I. INTRODUCTION

While seeking the exciton Bose-Einstein condensation in double quantum wells, the several non-trivial effects were observed. A very large exciton lifetime is a special feature of excitons in double quantum well in the presence of an electric field directed in parallel to the normal of quantum well plane. The effect occurs due to the separation of electrons and holes into the different wells, which causes a very weak overlapping their wave function and the damping of the mutual recombination. The large lifetime allows one to create a high concentration of the excitons at small pumping and to study a manifestation of the effects of exciton-exciton interaction. Excitons with electrons and holes localized in different wells are called 'indirect excitons'. The nonzero dipole moment of the indirect excitons should cause their mutual repulsion that complicates the creation of the exciton condensed phase. These properties and the facts, that excitons have the integer spin and the small mass, stimulated the search of the exciton Bose-Einstein condensation in double quantum wells. These investigations gave a number of new results. So, the narrow band with unusual properties was observed and studied by Timofeev’s group in the emission spectra of the indirect excitons in AlGaAs system. It was shown that this band appears at the some threshold pumping, also the peculiarities of the temperature and pumping dependencies were revealed. The authors built a phase diagram "threshold pumping - temperature".

The nontrivial results were found in a spatial distribution of exciton emission from quantum well. In the papers an appearance of a ring outside the laser spot was observed in the emission from double quantum well. The ring radius exceeded significantly the exciton diffusion length. The explanation of the appearance of the ring was given in the papers under the assumption, that holes are captured by the well more effectively than electrons, and, in addition, there are donors in the crystal, which create some concentration of free electrons. As a result a region rich by holes arises in a vicinity of the laser spot. Outside of this region the quantum well is enriched by free electrons. On the boundary of the region the processes of a recombination take place, which causes the a creation of excitons on the ring and the appearance of a spatial distribution of the emission in the form of the ring.

Intriguing facts appear under an investigation of the spatial distribution of the exciton density. Different spatial nonhomogeneous structures were observed in the emission of the indirect excitons at the pumping greater a critical value. Thus, in the paper a division of the emission ring was observed on separate fragments periodically localized along the ring. In the paper, in which the excitation of the quantum well carried out through a window in metallic electrode, the authors found in the luminescence spectrum a periodical structure of the islands situated along the ring under perimeter of the window. The number of the islands growths with increasing of the radius of the window. The similar results were obtained not only in double quantum wells, but also in a wide quantum well in the presence of a strong electric field, which divides electrons and holes between different sides of the quantum well. As the result the excitons with charges, strong separated in the space, and with large lifetime are created. On such dipole excitons in the wide quantum well, the effects similar to those on excitons in double quantum wells were observed, namely the appearance of periodical structures of the islands under the window in a electrode and on the ring outside the laser spot. Recently Timofeev and coauthors presented examples of the structures in the emission spectra at the different forms of windows in the electrode: the rectangle, two circles and others. In the paper the authors, choosing the form of the electrode, created additional periodical potential for excitons. It was found, that besides the periodical structure imposed by external conditions, the partition of the emission into fragments was observed in the direction, in which the potential is almost uniform.

The phenomenon of the symmetry loss and the creation of structures in the emission spectra of indirect excitons stimulated a series of theoretical investigations. The authors of the work considered the instability, which arises under kinetics of level occupations by the particles with the Bose-Einstein statistics. Namely, the growth of the occupation of the level with zero moment should stimulate the transitions of excitons to this level. But the density of excitons was found greater, and the temperature was found lower than these values ob-
served on the experiments. Some authors explain the appearance of the periodicity by Bose condensation of excitons\textsuperscript{16,17}. There is a suggestion to describe the system by a nonlinear Schrödinger equation\textsuperscript{18,19}. Also a possibility of the Mott transition in considered systems was studied\textsuperscript{20}. The periodicity appearance in the exciton system was investigated in\textsuperscript{21} using Bogolyubov’s equations under some approximations of inter-exciton interaction. Let note the paper\textsuperscript{22} in which the observed in\textsuperscript{23} appearance of the islands was explained from classical model diffusion equation with Coulomb interaction between electrons, holes, and excitons. In the listed above works the main efforts were applied for the ascertain- ment of the principal possibility of the appearance of the periodicity of the exciton density distribution. A specific application of the results for an explanation of the numerous experiments of different kind (at different pumping and temperature, nonhomogeneous external fields) was not employed.

An appearance of an instability of the uniform distribution of the exciton density and a formation of the periodical distribution were shown in the work\textsuperscript{24} from a position of self-organization processes in non-equilibrium systems for excitons with attractive interaction. The phenomenon has the threshold behavior with respect to a pumping. After the successful observation of periodical structures in the system of indirect excitons in double quantum well on the base AlGaAs crystal by Timofeev’s and Butov’s groups, the several works were fulfilled\textsuperscript{24–30} in this model devoted to explanation of the experiments. The theoretical approaches of the works\textsuperscript{24–30} are based on the following assumptions.

1. There is an exciton condensed phase caused by the attractive interaction between excitons. As was mentioned, there are the dipole-dipole repulsion interaction between excitons. But the simple calculations show, that the exchange and van der Waals interactions exceed the dipole-dipole repulsion at certain distances between excitons, if the distance between quantum wells is not very far and the exciton dipole moment is not too large. An existence of attractive interaction between excitons is confirmed by the calculations of biexcitons\textsuperscript{31–34}, and under investigation of many-exciton systems\textsuperscript{35}.

