Analysis of D Dimensional Dirac equation for \( q \)-deformed Posch-Teller combined with \( q \)-deformed trigonometric Manning Rosen Non-Central potential using Asymptotic Iteration Method (AIM)

Y. Alam, Suparmi, Cari, F. Anwar
Physics Department, Graduate Program, Universitas Sebelas Maret, Jl. Ir. Sutami 36A Kentingan, Surakarta

E-mail: yuniar.alam08@gmail.com

Abstract. In this study, we used asymptotic iteration method (AIM) to obtain the relativistic energy spectra and wavefunctions for D Dimensional Dirac equation. Solution of the D Dimensional Dirac equation using asymptotic iteration method was done by four steps. The first step, we substituted \( q \) deformed Poschl-Teller potential plus \( q \)-deformed Manning Rosen Non-Central potential into D dimensional Dirac equation. And then, general term of D dimensional Dirac equation for \( q \) deformed Poschl-Teller potential plus \( q \)-deformed Manning Rosen Non-Central potential was reduced into one dimensional Dirac equation, consist of radial part and angular part. The second step, both of one dimensional part must be reduced to hypergeometric type differential equation by suitable parameter change. And then, hypergeometric type differential equation was transformed into AIM type differential equation. For the last step, AIM type differential equation can be solved to obtain the relativistic energy and wavefunctions of Dirac equation. Relativistic energy and wavefunctions were visualized by using Matlab software.

1. Introduction

P. A. M Dirac submit the equation known as the Dirac equation. The Dirac equation has a probability density which is always a positive value, but the solution still provide information of a free particle of negative energy. The Dirac equation describes the basic correspondent for particle spin \( \frac{1}{2} \) in electron[1]. Dirac found that a particle that has mass and charge as an electron must have an intrinsic angular momentum and magnetic moment[2]. The Dirac equation is used when a particle is exposed to potential field strong relativistic effects must be considered that emberikan corrections to the nonrelativistic quantum mechanics[3]. In a spin \( \frac{1}{2} \) particle, there is the concept of spin symmetry and symmetry pseudospin. Spin symmetry and pseudospin symmetry occurs vector potential \( V (r) \) and a scalar potential \( S (r) \) is a constant. Spin symmetry occurs when \( V (r) = S (r) \) and pseudospin symmetry occurs when \( V (r) = - S (r) \). The concept of spin symmetric has been applied to a spectrum of machinery and antinukleon, and the concept of symmetry pseudospin used to explain quasi-degeneration of double nucleons, superdeportmasi in nuclei[6], exotic nuclei[7], and to establish an affective nuclear shell-model scheme[8].
This study presents the completion of D-Dimensional Dirac equation in the case of spin symmetry for q-deformed Posch-Teller potential and q-deformed non-central Manning Rosen using asymptotic iteration method. Completion of the D-Dimensional Dirac equation by reduction into hypergeometric equations with variable substitution. Energy spectrum and the wave functions obtained from asymptotic iteration method.

2. Asymptotic Iteration Method (AIM)
This method is used to solve differential equation in the following form:

$$y_n(x) - \lambda_0(x)y_n(x) - s_o(x)y_n'(x) = 0$$  \hspace{1cm} (1)

The one-dimensional Dirac equation can be reduced into hypergeometric or confluent hypergeometric type differential equation by suitable changes of variables, and then changes it into the differential equation which has the form in Eq.(1). The solution of Eq.(1) can be obtained by using iteration of $\lambda_i$ and $s_i$,

$$\lambda_i(x) = \lambda_{i-1} + \lambda_{i-1} + s_{i-1}$$
$$s_i(x) = s_{i-1} + s_0 \lambda_{i-1}$$
$$i = 1,2,3,...$$  \hspace{1cm} (2)

Eigenvalues can be obtained using equation:[9]

$$\lambda_i(x)s_{i-1}(x) - \lambda_{i-1}(x)s_i(x) = 0 = \Delta_i, i = 1,2,3,...$$  \hspace{1cm} (3)

