Continuous Cluster Expansion for Field Theories

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Abstract

A new version of the cluster expansion is proposed without breaking the translation and rotation invariance. As an application of this technique, we construct the connected Schwinger functions of the regularized \( \phi^4 \) theory in a continuous way.

keywords Constructive Field Theory; Cluster Expansion; Translation and Rotation Invariance; Connected Schwinger Function

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1. Introduction

As a constructive tool for field theories, the cluster expansion [19, 22] can be used to analyze the infinite volume limit in a rigorous manner. Contrary to the formal perturbative treatment, it is indeed convergent at least for weak coupling. Also its multiscale version, called phase cell expansion [3, 4], plays the central role in the constructive renormalization.

However, the traditional cluster expansion breaks the translation and rotation invariance explicitly, since it is based on a fixed discretization of space-time [7, 11, 18] or a wavelet decomposition of the fields [16]. Although some remedies [5, 9, 10] have been invented, none of them is fully satisfactory and a manifestly Euclidean invariant expansion is desired. Thanks to the Pauli’s principle, Fermionic field theories may be constructed continuously by some rearrangements and subtractions of the perturbative series [15, 17]. But continuous constructions of the Bosonic ones are more difficult, expect for some special cases such as (Sine-Gordon)\(_2\) with \( \beta < 4\pi \) [6, 12]. Recently, a new constructive technique for Bosonic field theories, called loop vertex expansion [20], has been well developed. As one of its successes, the new technique has been used to construct the regularized \( \phi^4 \) theory in a continuous way [21]. Unfortunately, when it is applied to the unregularized \( \phi^4 \) theory, the cluster expansion is still required to remove the volume cut-off [23].

In this paper, we provide a continuous version of the cluster expansion, which is based on dynamical discretizations of space-time instead of a fixed one and thus retains the translation and rotation invariance explicitly. As an example, we construct the connected Schwinger functions of the regularized \( \phi^4 \) theory in this way. Generalizations to other stable interactions of polynomial type are straightforward. Moreover, the phase cell expansion may be realized in a
similar way and then it seems promising to construct unregularized Bosonic field theories continuously. The progress in this direction will appear in future publications.

2. Expansion

Let us denote by $B$ the collection of all Borel subsets of $\mathbb{R}^d$ and by $B'$ the collection of all bounded ones in $B$. We consider a regularized $\phi^4$ theory with the interacting part restricted in $\Lambda \in B$. The generating functional of this theory is

$$Z_{\Lambda}[J] = \int d\mu(\phi) e^{-\lambda f_\Lambda dx \phi^4 + f_\Lambda dx J_x \phi} , \ J \in \mathcal{S}(\mathbb{R}^d).$$

(2.1)

Here $d\mu$ is the Gaussian measure on the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ with covariance

$$C_{x,y} = \frac{1}{(4\pi)^{d/2}} \int_0^1 \frac{d\alpha}{\alpha^{d/2}} e^{-\alpha - |x-y|^2/4\alpha},$$

(2.2)

which is a regularized version of the full covariance

$$((-\Delta+1)^{-1})_{x,y} = \frac{1}{(4\pi)^{d/2}} \int_0^\infty \frac{d\alpha}{\alpha^{d/2}} e^{-\alpha - |x-y|^2/4\alpha}. $$

(2.3)

The connected Schwinger functions of the theory are

$$S_{r;\Lambda;\omega_1,\ldots,\omega_r}^c = \frac{d\mu}{d\mu} \ln Z_{\Lambda} |_{J=0} \omega_1, \ldots, \omega_r \in \mathbb{R}^d.$$  

(2.4)

(The $Z_2$ symmetry of the theory will not be used in the following, since the purpose of this paper is to provide a new expansion for general cases.) What we are really interested in is the limit theory as $\Lambda$ approaches $\mathbb{R}$, which can be seen as a simplified model for a single slice in the renormalization group of the full $\phi^4$ theory [8, 13, 14].

For $n \geq 0$, we denote by $\mathcal{I}_n$ the set of all sequences $(t_j)_{j=1}^n$ with $t_1, \ldots, t_n \in [0,1]$ and by $\mathcal{X}_n$ the set of all sequences $(x_j)_{j=1}^n$ with $x_1, \ldots, x_n \in \mathbb{R}^d$ satisfying $|x_j - x_j'| > 1$ for $j \neq j'$. For $x \in \mathcal{X}_0$, let $B_x = B_x^c = \emptyset$. For $(x_j)_{j=1}^n \in \mathcal{X}_n$ with $n \geq 1$, let

$$B_{x_1,\ldots,x_n} = \{ y \in \mathbb{R}^d : \min_{1 \leq j \leq n} |y-x_j| \leq 1 \}$$

and $B_{x_1,\ldots,x_n}^c = B_{x_1,\ldots,x_n} \setminus B_{x_1,\ldots,x_{n-1}}$. We recursively define $C_{t,x}$ for $t \in \mathcal{I}_n$ and $x \in \mathcal{X}_n$ as follows. If $n = 0$, $C_{t,x} = C$. Otherwise,

$$C_{t,x} = t_n C_{t';x'} + (1-t_n) \left( \chi_{B_x} C_{t';x'} \chi_{B_x} + \chi_{(B_x)^c} C \chi_{(B_x)^c} \right)$$

(2.5)

for $t = (t',t_n)$ and $x = (x',x_n)$, where $\chi_A$ denotes the characteristic function of subset $A$ of $\mathbb{R}^d$. By the convexity of the linear combination (2.6), the interpolated covariance $C_{t,x}$ remains positive and has the bound

$$0 \leq (C_{t,x})_{x,y} \leq C_1 e^{-2|x-y|}$$

(2.7)
from (2.2). Let \( d\mu_{t,x} \) be the Gaussian measure on \( \mathcal{S}'(\mathbb{R}^d) \) with covariance \( C_{t,x} \) and let \( (F)_{t,x}^\phi \) be an abbreviation for \( \int d\mu_{t,x}(\phi) F[\phi] \). For \( t = (t_j)_{j=1}^n \in \mathcal{T}_n \) and \( x = (x_j)_{j=1}^n \in \mathcal{X}_n \) with \( n \geq 1 \),

\[
\frac{\partial C_{t,x}}{\partial t_n} = \sum_{k=1}^n \left( \prod_{l=k}^{n-1} t_l \right) \left( \chi_{B_{x_1} \ldots x_k} C_{\chi(B_{x_1} \ldots x_k)} + \chi(B_{x_1} \ldots x_k) C\chi_{B_{x_1} \ldots x_k} \right) \tag{2.8}
\]

and then, by the formula for infinitesimal change of covariance [7],

\[
\frac{\partial}{\partial t_n} (F)_{t,x}^\phi = \int_{(B_x)^c} dx \left( \int_{(B_x)^c} dy \left( \frac{\delta}{\delta \phi_x} \frac{\delta}{\delta \phi_y} F \right)_{t,x} \right) = \int_{(B_x)^c} dx \left( \sum_{k=1}^n \left( \prod_{l=k}^{n-1} t_l \right) \Delta_{x_1, \ldots, x_k}^\phi F \right)_{t,x}, \tag{2.9}
\]

where \( \Delta_{x_1, \ldots, x_k}^\phi \) is written simply for \( \int_{(B_x)^c} dy \frac{\delta}{\delta \phi_x} C_{x,y} \frac{\delta}{\delta \phi_y} \).

