REMARKS ON DIFFERENTIABILITY IN THE INITIAL DATA FOR STOCHASTIC REFLECTING FLOW

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Abstract. Stochastic flows generated by reflected SDEs in a half-plane with an additive diffusion term are considered. A derivative in the initial data is represented a.s. as an infinite product of matrices. We use this representation and construct an example of a reflecting flow with a linear drift such that it is not locally continuously differentiable.

Introduction

Differentiability in the initial data of flows generated by SDEs with smooth coefficients is well-studied subject of stochastic analysis (see for example [9, 11]). Equations for derivatives can be obtained by formal differentiation of initial equations. A problem of differentiability of flows generated by reflected SDEs (RSDEs) in a domain is more complicated. The corresponding results on differentiability appeared comparatively recently.

The first paper on reflecting flows differentiability is due to Deuschel and Zambotti [6]. They considered reflecting flows in an orthant with additive diffusion term. The approach of [6] was generalized by Andres [3, 4] to SDEs (with additive noise) in a polyhedron or a domain with a smooth boundary. Another approach was developed by Pilipenko in [14, 15, 16, 17], where the Sobolev derivatives were studied. It also was noticed that if we are able to prove that the flow satisfies the Lipschitz property with respect to the initial data, then this implies not only Sobolev but also Frechet differentiability [17]. Usually the Lipschitz property is satisfied if a diffusion term is constant and a drift is Lipschitzian. If the diffusion term is not constant then the problem of Frechet differentiability is open even for $C^\infty$ coefficients. The third approach of investigation was proposed by Burdzy [5], who used excursion theory to study a reflected Brownian flow in a domain with a smooth boundary. It is worth to mention that the curvature of the boundary gives some new interesting terms to a representation for the derivative, that contain a local time of a process at the boundary.

Consider an SDE in a half-space $\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times [0, \infty)$ with a normal reflection at the boundary $\partial \mathbb{R}^d_+ = \mathbb{R}^{d-1} \times \{0\}$:

$$d\varphi_t(x) = a(\varphi_t(x))dt + dw(t) + nL(dt, x), \quad t \geq 0,$$

$$\varphi_0(x) = x, \quad x \in \mathbb{R}^d_+, \quad \varphi_t(x) \in \mathbb{R}^d_+, \quad t \geq 0,$$

where $n = (0, \ldots, 0, 1)$ is a normal vector to the hyperplane $\partial \mathbb{R}^d_+$, $w(t)$ is a Wiener process in $\mathbb{R}^d$,

$$\{L(t, x), t \geq 0\} \text{ is continuous and non-decreasing in } t \text{ process,}$$

$$L(0, x) = 0,$$

$$\int_0^t \mathbb{1}_{\varphi_s(x) \in \partial \mathbb{R}^d_+} L(ds, x) = L(t, x), \quad t \geq 0.$$ 

The last condition means that $L(t, x)$ does not increase in $t$ when $\varphi_t(x) \in \mathbb{R}^d_+ \setminus \partial \mathbb{R}^d_+$. So, a solution of the RSDE behaves as a solution of an SDE without reflection inside the upper half-space.
Let us give informal explanation how to guess a form of an equation for the derivative in initial data (the proof of the corresponding fact is non-trivial). Since inside the upper half-space the equation behaves as usual SDE, the derivative should be obtained by formal differentiation of (1) with respect to \(x\), i.e., \(\frac{\partial \varphi_t(x)}{\partial t} = \nabla a(\varphi_t(x)) \nabla \varphi_t(x)\) if \(\varphi_t(x) \notin \partial \mathbb{R}^d_+\). If \(\varphi_t(x) \in \partial \mathbb{R}^d_+\), then the \(d\)-th coordinate of the process \(\varphi_t(x)\) attains a minimum (it equals zero). So, the derivative of \(d\)-th coordinate should be equal to 0. This requirement and some technical details are enough to determine uniquely the derivative [3, 4, 6, 15, 17] (see §1 for strict statement).

Underline one important circumstance. It was proved in all papers cited above that

\[
\forall \, x \, \forall \, t \quad P(\exists \nabla \varphi_t(x)) = 1. \tag{6}
\]

Note that the statements

\[
\forall \, x \quad P(\forall \, t \geq 0 \quad \exists \nabla \varphi_t(x)) = 1
\]

and

\[
\forall \, t \geq 0 \quad P(\forall \, x \quad \exists \nabla \varphi_t(x)) = 1,
\]

are, generally, incorrect. This fact is easy to explain in one-dimensional case. Let \(\varphi_t(x)\) be a reflected Brownian motion. It can be checked that

\[
\varphi_t(x) = \begin{cases} 
\varphi_0(0) = w(t) - \min_{0 \leq s \leq t} w(s), t > \sigma(x), \\
x + w(t), t \leq \sigma(x),
\end{cases}
\]

where \(\sigma(x)\) is the first instant when \(x + w(\cdot)\) hits zero:

\[
\sigma(x) = \inf\{t \geq 0 : x + w(t) = 0\}.
\]

Then

\[
\frac{\partial \varphi_t(x)}{\partial x} = \begin{cases} 
0, x < - \min_{0 \leq s \leq t} w(s), \\
1, x > - \min_{0 \leq s \leq t} w(s), \\
does \not \text{exist, } x = - \min_{0 \leq s \leq t} w(s).
\end{cases}
\]

However, for any fixed \(t > 0, x > 0\):

\[
P(\varphi_t \text{ is continuously differentiable in a neighborhood of } x) = 1. \tag{7}
\]

