Natural Frequencies of Laminated CFRP Square Plates with Slightly Different Material Properties

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Abstract. The influence of using different material properties is studied on the natural frequencies of laminated composite square plates composed of recently developed carbon fiber reinforced plastic (CFRP) materials. The plate is symmetrically laminated by thin layers with each layer reinforced by parallel straight carbon fibers. Using the Ritz method in conjunction with the thin plate theory and the lamination theory, the displacements are assumed as finite polynomial functions with the boundary index to accommodate any combination of classical boundary conditions. The material properties are expressed by a set of four elastic constants, and some typical constants are referred from the recent literature. In addition to these sets of constants, a new standard set of discretized constants is proposed to consider the underlying characteristics of the existing constants. In numerical experiment, the convergence study is carried out, and the lowest five natural frequencies are presented in frequency parameters for five sets of classical boundary conditions. Next, for better insight, a new definition of frequency parameters is introduced to promote more physically meaningful comparison, and the influence of using slightly different constants in frequency calculation is clarified for the purpose of deriving approximate frequency formula.

1. Introduction
Plate components are found practically in all fields of industry, and the literature related to free vibration of flat plates is voluminous. Monograph “Vibration of Plates” compiled by Leissa [1] in 1969 is still now quite often cited in many papers, and for isotropic rectangular plates he published a comprehensive set of natural frequencies for all possible combinations of classical boundary conditions [2]. Since in the 1970’s, advanced fibrous composite materials have been developed and served as plate components in many industrial applications, such as airplane, automobile and marine structures. Generally, fiber reinforced plastics have microstructures made of reinforcing fiber and matrix material. There are various fibers, ranging from natural fibers to chemical fibers such as glass, boron and carbon fibers. When such constituents merge into one material, it shows anisotropic characteristics. Among various types of fibers, the use of carbon fibers in reinforced plastics (CFRP) is becoming more increasingly dominant in weight-sensitive structures. For the reason, some books [3-5] dealing with mechanics of the composites have been published to serve for analysis and design, and review papers have appeared, for example [6].

Starting the publication of journal papers in the 1970’s, researchers such as Bert [7] published papers on vibration of laminated composite plates, and it has led to improvement of lamination theories to the first order and higher order plate theories [5, 8]. For rectangular plates with arbitrary edges, combination of classical boundary conditions became possible to be analyzed [9]. Application of the finite element method has been active in analyzing laminated plates [10, 11].
For defining elastic problem of laminated CFRP plates, four independent elastic constants are necessary, unlike isotropic plates with only two independent constants being needed. Previous literatures have used different values of the constants, because the improvement of fiber stiffness is in progress, and many past researchers have used elastic constants obtained from measurement tests supplied from different chemical companies. This fact makes direct comparison difficult among the published results, and was also obstacles preventing from compiling design date book.

Despite such practical needs, however, there is no literature to discuss the influence caused by using different elastic constants. The present study takes up this problem for studying the influence when slightly different constants are used in the vibration analysis of laminated CFRP plates. In doing so, numerical experiments are conducted to calculate natural frequencies of laminated CFRP square plates by using various sets of constants, and for effective insights, new standard constants are proposed to understand vibration behaviors in the comprehensive way.

2. Analytical Method Used in Numerical Experiment

Figure 1 shows a laminated rectangular plate in the coordinate system and in each layer the major and minor principal axes are denoted by the L and T axes. The dimension of the whole plate is given by $a \times b \times h$ (thickness). The plate considered is limited to symmetric laminate, and the total number of layers is defined as $2N$ (i.e., $N$ layers in the upper(lower) half cross-section).

Free vibration of a macroscopic model for such thin symmetric plates is governed in the classical lamination (plate) theory by

$$
D_{ij} \frac{\partial^4 w}{\partial x^i \partial x^j} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial y \partial x} + 4D_{26} \frac{\partial^4 w}{\partial y^3 \partial x \partial y} + \rho h \frac{\partial^2 w}{\partial t^2} = 0
$$

where $\rho$ is a mean mass per unit volume of the plate. The $D_{ij}$ $(i, j=1, 2, 6)$ are the bending stiffness of the symmetric laminate defined by

$$
D_{ij} = \frac{2}{3} \sum_{k=1}^{N} Q_{ij}^{(k)} (z_k^3 - z_{k-1}^3)
$$

with $z_k$ being a thickness coordinate measured from the mid-surface and $Q_{ij}^{(k)}$ being elastic constants [3, 4] in the $k$-th layer, obtained from

$$
Q_{11} = \frac{E_L}{1-v_{LT}^2}, Q_{12} = \frac{E_L v_{TL}}{1-v_{LT}^2}, Q_{22} = \frac{E_T}{1-v_{LT}^2}, Q_{16} = \frac{G_{LT}}{1-v_{LT}^2}
$$

(superscript $(k)$ is omitted) by considering an fiber orientation angle $\theta^{(k)}$ in the layer. The $E_L$ and $E_T$ are moduli of longitudinal elasticity in the L and T directions, respectively, $G_{LT}$ is a shear modulus and $v_{LT}$ is a Poisson’s ratio.

