Efficient Algorithms for k-Regret Minimizing Sets

Pankaj K. Agarwal†  Nirman Kumar‡  Stavros Sintos§  Subhash Suri¶

Abstract

A regret minimizing set $Q$ is a small size representation of a much larger database $P$ so that user queries executed on $Q$ return answers whose scores are not much worse than those on the full dataset. In particular, a $k$-regret minimizing set has the property that the regret ratio between the score of the top-1 item in $Q$ and the score of the top-$k$ item in $P$ is minimized, where the score of an item is the inner product of the item’s attributes with a user’s weight (preference) vector. The problem is challenging because we want to find a single representative set $Q$ whose regret ratio is small with respect to all possible user weight vectors.

We show that $k$-regret minimization is NP-Complete for all dimensions $d \geq 3$. This settles an open problem from Chester et al. [VLDB 2014], and resolves the complexity status of the problem for all $d$: the problem is known to have polynomial-time solution for $d \leq 2$. In addition, we propose two new approximation schemes for regret minimization, both with provable guarantees, one based on coresets and another based on hitting sets. We also carry out extensive experimental evaluation, and show that our schemes compute regret-minimizing sets comparable in size to the greedy algorithm proposed in [VLDB 14] but our schemes are significantly faster and scalable to large data sets.

1 Introduction

Multi-criteria decision problems pose a unique challenge for databases systems: how to present the space of possible answers to a user. In many instances, there is no single best answer, and often a very large number of incomparable objects satisfy the user’s query. For instance, a database query for a car or a smartphone can easily produce an overwhelming number of potential choices to present to the user, with no obvious way to rank them. Top-$k$ and the skyline operators are among the two main techniques used in databases to manage this kind of complexity, but each has its own shortcoming.

---

*Work by Agarwal and Sintos is supported by NSF under grants CCF-15-13816, CCF-15-46392, and IIS-14-08846, by ARO grant W911NF-15-1-0408, and by Grant 2012/229 from the U.S.-Israel Binational Science Foundation. Work by Suri and Kumar is supported by NSF under grant CCF-15-25817.

†Department of Computer Science, Duke University, Durham, NC 27708-0129, USA; pankaj@cs.duke.edu.
‡Department of Computer Science, University of Memphis, Memphis, TN 38152, USA; nkumar8@memphis.edu.
§Department of Computer Science, University of Memphis, Memphis, TN 38152, USA; ssintos@cs.duke.edu.
¶Department of Computer Science, University of California, Santa Barbara, CA 93106, USA; suri@cs.ucsb.edu.
The top-$k$ operator relies on the existence of a utility function that is used to rank the objects satisfying the user’s query, and then selecting the top $k$ by score according to this function. A commonly used utility function takes the inner product of the object attributes with a weight vector, also called the user’s preference, thus forming a weighted linear combination of the different features. However, formulating the utility function is complicated, as users often do not know their preferences precisely, and, in fact, exploring the cost-benefit tradeoffs of different features is often the goal of database search.

The second approach of skylines is based on the principle of pareto optimality: if an object $p$ is better than another object $q$ on all features, then $p$ is always preferable to $q$ by any rational decision maker. This coordinate-wise dominance is used to eliminate all objects that are dominated by some other object. The skyline is the set of objects not dominated by any other object, and has proved to be a powerful tool in multi-criteria optimization. Unfortunately, while skylines are extremely effective in reducing the number of objects in low dimensions, their utility drops off quickly as the dimension (number of features) grows, especially when objects in the database have anti-correlated features (attributes). Indeed, theoretically all objects of the database can appear on the skyline even in two dimensions. Furthermore, the skyline does not necessarily preserve “top-$k$” objects as $k$ increases, in which case one uses $k$-skybands – the subset of points each of which is dominated by at most $k$ points. The size of the skyband grows even more rapidly.

Regret minimization is a recent approach, proposed initially by Nanongkai et al. [29], to address the shortcomings of both the top $k$ and skylines. The regret minimization hybridizes top $k$ and skylines by computing a small representative subset $Q$ of the much larger database $P$ so that for any preference vector the top ranked item in $Q$ is a good approximation of the top ranked item in $P$. The hope is that the size of $Q$ is much smaller than that of the skyline of $P$. In fact, it is known that for a given regret ratio, there is always a regret minimizing set whose size depends only on the regret ratio and the dimension, and not on the size of $P$. In contrast, as mentioned above, the skyline size can be as large as $|P|$.

The goal is to find a subset $Q$ of small size whose approximation error is also small: posed in the form of a decision question, is there a subset of $r$ objects so that every user’s top-1 query can be answered within error at most $x\%$? In general, this is too stringent a requirement and motivated Chester et al. [10] to propose a more relaxed version of the problem, called the $k$-regret minimization. In $k$-regret minimization, the quality of approximation is measured as the gap between the score of the top 1 item in $Q$ and the top $k$ item in $P$ expressed as a ratio, so that the value is always between 0 and 1.

In this paper, we make a number of contributions to the study of $k$-regret minimizing sets. As a theoretical contribution, we prove that the $k$-regret minimization problem is NP-Complete for any dimension $d \geq 3$. This resolves an open problem of Chester et al. [10] who presented a polynomial-time algorithm for $d = 2$ and showed NP-hardness for dimension $d \geq n$, leaving open the tantalizing question of whether the problem was in class P for low dimensions — the dimension being a fixed constant. Our result shows otherwise and settles the complexity landscape of the problem for all dimensions. On more practical side,\footnote{We should point out that the term $k$-regret is used to denote different things by Nanongkai et al. [29] and Chester et al. [10]. In the former, $k$-regret is the representative set of $k$ objects, whereas in the latter, $k$-regret is used to denote the regret ratio between the scores of top 1 and top $k$. In our paper, we follow the convention of Chester et al. [10].}
we present simple and efficient algorithms that are guaranteed to compute small regret minimizing sets and that are scalable to large datasets even for larger values of \( k \) and even when the size of skyline is large.

**Our Model.** An object is represented as a point \( p = (p_1, \ldots, p_d) \) in \( \mathbb{R}^d \) with non-negative attributes, i.e., \( p_i \geq 0 \) for every \( i \leq d \). Let \( \mathcal{X} = \{(p_1, \ldots, p_d) \in \mathbb{R}^d \mid p_i \geq 0 \ \forall i \} \) denote the space of all objects, and let \( \mathcal{P} \subset \mathcal{X} \) be a set of \( n \) objects. A user preference is also represented as a point \( u = (u_1, \ldots, u_d) \in \mathcal{X} \), i.e., all \( u_i \geq 0 \). Given a preference \( u \in \mathbb{R}^d \), we define the score of an object \( p \) to be \( \omega(u, p) = \langle u, p \rangle = \sum_{i=1}^d u_i p_i \).

![Figure 1. Left: top 3 points in two different preferences. Right: Set of points in the red circles is a \((1,0)\)-regret set. Set of points in the blue circles is a \((3,0)\)-regret set.](image)

For a preference \( u \in \mathcal{X} \) and an integer \( k \geq 1 \), let \( \varphi_k(u, \mathcal{P}) \) denote the point \( p \in \mathcal{P} \) with the \( k \)-th largest score (i.e., there are less than \( k \) points of \( \mathcal{P} \) with larger score than \( \omega(u, p) \) and there are at least \( k \) points with score at least \( \omega(u, p) \)), and let \( \omega_k(u, \mathcal{P}) \) denote its score. Set \( \Phi_k(u, \mathcal{P}) = \{\varphi_j(u, \mathcal{P}) \mid 1 \leq j \leq k\} \) to be the set of \( k \) top points with respect to preference \( u \). For brevity, we set \( \omega(u, \mathcal{P}) = \omega_1(u, \mathcal{P}) \). If \( \mathcal{P} \) is obvious from the context, we drop \( \mathcal{P} \) from the list of the arguments, i.e., we use \( \omega_k(u) \) to denote \( \omega_k(u, \mathcal{P}) \) and so on.

For a subset \( \mathcal{Q} \subseteq \mathcal{P} \) and a preference \( u \) (w.r.t. \( \mathcal{P} \)), denoted by \( \ell_k(u, \mathcal{Q}, \mathcal{P}) \), as

\[
\ell_k(u, \mathcal{Q}, \mathcal{P}) = \max \left\{ 0, \frac{\omega_k(u, \mathcal{Q}) - \omega(u, \mathcal{Q})}{\omega_k(u, \mathcal{P})} \right\}.
\]

That is, \( \ell_k(u, \mathcal{Q}, \mathcal{P}) \) is the relative loss in the score of the \( k \)-th topmost object if we replace \( \mathcal{P} \) with \( \mathcal{Q} \). We refer to the maximum regret of \( \mathcal{Q} \)

\[
\ell_k(\mathcal{Q}, \mathcal{P}) = \max_{u \in \mathcal{X}} \ell_k(u, \mathcal{Q}, \mathcal{P})
\]

as the *regret ratio* of \( \mathcal{Q} \) (w.r.t. \( \mathcal{P} \)). If \( \ell_k(\mathcal{Q}) \leq \epsilon \), we refer to \( \mathcal{Q} \) as a \((k, \epsilon)\)-regret set (see Figure 1). By definition, a \((k, \epsilon)\)-regret set is also a \((k', \epsilon)\)-regret set for any \( k' \geq k \). In particular, a \((1, \epsilon)\)-regret set is a \((k, \epsilon)\)-regret set for any \( k \geq 1 \). However, there may exist a \((k, \epsilon)\)-regret set whose size is much smaller than any \((k-1, \epsilon)\)-regret set, so the notion of \((k, \epsilon)\)-regret set is useful for all \( k \).

Notice that \( \ell_k(\mathcal{Q}) \) is a monotonic decreasing function of its argument, i.e., if \( \mathcal{Q}_1 \subseteq \mathcal{Q}_2 \), then \( \ell_k(\mathcal{Q}_1) \geq \ell_k(\mathcal{Q}_2) \). Furthermore, for any \( t > 0 \), \( \omega(tu, p) = t \omega(u, p) \) but \( \varphi_k(tu, \mathcal{P}) = \varphi_k(u, \mathcal{P}) \), \( \Phi_k(tu, \mathcal{P}) = \Phi_k(u, \mathcal{P}) \), and \( \ell_k(tu, \mathcal{Q}, \mathcal{P}) = \ell_k(u, \mathcal{Q}, \mathcal{P}) \) (scale invariance).