We define, for \( \Lambda \in \mathcal{B}' \) and \( (x_j)_{j=1}^n \in \mathcal{X}_n \) with \( n \geq 1 \),

\[
\tilde{Z}_{\Lambda; x_1, \ldots, x_n} = \left( \prod_{j=1}^{n-1} \int_{0}^{t_j} dt_j \right) \left( \prod_{j=2}^{n} \left( \prod_{k=1}^{j-1} t_k \right) \Delta_{x_1, \ldots, x_k}^\phi \right) e^{-\lambda f_{x_1} dx \phi_x^1 + \int_{A} dx J_x \phi_x} \nonumber
\]

\[
\bigg( (\prod_{j=2}^{n} \Delta_{x_1, \ldots, x_{\eta(j)}}^\phi) e^{-\lambda f_{x_1} dx \phi_x^1 + \int_{A} dx J_x \phi_x} \bigg)^{\phi}_{(t_j)_{j=1}^{n-1};(x_j)_{j=1}^{n-1}} \tag{2.10}
\]

In particular, \( \tilde{Z}_{\Lambda; x_1} = Z_{\Lambda} := \int d\mu(\phi) e^{-\lambda f_{x_1} dx \phi_x^1 + \int_{A} dx J_x \phi_x} \), which is actually independent of \( x_1 \). For \( n \geq 2 \), we call a map \( \eta: \{2, 3, \ldots, n\} \rightarrow \{1, 2, \ldots, n-1\} \) with \( \eta(j) < j \) an ordered tree with \( n \) vertices and denote by \( \mathcal{T}_n \) the set of all ordered trees with \( n \) vertices. Then \( \tilde{Z}_{\Lambda; x_1, \ldots, x_n} \) can be rewritten as

\[
\sum_{\eta \in \mathcal{T}_n} \left( \prod_{j=1}^{n-1} \int_{0}^{t_j} dt_j \right) \left( \prod_{j=2}^{n} \left( \prod_{k=1}^{j-1} t_k \right) \right) \left( \prod_{j=2}^{n} \Delta_{x_1, \ldots, x_{\eta(j)}}^\phi \right) e^{-\lambda f_{x_1} dx \phi_x^1 + \int_{A} dx J_x \phi_x} \nonumber
\]

\[
\bigg( (\prod_{j=2}^{n} \Delta_{x_1, \ldots, x_{\eta(j)}}^\phi) e^{-\lambda f_{x_1} dx \phi_x^1 + \int_{A} dx J_x \phi_x} \bigg)^{\phi}_{(t_j)_{j=1}^{n-1};(x_j)_{j=1}^{n-1}} \tag{2.11}
\]

Write simply \( Z_{\Lambda; x} = \tilde{Z}_{\Lambda \cap B_{x}; x} \) for \( \Lambda \in \mathcal{B} \). By taking

\[
F[\phi] = \left( \prod_{j=2}^{n} \sum_{k=1}^{j-1} \left( \prod_{l=k}^{j-2} t_l \right) \Delta_{x_1, \ldots, x_k}^\phi \right) e^{-\lambda f_{x_1} dx \phi_x^1 + \int_{A} dx J_x \phi_x} \tag{2.12}
\]

in (2.9) and performing the integrations over \( t_1, \ldots, t_n \), we have

\[
\tilde{Z}_{\Lambda; x} = Z_{\Lambda; x} + \int_{(B_x)^c} dz \tilde{Z}_{\Lambda; x, z}, \tag{2.13}
\]
where the first summand factorizes since both of the Gaussian measure at $t_n = 0$ and the integrand factorize. Applying (2.13) successively, we obtain, for $z_1 \notin B_x$,

$$Z_A \setminus B_x = Z_A \setminus B_x; z_1 = Z_A \setminus B_x; z_1 + \int_{(B_x)^c} dz_2 Z_A \setminus B_x; z_1, z_2 = \cdots$$

$$= \sum_{m \geq 1} \left( \prod_{j=1}^{m-1} \int_{(B_x)^c} dz_{j+1} \right) Z_A \setminus B_x; z_1, \ldots, z_m Z_A \setminus B_x; z_1, \ldots, z_m,$$

(2.14)

where the summands vanish for $m$ sufficiently large by the boundedness of $\Lambda$ and the ranges of integrations $(B_{x_1}, \ldots, z_i)^c$ can be replaced by $(B_{x_1}, \ldots, z_i)^c$. Then, dividing both sides of (2.14) by $Z_A$, we have the equation of Kirkwood-Salzburg type [1]

$$\frac{Z_A \setminus B_x; z_1}{Z_A} = \frac{Z_A \setminus B_x}{Z_A} - \left( Z_A \setminus B_x; z_1 - 1 \right) \frac{Z_A \setminus B_x; z_1}{Z_A} - \sum_{m \geq 2} \left( \prod_{j=1}^{m-1} \int_{(B_x)^c} dz_{j+1} \right) \frac{Z_A \setminus B_x; z_1, \ldots, z_m}{Z_A} \frac{Z_A \setminus B_x; z_1, \ldots, z_m}{Z_A},$$

(2.15)

which can be rewritten simply as $f_A = e + A_A f_A$. Here $e$ and $f_A$ are regarded as Borel measurable functions on $\bigcup_{n \geq 1} \mathcal{X}_n$ with $e_x = 1_{x \notin \mathcal{X}_n}$ and $f_A x = \frac{Z_A \setminus B_x}{Z_A}$, while $A_A$ is regarded as a linear operator with

$$(A_A f)_x, z_1 = 1_{x \notin \mathcal{X}_n} f_x - \left( Z_A \setminus B_x; z_1 - 1 \right) f_x, z_1 - \sum_{m \geq 2} \left( \prod_{j=1}^{m-1} \int_{(B_x)^c} dz_{j+1} \right) f_A x, z_1, \ldots, z_m.$$

(2.16)

(We say a function $f$ on $\bigcup_{n \geq 1} \mathcal{X}_n$ is Borel measurable iff $f|\mathcal{X}_n$ is Borel measurable for each $n \geq 1$.)

