A definable Henselian valuation with high quantifier complexity

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We give an example of a parameter-free definable Henselian valuation ring which is neither definable by a parameter-free $\forall\exists$-formula nor by a parameter-free $\exists\forall$-formula in the language of rings. This answers a question of Prestel.

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1 Introduction

There have been several recent results concerning definitions of Henselian valuation rings in the language of rings, mostly using formulae of low quantifier complexity (cf. [1, 2, 4–8, 11]). After a number of these results had been proven, Prestel showed a Beth-like Characterization Theorem which gives criteria for the existence of low-quantifier definitions for Henselian valuations:

\textbf{Theorem 1.1} (Prestel, [11, Characterization Theorem]). Let $\Sigma$ be a first order axiom system in the ring language $\mathcal{L}_{\text{ring}}$ together with a unary predicate $O$. Then there exists an $\mathcal{L}_{\text{ring}}$-formula $\varphi(x)$, defining uniformly in every model $(K, O)$ of $\Sigma$ the set $O$, of quantifier type

$\exists$ iff $(K_1 \subseteq K_2 \Rightarrow O_1 \subseteq O_2)$
$\forall$ iff $(K_1 \subseteq K_2 \Rightarrow O_2 \cap K_1 \subseteq O_1)$
$\exists\forall$ iff $(K_1 \prec \exists K_2 \Rightarrow O_1 \subseteq O_2)$
$\forall\exists$ iff $(K_1 \prec \exists K_2 \Rightarrow O_2 \cap K_1 \subseteq O_1)$

for all models $(K_1, O_1), (K_2, O_2)$ of $\Sigma$. Here $K_1 \prec \exists K_2$ means that $K_1$ is existentially closed in $K_2$, i.e., every existential $\mathcal{L}_{\text{ring}}$-formula $\varphi(x_1, \ldots, x_m)$ with parameters from $K_1$ that holds in $K_2$ also holds in $K_1$.

Applying the conditions in Theorem 1.1, it is easy to see that most known parameter-free definitions of Henselian valuation rings in $\mathcal{L}_{\text{ring}}$ are in fact equivalent to $\varnothing\forall\exists$-formulae or $\varnothing\exists\forall$-formulae. Consequently, Prestel asked the following:

\textbf{Question 1.2} Let $(K, w)$ be a Henselian valued field such that $O_w$ is a $\varnothing$-definable subset of $K$ in the language $\mathcal{L}_{\text{ring}}$. Is there already a $\varnothing\forall\exists$-formula or a $\varnothing\exists\forall$-formula which defines $O_w$ in $K$?

The aim of this note is to provide a counterexample to Prestel’s question. More precisely, we show:

\textbf{Theorem 1.3} There are ordered abelian groups $\Gamma_1$ and $\Gamma_2$ such that for any PAC field $k$ with $k \neq k^\text{sep}$ the Henselian valuation ring $O_w = k((\Gamma_1))[[\Gamma_2]]$ is $\varnothing$-definable in the field $K = k((\Gamma_1))((\Gamma_2))$. However, $O_w$ is neither definable by a $\varnothing\forall\exists$-formula nor by a $\varnothing\forall\forall$-formula in $K$.

Moreover, we consider a specific example, namely the case $k = \mathbb{Q}_{\text{tot}}^\text{tot}(\sqrt{-1})$. Here, $\mathbb{Q}_{\text{tot}}^\text{tot}$ denotes the totally real numbers, that is the maximal extension of $\mathbb{Q}$ such that for every embedding of the field into the complex...
numbers the image lies inside the real numbers. By [9, Example 5.10.7], the field \( \mathbb{Q}_{\text{tot}}(\sqrt{-1}) \) is an example of a PAC field. From the results contained in this paper, it is easy to obtain an explicit \( \mathcal{L}_{\text{ring}} \)-formula which defines \( O_w \) in the field

\[
K = \mathbb{Q}_{\text{tot}}(\sqrt{-1})((\Gamma_1))((\Gamma_2))
\]

and which—by Theorem 1.3—is not equivalent to a \( \emptyset \)-\( \forall \emptyset \)-formula or a \( \emptyset \)-\( \forall \emptyset \)-formula modulo \( \text{Th}(K) \).

