Abstract

Every Grothendieck fibration gives rise to a vertical/cartesian orthogonal factorization system on its domain. We define a cartesian factorization system to be an orthogonal factorization in which the left class satisfies 2-of-3 and is closed under pull-back along the right class. We endeavor to show that this definition abstracts crucial features of the vertical/cartesian factorization system associated to a Grothendieck fibration, and give comparisons between various 2-categories of factorization systems and Grothendieck fibrations to demonstrate this relationship. We then give a construction which corresponds to the fiberwise opposite of a Grothendieck fibration on the level of cartesian factorization systems.

A Grothendieck fibration $p : E \to B$ is a way of packaging the data of an indexed category $E_B$ varying with $B \in B$. There is an involution on indexed categories giving by taking the pointwise dual: namely, $E_B \mapsto E_B^{op}$. What does this involution look like on the corresponding Grothendieck fibrations?

The Grothendieck construction of the fiberwise opposite of an indexed category is called the category of generalized lenses by Spivak [5]. It has uses in a general theory of open dynamical systems [3].

A hint of how this construction should proceed is given by the theory of polynomial functors [2]. A polynomial functor $P : \text{Set} \to \text{Set}$ is a functor of the form

$$P(X) = \sum_{b \in B} X^{E_b}$$

for some family of sets $E_b$ varying with $b \in B$. We may see this family as a function $p : E \to B$, and the data of a polynomial functor is precisely the data of such a function. Any natural transformation between polynomial functors $P$ and $P'$ may be represented by an odd diagram of the following sort, called a morphism of polynomials:

$$
\begin{array}{c}
\begin{tikzpicture}
  \node (A) at (0,0) {$E$};
  \node (B) at (0,-1) {$B$};
  \node (C) at (1,0) {$E'$};
  \node (D) at (1,-1) {$B'$};
  \draw[->] (A) -- (C);
  \draw[->] (B) -- (D);
  \draw[->] (A) -- (B) node [midway, left] {$p$};
  \draw[->] (C) -- (D) node [midway, right] {$p'$};
\end{tikzpicture}
\end{array}
$$

Thinking only of the functions $p : E \to B$, we would expect a morphism $p \to p'$ would

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be a square as below right, which is equivalent to a diagram as below right:

\[ \begin{align*}
E & \longrightarrow \bullet \longrightarrow E' \\
\downarrow p & \quad \downarrow r \quad \downarrow p' \\
B & \longrightarrow B' \quad B & \longrightarrow B'
\end{align*} \]

The shape of this diagram on the right is explained by the vertical/cartesian factorization system on \( \text{Set}^\downarrow \) associated to the codomain fibration \( \text{cod} : \text{Set}^\downarrow \rightarrow \text{Set} \). Namely, the left class of this factorization system consists of all commuting squares whose bottom face is an isomorphism, and the right class consists of all pullback squares. The morphisms of polynomials are precisely the spans in \( \text{Set}^\downarrow \) whose left leg is vertical and whose right leg is cartesian.

Thinking fiberwise, in terms of indexed categories, a morphism of functions \( p \rightarrow p' \) can be expressed as a pair \( f : B \rightarrow B' \) and a family \( f^*_b : E_b \rightarrow E'_{f(b)} \) for \( b \in B \). On the other hand, a morphism of polynomials is given by a pair of a function \( f : B \rightarrow B' \) and a family \( f^*_b : E'_{f(b)} \rightarrow E_b \) for \( b \in B \). We can see, therefore, that forming these spans of squares whose left leg is vertical and right leg is cartesian (the polynomial morphisms) corresponds to taking the fiberwise opposite. We will see that this construction works generally.

Every Grothendieck fibration \( p : E \rightarrow B \) gives rise to such an orthogonal factorization system, where the vertical maps are those sent to isomorphisms by \( p \) and the cartesian maps are those satisfying the cartesian lifting property. We will show that we can perform a construction analogous to the construction of the category of polynomials as spans whose left leg is vertical and whose right leg is cartesian from the codomain fibration \( \text{cod} : \text{Set}^\downarrow \rightarrow \text{Set} \) for any Grothendieck fibration — without any extraneous assumptions on the categories \( E \) and \( B \).

In this paper, we will abstract the crucial features of the vertical/cartesian factorization systems associated to Grothendieck fibrations and define a cartesian factorization system to be an orthogonal factorization system \( (\rightarrow, \rightarrow) \) satisfying two additional properties:

1. The left class \( \rightarrow \) satisfies 2-of-3.

2. Pullbacks of the left class \( \rightarrow \) along the right class \( \rightarrow \) exist and are in \( \rightarrow \).

Every vertical/cartesian factorization system arising from a Grothendieck fibration is a cartesian factorization system. While we do not show that every cartesian factorization system arises this way, we will show that under mild conditions (such as the existence of a terminal object) they do. In particular, we will show that if a cartesian factorization system admits enough \( \rightarrow \)-injectives (Definition 2.5), then we can construct a Grothendieck fibration with a right adjoint right inverse from it. We prove in Theorem 2.14 that this construction gives an equivalence of the a 2-category of cartesian factorization systems with enough injectives and a 2-category of Grothendieck fibrations with right adjoint right inverses.

We will then describe the fiberwise dual construction, which mimics the definition of polynomial morphisms in a general cartesian factorization system. Namely, the fiberwise
dual $\mathcal{C}'$ of a category $\mathcal{C}$ equipped with a cartesian factorization system is the category of spans whose left leg is in the left class and whose right leg is in the right class. We show that $\mathcal{C}'$ may be equipped with a cartesian factorization system such that $\mathcal{C}' \simeq \mathcal{C}$. This gives an involution $(-)^\vee : \text{Cart} \to \text{Cart}$ on the category of cartesian factorization systems (it does not behave well with natural transformations, although we can rectify this issue by passing to double categories, as in Theorem 3.10).

