Polaron residue and spatial structure in a Fermi gas

Christian Trefzger and Yvan Castin

Laboratoire Kastler Brossel, École Normale Supérieure and CNRS, UPMC
24 rue Lhomond, 75231 Paris, France, EU

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Abstract – We consider a mobile impurity of mass \( M \) interacting via a \( s \)-wave broad or narrow Feshbach resonance with a Fermi sea of particles of mass \( m \). Truncating the Hilbert space to at most one pair of particle-hole excitations of the Fermi sea, we study ground-state properties of the polaronic branch other than energy, namely the quasiparticle residue \( Z \), and the impurity-to-fermion pair correlation function \( G(x) \). We show that \( G(x) \) – 1 vanishes at large distances as \( -\frac{(A_4 + B_4 \cos 2k_F x)/(4k_F x)^4}{\text{with} \ k_F \ \text{the Fermi wave vector; since} \ A_4 > 0 \ \text{and} \ B_4 > 0, \ \text{the polaron has a diverging rms radius and shows Friedel-like oscillations. For weak attractions, we obtain analytical results. They detect the failure of Hilbert space truncation for diverging} \ M/m, \ \text{as expected from the Anderson orthogonality catastrophe}; \ \text{at distances from} \sim 1/k_F \ \text{to the asymptotic distance where the} 1/x^4 \ \text{law applies, they reveal an intriguing multiscale structure of} \ G(x). \)

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Introduction. – The physics of atomic Fermi gases is rapidly developing, thanks to the Feshbach resonance tuning of the \( s \)-wave scattering length \( \alpha \) of the interaction, leading to highly degenerate strongly interacting Fermi gases [1,2]. The first realizations of spin-polarized gases [3,4] asked for theoretical interpretation. It was proposed that, in the strongly polarized case, the minority atoms dressed by the Fermi sea of the majority atoms form a normal gas of quasiparticles called polarons [5,6], which agrees with the experimental phase diagram [3,7].

Strictly speaking, such polarons should be called “Fermi polarons”, to distinguish from the condensed-matter-physics ones, i.e. electrons coupled to a bosonic bath of phonons [8].

Basic single-polaron properties, such as its binding energy to the Fermi sea and its effective mass, are well understood, both for broad [5,6,9,10] and narrow [11–13] Feshbach resonances. Here we study the polaron ground-state properties beyond the energy search, for both types of resonances. We study the quasiparticle residue \( Z \), already investigated experimentally in [14,15] and theoretically in [12,16], and the intriguing issue of the density distribution that surrounds the impurity, as characterized by the pair correlation function: This gives access to the spatial extension of the polaron, of potential relevance for the collective properties of the polaronic gas.

Model. – We consider in three dimensions an ideal gas of \( N \) same-spin-state fermions of mass \( m \), enclosed in a cubic box \( V \) with periodic boundary conditions. The gas is perturbed by an impurity, a distinguishable particle, of mass \( M \). The impurity interacts with each fermion resonantly on a \( s \)-wave broad or narrow Feshbach resonance, in a two-channel model [17–19], with a Hamiltonian \( \hat{H} \) written in [11]: The particles exist either in the form of fermions or impurity in the open channel, or in the form of a tightly bound fermion-to-impurity molecule in the closed channel. These two forms are coherently interconverted by the interchannel coupling of amplitude \( \Lambda \). We restrict to the zero-range limit, with interactions characterized by the \( s \)-wave scattering length \( \alpha \) and the non-negative Feshbach length \( R_* \) [20]. In terms of the effective coupling constant \( g \) and of the interchannel coupling \( \Lambda \), one has

\[
g = \frac{2 \pi \hbar^2 \alpha}{\mu} \quad \text{and} \quad R_* = \frac{\pi \hbar^4}{\Lambda^2 \mu^2},
\]

where \( \mu = mM/(m + M) \) is the reduced mass. A broad Feshbach resonance corresponds to \( R_* = 0 \).