Note that in all examples constructed, \( w \) admits proper Henselian refinements and hence is not the canonical Henselian valuation of \( K \). Thus, our results do not contradict [5, Theorem 1.1] which states that the canonical Henselian valuation is in most cases \( \emptyset \)-\( \forall \emptyset \)-definable or \( \emptyset \)-\( \forall \emptyset \)-definable as soon as it is \( \emptyset \)-definable at all (cf. also [5] for the definition of the canonical Henselian valuation of a field).

2 The construction

2.1 The value group

In this section, we consider examples of (Hahn) sums of ordered abelian groups. For \( H \) and \( G \) ordered abelian groups, consider the lexicographic sum \( G \oplus H \), that is the ordered group with underlying set \( G \times H \) and equipped with the lexicographic order such that \( G \) is more significant. More generally, recall that for a totally ordered set \( (I, \prec) \) and a family \( (G_i)_{i \in I} \) of ordered abelian groups, there is a corresponding Hahn sum \( G := \bigoplus_{i \in I} G_i \) consisting of all sequences \( (g_i)_{i \in I} \in \prod_{i \in I} G_i \) with finite support. Componentwise addition and the lexicographic order (where \( G_i \) is more significant than \( G_j \) if \( i \prec j \)) give \( G \) the structure of an ordered abelian group. For any \( k \in I \), the final segment \( \bigoplus_{i \in I, i \geq k} G_i \) is a convex subgroup of \( G \) and the quotient of \( G \) by said subgroup is isomorphic to the corresponding initial segment \( \bigoplus_{i \in I, i \leq k} G_i \).

We consider the ordered abelian groups

\[
X := \mathbb{Z}_{(2)} = \left\{ \frac{a}{b} \in \mathbb{Q} \mid a, b \in \mathbb{Z}, 2 \nmid b \right\} \quad \text{and} \quad Y := \mathbb{Z}_{(3)} = \left\{ \frac{a}{b} \in \mathbb{Q} \mid a, b \in \mathbb{Z}, 3 \nmid b \right\}
\]

as building blocks in the construction of Hahn sums. All ordered abelian groups considered in this note are of the form \( \bigoplus_{i \in J} G_i \) for some ordered index set \( J \) with \( G_j \in \{X, Y\} \) for all \( j \in J \). Let \((\mathbb{N}, <)\) denote the natural numbers with their usual ordering and \((\mathbb{N}', <)\) the natural numbers in reverse order. Define

\[
\Gamma_1 := \bigoplus_{\mathbb{N}} X \quad \text{and} \quad \Gamma_2 := \bigoplus_{\mathbb{N}'} (X \oplus Y).
\]

Then, the ordered abelian group \( Y \) is the quotient of \( \Gamma_1 \) by its convex subgroup

\[
\Lambda_1 := \left( \bigoplus_{\mathbb{N}, [0]} (X \oplus Y) \right) \oplus \left( \bigoplus_{\mathbb{N}, [0]} \left( \bigoplus_{\mathbb{N}} (X \oplus Y) \right) \right).
\]

Note that there is an isomorphism \( f_1 : \Lambda_1 \cong \Gamma_1 \) of ordered abelian groups induced by the (unique) isomorphism of the index sets. Furthermore, \( X \oplus Y \) is a convex subgroup of \( \Gamma_2 \), with corresponding quotient

\[
\Lambda_2 = \bigoplus_{\mathbb{N}', [0]} (X \oplus Y).
\]

Again, the (unique) isomorphism of the index sets induces an isomorphism \( g_2 : \Lambda_2 \cong \Gamma_2 \). We now consider the lexicographic sum

\[
\Gamma := \Gamma_2 \oplus \Gamma_1.
\]

Lemma 2.1 Let \( \Gamma \) be as above. Then, the convex subgroup \( \Gamma_1 \) is a parameter-free \( \mathcal{L}_{\text{ring}} \)-definable subgroup of \( \Gamma \).

