State-dependent graviton noise in the equation of geodesic deviation

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Abstract We consider an equation of the geodesic deviation appearing in the problem of gravitational wave detection in an environment of gravitons. We investigate a state-dependent graviton noise (as discussed in a recent paper by Parikh, Wilczek and Zahariade) from the point of view of the Feynman integral and stochastic differential equations. The evolution of the density matrix and the transition probability in an environment of gravitons is obtained. We express the time evolution by a solution of a stochastic geodesic deviation equation with a noise dependent on the quantum state of the gravitational field.

1 Introduction

The study of the effect of an environment, which is not directly observable, on the motion of macroscopic bodies, began with the phenomenon of Brownian motion. The mathematical theory of Brownian motion allowed to confirm the presence of invisible molecules (see [1]). In a recent paper Parikh, Wilczek and Zahariade (PWZ) [2] suggest that in a similar way we can detect the environment of gravitons in spite of the negative conclusions of Dyson [3]. According to PWZ the noise resulting from the ground state or coherent state of gravitons is weak and practically undetectable. However, the high temperature states and squeezed states of the graviton can lead to a substantial increase of noise which can be detected in the gravitational wave detection experiments. The authors [2] derive their results by an investigation of the geodesic deviation equation in the environment of the quantized gravitational field. They apply the influence functional method [4] in order to transform the evolution of the transition probability of macroscopic bodies into an expectation value with respect to the noise disturbing the geodesic deviation equation. Studies on the effect of gravitons on the motion of other particles appeared earlier [5–10], but these papers did not tackle directly the geodesic deviation equation. The quantum noise from the environment has been investigated in other fields of physics [11–13]. It can be detected in quantum optics (Ref. [14], sec. 14) and solid state physics.

The evolution of the density matrix in an environment of unobservable particles is usually treated by means of the influence functional [4] which starts from the Feynman integral. As a result one can express the evolution of the density matrix as a solution of the master equation. One can also express the density matrix by the Wigner function and derive a stochastic equation for the evolution of the Wigner function. We have derived such equations for the thermal graviton environment by means of the Feynman integral in [7]. In this paper we apply the method to the quantum evolution of a particle following the geodesic deviation equation when the quantum gravitational field is in an arbitrary Gaussian state. We apply a method of transforming the Feynman integral into an expectation value over a state-dependent noise. In Sect. 3 we show how this transformation applies to quantum field theory. In Sect. 4 we discuss in detail perturbations by noise resulting from the time-dependent Gaussian states. We explain the calculations of the density matrix and the transition probability in a linear coupling to the environment in Sect. 5. We apply the method to the PWZ model in Sect. 6 (as introduced in [2]) in the one mode approximation. In Sect. 7 the influence of infinite number of modes of the thermal gravitational field upon the geodesic deviation equation is treated by the method of the forward-backward Feynman integral as previously applied in [7] to a geodesic motion. In Sect. 8 we apply our method of the state-dependent noise to derive a stochastic deviation equa-
tion which governs the evolution of the density matrix and the transition probability. Section 9 contains a summary of the results. In the Appendix we discuss the stochastic deviation equation which results from the assumption that the initial values of the classical gravitational field have the thermal Gibbs distribution.

2 State-dependent transformation of the Feynman integral

Let us consider first a simple model of the Schrödinger equation of quantum mechanics in one dimension with the time-dependent potential \( U_t \)

\[
i\hbar \partial_t \psi_t = \left( -\frac{\hbar^2}{2m} \nabla^2_x + U_t \right) \psi_t.
\] (1)

Assume that we have a solution \( \psi^g_t \) of another Schrödinger equation (with a potential \( U_t \))

\[
i\hbar \partial_t \psi^g_t = \left( -\frac{\hbar^2}{2m} \nabla^2_x + \tilde{U}_t \right) \psi^g_t.
\] (2)

Let us write the solution of Eq. (1) in the form

\[
\psi_t = \psi^g_t \chi_t.
\] (3)

Inserting \( \chi_t \) from Eq. (3) into Eqs. (1)–(2) we find that \( \chi_t \) satisfies the equation

\[
\partial_t \chi_t = \frac{i\hbar}{2m} \nabla^2_x \chi_t + \frac{i\hbar}{m} (\nabla_x \ln \psi^g_t) \nabla_x \chi_t - \frac{i}{\hbar} (U_t - \tilde{U}_t) \chi_t
\] (4)

with the initial condition

\[
\chi_0 = \psi_0(\psi^g_0)^{-1}
\] (5)

expressed by the initial conditions for \( \psi_t \) and \( \psi^g_t \).

We apply Eq. (4) either with \( \tilde{U}_t = U_t \) or with \( U_t = V_t + \frac{m\omega^2x^2}{2} \) and \( \tilde{U}_t = m\omega^2x^2 \). In the first case Eq. (4) reads

\[
\partial_t \chi_t = \frac{i\hbar}{2m} \nabla^2_x \chi_t + \frac{i\hbar}{m} (\nabla_x \ln \psi^g_t) \nabla_x \chi_t - \frac{i}{\hbar} (U_t - \tilde{U}_t) \chi_t.
\] (6)

In the second case

\[
\partial_t \chi_t = \frac{i\hbar}{2m} \nabla^2_x \chi_t + \frac{i\hbar}{m} (\nabla_x \ln \psi^g_t) \nabla_x \chi_t - \frac{i}{\hbar} V_t \chi_t
\] (7)

Equation (6) can be considered as the diffusion equation with the imaginary diffusion constant \( \frac{i\hbar}{m} \) and a time-dependent drift \( \frac{i\hbar}{m} \nabla_x \ln \psi^g_t \). If we extend the diffusion theory [1,17] to a complex domain then we can conclude that the solution of Eq. (6) is determined by the solution of the Langevin equation

\[
dq_t = \frac{i\hbar}{m} \nabla \ln \psi^g_{t-s}(q_s) ds + \sqrt{\frac{i\hbar}{m}} dq_t.
\] (8)

Here, the Brownian motion \( dq_t \) is defined as the Gaussian process with the covariance

\[
E[q_tq_s] = min(t,s).
\] (9)

The solution of Eq. (6) is

\[
\chi_t(x) = E[\chi_0(q_t(x))],
\] (10)

where \( q_t(x) \) is the solution of Eq. (8) with the initial condition \( q_0(x) = x \) and the expectation value is over the paths of the Brownian motion.

The solution of Eq. (7) is expressed [17] by the Feynman–Kac formula (where the Feynman paths are replaced by the paths of the diffusion process \( q_t(x) \) of Eq. (8))

\[
\chi_t(x) = E\left[ \exp \left( \frac{-i}{\hbar} \int_0^t ds V_{t-s}(q_s(x)) \right) \chi_0(q_t(x)) \right]
\] (11)

We could derive Eq. (10) from the Feynman integral. The solution \( \psi_t \) of Eq. (1) with the initial condition \( \psi_0 \) at \( t = 0 \) can be expressed by the Feynman integral

\[
\psi_t = \int Dq \times \exp \left( \frac{i}{\hbar} \int_0^t \left( \frac{1}{2} \frac{dq_s}{ds} \right)^2 - U_t(q_s) \right) \psi_0(q_t(x)),
\] (12)

where the integral is over paths \( q_t(x) \) starting from \( x \) (i.e. \( q_0(x) = x \); we shall also denote \( q_t(x) \) by \( q_t \) when there is no danger of confusion).

We show (first for the formula (10)) that if in the Feynman formula (12) we take as Feynman paths the paths of the diffusion process (8) then we obtain Eq. (10). The proof is based on the representation of \( \psi_0^g(q_t) = \exp \ln \left( \psi_0^g(q_t) \right) \) as an integral

\[
\ln \psi_0^g(q_t(x)) = \ln \psi^g_t(x) + \int_0^t ds \ln \psi_{t-s}(q_s(x)) + \ln \psi_0^g(q_t(x))
\] (13)

and the identity (for the process (8))

\[
\frac{i}{\hbar} \int_0^t \left( \frac{1}{2} m \frac{dq_s}{ds} \right)^2 - U_t(q_s) \right) + \ln \psi_0^g(q_t(x)) = -\frac{1}{2} \int_0^t \left( \frac{db}{ds} \right)^2 + \ln \psi^g_t(x).
\] (14)
Equation (14) shows that the integration over $q$ in the Feynman formula (12) can be replaced by an average over Brownian paths $b_t$ because we arrive at Eq. (10) (which still will be proved in another way by means of the stochastic calculus).