On the other hand, Eq.(1) can be written in term:

$$y''(x) = 2\left(\frac{tx^{N+1}}{1-bx^{N+2}} - \frac{c+1}{x}\right)y'(x) - \frac{wx^N}{1-bx^{N+2}}$$  \hspace{1cm} (4)

Eq.(4) is AIM-type differential equation which is solved by using Eq.(5)[10,11]

$$y_n(x) = (-1)^n C'(N+2)^n \frac{1}{2} \frac{\Gamma(n+p+n)}{\Gamma(n)} F_1(-n,p+n,\sigma,bx^{N+2})$$  \hspace{1cm} (5)

where

$$\sigma_n = \frac{\Gamma(\sigma+n)}{\Gamma(\sigma)}$$
$$\sigma = \frac{2c+N+3}{N+2}$$
$$p = \frac{(2c+1)b+2t}{(N+2)b}$$

$C'$ is normalization constant and $\frac{1}{2} F_1$ is hypergeometric function.[11]

3. Solution of Dirac Equation in D Dimension
The dirac equation with scalar potential $S(r)$ and vector potential $V(r)$ ($\hbar = 1, c = 1$)[16],

$$\{\hat{a}, \hat{p} + \beta(M + S(\hat{r}))\} \psi(\hat{r}) = (E - V(\hat{r})) \psi(\hat{r})$$  \hspace{1cm} (9)

which $E$ is relativistic energy of system and $\hat{p}$ is momentum operator ($\hat{p} = -i\nabla$), while $\hat{a}$ and $\beta$ is matrix in term:

$$\hat{a}_i = \begin{pmatrix} 0 & \sigma \end{pmatrix}$$
$$\beta_i = \begin{pmatrix} 0 & 0 \\ \sigma & 0 \end{pmatrix}$$

Where $\sigma_i$ are Pauli’s matrices and $I$ is the $2 \times 2$ unit matrix. Here we have used the relation for the Pauli’s matrices as follow
\[ \hat{\sigma}_i \hat{\sigma}_j + \hat{\sigma}_j \hat{\sigma}_i = 2 \delta_{ij} \mathbf{1} \]  
(12)

The wave function of spinor Dirac can be classified in two form, spinor upper \( \chi(\tilde{r}) \) and spinor lower \( \varphi(\tilde{r}) \) as follows:[12,13]

\[ \psi(r) = \left( \begin{array}{c} f_{\alpha}(r) \\ g_{\alpha}(r) \end{array} \right) = \left( \begin{array}{c} \frac{F_{\alpha}(r)}{\sqrt{\frac{n_k}{l_{\alpha-1}}}} y_{l_{\alpha-1}n_{\alpha}}^\prime (\tilde{\kappa} = \theta_1, \theta_2, \ldots, \theta_{D-1}) \\ \frac{G_{\alpha}(r)}{\sqrt{\frac{n_k}{l_{\alpha-1}}}} y_{l_{\alpha-1}n_{\alpha}}^\prime (\tilde{\kappa} = \theta_1, \theta_2, \ldots, \theta_{D-1}) \end{array} \right) \]  
(13)

we have

\[ c\tilde{\sigma} \tilde{\vec{p}} g_{nk}(r) = [E - V(\tilde{r}) - Mc^2 - S(\tilde{r})] f_{nk}(r) \]  
(14a)

\[ c\tilde{\sigma} \tilde{\vec{p}} f_{nk}(r) = [E - V(\tilde{r}) + Mc^2 + S(\tilde{r})] g_{nk}(r) \]  
(14b)

For spin symmetry, equation (14a) becomes

\[ g_{nk}(r) = \frac{\tilde{\sigma} \tilde{\vec{p}}}{[E + M]} f_{nk}(r) \]  
(15a)

\[ \tilde{\sigma} \tilde{\vec{p}} g_{nk}(r) = [E - 2V(\tilde{r}) - M] f_{nk}(r) \]  
(15b)

Substituting Eq.(15a) into Eq.(15b) yields

\[ [\mathbf{p}^2 + 2V(\tilde{r})(E + M)] f_{nk}(\tilde{r}) = [E^2 - M^2] f_{nk}(\tilde{r}) \]  
(16)