Polaronic ansatz. – Whereas the ground state of the system presents a polaronic branch and a dimeronic branch [9,11–13,16,21,22], we restrict to the polaronic branch. The ground state of the \( N \) fermions is the usual Fermi sea (“FS”), of energy \( E_{\text{FS}}(N) \). We determine the ground state of a single impurity interacting with the \( N \) fermions using the unexpectedly accurate approximation proposed in [6] (and generalized to the narrow resonance case in [11,13]), that truncates the Hilbert space to at most
one pair of particle-hole excitations of the Fermi sea. For a zero total momentum, this corresponds to the ansatz

\[ |\psi_{\text{pol}}\rangle = \left( \phi \hat{d}_q^\dagger \phi_{\text{q}} + \sum \phi_{\text{q}} \hat{d}_q^\dagger \phi_{\text{q}} + \sum \phi_{\text{kq}} \hat{d}_q^\dagger \phi_{\text{kq}} \right) |\text{FS}\rangle, \tag{2} \]

where \( \hat{d}_q^\dagger \) and \( \hat{d}_q \) are the creation operators of an impurity, a fermion and a closed-channel molecule of wave vector \( \mathbf{k} \). Primed sums are restricted to \( \mathbf{q} \) belonging to the Fermi sea, and to \( \mathbf{k} \) not belonging to that Fermi sea. The successive terms in (2) correspond to that order to the ones generated by repeated action of the Hamiltonian \( \hat{H} \) on the \( \Lambda = 0 \) polaronic ground state. One then has to minimize the expectation value of \( \hat{H} \) within the ansatz (2), with respect to the variational parameters \( \phi, \phi_{\text{q}} \) and \( \phi_{\text{kq}} \), with the constraint \( \langle \psi_{\text{pol}}|\psi_{\text{pol}}\rangle = 1 \). Expressing \( \phi_{\text{kq}} \) in terms of \( \phi_{\text{q}} \) and \( \phi_{\text{q}} \) in terms of \( \phi \), as in [11], one is left with a scalar implicit equation for the polaron energy counted with respect to \( E_{\text{FS}}(N) \); in the thermodynamic limit:

\[ \Delta E_{\text{pol}} \equiv E_{\text{pol}} - E_{\text{FS}}(N) = \int d^3q \frac{1}{(2\pi)^3} \frac{1}{D_q}, \tag{3} \]

where the prime on the integral over \( \mathbf{q} \) means \( 0 < q < q_F \), with \( k_F = (6\pi^2\rho)^{1/3} \) the Fermi wave vector and \( \rho = N/V \) the mean Fermi sea density. The function of the energy in the denominator of the integrand is

\[ D_q = \frac{1}{g} - \frac{\mu k_F}{\pi^2 \hbar^2} + \frac{\mu^2 R_c}{\pi^2 \hbar^2} \left( \Delta E_{\text{pol}} + \frac{\mu}{m} \epsilon_{\text{q}} \right) + \int d^3k' \left( 1 \frac{1}{(2\pi)^3} \left( E_{\text{q-k'}} - \epsilon_{\text{k'}} - \epsilon_{\text{q}} - \Delta E_{\text{pol}} \right) - \frac{2\mu}{\hbar^2 k'^2} \right), \tag{4} \]

where \( \epsilon_{\text{k}} = \frac{k^2}{2m} \) for the fermions, \( E_{\text{k}} = \frac{k^2}{2\mu} \) for the impurity, and the prime on the integral over \( \mathbf{k}' \) means that it is restricted to \( k' > k_F \).

**Quasiparticle residue.** – The polaron is a well-defined quasiparticle if it has a non-zero residue \( Z \), which is defined from the long imaginary-time decay of the Green’s function [9]. Within the polaronic ansatz (2), it was shown in [16] that simply \( Z = |\phi|^2 \). Writing the amplitude \( \phi_{\text{q}} \) in terms of the denominator (4) as in [11], \( \phi_{\text{q}} = \mathcal{A}/D_q \) where \( \mathcal{A} \) is a normalization factor, and using the coupled equations for \( \phi, \phi_{\text{q}} \) and \( \phi_{\text{kq}} \) [11], we get

\[ Z \equiv |\phi|^2 = \left[ 1 + \frac{1}{\mathcal{A}} \int d^3q \frac{1}{(2\pi)^3} \frac{1}{D_q} \right. \]

\[ \left. + \int d^3x d^3q \frac{1}{(2\pi)^3} \left( \frac{1}{D_q} \left( E_{\text{q-k-x}} - \epsilon_{\text{k-x}} - \epsilon_{\text{q}} - \Delta E_{\text{pol}} \right) \right)^2 \right]^{-1}. \tag{5} \]

