Spin networks, quantum automata and link invariants

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Abstract
The spin network simulator model represents a bridge between (generalized) circuit schemes for standard quantum computation and approaches based on notions from Topological Quantum Field Theories (TQFT). More precisely, when working with purely discrete unitary gates, the simulator is naturally modelled as families of quantum automata which in turn represent discrete versions of topological quantum computation models. Such a quantum combinatorial scheme, which essentially encodes $SU(2)$ Racah–Wigner algebra and its braided counterpart, is particularly suitable to address problems in topology and group theory and we discuss here a finite states–quantum automaton able to accept the language of braid group in view of applications to the problem of estimating link polynomials in Chern–Simons field theory.
1 Introduction

In the past few years there has been a tumultuous activity aimed at introducing novel conceptual schemes for quantum computing. The approach proposed in [1], [2] relies on the (re)coupling theory of $SU(2)$ angular momenta (Racah–Wigner tensor category) and might be viewed as a generalization to arbitrary values of the spin variables of the usual quantum circuit model based on qubits and Boolean gates. Computational states belong to finite–dimensional Hilbert spaces labelled by both discrete and continuous parameters, and unitary gates may depend on quantum numbers ranging over finite sets of values as well as continuous (angular) variables. When working with purely discrete unitary gates, the computational space of the simulator is naturally modelled as families of quantum automata which in turn represent discrete versions of topological models of quantum computation based on modular functors of $SU(2)$ Chern–Simons theory (cfr. [3] and references therein). The discretized quantum theory underlying our scheme actually belongs to the class of $SU(2)$ ‘state sum models’ introduced in [4] and widely used in 3–dimensional simplicial quantum gravity (cfr. section 5 of [2] and references therein). From the computational viewpoint, we are in the presence of finite–states and discrete–time machines able to accept any (quantum) language compatible with the Racah–Wigner algebra on the one hand, and as powerful as Freedman’s quantum field computer on the other. As is getting more and more evident, the exponential efficiency that quantum algorithms may achieve vs. classical ones might prove especially relevant in addressing problems in which the space of solutions is not only endowed with a numerical (‘digital’) representation but is itself characterized by some additional ‘combinatorial’ structure, definable in terms of the grammar and the syntax of some ‘language’. Our scheme can be easily adapted –by braiding the Racah–Wigner tensor category as usually done in $SU(2)$ Chern–Simons framework– to make it accepting the language of the braid group $B_n$ and to handle with (colored) link polynomials expressed as expectation values of composite Wilson loop operators. In the last section we are going to exhibit a quantum automaton calculation which processes efficiently –linearly in the number of pairwise braidings– the language of the braid group. This does not mean that we have really got a ‘quantum algorithm’ since we should show that each ‘elementary’ (in the sense of the spin network simulator) unitary gate can be evaluated (or approximated) efficiently. This could be achieved by exploiting both the combinatorial properties of the spin network graph.
and suitable recursion relations which hold for hypergeometric–type polynomials of discrete variables (work is in progress in such directions).

Finally, let us point out that the idea of using braiding operators to implement quantum gates actually dates back to Freedman and collaborators [3] and has been exploited also by Kauffman and Lomonaco [5]. The common challenge of all approaches is, on the one hand, the search for suitable unitary representations of the braid group and, on the other, the selection of suitable encoding maps into quantum circuits (or, eventually, into new quantum computing schemes). Aharonov, Jones and Landau have recently provided in [6] an efficient quantum algorithm which approximates the Jones polynomial built from Markov traces arising from representations of the braid group in the Temperley–Lieb algebra.

2 Complexity of braids and links

Let us recall some basic properties of the Artin braid group $B_n$. $B_n$ has $n$ generators, denoted by $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ plus the identity $e$, which satisfy the relations

\[
\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if} \quad |i - j| > 1 \quad (i, j = 1, 2, \ldots, n - 1)
\]

\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad (i = 1, 2, \ldots, n - 2).
\] (1)

An element of the braid group is a ‘word’ $w$ in the standard generators of $B_n$, e.g. $w = \sigma_3^{-1} \sigma_2 \sigma_3^{-1} \sigma_2^{-1} \sigma_3^{-1} \sigma_2^{-1} \sigma_1 \sigma_2^{-2} \in B_4$; the length $|w|$ of the word $w$ is the number of its ‘letters’. The group acts naturally on topological sets of $n$ disjoint strands – running downward and labeled from left to right – in the sense that each generator $\sigma_i$ corresponds to the over–crossing of the $i$th strand on the $(i + 1)$th, and $\sigma_i^{-1}$ represents the inverse operation (under–crossing) according to $\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = e$.

As is well known, representation theory of Artin braid group enters heavily into many physical applications, ranging from statistical mechanics to (topological) quantum field theories. Motivated by this remark, the search for algorithms addressing computational problems arising in braid group context is becoming more and more compelling. Historically, three fundamental decision problems for any finitely presented group $G$ were formulated by Max Dehn in 1911:

- word problem: does there exist an algorithm to determine, for any arbitrary word $w$ in the generators of $G$, whether or not $w = \text{identity in } G$?
conjugacy problem: does there exist an algorithm to decide whether any pair of words in the generators of $G$ are conjugate to each other?

isomorphism problem: given an arbitrary pair of finite presentations in some set of generators, does there exist an algorithm to decide whether the groups they present are isomorphic?

