Scalar Kinks

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Abstract

We determine the excitations and $S$ matrix of an integrable isotropic antiferromagnetic quantum spin chain of alternating spin $1/2$ and spin $1$. There are two types of gapless one-particle excitations: the usual spin $1/2$ (“spinor”) kink, and a new spin $0$ (“scalar”) kink. Remarkably, the scalar-spinor scattering is nontrivial, yet the spinor-spinor scattering is the same as if the scalar kinks were absent. Moreover, there is no scalar-scalar scattering.

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1. Introduction

One-dimensional integrable quantum spin chains are among the few many-body quantum systems for which some physical quantities can be calculated exactly. The prototypical example is the antiferromagnetic \( su(2) \)-invariant spin 1/2 Heisenberg chain\(^1\). At zero temperature \( (T) \) and zero magnetic field \( (H) \), the ground state is described by an infinite filled Fermi sea, and the excitations consist of an even number of spin 1/2 quasiparticles, called kinks. The dispersion relation and \( S \) matrix for the kinks have been calculated\(^2\,^3\). Moreover, the thermodynamic properties of the model for low \( T \) and small \( H \) have been obtained\(^4\,^5\,^6\,^7\). It was conjectured in Ref. 7, and subsequently established, that the specific heat \( (C_H) \) has the following property:

\[
\lim_{T \to 0} \lim_{H \to 0} \frac{C_H}{T} = \lim_{H \to 0} \lim_{T \to 0} \frac{C_H}{T}.
\]

Indeed, the LHS can be evaluated by the method of Filyov, \textit{et al.}\(^8\) (see, e.g., Ref. 15), while the RHS can be evaluated by the method of Johnson and McCoy\(^6\).

This model is critical. It can be regarded as a lattice field theory, which in the continuum limit is a conformal field theory\(^9\). The central charge \( (c) \) is proportional\(^10\) to \( \lim_{T \to 0} C_H / T \), and thus has the value \( c = 1 \). The conformal field theory in question has been identified\(^10\) as the level-one \( SU(2) \) WZW model\(^11\). We emphasize that, in contrast to the bootstrap approach\(^12\) for integrable quantum field theories, here the \( S \) matrix of the (gapless) kinks is obtained by starting from the “microscopic” spin-chain Hamiltonian and the corresponding Bethe Ansatz equations.

A major triumph of the quantum inverse scattering method is the construction and solution of integrable isotropic higher-spin generalizations of the Heisenberg chain, to which we shall refer as spin \( s \) chains\(^13\,^14\). For \( s > 1/2 \), \( C_H \) does \textit{not} have the property (1). The LHS (evaluated\(^15\) by the method of Filyov, \textit{et al.}) gives \( c = 3s/(s + 1) \), while the RHS
(evaluated\textsuperscript{17} by the method of Johnson-McCoy) gives $c = 1$. *

Evidently, thermodynamic properties of $s > 1/2$ chains at the point $T = H = 0$ depend on how the point is approached in the $(T, H)$ plane, e.g., $H = 0, T = 0^+$ versus $T = 0, H = 0^+$. Moreover, there is a discrepancy between the results of Takhtajan\textsuperscript{16} (see also Ref. 20) and Reshetikhin\textsuperscript{22} for the two-body $S$ matrix. These facts strongly suggest that there are (at least) two continuous field theories in the $(T, H) = (0, 0)$ limit of the spin $s$ isotropic chain. A field theory with Takhtajan’s $S$ matrix corresponds to the limit $T = 0, H = 0^+$; while a field theory with Reshetikhin’s $S$ matrix corresponds to the limit $H = 0, T = 0^+$. We hope to return to this matter in the future.

The singular behavior at $T = H = 0$ is presumably related to the RSOS structure of the space of states, which appears\textsuperscript{21,22} for $s > 1/2$ in both of the limits discussed above.

