New Solutions of the 2 + 1 Dimensional Einstein Gravity Coupled to Maxwell Power type Non Linear Electric field with Dilaton field

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Abstract

New solutions are derived in the 2 + 1 gravity which is coupled to $|F|^k$ type non-linear electric field in Maxwell Power theory with dilaton field. We obtain consistent solutions in general $k$ case. We also investigate the behavior of the metric function with the space-time singularity. Then, we found some black hole solutions when the space-time has a singular point at $r = 0$. Addition, we derive the Brown-York mass when the space-time represents black hole.

1 Introduction

Recently, one of the most standard theory of gravitation is the Einstein’s general theory of relativity (GR). However, GR is the classical theory, so we have to quantize the theory. To construct the quantum GR, many modified theories of gravitation had suggested. As there is a Schwarzschild black hole (BH) solution in the GR, modified gravity also have characteristic solutions with theory to theory. That is why to find a new solution in the theory of gravitation is important.

The 2 + 1 gravity has been suggested by Deser and Jackiw [1]. This theory has the characteristic structure as follows. At first, lower dimensional theory of gravitation has fewer unknown functions than higher dimensional theory. That is why to derive new exact solutions in 2 + 1 space-time is more easier than higher dimensional one. Next, it is conformal flat theory because Weyl tensor always vanishes. This point is compatible to the conformal flat quantum theories (CFT). In addition, the lack of degrees of freedom tends to gravitational wave does not propagate. Carlip reviewed such as interesting properties of the 2 + 1 gravity [2].

This theory came to attract attention after the discovery of BTZ solution [3]. A solution represents the black hole which is in the AdS space-time. Note that BTZ black hole is not asymptotically flat like Schwarzschild black hole. After their research, a lot of solutions were derived with many types of matter fields in the 2 + 1 space-time.

In the present article, let us treat non linear electric field (NLEF) and dilaton field as matters. To the past research, Cataldo find exact solutions with Born-Infeld [4] type and $|F^{3/4}|$ type NLEF [5], [6]. Note that $\mathcal{F}$ is the Maxwell invariant which is defined as $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electric tensor in which $A_\mu$ is vector potential. Gurtug found an exact solution with $|F^k|$ ($k$ is a real number) type NLEF which is called Maxwell Power (MP) theory [7]. They are circular symmetric static solution with electric field. Moreover, solutions in the general dimension are derived by some authors [8], [9], [10]. Then, Hendi found that MP type NLEF is related to the spherically symmetric solution of the $F(R) = R^k$ type modified gravity [11]. Recently, new solutions were derived with scalar hair in the AdS space-time [12].

To treat the dilaton field is meaning full because it is related to the open and the closed bosonic string theory (on our paper, we however neglect the Kolb-Ramond antisymmetric tensor field). Dilaton field is also important for the theory of gravitation because it has correspondence to the modified gravity. Since to obtain solutions of the theory is important, researchers find solutions in the theory of gravitation with dilaton field as follows. Gibbons and Maeda found a electrically charged black hole solution in 3 + 1 space-time [13]. In the 2 + 1 gravity, Sá and Lemos found the black hole solution without electric
field [14]. Then, Chan and Mann discovered some solutions with linear electric field [15]. After their research, Yamazaki and Ida introduced Born-Infeld type NLEF and found a new solution [16]. However, MP type NLEF with dilaton field had not considered. That is why we try to find new solutions of the 2+1 gravity which is coupled to MP type NLEF with dilaton field.

In this article, we take natural unit which is \( c = 8\pi G = 1 \), where \( c \) is speed of light, \( G \) is Newton constant in the 2+1 space-time. Additionally, we set the metric signature as \((-+++)\).

## 2 Circular symmetric static solutions of the 2+1 gravity which is coupled to \( F^k \) type NLEF with dilaton field

The action of the 2+1 gravity which is coupled to \( F^k \) type NLEF with dilaton field is

\[
I = \frac{1}{2} \int d^3x \sqrt{-g} \times \left\{ R - \alpha (\partial \phi)^2 - e^{-\phi} |F|^k - 2e^{\beta \phi} \Lambda \right\},
\]

(1)

Where \( g \) is determinant of the metric \( g_{\mu\nu} \), \( R \) is Ricci scalar, \( \phi \) is the dilaton field, \( \alpha \) and \( \beta \) are constants. We get Einstein equation when we take the variation of \( g_{\mu\nu} \),

\[
G^\mu_\nu + \Lambda e^{\beta \phi} \delta^\mu_\nu = T^\mu_\nu,
\]

