A Symmetric Lambda-Calculus Corresponding to the Negation-Free Bilateral Natural Deduction

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Abstract
Filinski constructed a symmetric lambda-calculus consisting of expressions and continuations which are symmetric, and functions which have duality. In his calculus, functions can be encoded to expressions and continuations using primitive operators. That is, the duality of functions is not derived in the calculus but adopted as a principle of the calculus. In this paper, we propose a simple symmetric lambda-calculus corresponding to the negation-free natural deduction based bilateralism in proof-theoretic semantics. In our calculus, continuation types are represented as not negations of formulae but formulae with negative polarity. Function types are represented as the implication and but-not connectives in intuitionistic and paraconsistent logics, respectively. Our calculus is not only simple but also powerful as it includes a call-value calculus corresponding to the call-by-value dual calculus invented by Wadler. We show that mutual transformations between expressions and continuations are definable in our calculus to justify the duality of functions. We also show that every typable function has dual types. Thus, the duality of function is derived from bilateralism.

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1 Introduction

A function of the type $A_0 \rightarrow A_1$ from expressions of the type $A_0$ to expressions of the type $A_1$ can be regarded as a function from continuations of the type $A_1$ to continuations of the type $A_0$. This property of functions is called duality.

Filinski constructed a symmetric $\lambda$-calculus based on the duality of functions \cite{Filinski1994}. His calculus consists of expressions $E$, continuations $C$, and functions $F$. Expressions and continuations are symmetric. Functions are neutral, that is, functions can be encoded to expressions and continuations like $\lceil F \rceil$ and $\lfloor F \rfloor$, respectively. Expressions and continuations can be decoded to functions by operators $\bar{F}$ and $\bar{C}$. The operators $\lceil, \rceil, \lfloor, \rfloor, \tau,$ and $\gamma$ are primitive since the duality of functions is adopted as a principle of his calculus.

The duality allows the call-with-current-continuation operator (call/cc) to have a type $((A_0 \rightarrow A_1) \rightarrow A_0) \rightarrow A_0$. In a traditional interpretation of function types, the type means that call/cc takes an expression of the type $(A_0 \rightarrow A_1) \rightarrow A_0$ and returns an expression of the type $A_0$. However, in the symmetric $\lambda$-calculus, call/cc takes a continuation of the type $A_0$ and becomes a function of the type $(A_0 \rightarrow A_1) \rightarrow A_0$, which takes an expression of the type $A_0 \rightarrow A_1$ and returns an expression of the type $A_0$. 


The duality of functions seems to be one of the most significant reasons that it is possible for the symmetric λ-calculus to have the provability of classical logic, because the type
$$(A_0 \rightarrow A_1) \rightarrow A_0$$
corresponds to the Peirce formula on the formulae-as-types notion [1] [22], which strengthens the λ-calculus corresponding to the minimal logic having the provability of classical logic [21].

In this paper, we justify the duality of functions in the symmetric λ-calculus using bilateralism in proof-theoretic semantics. In proof-theoretic semantics there exists an idea that meanings of logical connectives are given by the contexts in which the logical connectives occur. In this idea, a meaning of a logical connective is considered to be defined by its introduction rule of a natural deduction and its elimination rule is naturally determined to be in harmony with the introduction rule.

Rumfitt suggested that the original natural deduction invented by Gentzen [15] [16] is not harmonious, and constructed a natural deduction based on bilateralism [32]. Within the notion of bilateralism, provability is not defined for a plain formula $A$ but a formula with polarity $+A$ and $-A$. Provability of $+A$ means that $A$ is accepted, and provability of $-A$ means that $A$ is rejected. The traditional formulation for which provability of $A$ means that $A$ is accepted is based on the notion of unilateralism rather than bilateralism. Bilateralism does not permit anything neutral and forces everything to have either positive or negative polarity. Rumfitt showed that a natural deduction of classical logic that is constructed on unilateralism can be reconstructed on bilateralism.

In this paper, we construct a symmetric λ-calculus corresponding to the negation-free bilateral natural deduction. A distinguishing aspect of our calculus is that we adopt the but-not connective as a constructor for functions between continuations. Another distinguishing aspect is that reductio ad absurdum is a construction of a configuration also known as a command. In our calculus, continuations and commands are first-class citizens.

Our bilateral λ-calculus contains a computationally consistent call-by-value calculus. The calculus corresponds to the sub-calculus of the call-by-value dual calculus invented by Wadler [38] [39] obtained by adding the but-not connective and removing the negation connective. The equivalence is formally obtained by giving mutual translations between these calculi. In other words, the translation provides a strong relationship between a bilateral natural deduction and a sequent calculus including proofs on the formulae-as-types notion.

The translations clarify a significant difference between the bilateral natural deduction and the sequent calculus. The negation of the dual calculus is not involutive, that is, $\neg\neg A$ is not isomorphic to $A$. Although the dual calculus also has the involutive duality as the metalevel operation that comes from the left-hand-side and right-hand-side duality of the classical sequent-calculus framework, there exists no inference rule to operate the involutive duality in the calculus. In the bilateral λ-calculus, the negation is represented using inversions of polarities, and is involutive by definition.

A symmetric λ-calculus which was constructed by Lovas and Crary is the only similar calculus based on bilateralism [25]. However, they adopted the negation connective $\neg$ as a primitive logical connective, and function type $\rightarrow$ is defined as syntactic sugar. In Lovas and Crary’s calculus it is necessary to use reductio ad absurdum, although it is generally easy to define functions between expressions. This means that it is not easy to define a sub-calculus corresponding to the minimal logic. Our calculus does not include the negation connective. Our work claims that the negation connective is not necessary but negative polarity is sufficient to define a symmetric λ-calculus based on bilateralism.

Using our calculus, we justify the duality which Filinski adopted as a principle in con-
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constructing his calculus. Specifically, the encodings to expressions and continuations are definable in our calculus. More correctly, mutual transformations between expressions and continuations of function types are definable in our calculus. We also show that every typable function has dual types about expressions and continuations. We clarify that bilateralism naturally raises the duality of functions.

Finally, we note that one of our goals is to construct a simple and powerful calculus in which the duality of functions is definable. We do not intend to clarify anything unknown in classical logic by assigning \( \lambda \)-terms to proofs, as seen in existing work in structural proof theory. Actually, our calculus is a sub-calculus of a natural extension of the dual calculus.

The remainder of this paper is organized as follows: In Section 2, we introduce bilateral natural deductions. In Section 3, we add proofs to nodes in derivation trees. In Section 4, we construct a symmetric \( \lambda \)-calculus corresponding to the negation-free bilateral natural deduction. In Section 5, we justify the duality of functions using our calculus. In Section 6, we discuss related work to clarify the contributions of this paper. In Section 7, we conclude the paper by identifying future research directions.

2 Bilateral Natural Deductions

In this section, we introduce bilateralism, which was proposed by Rumfitt [32], and define a few variants of Rumfitt’s bilateral natural deduction.