This reproduces the diagrammatic ladder approximation for \( Z \) of [25] for \( R_* = 0 \) and of [12] for \( R_* > 0 \). In fig. 1 we plot \( Z \) as a function of \( 1/k_Fa \) for various mass ratios \( M/m \) and reduced Feshbach lengths \( k_FR_* \). We find that \( Z \) tends to 1 when \( a \to 0^- \), as expected, and to 0 when \( a \to 0^+ \). The accuracy of the polaronic ansatz \( a \text{ priori} \) becomes questionable when \( Z \to 0 \). In fig. 1, we thus plot \( Z \) as a dashed line for a predicted value below 1/2. In the weakly attractive limit, we shall give a systematic expansion of \( Z \) up to second order in \( k_Fa \).

**Spatial structure.** – The pair correlation function \( G(x_\mu-x_d) \) is proportional to the probability density of finding a fermion at position \( x_\mu \), knowing that the impurity is located at \( x_d \). It can be extracted from a measurement of the positions of all particles, and a further average over many realizations\(^2\). In terms of the fermionic and impurity field operators \( \hat{\psi}_{\mu}(x_\mu) \) and \( \hat{\psi}_d(x_d) \), and the impurity density \( \rho_d = \langle \hat{\psi}_d^\dagger(x_d) \hat{\psi}_d(x_d) \rangle \), one has

\[ G(x_\mu-x_d) = \left( \frac{\hat{\psi}_d^\dagger(x_d) \hat{\psi}_d^\dagger(x_d) \hat{\psi}_d(x_d) \hat{\psi}_d(x_d) \hat{\psi}_d(x_u)}{\rho_d} \right). \tag{6} \]

Due to the interchannel coupling, the impurity has a non-zero probability \( \pi_{\text{closed}} \) (studied in [11]) to be bound within a closed-channel molecule, where it cannot contribute to \( G(x) \) and \( \rho_d \). In terms of the probability \( \pi_{\text{open}} = 1 - \pi_{\text{closed}} \)

\(^1\)For weak interactions, this truncation will be validated by comparison to perturbative expansion. For strong interactions, one could include a second pair of particle-hole excitation, as in [10]. We rather compare to diagrammatic Monte Carlo [9]: For \( R_* = 0 \), \( M = m, Z \) obtained from (2) well agrees with exact \( Z \) even at unitarity [29].

\(^2\)Single-shot spatial cold-atom distributions integrated over some direction \( z \) can now be accurately measured by absorption imaging, giving access to spatial noise and its correlations [26]. The unwanted integration over \( z \) can be undone by an inverse Abel transform [27].
for the impurity to be in the open channel, one finds \( \rho_{\text{imp}} = \rho_{\text{open}} / V \). In the thermodynamic limit one has

\[
\frac{\rho_{\text{open}}}{Z} = 1 + \int \frac{d^3k}{(2\pi)^3} \frac{1/D_q}{E_q - k + \varepsilon_k - \varepsilon_q - \Delta E_{\text{pol}}} ^2 , \tag{7}
\]

\[
G(x) = 1 + \frac{Z}{\rho \rho_{\text{open}}} \left[ -2f(x) + \int \frac{d^3q}{(2\pi)^3} |f_q(x)|^2 \right.
- \int \frac{d^3k}{(2\pi)^3} \tilde{f}_k(x) \right] ^2 , \tag{8}
\]

where we have introduced the functions

\[
f_q(x) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{ikx} / D_q}{E_q - k + \varepsilon_k - \varepsilon_q - \Delta E_{\text{pol}}} , \tag{9}
\]

\[
\tilde{f}_k(x) = \int \frac{d^3q}{(2\pi)^3} \frac{e^{ikx} / D_q}{E_q - k + \varepsilon_k - \varepsilon_q - \Delta E_{\text{pol}}} , \tag{10}
\]

\[
f(x) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{ikx} / D_q}{E_q - k + \varepsilon_k - \varepsilon_q - \Delta E_{\text{pol}}} . \tag{11}
\]