(2)

where \( T^\mu_\nu \) is energy-momentum tensor

\[
T^\mu_\nu = \alpha \partial^\mu \partial_\nu \phi - \frac{\alpha}{2} (\partial \phi)^2 \delta^\mu_\nu - |F|^k e^{-\phi} \left( \delta^\mu_\nu - \frac{4kF^{\mu\nu}F_{\mu\nu}}{|F|^k} \right).
\]

(3)

On the other hand, let us take the variation of (1) respect to \( \phi \), we get the following dilaton equation

\[
\alpha \nabla^\mu (\partial_\mu \phi) + \frac{e^{-\phi}}{2} |F|^k - \beta e^{\beta \phi} \Lambda = 0,
\]

(4)

In addition, electric field equation is derived from the variation of (1) which is respect to \( A_\mu \)

\[
\partial_\mu \left( \sqrt{-g}e^{-\phi} F^{\mu\nu} |F|^{k-1} \right) = 0.
\]

(5)

### 2.1 Solution of general \( k \) case

Now, let us solve above equations under the circular symmetric static ansatz. We set metric ansatz as

\[
ds^2 = -f(r) dt^2 + \frac{e^{2\delta(r)} dr^2}{f(r)^2} + r^2 d\theta^2,
\]

(6)

where \( f(r) \), \( \delta(r) \) are unknown functions which is depended only radial coordinate \( r \). Following discussion, we assume dilaton field is depended only \( r \). Beside, electric field ansatz is

\[
F = e^{D\phi(r)+\delta(r)} E(r) dt \wedge dr,
\]

(7)

where \( E(r) \) is unknown function which is depended only \( r \) and \( D \) is a constant.

Then, Einstein equations will be

\[
e^{-2\delta(r)} \left( f^2 \delta_{\nu} - f f_{\nu} \right) = \frac{\alpha}{2} e^{-2\delta(r)} f f_{\nu} = \frac{\alpha}{2} e^{-2\delta(r)} f f_{\nu} \left( \frac{d\phi}{dr} \right)^2 - \frac{1}{2} \frac{2k}{r} e^{-\phi} |F|^k - e^{\beta \phi} \Lambda,
\]

(8)

\[
e^{-2\delta(r)} (ff_{\nu}\partial_\nu - f f_{\nu\nu} - \left( f f_{\nu} \right)^2) = \frac{\alpha}{2} e^{-2\delta(r)} f f_{\nu} \left( \frac{d\phi}{dr} \right)^2 - \frac{1}{2} \frac{2k}{r} e^{-\phi} |F|^k + e^{\beta \phi} \Lambda.
\]

(9)

When we subtract (9) from (8), we get

\[
\frac{1}{r} \frac{d\delta}{dr} = \alpha \left( \frac{d\phi}{dr} \right)^2.
\]

(10)

In this paper, we assume function \( \delta(r) \) as

\[
\delta(r) = n \ln \left( \frac{r}{r_*} \right),
\]

(11)

which is embraced in article [16]. Here \( n \) and \( r_* \) are constants. Thus, we can determinant the dilaton field \( \phi \) as

\[
\phi(r) = \pm \sqrt{\frac{n}{\alpha}} \ln \left( \frac{r}{r_0} \right),
\]

(12)

where \( r_0 \) is some constant.

Next, let us solve the electric field equation. We can solve equation (10) analytically, then

\[
E(r) = \left( \frac{2}{r} \right)^{2n-(2nD-1)(2k-1)}
\]

(14)

\[g\] is an integration constant which is related to the charge of the space-time. The \( k = 1 \) case, when the electric field \( E(r) \) proportional to \( \ln(r) \), was already discussed by Chan [13].

To determine the relation of constants, we calculate Einstein equations and dilaton field. At first, take the summation (8) and (9), then

\[
e^{-2\delta(f^2 \delta_{\nu} - 2ff_{\nu})} = (1 - 2k)e^{-\phi} |F|^k + 2e^{\beta \phi} \Lambda.
\]

(15)
On the other hand, from the dilaton field equation, we get

$$
\alpha e^{-2\delta} \left\{ \frac{f^2}{r} - f^2 \delta r + 2f f_r \right\} \left( \frac{d\phi}{dr} \right) + \alpha e^{-2\delta} f^2 e^2 \phi \frac{d^2\phi}{dr^2} + \frac{e^{-\phi}}{2} |F|^k - \beta e^2 \phi \Lambda = 0. \tag{16}
$$

