ON THE TRANSVERSE INVARIANT AND BRAID DYNAMICS

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Abstract. Suppose \((B, \pi)\) is an open book supporting \((Y, \xi)\), where the binding \(B\) is possibly disconnected, and \(K\) is a braid about this open book. Then \(B \cup K\) is naturally a transverse link in \((Y, \xi)\). We prove that the transverse link invariant in knot Floer homology,

\[ \hat{t}(B \cup K) \in \widehat{HF}(Y, B \cup K), \]

defined in [BVVV13] is always nonzero. This generalizes the main results of Etnyre and Vela-Vick in [VV11, EVV10]. As an application, we show that if \(K\) is braided about an open book with connected binding, and has fractional Dehn twist coefficient greater than one, then \(\hat{t}(K) \neq 0\). This generalizes a result of Plamenevskaya [Pla15] for classical braids.

1. Introduction

A central problem in 3-dimensional contact geometry involves distinguishing and classifying transverse links. There is a classical invariant of transverse links called the self-linking number. Distinguishing transverse links that are smoothly isotopic and have the same self-linking number can be quite difficult. In this paper, we study a transverse link invariant in knot Floer homology which has proven useful in this regard. We recall this and related invariants below.

Suppose \((B, \pi)\) is an open book supporting a contact manifold \((Y, \xi)\) and \(K\) is a link braided about this open book. Then \(K\) can be viewed naturally as a transverse link in \((Y, \xi)\). Baldwin, Vela-Vick, and Vértesi used this viewpoint in [BVVV13] to define an invariant of transverse links, the so-called BRAID invariant, which assigns to such \(K\) a class in knot Floer homology which has proven useful in this regard. We recall this and related invariants below.

\[ \hat{t}(K) \in \widehat{HF}(Y, K). \]

They then proved that \(\hat{t}\) agrees with the LOSS invariant \(\hat{T}\) defined by Lisca, Ozsváth, Stipsicz, and Szabó [LOSS09] for transverse knots, and with the GRID invariant \(\hat{\theta}\) defined by Ozsváth, Szabó, and Thurston [OST08] for transverse links in the tight contact 3-sphere \((S^3, \xi_{std})\).

Our first goal in this paper is to prove a nonvanishing result for the BRAID invariant of a braid together with the binding. To set the stage, recall that the binding \(B\) of an open book \((B, \pi)\) is naturally a transverse link in the supported contact manifold \((Y, \xi)\).

Theorem 1.1. Suppose \((B, \pi)\) is an open book supporting \((Y, \xi)\) and \(K\) is a transverse link braided about this open book. Then the BRAID invariant \(\hat{t}(B \cup K)\) is nonzero.

There are two notable antecedents to Theorem 1.1. First, Vela-Vick proved in [VV11] that the LOSS invariant \(\hat{T}(B)\) is nonzero when the binding \(B\) is connected. Second, Etnyre and
Vela-Vick proved in [EVV10] that $\bar{c}(B)$ is nonzero even when $B$ is disconnected, where
$$
\bar{c}(B) \in SFH(-Y(B), \Gamma_\mu) \cong \hat{HF}(\Gamma, Y(B)),
$$
is the transverse invariant defined in [SV09] using a partial open book for the link complement.

Theorem 1.1 can be viewed as a generalization of both of these results. Indeed, restricting Theorem 1.1 to the case where $K$ is the empty link and $B$ is connected recovers the first result, since the BRAID and LOSS invariants agree for transverse knots. Moreover, it is shown in [SV09] that $\bar{c} = \hat{f}$ for transverse knots, and the arguments of [BVVV13, SV09] should extend to show that $\bar{c} = \hat{t}$ for transverse links as well. Therefore, restricting Theorem 1.1 to the case where $K$ is empty should recover the second result above.

Below, we discuss an application of Theorem 1.1 to braid dynamics.

Given a classical braid $K \subset S^3$ — a braid with respect to the open book $(U, \pi)$ corresponding to the fibration of the unknot complement by disks—the fractional Dehn twist coefficient $c(K)$ is a measure of how much the braid $K$ twists. This notion generalizes naturally to closed braids about arbitrary open books [IK17a].

It is shown in [BVVV13] that if $c(K) < 0$, which is to say that $K$ is not right-veering, then the BRAID invariant of $K$ vanishes. This was later exploited by Baldwin and Grigsby in [BG15] to give a new solution to the word problem in the classical braid group. More recently, Plamenevskaya [Pla15] proved the following result in the opposite direction for the GRID invariant:

**Theorem 1.2** (Plamenevskaya). If $K$ is a classical braid with $c(K) > 1$ then $\hat{\vartheta}(K) \neq 0$.

The above is proven by applying comultiplication [Ball] of the invariant $\hat{\vartheta}$ to reduce to the case of studying a reference braid $T$ having $\hat{\vartheta}(T) \neq 0$. Reducing to the reference braid also uses the notion of $\sigma$-positivity, coming from the Dehornoy ordering on the braid group. The nonvanishing of $\hat{\vartheta}(T)$ is proven by constructing a grid diagram for $T$ and studying the combinatorial chain complex. It is interesting to observe that the reference braid $T$ used is transversely isotopic to $K \cup U$, where $U$ is the binding of the disk open book for $S^3$ and $K$ is some braid about $U$.

Studying properties of the transverse invariant coming from the dynamics of the braid monodromy of $K$, is more natural from the perspective of the BRAID invariant. Indeed, using Theorem 1.1 one can generalize the above theorem:

**Theorem 1.3.** If $K$ is a braid about an open book having connected binding with $c(K) > 1$ then $\hat{t}(K) \neq 0$.

We remark that the fractional Dehn twist coefficients of $K$ and of the monodromy of the underlying open book decomposition can differ by arbitrary amounts, see [IK17a].

One feature of our proof is that it does not appeal to any ordering of the braid group, for there is no known interesting notion of $\sigma$-positivity for braids about general open books.

\footnote{We often denote a braid and its closure by the same symbol. The fractional Dehn twist coefficient is an invariant of closed braids.}
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2. Preliminaries

2.1. Contact Geometry. We assume the reader has a certain knowledge of contact geometry. For an introduction to the Giroux correspondence and open books consult the wonderful notes [Etn05]. For a reference on transverse and Legendrian links we point the reader to [Etn04].

In this section we explain the connection between transverse links and braids, and illustrate how to braid the binding of an open book about itself.

Let \((Y, \xi)\) be a contact 3-manifold. Suppose that \((B, \pi)\) is an open book supporting \((Y, \xi)\). B sits naturally as a transverse link. Any link braided about \(B\) is also naturally a transverse link, as the contact plane field is nearly tangent to the pages away from the binding \(B\). The following is a generalization of a theorem of Bennequin [Ben83]

\[\text{Theorem 2.1. [Pav11]} \quad \text{Suppose } (B, \pi) \text{ is an open book supporting } (Y, \xi). \text{ Every transverse link in } (Y, \xi) \text{ is transversely isotopic to a braid with respect to } (B, \pi).\]

There is a notion of positive Markov stabilization for braids with respect to an arbitrary open book, defined in [Pav11]. This operation increases the braid index by one, but preserves the transverse isotopy class of the braid. The following is a generalization of the transverse Markov theorem of Wrinkle [Wri02].

\[\text{Theorem 2.2. [Pav11]} \quad \text{Suppose } K_1 \text{ and } K_2 \text{ are braids with respect to an open book } (B, \pi) \text{ supporting } (Y, \xi). \text{ } K_1 \text{ and } K_2 \text{ are transversely isotopic if and only if they admit positive Markov stabilizations } K_1^+ \text{ and } K_2^+ \text{ which are braid isotopic with respect to } (B, \pi).\]

Since the binding \(B\), having \(n\) components, of an open book supporting \((Y, \xi)\) sits naturally as a transverse link, a copy of \(B\) may be braided about itself, resulting in a braid of index \(n\).