The set of formulae is defined as follows:

\[
\text{(formulae)} \quad \mathcal{A} ::= o \mid (\neg \mathcal{A}) \mid (\mathcal{A} \rightarrow \mathcal{B}) \mid (\mathcal{A} \land \mathcal{B}) \mid (\mathcal{A} \lor \mathcal{B})
\]

where \( o \) ranges over propositional variables. We note that \( \bot \) is not contained by the set of formulae. The connective power of \( \neg \) is stronger than that of \( \land, \lor, \) and \( \rightarrow \). The connective powers of \( \land \) and \( \lor \) are stronger than that of \( \rightarrow \). We omit parentheses when the context renders them obvious.

We recall the natural deduction invented by Gentzen [15, 16] and consider its propositional fragment \( \text{ND}_{\text{prop}} \), as shown in Figure 1. At each inference rule, formulae or \( \bot \) above a line are assumptions and a formula or \( \bot \) below a line is a conclusion. A derivation is a tree that has exactly one root. Symbol \( \vdash \) denotes a transitive connection between a leaf and

\[
\begin{align*}
\frac{\mathcal{A} \quad \neg \mathcal{A}}{\bot} & \quad (\bot \text{-I}) & \frac{\bot}{\mathcal{A}} & \quad (\bot \text{-E}) \\
\frac{[\mathcal{A}_0]}{[\mathcal{A}_n]} & \quad (\rightarrow \text{-I}) & \frac{\mathcal{A}_n \rightarrow \mathcal{A}_1}{\mathcal{A}_n} & \quad (\rightarrow \text{-E}) \\
\frac{\mathcal{A}_0 \rightarrow \mathcal{A}_1}{\mathcal{A}_0 \land \mathcal{A}_1} & \quad (\land \text{-I}) & \frac{\mathcal{A}_0 \land \mathcal{A}_1}{\mathcal{A}_0} & \quad (\land \text{-E}_0) \\
\frac{\mathcal{A}_0 \land \mathcal{A}_1}{\mathcal{A}_1} & \quad (\land \text{-E}_1) & \frac{\mathcal{A}_0}{\mathcal{A}_0 \lor \mathcal{A}_1} & \quad (\lor \text{-I}_0) \\
\frac{\mathcal{A}_1}{\mathcal{A}_0 \lor \mathcal{A}_1} & \quad (\lor \text{-I}_1) & \frac{\mathcal{A}_0 \lor \mathcal{A}_1}{\mathcal{A}_0} & \quad (\lor \text{-E}) \\
\frac{[\mathcal{A}_0]}{[\mathcal{A}_0]} & \quad (\lor \text{-E}) & \frac{[\mathcal{A}_1]}{[\mathcal{A}_1]} \\
\end{align*}
\]

Figure 1 Natural deduction \( \text{ND}_{\text{prop}} \).
a node, and \( A \) means that \( A \) is discharged from assumptions in a standard manner. Rules \( (\bot-E) \) and \( (\neg-E) \) are also known as explosion and reductio ad absurdum, respectively. A judgment is defined as \( \Gamma \vdash A \) or \( \Gamma \vdash \bot \), where \( \Gamma \) is a multiset of formulae.

There exists an idea that meanings of logical connectives are defined by their introduction rules and their elimination rules should be defined in harmony with their introduction rules in proof-theoretic semantics. Rumfitt attempted to justify logical connectives and inference rules using a notion of harmony which was proposed by Dummett [8]. We consider a logical connective \textit{tonk} which was proposed by Prior [29]. Its introduction rule \( (\textit{tonk-I}) \) and elimination rule \( (\textit{tonk-E}) \) are as follows:

\[
\frac{A_0}{A_0 \textit{tonk} A_1} \quad \frac{A_0 \textit{tonk} A_1}{A_1} \quad \frac{A_0}{A_0 \textit{tonk} A_1} \quad \frac{A_0 \textit{tonk} A_1}{A_1}.
\]

A pair of contiguous introduction and elimination rules is called harmonious if the residue after removing the pair is also a derivation. Such a procedure is called normalization. In this section, we let \( \rightsquigarrow \) denote the normalization procedure. The pair of \( (\textit{tonk-I}) \) and \( (\textit{tonk-E}) \) is not harmonious because the right-hand side of the following \( \rightsquigarrow \) relation is not a derivation:

\[
\frac{A_0}{A_0 \textit{tonk} A_1} \quad \frac{A_0 \textit{tonk} A_1}{A_1} \rightsquigarrow \frac{A_0}{A_1}.
\]

Rumfitt suggested that \( \text{ND}_{\text{prop}} \) also does not enjoy the harmony condition and proposed a notion of bilateralism to construct a harmonious natural deduction.

Bilateralism is based on two notions of acceptance and rejection of formulae. They are also called verification and falsification, respectively, by Wansing [41, 42]. Formulae \( A \) with polarity are defined as \( + A \) and \( - A \). A derivation of root \( + A \) means that \( A \) is accepted. A derivation of root \( - A \) means that \( A \) is rejected.

Let \( A \) be a formula with polarity. Conjugates \((+ A)^*\) and \((- A)^*\) are defined as \(- A\) and \(+ A\), respectively.

Rumfitt adopted (Non-contradiction) and (Reductio) which are called coordination principles and defined inference rules of logical connectives, as shown in Figure 2, which are naturally derived from the standard boolean semantics. In this paper, we call this logic a bilateral natural deduction \( \text{Bi-ND}_{\text{prop}} \).

\( \text{ND}_{\text{prop}} \) is based on the notion of unilaterism rather than bilateralism. A derivation of root \( A \) in \( \text{ND}_{\text{prop}} \) means that \( A \) is accepted. There exists the following relation between \( \text{ND}_{\text{prop}} \) and \( \text{Bi-ND}_{\text{prop}} \):

\[\textbf{Theorem 2.1 (Rumfitt [32]).} \text{ For any } n \geq 0, A_0, \ldots, A_{n-1} \vdash A \text{ is provable in } \text{ND}_{\text{prop}} \text{ if and only if } +A_0, \ldots, +A_{n-1} \vdash +A \text{ is provable in } \text{Bi-ND}_{\text{prop}}.\]

\[\text{Remark.} \text{ It is controversial that explosion and reductio ad absurdum are regarded as elimination rules of the logical connectives } \bot \text{ and } \neg, \text{ respectively. Rumfitt’s bilateralism is also criticized in a paper [24]. That is, bilateralism is called a work in progress. However, the subject of this paper is not a justification of bilateralism in proof-theoretic semantics.} \]

The natural deduction \( \text{Bi-ND}_{\text{prop}} \) is not symmetric. We extend the language by adding a logical connective \( \leftarrow \):

\[
\text{(formulae) } A ::= o \mid (\neg A) \mid (A \rightarrow A) \mid (A \leftarrow A) \mid (A \land A) \mid (A \lor A).
\]

We will use the logical connective as function types of continuations in the following.

The connective \( \leftarrow \) is called the \textit{but-not} connective because \( A_0 \leftarrow A_1 \) is logically equivalent to \( A_0 \land \neg A_1 \) in classical logic. The but-not connective is also written as pseudo-difference
See Appendix B.

