COHOMOLOGY, FUSION AND A P-NILPOTENCY CRITERION

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Abstract. Let $G$ be a finite group, $p$ a fix prime and $P$ a Sylow $p$-subgroup of $G$. In this short note we prove that if $p$ is odd, $G$ is $p$-nilpotent if and only if $P$ controls fusion of cyclic groups of order $p$. For the case $p = 2$, we show that $G$ is $p$-nilpotent if and only if $P$ controls fusion of cyclic groups of order 2 and 4.

1. Introduction

Throughout the text let $p$ denote a fix prime. Let $G$ be a finite group and $P$ a Sylow $p$-subgroup of $G$. We denote by $H^\bullet(G,F_p)$ the mod $p$ cohomology algebra. It is well known that the restriction map in cohomology

(1) $H^\bullet(G,F_p) \rightarrow H^\bullet(P,F_p)$

is injective (see [2, Proposition 4.2.2]). Suppose that $G$ is $p$-nilpotent, i.e., $P$ has a normal complement $N$ in $G$. In this situation the composition

(2) $P \rightarrow G \rightarrow G/N,$

is an isomorphism. Therefore the composition

(3) $H^\bullet(G/N,F_p) \xrightarrow{\text{inf}_{G/N}} H^\bullet(G,F_p) \xrightarrow{\text{res}^G_P} H^\bullet(P,F_p)$

is also an isomorphism. This together with [1] implies that, if $G$ is $p$-nilpotent, then the restriction map in cohomology $\text{res}^G_P : H^\bullet(G,F_p) \rightarrow H^\bullet(P,F_p)$ is an isomorphism. The following result of M. Atiyah shows that the converse is also true.

**Theorem 1 (Atiyah).** If $\text{res}^G_P : H^1(G,F_p) \rightarrow H^1(P,F_p)$ are isomorphisms for all $i$ big enough, then $G$ is $p$-nilpotent. In particular $G$ is $p$-nilpotent if and only if $\text{res}^G_P : H^\bullet(G,F_p) \rightarrow H^\bullet(P,F_p)$ is an isomorphism.

**Proof.** A proof of this can be found in the introduction of [8].

Atiyah’s $p$-nilpotency criterion uses the cohomology in high dimension. Another cohomological criterion for $p$-nilpotency using cohomology in dimension 1 was provided by J. Tate ([10]).

**Theorem 2 (Tate).** If $\text{res}^G_P : H^1(G,F_p) \rightarrow H^1(P,F_p)$ is an isomorphism, then $G$ is $p$-nilpotent.

**Proof.** See [10].

D. Quillen generalized Atiyah’s $p$-nilpotency criterion for odd primes ([8]).

**Theorem 3 (Quillen).** Let $p$ be an odd prime. Then $G$ is $p$-nilpotent if and only if $\text{res}^G_P : H^\bullet(G,F_p) \rightarrow H^\bullet(P,F_p)$ is an $F$-isomorphism.

**Proof.** See [8].
Atiyah’s $p$-nilpotency criterion can be reinterpreted in terms of $p$-fusion. We recall that a subgroup $H$ of $G$ controls $p$-fusion in $G$ if

(a) $H$ contains a Sylow $p$-subgroup of $G$ and
(b) for any subgroup $A$ of $G$ and for any $g \in G$ such that $A, A^g \leq H$, there exists $x \in H$ such that for all $a \in A$, $a^g = a^x$.

By a result of G. Mislin [7], a subgroup $H$ of $G$ controls $p$-fusion in $G$ if and only if $\text{res}_H^G : H^*(G, \mathbb{F}_p) \rightarrow H^*(H, \mathbb{F}_p)$ is an isomorphism. Using Mislin’s result Atiyah’s $p$-nilpotency criterion follows from Frobenius $p$-nilpotency criterion.

