CLASSIFICATION OF LINKAGE SYSTEMS

RAFAEL STEKOLSHCHIK

Abstract. In 1972, R. Carter introduced admissible diagrams to classify conjugacy classes in a finite Weyl group \( W \). For any two non-orthogonal roots \( \alpha \) and \( \beta \) corresponding to vertices of admissible diagram, we draw the dotted (resp. solid) edge \( \{\alpha, \beta\} \) if \( \langle \alpha, \beta \rangle > 0 \) (resp. \( \langle \alpha, \beta \rangle < 0 \)). The diagram with properties of admissible diagram and possibly containing dotted edges are said to be Carter diagrams. For any Carter diagram \( \Gamma \), we introduce the partial Cartan matrix \( B_\Gamma \), which is analogous to the Cartan matrix associated with a Dynkin diagram. A linkage diagram is obtained from \( \Gamma \) by adding an extra root \( \gamma \), together with its bonds, so that the resulting subset of roots is linearly independent. With every linkage diagram we associate the linkage label vector \( \gamma^\nabla \), similar to the “numerical labels” introduced by Dynkin for the study of irreducible linear representations of the semisimple Lie algebras. The linkage diagrams connected under the action of dual partial Weyl group \( W_\nabla \) (associated with \( B_\Gamma \)) constitute the set \( \mathcal{L}(\Gamma) \) dubbed the linkage system. For any simply-laced Carter diagram \( \Gamma \), the linkage system \( \mathcal{L}(\Gamma) \) is explicitly constructed. To obtain linkage diagrams \( \theta^\nabla \in \mathcal{L}(\Gamma) \), we use an easily verifiable criterion: \( B_\nabla ^\nabla (\theta^\nabla ) < 2 \), where \( B_\nabla ^\nabla \) is the quadratic form associated with \( B_\nabla -1 \). A Dynkin diagram \( \Gamma \) such that rank(\( \Gamma \)) = rank(\( \Gamma \)) + 1 and any \( \Gamma \)-associated root subset \( S \) lies in the root system \( \Phi(\Gamma) \), will be called the Dynkin extension of the Carter diagram \( \Gamma \) and denoted by \( \Gamma <_D \Gamma \). The linkage system \( \mathcal{L}(\Gamma) \) is the union of \( \Gamma \)-components \( \mathcal{L}_\Gamma(\Gamma) \) taken for all Dynkin extensions of \( \Gamma <_D \Gamma \). The subset \( \Phi(S) \) of roots of \( \Phi(\Gamma) \) linearly dependent on roots of \( S \) is said to be a partial root system. The size of \( \mathcal{L}_\Gamma(\Gamma) \) is estimated as follows: \( |\mathcal{L}_\Gamma(\Gamma)| \leq |\Phi(\Gamma')| - |\Phi(S)| \).

Carter diagrams of the same type and the same index are said to be covalent. For any pair \( \{\Gamma, \tilde{\Gamma}\} \) of covalent Carter diagrams, where \( \Gamma \) is the Dynkin diagram, we explicitly construct the invertible linear map \( M : \mathcal{P} \to \mathcal{R} \), where \( \mathcal{R} \) (resp. \( \mathcal{P} \)) is the root system (resp. partial root system) corresponding to \( \Gamma \) (resp. \( \tilde{\Gamma} \)). This allows us to derive some results for the Carter diagrams from analogous results for the Dynkin diagrams. In particular, we have \( |\mathcal{L}(\tilde{\Gamma})| = |\mathcal{L}(\Gamma)| \).

For the simple-laced Dynkin diagram \( \Gamma \), every \( A, D, E \)-component of the linkage system \( \mathcal{L}(\Gamma) \) coincides with a weight system of one of fundamental representations of the simple Lie algebra \( \mathfrak{g} \) associated with \( \Gamma \).

The 8-cell “spindle-like” subsystems in \( \mathcal{L}(\Gamma) \), called loctets, play the essential role in describing the linkage systems.

Contents
1. Preview of the basic notions
1.1. Admissible and Carter diagrams, solid and dotted edges
1.1.1. The primary root system
1.1.2. The admissible diagram
1.1.3. Solid and dotted edges
1.1.4. The \( \Gamma \)-associated root subset
1.1.5. The Carter diagrams
1.1.6. Three classes of Carter diagrams
1.2. Linkage diagrams and linkage systems
1.2.1. The partial Cartan matrix \( B_\Gamma \)
1.2.2. The linkage diagrams
1.2.3. The linkage systems
1.3. Dynkin extensions, partial root systems and linkage system components
1.3.1. \( A, D, E \)-types of Dynkin and Carter diagram
1.3.2. Partial root system
1.3.3. Covalent Carter diagrams
1.3.4. Dynkin extensions
1.3.5. Root stratum
1.3.6. Linkage system components

2. **Introduction and main results**
2.1. The partial Cartan matrix, inverse quadratic form and linkages
2.1.1. Linkages and linkage diagrams
2.1.2. The projection of the linkage root
2.2. The linkage systems and loctets
2.2.1. The starlike numbering of vertices
2.2.2. Loctets
2.3. The Carter diagrams and connection diagrams
2.3.1. The Dynkin diagrams
2.3.2. The connection diagrams
2.4. The main results
2.4.1. Partial Weyl group and dual partial Weyl group
2.4.2. Bijection of covalent root systems
2.4.3. Whether a vector is a linkage root?
2.4.4. Three loctet types
2.4.5. The structure of loctets
2.4.6. On tables and diagrams
2.4.7. Linkage systems for Dynkin diagrams and weight systems
2.4.8. The structure and sizes of ADE components of linkage systems
2.4.9. The invariant of the linkage system component
2.5. On connection with the Carter theorem

3. **Theorem on covalent Carter diagrams**
3.1. The size of linkage system \( L(\Gamma) \)
3.2. Transition between a partial root system \( P \) and a root system \( R \)
3.3. Relation of linkage diagrams lying in \( L(\Gamma) \) and in \( L(\tilde{\Gamma}) \)
3.4. Linkage system components

4. **Theorem on a linkage root**
4.1. Linear dependence and maximal roots
4.2. The inverse quadratic form \( B_\Gamma \)
4.2.1. Linkage roots
4.2.2. The projection of the linkage root
4.3. The criterion of a linkage root

5. **Loctets**
5.1. The dual partial Weyl group associated with a root subset
5.1.1. Groups \( W_S \) and \( W_{\tilde{S}} \)
5.2. On 4-cycles and 4-cycles with a diagonal
5.2.1. How many endpoints may a linkage diagram have?
5.2.2. The diagonal in a square
5.3. Loctets and unicolored linkage diagrams

6. **Enumeration of linkage diagrams, loctets and linkage systems**
6.1. Calculation of linkage diagrams \( \gamma^N(8) \)
6.1.1. Calculation example for diagram \( E_6(a_1) \)
6.2. Calculation of the &beta;-unicolored linkage diagrams
6.2.1. &beta;-unicolored linkage diagrams in \( L(E_6(a_1)) \) and \( L(E_6(a_2)) \)
6.2.2. \( \beta \)-unicolored linkage diagrams in \( L(E_7(a_1)) \)
6.3. Linkage systems for simply-laced Dynkin diagrams
6.3.1. Asymmetric relations between \( A_, D_, E_-types of Dynkin diagrams

7. **Theorem on coincidence of linkage and weight systems for Dynkin diagrams**
7.1. Dominant weights and Dynkin labels
7.2. Relationship between linkage system and weight system for Dynkin diagrams

8. **D-type linkage systems**
   8.1. The linkage systems \( \mathcal{L}(D_l) \) and \( \mathcal{L}(D_l(a_k)) \) for \( l \geq 8 \)
   8.2. The linkage systems \( \mathcal{L}(D_l) \) and \( \mathcal{L}(D_l(a_k)) \) for \( l = 5, 6, 7 \)
   8.2.1. The D- and E-components \( \mathcal{L}_{D_k}(D_5), \mathcal{L}_{E_k}(D_5), \mathcal{L}_{D_k}(D_5(a_1)) \) and \( \mathcal{L}_{E_k}(D_5(a_1)) \)
   8.2.2. The D- and E-components \( \mathcal{L}_{D_r}(D_6), \mathcal{L}_{D_r}(D_6(a_k)), \mathcal{L}_{E_r}(D_6) \) and \( \mathcal{L}_{E_r}(D_6(a_k)) \)
   8.2.3. The D- and E-components \( \mathcal{L}_{D_k}(D_7), \mathcal{L}_{D_k}(D_7(a_k)), \mathcal{L}_{E_k}(D_7), \) and \( \mathcal{L}_{E_k}(D_7(a_k)) \)

9. **E-type linkage systems**
   9.1. The linkage systems \( \mathcal{L}(E_l) \) and \( \mathcal{L}(E_l(a_k)) \) for \( l = 6, 7 \)
   9.1.1. The linkage systems \( \mathcal{L}(E_6), \mathcal{L}(E_6(a_k)) \) for \( k = 1, 2 \)
   9.1.2. The linkage systems \( \mathcal{L}(E_7), \mathcal{L}(E_7(a_k)) \) for \( k = 1, 2, 3, 4 \)

10. **A-type linkage systems**
    10.1. The linkage systems \( \mathcal{L}(A_l) \) for \( l = 5, 6, 7 \)
    10.1.1. The linkage systems \( \mathcal{L}(A_3) \)
    10.1.2. The linkage systems \( \mathcal{L}(A_4) \)
    10.1.3. The linkage systems \( \mathcal{L}(A_5) \)
    10.1.4. The linkage systems \( \mathcal{L}(A_6) \)
    10.1.5. The linkage systems \( \mathcal{L}(A_7) \)
    10.2. The linkage systems \( \mathcal{L}(A_l) \) for \( l \geq 8 \)

Appendix A. **Some properties of Carter and connection diagrams**
   A.1. Similarity of Carter diagrams
   A.2. The ratio of lengths of roots
   A.3. Basic lemmas
   A.4. The ordered tree of Carter diagrams
   A.5. \( \Gamma \)-associated root subsets and conjugacy classes
   A.5.1. Two \( \Gamma \)-associated conjugacy classes
   A.5.2. Two non-conjugate \( \Gamma \)-associated sets
   A.5.3. Example of equivalent 4-cycles
   A.6. The diagonal elements of \( B_{A_l}^{-1} \) for \( A_l, D_l, D_l(a_k) \)
   A.7. Simply extendable Carter diagrams
   A.7.1. Simple extensions for the Carter diagram \( A_l \)
   A.8. The partial Cartan matrix \( B_{\Gamma}^{-1} \) and the inverse matrix \( B_{\Gamma}^{-1} \)

Appendix B. **Linkage diagrams \( \gamma^\vee(8) \) and inequality \( \mathcal{B}^\vee_\Gamma(\gamma^\vee) < 2 \)**
   B.1. The linkage diagrams \( \gamma^\vee_1(8) \) and solutions of inequality \( \mathcal{B}^\vee_\Gamma(\gamma^\vee_1(8)) < 2 \)
   B.2. \( \beta \)-unicolored linkage diagrams. Solutions of inequality \( \mathcal{B}^\vee_\Gamma(\gamma^\vee) < 2 \)
   B.3. Linkage diagrams \( \gamma^\vee_1(6) \) per loctets and components

Appendix C. **The linkage systems for the Carter diagrams of rank \( l < 8 \)**
   C.1. The linkage systems \( \mathcal{L}(D_4(a_1)), \mathcal{L}(D_5(a_1)), \mathcal{L}(D_6(a_1)), \mathcal{L}(D_6(a_2)) \)
   C.2. The linkage systems \( \mathcal{L}(E_6(a_1)), \mathcal{L}(E_6), \mathcal{L}(E_7(a_1)), \mathcal{L}(D_5), \mathcal{L}(D_6) \)
   C.3. The linkage systems \( \mathcal{L}(D_7(a_1)), \mathcal{L}(D_7(a_2)), \mathcal{L}(D_7) \)
   C.4. The linkage systems \( \mathcal{L}(D_4) \)

List of Figures
References
Index
1. Preview of the basic notions

1.1. Admissible and Carter diagrams, solid and dotted edges.

1.1.1. The primary root system. We consider a root system $\Phi$, the finite Weyl group $W$ acting on $\Phi$, the diagram Dynkin $\Gamma_D$ associated with $\Phi$ and the corresponding Cartan matrix $B$. The quadratic form $B$ is associated with the Cartan matrix $B$ and $B$ determines the inner product $(\cdot, \cdot)$ on the linear space $V$ spanned by simple roots of $\Phi$. Inside the root system $\Phi$ we will consider different root subsets and root subsystems, so we call $\Phi$ the primary root system. There exist root subsets which can be embedded into different primary root systems. For example, $\Phi(A_7) \subset \Phi(A_8), \Phi(D_8), \Phi(E_8)$. In this paper, besides the quadruple of objects $\{\Phi, W, \Gamma_D, B\}$ we will consider also a certain its generalization, the quadruple $\{P, W^\vee, \Gamma, B^T\}$, where $P$ (resp. $W^\vee$, resp. $B^T$) is the partial root system, (resp. partial Weyl group, partial Cartan matrix) and $\Gamma$ is the Carter diagram. These notions will be explained shortly. The generalized quadruple is our object of interest. And all this is studied in the area which is the primary root system $\Phi$.

1.1.2. The admissible diagram. In 1972, R. Carter introduced admissible diagrams to classify conjugacy classes in a finite Weyl group $W$. These diagrams are also used to characterize elements of the Weyl group, see definition in [L1]. Each element $w \in W$ can be expressed in the form

$$w = s_{\tau_1} s_{\tau_2} \ldots s_{\tau_k}, \text{ where } \tau_i \in \Phi,$$

and $s_{\tau} \in W$ are reflections corresponding to not necessarily simple roots $\tau_i \in \Phi$. Carter proved that $k$ in the decomposition $[\text{1.1}]$ is the smallest if and only if the subset of roots $\{\tau_1, \tau_2, \ldots, \tau_k\}$ is linearly independent; such a decomposition is said to be reduced. We denote by $l_C(w)$ the smallest value $l$ in any expression like $[\text{1.1}]$, we have $l_C(s_{\tau_1} s_{\tau_2} \ldots s_{\tau_l}) = l$. The set $S = \{\tau_1, \ldots, \tau_l\}$ consists of linearly independent and not necessarily simple roots, see [Ca72 Lemma 3]. We always have $l_C(w) \leq l(w)$, where $l(w)$ is the smallest value $k$ in any expression like $[\text{1.1}]$ such that all roots $\alpha_i$ are simple.

**Lemma 1.1.** [Ca72 Lemma 3] Let $\alpha_1, \alpha_2, \ldots, \alpha_k \in \Phi$. Then $s_{\alpha_1} s_{\alpha_2} \ldots s_{\alpha_k}$ is reduced if and only if $\alpha_1, \alpha_2, \ldots, \alpha_k$ are linearly independent. □

We associate with the element $w \in W$ (and with conjugacy class $C(w)$ containing $w$) the diagram $\Gamma$ with nodes corresponding to roots in $[\text{1.1}]$, nodes $\alpha$ and $\beta$ are joined by $n_{\alpha\beta} \cdot n_{\beta\alpha}$ bonds, where

$$n_{\alpha\beta} = 2\frac{(\alpha, \beta)}{\langle \alpha, \alpha \rangle}, \quad n_{\beta\alpha} = 2\frac{(\beta, \alpha)}{\langle \beta, \beta \rangle}. \quad (\text{1.2})$$

Number $n_{\alpha\beta}$ and $n_{\beta\alpha}$ are integer, and $n_{\alpha\beta} \cdot n_{\beta\alpha} \in \{0, 1, 2, 3\}$, see [Bo02] Ch.6, §1, $\langle n \rangle 3$.

A diagram $\Gamma$ is said to be admissible, see [Ca72 p. 7], if

(a) The nodes of $\Gamma$ correspond to a set of linearly independent roots in $\Phi$.

(b) If a subdiagram of $\Gamma$ is a cycle, then it contains an even number of nodes. \quad (\text{1.3})

Surprisingly, the admissible diagrams can contain cycles though the extended Dynkin diagram $\tilde{A}_l$ cannot be a part of any admissible diagram, see Lemma \ref{L1}. It turned out that the cycles in admissible diagrams essentially differ from the cycle $\tilde{A}_l$. Namely, in such a cycle, there exist necessarily two pairs of roots: A pair with a positive inner product and a pair with a negative inner product. This does not happen for $\tilde{A}_l$. 


1.1.3. **Solid and dotted edges.** This observation motivated us to distinguish such pairs of roots: Let us draw the dotted (resp. solid) edge \{\alpha, \beta\} if \((\alpha, \beta) > 0\) (resp. \((\alpha, \beta) < 0\)), see Fig. 1.1. The diagrams with properties of admissible diagrams and containing dotted edges are said to be **Carter diagrams**. Up to dotted edges, the classification of Carter diagrams coincides with the classification of admissible diagrams.

Recall that \((\alpha, \beta) > 0\) (resp. \((\alpha, \beta) < 0\)) means that the angle between roots \(\alpha\) and \(\beta\) is acute (resp. obtuse). Recall that, for the Dynkin diagrams, all angles between simple roots are obtuse, all edges are solid, and no special designation is necessary. A **solid edge** indicates an obtuse angle between roots exactly as for simple roots in the case of Dynkin diagrams. A **dotted edge** indicates an acute angle between the roots considered.

1.1.4. **The \(\Gamma\)-associated root subset.** For any Carter diagram \(\Gamma\) and a primary root system \(\Phi\), let \(\Gamma'\) be a Dynkin diagram such that \(\text{rank}(\Gamma') = \text{rank}(\Gamma) + 1\), \(\Phi(\Gamma') \subseteq \Phi\), where \(\Phi(\Gamma')\) is the root system associated with \(\Gamma'\). Consider a subset \(S \subseteq \Phi(\Gamma')\) of linearly independent roots with following properties: Vertices of \(\Gamma\) are in one-to-one correspondence with roots of \(S\); solid (resp. dotted) edges of \(\Gamma\) are in one-to-one correspondence with pairs \{\(\alpha, \beta\)\}, where \(\alpha, \beta \in S\), such that \((\alpha, \beta) < 0\) (resp. \((\alpha, \beta) > 0\)). The root subset \(S\) is said to be \(\Gamma\)-**associated**.

**Remark 1.2.** (i) Emphasize that whenever we say that \(S\) is the \(\Gamma\)-associated root subset, we mean that elements of \(S\) are taken from \(\Phi(\Gamma') \subseteq \Phi\), i.e., they are taken from the primary root system \(\Phi\), see \([\text{Ca}72, \text{Ca}73]\).

(ii) Not necessarily for every Carter diagram \(\Gamma\) there exists a set \(S\) with a given properties. For example, there is no \(E_6\)-associated root subset in the primary root subset \(\Phi = \Phi(D_n)\) for any integer \(n\), see Lemma \([A.5]\).

1.1.5. **The Carter diagrams.** Any admissible diagram \(\Gamma\) is said to be a **Carter diagram** if any edge connecting a pair of roots \{\(\alpha, \beta\)\} with inner product \((\alpha, \beta) > 0\) (resp. \((\alpha, \beta) < 0\)) is drawn as dotted (resp. solid) edge. There is no edge for the inner product \((\alpha, \beta) = 0\). Let

\[
\text{If } \alpha, \beta \in \Phi, \text{ then } \alpha \perp \beta \text{ if } \langle \alpha, \beta \rangle = 0.
\]

be any \(\Gamma\)-associated set (of not necessarily simple roots), where roots of the set \(S_\alpha := \{\alpha_i \mid i = 1, \ldots, k\}\) (resp. \(S_\beta := \{\beta_j \mid j = 1, \ldots, h\}\)) are mutually orthogonal. According to \([1.3(a)]\), there exists the set \([1.3]\) of linearly independent roots. Thanks to \([1.3(b)]\), such a partition into the sum of two mutually orthogonal sets \(S_\alpha\) and \(S_\beta\) is possible. The partition \([1.3]\) is said to be **bicolored partition**. Let \(w = w_1w_2\) be the decomposition of \(w\) into the product of two involutions. By \([\text{Ca}72, \text{Ca}73]\) Lemma 5 each of \(w_1\) and \(w_2\) can be expressed as a product of reflections corresponding to mutually orthogonal roots as follows:

\[
w = w_1w_2, \quad \text{where} \quad w_1 = s_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_k}, \quad w_2 = s_{\beta_1}s_{\beta_2}\ldots s_{\beta_h}.
\]

Since \(S\) is linearly independent, the decomposition \([1.5]\) is reduced, see Lemma \([1.1]\) and \(k + h = \ell_C(w)\). The decomposition \([1.5]\) is said to be a **bicolored decomposition**. The subset of roots corresponding to \(w_1\) (resp. \(w_2\)) is said to be \(\alpha\)-**set** (resp. \(\beta\)-**set**):

\[
\alpha\text{-set} = \{\alpha_1, \alpha_2, \ldots, \alpha_k\} , \quad \beta\text{-set} = \{\beta_1, \beta_2, \ldots, \beta_h\}.
\]

Up to the difference between solid and dotted edges, the Carter diagram \(\Gamma\) is the admissible diagram. Any coordinate of a given vector of linkage labels from an \(\alpha\)-**set** (resp. a \(\beta\)-**set**) we call an **\(\alpha\)-label** (resp. a **\(\beta\)-label**). Let \(L \subseteq V\) be the linear subspace spanned by root subsets \([1.4]\). We denote by \(\Pi_w\) the corresponding root basis, whose elements are not necessarily simple roots:

\[
\Pi_w = \{\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_h\}.
\]

1.1.6. **Three classes of Carter diagrams.** We consider three classes of simply-laced connected Carter diagrams: Denote the class of diagrams containing 4-cycle \(D_4(a_1)\) by \(C_4\); the class of diagrams without cycles and containing \(D_4\) (as a subdiagram) by \(DE_4\), i.e., \(DE_4\) is the class consisting of Dynkin diagrams \(E_6, E_7, E_8\) and \(D_l\) for \(l \geq 4\); the class of Dynkin diagrams \(A_l\) by \(A\). Conjugate
elements in the Weyl group $W$ are associated with the same Carter diagram $\Gamma$. For any $\Gamma \in C_4 \cup DE_4$, the uniqueness of the associated conjugacy class takes place, see [Sh10 Theorem 6.5].

**Remark 1.3.** Note that the uniqueness theorem does not hold for $A_4$. There are two conjugacy classes of type $A_3$ in $W(E_7)$ and in $W(D_6)$, two conjugacy classes of type $A_7$ in $W(E_8)$, see [Ca72 p. 31] and [Ca72 Lemma 27]. Moreover, for the Carter diagram $A_3$, there exist three conjugacy classes in $W(D_4)$. □

1.2. **Linkage diagrams and linkage systems.**

1.2.1. The partial Cartan matrix $B_\Gamma$. Similarly to the Cartan matrix associated with Dynkin diagrams, we determine the Cartan matrix for each Carter diagram $\Gamma$ as follows

$$B_\Gamma := \begin{pmatrix}
(\alpha_1, \alpha_1) & \ldots & (\alpha_i, \alpha_k) & (\alpha_1, \beta_1) & \ldots & (\alpha_1, \beta_h) \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
(\alpha_k, \alpha_1) & \ldots & (\alpha_k, \alpha_k) & (\alpha_k, \beta_1) & \ldots & (\alpha_k, \beta_h) \\
(\beta_1, \alpha_1) & \ldots & (\beta_1, \alpha_k) & (\beta_1, \beta_1) & \ldots & (\beta_1, \beta_h) \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
(\beta_h, \alpha_1) & \ldots & (\beta_h, \alpha_k) & (\beta_h, \beta_1) & \ldots & (\beta_h, \beta_h)
\end{pmatrix}, \quad (1.8)$$

where $S = \{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_h\}$. Here, subsets of roots $\alpha$-set and $\beta$-set, see (1.6), match the bicolored partition (1.4) and bicolored decomposition of a certain element $w \in W$ corresponding to $\Gamma$. We also write $S = \{\tau_1, \ldots, \tau_{k+h}\}$ if the bicolored partition like (1.4) does not matter.

We call the matrix $B_\Gamma$ a partial Cartan matrix corresponding to the Carter diagram $\Gamma$. The partial Cartan matrix $B_\Gamma$ is well-defined since $(\tau_1, \tau_1)$ in (1.8) do not depend on the choice of $\Gamma$-associated root subset. If $\Gamma$ is a Dynkin diagram, the partial Cartan matrix $B_\Gamma$ is the Cartan matrix associated with $\Gamma$. The matrix $B_\Gamma$ is positive definite (Proposition 1.1). The symmetric bilinear form associated with the partial Cartan matrix $B_\Gamma$ is denoted by $(\cdot, \cdot)_\Gamma$ and the corresponding quadratic form is denoted by $\mathcal{B}_\Gamma$. Let $L \subset V$ be the subspace spanned by the root subset $S$, the subspace $L$ is said to be $S$-associated subspace. For $S = \{\tau_1, \ldots, \tau_l\}$, we write $L = [\tau_1, \ldots, \tau_l]$.

**Remark 1.4.** In all cases where we consider the partial Cartan matrix $B_\Gamma$ for simply-laced Carter diagrams, we assume that $(\tau_1, \tau_1)$ takes values in $\{-1, 0, 1\}$, not in $\{-\frac{1}{2}, 0, \frac{1}{2}\}$. In other words, we consider the bilinear form $(\cdot, \cdot)$ obtained by doubling the usual bilinear form related to the Cartan matrix. Then linkage labels introduced in (1.2.1) are integer.

1.2.2. The linkage diagrams. We consider a class of diagrams, called linkage diagrams, which constitute a subclass of the class of connection diagrams, see (2.3.2) and generalize the class of Carter diagrams, see (1.1.5). Any linkage diagram is obtained from a Carter diagram $\Gamma$ by adding one extra root $\gamma$, with its bonds, so that the roots corresponding to vertices of $\Gamma$ together with $\gamma$ form some linearly independent root subset. The extra root $\gamma$ added to the Carter diagram $\Gamma$ is said to be a linkage root, see (2.1.1). Any linkage diagram constructed in this way may also be a Carter diagram but this is not necessarily so. The following inclusions hold:

| Dynkin diagrams of $CCl$ | Carter diagrams | Linkage diagrams | Connection diagrams |
|-------------------------|----------------|-----------------|--------------------|

With every linkage diagram we associate the vector of linkage labels. The linkage labels are similar to the Dynkin labels, see [Si81], which are the “numerical labels” introduced by Dynkin in [Dy50] for the study of irreducible linear representations of the semisimple Lie algebras, [GOV90, KOV95, Ch84].

For any simply-laced Carter diagram $\Gamma$, a linkage label takes one of three values $\{-1, 0, 1\}$. There is one-to-one correspondence between linkage diagrams obtained from the given simply-laced Carter

\[\text{Remark 1.3.} \quad \text{Note that the uniqueness theorem does not hold for } A_4. \text{ There are two conjugacy classes of type } A_3 \text{ in } W(E_7) \text{ and in } W(D_6), \text{ two conjugacy classes of type } A_7 \text{ in } W(E_8), \text{ see [Ca72 p. 31] and [Ca72 Lemma 27]. Moreover, for the Carter diagram } A_3, \text{ there exist three conjugacy classes in } W(D_4). \square\]

**Remark 1.4.** In all cases where we consider the partial Cartan matrix $B_\Gamma$ for simply-laced Carter diagrams, we assume that $(\tau_1, \tau_1)$ takes values in $\{-1, 0, 1\}$, not in $\{-\frac{1}{2}, 0, \frac{1}{2}\}$. In other words, we consider the bilinear form $(\cdot, \cdot)$ obtained by doubling the usual bilinear form related to the Cartan matrix. Then linkage labels introduced in (1.2.1) are integer.
diagram $\Gamma$ and vectors of linkage labels taking coordinates from the set $\{-1, 0, 1\}$. For this reason, we use terms linkage labels and linkage diagrams as synonyms. Some linkage diagrams and their linkage labels for the Carter diagram $E_6(a_1)$ are depicted in Fig. 1.1. The linkage diagrams are

\[ \text{Figure 1.1. Examples of linkage diagrams and vectors of linkage labels for } E_6(a_1). \]

the main characters of this paper. **We give a complete description of linkage diagrams constructed for every simply-laced Carter diagram.** By abuse of notation, this description is essentially based on the answer to the following question:

What linkage roots can be added to a given irreducible linearly independent root subset, so that the resulting set would also be an irreducible linearly independent root subset?

It turns out that the answer to this question is very simple within the framework of the quadratic form associated with the partial Cartan matrix $B_\Gamma$, see [2.4.3].

1.2.3. The linkage systems. The group $W_\gamma$, named the dual partial Weyl group, acts on linkage diagrams, see [2.1]. Under the action of $W_\gamma$ the set of linkage diagrams (= vectors of linkage labels) constitute the diagram called the linkage system similarly to the weight system in the representation theory of semisimple Lie algebras, [SIT, p. 30]. We denote the linkage system associated with the Carter diagram $\Gamma$ by $\mathcal{L}(\Gamma)$.

1.3. Dynkin extensions, partial root systems and linkage system components.

1.3.1. A-, D-, E-types of Dynkin and Carter diagram. The Dynkin diagram $A_l$, where $l \geq 1$ (resp. $D_l$, where $l \geq 4$; resp. $E_l$, where $l = 6, 7, 8$) is said to be the Dynkin diagram of A-type (resp. D-type, resp. E-type). The Carter diagram $A_l$, where $l \geq 1$ (resp. $D_l$, $D_l(ak_l)$, where $l \geq 4$, $1 \leq k \leq \left\lfloor \frac{l}{2} \right\rfloor$; resp. $E_l$, $E_l(ak_l)$, where $l = 6, 7, 8$, $k = 1, 2, 3, 4$) is said to be the Carter diagram of A-type (resp. D-type, resp. E-type).

1.3.2. Partial root system. For a $\Gamma$-associated root subset $S$, we denote by $\Phi(S)$ the subset of roots of $\Phi$ linearly dependent on roots of $S$. The subset of roots $\Phi(S)$ is said to be a partial root system. There are examples of two non-conjugate $\Gamma$-associated root subsets $S_1$ and $S_2$, see [A53.3]. This is a reason why I prefer denote the partial root system by $\Phi(S)$, not by $\Phi(\Gamma)$.

1.3.3. Covalent Carter diagrams. Carter diagrams of the same type and the same index are said to be covalent Carter diagrams. The corresponding root systems (usual or partial) are said to be covalent root systems. For example, the E-type covalent diagrams of index 6 are $\{E_6, E_6(a_1), E_6(a_2)\}$ and the E-type covalent diagrams of index 7 are $\{E_7, E_7(a_1), E_7(a_2), E_7(a_3), E_7(a_4)\}$, the D-type covalent diagrams of index l are $\{D_l, D_l(a_1), D_l(a_2), \ldots, D_l(a_{\left\lfloor \frac{l-2}{2} \right\rfloor})\}$.

We are interested in the following covalent pairs of diagrams $\{\Gamma, \tilde{\Gamma}\}$, where $\Gamma$ is a Dynkin diagram and $\tilde{\Gamma}$ is a Carter diagram covalent to $\Gamma$:

\[
\begin{align*}
\{D_l(ak_l), D_l\} & \text{ for } l \geq 4 \text{ and } 1 \leq k \leq \left\lfloor \frac{l-2}{2} \right\rfloor, \\
\{E_6(ak_l), E_6\} & \text{ for } k = 1, 2, \\
\{E_7(ak_l), E_7\} & \text{ for } k = 1, 2, 3, 4.
\end{align*}
\]
Let \( \{ \tilde{\Gamma}, \Gamma \} \) be a covalent pair of Carter diagrams out of list \([1, 9]\) and \( \tilde{S} \) (resp. \( S \)) be \( \tilde{\Gamma} \)-associated (resp. \( \Gamma \)-associated) subset, \( \mathcal{P} \) (resp. \( \mathcal{R} \)) be the partial root system (resp. root system) spanned by \( \tilde{S} \) (resp. \( S \)).

In \([8]\) we show that there exists the invertible linear map \( M : \mathcal{P} \rightarrow \mathcal{R} \). Moreover, \( \tilde{\tau} \) is a root of \( \mathcal{P} \) if and only if \( \tilde{\tau} \) is a root of \( \mathcal{R} \) (Theorem \( 3.4 \)). For any covalent pair \([1, 9]\), we have \( | \mathcal{R} | = | \mathcal{P} | \). (Corollary \( 3.3 \)).

| \( \Gamma \) | Number of components | \( E \)-components | Number of linkage diagrams |
|---|---|---|---|
| \( E_6, E_6(a_1), E_6(a_2) \) | 2 | \( \frac{4}{3} \) | \( 2 \times 27 = 54 \) |
| \( E_7, E_7(a_1), E_7(a_2), E_7(a_3), E_7(a_4) \) | 1 | \( \frac{3}{2} \) | 56 |

| \( \Gamma \) | Number of components | Number of linkage diagrams, and \( p = B_{\tilde{\Gamma}}^{E}(\gamma^\vee) \) |
|---|---|---|
| \( D_4, D_4(a_1) \) | 3 | \( 3 \times 8 = 24 \) | - | 24 |
| \( D_5, D_5(a_1) \) | 3 | 10 | \( 2 \times 16 = 32 \) | 42 |
| \( D_6, D_6(a_1), D_6(a_2) \) | 3 | 12 | \( 2 \times 32 = 64 \) | 76 |
| \( D_7, D_7(a_1), D_7(a_2) \) | 3 | 14 | \( 2 \times 64 = 128 \) | 142 |
| \( D_l, D_l(a_1), l > 7 \) | 1 | \( 2l \) | - | \( 2l \) |

| \( \Gamma \) | Number of linkage diagrams, and the number \( p = B_{\tilde{\Gamma}}^{E}(\gamma^\vee) \) |
|---|---|---|---|
| \( A_3 \) | 2 \times 4 = 8 | 6 | - | 14 |
| \( A_4 \) | 2 \times 5 = 10 | 2 \times 10 = 20 | - | 30 |
| \( A_5 \) | 2 \times 6 = 12 | 2 \times 15 = 30 | 20 | 62 |
| \( A_6 \) | 2 \times 7 = 14 | 2 \times 21 = 42 | \( 2 \times 35 = 70 \) | 126 |
| \( A_7 \) | 2 \times 8 = 16 | 2 \times 28 = 56 | \( 2 \times 56 = 112 \) | 184 |
| \( A_8 \) | 2 \times 9 = 18 | 2 \times 36 = 72 | - | 90 |
| \( A_l, l > 8 \) | 2 \times (l + 1) | \( 2 \times \frac{(l + 1)(l + 2)}{2} = l(l + 1) \) | - | \( (l + 1)(l + 2) \) |

Table 1.1. The \( ADE \) components of the linkage system \( \mathcal{L}(\Gamma) \), the number of linkage diagrams in every component and values \( B_{\tilde{\Gamma}}^{E}(\gamma^\vee) \).

1.3.4. Dynkin extensions. We show in Corollary \( 3.6 \) that for a Carter diagram \( \Gamma \) (\( \text{rank}(\Gamma) < 8 \)) and any \( \Gamma \)-associated root subset \( S \) there exists the Dynkin diagram \( \Gamma' \) such that \( \text{rank}(\Gamma') = \text{rank}(\Gamma) + 1 \) and \( \Phi(S) \subset \Phi(\Gamma') \), where \( \Phi(S) \) is the partial root system and \( \Phi(\Gamma') \) is the root system associated with \( \Gamma' \). This pair \( \{ \Gamma, \Gamma' \} \) is said to be the Dynkin extension of the Carter diagram \( \Gamma \) and is denoted by \( \Gamma' <_{D} \Gamma' \).

For any pair \( \{ \tilde{\Gamma}, \Gamma \} \) out of list \([1, 9]\), we have \( | \mathcal{L}(\tilde{\Gamma}) | = | \mathcal{L}(\Gamma) | \) (Corollary \( 3.9 \)).

1.3.5. Root stratum. Consider the connection diagram \( \tilde{\Gamma} \), see \([2.3.2]\) obtained from a certain Carter diagram \( \Gamma \) by adding only one vertex \( \alpha \), where \( \alpha \) is connected to \( \Gamma \) at \( v \) points, where \( v = 1, 2 \) or 3.
If \( \tilde{\Gamma} \) is also a Carter diagram, this extension \( \Gamma \) to \( \tilde{\Gamma} \) is said to be a regular extension and is denoted by
\[
\Gamma < \tilde{\Gamma} \quad \text{or} \quad \Gamma \overset{\alpha}{\sim} \tilde{\Gamma}.
\]
The Dynkin extension is not necessary regular. For example, \( D_n(a_k) <_D D_{n+1} \) is a Dynkin extension but \( D_n(a_k) \) is not a subdiagram of \( D_{n+1} \).

The subset of roots \( \Phi(\Gamma') \setminus \Phi(S) \) is said to be the root stratum; it consists of roots of \( \Phi(\Gamma') \) linearly independent of roots of \( S \).

1.3.6. Linkage system components. Let \( \mathcal{L}_{\Gamma'}(\Gamma) := \{ \gamma^\vee \mid \gamma \in \Phi(\Gamma') \setminus \Phi(S) \} \). Here, \( \gamma^\vee \) is the vector of linkage labels associated with \( \gamma \). The set \( \mathcal{L}_{\Gamma'}(\Gamma) \) is said to be the linkage system component (or, \( \Gamma'-\text{component of } \mathcal{L}(\Gamma) \)). The linkage system \( \mathcal{L}(\Gamma) \) is the union of the sets \( \mathcal{L}_{\Gamma'}(\Gamma) \) taken for all Dynkin extensions of \( \Gamma \):
\[
\mathcal{L}(\Gamma) = \bigcup_{\Gamma <_D \Gamma'} \mathcal{L}_{\Gamma'}(\Gamma).
\]

There are several Dynkin extensions and, accordingly, several linkage system components for the given Carter diagram \( \Gamma \). The number of roots in the root stratum \( \Phi(\Gamma') \setminus \Phi(S) \) is said to be the stratum size. We have
\[
| \Phi(\Gamma') \setminus \Phi(S) | = | \Phi(\Gamma') | - | \Phi(S) | \quad \text{and} \quad | \mathcal{L}_{\Gamma'}(\Gamma) | \leq | \Phi(\Gamma') | - | \Phi(S) |,
\]
see \([32]\). From the latter inequality we get the following proposition:

**Proposition** (Proposition 3.1) For any Carter diagram \( \Gamma \), we have \( | \mathcal{L}(\Gamma) | \leq \mathcal{E} \), where \( \mathcal{E} \) is given by Table 3.2.

It is checked in \([3] [10]\) and \([11]\) that there exists at least \( \mathcal{E} \) linkage diagrams in the linkage system \( \mathcal{L}(\Gamma) \). Together with Proposition 3.1 we get the following

**Corollary** (Corollary 3.2) For any Carter diagram \( \Gamma \), we have \( | \mathcal{L}(\Gamma) | = \mathcal{E} \), see Table 3.2.

For each Carter diagrams \( \Gamma \in C4 \coprod DE4 \coprod A \), the values \( \mathcal{E}(\gamma^\vee) \), the number of components, and number of linkage diagrams are presented in Table 1.1 see Theorem 2.3.

Every component of \( \mathcal{L}(\Gamma) \) is determined by the type of Dynkin extension \( \Gamma <_D \Gamma' \). There are \( A, D, E \)-types of Dynkin extensions. The corresponding components of the linkage systems are called \( A, D, E \)-components. For description of components of the linkage system for each Carter diagram \( \Gamma \), see Table 1.1 and Theorem 2.3.

2. Introduction and main results

2.1. The partial Cartan matrix, inverse quadratic form and linkages.

2.1.1. Linkages and linkage diagrams. Let \( w = w_1 w_2 \) be the bicolored decomposition of some element \( w \in W \), where \( w_1, w_2 \) are two involutions associated, respectively, with an \( \alpha \)-set \( \{ \alpha_1, \ldots, \alpha_k \} \) and a \( \beta \)-set \( \{ \beta_1, \ldots, \beta_h \} \) of roots from the root system \( \Phi \), see \((1.5), (1.6) \), and let \( \Gamma \) be the Carter diagram associated with this bicolored decomposition. We consider the extension of the root basis \( \Pi_w \) by means of the root \( \gamma \in \Phi \), so that the set of roots
\[
\Pi_w(\gamma) = \{ \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_h, \gamma \}
\]
is linearly independent. Let us multiply \( w \) on the right by the reflection \( s_\gamma \) corresponding to \( \gamma \) and consider the diagram \( \Gamma' = \Gamma \cup \gamma \) together with new edges. By \([11][3]\) these edges are solid (resp. dotted) for \( (\gamma, \tau) = -1 \) (resp. \( (\gamma, \tau) = 1 \)), where \( \tau \in \Pi_w \), see \((1.7)\).
The diagram $\Gamma'$ is said to be a *linkage diagram* and the root $\gamma$ is said to be a *linkage root*. The roots $\tau$ corresponding to the new edges ($\langle \gamma, \tau \rangle \neq 0$) are said to be *endpoints* of the linkage diagram; endpoints lying in an $\alpha$-set (resp. $\beta$-set) are said to be $\alpha$-endpoints (resp. $\beta$-endpoints). Consider vector $\gamma^\nabla$ defined by (2.2). This vector is said to be the *linkage label vector*. There is, clearly, the one-to-one correspondence between linkage label vectors $\gamma^\nabla$ (with labels $\gamma_i^\nabla \in \{-1,0,1\}$) and simply-laced linkage diagrams (i.e., linkage diagrams such that $(\gamma, \tau) \in \{-1,0,1\}$).