It is crucial for the derivation that functionals of the Brownian motion satisfy a modified differential formula (Ito formula [18], for an elementary version see [19]) following from the non-differentiability of the Brownian paths $b_t$

$$df_s(b) = \nabla f_s \circ db + \partial_t f_s ds$$

$$= \nabla f_s db + \frac{1}{2} \nabla^2 f_s (db)^2 + \partial_t f_s ds$$

$$= \nabla f_s db + \frac{1}{2} \nabla^2 f_s ds + \partial_t f_s ds,$$  \hspace{1cm} (15)

where $\circ$ denotes the Stratonovitch differential (the differential $db$ without circle is called Ito differential [18]) and on the rhs of Eq. (15) we insert $(db)^2 = ds$ (we denote $E[.]$ by $\langle . \rangle$ as a shorthand). If the Langevin equation (8) is satisfied and if $f_s$ is a function of $q$ then

$$df_s = \nabla f_s \circ dq + \partial_t f_s ds$$

$$= \nabla f_s dq + \frac{i\hbar}{2m} \nabla^2 f_s ds + \partial_t f_s ds.$$  

We insert $f_s = \ln \psi^{\xi -}_{t-s}$. Then

$$d \ln \psi^{\xi -}_{t-s}(q_t) = \nabla \ln \psi^{\xi -}_{t-s} dq_t + \frac{i\hbar}{2m} \nabla^2 \ln \psi^{\xi -}_{t-s} ds$$

$$+ \partial_t \ln \psi^{\xi -}_{t-s} ds.$$ \hspace{1cm} (16)

We have

$$\nabla^2 \ln \psi^{\xi -}_{t-s} = (\nabla^2 \psi^{\xi -}_{t-s})(\psi^{\xi -}_{t-s})^{-1} - (\psi^{\xi -}_{t-s})^{-2}(\nabla \psi^{\xi -}_{t-s})^2.$$  

We use the fact that $\psi^{\xi -}_{t-s}$ satisfies the Schrödinger equation (2) (with $\bar{U} = U$) . Then, in equation (16) we can express $\nabla^2 \psi_{t-s}$ by $U_{t-s}$ and $\partial_t \psi_{t-s}$. We can check after an insertion of Eqs. (13) and (16) that in the exponential of the Feynman integral there remains solely $-\frac{1}{2} \int ds (\frac{dB_s}{ds})^2$. This means that the functional integral in the Feynman formula (12) is reduced to an average over the Brownian motion as in Eq. (10).

The formula (10) can be proved directly by differentiation (using the Ito formula). As $q_t$ (8) is a Markov process Eq. (10) defines a semigroup. Then, in order to show that the formula (10) solves Eq. (6) it is sufficient to calculate the generator of the semigroup at $t = 0$. We have from Eq. (10)

$$d\chi_t = E[\chi_0 dq_t + \frac{i}{2} \nabla^2 \chi_0 dq_t dq_t]$$ \hspace{1cm} (17)

In this formula we insert $dq_t$ from Eq. (8). Note that $E[\bar{U} db_t] = 0$ (if $F$ depends on time less or equal to $t$) and $dq_t dq_t = \frac{\hbar}{m} dt$. After the calculation of the differentials we may let $t \to 0$ to convince ourselves that the rhs of $d\chi$ is $\frac{i\hbar}{2m} \nabla^2 \chi + \frac{i\hbar}{m} \nabla \ln \psi^{\xi -}_{t} \nabla \chi$.

If in Eq. (2) we used $\bar{U} = \frac{m\omega^2}{2} q^2$ in Eq. (16) with $U = V + \frac{m\omega^2}{2} q^2$ then after an insertion of Eq. (16) in Eq. (12) the potential $U - \bar{U} = V$ would remain in the Feynman integral. Hence, we would obtain the result (11). This equation can be proved directly using the stochastic calculus (Eq. (11) defines a semigroup, hence it is sufficient to calculate its generator at $t = 0$)

$$d\chi_t = E \left[ -\frac{i}{\hbar} V_t(q_t) \chi_0(q_t) dt + \nabla \chi_0 dq_t + \frac{i}{2} \nabla^2 \chi_0 dq_t dq_t \right].$$ \hspace{1cm} (18)

When we insert $dq_t$ from Eq. (8) then we obtain the rhs of Eq. (7).

The imaginary time version of our formulas belongs to the standard theory of stochastic differential equations [17]. The derivation of the Feynman integral has been discussed earlier in a different form in [15, 16]. The stochastic formulas for the Feynman integral are also discussed in [20–23]. There are some mathematical subtleties concerning analytic properties of functions of the stochastic processes as our formalism is an analytic continuation of the one well known for the imaginary time. For a large class of analytic potentials the replacement of the real paths of Feynman by complex diffusion processes driven by Brownian paths is completely equivalent (this is a mathematical theory of the Feynman integral). As follows from the derivation of the Schrödinger equation by means of the stochastic calculus the formula (11) holds true for arbitrary particular solution $\psi_0^\xi$ and arbitrary initial condition $\psi_0^\xi \chi_0$ if these functions have a holomorphic extension from the real line.

If we take $\psi_0^\xi$ in the WKB form $\exp(\frac{1}{\hbar} S_0)$ then in the limit $\hbar \to 0$ of the Schrödinger equation the function $S$ satisfies the Hamilton–Jacobi equation. The stochastic equation (8)

$$dq_s = -\frac{1}{m} \nabla S_{t-s}(q_s) ds + \sqrt{\frac{i\hbar}{m}} db_s$$

together with Eq. (10) can be considered as a Hamilton–Jacobi version of a sum over trajectories.

The formalism can be extended to arbitrary number of dimensions and to Lagrangians of the form

$$L = \frac{1}{2} g_{ik} \frac{dy^i}{ds} \frac{dy^k}{ds} - U.$$  \hspace{1cm} (19)

Then the Schrödinger equation (1) holds with the Hamiltonian

$$H = \frac{1}{2} g_{ik} \frac{1}{g^j} g^{jk} p_k + U.$$  \hspace{1cm} (20)
where \( g = \det[g_{jk}] \), \( g^{jk} \) are the matrix elements of the inverse matrix and

\[
p_j = g_{jk} \frac{d^k_j}{ds}
\]

is the momentum in classical mechanics. In quantum mechanics

\[
p_j = -i\hbar \frac{\partial}{\partial y^j}.
\]

The corresponding stochastic equation (an analog of Eq. (8)) reads

\[
dq^j = i\hbar m g^{jk} \partial_k \ln(\psi^*_{t-x}) ds + \sqrt{\frac{i\hbar}{m}} e^i_k \circ db^k,
\]

where \( e^i_k \circ db^k = g^{jk} \).

We consider as a simple example \( \tilde{U} = m\omega^2 x^2 \). Then the ground state solution of Eq. (2) is

\[
\psi_g(x) = \left( \frac{\pi\hbar}{m\omega} \right)^{-\frac{1}{4}} \exp\left( -\frac{m\omega x^2}{2\hbar} \right).
\]

The stochastic equation (8) reads

\[
dq = -i\omega q dt + \sqrt{\frac{i\hbar}{m}} db.
\]

A simple calculation gives

\[
\int dx |\psi_g^*(x)|^2 E[q_s(x) q_{s'}(x)] = \frac{\hbar}{2m\omega} \exp(-i\omega|t - t'|).
\]

The rhs of Eq. (26) is the expectation value in the ground state of the time-ordered products of Heisenberg picture position operators of the harmonic oscillator. In this sense the stochastic process \( q_t \) has a physical meaning as its correlation functions coincide with quantum expectation values.

