Fractional powers of sectorial operators via the Dirichlet-to-Neumann operator

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ABSTRACT
In the very influential paper [4] Caffarelli and Silvestre studied regularity of \((-\Delta)^s\), \(0 < s < 1\), by identifying fractional powers with a certain Dirichlet-to-Neumann operator. Stinga and Torrea [15] and Galé et al. [7] gave several more abstract versions of this extension procedure. The purpose of this paper is to study precise regularity properties of the Dirichlet and the Neumann problem in Hilbert spaces. Then the Dirichlet-to-Neumann operator becomes an isomorphism between interpolation spaces and its part in the underlying Hilbert space is exactly the fractional power.

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1. Introduction

In the very influential article [4] Caffarelli and Silvestre study the fractional powers \((-\Delta)^s\), \(0 < s < 1\), on \(\mathbb{R}^N\) of the operator \(-\Delta\) by identifying the operator \((-\Delta)^s\) with a Dirichlet-to-Neumann operator with respect to an extension to the upper half-plane. Subsequently, such extensions have been investigated for general selfadjoint lower bounded operators (instead of \(-\Delta\)) by Stinga and Torrea [15] and for generators of semigroups and even integrated semigroups by Galé et al. [7]. We also refer to [5, 13] and their references for the case of symmetric second-order elliptic operators in divergence form with smooth coefficients on bounded open sets \(\Omega \subset \mathbb{R}^N\) subject to Dirichlet and Neumann boundary conditions on the boundary \(\partial \Omega\).

In the context of these investigations given an operator \(A\) on a Banach space and \(0 < s < 1\) a Bessel kind of equation

\[-u''(t) - \frac{1-2s}{t} u'(t) + Au(t) = 0, \quad t \in (0, \infty)\]

plays an important role, where in a concrete example \(A\) may be the Laplacian acting on the first-order Sobolev space \(H^1_0(\Omega)\) for some open set \(\Omega \subset \mathbb{R}^N\).

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The purpose of our paper is to study Eq. (1.1) for operators on a Hilbert space $H$ associated with a form. We will prove well-posedness of Eq. (1.1) subject to Dirichlet boundary conditions at $t = 0$ and also to (a special kind of) Neumann boundary conditions at $t = 0$. Let us explain our main results with the necessary precision.

Our starting point are Hilbert spaces $V$ and $H$ such that $V \hookrightarrow H$ (i.e. $V$ is continuously and densely embedded in $H$). Identifying each $y \in H$ with the anti-linear form $v \mapsto (v, y)_H$ we obtain the Gelfand triple

$$V \hookrightarrow H \hookrightarrow V'.$$

Let $E : V \times V \rightarrow \mathbb{C}$ be a coercive, continuous and sesquilinear form. Then

$$\langle Au, v \rangle_{V', V} = E(u, v), \quad \forall u, v \in V,$$

defines an operator $A \in \mathcal{L}(V, V')$ which we consider in Eq. (1.1). Here $\mathcal{L}(V, V')$ is the space of all linear and bounded operators from $V$ to $V'$. The operator $A$ is not really the point of interest, it is rather its part $m$-sectorial with vertex 0 in the sense of Kato [9, Section V3.10]. Thus its fractional power $A^s$ is again an $m$-sectorial operator with vertex 0. It is this operator we want to identify.

If $E$ is symmetric, then $A$ is self-adjoint with lower bound greater than 0. A concrete example is $A = -\Delta_D$, where $\Delta_D$ is the Dirichlet Laplacian on $L^2(\Omega)$. In fact each self-adjoint operator $A$ with lower bound greater than 0 comes from a form in this way. But we emphasize that symmetry is not needed for our investigation.

The correct spaces for well-posedness of (1.1) are Sobolev spaces on $(0, \infty)$ as they also occur in interpolation theory (see e.g. [11]). For all $0 < s < 1$, we let

$$L^s_2(V) := L^2 \left( (0, \infty); V, \frac{dt}{t} \right)$$

and define

$$W_s(H, V) := \left\{ u \in L^1_{\text{loc}}((0, \infty); V) : u' \in L^1_{\text{loc}}((0, \infty); H),
\right.\left. t^s u \in L^s_2(V) \text{ and } t^s u' \in L^s_2(H) \right\}.$$ 

We say that a function $u : (0, \infty) \rightarrow V$ is $s$-harmonic, if $u \in W_{1-s}(H, V), t^{1-2s} u \in W_s(V', H)$ and $u$ solves (1.1). Such an $s$-harmonic function $u$ always has a trace

$$u(0) = \lim_{t \downarrow 0} u(t) \quad \text{(with convergence in } H)$$

and $u(0) \in [H, V]_s$ (the complex interpolation space), and it also has an $s$-normal derivative defined by

$$u'(0) := -\lim_{t \downarrow 0} t^{1-2s} u'(t) \quad \text{(with weak convergence in } V')$$

and it turns out that $u'(0) \in [H, V']_s$.

Having introduced this notion, our main results are easy to formulate. First of all we establish two well-posedness results (Theorem 3.4):

- **Dirichlet Problem.** For each $x \in [H, V]_s$ there exists a unique $s$-harmonic function $u$ satisfying $u(0) = x$. 
• **Neumann Problem.** For each \( y \in [H, V'] \), there exists a unique \( s \)-harmonic function \( u \) satisfying \( u'_s(0) = y \).

These well-posedness results are an invitation to consider the Dirichlet-to-Neumann operator \( D_s \) which acts as follows. Given \( x \in [H, V] \), consider the \( s \)-harmonic function \( u \) satisfying \( u(0) = x \) and define \( D_s x := u'_s(0) \). Then we prove that \( D_s : [H, V] \rightarrow [H, V'] \) is an isomorphism (Theorem 3.6). Note that

\[
[H, V] \subset H \subset [H, V'].
\]

Now it is natural to consider the part \( D_s \) of \( D_s \) in \( H \), which is a closed unbounded operator on \( H \).

Our main result (Theorem 4.1) says that

\[
D_s = c_s A^s,
\]

where \( c_s \) is a universal constant.

The study of \( s \)-harmonic functions in suitable Sobolev and interpolation spaces and the corresponding well-posedness results are the novelty of the present article. We also obtain \( C^\infty \)-regularity making use of a Poisson formula discovered by Stinga and Torrea in [15] on Hilbert spaces, then extended to Banach spaces in [7]. A different Poisson formula was also used in [4]. Finally we extend our results to the case where \( E \) is not coercive obtaining less precise regularity results though.

The paper is structured as follows. In Section 2, we put together some properties of the mixed Sobolev spaces related to fractions. The Dirichlet and Neumann problem is studied in Section 3. The main result on the identification of the Dirichlet-to-Neumann map with the fractional power in the coercive case is obtained in Section 4. In Section 5, we drop the condition that \( E \) is coercive and assume merely that \( E \) is sectorial with vertex zero.

### 2. Sobolev spaces

The Dirichlet and the Neumann problems we have in mind are well posed in mixed Sobolev spaces which are known from interpolation theory. We give the definition, cite results we shall need and prove an integration by parts formula.

Let \( X \) be a Hilbert space. We will consider spaces of integrable functions on \((0, \infty)\) with values in \( X \). Derivatives will be taken in the distributional sense; i.e. using the elements of the scalar space \( C^\infty_c((0, \infty)) \) of all infinitely differentiable \( \mathbb{C} \)-valued functions with compact support as test functions. Here is the precise definition.

**Definition 2.1.**

(a) Let \( u, v \in L^1_{\text{loc}}(X) := L^1_{\text{loc}}((0, \infty); X) \). We say that \( v \) is the **weak derivative** of \( u \) if

\[
- \int_0^\infty \varphi'(t) u(t) \ dt = \int_0^\infty \varphi(t) v(t) \ dt
\]

for all \( \varphi \in C^\infty_c((0, \infty)) \). In that case we write \( u' := v \).

(b) Let \( E \) be a subspace of \( L^1_{\text{loc}}(X) \) and let \( u \in L^1_{\text{loc}}(X) \). We say that \( u' \in E \) if there exists a \( v \in E \) such that \( v \) is the weak derivative of \( u \).
The weak derivative is unique if it exists and for all $u \in C^1((0, \infty); X)$ the weak and classical derivatives coincide.

Let $X$, $Y$ be Hilbert spaces such that $Y \hookrightarrow X$. This means that $Y$ is a dense subspace of $X$ and the injection of $Y$ into $X$ is continuous. Fix $0 < s < 1$. We define the space

$$W_s(X, Y) := \left\{ u \in L^1_{\text{loc}}(Y) : u' \in L^1_{\text{loc}}(X), \left( t \mapsto t^s u(t) \right) \in L^s_2(Y) \text{ and} \right\}$$

where for $Z = X$ or $Z = Y$,

$$L^s_2(Z) := L^2 \left( Z, \frac{dt}{t} \right) = L^2 \left( (0, \infty); Z, \frac{dt}{t} \right).$$

To avoid clutter we write $t^s$ for the function $t \mapsto t^s$. It is clear that $W_s(X, Y)$ endowed with the norm

$$\|u\|_{W_s(X,Y)} = \left( \|t^s u\|_{L^s_2(Y)}^2 + \|t^s u'\|_{L^s_2(X)}^2 \right)^{\frac{1}{2}}$$

$$= \left( \int_0^\infty \left( \|u(t)\|^2_Y + \|u'(t)\|^2_X \right) t^{2s-1} dt \right)^{\frac{1}{2}}$$

is a Banach space and it is even a Hilbert space.

We quote the following result from [11, Proposition 1.2.10].

**Proposition 2.2.** Let $u \in W_s(X, Y)$. Then $u(0) := \lim_{t \downarrow 0} u(t)$ exists in the norm on $X$. Moreover, $u(0) \in [X, Y]_{1-s}$. The map $u \mapsto u(0)$ from $W_s(X, Y)$ into $[X, Y]_{1-s}$ is continuous and surjective.

Recall that $[X, Y]_\theta$ is the complex interpolation space between $X$ and $Y$ for all $0 < \theta < 1$. Note that the complex interpolation space $[X, Y]_\theta$ coincides with the trace-method real interpolation space $(X, Y)_{\theta,2}$ since we restrict ourselves to Hilbert spaces.

Let

$$C^\infty_c((0, \infty); Y) := \left\{ u : [0, \infty) \rightarrow Y : u \text{ is infinitely differentiable and} \right.$$  

$$\text{suppu is compact in } [0, \infty) \}.$$  

Clearly $C^\infty_c((0, \infty); Y)$ is a subspace of $W_s(X, Y)$.

We need the following density result.