Mislin’s type of result can also be provided for the concept of $F$-isomorphism. In order to do this we introduce the following concept. Let $C$ be a class of finite $p$-groups. We say that a subgroup $H$ of $G$ controls fusion of $C$-groups in $G$ if

(a) Any $C$-subgroup of $G$ is conjugated to a subgroup of $H$ and
(b) for any $C$-subgroup $A$ of $G$ and for any $g \in G$ such that $A, A^g \leq H$, there exists $x \in H$ such that for all $a \in A$, $a^g = a^x$.

The condition (b) can be rewritten as

(b') if $A$ is a $C$-subgroup of $H$ and $g \in G$ satisfies that $A^g \leq H$, then $g \in C_G(A).H$.

Theorem A bellow, which will be proved in Section 2, follows naturally from Quillen’s work on cohomology (see [8] and [9]). Note that the “if” was proved in [4] and it is a direct consequence of Quillen’s stratification ([9]). The converse follows from a careful reading of [8] Section 2.3.

**Theorem A.** Let $G$ be a finite group and $H$ a subgroup of $G$. Then $\text{res}_H^G : H^*(G, \mathbb{F}_p) \rightarrow H^*(H, \mathbb{F}_p)$ is an $F$-isomorphism if and only if $H$ controls fusion of elementary abelian $p$-subgroups of $G$.

In Section 3 we will prove the following $p$-nilpotency criterion that can be seen as a generalization of Quillen $p$-nilpotency criterion (Theorem 3 above) to the prime $p = 2$.

**Theorem B.** Let $G$ be a finite group and $P$ a Sylow $p$-subgroup of $G$. Then the following two conditions are equivalent

1. $G$ is $p$-nilpotent.
2. $P$ controls fusion of cyclic subgroups of order $p$ in case $p$ is odd, and cyclic subgroups of order 2 and 4 in case $p = 2$.

Note that Theorem A and Theorem B imply Quillen’s $p$-nilpotency criterion. We will finish this short note by giving two applications of Theorem B. The first application will consist on reproving a result of H-W. Henn and S. Priddy that implies that “most” finite groups are $p$-nilpotent (see [4]). The second application is a generalization to the prime $p = 2$ of the following fact: if all elements of order $p$ of a finite group $G$ are in some upper center of $G$ and $p$ is an odd prime, then $G$ is $p$-nilpotent (see [12] and [4]). For the prime $p = 2$ we will show that if all elements of order 2 and 4 are in some upper center of $G$, then $G$ is 2-nilpotent.

We would like to end this introduction with an example of Quillen [8] where the necessity of considering cyclic groups of order 2 and 4 for the case $p = 2$ in Theorem B is illustrated.

**Example 4.** Consider $Q = \{1, -1, i, -i, j, -j, k, -k\}$ the quaternion group and $\alpha$ an automorphism of order 3 that permutes $i, j$ and $k$. Let $G$ be the semidirect product between $Q$ and $\langle \alpha \rangle$ given by the action of $\alpha$ in $Q$. $A = \{1, -1\}$ is the only subgroup of exponent 2 in $G$. Clearly $Q$ controls fusion of cyclic subgroup of order 2. However $G$ is not 2-nilpotent.
2. Cohomology and fusion

The aim of this section is to sketch the proof of Theorem A. In subsections 2.1, 2.2 and 2.3 we will recall Quillen work in the mod $p$ cohomology algebra of a finite group. This will be used in subsection 2.4 to prove Theorem A.

For a finite group $G$ the mod $p$ cohomology algebra

\[ H^\bullet(G) = H^\bullet(G, \mathbb{F}_p) \]

is a finitely generated, connected, anti-commutative, $\mathbb{N}_0$-graded $\mathbb{F}_p$-algebra.

Let $\alpha_\bullet: A_\bullet \to B_\bullet$ be a homomorphism of finitely generated, connected, anti-commutative, $\mathbb{N}_0$-graded $\mathbb{F}_p$-algebras. Then $\alpha_\bullet$ is called an $F$-isomorphism if $\ker(\alpha_\bullet)$ is nilpotent, and for all $b \in B_n$ there exists $k \geq 0$ such that $b^{p^k} \in \text{im}(\alpha_\bullet)$.