In spherical coordinates, modified Pöschl-Teller potential combined with trigonometric Manning Rosen non-central potential is defined as

\[ V(r, \theta) = \left( \frac{\kappa(\kappa - 1)}{\sin^2 \alpha r} + \frac{\eta(\eta + 1)}{\cosh^2 \alpha r} \right) + \frac{1}{r^2} \left( \frac{\nu(\nu - 1)}{\sin^2 \theta} - 2q \cot \theta \right) \]  
(17)

Substitute Eq.(17) into Eq.(16) ans simplify the resulting equation, and let

\[ f_{nk} = \frac{r_{nk}(r)}{\theta(\theta)} \phi(\varphi) \]  
(18)

Then we have,

\[ \frac{r^2}{r_{nk}(r)} \frac{\partial}{\partial r} \left( \frac{l_{D-1} + D - 1}{r^3} \right) F_{nk}(r) + \left( \frac{\kappa(\kappa - 1)}{\sin^2 \alpha r} + \frac{\eta(\eta + 1)}{\cosh^2 \alpha r} \right) [E + M] - \]  
\[ [E^2 + M^2] \right)^2 = l_{D-1}(l_{D-1} + D - 2) \]  
(19)

Separating the variables in Eq.(20), we obtain

\[ \frac{d^2 F_{nk}(r)}{dr^2} - \left( \frac{l_{D-1} + D - 1}{r^2} \right) F_{nk}(r) + \left( E + M \left( \frac{\kappa(\kappa - 1)}{\sin^2 \alpha r} + \frac{\eta(\eta + 1)}{\cosh^2 \alpha r} \right) - [E^2 - M^2] \right) F_{nk}(r) = 0 \]  
(20)

\[ \frac{1}{y_s(\theta)} \frac{\partial}{\partial \theta_s} \sin^2 \theta_s \frac{\partial}{\partial \theta_s} V(\theta) + \lambda_3 - \frac{\lambda_2}{\sin^2 \theta_i} - [E + M] V(\theta) = 0 \]  
(21)

4. Analytical solution of radial and angular parts of the dirac equation

4.1 Solution of the radial part

The radial part of the Dirac equation in Eq. (21) we use the approximation value for the centrifugal term[14,15].
\[
\frac{1}{r^2} \approx \frac{\mu^2}{4\sinh^2 \mu r} \quad (22)
\]

Substituting Eq. (22) into Eq. (20) and simplifying the equation by substituting variable \( \cosh^2 q = z \), we have
\[
z(q-z) \frac{d^2 F_n(z)}{dz^2} + \frac{1}{4} (q - 2z) \frac{dF_n(z)}{dz} - \left( \frac{A_s}{4(q-z)} - \frac{B_s}{4z} + \frac{E_s}{4} \right) F_n(z) = 0 \quad (23)
\]

by substituting,
\[
F_{nq}(z) = z^n (q-z)^{-\frac{1}{2}} \quad (24)
\]

Into Eq. (23), we have the second-order differential
\[
z(q-z) f'' + \left( 2\delta + \frac{1}{2} \right) q + z(2\delta + 2\gamma + 1) f' - \left( \delta + \gamma + \frac{E_s}{4} \right) f = 0 \quad (25)
\]

Where \( 4\delta^2 q = 2\delta q + B_s \) and \( 4\gamma^2 q = 2\gamma q + A_s \)

Eq. (25) can be transformed to differential equation type AIM,
\[
f'' + \left( 2\delta + 2\gamma + 1 \right) f + \left( \delta + \gamma + \frac{E_s}{4} \right) f = 0 \quad (26)
\]

From Eq. (26), we have
\[
\lambda_0 = \frac{(2\delta + 2\gamma + 1)z - (2\delta + \frac{1}{2})q}{z(q-z)} \quad (27)
\]

\[
s_0 = \frac{(\delta + \gamma)^2 + E_s^2}{4} \quad (28)
\]