Recall that the neighborhood of a transverse knot in any contact manifold is standard. If \(K\) is transverse it admits a neighborhood contactomorphic to

\[N_\epsilon = \{(r, \theta, z) : r < \epsilon\} \subset \mathbb{R}^2 \times S^1\]

where \(\xi = \ker(\alpha) = \ker(dz + r^2d\theta)\), and \(K\) is identified with \((0,0) \times S^1\). In these coordinates, \(K\) admits a parametrization \(\gamma(t) = (0, t, t)\), where \(t \in [0, 2\pi)\). Consider the following transverse isotopy

\[\Gamma_s(t) = (s, t, t)\]

from \(\gamma_0(t)\) to \(\gamma_{\epsilon/2}(t)\). Applying this isotopy to each component of \(B\) realizes a copy of \(B\) as an index \(n\) braid.

2.2. The BRAID invariant. In this subsection we review the definition of the BRAID invariant, defined in [BVVV13]. The definition is reminiscent of the definition of the contact invariant given in [HKM09].

As before let \((B, \pi)\) be an open book supporting \((Y, \xi)\). Let \((S, \phi)\) be the abstract open book corresponding to \((B, \pi)\). Suppose \(S\) has genus \(g\) and \(m\) boundary components.
Let $K$ be an index $n$ braid with respect to $(B, \pi)$. We may assume, via a braid isotopy, that $K \cap S_0$ consists of $n$ points, $P = \{p_1, \ldots, p_n\}$. Let $\hat{\phi} \in \text{MC}(S \setminus P, \partial S)$ denote some lift of $\phi$ which determines $K$.

A basis of arcs $\{a_i\}_{i=1}^{2g+n+m-2} \subset S \setminus P$ is a collection of properly embedded disjoint arcs which cut $S \setminus P$ into $n$ discs, each having precisely one point of $P$. Let $\{b_i\}_{i=1}^{2g+n+m-2}$ be another basis of arcs obtained by slightly moving the endpoints of $a_i$ in the oriented direction of $\partial S$, and isotoping in $S \setminus P$ so that $a_i$ intersects $b_i$ transversely in a single point with positive sign.

The surface $S$, marked points $P$, and bases of arcs $\{a_i\}$ and $\{b_i\}$ specify a Heegaard diagram $\mathcal{H} = (\Sigma, \beta, \alpha, w_K, z_K)$ encoding $(-Y, K)$:

- $\Sigma = S_{1/2} \cup -S_0$
- $\alpha_i = a_i \times \{0, 1/2\}$, $\beta_i = b_i \times \{1/2\} \cup \hat{\phi}(b_i) \times \{0\}$
- $z_i = p_i \times \{0\}$, $w_i = p_i \times \{1/2\}$

For each $i$, $\alpha_i$ intersects $\beta_i$ in a single point in the region $S_{1/2}$ denoted $x_i$. Let $x \in \mathbb{T}_\beta \cap \mathbb{T}_\alpha$ denote the generator having component $x_i$ on $\alpha_i$. The homology class $[x] \in \text{HFK}^-(-Y, K)$ is an invariant of the transverse isotopy class of $K$ (Theorem 3.1 of [BVVV13]). The invariant is denoted $t(K)$. The natural map $\text{HFK}^-(-Y, K) \to \widehat{\text{HFK}}(-Y, K)$ sends $t(K)$ to $\hat{t}(K)$.

The invariant $t(K)$ is equivalent to that of $\theta(K)$ defined in [OST08], and also equivalent to $T(K)$ defined in [LOSS09]. See [BVVV13] for proofs of equivalence.

2.3. Comultiplicativity of the BRAID invariant. In this subsection we establish the comultiplicativity of the BRAID invariant, a key ingredient in our proof of Theorem 1.3. The proof is a completely straightforward adaptation of the argument for the comultiplicativity of the contact class [Bal08], and also generalizes the comultiplicativity of the GRID invariant [Bal10].

Let $S$ be a surface with boundary and $P = \{p_1, \ldots, p_n\} \subset S$ be a collection of marked points. Let $K_g \subset (Y_g, \xi_g)$, $K_h \subset (Y_h, \xi_h)$, and $K_{hg} \subset (Y_{hg}, \xi_{hg})$ be transverse links specified by elements $g, h \in \text{MC}(S \setminus P, \partial S)$ as in subsection 2.1.

**Theorem 2.3.** There exists a natural comultiplication map

$$\hat{\mu}: \widehat{\text{HFK}}(-Y_{hg}, K_{hg}) \to \widehat{\text{HFK}}(-Y_g, K_g) \otimes \widehat{\text{HFK}}(-Y_h, K_h)$$

sending $\hat{t}(K_{hg})$ to $\hat{t}(K_g) \otimes \hat{t}(K_h)$. In particular, $\hat{t}(K_g), \hat{t}(K_h) \neq 0$ implies that $\hat{t}(K_{hg}) \neq 0$.

Let $\{a_i\}$ and $\{b_i\}$ denote the bases of arcs described in Subsection 2.2. We construct a third basis of arcs $\{c_i\}$ from by applying a small isotopy moving the endpoints of arcs in $\{b_i\}$ along $\partial S$ in the direction given by the boundary orientation. We require that $a_i$ and $b_i$ intersect $c_i$ transversely in a single point with positive sign.

We construct three sets of curves and two sets of basepoints on the Heegaard surface $\Sigma = S_{1/2} \cup -S_0$:

- $\alpha = \{a_i\} = \{a_i \times \{0, 1/2\}\}$
- $\beta = \{\beta_i\} = \{b_i \times \{1/2\} \cup g(b_i) \times \{0\}\}$
Theorem 2.3: Let $D_1, \ldots, D_m$ denote the connected regions of $\Sigma \setminus \{\alpha \cup \beta \cup \gamma\}$. A triply-periodic domain of a pointed Heegaard triple diagram $(\Sigma, \alpha, \beta, \gamma, z, w)$ is a formal linear combination $P = \sum_i p_i D_i$ such that $n_\alpha(P) = n_\beta(P) = n_\gamma(P) = 0$ and $\partial P = \sum_i p_i \partial D_i$ is a linear combination of complete $\alpha$, $\beta$ and $\gamma$ curves.

Definition 2.4. A pointed Heegaard triple-diagram is said to be weakly-admissible if every non-trivial triply-periodic domain $P$ has both positive and negative coefficients.

For each $i$, the triple of curves $\alpha_i$, $\beta_i$ and $\gamma_i$ intersect form the arrangement on $S_{1/2}$ pictured in Figure 1.

Suppose that $P = \sum_i p_i D_i$ is a triply periodic domain of $(\Sigma, \alpha, \beta, \gamma, z, w)$. The regions $D_1$ and $D_6$ will always contain points of $w$ (for some values of $i$ they are actually the same region), therefore $p_1 = p_6 = 0$. Since $\partial P$ contains some total number of $\alpha$ curves, we have that $p_7 = p_4 - p_5 = -p_2$. If $p_7 \neq 0$ we have that the domain $P$ has both positive and negative

- $\gamma = \{\gamma_i\} = \{c_i \times \{1/2\} \cup h(g(c_i)) \times \{0\}\}$
- $w = \{w_i\} = \{p_i \times \{1/2\}\}$ and $z = \{z_i\} = \{p_i \times \{0\}\}$.