2.1.2. The *projection of the linkage root*. Let $L \subset V$ be the linear space spanned by the $\Gamma$-associated root subset, $\tau L$ be the projection of the linkage root $\gamma$ on $L$. For any $\tau_i \in S$, we define vectors $\tau_i^\nabla$ as follows:

$$\tau_i^\nabla := B_\Gamma \tau_i,$$

where $B_\Gamma$ is the partial Cartan matrix, for details see §5.1. Consider the linear space $L^\nabla$ spanned by vectors $\tau_i^\nabla$, where $\tau \in \Pi_w$, see §2.1.1. For any linkage root $\gamma \in \Phi$, we denote by $\gamma^\nabla \in L^\nabla$ the vector obtained as follows:

$$\gamma^\nabla = B_\Gamma \gamma L, \quad B_\Gamma^{-1} \gamma^\nabla = \gamma L,$$

see Proposition 5.2. The linear space $L^\nabla$ is said to be the *space of linkage labels*. The vector $\gamma^\nabla$ is said to be the *linkage label vector*. In the space $L^\nabla$, we take the *inverse quadratic form* $\mathcal{B}_\Gamma^\nabla$ associated with the inverse matrix $B_\Gamma^{-1}$. The quadratic form $\mathcal{B}_\Gamma^\nabla$ provides an easily verifiable criterion that the vector $u$ is the linkage label vector for a certain linkage root $\gamma \notin L$ (i.e., $u = B_\Gamma \gamma L$ for a certain $\gamma \notin L$). This criterion (Theorem 4.5) is the following inequality:

$$\mathcal{B}_\Gamma^\nabla(u) < 2.$$

2.2. The linkage systems and loctets.

2.2.1. The *starlike numbering of vertices*. Throughout this article, we use a special numbering of vertices adjacent to the branch point. Such a numbering is said to be *starlike*. Let $\Gamma$ be the Carter diagram such that $\Gamma \in C_4 \square DE_4$ and $\Gamma \neq D_4(a_1)$. Then $\Gamma$ contains $D_4$ as subdiagram. We use the numbering of the vertices shown in Fig 2.2. Namely, the branch point is denoted by $\beta_1$ and the neighboring vertices are denoted by \{ $\alpha_1, \alpha_2, \alpha_3$ \}. The starlike numbering is presented, for example, in Figs. 5.13-5.15. This numbering is very important for unification construction of loctets.

![Figure 2.2](image-url)

**Figure 2.2.** The starlike numbering of vertices adjacent to the branch point

2.2.2. **Loctets.** The group $W^\nabla_S$, named the dual partial Weyl group, see §5.1 acts on vectors of linkage labels as follows:

$$(w \gamma)^\nabla = w^* \gamma^\nabla \text{ for any } \gamma^\nabla \in L^\nabla, \quad w^* \in W^\nabla_S,$$

see Proposition 5.2. Under the action of $W^\nabla_S$ the set of linkage diagrams (= linkage label vectors) constitute the diagram called the *linkage system* similarly to the weight system in the representation theory of semisimple Lie algebras, [SI81] p. 30. We denote the linkage system associated with the Carter diagram $\Gamma$ by $\mathcal{L}(\Gamma)$.

The linkage systems $\mathcal{L}(E_6(a_1))$ and $\mathcal{L}(E_6(a_2))$ are depicted in Fig. 4.3. The linkage systems for all Carter diagrams are presented in Figs. C.42 C.51 E.19 C.52 top), C.53 C.55 C.57 C.59 C.65.
Every linkage diagram containing at least one non-zero $\alpha$-label $\alpha_1$, $\alpha_2$ or $\alpha_3$ (see §1.1.5) belongs to a certain 8-cell "spindle-like" linkage subsystem called octet (= linkage octet). **The octets are the main construction blocks for every linkage system.**

Let $a_i = (\alpha_i, \gamma)$ (resp. $b_i = (\beta_i, \gamma)$), where $i = 1, 2, 3$, be coordinates of the linkage label vector $\gamma$. If all $a_i = 0$ (resp. all $b_i = 0$), the linkage diagram $\gamma$ is said to be a $\beta$-unicolored (resp. $\alpha$-unicolored) linkage diagram. **Every linkage system is the union of several octets and several $\beta$-unicolored linkage diagrams**, see [6]. There are exactly 6 octets in the linkage system $\mathcal{L}(E_6(a_k))$ for $k = 1, 2$, see Figs. [C.46] and [C.47] and in the linkage system $\mathcal{L}(E_7(a_k))$ for $k = 1, 2, 3, 4$, see Figs. [C.48] - [C.51]. For $E_6(a_1)$ and $E_6(a_2)$, the linkage system is the union of 6 octets and 6 $\beta$-unicolored linkage diagrams, altogether $54 = 2 \times 27 = 6 \times 8 + 6$ linkage diagrams, see Fig. 2.3.

![Linkage systems](image)

**Figure 2.3.** Linkage systems $\mathcal{L}(E_6(a_1))$ and $\mathcal{L}(E_6(a_2))$, see Fig. [C.46] and Fig. [C.47]. The 8-cell bold subdiagrams are octets, see Fig. 2.4.

2.3. The Carter diagrams and connection diagrams. In this paper we consider several types of diagrams. The **Carter diagrams** are obtained by a small deformation of **admissible diagrams** introduced by R. Carter [Ca72], they describes a bicolored decomposition of some element $w \in W$, see [1.1.5]. **The connection diagrams** generalize the Carter diagram; each connection diagram describes a decomposition (not necessary bicolored) of a certain element $w \in W$, and this diagram is supplied with a certain order of reflections, see [2.3.2]. In both cases all reflections are associated with roots which are not necessary simple. The **linkage diagram** is a particular case of the connection diagram obtained from a certain Carter diagram by adding one extra vertex with its bonds, see [2.1.1]. The linkage diagrams are the focus of this paper.
2.3.1. The Dynkin diagrams. Let \( \Phi \) be a certain classic root system such that all roots of the same length. Then the corresponding Dynkin diagram \( \Gamma_D \) is simply-laced. Let \( \Pi \) the set of all simple roots in \( \Phi \), \( V \) be the linear space spanned by all roots, \( W \) the finite Weyl group acting on \( \Phi \). Let \( B \) be the corresponding Cartan matrix, \( (\cdot, \cdot) \) the corresponding symmetric bilinear form (= inner product on \( V \)) and \( B \) the quadratic Tits form associated with \( B \). We suppose that each diagonal element of \( B \) is equal to 2, see Remark 2.2. The following relation is the well-known property connecting roots and the quadratic Tits forms¹:

\[
B(\alpha) = 2 \iff \alpha \in \Phi. \tag{2.3}
\]

For any two non-orthogonal simple roots \( \alpha \neq \beta \), we have

\[
(\alpha, \beta) = \|\alpha\|\|\beta\| \cos(\alpha, \beta) = \sqrt{2} \cdot \sqrt{2} \left(\frac{1}{2}\right) = -1. \tag{2.4}
\]

2.3.2. The connection diagrams. Let \( \Gamma \) be the diagram characterizing connections between roots of a certain set \( S \) of linearly independent and not necessarily simple roots, \( o \) be the order of reflections in the decomposition \( \Gamma \). The pair \( (\Gamma, o) \) is said to be a connection diagram. Here, \( \Gamma \) is the diagram describing connections between roots as it is described by the Dynkin diagrams or by the Carter diagrams, and \( o \) is the order of elements in the (not necessarily bicolored) decomposition \( \Gamma \), see \[A.1] [A.3]

2.4. The main results.

2.4.1. Partial Weyl group and dual partial Weyl group. Let \( S = \{\tau_1, \ldots, \tau_\ell\} \) be a certain \( \Gamma \)-associated root subset, where \( \Gamma \) is a certain Carter diagram. We introduce the partial Weyl group \( S_\Gamma \) generated by reflections \( \{s_{\tau_1}, \ldots, s_{\tau_\ell}\} \), and the dual partial Weyl group \( S_\Gamma^\vee \) generated by dual reflections \( \{s_{\tau_1}^*, \ldots, s_{\tau_\ell}^*\} \), see \[B.1\]

The linkage diagrams \( \gamma^\vee \) and \( (w^\gamma)^\vee \) are related as follows:

\[
(w^\gamma)^\vee = w^* \gamma^\vee, \quad \text{where } w^* \in W_\Gamma \text{ (Proposition 5.2).}
\]

The quadratic form \( B_\Gamma \) takes a certain constant value for all elements \( w^\gamma \), where \( w \) runs over \( W_\Gamma \) (Proposition 4.4).

2.4.2. Bijection of covalent root systems. Let \( \Gamma \subset \Gamma' \) be a certain Dynkin extension, see \[C.1\] \( S \) be a \( \Gamma \)-associated root subset. Recall that the partial root system \( \Phi(S) \) is the subset of roots of \( \Phi(\Gamma') \) which are linearly dependent on roots of \( S \), and the root stratum \( \Phi(\Gamma') \setminus \Phi(S) \) consists of roots of \( \Phi(\Gamma') \) which are linearly independent of roots of \( S \), see \[C.3\]

Let \( \mathcal{L}_\Gamma(\Gamma) := \{ \gamma^\vee \mid \gamma \in \Phi(\Gamma') \setminus \Phi(S) \} \). The set \( \mathcal{L}_\Gamma(\Gamma) \) is said to be the linkage system component (or, \( \Gamma' \)-component of \( \mathcal{L}(\Gamma) \)). The linkage system \( \mathcal{L}(\Gamma) \) is the union of the sets \( \mathcal{L}_\Gamma(\Gamma) \) taken for all Dynkin extensions of \( \Gamma \):

\[
\mathcal{L}(\Gamma) = \bigcup_{\Gamma < \Gamma'} \mathcal{L}_\Gamma(\Gamma').
\]

There are several Dynkin extensions and, accordingly, several linkage system components for the given Carter diagram \( \Gamma \). The number of roots in the root stratum \( \Phi(\Gamma') \setminus \Phi(S) \) is said to be the stratum size. We have

\[
|\Phi(\Gamma') \setminus \Phi(S)| = |\Phi(\Gamma')| - |\Phi(S)| \quad \text{and} \quad |\mathcal{L}_\Gamma(\Gamma)| \leq |\Phi(\Gamma')| - |\Phi(S)|.
\]

From the latter inequality we get the following proposition:

**Proposition** (Proposition 5.1) For any Carter diagram \( \Gamma \), we have \( |\mathcal{L}(\Gamma)| \leq \delta \), where the estimate \( \delta \) is given by Table 5.2.

It is checked in \[D.1\] and \[D.2\] that there exists at least \( \delta \) linkage diagrams in the linkage system \( \mathcal{L}(\Gamma) \). Together with Proposition 5.1 we get the following

¹In order for the values of the linkage labels (see \[C.1\]) be integer, as in \[C.1\], we choose the diagonal elements equal 2. Often, the diagonal elements are chosen equal 1; then \( \mathcal{L}(\Gamma) \) looks as follows: \( B(\alpha) = 1 \iff \alpha \in \Phi \), see \[Kac80\].
Corollary (Corollary 3.2) For any Carter diagram $\Gamma$, we have $|\mathcal{L}(\Gamma)| = 6$, see Table 3.2.

By Proposition 3.1 we get also the following

Corollary (Corollary 3.9) For the diagram $E_i(a_k)$ and $D_i(a_k)$, we have
\[
|\mathcal{L}(E_i(a_k))| = |\mathcal{L}(E_i)| \quad \text{where } k = 1, 2,
|\mathcal{L}(D_i(a_k))| = |\mathcal{L}(D_i)| \quad \text{where } l \geq 4 \text{ and } 1 \leq k \leq \left\lfloor \frac{l-2}{2} \right\rfloor.
\]

For each Carter diagrams $\Gamma \in \mathbb{C}^4 \sqcup \mathbb{D}^4 \sqcup \mathbb{A}$, the values $\mathcal{B}^\vee(\gamma^\vee)$, the number of components, and number of linkage diagrams are presented in Table 3.1, see Theorem 2.3.

The essential part of $\mathbb{A}$ is the bijection of the partial root system and the root system:

Theorem (Theorem 3.4) Consider a covalent pair of Carter diagrams $\{\Gamma, \Gamma\}$, see $\{1.3.3\}$. Let $\mathcal{R}$ (resp. $\mathcal{P}$) be the root system (resp. partial root system) corresponding to $\Gamma$ (resp. $\Gamma\$). There exists the map $M : \mathcal{P} \rightarrow \mathcal{R}$ given by Table 3.3 is the invertible linear map.

Corollary (Corollary 3.5) (i) For any pair $\{\Gamma\}$, the root system $\Phi(\Gamma')$ contains the root subsystem $\mathcal{R}$ if and only if $\Phi(\Gamma')$ contains the partial root subsystem $\mathcal{P}$.

(ii) For any covalent pair $\{\Gamma\}$, we have $|\mathcal{R}| = |\mathcal{P}|$.

Corollary (Corollary 3.6) For any Carter diagram out of the list of $\{1.3.3\}$ there exists the Dynkin extension.

2.4.3. Whether a vector is a linkage root? Let $S = \{\tau_1, \ldots, \tau_l\}$ be a certain $\Gamma$-associated root subset, where roots $\tau$ are not necessarily simple. Let $L \subseteq E$ be the linear subspace spanned by all roots of $S$, let $\gamma_L$ be the projection of the root $\gamma$ on $L$. The main result of $\{1\}$ is the following theorem which verifies whether or not a given vector is a linkage root:

Theorem (Theorem 4.5) A root $\theta \in \Phi$ is a linkage root (i.e., $\theta$ is linearly independent of roots of $L$) if and only if
\[
\mathcal{B}^\vee(\theta^\vee) < 2,
\]
where $\theta^\vee = B^\Gamma \theta_L$ and $\theta_L$ is the projection of $\theta$ on $L$.

2.4.4. Three loctet types. In $\{3\}$ by means of inequality 2.5, we obtain a complete description of linkage diagrams for all linkage systems. We introduce the 8-cell linkage subsystem called loctet (= linkage octet) as depicted in Fig. 2.4. Recall that we adhere to a starlike numbering of $\{2.2\}$ so every linkage label vector in Fig. 2.4 looks as $\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \ldots\}$. Here, in Fig. 2.4 we consider the case where $\alpha$-set contains only 3 coordinates, but in the common case the structure of loctets does not change. Loctets are the main construction blocks in the structure of the linkage systems. Consider roots $\gamma^\vee_i(n)$ depicted in Fig. 2.4 where $\{ij\}$ is associated with type $L_{ij}$ and $n$ is the order number of the linkage diagrams in the vertical numbering in Fig. 2.4. We call the octuple of linkages depicted in every connected component in Fig. 2.4 the loctet of type $L_{12}$ (resp. $L_{13}$, resp. $L_{23}$).

2.4.5. The structure of loctets.

Corollary (on the structure of loctets and linkage diagrams (Corollary 5.1)). (i) Any linkage diagram containing a non-zero $\alpha$-label $\alpha_1$, $\alpha_2$ or $\alpha_3$ belongs to one of the loctets of the linkage system.

(ii) Any linkage diagram of the loctet uniquely determines the whole loctet.
(iii) If two loctets have one common linkage diagram, they coincide.

(iv) Every linkage diagram from the linkage system either belongs to one of the loctets or is $\beta$-unicolored.

Remark 2.1. The Carter diagrams $E_8(a_k)$ for $k = 1, \ldots, 4$ and $E_8$ do not represent any conjugacy classes in $W(D_n)$ and what is why:

(a) Among Carter diagrams of $DE4$ class, see §1.1.6, only Carter diagrams $D_l(a_k)$ (for some $l$) represent conjugacy classes in $W(D_n)$; the $E_l(a_k)$-associated root subsets, where $l = 6, 7, 8$, cannot be mapped into $W(D_n)$, see Lemma A.5.

(b) Among Carter diagrams of $C4$ class, only Carter diagrams $D_l(a_k)$ (for some $l, k$) represent conjugacy classes in $W(D_n)$; the $E_l(a_k)$-associated root subsets, where $l = 6, 7, 8$, cannot be mapped into $W(D_n)$, see Lemma A.6.

Hence, there is no a linearly independent 9-element root subset containing any $E_8(a_k)$-associated or $E_8$-associated root subset, i.e., there are no linkages for any Carter diagram of type $E_8(a_k)$ or $E_8$. For this reason, among simply-laced Carter diagrams of $E$-type it suffices to consider only diagrams with a number of vertices $l < 8$. □

2.4.6. On tables and diagrams. In §6.1 the calculation technique for loctet diagrams $\gamma_{\Gamma}^{ij}(8)$ ($= 8$th linkage diagram of the loctet) is explained. According to Corollary A.4, the whole loctet is uniquely determined from $\gamma_{\Gamma}^{ij}(8)$. From Tables B.16–B.21 one can recover the calculation of $\gamma_{\Gamma}^{ij}(8)$ for Carter diagrams $\Gamma \in C4 \sqcup DE4$. In §6.2 the calculation technique for $\beta$-unicolored linkage diagrams is similarly explained. From Tables B.22–B.24 one can recover the calculation of $\beta$-unicolored linkage diagrams for Carter diagrams $\Gamma \in C4 \sqcup DE4$. In §B.3 the loctets per component for all linkage systems are listed, see Table B.25. In §A.8 the partial Cartan matrix $B_{\Gamma}$, and the matrix $B_{\Gamma}^{-1}$, for Carter diagrams $D_l(a_k)$, $D_l$, where $l \leq 8$, and for $E_l(a_k)$, $E_l$, where $l \leq 7$, are listed, see Tables A.11–A.14. In §C all linkage systems are depicted, see Figs. C.42–C.66. The description of all linkage systems is presented in Theorem 2.3, see §2.4.8.

2.4.7. Linkage systems for Dynkin diagrams and weight systems. The linkage system and the weight system for $E_6$ coincide, see Fig. C.52 (top) and Fig. C.52 (bottom). It becomes obvious after recognizing loctets in both diagrams. The comparative figure containing both the linkage systems and the weight systems together with all their loctets can be seen in Fig. 2.6.

Similarly, the linkage system and the weight system for $E_7$ coincide, see Fig. C.53 and Fig. C.54. We observe that sizes of components in linkage systems for Carter diagrams $E_6$, $E_7$ and $D_l$ (and for their covalent $E_6(a_i)$, $E_7(a_i)$ and $D_l(a_i)$), see §2.4.2, Corollary 3.9 are, respectively, 27, 56 and $2l$ which coincide with the dimensions of the smallest fundamental representations of simple Lie algebras $E_6$, $E_7$, and $D_l$, respectively. Similarly, sizes of components of the Carter diagram $A_l$,
where \( l \geq 8 \), are, respectively, \( l + 1 \), \( \frac{(l+1)}{2} \): The \( A \)-component (resp. \( D \)-component) coincides with the weight system \((A_l, \omega_1)\) (resp. \((A_l, \omega_2)\)), see [SV98] Figs. 1 and 5. These facts get \( \text{a priori} \) reasoning in the following theorem:

**Theorem** (Theorem 7.3) Let \( \Gamma \) be a simply-laced Dynkin diagram, \( \mathfrak{g} \) the simple Lie algebra associated with \( \Gamma \). Every \( A \)-, \( D \)- or \( E \)-component of the linkage system \( \mathcal{L}(\Gamma) \) coincides with a weight system \( \Upsilon(\mathfrak{g}) \) of one of fundamental representations of \( \mathfrak{g} \).

**Remark 2.2.** (i) The weight system corresponding to the highest weight \( \omega_1 \) for type \( D_l \) is taken from [PSV98] Fig. 4], see Fig. 2.5. The number of weights is \( 2l \). By Theorem 2.3, the linkage system has the same diagram. Compare with the linkage systems \( \mathcal{L}(D_l(a_k)) \), see Fig. 8.18, 8.19.

![Figure 2.5. The weight system \((D_l, \omega_1)\).](image)

2.4.8. The structure and sizes of \( ADE \) components of linkage systems.

**Theorem 2.3** (The aggregate description of the structure and sizes of \( ADE \) components). For each Carter diagrams \( \Gamma \in C_4 \cup [D_4 \cup A_4 \), the number of linkage system components, the number of linkage diagrams in every component, the invariant characterizing every component (values \( \Bbbk_l(\gamma) \)) are presented in Table 1.3.

**Proof.** The number of linkage diagrams is obtained from the results of stratification of root systems in \([3] \) and enumeration of loctets and \( \beta \)-unicolored linkage diagrams for every Carter diagram from Table 1.3. The number of components is obtained from the shape of linkage systems. For the Carter diagrams of \( C_4 \) class, see Figs. C.43, C.51, C.69, C.62, C.65. For the Carter diagrams from \( D_4 \) class, see Figs. C.52 (top), C.53, C.55, C.57, C.63, C.65.

For \( D_l(a_k) \) and \( D_l \), where \( l \geq 8 \), the statement is proved in \([8.1] \). For \( D_l(a_k) \) and \( D_l \), where \( l > 8 \), the statement is proved in \([8.2] \) see also \([4] \).

For Carter diagrams \( A_l \), the technique of loctets does not work. The number of linkage diagrams and number of components are obtained in \([10.1] \) for \( l < 8 \) and in \([10.2] \) for \( l \geq 8 \).

For the rational number \( p \) characterizing every component, see \([2.4.9] \).

**Remark 2.4** (On linkage systems \( \mathcal{L}(D_4) \) and \( \mathcal{L}(D_4(a_1)) \)). (i) For \( D_4 \) and \( D_4(a_1) \), all 3 components are \( D \)-components, and for each component \( \Bbbk_4(\gamma) = 1 \). For \( D_4 \), see Fig. C.60 for \( D_4(a_1) \), see Fig. C.42.

(ii) To obtain each component of \( D_3 \) one can take only 4 first coordinates for any linkage diagrams in Fig. 2.4. For example, see Fig. C.60. Every component is exactly the loctet. These components coincide with 3 weight systems of 3 fundamental representations of semisimple Lie algebra \( D_4 \): \((D_4, \omega_1), (D_4, \omega_3), (D_4, \omega_4)\), see [PSV98] Fig. 10].
Figure 2.6. Loctets in the weight system and in the linkage system $E_6$
2.4.9. The invariant of the linkage system component. The linkage system component $\mathcal{L}(\Gamma)$, where $\Gamma < D \Gamma$ is a Dynkin extension, is the $W_{\mathcal{L}}$-orbit on the set of linkage diagrams, see [5.11].

The rational number $p = B_\mathcal{L}(\mathcal{L}(\Gamma))$ is the invariant characterizing this linkage system component. We call this component the extension of the Carter diagram $\Gamma$ by $p$. We denote this extension by $(\Gamma, p)$.

We will see that the Carter diagrams $D_5(a_1), D_6(a_1), D_6(a_2), D_7(a_1), D_7(a_2), D_5, D_6, D_7$ have two Dynkin extensions corresponding to the $D$-component and the $E$-component of the linkage system component $\mathcal{L}(\Gamma)$. For example, $D_5(a_1)$ has extension $\{D_5(a_1), 1\}$ containing 10 linkages, and extension $\{D_5(a_1), \frac{1}{2}\}$ containing 32 linkages, see Fig. [C.43 Table I.1]

We will see also that Carter diagrams $A_5, A_6, A_7$ have three Dynkin extensions corresponding to $A$-, $D$- and $E$-component of the linkage system $\mathcal{L}(\Gamma)$. For example, $A_7$ has extension $\{A_7, \frac{1}{2}\}$ containing 16 linkages, $\{A_7, \frac{3}{2}\}$ containing 56 linkages and $\{A_7, \frac{15}{8}\}$ containing 112 linkages, see Fig. [10.29] Fig. [10.33] (bottom), Table [1.1].

2.5. On connection with the Carter theorem. Denote the set of elements $w \in W$, each of which corresponds to an admissible diagram, by $W_0$. The existence of an admissible diagram for the element $w$ means that $w$ can be decomposed into the product of two involutions as follows:

$$w = w_1w_2, \quad \text{where} \quad w_1 = s_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_k}, \quad w_2 = s_{\beta_1}s_{\beta_2}\ldots s_{\beta_h},$$

(2.6)

the roots $\{\alpha_i \mid i = 1, \ldots, k\}$ being mutually orthogonal, and the roots $\{\beta_j \mid j = 1, \ldots, h\}$ being also mutually orthogonal.

Thus, $W_0$ is the subset of elements $w \in W$ that can be decomposed as (2.6). It turns out that $W_0 = W$. This fact is one of the main results of [C.72] Theorem C]. I call this result the Carter theorem. The proof of the Carter theorem is based on the classification of conjugacy classes. (By p. G-21], N. Burgoyne carried out a check of classification for $E_7$ and $E_8$ with a computer aid).

I would like to quote Carter from [Ca00] p. 4):

“One remarkable feature of the theory of Coxeter groups and Iwahori-Hecke algebras is the number of key properties which at present have no uniform proof and can only be proved in a case-by-case manner. We mention three examples of such properties. In the first place every element of a Weyl group is a product of two involutions. This is a key property in Carter’s description of the conjugacy classes in the Weyl group [Ca72]. Secondly, every element of a finite Coxeter group can be transformed into an element of minimal length in its conjugacy class by a sequence of conjugations by simple reflections such that the length does not increase at any stage. This property is basic to the Geck-Pfeiffer approach to the conjugacy classes of Coxeter groups. Thirdly, there are basic properties of Lusztig’s a-function which at present have only case-by-case proofs. The a-function is an important invariant of irreducible characters of Coxeter groups. One cannot be satisfied with the theory of Coxeter groups until such case-by-case proofs are replaced by uniform proofs of a conceptual nature.”

In [SU], using the classification of linkage system, we give the proof of the Carter theorem, which does not use the classification of conjugacy classes.

3. Theorem on covalent Carter diagrams

3.1. The size of linkage system $\mathcal{L}(\Gamma)$. The following proposition together with corollaries constitute the main result of this section:

Proposition 3.1. For the Carter diagram $\Gamma$, we have $|\mathcal{L}(\Gamma)| \leq \mathcal{E}$, where $\mathcal{E}$ is given by Table [3.2].

In sections [8.10] we check that for each Carter diagram $\Gamma$, there exists at least $\mathcal{E}$ linkage diagrams in the linkage system $\mathcal{L}(\Gamma)$. Together with Proposition 3.1 we get the following

Corollary 3.2. For the Carter diagram $\Gamma$, we have $|\mathcal{L}(\Gamma)| = \mathcal{E}$ (where $\mathcal{E}$ is given by Table [3.2]).
By Proposition 3.1 and calculations of this section we get

**Corollary 3.3.** For the Carter diagram \( E_6(a_k) \) and \( D_l(a_k) \), we have

- \(|\mathcal{L}(E_6(a_k))| = |\mathcal{L}(E_6)| = 54\), where \( k = 1, 2\),
- \(|\mathcal{L}(E_7(a_k))| = |\mathcal{L}(E_7)| = 56\), where \( k = 1, 2, 3, 4\),
- \(|\mathcal{L}(D_l(a_k))| = |\mathcal{L}(D_l)|\), where \( l \geq 4 \) and \( k = 1 \leq k \leq \left\lfloor \frac{l-2}{2} \right\rfloor\).

For each Carter diagram \( \Gamma \in \mathcal{C}_4 \coprod \mathcal{D}_E \coprod \mathcal{A} \), the values \( \mathcal{R}(\gamma \nabla)\), the number of components, and number of linkage diagrams are presented in Table 1.1 see Theorem 2.3.

### 3.2. Transition between a partial root system \( \mathcal{P} \) and a root system \( \mathcal{R} \).

**Theorem 3.4** (On bijection of covalent root systems). (i) Consider a covalent pair of Carter diagrams \( \{\tilde{\Gamma}, \Gamma\} \) out of the following list:

\[
\begin{align*}
\{D_l(a_k), D_l\} & \text{ for } l \geq 4 \text{ and } 1 \leq k \leq \left\lfloor \frac{l-2}{2} \right\rfloor, \\
\{E_6(a_k), E_6\} & \text{ for } k = 1, 2, \\
\{E_7(a_k), E_7\} & \text{ for } k = 1, 2, 3, 4.
\end{align*}
\]

Let \( \mathcal{R} \) (resp. \( \mathcal{P} \)) be the root system (resp. partial root system) corresponding to \( \Gamma \) (resp. \( \tilde{\Gamma} \)). There exists the map \( M : \mathcal{P} \longrightarrow \mathcal{R} \) given by Table 3.3 is the invertible linear map.

(ii) Let \( \tilde{\tau} \) be a root in \( \Phi \), where \( \Phi \) is the primary root system. Then \( \tilde{\tau} \) is a root of the partial root system \( \mathcal{P} \) if and only if \( \tilde{\tau} \) is a root of the root system \( \mathcal{R} \).

**Proof.** (i) For all pairs \( \{\mathcal{P}, \mathcal{R}\} \) from Table 3.3, it is easy to check by (2.3) that vectors images under the map \( M \) are roots. For example, for case (2), the value \( \mathcal{R}(\beta_{k+1}) \) is as follows:

\[
\begin{align*}
\mathcal{R}(\beta_{k+1}) &= \mathcal{R}(\tau_1) + 4 \sum_{i=2}^{k} \mathcal{R}(\tau_i) + \mathcal{R}(\tau_{k+1}) + \mathcal{R}(\tau_{k+1}) + \\
&= 4(\tau_1, \tau_2) + 8 \sum_{i=2}^{k-1} (\tau_{k-1}, \tau_k) + 4(\tau_k, \tau_{k+1}) + 4(\tau_k, \tau_{k+1}) = \\
&= 2 + 8(k-1) + 4 + 2 - 4 - 8(k-2) - 4 - 4 = 2 + 8(k-1) - 8(k-1) = 2.
\end{align*}
\]

Thus, \( \beta_{k+1} \) is the root. It only remains to check some inner products.
| Partial root system | Root system | Linear maps |
|---------------------|-------------|-------------|
| \(P\) | \(R\) | \(M : P \hookrightarrow R\) and \(M^{-1} : R \hookrightarrow P\) |
| 1 | | \(M := \{ \alpha_3 = - (\alpha_1 + \beta_1 + \beta_2) \} \) \(M^{-1} := \{ \beta_2 = - (\alpha_1 + \alpha_3 + \beta_1) \} \) |
| 2 | | \(M := \{ \beta_{k+1} = -(\tau_1 + 2 \sum_{i=2}^{k} \tau_i + \tau_{k+1} + \tau_{k+1}), \quad k \geq 2, \) \(\beta_2 = -(\tau_1 + \tau_2 + \tau_2), \) \(M^{-1} := \{ \tau_{k+1} = -(\tau_1 + 2 \sum_{i=2}^{k} \tau_i + \tau_{k+1} + \beta_{k+1}), \quad k \geq 2, \) \(\tau_2 = -(\tau_1 + \tau_2 + \beta_2) \} \) |
| 3 | | \(M := \{ \beta_3 = -(\alpha_1 + \alpha_3 + \beta_1) \} \) \(M^{-1} := \{ \beta_3 = -(\alpha_1 + \alpha_3 + \beta_1) \} \) |
| 4 | | \(M := \{ \beta_3 = -(\alpha_1 + \alpha_3 + \beta_1) \} \) \(M^{-1} := \{ \beta_3 = -(\alpha_1 + \alpha_3 + \beta_1) \} \) |
| 5 | | \(M := \{ \alpha_3 = -(\beta_2 + \beta_3 + \beta_1) \} \) \(M^{-1} := \{ \alpha_3 = -(\beta_2 + \beta_3 + \beta_1) \} \) |
| 6 | | \(M := \{ \beta_3 = 2(\beta_3 + \beta_1 + \beta_3), \quad \beta_3 = 2(\alpha_3 + \beta_1 + \beta_3 + \beta_2), \quad \beta_3 = \alpha_3, \quad \beta_3 = -\beta_3 \} \) \(M^{-1} := \{ \alpha_3 = \alpha_3 + \beta_2 + \beta_3 + \beta_1, \quad \beta_3 = \beta_3 + \beta_2 + 2\alpha_2 + \beta_1 + \beta_3, \quad \beta_3 = -\alpha_3 \} \) |
| 7 | | \(M := \{ \alpha_3 = \alpha_3 + \beta_2 + \beta_3 + \beta_1, \quad \alpha_3 = \beta_3 + \beta_2 + \beta_3 + \beta_1, \quad \beta_3 = \beta_3 + \beta_2 + 2\alpha_2 + \beta_1 + \beta_3, \quad \beta_3 = -\alpha_3 \} \) \(M^{-1} := \{ \beta_4 = \alpha_4 + \beta_3 + \alpha_2 + \beta_1 - \beta_1 - \alpha_1, \quad \beta_3 = \alpha_3 + \beta_3 + \alpha_2 - \alpha_2 - \beta_2, \quad \alpha_3 = \beta_3 = -\beta_3 \} \) |
| 8 | | \(M := \{ \beta_4 = -(2\alpha_4 + \beta_3 + \beta_4 - \beta_2), \quad \alpha_3 = -(\beta_1 + \beta_3 + \alpha_2 + \beta_2), \quad \alpha_3 = -(\beta_1 + \alpha_2 + \beta_2 - \alpha_2 - \beta_1), \quad \beta_1 = -(\alpha_3 + \beta_3 + \alpha_2 + \beta_3) \} \) \(M^{-1} := \{ \alpha_4 = -(\beta_4 + \alpha_1 + \beta_1 + \alpha_2 + \beta_3), \quad \beta_4 = \beta_4 + 2\alpha_1 + 2\beta_1 + 2\alpha_2 + \beta_2 + \beta_3 \} \) |

Table 3.3. The bijective linear maps between root systems and partial root systems.
(1) **Pair** $\{D_1(a_1), D_1\}$. Let us check relations for $\alpha_3$:

\[
\begin{align*}
(\alpha_3, \alpha_1) &= -(\mathcal{R}(\alpha_1) + (\beta_1, \alpha_1) + (\beta_2, \alpha_1)) = -(2 - 2) = 0, \\
(\alpha_3, \alpha_2) &= -(\alpha_1, \alpha_2) + (\beta_1, \alpha_2) + (\beta_2, \alpha_2) = -1 + 1 = 0, \\
(\alpha_3, \beta_1) &= -(\alpha_1, \beta_1) + (\beta_1, \beta_1) + (\beta_2, \beta_1) = 1 - 2 = -1.
\end{align*}
\]

(2) **Pair** $\{D_1(a_2), D_1\}$. The root $\beta_{k+1}$ is connected with $S = \{\tau_1, \ldots, \tau_l\}$ at point $\tau_2$:

For $k \geq 2$,
\[
\begin{align*}
\beta_{k+1} &\perp \tau_1, \text{ since } (\tau_1, \tau_1 + 2\tau_2) = 0, \\
(\beta_{k+1}, \tau_2) &= -(\tau_2, \tau_1) + 2\mathcal{R}(\tau_2) + 2(\tau_2, \tau_3) = -(1 + 4 - 2) = -1, \\
\beta_{k+1} &\perp \tau_i \text{ for } 3 \leq i \leq k - 1, \text{ since } (\tau_i, \tau_{i-1} + \tau_i + \tau_{i+1}) = 0, \\
\beta_{k+1} &\perp \tau_k, \text{ since } (\tau_k, 2\tau_{k-1} + 2\tau_k + \tau_{k+1} + \tau_{k+1}^2) = 0, \\
\beta_{k+1} &\perp \tau_{k+1}, \text{ since } (\tau_{k+1}, 2\tau_k + \tau_{k+1}) = 0, \\
\beta_{k+1} &\perp \tau_{k+2}, \text{ since } (\tau_{k+2}, \tau_{k+1} + \tau_{k+1}^2) = 0, \\
\beta_{k+1} &\perp \tau_i \text{ for } i > k + 2.
\end{align*}
\]

For $k = 1$,
\[
\begin{align*}
\beta_2 &\perp \tau_1, \text{ since } (\tau_1, \tau_1 + 2\tau_2) = 2 - 1 - 1 = 0, \\
(\beta_2, \tau_2) &= -((\tau_2, \tau_1) + \mathcal{R}(\tau_2) + (\tau_2, \tau_3)) = -(1 + 2) = -1, \\
\beta_2 &\perp \tau_3, \text{ since } (\tau_2, \tau_3) = -1 + 1 = 0, \\
\beta_2 &\perp \tau_3 \text{ for } i > 3.
\end{align*}
\]

By Remark 13, the root $\beta_{k+1}$ is uniquely determined, so the root $\tau_{k+1}$ is also uniquely determined by $\beta_{k+1}$.

(3) **Pair** $\{E_6(a_1), E_6\}$. For the root $\beta_3 = -(\alpha_1 + \beta_1 + \alpha_3 + \bar{\beta}_3)$, we have $\beta_3 \perp \beta_1, \alpha_2, \alpha_3, \beta_2$ and $\beta_3 = -(\alpha_1 + \beta_1 + \alpha_3 + \bar{\beta}_3)$, the following relations hold:
\[
\begin{align*}
\beta_3 &\perp \alpha_1, \alpha_3, \beta_1, \\
(\beta_3, \alpha_1) &= -(\mathcal{R}(\alpha_1) + (\alpha_1, \alpha_1)) = -1, \\
(\beta_3, \alpha_2) &= -(\alpha_2, \alpha_2) + (\alpha_2, \beta_1) = -(2 - 1) = -1, \\
(\beta_3, \beta_2) &= -(\alpha_2 + \beta_1 + \alpha_3 + \bar{\beta}_3, 2\alpha_3 + \beta_1 + \bar{\beta}_3 + \bar{\beta}_2) \\
&= -(\alpha_2 + \beta_1 + \alpha_3 + \bar{\beta}_3, 3\alpha_3 + \beta_1 + \bar{\beta}_3 + \bar{\beta}_2) \\
&= -(\alpha_2 + \beta_1 + \alpha_3 + \bar{\beta}_3, 3\alpha_3 + \beta_1 + \bar{\beta}_3 + \bar{\beta}_2) + (\alpha_3, \beta_3) = 0.
\end{align*}
\]

(4) **Pair** $\{E_6(a_2), E_6\}$. For roots $\beta_2 = -(\alpha_2 + \beta_1 + \alpha_3 + \bar{\beta}_3)$ and $\beta_3 = -(\alpha_1 + \beta_1 + \alpha_3 + \bar{\beta}_3)$, the following relations hold:
\[
\begin{align*}
(\beta_2, \alpha_3) &= -(\beta_3, \alpha_3), \\
(\beta_2, \alpha_2) &= -(\alpha_2, \alpha_2) + (\beta_3, \alpha_1) = -(2 - 1) = -1, \\
(\beta_2, \beta_3) &= -(\alpha_2 + \beta_1 + \alpha_3 + \bar{\beta}_3, 2\alpha_3 + \beta_1 + \bar{\beta}_3 + \bar{\beta}_2) \\
&= -(\alpha_2 + \beta_1 + \alpha_3 + \bar{\beta}_3, 3\alpha_3 + \beta_1 + \bar{\beta}_3 + \bar{\beta}_2) + (\alpha_3, \beta_3) = -1 - 2 - 4 + 1 + 1 - 2 + 2 = 0.
\end{align*}
\]

(5) **Pair** $\{E_7(a_1), E_7\}$. Here, $\alpha_3 = -(\beta_2 + \alpha_2 + \beta_3 + \bar{\alpha}_3)$. We have $\alpha_3 \perp \alpha_1, \beta_3, \beta_1, \alpha_2, \beta_4$ and $\alpha_3, \beta_2 = -(\mathcal{R}(\beta_2) + (\alpha_2, \alpha_2)) = -1$.

(6) **Pair** $\{E_7(a_2), E_7\}$. For roots $\beta_2 = -(\alpha_2 + \beta_1 + \alpha_3 + \bar{\beta}_3)$ and $\beta_3 = 2\alpha_3 + \beta_1 + \bar{\beta}_3 + \bar{\beta}_2$, we have $\beta_2 \perp \beta_1, \alpha_1, \beta_4$ and $\beta_3 \perp \beta_1, \alpha_1, \beta_4$. Additionally, the following relations hold:
\[
(\beta_2, \alpha_3) = (\beta_3, \alpha_3) = -1, \\
(\beta_2, \alpha_2) = -(\alpha_2, \alpha_2) + (\beta_3, \alpha_1) = -(2 - 1) = -1, \\
(\beta_2, \beta_3) = -(\alpha_2 + \beta_1 + \alpha_3 + \bar{\beta}_3, 2\alpha_3 + \beta_1 + \bar{\beta}_3 + \bar{\beta}_2) \\
= -(\alpha_2 + \beta_1 + \alpha_3 + \bar{\beta}_3, 3\alpha_3 + \beta_1 + \bar{\beta}_3 + \bar{\beta}_2) + (\alpha_3, \beta_3) = 0.
\]

(7) **Pair** $\{E_7(a_3), E_7\}$. Here, $\beta_4 \perp \beta_3, \alpha_2, \alpha_3, \beta_1$ and $\beta_4, \alpha_1 = -1$. For example,
\[
(\beta_4, \beta_1) = (\alpha_4 + \bar{\beta}_3 - \alpha_2 - \beta_1 - \alpha_1, \beta_1) = -(\alpha_2 + \beta_1 + \alpha_1, \beta_1) = 0, \\
(\beta_4, \alpha_2) = (\alpha_4 + \bar{\beta}_3 - \alpha_2 - \beta_1 - \alpha_1, \alpha_2) = (\alpha_4 - \alpha_2, \beta_2) = 0, \\
(\beta_4, \alpha_3) = (\alpha_4 + \bar{\beta}_3 - \alpha_2 - \beta_1 - \alpha_1, \alpha_3) = -(\beta_1 - \alpha_2 + \bar{\beta}_3, \alpha_2) = 0.
\]

Further, $\alpha_3 \perp \beta_3, \alpha_2, \beta_1, \alpha_1$ and $(\alpha_3, \beta_2) = -1$. For example,
\[
(\alpha_3, \alpha_2) = (\bar{\alpha}_3 + \bar{\beta}_3 - \alpha_2 - \beta_2, \alpha_2) = -(\alpha_2, \alpha_2) - (\beta_2, \alpha_2) + (\beta_3, \alpha_2) + 0, \\
(\alpha_3, \beta_1) = (\bar{\alpha}_3 + \bar{\beta}_3 - \alpha_2 - \beta_2, \beta_1) = (\alpha_2, \beta_1) = 0.
\]

Similarly, $\beta_3 \perp \alpha_2, \alpha_1, \beta_1$ and $(\beta_3, \alpha_2) = -1$. 

