A RIGIDITY RESULT FOR SOME PARABOLIC GERMS

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Abstract. The goal of this note is to prove a rigidity result for unicritical polynomials with parabolic cycles.

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1. INTRODUCTION

Let $f_c(z) = z^d + c$. We will denote the Julia and filled-in Julia sets of $f_c$ by $J(f_c)$ and $K(f_c)$ respectively. The degree $d$ multibrot set $M_d$ is defined as the set of all complex parameters $c$ for which $K(f_c)$ is connected.

A parameter $c$ of $M_d$ is called a parabolic parameter if $f_c$ has a periodic cycle with multiplier a root of unity. We will prove a rigidity principle concerning parabolic germs obtained as restrictions of suitable iterates of $f_c$ where $c$ is a parabolic parameter of $M_d$. Every parabolic parameter of $M_d$ is the root point of a hyperbolic component. For $i = 1, 2$, let $c_i$ be the root point of a hyperbolic component $H_i$ of period $n_i$ of $M_d$, $z_i$ be the characteristic parabolic point (i.e. the parabolic periodic point on the boundary of the critical value Fatou component) of $f_{c_i}$. It is worthwhile to note that under the above assumptions, $(f_{c_i}^{n_i})'(z_i) = 1$; i.e. the restriction of $f_{c_i}^{n_i}$ in a neighborhood of $z_i$ is a parabolic germ with multiplier 1. We show that:

**Theorem 1.1** (Parabolic Germs Determine Parabolic Parameters). *If there exist small neighborhoods $N_1$ and $N_2$ of $z_1$ and $z_2$ (in the dynamical planes of $c_1$ and $c_2$ respectively) such that $f_{c_1}^{n_1}|_{N_1}$ and $f_{c_2}^{n_2}|_{N_2}$ are conformally conjugate, then $f_{c_1}$ and $f_{c_2}$ are affinely conjugate.*

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By the classical theory of conformal conjugacy classes of parabolic germs \cite{Eca75, Vor81}, there is an infinite-dimensional family of conformally different parabolic germs. On the other hand, there is only a one-parameter family of unicritical complex polynomials of a given degree. Hence it seems extremely unlikely that two different unicritical polynomials (of the same degree) would have conformally conjugate parabolic germ restrictions (compare \cite[§3]{CEP15}). The above theorem confirms this suspicion by showing that if suitable iterates of two unicritical parabolic polynomials have conformally conjugate ‘tangent to identity’ parabolic germ restrictions, then the two polynomials are indeed affinely conjugate.

The paper is organized as follows. We will review some theory about parabolic germs in Section 2 and we will review the theory of parabolic-like maps (a parabolic analogue of polynomial-like maps introduced by the first author in \cite{Lom15}) in Section 3. We will prove Theorem 1.1 in Section 4 by combining a rigidity result about parabolic-like maps (see Lemma 4.1) and a local to global principle that concludes a global statement about two polynomials from a local conformal conjugacy information between them (see Lemma 4.2).

2. Preliminaries

Recall that for a parabolic germ, the attracting and repelling Fatou coordinates that conjugate the dynamics of the germ to translation by +1, are defined in the various attracting and repelling petals respectively. For a general parabolic germ, these various Fatou coordinates do not agree on their common domains of definition. Loosely speaking, horn maps are objects that record the difference between a pair of (adjacent) attracting and repelling Fatou coordinates. Horn maps completely determine the conformal conjugacy classes of parabolic germs, and conversely, if two parabolic germs are conformally conjugate, then they have the same horn maps \cite{Eca75, Vor81}. In our setting, where the parabolic germs are restrictions of globally defined polynomials, the horn maps have a natural maximal domain of definition (the extension is obtained by iterating the dynamics). One of the crucial properties of these extended horn maps that we will have need for is that they are finite-type maps whose critical values are completely determined by the conformal positions of the critical points of the polynomial \cite[Proposition 4]{BE02, Eps93}.