This example makes reasonable a conjecture: equality (7) is always satisfied if coefficients of the RSDE are smooth. A result of paper [6] gives another argument in favor of this hypothesis. It was proved (additive noise, \(\mathcal{C}^1\) drift, normal reflection at hyperplanes) that there exists a modification \(\psi_t(x)\) of the derivative \(\nabla \varphi_t(x)\) such that for all \(t, x_0\)

\[
P(\psi_t(x_0) = \nabla \varphi_t(x_0)) = 1, \tag{8}
\]

and

\[
\psi_t(x) \to \psi_t(x_0), \; x \to x_0 \; \text{a.s.} \tag{9}
\]

In §2 we give an example of a flow \(\varphi_t(x), t \geq 0, x \in \mathbb{R}^d_+\) generated by RSDE in a half-plane \(\mathbb{R}^d_+\) with normal reflection at the boundary \(\partial \mathbb{R}^d_+\), additive diffusion and \(\mathcal{C}^\infty\) drift such that it is not locally continuously differentiable a.s. It will be shown that this flow is not even locally differentiable a.s. Moreover

\[
\forall \, x \quad P(\left\{\sigma(x) < t\right\} \cap \{\exists \text{ a neighbourhood } U(x) \forall \, y \in U(x) \exists \nabla \varphi_t(y)\}) = 0,
\]

where \(\sigma(x) = \inf\{t \geq 0 : \varphi_t(x) \in \partial \mathbb{R}^d_+\}\).

Note that this statement neither contradicts (6) nor contradicts (8), (9).

In §1 we give some preliminary formulas, in particular, the derivative \(\nabla \varphi_t\) will be represented as an infinite product of matrices.

§1. Representation of the derivative on initial value for reflecting flow

To give a representation of the derivative on the initial value for a reflecting stochastic flow, we need to introduce one type of integral equation (see Theorem 1 below). An equation for the derivative on the initial value is given in Theorem 2. The main result of this Section is Theorem 4, where we express the derivative as an infinite products of matrices.

Consider \(d \times d\) matrices
Then there exists a unique function \( \gamma \) we can solve (11), (12) successively on intervals \([0, \infty)\).

**Theorem 1.** Let \( \alpha : [0, \infty) \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d \) be a continuous and bounded function taking values in the space of \(d \times d\) matrices, and \( \beta : [0, \infty) \rightarrow \mathbb{R} \) a continuous function, \( \beta(0) \neq 0 \). Denote

\[
\sigma = \inf\{t \geq 0 : \beta(t) = 0\}, \quad \tau(t) = \sup\{s \in [0, t] : \beta(s) = 0\}.
\]

Then there exists a unique function \( \gamma : [0, \infty) \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d \) satisfying the system

\[
P\gamma(t) = P + \int_0^t P\alpha(s)\gamma(s)ds, \quad t \geq 0,
\]

\[
Q\gamma(t) = \begin{cases} 
Q + \int_0^t Q\alpha(s)\gamma(s)ds, & t < \sigma, \\
\int_{\tau(t)}^t Q\alpha(s)\gamma(s)ds, & t \geq \sigma.
\end{cases}
\]

Theorem on existence and uniqueness for a solution of such type equation was proved in more general setting in [4, 9], see also [1], where such equations were introduced for the first time.

So, we only give a sketch of a proof in order to explain difficulties that may arise and a form of representations that we will obtain.

Assume at first that the function \( \beta \) has only finite number of zeroes \( \sigma = \sigma_0 < \sigma_1 < \sigma_2 < \ldots < \sigma_n \), then we can solve (11), (12) successively on intervals \([0, \sigma_0], [\sigma_0, \sigma_1], [\sigma_1, \sigma_2] \) and so on. It is easy to see that in this case \( \gamma(t) \) is obtained by successive solution of the following linear equations

\[
\gamma(t) = E + \int_0^t \alpha(s)\gamma(s)ds, \quad t < \sigma_0,
\]

\[
\gamma(t) = P\gamma(\sigma_k -) + \int_{\sigma_k}^t \alpha(s)\gamma(s)ds, \quad t \in [\sigma_k, \sigma_{k+1}).
\]

Denote by \( E_{st} \) a solution of the following matrix-valued equation

\[
\begin{cases}
\partial_{ss} E_{st} = \alpha(t)E_{st}, & t \geq s, \\
E_{ss} = E.
\end{cases}
\]

It follows from (13), (14) that

\[
\gamma(t) = E_{0t}, \quad t < \sigma_0.
\]

If \( t \in [\sigma_k, \sigma_{k+1}) \), then

\[
\gamma(t) = E_{\sigma_k t}P\gamma(\sigma_{k-}) = E_{\sigma_k t}PE_{\sigma_{k-1} \sigma_k}P\gamma(\sigma_{k-1}-) = \ldots = E_{\sigma_k t}PE_{\sigma_{k-1} \sigma_k}P\ldots P\xi_{\sigma_1 \sigma_k}PE_{\sigma_0 \sigma_k}.
\]

Assume now that \( \beta \) has infinite number of zeroes, for example, let \( \beta \) be a typical trajectory of a Wiener process. Then we should use more delicate methods to solve (11), (12). It can be done as follows. Introduce a function \( \pi \) that takes a matrix-valued function \( x = x(t), t \geq 0 \), to

\[
(\pi x)(t) = \begin{cases} 
x(t), & t < \sigma, \\
Px(t) + Q(x(t) - x(\tau(t))), & t \geq \sigma,
\end{cases}
\]

where \( \sigma \) and \( \tau(t) \) are from (10).

Observe that system (11), (12) is equivalent to the following

\[
\gamma(t) = \pi \left( E + \int_0^t \alpha(s)\gamma(s)ds \right)(t), \quad t \geq 0.
\]
Moreover there exists a modification uniformly Lipschitzian in $x$

For all $\omega$ one can find a unique solution of $\text{RSDE}$, and Lipschitz property for any $\omega$.

