Application of higher order holonomy corrections to the perturbation theory of cosmology

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Abstract

Applying the higher order holonomy corrections to the perturbation theory of cosmology, the lattice power law of loop quantum cosmology, \( \tilde{\mu} \propto p^\beta \), is analyzed and the range of \( \beta \) is decided to be \([-1,0]\) which is different from the conventional range \(-0.1319 > \beta \geq -5/2\) (Bojowald M and Hossain GM 2008 Phys. Rev. D 77 023508). At the same time, we find that there is an anomaly-free condition in this theory, and we obtain this condition in the vector and tensor mode. We also find that the nonzero mass of a gravitational wave essentially results from the quantum nature of the Riemannian geometry of loop quantum gravity.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The spacetime metric of big bang cosmology is a homogeneous and isotropic Friedmann–Robertson–Walker (FRW) metric. However, this model is just an approximation of ‘zero-order’ universe [2]. If we only focus on the FRW metric, we will ignore many interesting things in the universe such as galaxy clusters, galaxies, stars, etc. So it is necessary to introduce the inhomogeneous and anisotropy perturbation to describe these things [3]. On the other hand, the effects of quantum gravity should be significant in the very early Universe. Therefore, it is interesting to study possible quantum gravity effects in cosmological perturbation theory.

At present, the problem of finding the quantum theory of the gravitational field is still open. One of the most active current approaches is loop quantum gravity (LQG). LQG [4–6] is a mathematically well-defined, non-perturbative and background-independent quantization of general relativity. Its cosmological version, the loop quantum cosmology (LQC) [7], has achieved many successes. A major success of LQC is the resolution of the big bang singularity [8, 9, 12]; this result depends crucially on the discreteness of spacetime geometry. With such a result, the big bang singularity will be avoided through a big-bounce mechanism.
in the high-energy region. In addition, LQC can also set up suitable initial conditions for successful inflation \[13, 14\] as well as possibly leaving an imprint in the cosmic microwave background \[14\].

In LQG, spacetime is quantized. The geometric operators, such as the area operator and the volume operator, have discrete eigenvalues. So there is a smallest area gap $\Delta_1$ \[15, 16\]. In LQC, the coordinate size of a loop is $\tilde{\mu}$. $\tilde{\mu}$ is the function of $p = a^2$ (where $a$ is the scale factor of the universe), i.e. $\tilde{\mu} = \tilde{\mu}(p)$. In the early literature \[9, 10\], the work is always based on the simplest choice of $\tilde{\mu}(p) = \mu_0 = \text{const}$. However, this form can lead to some unusual features. As pointed out by Ashtekar in \[12\] the choice of $\tilde{\mu}(p) = \text{const}$ can lead to the big bounce and occurs at classical matter density like water, so he suggests to select the function as $\tilde{\mu}(p) \propto p^{-1/2}$. From this time forth, in most current works \[11\], $\tilde{\mu}(p) \propto p^{-1/2}$ has been applied. And it was shown that the choice $\tilde{\mu}(p) \propto p^{-1/2}$ is physically and mathematically consistent \[12\]. Up to now, however, there is still no theory to decide the function of $\tilde{\mu}(p)$. As research continues, there may be some other form of $\tilde{\mu}(p)$ which can give better physics. Therefore, finding out if the form of this function has theoretical significance.

An ansatz for the form of this function can be taken as $\tilde{\mu}(p) \propto p^{\beta}$. In \[1\], the range of $\beta$ has been decided as $-0.1319 > \beta \geq -5/2$. However, it is just the conclusion of the first-order holonomy corrections. If we want a more accurate determination of the range of $\beta$, we must consider the higher order corrections.

Even in the case of homogeneous and isotropic models, the quantum equation of state is very difficult to analyze. Fortunately, there is a powerful tool, i.e. effective theory, which allows us to include loop quantum effects by correction terms in equations of the classical type \[17\]. There are two types of quantum corrections that are expected from the Hamiltonian of LQG. One correction arises for inverse powers of the densitized triad, which when quantized becomes an operator with zero in the discrete part of its spectrum thus lacking a direct inverse. The other comes from the fact that a loop quantization is based on holonomies, i.e. exponentials of the connection rather than direct connection components \[18\].

In LQC, there is no well-defined quantum operator corresponding to $c = \gamma k$. So we should find a well-defined operator to replace it. The conventional way is replacing the $c$ by $\sin \tilde{\mu}c/\tilde{\mu}$.