(8) Pair \{E_7(a_4), E_7\}, We have \( \beta_4 \perp \beta_1, \alpha_2, \beta_3, \beta_2 \). The remaining cases for \( \beta_4 \):

\[
(\beta_4, \alpha_1) = (2\alpha_4 - \alpha_4 + \beta_3 - \beta_4) + (\beta_4, \alpha_2 - \alpha_4) - (\beta_2, \beta_2 + \alpha_2 - \alpha_4) = 0 + 0 - 1 + 0 = -1.
\]

\[
(\beta_4, \alpha_3) = (2\alpha_4, \beta_3 + \beta_2) + (\beta_3, \beta_3 + \alpha_1 + \alpha_2) + (\beta_4, \alpha_1) + (-\beta_2, \beta_2 + \alpha_2) = 0 + 0 + 1 - 1 = 0.
\]

Further, \( \alpha_3 \perp \alpha_2, \beta_1, \beta_3 \) and remaining cases for \( \alpha_3 \) are as follows:

\[
(\alpha_3, \beta_1) = (\beta_2, \beta_2) - (\beta_2, \alpha_1) = -2 + 1 = -1,
\]

\[
(\alpha_3, \alpha_1) = (\beta_1, \beta_3 + \alpha_1) + (\beta_3, \alpha_2 - \alpha_4) + (\alpha_2, \beta_1 + \beta_2 + \alpha_2) + (\beta_2, \beta_2 + \alpha_2 - \alpha_4) = 0.
\]

Finally, \( \alpha_1 \perp \alpha_2, \beta_2, \beta_3 \) and \( (\alpha_1, \beta_1) = -(\beta_1, \beta_1) - (\beta_1, \alpha_2) = -1 \).

(ii) Let \( \tilde{S} = \{\tilde{\tau}_1, \ldots, \tilde{\tau}_l\} \) (resp. \( S = \{\tau_1, \ldots, \tau_l\} \)) be the \( \tilde{\Gamma} \)-associated (resp. \( \Gamma \)-associated) root subset generating the partial root system \( \mathcal{P} \) (resp. root system \( \mathcal{R} \)). Let \( M \) be matrix given by Table 3.3. \( M \) acts as follows:

\[
M\tilde{\tau}_i = \tau_i. \tag{3.2}
\]

Let \( \tilde{\tau} \) be a certain root in \( \mathcal{P} \). Then

\[
\tilde{\tau} = \sum_{i=0}^{l} t_i \tilde{\tau}_i,
\]

where \( t_i \) are some integer coefficients. Consider the vector \( \tau := M\tilde{\tau} \). We have

\[
\tau = M\tilde{\tau} = \sum_{i=0}^{l} t_i M\tilde{\tau}_i = \sum_{i=0}^{l} t_i \tau_i.
\]

Thus, \( \tau \) belongs to the integer lattice spanned by \( S \). Since \( M^{-1} \) is the integer matrix then the vector \( \tilde{\tau} = M^{-1}\tau \) also belongs to the integer lattice spanned by \( S \). In addition, \( \tilde{\tau} \in \mathcal{P} \subset \Phi \), i.e., \( \tilde{\tau} \) is the root in \( \Phi \). Therefore, \( \tilde{\tau} \) is the root in the root system \( \mathcal{R} \). The converse is similar.

Thus, linear transformations \( M : \mathcal{P} \mapsto \mathcal{R} \) and \( M^{-1} : \mathcal{R} \mapsto \mathcal{P} \) map roots of \( \mathcal{P} \) onto roots of \( \mathcal{R} \) and realize the bijection of \( \mathcal{P} \) and \( \mathcal{R} \).

The matrix \( M \) maps is called the transition matrix from \( \tilde{S} \) onto \( S \).

**Corollary 3.5.** (i) For any pair \( [3.1] \), the root system \( \Phi(\Gamma') \) contains the root subsystem \( \mathcal{R} \) if and only if \( \Phi(\Gamma) \) contains the partial root subsystem \( \mathcal{P} \).

(ii) For any pair \( [3.1] \),

\[
| \mathcal{R} | = | \mathcal{P} | . \tag{3.3}
\]

**Corollary 3.6.** For any Carter diagram out of the list \( [3.1] \), there exists the Dynkin extension.

**Proof.** For Dynkin diagrams \( \Gamma = A_l, D_l, E_l \), the Dynkin extensions exist since \( \Phi(E_l) \subset \Phi(E_{l+1}) \) for \( l < 8 \), \( \Phi(A_l) \subset \Phi(A_{l+1}) \) and \( \Phi(D_l) \subset \Phi(D_{l+1}) \). For non-Dynkin Carter diagrams \( \tilde{\Gamma} = E_l(a_k), D_l(a_k) \), the Dynkin extensions exist by Corollary 3.5(i): Since \( \Phi(\Gamma) \subset \Phi(\Gamma') \) then \( \Phi(\tilde{\Gamma}) \subset \Phi(\Gamma') \). □

**Corollary 3.7.** (i) Let \( S = \{\tau_1, \ldots, \tau_{l-1}\} \) be an \( A_{l-1} \)-associated subset of roots in \( \Phi(D_l) \). There is the only root \( \tau_k \) such \( S \cup \tau_k \) is the \( D_l(a_{k-1}) \)-associated subset, see Fig. 3.7(a).

(ii) If \( \tau_k \) (resp. \( \tau_{k+1} \)) is a root such that the subset \( S \cup \tau_k \) (resp. \( S \cup \tau_{k+1} \)) constitutes the \( D_l(a_{k-1}) \)-associated (resp. \( D_l(a_k) \)-associated) subset, then

\[
\tau_k + \tau_{k+1} + \tau_{k+1} - \tau_k = 0, \tag{3.4}
\]

see Fig. 3.7(d).
Proof. (i) By Remark 4.3, the maximal root \( \mu_{\text{max}} \) is uniquely determined. The vector \( \beta_k \) defined in Table 3.3 (case (2)) coincides with \( \mu_{\text{max}} \), i.e., there exists only one root \( \beta_k \) connected with \( \tau_2 \) and orthogonal to roots \( \{ \tau_1, \tau_3, \ldots, \tau_l \} \). Thus, the vector

\[
\tau_k = -(\tau_1 + 2 \sum_{i=2}^{k-1} \tau_i + \tau_k + \beta_k) \tag{3.5}
\]

is also uniquely determined:

\[
\beta_k = \beta_{k+1}. \tag{3.6}
\]

(ii) By (3.6), we have

\[
-(\tau_1 + 2 \sum_{i=2}^{k-1} \tau_i + \tau_k + \tau_k) = -(\tau_1 + 2 \sum_{i=2}^{k} \tau_i + \tau_{k+1} + \tau_{k+1}).
\]

In other words,

\[
\tau_k = \tau_k + \tau_{k+1} + \tau_{k+1},
\]

i.e., (3.4) holds. Further,

\[
(\tau_k, \tau_{k+1}) = (\tau_k, \tau_{k+1}) + (\tau_{k+1}, \tau_{k+1}) = -1 + 2 = 1.
\]

Therefore, \( \tau_k \) and \( \tau_{k+1} \) are connected by a dotted edge, see Fig. 3.7(d).

\[
\square
\]

3.3. Relation of linkage diagrams lying in \( \mathcal{L}(\tilde{\Gamma}) \) and in \( \mathcal{L}(\Gamma) \). Denote by \( \gamma_{\tilde{\Gamma}} \) and \( \gamma_{\Gamma} \) the following vectors:

\[
\gamma_{\tilde{\Gamma}} = \left( \begin{array}{c} (\gamma, \tau_1) \\ \vdots \\ (\gamma, \tau_l) \end{array} \right), \quad \gamma_{\Gamma} = \left( \begin{array}{c} (\gamma, \tilde{\tau}_1) \\ \vdots \\ (\gamma, \tilde{\tau}_l) \end{array} \right). \tag{3.7}
\]

Lemma 3.8. Let \( \{ \tilde{\Gamma}, \Gamma \} \) be the covalent pair of Carter diagrams from (3.1).

(i) The vector \( \gamma_{\tilde{\Gamma}} \) is the linkage label vector in the linkage system \( \mathcal{L}(\tilde{\Gamma}) \) if and only if \( \gamma_{\Gamma} \) is the linkage label vector in the linkage system \( \mathcal{L}(\Gamma) \).

(ii) Let \( \gamma_{\tilde{\Gamma}}, \theta_{\tilde{\Gamma}} \in \mathcal{L}(\tilde{\Gamma}) \) and \( \gamma_{\Gamma}, \theta_{\Gamma} \in \mathcal{L}(\Gamma) \). Then \( \gamma_{\tilde{\Gamma}}, \theta_{\tilde{\Gamma}} \) coincide if and only if \( \gamma_{\Gamma}, \theta_{\Gamma} \) coincide.

Proof. Let \( L \) be the subspace spanned by roots of \( \tilde{S} = \{ \tilde{\tau}_1, \ldots, \tilde{\tau}_l \} \), we will write \( L = [\tilde{\tau}_1, \ldots, \tilde{\tau}_l] \).

By Theorem 3.4 we have

\[
L = [\tilde{\tau}_1, \ldots, \tilde{\tau}_l] = [\tau_1, \ldots, \tau_l]. \tag{3.8}
\]

(i) The fact that \( \gamma_{\tilde{\Gamma}} \) is the linkage label vector in the linkage system \( \mathcal{L}(\tilde{\Gamma}) \) means that \( \gamma \) is the linkage root linearly independent of roots of \( \tilde{\Gamma} \), see (2.11). In turn, by (3.8), this means, that \( \gamma \) is also linearly independent of roots of \( S \) and \( \gamma_{\Gamma} \) is the linkage label vector in the linkage system \( \mathcal{L}(\Gamma) \).
of roots $\gamma$ (resp. $\theta$) on $L$. Note that projections of $\gamma$ and $\theta$ on $L$ do not depend on choice of the diagram $\tilde{\Gamma}$. By (3.9) from $\gamma^\nabla = \theta^\nabla$ we get $B_{\tilde{\Gamma}}\gamma_L = B_{\tilde{\Gamma}}\theta_L$. Since $B_{\tilde{\Gamma}}$ is positive definite (Proposition 4.1), we have $\gamma_L = \theta_L$. Again, by (3.9) we have $\gamma^\nabla = \theta^\nabla$. \hfill $\square$

Corollary 3.9. For any covalent pair of Carter diagrams $\{\tilde{\Gamma}, \Gamma\}$ out of (3.1) we have

$$| \mathcal{L}(\tilde{\Gamma}) | = | \mathcal{L}(\Gamma) |.$$  

This follows directly from Lemma 3.8 \hfill $\square$

3.4. Linkage system components. Consider a Carter diagram $\Gamma$. Let $\Gamma <_D \Gamma'$ be a Dynkin extension, $S$ a certain $\Gamma$-associated root subset, $S \subset \Phi(\Gamma')$, where $\Phi(\Gamma')$ is the root system associated with $\Gamma'$.

Recall that the root subset $\Phi(S)$ consisting of roots of $\Phi$ linearly dependent on roots of $S$ is said to be a partial root system. The subset of roots $\Phi(\Gamma') \setminus \Phi(S)$ is said to be the root stratum, see 1.3.5. The number of roots in the root stratum $\Phi(\Gamma') \setminus \Phi(S)$ is said to be the stratum size:

$$| \Phi(\Gamma') \setminus \Phi(S) | = | \Phi(\Gamma') | - | \Phi(S) |.$$  

see (3.10). We define the set of linkage diagrams $\mathcal{L}_{\Gamma'}(\Gamma)$ as follows:

$$\mathcal{L}_{\Gamma'}(\Gamma) = \{ \gamma^\nabla | \gamma \in \Phi(\Gamma') \setminus \Phi(S) \}.$$  

(3.11)

The set $\mathcal{L}_{\Gamma'}(\Gamma)$ is said to be the linkage system component (or, $\Gamma'$-component of $\mathcal{L}(\Gamma)$). There are several Dynkin extensions and, accordingly, several linkage system components for the given Carter diagram $\Gamma$:

$$\Gamma <_D \Gamma_1, \ldots, \Gamma <_D \Gamma_n.$$  

The linkage system $\mathcal{L}(\Gamma)$ is the union of the sets $\mathcal{L}_{\Gamma_i}(\Gamma)$ taken for all Dynkin extensions of $\Gamma$:

$$\mathcal{L}(\Gamma) = \bigcup_{\Gamma <_D \Gamma_i} \mathcal{L}_{\Gamma_i}(\Gamma).$$

For example, the Dynkin extensions for $\Gamma = A_7$ are as follows:

$$A_7 <_D A_8, \quad A_7 <_D D_8, \quad A_7 <_D E_8,$$

corresponding linkage system components are $\mathcal{L}_{A_8}(A_7)$, $\mathcal{L}_{D_8}(A_7)$, $\mathcal{L}_{E_8}(A_7)$ and the linkage system $\mathcal{L}(A_7)$ is the union of linkage system components of $A$, $D$, and $E$-type:

$$\mathcal{L}(A_7) = \mathcal{L}_{A_8}(A_7) \cup \mathcal{L}_{D_8}(A_7) \cup \mathcal{L}_{E_8}(A_7).$$

For every root $\gamma \in \Phi(\Gamma') \setminus \Phi(S)$, there exist the linkage diagram $\gamma^\nabla \in \mathcal{L}_{\Gamma'}(\Gamma)$. There exist pairs of roots $\gamma_1, \gamma_2 \in \Phi(\Gamma') \setminus \Phi(S)$ such that $\gamma_1^\nabla = \gamma_2^\nabla$, see Lemma 3.1. Thus, from (3.10) we deduce that:

$$| \mathcal{L}_{\Gamma'}(\Gamma) | \leq | \Phi(\Gamma') | - | \Phi(S) |.$$  

(3.12)
4. **Theorem on a linkage root**

Let $\Gamma$ be the Carter diagram corresponding to the bicolored decomposition of $w$ given as in eq. $\Gamma_{1.3}$, and $\Pi_w = \{\tau_1, \ldots, \tau_l\}$ the corresponding root basis, see $\Gamma_{1.7}$. Recall that $L$ is the $\Pi_w$-associated subspace:
\[
L = [\tau_1, \ldots, \tau_l],
\]
see $\Gamma_{2.1.2}$.

Let $B$ be the Cartan matrix corresponding to the primary root system $\Phi$, see $\Gamma_{1.1.1}$.

**Proposition 4.1.** (i) The restriction of the bilinear form associated with the Cartan matrix $B$ on the subspace $L$ coincides with the bilinear form associated with the partial Cartan matrix $B_{\Gamma}$, i.e., for any pair of vectors $v, u \in L$, we have
\[
(v, u)_B = (v, u), \text{ and } B_B(v) = B(v).
\]

(ii) For every Carter diagram, the matrix $B_{\Gamma}$ is positive definite.

**Proof.** (i) From $\Gamma_{1.8}$ we deduce:
\[
(v, u)_B = \left(\sum_i t_i \tau_i, \sum_j q_j \tau_j\right)_B = \sum_{i,j} t_i q_j (\tau_i, \tau_j)_B = \sum_{i,j} t_i q_j (\tau_i, \tau_j)_B = (v, u).
\]

(ii) This follows from (i). ∎

**Remark 4.2** (The classical case). Recall that the $n \times n$ matrix $K = (k_{ij})$, where $1 \leq i, j \leq n$, such that
\[
\begin{align*}
(C1) \quad & k_{ii} = 2 \text{ for } i = 1, \ldots, n, \\
(C2) \quad & -k_{ij} \in \mathbb{Z} = \{0, 1, 2, \ldots\} \text{ for } i \neq j, \\
(C3) \quad & k_{ij} = 0 \text{ implies } k_{ji} = 0 \text{ for } i, j = 1, \ldots, n
\end{align*}
\]
is called a generalized Cartan matrix. $\Gamma_{1.8}, \Gamma_{2.1.2}, \Gamma_{1.1.1}$ [Kac80], [St08] §2.1]. For the Carter diagram $\Gamma$, which is not a Dynkin diagram, the condition (C2) fails: The elements $k_{ij}$ associated with dotted edges are positive.

If the Carter diagram does not contain any cycle, then the Carter diagram is the Dynkin diagram, the corresponding conjugacy class is the conjugacy class of the Coxeter element, and the partial Cartan matrix is the classical Cartan matrix, which is the particular case of a generalized Cartan matrix.

4.1. **Linear dependence and maximal roots.** Let $S = \{\tau_1, \ldots, \tau_l\}$ be a $\Gamma$-associated subset. The matrix $B_{\Gamma}$ is well-defined, since $(\tau_i, \tau_j)$ is 0 (resp. $-1$, resp. 1) if the corresponding connection $\{\tau_i, \tau_j\}$ does not exist (resp. solid edge, resp. dotted edge). Let $\gamma$ be a root linearly dependent on $S$ as follows:
\[
\gamma = t_1 \tau_1 + \cdots + t_l \tau_l. \tag{4.3}
\]

Then we have
\[
\begin{pmatrix}
(\gamma, \tau_1) \\
\vdots \\
(\gamma, \tau_l)
\end{pmatrix} = B_{\Gamma} \begin{pmatrix}
t_1 \\
\vdots \\
t_l
\end{pmatrix} = B_{\Gamma} \gamma, \text{ and } \begin{pmatrix}
t_1 \\
\vdots \\
t_l
\end{pmatrix} = B_{\Gamma}^{-1} \begin{pmatrix}
(\gamma, \tau_1) \\
\vdots \\
(\gamma, \tau_l)
\end{pmatrix}. \tag{4.4}
\]

Replacing $\tau_i$ with $-\tau_i$, we get the coefficient $-t_i$ instead of $t_i$ in the decomposition $\Gamma_{1.3}$.

**Remark 4.3.** Let the vector $\gamma$ be linearly dependent on roots of $S = \{\tau_1, \ldots, \tau_l\}$, and let $\gamma$ be connected with only one $\tau_i \in S$. We have two frequently occurring cases:

(i) Suppose $\gamma$ is connected to the same point as the maximal (or minimal) root in the root system $S$. In other words, the orthogonality relations $(\gamma, \tau_i)$ in eq. $\Gamma_{4.3}$ coincide with orthogonality relations for the maximal (resp. minimal) root while the edge connecting with $\gamma$ is dotted (resp. solid). Since equation $\Gamma_{4.4}$ has a unique solution
\[
\gamma = B_{\Gamma}^{-1} \gamma^\nabla, \tag{4.5}
\]
we deduce that \( \gamma \) coincides with the maximal (resp. minimal) root.

(ii) Consider the necessary condition that \( \gamma \) is a root. We have

\[
\gamma^\nabla := \begin{pmatrix}
(\gamma, \tau_1) \\
\vdots \\
(\gamma, \tau_i) \\
\vdots \\
(\gamma, \tau_l)
\end{pmatrix} = \begin{pmatrix}
0 \\
\vdots \\
\pm 1 \\
\vdots \\
0
\end{pmatrix},
\]

(4.6)

where the sign + (resp. −) corresponds to the dotted (resp. solid) edge connecting \( \gamma \) with \( \tau_i \). Let \( \mathcal{B}_\Gamma \) be the quadratic form associated with the partial Cartan matrix \( B_\Gamma \). By (4.5) and (4.6) the value of \( \mathcal{B}_\Gamma \) on the root \( \gamma \) is as follows

\[
\mathcal{B}_\Gamma(\gamma) = \langle B_\Gamma \gamma, \gamma \rangle = \langle \gamma^\nabla, B_\Gamma^{-1} \gamma^\nabla \rangle = b^\vee_{i,i},
\]

(4.7)

where \( b^\vee_{i,i} \) is the \( i \)th diagonal element of \( B_\Gamma^{-1} \). If \( \gamma \) is a root, then \( \mathcal{B}(\gamma) = 2 \), and the necessary condition that \( \gamma \) ia a root, which is connected with only one \( \tau_i \in S \), is the following simple equality:

\[
b^\vee_{i,i} = 2.
\]

(4.8)

4.2. The inverse quadratic form \( \mathcal{B}_\Gamma^\vee \).

4.2.1. Linkage roots. Recall that the linkage diagram is obtained from a Carter diagram \( \Gamma \) by adding one extra root \( \gamma \), with its bonds, so that the roots corresponding to vertices of \( \Gamma \) together with \( \gamma \) form a linearly independent root subset. This extra root \( \gamma \) is said to be the linkage root. Any linkage diagram constructed in this way might also be a Carter diagram but this is not necessarily so. With every linkage diagram we associate the linkage label vector, we use terms linkage label vectors and linkage diagrams as synonyms, see §1.2.2. The linkage labels are similar to the Dynkin labels, see §7.1. Two linkage diagrams and their linkage label vectors for the Carter diagram \( E_6(a_1) \) are depicted in Fig. 1.1.

We would like to get an answer to the following question:

\[
\text{What linkage roots can be added to the irreducible linearly independent root subset?}
\]

(4.9)

It turns out that the answer to this question is very simple within the framework of the quadratic form associated with the partial Cartan matrix. Let \( \gamma \) be a linkage root for \( \Gamma \), and let

\[
\gamma^\nabla := \begin{pmatrix}
(\gamma, \tau_1) \\
\vdots \\
(\gamma, \tau_i) \\
\vdots \\
(\gamma, \tau_l)
\end{pmatrix}
\]

(4.10)

be the linkage label vector. We denote the space spanned by \( L \) and \( \gamma \) by \( L(\gamma) \), and write

\[
L = [\tau_1, \ldots, \tau_l], \quad L(\gamma) = [\tau_1, \ldots, \tau_i, \gamma].
\]

(4.11)

Let \( \mathcal{B}_\Gamma \) be the quadratic form associated with the partial Cartan matrix \( B_\Gamma \). Then

\[
\mathcal{B}_\Gamma(\gamma) = \langle B_\Gamma \gamma, \gamma \rangle = \langle \gamma^\nabla, B_\Gamma^{-1} \gamma^\nabla \rangle = b^\vee_{i,i},
\]

(4.12)

Since \( B_\Gamma \) is positive definite, the eigenvalues of \( B_\Gamma \) are positive. Hence the eigenvalues of \( B_\Gamma^{-1} \) are also positive, and the matrix \( B_\Gamma^{-1} \) is positive definite. We call the quadratic form \( \mathcal{B}_\Gamma^\vee \) corresponding to the matrix \( B_\Gamma^{-1} \) the inverse quadratic form. The form \( \mathcal{B}_\Gamma^\vee \) is positive definite.
4.2.2. The projection of the linkage root. Let $L^\perp$ be the orthogonal complement of $L$ to $L(\gamma)$ in the sense of the symmetric bilinear form $(\cdot, \cdot)$ associated with the primary root system $\Phi$:

$$L(\gamma) = L \oplus L^\perp. \quad (4.13)$$

Let $\gamma_L$ be the projection of the linkage root $\gamma$. For any root $\theta \in L(\gamma)$ such that $\theta \notin L$, we have $L(\theta) = L(\gamma)$, and $\theta$ is uniquely decomposed into the following sum:

$$\theta = \theta_L + \mu, \quad \text{where} \quad \theta_L \in L, \quad \mu \in L^\perp. \quad (4.14)$$

Given any vector $\theta$ by decomposition (4.14), we introduce also the conjugate vector $\overline{\theta}$ as follows:

$$\theta = \theta_L + \mu, \quad \overline{\theta} = \theta_L - \mu, \quad \text{where} \quad \theta_L \in L, \quad \mu \in L^\perp. \quad (4.15)$$

![Figure 4.8](image)

**Figure 4.8.** The roots $\gamma = \gamma_L + \mu$ and $\overline{\gamma} = \gamma_L - \mu$.

The vector $\mu$ is said to be the normal extending vector, see Fig. 4.8. The following proposition collects a number of properties of the projection $\gamma_L$ of the linkage root $\gamma$, the linkage label vector $\gamma^\nabla$, and the normal extending vector $\mu$:

**Proposition 4.4.** (i) The linkage label vector $\gamma^\nabla$ and the projection $\gamma_L$ are related as follows:

$$\gamma^\nabla = \gamma_L^\nabla = B_\Gamma \gamma_L. \quad (4.16)$$

(ii) The component $\mu$ is, up to sign, a fixed vector for any root $\theta \in L(\gamma) \setminus L$.

(iii) The value $\mathcal{B}_\Gamma(\gamma_L)$ is constant for any root $\theta \in L(\gamma) \setminus L$.

(iv) The vector $\theta = \theta_L + \mu$ is a root in $L(\gamma) \setminus L$ if and only if $\overline{\theta} = \theta_L - \mu$ is a root in $L(\gamma) \setminus L$.

(v) If $\delta$ is a root in $L(\gamma) \setminus L$ such that $\delta^\nabla = \gamma^\nabla$, then $\delta = \gamma_L + \mu$ or $\gamma_L - \mu$, i.e. $\delta = \gamma$ or $\delta = \overline{\gamma}$.

(vi) For any root $\theta \in L(\gamma) \setminus L$, we have

$$\mathcal{B}_\Gamma(\theta^\nabla) = \mathcal{B}_\Gamma(\theta_L), \quad (4.17)$$

and, $\mathcal{B}_\Gamma(\theta^\nabla)$ is a constant for all roots $\theta \in L(\gamma) \setminus L$.

**Proof.** (i) Because $(\tau_i, \mu) = 0$ for any $\tau_i \in S$, we have by (4.10) and (4.14)

$$\gamma^\nabla = \begin{pmatrix} (\gamma, \tau_1) \\ \vdots \\ (\gamma, \tau_l) \end{pmatrix} = \begin{pmatrix} (\gamma_L + \mu, \tau_1) \\ \vdots \\ (\gamma_L + \mu, \tau_l) \end{pmatrix} = \gamma_L^\nabla.$$
By (4.4), since $\gamma_L \in L$, it follows that $\gamma_L^\nabla = B_T \gamma_L$. Therefore, $\gamma^\nabla = \gamma_L^\nabla = B_T \gamma_L$.

(ii) Any root $\theta \in L(\gamma)$ such that $\theta \notin L$ looks as $\tau = \pm \gamma$ for some $\tau \in L$. If $\gamma = \gamma_L + \mu$, where $\gamma_L \in L$ then $\theta = \tau = \pm \gamma_L \pm \mu = \theta_L \pm \mu$, where $\theta_L = \tau \pm \gamma_L \in L$.

(iii) Let $B$ be the quadratic form associated with the primary root system $\Phi$, see §1.1. By (4.14) we have $\theta_L \perp \mu$, and $B(\theta) = B(\theta_L) + B(\mu)$. Here, $B(\theta) = 2$ since $\theta$ is the root, and by (ii), we have $B(\theta_L)$ is constant. By (4.12), we have $B_T(\theta_L) = B(\theta_L)$, i.e., $B_T(\theta_L)$ is also constant for all $\theta \in L(\gamma)$.

(iv) Let $\theta$ be a root, i.e., $B(\theta) = B(\theta_L) + B(\mu) = 2$. Then for $\theta$, we have $B(\theta) = B(\theta_L) + B(-\mu) = 2$, and $\theta$ is a root as well.

(v) By $\delta^\nabla = \gamma^\nabla$ and (4.16), we have $B_T \delta_L = B_T \gamma_L$, i.e., $\delta_L = \gamma_L$. By heading (ii), we have $\delta = \gamma_L + \mu$ or $\gamma_L - \mu$.

(vi) By heading (i), since $\theta_L \in L$, we have $\theta_L^\nabla = B_T \theta_L$. Thus,

$$B_T^\nabla(\theta^\nabla) = (B_T^{-1} \theta^\nabla, \theta^\nabla) = (\theta_L, B_T \theta_L) = B_T(\theta_L).$$

(4.18)

4.3. The criterion of a linkage root. In this section we give a criterion that a given vector is a linkage root.

Theorem 4.5 (Criterion of a linkage root). (i) Let $\theta^\nabla$ be the linkage label vector corresponding to a certain root $\theta \in L(\gamma)$, i.e., $\theta^\nabla = B_T \theta_L$. The root $\theta$ is a linkage root, (i.e., $\theta$ is linearly independent of roots of $L$) if and only if

$$B_T^\nabla(\theta^\nabla) < 2.$$  

(4.19)

(ii) Let $\theta \in L(\gamma)$ be a root connected only with one $\tau_i \in S$. The root $\theta$ is a linkage root if and only if

$$b^\nabla_{i,i} < 2,$$

(4.20)

where $b^\nabla_{i,i}$ is the $i$th diagonal element of $B_T^{-1}$.

Proof. (i) Let $B_T^\nabla(\theta^\nabla) = 2$. By Proposition 4.4 (vi) we have $B_T(\theta_L) = 2$, and by Proposition 4.1 we have also $B(\theta_L) = 2$. Since $\theta \in L(\gamma)$ is a root, then $B(\theta) = 2$ and $B(\mu) = B(\theta) - B(\theta_L) = 0$. Therefore, $\mu = 0$, and by (4.14) $\theta$ coincides with its projections on $L$: $\theta = \theta_L$. Thus $\theta$ is linearly depends on vectors of $L$.

Conversely, let $B_T^\nabla(\theta^\nabla) < 2$, i.e., $B_T(\theta_L) < 2$. As above, we have $B(\theta_L) < 2$ and $B(\mu) = B(\theta) - B(\theta_L) \neq 0$, i.e., $\mu \neq 0$ and $\theta$ is linearly independent of roots of $L$.

(ii) We have

$$\theta^\nabla = \begin{pmatrix} (\theta, \tau_1) \\ \vdots \\ (\theta, \tau_i) \\ \vdots \\ (\theta, \tau_r) \end{pmatrix} = \begin{pmatrix} 0 \\ \cdots \\ \pm 1 \\ \cdots \\ 0 \end{pmatrix},$$

and $B_T^\nabla(\theta^\nabla) = b^\nabla_{i,i}$. Thus, statement (ii) follows from (i).

□

Remark 4.6. There are cases where a given root subset $L$ can be extended to two different subsets $L(\gamma) \neq L(\delta)$ such that $B_T^\nabla(\gamma^\nabla) \neq B_T^\nabla(\delta^\nabla)$. For example, there are two extensions for $D_5$ with different values of $B_T^\nabla(\theta^\nabla)$:

$$B_T^\nabla(\gamma^\nabla) = \frac{5}{4} \text{ for } D_5 \gtrsim E_6, \quad B_T^\nabla(\delta^\nabla) = 1 \text{ for } D_3 \gtrsim D_6.$$  

(4.21)
see Fig. 4.9. The matrix $B^{-1}_\Gamma$ for $\Gamma = D_5$ is given in Table A.14. For more examples of extensions with different values of $B^\vee(\theta^\vee)$, see Table 1.1.

Figure 4.9. Two extensions: $D_5 \lessgtr E_6$ and $D_5 \lessgtr D_6$

5. Loctets

5.1. The dual partial Weyl group associated with a root subset. As in §4.2, let $\Gamma$ be the Carter diagram corresponding to the bicolored decomposition of $w$ given as in eq. (1.5), and $\Pi_w = \{\tau_1, \ldots, \tau_l\}$ is the corresponding root basis, see (1.7). Let $L$ be the $\Pi_w$-associated subspace:

$L = [\tau_1, \ldots, \tau_l].$

For any $\tau_i \in \Pi_w$, we define vectors $\tau_i^\vee$ as follows:

$\tau_i^\vee := B_\Gamma \tau_i.$ (5.1)

Let $L^\vee$ be the linear space spanned by vectors $\tau_i^\vee$, where $\tau_i \in \Pi_w$, see [2.1.1]. The linear space $L^\vee$ is said to be the space of linkage labels. Eq. (5.1) is consistent with (2.2). The map $\tau_i \rightarrow \tau_i^\vee$ given by (5.1) is expanded to the linear mapping $L \rightarrow L^\vee$, and

$u^\vee = B_\Gamma u = \left( \begin{array}{c} (u, \tau_1) \\ \vdots \\ (u, \tau_l) \end{array} \right)$ for any $u \in L.$ (5.2)

Consider the restriction of the reflection $s_{\tau_i}$ on the subspace $L$. For any $v \in L$, by Proposition 4.1 we have:

$s_{\tau_i} v = v - 2\frac{(\tau_i, v)}{(\tau_i, \tau_i)} \tau_i = v - (\tau_i, v) \tau_i = v - (B_\Gamma \tau_i, v) \tau_i = v - (\tau_i^\vee, v) \tau_i.$ (5.3)

We define the dual reflection $s^*_{\tau_i}$ acting on a vector $u^\vee \in L^\vee$ as follows:

$s^*_{\tau_i} u^\vee := u^\vee - (u^\vee, \tau_i) \tau_i^\vee.$ (5.4)

Let $W_S$ (resp. $W^\vee_S$) be the group generated by reflections $\{s_{\tau_i} \mid \tau_i \in S\}$ (resp. $\{s^*_{\tau_i} \mid \tau_i \in S\}$), where $S = \{\tau_1, \ldots, \tau_l\}$.

**Proposition 5.1.** (i) For any $\tau_i \in \Pi_w$, we have

$s^*_{\tau_i} = \iota s_{\tau_i} = \iota s_{\tau_i}^{-1}.$ (5.5)

(ii) The mapping

$w \rightarrow \iota w^{-1}$ (5.6)

determines an isomorphism of $W_S$ onto $W^\vee_S$.

**Proof.** (i) By [4.3] and [5.4], for any $v \in L$, $u^\vee \in L^\vee$ we have:

$\langle s^*_{\tau_i} u^\vee, v \rangle = \langle u^\vee - (u^\vee, \tau_i) \tau_i^\vee, v \rangle = \langle u^\vee, v \rangle - \langle u^\vee, \tau_i \rangle (v, \tau_i^\vee),$

$\langle u^\vee, s_{\tau_i} v \rangle = \langle u^\vee, v - (\tau_i^\vee, v) \tau_i \rangle = \langle u^\vee, v \rangle - \langle \tau_i^\vee, v \rangle (u^\vee, \tau_i).$ (5.7)
Thus,
\[
(s_{\tau_i}^* u^\nabla, v) = \langle u^\nabla, s_{\tau_i} v \rangle, \quad \text{for any } v \in L, u^\nabla \in L^\nabla,
\]
and (5.8) holds.

\[\square\]

5.1.1. Groups $W_S$ and $W_N^\nabla$. One should note that $W_S$ (and, therefore, $W_N^\nabla$) is not necessarily Weyl group, since the roots $\tau_i \in \Pi_w$ are not necessarily simple: They do not constitute a root subsystem. We call $W_S$ the partial Weyl group, and $W_N^\nabla$ the dual partial Weyl group associated with the root subset $S$, or, for short, the dual partial Weyl group. By (5.1), (5.2) and (5.4), for any $u^\nabla \in L^\nabla$, we have
\[
(s_{\tau_i}^* u^\nabla) \tau_k = u^\nabla_k - u_{\tau_i}(B_{\Gamma^*} \tau_i),
\]
and (5.9) holds.

Then, by (5.9) we get
\[
\begin{cases}
-u_{\tau_i}, & \text{if } k = i, \\
u_{\tau_i} + u_{\tau_i}, & \text{if } \{\tau_k, \tau_i\} \text{ is a solid edge, i.e., } (\tau_k, \tau_i) = -1, \\
u_{\tau_i} - u_{\tau_i}, & \text{if } \{\tau_k, \tau_i\} \text{ is a dotted edge, i.e., } (\tau_k, \tau_i) = 1, \\
u_{\tau_i}, & \text{if } \tau_k \text{ and } \tau_i \text{ are not connected, i.e., } (\tau_k, \tau_i) = 0.
\end{cases}
\]

(5.10)

Now, we will show that values $(s_{\tau_i}^* u^\nabla) \tau_k$ belong to the set $\{-1, 0, 1\}$, i.e., $s_{\tau_i}^*$ acts in the linkage system $\mathcal{L}(\Gamma)$, and
\[
W^\nabla(S) : \mathcal{L}(\Gamma) \rightarrow \mathcal{L}(\Gamma).
\]

Let $\{\tau_k, \tau_i\}$ be a solid edge. If $u^\nabla_k = u^\nabla_i = 1$ (resp. $u^\nabla_k = u^\nabla_i = -1$) then roots $\{u, \tau_k, \tau_i\}$ (resp. $\{u, \tau_k, \tau_i\}$) constitute the root system corresponding to the extended Dynkin diagram $A_2$, which is impossible, see Lemma [A.2]. For remaining pairs $\{u^\nabla_k, u^\nabla_i\}$, we have $-1 \leq u^\nabla_k + u^\nabla_i \leq 1$. Now, let $\{\tau_k, \tau_i\}$ be a dotted edge. If $u^\nabla_k = 1$ and $u^\nabla_i = -1$ (resp. $u^\nabla_k = -1$ and $u^\nabla_i = 1$) then roots $\{u, -\tau_k, \tau_i\}$ (resp. $\{u, \tau_k, -\tau_i\}$) constitute the root system $A_2$, which is impossible. For remaining pairs $\{u^\nabla_k, u^\nabla_i\}$, we have $-1 \leq u^\nabla_k - u^\nabla_i \leq 1$.

**Proposition 5.2.** (i) For dual reflections $s_{\tau_i}^*$, the following relations hold
\[
B_{\Gamma^*} s_{\tau_i} = s_{\tau_i} B_{\Gamma^*},
\]
and (5.12)
\[
(s_{\tau_i} \gamma)^\nabla = s_{\tau_i} B_{\Gamma^*} \gamma L = s_{\tau_i} \gamma^\nabla.
\]
\[
(5.13)
\]
(ii) For $w^* \in W_N^\nabla$ ($w^* := t_w^{-1}$), the dual element of $w \in W$, we have
\[
(w^* \gamma)^\nabla = w^* \gamma^\nabla \quad (= t_w^{-1} \gamma^\nabla).
\]
\[
(5.14)
\]
(iii) The following relations hold
\[
\mathcal{B}_\Gamma(s_{\tau_i} v) = \mathcal{B}_\Gamma(v) \text{ for any } v \in L,
\]
\[
\mathcal{B}_\Gamma'(s_{\tau_i}^* u^\nabla) = \mathcal{B}_\Gamma'(u^\nabla) \text{ for any } u^\nabla \in L^\nabla.
\]
\[
(5.15)
\]
**Proof.** (i) The equality (5.12) holds since the following is true for any $u, v \in L$:
\[
(s_{\tau_i} u, v)_{\Gamma^*} = (u, s_{\tau_i} v)_{\Gamma^*}, \text{ i.e., } \langle B_{\Gamma^*} s_{\tau_i} u, v \rangle = \langle B_{\Gamma^*} u, s_{\tau_i} v \rangle = \langle s_{\tau_i} B_{\Gamma^*} u, v \rangle, \text{ and }
\]
\[
(\langle B_{\Gamma^*} s_{\tau_i} - s_{\tau_i} B_{\Gamma^*} u, v \rangle) = 0.
\]
Let us consider eq. (5.13). Since $(\tau_i, \mu) = 0$ for any $\tau_i \in S$, and $s_{\tau_i} \mu = \mu$, we have
\[
(s_{\tau_i} \gamma)^\nabla = (s_{\tau_i} \gamma, \tau_1) \ldots (s_{\tau_i} \gamma, \tau_l) = (s_{\tau_i} \gamma L + \mu, \tau_1) \ldots (s_{\tau_i} \gamma L + \mu, \gamma) = (s_{\tau_i} \gamma L, \tau_1) \ldots (s_{\tau_i} \gamma L, \gamma) = (s_{\tau_i} \gamma L)^\nabla.
\]
Then, by (5.12) and (4.16)

$$(s_{	au_{i}}\gamma)^\nabla = (s_{	au_{i}}\gamma L)^\nabla = B_{\Gamma} s_{\tau_{i}} \gamma L = s_{\tau_{i}}^* B_{\Gamma} \gamma L = s_{\tau_{i}}^* \gamma^\nabla.$$  

(ii) Let $w = s_{\tau_{1}} s_{\tau_{2}} \ldots s_{\tau_{m}}$ be a decomposition of $w \in W$. Since $s_{\tau}^* = t s_{\tau}^{-1} = t s_{\tau}$, we deduce from (5.13) the following:

$$\begin{align*}
(w\gamma)^\nabla &= (s_{\tau_{1}} s_{\tau_{2}} \ldots s_{\tau_{m}} \gamma)^\nabla = s_{\tau_{1}}^* (s_{\tau_{2}} \ldots s_{\tau_{m}} \gamma)^\nabla = s_{\tau_{1}}^* s_{\tau_{2}}^* (s_{\tau_{3}} \ldots s_{\tau_{m}} \gamma)^\nabla = \cdots = s_{\tau_{1}}^* s_{\tau_{2}}^* \cdots s_{\tau_{m}}^* \gamma^\nabla = (s_{\tau_{1}} s_{\tau_{2}} \ldots s_{\tau_{m}})^{-1} \gamma^\nabla = w^* \gamma^\nabla.
\end{align*}$$

(iii) Further, (5.15) holds since

$$\begin{align*}
\mathcal{B}_{\Gamma}(s_{\tau_{i}} v) &= (B_{\Gamma} s_{\tau_{i}} v, s_{\tau_{i}} v) = (s_{\tau_{i}}^* B_{\Gamma} v, s_{\tau_{i}} v) = (B_{\Gamma} v, v) = \mathcal{B}_{\Gamma}(v),
\end{align*}$$

$$\begin{align*}
\mathcal{B}_{\Gamma}^*(s_{\tau_{i}}^* u^\nabla) &= (B_{\Gamma}^* s_{\tau_{i}}^* u^\nabla, s_{\tau_{i}}^* u^\nabla) = (s_{\tau_{i}} B_{\Gamma}^* u^\nabla, s_{\tau_{i}}^* u^\nabla) = (B_{\Gamma}^* u^\nabla, u^\nabla) = \mathcal{B}_{\Gamma}^*(u^\nabla).
\end{align*}$$

The linkage system component $\mathcal{L}_{\Gamma}(\Gamma)$, where $\Gamma < \Gamma'$ is a Dynkin extension, is the $W_{\Gamma'}$-orbit on the set of linkage diagrams. We mentioned in (5.13) that the rational number $p = \mathcal{B}_{\Gamma}^*(u^\nabla)$ is the invariant of the linkage system component. We call this component the extension of the Carter diagram $\Gamma$ by $p$ and denote this extension by $\{\Gamma, p\}$.

5.2. On 4-cycles and 4-cycles with a diagonal. In further considerations we need some facts on 4-cycles and 4-cycles with a diagonal.