2.1. Quillen’s stratification. Let $G$ be a finite group. Let $\mathcal{E}_G$ denote the category whose objects are the elementary abelian $p$-subgroups of $G$ and whose morphisms are given by conjugation, i.e., for $E, E' \in \text{ob}(\mathcal{E}_G)$ one has

\[ \text{mor}_G(E, E') = \{ i_g: E \to E' \mid g \in G, g E g^{-1} \leq E' \}, \]

where $i_g(e) = g e g^{-1}, e \in E$. Then

\[ H^\bullet(\mathcal{E}_G) = \varprojlim_{E \in \text{ob}(\mathcal{E}_G)} H^\bullet(E) \]

is a finitely generated, connected, anti-commutative, $\mathbb{N}_0$-graded $\mathbb{F}_p$-algebra. Moreover, the restriction maps $\text{res}_E^G$ yield a map

\[ q_G = \prod_{E \in \text{ob}(\mathcal{E}_G)} \text{res}_E^G: H^\bullet(G) \to H^\bullet(\mathcal{E}_G). \]

The following result is known as Quillen stratification.

**Theorem 5** (Quillen). Let $G$ be a finite group. Then $q_G: H^\bullet(G) \to H^\bullet(\mathcal{E}_G)$ is an $F$-isomorphism.

**Proof.** See [1, Cor. 5.6.4] or [9].

2.2. Cohomology of elementary abelian $p$-groups. One can easily deduce the cohomology of an elementary abelian $p$-group from the cohomology of the cyclic group of exponent $p$ and the Kunneth formula.

**Lemma 6.** Let $A$ be an elementary abelian $p$-group. Then

\[ H^\bullet(A, \mathbb{F}_p) \cong \begin{cases} \Lambda(A^*) \otimes S(\beta(A^*)) & \text{if } p \text{ is odd} \\ S(A^*) & \text{if } p = 2, \end{cases} \]

where $\Lambda$ denotes the exterior algebra functor, $S$ the symmetric algebra functor, $A^* = \text{Hom}(A, \mathbb{F}_p) = H^1(A, \mathbb{F}_p)$ and $\beta$ the Bockstein homomorphism from $H^1(A, \mathbb{F}_p)$ to $H^2(A, \mathbb{F}_p)$.

**Proof.** See [2] Chap. 3 Section 5].

From the previous lemma one can easily deduces that

\[ H^\bullet(A, \mathbb{F}_p)/\sqrt{0} \cong S(A^*). \]
2.3. The spectrum of $H(G)$. Let $G$ be a finite group. Following Quillen (8) we define
\begin{equation}
H(G) = \begin{cases} 
\oplus_{i \geq 0} H^{2i}(G, \mathbb{F}_p) & \text{if } p \text{ is odd} \\
\oplus_{i \geq 0} H^{i}(G, \mathbb{F}_p) & \text{if } p = 2. 
\end{cases}
\end{equation}

$H(G)$ is a graded commutative ring. For an elementary abelian $p$-subgroup $A$ of $G$, denote by $g_A$ the ideal of $H(G)$ consisting of elements $u$ such that $u|_A$ is nilpotent. From [8], $\text{res}^G_A : H(G) \rightarrow H(A)$ induces a monomorphism
\begin{equation}
H(G)/g_A \cong S(A^*).
\end{equation}

In particular, the ideal $g_A$ is a prime ideal of $H(G)$. Furthermore,

**Theorem 7** (Quillen). Let $A, A' \subseteq G$ be elementary abelian subgroups of $G$. Then $g_A \subseteq g_{A'}$ if and only if $A'$ is conjugated to a subgroup of $A$. In particular $g_A = g_{A'}$ if and only if $A$ and $A'$ are conjugated in $G$.

