ON THE HARDY–LITTLEWOOD MAJORANT PROBLEM
FOR ARITHMETIC SETS

BEN KRAUSE, MARIUSZ MIREK, AND BARTOSZ TROJAN

Abstract. The aim of this paper is to exhibit a wide class of sparse deterministic sets, \( B \subseteq \mathbb{N} \), so that
\[ \limsup_{N \to \infty} \frac{N-1}{N} |B \cap [1,N]| = 0, \]
for which the Hardy–Littlewood majorant property holds:
\[ \sup_{|a_n| \leq 1} \left\| \sum_{n \in B \cap [1,N]} a_n e^{2\pi in\xi} \right\|_{L^p(T,d\xi)} \leq C_p \left( \sum_{n \in B \cap [1,N]} |a_n| \right)^{1/p} \]
where \( p \geq p_B \) is sufficiently large, the implicit constant \( C_p \) is independent of \( N \), and the supremum is taken over all complex sequences \( (a_n : n \in \mathbb{N}) \) such that \( |a_n| \leq 1 \).

1. Introduction

In 1937, Hardy and Littlewood [7] conjectured that for each \( p \geq 2 \) there is a constant \( C_p > 0 \) such that for every finite set \( A \subset \mathbb{N} \) and every sequence \( (a_n : n \in A) \) of complex numbers satisfying \( \sup_{n \in A} |a_n| \leq 1 \) we have
\[ \left\| \sum_{n \in A} a_n e^{2\pi in\xi} \right\|_{L^p(T,d\xi)} \leq C_p \left( \sum_{n \in A} |a_n| \right)^{1/p} . \]
This conjecture, known as the Hardy–Littlewood majorant problem, was suggested by a simple observation, based on Parseval’s identity, which implies that \( C_p = 1 \) for every even integer \( p \geq 2 \). It was also noticed by Hardy and Littlewood that \( C_2 > 1 \). In 1962, Boas [2] showed that \( C_p > 1 \) for any \( p \notin \{2k : k \in \mathbb{N}\} \).

Finally, in early seventies Bachelis [1] disproved the Hardy–Littlewood conjecture showing unboundedness of \( C_p \) for every \( p \notin \{2k : k \in \mathbb{N}\} \) as \( |A| \to \infty \).

Although inequality (1) fails to hold in general, recently some attention has been paid to quantify this failure. To do so, for \( N \in \mathbb{N} \) we consider
\[ C_p(N) = \sup_{A \subseteq \{1, \ldots, N\}} C_p(A,N) \]
where for \( A \subseteq \{1, \ldots, N\} \) we have set
\[ C_p(A,N) = \sup_{|a_n| \leq 1} \left\| \sum_{n \in A} a_n e^{2\pi in\xi} \right\|_{L^p(T,d\xi)} \cdot \left( \sum_{n \in A} |e^{2\pi in\xi}| \right)^{-1} . \]
It was proven in [10] that for every \( p \in (2,4) \) there is a constant \( C > 0 \) such that
\[ \log C_p(N) \geq C \frac{\log N}{\log \log N} . \]

Consequently, the Hardy–Littlewood majorant problem was reformulated to a slightly weaker statement. Namely, it was conjectured that for every \( p \geq 2 \) and \( \varepsilon > 0 \) there is a constant \( C_{p,\varepsilon} > 0 \) such that for every \( N \in \mathbb{N} \)
\[ C_p(N) \leq C_{p,\varepsilon} N^\varepsilon . \]

(2)
It is worth mentioning that (2) implies the restriction conjecture for the Fourier transform on $\mathbb{R}^d$, i.e. that for every $p > 2d/(d - 1)$ there exists a constant $C_{p,d} > 0$ such that

$$\|f \hat{d}\sigma\|_{L^p(\mathbb{R}^d)} \leq C_{p,d} \|f\|_{L^\infty(S^{d-1},d\sigma)}$$

where $\sigma$ is the spherical measure on the unit sphere $S^{d-1}$ in $\mathbb{R}^d$. In [11] it was stated that for suitable sets $A$ the inequality (1) may be treated as a restatement of (3). However, Mockenhaupt and Schlag [11] disproved (2) by showing that for all $p > 2$ which is not an even integer, there are constants $\eta > 0$ and $C > 0$ such that $C_p(N) \geq CN^\eta$. For $p = 3$ the same result was obtained by Green and Ruzsa [5].

In view of the restriction conjecture one may ask whether there are sets $A \subseteq \{1, \ldots, N\}$ such that for every $p \geq 2$ and $\varepsilon > 0$ there exists a constant $C_{p,\varepsilon} > 0$ for which we have

$$C_p(A, N) \leq C_{p,\varepsilon}N^\varepsilon.$$  

The question above has been extensively studied by Mockenhaupt and Schlag in [11] where the authors proved that for every $g \in (0, 1)$ and $p \geq 2$ there are random sets $A \subseteq \{1, \ldots, N\}$ with cardinality $N^\varepsilon$ satisfying (4) with a large probability.

The Hardy–Littlewood majorant property plays an important role in combinatorial problems. In [4] Green used a variant of the inequality (1) for the set of prime numbers $\mathbb{P}$ to deduce that every subset of $\mathbb{P}$ with non-vanishing relative upper-density contains at least one arithmetic progression of length three. Specifically, Green proved that for every $p \geq 2$ there is a constant $C_p > 0$ such that for all $N \in \mathbb{N}$

$$C_p(\mathbb{P}_N, N) \leq C_p$$

where $\mathbb{P}_N = \mathbb{P} \cap [1, N]$, the set of primes less than or equal to $N$. Generally speaking, in problems of this kind it is critical to know whether the majorant property (1) holds for some $p \in (2, 3)$ with the uniform constant $C_p$, independent of the cardinality of the set $A$ (see [9, 12]).