2. The finite value of the exciton lifetime plays an important role in the formation of a spatial distribution of exciton condensed phases. As usually, the exciton lifetime exceeds significantly the duration of the establishment of a local equilibrium. By this reason, the lifetime of excitons is suggested to be equal infinity under the solutions of many exciton problems. But, the taking into account the finiteness of the exciton lifetime is necessary in the study of the spatial distribution phases in two-phase systems, because the exciton lifetime is less than the time of the establishment of the equilibrium between phases. The last time is determined by slow diffusion processes and is great. Just the finite exciton lifetime restricts the maximal size of the exciton condensed phase and it causes the existence of a correlation in positions of separate regions of the condensed phase. As the result the spatial structures of the condensed phase appear in the shape of separate islands (in two dimensional case), parameters of which and the mutual position depend on the exciton lifetime. Thus, the created spatial structures are non-equilibrium and they are a consequence of a self-organization in non-equilibrium systems.

The theory, developed in the works\textsuperscript{34–39}, has explained many features of the indirect exciton manifestation in the double quantum wells on the base AlGaAs crystal. So, the observed in\textsuperscript{34} behavior of narrow band as a function of the pumping and the temperature is presented in\textsuperscript{34}. It is follow from these calculations in accordance with the experimental\textsuperscript{4}, that the intensity of the emission of the narrow band decreases linearly with temperature at fixed pumping and rises ultralinearly with the growth of the pumping at fixed temperature. The theory\textsuperscript{25} has explained the experimental "threshold pumping - temperature" phase diagram obtained in\textsuperscript{25}. The works\textsuperscript{24,25–28} were dedicated to the interpretation of the appearance of the periodically situated islands in the emission of spectra from the both the ring outside the laser spot, found in\textsuperscript{24}, and from the region of the double quantum well under perimeter of the window in electrode, found in\textsuperscript{25}. The theory described the sizes of the islands, the distances between them, their appearance and vanishing depending on pumping and temperature. The calculated behavior of the condensed phase islands around two laser spots observed in\textsuperscript{25} is presented in\textsuperscript{30}. At approaching the centers of lasers spots, the rings of the emission around two spots transform from two rings around two centers to the deformed single ring with two laser spots inside the ring in according with experiment\textsuperscript{4}. The theory explained an appearance of a spike observed in\textsuperscript{25} in emission of the indirect excitons after the shutdown of a pumping by the increasing of the exciton lifetime in a consequence of the removal of an Auger processes. In the paper\textsuperscript{24} the fragmentation of the inner ring in the laser spot was predicted. These fragmentation was observed recently in the experiment\textsuperscript{25}.

Several theoretical investigations were made for new systems, which were not studied by experiments so far. It was shown, that a periodical structure of the condensed phase islands arise at the light irradiation in the quantum well under electrode with a slot\textsuperscript{29}. The chain of the islands moves in the presence of a linear potential along the slot. The process reminds the Gunn effect in semiconductors and can be called "excitonic Gunn effect". The pumping dependence study of exciton density distribution in the well under the electrode in the shape of a disk presents all stages of the phase transitions: from the islands of the condensed phase in a gas phase till the islands of the gas phase in an environment of the condensed phase\textsuperscript{30}. The investigation of excitonic pulse moving in external fields\textsuperscript{40} shown that maximum exciton density remains constant during the exciton lifetime and there is a possibility of a control of pulse moving by an another laser if the formation of the pulse occurs by
excitons in the condensed phase.

While developing the theory, two approaches of the theory of phase transitions were used: the model of nucleation (Lifshits-Slyozov) and the model of spinodal decomposition (Cahn-Hillart). These models were generalized on the particles with the finite lifetime, that is important for interpretation of the experimental results. The involvement of Bose-Einstein condensation for excitons was not required for the explanation of the experiments, the considered condensation is the condensation in real space. Among many experiments, explained by the theory, there are two ones, presented below, which did not considered jet in the framework of the presented theory.

A. It was shown in the work\cite{45}, that maximum of the frequency dependence of the emission from the region between the islands is lower, than maximum of the emission frequency from the islands, so from the region, where the exciton density is large. The difference of the maxima is small, it is less than the width of the emission band. But on the base of this date the authors came to the conclusion, that there is the repulsion interaction between excitons only. This result contradicts the main assumptions of the model of the works\cite{29} about the presence of attractive interaction between excitons, that exist at some distance between excitons and causes the creation of a condensed phase. In the Appendix we present a possible explanation of such effect in the case of the attractive interaction between excitons\cite{30}. The explanation is based on the presence in the well of localized exciton states, levels of which are situated lower than the exciton levels and become saturated with increasing pumping. The emission band is determined by the free and localized exciton states. The exciton states form the upper part of the band. With increasing pumping the number of excitons and the blue part of the band emission growth. In the case of an attractive interaction, a lowering of the exciton levels caused by the exciton condensation is small in comparison with the whole width of the band. As the result the maximum of the emission shifts to higher side with increasing the pumping, if the emissions from the exciton condensed phase and from the localized states are not separated.

B. In the works\cite{29,45,46} a coherence was observed in the emission spectra from island\cite{45}, or even from different islands\cite{29,45}. The coherence was revealed in the interference of the emission from the different spatial points.