* We remark that the central charge can also be determined from finite-size corrections of the ground-state energy. As shown in Ref. 18, an analytic computation of finite-size corrections for the spin $s$ chain which makes use of the string hypothesis gives $c = 1$, which is consistent with the results for $T = 0, H = 0^+$; while the corresponding numerical calculation (which does not make use of the string hypothesis) gives $c = 3s/(s + 1)$, which is consistent with the results for $H = 0, T = 0^+$. For $s = 1$, analytic calculations not using the string hypothesis also give\textsuperscript{19} $c = 3/2$. It is therefore tempting to conjecture that the use of the string hypothesis in the analytic finite-size scaling computation has the implicit assumption $H = 0^+$.

† Takhtajan performs a quantum mechanical (as opposed to thermodynamical) calculation, and thus, he clearly works at $T = 0$. The magnetic field $H$ does not appear explicitly in his calculation. However, his calculation makes essential use of the string hypothesis, which is not strictly correct for $T = H = 0$. (See, e.g., Ref. 23.) Since the string hypothesis is believed to be true for max $(T, H) > 0$, his calculation is valid presumably only for $H = 0^+$. 

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Evidence for such a structure is readily seen by counting states. Indeed, since the kinks have spin 1/2, naively one expects that the number of $n$-kink states is $2^n$. However, the Bethe Ansatz implies that the actual number of such states is larger. For instance, for $s = 1$, there are $2^{3n-2}/2$ such states.

A new type of integrable quantum spin chain has recently been constructed, involving spins of different types. In Ref. 25 we have investigated thermodynamic properties for the particular case of an $su(2)$-invariant chain of alternating spin 1/2 and spin 1. More general cases have also been studied. In this Letter, we perform a direct Bethe-Ansatz calculation of the $S$ matrix for the excitations of the $su(2)$-invariant alternating spin 1/2 and spin 1 chain, at the conformal point.* At first thought, one expects that – as for the $s > 1/2$ chains – there may be more than one possible field theory in the $(T, H) = (0, 0)$ limit; and hence, there may be more than one possible result for the $S$ matrix. However, for this chain, $C_H$ does satisfy the property (1). Indeed, the limits on the RHS and LHS have been computed in Refs. 25 and 26, respectively, and give the same value of the central charge ($c = 2$). This suggests that the $S$ matrix for the particular case of alternating spin 1/2 and spin 1 is unique.

Our results are surprising. In looking for 2-particle excitations, we find, in addition to the expected singlet and triplet states, an additional singlet state. We interpret this to mean that there are two types of one particle excitations: the usual spin 1/2 (“spinor”) kink, and a new spin 0 (“scalar”) kink. The kinks obey certain super-selection rules. The scattering between spinor kinks is the same as for the Heisenberg chain, and there is no scattering between scalar kinks. Nevertheless, there is nontrivial scattering between the spinor kinks and the scalar kinks.

* In Ref. 25, we consider a two-parameter $(\tilde{c}, \tilde{\bar{c}})$ family of Hamiltonians. For simplicity, we restrict our attention here to the case $\tilde{c} = \tilde{\bar{c}} > 0$, for which there is a single speed of sound.
2. Ground state and excitations

We consider a system with a strictly alternating arrangement of $2N$ spins, with spins $1/2$ at even sites and spins $1$ at odd sites. That is, there are $N$ spins $\vec{\sigma}_2, \vec{\sigma}_4, \cdots, \vec{\sigma}_{2N}$ of spin $1/2$ and $N$ spins $\vec{s}_1, \vec{s}_3, \cdots, \vec{s}_{2N-1}$ of spin $1$. The $su(2)$-invariant Hamiltonian $H$ is given by

$$H = -\frac{1}{18} \sum_{n=1}^{N} \left\{ (2\vec{\sigma}_{2n} \cdot \vec{s}_{2n+1} + 1) (2\vec{\sigma}_{2n+2} \cdot \vec{s}_{2n+1} + 3) + (2\vec{\sigma}_{2n} \cdot \vec{s}_{2n-1} + 1) [ (1 + \vec{s}_{2n-1} \cdot \vec{s}_{2n+1}) (2\vec{\sigma}_{2n} \cdot \vec{s}_{2n+1} + 1) + 2] \right\}.$$  (2)

Note that the Hamiltonian contains both nearest and next-to-nearest neighbor interactions. We assume periodic boundary conditions: $\vec{\sigma}_{2n} \equiv \vec{\sigma}_{2n+2N}$ and $\vec{s}_{2n+1} \equiv \vec{s}_{2n+1+2N}$.

