Quantum criticality on a chiral ladder: an $SU(2)$ iDMRG study

Philipp Scholl,$^{1,2}$ Andreas Haller,$^{1,2}$ Matteo Rizzi,$^1$ and Román Orús$^{1,3,4}$

$^1$Institute of Physics, Johannes Gutenberg University, 55099 Mainz, Germany
$^2$Graduate School Materials Science in Mainz, Staudingerweg 9, 55128 Mainz, Germany
$^3$Donostia International Physics Center, Paseo Manuel de Lardizabal 4, E-20018 San Sebastián, Spain
$^4$Ikerbasque Foundation for Science, María Díaz de Haro 3, E-48013 Bilbao, Spain

In this paper we study the ground state properties of a ladder Hamiltonian with chiral $SU(2)$-invariant spin interactions, a possible first step towards the construction of truly two-dimensional non-trivial systems with chiral properties starting from quasi-one dimensional ones. Our analysis uses a recent implementation by us of $SU(2)$ symmetry in tensor network algorithms, specifically for infinite Density Matrix Renormalization Group (iDMRG). After a preliminary analysis with Kadanoff coarse-graining and exact diagonalization for a small size system, we discuss its bosonization and the continuum limit of the model and show that it corresponds to a conformal field theory, in agreement with our numerical findings. In particular, the scaling of the entanglement entropy as well as finite-entanglement scaling data show that the ground state properties match those of the universality class of a $c = 1$ conformal field theory (CFT) in $(1+1)$ dimensions. We also study the algebraic decay of spin-spin and dimer-dimer correlation functions, as well as the algebraic convergence of the ground state energy with the bond dimension, and the entanglement spectrum of half an infinite chain. Our results for the entanglement spectrum are remarkably similar to those of the spin-1/2 Heisenberg chain, which we take as an indication that both systems are described by the same CFT at low energies, i.e., an $SU(2)_1$ Wess-Zumino-Witten theory. In our numerical study we arrive up to total bond dimension $\chi \sim 1200$. Moreover, we explain in detail how to construct Matrix Product Operators for $SU(2)$-invariant three-spin interactions, something that had not been addressed with sufficient depth in the literature.

I. INTRODUCTION

The study of quantum criticality with Density Matrix Renormalization Group [1] has a long history. As such, quantum critical systems have an infinite correlation length, and it is well-known that this cannot be exactly captured by DMRG, which is based on Matrix Product States [2], but perhaps rather by other tensor networks [3] such as the Multiscale Entanglement Renormalization Ansatz [4] and Tree Tensor Networks [5]. Nevertheless, DMRG is very efficient and simple to program, and this is the reason why it is the preferred option to study criticality, both in its finite-size and infinite-size (iDMRG) [6] versions. The approach, then, is to push forward as much as possible the MPS bond dimension, and do appropriate finite-size and/or finite-entanglement [7] scalings to extract critical properties. To push the bond dimension, one of the best ideas is to implement symmetries. In particular, for $SU(2)$-invariant systems, the use of $SU(2)$ at the level of the MPS has proven remarkably useful in simulations of, e.g., Heisenberg quantum spin chains and quasi-1d systems [8]. Using symmetries in the algorithm is thus a good idea to increase the bond dimension and extract accurate critical properties.

In this paper we use our own implementation of $SU(2)$-invariant iDMRG [9] to study the ground-state properties of a spin-1/2 2-leg ladder with chiral 3-spin interactions. The model is similar to the system in Ref. [10], defined on the two-dimensional Kagome lattice and with a Hamiltonian made of purely-chiral 3-spin terms [33]. In that model, a ground state analysis using 2d DMRG in cylinders unveiled a ground state with chiral topological order. Here our motivation for studying the ladder is multi-fold. First, it allows us to study the crossover from 1d to 2d for chiral interactions. In particular, we find that the ladder has chiral properties, and a critical ground state with a central charge $c = 1$ and other critical exponents that we characterise numerically. Our findings are also compatible with previous studies showing that the continuum limit of the model is a $(1+1)$-dimensional conformal field theory [11–13], and show in particular that the entanglement spectrum matches the expected behaviour of an $SU(2)_1$ Wess-Zumino-Witten (WZW) theory at low energies. Moreover, the model is the perfect test-bed for our own implementation of $SU(2)$ symmetry in TNs, and allows us to understand, as well, how to implement $SU(2)$-invariant Matrix Product Operators (MPO) for Hamiltonians with 3-spin chiral interactions: something that, to our surprise, had not yet been discussed with enough detail in the literature. Finally, the model is one of the simplest $SU(2)$-generalizations of a quite widespread strategy in trying to access non-trivial two dimensional systems with chiral properties, starting from quasi-one dimensional ones (often dubbed as “wire deconstructionism”) [14, 15]: the key idea being to gap out right movers of a wire with left movers (or vice versa, of course) of the neighbouring one by means of suitable interactions, remaining at the end with a chiral edge current on the external legs of the ladder. Such an approach is experiencing a growing application in cold atoms, photonics and nanowire experiments [16].

The structure of this paper is as follows. In Sec. II we explain the details of the model Hamiltonian for the chiral ladder with 3-spin interactions, explain briefly...
the expected behaviour from Kadanoff coarse-grainings and small-size exact diagonalization results, and discuss bosonization, its continuum limit, and expected behaviour of some correlation functions. In Sec. III we explain some details about the implementation of our numerical method, namely, SU(2)-invariant iDMRG. In Sec. IV we present the results of our simulation, where we show that the ground state of the system corresponds to a conformal field theory (CFT) with $c = 1$. We additionally study the algebraic decay of spin-spin and dimer-dimer correlation functions, as well as the entanglement spectrum and the convergence of the ground state energy. Our results for the entanglement spectrum are remarkably similar to those of the spin-$1/2$ Heisenberg chain, which we take as a strong indication that the CFT at low energies is a SU(2)$_1$ WZW theory. We wrap up our conclusions in Sec. V. Finally, in Appendix A we spend some time explaining the details of how to construct the SU(2)-invariant MPO for the Hamiltonian that we want to simulate, focusing on chiral 3-spin interactions, and in Appendix B we provide numerical data on the finite-entanglement scaling of the entanglement spectrum.