\(^2\)If there are multiple objects with score \( \omega_j(u, \mathcal{P}) \), then either we include all such points in \( \Phi_k(u, \mathcal{P}) \) or break the tie in a consistent manner.
Our goal is to compute a small subset $Q \subseteq P$ with small regret ratio, which we refer to as the regret minimizing set (RMS) problem. Since the regret ratio can be decreased by increasing the size of the subset, there are two natural formulations of the RMS problem.

(i) min-error: Given a set $P$ of objects and a positive integer $r$, compute a subset of $P$ of size $r$ that minimizes the regret ratio, i.e., return a subset $Q^* = \arg\min_{Q \subseteq P: |Q| \leq r} \ell_k(Q)$, and let $\ell(r) = \ell_k(Q^*)$.

(ii) min-size: Given a set $P$ of objects and a parameter $\epsilon > 0$, compute a smallest size subset with regret ratio at most $\epsilon$, i.e., return $Q^# = \arg\min_{Q \subseteq P: \ell_k(Q) \leq \epsilon} |Q|$, and set $s(\epsilon) = |Q|$.

Our results. We present the following results in this paper:

(I) We show that the RMS problem is NP-Complete even for $d = 3$ and $k > 1$. The previous hardness proof [10] requires the dimension $d$ to be as large as $n$, and it was an open question whether the problem was NP-complete in low dimensions. Since a polynomial-time algorithm exists for both formulations of the regret minimizing set problem in 2D, our result settles the problem for $k > 1$. Proving hardness in small dimensions, $d = 3$, requires a different proof technique. In fact, it is not trivial to check whether $\ell_k(Q) \leq \epsilon$ for given $\epsilon > 0$, i.e., it is not obvious that the RMS problem is in NP. Using a few results from discrete geometry, we present an efficient algorithm for computing $\ell_k(Q)$.

(II) We show that for any $P \subset X$ and for any $\epsilon > 0$ there exists a $(1, \epsilon)$-regret set, and thus a $(k, \epsilon)$-regret set for any $k \geq 1$, of $P$ whose size is independent of the size of $P$. By establishing a connection between $(k, \epsilon)$-regret sets and the so-called core sets [2], we show that for any $P \subset X$ and for any $\epsilon > 0$, a $(1, \epsilon)$-regret set of size $O(\frac{1}{\epsilon^{d-1}})$ can be computed in time $O(n + \frac{1}{\epsilon^{d-1}})$. Notice that for the min-error problem Nanongkai et al. [29] give an algorithm that returns a set $Q$ such that $\ell(r) \leq \frac{d-1}{(r-d+1)\frac{d-1}{2} + d-1}$. Solving for $r$, we get for a fixed error $\epsilon$ a $(1, \epsilon)$-regret set of size $O(\frac{1}{\epsilon^{d-1}})$. Our result improves this bound significantly and it is optimal in the worst case. Furthermore, we can maintain our $(1, \epsilon)$-regret set under insertion/deletion of points in $O(\frac{\text{polylog}(n)}{\epsilon^{d-1}})$ time per update. The efficient maintenance of a regret set is important in various applications and it has not been considered before.

(III) For a given $P$ and $\epsilon > 0$, there may exist a $(k, \epsilon)$-regret set of $P$ of size much smaller than $1/\epsilon^{\frac{d-1}{2}}$. We complement our NP-Completeness result by presenting approximation algorithms for the RMS problem. Given $P \subset X$ of size $n$ and $\epsilon > 0$, we can compute
a \((k, 2\epsilon)\)-regret set of \(P\) of size \(O(s(\epsilon) \log(s(\epsilon)))\) in time \(O\left(\frac{n}{\epsilon^d} \log(n) \log(\frac{1}{\epsilon})\right)\). Roughly speaking, we formulate the regret-minimizing set problem as a classical hitting-set problem and use a greedy algorithm to compute a small size hitting set.

By plugging the above algorithm into a binary search, we also obtain an algorithm for the min-error version of the problem: given a parameter \(r\), we compute a set \(Q \subseteq P\) of size \(O(r \log r)\) such that \(\ell_{cr} \log r \leq \ell_k(Q) \leq 2\ell(r)\) for a sufficiently large constant \(c\). The algorithm runs in \(O\left(\frac{n}{\ell_k(Q)^{d-1}} \log(n) \log(\frac{1}{\ell_k(Q)})\right)\) time. If \(\ell_k(Q)\) is very small the algorithm runs in \(O(n^d)\) time. The expected running time of this algorithm is much smaller if the objects are uniformly distributed or drawn from some other nice distribution.

(IV) We present experimental results to evaluate the efficacy and the efficiency of our algorithms on both synthetic and real data sets. We compare our algorithms with the state of the art greedy algorithm for the \(k\)-regret minimization problem presented in [10]. Our hitting-set based algorithm is significantly faster than the previous known algorithms and the maximum regret ratios of the returned sets are very close, if not better, than the maximum regret ratios of the greedy algorithm. The core set algorithm is significantly faster than hitting set and greedy algorithms. Although the (maximum) regret ratio of the set returned by the core-set based algorithm is worse than those of other algorithms, the regret in 90% – 95% directions is roughly the same as that of the other two algorithms.

2 3D RMS is NP-Complete

In this section we show that the \(k\)-RMS problem is NP-Complete for \(d \geq 3\) and \(k \geq 2\). More precisely, given a set \(P \subset X\) in \(\mathbb{R}^3\), a parameter \(\epsilon > 0\), and an integer \(r\), the problem of determining whether there is a \((k, \epsilon)\)-regret set of \(P\) of size at most \(r\) is NP-Complete. We first show its membership in NP. We then show NP-hardness for \(k = 2\) and later show how to extend the argument to higher values of \(k\).

2.1 RMS problem is in NP

Given a subset \(Q \subseteq P\) of objects, we describe a polynomial-time algorithm for computing the regret ratio of \(Q\). For simplicity, we describe the algorithm for \(d = 3\) but it extends to \(d > 3\).

Let \(\Omega = \{p - q \mid p, q \in P, p \neq q\}\) be the set of vectors in directions passing through a pair of points of \(P\). For a vector \(w \in \Omega\), let \(h_w : \langle x, w \rangle = 0\), be the plane normal to \(w\) passing through the origin. By construction, for \(w = p - q\) the score of \(p\) is higher than that of \(q\) for all preferences in one of the open halfspaces bounded by \(h_w\) (namely, \(\langle x, w \rangle > 0\)), lower in the other halfspace, and equal for all preferences in \(h_w\). Set \(H = \{h_w \mid w \in \Omega\} \cup \{x_i = 0 \mid 1 \leq i \leq 3\}\), i.e., \(H\) includes all the planes \(h_w\) along with the coordinate planes. \(H\) induces a decomposition \(A(H)\) of \(\mathbb{R}^3\) into cells of various dimensions, where each cell is a maximal connected region of points lying in the same subset of hyperplanes of \(H\) (see Figure 2). It is well known that

\[\text{...}\]

\[\text{...}\]

\[\text{...}\]

\[\text{...}\]
Lemma 2.1. For each cell \( C \) of \( A(H) \), let \( \ell(C, Q) = \max_{u \in C} \ell_k(u, Q) \) be the regret ratio of \( Q \) within \( C \). Then \( \ell_k(Q) = \max_{C \in \mathcal{C}} \ell_k(C, Q) \). The following lemma is useful in computing \( \ell(C, Q) \).

**Lemma 2.1** For each cell \( C \in A(H) \) and for any \( i \leq n \), \( \varphi_i(u, P) \) (and thus \( \varphi_i(u, Q) \)) is the same for all \( u \in C \).

**Proof:** Suppose on the contrary, there are two points \( u_1, u_2 \in C \) and \( j \geq 0 \) such that \( \varphi_j(u_1, Q) \neq \varphi_j(u_2, Q) \). Hence, there are two points \( p_1, p_2 \in Q \) such that \( \langle u_1, p_1 \rangle \geq \langle u_1, p_2 \rangle \) and \( \langle u_2, p_1 \rangle \leq \langle u_2, p_2 \rangle \), and at least one of the inequalities is strict. Let \( h_w \in H \) be the plane that is normal to \( p_1 - p_2 \) and passes through the origin. It divides \( \mathbb{R}^3 \) into two halfspaces. Reference vectors \( u_1, u_2 \) lie in the opposite halfspaces of \( h_w \), and at least one of the \( u_1, u_2 \) lies in the open halfspace. However, this is a contradiction because \( u_1, u_2 \) lie in the same cell of \( A(H) \) and thus lie on the same side of each plane in \( H \).

Fix a cell \( C \). Let \( p_i = \varphi_i(u, Q) \) and \( p_j = \varphi_j(u, P) \) for any \( u \in C \) (from Lemma 2.1 we have that the ordering inside a cell is the same). Furthermore, let \( h_j \) be the plane \( \langle x, p_j \rangle = 1 \) and let \( C^j = h_j \cap C \). \( C^j \) is a 2D polygon and each ray \( p \) in \( C \) intersects \( C^j \) at exactly one point \( p^j \). Since \( \ell(u, Q) \) is the same for all points on \( p \), \( \ell_k(C, Q) = \ell_k(C^j, Q) \). Furthermore, by Lemma 2.1, \( \ell_k(C^j, Q) \) is either 0 for all \( u \in C^j \) or

\[
\ell_k(C^j, Q) = \max_{u \in C^j} \frac{\omega(u, p_j) - \omega(u, p_i)}{\omega(u, p_j)} = \max_{u \in C^j} 1 - \omega(u, p_i)
\]

Since \( C^j \) is convex and \( \langle u, p_i \rangle \) is a linear function, it is a minimum within \( C^j \) at a vertex of \( C^j \), so we compute \( \langle u, p_i \rangle \) for each vertex \( u \in C^i \) and choose the one with the minimum value. Repeating this step for all cells of \( \mathcal{C} \) we compute \( \ell_k(Q) \).
By a well known result in discrete geometry [3], the total number of vertices in \( C \) over all cells \( C \in \mathcal{C} \) is \( O(|\mathcal{C}|^2) = O(n^q) \). Furthermore, if \( b \) bits are used to represent the coordinates of each point in \( P \), each vertex of \( C \) requires \( O(b) \) bits. Finally, the algorithm extends to higher dimensions in a straightforward manner. The total running time in \( \mathbb{R}^d \) is \( O(n^{2d-1}) \). We thus conclude the following.

**Lemma 2.2** Given a set \( P \) of \( n \) points in \( \mathbb{R}^d \) and a subset \( Q \subseteq P \), \( \ell_k(Q) \) can be computed in \( O(n^{2d-1}) \) time.