The corresponding energy, momentum, and spin eigenvalues are given by\textsuperscript{24}

$$E = \frac{7}{12} N - \sum_{j=1}^{M} \left( \frac{1}{2} \lambda_j^2 + \frac{1}{4} + \frac{1}{\lambda_j^2 + 1} \right),$$  (3)

$$P = \frac{1}{2i} \sum_{j=1}^{M} \log \left( \frac{\lambda_j + \frac{i}{2} \lambda_j + i}{\lambda_j - \frac{i}{2} \lambda_j - i} \right),$$  (4)

$$S^z = \frac{3}{2} N - M,$$  (5)

where the variables $\lambda_j$ satisfy the Bethe Ansatz (BA) equations

$$\left( \frac{\lambda_j + \frac{i}{2} \lambda_j + i}{\lambda_j - \frac{i}{2} \lambda_j - i} \right)^N = -\prod_{k=1}^{M} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i}, \quad j = 1, \cdots, M.$$  (6)

The momentum operator is defined as one-half the log of the two-site shift operator, and hence the factor $1/2$ in Eq. (4). The Bethe Ansatz states are highest weight vectors of $su(2)$ (see, e.g., Ref. 2), and thus have spin quantum numbers $S = S^z \geq 0$.

We assume the string hypothesis, which states that the solutions of the BA equations (6) for $N \to \infty$ are collections of $M_n$ strings of length $n$ of the form (for $M_n > 0$)

$$\lambda_{\alpha}^{(n,j)} = \lambda_{\alpha}^n + i \left( \frac{n+1}{2} - j \right),$$  (7)
where \( j = 1, \ldots, n; \alpha = 1, \ldots, M_n; n = 1, \ldots, \infty \); and the centers \( \lambda_n^\alpha \) are real. The total number of \( \lambda \) variables is \( M = \sum_{n=1}^{\infty} n M_n \). This hypothesis is believed to be true for \( H = 0^+ \). (See, e.g., Ref. 23.) As discussed in the Introduction, we expect that our calculation, which is analogous to the one of Takhtajan\(^{16}\), will lead to the unique \( S \) matrix.

Forming products of the BA equations over the imaginary parts of the strings (following Takahashi\(^{4}\) and Gaudin\(^{5}\)), and then taking the logarithm, we obtain the following equations for the string centers:

\[
h_n(\lambda_n^\alpha) = J_n^\alpha, \quad \alpha = 1, \ldots, M_n, \quad n = 1, \ldots, \infty, \tag{8}
\]

where the functions \( h_n(\lambda) \) are defined by

\[
h_n(\lambda) = \frac{1}{2\pi} \left\{ [Nq_n(\lambda) + \Xi_{n1}(\lambda)] - \sum_{m=1}^{\infty} \sum_{\beta=1}^{M_m} \Xi_{nm}(\lambda - \lambda_{\beta}^m) \right\}, \tag{9}
\]

\( q_n(\lambda) \) are odd monotonic-increasing functions defined by

\[
q_n(\lambda) = \pi + i \log \left( \frac{\lambda + \frac{in}{2}}{\lambda - \frac{in}{2}} \right), \quad -\pi < q_n(\lambda) \leq \pi, \tag{10}
\]

and \( \Xi_{nm}(\lambda) \) are given by

\[
\Xi_{nm}(\lambda) = (1 - \delta_{nm})q_{|n-m|}(\lambda) + 2q_{|n-m|+2}(\lambda) + \cdots + 2q_{n+m-2}(\lambda) + q_{n+m}(\lambda). \tag{11}
\]