**Proposition 2.3.** Let $s \in (0, 1)$. Then the following assertions hold.

(a) If $s \geq \frac{1}{2}$, then the space $C^\infty_c((0, \infty); Y)$ is dense in $W_s(X, Y)$.

(b) If $s < \frac{1}{2}$, then the space

$$\left\{ u \in W_s(X, Y) \cap C^\infty((0, \infty); Y) : \text{suppu is a bounded set in } (0, \infty) \right\}$$

is dense in $W_s(X, Y)$.

The proof of Proposition 2.3 requires quite some preparation.
Lemma 2.4. Let \( \rho \) be a Hilbert space and \( \theta \in (0, 1) \). Define the space
\[
W^\theta(Z) = \{ u \in L^1_{\text{loc}}(Z) : t^\theta u \in L^2_Z(Z) \},
\]
with the norm \( \| u \|_{W^\theta(Z)} = \| t^\theta u \|_{L^2_Z(Z)} \). Note that \( W^\theta(Z) = L^2((0, \infty); Z, t^{2\theta-1} dt) \).

**Lemma 2.4.** Let \( \theta \in (0, 1) \). Let \( u \in W^\theta(Z) \) and \( r \in (0, \infty) \). Define \( L_r u : (0, \infty) \to Z \) by
\[
(L_r u)(t) = u(r^{-1} t).
\]
Then \( L_r u \in W^\theta(Z) \) and \( \| L_r u \|_{W^\theta(Z)} = r^\theta \| u \|_{W^\theta(Z)} \). Moreover, \( \lim_{r \to 1} L_r u = u \) in \( W^\theta(Z) \) for all \( u \in W^\theta(Z) \).

**Proof.** Let \( u \in W^\theta(Z) \) and \( r \in (0, \infty) \). Clearly \( L_r u \in L^1_{\text{loc}}(Z) \). Moreover,
\[
\| t^\theta L_r u \|_{L^2_Z(Z)}^2 = \int_0^\infty \| t^\theta u(r^{-1} t) \|_Z^2 \frac{dt}{t} = r^{2\theta} \int_0^\infty \| t^\theta u(t) \|_Z^2 \frac{dt}{t} = r^{2\theta} \| u \|_{W^\theta(Z)}^2.
\]
This proves the first two claims.

If \( u \) is a step function, then it is easy to see that \( \lim_{r \to 1} L_r u = u \) in \( W^\theta(Z) \). Since the step functions are dense in \( L^2((0, \infty); Z, t^{2\theta-1} dt) \) by [1, Lemma 3.26(1)], the lemma follows. \( \square \)

**Remark 2.5.** The space \((0, \infty)\) with multiplication is a one-dimensional Lie group. The Haar measure is \( \frac{dt}{t} \) and the corresponding \( L_2 \)-space is \( L_2^+ \). Lemma 2.4 states that the vector valued left representation in \( W^\theta(Z) \) is well-defined and is a continuous representation of the group \((0, \infty)\) in \( W^\theta(Z) \).

For the remainder of this section fix for all \( n \in \mathbb{N} \) a function \( \rho_n \in C^\infty_c((0, \infty)) \) such that \( \rho_n \geq 0 \), \( \text{supp} \rho_n \subset (1 - \frac{1}{2n}, 1 + \frac{1}{2n}) \) and \( \lim_{n \to \infty} \int_0^\infty \rho_n(t) \frac{dt}{t} = 1 \). For all \( \chi \in C^\infty_c((0, \infty)) \), \( \theta \in (0, 1) \) and \( u \in W^\theta(Z) \) define \( \chi * u : (0, \infty) \to Z \) by
\[
(\chi * u)(t) = \int_0^\infty \chi(r) u(r^{-1} t) \frac{dr}{r}.
\]
Clearly \( \chi * u \in C^\infty_c((0, \infty); Z) \).

**Lemma 2.6.** Let \( \theta \in (0, 1) \) and \( u \in W^\theta(Z) \). Then the following assertions hold.
(a) If \( \chi \in C^\infty_c((0, \infty)) \), then \( \chi * u \in W^\theta(Z) \cap C^\infty((0, \infty); Z) \).
(b) \( \lim_{n \to \infty} \rho_n * u = u \) in \( W^\theta(Z) \).

**Proof.** (a). Let \( t \in (0, \infty) \). Then
\[
t^\theta \| (\chi * u)(t) \|_Z \leq \int_0^\infty r^\theta |\chi(r)| (r^{-1} t)^\theta \| u(r^{-1} t) \|_Z \frac{dr}{r}.
\]
So
\[
\left( \int_0^\infty \| t^\theta (\chi * u)(t) \|_Z^2 \frac{dt}{t} \right)^{\frac{1}{2}} \leq \int_0^\infty t^\theta |\chi(t)| \frac{dt}{t} \cdot \left( \int_0^\infty \| t^\theta u(t) \|_Z^2 \frac{dt}{t} \right)^{\frac{1}{2}}.
\]
Therefore, \( \chi * u \in W^\theta(Z) \).
Lemma 2.6(b), this time applied with \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} \supp u \subseteq X \). Let \( C \subseteq W \).

Proof of Proposition 2.3. Let \( \theta \in (0, 1) \). Then the space \( W^\theta(Z) \cap C^\infty((0, \infty); Z) \) is dense in \( W^\theta(Z) \).

As an immediately consequence we obtain the next proposition.

**Proposition 2.7.** Let \( \theta \in (0, 1) \). Then the space \( W^\theta(Z) \cap C^\infty((0, \infty); Z) \) is dense in \( W^\theta(Z) \).

Now we are able to prove Proposition 2.3.

**Proof of Proposition 2.3.** The proof is in several steps.

**Step 1.** Let \( s \in (0, 1) \). We claim that the space \( W_s(X, Y) \cap C^\infty((0, \infty); Y) \) is dense in \( W_s(X, Y) \). Indeed, let \( u \in W_s(X, Y) \). Then \( \rho_n * u \in C^\infty((0, \infty); Y) \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} \rho_n * u = u \) in \( W^s(Y) \) by Lemma 2.6(b). For all \( n \in \mathbb{N} \) define \( \psi_n \in C^\infty_c((0, \infty)) \) by \( \psi_n(r) = \frac{1}{r} \rho_n(r) \). Then \( \psi_n \geq 0 \) and \( \supp \psi_n \subseteq (1 - \frac{1}{2n}, 1 + \frac{1}{n}) \). Moreover, \( \lim_{n \to \infty} \int_0^\infty \psi_n(r) \frac{dr}{r} = 1 \). Hence \( \lim_{n \to \infty} (\rho_n * u)' = \lim_{n \to \infty} \psi_n * (u') = u' \) in \( W^s(X) \) by Lemma 2.6(b), this time applied with \( \rho_n \) replaced by \( \psi_n \). So \( \lim_{n \to \infty} \rho_n * u = u \) in \( W_s(X, Y) \)

**Step 2.** Let \( s \in (0, 1) \). We show that the space \( \{ u \in W_s(X, Y) \cap C^\infty((0, \infty); Y) : \supp u \) is a bounded set in \((0, \infty)\} \) is dense in \( W_s(X, Y) \). In fact, since \( Y \) is continuously embedded into \( X \) there exists a constant \( c > 0 \) such that \( \| y \|_X \leq c \| y \|_Y \) for all \( y \in Y \). Let \( u \in W_s(X, Y) \cap C^\infty((0, \infty); Y) \). Let \( \chi \in C^\infty((0, \infty)) \) be such that \( \I_{[0,1]} \leq \chi \leq \I_{[0,2]} \). For all \( n \in \mathbb{N} \) define \( \chi_n : (0, \infty) \to \mathbb{R} \) by \( \chi_n(t) = \chi(\frac{t}{n}) \). Moreover, define \( u_n = \chi_n u \). Then \( u_n \in W_s(X, Y) \). If \( n \in \mathbb{N} \), then

\[
\| t^2(u - u_n) \|_{L^2_s(Y)}^2 = \int_0^\infty \left| t^2(1 - \chi_n(t)) \| u(t) \|_Y \right|^2 \frac{dt}{t} \leq \int_0^\infty \| t^2u(t) \|_Y^2 \frac{dt}{t}.
\]

So \( \lim_{n \to \infty} \| t^2(u - u_n) \|_{L^2_s(Y)} = 0 \). Next, \( u_n' = \chi_n' u + \chi_n u' \) for all \( n \in \mathbb{N} \). It follows similarly that \( \lim_{n \to \infty} \| t^2(u' - \chi_n u') \|_{L^2_s(X)} = 0 \). We shall show that \( \lim_{n \to \infty} \| t^2 \chi_n' u \|_{L^2_s(X)} = 0 \). Let \( n \in \mathbb{N} \). Then

\[
\| t^2 \chi_n' u \|_{L^2_s(X)}^2 = \frac{1}{n^2} \int_0^\infty \left| t^2 \chi_n'(\frac{t}{n}) \| u(t) \|_X \right|^2 \frac{dt}{t} \leq \frac{\| \chi' \|_2^2}{n^2} \int_0^\infty \| t^2u(t) \|_X^2 \frac{dt}{t} \leq \frac{c^2 \| \chi' \|_2^2}{n^2} \| t^2u \|_{L^2_s(Y)}^2.
\]

So \( \lim_{n \to \infty} \| t^2 \chi_n' u \|_{L^2_s(X)} = 0 \) and hence \( \lim_{n \to \infty} u_n = u \) in \( W_s(X, Y) \). Then Step 2 follows by an application of Step 1.
Step 3. We prove the two statements of Proposition 2.3.
(b). This is a special case of Step 2.
(a). Let $u \in W_s(X, Y) \cap C^\infty(\mathbb{R} \cap (0, \infty); Y)$ and suppose that \( \text{supp} u \) is a bounded set in \( (0, \infty) \). For all $n \in \mathbb{N}$ define $u_n \colon [0, \infty) \to Y$ by $u_n(t) = u(t + \frac{1}{n})$. Then $u_n \in C_c^\infty([0, \infty); Y)$. Moreover, if $n \in \mathbb{N}$, then

$$\|t^n u_n\|_{L^2_s(Y)}^2 = \int_0^\infty t^{2s-1} u(t + \frac{1}{n}) \|u(t)\|_Y^2 \, dt = \int_{\frac{1}{n}}^\infty \left( t - \frac{1}{n} \right)^{2s-1} \|u(t)\|_Y^2 \, dt \leq \int_{\frac{1}{n}}^\infty t^{2s-1} \|u(t)\|_Y^2 \, dt \leq \|u\|_{W_s(X, Y)}^2,$$

where we have used that $2s - 1 \geq 0$ in the first inequality. Similarly, $\|t^n u_n'\|_{L^2_s(X)} \leq \|u\|_{W_s(X, Y)}$ for all $n \in \mathbb{N}$. Hence the sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in $W_s(X, Y)$. Therefore, it has a subsequence which converges weakly in $W_s(X, Y)$. So $u$ is in the weak closure of $C_c^\infty([0, \infty); Y)$ in $W_s(X, Y)$. Together with Step 2 it follows that $C_c^\infty([0, \infty); Y)$ is weakly dense in $W_s(X, Y)$. Since $C_c^\infty([0, \infty); Y)$ is convex, it is then also norm dense in $W_s(X, Y)$.

