A Borel–Weil Theorem for the Quantum Grassmannians

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Abstract
We establish a noncommutative generalisation of the Borel–Weil theorem for the Heckenberger–Kolb calculi of the quantum Grassmannians. The result is presented in the framework of noncommutative complex and holomorphic structures and generalises previous work of a number of authors on quantum projective space. As a direct consequence we get a novel holomorphic presentation of the twisted Grassmannian coordinate ring studied in noncommutative algebraic geometry.

1 Introduction

The Borel–Weil theorem [43] is an elegant geometric procedure for constructing all unitary irreducible representations of a compact Lie group. The construction realises each representation as the space of holomorphic sections of a line bundle over a flag manifold. It is a highly influential result in the representation theory of Lie groups and since the discovery of quantum groups has inspired a number of noncommutative generalisations. Important examples include the homological algebra approaches of Polo, Andersen, and Wen [1], and of Parshall and Wang [42], the representation theoretic approach of Mimachi, Noumi, and Yamada [33, 34], the coherent state approach of Biedenharn and Lohe [5], and the equivariant vector bundle approach of Gover and Zhang [16]. Moreover, the question of whether these examples can be understood in terms of a general Borel–Weil theorem for compact quantum groups is an important open problem [48, Problem 1.5].

In the classical setting the holomorphic sections of a line bundle over a flag manifold are the same as its parabolic invariant sections. In the above examples the parabolic invariant description is generalised without introducing any noncommutative notion of holomorphicity. In recent years, however, the study of differential calculi over the quantum flag manifolds $C_q[G_0/L_0]$ has yielded a much better understanding of their noncommutative complex geometry. Subsequent work on the Borel–Weil theorem for quantum groups has used the notion of a complex structure on a differential $*$-calculus to generalise the...
Koszul–Malgrange presentation of holomorphic vector bundles [28]. This direction of research was initiated by Majid in his influential paper on the Podleś sphere [30]. It was continued by Khalkhali, Landi, van Suijlekom, and Moatadelro in [24, 25, 26] where the definitions of complex structure and noncommutative holomorphic vector bundle were introduced and the family of examples extended to include quantum projective space $\mathbb{C}_q[\mathbb{C}P^n]$.

The differential calculi used in the above work are those identified by Heckenberger and Kolb in their remarkable classification of calculi over the irreducible quantum flag manifolds [21]. In this paper we show that for the $A$-series irreducible quantum flag manifolds, which is to say the quantum Grassmannians, the Heckenberger–Kolb calculus $\Omega^1_q(\text{Gr}_{n,r})$ has an associated $q$-deformed Borel–Weil theorem generalising the case of quantum projective space. This is done using the framework of quantum principal bundles together with a realisation of the Heckenberger–Kolb calculus as the restriction of a quotient of the standard bicovariant calculus on $\mathbb{C}_q[\text{SU}_n]$. This means that the paper is closer in form to [30, 24] rather than [25, 26], where Dąbrowski and D’Andrea’s spectral triple presentation of the quantum projective space calculus was used.

In addition to the theory of quantum principal bundles, the paper uses two novel mathematical objects. The first is a principal $\mathbb{C}[U_1]$-bundle over the quantum Grassmannians which generalises the well-known presentation of the odd-dimensional quantum spheres as the total space of a $\mathbb{C}[U_1]$-bundle over quantum projective space. Just as for quantum projective space, this presents a workable description of the line bundles over $\mathbb{C}_q[\text{Gr}_{n,r}]$. The second is a sequence of twisted derivation algebras which serve as a crucial tool for demonstrating non-holomorphicity of line bundle sections in the proof of the Borel–Weil theorem.

One of our main motivations for extending the Borel–Weil theorem to the quantum Grassmannians is to further explore the connections between quantum flag manifolds $\mathbb{C}_q[G_0/L_0]$ and their twisted homogeneous coordinate ring counterparts $\mathcal{H}_q(G_0/L_0)$ in noncommutative algebraic geometry [44, 3, 10]. These rings are deformations of the homogeneous coordinate rings of the classical flag manifolds and are important examples in the theory of quantum cluster algebras [18, 17]. As was shown in [13], every quantum flag manifold can be constructed as a coinvariant subalgebra of $\mathcal{H}_q(G_0/L_0) \otimes \mathcal{H}_q(G_0/L_0)^\ast$, where $\mathcal{H}_q(G_0/L_0)^\ast$ denotes the dual comodule of $\mathcal{H}_q(G_0/L_0)$. An important consequence of the holomorphic approach to the Borel–Weil theorem for $\mathbb{C}_q[\text{Gr}_{n,r}]$ is that one can mimic the classical ample bundle presentation of $\mathcal{H}((\text{Gr}_{n,r})$ to go in the other direction, that is, to construct the twisted homogeneous coordinate ring $\mathcal{H}_q(\text{Gr}_{n,r})$ from the quantum coordinate algebra $\mathbb{C}_q[\text{Gr}_{n,r}]$. This directly generalises the work of [24, 25, 26] for quantum projective space, which is an essential ingredient in the construction of the category of coherent sheaves over $\mathbb{C}_q[\mathbb{C}P^n]$ [40].

The paper naturally leads to a number of future projects. First is the Borel–Weil theorem for the $C$-series irreducible quantum flag manifolds and the full quantum flag manifold of $\mathbb{C}_q[\text{SU}_3]$ [39]. Second is the formulation of a quantum Borel–Weil–Bott theorem for $\mathbb{C}_q[\text{Gr}_{n,r}]$. Third, the standard circle bundle introduced in §3 allows for a natural
definition of weighted quantum Grassmannians generalising the definition of weighted quantum projective space [7]. This and the connections with Cuntz–Pimsner algebras [2] will be discussed in [41]. Finally, this paper is a first step towards the longer term goals of understanding the noncommutative Kähler geometry of $C_q[Gr_{n,r}]$ [37], constructing spectral triples for $C_q[Gr_{n,r}]$ [12], and defining the category of coherent sheaves [40] of $C_q[Gr_{n,r}]$.

The paper is organised as follows: In Section 2, we recall some well-known material about cosemisimple and coquasi-triangular Hopf algebras, Hopf–Galois extensions, principal comodule algebras, and quantum homogeneous spaces. We also recall the basics of the theory of differential calculi and quantum principal bundles, as well as the more recent notions of noncommutative complex and holomorphic structures.

In Section 3, we recall the definition of the quantum Grassmannians $C_q[Gr_{n,r}]$ and introduce $C_q[S^{n,r}]$ the total space of the $C[U_1]$-circle bundle discussed above. We show that $C_q[S^{n,r}]$ is equal to the direct sum of the line bundles over $C_q[Gr_{n,r}]$, deduce a set of generators for $C_q[S^{n,r}]$, and give the defining coaction of the $C[U_1]$-bundle. Moreover, we construct an explicit strong principal connection for the bundle and discuss the special case of quantum projective space.

In Sections 4 and 5, we use the quantum Killing form, and its associated bicovariant calculus, to construct a calculus $\Omega^1_q(SU_n, r)$ which restricts to the Heckenberger–Kolb calculus on $C_q[Gr_{n,r}]$. Moreover, $\Omega^1_q(SU_n, r)$ is shown to induce a quantum principal bundle structure on the Hopf–Galois extensions $C_q[Gr_{n,r}] \hookrightarrow C_q[S^{n,r}]$ and $C_q[Gr_{n,r}] \hookrightarrow C_q[SU_n]$. The universal principal connection introduced in Section 3 is then shown to restrict to a principal connection for the bundle, and the induced covariant holomorphic structures on the line bundles are shown to be the unique such structures.

In Section 6, we prove the main result of the paper.

**Theorem 1.1 (Borel–Weil)** It holds that

$$H^0(E_k) = V(r,k), \quad H^0(E_{-k}) = 0, \quad k \in \mathbb{N}_0,$$

where $V(r,k)$ is the $q$-analogue of the irreducible corepresentation occurring in the classical Borel–Weil theorem.

Finally, we use the theorem to give a novel presentation of $\mathcal{H}_q(Gr_{n,r})$ generalising the classical ample bundle presentation of the homogeneous coordinate ring.

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2 Preliminaries

The preliminaries are divided into three subsections. The first deals with Hopf algebra theory, the second deals with differential calculi and complex structures, and the third recalls the corepresentation theory of the quantum group $\mathbb{C}_q[SU_n]$. This reflects the subject of the paper which shows how, for the quantum Grassmannians, the interaction of Hopf–Galois theory and differential calculi gives a geometric realisation for a special class of corepresentations of $\mathbb{C}_q[SU_n]$.

2.1 Preliminaries on Hopf Algebras and Hopf–Galois Extensions

In this subsection we recall some well-known material about cosemisimple and coquasitriangular Hopf algebras, Hopf–Galois extensions, and principal comodule algebras, as well as Takeuchi’s categorical equivalence for quantum homogeneous spaces.

2.1.1 Cosemisimple and Coquasi-triangular Hopf Algebras

Throughout the paper all algebras are assumed to be unital and defined over the complex numbers. The letters $G$ or $H$ will always denote Hopf algebras, all antipodes are assumed to be bijective, and we use Sweedler notation. Moreover, we denote $g^+ := g - \varepsilon(g)1$, for $g \in G$, and $V^+ := V \cap \ker(\varepsilon)$, for $V$ a subspace of $G$.

For any left $G$-comodule $(V, \Delta_L)$, its space of matrix elements is the coalgebra

$$\mathcal{C}(V) := \text{span}_\mathbb{C}\{(id \otimes f)\Delta_L(v) \mid f \in \text{Hom}_\mathbb{C}(V, \mathbb{C}), v \in V\} \subseteq G.$$ (The matrix coefficients of a right $G$-comodule are defined similarly.) A comodule is irreducible if and only if its coalgebra of matrix elements is irreducible, and $\mathcal{C}(V) = \mathcal{C}(W)$ if and only if $V$ is isomorphic to $W$. Furthermore, an isomorphism is given by

$$(ev \otimes id) \circ (id \otimes \Delta_R) : V^* \otimes V \to \mathcal{C}(V), \quad \text{for all} \quad V \in \hat{G}, \quad (1)$$

where $V^*$ denotes the dual left $G$-comodule, $ev : V^* \otimes V \to \mathbb{C}$ is the evaluation map, and $\hat{G}$ denotes the isomorphism classes of irreducible left $G$-comodules [27, Theorem 11.8].

**Definition 2.1.** A Hopf algebra $G$ is called **cosemisimple** if $G \simeq \bigoplus_{V \in \hat{G}} \mathcal{C}(V)$. We call this decomposition the *Peter–Weyl* decomposition of $G$.

We finish with another Hopf algebra structure of central importance in the paper.

**Definition 2.2.** We say that a Hopf algebra $G$ is **coquasi-triangular** if it is equipped with a convolution-invertible linear map $r : G \otimes G \to \mathbb{C}$ obeying, for all $f, g, h \in G$, the relations

$$r(fg \otimes h) = r(f \otimes h_{(1)})r(g \otimes h_{(2)}), \quad r(f \otimes gh) = r(f_{(1)} \otimes h)r(f_{(2)} \otimes g),$$

$$g_{(1)}f_{(1)}r(f_{(2)} \otimes g_{(2)}) = r(f_{(1)} \otimes g_{(1)})f_{(2)}g_{(2)}, \quad r(f \otimes 1) = r(1 \otimes f) = \varepsilon(f).$$
2.1.2 Principal Comodule Algebras

For $H$ a Hopf algebra, and $V$ a right $H$-comodule with coaction $\Delta_R$, we say that an element $v \in V$ is coinvariant if $\Delta_R(v) = v \otimes 1$. We denote the subspace of all coinvariant elements by $V^{\text{co}H}$ and call it the coinvariant subspace of the coaction. We define a coinvariant subspace of a left-coaction analogously. For a right $H$-comodule algebra $P$ with multiplication $m$, its coinvariant subspace $M := P^{\text{co}H}$ is clearly a subalgebra of $P$. Throughout the paper we will always use $M$ in this sense.

We now recall a generalisation of the classical notion of an associated vector bundle. For $V$ and $W$ respectively a right, and left, $H$-comodule, the cotensor product of $V$ and $W$ is the vector space $V \Box_H W := \ker(\Delta_R \otimes \text{id} - \text{id} \otimes \Delta_L : V \otimes W \to V \otimes H \otimes W)$.

For $P$ a right $H$-comodule algebra, $P \Box_H V$ is clearly a left $M$-module. We call any such $M$-module an associated vector bundle, or simply an associated bundle. In the special case when $V$ is 1-dimensional, we call the module an associated line bundle.

If the mapping
$$\text{can} := (m_P \otimes \text{id}) \circ (\text{id} \otimes \Delta_R) : P \otimes_M P \to P \otimes H,$$
is an isomorphism, then we say that $P$ is a $H$-Hopf–Galois extension of $M$. We sometimes find it convenient to denote a Hopf–Galois extension by $M \hookrightarrow P$ without explicit reference to $H$.

If the functor $P \otimes_M - : \mathcal{M}\text{Mod} \to \mathcal{C}\text{Mod}$, from the category of left $M$-modules to the category of complex vector spaces, preserves and reflects exact sequences, then we say that $P$ is faithfully flat as a right module over $M$. The definition of faithful flatness of $P$ as a left $M$-module is analogous.

Definition 2.3. A principal right $H$-comodule algebra is a right $H$-comodule algebra $(P, \Delta_R)$ such that $P$ is a Hopf–Galois extension of $M := P^{\text{co}H}$ and $P$ is faithfully flat as a right and left $M$-module. We call $P$ the total space and $M$ the base.

Directly verifying the conditions of this definition can in general be quite difficult. The following theorem gives a more practical reformulation.

Theorem 2.4 [8, Theorem 3.20] A right $H$-comodule algebra $(P, \Delta_R)$ is principal if and only if there exists a linear map $\ell : H \to P \otimes P$, called a principal $\ell$-map, such that

1. $\ell(1_H) = 1_P \otimes 1_P$,
2. $m_P \circ \ell = \varepsilon_H 1_P$,
3. $(\ell \otimes \text{id}_H) \circ \Delta_H = (\text{id}_P \otimes \Delta_R) \circ \ell$,
4. $(\text{id}_H \otimes \ell) \circ \Delta_H = (\Delta_L \otimes \text{id}_P) \circ \ell$. 

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2.1.3 Quantum Homogeneous Spaces and Takeuchi’s Categorical Equivalence

In this subsection, we recall the basic theory of a special type of comodule algebra. A quantum homogeneous space is a Hopf algebra $G$ endowed with a right $H$-coaction of the form $(\text{id} \otimes \pi) \circ \Delta$, where $\pi : G \to H$ is a surjective Hopf algebra map. It was shown in [32, Corollary 1.5] that if $H$ is cosemisimple then $G$ is faithfully flat as a left and right $M$-module, where $M := G^{coH}$.

In the homogeneous case the category of associated vector bundles has a particularly nice form. Consider the abelian categories $\overline{G}_M \text{Mod}_M$ and $H M \text{Mod}_M$: The objects in $\overline{G}_M \text{Mod}_M$ are $M$-bimodules $F$ (with left and right actions denoted by juxtaposition) endowed with a left $G$-coaction $\Delta_L$ such that

$$\Delta_L(mf) = m_{(1)} f_{(-1)} m'_{(1)} \otimes m_{(2)} f_{(0)} m'_{(2)}, \quad \text{for all } m, m' \in M, f \in \mathcal{F}. \quad (2)$$

The morphisms in $\overline{G}_M \text{Mod}_M$ are the $M$-bimodule and left $G$-comodule maps. The objects in $H M \text{Mod}_M$ are left $H$-comodules and right $M$-modules for which it holds that $\Delta_L(vm) = \Delta_L(v) ((\pi \otimes \text{id}) \Delta_L)(m)$. The morphisms in $H M \text{Mod}_M$ are the left $H$-comodule, right $M$-module, maps.