We say that a parabolic parameter $c$ is primitive (or simple) if each parabolic periodic point of $f_c$ has a single attracting petal. Equivalently, a primitive parabolic parameter $c$ lies on the boundary of a unique hyperbolic component of $\mathcal{M}_d$, and is the root thereof. In this case, the period of the corresponding hyperbolic component (on whose boundary $c$ lies) is equal to the period of the parabolic cycle of $f_c$. 
A parabolic parameter $c$ of $\mathcal{M}_d$ is called satellite if each parabolic periodic point of $f_c$ has at least two attracting petals. Equivalently, a satellite parabolic parameter $c$ is the (unique) common boundary point of two hyperbolic components $H$ and $H'$ of $\mathcal{M}_d$, one of which, say $H'$, has $c$ as its root. In this case, the period of $H'$ is strictly greater than the period of the parabolic cycle of $f_c$ (compare [EMS16]).

Since the proof of Theorem 1.1 in the case when both $c_1$ and $c_2$ are primitive parameters is essentially present in [EMS16 Theorem 1.4], we will be concerned here with the satellite case. Let the period of the parabolic cycle of $f_{c_i}$ be $k_i$ (so $c_i$ sits on the boundary of a hyperbolic component of period $k_i$ and a hyperbolic component of period $n_i$). Set $q_i := n_i/k_i$. Recall that $f_{c_i}^{q_i}(z) = z + a_i(z - z_i)^{q_i} + O((z - z_i)^{q_i+2})$ for some $a_i \in \mathbb{C}$, as the Taylor expansion of $f_{c_i}^{q_i}$ near $z_i$. Furthermore, $q_i$ is the number of attracting petals at the parabolic point $z_i$, and these petals are permuted by $f_{c_i}^{k_i}$. If the parabolic germs of $f_{c_i}^{q_i}$ (for $i = 1, 2$) are conformally conjugate, then we have $q_1 = q_2$ (in fact, $q_i$ is a topological conjugacy invariant of the parabolic germ). Therefore, the two polynomials $f_{c_1}$ and $f_{c_2}$ have the same number of petals at each parabolic periodic point.

3. PARABOLIC-LIKE MAPS

The theory of parabolic-like maps extends the theory of polynomial-like maps to objects with a parabolic external class. For any polynomial map $P$ on the Riemann sphere $\hat{\mathbb{C}}$, infinity is a superattracting fixed point, and the filled Julia set $K_P$ is the complement of the basin of attraction of infinity $\mathcal{A}(\infty)$, that is $K_P = \hat{\mathbb{C}} \setminus \mathcal{A}(\infty)$. So the preimage of a topological disk $U$ (nice enough, for example bounded by an equipotential) containing $K_P$ is a topological disk $U'$ compactly contained in $U$, and $P_{|U'} : U' \to U$ is a proper holomorphic map of degree $d = \deg(P)$. The triple $(P, U', U)$ is a (trivial) example of a polynomial-like map. Formally, a (deg $d$) polynomial-like map is a triple $(f, U', U)$ where $U'$ and $U$ are topological disks, $U' \subset \subset U$ and $f : U' \to U$ is a (deg $d$) proper holomorphic map [DH85]. The filled Julia set $K_f$ of a polynomial-like map is the set of points which never leave $U'$ under iteration (for a polynomial $P$, this is just $K_P$). With any degree $d$ polynomial-like map, one can associate a degree $d$ covering of the unit circle $h_f : \mathbb{S}^1 \to \mathbb{S}^1$ which encodes the dynamics of the polynomial-like map outside its filled Julia set. The map $h_f$ is called the external map of the polynomial-like map $f$. The external map of a polynomial-like map is strictly expanding, with all periodic points repelling, and it is defined up to real-analytic diffeomorphisms of the circle. In this way a polynomial-like map can be considered as a union of two different dynamical systems: the filled Julia set $K_P$ and the external map $h_f$. By replacing the external map of a degree $d$ polynomial-like map with the map $z \to z^d$ (which is an external map for a degree $d$ polynomial), Douady and Hubbard proved that every degree $d$ polynomial-like map is hybrid equivalent to a polynomial of the
same degree (where a hybrid equivalence is a quasiconformal conjugacy \( \varphi \) with \( \partial \varphi = 0 \) on \( K_f \)), and that this polynomial is unique if \( K_f \) is connected.

