The ternary Goldbach problem with prime numbers of a mixed type

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2017

Abstract

In the present paper we prove that every sufficiently large odd integer $N$ can be represented in the form

$$N = p_1 + p_2 + p_3,$$

where $p_1, p_2, p_3$ are primes, such that $p_1 = x^2 + y^2 + 1$, $p_2 = [n^c]$.

Keywords: Goldbach problem, Prime numbers, Circle method.

1 Notations

Let $N$ be a sufficiently large odd integer. The letter $p$, with or without subscript, will always denote prime numbers. Let $A > 100$ be a constant. By $\varepsilon$ we denote an arbitrary small positive number, not the same in all appearances. The relation $f(x) \ll g(x)$ means that $f(x) = O(g(x))$. As usual $[t]$ and $\{t\}$ denote the integer part, respectively, the fractional part of $t$. Instead of $m \equiv n \pmod{k}$ we write for simplicity $m \equiv n (k)$. As usual $e(t) = \exp(2\pi it)$. We denote by $(d, q), [d, q]$ the greatest common divisor and the least common multiple of $d$ and $q$ respectively. As usual $\varphi(d)$ is Euler’s function; $\mu(d)$ is Möbius’ function; $r(d)$ is the number of solutions of the equation $d = m_1^2 + m_2^2$ in integers $m_j$; $\chi(d)$ is the non-principal character modulo 4 and $L(s, \chi)$ is the corresponding Dirichlet’s $L$-function. By $c_0$ we denote some positive number, not necessarily the same in different occurrences. Let $c$ be a real constant such that $1 < c < 73/64$. 
Denote

\[ \gamma = \frac{1}{c}; \quad (1) \]
\[ D = \frac{N^{1/2}}{(\log N)^{A}}; \quad (2) \]
\[ \psi(t) = \{t\} - \frac{1}{2}; \quad (3) \]
\[ \theta_0 = \frac{1}{2} - \frac{1}{4}c \log 2 = 0.0289...; \quad (4) \]

\[ S_{d,l}(N) = \prod_{p \mid d \atop p \nmid N} (1 - \frac{1}{(p-1)^2}) \prod_{p \mid d \atop p \nmid N-l} (1 - \frac{1}{(p-1)^2}) \]
\[ \times \prod_{p \mid d \atop p \nmid N} (1 + \frac{1}{(p-1)^3}) \prod_{p \mid d \atop p \nmid N-l} (1 + \frac{1}{p-1}); \quad (5) \]

\[ S(N) = \prod_{p \mid N} (1 - \frac{1}{(p-1)^2}) \prod_{p \mid N} (1 + \frac{1}{(p-1)^3}); \quad (6) \]

\[ \mathfrak{S}_r(N) = \pi S(N) \prod_{p \mid (N-1)} \left(1 + \chi(p) \frac{p-3}{p(p^2 - 3p + 3)} \right) \prod_{p \mid N} \left(1 + \chi(p) \frac{1}{p(p-1)} \right) \]
\[ \times \prod_{p \mid N-1} \left(1 + \chi(p) \frac{2p-3}{p(p^2 - 3p + 3)} \right); \quad (7) \]

\[ \Delta(t, h) = \max_{y \leq t} \max_{(l, h) = 1} \left| \sum_{p \leq y \atop p \equiv l \atop \phi(h)} \log p - \frac{y}{\varphi(h)} \right|. \quad (8) \]

## 2 Introduction and statement of the result

In 1937 I. M. Vinogradov [15] solved the ternary Goldbach problem. He proved that for a sufficiently large odd integer \( N \)

\[ \sum_{p_1 + p_2 + p_3 = N} \log p_1 \log p_2 \log p_3 = \frac{1}{2} \mathfrak{S}(N) N^2 + \mathcal{O} \left( \frac{N^2}{\log^4 N} \right), \]

where \( \mathfrak{S}(N) \) is defined by [9] and \( A > 0 \) is an arbitrarily large constant.

In 1953 Piatetski-Shapiro proved that for any fixed \( c \in (1, 12/11) \) the sequence

\[ ([n^c])_{n \in \mathbb{N}} \]

contains infinitely many prime numbers. Such prime numbers are named in honor of
Piatetski-Shapiro. The interval for $c$ was subsequently improved many times and the best result up to now belongs to Rivat and Wu \[10\] for $c \in (1, 243/205)$. 