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Proof. We write $\Gamma$ as a Hahn sum $\Gamma = \bigoplus_{j \in J} G_j$ with $G_j \in \{X, Y\}$. There is a smallest element $k \in J$ which has a successor $k'$ such that $G_{k'} = G_k = Y$. For that $k$, one has

$$\Gamma_1 = \bigoplus_{j \in J, j > k} G_j;$$

the idea of this proof is to express this as a formula, using that $(J, <)$ is interpretable in $\Gamma$; cf., e.g., [3, 12]. We now explain this interpretation in some detail.

Fix $r \in \mathbb{N}$ (we shall only consider $r = 3, 6$). For $x \in \Gamma \setminus r \Gamma$, let $F_r(x)$ be the largest convex subgroup of $\Gamma$ which is disjoint from $x + r \Gamma$. For fixed $r$, $F_r(x)$ is definable uniformly in $x$ by [12, Lemma 2.11] or [3, Lemma 2.1], namely:

$$y \in F_r(x) \iff [0, r \max \{-y, y\}] \cap x + r \Gamma = \emptyset.$$ 

Using that all $G_j$ are Archimedean, one can check that the set of groups of the form $F_r(x) (x \in \Gamma \setminus r \Gamma)$ is exactly equal to the set of groups of the form

$$\bigoplus_{j \in J, j > j_0} G_j,$$

where $j_0$ runs over those $j \in J$ for which $G_j$ is not $r$-divisible; cf. [12, Example 2.3] or a combination of the examples in [3, §§ 4.1 & 4.2] for details.

Thus we have the interpretation $J = (\Gamma \setminus 6\Gamma) / \sim_6$, where $x \sim_r x'$ iff $F_r(x) = F_r(x')$, and

$$J_F := (\Gamma \setminus 3\Gamma) / \sim_3 = \{ j \in J \mid G_j = Y \}.$$ 

The order on $J$ is also definable, since we have

$$x / \sim_6 \leq x' / \sim_6 \iff F_6(x) \supseteq F_6(x').$$

Now our $k$ from above is a $\emptyset$-definable element of $J$ and we have $F_6(x) = \Gamma_1$ for any $x \in \Gamma \setminus 6\Gamma$ with $x / \sim_6 = k$, as desired.

Next, we give different existentially closed embeddings of $\Gamma$ into itself which we shall use to apply Prestel’s Theorem. We use the following facts:

**Theorem 2.2** (Weispfening, [13, Corollaries 1.4 & 1.7]). Let $G_1$ and $G_2$ be ordered abelian groups.

1. If $G_1$ is a convex subgroup of $G_2$, then $G_1$ is existentially closed in $G_2$.
2. Consider the Hahn sum $G = G_2 \oplus G_1$. Let $G'_1$ (resp. $G'_2$) be an ordered subgroup of $G_1$ (resp. $G_2$) that is existentially closed in $G_1$ (resp. $G_2$), and put $G' := G'_2 \oplus G'_1$. Then $G'$ is existentially closed in $G$.

The first embedding $f_3: \Gamma \to \Gamma$ which we want to consider is given by $f_1: \Lambda_1 \to \Gamma_1$ (defined above) and $f_2: \Gamma_2 \oplus Y \to \Gamma_2$ which maps $\Gamma_2$ isomorphically to $\Lambda_2$ via $g_2^{-1}$ (defined above) and which embeds $Y$ into $X \oplus Y$ as a convex subgroup:

$$f_3: \Gamma_2 \oplus \Gamma_1 = \Gamma_2 \oplus Y \oplus \Lambda_1 \cong \frac{f_2(\Gamma_2 \oplus Y) \oplus f_1(\Lambda_1)}{<3 \Gamma_2} = \Gamma_1$$

$$<3 \Gamma_2 \oplus \Gamma_1.$$ 

The second embedding is $g_3: \Gamma \to \Gamma$ given by $g_2: \Lambda_2 \to \Gamma_2$ (defined above) and $g_1 : (X \oplus Y) \oplus \Gamma_1 \to \Gamma_1$ which embeds it as a convex subgroup. More precisely, we consider the isomorphism

