PERFECT POWERS THAT ARE SUMS OF SQUARES IN A THREE TERM ARITHMETIC PROGRESSION

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Abstract. We determine primitive solutions to the equation \((x-r)^2 + x^2 + (x+r)^2 = y^n\) for \(1 \leq r \leq 5,000\), making use of a factorization argument and the Primitive Divisors Theorem due to Bilu, Hanrot and Voutier.

1. Introduction

Perfect powers that are sums of powers of consecutive terms in an arithmetic progression have attracted considerable attention. For example, Dickson’s History of the Theory of Numbers [Volume II, 582–588] surveys the contributions of several prominent mathematicians (including Cunningham, Catalan, Genocchi and Lucas) during the early 19th and 20th century towards solving specific cases of the Diophantine equation

\[(1) \quad x^k + (x+r)^k + \cdots + (x+(d-1)r)^k = y^n \quad x, y, d, k, r, n \in \mathbb{Z}, \quad n \geq 2.\]

This is still a remarkably active field, with recent results due to [ZB13], [Zha14], [Haj15], [BPS16], [BPSS18], [PS17], [Zha17], [Kou17] and [AGP17].

In this paper, we consider the case \(d = 3\) and \(k = 2\), namely the equation

\[(2) \quad (x-r)^2 + x^2 + (x+r)^2 = y^n, \quad x, y, r, n \in \mathbb{Z}, \quad n \geq 2.\]

In [Kou17], the first author studies equation \(2\) where \(r\) is of the form \(p^b\) with \(p\) a suitable prime. In this paper, we completely solve \(2\) for all values of \(1 \leq r \leq 5,000\), under the natural assumption \(\gcd(x, y) = 1\), using the characterization of primitive divisors in Lehmer sequences due to Bilu, Hanrot and Voutier [BHV01].

An integer solution \((x, y)\) of \(2\) is said to be primitive if \(\gcd(x, y) = 1\). This is equivalent to \(x, y, r\) being pairwise coprime. A solution where \(xy = 0\) is called a trivial solution.

Theorem 1. Let \(1 \leq r \leq 5,000\). All non-trivial primitive solutions to equation \(2\) with prime exponent \(n\) are given in Table 1.

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are non–zero coprime rational integers and \(\alpha/\beta\) and exponent \(n\) that is composite can also be recovered from Table 1. This can be done by simply checking whether \(y\) is a perfect power.

Remark: Non-trivial primitive solutions to equation (2), with \(1 \leq r \leq 5,000\) and exponent \(n\) that is composite can also be recovered from Table 1. This can be done by simply checking whether \(y\) is a perfect power.

2. Prime divisors of Lehmer sequences

A Lehmer pair is a pair \(\alpha, \beta\) of algebraic integers such that \((\alpha + \beta)^2\) and \(\alpha\beta\) are non–zero coprime rational integers and \(\alpha/\beta\) is not a root of unity. The Lehmer sequence associated to the Lehmer pair \((\alpha, \beta)\) is

\[
\tilde{u}_n = \tilde{u}_n(\alpha, \beta) = \begin{cases} 
\frac{\alpha^n - \beta^n}{\alpha - \beta} & \text{n odd} \\
\frac{\alpha^n - \beta^n}{\alpha^n - \beta^n} & \text{n even}
\end{cases}
\]

A prime \(p\) is called a primitive divisor of \(\tilde{u}_n\) if it divides \(\tilde{u}_n\) but does not divide \((\alpha^2 - \beta^2)^2 \cdot \tilde{u}_1 \cdots \tilde{u}_{n-1}\). We shall make use of the following celebrated theorem [BHVD].