Denoting $Z_{r, A; w_1, \ldots, w_r} = \delta_{r, A} \cdot \frac{Z_A}{Z_A} |_{j=0}$ and $f_{r, A; w_1, \ldots, w_r} = \frac{Z_{r, A; w_1, \ldots, w_r}}{\delta_{r, A} \cdot \delta_{A}} f_A |_{j=0}$, we have

$$(1 - A_{0, A}) f_{r, A} = e \delta_{r, 0} + \sum_{0 < s \leq r} A_{s, A} f_{r - s, A}.$$  

(2.17)

Here $f_{r, A}$ is regarded as a Borel measurable function on $\mathbb{R}^{d \times r} \times \bigcup_{n \geq 1} \mathcal{X}_n$, while $A_{s, A}$ is regarded as a linear operator with $A_{0, A} = A_A |_{0}$ and, for $0 < s \leq r$, $A_{s, A}$ is regarded as a linear operator with $A_{0, A} = A_A |_{0}$ and, for $0 < s \leq r$,

$$(A_{s, A} f)_{w_1, \ldots, w_r; x, z_1} = - \sum_{I, I' \subset \{1, \ldots, r\}} \sum_{|I| = s, |I'| = \{1, \ldots, r\} \setminus I} \left( \prod_{j=1}^{m-1} \int_{(B_x)^c} dz_{j+1} \right) \frac{Z_{s, A; B_{x}^c; (w_i)_{i \in I}; z_1, \ldots, z_m f(w_{I'})_{i \in I' \setminus I}; x, z_1, \ldots, z_m}.$$  

(2.18)

Now letting $f_{r, A}$ and $A_r$ be the formal limits of $f_{r, A}$ and $A_{r, A}$ respectively as $A \to \mathbb{R}^d$, we have that

$$(1 - A_0) f_r = e \delta_{r, 0} + \sum_{0 < s \leq r} A_{s} f_{r - s}.$$  

(2.19)
Also we have the inequality

\[
\ell(T) \leq \min_{x} (Z_0(B_x); z_{1}, \ldots, z_{m} f_{x}, z_{1}, \ldots, z_{m})
\]

where the minimum is taken over all trees \(x\). We review the following two tree-lengths for equivalence of \(\ell\) with \(R\):

- For \(0 < s \leq r\),

\[
(A_s f)_{w_1, \ldots, w_r; x, z_1} = -\sum_{|I|=s, |I'|=r} \sum_{m \geq 2} (m-1) \prod_{j=1}^{m-1} \int_{(B_x, z_{1}, \ldots, z_{j})^c} dz_{j+1}^s Z_0(B_x); z_{1}, \ldots, z_{m} f_{x}, z_{1}, \ldots, z_{m} f_{x}, z_{1}, \ldots, z_{m}.
\]

(2.20)

and, for \(0 < s \leq r\),

\[
(A_s f)_{w_1, \ldots, w_r; x, z_1} = -\sum_{|I|=s, |I'|=r} \sum_{m \geq 2} (m-1) \prod_{j=1}^{m-1} \int_{(B_x, z_{1}, \ldots, z_{j})^c} dz_{j+1}^s Z_s(B_x); (w_i)_{i \in I}; z_{1}, \ldots, z_{m} f_{(w_i)_{i \in I'}}, z_{1}, \ldots, z_{m} f_{x}, z_{1}, \ldots, z_{m}.
\]

(2.21)

To exhibit the exponential decay of the connected Schwinger functions, we review the following two tree-lengths for \(x_1, \ldots, x_n \in \mathbb{R}^d\) [2]. One is

\[
\ell'_{x_1, \ldots, x_n} = \min_T \sum_{(i,j) \in T} |x_i - x_j|,
\]

(2.22)

where the minimum is taken over all trees \(T\) connecting \(x_1, \ldots, x_n\). The other is \(\ell_{x_1, \ldots, x_n} = \inf \ell'_{x_1, \ldots, x_n, y_1, \ldots, y_m}\), where the infimum is taken over all finite subsets \(\{y_1, \ldots, y_m\}\) of \(\mathbb{R}^d\) (including \(\emptyset\)). In particular, \(\ell'_{x_1} = \ell_{x_1} = 0\) and \(\ell'_{x_1, x_2} = \ell_{x_1, x_2} = |x_1 - x_2|\). It is easy to see that \(\ell'_{x_1, \ldots, x_n}\) and \(\ell_{x_1, \ldots, x_n}\) are symmetric in \(x_1, \ldots, x_n\). Also we have the inequality \(\frac{1}{s} \ell'_{x_1, \ldots, x_n} \leq \ell'_{x_1, \ldots, x_n} \leq \ell_{x_1, \ldots, x_n}\) [2], indicating the equivalence of \(\ell'_{x_1, \ldots, x_n}\) and \(\ell_{x_1, \ldots, x_n}\) in some sense. We can also define these two tree-lengths for nonempty subsets \(X_1, \ldots, X_n \subset \mathbb{R}^d\) as

\[
\ell'_{X_1, \ldots, X_n} = \min_T \sum_{(i,j) \in T} \inf_{x \in X_i, x' \in X_j} |x - x'|
\]

(2.23)

and \(\ell_{X_1, \ldots, X_n} = \inf \ell'_{X_1, \ldots, X_n, \{y_1\}, \ldots, \{y_m\}}\) with the infimum also taken over all finite subsets \(\{y_1, \ldots, y_m\}\) of \(\mathbb{R}^d\). Writing simply \(\ell_{x_1, \ldots, x_n; y_1, \ldots, y_m} = \ell\{x_1, \ldots, x_n\}; \{y_1, \ldots, y_m\}\), we list the following useful inequalities for \(\ell\) and leave the proofs of them to the reader:

\[
\ell_{x_1, \ldots, x_n} \leq \ell_{x_1, \ldots, x_n, x_{n+1}} \leq \ell_{x_1, \ldots, x_n} + \ell_{x_{n+1}},
\]

(2.24)

\[
\ell_{x_1, \ldots, x_n; y_1, \ldots, y_m} \geq \ell_{x_1, \ldots, x_n; y_{1}, \ldots, y_{2m}} \geq \ell_{x_1, \ldots, x_n; y_1, \ldots, y_m} - \ell_{y_m, \ldots, y_{2m}}.
\]

(2.25)

For \(r \geq 0\), let \(\mathcal{F}_r\) be the Banach space of Borel measurable functions on \(\mathbb{R}^{dr} \times \bigcup_{n \geq 1} \mathcal{X}_n\) with the norm

\[
\|f\|_r = \frac{1}{r} \sup_{n \geq 1} \sup_{x \in \mathcal{X}_n} \frac{e^{r e^\ell f}}{e^{r e^\ell f}} |f_{w, x}|.
\]

(2.26)
where $t_{w,x}$ is used to show the exponential decay of $f_{w,x}$ in $w$. In particular,

$$
\|f\|_0 = \sup_{n \geq 1} \sup_{x \in X_n} 2^{1-n} |f_x|.
$$

(2.27)

Also let $\|A\|_{s \to r} = \sup_{\|f\|_s = 1} \|Af\|_r$ for linear operator $A : F_s \to F_r$.