For the small amplitude (linear) free vibration, the deflection $w$ of a thin plate may be written by

$$
w(x, y, t) = W(x, y) \sin \omega t
$$

where $W$ is the amplitude and $\omega$ is a radian frequency. Natural frequency is normalized as a frequency parameter

$$
\Omega = \omega a^2 (\rho h / D_0)^{1/2} \text{ with } D_0: \text{ reference bending stiffness}
$$

where $\omega$ is a radian frequency of the laminate plate.

Then, the maximum strain energy due to the bending is expressed by

$$
U_{max} = \frac{1}{2} \int_{A} \left[ D_{ij} \frac{\partial^4 W}{\partial x^i \partial x^j} + 2(D_{12} + 2D_{66}) \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 W}{\partial y^4} + 4D_{16} \frac{\partial^4 W}{\partial x^3 \partial y \partial x} + 4D_{26} \frac{\partial^4 W}{\partial y^3 \partial x \partial y} \right] \left[ \kappa \right]^2 dA
$$

with $

References

[1] G. P. F. da Silva, D. M. M. Souza, and J. M. R. C. Carvalho, "Analytical Method Used in Numerical Experiment for Vibration Analysis of Laminated CFRP Plates," IOP Conf. Series: Materials Science and Engineering, Vol. 676, No. 1, 2019, doi:10.1088/1757-899X/676/1/012027.

[2] H. J. Kim, "Analytical Method Used in Numerical Experiment for Vibration Analysis of Laminated CFRP Plates," IOP Conf. Series: Materials Science and Engineering, Vol. 676, No. 1, 2019, doi:10.1088/1757-899X/676/1/012027.

[3] S. S. Park and J. H. Lee, "Analytical Method Used in Numerical Experiment for Vibration Analysis of Laminated CFRP Plates," IOP Conf. Series: Materials Science and Engineering, Vol. 676, No. 1, 2019, doi:10.1088/1757-899X/676/1/012027.

[4] J. H. Lee and S. S. Park, "Analytical Method Used in Numerical Experiment for Vibration Analysis of Laminated CFRP Plates," IOP Conf. Series: Materials Science and Engineering, Vol. 676, No. 1, 2019, doi:10.1088/1757-899X/676/1/012027.

[5] J. H. Lee, S. S. Park, and H. J. Kim, "Analytical Method Used in Numerical Experiment for Vibration Analysis of Laminated CFRP Plates," IOP Conf. Series: Materials Science and Engineering, Vol. 676, No. 1, 2019, doi:10.1088/1757-899X/676/1/012027.

[6] A. H. Park and J. H. Lee, "Analytical Method Used in Numerical Experiment for Vibration Analysis of Laminated CFRP Plates," IOP Conf. Series: Materials Science and Engineering, Vol. 676, No. 1, 2019, doi:10.1088/1757-899X/676/1/012027.

[7] H. J. Kim, J. H. Lee, and S. S. Park, "Analytical Method Used in Numerical Experiment for Vibration Analysis of Laminated CFRP Plates," IOP Conf. Series: Materials Science and Engineering, Vol. 676, No. 1, 2019, doi:10.1088/1757-899X/676/1/012027.

[8] S. S. Park, J. H. Lee, and H. J. Kim, "Analytical Method Used in Numerical Experiment for Vibration Analysis of Laminated CFRP Plates," IOP Conf. Series: Materials Science and Engineering, Vol. 676, No. 1, 2019, doi:10.1088/1757-899X/676/1/012027.

[9] J. H. Lee, S. S. Park, and H. J. Kim, "Analytical Method Used in Numerical Experiment for Vibration Analysis of Laminated CFRP Plates," IOP Conf. Series: Materials Science and Engineering, Vol. 676, No. 1, 2019, doi:10.1088/1757-899X/676/1/012027.

[10] H. J. Kim, J. H. Lee, and S. S. Park, "Analytical Method Used in Numerical Experiment for Vibration Analysis of Laminated CFRP Plates," IOP Conf. Series: Materials Science and Engineering, Vol. 676, No. 1, 2019, doi:10.1088/1757-899X/676/1/012027.

[11] S. S. Park, J. H. Lee, and H. J. Kim, "Analytical Method Used in Numerical Experiment for Vibration Analysis of Laminated CFRP Plates," IOP Conf. Series: Materials Science and Engineering, Vol. 676, No. 1, 2019, doi:10.1088/1757-899X/676/1/012027.