If $\mathcal{F} \in \overline{G}_M \text{Mod}_M$, then $\mathcal{F}/(M^+ \mathcal{F})$ becomes an object in $H M \text{Mod}_M$ with the obvious right $M$-action and the right $H$-coaction $\Delta_L[f] = \pi(f_{(-1)}) \otimes [f_{(0)}]$, for $f \in \mathcal{F}$, where $[f]$ denotes the coset of $f$ in $\mathcal{F}/(M^+ \mathcal{F})$. We define a functor $\Phi : \overline{G}_M \text{Mod}_M \to H M \text{Mod}_M$ as follows: $\Phi(\mathcal{F}) := \mathcal{F}/(M^+ \mathcal{F})$, and if $g : \mathcal{F} \to \mathcal{F}'$ is a morphism in $\overline{G}_M \text{Mod}_M$, then $\Phi(g) : \Phi(\mathcal{F}) \to \Phi(\mathcal{F}')$ is the map to which $g$ descends on $\Phi(\mathcal{F})$.

If $V \in H M \text{Mod}_M$, then $G \Box_H V$ becomes an object in $\overline{G}_M \text{Mod}_M$ by defining

$$m' \left( \sum_i g_i \otimes v_i \right) m := \sum_i m' g_i m_{(1)} \otimes v_i m_{(2)},$$

$$\Delta_L \left( \sum_i g_i \otimes v_i \right) := \sum_i (g_i)_{(1)} \otimes (g_i)_{(2)} \otimes v_i.$$

We define a functor $\Psi : H M \text{Mod}_M \to \overline{G}_M \text{Mod}_M$ as follows: $\Psi(\mathcal{V}) := G \Box_H \mathcal{V}$, and if $\gamma$ is a morphism in $\overline{G}_M \text{Mod}_M$, then $\Psi(\gamma) := \text{id} \otimes \gamma$.

It was shown in [47, Theorem 1] and [23, Corollary 6.3] that a unit-counit equivalence of the categories $\overline{G}_M \text{Mod}_M$ and $H M \text{Mod}_M$, which we call Takeuchi’s equivalence, is given by the functors $\Phi$ and $\Psi$ and the natural transformations

$$C : \Phi \circ \Psi(\mathcal{V}) \to \mathcal{V}, \quad \left[ \sum_i g_i \otimes v_i \right] \mapsto \sum_i \varepsilon(g_i) v_i,$$

$$U : \mathcal{F} \to \Psi \circ \Phi(\mathcal{F}), \quad f \mapsto f_{(-1)} \otimes [f_{(0)}].$$

Using the equivalence, we define the dimension of an object $\mathcal{F} \in \overline{G}_M \text{Mod}_M$ to be the dimension of $\Phi(\mathcal{F})$ as vector space.
Faithful flatness of $G$ implies $\ker(\pi) = M^+ G$ [32, Theorem 1.1] (see also [38, Lemma 4.2]), and so, $\pi$ restricts to an isomorphism $\Phi(G) \simeq H$. This in turn implies that an explicit inverse to the canonical map is given by
\[
\text{can}^{-1} : G \otimes \Phi(G) \to G \otimes_M G, \quad g' \otimes [g] \mapsto g' S(g_{(1)}) \otimes_M g_{(2)}.
\]
Hence, if $G$ is faithfully flat as a module over a quantum homogeneous space $M$, then $G$ is a Hopf–Galois extension of $M$, and so, it is a principal comodule algebra.

2.2 Preliminaries on Complex and Holomorphic Structures

In this subsection we give a concise presentation of the basic theory of differential calculi and quantum principal bundles. We also recall the more recent notions of complex and holomorphic structures as introduced in [24] and [4]. The definition of a holomorphic structure is motivated by the classical Koszul–Malgrange theorem [28] which establishes an equivalence between holomorphic structures on a vector bundle and flat $\partial$-operators on the smooth sections of the vector bundle. A more detailed discussion of differential calculi and complex structures can be found in [36].

2.2.1 First-Order Differential Calculi

A first-order differential calculus over a unital algebra $A$ is a pair $(\Omega^1, d)$, where $\Omega^1$ is an $A$-$A$-bimodule and $d : A \to \Omega^1$ is a linear map for which the Leibniz rule holds
\[
d(ab) = a(db) + (da)b, \quad a, b \in A,
\]
and for which $\Omega^1 = \text{span}_\mathbb{C}\{adb | a, b \in A\}$. A morphism between two first-order differential calculi $(\Omega^1(A), d_\Omega)$ and $(\Gamma^1(A), d_\Gamma)$ is a bimodule map $\varphi : \Omega^1(A) \to \Gamma^1(A)$ such that $\varphi \circ d_\Omega = d_\Gamma$. Note that when a morphism exists it is unique. The direct sum of two first-order calculi $(\Omega^1(A), d_\Omega)$ and $(\Gamma^1(A), d_\Gamma)$ is the calculus $(\Omega^1(A) \oplus \Gamma^1(A), d_\Omega + d_\Gamma)$. We say that a first-order calculus is connected if $\ker(d) = \mathbb{C}1$.

The universal first-order differential calculus over $A$ is the pair $(\Omega^1_u(A), d_u)$, where $\Omega^1_u(A)$ is the kernel of the product map $m : A \otimes A \to A$ endowed with the obvious bimodule structure, and $d_u$ is defined by
\[
d_u : A \to \Omega^1_u(A), \quad a \mapsto 1 \otimes a - a \otimes 1.
\]
By [49, Proposition 1.1] there exists a surjective morphism from $\Omega^1_u(A)$ onto any other calculus over $A$.

We say that a first-order differential calculus $\Omega^1(M)$ over a quantum homogeneous space $M$ is left-covariant if there exists a (necessarily unique) left $G$-coaction $\Delta_L : \Omega^1(M) \to G \otimes \Omega^1(M)$ such that $\Delta_L(mdn) = \Delta(m)(\text{id} \otimes d)\Delta(n)$, for $m, n \in M$. Any covariant calculus $\Omega^1(M)$ is naturally an object in $G M \text{Mod}_M$. Using the surjection $\Omega^1_u(M) \to \Omega^1(M)$, it can be shown that there exists a subobject $I \subseteq M^+$ (where $M^+$
is considered as an object in \( H\text{Mod}_M \) in the obvious way) such that an isomorphism is given by
\[
\Phi(\Omega^1(M)) \rightarrow M^+/I =: V_M, \quad [dm] \mapsto [m^+].
\]
Moreover, this association defines a bijection between covariant first-order calculi and sub-objects of \( M^+ \). We tacitly identify \( \Phi(\Omega^1(M)) \) and \( M^+/I \) throughout the paper, and use the well-known formula \( dg = g_1 \otimes g_2^{(2)} \) for the implied map \( d : G \rightarrow G \square_H V_M \).

Finally, let us consider the case of the trivial quantum homogeneous space, that is, the quantum homogeneous space corresponding to the Hopf algebra map \( \varepsilon : G \rightarrow \mathbb{C} \). Here we also have an obvious notion of right covariance for a calculus with respect to a (necessarily unique) right \( G\)-coaction \( \Delta_R \). If a calculus is both left and right-covariant and satisfies \( (\text{id} \otimes \Delta_R) \circ \Delta_L = (\Delta_L \otimes \text{id}) \circ \Delta_R \), then we say that it is bicovariant. It was shown in \([49, \text{Theorem 1.8}]\) that a left-covariant calculus is bicovariant if and only if the corresponding ideal \( I \) is a subcomodule of \( G \) with respect to the \( (right) \) adjoint coaction \( \text{Ad} : G \rightarrow G \otimes G \), defined by \( \text{Ad}(g) := g_{(2)} \otimes S(g_{(1)})g_{(3)} \). Finally, when considering covariant calculi over Hopf algebras we use the notation
\[
\Lambda^1 := G^+/I. \tag{3}
\]

### 2.2.2 Differential Calculi

A graded algebra \( A = \bigoplus_{k \in \mathbb{N}_0} A_k \), together with a degree 1 map \( d \), is called a differential graded algebra if \( d \) is a graded derivation, which is to say, if it satisfies the graded Leibniz rule
\[
d(\alpha \beta) = (d\alpha)\beta + (-1)^k \alpha d\beta, \quad \text{for all } \alpha \in A^k, \beta \in A.
\]
The operator \( d \) is called the differential of the differential graded algebra.

**Definition 2.5.** A differential calculus over an algebra \( A \) is a differential graded algebra \((\Omega^*, d)\) such that \( \Omega^0 = A \) and \( \Omega^k = \text{span}_\mathbb{C}\{a_0 \Lambda a_1 \wedge \cdots \wedge da_k \mid a_0, \ldots, a_k \in A\} \), where \( \wedge \) denotes multiplication in \( \Omega^* \).

We say that a differential calculus \((\Gamma^*, d_\Gamma)\) extends a first-order calculus \((\Omega^1, d_\Omega)\) if there exists an isomorphism \( \varphi : (\Omega^1, d_\Omega) \rightarrow (\Gamma^1, d_\Gamma) \). It can be shown that any first-order calculus admits an extension \( \Omega^* \) which is maximal in the sense that there there exists a unique morphism from \( \Omega^* \) onto any other extension of \( \Omega^1 \), where the definition of differential calculus morphism is the obvious extension of the first-order definition \([36, \text{§2.5}]\). We call this extension the maximal prolongation of the first-order calculus.

### 2.2.3 Complex Structures

We call a differential calculus \((\Omega^*, d)\) over a \(*\)-algebra \( A \) a differential \(*\)-calculus if the involution of \( A \) extends to an involutive conjugate-linear map on \( \Omega^* \), for which
(dω)* = dω*, for all ω ∈ Ω•, and

(ω_k ∧ ω_l)* = (-1)^kl ω_l ∧ ω_k*, for all ω_k ∈ Ω^k, ω_l ∈ Ω^l.

**Definition 2.6.** An **almost complex structure** for a differential ∗-calculus Ω• is an N_0^2-algebra grading \( \bigoplus_{(a,b) \in N_0^2} \Omega^{(a,b)} \) for Ω• such that, for all (a, b) \in N_0^2,

1. \( \Omega^k = \bigoplus_{a+b=k} \Omega^{(a,b)} \),
2. \( (\Omega^{(a,b)})^* = \Omega^{(b,a)} \).

Let \( \partial \) and \( \overline{\partial} \) be the unique homogeneous operators of order (1, 0), and (0, 1) respectively, defined by

\[ \partial|_{\Omega^{(a,b)}} := \text{proj}_{\Omega^{(a+1,b)}} \circ d, \quad \overline{\partial}|_{\Omega^{(a,b)}} := \text{proj}_{\Omega^{(a,b+1)}} \circ d, \]

where \( \text{proj}_{\Omega^{(a+1,b)}} \), and \( \text{proj}_{\Omega^{(a,b+1)}} \), are the projections from \( \Omega^{a+b+1} \) onto \( \Omega^{a+1,b} \), and \( \Omega^{a,b+1} \) respectively.

**Definition 2.7.** A **complex structure** is an almost complex structure for which \( d = \partial + \overline{\partial} \), or equivalently, for which \( \partial^2 = \overline{\partial^2} = 0 \). The **opposite almost-complex structure** of an almost-complex structure \( \Omega^{(••)} \) is the N_0^2-algebra grading \( \overline{\Omega^{(••)}} \), defined by setting \( \overline{\Omega^{(a,b)}} := \Omega^{(b,a)} \). Note that the ∗-map of the calculus sends \( \Omega^{(a,b)} \) to \( \overline{\Omega^{(a,b)}} \) and vice-versa. Moreover, it is clear that an almost-complex structure is integrable if and only if its opposite almost-complex structure is integrable.

If \( G \) and \( H \) are Hopf ∗-algebras and \( \pi : G \rightarrow H \) is a ∗-algebra map, then \( M = G^{\text{co}H} \) is clearly a ∗-algebra. We say that a complex structure on a differential ∗-calculus over \( M \) is **covariant** if the decomposition in Definition 2.6.1 is a decomposition in the category \( \mathcal{G}_M \text{-Mod}_M \).

### 2.2.4 Connections and Holomorphic Structures

For \( \mathcal{F} \) a left module over an algebra \( A \) and \( \Omega^•(A) \) a differential calculus over \( A \), a **connection** for \( \mathcal{F} \) is a \( \mathbb{C} \)-linear mapping \( \nabla : \mathcal{F} \rightarrow \Omega^1(\mathcal{F}) \otimes_A \mathcal{F} \) such that

\[ \nabla(af) = a\nabla(f) + da \otimes_A f, \quad f \in \mathcal{F}, \ a \in A. \]

For any connection \( \nabla : \mathcal{F} \rightarrow \Omega^1 \otimes A \mathcal{F} \), setting

\[ \nabla(\omega \otimes_A f) := d\omega \otimes_A f + (-1)^k \omega \wedge \nabla(f), \quad f \in \mathcal{F}, \ \omega \in \Omega^k, \]

defines an extension of \( \nabla \) to a \( \mathbb{C} \)-linear map \( \nabla : \Omega^•(\mathcal{F}) \otimes_A \mathcal{F} \rightarrow \Omega^•(\mathcal{F}) \otimes_A \mathcal{F} \). We say that \( \nabla \) is **flat** if the curvature operator \( \nabla^2 : \mathcal{F} \rightarrow \Omega^2 \otimes_A \mathcal{F} \) is the zero map. It is easily checked that flatness of \( \nabla \) implies that \( \nabla : \Omega^•(\mathcal{F}) \otimes_A \mathcal{F} \rightarrow \Omega^•(\mathcal{F}) \otimes_A \mathcal{F} \) is a complex. Note that the curvature operator is always a left \( A \)-module map, as is the difference of any two connections.
Definition 2.8. For $\Omega^\bullet(\mathcal{A})$ a $*$-calculus endowed with a choice of complex structure, a $\overline{\partial}$-operator for an $\mathcal{A}$-module $\mathcal{F}$ is a connection for the calculus $\Omega^{(0,\bullet)}$. A holomorphic structure for $\mathcal{F}$ is a flat $\overline{\partial}$-operator.

Since a complex structure is a bimodule decomposition, composing a connection for $\mathcal{F}$ with the projection $\Pi^{(0,1)} : \Omega^1 \otimes \mathcal{A} \mathcal{F} \rightarrow \Omega^{(0,1)} \otimes \mathcal{A} \mathcal{F}$ gives a $\overline{\partial}$-operator for $\mathcal{F}$. For a quantum homogeneous space $\mathcal{M} := G^\text{co}(H)$, a connection for $\mathcal{F} \in G^\text{co}\mathcal{M}$ is said to be covariant if it is a left $G$-comodule map. For a covariant complex structure, a covariant connection will clearly project to covariant holomorphic structure.

2.2.5 Quantum Principal Bundles and Principal Connections

For a right $H$-comodule algebra $(\mathcal{P}, \Delta_R)$ with $\mathcal{M} := \mathcal{P}^\text{co}(H)$, it can be shown [8, Proposition 3.6] that $\mathcal{M} \hookrightarrow \mathcal{P}$ is a Hopf–Galois extension if and only if an exact sequence is given by

$$0 \rightarrow P \Omega^1_u(M)P \xrightarrow{i} \Omega^1_u(P) \xrightarrow{\text{ver}} P \otimes H^+ \rightarrow 0,$$

where $\Omega^1_u(M)$ is the restriction of $\Omega^1_u(P)$ to $M$, $i$ is the inclusion map, and ver is the restriction of can to $\Omega^1_u(P)$. The following definition presents sufficient criteria for the existence of a non-universal version of this sequence.

