence $\{e_k^{(i')}\}$ for each $1 \leq k \leq N$ which is strongly $L^2$-convergent to $e_k$ as $i' \to \infty$. Let $s_{i'}$ be an element of $E^{(i')}$ with $\|s_{i'}\|_M = 1$ which is $L^2$-orthogonal to all $e_k^{(i')}$ ($1 \leq k \leq N$). Then the rest of the proof remains valid.

**Corollary 3.18.** For each degree of $\mathbb{Z}/2$ we have the inequality

$$\dim \oplus_{\lambda \leq \lambda_1} E_{\lambda}(D_0) \geq \limsup_{i \to 0} \dim \oplus_{\lambda \leq \lambda_1} E_{\lambda}(D_i)$$

**Proposition 3.19.** Suppose $\lambda_1 < \lambda_0$. For each degree of $\mathbb{Z}/2$ we have the inequality

$$\dim \oplus_{\lambda \leq \lambda_1} E_{\lambda}(D_0) \leq \liminf_{i \to 0} \dim \oplus_{\lambda \leq \lambda_1} E_{\lambda}(D_i)$$

**Proof.** Let $\epsilon_0 > 0$ be a sufficiently small number, which we fix later. Let $e_1, e_2, \ldots, e_N$ be an $L^2$-orthonormal basis of the finite dimensional space $E := \oplus_{\lambda \leq \lambda_1} E_{\lambda}(D_0)$ consisting of eigenvectors of $D_0$ with eigenvalues $\mu_1, \mu_2, \ldots, \mu_N$ respectively. For a sufficiently large $\alpha$ and sufficiently small $\epsilon > 0$, the truncated sections $e_i' = \rho_{\alpha\epsilon\epsilon} e_i$ satisfy

$$|\delta_{ij} - \int_M (e_i', e_j')| < \epsilon_0,$$

$$|\mu_i \mu_j \delta_{ij} - \int_M (D_0 e_i', D_0 e_j')| < \epsilon_0$$

for every $1 \leq i, j \leq N$ as in the proof of Lemma 3.8 or Lemma 3.9. Here $\delta_{ij}$ is Kronecker’s delta. Let $E'$ be the vector space spanned by $e_i'$. Since the support of all the $e_i'$ are contained in the compact support of $\rho_{\alpha\epsilon\epsilon}$, Assumption 3.5 implies that

$$|\mu_i \mu_j \delta_{ij} - \int_M (D_t e_i', D_t e_j')| < \epsilon_0.$$
for every $t$ sufficiently closed to 0. Let $E'$ be the vector space spanned by $e'_i$. Then if $e_0$ is sufficiently small, each element $s'$ of $E'$ satisfies $\|D_i(s')\|_M^2 \leq \lambda_1 \|s'\|_M^2$. Corollary 3.15 implies $\dim E' \leq \dim \oplus_{\lambda < \lambda_1} E_\lambda(D_i)$. It is easy to check that the above inequality holds for each degree of $\mathbb{Z}/2$.

\[ \qed \]

**Proof of Theorem 3.21.** From Proposition 3.14 there is $0 < \lambda_1 < \lambda_0$ such that $E_{\lambda_1} = 0$. Then Corollary 3.15 and Proposition 3.19 imply

\[ \dim \oplus_{\lambda < \lambda_1} E_\lambda(D_0) = \lim_{t \to 0} \dim \oplus_{\lambda < \lambda_1} E_\lambda(D_t) \]

and the equality holds for each degree, from which the claim follows. \[ \qed \]

### 3.5. Gluing formula

In this subsection we generalize Theorem 3.6. We first need to generalize Assumption 3.5.

Let $(M, W, \sigma, D, V)$ be as in Section 3.4. For $i = 1, 2, \ldots$ we further suppose that the data $(M_i, W_i, \sigma_i, D_i, V_i)$ satisfy the setting in Section 3.1.

Suppose $M$ has a sequence of closed open subsets $\{K_i\}$ ($i = 1, 2, \ldots$) satisfying $K_i \cup V = M$, $K_i \subset \text{int}(K_{i+1})$ and $M = \bigcup_i K_i$. Suppose there exists an isometric open embedding $\iota_i : \text{int}(K_i) \to M_i$ and an isomorphism $\tilde{\iota}_i : W|_{\text{int}(K_i)} \cong \iota_i^* W_i$ as $\mathbb{Z}/2$-graded Hermitian vector bundle over $\text{int}(K_i)$ for each $i$.

If $s$ is a compactly supported section of $W$, then $\tilde{\iota}_i s|_{\text{int}(K_i)}$ makes sense as a compactly supported section of $W_i$ for large $i$ with $\text{supp } s \subset \text{int}(K_i)$ by extending as 0 outside $\iota_i(\text{supp } s)$. We simply write $\tilde{\iota}_i s$ for this section on $W_i$ for large $i$.

If $K$ is a compact subset of $M$ and $s_i$ is a section of $W_i$ for large $i$ with $K \subset \text{int}(K_i)$, then $\tilde{\iota}_i^{-1}(s_i|_{\text{int}(K_i)})|_K$ is a section of $W|_K$. We simply write $\tilde{\iota}_i^{-1}s_i|_K$ for this section.

Then $s \mapsto \{\tilde{\iota}_i^{-1}(D_i s)|_K\}$ is a differential operator on $K$ for large $i$. We write $\iota_i^* D_i$ for this operator on $K$.

**Assumption 3.20.**

1. $\iota_i(M \setminus V) = M \setminus V_i$.
2. The data $(M_i, W_i, \sigma_i, D_i, V_i)$ satisfy Assumption 3.5 with the same constant $\lambda_0$ for $(M, W, \sigma, D, V)$.
3. On each compact subset $K$ of $M$ the coefficients of the differential operator $\iota_i^* D_i$ is $C^\infty$-convergent to those of $D$ on $K$ as $i \to \infty$.

We do not assume that the propagation speed of $\{D_i\}$ is uniformly bounded with respect to $i$.

**Theorem 3.21.** Under Assumption 3.20 ind $D_i$ is equal to ind $D$ for large $i$.

**Proof.** The most of the arguments in Sections 3.3 and 3.4 go through.

The statement and proof of Proposition 3.13 is straightforwardly generalized with replacement of $D_t$ by $D_i$.

The statement and proof of Proposition 3.19 is also straightforwardly generalized.

To generalize Proposition 3.16 we need the construction in Remark 3.17. As for the statement we replace $D_t$ with $D_i$, and let $E^{(i)}$ be a $\mathbb{Z}/2$ graded subspace of $L^2(M_i, W_i)$. As for the proof take $e_k^{(i)}$ with support contained in $\text{int}(K_i)$. Let $s^{(i)}$ be an element of $E^{(i)}$ with $\|s^{(i)}\|_{M_i} = 1$ which is $L^2$-orthogonal to all $\tilde{\iota}_i e_k^{(i)}$ ($1 \leq k \leq N$). Then the rest of the proof remain valid.

Then the argument of Section 3.4 can be straightforwardly generalized to show Theorem 3.21. \[ \qed \]
When $M$ has two (or more) connected components while each $M_i$ is connected, Theorem 3.21 is regarded as a gluing formula of index as explained below.

**Proposition 3.22.** Let $(M, W, \sigma, D, V)$ be a data satisfying Assumption 3.7. Suppose $M$ is the disjoint union of $M'$ and $M''$. Let $(W', \sigma', D', V')$ and $(W'', \sigma'', D'', V'')$ be the restrictions of $(W, \sigma, D, V)$ to $M'$ and $M''$ respectively. Then we have
\[ \text{ind } D_i = \text{ind } D' + \text{ind } D'', \]
for sufficiently large $i$.

**Proof.** Since $\text{ind } D = \text{ind } D' + \text{ind } D''$, Theorem 3.21 implies the required gluing formula. 

The following vanishing lemma follows from the partial integration formula Lemma 3.8 and the second inequality in Assumption 3.1.

**Lemma 3.23.** Let $(M, W, \sigma, D, V)$ be a data satisfying Assumption 3.7. If $M = V$, then we have $\ker D \cap L^2(W) = 0$.

Using the gluing formula Proposition 3.22 and the above Lemma 3.23, we have the following excision formula of index.

**Proposition 3.24.** Let $(M, W, \sigma, D, V)$ be a data satisfying Assumption 3.7. Suppose $M$ is the disjoint union of $M'$ and $M''$, and $M''$ is contained in $V$. Let $(W', \sigma', D', V')$ be the restriction of $(W, \sigma, D, V)$ to $M'$. Then we have
\[ \text{ind } D_i = \text{ind } D' \]
for sufficiently large $i$.

3.6. **Product formula.** Following Atiyah and Singer [1], we formulate a product formula for elliptic operators. Except that we need Lemma 3.8 for partial integration on complete Riemannian manifolds, the argument is exactly the same as in [1]. The main purpose of this subsection is to formulate Assumption 3.26 below, which is crucial for the case of complete manifolds. To apply the product formula it is necessary to check the assumption for specific operators, which is our another task and is carried out in Section 5.

For $k = 0, 1$ let $M_k$ be a complete Riemannian manifold, $W_k$ a $\mathbb{Z}/2$-graded Hermitian vector bundle over $M_k$, and $D_k : \Gamma(W_k) \to \Gamma(W_k)$ a degree-one formally self-adjoint order-one elliptic operator with principal symbol $\sigma_k$.

Let $G$ be a compact Lie group and $P \to M_0$ a principal $G$-bundle. Suppose $G$ acts on $M_1$ isometrically, $W_1$ is $G$-equivariant $\mathbb{Z}/2$-graded Hermitian vector bundle, and $D_k$ is a $G$-invariant operator.

Then $M = P \times_G M_1$ is a fiber bundle over $M_0$ with fiber $M_1$. We write $\pi : M \to M_0$ for the projection map. Let $\widetilde{W}_0$ and $\widetilde{W}_1$ be the vector bundles over $M$ defined by $\widetilde{W}_0 = \pi^*W_0$ and $\widetilde{W}_1 = P \times_G W_1$, and we put $W = \widetilde{W}_0 \otimes \widetilde{W}_1$.

We would like to lift $D_0$ and $D_1$ as operators on $W$. The lift of $D_1$ is given straightforward: Defining the operator $\widetilde{D}_1$ on $\Gamma(\widetilde{W}_0 \otimes \widetilde{W}_1)$ by $\epsilon \otimes D_1$ on each fiber of $\pi : M \to M_0$, where $\epsilon : W_0 \to W_0$ is equal to $+id$ on the degree 0 part of $W_0$, and to $-id$ on the degree 1 part of $W_1$.

We next construct $\widetilde{D}_0 : \Gamma(W) \to \Gamma(W)$. Let $\{V_\alpha\}$ be an open covering of $M_0$ and $\{p_\alpha^2\}$ a partition of unity. Suppose we have local trivializations $P|_{V_\alpha} \cong V_\alpha \times G$ with transition functions $g_{\alpha \beta}$. Using the local trivialization on $V_\alpha$ we have the identifications.
\[
\pi^{-1}(V_\alpha) \cong V_\alpha \times M_1 \text{ and } W|_{\pi^{-1}(V_\alpha)} \cong W_0|_{V_\alpha} \times W_1. \]
Let \( \tilde{D}_{0,\alpha} \) be the operator on \( W|_{\pi^{-1}(V_\alpha)} \) defined by \( D_0 \) using the product structure. We put \( \tilde{D}_0 := \sum_\alpha \rho_\alpha \tilde{D}_{0,\alpha} \rho_\alpha. \)

**Lemma 3.25.** \( \tilde{D}_0 \tilde{D}_1 + \tilde{D}_1 \tilde{D}_0 = 0. \)

**Proof.** It follows from \( \tilde{D}_1 \tilde{D}_{0,\alpha} + \tilde{D}_{0,\alpha} \tilde{D}_1 = 0 \) and \( \tilde{D}_1 \rho_\alpha - \rho_\alpha \tilde{D}_1 = 0. \)

Let \( R \) be a \( G \)-invariant \( \mathbb{Z}/2 \)-graded finite dimensional subspace of \( \Gamma(M_1, W_1) \), and \( \tilde{R} \) the fiber bundle \( P \times_G R \) over \( M_0 \). Then we have an embedding

\[
(10) \quad \Gamma(M_0, W_0 \otimes \tilde{R}) \to \Gamma(M, \tilde{W})
\]

which is preserved by the action of \( \tilde{D}_0 \). Let \( \tilde{D}_R \) be the restriction of \( \tilde{D}_0 \) on \( \Gamma(M_0, W_0 \otimes \tilde{R}) \). Then \( \tilde{D}_R \) is a differential operator on \( W_0 \otimes \tilde{R} \) with principal symbol \( \sigma_0 \otimes \text{id}_{\tilde{R}}. \)

**Assumption 3.26.**

1. \( D_0 \) has finite propagation speed, i.e., \( \sigma_0 \) is \( L^\infty \)-bounded.
2. The data \( (M_1, W_1, D_1) \) satisfies Assumption 3.24.
3. \( R = E_0(D_1) \).
4. The data \( (M, W, \tilde{D}_0 + \tilde{D}_1) \) satisfies Assumption 3.24.

We do not assume the second condition of Assumption 3.24 for the data \( (M_0, W_0, D_0) \).

Recall that \( \tilde{D}_R \) is given by \( \tilde{D}_0 \) via the embedding (10). Since \( \tilde{D}_1 = 0 \) on the image of the embedding, Assumption 3.24 implies that the data \( (M_0, W_0 \otimes \tilde{R}, \tilde{D}_R) \) satisfies Assumption 3.24 as well.

**Theorem 3.27** (product formula). \( \text{ind}(\tilde{D}_0 + \tilde{D}_1) = \text{ind} \tilde{D}_R \)

**Proof.** We show that the embedding (10) gives the isomorphism \( E_0(\tilde{D}_R) \cong E_0(\tilde{D}_0 + \tilde{D}_1) \). If \( s \) is in the image of \( E_0(D_R) \), then the construction of \( D_R \) implies that \( s \) is obviously in \( E_0(\tilde{D}_0 + \tilde{D}_1) \). From Lemma 3.24 and Lemma 3.25 if \( s \) is an element of \( E_0(\tilde{D}_0 + \tilde{D}_1) \) we have

\[
0 = \int_M (|\tilde{D}_0 + \tilde{D}_1|^2 s, s) = \int_M (\tilde{D}_0^2 s + \tilde{D}_1^2 s, s) = ||\tilde{D}_0 s||^2 + ||\tilde{D}_1 s||^2,
\]

i.e., \( \tilde{D}_0 s = \tilde{D}_1 s = 0 \). In particular \( \tilde{D}_1 s = 0 \) implies that \( s \) is in the image of the embedding (10). Moreover \( \tilde{D}_0 s = 0 \) implies \( s \) is in the image of \( E_0(D_R) \). \( \square \)

4. **Local Index**

In this section we first define a class of Riemannian manifolds, and consider compatible fibrations and compatible systems on them which satisfy some natural conditions. For such a class we will define the local index of a strongly acyclic compatible system in this section and prove the product formula in the next section.