The connectives \( \Gamma \) and \( \Delta \) connective in intuitionistic logic because \( \not\Gamma \) is a primitive connective in paraconsistent logic, whereas intuitionistic logic can be defined by sequents \( \Gamma \). In paraconsistent logic, sequent calculus consists of sequents \( \Gamma \rightarrow \Delta \), where \( \Gamma \) is empty or a singleton formula, whereas intuitionistic logic can be defined by sequents \( \Gamma \rightarrow \Delta \), where \( \Delta \) is empty or a singleton formula.

We define a natural deduction Bi-ND\(_{prop}\) by adding inference rules, as shown in Figure 2. The connectives \( \rightarrow \) and \( \leftarrow \) are symmetrically located in Bi-ND\(_{prop}\), as follows:

**Proposition 2.2.** \( +A_0 \rightarrow A_1 \vdash +A_0 \leftarrow A_1, +A_0 \rightarrow A_1 \vdash +A_0 \leftarrow A_1, +A_0 \leftarrow A_1 \vdash -A_0 \rightarrow A_1, and -A_0 \rightarrow A_1 \vdash +A_0 \rightarrow A_1 \) are provable in Bi-ND\(_{prop}\).

**Proof.** See Appendix B.

Let \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \) be languages such that \( \mathcal{L}_0 \subseteq \mathcal{L}_1 \), and \( \mathcal{S}_0 \) and \( \mathcal{S}_1 \) be logics on the languages \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \), respectively. We define \( \mathcal{S}_1 \) as an extension of \( \mathcal{S}_0 \) if any formula \( \varphi \) that is provable in \( \mathcal{S}_0 \) is also provable in \( \mathcal{S}_1 \). We define that an extension \( \mathcal{S}_1 \) of \( \mathcal{S}_0 \) is conservative if any formula \( \varphi \) on the language \( \mathcal{L}_0 \) that is provable in \( \mathcal{S}_1 \) is also provable in \( \mathcal{S}_0 \).

**Proposition 2.3.** Bi-ND\(_{prop}\) is a conservative extension of Bi-ND\(_{prop}\).
The inference rules

\[ \begin{align*}
&+ A_0 - A_1 (\leftarrow I_+) \\
&+ A_0 \leftarrow A_1 \leftarrow A_1 (\leftarrow I) \\
&- A_0 \leftarrow A_1 (\leftarrow E_+) \\
&- A_0 \leftarrow A_1 (\leftarrow E_-)
\end{align*} \]

\[ \text{Figure 3 Inference rules for } \leftarrow. \]

**Proof.** It is obvious because Bi-ND\textsubscript{prop} is complete to the standard two-value semantics, and Bi-ND\textsuperscript{−}\textsubscript{prop} is sound to the semantics.

Bi-ND\textsubscript{prop} and Bi-ND\textsuperscript{−}\textsubscript{prop} include sub-logics as follows:

**Proposition 2.4.** 1. The inference rules \((\land I_0), (\land I_1), (\land E_0), (\land E_1), (\lor E_0), (\lor E_1), (\lor I_0), (\lor I_1), (\lor E_0), (\lor E_1), (\leftrightarrow I_0), (\leftrightarrow I_1), (\leftrightarrow E_0), (\leftrightarrow E_1)\) are derivable in Bi-ND\textsubscript{prop}, and

2. The inference rules \((\to I_0), (\to I_1), (\to E_0), (\to E_1)\) are derivable in Bi-ND\textsuperscript{−}\textsubscript{prop}.

**Proof.** See Appendix B.

### 3 Derivation Trees with Proofs in Their Nodes

In this section, we introduce derivation trees with proofs in their nodes to mediate between natural deductions and \(\lambda\)-calculi introduced in Sections 2 and 4, respectively.

We add proofs to polarised formulae in the \(\neg\)-free fragment of Bi-ND\textsuperscript{−}\textsubscript{prop}, that is, the (Non-contradiction), (Reductio), \((\to I_0), (\to I_1), (\to E_0), (\to E_1), (\land I_0), (\land I_1), (\land E_0), (\land E_1), (\lor I_0), (\lor I_1), (\lor E_0), (\lor E_1), (\leftrightarrow I_0), (\leftrightarrow I_1), (\leftrightarrow E_0), (\leftrightarrow E_1)\) fragment, to construct a symmetric \(\lambda\)-calculus. We note that the other inference rules are derivable by Proposition 2.4.

We assume a set of proof variables. We write \(\alpha\) for a proof variable. We define that nodes \(+ A\) and \(- A\) in the natural deduction respectively denote that \(t\) is a proof for acceptance and rejection of \(A\). We also define that a node \(\bot\) in the natural deduction denotes that \(T\) is a proof for contradiction.

Node \(\lambda \alpha . t\): \(+ A_0 \to A_1\) denotes that \(\lambda \alpha . t\) is a proof for acceptance of \(A_0 \to A_1\) if \(\alpha\) is a proof variable for acceptance of \(A_0\) and \(t\) is a proof for acceptance of \(A_1\). Node \(\lambda \alpha . t\): \(- A_0 \leftarrow A_1\) denotes that \(\lambda \alpha . t\) is a proof for acceptance of \(A_0 \to A_1\) if \(\alpha\) is a proof variable for acceptance of \(A_1\) and \(t\) is a proof for acceptance of \(A_0\).

Node \(t_0 t_1\): \(+ A_1\) denotes that \(t_0 t_1\) is a proof for acceptance of \(A_1\) if \(t_0\) is a proof for acceptance of \(A_0 \to A_1\) and \(t_1\) is a proof for acceptance of \(A_0\). Node \(t_0 t_1\): \(- A_0 \leftarrow A_1\) denotes that \(t_0 t_1\) is a proof for rejection of \(A_1\) if \(t_0\) is a proof for rejection of \(A_0 \leftarrow A_1\) and \(t_1\) is a proof for rejection of \(A_1\).

Node \((t_0, t_1)\): \(+ A_0 \land A_1\) denotes that \((t_0, t_1)\) is a proof for acceptance of \(A_0 \land A_1\) if \(t_0\) is a proof for acceptance of \(A_0\) and \(t_1\) is a proof for acceptance of \(A_1\). Node \((t_0, t_1)\): \(- A_0 \lor A_1\) denotes that \((t_0, t_1)\) is a proof for rejection of \(A_0 \lor A_1\) if \(t_0\) is a proof for rejection of \(A_0\) and \(t_1\) is a proof for rejection of \(A_1\).

Node \(\pi_0(t)\): \(+ A_0\) denotes that \(\pi_0(t)\) is a proof for acceptance of \(A_0\) if \(t\) is a proof for acceptance of \(A_0 \land A_1\). Node \(\pi_0(t)\): \(- A_0\) denotes that \(\pi_0(t)\) is a proof for rejection of \(A_0\) if \(t\) is a proof for rejection of \(A_0 \lor A_1\). Nodes \(\pi_1(t_0)\): \(+ A_1\) and \(\pi_1(t_0)\): \(- A_1\) are similar.
(proofs) \quad t ::= c \mid \alpha \mid \lambda \alpha.t \mid tt \mid (t, t) \mid \pi_0(t) \mid \pi_1(t) \mid \mu \alpha.T

\quad T ::= (t \mid t)

\textbf{Figure 4} Proofs of the negation-free natural deduction.