The application of inverse triad corrections and conventional holonomy corrections to the scalar mode of perturbation can be viewed in \[19\], the vector mode in \[20\] and the tensor mode in \[1\].

In this paper, we focus on the higher holonomy corrections rather than the conventional correction. We apply these higher corrections to the vector and the tensor mode and see whether the mass of a gravitational wave results from the nature of discrete geometry. We will also analyze the range of $\beta$ with high-order holonomy corrections.

This paper is organized as follows. At first, the perturbed variables are introduced in section 2. Then in section 3, we apply the high-order holonomy corrections to obtain the effective Hamiltonian constraint. Detailed analysis of the range of $\beta$ is given in section 4. The discussion is given in section 5.

2. Background and perturbed constraint

In Ashtekar’s formalism of general relativity \[21, 22\], the spatial metric as a canonical field is replaced by the densitized triad $E^a_i$, defined as

$$E^a_i := \left| \det \left( e^i_k \right) \right| e^a_i,$$

and the spin connection $\Gamma^i_a$ which is

$$\Gamma^i_a = -\epsilon^{ijk} e^b_j \left( \partial_i e^b_k + \frac{1}{2} e^c_k e^l_a \partial_l e^b_i \right).$$

2
The canonical variables are the densitized triad $E^a_i$ and the Ashtekar connection $A^a_i = \Gamma^a_i + \gamma K^a_i$, where $K^a_i$ is the extrinsic curvature and $\gamma$ is the Barbero–Immirzi parameter.

The canonical variables reduced to spatially homogeneous and isotropic cosmology are

$$E^a_i = \bar{p}\delta a_i, \quad K^a_i = \bar{k}\delta a_i, \quad \Gamma^a_i = 0.$$  \(3\)

They are background variables, and the perturbation will be added based on these variables.

In the perturbation theory, we denote the background variables by a bar:

$$\bar{E}^a_i = \bar{p}\delta a_i, \quad \bar{K}^a_i = \bar{k}\delta a_i, \quad \bar{\Gamma}^a_i = \bar{0}.$$  \(4\)

They are background variables, and the perturbation will be added based on these variables.

As described in \([1, 19, 20]\), the symplectic structure splits into two parts: one for the background variables and the other for perturbations, i.e.

$$\{\bar{k}, \bar{p}\} = \frac{8\pi G}{3V_0}.$$  \(6\)

and

$$\{\delta K^a_i (x), \delta E^a_j (y)\} = \frac{8\pi G}{3\sqrt{\bar{p}}} \delta(x,y) \delta a_i \delta j.$$  \(7\)

Here, $G$ is the gravitational constant and $V_0$ is a fiducial volume.

In a vector mode, the gravity part of the perturbed Hamiltonian constraint (up to quadratic terms) is \([20]\)

$$H_G[N] = \frac{1}{16\pi G} \int d^3 x \bar{N} \left[ \bar{k}^2 \left( -6\sqrt{\bar{p}} - \frac{\delta E^a_i \delta E^d_j \delta k^i \delta^d_j}{2\bar{p}^{3/2}} \right) \right.$$  
$$+ \frac{\sqrt{\bar{p}}}{2} (\delta K^a_i \delta K^d_j \delta a^i \delta^d_j) - \frac{2\bar{k}}{\sqrt{\bar{p}}} (\delta E^a_j \delta K^d_j) \left. \right].$$  \(8\)

On the other hand, when we introduce the inhomogeneous perturbation, the diffeomorphism constraint does not vanish any more. So the gravitational part of the diffeomorphism constraint is changed into \([20]\)

$$D_G[N^a] = \frac{1}{8\pi G} \int d^3 x \delta N^a \left[ -\bar{p} (\partial_a \delta K^a_i) - \bar{k} \delta a^i (\partial_d \delta E^d_i) \right].$$  \(9\)

Using equations (6) and (7), we can testify the following relation easily:

$$\{H_G, D_G\} = 0.$$  \(10\)