**Theorem 1.** For $\lambda > 0$ sufficiently small, we have $\|A_0\|_{r \to r} \leq \frac{3}{4}$ and, for $0 < s \leq r$, $\|A_s\|_{r-s \to r} \leq e^s$.

By Theorem 1, $(2.19)$ has a unique solution $(f^*_r)_{r \geq 0}$ with

$$
f^*_r = (1 - A_0)^{-1} (e \delta, 0 + \sum_{0 < s \leq r} A_s f^*_{r-s}) \in \mathcal{F}_r
$$

(2.28)

for $\lambda > 0$ sufficiently small. Since $\|e\|_0 = 1$, we have

$$
\|f^*_r\|_r \leq 4 (\delta, 0 + \sum_{0 < s \leq r} e^s \|f^*_{r-s}\|_{r-s})
$$

(2.29)

and then can obtain inductively $\|f^*_r\|_r \leq 4(5e)^r$.

By (2.14), we have

$$
\frac{\delta}{\delta J_{w_1}} Z_{\lambda} = \sum_{n \geq 1} \left( \prod_{j=1}^{n-1} \int_{(B_{x_1, \ldots, x_j})^c} dz_{j+1} \right) Z_{\lambda;x_1, \ldots, x_n} Z_{\lambda \setminus B_{x_1, \ldots, x_n}} |_{x_1 = w_1}
$$

$$
= \sum_{n \geq 1} \left( \prod_{j=1}^{n-1} \int_{(B_{x_1, \ldots, x_j})^c} dz_{j+1} \right) Z_{\lambda \setminus B_{x_1, \ldots, x_n}} Z_{\lambda \setminus B_{x_1, \ldots, x_n}} |_{x_1 = w_1},
$$

(2.30)

where we have used the fact $\frac{\delta}{\delta J_{w_1}} Z_{\lambda \setminus B_{x_1, \ldots, x_n}} |_{x_1 = w_1} = 0$. Then, for $w_1, \ldots, w_r \in \mathbb{R}^d$,

$$
S^c_{r, \lambda; w_1, \ldots, w_r} = \frac{\delta^{r-1}}{\delta J_{w_2} \cdots \delta J_{w_r}} \left( \frac{1}{Z_{\lambda}} \frac{\delta}{\delta J_{w_1}} Z_{\lambda} \right) |_{j=0}
$$

$$
= \sum_{n \geq 1} \left( \prod_{j=1}^{n-1} \int_{(B_{x_1, \ldots, x_j})^c} dx_{j+1} \right) \frac{\delta^{r-1}}{\delta J_{w_2} \cdots \delta J_{w_r}} \left( f_{\lambda;x_1, \ldots, x_n} \frac{\delta}{\delta J_{w_1}} Z_{\lambda;x_1, \ldots, x_n} \right) |_{j=0, x_1 = w_1}
$$

$$
= \sum_{1 \leq I \subseteq \{1, \ldots, r \}} \sum_{I' \subseteq \{1, \ldots, r \} \setminus I} \sum_{n \geq 1} \left( \prod_{j=1}^{n-1} \int_{(B_{x_1, \ldots, x_j})^c} dx_{j+1} \right) Z_{|I|, \lambda; (w_i)_{i \in I}; x_1, \ldots, x_n} |_{x_1 = w_1},
$$

(2.31)

which can be rewritten as $S^c_{r, \lambda} = \sum_{1 \leq s \leq r} T_{s, \lambda} f^*_{r-s, \lambda}$ with

$$(T_{s, \lambda} f)_{w_1, \ldots, w_r} = \sum_{1 \leq I \subseteq \{1, \ldots, r \}} \sum_{I' \subseteq \{1, \ldots, r \} \setminus I} \sum_{n \geq 1} \left[ \prod_{j=1}^{n-1} \int_{(B_{x_1, \ldots, x_j})^c} dx_{j+1} \right] Z_{|I|, \lambda; (w_i)_{i \in I'; x_1, \ldots, x_n}} |_{x_1 = w_1},$$

(2.32)
\[
\left( \prod_{j=1}^{n-1} \int_{(B_{x_1},...,x_j) \in \mathbb{R}^d} dx_{j+1} \right) Z_{s, \Lambda; (w_i)_{i \in I}} x_{j+1} f(w_i) x_{j+1} \bigg| x_{j+1} = w_{j+1} \right). \tag{2.32}
\]

The formal limit of \( S_t^c \) as \( \Lambda \to \mathbb{R}^d \) is \( S_t^c = \sum_{1 \leq s \leq r} T_s f_{r-s}^s \)
with
\[
(T_s f)_{w_1,...,w_r} = \sum_{1 \leq |I| \leq s, |I'| \leq r} \sum_{n \geq 1} \left( \prod_{j=1}^{n-1} \int_{(B_{x_1},...,x_j) \in \mathbb{R}^d} dx_{j+1} \right) Z_{s, \Lambda; (w_i)_{i \in I}} x_{j+1} f(w_i) x_{j+1} \bigg| x_{j+1} = w_{j+1} \right).
\tag{2.33}
\]

For \( r \geq 1 \), let \( \mathcal{F}_r \) be the Banach space of Borel measurable functions \( f: \mathbb{R}^d \to \mathbb{R} \)
with the norm
\[
\|f\|'_r = \frac{1}{\mathcal{F}_r} \text{ess sup} e^{\frac{\ell}{\mathcal{F}_r}}|f_w|,
\tag{2.34}
\]
where \( \ell_w \) is used to show the exponential decay of \( f_w \) in \( w \). Also let
\[
\|T\|'_{s \to r} = \sup_{\|f\|_s = 1} \|Tf\|'_r.
\tag{2.35}
\]

for linear operator \( T: \mathcal{F}_s \to \mathcal{F}_r' \).