**Definition 4.1.** Let \( M \) be a Riemannian manifold. If there exists an open subset \( V \) of \( M \) which satisfies the following properties, then we call \( M \) a manifold with the generalized cylindrical end \( V \).

1. The complement \( M \setminus V \) is compact.
2. There exists a finite open covering \( \{V_\beta\}_{\beta \in B} \) of \( V \).
3. There exist families of Riemannian manifolds \( \{N_\beta\}_{\beta \in B} \) and non-negative integers \( \{m_\beta\}_{\beta \in B} \) such that each \( N_\beta \) is an open subset of a closed Riemannian manifold and each \( V_\beta \) is isometric to \( N_\beta \times \mathbb{R}^{m_\beta}_{>0} \), where \( \mathbb{R}_{>0} \) is the set of positive real numbers.
We identify $V_\beta$ with $N_\beta \times \mathbb{R}^m_\beta$ and write $V = \cup_{\beta} N_\beta \times \mathbb{R}^m_{\beta}$. Let $M$ be a manifold with the generalized cylindrical end $V = \cup_{\beta \in B} N_\beta \times \mathbb{R}^m_{\beta}$. Let $\{\pi_\alpha, V_\alpha\}_{\alpha \in A}$ be a compatible fibration on $V$. Hereafter we take and fix a relatively compact open neighborhood $V_\infty$ of $M \setminus V$ and put $\bar{A} := A \cup \{\infty\}$. Let $\{\rho^2_\alpha\}_{\alpha \in \bar{A}}$ be an admissible partition of unity of $M = \cup_{\alpha \in \bar{A}} V_\alpha$. Here $V_\infty$ has the fiber bundle structure whose fiber consists of a single point. Let $\{W, D_\alpha\}_{\alpha \in A}$ be a compatible system on $V$. In this section we consider the above data with the following translationally invariance conditions on the end.

**Condition (TI)**

(1) There is a bijective map between $A$ and $B$.

(2) Each $V_\alpha$ is isometric to $N_\alpha \times \mathbb{R}^m_{\alpha}$ under the identification in (1).

(3) Each $N_\alpha$ has a fiber bundle structure $N_\alpha \to U_\alpha$ such that the fiber bundle structure $V_\alpha \to U_\alpha$ can be identified with the induced fiber bundle structure $N_\alpha \times \mathbb{R}^m_{\alpha} \to U_\alpha \times \mathbb{R}^m_{\alpha}$.

(4) For each $\alpha \in A$, the $\mathbb{Z}/2$-graded Clifford module bundle $W|_{V_\alpha}$ and the operator $D_\alpha$ are invariant under the translation action of the $\mathbb{R}^m_{\alpha}$-direction.

(5) For each $\alpha \in A$, there exist an open subset $N'_\alpha$ of $N_\alpha$ and a positive real number $\epsilon_\alpha$ such that $N'_\alpha \subseteq N_\alpha$ and $M = V_\infty \cup (\cup_{\alpha \in A} N'_\alpha \times \mathbb{R}^m_{\alpha})$, where $R_{\text{r},\alpha}$ is the set of real numbers greater than $\epsilon_\alpha$.

(6) For each $\alpha \in A$ there exists a compactly supported function on $N'_\alpha$ such that $\rho_\alpha|_{N'_\alpha \times \mathbb{R}^m_{\alpha}}$ is the pull back of it via the natural projection $N'_\alpha \times \mathbb{R}^m_{\alpha} \to N'_\alpha$.

A typical example of a data with Condition (TI) is a data on a manifold with cylindrical end $V = N \times \mathbb{R}_+$ which is invariant under the translation on the $\mathbb{R}_+$-direction. We can define a product of two data with Condition (TI) in an appropriate and obvious way. The product also satisfies the same condition, which is crucial in the proof of the product formula. See Section 5 for details.

We first show the following technical lemma for the above data which is used to show Theorem 4.7.

**Lemma 4.2.** Suppose that $A$ is non-empty and $V$ is connected. There exists an admissible partition of unity $\{\chi^2_\alpha\}_{\alpha \in \bar{A}}$ which satisfies the condition (5),(6) in Condition (TI) and $\text{supp}(\chi_\alpha) \subseteq \text{supp}(\rho_\alpha)$ for each $\alpha \in A$.

**Proof.** Since $A$ is a finite set it is enough to show that when we fix $\alpha \in A$, there exists an admissible partition of unity $\{\chi^2_\alpha'\}_{\alpha' \in \bar{A}}$ of $M = \cup_{\alpha' \in \bar{A}} V_{\alpha'}$ which satisfies the condition (5) and (6) in Condition (TI), $\text{supp}(\chi_\alpha) \subseteq \text{supp}(\rho_\alpha)$ and $\text{supp}(\chi_{\alpha'}) = \text{supp}(\rho_{\alpha'})$ for all $\alpha' \in \bar{A} \setminus \{\alpha\}$.

We first take and fix a non-decreasing function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that

$$
\varphi(r) = \begin{cases} 
0 & (0 \leq r \leq 1/2) \\
r & (1 \leq r),
\end{cases}
$$

and $\text{supp}(\varphi) = [1/2, \infty)$. We define $\hat{\rho}_\alpha : M \to \mathbb{R}$ by the composition $\hat{\rho}_\alpha := \varphi \circ \rho_\alpha$. Then, by definition, we have $\text{supp}(\hat{\rho}_\alpha) \subseteq \text{supp}(\rho_\alpha)$ and

$$
\hat{\rho}_\alpha(x) + \sum_{\alpha' \neq \alpha} \rho_{\alpha'}(x) > 0
$$
for all \( x \in M \). By normalizing the family of functions \( \{ \hat{p}_a \} \cup \{ \rho_a' \} \) we obtain the required admissible partition of unity \( \{ \chi_{\alpha'}^a \}_{\alpha \in A} \).

\[ \square \]

4.1. Vanishing theorem. Let \( M \) be a Riemannian manifold and \( V \) an open subset of \( M \). Let \( \{ \pi_\alpha : V_\alpha \to U_\alpha \}_{\alpha \in A} \) be a compatible fibration and \( \{ W, D_\alpha \}_{\alpha \in A} \) a compatible system on \( V \). Let \( \{ \rho_\alpha^2 \}_{\alpha \in A} \) be an admissible partition of unity. In this subsection we assume the following three conditions.

- \( M \) has a generalized cylindrical end \( V' \).
- \( \{ \pi_\alpha : V_\alpha \to U_\alpha, W, D_\alpha, \rho_\alpha \} \) satisfies Condition (TI).
- \( \{ W, D_\alpha \}_{\alpha \in A} \) is strongly acyclic.

For each \( \alpha \in A \) we put \( D_\alpha' := \rho_\alpha D_\alpha \rho_\alpha \). Take any Dirac-type operator \( D \) on \( \Gamma(W) \) which is translationally invariant on the end and a positive real number \( t \). We define the operator acting on \( \Gamma(W) \) by \( D_t := D + t \sum_{\alpha \in A} D_\alpha' \).

Remark 4.3. Note that \( D_\alpha \) and \( \rho_\alpha \) commute each other because \( \rho_\alpha \) is a pull-back of a function on \( U_\beta \) and \( D_\alpha \) contains only the derivatives along fibers.

Proposition 4.4. \( D_t \) is elliptic for all \( t \geq 0 \).

Proof. With the identification of tangent bundles and cotangent bundles via the Riemannian metric, the principal symbol \( \sigma_t \) of \( D_t \) is written as

\[ \sigma_t = c \circ (id_{TM} + t \sum_\alpha \rho_\alpha p_\alpha \rho_\alpha), \]

where \( p_\alpha : TV_\alpha \to T[\pi_\alpha] \) is the orthogonal projection. We put

\[ S := id_{TM} + t \sum_\alpha \rho_\alpha p_\alpha \rho_\alpha. \]

\( S \) is a symmetric operator. In order to prove the proposition it is sufficient to show that for each \( x \in M S_x : T_x M \to T_x M \) is a positive operator. By definition, for \( u \in T_x M \) we have

\[ (S_x u, u) = |u|^2 + t \sum_\alpha |\rho_\alpha(x) p_\alpha(u)|^2 \geq 0, \]

where we think \( p_\alpha(u) = 0 \) if \( x \notin V_\alpha \). Moreover, \( (S_x u, u) = 0 \) if and only if \( u = 0 \). This implies \( S_x \) is positive.

We show the following fundamental lemma in our argument.

Lemma 4.5. For each \( \alpha \) the anti-commutator \( DD'_\alpha + D'_\alpha D \) is a differential operator along fibers of \( \pi_\alpha \) of order at most 2.

Proof. Recall that, for each \( \alpha \) the principal symbol of horizontal direction \( D^{H_\alpha} \) of \( D \) with respect to \( \pi_\alpha \) anti-commutes not only with the symbol of \( D'_\alpha \), but also with the whole operator \( D'_\alpha \). The statement follows from this property. It is straightforward to check it using local description. Instead of giving the detail of the local calculation, however, we here give an alternative formal explanation for the above lemma. For \( b \in U_\alpha \) let \( W_b \) be the sections of the restriction of \( W \) on the fiber \( \pi_\alpha^{-1}(b) \). Then \( W = \bigcup W_b \) is formally an infinite dimensional vector bundle over \( U_\alpha \). We can regard \( D'_\alpha \) as an endomorphism on \( W \). Then \( D'_\alpha \) is a order-zero differential operator on \( W \) whose principal symbol is equal to \( D'_\alpha \) itself. Then, as a differential operator on \( W \), the anti-commutator \( D'_\alpha D^{H_\alpha} + D^{H_\alpha} D'_\alpha \) is an (at most) order-one operator whose principal symbol is given by the anti-commutator...
between the Clifford action by $TU_{\alpha}$ and $D'_{\alpha}$. This principal symbol vanishes, which implies that the anti-commutator is order-zero as a differential operator on $W$, i.e., it does not contain derivatives of $U_{\alpha}$-direction.

For an operator appearing in the above lemma we have the following a priori estimate.

**Lemma 4.6.** For each fiber $F$ of $\pi_{\alpha}$ and arbitrary differential operator $Q$ of order at most 2 along $F$, there exists a constant $C_Q$ such that the inequality
\[
|\int_F (s_F, Qs_F)| \leq C_Q \int_F |D_{\alpha}s_F|^2
\]
holds for all sections $s_F$.

The following vanishing theorem is a main theorem in this subsection.

**Theorem 4.7.** Let $M$ be a Riemannian manifold with the generalized cylindrical end $V$. Suppose that $M$ is equipped with the data $\{\pi_{\alpha} : V_{\alpha} \to U_{\alpha}, W, D_{\alpha}, \rho_{\alpha}\}$ which satisfies Condition (TI). Let $D$ be a Dirac-type operator acting on $\Gamma(W)$ which is translationally invariant on the end. Put $D'_{\alpha} := \rho_{\alpha}D_{\alpha}\rho_{\alpha}$ and $D_t := D + t\sum_{\alpha} D'_{\alpha}$ for a positive number $t > 0$. Then the space of $L^2$-solutions of the equation $D_t s = 0$ is trivial for all $t \gg 1$.

To show this theorem we make several preparations. Let $\{\chi_{\alpha}^2\}_{\alpha \in A}$ be an admissible partition of unity constructed in Lemma 4.2 and put $K_{\alpha} := \text{supp } \chi_{\alpha}$.

**Lemma 4.8.**
\begin{enumerate}
\item $D'_{\alpha}$ is an elliptic operator on each fiber of $\pi_{\alpha}$ which is contained in $K_{\alpha}$.
\item If $Q$ is a differential operator along fibers of order at most 2, then there exists a constant $C_Q$ such that for each section $s_{\alpha}$ satisfying $\text{supp } s_{\alpha} \subset K_{\alpha}$ and $D'_{\alpha}s_{\alpha} \in L^2(W)$ we have an estimate
\[
|\int_M (s_{\alpha}, Qs_{\alpha})| \leq C_Q \int_M |D'_{\alpha}s_{\alpha}|^2.
\]
\end{enumerate}

**Proof.** The first statement follows from the fact that $\rho_{\alpha}$ takes positive values on $K_{\alpha}$. Note that since the data satisfies Condition (TI) we can choose the constants $C_Q$ in Lemma 4.6 uniformly for all fibers of $\pi_{\alpha}$ contained in $K_{\alpha}$. The second statement follows from this fact.

**Lemma 4.9.** There exists an operator $Z$ which does not contain any differential terms and satisfying
\[
D_t^2 = \sum_{\alpha} \chi_{\alpha}D_t^2\chi_{\alpha} + Z.
\]
Moreover $Z$ does not depend on $t$.

**Proof.** If $\chi$ is an admissible function then it commutes with $D_{\alpha}$ and hence we have $[D_t, \chi] = [D, \chi]$. Using this equality and the fact $[D, \chi]$ does not contain any differential operators we have
\[
[[D_t^2, \chi], \chi] = [[D_t[D_t, \chi] + [D_t, \chi]D_t], \chi] = [[D_tD_t, \chi] + [D_t, \chi]D_t, \chi] = [D_t, \chi][D_t, \chi] + [D_t, \chi][D_t, \chi] = 2[D_t, \chi]^2.
\]
Put $\chi := \chi_\alpha$ and take summation for all $\alpha$ we have

$$2D_t^2 - 2 \sum_{\alpha} \chi_\alpha D_t^2 \chi_\alpha = \sum_{\alpha} [[D_t^2, \chi], \chi] = 2 \sum_{\alpha} [D, \chi_\alpha]^2.$$  

Then $Z := \sum_{\alpha} [D, \chi_\alpha]$ is the required operator of order 0.

**Proposition 4.10.** If $s_\alpha$ is an $L^2$-bounded section of $W$ such that $Ds_\alpha$ and $D'_\alpha s_\alpha$ are also $L^2$-bounded section for all $\alpha \in A$ and supp $s_\alpha \subset K_\alpha$, then we have the inequality

$$\int_M |D_t s_\alpha|^2 \geq \int_M (Zs_\alpha, s_\alpha) + \int_M |s_\alpha|^2$$

for all $t \gg 1$.

We use the above proposition and lemmas to show Theorem 4.7 as follows.