Similarly, in a tensor mode, the gravity part of the perturbed Hamiltonian constraint (up to quadratic terms) is \([1]\)

$$H_G[N^a] = \frac{1}{16\pi G} \int d^3 x \bar{N} \left[ \bar{k}^2 \left( -6\sqrt{\bar{p}} - \frac{\delta E^a_i \delta E^d_j \delta k^i \delta^d_j}{2\bar{p}^{3/2}} \right) \right.$$  
$$+ \frac{\sqrt{\bar{p}}}{2} (\delta K^a_i \delta K^d_j \delta a^i \delta^d_j) - \frac{2\bar{k}}{\sqrt{\bar{p}}} (\delta E^a_j \delta K^d_j) \left. \right].$$  \(11\)

where $\delta E^a_i = -\frac{1}{4}\bar{p} h^a_i$; here $h^a_i := \delta^{ij} h_{ij}$, and $h_{ab}$ is the symmetric metric perturbation field. It is transverse and traceless, i.e. it satisfies $\partial^a h_{ab} = 0$ and $\delta^{ab} h_{ab} = 0$. \([3]\).
3. Vector and tensor mode with higher order holonomy corrections

3.1. Higher order holonomy corrections

Instead of the conventional way of introducing the holonomy corrections, in this paper, we focus on the higher order holonomy corrections [23].

At first, let us consider the Taylor series

\[ \sin^{-1} x = \sum_{l=0}^{\infty} \frac{(2l)!}{2^l (l!)^2 (2l + 1)} x^{2l+1} \]

for \(-1 \leq x \leq 1\), and setting \( x = \sin(\tilde{\mu} \gamma \hat{k}) \), we have

\[ \gamma \hat{k} = \frac{1}{\tilde{\mu}} \sum_{l=0}^{\infty} \frac{(2l)!}{2^l (l!)^2 (2l + 1)} \sin(\sin(\tilde{\mu} \gamma \hat{k}))^{2l+1}, \]

(13)

This inspires us to define an \( n \)th-order holonomized connection variable as

\[ c_{\text{h}}^{(n)} := \frac{1}{\tilde{\mu}} \sum_{l=0}^{n} \frac{(2l)!}{2^l (l!)^2 (2l + 1)} \sin(\sin(\tilde{\mu} \gamma \hat{k}))^{2l+1}, \]

(14)

which can be made arbitrarily close to \( \gamma \hat{k} \) as \( n \to \infty \). We can see that \( c_{\text{h}}^{(n)} \) is a function of the holonomy \( \sin(\tilde{\mu} \gamma \hat{k}) \) and the discreteness variable \( \tilde{\mu} \). Therefore, we can replace \( \gamma \hat{k} \) by \( c_{\text{h}}^{(n)} \) to implement the underlying structure of LQC. When \( n = 0 \), \( c_{\text{h}}^{(0)} = \sin(\sin(\tilde{\mu} \gamma \hat{k})/\tilde{\mu} \) is the same with the conventional holonomy corrections.

There is an ambiguity in this replacement. If we set \( x = \sin(m \tilde{\mu} \gamma \hat{k}) \) in equation (12), where \( m \) is an arbitrary constant, equation (13) changes to

\[ m \gamma \hat{k} = \frac{1}{m \tilde{\mu}} \sum_{l=0}^{\infty} \frac{(2l)!}{2^l (l!)^2 (2l + 1)} \sin(\sin(m \tilde{\mu} \gamma \hat{k}))^{2l+1}, \]

(15)

and we have

\[ \gamma \hat{k} = \frac{1}{m \tilde{\mu}} \sum_{l=0}^{\infty} \frac{(2l)!}{2^l (l!)^2 (2l + 1)} \sin(\sin(m \tilde{\mu} \gamma \hat{k}))^{2l+1}. \]

(16)

So we can define a more general \( n \)th-order holonomized connection variable \( c_{\text{h}}^{(n)} \),

\[ c_{\text{h}}^{(n)} := \frac{1}{m \tilde{\mu}} \sum_{l=0}^{n} \frac{(2l)!}{2^l (l!)^2 (2l + 1)} \sin(\sin(m \tilde{\mu} \gamma \hat{k}))^{2l+1}, \]

(17)

where \( m \) is an ambiguity parameter.