The effect B is not considered in the presented paper. For its study, the microscopic model of the condensed phase is needed. But a qualitative explanation may be given. The interference of the wave functions does not observed on the experiment directly, the interference of electromagnetic waves is shown on the experiment. Because the electromagnetic field and scattered field are coherent, the interference of the emission from two islands may arise as a result of an imposition of electromagnetic field emitted by some island and the scattered field by other island. It was shown in the papers\cite{47,48,49}, that the strong correlation between exciton densities at different points takes place in the case when the exciton condensed phase exists. Also there is a sharp maximum of the Fourier transformation of the two point correlation function. It is reason of a mutual connection of the wave emitted from some point with the wave scattered by other region. But the quantitative calculations require the date of microscopic model of the exciton condensed phase, particularly, the numerical value of the polarizability is needed.

In the presented paper the hydrodynamics equation for excitons is obtained for the case, if excitons are in condensed phase. The equations allow to describe the moving of the complicated system composed of two phases: gas and condensed ones. We have studied the spatial distribution of both the exciton density and exciton flux in the case of condensation at steady-state pumping. From analysis of different solution of the equation for exciton density it was shown the existence of exciton autosolitons at some parameters of the system. Also the possible explanation of the effect A is given taking into account the presence of localized states, which become totally occupied with increasing pumping.

It is necessary to emphasize that besides the system, we investigated, there is another one, in which the exciton condensation is studied. It is the exciton condensation in bilayer quantum Hall system. In this system the two layers are filled by electrons in the presence of strong magnetic field with total (for two wells) Landau level filling factor $\nu = 1$ and with total density $\nu_T = n_1 + n_2$, where $n_i$ is the density in the separate well ($i = 1, 2$). An electron in the lowest Landau level of one layer bounded to hole in the lowest Landau level of the other layer is considered as exciton. The review of collective effects in such system is presented in\cite{31}. The bilayer quantum Hall system differs from the system that we consider. The electrons are created in the bilayer quantum Hall system by donors and the system is equilibrium. We studied the system in which the excitons are created by the light, the system is non-equilibrium. The excitons have the finite value of the lifetime and this fact influences significantly on the behavior of the collective states, particularly, on the formation of the spatial structures.

### II. HYDRODYNAMICS OF EXCITON CONDENSED PHASE

The hydrodynamic equations of excitons were obtained and analyzed in the works\cite{32}. In comparison with this paper, we obtain\cite{31} the hydrodynamic equations of excitons generalizing the Navier-Stokes equations taking into account the finite exciton lifetime, the pumping of exciton and the existence of an exciton condensed phase caused by interaction between excitons. The system is described by the exciton density $n = n(\vec{r}, t)$ and by the velocity of the exciton liquid $\vec{u} = \vec{u}(\vec{r}, t)$. The equation of the conti-
nity of the exciton density is rewritten in the form
\[ \frac{\partial n}{\partial t} + \text{div}(nu) = G - \frac{n}{\tau_{ex}}, \tag{1} \]
where \( G \) is the pumping (the number of excitons created for unit time in unit area of the quantum well), \( \tau_{ex} \) is the exciton lifetime. In the comparison with the typical equation for a liquid, the presented equation for excitons contains the terms, that describe the pumping and the finite lifetime of the excitons.

The equation for the movement of the unit volume of an exciton liquid is rewritten in the form
\[ \frac{\partial mnu_i}{\partial t} = - \frac{\partial \Pi_{ik}}{\partial x_k} - \frac{mnu_i}{\tau_{sc}}, \tag{2} \]
where \( m \) is the exciton mass, \( \Pi_{ik} \) is the tensor of the density of the exciton flux.

\[ \Pi_{ik} = P_{ik} + mnu_iu_k - \sigma'_{ik}, \tag{3} \]
where \( P_{ik} \) is the pressure tensor, \( \sigma'_{ik} \) is the viscosity tensor of a tension.

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In comparison with the typical Navier-Stokes equation a braking of exciton liquid by phonons and by defects is introduced in Eq. (2). In the equation (2) we neglected by the momentum change caused by the creation and the annihilation of excitons. Indeed, the momentum change in the unit time and in the unit volume owing to an disappearance of the excitons has the order \( mnu/\tau_{ex} \). Since \( \tau_{ex} \gg \tau_{sc} \) this value is much less the last term in the formula (2). The momentum change due to the addition of new excitons by pumping is small too, because mean exciton moment, created by external light, is close to zero.

Introducing coefficients of the viscosity and using Eq. (1), Eq. (2) may be rewritten in the form
\[ \rho \left( \frac{\partial u_i}{\partial t} + u_k \frac{\partial}{\partial x_k} u_i \right) = - \frac{\partial P_{ik}}{\partial x_k} + \eta \Delta u_i + \frac{\rho^2 u_i}{\tau_{sc}}, \tag{4} \]
\rho = mn \text{ is the mass of excitons in unit volume.}

Let us consider the tensor of pressure. To find the connection between the tensor and others parameters it is needed to use the equation of the state. We suggest that the state of the local equilibrium is realized and the state of the system may be described by a free energy, which depends on spatial coordinate. Let us present the functional of the free energy in the form
\[ F = \int d\vec{r} \left( \frac{K}{2} (\nabla n)^2 + f(n) \right). \tag{5} \]

The first term in the integrand describes the energy of non-homogeneity.