**Proof of Theorem 4.7 assuming Proposition 4.10.** We take $t \gg 1$ so that the inequality in Proposition 4.10 holds. We first note that since $M$ has a generalized cylindrical end and the data satisfies Condition (TI), an $L^2$-bounded section $s$ which satisfies $D_t s = 0$ is an element in the Sobolev space $L^2_k(W)$ for arbitrary $k \in \mathbb{N}$ by the elliptic estimate, and hence $s_\alpha = \chi_\alpha s$ is. Moreover since $D_t$ is translationally invariant it has the bounded extension $D_t : L^2_k(W) \to L^2(W)$. Then we can use the partial integration formula in Lemma 3.8 for $D_t s_\alpha$, and we have $s = 0$ as in the following:

$$0 = \int_M (D_t^2 s, s) = \sum_{\alpha} \int_M (\chi_\alpha D_t^2 \chi_\alpha s, s) + \int_M (Zs, s) \quad \text{(Lemma 4.9 1)}$$

$$= \sum_{\alpha} \left( \int_M |D_t s_\alpha|^2 + \int_M (Zs_\alpha, s_\alpha) \right) \quad \text{(}s_\alpha := \chi_\alpha s)$$

$$\geq \sum_{\alpha} \left( \int_M |(Zs_\alpha, s_\alpha)| + \int_M |s_\alpha|^2 + \int_M (Zs_\alpha, s_\alpha) \right) \quad \text{(Proposition 4.10)}$$

$$\geq \sum_{\alpha} \int_M |s_\alpha|^2 = \int_M |s|^2.$$

\hfill \Box

**4.1.1. Proof of Proposition 4.10.** For each fixed $\alpha \in A$, we can write $D_t$ as

$$D_t = D_{\neq \alpha} + tD'_\alpha$$

on $V_\alpha$, where we put

$$D_{\neq \alpha} := D + t \sum_{\beta \neq \alpha} D'_{\beta}.$$  

Using these notations together with $Q_\alpha := DD'_\alpha + D'_\alpha D$ and $Q_{\beta \alpha} := D'_{\beta} D'_\alpha + D'_\alpha D_{\beta}$, we have

$$D_t^2 = D_{\neq \alpha}^2 + tQ_\alpha + t^2 \sum_{\beta \neq \alpha} Q_{\beta \alpha} + t^3 D'^2_\alpha,$$
and since \( \int_M (D^2_{\alpha} s, s) \) and \( \int_M (Q_\alpha s, s) \) are non-negative for an \( L^2 \)-section \( s \) satisfying the assumptions we also have
\[
\int_M |D_t s|^2 \geq t^2 \int_M |D'_\alpha s|^2 - t \left( \int_M (Q_\alpha s, s) \right).
\]

From Lemma 4.8.2, \( Q_\alpha \) is a differential operator along fibers of \( \pi_\alpha \) of order at most 2. Then from Lemma 4.8.2 again there exists a constant \( C' \) such that
\[
\left|\int_M (Q_\alpha s, s)\right| \leq C' \int_M |D'_\alpha s|^2.
\]
Combining these inequalities we have
\[
\int_M |D_t s|^2 \geq (t^2 - C't) \int_M |D'_\alpha s|^2.
\]
On the other hand using Lemma 4.8.2 again there exists a constant \( C'' \) such that
\[
\left|\int_M (Zs_\alpha, s_\alpha)\right| + \int_M |s_\alpha|^2 \leq C'' \int_M |D'_\alpha s|^2,
\]
and hence if we take \( t \gg 1 \) so that \( t^2 - C't \geq C'' \), then we have
\[
\int_M |D_t s|^2 \geq \left|\int_M (Zs_\alpha, s_\alpha)\right| + \int_M |s_\alpha|^2.
\]
Note that since \( A \) is a finite set we may assume that \( C' \) and \( C'' \) do not depend on \( \alpha \), and we complete the proof.

4.2. Definition of the local index - Generalized cylindrical end case. In this subsection we give the definition of the local index of a strongly acyclic compatible system on a manifold \( M \) with the generalized cylindrical end \( V \). Consider the data on \( V \) which satisfies the Condition (TI) and take any Dirac-type operator \( D \) acting on \( \Gamma(W) \) which is translationally invariant on the end. We assume that the compatible system on \( V \) is strongly acyclic. As in the previous Subsection we put \( D_t := D + t \sum_\alpha \rho_\alpha D_\alpha \rho_\alpha \).

**Lemma 4.11.** If \( t \) is large enough so that the inequality in Proposition 4.10 holds, then \( D_t \) satisfies the Assumption 3.1.

**Proof.** Since the principal symbol of \( D_t \) is given by a linear combination of the Clifford multiplication of \( TM \) and that of fiber directions of \( \{\pi_\alpha\}_{\alpha \in A} \), (i) of Assumption 3.1 is satisfied. We show the condition (ii) of Assumption 3.1. Let \( s \) be a smooth compactly-supported section of \( W \) with \( \text{supp } s \subset V \). Let \( \{\chi_\alpha^2\}_{\alpha \in A} \) be the admissible partition of unity constructed in Lemma 4.12. For each \( s_\alpha := \chi_\alpha s \) we can apply Proposition 4.10 and hence, we have \( ||D_t s||_\pi^2 \geq ||s||_\pi^2 \) as in the same way in the proof of Theorem 4.7.

Results in Section 3 imply the following.

**Proposition 4.12.** If \( t \) is large enough so that the inequality in Proposition 4.10 holds, then the space of \( L^2 \)-solutions of \( D_t s = 0 \) is finite dimensional and its super-dimension is independent for \( t \gg 1 \) and any other continuous deformations of data.

**Definition 4.13.** We define the local index \( \text{ind}(M, V, W) \) as the index of \( D_t \) in the sense of Section 3.

In the case of cylindrical end we have the following sum formula of local indices.
Lemma 4.14. For $i = 1, 2$ let $M_i$ be manifolds with cylindrical ends $V_i = N_i \times \mathbb{R}_{>0}$ and $N_i^0$ be connected components of $N_i$. Suppose that there is an isometry $\phi : N_i^0 \to N_j^0$, and for some $R > 0$ the map $\phi : N_i^0 \times (0, R) \to N_j^0 \times (0, R)$ given by $(x, r) \mapsto (\phi(x), R - r)$ induces the isomorphism between the strongly acyclic compatible systems on them. Then we can glue $M_1 \setminus (N_1^0 \times [R, \infty))$ and $M_2 \setminus (N_2^0 \times [R, \infty))$ to obtain a new manifold $\hat{M}$ with cylindrical end $\hat{V} = \hat{N} \times (0, \infty)$ for $\hat{N} = (N_1 \setminus N_1^0) \cup (N_2 \setminus N_2^0)$, and we also have a Clifford module bundle $\hat{W}$ obtained by gluing $W$ and $W'$ on $N_1^0 \times (0, R) \cong N_2^0 \times (0, R)$. Then we have

$$\text{ind}(\hat{M}, \hat{V}, \hat{W}) = \text{ind}(M_1, V_1, W_1) + \text{ind}(M_2, V_2, W_2).$$

4.3. Definition of the local index - general case. Let $V$ be an open subset of $M$ such that $M \setminus V$ is compact. Assume that $V$ has a strongly acyclic compatible system. We would like to define the local index for such a general case. The way to define it is almost same in [4]. To verify the construction we have to check the following.

Proposition 4.15. For given $(M, V, W)$ and the strongly acyclic compatible system $(V_\alpha \xrightarrow{C_\alpha} U_\alpha, D_\alpha)$ on $V$ we can deform them to $(M', V', W')$ so that it has a cylindrical end with a translationally invariant strongly acyclic compatible system.

To prove the proposition, it is enough to show the following lemma.

Lemma 4.16. There exists a smooth admissible function $f : M \to \mathbb{R}$ and a regular value $c$ of $f$ such that $f^{-1}(-\infty, c]$ is a compact subset containing $M \setminus V$.

To show this lemma, we first note that the following.

Lemma 4.17. Let $\{V_\alpha''\}_{\alpha \in A}$ be the open covering of $M$ which appears in the 4th property in Definition 2.11. For any subset $D$ of $M$, let $K(D)$ be

$$K(D) = \bigcup_\alpha \pi_\alpha^{-1}(D \cap V_\alpha'').$$

Let $h : M \to \mathbb{R}$ be a smooth function. Let $J$ be a connected interval. If $K(D) \subset h^{-1}(J)$, then we have $D \subset I(h)^{-1}(J)$, where $I : C^\infty(M) \to C^\infty(M)$ is the averaging operation.

Proof. Use the fact $\pi_\alpha^{-1}(V_\alpha'') = V_\alpha''$ and the inequality in Definition 2.11. □

Proof of Lemma 4.16. Since each $\pi_\alpha$ is a proper map, the subset $K(M \setminus V)$ defined in Lemma 4.17 is compact. Let $f_0 : M \to \mathbb{R}$ be the distance function from the compact subset $K(M \setminus V)$. Take a real number $r > 0$ so that $f_0^{-1}[0, r]$ is a compact neighborhood of $K(M \setminus V)$. Let $\epsilon > 0$ be a positive real number satisfying $2\epsilon < r$. Let $h : M \to \mathbb{R}$ be a smooth function such that $|f_0(x) - h(x)| < \epsilon$ for all $x \in M$, and we put $f := I(h)$. Let $c$ be a regular value of $f$ satisfying $\epsilon < c < r - \epsilon$. Then we have

$$K(M \setminus V) = f_0^{-1}(0) \subset h^{-1}(-\epsilon, \epsilon).$$

By Lemma 4.17 we have

$$M \setminus V \subset f^{-1}(-\epsilon, \epsilon) \subset f^{-1}(-\infty, c].$$

Suppose that $x \notin K(h^{-1}(-\infty, c])$. Then one has $K(\{x\}) \subset h^{-1}(c, \infty)$, and it implies $x \notin f^{-1}(-\infty, c]$ by Lemma 4.17. Then we have

$$f^{-1}(-\infty, c] \subset K(h^{-1}(-\infty, c]) \subset K(f_0^{-1}(-\infty, c + \epsilon)) \subset K(f_0^{-1}[0, r]).$$

In particular $f^{-1}(-\infty, c]$ is compact. □
Definition 4.18. We define the local index $\text{ind}(M, V, W)$ to be the local index for the deformed data $(M', V', W')$.

Note that the local index $\text{ind}(M, V, W)$ is well-defined, i.e., it does not depend on various choice of the construction. It follows from the sum formula (Lemma 4.14) and Theorem 4.7 as in the same way in [4]. The well-definedness means the excision property of local index.

Theorem 4.19. Let $M$ be a Riemannian manifold and $V$ an open subset such that $M \setminus V$ is compact. Let $W$ be a $\mathbb{Z}_2$-graded $\text{Cl}(TM)$-module bundle on $M$ and suppose that the metric on $V$ is a compatible metric and $V$ has a strongly acyclic compatible system. Let $M'$ be an open neighborhood of $M \setminus V$ with $V \cap M'$ admissible. Then we have

$$\text{ind}(M, V, W) = \text{ind}(M', V \cap M', W|_{M'}).$$

The local index $\text{ind}(M, V, W)$ also has the following property.

Theorem 4.20. Let $M$, $V$, and $W$ be the data as in Theorem 4.19. Suppose that $V'$ is an admissible open subset of $V$ with $M \setminus V'$ compact. Then we have

$$\text{ind}(M, V, W) = \text{ind}(M', V', W).$$

Note that if $M$ is closed, then $\text{ind}(M, V, W)$ is equal to the index of the Dirac-type operator $D$ because of the homotopy invariance of indices. Using the excision property, additivity for disjoint unions and vanishing theorems we have the localization theorem.

Theorem 4.21. Let $M$ be a closed Riemannian manifold and $V$ an open subset. Let $W$ be a $\text{Cl}(TM)$-module bundle on $M$ and $D$ a Dirac-type operator acting on $\Gamma(W)$. Suppose that $V$ has a strongly acyclic compatible system. Let $\bigcup_{i=1}^N V_i$ be an open neighborhood of $M \setminus V$ such that $V_i \cap V_j = \emptyset$ if $i \neq j$. Then we have the following equality.

$$\text{ind} D = \sum_{i=1}^N \text{ind}(V_i, V_i \cap V, W|_{V_i \cap V}).$$

Remark 4.22. The arguments in this section are valid in orbifold category.

5. Product formula of local indices

In this section we formulate the product of acyclic compatible systems. Once we have an appropriate formulation of the product, then we obtain the product formula of local indices of the strongly acyclic compatible systems by results in Section 3.

5.1. Product of compatible fibrations. In this subsection we formulate a product of compatible fibrations. The product is defined for the following collection of data for $i = 0, 1$ which satisfy the Assumption 5.1.

1. $M_i$ : a manifold.
2. $V_i$ : an open set of $M_i$.
3. $\{\pi_{i,\alpha} : V_{i,\alpha} \to U_{i,\alpha}, U_{i,\alpha\beta} \mid \alpha, \beta \in A_i\}$ : a compatible fibration on $V_i$.
4. $G$ : a compact Lie group which acts smoothly on $M_1$.
5. $\pi_P : P \to M_0$ : a principal $G$-bundle over $M_0$.

Assumption 5.1. (1) $V_i$ is $G$-invariant and the fibrations $V_{i,\alpha} \to U_{i,\alpha}, V_{i,\alpha} \cap V_{i,\beta} \to U_{i,\alpha\beta}$ and $U_{i,\alpha\beta} \to \pi_{1,\alpha}(V_{i,\alpha} \cap V_{i,\beta})$ are $G$-equivariant fiber bundles for all $\alpha, \beta \in A_i$. 
Together with Assumption 5.1 we assume the following.

(2) there exist principal $G$-bundles $P_\alpha \to U_{0,\alpha}$, $P_{\alpha\beta} \to U_{0,\alpha\beta}$ and bundle maps $P|_{V_{0,\alpha}\cap V_{0,\beta}} \to P|_{\pi_{0,\alpha}(V_{0,\alpha}\cap V_{0,\beta})}$, $P|_{V_{0,\alpha}\cap V_{0,\beta}} \to P_{\alpha\beta}$, $P_{\alpha\beta} \to P_{\alpha}|_{\pi_{0,\alpha}(V_{0,\alpha}\cap V_{0,\beta})}$ for all $\alpha, \beta \in A_0$ such that the following diagrams commute;

\[
\begin{array}{c}
P|_{V_{0,\alpha}\cap V_{0,\beta}} \\ \downarrow \\
\pi_{0,\alpha}(V_{0,\alpha}\cap V_{0,\beta}) \quad \pi_{0,\alpha}(V_{0,\alpha}\cap V_{0,\beta}) \\
\downarrow p_{0,\alpha} \\
U_{0,\alpha} \quad U_{0,\alpha}
\end{array}
\]

\[
\begin{array}{c}
P|_{V_{0,\alpha}\cap V_{0,\beta}} \\ \downarrow \\
\pi_{0,\alpha}(V_{0,\alpha}\cap V_{0,\beta}) \quad \pi_{0,\alpha}(V_{0,\alpha}\cap V_{0,\beta}) \\
\downarrow p_{\alpha\beta} \\
U_{0,\alpha\beta} \quad U_{0,\alpha\beta}
\end{array}
\]

For later convenience we take an open neighborhood $V_{i,0}$ of $M_i \setminus V_i$ and consider the trivial fiber bundle structure $\pi_{i,0} : V_{i,0} \to V_{i,0}$. In other words we consider a compatible fibration $\{\pi_{i,\alpha} : V_{i,\alpha} \to U_{i,\alpha} \mid \alpha \in A_i \cup \{0\}\}$ on $M_i = V_{i,0} \cup (\cup_{\alpha} V_{i,\alpha})$. Let $M$ be the quotient manifold by the diagonal action of $G$ on $P \times M_1$. Then the natural map $\pi : M \to M_0$ is a fiber bundle whose fiber is equal to $M_1$. To define a structure of compatible fibration on $M$ we first prepare several notations for $i = 0, 1$.