**Theorem 2.** For \( \lambda > 0 \) sufficiently small and \( 1 \leq s \leq r \), \( \|T\|'_{s \to r} \leq c^s \).

By Theorem 2, we have, for \( \lambda > 0 \) sufficiently small,
\[
\|S_t^c\|'_r \leq \sum_{1 \leq s \leq r} \|T_s\|'_r \leq \sum_{1 \leq s \leq r} c^s \cdot 4(5c)^{r-s} \leq (5c)^r,
\tag{2.36}
\]
which is equivalent to
\[
|S_t^c(x)_{w_1,...,w_r}| \leq (5c)^r e^{-\ell_{w_1,...,w_r}} w_1,...,w_r \in \mathbb{R}^d.
\tag{2.37}
\]

**3. Estimation**

Let \( \mathcal{H}_n \) be the set of all sequences \( (h_j)_{j=1}^n \) with \( h_j \in \mathbb{N} \). For \( (h_j)_{j=1}^n, (h_j')_{j=1}^n \in \mathcal{H}_n \),
we write \( (h_j)_{j=1}^n \leq (h_j')_{j=1}^n \) iff \( h_j \leq h_j' \) for \( 1 \leq j \leq n \) and \( (h_j)_{j=1}^n \pm (h_j')_{j=1}^n := (h_j \pm h_j')_{j=1}^n \). We define, for \( x = (x_j)_{j=1}^n \in \mathcal{X}_n \) and \( h = (h_j)_{j=1}^n \in \mathcal{H}_n \),
\[
\|F\|_{\mathcal{F}_s}^\phi = \left( \prod_{j=1}^n \frac{1}{h_j} \prod_{k=1}^n \int_{B_{x_j}} dy_{j,k} \right) \left( \prod_{j=1}^n \prod_{k=1}^n \frac{\delta}{\phi_{y_{j,k}}} \right) F \tag{3.1}
\]

Then we have
\[
\|F_1 F_2\|_{\mathcal{F}_s}^\phi = \left( \prod_{j=1}^n \frac{1}{h_j} \prod_{k=1}^n \int_{B_{x_j}} dy_{j,k} \right) \left( \prod_{j=1}^n \prod_{k=1}^n \frac{\delta}{\phi_{y_{j,k}}} \right) F_1 F_2.
\]
and in general

\[ \| F_1 \cdots F_r \|_{x:h}^\phi \leq \sum_{h_1, \ldots, h_r \in H_n} \| F_1 \|_{x:h_1}^\phi \cdots \| F_r \|_{x:h_r}^\phi \]  

(3.3)

In order to deal with some singular functional conveniently in the following, we regard them as signed measures on \( \mathbb{R}^d \). (For signed measure \( \nu \), we have the Jordan decomposition \( \nu = \nu^+ - \nu^- \), where \( \nu^+ \), \( \nu^- \) are the positive and negative variations of \( \nu \), and we have the total variation of \( \nu \) as \( |\nu| = \nu^+ + \nu^- \). Furthermore, for signed measures \( \nu_1 \), \( \nu_2 \), we write \( \nu_1 \geq \nu_2 \) iff \( \nu_1 - \nu_2 \) is a positive measure. In particular, we have \( \delta_z(x) = \delta_z(x) \geq 0 \).)

Then it is easy to show that \( \sum_{h \in H_n} \| \phi_z \|_{x:h} \leq 1 + |\phi_z| \) and, by (3.3),

\[ \sum_{h \in H_n} \| \prod_{k=1}^r \phi_{z_k} \|_{x:h}^\phi \leq \sum_{h_1, \ldots, h_r \in H_n} \sum_{h_1 + \cdots + h_r = h} \prod_{k=1}^r \| \phi_{z_k} \|_{x:h_k}^\phi \]

(3.4)

Also we have the following bound:

**Lemma 1.** For \( x \in X_n \), \( h \in H_n \) and \( \lambda \geq 0 \),

\[ \left\| e^{-\lambda f_{1,n} x} \phi_1^4 \right\|_{x:h} \leq e^{c_2 \lambda n}. \]  

(3.5)

**Proof.** For any \( D \in B' \), we have

\[
\left( \prod_{k=1}^h \int_D dy_k \right) \left( \prod_{k=1}^h \delta \phi_{y_k} \right) e^{-\lambda f_D x} \phi_1^4 \left( \sum_{p_1, \ldots, p_4 \geq 0} \prod_{r=1}^4 \frac{h^!}{p_r!} \lambda^{p_r} \prod_{r=1}^4 \frac{4!}{(4-r)!} \right) \]

(3.6)
with \( c = \sup_{\xi \geq 0} \left( (1+\xi)^4 - 2\xi^4 \right) \). Then, for \( x = (x_j)_{j=1}^n \in \mathcal{X} \) and \( h = (h_j)_{j=1}^n \in \mathcal{H} \),

\[
\|e^{-\lambda f_{\alpha} dx} \|_{x,h}^\phi = e^{-\lambda f_{\alpha} dx} \|_{x,h}^\phi \prod_{j=1}^n \frac{1}{h_j!} \int_{\Lambda \cap B'_{x_1,\ldots,x_j}} dy_{j,k} \left( \prod_{k=1}^{h_j} \delta_{\phi_{y_{j,k}}} \right) e^{-\lambda f_{\alpha} dx} \|_{x,h}^\phi \leq e^{\text{csv} f_{\alpha}} \quad (3.7)
\]

where we use the facts that the range of the integration over \( y_{j,k} \) can be replaced by \( \Lambda \cap B'_{x_1,\ldots,x_j} \) here and \( |\Lambda \cap B'_{x_1,\ldots,x_j}| \) can be bounded by \( v_d \), the volume of the unit ball in \( d \) dimensions.