Next, we want to specify our settings to Gelfand triples; i.e. we consider two Hilbert spaces $H, V$ such that $V \xleftarrow{d} H$. Let $i$ be the inclusion from $V$ into $H$. Then the dual map $i^*$ is a continuous map from $H'$ into $V'$, where $H'$ and $V'$ denote the antidual of $H$ and $V$, respectively. Since $i$ has dense image, the map $i^*$ is injective. Moreover, it also has a dense image. By the Riesz representation theorem one can identify $H$ with $H'$. We call $H$ the pivot space. Thus one has the chain

$$V \hookrightarrow H \cong H' \hookrightarrow V',$$

which is known as a Gelfand triple. Therefore, one has the following continuous and dense embeddings

$$V \xleftarrow{d} H \xleftarrow{d} V'.$$

Remark 2.8. By the spectral theorem up to unitary equivalence one can assume that $H = L^2(\Gamma, \sigma)$ for some measure space $(\Gamma, \Sigma, \sigma)$, and $V = L^2(\Gamma, m \, d\sigma)$ for some measurable function $m \colon \Gamma \to [1, \infty)$. Then $V' = L^2(\Gamma, \frac{d\sigma}{m})$ and the duality is given by

$$\langle f, g \rangle_{V', V} = \int_\Gamma f(x) \overline{g(x)} \, d\sigma(x)$$

for all $f \in V'$ and $g \in V$. Thus $\langle f, g \rangle_{V', V}$ is written in terms of the measure $\sigma$ without weight. This is the reason for calling $H$ the pivot space. In this unitary equivalent situation the complex interpolation spaces become

$$[H, V]_s = L^2(\Gamma, m^s \, d\sigma) \quad \text{and} \quad [H, V']_s = L^2(\Gamma, m^{-s} \, d\sigma).$$

In particular, $[H, V]' = [H, V]_1$ and we have the new Gelfand triple

$$[H, V]_s \xleftarrow{d} H \xleftarrow{d} [H, V]'_s,$$

with again $H$ as pivot space.
The following integration by parts formula will be crucial for us.

**Proposition 2.9.** Let $0 < s < 1$. Let $w \in W_s(V', H)$ and $v \in W_{1-s}(H, V)$. Then $t \mapsto \langle w'(t), v(t) \rangle_{V', V}$ and $t \mapsto \langle w(t), v'(t) \rangle_H$ are elements of $L^1((0, \infty))$. Moreover,

$$- \int_0^\infty \langle w'(t), v(t) \rangle_{V', V} \, dt = \int_0^\infty \langle w(t), v'(t) \rangle_{H} \, dt + \langle w(0), v(0) \rangle_{[H, V']_b, [H, V]_b}.$$  

**Proof.** Let $w \in W_s(V', H)$ and $v \in W_{1-s}(H, V)$. By definition $t^s w' \in L^1_s(V')$ and $t^{1-s} v \in L^\infty_s(V)$. So $t \mapsto \langle w'(t), v(t) \rangle_{V', V}$ is an element of $L^1((0, \infty))$. Similarly, $t^s w \in L^1_s(H)$ and $t^{1-s} v' \in L^\infty_s(H)$. Consequently $t \mapsto \langle w(t), v'(t) \rangle_H$ is an element of $L^1((0, \infty))$.

Together with Proposition 2.2 it follows that the map

$$(w, v) \mapsto \int_0^\infty \left( \langle w'(t), v(t) \rangle_{V', V} + \langle w(t), v'(t) \rangle_H \right) \, dt + \langle w(0), v(0) \rangle_{[H, V']_b, [H, V]_b}$$

is continuous from $W_s(V', H) \times W_{1-s}(H, V)$ into $C$. Hence it suffices to show that

$$\int_0^\infty \left( \langle w'(t), v(t) \rangle_{V', V} + \langle w(t), v'(t) \rangle_H \right) \, dt + \langle w(0), v(0) \rangle_{[H, V']_b, [H, V]_b} = 0 \tag{2.1}$$

for all $(w, v)$ in a dense subset of $W_s(V', H) \times W_{1-s}(H, V)$.

Let $w \in W_s(V', H) \cap C^\infty((0, \infty); H)$, $v \in W_{1-s}(H, V) \cap C^\infty((0, \infty); H)$ and suppose that both $\text{supp} w$ and $\text{supp} v$ are bounded sets in $(0, \infty)$. Then

$$\int_0^\infty \left( \langle w'(t), v(t) \rangle_{V', V} + \langle w(t), v'(t) \rangle_H \right) \, dt$$

$$= \lim_{\varepsilon \to 0} \int_0^\infty \left( \langle w'(t), v(t) \rangle_{V', V} + \langle w(t), v'(t) \rangle_H \right) \, dt$$

$$= \lim_{\varepsilon \to 0} \int_0^\infty \left( \langle w'(t), v(t) \rangle_{V', V} + \langle w(t), v'(t) \rangle_H \right) \, dt$$

$$= \lim_{\varepsilon \to 0} - \langle w(\varepsilon), v(\varepsilon) \rangle_H. \tag{2.2}$$

We distinguish two cases.

**Case 1.** Suppose that $s \geq \frac{1}{2}$.

Let $w \in C^\infty_c((0, \infty); H)$ and $v \in W_{1-s}(H, V) \cap C^\infty((0, \infty); H)$ be such that $\text{supp} v$ is a bounded set in $(0, \infty)$. Then $\lim_{\varepsilon \to 0} v(\varepsilon) = v(0)$ in $H$ by Proposition 2.2. So

$$\lim_{\varepsilon \to 0} \langle w(\varepsilon), v(\varepsilon) \rangle_H = \langle w(0), v(0) \rangle_H = \langle w(0), v(0) \rangle_{[H, V']_b, [H, V]_b}.$$

Hence (2.1) is valid using (2.2). Since $C^\infty_c((0, \infty); H)$ is dense in $W_s(V, H)$ by Proposition 2.3(b) and the space

$$\left\{ v \in W_{1-s}(H, V) \cap C^\infty((0, \infty); V) : \text{supp} v \text{ is a bounded set in } (0, \infty) \right\}$$

is dense in $W_{1-s}(H, V)$ by Proposition 2.3(a), the proposition follows in this case.

**Case 2.** Suppose that $s < \frac{1}{2}$.

Obviously $1 - s \geq \frac{1}{2}$, so now the space $C^\infty_c((0, \infty); V)$ is dense in $W_{1-s}(H, V)$ by Proposition 2.3(b). Let $v \in C^\infty_c((0, \infty); V)$ and $w \in W_s(V, H) \cap C^\infty((0, \infty); V)$ with $\text{supp} w$ a
bounded set in $(0, \infty)$. Then Proposition 2.2 implies that $\lim_{\varepsilon \downarrow 0} w(\varepsilon) = w(0)$ in $V'$. Also $\lim_{\varepsilon \downarrow 0} v(\varepsilon) = v(0)$ in $V$ since $v \in C_c^\infty((0, \infty); V)$. So

$$\lim_{\varepsilon \downarrow 0} \langle w(\varepsilon), v(\varepsilon) \rangle_H = \lim_{\varepsilon \downarrow 0} \langle w(\varepsilon), v(\varepsilon) \rangle_{V', V} = \langle w(0), v(0) \rangle_{V', V} = \langle w(0), v(0) \rangle_{[H, V]_s, [H, V]_s}.$$  

By (2.2) one deduces (2.1) and the density of Proposition 2.3 completes the proof in this case. \hfill \Box

3. The Dirichlet and Neumann problem

The aim of this section is to prove well-posedness and regularity of solutions of a Dirichlet and a Neumann problem.

Let $V, H$ be Hilbert spaces such that $V \overset{d}{\rightarrow} H$ and let $\mathcal{E} : V \times V \rightarrow \mathbb{C}$ be a continuous and coercive sesquilinear form. So there are constants $\mu, M > 0$ such that $|\mathcal{E}(u, v)| \leq M\|u\|_V\|v\|_V$ and $\text{Re}\ \mathcal{E}(u, u) \geq \mu\|u\|_V^2$ for all $u, v \in V$. Denote by $\mathcal{A} \in \mathcal{L}(V, V')$ the operator given by

$$\langle \mathcal{A}u, v \rangle_{V', V} = \mathcal{E}(u, v)$$

for all $u, v \in V$. Throughout the remainder of the paper, we shall use the notation $\mathcal{E}(u) := \mathcal{E}(u, u)$. Let $0 < s < 1$ be fixed throughout this section. We are interested in the equation

$$u''(t) + \frac{1 - 2s}{t}u'(t) - \mathcal{A}u(t) = 0, \quad t \in (0, \infty). \quad (3.1)$$

We shall see in Theorem 3.4 that the Sobolev space in the next definition is the correct space for the well-posedness of the Dirichlet problem.

**Definition 3.1.** An $(\mathcal{E}, s)$-harmonic function (or shortly $s$-harmonic function) is a function $u \in W_{1-s}(H, V)$ such that $t^{1-2s}u' \in W_s(V', H)$ and

$$-(t^{1-2s}u')'(t) + t^{1-2s}\mathcal{A}u(t) = 0 \text{ in } V' \text{ for a.e. } t \in (0, \infty). \quad (3.2)$$

Note that both functions $(t^{1-2s}u')'$ and $t^{1-2s}\mathcal{A}u$ are in $L^2(V', t^{2s}\text{d}t)$ so that we actually obtain an identity in this space. Note also that (3.2) is equivalent to (3.1).

If $u$ is $s$-harmonic, then Proposition 2.2 implies that

$$u(0) := \lim_{t \downarrow 0} u(t)$$

exists in $H$ and is an element of $[H, V]_s$. Similarly,

$$-\lim_{t \downarrow 0} t^{1-2s}u'(t)$$

exists in $V'$ and is an element of $[H, V'_s]$. We consider this limit as an $s$-normal derivative. If $s = \frac{1}{2}$ then it equals $-u'(0)$.