In 1992, A. Balog and J. P. Friedlander \[1\] considered the ternary Goldbach problem with variables restricted to Piatetski-Shapiro primes. They proved that, for any fixed $1 < c < 21/20$, every sufficiently large odd integer $N$ can be represented in the form

$$N = p_1 + p_2 + p_3,$$

where $p_1, p_2, p_3$ are primes, such that $p_k = [nc^k], k=1,2,3$. Rivat \[10\] extended the range to $1 < c < 199/188$; Kumchev \[7\] extended the range to $1 < c < 53/50$. Jia \[5\] used a sieve method to enlarge the range to $1 < c < 16/15$. Furthermore Kumchev \[7\] proved that for any fixed $1 < c < 73/64$ every sufficiently large odd integer may be written as the sum of two primes and prime number of type $p = [nc]$.

On the other hand in 1960 Linnik \[8\] showed that there exist infinitely many prime numbers of the form $p = x^2 + y^2 + 1$, where $x$ and $y$ are integers. In 2010 Tolev \[14\] proved that every sufficiently large odd integer $N$ can be represented in the form

$$N = p_1 + p_2 + p_3,$$

where $p_1, p_2, p_3$ are primes, such that $p_k = x_k^2 + y_k^2 + 1$, $k=1,2$. In 2017 Teräväinen \[12\] improved Tolev’s result for primes $p_1, p_2, p_3$, such that $p_k = x_k^2 + y_k^2 + 1$, $k=1,2,3$.

Recently the author \[2\] proved that there exist infinitely many arithmetic progressions of three different primes $p_1, p_2, p_3 = 2p_2 - p_1$ such that $p_1 = x^2 + y^2 + 1$, $p_3 = [nc]$.

Define

$$\Gamma(N) = \sum_{p_1 + p_2 + p_3 = N} r(p_1 - 1)p_2^{1-\gamma} \log p_1 \log p_2 \log p_3. \quad (9)$$

Motivated by these results we shall prove the following theorem.

**Theorem 1.** Assume that $1 < c < 73/64$. Then the asymptotic formula

$$\Gamma(N) = \frac{\gamma}{2} \mathcal{S}_\Gamma(N) N^2 + O \left( N^2 (\log N)^{-\theta_0} (\log \log N)^6 \right),$$

holds. Here $\gamma$, $\theta_0$ and $\mathcal{S}_\Gamma(N)$ are defined by (1), (4) and (7).

Bearing in mind that $\mathcal{S}_\Gamma(N) \gg 1$ for $N$ odd, from Theorem 1 it follows that for any fixed $1 < c < 73/64$ every sufficiently large odd integer $N$ can be written in the form

$$N = p_1 + p_2 + p_3,$$

where $p_1, p_2, p_3$ are primes, such that $p_1 = x^2 + y^2 + 1$, $p_2 = [nc]$. 

3
The asymptotic formula obtained for $\Gamma(N)$ is the product of the individual asymptotic formulas

$$
\sum_{p_1+p_2+p_3=N} r(p_1 - 1) \log p_1 \log p_2 \log p_3 \sim \frac{1}{2} \mathcal{S}_\Gamma(N) N^2
$$

and

$$
\frac{1}{N} \sum_{\substack{p \leq N \atop p = \lfloor \alpha \rfloor}} p^{1-\gamma} \log p \sim \gamma.
$$

The proof of Theorem 1 follows the same ideas as the proof in [2].

## 3 Outline of the proof

Using (9) and well-known identity $r(n) = 4 \sum_{d|n} \chi(d)$ we find

$$
\Gamma(N) = 4 \left( \Gamma_1(N) + \Gamma_2(N) + \Gamma_3(N) \right),
$$

where

$$
\Gamma_1(N) = \sum_{\substack{p_1+p_2+p_3=N \atop p_2=\lfloor \alpha \rfloor}} \left( \sum_{d|p_1 \atop d \leq D} \chi(d) \right) p_2^{1-\gamma} \log p_1 \log p_2 \log p_3,
$$

$$
\Gamma_2(N) = \sum_{\substack{p_1+p_2+p_3=N \atop p_2=\lfloor \alpha \rfloor}} \left( \sum_{d|p_1 \atop D < d < N/D} \chi(d) \right) p_2^{1-\gamma} \log p_1 \log p_2 \log p_3,
$$

$$
\Gamma_3(N) = \sum_{\substack{p_1+p_2+p_3=N \atop p_2=\lfloor \alpha \rfloor}} \left( \sum_{d|p_1 \atop d \geq N/D} \chi(d) \right) p_2^{1-\gamma} \log p_1 \log p_2 \log p_3.
$$