Following the development of the classical theory of algorithms (recursive functions and Turing machine) it is reasonable to expect that Dehn’s problems might be recursively solvable or, at least, that some ‘local’ ones (the word and the conjugacy problems) be so. It turns out, instead, that not only these problems, but a host of local and global decision problems sharing a combinatorial flavor are unsolvable within such scheme. As for the braid group, and referring to [7] (which contains an exhaustive review together with an up-to-date bibliography), the word problem has been recently solved. More precisely, given two words $w, w' \in B_n$ there exists an efficient algorithm whose time complexity function (namely the time required to perform the computation as a function of the input ‘size’) grows as $|w|^2 n \log n$, where $|w|$ is the length of the word $w$ and $n$ the index of the braid group. Thus this problem belongs to the (classical) complexity class $P$, which contains languages accepted by a Turing machine in polynomial time. The word problem seems to be one short step away from the non-minimal braid problem: given a word $w$ in the generators $\mathbb{I}$ and their inverses, determine whether there is a shorter word $w'$ in the same generators which represents the same element of $B_n$. The decision process can be described as follows: list all the words that are shorter than the given one, drop from the list as many candidates as possible by simple criteria, and then test, one by one, whether the survivors represent the same element as $w$. Surprisingly enough, this problem turns out to be $\text{NP}$–complete, namely belongs to the complexity class of Non deterministic Polynomial algorithm, for which a proposed solution can be checked efficiently ( polynomially). $\text{NP}$–complete problems are in $\text{NP}$ and can be characterized by saying that a polynomial algorithm for one of them provides automatically an efficient solution for all $\text{NP}$ problems. Finally, the best known classical algorithm for the conjugacy problem in the braid group is exponential in both $n$ and $|w|$.

Knot theory is closely related to braid group owing to Alexander’s Theorem, which states that every knot (or link) $L$ in the 3–sphere $S^3$ can be represented (not uniquely) as a closed braid for some suitable $n$. Moreover, the reduction from a planar diagram of the link $L$ to the closed braid $\hat{w}_{(n)}(L)$, with $w_{(n)} \in B_n$, has been recently shown to be polynomial in the number of
strands (see the original references in [4]). The major challenge in knot theory is of course the detection and classification of all possible knots and links up to ambient isotopy. There are a number of topological, combinatorial and algebraic invariants, starting from the group of the knot (the fundamental group of the complement of the knot in $S^3$) and numerical invariants (linking number, crossing number, ..) up to invariants of polynomial type. This last class of invariants is related to the braid group, in the sense that knot polynomials are actually associated with characters (technically, Markov traces) of (suitable) representations of $B_n$ for some $n$. A crucial role is played by the Jones polynomial [8], which, when evaluated at particular roots of unity, turns out to be associated with suitable expectation values of observables in Chern–Simons topological field theory [9]. From the computational side, it has been proved that the exact evaluation of the polynomial $V_L(\omega)$ at $\omega = \text{root of unity}$ can be performed in polynomial time in terms of the number of crossings of planar diagram of $L$ if $\omega$ is a 2nd, 3rd, 4th, 6th root of unity. Otherwise, the problem is $\#P$–hard [10] (the computational complexity class $\#P$–hard is the enumerative analog of $NP$–complete problems).

Generally speaking, the exponential efficiency that quantum algorithms may achieve vs. classical ones might prove especially relevant in addressing problems in which the space of solutions is not only endowed with a numerical representation but is itself characterized by some additional ‘combinatorial’ structure, definable in terms of a grammar and a syntax and thus suitable to be encoded naturally in the spin network computing framework (which will be addressed in the following). There are a number of problems that are not easily formulated in numerical terms and that are quite often intractable in classical complexity theory (cfr. [11]). In combinatorial and algebraic topology typical issues are: the construction of presentations of the fundamental group (or the first homology group) of compact 3–manifolds decomposed as handlebodies; the study of equivalence classes of knots/links in the three–sphere, related in turn to the classification of hyperbolic 3–manifolds; the enumeration of inequivalent triangulations of $D$–dimensional compact manifolds.

Of course efficient solutions of this kind of problems would be interesting also in view of applications to (discretized) models for quantum gravity at least in low spacetime dimensions.
3 Spin network quantum simulator

The theory of binary coupling of \( N = n + 1 \) SU(2) angular momenta represents the generalization to an arbitrary \( N \) of the coupling of two angular momentum operators \( J_1, J_2 \) which involves Clebsch–Gordan (or Wigner) coefficients in their role of unitary transformations between uncoupled and coupled basis vectors, \(| j_1 m_1 \rangle \otimes | j_2 m_2 \rangle \) and \(| j_1 j_2; JM \rangle \) respectively. The quantum numbers \( j_1, j_2 \) associated with \( J_1, J_2 \) label irreducible representations of \( SU(2) \) ranging over \{0, \( 1/2 \), \( 1 \), \( 3/2 \), \ldots\}; \( m_1, m_2 \) are the magnetic quantum numbers, \( -j_i \leq m_i \leq j_i \) in integer steps; \( J \) is the spin quantum number of the total angular momentum operator \( J = J_1 + J_2 \) whose magnetic quantum number is \( M = m_1 + m_2 \), \( -J \leq M \leq J \). Here units are chosen for which \( \hbar = 1 \) and we refer to [12] for a complete account on the theory of angular momentum in quantum physics. On the other hand, \( SU(2) \) ‘recoupling’ theory—which deals with relationships between distinct binary coupling schemes of \( N \) angular momentum operators—is a generalization to any \( N \) of the simplest case of three operators \( J_1, J_2, J_3 \) which calls into play unitary transformations known as Racah coefficients or \( 6j \) symbols. A full fledged review on this advanced topic in the general framework of Racah–Wigner algebra can be found in [13].