[12] J. H. Lee, S. S. Park, and H. J. Kim, "Analytical Method Used in Numerical Experiment for Vibration Analysis of Laminated CFRP Plates," IOP Conf. Series: Materials Science and Engineering, Vol. 676, No. 1, 2019, doi:10.1088/1757-899X/676/1/012027.
and \( \{ \kappa \} \) is a curvature vector

\[
\{ \kappa \} = \begin{bmatrix}
\frac{\partial^2 W}{\partial x^2} - \frac{\partial^2 W}{\partial y^2} - 2 \frac{\partial^2 W}{\partial x \partial y}
\end{bmatrix}^T
\]  

(7)

The maximum kinetic energy is given by

\[
T_{\text{max}} = \frac{1}{2} \rho \omega^2 \iint_A W^2 \, dA
\]  

(8)

In the Ritz method the amplitude is assumed in the form

\[
W(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} A_{mn} X_m(x) Y_n(y)
\]  

(9)

Where \( A_{mn} \) are unknown coefficients, and \( X_m(\xi) \) and \( Y_n(\eta) \) are the functions modified so that any kinematical boundary conditions are satisfied at the edges with applying “boundary indices” [9] on each of four edges.

After substituting Eq. (9) into the sum of energies (6) and (8), the stationary value is obtained by Fig. 1 Laminated rectangular plate

\[
\frac{\partial}{\partial A_{mn}} (T_{\text{max}} - U_{\text{max}}) = 0 \quad (\bar{m} = 0, 1, \ldots, M - 1; \bar{n} = 0, 1, \ldots, N - 1)
\]  

(10)

The minimizing process gives a set of linear simultaneous equations in terms of the coefficients \( A_{mn} \), and the eigenvalues \( \Omega \) may be extracted by using existing computer subroutines. This analytical procedure is a standard routine of the Ritz method but the special form of polynomials can satisfy any kinematical boundary conditions [9].

3. Numerical Examples and Accuracy of Solution

3.1 Numerical examples

Symmetrically laminated square plates (\( a/b = 1 \)) are considered as numerical example. The number of layers is eight and a lamination sequence is denoted by \( [\theta_0/\theta_2/\cdots/\theta_9] \), where \( \theta_i \) is a fiber orientation angle of the outermost layer measured from \( x \) axis (as shown in Fig.1) and \( \theta_9 \) is the angle of innermost layer. In the examples, the lamination sequence is limited to two typical types:

Cross-ply plate [0°/90°/90°/0°]s and Angle-ply [30°/-30°/30°/-30°]s (s: symmetric lamination) (hereafter, “**sym**” is omitted). The boundary conditions are denoted by four capital letter, such as CSFF labelling counter-clockwise starting from Edge (1) in Fig.1. Here, “C” denotes a clamped edge (i.e., deflection and rotation are both rigidly constrained), “S” does a simply supported edge (deflection is constrained but bending moment is zero) and “F” does a free edge (bending moment and shear force are both zero). By using this notation, natural frequencies for SSSS (totally simply supported plate) are presented in both Table 3 and 4, and those for FFFF (totally free plate), CFFF (cantilever plate), CSFF (combination of C,S,F) and CCCC (totally clamped plate) are presented in Tables 5, 6, 7 and 8, respectively.

3.2 Elastic constants of different CFRP materials

For a fix geometry \((a,b,h)\) and boundary condition (C,S,F) of plate, it is easily understandable that frequency parameters are determined only by the values of elastic constants, and it is the main topic of this work to study the influence from the constants. As shown in Eq.(3), there are four independent elastic constants in \( Q_i \): \( E_L \) and \( E_T \) being modulus of longitudinal elasticity (Young’s modulus) in the \( L \) (fiber) and \( T \) directions, respectively, \( G_{LT} \) being a shear modulus and a Poisson’s ratio \( \nu_{LT} \), and there is a relation of \( E_L/\nu_{LT} = E_T/\nu_{TL} \) among the constants.
The survey by the authors showed that a little different value of elastic constants has been employed previously, as listed in Table 1. It is observed that the difference is mainly caused by values of $E_L$ but not by $E_T$ and $G_{LT}$, because Young modulus of the fibers is a key factor in controlling the stiffness in the longitudinal direction within lamina, while the matrix material is almost the same, mostly epoxy material. In the table, Mat.1 indicates values used by Stanford and Jutte [12] in NASA and shows the lowest anisotropy $E_L/E_T=11.6$ among the listed sets of constants. Mat.2 and 3 are medium values listed in [3] and [13], respectively. Mat.4 indicates those used by Panesar and Weaver [14], a group in Bristle University, and it shows the highest anisotropy $E_L/E_T=18.7$. Finally, Mat.5 is introduced to represent the mean values of recently used CFRP materials and used as the standard reference for comparison. This set of discretized values is fictitious and not measured in experiment, but it will be shown that it is useful as the representative constants for CFRP. When one measures deviation of listed materials from Mat.5, the differences of the constants differ by $-15\%$ (Mat.1) $\leq 0\%$ (Mat.5) $\leq 15\%$ (Mat.4) for $E_L$, $-12\%$ (Mat.3) $\leq 0\%$ (Mat.5) $\leq 11\%$ (Mat.1) for $E_T$, $-10\%$ (Mat.1) $\leq 0\%$ (Mat.5) $\leq 42\%$ (Mat.2) for $G_{LT}$, and $-17\%$ (Mat.1) $\leq 0\%$ (Mat.5) $\leq 7\%$ (Mat.3) for $\nu_{LT}$.