**Proof.** See [8] Theorem 2.7].

Let us consider the extension of quotient fields associated to the monomorphism in (11).
\begin{equation}
k(g_A)^* \hookrightarrow k(A).
\end{equation}

We have that

**Theorem 8** (Quillen). The extension $k(A)/k(g_A)$ is a normal extension and
\begin{equation}
\text{Aut}(k(A)/k(g_A)) \cong N_G(A)/G_G(A).
\end{equation}

**Proof.** See [8] Theorem 2.10].

2.4. F-isomorphisms and fusion. The following lemma is a standard result in commutative algebra.

**Lemma 9.** Let $A$ and $B$ be commutative $\mathbb{F}_p$-algebras and $f : A \rightarrow B$ an F-isomorphism. Then $f^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$ is a homeomorphism.

**Proof.** Since the kernel of $f$ is nilpotent, then for any radical ideal $a$ of $A$ one has that $f^{-1}(\sqrt{f(a)}) = a$. Since for any $x \in B$ there exits $y \in A$ and $n \geq 0$ such that $f(y) = x^{p^n}$, then for any radical ideal $b$ of $B$ one has that $\sqrt{f(f^{-1}(b))} = b$. Therefore
\begin{align}
a & \longrightarrow f(a) \\
b & \longrightarrow f^{-1}(b)
\end{align}
is a bijection between the radical ideals of $A$ and the radical ideals of $B$. In particular $f^*$ is an isomorphism of varieties.

We are now ready to prove Theorem A.

**Proof of Theorem A.** Suppose first that $H$ controls fusion of elementary abelian $p$-subgroups of $G$. Then the embedding functor
\begin{equation}
j_{H,G} : \mathcal{E}_H \rightarrow \mathcal{E}_G
\end{equation}
is an equivalence of categories. Therefore
\begin{equation}
H^*(j_{H,G}) : H^*(\mathcal{E}_G) \longrightarrow H^*(\mathcal{E}_H)
\end{equation}
is an isomorphism. Consider the commutative diagram

\[
\begin{array}{ccc}
H^\ast(G) & \xrightarrow{\text{res}_G^H} & H^\ast(C_G) \\
\downarrow & & \downarrow \\
H^\ast(H) & \xrightarrow{\text{res}_H^G \cdot h} & H^\ast(C_H).
\end{array}
\]

By Theorem 5 and equation (17) it follows that \(\text{res}_G^H\) is an \(F\)-isomorphism.

Subclaim 1: If \(A\) and \(A'\) are conjugated in \(G\), then they are conjugated in \(H\).

Subproof. By Lemma 6, \(f^* : \text{Spec}(H(H)) \rightarrow \text{Spec}(H(G))\) provides a bijection between the prime ideals of \(H(H)\) and the prime ideals of \(H(G)\). Furthermore, if \(A\) is an elementary abelian \(p\)-subgroup of \(H\), then \(g_A = f^*(h_A)\). By Theorem 7 if \(A\) and \(A'\) are conjugated in \(G\), then \(g_A = g_{A'}\). In particular, \(f^*(h_A) = g_A = g_{A'} = f^*(h_A)\). Therefore \(h_A = h_{A'}\). Hence, by Theorem 7 \(A\) and \(A'\) are conjugated in \(H\).

Subclaim 2: \(N_G(A) = C_G(A)N_H(A)\).

Subproof. Since \(k(h_A)\) is a purely inseparable extension of \(k(g_A)\), then

\[
\text{Aut}(k(A)/k(h_A)) \cong \text{Aut}(k(A)/k(g_A)).
\]

Therefore, by Theorem 5 \(N_H(A)/C_H(A) \cong N_G(A)/C_G(A)\). □

Subclaim 3: \(H\) controls fusion of elementary abelian \(p\)-subgroups of \(G\).

Subproof: Let \(A\) be an elementary abelian \(p\)-subgroup of \(H\) and \(g \in G\) such that \(A^g \leq H\). Then, by Subclaim 1 there exists \(h \in H\) such that \(A^g = A^h\). In particular, by Subclaim 2, \(gh^{-1} \in N_G(A) = C_G(A)\cdot N_H(A)\). Therefore \(g \in C_G(A)\cdot H\) □

3. A \(p\)-Nilpotency Criterion

In this section we will prove our main result Theorem B. To ease the notation we denote by \(C_p\) the class of cyclic groups of order \(p\) in case \(p\) is odd and cyclic groups of order \(2\) and \(4\) in case \(p = 2\). Put \(p = p\) if \(p\) is odd and \(p = 4\) in case \(p = 2\).