Definition 2.9. A quantum principal $H$-bundle is a Hopf–Galois extension $\mathcal{M} = \mathcal{P}^\text{co}(H)$ together with a sub-bimodule $N \subseteq \Omega^1_u(P)$ which is coinvariant under the right $H$-coaction $\Delta_R$ and for which $\text{ver}(N) = G \otimes I$, for some Ad-subcomodule right ideal $I \subseteq H^+$.

Denote by $\Omega^1(P)$ the first-order calculus corresponding to $N$, denote by $\Omega^1(M)$ the restriction of $\Omega^1(P)$ to $M$, and finally as a special case of 3, we denote $\Lambda^1_H := H^+/I$.

The quantum principal bundle definition implies [19] that an exact sequence is given by

$$0 \rightarrow P \Omega^1(M)P \xrightarrow{i} \Omega^1(P) \xrightarrow{\text{ver}} P \otimes \Lambda^1_H \rightarrow 0.$$  

A principal connection for a quantum principal $H$-bundle $\mathcal{M} \hookrightarrow \mathcal{P}$ is a right $H$-comodule, left $\mathcal{P}$-module, projection $\Pi : \Omega^1(P) \rightarrow \Omega^1(P)$ such that $\ker(\Pi) = P \Omega^1(M)P$. A principal connection $\Pi$ is called strong if $(\text{id} - \Pi)(dP) \subseteq \Omega^1(M)P$.

Takeuchi’s equivalence implies that for any homogeneous space endowed with a quantum principal bundle structure, left $G$-covariant principal connections are in bijective correspondence with right $H$-comodule complements to $V_M G$ in $\Lambda^1$. Explicitly, for such a complement $K$ the corresponding principal connection is given by

$$\Pi : G \otimes \Lambda^1 \rightarrow G \otimes \Lambda^1, \quad g \otimes v \mapsto g \otimes \text{proj}_K(v),$$

where $\text{proj}_K : \Lambda^1 \rightarrow K$ denotes the obvious connection.
For any associated vector bundle $\mathcal{F} = P \square_H V$, let us identify $\Omega^1(M) \otimes_M \mathcal{F}$ with its canonical image in $\Omega^1(M)P \otimes V$. A strong principal connection $\Pi$ induces a connection $\nabla$ on $\mathcal{F}$ by

$$\nabla : \mathcal{F} \to \Omega^1(M) \otimes_M \mathcal{F}, \quad \sum_i g_i \otimes v_i \mapsto (\text{id} - \Pi) \otimes \text{id} \left( \sum_i d g_i \otimes v_i \right).$$ (6)

The following theorem shows that strong connections and principal comodule algebras have an intimate relationship.

**Theorem 2.10** [8, Theorem 3.19] For a Hopf–Galois extension $M = P^{\co(H)}$, it holds that:

1. For $\ell : H \to P \otimes P$ a principal $\ell$-map, a strong principal connection is defined by
   $$\Pi_\ell := (m_P \otimes \text{id}) \circ (\text{id} \otimes \ell) \circ \text{ver} : \Omega^1_u(P) \to \Omega^1_u(P).$$ (7)
2. This defines a bijective correspondence between principal $\ell$-maps and strong principal connections.

### 2.3 The Quantum Group $\mathbb{C}_q[SU_n]$ and its Corepresentations

In this subsection we recall the definitions of the quantum groups $\mathbb{C}_q[U_n]$ and $\mathbb{C}_q[SU_n]$, as well as the coquasi-triangular structure of the latter. The definition of the quantised enveloping algebra $U_q(\mathfrak{sl}_n)$ is then presented, along with its dual pairing with $\mathbb{C}_q[SU_n]$. Finally, Noumi, Yamada, and Mimachi’s quantum minor presentation of the corepresentation theory of $\mathbb{C}_q[SU_n]$ is recalled. Where proofs or basic details are omitted we refer the reader to [27, §9.2] and [33, 34].

#### 2.3.1 The Quantum Groups $\mathbb{C}_q[U_n]$ and $\mathbb{C}_q[SU_n]$

For $q \in \mathbb{R}_{>0}$, let $\mathbb{C}_q[GL_n]$ be unital complex algebra generated by the elements $u^i_j, u^{-1}_i$, for $i, j = 1, \ldots, n$ satisfying the relations

- $u^k_i u^j_k = qu^j_i u^k_k$, $u^k_i u^j_k = qu^j_ik^k_k$, $1 \leq i < j \leq n; 1 \leq k \leq n$,
- $u^i_k u^j_i = u^j_k u^i_i, \quad u^i_k u^j_i = u^j_i u^i_k + (q - q^{-1})u^j_i u^j_k$, $1 \leq i < j \leq n; 1 \leq k < l \leq n$,
- $\det_n\det_n^{-1} = 1, \quad \det_n^{-1}\det_n = 1$,

where $\det_n$, the quantum determinant, is the element

$$\det_n := \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} u^1_{\sigma(1)} u^2_{\sigma(2)} \cdots u^n_{\sigma(n)},$$

with summation taken over all permutations $\sigma$ of the set $\{1, \ldots, n\}$, and $\ell(\sigma)$ is the number of inversions in $\sigma$. As is well known [27, §9.2.2], $\det_n$ is a central element of $\mathbb{C}_q[GL_n]$. 

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A coquasi-triangular structure for $\mathbb{C}[SU_n]$, with coproduct $\Delta$ and counit $\varepsilon$, is uniquely determined by $\Delta(u_j^i) := \sum_{k=1}^n u_j^i \otimes u_j^k$; $\Delta(\det^{-1}) := \det^{-1} \otimes \det^{-1}$; $\varepsilon(u_j^i) := \delta_{ij}$; and $\varepsilon(\det^{-1}) = 1$. Moreover, we can endow $\mathbb{C}[GL_n]$ with a Hopf algebra structure by defining

$$S(\det^{-1}) := \det, \quad S(u_j^i) := (-q)^{i-j} \sum_{\sigma \in S_{n-1}} (-q)^{\ell(\sigma)} u_j^{k_1} \sigma(l_1) u_j^{k_2} \sigma(l_2) \cdots u_j^{k_{n-1}} \det^{-1}$$

where $\{k_1, \ldots, k_{n-1}\} := \{1, \ldots, n\}\{j\}$, and $\{l_1, \ldots, l_{n-1}\} := \{1, \ldots, n\}\{i\}$ as ordered sets. A Hopf $*$-algebra structure is defined on $U$ along with the quantum Serre relations which we omit (see [27, §10.1.3]). Hence there exists an associated bicovariant calculus $\Omega_{q, bc}(SU_n)$ which we call the quantum special unitary group of order $n$. We denote the Hopf $*$-algebra by $\mathbb{C}/[U_n]$, and call it the quantum unitary group of order $n$.

A coquasi-triangular structure for $\mathbb{C}[SU_n]$ is defined by

$$r(u_j^i \otimes u_j^k) := q^{-\frac{1}{2}} (q^{\delta_{ij}} \delta_{kl} + (q - q^{-1})\theta(i - k)\delta_{il}\delta_{kj}), \quad i, j, k, l = 1, \ldots, n,$$

where $\theta$ is the Heaviside step function [27, Theorem 10.9]. From $r$ we can produce a family of linear maps

$$Q_{ij} : \mathbb{C}[SU_n] \to \mathbb{C}, \quad g \mapsto \sum_{a=1}^n r(u_a^i \otimes g(1))r(g(2) \otimes u_a^j), \quad \text{for } i, j = 1, \ldots, n.$$

The quantum Killing form of $r$ is the linear map

$$Q : \mathbb{C}[SU_n] \to M_n(\mathbb{C}), \quad h \mapsto [Q_{ij}(h)]_{(ij)}.$$

It is easily seen that $\ker(Q)^+ = \text{ker}(Q)$ is a right ideal of $\mathbb{C}[SU_n]^+$. Moreover, it is an Ad-subcomodule of $\mathbb{C}[SU_n]^+$ [27, §10.1.3]. Hence there exists an associated bicovariant calculus $\Omega_{q, bc}(SU_n)$ which we call the standard bicovariant calculus.

### 2.3.2 The Quantised Enveloping Algebra $U_q(\mathfrak{sl}_n)$

Recall that the Cartan matrix of $\mathfrak{sl}_n$ is the array $a_{ij} = 2\delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1}$, where $i, j = 1, \ldots, n - 1$. The quantised enveloping algebra $U_q(\mathfrak{sl}_n)$ is the noncommutative algebra generated by the elements $E_i, F_i, K_i, K_i^{-1}$, for $i = 1, \ldots, n - 1$, subject to the relations

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$K_iE_jK_i^{-1} = q^{a_{ij}}E_j, \quad K_iF_jK_i^{-1} = q^{-a_{ij}}F_j,$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{(q - q^{-1})},$$

along with the quantum Serre relations which we omit (see [27, §6.1.2] for details). A Hopf algebra structure is defined on $U_q(\mathfrak{sl}_n)$ by setting

$$\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \quad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$

$$S(E_i) = -E_iK_i^{-1}, \quad S(F_i) = -K_iF_i, \quad S(K_i) = K_i^{-1}, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1.$$
A non-degenerate dual pairing between $\mathbb{C}_q[SU_n]$ and $U_q(\mathfrak{sl}_n)$ is uniquely defined by
\[ \langle K_i, u_i^j \rangle = q^{-1}, \quad \langle K_{i-1}, u_i^j \rangle = q, \quad \langle K_j, u_i^j \rangle = 1, \quad \text{for } j \neq i, \]
\[ \langle E_i, u_i^{j+1} \rangle = 1, \quad \langle F_i, u_i^{j+1} \rangle = 1, \]
with all other pairings of generators being zero.

### 2.3.3 Quantum Minor Determinants and Irreducible Corepresentations

For $I := \{i_1, \ldots, i_r\}$ and $J := \{j_1, \ldots, j_r\}$ two subsets of $\{1, \ldots, n\}$, the associated quantum minor $z_J^I$ is the element
\[ z_J^I := \sum_{\sigma \in S_r} (-q)^{\ell(\sigma)} u_{j_1}^{\sigma(i_1)} \cdots u_{j_r}^{\sigma(i_r)} = \sum_{\sigma \in S_r} (-q)^{\ell(\sigma)} u_{\sigma(j_1)}^{i_1} \cdots u_{\sigma(j_r)}^{i_r}, \]
Note that when $I = J = \{1, \ldots, n\}$ we get back the determinant. For $I, J \subseteq \{1, \ldots, n\}$ with $|I| = r$ and $|J| = n - r$, and denoting $R := \{1, \ldots, r\}$, we adopt the conventions
\[ z_r^I := z_R^I, \quad \bar{z}_r^I := \bar{z}_R^I, \quad z := z_R^I, \quad \bar{z} := z_R^I, \]
where $R^c$ denotes the complement to $R$ in $\{1, \ldots, n\}$. The $*$-map of $\mathbb{C}_q[SU_n]$ acts on quantum minors as
\[ (z_J^I)^* = S(z_J^I) = (-q)^{\ell(J,I^c) - \ell(I,R)} z_J^{I^c}, \quad (8) \]
where $\ell(S, T) := |\{(s, t) \in S \times T \mid s > t\}|$, for $S, T \subseteq \{1, \ldots, n\}$ [33, §3.1]. Moreover, the coproduct acts on $z_J^I$ according to
\[ \Delta(z_J^I) = \sum_K z_K^I \otimes z_J^K, \quad (9) \]
where summation is over all ordered subsets $K \subseteq \{1, \ldots, n\}$ with $|I| = |J| = |K|$ [33, §1.2].

A Young diagram is a finite collection of boxes arranged in left-justified rows, with the row lengths in non-increasing order. Young diagrams with $p$ rows are clearly equivalent to dominant weights of order $p$, which is to say elements
\[ \lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{N}^p, \quad \text{such that } \lambda_1 \geq \cdots \geq \lambda_p. \]

We denote the set of dominant weights of order $p$ by $\text{Dom}(p)$. A semi-standard tableau of shape $\lambda$ with labels in $\{1, \ldots, n\}$ is a collection $T = \{T_{r,s}\}_{r,s}$ of elements of $\{1, \ldots, n\}$ indexed by the boxes of the corresponding Young diagram, and satisfying, whenever defined, the inequalities
\[ T_{r-1,s} < T_{r,s}, \quad T_{r,s-1} \leq T_{r,s}. \]
We denote by \( \text{SSTab}_n(\lambda) \) the set of all semi-standard tableaux for a given dominant weight \( \lambda \).

For a Young diagram with \( n - 1 \) rows, the **standard monomial** associated to a semi-standard tableau \( T \) is

\[
z^T := z^{T_1} \cdots z^{T_{\lambda_1}} \in \mathbb{C}_q[SU_n],
\]

where \( T_s := \{T_{1,s}, \ldots, T_{m_s,s}\} \) as an ordered set, for \( 1 \leq s \leq \lambda_1 \), and \( m_s \) is the length of the \( s \)th column. It can be shown that the elements \( z^T \) are linearly independent, that the space

\[
V_L(\lambda) := \text{span}_\mathbb{C}\{z^T \mid T \in \text{SSTab}_n(\lambda)\}
\]

is an irreducible left \( \mathbb{C}_q[SU_n] \)-comodule, and that every irreducible left comodule of \( \mathbb{C}_q[SU_n] \) is isomorphic to \( V(\lambda) \) for some \( \lambda \) (see [33, §2] and [34] for details). Similarly, for \( z_T := z^{T_1} \cdots z^{T_{\lambda_n}} \in \mathbb{C}_q[SU_n] \), the space \( V_R(\lambda) := \text{span}_\mathbb{C}\{z_T \mid T \in \text{SSTab}_n(\lambda)\} \) is an irreducible right \( \mathbb{C}_q[SU_n] \)-comodule and all irreducible right \( \mathbb{C}_q[SU_n] \)-comodules are of this form. Moreover, \( \mathbb{C}_q[SU_n] \) is cosemisimple [34].

Using the dual pairing between \( U_q(\mathfrak{sl}_n) \) and \( \mathbb{C}_q[SU_n] \), every left comodule \( V(\lambda) \) can be given a left \( U_q(\mathfrak{sl}_n) \)-module action \( \triangleright \) in the usual way. The basis vectors \( z^T \) are all *weight vectors*, which is to say, for any dominant weight \( \lambda \) and any \( T \in \text{SSTab}_n(\lambda) \), we have \( K_i \triangleright z^T = q^{\lambda_i} z^T \), for some \( \lambda_i \in \mathbb{Z} \). We call \( (\lambda_1, \ldots, \lambda_n-1) \) the *weight* of \( z^T \). With respect to the natural ordering on weights, it is clear that each \( V(\lambda) \) has a vector of highest weight which is unique up to scalar multiple. Moreover, the irreducible corepresentations of \( \mathbb{C}_q[SU_n] \) are uniquely identified by their highest weights. The corresponding statements for right comodules also hold.