For \( (x_j)_{j=1}^n \in \mathcal{X} \) and \( \eta \in \mathcal{T} \), we have

\[
\left\| \left( \prod_{j=1}^n \Delta_{x_1}^{\phi} \right) e^{-\lambda f_{\alpha} dx} \|_{x,h}^\phi \right\|_{(x_j)_{j=1}^n,1} \leq \left( \prod_{j=1}^{n-1} d_{\eta}(j)! \right) \left( \prod_{j=2}^n \sup_{y \in B_{x_1}(j)} C_{x_1,y} \right) \left\| \left( \prod_{j=2}^n \delta_{\phi_{x_1,j}} \right) e^{-\lambda f_{\alpha} dx} \|_{x,h}^\phi \right\|_{(x_j)_{j=1}^n,1} d_{\eta} \quad (3.8)
\]

with \( d_{\eta}(j) = |\eta^{-1} \{ j \}| \) and \( d_{\eta} = (d_{\eta}(j))_{j=1}^n \). Then, by (3.2), Lemma 1 and the fact \( \sup_{y \in B_{x_1}(j)} C_{x_1,y} \leq c_1 e^{2|y_j - x_{\eta}(j)|} \), we can continue with

\[
\left\| \sum_{L,L' \subseteq \{2,\ldots,n\}} (4\lambda)^{|L'|} \left( \prod_{l \in L'} \phi_{x_1,l}^3 \right) \left( \prod_{l \in L} \delta_{\phi_{x_1,l}} \right) F \right\|_{(x_j)_{j=1}^n,1} d_{\eta} \leq \left( c_1 e^{2+2\lambda} \right)^{n-1} \left( \prod_{j=1}^{n-1} d_{\eta}(j)! \right) e^{-\sum_{j=1}^n |x_j - x_{\eta}(j)|} U_{x_1,\ldots,x_n}^\phi(F),
\]

where \( U_{x_1,\ldots,x_n}^\phi(F) \) is an abbreviation for

\[
\sum_{L,L' \subseteq \{2,\ldots,n\}} (4\lambda)^{|L'|} \left( \prod_{l \in L'} \phi_{x_1,l}^3 \right) \left( \prod_{l \in L} \delta_{\phi_{x_1,l}} \right) F \right\|_{(x_j)_{j=1}^n,1} d_{\eta} \quad (3.10)
\]

We now deal with the three factors \( U_{x_1,\ldots,x_n}^\phi(F) \), \( e^{-\sum_{j=1}^n |x_j - x_{\eta}(j)|} \) and \( \prod_{j=1}^{n-1} d_{\eta}(j)! \) in (3.9) one by one. First, let us consider \( U_{x_1,\ldots,x_n}^\phi(F) \) with \( F = \phi_{x_1} \cdots \phi_{x_n} \):

\[
U_{x_1,\ldots,x_n}^\phi(\phi_{x_1} \cdots \phi_{x_n}) \leq \sum_{L,L' \subseteq \{2,\ldots,n\}} \sum_{\tau: L \rightarrow \tau(L))} (4\lambda)^{|L'|} \left( \prod_{l \in L} \delta_{\phi_{x_1,l}} \right)
\]

9
The inductive assumption, we have without loss of generality. By performing integration by parts and applying the following lemma, which is needed to bound the expectations $(U_{x_1,\ldots,x_n}^\phi(\phi_{w_1} \cdots \phi_{w_r}))_{t:T}$.

**Lemma 2.** For $w_1,\ldots, w_r \in \mathbb{R}^d$ and $(x_1, \ldots, x_n) \in \mathcal{X}_n$, 

$$\left\langle \prod_{i=1}^r (1+|\phi_{w_i}|) \prod_{j=1}^n (1+|\phi_{x_j}|)^3 \right\rangle_{t:T} \leq c_3 r! (r!)^{1/2}. \quad (3.12)$$

**Proof.** First, we claim that, for $\sum_{j=1}^n s_j = 2s$ even and $(x_1, \ldots, x_n) \in \mathcal{X}_n$, 

$$\left| \left\langle \prod_{j=1}^n \phi_{x_j}^{s_j} \right\rangle_{t:T} \right| \leq c^s \prod_{j=1}^n (s_j!)^{1/2}, \quad (3.13)$$

which is proved by induction on $s$ and is trivial for $s = 0$. Assuming it holds for $s = \bar{s} - 1$, we consider the case $s = \bar{s}$ and assume further $s_1 = \max_{j=1}^n s_j$ without loss of generality. By performing integration by parts and applying the inductive assumption, we have

$$\left| \left\langle \prod_{j=1}^n \phi_{x_j}^{s_j} \right\rangle_{t:T} \right| = \left| \int_{\mathbb{R}^d} dx \left( C_{t:t} \right)_{x_1,x} \delta_{\phi_{x_1}^{s_1-1}} \left( \prod_{j=2}^n \phi_{x_j}^{s_j} \right)_{t:T} \right|$$

$$\leq (s_1 - 1) \left( C_{t:t} \right)_{x_1,x_1} \left| \left\langle \phi_{x_1}^{s_1-2} \prod_{j=2}^n \phi_{x_j}^{s_j} \right\rangle_{t:T} \right| + \sum_{k=2}^n s_k \left( C_{t:t} \right)_{x_1,x_k} \left| \left\langle \phi_{x_1}^{s_1-1} \phi_{x_k}^{s_k-1} \prod_{j=2}^n \phi_{x_j}^{s_j} \right\rangle_{t:T} \right|$$

$$\leq c^{\bar{s}-1} \prod_{j=1}^n (s_j!)^{1/2} \sum_{j=1}^n \left( C_{t:t} \right)_{x_1,x_j}, \quad (3.14)$$

where $\phi_{x}^{-1}$ is temporarily defined as 0. Since 

$$\sum_{j=1}^n \left( C_{t:t} \right)_{x_1,x_j} \leq c_1 \sum_{j=1}^n e^{-2|x_1-x_j|} \leq \frac{2d c_1 e}{v_d} \sum_{j=1}^n$$

$$\int_{|x-x_j| \leq 1/2} dx \ e^{-2|x_1-x|} \leq \frac{2d c_1 e}{v_d} \int_{\mathbb{R}^d} dx \ e^{-2|x|} < \infty, \quad (3.15)$$
we can advance the induction for $c$ sufficiently large and thus complete the proof of the claim.