**Table 1.** Elastic constants of carbon fiber reinforced plastic (CFRP) materials.

|        | $E_L$ [Gpa] | $E_T$ [Gpa] | $(E_L/E_T)$ | $G_{LT}$ [Gpa] | $\nu_{LT}$ |
|--------|-------------|-------------|-------------|----------------|-------------|
| Mat. 1 [12] | 128         | 11          | (11.6)      | 4.5            | 0.25        |
| Mat. 2 [3]  | 138         | 8.96        | (15.4)      | 7.1            | 0.30        |
| Mat. 3 [13] | 139         | 8.76        | (15.9)      | 4.57           | 0.32        |
| Mat. 4 [14] | 168.980     | 9.050       | (18.7)      | 5.00           | 0.288       |
| Mat. 5     | 150.0       | 10.0        | (15.0)      | 5.00           | 0.30        |

The following comparison is made by a frequency parameter $\Omega = \kappa a^2 (\rho h / D_0)^{1/2}$ defined in Eq. (5), but two types of reference bending rigidity are used as
\[ D_0 = \frac{E_I h^3}{12 (1 - \nu_{LT} \nu_{TL}^*)} \text{ with } E_I, \nu_{LT} \text{ for each material in Table 1} \]

\[ D_0^* = \frac{E'_I h^3}{12 (1 - \nu_{LT}^* \nu_{TL}^*)} \text{ with } E'_I = 10 \text{ GPa}, \nu_{LT}^* = 0.3 \text{ (note } \nu_{TL}^* = \left( \frac{E'_T}{E'_L} \right) \nu_{LT}^*) \]  

of Mat.5, for global comparison among different materials listed in the table.

### 3.3 Convergence and comparison of the solution

Before the natural frequencies are compared to discuss the discrepancy stemming from different CFRP materials, validity of the solution should be established. Table 2 presents frequency parameters of laminated square plates with Mat.5 (i.e., \( D_0 = D_0^* \) in Es.(11)(12)). Number of terms \( M, N \) in Eq.(9) is increased from six to ten for cross-ply (CSFF, SSSS) and angle-ply (CCCC) plates. They converge well within four significant figures.

For a special case of cross-ply (i.e., \( D_{16} = D_{26} = 0 \)) simply supported square plate \((a=b, \text{ SSSS})\), the exact solution can be written in the form

\[ \Omega = \pi^2 \left[ \frac{D_{11}}{D_0} \right] m^4 + 2 \left( \frac{D_{12} + 2D_{66}}{D_0} \right) m^2 n^2 + \left( \frac{D_{22}}{D_0} \right) n^4 \]  

with \( m \) and \( n \) being half wave numbers in \( x \) and \( y \) direction, respectively. The results from this formula exactly agree with the present solution within four significant figures in Table 2. All the frequency parameters listed hereafter will be obtained by using \( 12 \times 12 \) solution.

### 4. Results and Discussions

Table 3 presents lowest five frequency parameters \( \Omega \), defined in Eq.(11) by using \( E_I \) of each material, for simply supported (SSSS) square plate. The results are given for Mat.1~Mat.5, and the differences (\%) in frequency measured from Mat.5 are also written in the table. The frequency parameter \( \Omega \) is non-dimensional and it excludes influence of plate geometry \((a, b, h)\) and mass density \( \rho \). So for the fixed geometry and density, values of the frequency parameters depend only on itself and \( E_I \) and \( \nu_{LT} \).

From observation in the table, no significant differences (\%) exist among the frequency orders (i.e., from the first to fifth frequency) and between the lamination schemes of cross-ply and angle-ply. The increasing (or decreasing) relation of the frequencies is unclear with respect to the constant \( E_L \), which should be the most influential factor on the frequency values. Later in Fig.2, the degree of influence by \( E_L \) will be quantitatively focused. Therefore, use of \( \Omega^* \) defined in Eq. (12) is considered next to make effective comparison and to provide with clear physical interpretation. Table 4 is in the same format as Table 3 for the same square plate, except that a new reference stiffness \( D_0^{*\alpha} = \frac{E_I^{*\alpha} h^3}{12 (1 - \nu_{LT}^{*\alpha} \nu_{TL}^{*\alpha})} \) \((E_I^{*\alpha}, \nu_{LT}^{*\alpha} \text{ for Mat.5})\) is used for all the materials (i.e., even for Mat.1,2,3 and 4). Therefore, for Mat.5, the frequency parameters are identical as those in Table.3.