- $\tilde{A}_i := A_i \cup \{\infty\}$.
- $A := \tilde{A}_0 \times \tilde{A}_1$.
- $\overset{\sim}{A} := A \setminus (\infty, \infty)$.
- $V_{\alpha_0,\alpha_1} := P|_{V_{0,\alpha_0}} \times_G V_{1,\alpha_1}$ for $\alpha \in \tilde{A}_i$.
- $U_{\alpha_0,\alpha_1} := P_{\alpha_0} \times_G U_{1,\alpha_1}$ for $\alpha \in \tilde{A}_i$.
- $U_{(\alpha_0,\alpha_1)(\beta_0,\beta_1)} := P_{\alpha_0,\beta_0} \times_G U_{1,\alpha_1,\beta_1}$.
- $V := \bigcup_{(\alpha_0,\alpha_1) \in A} V_{\alpha_0,\alpha_1}$.

Then we have the following.

**Proposition 5.2.** A collection of data

\[\{\pi_{\alpha_0,\alpha_1} : V_{\alpha_0,\alpha_1} \to U_{\alpha_0,\alpha_1}, \; U_{(\alpha_0,\alpha_1)(\beta_0,\beta_1)} \mid (\alpha_0, \alpha_1), (\beta_0, \beta_1) \in A\}\]

is a compatible fibration on $V = \bigcup_{(\alpha_0,\alpha_1) \in A} V_{\alpha_0,\alpha_1}$.

**5.2. Product of acyclic compatible systems.** In this subsection we define a product of acyclic compatible systems. To define the product we consider the following data together with the data 1, 2, 3, 4 and 5 in Subsection 5.1.

- (6) a compatible Riemannian metric on $M_i$.
- (7) $W_i$ : a compatible $Cl(TM_i)$-module bundle over $M_i$.
- (8) $\{D_{i,\alpha} \mid \alpha \in A_i\}$ : strongly acyclic compatible system over $V_i = \cup_{\alpha \in A_i} V_{i,\alpha}$.

Together with Assumption 5.1 we assume the following.
Assumption 5.3. The metric on $M_1$ is $G$-invariant, and $W_1 \to M_1$ and $\{D_{1,\alpha}\}$ are $G$-equivariant.

From the Assumption 5.1 (2), the restrictions of $P$ at each fibers of $\pi_{0,\alpha}$ are trivial. Moreover we have the following.

Lemma 5.4. There exists a connection on $P$ which is trivial flat over each fibers of $\pi_{0,\alpha}$ for all $\alpha \in A_0$.

Proof. Take connections $\nabla_\alpha$ for each $P_\alpha \to U_{0,\alpha}$. Let $\nabla_\alpha$ be the pull-back connections of them to $P|_{U_{0,\alpha}} \to V_{0,\alpha}$ by $\pi_{0,\alpha}$. Define a connection $\nabla$ on $P$ by patching $\{\nabla_\alpha\}_\alpha$ by an admissible partition of unity $\{\rho_\alpha^2\}_\alpha$, which satisfies the required property. \hfill $\square$

Using this connection on $P$ and the compatible metric on $M_0$ we have the metric on $P$, and hence the metric on $M$. Moreover since the connection is trivial along fibers of $\{\pi_{0,\alpha}\}$ it induces a family of connections of $\{P_\alpha\}$ and $\{P_{\alpha\beta}\}$, and hence a family of metrics on them. Combining them with the $G$-invariant compatible metric on $M_1$ we have a family of metrics of $\{T[\pi_{0,\alpha}]\}$, $\{TU_{0,\alpha}\}$ and so on. It defines a compatible metric of a compatible fibration $M = \cup_{(\alpha_0,\alpha_1)} V_{\alpha_0,\alpha_1}$. We put $\tilde{W}_0 := \pi^*W_0 = \pi^*_pW_0 \times_G M_1$. $\tilde{W}_1 := P \times_G W_1$ and $W := \tilde{W}_0 \otimes \tilde{W}_1$. Then $W \to M$ is a compatible Clifford module bundle of $M = \cup_{(\alpha_0,\alpha_1)} V_{\alpha_0,\alpha_1}$ with respect to the above induced compatible metric. Now we define differential operators $\tilde{D}_{0,\alpha_0}$ and $\tilde{D}_{1,\alpha_1}$ for each $\alpha_0 \in A_0$ and $\alpha_1 \in A_1$ which act on $\Gamma(\tilde{W}_0|_{V_{\alpha_0,\alpha_1}})$ and $\Gamma(\tilde{W}_1|_{V_{\alpha_0,\alpha_1}})$ respectively. The operator $\tilde{D}_{1,\alpha_1}$ is the one induced from the $G$-equivariant operator $D_{1,\alpha_1}$ on $\Gamma(\tilde{W}_1|_{V_{\alpha_0,\alpha_1}})$. On the other hand $\tilde{D}_{0,\alpha_0}$ is the operator defined as follows: Since $D_{0,\alpha_0}$ is a differential operator along fibers of $\pi_{0,\alpha_0}$ and $P$ is trivial at each fiber of $\pi_{0,\alpha_0}$, we can define the operator acting on the restriction $\Gamma(\pi^*_pW_0|_{\text{fiber}} \times_G M_1)$ using $D_{0,\alpha_0}|_{\text{fiber}}$ and a trivialization of $P|_{\text{fiber}}$. Since such operators along fibers do not depend on trivialization we have a differential operator $\tilde{D}_{0,\alpha_0}$ acting on $\Gamma(\tilde{W}_0|_{V_{\alpha_0,\alpha_1}})$. Using these operators we define an operator acting on $\Gamma(W|_V)$ by $D_{\alpha_0,\alpha_1} := \tilde{D}_{0,\alpha_0} \circ \text{id}_{\tilde{W}_1} + \epsilon_{\tilde{W}_0} \otimes \tilde{D}_{1,\alpha_1}$, where $\epsilon_{\tilde{W}_0}$ is a map on $\tilde{W}_0$ defined by $\epsilon_{\tilde{W}_0}(v) := (-1)^{\text{deg} v} v$. For later convenience we put $\tilde{D}_{\alpha_0,\infty} = \tilde{D}_{\infty,\alpha_1} = 0$.

Proposition 5.5. A collection of differential operators $\{D_{\alpha_0,\alpha_1} \mid (\alpha_0, \alpha_1) \in A\}$ is a strongly acyclic compatible system on $(V, W|_V)$.

Proof. Since $\tilde{D}_{0,\alpha_0} \otimes \text{id}_{\tilde{W}_1}$ and $\epsilon_{\tilde{W}_0} \otimes \tilde{D}_{1,\alpha_1}$ anti-commute each other we have

$$
(\sum_{\alpha_0} \sum_{\alpha_1} t_{\alpha_0,\alpha_1} D_{\alpha_0,\alpha_1})^2 = \left(\sum_{\alpha_0} \left(\sum_{\alpha_1} t_{\alpha_0,\alpha_1}\right) \tilde{D}_{0,\alpha_0}\right)^2 \otimes \text{id}_{\tilde{W}_1} + \epsilon_{\tilde{W}_0} \otimes \left(\sum_{\alpha_1} \left(\sum_{\alpha_0} t_{\alpha_0,\alpha_1}\right) \tilde{D}_{1,\alpha_1}\right)^2
$$

for any family of non-negative numbers $(t_{\alpha_0,\alpha_1})$, and the equality among anti-commutators

$$
\{D_{\alpha_0,\alpha_1}, D_{\alpha_0',\alpha_1'}\} = \{\tilde{D}_{0,\alpha_0}, \tilde{D}_{0,\alpha_0'}\} \otimes \text{id}_{\tilde{W}_1} + \epsilon_{\tilde{W}_0} \otimes \{\tilde{D}_{1,\alpha_1}, \tilde{D}_{1,\alpha_1'}\}.
$$

These equalities imply that if $\{D_{1,\alpha_i}\}_{\alpha_i}$ are strongly acyclic compatible systems, then $\{D_{\alpha_0,\alpha_1}\}_{(\alpha_0, \alpha_1)}$ is so. \hfill $\square$
5.3. Product formula. To apply results in Subsection 4.1 and 4.2, we have to deform the end of $M_i$ and $M$ as in the following way: we can deform $M_i$ into $\tilde{M}_i$ so that $\tilde{M}_i$ has the generalized cylindrical end, and, as we showed in Subsection 4.3, we may assume that the strongly acyclic compatible systems and admissible partitions of unity on $M_i$ satisfy Condition (TI). We may also assume that the deformation $\tilde{M}_1$ for $M_1$ is $G$-equivariant. In addition we can deform $P$ into $\tilde{P}$ together with its connection so that it is translationally invariant on the end. Then one can check that the product $\tilde{M} = \tilde{P} \times_G \tilde{M}_1$ has the generalized cylindrical end. Because of the excision property of the local index, we have that $M_i$ (resp. $M$) and $\tilde{M}_i$ (resp. $\tilde{M}$) have the same local index. So hereafter we assume that $M_i$ and $M$ has the generalized cylindrical end and all the data satisfy Condition (TI).

Let $\{\rho^2_{i,1}\}_{i \in A_i}$ be admissible partitions of unity of $M_i$. We may assume that $\{\rho^2_{i,1}\}$ is $G$-invariant. Using these partitions of unity we have an admissible partition of unity $\{\rho^2_{a_0,a_1}\}_{(a_0,a_1) \in \tilde{A}}$ on $M = \bigcup_{(a_0,a_1)} V_{a_0,a_1}$ which is defined by $\rho_{a_0,a_1}(\{u,y\}) := \rho_{0,a_0}(\pi(u))\rho_{1,a_1}(y)$ for $[u,y] \in M$.

For any translationally invariant Dirac-type operators $D_i$ on $\Gamma(W_i)$, using a local trivialization of $P$ we have their lifts $\tilde{D}_0 \otimes \text{id}_{\tilde{W}_1}$ and $\epsilon_{\tilde{W}_0} \otimes \tilde{D}_1$ on $\Gamma(W)$ as in Subsection 3.6. Note that $D := \tilde{D}_0 \otimes \text{id}_{\tilde{W}_1} + \epsilon_{\tilde{W}_0} \otimes \tilde{D}_1$ is a translationally invariant Dirac-type operator on $\Gamma(W)$.

Because of Lemma 5.5 if we take a positive number $t$ large enough, then the inequality in Proposition 4.10 holds for deformed operators $D_{i,t}$ and $D_t$ on $M_i$ and $M$. On the other hand we have a decomposition $D_t = D^B_t + D^F_t$, where

$$D^B_t := \left( \tilde{D}_0 + t \sum_{a_0} \left( \sum_{a_1} \rho^2_{a_1} \right) \pi^* \rho_{0,a_0} \tilde{D}_{0,a_0} \pi^* \rho_{0,a_0} \otimes \text{id}_{\tilde{W}_1} \right)$$

$$D^F_t := \left( \tilde{D}_1 + t \sum_{a_0} \left( \sum_{a_1} \rho^2_{a_0} \right) \rho_{1,a_1} \tilde{D}_{1,a_1} \rho_{1,a_1} \otimes \text{id}_{\tilde{W}_1} \right).$$

Note that $D^F_t$ is a differential operator along fibers of $\pi : M \to M_0$. They anti-commutes each other. Namely,

**Lemma 5.6.** $D^B_t D^F_t + D^F_t D^B_t = 0$.

Moreover since $M_1$ and $M$ have the generalized cylindrical end structure, we have the following by Lemma 4.12.

**Lemma 5.7.** $D_t$ and $D^F_t$ satisfy Assumption 3.26.

When we write $\ker D_{i,t} = E^0 \oplus E^1$ as the $G$-equivariant $\mathbb{Z}/2$-graded vector space, $G$-equivariant local index of $(M_i, V_i, W_i)$ can be written as $\text{ind}_G(M_i, V_i, W_i) = [E^0] - [E^1] \in R(G)$. Let $E^i$ be the vector bundle over $M_0$ defined by $E^i = P \times_G E^i$. Then the strongly acyclic compatible system on $(M_0, V_0, W_0)$ induces another strongly acyclic compatible
systems on \((M_0, V_0, W_0 \otimes E^i)\) via \(\{D_{0,i} \otimes \text{id}_{E^i}\}\) for \(i = 0, 1\). Lemma 5.6, Lemma 5.7 and the product formula in Section 3 imply the following product formula of local indices.

**Theorem 5.8.** We have the following product formula.

\[
\text{ind}(M_0, V_0, W_0 \otimes E^0) - \text{ind}(M_0, V_0, W_0 \otimes E^1) = \text{ind}(M, V, W) \in \mathbb{Z}.
\]

6. **FOUR-DIMENSIONAL CASE**

6.1. **Local indices for elliptic singularities.** A critical point of a 2n-dimensional singular Lagrangian fibration \(\mu: (M, \omega) \to B\) is called a *nondegenerate elliptic singular point of rank \(k\) \((\leq n)\)* if there exists a symplectic coordinates \(x_1, \ldots, x_n, y_1, \ldots, y_n\) such that in these coordinate, \(\mu\) is written as \(\mu = (x_1, \ldots, x_k, x_{k+1}^2 + y_{k+1}^2, \ldots, x_n^2 + y_n^2)\). See [12, 11, 9].

In this subsection we calculate local indices for elliptic singularities in four-dimensional case.

6.1.1. **Definition of \(RR_0(a_1, a_2)\).** Let \(D := \{z \in \mathbb{C} \mid |z| < 1\}\) be the unit open disc in \(\mathbb{C}\). Let \(X_0\) be the product of two copies of \(D\) with symplectic structure

\[
\omega_0 := \frac{-i}{2\pi} \sum_{k=1}^{2} dz_k \wedge d\overline{z}_k,
\]

and \((L_0, \nabla^{L_0})\) a prequantizing line bundle on \((X_0, \omega_0)\).