We will show in Theorem 3.7 that $(-)^\vee$ corresponds to the fiberwise opposite in the sense that the following square of functors commutes (up to equivalence):

$$
\begin{array}{ccc}
\text{Groth} & \xrightarrow{\Phi} & \text{Cart} \\
\downarrow \text{(-)fibop} & & \downarrow (-)^\vee \\
\text{Groth} & \xrightarrow{\Phi} & \text{Cart}
\end{array}
$$

where $(-)^\text{fibop} : \text{Groth} \to \text{Groth}$ is the fiberwise opposite of a Grothendieck fibration, constructed by passing through indexed categories. The variance of these functors is rather tricky to extend to the level of 2-categories; we only consider their action on morphisms and not natural transformations here.

We then show that the natural double category one might construct out of $\mathcal{E}$ and $\mathcal{E}'$ (in the usual way one makes a double category out of spans) is equivalent to a Grothendieck double construction of the indexed category $\mathcal{E}_{(-)} : \mathcal{B}^{\text{op}} \to \mathcal{C}$ whose horizontal category is the Grothendieck construction of $\mathcal{E}_{(-)}$ and whose vertical category is the Grothendieck construction of the pointwise opposite. We will show in Theorem 3.10 that this construction is 2-functorial, landing in the 2-category of double categories, functors, and horizontal transformations. This construction plays a role in a double categorical theory of open dynamical systems [3].

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## 1 Orthogonal Factorization Systems

First, we recall the definition of an orthogonal factorization system, and a few elementary lemmas. For proofs, see [4].

**Definition 1.1.** A orthogonal factorization system $(\to, \twoheadrightarrow)$ on a category $\mathcal{C}$ is a pair of collections of arrows $\bullet \to \bullet$ and $\bullet \twoheadrightarrow \bullet$ such that

1. $\to$ and $\twoheadrightarrow$ both contain all isomorphisms and are closed under composition.

2. Every map $f : A \to B$ factors as $f = me$ with $e \in \to$ and $m \in \twoheadrightarrow$.

\[\bullet \xrightarrow{m} \bullet \xrightarrow{e} \bullet\]
3. This factorization is *uniquely functorial* in the sense that whenever we have a solid diagram:

![Diagram]

We have a unique dashed arrow making the diagram commute.

We recall that this definition implies that \( \mathcal{L} \) is *orthogonal* to \( \mathcal{R} \) in the sense that every lifting problem:

![Lifting Problem]

has a *unique* solution, given by the dashed arrow. We now recall a few elementary lemmas in the theory of orthogonal factorization systems.

**Lemma 1.2** (Saturation). Let \((\rightarrow, \Rightarrow)\) be an orthogonal factorization system. Then a map is in \(\rightarrow\) (resp. \(\Rightarrow\)) if and only if it satisfies the unique left (resp. right) lifting property against \(\Rightarrow\)s (resp. \(\rightarrow\)s).

**Lemma 1.3** (Cancellation). Let \((\rightarrow, \Rightarrow)\) be an orthogonal factorization system. For any composable arrows \(f\) and \(g\):

- if \(f\) and \(gf\) are in \(\rightarrow\), then so is \(g\), and
- if \(g\) and \(gf\) are in \(\Rightarrow\), then so is \(f\).

**Lemma 1.4.** Let \((\rightarrow, \Rightarrow)\) be an orthogonal factorization system. Then the right class \(\Rightarrow\) is preserved under pullback.

**Lemma 1.5.** Let \((\rightarrow, \Rightarrow)\) be an orthogonal factorization system. Then a map is an isomorphism if and only if it is in both \(\rightarrow\) and \(\Rightarrow\).

## 2 Cartesian Factorization Systems

Now we come to our main definition.

**Definition 2.1.** An orthogonal factorization system \((\rightarrow, \Rightarrow)\) is *cartesian* if

1. (Left 2-of-3) The class \(\rightarrow\) satisfies 2-of-3: if \(g\) and \(gf\) are in \(\rightarrow\), then so is \(f\).

2. (Right Stability) Pullbacks of \(\rightarrow\)s along \(\Rightarrow\)s exist and are in \(\rightarrow\).

We will refer to an orthogonal factorization system that satisfies (2) — Right Stability — as a right stable orthogonal factorization system.

We define the 2-category \(\text{Cart}\) of cartesian factorization systems to consist of categories with cartesian factorization systems, functors which preserve both classes, and natural transformations between such functors.
In this section, we will see a few basic properties of cartesian factorization systems, and then relate them to Grothendieck fibrations. We will end the section by seeing that certain cartesian factorization systems are equivalent to Grothendieck fibrations with a right adjoint right inverse.

**Lemma 2.2.** Let \((\rightarrow, \Rightarrow)\) be a cartesian factorization system on a category \(C\). Then every square of the form

\[
\begin{array}{ccc}
\circ & \rightarrow & \circ \\
\downarrow & & \downarrow \\
\circ & \rightarrow & \circ
\end{array}
\]

is a pullback.

**Proof.** Consider the comparison map to the pullback:

\[
\begin{array}{ccc}
\circ & \rightarrow & \circ \\
\downarrow & & \downarrow \\
\circ & \rightarrow & \circ
\end{array}
\]

The comparison map is in \(\Rightarrow\) by cancelation for the right class, and it is in \(\rightarrow\) because \(\rightarrow\) satisfies 2-of-3 by hypothesis. Therefore, it is an isomorphism, and the outer square is a pullback. \(\square\)

**Remark 2.3.** A *modality* is an orthogonal factorization system on a category with finite limits in which the left class is stable under pullback. Clearly, a modality is in particular a right stable orthogonal factorization system. A modality is *lex* if and only if the left class satisfies 2-of-3; that is, if and only if it is a cartesian factorization system, in addition to a modality.