By using Eq. (3) and using Matlab 2011 software, energy eigenvalue can be obtained, with \( \epsilon = \frac{E_s^2}{4} \)

\[
s_0\lambda_i - s_i\lambda_0 = 0 \rightarrow \epsilon_i = (2\delta + 2\gamma + 1) + \left( \delta + \gamma \right) \epsilon_i = \left( \delta + \gamma + 1 \right)^2
\]

\[
s_i\lambda_j - s_j\lambda_0 = 0 \rightarrow \epsilon_j = (2\delta + 4\gamma + 4) + \left( \delta + \gamma \right) \epsilon_j = (\delta + \gamma + 2)^2
\]

\[
s_s\lambda_s - s_s\lambda_0 = 0 \rightarrow \epsilon_s = (6\delta + 6\gamma + 9) + \left( \delta + \gamma \right) \epsilon_s = (\delta + \gamma + 3)^2
\]

can be generalized as follow
\[
\epsilon = \left( \delta + \gamma + n_r \right)^2 \quad (29)
\]

From Eq. (63) we get relativistic energy Eq. of this system is
\[
\left[ E_s^2 - M^2 \right] \frac{1}{2} = \left( \frac{1}{2} q \right) \left( B_s + \frac{1}{4} q \pm \frac{1}{2} q \sqrt{A_s + \frac{1}{4} q + n_r + \frac{1}{2} q^2} \right)^2 \quad (30)
\]

where \( n_r \) is radial quantum numbers \( (n_r = 0, 1, 2 ... ) \), \( l \) is orbital quantum numbers which is obtained from angular part solution. And then, radial wavefunction can be obtained by using Eq. (5), Eq. (6), Eq. (7) and Eq. (8), we have:
\[
c = \gamma + \frac{1}{4}, N = -1, t = \delta - \frac{3}{2}, b = 1,
\]

so, \( \sigma = \frac{2c+N+3}{N+2} = 2\delta + \frac{1}{2} \) and \( p = \frac{(2c+1)b+2t}{(N+2)b} = 2\delta + 2\gamma \)

From Eq. (5), we have,
\[
f_{n_r}(z) = (-1)^n_r C_2(1)^{n_r} (2\delta + \frac{1}{2})_{n_r} F_{1\left(-n_r, 2\delta + 2\gamma + n_r, 2\delta + \frac{1}{2}, z\right)} \quad (31)
\]

By substituting Eq. (31) to Eq. (24), we have radial wavefunction,
\( f_{nr}(z) = z^\delta (q - z)^\gamma (-1)^n r C_2(n) \left( 2\delta + \frac{1}{2} \right) n_r \frac{1}{2} F_1 \left( -n_r, 2\delta + 2\gamma + n_r, 2\delta + \frac{1}{2}, z \right) \) (32)

which \( z = \cosh^2 q \), so

\( f_{nr}(z) = z^\delta (q - z)^\gamma (-1)^n r C_2(n) \left( 2\delta + \frac{1}{2} \right) n_r \frac{1}{2} F_1 \left( -n_r, 2\delta + 2\gamma + n_r, 2\delta + \frac{1}{2}, z \right) \) (33)

where \( C(n_r) \) is radial normalization constant, \( \frac{1}{2} F_1 \) is hypergeometric function.

4.2 Solution of the angular part

For angular part in Eq.(21), can be obtain by using AIM to find orbital quantum number.

\[
\frac{\partial^2 Y_i(\theta)}{\partial \theta^2} = \left[ E + M \left( \frac{\sin^2 \theta}{\sin \theta} - 2q \cot \theta \right) \right] Y_i(\theta) + \lambda_i Y_i(\theta) = 0
\] (34)

Equation (65) can be written as, \( \cot \theta \) is a solution of \( \theta \):

\[
z_i (1 - z_i) \frac{d^2 Y_i(z_i)}{dz_i^2} + \left( 1 - 2z_i \right) \frac{dY_i(z_i)}{dz_i} + \left[ (E + M) \nu(v - 1) - \frac{2qM}{4z_i} + \frac{2qM}{4(1 - z_i)} \right] Y_i(z_i) = 0
\] (35)

by using

\[
Y_i(z_i) = z_i^n(1 - z_i)^\rho f_i(z_i)
\] (36)

and simplying it, Eq.(35) can be transform to hypergeometric differential equation:

\[
z_i (1 - z_i) f_i''(z_i) + \left( (2\alpha + 1) - z_i(2\alpha + 2\beta + 2) \right) f_i'(z_i) + \left[ (E + M) \nu(v - 1) - (\alpha + \beta)(\alpha + \beta + 1) \right] f_i(z_i) = 0
\] (37)