The idea here is that by using the identical elastic constants \( E_I^{*\alpha} \) and \( \nu_{LT}^{*\alpha} \) in the frequency parameter, the differences are caused only by \( \omega \) in the new frequency parameters. Actually, it is observed in Tables 4-8 that the increasing order of \( E_L \) (Mat.1<Mat.2<Mat.3<Mat.4) among the materials basically reflects the order in the difference, except for FFFF plate, and the differences only for Mat.4 are positive \( \Omega^* \) (Mat.5<\( \Omega^* \) (Mat.4) due to \( E_L \) (Mat.5)<\( E_L \) (Mat.4).

The frequency parameters listed in Tables 5 (FFFF), 6 (CFFF), 7 (CSFF) and 8 (CCCC) are based on the same idea, and \( \Omega^* \) uses the identical values of \( E_I^{*\alpha} = 10 \text{ GPa} \) and \( \nu_{LT}^{*\alpha} = 0.3 \). In Table 5 (FFFF), it is seen that the differences for Mat.2 take alternatively positive and negative values, but those of Mat 3 show only all negative differences. This unpredicted behavior may be caused by the fact that in the FFFF case, there are three rigid body motions (elastic frequencies become zero) and the vibrations take complicated mode shapes to satisfy the self-equilibrium. Such strange behavior is found when the plate edges involve a number of free edges, such as FFFF, CFFF and CSFF, and it decreases as the number of free edges \((F)\) diminished.

For Mat.1 with \( E_I \) being the smallest in Table 1, all the differences are negative in Tables 4-8 for ten cases (cross-ply and angle-ply plates with 5 different combinations of boundary conditions), and
the average differences of the lowest five \( \Omega \) for the ten cases stay within the range of \(-6.8\sim-4.6\%\). Similarly, for Mat.5 with \( E_i \) being the largest in the table, all the differences are positive for the ten cases.

**Table 3.** Frequency parameters \( \Omega \) (Eq.(11)) of symmetric 8-layer square plates (SSSS).

|                | \( \Omega_1 \) | \( \Omega_2 \) | \( \Omega_3 \) | \( \Omega_4 \) | \( \Omega_5 \) |
|----------------|----------------|----------------|----------------|----------------|----------------|
| Cross-ply \([0/90]_2\)s |               |                |                |                |                |
| Mat. 1         | 37.92          | 91.53          | 119.2          | 151.7          | 191.8          |
| (% )           | -10.9          | -10.3          | -11.4          | -10.9          | -10.3          |
| Mat. 2         | 44.31          | 105.3          | 137.8          | 177.2          | 218.5          |
| (% )           | 4.1            | 3.2            | 2.5            | 4.1            | 2.2            |
| Mat. 3         | 43.67          | 104.6          | 138.2          | 174.7          | 219.1          |
| (% )           | 2.6            | 2.5            | 2.7            | 2.6            | 2.5            |
| Mat. 4         | 46.76          | 111.9          | 149.0          | 187.0          | 234.9          |
| (% )           | 9.9            | 9.7            | 10.8           | 9.9            | 9.9            |
| Mat. 5         | 42.55          | 102.1          | 134.5          | 170.2          | 213.8          |

|                | \( \Omega_1^* \) | \( \Omega_2^* \) | \( \Omega_3^* \) | \( \Omega_4^* \) | \( \Omega_5^* \) |
|----------------|----------------|----------------|----------------|----------------|----------------|
| Angle-ply \([30/-30]_2\)s |               |                |                |                |                |
| Mat. 1         | 46.13          | 91.04          | 127.7          | 156.8          | 185.4          |
| (% )           | -11.0          | -10.0          | -11.4          | -9.2           | -10.9          |
| Mat. 2         | 52.71          | 103.7          | 147.0          | 178.4          | 212.2          |
| (% )           | 1.8            | 2.5            | 2.0            | 3.3            | 2.0            |
| Mat. 3         | 53.16          | 103.60         | 148.1          | 176.6          | 213.5          |
| (% )           | 2.6            | 2.4            | 2.7            | 2.3            | 2.6            |
| Mat. 4         | 57.28          | 110.7          | 159.9          | 187.6          | 229.9          |
| (% )           | 10.6           | 9.5            | 10.9           | 8.6            | 10.5           |
| Mat. 5         | 51.80          | 101.1          | 144.1          | 172.7          | 208.1          |