Really, the Skorokhod map in a half-plane is Lipschitzian. Since the noise is additive, the existence, uniqueness for a solution of $\text{RSDE}$, and Lipschitz property for any $\omega$ can be proved by standard arguments, see for example from [2].

Denote

$$
\sigma(x) = \inf \{ t \geq 0 : \, \psi_t(x) \in \partial \mathbb{R}^d_+ \}. 
$$ (21)

**Theorem 2.** For all $x \in \mathbb{R}^{d-1} \times (0, \infty), t \geq 0$

$$
P(\text{Frechet derivative } \nabla \psi_t(x) \text{ exists}) = 1.
$$

Moreover there exists a modification $\psi_t(x)$ of the derivative, i.e.

$$
P(\psi_t(x) = \nabla \psi_t(x)) = 1, \, t \geq 0, \, x \in \mathbb{R}^{d-1} \times (0, \infty),
$$

such that

1) for any $x$ the process $\psi_t(x), t \geq 0$ is càdlàg,

2)

$$
P\psi_t(x) = P + \int_0^t P\nabla a(\psi_s(x))\psi_s(x)\, ds, \, t \geq 0,
$$

$$
Q\psi_t(x) = \begin{cases} Q + \int_0^t Q\nabla a(\psi_s(x))\psi_s(x)\, ds, & t < \sigma(x), \\
\int_{\tau(t,x)}^t Q\nabla a(\psi_s(x))\psi_s(x)\, ds, & t \geq \sigma(x), \end{cases}
$$ (23)

where $P$ and $Q$ are the same as in Theorem 1,

$$
\tau(t, x) = \sup \{ s \in [0, t] : \, \varphi_s(x) \in \partial \mathbb{R}^d_+ \}
$$ (24)

is the last instant before $t$ when the process $\varphi(x)$ visit the hyperplane.

The proof of a differentiability and representation (22), (23) see for example in [17, 4].

**Remark.** System (22), (23) is a particular case of (11), (12), where $\alpha(t) = \nabla a(\varphi_t(x))$, and $\beta(t)$ is $\varphi^d_t(x)$ (the $d$-th coordinate of the process $\varphi_t(x)$).

**Remark.** Assume that $\varphi_t(x) \notin \partial \mathbb{R}^d_+ \times t \in [t_1, t_2]$. Then there exists a neighborhood $U(x)$ of $x$ such that $\varphi_t(y) \notin \partial \mathbb{R}^d_+, t \in [t_1, t_2], y \in U(x)$. So, $\varphi_t(y)$ satisfies the following integral equation

$$
\varphi_t(y) = \varphi_{t_1}(y) + \int_{t_1}^t a(\varphi_s(y))\, ds + w(t) - w(t_1), \, t \in [t_1, t_2], \, y \in U(x).
$$ (25)
Representations (22), (23) imply that
\[
\frac{\partial \psi_t(y)}{\partial t} = \nabla a(\varphi_t(y))\psi_t(y), \quad t \in [t_1, t_2].
\]
In particular,
\[
\psi_t(x) = E + \int_0^t \nabla a(\varphi_s(x))\psi_s(x)ds, \quad t < \sigma(x),
\]
as it should be for a derivative in the initial data of integral equation (25).

The main aim of this Section is to obtain a representation of (22), (23) solution, which is similar to (16), (17). We prove the corresponding result in general settings for equations (11), (12).

Note that a product of matrices depends on the order of the product. Thus we need a formal definition and sufficient condition for convergence of infinite product.

Let \( K \) be a countable set with a linear order \( \leq \). Let us introduce a partial order on finite subsets of \( K \) as follows
\[
L_1 \preceq L_2 \iff L_1 \subset L_2.
\]
Let \( X \) be a Banach space, \( \{A_k, k \in K\} \) be a collection of linear continuous operators on \( X \) (for example, \( X = \mathbb{R}^n \), \( A_k \) is \( d \times d \) matrix).

Let \( L = \{l_1, \ldots, l_n\} \subset K \), \( l_1 \geq \ldots \geq l_n \). By \( \prod_{k \in L} A_k \) denote a product \( A_{l_1}A_{l_2}\ldots A_{l_n} \) (operators with greater indices are on the left).

**Definition 1.** An infinite product \( \prod_{k \in K} A_k \) converges and equals a linear continuous operator \( U \) if
\[
\forall \varepsilon > 0 \exists L_0 \subset K, |L_0| < \infty \forall L \ni L_0 : \| \prod_{k \in L} A_k - U \| < \varepsilon,
\]
where \( | \cdot | \) is a number of elements in a set, \( \| \cdot \| \) is a norm of a linear operator.

**Remark.** Definition 1 means a convergence of generalized sequence of matrices \( \{\prod_{k \in L} A_k, |L| < \infty, L \subset K\} \), where partial order is \( \preceq \).

**Remark.** We do not require the non-degeneracy of a limit in contrast to the usual definition of infinite product of numbers.

**Theorem 3.** Assume that \( A_k = E + B_k, k \in K \), where linear operators \( B_k \) are such that
\[
\sum_{k \in K} \|B_k\| < \infty.
\]
Then the infinite product \( \prod_{k \in K} A_k \) converges.