Let us consider the structure of a singular Lagrangian fibration \(\mu_0: (X_0, \omega_0) \to [0, 1) \times [0, 1)\) on \(X_0\) which is defined by

\[
\mu_0(z) := (|z_1|^2, |z_2|^2).
\]

We put the following assumption.

**Assumption 6.1.** The cohomology groups \(H^* \left( \mu_0^{-1}(b); (L_0, \nabla^{L_0})|_{\mu_0^{-1}(b)} \right)\) vanish for all points \(b \in [0, 1) \times [0, 1)\) except for \(b = (0, 0)\).

Let \(a_1\) and \(a_2\) be arbitrary integers. We define a good compatible fibration on \(X_0 \setminus \{(0, 0)\}\) consisting of three quotient maps of the torus actions by

\[
\begin{align*}
\pi^0_0: V^0_0 &:= X_0 \cap (\mathbb{C}^* \times \mathbb{C}^*) \to U^0_0 := V^0_0/T^2, \\
\pi^1_0: V^0_1 &:= \{(z_1, z_2) \in X_0 \mid |z_1| > |z_2|, |z_1|^2 + a_1|z_2|^2 \notin \mathbb{Z}\} \to U^0_1 := V^0_1/S^1, \\
\pi^2_0: V^0_2 &:= \{(z_1, z_2) \in X_0 \mid |z_1| < |z_2|, a_2|z_1|^2 + |z_2|^2 \notin \mathbb{Z}\} \to U^0_2 := V^0_2/S^1,
\end{align*}
\]

where the \(T^2\)-action on \(V^0_0\) is the standard one, the \(S^1\)-action on \(V^0_1\) is defined by

\[
t(z_1, z_2) := (tz_1, t^{a_1}z_2),
\]

and the \(S^1\)-action on \(V^0_2\) is defined by

\[
t(z_1, z_2) := (t^{a_2}z_1, t^{a_2}z_2).
\]

We take and fix an arbitrary Hermitian structure \((g_0, J_0)\) invariant under the standard \(T^2\)-action on \(X_0\) and compatible with \(\omega_0\). Since \(g_0\) is \(T^2\)-invariant \(g_0\) induces a compatible Riemannian metric of this compatible fibration.

Let \(W_0\) be the Hermitian vector bundle on \(X_0\) which is defined by

\[
W_0 := \wedge^\bullet_c(TX_0) \otimes L_0.
\]
$W_0$ is a $\mathbb{Z}_2$-graded Clifford module bundle with respect to the Clifford module structure $\{0\}$. We take a compatible system $\{D_i\}_{i=0,1,2}$ to be the family of de Rham operators along fibers of $\pi^0_i$ ($i = 0, 1, 2$) which is defined by the same way as in Example 2.38. By Assumption 6.4 and the definition of $V^0_i$ ($i = 1, 2$) the kernel of all $D_i$ vanish. We should notice that for $z \in V^0_i$ the cohomology groups $H^* \left( \left( \pi^0_i \right)^{-1} (\pi^0_i(z)) ; (L_0, \nabla_{L_0}) |_{\left( \pi^0_i \right)^{-1} (\pi^0_i(z))} \right)$ vanish if and only if $z$ satisfies the equality $|z_1|^2 + a_1 |z_2|^2 \notin \mathbb{Z}$ for $i = 1$ and $a_2 |z_1|^2 + |z_2|^2 \notin \mathbb{Z}$ for $i = 2$. Hence $\{D_i\}$ is strongly acyclic.

**Definition 6.2.** Let $D$ be a Dirac-type operator on $W_0$. We define $RR_0(a_1, a_2)$ to be the local index in the sense of Definition 4.18 with respect to $D$ and the above data.

**Remark 6.3.** $RR_0(a_1, a_2)$ does not depend on the choice of a compatible Hermitian structure $(g_0, J_0)$ since it is deformation invariant.

6.1.2. **Definition of $RR_1(a_+, a_-)$.** Let $X_1 := (0, 1) \times S^1 \times D$ be the product of $(0, 1) \times S^1$ and $D$ with symplectic structure

$$\omega_1 := dr \wedge d\theta + \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z}$$

for $(r, e^{2\sqrt{-1} \theta}, z) \in X_1$, and $(L_1, \nabla_{L_1})$ a prequantizing line bundle on $(X_1, \omega_1)$.

Let us consider the structure of singular Lagrangian fibration $\mu_1 : (X_1, \omega_1) \to (0, 1) \times [0, 1)$ which is defined by

$$\mu_1(r, u, z) := (r, |z|^2).$$

We put the following assumption.

**Assumption 6.4.** For all points $b \in (0, 1) \times [0, 1)$, $H^* (\mu_1^{-1}(b) ; (L_1, \nabla_{L_1})|_{\mu_1^{-1}(b)})$ vanish.

Let $a_+$ and $a_- \in \mathbb{Z}$ be arbitrary integers. We take an element $r_1 \in (0, 1)$ and fix it. Then, we define a good compatible fibration on $X_1 \setminus \mu_1^{-1}(r_1, 0)$ consisting of three quotient maps of the torus actions by

$$\pi^1_0 : V^1_0 := (0, 1) \times S^1 \times (D \setminus \{0\}) \to U^1_0 := V^1_0 / T^2,$$

$$\pi^1_1 : V^1_1 := \{(r, u, z) \in (r_1, 1) \times S^1 \times D \mid r + a_+ |z|^2 \notin \mathbb{Z}\} \to U^1_1 := V^1_1 / S^1,$$

$$\pi^1_2 : V^1_2 := \{(r, u, z) \in (0, r_1) \times S^1 \times D \mid r + a_- |z|^2 \notin \mathbb{Z}\} \to U^1_2 := V^1_2 / S^1,$$

where the $T^2$-action on $V^1_0$ is defined by

$$t(r, u, z) := (r, t_1 u, t_2 z),$$

the $S^1$-action on $V^1_1$ is defined by

$$t(r, u, z) := (r, tu, t^{a_+} z),$$

and the $S^1$-action on $V^1_2$ is defined by

$$t(r, u, z) := (r, tu, t^{a_-} z).$$

We take an arbitrary Hermitian structure $(g_1, J_1)$ which is invariant under the standard $T^2$-action on $X_1$ and compatible with $\omega_1$ and fix it. We define the $\mathbb{Z}_2$-graded Clifford module bundle $W_1$ and the strongly acyclic compatible system in the same way as in Section 6.1.1.
Definition 6.5. Let $D$ be a Dirac-type operator on $W_1$. Then, we define $RR_1(a_+, a_-)$ to be the local index in the sense of Definition 4.18 with respect to $D$ and the above data.

Remark 6.6. $RR_1(a_+, a_-)$ does not depend on the choice of a compatible Hermitian structure $(g_1, J_1)$ since it is deformation invariant.

6.1.3. Computation. First we can show the following lemma.

Lemma 6.7. For integers $a, b, c \in \mathbb{Z}$ we have

$$RR_0(a, b) = RR_0(b, a), \quad RR_1(a, b) = RR_1(a + c, b + c).$$

Proof. We prove the latter equation. The proof of the former equation is similar. Let $\varphi: (0, 1) \times S^1 \times D \to (0, 1) \times S^1 \times D$ be the diffeomorphism which is defined by

$$\varphi(r, u, z) = (r, u, a'u'z).$$

On the target space of $\varphi$ we consider the same compatible fibration as $\{z_i\}_{i=0, 1, 2}$ except that $a$ and $b$ are replaced by $a + c$ and $b + c$, respectively. Then $\varphi$ induces an isomorphism between compatible fibrations.

As the other data on the target space of $\varphi$ we consider the data which are induced from those on the source space by $\varphi^{-1}$. Then the local index for the induced data on the target space is nothing but $RR_1(a, b)$.

On the other hand, the data $(\varphi^{-1})^*\omega_1$ and $(\varphi^{-1})^*\nabla^{L_1}$ can be deformed to $\omega_1$ and $\nabla^{L_1}$ by linear deformations. Since the local index is invariant under continuous deformation this implies that the latter equation.

Moreover, we can also show the following lemma by Theorem 4.19.

Lemma 6.8.

$$RR_0(a, b) = RR_0(a', b) + RR_1(a', a), \quad RR_1(a, c) = RR_1(a, b) + RR_1(b, c).$$

Then we can calculate $RR_0(a_1, a_2)$ and $RR_1(a_+, a_-)$.

Theorem 6.9.

$$RR_0(a_1, a_2) = 1, \quad RR_1(a_+, a_-) = 0.$$ 

Proof. We show $RR_0(0, 1) = 1$ and $RR_0(0, 0) = 1$. Then the theorem follows from these equalities and Lemma 6.7 and 6.8.

First we show $RR_0(0, 1) = 1$. Let us consider the standard toric action on $\mathbb{C}P^2$ with hyperplane bundle as a prequantizing line bundle. We adopt the moment map $\mu$ of this action as a singular Lagrangian fibration. The image $B$ of $\mu$ is the triangle in $\mathbb{R}^2$ with vertices $(0, 0), (1, 0), (0, 1)$, and $\mu$ has three Bohr-Sommerfeld fibers which corresponds one-to-one to three fixed points $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$ of the toric action.

We construct a compatible fibration on $\mathbb{C}P^2 \setminus \{(1 : 0 : 0), [0 : 1 : 0], [0 : 0 : 1]\}$. For each $k \in \mathbb{Z}/3$ let $V_k$ be a pairwise disjoint $T^2$-invariant open neighborhood of $\{[z_0 : z_1 : z_2] \in \mathbb{C}P^2 \mid z_k = 0\} \setminus \{(1 : 0 : 0), [0 : 1 : 0], [0 : 0 : 1]\}$, and $G_k$ the stabilizer of $\{[z_0 : z_1 : z_2] \in \mathbb{C}P^2 \mid z_k = 0\} \setminus \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$. Each $G_k$ is a circle subgroup in $T^2$ and $G_{k-1}$ acts on $V_k$ freely. Then we put $U_k := V_k/G_{k-1}$ and define $\pi_k: V_k \to U_k$ to be the quotient map. We also put $V_4 := U_4 := B \setminus \partial B$ and define $\pi_4: V_4 \to U_4$ to be the identity map. These data define a good compatible fibration on $\mathbb{C}P^2 \setminus \{(1 : 0 : 0), [0 : 1 : 0], [0 : 0 : 1]\}$.
The $\mathbb{Z}_2$-graded Clifford module bundle and the strongly acyclic compatible system are defined by the same way as in Section 6.1.1.

Then by Theorem [4.21] the Riemann-Roch number is localized at $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$, and the contribution of each fixed point is equal to $RR_0(0,1)$.

On the other hand it is well-known that the Riemann-Roch number of $\mathbb{C}P^2$ is 3. Thus we obtain $RR_0(0,1) = 1$.

Next we show $RR_0(0,0) = 1$. It is a direct consequence of the product formula [5.8] and the fact $[D^+] = 1$ (see [3, Theorem 6.7]).

We can also show $RR_0(0,0) = 1$ in the following way. We consider $\mathbb{C}P^1 \times \mathbb{C}P^1$ with standard toric action. The image of the moment map is a square. By the similar construction as above the Riemann-Roch number is localized at four vertices and the contribution of any vertex is $RR_0(0,0)$. On the other hand the Riemann-Roch number of $\mathbb{C}P^1 \times \mathbb{C}P^1$ is four. This implies $RR_0(0,0) = 1$. □

6.2. Application to locally toric Lagrangian fibrations. In this subsection we apply the localization formula (Theorem 4.21), the product formula (Theorem 5.8), and Theorem 6.9 to show that for a four-dimensional closed locally toric Lagrangian fibration the Riemann-Roch number is equal to the number of Bohr-Sommerfeld fibers (Theorem 6.23).

6.2.1. Locally toric Lagrangian fibrations. Let $\omega_{\mathbb{C}^n}$ be the standard symplectic structure on $\mathbb{C}^n$.

$$\omega_{\mathbb{C}^n} := \frac{\sqrt{-1}}{2\pi} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k.$$  

The standard action of $T^n$ on $\mathbb{C}^n$ preserves $\omega_{\mathbb{C}^n}$ and the map $\mu_{\mathbb{C}^n} : \mathbb{C}^n \rightarrow \mathbb{R}^n$ which is defined by

$$\mu_{\mathbb{C}^n}(z) := (|z_1|^2, \ldots, |z_n|^2)$$

for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ is a moment map of the standard $T^n$-action. Note that the image of $\mu_{\mathbb{C}^n}$ is the $n$-dimensional standard positive cone

$$\mathbb{R}^n_+ := \{ r = (r_1, \ldots, r_n) \in \mathbb{R}^n : r_i \geq 0 \; i = 1, \ldots, n \}.$$

Let $(M, \omega)$ be a 2n-dimensional symplectic manifold and $B$ an n-dimensional manifold with corners.

**Definition 6.10** ([13]). A map $\mu : (M, \omega) \rightarrow B$ is called a **locally toric Lagrangian fibration** if there exists a system $\{(U_\alpha, \varphi_\alpha^B)\}$ of coordinate neighborhoods of $B$ modeled on $\mathbb{R}^n_+$, and for each $\alpha$ there exists a symplectomorphism $\varphi_\alpha^M : (\mu^{-1}(U_\alpha), \omega) \rightarrow (\mu_{\mathbb{C}^n}^{-1}(\varphi_\alpha^B(U_\alpha)), \omega_{\mathbb{C}^n})$ such that $\mu_{\mathbb{C}^n} \circ \varphi_\alpha^M = \varphi_\alpha^B \circ \mu$.

Note that a locally toric Lagrangian fibration is a singular Lagrangian fibration that allows only elliptic singularities.

By the definition of a manifold with corners, $B$ is equipped with a natural stratification. We denote by $S^{(k)}B$ the k-dimensional part of $B$, namely, $S^{(k)}B$ consists of those points which have exactly $k$ nonzero components in a local coordinate system. Then, it is easy to see that the fiber of $\mu$ at a point in $S^{(k)}B$ is a k-dimensional torus. In particular, all fibers of $\mu$ are smooth.

**Example 6.11** (Projective toric variety). The moment map of a nonsingular projective toric variety is a locally toric Lagrangian fibration.
Example 6.12 (Non toric example). Let $c \in \mathbb{N}$ be a positive integer. We consider the diagonal Hamiltonian $S^1$-action on $(\mathbb{C}^2, \omega_{\mathbb{C}^2})$ with moment map

$$\Phi(z) := ||z||^2 - c.$$ 

It is well-known that the symplectic quotient $(\Phi^{-1}(0), \omega_{\mathbb{C}^2}|_{\Phi^{-1}(0)}) / S^1$ is $\mathbb{C}P^1$ with $c$ times Fubini-Study form $\omega_{FS}$. In the rest of this example we identify $(\mathbb{C}P^1, c\omega_{FS})$ with $(\Phi^{-1}(0), \omega_{\mathbb{C}^2}|_{\Phi^{-1}(0)}) / S^1$.