(types) \quad A ::= o \mid (A \rightarrow A) \mid (A \leftarrow A) \mid (A \land A) \mid (A \lor A)

(expressions) \quad E ::= \text{cst}^o \mid x^A \mid \lambda x^A.E \mid EE \mid (E, E) \mid \pi_0(E) \mid \pi_1(E) \mid \mu \alpha^A.N

(continuations) \quad C ::= \bullet^o \mid a^A \mid \lambda a^A.C \mid CC \mid (C, C) \mid \pi_0(C) \mid \pi_1(C) \mid \mu x^A.N

(commands) \quad N ::= \langle E \mid C \rangle

(syntactical objects) \quad D ::= E \mid C \mid N

\textbf{Figure 5} The bilateral lambda-calculus BLC.

Node \langle t_0 \mid t_1 \rangle : \bot denotes that \langle t_0 \mid t_1 \rangle is a proof of contradiction if \( t_0 \) is a proof for acceptance of \( A \) and \( t_1 \) is a proof for rejection of \( A \).

Node \( \mu \alpha.T : + A \) denotes that \( \mu \alpha.T \) is a proof for acceptance of \( A \) if \( \alpha \) is a proof variable for rejection of \( A \) and \( T \) is a proof of contradiction. Node \( \mu \alpha.T : - A \) denotes that \( \mu \alpha.T \) is a proof for rejection of \( A \) if \( \alpha \) is a proof variable for acceptance of \( A \) and \( T \) is a proof of contradiction.

We formally define the set of proofs in a Curry-style bilateral \( \lambda \)-calculus, as shown in Figure 4 where \( c \) ranges over constants for adding logical axioms.

Let \( \Gamma \) be a set of nodes. Judgment \( \Gamma \vdash t : + A \) denotes that \( t \) is a proof for acceptance of \( A \) under \( \Gamma \). Judgment \( \Gamma \vdash t : - A \) denotes that \( t \) is a proof for rejection of \( A \) under \( \Gamma \). Judgment \( \Gamma \vdash T : \bot \) denotes that \( T \) is a proof for contradiction under \( \Gamma \).

\section{4 Bilateral Lambda-Calculi}

In this section, we construct a Church-style symmetric \( \lambda \)-calculus based on bilateralism and define a call-by-value sub-calculus.

\subsection{4.1 Definition and Basic Properties}

We respectively call proofs for acceptance and rejection \textit{expressions} and \textit{continuations}. We distinguish proof variables for acceptance from those for rejection. We construct an alternative symmetric \( \lambda \)-calculus called a bilateral \( \lambda \)-calculus (BLC).

We define types, polarized types, expressions, continuations, commands, and syntactical objects as shown in Figure 5.

Expression \( \text{cst}^o \) denotes a constant. Expression \( x^A \) denotes an expression variable. Expression \( \lambda x^A.E \) denotes a \( \lambda \)-abstraction of expression \( E \) by \( x^A \). Expression \( E_0.E_1 \) denotes an application of function \( E_0 \) to expression \( E_1 \). Expression \( \langle E_0, E_1 \rangle \) denotes a pair of expressions \( E_0 \) and \( E_1 \). Expressions \( \pi_0(E) \) and \( \pi_1(E) \) are projections.

Continuations are defined symmetrically to expressions. Continuation \( \bullet^o \) denotes the unique constant denoting a continuation of \( o \). By the definition based on bilateralism, the calculus is involutive on the notion of polarities.
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\[
\frac{\Gamma \vdash + E : A \quad \Gamma \vdash - C : A}{\Gamma \vdash \text{ N } (E | C)} \quad \text{(Non-contradiction)}
\]

\[
\frac{\Pi; \Sigma, a : A \vdash_\alpha N}{\Pi; \Sigma \vdash_+ a^A.N : A} \quad \text{ (Reductio+)}
\]

\[
\frac{\Pi; \Sigma \vdash - \mu x^A.N : A}{\Pi; \Sigma \vdash_\alpha N} \quad \text{ (Reductio−)}
\]

\[
\frac{\Gamma \vdash + \text{ cst}^o : o}{\Gamma \vdash - \text{ o} : o} \quad \text{(Constant+)}
\]

\[
\frac{\Pi, x : A_0 ; \Sigma \vdash_+ E : A_1}{\Pi; \Sigma \vdash_+ \lambda x^a_0.E : A_0 \rightarrow A_1} \quad \text{ (→I+)}
\]

\[
\frac{\Pi, x : A_0 ; \Sigma \vdash - \lambda x^a_0.C : A_0}{\Pi; \Sigma \vdash - \lambda x^a_0.C : A_0 \leftarrow A_1} \quad \text{ (←I−)}
\]

\[
\frac{\Pi \vdash + E_0 : A_0 \rightarrow A_1 \quad \Gamma \vdash + E_1 : A_0}{\Gamma \vdash + E_0.E_1 : A_0 \land A_1} \quad \text{ (∧I+)}
\]

\[
\frac{\Pi \vdash + C_0 : A_0 \leftarrow A_1 \quad \Gamma \vdash - C_1 : A_1}{\Gamma \vdash + C_0.C_1 : A_0} \quad \text{ (∧E−)}
\]

\[
\frac{\Gamma \vdash + E : A_0 \land A_1}{\Gamma \vdash + \pi_0(E) : A_0} \quad \text{ (∧-E+)}
\]

\[
\frac{\Gamma \vdash + C : A_0 \lor A_1}{\Gamma \vdash - \pi_0(C) : A_0} \quad \text{ (∧-E−)}
\]

\[
\frac{\Gamma \vdash + E : A_0 \land A_1}{\Gamma \vdash + \pi_1(E) : A_1} \quad \text{ (∧-E+1)}
\]

\[
\frac{\Gamma \vdash + C : A_0 \lor A_1}{\Gamma \vdash - \pi_1(C) : A_1} \quad \text{ (∧-E−1)}
\]

\[\text{Figure 6} \quad \text{A type system of BLC.}\]

Commands are first-class citizens. A command can be abstracted by expression variable \(x^A\) or continuation variable \(a^A\). Command \(N\) abstracted by \(a^A\) is expression \(\mu a^A.N\). A command abstracted by \(x^A\) is continuation \(\mu x^A.N\). A similar idea can be seen in \(\tilde{\lambda}\mu\tilde{\mu}\)-calculus which was proposed by Curien and Herbelin \[3\].

Expressions, continuations, and commands are called syntactical objects.

We assume that the connective powers of applications are stronger than those of \(\lambda\)-abstractions. We omit superscripts that denote types when the context renders them obvious.