This definition is meant to abstract some crucial features of the vertical/cartesian orthogonal factorization system associated to a Grothendieck fibration.

**Proposition 2.4.** Let \(p : E \rightarrow B\) be a Grothendieck fibration. The induced vertical/cartesian orthogonal factorization system \((\rightarrow, \Rightarrow)\) is cartesian. This gives a 2-functor \(\Phi : \text{Groth} \rightarrow \text{Cart}\) from the 2-category of Grothendieck fibrations to the 2-category of cartesian factorization systems.

**Proof.** The vertical morphisms satisfy 2-of-3 since they are, by definition, those morphisms sent by \(p\) to isomorphisms. It remains to show, then, that pullbacks of vertical maps over cartesian maps exist and are vertical. Suppose we have a diagram as follows:

\[
\begin{array}{ccc}
B & \downarrow^g \\
A & \rightarrow_f & C
\end{array}
\]
We may complete this into a square:

\[ (pf)^* B \xrightarrow{k} B \]
\[ \downarrow^j \quad \downarrow^g \]
\[ A \xrightarrow{f} C \]

We do this in the usual way that one defines the base change operation on fibers: namely, we take \( k \) to be a cartesian lift of \( (pg)^{-1}(pf) \) and \( j \) is the unique morphism determined by the universal property of \( f \). We note that \( pj = \text{id}_{pA} \), so that it is vertical. It remains to show that this square is a pullback.

Given a solid diagram like so:

\[ Z \quad \quad \quad \quad \quad \quad \]
\[ (pf)^* B \xrightarrow{k} B \]
\[ \downarrow^j \quad \downarrow^g \]
\[ A \xrightarrow{f} C \]

The dashed arrow exists uniquely by the universal property of \( k \), using \( (pj)^{-1} \) to create the triangle in \( B \) to lift.

We can quickly show that this assembles into a 2-functor. Given a cartesian functor \( p \rightarrow q \):

\[ \mathcal{E} \xrightarrow{F} \mathcal{E}^\prime \]
\[ \downarrow^p \quad \downarrow^q \]
\[ B \xrightarrow{G} B^\prime \]

\( F \) preserves cartesian morphisms by hypothesis, and we can see that \( F \) preserves vertical morphisms by the commutativity, up to isomorphism, of this square.

There is, in fact, a strong relation between cartesian factorization systems and Grothendieck fibrations. We will now explore this relationship, beginning with a few preliminary definitions and lemmas.

**Definition 2.5.** Let \( (\rightarrow, \rightarrow) \) be an orthogonal factorization system. An object \( X \) is \( \rightarrow \)-injective (or just, injective) if every extension problem like so:

\[ \bullet \xrightarrow{\rightarrow} X \]
\[ \downarrow^\exists ! \]
\[ \bullet \]

admits a unique solution. We say that \( (\rightarrow, \rightarrow) \) has enough \( \rightarrow \)-injectives if for every \( X \) there is a map \( X \rightarrow Y \) (in \( \rightarrow \)) into an injective object \( Y \).
If our category $C$ has a terminal object, then this theory of injective objects becomes particularly simple.

**Lemma 2.6.** If $C$ has a terminal object, then an object $X$ is injective if and only if the terminal map $X \rightarrow \ast$ is in the right class.

**Remark 2.7.** A modality, which is a stable orthogonal factorization system on a category with finite limits, has enough $\rightarrow$-injectives given by factoring the terminal morphism. In this case, we call the operation of injective replacement the *modal operator*.

However, all of the basic facts which follow immediately from this characterization of injective objects — that every map between them is in the right class $\rightarrow$, that if $Y$ is injective and $X \rightarrow Y$ then so is $X$ — follow as well without the presence of a terminal object.

**Lemma 2.8.** Every map between injective objects is in the right class $\rightarrow$.

*Proof.* We show that if $X$ and $Y$ are injective and $f : X \rightarrow Y$, then every lifting problem

\[
\begin{array}{ccc}
A & \rightarrow & X \\
\downarrow & & \downarrow f \\
B & \rightarrow & Y
\end{array}
\]

admits a unique solution, so that $f$ is in $\rightarrow$ by the saturation of that class. Since $X$ is injective, there is a unique solution to the following extension problem:

\[
\begin{array}{ccc}
A & \rightarrow & X \\
\downarrow & & \\
B
\end{array}
\]

We just need to show that this solves the above lifting problem. Since $Y$ is also injective, there is a unique solution to the following extension problem which is already given by the map $B \rightarrow Y$:

\[
\begin{array}{ccc}
A & \rightarrow & X & \rightarrow & Y \\
\downarrow & & \downarrow f & & \\
B
\end{array}
\]

therefore the composite $B \rightarrow X \rightarrow Y$ is equal to the map $B \rightarrow Y$. Uniqueness of this result follows because, as we just saw, any solution of the extension problem (2) gives a solution of the lifting problem (1). \qed

**Lemma 2.9.** If $Y$ is injective and $k : X \rightarrow Y$ is a map in the right class, then $X$ is injective.
Proof. Consider an extension problem of the following form, for which we will produce a unique solution:

\[
\begin{array}{ccc}
A & \xrightarrow{g} & X \\
\downarrow{f} & & \downarrow{k} \\
B & & Y
\end{array}
\]

Now since \( Y \) is injective, we have a unique solution to this extension problem:

\[
\begin{array}{ccc}
\bullet & \xrightarrow{g} & X \\
\downarrow{f} & & \downarrow{k} \\
\bullet & & Y
\end{array}
\]

Re-arranging, we get a square

\[
\begin{array}{ccc}
\bullet & \xrightarrow{g} & X \\
\downarrow{f} & & \downarrow{k} \\
\bullet & \xrightarrow{=} & Y
\end{array}
\]

which has a unique filler by orthogonality. Therefore, we have a unique solution to the extension problem we started with.