The contribution involving \( f(x) \) is an interference effect between the subspaces with zero and one pair of particle-hole excitations in the Fermi sea. In fig. 2 we plot the numerically obtained deviation \( G(x) - 1 \) of the pair correlation function from unity (which is both its large distance limit and its non-interacting limit)\(^3\). The presence of the impurity induces oscillations in the fermionic density that are still significant at distances of several \( 1/k_F \).

**Properties of the pair correlation function.** – A first property in the thermodynamic limit is the sum rule:

\[
\int d^3x [G(x) - 1] = 0 , \tag{12}
\]

which follows directly from the integral representation of the Dirac delta distribution \( \int d^3x \exp(ik \cdot x) = (2\pi)^3 \delta(k) \).

A second property is that, in the limit where \( x \to +\infty \),

\[
G(x) - 1 \sim - \frac{A_4 + B_4 \cos(2k_F x)}{(k_F x)^4} . \tag{13}
\]

The prefactor \( 1/(k_F x)^4 \) is thus a periodic function of \( x \) of period \( \pi/k_F \), with a mean value \( A_4 \) and a cosine contribution (of amplitude \( B_4 \)) reminiscent of the Friedel oscillations. The fact that the mean value \( A_4 \) differs from zero has an important physical consequence: It shows that the polaron is a spatially extended object, since even the first moment \( \langle x \rangle \) of \( G(x) - 1 \) diverges (logarithmically) in the thermodynamic limit.

Equation (13) results from an asymptotic expansion of (9), (10), (11) in powers of \( 1/x \), obtained by repeated integration by parts as in [29], always integrating the exponential function \( e^{ikx} \) or \( e^{\pm ikx} \) to pull out a \( 1/x \) factor:

\[
f_q(x) \sim \frac{\mu/(2\pi^2 \hbar^2)}{k_F^3 D_q} \sum_{u = \pm 1} F(k_F, u; q, u') e^{ik_F x u} , \tag{14}
\]

\[
\tilde{f}_k(x) \sim - \frac{\mu k_F/(2\pi^2 \hbar^2)}{x^2 k_F^2 D_k e_z} \sum_{u = \pm 1} F(k, u; k_F, u') e^{ik_F x u'} , \tag{15}
\]

\[
f(x) \sim \frac{-\mu/(8\pi^2 \hbar^2)}{x^4 D_k e_z} \sum_{u, u' = \pm 1} F(k_F, u; k_F, u') e^{ik_F x (u - u')} , \tag{16}
\]

Here \( e_z \) is the unit vector along \( z \), \( u \) is the cosine of the angle between \( x \) and \( k \), \( u' \) is the cosine of the angle between \( x \) and \( q \), and the function \( F \) is defined as follows:

\[
F(k, u; q, u') = \left[ - \frac{4m^2}{(m + M)^2} (1 - u^2)(1 - u'^2) \frac{q^2}{k^2} + \left( 1 + \frac{m - M}{m + M} \frac{q^2}{k^2} - \frac{2 \mu q}{M k u' - \Delta E_{\text{pol}}} \frac{2 \mu}{k^3 \hbar^2} \right)^2 \right]^{-1/2} . \tag{17}
\]

Therefore, in eq. (8) the integrals containing \( |f_q(x)|^2 \) and \( |\tilde{f}_k(x)|^2 \) provide a \( 1/x^4 \) contribution with an oscillating prefactor, as \( f(x) \) does. In fig. 3 we plot \( A_4 \) and \( B_4 \) as functions of \( 1/k_F a \) for various values of \( M/m \) and \( k_F R_a \).

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\(^3\)The function \( x^2 G(x) \) has a finite limit in \( x = 0 \) since the impurity-to-fermion wave function diverges as the inverse relative distance. For \( R_a = 0 \) and \( m = M \), this limit is called the contact [28].