In order to estimate $\Gamma_1(N)$ and $\Gamma_3(N)$ we have to consider the sum

$$
I_{d,l,J}(N) = \sum_{\substack{p_1+p_2+p_3=N \atop p_1 \equiv (d) \atop p_2 \in J \atop p_3 = \lfloor \alpha \rfloor}} p_2^{1-\gamma} \log p_1 \log p_2 \log p_3,
$$

where $d$ and $l$ are coprime natural numbers, and $J \subset [1, N]$. If $J = [1, N]$ then we write for simplicity $I_{d,l}(N)$. We apply the circle method. Clearly

$$
I_{d,l,J}(N) = \int_0^1 S_{d,l,J}(\alpha)S(\alpha)S_c(\alpha)e(-N\alpha) d\alpha,
$$
where

\[ S_{d,l}(\alpha) = \sum_{p \in J_{(d)}} e(\alpha p) \log p, \quad (16) \]

\[ S(\alpha) = S_{1,1:1,N}(\alpha), \quad (17) \]

\[ S_c(\alpha) = \sum_{p \leq N} p^{1-\gamma} e(\alpha p) \log p. \quad (18) \]

We define major and minor arcs by

\[ E_1 = \bigcup_{q \leq Q} \bigcup_{\alpha = 0}^{q-1} \left[ \frac{a}{q} - \frac{1}{q\tau}, \frac{a}{q} + \frac{1}{q\tau} \right] \quad \text{and} \quad E_2 = \left[ \frac{1}{\tau}, 1 + \frac{1}{\tau} \right] \setminus E_1, \quad (19) \]

where

\[ Q = (\log N)^B, \quad \tau = NQ^{-1}, \quad A > 4B + 3, \quad B > 14. \quad (20) \]

Then we have the decomposition

\[ I_{d,l;J}(N) = I_{d,l;J}^{(1)}(N) + I_{d,l;J}^{(2)}(N), \quad (21) \]

where

\[ I_{d,l;J}^{(i)}(N) = \int_{E_i} S_{d,l;J}(\alpha)S(\alpha)S_c(\alpha)e(-N\alpha) d\alpha, \quad i = 1, 2. \quad (22) \]

We shall estimate \( I_{d,l;J}^{(1)}(N), \Gamma_3(N), \Gamma_2(N) \) and \( \Gamma_1(N) \) respectively in the sections 4, 5, 6 and 7. In section 8 we shall complete the proof of the Theorem.

4 \textbf{Asymptotic formula for } I_{d,l;J}^{(1)}(N) \textbf{ }

We have

\[ I_{d,l;J}^{(1)}(N) = \sum_{q \leq Q} \sum_{\alpha = 0}^{q-1} H(a, q), \quad (23) \]

where

\[ H(a, q) = \int_{-1/q\tau}^{1/q\tau} S_{d,l;J} \left( \frac{a}{q} + \alpha \right) S \left( \frac{a}{q} + \alpha \right) S_c \left( \frac{a}{q} + \alpha \right) e \left( -N \left( \frac{a}{q} + \alpha \right) \right) d\alpha. \quad (24) \]
On the other hand

\[ S_{d,l,J} \left( \frac{a}{q} + \alpha \right) = \sum_{1 \leq m \leq q \atop (m,q)=1} e \left( \frac{am}{q} \right) T(\alpha) + \mathcal{O}(q \log N) , \]  

(25)

where

\[ T(\alpha) = \sum_{p \in J \atop p \equiv f (d,q)} e(\alpha p) \log p . \]

According to Chinese remainder theorem there exists integer \( f = f(l,m,d,q) \) such that \((f, [d,q]) = 1\) and

\[ T(\alpha) = \sum_{p \in J \atop p \equiv f (d,q)} e(\alpha p) \log p . \]

Applying Abel’s transformation we obtain

\[
T(\alpha) = - \int_{J_1}^{J_2} \left( \sum_{p \in J \atop p \equiv f ([d,q])} \log p \right) \frac{d}{dt} (e(\alpha t)) dt + \left( \sum_{p \in J \atop p \equiv f ([d,q])} \log p \right) e(\alpha J_2) \\
= - \int_{J_1}^{J_2} \left( \frac{t - J_1}{\varphi([d,q])} + \mathcal{O}(\Delta(J_2, [d,q])) \right) \frac{d}{dt} (e(\alpha t)) dt \\
+ \left( \frac{J_2 - J_1}{\varphi([d,q])} + \mathcal{O}(\Delta(J_2, [d,q])) \right) e(\alpha J_2) \\
= \frac{1}{\varphi([d,q])} \int_{J_1}^{J_2} e(\alpha t) dt + \mathcal{O}( (1 + |\alpha|(J_2 - J_1)) \Delta(J_2, [d,q]) ) . \]  

(26)

We use the well known formula

\[
\int_{J_1}^{J_2} e(\alpha t) dt = M_J(\alpha) + \mathcal{O}(1) , \]  

(27)

where

\[ M_J(\alpha) = \sum_{m \in J} e(\alpha m) . \]