The present article is devoted to study a wide class of deterministic infinite sets $A \subseteq \mathbb{N}$ with vanishing Banach density, i.e.

$$\limsup_{N \to \infty} \frac{|A \cap [1, N]|}{N} = 0,$$

and obeying the Hardy–Littlewood majorant property. In particular, we will be concerned with the sets

$$A = \{ \lfloor h(n) \rfloor : n \in \mathbb{N} \}$$

where $h$ is a regularly varying function of the form $h(x) = x\ell(x)$, for a suitably chosen slowly varying function $\ell$, e.g.

$$\ell(x) = (\log x)^B, \quad \text{or} \quad \ell(x) = \exp\left( B(\log x)^C \right), \quad \text{or} \quad \ell(x) = l_m(x),$$

where $B > 0$, $C \in (0, 1)$, $l_1(x) = \log x$ and $l_{m+1}(x) = \log(l_m(x))$, for $m \in \mathbb{N}$. We show that for every $p \geq 2$ there exists a constant $C_p > 0$ such that for every $N \in \mathbb{N}$ we have

$$C_p(A_N, N) \leq C_p$$

where $A_N = A \cap [1, N]$. We also consider the sets (5) with

$$h(x) = x^c\ell(x)$$

for some $c > 1$ sufficiently close to 1. In this case we show that it is possible to find $p_c > 2$ such that for every $p > p_c$ there exists a constant $C_{c,p} > 0$ such that for every $N \in \mathbb{N}$

$$C_p(A_N, N) \leq C_{c,p}.$$

Moreover, $\lim_{c \to 1} p_c = 2$.  

1.1. **Statement of the results.** Before we precisely formulate the main results we need to introduce some definitions.

**Definition 1.1.** Let $\mathcal{L}$ be a family of slowly varying functions $\ell : [x_0, \infty) \to (0, \infty)$ such that

$$\ell(x) = \exp \left( \int_{x_0}^x \frac{\vartheta(t)}{t} \, dt \right)$$

where $\vartheta \in C^2([x_0, \infty))$ is a real function satisfying

$$\lim_{x \to \infty} \vartheta(x) = 0, \quad \lim_{x \to \infty} x\vartheta'(x) = 0, \quad \lim_{x \to \infty} x^2\vartheta''(x) = 0.$$

We also distinguish a subfamily $\mathcal{L}_0$ of $\mathcal{L}$.

**Definition 1.2.** Let $\mathcal{L}_0$ be a family of slowly varying functions $\ell : [x_0, \infty) \to (0, \infty)$ such that

$$\ell(x) = \exp \left( \int_{x_0}^x \frac{\vartheta(t)}{t} \, dt \right)$$

where $\vartheta \in C^2([x_0, \infty))$ is positive decreasing real function satisfying

$$\lim_{x \to \infty} \vartheta(x) = 0, \quad \lim_{x \to \infty} x\vartheta'(x) = 0, \quad \lim_{x \to \infty} x^2\vartheta''(x) = 0,$$

and for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that $1 \leq C_\varepsilon \vartheta(x) x^\varepsilon$ and $\lim_{x \to \infty} \ell(x) = \infty$.

Finally, we define the subfamily $\mathcal{R}_c$ of regularly varying functions.

**Definition 1.3.** For every $c \in (0, 2) \setminus \{1\}$ let $\mathcal{R}_c$ be a family of increasing, convex, regularly-varying functions $h : [x_0, \infty) \to [1, \infty)$ of the form

$$h(x) = x^c L(x)$$

where $L \in \mathcal{L}$. If $c = 1$ we impose that $L \in \mathcal{L}_0$.

We fix two functions $h_1 \in \mathcal{R}_{c_1}$ and $h_2 \in \mathcal{R}_{c_2}$ for $c_1 \in [1, 2)$ and $c_2 \in [1, 6/5)$. Let $\varphi_1$ and $\varphi_2$ be the inverse of $h_1$ and $h_2$, respectively. We consider a function $\psi : [x_0, \infty) \to (0, \infty)$ such that for all $x \geq x_0$, $\psi(x) \leq 1/2$ and

\[ \lim_{x \to +\infty} \frac{\psi(x)}{\varphi_2^2(x)} = 1, \quad \lim_{x \to +\infty} \frac{\psi'(x)}{\varphi_2^2(x)} = 1, \quad \lim_{x \to +\infty} \frac{\psi''(x)}{\varphi_2^2(x)} = 1. \] (6)

Finally, we define two sets

$$\mathbf{B}_+ = \{ n \in \mathbb{N} : \{ \varphi_1(n) \} < \psi(n) \}, \quad \mathbf{B}_- = \{ n \in \mathbb{N} : \{ -\varphi_1(n) \} < \psi(n) \}.$$ 

Let us observe that if $h_1 = h_2 = h$ is the inverse function $\varphi$ and $\psi(x) = \varphi(x + 1) - \varphi(x)$ then $\mathbf{B}_- = \mathbf{A}$. Indeed, we have the following chain of equivalences

- $m \in \mathbf{A} \iff m = \lfloor h(n) \rfloor$ for some $n \in \mathbb{N}$
- $\iff h(n) - 1 < m \leq h(n) < m + 1$
- $\iff \varphi(m) \leq n < \varphi(m + 1)$, since $\varphi$ is well-defined and monotonically increasing
- $\iff 0 \leq n - \varphi(m) < \varphi(m + 1) - \varphi(m) = \psi(m) < 1/2$
- $\iff 0 \leq \{ -\varphi(m) \} < \psi(m)$
- $\iff m \in \mathbf{B}_-$