Moreover, \( J_n^\alpha \) are integers or half-odd integers which satisfy

\[
|J_n^\alpha| \leq J_{max}^n = N \left( \frac{3}{2} - \frac{1}{2} \delta_{n1} \right) + \frac{1}{2} (M_n - 1) - \sum_{m=1}^{\infty} \min(m, n) M_m. \tag{12}
\]

We obtained the values of \( J_{max}^n \) using the prescription of Faddeev and Takhtajan\(^{2}\). We assume that the numbers \( \{J_n^\alpha\} \) can be regarded as quantum numbers of the model: for every set \( \{J_n^\alpha\} \) in the range (12) (no two of which are identical), there is a unique solution \( \{\lambda_n^\alpha\} \) (no two of which are identical) of Eq. (8).
For the ground state, $M_1 = M_2 = N/2$ and $M_n = 0$ for $n > 2$. Evidently, this spin-singlet state consists of a sea of 1-strings and a sea of 2-strings. The seas are filled (i.e., there are no holes), and hence are described for $N \to \infty$ by root densities $\rho_n(\lambda) = N^{-1}dh_n(\lambda)/d\lambda$, $n = 1, 2$, which can be shown to be given by

$$\rho_1(\lambda) = \rho_2(\lambda) = s(\lambda) \quad (13)$$

where $s(\lambda) = (2 \cosh \pi \lambda)^{-1}$. (See also Ref. 25.)

Excited states of $\nu$ kinks consist of $\nu_1$ holes in the sea of 1-strings, $\nu_2$ holes in the sea of 2-strings, with $\nu = \nu_1 + \nu_2$, and a finite number of strings of length $n \geq 3$. The numbers $\nu_1$ and $\nu_2$ are restricted to be even non-negative integers. The hole rapidities $\tilde{\lambda}_\alpha^n$ are given by

$$h_n(\tilde{\lambda}_\alpha^n) = \tilde{J}_\alpha^n, \quad \alpha = 1, \cdots, \nu_n, \quad n = 1, 2 \quad (14)$$

where $\tilde{J}_\alpha^n$ are integers or half-odd integers which satisfy $|\tilde{J}_\alpha^n| \leq J_{\text{max}}^n$ and which are not in the set $\{J_\alpha^n\}$. Correspondingly, we introduce hole densities

$$\tilde{\rho}_n(\lambda) = \frac{1}{N} \sum_{\alpha=1}^{\nu_n} \delta(\lambda - \tilde{\lambda}_\alpha^n), \quad n = 1, 2 \quad (15)$$

and string densities

$$\rho_n(\lambda) = \frac{1}{N} \sum_{\alpha=1}^{M_n} \delta(\lambda - \lambda_\alpha^n), \quad n \geq 3 \quad (16)$$

The densities $\rho_1$ and $\rho_2$ now satisfy

$$\rho_n(\lambda) + \tilde{\rho}_n(\lambda) = \frac{1}{N} \frac{dh_n(\lambda)}{d\lambda}, \quad n = 1, 2 \quad (17)$$

or equivalently$^{25}$

$$\tilde{\rho}_n + \sum_{m=1}^{\infty} A_{nm} \ast \rho_m = a_n + \sum_{l=1}^{\min(n,2)} a_{n+3-2l}, \quad n = 1, 2 \quad (18)$$

where

$$a_n(\lambda) = \frac{1}{2\pi} \frac{d\eta_n(\lambda)}{d\lambda} = \frac{1}{2\pi} \frac{n}{\lambda^2 + \frac{n^2}{4}} \quad (19)$$
\[ A_{nm}(\lambda) = \delta_{nm} \delta(\lambda) + \frac{1}{2\pi} d\Xi_{nm}(\lambda), \]  
(20)

and * denotes the convolution \((f \ast g)(\lambda) = \int_{-\infty}^{\infty} d\lambda' \ f(\lambda - \lambda')g(\lambda')\). The linear integral equations (18) can be solved with the help of Fourier transforms. The solution is

\[ \rho_1 = s - \tilde{\rho}_1 + s \ast \tilde{\rho}_2, \]
\[ \rho_2 = s + s \ast \tilde{\rho}_1 + \sigma \ast \tilde{\rho}_2 - \sum_{m=3}^{\infty} a_{m-2} \ast \rho_m, \]
(21)