An immediate corollary of the above lemma is the following:

**Corollary 2.3** The RMS problem is in NP.

### 2.2 NP-Hardness Reduction

We first show the hardness for \( k = 2 \). Recall that a preference vector has only non-negative coordinates. For simplicity, however, we first consider all points in \( \mathbb{R}^3 \) as preference vectors and define \( \ell_k(Q) = \max_{u \in \mathbb{R}^3} \ell_k(u, Q) \), and later we describe how to restrict the preference vectors to \( X \).

Recall that the RMS problem for \( \epsilon = 0 \) and \( k = 2 \) asks: Is there a subset \( Q \subseteq P \) of size \( r \) such that in every direction \( u \), the point in \( Q \) with the highest score along \( u \), i.e., \( q_1(u, S) \), has score at least as much as that of the second best in \( P \) along \( u \), i.e., of the point \( q_2(u, P) \)?

Let \( \Pi \) be a strictly convex polytope in \( \mathbb{R}^3 \). The 1-skeleton of \( \Pi \) is the graph formed by the vertices and edges of \( \Pi \). Given \( \Pi \) and an integer \( r > 0 \), the convex-polytope vertex-cover (CPVC) asks whether the 1-skeleton of \( \Pi \) has a vertex cover of size at most \( r \), i.e., whether there is a subset \( C \) of vertices of \( \Pi \) of size \( r \) such that every edge is incident on at least one vertex of \( C \). The CPVC problem is NP-Complete, as shown by Das and Goodrich [11].

Given \( \Pi \) with \( V \) as the set of its vertices, we construct an instance of the RMS problem for \( k = 2 \), as follows. First we translate \( \Pi \) so that the origin lies inside \( \Pi \). Next we set \( P = V \). The next lemma proves the NP-hardness of the RMS problem for \( k = 2 \) and \( \epsilon = 0 \).

**Lemma 2.4** A subset \( Q \subseteq V \) is a vertex cover of \( \Pi \) if and only if \( Q \) is a \((2, 0)\)-regret set for \( P \).

**Proof:** If \( Q \) is a vertex cover of \( \Pi \), we show that \( Q \) is also a \((2, 0)\)-regret set. Take a vector \( u \in \mathbb{R}^3 \) and assume that \( q = q_1(u, P) \) (if there is more than one point with rank one, we can let \( q \) be any one of them). If \( q \in Q \) then obviously \( \omega_1(u, Q) = \omega_1(u, P) \geq \omega_2(u, P) \).

Now, assume that \( q \notin Q \). Let \( (q, q_1), \ldots, (q, q_g) \) be the edges in \( \Pi \) incident on \( q \). Set \( N_q = \{ q_i \mid 1 \leq i \leq g \} \). Since \( Q \) is a vertex cover of \( \Pi \) and \( q \notin Q \), \( N_q \subseteq Q \). We claim that \( q_2(u, P) \in N_q \), which implies that \( \omega(u, Q) \geq \omega_2(u, P) \). Hence, \( Q \) is a \((2, 0)\)-regret set.

Indeed, since \( \Pi \) is convex, and \( q \) is maximal along direction \( u \), the plane \( h \) on \( q \) vertical to \( u \) is a supporting hyperplane for \( \Pi \). A plane \( h' \) parallel to \( h \) is translated toward the origin starting with its initial position at \( h \). There are two cases. In the first case, where \( q \) and \( q_2(u, P) \) have the same score, they belong to the same face of \( \Pi \) that must be contained in \( h \) itself — in this case \( h \) also contains a point from \( N_q \), since every face containing \( q \) and points other than \( q \) must contain a 1 dimensional face as well, and therefore a point in \( N_q \).

In the second case, as \( h' \) is translated, it must first hit one of the neighbors of \( q \), by convexity. As a result, in any case, there will be a point in \( N_q \) that gives the rank-two point on \( u \).
Next, if \( Q \) is a \((2,0)\)-regret set, we show that \( Q \) is a vertex cover of \( \Pi \). Suppose to the contrary \( Q \) is not a vertex cover of \( \Pi \), i.e., there is an edge \((q_1, q_2)\) in \( \Pi \) but \( q_1, q_2 \notin Q \). Since \( \Pi \) is a strictly convex polytope, there is a plane \( h \) tangent to \( \Pi \) at the edge \((q_1, q_2)\) that does not contain any other vertex of \( \Pi \). If we take the direction \( u \) normal to \( h \) then \( \Phi_2(u, P) = \{q_1, q_2\} \). If \( q_1, q_2 \notin Q \) then \( \omega_1(u, Q) < \omega_2(u, P) \), which contradicts the assumption that \( Q \) is a \((2,0)\)-regret set of \( P \).

Restricting to \( X \). In order to show that the RMS problem is NP-hard even when preferences are restricted to \( X \), polytope \( \Pi \) needs to have two additional properties:

(i) All vertices of \( \Pi \) must lie in the first orthant.

(ii) For any edge \((v_1, v_2)\) of \( \Pi \), where \( v_1, v_2 \) are vertices of \( P \), there is a direction \( u \in X \) such that \( v_1, v_2 \) are the top vertices in direction \( u \).

It is easy to satisfy property (i) because the translation of the vertices of a polytope does not change the rank of the points in any direction. On the other hand, property (ii) is not guaranteed by the construction in [11].

We show that there is an affine transformation of \( \Pi \) that can be computed and applied in polynomial time, to get a polytope \( \Pi' \) with the same combinatorial structure as \( \Pi \), but that also satisfies properties (i), and (ii). The fact that the polytope has the same combinatorial structure implies that the underlying graph is the same, and therefore a vertex cover will also be a \((2,0)\)-regret set of \( \Pi \). The details of the transformation can be found in Appendix [A]. The first part of the NP-hardness proof is the same with the case of all directions in \( \mathbb{R}^3 \), if \( Q \) is a vertex cover of \( \Pi' \) then it is also a \((2,0)\)-regret set. Using property (ii) of \( \Pi' \), it is straightforward to show the other direction, as well.

Choosing \( \epsilon > 0 \). While the above suffices to prove the hardness of the RMS problem for \( \epsilon = 0 \), it is possible that when \( \epsilon > 0 \) the problem is strictly easier. However, we show the stronger result that the RMS problem is NP-hard even when \( \epsilon \) is required to be strictly positive. In order to get the NP-hardness of the RMS problem for \( \epsilon > 0 \) and \( k = 2 \), we find a small enough strictly positive value of \( \epsilon \) with bounded bit complexity such that any \((2,\epsilon)\)-regret set is also a \((2,0)\)-regret set, and vice versa. For each cell \( C \in \mathcal{C} \), we take a direction \( u_C \in C \) and let \( \lambda_C = 1 - \omega_3(u_C, P)/\omega_2(u_C, P) > 0 \). By defining \( \epsilon = \frac{1}{2} \min_C \lambda_C \) we can conclude the result.

Larger values of \( k \). By making \( k - 1 \) copies of each point in the above construction it is straightforward to show that the RMS problem is NP-complete for any \( k \geq 2 \) and \( d \geq 3 \).

**Theorem 2.5** The RMS problem is NP-complete for \( d \geq 3 \) and for \( k \geq 2 \).

### 3 Coreset-based Approximation

In this section, we present an approximation scheme for the RMS problem using coresets. The general idea of a coreset is to approximately preserve some desired characteristics of the full data set using only a tiny subset [2]. The particular geometric characteristic most relevant to our problem is the extent of the input data in any direction, which can be
formalized as follows. Given a set of points $P$ and a direction $u \in \mathbb{R}^d$, the directional width of $P$ along $u$, denoted $\text{width}(u, P)$, is the distance between the two supporting hyperplanes of $\mathbb{R}^d$, one in direction $u$ and the other in direction $-u$. The connection between $k$-regret and the directional width comes from the fact that the supporting hyperplane in a direction $u$ is defined by the extreme point in that direction, and its distance from the origin is simply its score. Therefore, we have the equality:

$$\text{width}(u, P) = \omega(u, P) + \omega(-u, P).$$

We use coresets that approximate directional width to approximate $k$-regret sets. In particular, a subset $Q$ is an $\epsilon$-kernel coreset if $\text{width}(u, Q) \geq (1 - \epsilon) \text{width}(u, P)$, for all directions $u \in \mathbb{R}^d$.

**Lemma 3.1** If $Q \subseteq P$ is an $\epsilon$-kernel coreset of $P$ then $Q$ is also $(1, \epsilon)$-regret set of $P$.

**Proof**: If $Q$ is an $\epsilon$-kernel coreset of $P$ then $\text{width}(u, P) - \text{width}(u, Q) \leq \epsilon \text{width}(u, P) \leq \epsilon \omega(u, P)$. The last inequality follows because $\omega(-u, P) \leq 0$. Furthermore $\omega(-u, Q) \leq \omega(-u, P)$. We thus have

$$\omega(u, P) - \omega(u, Q) = \omega(u, P) + \omega(-u, P) - \omega(u, Q) + \omega(-u, Q)$$

$$\leq \text{width}(u, P) - \omega(u, Q) - \omega(-u, Q)$$

$$= \text{width}(u, P) - \text{width}(u, Q)$$

$$\leq \epsilon \omega(u, P).$$

Hence, $\omega(u, Q) \geq (1 - \epsilon) \omega(u, P)$. We use the results of [1, 8] that compute small $\epsilon$-kernel coresets efficiently, as well as allows dynamic updates, and prove the following result.

**Theorem 3.2** Given a set $P$ of $n$ points in $\mathbb{R}^d$, $\epsilon > 0$ and an integer $k > 0$, we can compute in time $O(n + \frac{1}{\epsilon^{d-1}})$ a subset $Q \subseteq P$ of size $O\left(\frac{1}{\epsilon^{d-1/2}}\right)$ whose $k$-regret ratio is at most $\epsilon$, i.e. $\ell_k(Q, P) \leq \epsilon$.

**Proof**: We choose $Q$ as an $\epsilon$-kernel coreset of $P$. By Lemma 3.1, $Q$ is a $(1, \epsilon)$-regret set of $P$ and thus also a $(k, \epsilon)$-regret set of $P$ for any $k \geq 1$. Chan [8] has described an algorithm for computing $\epsilon$-kernel of size $O\left(\frac{1}{\epsilon^{d-1/2}}\right)$ in time $O(n + \frac{1}{\epsilon^{d-1}})$. Hence, the theorem follows.

We conclude this section by making two remarks:

The size of $Q$ in the preceding theorem is asymptotically optimal: there exist point sets for which no smaller subset satisfies this property. The size optimality follows from the construction described below for 2D—generalization to higher dimensions is straightforward.