Bearing in mind that \(|\alpha| \leq 1/q\tau\) and \( J \subset (1, N) \), from (20), (26) and (27) we get

\[
T(\alpha) = \frac{M_J(\alpha)}{\varphi([d,q])} + \mathcal{O} \left( \left( 1 + \frac{Q}{q} \right) \Delta(N, [d,q]) \right) . \]  

(28)
From (25) and (28) it follows
\[ S_{d,l,J} \left( \frac{a}{q} + \alpha \right) = \frac{c_d(a,q,l)}{\varphi([d,q])} M_J(\alpha) + O(Q(\log N)\Delta(N, [d,q])), \]
(29)
where
\[ c_d(a,q,l) = \sum_{\substack{1 \leq m \leq q \\text{gcd}(m,q)=1 \\text{gcd}(m,l)=1 \\text{gcd}(d,q)}} e \left( \frac{am}{q} \right). \]

We shall find asymptotic formula for \( S_c \left( \frac{a}{q} + \alpha \right) \). From (13) we have
\[ S_c(\alpha) = \sum_{p \leq N} p^{1-\gamma}([-p^\gamma] - [-(p+1)^\gamma])e(\alpha p) \log p \]
\[ = \Omega(\alpha) + \Sigma(\alpha), \]
(30)
where
\[ \Omega(\alpha) = \sum_{p \leq N} p^{1-\gamma}((p+1)^\gamma - p^\gamma)e(\alpha p) \log p, \]
\[ \Sigma(\alpha) = \sum_{p \leq N} p^{1-\gamma}(\psi(-(p+1)^\gamma) - \psi(-p^\gamma))e(\alpha p) \log p. \]
(31) (32)
According to Kumchev ([7], Theorem 2) for \( 64/73 < \gamma < 1 \) uniformly in \( \alpha \) we have that
\[ \Sigma \left( \frac{a}{q} + \alpha \right) \ll N^{1-\varepsilon}. \]
(33)
On the other hand
\[ (p+1)^\gamma - p^\gamma = \gamma p^{\gamma-1} + O(p^{\gamma-2}). \]
(34)
The formulas (31) and (34) give us
\[ \Omega(\alpha) = \gamma S(\alpha) + O(N^\varepsilon), \]
(35)
where \( S(\alpha) \) is defined by (17).
According to ([6], Lemma 3, §10) we have
\[ S \left( \frac{a}{q} + \alpha \right) = \frac{\mu(q)}{\varphi(q)} M(\alpha) + O \left( N e^{-c_\alpha \sqrt{\log N}} \right), \]
where
\[ M(\alpha) = \sum_{m \leq N} e(\alpha m). \]
Bearing in mind (30), (33), (35) and (36) we obtain
\[ S_c \left( \frac{a}{q} + \alpha \right) = \gamma \frac{\mu(q)}{\varphi(q)} M(\alpha) + O \left( Ne^{-c_0 \sqrt{\log N}} \right). \] (37)

Furthermore, we need the trivial estimates
\[ \left| S_{d,l,J} \left( \frac{a}{q} + \alpha \right) \right| \ll \frac{N \log N}{d}, \quad \left| S \left( \frac{a}{q} + \alpha \right) \right| \ll N, \quad |M(\alpha)| \ll N, \quad |\mu(q)| \ll 1. \] (38)

By (29), (36) – (38) and the well-known inequality \( \varphi(n) \gg n (\log \log n)^{-1} \) we find
\[
\begin{align*}
S_{d,l,J} \left( \frac{a}{q} + \alpha \right) S \left( \frac{a}{q} + \alpha \right) S_c \left( \frac{a}{q} + \alpha \right) e \left( -N \frac{a}{q} + \alpha \right) \\
= \gamma \frac{c_d(a,q,l)\mu^2(q)}{\varphi([d,q])\varphi^2(q)} M_J(\alpha) M^2(\alpha) e \left( -N \frac{a}{q} + \alpha \right) + O \left( \frac{N^3}{d} e^{-c_0 \sqrt{\log N}} \right) \\
+ O \left( \frac{N^2 Q \log^2 N}{q^2} \Delta(N,[d,q]) \right).
\end{align*}
\] (39)

Having in mind (20), (24) and (39) we get
\[
H(a,q) = \gamma c_d(a,q,l)\mu^2(q) \varphi([d,q])\varphi^2(q) e \left( -N \frac{a}{q} \right) \int_{-1/q^\tau}^{1/q^\tau} M_J(\alpha) M^2(\alpha) e(-N\alpha) d\alpha \\
+ O \left( \frac{N^2}{q^d} e^{-c_0 \sqrt{\log N}} \right) + O \left( \frac{NQ^2 \log^2 N}{q^3} \Delta(N,[d,q]) \right). \] (40)