The architecture of the ‘spin network’ simulator worked out in [2] relies extensively on recoupling theory and can be better summarized by resorting to a combinatorial setting where the computational space is modelled as an \( SU(2) \)–fiber space structure over a discrete base space \( V \)

\[
(V, C^{2J+1}, SU(2)^{J})_n
\]

which encodes all possible computational Hilbert spaces as well as unitary gates for any fixed number \( N = n + 1 \) of incoming angular momenta.

- The base space \( V \equiv \{v(b)\} \) represents the vertex set of a regular, 3–valent graph \( G_n(V, E) \) whose cardinality is \(|V| = (2n)!/n!\). There exists a one–to–one correspondence

\[
\{v(b)\} \leftrightarrow \{H_n^J(b)\}
\]

between the vertices of \( G_n(V, E) \) and the computational Hilbert spaces of the simulator.

The label \( b \) above has the following meaning—in which we shall extensively return later on: for any given pair \((n, J)\), all binary coupling schemes
of the \(n+1\) angular momenta \(\{J_j\}\), identified by the quantum numbers \(j_1, \ldots, j_{n+1}\) plus \(k_1, \ldots, k_{n-1}\) (corresponding to the \(n-1\) intermediate angular momenta \(\{K_i\}\)) and by the brackets defining the binary couplings, provide the ‘alphabet’ in which quantum information is encoded (the rules and constraints of bracketing are instead part of the ‘syntax’ of the resulting coding language). The Hilbert spaces \(\mathcal{H}^J_n (k_1, \ldots, k_{n-1})\) thus generated, each \((2J + 1)\)-dimensional, are spanned by complete orthonormal sets of states with quantum number label set \(\mathcal{B}\) such as, e.g. for \(n = 3\), \(\{(j_1(j_2j_3)k_1)k_2j_4\}_J \cup \{(j_1j_2)k_1(j_3j_4)k_2\}_J\).

More precisely, for a given value of \(n\), \(\mathcal{H}^J_n (b)\) is the simultaneous eigenspace of the squares of \(2(n+1)\) Hermitean, mutually commuting angular momentum operators \(J_1, J_2, J_3, \ldots, J_{n+1}\) with fixed sum \(J_1 + J_2 + J_3 + \ldots + J_{n+1} = J\), of the intermediate angular momentum operators \(K_1, K_2, K_3, \ldots, K_{n-1}\) and of the operator \(J_z\) (the projection of the total angular momentum \(J\) along the quantization axis). The associated quantum numbers are \(j_1, j_2, \ldots, j_{n+1}\); \(j_1, k_1, j_2, k_2, \ldots, j_{n-1}\) and \(M\), where \(-J \leq M \leq J\) in integer steps. If \(\mathcal{H}^{j_1} \otimes \mathcal{H}^{j_2} \otimes \cdots \otimes \mathcal{H}^{j_n} \otimes \mathcal{H}^{j_{n+1}}\) denotes the factorized Hilbert space, namely the \((n+1)\)-fold tensor product of the individual eigenspaces of the \((J_i)^2\)’s, the operators \(K_i\)'s represent intermediate angular momenta generated, through Clebsch–Gordan series, whenever a pair of \(J_i\)'s are coupled. As an example, by coupling sequentially the \(J_i\)'s according to the scheme \((\cdots ((J_1 + J_2) + J_3) + \cdots + J_{n+1}) = J\) – which generates \((J_1 + J_2) = K_1, (K_1 + J_3) = K_2\), and so on – we should get a binary bracketing structure of the type \((\cdots (((\mathcal{H}^{j_1} \otimes \mathcal{H}^{j_2})_{k_1} \otimes \mathcal{H}^{j_3})_{k_2} \otimes \cdots \otimes \mathcal{H}^{j_{n+1}})_{k_{n-1}})_{J}\), where for completeness we add an overall bracket labelled by the quantum number of the total angular momentum \(J\). Note that, as far as \(j_i\)’s quantum numbers are involved, any value belonging to \(\{0, 1/2, 1, 3/2, \ldots\}\) is allowed, while the ranges of the \(k_i\)’s are suitably constrained by Clebsch–Gordan decompositions (e.g. if \((J_1 + J_2) = K_1 \Rightarrow [j_1 - j_2] \leq k_1 \leq j_1 + j_2\). We denote a binary coupled basis of \((n+1)\) angular momenta in the \(JM\)-representation and the corresponding Hilbert space introduced in (3) as

\[
\{ | [j_1, j_2, j_3, \ldots, j_{n+1}]^b; k_1^b, k_2^b, \ldots, k_{n-1}^b; JM \rangle, -J \leq M \leq J \}
\]

\[
= \mathcal{H}^J_n (b) \doteq \text{span} \{ | b; JM \rangle_n \}, \quad (4)
\]