The pressure tensor will be obtained from equation of the state of the system. At the given presentation of the free energy, the pressure tensor is determined by the formula
\[ P_{\alpha\beta} = \left( \rho - \frac{K}{2} (\nabla n)^2 - K_n \Delta n \right) \delta_{\alpha\beta} + \frac{\partial n}{\partial x_{\alpha}} \frac{\partial n}{\partial x_{\beta}}, \tag{6} \]
where \( p = n f'(n) - f(n) \) is the equation of the state, \( p \) is the isotropic pressure.

Taking into account (6), we rewrite the equation (4) finally in the form
\[ \frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} + \frac{1}{m} \frac{\partial}{\partial x_i} \left( -K \Delta n + \frac{\partial f}{\partial n} \right) + \nu \Delta u_i + \left( \frac{\partial}{\partial x_i} \right) \text{div} \vec{u} - \frac{u_i}{\tau_{sc}} = 0. \tag{7} \]

Eqs. (17) are the equations of hydrodynamics for an exciton system. These equations differ from the hydrodynamic equations, investigated in [22], by the presence of the terms of the right side in Eq. (11), which take into account the lifetime and pumping, and by the presence of the third term in Eq. (7), which describes a condensed phase. It follows from the estimations, made in the work [22], that the terms with the viscosity coefficients are small and we shall neglect them.

In the case of a steady state irradiation of the system, the Eqs. (11) and (7) have the solution \( n = G \tau, u = 0 \). In order to investigate the stability of this solution we consider the behavior of a small fluctuation of the exciton density.

\[ \lambda_{\pm}(\vec{k}) = \frac{1}{2} \left( -(1/\tau_{sc} + 1/\tau_{ex}) \pm \left( (1/\tau_{sc} - 1/\tau_{ex})^2 - \frac{4k^2 n}{m} (k^2 K + \frac{\partial^2 f}{\partial n^2}) \right) \right), \tag{8} \]

It is follow from (8), that both parameters \( \lambda_{\pm}(\vec{k}) \) have a negative real part at small and large values of vector \( \vec{k} \) and, therefore, the uniform solution of hydrodynamics equation is stable. But the value \( \lambda_+(\vec{k}) \) may be positive in some interval of vector \( \vec{k} \), when \( \frac{\partial^2 f}{\partial n^2} \) becomes negative. In these case the uniform distribution of the exciton density is unstable with respect to a formation of non-homogeneous structures. The instability arises at some threshold value of exciton density and at some critical value of the wave vector. Analysis of the equation (8) gives the following expression for the critical values of the wave vector \( k_c \) and the exciton density \( n_c \)
\[ k_c^4 = \frac{m}{K_n \tau_{sc} \tau_{ex}}, \tag{9} \]
\[ \frac{k_c^2 n_c}{m} \left( k_c^2 K + \frac{\partial f(n_c)}{\partial n_c^2} \right) + \frac{1}{\tau_{sc} \tau_{ex}} = 0. \tag{10} \]
For stable particles (\(\tau_{ex} \to \infty\)) the equations \(\text{Eq.}(9,10)\) give the condition \(\frac{\partial f}{\partial \tau_{ex}} = 0\), that is condition for spinodal decomposition for a system in the equilibrium case.

Depending on parameters the Eqs. \(\text{Eq.}(11,12)\) describe the ballistic and diffusion movement of the exciton system. The relaxation time \(\tau_{sc}\) plays the important role in a formation of the exciton moving. Due to arising of the nonhomogeneous structures, the exciton currents in the system exist \((\vec{j} = n \vec{u} \neq 0)\) even under the uniform steady-state pumping. Excitons are moving from regions with the small exciton density to the regions with the high density. In the presented paper we shall consider the spatial distribution of the exciton density at exciton condensation in the double quantum well under steady-state pumping. In this case the exciton current is small and we suggest the existence of the next conditions

\[
\frac{\partial u_i}{\partial t} < < u_i/\tau_{sc}. \quad (11)
\]

\[
\frac{\partial u_i}{\partial x} < < u_i/\tau_{sc}. \quad (12)
\]

Particularly, the Eq. \(\text{Eq.}(11)\) holds under the study of the steady-state exciton distribution. The fulfillment of Eq. \(\text{Eq.}(12)\) will be shown later after some numerical calculations.

Using the conditions \(\text{Eq.}(11)\) and \(\text{Eq.}(12)\) we obtain from Eq. \(\text{Eq.}(7)\) the value of the velocity \(\vec{u}\)

\[
\vec{u} = -\frac{\tau_{sc}}{m} \vec{\nabla} \left( -K \Delta n + \frac{\partial f}{\partial n} \right), \quad (13)
\]

As the result the equation for the exciton density current may be presented in the form

\[
\vec{j} = n \vec{u} = -M \nabla \mu, \quad (14)
\]

where \(\mu = \delta F/\delta n\) is the chemical potential of the system, \(M = nD/\kappa T\) is the mobility, \(D = \kappa T \tau_{sc}/m\) is the diffusion coefficient of excitons.