Proposition 2.10. Let \( \mathcal{C}_R \) denote the full subcategory of \( \to \)-injective objects in an orthogonal factorization system \((\to, \mapsto)\). Then \((\to, \mapsto)\) has enough injectives if and only if the inclusion \( \mathcal{C}_R \to \mathcal{C} \) has a left adjoint.

Proof. Suppose that \((\to, \mapsto)\) has enough injectives. We will show that the slice category \( X \downarrow \mathcal{C}_R \) of \( X \) over the injective objects admits an initial object. Suppose that \( X \to Y \) is an injective replacement of \( X \). Then the unique extension condition of injective objects says that this is initial in \( X \downarrow \mathcal{C}_R \).

Now suppose that \( X \downarrow \mathcal{C}_R \) has an initial object \( X \to Y \). We will show that this is in the left class, and is therefore an injective replacement of \( X \). Let \( X \mapsto M \mapsto Y \) be the factorization of \( X \to Y \), seeking to show that \( M \mapsto Y \) is an isomorphism. Note that by Lemma 2.9, \( M \) is also injective. Therefore, by the initiality of \( X \to Y \), we have a unique map \( Y \to M \) under \( X \). Since \( X \to Y \) is initial, the induced endomorphism \( Y \to M \mapsto Y \) is the identity. On the other hand, by cancellation for \( \mapsto \), the endomorphism \( M \mapsto Y \to M \) is in \( \mapsto \); but by Lemma 2.8 maps between injective objects are in \( \mapsto \), and therefore this endomorphism is an isomorphism. It follows that \( M \mapsto Y \) is an isomorphism as well.

Proposition 2.11. Suppose that \((\to, \mapsto)\) is a right stable orthogonal factorization system on \( \mathcal{C} \) which has enough injectives. Then the injective replacement functor \( \mathcal{R} : \mathcal{C} \to \mathcal{C}_R \) is a Grothendieck fibration.

Furthermore, \((\to, \mapsto)\) is cartesian if and only if and \( \to \) is the class of vertical morphisms and \( \mapsto \) the class of cartesian morphisms for this Grothendieck fibration.
Proof. We show that every map in $C_R$ has a cartesian lift. That is, suppose we have a diagram like so:

\[
\begin{array}{ccc}
X & \xrightarrow{e} & \ast \\
\downarrow & & \downarrow \\
A & \xrightarrow{m} & B
\end{array}
\]

Define the lift of the map $A \rightarrow B$ to be the pullback:

\[
\begin{array}{ccc}
X' & \xleftarrow{e^*m} & X \\
\downarrow & & \downarrow \\
A & \xrightarrow{m} & B
\end{array}
\]

That this is cartesian follows quickly from the universal property of the pullback:

Given a triangle with vertex $RZ$, we get the above solid diagram and so a unique dashed map.

Now, we show that the induced vertical/cartesian factorization system is $\langle \rightarrow, \rightarrow \rangle$ if and only if the factorization system is cartesian. Now, if $\rightarrow$ is the vertical class, then it satisfies 2-of-3. So, suppose that $\rightarrow$ satisfies 2-of-3, and we will show that a map is in $\rightarrow$ if and only if it is sent to an isomorphism by $R$. Let $f: X \rightarrow Y$, and consider the following naturality square for $R$:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
RX & \xrightarrow{Rf} & RY
\end{array}
\]

By cancellation, $Rf: RX \rightarrow RY$ is also in $\rightarrow$, and is therefore an isomorphism. Similarly, if $Rf$ is an isomorphism for arbitrary $f: X \rightarrow Y$, then by 2-of-3, $f$ is in $\rightarrow$.

Remark 2.12. Working in homotopy type theory, Rijke and Cherubini [1] have associated to every modality another modal-equivalence/modal-étale factorization system. This factorization system is the vertical/cartesian factorization system induced by the modal operator $!: \text{Type} \rightarrow \text{Type}$. 

Theorem 2.13. Sending a right stable orthogonal factorization system with enough $\rightarrow$-injectives to the Grothendieck fibration $R: C \rightarrow C_R$ given by injective replacement constitutes a pseudo-functor $\Xi: \text{RSOFS}_R \rightarrow \text{Groth}_R$ from the 2-category of right stable orthogonal factorization systems with enough injectives and functors preserving both classes to the 2-category of Grothendieck fibrations with right adjoint right inverses and cartesian functors between them.
Proof. We begin by construction $\Xi$. Given a cartesian factorization system $(\to, \rightrightarrows)$ on a category $C$ enough injectives, we send it to the injective replacement $R : C \to C_R$, which is a Grothendieck fibration with a right adjoint right inverse by Proposition 2.11. Given a functor $F : C \to D$ which preserves the left class, take the square

$$
\begin{array}{ccc}
C & \xrightarrow{F} & D \\
\downarrow R & & \downarrow R \\
C_R & \xrightarrow{RF} & D_R
\end{array}
$$

We can see that this square commutes up to isomorphism by considering the injective replacement $X : \to RX$; its image $FX : \to FRX$ gives rise to an isomorphism $RFX \sim \sim RFRX$. Given a natural transformation $\theta : F \to G$ between such functors, we can take $R\theta : RF \to RG$ to get an appropriate 2-cell in $\text{Groth}$.