where

\[ \sigma(\lambda) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega \lambda}}{1 + e^{-|\omega|}} d\omega, \]
(22)

and the hole \((\tilde{\rho}_n)\) and string \((\rho_n, n \geq 3)\) densities are given by Eqs. (15) and (16), respectively. The string positions \(\lambda^n_n, n \geq 3\), are still to be determined in terms of the hole positions \(\tilde{\lambda}^{1}_{\alpha}, \tilde{\lambda}^{2}_{\alpha}\), using Eq. (8) for \(n \geq 3\).

The energy and momentum of the excitations follows from Eqs. (3) and (4), respectively:

\[
E = N \left\{ \frac{7}{12} - \pi \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \left[ a_n + \sum_{l=1}^{\min(n,2)} a_{n+3-2l} \right] \rho_n \ d\lambda \right\}
= E_0 + \sum_{n=1}^{2} \sum_{\alpha=1}^{\nu_n} \varepsilon(\tilde{\lambda}^{n}_{\alpha}),
\]
(23)

and

\[
P = -\frac{N}{2} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \left[ (q_n - \pi) + \sum_{l=1}^{\min(n,2)} (q_{n+3-2l} - \pi) \right] \rho_n \ d\lambda
= P_0 + \sum_{n=1}^{2} \sum_{\alpha=1}^{\nu_n} p(\tilde{\lambda}^{n}_{\alpha}),
\]
(24)

where \(E_0\) and \(P_0\) are the energy and momentum of the ground state, respectively, and \(\varepsilon(\lambda)\) and \(p(\lambda)\) are the energy and momentum of a kink with rapidity \(\lambda\),

\[
\varepsilon(\lambda) = \pi s(\lambda) = \frac{\pi}{2 \text{ch} \pi \lambda}, \quad p(\lambda) = -\frac{\pi}{4} + \frac{1}{2} \tan^{-1} \text{sh} \pi \lambda.
\]
(25)
We observe that
\[
\frac{dp}{d\lambda} = \varepsilon(\lambda),
\] (26)
and that the kink dispersion relation is
\[
\varepsilon = -\frac{\pi}{2} \sin 2p,
\] (27)
implying that the (unique) speed of sound is \(v_s = \pi\).

A useful formula for the spin is
\[
S^z = \frac{\nu_2}{2} - \sum_{n=3}^{\infty} (n-2)M_n,
\] (28)
which follows from (5), and from the fact that
\[
M_1 = \frac{N}{2} - \nu_1 + \frac{\nu_2}{2}, \quad M_2 = \frac{N}{2} + \frac{\nu_1 - \nu_2}{2} - \sum_{n=3}^{\infty} M_n.
\] (29)
Notice that the expression (28) for \(S^z\) is independent of \(\nu_1\), which implies that the kinks corresponding to holes in the sea of 1-strings have spin 0. We shall discuss this point further below.

We particularize now to the case of two-kink excitations \((\nu = 2)\). There are three possibilities:
(i) *triplet* \((S^z = 1)\)

This state is characterized by \(M_1 = \frac{N}{2} + 1; M_2 = \frac{N}{2} - 1; M_n = 0\) for \(n > 2\), which corresponds to two holes in the sea of 2-strings \((\nu_1 = 0, \nu_2 = 2)\). The root densities are given by (see Eq. (21))
\[
\rho_1(\lambda) = s(\lambda) + \frac{1}{N} \sum_{\alpha=1}^{2} s(\lambda - \tilde{\lambda}_\alpha^2),
\]
\[
\rho_2(\lambda) = s(\lambda) + \frac{1}{N} \sum_{\alpha=1}^{2} \sigma(\lambda - \tilde{\lambda}_\alpha^2).
\] (30)
(ii) *singlet* \((S^z = 0)\)
This state is characterized by \( M_1 = \frac{N}{2} + 1; M_2 = \frac{N}{2} - 2; M_3 = 1; M_n = 0 \) for \( n > 3 \), which corresponds to two holes in the sea of 2-strings \((\nu_1 = 0, \nu_2 = 2)\) and one 3-string. The root densities are given by