**Remark.** A sum of real-valued series with non-negative terms is independent of the order of summation.

**Proof of Theorem 3.** Note that for any collections of operators \( \{C_k\}, \{D_k\} \) the following inequality holds
\[
\| (E + D_0)(E + C_1)(E + D_1)(E + C_2)\ldots (E + C_n)(E + D_n) -
\quad - (E + C_1)\ldots (E + C_n)\| =
\quad = \| \sum_{k} \left[ (E + D_0)(E + C_1)(E + D_1)\ldots (E + C_{k-1})(E + D_{k-1}) -
\quad - (E + D_0)(E + C_1)(E + D_1)\ldots (E + C_{k-1}) \right] (E + D_k)\ldots (E + D_n) \| \leq
\quad \leq \left( \sum_{k=0}^{n} \|D_k\| \right) \prod_{k=0}^{n} (1 + \|D_k\|) \prod_{k=1}^{n} (1 + \|C_k\|) \leq
\quad \leq \left( \sum_{k=0}^{n} \|D_k\| \right) \exp \left\{ \sum_{k=0}^{n} \|D_k\| + \sum_{k=1}^{n} \|C_k\| \right\}.
\]
Let
\[
L_0 \subset L_1, \quad L_0 = \{l_1, \ldots, l_n\},
\]
Thus (27) implies that for any $L \subseteq \mathbb{R}$, so

$$\mathcal{L}(\{\sigma_k, \tau_k\}) \leq \sum_{k \in K} \|B_j\| \cdot \exp\left\{ \sum_{k \in K} \|B_k\| \right\}.$$  

Let $\varepsilon > 0$ be fixed. Choose $L_0$ such that

$$\sum_{j \notin L_0} \|B_j\| \exp\left\{ \sum_{k \in K} \|B_k\| \right\} < \varepsilon.$$  

Thus (27) implies that for any $L_1, L_0 \subseteq L_1$

$$\|\prod_{k \in L_0} A_k - \prod_{k \in L_1} A_k\| \leq \varepsilon.$$  

So $\{\prod_{k \in L} A_k, |L| < \infty, L \subseteq K\}$ is a generalized Cauchy sequence. This implies a convergence of the product (see [7, Ch.1, §7]).

Theorem 3 is proved.  

Let us consider equations (11), (12). Represent a set \( \{t \geq 0 : \beta(t) \neq 0\} \) as a denumerable union of disjoint sets $[0, \sigma_0) \cup \cup_k (\sigma_k, \tau_k)$, where possibly $\sigma_0 = \infty$ or $\tau_k = \infty$ for some $k$. Introduce a linear order in the set $K = \{(\sigma_k, \tau_k)\}$ of intervals:

$$\sigma_i, \tau_i < (\sigma_j, \tau_j) \Leftrightarrow \tau_i < \sigma_j.$$  

The main results of this Section is the next Theorem and Corollary.

**Theorem 4.** Assume that

$$\lambda\{\{t \geq 0 : \beta(t) = 0\}\} = 0,$$  

where $\lambda$ is a Lebesgue measure. Then a solution of system (11), (12) is of the form

$$\gamma(t) = \begin{cases} \mathcal{E}_{0\alpha}, & t < \sigma_0, \\ \mathcal{E}_{\tau(t)} t P \left( \prod_{(\sigma, \tau) \subset [0, \tau]} (P \mathcal{E}_{\sigma, \tau} P) \right) \mathcal{E}_{0\sigma_0}, & t \geq \sigma_0, \end{cases}$$  

where $\mathcal{E}_{st}$ is defined in (15).

**Remark.** Notice that \( \lambda\{\{t \geq 0 : \varphi^d_t(x) = 0\}\} = 0 \) a.s., where $\varphi^d_t(x)$ is $d$-th coordinate of the process $\varphi_t(x)$. So, conditions of Theorem 4 are satisfied for the solution of (22), (23) for a.a. $\omega$. Combining Theorems 2 and 4 we obtain the following statement.

**Corollary 1.** Let $\varphi_t(x)$ be a solution of (1) - (5). Then for all $t \geq 0, x \in \mathbb{R}^{d-1} \times (0, \infty)$ with probability 1 we have

$$\nabla \varphi_t(x) = \begin{cases} \mathcal{E}_{0\sigma}, & t < \sigma(x), \\ \mathcal{E}_{\tau(t)} t P \left( \prod_{(\sigma, \tau) \subset [0, \tau]} (P \mathcal{E}_{\sigma, \tau} P) \right) \mathcal{E}_{0\sigma_0}, & t \geq \sigma(x), \end{cases}$$  

where $\mathcal{E}_{st}(x)$ is a solution of

$$\begin{cases} \frac{\partial}{\partial t} \mathcal{E}_{st}(x) = \nabla a(\varphi_t(x)) \mathcal{E}_{st}(x), & t \geq s, \\ \mathcal{E}_{ss}(x) = E, \end{cases}$$  

where $\sigma(x), \tau(t, x)$ are defined in (21) and (24) respectively, and $\{(\sigma_k(x), \tau_k(x))\}$ is a collection of disjoint intervals such that

$$\cup_k (\sigma_k(x), \tau_k(x)) = \{t > \sigma(x) : \varphi_t(x) \notin \partial \mathbb{R}^d_+\}.$$  

\[L_1 = \{m_0, 1, \ldots, m_0, k, l_1, m_1, 1, \ldots, m_1, k_1, l_2, m_2, 1, \ldots, m_2, k_2, l, 1, \ldots, m_n, 1, \ldots, m_n, k_n\},\]  

where elements in parenthesis are in decreasing order. It follows from (26) that

$$\|\prod_{k \in L_0} (E + B_k) - \prod_{k \in L_1} (E + B_k)\| \leq$$  

$$\leq \left( \sum_{i=0}^n \sum_{j=0}^k \|B_{m_i, j}\| \right) \exp\left( \sum_{i=0}^n \sum_{j=0}^k \|B_{m_i, j}\| + \sum_{i=1}^n \|B_i\| \right) \leq$$  

$$\leq \sum_{j \notin L_0} \|B_j\| \exp\left( \sum_{k \in K} \|B_k\| \right).$$
Proof of Theorem 4. The result of Theorem 4 is obvious if a number of intervals \((\sigma_k, \tau_k)\) is finite (see representation (17) and observe that \(P^{\mathbb{R}} = P\)). Therefore, further we consider only the case when the corresponding number of intervals is countable.