Taking into account (23), (40) and following the method in [13] we obtain
\[
I_{d,l,J}(N) = \gamma \frac{\mathcal{G}_{d,l}(N)}{\varphi(d)} \sum_{\substack{m_1 + m_2 + m_3 = N \atop m_1 \in J}} 1 + O \left( \frac{N^2}{d} (\log N) \sum_{q > Q} \frac{(d,q) \log q}{q^2} \right) \\
+ O \left( \tau^2 \log N \sum_{q \leq Q} \frac{q}{[d,q]} \right) + O \left( NQ^2 (\log N)^2 \sum_{q \leq Q} \frac{\Delta(N,[d,q])}{q^2} \right) \\
+ O \left( \frac{N^2}{d} e^{-c_0 \sqrt{\log N}} \right), \] (41)

where \( \mathcal{G}_{d,l}(N) \) is defined by [5].
5 Upper bound for $\Gamma_3(N)$

Consider the sum $\Gamma_3(N)$.

Since
\[
\sum_{\substack{d | p_1 - 1 \\ d \geq N/D}} \chi(d) = \sum_{\substack{m | p_1 - 1 \\ m \leq (p_1 - 1)D/N}} \chi\left(\frac{p_1 - 1}{m}\right) = \sum_{j = \pm 1} \chi(j) \sum_{\substack{m | p_1 - 1 \\ m \leq (p_1 - 1)D/N}} 1
\]
then from (13) and (14) it follows
\[
\Gamma_3(N) = \sum_{m < D} \sum_{j = \pm 1} \chi(j) I_{4m,1+jm;J_m}(N),
\]
where $J_m = [1 + mN/D, N]$. Therefore from (21) we get
\[
\Gamma_3(N) = \Gamma_3^{(1)}(N) + \Gamma_3^{(2)}(N),
\]
where
\[
\Gamma_3^{(\nu)}(N) = \sum_{m < D} \sum_{j = \pm 1} \chi(j) I_{4m,1+jm;J_m}(N), \quad \nu = 1, 2.
\]

Let us consider first $\Gamma_3^{(2)}(N)$. Bearing in mind (22) for $i = 2$ and (43) for $\nu = 2$ we have
\[
\Gamma_3^{(2)}(N) = \int_{E_2} K(\alpha) S(\alpha) S_c(\alpha) e(-N\alpha) d\alpha,
\]
where
\[
K(\alpha) = \sum_{m < D} \sum_{j = \pm 1} \chi(j) S_{4m,1+jm;J_m}(\alpha).
\]

Using Cauchy’s inequality we obtain
\[
\Gamma_3^{(2)}(N) \ll \sup_{\alpha \in E_2 \setminus \{1\}} |S_c(\alpha)| \int_{E_2} |K(\alpha) S(\alpha)| d\alpha + O(N^\varepsilon)
\]
\[
\ll \sup_{\alpha \in E_2 \setminus \{1\}} |S_c(\alpha)| \left( \int_0^1 |K(\alpha)|^2 d\alpha \right)^{1/2} \left( \int_0^1 |S(\alpha)|^2 d\alpha \right)^{1/2} + O(N^\varepsilon). \quad (45)
\]
From (30) and (35) we have
\[
S_c(\alpha) = \gamma S(\alpha) + \Sigma(\alpha) + O(N^\varepsilon), \quad (46)
\]
where $S(\alpha)$ and $\Sigma(\alpha)$ are defined by (17) and (32).

Using (19) and (20) we can prove in the same way as in ([6], Ch.10, Th.3) that

$$
\sup_{\alpha \in E_2 \setminus \{1\}} |S(\alpha)| \ll \frac{N}{(\log N)^{B/2-4}}. \quad (47)
$$

According to Kumchev ([7], Theorem 2) we have that

$$
\sup_{\alpha \in E_2 \setminus \{1\}} |\Sigma(\alpha)| \ll N^{1-\varepsilon}. \quad (48)
$$

Bearing in mind (46) – (48) we get

$$
\sup_{\alpha \in E_2 \setminus \{1\}} |S_c(\alpha)| \ll \frac{N}{(\log N)^{B/2-4}}. \quad (49)
$$