We now need to show the pseudofunctoriality of this construction. We define

- the unit $u_C : R \to C$ given by noting any injective replacement $X : \to RX$ is also in $\rightrightarrows$ and therefore is an isomorphism.
- the compositor $f_{F,G} : RGF \sim \sim RGF$ given at $X$ by taking the injective replacement $rFX : FX \to RFX$ and applying $RG$ to it to get an isomorphism $RGrFX : RFGX \sim \sim RGRFX$, and taking the inverse $f_{F,G} := (RGFX)^{-1}$.

The coherence conditions follow from the uniqueness of injective replacements. □

Theorem 2.14. The pseudo-functors $\Phi$ and $\Xi$ of Proposition 2.4 and Theorem 2.13 assemble into an equivalence:

$$
\begin{array}{ccc}
\text{Groth}_R & \sim & \text{Cart}_R \\
\Xi & \bowtie & \Phi
\end{array}
$$

Proof. First, we should show that if a Grothendieck fibrations $p : \mathcal{E} \to \mathcal{B}$ admits a right adjoint right inverse $r$, then $\mathcal{E}$ admits vertical replacements.

For the unit, we take the identity $C \to \Phi \Xi C$, which preserves both classes by Proposition 2.11. For the counit, we take the cartesian functor

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{rp} & \mathcal{E}_R \\
\downarrow p & & \downarrow p \\
\mathcal{B} & \xrightarrow{p} & \mathcal{B}
\end{array}
$$

We note that we may take $rp : \mathcal{E} \to \mathcal{E}_R$ for the injective replacement where $r : \mathcal{B} \to \mathcal{E}$ is the right adjoint to $p$ since $rB$ is $\rightarrow$-injective for all $B \in \mathcal{B}$. This square commutes up to
natural isomorphism since, by hypothesis, the unit \( \eta : X \to rpX \) is vertical, and therefore \( p\eta : pX \xrightarrow{\sim} prpX \) gives a witness to commutativity. It remains to show that this counit is invertible. We will show that \( p : \mathcal{E}_R \to \mathcal{B} \) is an equivalence with inverse \( r : \mathcal{B} \to \mathcal{E}_R \). We note that the unit \( \eta : X \to rpX \) is an isomorphism since as a map between \( \implies \)-injectives, it is in \( \implies \) and it is by hypothesis in \( \leftarrow \). The counit \( \epsilon : prX \to X \) is the identity by assumption.

3 The Fiberwise Dual

As we saw in the last section, there is a close relationship between cartesian factorization systems and Grothendieck fibrations. In this section, we will give a construction which corresponds to taking the fiberwise opposite of a indexed category.

**Definition 3.1.** Let \((\implies, \leftarrow)\) be a cartesian factorization system on a category \(\mathcal{C}\). Define the fiberwise dual of \(\mathcal{C}\) to be the category \(\mathcal{C}^\vee\) with the same objects as \(\mathcal{C}\), but where a morphism \(A \to B\) is a span of the following form:

\[
\begin{array}{ccc}
A & \xleftarrow{\cdot} & B
\end{array}
\]

Composition is given by pullback, as is usual with spans. We also define the dual classes \(\implies^\vee\) and \(\leftarrow^\vee\) to be the spans of the following forms respectively:

\[
\begin{array}{ccc}
A & \xleftarrow{\cdot} & B
\end{array}
\]

Though this is naturally a bicategory, there is at most one morphism between any two such spans and it must be an isomorphism. Consider a morphism between these spans as follows:

\[
\begin{array}{ccc}
A & \xleftarrow{\cdot} & B
\end{array}
\]

By cancellation for \(\leftarrow\), the middle arrow is in \(\leftarrow\); by 2-of-3 for \(\leftarrow\), it is also in \(\rightarrow\). But it is therefore an isomorphism. It is unique by the following lemma:

**Lemma 3.2.** Let \((\leftarrow, \rightarrow)\) be a cartesian factorization system. Then given a solid diagram as below, there may be at most one dashed arrow making the diagram commute:

\[
\begin{array}{ccc}
\bullet & \xleftarrow{\cdot} & \bullet
\end{array}
\]
Proof. We will show that any two \((\rightarrow, \rightarrow)\)-factorizations of such a dashed arrow will agree. Consider a factorization of the top square and note that by unique functoriality, there is a unique dotted arrow filling the diagram

\[
\begin{array}{c}
\bullet \rightarrow \bullet \rightarrow \bullet \\
\downarrow \quad \exists \quad \downarrow \\
\bullet \rightarrow \bullet \rightarrow \bullet
\end{array}
\]

By Lemma 2.2, the right square is a pullback. Therefore, the \((\rightarrow, \rightarrow)\)-factorization of any dashed arrow in the original diagram is given by the universal property of this pullback, and is therefore uniquely determined by the data of the solid diagram.

Therefore, the natural map from the bicategory of such spans to the locally discrete bicategory given by taking the category of isomorphism classes of spans and reincluding as a bicategory is an equivalence. We are therefore justified in thinking of \(C^\vee\) as a category.

\[\text{Remark 3.3.}\] In univalent foundations, \(C^\vee\) would be proven to be a category, so long as \(C\) is a (univalent) category. That is, showing that there is at most a single isomorphism between any two such spans shows that the type of such spans is in fact a set.

\[\text{Theorem 3.4.}\] For \((\rightarrow, \rightarrow)\) a cartesian factorization system on a category \(C\), the dual classes \((\rightarrow^\vee, \rightarrow^\vee)\) form a cartesian factorization system on \(C^\vee\).

\[\text{Proof.}\] It is evident that every morphism in \(C^\vee\) factors as a morphism in \(\rightarrow^\vee\) followed by a morphism in \(\rightarrow^\vee\). We will now show the unique functoriality condition. Given a commuting square in \(C^\vee\) like so:

\[
\begin{array}{c}
\bullet \rightarrow \bullet \rightarrow \bullet \\
\downarrow \quad \exists \quad \downarrow \\
\bullet \rightarrow \bullet \rightarrow \bullet
\end{array}
\]

We note that the middle arrows are in the classes described above by pullback preservation. We may therefore expand this diagram to show a functorial factorization.
Since these diagrams are merely re-arrangements of each other, we find that this functoriality is unique.