Fix a positive integer $k \leq n \sqrt{\epsilon}$. Consider a set of $n/k$ points $P \subset S^1_+$ uniformly distributed on the unit circle, on the arc in the first quadrant. Make $k$ copies of these points to get a set of $n$ points. This construction ensures that the top-$k$ points in any direction $u$ have exactly the same score; that is, $\omega_1(u, P) = \omega_k(u, P)$ for all $u \in \mathbb{R}^2$. It is known that when points are uniformly distributed on a circle, an $\epsilon$-kernel coreset gives the optimum (asymptotically) coreset with size $\Theta\left(\frac{1}{\sqrt{\epsilon}}\right)$. Since $q_1(u, P')$ and $q_k(u, P')$ lie at the same position for any $u \in \mathbb{R}^2$, we also have that the optimal $(k, \epsilon)$-regret set has size $\Theta\left(\frac{1}{\sqrt{\epsilon}}\right)$.

The set $Q$ can also be maintained under insertion/deletion of points in $P$ in time $O\left(\frac{\log^d n}{\epsilon^{d-1}}\right)$ per update. The dynamic update performance follows from the construction in [11].
4 Regret Approximation using Hitting Sets

Theorem 3.2 shows that a \((k, \epsilon)\)-regret set of size \(O\left(\frac{1}{\epsilon^d} \right)\) can be computed quickly. However, given \(P\) and \(\epsilon > 0\), there may be a \((k, \epsilon)\)-regret set of much smaller size. In this section, we describe an algorithm that computes a \((k, \epsilon)\)-regret set of size close to \(s_\epsilon := s(\epsilon)\), the minimum size of a \((k, \epsilon)\)-regret set, by formulating the RMS problem as a hitting-set problem.

A range space (or set system) \(\Sigma = (X, \mathcal{R})\) consists of a set \(X\) of objects and a family \(\mathcal{R}\) of subsets of \(X\). A subset \(H \subseteq X\) is a hitting set of \(\Sigma\) if \(H \cap R \neq \emptyset\) for all \(R \in \mathcal{R}\). The hitting set problem asks to compute a hitting set of the minimum size. The hitting set problem is a classical NP-Complete problem, and a well-known greedy \(O(\log n)\)-approximation algorithm is known.

We construct a set system \(\Sigma = (P, \mathcal{R})\) such that a subset \(Q \subseteq P\) is a \((k, \epsilon)\)-regret set if and only if \(Q\) is a hitting set of \(\Sigma\). We then use the greedy algorithm to compute a small-size hitting set of \(\Sigma\). A weakness of this approach is that the size of \(\mathcal{R}\) could be very large and the greedy algorithm requires \(\mathcal{R}\) to be constructed explicitly. Consequently, the approach is expensive even for moderate inputs say \(d \sim 5\).

Inspired by the above idea, we propose a bicriteria approximation algorithm: given \(P\) and \(\epsilon > 0\), we compute a subset \(Q \subseteq P\) of size \(O(s_\epsilon \log s_\epsilon)\) that is a \((k, 2\epsilon)\)-regret set of \(P\); the constant 2 is not important, it can be made arbitrarily small at the cost of increasing the running time. By allowing approximations to both the error and size concurrently, we can construct a much smaller range space and compute a hitting set of this range space.

The description of the algorithm is simpler if we assume the input to be well conditioned. We therefore transform the input set, without affecting an RMS, so that the score of the topmost point does not vary too much with the choice of preference vectors, i.e., the ratio \(\max_{u \in X} \omega(u, P) / \min_{u \in X} \omega(u, P)\) is bounded by a constant that depends on \(d\).

We transform \(P\) into another set \(P'\), so that (i) for any \(u \in X\), \(\varphi_1(u, P')\) does not lie close to the origin and (ii) for any \((k, \epsilon)\)-regret set \(Q \subseteq P\), the subset \(Q' \subseteq P'\) is a \((k, \epsilon)\)-regret set in \(P'\), and vice versa. Nanongkai et al. [29] showed that a non-uniform scaling of \(P\) satisfies (ii). In the following lemma, we show a stronger result.

**Lemma 4.1** Let \(P\) be a set of \(n\) points in \(\mathbb{R}^d\), and let \(M\) be a full rank \(d \times d\) matrix. A subset \(Q \subseteq P\) is a \((k, \epsilon)\)-regret set of \(P\) if and only if \(Q' = MQ\) is a \((k, \epsilon)\)-regret set of \(P' = MP\).

**Proof:** First, observe that \(\langle u, Mp \rangle = u^T M p = (M^T u)^T p = (M^T u, p)\), and so \(\omega_k(u, MP) = \omega_k(M^T u, P)\). We define a mapping \(F : X \rightarrow X\) and its inverse \(F^{-1} : X \rightarrow X\) as \(F(u) = (M^{-1})^T u\) and \(F^{-1}(u) = M^T u\). Our proof now follows easily from these mappings.

If \(Q\) is a \((k, \epsilon)\)-regret set for \(P\), then for any \(u \in S_{\epsilon}^{-1}\) we have \(\omega_1(u, MQ) = \omega_1(M^T u, Q) = \omega_1(F^{-1}(u), Q) \geq (1 - \epsilon) \omega_k(F^{-1}(u), P) = (1 - \epsilon) \omega_k(u, MP) = (1 - \epsilon) \omega_k(u, P')\)

Conversely, if \(Q'\) is a \((k, \epsilon)\)-regret set for \(P'\), then for any \(u \in S_{\epsilon}^{-1}\), \(\omega_1(u, Q) = \omega_1(u, M^{-1} Q') = \omega_1((M^{-1})^T u, Q') = \omega_1(F(u), Q') \geq (1 - \epsilon) \omega_k(F(u), P') = (1 - \epsilon) \omega_k(u, M^{-1} P') = (1 - \epsilon) \omega_k(u, P)\). This completes the proof.

We now describe the transformation of the input points, which is a non-uniform scaling of \(P\). Specifically, for each \(1 \leq j \leq d\), let \(m_j = \max_{p_i \in P} p_{ij}\) be the maximum value of the \(j\)th
coordinate among all points. Let \( B \subseteq P \) be the subset of at most \( d \) points, one per coordinate, corresponding to these \( m_j \) values. We refer to \( B \) as the basis of \( P \), and let \( \text{Basis}(P) \) be the method to find the basis \( B \). We divide the \( j \)-th coordinate of all points by \( m_j \), for all \( j = 1, 2, \ldots, d \). Let \( P' \) be the resulting set, and let \( B' \) be the transformation of \( B \). We note that for each coordinate \( j \) there is a point \( p'_j \in B' \) with \( p'_{ij} = 1 \). The different scaling factor in each coordinate can be represented by the diagonal matrix \( M \) where \( M_{jj} = 1/m_j \), and so \( P' = MP \). Let \( \text{Scale}(P) \) be the procedure that scales the set \( P \) according to the above transformation. The key property of this affine transformation is the following lemma.

**Lemma 4.2** Let \( M \) be the affine transform described above and let \( P' = MP \). Then, for all \( u \in X \),

\[
\sqrt{d} \cdot \|u\| \geq \omega(u, P') \geq \frac{1}{\sqrt{d}} \cdot \|u\|.
\]

**Proof:** Since \( \omega(\cdot, \cdot) \) is a linear function, without loss of generality consider a vector \( u \in X \) with \( \|u\| = 1 \). After the transformation \( M \), for each coordinate \( j \), we have \( p'_j \leq 1 \). Therefore, \( \|p'\| \leq \sqrt{d} \) and also \( \sqrt{d} \geq \omega(u, P') \) because \( u \) is a unit vector. For the second inequality, we note that for any unit norm vector \( u \) we must have \( u_j \geq \frac{1}{\sqrt{d}} \), for some \( j \). Since our transform ensures the existence of a point \( p' \in B' \) with \( p'_j = 1 \), we must have \( \omega(u, P') \geq \langle u, p' \rangle \geq \frac{1}{\sqrt{d}} \).

This completes the proof. \( \blacksquare \)

In the following, without loss of generality, we assume that \( P \subset [0,1]^d \) and there is a set \( B \subseteq P \) of at most \( d \) points, such that for any \( 1 \leq j \leq d \), there is a point \( p \in B \) with \( p_j = 1 \).

### 4.1 Approximation Algorithms

We first show how to formulate the min-size version of the RMS problem as a hitting set problem. Let \( P \), \( k \), and \( \epsilon \) be fixed. For a vector \( u \in X \), let \( R_u = \{ p \in P \mid \omega(u, p) \geq (1 - \epsilon)\omega_k(u,p) \} \). Note that if \( \epsilon = 0 \), then \( R_u = \Phi_k(u) \), the set of top-\( k \) points of \( P \) in direction \( u \). Set \( R_u = \{ R_u \mid u \in X \} \). Although there are infinitely many preferences we show below that \( |R_u| \) is polynomial in \( |P| \). We now define the set system \( \Sigma = (P, R_u) \).

**Lemma 4.3**

(i) \( |R_u| = O(n^d) \).

(ii) A subset \( Q \subseteq P \) is a hitting set of \( \Sigma \) if and only if \( Q \) is a \((k, \epsilon)\)-regret set of \( P \).

**Proof:** (i) Note that \( R_u \) is a subset of \( P \) that is separated from \( P \setminus R_u \) by the hyperplane \( h_u : \langle u, x \rangle \geq (1 - \epsilon)\omega_k(u, p) \). Such a subset is called linearly separable. A well-known result in discrete geometry \[3\] shows that a set of \( n \) points in \( \mathbb{R}^d \) has \( O(n^d) \) linearly separable subsets. This completes the proof of (i).

(ii) First, by definition any \((k, \epsilon)\)-regret set \( Q \) has to contain a point of \( R_u \) for all \( u \in X \) because otherwise \( \ell_k(u, Q) > \epsilon \). Hence, \( Q \) is a hitting set of \( \Sigma \). Conversely, if \( Q \cap R_u \neq \emptyset \), then \( \ell_k(u, Q) \leq \epsilon \). If \( Q \) is a hitting set of \( \Sigma \), then \( Q \cap R_u \neq \emptyset \) for all \( u \in X \), so \( Q \) is also a \((k, \epsilon)\)-regret set. \( \blacksquare \)
HS is the greedy algorithm in [6] for (c) Sphere

(d) ElNino

Figure 3. Running time for k = 1.

Figure 4. Running time for k = 10.

We can thus compute a small-size \((k, \epsilon)\)-regret set of \(P\) by running the greedy hitting set algorithm on \(\Sigma\). In fact, the greedy algorithm in [6] returns a hitting set of size \(O(s_{\epsilon} \log s_{\epsilon})\). As mentioned above, the challenge is the size of \(\mathcal{R}_u\). Even for small values of \(k\), \(|\mathcal{R}_u|\) can be \(\Omega(n^{d/2})\) [3]. Next, we show how to construct a much smaller set system.