II. CHIRAL LADDER

A. The model

The model that we analyze in this paper is a 2-leg ladder with chiral interactions on triangles. Specifically, it is a model of spins-$1/2$ on the sites of the ladder of Fig. 1 via the three-spin interaction Hamiltonian

$$H = \sum_{i=1}^{L-2} J_i \mathbf{S}_i \cdot (\mathbf{S}_{i+1} \times \mathbf{S}_{i+2}),$$

with $\mathbf{S}_i$ the spin-$1/2$ operator at site $i$. The sites of the ladder are labeled in a snake-like pattern as shown in the details of the ladder in Fig. 2 and both triangles follow this snake-like labelling (upper triangles 1-2-3, lower triangles 2-3-4). We will consider the cases where $J_i \in \{\pm 1\}$ in which the coupling coefficients depend on the traversal of the triangle. A triangle formed by sites $i, i+1, i+2$ is traversed in (against) the direction of the labels if $J_i = 1$ ($J_i = -1$). This can be rephrased to clockwise or anticlockwise configurations for each triangle. The triangles of the full ladder are all clockwise configured if $J_i = (-1)^i$ (anticlockwise if $J_i = -(1)^i$). Mixing the two scenarios gives rise to $J_i = 1 \forall i$ ($J_i = -1 \forall i$), which leads to a staggered, anticlockwise-clockwise (clockwise/anticlockwise) configuration pattern.

Playing with different clockwise/anticlockwise configurations of the triangles, we can indeed get different Hamiltonians. For instance, for a unit cell of two triangles we can get the four configurations in Fig. 2. In the figure, two of the configurations ($H_1$ and $H_2$) have the same orientation of the triangles (i.e., both clockwise or both anticlockwise), and two ($H_3$ and $H_4$) have opposite orientation (i.e., one clockwise and one anticlockwise). For the sake of the properties of the system, only the relative orientation between the two triangles matters since the two options can be mapped between each other by a mirror symmetry. The main difference between the two cases with different relative triangle orientations (e.g. $H_1$ and $H_3$ in Fig. 2) is, intuitively, that the expected behavior of the Hamiltonian under time-reversal is different. While edge states for $H_1$ are expected to be counter-propagating, they propagate in the same direction for $H_3$, see the arrows in Fig. 2. In what follows we show that this intuition is indeed true.

B. First intuition with Kadanoff coarse-graining

The first approach we take to understand the dominant physics of the model consists in a Kadanoff-like coarse-graining procedure of the triangles into effective spin-$1/2$’s. In particular, we simply (i) project the $2^3$-dimensional Hilbert space of the triangle $n$ starting at

**FIG. 1: 2-leg ladder made of triangles, for the model in Eq. (1).**

**FIG. 2: Different orientations of the chiral triple product result in different models. In the first two cases the orientation is chosen to be the same whereas it is opposite in the last two cases.**
site $i = 3n - 2$ onto the 2-dimensional subspace of lowest energy via the isometry $W_n : \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \rightarrow \frac{1}{2}$, (ii) construct the representation of the operators $W_n S_i W_n^\dagger$ and $W_n (S_j \times S_{j+1}) W_n^\dagger$ in this subspace, and then (iii) look for the emerging Hamiltonian.

The first step is rather easy, once we recall that the $SU(2)$-invariant Hamiltonian triangle term of Eq. (1) has to be proportional to the identity in the different subspaces with definite total spin, that it has null trace and that it will be vanishing once the three spins are all parallel arranged. Indeed, a bit of algebra with Pauli matrices and Levi-Civita symbols leads to the expression:

$$S_1 \cdot (S_2 \times S_3) = \sum_{\alpha = \pm} \frac{\sqrt{3}}{4} \mathbb{P}_{1/2, \alpha} + 0 \mathbb{P}_{3/2}, \quad (2)$$

where $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}$ and $\mathbb{P}$ are the corresponding projectors. The searched isometry will then be depending on the sign of the triangle coupling, i.e.,

$$W_n W_n^\dagger = \mathbb{P}_{1/2, -\text{sgn}(J_{3n-2})}. \quad (3)$$

Next we have to construct the coarse-grained expressions of the spin operators involved in the interaction between triangles $n$ and $n + 1$. It turns out that we can choose the projectors such that $\forall j \in \{3n - 2, 3n - 1, 3n\}$ and $\alpha \in \{\pm\}$:

$$W_{1/2, \alpha} S_j W_{1/2, \alpha} = \frac{1}{3} \bar{S}_n \quad (4)$$

$$W_{1/2, \alpha} (S_j \times S_{j+1}) W_{1/2, \alpha} = \frac{\alpha}{\sqrt{3}} \bar{S}_n \quad (5)$$

with $\bar{S}_n$ the new effective spin $1/2$. The resulting effective Hamiltonian then reads:

$$H_{\text{eff}} = -\text{sgn} (J_1 J_2) \frac{|J_1| + |J_2|}{3 \sqrt{3}} \sum_{n=1}^N \bar{S}_n \cdot \bar{S}_{n+1}, \quad (6)$$

where $N \approx L/3$ is the total number of effective triangles, and we neglected an additive term $-\frac{\sqrt{3}}{4} \frac{|J_1| + |J_2|}{2} N$. We thus obtained an emerging spin-1/2 Heisenberg chain, whose magnetic character (ferro- or antiferro-) depends on the mutual signs of the triangle couplings $J_1$ and $J_2$, and we can resort to a wealth of known facts to foresee the behaviour of our triangle ladder.

If the triangles are all (anti-)clockwise oriented ($H_1$ and $H_2$), then the effective model (6) is antiferromagnetic: we therefore predict that it will be gapless, with central charge $c = 1$, and that its ground state would tend to minimise the total-spin of the chain, i.e., for even $N$ will be in the zero total spin sector. Conversely, if the triangles have mixed character ($H_3$ and $H_4$), then the effective model (6) is ferromagnetic: we have thus good reasons to expect that the system will try to maximise its total spin, giving rise to a macroscopic degeneracy of the ground state manifold, and without a well-defined CFT character. Of course, such low-energy projection is a very strong simplification and further corrections would be needed to describe the full richness of the model (e.g., the degeneracy counting of the case $J_1 = J_2$ in finite systems will be non-trivial). But still, we will see below that the main results obtained by this simple analysis are in fact confirmed by more sophisticated theoretical and numerical approaches.