Figure 6 shows the type system of BLC consisting of judgments \(\Gamma \vdash_+ E : A, \Gamma \vdash_− C : A\), and \(\Gamma \vdash_\alpha N\), where type environments \(\Gamma\) are defined as follows:

(type environments) \( \Gamma := \Pi; \Sigma \quad \Pi := \emptyset \mid \Pi, x : A \quad \Sigma := \emptyset \mid \Sigma, a : A \).

Judgments \(\Pi; \Sigma \vdash_+ E : A, \Pi; \Sigma \vdash_− C : A\), and \(\Pi; \Sigma \vdash_\alpha N\) correspond to \(+ A | A \in \Pi\), \(- A | A \in \Sigma\) \(+ A, \{ + A | A \in \Pi\}, \{ - A | A \in \Sigma\} \vdash A\), and \(+ A | A \in \Pi\), \(- A | A \in \Sigma\) \(\vdash\), respectively.

The type system contains rules about commands. Rule (Non-contradiction) defines a command from an expression and a continuation. Additionally, even if a command occurs in a derivation, the derivation does not necessarily end and may be continued by (Reductio+) or (Reductio−). The other inference rules about expressions are defined in a standard manner. The inference rules about continuations are defined symmetrically to expressions.
(values) \( V ::= \text{cst} \mid x \mid \lambda x. E \mid (V, V) \mid \pi_0(V) \mid \pi_1(V) \mid \mu a. \langle V \mid \pi_0(a) \rangle \mid \mu a. \langle V \mid \pi_1(a) \rangle \)

(contexts) \( \mathcal{E} ::= \{ - \} \mid \mathcal{E} E \mid \mathcal{E} V \mid (\mathcal{E}, \mathcal{E}) \mid (V, \mathcal{E}) \mid \pi_0(\mathcal{E}) \mid \pi_1(\mathcal{E}) \)

Figure 7 Values and contexts of CbV-BLC.

The bilateral \( \lambda \)-calculus is well designed. The so-called weakening holds as follows:

\[ \text{Proposition 4.1.} \quad \text{1.} \quad \Pi; \Sigma \vdash_+ E \quad \text{implies} \quad \Pi, x: A; \Sigma \vdash_+ E; A_0 \quad \text{and} \quad \Pi; \Sigma, a: A \vdash_+ E; A_0, \]
\[ \text{2.} \quad \Pi; \Sigma \vdash_- C; A_0 \quad \text{implies} \quad \Pi, x: A; \Sigma \vdash_- C; A_0 \quad \text{and} \quad \Pi; \Sigma, a: A \vdash_- C; A_0, \]
\[ \text{3.} \quad \Pi; \Sigma \vdash_\circ N \quad \text{implies} \quad \Pi, x: A; \Sigma \vdash_\circ N \quad \text{and} \quad \Pi; \Sigma, a: A \vdash_\circ N. \]

Proof. By induction on derivation.

The substitution lemma definitely holds as follows:

\[ \text{Lemma 4.2.} \quad \text{1.} \quad \text{Assume} \quad \Pi, x: A_0; \Sigma \vdash_+ E'; A_1 \quad \text{and} \quad \Pi; \Sigma \vdash_+ E; A_0. \quad \text{Then,} \quad \Pi; \Sigma \vdash_+ [E/x]E'; A_1 \quad \text{holds.} \]
\[ \text{2.} \quad \text{Assume} \quad \Pi, x: A_0; \Sigma \vdash_- C; A_1 \quad \text{and} \quad \Pi; \Sigma \vdash_+ E; A_0. \quad \text{Then,} \quad \Pi; \Sigma \vdash_- [E/x]C; A_1 \quad \text{holds.} \]
\[ \text{3.} \quad \text{Assume} \quad \Pi, x: A; \Sigma \vdash_\circ N \quad \text{and} \quad \Pi; \Sigma \vdash_+ E; A. \quad \text{Then,} \quad \Pi; \Sigma \vdash_\circ [E/x]N \quad \text{holds.} \]
\[ \text{4.} \quad \text{Assume} \quad \Pi; \Sigma, a: A_0 \vdash_+ E; A_1 \quad \text{and} \quad \Pi; \Sigma \vdash_- C; A_0. \quad \text{Then,} \quad \Pi; \Sigma \vdash_+ [C/a]E; A_1 \quad \text{holds.} \]
\[ \text{5.} \quad \text{Assume} \quad \Pi; \Sigma, a: A_0 \vdash_- C'; A_1 \quad \text{and} \quad \Pi; \Sigma \vdash_- C; A_0. \quad \text{Then,} \quad \Pi; \Sigma \vdash_- [C/a]C'; A_1 \quad \text{holds.} \]
\[ \text{6.} \quad \text{Assume} \quad \Pi; \Sigma, a: A \vdash_\circ N \quad \text{and} \quad \Pi; \Sigma \vdash_- C; A. \quad \text{Then,} \quad \Pi; \Sigma \vdash_\circ [C/a]N \quad \text{holds.} \]

Proof. By induction on derivation.

The bilateral \( \lambda \)-calculus enjoys the type uniqueness property, that is, every expression and continuation has a unique positive and negative type, respectively, as follows:

\[ \text{Proposition 4.3.} \quad \text{1.} \quad \text{If} \quad \Gamma \vdash_+ E; A_0 \quad \text{and} \quad \Gamma \vdash_+ E; A_1, \quad \text{then} \quad A_0 \quad \text{and} \quad A_1 \quad \text{are the same.} \]
\[ \text{2.} \quad \text{If} \quad \Gamma \vdash_- C; A_0 \quad \text{and} \quad \Gamma \vdash_- C; A_1, \quad \text{then} \quad A_0 \quad \text{and} \quad A_1 \quad \text{are the same.} \]

Proof. The proposition holds immediately from the definition of the type system.

4.2 The Call-by-Value Lambda-Calculus CbV-BLC

We define a call-by-value bilateral \( \lambda \)-calculus CbV-BLC. Types, expressions, continuations, commands, and typing rules are the same as those of BLC. The values and the call-by-value evaluation contexts for expressions of CbV-BLC are defined as shown in Figure 8.

An evaluation context \( \mathcal{E} \) is an expression with a hole \( \{ - \} \). The expression obtained by filling the hole of \( \mathcal{E} \) with an expression \( E \) is denoted by \( \mathcal{E}(E) \). The equations of CbV-BLC are shown in Figure 8.