Recall that \(\ell_k(u, Q)\) is independent of \(\|u\|\) so we focus on unit preference vectors, i.e., we assume \(\|u\| = 1\). Let \(\mathcal{U} = \{u \in \mathcal{X} \mid \|u\| = 1\}\) be the space of all unit preference vectors; \(\mathcal{U}\) is the portion of the unit sphere restricted to the positive orthant. For a given parameter \(\delta > 0\), a set \(N \subseteq \mathcal{U}\) is called a \(\delta\)-net if the spherical caps of radius \(\delta\) around the points of \(N\) cover \(\mathcal{U}\), i.e., for any \(u \in \mathcal{U}\), there is a point \(v \in N\) with \(\langle u, v \rangle \geq \cos(\delta)\). A \(\delta\)-net of size \(O(\frac{1}{\delta^{d-1}})\) can be computed by drawing a "uniform" grid on \(\mathcal{U}\). In practice, it is simpler and more efficient to simply choose a random set of \(O(\frac{1}{\delta^{d-1}} \log \frac{1}{\epsilon})\) directions — this will be a \(\delta\)-net with probability at least 1/2. Set \(\delta = \frac{\epsilon}{2^d}\). Let \(N\) be be a \(\delta\)-net of \(\mathcal{U}\), and let \(\mathcal{R}_N = \{R_u \mid u \in N\}\).

Set \(\Sigma_N = (P, \mathcal{R}_N)\). Note that \(|\mathcal{R}_N| = O(\frac{1}{\delta^{d-1}})\). Our main observation, stated in the lemma below, is that it suffices to compute a hitting set of \(\Sigma_N\). That is, a subset \(Q \subseteq P\) that has a small regret with respect to vectors in \(N\) has small regret for all preferences.

**Lemma 4.4** Let \(Q'\) be a hitting set of \(\Sigma_N\), and let \(B\) be the basis of \(P\). Then \(Q = Q' \cup B\) is a \((k, 2\epsilon)\)-regret set of \(P\).

Algorithm 1 summarizes the algorithm. GREEDY_HS is the greedy algorithm in [6] for computing a hitting set.

**Algorithm 1.** RMS_HS

Input: \(P\): Input points, \(k \geq 1\): rank, \(\epsilon > 0\): error parameter.
Output: \(Q\) a \((k, 2\epsilon)\)-regret set

1. \(B := \text{Basis}(P)\)
2. \(P := \text{Scale}(P)\).
3. \(\delta := \frac{\epsilon}{2^d}\)
We now prove Lemma 4.4. First, consider the case when $\omega_k(u, P) \geq \frac{1}{(1-\epsilon)\sqrt{d}}$. In this case, by Lemma 4.2 the set $B$ is guaranteed to contain a point $q$ with $\omega(u, q) \geq \frac{1}{\sqrt{d}}$, which proves the claim. So let us now assume that $\omega_k(u, P) > \frac{1}{(1-\epsilon)\sqrt{d}}$. Let $\hat{u}, \bar{u} \in \mathbb{N}$ be a direction in the net $N$ such that, 

\[ (\hat{u}, \bar{u}) \leq \epsilon/2d, \]

where $\hat{u}, \bar{u}$ is the angle between $u$ and $\bar{u}$. Such a direction exists because $N$ is a $\frac{\epsilon}{2d}$-net on $U$. Observe that,

\[ \| u - \bar{u} \| = \sqrt{2 - 2\cos((\hat{u}, \bar{u}))} = 2\sin\left(\frac{\hat{u}}{2}\right) \leq \frac{\epsilon}{2d}, \]

where we have used first the cosine rule, the identity $1 - \cos \theta = 2\sin^2\left(\frac{\theta}{2}\right)$, as well as the inequality $\sin \theta \leq \theta$ for $\theta \geq 0$ in the final step. Also, observe that for any $p \in P$ we have,

\[ |\omega(u, p) - \omega(\hat{u}, p)| \leq \frac{\epsilon}{2\sqrt{d}}. \]  

This follows because,

\[ |\omega(u, p) - \omega(\hat{u}, p)| = |\langle u, p \rangle - \langle \hat{u}, p \rangle| = |\langle u - \hat{u}, p \rangle| \leq \| u - \hat{u} \| \| p \| \leq \frac{\epsilon}{2d} \times \sqrt{d} = \frac{\epsilon}{2\sqrt{d}}, \]

where we have used the Cauchy-Schwarz inequality for the first inequality, the upper bound on $\| u - \hat{u} \|$ derived earlier, along with $\| p \| \leq \sqrt{d}$ for the second inequality.

Let $x_1, x_2, \ldots, x_k \in P$ be the top-$k$ points along direction $u$, i.e., $x_i = \varphi_i(u, P)$. Also, let $y_k$ be the top-$k$ point along direction $\bar{u}$. As remarked we can assume, $\omega(u, x_i) \geq \omega(u, x_k) \geq \omega(u, y_k)$.

Analysis. The correctness of the algorithm follows from Lemma 4.4. Since a hitting set of $\Sigma$ is also a hitting set of $\Sigma_N$, $\Sigma_N$ has a hitting set of size at most $s_\epsilon$. The greedy algorithm in [6] returns a hitting set of size $O(s_\epsilon \log s_\epsilon)$ for $d \geq 4$ and of size $O(s_\epsilon)$ for $d = 3$. Therefore $|Q| = O(s_\epsilon \log s_\epsilon)$ for $d \geq 4$ and $O(s_\epsilon)$ for $d = 3$. Computing the set $B$ takes $O(n)$ time. We can construct $R_u$ for each $u \in N$ in $O(n)$ time. The greedy algorithm in [6] takes $O\left(\frac{n}{\epsilon^2} \log n \log \frac{1}{\epsilon}\right)$ expected time (the bound on the running time also holds with high probability).

We now prove Lemma 4.4.

Proof of Lemma 4.4. It suffices to show that for any direction $u \in U$ there is a point $q \in Q$ for which $\omega(u, q) \geq (1 - \epsilon)\omega_k(u, P)$.

Let us first consider the case when $\omega_k(u, P) \leq \frac{1}{(1-\epsilon)\sqrt{d}}$. In this case, by Lemma 4.2 the set $B$ is guaranteed to contain a point $q$ with $\omega(u, q) \geq \frac{1}{\sqrt{d}}$, which proves the claim. So let $\epsilon \leq \epsilon/2d$. Let $\bar{u} \in \mathbb{N}$ be a direction in the net $N$ such that, 

\[ (\hat{u}, \bar{u}) \leq \epsilon/2d, \]

where $\hat{u}, \bar{u}$ is the angle between $u$ and $\bar{u}$. Such a direction exists because $N$ is a $\frac{\epsilon}{2d}$-net on $U$. Observe that,

\[ \| u - \bar{u} \| = \sqrt{2 - 2\cos((\hat{u}, \bar{u}))} = 2\sin\left(\frac{\hat{u}}{2}\right) \leq \frac{\epsilon}{2d}, \]

where we have used first the cosine rule, the identity $1 - \cos \theta = 2\sin^2\left(\frac{\theta}{2}\right)$, as well as the inequality $\sin \theta \leq \theta$ for $\theta \geq 0$ in the final step. Also, observe that for any $p \in P$ we have,

\[ |\omega(u, p) - \omega(\hat{u}, p)| \leq \frac{\epsilon}{2\sqrt{d}}. \]  

This follows because,

\[ |\omega(u, p) - \omega(\hat{u}, p)| = |\langle u, p \rangle - \langle \hat{u}, p \rangle| = |\langle u - \hat{u}, p \rangle| \leq \| u - \hat{u} \| \| p \| \leq \frac{\epsilon}{2d} \times \sqrt{d} = \frac{\epsilon}{2\sqrt{d}}, \]

where we have used the Cauchy-Schwarz inequality for the first inequality, the upper bound on $\| u - \hat{u} \|$ derived earlier, along with $\| p \| \leq \sqrt{d}$ for the second inequality.

Let $x_1, x_2, \ldots, x_k \in P$ be the top-$k$ points along direction $u$, i.e., $x_i = \varphi_i(u, P)$. Also, let $y_k$ be the top-$k$ point along direction $\bar{u}$. As remarked we can assume, $\omega(u, x_i) \geq \omega(u, x_k) \geq \omega(u, y_k)$. 

6. $R_N := \{ R_u \mid u \in N \}$
7. $Q' := \text{GREEDY_HS}(P, R_N)$
8. Return $Q := Q' \cup B$
\[ \frac{1}{(1-\epsilon)\sqrt{d}}. \] Now, for any \( i = 1, 2, \ldots, k \) we have that,

\[ \omega(u, x_i) \geq \omega(u, x_i) - \frac{\epsilon}{2\sqrt{d}} \geq \omega(u, x_i) - \frac{(1-\epsilon)\epsilon}{2} \omega(u, x_i) = \omega(u, x_i) \left( 1 - \frac{(1-\epsilon)\epsilon}{2} \right) \geq \omega(u, x_k) \left( 1 - \frac{(1-\epsilon)\epsilon}{2} \right). \]

The first inequality follows by Equation 1 and the second inequality holds since \( \omega(u, x_i) \geq \omega(u, x_k) > \frac{1}{(1-\epsilon)\sqrt{d}} \). This implies that there are \( k \) points whose scores are each at least \( \omega(u, x_k) \left( 1 - \frac{(1-\epsilon)\epsilon}{2} \right) \), and therefore the \( k \)-th best score along \( \bar{u} \), i.e., \( \omega(\bar{u}, y_k) \), is at least \( \omega(u, x_k) \left( 1 - \frac{(1-\epsilon)\epsilon}{2} \right) \). Now, the algorithm guarantees that there is a point \( q \in Q \) such that \( \omega(\bar{u}, q) \geq (1-\epsilon)\omega(\bar{u}, y_k) \). We claim that this \( q \) “settles” direction \( u \) as well, up-to the factor \((1-\epsilon)^2\). Indeed,

\[ \omega(u, q) \geq \omega(u, q) - \frac{\epsilon}{2\sqrt{d}} \geq (1-\epsilon)\omega(u, y_k) - \frac{\epsilon}{2\sqrt{d}} \geq (1-\epsilon) \left( 1 - \frac{(1-\epsilon)\epsilon}{2} \right) \omega(u, x_k) - \frac{\epsilon}{2\sqrt{d}} \geq (1-\epsilon) \left( 1 - \frac{(1-\epsilon)\epsilon}{2} \right) \omega(u, x_k) - \frac{(1-\epsilon)\epsilon}{2} \omega(u, x_k) \geq (1-\epsilon)^2 \omega(u, x_k) \geq (1-2\epsilon)\omega(u, x_k). \]

This completes the proof.