3 The relativistic quantum field theory

In the canonical field theory with the Hamiltonian

\[
\mathcal{H} = \frac{1}{2} \int dx (\Pi^2 + (\omega\Phi)^2) + \int dx V(\Phi),
\]

where \( \Pi(x) \) is the canonical momentum

\[
\omega = \sqrt{-\Delta + m^2}
\]

and

\[
[\Phi(x), \Pi(y)] = i\hbar \delta(x - y)
\]

consider the Schrödinger equation

\[
i\hbar \partial_t \psi = \mathcal{H} \psi
\]

and its imaginary time version

\[
-i\hbar \partial_t \psi = \mathcal{H} \psi.
\]

Let

\[
\psi_0 = \psi_g \chi_0,
\]

where \( \psi_g \) is the ground state. Then, \( \chi \) satisfies the equation (an infinite dimensional version of Eq. (6))

\[
\hbar \partial_t \chi = \frac{1}{2} \int dx \left( -\Pi^2 - 2(\Pi \ln \psi_g) \Pi \right) \chi,
\]

where

\[
\Pi(x) = -i\hbar \frac{\delta}{\delta \Phi(x)}.
\]

Equation (31) is a diffusion equation in infinite dimensional spaces. An approach to Euclidean field theory based on this equation has been developed in [24]. It follows that the solution of Eq. (31) can be expressed as

\[
\chi_t(\Phi) = E[\chi_0(\Phi(\Phi))],
\]

where \( \Phi(\Phi) \) is the solution of the stochastic equation

\[
d\Phi(x) = \hbar \frac{\delta}{\delta \Phi(x)} \ln \psi_g dt + \sqrt{\hbar} dW_t(x)
\]

with the initial condition \( \Phi \). \( E[...] \) denotes an expectation value with respect to the Wiener process (Brownian motion) defined by the covariance

\[
E[W_t(x) W_s(y)] = \min(t, s) \delta(x - y).
\]

The correlation functions of the Euclidean field in the ground state \( \psi_g \) can be expressed by the correlation functions of the stochastic process \( \Phi_t \).

Let us consider the simplest example: the free field. Then, the ground state is

\[
\psi_g = \left( \det \left( \frac{\pi\hbar}{\omega} \right) \right)^{-\frac{1}{4}} \exp\left( -\frac{1}{2\hbar} \Phi_0 \Phi \right).
\]
Equation (34) reads
\[ d\Phi = -\omega \Phi \, dt + \sqrt{\hbar} dW \]  
(37)
with the solution (with the initial condition \( \Phi \) at \( t_0 \))
\[ \Phi_t = \exp(-\omega (t-t_0)) \Phi + \sqrt{\hbar} \int_{t_0}^t \exp(-\omega (t-s)) dW_s. \]  
(38)

We can calculate
\[
\int \mathcal{D}\phi \phi_\gamma(\Phi)^2 E[\Phi_t(x)\Phi_t'(x')]
= \hbar \left( \frac{1}{2\omega} \exp(-|t - t'|/\omega) \right)(x, x').
\]  
(39)

In Eq. (39) the rhs denotes the kernel of the operator.

If the ground state has some analyticity property then we can extend the time in Eqs. (28)–(38) \( t \to it \) from imaginary time to the real time [15,16]. An analytic continuation to real time of Eq. (37) reads
\[ d\Phi = -i\omega \Phi_t \, dt + \sqrt{\hbar} dW. \]  
(40)

The solution is
\[ \Phi_t(\Phi) = \exp(-i\omega (t-t_0)) \Phi + \sqrt{\hbar} \int_{t_0}^t \exp(-i\omega (t-s)) dW_s. \]  
(41)

The solution of the Schrödinger equation (28) has the same form (33). We can see that
\[
\left( \psi_\gamma, T \left( \exp \left( \int dt \Phi_t^q(f_t) \right) \psi_\gamma \right) \right)
= \int \mathcal{D}\phi \exp \left( -\frac{1}{\hbar} \phi_\omega \phi \right) E \left[ \exp \left( \int dt dx f_t(x) \Phi_t(x) \right) \right]
= \exp \left( \frac{\hbar}{2} \int dt d't' \left( f_t, (2\omega)^{-1} \exp(-i\omega|t - t'|) f_{t'} \right) \right),
\]  
(42)

where \( \Phi_t^q(f) = \int dx \Phi_t^q(x) f(x) \) is the quantum field and \( T \) denotes the time-ordered product.

We can transform the complex equation (40) into two real equations defining
\[ \Phi = \phi_1 + i\phi_2 \]  
(43)
and
\[ \phi_+ = \phi_1 + \phi_2 \]  
(44)
\[ \phi_- = \phi_1 - \phi_2. \]  
(45)

Then it follows from Eq. (40) that
\[ \phi_+ = \omega^{-1} \phi_- \]  
(46)
and \( \phi_- \) satisfies the random wave equation
\[ \partial_+^2 \phi_- = -\omega^2 \phi_- + \sqrt{2\hbar} \omega \partial_+ W, \]  
(47)
which has the solution
\[ \phi_-(t) = \cos(\omega t) \phi_-(0) + \omega^{-1} \sin(\omega t) \partial_+ \phi_-(0) + \sqrt{2\hbar} \int_0^t \sin(\omega (t-s)) dW_s, \]  
(48)
where \( \partial_+ \phi_-(0) = \omega \phi_+ (0) \).

We further develop the theory of stochastic wave equations for the quantum theory of fields in expanding universes in [26] as an extension of Starobinsky stochastic inflation [27].

### 4 Time-dependent reference state

There are time-dependent solutions in the Gaussian form of the Schrödinger equation for the free field theory
\[ \psi_t^g(\Phi) = A(t) \exp \left( \frac{i}{2\hbar} (\Phi \Gamma(t) \Phi + 2I, \Phi) \right). \]  
(49)

In the time-dependent case Eq. (8) in quantum field theory takes the form [17]
\[ d\Phi_t = -\Gamma(t-s) \Phi_t ds - J_{1-s} ds + \sqrt{\hbar} dW_s. \]  
(50)

Let \( \Phi_t(\Phi) \) be the solution of Eq. (50) with the initial condition \( \Phi \) then the solution of the Schrödinger equation (28) with the initial condition \( \psi_0^g \chi \) is
\[ \psi_t = \psi_t^g E[\chi(\Phi_t(\Phi))]. \]  
(51)

In the model (27) with a time-dependent interaction \( V_t \) it would be difficult to find any explicit solution \( \psi_t^g \). However, we can use the solution \( \psi_t^g \) of the free field theory and take interaction into account by means of the Feynman–Kac formula. According to Eq. (11) the solution of the Schrödinger equation with the interaction \( V_t \) reads
\[ \psi_t(\Phi) = \psi_t^g E \left[ \exp \left( -\frac{i}{\hbar} \int_0^t V_{t-s} \Phi_s ds \right) \chi(\Phi_t(\Phi)) \right]. \]  
(52)

We shall apply Eq. (52) for \( V \) which is linear in \( \Phi \) \( (\Phi \) could be the gravitational potential in de Donder gauge as in [25]; it will have many components when describing gravitons in the transverse-traceless gauge in Sects. 7, 8).

As an application to the particle motion in a (quantized squeezed) gravitational wave let us consider the Hamiltonian
of a harmonic oscillator ($\hat{U} = \frac{\omega^2 x^2}{2}$ in Eq. (2) corresponding to a single mode of the gravitational field)

$$H = -\frac{\hbar^2}{2} \nabla^2 + \frac{1}{2} \omega^2 x^2.$$  \hspace{1cm} (53)

Let us look for a time-dependent solution of the Schrödinger equation (2) in the form

$$\psi^s = A(t) \exp \left( \frac{i}{\hbar} (x \Gamma(t) + 2J_t x) \right).$$  \hspace{1cm} (54)

Then, $\psi^s$ is a solution of the Schrödinger equation (2) if

$$i \hbar \partial_t \ln A = - \frac{i}{2} \Gamma^2 + \frac{1}{2} J^2,$$

$$- \partial_s J = \Gamma J$$  \hspace{1cm} (55, 56)

and

$$\partial_t \Gamma + \Gamma^2 + \omega^2 = 0.$$  \hspace{1cm} (57)

It can be seen that the Riccatti equation (57) is equivalent to

$$\frac{d^2 u}{dt^2} + \omega^2 u = 0,$$  \hspace{1cm} (58)

where

$$u = \sigma \cos(\omega t) + \delta \sin(\omega t),$$  \hspace{1cm} (60)

where $\sigma, \delta$ are complex numbers.