A parabolic-like map is an object similar to a polynomial-like map, in the sense that it can be considered as the union of two different dynamical systems: the filled Julia set and the external map [Lom15]. However, the external map of a parabolic-like map contains a parabolic fixed point, which complicates the setting considerably.

![Figure 1](image.png)

**Figure 1.** For a parabolic-like map \((f, U', U, \gamma)\) the arc \( \gamma \) divides \( U' \) and \( U \) into \( \Omega', \Delta' \) and \( \Omega, \Delta \) respectively. These sets are such that \( \Omega' \) is compactly contained in \( U \), \( \Omega' \subset \Omega \), \( f : \Delta' \to \Delta \) is an isomorphism and \( \Delta' \) contains at least one attracting fixed petal of the parabolic fixed point.

**Definition. (Parabolic-like maps)** A parabolic-like map of degree \( d \geq 2 \) is a 4-tuple \((f, U', U, \gamma)\) where

- \( U' \) and \( U \) are open subsets of \( \mathbb{C} \), with \( U' \), \( U \) and \( U \cup U' \) isomorphic to a disc, and \( U' \) not contained in \( U \),
- \( f : U' \to U \) is a proper holomorphic map of degree \( d \geq 2 \) with a parabolic fixed point at \( z = z_0 \) of multiplier 1,
- \( \gamma : [-1, 1] \to \overline{U} \) is an arc with \( \gamma(0) = z_0 \), forward invariant under \( f \), \( C^1 \) on \([-1, 0]\) and on \([0, 1]\), and such that

\[
f(\gamma(t)) = \gamma(dt), \quad \forall -\frac{1}{d} \leq t \leq \frac{1}{d};
\]

\[
\gamma([\frac{1}{d}, 1) \cup (-1, -\frac{1}{d}]) \subseteq U \setminus U', \quad \gamma(\pm 1) \in \partial U.
\]
It resides in repelling petal(s) of \( z_0 \) and it divides \( U' \) and \( U \) into \( \Omega', \Delta' \) and \( \Omega, \Delta \) respectively, such that \( \Omega' \subset U \) (and \( \Omega \subset \Omega' \)), \( f : \Delta' \to \Delta \) is an isomorphism (see Figure 1), and \( \Delta' \) contains at least one attracting fixed petal of \( z_0 \). We call the arc \( \gamma \) a dividing arc.

The filled Julia set \( K_f \) of a parabolic-like map \( (f,U',U,\gamma) \) is the set of points which never leave \( \Omega' \cup \{ z_0 \} \) under iteration. The model family in degree 2 is given by the family of quadratic rational maps with a parabolic fixed point of multiplier 1 at infinity (normalized by having critical points at degree 2 connected. For more detailed studies on parabolic-like maps, consult \([\text{Lom15}, \text{Proposition 4.2}]\). By replacing the external map of a degree 2 parabolic-like map with the map \( h_2(z) = \frac{z^2+1/3}{z^2+3+1} \) is an external map for every \( P_A, A \in \mathbb{C} \) \([\text{Lom15}]\) for a dynamical description, \([\text{Lom14a}]\) for a parameter space (of degree 2 analytic families of parabolic-like maps) description, and \([\text{Lom14b}]\) for an easy discussion on the results contained in the previous two articles.

4. Proof of the Theorem

The root of every satellite hyperbolic component of the Multibrot set \( M_d \) admits a parabolic-like restriction (see \([\text{Lom15}] \) §3.1, Example 3) for details of the construction in the case \( d = 2 \), the case \( d > 2 \) being similar).

**Lemma 4.1** (Rigidity of Parabolic-like Mappings). Let \( c_1 \) and \( c_2 \) be the root points of two satellite hyperbolic components \( H_1 \) and \( H_2 \) (of period \( n_1 \) and \( n_2 \) respectively) of the Multibrot set \( M_d \). If the parabolic-like mappings defined by the restrictions of \( f_{c_1}^{n_1} \) and \( f_{c_2}^{n_2} \) (around their characteristic Fatou components) are conformally conjugate, then \( c_1 = c_2 \) up to affine conjugacy.