Now, \( \rightarrow^\vee \) satisfies 2-of-3 since \( \rightarrow \) does and \( \rightarrow^\vee = \rightarrow^{\text{op}} \) as categories. It remains then to show that pullbacks of \( \rightarrow^\vee \)'s along \( \rightarrow^\vee \)'s exist and are in \( \rightarrow^\vee \). Suppose we have such a cospan:

\[
\begin{array}{ccc}
B & \rightarrow & C \\
\uparrow & & \uparrow \\
A & \rightarrow & C
\end{array}
\]

We see that this data consists of morphisms \( A \rightarrow C \rightarrow B \). Let \( A \rightarrow D \rightarrow B \) be the factorization of the composite of these morphisms. Then we may form the following square:

\[
\begin{array}{ccc}
D & \rightarrow & D \\
\uparrow & \rightarrow & \uparrow \\
A & \rightarrow & C \\
\uparrow & \rightarrow & \uparrow \\
A & \rightarrow & C
\end{array}
\]

The top left square is a pullback by Lemma 2.2, so this is a commuting square in \( C^\vee \).

Now, suppose we have a square as follows:

\[
\begin{array}{ccc}
X & \leftarrow & Y \\
\uparrow & \beta & \uparrow \\
Z & \rightarrow & C \\
\uparrow & \alpha & \uparrow \\
A & \rightarrow & C
\end{array}
\]

The data of the terminal map into the limit cone consists of a diagram:

\[
\begin{array}{cc}
& X \\
\uparrow & \star \\
Z & \leftarrow & Y \\
\uparrow & \rightarrow & \uparrow \\
A & \rightarrow & D & \rightarrow & B
\end{array}
\]

Many of our choices are fixed by commutativity and the pullback conditions. We end
up with a diagram like this:

which ultimately depends only on the data of the dashed map $Y \rightrightarrows D$ making the square on the left and the triangle on the right commute. There is a small point to focus on here: the choice of $\beta: Z \to Y$. Since the square will be a pullback, we need a map $Z \to Y$ such that the composite $Z \to Y \to D$ is equal to $Z \to A \to D$; but such factorizations are unique, so if we can provide such a composite with left component $\beta: Z \to Y$, then every choice will be isomorphic to this choice.

Luckily, asking that the square on the left and the triangle on the right commute can be arranged into the single square:

This square has a unique filler by orthogonality, which is in the right class by cancellation.

\textbf{Proposition 3.5.} For any cartesian factorization system $(\to, \rightrightarrows)$ on a category $C$, the functor $C \to C^{\vee\vee}$ sending a map $f: X \to Y$ which factors as $X \to M \rightrightarrows Y$ to the span-of-spans

is an equivalence in $\text{Cart}$.\hfill\Box

\textit{Proof.} We will show that this is a functor, since it is immediately seen to be fully faithful and essentially surjective. It quite clearly preserves both classes.

Identities are sent to identities since $X \xrightarrow{\text{id}} X \xrightarrow{\text{id}} X$ is a $(\to, \rightrightarrows)$-factorization of $\text{id}$. As for preserving composition, we note that composition in $C^{\vee\vee}$ is given by taking the pullback in $C^\vee$ which is given by factoring in $C$. For any composite $X \xrightarrow{f} Y \xrightarrow{g} Z$, its factorization is given by the factorization of $M \rightrightarrows Y \rightrightarrows N$ where $X \to M \rightrightarrows Y$ and $Y \to N \rightrightarrows Z$ are
the factorizations of \( f \) and \( g \) respectively in the following way:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
M & \xrightarrow{g} & N \\
\downarrow & & \downarrow \\
K & \xrightarrow{} & Z
\end{array}
\]

\[ \square \]

**Proposition 3.6.** The fiberwise dual construction \( C \mapsto C^\lor \) gives a functor \( (-)^\lor : \text{Cart} \to \text{Cart} \).

**Proof.** Given any functor \( F : C \to D \) which preserves both classes of the cartesian factorization system on \( C \), we note that \( F \) preserves any pullback square of the form:

\[
\begin{array}{ccc}
\bullet & \xrightarrow{} & \bullet \\
\downarrow & & \downarrow \\
\bullet & \xrightarrow{} & \bullet
\end{array}
\]

by Lemma 2.2. Therefore, we can define \( F^\lor : C^\lor \to D^\lor \) by

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{g} & D \\
\downarrow & & \downarrow \\
E & \xrightarrow{Ff} & F \circ B \\
\downarrow & & \downarrow \\
F \circ C & \xrightarrow{Fg} & F \circ D
\end{array}
\]

This is evidently functorial. \[ \square \]

We now show that the fiberwise dual \( C^\lor \) does in fact represent the fiberwise dual of a Grothendieck fibration.

**Theorem 3.7.** The following square of functors commutes up to natural equivalence:

\[
\begin{array}{ccc}
\text{Groth} & \xrightarrow{\Phi} & \text{Cart} \\
\downarrow & \downarrow \Phi \circ (\_) & \downarrow \Phi \\
\text{Groth} & \xrightarrow{\Phi} & \text{Cart}
\end{array}
\]

**Proof.** We will show that if \( p : E \to B \) is the Grothendieck construction of an indexed category \( E_{(-)} : B^{\text{op}} \to \text{Cat} \), then \( E^\lor \simeq \int_{B: \text{B}} E_{B}^{\text{op}} \) is the Grothendieck construction of the fiberwise opposite. First, let's set some notation:

- We will denote objects of the Grothendieck constructions \( \int_{B: \text{B}} E_{B}^{\text{op}} \) by \( \left[ \begin{array}{c} E \\ B \end{array} \right] \) where \( B \in \text{B} \) and \( E \in E_{B} \).