Equation (8) reads

$$dq_s = - \partial_s \ln u_{t-s} dq_{t-s} ds - J_{t-s} ds + \sqrt{\hbar} db_s,$$  \hspace{1cm} (61)

with the solution

$$q_s(q) = \frac{u_{t-s}}{u_t} q + \sqrt{\hbar} u_{t-s} \int_0^s u_{t-s}^{-1} db_t$$

$$- u_0 u_{t-s} J_0 \int_0^s u_{t-s}^{-2} d\tau.$$  \hspace{1cm} (62)

We have (as $E[(\int f db_s)^2] = \int f^2 ds$)

$$E[(q_s - \langle q_s \rangle)(q_s' - \langle q_s' \rangle)]$$

$$= i \hbar u_{t-s} u_{t-s'} \int_0^{\min(s,s')} u_{t-s}^{-2} d\tau$$

$$= (-i \hbar) \omega^2 u_{t-s} u_{t-s'}$$

$$\times (\sigma^2 + \delta^2)^{-1} \left( \Gamma(t) - \Gamma(t - \min(s, s')) \right).$$  \hspace{1cm} (63)

If $\delta a = i \sigma$ then $\Gamma(0) = i a^{-1} \omega$ and the solution $\psi_s$ starts from a real $i \Gamma$. Choosing $J_0 = a^{-1} \omega x_0 + ip$ we obtain the squeezed state

$$\psi_0^s = \exp \left( \frac{-(\omega x - x_0)^2}{2ah} + \frac{ip}{\hbar} x \right)$$

with the squeezing $\alpha$ of the coordinate (for a time evolution of squeezed states see [28]). The case $\delta = i \sigma$ corresponds to the ground state (24) ($m = 1$). It can be shown that the formula (63) is continuous with respect to the limit $\delta \to i \sigma$.

5 Linear coupling to an oscillator environment

Let us consider a model of a system with a Lagrangian $L_\xi$ described by a coordinate $\xi$ linearly interacting with an oscillator. We have the Lagrangian

$$L = L_\xi + \frac{1}{2} \left( \left( \frac{dq}{ds} \right)^2 - \omega^2 q^2 \right) + q f_s(\xi).$$  \hspace{1cm} (64)

In the model with a linear coupling and the quadratic Lagrangian for the $q$ variable (64) the functional integral in Eq. (11) can be reduced to the Gaussian integral

$$\chi_x(x) = \int dk \chi_0(k) E\left[ \exp \left( \frac{i}{\hbar} \int_0^s ds q_s f_{r-s}(\xi_s) + ik q_s(\xi) \right) \right]$$  \hspace{1cm} (65)

where we used the Fourier representation of $\chi_0(x)$ (we use the same notation for a function and its Fourier transform)

$$\chi_0(x) = \int dk \chi_0(k) \exp(ikx)$$

For a Gaussian variable $q_s$ we have (for any number $a_s$)

$$E[\exp(a_s q_s)]$$

$$= \exp \left( a_s \langle q_s \rangle + \frac{1}{2} (a_s q_s - a_s \langle q_s \rangle)^2 \right)$$  \hspace{1cm} (66)

The formula (66) can easily be generalized to $\int ds a_s q_s$ with $q_s$ of Eq. (62)) and to an infinite number of modes $q$ (as will be done in Sects. 7, 8).

We are interested in the scattering amplitude from an initial state $\psi^s_0(x) \chi_i(z) \phi_s(\xi)$ to the final state $\psi^s_0(x) \chi_f(z) \phi_f(\xi)$. We apply the transformation of Sect. 2 only to the oscillator path integral. Then, according to Eq. (11) the amplitude $a_{ij}$ is (where $(\psi^s_0 \chi_i(z) \phi_i)_t$ means the unitary evolution of the wave function)
\[ a_{fi} = (\psi_f^* \chi_f \phi_f, (\psi_0^* \chi_i(x) \phi_i)_f) \]

\[ = \int dxd\xi d\xi' D\xi D\xi' |\psi_f^*(x)\chi_f(x)\phi_f(\xi)\psi_f^*(x')\chi_f(x')|^2 \]

\[ \times \psi_f^* \chi_f \phi_f(\xi) E \left[ \exp \left( \frac{i}{\hbar} \int_0^t q_i f_{i-s}(\xi_s) \chi_i(q_t(x)) \right) \right] \]

(67)

where the initial state of the oscillator is \( \psi_0^* \chi_i \).

If we do not observe the final states of the oscillator and average the probability \( P(i, f) \) of the transition \( \phi_i \rightarrow \phi_f \) over these states using the completeness relation

\[ \sum_f \psi_f^* \chi_f(x) \psi_f^* \chi_f(x') \chi_f(x') = \delta(x - x') \]  

(68)

then we obtain

\[ P(i, f) = \sum_f |a_{fi}|^2 = \int dxd\xi d\xi' D\xi D\xi' |\psi_f^*(x)\chi_f(x)\phi_f(\xi)\psi_f^*(x')\chi_f(x')|^2 \]

\[ \times \phi_f(\xi) \phi_f(\xi') \psi_f^*(\xi, \xi') \phi_f(\xi) \phi_f(\xi') \]

\[ \times \exp \left( -\frac{i}{\hbar} \int ds L_\xi + \frac{i}{\hbar} \int ds L_{\xi'} \right) \]

\[ \times E \left[ \exp \left( \frac{i}{\hbar} \int_0^t q_i f_{i-s}(\xi_s) \chi_i(q_t(x)) \right) \right] \]

\[ \times \exp \left( -\frac{i}{\hbar} \int_0^t q_i^* f_{i-s}(\xi_s) \chi_i(q_t^*(x)) \right) \].

(69)

where \( q^* \) denotes a complex conjugation of an independent version of the process \( q_t \) and \( \xi'(\xi') \) is another realization of the path \( \xi \).

If we define the density matrix \( \rho \) as an average over the environment of the oscillator then the density matrix of the \( \xi \) system is

\[ \rho(\xi, \xi') = \int dxd\xi d\xi' D\xi D\xi' |\psi_f^*(x)\chi_f(x)\phi_f(\xi)\psi_f^*(x')\chi_f(x')|^2 \]

\[ \times \psi_f^* \chi_f \phi_f(\xi) E \left[ \exp \left( \frac{i}{\hbar} \int_0^t q_i f_{i-s}(\xi_s) \chi_i(q_t(x)) \right) \right] \]

\[ \times \exp \left( -\frac{i}{\hbar} \int_0^t q_i^* f_{i-s}(\xi_s) \chi_i(q_t^*(x)) \right) \].

(70)

It can be seen from Eqs. (69), (70) that the calculations of the density matrix and the transition probability are closely related. When calculating the transition probability we need to perform an extra (\( \xi, \xi' \)) integral over the final states \( \phi_f(\xi) \phi_f(\xi') \) of the \( \xi \) system in comparison to the calculation of the density matrix in Eq. (70).

If \( \chi_i = 1 \) then according to Eq. (66) the expectation value (67) is

\[ E \left[ \exp \left( \frac{i}{\hbar} \int_0^t q_i f_{i-s}(\xi_s) \right) \right] \]

\[ = \exp \left( -\frac{1}{2\hbar^2} \int_0^t ds ds' \right) \]

\[ \times E \left[ (q_i - \langle q_i \rangle)(q_i' - \langle q_i' \rangle \right] f_{i-s}(\xi_s) f_{i-s'}(\xi_s') \right). 