where the string inside \([j_1, j_2, j_3, \ldots, j_{n+1}]^b\) is not necessarily an ordered one, \(b\) \(\mathcal{B}\) indicates the current binary bracketing structure and the \(k_i\)'s are uniquely associated with the chain of pairwise couplings selected by \(b\).
For a given value of $J$ each $\mathcal{H}_n^J(b)$ has dimension $(2J + 1)$ over $C$ and thus there exists one isomorphism

$$\mathcal{H}_n^J(b) \cong C^{2J+1}$$

for each admissible binary coupling scheme $b$ of $(n + 1)$ incoming spins. The vector space $C^{2J+1}$ is naturally interpreted as the typical fiber attached to each vertex $v(b) \in V$ of the fiber space structure through the isomorphism $(5)$. In other words, Hilbert spaces corresponding to different bracketing schemes, although isomorphic, are not identical since they actually correspond to (partially) different complete sets of physical observables, namely for instance $\{J_1^2, J_2^2, J_3^2, J_z\}$ and $\{J_1^2, J_2^2, J_3^2, J_2^3, J_3, J_z\}$ respectively (in particular, $J_1^2$ and $J_2^3$ cannot be measured simultaneously).

On the mathematical side this remark reflects the fact that the tensor product $\otimes$ is not an associative operation.

- For what concerns unitary operations acting on the computational Hilbert spaces $\mathcal{H}_n$, we examine first unitary transformations associated with recoupling coefficients ($3nj$ symbols) of $SU(2)$ ($j$–gates in the present quantum computing context). As shown in Ref. 13 any such coefficient can be split into ‘elementary’ $j$–gates, namely Racah and phase transforms. A Racah transform applied to a basis vector is defined formally as $\mathcal{R} : |(a b)_{d} c \ldots ; JM\rangle \mapsto |(a b d)_{c} e \ldots ; JM\rangle$, where Latin letters $a, b, c, \ldots$ are used here to denote generic, both incoming ($j_\ell$’s in the previous notation) and intermediate ($k_i$’s) spin quantum numbers. Its explicit expression reads

$$|((a b)_{c} e)_{f} ; M\rangle = \sum_d (-1)^{a+b+c+f} [(2d+1)(2e+1)]^{1/2} \begin{pmatrix} a & b & d \\ c & f & e \end{pmatrix} |((a b)_{d} c)_{f} ; M\rangle,$$

where there appears the $6j$ symbol of $SU(2)$ and $f$ plays the role of the total angular momentum quantum number. Note that, according to the Wigner–Eckart theorem, the quantum number $M$ (as well as the angular part of wave functions) is not altered by such transformations, and that the same happens with $3nj$ symbols. On the other hand, the effect of a phase transform amounts to introducing a suitable phase whenever two spin labels are swapped

$$|(\ldots (a b)_{c} \ldots ; JM\rangle = (\ldots |(b a)_{c} \ldots ; JM\rangle.$$

8
These unitary operations are combinatorially encoded into the edge set $E = \{e\}$ of the graph $G_n(V, E)$: $E$ is just the subset of the Cartesian product $(V \times V)$ selected by the action of these elementary $j$–gates. More precisely, an (undirected) arc between two vertices $v(b)$ and $v(b')$ 

$$e(b, b') = (v(b), v(b')) \in (V \times V)$$

exists if, and only if, the underlying Hilbert spaces are related to each other by an elementary unitary operation (6) or (7). Note also that elements in $E$ can be considered as mappings $(V \times \mathbb{C}^{2J+1})_n \rightarrow (V \times \mathbb{C}^{2J+1})_n$ 

$$(v(b), \mathcal{H}^J_n(b)) \mapsto (v(b'), \mathcal{H}^J_n(b'))$$

connecting each given decorated vertex to one of its nearest vertices and thus define a ‘transport prescription in the horizontal sections’ belonging to the total space $(V \times \mathbb{C}^{2J+1})_n$ of the fiber space (2). The crucial feature that characterizes the graph $G_n(V, E)$ arises from compatibility conditions satisfied by 6$j$ symbols in (6), cfr. [12].

The Racah (triangular) identity, the Biedenharn–Elliott (pentagon) identity and the orthogonality conditions for 6$j$ symbols ensure indeed that any simple path in $G_n(V, E)$ with fixed endpoints can be freely deformed into any other, providing identical quantum transition amplitudes at the kinematical level.

To complete the description of the structure $(V, \mathbb{C}^{2J+1}, SU(2)_J)_n$ we should call into play $M$–gates which act on the angular dependence of vectors in $\mathcal{H}^J_n(b)$ by rotating them. The shorthand notation $SU(2)_J$ employed in (2) actually refers to the group of Wigner rotations, which in turn can be interpreted as actions of the automorphism group of the fiber $\mathbb{C}^{2J+1}$. Since rotations in the $JM$ representation do not alter the binary bracketing structure of vectors in computational Hilbert spaces we might interpret these actions as ‘transport prescriptions along the fiber’. However, we are going to exploit here only computational degrees of freedom associated with $j$–gates, and thus we do not need to explicitate the action of gates along the fiber. For this reason, we switch to the notation $G_n(V, E) \times \mathbb{C}^{2J+1}$, instead of (2), to represent the computational space of the spin network simulator.