The Poisson brackets between the canonical variables and the \( c_{\text{h}}^{(n)} \) are

\[ \left\{ \bar{p}, c_{\text{h}}^{(n)} \gamma \right\} = -\frac{8\pi G}{3V_0} \cos(\tilde{\mu} \gamma \hat{k}) \mathcal{G}_n(\tilde{\mu} \gamma \hat{k}), \]

(18)

and

\[ \left\{ \bar{k}, c_{\text{h}}^{(n)} \gamma \right\} = \frac{8\pi G}{3V_0} \frac{\partial \tilde{\mu}}{\partial \bar{p}} \left[ \cos(\tilde{\mu} \gamma \hat{k}) \mathcal{G}_n(\tilde{\mu} \gamma \hat{k}) \bar{k} - \frac{c_{\text{h}}^{(n)}}{\gamma} \right], \]

(19)

where

\[ \mathcal{G}_n(\tilde{\mu} \gamma \hat{k}) = \sum_{l=0}^{n} \frac{(2l)!}{2^l (l!)^2} \sin(\sin(\tilde{\mu} \gamma \hat{k}))^{2l+1}, \]

(20)

From [23] we can see that the role of higher order holonomy corrections is like a filter, which excludes the impact of human factors on the theory and leaves a pure quantum effect.
3.2. Vector mode

In classical perturbation theory of cosmology, the gauge-invariant variables of the vector mode will decay quickly. Therefore, there is a small role for the vector mode perturbation for a universe [3]. However, once we introduce the quantum correction, we must consider whether the perturbation theory is anomaly free [20]. The requirement of an anomaly-free condition can reduce some ambiguities of LQC. Inserting the higher holonomy corrections in equation (8), we can obtain the effective gravity part of the perturbed Hamiltonian constraint

$$H^0_{G}[\mathcal{N}] = \frac{1}{16\pi G} \int_{\Sigma} d^3x \sqrt{\mathcal{N}} \left\{ \left( \frac{c^m_{(n)}}{\gamma} \right)^2 \left[ -6\sqrt{\bar{p}} - \frac{1}{2\bar{p}^{3/2}} (\delta E^c_{j}\delta E^d_{k}\delta^i_{\bar{k}}) \right] \right. $$

$$+ \sqrt{\bar{p}} \left( \delta K^i_{j}\delta K^k_{d}\delta^d_{\bar{k}} \right) - \frac{2}{\sqrt{\bar{p}}} \frac{c^m_{(n)}}{\gamma} \delta E^c_{j} \delta K^i_{j} \right\} .$$

(21)

Generally speaking, we should replace all the \( \gamma \bar{k} \) by \( c^{m}_{(n)} \). But in order to obtain a homogeneous limit which is in agreement with what has been used in isotropic models, we set the parameter \( m \) in the first term to be equal to 1 [1]. The parameter \( m \) in the last term should lead to an anomaly-free constraint algebra, so we do not fix it at first. In the following discussion, we will determine the correct value of \( m \) in the last term by requiring an anomaly-free constraint algebra in the presence of quantum corrections.

In a homogeneous and isotropic model, there is no diffeomorphism constraint. So the algebra of constraints is closed. When we consider the inhomogeneous perturbation, the diffeomorphism constraint will turn up. From equation (10) we can see that, in classical theory, the algebra of constraints is still closed. So when we write down the constraints with the quantum corrections, we need to ensure that the constraints are still closed. In other words, the anomaly terms, which cannot be expressed by the linear combination of the Hamiltonian constraint and the diffeomorphism constraint, should be vanished. On the other hand, the diffeomorphism constraint does not receive quantum corrections in the full theory [24], so equation (9) does not change.

The Poisson bracket between the two constraints is

$$\{H^0_G, D_G\} = \frac{\sqrt{\bar{p}}}{8\pi G} \left[ \bar{k} + \frac{c^m_{(n)}}{\gamma} - \frac{c^h_{(n)}}{\gamma} \cos(\tilde{\mu}\gamma \bar{k}) \mathcal{L}_c(\tilde{\mu}\gamma \bar{k}^c) \right] D_G$$

$$+ \frac{1}{8\pi G} \int_{\Sigma} d^3x \bar{p}(\bar{\alpha}\delta N^j) A^{(n)c}_j, \quad (22)$$

where the anomaly part is

$$A^{(n)c}_j = \sqrt{\bar{p}} \left[ \bar{p} \frac{\partial}{\partial \bar{p}} \left( \frac{c^m_{(n)}}{\gamma} \right)^2 + \left( \frac{c^h_{(n)}}{\gamma} \right)^2 - \bar{k}^2 \right. $$

$$+ 2 \left[ \frac{c^m_{(n)}}{\gamma} \cos(\tilde{\mu}\gamma \bar{k}) \mathcal{L}_c(\tilde{\mu}\gamma \bar{k}) - \frac{c^m_{(n)}}{\gamma} \right] \bar{k} \left( \delta E^c_{j} \right) \right\} .$$