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• A morphism in the Grothendieck construction $\mathcal{E} = \int_{B:B} \mathcal{E}_B$ will be denoted by

$$\left( f \right)_f : \left( \begin{array}{c} E \\ B \end{array} \right) \Rightarrow \left( \begin{array}{c} E' \\ B' \end{array} \right)$$

where $f : B \to B'$ and $f_2 : E \to f^*E'$.

• A morphism in the Grothendieck construction $\int_{B:B} \mathcal{E}_B \text{op}$ of the pointwise opposite will be denoted by

$$\left( f^\sharp \right)_f : \left( \begin{array}{c} E \\ B \end{array} \right) \Leftarrow \left( \begin{array}{c} E' \\ B' \end{array} \right)$$

where $f : B \to B'$ and $f^\sharp : f^*E' \to E$. We will refer to these morphisms as lenses, following [5].

We will construct a functor $\varphi : \int_{B:B} \mathcal{E}_B \text{op} \to \mathcal{E}'$ as follows:

• $\varphi$ acts as the identity on objects.

• A lens $\left( f^\sharp \right)_f : \left( \begin{array}{c} E \\ B \end{array} \right) \Leftarrow \left( \begin{array}{c} E' \\ B' \end{array} \right)$ gets sent to the span:

$$\begin{array}{c}
\left( \begin{array}{c} f^\sharp \\ \text{id} \end{array} \right) \\
\left( \begin{array}{c} f^\sharp E' \\ B \end{array} \right)
\end{array} \Rightarrow \begin{array}{c}
\left( \begin{array}{c} \text{id} \\ f \end{array} \right) \\
\left( \begin{array}{c} E' \\ B' \end{array} \right)
\end{array}$$

This clearly sends identities to identities and preserves both classes of the cartesian factorization system; we need to show that it preserves composition. Suppose we have $\left( f^\sharp \right)_f : \left( \begin{array}{c} E \\ B \end{array} \right) \Leftarrow \left( \begin{array}{c} E' \\ B' \end{array} \right)$ and $\left( g^\sharp \right)_g : \left( \begin{array}{c} E' \\ B' \end{array} \right) \Leftarrow \left( \begin{array}{c} E'' \\ B'' \end{array} \right)$ with composite $\left( f^\sharp \circ f^*g^\sharp \right)_{g \circ f}$. We may then form the following diagram in $\mathcal{E}$: 
The square is a pullback by Lemma 2.2, and the outer span is the span associated to the composite \((f^* \circ f^* g^*)\).

It remains to show that \(\varphi\) is an equivalence. It is clearly essentially surjective; we will show it is fully faithful. Every span with left leg vertical and right leg cartesian is of the form

\[
\begin{array}{c}
\left( \begin{array}{c}
E \\
B
\end{array} \right) \\
\downarrow
\end{array}
\xRightarrow{\left( \begin{array}{c}
f^* \id \\
f
\end{array} \right)}
\begin{array}{c}
\left( \begin{array}{c}
f^* E' \\
B'
\end{array} \right) \\
\downarrow
\end{array}
\xRightarrow{\left( \begin{array}{c}
\id \\
f
\end{array} \right)}
\begin{array}{c}
\left( \begin{array}{c}
E' \\
B'
\end{array} \right)
\end{array}
\]

and is therefore involves precisely the data of a lens \((f^* : E \Rightarrow f^* E')\). \(\square\)

Since the morphisms of \(C^\vee\) are certain sorts of spans, we can form a double category whose vertical category is \(C^\vee\) and whose horizontal category is \(C\) and where squares are the usual squares in the double category of spans. This construction mimics, in terms of cartesian factorization systems, a Grothendieck double construction which produces a double category from an indexed category.

**Definition 3.8.** Let \(\mathcal{E} : B^{\text{op}} \to \text{Cat}\) be an indexed category, and let \(f^*\) denote \(\mathcal{E}(f)\). Its *Grothendieck double construction* \(\mathcal{E} \dashv \mathcal{E}^{\text{op}}\) is the double category with:

- **Objects** pairs \((\begin{array}{c}E \\
B\end{array})\) with \(B \in B\) and \(E \in \mathcal{E}(B)\).

- **Vertical morphisms** \((\begin{array}{c}f^* \\
f\end{array}) : \begin{array}{c}E \\
B\end{array} \Rightarrow \begin{array}{c}E' \\
B'\end{array}\) are morphisms in the Grothendieck construction \(\int \mathcal{E}^{\text{op}}\) of the pointwise opposite of \(\mathcal{E}\), namely pairs \(f : B \to B'\) and \(f^* : f^* B' \to B\).

- **Horizontal morphisms** \((\begin{array}{c}g^* \\
g\end{array}) : \begin{array}{c}E \\
B\end{array} \Rightarrow \begin{array}{c}E' \\
B'\end{array}\) are morphisms in the Grothendieck construction \(\int \mathcal{E}\) of \(\mathcal{E}\), namely pairs \(g : B \to B'\) and \(g^* : E \to g^* E'\).