The Hopf algebra \( \mathbb{C}_q[U_n] \) is also cosemisimple [33, §Theorem 2.11]. Its irreducible comodules are indexed by pairs \( (\lambda, k) \), where \( \lambda \) identifies a Young diagram with \( n - 1 \) rows and \( k \in \mathbb{Z} \). Each comodule has a concrete realisation in terms of the standard monomials of the tableaux for \( \lambda \) (defined in analogy with the \( \mathbb{C}_q[SU_n] \) case) multiplied by \( \text{det}^k \). With respect to the dual pairing between \( \mathbb{C}_q[U_n] \) and the quantised enveloping algebra \( U_q(\mathfrak{sl}_n) \), there is an obvious analogous notion of weights. Moreover, the irreducible comodules are classified by their highest weights [33, §1.4, §2.2].

### 2.4 Some Identities

In this subsection we recall two families of technical identities which prove very useful throughout the paper. The first is expressed in terms of the following notation: for \( I \subseteq \{1, \ldots, n\} \), and \( i, j \in \{1, \ldots, n\} \),

\[
I_{ij} := \begin{cases} 
(I \setminus \{i\}) \cup \{j\} & \text{if } i \neq j \text{ and } i \in I \\
I & \text{if } i = j.
\end{cases}
\]

For \( i \neq j \), when \( i \notin I \) or \( j \in I \) we do not assign a direct meaning to the symbol \( I_{ij} \), however we do denote \( z^J_{IJ} = z^J_{Iij} := 0 \), for all index sets \( J \).
Proposition 2.11 (Goodearl formulae) \cite[§2]{2} It holds that, for $i, j = 1, \ldots, n$,

1. $r(u^i_j, z^j_I) \neq 0$ only if $i \geq j$ and $J = I_{ji}$,
2. $r(z^j_I, u^i_j) \neq 0$ only if $i \leq j$ and $J = I_{ji}$.

We now recall the $q$-deformation of the classical Laplace expansion of a matrix minor.

Lemma 2.12 (Laplacian expansions) For $I, J \subseteq \{1, \ldots, n\}$ with $|I| = |J|$, and $J_1$ a choice of non-empty subset of $J$, it holds that

$$(-q)^\ell(I_1, I_1^c)z^j_I = \sum_{I_1} (-q)^\ell(I_1, I_1^c)z^j_{I_1}z^j_{I_1^c}, \quad (10)$$

where summation is over all subsets $I_1 \subseteq I$ such that $|I_1| = |J_1|$.

Remark 2.13 It should be noted that the proof of Proposition 2.11 referred to above is for the bialgebra $C_q[M_n]$ endowed with the coquasi-triangular structure unscaled by the factor $q - 1$. However, the given identities are directly implied by this result. Moreover, the proof gives exact values are given for $r(u^i_j, z^j_I)$ and $r(z^j_I, u^i_j)$ in the non-zero cases.

3 Three Principal Comodule Algebras

In this section we recall the definition of the quantum Grassmannian $C_q[Gr_{n,r}]$ as the quantum homogeneous space associated to a certain Hopf algebra map from $C_q[SU_n]$ onto $C_q[SU_r] \otimes C_q[U_{n-r}]$ as introduced in \cite{31}. Moreover, we introduce a novel quantum homogeneous space associated to a Hopf algebra map $C_q[SU_n] \rightarrow C_q[SU_r] \otimes C_q[U_{n-r}]$. We then use this homogeneous space to give a workable description of the line bundles over $C_q[Gr_{n,r}]$ central to our later proof of the Borel–Weil theorem below. Finally, we construct the standard circle bundle over $C_q[Gr_{n,r}]$ which generalises the relationship between the odd-dimensional quantum spheres and quantum projective space to the Grassmannian setting.

3.1 Quantum Grassmannians

The quantum Grassmannians are defined in terms of three Hopf algebra maps which we now recall. Let $\alpha_r : C_q[SU_n] \rightarrow C_q[SU_r]$ be the surjective Hopf $*$-algebra map defined by

$$\alpha_r(u^n_i) = \det^{-1}, \quad \alpha_r(u^i_j) = \delta_{ij} 1, \quad \text{for } i, j = 1, \ldots, n; \ (i, j) \notin R \times R,$$
$$\alpha_r(u^i_j) = u^i_j, \quad \text{for } (i, j) \in R \times R.$$

Moreover, let $\beta_r : C_q[SU_n] \rightarrow C_q[SU_{n-r}]$ be the surjective Hopf $*$-algebra map

$$\beta_r(u^i_j) = \delta_{ij} 1, \quad \text{for } i, j = 1, \ldots, n; \ (i, j) \notin R^c \times R^c,$$
$$\beta_r(u^i_j) = u^i_{j-r}, \quad \text{for } (i, j) \in R^c \times R^c.$$
Definition 3.1. The quantum Grassmannian $\mathbb{C}_q[\text{Gr}_{n,r}]$ is the quantum homogeneous space associated to the Hopf $\ast$-algebra surjection

$$\pi_{n,r} : \mathbb{C}_q[SU_n] \to \mathbb{C}_q[U_r] \otimes \mathbb{C}_q[SU_{n-r}], \quad \pi_{n,r} := (\alpha_r \otimes \beta_r) \circ \Delta.$$  

Since $\mathbb{C}_q[SU_r] \otimes \mathbb{C}_q[U_{n-r}]$ is the product of two cosemisimple Hopf algebras it is itself cosemisimple, and so, as discussed in §2.1.3 the extension $\mathbb{C}_q[\text{Gr}_{n,r}] \hookrightarrow \mathbb{C}_q[SU_n]$ is a principal comodule algebra.

We should also recall the alternative description of $\mathbb{C}_q[\text{Gr}_{n,r}]$ in terms of the quantised enveloping algebra $U_q(\mathfrak{sl}_n)$. Consider the subalgebra $U_q(\mathfrak{l}_r) \subseteq U_q(\mathfrak{sl}_n)$ generated by the elements

$$\{E_i, F_i, K_j \mid i, j = 1, \ldots, n-1; i \neq r\}.$$  

We omit the proof of the following lemma which is a straightforward technical exercise.

Lemma 3.2 It holds that

$$\mathbb{C}_q[\text{Gr}_{n,r}] = \{g \in \mathbb{C}_q[SU_n] \mid g \circ l = z(l)g, \text{ for all } l \in U_q(\mathfrak{l}_r)\}.$$  

We now present a set of generators for the algebra $\mathbb{C}_q[\text{Gr}_{n,r}]$. This result is a special case of more general results of Stokman [46, Theorem 2.5], and Heckenberger and Kolb [21, Proposition 3.2] which give generating sets for all the quantum flag manifolds.

Theorem 3.3 The quantum Grassmannian $\mathbb{C}_q[\text{Gr}_{n,r}]$ is generated as an algebra by the elements

$$\{z^{IJ} := z^Iz^J \mid |I| = r, |J| = n-r\}.$$  

Example 3.4. For the special case of $r = 1$, we see that $\mathbb{C}_q[SU_1] \simeq \mathbb{C}$, and $\pi_{n,r}$ reduces to $\alpha_1$. Hence the associated quantum homogeneous space is quantum projective space $\mathbb{C}_q[\mathbb{C}P^n]$ as introduced in [31]. In the special case $n = 1$, this reduces to the standard Podleś sphere.

As is easily seen, the algebra $\mathbb{C}_q[\text{Gr}_{n,r}]$ is isomorphic to $\mathbb{C}_{q-1}[\text{Gr}_{n,n-r}]$ under the restriction of the map $\varphi : \mathbb{C}_q[SU_n] \to \mathbb{C}_{q-1}[SU_n]$ defined by $\varphi(u^i_j) := u^{n-i+1}_j$. This isomorphism generalises the classical isomorphism of Grassmannians corresponding to the symmetry of the Dynkin diagram of $\mathfrak{sl}_n$. As is readily checked, the isomorphism extends to all our later constructions, and so, for convenience we restrict to the case of $r \geq n-r$.

3.2 The Quantum Homogeneous Space $\mathbb{C}_q[S^{n,r}]$

To define the quantum homogeneous space $\mathbb{C}_q[S^{n,r}]$, we need to introduce another Hopf algebra map. Let $\alpha'_r : \mathbb{C}_q[SU_n] \to \mathbb{C}_q[SU_r]$ be the surjective Hopf $\ast$-algebra map defined by setting

$$\alpha'_r(u^i_j) = \delta_{ij}1, \quad \alpha'_r(u^i_j) = u^i_j, \quad \text{for } i, j = 1, \ldots, n; \ (i, j) \notin R \times R,$$
$$\text{for } (i, j) \in R \times R.$$
Definition 3.5. We denote by $C_q[S^{n,r}]$ the quantum homogeneous space associated to the surjective Hopf $*$-algebra map $\sigma_{n,r} : C_q[SU_n] \to C_q[SU_r] \otimes C_q[SU_{n-r}]$ for $\sigma_{n,r} := (\alpha_r \otimes \beta_r) \circ \Delta$.

Just as for the quantum Grassmannians, we present an alternative quantised enveloping algebra version of the definition. Denote by $U_q(t_r) \subseteq U_q(sl_n)$ the subalgebra of $U_q(sl_n)$ generated by the set of elements

$$\{E_i, F_i, K_i \mid i = 1, \ldots, n-1\}.$$  

An analogous result to Lemma 3.2 holds in this case. We omit the proof which is again a technical exercise.

Lemma 3.6 It holds that

$$C_q[S^{n,r}] = \{g \in C_q[SU_n] \mid g \circ k = \varepsilon(k)g, \text{ for all } k \in U_q(t_r)\}.$$  

Example 3.7. For the case of $r = 1$, the map $\sigma_{n,r}$ reduces to $\beta_1$, and so, the associated quantum homogeneous space is the well known quantum sphere $C_q[S^{2n+1}]$ of Vaksmann and Soibel’man [35, 31, 45], which in the case $n = 1$ reduces to $C_q[SU_2]$.

3.3 Line Bundles over the Quantum Grassmannians

Since $C_q[U_r]$ and $C[SU_{n-r}]$ are both cosemisimple, the 1-dimensional corepresentations of $C_q[U_r] \otimes C[SU_{n-r}]$ will be exactly those which are tensor products of a 1-dimensional corepresentation of $C_q[U_r]$ and a 1-dimensional corepresentation of $C[SU_{n-r}]$. The classification of comodules presented in §2.3.3 now implies that all such comodules are of the form

$$V_k \to V_k \otimes C_q[U_r] \otimes C_q[SU_{n-r}], \quad v \mapsto v \otimes \det_r^k \otimes 1, \quad k \in \mathbb{Z}.$$  

We denote the line bundle corresponding to $V_k$ by $E_{-k}$, and identify it with its canonical image in $C_q[SU_n]$.

Lemma 3.8 It holds that $C_q[S^{n,r}] \simeq \bigoplus_{k \in \mathbb{Z}} E_k$. Moreover, this decomposition is an algebra grading of $C_q[S^{n,r}]$.

Proof. For $\text{proj} : C_q[U_{n-1}] \to C_q[SU_n]$ the canonical projection, the definitions of $\pi_{n,r}$ and $\sigma_{n,r}$ imply commutativity of the diagram

$$\begin{array}{ccc}
C_q[SU_n] & \xrightarrow{\pi_{n,r}} & C_q[U_r] \otimes C_q[SU_{n-r}] \\
\sigma_{n,r} & & \downarrow \text{proj} \otimes \text{id} \\
& & C_q[SU_r] \otimes C_q[SU_{n-r}].
\end{array}$$
For $W$ an irreducible direct summand in the Peter–Weyl decomposition of $\mathbb{C}_q[U_n]$, and $v \in W \cap \mathbb{C}_q[S^{n,r}]$, it holds that

$$\Delta_{\pi_n,r}(v) \in W \otimes \alpha_r(W) \otimes 1.$$ 

A basic weight argument will confirm that $\alpha_r(W)$ is irreducible as a coalgebra, and so, $\text{proj}(\alpha_r(W)) = \mathbb{C}1$ if only if $\dim(W) = 1$. It follows from cosemisimplicity of $\mathbb{C}_q[U_n]$, and the classification of its irreducible comodules in §2.3.3, that all such sub-coalgebras are of the form $\mathbb{C} \text{det}^k_{n-1}$, for $k \in \mathbb{Z}$. Thus, we see that $v \in \mathbb{C}_q[S^{n,r}]$ only if it is contained in $\bigoplus_{k \in \mathbb{Z}} \mathcal{E}_k$. Moreover, since $\mathbb{C}_q[S^{n,r}]$ is clearly homogeneous with respect to to the Peter–Weyl decomposition, we can conclude that $\mathbb{C}_q[S^{n,r}] \subseteq \bigoplus_{k \in \mathbb{Z}} \mathcal{E}_k$. Since the opposite inclusion is clear, we have the required vector space decomposition $\mathbb{C}_q[S^{n,r}] \simeq \bigoplus_{k \in \mathbb{Z}} \mathcal{E}_k$. Finally, the fact that $\Delta_{\pi_n,r}$ is an algebra map implies that this decomposition is an algebra grading. \hfill \square

**Proposition 3.9** The algebra $\mathbb{C}_q[S^{n,r}]$ is generated by the elements

$$\{z^I, \overline{z}^J | I, J \subseteq \{1, \ldots, r\}; |I| = r, |J| = n - r\}. \tag{11}$$

Moreover, $z^I \in \mathcal{E}_1$ and $\overline{z}^J \in \mathcal{E}_{-1}$, and $\mathcal{E}_0 = \mathbb{C}_q[\text{Gr}_{n,r}]$.

**Proof.** Note first that

$$\Delta_{\pi_n,r}(z^I) = z^I_A \otimes \alpha_r(z^A_B) \otimes \beta_r(z^B)$$

$$= z^I_A \otimes \alpha_r(z^A) \otimes \beta_r(z)$$

$$= z^I \otimes \alpha_r(z) \otimes 1$$

$$= z^I \otimes \det_r \otimes 1.$$ 

An analogous calculation shows that $\Delta_{\pi_n,r}(\overline{z}^J) = \overline{z}^J \otimes \text{det}^{-1}_{n-r} \otimes 1$. Thus we see that $z^I \in \mathcal{E}_1$, $\overline{z}^J \in \mathcal{E}_{-1}$, and the algebra generated by $z^I$ and $\overline{z}^J$ is contained in $\mathbb{C}_q[S^{n,r}]$. Moreover, this algebra is homogeneous with respect to the decomposition $\mathbb{C}_q[S^{n,r}] \simeq \bigoplus_{k \in \mathbb{Z}} \mathcal{E}_k$. We denote the space of homogeneous elements of degree $k$ by $\mathcal{E}_k$. It follows from Theorem 3.3 and (9) that $\mathcal{E}_k$ is a subobject of $\mathcal{E}_k$ in the category $\mathcal{G} \text{Mod}_M$. Hence, since the latter is clearly irreducible, we must have $\mathcal{E}_k = \mathcal{E}_k$. It now follows that $\mathbb{C}_q[S^{n,r}]$ is generated by the elements $z^I$ and $\overline{z}^J$. The fact that $\mathcal{E}_0 = \mathbb{C}_q[\text{Gr}_{n,r}]$ is simply a restatement of Theorem 3.3. \hfill \square

**Example 3.10.** For the special case of $r = n - 1$, the generating set (11) reduces to the well-known set of generators for $\mathbb{C}_q[S^{2n-1}]$

$$\{z_i := S(u^i_n), \overline{z}_i := u^i_n | i = 1, \ldots, n\}.$$ 

Moreover, we recover the well-known description of $\mathbb{C}_q[S^{2n-1}]$ as the direct sum of the line bundles over $\mathbb{C}_q[\mathbb{C}P^n]$. 18
3.4 The Standard Circle Bundle

Corresponding to the algebra grading $\mathbb{C}_q[S^{n,r}] \simeq \bigoplus_k \mathbb{C}_k$, we have a right $\mathbb{C}[U_1]$-coaction $\Delta_R : \mathbb{C}_q[S^{n,r}] \to \mathbb{C}_q[S^{n,r}] \otimes \mathbb{C}[U_1]$ which acts as $\Delta_R(z^I) = z^I \otimes t$ and $\Delta_R(\tau^I) = \tau^I \otimes t^{-1}$. With respect to $\Delta_R$ we clearly have $\mathbb{C}_q[S^{n,r}]^{\text{co}(\mathbb{C}[U_1])} = \mathbb{C}_q[\text{Gr}_{n,r}]$.