Eq. (37) is hypergeometric type Eq. and we can solve it by AIM as follow

\[
\lambda_0 = \frac{x_i(2\alpha + 2\beta + 2) - 2(\alpha + 1)}{x_i(1 - z_i)}
\] (38)

\[
s_0 = \frac{(4\nu + 1)(\alpha + \beta + 1) - (E + M)\nu(v + 1)}{x_i(1 - z_i)}
\] (39)

one can generalize Eq. (39).

\[
L_i = \sqrt{-\frac{(qM)^2}{(E + M)\nu(v - 1) - (E + M)\nu(v + 1) + \frac{1}{4} - n_i - \frac{1}{2}}} - \left(\sqrt{(E + M)\nu(v + 1) + \frac{1}{4} - n_i - \frac{1}{2}}\right)^2
\] (40)

Where \( L \) is orbital quantum number and \( n_l \) is angular quantum number. From Eq.(5), we have

\[
Y(z) = (-1)^n C_2(1)^n n_l \frac{1}{2} F_1 (-n_l, \rho + n_l, \alpha, b z^{N+2})
\] (41)

\( C_2 \) is angular normalization constant.

Solution for \( \theta \) and \( \theta_i \) can be determined by the same way with solution for \( \theta_i \). And we get solution for orbital quantum number for \( \theta_2 \), and \( \theta_3 \) respectively as follows:

\[
L_i = \sqrt{-\frac{(qM)^2}{(E + M)(v - i) + \frac{1}{4} - n_i - \frac{1}{2}}} - \left(\sqrt{(E + M)(v - i) + \frac{1}{4} - n_i - \frac{1}{2}}\right)^2
\] (42)

\[
L_i = \sqrt{-\frac{(qM)^2}{(E + M)(v - i) + \frac{1}{4} - n_i - \frac{1}{2}}} - 1
\] (43)

5. Result and Discussion

In this section, we discuss several results which were obtained in the previous section. From relativistic energy equation in Eq.(30) and orbital quantum number equation in Eq.(41), and by using Matlab software we have numeric solution of relativistic energy are listed in Table 1 with parameters \( \kappa = 2, \eta = 2, \nu = 3, q = 2 \) and \( M = 5 \text{ fm}^{-1} \), the negative value of relativistic energy is taken due to
the pseudospin symmetric limit[19] By inspecting Table 1, show that increase of value \( \alpha \) and \( n_r \) in the same quantum state causes decrease energy eigenvalue.

**Table 1. Relativistic energy corresponding to several states of a particle under the influence of modified Pöschl-Teller potential and trigonometric Scarf II potential.**

| \( n_r \) | \( n_l \) | \( k \) | \( \alpha \) | \( E_{n_r n_l} \ (fm^{-1}) \) |
|---|---|---|---|---|
| 0 | 0 | 0 | 0.1 \( fm^{-1} \) | -5.0805 |
| 1 | 1 | 0 | 0.2 \( fm^{-1} \) | -5.1083 |
| 2 | 2 | 0 | 0.3 \( fm^{-1} \) | -5.1411 |

By varying parameter which corresponding value \( \delta \) and \( \gamma \), some of the radial wavefunctions are listed in Table 2. Radial wavefunctions for particle under the influence of modified Pöschl-Teller potential and Manning Rosen potential are affected by potential constants \( k, \eta, q, v \) and by \( \alpha \). The parameter \( \alpha \) has a dimension inverse of distance in space that describes the reach of Pöschl-Teller potential. If \( \alpha \) is enlarged, physically means that the potential reach is smaller in a space. By inspecting Table 2, due to the increase in the value of \( \alpha \) causes particles move further away from the nucleus and show that change in radial wavefunctions are affected of potential constants \( k, \eta, q \) and \( v \).