Let $\widehat{\mu} : (\wideparen{M}, \omega) \to \wideparen{B}$ be the singular Lagrangian fibration which is defined by

\begin{align*}
\wideparen{M}, \omega &= (\mathbb{R} \times S^1 \times \mathbb{C}P^1, dr \wedge d\theta \oplus c\omega_{FS}), \\
\wideparen{B} &= \mathbb{R} \times [0, c], \\
\widehat{\mu}(r, u, [z_0 : z_1]) &= (r, |z_1|^2),
\end{align*}

where we use the coordinate $(r, e^{2\pi \sqrt{-1}\theta}) \in \mathbb{R} \times S^1$. For a negative integer $a \in \mathbb{Z}$ ($a < 0$) and a positive integer $b \in \mathbb{N}$, we define the $\mathbb{Z}$-actions on $\wideparen{M}$ and $\wideparen{B}$ by

\begin{align*}
n(r, u, [z_0 : z_1]) &= (r + n(-a|z_1|^2 + b), u, [z_0 : u^{na}z_1]), \\
n(r_1, r_2) &= (r_1 + n(-ar_2 + b), r_2).
\end{align*}

It is easy to see that (13) and (14) are free $\mathbb{Z}$-actions and (13) preserves $\omega$. Then we put

\begin{align*}
(M, \omega) &= (\wideparen{M}, \omega)/\mathbb{Z}, \\
B &= \wideparen{B}/\mathbb{Z}.
\end{align*}

It is also easy to see that $\widehat{\mu}$ is equivariant with respect to (13) and (14). Hence $\widehat{\mu}$ induces the map from $M$ to $B$ which we denote by $\mu : (M, \omega) \to B$. By construction, $B$ is a cylinder and $\mu$ is a locally toric Lagrangian fibration which has singular fibers on $\partial B$.

Let $\mu : (M^{2n}, \omega) \to B$ be a locally toric Lagrangian fibration. By definition, for each $\alpha$ there is a symplectomorphism $\varphi^M_\alpha : \mu^{-1}(U_\alpha) \to \mu^{-1}(\varphi^B_\alpha(U_\alpha))$, and $\mu^{-1}(\varphi^B_\alpha(U_\alpha))$ has a $T^n$-action which is obtained by restricting the standard $T^n$-action on $\mathbb{C}^n$. Then, it is known by [13, Proposition 3.13] that on each nonempty overlap $U_\alpha \cap U_\beta$ there exists an automorphism $\rho_{\alpha\beta} \in \text{Aut}(T^n)$ of $T^n$ such that $\varphi^M_\alpha := \varphi^M_\alpha \circ (\varphi^B_\beta)^{-1}$ is $\rho_{\alpha\beta}$-equivariant, namely,

$$\varphi^M_{\alpha\beta}(tx) = \rho_{\alpha\beta}(t)\varphi^M_{\alpha\beta}(x)$$

for $t \in T^n$ and $x \in \mu^{-1}(\varphi^B_\beta(U_\alpha \cap U_\beta))$. Moreover, we can show that $\rho_{\alpha\beta}$’s form a Čech one-cocycle $\{\rho_{\alpha\beta}\}$ on $\{U_\alpha\}$ with coefficients in $\text{Aut}(T^n)$. Hence it defines an element $\{\rho_{\alpha\beta}\}$ in the Čech cohomology $H^1(B; \text{Aut}(T^n))$. Then we have the following lemma.

Lemma 6.13 ([13]). The Čech cohomology class $\{\rho_{\alpha\beta}\}$ is the obstruction class in order that the $T^n$-actions on $\mu^{-1}(\varphi^B_\alpha(U_\alpha))$ for all $\alpha$ can be patched together to obtain a global $T^n$-action on $M$.

For more detail see [13].

Let $q_B : \wideparen{B} \to B$ be the universal covering of $B$. Since the Čech cohomology $H^1(B; \text{Aut}(T^n))$ is identified with the moduli space of representations of the fundamental group $\pi_1(B)$ of $B$ to $\text{Aut}(T^n)$, the fiber product $q_B^* M := \{(\widetilde{b}, x) \in \wideparen{B} \times M \mid q_B(\widetilde{b}) = \mu(x)\}$ admits a $T^n$-action.
We take a representative \( \rho : \pi_1(B) \to \text{Aut}(T^n) \) of the equivalence class of representations corresponding to \( \{[\rho_{\alpha\beta}]\} \). Then the \( T^n \)-action on \( q_B^*M \) can be written explicitly. See [14, Lemma 3.1] for the explicit description.

On the other hand, by the construction, \( \pi_1(B) \) acts on \( q_B^*M \) from the left by the inverse of the deck transformation, and it is shown that the \( T^n \)-action and the \( \pi_1(B) \)-action satisfy the following relationship

\[
(t(a\tilde{x})) = a(\rho (a^{-1})(t)\tilde{x})
\]

for \( t \in T^n \), \( a \in \pi_1(B) \), and \( \tilde{x} \in q_B^*M \). Let \( T^n \ltimes \pi_1(B) \) be the semidirect product of \( T^n \) and \( \pi_1(B) \) with respect to \( \rho \). Then, [15] implies that these actions form an action of \( T^n \ltimes \pi_1(B) \) on \( q_B^*M \). For more details see [14].

Let \( q_M : q_B^*M \to M \) be the natural projection. Note that \( q_M^*\omega \) is \( T^n \ltimes \pi_1(B) \)-invariant since \( \omega \) is invariant under the \( T^n \)-action on \( \mu^{-1}(U_\alpha) \) induced by the standard \( T^n \)-action on \( \mathbb{C}^n \) for each \( \alpha \). Now we show the following lemma.

**Lemma 6.14.** There exists a Hermitian structure \( (\tilde{g}, \tilde{J}) \) on \( q_B^*M \) compatible with \( q_M^*\omega \) which is invariant under the action of \( T^n \ltimes \pi_1(B) \).

**Proof.** It is sufficient to show that the existence of an invariant Riemannian metric. Let \( g' \) be a Riemannian metric on \( M \). We define the Riemannian metric \( \tilde{g} \) on \( q_B^*M \) by

\[
\tilde{g}_{\tilde{x}}(u, v) := \int_{T^n} (\varphi^*(q_M^*g'))_{\tilde{x}}(u, v)dt,
\]

where \( \varphi_t \) implies the \( T^n \)-action for \( t \in T^n \). It is sufficient to show that \( \tilde{g} \) is \( \pi_1(B) \)-invariant. For \( a \in \pi_1(B) \) we denote the \( \pi_1(B) \)-action by \( \phi_a \). Then we have

\[
(\phi_a^*\tilde{g})_{\tilde{x}}(u, v) = \int_{T^n} (\varphi^*(q_M^*g'))_{\tilde{x}}(u, v)dt
\]

\[
= \int_{T^n} (\varphi^*_{\rho(a^{-1})(t)}(q_M^*g'))_{\tilde{x}}(u, v)dt
\]

\[
= \int_{T^n} (\varphi^*_{\rho(a^{-1})(t)}(q_M^*g'))_{\tilde{x}}(u, v)dt
\]

\[
= \det \rho(a^{-1}) \int_{T^n} (\varphi^*_{\rho(a^{-1})(t)}(q_M^*g'))_{\tilde{x}}(u, v)\rho(a^{-1})dt
\]

\[
= \int_{T^n} (\varphi^*_{\rho(a^{-1})(t)}(q_M^*g'))_{\tilde{x}}(u, v)dt.
\]

Here we remark that \( \det \rho(a^{-1}) = \pm 1 \) since \( \rho(a^{-1}) \in \text{Aut}(T^n) \).

**Corollary 6.15** (the existence of an invariant Hermitian structure). There exists a Hermitian structure \( (g, J) \) on \( M \) compatible with \( \omega \) such that on each \( \mu^{-1}(U_\alpha) \) \( (g, J) \) is invariant under the \( T^n \)-action on \( \mu^{-1}(U_\alpha) \) which is induced from the \( T^n \)-action on \( \mu_{\mathbb{C}^n}^{-1}(\varphi^*_\alpha(U_\alpha)) \) with the identification \( \varphi^*_\alpha \).

**Proof.** By Lemma 6.14 there is a \( T^n \ltimes \pi_1(B) \)-invariant Hermitian structure \( (\tilde{g}, \tilde{J}) \) on \( q_B^*M \) compatible with \( q_M^*\omega \). In particular, since \( (\tilde{g}, \tilde{J}) \) is \( \pi_1(B) \)-invariant, \( (\tilde{g}, \tilde{J}) \) induces an \( \omega \)-compatible Hermitian structure on \( M \) which is denoted by \( (g, J) \). Then, \( (g, J) \) is the required one.
Lemma 6.16 (The existence of an averaging operation). Suppose that there exists a compatible fibration \( \{ \pi_\alpha : V_\alpha \to U_\alpha \} \) on \( M \) such that for each \( \alpha \) a fiber of \( \pi_\alpha \) is contained in that of \( \mu \), namely, \( \pi_\alpha^{-1}\pi_\alpha(x) \subset \mu^{-1}\mu(x) \) for \( x \in V_\alpha \). There exists an averaging operation \( I : C^\infty(M) \to C^\infty(M) \) with respect to \( \{ \pi_\alpha : V_\alpha \to U_\alpha \} \).

Proof. For \( f \in C^\infty(M) \) let \( \tilde{f} \in C^\infty(q_B^*M) \) be the function on \( q_B^*M \) which is defined by

\[
\tilde{f}(\bar{x}) := \int_{T^n} (f \circ q_M)(t\bar{x})dt.
\]

Then, by the similar way to that in the proof of Lemma 6.14, we can show that \( \tilde{f} \) is \( T^n \rtimes_R \pi_1(B) \)-invariant. Hence it descends to the function on \( M \). We denote it by \( I(f) \).

Then, it is clear that \( I(f) \) satisfies the properties in Definition 2.11 \( \square \)

6.2.2. Bohr-Sommerfeld fibers and the Riemann-Roch number. Let \( \mu : (M, \omega) \to B \) be a prequantized locally toric Lagrangian fibration with prequantizing line bundle \( (L, \nabla) \).

Recall that, as described above, all fibers are smooth.

Definition 6.17. A fiber \( F \) of \( \mu \) is said to be **Bohr-Sommerfeld** if the restriction \( (L, \nabla)|_F \) is trivially flat. A point \( b \) of \( B \) is also said to be **Bohr-Sommerfeld** if the fiber \( \mu^{-1}(b) \) is Bohr-Sommerfeld.

Remark 6.18. A fiber \( F \) of \( \mu \) is Bohr-Sommerfeld if and only if the cohomology \( H^*(F ; (L, \nabla)|_F) \) does not vanish, see Lemma 2.29. This is also equivalent to the condition that the de Rham operator on \( F \) with coefficients in \( (L, \nabla)|_F \) has non zero kernel.

First we specify Bohr-Sommerfeld points for the local model.

Proposition 6.19. Let \( (L, \nabla) \) be a prequantizing line bundle on \( (\mathbb{C}^n, \omega_{\mathbb{C}^n}) \). Then, a point \( b \in \mathbb{R}^n_+ \) is Bohr-Sommerfeld if and only if \( b \in \mathbb{R}^n_+ \cap \mathbb{Z}^n \).

Proof. Since \( \mathbb{C}^n \) is contractible \( L \) is trivial as a complex line bundle. Then we can assume that \( L \) is of the form \( L = \mathbb{C}^n \times \mathbb{C} \) without loss of generality. Then, \( \nabla \) can be written as

\[
\nabla = d - 2\pi \sqrt{-1} A
\]

for some one form on \( \mathbb{C}^n \) with \( dA = \omega_{\mathbb{C}^n} \). Moreover \( A \) is unique up to exact one form since \( \mathbb{C}^n \) is contractible. In particular, \( A \) is of the form

\[
A = \frac{\sqrt{-1}}{4\pi} \sum_{i=1}^{n} (z_i d\bar{z}_i - \bar{z}_i dz_i) + df
\]

for some smooth function \( f \) on \( \mathbb{C}^n \).

By using the polar coordinate \( z_i = r_i e^{2\pi \sqrt{-1} \theta_i} \), we can write \( \mu_{\mathbb{C}^n} \) and \( A \) in the following forms

\[
\mu_{\mathbb{C}^n} = (r_1^2, \ldots, r_n^2), \quad A = \sum_{i} r_i^2 d\theta_i + df.
\]

In particular, we see that the tangent space along a nonsingular fiber of \( \mu_{\mathbb{C}^n} \) is spanned by \( \partial \theta_i \)'s. Thus a direct computation shows that a point \( b \in \mathbb{R}^n_+ \) is Bohr-Sommerfeld if and only if \( b \in \mathbb{R}^n_+ \cap \mathbb{Z}^n \). \( \square \)

By the above proposition and the definition of a locally toric Lagrangian fibration we can obtain the following corollary.
Corollary 6.20. For a locally toric Lagrangian fibration Bohr-Sommerfeld fibers appear discretely.

Example 6.21. For a nonsingular projective toric variety it is well-known that Bohr-Sommerfeld fibers correspond one-to-one to the integral points in the moment polytope. For example see [3].

Example 6.22. We consider the locally toric Lagrangian fibration $\mu: (M, \omega) \to B$ in Example 6.12. We show that $(M, \omega)$ is prequantizable.

Let $(H_c, \nabla H_c)$ be the $c$ times tensor power of the hyperplane bundle on $\mathbb{C}P^1$. With the identification of $(\mathbb{C}P^1, c\omega_{FS})$ and the symplectic quotient $(\Phi^{-1}(0), \omega_{\mathbb{C}^2|_{\Phi^{-1}(0)}})/S^1$ in Example 6.12 $(H_c, \nabla H_c)$ can be written in the following explicit way

$$(H_c, \nabla H_c) = \left( \Phi^{-1}(0) \times \mathbb{C}, d + 1/2 \sum_i (z_i dz_i - \bar{z}_id\bar{z}_i) \right) / S^1,$$

where the $S^1$-action is defined by

$$t \cdot (z_0, z_1, w) := (t z_0, t z_1, t^c w).$$

Now let us define the prequantizing line bundle $(\tilde{L}, \tilde{\nabla})$ on $(\tilde{M}, \tilde{\omega})$ by

$$(\tilde{L}, \tilde{\nabla}) := \left( \text{pr}^*_1(\mathbb{R} \times S^1 \times \mathbb{C}, d - 2\pi \sqrt{-1} rd\theta) \otimes_{\mathbb{C}} \text{pr}^*_2(H_c, \nabla H_c) \right).$$

We also define the lift of the $\mathbb{Z}$-action (13) on $\tilde{M}$ to $\tilde{L}$ by

$$(r, u, [z_0 : z_1, w]) := (r + n(-a|z_1|^2 + b), u, [z_0 : u^{na}z_1, u^{nb}w]).$$

It is easy to see that (16) preserves $\tilde{\nabla}$ and the standard Hermitian metric. We put

$$(L, \nabla) := (\tilde{L}, \tilde{\nabla})/\mathbb{Z}.$$

Then $(L, \nabla)$ is a prequantizing line bundle on $(M, \omega)$.