The main result of this paper is the following theorem.
Theorem 1. Assume that $c_1 \in [1, 2)$ and $c_2 = 1$. Then for every $p \geq 2$ there exists a constant $C_p > 0$ such that for every $N \in \mathbb{N}$ and any sequence of complex numbers $(a_n : n \in \mathbb{N})$ satisfying $\sup_{n \in \mathbb{N}} |a_n| \leq 1$ we have

\begin{equation}
\left\| \sum_{n \in \mathcal{B}_N} a_n e^{2\pi i n \xi} \right\|_{L^p(T, d\xi)} \leq C_p \left\| \sum_{n \in \mathcal{B}_N} e^{2\pi i n \xi} \right\|_{L^p(T, d\xi)} \tag{7}
\end{equation}

where $\mathcal{B}_N^\pm = \mathcal{B}_\pm \cap [1, N]$.

We observe that by the Hausdorff–Young inequality for every $p \geq 2$ we obtain

\[ \left\| \sum_{n \in \mathcal{B}_N} a_n e^{2\pi i n \xi} \right\|_{L^p(T, d\xi)} \leq |\mathcal{B}_N^\pm|^{1/p'} . \]

Moreover, by integrating over frequencies $|\xi| \leq 1/(100N)$, we have the following lower bound

\[ \left\| \sum_{n \in \mathcal{B}_N} e^{2\pi i n \xi} \right\|_{L^p(T, d\xi)} \gtrsim |\mathcal{B}_N^\pm|^{-1/p} . \]

These inequalities combined together yield

\begin{equation}
\left\| \sum_{n \in \mathcal{B}_N} a_n e^{2\pi i n \xi} \right\|_{L^p(T, d\xi)} \lesssim |\mathcal{B}_N^\pm|^{1/p} N^{-1/p} \left\| \sum_{n \in \mathcal{B}_N} e^{2\pi i n \xi} \right\|_{L^p(T, d\xi)} . \tag{8}
\end{equation}

By Proposition 2.1 for $c_2 = 1$, we have $|\mathcal{B}_N^\pm| \sim \varphi_2(N)$ where $\varphi_2(N) = NL\varphi_2(N)$ for some slowly varying function $L\varphi_2 \in \mathcal{L}_0$. Therefore, applying inequality (8), we obtain

\[ C_p(\mathcal{B}_N^\pm, N) \lesssim L\varphi_2(N)^{1/p} \lesssim N^\varepsilon \]

for any $\varepsilon > 0$. Hence, the main difficulty in proving Theorem 1 is to show that the constant in (7) is independent of $N$.

Next, we would like to relax the hypothesis in Theorem 1 to allow any $c_2 \in [1, 6/5)$. It is possible at the expense of a slightly worse range of $p$. Let us introduce

\[ p(c_1, c_2) = \frac{2/c_1 - 6/c_2 + 6}{1/c_1 + 3/c_2 - 3} = 2 + \frac{12 - 12/c_2}{1/c_1 + 3/c_2 - 3} . \]

We observe that if $c_1 \in [1, 2)$ and $c_2 \in [1, 6/5)$ then $1 < \frac{1}{c_1} + \frac{1}{c_2}$, thus

\[ \frac{12 - 12/c_2}{1/c_1 + 3/c_2 - 3} \geq 0 . \]

Also notice that

\[ \lim_{c_2 \to 1} p(c_1, c_2) = 2 . \]

The extended version of Theorem 1 has the following form.

Theorem 2. Assume that $c_1 \in [1, 2)$ and $c_2 \in [1, 6/5)$. Then for every $p \geq p(c_1, c_2)$ there exists a constant $C_p > 0$ such that for every $N \in \mathbb{N}$ and any sequence of complex numbers $(a_n : n \in \mathbb{N})$ satisfying $\sup_{n \in \mathbb{N}} |a_n| \leq 1$ we have

\[ \left\| \sum_{n \in \mathcal{B}_N} a_n e^{2\pi i n \xi} \right\|_{L^p(T, d\xi)} \leq C_p \left\| \sum_{n \in \mathcal{B}_N} e^{2\pi i n \xi} \right\|_{L^p(T, d\xi)} \]

where $\mathcal{B}_N = \mathcal{B}_\pm \cap [1, N]$. 


We were inspired to study Hardy–Littlewood majorant property by the paper of Mockenhaupt and Schlag \cite{Schlag} where the authors considered sparse random subsets of the integers. The desire to better understand structure of deterministic sets which satisfy \cite{Schlag} was our principal motivation.

Before turning to the arguments, let us begin with some preliminary remarks. The heart of the matter lies in proving our Proposition \cite{I} which can be thought of as a restriction estimate for our sets $B^\pm_N$. We accomplish this using a Tomas–Stein $TT^*$ argument, which forces us to estimate certain exponential sums, see Section 3 below. These estimates are quite delicate, and lead to the technical restriction on the range of $L^p$ spaces which we are able to handle; in particular, we do not yet know how to extend Theorem 2 to the full regime $2 < p < p(c_1, c_2)$. Finally, it is worth calling attention to the explicit construction of the sets $B^\pm_N$ for which the full strength of the Hardy–Littlewood property holds. To the best of the authors knowledge it is the first treatment where such a wide family of subsets of the integers satisfies property \cite{I}.