(23)

To obtain this, we need the Poisson bracket

$$\{\delta K^i_{j}(x), \bar{\alpha}_d \delta E^d_{k}(y)\} = 8\pi G \delta^i_d \delta^d_{k} \bar{\alpha}_d \delta(x, y), \quad (24)$$

$$\{\delta E^i_{j}(x), \bar{\alpha}_d \delta K^d_{k}(y)\} = -8\pi G \delta^i_d \delta^d_{k} \bar{\alpha}_d \delta(x, y). \quad (25)$$
To cancel the anomaly part, it must be requested that $A_{ij}^{(n)c} = 0$, i.e.

$$
n_{mc} = (\beta + 1) c_{nh} \cos(\tilde{\mu} k \bar{k}) \mathcal{S}_n (\tilde{\mu} \gamma \bar{k}) + 1 - \frac{2\beta}{2} \left( \frac{c_{nh}}{\gamma k} \right)^2 - \frac{1}{2}, \tag{26}
$$

From [23] we know that the big bounce occurs when $\tilde{\mu} c = \frac{\pi}{2}$, so the maximum of $c(n)mh\tilde{\mu}$ is $\frac{\pi}{2}$. According to equation (26), we have

$$
(\beta + 1) c_{nh} \cos(\tilde{\mu} k \bar{k}) \mathcal{S}_n (\tilde{\mu} \gamma \bar{k}) + \left[ 1 - \frac{2\beta}{2} \left( \frac{c_{nh}}{\gamma k} \right)^2 - \frac{1}{2} \right] \tilde{\mu} k \bar{k} \leq \frac{\pi}{2}. \tag{27}
$$

Equation (27) can be seen as a limit to the evolution of $c_{mh}^{(n)}$, and we can restrict the range of $\beta$ through this limit.

### 3.3. Tensor mode

In classical perturbation theory of cosmology, there is only one equation in the tensor mode, i.e. the gravitational wave equation. From this equation, we know that the gravitational waves are massless. However, when quantum corrections are taken into account, a mass term will appear in this equation [1]. It is the only conclusion calculated in the first-order correction. We extend this method to the higher holonomy corrections, and take the limit of $n \to \infty$. In this way, we can find that the mass of gravitational waves essentially results from the quantum nature of the Riemannian geometry of LQG.

Inserting the higher holonomy corrections in equation (11), the effective gravity part of the perturbed Hamiltonian constraint can be expressed as

$$
H_{G}^O[N] = \frac{1}{16\pi G} \int_{\Sigma} d^3x \bar{N} \left\{ \left( c_{nh}^{(n)} \gamma k \right)^2 \left[ -6\sqrt{\bar{\rho}} \frac{1}{2\beta^{3/2}} \left( \delta E_a^j \delta E_b^i \delta_i^j \delta_f^d \right) \right] + \sqrt{\bar{\rho}} \left( \delta K_{ij} \delta K_{ji} \delta_i^j \delta_f^d \right) \right\}.
$$

From this Hamiltonian, one can obtain the time derivative of the background variables

$$
\dot{\bar{p}} = 2\bar{p} \frac{c_{nh}^{(n)}}{\gamma} \cos(\tilde{\mu} k \bar{k}) \mathcal{S}_n (\tilde{\mu} \gamma \bar{k}) \tag{29}
$$

$$
\dot{\bar{k}} = -\frac{1}{2} \left( \frac{c_{nh}^{(n)}}{\gamma} \right)^2 - 2\bar{p} \frac{c_{nh}^{(n)}}{\gamma} \frac{\tilde{\mu} \bar{k} \bar{p}}{\gamma} \left[ \cos(\tilde{\mu} k \bar{k}) \mathcal{S}_n (\tilde{\mu} \gamma \bar{k}) \tilde{k} - \frac{c_{nh}^{(n)}}{\gamma} \right], \tag{30}
$$

and the time derivative of the perturbed variable $\delta E^a_i$:

$$
\delta \dot{E}_i^a = [\delta E_i^a, H_{G}^O] = -\bar{p} \delta_{ij}^a \delta^b_0 \delta K_{ij} - \frac{1}{2} \bar{p} \frac{c_{nh}^{(n)}}{\gamma} \delta h_i^a. \tag{31}
$$