**Theorem 10.** Let \(G\) be a finite group and \(P\) a Sylow \(p\)-subgroup of \(G\). Then the following two conditions are equivalent

1. \(G\) is \(p\)-nilpotent.
2. \(P\) controls fusion of \(C_p\)-groups.

**Proof.** It is clear that if \(G\) is \(p\)-nilpotent, then \(P\) controls fusion of \(C_p\)-groups.

Let us show the converse. Using Frobenius \(p\)-nilpotency criterion it is enough to prove that for any subgroup \(B\) of \(P\) and for any \(p'\)-element \(g \in N_G(B)\), then \(g\) centralizes \(B\). The subgroup \(B\) is contained in \(Z_l(P)\) for some \(l \geq 1\) where \(Z_l(P)\) denotes the \(l\)-upper center of \(P\). We will show by induction on \(l\) that \(g \in C_G(B)\).

Suppose first that \(B \leq Z(P)\) and consider \(a \in B\) such that \(a^p = 1\). Since \(P\) controls fusion of \(C_p\)-groups, there exists \(x \in P\) such that \(a^x = a^e\) and since \(a \in Z(P)\), then \(x = a\). Hence we have that \(g\) centralizes all elements of order \(p\) (2 and 4 in case \(p = 2\)) in \(B\). Thus, by [6] Chap. V Lemma 5.12, \(g\) centralizes \(B\).

For the general case, consider \(B \leq Z_l(P)\) and suppose the assumption to be true for any subgroup contained in \(Z_{l-1}(P)\).

Subclaim 1: For \(a \in B\) such that \(a^p = 1\), we have that \([a, g, g] = 1\).
Subproof. We have that \( g \) normalizes the subgroups \( K = \langle a \in B \mid a^p = 1 \rangle \) and \([K, g]\). We also have that

\[ (20) \quad [K, g] = \langle [a, g]^b \mid a, b \in K \text{ and } a^p = 1 \rangle. \]

Take \( a \in B \) such that \( a^p = 1 \). Since \( P \) controls fusion of \( C_p \)-groups, there exists \( x \in P \) such that \( a^g = a^x \). In particular \( [a, g] = [a, x] \in Z_{i-1}(P) \). Therefore, by (20), \([K, g] = Z_{i-1}(P)\). Since \( g \) normalizes \([K, g]\) and by induction hypothesis we have that \([K, g, g] = 1\). 

\( \square \)

Subclaim 2: \( g \in C_G(B) \).

Subproof. Take \( a \in B \) such that \( a^p = 1 \) and put \( p^e \) the exponent of \( B \). Consider the subgroup \( H = \langle a, [a, g] \rangle \). By the Subclaim 1, \( \gamma_2(H) = 1 \). Then, by [5, Chap. III, Theorem 9.4], we have that \( [a, g^{p^e}] = [a, g]^{p^e} = 1 \). Since \( g \) is a \( p^e \)-element of \( G \), \( g \) centralizes all elements of order \( p \) (2 and 4 in case \( p = 2 \)) in \( B \). Thus, by [6, Chap. V Lemma 5.12], \( g \) centralizes \( B \).

This ends the proof. \( \square \)

As a consequence to this we have the following corollary.

**Corollary 11.** Let \( G \) a finite group and \( P \) a Sylow \( p \)-subgroup of \( G \) such that

1. \( N_G(P) \) controls fusion of \( C_p \)-groups and
2. \( N_G(P) = C_G(P).P \).

Then \( G \) is \( p \)-nilpotent.