There are unique generators $x_g \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $x_h \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$ and $x_{hg} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ having all components in the region $S_{1/2}$. By construction, their homology classes in $\widehat{HFK}(-Y_g, K_g)$, $\widehat{HFK}(-Y_h, K_h)$ and $\widehat{HFK}(-Y_{hg}, K_{hg})$ are their respective BRAID invariants.

If the triple Heegaard diagram $(\Sigma, \alpha, \beta, \gamma, w, z)$ is weakly-admissible there is a multiplication chain map

$$m : \widehat{CFK}(Y_g, -K_g) \otimes \widehat{CFK}(Y_h, -K_h) \to \widehat{CFK}(Y_{hg}, -K_{hg})$$

defined on the generators by counting holomorphic triangles

$$m(a \otimes b) = \sum_{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma} \sum_{\phi \in \pi_2(a, b, x)} \mu(\phi) = 0 n_a(\phi) = n_w(\phi) = 0$$

Applying the $\text{Hom}_{\mathbb{Z}_2}(\cdot, \mathbb{Z}_2)$ functor to the above yields a comultiplication chain map

$$\mu : \widehat{CFK}(-Y_{hg}, K_{hg}) \to \widehat{CFK}(-Y_g, K_g) \otimes \widehat{CFK}(-Y_h, K_h)$$

defined on generators by

$$\mu(x) = \sum_{a \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{b \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma} \phi \in \pi_2(a, b, x) \mu(\phi) = 0 n_a(\phi) = n_w(\phi) = 0$$

To prove Theorem 2.3 it suffices to show that the triple diagram above is weakly admissible, and that $\mu(x_{hg}) = x_g \otimes x_h$. For the latter, there is a unique pair $(a \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta, b \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma)$ for which there exists a Whitney triangle $\phi \in \pi_2(a, b, x_{hg})$ satisfying $n_a(\phi) = n_w(\phi) = 0$ and having a holomorphic representative. We will see that $a = x_g, b = x_h$, that the Whitney triangle is unique, and that the triangle has a single holomorphic representative contributing to $m(a \otimes b)$.
coefficients, so assume that $p_7 = p_2 = 0$ and $p_4 = p_5$. Because $\partial P$ contains some total number of $\beta$ curves it now follows that $p_4 = p_5 = -p_3$. Either $P$ is to have both positive and negative coefficients, or we have that $p_1 = p_2 = \cdots = p_7 = 0$. Since all $\alpha$, $\beta$ and $\gamma$ curves meet the region $S_{1/2}$ it is clear that any non-trivial triply periodic domain must have both positive and negative coefficients. We conclude that the diagram is weakly-admissible.

We turn to the computation of $\mu(x_{hg})$.

Let $(x_g)_i$, $(x_h)_i$, and $(x_{hg})_i$ denote the components of $x_g$, $x_h$, and $x_{hg}$ on $\alpha_i$, $\beta_i$, and $\gamma_i$, respectively. Suppose that $\phi \in \pi_2(\mathbf{a}, \mathbf{b}, x_{hg})$ is a Whitney triangle admitting a holomorphic representative with $n_\mathbf{a}(\phi) = n_\mathbf{w}(\phi) = 0$. Let $D = \sum_i p_i D_i$ denote the domain of $\phi$. $n_\mathbf{w}(\phi) = 0$ immediately implies that $p_1 = p_6 = 0$.

Because $(x_{hg})_i$ is a corner of $\phi$, a standard holomorphic polygon counting argument gives that $p_5 = p_2 + p_4 + 1$. Suppose that $\phi$ has no corner at $(x_g)_i$; in this case it follows that $p_5 + p_7 = p_4$. Subtracting the second equation from the first yields $-p_7 = p_2 + 1$, in which case either $p_7$ or $p_2$ must be negative, and $\phi$ can not admit a holomorphic representative. Thus $\phi$ has a corner at $(x_g)_i$. A similar argument shows that $\phi$ must have a corner at $(x_h)_i$, and that $p_5 = 1$ is the only nonzero multiplicity of $D$ pictured in Figure 1.

Applying this argument for each local configuration as in the figure we see that $\mathbf{a} = x_g$, $\mathbf{b} = x_h$, and that $D$ is a union of small triangles, in which case $\phi$ must have a unique holomorphic representative. We conclude that $\mu(x_{hg}) = x_g \otimes x_h$.

2.4. Right-veeringness and the fractional Dehn twist coefficient. The monoid of right-veering diffeomorphisms in the mapping class group, along with the notion of fractional Dehn twist coefficient, was introduced and studied by Honda, Kazez, and Matić (HKM07). We do not give precise definitions, as by now the notions are standard in the literature. See sections 2 and 3 of [HKM07] for detailed discussions.
For two properly embedded arcs $a, b \subset S \setminus P$ such that $a \cap \partial S = b \cap \partial S \neq \emptyset$, we denote the condition that $b$ is to the right of $a$ by $a \leq b$. We say that $\psi \in \text{MCG}(S \setminus P, \partial S)$ is \textit{right-veering} if $\psi(a) \geq a$ for every properly embedded arc $a \subset S \setminus P$ meeting $\partial S$.

We denote the \textit{fractional Dehn twist coefficient} of $\psi \in \text{MCG}(S \setminus P, \partial S)$ about a boundary component $C \subset \partial S$ by $c(\psi, C)$. The definition involves the Nielsen-Thurston classification of surface diffeomorphisms.

\textbf{Remark 2.6.} Although the fractional Dehn twist coefficient was originally defined in the case that $P = \emptyset$, the definition carries over naturally when punctures are introduced. This is further studied in [IK17a].

Let $\tau_C \in \text{MCG}(S \setminus P, \partial S)$ denote a positive Dehn twist about a small push-off of $C$ into the interior of $S$. The following Lemma follows from the definition of the fractional Dehn twist coefficient:

\textbf{Lemma 2.7.} (see Proposition 4.10 [IK17a]) Let $\psi \in \text{MCG}(S \setminus P, \partial S)$, and suppose $C = \partial S$ is a boundary component. Then we have that $c(\tau_C^N \circ \psi, C) = N + c(\psi, C)$.

Stronger statements are given in section 3 of [HKM07], but we will only need the following:

\textbf{Lemma 2.8.} Let $\psi \in \text{MCG}(S \setminus P, \partial S)$. If $c(\psi, C) > 0$ for every component $C$ of $\partial S$, then $\psi$ is right-veering.

\section{The Transverse Invariant of a Braid and Its Axis}

Suppose that $(B, \pi)$ is an open book decomposition, with $B$ having $n$ components, supporting $(Y, \xi)$. Let $(S, \phi)$ be the abstract open book corresponding to $(B, \pi)$, where $S$ has genus $g$. As discussed in subsection 2.1, the binding $B$ is naturally a transverse link that may be braided about the open book via a transverse isotopy. Abusing notation, we denote the resulting $n$-braid $B$.

$B$ is specified by a lift $\tilde{\phi} \in \text{MCG}(S \setminus \{p_1, \ldots, p_n\}, \partial S)$ of $\phi$. Thinking of $\phi$ as fixing a collar neighborhood $\nu(\partial S)$ of the boundary, one obtains $\tilde{\phi}$ by composing $\phi$ with $n$ push maps supported in $\nu(\partial S)$, each going once in the oriented direction of the nearby boundary component. See Figure 2 for the push maps and a basis of arcs $\{a_1\}_{1}^{2g+n-1} \cup \{a_{2,i}\}_{2g+1}^{2g+n-1}$ for $S \setminus \{p_1, \ldots, p_n\}$.

The basis of arcs along with $\tilde{\phi}$ specify a Heegaard diagram $D = (\Sigma, \beta, \alpha, w_B, z_B)$, along with a generator $x_D$, shown in Figure 3 for $(Y, B)$, as in subsection 2.2. The labelling of the basis arcs induces a labelling of the $\beta$ and $\alpha$ curves. The homology class $[x_D]$, in $HFK(D)$, is the braid invariant $\hat{t}(B)$.