Although careful readers will wonder why \( \pi_0(V) \) and \( \pi_1(V) \) are values, they can often be seen in \( \lambda \)-calculi based on categorical semantics (cf. Definition 7.7 in Selinger’s paper [33] and Figure 2 in Wadler’s paper [39]). We also note that \( \mu a. \langle V \mid \pi_0(a) \rangle \) and \( \mu a. \langle V \mid \pi_1(a) \rangle \)
A Symmetric Lambda-Calculus Corresponding to a Bilateral Natural Deduction

We can define case expressions using pairs of continuations as follows:

\[
\lambda x. V \equiv [V/x] E \\
\lambda a. C_0 \equiv [C_0/a] C_0
\]

\[
\lambda x. V \equiv V \quad \text{if } x \notin \text{fv}(V) \\
\lambda a. C \equiv C \quad \text{if } a \notin \text{fv}(C)
\]

\[
\pi_0((V_0, V_1)) \equiv V_0 \\
\pi_1((V_0, V_1)) \equiv V_1 \\
\pi_0(V) \equiv V \\
\pi_1(V) \equiv V \\
\mu a. (E | a) \equiv E \quad \text{if } a \notin \text{fv}(E) \\
\mu x. (x | C) \equiv C \quad \text{if } x \notin \text{fv}(C)
\]

\[
\langle V | \mu x. N \rangle = [V/x] N \\
\langle \mu a. N | C \rangle = [C/a] N
\]

\[
\langle \chi | \langle A \land A \rangle | \langle A \lor A \rangle | (\neg A) \rangle
\]

\[
A ::= \chi \mid \langle A \land A \rangle \mid \langle A \lor A \rangle \mid (\neg A)
\]

\[
M ::= x \mid \langle M, M \rangle \mid \langle M \rangle \text{inl} \mid \langle M \rangle \text{inr} \mid \{K\} \text{not} \mid \{S\} \alpha
\]

\[
K ::= \alpha \mid \{K, K\} \mid \text{fst}[K] \mid \text{snd}[K] \mid \text{not}(M) \mid x.(S)
\]

\[
S ::= M \bullet K
\]

\[
O ::= M \mid K \mid S
\]

Figure 8 The equations of CbV-BLC.

Figure 9 The syntax of the dual calculus.

are values for \( A \lor B \), namely, they mean the left and the right injections of \( V \), respectively. We can define case expressions using pairs of continuations as follows:

\[
\text{inl}(E) \equiv \mu a. (E | \pi_0(a)) \\
\text{inr}(E) \equiv \mu a. (E | \pi_1(a)) \\
\text{case}(E, x_0, E_0, x_1, E_1) \equiv \mu a. (E | (\mu x_0. (E_0 | a), \mu x_1. (E_1 | a)))
\]

\[
\begin{align*}
\Gamma \vdash E : A_0 & \quad \Gamma \vdash E : A_1 \\
\vdash \text{inl}(E) : A_0 \lor A_1 & \quad \vdash \text{inr}(E) : A_0 \lor A_1
\end{align*}
\]

\[
\begin{align*}
\Pi; \Sigma \vdash E : A_0 \lor A_1 & \quad \Pi, x_0 : A_0; \Sigma \vdash E_0 : A \\
\Pi, x_1 : A_1; \Sigma \vdash E_1 : A
\end{align*}
\]

\[
\vdash \text{case}(\text{inl}(V), x_0, E_0, x_1, E_1) : C
\]

\[
\equiv \langle \mu a. \mu a_2. (V | \pi_0(a_2)) | (\mu x_0. (E_0 | a), \mu x_1. (E_1 | a)) \rangle | C
\]

\[
\equiv \langle \mu a. \mu a_2. (V | \pi_0(a_2)) | (\mu x_0. (E_0 | C), \mu x_1. (E_1 | C)) \rangle
\]

\[
= \chi \langle V | \pi_0((\mu x_0. (E_0 | C), \mu x_1. (E_1 | C))) \rangle = \chi \langle V | \mu x_0. (E_0 | C) \rangle = \chi \langle V/x_0, E_0 | C \rangle.
\]

The calculus CbV-BLC is CbV-DC++ which is a sub-calculus of an extension with the but-not connective of the call-by-value dual calculus by Wadler \[59\]. Types, terms, coterms, statements, and syntactical objects are shown in Figure 8. A key difference from BLC is that the dual calculus adopts \( \neg \) as a primitive connective and function types are syntactic sugar. See Wadler’s papers \[59\] or Appendix A for the details. We can define a translation from CbV-BLC. Consequently, the consistency of our call-by-value calculus is obtained from the consistency of the call-by-value dual calculus. Specifically, we can obtain the following:
\begin{theorem}
There exist translations \((-)^\sharp\) from \(CbV-BLC\) into \(CbV-DC_{\rightarrow \leftarrow}\) and \((-)^\flat\) from \(CbV-DC_{\rightarrow \leftarrow}\) into \(CbV-BLC\), which satisfy:
\begin{itemize}
  \item \(D_0 \equiv_v D_1\) implies \((D_0)^\sharp \equiv_{dcv} (D_1)^\flat\),
  \item \(O_0 \equiv_{dcv} O_1\) implies \((O_0)^\flat = (O_1)^\sharp\),
  \item \((D(D)^\flat)^\flat = v D\) holds, and
  \item \((O(O)^\flat)^\flat = dcv O\) holds.
\end{itemize}
where \(\equiv_{dcv}\) is the equality relation of \(CbV-DC_{\rightarrow \leftarrow}\).
\end{theorem}

\textbf{Proof.} See Appendix A \hfill ⬇

The theorem reasons about the call-by-value variant of BLC via the call-by-value dual calculus. Furthermore, the theorem shows that the but-not type \(A \leftarrow B\) in the call-by-value dual calculus is considered as the function type for continuations. The theorem also reveals the difference between the dual calculus, whose negation type \(\neg A\) is not involutive, and BLC, whose polarities \(+A\) and \(\neg A\) are involutive.

The negation type of the dual calculus can appear anywhere in a type. The negation type enables encoding of a coterm, say \(K\), of type \(A\) to a term \(K\not\text{not}\) of type \(\neg A\), and handling of the encoded coterms as a part of terms. For instance, \([[K_1, K_2] \not\text{not}\) of type \(\neg (A_1 \vee A_2)\) is a term which encodes the pair of coterms \(K_1\) and \(K_2\), and functions, such as \(\lambda x_2. [x_1. (x_2 \not\text{not}(\text{inl}))\not\text{not}\) of type \(\neg (A_1 \vee A_2) \rightarrow \neg A_1\) that handles such terms, are definable in the dual calculus. The expressive power of BLC is strictly weaker than the dual calculus, since BLC does not permit defining such functions. The theorem also raises a question whether BLC offers an adequate theoretical framework for expressing practical control operators. We conjecture that the polarities of BLC are enough for this purpose. This is future work.

\section{Justifying the Duality of Functions}

In this section, we reason about the duality of functions in Filinski’s symmetric \(\lambda\)-calculus using the bilateral \(\lambda\)-calculus.

\subsection{Filinski’s Symmetric Lambda-Calculus}

A function of the type \(A_0 \rightarrow A_1\) from expressions of the type \(A_0\) to expressions of the type \(A_1\) can be regarded as a function from continuations of the type \(A_0\) to continuations of the type \(A_1\), and vice versa. This property of functions is called the duality of functions.