2. Some properties of the sets $B^\pm$

As it has been observed, when $c_1 \in [1, 2)$ and $c_2 \in [1, 6/5)$ we have $1 < \frac{1}{3c_1} + \frac{1}{c_2}$, or equivalently

$$3(1 - \gamma_2) + (1 - \gamma_1) < 1,$$

where $\gamma_1 = 1/c_1$ and $\gamma_2 = 1/c_2$. Under this assumption, we prove the asymptotic formula for the cardinality of sets $B^\pm$.

**Proposition 2.1.** For every $\epsilon > 0$

$$|B^\pm_N| = \varphi_2(N)(1 + O(N^{-\epsilon})).$$

From now on we only work with the sets $B_+$ because all the results remain valid for $B_-$ with similar proofs. To simplify the notation we write

$$B = B_+ = \{n \in \mathbb{N} : \{\varphi_1(n)\} < \psi(n)\}.$$  

We need the following working characterizations of the sets $B$.

**Lemma 2.2.** $n \in B$ if and only if $|\varphi_1(n)| - |\varphi_1(n) - \psi(n)| = 1$.

**Proof.** We begin with the forward implication; it suffices to show that if $n \in B$, the integer

$$|\varphi_1(n)| - |\varphi_1(n) - \psi(n)|$$

belongs to $(0, 3/2)$. By definition, if $n \in B$ then $0 \leq \varphi_1(n) - |\varphi_1(n)| < \psi(n)$, thus

$$-\varphi_1(n) \leq -|\varphi_1(n)| < \psi(n) - \varphi_1(n)$$

if and only if

$$\varphi_1(n) \geq |\varphi_1(n)| > \varphi_1(n) - \psi(n),$$

from where it follows that

$$|\varphi_1(n)| - |\varphi_1(n) - \psi(n)| > \{\varphi_1(n) - \psi(n)\} \geq 0.$$  

In view of $|\varphi_1(n) - \psi(n)| \geq \varphi_1(n) - \psi(n) - 1$, we obtain

$$|\varphi_1(n)| - |\varphi_1(n) - \psi(n)| \leq |\varphi_1(n)| - \varphi_1(n) + \psi(n) + 1 \leq \psi(n) + 1 < 3/2.$$  

We now turn to the reverse implication; if $|\varphi_1(n)| = 1 + |\varphi_1(n) - \psi(n)|$, we have

$$0 \leq \varphi_1(n) - |\varphi_1(n)| = \varphi_1(n) - 1 - |\varphi_1(n) - \psi(n)|$$

$$< \varphi_1(n) - 1 + 1 + \psi(n) - \varphi_1(n) = \psi(n).$$

Consequently, we get $\{\varphi_1(n)\} < \psi(n)$, as desired. \hfill $\Box$
Our next task is to show that for every $\delta \geq 0$ satisfying $3(1 - \gamma_2) + (1 - \gamma_1) + 6\delta < 1$ there is $\delta' > 0$ such that

$$\sum_{n \in B_N} e^{2\pi i \xi n} = \sum_{n=1}^N \psi(n) e^{2\pi i \xi n} + O\left(\varphi_2(N) N^{-\delta - \delta'}\right)$$

where the implied constant is independent of $\xi$ and $N$. Let us observe that the asymptotic formula (9) follows from (11) by taking $\xi = 0$. Indeed, we have

$$|B_N| = \sum_{n \in B_N} 1 = \sum_{n=1}^N \psi(n) + O\left(\varphi_2(N) N^{-\epsilon}\right)$$

and summation by parts yields

$$\frac{1}{\varphi_2(N)} \sum_{n=1}^N \psi(n) = \frac{N \psi(N)}{\varphi_2(N)} - \frac{1}{\varphi_2(N)} \int_1^N x \psi'(x) \, dx = \frac{1}{\varphi_2(N)} \int_1^N \psi(x) \, dx = 1 + o(1).$$

Although, for the proof of (9) we only needed (10) with $\xi = 0$, the more general version will be used in our future works.

For the proof of (11), let us introduce the “sawtooth” function $\Phi(x) = \{x\} - 1/2$. Notice that

$$|\varphi_1(n)| - |\varphi_1(n) - \psi(n)| = \psi(n) + \Phi(\varphi_1(n) - \psi(n)) - \Phi(\varphi_1(n)).$$

With this in mind, we may write

$$\sum_{n \in B_N} e^{2\pi i \xi n} = \sum_{n=1}^N \psi(n) e^{2\pi i \xi n} + \sum_{n=1}^N \left(\Phi(\varphi_1(n) - \psi(n)) - \Phi(\varphi_1(n))\right) e^{2\pi i \xi n}.$$

The second sum we absorb into an error term of the order $O(\varphi_2(N) N^{-\epsilon})$. To do so, see [3], we expand $\Phi$ into its Fourier series, i.e.

$$\Phi(x) = \sum_{0 < |m| \leq M} \frac{1}{2\pi i m} e^{-2\pi i m x} + O \left( \min\left\{ 1, \frac{1}{M \|x\|} \right\} \right),$$

for some $M > 0$ where $\|x\| = \min\{|x - n| : n \in \mathbb{Z}\}$ is the distance of $x \in \mathbb{R}$ to the nearest integer. Next, we expand

$$\min\left\{ 1, \frac{1}{M \|x\|} \right\} = \sum_{m \in \mathbb{Z}} b_m e^{2\pi i m x}$$