5.2.1. How many endpoints may a linkage diagram have? In what follows, we show that the number of endpoints in any linkage diagram is $\leq 6$ (see Proposition (5.3)(i)), and in some cases this number is $\leq 4$ (see Proposition (5.3)(iii)).

**Proposition 5.3.** Let $w = w_\alpha w_\beta$ be a bicolored decomposition of $w$ into the product of two involutions, and $\Gamma$ the Carter diagram corresponding to this decomposition.

(i) Let $\gamma^\nabla$ be any linkage diagram obtained from $\Gamma$. The number of $\alpha$-endpoints in $\gamma^\nabla \leq 3$. The same applies to $\beta$-endpoints.

(ii) Let an $\alpha$-set in $\Gamma$ contain 3 roots $\{\alpha_1, \alpha_2, \alpha_3\}$. Suppose there exist non-connected roots $\beta$ and $\gamma$ connected to every $\alpha_i$ from $\alpha$-set. Then roots $\{\alpha_1, \alpha_2, \alpha_3, \beta, \gamma\}$ are linearly dependent.

(iii) Let $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ constitute a square in $\Gamma$. Suppose $\gamma$ connected to all vertices of the square. Then roots $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma\}$ are linearly dependent.

**Proof.** (i) If the linkage diagram $\gamma^\nabla$ has 4 $\alpha$-endpoints, then $\gamma^\nabla$ contains the diagram $\widetilde{D}_4 = \{\gamma, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. The vector\(^1\) $\varphi = 2\gamma + \sum_{i=1}^{4} \alpha_i$ has zero length, since

$$\begin{align*}
(\varphi, \varphi) &= 4(\gamma, \gamma) + 4 \sum_{i=1}^{4} (\alpha_i, \alpha_i) + 4 \sum_{i=1}^{4} (\gamma, \alpha_i) = 4 \cdot 2 + 4 \cdot 2 - 16 \cdot 1 = 0.
\end{align*}$$

Hence, $\varphi = 0$, contradicting the linear independence of roots $\{\gamma, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

(ii) Suppose there exist roots $\beta$ and $\gamma$ connected to every $\alpha_i$ such that the vectors of the quintuple $\{\alpha_1, \alpha_2, \alpha_3, \beta, \gamma\}$ are linearly independent, see Fig. 5.10(a). Then we have three cycles: $\{\alpha_i, \gamma, \alpha_j, \beta\}$, where $1 \leq i < j \leq 3$. Every cycle should contain an odd number of dotted edges. Let $n_1, n_2, n_3$ be the odd numbers of dotted edges in every cycle, therefore $n_1 + n_2 + n_3$ is odd. On the other hand, every dotted edge enters twice, so $n_1 + n_2 + n_3$ is even, which is a contradiction.

(iii) Suppose $\gamma$ is connected to all vertices of the square and $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma\}$ are linearly independent. Then we have 5 cycles: Four triangles $\{\alpha_i, \beta_j, \gamma\}$, where $i = 1, 2$ and $j = 1, 2$, and the square $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$, see Fig. 5.10(b). Every cycle should contain an odd number of dotted edges.

\(^1\)We denote vertices and the corresponding roots by the same letters.
Every cycle should contain an odd number of dotted edges, a case which cannot happen.

Let \( n_1, n_2, n_3, n_4, n_5 \) be the numbers of dotted edges in every cycle, therefore \( n_1 + n_2 + n_3 + n_4 + n_5 \) is odd. On the other hand, every dotted edge enters twice, so \( n_1 + n_2 + n_3 + n_4 + n_5 \) is even, which is a contradiction. For example, the left square in Fig. 5.10(b) is transformed to the right one by the reflection \( s_{\alpha_1} \), then the right square contains the cycle \( \{ \alpha_1, \beta_2, \gamma \} \) with 3 solid edges, i.e., the extended Dynkin diagram \( \tilde{A}_2 \), contradicting Lemma A.2.

**5.2.2. The diagonal in a square.**

The following proposition describes the structure of squares with and without diagonal. The squares arise during consideration of linkage diagrams near to the branch point of any Carter diagram, see Figs. 5.13 - 5.15. The following statement is essential to the proof of Proposition 5.6 determining the structure of l octets.

**Proposition 5.4 (On squares with a diagonal).** Let \( \gamma \) form a linkage diagram containing the square \( \{ \alpha_i, \beta_k, \alpha_j, \gamma \} \) without the diagonal \( \{ \alpha_i, \alpha_j \} \), i.e.,

\[
(\alpha_i, \beta_k) \neq 0, \quad (\beta_k, \alpha_j) \neq 0, \quad (\alpha_j, \gamma) \neq 0, \quad (\gamma, \alpha_i) \neq 0, \quad \text{and} \quad (\alpha_i, \alpha_j) = 0.
\]

Let the roots \( \{ \alpha_i, \beta_k, \alpha_j, \gamma \} \) be linearly independent. If the number of dotted edges in the square is even (resp. odd) then the square has a diagonal (resp. has no diagonals). Namely:

(a) If the square has no dotted edge, then \( (\gamma, \beta_k) = 1 \), i.e., there exists the dotted diagonal \( \{ \gamma, \beta_k \} \), see Fig. 5.11(a).

(b) If the square has two dotted edges \( \{ \gamma, \alpha_i \} \) and \( \{ \gamma, \alpha_j \} \) and remaining edges are solid, then \( (\gamma, \beta_k) = -1 \) i.e., there exists the solid diagonal \( \{ \gamma, \beta_k \} \), see Fig. 5.11(b).

(c) If the square has two dotted edges \( \{ \alpha_j, \gamma \} \) and \( \{ \beta_k, \alpha_i \} \) and remaining edges are solid, then \( (\gamma, \beta_k) = -1 \), i.e., there exists the solid diagonal \( \{ \gamma, \beta_k \} \), see Fig. 5.11(c).

(d) If the square has two dotted edges \( \{ \alpha_i, \gamma \} \) and \( \{ \beta_k, \alpha_j \} \) and remaining edges are solid, then \( (\gamma, \beta_k) = 1 \), i.e., there exists the dotted diagonal \( \{ \gamma, \beta_k \} \), see Fig. 5.11(d).

(e) If there is only one dotted edge \( \{ \gamma, \alpha_j \} \) and remaining edges are solid, then \( (\gamma, \beta_k) = 0 \), i.e., there are no diagonals, see Fig. 5.11(e).

(f) If there are three dotted edges \( \{ \gamma, \alpha_i \}, \{ \gamma, \alpha_j \}, \{ \beta_k, \alpha_j \} \) and the remaining edge is solid, then \( (\gamma, \beta_k) = 0 \), i.e., there are no diagonals, see Fig. 5.11(f).

**Figure 5.11.** Linkage diagrams containing a square.

**Proof.** If there is no diagonal \( \{ \gamma, \beta_k \} \) for one of cases (a), (b), (c) or (d), we get the extended Dynkin diagram \( \tilde{A}_3 \) by following changes:

(a) no changes, \quad (b) \( \gamma \rightarrow \gamma \), \quad (c) \( \gamma \rightarrow \gamma \), \quad (d) \( \alpha_i \rightarrow \alpha_i \),

\( \alpha_i \rightarrow \alpha_i \),
5.3. **Loctets and unicolored linkage diagrams.** In this section we give the complete description of linkage diagrams for every linkage system. It turns that each linkage diagram containing at least one non-zero \( \alpha \)-label \( \alpha_1, \alpha_2 \) or \( \alpha_3 \) belongs to a certain 8-cell linkage subsystem which we call the loctet, see Fig. 2.4. Every linkage system is the union of several loctets and several \( \beta \)-unicolored linkage diagrams. For the exact description, see Tables B.2–5, Figs. C.42–C.66, 8.18, 8.19, Theorem 2.3.

**Proposition 5.5.** Let \( \Gamma \) be a Carter diagram in \( C_4 \) class, \( \Gamma \neq D_4(a_1) \). Let \( \gamma^\nabla \) be a linkage diagram from \( \mathcal{L}(\Gamma) \),

\[
\gamma^\nabla = \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\vdots \\
b_1 \\
\vdots \\
\end{pmatrix} = \begin{pmatrix}
(\alpha_1, \gamma) \\
(\alpha_2, \gamma) \\
(\alpha_3, \gamma) \\
\vdots \\
(\beta_1, \gamma) \\
\vdots \\
\end{pmatrix} \quad (5.17)
\]

Among labels \( a_i \) of the linkage diagram \( \gamma^\nabla \), where \( i = 1, 2, 3 \), at least one label \( a_i \) is equal to 0.

**Proof.** For \( \Gamma \neq D_4(a_1) \), any Carter diagram out \( C_4 \) contains \( D_5(a_1) \) as a subdiagram. In the diagram \( D_5(a_1) \), the vertices \( \alpha_1, \alpha_2, \alpha_3 \) are connected to \( \beta_1 \), see Fig. 2.2. Thus, no root \( \gamma \) can be connected to all \( \alpha_i \), where \( i = 1, 2, 3 \), otherwise we get the contradiction with Proposition 5.3(ii). \( \square \)

The following proposition explains relations between linkage diagrams depicted in Fig. 2.4 and shows that every linkage diagram containing at least one non-zero \( \alpha \)-label belongs to one of the loctets \( L_{12}, L_{13}, L_{23} \).

**Proposition 5.6** (Structure of the loctet). (i) The linkage label vector \( \gamma^\nabla(n) \) depicted in Fig. 5.12 (see also Fig. 2.4) are connected by reflections \( s^*_\alpha \) where \( i = 1, 2, 3 \), and reflection \( s^*_\beta \) as follows:

\[
\begin{align*}
s^*_\alpha \gamma^\nabla(8) &= \gamma^\nabla(7), & s^*_\alpha \gamma^\nabla(1) &= \gamma^\nabla(2), \\
s^*_\beta \gamma^\nabla(7) &= \gamma^\nabla(6), & s^*_\beta \gamma^\nabla(2) &= \gamma^\nabla(3), \\
s^*_\alpha \gamma^\nabla(6) &= s^*_\alpha \gamma^\nabla(3) = \gamma^\nabla(4), & s^*_\alpha \gamma^\nabla(6) &= s^*_\alpha \gamma^\nabla(3) = \gamma^\nabla(5),
\end{align*}
\]

where \( \{i, j, k\} = \{1, 2, 3\} \). Relations of the last line in 5.18 hold up to permutation of indices \( n = 4 \) and \( n = 5 \) in \( \gamma^\nabla(n) \).

(ii) If \( \gamma^\nabla \) contains exactly two non-zero labels \( a_i, a_j \) (corresponding to \( \alpha_i, \alpha_j \)), where \( 1 \leq i, j \leq 3 \), then \( \gamma^\nabla \) is one of the following linkage diagrams:

\[
\gamma^\nabla(3), \quad \gamma^\nabla(4), \quad \gamma^\nabla(5), \quad \gamma^\nabla(6).
\]

(iii) If \( \gamma^\nabla \) contains exactly one non-zero label \( a_i \) (corresponding to \( \alpha_i \)), where \( 1 \leq i \leq 3 \), then \( \gamma^\nabla \) is one of the following linkage diagrams:

\[
\gamma^\nabla(1), \quad \gamma^\nabla(2), \quad \gamma^\nabla(7), \quad \gamma^\nabla(8).
\]
We show (5.18) only for \( \gamma \). By (5.10) we have

\[
\alpha \times \gamma
\]

for coordinates \(\alpha\) by changing the sign of coordinates \(\gamma\). Thus, we have the first line of (5.18) as follows:

\[
\gamma = \sigma \gamma
\]

for \(\sigma\) and \(\gamma\).

Proof. (i) Recall that vertices \(\alpha_1, \alpha_2, \alpha_3\) are connected with \(\beta_1\) by starlike shape, see Fig. 2.2. By (5.10) we have

\[
(s^*_{\alpha_i} \gamma \nabla)_{\alpha_k} = \begin{cases} -\gamma_{\alpha_i} & \text{for } k = i, \\ \gamma_{\alpha_i} & \text{for } k \neq i, \end{cases}
\]

and

\[
(s^*_{\beta_i} \gamma \nabla)_{\beta_k} = \begin{cases} -\gamma_{\beta_i} & \text{for } k = i, \\ \gamma_{\beta_i} & \text{for } k \neq i. \end{cases}
\]

Further, we use the “skew-symmetric relations” for coordinates \(\alpha_1, \alpha_2, \alpha_3\) and \(\beta_1\) for the following pairs of linkage diagrams \(\{\gamma^\nabla_{ij}(1), \gamma^\nabla_{ij}(8)\}\), \(\{\gamma^\nabla_{ij}(2), \gamma^\nabla_{ij}(7)\}\) and \(\{\gamma^\nabla_{ij}(3), \gamma^\nabla_{ij}(6)\}\), see Fig. 5.12. One proves the remaining cases \(n = 1, 2, 3\) by changing the sign of coordinates \(\alpha_1, \alpha_2, \alpha_3\) and \(\beta_1\) in \(\gamma^\nabla_{ij}(n)\). Applying \(s^*_{\alpha_k}\) to \(\gamma^\nabla_{ij}(8)\), where \(\{i, j, k\} = \{1, 2, 3\}\), we have the first line of (5.18) as follows:

\[
s^*_{\alpha_k} \gamma^\nabla_{ij}(8) = \gamma^\nabla_{ij}(7)
\]
Applying $s_{\beta_1}^*$ to $\gamma_{ij}^\gamma(7)$, we have the second line of (5.18):

$$s_{\beta_1}^* \gamma_{ij}^\gamma(7) = \gamma_{ij}^\gamma(6)$$

$$s_{\beta_1}^* \left( \begin{array}{c} 0 \\ 0 \\ 1 \\ \vdots \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \\ 1 \\ \vdots \end{array} \right), s_{\beta_1}^* \left( \begin{array}{c} 1 \\ -1 \\ \beta_1 \\ \vdots \end{array} \right) = \left( \begin{array}{c} 1 \\ -1 \\ \beta_1 \\ \vdots \end{array} \right), s_{\beta_1}^* \left( \begin{array}{c} 0 \\ 1 \\ \vdots \end{array} \right) = \left( \begin{array}{c} 0 \\ 1 \\ \vdots \end{array} \right), s_{\beta_1}^* \left( \begin{array}{c} 1 \\ 0 \\ \beta_1 \\ \vdots \end{array} \right) = \left( \begin{array}{c} 1 \\ 0 \\ \beta_1 \\ \vdots \end{array} \right).$$

Applying $s_{\alpha_i}^*$, $s_{\alpha_j}^*$ to $\gamma_{ij}^\gamma(6)$ we have the last line of (5.18):

$$s_{\alpha_i}^* \gamma_{ij}^\gamma(6) = \gamma_{ij}^\gamma(4), s_{\alpha_j}^* \gamma_{ij}^\gamma(6) = \gamma_{ij}^\gamma(5)$$

(ii) Here, it suffices to prove that the label $b_1$ corresponding to the coordinate $\beta_1$ is uniquely determined by $\alpha_i, \alpha_j$. For the linkage diagram $\gamma_{ij}^\gamma(3)$, the statement follows from Proposition 5.4(a), see Fig. 5.11(a). For example, for $E_7(a_1)$, the linkage diagrams $\gamma_{ij}^\gamma(3)$, where $\{ij\} \in \{\{12\}, \{13\}, \{23\}\}$, depicted

![Linkage Diagrams](image)

**Figure 5.13.** The linkage diagrams $\gamma_{ij}^\gamma(3)$ for $E_7(a_1)$, loctets $L_{ij}^b$.

In Fig. 5.13 see the linkage system $E_7(a_1)$, loctets $L_{ij}^b$ in Fig. C.48. For the linkage diagram $\gamma_{ij}^\gamma(6)$, the statement follows from Proposition 5.4(b), see Fig. 5.11(b). For example, for $E_7(a_2)$, the linkage diagrams $\gamma_{ij}^\gamma(6)$, where $\{ij\} \in \{\{12\}, \{13\}, \{23\}\}$, depicted in Fig. 5.13 see the linkage system $E_7(a_2)$, loctets $L_{ij}^b$ in Fig. C.49.
Figure 5.14. The linkage diagrams \( \gamma_{ij}(6) \) for \( E_7(a_2) \), loctets \( L_{ij}^b \).

For the linkage diagram \( \gamma_{ij}(4) \) and \( \gamma_{ij}(5) \), the statement follows from Proposition 5.4(e), see Fig. 5.11(e). For example, for \( E_7(a_3) \), the linkage diagrams \( \gamma_{ij}(4) \), where \( \{ij\} \in \{\{12\}, \{13\}, \{23\}\} \), depicted in Fig. 5.15, see the linkage system \( E_7(a_3) \), loctets \( L_{ij}^b \) in Fig. C.50.

Figure 5.15. The linkage diagrams \( \gamma_{ij}(4) \) for \( E_7(a_3) \), loctets \( L_{ij}^b \).

(iii) The label \( b_1 \) corresponding to the coordinate \( \beta_1 \) may take two values in \( \{-1, 0, 1\} \) depending on the value of \( a_i \), see Fig. 2.4. Indeed, if \( a_i = 1 \), then \( b_1 = (\gamma, \beta_1) \neq 1 \), otherwise the triangle \( \{\alpha_i, \beta_1, \gamma\} \) contains exactly two dotted edges, i.e., contains \( \tilde{A}_2 \), contradicting Lemma A.2. Thus, \( b_1 = -1 \) or \( b_1 = 0 \). Respectively, we have linkage diagrams \( \gamma_{ij}(7) \) or \( \gamma_{ij}(1) \): □

Corollary 5.7. (i) Any linkage diagram containing a non-zero \( \alpha \)-label \( \alpha_1, \alpha_2 \) or \( \alpha_3 \) belongs to one of the loctets of the linkage system.

(ii) Any linkage diagram of the loctet uniquely determines the whole loctet.

(iii) If two loctets have one common linkage diagram, they coincide.

(iv) Every linkage diagram from the linkage system either belongs to one of the loctets or is \( \beta \)-unicolored.

Proof. Statements (i) and (iv) follow from headings (ii), (iii) of Proposition 5.6; statements (ii) and (iii) follow from heading (i) of Proposition 5.6. □

The loctets of types \( L_{12}, L_{13}, L_{23} \) are the main construction blocks used for every linkage system, see all figures in Figs. C.42–C.66, 8.18, 8.19. By Proposition 5.6, any linkage diagram of a loctet gives rise to the whole loctet. By Theorem 4.5, to obtain all loctets associated with the given Carter diagram it suffices to find linkage diagrams \( \gamma_{ij}(n) \) for a certain fixed \( n \in \{1, 2, \ldots, 8\} \) satisfying the inequality:

\[ \mathcal{B}^\Gamma(\gamma_{ij}(n)) < 2. \]

The number of different loctets is defined by a number of different linkage diagrams \( \gamma_{ij}(n) \) for a given fixed \( n \), where \( 1 \leq n \leq 8 \). In what follows, we enumerate loctets by linkage diagrams \( \gamma_{ij}(8) \). In §6.1, as an example, we show how to calculate linkage diagrams \( \gamma_{ij}(8) \) of all loctets of \( E_6(a_1) \). From Tables B.10, B.21 one can recover the complete calculation of all linkage diagrams \( \gamma_{ij}(8) \) of all loctets for Carter diagrams \( \Gamma \in C_4 \bigcup DE_4 \). The linkage diagrams \( \gamma_{ij}(6) \) for every component and every loctet are listed in Table B.25 for all linkage systems with \( l < 8 \).

The partial Cartan matrices \( B_{\Gamma} \) and the inverse matrices \( B_{\Gamma}^{-1} \) for all Carter diagrams are presented in Tables A.11, A.13.
For Carter diagrams $A_3$, the technique of loctets does not work. In this case we construct the linkage system $\mathcal{L}(A_3)$ by induction, see \[10\]

6. Enumeration of linkage diagrams, loctets and linkage systems

In this section we demonstrate some calculation examples of linkage diagrams, loctets and linkage systems. These calculations are based on findings of \[5.3\]

6.1. Calculation of linkage diagrams $\gamma^\nabla(8)$. In this section, we consider only case $l < 8$. For case $D_l$, $l \geq 8$, see \[8.1\] and for case $A_l$, $l \geq 8$, see \[10.2\]

It seems a little easier to calculate the 8th linkage diagram (we calculate it for every Carter diagram loctet) rather than to calculate any other linkage diagram of a loctet since 8th linkage diagram contains 3 zeroes among coordinates $\{\alpha_1, \alpha_2, \alpha_3, \beta_1\}$. We have

$$\gamma^\nabla(8) = \begin{cases} 
\{a_1, a_2, a_3, 0, b_2\} & \text{for } D_5(a_1), \\
\{a_1, a_2, a_3, 0, b_2, b_3\} & \text{for } D_6(a_1), E_6(a_1), E_6(a_2), \\
\{a_1, a_2, a_3, 0, b_2, b_3\} & \text{for } D_6(a_2), \\
\{a_1, a_2, a_3, 0, b_2, b_3\} & \text{for } D_7(a_1), D_7(a_2), \\
\{a_1, a_2, a_3, 0, b_2, b_3, b_4\} & \text{for } E_7(a_1), E_7(a_2), E_7(a_3), E_7(a_4),
\end{cases}$$

(6.1)

where $a_i = 0$, $a_j = 0$, $a_k = 1$, $b_1 = 0$ and $\{i, j, k\} = \{1, 2, 3\}$ for type $L_{ij}$. The quadratic form $B_\Gamma^\nabla$ applied to the linkage label vector $\gamma^\nabla(8)$ gives rise to a function $F$ of several variables $b_2, b_3, \ldots$, where three dots mean $a_i$, $b_i$ for $i \geq 4$ (if the $\alpha$-set (resp. $\beta$-set) contains $> 3$ labels), i.e.,

$$F(b_2, b_3, \ldots) := B_\Gamma^\nabla(\gamma^\nabla(8)).$$

Let $q(b_2, b_3, \ldots)$ be the quadratic part of $F$ (containing only quadratic terms), $l(b_2, b_3, \ldots)$ the linear part of $F$, and $f$ the free term:

$$F(b_2, b_3, \ldots) = q(b_2, b_3, \ldots) + l(b_2, b_3, \ldots) + f.$$

It is easily to see that $q(b_2, b_3, \ldots)$ is the same for loctet types $L_{12}$, $L_{13}$, $L_{23}$. Consider the case, where the $\alpha$-set (resp. $\beta$-set) contains $\leq 3$ labels. The quadratic form $q(b_2, b_3)$ is determined by the principal submatrix of $B_\Gamma^{-1}$ associated with coordinates $\beta_2, \beta_3$:

$$q(b_2, b_3) = d_{\beta_2 \beta_2} b_2^2 + 2 d_{\beta_2 \beta_3} b_2 b_3 + d_{\beta_3 \beta_3} b_3^2,$$

(6.2)

where $d_{ij}$ is the $\{i, j\}$th element of the inverse matrix $B_\Gamma^{-1}$. The quadratic terms related to coordinates $\alpha_4$ or $\beta_4$ should be supplemented in the respective cases, see \[6.1\]. The linear part $l(b_2, b_3)$ and the free term $f$ are as follows:

$$l(b_2, b_3) = \begin{cases} 
2(d_{\alpha_1 \beta_2} b_2 + d_{\alpha_1 \beta_3} b_3) & \text{for } L_{23}, \\
2(d_{\alpha_2 \beta_2} b_2 + d_{\alpha_2 \beta_3} b_3) & \text{for } L_{13}, \\
2(d_{\alpha_3 \beta_2} b_2 + d_{\alpha_3 \beta_3} b_3) & \text{for } L_{12},
\end{cases}$$

$$f = \begin{cases} 
d_{\alpha_1 \alpha_1} & \text{for } L_{23}, \\
d_{\alpha_2 \alpha_2} & \text{for } L_{13}, \\
d_{\alpha_3 \alpha_3} & \text{for } L_{12}.
\end{cases}$$

In section \[6.1.1\] we give an example of the calculations for the case $E_6(a_1)$. By Tables \[A.16\] \[B.19\] one can recover calculations for the remaining diagrams $\Gamma \in \mathbb{C}4$.

6.1.1. Calculation example for diagram $E_6(a_1)$. The matrix $B_\Gamma^{-1}$ for $\Gamma = E_6(a_1)$ is given in Table \[A.11\] Here, by \(6.2\):

$$q(b_2, b_3) = \frac{4}{3}(b_2^2 + b_2 b_3 + b_3^2).$$

(6.3)

In all cases below we use the fact that $b_2$, $b_3$ take values in $\{-1, 0, 1\}$, see Remark \[1.4\]
a) Loctets $L_{12}$, $\gamma_{12}^\nabla(8) = \{0, 0, 1, 0, b_2, b_3\}$.

\[
R^\nabla(\gamma_{12}^\nabla(8)) = \frac{1}{3}(10 + 2(4b_2 + 5b_3) + 4(b_2^2 + b_2b_3 + b_3^2)) < 2, \quad \text{i.e.,}
\]
\[
\frac{1}{3}(2(b_2 + b_3)^2 + 2(b_2 + 2)^2 + 2(b_3 + \frac{5}{2})^2 - \frac{21}{2}) < 2,
\]
\[
\frac{2}{3}(b_2 + b_3)^2 + (b_2 + 2)^2 + (b_3 + \frac{5}{2})^2 < 2 + \frac{7}{2} = \frac{11}{2},
\]
\[
(b_2 + b_3)^2 + (b_2 + 2)^2 + (b_3 + \frac{5}{2})^2 < \frac{33}{4} = 8\frac{1}{4}.
\]

Recall that $b_2, b_3 \in \{-1, 0, 1\}$. By (6.3) we have $b_2 \neq 1$, i.e., $b_2 = -1$ or $b_2 = 0$. For $b_2 = 0$ we get

\[
b_3^2 + (b_3 + \frac{5}{2})^2 < 4\frac{1}{4}.
\]

Only $b_3 = -1$ is suitable for this case. For $b_2 = -1$ we get

\[
(b_3 - 1)^2 + (b_3 + \frac{5}{2})^2 < 4\frac{1}{4}.
\]

Again, $b_3 = -1$ is the single suitable solution. We get

\[
\gamma_{12}^\nabla(8) = \{0, 0, 1, 0, 0, -1\} \text{ or } \gamma_{12}^\nabla(8) = \{0, 0, 1, 0, -1, -1\}.
\]

b) Loctets $L_{13}$, $\gamma_{13}^\nabla(8) = \{0, 1, 0, 0, b_2, b_3\}$.

\[
R^\nabla(\gamma_{13}^\nabla(8)) = \frac{1}{3}(4 + 2(-b_2 + b_3) + 4(b_2^2 + b_2b_3 + b_3^2)) < 2, \quad \text{i.e.,}
\]
\[
\frac{1}{3}(2(b_2 + b_3) + 2(b_2 - \frac{1}{2})^2 + 2(b_3 + \frac{1}{2})^2 + 4 - \frac{1}{2} - \frac{1}{2}) < 2,
\]
\[
(b_2 + b_3)^2 + (b_2 - \frac{1}{2})^2 + (b_3 + \frac{1}{2})^2 < \frac{3}{2}.
\]

We get

\[
\gamma_{13}^\nabla(8) = \{0, 1, 0, 0, 0, 0\} \text{ or } \gamma_{13}^\nabla(8) = \{0, 1, 0, 0, 1, -1\}.
\]

c) Loctets $L_{23}$, $\gamma_{23}^\nabla(8) = \{1, 0, 0, 0, b_2, b_3\}$.

\[
R^\nabla(\gamma_{23}^\nabla(8)) = \frac{1}{3}(4 + 2(b_2 + 2b_3) + 4(b_2^2 + b_2b_3 + b_3^2)) < 2, \quad \text{i.e.,}
\]
\[
(b_2 + b_3)^2 + (b_2 + \frac{1}{2})^2 + (b_3 + 1)^2 < \frac{9}{4}.
\]

Here,

\[
\gamma_{23}^\nabla(8) = \{0, 1, 0, 0, 0, 0\} \text{ or } \gamma_{23}^\nabla(8) = \{0, 1, 0, 0, 0, -1\}.
\]

The corresponding 6 loctets of the linkage system $E_6(a_1)$ are depicted in Fig. C.46.

6.2. Calculation of the $\beta$-unicolored linkage diagrams. Now we consider $\beta$-unicolored linkage diagrams. Let $a_i = (\alpha_i, \gamma_i)$ (resp. $b_i = (\beta_i, \gamma_i)$) be coordinates of linkage label vector $\gamma^\nabla$. For $\beta$-unicolored linkage diagram $\gamma^\nabla$, we have $a_i = 0$ for $i = 1, 2, 3$. In addition, we note that

\[
b_1 = (\gamma, \beta_1) = 0, \quad (6.5)
\]

otherwise roots $\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \gamma\}$ constitute the extended Dynkin diagram $\tilde{D}_4$, which is impossible, see Lemma A.3. Eq. (6.5) holds for any Carter diagrams out $C4 \coprod DE4$ except for $D_4(a_1)$. Note that

\[
R^\nabla(\gamma^\nabla) = R^\nabla(-\gamma^\nabla) = q(b_2, b_3, \ldots),
\]
and solving the inequality $\mathcal{B}^\vee(\gamma^\vee) < 2$ we can assume that $b_2 > 0$, or $b_2 = 0, b_3 > 0$, etc. We present here calculations only for the cases $E_6(a_1), E_6(a_2)$ and $E_7(a_1)$. From Tables 3.22-3.24 one can recover calculations of the remaining cases.

6.2.1. $\beta$-unicolored linkage diagrams in $\mathcal{L}(E_6(a_1))$ and $\mathcal{L}(E_6(a_2))$. For Carter diagrams $E_6(a_1)$ and $E_6(a_2)$, the $\beta$-unicolored linkage diagrams look as $\{0, 0, 0, b_2, b_3\}$. For both diagrams $E_6(a_1)$ and $E_6(a_2)$, the $\beta$-unicolored linkage diagrams coincide since the principal $2 \times 2$ submatrices of $B_7^{-1}$ associated with coordinates $\beta_2, \beta_3$ coincide and are as follows:

$$\frac{1}{3} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix},$$

see Table A.11. By (6.3) and Theorem 4.5 we have

$$q(b_2, b_3) = \frac{4}{3}(b_2^2 + b_3^2 + b_2 b_3) = \frac{4}{3} \left( \frac{1}{2} b_2^2 + \frac{1}{2} b_3^2 + \frac{1}{2} (b_2 + b_3)^2 \right) < 2, \quad \text{i.e.,}$$

$$b_2^2 + b_3^2 + (b_2 + b_3)^2 < 3. \quad (6.6)$$

There are exactly 6 solutions of the inequality (6.6), the corresponding linkage diagrams are:

$$\{0, 0, 0, 0, 0, 1\}, \, \{0, 0, 0, 0, 0, -1\},$$
$$\{0, 0, 0, 0, 1, 0\}, \, \{0, 0, 0, 0, -1, 0\},$$
$$\{0, 0, 0, 0, 1, -1\}, \, \{0, 0, 0, 0, -1, 1\}. \quad (6.7)$$

These 6 linkage diagrams are located outside of the loctets in the linkage systems $\mathcal{L}(E_6(a_1))$, $\mathcal{L}(E_6(a_2))$, see Figs. C.46-47.

6.2.2. $\beta$-unicolored linkage diagrams in $\mathcal{L}(E_7(a_1))$. Here, the $\beta$-unicolored linkage diagrams look as $\{0, 0, 0, b_2, b_3, b_4\}$. The principal $3 \times 3$ submatrices of $B_7^{-1}$ associated with coordinates $\beta_2, \beta_3, \beta_4$ is as follows:

$$\frac{1}{2} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix},$$

see Table A.12. Then we have

$$\mathcal{B}^\vee(\gamma^\vee) = \frac{1}{2}(3b_2^2 + 4b_3^2 + 3b_4^2 + 4b_2b_3 + 2b_2b_4 + 4b_3b_4) < 2, \quad \text{i.e.,}$$

$$2(b_2 + b_3)^2 + 2(b_3 + b_4)^2 + (b_2 + b_4)^2 < 4. \quad (6.8)$$

There are exactly 8 solutions of the inequality (6.8), the corresponding linkage diagrams are:

$$\{0, 0, 0, 0, 1, -1, 0\}, \, \{0, 0, 0, 0, -1, 1, 0\},$$
$$\{0, 0, 0, 0, 1, -1\}, \, \{0, 0, 0, 0, -1, 1\},$$
$$\{0, 0, 0, 0, 0, 1\}, \, \{0, 0, 0, 0, 0, -1\},$$
$$\{0, 0, 0, 0, 1, 0\}, \, \{0, 0, 0, 0, -1, 0\}. \quad (6.9)$$

These 8 linkage diagrams are located outside of the loctets in the linkage systems $\mathcal{L}(E_7(a_1))$, see Fig. C.48.

6.3. Linkage systems for simply-laced Dynkin diagrams. In order to build the linkage systems for Dynkin diagrams $E_n$, $D_n$, we can use the technique of the partial Cartan matrix, linkage diagrams and loctets from [5.3]. Note that the partial Cartan matrix for Dynkin diagrams coincides with the usual Cartan matrix $B$ associated with the given Dynkin diagram. Since $E_8$ does not have linkage diagrams, see Remark 2.4, we are interested only in $E_6$, $E_7$, $D_n$. In cases $E_6$, $E_7$, $D_5$, $D_6$, $D_7$, for the Cartan matrix $B$ and its inverse $B^{-1}$, see Table A.14. One can recover the complete
calculation of linkage diagrams $\gamma(\delta)$ of all loctets of $E_6$, $E_7$, $D_5$, $D_6$, $D_7$ by means of Tables [\ref{tab:linkage}]. The $\beta$-unicolored linkage diagrams look as follows:

$$
\gamma = \begin{cases} 
\{0,0,0,b_2,b_3\} & \text{for } E_6, \\
\{0,0,0,a_4,0,b_2,b_3\} & \text{for } E_7, \\
\{0,0,0,0,b_2\} & \text{for } D_5, \\
\{0,0,0,a_4,0,b_2\} & \text{for } D_6, \\
\{0,0,0,a_4,0,0,b_2,b_3\} & \text{for } D_7,
\end{cases}
$$

see Tables [\ref{tab:linkage}]. The $\beta$-unicolored linkage diagrams are located outside of all loctets, see Figs. [\ref{fig:linkage}]. (top), [\ref{fig:linkage}].

Note that for $E_n$, the principal matrix associated with coordinates $\beta_2$, $\beta_3$ coincide with the principal matrix for the Carter diagrams $E_n(a_1)$, $E_n(a_2)$, see [\ref{tab:linkage}]. and, consequently, $\beta$-unicolored linkage diagrams coincide with these diagrams for $E_6(a_1)$, $E_6(a_2)$, see 6 solutions [\ref{tab:linkage}]. of the inequality $0.6$.

For $A_1$, the technique of loctets cannot be applied. The reason of this is lack of branch point. For $l = 5, 6, 7$, the linkage system $L(A_l)$ consists of $A$-, $D$- and $E$-components, see $\ref{tab:linkage}$. For $l \leq 4$ and $l \geq 8$, the linkage system $L(A_l)$ consists only of $A$- and $D$-components, see $\ref{tab:linkage}$. The linkage system components $L_{A_1}(A_l)$ and $L_{A_1}(A_l)$ are constructed by means of induction process, see $\ref{tab:linkage}$ and Fig. $\ref{fig:linkage}$.

6.3.1. Asymmetric relations between $A$-, $D$-, $E$-types of Dynkin diagrams. The relation between the Dynkin diagrams of $E$-type and $D$-type is asymmetric in the following sense: The root systems $E_n$ contains the root subsystem $D_n-1$ for $n = 6, 7, 8$; however, $E_6$ is not contained in $D_n$ for any $n$, see Remark $\ref{rem:asymmetry}$.

Similarly, the relation between the Dynkin diagrams of $A$-type and Dynkin diagrams of $D$- and $E$-types is asymmetric, namely: The root systems of the $D$- and the $E$-type contain root subsystems $A_n$ for a certain $n$. The converse statement is not true: The root system of $A$-type does not contain root subsystems of $D$- and $E$-types, see Lemma $\ref{lem:asymmetry}$.ii).

The asymmetric relations between $A$-, $D$-, $E$-types of Dynkin diagrams can be observed by means of the Table $\ref{tab:asymmetry}$. By abuse of language we can say that $E$-type contains $ADE$-types; $D$-type contains $AD$-types; $A$-type contains only $A$-type.

7. Theorem on coincidence of linkage and weight systems for Dynkin diagrams

7.1. Dominant weights and Dynkin labels. Let $\Phi$ be the root system in the linear space $V$ with the Weyl group $W$ (associated with the given Dynkin diagram $\Gamma$). The set of vectors $\Lambda = \{\lambda \mid \lambda \in V\}$ such that

$$
\langle \lambda, \alpha \rangle := \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \text{ for any } \alpha \in \Phi
$$

(7.1)

is said to be the weight lattice in $V$, and vectors $\lambda \in \Lambda$ are called weights. Clearly, the weight lattice contains the root system $\Phi$, see [\ref{ref:weight}, Bo02]. Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ be the set of positive simple roots of $\Phi$, and $\lambda$ a weight in $\Lambda$. If $\langle \lambda, \alpha_1 \rangle \geq 0$ for any $\alpha_1 \in \Delta$, the weight $\lambda$ is said to be dominant. The set of all dominant weights is denoted by $\Lambda^+$, this set lies in the fundamental Weyl chamber $\mathcal{C}$. The dominant weight $\vec{\omega}_i \in \Lambda^+$ is said to be a fundamental dominant weight if

$$
\frac{2(\vec{\omega}_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}.
$$

(7.2)

The coefficients $\{\lambda_i\}$ of the decomposition of the weight $\lambda \in \Lambda$ in the basis of fundamental weights $\vec{\omega}_i$ are said to be Dynkin labels. In other words, for

$$
\lambda = \sum_{i=1}^{l} \lambda_i \vec{\omega}_i, \quad \text{we have} \quad \langle \lambda, \alpha_j \rangle = \lambda_j.
$$

(7.3)
For the simply-laced Dynkin diagrams\(^1\), by (7.1) we have \((\lambda, \alpha_j) = \lambda_j\). In this case, the vector of Dynkin labels looks as follows:

\[
\lambda^\vee := \begin{pmatrix}
(\lambda, \alpha_1) \\
\ldots \\
(\lambda, \alpha_l)
\end{pmatrix} = B \lambda,
\]  
(7.4)

where \(B\) is the Cartan matrix associated with \(\Gamma\). Let \(E\) be a module over the Weyl group \(W\), \(E^*\) be the dual space, \(W^\vee\) be the contragredient representation of \(W\) in \(E^*\). The element \(w^* \in W^\vee\) is defined as follows:

\[
w^* := t^i w^{-1}, \quad \text{in particular,} \quad s^*_\alpha = t s_\alpha.
\]  
(7.5)

Then we have

\[
w^* B = B w \implies B w^{-1} = (w^*)^{-1} B \implies w B^{-1} = B^{-1} w^*.
\]  
(7.6)

In particular,

\[
w^* \lambda^\vee = w^* B \lambda = B w \lambda = (w \lambda)^\vee,
\]  
(7.7)

see Propositions 5.1, 5.2. By (7.1) \(W\) acts on the weight lattice \(\Lambda\). By (7.7) \(W^\vee\) acts on the lattice spanned by vectors of Dynkin labels \(\lambda^\vee\), where \(\lambda \in \Lambda\).

We say that a weight \(\lambda\) is conjugate to a weight \(\eta\), if they lie in the same orbit of \(W\). The following proposition is one of central statements in the theory of root systems and weights.

**Proposition 7.1.** (\([\text{Bo02}\text{, Ch.VI, §1, n°10]}\))^2 For every weight \(\lambda \in \Lambda\), there is one and only one fundamental weight \(\varpi_i\) conjugate to \(\lambda\).  
\(\Box\)

The fundamental weights \(\varpi_i\), where \(i = 1, \ldots, l\), generate \(l\) non-intersecting \(W\)-orbits. These orbits are the components of the diagram we call weight system\(^3\). The weight systems arise in the representation theory of semisimple Lie algebras, see \([\text{Bo05}\text{, Si81]}\). Let \(g\) be the simple Lie algebra associated with the Dynkin diagram \(\Gamma\). The irreducible finite-dimensional representation \(V\) of \(g\) is said to be fundamental representation if the highest weight of \(V\) is a fundamental weight, see \([\text{Bo05}]\).

For weight systems of two dual 27-dimensional fundamental representations of the semisimple Lie algebra \(E_6\), see Fig. C.52. In particle physics these representations are denoted by \(27\) and \(\overline{27}\), see \([\text{Si81}]\).

### 7.2. Relationship between linkage system and weight system for Dynkin diagrams.

Let \(\varpi_i\) be the unit vector with a 1 in the \(i\)th slot: \((\varpi_i)_j = \delta_{ij}\).

**Lemma 7.2.** (i) There is a one-to-one correspondence between the \(W\)-orbit of the fundamental weight \(\varpi_i\) and the \(W^\vee\)-orbit of the unit vector \(\varpi_i\).

(ii) The correspondence between \(W\)- and \(W^\vee\)-orbits is carried out by Cartan matrix \(B\) as follows:

\[
B(\varpi_i) = \varpi_i,  
B(w_1 \varpi_i) = w_1^\ast \varpi_i,  
B(w_2 w_1 \varpi_i) = w_2^\ast w_1^\ast \varpi_i,
\]  
(7.8)

where \(w_1\) and \(w_2\) are any elements in \(W\), see Fig. C.7.10.