$$g_{1,1} : \Gamma_1 \to \left( \bigoplus_{N \setminus \{0\}} Y \right) \oplus X \oplus \bigoplus_{N \setminus \{0, 1\}} \left( \bigoplus_{N \setminus \{0\}} Y \right) \oplus X$$

induced by the (unique) order isomorphism of the index sets, and the embedding

$$g_{1,2} : X \oplus Y \to \left( \bigoplus_{N \setminus \{0\}} Y \right) \oplus X \oplus Y$$
as a convex subgroup which maps $X \oplus Y$ onto itself as a final segment of the Hahn sum on the right. Overall, we obtain the following embedding of $\Gamma$ into itself:

$$g_3 : \Gamma_2 \oplus \Gamma_1 = \Lambda_2 \oplus (X \oplus Y) \oplus \Gamma_1 \cong g_2(\Lambda_2) \oplus g_1((X \oplus Y) \oplus \Gamma_1) \sim_{\Gamma_2 \oplus \Gamma_1}$$

2.2 The residue field

Let $k$ be a PAC field which is not separably closed. Then, any Henselian valuation with residue field $k$ is $\emptyset$-definable ([8, PAC 3.5 & Theorem 3.6]). Moreover, assume that $k$ is a PAC field of characteristic 0 such that the algebraic part $k_0$ of $k$ is not algebraically closed, i.e., $k_0 := \mathbb{Q}^{alg} \cap k \subseteq \mathbb{Q}^{alg}$. By [4, Theorem 3.5] and its proof, any Henselian valuation with residue field $k$ is $\emptyset$-3-definable: In fact, for any monic and irreducible $f \in k_0[X]$ with $\deg(f) > 1$, [4, §3] gives a parameter-free $L_{ring}$-formula depending on $f$ which defines the valuation ring of $v$ in any Henselian valued field $(K, v)$ with residue field $k$.

In order to get an explicit example, we consider the maximal totally real extension $Q^{intR}$ of $\mathbb{Q}$. As mentioned in the introduction, $k := Q^{intR}(\sqrt{-1})$ is a PAC field by [9, Example 5.10.7]. Furthermore, as $\sqrt{2}$ is not totally real, $f = X^2 - 2$ is a monic and irreducible polynomial with coefficients in the algebraic part $k_0$ of $k$. Thus, by [4, Proposition 3.3], the formula

$$\eta(x) \equiv (\exists u, t)(x = u + t \land (\exists y, z, y_1, z_1)(u = y_1 - z_1 \land y_1(y^3 - 2) = 1 \land z_1(z^3 - 2) = 1) \land (\exists y, z, y_1, z_1)(t = 0 \lor (t = y_1z_1 \land y_1(y^3 - 2) = 1 \land z_1(z^3 - 2) = 1)))$$

defines the valuation ring of $v$ in any Henselian valued field $(K, v)$ with residue field $k$.

2.3 Power series fields

Now, define $K := k((\Gamma_1))((\Gamma_2)) = k((\Gamma_2 \oplus \Gamma_1))$ for $K$ PAC but not separably closed. Then, the valuation ring of the Henselian valuation $v$ on $K$ with value group $\Gamma_2 \oplus \Gamma_1$ and residue field $k$ is $\emptyset$-definable by the results discussed in the previous section. Moreover, for $k = Q^{intR}$, $O_v$ is $\emptyset$-3-definable by the formula $\eta(x)$ (as above). Let $w$ be the coarsening of $v$ with value group $\Gamma_1$ and residue field $k((\Gamma_1))$. Recall that by Lemma 2.1, the convex subgroup $\Gamma_1$ is $\emptyset$-definable in the ordered abelian group $\Gamma_2 \oplus \Gamma_1$. Thus, $w$ is $\emptyset$-definable on $K$.