Select a sequence \(\{K_n\}_{n \geq 1} \subset K\), \(K_n = \{(\sigma_i^{(n)}, \tau_i^{(n)}), i = 1, n\}\) such that

\[
K_n \subset K_{n+1}, \quad \cup_n K_n = K, \quad \sigma_1^{(n)} < \sigma_2^{(n)} < \ldots < \sigma_n^{(n)}.
\]

Set

\[
\gamma_n(t) = E_{\tau_n(t)} \times P \prod_{(\sigma_i^{(n)}, \tau_i^{(n)}) \in [0, t]} (PE_{\sigma_i^{(n)}, \tau_i^{(n)}} P) E_{0_{\gamma_0}}, \quad t \geq \sigma_0,
\]

and \(\gamma_n(t) = E_{0t}\) if \(t \in [0, \sigma_0]\). Here \(\tau_n(t) := \max\{\tau_k^{(n)} : \tau_k^{(n)} \leq t\}\).

By \(\gamma(t)\) denote the right-hand side of (29). Let us verify that \(\gamma(t)\) is well-defined and

\[
\lim_{n \to \infty} \gamma_n(t) = \gamma(t).
\]

At first, observe that (28) implies the convergence \(\tau_n(t) \to \tau(t), \ n \to \infty, \) as \(t \geq \sigma_0\). Thus

\[
E_{\tau_n(t)} t \to E_{\tau(t)} t, \quad n \to \infty.
\]

Let us prove that

\[
\prod_{(\sigma_i^{(n)}, \tau_i^{(n)}) \in [0, t]} (PE_{\sigma_i^{(n)}, \tau_i^{(n)}} P) \to \prod_{(\sigma, \tau) \in [0, t]} (PE_{\sigma, \tau} P), \quad n \to \infty.
\]

Observe that for any \(d \times d\)-matrix \(A\) the matrix \(PAP\) can be considered as a linear operator from \(\mathbb{R}^{d-1} \times \{0\}\) to \(\mathbb{R}^{d-1} \times \{0\}\). In particular, \(P\) acts as an identity operator in \(\mathbb{R}^{d-1} \times \{0\}\), and

\[
\|PAP - P\| = \|P(A - E)P\| \leq \|A - E\|.
\]

For all \(s \leq t\) we have an estimate

\[
\|E_{st}\| \leq \exp\{c(t - s)\},
\]

where \(\| \cdot \|\) is a norm of matrix considered as a linear operator in \(\mathbb{R}^d\), \(c = \sup_{r \geq 0} \|\alpha(r)\|\). So, for any \(T > 0\) there is a constant \(K = K(T)\) such that

\[
\forall s, t \in [0; T], s \leq t : \quad \|E_{st} - E\| = \|\int_s^t \alpha(z) E_{sz} dz\| \leq K(t - s).
\]

Now Theorem 3, (35), and (36) imply (34). Hence (33) is proved.

Let us prove now that \(\gamma(t)\) satisfies (11).

Denote \(\alpha_n(t) = \alpha(t) \mathbb{I}_{\{t \in \cup_k [\sigma_k^{(n)}, \tau_k^{(n)}) \cup [0, \sigma_0]\}}\). Observe that

\[
\frac{d\gamma_n(t)}{dt} = \alpha(t) \gamma_n(t), \quad t \in \cup_{k=1}^n (\sigma_k^{(n)}, \tau_k^{(n)}) \cup [0, \sigma_0)
\]

and

\[
\gamma_n(\sigma_{k+1}^{(n)}) = \gamma_n(\tau_k^{(n)}), \gamma_n(\sigma_{k+1}^{(n)}) = P\gamma_n(\tau_k^{(n)}).
\]

So

\[
P\gamma_n(t) = P + P \int_0^t \alpha_n(s) \gamma_n(s) ds, \quad t \geq 0,
\]

\[
Q\gamma_n(t) = \begin{cases} Q + Q \int_0^t \alpha_n(s) \gamma_n(s) ds, & t < \sigma_0, \\ Q \int_{\tau_n(t)}^t \alpha_n(s) \gamma_n(s) ds, & t \geq \sigma_0. \end{cases}
\]

It follows from (28) that \(\alpha_n(t) \to \alpha(t), \ n \to \infty\), for \(\lambda\) a.a. \(t \geq 0\). Thus, the Lebesgue dominated convergence theorem and (33) yield

\[
P\gamma(t) = \lim_{n \to \infty} P\gamma_n(t) = \lim_{n \to \infty} (P + P \int_0^t \alpha_n(s) \gamma_n(s) ds) = \ldots
\]
\[
= P + P \int_0^t \alpha(s) \gamma(s) ds,
\]

i.e. \( \gamma(t) \) satisfies (11).