**Proof.** By (7.2)

\[
\varpi_i = B^{-1} \varpi_i.
\]  
(7.9)

By (7.3), we have

\[
w \varpi_i = w B^{-1} \varpi_i = B^{-1} (w^\ast \varpi_i).
\]  
(7.10)

---

\(^1\)Recall that in this case, \((\alpha, \alpha) = 2\) for any simple root \(\alpha\), see (2.3).  
\(^2\)This proposition follows from the fact the Weyl group \(W\) acts simply transitively on the set of all Weyl chambers, see \([\text{Bo02}\text{, Ch.VI, §1, n°5, Theorem 2]}\), \([\text{Si81}]\).  
\(^3\)In the literature (see, for example, \([\text{Va00}\text{, Ch84}]\)), the term weight diagram is often used instead of the term weight system. However, the term “diagram” is heavily overloaded in our context.
Relation (7.10) yields the desired correspondence. The first and second relations from (7.8) follow from (7.9) and (7.10). Further, by (7.6)

\[ B(w_2^{-1} \bar{\omega}_i) = w_2 B w_1 \bar{\omega}_i = w_2^* w_1^* B \bar{\omega}_i = w_2^* w_1^* \bar{\omega}_i. \]

\[ \square \]

Figure 7.16. The one-to-one correspondence between \( W \)-orbit and \( \mathcal{W}^\vee \)-orbit

**Theorem 7.3** (On linkage systems for Dynkin diagrams). Let \( \Gamma \) be a simply-laced Dynkin diagram, \( g \) the simple Lie algebra associated with \( \Gamma \). Every \( A \)-, \( D \)- or \( E \)-component of the linkage system \( \mathcal{L}(\Gamma) \) coincides with a weight system of one of fundamental representations of \( g \).

**Proof.** The fact \( \gamma^\nabla \in \mathcal{L}(\Gamma) \) means that

\[ \gamma^\nabla = \begin{pmatrix} (\gamma, \tau_1) \\ \vdots \\ (\gamma, \tau_l) \end{pmatrix} \]

for a certain root \( \gamma \) not lying in the space \( L \) spanned by the \( \Gamma \)-associated root subset \( S = \{ \tau_1, \ldots, \tau_l \} \), see (2.2). In addition,

\[ \gamma = \gamma_L + \mu, \quad \gamma^\nabla = B \gamma_L. \]

where \( \gamma_L \) is the projection of \( \gamma \) on the linear subspace \( L = \langle \tau_1, \ldots, \tau_l \rangle \) and \( \mu \) depends on the type of the extension, see [4.1] and Proposition 4.4. It suffices to prove that any \( \gamma^\nabla \) lies in the \( \mathcal{W}^\vee \)-orbit of a certain unit vector \( \bar{e}_i \). For every simply root \( \tau_i \in \Phi \), the label \( \gamma^\nabla = (\gamma, \tau_i) = (\gamma_L, \tau_i) \) takes value in \( \{-1, 0, 1\} \), see [2.11]. Then \( (\gamma_L, \alpha) \in \mathbb{Z} \) for any root \( \alpha \in \Phi \). By (7.1), since \( (\alpha, \alpha) = 2 \) we have \( (\gamma_L, \alpha) \in \mathbb{Z} \) for any root \( \alpha \in \Phi \). Thus, the projection \( \gamma_L \) of any linkage root \( \gamma \) is a weight. Then,

by Proposition 7.1 \( \gamma_L \) is conjugate to some fundamental weight \( \omega_i \):

\[ \gamma_L = w \bar{\omega}_i. \]

Finally, by (7.10)

\[ \gamma^\nabla = B \gamma_L = B w \bar{\omega}_i = B B^{-1} w^* \bar{\omega}_i = w^* \bar{\omega}_i, \]

i.e., \( \gamma^\nabla \) lies in the \( \mathcal{W}^\vee \)-orbit of \( \bar{e}_i \):

\[ \gamma^\nabla = w^* \bar{\omega}_i, \]

(7.12)

and the weight \( w \bar{\omega}_i \) (on the \( \mathcal{W} \)-orbit of the fundamental weight \( \bar{\omega}_i \)) uniquely corresponds to \( \gamma^\nabla \):

\[ w \bar{\omega}_i = B^{-1} \gamma^\nabla. \]

\[ \square \]

**Corollary 7.4.** For any simply-laced Dynkin diagram \( \Gamma \), every \( A \)-, \( D \)- or \( E \)-component of the linkage system \( \mathcal{L}(\Gamma) \) contains a unique \( \beta \)-unicolored or \( \alpha \)-unicolored linkage diagram \( \gamma^\nabla \) such that \( \gamma_L = B^{-1} \gamma^\nabla \) coincides with one of fundamental weights \( \bar{\omega}_i \).

**Proof.** Let us take \( \gamma^\nabla = \bar{e}_i \), see (7.12). The vector \( \bar{e}_i \) has only one non-zero coordinate, i.e., \( \bar{e}_i \) is \( \beta \)-unicolored or \( \alpha \)-unicolored. By Theorem 7.3 and Proposition 7.1 \( \gamma^\nabla \) is unique.  \[ \square \]
Remark 7.5. If $\Gamma$ is a Carter diagram but not a Dynkin diagram, Corollary 7.4 does not hold. For example, for the linkage system $\mathcal{L}(D_6(a_1))$, the first 64-element component does not contain such linkage diagrams at all, see Fig. (C.44) left, and the second 64-element component contains 2 such linkage diagrams:

$$\gamma^{\nabla}_{13}(8) = \{0, 1, 0, 0, 0, 0\} \in L^{\nabla}_{13}, \quad \gamma^{\nabla}_{12}(8) = \{0, 0, 1, 0, 0, 0\} \in L^{\nabla}_{12},$$

see Fig. (C.44) right.

8. $D$-type linkage systems

8.1. The linkage systems $\mathcal{L}(D_l)$ and $\mathcal{L}(D_l(a_k))$ for $l \geq 8$. In this section, we assume that $\Gamma$ is one of the Carter diagrams $D_l$ or $D_l(a_k)$, see Fig. 8.17(a), (b). Let $S$ be a $\Gamma$-associated root subset.

Lemma 8.1. Let $D_{l+1}$ be the Dynkin extension of the Carter diagram $D_l$, and $\tau_{l+1}$ a simple positive root from the root stratum $\Phi(D_{l+1}) \backslash \Phi(S)$:

$$\Phi(S) = \{\tau_1, \ldots, \tau_l\}, \quad \Phi(D_{l+1}) = \{\tau_1, \ldots, \tau_l, \tau_{l+1}\}. \quad (8.1)$$

Let $\varphi$ be a positive root in the root stratum $\Phi(D_{l+1}) \backslash \Phi(S)$, and $\mu_{\max}$ be maximal root in $\Phi(D_{l+1})$.

(i) The vector

$$\delta = \mu_{\max} - \varphi + \tau_{l+1} \quad (8.2)$$

is also a root in $\Phi(D_{l+1}) \backslash \Phi(S)$.

(ii) The linkage label vectors $\varphi^{\nabla}$ and $-\delta^{\nabla}$ coincide:

$$\delta^{\nabla} = -\varphi^{\nabla}. \quad (8.3)$$

Figure 8.17. The Dynkin extension $D_l < D_{l+1}$

Proof. Let $\mathcal{B}$ be the quadratic Tits form associated with $D_{l+1}$.

(i) By (2.3) it suffices to prove that

$$\mathcal{B}(\delta) = 2. \quad (8.4)$$

If $\varphi = \mu_{\max}$ (resp. $\tau_{l+1}$), then $\delta = \tau_{l+1}$ (resp. $\mu_{\max}$). In both cases, $\delta$ is the root in $\Phi(D_{l+1}) \backslash \Phi(S)$. Suppose $\varphi \neq \mu_{\max}, \tau_{l+1}$. We need to prove that

$$\mathcal{B}(\mu_{\max} - \varphi + \tau_{l+1}) = 2, \quad \text{i.e.,}$$

$$\mathcal{B}(\mu_{\max}) + \mathcal{B}(\varphi) + \mathcal{B}(\tau_{l+1}) + 2(\mu_{\max}, \tau_{l+1}) - 2(\varphi, \mu_{\max} + \tau_{l+1}) = 2. \quad (8.5)$$

Since $\varphi, \tau_{l+1}, \mu_{\max}$ are roots, we have $\mathcal{B}(\varphi) = \mathcal{B}(\tau_{l+1}) = \mathcal{B}(\mu_{\max}) = 2$. Then the last equality in (8.5) is equivalent to

$$4 + 2(\mu_{\max}, \tau_{l+1}) - 2(\varphi, \mu_{\max} + \tau_{l+1}) = 0, \quad \text{i.e.,}$$

$$(\varphi, \mu_{\max} + \tau_{l+1}) - (\mu_{\max}, \tau_{l+1}) = 2.$$

Further, $(\mu_{\max}, \tau_i) = 0$ for any $i \neq l$. In particular, $(\mu_{\max}, \tau_{l+1}) = 0$ and it suffices to prove that

$$(\varphi, \mu_{\max} + \tau_{l+1}) = 2 \text{ for any } \varphi \in \Phi(D_{l+1}) \backslash \Phi(S). \quad (8.6)$$

We have

$$(\gamma, \mu_{\max} + \tau_{l+1}) = \begin{cases} 2 & \text{for } \gamma = \tau_{l+1}, \text{ since } \mu_{\max} \perp \tau_{l+1}, \\ 0 & \text{for } \gamma = \tau_i, \text{ since } (\tau_i, \mu_{\max}) = 1, (\tau_i, \tau_{l+1}) = -1, \\ 0 & \text{for } \gamma = \tau_i, \text{ where } i < l. \end{cases} \quad (8.7)$$
The fact that $\varphi \in \Phi(D_{l+1}) \setminus \Phi(S)$ and $\varphi$ is positive means that $\tau_{l+1}$ enters with coefficient 1 into the decomposition of $\varphi$ with relation to $\{\tau_1, \ldots, \tau_{l+1}\}$. Then, by (8.7) the relation (8.6) holds for any root $\varphi \in \Phi(D_{l+1}) \setminus \Phi(S)$. Therefore, (8.4) holds. In other words, $\delta$ is also a root. By (8.2) $\tau_{l+1}$ also enters with coefficient 1 into the decomposition of $\delta$ with relation to $\{\tau_1, \ldots, \tau_{l+1}\}$. Thus, $\delta \in \Phi(D_{l+1}) \setminus \Phi(S)$.

(ii) By (8.2) we have
\[ (\delta, \tau_i) = (\mu_{\text{max}} + \tau_{l+1}, \tau_i) - (\varphi, \tau_i), \] where $1 \leq i \leq l$.

By (8.4) for $i \leq l$, we have
\[ \delta_i^- = (\delta, \tau_i) = - (\varphi, \tau_i) = - \varphi_i^- , \] where $1 \leq i \leq l$.

Since $L = [\tau_1, \ldots, \tau_l]$, we have $\delta_i^- = - \varphi_i^-$. \qed

Now, we can calculate the size of the linkage systems $L(D_l)$ and $L(D_l(a_k))$ for $l \geq 8$. For sizes of the linkage systems $L(D_5), L(D_6), L(D_7)$, and $L(D_5(a_k)), L(D_6(a_k)), L(D_7(a_k))$, see (8.2).

**Corollary 8.2.** (i) For any $l$, the size of the linkage system component $L_{D_{l+1}}(D_l)$ (resp. $L_{D_{l+1}}(D_l(a_k))$, where $1 \leq k \leq \left\lceil \frac{l-2}{2} \right\rceil$, is equal to $2l$.

(ii) For $l = 4$ and $l \geq 8$, the size of the linkage system $L(D_l)$ (resp. $L(D_l(a_k))$, where $1 \leq k \leq \left\lceil \frac{l-2}{2} \right\rceil$, is equal to $2l$.

**Proof.** (i) The number of roots of $\Phi(D_l)$ is $2l(l-1)$, see [Bo02, Table IV]. Then by (3.12) and Corollary 3.9 we have
\[ |\mathcal{L}_{D_{l+1}}(D_l(a_k))| = |\mathcal{L}_{D_{l+1}}(D_l)| \leq 2(l+1)l - 2l(l-1) = 4l. \]

By Lemma (iii), this size is twice less:
\[ |\mathcal{L}_{D_{l+1}}(D_l(a_k))| = |\mathcal{L}_{D_{l+1}}(D_l)| \leq 2l. \]

In Fig. (8.20) we present $2l$ linkage label vectors for the Carter diagram $D_l$. So, we conclude that there are exactly $2l$ linkage label vectors in the $D$-component $L_{D_{l+1}}(D_l)$ and, therefore, in the $D$-component $L_{D_{l+1}}(D_l(a_k))$.

(ii) For $l = 4$ and $l \geq 8$, we have $L(D_l) = L_{D_{l+1}}(D_l)$ and $L(D_l(a_k)) = L_{D_{l+1}}(D_l(a_k))$. So, $|L(D_l)| = 2l$ and $|L(D_l(a_k))| = 2l$. For $l = 5, 6, 7$, this is not so, since
\[ L(D_l) = L_{E_{l+1}}(D_l) \cup L_{D_{l+1}}(D_l) \text{ and } L(D_l(a_k)) = L_{E_{l+1}}(D_l(a_k)) \cup L_{D_{l+1}}(D_l(a_k)). \]

8.2. **The linkage systems** $L(D_l)$ and $L(D_l(a_k))$ for $l = 5, 6, 7$. For $l = 5, 6, 7$, in addition to Dynkin extensions $D_l \subset D_{l+1}$, there are Dynkin extensions of $E$-type, namely
\[ D_5 \subset D_6 \subset D_7 \subset D_8. \]
The corresponding $E$-components are $L_{E_6}(D_5), L_{E_7}(D_6), \text{ and } L_{E_8}(D_7)$.

8.2.1. **The $D$- and $E$-components** $L_{D_6}(D_5), L_{E_6}(D_5), L_{D_6}(D_5(a_1)) \text{ and } L_{E_6}(D_5(a_1))$. By (3.12) and Corollary (3.9) we have
\[ |L_{E_6}(D_5(a_1))| = |L_{E_6}(D_5)| \leq |\Phi(E_6)| - |\Phi(D_5)| = 72 - 40 = 32. \]

The linkage system $L(D_5)$ (resp. $L(D_5(a_1))$ consists of two parts:
\[ L(D_5) = L_{D_6}(D_5) \cup L_{E_6}(D_5), \]
\[ L(D_5(a_1)) = L_{D_6}(D_5(a_1)) \cup L_{E_6}(D_5(a_1)). \]
By Corollary 8.2(i), we have
\[
|\mathcal{L}_{D_6}(D_5)| = |\mathcal{L}_{D_6}(D_5(a_1))| = 2 \times 5 = 10,
|\mathcal{L}(D_5)| \leq 10 + 32 = 42,
|\mathcal{L}(D_5(a_1))| \leq 10 + 32 = 42.
\]
(8.10)

There are 42 linkage diagrams of \(\mathcal{L}(D_5)\) presented in Fig. C.35; so we conclude that there are exactly 42 diagrams in \(\mathcal{L}(D_5)\), and, therefore, in \(\mathcal{L}(D_5(a_1))\), see Fig. C.43.

8.2.2. The D- and E-components \(\mathcal{L}_{D_7}(D_6), \mathcal{L}_{D_7}(D_6(a_k)), \mathcal{L}_{E_7}(D_6)\) and \(\mathcal{L}_{E_7}(D_6(a_k))\). Let \(S_1\) (resp. \(S_2\)) be \(D_6(a_1)\)-associated (resp. \(D_6(a_2)\)-associated) root subset. As above, by (3.12) and Corollary 3.9 we have
\[
|\mathcal{L}_{E_7}(D_6(a_2))| = |\mathcal{L}_{E_7}(D_6(a_1))| = |\mathcal{L}_{E_7}(D_6)| \leq |\Phi(E_7)| - |\Phi(D_6)| = 126 - 60 = 66.
\]

| \(\varphi \in \Phi(D_8)\backslash \Phi(\tau)\) | \(\delta \in \Phi(D_8)\backslash \Phi(\tau)\) | \(\delta^\nabla = -\varphi^\nabla \in \mathcal{L}_{D_7}(D_7)\) |
|---|---|---|
| 1 | 0 0 0 0 0 0 1 | 1 2 2 2 2 2 1 | 0 0 0 0 0 0 -1 |
| 0 | 0 0 0 0 1 1 | 1 2 2 2 2 1 1 | 0 0 0 0 0 -1 1 |
| 2 | 0 0 0 0 0 1 1 | 1 2 2 2 1 1 1 | 0 0 0 0 1 0 |
| 0 | 0 0 0 1 1 1 1 | 1 2 2 1 1 1 1 | 0 0 -1 0 0 0 |
| 3 | 0 0 0 1 1 1 1 | 1 2 2 1 1 1 1 | 0 0 1 0 0 0 |
| 0 | 0 0 1 1 1 1 1 | 1 2 1 1 1 1 1 | 0 -1 0 0 0 0 |
| 4 | 0 1 1 1 1 1 1 | 1 1 1 1 1 1 1 | -1 1 0 0 0 0 |
| 0 | 1 1 1 1 1 1 1 | 1 1 1 1 1 1 1 | -1 0 0 0 |
| 5 | 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 | 1 0 0 0 0 |
| 0 | 1 1 1 1 1 1 1 | 1 1 1 1 1 1 1 | -1 0 0 0 |

Table 8.4. 7 pairs of positive roots \(\varphi, \delta \in D_8\) such that \(\delta^\nabla = -\varphi^\nabla\). There are also 7 pairs of negative roots \(\varphi, \delta \in D_8\) such that \(\delta^\nabla = -\varphi^\nabla\).

The linkage system \(\mathcal{L}(D_6)\), (resp. \(\mathcal{L}(D_6(a_1))\) and \(\mathcal{L}(D_6(a_2))\)) consists of two parts:
\[
\mathcal{L}(D_6) = \mathcal{L}_{D_7}(D_6) \cup \mathcal{L}_{E_7}(D_6),
\]
\[
\mathcal{L}(D_6(a_1)) = \mathcal{L}_{D_7}(D_6(a_1)) \cup \mathcal{L}_{E_7}(D_6(a_1)),
\]
\[
\mathcal{L}(D_6(a_2)) = \mathcal{L}_{D_7}(D_6(a_2)) \cup \mathcal{L}_{E_7}(D_6(a_2)).
\]
(8.11)

By Corollary 8.2(i), we have
\[
|\mathcal{L}_{D_7}(D_6)| = |\mathcal{L}_{D_7}(D_6(a_1))| = |\mathcal{L}_{D_7}(D_6(a_2))| = 2 \times 6 = 12.
\]
By (8.11),
\[
|\mathcal{L}(D_6(a_1))| = |\mathcal{L}(D_6(a_2))| = |\mathcal{L}(D_6)| \leq 12 + 66 = 78.
\]
Let the coordinates of roots of $E_7$ be as follows
\[ \tau_6 \tau_1 \tau_2 \tau_3 \tau_4 \tau_5 \beta_2 \]

Consider the maximal (resp. minimal) root in $E_7$:
\[ \mu_{\text{max}} = \begin{pmatrix} 2 & 3 & 4 & 3 & 2 & 1 \\ 2 \end{pmatrix}, \quad \mu_{\text{min}} = -\begin{pmatrix} 2 & 3 & 4 & 3 & 2 & 1 \\ 2 \end{pmatrix}. \]

Roots $\pm \mu_{\text{max}}$ are orthogonal to any simple root except for $\tau_6$. Further, $\pm \mu_{\text{max}} \not\in \Phi(D_6)$, where $\Phi(D_6)$ is spanned by $\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \beta_2\}$. Vectors $\pm \mu_{\text{max}}$ from $L_{E_7}(D_6)$ are zero, i.e.,
\[ |L(D_6(a_1))| = |L(D_6(a_2))| = |L(D_6)| \leq 78 - 2 = 76. \]

There are 76 linkage diagrams of $L(D_6)$ (resp. $L(D_6(a_1))$, resp. $L(D_6(a_2))$) presented in Fig. C.57 (resp. Fig. C.44, resp. Fig. C.45). We conclude that there are exactly 76 linkage diagrams in each of linkage systems $L(D_6)$, $L(D_6(a_1))$, and $L(D_6(a_2))$.

8.2.3. The $D$- and $E$-components $L_{D_8}(D_7)$, $L_{D_8}(D_7(a_k))$, $L_{E_8}(D_7)$, and $L_{E_8}(D_7(a_k))$. By (3.12) and Corollary 3.9
\[ |L_{E_8}(D_7(a_2))| = |L_{E_8}(D_7(a_1))| = |L_{E_8}(D_7)| \leq |\Phi(E_8)| - |\Phi(D_7)| = 240 - 84 = 156. \] (8.12)

| $\eta \in \Phi(E_8) \setminus \Phi(D_7)$ | $\lambda \in \Phi(E_8) \setminus \Phi(D_7)$ | $\eta^{\nabla} = -\lambda^{\nabla} \in L_{E_8}(D_7)$ |
|---|---|---|
| 1 2 3 4 3 2 1 0 2 | 2 4 6 5 4 3 2 3 | 0 0 0 0 0 0 -1 0 |
| 2 3 4 3 2 1 1 2 | 2 4 6 5 4 3 1 3 | 0 0 0 0 -1 1 0 0 |
| 2 3 4 3 2 2 1 2 | 2 4 6 5 4 2 1 3 | 0 0 0 -1 1 0 0 0 |
| 2 3 4 3 3 2 1 2 | 2 4 6 5 3 2 1 3 | 0 0 -1 1 0 0 0 0 |
| 2 3 4 4 3 2 1 2 | 2 4 6 4 3 2 1 3 | 0 -1 1 0 0 0 0 0 |
| 2 3 5 4 3 2 1 2 | 2 4 5 4 3 2 1 3 | -1 1 0 0 0 0 0 0 |
| 2 3 5 4 4 3 1 3 | 2 4 5 4 3 2 1 3 | -1 0 0 0 0 0 0 0 |

Table 8.5. 7 pairs of positive roots $\eta, \lambda \in \Phi(E_8)$ such that $\eta^{\nabla} = -\lambda^{\nabla}$. There are also 7 pairs of negative roots $\eta, \lambda \in \Phi(E_8)$ such that $\eta^{\nabla} = -\lambda^{\nabla}$.
Figure 8.18. $D_l(a_k)$ for $l > 7$, 1 loctet, 2l linkage diagrams
Figure 8.19. The linkage system $D_l(\alpha_k)$ for $l > 7$ (wind rose of linkages). The single loctet $L_{23}$ is depicted in the shaded area.
Figure 8.20. The linkage system $D_l$ for $l > 7$, $2l$ linkages. The single loctet $L_{23}$ is depicted in the shaded area.
Let $L$ be the linear space spanned by the simple roots $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3\}$ of the root system $\Phi(D_7)$, and $L^\vee$ is the space of linkage labels, see [2,1,2]. We have

\[
\mathcal{L}_{D_8}(D_7) = \{\gamma \in \Phi(D_8) \setminus \Phi(D_7)\} \subset L^\vee,
\]
\[
\mathcal{L}_{E_8}(D_7) = \{\gamma \in \Phi(E_8) \setminus \Phi(D_7)\} \subset L^\vee.
\]

For each of the seven pairs of $\{\eta, \lambda\}$ of positive roots in $\Phi(E_8)$ given in Table 8.5, we have $\eta^\vee = -\lambda^\vee$. There are also seven pairs of negative roots such that every pair gives the same linkage label vector in $\Phi(E_8) \setminus \Phi(D_7)$. Thus, we have to subtract 14 in the estimation (8.12) for $\mathcal{L}_{E_8}(D_7)$:

\[
|\mathcal{L}_{E_8}(D_7(a_2))| = |\mathcal{L}_{E_8}(D_7(a_1))| = |\mathcal{L}_{E_8}(D_7)| \leq 156 - 14 = 142.
\]

In addition, we have a new phenomenon: components $\mathcal{L}_{E_8}(D_7)$ and $\mathcal{L}_{D_8}(D_7)$ overlap. Namely, roots of Tables 8.4 and 8.5 yield the same linkage label vectors in $L^\vee$, i.e.,

\[
\mathcal{L}_{D_8}(D_7) \subset \mathcal{L}_{E_8}(D_7).
\]

Hence,

\[
\mathcal{L}(D_7) = \mathcal{L}_{D_8}(D_7) \cup \mathcal{L}_{E_8}(D_7) = \mathcal{L}_{E_8}(D_7), \text{ i.e., } |\mathcal{L}(D_7)| = |\mathcal{L}_{E_8}(D_7)|.
\]

Thus, by (8.13), (8.15)

\[
|\mathcal{L}(D_7(a_1))| = |\mathcal{L}(D_7(a_2))| = |\mathcal{L}(D_7)| \leq 142.
\]

In Figs. C.63, C.64, C.65(a) we have $2 \times 64 + 14 = 142$ linkage label vectors in the linkage system $\mathcal{L}(D_7)$. Then we conclude that there are exactly 142 elements in each of linkage systems $\mathcal{L}(D_7)$, $\mathcal{L}(D_7(a_1))$ and $\mathcal{L}(D_7(a_2))$.

9. E-type linkage systems

9.1. The linkage systems $\mathcal{L}(E_l)$ and $\mathcal{L}(E_l(a_k))$ for $l = 6, 7$. In this section, we suppose that $\Gamma$ is one of the Carter diagrams

\[
E_6, E_6(a_k), \text{ where } k = 1, 2,
\]
\[
E_7, E_7(a_k), \text{ where } k = 1, 2, 3, 4,
\]

see Table 3.3. Let $S$ be a $\Gamma$-associated root subset.

9.1.1. The linkage systems $\mathcal{L}(E_6)$, $\mathcal{L}(E_6(a_k))$ for $k = 1, 2$. Since the root system $\Phi(E_6)$ is not contained in the root system $\Phi(D_n)$, see Lemma A.1, then by means of Theorem 3.4 and bijective maps of Table 8.3, we deduce that $\Phi(E_6(a_k))^1$, where $k = 1, 2$, are also not contained in $\Phi(D_n)$.

Since $E_6 \subset E_7, E_7(a_1), E_7(a_2)$ and $E_6(a_1) \subset E_7(a_3), E_7(a_4)$, we derive that $\Phi(E_7)$ and $\Phi(E_7(a_k))$ are not contained in $\Phi(D_n)$ for $k = 1, 2, 3, 4$. Thus, for Carter diagrams 9.1.1, a Dynkin extension can be obtained only by means of $E_{l+1}$:

\[
E_6 <_D E_7, \quad E_6(a_k) <_D E_7, \text{ where } k = 1, 2
\]
\[
E_7 <_D E_8, \quad E_7(a_k) <_D E_8, \text{ where } k = 1, 2, 3, 4.
\]

Then the linkage systems for $E_l$ (resp. $E_l(a_k)$) are

\[
\mathcal{L}(E_6) = \mathcal{L}_{E_6}(E_6), \quad \mathcal{L}(E_6(a_k)) = \mathcal{L}_{E_7}(E_6(a_k)), \quad k = 1, 2,
\]
\[
\mathcal{L}(E_7) = \mathcal{L}_{E_8}(E_7), \quad \mathcal{L}(E_7(a_k)) = \mathcal{L}_{E_8}(E_7(a_k)), \quad k = 1, 2, 3, 4.
\]

By (3.12) and Corollary 3.9

\[
|\mathcal{L}(E_6(a_1))| = |\mathcal{L}(E_6(a_2))| = |\mathcal{L}(E_6)| \leq |\Phi(E_7)| - |\Phi(E_6)| = 126 - 72 = 54.
\]

\[\text{By abuse of notation, we write } \Phi(E_6(a_k)) \text{ meaning the partial root system } \Phi(S), \text{ where } S \text{ is the } E_6(a_k)-\text{associated subset; we write } \Phi(E_7(a_k)) \text{ meaning the partial root system } \Phi(S), \text{ where } S \text{ is the } E_7(a_k)-\text{associated subset.}\]
In Figs. C.46, C.47, C.52 (top) we have 54 linkage label vectors in the linkage system \( L(E_6) \), (resp. \( L(E_6(a_1)) \), resp. \( L(E_6(a_2)) \)). We conclude that there are exactly 54 elements in each of linkage systems \( L(E_6) \), \( L(E_6(a_1)) \), \( L(E_6(a_2)) \).

9.1.2. The linkage systems \( L(E_7) \), \( L(E_7(a_k)) \) for \( k = 1, 2, 3, 4 \). Throughout this subsection we assume that \( k = 1, 2, 3, 4 \).

**Lemma 9.1.** Consider the Dynkin extension \( E_7 <_D E_8 \). Let \( S = \{ \tau_1, \ldots, \tau_8 \} \) be \( E_7 \)-associated root subset. Let \( \varphi \) be a positive root in the root stratum \( \Phi(E_8) \setminus \Phi(S) \), and \( \mu_{\text{max}} \) be the maximal root in \( \Phi(E_8) \):

\[
\varphi = \tau_1, \tau_3, \tau_4, \tau_5, \tau_7, \tau_8, \mu_{\text{max}} = 2 4 6 5 4 3 2
\]

(i) The vector \( \delta = \mu_{\text{max}} - \varphi \), where \( \varphi \neq \mu_{\text{max}} \), is also the root in \( \Phi(E_8) \setminus \Phi(S) \).

(ii) The linkage label vectors \( \varphi^\top \) and \( -\delta^\top \) coincide:

\[
\delta^\top = -\varphi^\top.
\]

**Proof.** (i) As above, in Lemma 8.1 we need to prove that

\[
B(\delta) = 2, \text{ i.e., } B(\mu_{\text{max}}) = B(\varphi) - 2(\varphi, \mu_{\text{max}}) = 2.
\]

Eq. (9.7) is equivalent to

\[
(\varphi, \mu_{\text{max}}) = 1.
\]

We have

\[
(\gamma, \mu_{\text{max}}) = \begin{cases} 1 & \text{for } \gamma = \tau_8, \\ 0 & \text{for } \gamma = \tau_i, \text{ where } i < 8. \end{cases}
\]

The fact that \( \varphi \in \Phi(E_8) \setminus \Phi(S) \), where \( \varphi \neq \mu_{\text{max}} \) and \( \varphi \) is positive, means that \( \tau_8 \) enters with coefficient 1 into the decomposition of \( \varphi \) with relation to \( \{ \tau_1, \ldots, \tau_8 \} \), see [Bo02, Table VII]. Then, by (9.9) the relation \( \text{(IX)} \) holds for any root \( \varphi \in \Phi(E_8) \setminus \Phi(S) \). Therefore, \( \text{(IX)} \) holds, i.e., \( \delta \) is also a root. By [Bo02, Table VII] the coordinate \( \tau_8 \) of \( \mu_{\text{max}} \) is 2. By \( \text{(I.3)} \) \( \tau_8 \) also enters with coefficient 1 into the decomposition of \( \delta \) with relation to \( \{ \tau_1, \ldots, \tau_8 \} \). Thus \( \delta \) also belongs to \( \Phi(E_8) \setminus \Phi(S) \).

(ii) For \( \Gamma = E_7 \), by \( \text{(IX)} \) for \( \tau_i \neq \tau_8 \), we have

\[
\delta^\top = (\delta, \tau_i) = (\mu_{\text{max}} - \varphi, \tau_i) = -1(\varphi, \tau_i) = -\varphi^\top, \text{ where } \delta \in \Phi(E_8) \setminus \Phi(E_7).
\]

Since \( L = [\tau_1, \ldots, \tau_7] \). we have \( \delta^\top = -\varphi^\top \). \( \square \)

**Corollary 9.2.** The size of the linkage system \( L(E_7) \) (resp. \( L(E_7(a_k)) \)) is equal to 56.

**Proof.** By (8.12) and Corollary 8.9 we have

\[
| L(E_7(a_k)) | = | L(E_7) | \leq | \Phi(E_8) | - | \Phi(E_7) | = 240 - 126 = 114.
\]

Further, consider the maximal root \( \mu_{\text{max}} \) and minimal root \( \mu_{\text{min}} = -\mu_{\text{max}} \) in \( \Phi(E_8) \):

\[
\pm\mu_{\text{max}} = \pm 2 4 6 5 4 3 2
\]

We have \( \pm\mu_{\text{max}} \in \Phi(E_8) \setminus \Phi(E_7) \) and \( \pm\mu_{\text{max}} \notin \Phi(E_7) \). Roots \( \pm\mu_{\text{max}} \) are orthogonal to all \( \tau_i \) except for \( \tau_8 \), see (9.3). Then vectors \( \pm\mu_{\text{max}}^\top \) are zero. Thus, by (9.11) we have

\[
| L(E_7(a_k)) | = | L(E_7) | \leq 112.
\]

Finally, by Lemma 9.1 roots \( \varphi \) and \( \varphi - \mu_{\text{max}} \) give the same linkage label vectors. Therefore,

\[
| L(E_7(a_k)) | = | L(E_7) | \leq 56.
\]
In Figs. C.48, C.49, C.50, C.51, C.53 we have 56 linkage label vectors in the linkage system $\mathcal{L}(E_7)$, (resp. $\mathcal{L}(E_7(a_k))$ for $k = 1, 2, 3, 4$). We conclude that there are exactly 56 elements in each of linkage systems $\mathcal{L}(E_7)$ and $\mathcal{L}(E_7(a_k))$ for $k = 1, 2, 3, 4$. \hfill $\square$ 

10. A-type linkage systems

10.1. The linkage systems $\mathcal{L}(A_l)$ for $l = 5, 6, 7$. In this section, we suppose that $\Gamma = A_l$ with $l = 5, 6, 7$. Let $S$ be a $\Gamma$-associated root subset. First, we have

$$\mathcal{L}(A_l) = \mathcal{L}_{A_{l+1}}(A_l) \cup \mathcal{L}_{D_{l+1}}(A_l) \text{ for } l \neq 5, 6, 7,$$

$$\mathcal{L}(A_l) = \mathcal{L}_{A_{l+1}}(A_l) \cup \mathcal{L}_{D_{l+1}}(A_l) \cup \mathcal{L}_{E_{l+1}}(A_l) \text{ for } l = 5, 6, 7. \tag{10.1}$$

By (3.12) the sizes of $A$-, $D$- and $E$-components are estimated as follows:

$$|\mathcal{L}_{A_{l+1}}(A_l)| \leq |\Phi(A_{l+1})| - |\Phi(A_l)| = (l+1)(l+2) - l(l+1) = 2(l+1),$$

$$|\mathcal{L}_{D_{l+1}}(A_l)| \leq |\Phi(D_{l+1})| - |\Phi(A_l)| = 2(l+1) - l(l+1) = l(l+1),$$

$$|\mathcal{L}_{E_{l+1}}(A_l)| \leq |\Phi(E_{l+1})| - |\Phi(A_l)| = 2[0.3cm]$$

$$\begin{align*}
|\mathcal{L}_{A_{l+1}}(A_5)| &\leq |\Phi(A_6)| - |\Phi(A_5)| = 72 - 30 = 42, \\
|\mathcal{L}_{A_{l+1}}(A_6)| &\leq |\Phi(A_7)| - |\Phi(A_6)| = 126 - 42 = 84, \\
|\mathcal{L}_{A_{l+1}}(A_7)| &\leq |\Phi(A_8)| - |\Phi(A_7)| = 240 - 56 = 184.
\end{align*} \tag{10.2}$$

10.1.1. The linkage systems $\mathcal{L}(A_3)$. By (10.1) and (10.2) we have

$$\mathcal{L}(A_3) = \mathcal{L}_{A_4}(A_3) \cup \mathcal{L}_{D_4}(A_3), \text{ where } |\mathcal{L}_{A_4}(A_3)| \leq 8, \quad |\mathcal{L}_{D_4}(A_3)| \leq 12. \tag{10.3}$$

In Fig. 10.21 we have 8 linkage label vectors in the $A$-component of the linkage system $\mathcal{L}(A_3)$. By (10.3) we conclude that there are exactly 8 linkages in the $A$-component $\mathcal{L}_{A_4}(A_3)$. For the component $\mathcal{L}_{D_4}(A_3)$, there is a different picture: By Table 10.6 the size of $\mathcal{L}_{A_4}(A_3)$ is twice less than the estimate in (10.3), i.e., $|\mathcal{L}_{D_4}(A_3)| \leq 6$. Thus, $|\mathcal{L}(A_3)| = 6 + 8 = 14$, see Fig. 10.21.
Table 10.6. 6 pairs of roots $\varphi, \delta \in \Phi(D_4) \setminus \Phi(A_3)$ such that $\delta^\nabla = -\varphi^\nabla$

### 10.1.2. The linkage systems $\mathcal{L}(A_4)$

By (10.1) and (10.2) we have

$$\mathcal{L}(A_4) = \mathcal{L}_{A_5}(A_4) \cup \mathcal{L}_{D_5}(A_4), \quad \text{where} \quad |\mathcal{L}_{A_5}(A_4)| \leq 10, \quad |\mathcal{L}_{D_5}(A_4)| \leq 20. \quad (10.4)$$

The linkage system $\mathcal{L}(A_4)$ contains 4 parts. One of the two parts of the $D$-component (resp. $A$-component) of the linkage system $\mathcal{L}(A_4)$ corresponds to the positive roots of $\Phi(D_5) \setminus \Phi(A_4)$ (resp. $\Phi(A_5) \setminus \Phi(A_4)$), another part of the $D$-component (resp. $A$-component) of the linkage system $\mathcal{L}(A_4)$ corresponds to the negative roots of $\Phi(D_5) \setminus \Phi(A_4)$ (resp. $\Phi(A_5) \setminus \Phi(A_4)$). In all, there are $10 + 20 = 30$ elements in $\mathcal{L}(A_4)$.

#### Figure 10.22.

One of the two parts of the $D$-component and one of two parts of the $A$-component of the linkage system $\mathcal{L}(A_4)$

### 10.1.3. The linkage systems $\mathcal{L}(A_5)$

By (10.1) and (10.2) we have

$$\mathcal{L}(A_5) = \mathcal{L}_{A_6}(A_5) \cup \mathcal{L}_{D_6}(A_5) \cup \mathcal{L}_{E_6}(A_5), \quad \text{where} \quad |\mathcal{L}_{A_6}(A_5)| \leq 12, \quad |\mathcal{L}_{D_6}(A_5)| \leq 30, \quad |\mathcal{L}_{E_6}(A_5)| \leq 72 - 30 = 42. \quad (10.5)$$
Let the coordinates of roots of $E_6$ be as follows:

$$
\begin{array}{ccccccc}
\tau_1 & \tau_3 & \tau_4 & \tau_5 & \tau_6 & \tau_2
\end{array}
$$

Consider the maximal and minimal roots in $E_6$:

$$
\mu_{\text{max}} = \left(\begin{array}{cccccc}
1 & 2 & 3 & 2 & 1 \\
2 & 2 & & & &
\end{array}\right), \quad \mu_{\text{min}} = -\left(\begin{array}{cccccc}
1 & 2 & 3 & 2 & 1 \\
2 & 2 & & & &
\end{array}\right)
$$

The roots $\mu_{\text{max}}$ (resp. $\mu_{\text{min}}$) is orthogonal to any simple root except for $\tau_2$. The vectors $\pm \mu_{\nabla}$ are zero. Then by (10.5)

$$
| \mathcal{L}_{E_6}(A_5) | \leq 42 - 2 = 40.
$$

(10.6)

Lemma 10.1. Let $E_6$ be the Dynkin extension of the Carter diagram $\Gamma = A_5$, $\varphi$ be a certain positive root in $\Phi(E_6) \setminus \Phi(A_5)$.

(i) The vector

$$
\delta = \mu_{\text{max}} - \varphi,
$$

where $\varphi \neq \mu_{\text{max}}$, is also a root in $\Phi(E_6) \setminus \Phi(A_5)$.

(ii) The linkage label vectors $\varphi_{\nabla}$ and $-\delta_{\nabla}$ coincide:

$$
\delta_{\nabla} = -\varphi_{\nabla}.
$$

Proof. (i) As above, in Lemmas 8.1 and 9.1, we need to prove that

$$
\mathcal{B}(\mu_{\text{max}}) + \mathcal{B}(\varphi) - 2(\varphi, \mu_{\text{max}}) = 2.
$$

(10.9)

In other words, we need to prove that

$$
(\varphi, \mu_{\text{max}}) = 1.
$$

(10.10)

We have

$$
(\gamma, \mu_{\text{max}}) = \begin{cases} 
1 & \text{for } \gamma = \tau_2, \\
0 & \text{for } \gamma = \tau_i, \text{ where } i \neq 2.
\end{cases}
$$

(10.11)

The fact that $\varphi \in \Phi(E_6) \setminus \Phi(A_5)$, $\varphi \neq \mu_{\text{max}}$ and $\varphi$ is positive means that $\tau_2$ enters with coefficient 1 into the decomposition of $\varphi$ with relation to $\{\tau_1, \ldots, \tau_6\}$. By (10.11) the relation (10.10) holds for any root $\varphi \in \Phi(E_6) \setminus \Phi(A_5)$. Therefore, $\mathcal{B}(\mu_{\text{max}})$ holds, i.e., $\delta$ is also a root. By [Bo02, Table V] the coordinate $\tau_2$ of $\mu_{\text{max}}$ is 2. By (10.7) $\tau_2$ also enters with coefficient 1 into the decomposition of $\delta$ with relation to $\{\tau_1, \ldots, \tau_6\}$. Thus, $\delta$ also belongs to $\Phi(E_6) \setminus \Phi(A_5)$.

(ii) By (10.11) for $\varphi \neq \tau_2$, we have

$$
\delta_{\nabla}^i = (\delta, \tau_i) = - (\varphi, \tau_i) = -\varphi_{\nabla}^i.
$$

(10.12)

Since $L = [\tau_1, \tau_3, \tau_4, \tau_5, \tau_6]$, we have $\delta_{\nabla} = -\varphi_{\nabla}$.