We now give two different existentially closed embeddings of $K$ into itself which combined with Prestel’s Characterization Theorem show that $w$ is neither $\emptyset$-$\forall$-definable nor $\emptyset$-$\exists$-$\forall$-definable.

**Theorem 2.3** (Ax-Kochen and Eršov, cf. [10, p. 183]). Let $(K, w)$ be a Henselian valued field of equicharacteristic 0. Let $(K, w) \subseteq (L, u)$ be an extension of valued fields. If the residue field of $(K, w)$ is existentially closed in the residue field of $(L, u)$ and the value group of $(K, w)$ is existentially closed in the value group of $(L, u)$, then $(K, w)$ is existentially closed in $(L, u)$.

**Construction 2.4** Let $K = k((\Gamma_1))((\Gamma_2))$ with $\Gamma_1$ and $\Gamma_2$ as before. Let $w$ denote the power series valuation on $K$ with valuation ring $k((\Gamma_1))[[\Gamma_2]]$ and value group $\Gamma_2$.

1. Consider the existential embeddings $f_0 = \text{id}_K$, as well as $f_3$ as defined in Equation (*). By Theorem 2.3, there is an existential embedding $f : K \to K$ which prolongs $f_0$ and $f_3$. Then, as the embedding maps more than just $\Gamma_1$ into $\Gamma_2$, we have $f(O_w) \not\subseteq O_w$.

2. On the other hand, consider the existential embeddings $g_0 = \text{id}_K$, as well as $g_3$ as defined in Equation (†). Once again, there is an existential embedding $g : K \to K$ which prolongs $g_0$ and $g_3$. Then, as the embedding maps more than just $\Gamma_1$ into $\Gamma_2$, we have $g(O_w) \not\subseteq O_w$.

In particular, the Henselian valuation $w$ with value group $\Gamma_2$ is $\emptyset$-definable on

$K = Q^{intR}(\sqrt{-1})((\Gamma_1))((\Gamma_2))$

but neither $\emptyset$-$\forall$-definable nor $\emptyset$-$\exists$-$\forall$-definable by Theorem 1.1. This finishes the proof of Theorem 1.3.
References

[1] W. Anscombe and J. Koenigsmann, An existential $\emptyset$-definition of $F_q[[t]]$ in $F_q((t))$, J. Symb. Log. 79(4), 1336–1343 (2014).
[2] R. Cluckers, J. Derakhshan, E. Leenknegt, and A. Macintyre, Uniformly defining valuation rings in Henselian valued fields with finite or pseudo-finite residue fields, Ann. Pure Appl. Log. 164(12), 1236–1246 (2013).
[3] R. Cluckers and I. Halupczok, Quantifier elimination in ordered abelian groups, Confluentes Math. 3(4), 587–615 (2011).
[4] A. Fehm, Existential $\emptyset$-definability of Henselian valuation rings, J. Symb. Log. 80(1), 301–307 (2015).
[5] A. Fehm and F. Jahnke, On the quantifier complexity of definable canonical Henselian valuations, Math. Log. Q. 61(4–5), 347–361 (2015).
[6] A. Fehm and A. Prestel, Uniform definability of Henselian valuation rings in the Macintyre language, to appear in Bull. Lond. Math. Soc.
[7] J. Hong, Definable non-divisible Henselian valuations, Bull. Lond. Math. Soc. 46(1), 14–18 (2014).
[8] F. Jahnke and J. Koenigsmann, Definable Henselian valuations, J. Symb. Log. 80(1), 85–99 (2015).
[9] M. Jarden, Algebraic Patching, Springer Monographs in Mathematics (Springer-Verlag, 2011).
[10] F.-V. Kuhlmann and A. Prestel, On places of algebraic function fields, J. Reine Angew. Math. 353, 181–195 (1984).
[11] A. Prestel, Definable Henselian valuation rings, to appear in J. Symb. Log.
[12] P. H. Schmitt, Model theory of ordered abelian groups, Habilitationsschrift (Universität Heidelberg, 1982).
[13] V. Weispfenning, Existential equivalence of ordered abelian groups with parameters, Arch. Math. Log. 29(4), 237–248 (1990).