| \(r\) | \((x, y, n)\) | \(r\) | \((x, y, n)\) | \(r\) | \((x, y, n)\) |
|---|---|---|---|---|---|
| 2 | (21, 11, 3) | 952 | (381, 131, 3) | 2788 | (1232, 275, 3) |
| 7 | (3, 5, 3) | 1087 | (3927, 365, 3) | 3026 | (6279, 515, 3) |
| 11 | (31, 5, 5) | 1136 | (9723, 659, 3) | 3098 | (3333, 35, 5) |
| 70 | (862389, 13067, 3) | 1190 | (1719, 227, 3) | 3109 | (627, 29, 5) |
| 79 | (63, 29, 3) | 1316 | (54561, 2075, 3) | 3193 | (76365, 2597, 3) |
| 92 | (93, 35, 3) | 1339 | (6069, 485, 3) | 3247 | (20463, 1085, 3) |
| 119 | (801, 125, 3) | 1420 | (19413, 1043, 3) | 3341 | (2961, 365, 3) |
| 133 | (17307, 965, 3) | 1442 | (1971, 251, 3) | 3395 | (837, 293, 3) |
| 146 | (9, 35, 3) | 1469 | (4695, 413, 3) | 3472 | (11637, 755, 3) |
| 155 | (369, 77, 3) | 1519 | (6513, 509, 3) | 3859 | (29406489, 137405, 3) |
| 187 | (3255, 317, 3) | 1636 | (357, 179, 3) | 3967 | (27657, 1325, 3) |
| 196 | (207, 59, 3) | 1771 | (2097, 269, 3) | 4025 | (37257, 1613, 3) |
| 197 | (13, 5, 7) | 1910 | (597, 203, 3) | 4034 | (9765, 683, 3) |
| 205 | (147, 53, 3) | 1955 | (21, 197, 3) | 4228 | (2937, 395, 3) |
| 223 | (345, 77, 3) | 1960 | (161823, 4283, 3) | 4268 | (216153, 5195, 3) |
| 262 | (89, 11, 5) | 2009 | (294837, 6389, 3) | 4277 | (1011, 341, 3) |
| 371 | (374475, 7493, 3) | 2023 | (10035, 677, 3) | 4354 | (3447, 419, 3) |
| 376 | (1071, 155, 3) | 2162 | (3729, 371, 3) | 4417 | (499, 341, 3) |
| 434 | (255, 83, 3) | 2189 | (4053, 389, 3) | 4529 | (680936595, 1116293, 3) |
| 436 | (4169, 35, 5) | 2329 | (11109, 725, 3) | 4592 | (7305, 587, 3) |
| 439 | (987, 149, 3) | 2338 | (5505, 467, 3) | 4633 | (171057, 4445, 3) |
| 623 | (291, 101, 3) | 2378 | (1651, 11, 7) | 4669 | (59007, 2189, 3) |
| 713 | (30921, 1421, 3) | 2378 | (33808666101, 15079691, 3) | 4687 | (1277, 5, 11) |
| 727 | (2133, 245, 3) | 2392 | (2826957, 28835, 3) | 4712 | (1530639, 371, 5) |
| 736 | (82035, 2723, 3) | 2410 | (3171, 347, 3) | 4718 | (8397, 635, 3) |
| 772 | (105, 107, 3) | 2504 | (1659, 275, 3) | 4759 | (36363, 1589, 3) |
| 776 | (1545, 203, 3) | 2563 | (723, 245, 3) | 4808 | (1269, 371, 3) |
| 866 | (861, 155, 3) | 2567 | (13419, 821, 3) | 4961 | (4451643, 39029, 3) |
| 889 | (1095, 173, 3) | 2599 | (14637, 869, 3) | | |

Table 1. Triples of non-trivial primitive solutions \((x, y, n)\) of (2) for the values of \(1 \leq r \leq 5,000\).
Theorem 2 (Bilu, Hanrot and Voutier). Let \( \alpha, \beta \) be a Lehmer pair. Then \( \tilde{u}_n(\alpha, \beta) \) has a primitive divisor for all \( n > 30 \), and for all prime \( n > 7 \).