\(^4\)One also uses the fact that, uniformly in the integration domain: \( \forall n_1, n_2, n_3, n_4 \in \mathbb{N} \), there exists \( C_{n_1, n_2, n_3, n_4} \in \mathbb{R}^+ \) such that \( \left| \partial^{n_3} \partial^{n_3} \partial^{n_4} F(k, u; q, u') \right| \leq C_{n_1, n_2, n_3, n_4} q^{n_3 + n_4} \).
In the weakly attractive limit. — Following ref. [11], we define for \( a < 0 \)
\[
s \equiv \frac{\mu}{m} (-a R_e)^{1/2} k_F, \tag{18}
\]
and we take the limit \( a \to 0^- \) for fixed \( s < 1 \) (which implies \( R_e \to \infty \)). Then \( \Delta E_{\text{pol}} \) tends to \( 0^- \), so that the integral appearing in (4) is bounded \([11]\) and
\[
g D_q \to 1 - (sq/k_F)^2. \tag{19}
\]
After integration over \( q \) in eq. (3), we get as in \([11]\):
\[
\Delta E_{\text{pol}} \sim \frac{\hbar^2 k_F^2}{\mu} k_F a \frac{1}{4 \pi} s^2 \left[ \text{arctanh} s - 1 \right]. \tag{20}
\]
Also, the interchannel coupling amplitude \( \Lambda \) scales as \((-k_F a)^{1/2}\), see eq. (1), and a systematic expansion of observables in powers of \( k_F a \) may be performed, treating the interchannel coupling \( \Lambda \bar{u} \bar{d} + \text{h.c.} \) within perturbation theory. The terms neglected in the ansatz (2) have an amplitude \( O(\Lambda^3) \), that is a probability \( O(\Lambda^6) \). One can thus extract from eq. (5) the exact value of \( Z \) up to \( \Lambda^4 \):
\[
\frac{1}{Z} \to 1 + \frac{m^2}{\mu^2} \left[ c_1 \frac{k_F a}{\pi} + c_2 \left( \frac{k_F a}{\mu} \right)^2 + O(k_F a)^3 \right]. \tag{21}
\]
The first coefficient of the expansion (21) is given by
\[
c_1 = - \left( \frac{1}{1 - s^2} - \text{arctanh} s \right), \tag{22}
\]
it originates from the second term in eq. (5), a closed-channel contribution. This is why \( c_1 = 0 \) on a broad
Feshbach resonance where \( s = 0 \). The second coefficient is
\[
c_2 = \frac{1}{2} \frac{(1 + \alpha)^2}{1 - s^2} \left( \text{arctanh} s - \alpha \text{arctanh} \alpha \right) + \frac{1}{2} \left[ 1 + \frac{2m^2}{\mu^2} \left( \text{arctanh} s - 1 \right) \right] \left[ 1 + \frac{s^2}{(1 - s^2)^2} - \text{arctanh} s \right] + \frac{1}{2(1 - s^2)} + s^2(1 - \alpha)(1 + \alpha)^2 \text{arctanh} \alpha (1 - s^2)^2 (s^2 - \alpha^2) \frac{s^4 + (1 + \alpha)^2 s^2 - \alpha^2}{2s(1 - s^2)(s^2 - \alpha^2)} \left( \text{arctanh} s \right)^2, \tag{23}
\]
where the mass contrast \( \alpha = (M - m)/(M + m) \) also obeys 2 \( \text{arctanh} \alpha = \ln(M/m) \) (see footnote \(^5\)). The first term in eq. (23) originates from the last term in (5), an open-channel contribution. For a broad Feshbach resonance \( (s = 0) \), it is non-zero, whereas the sum of the other terms of (23), originating from the closed channel, vanishes, and
\[
c_2 = \frac{\ln(M/m)}{1 - s^2}. \tag{24}
\]
It is instructive to analyze the perturbative expansion in the exactly solvable limit of \( M/m \to +\infty \): The impurity can then be considered as a pointlike scatterer of fixed position, for convenience at the center of a spherical cavity of arbitrarily large radius, imposing contact conditions of scattering length \( a \) and effective range \( -2R_e \) on the fermionic wave function \([20]\). In the thermodynamic limit, one can then construct the Fermi sea of exactly calculable single-particle eigenstates in this scatterer-pluscavity problem. As shown in \([10]\) for \( R_e = 0 \), the truncated ansatz (2) provides a good estimate of \( \Delta E_{\text{pol}} \) for \( M/m \to \infty \). On the contrary, we find that it is qualitatively wrong for the quasiparticle residue: From eq. (5), it predicts a non-zero value of \( Z \), whereas the exact \( Z \) vanishes for \( M/m \to \infty \), which proves the disappearance of the polaronic character. For an infinite-mass impurity, indeed, \( Z \) is the modulus squared of the overlap between the ground state of the free Fermi gas and the ground state of the Fermi gas interacting with the scatterer. This overlap was studied in \([30]\), and vanishes in the thermodynamic limit, a phenomenon called the Anderson orthogonality catastrophe. Satisfactorily, the perturbative expansion (21) is able to detect this catastrophe:
\[
c_2 \sim \frac{\ln(M/m)}{(1 - s^2)^2}. \tag{24}
\]
Such a logarithmic divergence with the mass ratio was already encountered in the context of the sudden coupling of a Fermi sea to a finite-mass impurity, see the unnumbered equation below (4.3) in \([31]\). It originates from the first term of (23), that is from the last term of (5), where it is apparent that \( f^d d^k d'q/(k^2 - q^2)^2 \) diverges logarithmically at the Fermi surface.