**Proof.** Let \( c_i \) be the root of a satellite component of period \( n_i \) (attached at \( c_1 \) to an hyperbolic component of period \( k_i \)). Then \( q_i = n_i/k_i \) is the number of attracting petals at the parabolic point \( z_i \) of \( f_{c_i} \). Let us start by noticing that if the parabolic-like mappings defined by the restrictions of \( f_{c_1}^{n_1} \) and \( f_{c_2}^{n_2} \) are conformally conjugate, then the parabolic germs of \( f_{c_i}^{n_i} \) (for \( i = 1, 2 \)) are conformally conjugate, and hence \( q_1 = q_2 = q \).

We label the Fatou components of \( f_{c_i} \) touching at the characteristic parabolic point \( z_i \) counter-clockwise such that \( U_i^k \) is the Fatou component of \( f_{c_i} \) containing the critical value \( c_i \). Since \( c_i \) is the root of a satellite component attached to another hyperbolic component of period \( k_i \), the polynomial \( f_{c_i}^{k_i} \) has a polynomial-like restriction \( (h_i, V_i', V_i) \) that is hybrid equivalent to...
some (degree $d$) $\frac{d}{q}$-rabbit (basilica if $q = 2$) parameter on the boundary of the principal hyperbolic component of $\mathcal{M}_d$ (more precisely, $f^{c_{2k}}$ has a polynomial-like restriction $h_1$ that is hybrid equivalent to some polynomial $f_{c_1}$ with a fixed point of multiplier $e^{2\pi i q/d}$).

Let $\eta$ be a conformal conjugacy between the parabolic-like restrictions of $f^{c_{n_1}}$ and $f^{c_{n_2}}$ in neighborhoods of $U_1$ and $U_2$ (respectively). A priori, $\eta$ is defined only in a neighborhood $W$ of $U_1$. We will show that using the dynamics, $\eta$ can be extended as a conformal conjugacy from a neighborhood of $K(h_1)$ to a neighborhood of $K(h_2)$.

Choose an neighborhood $W_0$ of $z_1$ with $W_0 \subset W$ and such that $W_0$ does not contain any critical point of $f^{c_{n_1}}$. Since $W_0$ intersects the Julia set of $f_{c_1}$, we have that $\bigcup_{s=1}^{\infty} f^{s\circ\eta_1}(W_0) \supset K(h_1)$. Now fix $s \in \mathbb{N}$. Since the domain of $\eta$ contains $W_0$, we can use the functional equation $\eta \circ f^{c_{n_1}} = f^{c_{n_2}} \circ \eta$ to push forward $\eta$ to a holomorphic conjugacy $\eta_s : f^{c_{n_1}}(W_0) \to f^{c_{n_2}}(\eta(W_0))$ between $f^{c_{n_1}}$ and $f^{c_{n_2}}$ (possibly after shrinking $W_0$ to ensure that $f^{c_{n_1}}$ has no critical point in $W_0$). Since $W_0 \cap U_1 = \emptyset$, $f^{c_{n_1}}(U_1) = U_1$, and $\eta$ is already a conjugacy on $U_1$, it follows from the construction that $\eta_s \equiv \eta$ on the non-empty open set $f^{c_{n_1}}(W_0) \cap U_1$. This proves that $\eta_s$ extends the map $\eta$. Since we can perform this operation for each $s \in \mathbb{N}$, uniqueness of analytic continuation yields a holomorphic map $\eta$ defined on a neighborhood of $K(h_1)$ such that it conjugates $f^{c_{n_1}}$ to $f^{c_{n_2}}$.

Therefore, $\eta$ is a conformal conjugacy between the polynomial-like maps $h_1^{c_q}$ and $h_2^{c_q}$. By [LM16] Corollary 10.2, we have $c_1 = c_2$ up to affine conjugacy.

We now proceed to promote the germ conjugacy between $f^{c_{n_1}}$ and $f^{c_{n_2}}$ to a conformal conjugacy between two suitable parabolic-like mappings. We label the Fatou components of $f_{c_1}$ touching the parabolic point $z_i$ counterclockwise such that $U_i$ is the Fatou component of $f_{c_1}$ containing the critical value $c_i$. In order to investigate the consequences of such a conformal conjugacy between two parabolic germ restrictions, we will need to use the concept of extended horn maps, which are the natural maximal extensions of horn maps. A comprehensive account of horn maps, and their mapping properties can be found in [BE02, Shi00], so we do not include the definition here.