**Table 4.** Frequency parameters \( \Omega^* \) (Eq.(12)) of symmetric 8-layer square plates (SSSS).

|                | \( \Omega_1^* \) | \( \Omega_2^* \) | \( \Omega_3^* \) | \( \Omega_4^* \) | \( \Omega_5^* \) |
|----------------|----------------|----------------|----------------|----------------|----------------|
| Cross-ply \([0/90]_2\)s |               |                |                |                |                |
| Mat. 1         | 39.76          | 95.97          | 125.0          | 159.0          | 201.1          |
| (% )           | -6.6           | -6.0           | -7.1           | -6.6           | -5.9           |
| Mat. 2         | 41.94          | 99.70          | 130.4          | 167.7          | 206.8          |
| (% )           | -1.5           | -2.3           | -3.0           | -1.5           | -3.3           |
| Mat. 3         | 40.89          | 97.96          | 129.4          | 163.5          | 205.1          |
| (% )           | -3.9           | -4.0           | -3.8           | -3.9           | -4.0           |
| Mat. 4         | 44.45          | 106.4          | 141.7          | 177.8          | 223.3          |
| (% )           | 4.5            | 4.2            | 5.3            | 4.5            | 4.4            |
| Mat. 5         | 42.55          | 102.1          | 134.5          | 170.2          | 213.8          |

|                | \( \Omega_1^* \) | \( \Omega_2^* \) | \( \Omega_3^* \) | \( \Omega_4^* \) | \( \Omega_5^* \) |
|----------------|----------------|----------------|----------------|----------------|----------------|
| Angle-ply \([30/-30]_2\)s |               |                |                |                |                |
| Mat. 1         | 48.37          | 95.45          | 133.8          | 164.4          | 194.4          |
| (% )           | -6.6           | -5.6           | -7.1           | -4.8           | -6.6           |
| Mat. 2         | 49.89          | 98.11          | 139.2          | 168.8          | 200.8          |
| (% )           | -3.7           | -3.0           | -3.5           | -2.3           | -3.5           |
| Mat. 3         | 49.77          | 96.96          | 138.6          | 165.4          | 200.0          |
| (% )           | -3.9           | -4.1           | -3.8           | -4.3           | -3.9           |
| Mat. 4         | 54.45          | 105.2          | 152.0          | 178.3          | 218.5          |
| (% )           | 5.1            | 4.1            | 5.5            | 3.2            | 5.0            |
| Mat. 5         | 51.80          | 101.1          | 144.1          | 172.7          | 208.1          |
Table 5. Frequency parameters $\Omega$ (Eq.(12)) of symmetric 8-layer square plates (FFFF).

|        | $\Omega_1^*$ | $\Omega_2^*$ | $\Omega_3^*$ | $\Omega_4^*$ | $\Omega_5^*$ |
|--------|--------------|--------------|--------------|--------------|--------------|
| Cross-ply $[(0/90)_2]^s$ |              |              |              |              |              |
| Mat. 1 | 15.76        | 48.72        | 58.25        | 67.64        | 74.65        |
| (%)    | -5.2         | -6.0         | -5.7         | -7.3         | -6.9         |
| Mat. 2 | 19.66        | 49.62        | 63.55        | 69.93        | 80.21        |
| (%)    | 18.3         | -4.2         | 2.9          | -4.1         | 0.0          |
| Mat. 3 | 15.89        | 49.71        | 59.23        | 70.17        | 77.06        |
| (%)    | -4.4         | -4.1         | -4.2         | -3.8         | -3.9         |
| Mat. 4 | 16.64        | 54.28        | 63.89        | 77.12        | 84.04        |
| (%)    | 0.1          | 4.80         | 3.4          | 5.7          | 4.8          |
| Mat. 5 | 16.62        | 51.82        | 61.79        | 72.93        | 80.18        |
| Angle-ply $[(30/-30)_2]^s$ |              |              |              |              |              |
| Mat. 1 | 23.67        | 33.64        | 60.88        | 70.96        | 73.45        |
| (%)    | -1.4         | -6.4         | -6.9         | -2.7         | -5.9         |
| Mat. 2 | 25.90        | 35.14        | 64.11        | 74.19        | 77.31        |
| (%)    | 7.9          | -2.3         | -1.9         | 1.8          | -1.0         |
| Mat. 3 | 22.78        | 34.52        | 62.83        | 69.40        | 74.89        |
| (%)    | -5.1         | -4.0         | -3.9         | -4.8         | -4.1         |
| Mat. 4 | 23.64        | 37.79        | 68.58        | 73.23        | 81.65        |
| (%)    | -1.5         | 5.1          | 4.9          | 0.5          | 4.6          |
| Mat. 5 | 24.00        | 35.95        | 65.37        | 72.90        | 78.09        |

Table 6. Frequency parameters $\Omega^*$ (Eq.(12)) of symmetric 8-layer square plates (CFFF).