Let us point out that our model of quantum simulator actually complies with a variety of computing schemes, ranging from circuit–type models and finite states–automata up to discretized versions of ‘topological’ quantum computation. Inside each of these classes we may also think of different encoding schemes to deal with particular problems. In this respect, problems
from low dimensional topology, geometry, group theory and graph theory (rather than from number theory) turn out to be particularly suitable to be addressed in this ‘quantum–combinatorial’ framework.

Generally speaking, a computation is a collection of step–by–step transition rules (gates), namely a family of ‘elementary unitary operations’ and we assume that it takes one unit of the intrinsic discrete time variable to perform anyone of them. In the combinatorial setting described above such prescriptions amount to select (families of) ‘directed paths’ in $G_n(V, E) \times \mathbb{C}^{2^{J+1}}$ all starting from the same input state and ending in an admissible output state. A single path in this family is associated with a particular algorithm supported by the given program. By a directed path $\mathcal{P}$ with fixed endpoints we mean a (time) ordered sequence

$$|v_{\text{in}}\rangle_n \equiv |v_0\rangle_n \rightarrow |v_1\rangle_n \rightarrow \cdots \rightarrow |v_s\rangle_n \rightarrow \cdots \rightarrow |v_L\rangle_n \equiv |v_{\text{out}}\rangle_n,$$

where we use the shorthand notation $|v_s\rangle_n$ for computational states and $s = 0, 1, 2, \ldots, L(\mathcal{P})$ is the lexicographical labelling of the states along the path. $L(\mathcal{P})$ is the length of the path $\mathcal{P}$ and $L(\mathcal{P}) \cdot \tau = T$ is the time required to perform the process in terms of the discrete time unit $\tau$.

A computation consists in evaluating the expectation value of the unitary operator $U_{\mathcal{P}}$ associated with the path $\mathcal{P}$, namely

$$\langle v_{\text{out}} | U_{\mathcal{P}} | v_{\text{in}}\rangle_n.$$

By taking advantage of the possibility of decomposing $U_{\mathcal{P}}$ uniquely into an ordered sequence of elementary gates, (11) becomes

$$\langle v_{\text{out}} | U_{\mathcal{P}} | v_{\text{in}}\rangle_n = \left\lfloor \prod_{s=0}^{L-1} \langle v_{s+1} | U_{s,s+1} | v_s\rangle_n \right\rfloor_{\mathcal{P}}$$

with $L \equiv L(\mathcal{P})$ for short. The symbol $\left\lfloor \prod \right\rfloor_{\mathcal{P}}$ denotes the ordered product along the path $\mathcal{P}$ and each elementary operation is rewritten as $U_{s,s+1}$ ($s = 0, 1, 2, \ldots L(\mathcal{P})$) to stress its ‘one–step’ character.

4 Finite states–automata

The theory of automata and formal languages addresses in a rigorous way the notions of computing machines and computational processes. If $A$ is an
alphabet, made of letters, digits or other symbols, and $A^*$ denotes the set of all finite sequences of words over $A$, a language $L$ over $A$ is a subset of $A^*$.

The empty word is $\epsilon$, the concatenation of two words $u$ and $v$ is simply denoted by $uv$. In the sixties Noam Chomsky introduced a four level–hierarchy describing formal languages according to their internal structure, namely regular languages, context–free languages, context–sensitive languages and recursively enumerable languages (see e.g. [14]). The processing of each language is inherently related to a particular computing model. Here we are interested in finite states–automata, the machines able to accept regular languages. A deterministic finite states–automaton (DFA) consists of a finite set of states $S$, an input alphabet $A$, a transition function $F : S \times A \rightarrow S$, an initial state $s_{in}$ and a set of accepting states $S_{acc} \subset S$. The automaton starts in $s_{in}$ and reads an input word $w$ from left to right. At the $i$–th step, if the automaton reads the word $w_i$, it updates its state to $s' = F(s, w_i)$, where $s$ is the state of the automaton reading $w_i$. The word has been accepted if the final state reached after reading $w$ is in $S_{acc}$.

In the case of non–deterministic finite states–automaton (NFA) the transition function is defined as a map from $S \times A$ in $P(S)$, where $P(S)$ is the power set of $S$. After reading a particular symbol the transition can lead to different states according to some assigned probability distribution. If an NFA has $n$ internal states, for each symbol $a \in A$ there is an $n \times n$ transition matrix $M_a$ for which $(M_a)_{ij} = 1$ if and only if the transition from the state $i$ to the state $j$ is allowed once the symbol $a$ has been read.

From a general point of view, ‘quantum’ finite states–automata (QFA) are obtained from their classical probabilistic counterparts by moving from the notion of (classical) probability associated with transitions to quantum probability amplitudes. Computation takes place inside a suitable Hilbert space through unitary matrices. Following [15], the measure-once quantum finite–automaton (MOQFA) is defined as a 5-tuple $M = (Q, \Sigma, \delta, q_0, q_f)$, where: $Q$ is a finite set of states; $\Sigma$ is a finite input alphabet with an end–marker symbol $\#$; $\delta : Q \times \Sigma \rightarrow Q$ is the transition function; $\delta(q, \sigma, q')$ is the probability amplitude for the transition from the state $q$ to the state $q'$ upon reading symbol $\sigma$; the state $q_0$ is the initial configuration of the system, and $q_f$ is the accepted final state. For all states and symbols the function $\delta$ must be unitary. At the end of the computational process the automaton measures its configuration: if it is in an accepted state then the input is accepted, otherwise is rejected. The configuration of the automaton is in general a superposition of states in the Hilbert space where the automaton lives. The
transition function is represented by a set of unitary matrices $U_\sigma (\sigma \in \Sigma)$, where $U_\sigma$ represents the unitary transition of the automaton reading the symbol $\sigma$. The probability amplitude for the automaton $M$ to accept the string $w$ is given by

$$f_M(w) = \langle q_f | U_w | q_0 \rangle,$$

(13)

where explicit form of $f_M(w)$ defines the language $L$ accepted by the automaton $M$.