Next we see the Bohr-Sommerfeld fibers of $\mu$ with respect to $(L, \nabla)$. The direct computation shows that Bohr-Sommerfeld fibers of $\tilde{\mu}$ correspond one-to-one to the elements in $B \cap \mathbb{Z}^2$. Let $F$ be a fundamental domain of the $\mathbb{Z}$-action (14) on $B$. $F$ is written as

$$F := \{(r_1, r_2) \in \tilde{B} \mid 0 \leq r_2 \leq c, -1/2 \leq r_1 < -ar_2 + b - 1/2\}.$$

Then, Bohr-Sommerfeld fibers of $\mu$ correspond one-to-one to the elements in $F \cap \mathbb{Z}^2$. See Figure [1].

In the rest of this section we assume that $M$ is closed. Let $(g, J)$ be a Hermitian structure on $M$ compatible with $\omega$ as in Corollary 6.15. We define the Hermitian vector bundle $W$ on $M$ by

$$(17) \quad W := \wedge^\bullet TM_C \otimes L.$$

$W$ is a $\mathbb{Z}_2$-graded Clifford module bundle with respect to the Clifford module structure [15]. Let $D$ be the Dirac-type operator on $W$. We define the Riemann-Roch number to be the index of $D$.

The purpose of this section is to show the following theorem.

Theorem 6.23. Let $\mu: (M, \omega) \to B$ be a four-dimensional prequantizable locally toric Lagrangian fibration with prequantizing line bundle $(L, \nabla)$. Then the Riemann-Roch number is equal to the number of both nonsingular and singular Bohr-Sommerfeld fibers.
Proof. Let $B_{BS}$ be the set of Bohr-Sommerfeld points of $\mu$ in $B$. We put $V := \mu^{-1}(B \setminus B_{BS})$. In order to prove Theorem 6.23 we define a good compatible fibration on $V$ as follows.

On the regular non Bohr-Sommerfeld points $U_0 := S^{(2)}B \setminus B_{BS}$ of $\mu$ we define the fibration by

$$\pi_0 := \mu|_{U_0} : V_0 := \mu^{-1}(U_0) \to U_0.$$ 

Since $B$ is compact, there are only finitely many Bohr-Sommerfeld points in $S^{(1)}B$. Suppose we have exactly $k$ Bohr-Sommerfeld points $p_1, \ldots, p_k$ in $S^{(1)}B$, namely,

$$\{p_1, \ldots, p_k\} = B_{BS} \cap S^{(1)}B.$$ 

For each $i$ we take a contractible open neighborhood $W_i$ of $p_i$ in $B$ which satisfies the following properties.

1. For each $i$ $W_i$ contains no Bohr-Sommerfeld points except for $p_i$.
2. For each $i$ $W_i \cap S^{(1)}B$ is connected.
3. For each $i$ $W_i$ does not intersect $S^{(0)}B$, namely, $W_i \cap S^{(0)}B = \emptyset$.
4. $W_i$'s are pairwise disjoint, namely, $W_i \cap W_j = \emptyset$ for all $i \neq j$.
5. There exist finitely many non Bohr-Sommerfeld points in $S^{(1)}B$, say $q_1, \ldots, q_l$, such that we have

$$\bigcup_{i=1}^k W_i \cap S^{(1)}B = S^{(1)}B \setminus \{q_1, \ldots, q_l\}.$$ 

It is possible to take such neighborhoods since a connected component of $\partial B$ is compact.

We put $V_i' := \mu^{-1}(W_i)$. Since $W_i$ is contractible, by [13, Proposition 3.5], there exists a $T^2$-action on $V_i'$. Moreover, there exist a coordinate neighborhood $(U_{\alpha_i}, \phi_{\alpha_i}^B)$ of $B$ containing $p_i$, a diffeomorphism $\phi_{\alpha_i}^M : \mu^{-1}(U_{\alpha_i}) \to \mu^{-1}_C(\phi_{\alpha_i}^B(U_{\alpha_i}))$ in Definition 6.10 and an automorphism $\rho_{\alpha_i} \in \text{Aut}(T^2)$ which satisfy the following properties.

- $\mu_C \circ \phi_{\alpha_i}^M = \phi_{\alpha_i}^B \circ \mu$.
- On $V_i' \cap \mu^{-1}(U_{\alpha_i}) \phi_{\alpha_i}^M$ is $\rho_{\alpha_i}$-equivariant with respect to the $T^2$-action on $V_i'$ and the standard $T^2$-action on $\mathbb{C}^2$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example6.12}
\caption{Bohr-Sommerfeld points in Example 6.12}
\end{figure}
Let \( \varphi_B^\alpha(p_i) = (r_1, r_2) \in \mathbb{R}_+^2 \). Since \( p_i \in S(1)B \) there exists a unique coordinate \( r_{j_i} \), such that \( r_{j_i} = 0 \). We define the circle subgroup \( T_i \) of \( T^2 \) by

\[
T_i := \rho^{-1}_\alpha \left( \{ t = (t_1, t_2) \in T^2 \mid t_{j_i} = e \} \right).
\]

By the properties 2 and 3 of \( W_i \), \( T_i \) acts freely on \( V'_i \). Then, for each \( i \) we define the open set \( V_i \) to be the complement of all \( T_i \)-orbits in \( V'_i \) on each of which \( (L, \nabla) \) has a non-trivial global parallel section. We also define the fibration \( \pi_i : V_i \to U_i \) to be the natural projection with respect to the \( T_i \)-action on \( V_i \).

In order to show this equality, it is sufficient to prove the following claim: for a point \( x \in \mu^{-1}(W_i \cap S(1)B) \) the restriction of \( (L, \nabla) \) to the fiber \( \mu^{-1}(\mu(x)) \) of \( \mu \) is trivially flat if and only if the restriction of \( (L, \nabla) \) to the \( T_i \)-orbit \( T_i x \) through \( x \) is trivially flat. This can be shown as follows. By the property 2 of \( W_i \), the \( T^2 \)-action on \( \mu^{-1}(W_i \cap S(1)B) \) has the unique one-dimensional stabilizer \( S \subset T^2 \). For example, in case of \( j_i = 2 \), \( S \) is written as

\[
S = \rho^{-1}_\alpha \left( \{ t = (t_1, t_2) \in T^2 \mid t_1 = e \} \right).
\]

In any case, by the definition of \( T_i \), \( T^2 \) is decomposed as \( T^2 = T_i \times S \). Then, the above claim follows from the fact that in the \( T^2 \)-action on \( \mu^{-1}(\mu(x)) \) \( S \) acts trivially and \( T_i \) acts freely on \( \mu^{-1}(\mu(x)) \).

By construction \( \{ \pi_i : V_i \to U_i \mid i = 0, \ldots, k \} \) is a good compatible fibration on \( V \). Moreover, by Lemma 6.16 there is an averaging operation with respect to \( \{ \pi_i : V_i \to U_i \mid i = 0, \ldots, k \} \).

Recall that \( (g, J) \) is a Hermitian structure on \( M \) compatible with \( \omega \) as in Corollary 6.15. Then, as in the case of usual torus actions, \( g \) defines the compatible Riemannian metric of \( \{ \pi_i : V_i \to U_i \mid i = 0, \ldots, k \} \) whose restriction to each fiber of \( \mu \) is flat, and the \( \mathbb{Z}_2 \)-graded Clifford module bundle \( W \) defined by (17) becomes a compatible Clifford module bundle in the sense of Definition 2.19. We define the strongly acyclic compatible system.
in the same way as in Section 6.1.1. Then by Theorem 4.21 the Riemann-Roch number is localized at Bohr-Sommerfeld fibers and the fibers at \( q_1, \ldots, q_l \).

We consider their contributions. Since a fiber of \( \mu \) is connected, by Theorem 6.11, the contribution of a regular Bohr-Sommerfeld fiber is equal to one.

Next we consider the contributions of singular Bohr-Sommerfeld fibers. By Definition 6.11 each fiber on \( S^{(0)}B \) is Bohr-Sommerfeld, and its contribution is \( RR_0(a_1, a_2) \) for some \( a_1 \) and \( a_2 \). By Theorem 6.9 it is equal to one.

By the construction of the compatible fibration the local Riemann-Roch number for each singular Bohr-Sommerfeld fiber on \( S \) is localized at Bohr-Sommerfeld fibers and the fibers at \( q_1, \ldots, q_l \). Then by Theorem 4.21, the Riemann-Roch number

\[
\text{RR}_V(a) \quad \text{for each singular Bohr-Sommerfeld fiber on } S \text{ on } M, \omega
\]

is equal to the number of the lattice points in the moment polytope, in the four-dimensional case.

Example 6.24. Theorem 6.23 recovers Danilov’s result [3], which says that for a non-singular projective toric variety the Riemann-Roch number is equal to the number of the lattice points in the moment polytope, in the four-dimensional case.

Example 6.25. As we described in Example 6.22, the Bohr-Sommerfeld fibers correspond one-to-one to the elements in \( F \cap \mathbb{Z}^2 \). Then by Theorem 6.23 the Riemann-Roch number of \((M, \omega)\) is equal to the number of the elements in \( F \cap \mathbb{Z}^2 \) which is \((c + 1)(2b - ac)/2\).

Appendix A. Proof of Proposition 2.15

In this appendix we give a proof of Proposition 2.15. We first give the precise statement.

Let \( \{\pi_\alpha : V_\alpha \to U_\alpha\}_\alpha \) be a good compatible fibration on \( M \). Let \( \{V'_\alpha\}_\alpha \) be an admissible open covering of \( M \) in the second condition in Assumption 2.12. For each \( \alpha \in A \) take an open subset \( U'''_\alpha \) of \( U_\alpha \) such that \( \pi_\alpha(V'_\alpha) \subseteq U'''_\alpha \subseteq U_\alpha \). We put \( V''_\alpha := \pi_\alpha^{-1}(U''') \). For each \( \alpha \) take a smooth function \( \tau_\alpha : M \to \mathbb{R} \) such that \( \tau_\alpha \equiv 1 \) on \( V'_\alpha \) and \( \tau_\alpha \equiv 0 \) on \( M \setminus V''_\alpha \). In this appendix we show the following.

Proposition A.1. If \( \{\pi_\alpha\} \) is a good compatible fibration, then there exists an averaging operation \( I : C^\infty(M) \to C^\infty(M) \) such that for all \( f \in C^\infty(M) \) and \( x \in M \) we have

\[
\min_{y \in \pi_\alpha^{-1}(\pi_\alpha'(x))} f(y) \leq I(f)(x) \leq \max_{y \in \pi_\alpha^{-1}(\pi_\alpha'(x))} f(y),
\]

where \( \alpha'_\alpha := \max\{\alpha \in A \mid x \in V''_\alpha\} \).

Proof. For each \( \alpha \in A \) we define a linear map \( I_\alpha : C^\infty(M) \to C^\infty(M) \) by

\[
I_\alpha(f)(x) := (1 - \tau_\alpha(x))f(x) + \tau_\alpha(x)I_\alpha^0(f)(x),
\]

where \( I_\alpha^0(f) \) is the integration along fibers of \( \pi_\alpha : V_\alpha \to U_\alpha \) with the normalization condition \( I_\alpha^0(1) \equiv 1 \). Fix any total order of \( A = \{\alpha_1, \cdots, \alpha_n\} \) so that if \( V_{\alpha_i} \cap V_{\alpha_j} \neq \emptyset \) and the dimension of the fiber of \( \pi_{\alpha_j} \) is bigger than that of \( \pi_{\alpha_i} \) then \( i > j \). Using this total order we may write \( A = \{1, 2, \cdots, n\} \), \( \pi_{\alpha_k} = \pi_k \) and so on. We define a linear map \( I : C^\infty(M) \to C^\infty(M) \) by

\[
I(f) := I_1I_2 \cdots I_{n-1}I_n(f).
\]

Note that \( I \) satisfies the conditions (2) and (3) in Definition 2.11. We show that \( I \) satisfies the conditions (1), (4), and (5) respectively in the following three lemmas.
Lemma A.2. \(I(f)\) is an admissible function for all \(f \in C^\infty(M)\). More precisely, if \(x, y \in V'_\alpha\) and \(\pi_k(x) = \pi_k(y)\) for some \(k \in A\), then we have \(I(f)(x) = I(f)(y)\).

Proof. Note that, by definition, if \(x, y \in V'_k\) and \(\pi_k(x) = \pi_k(y)\) for some \(k \in A\), then we have \(x = x'\) and \(\pi_k(x) = \pi_k(y)\) for any \(\alpha \leq \alpha'' \leq \alpha' \) and \(x, y \in V'_\alpha\). If \(\alpha > \alpha'' \) and \(x' \in \pi_{\alpha'-1}(x)\), then we have \(x' \notin V'_\alpha\), and hence, \(I_\alpha(g)(x') = g(x')\) for any smooth function \(g\). In particular we have \(I_{\alpha''}(g)(x) = I_{\alpha''}(g)(x)\). Put \(\alpha_x'' := \max\{\alpha \in A \mid x \in V'_\alpha\}\). Note that \(\alpha_x'' \geq \alpha''\) and \(x \in V''_\beta\), then since \(I_{\alpha''}(g)\) is constant along \(\pi^{-1}_\beta \pi(x) \subset V''_\alpha\), we have \(I_\beta I_{\alpha''}(g)(x) = I_{\alpha''}(g)(x)\). By induction we have

\[
I(f)(x) = I_{\alpha_x''} \cdots I_{\beta}(f)(x),
\]

where we may assume that each \(\beta \in A\) satisfies \(\alpha'' \leq \beta \leq \alpha''\) and \(x, y \in V''_\beta\). Since \(x, y \in V'_{\alpha_x''}\) and \(\pi_{\alpha_x''}(x) = \pi_{\alpha_x''}(y)\) we have

\[
I(f)(x) = I_{\alpha''}^0(\cdots I_{\beta}(f)) = I_{\alpha''}^0(\cdots I_{\beta}(f))(y) = I(f)(y).
\]

Lemma A.3. Let \(f : M \to \mathbb{R}\) be a smooth function and \(x \in M\). Then we have

\[
\min_{y \in \pi^{-1}_{\alpha''} \pi_{\alpha''}} f(y) \leq I(f)(x) \leq \max_{y \in \pi^{-1}_{\alpha''} \pi_{\alpha''}} f(y),
\]

where \(\alpha'' := \max\{\alpha \in A \mid x \in V'_\alpha\}\).