Putting everything together, we obtain the following:

**Theorem 4.5** Let \( P \subset \mathbb{X} \) be a set of \( n \) points in \( \mathbb{R}^d \), \( k \geq 1 \) an integer, and \( \epsilon > 0 \) a parameter. Let \( s_\epsilon \) be the minimum size of a \((k, \epsilon)\)-regret set of \( P \). A subset \( Q \subseteq P \) can be computed in \( O \left( \frac{n}{\epsilon^2 r \log(n) \log \left( \frac{1}{\epsilon} \right)} \right) \) expected time such that \( Q \) is a \((k, 2\epsilon)\)-regret set of \( P \). The size of \( Q \) is \( O(s_\epsilon \log s_\epsilon) \) for \( d \geq 4 \) and \( O(s_\epsilon) \) for \( d \leq 3 \).

**Min-error RMS.** Recall that the min-error problem takes as input a parameter \( r \), and returns a subset \( Q \subseteq P \) of size at most \( r \) such that \( \ell_k(Q) = \ell(r) \). We propose a bicriteria approximation algorithm for the min-error problem by using Algorithm 1 for the min-size version of the problem. Let \( c \) be a sufficiently large constant. We perform a binary search on the values of the regret ratio \( \bar{\epsilon} \) in the range \([0, 1]\). For each value of the \( \bar{\epsilon} \) we run Algorithm 1 with parameter \( \epsilon = \bar{\epsilon} \) and let \( Q_{\bar{\epsilon}} \) be the returned set. If \( |Q_{\bar{\epsilon}}| > cr \log r \) we set \( \bar{\epsilon} \leftarrow 2\bar{\epsilon} \), otherwise \( \bar{\epsilon} \leftarrow \bar{\epsilon} / 2 \) and we continue the binary search with the new parameter. We stop when we find a set \( Q_{\epsilon} \) such that \( |Q_{\epsilon}| < cr \log r \) and \( |Q_{\epsilon/2}| > cr \log r \). The stopping condition
satisfies that $\ell (cr \log r) \leq \hat{c} \leq 2\ell (r)$. The following theorem summarizes the results of the min-error version of the problem.

**Theorem 4.6** Let $P \subset \mathbb{X}$ be a set of $n$ points in $\mathbb{R}^d$, $k \geq 1$ an integer, and $r > 0$ a parameter. Let $\ell (r)$ be the minimum regret ratio of a subset of $P$ of size at most $r$. A subset $Q \subseteq P$ can be computed in $O \left( \min \left\{ \frac{n}{\ell_k(Q)^{\frac{d}{2}}} \log (n) \log \left( \frac{1}{\ell_k(Q)} \right), n^d \right\} \right)$ expected time such that $\ell (cr \log r) \leq \ell_k(Q) \leq 2\ell (r)$ for $d \geq 4$ and $\ell (cr) \leq \ell_k(Q) \leq 2\ell (r)$ for $d \leq 3$ for a sufficiently large constant $c$. The size of $Q$ is $O(r \log r)$ for $d \geq 4$ and $O(r)$ for $d \leq 3$.

**Remarks.**

(i) For $k = 1$ the optimum solution of the RMS problem will always be a subset of the skyline of $P$. Hence, to reduce the running time we can only run our algorithms on skyline points. We can show that we still get the same approximation factors.

(ii) Instead of choosing $O\left( \frac{1}{\epsilon^{d-1}} \right)$ directions in one step and find a Hitting Set, we can sample in stages and maintain a subset $Q$ until $\ell_k(Q) \leq \epsilon$.

5 Experiments

We have implemented our algorithms as well as the current state of the art, namely, the greedy algorithms described in [29][10], and experimentally evaluated their relative performance.

**Algorithms.** In particular, the four algorithms we evaluate are the following:

- **RRS** is the Randomized Regret Set algorithm, based on coresets, described in Section 3. In our implementation, rather than choosing $O\left( \frac{1}{\epsilon^{d-1}r^{\frac{d}{2}}} \right)$ random preferences all at once, we choose them in stages and maintain a subset $Q$ until $\ell_k(Q) \leq \epsilon$. 

Figure 5. log10-scale running time.

Figure 6. Regret ratio and running time of AntiCor, $\sigma = 0.01$, $k = 10$. 

**HS** is the Hitting Set algorithm presented in Section 4 and our implementation incorporates the remarks made at the end of Section 4.

**NSLLX** is the greedy algorithm for 1-RMS problem described in [29], which iteratively finds the preference \( u \) with the maximum regret using an LP algorithm and adds \( q_1(u, P) \) to the regret set. We use Gurobi software [15] to solve the LP problems efficiently. We remark that this algorithm, as a preprocessing, removes all data points that are not on the skyline.

**CTVW** is the extension of the NSLLX algorithm for \( k > 1 \), proposed by [10], and it is the state of the art for the \( k \)-RMS problem. In [10] they discard all the points not on the skyline as preprocessing to run the experiments. The CTVW algorithm solves many (in the worst-case, \( \Omega(n) \)) instances of large LP programs to add the next point to the regret set. The number of LP programs is controlled by a parameter \( T \)—a larger \( T \) increases the probability of adding a good point to the regret set, but also leads to a slower algorithm. In the original paper, the authors suggested a value of \( T \) that is exponential in \( k \); for instance, \( T \geq 2.4 \cdot 10^7 \) for \( k = 10 \), which is clearly not practical. In practice, Chester et al. [10] used \( T = 54 \) for \( k = 4 \), which is also the value we adopted in our experiments for comparison. Indeed, using \( T > 54 \) increases the running time but does not lead to significantly better regret sets.

The algorithms are implemented in C++ and we run on a 64-bit machine with four 3600 MHz cores and 16GB of RAM with Ubuntu 14.04.

In evaluating the quality \( \ell_k(Q) \) of a regret set \( Q \subseteq P \), we compute the regret for a large set of random preferences (for example for \( d = 3 \) we take 20000 preferences), and use the maximum value found as our estimate. In fact, this approach gives us the distribution of the regret over the entire set of preference vectors.

**Datasets.** We use the following datasets in our experiments, which include both synthetic and real-world.

**BB** is the basketball dataset[^4] that has been widely used for testing algorithms for skyline computation, top-\( k \) queries, and the \( k \)-RMS problem problem, [10, 19, 22, 23, 37]. Each point in this dataset represents a basketball player and its coordinates contain five statistics (points, rebounds, blocks, assists, fouls) of the player.

**ElNino** is the ElNino dataset[^5] containing oceanographic data such as wind speed, water temperature, surface temperature etc. measured by buoys stationed in the Pacific ocean, and also used in [10]. This dataset has some missing values, which we have filled in with the minimum value of a coordinate for the point. If some values are negative (where it does not make sense to have negative values) they are replaced by the absolute value.

**Colors** is a data set containing the mean, standard deviation, and skewness of each \( H, S, \) and \( V \) in the HSV color space of a color image[^6]. This set is also a popular one for evaluating skylines and regret sets (see [4, 29]).

**AntiCor** is a synthetic set of points with *anti-correlated* coordinates. Specifically, let \( h \) be the hyperplane with normal \( n = (1, \ldots, 1) \), and at distance 0.5 from the origin. To generate a point \( p \), we choose a random point \( \bar{p} \) on \( h \cap X \), a random number \( t \sim N(0, \sigma^2) \), for a

[^4]: databasebasketball.com
[^5]: archive.ics.uci.edu/ml/datasets/El+Nino
[^6]: www.ics.uci.edu/ mlearn/MLRepository.html
Table 1. Summary of datasets used in experiments.

| ID   | Description                  | d | n             | Skyline |
|------|------------------------------|---|---------------|---------|
| BB   | Basketball                   | 5 | 21961         | 200     |
| ElNino | Oceanographic               | 5 | 178080        | 1183    |
| Colors | Colors                     | 9 | 68040         | 674     |
| AntiCor | Anti-correlated points     | 4 | 10000         | 657     |
| Sphere | Points on unit sphere     | 4 | 15000         | 15000   |
| SkyPoints | Many points close to skyline | 3 | 500           | 100     |

Figure 7. Maximum regret ratio for \( k = 1 \).

small standard deviation \( \sigma \), and \( p = \bar{p} + tn \). If \( p \in \mathcal{X} \) we keep it, otherwise discard \( p \). By design, many points lie on the skyline and the top-\( k \) elements can differ significantly even for nearby preferences. This data set is also widely used for testing top-\( k \) queries or the skyline computation (see \cite{5, 29, 37, 27}). For our experiments we set \( \sigma = 0.1 \) and generate 10000 points.

**Sphere** is a set of points uniformly distributed on the unit sphere inside \( \mathcal{X} \), in which clearly all the points lie on the skyline. We generate the Sphere dataset with 15000 points for \( d = 4 \) (all points lie on the skyline).

**SkyPoints** is a modification of the Sphere data set. We choose a small fraction of points from the Sphere data set and add, say, 20 points that lie very close to \( p \) but are dominated by \( p \). For larger value of \( k \), say \( k > 5 \), considering only the skyline points is hard to decide which point is going to decrease the maximum regret ratio in the original set. We generate SkyPoints data set for \( d = 3 \), 500 points; with 100 points on the skyline.

In evaluating the performance of algorithms, we focus on two main criteria, the runtime and the regret ratio, but also consider a number of other factors that influence their performance such as the value of \( k \), the size of the skyline etc.

RRS and HS are both randomized algorithms so we report the average size of the regret sets and the average running time computed over 5 runs. For \( k = 1 \), we use the NSLLX algorithm, and for \( k = 10 \), we use its extension, the CTVW algorithm. In some plots there are missing values for the CTVW algorithm, because we stopped the execution after running it on a data set for 2 days.

Figure 8. Maximum regret ratio for \( k = 10 \).
Running time. We begin with the runtime efficiency of the four algorithms, which is measured in the number of seconds taken by each to find a regret set, given a target regret ratio. Figure 3 shows the running times of NSLLX, HS, and RRS for \( k = 1 \). The algorithm RRS is the fastest. For some instances, the running time of HS and RRS are close but in some other instances HS is up to three times slower. The NSLLX algorithm is the slowest, especially for smaller values of the regret ratio. The relative advantage of our algorithms is quite significant for datasets that have large skylines, such as AntiCor and Sphere. Even for \( k = 1 \), NSLLX is 7 times slower than HS on AntiCor data set and 480 times slower on Sphere data set, for regret ratio \( \leq 0.01 \).