On the other hand, one can also obtain $\delta \dot{E}_i^a$ from $\delta E_i^a = -\frac{1}{2} \bar{p} \delta h_i^a$, i.e.

$$
\delta \dot{E}_i^a = -\frac{1}{2} \left( \bar{p} \delta h_i^a + \bar{p} \delta^a_0 \right)
$$

$$
= -\frac{1}{2} \left( 2\bar{p} \frac{c_{nh}^{(n)}}{\gamma} \cos(\tilde{\mu} k \bar{k}) \mathcal{S}_n (\tilde{\mu} \gamma \bar{k}) \delta h_i^a + \bar{p} \delta^a_0 \right). \tag{32}
$$
From equations (31) and (32), we have
\[ \delta K_a^i = \frac{1}{2} \left[ \dot{h}_a^i + \left( 2 \frac{c_h^{(n)}}{\gamma} \cos(\bar{\mu}_g k \bar{B}_a) (\bar{\mu}_g k \bar{B}_a) - \frac{c_{mh}^{(n)}}{\gamma} \right) \dot{h}_a^i \right]. \] (33)

So, the \( \dot{K}_a^i \) will be
\[ \delta \dot{K}_a^i = \frac{1}{2} \left[ \dot{h}_a^i + h_a^i \partial_t \left( 2 \frac{c_h^{(n)}}{\gamma} \cos(\bar{\mu}_g k \bar{B}_a) (\bar{\mu}_g k \bar{B}_a) - \frac{c_{mh}^{(n)}}{\gamma} \right) + \left( 2 \frac{c_h^{(n)}}{\gamma} \cos(\bar{\mu}_g k \bar{B}_a) (\bar{\mu}_g k \bar{B}_a) - \frac{c_{mh}^{(n)}}{\gamma} \right) \dot{h}_a^i \right]. \] (34)

Again, we can also obtain \( \delta \dot{K}_a^i \) from the Hamiltonian equation
\[ \delta \dot{K}_a^i = \left\{ \delta K_a^i, H_{matter} \right\} + \left\{ \delta K_a^i, H_{matter} \right\} \]
\[ = \left\{ \delta K_a^i, H_{matter} \right\} + \frac{1}{4} \left( \frac{c_h^{(n)}}{\gamma} \right)^2 h_a^i - \frac{1}{2} \frac{c_{mh}^{(n)}}{\gamma} h_a^i + \frac{1}{2} \nabla^2 h_a^i. \] (35)

From equations (34) and (35), one can obtain the gravitational wave equation
\[ \frac{1}{2} \left[ \dot{h}_a^i + 2 \frac{c_h^{(n)}}{\gamma} \cos(\bar{\mu}_g k \bar{B}_a) (\bar{\mu}_g k \bar{B}_a) h_a^i - \frac{1}{2} \nabla^2 h_a^i + T^{(n)}_Q h_a^i \right] = 8\pi G \Pi_{Q, a}, \] (36)
where \( \Pi_{Q, a} \) is the source term from the matter Hamiltonian and
\[ T^{(n)}_Q = -2 \frac{\partial \bar{\mu}_g}{\partial \bar{\mu}_g} \left\{ \frac{c_h^{(n)}}{\gamma} \left[ \cos(\bar{\mu}_g k \bar{B}_a) (\bar{\mu}_g k \bar{B}_a) - \cos(m \bar{\mu}_g k \bar{B}_a) (m \bar{\mu}_g k \bar{B}_a) \right] - \frac{c_{mh}^{(n)}}{\gamma} \left[ \cos(\bar{\mu}_g k \bar{B}_a) (\bar{\mu}_g k \bar{B}_a) - \cos(m \bar{\mu}_g k \bar{B}_a) (m \bar{\mu}_g k \bar{B}_a) \right] \right\} \]
\[ + \frac{1}{2} \left( \frac{c_h^{(n)}}{\gamma} \right)^2 \left[ 2 \frac{c_h^{(n)}}{\gamma} \cos(\bar{\mu}_g k \bar{B}_a) (\bar{\mu}_g k \bar{B}_a) - \cos(m \bar{\mu}_g k \bar{B}_a) (m \bar{\mu}_g k \bar{B}_a) \right] \]
\[ - 1 + \cos(m \bar{\mu}_g k \bar{B}_a) (m \bar{\mu}_g k \bar{B}_a) \left( \frac{c_h^{(n)}}{\gamma} \cos(\bar{\mu}_g k \bar{B}_a) - \frac{c_{mh}^{(n)}}{\gamma} \right)^2, \] (37)
where
\[ \mathcal{B}_n(\bar{\mu}_g k \bar{B}_a) = \sum_{l=0}^{n} \frac{2l(2l)!}{2^l(l!)} \sin^l(\bar{\mu}_g k \bar{B}_a). \] (38)