□

By (10.5), (10.6) the two parts of the $A$-component contain $2 \times 6 = 12$ elements; the two parts of the $D$-component contain $2 \times 15 = 30$ elements, see Fig. 10.23. By Lemma (10.1) and eq. (10.6) we see that the $E$-component contains $\leq 40/2 = 20$ elements. In Fig. 10.23 we see that the $E$-component contains exactly 20 elements. In all, the linkage system $\mathcal{L}(A_5)$ contains $12 + 30 + 20 = 62$ elements.
10.1.4. The linkage systems $L(A_6)$. By \[10.1\] and \[10.2\] we have

$$L(A_6) = L_{A_7}(A_6) \cup L_{D_7}(A_6) \cup L_{E_7}(A_6),$$
where

$$|L_{A_7}(A_6)| \leq 14, \quad |L_{D_7}(A_6)| \leq 42, \quad |L_{E_7}(A_6)| \leq |240 - 42 = 184.\]

However, there are 7 pairs of positive (resp. 7 pairs of negative) roots $\{\eta, \lambda\}$, where $\eta \in \Phi(E_7) \setminus \Phi(A_6)$ and $\lambda \in \Phi(A_7) \setminus \Phi(A_6)$ such that the corresponding linkage label vectors coincide up-to-sign:

$$\eta^\nabla = -\lambda^\nabla,$$
see Table \[10.7\]. In other words, we have to subtract 14 roots from the estimate of $|L_{E_7}(A_6)|$:

$$|L_{A_7}(A_6)| \leq 14, \quad |L_{D_7}(A_6)| \leq 42, \quad |L_{E_7}(A_6)| \leq 70.\]

There are $2 \times 7 + 2 \times 21 + 2 \times 35 = 14 + 42 + 70 = 126$ linkage diagrams in $L(A_6)$ presented in Fig. \[10.27\]. We conclude that there are exactly 126 linkage diagrams in $L(A_6)$.

10.1.5. The linkage systems $L(A_7)$. By \[10.1\] and \[10.2\] we have

$$L(A_7) = L_{A_8}(A_7) \cup L_{D_8}(A_7) \cup L_{E_8}(A_7),$$
where

$$|L_{A_8}(A_7)| \leq 16, \quad |L_{D_8}(A_7)| \leq 56, \quad |L_{E_8}(A_7)| \leq 240 - 56 = 184.\]
However, there are 8 pairs of positive (resp. 8 pairs of negative) roots \( \{ \eta, \lambda \} \), where \( \eta \in \Phi(E_8) \setminus \Phi(A_7) \) and \( \lambda \in \Phi(A_8) \setminus \Phi(A_7) \) such that the corresponding linkage label vectors coincide up-to-sign:

\[
\eta^\nabla = -\lambda^\nabla.
\]

The pairs \( \{ \eta_i, \lambda_i \} \), where \( 1 \leq i \leq 8 \) are depicted in Table 10.8. In other words, we have to subtract 16 linkage label vectors from the estimate of \( | \mathcal{L}_{E_8}(A_7) | \):

\[
| \mathcal{L}_{E_8}(A_7) | \leq 184 - 16 = 168.
\]  

**Remark 10.2.** For any root \( \gamma \) from the root system \( \Phi(A_l) \), we consider the sequence of coordinates of the linkage label vector \( \gamma^\nabla \) ordered as they lie in the diagram \( A_l \), see \( A \)-components in Figs. 10.27, 10.28 i.e.,

\[
\gamma^\nabla = \{ a_1, b_1, a_2, b_2, \ldots, a_i, b_i, \ldots \}, \text{ where } a_i = (\alpha_i, \gamma), b_i = (\beta_i, \gamma).
\]  

Then we say that the linkage diagram \( \gamma^\nabla \) is given in a **linear order**. Note that usually we consider coordinates of the linkage label vector \( \gamma^\nabla \) in a **bicolored order** given as follows:

\[
\gamma^\nabla = \{ a_1, a_2, \ldots, b_1, b_2, \ldots \}.
\]

In Table 10.8 the linkage label vectors \( \lambda^\nabla \) are given in a linear order. The same vectors in the \( A \)-component in Fig. 10.28 are given in the bicolored order. 

In addition, there are 28 pairs of positive (resp. 28 pairs of negative) roots \( \{ \eta, \lambda \} \), where \( \eta \in \Phi(E_8) \setminus \Phi(A_7) \) and \( \lambda \in \Phi(D_8) \setminus \Phi(A_7) \) such that the corresponding linkage label vectors coincide:

\[
\eta^\nabla = \lambda^\nabla.
\]

We do not present all of the 28 pairs \( \{ \eta, \lambda \} \): It suffices to produce one pair, the remaining 27 pairs are obtained by action of the Weyl group. Namely, let us take roots \( \eta_1 \in \Phi(E_8) \setminus \Phi(A_7) \) and \( \lambda_1 \in \Phi(D_8) \setminus \Phi(A_7) \) such that \( \eta_1^\nabla = \lambda_1^\nabla \); let \( w \in W \), where \( W \) is the Weyl group associated with \( \Phi(A_7) \). Though \( w\eta_1 \) and \( w\lambda_1 \) lie in different root systems, \( w^*\eta_1^\nabla \) and \( w^*\lambda_1^\nabla \) coincide:

\[
(w\eta_1)^\nabla = w^*\eta_1^\nabla = w^*\lambda_1^\nabla = (w\lambda_1)^\nabla,
\]
see Proposition 5.2(ii). Consider the following pair \( \{\eta_1, \lambda_1\} \):

\[
\eta_1 = \begin{array}{cccccccc}
2 & 4 & 5 & 4 & 3 & 2 & 1 \\
2
\end{array} \in \Phi(E_8) \setminus \Phi(A_7), \quad \lambda_1 = \begin{array}{cccccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array} \in \Phi(D_8) \setminus \Phi(A_7).
\]

Then

\[
\eta_1^\nabla = \lambda_1^\nabla = \begin{cases}
0 & 1 & 0 & 0 & 0 & 0 & 0 & \text{in linear order,} \\
\alpha_1 & \beta_1 & \alpha_2 & \beta_2 & \alpha_3 & \beta_3 & \alpha_4 & \text{in bicolored order.}
\end{cases}
\]

The linkage label vector \( \lambda_1^\nabla \) is the top element in one of two 28-element parts of the \( D \)-component of \( \mathcal{L}(A_7) \), see Fig. 10.28. Therefore, we have to subtract \( 2 \times 28 = 56 \) linkage label vectors from the estimate of

\[
| \mathcal{L}_{E_8}(A_7) | \leq 168 - 56 = 112.
\]

By (10.15), (10.18), we have

\[
| \mathcal{L}_{A_8}(A_7) | \leq 16, \quad | \mathcal{L}_{D_8}(A_7) | \leq 56, \quad | \mathcal{L}_{E_8}(A_7) | \leq 112.
\]

In accordance to Figs. 10.28 and 10.29 we have exactly \( 16 + 56 + 112 = 184 \) linkages in \( \mathcal{L}(A_7) \).

| \( \eta \in \Phi(E_8) \setminus \Phi(A_7) \) | \( \lambda \in \Phi(A_8) \setminus \Phi(A_7) \) | \( \lambda^\nabla = \eta^\nabla \) |
|----------------|----------------|----------------|
| \( \begin{array}{cccccccc}
\alpha_1 & \beta_1 & \alpha_2 & \beta_2 & \alpha_3 & \beta_3 & \alpha_4 \\
1 & 3 & 5 & 4 & 3 & 2 & 1
\end{array} \) | \( \begin{array}{cccccccc}
\alpha_1 & \beta_1 & \alpha_2 & \beta_2 & \alpha_3 & \beta_3 & \alpha_4 \\
1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \) | \( \begin{array}{cccccccc}
\alpha_1 & \beta_1 & \alpha_2 & \beta_2 & \alpha_3 & \beta_3 & \alpha_4 \\
0 & 0 & 0 & 1 & 0 & 0 & -1
\end{array} \) |

Table 10.8. 8 pairs of positive roots \( \{\eta, \lambda\} \) such that \( \eta^\nabla = \lambda^\nabla \). There are also 8 pairs of negative roots \( \{\eta, \lambda\} \) such that \( \eta^\nabla = \lambda^\nabla \).

10.2. The linkage systems \( \mathcal{L}(A_l) \) for \( l \geq 8 \). For convenience, we introduce coordinates \( \{x_i, y_j\} \), where \( 1 \leq i, j \leq l \), for every linkage diagram in the \( D \)-component of the linkage system \( \mathcal{L}_{D_{i+1}}(A_l) \). In Fig. 10.26, the path \( y_i \) (resp. \( x_i \)), where \( 1 \leq i \leq l \), consists of reflections acting in the direction from Nord-West to South-East (resp. from Nord-East to South-West). The linkage diagram lying in the intersection node of paths \( x_i \) and \( y_j \) has coordinates \( \{x_i, y_j\} \).
Proposition 10.3 (On the D-component of the linkage systems \( \mathcal{L}(A_l) \)). Consider the Dynkin extension \( A_l <_D D_{l+1} \).

(i) For \( l > 3 \), the D-component of the linkage systems \( \mathcal{L}(A_l) \) consists of two non-connected parts \( \mathcal{L}^+ \) and \( \mathcal{L}^- \) corresponding to the positive (resp. negative) linkage roots in the root stratum \( \Phi(D_{l+1}) \backslash \Phi(A_l) \). For \( l = 3 \), components \( \mathcal{L}^+ \) and \( \mathcal{L}^- \) coincide, see Fig. 10.24.

For \( l = 2n \) (resp. \( l = 2n + 1 \), where \( n \geq 2 \) (resp. \( n \geq 1 \)), reflections acting on linkage diagrams of \( \mathcal{L}^+ \) are given in Fig. 10.26 (resp. Fig. 10.26; the linkage diagram \( \gamma^\nabla \) in the top of \( \mathcal{L}^+ \) (vertex \( \{x_{2n}, y_1\} \) (resp. \( \{x_{2n+1}, y_1\} \)) depicted in Fig. 10.23(a) (resp. (b)).

\[
\gamma^\nabla = \{0, \ldots, 0, 1, 0, \ldots, 0\} \quad \gamma^\nabla = \{0, \ldots, 0, 1, 0, \ldots, 0\}
\]

In Fig. 10.24. The three last lines of \( \mathcal{L}^+ \) corresponding to coordinates \( y_{2n-1}, y_{2n-1} \) and \( y_{2n} \) (resp. \( y_{2n-1}, y_{2n} \) and \( y_{2n+1} \)) are given in Fig. 10.31 (resp. Fig. 10.32). (For \( 3 \leq l \leq 7 \), see Figs. 10.27 - 10.28).

(ii) For \( l > 3 \), we have

\[
| \mathcal{L}_{D_{l+1}}(A_l) | = l(l + 1).
\]  

Proof. (i) For transitions \( \mathcal{L}(A_5) \Rightarrow \mathcal{L}(A_6) \) and \( \mathcal{L}(A_6) \Rightarrow \mathcal{L}(A_7) \), see Fig. 10.33. For the transition \( \mathcal{L}(A_7) \Rightarrow \mathcal{L}(A_8) \), see Fig. 10.34. We prove the proposition by induction.

Transition \( 2n - 1 \Rightarrow 2n \). Consider the triangle \( \Delta_{2n-2} = \{\{y_1, x_{2n}\}, \{y_{2n-2}, x_{2n}\}, \{y_{2n-2}, x_3\}\} \) is the linkage subsystem containing linkage diagrams with the coordinate \( y_i \) for \( i \leq 2n - 2 \), see Fig. 10.26(a). To transit from \( 2n - 1 \) to \( 2n \) coordinates, we extend any linkage diagram \( \gamma^\nabla \) in the triangle \( \Delta_{2n} \) and in the line \( y_{2n-2} \) in Fig. 10.30 by the vertex \( \beta_n \) non-connected with \( \gamma \). In other words, we have \( \gamma^\nabla_{2n} = (\gamma, \beta_n) = 0 \), see Figs. 10.33, 10.26(a). Remember that any linkage diagram is equivalent to a certain linkage label vector, see Fig. 10.22. For the extension of the linkage diagram and the equivalent linkage label vectors, see Fig. 10.23(a).
By induction hypothesis, any reflection \( s^* \), which acts within the triangle \( \Delta_{2n-2} \), differs from \( s^*_\alpha \) and \( s^*_\beta \), see Fig. 10.26. So \( s^*_\varphi \) do not touch the coordinate \( \gamma^\varphi_{\beta_n} \). We put \( \gamma^\varphi_{\beta_n} = 0 \) to every linkage \( \gamma^\varphi \) from \( \Delta_{2n-2} \) and the subsystem given by the triangle \( \Delta_{2n-2} \) is saved in the transition \( 2n - 1 \Rightarrow 2n \). We see this process of the extension by the vertex \( \beta_n \) in lines \( y_{2n-2} \) of diagrams in Fig. 10.30 and Fig. 10.31.

By induction hypothesis, we have \( \gamma^\varphi_{\alpha_n} = 1 \) for any linkage diagram \( \gamma^\varphi \) in the line \( y_{2n-2} \) in \( \Delta_{2n-2} \).
see Fig. [10.30] The same holds for extensions of these diagrams in the line \( y_{2n-2} \) in Fig. [10.31] Consider now the last lines of the linkage system \( L(A_{2n}) \) in Fig. [10.31] By acting \( s_{\alpha_n}^* \) on every linkage diagram \( \gamma \nabla \) in the line \( y_{2n-2} \), we get linkage diagrams with \( \gamma_{\beta_n} = 1 \) in the line \( y_{2n-1} \). Really, by \((5.10)\) coordinates \( \gamma_{\alpha_n} \) and \( \gamma_{\beta_n} \) for any linkage diagram \( \gamma \nabla \) in the line \( y_{2n-2} \) are transformed as follows:

\[
(s_{\alpha_n}^* \gamma \nabla)_{\beta_n} = \gamma_{\beta_n}^+ + \gamma_{\alpha_n}, \quad (s_{\alpha_n}^* \gamma \nabla)_{\alpha_n} = -\gamma_{\alpha_n}.
\]

see \((5.19)\). The linkage diagram \( \{x_2, y_{2n-1}\} \) in the line \( y_{2n-1} \) in Fig. [10.31] is obtained by acting \( s_{\beta_n-1}^* \) on the linkage diagram \( \{x_3, y_{2n-1}\} \). Further, by acting \( s_{\beta_n}^* \) on every linkage diagram (except for linkage diagrams \( \{x_2, y_{2n}\} \) and \( \{x_1, y_{2n}\} \) in the line \( y_{2n-1} \) we get the last line with \( \gamma_{\alpha_n} = 0 \) and \( \gamma_{\beta_n} = -1 \). The linkage diagram \( \{x_2, y_{2n}\} \) (resp. \( \{x_1, y_{2n}\} \)) is obtained by acting \( s_{\beta_n-1}^* \) (resp. \( s_{\alpha_n}^* \)) on the linkage diagram \( \{x_3, y_{2n}\} \) (resp. \( \{x_2, y_{2n}\} \)).

Transition \( 2n \Rightarrow 2n+1 \) is considered in a similar way.

(ii) It is easy to see from Fig. [10.26] that

\[
| \mathcal{L}^+ | = | \mathcal{L}^- | = \frac{l(l+1)}{2}.
\]

Since \( \mathcal{L}_{D_{l+1}(A_l)} = \mathcal{L}^+ \sqcup \mathcal{L}^- \) the statement is proved.

\[\square\]

**Proposition 10.4** (On the A-component of the linkage systems \( \mathcal{L}(A_l) \)). The A-component of the linkage systems \( \mathcal{L}(A_l) \) consists of two non-connected parts \( \mathcal{L}^+ \) and \( \mathcal{L}^- \) corresponding to the positive (resp. negative) linkage roots in the Dynkin extension \( A_l < D A_{l+1} \). The number of linkage diagrams in the A-component of the linkage systems \( \mathcal{L}(A_l) \) is equal to \( 2(l+1) \).

**Proof.** This statement is also proved by induction. Consider, the transition \( 2n-1 \Rightarrow 2n \). For \( n = 3 \) this process is shown in Fig. [10.33] see \( A_5 \Rightarrow A_6 \). The first \( 2n-1 \) linkage diagrams are extended by the vertex \( \gamma_{\beta_n}^\nabla = 0 \). Under the action of \( s_{\alpha_n}^* \) the \( (2n) \)th linkage diagram is extended by the coordinate \( \gamma_{\beta_n}^\nabla = 1 \). Under the action of \( s_{\beta_n}^* \) the \( (2n+1) \)th linkage diagram is added, the coordinate \( \gamma_{\beta_n}^\nabla \) of this linkage diagram is equal \(-1\). \[\square\]
Figure 10.27. The linkage system $A_6$ has 6 components: The two parts of the $A$-component, the two 21-element parts of the $D$-component, and the two 35-element parts of the $E$-component.
Figure 10.28. The linkage system $A_7$. One of the two 8-element parts of the $A$-component (see Table 10.8) and one of the two 28-element parts of the $D$-component.
Figure 10.29. The linkage system $A_7$ (cont). One of the two 56-element parts of the $E$-component.
Figure 10.30. The last 2 lines of the $D$-component of $A_{2n-1}$

Figure 10.31. The last 3 lines of the $D$-component of $A_{2n}$

Figure 10.32. The last 3 lines of the $D$-component of $A_{2n+1}$
Figure 10.33. The transition $\mathcal{L}(A_5) \rightarrow \mathcal{L}(A_6)$ and $\mathcal{L}(A_6) \rightarrow \mathcal{L}(A_7)$. The new coordinate $\beta_3$ for the case $\mathcal{L}(A_6)$ (resp. $\alpha_4$ for the case $\mathcal{L}(A_7)$) and new linkage diagrams are in bold.
Figure 10.34. The transition from $A$-components and $D$-components in the linkage system $A_7$ to the linkage system $A_8$. The new coordinate $\beta_4$ and new linkage diagrams are marked in bold.
Appendix A. Some properties of Carter and connection diagrams

A.1. Similarity of Carter diagrams. Talking about a certain diagram \( \Gamma \) we actually have in mind a set of roots with orthogonality relations as it is prescribed by the diagram \( \Gamma \). We try to find some common properties of sets of roots (from the root systems associated with the simple Lie algebras) and diagrams associated with these sets. These diagrams are not necessarily Dynkin diagrams since sets of roots we study are not necessarily sets of simple roots and are not root subsystems. We use the term “Dynkin diagram” to describe connected sets of linearly independent simple roots in the root system. Similarly, “Carter diagrams” describe connected sets of linearly independent roots, not necessarily simple, and such that any cycle is even.

Two connection diagrams obtained from each other by a sequence of reflections \((A.1)\), are said to be similar connection diagrams, see Fig. A.35.

\[
\alpha \mapsto -\alpha. \quad (A.1)
\]

A transformation of connection diagrams obtained by a sequence of reflections \((A.1)\) is said to be a similarity transformation or similarity.

\[
\begin{array}{c}
\begin{array}{c}
\alpha \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\beta \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\gamma \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\delta \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\alpha \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\beta \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\gamma \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\delta \\
\end{array}
\end{array}
\end{array}
\]

Figure A.35. Eight similar 4-cycles equivalent to \( D_4(a_1) \)

By applying similarity \((A.1)\) any solid edge with an endpoint vertex being \( \alpha \) can be changed to a dotted one and vice versa; this does not change, however, the corresponding reflection:

\[
s_\alpha = s_{-\alpha}.
\]

Remark A.1 (On trees). For the set \( \{\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_n\} \) forming a tree, we may assume that, up to the similarity, all non-zero inner products \( (\alpha_i, \alpha_j) \) are negative. Indeed, if \( (\alpha_i, \alpha_j) > 0 \), we apply similarity transformation \( \alpha_j \mapsto -\alpha_j \), consider all inner products \( (\alpha_k, \alpha_j) > 0 \) and repeat similarity transformations \( \alpha_k \mapsto -\alpha_k \) if necessary. This process converges since the diagram is a tree.

A.2. The ratio of lengths of roots. Let \( \Gamma \) be a Dynkin diagram, and \( \sqrt{t} \) be the ratio of the length of any long root to the length of any short root. The inner product between two long roots is

\[
(\alpha, \beta) = \sqrt{t} \cdot \sqrt{t} \cdot \cos(\overline{\alpha, \beta}) = \sqrt{t} \cdot \sqrt{t} \cdot (\pm \frac{1}{2}) = \pm \frac{t}{2}.
\]

By Remark A.1 we may put \( (\alpha, \beta) = -\frac{t}{2} \). The inner product between two short roots is

\[
(\alpha, \beta) = \cos(\overline{\alpha, \beta}) = \pm \frac{1}{2}.
\]

Again, by Remark A.1 we may put \( (\alpha, \beta) = -\frac{1}{2} \). The inner product \( (\alpha, \beta) \) between roots of different lengths is

\[
(\alpha, \beta) = 1 \cdot \sqrt{t} \cdot \cos(\overline{\alpha, \beta}) = 1 \cdot \sqrt{t} \cdot (\pm \frac{\sqrt{t}}{2}) = \pm \frac{t}{2}.
\]
As above, we choose the obtuse angle and put \((\alpha, \beta) = \frac{t}{2}\).

We can summarize:

\[(\alpha, \beta) = \begin{cases} 
-\frac{1}{2} & \text{for } \|\alpha\| = \|\beta\| = 1, \\
-1 & \text{for } \|\alpha\| = \|\beta\| = 2, \text{ or } \|\alpha\| = 1, \|\beta\| = 2, \\
-\frac{3}{2} & \text{for } \|\alpha\| = \|\beta\| = 3, \text{ or } \|\alpha\| = 1, \|\beta\| = 3,
\end{cases}\]  

(A.2)

where all angles \(\hat{\alpha}, \hat{\beta}\) are obtuse.

A.3. Basic lemmas.

**Lemma A.2.** There is no root subset (in the root system associated with a Dynkin diagram) forming a simply-laced cycle containing only solid edges. Every cycle in the Carter diagram or in the connection diagram contains at least one solid edge and at least one dotted edge.

*Proof.* Suppose a subset \(S = \{\alpha_1, \ldots, \alpha_n\} \subset \Phi\) forms a cycle containing only solid edges. Consider the vector

\[v = \sum_{i=1}^{n} \alpha_i.\]

The value of the quadratic Tits form \(\mathcal{B}\) (see [St08]) on \(v\) is equal to

\[\mathcal{B}(v) = \sum_{i \in \Gamma_0} 1 - \sum_{i \in \Gamma_1} 1 = n - n = 0,\]

where \(\Gamma_0\) (resp. \(\Gamma_1\)) is the set of all vertices (resp. edges) of the diagram \(\Gamma\) associated with \(S\). Therefore, \(v = 0\) and elements of the root subset \(S\) are linearly dependent. \(\square\)

The following proposition is true only for trees.

**Lemma A.3** (Lemma 8, [Ca72]). Let \(S = \{\alpha_1, \ldots, \alpha_n\}\) be a subset of linearly independent (not necessarily simple) roots of a certain root system \(\Phi\), and \(\Gamma\) be the Dynkin diagram corresponding to \(\Phi\). Let \(\Gamma_S\) the Carter diagram or the connection diagram associated with \(S\). If \(\Gamma_S\) is a tree, then \(\Gamma_S\) is a Dynkin diagram.

*Proof.* If \(\Gamma_S\) is not a Dynkin diagram, then \(\Gamma_S\) contains an extended Dynkin diagram \(\tilde{\Gamma}\) as a subdiagram. Since \(\Gamma_S\) is a tree, we can turn all dotted edges to solid ones\(^1\), see Remark [A.1]. Further, we consider the vector

\[v = \sum_{i \in \tilde{\Gamma}_0} t_i \alpha_i,\]  

(A.3)

where \(\tilde{\Gamma}_0\) is the set of all vertices of \(\tilde{\Gamma}\), and \(t_i\) (where \(i \in \tilde{\Gamma}_0\)) are the coefficients of the nil-root, see [Kac80]. Let the remaining coefficients corresponding to \(\Gamma_S \setminus \tilde{\Gamma}\) be equal to 0. Let \(\mathcal{B}\) be the positive definite quadratic Tits form (see [St08]) associated with the diagram \(\Gamma\), and \((\cdot, \cdot)\) the symmetric bilinear form associated with \(\mathcal{B}\). Let \(\{\delta_i \mid i \in \tilde{\Gamma}_0\}\) be the set of simple roots associated with vertices \(\tilde{\Gamma}_0\). For all \(i, j \in \tilde{\Gamma}_0\), we have \((\alpha_i, \alpha_j) = (\delta_i, \delta_j)\), since this value is described by edges of \(\tilde{\Gamma}\). Therefore,

\[\mathcal{B}(v) = \sum_{i,j \in \tilde{\Gamma}_0} t_i t_j (\alpha_i, \alpha_j) = \sum_{i,j \in \tilde{\Gamma}_0} t_i t_j (\delta_i, \delta_j) = \mathcal{B}(\sum_{i \in \tilde{\Gamma}_0} t_i \delta_i) = 0.\]

Since \(\mathcal{B}\) is a positive definite form, we have \(v = 0\), i.e., vectors \(\alpha_i\) are linearly dependent. This contradicts the definition of the set \(S\). \(\square\)

\(^1\)This fact is not true for cycles, since by Lemma [A.2], we cannot eliminate all dotted edges.
**Lemma A.4.** The root system $A_n$ does not contain $\Gamma$-associated root subsets for

(i) $\Gamma$ is a cycle of length $\geq 4$,

(ii) $\Gamma$ is $D_4$.

**Proof.** (i) Recall that any root in $A_n$ is of the form $\pm(e_i - e_j)$, where $1 \leq i < j \leq n + 1$. Then, up to the similarity $\alpha \mapsto -\alpha$, a cycle of roots is one of the following forms:

\[
\{e_{i_1} - e_{i_2}, e_{i_2} - e_{i_3}, \ldots, e_{i_k - 1} - e_{i_k}, e_{i_k} - e_{i_1}\},
\{e_{i_1} - e_{i_2}, e_{i_2} - e_{i_3}, \ldots, e_{i_k - 1} - e_{i_k}, -(e_{i_k} - e_{i_1})\}.
\]

In the first case, the sum of all these roots is equal to 0, and roots are linearly dependent. In the second case, the sum of the $k - 1$ first roots is equal to the last one, and roots are also linearly dependent. Thus, for $A_n$, there are no cycles of linearly independent roots.

Note that $A_n$ does have cycles of length 3 containing linearly independent roots, for example:

\[
\{e_{i_1} - e_{i_2}, e_{i_2} - e_{i_3}, e_{i_2} - e_{i_4}\}.
\]

(ii) If a certain root subset contains $D_4$-associated root subset then it contains one of the following 4-element subsets:

\[
S_1 = \{e_{i_1} - e_{i_2}, e_{i_2} - e_{i_3}, e_{i_3} - e_{i_4}, e_{i_2} - e_{i_3}\}
\]

\[
S_2 = \{e_{i_1} - e_{i_2}, e_{i_2} - e_{i_3}, e_{i_3} - e_{i_4}, e_{i_3} - e_{i_5}\}
\]

In both cases $e_{i_2} - e_{i_3}$ is the branch point. In the first (resp. second) case $e_{i_2} - e_{i_5}$ (resp. $e_{i_3} - e_{i_5}$) is connected to $e_{i_1} - e_{i_2}$ and $e_{i_2} - e_{i_3}$ (resp. $e_{i_2} - e_{i_3}$ and $e_{i_3} - e_{i_4}$). Then \(\{e_{i_1} - e_{i_2}, e_{i_2} - e_{i_3}, e_{i_2} - e_{i_3}\}\) (resp. \(\{e_{i_2} - e_{i_3}, e_{i_3} - e_{i_4}, e_{i_3} - e_{i_5}\}\)) forms the 3-cycle and $S_1$ (resp. $S_2$) is not $D_4$.

**Lemma A.5.** The root systems $E_6$, $E_7$, $E_8$ are not contained in the root system $D_n$.

**Proof.** Roots of the root systems $E_6$ and $D_n$ are as follows:

\[
D_n = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq l\},
\]

\[
E_6 = \left\{\begin{array}{ll}
\pm e_i \pm e_j & (1 \leq i < j \leq 5), \\
\frac{1}{2} e_8 - e_7 - e_6 + \sum_{i=1}^{5} (-1)^{\nu(i)} e_i, & \text{where } \sum_{i=1}^{5} \nu(i) \text{ is even}
\end{array}\right\}.
\]  

(A.4)

See [Bo02], Tables IV and V. It is clear that some roots of $E_6$ cannot be obtained as roots of $D_n$. Thus, $\Phi(E_6) \not\subset \Phi(D_n)$ and, therefore, also $\Phi(E_7) \not\subset \Phi(D_n)$ and $\Phi(E_8) \not\subset \Phi(D_n)$.

**Lemma A.6.** The partial root systems

\[
\begin{cases}
E_6(a_k) & \text{for } k = 1, 2, \\
E_7(a_k) & \text{for } k = 1, 2, 3, 4, \\
E_8(a_k) & \text{for } 1 \leq k \leq 8
\end{cases}
\]  

(A.5)

are not contained in the root system $D_n$.

**Proof.** Since any $E_7(a_k)$ and $E_8(a_k)$ in (A.5) contains $E_6(a_1)$ or $E_6(a_2)$ as a subdiagram, see Fig. A.36 it suffices to prove that $E_6(a_1) \not\subset D_n$ and $E_6(a_2) \not\subset D_n$. This follows from Corollary 3.5(i) and Lemma A.5.
A.4. The ordered tree of Carter diagrams.

![Diagram of Carter diagrams](image)

**Figure A.36.** The ordered tree of Carter diagrams from $C_4 \mathbin{\Box} DE_4$
A.5. Γ-associated root subsets and conjugacy classes.

A.5.1. Two Γ-associated conjugacy classes. There exist Γ-associated elements $w_1$ and $w_2$ such that $w_1 \not\sim w_2$. For example, the Carter diagram $A_3$ determines two different conjugacy classes in $D_l$, see Fig. A.37 for details, see [St10] §B.2.2.

![Figure A.37](image)

**Figure A.37.** Elements $s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}$ and $s_{\alpha_{l-1}}s_{\alpha_1}s_{\alpha_{l-2}}$ are not conjugate

A.5.2. Two non-conjugate Γ-associated sets. Let $S_1 = \{\varphi_1, \ldots, \varphi_n\}$ and $S_2 = \{\delta_1, \ldots, \delta_n\}$ be two Γ-associated sets of roots. The sets $S_1$ and $S_2$ are said to be conjugate if there exists an element $T \in W$ such that $T : \varphi_i \mapsto \delta_i$ for $i = 1, \ldots, n$. In this case, we write $S_1 \simeq S_2$ and $TS_1 = S_2$.

Let $w_1$ (resp. $w_2$) be any $S_1$-associated (resp. $S_2$-associated) element. If $S_1 \simeq S_2$, then $w_1 \simeq w_2$.

There exist, however, conjugate elements $w_1$ and $w_2$ such that $S_1 \not\simeq S_2$. Consider two 4-cycles in $D_6$:

$$C_1 = \{e_1 + e_2, e_4 - e_1, e_1 - e_2, e_2 - e_3\},$$

$$C_2 = \{e_1 + e_2, e_4 - e_1, e_3 - e_4, e_2 - e_3\}.$$

These sets are non-conjugate: $C_1 \not\simeq C_2$, see Fig. A.38 and §A.5.3, but the $C_1$-associated element $w_1 = s_{e_1+e_2}s_{e_1-e_2}s_{e_4-e_1}s_{e_2-e_3}$ and the $C_2$-associated element $w_2 = s_{e_1+e_2}s_{e_3-e_4}s_{e_4-e_1}s_{e_2-e_3}$ are conjugate.

![Figure A.38](image)

**Figure A.38.** Equivalence of the $C_1$-associated element $w_1$ and the $C_2$-associated element $w_2$

A.5.3. Example of equivalent 4-cycles. Let us take the diagram $D_6$ with simple roots:

$$\begin{align*}
\alpha_1 &= e_1 - e_2, & \alpha_2 &= e_2 - e_3, & \alpha_3 &= e_3 - e_4, \\
\alpha_4 &= e_4 - e_5, & \alpha_5 &= e_5 - e_6, & \alpha_6 &= e_5 + e_6,
\end{align*}$$

(A.6)

and the maximal root $\alpha = e_1 + e_2$, see Fig. A.38(a). Consider the following two 4-cycles:

$$\begin{align*}
C_1 &= \{e_1 + e_2, e_4 - e_1, e_1 - e_2, e_2 - e_3\}, \\
C_2 &= \{e_1 + e_2, e_4 - e_1, e_3 - e_4, e_2 - e_3\}.
\end{align*}$$

(A.7)
Since the 4-index dipole can be mapped only onto a 4-index dipole, see [St10, Lemma B.1], and $C_2$ consists of two 4-index dipoles, it follows that $C_1$ and $C_2$ can not be conjugate. However, $C_1$ and $C_2$ have the common second dipole $\{e_4 - e_1, e_2 - e_3\}$. Moreover, let $w_1$ (resp. $w_2$) be $C_1$-associated (resp. $C_2$-associated):

$$w_1 = s_{e_1+e_2}s_{e_1-e_2}s_{e_4-e_1}s_{e_2-e_3},$$
$$w_2 = s_{e_1+e_2}s_{e_3-e_4}s_{e_4-e_1}s_{e_2-e_3},$$

then elements $w_1$ and $w_2$ are conjugate:

$$w_1 = s_{e_1+e_2}s_{e_1-e_2}s_{e_4-e_1}s_{e_2-e_3} = s_{e_1+e_2}(s_{e_1-e_2}s_{e_2-e_3})s_{e_4-e_1} =
\begin{align*}
&= s_{e_1+e_2} s_{e_2-e_3} s_{e_4-e_1} + s_{e_1+e_2} s_{e_2-e_3} s_{e_4-e_1} = s_{e_1+e_2} s_{e_2-e_3} s_{e_4-e_1} \approx s_{e_4-e_3} \\
&= s_{e_1+e_2} s_{e_3-e_4} s_{e_4-e_1} s_{e_2-e_3} = w_2. \quad \square
\end{align*}$$

(A.8)

(A.9)
A.6. The diagonal elements of $B_{\Gamma}^{-1}$ for $A_l$, $D_l$, $D_l(a_k)$. Let $\eta$ be one of vertices of $\Gamma = D_l(a_k)$ (resp. $\Gamma = D_l$) such that $\eta \neq \alpha_2, \alpha_3$, see Fig. A.39.

\[
\eta \in \begin{cases} 
\{\tau_1, \ldots, \tau_{k-1}, \varphi_1, \ldots, \varphi_{l-k-3}, \alpha_2, \alpha_3, \beta_1, \beta_2\} & \text{for } \Gamma = D_l(a_k), \\
\{\tau_1, \ldots, \tau_{l-3}, \alpha_2, \alpha_3, \beta_1, \beta_2\} & \text{for } \Gamma = D_l,
\end{cases} \tag{A.10}
\]

Removing the vertex $\eta$ with its bonds from $\Gamma$ we get the diagram $\Gamma'$ which is decomposed, except for $\eta = \tau_{k-1}$ and $\eta = \varphi_{l-k-3}$, into the union of two connected subdiagrams:

\[
\Gamma' = \begin{cases} 
A_{d-1} \oplus D_{l-k-1}, & \text{where } d = l - k - 2 \text{ (resp. } d = k), \\
A_{d-1} \oplus D_{l-k-1}(a_k), & \text{where } d = k - i, \text{ (resp. } d = l - k - 2 - i),
\end{cases} \tag{A.11}
\]

For $\eta = \tau_{k-1}$ (resp. $\eta = \varphi_{l-k-3}$), we have $d = 1$ and $\Gamma' = D_{l-1}(a_k)$ (resp. $\Gamma' = D_{l-1}(a_{l-k-3})$). In Fig. A.39(b) and (d), we associate the numerical label $d$ with the corresponding vertex $\eta$.

\[\text{Figure A.39. The numerical labels in the right hand side are the diagonal elements of } B_{\Gamma}^{-1}.\]

**Proposition A.7.** (i) The determinant of the partial Cartan matrix $B_{\Gamma}$ is as follows:

\[
\det B_{\Gamma} = \begin{cases} 
4 & \text{for } A_l, \text{ where } l \geq 2; \text{ here } B_{\Gamma} = B, \\
4 & \text{for } D_l \text{ and } D_l(a_k), \text{ where } l \geq 4.
\end{cases} \tag{A.12}
\]

(ii) Let $b_{\eta,\eta}^\vee$ be diagonal elements of $B_{\Gamma}^{-1}$, where $\eta$ is given by (A.10). For $\Gamma = D_l(a_k)$ or $\Gamma = D_l$,

\[
b_{\eta,\eta}^\vee = \begin{cases} 
\frac{1}{4} & \text{for } \eta = \alpha_2, \text{ or } \eta = \alpha_3, \\
\frac{d}{4} & \text{for } \eta \text{ given in (A.10)},
\end{cases} \tag{A.13}
\]

where $d$ is given in the vertex $\eta$ in Fig. A.39(b) and (d), and by the relation (A.11).

(iii) For $\Gamma = A_l$, let $b_{\eta,\eta}^\vee$ be diagonal elements of $B_{\Gamma}^{-1}$, where

\[
\eta \in \begin{cases} 
\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_n, \beta_n, & \text{for } l = 2n, \\
\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_n, \beta_n, \alpha_{n+1}, & \text{for } l = 2n + 1,
\end{cases} \tag{A.14}
\]

see Fig. A.40. Then

\[
b_{\eta,\eta}^\vee = d - \frac{d^2}{l+1}, \tag{A.15}
\]

where $d$ is the sequential number of the vertex in eq. (A.14) or in Fig. A.40.

\[\text{Figure A.40. The Dynkin diagrams } A_l.\]
The vertex \( k < \infty \).

In the case \( B \) (where \( l = 1 \)), we have

\[
\det B(A_{l+1}) = 2 \det B(A_l) - \det B(A_{l-1}) = 2(l+1) - l = l + 2,
\]

\[
\det B(D_{l+1}) = 2 \det B(D_l) - \det B(D_{l-1}) = 4,
\]

where \( B(A_l) \) (resp. \( B(D_l) \)) is the Cartan matrix for \( A_l \) (resp. \( D_l \)).

To get the determinant of the partial Cartan matrix \( B \) for the diagram \( \Gamma = D_{l+1}(a_k) \), we expand \( \det B(D_{l+1}(a_k)) \) with respect to the minors corresponding to the \( i \)-th line (associated with the vertex \( i \) of \( \Gamma \)). By induction, we have

\[
\det B(D_{l+1}(a_k)) = 2 \det B(D_l(a_k)) - \det B(D_{l-1}(a_k)) = 4
\]

for \( k > 2 \),

\[
\det B(D_{l+1}(a_{l-k})) = 2 \det B(D_l(a_{l-k})) - \det B(D_{l-1}(a_{l-k})) = 4
\]

for \( l - k > 3 \),

\[
\det B(D_{l+1}(a_2)) = 2 \det B(D_l(a_2)) - \det B(D_{l-1}(a_2)) = 4
\]

for \( k = 2 \) or \( l - k = 3 \).

In the case \( k < 2 \) and \( l - k < 3 \), we have \( l < 5 \), i.e., \( \Gamma = D_4(a_1) \).

(ii) For \( \Gamma = D_l(a_k) \), we have

\[
B_{\eta, \eta} = \begin{cases} 
\det B(D_l(a_k)) = 4 & \text{for } d = 1, \\
\det B(D_{l+1}(a_k)) = 4 & \text{for } d > 1, \\
\det B(D_{l+1}(a_k)) = 4 & \text{for } \eta = \alpha_2, \alpha_3.
\end{cases}
\]

For \( \Gamma = D_l \), we have

\[
B_{\eta, \eta} = \begin{cases} 
\det B(D_l) = 4 & \text{for } d = 1, \\
\det B(D_{l+1}) = 4 & \text{for } d > 1, \\
\det B(D_{l+1}) = 4 & \text{for } \eta = \alpha_2, \alpha_3.
\end{cases}
\]

(iii) For \( \Gamma = A_l \), we have:

\[
b_{\eta, \eta}^\vee = \frac{\det B(B_{n, n})}{\det B(A_l)} \tag{A.18}
\]

where \( B_{n, n} \) is the matrix obtained from \( B \) by deleting the \( \eta \)-th column and \( \eta \)-th row. The matrix \( B_{n, n} \) splits into the direct sum:

\[
B_{\eta, \eta} = B(A_{l-1}) \oplus B(A_{l-d}).
\]

Hence,

\[
b_{\eta, \eta}^\vee = \frac{\det B(A_{l-1}) \times \det B(A_{l-d})}{\det B(A_l)} \tag{A.19}
\]

By \( A.12 \) \( \det B(A_{l-1}) = l \) and by \( A.19 \) we have

\[
b_{\eta, \eta}^\vee = \frac{d \times (l+1-d)}{l + 1} = d - \frac{d^2}{l + 1}. \quad \square
\]

For expressions \( B^{-1} \) for the Dynkin diagram \( A_l \), see [OV90] p. 295. We give the partial Cartan matrices and its inverse for Carter diagrams \( E_l(a_1) \) (where \( l < 7 \)), for \( A_l \) (where \( l \leq 8 \)), for \( D_l(a_k) \) (where \( l \leq 8 \)) in Tables \( A.11 \ A.12 \ A.13 \ A.14 \ A.15 \ A.16 \).
A.7. Simply extendable Carter diagrams. We say that the Carter diagram $\Gamma$ is simply extendable in the vertex $\tau_p$ if the new diagram $\tilde{\Gamma}$ is obtained by adding an extra vertex $\tau_{l+1}$ and only one connection edge $\{\tau_p, \tau_{l+1}\}$, and $\tilde{\Gamma}$ is also the Carter diagram. In this case, the extra vertex $\tau_{l+1}$ is called simply linked, the vertex $\tau_p$ is called simply linkable, and $\tilde{\Gamma}$ is called simply extended, see Fig. A.41. We will show that extensibility in the vertex is closely associated with the value of the diagonal element $b^\vee_{\tau_p, \tau_p}$ of the matrix $B^{-1}_\Gamma$, the inverse of the partial Cartan matrix $B_\Gamma$.

**Proposition A.8.** The Carter diagram $\Gamma$ is simply extendable in the vertex $\tau_p$ if and only if

$$b^\vee_{\tau_p, \tau_p} < 2,$$

where $b^\vee_{\tau_p, \tau_p}$ is the diagonal element (corresponding to the vertex $\tau_p$) of the matrix $B^{-1}_\Gamma$.

**Proof.** This is a direct consequence of Theorem 4.5. Indeed, the linkage label vector $\gamma_\nabla$ corresponding to the simply linked vertex $\tau_p$ is the vector with 1 or $-1$ in the place $\tau_p$ and zeros in remaining places. Then $B^\vee_{\Gamma}(\gamma_\nabla) = b^\vee_{\tau_p, \tau_p}$.