Filinski adopted the duality as a principle and constructed a symmetric \(\lambda\)-calculus \cite{filinski97}
\cite{filinski98}. The symmetric \(\lambda\)-calculus consists of functions \(F\), expressions \(E\), and continuations \(C\). Functions consist of \(\lambda\)-abstractions of expressions, decodings of expressions, \(\lambda\)-abstractions of continuations, and decodings of continuations as follows:

\[
\begin{align*}
\text{(functions) } & F_{A_0}^{A_1} := X_{A_0} \Rightarrow E_{A_1} \mid \overline{E_{[A_0 \rightarrow A_1]}} \mid \ Y_{A_1} \Leftrightarrow C_{A_0} \mid \overline{C_{[A_1 \leftarrow A_0]}}.
\end{align*}
\]

Let \(A_0 \rightarrow A_1\) be a function type. Filinski defined a function type \([A_0 \rightarrow A_1]\) for an expression, which denotes an exponential object \(A_1^{A_0}\) in categorical semantics, where \(A_0\) and \(A_1\) are objects that correspond to types \(A_0\) and \(A_1\). We note that \(A_2 \times A_0 \rightarrow A_1\) is bijective to \(A_2 \rightarrow A_1^{A_0}\) in categorical semantics. Similarly, Filinski defined a function type \([A_1 \leftarrow A_0]\) for a continuation, which denotes a coexponential object \(A_{0A_1}\), and \(A_0 \rightarrow A_2 + A_1\) is bijective to \(A_{0A_1} \rightarrow A_2\).
Expressions and continuations consist of constants, variables, applications of functions, and encodings of functions as follows:

(expressions) \[ E_0 := \text{ cst}_0 \mid x_0 \mid F^A_0 E_A \quad E_{[A_0 \rightarrow A_1]} := x_{[A_0 \rightarrow A_1]} \mid F^A_{[A_0 \rightarrow A_1]} E_A \mid \pi F^A_{A_1} \]

(continuations) \[ C^0 := \bullet_0 \mid a^0 \mid F^A_0 C^A \quad C_{[A_1 \leftarrow A_0]} := a_{[A_1 \leftarrow A_0]} \mid F^A_{A_1 \leftarrow A_0} C^A \mid \lambda F^A_{A_1 \leftarrow A_0} \]

We note that the encodings and decodings are defined to be primitive operators because the duality is adopted as a principle.

We explain commands in Filinski’s symmetric \( \lambda \)-calculus, which is a triple called a configuration:

\[
\Gamma \vdash E : + A_0 \quad \Gamma \vdash F : A_0 \rightarrow A_1 \quad \Gamma \vdash C : \neg A_1
\]

\[
\vdash \langle E \mid F \mid C \rangle
\]

for the symmetric \( \lambda \)-calculus where \( \vdash E : + A_0 \) and \( \vdash C : \neg A_1 \) for expression \( E \) of type \( A_0 \) and configuration \( C \) of type \( A_1 \), respectively. The notation was introduced by Ueda and Asai [36]. A difference from commands in the bilateral \( \lambda \)-calculus is that configurations are not pairs consisting of expressions and continuations, but triples. Another difference is that any configuration cannot be abstracted by expression or continuation variables. One other difference is that the continuation types are represented using the negation connective in Filinski’s calculus.

We can see that the configuration notion is also based on the duality principle. If \( F \) is regarded as a function from expressions of the type \( A_0 \) to expressions of the type \( A_1 \), then \( F \) is applied to \( E \) and an expression of the type \( A_1 \) that is consistent with \( C \) of the type \( A_1 \) is generated. Similarly, if \( F \) is regarded as a function from continuations of the type \( A_1 \) to continuations of the type \( A_0 \), then \( F \) is applied to \( C \) and a continuation of the type \( A_0 \) that is consistent with \( E \) of the type \( A_0 \) is generated. The configuration notion includes both cases.

5.2 Mutual Transformations between Functions

Let us see how the duality occurs in the bilateral \( \lambda \)-calculus. The bilateral \( \lambda \)-calculus does not permit anything neutral that is neither expression nor continuation. Even if we want to define a neutral function, we must decide whether the type of the function is either \( + A_0 \rightarrow A_1 \) or \( \neg A_0 \leftarrow A_1 \). If we define a function between expressions which is applied to a continuation, then the function cannot be as-is applied to the continuation, and vice versa.

However, we can define encodings \( \iota E, \iota x, \iota C = \lambda x.\mu y. \langle \text{Ex} \mid a \rangle \) and \( \gamma C^* = \lambda x.\mu y. \langle x \mid C a \rangle \) to continuations and expressions in the bilateral \( \lambda \)-calculus, respectively, and the encodings are mutual transformations as follows:

\textbf{Theorem 5.1.} The following inferences are derivable:

\[
\begin{align*}
\Gamma \vdash E : A_0 \rightarrow A_1 & \quad \Gamma \vdash C : A_0 \leftarrow A_1 \\
\Gamma \vdash \iota E : A_0 \leftarrow A_1 & \quad \Gamma \vdash \gamma C : A_0 \rightarrow A_1
\end{align*}
\]

\textbf{Proof.} See Appendix B.

The mutual transformations enjoy the following property:

\textbf{Theorem 5.2.} 1. \( \langle \gamma C^0 \mid \nu C_1 \rangle =_v \langle \nu C \mid C_0 C_1 \rangle \) holds,
2. \( \langle \nu C \mid \iota E, \iota C \rangle =_v \langle \nu E \mid C \rangle \) holds,
3. \( \langle \iota E \mid \nu C \mid \nu C \rangle =_v \langle \nu C \mid C \rangle \) holds, and
4. \( \langle \nu C \mid \iota C^0 \mid \iota C_1 \rangle =_v \langle \nu C \mid C_0 C_1 \rangle \) holds.
The first and second statements hold immediately from the definition of $\sim$, $\Gamma C$, and $\triangleleft E, \triangleright$ as follows:

$$\Gamma C \triangleright V | C_1 \equiv (\langle \lambda x.\mu a. (x | C_0 a) \rangle V | C_1) =_v (\langle \mu a. (V | C_0 a) | C_1 \rangle =_v (V | C_0 C_1)$$

$$\langle V | \triangleleft E, \triangleright C \rangle \equiv \langle V | (\lambda a.\mu x. (Ex | a)) C \rangle =_v \langle V | \mu x. (Ex | C) \rangle =_v (E V | C) .$$

The third and fourth statements hold from the first and second statements. ▶

Theorems 5.1 and 5.2 ensure that we can always recover to define functions between expressions (and continuations) from functions between continuations (resp. expressions) using the mutual transformations. Thus, we confirm that the duality of functions is derived from definability of the mutual transformations in the bilateral $\lambda$-calculus.

### 5.3 Dual Proofs for Functions

We also provide an alternative justification of the duality using derivation trees with proofs in their nodes introduced in Section 3.

We define a polarization, which is a function from proof variables and proof constants to expression or continuation variables with types and expression or continuation constants, respectively. A polarization for proofs is defined by

$$p(\lambda \alpha . t) = \lambda p(\alpha), p(t) \quad p(t_0 t_1) = p(t_0) p(t_1) \quad p(t_0, t_1) = (p(t_0), p(t_1))$$

$$p(\pi_0 (t)) = \pi_0 (p(t)) \quad p(\pi_1 (t)) = \pi_1 (p(t)) \quad p(\mu \alpha . T) = \mu p(\alpha), p(T)$$

$$p(t_0 | t_1) = (p(t_0) | p(t_1)).$$

Let $V$ be a set of proof variables. We define $p(V)$ as the concatenation of the positive type environments and the negative type environments of $V$ by $p$.