- **There is a square**

\[
\begin{array}{c}
\begin{array}{c}
E_1 \\
B_1
\end{array} \\
\downarrow
\end{array}
\xRightarrow{\left( \begin{array}{c}
g_1 \\
g
\end{array} \right)}
\begin{array}{c}
\begin{array}{c}
E_2 \\
B_2
\end{array} \\
\downarrow
\end{array}
\end{array}
\xRightarrow{\left( \begin{array}{c}
f_2 \\
f
\end{array} \right)}
\begin{array}{c}
\begin{array}{c}
E_3 \\
B_3
\end{array} \\
\downarrow
\end{array}
\xRightarrow{\left( \begin{array}{c}
g_2 \\
g
\end{array} \right)}
\begin{array}{c}
\begin{array}{c}
E_4 \\
B_4
\end{array}
\end{array}
\]

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if and only if the following diagrams commute:

\[
\begin{array}{c}
B_1 \xrightarrow{g_1} B_2 \\
f_1 & \downarrow \quad \downarrow f_2 \\
B_3 \xrightarrow{g_2} B_4 \\
\end{array}
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\begin{array}{c}
f_1^* E_3 \xrightarrow{f_1^*} E_1 \\
f_1^* g_2^* E_4 \xrightarrow{g_1^* g_2^*} g_1^* E_2 \\
\end{array}
\]

We will call the squares in the Grothendieck double construction \textit{commuting squares}, since they represent the proposition that the “lower” and “upper” squares appearing in their boundary commute.

Composition is given as in the appropriate Grothendieck constructions. It just remains to show that commuting squares compose.

- For vertical composition we appeal to the following diagram:

\[
\begin{array}{c}
f_1^* f_3^* E_5 \xrightarrow{f_1^* f_3^*} f_1^* E_3 \xrightarrow{f_1^*} E_1 \\
f_1^* f_3^* g_3^* E_6 \xrightarrow{g_1^* f_3^* g_3^*} g_1^* f_3^* E_6 \xrightarrow{g_1^* f_3^*} g_1^* E_2 \\
\end{array}
\]

The outer diagram is the “upper” square of the composite, while the “upper” squares of each factor appear in the top left and right respectively.

- For horizontal composition we appeal to the following diagram:

\[
\begin{array}{c}
f_1^* E_3 \xrightarrow{f_1^*} E_1 \\
f_1^* g_2^* E_4 \xrightarrow{g_1^* f_3^*} g_1^* E_2 \\
\end{array}
\]

Theorem 3.9. For an indexed category \( \mathcal{E} : \mathcal{B}^{op} \rightarrow \mathbf{Cat} \), there is an equivalence of double categories between the Grothendieck double construction of \( \mathcal{E} \) and the double category of spans in its Grothendieck construction with left leg vertical and right leg horizontal.
Proof. By Theorem 3.7, the vertical and horizontal categories of these two double categories are equivalent. It remains to show that there exists a map of spans

\[
\begin{array}{c}
(E_1) \\ (B_1)
\end{array}
\Rightarrow
\begin{array}{c}
(E_2) \\ (B_2)
\end{array}
\]

\[
\begin{array}{c}
(f^*_1) \\ (g_1)
\end{array}
\Rightarrow
\begin{array}{c}
(f^*_2) \\ (g_2)
\end{array}
\]

if and only if the appropriate diagrams as in Diagram 3 commute. We note that, by Lemma 3.2, such a map of spans is unique if it exists.

A dashed map as in Diagram 4 consists of a component \( x : B_1 \to B_2 \) and \( y : f^*_1 E_1 \to x^* f^*_2 E_4 \). We consider what the diagram says about \( x \) first. By the commutativity of the top square, \( x \) must equal \( g_1 \), and therefore the bottom square expresses the commutativity of the left square of Diagram 3.

Now, \( y \) must have signature \( f^*_1 E_3 \to g_1^* f^*_2 E_4 \), or equivalently \( f^*_1 E_1 \to f^*_1 g^*_2 E_4 \) by the fact that \( f_2 g_1 = g_2 f_1 \). The bottom square now says that \( y = f^*_1 g^*_2 \), so that the top square now gives us the right square of Diagram 3.

Of course, if the squares of Diagram 3 commute, then we can make these choice of \( x \) and \( y \) in order to give such a morphism of spans.

There is a useful corollary of this result: any pullback preserving functor \( F : \mathcal{E} \to \mathcal{C} \) can extend to a double functor \( \mathcal{E} \times \mathcal{E}^\vee \to \text{Span}(\mathcal{C}) \).

Finally, we show that this construction is 2-functorial.

**Theorem 3.10.** The assignment \( \mathcal{C} \mapsto \mathcal{C} \times \mathcal{C}^\vee \) sending a cartesian factorization system to the double category of spans with left leg vertical and right leg cartesian gives a 2-functor \( \text{Cart} \to \text{Dbl} \) from the 2-category of cartesian factorization systems and functors which preserve both classes to the 2-category of double categories, functors, and horizontal transformations.

**Proof.** A functor \( F : \mathcal{C}_1 \to \mathcal{C}_2 \) which preserves both classes with therefore preserve pullbacks of the form

\[
\begin{array}{c}
\bullet \\
\downarrow
\end{array}
\Rightarrow
\begin{array}{c}
\bullet \\
\downarrow
\end{array}
\]

by Lemma 2.2. The assignment \( \mathcal{C} \times \mathcal{C}^\vee \) is therefore transparently functorial.
Given a natural transformation $\alpha : F \Rightarrow G : C_0 \to C_1$, we may construct a horizontal natural transformation $\overline{\alpha} : F \wr F^\vee \Rightarrow G \wr G^\vee$ using the action of $\alpha$. To every object $C$ of $C_0 \wr C_0^\vee$, we assign the horizontal map $\alpha_C : FC \to GC$. To every span $C \xleftarrow{Z} C' \xrightarrow{C'}$ we assign the map of spans:

\[
\begin{array}{c}
FC \xrightarrow{\alpha_C} GC \\
FZ \xrightarrow{\alpha_Z} GZ \\
FC' \xrightarrow{\alpha_{C'}} GC'
\end{array}
\]

These satisfy the required laws quite straightforwardly from the naturality of $\alpha$ and the uniqueness part of the universal property of pullbacks. 

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