Thus we get the following relation when we use equation (12), (13) and (15)

$$
\left\{ \pm \sqrt{n\alpha(1 - 2k)} - \frac{1}{2} \right\} e^{-\phi} |F|^k + \left( \pm 2\sqrt{n\alpha + \beta} \right) e^2 \phi \Lambda = 0. \tag{17}
$$

This equation implies that

$$
- \beta = \pm 2\sqrt{n\alpha}, \quad \tag{18}
$$

$$
\beta (1 - 2k) + 1 = 0. \quad \tag{19}
$$

This choice is not consistent to k = 1/2 type MP theory. Let us discuss this issue in the next subsection.

Finally, we have to calculate unknown function f(r). Calculation of until now, we get the following differential equation

$$
\frac{2f}{r} \frac{df}{dr} - \frac{n f^2}{r^2} = 2^k (2k - 1) \left( \frac{r}{r_s} \right)^{2n} \left( \frac{r}{r_0} \right)^{2n(2k-1)(2D-1)} \tag{20}
$$

$$
\times \left( \frac{q}{r} \right)^{2k(2k-1)(2D-1) - 2} \left( \frac{r_0}{r_s} \right)^{2n} \Lambda.
$$

We can solve the equation analytically, then

$$
f^2(r) = \left( \frac{r_0}{r_s} \right)^{2n} \left\{ - \frac{\Lambda r^2}{2 - n} \right. 
$$

$$
- \frac{2^k (2k - 1)}{2(1 + k(1 - 2k)) - n(1 - 4D(k - 1)(2k - 1))} \left( \frac{q}{r} \right)^{2k(2k-1)+2n(1+D(2k-1))} \right\} - m r^n, \tag{21}
$$

where m is a constant.

2.2 Solution of k = 1/2 case

From the relation (19), β becomes infinity at k = 1/2. It was already reported that k = 1/2 MP theory has anomaly without dilaton field case [7]. Then, Mazharimousavi get the wormhole solution with a negative cosmological constant when we take a special electric ansatz like F = E dtd + dθ [17]. However, no one considered with the dilaton case.

We take same ansatzs as general k case. At first, we attend to the electric field equation, then

$$
\frac{d}{dr} (r e^{-\phi}) = 0. \tag{22}
$$

Without dilaton case, it was reported that the electric field equation has anomaly [7]. However, in our case, we can determinant dilaton field as

$$
\phi = \ln \left( \frac{r}{q} \right), \tag{23}
$$

from the equation (22).

Next, we solve the Einstein equations. From the ansatz, we get

$$
e^{-2\delta(r)} \left( \frac{f^2 \delta r - f f_r}{r} \right) = - \frac{\alpha}{2r^2} e^{-2\delta} f^2 - e^{2\beta \phi} \Lambda, \tag{24}
$$

$$
e^{-2\delta(r)} f f_r = \frac{\alpha}{2r^2} e^{-2\delta} f^2 - e^{2\beta \phi} \Lambda, \tag{25}
$$

$$
e^{-2\delta(r)} (f f_r \delta r - f f_{rr} - (f_r)^2) = - \frac{\alpha}{2r^2} e^{-2\delta} f^2 - e^{2\beta \phi} \Lambda + E(r) \sqrt{2}. \tag{26}
$$

When we take summation (25) and (24)

$$
\frac{d \delta}{dr} = \frac{\alpha}{r}. \tag{27}
$$

Then, the unknown function δ becomes

$$
\delta = \alpha \ln \left( \frac{r}{r_s} \right). \tag{28}
$$

Now let us substitute (28) and (23) to (24) or (25). Then, we get

$$
\left( \frac{r}{r_s} \right)^{2\alpha} \left( \frac{\alpha f^2}{2r^2} + \frac{f \frac{df}{dr}}{r} \right) + \left( \frac{r}{q} \right)^{\beta} \Lambda = 0. \tag{29}
$$

This equation can solve analytically, then

$$
f^2 = - \frac{2\Lambda r^2}{2 + \alpha + \beta} \left( \frac{r}{r_s} \right)^{2\alpha} \left( \frac{r}{q} \right)^{\beta} - m r^n. \tag{30}
$$

Electric field can determine from the dilaton equation (4) or (26). Thus we get

$$
E(r) = - 2\sqrt{2} (2\alpha + \beta) \left( \frac{r}{q} \right)^{\beta} \Lambda. \tag{31}
$$

So we conclude that we can get consistent solutions of the 2 + 1 gravity witch is coupled to k = 1/2 MP theory with dilaton field under the circular symmetric static ansatz.
3 Fundamental structure of the space-time

In this chapter, let us consider the fundamental geometric quantity such as scalar invariant. Addition, we investigate the behavior of the metric function $f(r)$. Another essential quantity that called Brown-York mass [18] in AdS space time [19] will be derived.