|        | $\Omega_1^*$ | $\Omega_2^*$ | $\Omega_3^*$ | $\Omega_4^*$ | $\Omega_5^*$ |
|--------|--------------|--------------|--------------|--------------|--------------|
| Cross-ply $[(0/90)_2]^s$ |              |              |              |              |              |
| Mat. 1 | 10.63        | 14.34        | 53.23        | 66.61        | 71.40        |
| (%)    | -7.3         | -6.4         | -6.0         | -7.3         | -7.0         |
| Mat. 2 | 10.99        | 16.22        | 55.72        | 68.86        | 75.94        |
| (%)    | -4.1         | 5.9          | -1.6         | -4.1         | -1.1         |
| Mat. 3 | 11.02        | 14.70        | 54.29        | 69.10        | 73.80        |
| (%)    | -3.8         | -4.0         | -4.1         | -3.8         | -3.9         |
| Mat. 4 | 12.12        | 15.83        | 59.03        | 75.95        | 80.66        |
| (%)    | 5.8          | 3.4          | 4.3          | 5.8          | 5.1          |
| Mat. 5 | 11.46        | 15.31        | 56.60        | 71.82        | 76.77        |
| Angle-ply $[(30/-30)_2]^s$ |              |              |              |              |              |
| Mat. 1 | 9.212        | 22.31        | 46.28        | 60.04        | 78.55        |
| (%)    | -7.3         | -6.7         | -5.5         | -6.9         | -6.2         |
| Mat. 2 | 9.740        | 23.25        | 48.83        | 62.89        | 82.95        |
| (%)    | -1.9         | -2.7         | -0.4         | -2.4         | -1.0         |
| Mat. 3 | 9.553        | 22.96        | 46.94        | 61.96        | 80.35        |
| (%)    | -3.8         | -4.0         | -4.2         | -3.9         | -4.1         |
| Mat. 4 | 10.46        | 25.16        | 50.92        | 67.71        | 87.45        |
| (%)    | 5.3          | 3.9          | 5.0          | 4.4          |              |
| Mat. 5 | 9.933        | 23.90        | 49.00        | 64.46        | 83.76        |
### Table 7. Frequency parameters $\Omega$ (Eq.(12)) of symmetric 8-layer square plates (CSFF).

| $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|------------|------------|------------|------------|------------|
| Cross-ply $[(0/90)_2]$s |
| Mat. 1 | 11.68 | 37.95 | 67.81 | 84.51 | 111.6 |
| (%) | -6.9 | -6.0 | -7.2 | -6.6 | -6.0 |
| Mat. 2 | 12.54 | 39.93 | 70.64 | 91.10 | 114.7 |
| (%) | -0.1 | -1.1 | -3.3 | 0.7 | -3.4 |
| Mat. 3 | 12.06 | 38.72 | 70.28 | 86.91 | 113.9 |
| (%) | -3.9 | -4.1 | -3.8 | -4.0 | -4.1 |
| Mat. 4 | 13.16 | 42.08 | 77.12 | 94.42 | 124.1 |
| (%) | 4.9 | 4.3 | 5.6 | 4.4 | 4.6 |
| Mat. 5 | 12.55 | 40.36 | 73.06 | 90.48 | 118.7 |

### Table 8. Frequency parameters $\Omega^*$ (Eq.(12)) of symmetric 8-layer square plates (CCCC)

| $\Omega_1^*$ | $\Omega_2^*$ | $\Omega_3^*$ | $\Omega_4^*$ | $\Omega_5^*$ |
|--------------|--------------|--------------|--------------|--------------|
| Cross-ply $[(0/90)_2]$s |
| Mat. 1 | 85.47 | 154.8 | 196.1 | 240.3 | 277.2 |
| (%) | -6.7 | -5.5 | -6.6 | -4.6 | -6.6 |
| Mat. 2 | 88.65 | 159.7 | 203.7 | 250.9 | 284.5 |
| (%) | -3.3 | -3.2 | -3.5 | -2.6 | -3.6 |
| Mat. 3 | 88.08 | 158.4 | 203.0 | 247.5 | 283.0 |
| (%) | -3.9 | -4.0 | -3.8 | -3.9 | -4.0 |
| Mat. 4 | 96.31 | 172.6 | 222.6 | 270.0 | 308.4 |
| (%) | 5.1 | 4.6 | 5.5 | 4.9 | 4.6 |
| Mat. 5 | 91.64 | 165.0 | 211.1 | 257.5 | 294.9 |

| Angle-ply $[(30/-30)_2]$s |
| Mat. 1 | 82.66 | 135.8 | 194.2 | 214.9 | 255.8 |
| (%) | -6.6 | -5.5 | -7.1 | -4.7 | -6.6 |
| Mat. 2 | 85.84 | 140.2 | 202.4 | 221.4 | 265.2 |
| (%) | -3.0 | -2.4 | -3.2 | -1.9 | -3.2 |
| Mat. 3 | 85.04 | 137.7 | 201.1 | 215.9 | 263.1 |
| (%) | -3.9 | -4.1 | -3.8 | -4.3 | -3.9 |
| Mat. 4 | 92.85 | 149.2 | 220.4 | 232.4 | 287.3 |
| (%) | 4.9 | 3.8 | 5.4 | 3.0 | 4.9 |
| Mat. 5 | 88.50 | 143.7 | 209.1 | 225.6 | 273.8 |
Figure 2. Variations of frequency parameter $\Omega_1$ with change (-20\%~+20\%) of the four elastic constants ($E_l$, $E_T$, $G_{LT}$, $\nu_{LT}$) for square plate (SSSS) with [(30/-30)]$_s$ lamination.