where

$$|b_m| \lesssim \min\left\{ \frac{\log M}{M}, \frac{1}{|m|}, \frac{M}{|m|^2} \right\}.$$
Now, our aim is to show that each part $I_1$, $I_2$, and $I_3$ is $O(\phi_2(N)N^{-\varepsilon})$. In the proof we use the estimates for the following trigonometric sums: for $m \in \mathbb{Z} \setminus \{0\}$, $l \in \{0, 1\}$, and $X \leq X' \leq 2X$ we consider
\[
\sum_{X \leq n \leq X' \leq 2X} e^{2\pi i (\xi_n + m(\varphi_1(n) - l\psi(n)))}
\]
By [9] Lemma 2.14, if $c_1 = 1$ then there is a positive decreasing real function $\sigma_1$ satisfying $\sigma_1(2x) \simeq \sigma_1(x)$ and $\sigma_1(x) \gtrsim x^{-\varepsilon}$ for any $\varepsilon > 0$, such that
\[
\varphi_1''(x) \simeq \frac{\varphi_1(x)\sigma_1(x)}{x^2}.
\]
We set $\sigma_1 \equiv 1$ whenever $c_1 > 1$. Similarly, by [9] Lemma 2.14 for $\varphi_2''$ we obtain
\[
\varphi_2''(x) \simeq \frac{\varphi_2(x)}{x^3}.
\]
Therefore, by [9] we may write
\[
|\psi''(x)| \sim |\varphi_2''(x)| \simeq \frac{\varphi_2(x)}{x^3}.
\]
Since $1/c_2 \leq 1 < 1 + 1/c_1$, we get
\[
\frac{\varphi_2(x)}{x\sigma_1(x)\varphi_1(x)} = o(1),
\]
thus
\[
|\psi''(x)| = o\left(\frac{\sigma_1(x)\varphi_1(x)}{x^2}\right).
\]
Let $F(x) = \xi x + m(\varphi_1(x) - l\psi(x))$. By (15) and (16), for any $X \leq x \leq X' \leq 2X$ we may write
\[
|F''(x)| = |m| \cdot |\psi''(x)| \simeq \frac{|m|\sigma_1(X)\varphi_1(X)}{X^2}.
\]
Therefore, the Van der Corput lemma (see [9] Theorem 2.2) yields
\[
\left| \sum_{X < n \leq 2X} e^{2\pi i (\xi_n + m(\varphi_1(n) - l\psi(n)))} \right| \lesssim X \left( \frac{m\sigma_1(X)\varphi_1(X)}{X^2} \right)^{1/2} + \left( \frac{X^2}{m\sigma_1(X)\varphi_1(X)} \right)^{1/2}
\]
(17)
\[
\lesssim m^{1/2} X \sigma_1(X) \varphi_1(X)^{-1/2}.
\]
Finally, we get
\[
\left| \sum_{j=1}^N e^{2\pi i (\xi_n + m(\varphi_1(n) - l\psi(n)))} \right| \leq \left| \sum_{j=0}^{\lfloor \log N \rfloor} \sum_{2^j < n \leq 2^{j+1}} e^{2\pi i (\xi_n + m(\varphi_1(n) - l\psi(n)))} \right| \lesssim m^{1/2} N (\log N) \sigma_1(N) \varphi_1(N)^{-1/2}
\]
since the function $x \mapsto x(\sigma_1(x)\varphi_1(x))^{-1/2}$ is increasing. In particular, we have proven the following lemma.

**Lemma 2.3.** There is a positive decreasing real function $\sigma_1$ satisfying $\sigma_1(2x) \simeq \sigma_1(x)$ and $\sigma_1(x) \gtrsim x^{-\varepsilon}$, for any $\varepsilon > 0$, such that for every $m \in \mathbb{Z} \setminus \{0\}$, $l \in \{0, 1\}$, and $N \geq 1$ we have
\[
\left| \sum_{1 \leq n \leq N} e^{2\pi i (\xi_n + m(\varphi_1(n) - l\psi(n)))} \right| \lesssim |m|^{1/2} N (\log N) \sigma_1(N) \varphi_1(N)^{-1/2}.
\]
If $c_1 > 1$ then $\sigma_1 \equiv 1$. The implied constant is independent of $m$, $N$ and $\xi$.

Next, we return to bounding $I_1$, $I_2$ and $I_3$. 


2.1. The estimate for $I_1$. Let
\[
S(x) = \sum_{x \leq n \leq x' < 2x} e^{2\pi i(n\xi - m\varphi_1(nx))}
\]
and $\phi_m(x) = e^{2\pi i m\psi(x)} - 1$. We observe that
\[
|\phi_m(x)| \lesssim mx^{-1} \varphi_2(x)
\]
and
\[
|\phi'_m(x)| \lesssim mx^{-2} \varphi_2(x).
\]
Applying to the inner sum in $I_1$ summation by parts together with (17) we obtain
\[
\left| \sum_{n=1}^{N} e^{2\pi i(n\xi - m\varphi_1(n))}\phi_m(n) \right|
\leq (\log N) \sup_{x \in [1,N]} \left( |S(2X)| \cdot |\phi_m(2X)| + |S(X)| \cdot |\phi_m(X)| + \int_{X}^{2X} |S(x)| \cdot |\phi'_m(x)| \, dx \right)
\lesssim (\log N) \sup_{x \in [1,N]} m^{3/2} \varphi_2(X) (\sigma_1(X) \varphi_1(X))^{-1/2}
\leq m^{3/2} \varphi_2(N)(\log N)(\sigma_1(N) \varphi_1(N))^{-1/2}.
\]
Therefore,
\[
|I_1| \lesssim \sum_{m=1}^{M} m^{1/2} (\log N) \varphi_2(N) (\sigma_1(N) \varphi_1(N))^{-1/2}
\lesssim M^{3/2} (\log N) \varphi_2(N) (\sigma_1(N) \varphi_1(N))^{-1/2}.
\]