C. Exact diagonalization of small systems

The intuition obtained from the Kadanoff blocking in the previous section can be further corroborated by a simple exact diagonalization exercise. Specifically, here we perform exact diagonalization for small sizes, in particular for 16 spins. For this case, we compute the ground state and low-energy excited states and evaluate some observables. Of particular interest in order to assess chirality are spin-current operators, which are defined by components of the spin vector product. Here we focus on the quantity

$$J_{ij}^z = \frac{i}{2} (\bar{S}_i^+ S_j^- - \bar{S}_i^- S_j^+), \quad (7)$$

which describes the flow of the $z$-component of magnetization between sites $i$ and $j$ (notice that, by $SU(2)$ invariance, there is not a preferred direction). For the wave functions obtained from our small-size exact diagonalization, we evaluate the expectation value of this current operator for the up, down, rung and slash pairs of sites, see Fig. 3.

For the Hamiltonian configuration $H_1$ (or equivalently $H_2$), we find from our results that the ground state is a singlet of $SU(2)$ with total spin zero, i.e., $\langle S^2 \rangle = 0$ with $S$ the total spin vector operator. Thus, we find that the ground state does not carry any currents. However, in the first excited state (an $SU(2)$ triplet) the pattern of currents for every pair of sites corresponds to the one in Fig. 4, where we show both open and periodic boundary conditions. The chirality of the currents in the bulk is clear, and matches the intuition from Fig. 2.

Complementarily, for the Hamiltonian configuration $H_3$ (or equivalently $H_4$), we find that the ground state...
has a well-defined total spin and is also degenerate, according to the data in Table I. In this case, we diagonalize the subspace of degenerate ground states in the \( S^z \) and the \( S^z \) basis and pick states with fixed total and \( z \)-component of the spin. We find that in such ground states there is a non-trivial current behaviour, as shown in Fig. 5. Again, the observed pattern also matches our intuitive picture from Fig. 2.

From our small-size study with exact diagonalization we learn a couple of important things in order to study this model. First, configurations \( H_3 \) and \( H_4 \) do not have counter-propagating chiral edge modes and, moreover, have a covariant ground state with a well-defined, non-zero, total spin \( S \) – therefore the ground manifold is a \( (2S + 1) \)-plet. Second, configurations \( H_1 \) and \( H_2 \) have counter-propagating chiral edge modes as well as a singlet ground state, which is thus \( SU(2) \) invariant.

As a matter of fact, the reason for \( H_3 \) breaking \( SU(2) \) symmetry down to \( U(1) \) lies in the different overall character of the Hamiltonian: as readily visible on short chains, \( H_1 \) is globally anti-ferromagnetic, while \( H_3 \) is instead ferromagnetic. This means, practically, that the ground state of \( H_1 \) belongs to the singlet sector (total spin zero), while the energy for \( H_3 \) would be minimized by a state with large spin. If we do not make use of symmetries, the multiplicity of the \( H_3 \) ground state manifold diverges at the targeted thermodynamic limit, thus making it burdensome for numerical algorithms to converge.

Given the above, in this paper we choose to analyze in detail the ground state of configuration \( H_1 \) (or equivalently \( H_2 \)), which is an \( SU(2) \) singlet, with an \( SU(2) \)-invariant infinite-DMRG code. The case of configuration \( H_3 \) (and \( H_4 \)) could be better assessed by an MPS code that incorporates \( U(1) \) symmetry instead, and/or an \( SU(2) \)-code that can target generic covariant states. We leave the study of this case for a separate work [17], and focus here entirely on the configuration providing an \( SU(2) \)-invariant ground state.

### D. Bosonization

A wide class of interactions can be treated by Jordan-Wigner transformation followed by bosonization of the fermionic modes [18, 19]. We start with the following set of identities

\[
S^+_j = e^{i\phi_j} c_j^\dagger, \quad S^-_j = e^{-i\phi_j} c_j, \quad S^z_j = n_j - 1/2,
\]

where \( \phi_j = \pi \sum_{k<j} S^+_k S^-_k \) is the Jordan-Wigner string. A term \( h(j) = J_j S_j \cdot (S_{j+1} \times S_{j+2}) \) of three consecutive spins with coupling \( J_j \) will transform to \( h(j) \rightarrow J_j (T(j) + V(j)) \). The kinetic and interacting terms read

\[
T(j) = -\frac{i}{4} \left( c_j^\dagger c_{j+1} - c_j^\dagger c_{j+2} + c_{j+1}^\dagger c_{j+2} \right) + h.c.,
\]

\[
V(j) = +\frac{i}{2} \left( c_j^\dagger c_{j+1} n_{j+1} - n_{j} c_{j+1}^\dagger c_{j+2} \right) + h.c.,
\]

with density operator \( n_j = c_j^\dagger c_j \). For convenience, we introduce two different species \( A \) and \( B \) for even/odd site
labels \( j \). The full kinetic part of the two-site invariant Hamiltonian in Eq. (1) reads

\[
H_{\text{kin}} = \sum_{j \text{ odd}} J_1 T(j) + J_2 T(j+1),
\]

and is diagonalized by means of a Fourier transformation. We can introduce a spinor \( d_k = (c_{A,k} \ c_{B,k})^T \) such that the Hamiltonian \( H_{\text{kin}} = \sum_k d_k^H(k)d_k \) reduces to a sum of \( 2 \times 2 \) matrices given by

\[
h(k) = -\frac{1}{4} \left( \begin{array}{cc} J_1 \sin k & -ij_+ (1 + e^{-ik}) \\ -ij_+ (1 + e^{+ik}) & J_2 \sin k \end{array} \right) \]

and \( J_+ = J_1 + J_2 \). Without further restrictions, we assume \( J_1 \geq 0 \) to avoid unnecessary ambiguities in the ordering of the bands and show the resulting dispersion in Fig. 6. In case \( J_2/J_1 = -1 \), we have two Fermi momenta, hence a central charge \( c = 2 \). This changes as soon as we allow \( J_2 \gtrsim -J_1 \): \( A, B \) scattering is allowed and a gap at zero momentum is formed. We start from the bare Hamiltonian \( H_0 = \sum_k d_k^H h_0(k) d_k \) with

\[
h_0(k) = -\frac{1}{2} \left( \begin{array}{cc} J_1 \sin k & 0 \\ 0 & J_2 \sin k \end{array} \right)
\]

which is a Luttinger liquid. We then rewrite the full Hamiltonian using \( c_{\alpha,j} \approx e^{i\phi_B} R_\alpha(j) + e^{-i\phi_A} L_\alpha(j) \), i.e. a decomposition into a set of two right and left moving fields \( R_\alpha(j) \) and \( L_\alpha(j) \) which are confined around the Fermi momenta of the two bands. At half filling, we find terms such as

\[
H \approx H_0 + \sum_j \Delta h(j) (J_1 + J_2 - 2n_{AB}(j)),
\]

for which we identify

\[
\Delta h(x) = \frac{1}{2i} L_A^+(x) R_B(x) + \text{h.c.},
\]

\[
n_{AB}(x) = J_1 [n_{R,A}(x) + n_{L,A}(x)] + J_2 [n_{R,B}(x) + n_{L,B}(x)].
\]