It will be more convenient to consider the following diagram, in Figure 4, isotopic to $D$, still denoted $D$.

The following notion was introduced in [HPT13] and is very useful for studying the relative Alexander grading.
Figure 2. The basis of arcs \( \{a_i\}_{1}^{2g+n-1} \cup \{a_{2,i}\}_{2g+1} \) for the case \( n = 3 \) and \( g = 2 \) is depicted in red. The push maps, supported in the shaded neighborhood \( \nu(\partial S) \), go in the orientation of \( \partial S \) and are depicted in blue.

Figure 3. A portion of the Heegaard diagram \( D \) for \((-Y,B)\) in the case \( g = 1 \) and \( n = 3 \). The homology class of the generator depicted by orange dots, in \( \widehat{HFK}(D) \), is equal to the transverse invariant \( \widehat{t}(B) \). The indexing of the basis in Figure 2 induces an indexing of the \( \alpha \) and \( \beta \) curves in \( D \).

Definition 3.1. Let \( L_1 \cup \cdots \cup L_l = L \subset Y \) be an \( l \) component link, and let

\[(\Sigma, \alpha, \beta, z_1 \cup \cdots \cup z_l, w_1 \cup \cdots \cup w_l)\]

be a Heegaard diagram for \((Y,L)\) where the basepoints \( z_{L_i} \) and \( w_{L_i} \) encode the link component \( L_i \). Suppose that \([L_i]\) has order \( p \) in \( H_1(Y) \).
Let $\lambda_i \subset \Sigma$ be a longitude for $L_i$ constructed by connecting the points of $z_{L_i}$ to the $w_{L_i}$ by oriented arcs $\{\gamma_z|z \in z_{L_i}\}$ in the complement of $\alpha$ curves and gluing together with arcs $\{\gamma_w|w \in w_{L_i}\}$ connecting the $w_{L_i}$ with the $z_{L_i}$ in the complement of the $\beta$ curves.

Let $D_1, \ldots, D_r$ denote the closures of components of $\Sigma \setminus (\lambda_i \cup \alpha \cup \beta)$. A relative periodic domain is a 2-chain $P = \Sigma a_i D_i$, whose boundary satisfies

$$\partial P = p\lambda_i + \sum n_i \alpha_i + \sum m_i \beta_i.$$ 

A relative periodic domain $P$ naturally corresponds to a homology class in $H_2(Y \setminus \nu(L_i), \partial(Y \setminus \nu(L_i)))$.

**Lemma 3.2.** (see Lemma 2.3 of [HP13]) Let $L_i \subset L$ be as in the definition above. Let $P$ be a relative periodic domain whose homology class agrees with that of some rational Seifert surface $F$ for $L_i$. For $x, y \in T_\alpha \cap T_\beta$, we have

$$A_{L_i}(x) - A_{L_i}(y) = n_x(P) - n_y(P)$$

where the Alexander grading above is defined using the surface $F$.

In this paper we consider only the Alexander grading induced by the binding of an open book. The choice of Seifert surface will always be a page of the open book, so we suppress the surface from the notation.

![Figure 4](image-url)

**Figure 4.** A portion of a Heegaard diagram isotopic to $D$, still denoted $D$. The generator $x_D$, whose homology class is $\hat{t}(B)$, is depicted by orange dots. The black basepoints are $z'$s and $w'$s, alternating when read from left to right. The purple depicts an oriented (as $\partial S$) longitude for $B$.

**Lemma 3.3.** The Alexander grading of a generator of $\overline{CFK}(D)$ is the number of its components in the region $S_{1/2}$ minus $(g + n - 1)$. In particular, a generator has maximal $(g + n - 1)$ Alexander grading if and only if all of its components lie in the region $S_{1/2}$.
Proof. Consider the relative periodic domain $S_{1/2}$ having boundary consisting only of a longitude for $B$ (See Figure 4). Using Lemma 3.2 we see that a generator has all components in $S_{1/2}$ if and only if it has maximal Alexander grading. Symmetry determines the absolute grading. \[\square\]

If $\phi$ is the identity monodromy, then $Y \cong \#^{2g+n-1}S^1 \times S^2$. The homology classes of oriented curves $\{A_i\}_{1}^{g+n-1} \cup \{B_i\}_{1}^{g} \subset S$ pictured in Figure 5 freely generate $H_1(Y; \mathbb{Z})$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{A basis for $H_1(Y; \mathbb{Z})$, when $\phi = 1_{(S, \partial S)}$}
\end{figure}

Proposition 3.4. If $\phi$ is the identity monodromy, then $x_D$ is the unique generator of maximal Alexander grading having $\text{Spin}^C$ structure $s_{w_B}(x_D) = s(\xi)$. In particular, $[x_D] = \hat{t}(B) \neq 0$.

Proof. Let $y \in \overline{CFK}(D, \text{top})$, and let $(y)^i, (y)_i$, denote the component of $y$ on $\beta_i, \alpha_i$ respectively.

By applying the simple homology obstruction given in Lemma 2.19 of [OS04b], we will show that the difference $s_{w_B}(y) - s_{w_B}(x_D)$, a linear combination of Poincaré duals of basis elements for $H_1(Y; \mathbb{Z})$, is equal to zero if and only if $y = x_H$.

We will recursively construct a sequence of generators $\{y_j\}$

\[y = y_{g+n}, y_{g+n-1}, \ldots, y_1 = x_D\]

in $\overline{CFK}(D, \text{top})$ such that

- $(y_j)^i = (x_D)^i$ for $j > g$ and $i \geq g + j$,
- $(y_j)^i = (x_D)^i$ and $(y_j)^{2k} = (x_D)^{2k}$ for $j \leq g$, $i \geq 2j - 1$, and any $k$.
- $s_{w_B}(y_{j+1}) - s_{w_B}(y_j)$ is a linear combination of $PD([A_j])$ and $PD([B_j])$ (the latter only appears for $j \geq g$). The difference is equal to zero if and only if $y_{j+1} = y_j$.

STEP 1

We begin by constructing $y_{g+n-1}$. If $(y)^{2g+n-1} = (x_D)^{2g+n-1}$ then we set $y_{g+n-1} = y$, so assume $(y)^{2g+n-1} \neq (x_D)^{2g+n-1}$. There are two other intersection points involving $\beta_{2g+n-1}$ in the region $S_{1/2}$, one with $\alpha_{2g+n-1}$ and another with $\alpha_{2,2g+n-1}$.
STEP 1a

If \((y)^{2g+n-1}=(y)_{2g+n-1}\) let \(y_{g+n-1}\) be obtained from \(y\) by replacing \((y)^{2g+n-1}\) with \((x_D)^{2g+n-1}\). We see that \(s_{w_B}(y)-s_{w_B}(y_{g+n-1})=PD([A_{g+n-1}])\).

STEP 1b

If \((y)^{2g+n-1}=(y)_{2,2g+n-1}\) then it must be the case that \((y)^{2,2g+n-1}=(y)_{2g+n-1}\). Let \(y_{g+n-1}\) be obtained from \(y\) by replacing \((y)^{2g+n-1}\), \((y)^{2,2g+n-1}\), with \((x_D)^{2g+n-1}\), \((x_D)^{2,2g+n-1}\), respectively. We see that \(s_{w_B}(y)-s_{w_B}(y_{g+n-1})=PD([A_{g+n-1}])\).