The speed advantage of RRS and HS algorithms over CTVW becomes much more pronounced for \( k = 10 \), as shown in Figure 4. Recall that CTVW discards all points that are not on the skyline. The running time is significantly larger if one runs this algorithm on the entire point set or when the skyline is large. For example, for the AntiCor and Sphere data sets, which have large size skylines, the CTVW algorithm is several orders of magnitude slower than ours. For example, for the AntiCor and Sphere data sets, which have large size skylines, the CTVW algorithm is several orders of magnitude slower than ours. If we set the parameter \( \sigma = 0.01 \) for AntiCor data set, and generate 10000 points (the skyline has 8070 points in this case) the running time of CTVW is much higher as can be seen in Figure 6b.

Because of the high running time of NSLLX and CTVW algorithms, in Figure 5, we show the running time in the log scale with base 10.

Regret ratio. We now compare the quality of the regret sets (size) computed by the four algorithms. Figures 7 and 8 show the results for \( k = 1 \) and for \( k = 10 \), respectively. The experiments show that in general the HS algorithm finds regret sets comparable in size to NSLLX and CTVW. This is also the case for AntiCor data set if we set \( \sigma = 0.01 \) as can be seen in Figure 6a. The RRS algorithm tends to find the largest regret set among the four algorithms, but it does have the advantage of dynamic updates: that is, RRS can maintain a regret set under insertion/deletion of points. However, since the other algorithms do not allow efficient updates, we do not include experiments on dynamic updates.

The sphere data set is the worst-case example for regret sets since every point has the highest score for some direction. As such, the size of the regret set is much larger than for the other data sets. HS and RRS algorithm rely on random sampling on preference vectors instead of choosing vectors adaptively to minimize the maximum regret, it is not surprising that for Sphere data sets CTVW does 1.5-3 times better than the HS algorithm. Nevertheless,
as we will see below the regret of HS in 95% directions is close to that of CTVW.

**Regret distribution.** The regret ratio only measures the largest relative regret over all preference vectors. A more informative measure could be to look at the entire distribution of the regret over all preference vectors. On all three real datasets, we found that 95% of the directions had 0-regret ratio for all four algorithms. Therefore, we only show the results for the two synthetic main data sets, namely, Sphere and AntiCor. See Figure 9. In this experiment, we fixed the regret set size to 20 for the Sphere dataset and 10 for the AntiCor dataset. We observe that the differences in the regret ratios in 95% of the directions are much smaller than the differences in the maximum regret ratios. For example, the difference of the maximum regret ratio between RRS and NSLLX in Sphere data set is 0.048, while the difference in the 95% (85%) of the directions is 0.019 (0.0096). Similarly, for AntiCor data set the difference in the maximum regret ratio is 0.018 but the difference in the 95% of the directions is 0.0021.

**Impact of larger $k$.** We remarked in the introduction that the size of $(k, \epsilon)$-regret set can be smaller for some datasets than their $(1, \epsilon)$-regret set, for $k > 1$. We ran experiments to confirm this phenomenon, and the results are shown in Figure 10. As Figure 10a shows, the size of 1-regret set is 3.5 times larger than 10-regret sets for some values of the regret ratio. Figure 10b shows how the size of the regret set computed by the HS algorithm decreases with $k$, for a fixed value of the regret ratio 0.01.

**Skyline effect.** In order to improve its running time, the algorithm CTVW [10] removes all the non-skyline points, as a preprocessing step, before computing the regret set. While expedient, this strategy also risks losing good candidate points, and as a result may lead to worse regret set. In this experiment, we used the Skypoint dataset to explore this cost/benefit tradeoff. In particular, the modified version of CTVW that does not remove non-skyline points is called CTVW*.

The results are shown in Figure 11a which confirm that removal of non-skyline points can

![Figure 10. Regret ratio and size of the regret set as a function of $k$ for Color data set.](image1)

(a) Regret ratio for $k = 1, 10$.  
(b) Size of the $(k, 0.01)$-regret set as a function of $k$.

![Figure 11. Regret ratio and running time of SkyPoints.](image2)

(a) Regret ratio  
(b) Running time
cause significant increase in the size of the regret set, for a given target regret ratio. The
experiment shows that the regret set computed by CTVW is about 3 times larger than the
one computed by either HS or CTVW*. (In this experiment, the regret size differences are
most pronounced for small values of regret ratio. When large values of regret ratio are
acceptable, the loss of good candidate points is no longer critical.) Of course, while CTVW*
finds nearly as good a regret set as HS, its running time is much worse than that of HS, or
CTVW, because of this change, as shown in Figure 11b.

6 Related Work

The work on regret minimization was inspired by preference top-k and skyline queries.
Both of these research topics try to help a user find the “best objects” from a database.
Top-k queries assign scores to objects by some method, and return the objects with the
topmost k scores while the skyline query finds the objects such that no other object can
be strictly better. Efficiently answering top-k queries has seen a long line of work, see e.g.
[12, 14, 16, 17, 24, 26, 33, 34, 38, 39, 41] and the survey [18]. In earlier work, the ranking
of points was done by weight, i.e., ranking criterion was fixed. Recent work has considered
the specification of the ranking as part of the query. Typically, this is specified as a preference
vector u and the ranking of the points is by linear projection on u see e.g. [12, 17, 39].

Another ranking criterion is based on the distance from a given query point in a metric
space i.e., the top-k query is a k-nearest neighbor query [35].

In general, preference top-k queries are hard, and this has led to approximate query an-
swering [9, 39, 40]. Motivated by the need of answering preference top-k queries, Nanongkai et.
al. [29] introduced the notion of a 1-regret minimizing set (RMS) query. Their definition
attempted to combine preference top-k queries and the concept of skylines. They gave
upper and lower bounds on the regret ratio if the size of the returned set is fixed to r.

Moreover, they proposed an algorithm to compute a 1-regret set of size r with regret ratio
\( O\left(\frac{d-1}{[r-d+1]^{1/(d-1/d-1)}}\right) \), as well as a greedy heuristic that works well in practice.

Chester et. al. [10] generalized the definition of 1-RMS to the k-RMS for any \( k \geq 1 \). They
showed that the k-RMS problem is NP-hard when the dimension \( d \) is also an input to the
problem, and they provided an exact polynomial algorithm for \( d = 2 \). There has been more
work on the 1-RMS problem see [7, 28, 32], including a generalization by Faulkner et. al.
[20] that considers non-linear utility functions.

The 1-regret problem can be easily addressed by the notion of \( \epsilon \)-kernel coresets, first
introduced by Agarwal et al. [1]. Later, faster algorithms were proposed to construct a
coreset [8].

The 1-RMS problem is also closely related to the problem of approximating the Pareto
curve (or skyline) of a set of points. Papadamitriou and Yannakakis [30, 31] considered
this problem and defined an approximate Pareto curve as a set of points whose \((1 + \epsilon)\)
scaling dominates every point on the skyline. They showed that there exists such a set of
polynomial size [30, 31]. However, computing such a set of the smallest size is NP-Complete
[21]. See also [36].
7 Conclusion

In this paper, we studied the RMS problem. More specifically we showed that the RMS problem is NP-Complete even in $\mathbb{R}^3$, which is a stronger result than the NP-hardness proof in [10] where the dimension is an input to the problem. Furthermore, we give bicriteria approximation algorithms for the RMS problem with theoretical guarantees, using the idea of coresets and by mapping the problem to the well known hitting set problem. Finally, we run experiments comparing the efficacy and the efficiency of our algorithms with the greedy algorithms presented in [10][29].

There are still some interesting problems for future work. In terms of the complexity, our NP-completeness proof holds for $k > 1$. Is the 1-regret minimization problem NP-Complete in $\mathbb{R}^3$? In terms of the approximation algorithms, is it possible to find algorithms with theoretical guarantees where the running time does not have an exponential dependence on $d$, i.e., terms like $\frac{1}{\epsilon^{\Omega(d)}}$ do not occur? This is important, because in practice, the factor $\frac{1}{\epsilon^{\Omega(d)}}$ can be very large even for moderately small $d$ (say $d > 20$), thus severely limiting the practical utility of these algorithms.

References

[1] P. K. Agarwal, S. Har-Peled, and K. R. Varadarajan. Approximating extent measures of points. *Journal of the ACM (JACM)*, 51(4):606–635, 2004.

[2] P. K. Agarwal, S. Har-Peled, and K. R. Varadarajan. Geometric approximation via coresets. *Combinatorial and computational geometry*, 52:1–30, 2005.

[3] P. K. Agarwal and M. Sharir. Arrangements and their applications. *Handbook of computational geometry*, pages 49–119, 2000.

[4] I. Bartolini, P. Ciaccia, and M. Patella. Efficient sort-based skyline evaluation. *ACM Transactions on Database Systems (TODS)*, 33(4):31, 2008.

[5] S. Bőrzsönyi, D. Kossmann, and K. Stocker. The skyline operator. In *Proc. 17th Int. Conf. Data Eng.*, pages 421–430, 2001.

[6] H. Brönnimann and M. T. Goodrich. Almost optimal set covers in finite vc-dimension. *Discrete & Computational Geometry*, 14(4):463–479, 1995.

[7] I. Catallo, E. Ciceri, P. Fraternali, D. Martinenghi, and M. Tagliasacchi. Top-k diversity queries over bounded regions. *ACM Transactions on Database Systems (TODS)*, 38(2):10, 2013.

[8] T. M. Chan. Faster core-set constructions and data stream algorithms in fixed dimensions. In *Proceedings of the twentieth annual symposium on Computational geometry*, pages 152–159. ACM, 2004.

[9] D. Chen, G.-Z. Sun, and N. Z. Gong. Efficient approximate top-k query algorithm using cube index. In *Asia-Pacific Web Conference*, pages 155–167. Springer, 2011.
[10] S. Chester, A. Thomo, S. Venkatesh, and S. Whitesides. Computing k-regret minimizing sets. *Proceedings of the VLDB Endowment*, 7(5):389–400, 2014.

[11] G. Das and M. T. Goodrich. On the complexity of optimization problems for 3-dimensional convex polyhedra and decision trees. *Computational Geometry*, 8(3):123 – 137, 1997.

[12] G. Das, D. Gunopulos, N. Koudas, and N. Sarkas. Ad-hoc top-k query answering for data streams. In *Proceedings of the 33rd International Conference on Very Large Data Bases, VLDB’07*, pages 183–194, 2007.