From (17) after straightforward computations we find

$$
\int_0^1 |S(\alpha)|^2 d\alpha \ll N \log N. \quad (50)
$$

On the other hand from (16) and (44) we obtain

$$
\int_0^1 |K(\alpha)|^2 d\alpha = \sum_{m_1, m_2 < D} \sum_{j_1 = \pm 1} \sum_{j_2 = \pm 1} \chi(j_1)\chi(j_2)
$$

$$
\times \int_0^1 S_{4m_1,1+j_1m_1;J_{m_1}}(\alpha)S_{4m_2,1+j_2m_2;J_{m_2}}(-\alpha) d\alpha
$$

$$
= \sum_{m_1, m_2 < D} \sum_{j_1 = \pm 1} \sum_{j_2 = \pm 1} \chi(j_1)\chi(j_2)
$$

$$
\times \sum_{p_1 \equiv 1+j_1 m_1 (4m)} \log p_1 \log p_2 \int_0^1 e(\alpha(p_1 - p_2)) d\alpha
$$

$$
= \sum_{m < D} \sum_{j = \pm 1} \sum_{p \equiv 1+j m (4m)} (\log p)^2
$$

$$
<< (\log N)^2 \sum_{m < D} \sum_{p \equiv 1+j m (4m)} 1
$$

$$
<< N(\log N)^2 \sum_{m < D} \frac{1}{m}
$$

$$
<< N \log^3 N. \quad (51)
$$
Thus from (45), (49) – (51) it follows
\[ \Gamma_3^{(2)}(N) \ll \frac{N^2}{(\log N)^{B/2 - 6}}. \] (52)

Now let us consider \( \Gamma_3^{(1)}(N) \). From (41) and (43) for \( \nu = 1 \) we get
\[
\Gamma_3^{(1)}(N) = \Gamma^* + \mathcal{O}(N^2(\log N)\Sigma_1) + \mathcal{O}(\tau^2(\log N)\Sigma_2) \\
+ \mathcal{O}(NQ^2(\log N)^2\Sigma_3) + \mathcal{O}(N^2e^{-c_0\sqrt{\log N}\Sigma_4}),
\] (53)

where
\[
\Gamma^* = \gamma \left( \sum_{\substack{m_1 + m_2 + m_3 = N \\
m_1 \in J_m}} 1 \right) \sum_{m < D} \frac{1}{2m} \sum_{j = \pm 1} \chi(j) \mathcal{G}_{4m,1+jm}(N),
\]
\[
\Sigma_1 = \sum_{m < D} \sum_{q > Q} \frac{(4m, q) \log q}{mq^2},
\]
\[
\Sigma_2 = \sum_{m < D} \sum_{q \leq Q} \frac{q}{[4m, q]},
\]
\[
\Sigma_3 = \sum_{m < D} \sum_{q \leq Q} \frac{\Delta(N, [4m, q])}{q^2},
\]
\[
\Sigma_4 = \sum_{m < D} \frac{1}{m}.
\]

From the definition (5) it follows that \( \mathcal{G}_{4m,1+jm}(N) \) does not depend on \( j \). Then we have
\[
\sum_{j = \pm 1} \chi(j) \mathcal{G}_{4m,1+jm}(N) = 0 \text{ and that leads to}
\]
\[ \Gamma^* = 0. \] (54)

Arguing as in [13] and using Bombieri – Vinogradov’s theorem we find the following estimates
\[ \Sigma_1 \ll \frac{\log^3 N}{Q}, \quad \Sigma_2 \ll Q \log^2 N, \] (55)
\[ \Sigma_3 \ll \frac{N}{(\log N)^{A-B-5}}, \quad \Sigma_4 \ll \log N. \] (56)

Bearing in mind (20), (53) – (56) we obtain
\[ \Gamma_3^{(1)}(N) \ll \frac{N^2}{(\log N)^{B-4}}. \] (57)

Now from (42), (52) and (57) we find
\[ \Gamma_3(N) \ll \frac{N^2}{(\log N)^{B/2 - 6}}. \] (58)
6 Upper bound for $\Gamma_2(N)$

Consider the sum $\Gamma_2(N)$ defined by (12). We denote by $\mathcal{F}$ the set of all primes $p \leq N$ such that $p - 1$ has a divisor belongs to the interval $(D, N/D)$. Using the inequality $uv \leq u^2 + v^2$ and taking into account the symmetry with respect to $d$ and $t$ we get

$$
\Gamma_2(N)^2 \ll (\log N)^6 N^{2-2\gamma} \left| \sum_{p_1 + p_2 + p_3 = N} \sum_{p_1 \leq N} \sum_{d | p_1 - 1} \chi(d) \right| \left| \sum_{p_4 + p_5 + p_6 = N} \sum_{p_4 \in \mathcal{F}} \sum_{t | p_4 - 1} \chi(t) \right|
$$

$$
\ll (\log N)^6 N^{2-2\gamma} \sum_{p_1 + p_2 + p_3 = N} \sum_{p_1 \leq N} \sum_{p_4 \in \mathcal{F}} \sum_{d | p_1 - 1} \chi(d)^2.
$$

(59)

Further we use that if $n$ is a natural such that $n \leq N$, then the number of solutions of the equation $p_1 + p_2 = n$ in primes $p_1, p_2 \leq N$ such that $p_1 = [m^{1/\gamma}]$ is $O\left(N^\gamma (\log N)^{-2} \log \log N\right)$, i.e.

$$
\#\{p_1 : p_1 + p_2 = n, \ p_1 = [m^{1/\gamma}], \ n \leq N\} \ll \frac{N^\gamma \log \log N}{\log^2 N}.
$$

(60)

This follows for example from ([3], Ch.2, Th.2.4).