3. Proof of Theorem 1

We can rewrite (2) as
\[
3x^2 + 2r^2 = y^n.
\]
Suppose \( \gcd(x, y) = 1 \); this implies that \( x, y, r \) are pairwise coprime. Note that \( n \neq 2 \) as 2 is a quadratic non-residue modulo 3. We shall henceforth suppose that \( n \) is an odd prime. We rewrite (4) as
\[
(3x^2 + 6r^2 = 3y^n,
\]
Let \( K = \mathbb{Q}(\sqrt{-6}) \) and write \( \mathcal{O}_K = \mathbb{Z}[\sqrt{-6}] \) for its ring of integers. This has class group isomorphic to \( \mathbb{Z}/2\mathbb{Z} \). We factorize the left-hand side of equation (5) as
\[
(3x + r\sqrt{-6})(3x - r\sqrt{-6}) = 3y^n.
\]
It follows that
\[
(3x + r\sqrt{-6})\mathcal{O}_K = p_3 \cdot s^n
\]
where \( p_3 \) is the unique prime of \( \mathcal{O}_K \) above 3 and \( s \) is an ideal of \( \mathcal{O}_K \). The ideal \( p_3 \) is not principal, thus \( s \) is not either, and \( p_3^2 = (3) \). We write
\[
(3x + r\sqrt{-6})\mathcal{O}_K = p_3^{1-n} \cdot (p_3\sqrt{-6})^n = (3^{(1-n)/2})(p_3\sqrt{-6})^n.
\]
It follows that \( p_3\sqrt{-6} \) is a principal ideal. Write \( p_3\sqrt{-6} = (\gamma)\mathcal{O}_K \) where \( \gamma = u + v\sqrt{-6} \in \mathcal{O}_K \) with \( u, v \in \mathbb{Z} \). After possibly changing the sign of \( \gamma \) we obtain,
\[
3x + r\sqrt{-6} = \frac{\gamma^n}{3^{(n-1)/2}}.
\]
Subtracting the conjugate equation from this equation, we obtain
\[
\frac{\gamma^n}{3^{(n-1)/2}} - \frac{\bar{\gamma}^n}{3^{(n-1)/2}} = 2 \cdot r\sqrt{-6},
\]
or equivalently,
\[
\frac{\gamma^n}{3^{n/2}} - \frac{\bar{\gamma}^n}{3^{n/2}} = 2 \cdot r\sqrt{-2}.
\]
Let \( L = \mathbb{Q}(\sqrt{-6}, \sqrt{3}) = \mathbb{Q}(\sqrt{-2}, \sqrt{3}) \). Write \( \mathcal{O}_L \) for the ring of integers of \( L \) and let
\[
\alpha = \frac{\gamma}{\sqrt{3}} \quad \text{and} \quad \beta = \frac{\bar{\gamma}}{\sqrt{3}}.
\]
Lemma 3.1. Let \( \alpha, \beta \) be as above. Then, \( \alpha \) and \( \beta \) are algebraic integers. Moreover, \( (\alpha + \beta)^2 \) and \( \alpha \beta \) are non-zero coprime rational integers and \( \alpha/\beta \) is not a unit.

Proof. Let \( \gamma = u + v\sqrt{-6} \) with \( u, v \in \mathbb{Z} \). Then
\[
(\alpha + \beta)^2 = \frac{4u^2}{3},
\]
Since \( p_3\sqrt{-6} = (\gamma)\mathcal{O}_K \) and \( p_3 | \sqrt{-6} \) we conclude that \( p_3 | u \) and so \( 3 | u \). So, \( (\alpha + \beta)^2 \) is a rational integer. If \( (\alpha + \beta)^2 = 0 \) then we have \( u = 0 \). However, from (11) and the fact that \( n \) is odd we understand that this cannot happen. Clearly, \( \alpha\beta = \frac{\gamma\bar{\gamma}}{3} \) is a non-zero rational integer.
We have to check that \((\alpha + \beta)^2\) and \(\alpha \beta\) are coprime. Suppose they are not coprime. Then there exist a prime \(q\) of \(\mathcal{O}_L\) dividing both. Then \(q\) divides \(\alpha, \beta\) and from equations (4) and (9) we understand that \(q\) divides \((y)\mathcal{O}_L\) and \((2r\sqrt{-2})\mathcal{O}_L\) which contradicts the assumption that \((x, y)\) is a non–trivial trivial solution.