[13] Z. Gong, G.-Z. Sun, J. Yuan, and Y. Zhong. Efficient top-k query algorithms using k-skyband partition. In *International Conference on Scalable Information Systems*, pages 288–305. Springer, 2009.

[14] U. Güntzer, W. Balke, and W. Kiessling. Optimizing multi-feature queries for image databases. In *Proceedings of the 26th International Conference on Very Large Data Bases, VLDB’00*, pages 419–428, 2000.

[15] I. Gurobi Optimization. Gurobi optimizer reference manual, 2015.

[16] J.-S. Heo, J. Cho, and K.-Y. Whang. The hybrid-layer index: A synergic approach to answering top-k queries in arbitrary subspaces. In 2010 *IEEE 26th International Conference on Data Engineering (ICDE 2010)*, pages 445–448. IEEE, 2010.

[17] V. Hristidis, N. Koudas, and Y. Papakonstantinou. Prefer: A system for the efficient execution of multi-parametric ranked queries. *SIGMOD Rec.*, 30(2):259–270, May 2001.

[18] I. F. Ilyas, G. Beskales, and M. A. Soliman. A survey of top-k query processing techniques in relational database systems. *ACM Computing Surveys (CSUR)*, 40(4):11, 2008.

[19] S. Jasna and M. J. Pillai. An algorithm for retrieving skyline points based on user specified constraints using the skyline ordering. *International Journal of Computer Applications*, 104(11), 2014.

[20] T. Kessler Faulkner, W. Brackenbury, and A. Lall. k-regret queries with nonlinear utilities. *Proceedings of the VLDB Endowment*, 8(13):2098–2109, 2015.

[21] V. Koltun and C. H. Papadimitriou. Approximately dominating representatives. *Theor. Comput. Sci.*, 371(3):148–154, Feb. 2007.

[22] R. Kulkarni and B. Momin. Skyline computation for frequent queries in update intensive environment. *Journal of King Saud University-Computer and Information Sciences*, 2015.

[23] R. Kulkarni and B. Momin. Parallel skyline computation for frequent queries in distributed environment. In *Computational Techniques in Information and Communication Technologies (ICCTICT), 2016 International Conference on*, pages 374–380. IEEE, 2016.
[24] C. Li, K. K. C.-C. Chang, and I. F. Ilyas. Supporting ad-hoc ranking aggregates. In Proceedings of the 2006 ACM SIGMOD International Conference on Management of Data, SIGMOD ’06, pages 61–72, 2006.

[25] Q. Liu, Y. Gao, G. Chen, Q. Li, and T. Jiang. On efficient reverse k-skyband query processing. In International Conference on Database Systems for Advanced Applications, pages 544–559. Springer, 2012.

[26] A. Marian, N. Bruno, and L. Gravano. Evaluating top-k queries over web-accessible databases. ACM Trans. Database Syst., 29(2):319–362, June 2004.

[27] M. Morse, J. M. Patel, and W. I. Grosky. Efficient continuous skyline computation. Information Sciences, 177(17):3411–3437, 2007.

[28] D. Nanongkai, A. Lall, A. Das Sarma, and K. Makino. Interactive regret minimization. In Proceedings of the 2012 ACM SIGMOD International Conference on Management of Data, pages 109–120. ACM, 2012.

[29] D. Nanongkai, A. D. Sarma, A. Lall, R. J. Lipton, and J. Xu. Regret-minimizing representative databases. Proceedings of the VLDB Endowment, 3(1-2):1114–1124, 2010.

[30] C. H. Papadimitriou and M. Yannakakis. On the approximability of trade-offs and optimal access of web sources. In Proceedings of the 41st Annual Symposium on Foundations of Computer Science, FOCS ’00, pages 86–92, 2000.

[31] C. H. Papadimitriou and M. Yannakakis. Multiobjective query optimization. In Proceedings of the Twentieth ACM SIGMOD-SIGACT-SIGART Symposium on Principles of Database Systems, PODS ’01, pages 52–59, 2001.

[32] P. Peng and R. C.-W. Wong. Geometry approach for k-regret query. In 2014 IEEE 30th International Conference on Data Engineering, pages 772–783. IEEE, 2014.

[33] S. Rahul and Y. Tao. Efficient top-k indexing via general reductions. In Proceedings of the 35th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems, PODS ’16, pages 277–288, 2016.

[34] M. Theobald, G. Weikum, and R. Schenkel. Top-k query evaluation with probabilistic guarantees. In Proceedings of the 300th International Conference on Very Large Data Bases, VLDB ’04, pages 648–659, 2004.

[35] E. Tiakas, G. Valkanas, A. N. Papadopoulos, Y. Manolopoulos, and D. Gunopulos. Processing top-k dominating queries in metric spaces. ACM Trans. Database Syst., 40(4):23:1–23:38, Jan. 2016.

[36] S. Vassilvitskii and M. Yannakakis. Efficiently computing succinct trade-off curves. Theor. Comput. Sci., 348(2):334–356, Dec. 2005.

[37] A. Vlachou, C. Doukeridis, Y. Kotidis, and K. Nørvåg. Reverse top-k queries. In 2010 IEEE 26th International Conference on Data Engineering (ICDE 2010), pages 365–376. IEEE, 2010.
[38] D. Xin, C. Chen, and J. Han. Towards robust indexing for ranked queries. In Proceedings of the 32Nd International Conference on Very Large Data Bases, VLDB ’06, pages 235–246, 2006.

[39] A. Yu, P. K. Agarwal, and J. Yang. Processing a large number of continuous preference top-k queries. In Proceedings of the 2012 ACM SIGMOD International Conference on Management of Data, pages 397–408. ACM, 2012.

[40] A. Yu, P. K. Agarwal, and J. Yang. Top-k preferences in high dimensions. IEEE Trans. Knowl. Data Eng., 28(2):311–325, 2016.

[41] Z. Zhang, S. w. Hwang, K. C.-C. Chang, M. Wang, C. A. Lang, and Y. c. Chang. Boolean + ranking: Querying a database by k-constrained optimization. In Proceedings of the 2006 ACM SIGMOD International Conference on Management of Data, SIGMOD ’06, pages 359–370, 2006.

A Transform Polytope

We will present the transformation as a composition of transformations.

Construction. First, we translate $\Pi$ such that the origin $o$ is inside $\Pi$. Then, we compute the polar dual $\Pi^*$. (The polar dual of a polytope containing the origin $o$ is defined as the intersection of all hyperplanes $\langle x, p \rangle \leq 1$ where $p \in P$, and it can be equivalently defined as the intersection of the dual hyperplanes $\langle x, v \rangle \leq 1$ for all the vertices $v$ of $P$). Let $v$ be a vertex of $\Pi^*$. Translate $\Pi^*$ such that $v$ becomes the origin. Then take a rotation such that polytope $\Pi^*$ does not intersect the negative orthant — i.e., the set of points in $\mathbb{R}^3$ which have all coordinates strictly negative; we can always do it because $\Pi^*$ is convex. Let $u_1, u_2, u_3$ be the three directions emanating from the origin such that the cone defined by them, contains the entire polytope $\Pi^*$. Such directions always exist and can be found in polynomial time. It is known that we can find in polynomial time an affine transformation such that $u_1$ is mapped to the direction $e_1 = (1, 0.01, 0.01)$, $u_2$ to direction $e_2 = (0.01, 1, 0.01)$ and $u_3$ to $e_3 = (0.01, 0.01, 1)$ (we can do it by first transforming $u_1, u_2, u_3$ to the unit axis vectors and then transform them to $e_1, e_2, e_3$). Apply this affine transformation to $\Pi^*$ to get $\hat{\Pi}^*$. Polytope $\hat{\Pi}^*$ lies in the first orthant, except for vertex $v$ which is at the origin. Shift this polytope slightly (such a shift can be easily computed in polynomial time) such that the origin lies in the interior of the polytope and very close to $v$ which now lies in the negative orthant, and all the other vertices are still in the first orthant. Finally we compute the polar dual of $\hat{\Pi}^*$; call this $\hat{\Pi}$. Translate $\hat{\Pi}$ until all vertices have positive coordinates, and let $\Pi'$ denote the new polytope.

Lemma A.1 Polytope $\Pi'$ is combinatorially equivalent to $\Pi$ and satisfies properties (i), (ii).

Proof: We start by mapping property (ii) in the dual space. Consider a polytope $G$ and its dual $G^*$ (where the origin lies inside them). It is well known that any vertex $v$ of $G$ corresponds to a hyperplane $h_v$ in the dual space that defines a facet of $G^*$. An edge between two vertices in $G$ corresponds to an edge between the two corresponding faces
in \( G^* \). Furthermore, if a vertex \( v \) of \( G \) is the top-\( k \) vertex of \( G \) in a direction \( u \), then the corresponding hyperplane \( h_v \) is the \( k \)-th hyperplane (among the \( n \) dual hyperplanes) that is intersected by the ray \( ou \), where \( o \) is the origin. From the above it is straightforward to map property (ii) in the dual space: (ii’) For any edge \( (f_1, f_2) \) where \( f_1, f_2 \) are faces of \( G^* \) there is a direction \( u \in X \) such that the first two hyperplanes that are intersected by the ray \( ou \) are \( h_1, h_2 \), where \( h_1 \) is the hyperplane that contains \( f_1 \) and \( h_2 \) the hyperplane that contains the face \( f_2 \).

We now show how these properties can be guaranteed in \( \hat{\Pi}^* \). Notice that from the construction of \( \hat{\Pi}^* \), the origin lies inside \( \hat{\Pi}^* \) and all faces of \( \hat{\Pi}^* \) have non empty intersection with the positive octant. By convexity, \( \hat{\Pi}^* \) satisfies property (ii’) because for any edge \( e = (f_1, f_2) \) of \( \hat{\Pi}^* \) there is a ray emanating from the origin that first intersects the edge \( e \), and hence the hyperplanes \( h_1, h_2 \) are the first hyperplanes that are intersected by the ray. So, its dual polytope \( \hat{\Pi} \) satisfies property (ii). In addition, \( \Pi^* \) is combinatorially equivalent to \( \Pi \), by duality. Since we apply an affine transformation \( \hat{\Pi}^* \) is also combinatorially equivalent to \( \Pi^* \). Finally, the polytope \( \hat{\Pi} \) is combinatorially equivalent to \( \hat{\Pi}^* \) (its dual). Notice that translation does not change the combinatorial structure of a polytope or the ordering of the points in any direction, so \( \Pi' \) satisfies property (ii), property (i) by definition, and is also combinatorially equivalent to \( \Pi \).