When \( n = 0, \mathcal{B}_n(\bar{\mu}_g k \bar{B}_a) = 0 \). So it will not appear in a conventional way. The definition of the effective mass is
\[ m^2 = \frac{T_Q}{a^2}. \] (39)

When \( \bar{\mu}_g k \rightarrow \frac{2}{\gamma}, \) and \( n \rightarrow \infty \), the mass term will never vanish. From this, we can confirm that the nonzero mass of the gravitational wave results from the quantum nature of the Riemannian geometry of LQG.
From the definition of the gravitational wave [1], we should require that $T_Q^{(n)} \geq 0$, and this is another condition to restrict the range of $\beta$.

4. Lattice refinements

In the process of obtaining the range of $\beta$ in [1], the authors expand the ‘sin’ (and ‘cos’) of equation (26) ($n = 0$) and take only the first few terms of it. So in his paper, the first non-zero term of the anomaly part is $k^4$. By this way, one can obtain the relationship between $m$ and $\beta$ from equation (26) ($n = 0$), and the range of $\beta$ by requiring that $T_Q^{(n)} > 0$ from equation (37) ($n = 0$).

However, there are some problems in this method. First of all, the relation $m^2 = 5 + 2\beta$ in [1] is obtained by requiring that the term $k^4$ is vanished. But it cannot ensure that the whole anomaly part can be canceled, because the terms $k^6, k^8$, etc. still appear. Secondly, when we consider the higher corrections, some high-order terms like $\sin^3(x)$ will appear. If we keep more terms of ‘sin’ (and ‘cos’), we will find that the first non-zero term is not $k^4$, maybe, it will be $k^5$; it depends on how many terms you kept. It decides the different relation between $m$ and $\beta$. So if we want to obtain the more accurate range of $\beta$, we should not expand ‘sin’ (and ‘cos’) in the equations; in other words, we keep all its terms.

From the discussions above we can know that equation (27) and $T_Q^{(n)} \geq 0$ are two restrictions to $\beta$, so we analyze these two restrictions respectively. At first, we note that the products $\tilde{\mu}\gamma k$ always appear together in the expression of $c(n)m\tilde{\mu}$, so we set $x = \tilde{\mu}\gamma k$, and then $c(n)m\tilde{\mu}$ is the function of $x$. When the big bounce occurs at $x = \frac{\pi}{2}$, and $x \to 0$ with the expansion of the Universe, we can draw the graphs of this function between 0 and $x = \frac{\pi}{2}$ with different values of $n$.

From figure 1 we can see that, when $n = 0$ (it corresponds with the conventional holonomy corrections), the case of $\beta = -\frac{5}{2}$ (which was lower bound in [1]) can fulfill the condition $c(n)m\tilde{\mu} \leq \frac{\pi}{2}$. But it can be even lower than that because $\beta = -2.7$ can also fulfill the condition. However, we will be dealing with the higher corrections, so let us analyze the case of large $n$.

From figure 1, we can see that the larger $n$ leads to the bigger lower bound of $\beta$. Because equation (27) should be kept everywhere, it includes the point of the big bounce, i.e. $x = \frac{\pi}{2}$. Inserting $x = \frac{\pi}{2}$ into (27), it will become

$$\left( \frac{1 - 2\beta}{2} - \frac{1}{2} \right) \frac{\pi}{2} \leq \frac{\pi}{2},$$

which leads to

$$\beta \geq -1.$$  \hspace{1cm} (41)

On the other hand, equation (26) is the relationship for $m$ and $\beta$. We insert the $c(n)m\tilde{\mu}$ into equation (37). On the right-hand side of equation (37), there are still some terms which contain $m$, so we can use the following relation to replace these terms:

$$\cos(m\tilde{\mu}\gamma k)\Sigma_m(m\tilde{\mu}\gamma k) = \tilde{\mu}\gamma k \frac{\partial}{\partial(\tilde{\mu}\gamma k)} c^{(n)} + c^{(n)}.$$  \hspace{1cm} (42)

From equations (26), (37) and (42) we can see that $c^{(n)}$ is also the function of $x = \tilde{\mu}\gamma k$, and there are two parameters $n$ and $\beta$ in it.