Let us show that \( \gamma(t) \) satisfies (12). Let \( t \geq \sigma_0 \) (the case \( t \in [0; \sigma_0) \) is trivial). Since \( \lim_{n \to \infty} \tau_n(t) = \tau(t) \), using the Lebesgue theorem again, we get

\[
Q \gamma_n(t) = Q \int_{\tau_n(t)}^t \alpha_n(s) \gamma_n(s) ds = \frac{Q \int_{\tau_n(t)}^t \Pi_{s \in [\tau_n(t), t]} \alpha_n(s) \gamma_n(s) ds}{\lim_{n \to \infty} \int_{\tau_n(t)}^t \Pi_{s \in [\tau(t), t]} \alpha(s) \gamma(s) ds} = Q \int_{\tau(t)}^t \alpha(s) \gamma(s) ds,
\]

i.e.

\[
Q \gamma(t) = Q \int_{\tau(t)}^t \alpha(s) \gamma(s) ds.
\]

Thus, \( \gamma(t) \) satisfies (11), (12). Uniqueness of (11), (12) solution implies the equality \( \gamma(t) = \gamma(t) \). Theorem 4 is proved.

**Remark.** Let \( \varphi_1(x) \) be a solution of reflected SDE

\[
d\varphi_1(x) = a(\varphi_1(x)) dt + \sum_{k=1}^m \sigma_k(\varphi_1(x)) dw_k(t) + nL(dt, x), \ t \geq 0, x \in \mathbb{R}^d,
\]

where conditions (2) – (5) are also satisfied. Assume that functions \( a, \sigma_k \) are continuously differentiable and have bounded derivatives. Then the Sobolev derivative \( \nabla \varphi_1(\cdot) \) exists a.s. (see [15]) and there is a modification \( \psi_1(x) \) of the derivative such that

\[
\psi_1(x) = \pi \left( E + \int_0^t \nabla a(\varphi_s(x)) \psi_s(x) ds + \sum_{k=1}^m \int_0^t \nabla \sigma_k(\varphi_s(x)) \psi_s(x) dw_k(s) \right) (t), \tag{39}
\]

where \( \pi \) is defined in (18).

The author does not know a result on representation of (39) solution as a product (30), where \( \mathcal{E}_{st}(x) \) is a stochastic exponent,

\[
\mathcal{E}_{st}(x) = E + \int_s^t \nabla a(\varphi_z(x)) \mathcal{E}_{sz}(x) dz + \sum_{k=1}^m \int_s^t \nabla \sigma_k(\varphi_z(x)) \mathcal{E}_{sz}(x) dw_k(z).
\]

In this case Theorem 3 is inapplicable. It is possible that representation (30) is not satisfied.

\[\square\]

**§ 2. An example of a reflecting flow that is not locally differentiable**

Consider a reflecting flow \( \varphi_1(x), t \geq 0, x = (x_1, x_2) \in \mathbb{R}_+^2 \) in a half-plane that satisfies (1)–(5) with \( a(x) = Ax \), where \( A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \):

\[
d\varphi_1(x) = A \varphi_1(x) dt + dw(t) + nL(dt, x).
\]

In coordinate form equation (40) can be written as follows

\[
d\varphi_1^1(x) = (\varphi_1^1(x) + \varphi_1^2(x)) dt + dw_1(t), \tag{41}
\]

\[
d\varphi_1^2(x) = (\varphi_1^1(x) + \varphi_1^2(x)) dt + dw_2(t) + L(dt, x). \tag{42}
\]

In this case the operator \( \mathcal{E}_{st} \) from (31) is non-random and it is equal to \( \mathcal{E}_{st} = \mathcal{E}_{t-s} \), where

\[
\mathcal{E}_t = e^{At} = \begin{pmatrix} \frac{e^{at} + 1}{2} & \frac{e^{at} - 1}{2} \\ \frac{e^{at} - 1}{2} & \frac{e^{at} + 1}{2} \end{pmatrix}.
\]
Let $\sigma = \sigma(x), \tau(t) = \tau(t,x)$, $[\sigma_k, \tau_k] = [\sigma_k(x), \tau_k(x)]$ are the same as in (32). Denote $\Delta_k = \Delta_k(x) = \tau_k(x) - \sigma_k(x)$. Then for a.a. $\omega \in \{ t \geq \sigma(x) \}$ representation (30) has a form

$$
\nabla \varphi_1(x) = \begin{pmatrix}
\frac{e^{2(t-\tau(t,x)) + 1}}{2} & 0 \\
\Pi_{[\sigma_k, \tau_k] \subset (0,t]} \frac{2^{\Delta_k} + 1}{2} & 0 \\
0 & \frac{e^{2\sigma(x) + 1}}{2} & 0 \\
0 & \frac{e^{2\sigma(x) - 1}}{2} & 0
\end{pmatrix}.
$$

In particular,

$$\frac{\partial \varphi_1}{\partial x_1}(x) = \frac{e^{2(t-\tau(t))} + 1}{2} e^{2t \sigma} + 1 \prod_{[\sigma_k, \tau_k] \subset (0,t]} \frac{e^{2\Delta_k} + 1}{2}.$$  

(43)

**Remark.** The product in the right-hand side of (43) is a product of numbers (and not a product of matrices as in general case of §1). So, generally, the order of the product is inessential.

By $f_i(x) = f_i(x_1, x_2)$, $t \geq \sigma(x)$, denote the right-hand side of (43). Set $f_i(x) = (e^{2t} + 1)/2$ for $t < \sigma(x)$.

The main result of this Section is contained in the following two theorems.

**Theorem 5.** For all $t > 0$

1) For all $x_2 > 0$ and a.a. $\omega \in \Omega$ the function $x_1 \mapsto f_i(x_1, x_2)$ is nondecreasing in $x_1$.

2) For any $x = (x_1, x_2)$, $x_2 > 0$ and a.a. $\omega \in \{ \sigma(x) < t \}$ a function $f_i(\cdot)$ is discontinuous in any neighborhood of $x$.