We now use abelian Bosonization, i.e. one field per species related to the density \( \phi_\alpha(x) \) and its conjugate field \( \phi_\alpha^*(x) \). The right and left moving modes can be expressed by the two bosonic fields

\[
R_\alpha(x) \sim e^{i(\phi_\alpha(x) - \phi_\alpha^*(x))}, \quad L_\alpha(x) \sim e^{i(\phi_\alpha(x) + \phi_\alpha^*(x))},
\]

and we can rewrite the expressions

\[
\Delta h(x) \sim \sin [\theta_B(x) - \phi_B(x) - \theta_A(x) - \phi_A(x)],
\]

\[
n_{AB}(x) \sim J_1 \partial_x \phi_A(x) + J_2 \partial_x \phi_B(x).
\]

\( \Delta h(x) \) is always a relevant perturbation and induces a gap at zero momentum, which is what we observe in Fig. 6. At the fine-tuned point \( J_1 = -J_2 \), there is no such term present in the Hamiltonian. However, scattering will be restored by \( \Delta h(x) n_{AB}(x) \): albeit being less relevant than \( \Delta h(x) \) alone, it induces a similar gapping mechanism by coupling left movers of wire \( A \) and right movers of wire \( B \). This is consistent with the observed value of the central charge \( c = 1 \).

### E. Continuum limit

The ladder Hamiltonian in Eq. (1) has been considered in the literature in the presence of extra spin-spin Heisenberg-like interactions [11–13]. In these works, the low-energy continuum limit of our purely-chiral lattice model \( H_1 \) (equivalently \( H_2 \)) has also been computed (see, e.g., Appendix A of Ref. [11]). Here we sketch briefly the main points of this derivation, and discuss some implications.

The key idea to derive the continuum limit is to use the same formalism as in Ref. [20] (see also Ref. [22]) to deal with the continuum limit of the antiferromagnetic Heisenberg quantum spin chain. Starting from a Hubbard-like Hamiltonian for fermions with spin (say, electrons), one considers operator \( c_{\alpha,i} \) which annihilates an electron of spin \( \alpha \) at site \( i \). Spin operators are then written in terms of these fermionic operators as

\[
\mathbf{S}_i = \frac{1}{2} c_{\alpha,i}^{\dagger} \sigma_{\alpha\beta} c_{\beta,i},
\]

with \( \sigma \) a vector of Pauli matrices. As explained in Ref. [20], in the continuum limit the fermionic field is expanded around the Fermi points \( k \approx \pm \pi/2 \)

\[
c_{\alpha,i} \rightarrow \psi_\alpha(x) \sim e^{-i\pi x/2} \psi_{L,\alpha}(x) + e^{i\pi x/2} \psi_{R,\alpha}(x),
\]

with \( \psi_{L/R,\alpha}(x) \) slowly-varying fields on the scale of lattice spacing \( a \), which annihilate left/right moving electrons. These chiral fermions can be bosonized as

\[
\varphi_{L/R,\alpha}(x) \sim e^{-i\pi x/a} \varphi_{L/R,\alpha}(x),
\]

with \( \varphi_{L/R,\alpha}(x) \) chiral bosons. Introducing charge and spin degrees of freedom as

\[
\varphi_{L/R,c}(x) = \varphi_{L/R,\uparrow}(x) + \varphi_{L/R,\downarrow}(x),
\]

\[
\varphi_{L/R,s}(x) = \frac{\varphi_{L/R,\uparrow}(x) - \varphi_{L/R,\downarrow}(x)}{\sqrt{2}},
\]

one can see that at half filling (or more restrictively, for one electron per lattice site), a small Hubbard interaction gaps out the charge mode. This can then be integrated out, and the low-energy properties are then described by spin physics. Moreover, for the Heisenberg model, the \( SU(2) \) spin symmetry is independently conserved for fields \( \psi_{L/R,\alpha}(x) \) in Ref. [20] which, following Noether’s theorem implies that one can write the conserved (chiral) currents

\[
J_{L/R} = \frac{1}{2} \psi_{L/R,\alpha}^{\dagger} \sigma_{\alpha\beta} \psi_{L/R,\beta}.
\]

Thus, in the continuum limit, the lattice spin operators can be written as

\[
a^{-1} \mathbf{S}_i \approx (J_L + J_R) + \frac{1}{2} (-1)^i \left( \psi_L,\alpha \sigma_{\alpha\beta} \psi_{R,\beta} + \text{h.c.} \right).
\]

\[
(22)
\]
Bosonizing the fermionic fields in the second term and integrating out the charge boson [20] one arrives at the expression

\[ a^{-1}S_i \approx (J_L + J_R) + (-1)^i \Theta \text{tr}(g_W \cdot \sigma), \]  

with \( g_W(x) \) the Wess-Zumino-Witten (WZW) field (i.e., an \( SU(2) \) matrix) and \( \Theta \) a non-universal constant. Matrix \( g_W(x) \) is explicitly given by

\[ g_W(x) = \left( \begin{array}{cc} e^{i\sqrt{2}\varphi_s} & e^{i\sqrt{2}\varphi_s} \\ e^{-i\sqrt{2}\varphi_s} & e^{-i\sqrt{2}\varphi_s} \end{array} \right), \]

with \( \varphi_s \equiv \varphi_{L,s} + \varphi_{R,s} \equiv (\varphi_{L,\uparrow} + \varphi_{R,\uparrow} - \varphi_{L,\downarrow} - \varphi_{R,\downarrow})/\sqrt{2} \) and \( \varphi_s \equiv \varphi_{L,s} - \varphi_{R,s} \equiv (\varphi_{L,\uparrow} - \varphi_{R,\uparrow} - \varphi_{L,\downarrow} + \varphi_{R,\downarrow})/\sqrt{2} \) following notation from Ref. [21]. Usually one defines \( n \equiv \Theta \text{tr}(g_W \cdot \sigma) \), which physically amounts to a quantum field for the staggered magnetization. Thus one finally arrives at the usual expression

\[ a^{-1}S_i \approx (J_L + J_R) + (-1)^i n. \]  