\[
\rho_1(\lambda) = s(\lambda) + \frac{1}{N} \sum_{\alpha=1}^{2} s(\lambda - \tilde{\lambda}_1^2),
\]
\[
\rho_2(\lambda) = s(\lambda) + \frac{1}{N} \left[ \sum_{\alpha=1}^{2} \sigma(\lambda - \tilde{\lambda}_1^2) - a_1(\lambda - \lambda_1^3) \right], \tag{31}
\]

where \( \lambda_1^3 = (\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2)/2 \).

(iii) singlet \((S^z = 0)\)

This state is characterized by \( M_1 = \frac{N}{2} - 2; M_2 = \frac{N}{2} + 1; M_n = 0 \) for \( n > 2 \), which corresponds to two holes in the sea of 1-strings \((\nu_1 = 2, \nu_2 = 0)\). The root densities are given by

\[
\rho_1(\lambda) = s(\lambda) - \frac{1}{N} \sum_{\alpha=1}^{2} \delta(\lambda - \tilde{\lambda}_1^1),
\]
\[
\rho_2(\lambda) = s(\lambda) + \frac{1}{N} \sum_{\alpha=1}^{2} s(\lambda - \tilde{\lambda}_1^1). \tag{32}
\]

All together, there are 5 two-kink states. Taking into account degeneracies from the possible values of the string positions, we find that the number of \( \nu \)-kink states is given by

\[
\sum_{m=0}^{\nu/2} 2^{2m}. \tag{33}
\]

In comparison, for the spin 1/2 Heisenberg chain, there are only 4 two-kink states (triplet plus singlet); and the number of \( \nu \)-kink states is only \( 2^\nu \).

Our interpretation of these results is that there are two types of kinks: the usual spin 1/2 kink and a new spin 0 kink. We refer to these as spinor kinks and scalar kinks, respectively. The number of scalar kinks is \( \nu_1 \), and the number of spinor kinks is \( \nu_2 \). As already remarked, multi-kink states are subject to the super-selection rule that both \( \nu_1 \)
and $\nu_2$ must be even. Evidently, configurations of kinks which differ only by interchanges of scalars with spinors are not distinct. Hence, for given values of $\nu_1$ and $\nu_2$, the number of states is $2^{\nu_2}$. With the above assignment of spin quantum numbers to the kinks, and with the super-selection rule, we reproduce the counting (33) of $\nu$-kink states. There is no evidence of RSOS structure.

3. $S$ matrix

We compute the $S$ matrix following the method of Korepin$^3$ in a formulation given by Andrei and Destri$^{20}$. The basic idea can be understood by considering a particle moving in one dimension with momentum $p$ in an even finite-range potential. The $S$ matrix is diagonal in the basis of parity eigenstates, with matrix elements $S = e^{i\phi}$. Putting the system in a periodic box of length $L$ leads to the quantization condition

$$e^{ipL}S = 1.$$ (34)

Therefore, the momentum of the interacting particle is related to the phase shift $\phi$ by

$$p = \frac{2\pi}{L} m - \frac{1}{L} \phi,$$ (35)

where $m$ is an integer. With these conventions, the phase shift for an attractive potential is positive.

The same formula holds for the scattering of kinks, with $L = 2N$ being the number of spins in the chain. In order to obtain the phase shifts, we must identify in the expression for the momentum $p(\tilde{\lambda}_n^\alpha)$ of a kink with rapidity $\tilde{\lambda}_n^\alpha$ the free term $2\pi m/L = \pi m/N$. We can accomplish this with the help of the counting function $h_n(\lambda)$, a convenient expression for which is obtained by integrating Eq. (17)