Moreover, for any $x = (x_1, x_2)$, $x_2 > 0$ and $\delta > 0$:

$$P \left( \{ a \text{ function } f_i(\cdot, x_2) \text{ does not have a jump discontinuity on } (x_1 - \delta, x_1 + \delta) \} \cap \{ \sigma(x) < t \} \right) = 0.$$

**Theorem 6.** For any $t > 0$

$$P \left( \text{derivative } \frac{\partial \varphi_1}{\partial x_1} \text{ exists for all points of some non-empty open subset of } \{ x \in \mathbb{R}^2_+ : \sigma(x) < t \} \right) = 0.$$

To prove the Theorems we need the following statement on the monotonicity of the flow $\{ \varphi_i(x) \}$.

**Lemma 1.** The flow $\varphi_i(\cdot)$ is monotonous in $x$ in the following sense.

1) If $x^i = (x_1^{(i)}, x_2^{(i)}) \in \mathbb{R}_+^2$, $i = 1, 2$ are such that $x_1^{(1)} \leq x_2^{(1)}$, $x_2^{(1)} \leq x_2^{(2)}$, then with probability 1

$$\varphi_1^i(x^1) \leq \varphi_1^j(x^2), \quad \varphi_2^i(x^1) \leq \varphi_2^j(x^2)$$  

(44)

for all $t \geq 0$.

2) If we have at least one strict inequality $x_1^{(1)} < x_2^{(2)}$ or $x_2^{(1)} < x_1^{(2)}$, then

$$\varphi_1^i(x^1) < \varphi_1^j(x^2), \quad t \geq 0,$$

$$\sigma(x^1) < \sigma(x^2),$$

and

$$\varphi_2^i(x^1) < \varphi_2^j(x^2)$$  

(46)

for all $t$ such that $\varphi_2^i(x^2) > 0$.

The proof of the first statement can be done similarly to [12].

The proof of (45) follows from the next obvious lemma.

**Lemma 2.** Assume that continuous functions $v, \xi_i, g_i$, $i = 1, 2$, are such that

$$\xi_i(t) = \xi_i(0) + \int_0^t g_i(s)ds + v(t), \quad t \geq 0,$$

and $\xi_1(0) \leq \xi_2(0)$, $g_1(s) \leq g_2(s), s \in [0; t]$. Suppose that either $\xi_1(0) < \xi_2(0)$ or there exists a point $s_0 \in [0; t]$ such that $g_1(s_0) < g_2(s_0)$. Then $\xi_1(t) < \xi_2(t)$. 

Lemma 3. 1) For all $t \in (t_0; t_1)$, then $\varphi^2_t(x^2) > 0$ if $z \in (t_0; t_1)$, then $\varphi^2_t(x^2)$ satisfies the integral equation without reflection on $[t_0, t_1]$:

$$\varphi^2_t(x) = \varphi^2_{t_1}(x) + \int_{t_0}^{t} (\varphi^1_s(x) + \varphi^2_s(x))dz + (w_2(t) - w_2(t_0)), \quad t \in [t_0, t_1].$$

Remark. Generally speaking, it is not difficult to prove deterministic analogue of Lemma 1, where $w$ is an arbitrary continuous function in equation (40).

Lemma 1 yields the following.

Corollary 2. Let $x_1 < y_1$. Then for any $x_2 > 0, t \geq 0$, we have inclusion of sets

$$\{s \in [0; t]: \varphi^2_s(x_1, x_2) = 0\} \supset \{s \in [0; t]: \varphi^2_s(y_1, x_2) = 0\}.$$  

Moreover, if $t > \sigma(x_1, x_2)$, then this inclusion is strict.

Lemma 3. 1) For all $a_1, a_2 > 0$:

$$\frac{e^{a_1} + 1}{2} \cdot \frac{e^{a_2} + 1}{2} < \frac{e^{a_1+a_2} + 1}{2}. \quad (47)$$

2) For all $\{a_n, n \geq 1\} \subset (0, \infty)$, $\sum_{n\geq1} a_n < \infty$:

$$\prod_{n\geq1} \frac{e^{a_n} + 1}{2} < \exp\{\sum_{n\geq1} a_n\} + 1. \quad (48)$$

The first statement is trivial. Inequality (48) follows from (47) by passing to a limit. It is easy to see that we obtain the strict inequality in a limit.

Corollary 2 and Lemma 3 yield the following statement.

Corollary 3. Let $x_1 < y_1$ and $t > 0$. Then

$$f_t(x_1, x_2) \leq f_t(y_1, x_2),$$

where $f_t$ is the right-hand side of (43).

If $t > \sigma(x_1, x_2)$, then

$$f_t(x_1, x_2) < f_t(y_1, x_2). \quad (49)$$

Therefore the first part of Theorem 5 is proved. Let us verify the second part.

Let $x^0 = (x_1^0, x_2^0) \in \mathbb{R} \times (0, \infty)$ and $t > 0$ be arbitrary. At first let us prove that for a.a. $\omega \in \{\sigma(x^0) < t\}$ and for any $\varepsilon > 0$ there exists a point $\bar{x}_1 \in (x_1^0 - \varepsilon, x_1^0)$ and an interval $\bar{t} \in (0; t)$ such that $\varphi(\bar{x}_1, x_2^0)$ touches the abscissa axis at $\bar{t}$, i.e.

$$\varphi^2(\bar{x}_1, x_2^0) = 0, \quad \varphi^2(x_1, x_2^0) = 0, \quad \text{for } x_1 > \bar{x}_1. \quad (50)$$