Therefore, the equation for the exciton density \(\text{Eq.}(11)\) equals

\[
\frac{\partial n}{\partial t} = \frac{D}{\kappa T} \left( -K \Delta^2 n - K \vec{\nabla} n \cdot \vec{\nabla} \Delta n \right) + \frac{D}{\kappa T} \cdot \left( \frac{\partial^2 f}{\partial n^2} \vec{\nabla} n \right) + G - \frac{n}{\tau_{ex}}. \quad (15)
\]

Just in the form of \(\text{Eq.}(15)\) we investigated a spatial distribution of the exciton density at exciton condensation at different dependencies \(f\) on \(n\) \([22, 23, 33, 38, 39]\). So, our previous consideration of the problem corresponds to the diffusion movement of hydrodynamics equations \(\text{Eq.}(11,12)\). At some conditions, applied to the functional \(F\) the uniform solution is unstable, and the spatial structure arises in the system. For the system under study the condensed phase appears, if the function \(f(n)\) describes a phase transition. In the papers mentioned above the examples of such dependencies were given. Here, we analyze an other dependence \(f(n)\), which often is also used in the theory of phase transitions. We shall approximate the density of the free energy in the form

\[
f = \kappa T n(\ln(n/n_0) - 1) + a \frac{n^2}{2} + b \frac{n^4}{4} + c \frac{n^6}{6}, \quad (16)
\]

where \(a, b, c\) are the constant values. Three last terms in the formula \(\text{Eq.}(16)\) are the main terms, they arise due to an exciton-exciton interaction and describe the phase transition. The first term was introduced in order to describe the system in a space, where the exciton concentration is small (if such region exists in the system). With increasing the exciton density the term \(a \frac{n^2}{2}\) manifests itself firstly. It gives the contribution the an value to chemical potential. The origin of this term is connected in our system with the dipole-dipole interaction that should become apparent at the beginning due to its long-range nature. For estimations of \(a\) we may use for the dipole-exciton interaction in double quantum well the plate capacitor formula \(an = 4\pi e^2 d n / \epsilon\), where \(d\) is the distance between wells, \(\epsilon\) is the dielectric constant. This formula is used usually for a determination of the exciton density from the the experimental meaning of the blue shift of the frequency of the exciton emission with the rise of the density. It is follow from the formula that \(a = 4\pi e^2 / \epsilon\). This expression is approximate because it does not take into account the exciton-exciton correlations \([22, 23]\). When the exciton density grows the last two terms in \(\text{Eq.}(14)\) begin to play a role. An existence of the condensed phase requires that the value \(b\) was negative \((b < 0)\). For stability of the system at large \(n\) the parameter \(c\) should be positive \((c > 0)\). It is suggested in the model, that the condensed phase arises due to the exchange and van der Waals interactions. For the system with the large distance between wells the dipole-dipole interaction exceeds attractive interaction and the condensed phase does not arise. The disappearance of chemical potential minimum as a function of \(n\) with increasing of the parameter \(a\) (with increasing the distance between the quantum wells) is presented in Fig1.

Let us introduce the dimensionless parameters: \(\tilde{n} = n/n_0\), where \(n_0 = (a/c)^{1/4}, \tilde{b} = b/(ac)^{1/2}, \tilde{r} = r/\xi\), where \(\xi = (K/a)^{1/2}\) is the coherence length, \(\tilde{t} = t/t_0\), where \(t_0 = \sqrt{\tau_{ex} \mu_0 / \epsilon_0}\), \(D_1 = \frac{\tau_{ex} \epsilon_0}{\mu_0} G = G t_0 / n_0, \tau_{ex} = \tau / t_0\). As the result the equation \(\text{Eq.}(15)\) is reduced to the form (hereinafter the symbol \(\sim\) will be omitted in the equation)

\[
\frac{\partial n}{\partial \tilde{t}} = D_1 \Delta n - n \Delta^2 n + n \Delta n(1 + 3bn^2 + 5n^4)
\]

\[
- \vec{\nabla} n \cdot \vec{\nabla} \Delta n + (\vec{\nabla} n)^2 (1 + 9bn^2 + 25n^4) + G - \frac{n}{\tau_{ex}}. \quad (17)
\]

The solutions of the equation \(\text{Eq.}(17)\) are presented in Fig.2 in the one-dimensional case \((n(\tilde{r}, \tilde{t}) \equiv n(z, t))\) for three values of the steady-state pumping. The solutions are obtained at the initial conditions \(n(z, 0) = 0\) and the boundary conditions \(n'(0, t) = n'(L, t) = \)
the boundary conditions. At the given parameters the amplitude of the lattice) does not depend on zero in the centers of the condensed and gas phases and the current equals 10⁴cm/s. In order to verify the fulfilment of the condition \( \frac{\partial n}{\partial t} \), let us suppose that \( \frac{\partial u}{\partial x} \sim \frac{u}{l} \), where \( l \) is the period of the structure. It follows from experiments\(^{20}\) that \( l \sim (5 \div 10) \mu m \). Using these date we obtain that the condition \( \tau_{sc} \geq \frac{u_n}{n} \) is satisfied very good. This condition is violated at \( \tau_{sc} \geq \frac{u_n}{n} \). It is the very large value. The calculation using uncertainty principle from the band width of the narrow line \( 2 \cdot 10^{-4} \) gives more less the magnitude \( 3 \cdot 10^{-12} \). Therefore, the formation of nonuniform exciton dissipative structures in the double quantum well occurs by the diffusion movement of excitons. For a proof of the main equation \( \text{(15)} \), the last term in Eq. \( \text{(2)} \), which describes the loss of the momentum due to scattering on defects and phonons, is of importance. Just this term describes the processes, which cause a decay of the exciton flux. From the viewpoint of the possibility of the appearance of superfluidity, the situation for excitons is more complicated than that for the liquid helium and for the atoms of alkali metals at ultralow temperatures. In the last systems, the phonons...
of exciton superfluidity it is needed, that the value of $\tau_{sc}$ grows significantly. It is possible for exciton polaritons, which slightly interact with phonons; and there is a certain experimental evidence on an observation of the polariton condensation\cite{5,54}. For the indirect excitons the critical temperature of the superfluid transition is strongly lowered by inhomogeneities\cite{31,54}. So, the question about the possibility of the superfluidity existence for the indirect excitons on the base AlGaAs system is open.