In this section we are interested in the following two problems.

- **Given** $x \in [H, V]_s$, the **Dirichlet problem** consists in finding an $s$-harmonic function $u$ such that $u(0) = x$.
• Given \( y \in [H, V'] \), the **Neumann problem** consists in finding an \( s \)-harmonic function \( u \) such that \( y = -\lim_{t \to 0} t^{1-2s} u'(t) \).

We will see that both problems are well-posed.

We define the sesquilinear form \( b_s : W_{1-s}(H, V) \times W_{1-s}(H, V) \to \mathbb{C} \) by

\[
b_s(u, v) := \int_0^\infty \left( \langle u'(t), v'(t) \rangle_H + \mathcal{E}(u(t), v(t)) \right) t^{2(1-s)} \frac{dt}{t}.
\]  

(3.4)

Then \( b_s \) is continuous and coercive.

**Lemma 3.2.** Let \( u \) be \( s \)-harmonic. Write \( y := -\lim_{t \to 0} t^{1-2s} u'(t) \) in \( V' \). Then

\[
b_s(u, v) = \langle y, v(0) \rangle_{[H, V']_s, [H, V]_s}
\]  

(3.5)

for all \( v \in W_{1-s}(H, V) \). In particular,

\[
b_s(u) = \langle y, u(0) \rangle_{[H, V']_s, [H, V]_s}.
\]  

(3.6)

**Proof.** Note that \( u \in W_{1-s}(H, V) \) since \( s \)-harmonic. Set \( w := t^{1-2s} u' \). Then \( w \in W_s(V', H) \).

Let \( v \in W_{1-s}(H, V) \). Then Proposition 2.9 gives

\[
\int_0^\infty \langle w'(t), v(t) \rangle_{V', V} dt = -\int_0^\infty \langle w(t), v'(t) \rangle_H dt + \langle y, v(0) \rangle_{[H, V']_s, [H, V]_s}
\]  

\[
= -\int_0^\infty \langle u'(t), v'(t) \rangle_H t^{1-2s} dt + \langle y, v(0) \rangle_{[H, V']_s, [H, V]_s}.
\]

Since \( w'(t) = t^{1-2s} Au(t) \) in \( V' \) for a.e. \( t \in (0, \infty) \), it follows that

\[
\langle w'(t), v(t) \rangle_{V', V} = t^{1-2s} \mathcal{E}(u(t), v(t))
\]

for a.e. \( t \in (0, \infty) \). This proves (3.5). \( \Box \)

Conversely, we may use the form \( b_s \) to prove \( s \)-harmonicity using only a small space of test functions.

**Lemma 3.3.** Let \( u \in W_{1-s}(H, V) \). Assume \( b_s(u, v) = 0 \) for all \( v \in C_c^\infty((0, \infty); V) \). Then \( u \) is \( s \)-harmonic.

**Proof.** Let \( \varphi \in C_c^\infty((0, \infty)) \). For all \( v \in V \) define \( \tilde{v} \in C_c^\infty((0, \infty); V) \) by \( \tilde{v}(t) = \overline{\varphi(t)v} \). Then by assumption

\[
0 = b_s(u, \tilde{v}) = \int_0^\infty \left( \langle u'(t), \tilde{v}'(t) \rangle_H + \mathcal{E}(u(t), \tilde{v}(t)) \right) t^{1-2s} dt
\]  

\[
= \int_0^\infty \langle u'(t), v \rangle_H \varphi'(t) t^{1-2s} dt + \int_0^\infty \langle Au(t), v \rangle_{V', V} \varphi(t) t^{1-2s} dt
\]  

\[
= \int_0^\infty \varphi'(t) \langle w(t), v \rangle_{V', V} dt + \int_0^\infty \varphi(t) \langle t^{1-2s} Au(t), v \rangle_{V', V} dt,
\]

where \( w = t^{1-2s} u' \). Since \( v \in V \) is arbitrary, Definition 2.1 implies that

\[
-w' + t^{1-2s} Au = 0
\]
in $L^1_{\text{loc}}(V')$. Hence
\[-(t^{1-2s}u')(t) + t^{1-2s}Au(t) = 0\]
in $V'$ for almost every $t \in (0, \infty)$. Because $u \in W_{1-s}(H, V)$, one has $t^{1-s}u \in L^2_s(V)$, so $t^{1-s}Au \in L^2_s(V')$. Hence $t^s(t^{1-2s}Au) \in L^2_s(V')$ and this implies that $t^sW' \in L^2_s(V')$. In addition $t^sW = t^s(t^{1-2s}u') = t^{1-s}u' \in L^2_s(H)$, since $u \in W_{1-s}(H, V)$. Therefore, $w \in W_s(V', H)$. We proved that $u$ is $s$-harmonic. 

We can now prove well-posedness of the Dirichlet problem and the Neumann problem.

**Theorem 3.4.** The following assertions hold.

(a) **(Dirichlet Problem).** Let $x \in [H, V]$. Then there exists a unique $s$-harmonic function $u$ such that $u(0) = x$.

(b) **(Neumann Problem).** Let $y \in [H, V']$. Then there exists a unique $s$-harmonic function $u$ such that $\lim_{t \downarrow 0} -t^{1-2s}u'(t) = y$.

**Proof.** (a). By Proposition 2.2 there exists a $\phi \in W_{1-s}(H, V)$ such that $\phi(0) = x$. Define $L: W^0_{1-s}(H, V) \to \mathbb{C}$ by $L \varphi := b_s(\varphi, v)$, where $W^0_{1-s}(H, V) := \{v \in W_{1-s}(H, V) \mid v(0) = 0\}$, which is a closed subspace of $W_{1-s}(H, V)$. Then $L$ is continuous and anti-linear. Since the form $b_s$ is coercive, there exists a unique $w \in W^0_{1-s}(H, V)$ such that $b_s(w, v) = Lv$ for all $v \in W^0_{1-s}(H, V)$. Let $u := \phi - w$. Then $u \in W_{1-s}(H, V)$, $u(0) = \phi(0) = x$ and $b_s(u, v) = 0$ for all $v \in W^0_{1-s}(H, V)$. It follows from Lemma 3.3 that $u$ is $s$-harmonic. This proves existence. Uniqueness follows from Lemma 3.2.

(b). Define $L: W_{1-s}(H, V) \to \mathbb{C}$ by $L \varphi := \langle y, \varphi(0) \rangle_{[H, V'], [H, V]}$. Then $L$ is continuous and anti-linear by Proposition 2.2. By the Lax–Milgram Lemma there exists a unique $u \in W_{1-s}(H, V)$ such that $b_s(u, v) = Lv$ for all $v \in W_{1-s}(H, V)$. In particular, $b_s(u, v) = 0$ for all $v \in C^\infty_c((0, \infty); V)$. It follows from Lemma 3.3 that $u$ is $s$-harmonic. Let $z := -\lim_{t \downarrow 0} t^{1-2s}u'(t)$ in the sense of $V'$. Then $z \in [H, V']$ by Proposition 2.2. From Lemma 3.2, we deduce that
\[\langle y, \varphi(0) \rangle_{[H, V'], [H, V]} = b_s(u, v) = \langle z, \varphi(0) \rangle_{[H, V'], [H, V]}\]
for all $v \in W_{1-s}(H, V)$. Hence $y = z$ by the surjectivity in Proposition 2.2. This shows that $u$ solves the Neumann problem. Uniqueness follows also from Lemma 3.2. 

Theorem 3.4 and Proposition 2.2 allow us to define the Dirichlet-to-Neumann operator $D_s$ in the following way.

**Definition 3.5.** Define $D_s : [H, V] \to [H, V']$ as follows. Let $x \in [H, V]$. Let $u$ be the unique $s$-harmonic function satisfying $u(0) = x$. Then $D_s x = y$, where $y = -\lim_{t \downarrow 0} t^{1-2s}u'(t)$ in $V'$. We call $D_s$ the **Dirichlet-to-Neumann operator** (with respect to $s$ and $\mathcal{E}$).

**Theorem 3.6.** The operator $D_s$ is an isomorphism from $[H, V]$ onto $[H, V']$.

**Proof.** It follows from Theorem 3.4 that $D_s$ is linear and bijective. We show that $D_s^{-1}$ is continuous. Let $y \in [H, V']$ and set $x := D_s^{-1}y$. Let $u$ be the $s$-harmonic function satisfying $u(0) = x$ and $-\lim_{t \downarrow 0} t^{1-2s}u'(t) = y$. Then $b_s(u) = \langle y, u(0) \rangle_{[H, V'], [H, V]}$ by (3.6).
By Proposition 2.2, there exists a constant \( c > 0 \) such that \( \|v(0)\|_{[H, V]} \leq c\|v\|_{W_1,2,(H, V)} \) for all \( v \in W_s(H, V) \). Let \( \mu \in (0, 1] \) be a coercivity constant for \( \mathcal{E} \). Then
\[
\mu \|u\|_{W_1,2,(H, V)}^2 \leq \Re b_2(u) = \Re \langle y, u(0) \rangle_{[H, V'], [H, V]},
\]
\[
\leq \|y\|_{[H, V']} \|u(0)\|_{[H, V]} \leq c\|y\|_{[H, V']} \|u\|_{W_1,2,(H, V)}.
\]
Hence
\[
\|u\|_{W_1,2,(H, V)} \leq c\mu^{-1}\|y\|_{[H, V']}.
\] (3.7)

Therefore,
\[
\|x\|_{[H, V]} = \|u(0)\|_{[H, V]} \leq c\|u\|_{W_1,2,(H, V)} \leq c^2\mu^{-1}\|y\|_{[H, V']}.
\]
This shows that \( D_s^{-1} \) is continuous. Then also \( D_s \) is continuous, by the bounded inverse theorem.

The next proposition combines several results of this section.

**Proposition 3.7.** The set
\[
\mathcal{H} = \{ u \in W_1,2(H, V) : u \text{ is } s\text{-harmonic} \}
\]
is a closed subspace of \( W_1,2(H, V) \). We provide \( \mathcal{H} \) with the induced norm of \( W_1,2(H, V) \). Then the mappings \( u \mapsto u(0) \) from \( \mathcal{H} \) into \( [H, V] \) and \( u \mapsto -\lim_{t \to 0} t^{1-2s}u'(t) \) from \( \mathcal{H} \) into \([H, V']\) are both isomorphisms.