As we mentioned above, hereafter we will investigate negative cosmological constant case, then

$$\ell = \frac{1}{\sqrt{-\Lambda}}.$$  \hspace{1cm} (32)

where $\ell$ is cosmological radius. Why we consider negative $\Lambda$ case? At first, asymptotic behavior of the fundamental black hole solution in the $2 + 1$ gravity that called BTZ black hole is AdS space-time. Moreover, because of the AdS/CFT correspondence [20], it is believed that AdS space-time itself interesting. So, we hope that it is meaningful to include the negative cosmological constant.

3.1 general $k$ case

The solution (21) is constructed by power term of $r$. Then, we can rewrite (21) as

$$f(r) = Ar^2 - mr^n - B \left(\frac{q}{r}\right)^b,$$  \hspace{1cm} (33)

where

$$A = \left(\frac{r_0}{r_*}\right)^{2n} \frac{1}{F^2},$$

$$B = \frac{2^k(2k - 1)}{2(1 + k(1 - 2k)) - n(1 - 4D(k - 1)(2k - 1))},$$

$$b = 2k[(2k - 1) + 2n(1 + D(2k - 1))].$$ \hspace{1cm} (34)

Then, Fundamental geometric quantity such as Ricci scalar of the solution (21) reads to

$$R = \frac{r^2}{F^4} \left\{ 2(2n - 3)Ar^2 - nm r^n - ((b + 2)n - b(b + 1))B \left(\frac{q}{r}\right)^b \right\}.$$ \hspace{1cm} (35)

The singular behavior of the Ricci scalar is depended on $n$ and $k$ (equal to $\alpha$ and $\beta$) becomes as follows. i) a singular point at origin. ii) a singular point at infinity. iii) two singular points at origin and infinity. Higher order curvature invariant such as $R^\mu\nu R_{\mu\nu}$ and $R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta}$ also has same structure. Thus, our new solution (21) has physical singular point at $r = 0$ or $r = \infty$ that depends on $n$ and $k$. In addition, it is possible to create a black hole solution with some fine parameter sets. To get a black hole solution, we will choice i) type physical singular behavior. Then, parameters must satisfy

$$0 \leq n, \frac{3k + 1}{2k - 1} \leq n.$$  \hspace{1cm} (36)

Figure 1: The behavior of the metric function when singular point lies at $r = 0$. The horizontal line represents radial position $r$, and vertical line represents the metric function $f(r)$. The parameter sets are $\ell = n = D = 1$ with $k = 3/4$. Red line, green line and blue line means $m = 0.5, 1.78, 3$ respectively.

Figure 2: The behavior of the metric function when singular point lies at $r = 0$. The parameter sets are $\ell = 1, n = D = 0.1$ with $k = 3/4$. Red line, green line and blue line means $m = 0.1, 1, 2$ respectively.

The behavior of the metric function in this case is shown in Figure 1 and Figure 2. We cannot derive the horizon radius with analytical method in generally, however, there are horizons at $f(r_H) = 0$ where $r_H$ is its radius at some parameter sets. From Figure 1 we notice that there are two horizons and the space-time is similar to the R-N black hole. Thus, the parameter sets in Figure 1 could generate R-N like black holes. On the other hand, some parameter sets construct the metric function like Figure 2. Then, the space-time is similar to the Schwarzschild-AdS black-hole. That is why we conclude that our
new solutions could generate some different types of black hole.

Next, let us calculate space-time mass that called Brown-York mass when physical singular point lies at $r = 0$. It is correspond to the black hole mass. According to Brown and York, the mass function of the space-time becomes to

$$m(r) = 2\pi f(f_0 - f)e^{-\delta}, \quad (37)$$

where $f_0$ is the metric function at $m = 0$, thus

$$f_0(r) = \sqrt{Ar^2 + B}\left(\frac{q}{r}\right)^{\frac{1}{n}}. \quad (38)$$

Then, the mass of the space-time is defined as

$$M \equiv \lim_{r \to \infty} m(r). \quad (39)$$

We can calculate the space-time mass as

$$M = \pi r_0^n m, \quad (40)$$

when $n < 2$. However, when $n < 2$, the mass will diverge. It is correspond to the metric function which is represented by purple line of Figure 3 that does not be black holes.