Figure 6. **STEP 1**: The construction of \(y_{g+n-1}\), both cases. The orange dots (only the relevant ones are shown in each case) are components of \(x_H\) and \(y_{g+n-1}\) (they agree here by construction). The brown squares represent \(y\). The highlighted path \(e(y,y_{g+n-1})\) is homologous to \([A_{g+n-1}]\) in both cases.

STEP 2

Assuming that we have constructed \(y_{j+1}\) for some \(j > g\) satisfying the hypotheses, we construct \(y_j\). If \((y_{j+1})^{g+j}=(x_D)^{g+j}\) then we set \(y_j=y_{j+1}\), so assume \((y_{j+1})^{g+j}\neq (x_D)^{g+j}\). There are three other intersection points involving \(\beta_{g+j}\) in the region \(S_{1/2}\), one with \(\alpha_{g+j}\), another with \(\alpha_{g+j+1}\), and a third with \(\alpha_{2,g+j}\).

STEP 2a

If \((y_{j+1})^{g+j}=(y_{j+1})_{g+j}\), let \(y_j\) be obtained from \(y_{j+1}\) by replacing \((y_{j+1})^{g+j}\) with \((x_D)^{g+j}\). As in the first case of the construction of \(y_{g+n-1}\) we see that \(s_{w_B}(y_{j+1})-s_{w_B}(y_j)=PD([A_j])\).

STEP 2b

If \((y_{j+1})^{g+j}=(y_{j+1})_{2,g+j}\) there are two further sub-cases. Either \((y_{j+1})_{g+j}=(y_{j+1})_{2,g+j+1}\), in which case it follows that \((y_{j+1})^{2,g+j}=(y_{j+1})_{2,g+j+1}\), or \((y_{j+1})_{g+j}=(y_{j+1})_{2,g+j}\).
STEP 2bi
In the former case, let $y_j$ be obtained from $y_{j+1}$ by replacing $(y_{j+1})^{g+j}$ with $(x_D)^{g+j}$, $(y_{j+1})^{2g+j}$ with $(x_D)^{2g+j}$, and $(y_{j+1})^{2g+j+1}$ with $(x_D)^{2g+j+1}$.

STEP 2bii
In the latter case, let $y_j$ be obtained from $y_{j+1}$ by replacing $(y_{j+1})^{g+j}$ with $(x_D)^{g+j}$ and $(y_{j+1})^{2g+j}$ with $(x_D)^{2g+j}$. In either case we have that $s_{w_B}(y_{j+1}) - s_{w_B}(y_j) = PD([A_j])$. See Figure 7.

**Figure 7.** STEP 2b: The construction of $y_j$. The orange dots (only the relevant ones are shown) are components of $x_H$ and $y_j$ (they agree here by construction). The brown squares are components of $y_{j+1}$. The highlighted path $\epsilon(y_{j+1},y_j)$ is homologous to $[A_j]$ in both cases.

STEP 2c
If $(y_{j+1})^{g+j} = (y_{j+1})^{2g+j+1}$ then it must be the case that $(y_{j+1})^{g+j} = (y_{j+1})^{2g+j+1}$. Let $y_j$ be obtained from $y_{j+1}$ by replacing $(y_{j+1})^{g+j}$, $(y_{j+1})^{2g+j+1}$, with $(x_D)^{g+j}$, $(x_D)^{2g+j+1}$, respectively. The picture is similar to that of the second case of the construction of $y_{g+n-1}$ (Step 1b). We see that $s_{w_B}(y_{j+1}) - s_{w_B}(y_j) = PD([A_j])$.

STEP 3
Assuming that we have constructed $y_{j+1}$ for some $j \leq g$ satisfying the hypotheses, we construct $y_j$. For each $i > 2g$, $\beta_{2,i}$ intersects only $\alpha_{2,i}$, among the alpha curves whose components of $y_{j+1}$ are not fixed, in the domain $S_{1/2}$ we have that $(y_{j+1})^{2i} = (y_{j+1})^{2,2i}$. Of the remaining alpha curves, $\beta_{2j}$ and $\beta_{2j-1}$ intersect only $\alpha_{2j}$ and $\alpha_{2j-1}$. Thus there are two cases, with three nontrivial sub-cases each. Either $(y_{j+1})^{2j} = (y_{j+1})^{2j}$ and $(y_{j+1})^{2j-1} = (y_{j+1})^{2j-1}$, or $(y_{j+1})^{2j} = (y_{j+1})^{2j-1}$ and $(y_{j+1})^{2j-1} = (y_{j+1})^{2j}$. 

The trivial sub-case is that \((y_{j+1})^{2j} = (x_D)^{2j}\) and \((y_{j+1})^{2j-1} = (x_D)^{2j-1}\), in which case we let \(y_j = y_{j+1}\).

**STEP 3a**

Suppose that \((y_{j+1})^{2j} = (y_{j+1})_{2j}\) and \((y_{j+1})^{2j-1} = (y_{j+1})_{2j-1}\).

**STEP 3ai**

If \((y_{j+1})^{2j} \neq (x_D)^{2j}\) and \((y_{j+1})^{2j-1} = (x_D)^{2j-1}\) then, obtaining \(y_j\) from \(y_{j+1}\) by replacing \((y_{j+1})^{2j}\) with \((x_D)^{2j}\), we have that \(s_{w_B}(y_{j+1}) - s_{w_B}(y_j) = PD([B_j])\). See Figure 8.

**STEP 3aii**

If \((y_{j+1})^{2j} = (x_D)^{2j}\) and \((y_{j+1})^{2j-1} \neq (x_D)^{2j-1}\) then, obtaining \(y_j\) from \(y_{j+1}\) by replacing \((y_{j+1})^{2j-1}\) with \((x_D)^{2j-1}\), we have that \(s_{w_B}(y_{j+1}) - s_{w_B}(y_j) = PD([A_j])\).

**STEP 3aiii**

If \((y_{j+1})^{2j} \neq (x_D)^{2j}\) and \((y_{j+1})^{2j-1} \neq (x_D)^{2j-1}\) then, we obtain \(y_j\) from \(y_{j+1}\) by replacing \((y_{j+1})^{2j}\) with \((x_D)^{2j}\) and \((y_{j+1})^{2j-1}\) with \((x_D)^{2j-1}\). Combining the paths from the two previous sub-cases, we have that \(s_{w_B}(y_{j+1}) - s_{w_B}(y_j) = PD([A_j] + [B_j])\).

**STEP 3b**
Now, assume we are in the case that \((y_{j+1})^{2j} = (y_{j+1})^{2j-1}\) and \((y_{j+1})^{2j} = (y_{j+1})^{2j-1}\). The intersection point \((y_{j+1})^{2j}\) is fixed, and there are three possibilities for \((y_{j+1})^{2j-1}\). In each of these sub-cases we obtain \(y_j\) from \(y_{j+1}\) by replacing \((y_{j+1})^{2j}\) with \((x_D)^{2j}\) and \((y_{j+1})^{2j-1}\) with \((x_D)^{2j-1}\). Figure 9 shows the three possible sub-cases. We see that \(s_{w_B}(y_{j+1}) - s_{w_B}(y_j)\) is a linear combination of \(PD([A_j])\) and \(PD([B_j])\).

Figure 9. **STEP 3b**: The construction of \(y_j\), in the case that \((y_{j+1})^{2j} = (y_{j+1})^{2j-1}\). To avoid overcrowding the figure, only the relevant alpha and beta curves are shown. The orange dots are components of \(x_H\) and \(y_j\) (they agree here by construction). The brown squares are components of \(y_{j+1}\). The highlighted path \(\epsilon(y_{j+1}, y_j)\) is homologous to \([B_j]\) in the first sub-case, \([A_j]\) in the second, and \([A_j] + [B_j]\) in the third.