By Girsanov’s theorem, the distribution of the process $\varphi^2_t(x), t \in [0, T]$, is absolutely continuous with respect to the distribution of reflected Wiener process that started from $x_2$. Hence for a.a. $\omega$ the set $\{t \in [0, T]: \varphi^2_t(x^0) = 0\}$ is a compact set of zero Lebesgue measure, and it does not have inner and isolated points. By Corollary 2, Lemma 1, and the absence of isolated points in the set $\{t \in [0, T]: \varphi^2_t(x^0) = 0\}$, it follows that for a.a. $\omega \in \{\sigma(x^0) < t\}$ and any $x = (x_1, x_2^0) \subset (0, \infty)$, there are non-empty intervals $(\sigma_k(x^0), \tau_k(x^0)), (\sigma_j(x), \tau_j(x))$ (see (32) for the definition of $(\sigma_j(x), \tau_j(x))$) such that

$$\sigma_k(x_0) < \sigma_j(x) < \tau_j(x) < \sigma_l(x) < \tau_l(x) < \tau_k(x^0) < t. \quad (51)$$

We will show the existence of $\bar{x}_1$ and $\bar{t}$ from (50) such that $\bar{x}_1 \in (x_1^0, x_1^0)$ and $\bar{t} \in [\tau_j(x), \sigma_l(x)]$.

Put

$$K_\alpha = \{s \in [0, t]: \varphi^2_s(\alpha, x_2^0) = 0\} \cap [\tau_j(x), \sigma_l(x)].$$

Note that

1) $K_{x_1} \subset K_{x_2}, \alpha_1 \geq \alpha_2$ (see Corollary 2);
2) $K_{x_1} = \emptyset$;
3) $K_{x_1} \neq \emptyset$. 


Denote $\tau_1 = \sup\{\mu : K_\mu \neq \emptyset\}$. Since the intersection of centered compact sets is non-empty [10], we have $\cap_{\mu<\tau_1} K_\mu \neq \emptyset$.

Suppose $\tilde{t} \in \cap_{\mu<\tau_1} K_\mu$. Then $\varphi_\tilde{t}^2(\mu, x_0^1) = 0, 0 < \tau_1$, and $\varphi_\tilde{t}^2(\mu, x_0^2) > \varphi_\tilde{t}(\mu, x_0^1)$. From the continuity of the flow $\varphi$ in a spatial argument it follows that $\varphi_\tilde{t}^2(\tau_1, x_0^2) = 0$.

Denote $\tau = (\tau_1, x_0^1), \tau = (\tau_1 + \varepsilon, x_0^0)$. Let $[\tau, \tau'] \subset \tau_1, \tau_1$, $\varphi_\tilde{t}^2(\tau, \tau')$ be a segment from from the collection $\left\{\{\sigma_m(\tau), \tau_m(\tau')\}\right\}$ such that $\bar{\tau} > 1$. Therefore, by Lemma 3

$$
\begin{align*}
\frac{f_1(\tau) - f_1(\tau')}{\varepsilon} & \geq \frac{e^{2(\tau - \tau') + 1}}{2} - \frac{1}{2} \cdot \frac{e^{2(\tau - \tau') + 1}}{2} = \\
& = \frac{(e^{2(\tau - \tau') - 1}(e^{2(\tau - \tau') - 1}) - 1)}{4} \geq \frac{e^{2(\tau - \tau') - 1}(e^{2(\tau - \tau') - 1}) - 1}{4} > 0.
\end{align*}
$$

Thus

$$f(\tau_1 + x_0^0) - f(\tau_1, x_0^2) > 0.$$

Theorem 5 is proved.

Proof of Theorem 6. Let us remember that for any $t$ and $x$ we have the equality

$$f_t(x) = \frac{\partial \varphi_t^1(x)}{\partial x_1} \tag{52}$$

for a.a. $\omega \in \{\sigma(x) \leq t\}$. A derivative cannot has a jump discontinuity, but a function $f_t(\cdot, x_2)$ has jump discontinuities in any neighborhood of $x$ and a.a. $\omega \in \{\sigma(x) \leq t\}$ because of Theorem 5. So, if equality (52) be satisfied simultaneously for all $x$ (independently of $\omega$), then this contradiction would immediately imply the proof of the Theorem. Generally, a set of appropriate $\omega$ depends on $x$. So, to be accurate, we need several additional arguments.

Fubini’s theorem yields that for any $x_0 = (x_0^0, x_0^2), \varepsilon > 0$, and for a.a. $\omega \in \{\sigma(x) < t\}$ there are $x_2, |x_2 - x_0^2| < \varepsilon$, and $\delta > 0$ such that $\sigma(x) < t, f_t(x_1, x_2) = \frac{\partial \varphi_t^1(x_1, x_2)}{\partial x_1}$ for a.a. $x_2 \in [x_0^2 - \delta; x_1^0 + \delta]$ with respect to the Lebesgue measure, and $f_t(\cdot, x_2)$ is discontinuous on $(x_0^2 - \delta, x_1^0 + \delta)$. The function $f_t(\cdot, x_2)$ is non-monotone on $[x_0^2 - \delta; x_1^0 + \delta]$. Hence $\varphi_t^1(x_2)$ is concave on $[x_0^2 - \delta; x_1^0 + \delta]$. So, left and right derivatives of $\varphi_t^1(x_2)$ exist and are non-decreasing. Assume that the derivative in $x_1$ exists for all $x_1 \in [x_0^2 - \delta; x_1^0 + \delta]$. Then it must be monotone and discontinuous, because the function $f_t(\cdot, x_2)$ is monotone and discontinuous. This contradiction proves Theorem 6.

Remark. It follows from the proof of Theorem 5 that the absence of the derivative $\nabla \varphi_t(\cdot)$ on a dense subset of $\{x : \sigma(x) < t\}$ is rather a rule than an exception.

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