$$\frac{1}{N} h_n(\lambda) = \frac{1}{2} \int_{-\infty}^{\infty} \epsilon(\lambda - \lambda') \left[ \rho_n(\lambda') + \tilde{\rho}_n(\lambda') \right] d\lambda' + c_n, \quad n = 1, 2,$$ (36)
where \( \epsilon(x) = \text{sign } x = x/|x| \). We henceforth drop the integration constants \( c_n \), which contribute only additive constants to the phase shifts. Evaluating the counting function at \( \tilde{\lambda}_n^\alpha \) gives the integer or half-odd integer \( \tilde{J}_n^\alpha \), as we see from Eq. (14). Therefore,

\[
\frac{1}{N} \tilde{J}_n = \frac{1}{2} \int_{-\infty}^{\infty} \epsilon(\tilde{\lambda}_n^\alpha - \lambda) \ s(\lambda) \ d\lambda + \frac{1}{2} \int_{-\infty}^{\infty} \epsilon(\tilde{\lambda}_n^\alpha - \lambda) \ r_n(\lambda) \ d\lambda, \quad n = 1, 2, \quad (37)
\]

where the quantities \( r_n(\lambda) \) are defined by

\[
\rho_n(\lambda) + \tilde{\rho}_n(\lambda) = s(\lambda) + r_n(\lambda), \quad n = 1, 2. \quad (38)
\]

There remains to relate \( \tilde{J}_n^\alpha \) to the momentum \( p(\tilde{\lambda}_n^\alpha) \) of a kink. From Eqs. (25) and (26), we have that

\[
p(\tilde{\lambda}_n^\alpha) = \frac{\pi}{2} \int_{-\infty}^{\infty} \epsilon(\tilde{\lambda}_n^\alpha - \lambda) \ s(\lambda) \ d\lambda - \frac{\pi}{4}. \quad (39)
\]

We conclude from Eqs. (35), (37), (39) that the phase shifts are given (up to additive constants) by

\[
\phi(\tilde{\lambda}_n^\alpha) = \pi N \int_{-\infty}^{\infty} \epsilon(\tilde{\lambda}_n^\alpha - \lambda) \ r_n(\lambda) \ d\lambda, \quad n = 1, 2. \quad (40)
\]

For two-kink states, this expression is a function of \( \tilde{\lambda}_{12}^n \equiv \tilde{\lambda}_1^n - \tilde{\lambda}_2^n \), which we shall take to be positive. In order to evaluate the above integral, we follow Faddeev and Takhtajan, and first compute its derivative:

\[
\frac{d\phi(\tilde{\lambda}_{12}^n)}{d\tilde{\lambda}_{12}^n} = 2\pi N r_n(\tilde{\lambda}_1^n), \quad (41)
\]

This procedure introduces further integration constants in the expression for the phase shift, which could in principle be determined by directly evaluating (40). The evaluation of (41) is accomplished with the help of Fourier transforms of \( s(\lambda) \) and \( a_n(\lambda) \), the only nontrivial integral which appears being

\[
\int_0^\infty \frac{\cos \lambda \omega}{e^{\omega} + 1} \ d\omega = \frac{1}{4} \left[ \psi\left(\frac{i\lambda}{2}\right) + \psi\left(-\frac{i\lambda}{2}\right) - \psi\left(\frac{1}{2} + \frac{i\lambda}{2}\right) - \psi\left(\frac{1}{2} - \frac{i\lambda}{2}\right) \right], \quad (42)
\]
where $\psi(z) = d \log \Gamma(z)/dz$.

For the triplet state (30), we obtain the $S$ matrix

$$S_t(\tilde{\lambda}_{12}^2) = \frac{\Gamma(i\tilde{\lambda}_{12}^2)\Gamma(1/2 - i\tilde{\lambda}_{12}^2)}{\Gamma(-i\tilde{\lambda}_{12}^2)\Gamma(1/2 + i\tilde{\lambda}_{12}^2)}; \tag{43}$$

for the first singlet state (31), we obtain

$$S_{s1}(\tilde{\lambda}_{12}^2) = \frac{\tilde{\lambda}_{12}^2 + i}{\tilde{\lambda}_{12}^2 - i} S_t(\tilde{\lambda}_{12}^2); \tag{44}$$

and for the second singlet state (32), we obtain

$$S_{s2}(\tilde{\lambda}_{12}^2) = 1; \tag{45}$$

up to constant (rapidity-independent) phase factors.