Returning to the general case, let \((B, \pi)\) be an open book supporting \((Y, \xi)\) with abstract open book \((S, \phi)\). Let \(\delta \subset S\) be a nonseparating curve. \(\delta\) is naturally a Legendrian knot in \((Y, \xi)\), and the page framing of \(\delta\), denoted \(tb\), is the Thurston-Bennequin (contact) framing. Performing contact (+1) surgery along \(\delta\) is equivalent to composing the monodromy \(\phi\) with a negative Dehn twist along \(\delta\). Let \(\mu\) denote a meridian for \(\delta \subset Y\). Let \(B', B''\), denote the image of \(B\) under the Dehn surgery to \(-Y_{tb+\mu}(\delta), -Y_{tb}(\delta)\), respectively. Since \(\delta\) can be made disjoint from \(S\) (within \(Y\)), a Seifert surface for \(B\), the ordered triple \{\((-Y, B), (-Y_{tb+\mu}(\delta), B'), (-Y_{tb}(\delta), B'')\)\} is a distinguished triangle of knots ([OS10] sec. 2).

Theorem 8.2 of [OS04a] tells us that any distinguished triangle of knots induces an exact triangle of knot Floer homology groups, where the maps preserve the Alexander grading. In particular, we have:
\[
\begin{array}{ccc}
\widehat{HFK}(Y, B, g + n - 1) & \xrightarrow{F} & \widehat{HFK}(Y_{tb+\mu}(\delta), B', g + n - 1) \\
& \xrightarrow{} & \widehat{HFK}(Y_{tb}(\delta), B'', g + n - 1)
\end{array}
\]

\(S\) is a minimal genus Seifert surface for the link \(B\). The genus of \(B''\) is strictly less than the genus of \(B\). Let \(S'' \subset Y_{tb}(\delta)\) denote the Seifert surface obtained from \(S\) by deleting a neighborhood of \(\delta\) and capping off the two new boundary components. In particular, we have that \(g + n - 1 > \frac{n - \chi(S'')}{2} = \frac{|B| - \chi(S'')}{2}\). An adjunction inequality (Proposition 2.1 of [Ni06]) tells us that \(\widehat{HFK}(Y_{tb}(\delta), B'', g + n - 1) = 0\). It follows that the map

\[F : \widehat{HFK}(Y, B, g + n - 1) \rightarrow \widehat{HFK}(Y_{tb+\mu}(\delta), B', g + n - 1),\]

in the exact triangle above, is an isomorphism.

Any monodromy \(\phi \in \text{MCG}(S, \partial S)\) is related to the identity map on \(S\) by addition and removal of negative Dehn twists along nonseparating curves. Combining these facts with Proposition 3.3 and the following remark we have proven the first half of Theorem 1.1.

**Remark 3.5.** In Proposition 3.7 of [HKM09] it is proven that the contact invariant in \(\widehat{HF}\) is functorial under such contact (+1) surgeries. Their argument carries over directly and shows that the BRAID invariant \(\widehat{\iota}(K)\) is functorial under contact (+1) surgeries as well, a result first proven in the LOSS context [OS10] (only in the case that \(K\) is connected). In particular we have that for the map \(F\) above, \(F(\widehat{\iota}(B)) = \widehat{\iota}(B')\).

**Theorem 3.6.** If \((B, \pi)\) is an open book supporting \((Y, \xi)\), then \(\widehat{\iota}(B) \in \widehat{HFK}(-Y, B)\) is nonzero.

Now let \(K\) be a link braided about \(B\) having braid index \(k\). We may add \(k\) pairs of basepoints and curves to \(\mathcal{D}\) to obtain a new diagram \(\mathcal{H} = (\Sigma, \beta, \alpha, w_K \cup w_B, z_K \cup z_B)\), see Figure 10 which encodes \((-Y, K \cup B)\). We reindex the \(\beta\) and \(\alpha\) curves for our convenience. This diagram can be obtained by considering a basis of arcs for \(S \setminus \{p_1, \ldots, p_{n+k}\}\); it is isotopic to the usual diagram appearing in the definition of the braid invariant of \(K \cup B\).

By construction, the homology class of \(x_H\) in \(\widehat{HFK}(\mathcal{H})\) is \(\widehat{\iota}(B \cup K)\). Let \(\widehat{CFK}(\mathcal{H}, A^{\text{top}}_B)\) denote the summand of \(\widehat{CFK}(\mathcal{H})\) in the top Alexander grading induced by \(B\). An identical argument as in Lemma 3.3 tells us that a generator of \(\widehat{CFK}(\mathcal{H})\) is in \(\widehat{CFK}(\mathcal{H}, A^{\text{top}}_B)\) if and only if all of its components are in the region \(S_{1/2}\).

**Lemma 3.7.** As complexes we have \(\widehat{CFK}(\mathcal{H}, A^{\text{top}}_B) \simeq (V_1 \otimes V_2 \otimes \cdots \otimes V_k) \otimes \widehat{CFK}(\mathcal{D}, \text{top})\), where each \(V_i\) is a free rank two \(\mathbb{F}\)-module with basis \(\{x_i, y_i\}\). The differential on \(V_i\) is zero.

**Proof.** Observe that in \(S_{1/2}\) for any \(i \leq k\) the curve \(\alpha_i\) only intersects curves \(\beta_j\) for \(j \leq i\), and no other \(\beta\) curves. It is immediate that as complexes we have \(\widehat{CFK}(\mathcal{H}, A^{\text{top}}_B) \simeq (V_1 \otimes V_2 \otimes \cdots \otimes V_k) \otimes \widehat{CFK}(\mathcal{D}, \text{top})\).
Figure 10. A portion of the diagram $\mathcal{H}$ in the case $k = 2$, $g = 1$, and $n = 2$. The $k$ pairs of alpha/beta curves introduced to encode $K$ are indexed 1,...,$k$ left to right. The rest of the curves are indexed as before (see Figures 2 and 3) with a shift of $k$ in the first coordinate. The orange dots are components of $x_H$.

$$\cdots \otimes V_k \otimes \widehat{CFK}(D, \text{top})$$ For each $i \leq k$ there are no disks leaving $x_i$ nor $y_i$ which contribute to the differential. \hfill \Box

Note that $x_H$ is identified with $(x_1 \otimes \cdots \otimes x_k) \otimes x_D$. Combining Theorem 3.6 with Lemma 3.7 proves Theorem 1.1.

**Theorem 1.1** Let $(B, \pi)$ be an open book supporting $(Y, \xi)$. If $K$ is braided about $B$, then $\tilde{\ell}(B \cup K) \in \widehat{HFK}(-Y, B \cup K)$ is nonzero.

### 4. Elementary non-vanishing results

In this short section we establish some elementary non-vanishing results for $\tilde{\ell}(K)$ which we will use in conjunction with Theorem 2.3 for the proof of Theorem 1.3.

As before, let $S$ be a surface with boundary and $P = \{p_1, \ldots, p_n\} \subset S$ a collection of marked points. Let $\gamma \subset S$ denote an embedded arc connecting two points $p_i \neq p_j$ of $P$. We let $\sigma_\gamma$ denote the right-handed half twist along $\gamma$. The element $\sigma_\gamma$ has support in a small neighborhood of $\gamma$, and exchanges the points $p_i$ and $p_j$, see Figure 11.

If $\delta$ is a simple closed curve in $S \setminus P$, we let $\tau_\delta$ denote a positive Dehn twist along $\delta$.