**Proof.** It follows from Lemmas 3.3 and 3.2 that \( \mathcal{H} \) is a closed subspace. The surjectivity of both maps is proved in Theorem 3.4 and the injectivity in Lemma 3.2. The continuity of the first mapping follows from Proposition 2.2 and the continuity of the second follows from (3.7). The continuity of the inverses is a consequence of the closed graph theorem.

We conclude this section by specifying to the case \( s = \frac{1}{2} \), which is much simpler.

**Proposition 3.8.** A function \( u \) is \( \frac{1}{2} \)-harmonic if and only if we have that \( u \in W^{2,2}((0, \infty); V') \cap L^2((0, \infty); V) \) and \(-u'' + Au = 0 \) in \( L^2((0, \infty); V') \).

**Proof.** By definition \( u \) is \( \frac{1}{2} \)-harmonic if and only if \( u \in W^{1,2}_s(H, V), u' \in W^{1,2}_s(V', H) \) and \(-u''(t) + Au(t) = 0 \) in \( V' \) for a.e. \( t \in (0, \infty) \). The latter is equivalent to \( u \in L^2((0, \infty); V), u' \in L^2((0, \infty); H), u'' \in L^2((0, \infty); V') \) and \(-u''(t) + Au(t) = 0 \) in \( V' \) for a.e. \( t \in (0, \infty) \). By [10, Chapter 1, Proposition 2.2] one has the inclusion \( W^{2,2}((0, \infty); V') \cap L^2((0, \infty); V) \subset W^{1,2}((0, \infty); H) \). Then the proposition follows.

Proposition 3.7 has the following form if \( s = \frac{1}{2} \).

**Theorem 3.9.** The set
\[
\mathcal{H}_s = \left\{ u \in W^{1,2}((0, \infty); H) \cap L^2((0, \infty); V) : u \text{ is } \frac{1}{2}\text{-harmonic} \right\}
\]
is a closed subspace of $W^{1,2}((0,\infty);H) \cap L^2((0,\infty);V)$. Moreover, the mappings $u \mapsto u(0)$ from $\mathcal{H}_{ar} \downarrow$ into $[H,V]_\frac{1}{2}$ and $u \mapsto -u'(0)$ from $\mathcal{H}_{ar} \downarrow$ into $[H,V']_\frac{1}{2}$ are both isomorphisms.

Denote by $A$ the part of the operator $\mathcal{A}$ in $H$; i.e.

$$D(A) = \{x \in V : Ax \in H\} \quad \text{and} \quad Ax = \mathcal{A}x.$$ 

Then $A$ is an $m$-sectorial operator with vertex $\gamma > 0$ in the sense of Kato [9, Section V.3.10]. Moreover, $A$ has bounded imaginary powers, see for example [8, Corollary 7.1.8]. Hence $[H,D(A)]_\theta = D(A^\theta)$ for all $\theta \in (0,1)$ by [16, Theorem 1.15.3].

Finally, we mention the following $W^{2,2}\text{-regularity}.

**Proposition 3.10.** Let $u$ be $\frac{1}{2}$-harmonic. Then the following assertions are equivalent.

(i) $u \in W^{2,2}((0,\infty);H)$;

(ii) $u(0) \in [H,D(A)]_\frac{1}{4} = D(A^\frac{1}{4})$;

(iii) $u'(0) \in [H,D(A)]_\frac{1}{4} = [H,V]_\frac{1}{2}$.

The proof of Proposition 3.10 is based on the following properties of traces [10, Chapter 1, Theorems 3.1 and 3.2].

**Proposition 3.11.** Let $Y \preceq X$, where $X,Y$ are Hilbert spaces. Then the mapping $u \mapsto (u(0),u'(0))$ maps $W^{2,2}((0,\infty);X) \cap L^2((0,\infty);Y)$ into $[X,Y]_\frac{1}{4} \times [X,Y]_\frac{1}{4}$ and is surjective.

**Proof of Proposition 3.10.** Suppose (i) is valid. Since $u'' = Au$, it follows that $Au(t) = u''(t) \in H$ and hence $u(t) \in D(A)$ for almost all $t > 0$. So $u \in L^2((0,\infty);D(A)) \cap W^{2,2}((0,\infty);H)$. Then the implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) follow directly from Proposition 3.11.

(ii) $\Rightarrow$ (i). Our proof is based on the Dore–Venni Theorem. We consider the negative Dirichlet Laplacian $B^D$ on $L^2((0,\infty);H)$ given by

$$D(B^D) = \{w \in W^{2,2}((0,\infty);H) : w(0) = 0\} \quad \text{and} \quad B^Dw = -w''.$$ 

This is a selfadjoint and positive operator. In fact it is associated with the closed form

$$b^D(w,v) = \int_0^\infty (w'(t),v'(t))_H dt \quad \text{and} \quad D(b^D) = W^{1,2}_0((0,\infty);H).$$

The other operator in $L^2((0,\infty);H)$ which we consider is the operator $\mathcal{A}_2$ with domain $D(\mathcal{A}_2) = L^2((0,\infty);D(A))$ given by $(\mathcal{A}_2w)(t) = A(w(t))$. Then the operators $-B^D$ and $-\mathcal{A}_2$ generate bounded holomorphic $C_0$-semigroups on $L^2((0,\infty);H)$ which commute. Moreover, $\mathcal{A}_2$ is invertible. It follows from a version of the Dore–Venni Theorem [6, Theorem 2.1] (see also [14, Theorem 8.4] and [12, Corollary 4.7]) that the operator $B^D + \mathcal{A}_2$ with usual domain $D(B^D + \mathcal{A}_2) = D(B^D) \cap D(\mathcal{A}_2)$ is invertible.

By assumption we have $u(0) \in [H,D(A)]_\frac{1}{4}$. Then by Proposition 3.11 there exists a $\phi \in W^{2,2}((0,\infty);H) \cap L^2((0,\infty);D(A))$ such that $\phi(0) = u(0)$. Let $f := -\phi'' + \mathcal{A}_2\phi \in L^2((0,\infty);H)$. Since $B^D + \mathcal{A}_2$ is invertible, there exists a $w \in D(B^D) \cap D(\mathcal{A}_2)$ such that $-w'' + \mathcal{A}_2w = f$. Let $\tilde{u} = \phi - w$. Then $\tilde{u} \in W^{2,2}((0,\infty);H) \cap L^2((0,\infty);D(A))$
\[ W. \, ARENDT \, ET \, AL. \]

\[ \text{and } -\tilde{u}'' + A_2 \tilde{u} = 0. \text{ Therefore, Proposition 3.8 implies that } \tilde{u} \text{ is } \frac{1}{2} \text{-harmonic. Moreover, } \\
\tilde{u}(0) = \phi(0) = u(0). \] Then Theorem 3.4(a) gives \( u = \tilde{u} \in W^{2,2}((0, \infty); H) \). This proves (i).

(iii) \( \Rightarrow \) (i). The proof is similar, but here we consider the negative Laplacian with Neumann boundary conditions \( B^N \) on \( L^2((0, \infty); H) \); that is

\[ D(B^N) = \{ w \in W^{2,2}((0, \infty); H) : w'(0) = 0 \} \quad \text{and} \quad B^N w = -w''. \]

This operator is associated with the closed form \( b^N \) given by

\[ b^N(w, v) = \int_0^\infty (w'(t), v'(t))_H \, dt \quad \text{and} \quad D(b^N) = W^{1,2}((0, \infty); H). \]

Then again by the Dore–Venni Theorem the operator \( B^N + A_2 \) with usual domain \( D(B^N + A_2) = D(B^N) \cap D(A_2) \) is invertible, where \( A_2 \) is as above.

By assumption we have \( u'(0) \in [H, D(A)]_{\frac{1}{4}} \). Then by Proposition 3.11 there exists a \( \phi \in W^{2,2}((0, \infty); H) \cap L^2((0, \infty); D(A)) \) such that \( \phi'(0) = u'(0) \). Let \( f := -\phi'' + A_2 \phi \in L^2((0, \infty); H) \). Since \( BD + A_2 \) is surjective, there exists a \( w \in D(B^N) \cap D(A_2) \) such that

\[ -w'' + A_2 w = f. \]

Let \( \tilde{u} = \phi - w \). Then \( \tilde{u} \in W^{2,2}((0, \infty); H) \cap L^2((0, \infty); D(A)) \) and \( -\tilde{u}'' + A_2 \tilde{u} = 0 \). Moreover, \( \tilde{u}'(0) = \phi'(0) = u'(0) \). Thus \( \tilde{u} \) is the \( \frac{1}{2} \)-harmonic function satisfying \( \tilde{u}'(0) = u'(0) \). So \( u = \tilde{u} \in W^{2,2}((0, \infty); H) \) and we have shown (i).

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**4. The fractional powers via the D-t-N operator**

We adopt the notation and assumptions of Section 3; that is \( V \) and \( H \) are Hilbert spaces, \( V \hookrightarrow H \), the sesquilinear form \( \mathcal{E} : V \times V \to \mathbb{C} \) is continuous, coercive and \( A \in \mathcal{L}(V, V') \) is given by \( (Au, v)'_V' = \mathcal{E}(u, v) \). The number \( s \in (0, 1) \) is fixed and \( D_s : [H, V]_s \to [H, V']_s \) is the Dirichlet-to-Neumann operator (see Definition 3.5). Note that

\[ [H, V]_s \xleftarrow{d} H \xrightarrow{d} [H, V']_s = [H, V]_s. \]

Let \( D_s \) be the part of the operator \( D_s \) in \( H \). So \( D_s \) is the operator in \( H \) given by

\[ D(D_s) := \{ x \in [H, V]_s : D_s x \in H \} \quad \text{and} \quad D_s x = D_s x. \]

Therefore, the graph of \( D_s \) is given by

\[ \text{graph}(D_s) = \left\{ (x, y) \in H \times H : \text{there exists an } s\text{-harmonic map } u \text{ such that } \\
\quad u(0) = x \text{ and } y = -\lim_{t \downarrow 0} t^{1-2s}u'(t) \text{ in } V' \right\}. \]

Recall that \( A \) is the part of the operator \( A \) in \( H \). Denote by \( A^s \) the fractional power of \( A \). Our main result of this paper is the following. Define \( c_s := 2^{1-2s} \Gamma(1-2s)/\Gamma(3) \).