(71)

6 Particle interacting with gravitons: PWZ model in one mode approximation

Parikh, Wilczek and Zaharia [2] consider two masses \( M \) and \( m_0 \) interacting with gravitational field \( q_{\omega} \). In a one mode approximation the Lagrangian describing the geodesic deviation of the \( m_0 \) mass (in the free falling frame and \( M >> m_0 \)) is [2, 29]

\[ L = \frac{1}{2} (\frac{dq_{\omega}}{ds})^2 - \omega^2 q_{\omega}^2 + \frac{1}{2} m_0 (\frac{d\xi}{ds})^2 \]

\[ - m_0 \lambda (1 - \omega^2 q_{\omega}^2) - V(\xi) \]

\[ = \frac{1}{2} g_{jk} \frac{dy^j}{ds} \frac{dy^k}{ds} - \frac{1}{2} \omega^2 q_{\omega}^2 - V(\xi) \exp \left( \frac{i}{\hbar} \int_0^t q_i f_{i-s}(\xi_s) \chi_i(q_t(x)) \right) \]

(72)

where \( \lambda = \sqrt{8\pi g} \), \( y = (q_{\omega}, \xi) \), \( \xi \) is the geodesic deviation (the potential \( V(\xi) \) is absent in [2] but we add it here for further applications in the next sections).

\[ (g_{jk}) = \begin{bmatrix} 1 & \frac{1}{m_0 \lambda} \xi \\ -\frac{1}{m_0 \lambda} \xi & m_0 \end{bmatrix} \].

(73)

The Lagrange equation is

\[ \frac{d^2 \xi}{ds^2} = m_0 \lambda \frac{d^2 q_{\omega}}{ds^2} \xi \exp \left( \frac{i}{\hbar} \int_0^t q_i f_{i-s}(\xi_s) \chi_i(q_t(x)) \right) \]

The Hamiltonian is determined by Eq. (20) with

\[ [g_{jk}] = (m_0 - \lambda^2 \xi^2)^{-1} \begin{bmatrix} m_0 & m_0 \lambda \xi \\ m_0 \lambda \xi & 1 \end{bmatrix} \].

(74)

We could quantize the model (72) with the explicit quantum version (20) of the Hamiltonian. However, the averaging over gravitons is simple only in the influence functional approach.

In the PWZ model (72), where the gravitational field is described by one mode \( q_{\omega} \), when we apply the transformation
(25) to \(q_t\) with the ground state (24) \((m = 1)\) then from Eq. (70) we obtain
\[
\rho_t(\xi, \xi') = \int ds \exp \left( - i \frac{\omega}{\hbar} s^2 \right) \times E \left[ \exp \left( i \int_0^t \frac{m_0}{2} \frac{d\xi}{ds} \frac{d\xi}{ds} - \frac{m_0}{2} \frac{d\xi'}{ds} \frac{d\xi'}{ds} - V(\xi) + V(\xi') \right) \right.
\]
\[
\times \exp \left( - \frac{i \lambda_0}{\hbar} \int_0^t (q_{t-s} - \xi') \frac{d^2\xi}{ds^2} \right) \chi_t(q_t(x)) \overline{\chi_t}(\xi) \psi(\xi(\xi')) \right] = \mathcal{K}_t \rho_0, (75)
\]
where \(q^*\) is another realization of the process \(q\) (and the complex conjugation of this realization). We have changed \(s \to t - s\) in the integral in the exponential and inserted \(f_s = \frac{m_0}{2} \frac{d^2\xi}{ds^2} \). We have denoted the evolution kernel of the density matrix by \(\mathcal{K}_t\).

The calculation of the expectation value (75) according to Eq. (71) gives (we use the solution (41), where \(\Phi \to \chi_t\) of Eq. (25) and assume that the initial condition \(\chi_t = 1\))
\[
\rho_t \simeq \int d\xi d\xi' D\xi D\xi' \exp \left(- i \omega \frac{\alpha^2}{\hbar} \right) \times E \left[ \exp \left( i \int_0^t \left( \frac{m_0}{2} \frac{d\xi}{ds} \frac{d\xi}{ds} - \frac{m_0}{2} \frac{d\xi'}{ds} \frac{d\xi'}{ds} \right) - V(\xi) + V(\xi') - \frac{i \lambda}{\hbar} \int_0^t (q_{t-s} - \xi') ds \right] \right.
\]
\[
\times \exp \left( - i \omega (t - s) f_s \right) \left. - q \exp (i \omega (t - s) f'_s) ds \right) \times \exp \left( - \frac{\lambda^2}{2\hbar^2} \int_0^t dsds' \times \left( \chi_t(q_{t-s} - \xi') (q_{t'-s} - \xi') f_s f'_s \right) \right)
\]
\[
\exp \left( \left. E[(q_{t-s} - \xi')(q_{t'-s} - \xi')] f_s f'_s \right) \right) = \mathcal{K}_t \rho_0, (76)
\]
where \(f = \frac{m_0}{2} \frac{d^2\xi}{ds^2}, f' = \frac{m_0}{2} \frac{d^2\xi'}{ds^2}\). In Eq. (76) we have (this is the special case of Eq. (63) with \(u_s = \exp(i\omega s)\))
\[
E[(q_{t-s} - \xi')(q_{t'-s} - \xi')]
\]
\[
= \frac{\hbar}{2\omega} \exp(-i\omega|s - s'|) - \exp(-i\omega(2t - s - s')). (77)
\]
If the oscillator is in a time-dependent state then we should insert the solution (62) in the Feynman formula (75). Hence, instead of Eq. (76) we have (for typographical reasons from now on we identify \(\mathcal{T} = \Gamma^*\), when acting on \(q_t^*\) the star *)
\[
\int \exp(i \frac{\Gamma(t)^2}{\hbar}) \left( \exp \left( \frac{i \lambda}{\hbar} \int_0^t \chi_t(q_{t-s} - \xi') ds \right) \right) \times \exp \left( - \frac{\lambda^2}{4\hbar^2} \int_0^t dsds' \left( f_s f'_s \exp(-i\omega|s - s'|) + f'_s f_s \exp(i\omega|s - s'|) \right) \right).
\]
\[
\text{where}
\]
\[
E[(q_{t-s} - \xi')(q_{t'-s} - \xi')]
\]
\[
= \chi_t(q_{t-s} + \xi')(q_{t'-s} + \xi') f_s f'_s \left( \right) = \mathcal{K}_t \rho_0, (77)
\]
\[
\text{We write}
\]
\[
X = \frac{1}{2} (\xi + \xi')
\]
\[
y = \xi - \xi'. (81)
\]
\[
\text{We expand the exponential in Eq. (80) in } y. \text{ Then, the terms independent of } y \text{ cancel and there remains (till the terms quadratic in } y)\)
\[
\rho_t \simeq \int D\xi D\xi' \exp \left( i \int_0^t \frac{m_0}{2} \frac{d\xi}{ds} \frac{d\xi}{ds} - V(\xi) \right) \exp \left( - \frac{\lambda^2}{8\hbar^2} \int_0^t dsds' \left( - \sin(\omega|s - s'|) \right) d^2\xi d^2\xi' \cos(\omega|s - s'|) \right) \rho_0(X, y). (82)
\]
The term linear in \( y \) gives a modification of the equation of motion of the \( \xi \) coordinate whereas the term quadratic in \( y \) is a noise acting upon the particle [7].