**Lemma 4.2.** Let $c_1$ and $c_2$ be the root points of two satellite hyperbolic components $H_1$ and $H_2$ (of period $n_1$ and $n_2$ respectively) of the Multibrot set $\mathcal{M}_d$, and $z_1$ and $z_2$ be the characteristic parabolic points of $f_{c_1}$ and $f_{c_2}$ (respectively). Then the following are equivalent.

- The parabolic-like mappings defined by the restrictions of $f^{c_{n_1}}$ and $f^{c_{n_2}}$ (around $U_1$ and $U_2$ respectively) are conformally conjugate.
The (tangent-to-identity) parabolic germs given by the restrictions of \( f_{c_1}^{on_1} \) and \( f_{c_2}^{on_2} \) (around \( z_1 \) and \( z_2 \) respectively) are conformally conjugate.

Proof. Conformal conjugacy of the parabolic-like maps clearly implies conformal conjugacy of the corresponding germs. So we only need to show that when \( g_1 := f_{c_1}^{on_1}|_{N_1} \) and \( g_2 := f_{c_2}^{on_2}|_{N_2} \) are conformally conjugate by some local biholomorphism \( \varphi : N_1 \to N_2 \) (where \( N_i \) is a small neighborhood of \( z_i \)), the parabolic-like maps \( f_{c_1}^{on_1} \) and \( f_{c_2}^{on_2} \) (around \( U_1^1 \) and \( U_2^1 \) respectively) are also conformally conjugate. We now proceed to prove this.

Note that \( \varphi_1 \) must map an attracting petal \( P_{c_1}^{att,1} \subset N_1 \cap U_1^1 \) to some attracting petal \( P_{c_2}^{att,k} \subset N_2 \cap U_2^k \), and so \( \varphi := f_{c_2}^{on_2} \circ \varphi_1 \) is a conformal conjugacy between \( g_1 \) and \( g_2 \) such that it maps \( P_{c_1}^{att,1} \) to a petal \( P_{c_2}^{att,1} \) (this is no more than a matter of convenience).

For \( k \in \mathbb{Z}/q \mathbb{Z} \), if \( \psi_{c_2}^{att,k} \) is an extended attracting Fatou coordinate for \( f_{c_2}^{on_2} \) in \( U_2^k \), then there exist extended attracting Fatou coordinate \( \psi_{c_1}^{att,k} \) for \( f_{c_1}^{on_1} \) in \( U_1^k \) such that \( \psi_{c_2}^{att,k} = \psi_{c_1}^{att,k} \circ \varphi \) in their common domain of definitions. Similarly, we can transport the extended repelling Fatou coordinates of \( f_{c_2}^{on_2} \) at \( z_2 \) by \( \varphi \) to define extended repelling Fatou coordinates of \( f_{c_1}^{on_1} \) at \( z_1 \). It follows that with these choices of Fatou coordinates the extended horn maps \( h_{c_1,k}^{\pm} \) of \( f_{c_1}^{on_1} \) at \( z_i \) coincide, this is \( h_{c_1,k}^{\pm} = h_{c_1,k}^{\pm} \).

\( ^1 \)Here is an alternative route to extend \( \eta \) to the entire Fatou component. We can choose Riemann maps \( \varphi_{c_1} : U_1^1 \to \mathbb{D} \) with \( \varphi_{c_1}(c_1) = 0 \) such that \( \varphi_{c_1} \) conjugates \( f_{c_1}^{on_1}|_{U_1^1} \) to the Blaschke product \( B(z) = \frac{z^n}{\prod_{i=1}^{n} (z - c_i)} \). An easy computation in Fatou coordinates now shows that \( \varphi_{c_2}^{-1} \circ \varphi_{c_1} \) extends the local conjugacy \( \eta \) to the entire immediate basin \( U_1^1 \) such that it conjugates \( f_{c_1}^{on_1} \) on \( U_1^1 \) to \( f_{c_2}^{on_2} \) on \( U_2^1 \).
conjugacy \( \eta \) extends as a homeomorphism from \( \partial U_1 \) onto \( \partial U_2 \). Note also that by definition, \( \eta = g_2^{c(-r)} \circ \varphi \) in their common domain of definition. Therefore, \( \eta \) is defined in a neighborhood \( V \) of the point \( z_1 \), and continues to be a conjugacy between the germs \( g_1 \) and \( g_2 \).