Figure A.41. Simply linkable vertices are marked by numerical values, which are the diagonal elements of $B^{-1}_\Gamma$, see Tables A.11 - A.14.

**Remark A.9.** For $D_l(a_k)$ (resp. $D_l$), where $l \geq 8$, we have $b^\vee_{\eta, \eta} < 2$ only for the endpoints $\tau_{k-1}$ and $\varphi_{l-3}$ (resp. only for the endpoint $\tau_{l-3}$), see Fig. A.39(a),(b) (resp. Fig. A.39(c),(d)). For $D_4(a_k), D_5(a_k), D_6(a_k), D_7(a_k)$ and $D_4, D_5, D_6, D_7$, we also have $b^\vee_{\eta, \eta} < 2$ for $\eta = \alpha_2, \alpha_3$.

A.7.1. Simple extensions for the Carter diagram $A_l$.

**Proposition A.10.** (i) For $l \leq 4$ or $l \geq 8$, the simply linkable vertices $\tau$ are only vertices with $d \in \{1, 2, l-1, l\}$, see Table A.9.

(ii) For $A_5, A_6$ and $A_7$, except for $d \in \{1, 2, l-1, l\}$, there are some additional simply linkable vertices $\tau$, see Table A.10 and Table A.15.

| $d$ | Simply linkable vertex $\tau$ | Simply extended diagram |
|-----|-------------------------------|-------------------------|
| 1   | $\alpha_1$                    | $A_{l+1}$               |
| 2   | $\beta_1$                     | $D_{l+1}$               |
| $l-1$ | $\tau_{l-1} = \begin{cases} \alpha_n & \text{for } l = 2n \\ \beta_n & \text{for } l = 2n+1 \end{cases}$ | $\tau_{l-1}$ $A_{l+1}$ |
| $l$  | $\tau_l = \begin{cases} \beta_n & \text{for } l = 2n \\ \alpha_{n+1} & \text{for } l = 2n+1 \end{cases}$ | $\tau_l$ $D_{l+1}$ |

**Table A.9.** The simply extendable diagrams for $A_l$, where $l \leq 4$ or $l \geq 8$.
| $\Gamma$ | Simply linkable vertex $\tau$ | $b^{\vee}_{\eta,0}$ | Simply extended diagram |
|----------|-------------------------------|------------------|---------------------|
| $A_5$    | $\alpha_2$                   | $b^{\vee}_{\alpha_2,\alpha_2} = \frac{2}{2}$ | $E_6$ |
| $A_6$    | $\alpha_2, \beta_2$          | $b^{\vee}_{\alpha_2,\alpha_2} = b^{\vee}_{\beta_2,\beta_2} = \frac{12}{4}$ | $E_7$ |
| $A_7$    | $\alpha_2, \alpha_3$         | $b^{\vee}_{\alpha_2,\alpha_2} = b^{\vee}_{\alpha_3,\alpha_3} = \frac{15}{8}$ | $E_8$ |

Table A.10. Additional simply extendable diagrams for $A_5$, $A_6$, $A_7$

Proof. (i) The statement holds for $l \leq 4$ since in this case any $d$ is from $\{1, 2, l-1, l\}$, and the simple extensions in these cases are as follows: $A_3$ for $l = 2$; $A_4$ or $D_4$ for $l = 3$; $A_5$ or $D_5$ for $l = 4$.

Let $l \geq 8$. By Proposition A.8, the Carter diagram $\Gamma$ is simply extendable in the vertex $\tau_d$ if and only if $b^{\vee}_{\eta,0} < 2$. By eq. (A.15) from Proposition A.7, we have to solve the inequality

$$d - \frac{d^2}{l+1} < 2, \text{ i.e. } d^2 - d(l+1) + 2(l+1)^2 > 0,$$

so,

$$d < d_- \text{ or } d > d_+, \text{ where } d_{\pm} = \frac{(l+1) \pm \sqrt{(l+1)^2 - 8(l+1)}}{2}.$$

Since $l \geq 8$, we have $l - 5 \leq \sqrt{(l+1)^2 - 8(l+1)}$. Then

$$l - 2 \leq \frac{l+1 + \sqrt{(l+1)^2 - 8(l+1)}}{2} = d_+ < d,$$

$$3 \geq \frac{l+1 - \sqrt{(l+1)^2 - 8(l+1)}}{2} = d_- > d,$$

By, (A.21)

$$d < d_- \leq 3 \text{ and } d > d_+ \geq l - 2.$$

In other words,

$$d \in \{1, 2, l-1, l\},$$

see Table A.9.

(ii) This statement is immediately verified by Table A.15. □
A.8. The partial Cartan matrix $B_\Gamma$ and the inverse matrix $B^{-1}_\Gamma$.

| The Carter diagram | The partial Cartan matrix $B_\Gamma$ | The inverse matrix $B^{-1}_\Gamma$ |
|--------------------|-------------------------------------|----------------------------------|
| $D_4(a_1)$         | $\begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & 1 & -1 \\ -1 & 1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & -1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$ |
| $D_5(a_1) = D_5(a_2)$ | $\begin{bmatrix} 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 & 1 \\ 0 & 0 & 2 & -1 & -1 \\ -1 & -1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 4 & 2 & 2 & 4 & 0 \\ 2 & 5 & 1 & 4 & -2 \\ 2 & 1 & 5 & 4 & 2 \\ 4 & 4 & 4 & 8 & 0 \\ 0 & -2 & 2 & 0 & 4 \end{bmatrix}$ |
| $E_6(a_1)$         | $\begin{bmatrix} 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & -1 & -1 \\ -1 & -1 & -1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 4 & 2 & 4 & 5 & 1 & 2 \\ 2 & 4 & 2 & 4 & 5 & 1 \\ 5 & 4 & 8 & 10 & 2 & 4 \\ 1 & -1 & 4 & 2 & 4 & 2 \\ 2 & 1 & 5 & 4 & 2 & 4 \end{bmatrix}$ |
| $E_6(a_2)$         | $\begin{bmatrix} 2 & 0 & 0 & -1 & 0 & 1 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & -1 & -1 \\ -1 & -1 & -1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2 & 0 \\ 1 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 4 & 2 & 0 & 3 & -1 & -2 \\ 2 & 4 & 0 & 3 & -2 & -1 \\ 0 & 0 & 6 & 3 & 3 & 3 \\ 3 & 3 & 3 & 6 & 0 & 0 \\ -1 & -2 & 3 & 0 & 4 & 2 \\ -2 & -1 & 3 & 0 & 2 & 4 \end{bmatrix}$ |
| $D_6(a_1) = D_6(a_3)$ | $\begin{bmatrix} 2 & 0 & 0 & -1 & 0 & -1 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & -1 & -1 \\ -1 & -1 & -1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 4 & 2 & 2 & 4 & 0 & 2 \\ 2 & 3 & 1 & 3 & 0 & -1 \\ 2 & 1 & 3 & 3 & 1 & 1 \\ 4 & 3 & 3 & 6 & 0 & 2 \\ 0 & -1 & 0 & 2 & 0 & 2 \\ 2 & 1 & 1 & 2 & 0 & 2 \end{bmatrix}$ |
| $D_6(a_2)$         | $\begin{bmatrix} 2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 & 2 & 0 \\ 0 & 1 & -1 & -1 & 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 2 & 1 & 1 & 0 & 2 & 0 \\ 1 & 3 & 0 & -1 & 2 & 2 \\ 1 & 0 & 3 & 1 & 2 & 2 \\ 0 & -1 & 1 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 & 4 & 0 \\ 0 & -2 & 2 & 2 & 0 & 4 \end{bmatrix}$ |

Table A.11. The partial Cartan matrix $B_\Gamma$ and the inverse matrix $B^{-1}_\Gamma$ for Carter diagrams with the number of vertices $l < 7$. 
### Table A.12.

| The Carter diagram | The partial Cartan matrix $B_T$ | The inverse matrix $B_T^{-1}$ |
|-------------------|---------------------------------|-----------------------------|
| $\Gamma_2$ | $\begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 8 & 4 & 8 & 10 & 2 & 4 & 4 \\ 4 & 4 & 4 & 6 & 0 & 2 & 2 \\ 8 & 4 & 12 & 12 & 4 & 6 & 4 \\ 10 & 6 & 12 & 15 & 3 & 6 & 5 \\ 2 & 0 & 4 & 3 & 3 & 2 & 1 \\ 4 & 2 & 6 & 6 & 2 & 4 & 2 \\ 4 & 2 & 4 & 5 & 1 & 2 & 3 \end{bmatrix}$ |

| $\Gamma_2$ | $\begin{bmatrix} 2 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 4 & 4 & 4 & 6 & 0 & 2 & 2 \\ 4 & 8 & 8 & 8 & -2 & 2 & 4 \\ 4 & 4 & 8 & 8 & 2 & 4 & 2 \\ 6 & 8 & 8 & 12 & 0 & 4 & 4 \\ 0 & -2 & 2 & 0 & 3 & 1 & -1 \\ 2 & 2 & 4 & 4 & 1 & 3 & 1 \\ 2 & 4 & 2 & 4 & -1 & 1 & 3 \end{bmatrix}$ |

| $\Gamma_3$ | $\begin{bmatrix} 2 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 & -1 \\ 0 & 0 & 2 & -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 4 & 4 & 0 & 0 & 4 & -2 & -2 & 2 \\ 4 & 8 & 0 & 6 & -4 & -2 & 4 \\ 0 & 0 & 4 & 2 & 2 & 2 & 0 \\ 4 & 6 & 2 & 7 & -2 & -1 & 3 \\ -2 & -4 & 2 & -2 & 4 & 2 & -2 \\ -2 & -2 & 2 & -1 & 2 & 3 & -1 \\ 2 & 4 & 0 & 3 & -2 & -1 & 3 \end{bmatrix}$ |

| $\Gamma_4$ | $\begin{bmatrix} 2 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 2 & 0 & -1 & 1 & 0 & -1 \\ 0 & 2 & -1 & -1 & 1 & 0 & -1 \\ -1 & -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 4 & 0 & 0 & 0 & 2 & 0 & 2 & -2 \\ 0 & 4 & 0 & 2 & -2 & 0 & 2 \\ 0 & 0 & 4 & 2 & 2 & -2 & 0 \\ 2 & 2 & 2 & 4 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 3 & -1 & -1 \\ 2 & 0 & -2 & 0 & -1 & 3 & -1 \\ -2 & 2 & 0 & 0 & -1 & -1 & 3 \end{bmatrix}$ |

| $\Gamma_5$ | $\begin{bmatrix} 2 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 2 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & -1 \\ -1 & -1 & -1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 12 & 6 & 6 & 4 & 12 & 0 & 8 \\ 6 & 7 & 3 & 2 & 8 & -2 & 4 \\ 6 & 3 & 7 & 2 & 8 & 2 & 4 \\ 4 & 2 & 4 & 4 & 0 & 4 & 0 \\ 12 & 8 & 8 & 4 & 16 & 0 & 8 \\ 8 & 4 & 4 & 4 & 8 & 0 & 8 \end{bmatrix}$ |

| $\Gamma_6$ | $\begin{bmatrix} 2 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 2 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 2 & 0 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 8 & 4 & 4 & 0 & 8 & 0 & 4 \\ 4 & 7 & 1 & -2 & 6 & -4 & 2 \\ 4 & 1 & 7 & 2 & 6 & 4 & 2 \\ 0 & -2 & 2 & 4 & 0 & 4 & 0 \\ 8 & 6 & 6 & 0 & 12 & 0 & 4 \\ 0 & -4 & 4 & 4 & 0 & 8 & 0 \\ 4 & 2 & 2 & 0 & 4 & 0 & 4 \end{bmatrix}$ |
| The Carter diagram | The partial Cartan matrix $B_{1\Gamma}$ | The inverse matrix $B_{1\Gamma}^{-1}$ |
|-------------------|----------------------------------------|-------------------------------------|
| $\mathcal{D}_8(a_1) = \mathcal{D}_8(a_5)$ | \[
\begin{bmatrix}
2 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
0 & 2 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & -1 & -1 \\
-1 & -1 & -1 & 0 & 0 & 2 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 2 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 2
\end{bmatrix}
\] | \[
\begin{bmatrix}
8 & 4 & 4 & 4 & 8 & 0 & 6 & 2 \\
4 & 4 & 2 & 5 & -1 & 3 & 1 & 0 \\
4 & 2 & 4 & 2 & 5 & 1 & 3 & 1 \\
4 & 2 & 2 & 4 & 0 & 4 & 2 & 0 \\
8 & 5 & 5 & 4 & 10 & 0 & 6 & 2 \\
0 & -1 & 1 & 0 & 0 & 2 & 0 & 0 \\
6 & 3 & 3 & 4 & 6 & 0 & 6 & 2 \\
2 & 1 & 1 & 2 & 0 & 2 & 0 & 2
\end{bmatrix}
\] |
| $\mathcal{D}_8(a_2) = \mathcal{D}_8(a_4)$ | \[
\begin{bmatrix}
2 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & 2 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 2 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 2 & 0 & -1 & 0 \\
-1 & -1 & -1 & 0 & 0 & 2 & 0 & 0 \\
0 & 1 & -1 & 0 & -1 & 0 & 2 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & 0 & 2
\end{bmatrix}
\] | \[
\begin{bmatrix}
6 & 3 & 3 & 2 & 0 & 6 & 0 & 4 \\
3 & 4 & 1 & 1 & -1 & 4 & -2 & 2 \\
3 & 1 & 4 & 1 & 1 & 4 & 2 & 2 \\
2 & 1 & 1 & 2 & 0 & 2 & 0 & 2 \\
0 & -1 & 1 & 0 & 2 & 0 & 2 & 0 \\
6 & 4 & 4 & 2 & 0 & 8 & 0 & 4 \\
0 & -2 & 2 & 0 & 2 & 0 & 4 & 0 \\
4 & 2 & 2 & 2 & 0 & 4 & 0 & 4
\end{bmatrix}
\] |
| $\mathcal{D}_8(a_3)$ | \[
\begin{bmatrix}
2 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
0 & 2 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & -1 & 0 & -1 \\
-1 & -1 & -1 & 0 & 0 & 2 & 0 & 0 \\
0 & 1 & -1 & -1 & 0 & 2 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 2
\end{bmatrix}
\] | \[
\begin{bmatrix}
4 & 2 & 2 & 0 & 4 & 0 & 2 & 0 \\
2 & 4 & 0 & -2 & 3 & -3 & 1 & -1 \\
2 & 0 & 4 & 2 & 3 & 3 & 1 & 1 \\
0 & -2 & 2 & 4 & 0 & 4 & 0 & 2 \\
4 & 3 & 3 & 0 & 6 & 0 & 2 & 0 \\
0 & -3 & 3 & 4 & 0 & 6 & 0 & 2 \\
2 & 1 & 1 & 0 & 2 & 0 & 2 & 0 \\
0 & -1 & 1 & 2 & 2 & 0 & 2 & 0
\end{bmatrix}
\] |
| $\mathcal{D}_8$ | \[
\begin{bmatrix}
2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & -1 & 0 \\
-1 & -1 & -1 & 0 & 0 & 2 & 0 & 0 \\
0 & 1 & -1 & -1 & 0 & 2 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 2
\end{bmatrix}
\] | \[
\begin{bmatrix}
8 & 4 & 4 & 4 & 8 & 0 & 6 & 2 \\
4 & 4 & 2 & 5 & -1 & 3 & 1 & 0 \\
4 & 2 & 4 & 2 & 5 & 1 & 3 & 1 \\
4 & 2 & 2 & 4 & 0 & 4 & 2 & 0 \\
8 & 5 & 5 & 4 & 10 & 0 & 6 & 2 \\
0 & -1 & 1 & 0 & 0 & 2 & 0 & 0 \\
6 & 3 & 3 & 4 & 6 & 0 & 6 & 2 \\
2 & 1 & 1 & 2 & 2 & 0 & 2 & 2
\end{bmatrix}
\] |

Table A.13. (cont.) The partial Cartan matrix $B_{1\Gamma}$ and the inverse matrix $B_{1\Gamma}^{-1}$ for Carter diagrams $\mathcal{D}_8(a_1), \mathcal{D}_8(a_2), \mathcal{D}_8(a_3), \mathcal{D}_8$
Table A.14. (cont.) The Cartan matrix $B$ and the inverse matrix $B^{-1}$ for the conjugacy classes $E_6, E_7, D_4, D_5, D_6, D_7$
| $A_i$ | The Cartan matrix $B$ | $B^{-1}$ | S.c. in $\tau$ |
|-------|----------------------|----------|--------------|
| $\alpha_1$ | $\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$ | $\begin{pmatrix} 3 & 1 & 2 \\ 1 & 3 & 2 \\ 2 & 2 & 4 \end{pmatrix}$ | $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \end{pmatrix}$ | $\begin{pmatrix} A_4 \\ A_7 \\ D_4 \end{pmatrix}$ |
| $\alpha_2$ | $\begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$ | $\begin{pmatrix} 4 & 2 & 3 & 1 \\ 2 & 6 & 4 & 3 \\ 3 & 4 & 6 & 2 \\ 1 & 3 & 2 & 4 \end{pmatrix}$ | $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{pmatrix}$ | $\begin{pmatrix} A_5 \\ D_5 \\ D_5 \\ A_5 \end{pmatrix}$ |
| $\alpha_3$ | $\begin{pmatrix} 2 & 0 & 0 & -1 & 1 & 0 \\ 0 & 2 & 0 & -1 & -1 & 0 \\ -1 & -1 & 2 & 0 & 2 & 0 \\ 0 & -1 & -1 & 0 & 2 & 0 \end{pmatrix}$ | $\begin{pmatrix} 4 & 6 & 2 & 8 & 4 \\ 2 & 6 & 4 & 4 & 8 \\ 3 & 9 & 8 & 6 & 12 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$ | $\begin{pmatrix} \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \end{pmatrix}$ | $\begin{pmatrix} D_6 \\ A_6 \\ D_6 \\ D_6 \end{pmatrix}$ |
| $\alpha_4$ | $\begin{pmatrix} 2 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 & 0 \end{pmatrix}$ | $\begin{pmatrix} 7 & 5 & 3 & 1 & 6 & 4 & 2 \\ 4 & 12 & 6 & 8 & 9 & 3 & 5 \\ 3 & 9 & 15 & 5 & 6 & 10 & 12 \\ 1 & 3 & 5 & 7 & 2 & 4 & 6 \\ 6 & 10 & 6 & 2 & 12 & 8 & 4 \\ 4 & 12 & 12 & 4 & 8 & 12 & 14 \end{pmatrix}$ | $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_4 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$ | $\begin{pmatrix} A_7 \\ A_8 \\ E_4 \\ D_7 \\ E_7 \\ A_7 \end{pmatrix}$ |
| $\alpha_5$ | $\begin{pmatrix} 2 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 & 0 \end{pmatrix}$ | $\begin{pmatrix} 2 & 6 & 10 & 6 & 4 & 8 & 12 \\ 7 & 12 & 8 & 4 & 14 & 10 & 6 \\ 6 & 10 & 6 & 2 & 12 & 8 & 4 \\ 2 & 6 & 10 & 6 & 4 & 8 & 12 \\ 2 & 6 & 10 & 6 & 4 & 8 & 12 \end{pmatrix}$ | $\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$ | $\begin{pmatrix} D_8 \\ D_8 \\ D_8 \\ D_8 \end{pmatrix}$ |

Table A.15. (cont.) The Cartan matrix $B$ and its inverse matrix $B^{-1}$ for the Carter diagram $A_i$. The diagonal elements $b_{\tau,\tau}'$ in $B^{-1}$ correspond to simple extension in the point $\tau$ if $b_{\tau,\tau}' < 2$. In frames: $b_{\tau,\tau}' \geq 2$. 

RAFAEL STEKOLSCHIK
### APPENDIX B. Linkage diagrams $\gamma^\nabla(8)$ and inequality $\mathcal{E}_i^\nabla(\gamma^\nabla) < 2$

#### B.1. The linkage diagrams $\gamma^\nabla_{ij}(8)$ and solutions of inequality $\mathcal{E}_i^\nabla(\gamma^\nabla_{ij}(8)) < 2.$

| The Carter diagram | $L_{ij}$ | $\mathcal{E}_i^\nabla(\gamma^\nabla_{ij}(8)) = p < 2$ | The linkage diagram $\gamma^\nabla_{ij}(8)$ | $p$ |
|--------------------|---------|---------------------------------|-------------------------------|------|
| $E_6(a_1)$ | $L_{12}$ | $\frac{1}{2}(10 + 2(4b_2 + 5b_3) + q) < 2,$ or $(b_2 + b_3)^2 + (b_2 + 2)^2 + (b_3 + \frac{3}{2})^2 < \frac{31}{4}$ | $\{0, 0, 1, 0, 0, -1\}$ | $\frac{4}{3}$ |
| | | $\frac{1}{3}(4 + 2(-b_2 + b_3) + q) < 2,$ or $(b_2 + b_3)^2 + (b_2 - \frac{1}{2})^2 + (b_3 + \frac{1}{2})^2 < \frac{3}{2}$ | $\{0, 1, 0, 0, 1, -1\}$ | | |
| | | $\frac{1}{3}(4 + 2(b_2 + 2b_3) + q) < 2,$ or $(b_2 + b_3)^2 + (b_2 + \frac{3}{2})^2 + (b_3 + 1)^2 < \frac{2}{4}$ | $\{1, 0, 0, 0, 0, 1\}$ | | |
| $E_6(a_2)$ | $L_{12}$ | $\frac{1}{2}(6 + 2(3b_2 + 3b_3) + q) < 2,$ or $(b_2 + b_3)^2 + (b_2 + \frac{3}{2})^2 + (b_3 + \frac{3}{2})^2 < \frac{7}{2}$ | $\{0, 0, 1, 0, 0, -1\}$ | | |
| | | $\frac{1}{3}(4 + 2(-2b_2 - b_3) + q) < 2,$ or $(b_2 + b_3)^2 + (b_2 - 1)^2 + (b_3 - \frac{1}{2})^2 < \frac{5}{4}$ | $\{0, 1, 0, 0, 1, 0\}$ | | |
| | | $\frac{1}{3}(4 + 2(-b_2 - 2b_3) + q) < 2,$ or $(b_2 + b_3)^2 + (b_2 - \frac{1}{2})^2 + (b_3 - 1)^2 < \frac{7}{4}$ | $\{1, 0, 0, 0, 0, 1\}$ | | |
| $E_7(a_1)$ | $L_{12}$ | $\frac{1}{2}(12 + 2(4b_2 + 6b_3 + 4b_4) + q) < 2,$ or $(b_2 + b_4 + 1)^2 + 2(b_3 + b_4 + \frac{1}{2})^2 + 2(b_2 + b_3 + \frac{3}{2})^2 < 2$ | $\{0, 0, 1, 0, -1, -1\}$ | | |
| | | $\frac{1}{3}(4 + 2(2b_3 + b_4) + q) < 2,$ or $(b_2 + b_4)^2 + 2(b_3 + b_4 + 1)^2 + 2(b_2 + b_3)^2 < 2$ | $\{0, 1, 0, 0, 1, -1\}$ | | |
| | | $\frac{1}{2}(8 + 2(2b_2 + 4b_3 + 4b_4)) + q) < 2,$ or $(b_2 + b_4 + 1)^2 + 2(b_3 + b_4 + \frac{1}{2})^2 + 2(b_2 + b_3 + \frac{1}{2})^2 < 2$ | $\{1, 0, 0, 0, 0, -1\}$ | | |
| $E_7(a_2)$ | $L_{12}$ | $\frac{1}{2}(8 + 2(2b_2 + 4b_3 + 2b_4) + q) < 2,$ or $(b_2 + b_3)^2 + (b_3 + b_4)^2 + (b_2 - b_4)^2 + (b_2 + 2)^2 + (b_3 + 4)^2 + (b_4 + 2)^2 < 20$ | $\{0, 0, 1, 0, -1, -1\}$ | | |
| | | $\frac{1}{2}(8 + 2(-2b_2 + 2b_3 + 4b_4) + q) < 2,$ or $(b_2 + b_3)^2 + (b_3 + b_4)^2 + (b_2 - b_4)^2 + (b_2 - 2)^2 + (b_3 + 2)^2 + (b_4 + 4)^2 < 20$ | $\{0, 1, 0, 0, 0, -1\}$ | | |
| | | $\frac{1}{2}(4 + 2(2b_3 + 2b_4) + q) < 2,$ or $(b_2 + b_3)^2 + (b_3 + b_4)^2 + (b_2 - b_4)^2 + (b_3 + 2)^2 + (b_4 + 2)^2 + b_3^2 < 8$ | $\{1, 0, 0, 0, 0, -1\}$ | | |

Table B.16. The linkage diagrams $\gamma^\nabla_{ij}(8)$ obtained as solutions of inequality $\mathcal{E}_i^\nabla(\gamma^\nabla_{ij}(8)) < 2.$
| Carter diagram | $L_{ij}$ | $\mathcal{B}_I^\gamma (\gamma^\gamma_{ij}(8)) = p < 2$ | $\mathcal{B}_I^\gamma (\gamma^\gamma_{ij}(8)) = p < 2$ | $\mathcal{B}_I^\gamma (\gamma^\gamma_{ij}(8)) = p < 2$ |
|----------------|---------|---------------------------------|---------------------------------|---------------------------------|
| $E_7(a_3)$     | $L_{12}$ | $q = 4b_2^2 + 3b_3^2 + 3b_4^2 + 4b_2b_3 - 4b_2b_4 - 2b_3b_4$ | $\begin{array}{l}
\frac{1}{2}(4 + 2(2b_2 + 2b_3) + q) < 2, \text{ or} \\
2(b_2 + b_3 + 1)^2 + (b_3 - b_4)^2 + 2(b_4 - b_2)^2 < 2
\end{array}$ | $\begin{array}{l}
\{0, 0, 1, 0, 0, -1, 0\} \\
\{0, 0, 1, 0, -1, 0, 0\}
\end{array}$ |
| $E_7(a_4)$     | $L_{12}$ | $q = 3b_2^2 + 3b_3^2 + 3b_4^2 - 2b_2b_3 - 2b_2b_4 - 2b_3b_4$ | $\begin{array}{l}
\frac{1}{2}(4 + 2(2b_2 - 2b_3) + q) < 2, \text{ or} \\
(b_2 - b_3)^2 + (b_3 - b_4)^2 + (b_2 - b_4)^2 + (b_2 + 2)^2 + (b_3 - 2)^2 + b_4^2 < 2
\end{array}$ | $\begin{array}{l}
\{0, 0, 1, 0, 1, 0, 0\} \\
\{0, 0, 1, 0, -1, 0, 0\}
\end{array}$ |
| $D_5(a_1)$     | $L_{12}$ | $q = 5 + 2(2b_2 + 4b_4^2) < 2, \text{ or}$ | $\begin{array}{l}
\frac{1}{4}(5 + 2(2b_2 + 4b_4^2) + q) < 2, \text{ or}
\frac{1}{2}(b_2 + 2)^2 < 1
\end{array}$ | $\begin{array}{l}
\{0, 0, 1, 0, 0\} \\
\{0, 0, 1, 0, -1\}
\end{array}$ |
| $D_6(a_1)$     | $L_{12}$ | $q = 2b_2^2 + 2b_3^2$ | $\begin{array}{l}
\frac{1}{4}(3 + 2(b_2 + b_3) + q) < 2, \text{ or} \\
(b_2 + b_3)^2 + (b_3 + b_4)^2 < 1
\end{array}$ | $\begin{array}{l}
\{0, 0, 1, 0, 0, 0\} \\
\{0, 0, 1, 0, -1, 0\} \\
\{0, 0, 1, 0, 0, -1\} \\
\{0, 0, 1, 0, -1, -1\}
\end{array}$ |
| $L_{13}$ | $\frac{1}{2}(3 + 2(-b_2 + b_3) + q) < 2, \text{ or}$ | $\begin{array}{l}
\frac{1}{2}(b_2 - b_3)^2 + (b_3 + b_4)^2 < 1
\end{array}$ | $\begin{array}{l}
\{0, 1, 0, 0, 0, -1\} \\
\{0, 1, 0, 0, 1\} \\
\{0, 1, 0, -1\} \\
\{0, 1, 0, -1\}
\end{array}$ |
| $L_{23}$ | $\frac{1}{2}(4 + 2b_3 + q) < 2, \text{ or}$ | $\begin{array}{l}
b_2^2 + (b_3 + 1)^2 < 1
\end{array}$ | $\begin{array}{l}
\{1, 0, 0, 0, -1\}
\end{array}$ |

Table B.17. (cont.) The linkage diagrams $\gamma^\gamma_{ij}(8)$ obtained as solutions of inequality $\mathcal{B}_I^\gamma (\gamma^\gamma_{ij}(8)) < 2$
| The Carter diagram | \( L_{ij} \) | \( \mathcal{R}_1^\gamma (\gamma_{ij}^\Nabla (8)) = p < 2 \) | The linkage diagram \( \gamma_{ij}^\Nabla (8) \) | \( p \) |
|---------------------|-------------|-------------------------------------------------|---------------------------------|-----|
| \( D_6(a_2) \)     | \( L_{12} \) | \( q = 2a_2^2 + 4b_2^2 + 4a_1b_2 \) \( \frac{1}{2}(3 + 2(a_4 + 2b_2) + q) < 2 \), or \( (a_4 + 2b_2 + 1)^2 + a_3^2 < 2 \) | \( \{0, 0, 1, 0, 0, 0\} \) \( \{0, 0, 1, 0, 0, -1\} \) \( \{0, 0, 1, 1, 0, -1\} \) \( \{0, 0, 1, -1, 0, 0\} \) | \( \frac{4}{3} \) |
|                    | \( L_{13} \) | \( \frac{1}{2}(3 + 2(-a_4 - 2b_2) + q) < 2 \), or \( (a_4 + 2b_2 - 1)^2 + a_3^2 < 2 \) | \( \{0, 0, 1, 0, 0, 0\} \) \( \{0, 0, 1, 0, 0, 1\} \) \( \{0, 1, 0, -1, 0, 1\} \) \( \{0, 1, 0, 1, 0, 0\} \) | \( 1 \) |
|                    | \( L_{23} \) | \( \frac{1}{2}(2 + q) < 2 \), or \( (a_4 + 2b_2)^2 + a_3^2 < 1 \) | \( \{1, 0, 0, 0, 0, 0\} \) | \( 1 \) |
| \( D_7(a_1) \)     | \( L_{12} \) | \( q = 4a_3^2 + 4b_3^2 + 8b_1^2 + 8a_1b_3 \) \( \frac{1}{4}(7 + 2(2a_4 + 2b_2 + 4b_4) + q) < 2 \), or \( (a_4 + b_3 + \frac{1}{2})^2 + (b_2 + \frac{1}{4})^2 + (b_3 + \frac{1}{2})^2 < 1 \) | \( \{0, 0, 1, 1, 0, 0, -1\} \) \( \{0, 0, 1, 1, 0, -1, -1\} \) \( \{0, 0, 1, 1, 0, -1, -1\} \) \( \{0, 0, 0, 0, 0, 0, 0\} \) \( \{0, 0, 1, -1, 0, 0, 0\} \) \( \{0, 0, 1, -1, 0, -1, 0\} \) | \( \frac{4}{3} \) |
|                    | \( L_{13} \) | \( \frac{1}{4}(7 + 2(2a_4 - 2b_2 + 4b_4) + q) < 2 \), or \( (a_4 + b_3 + \frac{1}{2})^2 + (b_2 - \frac{1}{4})^2 + (b_3 + \frac{1}{2})^2 < 1 \) | \( \{0, 0, 0, 1, 0, 0, 0\} \) \( \{0, 0, 0, 0, 0, 1, 0\} \) \( \{0, 0, 0, 0, 0, 1, 0\} \) \( \{0, 1, 0, 0, 0, 0, 0\} \) \( \{0, 1, 0, 0, 0, 1, 0\} \) \( \{0, 1, 0, 0, 1, -1, 0\} \) \( \{0, 1, 0, -1, 0, 0, 0\} \) \( \{0, 1, 0, -1, 0, -1, 0\} \) | \( 1 \) |
|                    | \( L_{23} \) | \( \frac{1}{4}(12 + 2(4a_4 + 8b_3) + q) < 2 \), or \( (a_4 + b_3 + 1)^2 + (b_3 + 1)^2 + b_2^2 < 1 \) | \( \{1, 0, 0, 0, 0, 0, -1\} \) | \( 1 \) |

Table B.18. (cont.) The linkage diagrams \( \gamma_{ij}^\Nabla (8) \) obtained as solutions of inequality \( \mathcal{R}_1^\gamma (\gamma_{ij}^\Nabla (8)) < 2 \)
The Carter diagram $L_{ij}$ 

| $L_{12}$ | \( q = 4a_4^2 + 8b_2^2 + 4b_3^2 + 8a_4b_2 \) | \( \frac{1}{4}(7 + 2(2a_4 + 4b_2 + 2b_3) + q) < 2 \), or \( (a_4 + b_3 + \frac{1}{2})^2 + (b_2 + \frac{1}{4})^2 + (b_3 + \frac{1}{2})^2 < 1 \) |
|---|---|---|
| \( D_7(a_2) \) | \( \frac{1}{4}(7 + 2(-2a_4 - 4b_2 + 2b_3) + q) < 2 \), or \( (a_4 + b_3 - \frac{1}{2})^2 + (b_2 - \frac{1}{4})^2 + (b_3 + \frac{1}{2})^2 < 1 \) |
| \( L_{23} \) | \( \frac{1}{4}(8 + 8b_3) + q < 2 \), or \( (a_4 + b_2)^2 + (b_3 + 1)^2 + b_2^2 < 1 \) |

Table B.19. (cont.) The linkage diagrams $\gamma_{ij}^\nabla(8)$ obtained as solutions of inequality $\mathcal{B}_1^\nabla(\gamma_{ij}^\nabla(8)) < 2$
| The Carter diagram | $L_{ij}$ | $\mathcal{R}^\gamma_i (\gamma^\gamma_j (8)) = p < 2$ | The linkage diagram | $\gamma^\gamma_{ij} (8)$ | $p$ |
|--------------------|---------|---------------------------------|--------------------|-----------------|-----|
| $E_6$              | $L_{12}$ | $q = 4b_2^2 + 4b_2b_3 + 4b_3^2$ | $\frac{1}{3}(6 + 2(3b_2 + 3b_3) + q) < 2$, or $(b_2 + b_3)^2 + (b_2 + \frac{1}{2})^2 + (b_3 + \frac{1}{2})^2 < \frac{9}{2}$ | $\{0, 0, 1, 0, 0, -1\}$ | $0, 0, 1, 0, 0, -1, 0$ | \(\frac{4}{3}\) |
|                   | $L_{13}$ | $\frac{1}{3}(10 + 2(4b_2 + 5b_3) + q) < 2$, or $(b_2 + b_3)^2 + (b_2 + 2)^2 + (b_3 + \frac{3}{2})^2 < \frac{33}{4}$ | $\{0, 0, 0, 0, -1, -1\}$ | $\{0, 0, 0, 0, -1\}$ | $0, 0, 0, 0, -1, 0$ | |
|                   | $L_{23}$ | $\frac{1}{3}(10 + 2(5b_2 + 4b_3) + q) < 2$, or $(b_2 + b_3)^2 + (b_2 + \frac{1}{2})^2 + (b_3 + 2)^2 < \frac{13}{2}$ | $\{1, 0, 0, 0, -1, -1\}$ | $\{1, 0, 0, 0, -1\}$ | $\{1, 0, 0, 0, -1, 0\}$ | |
| $E_7$              | $L_{12}$ | $q = 3a_2^2 + 4b_2^2 + 8b_3^2 + 4a_4b_2 + 8b_2a_4 + 8b_2b_3$ | $\frac{1}{3}(7 + 2(3a_4 + 4b_2 + 6b_3) + q) < 2$, or $(a_4 + 2b_2 + 2b_3 + 2)^2 + (a_4 + 2b_3 + 1)^2 + a_2^2 < 2$ | $\{0, 0, 0, 0, -1\}$ | $\{0, 0, 1, 0, 0, -1\}$ | |
|                   | $L_{13}$ | $\frac{1}{3}(15 + 2(5a_4 + 6b_2 + 10b_3) + q) < 2$, or $(a_4 + 2b_2 + 2b_3 + 3)^2 + (a_4 + 2b_3 + 2)^2 + a_2^2 < 2$ | $\{0, 1, 0, 0, -1, -1\}$ | $\{0, 1, 0, 0, -1\}$ | $\{0, 1, 0, 0, -1, 0\}$ | \(\frac{3}{2}\) |
|                   | $L_{23}$ | $\frac{1}{3}(12 + 2(4a_4 + 6b_2 + 3b_3) + q) < 2$, or $(a_4 + 2b_2 + 2b_3 + 3)^2 + (a_4 + 2b_3 + 1)^2 + a_2^2 < 2$ | $\{1, 0, 1, 0, -1, -1\}$ | $\{1, 0, 0, 1, -1\}$ | $\{1, 0, 0, 1, -1, 0\}$ | |
| $D_5$              | $L_{12}$ | $q = 4b_2^2$ | $\frac{1}{4}(5 + 2b_2 + (2 + 4b_2)b_2) < 2$, or $(2b_2 + 1)^2 < 4$ | $\{0, 0, 1, 0, -1\}$ | $\{0, 0, 1, 0, 0\}$ | \(\frac{5}{4}\) |
|                   | $L_{13}$ | $\frac{1}{4}(8 + 4b_2 + (4 + 4b_2)b_2) < 2$, or $(b_2 + 1)^2 < 1$ | $\{0, 1, 0, 0, -1\}$ | $\{0, 1, 0, -1\}$ | $\{1, 0, 0, 0, 0\}$ | |
|                   | $L_{23}$ | $\frac{1}{4}(5 + 2b_2 + (2 + 4b_2)b_2) < 2$, or $(2b_2 + 1)^2 < 4$ | $\{1, 0, 0, 0, -1\}$ | $\{1, 0, 0, 0\}$ | $\{1, 0, 0, 1, 0, 0\}$ | \(\frac{5}{4}\) |
| $D_6$              | $L_{12}$ | $q = 4b_2^2 + 4b_2a_4 + 2b_3b_4$ | $\frac{1}{3}(3 + 2(a_4 + 2b_2) + q) < 2$, or $(b_2 + a_4 + 1)^2 + a_2^2 < 2$ | $0, 0, 1, 0, 0, 0\}$ | $\{0, 0, 1, 0, 0, -1\}$ | \(\frac{5}{4}\) |
|                   | $L_{13}$ | $\frac{1}{2}(6 + 2(2a_4 + 4b_2) + q) < 2$, or $(b_2 + a_4 + 2)^2 + a_2^2 < 2$ | $\{0, 1, 0, 0, -1\}$ | $\{0, 1, 0, -1\}$ | $\{1, 0, 0, 0, 0\}$ | |
|                   | $L_{23}$ | $\frac{1}{2}(3 + 2(a_4 + 2b_2) + q) < 2$, or $(b_2 + a_4 + 1)^2 + a_2^2 < 2$ | $\{1, 0, 0, 0, 0\}$ | $\{1, 0, 0, -1\}$ | $\{1, 0, 0, 0, 0, -1\}$ | \(\frac{5}{4}\) |

Table B.20. (cont.) The linkage diagrams $\gamma^\gamma_{ij} (8)$ obtained as solutions of inequality $\mathcal{R}^\gamma_i (\gamma^\gamma_{ij} (8)) < 2$
| The Carter diagram | $L_{ij}$ | $\mathcal{R}_1^\vee (\gamma_{ij}^\vee (8)) = p < 2$ | The linkage diagram $\gamma_{ij}^\vee (8)$ | $p$ |
|-------------------|---------|---------------------------------|-----------------|-----|
|                   |         |                                 |                 |     |
| $D_7$             | $L_{12}$| $q = 8a_4^2 + 12b_2^2 + 4b_3^2 + 16b_2a_4 + 8b_2b_3 + 8b_3a_4$ | $\{0, 0, 1, 0, 0, 0, 0\}$ | $\frac{1}{4}$ |
|                   |         | $\frac{1}{q}(7 + 4(2a_4 + 3b_2 + b_3) + q) < 2$, or $\{0, 0, 1, 0, 0, -1, 1\}$ | $\{0, 0, 1, 1, 0, -1, -1\}$ |     |
|                   |         | $(a_4 + b_2 + b_3 + \frac{1}{2})^2 + (a_4 + b_2 + \frac{1}{2})^2 + (b_2 + \frac{1}{2})^2 < 1$ | $\{0, 0, 1, -1, 0, 0\}$ |     |
|                   |         | $\{0, 0, 1, 1, 0, -1, 0\}$ | $\{0, 0, 1, -1, 0, 0, 1\}$ |     |
|                   |         | $\{0, 0, 1, 0, 0, -1\}$ | $\{0, 0, 1, 0, 0, -1\}$ |     |
|                   |         | $\frac{1}{4}(16 + 2(8a_4 + 12b_2 + 4b_3) + q) < 2$, or $\{0, 1, 0, 0, 0, -1, 0\}$ | $\frac{1}{4}$ |     |
|                   | $L_{13}$| $(a_4 + b_2 + b_3 + 1)^2 + (a_4 + b_2 + 1)^2 + (b_2 + 1)^2 < 1$ | $\{0, 1, 0, 0, 0, 0\}$ |     |
|                   |         | $\{0, 1, 0, 0, 0, -1, 1\}$ | $\{0, 0, 1, 1, 0, -1, -1\}$ |     |
|                   |         | $\{0, 0, 1, 1, 0, -1\}$ | $\{0, 0, 1, -1, 0, 0, 0\}$ |     |
|                   |         | $\{0, 0, 1, 1, 0, -1\}$ | $\{0, 0, 1, -1, 0, 0, 1\}$ |     |
|                   |         | $\{0, 0, 1, 0, 0, -1\}$ | $\{0, 0, 1, 0, 0, -1\}$ |     |
|                   | $L_{23}$| $\frac{1}{q}(7 + 4(2a_4 + 3b_2 + b_3) + q) < 2$, or $\{1, 0, 0, 0, 0, 0\}$ | $\frac{1}{4}$ |     |
|                   |         | $(a_4 + b_2 + b_3 + \frac{1}{2})^2 + (a_4 + b_2 + \frac{1}{2})^2 + (b_2 + \frac{1}{2})^2 < 1$ | $\{1, 0, 0, 0, 0, 0, 1\}$ |     |
|                   |         | $\{1, 0, 0, 0, 0, -1, 1\}$ | $\{1, 0, 0, 1, 1, 0, -1, -1\}$ |     |
|                   |         | $\{1, 0, 1, 0, 0, -1\}$ | $\{1, 0, 0, -1, 0, 0, 1\}$ |     |
|                   |         | $\{1, 0, 0, 1, 0, -1, 0\}$ | $\{1, 0, 0, -1, 0, 0, 1\}$ |     |
|                   |         | $\{1, 0, 0, 0, 0, 0, -1\}$ | $\{1, 0, 0, 0, 0, 0, -1\}$ |     |