Then, by the Cauchy-Schwarz inequality,

$$\left\langle \prod_{i=1}^{r} |\phi_{w_i}| \prod_{j=1}^{n} |\phi_{x_j}| \right\rangle_{t,x}^{\phi} \leq \left( \left\langle \prod_{i=1}^{r} \phi_{w_i}^{2} \right\rangle_{t,x}^{\phi} \left\langle \prod_{j=1}^{n} \phi_{x_j}^{2} \right\rangle_{t,x}^{\phi} \right)^{1/2} \leq \left( (2c_1)^r r! \prod_{j=1}^{n} c_{j}^{1/2} (2r_j)^{1/2} \right)^{1/2}, \tag{3.16}$$

which yields

$$\left\langle \prod_{i=1}^{r} (1+|\phi_{w_i}|) \prod_{j=1}^{n} (1+|\phi_{x_j}|) \right\rangle_{t,x}^{3} \leq 2^{r+3n} \sup_{l \subset \{1, \ldots, r\}} \sup_{0 \leq r_j \leq 3} \left\langle \prod_{i \in l} |\phi_{w_i}| \prod_{j=1}^{n} |\phi_{x_j}| \right\rangle_{t,x}^{\phi} \leq 2^{r+3n} \left( (2c_1)^r (\sqrt{6} c^{3})^{n} r! \right)^{1/2}. \tag{3.17}$$

$\square$

The factor $e^{-2 \sum_{j=1}^{n} |x_j - x_{n(j)}|}$ will play two roles in our derivation. One half of the factor is used later to extract some exponential decay in external points $w_1, \ldots, w_r$. The other half is used to control the integrations over $x_2, \ldots, x_n$ as follows. By (3.11) and Lemma 2, we have, for $t \in T_n$ and $\eta \in T_n$,

$$\left( \prod_{j=1}^{n} \int_{(B_{x_1,\ldots,x_j})^{c}} dx_{j+1} \right) e^{-\sum_{j=2}^{n} |x_j - x_{n(j)}|} \left\langle U_{x_1,\ldots,x_n}^{\phi} (\phi_{w_1}^{1} \cdots \phi_{w_r}^{1}) \right\rangle_{t(x_j)}^{\phi} \leq \left( \prod_{j=1}^{n} \int_{(B_{x_1,\ldots,x_j})^{c}} dx_{j+1} \right) e^{-\sum_{j=2}^{n} |x_j - x_{n(j)}|} \sum_{L \subset \{2,\ldots,n\}} \sum_{L' \subset \{2,\ldots,n\} \setminus L} \sum_{\tau: [L] \rightarrow [1,\ldots,r]} \left( 4\lambda \right)^{|L|'} \left( \prod_{l \in L'} \delta_{x_l, x_{\tau(l)}} \right) \left( \prod_{l \in L} (1+|\phi_{x_l}|)^3 \right) \prod_{i \in \{1,\ldots,r\} \setminus \tau(L)} \left( 1+|\phi_{w_i}| \right)_{t(\tau)}^{\phi} \leq \sum_{L \subset \{2,\ldots,n\}} \sum_{|L| \leq r} \sum_{\tau: [L] \rightarrow [1,\ldots,r]} (4c_3 c_5 \lambda)^{n-1-|L|} c_{3}^{-1} \left( 4c_3 c_5 \lambda \right)^{|L|} \left( r-|L| \right)^{1/2} \leq c_3^{r} \left( 8c_3 c_5 \lambda \right)^{n-1} \max \left\{ (4c_3 c_4 c_5 \lambda)^{-\min\{r,n-1\}}, 1 \right\} \tag{3.18}$$

with $c_5 = \int_{\mathbb{R}^d} dx e^{-|x|}$. In the last line of (3.18), we use the facts that the number of subsets of $\{2,\ldots,n\}$ is $2^{n-1}$ and the number of injective maps from $L$ to $\{1,\ldots,r\}$ with $|L| \leq r$ is $\frac{r!}{(r-|L|)!}$. We assume from now on that $\lambda < (4c_3 c_4 c_5)^{-1}$.

To control the factor $\prod_{j=1}^{n} d_{\eta}(j)!$, we need the following lemma:
Lemma 3 (Speer [16]). For \( n \geq 2 \),
\[
\sum_{\eta \in \mathcal{T}_n} \left( \prod_{j=1}^{n-1} d_{\eta(j)}! \right) \left( \prod_{j=1}^{n-1} \int_0^1 dt_j \right) \prod_{j=2}^{n} \prod_{l=\eta(j)}^{j-2} t_l = \frac{1}{n} \left( \frac{2n-2}{n-1} \right) \leq 4^{n-1}. \tag{3.19}
\]

Proof. For completeness, we give a new proof for this lemma. Let
\[
\omega = x(1-x)^{-1} = \frac{1 - \sqrt{1 - 4x}}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{2n-2}{n-1} \right) x^n. \tag{3.20}
\]

Given \( \eta \in \mathcal{T}_n \), we have
\[
\prod_{j=1}^{n} \left( 1 - \omega \prod_{l=j}^{n-1} t_l \right)^{-d_{\eta(j)}-1} - 1 = \int_0^1 dt_n \frac{\partial}{\partial t_n} \prod_{j=1}^{n} \left( 1 - \omega \prod_{l=j}^{n} t_l \right)^{-d_{\eta(j)}-1} = \sum_{k=1}^{n} (d_{\eta(k)}+1) \int_0^1 dt_n \left( \prod_{j=1}^{n-1} t_l \right)^x (1-\omega)^{-1} \prod_{j=1}^{n} \left( 1 - \omega \prod_{l=j}^{n} t_l \right)^{-d_{\eta(j)}-\delta_{j,k}} - 1, \tag{3.21}
\]
where \( d_{\eta}(n) \) is defined as 0. Multiplying both sides of (3.21) by
\[
\left( \prod_{j=1}^{n-1} d_{\eta(j)}! \right) \left( \prod_{j=2}^{n} \prod_{l=\eta(j)}^{j-2} t_l \right), \tag{3.22}
\]
performing integrations over \( t_1, \ldots, t_{n-1} \) and summing over all \( \eta \in \mathcal{T}_n \), we obtain
\[
A_n(x) - B_n = x A_{n+1}(x) \text{ with}
\]
\[
A_n(x) = \sum_{\eta \in \mathcal{T}_n} \left( \prod_{j=1}^{n-1} d_{\eta(j)}! \right) \left( \prod_{j=1}^{n-1} \int_0^1 dt_j \right) \left( \prod_{j=2}^{n} \prod_{l=\eta(j)}^{j-2} t_l \right) \prod_{j=1}^{n} \left( 1 - \omega \prod_{l=j}^{n} t_l \right)^{-d_{\eta(j)}-1} \tag{3.23}
\]
and
\[
B_n = \sum_{\eta \in \mathcal{T}_n} \left( \prod_{j=1}^{n-1} d_{\eta(j)}! \right) \left( \prod_{j=1}^{n-1} \int_0^1 dt_j \right) \prod_{j=2}^{n} \prod_{l=\eta(j)}^{j-2} t_l, \tag{3.24}
\]
which yields
\[
\omega = x(1-x)^{-1} = x + x^2 \int_0^1 dt_1 (1-t_1\omega)^{-2} (1-\omega)^{-1} = x + x^2 A_2(x)
= x + x^2 B_2 + x^3 A_3(x) = \cdots = x + \sum_{n \geq 2} x^n B_n. \tag{3.25}
\]