Thus, the peculiarities, that are observed at large densities of the indirect excitons, may be explained by the phase transitions in the system of the particles with attractive interactions and the finite value of the lifetime without an involvement of the Bose-Einstein condensation.

III. EXCITONIC AUTOSOLITONS

As it was shown, at $n < n_{c1}(G < G_{c1})$ the uniform solution of Eq. (17) is stable. But, at some limits of a pumping at $G < G_{c1}$ there exists a stationary solution for the exciton density distribution localized in a space. For example, with the parameters used for calculations of the exciton distribution in Fig.3, the threshold value of the pumping equals $G_{c1} = 0.0055$; but, at a steady-state pumping there is the spatial nonuniform solution of the equation (17) at $G < G_{c1}$ in the form of an isolated spike. It may be be obtained solving Eq. (17) at the pumping, which consist of a constant value $G_0$ and an additional pulse $dG$ with the maxima in the some point of the space and in the time moment

$$dG = s \exp[-w(z-z_0)^2] \exp[-p(t-t_0)^2]$$

where $s$, $w$, $p$ are parameters. The formula describes a pulse of the pumping, which acts during some time interval with the maximum in the point $z_0$.

The solution of Eq. (17) obtained under an application of the addition pulse (18) in the region $z_0 = L/2$ has at $t \to \infty$ the form presented in Fig.4. The solution exists at $t \to \infty$, i.e. at the times, when the action of the addition pulse is absent already. The shape of the spike $n(z)$ does not depend on parameters $s$, $w$, $p$, except cases, when at least one of these parameters tends to zero and becomes less some value. In addition, the solution in the form, presented in Fig.4, arises also, if the additional pulse is absent, but there is some distribution of the exciton density in the initial time $t = 0$:

$$n(z, 0) = s_0 \exp(-w(z-z_0)^2).$$

$$n(z, 0) = s_0 \exp(-w(z-z_0)^2).$$

Fig.5 shows the distribution of exciton current in the vicinity of the localized solution. The current changes the sign in the center of the localized state of the density.

We have verified by direct calculations, that the solution presented in Fig.4 in the form of a localized distribution of the density is stable. We call the state, that describes this solution localized in space, by excitonic autosoliton. The spatial dependence of the exciton density will be designated by $n_{as}(z)$. The autosolitons exist in the some regions of the pumping $G_{cas} < G < G_{c1}$. At chosen parameters in Fig.4 the autosolitons arise under conditions $0.003 < G < 0.0055$. The solutions exist in the form of the autosoliton side by side with the uniform solutions. The lower boundary $G_{cas}$ depends on exciton parameters, particularly, on the exciton lifetime.

Excitonic autosolitons correspond to solitary solutions of nonlinear equations for the excitons (17). The name “auto” is introduced according to\cite{55}, it underlines, that the solitary solutions arise in the dissipative system in a contrast to “solitons”, which appear in conservative systems. The obtained here autosolitons correspond to the “static autosolitons” according to the classification\cite{55}. 
The solutions in the form of the autosolitons are degenerated: if there is a solitary solution \( n_{as}(z) \), then \( n_{as}(z - z_0) \) will be also solution at the arbitrary \( z_0 \) (in the infinite medium). But, if there is an external field in a system, which creates a spatial dependent additional potential for excitons, the solitary excitation moves. Thus, at linear spatial dependence of the additional potential energy in the formula of the free energy \( \delta V = -d z \) should be added. In this case the equation (17) has the solution in the form of autowaves \( n_{as}(z - vt) \), where \( v \) is the velocity of the autowave. In the region, in which the periodic solution of the exciton density takes place \( G(c1) < G < G(c2) \), such autowaves were investigated in the work\(^{38}\).

Localized solutions exist also in the same region at the pumping greater the value, at which the periodical structure arises \( G > G(c1) \). The dependence of the exciton density may be obtained from Eq. (17) choosing an additional pumping pulse in the form \( \delta V = -d z \), but at \( a < 0 \). An example of such solution is presented in Fig. 6. These structures appear in the form of a dip, and can be called ”dark autosolitons” by analogy with the soliton’s terminology.