The triplet and first singlet $S$ matrices (43), (44) coincide with those for the Heisenberg chain\(^2 \dagger\). That is, the spinor kinks of the alternating chain have the same scattering as the kinks of the spin 1/2 chain. The second singlet $S$ matrix (45) is trivial. Evidently, there is no scattering between scalar kinks.

There remains to determine whether there is scattering between the spinor kinks and the scalar kinks. Since a two-kink state consisting of a spinor kink and a scalar kink is not possible, one must consider states with at least four kinks in order to answer this question. We have analyzed the four-kink state characterized by $M_1 = N/2 - 1$; $M_2 = N/2$; and $M_n = 0$ for $n > 2$, for which $\nu_1 = \nu_2 = 2^*$. We find that the $S$ matrix for scalar - spinor scattering

\[\dagger\] Actually, the phase shifts of Faddeev-Takhtajan have the opposite sign. One can recover the Faddeev-Takhtajan results by invoking the prescription\(^3\) that the phase shift for the scattering of a hole with another hole (or with a particle) acquires an additional minus sign. Correspondingly, the physical strip for a hole becomes $-1 < \text{Im} \tilde{\lambda} \leq 0$.

\[^*\] Factorization implies that an analysis of other states with $\nu_1, \nu_2 \geq 2$ should give the same result.
is given by
\[ S(\tilde{\lambda}) = i \coth \frac{\pi}{2} \left( \tilde{\lambda} + \frac{i}{2} \right), \] (46)
where \( \tilde{\lambda} = \tilde{\lambda}_1^1 - \tilde{\lambda}_2^2 \). The pole at \( \tilde{\lambda} = -i/2 \) does not lie on the physical sheet \( 0 \leq \text{Im} \tilde{\lambda} < 1 \), and so does not correspond to a bound state. Evidently, there is nontrivial scattering between a scalar and a spinor.

The \( S \) matrix (43) - (46) is crossing-symmetric and unitary.

4. Discussion

We have emphasized that the property (1) is satisfied for the alternating spin 1/2 and spin 1 chain given by Eq. (2). There is no ambiguity (at least for the central charge) at the point \( T = H = 0 \). Therefore, it is plausible that results obtained at this point (even using the string hypothesis) should be unique. The property (1) is also satisfied by the \( su(n) \) chain in the fundamental representation (see Ref. 30). It would be interesting to find other models for which this property is satisfied.

We have seen that the chain (2) has gapless kinks (“spin waves”) which have spin 0, as well as spin 1/2. To the best of our knowledge, this is the first example of a magnetic chain with scalar kinks. Remarkably, the scalar-spinor scattering is nontrivial, yet the spinor-spinor scattering is the same as if the scalar kinks were absent. Moreover, there is no scalar-scalar scattering.

It would be interesting to determine the conformal field theory (CFT) to which this model corresponds. Certainly, the CFT must be \( su(2) \)-invariant, and have central charge \( c = 2 \). One can rule out the level-four \( SU(2) \) WZW model, since this CFT corresponds to the homogeneous \( s = 2 \) chain, whose excitations and \( S \) matrix\textsuperscript{16,22} are very different from what we have found here. Another possibility\textsuperscript{26,28} is a set of two level-one \( SU(2) \) WZW models. However, this CFT also does not seem to be compatible with our set of excitations.
The $S$ matrix (43) - (46) can also be obtained by starting with an anisotropic chain with anisotropy parameter $\eta$ and lattice spacing $a$, and then taking the continuum limit $a \to 0$ and the isotropic limit $\eta \to 0$ while keeping a mass parameter $m^2 \propto a^{-2} \exp(-\pi^2/\eta)$ fixed. (See Ref. 31.) Thus, this $S$ matrix should also be described by a massive $su(2)$-invariant integrable quantum field theory, which in the ultraviolet limit $m \to 0$ reduces to the above-mentioned CFT.

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