Thus the summands in the sum (59) for which $p_1 = p_4$ can be estimated with $O(N^{3+\varepsilon})$. Therefore

$$
\Gamma_2(N)^2 \ll (\log N)^6 N^{2-2\gamma} \Sigma_1 + N^{3+\varepsilon},
$$

(61)

where

$$
\Sigma_1 = \sum_{p_1 \leq N} \left| \sum_{d | p_1 - 1} \sum_{D < d < N/D} \chi(d) \right|^2 \sum_{p_4 \leq N} \sum_{p_2 \neq p_1} \sum_{p_5 = [n_3]} \sum_{p_6 = [n_4]} 1.
$$

We use again (60) and find

$$
\Sigma_1 \ll \frac{N^{2\gamma}}{\log^4 N} (\log \log N)^2 \Sigma_2 \Sigma_3
$$

(62)

where

$$
\Sigma_2 = \sum_{p \leq N} \left| \sum_{d | p - 1} \sum_{D < d < N/D} \chi(d) \right|^2, \quad \Sigma_3 = \sum_{p \leq N} 1.
$$
Arguing as in ([4], Ch.5) we obtain
\[
\Sigma_{2} \ll \frac{N (\log \log N)^7}{\log N}, \quad \Sigma_{3} \ll \frac{N (\log \log N)^{3}}{(\log N)^{1+2\theta_{0}}},
\]
(63)
where \(\theta_{0}\) is denoted by (4).

From (61) – (63) it follows
\[
\Gamma_{2}(N) \ll N^2 (\log N)^{-\theta_{0}} (\log \log N)^{6}.
\]
(64)

7  Asymptotic formula for \(\Gamma_{1}(N)\)

In this section our argument is a modification of Tolev’s [14] argument. Consider the sum \(\Gamma_{1}(N)\). From (11), (14) and (21) we get
\[
\Gamma_{1}(N) = \Gamma_{1}^{(1)}(N) + \Gamma_{1}^{(2)}(N),
\]
(65)
where
\[
\Gamma_{1}^{(1)}(N) = \sum_{d \leq D} \chi(d) I_{d,1}^{(1)}(N),
\]
\[
\Gamma_{1}^{(2)}(N) = \sum_{d \leq D} \chi(d) I_{d,1}^{(2)}(N).
\]

We estimate the sum \(\Gamma_{1}^{(2)}(N)\) by the same way as the sum \(\Gamma_{3}^{(2)}(N)\) and obtain
\[
\Gamma_{1}^{(2)}(N) \ll \frac{N^2}{(\log N)^{B/2-6}}.
\]
(66)

Now we consider \(\Gamma_{1}^{(1)}(N)\). We use the formula (41) for \(J = [1,N]\). The error term is estimated by the same way as for \(\Gamma_{3}^{(1)}(N)\). We have
\[
\Gamma_{1}^{(1)}(N) = \frac{\gamma}{2} \mathcal{G}(N) N^{2} \sum_{d \leq D} \frac{\chi(d) \mathcal{S}_{d,1}^{*}(N)}{\varphi(d)} + \mathcal{O}\left(\frac{N^2}{(\log X)^{B-4}}\right),
\]
(67)
where \(\mathcal{G}(N)\) is defined by (3) and
\[
\mathcal{S}_{d,1}^{*}(N) = \prod_{p \mid d, \ p \nmid N} \left(1 - \frac{1}{(p-1)^2}\right)^{-1} \prod_{p \mid d, \ p \nmid N-1} \left(1 - \frac{1}{(p-1)^2}\right) \times \prod_{p \mid d, \ p \nmid N} \left(1 + \frac{1}{(p-1)^3}\right)^{-1} \prod_{p \mid d, \ p \nmid N-1} \left(1 + \frac{1}{p-1}\right);
\]
(68)
Denote
\[ \Sigma = \sum_{d \leq D} f(d), \quad f(d) = \frac{\chi(d) \mathcal{S}_{d,1}(N)}{\varphi(d)}. \]  
(69)