**Table 2. Energy eigenvalue in \( fm^{-1} \) with \( n_r = 2 \) \( n_l = 2 \), \( v = 2 \), \( q = 4 \), \( M = 5 \), \( \alpha = 0.05 \) for particle under the influence of modified Pöschl-Teller potential and Manning-Rosen non-central potential variation \( k \).**

| \( N_r \) | \( E_{nr} \) | \( F_{nr} \) |
|---|---|---|
| 0 | -5.14162 | \((\cos^2 \alpha r)^2 (\sin^2 \alpha r) \) C \((2 \delta + \frac{1}{2}) \) |
| 1 | -5.09810 | \((\cos^2 \alpha r)^2 (\sin^2 \alpha r) \) C \((2 \delta + \frac{1}{2}) \) |
| 2 | -5.05656 | \((\cos^2 \alpha r)^2 (\sin^2 \alpha r) \) C \((2 \delta + \frac{1}{2}) \) |
| 3 | -5.03558 | \((\cos^2 \alpha r)^2 (\sin^2 \alpha r) \) C \((2 \delta + \frac{1}{2}) \) |

6. **Conclusion**

The Dirac equation in D dimensions of q-deformed trigonometric Poschl-Teller potential combined with Manning-Rosen non-central potential using Asymptotic Iteration Method (AIM). The radial part of D- dimensions of the Dirac equation reduces to one dimensional Schrodinger type equation in centrifugal approximation scheme. In the exact spin symmetric case, the relativistic energy equation reduces to the non-relativistic energy in the non-relativistic condition. The radial part of the wavefunction is obtained approximately from Eq.(33) and the angular part in Eq.(41). The results show that the disturbance of modified Pöschl-Teller Potential and trigonometric Manning-Rosen non-
central potential change in the wave function of the radial part and the angular part. Relativistic energy equa- tion can be obtained via AIM in Eq.(30) and equation of orbital quantum number \( l \) in Eq.(40), Eq.(42) and Eq.(43), where both are interrelated between quantum numbers. Relativistic energy also is solved numerically using Matlab software, where the increase in the radial quantum number \( n \), causes a decrease in the energy spectrum.

Acknowledgments
This Research was partly supported by Higher Education Project Grant with contract no. 632/UN27.21/LT/2016.

Reference
[1] Greiner W 2000 Relativistic Quantum Mechanics, Wave Equations (3rd Edn.) (Berlin:Springer-Verlag)
[2] Beiser A 1987 Konsep Fisika Modern (Jakarta: Erlangga)
[3] Ikhdair S M and Hamsavi M 2012. Nuclear Theory Few-Body Systems 1 487.
[4] Ginocchio J.N 2004 Phys.Rep.C 69 034313
[5] Ginocchio J.N 1997 Phys.Rev.Lett 78 436
[6] Meng J, Sugawara-Tanabe K, Yamaji S and Arima A 1999 Phys.Rev.C 59 154
[7] Dudek J, Nazarewicz W, Szymanski Z and Leander G A 1987 Phys.Rev.Lett 59 1405
[8] Troltenier D, Bahri C and Draayer J P 1995 Nucl.Phys.A 586 53
[9] Rostami A and Motavali H 2008 Progress in Electromagnetic Research B 4 171
[10] Soyulu A, Bayrak O and Boztosun I 2008 Journal of Physics A: Mathematical and Theoretical 41 065308
[11] Falaye B J, Hamzavi M and Ikhdair S M. 2012 arXiv:1207.1218v [nucl-th]
[12] Shi-Hai Dong 2011 Wave Equations in Higher Dimensions, Springer Dordrecht Heidelberg.
[13] Tapas Das arXive:1409.1457v1
[14] Greene R I and Aldrich C 1976 Phys. Rev. A 14 2363
[15] Ikhdair S M and Sever R 2008 arXiv:0801.4857v3 [quant-ph] 14 Feb 2008
[16] Hu X Q, Luo G, Wu Z M, Niu L B and Ma Y 2010 Commun. Theor. Phys. 53 242