This equation sets the connection between lattice spin operators and chiral spin-current fields. Thus, for a Hubbard-like system of electrons with exactly one electron per site, charge degrees of freedom are frozen and spin physics emerges entirely in terms of these operators. Importantly for our purposes, in Ref. [13] it was shown that with this substitution, the continuum limit of our chiral Hamiltonian \( H_1 \) is given by

\[ H_1 \approx g \int dx (J_{L,1} \cdot J_{R,2} - J_{R,1} \cdot J_{L,2}), \]

where \( g = 4a/\pi \) with \( a \) the lattice spacing, and \( J_{L/R,1/2} \) is the chiral \( L/R \) current for the upper (1) or lower (2) legs of the ladder. Here, the approximation \( \approx \) means that the equivalence holds up to irrelevant local perturbations, and in the limit of lattice spacing going to zero. The calculation arriving to this effective Hamiltonian, which we do not reproduce entirely here, makes use of operator product expansions and analyses the existence of irrelevant terms in the continuum Hamiltonian. Notice that, as expected from the lattice Hamiltonian, the obtained quantum field theory is odd with respect to time-reversal symmetry (i.e., the exchange \( L \leftrightarrow R \)). As a remark, let us also mention that, recently, it has been shown that Eq. (26) can also be written in terms of four Majorana fields for each leg [23].

In the CFT language, operators \( J_{L/R} \) are \( SU(2)_1 \) Kac-Moody chiral currents, and are the ones entering the \( SU(2)_1 \) WZW model as low-energy effective field theory of the spin-1/2 Heisenberg quantum spin chain [20]. Importantly for our purposes, in Ref. [13] it was shown that the quantum field theory in Eq. (26) for the chiral spin ladder is an RG fixed point, i.e., \( dq/dl = 0 \), with \( l \) the RG-flow parameter. In other words, the continuum limit of \( H_1 \) is a scale-invariant quantum field theory in \( (1+1) \) dimensions. In combination with unitarity, a theorem by Zamolodchikov and Polchinski [24] implies that this is indeed a \((1+1)\)-dimensional conformal field theory (CFT) [34]. Therefore, we expect a critical behaviour in the numerical simulations of the lattice Hamiltonian for a purely-chiral ladder in Eq. (1).
At this point it is worth mentioning that, using these results for the continuum limit, we can actually predict some of the expected behavior for the correlation functions that we will compute for the lattice model. In particular, let us consider here the spin-spin correlator \((S_i S_j)\) for, say, the upper leg. Rewriting the continuum limit of spin operators as in Eq. (25), one arrives at an expression for the correlator in terms of \(J_{L/R}\) and \(n\) fields. Using the operator product expansion for these fields, one can compute the decay of the asymptotic decay of the correlator. These operator product expansions can be found in, e.g., the Appendix of Ref. [11]. The relevant non-vanishing ones for our case are

\[
J_M^n(x_M)J_M^n(0) \sim \frac{1}{(2\pi)^2} \frac{\delta_{ab}/2}{x_M^2} + \frac{i\epsilon_{abc} J_R^n(0)}{2\pi x_M},
\]

\[
n^a(x)n^b(0) \sim \frac{1}{2\pi^2a} \frac{\delta_{ab}}{(x_L x_R)^{1/2}},
\]

\[
J_L^n(x_L)n^b(0) \sim \frac{i\epsilon_{abc}n^c(0) + \delta_{ab}\text{tr}(g(0))/2\pi a}{4\pi x_L},
\]

\[
J_R^n(x_R)n^b(0) \sim \frac{i\epsilon_{abc}n^c(0) - \delta_{ab}\text{tr}(g(0))/2\pi a}{4\pi x_R},
\]

with \(M = L/R\) and \(x_{L/R} = vt \pm ix\) holonomic/antiholonomic coordinates (where \(\tau\) is imaginary time and \(v\) the velocity of the spin mode). Expanding the correlator and computing the vacuum expectation value according to the above expressions, the leading contribution at long distances is given by \(\sum_a n^a(x)n^a(0)\), such that we obtain

\[
(S_i S_j) \propto \frac{1}{|j - i|}.
\]  

This confirms this asymptotic decay later with our numerical simulations, as well as compute a number of other lattice correlation functions.

### III. METHODS

The numerical method that we used to compute the ground state properties of the model is infinite-DMRG [6]. The iDMRG algorithm has been extensively discussed many times in the literature, see, e.g., Sec. III of Ref. [25]. In our specific implementation we used the 2-site update for an infinite MPS with a 2-site unit cell. Each physical index of the MPS has dimension 4, and describes the upper and lower spin-1/2 of each rung of the ladder. Moreover, we implemented \(SU(2)\) symmetry using the scheme that we described in Ref. [9], based on the formalism of fusion trees. Therefore, the physical index of the MPS carries the quantum numbers \(\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1\). We also tried different types of initial conditions for the variational optimization, such as product states or random states.

### IV. RESULTS

#### A. Energy convergence

Let us start by showing the results for the energy convergence of our \(SU(2)\) iDMRG code for the considered chiral Hamiltonian on a ladder. The convergence of the energy with the effective MPS bond dimension \(\chi\) is shown in Fig. 7, where we use up to \(\chi \approx 1200\). In order to give an overview of simulation parameters and the corresponding irreps on the virtual bonds, we listed some examples in Table II. As one can see from the plot, a large bond dimension of the order of one thousand is required in order to start seeing convergence, so the incorporation of \(SU(2)\) symmetry in the numerical algorithm is indeed crucial to extract the correct properties of the system. In Fig. 8 we show a similar plot, namely, the convergence of the ground state energy with the discarded weight in the iDMRG approximation. As is well known, this allows for a better extrapolation to the \(\chi \to \infty\) limit [26]. In our case, we obtain an estimate of

\[
E_0(\chi \to \infty) \approx -0.578978(2).
\]

In the insets of Fig. 7 and Fig. 8 we plot the convergence of the error \(\Delta E_0 = E_0 - E_0(\chi \to \infty)\) between the energy and its extrapolated value, as a function of the inverse effective MPS bond dimension \(1/\chi\) and the discarded weight respectively [35]. As shown in the plots, our results agree very well with an algebraic convergence in \(\chi\), which is typical of a critical system.