The spin network computational space $G_n(V, E) \times \mathbb{C}^{2J+1}$ (for a fixed $n$) naturally encodes the structures that define (families of) quantum automata. Any finite set of binary coupled states belonging to the computational Hilbert spaces [4] may represent states of some automaton, while combinations of the unitary operations introduced in (6) and (7) acting on such states are actually transition functions and the amplitude (13) complies with expectation values introduced in (12). The inherently step–by–step character of transition functions is related to the existence of an intrinsic discrete–time variable, denoted by $\tau$ in the previous section. All such features make the spin network the ideal candidate for handling with finite–states (discrete–time) automata able to accept any ‘quantum language’ compatible with the algebra of $SU(2)$ angular momenta. In the next section we shall provide an explicit example for this class of quantum language.

5 Quantum automaton calculation of link invariants

As we shall recall below, (colored) link invariants can be obtained as expectation values of Wilson–type operators in the 3D quantum $SU(2)$ Chern–Simons theory for each fixed value of the coupling constant $k$ (related to the deformation parameter of the universal enveloping algebra $U_q(sl(2))$, $q =$ root of unity) [9].

From a purely algebraic viewpoint, the basic ingredient for building up such link invariants is the ‘tensor structure’ naturally associated with the representation ring of the Lie algebra of any simple compact group. In the case of $SU(2)$ this structure is provided by (tensor products of) Hilbert spaces supporting irreducible representations together with unitary morphisms between them: these objects are collected in the so–called Racah–Wigner tensor category introduced in section 3 [13]. In order to deal with (planar diagrams
of) links we need also to specify the eigenvalues of the braiding matrix to be associated with the crossings of the links and this is easily achieved by ‘braiding’ the Racah–Wigner category. In the present context, it is natural to take advantage of quantum group techniques in order to ‘split’ phase transforms like that in (7) by assigning different weights depending on the deformation parameter $q$ to right and left handed twists. From the combinatorial viewpoint, this generalization corresponds to replace the spin network computational space $G_n(V, E) \times \mathbb{C}^{2J+1}$ (encoding the Racah–Wigner category) with its $q$–braided counterpart $(G_n(V, E) \times \mathbb{C}^{2J+1}) \times \mathbb{Z}_2$, where $6j$ symbols in the Racah transforms become $q$–deformed.

As a general remark, let us point out that the link invariants we are going to address are ‘universal’ in the sense that historically distinct approaches (R–matrix representations obtained with the quantum group method, monodromy representations of the braid group in 2D conformal field theories, the quasi tensor category approach by Drinfeld and the 3D quantum Chern–Simons theory, cfr. [16] for a review) are indeed different aspects of the same underlying algebraic structure. In this sense the Chern–Simons setting describes the universal structure of braid group representations shared by all the models quoted above, and in particular the ‘universal’ (colored) link polynomial essentially coincides with the Reshetikhin–Turaev invariant [17]. However, when the group is $SU(2)$, we may speak of ‘extended’ Jones polynomials since the Jones polynomial [8] is recovered by selecting the $j = \frac{1}{2}$ representation on each of the link component (or on each strand of the associated braid). On the other hand, the topological quantum field approach calls into play links embedded in three–dimensional manifolds, and thus, for instance, the invariants can be naturally interpreted as quantum invariants of hyperbolic manifolds (complements of links in the 3-sphere).

Recall that the $SU(2)$ Chern–Simons action for the 3–sphere $S^3$ is

$$k S(A) = \frac{k}{4\pi} \int_{S^3} \text{tr}(AdA + \frac{2}{3}A \wedge A \wedge A)$$  (14)

where $A$ is the connection one–form with value in the Lie algebra of the gauge group $SU(2)$ and $k$ is the coupling constant (constrained to be a positive integer by the gauge–invariant quantization procedure). The observables of the associated quantum field theory are Wilson loop operators defined, for an oriented knot $K$ carrying a spin–$j$ representation, as

$$W_j[K] = \text{tr}_j \text{Pexp} \oint_K A$$  (15)
and, for a link \( L \) made of \( s \) components \( K_i \), each labelled by a spin (coloring),

\[
W_{j_1 j_2 \ldots j_s} [L] = \prod_{l=1}^{s} W_{j_l} [K_l].
\]

(16)

The expectation values of the above operators are formally given by

\[
V_{j_1 \ldots j_s} [L] = \frac{\int_{S^3} [dA] W_{j_1 \ldots j_s} [L] e^{ikS(A)}}{\int_{S^3} [dA] e^{ikS(A)}}.
\]