**Proposition 4.1.** Let $S$ be a surface with boundary and $P = \{p_1, \ldots, p_n\} \subset S$ a collection of marked points. Let $K_\gamma$ denote the closure of $\sigma_\gamma$, a positive half twist along some embedded arc $\gamma$. Let $K_\delta$ denote the closure of $\tau_\delta$, a positive Dehn twist along some simple closed curve $\delta$. Then we have that...
\[ \hat{t}(K_\gamma), \hat{t}(K_\delta) \neq 0. \]

**Proof.** We first prove that \( \hat{t}(K_\gamma) \) is non-zero. One can choose a basis of arcs \( \{a_i\} \) for \( S \setminus P \) such that \( a_1 \) intersects \( \gamma \) in a single point, and all other basis arcs \( \{a_i | i > 1\} \) are disjoint from \( \gamma \). We use this basis of arcs to obtain a Heegaard diagram \( \mathcal{H} \) encoding \( (-Y, K_\gamma) \) as in Subsection 2.2. The resulting complex \( \hat{CFK}(\mathcal{H}) \) has no differential, so the result follows.

Next consider we consider a positive Dehn twist \( \tau_\delta \) along a simple closed curve \( \delta \). If \( \delta \) does not separate a region \( S \) from \( \partial S \), we may construct a basis of arcs \( \{a_i\} \) for \( S \setminus P \) such that \( a_1 \) intersects \( \delta \) in a single point, and all other basis arcs \( \{a_i | i > 1\} \) are disjoint from \( \delta \). The knot Floer complex coming from the corresponding diagram will again have no differential, so the result will follow.

Suppose that \( \delta \) separates a region \( R \subset S \) from \( \partial S \). If \( R \) contains no points of \( P \), we may write \( \tau_\delta \) as a product of positive Dehn twists along non-separating curves in \( R \), in which case \( \hat{t}(K_\delta) \neq 0 \) by Theorem 2.3 and the previous case above.

We are left to consider the case that \( \delta \) separates off a region \( R \subset S \) such that \( P \cap R = P' \neq \emptyset \) and \( \partial R = \delta \). We may find a properly embedded separating arc \( c \subset S \) so that there exists a component \( S' \) of \( S \setminus c \) which contains and is homeomorphic to \( R \) also satisfying \( S' \cap P = P' \), see Figure 12. The Dehn twist \( \tau_\delta \) restricts to an element \( \tau'_\delta \in MCG(S' \setminus P', \partial S') \), we let \( K'_\delta \) denote the closure of \( \tau'_\delta \).

We may now choose a basis of arcs \( \{a_i\} \cup \{b_i\} \) for \( S \setminus P \) so that \( \{a_i\} \) is a basis for \( S' \setminus P' \), and so that the arcs \( \{b_i\} \) are disjoint from \( S' \). Let \( \mathcal{H} \) denote the Heegaard diagram constructed using \( (S \setminus P, \{a_i\} \cup \{b_i\}) \) as in Subsection 2.2, and \( \mathcal{H}' \) denote the Heegaard diagram constructed using \( (S' \setminus P', \{a_i\}) \). Because \( \{a_i\} \) is a basis for \( S' \setminus P' \), there is some point \( p_j \in P' \) in the same component of \( S \setminus \{a_i\} \) as every point of \( P \setminus P' \). Recall that \( w_j = p_j \times \{1/2\} \).

Since any holomorphic disk \( \phi \) contributing to the differential of \( \hat{CFK}(\mathcal{H}) \) must satisfy \( n_{w_j}(\phi) = 0 \) we have
\[
\hat{CFK}(\mathcal{H}) = A \otimes \hat{CFK}(\mathcal{H}')
\]
where the complex $A$ has no differential. Let $x_{\mathcal{H}}$ and $x_{\mathcal{H}'}$ denote the generators whose homology classes represent the $\tilde{t}(K_\delta)$ and $\tilde{t}(K'_\delta)$ respectively. Under this isomorphism the generator $x_{\mathcal{H}}$ is identified with $a \otimes x_{\mathcal{H}'}$, for some $0 \neq a \in \mathcal{A}$.

Note that $\tilde{t}(K'_\delta) \neq 0$ by Theorem 1.1, see Figure 13. The result follows.

Figure 12. The arc $c$ separates off a subsurface $S'$ which contains and is homeomorphic to $R$.

Figure 13. The positive Dehn twist pictured on the left is equivalent to the one on the right composed with the boundary parallel push map of a point depicted in blue. Thus the braid closure is transversely isotopic to some braid union the binding of the open book.

5. Proof of Theorem 1.3

In this section we prove Theorem 1.3 by combining the following two Lemmas with Theorems 1.1 and 2.3 in addition to Proposition 4.1.
Lemma 5.1. Suppose that \( c \subset S \) is an embedded arc connecting \( \partial S \) to \( p_1 \in P \), disjoint from all other marked points. Let \( \phi \in \text{MCG}(S \setminus P, \partial S) \) be a monodromy such that \( \phi(c) \) is to the right of \( c \), and disjoint from \( c \) along its interior.

If \( \phi \) fixes \( p_1 \) there exists a simple closed curved \( \delta \in S \setminus P \) such that the positive Dehn twist \( \tau_\delta \) maps \( c \) to \( \phi(c) \). If \( \phi \) does not fix \( p_1 \) there exists an embedded arc \( \gamma \) from \( p_1 \) to \( \phi(p_1) \) such that the positive half twist \( \sigma_\gamma \) maps \( c \) to \( \phi(c) \).

Proof. Suppose first that \( \phi \) fixes \( p_1 \). Let \( \alpha = c \cup \phi(c) \). We apply an isotopy supported in a neighborhood of \( \alpha \cap \partial S \) to push \( \alpha \) into the interior of \( S \), still denoting the resulting curve \( \alpha \). The simple closed curve \( \alpha \in S \) admits an annular neighborhood \( S^1 \times I \simeq N(\alpha) \subset S \), so that \( N(\alpha) \cap P = p_1 \). \( N(\alpha) \) has two boundary components, one which is disjoint from \( c \), and one which intersects \( c \) once transversely. Let \( \delta \) denote the latter component of \( \partial N(\alpha) \). It is clear that \( \tau_\delta(c) = \phi(c) \).

Now suppose that \( \phi \) does not fix \( p_1 \). Let \( \gamma = c \cup \phi(c) \). We apply a small isotopy supported in a neighborhood of \( \gamma \cap \partial S \) to push \( \gamma \) into the interior of \( S \). It is clear that \( \sigma_\gamma(c) = \phi(c) \). \( \square \)

Remark 5.2. The following Lemma is similar to Lemma 5.2 of [HKM07], where a surface with possibly multiple boundary components, but no marked points, is considered. One may be tempted to say that the Lemma follows from Lemma 5.2 of [HKM07] by capping off boundary components with disks each having a single marked point. This procedure will not in general preserve the property that two arcs intersect efficiently, so one has to be careful. Moreover, the intermediate arc they construct (playing their role of \( b \), see the proof that follows) may begin and end at the same boundary component of the surface.

It is simpler to adapt their proof than try to apply their result, so this is the approach we take.

Recall, Subsection 2.4, that \( a \leq \phi(a) \) denotes that \( \phi(a) \) is to the right of \( a \).

Lemma 5.3. Suppose that \( e_0 \subset S \) is an embedded arc connecting \( \partial S \) to \( p_1 \in P \), disjoint from all other marked points. Let \( \phi \in \text{MCG}(S \setminus P, \partial S) \) be a monodromy taking the arc \( e_0 \) to the right. There exists a sequence of arcs

\[
e_0 \leq e_1 \leq e_2 \leq \cdots \leq e_m = \phi(e_0)
\]

each from \( e_0 \cap \partial S \) to a point of \( P \), such that \( e_i \) and \( e_{i+1} \) are disjoint along their interiors for each \( 0 \leq i \leq m \).