**Theorem 4.1.** One has \( c_s A^s = D_s \).

We first prove that \( D_s \) is \( m \)-sectorial. For that we use the following result [3, Theorem 2.1].
Proposition 4.2. Let $W$ be a Hilbert space and let $b: W \times W \to \mathbb{C}$ be a continuous sesquilinear form. Let $H$ be a Hilbert space and $j: W \to H$ be a continuous, linear map with dense image. Suppose there exist $\mu > 0$ and $\omega \in \mathbb{R}$ such that

$$\mu \|u\|^2_W \leq \text{Re} b(u) + \omega \|j(u)\|^2_H$$

for all $u \in W$. Then there exists a unique $m$-sectorial operator $B$ on $H$ such that

$$\text{graph}(B) = \{(x, y) \in H \times H : \text{there exists } u \in W \text{ such that } j(u) = x \text{ and } b(u, v) = \langle y, j(v) \rangle_H \text{ for all } v \in W\}.$$ 

We call the operator $B$ in Proposition 4.2 the operator associated with the pair $(b, j)$.

We wish to apply Proposition 4.2 with $W = W_{1-s}(H, V)$ and $b = b_t$ the sesquilinear form given by \((3.4)\). Recall that $b_t: W_{1-s}(H, V) \times W_{1-s}(H, V) \to \mathbb{C}$ is given by

$$b_t(u, v) := \int_0^\infty \left( \langle u'(t), v'(t) \rangle_H + E(u(t), v(t)) \right) t^{2(1-s)} \frac{dt}{t},$$

the form $b_t$ is continuous and coercive. Define $j: W_{1-s}(H, V) \to H$ by $j(u) = u(0)$. Note that also $j$ depends on $s$. Then $j$ is linear, continuous with dense image (see Proposition 2.2).

Proposition 4.3. The operator associated with $(b_s, j)$ is $D_s$. In particular, $D_s$ is $m$-sectorial.

Proof. Let $B$ be the operator associated with $(b_s, j)$. Let $(x, y) \in \text{graph}(B)$. Then there exists a $u \in W_{1-s}(H, V)$ such that $u(0) = x$ and $b_s(u, v) = \langle y, v(0) \rangle_H$ for all $v \in W_{1-s}(H, V)$.

Choosing in particular $v \in C_c^\infty((0, \infty); V)$, Lemma 3.3 shows that $u$ is $s$-harmonic. Let $z := D_s x = -\lim_{t \downarrow 0} t^{1-2s}u'(t)$ in $V'$. Then \((3.5)\) gives

$$\langle y, v(0) \rangle_H = b_s(u, v) = \langle z, v(0) \rangle_{[H, V'], [H, V]},$$

for all $v \in W_{1-s}(H, V)$. Consequently, $\langle y, w \rangle_H = \langle z, w \rangle_{[H, V'], [H, V]}$ for all $w \in [H, V]$ by Proposition 2.2. This implies that $z = y \in H$. Hence $x \in D(D_s)$ and $D_s x = Bx$. We have shown that $B \subset D_s$.

Conversely, let $x \in D(D_s)$. Write $y = D_s x \in H$. Let $u$ be $s$-harmonic such that $u(0) = x$. Then $y = -\lim_{t \downarrow 0} t^{1-2s}u'(t)$ in $V'$. Let $v \in W_{1-s}(H, V)$. Then \((3.5)\) gives

$$b_s(u, v) = \langle y, v(0) \rangle_{[H, V'], [H, V]} = \langle y, v(0) \rangle_H,$$

where we have used that $y \in H$. Hence $x = u(0) \in D(B)$ and $Bx = y$. This shows that $D_s \subset B$. \qed

Recall that the operator $-A$ generates a holomorphic $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $H$. In particular, the mapping $t \mapsto e^{-tA}$ is in $C^\infty((0, \infty); \mathcal{L}(H))$ and even in $C^\infty((0, \infty); D(A^k))$ for all $k \in \mathbb{N}$ if we provide $D(A^k)$ with the norm $\|x\|_{D(A^k)} = \|A^k x\|_H$. Moreover, $\|e^{-tA}\|_{\mathcal{L}(H)} \leq e^{-\mu t}$ for all $t > 0$, where $\mu > 0$ is a coercivity constant of the form $E$. Define the function $\mathcal{U}: [0, \infty) \to \mathcal{L}(H)$ by

$$\mathcal{U}(t) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-r^2 \frac{t}{s}} e^{-rA} \frac{dr}{r}.$$
Then
\[ \mathcal{U} \in C^\infty((0, \infty); \mathcal{L}(H)) \cap C([0, \infty); \mathcal{L}(H)). \]

The function \( \mathcal{U} \), which is somehow subordinated to the semigroup generated by \(-A\), was introduced by Stinga and Torrea [15, Theorem 2.1 and Proposition 2.3] (see also [7, Theorem 1.1]). We need the following properties which are similar to those established in [15].

**Proposition 4.4.** The following assertions hold.
(a) If \( t > 0 \), then \( \mathcal{U}(t)x \in D(A) \) for all \( x \in H \) and
\[ \mathcal{U}''(t) + \frac{1 - 2s}{t}\mathcal{U}'(t) = A\mathcal{U}(t). \]
(b) \( \mathcal{U}(0) = A^{-s}. \)
(c) Let \( x \in D(A^s) \). Define \( u \in C^\infty((0, \infty); H) \) by \( u(t) := \mathcal{U}(t)A^s x. \) Then
\[ \lim_{t \downarrow 0} -t^{1-2s}u'(t) = c_s A^s x \]
in \( H \).
(d) There exist \( \delta \in (0, \mu) \) and \( M \geq 0 \) such that \( \|\mathcal{U}(t)\|_{\mathcal{L}(H)} \leq Me^{-\delta t} \) for all \( t > 0 \).
(e) Let \( x \in D(A^2) \). Then \( \mathcal{U}(\cdot)A^s x \) is s-harmonic.

Now we are able to prove Theorem 4.1.

**Proof of Theorem 4.1.** Let \( x \in D(A^2) \). We shall show that \( x \in D(D_2) \) and \( D_2 x = c_s A^s x. \) In fact, by Proposition 4.4(e) the function \( u(\cdot) := \mathcal{U}(\cdot)A^s x \) is s-harmonic and \( u(0) = x \) by Proposition 4.4(b). Moreover, \( c_s A^s x = \lim_{t \downarrow 0} t^{1-2s}u'(t) \) by Proposition 4.4(c). Thus, by the definition of \( D_2 \), one has \( x \in D(D_2) \) and \( D_2 x = c_s A^s x. \) Since \( D(A^2) \) is a core of \( D(A^s) \) and \( D_2 \) is closed (as the operator \( D_2 \) is \( m \)-sectorial by Proposition 4.3), it follows that \( c_s A^s \subset D_2 \). Because \( c_s A^s \) is \( m \)-sectorial and \( D_2 \) is sectorial one concludes that \( c_s A^s = D_2. \)

Theorem 4.1 has the following corollary.

**Corollary 4.5.** Let \( u \) be s-harmonic, \( x = u(0) \) and \( y = -\lim_{t \downarrow 0} t^{1-2s}u'(t) \) in \( V' \). Then \( x \in D(A^s) \) if and only if \( y \in H \).

Moreover Proposition 4.4(e) extends to the following representation formula.

**Corollary 4.6.** Let \( u \) be s-harmonic. Then
\[ u(t) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-\frac{r^2}{2}} y^s A^s e^{-rA^s} x \frac{dr}{r} \] (4.1)
for all \( t > 0 \), where \( x = u(0). \) In particular, \( u \in C^\infty((0, \infty); H). \) Stronger, \( u \in C^\infty((0, \infty); D(A^k)) \) for all \( k \in \mathbb{N}. \)

**Proof.** Note that \( x \in [H, V], \) by Proposition 2.2. Since first \( D(A^2) \) is a core for \( A, \) secondly the domain \( D(A) \) with graph norm is densely and continuously embedded in \( V \) and thirdly \( V \) is densely and continuously embedded in \( [H, V], \) it follows that \( D(A^2) \) is dense in \( [H, V], \).
Hence there exists a sequence \((x_n)_{n \in \mathbb{N}}\) in \(D(A^2)\) such that \(x_n \to x\) in \([H, V]_\beta\). Now it follows from the second statement in Proposition 3.7 that \(u_n \to u\) in \(W_{1-s}(H, V)\), where \(u_n\) is the \(s\)-harmonic function satisfying \(u_n(0) = x_n\) for all \(n \in \mathbb{N}\). Since \(W_{1-s}(H, V) \subset C((0, \infty); H)\), the closed graph theorem implies that \(u_n(t) \to u(t)\) in \(H\) as \(n \to \infty\) for all \(t > 0\). We know from Proposition 4.4(e) and (b) that

\[
    u_n(t) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-\frac{r^2}{4}} r^s e^{-rA} A^s x_n \frac{dr}{r}
\]

for all \(t > 0\). Note that there exists a constant \(c > 0\) such that \(\|A^s e^{-rA} x\|_H \leq \frac{c}{r} \|x\|_H\) for all \(r \in (0, 1]\) and \(x \in H\). Letting \(n \to \infty\), we get (4.1) by Lebesgue’s Theorem. \(\square\)

As a consequence of these results, each \(s\)-harmonic function is a classical solution of the equation

\[
    u''(t) + \frac{1 - 2s}{t} u'(t) - Au(t) = 0, \quad t \in (0, \infty).
\]

Finally we specify the results for \(s = \frac{1}{2}\). Let \(x \in [H, V]_{\frac{1}{2}}\). By Theorem 3.4(a) there is a unique \(\frac{1}{2}\)-harmonic function \(u\) such that \(u(0) = x\). Then \(y := -u'(0) \in [H, V']_{\frac{1}{2}}\). Then by definition \(D_{\frac{1}{2}} x = y\). Moreover, \(D_{\frac{1}{2}} : [H, V]_{\frac{1}{2}} \to [H, V']_{\frac{1}{2}}\) is an isomorphism by Theorem 3.6. Also \(D_{\frac{1}{2}} x \in H\) if and only if \(x \in D(A_{\frac{1}{2}})\) by Theorem 4.1. In that case \(D_{\frac{1}{2}} x = A_{\frac{1}{2}} x\), since \(c_{\frac{1}{2}} = 1\).

**Proposition 4.7.** Let \(x \in D(A_{\frac{1}{2}})\). Then the unique \(\frac{1}{2}\)-harmonic function \(u\) satisfying \(u(0) = x\) is given by \(u(t) = e^{-tA_{\frac{1}{2}}} x\) for all \(t > 0\). Hence

\[
    e^{-tA_{\frac{1}{2}}} x = \mathcal{U}(t) A_{\frac{1}{2}} x
\]

for all \(t > 0\).