In the expression (78) of the time-dependent reference state we obtain

\[
\rho_t \simeq \exp \left( -\frac{i}{\hbar} \int_0^t (\Gamma(t) - \Gamma^*(t))^{-1} \times \left( \lambda \int_0^t (u_i - u_i - u_i - u_i) ds \right)^2 \right.
\]

\[
\left. - \frac{i \lambda^2}{2 \hbar \omega^2} \int_0^t (u_i u_i' - u_i' u_i) ds \right) ds dt.
\]

We expand (83) in \( y \) again. Let us consider the general expression appearing after an expansion in \( y \) till the second order terms in Eqs. (82), (83)

\[
\int D X D y \times \exp \left( \frac{i}{\hbar} \int_0^t \left( y m_0 \frac{d^2 X}{ds^2} + V'(X) + L(X) + \frac{i}{\hbar} M y \right) \right) \rho_0(X, y)
\]

\[
= \int D X D y \exp \left( \frac{i}{\hbar} \int_0^t \left( y \mathcal{L} + \frac{i}{\hbar} \right) \right) \rho_0(X, y)
\]

\[
= \int D X D y \exp \left( -\frac{1}{2 \hbar}(y - i \hbar M^{-1} \mathcal{L})(y - i \hbar M^{-1} \mathcal{L}) - \frac{1}{2} \mathcal{L} M^{-1} \mathcal{L} \right) \rho_0(X, y),
\]

where by \( L \) we denote a functional of \( X \), \( M \) is an operator and by \( \mathcal{L} \) we denote the term proportional to \( y \). If we introduce \( \mathcal{X} = M^{-\frac{1}{2}} \mathcal{L} \) then \( \mathcal{X} \) becomes a Gaussian variable with the white noise distribution which can be represented as \( \partial_b s \).

(85)

7 Infinite number of modes:thermal state of gravitons

We are to generalize the results of Sect. 6 to an infinite number of modes of the gravitational field. We could do it in the formulation of Sect. 6 by means of a representation of the thermal state as a sum over eigenstates with a proper weight factor. We have studied earlier [7] an analogous model of a particle geodesic motion in an environment of quantum gravitational waves in a thermal state. There are minor changes from the setting of Parikh, Wilczek and Zahariade [2] where the effect of gravitons on geodesic deviation is considered.

The Lagrangian (72) with an infinite number of modes in the coordinate space is [29]

\[
\mathcal{L} = \frac{1}{4} \int dx h_{\alpha}(s, x)(-\partial_\alpha^2 + \Delta) h_{\alpha}(s, x) + \frac{1}{2} m_0 \frac{d^2 \xi_\alpha}{ds^2} + \frac{1}{4} m_0 \lambda \partial_\alpha^2 h_{\alpha}(s, \xi_\alpha) \xi_\alpha^4 - \frac{1}{4} m_0 \lambda \partial_\alpha^2 h_{\alpha}(s, \xi_\alpha) \xi_\alpha^4 \xi_\alpha^4.
\]

(85)

where the metric perturbation \( h_{\alpha} \) of the Minkowski metric \( \eta_{\mu \nu} \rightarrow \eta_{\mu \nu} + h_{\mu \nu} \) is in the transverse-traceless gauge, the geodesic deviation \( \xi \) in a gravitational transverse traceless gauge has only the spatial components \( \xi_\alpha \). We apply the decomposition of \( h_{\alpha} = h_{\alpha}(s, \xi_\alpha) \) into the classical wave solution \( h_{\alpha} \) and the quantized (graviton) part \( h_{\alpha} \) is decomposed in the amplitudes \( h_{\alpha} \) (where \( \alpha = +, \times \), in the linear polarization) by means of the polarization tensors \( \epsilon_{\alpha \beta} \) (as \( h_{\alpha} = \epsilon_{\alpha \beta} h_{\beta} \) [30,31,37])

\[
h_{\alpha}(t, x) = h_{\alpha}(t, x) + (2\pi)^{-\frac{1}{2}} \int \frac{d \mathbf{k}}{\sqrt{\mathbf{k}}} h_{\alpha} \epsilon_{\alpha \beta} \exp(-i \mathbf{k} x) + h_{\alpha} \epsilon_{\alpha \beta} \exp(i \mathbf{k} x).
\]

With the infinite number of gravitational wave modes in the model (72) of Sect. 6 we shall have \( \omega = |\mathbf{k}| \) (we set the velocity of light \( c = 1 \)).\( H = H_x + H_x \) is the sum of the independent Hamiltonians (27) with \( V = m_0 = 0 \).

The \( h_{\alpha} \) are quantized at finite temperature \( T \) with the Gibbs distribution \( \exp(-\beta H) \), \( \beta^{-1} = k_B T \) where \( k_B \) is the Boltzman constant. The classical part of the particle-wave interaction can be considered as a time-dependent external potential \( V(\xi) \) with

\[
V = \frac{m_0}{4} \delta_\alpha^2 h_{\alpha}(s, \xi_\alpha) \xi_\alpha \xi_\alpha \xi_\alpha \xi_\alpha.
\]

(86)

Denote

\[
f'_{\alpha \beta} = \frac{m_0}{2} \frac{d^2}{ds^2} \xi_\alpha \xi_\beta
\]

(87)

\[
f''_{\alpha \beta} = \frac{m_0}{2} \frac{d^2}{ds^2} \xi_\alpha \xi_\beta.
\]

(88)

Then, as in [7] (there are some misprints of signs in [7]; for the derivation of the thermal formula see [32], sec. 18, see also [5,8,10]; the gravitational case is analogous to the
electromagnetic one treated in [33]) we obtain for the density matrix evolution kernel

\[
K_i(\xi, \xi') = \int D\xi D\xi' \exp\left(\frac{i m_0}{\hbar} \int_0^t ds \left(\frac{d\xi_r}{ds} \frac{d\xi'_r}{ds} - \frac{d\xi_r}{ds} \frac{d\xi'_r}{ds}\right)\right)
\times \exp\left(\frac{i}{\hbar} \int_0^t ds (V_{i-s}(\xi_s) - V_{i-s}(\xi'_s))\right)
\times \exp\left(-\frac{i}{\hbar} \int_0^t ds (q^{r'l} - q^{r'l}) \partial_s (q^{r'l} + q^{r'l})\right).
\]  

(93)

where

\[
q^{r'l} = \frac{1}{2} \frac{d^2}{ds^2} \left(\xi^{r'l} - \frac{1}{3} \xi^{r'i} \xi^{j} \xi^{j}\right)
\]  

(94)

results from a resummation of \(f^{r'l}\) with \((A_{ij; rl})\)

\[
\gamma = \frac{8\pi G m_0^2}{10\pi}
\]  

(95)

\[
w = \frac{2\gamma}{\beta}
\]  

(96)

We expand Eq. (93) around \(X (81)\) (now with three spatial indices)

\[
q^{r'l} - q^{r'l} = \frac{d^2}{ds^2} \left(X^{r'y} + X^{l'ly} - \frac{2}{3} q^{r'l} X^{l'y}\right)
\]

In the exponential (93) the term linear in \(y\) becomes

\[
y_n \left(-\frac{d^2 X_n}{ds^2} + \frac{\lambda}{2} \frac{d^2 h_n}{ds^2} X_r + \frac{8\pi G m_0}{10\pi} X_l \frac{d^5}{ds^3} \left(\frac{1}{3} X_r X_r \delta_{nl} - X_n X_l\right)\right)
\]

(97)

The term quadratic in \(y\) is the noise term. For low temperature we obtain in general the non-local and non-Markovian stochastic equation (84). In the high temperature limit \(\beta \hbar \to 0\) the calculation of the evolution kernel is reduced to an expectation value over the solutions of the stochastic equation

\[
-\frac{d^2 X_n}{ds^2} + \frac{\lambda}{2} h_n X_r + \frac{8\pi G m_0}{10\pi} X_l \frac{d^5}{ds^3} \left(\frac{1}{3} X_r X_r \delta_{nl} - X_n X_l\right)
\]

(98)

As explained in the derivation of Eq. (84) the term quadratic in \(y\) defines the operator \(M\). From Eq. (93) we obtain that \(M\) is an operator defined by the bilinear form (on the rhs of Eq. (98) we have the square root of the matrix \(M\))

\[
w y' M^{r'l} y' = 2 w \int ds \left(\frac{d^2}{ds^2} (X^{j'y}) \frac{d^2}{ds^2} (X^{j'y}) + \frac{d^2}{ds^2} (X^{l'ly}) \frac{d^2}{ds^2} (X^{l'ly})\right)
\]

\[
= \frac{\lambda^2}{4\pi} \beta^{-1} y' X^k M_{rk;ln} y' X^n
\]

(99)

We derive Eq. (98) with the noise (99) in the Appendix working directly with the classical model of thermal gravitational
waves. The modification of the deviation equation coincides with the one of Ref. [29] (up to an infinite renormalization term). It is discussed already in [37] (sec. 36.8) and in more detail in [38] (see in particular the last section of this paper). The spectrum of the noise (of the quadrupole $q^{il}$) as seen from Eqs. (89), (90) is $8\pi Gk\cosh\left(\frac{h_kk}{2}\right)$ which at low temperature is $8\pi Gk$ and at high temperature $8\pi G\beta^{-1}$ (the spectrum of the coordinates $\xi$, as considered by [29] in Eqs. (3.13)-(3.14), is multiplied by $k^4$ as a result of the fourth order derivative in Eq. (99)).