Finally, we need to show that \(\alpha/\beta = \gamma/\tilde{\gamma} \in \mathcal{O}_K\) is not a root of unity. Since the only roots of unity in \(K\) are \(\pm 1\) we conclude \(\gamma = \pm \tilde{\gamma}\). Then, either \(v = 0\) or \(u = 0\) which both cannot hold because of (9).

From Lemma 3.1 we have that the pair \((\alpha, \beta)\) is Lehmer pair and we denote by \(\tilde{u}_k\) the associate Lehmer sequence. Substituting into equation (9), we see that

\[
\left(\frac{\alpha - \beta}{2\sqrt{-2}}\right) \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) = r.
\]

Hence, we get:

\[
\frac{\alpha^n - \beta^n}{\alpha - \beta} = r/v = r'.
\]

**Lemma 3.2.** For a prime \(q \nmid 6\), let

\[
B_q = \begin{cases} 
q - 1 & \text{if } \left(\frac{-6}{q}\right) = 1 \\
q + 1 & \text{if } \left(\frac{-6}{q}\right) = -1.
\end{cases}
\]

Let

\[B := \max\left(\{7\} \cup \{B_q : q \text{ prime, } q \nmid r', \ q \nmid 6v\}\right).
\]

Then \(n \leq B\).

**Proof.** Recall that the exponent \(n\) is an odd prime. Suppose \(n > 7\). By the theorem of Bilu, Hanrot and Voutier, \(\tilde{u}_n = (\alpha^n - \beta^n)/(\alpha - \beta) = \gamma\) is divisible by a prime \(q\) that does not divide \((\alpha^2 - \beta^2)^2 = -32u^2v^2/3\) nor the terms \(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_{n-1}\). Note that this is a prime \(q\) divides \(r'\) but not \(6v\). Let \(q\) be a prime of \(K = \mathbb{Q}(\sqrt{-6})\) above \(q\). As \((\alpha + \beta)^2\) and \(\alpha \beta\) are coprime integers, and as \(\alpha, \beta\) satisfy (11) we see that \(\gamma, \tilde{\gamma}\) are not divisible by \(q\). We claim the multiplicative order of the reduction of \(\gamma/\tilde{\gamma}\) modulo \(\mathbb{F}_q\) divides \(B_q\). If \(-6\) is a square modulo \(q\), then \(\mathbb{F}_q = \mathbb{F}_q^*\) and so the multiplicative order divides \(q - 1 = B_q\). Otherwise, \(\mathbb{F}_q = \mathbb{F}_q^*\). However, \(\gamma/\tilde{\gamma}\) has norm 1, and the elements of norm 1 in \(\mathbb{F}_q^*\) form a subgroup of order \(q + 1 = B_q\). Thus in either case

\[
(\gamma/\tilde{\gamma})^{B_q} \equiv 1 \pmod{q},
\]

This implies that \(q \mid \tilde{u}_n\). As \(q\) is primitive divisor of \(\tilde{u}_n\) we see that \(n \leq B_q\), proving the lemma. \(\square\)

**Proof of Theorem 1.** We notice that since \(\gcd(x, r) = 1\) we can immediately deduce that \(3 \nmid r\). We wrote a simple Sage [Dev17] script which for each \(1 \leq r \leq 5,000\) such that \(3 \nmid r\), and for each \(v \mid r\) computed \(B\) as in Lemma 3.2. For each odd prime \(n \leq B\) we know from equation (8) that \(u\) is an integer solution of the polynomial equation

\[
1 = 2 \cdot r \cdot \sqrt{-6} \cdot 3^{(n-1)/2} \cdot \left((u + v\sqrt{-6})^n - (u - v\sqrt{-6})^n\right) - 1
\]

Computing the roots of these polynomials we are able to obtain the solutions \((|x|, y, n)\) as in Table 1. \(\square\)
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