2.2. The estimates for $I_2$ and $I_3$. We only treat $I_2$ because $I_3$ can be handled by a similar reasoning. By (13), (14) and Lemma (2.2) we have
\[
\sum_{n=1}^{N} \min \left\{ 1, \frac{1}{M} \frac{1}{\| \varphi_1(n) - \psi(n) \|} \right\} \leq \sum_{m \in \mathbb{Z}} |b_m| \left| \sum_{n=1}^{N} e^{2\pi i m(\varphi_1(n) - \psi(n))} \right|
\lesssim N (\log M) + \left( \sum_{0 < |m| \leq M} \frac{\log M}{M} + \sum_{|m| > M} \frac{M}{|m|^2} \right) |m|^{1/2} (\log N) \varphi_2(N) (\sigma_1(N) \varphi_1(N))^{-1/2}
\lesssim \frac{N (\log M)}{M} + M^{1/2} (\log M) (\log N) \varphi_2(N) (\sigma_1(N) \varphi_1(N))^{-1/2}.
\]

2.3. Concluding remarks. Based on Subsections 2.1 and 2.2 we get
\[
|I_1| + |I_2| + |I_3| \lesssim \frac{N (\log M)}{M} + M^{3/2} (\log M) (\log N) \varphi_2(N) (\sigma_1(N) \varphi_1(N))^{-1/2}.
\]

Therefore, by taking $M = N^{1+\delta} (\log N) \varphi_2(N)^{-1}$, we conclude
\[
|I_1| + |I_2| + |I_3| \lesssim \varphi_2(N) N^{-\delta} \left( 1 + N^{3/2+5\delta/2} (\log N) \sigma_1(N)^{-1/2} \varphi_1(N)^{-1/2} \varphi_2(N)^{-3/2} \right)
\lesssim \varphi_2(N) N^{-\delta} \left( 1 + N^{3/2+6\delta/2 - \gamma_1/2 - 3\gamma_2/2} \right)
\]
which is bounded by a constant multiple of $\varphi_2(N) N^{-\delta}$ since $3(1 - \gamma_2) + (1 - \gamma_1) + 6\delta < 1$. 

3. Proof of Theorem 1

Let
\[ F(f)(\xi) = \sum_{n \in \mathbb{Z}} f(n) e^{2\pi i \xi n} \]
denote the Fourier transform on \( \mathbb{Z} \), and
\[ \hat{f}(n) = \int_{\mathbb{T}} f(\xi) e^{-2\pi i \xi n} \, d\xi \]
denote the Fourier transform on \( \mathbb{T} \). For any measure space \( X \), let \( C(X) \) be the space of all continuous functions on \( X \). For \( N \in \mathbb{N} \) we introduce on \( \mathbb{Z} \) a measure \( \mu_N \) defined
\[ \mu_N(x) = N^{-1} \sum_{n \in B_N} \psi(n)^{-1} \delta_n(x). \]

Let \( T_N : C(B_N) \to C(\mathbb{T}) \) be the linear operator given by
\[ T_N(f) = F(f \mu_N). \]

We are going to prove the following proposition.

**Proposition 3.1.** For each
\[ p \geq 2 + \frac{12 - 12/c_2}{1/c_1 + 3/c_2 - 3} \]
there is a constant \( C_p > 0 \) such that for all \( N \in \mathbb{N} \) and \( f \in L^2(B_N, \mu_N) \)
\[ \|T_N f\|_{L^p(\mathbb{T})} \leq C_p N^{-1/p} \|f\|_{L^2(B_N, \mu_N)}. \]

Before embarking on the proof we show the following.

**Lemma 3.2.** For every \( \delta > 0 \) satisfying \( (1 - \gamma_1) + 3(1 - \gamma_2) + 6\delta < 1 \) there is \( \delta' > 0 \) such that
\[ \sum_{n \in B_N} \psi(n)^{-1} e^{2\pi i \xi n} = \sum_{n=1}^{N} e^{2\pi i \xi n} + \mathcal{O}(N^{1-\delta-\delta'}). \]
The implied constant is independent of \( \xi \) and \( N \).

**Proof.** For \( N \in \mathbb{N} \) and \( \xi \in \mathbb{T} \) we set
\[ S_N(\xi) = \sum_{k \in B_N} e^{2\pi i \xi k}. \]

Then, by the summation by parts we have
\[ \sum_{n \in B_N} \psi(n)^{-1} e^{2\pi i \xi n} = \sum_{n=1}^{N} \psi(n)^{-1} (S_n(\xi) - S_{n-1}(\xi)) \]
\[ = \psi(N+1)^{-1} S_N(\xi) + \sum_{n=1}^{N} (\psi(n)^{-1} - \psi(n+1)^{-1}) S_n(\xi). \]