To explain the appearance of the autosoliton-type solution we recall that the phase transition are investigated in the paper. As is known, there exists the region between spinodal and binodal, in which the creation of a nucleus of new phase is needed for phase transition. The size of the nucleus should exceed some critical value. The obtained critical value of \( n_c \) (Eqs. 10\(^{8}\)) is based under the consideration of the small fluctuations and corresponds really to the boundary of the spinodal region corrected by non-equilibrium state of the system. The autosoliton arises at pulses larger some critical value. So, the appearance of autosolitons corresponds to the appearance of the nucleus outside the spinodal boundary for phase transitions of the stable particles. A size of the new phase of the stable particles increases with time, while the distribution density of the unstable particles (excitons) does not depend on the time (at a steady-state pumping). It explains the existence of the localized states of exciton distribution outside the region of the instability of an uniform distribution. It should be noted, that in the phase transition approach of nucleation-growth with an generation on unstable particles\(^{24,26}\), the existence of the condensed phase islands outside the spinodal region is taking into account automatically, because this approach consider fluctuations, which do not presented in the equation (17).

The model of the nucleation-growth, which we used under the investigation of the exciton condensation in quantum well on the base AlGaAs crystal are considered.

The hydrodynamic equations for excitons were build with taking into account a possibility of the condensed phase formation, the finite lifetime of excitons, the scattering of excitons by defects. The analysis of the equations was fulfilled for the diffusive exciton flow. The equations explain many spatial excitonic structures obtained under experimental investigations of indirect excitons in quantum wells on the base AlGaAs without an involvement of the Bose-Einstein condensation of excitons.

Also, solutions of the equations in a form of solitary localized states (the bright and dark exciton autosolitons) were found. They exist side by side with the steady-state exciton density state and may be excited by the additional pulse greater some threshold value. In the model of the nucleation-growth these solutions correspond to the nuclei of the condensed phase in a gas phase (the bright autosolitons) and to the nuclei of the gas phase in the condensed phase (the dark autosolitons).

It was shown, that the results of the experiments\(^{41}\), in which has been obtained, that the emission spectrum from the condensed phase region is shifted to shortwave side in a comparison of the emission spectra from the region of the gas phase, do not contradict to the model with an attractive interaction between excitons. For the explanation it was taking into account, that a formation of the emission spectrum occurs by both free and trapped excitons.

![Exciton density](image.png)

**FIG. 6:** The spatial dependence of the exciton density of the dark autosoliton at \( G = 0.0095 > G_c2; D_1 = 0.03, b = -1.9 \).
Appendix A: Distribution of excitons between localized and delocalized states

According to\textsuperscript{41} the frequency of the emission from the islands on the ring around the laser spot is higher, than the frequency of emission from the region between islands. The authors made the conclusion\textsuperscript{22}, that interaction between excitons is repulsive, and, therefore, the formation of the condensed phase by attractive interaction between excitons is impossible. It contradicts the main assumption of the works\textsuperscript{22–28}, though these works explain many experiments. Now we remove this contradiction, taking into account the presence of localized excitons.

Residual donors and acceptors, defects, inhomogeneous thickness of the wells create an accidental fluctuating potential, which may be the reason of appearance of the localized levels. Till now the explanation of the creation of the localized states is not determined definitely, but their existence is confirmed by the presence of an emission in the region of the frequencies less the frequency of the exciton band emission and broadening of exciton lines. At the low temperature and at the small pumping the main part of the band consists of the emission from defect centers, the part of the exciton emission growths with increasing pumping. Let us consider the relation between the contribution to the emission band intensity from free excitons and the excitons localized on defects. Since the defect structure of the samples depends on their preparation, a solution of this problem can not be solved in general. We shall use some approximations.

We shall consider the energy distribution of electron-hole pairs at steady-state irradiation. Such pair may be the delocalized exciton in the exciton band and an electron and hole localized near a defect or in the region with a modified thickness of the well. For small density of the exciton (the electron-hole pairs) the interaction between them may be neglected. Due to long-range character the dipole-dipole interaction, which appear with increasing the excitation density, gives an identical shift of delocalized and localized levels. It means that this interaction do not influence on mutual distribution of the localized and delocalized states. We shall suggest that the localized states are saturable, namely, every center may capture a restricted number of excitations. In our calculations we shall assume that only single excitation may be localized on the defect. Another excitations are or absent or have very low binding energy and are unstable. The dependence of a density of localized states on energy was chosen in the exponential form, namely \( \rho(E) = \alpha N_l \exp(\alpha E), \) where \( N_l \) is the density of the defect centers, \( E \) is the depth of the trap level. The exciton states (free and localized) are distributed on levels after a creation of electrons and holes by an external irradiation and their subsequent recombination and relaxation. Because the time of the relaxation is much less than the exciton lifetime, the distribution of excitation between free and localized states corresponds to a state of thermodynamical equilibrium. In considered model we should obtain the distribution of electron-hole pairs, a population of which on a single level may be changed from zero to infinity for \( E > 0 \) (the free exciton states) and from zero to one for \( E < 0 \) (the localized states). Formally, in considered system the free excitons have Bose-Einstein statistic and localized excitations obey Fermi-Dirac statistic. At small exciton density Bose-Einstein and Boltzmann statistics give the similar results for the free excitons, but the application of Fermi-Dirac statistic for localized states with single level for one trap is important. The equation for the energy distribution may be find from minima of large canonical distribution

\[
\exp(\Omega + 2N\mu - E) = \frac{1}{1 + \exp(\Omega - 2N\mu - E)}.
\]

where \( N = \Sigma n_i + \Sigma k n_k, E = \Sigma n_i E_i + \Sigma k E_k, n_i = 0, 1, n_k = 0, 1, ..., k \) is the wave vector of the exciton, \( l \) designates the singular levels. \( \mu \) is the exciton chemical potential.