As is well known, for the special case of $r = 1$ the extension $\mathbb{C}_q[\mathbb{C}P^n] \hookrightarrow \mathbb{C}_q[S^{2n-1}]$ is a principal comodule algebra with $\mathbb{C}[U_1]$-fibre. The following proposition shows that this fact extends to all values of $r$.

**Proposition 3.11** A principal $\ell$-map $\ell : \mathbb{C}[U_1] \to \mathbb{C}_q[S^{n,r}] \otimes \mathbb{C}_q[S^{n,r}]$ is defined by

$$t^k \mapsto S(z^k_{(1)}) \otimes z^k_{(2)}, \quad t^{-k} \mapsto S(\tau^k_{(1)}) \otimes \tau^k_{(2)}, \quad k \in \mathbb{N}_0.$$  

Hence, $\mathbb{C}_q[\text{Gr}_{n,r}] \hookrightarrow \mathbb{C}_q[S^{n,r}]$ is a principal $\mathbb{C}[U_1]$-comodule algebra, which we call the standard circle bundle of $\mathbb{C}_q[\text{Gr}_{n,r}]$.

**Proof.** We begin by showing that $\ell$ is well defined, which is to say that its image is contained in $\mathbb{C}_q[S^{n,r}] \otimes \mathbb{C}_q[S^{n,r}]$. Note first that

$$S(z^k_{(1)}) \otimes z^k_{(2)} = \sum_{K_1, \ldots, K_k} S(z^R_{K_1}) \cdots S(z^R_{K_k}) \otimes z^{K_1} \cdots z^{K_k}. \quad (12)$$

It follows from (8) that each $S(z^R_{K_k})$ is proportional to $\tau^{K_k}$. Hence, for $k \geq 0$, we have that $\ell(t^k) \in \mathbb{C}_q[S^{2n-1}] \otimes \mathbb{C}_q[S^{2n-1}]$. The case of $k < 0$ is established analogously.

We now show that the requirements of Theorem 2.4 are satisfied. It is obvious that conditions 1 and 2 hold. For $k \geq 0$, condition 3 follows from

$$(\ell \otimes \text{id}_{\mathbb{C}[U_1]}) \circ \Delta_{\mathbb{C}[U_1]}(t^k) = \ell(t^k) \otimes t^k = S(z^k_{(1)}) \otimes z^k_{(2)} \otimes t^k$$

$$= (\text{id}_{\mathbb{C}_q[S^{n,r}]} \otimes \Delta_R)(S(z^k_{(1)}) \otimes z^k_{(2)})$$

$$= (\text{id}_{\mathbb{C}_q[S^{n,r}]} \otimes \Delta_R)(\ell(t^k)).$$

The fourth condition of Theorem 2.4 is demonstrated analogously, as are both conditions for the case of $k < 0$. 

We now recall the definition of a cleft comodule algebra, which can be viewed as a noncommutative generalisation of a trivial bundle.

**Definition 3.12.** A right $H$-comodule algebra $P$ is called cleft if there exists a convolution-invertible right $H$-comodule map $j : H \to P$.

**Lemma 3.13** The extension $\mathbb{C}_q[\text{Gr}_{n,r}] \hookrightarrow \mathbb{C}_q[S^{n,r}]$ is non-cleft.

**Proof.** For $r \geq 2$, by looking at the action of $\alpha_1$ on the generators $z^I$ and $\tau^I$, it is easily seen that $\alpha_1$ restricts to a map $\mathbb{C}_q[S^{n,r}] \to \mathbb{C}_q[S^{n-1,r-1}]$ which preserves the
Z-grading, or equivalently, is a right \( C[U_1] \)-comodule map. Moreover, \( \alpha' \) restricts to a right \( C[U_1] \)-comodule map \( C_q[S^{2n-1}] \to C_q[S^{2n-3}] \), for \( n \geq 3 \). By taking appropriate compositions of these maps one can produce a right \( C[U_1] \)-comodule map

\[
f : C_q[S^{n,r}] \to C_q[S^3] \simeq C_q[SU_2].
\]

Assume now that there exists a convolution-invertible right \( C[U_1] \)-comodule map \( j : C_q[U_1] \to C_q[S^{n,r}] \) with convolution inverse \( j \). Since \( f \) is clearly a unital algebra map, it would hold that

\[
(f \circ j(t^k))(f \circ j(t^k)) = f(j(t^k)j(t^k)) = f(1) = 1 = \varepsilon(t^k), \quad \text{for all } k \in \mathbb{Z},
\]

implying that \( f \circ j : C[U_1] \to C_q[SU_2] \) is convolution-invertible, and hence a cleaving map for the quantum Hopf fibration \( C_q[SU_2] \to C[U_1] \). However, it was shown in [20, Appendix] that no such map exists. Hence, we can conclude that the bundle is non-cleft. \( \square \)

**Remark 3.14** In [7] it was observed that the classical construction of weighting projective space (through a weighting of the \( U_1 \)-coaction on \( S^{2n-1} \)) can be directly generalised to the quantum setting. This construction can be directly extended to the \( C[U_1] \)-coaction on \( C_q[S^{n,r}] \) considered above allowing one to define quantum weighted Grassmannians. This will be discussed in greater detail in [41].

**Remark 3.15** It was shown in [2] that for a quantum principal \( C[U_1] \)-comodule algebra \( M \hookrightarrow P \), admitting a suitable \( C^* \)-completion \( \overline{M} \hookrightarrow \overline{P} \), the \( C^* \)-algebra \( \overline{P} \) is a Cuntz–Pimsner algebra over \( \overline{M} \). The compact quantum group completion of the extension \( C_q[Gr_{n,k}] \hookrightarrow C_q[S^{n,r}] \) is easily seen to satisfy the required conditions, and so, \( C_q[S^{n,r}] \) is a Cuntz–Pimsner algebra over \( C_q[Gr_{n,r}] \). This will be discussed in greater detail in [41].

## 4 A Restriction Calculus Presentation of \( \Omega^1_q(Gr_{n,r}) \)

In this subsection we generalise the work of [35] for the case of quantum projective space and realise the Heckenberger–Kolb calculus of \( C_q[Gr_{n,r}] \) as the restriction of a quotient of the standard bicovariant calculus over \( C_q[SU_n] \). This allows us to present the calculus as the base calculus of a quantum principal bundle. The universal principal connection, associated to the \( \ell \)-map considered in §3.3.2, is shown to descend to this quotient and to induce covariant holomorphic structures on the line bundles of \( C_q[Gr_{n,r}] \).

### 4.1 The Heckenberger–Kolb Calculus for the Quantum Grassmannians

We give a brief presentation of the Heckenberger–Kolb calculus of the quantum Grassmannians starting with the classification of first-order differential calculi over \( C_q[Gr_{n,k}] \). Recall that a first-order calculus is called *irreducible* if it contains no non-trivial sub-bimodules.
Theorem 4.1 [21, §2] There exist exactly two non-isomorphic irreducible left-covariant first-order differential calculi of finite dimension over $\mathbb{C}_q[Gr_{n,k}]$.

The direct sum of $\Omega^{(1,0)}$ and $\Omega^{(0,1)}$ is a $\ast$-calculus which we call it the Heckenberger–Kolb calculus of $\mathbb{C}_q[Gr_{n,k}]$ and which we denote by $\Omega^1_q(Gr_{n,k})$.

Lemma 4.2 [22] Denoting by $\Omega^\bullet_q(Gr_{n,r}) = \bigoplus_{k \in \mathbb{N}_0} \Omega^k_q(Gr_{n,r})$ the maximal prolongation of $\Omega^1_q(Gr_{n,r})$, each $\Omega^k_q(Gr_{n,r})$ has classical dimension, which is to say,

$$\dim(\Omega^k_q(Gr_{n,r})) = \binom{2r(n - r)}{k}, \quad k = 0, \ldots, 2r(n - r),$$

and $\Omega^k_q(Gr_{n,r}) = 0$, for $k > 2r(n - r)$. Moreover, the decomposition $\Omega^1_q(Gr_{n,r})$ into its irreducible sub-calculi induces a pair of opposite complex structures for the total calculus of $\Omega^1_q(Gr_{n,r})$.

4.2 A Family of Quotients of the Standard Bicovariant Calculus on $\mathbb{C}_q[SU_n]$ We begin by recalling the $n^2$-dimensional basis for the calculus, as originally constructed in [35, Lemma 4.2].

Lemma 4.3 For $q \neq 1$, a basis of $\Phi(\Omega^1_{q, bc}(SU_n))$ is given by

$$b_{ij} := q^2\nu^{-1}[u^j_i], \quad b_{ii} := -q^{2(i-1)\nu-2(1-\delta_{i1})}[u^i_1 S(u^1_i)], \quad i, j = 1, \ldots, n; \ i \neq j.$$ 

Moreover, a dual basis is given by the functionals

$$b^{ij} := Q_{ji}, \quad b^{ii} := Q_{ii}.$$ 

It is easily shown that the restriction of the standard bicovariant calculus $\Omega^1_{q, bc}(SU_n)$ to the quantum Grassmannians has non-classical dimension, and so, it cannot be isomorphic to the Heckenberger–Kolb calculus. We circumvent this problem by constructing a certain family of quotients of $\Omega^1_{q, bc}(SU_n)$. We omit the proof of the lemma which is a direct generalisation of [35, Lemma 4.3].

Lemma 4.4 A right submodule of $\Phi(\Omega^1_{q, bc}(SU_n))$ is given by

$$V_r := \text{span}_\mathbb{C}\{b_{ij} \mid i, j = r + 1, \ldots, n\}.$$ 

Hence, denoting by $B_r$ the sub-bimodule of $\Omega^1_{q, bc}(SU_n)$ corresponding to $V_r$, a differential calculus is given by the quotient

$$\Omega^1_{q}(SU_n, r) := \Omega^1_{q, bc}(SU_n)/B_r.$$ 21
We denote its corresponding ideal in \( \mathbb{C}_q[SU_n] \) by \( I_{\mathbb{C}_q[SU_n]} \).

Finally, we come to right covariance of the calculus, as defined in Definition 2.9. We omit the proof which is a standard argument using the corepresentation theory of \( \mathbb{C}_q[SU_n] \) and \( \mathbb{C}_q[U_n] \) presented in §2.3.3. We note that the lemma can also be proved using a direct generalisation of [35, Lemma 4.3].

**Lemma 4.5** The calculus \( \Omega^1_q(SU_n, r) \) is right \( \mathbb{C}_q[SU_n] \otimes \mathbb{C}_q[U_{n-r}] \)-covariant.

### 4.3 The Action of \( \mathcal{Q} \) on the Generators of \( \mathbb{C}_q[S^n] \) and \( \mathbb{C}_q[Gr_{n,r}] \)

In this subsection we establish necessary conditions for non-vanishing of the functions \( Q_{ij} \) on the generators of \( \mathbb{C}_q[S^n] \) and \( \mathbb{C}_q[Gr_{n,r}] \). These results are used in the next subsection to describe the restriction of \( \Omega^1_q(SU_n, r) \) to the quantum Grassmannians.

**Lemma 4.6** For \( i, j = 1, \ldots, n \), it holds that

1. \( Q_{ij}(z^I_j) \neq 0 \) only if \( I = I_{ij} \), or equivalently only if \( J = I_{ji} \),

2. \( Q_{ii}(z) = \lambda \) and \( Q_{ii}(\overline{z}) = \lambda^{-1} \), for a certain non-zero \( \lambda \in \mathbb{C} \).

**Proof.**

1. By Goodearl’s formulae

\[
Q_{ij}(z^I_j) = \sum_A \sum_{a=1}^n r(u^i_a \otimes z^I_A) r(z^A_j \otimes u^j_a) = \sum_{a=1}^n r(u^i_a \otimes z^I_{J_{ai}}) r(z^I_{J_{aj}} \otimes u^j_a).
\]

Now \( r(u^I_k \otimes z^I_{J_{ai}}) \) gives a non-zero answer only if \( I_{ai} = J_{aj} \), or equivalently only if \( I = I_{ij} \), or equivalently only if \( J = I_{ji} \). Hence \( Q_{ij}(z^I_j) \) gives a non-zero answer only in the stated cases. \( \square \)

2. It follows from Proposition 2.11 that, for \( i = 1, \ldots, r \),

\[
Q_{ii}(z) = \sum_{a=1}^n r(u^i_a \otimes z^I_k) r(z^K \otimes u^a_i) = r(u^i_i \otimes z) r(z \otimes u^i_i) = q^{2-\frac{2n}{m}} =: \lambda,
\]

where we have used the identity

\[
r(u^I_k \otimes z^I_j) = r(z^I_j \otimes u^I_k) = q^{\delta_{k,l}}, \quad \text{for } \delta_{k,l} = 1 \text{ if } k \in I, \text{ and } \delta_{k,l} = 0, \text{ if } k \notin I,
\]

proved in [15, Lemma 2.1]. An analogous argument shows that \( Q_{ii}(\overline{z}) = \lambda^{-1} \), for all \( i \).

\( \square \)

Building on this lemma, we next produce a set of necessary requirements for non-vanishing of the maps \( Q_{ij} \) on the generators of \( \mathbb{C}_q[Gr_{n,r}] \).
Lemma 4.7 For $i, j = 1, \ldots, n$, and $(i, j) \notin R^c \times R^c$, it holds that $Q_{ij}(z^{1J}) \neq 0$ only if $I = R_{ij}$ and $J = R^c$, or $I = R$ and $J = R^c_{ij}$.

Moreover, we have $Q_{ij}(z^{RR^c}) = Q_{ij}(1)$.

Proof. Note first that

$$Q_{ij}(z^{1J}) = \sum_{|K|=r} \sum_{|L|=n-r} \sum_{a=1}^n r(u^i_a \otimes z^J_{KL}) r(z^{KL} \otimes u^j_r)$$

$$= \sum_{|K|=r} \sum_{|L|=n-r} \sum_{a,b,c=1}^n r(u^i_a \otimes z^J_K) r(u^b_b \otimes z^L_J) r(z^K \otimes u^j_c) r(\sigma^L \otimes u^j_r)$$

$$= \sum_{|L|=n-r} \sum_{a=1}^i \sum_{c=1}^j r(u^i_a \otimes z^J_J) Q_{aj}(z^J) r(\sigma^L \otimes u^j_r).$$

Now $r(\sigma^L \otimes u^j_r) \neq 0$ only if $c \leq j$, which cannot happen if $c \neq j$ since $R^c_{ij}$ would contain a repeated element and could not be equal to $I$. Hence

$$Q_{ij}(z^{1J}) = \sum_{|L|=n-r} \sum_{a=1}^i \sum_{c=1}^j r(u^i_a \otimes z^J_J) Q_{aj}(z^J) r(\sigma^L \otimes u^j_r)$$

$$= \sum_{a=1}^i r(u^i_a \otimes z^J_J) Q_{aj}(z^J) r(\sigma^{RR} \otimes u^j_r).$$

If we now assume that $I \neq R$, then by the above lemma, the constant $Q_{aj}(z^J) \neq 0$ only if $(a, j) \in R \times R^c$. Since $r(u^i_a \otimes z^J_J) \neq 0$ only if $J = R^c_{ija}$, and we have assumed that $(i, j) \notin R^c \times R^c$, we can get a non-zero result only if $i = a$. Thus, we have shown that $Q_{ij}(z^{1J}) \neq 0$ only if $I = R_{ij}$ and $J = R^c$.