**Proof.** There exist \(\varepsilon > 0\) and \(M \geq 1\) such that \(\|e^{-tA_{\frac{1}{2}}} \|_{L(H)} \leq Me^{-\varepsilon t}\) for all \(t \geq 0\) (see for example [2, Theorem 5.1.12]). Let \(x \in D(A)\). Define \(u : (0, \infty) \to H\) by \(u(t) = e^{-tA_{\frac{1}{2}}} x\). Since \(D(A)\) is continuously embedded into \(V\), there exists a constant \(c > 0\) such that \(\|y\|_V \leq c\|Ay\|_H\) for all \(y \in D(A)\). Then \(\|u(t)\|_V \leq c\|e^{-tA_{\frac{1}{2}}} \|_{L(H)} \|Ax\|_H \leq cMe^{-\varepsilon t} \|Ax\|_H\) for all \(t > 0\) and \(u \in L_2((0, \infty); V)\). Moreover, \(u \in C^2((0, \infty); H)\) and \(u''(t) = Au(t) = e^{-tA_{\frac{1}{2}}} Ax\) for all \(t > 0\). Hence \(u \in W^{2,2}((0, \infty); H) \subset W^{2,2}((0, \infty); V')\) and \(-u'' + Au = 0 \in L_2((0, \infty); V')\). Then \(u\) is \(\frac{1}{2}\)-harmonic by Proposition 3.8. Now (4.2) follows from (4.1). Since \(D(A)\) is dense in \(D(A_{\frac{1}{2}})\) the identity (4.2) remains true for all \(x \in D(A_{\frac{1}{2}})\). \(\square\)

**5. The non-coercive case**

Up to now we used that the form \(\mathcal{E}\) is coercive. In this section we wish to replace this by the much weaker condition that \(\mathcal{E}\) is merely sectorial with vertex 0.
In general, if \(a: D(a) \times D(a) \to \mathbb{C}\) is a sesquilinear form, then we say that \(a\) is **sectorial with vertex** 0 if there exists a \(\theta \in [0, \frac{\pi}{2})\) such that \(a(u) \in \Sigma_0\) for all \(u \in D(a)\), where

\[
\Sigma_0 = \{re^{i\alpha} : r \in [0, \infty) \text{ and } \alpha \in [-\theta, \theta]\}.
\]

In Theorem 5.1 we associate an \(m\)-sectorial operator to a densely defined sectorial form with vertex 0. There is even a \(j\)-version of it like in Proposition 4.2 that turns out to be very useful in this section.

**Theorem 5.1.** Let \(H\) be a Hilbert space, \(a: D(a) \times D(a) \to \mathbb{C}\) a sectorial form with vertex 0 and \(j: D(a) \to H\) a linear map with dense image. Then there exists a unique \(m\)-sectorial operator \(B\) in \(H\) such that

\[
\text{graph}(B) = \left\{(x, y) \in H \times H : \text{there exists a sequence } (u_n)_{n \in \mathbb{N}} \text{ in } D(a) \text{ such that}
\begin{align*}
(i) \quad & \lim_{n \to \infty} j(u_n) = x \text{ in } H, \\
(ii) \quad & \sup \{\text{Re } a(u_n) : n \in \mathbb{N}\} < \infty, \text{ and} \\
(iii) \quad & \lim_{n \to \infty} a(u_n, v) = \langle y, j(v) \rangle_H \text{ for all } v \in D(a)\right\}.
\end{align*}
\]

**Proof.** This is a special case of [3, Theorem 3.2]. \(\square\)

Note that \(B\) is the same operator as in Proposition 4.2 if the domain \(D(a)\) is provided with a Hilbert space structure such that \(j\) is continuous and the form \(a\) is coercive and continuous. We call the operator \(B\) in Theorem 5.1 the **operator associated with** \((a, j)\). In particular, if \(a\) is a densely defined sectorial form with vertex 0 in a Hilbert space \(H\), then one can choose for \(j\) the identity map and we obtain an \(m\)-sectorial operator, which we call the **operator associated with** \(a\).

Now we extend the previous results for coercive forms to sectorial forms.

**Theorem 5.2.** Let \(H, V\) be Hilbert spaces such that \(V \rightleftharpoons H\) and let \(E: V \times V \to \mathbb{C}\) be a continuous sectorial form with vertex 0. Let \(A\) be the operator associated with \(E\). Further, let \(s \in (0, 1)\) and define \(b: W_{1-s}(H, V) \times W_{1-s}(H, V) \to \mathbb{C}\) by

\[
b(u, v) = \int_0^\infty \left( \langle u'(t), v'(t) \rangle_H + E(u(t), v(t)) \right) t^{2(1-s)} \frac{dt}{t}.
\]

Define \(j: W_{1-s}(H, V) \to H\) by \(j(u) = u(0)\). Then \(b\) is sectorial with vertex 0. Let \(B\) be the operator associated with \((b, j)\). Then \(B = c_s A^s\), where \(c_s = 2^{1-2s} \Gamma(1-s) \Gamma(s)\).

**Proof.** It is easy to see that \(b\) is sectorial with vertex 0. For all \(n \in \mathbb{N}\) define \(E_n: V \times V \to \mathbb{C}\) by

\[
E_n(u, v) = E(u, v) + \frac{1}{n} \langle u, v \rangle_V.
\]

Then \(E_n\) is continuous and coercive. Let \(A_n\) be the \(m\)-sectorial operator in \(H\) associated with \(E_n\). Then \(A_n\) is sectorial with vertex 0. Moreover, \(\lim_{n \to \infty} A_n = A\) in the strong resolvent sense by [3, Corollary 3.9]. Hence \(\lim_{n \to \infty} A_n^s = A^s\) in the strong resolvent sense by the representation formula [17, (6) in Section IX.11]. For all \(n \in \mathbb{N}\) define \(b_n: W_{1-s}(H, V) \times
for all \( u \) it is again a Dirichlet-to-Neumann map. For this we need quite some preparation.

Proposition 4.2. Then \( \lim_{n \to \infty} B_n = B \) in the strong resolvent sense gives

\[ T \]

Then \( b_n \) is continuous and coercive. Let \( B_n \) be the operator associated with \((b_n,j)\) as in Proposition 4.2. Then \( \lim_{n \to \infty} B_n = B \) in the strong resolvent sense again by [3, Corollary 3.9]. But \( B_n = c_n A_n^s \) for all \( n \in \mathbb{N} \) by Theorem 4.1. Taking the limit as \( n \to \infty \) and using the uniqueness of the limit in the strong resolvent sense gives \( B = c A^s \) as required.

Adopt the notation and assumptions as in Theorem 5.2. We suppose from now on in addition that \( \mathcal{E} \) is \( H \)-elliptic, that is there exists a constant \( \mu > 0 \) such that

\[ \text{Re} \mathcal{E}(u) + \| u \|_H^2 \geq \mu \| u \|_V^2 \]  

(5.1)

for all \( u \in V \). In this case we can give an explicit description of the operator \( B \) and show that it is again a Dirichlet-to-Neumann map. For this we need quite some preparation.

Recall that if \( X \) is a Banach space and \(-\infty < \alpha < \beta < \infty\), then \( W^{1,1}((\alpha, \beta); X) \subset C([\alpha, \beta]; X) \) and

\[ u(t) = u(\alpha) + \int_{\alpha}^{t} u'(r) \, dr \]

for all \( u \in W^{1,1}((\alpha, \beta); X) \) and \( t \in [\alpha, \beta] \). Conversely, if \( x \in X \), \( v \in L^1((\alpha, \beta); X) \) and \( u: (\alpha, \beta) \to X \) is given by \( u(t) = x + \int_{\alpha}^{t} v(r) \, dr \), then \( u \in W^{1,1}((\alpha, \beta); X) \) and \( u' = v \). For all \(-\infty \leq a < b \leq \infty \) and \( \beta \in [1, \infty] \) we let

\[ W^{1,p}_{\text{loc}}((a, b); X) := \left\{ u: (a, b) \to X : u|_{(\alpha, \beta)} \in W^{1,p}((\alpha, \beta); X) \right\} \]

and if \( a \neq -\infty \), then we define

\[ L^p_{\text{loc}}([a, b); X) = \left\{ u: [a, b) \to X : u|_{[a, \beta)} \in L^p([a, \beta); X) \text{ for all } \beta \in (a, b) \right\} . \]

We always identify \( u \in W^{1,p}_{\text{loc}}((a, b); X) \) with its continuous representative.

Recall that \( 0 < s < 1 \). We define the space

\[ W := \left\{ u \in C([0, \infty); H) \cap W^{1,2}_{\text{loc}}((0, \infty); H) \cap L^2_{\text{loc}}((0, \infty); V) : \right\} \]

\[ \int_{0}^{\infty} \left( \| u'(t) \|_H^2 + \text{Re} \mathcal{E}(u(t)) \right) t^{2(1-s)} \frac{dt}{t} < \infty \right\} . \]

We provide \( W \) with the norm

\[ \| u \|_W^2 = \| u(0) \|_H^2 + \int_{0}^{\infty} \left( \| u'(t) \|_H^2 + \text{Re} \mathcal{E}(u(t)) \right) t^{2(1-s)} \frac{dt}{t} . \]

We first prove that \( W \) is a Hilbert space. For the proof we need two lemmas.
Lemma 5.3. Let \( u \in W \). Then \( u' \in L^1_{\text{loc}}([0, \infty); H) \). Moreover,
\[
\int_0^t \| u'(r) \|_H \; dr \leq \| u \|_W \cdot \left( \frac{1}{2s} t^{2s} \right)^{\frac{1}{2}} \tag{5.2}
\]
and
\[
\| u(t) \|_H^2 \leq \frac{1}{s} \| u \|_W^2 (1 + t^{2s}) \tag{5.3}
\]
for all \( t > 0 \).

Proof. Let \( u \in W \) and \( t > 0 \). Then
\[
\int_0^t \| u'(r) \|_H \; dr \leq \left( \int_0^t \| u'(r) \|_H^2 r^{2(1-s)} \; \frac{dr}{r} \right)^{\frac{1}{2}} \left( \int_0^t r^{2s-1} \; dr \right)^{\frac{1}{2}} \leq \| u \|_W \cdot \left( \frac{1}{2s} t^{2s} \right)^{\frac{1}{2}}.
\]
This shows (5.2) and that \( u' \in L^1_{\text{loc}}([0, \infty); H) \). If \( a \in (0, t) \), then
\[
u(t) = u(a) + \int_a^t u'(r) \; dr.
\]
Letting \( a \downarrow 0 \) gives
\[
\nu(t) = u(0) + \int_0^t u'(r) \; dr. \tag{5.4}
\]
Moreover, (5.4) and (5.2) give
\[
\| \nu(t) \|_H^2 \leq \frac{1}{2} \| \nu(0) \|_H^2 + \frac{1}{2} \left( \int_0^t \| u'(r) \|_H \; dr \right)^2 \leq \frac{1}{2} \| \nu(0) \|_H^2 + \frac{1}{4s} \| \nu \|_W^2 t^{2s} \leq \| \nu \|_W^2 \left( 1 + \frac{1}{s} t^{2s} \right)
\]
and (5.3) follows. \( \square \)

Recall that \( \mu \) is defined in (5.1).