Turning back to a finite-mass ratio \( M/m \), we now calculate the coefficients \( A_4 \) and \( B_4 \) in eq. (13) to leading order in \( k_F a \). The integrals appearing in eq. (8) contribute to these coefficients to second order in \( k_F a \), see eqs. (14), (15), since \( 1/D_q \) scales as \( g \) (see footnote \(^6\)). Also \( Z/\pi^{\text{open}} \)

\(^5\)Contrary to a first impression, \( c_2 \) has a finite limit when \( s \to a \).

\(^6\)The integrals over \( k \) of the modulus squared of (14), and over \( q \) for the modulus squared of (15), have contributions scaling as \((k_F a)^2 \ln(k_F a)\) that cancel in their difference.
deviates from unity only to second order in \(k_F a\), see eq. (7). The leading-order contribution to \(A_4\) and \(B_4\) is thus given by the interference term \(f(x)\), and from eq. (16) we obtain
\[
A_4 \to_{a \to 0} \frac{3\pi^2}{1 - \pi^2} \left( \frac{\arctanh s}{s} - 1 \right) \hspace{1cm},
\]
\[
B_4 \to_{a \to 0} \frac{3}{2\pi} \left( 1 + \frac{M}{m} \right) \frac{k_F a}{1 - s^2}.
\]
These analytical predictions are satisfactorily compared to the numerical evaluation of \(A_4\) and \(B_4\) in fig. 3.

Taking again \(M/m \to +\infty\), we find a severe divergence in the small-\(k_F a\) expansion: The coefficient of the term linear in \(k_F a\) in \(B_4\) diverges linearly with the mass ratio. Instead, the coefficient of the term linear in \(k_F a\) in \(A_4\) (not given) diverges only logarithmically with the mass ratio \(\sim -A_4(0)\ln(M/m)\). This suggests that, for an impurity of infinite mass, the asymptotic behaviour of \(G(x) - 1\) is no longer \(O(1/x^4)\) and that it decreases more slowly. To confirm this expectation, restricting for simplicity to \(R_a \neq 0\), we have calculated the exact mean fermionic density in presence of a fixed pointlike scatterer, obtaining an expression equivalent to the one of §2.2.2 of [32] and leading at large distances to
\[
G(x) - 1 \sim_{x \to +\infty} \frac{3}{2(k_F x)^3} \text{Re} \left( e^{2ik_F x} \frac{k_F a}{1 + i k_F a} \right). 
\]
As this \(1/x^3\) law has a zero-mean oscillating prefactor, \(G(x) - 1\) has a non-diverging integral over the whole space.