The distribution of excitons over free and localized level is determined from minimum of the functional \([A1]\).

As the result we obtain the following conditions for the mean values of the free exciton density \( n \) and the density of the localized states \( n_L \)

\[
n_{ex} = \frac{g\nu}{4\pi E_{ex}a_{ex}^2} \int_0^\infty \frac{dE}{\exp(E/\kappa T) - 1},
\]

\[
n_L = \alpha N_l \int_0^\infty \frac{\exp(\alpha E)dE}{\exp(\alpha E/\kappa T) + 1},
\]

where \( a_{ex} = (h^2)/(\mu_{ex} e^2) \) and \( E_{ex} = (\mu_{ex} e^4)/(2e^2\hbar^2) \) are the radius and the energy of the exciton in the ground state in bulk material, \( g = 4 \), \( \mu_{ex} \) is the reduced mass of the exciton, \( \nu \) is the ratio of the reduced and the total mass of the exciton. The chemical potential \( \mu \) is determined from condition

\[
n_L + n = G_{\tau_{ex}},
\]

where \( G_{\tau_{ex}} \) is the whole number of excitation (free and localized) per unit surface.

The dependence of the distribution of the free and the localized exciton on pumping is presented in Fig.7 as function of whole number of the excitation presented in units of \( 1/a_{ex}^2 \). Let us the exciton radius equals 10nm. Then the concentration of the traps and the width of the distribution of trap levels, chosen under calculations of Fig.7, have the order of \( 10^4 \) cm\(^{-2} \) and 0.003eV, correspondingly.

As it is seen from Fig.7 that the number of the localized excitations exceeds at small pumping the number of free excitons and the emission band should be determined by the emission from the traps. With increasing pumping the occupation of the trap levels become saturated. For chosen parameters the concentration of excitations at the
saturation is a value of the order of $10^9$ cm$^{-2}$. Simultaneously with the saturation of the localized levels the exciton density growths. As the result, the shortwave part of the emission band should be increased with increasing pumping. When the exciton density becomes great, the collective exciton effects begin to manifest themselves. The equations \[ A2, A3 \] do not take into account the interactions between the excitations, and special models and theories are needed for descriptions of collective effects. The appearance of a narrow line was observed in the work \[ 59 \] with increasing pumping on the shortwave part on the exciton emission band. Simultaneously, the patterns arise in the emission spectra. The narrow line appeared after the localized states becomes occupied. According to our model, the islands of condensed phase arise, if the exciton density become higher than some threshold value, and the narrow band corresponds to the condensed phase, caused by the attractive interaction between excitons. The energy per single exciton in the condensed phase is less than the energy of free excitons (the thick line in Fig.8), but the gain of energy under condensation is less than the whole bandwidth, which are formed by the localized and delocalized states. The gain of energy is significantly less than the binding energy of the exciton to an electron -hole drop in silicon and germanium. According to our model, the narrow band is shifted to the red side with increasing pumping in the value less than 0.5 meV, while the whole bandwidth has the order of 2 meV. So, the energy of photons emitted from the islands of condensed phase is higher than the energy of photons emitted by traps. The excitons can not leave the condensed phase (the islands) and move to the traps (to the states with lower energy) since the levels of the traps are occupied already. Thus, the emission frequency of the condensed phase is larger, than the emission frequency of the gas phase, even at the attractive interaction between excitons. It may be reason of obtained in \[ 51 \] results in which the maximum of the frequency of the emission from the islands is higher than the maximum frequency from the regions between the islands.

The qualitative results coincide with the results obtained from the solution of kinetics equations in \[ 29 \] for level distributions. The similar behavior of the distribution of free and trapped excitons takes place for an other density dependence of the density of localized states. We confirmed the results for the gaussian density distribution.

The results may be applied for explanation of intensity and temperature dependence of the exciton emission of dipolar excitons in InGaAs coupled double quantum wells\[ 59 \]. The authors observed with increasing pumping the growth of the shortwave side of the emission band, the narrowing of the band. They obtained that the shortwave side of the band is very sharp. These results may be explained by suggestion, that the lower part of the band is formed by localized states. After the saturation of the localized states with increasing pumping, the excitations begin to occupy the levels of free excitons. Since the density of exciton state is much greater than the density of defect states the shortwave edge of the band is sharp. The exciton condensed phase may not arise in the system investigated in \[ 29 \] as the distance between wells (17nm) is greater than the distance in the system investigated by the Timofeev’s group (13nm), so the repulsive dipole-dipole interaction between excitons in \[ 29 \] may be larger than the attractive interaction.

It should be noted, that in the work \[ 3 \] the band of the emission from the condensed phase is wider than the narrow band that was observed in the work \[ 4 \], where the another method of the creation of excitons was applied. Maybe, it is related to the fact, that in the work \[ 3 \] excitons were created in the region of the p-n transition: the electrons approach this region from one side, and the
holes move to the region from other side. The excitons are situated in a space between the regions rich by electrons from one side, and by holes from other side. To combine into excitons, electrons and holes should surely go through the region of the condensed phase. Therefore, this region contains the charges, which create an electric field and cause the broadening of the emission band.
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