(17)

where the functional integration is over flat \( SU(2) \)-connections on \( S^3 \) and the generating functional in the denominator will be normalized to 1. Such expectation values depend on the isotopy type of the oriented link \( L \) and on the set of representations labelled by \((j_1, \ldots, j_s)\) associated with the link components (and do not depend on the metric of the ambient manifold). To evaluate explicitly such functional averages we have to go through the relationship between Chern–Simons theory on an oriented, compact three–manifold with boundary and the Wess–Zumino–Witten (WZW) conformal field theory which lives on the 2–dimensional boundary components. If \( \Sigma^{(1)}, \Sigma^{(2)}, \ldots, \Sigma^{(r)} \) are the boundary components, (a finite number of) Wilson lines (carrying labels \( j_l \)) intersect \( \Sigma^{(i)} \) at some points (punctures) \( P_l^{(i)} \). According to the axioms for TQFT, we associates with each boundary component \( \Sigma^{(i)} \) a finite–dimensional Hilbert space denoted by \( H^{(i)} \). In this case the Chern–Simons functional (represented geometrically by a cobordism between incoming and outgoing boundaries and algebraically by a functor between the associated Hilbert spaces) is a state belonging to the tensor product of the boundary Hilbert spaces. It can be shown that the conformal blocks of the \( SU(2)_\ell \) WZW field theory on the boundaries with punctures actually provide basis vectors for the Hilbert spaces \( H^{(i)} \) (the level \( \ell \) of the WZW model is related to the deformation parameter \( q \) according to \( q = \exp\{-2\pi i/(\ell + 2)\} \), and in turn \( \ell \) is related to the coupling constant \( k(\geq 3) \) of the CS theory in the bulk by \( \ell = k - 2 \).

Suppose that the oriented link appearing in the the expectation value (17) is endowed with a plat presentation \( \[18\] \), namely is actually the closure of an oriented braid with \( 2m \) strands. If we remove two open three–balls from \( S^3 \) we get two boundaries (both topologically \( S^2 \), but with opposite orientations) and we can always accomodate \( 2m \) ‘unbraided’ Wilson lines carrying labels \( j_1, j_2, \ldots, j_{2m} \) starting from the incoming (lower) boundary and ending
into the outgoing (upper) one. Following the formulation due to Kaul [19], let us start by denoting this ‘identity’ oriented colored oriented braid as

\[ \nu_I \left( \hat{j}_1^* \hat{j}_2^* \cdots \hat{j}_{2m}^* \right) , \] (18)

where \( \hat{j}_i = (j_i, \epsilon_i) \) \( i = 1, 2, \ldots 2m \) represents the spin \( j_i \) together with an orientation \( \epsilon_i, i = \pm 1 \) for a strand going into or away from the boundary, while stars over the symbols stand for the opposite choice of the orientation. A generic colored oriented braid on \( 2m \) strand can be generated from this identity braid by applying a suitable ‘braiding’ operator \( B \) (written in terms of generators \( B_1, B_2, \ldots, B_{2m-1} \)) starting from the lower boundary. The resulting colored oriented braid is denoted by

\[ \nu_B \left( \hat{j}_1 \hat{j}_1^* \cdots \hat{j}_m \hat{j}_m^* \right) , \] (19)

where labels have been ordered according to the requirement of having a plat presentation for the associated link. According to our previous remark, we must now select bases in the two boundary Hilbert spaces \( H^{(1)} \) and \( H^{(2)} \) and then built up the Chern–Simons functional which will belong to \( H^{(1)} \otimes H^{(2)} \).

There are basically two possible choices of such bases on the incoming Hilbert space \( H^{(1)} \), corresponding to two sets of correlators in WZW theory involving \( 2m \) primary fields with angular momentum assignments \((j_1, \ldots, j_{2m})\), namely

\[ | \langle ((j_1 j_2) \ldots (j_{2m-1} j_{2m}))^0_{[p; r]} \rangle ; | \langle (j_1 (j_2 j_3) \ldots (j_{2m-2} j_{2m-1} j_{2m}))^0_{[t; s]} \rangle \rangle . \] (20)

The shorthand notations \([p; r]\) and \([t; s]\) represent the strings of intermediate spin variables arising from the given pairwise couplings of the incoming \((j_1, \ldots, j_{2m}), \{p_0, p_1, \ldots; r_0, r_1, \ldots\} \) and \( \{t_0, t_1, \ldots; s_0, s_1, \ldots\} \) respectively. Moreover, the superscript \(^0\) corresponds to the requirement that all such pairings must generate \( SU(2) \) singlets (spin–0 representations). Unitary transformations between the two sets of vectors \( [20] \) are implemented by the so–called duality matrices, involving \( q \)–deformed \( 6j \) symbols (cfir. the appendix of [19] for their explicit expressions).