We can now extend this conformal conjugacy to a conformal conjugacy \( \eta \) between a neighborhood of \( U_1 \) and a neighborhood of \( U_2 \) following the proof of Lemma 4.3 in [IM16]. We include the details for the reader. By Montel’s theorem, \( \bigcup_{s \in \mathbb{N}} f_{c_1}^s (V \cap \partial U_1) = \partial U_1 \), and since none of the \( f_{c_1}^s \) has a critical point on \( \partial U_1 \), we can extend \( \eta \) in a neighborhood of each point of \( \partial U_1 \) by using the functional equation \( \eta \circ f_{c_1}^s = f_{c_2}^s \circ \eta \). Uniqueness of analytic continuation yields an analytic extension of \( \eta \) in a neighborhood of \( U_1 \). Moreover, the extension is a proper degree 1 holomorphic map, and is a conformal conjugacy between \( f_{c_1}^s \) and \( f_{c_2}^s \). This shows that the parabolic-like mappings defined by \( f_{c_1}^s \) and \( f_{c_2}^s \) in neighborhoods of \( U_1 \) and \( U_2 \) (respectively) are conformally conjugate.

Proof of Theorem 1.1. Since the multiplicity of a parabolic germ is a topological invariant, it follows that either both \( c_1 \) and \( c_2 \) are primitive, or both of them are satellite. In the former case, the period of \( H_1 \) is equal to the period of the parabolic cycle of \( f_{c_1} \). Therefore, the conclusion follows from [IM16, Theorem 1.4]. On the other hand, when both the \( c_i \) are satellite parabolic parameters, the result follows from Lemma 4.1 and Lemma 4.2. □

References

[BE02] Xavier Buff and Adam L. Epstein. A parabolic Pommerenke-Levin-Yoccoz inequality. Fund. Math., 172:249–289, 2002.

[CEP15] Arnaud Chéritat, Adam Lawrence Epstein, and Carsten Lunde Petersen. Perspectives on parabolic points in holomorphic dynamics. [http://www.birs.ca/workshops/2015/15w5082/report15w5082.pdf] 2015. Conference report.

[DH85] Adrien Douady and John H. Hubbard. On the dynamics of polynomial-like mappings. Ann. Sci. Ec. Norm. Sup., 18:287–343, 1985.

[Eca75] Jean Ecalle. Théorie itérative : introduction à la théorie des invariants holomorphes. J. Math. Pures Appl. (9), 54:183–25, 1975.

[EMS16] D. Eberlein, S. Mukherjee, and D. Schleicher. Rational parameter rays of the multibrot sets. In Dynamical Systems, Number Theory and Applications, chapter 3, pages 49–84. World Scientific, 2016. [http://dx.doi.org/10.1142/9789814699877_0003]

[Eps93] Adam Epstein. Towers of Finite Type Complex Analytic Maps. PhD thesis, The City University of New York, 1993.

[IM16] Hiroyuki Inou and Sabyasachi Mukherjee. Discontinuity of straightening in antiholomorphic dynamics. [https://arxiv.org/abs/1605.08061] 2016.

[Lom14a] Luna Lomonaco. Parameter space for families of parabolic-like mappings. Advances in Mathematics, 261C:200–219, 2014.

[Lom14b] Luna Lomonaco. Results about parabolic-like mappings. Analysis in Theory and Applications, 30:120–129, 2014.
[Lom15] Luna Lomonaco. Parabolic-like mappings. *Ergodic Theory and Dynamical Systems*, 35:2171–2197, 2015.

[Shi00] Mitsuhiro Shishikura. Bifurcation of parabolic fixed points. In *The Mandelbrot Set, Theme and Variations*, London Mathematical Society Lecture Note Series (No. 274), pages 325–364. Cambridge University Press, 2000.

[Vor81] S. M. Voronin. Analytic classification of germs of conformal mappings $(\mathbb{C},0) \to (\mathbb{C},0)$. *Funktsional. Anal. i Prilozhen*, 15:1–17, 1981.

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