From the evolution of $T_Q^{(n)}$ with $n = 0$ (see figure 1(a)), we can see that, if we require that $T_Q^{(0)} > 0$, the $\beta$ should be smaller than $-\frac{1}{2}$. With equation (41), the range of $\beta$ is

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Figure 1. The evolution of $y_1 = \frac{\theta_{n}}{k}$ and $y_2 = \epsilon_{n}^{(n)}\tilde{\mu}$ with $x = \tilde{\mu}y\bar{k}$ is shown in (a), (b), (c) and (d), (e), (f) respectively. The dashed lines in (d), (e) and (f) represent $y_2 = \frac{\pi}{2}$.

\[ -\frac{1}{3} \geq \beta \geq -1. \] This result is smaller than $-0.1319 > \beta \geq -5/2$. It is because we use the 'sin', not the first orders of the expanding term of 'sin'.

When $n > 0$, we find that the upper bound of $\beta$ is larger than $-\frac{1}{3}$, and when $n \to \infty$, the upper bound will be zero. We display $n = 100$ and $n = 1000$ in figure 1 also.

So, the final range of $\beta$ should be $[-1, 0]$. From this we can see that $\beta = 0$ is not eliminated as in [1] and $\beta = -\frac{1}{2}$ is also in this range.

5. Discussion

In this paper, we apply the higher order holonomy corrections to the perturbation theory of cosmology. When we take the limit of $n \to \infty$, the form of the LQC will be back to classical theory, but the effect of the quantum geometry will be kept. From the analyses above, we know that the mass of the gravitational wave will not disappear when $n \to \infty$. It will decrease to zero with the expansion of the Universe. So it is the 'pure' quantum effect that the gravitational wave has nonzero mass.

Other important effects are related to the discrete spacetime geometry. Discrete space means the existence of the area gap, and there is a function $\tilde{\mu}(p)$ related to this area gap. The form of the function $\tilde{\mu}(p)$ has an important impact on LQC. But now the framework of the theory is not perfect to decide this function, so we can only restrict the form of the function as $\tilde{\mu} \propto p^\beta$, from some other aspects such as effective theory and perturbation theory of cosmology.

In the effective LQC framework, we apply two conditions to limit the range of $\beta$. One is anomaly free, which means that the constraint algebra of the vector mode should be closed when we consider the quantum effect. It is the mathematical requirement of the theory. This can restrict $\beta$ to $[-1, +\infty)$. 

\[ -\frac{1}{3} \geq \beta \geq -1. \] This result is smaller than $-0.1319 > \beta \geq -5/2$. It is because we use the 'sin', not the first orders of the expanding term of 'sin'.

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In the effective LQC framework, we apply two conditions to limit the range of $\beta$. One is anomaly free, which means that the constraint algebra of the vector mode should be closed when we consider the quantum effect. It is the mathematical requirement of the theory. This can restrict $\beta$ to $[-1, +\infty)$.
The other condition is the requirement of the positive definite mass of the gravitational waves. This is the physical requirement. We cannot ensure that the mass of the gravitational wave is positive when $\beta > 0$. And from figure 1, we can see that the behavior of mass between $\beta > 0$ and $\beta \leq 0$ is very different. Therefore it can restrict the range of $\beta$ to be $(-\infty, 0]$. This requirement seems to be very natural. However, we do not yet understand the true meaning of the mass of gravitational waves, so this condition is only an assumption. The correctness of this assumption needs to be verified in future studies.

In conclusion, the range of $\beta$ should be $[-1, 0]$. But this range is only decided by the perturbation theory of cosmology. It cannot exclude $\beta = 0$. So the exclusion of $\beta = 0$ is based on the prediction of theory rather than theory itself. This may not be its final scope because only two conditions were discussed in this paper. Certainly there are many other conditions to limit the range of parameter. If we can restrict $\beta$ to a unique value, say $-1/2$, from the theory itself rather than from the predictive power of theory, then the theory will be more self-consistent. So, in future studies, we can compare different conditions on the parameter values to examine the self-consistency of the theory.

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