**Proof.** Let \( A \) be a \( C_p \)-group and \( g \in G \) such that \( A^g \leq P \). Since \( N_G(P) \) controls fusion \( C_p \)-groups, one has that \( g \in C_G(A).N_G(P) = C_G(A).P \). Then \( P \) controls fusion of \( C_p \)-groups and, by Theorem [10] \( G \) is \( p \)-nilpotent. \( \square \)

4. Some Applications

We now present the first application of Theorem [10]. In [5] H-W. Henn and S. Priddy proved that if a group \( G \) has a Sylow \( p \)-subgroup \( P \) such that

i) if \( p \) is odd, the elements of order \( p \) of \( P \) are in the center of \( P \) and, if \( p = 2 \), the elements of order 2 and 4 are in the center of \( P \),

ii) \( \text{Aut}(P) \) is a \( p \)-group,

then \( G \) is \( p \)-nilpotent. This implies that "most" finite groups are \( p \)-nilpotent (see [5]). The proof of Henn and Priddy is essentially topological. In [11] J. Thevenaz gave a group theoretical proof of this result using Alperin’s Fusion Theorem. In fact Thevenaz proved that if \( G \) satisfies condition i), then \( N_G(P) \) controls \( p \)-fusion in \( G \). This, together with condition ii) above implies that \( P \) controls \( p \)-fusion in \( G \) and therefore \( G \) is \( p \)-nilpotent. We now give a weaker version of Thevenaz result which also implies that a group satisfying i) and ii) is \( p \)-nilpotent.

**Proposition 12.** Let \( G \) be a finite group and \( P \) a Sylow \( p \)-subgroup of \( G \). Suppose that the elements of order dividing \( p \) in \( P \) (or 4 in case \( p = 2 \)) are in the center of \( P \). Then \( N_G(P) \) controls fusion of \( C_p \)-groups.

**Proof.** Let \( A \) be a \( C_p \)-group and \( g \in G \) such that \( A^g \leq N_G(P) \). In particular \( A^g \leq P \). Equivalently \( A \leq P^{g^{-1}} \). Hence, since the elements of \( P \) of order \( p \) (or 4 in case \( p = 2 \)) are in the center of \( P \), we have that \( P, P^{g^{-1}} \leq C_G(A) \). But, since \( P \) and \( P^{g^{-1}} \) are Sylow \( p \)-subgroups of \( C_G(A) \), there exists \( c \in C_G(A) \) such that \( P = P^{g^{-1}}.c \). Thus \( g^{-1}c \in N_G(P) \) and \( g \in N_G(P).C_G(A) \). \( \square \)

**Corollary 13.** Let \( G \) a finite group and \( P \) a Sylow \( p \)-subgroup of \( G \) such that
1. all elements of order dividing $p$ in $P$ (or 4 in case $p = 2$) are in the center of $P$ and
2. $N_G(P) = P.C_G(P)$.

Then $G$ is $p$-nilpotent.

Proof. It follows from Proposition 12 and Corollary 11. □

The second application of Theorem 10 is a generalization to $p = 2$ of the fact that if the elements of order $p$ of a finite group $G$ are in some upper center of $G$, then $G$ is $p$-nilpotent (see [12] and [4]).

Corollary 14. Let $G$ a finite group such that $K = \langle x \in G \mid x^p = 1 \rangle \leq Z_n(G)$ for some $n \geq 1$ (here $p$ means $p$ in case $p$ is odd and 4 in case $p = 2$). Then $G$ is $p$-nilpotent.

Proof. The subgroup $K$ is nilpotent of class at most $n$, and therefore a finite $p$-group. Let $p^e$ be the exponent of $K$. Then, by Hall-Petrescu collection formula (see [3, Theorem 2.1]), for any $y \in K$ and $x \in G$

\[(y, x^{p^e+n}) \in \prod_{0 \leq i \leq e+n} [K, G, \ldots, G]^{p^i+n-1} = 1.\]

Therefore one has that $G^{p^e+n} \leq C_G(K)$. Moreover, for any Sylow $p$-subgroup $P$ of $G$ one has $G = P.G^{p^e+n} = P.C_G(K)$. In particular $P$ controls fusion of $C_p$-groups. Hence, by Theorem 10 $G$ is $p$-nilpotent. □

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