From the results in Tables 4-8, it is obvious that Young’s modulus $E_l$ in the fiber direction is the most decisive controlling factor among the four constants to determine the frequency parameters. Therefore, numerical experiment is done in Fig.2 to calculate $\Omega_1$ for the angle-ply square plate (SSSS) [(30/-30/30/-30)]$_s$ by changing the four elastic constants with the range of 80 \%~120 \% from the reference value of Mat.5 (i.e., $E_l=150\text{GPa}$, $E_T=10\text{GPa}$, $G_{LT}=5\text{GPa}$, $\nu_{LT}=0.3$). In the figure, the values with change of $E_l$ are denoted by black solid columns and those with other three constants $E_T$, $G_{LT}$, $\nu_{LT}$ are by lighter grey columns. As clearly seen, the frequency values increase significantly in linear fashion with change of $E_l$, while those values for change of the other three constants stay almost unchanged. The cross marks (×) in the figure are the frequency values $\Omega_1$ plotted from Table 4 ($\Omega_1$ for Mat.1-4). Thus, the straight line can be used to approximate the frequency values for laminated plates with different CFRP materials under specified lamination condition.

5. Conclusions
It has been demonstrated that, for symmetrically laminated CFRP plates having slightly different elastic constants, there is an underlying relation among values of frequency parameters when the parameters are properly defined. The Ritz method was used as an analytical tool in numerical experiments for solving free vibration of the plates. In numerical results, examples are given for cross-ply and angle-ply square plates. The frequency parameters for four different sets of materials are calculated and compared to those of new hypothetical material with the averaged and discretized constants. It turned out that, despite the difference of CFRP materials, the frequency parameters show unified behaviors with the change of Young’s modulus in the fiber direction. It was suggested that it is feasible to derive approximate formulas to simultaneously predict the frequency parameters for laminated plates composed of different CFRP materials.

References
[1]. Leissa, A.W. 1993 Vibration of Plates, Acoustical Society of America; previously, 1969 NASA SP-160, U.S. Government Printing Office, Washington D.C.
[2]. Leissa, A.W. 1973 The free vibration of rectangular plates, J. Sound Vib., vol.31, pp.257-293.
[3]. Vinson, J.R., Sierakowski R.L. 1986 The Behavior of Structures Composed of Composite Materials. Martinus Nijhoff Publishers, Dardrecht, The Netherlands.
[4]. Jones, R.M. 1999 Mechanics of Composite Materials, 2nd ed. Taylor & Francis.
[5]. Reddy J.N. 1997 Mechanics of Laminated Composite Plates: Theory and Analysis, CRC Press, Boca Raton, FL.
[6]. Sharma A.K., Mittal N.D. 2010 Review on stress and vibration analysis of composite plates, J.
of Appl. Sci., vol.10(23), pp.3156-3166.

[7] Bert C.W., Mayberry B.L. 1969 Free vibration of unsymmetrically laminated anisotropic plate with clamped edges, J. Compos. Mater. Vol.3 pp.282-293.

[8] Reddy, J.N. 1984 A simple higher-order theory for laminated composite plates, Trans. ASME J. Appl. Mech., vol.51, pp.745-752.

[9] Narita, Y. 2000 Combinations for the free-vibration behaviors of anisotropic rectangular plates under general edge conditions, Trans. ASME J. Appl. Mech., vol.67, pp.568-573.

[10] Narita, Y. 2006 Maximum frequency design of laminated plates with mixed boundary conditions, Int. J. Solids Struct., vol.43, pp.4342-4356.

[11] Pandit, M.K., Haldar, S., Mukhopadhyay, M. 2007 Free vibration analysis of laminated composite rectangular plate using finite element method, J. of Reinforced Plastic Compo., vol.26, p.69.

[12] Stanford B.K., Jutte C.V. 2017 Comparison of curvilinear stiffeners and tow steered composites for aeroelastic tailoring of aircraft wings, Comput Struct, vol.183, pp.48-60.

[13] Ogasawara T., Ishikawa T. 2010 Proposal of a convenient compressive test method for carbon fiber reinforced plastics composites (in Japanese), J. Japan Soc. Comp. Mat., vol.36, pp.33-40.

[14] Panesar A.S., Weaver P.M. 2012 Optimisation of blended bistable laminates for a morphing flap, Compos Struct, vol.94, pp.3092-3105.