Similarly, we may write
\[ \sum_{n=1}^{N} e^{2\pi i \xi n} = \psi(N+1)^{-1} \sum_{n=1}^{N} \psi(n) e^{2\pi i \xi n} + \sum_{n=1}^{N} (\psi(n)^{-1} - \psi(n+1)^{-1}) \sum_{k=1}^{n} \psi(k) e^{2\pi i \xi k}. \]
Thus, subtracting (20) from (19) we may estimate
\[
\left| \sum_{n \in B_N} \psi(n)^{-1} e^{2\pi i \xi n} - \sum_{n=1}^{N} e^{2\pi i \xi n} \right| \leq \sum_{n=1}^{N} |\psi(n)^{-1} - \psi(n + 1)^{-1}| \cdot \left| S_n(\xi) - \sum_{k=1}^{n} \psi(k) e^{2\pi i \xi k} \right| + \psi(N + 1)^{-1} \left| S_N(\xi) - \sum_{k=1}^{N} \psi(k) e^{2\pi i \xi k} \right|.
\]
By (10), for any \( \delta > 0 \) satisfying \( 3(1 - \gamma_1) + (1 - \gamma_2) + 6\delta < 1 \) there is \( \delta' > 0 \) such that for all \( n \in \mathbb{N} \)
\[
\left| S_n(\xi) - \sum_{k=1}^{n} \psi(k) e^{2\pi i \xi k} \right| \leq C \varphi_2(n)n^{-\delta} - \delta'.
\]
Using (6) together with (9) Lemma 2.14 we obtain
\[
\psi'(n) \lesssim \varphi''(n) \lesssim \frac{\varphi_2(n)}{n^{2\delta}}.
\]
Therefore, again by (6) and the monotonicity of \( \varphi_2 \) we get
\[
|\psi(n)^{-1} - \psi(n + 1)^{-1}| \lesssim \sup_{t \in [n, n+1]} |\psi(t)^{-2}\psi'(t)| \lesssim \varphi_2(n)^{-1}.
\]
Hence,
\[
\sum_{n=1}^{N} |\psi(n)^{-1} - \psi(n + 1)^{-1}| \cdot \left| S_n(\xi) - \sum_{k=1}^{n} \psi(k) e^{2\pi i \xi k} \right| \lesssim \sum_{n=1}^{N} n^{-\delta} \lesssim N^{1-\delta-\delta'}.
\]