We have
\[ f(d) \ll d^{-1}(\log \log(10d))^2 \]  
(70)
with absolute constant in the Vinogradov’s symbol. Hence the corresponding Dirichlet series
\[ F(s) = \sum_{d=1}^{\infty} \frac{f(d)}{d^s} \]
is absolutely convergent in \( \text{Re}(s) > 0 \). On the other hand \( f(d) \) is a multiplicative with respect to \( d \) and applying Euler’s identity we find
\[ F(s) = \prod_{p} T(p, s), \quad T(p, s) = 1 + \sum_{l=1}^{\infty} f(p^l)p^{-ls}. \]  
(71)

From (68), (69) and (71) we establish that
\[ T(p, s) = \left( 1 - \frac{\chi(p)}{p^{s+1}} \right)^{-1} \left( 1 + \frac{\chi(p)}{p^{s+1}}E_d(p) \right), \]
where
\[ E_d(p) = \begin{cases} 
(p - 3)(p^2 - 3p + 3)^{-1} & \text{if } p \nmid N(N - 1), \\
(p - 1)^{-1} & \text{if } p \mid N, \\
(2p - 3)(p^2 - 3p + 3)^{-1} & \text{if } p \mid N - 1.
\end{cases} \]
Hence we find
\[ F(s) = L(s + 1, \chi)\mathcal{N}(s), \]  
(72)
where \( L(s + 1, \chi) \) is Dirichlet series corresponding to the character \( \chi \) and
\[ \mathcal{N}(s) = \prod_{p \mid N(N - 1)} \left( 1 + \chi(p) \frac{p - 3}{p^{s+1}(p^2 - 3p + 3)} \right) \prod_{p \mid N} \left( 1 + \chi(p) \frac{1}{p^{s+1}(p - 1)} \right) \times \prod_{p \mid N - 1} \left( 1 + \chi(p) \frac{2p - 3}{p^{s+1}(p^2 - 3p + 3)} \right). \]  
(73)

From the properties of the L-functions it follows that \( F(s) \) has an analytic continuation to \( \text{Re}(s) > -1 \). It is well known that
\[ L(s + 1, \chi) \ll 1 + |\text{Im}(s)|^{1/6} \quad \text{for } \text{Re}(s) \geq -\frac{1}{2}. \]  
(74)
Moreover
\[ \mathcal{N}(s) \ll 1. \] (75)

Using (72), (74) and (75) we get
\[ F(s) \ll N^{1/6} \quad \text{for} \quad \text{Re}(s) \geq -\frac{1}{2}, \quad |\text{Im}(s)| \leq N. \] (76)

We apply Perron’s formula given at Tenenbaum ([11], Chapter II.2) and also (70) to obtain
\[ \sum = \frac{1}{2\pi i} \int_{\kappa-iN}^{\kappa+iN} F(s) \frac{D^s}{s} ds + O \left( \sum_{t=1}^{\infty} \frac{D^\kappa \log \log(10t)}{t^{1+\kappa}} \right), \] (77)
where \( \kappa = 1/10 \). It is easy to see that the error term above is \( O \left( N^{-1/20} \right) \). Applying the residue theorem we see that the main term in (77) is equal to
\[ F(0) + \frac{1}{2\pi i} \left( \int_{1/10-iN}^{1/10+iN} + \int_{-1/2+iN}^{1/2+iN} \right) F(s) \frac{D^s}{s} ds. \]

From (76) it follows that the contribution from the above integrals is \( O \left( N^{-1/20} \right) \). Hence
\[ \sum = F(0) + O \left( N^{-1/20} \right). \] (78)

Using (72) we get
\[ F(0) = \frac{\pi}{4} \mathcal{N}(0). \] (79)

Bearing in mind (67), (69), (73), (78) and (79) we find a new expression for \( \Gamma_1(N) \)
\[ \Gamma_1^{(1)}(N) = \gamma \frac{S}{8} \mathcal{G}_T(N) N^2 + O \left( \frac{N^2}{(\log N)^{B-4}} \right), \] (80)
where \( \mathcal{G}_T \) is defined by (7).

From (65), (66) and (80) we obtain
\[ \Gamma_1(N) = \gamma \frac{S}{8} \mathcal{G}_T(N) N^2 + O \left( \frac{N^2}{(\log N)^{B/2-6}} \right). \] (81)

8 Proof of the Theorem

Therefore using (10), (58), (64) and (81) we find
\[ \Gamma(N) = \gamma \frac{S}{2} \mathcal{G}_T(N) N^2 + O \left( N^2 (\log N)^{-\theta_0} \log \log N \right)^6. \]
This implies that \( \Gamma(N) \to \infty \) as \( N \to \infty \).

The Theorem is proved.
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