8 General Gaussian state of the graviton

We generalize the results of Sect. 6 on averaging over the oscillator modes in a squeezed state to infinite number of modes of the gravitational field. The gravitational perturbation in the transverse-traceless gauge is decomposed into the amplitudes $h^\alpha$ by means of the polarizing tensors $e^\alpha_{rl}$ ($\alpha = +, \times$, $h_{rl} = e^\alpha_{rl}h_\alpha$). The gravitational Hamiltonian $H = H_+ + H_\times$ has the same form as for two independent scalar fields $\Phi \to h_\alpha$ (Eq. (27) with $m = U = 0$). In such a case we have two independent equations (57) for $\Gamma_\alpha$ and two independent $u_\alpha$. The Lagrangian (72) in multimode case takes the form of a sum over modes (we omit the classical part $h^w$ and neglect the dependence of $h^d(s, \mathbf{x})$ on spatial coordinates as in Eq. (93) of the previous section)

$$\mathcal{L} = \frac{1}{4} \int dk h^\alpha_{rl}(s, \mathbf{k})(-\delta^2 - k^2)h_\alpha(s, \mathbf{k}) + \frac{1}{2}m_0 \int d\xi_{rl} \frac{d\xi_{rl}}{ds} f_a^\alpha \cdot f_r^\alpha \frac{d\xi_{rl}}{ds},$$

$$-\frac{1}{4}m_0(2\pi)^{-\frac{3}{2}} \int \frac{d\xi_{rl} h^\alpha_{rl}(s, \mathbf{k})\xi_{rl}}{ds}.$$

The last term will be expressed as

$$\frac{\lambda m_0}{2}(2\pi)^{-\frac{3}{2}} \int d\xi_{rl} h^\alpha_{rl}(s, \mathbf{k})\xi_{rl} = \lambda(2\pi)^{-\frac{3}{2}} \int dk h^\alpha_{rl} f^\alpha_{rl},$$

where $f^\alpha = e^\alpha_{rl}f^r_{rl}$.

We consider a solution of the Schrödinger equation (28) ($V = 0$) in the Gaussian form (if we did not split the gravitational field into $h^w$ and $h^d$ then the classical part would be obtained from the term $J_h$ in Eq. (54)).

$$\psi^\alpha_t(h) = \exp\left(\frac{i}{2\hbar} \int dh^\alpha_{rl} \Gamma^\alpha_t h^\alpha_{rl}\right).$$

(100)

In Eq. (100) $(\Gamma^\alpha - \Gamma^{*\alpha})^{-1}$ has a physical meaning as the squeezing of the amplitude $h^\alpha$ in the uncertainty relations (squeezed states are produced during radiation [35,36]). In general, $\Gamma^+$ and $\Gamma^\times$ are independent solutions of Eq. (57). We restrict our discussion to the case $\Gamma^+ = \Gamma^{\times} \equiv \Gamma$ (and $u_+ = u_\times$). Then, the gravitational field resulting from the state (100) is covariant under rotations. In the expectation values (80) and (83) we shall have sums of the form

$$f^\alpha f^\beta = \lambda m_0 f^mn f^{rl}$$

(101)

which after averaging over $kk^{-1}$ will be expressed by $q^{rl}$ (94) in the way analogous to the thermal case of Sect. 7 ($\lambda m_0 f^mn f^{rl} \to \langle \lambda m_0 f^mn f^{rl}\rangle$). In detail

$$\rho_t \simeq \exp\left(-\frac{i\lambda^2 m_0^2}{2\hbar} \int dk \frac{4\pi}{5} \int \frac{dksds}{d\xi_{rl} h^\alpha_{rl}(s, \mathbf{k})\xi_{rl}}$$

$$\times (\Gamma(t) - \Gamma(t)^*)^{-1} \int_{0}^{t} \left(u_{i+1}^{(1)} - u_{i+1}^{(1)} d^w_s\right) ds^2 - \frac{i\lambda^2 m_0^2}{2\hbar} \int dk \int_{0}^{t} \left(u_{i+1}^{(1)} - u_{i+1}^{(1)} d^w_s\right) ds^2$$

$$\times (\Gamma(t) - \Gamma(t)\alpha^\alpha_{s'} q_{s'} q_{s'}^{rl} - q_{s'} q_{s'}^{rl} - u_{i+1}^{(1)} - u_{i+1}^{(1)} d^w_s\right) ds^2$$

$$- \exp(-ik(s+s'))(q_{s'}(s')q_{s'}(s') + q_{s'}(s)q_{s'}(s'))\right)$$

(102)

In Eq. (102) the $f^\alpha f^\beta$ term of Eq. (83) has been expressed by $q^{rl}$ and $u(k)$ is defined in Eq. (60) with $\omega = k$ (the coefficients $\delta$ and $\sigma$ may depend on $k$).

For the final result the integral over $k$ is crucial. Its exact value depends on the complex functions $\sigma(k)$ and $\delta(k)$ in Eq. (60). In the thermal case the effective action at high temperature in the exponential (93) was local in time (for a small $\beta$) owing to the $k^{-1}$ factor coming from $\cosh(\frac{1}{2}h/\beta)$. We do not have such a factor here. Nevertheless, we can see that the non-local final noise can be large owing to the squeezing factor $(\Gamma - \Gamma^*)^{-1}$ in Eq. (102). Explicitly, this term is

$$\exp\left(-\frac{i\lambda^2 m_0^2}{2\hbar} \int dksds \right)$$

$$\times (\Gamma(t) - \Gamma(t)\alpha^\alpha_{s'} q_{s'} q_{s'}^{rl} - q_{s'} q_{s'}^{rl} - u_{i+1}^{(1)} - u_{i+1}^{(1)} d^w_s\right) ds^2$$

$$- \exp(-ik(s+s'))(q_{s'}(s')q_{s'}(s') + q_{s'}(s)q_{s'}(s'))\right)$$

(103)

The exponential in Eq. (102) is of the form

$$\exp(iaq + ibq'i + iabq'q')$$

(104)

When $\frac{a}{a} = i, \Gamma = ik, u_+ = \exp(ikx)$ then in the integral (102) we obtain (this is an infinite mode version of Eq. (80))

$$\rho_t \simeq \exp\left(-\frac{\lambda^2 m_0^2}{2\hbar} \int dksds \right)$$

$$\times \left(\exp(-ik(s-s'))q_{s'}(s'q_{s'}(s') + q_{s'}(s)q_{s'}(s') + q_{s'}(s)q_{s'}(s'))\right)$$

(105)
Representing \(\sin(kt)\) as \(-k^{-1}\partial_t \cos(kt)\) we integrate over \(k\) obtaining \(\delta_\epsilon \delta(s - s')\) in a similar way as we did it in the thermal case arriving at the phase factor

\[
\exp\left(-i \frac{\gamma}{2\hbar} \int_0^t (q_{rl} - q'_{rl}) \partial_s(q_{rl}(s) + q'_{rl}(s))\right).
\]

We obtain the same lhs of the stochastic equation (98) as in the thermal state, but the noise resulting from Eq. (105) is different than the one of Eq. (98) (non-local in time and non-Markovian).