Proof. To simplify notation let \( a := e_0 \) and \( c := \phi(e_0) \). We let \( \#(a,c) \) denote the geometric intersection number of \( a \) with the interior of \( c \). It suffices to show that if \( a \leq c \), \( a \neq c \), and \( \#(a,c) \neq 0 \) then there exists a properly embedded arc \( b \) such that \( a \leq b \leq c \) and \( \#(a,b), \#(b,c) < \#(a,c) \). The desired sequence of arcs can be obtained by iterating the construction.

We endow \( S \setminus P \) with a hyperbolic metric and assume that \( a \) and \( c \) are geodesics intersecting in a collection of points \( \{x_1, \ldots, x_m\} = \{y_1, \ldots, y_m\} \) where

\[
x_i = a(t_i) \text{ for some } 0 = t_0 \leq t_1 \leq \cdots \leq t_m = 1
\]
\[
y_i = c(s_i) \text{ for some } 0 = s_0 \leq s_1 \leq \cdots \leq s_m = 1.
\]
The proof is a case by case analysis.

**CASE 1**

Suppose that $x_1 = y_r$, and that at $x_1$ the tangent vectors to $a$ and $c$, in this order, form a negative basis for $T_{x_1}(S)$. Consider the curve $b = a|_{[0,t_1]} * c|_{[s_r,1]}$. Clearly we have that $(a,b) = 0$ and $(b,c) = 0$. Note that $b$ is veers to the left of $c$ near $\partial S$, since they intersect efficiently it follows that $b \leq c$. When smoothed, the piecewise geodesic arc $b$ veers to the right of $a$.

**CASE 2**

Suppose now that $x_1 = y_r$, with $r > 1$, and that at $x_1$ the tangent vectors to $a$ and $c$ form a positive basis for $T_{x_1}(S)$. Let $x_r^* = y_r^*$ be the last point of $a$ to intersect $c|_{[0,s_r]}$. There are two sub-cases:

**SUB-CASE 2a**

Suppose that the tangent vectors to $a$ and $c$ form a negative basis for $T_{x_r^*}(S)$. Set $b = c|_{[0,s_{r^*}]} * a|_{[t_r,1]}$. It is clear that $(a,b), (b,c) < #(a,c)$.

We claim that $a \leq b \leq c$. We pass to the universal cover $\pi : \tilde{S} \to S$. Let $\tilde{a}$ and $\tilde{c}$ be lifts of $a$ and $c$ starting at some fixed lift $\tilde{x}_0$ of $x_0$. There is a natural lift $\tilde{b}$ of $b$ which consists of geodesic arcs along $\tilde{c}$ and some lift of $a$ different from $\tilde{a}$. In particular, $\tilde{b}$ starts along $\tilde{c}$ to the right of $\tilde{a}$, and then switches to some other lift of $a$. Because any two distinct lifts of $a$ are disjoint, it follows that $a \leq b$. Since $\tilde{b}$ starts along $\tilde{c}$ and diverges to the left along a lift of $a$, it is clear that $b \leq c$.

**SUB-CASE 2b**

Suppose that the tangent vectors to $a$ and $c$ form a positive basis for $T_{x_r^*}(S)$. We set $b = a|_{[0,t_1]} * (c|_{s_{r^*},s_r})^{-1} * a|_{[t_r,1]}$. Again, it is clear that $(a,b), (b,c) < #(a,c)$. Establishing $a \leq b \leq c$ involves passing to the universal cover as in the previous sub-case.

**CASE 3**

Suppose that $x_1 = y_1$, and that at $x_1$ the tangent vectors to $a$ and $c$ form a positive basis for $T_{x_1}(S)$. Let $\gamma = c|_{[0,s_1]} * (a|_{[0,t_1]})^{-1}$. There are two sub-cases.

**SUB-CASE 3a**

Assume that $\gamma$ is separating in $S \setminus P$. Let $R$ denote the subsurface of $S$ having oriented boundary $\gamma$. Since $a$ and $c$ are assumed to be intersecting efficiently, $R$ can not be a bigon containing no points of $P$. If $R$ contains some point $p \in P$ it is easy to connect $x_0$ to $p$ with an arc $b \subset R$ satisfying the desired properties.

If $R \cap P = \emptyset$, then there is an arc $\eta \subset R$, which starts at $x_0$, runs over a handle in $R$ and ends at $x_1$, and whose interior intersects neither $a$ nor $c$. Set $b = \eta * a|_{[t_1,1]}$. It is clear that $a \leq b \leq c$. Although we have not reduced the intersection number, $(a,b) = (a,c)$, we have reduced to the final sub-case:

**SUB-CASE 3b**

Assume that $\gamma$ is not separating. Let $R$ denote the connected component of $S \setminus a \setminus c$ having an oriented boundary component $\gamma$. Since $\gamma$ is not separating, $R$ must have at least one other
boundary component $\delta \neq \gamma$, which consists of a union of sub-arcs along $a$ and $b$ along with possibly $\partial S$.

We claim that for some $i > 0$, $a|_{[t_i, t_{i+1}]} \subset \delta$. If $\delta$ contains no such sub-arc of $a$ then it must then contain $a|_{[0, t_1]}$ as an oriented sub-arc. It follows that $\delta$ contains $c|_{[s_1, s_2]}$. Let $x_i = y_2$, then $\delta$ contains $c|_{[s_1, s_2]}$. Let $x_i = y_2$, then $\delta$ contains either $(a|_{[t_i-1, t_i]})^{-1}$ or $a|_{[t_i, t_{i+1}]}$, contradicting our assumption.

Now, we may connect $x_0$ to $x_{i+1}$ by an arc $\eta \subset R$ initially to the right of $a$ and left of $c$. Setting $b = \eta * a|_{[t_{i+1}, 1]}$ gives the desired arc.

We now prove Theorem 1.3:

**Theorem 1.3.** Let $K$ be braided about $(B, \pi)$, with $B$ connected. Let $S$ denote a page of the open book $(B, \pi)$. If $K$ is the closure of some $\phi \in \text{MCG}(S-P, \partial S)$ having $c(\phi) > 1$ then $\hat{\ell}(K) \neq 0$.

**Proof.** Via braid isotopy we may assume that $p_1 \in P$ lies in some collar neighborhood $\nu(\partial S)$ of the boundary. Let $e_0 \subset \nu(\partial S)$ be an embedded arc connecting $\partial S$ to $p_1 \in P$.

We let $C = \partial S$. By Lemma 2.7 we have that $c(\phi) > 1 \implies c(\tau_C^{-1} \circ \phi) > 0$. Lemma 2.8 tells us that $\tau_C^{-1} \circ \phi$ is right veering; in particular $\tau_C^{-1} \circ \phi$ takes the arc $e_0$ to the right.

Combining Lemmas 5.1 and 5.3 we see that there is a product of positive Dehn twists, and positive half twists, which take the arc $e_0$ to $\tau_C^{-1} \circ \phi(e_0)$; let $\psi$ denote this product.

Comultiplication (Theorem 2.3) implies that any composition of mapping classes having non-zero BRAID invariant will in turn have non-zero BRAID invariant, thus Proposition 4.1 gives $\hat{\ell}(\psi) \neq 0$.

The composition

$$\psi^{-1} \circ \tau_C^{-1} \circ \phi = \tau_C^{-1} \circ \psi^{-1} \circ \phi$$

fixes the arc $e_0$, which in turn implies that the braid $\psi^{-1} \circ \phi$ is precisely of the form braid union binding! By Theorem 1.1 we have that $\hat{\ell}(\psi^{-1} \circ \phi) \neq 0$. Finally, $\hat{\ell}(\psi) \neq 0$ and comultiplication imply that $\hat{\ell}(\phi) \neq 0$.

$\square$
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