Lemma 5.4. If \( u \in W \) and \( T \geq 1 \), then
\[
\mu \int_0^T \| u(t) \|_V^2 t^{2(1-s)} \; \frac{dt}{t} \leq \left( 1 + \frac{1}{s(1-s)} T^2 \right) \| u \|_W^2.
\]

Proof. By \( H \)-ellipticity and (5.3) one estimates
\[
\mu \int_0^T \| u(t) \|_V^2 t^{2(1-s)} \; \frac{dt}{t} \leq \int_0^T \text{Re} \, \mathcal{E}(u(t)) r^{2(1-s)} \; \frac{dt}{t} + \int_0^T \| u(t) \|_H^2 r^{2(1-s)} \; \frac{dt}{t} \\leq \| u \|_W^2 + \frac{1}{s} \| u \|_W^2 \int_0^T (1 + t^{2s}) t^{1-2s} \; dt = \| u \|_W^2 \left( 1 + \frac{1}{s} \left( \frac{1}{2(1-s)} T^{2(1-s)} + \frac{1}{2} T^2 \right) \right)
\]
and the lemma follows. \( \square \)
**Proposition 5.5.** The space \( W \) is a Hilbert space.

**Proof.** Let \((u_n)_{n \in \mathbb{N}}\) be a Cauchy sequence in \( W \). Then \( x := \lim_{n \to \infty} u_n(0) \) exists in \( H \). Moreover, there exists a \( v \in L^2((0, \infty); H, t^{2(1-s)} \frac{dt}{t}) \) such that \( \lim_{n \to \infty} u_n' = v \) in \( L^2((0, \infty); H, t^{2(1-s)} \frac{dt}{t}) \). It follows from \((5.2)\) that \((u_n')_{n \in \mathbb{N}}\) is a Cauchy sequence in \( L^1((0, T); H) \) for all \( T > 0 \). Hence \( v|_{(0,T)} \in L^1((0, T); H) \) and, moreover, \( \lim_{n \to \infty} u_n'|_{(0,T)} = v|_{(0,T)} \) in \( L^1((0, T); H) \). In particular, \( v \in L^1_{\text{loc}}((0, \infty); H) \). Define \( u : [0, \infty) \to H \) by

\[
\begin{align*}
u(t) &= x + \int_0^t v(r) \, dr. 
\end{align*}
\]

(5.5)

Let \( t > 0 \). By \((5.4)\) we have

\[
u_n(t) = u_n(0) + \int_0^t u'_n(r) \, dr
\]

for all \( n \in \mathbb{N} \). Hence

\[
\lim_{n \to \infty} u_n(t) = x + \int_0^t v(r) \, dr = u(t)
\]

in \( H \).

It follows from \((5.5)\) that \( u \in C([0, \infty); H) \cap W^{1,2}_{\text{loc}}((0, \infty); H) \) and \( u' = v \). Moreover, Lemma 5.4 implies that \((u_n)_{(a,b)} \in \mathbb{N}\) is also a Cauchy sequence in \( L^2((a, b); V) \) for all \( a, b \in (0, \infty) \) with \( a < b \). Since \( \lim_{n \to \infty} u_n = u \) in \( H \) pointwise, it follows that \( u|_{(a,b)} \in L^2((a, b); V) \) and \( \lim_{n \to \infty} u_n|_{(a,b)} = u|_{(a,b)} \) in \( L^2((a, b); V) \). Passing to a subsequence if necessary, we may assume that \( \lim_{n \to \infty} u_n(t) = u(t) \) in \( V \) for almost all \( t \in (0, \infty) \).

Let \( \varepsilon > 0 \). There exists an \( N_0 \in \mathbb{N} \) such that \( \|u_n - u_m\|_W^2 \leq \varepsilon \) for all \( n, m \geq N_0 \). Let \( n \in \mathbb{N} \) with \( n \geq N_0 \). Then \([9, \text{Lemma VIII.3.14a}]\) and Fatou's lemma give

\[
\begin{align*}
&\|u_n(0) - u(0)\|_H^2 + \int_0^\infty \left( \|u'_n(t) - u'(t)\|_H^2 + \text{Re} \mathcal{E}(u_n(t) - u(t)) \right) t^{2(1-s)} \frac{dt}{t} \\
&\leq \|u_n(0) - x\|_H^2 + \int_0^\infty \liminf_{m \to \infty} \left( \|u'_n(t) - u'_m(t)\|_H^2 + \text{Re} \mathcal{E}(u_n(t) - u_m(t)) \right) t^{2(1-s)} \frac{dt}{t} \\
&\leq \liminf_{m \to \infty} \|u_n - u_m\|_W^2 \leq \varepsilon.
\end{align*}
\]

Hence \( u_n \to u \) in \( W \). So \( \|u_n - u\|_W^2 \leq \varepsilon \) for all \( n \geq N_0 \) and \( \lim_{n \to \infty} u_n = u \) in \( W \). We have shown that the space \( W \) is complete. \( \square \)

We need one more lemma before we can give a Dirichlet-to-Neumann type description for the operator \( c_s A^s \).

**Lemma 5.6.** The space \( W_{1-s}(H, V) \) is dense in \( W \).

**Proof.** Lemma 5.4 implies that \( u \in W_{1-s}(H, V) \) for all \( u \in W \) with compact support in \([0, \infty)\). Let \( u \in W \). Let \( \eta \in C_c^\infty[0, \infty) \) be such that \( I_{[0,1]} \leq \eta \leq I_{[0,2]} \). Let \( n \in \mathbb{N} \). Define \( \eta_n \in C_c^\infty[0, \infty) \) by \( \eta_n(t) := \eta(t/n) \). Then \( I_{[0,n]} \leq \eta_n \leq I_{[0,2n]} \). Define \( u_n := \eta_n u \in W_{1-s}(H, V) \).
We shall show that $\sup_{n \in \mathbb{N}} \|u_n\|_W < \infty$. Obviously
\[
\int_0^\infty \text{Re} \mathcal{E}(u_n(t)) t^{2(1-s)} \frac{dt}{t} \leq \|u\|^2_W
\]
and
\[
\int_0^\infty \|(\eta_n u')(t)\|^2_H t^{2(1-s)} \frac{dt}{t} \leq \|u\|^2_W.
\]
It remains to show that $\sup_{n \in \mathbb{N}} \int_0^\infty \|(\eta_n u')(t)\|^2_H t^{2(1-s)} \frac{dt}{t} < \infty$. Using (5.3) one estimates
\[
\int_0^\infty \|(\eta_n u')(t)\|^2_H t^{2(1-s)} \frac{dt}{t} = \frac{1}{n^2} \int_0^{2n} \left( \eta\left(\frac{t}{n}\right) \right)^2 \|u(t)\|^2_H t^{1-2s} dt
\]
\[
\leq \frac{\|\eta\|^2_\infty \|u\|^2_W}{n^2} \int_0^{2n} (1 + t^{2s}) t^{1-2s} dt
\]
\[
= \frac{\|\eta\|^2_\infty \|u\|^2_W}{n^2} \left( \frac{(2n)^{2-2s}}{2 - 2s} + 2n^2 \right).
\]
Hence $\sup_{n \in \mathbb{N}} \|u_n\|_W < \infty$.

Passing to a subsequence if necessary, we have that there exists a $w \in W$ such that $\lim_{n \to \infty} u_n = w$ weakly in $W$. Then (5.3) implies that $\lim_{n \to \infty} u_n(t) = w(t)$ in $H$ for almost all $t > 0$. So $u = w \in W$. We have shown that $u$ is in the weak closure of $W_{1-s}(H, V)$ in $W$. Since $W_{1-s}(H, V)$ is convex, it is also in the strong closure. Hence $W_{1-s}(H, V)$ is dense in $W$.

Now we are able to show that the operator $B$ in Theorem 5.2 is a Dirichlet-to-Neumann map if $\mathcal{E}$ is $H$-elliptic.

**Theorem 5.7.** Adopt the assumptions and notation as in Theorem 5.2. Moreover, assume that $\mathcal{E}$ is $H$-elliptic. Define the form $\tilde{b} : W \times W \to \mathbb{C}$ by
\[
\tilde{b}(u, v) = \int_0^\infty \left( \langle u'(t), v'(t) \rangle_H + \mathcal{E}(u(t), v(t)) \right) t^{2(1-s)} \frac{dt}{t}.
\]
Let $x, y \in H$. Then the following assertions are equivalent.

(i) $x \in D(\mathcal{A}^\gamma)$ and $c_\gamma A^\gamma x = y$.

(ii) There exists a $u \in W$ such that $u(0) = x$ and $\tilde{b}(u, v) = \langle y, v(0) \rangle_H$ for all $v \in W$.

**Proof.** Define $\tilde{j} : W \to H$ by $\tilde{j}(u) = u(0)$. Then $\tilde{b}$ is continuous and
\[
\|u\|^2_W \leq \text{Re} \tilde{b}(u) + \|\tilde{j}(u)\|^2_H
\]
for all $u \in W$. Moreover, $\tilde{j}$ is continuous and has dense image. Obviously $\tilde{b}$ and $\tilde{j}$ are extensions of $b$ and $j$, respectively. In addition, $W$ is complete and $W_{1-s}(H, V)$ is dense in $W$ by Proposition 5.5 and Lemma 5.6. Hence by [3, Proposition 3.3] it follows that $B$ is the operator associated with $(\tilde{b}, \tilde{j})$ in the sense of Proposition 4.2. Then the equivalence follows immediately from Theorem 5.2 and the definition of the graph of $B$ in Proposition 4.2. \(\square\)
Let \( \tilde{b} \) be as in Theorem 5.7 and let \( u \in W \) be as in Condition (ii) in Theorem 5.7. Then \( u \in W^{2,2}_{\text{loc}}((0, \infty); V') \) and
\[
u''(t) + \frac{1-2s}{t}u'(t) - Au(t) = 0 \quad \text{in} \quad V' \quad \text{for a.e.} \quad t \in (0, \infty),
\]
where \( A : V \to V' \) is given by \( \langle Aw, v \rangle_{V', V} = \mathcal{E}(w, v) \) for all \( w, v \in V \).

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