If we instead assume that $I = R$, then an analogous argument will show that we get a non-zero result only when $J = R^c_{ij}$.

The identity $Q_{ij}(z^{RR^c}) = Q_{ij}(1)$ follows the Lemma 4.6.2 above. It can also be derived from Laplacian expansion as follows. In (10), setting $I = J = \{1, \ldots, n\}$ and $J_1 = R$ gives

$$1 = \det = \sum_{|I_1|=r} (-q)^{\ell(I_1, J_1)} z_R I_1 R^c = \sum_{|I_1|=r} (-q)^{\ell(I_1, J_1)} z_I I_1 R^c = \sum_{|I_1|=r} (-q)^{\ell(I_1, J_1)} z_I I_1 R^c.$$

As we just established, $Q_{ij}(z^{I_1 R^c}) = 0$ unless $I_1 = R$. Hence, as required

$$Q_{ij}(1) = \sum_{|I_1|=r} (-q)^{\ell(I_1, J_1)} Q_{ij}(z^{I_1 R^c}) = Q_{ij}(z^{RR^c}).$$
4.4 The Heckenberger–Kolb Calculus as the Restriction of $\Omega^1_q(SU_n, r)$

Using the results of the previous subsection, we present the Heckenberger–Kolb calculus as the restriction of the calculus $\Omega^1_q(SU_n, r)$, reproduce its decomposition into $\Omega^{(1,0)}$ and $\Omega^{(0,1)}$, and introduce a basis of $\Phi(\Omega^1_q(\text{Gr}_n, r))$.

**Lemma 4.8** The subspaces

\[
V^{(1,0)} := \{ b_{ij} \mid (i, j) \in R^c \times R \}, \quad V^{(0,1)} := \{ b_{ij} \mid (i, j) \in R \times R^c \}
\]

are non-isomorphic right $C_q[SU_r] \otimes C_q[U_{n-r}]$-subcomodules of $\Lambda^1_q[C_q[SU_r] \otimes C_q[U_{n-r}]]$. Moreover, they are right $C_q[SU_n]$-submodules.

**Proof.** A routine application of the right adjoint coaction will verify that $V^{(1,0)}$ and $V^{(0,1)}$ are subcomodules of $\Lambda^1_q[C_q[SU_r] \otimes C_q[U_{n-r}]]$. A direct comparison of weights confirms that the two comodules are non-isomorphic. The fact that $V^{(1,0)}$ is a submodule follows from the calculation

\[
Q_{ab}(u^I_j u^J_k) = 0, \quad \text{for all } (i, j) \in R^c \times R; (a, b) \notin R \times R^c; k, l = 1, \ldots, n.
\]

A similar argument shows that $V^{(0,1)}$ is a submodule.  \[\square\]

**Proposition 4.9** The restriction of $\Omega^1_q(SU_n, r)$ to $C_q[\text{Gr}_n, r]$ is the Heckenberger–Kolb calculus. Moreover, a choice of complex structure is given by

\[
\Omega^{(1,0)} := C_q[SU_n] \square_H V^{(1,0)}, \quad \Omega^{(0,1)} := C_q[SU_n] \square_H V^{(0,1)}.
\]

**Proof.** It follows from Lemma 4.7 that

\[
\text{span}_C\{ ([z^I_J]^+) \mid |I| = r, |J| = n - r \} \subseteq V^{(1,0)} \oplus V^{(0,1)}.
\]

Now it is clear that $C_q[\text{Gr}_n, r]^+$ is generated as a right $C_q[\text{Gr}_n, r]$-module by the set

\[
\{ ([z^I_J]^+) \mid |I| = r, |J| = n - r \}.
\]

Thus, since the above lemma tells us that $V^{(1,0)} \oplus V^{(0,1)}$ is a right submodule of $\Lambda^1$, we must have that

\[
\Phi(\Omega^1) = \{ [m^+] \mid m \in C_q[\text{Gr}_n, r] \} \subseteq V^{(1,0)} \oplus V^{(0,1)},
\]

where $\Omega^1$ denotes the restriction of $\Omega^1_q(SU_n, r)$ to $C_q[\text{Gr}_n, r]$.

Lemma 4.7 tells us that each non-zero $([z^I_J]^+)$ is contained in either $V^{(1,0)}$ or $V^{(0,1)}$. Thus, since $V^{(1,0)}$ and $V^{(0,1)}$ are submodules of $V^{(1,0)} \oplus V^{(0,1)}$, we must have a decomposition

\[
\Phi(\Omega^1) = W^{(1,0)} \oplus W^{(0,1)} := \Phi(\Omega^1) \cap V^{(1,0)} \oplus \Phi(\Omega^1) \cap V^{(0,1)}.
\]

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as well as a corresponding decomposition of calculi. Now Theorem 4.1 tells us that there can exist no non-trivial calculus of dimension strictly less than \( \dim(\Phi(\Omega^{(1,0)}) \). Thus the inequality

\[
\dim(W^{(1,0)}) \leq \dim(V^{(1,0)}) = \dim(\Phi(\Omega^{(1,0)})),
\]

implies that \( W^{(1,0)} = V^{(1,0)} \), or \( W^{(1,0)} = 0 \). To show that the former is true we need only show that there exists a non-zero \([m] \in V^{(1,0)}\), for some \( m \in \mathbb{C}_q[\text{Gr}_{n,r}] \). This is easily done by specialising the calculations of Lemma 4.6 and Lemma 4.7 to the simple case of \( z^{R_{R_{+1,1}}} \), and showing that \( Q_{r+1,1}(z^{R_{R_{+1,1}}}) \neq 0 \). A similar argument establishes that \( V^{(0,1)} = V^{(0,1)} \), and so, \( \Omega^1 \) is equal to \( \Omega^1_q(\text{Gr}_{n,r}), \Omega^{(1,0)} \oplus \Omega^{(1,0)} \), or \( \Omega^{(1,0)} \oplus \Omega^{(1,0)} \).

That the latter is true follows from the fact that \( V^{(1,0)} \) and \( V^{(0,1)} \) are non-isomorphic, and so, \( \Omega^1 \) is isomorphic to the Heckenberger–Kolb calculus as claimed. \( \square \)

**Corollary 4.10** For all \((i, j) \in R^c \times R\), there exist non-zero constants \( \lambda_{ij} \) such that

\[
[z^{R_{ij}R^c}] = \lambda_{ij}b_{ij}, \\
[z^{R_{ji}R^c}] = \lambda_{ji}b_{ij}.
\]

**Proof.** Since \( \Omega^1_q(\text{Gr}_{n,r}) \in H_{\text{Mod}_0} \), the elements \([z^{R_{ij}R^c}], [z^{R_{ji}R^c}]\), for \( i, j \in R \times R^c \), have to span \( V^{(1,0)} \oplus V^{(0,1)} \), and so, they must all be non-zero. The fact that \([z^{R_{ij}R^c}]\) is proportional to the basis element \( b_{ij} = [u^i_j] \), and \([z^{R_{ji}R^c}]\) is proportional to the basis element \( b_{ij} = [u^i_j] \), follows from Lemma 4.7 and the fact that \( Q_{ij}(u^i_j) \) is non-zero if and only if \( k = j \) and \( l = i \) [35, §4]. \( \square \)

5 **Two Quantum Principal Bundles**

In this section we use the calculus \( \Omega^1_q(SU_n, r) \) to give a quantum principal bundle presentation of the Heckenberger–Kolb calculus \( \Omega_q^1(\text{Gr}_{n,r}) \) in terms of the Hopf–Galois extensions \( \mathbb{C}_q[\text{Gr}_{n,r}] \hookrightarrow \mathbb{C}_q[S^{n,r}] \) and \( \mathbb{C}_q[\text{Gr}_{n,r}] \hookrightarrow \mathbb{C}_q[SU_n] \). We then construct principal connections for these bundles and show that the covariant holomorphic structures induced on the line bundles over \( \mathbb{C}_q[\text{Gr}_{n,r}] \) are the unique such structures.

5.1 **The Extension \( \mathbb{C}_q[\text{Gr}_{n,r}] \hookrightarrow \mathbb{C}_q[S^{n,r}] \)**

We begin by verifying the properties of a quantum principal bundle for the differential structure induced by \( \Omega_q^1(SU_n, r) \) on the extension \( \mathbb{C}_q[\text{Gr}_{n,r}] \hookrightarrow \mathbb{C}_q[S^{n,r}] \). We then show that the universal calculus principal connection associated to the \( \ell \)-map introduced in §3.4 descends to a principal connection for the bundle. We begin by introducing a Hopf algebra map useful throughout this section:

\[
\rho: \mathbb{C}_q[SU_n] \to \mathbb{C}[U_1], \quad u_1 \mapsto t, \quad u_1^n \mapsto t^{-1}, \quad u_j \mapsto \delta_{ij}1, \quad \text{for } (i, j) \neq (1, 1), (n, n).
\]

Throughout, to lighten notation, we denote \( H' := \mathbb{C}_q[U_1] \otimes \mathbb{C}_q[SU_{n-1}] \).
Lemma 5.1

1. A commutative diagram is given by

\[
\begin{array}{c}
\Omega^1_\mathfrak{u}(S^{n,r}) \\
\downarrow \text{ver} \\
\mathbb{C}_q[SU_n] \otimes \mathbb{C}[U_1]^+ \\
\downarrow \text{id} \otimes \rho \\
\mathbb{C}_q[SU_n] \otimes \mathbb{C}[U_1]^+ \\
\end{array}
\]

\[u : \mathbb{C}_q[SU_n] \otimes \mathbb{C}[U_1]^+ \rightarrow \mathbb{C}_q[SU_n] \square H' \left( \mathbb{C}_q[S^{n,r}] \right)^+.\]

2. The Hopf–Galois extension \(\mathbb{C}_q[Gr_{n,r}] \hookrightarrow \mathbb{C}_q[S^{n,r}]\) endowed with the restriction of \(\Omega^1_\mathfrak{u}(SU_n, r)\) to \(\mathbb{C}_q[S^{n,r}]\) is a quantum principal bundle.

Proof.

1. Recalling the isomorphism \(U : \Omega^1_\mathfrak{u}(S^{n,r}) \simeq \mathbb{C}_q[SU_n] \square H' \left( \mathbb{C}_q[S^{n,r}] \right)^+\), we see that

\[
\begin{align*}
(id \otimes \rho) \circ U(dz^I) &= z^I \otimes (t - 1) = \text{ver}(d_z z^I), \\
(id \otimes \rho) \circ U(d\overline{z}^I) &= \overline{z}^I \otimes (t^{-1} - 1) = \text{ver}(d_{\overline{z}} \overline{z}^I).
\end{align*}
\]

Let \(N \subseteq \Omega^1_\mathfrak{u}(S^{n,r})\) be the sub-bimodule corresponding to the restriction of the calculus \(\Omega^1_\mathfrak{u}(SU_n, r)\) to \(\mathbb{C}_q[S^{n,r}]\). Denoting \(J := I_{\mathbb{C}_q[SU_n]} \cap \mathbb{C}_q[S^{n,r}]^+\), Takeuchi’s equivalence implies that \(U(N) = \mathbb{C}_q[SU_n] \square H' J\).

2. Commutativity of the diagram above implies that

\[\text{ver}(N) = (id \otimes \rho)(\mathbb{C}_q[SU_n] \square H' J).\] (14)

Denote by \(T_1\) the subspace of \(\mathbb{C}_q[S^{n,r}]^+\) spanned by monomials in the elements \(z, \overline{z}\), and denote \(T_2\) the subspace spanned by monomials which contain as a factor at least one generator not equal to \(z\) or \(\overline{z}\). Clearly, \(\mathbb{C}_q[S^{n,r}]^+ \simeq T_1 \oplus T_2\). Moreover, Lemma 4.6 implies that \(J\) is homogeneous with respect to this decomposition allowing us to write \(J \simeq J_1 \oplus J_2\).

It is easily seen that \(T_1\) and \(T_2\) are sub-comodules of \(\mathbb{C}_q[S^{n,r}]^+\) with respect to the left \(\mathbb{C}_q[SU_r] \otimes \mathbb{C}_q[SU_{n-r}]\)-coaction of \(\mathbb{C}_q[S^{n,r}]\). Moreover, the coaction acts trivially on \(T_1\). Hence we have

\[
\begin{align*}
\text{ver}(N) &= (id \otimes \rho)(\mathbb{C}_q[SU_n] \square H' J) \\
&= (id \otimes \rho) \left( (\mathbb{C}_q[SU_n] \otimes J_1) \oplus (\mathbb{C}_q[SU_n] \square H' J_2) \right) \\
&= \mathbb{C}_q[SU_n] \otimes \rho(J_1) \\
&= \mathbb{C}_q[SU_n] \otimes \rho(J).
\end{align*}
\]

Finally, it follows from Lemma 4.5, and the fact that \(\rho\) is Hopf algebra map, that \(\rho(J)\) is Ad-coinvariant right ideal of \(\mathbb{C}[U_1]\), and so, we have a quantum principal bundle. \(\square\)
We now move on to showing that \( \ell \) induces a well-defined connection for the bundle. In the process we find a basis for \( \Lambda^1 \) and an alternative description of the action of \( \ell \).
Throughout, to lighten notation, we denote \( H' := C_q[SU_r] \otimes C_q[SU_{n-r}] \).

**Proposition 5.2**

1. \( \Lambda^1_{C[U_1]} = H \otimes C[t - 1] \).
2. Let \( i : C[U_1] \to C_q[S^{n-r}] \) be the linear map defined by \( i(t^k) := z^k, i(1) := 1 \), and \( i(t^{-k}) = z^{-k} \). A commutative diagram is given by

\[
\begin{array}{ccc}
\Omega^1_u(S^{n,r}) & \xrightarrow{\text{mo}(id \otimes \ell)} & C_q[U_1] \otimes C_q[S^{n,r}]^+ \\
& & \downarrow \text{U}^{-1} \\
C_q[SU_n] \otimes C[U_1]^+ & \xrightarrow{1 \otimes i} & C_q[SU_n] \square_H'(C_q[S^{n,r}])^+. \\
\end{array}
\]

3. The principal connection \( \Pi_\ell \) corresponding to \( \ell \) descends to a principal connection for the bundle.

**Proof.**

1. It follows from Lemma 4.6.1 that \( z - 1 \neq 0 \). Moreover, Proposition 4.9 implies that \( [z - 1] \in V_{C_q[S^{n,r}]} \) such that \( v \notin V_{C_q[Gr_{n,r}]} \). Hence \( \dim(V_{C_q[S^{n,r}]}) > \dim(V_{C_q[Gr_{n,r}]}). \)

Exactness of the sequence (5) now implies that \( \dim(\Lambda^1_{H}) \geq 1 \).