### 3.2 $k = 1/2$ case

Here, we consider the $k = 1/2$ case as same as general $k$ case. Ricci scalar of the solution is

$$R = -\frac{1}{r^2} \left(\frac{r_+}{r}\right)^{2\alpha} \left\{ \frac{2}{r^2} \left(\frac{r}{r_+}\right)^{2\alpha} \left(\frac{r}{q}\right)^{\beta} \right\} + \frac{\alpha m r^\alpha}{2 + \alpha + \beta}. \quad (41)$$

The feature of the Ricci scalar is as follows. i) To diverge at $r = 0$, power terms of $r$ must proportional to $1/r$. This reads to $-2 \leq \alpha$ and $\beta \leq 0$. This case our new solution has one physical singular point at origin. ii) To diverge at $r = \infty$, power terms of $r$ has to be proportional to $r$. This reads $\alpha \leq -2$ and $0 \leq \beta$. This case our new solution has one physical singular point at infinity. iii) In other case, Ricci scalar will diverge at $r = 0$ and $r = \infty$. Then, solution has two physical singular point at origin and infinity.

The behavior of the metric function of the $1/2$ case is shown in Figure 3. Note that $\alpha = 1/4, \beta = 1$ is correspond to the string theory. In this case, behavior of the solution is similar to the BTZ space-time. However, there is a naked singular point at $r = \infty$. Thus the space-time does not represent the “exact” black hole. On the other hand, the behavior of the metric function which is represented by green line in

Figure 3: The behavior of the metric function with $k = 1/2$ theory. The parameter sets are $r_H = 1, m = 1, \alpha = 1$. Red line, green line, blue line means $\beta = 1, -1, -4$ respectively.

However, behavior of the metric function like blue line of the Figure 3 does not similar to the black hole space-time. That is why we could not get black hole solutions in general sets of parameters.

Higher order curvature invariant such as $R^{\mu\nu}R_{\mu\nu}$ and $R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}$ also has same structure. Thus, our new solution has physical singular point at $r = 0$ or $r = \infty$ that depends on $\alpha$ and $\beta$.

The Brown-York mass is defined as $M = \pi r_0^n m$ with

$$f_0^2 = \frac{-2\Lambda r^2}{2 + \alpha + \beta} \left(\frac{r}{r_+}\right)^{2\alpha} \left(\frac{r}{q}\right)^{\beta}. \quad (43)$$

We can determinant the mass as

$$M = \pi r_0^n m, \quad (44)$$

when $0 < 2 + \alpha + \beta$. This is consistent to the existence of the black hole solution. However, $2 + \alpha + \beta < 0$ case, we cannot well define because of the Brown-York mass will be diverge. Then, we cannot include the mass. This case is correspond to the blue line of Figure 3 that does not represent the black hole space-time.

### 4 Conclusion and Remarks

We derive new solutions of the $2 + 1$ space-time that coupled to MP type NLEF with dilaton field. We could derive new circular symmetric static solutions in general $k$ case. Especially, when we include dilaton...
field, we could get consistent solution at \( k = 1/2 \) that has no consistent circular symmetric electric solution without dilaton field. From the singular analyze and behavior of the metric function, we notice that some parameter sets generate black holes. When space-time becomes black hole, we could include Brown-York mass.

The future work that we have to do as follows. At first, we have to investigate more detail structure of the solution. One of them is thermodynamics of the solution. In the BTZ black hole, thermodynamical consideration has been performed and the behavior of thermodynamic quantities are investigated. The space-time which we derived in our article should be studied along with the similar thermodynamical context. Consider the physical structure such as mechanics of our new solutions seems to interesting. Physical phenomena such as lensing effect is also interesting. Above analyze will derive physical viewpoints of our new solutions.

In addition, to derive more new solutions is important. We treat electric solutions in our article. Then, we did not derive magnetic solutions. So considering the magnetic type solution is meaningful. Second, to consider the rotating effect seems important. It is known that spinning case has more interesting property. That is why we have to treat point symmetric type ansatz in the future work. It seems meaningful to do the same works in higher dimensional case. We have to find solutions of higher dimensional Einstein gravity witch is coupled to non linear electromagnetic field with dilaton field.

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