Comparing (3.20) and (3.25), we complete the proof. \( \square \)
Now we are ready to perform the estimation for $Z_{r,\Lambda;w_1,\ldots,w_r;x_1,\ldots,x_n}$ with $n \geq 2$. First, we have

$$|Z_{r,\Lambda;w_1,\ldots,w_r;x_1,\ldots,x_n}| \leq \sum_{\eta \in T_n} \left( \prod_{j=1}^{n-1} \int_0 \int_{x_j}^{x_{j+1}} dt_j \right) \left( \prod_{j=2}^{n} \prod_{t_1=t_{(j)}}^{t_{j-1}} t_i \right)

\left\langle \left( \prod_{j=2}^{n-1} \prod_{t_1=t_{(j)}}^{t_{j-1}} t_i \right) \prod_{j=2}^{n} \prod_{t_1=t_{(j)}}^{t_{j-1}} t_i \right) \right) \left( c_1 e^{2+c_2 \lambda} \right)^{n-1}

\left( e^{-2 \sum_{j=2}^{n-1} \prod_{j=2}^{n} \prod_{t_1=t_{(j)}}^{t_{j-1}} t_i \right) \left( c_1 e^{2+c_2 \lambda} \right)^{n-1} \left( 4c_3 c_4 c_5 \right)^{n-1}. \quad (3.27)

Using (3.18), Lemma 3 and the fact $\sum_{j=2}^{n} \prod_{j=2}^{n} \prod_{t_1=t_{(j)}}^{t_{j-1}} t_i \right), we obtain that

$$\left( \prod_{j=1}^{n-1} \int_0 \int_{x_j}^{x_{j+1}} dx_{j+1} \right) \left( c_1 e^{2+c_2 \lambda} \right)^{n-1} \left( e^{-2 \sum_{j=2}^{n-1} \prod_{j=2}^{n} \prod_{t_1=t_{(j)}}^{t_{j-1}} t_i \right) \left( c_1 e^{2+c_2 \lambda} \right)^{n-1} \left( 4c_3 c_4 c_5 \right)^{n-1}. \quad (3.27)

Proof of Theorem 1. For $w \in \mathbb{R}^{d_r}$ and $(x, z_1) \in \mathcal{X}_{n+1}$,

$$|A_0 f|_{\mathbb{R}^{\mathbb{R}_w}} \leq 1_{x \in \mathcal{X}_n} |f|_{\mathbb{R}^{\mathbb{R}_w}} + |Z_{0,(B_\lambda)}:z_1-1| |f|_{\mathbb{R}^{\mathbb{R}_w}} + \sum_{m \geq 2} \left( \prod_{j=1}^{m-1} \int_{(B_\lambda)_j}^{(B_\lambda)_j+1} dt_{j+1} \right) |Z_{0,(B_\lambda)}:z_1,\ldots,z_m| |f|_{\mathbb{R}^{\mathbb{R}_w}} + \sum_{m \geq 2} \left( \prod_{j=1}^{m-1} \int_{(B_\lambda)_j}^{(B_\lambda)_j+1} dt_{j+1} \right) |Z_{0,(B_\lambda)}:z_1,\ldots,z_m| \left( e^{-\ell_{w,x,z_1}} + \sum_{m \geq 2} 2^{n+m-1} \right). \quad (3.28)

By the inequalities $\ell_{w,x,z_1} \geq \ell_{w,x,z_1} + \sum_{m \geq 2} 2^{n+m-1}$, we have

$$\|A_0 f\|_{r \to r} = \sup_{\|f\|_r = 1} \|A_0 f\|_r

\leq \operatorname{ess \ sup}_{(x, z_1) \in \mathcal{X}_{n+1}} \left( \prod_{j=1}^{m-1} \int_{(B_\lambda)_j}^{(B_\lambda)_j+1} dt_{j+1} \right) |Z_{0,(B_\lambda)}:z_1,\ldots,z_m| \left( e^{\ell_{z_1,\ldots,z_m}} \right). \quad (3.29)
Proof of Theorem 2. For $1 \leq s \leq r$ and $w_1, \ldots, w_r \in \mathbb{R}^d$, \[ |\mathbf{T}_s \mathbf{f}|_{w_1, \ldots, w_r} \leq (r-s)! \|\mathbf{f}\|_{r-s} \sum_{1 \in I, |I|=s, I'=(1, \ldots, r)} \sum_{n \geq 1} \lambda e^{\ell_{z_1, \ldots, z_m} + s} \leq 4^{r-s} \left(4 c_3 \left(1 + 32c_2^2c_3^{-1}\right)^{m-1}\right)^s \] for $\lambda$ sufficiently small. \[
abla
\]
\[
2^{n-1} \left( \prod_{j=1}^{n-1} \int_{(B_{x_1},...,x_j)^{d'}} dx_{j+1} \right) \left| Z_{s, \Lambda, \mathbb{R}^d; (w_i)_{i \in I}; x_1, ..., x_n} \right|_{x_1 = w_1},
\]

where

\[
\ell(w_i)_{i \in I'; x_1, ..., x_n} \geq \ell(w_i)_{i \in I; x_1, ..., x_n} - \ell(w_i)_{i \in I; x_1, ..., x_n}
\]
\[
\geq \ell(w_1, ..., w_{r} - \ell(x_1, ..., x_{n} + 1 - s)
\]

for \( x_1 = w_1 \) and \( \{w_i \mid i \in I\} \subset B_{x_1, ..., x_n} \). Then

\[
\| T_s \|_{r-s \to r} \leq \frac{(r-s)!}{r^r} \text{ess sup}_{w_1, ..., w_r \in \mathbb{R}^d} \sum_{I, I' \subset \{1, ..., r\}} \sum_{I \cap I' = \emptyset} \sum_{n \geq 1} 2^{n-1} \left( \prod_{j=1}^{n-1} \int_{(B_{x_1},...,x_j)^{d'}} dx_{j+1} \right) \left| Z_{s, \mathbb{R}^d; (w_i)_{i \in I}; x_1, ..., x_n} \right|_{x_1 = w_1}
\]
\[
\leq (ec)^s \sum_{n \geq 1} \left( 64c_1c_4c_5 \lambda e^{2+c_2\lambda} \right)^{n-1} (4c_3c_4c_5 \lambda)^{-\min\{s,n-1\}}
\]
\[
\leq (ec^3(1+32e^2c_1c_3^{-1}))^s (3.36)
\]

for \( \lambda \) sufficiently small.

\[ \square \]

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