**Proof of Proposition 4.2.** The \( TT^* \) argument will be critical in the proof. Firstly, let us calculate \( T_N^* \).
By Plancherel’s theorem we have
\[
\langle T_N f, g \rangle_{L^2(\mathbb{T})} = \int_{\mathbb{T}} \mathcal{F}(f \mu_N)(\xi) \overline{\mathcal{G}(\xi)} \, d\xi = \sum_{n \in \mathbb{Z}} f(n) \overline{g(n)} \mu_N(n) = \langle f, T_N^* g \rangle_{L^2(B_N, \mu_N)}.
\]
Therefore, the adjoint operator \( T_N^* : C(\mathbb{T})^* \to C(B_N)^* = C(B_N) \) is given by
\[
T_N^* g = \hat{g} \cdot 1_{B_N},
\]
and consequently, \( T_N T_N^* : C(\mathbb{T})^* \to C(\mathbb{T})^* \) may be written as
\[
T_N T_N^* f = f \ast \mathcal{F}(\mu_N).
\]
Let us observe that it is enough to show
\[
\| T_N T_N^* \|_{L^{p'}(\mathbb{T}) \to L^p(\mathbb{T})} \leq C_p N^{-2/p}.
\]
Indeed, for \( f \in L^2(B_N, \mu_N) \) and \( g \in L^{p'}(\mathbb{T}) \) we have
\[
|\langle T_N f, g \rangle_{L^2(\mathbb{T})}| = |\langle f, T_N^* g \rangle_{L^2(B_N, \mu_N)}| \leq \| f \|_{L^2(B_N, \mu_N)} \| T_N^* g \|_{L^2(B_N, \mu_N)}
\]
and since
\[
\| T_N^* g \|_{L^2(B_N, \mu_N)}^2 = \langle T_N T_N^* g, g \rangle_{L^2(\mathbb{T})} \leq \| T_N T_N^* \|_{L^{p'}(\mathbb{T}) \to L^p(\mathbb{T})} \| g \|_{L^{p'}(\mathbb{T})}^2,
\]
we obtain
\[
\| T_N f \|_{L^p(\mathbb{T})} \leq \| T_N T_N^* \|_{L^{p'}(\mathbb{T}) \to L^p(\mathbb{T})}^{1/2} \| f \|_{L^2(B_N, \mu_N)}.
\]
For the proof of (21), for \( N \in \mathbb{N} \), let us introduce an auxiliary measure \( \nu_N \) on \( \mathbb{Z} \) and the corresponding linear operator \( S_N : C(N_N) \to C(\mathbb{T}) \), by setting
\[
\nu_N(x) = N^{-1} \sum_{n \in N_N} \delta_n(x),
\]
and
\[ S_N f = \mathcal{F}(f \nu_N). \]
Here, \( N_N := N \cap [1, N] \). Reasoning similar to the above applied to the operator \( S_N \) leads to
\[ S_N S_N^* f = f \ast \mathcal{F}(\nu_N). \]
Since \( L^p(\mathbb{T}) \) can be embedded into \( C(\mathbb{T})^* \) for any \( p \geq 1 \) we may consider the operators \( T_N T_N^* \) and \( S_N S_N^* \) as mappings on \( L^p(\mathbb{T}) \) spaces. Next, we write
\[
\|T_N T_N^* f\|_{L^p(\mathbb{T})} = \|f \ast \mathcal{F}(\mu_N)\|_{L^p(\mathbb{T})} \\
\leq \|f \ast \mathcal{F}(\nu_N)\|_{L^p(\mathbb{T})} + \|f \ast \mathcal{F}(\mu_N - \nu_N)\|_{L^p(\mathbb{T})}.
\]
We are going to show that for each \( p \) satisfying (18) there is \( C_p > 0 \) such that
\[
\|f \ast \mathcal{F}(\nu_N)\|_{L^p(\mathbb{T})} \leq C_p N^{-2/p}\|f\|_{L^p(\mathbb{T})},
\]
(22)
\[
\|f \ast \mathcal{F}(\mu_N - \nu_N)\|_{L^p(\mathbb{T})} \leq C_p N^{-2/p}\|f\|_{L^p(\mathbb{T})}
\]
for all \( f \in L^p(\mathbb{T}) \). We start by proving (22) for \( p = 2 \). By Plancherel’s theorem we have
\[
\|f \ast \mathcal{F}(\nu_N)\|_{L^2(\mathbb{T})} = \|\hat{f}\nu_N\|_{L^2(\mathbb{T})} \leq \|\nu_N\|_{\ell^2(\mathbb{Z})}\|f\|_{L^2(\mathbb{T})} \\
\leq N^{-1}\|f\|_{L^2(\mathbb{T})}.
\]
On the other hand, for \( p = \infty \) we may write
\[
\|f \ast \mathcal{F}(\nu_N)\|_{L^\infty(\mathbb{T})} \leq \|\mathcal{F}(\nu_N)\|_{L^\infty(\mathbb{T})}\|f\|_{L^1(\mathbb{T})} \leq \|f\|_{L^1(\mathbb{T})}.
\]
Therefore, for \( p \geq 2 \) we use Riesz–Thorin interpolation theorem to obtain (22). To show (23), we apply analogous reasoning. Firstly, by Plancherel’s theorem we have
\[
\|f \ast \mathcal{F}(\mu_N - \nu_N)\|_{L^2(\mathbb{T})} = \|\hat{f}(\mu_N - \nu_N)\|_{L^2(\mathbb{T})} \leq \|\mu_N - \nu_N\|_{\ell^\infty(\mathbb{Z})}\|f\|_{L^2(\mathbb{T})} \\
\leq \varphi_2(N)^{-1}\|f\|_{L^2(\mathbb{T})}.
\]
Secondly, for \( p = \infty \) we get
\[
\|f \ast \mathcal{F}(\mu_N - \nu_N)\|_{L^\infty(\mathbb{T})} \leq \|\mathcal{F}(\mu_N) - \mathcal{F}(\nu_N)\|_{L^\infty(\mathbb{T})}\|f\|_{L^1(\mathbb{T})} \\
\leq N^{-\delta - \delta'}\|f\|_{L^1(\mathbb{T})}
\]
where in the last estimate we have used Lemma (24). Thus, again by Riesz–Thorin interpolation theorem, for \( p \geq 2 \) we get
\[
\|f \ast \mathcal{F}(\mu_N - \nu_N)\|_{L^p(\mathbb{T})} \leq \|\mu_N - \nu_N\|_{\ell^\infty(\mathbb{Z})}^{2/p}\|f\|_{L^p(\mathbb{T})}^{1-2/p} \}
\|\mathcal{F}(\mu_N - \nu_N)\|_{L^\infty(\mathbb{T})}^{2/p}\|f\|_{L^p(\mathbb{T})}^{1-2/p} \\
\leq \varphi_2(N)^{-2/p} N^{-(\delta + \delta')(1-2/p)}\|f\|_{L^p(\mathbb{T})}.
\]
Let us recall that for any \( \varepsilon > 0 \)
\[
\varphi_2(N) \geq \varepsilon N^{\gamma_2 - \varepsilon},
\]
Therefore, for the inequality (23) to hold true, we need to have \( \varepsilon > 0 \) and \( p > 2 \) to satisfy
\[-2(\gamma_2 - \varepsilon)/p - (\delta + \delta')(1 - 2/p) \leq -2/p.
\]
Hence,
\[
\varepsilon \leq -((1 - \gamma_2) + (\delta + \delta')(p - 2)/2.
\]
Because the right hand side has to be positive, we obtain the condition
\[(\delta + \delta')(p - 2)/2 - (1 - \gamma_2) > 0,
\]
which is equivalent to
\[ p > 2 + 2(1 - \gamma_2)/(\delta + \delta'). \]
Since $3(1 - \gamma_2) + (1 - \gamma_1) + 6\delta < 1$ we conclude
\[ p \geq 2 + \frac{12(1 - \gamma_2)}{\gamma_1 + 3\gamma_2 - 3}. \]
\[ \square \]

Next, we show Theorem 1 and Theorem 2.

**Proof of Theorem 1 and Theorem 2.** Let \((a_n : n \in \mathbb{N})\) be a sequence of complex numbers such that \(\sup_{n \in \mathbb{N}} |a_n| \leq 1\). Using Proposition \[\ref{prop:2}\] with \(f(n) = a_n \psi(n)\) we get
\[ \int_T \left| \sum_{n \in B_N} f(n) \psi(n)^{-1} e^{2\pi i \xi n} \right|^p d\xi \lesssim_p N^{p/2 - 1} \left( \sum_{n \in B_N} |f(n)|^2 \psi(n)^{-1} \right)^{p/2}, \]
thus, by \[\ref{eq:1}\],
\[ \int_T \left| \sum_{n \in B_N} a_n e^{2\pi i \xi n} \right|^p d\xi \lesssim_p N^{p/2 - 1} \left( \sum_{n \in B_N} \psi(n) \right)^{p/2} \lesssim_p N^{-1} \varphi_2(N)^p. \]

Finally, we may estimate
\[ \int_T \left| \sum_{n \in B_N} e^{2\pi i \xi n} \right|^p d\xi \gtrsim \int_{|\xi| \leq 1/(100N)} \left| \sum_{n \in B_N} e^{2\pi i \xi n} \right|^p d\xi \gtrsim N^{-1} \varphi_2(N)^p. \]
This completes the proof. \[\square\]