#### B. Entanglement

We have also studied several entanglement figures of merit in our system. First, in Fig. 9 we show the scaling
of the entanglement entropy $S(L)$ of a block of length $L$ with two open ends, for different values of the bond dimension $\chi$. The computational cost of this calculation is $O(\chi^5)$, as opposed to the $O(\chi^3)$ cost infinite-DMRG. Thus we are restricted to only moderate values of the bond dimension for the calculation of the entanglement entropy. Before reaching saturation due to finite-$\chi$ for large block sizes, this is compatible with a CFT scaling $S(L) \sim c/3 \log L$ \cite{b}. In our case, this implies that $c \approx 1$, as seen in the plot, where we also show our best fit including $O(1/L)$ subleading corrections. Furthermore, we also analyzed the scaling of the entanglement entropy of half an infinite chain with the MPS correlation length $\xi$ for up $\chi = 1184$.

In order to assess the consistency of our calculations, we computed the finite-entanglement scaling of the MPS correlation length $\xi$ with the bond dimension $\chi$ \cite{b}. In Fig. 11 we see that this follows a perfect algebraic fit $\xi \sim \chi^\kappa$, with exponent $\kappa \approx 1.16$, which following the approximate formula $\kappa \approx 6/(c \left(\sqrt{12/c} + 1\right))$ \cite{b}, is again compatible with $c \approx 1$. 

$S(L) = 1.723 + 0.981/3 \cdot \log(L) - 0.033/L$

$S(\xi) = 0.876 + 0.985/6 \cdot \log(\xi) - 0.006/\xi$

FIG. 7: Convergence of the ground state energy $E_0$ with the MPS bond dimension $\chi$. Here we use the discarded weight (see also Fig. 8) as an estimate of the relative error for the ground state energy in each simulation. The inset shows the convergence of the error $\Delta E_0 = E_0 - E_0(\chi \to \infty)$ between the energy and its extrapolated value.

FIG. 8: Convergence of the ground state energy $E_0$ with the iDMRG discarded weight. The inset shows the convergence of the error $\Delta E_0 = E_0 - E_0(\chi \to \infty)$ between the energy and its extrapolated value.

FIG. 9: Scaling of the entanglement entropy $S(L)$ for a block of size $L$ and different bond dimensions $\chi$. Computing the entanglement entropy of a block is restricted to only moderate bond dimensions due to a higher computational cost compared to the semi-infinite chain.

FIG. 10: Scaling of the entanglement entropy $S(\xi)$ for half an infinite chain with the MPS correlation length $\xi$ for up $\chi = 1184$. 

$S(L) = 1.723 + 0.981/3 \cdot \log(L) - 0.033/L$

$S(\xi) = 0.876 + 0.985/6 \cdot \log(\xi) - 0.006/\xi$
C. Correlation functions

In order to assess the criticality of the system, we computed a number of $SU(2)$-invariant correlation functions $C(i, j)$ in the system, with $i, j$ sites on the ladder. We observed algebraic decays and critical exponents, following $C(i, j) \sim r^{-\alpha}$ with $r$ the separation distance between $i$ and $j$ (notice that the sites can belong to different ladder legs). We computed all correlation functions for an MPS with bond dimension $\chi = 1184$ ($\chi_{\text{sym}} = 350$). Details are as follows.

1. Spin-spin correlator

First, we computed the spin-spin correlation functions $\langle S^u_i S^d_j \rangle$ and $\langle S^u_i S^u_j \rangle$, where $i, j$ are the rung indices and $u, d$ the leg indices, respectively. As shown in Fig. 12 and Fig. 13, we observe a similar algebraic decay in both cases with exponent $\alpha \approx 1$. This indeed matches our expectation from the continuum limit calculation in Eq. (28).

2. Dimer-dimer correlator

Next, we studied the dimer-dimer correlation function between vertical dimers $\langle (S^u_{i1} S^d_{i2})(S^u_{j1} S^d_{j2}) \rangle$. The four-body correlation is corrected by all possible disconnected parts, namely $\langle S^u_i S^d_i S^d_j S^u_j \rangle$, $\langle S^u_i S^u_j S^d_i S^d_j \rangle$ and $\langle S^u_i S^d_i S^u_j S^d_j \rangle$ with appropriate factors. The result is shown in Fig. 14, where the decay fits very well an algebraic decay, as expected for criticality, with decay exponent $\alpha \approx 5/4$. 

FIG. 11: Scaling of the MPS correlation length $\xi$ with the bond dimension $\chi$.

FIG. 12: Spin-spin correlation function between spins in the same leg.

FIG. 13: Spin-spin correlation function between spins in different legs.

FIG. 14: Dimer-dimer correlation function between vertical dimers.
the chiral ladder configurations that we did not consider here, and is currently work in progress [17]. Investigating similar chiral Hamiltonians in Kagome stripes would also be within reach and could lead to interesting physical insights. Finally, we expect that the simulations in this paper will help us to understand the procedure to simulate chiral quantum spin models in two spatial dimensions with SU(2)-invariant tensor networks such as Projected Entangled Pair States (PEPS) [30–32].

**V. CONCLUSIONS**

In this paper we have studied a chiral 2-leg ladder with SU(2) symmetry using an SU(2)-invariant iDMRG algorithm. The study allows us to benchmark our own implementation of SU(2) symmetry in tensor networks [9], as well as determine accurately the properties of the ladder system. After getting some intuition about the model by Kadanoff coarse-grainings, exact diagonalization, and bosonization, we find numerically that the ground state of the system agrees with a CFT with central charge $c \approx 1$, which is also compatible with previous studies of the continuum limit. In particular, we analyzed the scaling of the entanglement entropy of a block and of half an infinite system, as well as finite-entanglement scaling, ground-state energy convergence, entanglement spectrum, and different correlation functions showing algebraic decay at long separation distances. Our results for the entanglement spectrum are compatible with an SU(2)$_1$ WZW theory in the low-energy limit. Moreover, we explained in full detail how to obtain SU(2)-invariant MPOs for 3-spin interactions, something that so far had not been discussed in detail in the literature. Our procedure for constructing such MPOs can be generalized to arbitrary SU(2)-invariant interactions on 1d and quasi 1d systems.