Since \( \rho(u^i_j) = \rho(u^i_j S(u^i_j)) = 0 \), for \( i \neq j \), it is clear that \( \rho(I_{C_q[SU_n]}) = \rho(\ker(Q))^+ \). Thus, we have an injection

\[ \Lambda^1_{H} = H^+ / \rho(I_{C_q[SU_n]}) = H^+ / \rho(\ker(Q))^+ \hookrightarrow H / \rho(\ker(Q)), \]

implying the inequality

\[ 1 \leq \dim(\Lambda^1_{H}) \leq \dim(H / \rho(\ker(Q))). \] (15)

Lemma 4.6.2 now implies that \( z - \lambda 1 \in \ker(Q) \), and so, \( t - \lambda 1 \in \rho(\ker(Q)) \), which implies that \( \dim(H / \rho(\ker(Q))) \leq 1 \). Hence \( \Lambda^1_{C[U_1]} \) is a 1-dimensional subspace spanned by the element \( [z - 1] \).

2. This is easily confirmed by direct calculation.

3. Recalling (7) we see that \( \Pi_\ell \) descends to non-universal connection if

\[ i(\rho(I_{C_q[SU_n]})) \subseteq I_{C_q[SU_n]}. \]

It now follows from (15) that \( \dim(H / \rho(\ker(Q))) = 1 \), and that \( \rho(\ker(Q)) \) is spanned by the elements \( \{t^k - \lambda t^{k-1} | k \in \mathbb{Z}\} \). Now

\[ Q(i(t^k - \lambda t^{k-1})) = Q(z^k - \lambda z^{k-1}) = Q((z - \lambda 1)z^k) = 0, \quad \text{for all} \ k \geq 1. \]
Similarly, it can be shown that $Q \left( i(t^{-k} - \lambda t^{-k-1}) \right) = 0$, for all $k \geq 1$. Hence we have shown that $i(\rho(\ker(Q))) \subseteq \ker(Q)$. The fact that $\Pi_\ell$ descends to a well-defined non-universal connection now follows from
\[ i(\rho(I_{C_q}[SU_n])) = i(\rho(\ker(Q)^+)) \subseteq i(\rho(\ker(Q) \cap C_q[SU_n]^+)) \]
\[ = \ker(Q) \cap C_q[SU_n]^+ = \ker(Q)^+. \]

5.2 Holomorphic Structures

In this subsection we show that the covariant connections induced on $E_k$ by $\Pi_\ell$ restrict to covariant holomorphic structures for $E_k$, and that these are the unique such structures.

Following the same approach as in [40], we convert these questions into representation theory using the following two observations: Since the difference of two connections is always a left module map, the difference of two covariant $\partial$-operators for any object in $G M$Mod is always a morphism. Moreover, since the curvature operator of any connection is always a left module map, the curvature operator of a covariant $\partial$-operator is a morphism in $G M$Mod.

As explained in §2.2.4, for any line bundle $E_k$, the connection coming from the principal connection $\Pi_\ell$ induces a $\partial$-operator by projection. We denote this operator by $\partial E_k$.

Lemma 5.3 The $\partial$-operator $\partial E_k$ is the unique covariant $\partial$-operator for $E_k$. Moreover, it is a holomorphic structure.

Proof. If there exists a morphism $E_k \to \Omega^{(0,1)} \otimes_{C_q[Gr_{n,r}]} E_k$ in $G M$Mod, then it is clear that there exists a morphism
\[ C_q[Gr_{n,r}] = E_0 \to \Omega^{(0,1)} \otimes_{C_q[Gr_{n,r}]} E_0 \simeq \Omega^{(0,1)}. \]

However, since Theorem 4.1 implies that $\Phi(\Omega^{(0,1)})$ is irreducible as an object in $H$Mod, no such morphism exists. Thus, by the comments at the beginning of the subsection, these can exist no other covariant $\partial$-operator for $E_k$.

Similarly, there exists a morphism $E_k \to \Omega^{(0,2)} \otimes_{C_q[Gr_{n,r}]} E_k$ only if $\Phi(\Omega^{(0,2)})$ contains a copy of the trivial comodule. It follows from [36, Lemma 5.1] that $\Phi(\Omega^{(0,2)})$ is isomorphic to a quotient of $\Phi(\Omega^{(0,1)})^{\otimes 2}$. An elementary weight argument will show that $\Phi(\Omega^{(0,1)})^{\otimes 2}$ does not contain a copy of the trivial comodule, and so, $\partial E_k$ is a covariant holomorphic structure. □

Corollary 5.4 The principal connection $\Pi_\ell$ is the unique left $C_q[SU_n]$-covariant principal connection for the bundle $\Omega^1_q(Gr_{n,r}) \to \Omega^1_q(S^{n,r})$. 

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Proof. Assume that there exists a second covariant principal connection $\Pi'$. Extending the argument of the above lemma, it is easy to show that there exists only one left $C_q[SU_n]$-covariant connection for each $E_k$. Hence, for any $e \in E_k$,

$$(\text{id} - \Pi) \circ d(e) = \nabla'(e) = \nabla(e) = (\text{id} - \Pi_\ell) \circ d(e),$$

implying that $\Pi'(d(e)) = \Pi_\ell(d(e))$. Now since $C_q[S^{n,r}] \cong \bigoplus_{k \in \mathbb{Z}} E_k$, every element of $\Omega^1_q(S^{n,r})$ is a sum of elements of the form $gde$, for $g \in C_q[SU_n]$. Thus, since $\Pi_\ell$ and $\Pi'$ are both left $C_q[SU_n]$-module maps, they are equal. \hfill \square

5.3 The Extension $\mathbb{C}_q[Gr_{n,r}] \hookrightarrow \mathbb{C}_q[SU_n]$

In our proof of the Borel–Weil theorem in the next section, it proves very useful to have an alternative description of $\Pi_\ell$ as the restriction of a principal connection for the bundle $\mathbb{C}_q[Gr_{n,r}] \hookrightarrow \mathbb{C}_q[SU_n]$.

Proposition 5.5

1. The Hopf–Galois extension $\mathbb{C}_q[Gr_{n,r}] \hookrightarrow \mathbb{C}_q[SU_n]$ endowed with the first-order calculus $\Omega^1_q(SU_n, r)$ is a quantum principal bundle.

2. A $C_q[U_r] \otimes C_q[SU_{n-r}]$-comodule complement to $V_{\mathbb{C}_q[Gr_{n,r}]}$ in $\Lambda^1_{\mathbb{C}_q[U_r] \otimes C_q[SU_{n-r}]}$ is given by

$$V^0 := \text{span}_\mathbb{C}\{b_{ij} | i, j = 1, \ldots, r\}. \quad (16)$$

Moreover, it is the unique such complement, implying that the corresponding left $C_q[SU_n]$-covariant principal connection $\Pi$ is the unique such connection.

3. The principal connection $\Pi$ restricts to $\Pi_\ell$ on $\Omega^1_q(S^{n,r})$.

Proof.

1. This follows from an argument analogous to the proof of Lemma 5.1.

2. Note first that Lemma 4.8 implies that $V_{\mathbb{C}_q[Gr_{n,r}]} \mathbb{C}_q[SU_n] = V_{\mathbb{C}_q[Gr_{n,r}]}$. Moreover, it follows from Lemma 4.3 and Corollary 4.10 that $\Lambda^1 \cong V^0 \oplus V_{\mathbb{C}_q[Gr_{n,r}]}$. Uniqueness of the decomposition in (16) for the case of $r = 1$, that is the case of $\mathbb{C}_q[CP^n]$, is routine, see [35] for details. Hence, we concentrate on the case of $n \geq 2$. A direct examination shows that $V^0$ decomposes into a one-dimensional comodule and an irreducible $r^2 - 1$-dimensional comodule. A direct comparison of dimensions will show that these two comodules are pairwise non-isomorphic to $V^{(1,0)}$ and $V^{(0,1)}$. Hence the decomposition, and the corresponding covariant principal connection, are unique as claimed.
3. It follows from Lemma 5.1.1 and Proposition 5.2.2 that the principal connection \( \Pi \) acts on \( C_q[SU_n \Box H \Phi(\Omega_1^q(S^n,r))] \) as \( \text{id} \otimes (i \circ \rho) \). Thus to show that \( \Pi \) restricts to \( \Pi^r \) it suffices to show that \( \text{proj}_V \) restricts to \( i \circ \rho \) on \( \Phi(\Omega_1^q(S^n,r)) \). Since both maps vanish on \( V C_q[Gr_n,r] \), we need only show that they agree on a complement to \( V C_q[Gr_n,r] \). Proposition 5.2.1, and the fact that \( \rho(z - 1) = t - 1 \), imply that such a complement is given by \( C[z - 1] \), and that \( i \circ \rho \) acts on \( [z - 1] \) as the identity. Thus \( \Pi \) restricts to \( \Pi^r \) on \( C_q[S^n,r] \) as required.

\[ \square \]

**Remark 5.6** We have not considered here any lifting of \( \Pi \) to a principal connection at the universal level. Such a lifting, however, plays an important role in [38] where it is used to give a Nichols algebra description of the calculus \( \Omega_q^*(Gr_{n,r}) \).

\section{A Borel–Weil Theorem for the Quantum Grassmannians}

In this section we establish a \( q \)-deformation of the Borel–Weil theorem for the Grassmannians, the principal result of the paper. We then observe that the theorem gives a presentation of the twisted homogeneous coordinate ring \( H_q(Gr_{n,r}) \) deforming the classical ample bundle presentation of \( H(Gr_{n,r}) \). The proof of the theorem is expressed in terms of a sequence of twisted derivation algebras embedded in a sequence of modules over a ring generated by certain quantum minors. These novel sequences have many interesting properties and will be explored in further detail in a later work.

\subsection{A Sequence of Twisted Derivation Algebras}

For any \( j \in \{r + 1, \ldots, n\} \), denoting \( j' := n - j + 1 \), let \( Z_j \) be the subalgebra of \( C_q[SU_n] \) generated by the elements

\[ \{z_I | I \subseteq \{j', \ldots, n\}\} \]

Consider also \( T_j \) the two-sided ideal of \( Z_j \) generated by the elements

\[ \{z_J | J \cap \{r + 1, \ldots, j - 1\} \neq \emptyset\} \]

as well as the quotient algebras \( S_j := Z_j/T_j \). For any \( x \in Z_j \), we denote its coset in \( S_j \) by \( [x]_{S_j} \). Note that, for any \( j \), we have an obvious embedding \( Z_j \hookrightarrow Z_{j+1} \), which restricts to an embedding \( T_j \hookrightarrow T_{j+1} \). Thus we have a sequence

\[ S_r \xrightarrow{\varphi_r} S_{r+1} \xrightarrow{\varphi_{r+1}} \cdots \xrightarrow{\varphi_{n-1}} S_n \xrightarrow{\varepsilon} \mathbb{C} \]

Note that all of the maps \( \varphi_j \) have non-trivial kernel, and that the sequence does not form a complex.

A *twisted derivation algebra* is a triple \((A, \sigma, d)\), where \( A \) is an algebra, \( \sigma : A \to A \) is an algebra automorphism, and \( d \) is a a linear map, called the *twisted derivation*, satisfying

\[ d(ab) = d(a)b + \sigma(a)d(b), \quad \text{for all } a, b \in A. \]
We construct a twisted derivation for each $S_j$ from $T_{C_q[SU_n]} : C_q[SU_n] \to C_q[SU_n] \otimes V^{(0,1)}$, the holomorphic connection induced by $\Pi$ on $C_q[SU_n]$, where we consider $C_q[SU_n]$ as a trivial associated bundle in the obvious way. We find it useful to have the following explicit description of the action of $T_{C_q[SU_n]}$ on a general quantum minor.

**Lemma 6.1** For any quantum minor $z'_j$, we have

$$T_{C_q[SU_n]}(z'_j) = \sum_{(i,j) \in R \times R^c} Q_{j,i}(z''_{ji}) z'_{ji} \otimes b_{ij}.$$  

**Proof.** It follows from Lemma 4.3 and Lemma 4.6 that

$$T_{C_q[SU_n]}(z'_j) = \Pi^{(0,1)} \circ \Pi \circ d(z'_j) = \Pi^{(0,1)} \left( \sum_A z_A^I \otimes [z_A^J] \right) = \sum_A \sum_{(i,j) \in R \times R^c} Q_{j,i}(z_A^I) z_A^I \otimes b_{ij} = \sum_{(i,j) \in R \times R^c} Q_{j,i}(z''_{ji}) z'_{ji} \otimes b_{ij}. \quad \square$$

We also introduce the following projection map

$$\text{proj}_j : C_q[SU_n] \otimes V^{(0,1)} \to C_q[SU_n], \quad \sum_{(k,l) \in R \times R^c} f_{kl} \otimes b_{kl} \mapsto f'_{j,k}.$$  

**Proposition 6.2** The triple $(S_j, \sigma_j, \vartheta_j)$ is a twisted derivation algebra, where $\sigma_j$ is the algebra automorphism of $S_j$ defined by $\sigma_j[z_1]_{S_j} := q^{\delta_{j,j'} + \delta_{j,j'}} [z_1]_{S_j}$, and $\vartheta_j$ is the $\sigma_j$-twisted derivation of $S_j$ uniquely defined by

$$\vartheta_j[z_1]_{S_j} := \left[ \text{proj}_j \left( T_{C_q[SU_n]}(z'_j) \right) \right]_{S_j}. \quad (17)$$

**Proof.** As is easily checked, an algebra automorphism $\sigma_j$ of $C_q[SU_n]$ is defined by $\sigma_j(u_k^{ij}) = q^{\delta_{j,j'} + \delta_{j,j'}} u_k^{ij}$. This restricts to an automorphism of $Z_j$ which maps $T_j$ to itself, and hence induces the required algebra automorphism of $S_j$.

Consider now the obvious lifting of $\vartheta_j$ to a map $\vartheta'_j : Z_j \to S_j$. Let us show that this map is a twisted derivation. Since $T_{C_q}$ is a derivation, and $\text{proj}_j$ is a left $C_q[SU_n]$-module map, it is clear that we only need to show that $[\text{proj}_j(\omega z)]_{S_j} = [\text{proj}_j(\omega)]_{S_j} \sigma_j[z]_{S_j}$, for any $\omega \in C_q[SU_n] \otimes V^{(0,1)}$, $z \in Z_j$. Moreover, this equality is implied by the identities, for any $z_1 \in Z_j$,

$$\text{proj}_j((1 \otimes b_{kl}) \triangleleft z_1) \in T_j, \quad \text{for all } (k,l) \neq (j',j), \quad (18)$$

$$\text{proj}_j((1 \otimes b_{j'l'}) \triangleleft z_1) = \sigma_j(z_1). \quad (19)$$

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For the first identity note that Lemma 4.3 implies
\[
\text{proj}_j((1 \otimes b_{kl}) \triangleleft z_J) = \sum_A \text{proj}_j(z_A \otimes (b_{kl} \triangleleft z^A_J)) = q^2 \nu^{-1} \sum_A \text{proj}_j(z_A \otimes [u^k_J z^A_J])
\]
\[
= q^2 \nu^{-1} \sum_A \text{proj}_j\left( \sum_{(k,l) \in R \times R^c} Q_{ba}(u^k_J z^A_J) z_A \otimes b_{ab} \right)
\]
\[
= q^2 \nu^{-1} \sum_A Q_{jj'}(u^k_J z^A_J) z_A.
\]

A routine application of Goodearl’s formulae shows that
\[Q_{jj'}(u^k_J z^A_J) \neq 0\] only if \(A = (J_{k'l})_{jl}\) and \(k \leq j'; l \leq j\).

Since by assumption \(J \subseteq \{j', \ldots, n\}\), we must have \(k = j'\), and hence that \(A = J_{jl}\).
Next our assumption that \((k,l) \neq (j',j)\) means we must have \(l \in \{r + 1, \ldots, j - 1\}\). Hence, \(z_{J_{jl}} \in T_j\) as required. The identity (18) is proved similarly. Hence \(d_j'\) is a twisted derivation.