In Eq. (102) the exponential of the density matrix can be written in the form

\[
\rho_t \simeq \int D\xi D\xi' \exp\left(\frac{i}{2\hbar} \int_0^t ds (m_0 d\frac{\xi}{ds} - m_0 d\frac{\xi'}{ds}) - m t \right)
\]

\[
+ 2V(\xi) - 2V(\xi') + i \int ds ds' \left((q^{rl} - q'^{rl})C(q^{rl} + q'^{rl})
\right)
\]

\[
+(q^{rl} + q'^{rl}) A(q^{rl} + q'^{rl}) + (q^{rl} - q'^{rl}) B(q^{rl} - q'^{rl})\right).
\]

Equation (109) gives the spectrum of the noise of the quadrupole \(q^{rl}\) as \(8\pi Gak\) (defined in [39]) or the spectrum of the noise of the measured coordinate \(\bar{\xi}\) (if we perform the differentiation by parts over \(s\) contained in \(q^{rl}\)) as \(8\pi Gak\) in agreement with [29] (Eqs. (3.13)–(3.14)). The kinematic form of the noise (109) is the same as the one for the thermal noise discussed at the end of Sect. 7 and in the Appendix. However, because of a different spectrum of the noise (109) it is not of the local form (99).

9 Summary

We have calculated the density matrix resulting from an average over the gravitational field in a thermal state and in a general Gaussian state. The result shows that for gravitons in high temperature or in a highly squeezed state the modification of the geodesic deviation equation applied in a gravitational wave detection can have large noise amplitude. In general, the perturbation of the geodesic deviation equation is non-local and non-Markovian. Our results on the stochastic geodesic deviation equation show some differences in comparison to PWZ [2] which may come from approximations used by those authors. The formula for the noise (although quite complicated) can be useful in order to distinguish the contribution of the graviton noise from other sources of noise in the gravitational wave detection. Using Eq. (69) we can calculate the transition probability between the initial and final states of the detector. In the expansion in \(\hbar\) the calculation is reduced to expectation values of the noise. The noise could be detectable if gravitational waves come from an inflationary stage of universe evolution (squeezing) or from a merger of hot neutron stars.

Remark After a submission to arXiv of the first version of this paper there appeared extended versions [39,40] of PWZ paper [2]. Their modification of the deviation equation (Eq.(125) of Ref. [40]) is different from (98). The difference comes from the approximations discussed in [40] at the end of Sect. 2 where it is assumed that only one quantum mode of \(\xi'\) is excited.

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Data Availability Statement This manuscript has associated data in a data repository. [Authors’ comment: All data included in this manuscript are available upon request by contacting with the corresponding author (Zbigniew Haba).]

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Appendix: High temperature (classical) limit

We can treat the system in an environment of thermal grav-}
nons by means of the same procedure as applied in [34].

The equation for the gravitational field resulting from the
Lagrangian \( (85) \) is

\[
\frac{d^2 h_{rl}^w(t)}{dt^2} + k^2 h_{rl}^w(t) = (2\pi)^{-\frac{5}{2}} \lambda f_{rl}^w. \tag{110}
\]

We write the solution of Eq. (110) (assuming that when \( t \leq t_0 \), where \( t_0 \) is the initial time, \( h_{rl}^w \) behaves as free wave) in the form (in the transverse-traceless gauge)

\[
h_{rl}^w(t, x) = (2\pi)^{-\frac{5}{2}} \int dk \exp(ikx)h_{rl}^w(k, t)
\]

is the gravitational wave, \( h_{rl}^h \) describes the thermal modes distributed with the Gibbs equilibrium measure, \( h_{rl}^0 \) and \( \Pi_0^a \) are random initial conditions, \( h_{rl}^i \) is the gravitational field created by the motion of \( m_0 \). \( h_{rl}^w \) and \( h_{rl}^h \) satisfy the homogeneous equation

\[
\frac{d^2 h_{rl}^w(t)}{dt^2} + k^2 h_{rl}^w(t) = 0.
\]

The equation of motion for the coordinate \( \xi \) is

\[
\frac{d^2 \xi}{dt^2} = \frac{\lambda}{2} \frac{d^2 h_{rl}^w(t)}{dt^2} \xi + \frac{\lambda}{2} (2\pi)^{-\frac{5}{2}} \int d k \frac{d^2 h_{rl}^h(t)}{dt^2} \xi. \tag{112}
\]

We insert the gravitational field from Eq. (111) into Eq. (112). We obtain

\[
\frac{d^2 \xi}{dt^2} = \frac{\lambda}{2} \frac{d^2 h_{rl}^w(t)}{dt^2} \xi + N_{rl}(t) \xi + F_{rl}(\xi, t), \tag{113}
\]

where \( F \) is a non-linear interaction resulting from the interaction with the environment and

\[
N_{rl}(t) = \int d k N_{rl}^w(k, t) = -\frac{\lambda}{2} (2\pi)^{-\frac{5}{2}} \int d k k^2 e_{rl}^w
\times (h_{rl}^0 \cos(kt) + k^{-1} \Pi_0^a \sin(kt)). \tag{114}
\]

The classical Gibbs distribution (a classical limit of the quantum Gibbs distribution of Sect. 7) is \( (\Pi_0^a \) has the meaning of the initial canonical momentum)

\[
d\Pi_0^a dh_{rl}^0 \exp \left(-\frac{\beta}{2} \frac{lg^2}{2} + \frac{h_{rl}^0}{2} h_{rl}^0 \right) \tag{115}
\]

Calculating the correlation functions of the noise (assuming the Gibbs distribution of the initial values) we obtain

\[
\langle N_{rl}(k, t)N_{mn}(k', t') \rangle = \frac{\lambda^2}{4} \beta^{-1} (2\pi)^{-3} \delta(k + k') k^2 \Lambda_{rl;mn} \
\times \cos(k(t - t')). \tag{116}
\]

So that

\[
\langle N_{rl}(t)N_{mn}(t') \rangle = \langle \Lambda_{rl;mn} \rangle \frac{\lambda^2}{4} \beta^{-1} \delta_{t - t'} \tag{117}
\]

The non-linear force resulting from the graviton environment is

\[
F_t = \frac{1}{2} \lambda^2 (2\pi)^{-3} \delta_{t - t'} \int d k \frac{d^2 h_{rl}^w(t)}{dt^2} \xi + \frac{1}{2} \lambda^2 m_0 (2\pi)^{-3} \int d k d k' \times \langle \Lambda_{rl;mn} \rangle \int_0^t \cos(k(t - t')) f_{mn}(t') dt' \xi + f_{mn}(t) \xi', \tag{118}
\]

We write the last factor as

\[
\int_0^t \cos(k(t - t')) f_{mn}(t') dt' \xi + f_{mn}(t) \xi'(t)
\]

\[
= (- \int_0^t \partial_t \cos(k(t - t')) f_{mn}(t') dt' \xi + f_{mn}(t) \xi(t))
\]

\[
= \int_0^t \cos(k(t - t')) \partial_t f_{mn}(t') dt' \xi(t) + \cos(k(t - t)) f_{mn}(t) \xi(t). \tag{119}
\]

Then

\[
\int_0^t \int d k d k' \cos(k(t - t')) \partial_t f_{mn}(t') dt' \xi + \cos(k(t - t)) f_{mn}(t) \xi(t)
\]

\[
= -2\pi \int_0^t \partial_t^2 \delta(t - t') \partial_t f_{mn}(t') dt' = 2\pi \partial_t^2 f_{mn}(t). \tag{120}
\]

where we assumed that at \( t_0 \) the first and the second derivatives of \( f_{mn} \) are zero. If we assume that in (119) \( f_{mn}(t_0) = 0 \)
then we can write Eq. (113) in the form
\[
d^2\xi^r\over dt^2 = \lambda^2 \left( \frac{d^2}{dt^2} h_{rlw} \right) - \frac{\lambda^2}{5\pi} \left( \delta_{rm} \delta_{ln} - \frac{1}{3} \delta_{rl} \delta_{mn} \right) \xi^l \partial^3 f^{ml} + N^{rl}(t) \xi_l. \tag{121}
\]
Equation (121) coincides with Eq. (98) as the noise (117) is the same as the one defined by Eqs. (98), (99).

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