**Multiscale structure of \(G(x)\).** In the weakly attractive limit, it is apparent from eqs. (9), (10), (11) that the functions \(f_\delta, f_\epsilon, f\) are of order one in \(k_F a\), so that the leading order contribution to \(G(x) - 1\) in eq. (8) originates from the interference term \(\propto f(x)\) and is also of order one:
\[
G(x) - 1 \to_{a \to 0} O(k_F a).
\]
One then expects that, when \(k_F x \gg 1\), \(G(x) - 1\) drops according to the asymptotic \(1/x^4\) law with a coefficient of order one in \(k_F a\). This simple view is however, inferred by eq. (25), where \(A_4\) has a non-zero limit \(A_4(0)\) for \(a \to 0\). This suggests that the \(1/x^4\) law is only obtained at distances that diverge in the weakly attractive limit.

To confirm this expectation, one calculates \(f(x)\) to first order in \(k_F a\) for a fixed \(x\) (by using (19) and neglecting \(\Delta E_{\text{pol}}\) in the denominator of (11)), then one takes the limit \(k_F x \gg 1\). As shown in the appendix, this gives
\[
\lim_{a \to 0} \frac{G(x) - 1}{x} = \frac{A_4(0)}{k_F x} - \frac{M}{4m} \cos(2k_F x) + \frac{s^2}{2(1 - s^2)}
\]
\[
+ \frac{m}{2M} \left( 2 + \ln(2k_F x M/m) \right) + O(\frac{1}{(k_F x)^5})
\]
with the positive quantity \(\epsilon \equiv \Delta E_{\text{pol}}/E_F \ll 1\) and \(\gamma \approx 0.577215\) is Euler’s constant. For that order of taking limits, the oscillating bit still obeys a \(1/x^4\) law, with the same coefficient \(B_4\) as in (26); on the contrary, the non-oscillating bit obeys a different \(\ln x/x^4\) asymptotic law (dotted line in fig. 2(b)), which shows that the validity range of the \(1/x^4\) law is pushed to infinity when \(a \to 0\).

Remarkably, by keeping \(\Delta E_{\text{pol}}\) in the denominator of (11), one can obtain, see the appendix, an analytical expression for \(G(x) - 1\) that contains both the \(\ln x/x^4\) and the \(1/x^4\) laws as limiting cases, and that describes the crossover region with cosine- and sine-integral functions:
\[
G(x) - 1 = \frac{A_4(0)}{k_F x} \left( \frac{m}{M} \left[ \text{Ci}(k_F x/2) - \frac{1}{2} \text{Ci}(k_F x/2) \right] - \frac{1}{4} k_F^2 x^2 \left[ \left( \frac{\pi}{2} - \text{Si}(k_F x/2) \right) \sin(k_F x/2) \right] - \text{Ci}(k_F x/2) \right)
\]
\[
+ O(1),
\]
where the remainder \(O(1)\) is a uniformly bounded function of \(k_F x > 1\) and \(\epsilon \ll 1\). This formula satisfactorily reproduces the numerical results, see fig. 2(b), where \(\epsilon \approx 0.160\). It reveals that the pair correlation function has a multiscale structure for a weakly attractive interaction, with three spatial ranges: (i) the logarithmic range, \(1 < k_F x < \epsilon^{-1/2}\), (ii) the crossover range, \(\epsilon^{-1/2} < k_F x < \epsilon^{-1}\), and (iii) the asymptotic range, \(\epsilon^{-1} < k_F x\). The logarithmic range is immediately recovered from \(\text{Ci}(u) = \ln u + O(1)\). The \(\epsilon^{-1/2}\) scaling of its upper limit is intuitively recovered if one assumes that the relevant wave vectors in (11) obey \(|\kappa - q| \approx 1/x\): Neglecting \(\Delta E_{\text{pol}}\) with respect to \(E_{\text{pol}}\) in the denominator of (11) then indeed requires \(k_F x \approx (M\epsilon/m)^{-1/2}\).