Our work motivates further investigations along a number of directions. For instance, it would be interesting to dig deeper into the continuum limit of the model. The case of multi-leg and higher-spin ladders could also be analyzed with techniques similar to the ones in this paper. This would be particularly interesting in order to understand how two-dimensional physics emerges, and how the gapped/gapless nature of the *chiral* system depends on both the spin and the number of legs. An investigation of configuration $H_3$ with $U(1)$-invariant and/or SU(2)-covariant MPS methods would also be relevant in order to understand the overall physics of the chiral ladder configurations that we did not consider here, and is currently work in progress [17].
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Appendix A: Construction of SU(2)-invariant MPOs

Here we explain how to construct MPOs for different types of SU(2)-invariant interactions based on symmetry considerations only. We start with simple MPOs, such as the one for the Heisenberg quantum spin chain, and then move on to more complex interactions such as the three-spin chiral interactions that we consider in this paper.

The ultimate goal is to write the desired Hamiltonian in an SU(2)-invariant form, which implies a decomposition in terms of degeneracy tensors and structural tensors (Clebsch-Gordan coefficients) as described by the Wigner-Eckart theorem. In order to do this, we first consider the Clebsch-Gordan coefficients which are eligible for the MPO and determine afterwards which degeneracy factors are necessary in order to construct the correct Hamiltonian, with the constraint that the Hamiltonian must be an SU(2) scalar.

As a building block, we consider the generic MPO tensor in Fig. 16, which shows an internal structure due to the presence of four indices in the MPO tensor. The left/right indices are the MPO bond indices, and the up/down indices are the physical indices. Each trivalent tensor corresponds to an intertwiner of SU(2), i.e., a Clebsch-Gordan coefficient. Arrows show the direction of the legs (incoming/outgoing irreps, see Ref. [9]).

To show how the construction process works for the MPO, we consider two different Hamiltonians: a 2-site Heisenberg interaction $S \cdot S$, and a 3-spin chiral interaction $S \cdot (S \times S)$.

1. Heisenberg 2-spin interaction

Since the term $S \cdot S$ produces a scalar, we are interested in MPOs that transform as a scalar as well, which means that the bond indices at the left and right ends of the MPO need to have spin 0. For two physical spins 1/2, it is easy to check that this implies that the connecting bond index between two MPO sites can only have either spin 0 or 1. While the spin 0 channel is trivial and corresponds to the application of the identity operator on the physical spins, the spin 1 channel necessarily generates the dot product. This is because the only two scalar operators for two spins are $I \otimes I$ and $S \cdot S = S^z \otimes S^z + S^x \otimes S^x + S^y \otimes S^y$. For the left and right MPO tensors, we can now write down all the coefficients once all the irreducible representations are fixed. Focusing on the spin-1 channel for the connecting index the results are shown in Fig. 17 and Fig. 18. It is then straightforward to check that the contraction of the left and right MPO tensors from Figs. 17 and 18 gives back the desired dot product $S \cdot S$ of the 2-site Heisenberg Hamiltonian, with a prefactor of $-4/3$ as shown in Fig. 19.

Next, we can easily build the SU(2)-invariant MPO for the Heisenberg quantum spin chain with $H = \sum_i S_i \cdot S_{i+1}$ by adding one more spin zero channel to the MPO bond index. More specifically, the irrep with spin zero will have degeneracy two, i.e., we will have $0_1$ and $0_2$ (the subscript refers to the degeneracy). Combined with the non-degenerate spin channel $1_1$, the MPO can be written according to Fig. 20, where we show the details of the degeneracy tensors accompanying the structural (Clebsch-Gordan) part. As in the non-symmetric part, the irrep $0_1$ “propagates” through the bond indices until an interaction is hit (which is mediated by irrep $1_1$), and onwards propagates the irrep $0_2$. In describing this
\[ = -\left(\frac{2}{3}\right)^2 S^+_{i,s+1} - \left(\frac{2}{3}\right)^2 S^-_{i,s+1} = \frac{2}{3} S^+_{i,s+1} - \frac{2}{3} S^-_{i,s+1} = -\frac{4}{3} S^+_{i,s+1} \]

FIG. 19: The contraction of the MPO tensors in Figs. 17 and 18 produces the desired 2-site Heisenberg interaction with a \(-4/3\) prefactor. The sum is over the values of \(m\) for the spin-1 channel of the bond index.

\[
\text{full MPO} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\gamma S & 0 & 0 & 0 \\
\end{pmatrix}
\]

chargeSector \(j_{\text{int}} j_1 j_2 j_3 j_4\) dimensionality degeneracyTensor MPO

\(\frac{1}{2} \quad \frac{1}{2} \quad 0 \quad 0 \quad \frac{1}{2} \) \([1, 2, 2, 1]\) \[0_1 \quad 0_2 \]

\(\frac{1}{2} \quad \frac{1}{2} \quad 0 \quad 1 \quad \frac{1}{2} \) \([1, 2, 1, 1]\) \[0_1 \quad \gamma \]

\(\frac{1}{2} \quad \frac{1}{2} \quad 1 \quad 0 \quad \frac{1}{2} \) \([1, 1, 2, 1]\) \[0_1 \quad 0_2 \]

MPO tensor we have used the notation

\[ \vec{S} = \left( -\sqrt{\frac{2}{3}} S^+, \frac{2}{\sqrt{3}} S^z, \sqrt{\frac{2}{3}} S^- \right), \quad \vec{S} = \left( \frac{\sqrt{3}}{3} S^+, \frac{2}{\sqrt{3}} S^z, -\frac{\sqrt{3}}{3} S^- \right) \]

for a system of spin-1/2. Moreover, we also defined the factor \(\gamma \equiv i\sqrt{3/4}\) that compensates the unavoidable factor \(-4/3\), which appears due to the contraction of the Clebsch-Gordan tensors as shown in Fig. 19.

2. Chiral 3-spin interaction

As a second example of a scalar operator we consider the triple product \(S \cdot (S \times S)\). In this case, the MPO spans over three sites. The interaction on the second site can only be mediated by having a spin one representation on both virtual legs of the central MPO tensor. If this tensor had a spin zero representation on either leg, then it would not be possible to generate a three-body interaction. Moreover, a spin two representation is excluded since the 3-site MPO can then no longer be terminated by a spin zero after the third site, which we demand in order for it to be a scalar. Additionally, the central MPO tensor will have different internal spins 1/2 and/or 3/2, since

\[ \frac{1}{2} \otimes 1 = \frac{1}{2} \oplus \frac{3}{2}, \quad (A2) \]

where spin 1 will come from some bond index, and spin 1/2 from some physical index.

In order to start and terminate the interaction in the MPO, we can reuse the terms of the Heisenberg interaction from Fig. 17 and Fig. 18. The additional terms of the MPO for the triple product can be again constructed by evaluating the Clebsch-Gordan coefficients for fixed spin representations of the virtual legs for the central MPO tensor. These are given explicitly in Fig. 21, for the two possible values 1/2 and 3/2 of the internal leg.