Proof. Note that

\[
I_\alpha(g)(x) = (1 - \tau_\alpha(x))g(x) + \tau_\alpha(x)I^0_\alpha(g)(x) \leq \max_{\pi^{-1}_{\alpha''} \pi_\alpha} g
\]

for any smooth function \(g : M \to \mathbb{R}, \alpha \in A\) and \(x \in V'_\alpha\). On the other hand as we showed in the proof of Lemma A.2, we have

\[
I(f)(x) = I_{\alpha_x''} \cdots I_{\beta}(f)(x)
\]

for any \(x \in M\). Since \(\pi^{-1}_{\alpha''} \pi_{\alpha''} \subset \pi^{-1}_\beta \pi_\beta \subset \cdots \subset \pi^{-1}_{\alpha''} \pi_{\alpha''}\) we have

\[
I(f)(x) \leq \max_{\pi^{-1}_{\alpha''} \pi_{\alpha''}} \cdots \max_{\pi^{-1}_{\alpha''} \pi_{\alpha''}} f
\]

by (18) and induction.

Lemma A.4. Let \(f : M \to \mathbb{R}\) be a smooth function. If \(\text{supp}f\) is contained in \(V'_\alpha\) for some \(\alpha \in A\), then \(\text{supp}I(f)\) is also contained in \(V'_\alpha\).

Proof. Let \(f : M \to \mathbb{R}\) be a smooth function such that \(\text{supp}f \subset V'_\alpha\) for some \(\alpha \in A\). It is enough to show that \(\text{supp}I_\beta(f) \subset V'_\alpha\) for each \(\beta \in A\). Take \(x \in \text{supp}I_\beta(f)\) and a sequence \(\{x_n\}\) in \(M\) which satisfies \(I_\beta(f)(x_n) \neq 0\) and converges to \(x\). By definition of \(I_\beta(f)\) we have \(f(x_n) \neq 0\) for infinitely many \(n\) or \(\tau_\beta(x_n)I_\beta^0(f)(x_n) \neq 0\) for infinitely many \(n\). The former case implies that \(x \in \text{supp}f \subset V'_\alpha\). In the latter case by taking a subsequence we may assume that \(x_n \in V''_\beta\) and \(I_\beta^0(f)(x_n) \neq 0\) for all \(n\). In particular we have \(x \in V''_\beta \subset V'_\beta\). Since \(I^0_\beta\) is the integration along fibers there exist a sequence \(\{y_n \in \pi^{-1}_\beta \pi_\beta(x_n)\}\) such that \(f(y_n) \neq 0\) for all \(n\). By taking a subsequence we may assume \(\{y_n\}\) converges to some \(y \in \pi^{-1}_\beta \pi_\beta(x) \cap \text{supp}f \subset V'_\alpha\). Since \(\pi^{-1}_\beta \pi_\beta(V'_\alpha \cap V'_\beta) = V'_\alpha \cap V'_\beta\) we have \(x \in \pi^{-1}_\beta \pi_\beta(x) \subset V'_\alpha\).

\[\square\]
Appendix B. A sufficient condition for Assumption 2.12 (2)

We give a sufficient condition for Assumption 2.12 (2).

Lemma B.1. Let \( \{\pi_\alpha\}_{\alpha \in A} \) be a compatible fibration on \( M \) such that \( A \) is a finite set and the condition 5’ in Definition 2.14 is satisfied. If each \( \pi_\alpha \) has a continuous extension as a fiber bundle to the closure of \( V_\alpha \) with the condition

\[
V_\alpha \cap \overline{V_\beta} = \pi_\beta^{-1} \pi_\beta(V_\alpha \cap \overline{V_\beta})
\]

for all \( \beta \in A \), then \( \{\pi_\alpha\} \) satisfies the second condition in Assumption 2.12. Namely there exist an admissible open covering \( \{V'_\alpha \mid \alpha \in A\} \) of \( M \) such that \( \overline{V'_\alpha} \subset V_\alpha \).

Proof. Take and fix any open covering \( \{W_\alpha\}_{\alpha \in A} \) of \( M \) which satisfies \( \overline{W_\alpha} \subset V_\alpha \). Fix any total order of \( A = \{\alpha_1, \ldots, \alpha_n\} \) so that if \( V_{\alpha_i} \cap V_{\alpha_j} \neq \emptyset \) and the dimension of \( \pi_{\alpha_i} \) is bigger than that of \( \pi_{\alpha_j} \) then \( i > j \). Fix \( \alpha \in A \) and we define an increasing sequence of open sets \( V^{(k)}_\alpha \subset V_\alpha \) inductively in the following way:

\[
V^{(0)}_\alpha := W_\alpha
\]

\[
\vdots
\]

\[
V^{(k)}_\alpha := \pi_{\alpha_k}^{-1} \pi_{\alpha_k}(V^{(k-1)}_\alpha \cap V_{\alpha_k}) \cup V^{(k-1)}_\alpha
\]

\[
\vdots
\]

\[
V^{(n)}_\alpha := \pi_{\alpha_n}^{-1} \pi_{\alpha_n}(V^{(n-1)}_\alpha \cap V_{\alpha_n}) \cup V^{(n-1)}_\alpha
\]

By the construction \( \{V'_\alpha := V^{(n)}_\alpha\}_{\alpha \in A} \) is an admissible open covering of \( M \). We show \( \overline{V'_\alpha} \subset V_\alpha \) by induction on \( k \). Suppose that \( \{p_i\}_{i \in \mathbb{N}} \) is a sequence in \( V^{(k)}_\alpha \) which converges to \( p_\infty \) in \( M \). It is enough to show that if \( p_i \in \pi_{\alpha_k}^{-1} \pi_{\alpha_k}(V^{(k-1)}_\alpha \cap V_{\alpha_k}) \) for all \( i \) then \( p_\infty \in V_\alpha \).

In this case we have \( p_\infty \in \pi_{\alpha_k}^{-1} \pi_{\alpha_k}(V^{(k-1)}_\alpha \cap V_{\alpha_k}) \). On the other hand since the fibers are compact, \( \pi_{\alpha_k} \) is a closed map. Using the assumption (19) we have

\[
\pi_{\alpha_k}^{-1} \pi_{\alpha_k}(V^{(k-1)}_\alpha \cap V_{\alpha_k}) \subset \pi_{\alpha_k}^{-1} \pi_{\alpha_k}(V^{(k-1)}_\alpha \cap V_{\alpha_k}) \subset \pi_{\alpha_k}^{-1} \pi_{\alpha_k}(V^{(k-1)}_\alpha \cap V_{\alpha_k}) \subset V_\alpha \cap V_{\alpha_k}.
\]

In particular we have \( p_\infty \in V_\alpha \). \( \square \)

Remark B.2. Since \( \overline{V'_\alpha} \subset V_\alpha \), one can check that \( \{V'_\alpha\}_\alpha \) satisfies the same condition as in Assumption 19 i.e., \( V'_\alpha \cap \overline{V'_\beta} = \pi_\beta^{-1} \pi_\beta(V'_\alpha \cap \overline{V'_\beta}) \) for all \( \alpha, \beta \in A \).

Appendix C. Proof of Lemma 2.30

Proof of Lemma 2.30 Let \( H_1, H_2, \ldots, H_m \) be the elements of \( A \). Without loss of generality we assume that \( H_i \supset H_j \) implies \( i \leq j \). We construct a family of open sets \( V^{(j)}_i \) \((1 \leq i \leq j \leq m)\) by induction on \( 1 \leq i \leq m \). For the construction with \( i = i_0 \) we assume the following properties.

(C1) \( V^{(j)}_i \) contains the closure of \( V^{(j+1)}_i \) for all \( 1 \leq i < i_0 \) and \( i \leq j < m \).

(C2) If \( x \in V^{(i)}_i \) for \( 1 \leq i < i_0 \), then we have \( G_x \subset H_i \).

(C3) For \( x \in M \) with \( G_x = H_i \) for some \( 1 \leq i < i_0 \), we have

\[
x \in \bigcup_{\{j \mid H_j \supset H_i\}} V^{(m)}_j.
\]
Proof of Lemma 3.10.

If there is a function constructed as follows: For each $\{x_i\}$ disks centered in closed set $c$ is compact for any $V = \text{open neighborhood}$ required properties.

If $i_0 = 1$, then the above is the empty assumption. For $1 \leq i_0 \leq m$, using the above properties as the assumption of induction, we construct $V_{i_0}^{(i_0)} (i_0 \leq j \leq m)$ which satisfy the above properties with replacement of $i_0$ by $i_0 + 1$.

Suppose $1 \leq i_0 < m$ and assume (C1),(C2),(C3) and (C4). Then (C3) implies that the closed set

$$K_{i_0} := M^{H_{i_0}} \setminus \bigcup_{\{k \mid H_k \not\subset H_{i_0}\}} V_k^{(m)}$$

is contained in $\{x \in M \mid G_x = H_{i_0}\}$, where $M^{H_{i_0}}$ is the fixed point set $M^{H_{i_0}} = \{x \in M \mid G_x \supset H_{i_0}\}$. Hence (C2) implies that $K_{i_0}$ does not intersect with the open set

$$\bigcup_{\{j < i_0 \mid H_{i_0} \not\subset H_j\}} V_j^{(i_0 - 1)}.$$

Let $L_{i_0}$ be the closure of

$$\bigcup_{\{j < i_0 \mid H_{i_0} \not\subset H_j\}} V_j^{(i_0)}.$$

Then (C1) implies $K_{i_0} \cap L_{i_0} = \emptyset$. Since $K_{i_0}$ is a subset of $\{x \in M \mid G_x = H_{i_0}\}$, there is an open neighborhood $V$ of the closed set $K_{i_0}$ in the complement of $L_{i_0}$ such that for each $x \in V$ we have $G_x \subset H_{i_0}$. Now we take a decreasing sequence of open neighborhoods $V_{i_0}^{(j)} (i_0 \leq j \leq m)$ of $K_{i_0}$ so that $V_{i_0}^{(i_0)} = V$, $V_{i_0}^{(m)} \supset K_{i_0}$ and $V_{i_0}^{(j)}$ contains the closure of $V_{i_0}^{(j+1)}$ for $i_0 \leq j < m$. We can choose the decreasing sequence so that the open sets $V_{i_0}^{(j)} (i_0 \leq j \leq m)$ are $G$-invariant because the quotient space $M/G$ is a regular space.

Then it is straightforward to check (C1),(C2),(C3) and (C4) are satisfied with $i_0$ replaced by $i_0 + 1$.

The family of open sets $\{V_{i_0} := V_i^{(m)}\}^{i_0 \leq i \leq m}$ is an open covering of $M$ and satisfies the required properties.

Now if a family of open sets $\{V_x\}_{x \in M}$ satisfying $gV_x = V_{gx}$ for all $g \in G$ is given, then since $K_{i_0}$ is contained in $\bigcup_{\{x \mid G_x = H_{i_0}\}} V_x$ we can take a $G$-invariant open neighborhood $V = V_{i_0}$ of $K_{i_0}$ so that $V \subset \bigcup_{G_x = H_{i_0}} V_x$, and hence, we have the required open covering $\{V_H\}_{H \in A}$. $\square$

Appendix D. Proof of Lemma 3.10

Proof of Lemma 3.10. If there is a function $f$ satisfying the property in (1), then $\rho_{\epsilon,a}$ is constructed as follows: For each $\epsilon > 0$ let $\rho_\epsilon : \mathbb{R} \to [0, 1]$ be a smooth non-increasing function such that $\rho_\epsilon (l) = 1$ for $l \leq 0$, $\rho_\epsilon (l) = 0$ for $l \geq 2/\epsilon$ and $|d\rho_\epsilon (l)| < \epsilon$ for $i \in \mathbb{R}$. Then the composition $\tilde{\rho}_{\epsilon,a} (x) = \rho_\epsilon (f(x) - a)$ has the required properties.

Not we construct $f$ by smoothing the length function as follows. Fix a point $x_0 \in M$. Let $f_0 : M \to \mathbb{R}$ be the length from $x_0$. Then $f_0$ is a Lipschitz continuous function with Lipschitz constant 1. Since $M$ is complete, $f_0$ is a proper function such that $f_0^{-1} ((-\infty, c])$ is compact for any $c$. Let $\{\text{int}(D_{x_\gamma} (R_\gamma))\}$ be a locally finite open covering of $M$ by open disks centered in $x_\gamma$ with radius $R_\gamma$. Fix an isometry $TM_{x_\gamma} \cong \mathbb{R}^n$. We also assume that the exponential map centered in $x_\gamma$ gives a coordinate of $M_\gamma = \text{int}(D_{x_\gamma} (R_\gamma))$, and
the derivative of the exponential map and its inverse at any point has bounded by 2 with respect to operator norm. In particular $f_0$ has Lipschitz constant 2 for the standard metric on $\mathbb{R}^n$. We use this coordinate in the following local construction. Let $\{\rho_\gamma\}$ be a smooth partition of unity for it. Let $0 < r_\gamma < R_\gamma$ be the radius of the smallest disk centered in $x_\gamma$ containing the image of the support of $\rho_\gamma$. Let $C_\gamma$ be the maximal value of $|d\rho_\gamma|$ for the standard metric on $\mathbb{R}^n$. Let $n_\gamma$ be the number of open disks in the locally finite covering which intersects $D_{x_\gamma}(r_\gamma)$. Take a smooth function $K : \mathbb{R}^n \to \mathbb{R}$ satisfying $\int K(y) dy = 1$ and $K(y) = 0$ if $|y| > \min\{1, (R_\gamma - r_\gamma)/2, 1/(n_\gamma C_\gamma)\}$. Then the smoothing of $f$ defined by $f_\gamma(x) = \int K(x - y) f_0(y) dy$ ($x \in D_{x_\gamma}(r_\gamma)$) is Lipschitz continuous with Lipschitz constant 2 for the standard metric on $\mathbb{R}^n$, and satisfies $|f_\gamma(x) - f_0(x)| < \min\{1, 2/(n_\gamma C_\gamma)\}$ for $x \in D_{x_\gamma}(r_\gamma)$. Now define $f$ to be $\sum_\gamma \rho_\gamma f_\gamma$. Then $|f - f_0| \leq 1$. In particular $f$ is also a proper map and $f^{-1}((\infty, c])$ is compact for any $c$. Decompose $d f$ as follows: $d f = (\sum_\gamma \rho_\gamma d f_\gamma) + (\sum_\gamma (d \rho_\gamma) f_0) + (\sum_\gamma d \rho_\gamma (f_\gamma - f_0))$. Since the second term is zero, we have $|d f| \leq \sum_\gamma \rho_\gamma |d f_\gamma| + \sum_\gamma |d \rho_\gamma| |f_\gamma - f_0|$. Both terms are bounded from our construction. \qed

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