**Conclusion.** The Fermi polaron, composed of an impurity particle dressed by the particle-hole excitations of a Fermi sea close to a broad or narrow Feshbach resonance with zero-range interaction, is a spatially extended object: The density perturbation induced by the impurity in the Fermi gas asymptotically decays as the inverse quadratic distance, with a spatially modulated component reminiscent of the Friedel oscillations. In the weakly attractive limit, \(k_F a \to 0\) with \(|a|R_a\) fixed, where systematic analytical results are obtained, this density perturbation reaches its asymptotic regime over distances diverging as \(1/(k_F^2 |u|)\) and exhibits at intermediate distances a rich multiscale structure. This may have important consequences on the interaction between polarons [33]: A polaron should indeed be sensitive to the deformation of the underlying fermionic density profile induced by another polaron, since the impurity forming the polaron is coupled to that fermionic density. The long-range nature of \(G(x) - 1\) suggests that the resulting interaction may also have a long range.

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\footnote{The first relative deviation of \((k_F x)^4 |G(x) - 1|\) from its \(x \to 0\) limit is \(-24/(k_F x)^2 + \frac{2\pi x}{k_F x} \sin(k_F x)\). For \(mc/M < 6\), this is <10\% for \(k_F x > 16\). From a similar first-deviation analysis, being in the logarithmic range actually requires \(k_F x^{1/2} < (\mu/M)^{1/2}\).}
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APPENDIX

The integral over k and q in \(f(x)\) reduces to a triple integral over k, q and the angle \(\theta\) between k and q. Taking \(\lambda = |k - q|\) rather than \(\theta\) as the variable, and changing the integration order, we get

\[
\varphi(\lambda) = \int_{\max(1 - \lambda, 0)}^{\lambda + \eta} \, dq \int_{\max(1 - \eta, 0)}^{\lambda + \eta} \, dk \frac{q k}{(1 - s^2) q^2 + k^2 - \eta^2} + \epsilon
\]

and \(1/k_p\) is the unit of length. \(\varphi(\lambda)\) is a \(C^2\)-function over \([0, 2]\) and \([2, +\infty]\), with \(\varphi(0) = 0, \varphi''(0) = 1/[\epsilon(1 - s^2)]\), but with a jump \(J = \varphi''(+\epsilon) - \varphi''(-\epsilon) = M/[4\epsilon(1 - s^2)]\).

Triple integration by parts over each interval gives

\[
f(x) = \frac{F(x)}{x^2} \int J \cos 2x \varphi''(0) + \int_{0}^{\infty} d\lambda \varphi^{(3)}(\lambda) \cos \lambda x .
\]

The contribution of \(J\) reproduces eq. (27). We find that \(\varphi(\lambda)\) varies at three scales, \(\epsilon, \epsilon^{1/2}\) and \(\epsilon^3\). For \(0 < \lambda < \epsilon^{3/4}\), we use the scaling \(\lambda = \epsilon t\) and expand \(\varphi^{(3)}(\lambda)\) in powers of \(t\) at fixed \(x\). For \(\epsilon^{3/4} < \lambda < \epsilon^{1/4}\), we use the scaling \(\lambda = \epsilon^{1/2} x\) and expand \(\varphi^{(3)}(\lambda)\) in powers of \(x\) at fixed \(t\). For \(\epsilon^{1/4} < \lambda < 1\), we directly expand in powers of \(x\) at fixed \(\lambda\).

Finally, to obtain (29), one omits \(\epsilon\) in the denominator of \(\varphi(\lambda)\), so that it is no longer \(C^2\) at the origin: \(\varphi''(\lambda) = \{\eta [\ln(\eta/\lambda) + 3/2] - s^2/(1 - s^2)]\}/[2(1 - s^2)] + O(\lambda)\). We thus locally split \(\varphi(\lambda)\) as the sum of a singular part \(\propto \ln \lambda\) and a \(C^\infty\)-function. The only trick is then to take, in the last triple integration by parts over \(\lambda\) and in the bit involving the singular part, \((1 - \cos \lambda x)/x\) as a primitive of \(\sin \lambda x\).

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\(^8\)We have checked that the leading correction to (19) gives, as expected, a contribution also \(O(1)\) to the expression in between curly brackets in (30).