In order to construct the MPO we can now take linear combinations of matrices \(A\) and \(B\) in Fig. 21 together with the left and right tensors from Fig. 17 and Fig. 18. This is shown in 22. We also use the notation

\[ \vec{M} = \alpha \vec{A} + \beta \vec{B}. \]  

(A3)

Here the arrowheads indicate the center site of the MPO, in analogy to Fig. 20, where arrows were used to signal the start and end of the interaction.

The free parameters \(\alpha\) and \(\beta\) can now be chosen in order to reproduce the desired interaction. As a first option one could choose the superposition \(-A + B\). For this choice the MPO tensor at the central site simplifies...
such that the overall three-body MPO reproduces a next-to-nearest-neighbour Heisenberg interaction $S_1 \cdot S_3$ instead of the scalar triple product we aim for. As a matter of fact, this is also a valid scalar for three spins, where the second spin simply does not interact. By playing with this choice of $\alpha$ and $\beta$ it is also possible to construct MPOs for long-range interactions. In our case, though, we find that in order to generate the chiral triple product it is necessary to choose $\alpha = -i$ and $\beta = -i/2$, in which case the MPO tensor at the central site becomes

$$-A + B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$ (A4)

where

$$-i \left( A + \frac{1}{2} B \right) = i \begin{pmatrix} -S_1^z & -\sqrt{2} S_2^- & 0 \\ -\sqrt{2} S_2^+ & 0 & -\sqrt{2} S_3^- \\ 0 & -\sqrt{2} S_3^+ & S_2^z \end{pmatrix}.$$ (A5)

Evaluating the sum over the three MPO tensors in Fig. 22 yields

$$H = -\frac{4i}{6} \left( S_1^z S_2^+ S_3^- - S_2^+ S_3^- S_1^z + S_3^- S_1^z S_2^+ - S_1^z S_2^+ S_3^- \right) - S_1^z S_2^+ S_3^- + S_2^+ S_3^- S_1^z + S_3^- S_1^z S_2^+ - S_1^z S_2^+ S_3^-)$$

$$= -\frac{4}{3} S_1 \cdot (S_2 \times S_3).$$ (A6)

Notice that here again the factor $-4/3$ appears due to the relation between spin-1/2 operators and Clebsch-Gordan coefficients. From these tensors it is then easy to construct an MPO for a Hamiltonian made of a sum of chiral 3-spin interactions.

3. 2-spin and 3-spin interactions together

We can now construct an MPO for a Hamiltonian such as

$$H = J \sum_i S_i S_{i+1} + K \sum_i S_i \cdot (S_{i+1} \times S_{i+2}),$$ (A7)

i.e., a sum of 2-spin Heisenberg interactions and chiral 3-spin interactions. This Hamiltonian is an $SU(2)$ scalar, because it is constructed as a sum of scalar operators. $J$
and $K$ are parameters giving more weight to one term or the other.

As in the case of the plain Heisenberg model, we need two spin zero sectors in the bond dimensions of the MPO that take care of applying the identity to all sites to the left and to the right of the interacting sites. Moreover, here we also need two spin one sectors in the MPO bond dimension: one mediating the 2-spin interaction, and the other mediating the 3-spin interaction. The resulting MPO tensor is given in Fig. 23, where we specify the structural part, corresponding to the Clebsch-Gordan coefficients, as well as the degeneracy part. The structure of the MPO can also be represented by a Finite State Machine, as shown in Fig. 24.

4. Chiral 3-spin interactions on the ladder

For the purposes of this paper, we simulated a Hamiltonian with chiral 3-spin interactions of the triangles of the ladder in Fig. 1, in which we considered alternating orientations for the triangles as explained above. In order to construct an MPO, we considered the snake pattern from Fig. 25, and applied the techniques discussed previously to construct an MPO for the sum of the different 3-spin interactions. Then, as shown in the figure, we coarse-grained the two spins for each rung of the ladder into a single physical site (with irreps $I \oplus I$). In this way, the resulting MPO has a 2-site unit cell and is the one used in the iDMRG algorithm with a 2-site update.

Appendix B: Finite-entanglement scaling of the entanglement spectrum

In this appendix we show our results for the scaling with the bond dimension $\chi$ of the entanglement spectrum for the different spin sectors $S$. Our results are shown in Figs. 26 - 29 for integer spins $S = 0$ up to $S = 3$. Each dot in the plots is a $(2S + 1)$-plet. As we can see, the lowest part of the entanglement spectrum converges quickly with the bond dimension. The distribution of the lowest-lying entanglement energies tends to have an equidistant structure, typical of a CFT. The values for the largest possible bond dimension, which we take as essentially converged for the lowest-lying part of the spectrum, correspond to the ones shown in Fig. 15. We also notice that the convergence of the individual entanglement energies with the symmetric bond dimension seems to be algebraic as opposed to exponential. In practice, this means that from our plots one can extract the behavior

$$\varepsilon_\alpha \approx \frac{1}{\chi^{\mu_i}} \tag{B1}$$

for the $\alpha$th entanglement energy $\varepsilon_\alpha$, with $\mu$ an exponent controlling the behavior at large $\chi$. According to our data, the exponent $\mu$ may depend on the index $\alpha$ itself, i.e., be different for each one of the entanglement energies. Even if purely empirical, this behavior seems to hold well for all the studied values of the spin $S$. According to the results presented in this paper, we take this also as a strong indication that the system is critical and has an infinite correlation length.

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FIG. 27: Scaling of the entanglement spectrum for $S = 1$ with the MPS symmetric bond dimension $\chi$, in normalized units.

FIG. 28: Scaling of the entanglement spectrum for $S = 2$ with the MPS symmetric bond dimension $\chi$, in normalized units.

FIG. 29: Scaling of the entanglement spectrum for $S = 3$ with the MPS symmetric bond dimension $\chi$, in normalized units.

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[33] Here we use the word “chiral” in the sense that the Hamiltonian is not even under a time-reversal operation. As we will see later, the continuum limit is a field theory which is odd under time-reversal.
[34] As far as the authors know, this theorem only exists for two-dimensional CFTs, and its generalization to higher dimensions has proven remarkably hard.
[35] Recently it has been suggested that the convergence of the ground state energy with respect to the discarded weight should be fitted by a linear function. This however is in good agreement with our power-law fit that shows an exponent of 0.973.