When the outgoing Hilbert space is considered (corresponding to the boundary \( \Sigma^{(2)} \) endowed with the opposite orientation with respect to \( \Sigma^{(1)} \)) we have to introduce bases which are dual with respect to \( [20] \), suitably normalized according to

\[ \langle (j_1 j_2) \ldots (j_{2m-1} j_{2m}) \rangle^0_{[p; r]} | (j_1 j_2) \ldots (j_{2m-1} j_{2m}) \rangle^0_{[p'; r']} = \delta_{(p)(p')} \delta_{(r)(r')} \] (21)
and
\[ \langle \langle (j_1j_2j_3)(j_2m-2j_2m-1)j_2m \rangle \rangle_{[t,s]}^0 = \langle \langle (j_1j_2j_3)\ldots(j_2m-2j_2m-1)j_2m \rangle \rangle_{[t',s']}', \]
\[ (j_1j_2)\ldots(j_2t+1j_2t+2) \rangle \rangle_{[t,s]}^0 = \lambda_{pj}(\hat{j}_j, \hat{j}_{j'}) \langle \langle (j_1j_2)\ldots(j_2t+1j_2t+2) \rangle \rangle_{[t,s]}^0 \]
\[ B_{2l+1}|(j_2mj_1)\ldots(j_2t+1j_2t+2)\rangle \rangle_{[t,s]}^0 = \lambda_{m}(\hat{j}_j, \hat{j}_{j'}) \langle \langle (j_2mj_1)\ldots(j_2t+1j_2t+2) \rangle \rangle_{[t,s]}^0 \]

with eigenvalues given by the expressions
\[ \lambda_z(\hat{j}, \hat{j'}) = \lambda_z^{(+)}(j, j') = (-)^{j'+z}q^{(c_j+c_{j'})/2+c_{min(j,j')}^2} \text{ if } \epsilon \epsilon' = +1 \]
\[ \lambda_z(\hat{j}, \hat{j'}) = \lambda_z^{(-)}(j, j')^{-1} = (-)^{j'-z}q^{(c_j-c_{j'})/2-c_{z}^2} \text{ if } \epsilon \epsilon' = -1 \]

where \( q \) is the deformation parameter, \( z \in \{p_l, t_l\} \) and \( c_z \equiv z(z+1) \) is the quadratic Casimir for the spin \( z \) representation. Thus \( \lambda_z^{(+)}(j, j') \) is the eigenvalue of the matrix which performs a right handed half-twist in strands with the same orientation, while \( \lambda_z^{(-)}(j, j') \) is the eigenvalue of the matrix which performs a right handed half-twist in strands with the opposite orientations.

The expectation value of the Wilson operator (17) for any link \( L \) presented as the plat closure of an oriented braid over \( 2m \) strands can be recasted into the expectation value of the braiding operator \( B \) associated with the oriented braid (14) according to
\[ V_{j_1j_1^*\ldots j_1j_1^*} \langle [L] = \left( \prod_{i=1}^{m} [2j_i + 1] \right) \times \]
\[ \left( \langle \langle (\hat{i}_1^{l_1})\ldots(\hat{i}_m^{l_m}) \rangle \rangle_{[0,0]}^0 \right) B \left( \hat{j}_1 \hat{j}_1^* \ldots \hat{j}_m \hat{j}_m^* \right) \langle \langle (\hat{j}_1\hat{j}_1^*)\ldots(\hat{j}_m\hat{j}_m^*) \rangle \rangle_{[0,0]}^0 \]
\[ (25) \]

This expression (which is normalized according to the standard conventions) gives the value of the extended Jones polynomial of the link \( L \). The operator \( B \) is expressed in terms of (a finite sequence of) braid generators (23), suitably converted in terms of the current basis vectors by acting with \( q \)-Racah transforms on the even subset, while \([2j_i + 1]\) is the \( q \)-dimension of the representation \( j_i \).
Coming back to the definition of the quantum finite state–automaton (QFA) given in the previous section, namely a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$, we recognize that, given a plat presentation of a link $L$, we can always build up a QFA $M_L$ which recognizes the language of the braid group and whose associated probability amplitude (13) is given by the value of the extended Jones polynomial (25). More precisely: $q_0$ is a particular ket vector of the type (20); $\Sigma$ is the alphabet given by the $(2m - 1)$ braiding operators (23); $F$ is a set of final stated suitably selected among the bra vectors in (20). Note that, according to the expression given in (25), $F$ must contain (singlet) states which possess the same pairwise coupling structure as the initial state $q_0$ exhibits. Owing to the topological properties of the plat presentation, this means that final states may differ from $q_0$ by a permutation on the string $(j_1j_2\ldots j_m)$ and thus we can actually build up a family of $m!$ automata out of one initial state $q_0$. Once a final state $q_f$ has been selected (the right permutation can be singled out in a fast way even by a classical machine) the evaluation of (25) is carried out by the automaton linearly in the length of the ‘word’ $B$. The length of $B$ is the number of $\{ \text{braiding operators + } q\text{-Racah transforms} \}$ entering into the explicit expression of $B$ and equals the number of crossings of the plat presentation of $L$. It can be easily recognized that such an automaton can be associated with a particular path in the $q$–braided spin network computational space $(G_n(V, E) \times \mathbb{C}^{2^{(j+1)}} \times \mathbb{Z}_2)\times \mathbb{Z}_2$ for $n = 2m - 1$ since the process carried out here is a particular realization of the general computing procedure described at the end of section 3 (cfr. 12).

What we have really done is to ‘encode’ the combinatorial structure underlying quantum $SU(2)$ Chern–Simons theory (and the associated WZW boundary theory) at some fixed level $\ell$ into the above abstract $q$–braided $SU(2)$–decorated graph. This does not mean, of course, that we have got a quantum algorithm in the proper sense, since the encoding map could not be ‘efficient’ (nor efficiently approximated) with respect to standard models of computation (Boolean circuit, Turing machine). We are currently addressing such open problems.
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