Table B.21. (cont.) The linkage diagrams $\gamma_{ij}^\vee (8)$ obtained as solutions of inequality $\mathcal{R}_1^\vee (\gamma_{ij}^\vee (8)) < 2$
B.2. β-unicolored linkage diagrams. Solutions of inequality $\mathcal{U}_1(\gamma \nabla) < 2$.

| Diagram | $\mathcal{U}_1(\gamma \nabla) = p < 2$, $\gamma^\nabla - \beta$-unicolored linkage diagrams | $p$ |
|---------|---------------------------------|-----|
| $E_6(a_1), E_6(a_2)$ | $\frac{1}{2}(4b_2^2 + 4b_4^2 + 4b_2b_4) < 2$, or $(b_2 + b_3)^2 + b_4^2 + b_3^2 < 3$ | $\frac{4}{7}$ |
| | $\{0, 0, 0, 0, 0, 1\}$ | $\{0, 0, 0, 0, -1\}$ |
| | $\{0, 0, 0, 0, 1, 0\}$ | $\{0, 0, 0, -1, 0\}$ |
| | $\{0, 0, 0, 0, 1, -1\}$ | $\{0, 0, 0, -1, 1\}$ |
| $E_7(a_1)$ | $\frac{1}{2}(3b_2^2 + 4b_2^2 + 4b_2b_3 + 2b_2b_4 + 4b_2b_5) < 2$, or $2(b_2 + b_3)^2 + 2(b_3 + b_4)^2 + (b_2 + b_4)^2 < 4$ | $\frac{3}{7}$ |
| | $\{0, 0, 0, 0, 1, -1, 0\}$ | $\{0, 0, 0, -1, 1, 0\}$ |
| | $\{0, 0, 0, 0, 1, -1\}$ | $\{0, 0, 0, -1, 1\}$ |
| | $\{0, 0, 0, 0, 0, 1\}$ | $\{0, 0, 0, 0, -1\}$ |
| | $\{0, 0, 0, 0, 1, 0, 0\}$ | $\{0, 0, 0, -1, 0, 0\}$ |
| $E_7(a_2)$ | $\frac{1}{2}(3b_2^2 + 3b_4^2 + 3b_2 + 2b_2b_4 + 2b_2b_5) < 2$, or $(b_2 + b_3)^2 + (b_2 - b_4)^2 + (b_3 + b_4)^2 + b_2^2 + b_4^2 + b_3^2 < 4$ | $\frac{3}{7}$ |
| | $\{0, 0, 0, 0, 0, 1\}$ | $\{0, 0, 0, 0, -1, 1\}$ |
| | $\{0, 0, 0, 0, 0, 1\}$ | $\{0, 0, 0, 0, -1\}$ |
| | $\{0, 0, 0, 0, 0, 0, 1\}$ | $\{0, 0, 0, 0, 0, 1\}$ |
| | $\{0, 0, 0, 0, 1, 0, 0\}$ | $\{0, 0, 0, 0, -1, 0, 0\}$ |
| $E_7(a_3)$ | $\frac{1}{2}(4b_4^2 + 3b_4^2 + 4b_2b_3 - 4b_2b_4 - 2b_3b_4) < 2$, or $2(b_2 + b_3)^2 + 2(b_2 - b_4)^2 + (b_3 - b_4)^2 < 4$ | $\frac{3}{7}$ |
| | $\{0, 0, 0, 0, 0, 1\}$ | $\{0, 0, 0, 0, -1, 0\}$ |
| | $\{0, 0, 0, 0, 0, 0, 1\}$ | $\{0, 0, 0, 0, 0, -1\}$ |
| | $\{0, 0, 0, 0, 0, 0, -1\}$ | $\{0, 0, 0, 0, 0, -1\}$ |
| | $\{0, 0, 0, 0, 1, 0, -1\}$ | $\{0, 0, 0, -1, 1, 0\}$ |
| | $\{0, 0, 0, 0, -1, 0\}$ | $\{0, 0, 0, -1, 1, 0\}$ |
| $E_7(a_4)$ | $\frac{1}{2}(3b_2^2 + 3b_4^2 + 3b_2 - 2b_2b_3 - 2b_2b_4 - 2b_3b_4) < 2$, or $(b_2 - b_3)^2 + (b_2 - b_4)^2 + (b_3 - b_4)^2 + b_2^2 + b_3^2 + b_4^2 < 4$ | $\frac{3}{7}$ |
| | $\{0, 0, 0, 0, 1, 0\}$ | $\{0, 0, 0, -1, 0\}$ |
| | $\{0, 0, 0, 0, 0, 1\}$ | $\{0, 0, 0, 0, -1\}$ |
| | $\{0, 0, 0, 0, 0, 0, 1\}$ | $\{0, 0, 0, 0, 0, -1\}$ |
| | $\{0, 0, 0, 1, 1, 1\}$ | $\{0, 0, 0, -1, 1, 1\}$ |

Table B.22. (cont.) β-unicolored linkage diagrams obtained as solutions of inequality $\mathcal{U}_1(\gamma \nabla) < 2$
\[ \mathcal{P}^\vee (\gamma^\vee) = p < 2, \]

\[ \gamma^\vee - \beta\text{-unicolored linkage diagram} \]

| Diagram | Condition | Solution | \( p \) |
|---------|-----------|----------|--------|
| \( D_5(a_1) \) | \( \frac{1}{4}(4b_1^2) < 2, \) or \( b_2^2 < 2 \) | \( \{0,0,0,0,1\}, \{0,0,0,0,0\} \) | 1 |
| \( D_6(a_1) \) | \( \frac{1}{2}(2b_1^2 + 2b_2^2) < 2, \) or \( b_2^2 + b_3^2 < 2 \) | \( \{0,0,0,0,0,0\}, \{0,0,0,0,-1,0\} \) | 1 |
| \( D_6(a_2) \) | \( \frac{1}{2}(2a_3^2 + 4b_1^2 + 4b_2^2 + 4a_4b_2) < 2, \) or \( 2b_1^2 + b_2^2 + (a_4 + b_2)^2 < 2 \) | \( \{0,0,0,1,0,0\}, \{0,0,0,-1,0,0\} \) | 1 |
| \( D_7(a_1) \) | \( \frac{1}{4}(4a_3^2 + 4b_2^2 + 8b_3^2 + 8a_3b_3) < 2, \) or \( b_2^2 + b_3^2 + (a_4 + b_3)^2 < 2 \) | \( \{0,0,0,1,0,0,0\}, \{0,0,0,-1,0,0,0\} \) | 1 |
| \( D_7(a_2) \) | \( \frac{1}{4}(4a_3^2 + 8b_2^2 + 4b_3^2 + 8a_3b_2) < 2, \) or \( b_2^2 + b_3^2 + (a_4 + b_2)^2 < 2 \) | \( \{0,0,0,1,0,0,0,0\}, \{0,0,0,-1,0,0,0,0\} \) | 1 |

Table B.23. (cont.) \( \beta\text{-unicolored linkage diagrams obtained as solutions of inequality } \mathcal{P}^\vee (\gamma^\vee) < 2 \)
| Diagram | $\mathcal{B}_1^\gamma(\gamma^\nabla) = p < 2$ | $p$ |
|---|---|---|
| $E_6$ | $\frac{1}{3}(4b_2^2 + 4b_3^2 + 4b_2b_3) < 2$, or $\left( b_2 + b_3 \right)^2 + b_2^2 + b_3^2 < 3$ | $\frac{4}{3}$ |
| | (coincide with $E_6(a_1), E_6(a_2)$) | |
| $ \{0,0,0,0,0,1\} $ | $\{0,0,0,0,0,-1\}$ | |
| $ \{0,0,0,0,1,0\} $ | $\{0,0,0,0,0,-1,0\}$ | |
| $ \{0,0,0,0,1,-1\} $ | $\{0,0,0,0,0,-1,1\}$ | |
| $E_7$ | $\frac{1}{2}(3a_2^2 + 4b_2^2 + 8b_2b_3 + 8b_3b_4 + 4a_2b_2 + 8a_4b_3) < 2$, or $\left( a_4 + 2b_2 + 2b_3 \right)^2 + \left( a_4 + 2b_2 \right)^2 + a_4^2 < 4$ | $\frac{6}{2}$ |
| | | |
| $ \{0,0,0,1,0,0,0\} $ | $\{0,0,0,-1,0,0,0\}$ | |
| $ \{0,0,0,1,0,-1,0\} $ | $\{0,0,0,-1,0,1,0\}$ | |
| $ \{0,0,0,1,0,1,-1\} $ | $\{0,0,0,-1,0,-1,1\}$ | |
| $ \{0,0,0,1,0,0,-1\} $ | $\{0,0,0,-1,0,0,1\}$ | |
| $D_5$ | $\frac{1}{4}(4b_2^2) < 2$, or $b_2^2 < 2$ | $1$ |
| | $\{0,0,0,0,1\}$ | $\{0,0,0,0,-1\}$ | |
| $D_6$ | $\frac{1}{2}(2a_2^2 + 4b_2^2 + 4a_2b_2) < 2$, or $\left( 2b_2 + a_4 \right)^2 + a_4^2 < 2$ | $1$ |
| | | |
| $ \{0,0,0,1,0,0\} $ | $\{0,0,0,1,0,-1\}$ | |
| $ \{0,0,0,-1,0,0\} $ | $\{0,0,0,-1,0,1\}$ | |
| $D_7$ | $\frac{1}{4}(8a_2^2 + 16a_4b_2 + 8b_3a_4 + 12b_2^2 + 8b_2b_3 + 4b_3^2) < 2$, or $\left( a_4 + b_2 + b_3 \right)^2 + \left( a_4 + b_2 \right)^2 + b_2^2 < 2$ | $1$ |
| | | |
| $ \{0,0,0,0,0,0,1\} $ | $\{0,0,0,0,0,0,-1\}$ | |
| $ \{0,0,0,-1,0,0,1\} $ | $\{0,0,0,1,0,0,-1\}$ | |
| $ \{0,0,0,-1,0,1,0\} $ | $\{0,0,0,1,0,-1,0\}$ | |

Table B.24. (cont.) $\beta$-unicolored linkage diagrams obtained as solutions of inequality $\mathcal{B}_1^\gamma(\gamma^\nabla) < 2$
### B.3. Linkage diagrams $\gamma^N_i(6)$ per loctets and components.

| Diagram | Comp. | Linkage diagrams $\gamma^N_i(6)$ for the loctet of type $L_{ij}$ |
|---------|-------|---------------------------------------------------------------|
|         |       | Type $L_{ij}:(1,1,0,-1,...)$ | Type $L_{ij}:(1,0,1,-1,...)$ | Type $L_{ij}:(0,1,1,-1,...)$ |
| $D_5(a_1)$ | 1 | $(1,0,1,-1,0)$ | $(1,0,1,-1,0)$ | $L^a_{ij}$ | 
| | 2 | $(1,0,1,-1,1)$ | $(1,0,1,-1,1)$ | $L^b_{ij}$ | 
| | 3 | - | - | - |
| $D_6(a_1)$ | 1 | $(1,0,1,-1,1)$ | $(1,0,1,-1,1)$ | $L^a_{ij}$ | 
| | 2 | $(1,0,1,-1,0,0)$ | $(1,0,1,-1,0,0)$ | $L^b_{ij}$ | 
| | 3 | - | - | - |
| $D_6(a_2)$ | 1 | $(1,0,1,0,0)$ | $(1,0,1,0,0)$ | $L^a_{ij}$ | 
| | 2 | $(1,0,1,0,1)$ | $(1,0,1,0,1)$ | $L^b_{ij}$ | 
| | 3 | - | - | - |
| $I(a) = 4$ | 1 | $(1,0,1,0,0)$ | $(1,0,1,0,0)$ | $L^a_{ij}$ | 
| | 2 | $(1,0,1,0,0)$ | $(1,0,1,0,0)$ | $L^b_{ij}$ | 
| | 3 | - | - | - |
| $E_6(a_1)$ | 1 | $(1,0,1,0,0)$ | $(1,0,1,0,0)$ | $L^a_{ij}$ | 
| | 2 | $(1,0,1,0,0)$ | $(1,0,1,0,0)$ | $L^b_{ij}$ | 
| | 3 | - | - | - |
| $E_6(a_2)$ | 1 | $(1,0,1,0,0)$ | $(1,0,1,0,0)$ | $L^a_{ij}$ | 
| | 2 | $(1,0,1,0,0)$ | $(1,0,1,0,0)$ | $L^b_{ij}$ | 
| | 3 | - | - | - |
| $E_7(a_1)$ | 1 | $(1,0,1,0,0,0,0)$ | $(1,0,1,0,0,0,0)$ | $L^a_{ij}$ | 
| | 2 | $(1,0,1,0,1,1,1)$ | $(1,0,1,0,1,1,1)$ | $L^b_{ij}$ | 
| | 3 | - | - | - |
| $D_7(a_1)$ | 1 | $(1,0,1,0,0,1,0)$ | $(1,0,1,0,0,1,0)$ | $L^a_{ij}$ | 
| | 2 | $(1,0,1,0,0,1,0)$ | $(1,0,1,0,0,1,0)$ | $L^b_{ij}$ | 
| | 3 | - | - | - |

**Table B.25.** Linkage diagrams $\gamma^N_i(6)$ for the Carter diagrams from the class $C_4$, $n < 8$. For $D_6(a_2)$, $D_7(a_1)$, $D_7(a_2)$, the length of the $\alpha$-set is 4.
APPENDIX C. The linkage systems for the Carter diagrams of rank $l < 8$

The linkage systems are similar to the weight systems (= weight diagrams) of the irreducible representations of the semisimple Lie algebras.

C.1. The linkage systems $\mathcal{L}(D_4(a_1)), \mathcal{L}(D_5(a_1)), \mathcal{L}(D_6(a_1)), \mathcal{L}(D_6(a_2))$.

![Diagram](image)

**Figure C.42.** Three components of the linkage system $D_4(a_1)$. There are 24 linkage diagrams in the case $D_4(a_1)$.
Figure C.43. The linkage system $\mathcal{L}(D_3(a_1))$. There are one part of the $D$-component containing 10 linkage diagrams, and two parts of the $E$-component containing $2 \times 16 = 32$ elements.
The linkage system $\mathcal{L}(D_6(a_1))$. There are 12 linkage diagrams, 1 loctet in the single part of the $D$-component, and $2 \times 32 = 64$ linkage diagrams, 8 loctets in two parts of the $E$-component.
The linkage system $\mathcal{L}(D_6(a_2))$. There are 12 linkages, 1 loctet in the single part of the $D$-component, and $2 \times 32 = 64$ linkages, 8 loctets in two parts of the $E$-component.
C.2. The linkage systems $\mathcal{L}(E_6(a_1)), \mathcal{L}(E_6), \mathcal{L}(E_7(a_1)), \mathcal{L}(D_5), \mathcal{L}(D_6)$.

Figure C.46. The linkage system $E_6(a_1)$. The only component is the $E$-component containing two parts, 54 linkage diagrams, 6 loctets.
Figure C.47. The linkage system $E_6(a_2)$. The only component is the $E$-component containing two parts, 54 linkage diagrams, 6 loctets
Figure C.48. The linkage system $E_7(a_1)$. The only component is the $E$-component containing 56 linkage diagrams, 6 loctets.
Figure C.49. The linkage system $E_7(a_2)$. The only component is the $E$-component containing 56 linkage diagrams, 6 loctets.
Figure C.50. The linkage system $E_T(a_3)$, one component, 56 linkage diagrams, 6 loctets
Figure C.51. The linkage system $E_7(a_4)$, one component, 56 linkage diagrams, 6 loctets.
Figure C.52. $\mathcal{L}(E_6)$: The $E$-component contains 2 parts, each part contains 27 linkages (top). Weight systems of representations $27$ and $27$ of the semisimple Lie algebra $E_6$ (bottom)
Figure C.53. The linkage system $E_7$, one component, 56 elements
Figure C.54. Loctets in the weight system of the fundamental representations 56 of $E_7$. 
Figure C.55. The linkage system $\mathcal{L}(D_5)$: One part of the $D$-component containing 10 linkage diagrams and two $2 \times 16 = 32$ element parts in the $E$-component.

Figure C.56. Loctets in the weight system for 3 fundamental representations of $D_5$: $(D_5, \bar{\omega}_1)$, $(D_5, \bar{\omega}_4)$, and $(D_5, \bar{\omega}_5)$. 
FIGURE C.57. The linkage system $\mathcal{L}(D_6)$, one part of the $D$-component containing 12 elements and two parts of the $E$-component, each of which contains $2 \times 32 = 64$ elements.

FIGURE C.58. Loctets in the weight system for 3 fundamental representations of $D_6$: $(D_6, \omega_1)$, $(D_6, \omega_5)$ and $(D_6, \omega_6)$. 
C.3. The linkage systems $\mathcal{L}(D_7(a_1))$, $\mathcal{L}(D_7(a_2))$, $\mathcal{L}(D_7)$.

Figure C.59. The linkage system $\mathcal{L}(D_7(a_1))$, 1st part of the $E$-component, 64 linkage diagrams, 8 loctets
Figure C.60.  $D_7(a_1)$, second part of the $E$-component: 64 linkages, 8 loctets
Figure C.61. The linkage system $D_7(a_2)$, 1st part of the $E$-component, 64 linkage diagrams, 8 loctets
Figure C.62. The linkage system $D_7(a_2)$, 2nd part of the $E$-component, 64 linkage diagrams, 8 loctets
Figure C.63. The first component of the linkage system $D_7$, 64 linkages
Figure C.64. The second component of the linkage system $D_7$, 64 linkages
Figure C.65. The $D$-components for linkage systems $\mathcal{L}(D_7)$, $\mathcal{L}(D_7(a_1))$, $\mathcal{L}(D_7(a_2))$
C.4. The linkage systems $\mathcal{L}(D_4)$.

Figure C.66. The linkage system $D_4$, 24 linkages, 3 loctets
List of Figures

1.1 Examples of linkage diagrams and vectors of linkage labels for $E_6(a_1)$.

2.2 The starlike numbering of vertices adjacent to the branch point.

2.3 Linkage systems $\mathcal{L}(E_6(a_1))$ and $\mathcal{L}(E_6(a_2))$. The 8-cell bold subdiagrams are loctets.

2.4 The loctet types $L_{12}$, $L_{13}$ and $L_{23}$.

2.5 The weight system $(D_l, \omega)$.

2.6 Loctets in the weight system and in the linkage system $E_6$.

2.7 Relation between $\tau_k$ and $\tau_{k+1}$.

2.8 The roots $\gamma = \gamma_L + \mu$ and $\overline{\gamma} = \gamma_L - \mu$.

2.9 Two extensions: $D_5 \gamma < E_6$ and $D_5 \delta < D_6$.

2.10 Every cycle should contain an odd number of dotted edges, a case which cannot happen.

2.11 Linkage diagrams containing a square.

2.12 Linkage diagrams $\gamma_{ij}(n)$, where $1 \leq n \leq 8$, $1 \leq i < j \leq 3$, in the loctet $L_{ij}$.

2.13 The linkage diagrams $\gamma_{ij}(3)$ for $E_7(a_1)$, loctets $L_{ij}^b$.

2.14 The linkage diagrams $\gamma_{ij}(6)$ for $E_7(a_2)$, loctets $L_{ij}^b$.

2.15 The linkage diagrams $\gamma_{ij}(4)$ for $E_7(a_3)$, loctets $L_{ij}^b$.

2.16 The one-to-one correspondence between $W$-orbit and $W^\nu$-orbit.

2.17 The Dynkin extension $D_l < D_{l+1}$.

2.18 $D_l(a_k)$ for $l > 7$, 1 loctet, 2$l$ linkage diagrams.

2.19 The linkage system $D_l(a_k)$ for $l > 7$ (wind rose of linkages). The single loctet $L_{23}$ is depicted in the shaded area.

2.20 The linkage system $D_l$ for $l > 7$, 2$l$ linkages. The single loctet $L_{23}$ is depicted in the shaded area.

2.21 The single part of the $D$-component and two parts of the $A$-component of the linkage system $\mathcal{L}(A_3)$.

2.22 One of the two parts of the $D$-component and one of two parts of the $A$-component of the linkage system $\mathcal{L}(A_4)$.

2.23 One of the two parts of the $D$-component, one of the two parts of the $A$-component, and the $E$-component of the linkage system $\mathcal{L}(A_5)$.

2.24 Extension $\gamma^\nu \rightarrow {\overline{\gamma}}^\nu$ of linkage diagrams (a) and (b) and equivalent linkage label vectors. Additional coordinates are in bold.

2.25 Coordinates $\{x_i, y_j\}$ in the $D$-component of the linkage system $A_{2n+1}$.

2.26 The linkage system $A_6$ has 6 components: The two parts of the $A$-component, the two 21-element parts of the $D$-component, and the two 35-element parts of the $E$-component.

2.27 The linkage system $A_7$. One of the two 8-element parts of the $A$-component (see Table 10.8) and one of the two 28-element parts of the $D$-component.

2.28 The linkage system $A_7$ (cont). One of the two 56-element parts of the $E$-component.

2.29 The last 2 lines of the $D$-component of $A_{2n-1}$.

2.30 The last 3 lines of the $D$-component of $A_{2n}$.

2.31 The last 3 lines of the $D$-component of $A_{2n+1}$.

2.32 The transition $\mathcal{L}(A_5) \Longrightarrow \mathcal{L}(A_6)$ and $\mathcal{L}(A_6) \Longrightarrow \mathcal{L}(A_7)$. The new coordinate $\beta_3$ for the case $\mathcal{L}(A_6)$ (resp. $\alpha_4$ for the case $\mathcal{L}(A_7)$) and new linkage diagrams are in bold.
10.34 The transition from $A$-components and $D$-components in the linkage system $A_7$ to the 
linkage system $A_8$. The new coordinate $\beta_4$ and new linkage diagrams are marked in bold

A.35 Eight similar 4-cycles equivalent to $D_4(a_1)$

A.36 The ordered tree of Carter diagrams from $C_4 \coprod DE_4$

A.37 Elements $s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}$ and $s_{\alpha_{i-1}}s_{\alpha_{i-2}}$ are not conjugate

A.38 Equivalence of the $C_1$-associated element $w_1$ and the $C_2$-associated element $w_2$

A.39 The numerical labels in the right hand side are the diagonal elements of $B_{1}^{-1}$

A.40 The Dynkin diagrams $A_l$

A.41 Simply linkable vertices are marked by numerical values, which are the diagonal elements of $B_{1}^{-1}$, see Tables A.11 - A.14

C.42 Three components of the linkage system $D_4(a_1)$, 3 components

C.43 The linkage system $\mathcal{L}(D_5(a_1))$, 3 components, 5 loctets

C.44 The linkage system $\mathcal{L}(D_6(a_1))$, 3 components, 9 loctets

C.45 The linkage system $\mathcal{L}(D_6(a_2))$, 3 components, 9 loctets

C.46 The linkage system $E_6(a_1)$. The only component is the $E$-component containing two 
parts, 54 linkage diagrams, 6 loctets

C.47 The linkage system $E_6(a_2)$. The only component is the $E$-component containing two 
parts, 54 linkage diagrams, 6 loctets

C.48 The linkage system $E_7(a_1)$. The only component is the $E$-component containing 56 
linkage diagrams, 6 loctets

C.49 The linkage system $E_7(a_2)$. The only component is the $E$-component containing 56 
linkage diagrams, 6 loctets

C.50 The linkage system $E_7(a_3)$, one component, 56 linkage diagrams, 6 loctets

C.51 The linkage system $E_7(a_4)$, one component, 56 linkage diagrams, 6 loctets

C.52 $\mathcal{L}(E_6)$: The $E$-component contains 2 parts, each part contains 27 linkages (top). Weight systems of 
representations 27 and 27 of the semisimple Lie algebra $E_6$ (bottom)

C.53 The linkage system $E_7$, one component, 56 elements

C.54 Loctets in the weight system of the fundamental representations 56 of $E_7$

C.55 Linkage system for $\mathcal{L}(D_5)$, 3 components, $10 + 2 \times 16$ linkages

C.56 Loctets in the weight system for representations $(D_5, \omega_1)$, $(D_5, \omega_4)$ and $(D_5, \omega_5)$

C.57 Linkage system for $\mathcal{L}(D_6)$, 3 components, $12 + 2 \times 32$ linkages

C.58 Loctets in the weight system for representations $(D_6, \omega_1)$, $(D_6, \omega_5)$ and $(D_6, \omega_6)$

C.59 The linkage system $\mathcal{L}(D_7(a_1))$, first part of the $E$-component

C.60 $D_7(a_1)$, second part of the $E$-component: 64 linkages, 8 loctets

C.61 The linkage system $D_7(a_2)$, 1st part of the $E$-component, 64 linkage diagrams, 8 loctets

C.62 The linkage system $D_7(a_2)$, 2nd part of the $E$-component, 64 linkage diagrams, 8 loctets

C.63 The first component of the linkage system $D_7$, 64 linkages

C.64 The second component of the linkage system $D_7$, 64 linkages

C.65 The $D$-components for linkage systems $\mathcal{L}(D_7)$, $\mathcal{L}(D_7(a_1))$, $\mathcal{L}(D_7(a_2))$

C.66 The linkage system $D_4$, 24 linkages, 3 loctets
References

[Bo02] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4,5,6*. Translated from the 1968 French original by Andrew Pressley. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. xii+300 pp.

[Bo05] N. Bourbaki, *Lie groups and Lie algebras. Chapters 7,8,9*. Translated from the 1975 and 1982 French originals by Andrew Pressley. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2005. xii+434 pp.

[Ch84] R. N. Cahn, *Semi-Simple Lie Algebras and Their Representations*. Berkeley, Benjamin-Cummings publishing company, 1984.

[Ca00] R. W. Carter, *Book Review, Characters of finite Coxeter groups and Iwahori-Hecke algebras*, by M. Geck and G. Pfeiffer. Bull. Amer. Math. Soc. vol. 39, no. 2, 267–272, 2001.

[Ca70] R. W. Carter, *Conjugacy classes in the Weyl group*. 1970 Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69) pp. 297–318 Springer, Berlin.

[Ca72] R. W. Carter, *Conjugacy classes in the Weyl group*. Compositio Math. 25 (1972), 1–59.

[Dy50] E. B. Dynkin, *Some properties of the system of weights of a linear representation of a semisimple Lie group*. (Russian) Doklady Akad. Nauk SSSR (N.S.) 71, (1950). 221–224.

[Dy52] E. B. Dynkin, *Maximal subgroups of the classical groups*. (Russian) Trudy Moskov. Mat. Obsh. 1, (1952), 39–166.

[GOV90] V. V. Gorbatsevich, A. L. Onishchik, E. B. Vinberg, *Structure of Lie groups and Lie algebras*. (Russian) Current problems in mathematics. Fundamental directions, Vol. 41 (Russian), 5–259, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1990. English Translation: *Lie groups and Lie algebras III*, Encyclopediya of Mathematical Sciences, v. 41.

[H78] J. Humphreys, *Introduction to Lie algebras and representation theory*. Second printing, revised. Graduate Texts in Mathematics, 9. Springer-Verlag, New York-Berlin, 1978. xii+171 pp.

[H04] J. Humphreys, *Reflection groups and Coxeter groups*. Cambridge Studies in Advanced Mathematics, 29. Cambridge University Press, Cambridge, 1990. xii+204 pp.

[Kac80] V. Kac, *Infinite root systems, representations of graphs and invariant theory*. Invent. Math. 56 (1980), no. 1, 57-92.

[KOV95] F. I. Karpelevich, A. L. Onishchik, and E. B. Vinberg, *On the work of E. B. Dynkin in the theory of Lie groups in: Lie Groups and Lie Algebras: E. B. Dynkin's Seminar*, 1–12, Amer. Math. Soc. Transl. Ser. 2, 169, Amer. Math. Soc. 1995, 202 pp, vol. 169.

[Vo98] A. Kirillov, Jr. *An introduction to Lie groups and Lie algebras*. Cambridge Studies in Advanced Mathematics, 113. Cambridge University Press, Cambridge, 2008. xii+222 pp.

[Va00] A. L. Onishchik, E. B. Vinberg, *Lie groups and algebraic groups*. Translated from the Russian and with a preface by D. A. Leites. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1990. xx+328 pp.

[PSV98] E. Plotkin, A. Semenov, N. Vavilov, *Visual basic representations: an atlas*. Internat. J. Algebra Comput. 8 (1998), no. 1, 61–95.

[St08] R. Stekolshchik, *Notes on Coxeter Transformations and the McKay Correspondence*. Springer Monographs in Mathematics, 2008, XX, 240 p.

[St10] R. Stekolshchik, *Root systems and diagram calculus. I. Regular extensions of Carter diagrams and the uniqueness of conjugacy classes*, arXiv:1005.2769v6.

[St11] R. Stekolshchik, *Root systems and diagram calculus. III. Semi-Coxeter orbits of linkage diagrams and the Carter theorem*, arXiv:1105.2875v3.

[Va00] N. Vavilov, *A third look at weight diagrams*. (English summary) Rend. Sem. Mat. Univ. Padova 104 (2000), 201–250.
\( (\cdot, \cdot) \), symmetric bilinear form associated with \( B \), 6, 11, 29
\( (\cdot, \cdot)_r \), symmetric bilinear form associated with \( B_r \), 6, 26
\( B^{-1}_r \), inverse of the partial Cartan matrix, 73
\( B_r \), partial Cartan matrix, 6, 12, 35, 38, 72, 73
\( E \), linear space spanned by all roots of \( \Phi \), 11, 29
\( E^\circ \), dual space to \( E \), the representation space of \( W^\circ \), 39
\( L \), linear space
\( = [\tau_1, \ldots, \tau_l] \), 25, 28, 42
\( \Pi_r \), associated subspace, 25, 28
- spanned by the \( \Gamma \)-associated root subset, 10, 24, 27, 48, 49, 52
\( L^\circ \), space of linkage labels, 10, 48, 49
\( L_{ij} \), locet of type \((ij) \in \{(12), (13), (23)\}\), 13, 89
\( S = \{\tau_1, \ldots, \tau_l\} \), \( \Gamma \)-associated root subset, 6, 12, 21
\( S \) (set of linearly independent roots), 5
\( S \), \( \Gamma \)-associated root subset, 5
\( S \)-associated element, 5
\( S \)-associated subspace, 6
\( S_a \) (subset \( \{\alpha_i \mid i = 1, \ldots, k\} \)), 5
\( S_b \) (subset \( \{\beta_j \mid j = 1, \ldots, h\} \)), 5
\( W \), Weyl group, 5, 11, 29, 40, 55
\( W^\circ \), dual partial Weyl group, 12, 28
\( W_s \), partial Weyl group, 12, 28
\( \Gamma < \Gamma \), regular extension, 8, 12
\( \Gamma < \Gamma^\circ \), Dynkin extension, 17, 23, 41, 43, 49, 52, 57
\( \Gamma < \Gamma \), regular extension, 8, 12
\( \Gamma \), Carter diagram, 5, 6
\( \Gamma \)-associated element, 5
\( \Gamma \)-associated root subset, 5
\( \Gamma \)-associated set of roots, 5
\( \Lambda \), weight lattice, 39
\( \Lambda^+ \), set of all dominant weights in \( \Lambda \), 39
\( \Pi \), set of all simple roots in \( \Phi \), 11, 29
\( \Pi_r \), root subset associated with the bicolored decomposition of \( w \), 5, 25, 28
\( \Pi_r \)-associated subspace, 25, 28
\( \alpha \)-endpoints, 9, 30
\( \alpha \)-label, coordinate out of \( \alpha \)-set, 5, 11, 13, 35
\( \alpha \)-set, subset of roots corresponding to \( w_1 \), 5, 9, 30, 89
\( \beta \)-endpoints, 9, 30
\( \beta \)-label, coordinate out of \( \beta \)-set, 5, 11
\( \beta \)-set, subset of roots corresponding to \( w_2 \), 5, 9, 30
\( \gamma \), linkage root, 6, 10, 11
\( \gamma^\circ \), linkage diagram, 12, 15, 24–26
\( \gamma_1^\circ(8) \), linkage diagram, 14
\( \gamma_0^\circ(n) \), linkage diagram, 13
\( \lambda \), weight in the weight lattice \( \Lambda \), 39
\( \Lambda^\circ \), vector of Dynkin labels, 39
\( g \), simple Lie algebra, 14, 40
\( \mathcal{B} \), quadratic Tits form associated with the Cartan matrix \( B \), 11, 42
\( \mathcal{B}^\circ \), inverse quadratic form, 10, 25, 26
\( \mathcal{B}_r \), quadratic form associated with the partial Cartan matrix \( B_r \), 6
\( \mu \), normal extending vector, 26
\( \mathfrak{m} \), fundamental dominant weight, 39

**Index**

\( \Phi \), root system, 9, 29, 39
\( b_{i,j}^r \) (diagonal element of \( B^{-1}_r \)), 71
\( h \), number of coordinates in \( \beta \)-set (= number of \( \beta \)-labels), 5, 9
\( k \), number of coordinates in \( \alpha \)-set (= number of \( \alpha \)-labels), 5, 9
\( l \), number of vertices in the Carter diagram \( \Gamma \),
\( l = k + h \), 5, 9, 12
\( s_r^* \), dual reflection, 12, 28, 29
\( w = w_1 w_2 \), bicolored decomposition of \( w \); \( w_1, w_2 \) - involutions, 5, 9

\( \mathcal{B} \), Cartan matrix associated with a Dynkin diagram,

5, 11, 24, 38, 39, 76

\( \Phi(S) \), partial root system, 9, 12, 23, 50

admissible diagram, 4

bicolored decomposition, 5, 9, 11, 25, 28, 30

bicolored order, 55

branch point, 10

Carter matrix
- \( \mathcal{B} \), 5, 11, 24, 38, 39, 76
- generalized, 24
- inverse \( B^{-1}_r \) of the partial Cartan matrix, 6, 73
- partial Cartan matrix \( B_r \), 6, 35, 38, 72, 73

Carter diagram, 5

class of diagrams
- \( \Gamma \), Carter diagrams, 6
- \( C_4 \), 5, 13, 15, 32, 35
- \( DE_4 \), 5, 13, 35
- connection diagrams, 11, 12, 25
- simply-laced connected Carter diagrams, 5, 11

conjugate sets, 67

conjugate weight, 39

conjugation
- non-conjugate \( \Gamma \)-associated root subsets \( S_1 \) and \( S_2 \), 9, 12
- non-conjugate conjugacy classes, 14
- of bases of the root system, 9, 12
- of weights in the same \( W \)-orbit, 39

connection diagram, 6, 11, 12, 25, 65

Coxeter groups, 17

diagram
- admissible diagram, 4
- Carter diagram, 5

dominant weight, 39

dotted
- diagonal, 31
- edge, 4, 9, 22, 24, 29, 31, 35

dual partial Weyl group \( W^\circ \), 10
dual reflection \( s_r^* \), 12, 28, 29
dual space \( E^\circ \), 39

Dynkin extension
- \( \Gamma < \Gamma^\circ \), 17, 23
- \( A_5 < D_6 \), 52
- \( A_l < D_{l+1} \), 57
- \( D_l < D_{l+1} \) and \( D_l(a_k) < D_{l+1} \) for \( l < 8 \), 43
- \( D_l < D_{l+1} \) and \( D_l(a_k) < D_{l+1} \) for \( l \geq 8 \), 41
- $E_7 <_D E_8$, $E_7(a_1) <_D E_8$, 49
- Dynkin labels, 6, 25, 39
- fundamental representation, 40
- fundamental weight, 40
- highest weight of the representation, 40
- label
  - $\alpha$-label, 5, 11, 13, 35
  - $\beta$-label, 5, 11
  - Dynkin labels, 6, 25, 39
  - linkage label vector, 5, 6, 9, 13, 25, 26, 29, 32
- linear order for the linkage diagram, 55
- linkage diagrams
  - $\alpha$-unicolored, 11
  - $\beta$-unicolored, 11, 13, 14, 32, 86
  - $\beta$-unicolored, in $\mathcal{L}(E_6(a_1))$ and $\mathcal{L}(E_6(a_2))$, 37
  - $\beta$-unicolored, in $\mathcal{L}(E_7(a_1))$, 38
  - (= linkage label vectors), 25
  - calculation, $\beta$-unicolored, 37
  - calculation, $\gamma^\nabla(8)$, 36
  - containing a square, 31
  - examples for $E_6(a_1)$, 6, 36
  - examples, $\gamma^\nabla(3)$ for $E_7(a_1)$, 34
  - examples, $\gamma^\nabla(4)$ for $E_7(a_2)$, 34
  - examples, $\gamma^\nabla(6)$ for $E_7(a_2)$, 34
  - for Dynkin diagrams, 38
  - in $\mathcal{L}(A_4)$, 54
  - in $\mathcal{L}(D_3)$, $\mathcal{L}(D_4(a_1))$, 43
  - in $\mathcal{L}(\Gamma)$, 9, 12, 23
  - in the loctet: $\gamma^\nabla_i(n)$, $1 \leq n \leq 8$, 32
- linkage label vector, 9, 12, 25, 26, 50
- linkage label vector (= linkage diagram), 25
- linkage label vectors
  - $\mu^\nabla_{\text{max}}$ and $\mu^\nabla_{\text{min}}$ for $E_6$, 52
  - $\gamma^\nabla$, 10, 28
  - $\mathcal{L}^\nabla$ spanned by vectors $\gamma^\nabla_i$, $\gamma_i \in \Pi_w$, 10
  - for $D_7$, 48
  - in $\mathcal{L}(E_7)$, 50
- linkage root, 6, 10, 11, 25, 27
- linkage root criterion, 27
- linkage system
  - $\mathcal{L}(A_l)$ for $l = 5, 6, 7, 50$
  - $\mathcal{L}(A_l)$ for $l \geq 8$, 57
  - $\mathcal{L}(D_2)$ and $\mathcal{L}(D_2(a_1))$, 43, 90, 94
  - $\mathcal{L}(D_6)$ and $\mathcal{L}(D_6(a_1))$, 43, 90, 94
  - $\mathcal{L}(D_7)$ and $\mathcal{L}(D_7(a_1))$, 48, 105
  - $\mathcal{L}(D_l)$ and $\mathcal{L}(D_l(a_1))$ for $l < 8$, 43, 90
  - $\mathcal{L}(D_l)$ and $\mathcal{L}(D_l(a_1))$ for $l \geq 8$, 41
  - $\mathcal{L}(E_7)$ and $\mathcal{L}(E_7(a_1))$, 49, 94
- linkage system component, 9, 12, 17, 23
- loctet
  - 8th linkage diagram, 14, 36
  - (= linkage octet), 11, 13
  - of type $L_{ij}$, 13, 89
  - structure of loctets, 13
- non-conjugate $\Gamma$-associated sets, 67
- normal extending vector $\mu$, 26
- numbering near the branch point, 10
- obtuse angle between roots, 65

- partial root system, 9, 12, 23, 50
- primary root system, 4, 5, 24
- projection of the linkage root, 26
- quadratic form
  - $\mathcal{R}$, quadratic Tits form associated with the
    Cartan matrix $B$, 11, 42
  - $\mathcal{R}_\Gamma$, associated with the partial Cartan matrix
    $B_\Gamma$, 6, 24
  - inverse quadratic form $\mathcal{R}_\Gamma^\nabla$, 10, 25, 26
- regular extension $\Gamma < \tilde{\Gamma}$, 8, 12
- root stratum, 9, 12, 41, 49
- root system, 4, 9, 11, 12, 23, 29, 39, 54
- root system
  - partial, 9, 12, 23, 50
  - primary, 4, 5, 24
- similar Carter diagrams, 65
- similarity of Carter diagrams, 65
- similarity transformation for Carter diagrams, 66
- simply extendable Carter diagram, 73
- simply extended diagram, 73
- simply linkable vertex, 73
- simply linked vertex, 73
- solid
  - diagonal, 31
  - edge, 4, 9, 24, 29, 31
- space of linkage labels $L^\nabla$ for $\Phi(A_l)$, 55
- squares with a diagonal, 31
- starlike numbering, 10
- stratum size, 9, 12, 23
- symmetric bilinear form $\langle \cdot, \cdot \rangle$, 6, 11, 29
- symmetric bilinear form $\langle \cdot, \cdot \rangle_r$, 6, 26
- uniqueness of the associated conjugacy class, 5
- uniqueness theorem, 5
- weight
  - conjugation, 40
  - dominant weight, 39
  - fundamental dominant weight, 39
  - highest, 40
  - weight lattice, 39
- weight system
  - $(A_l, \mathcal{W}_1)$, 14
  - $(D_l, \mathcal{W}_1)$, 14
  - 56 of $E_7$, 101
  - (= weight diagram), 40
  - for $(D_4, \mathcal{W}_1)$, $(D_4, \mathcal{W}_3)$, $(D_4, \mathcal{W}_4)$, 15
  - for $E_6$, 16
- of representations $27$ and $\overline{27}$ for $E_6$, 40
- Weyl group, 5, 17
- Weyl group
  - $W$, 5, 11, 29, 40, 55
  - dual partial $W^\nabla$, 10, 12, 28
  - partial $W_S$, 12, 28
- primary root system, 4, 5, 24
- projection of the linkage root, 26