Finally we show that \(d_j'\) descends to a \(\sigma_j\)-twisted derivation on \(S_j\) by showing that it vanishes on the generators of \(T_j\). When \(J \cap \{r + 1, \ldots, j - 1\} \neq \emptyset\), we clearly have \(J_{j'l'} \cap \{r + 1, \ldots, j - 1\} \neq \emptyset\), and so, \([z_{J_{j'l'}}]_{S_j} = 0\). Hence, by Lemma 6.1, we have that \(d_j'\) vanishes on the generators of \(T_j\) as required. \(\square\)

For any \(k \in R^c\), denote \(P_k := \{k, \ldots, n\}\). Moreover, let \(P^l_k\) be the index set constructed from \(P_k\) by replacing each of its first \(l\) elements \(p\) by \(p'\) and then reordering. Finally, we denote by \(P(S_{r+1})\) the subset of \(S_{r+1}\) consisting of those elements which are products of elements of the form \([z_{P_k}]_{S_{r+1}}\). The following lemma is an easy consequence of the Goodearl formulae and Lemma 6.1, and so we omit its proof.

**Lemma 6.3**

1. \(d_j[z_J]_{S_j} = 0\), if \(j \notin J\),

2. \(d_j'[z_J]_{S_j} = 0\), for all \(J \subseteq \{1, \ldots, n\}\),

3. \(d_{k+l}[z_{P^l_k}]_{S_{k+l}}^{a} = a!(-q)^{-\frac{a(k+l)}{n}} \nu^a[z_{P^l_k}]_{S_{k+l}}^{a}\), for \(a \in \mathbb{N}\).

**Corollary 6.4** For every \(p \in P(S_{r+1})\), there exists a unique sequence \((a_{r+1}, \ldots, a_n)\) of non-negative natural numbers, at least one of which is non-zero, such that
\[
\varepsilon \circ d_n^{a_n} \circ \varphi_{n-1} \circ d_{n-1}^{a_{n-1}} \circ \cdots \circ \varphi_j \circ d_{r+1}^{a_{r+1}}(p) \neq 0.
\]
6.2 A Sequence of Modules

Consider $\tilde{Z}$ the subalgebra of $\mathbb{C}_q[SU_n]$ generated by the elements
\[ \{ z_J \mid \{1, \ldots, r\} \subseteq J \} . \]

We denote by $\tilde{Z}_j$ the $\tilde{Z}$-submodule of $\mathbb{C}_q[SU_n]$ generated by the elements of $Z_j$. Moreover, we denote by $\tilde{T}_j$ the $\tilde{Z}$-submodule of $\tilde{S}_j$ generated by the elements of $T_j$, and by $\tilde{S}_j$ the quotient module $\tilde{Z}_j/\tilde{T}_j$. Moreover, for any $x \in \tilde{Z}_j$, we denote its coset in $\tilde{S}_j$ by $[x]_{\tilde{S}_j}$.

**Corollary 6.5** A $\mathbb{C}$-linear inclusion $S \hookrightarrow \tilde{S}$ is defined by $[x]_S \mapsto [x]_{\tilde{S}}$. Moreover $\varnothing_j$ extends uniquely to a left $\tilde{Z}_j$-module map $\tilde{\varnothing}_j : \tilde{Z} \rightarrow \tilde{Z}$, and $\varphi_j$ extends uniquely to a left $\tilde{Z}_j$-module map $\varphi_j : \tilde{S}_j \rightarrow \tilde{S}_j$.

**Proof.** An elementary weight argument can be used to show that $S_j \cap \tilde{T}_j = T_j$, and so, the map $Z_j \hookrightarrow \tilde{Z}_j$ is an inclusion, for all $j$. It follows from Lemma 6.1, that for any $z_J \in \tilde{Z}$, we have
\[ \tilde{\partial}(z_J) = \sum_{(k,l) \in R \times R^c} Q_{lk}(z_J^{lk}) z_J^{lk} b_{kl} . \]

By assumption $\{1, \ldots, r\} \subseteq J$, and so, the index set $J_{lk}$ is not defined for any pair $(k, l) \in R \times R^c$, and so, $z_J^{lk} = 0$. This means that a left $\tilde{Z}$-module map is defined by
\[ \tilde{Z}_j \rightarrow \tilde{S}_j, \quad x \mapsto \text{proj}_j(\tilde{\partial}(x))_{\tilde{S}_j} . \]

Moreover, it is clear from the definition of $\tilde{T}_j$ that this map descends to a well-defined left $\tilde{Z}$-module map $\tilde{S}_j \rightarrow \tilde{S}_j$ extending $\varnothing_j$, and that it is the unique such map. \qed

**Remark 6.6** It is natural to consider the extension of the definition $\tilde{S}_j$ to a bimodule over $\tilde{Z}$. However, as is easily checked, $\varnothing_j$ does not have a natural extension to this space.

6.3 The Borel–Weil Theorem

In proving our $q$-deformation of the Borel–Weil theorem, the most difficult part is demonstrating non-holomorphicity of an element of any line bundle $\mathcal{E}_k$. Roughly, our approach is to decompose $\mathcal{E}_k$ into its irreducible left $\mathbb{C}_q[SU_n]$-comodules, choose a workable description of their highest weights, and then use the above sequence and Corollary 6.4 to show that they do not vanish under $\partial_{\mathbb{C}_q[SU_n]}$.

We begin with the workable description of the highest weight vectors. Since the details of the proof are for the most part completely classical applications of the corepresentation theory of §2.3.3, it is omitted.
Lemma 6.7 Each right comodule in the decomposition of $\mathbb{C}_q[S^{n,r}]$ into irreducible right $\mathbb{C}_q[SU_n]$-comodules contains a highest weight vector of the form

$$\sum_{k \geq 0} x_k s_k, \quad \text{for } x_k \in \hat{Z}, s_k \in \mathbb{Z}_{r+1}$$

such that $\varepsilon(x_0) \neq 0$, $\varepsilon(x_k) = 0$, for all $k \neq 0$, and $[s_0] s_{r+1} \in P(S_{r+1})$.

Proof. Note first that an element of $\mathbb{C}_q[SU_n]$-comodule is coinvariant under $\mathbb{C}_q[SU_r] \otimes \mathbb{C}_q[SU_{n-r}]$ if and only if it is coinvariant under $\mathbb{C}_q[SU_r]$ and $\mathbb{C}_q[U_{n-r}]$. Consider next a Young diagram admitting a partition of the form

$$\begin{array}{c}
A \\
B \\
C \\
D
\end{array}$$

(20)

where $A$ is an $(r \times k)$-array of boxes, and $C$ is an $((n-r) \times l)$-array of boxes, for some $k, l \in \mathbb{N}_0$. The space of $\mathbb{C}_q[U_r]$-coinvariant elements of the corresponding comodule $V$ is spanned by those standard monomials $z^T$, where $T$ is a semi-standard tableau for which the columns in $A$ are filled as $\{1, \ldots, r\}$.

Regarding $V$ now as a $\mathbb{C}_q[SU_{n-r}]$-comodule, we see that it is isomorphic to the tensor product of the comodules $V(B)$ and $V(D)$ corresponding to $B$ and $D$ respectively. Hence, the space of $\mathbb{C}_q[SU_{n-r}]$-coinvariant elements of the corresponding comodule is non-trivial if and only if $V(B)$ is dual to $V(D)$.

Moreover, we see that a Young diagram admits a $\mathbb{C}_q[SU_r] \otimes \mathbb{C}_q[SU_{n-r}]$-coinvariant element only if it admits a partition of the form (20). A standard highest weight argument using the Peter–Weyl decomposition and the isomorphism (1) now imply that each irreducible comodule in the decomposition of $\mathbb{C}_q[S^{n,r}]$ contains a highest weight comodule of the stated form. □

In the statement of the theorem below, as well as in Lemma 6.12, we find it useful to adopt the following conventions

$$\lambda(r, k) := (k, \ldots, k, 0, \ldots, 0) \in \text{Dom}(n-1), \quad V(r, k) := V(\lambda(r, k)).$$

Theorem 6.8 (Borel–Weil) It holds that

$$H^0(\mathcal{E}_k) = V(r, k), \quad H^0(\mathcal{E}_{-k}) = 0, \quad k \in \mathbb{N}_0.$$ 

Proof. It is clear from the construction of $\overline{\partial}_{\mathbb{C}_q[SU_n]}$, and the construction of the holomorphic structure for each line bundle, that the theorem would follow from a demonstration that

$$\ker(\overline{\partial}_{\mathbb{C}_q[SU_n]}|_{\mathbb{C}_q[SU_n]}|_{\mathbb{C}_q[S^{n,r}]}) : \mathbb{C}_q[S^{n,r}] \to \mathbb{C}_q[SU_n] \otimes V^{(0,1)} = \bigoplus_{k \in \mathbb{N}_0} V(r, k).$$

(21)

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The inclusion of $\bigoplus_{k \in \mathbb{N}_0} V(r,k)$ in the kernel of $\bar{\partial}_{C_q[SU_n]}$ is a direct consequence of Lemma 6.1. To establish the opposite inclusion, consider any irreducible right $C_q[SU_n]$-comodule $U$ in the decomposition of $C_q[S^m r]$ which is not equal to $V(r,k)$, for any $k \in \mathbb{N}_0$. Moreover, let $\sum_{k \geq 0} x_k s_k \in U$ be the highest weight vector presented in the above lemma. Proposition 6.4 and $\bar{\partial}$-linearity of $\partial_j$ imply that we have a unique set of integers $(a_{r+1}, \ldots, a_n)$, at least one of which is non-zero, such that

$$\epsilon \circ \partial_n^{a_n} \circ \cdots \circ \varphi_j \circ \partial_j^{a_j+1}[x_k s_k]_{S_{r+1}} = \epsilon(x_k)(\epsilon \circ \partial_n^{a_n} \circ \cdots \circ \varphi_j \circ \partial_j^{a_j+1}[s_k]_{S_{r+1}}) \neq 0.$$  

For any of the other summands $x_k s_k$ we have

$$\epsilon \circ \partial_n^{a_n} \circ \varphi_n \circ \cdots \circ \varphi_j \circ \partial_j^{a_j+1}[x_k s_k]_{S_{r+1}} = \epsilon(x_k)(\epsilon \circ \partial_n^{a_n} \circ \cdots \circ \varphi_j \circ \partial_j^{a_j+1}[s_k]_{S_{r+1}}) \neq 0.$$

From the definition of $\partial_j$, for each $j$, it is clear that this could not happen if $\sum_{k \geq 0} x_k s_k$ were contained in the kernel of $\bar{\partial}_{C_q[SU_n]}$. Hence, since $\bar{\partial}_{C_q[SU_n]}$ is a comodule map, we can conclude that $U$ is not contained in the kernel of $\bar{\partial}_{C_q[SU_n]}$, and that the identity in (21) holds as required. \qed

Corollary 6.9 The calculus $\Omega^1_q(Gr_{n,r})$ is connected.

**Proof.** Note that $m \in \ker(\partial : C_q[Gr_{n,r}] \to \Omega^1_q(Gr_{n,r}))$ if and only if $m \in \ker(\bar{\partial}) \cap \ker(\bar{\partial})$. Now the above theorem states that for the special case of $\mathcal{E}_0 = C_q[Gr_{n,r}]$, we have $H^0(\mathcal{E}_0) = \ker(\bar{\partial}) = \mathbb{C}$. Hence $\ker(\partial) = \mathbb{C}$ as required. \qed

6.4 The Twisted Homogeneous Coordinate Ring

The twisted homogeneous coordinate ring of a general flag manifold $G_0/L_0$ was introduced in [29, 44]. In the $q = 1$ case it reduces to the homogeneous coordinate ring of $G_0/L_0$ with respect to the Plücker embedding [14, §6.1]. We consider now the special case of the quantum Grassmannians. (These are subalgebras of the bialgebra $C_q[M_n]$ which is defined as $C_q[GL_n]$ without the generator $\det_n^{-1}$, see [35, §3].)

**Definition 6.10.** The twisted Grassmannian homogeneous coordinate ring $\mathcal{H}_q(Gr_{n,r})$ is the subalgebra of $C_q[M_n]$ generated by the quantum minors $z^j$, for $|I| = r$, which are defined just as in the $C_q[SU_n]$ case.

Clearly, $\mathcal{H}_q(Gr_{n,r})$ is a left subcomodule of $C_q[SU_n]$. The subcomodule $\mathcal{H}(Gr_{n,r})$ is defined to the subalgebra of $C_q[M_n]$ generated by the quantum minors $\pi^j$, for $|J| = n-r$, which are again defined just as in the $C_q[SU_n]$ case. The image of the multiplication map

$$m : \mathcal{H}(Gr_{n,r}) \otimes \mathcal{H}(Gr_{n,r})^* \to C_q[SU_n]$$

is clearly equal to $C_q[S^mr]$. Hence we have a presentation of the coordinate algebra $C_q[Gr_{n,r}]$ as the $\mathbb{C}[U_1]$-coinvariant part of a quotient of $\mathcal{H}(Gr_{n,r}) \otimes \mathcal{H}(Gr_{n,r})^*$. This
is the special Grassmannian case of the general flag manifold construction discussed in the introduction. As the following proposition shows, the theory of holomorphic structures allows us to go the other direction and construct $\mathcal{H}_q(Gr_{n,r})$ from $\mathbb{C}_q[Gr_{n,r}]$ by generalising the classical ample line bundle bundle presentation of $\mathcal{H}(Gr_{n,r})$.

**Theorem 6.11** The canonical projection $\text{proj} : \mathbb{C}_q[M_n] \to \mathbb{C}_q[SU_n]$ restricts to an algebra isomorphism

$$\mathcal{H}_q(Gr_{n,r}) \simeq \bigoplus_{k \in \mathbb{N}_0} H^0(\mathcal{E}_k).$$

**Proof.** The classification of comodules of $\mathbb{C}_q[SU_n]$ in §2.3.3 is easily seen to imply that canonical projection $\mathbb{C}_q[M_n] \to \mathbb{C}_q[SU_n]$ restricts to an injection on $\mathcal{H}_q(Gr_{n,r})$. Thus we can identify $\mathcal{H}_q(Gr_{n,r})$ with its image in $\mathbb{C}_q[SU_n]$. Since Theorem 6.8 tells us that $\bigoplus_{k \in \mathbb{N}_0} H^0(\mathcal{E}_k)$ is generated by $z^I$, for $|I| = r$, it is clear that the two algebras are isomorphic. \(\blacksquare\)

### 6.5 The Opposite Complex Structure

Let $\overrightarrow{\Omega}^{(\bullet, \bullet)}$ denote the complex structure for $\Omega^*(Gr_{n,r})$ opposite to the one introduced in Proposition 4.9. Using an argument analogous to the one above, each line bundle can be shown to have a unique covariant holomorphic structure with respect to this complex structure. This causes the Borel–Weil theorem to vary as follows.

**Lemma 6.12** With respect to the complex structure $\overrightarrow{\Omega}^{(\bullet, \bullet)}$, we have

$$H^0(\mathcal{E}_k) = 0, \quad H^0(\mathcal{E}_{-k}) = \{v^* \mid v \in V(r,k)\}.$$

Moreover, the canonical projection $\text{proj} : \mathbb{C}_q[M_n] \to \mathbb{C}_q[SU_n]$ restricts to an algebra isomorphism

$$\mathcal{H}_q(Gr_{n,r})^* \simeq \bigoplus_{k \in \mathbb{N}_0} H^0(\mathcal{E}_{-k}).$$

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