Polarizations $p$ and $p'$ are equivalent if

- for any proof variable $\alpha$, $p(\alpha)$ and $p'(\alpha)$ have the same polarity, and
- for any proof variables $\alpha$ and $\alpha'$, $p(\alpha) \equiv p(\alpha')$ implies $p'(\alpha) \equiv p'(\alpha')$, vice versa.

**Proposition 5.3.** Assume that $p$ and $p'$ are equivalent. Then,

1. $p(V) \vdash_{+} p(t) : A$ implies $p'(V) \vdash_{+} p'(t) : A$
2. $p(V) \vdash_{-} p(t) : A$ implies $p'(V) \vdash_{-} p'(t) : A$, and
3. $p(V) \vdash_{\circ} p(t)$ implies $p'(V) \vdash_{\circ} p'(t)$.

A polarization $p$ is a conjugate of a polarization $p'$ if for any variable $\alpha$, if $p(\alpha)$ is an expression variable, then $p'(\alpha)$ is a continuation variable, and vice versa.

A derivation tree denoting a function has two proofs for acceptance and rejection as follows:

**Theorem 5.4.** 1. $p(V) \vdash_{+} p(t) : A$ implies that there exists a conjugate $p'$ of $p$ such that $p'(V) \vdash_{+} p'(t) : A$,

2. $p(V) \vdash_{-} p(t) : A$ implies that there exists a conjugate $p'$ of $p$ such that $p'(V) \vdash_{+} p'(t) : A$, and
3. $p(V) \vdash_{\circ} p(t)$ implies that there exists a conjugate $p'$ of $p$ such that $p'(V) \vdash_{\circ} p'(t)$.

**Proof.** By induction on derivation. We note that Proposition 5.3 ensures differences between equivalent polarizations can be ignored. ▶
5.4 A Short Remark about the Two Justifications

Careful readers might think that

- no distinction of expression variables and continuation variables in derivation trees with proofs is better, and
- BLC, which distinguishes expressions and continuations and requires the mutual transformations, is unnecessarily delicate.

However, BLC and the mutual transformations have an advantage in cases that functions and arguments have common variables. For example, a function \( \lambda a_2.\mu x_1.(x_0 \mid a_2) \) cannot be applied to an argument \( x_0 \) under any assumption because the function and argument must have converse polarities to each other. Each other. Because the mutual transformations, which have no variable, can respectively transform functions between expressions and continuations to those between continuations and expressions in BLC, \( \langle\lambda a_2.\mu x_1.(x_0 \mid a_2)\rangle \) of the type \( +A \rightarrow A \) can be applied to \( x_0 \) of the type \( +A \) where \( \lambda a_2.\mu x_1.(x_0 \mid a_2) \) has the type \( -A \leftarrow A \).

6 Related Work and Discussion

In this section, we discuss related work from three viewpoints of symmetric \( \lambda \)-calculi on the formulae-as-types and approaches in structural proof theory.

6.1 Symmetric Lambda-Calculi

The first symmetric \( \lambda \)-calculus was proposed by Filinski \[12, 13\]. Filinski described functions between continuations as follows: “We can therefore equivalently view a function \( f : A \rightarrow B \) as a continuation accepting a pair consisting of an \( A \)-type value and a \( B \)-accepting continuation. Such a pair will be called the context of a function application, and its type written as \( [B \leftarrow A] \).” In our observation on bilateralism, his intuition is not only computationally but also proof-theoretic semantically reasonable. The underlying idea in defining our calculus is that Filinski’s \( [A_1 \leftarrow A_0] \) is regarded as \( -A_0 \leftarrow A_1 \). We elaborate his idea in proof-theoretic semantics and carefully use Rumfitt’s polarities and the but-not connective, instead of simply using the negation connective as Filinski did.

A symmetric \( \lambda \)-calculus proposed by Barbanera and Berardi was invented to extract programs from classical logic proofs. Their calculus contains the involutive negation \( A^\perp \) for each type \( A \) and has symmetric application similar to commands in BLC. The essential difference between their calculus and BLC is polarity, that is, the polarized type \( -(A \vee B) \) in BLC corresponds to \( (A \vee B)^\perp \), which is identified with \( A^\perp \wedge B^\perp \) in their calculus. This lack of polarity information makes it difficult to reason about functions of Filinski’s calculus.

A calculus which was proposed by Lovas and Crary is the only symmetric \( \lambda \)-calculus that corresponds to classical logic in which expressions and continuations are symmetric on the bilateralism \[25\]. They defined \( \lambda \)-terms similar to those of the dual calculus which was defined by Wadler \[38\], and did not analyze the duality of functions in Filinski’s symmetric \( \lambda \)-calculus. They also did not adopt the implication \( \rightarrow \) but the negation connective \( \neg \) as a primitive type constructor. An expression of function type \( A_0 \rightarrow A_1 \) has type \( \neg(\neg A_0 \wedge A_1) \). Therefore, it is necessary to use an inference rule that corresponds to reductio ad absurdum in classical logic just to define \( \lambda \)-abstractions and applications of expressions in the simply typed \( \lambda \)-calculus, unlike ours. We also show that the negative polarity is suitable for representing continuations rather than the negation connective on the notion of bilateralism.

Ueda and Asai investigated Filinski’s symmetric \( \lambda \)-calculus, and provided an explicit definition of commands by writing \( \neg A \) for a continuation type \( A \) \[36\]. However, they did not
attempts to reason about the neutrality of functions in the symmetric $\lambda$-calculus. Also, the use of the negation connective to represent continuations is not reasonable as we have shown in the present paper. Actually, they also used the negation connective at only the outermost position of formulae. This operator of formulae should not be the negation connective but the negative polarity on bilateralism.

Curien and Herbelin’s $\bar{\lambda}\mu\bar{\mu}$-calculus \[3\] corresponds to Gentzen’s sequent calculus LK as well as the dual calculus. This symmetric infrastructure, namely the duality of LK, exhibits the duality between continuations and programs. Its symmetricity corresponds to that of the polarities in BLC and to that of types $A$ and $\neg A$ in Ueda and Asai’s calculus. The calculi based on LK naturally contain the (not involutive) negation type, which provides a more expressive power than BLC, as noted in Section 6.2. This observation raises an interesting question: What is the role of the negation type in practical programming languages?

6.2 Approaches in Structural Proof Theory

Girard and Parigot constructed calculi corresponding to classical logic \[18, 28\] and analyzed classical logic proof-theoretically. Girard also invented linear logic \[17\], which is very useful for analyzing classical logic. Danos et al. confirmed that classical logic has well behaved fragments using the positive and negative polarities \[5, 6\]. The calculi invented through their approaches are larger than or incomparable to ours because their motivations are different from ours. A goal of our work is not to analyze classical logic but to construct a minimal calculus to justify the duality of functions and the computations that delimited continuations raise. Although analyzing negations is a topic of great interest in proof theory \[27, 11, 29, 2, 7, 10, 9, 31, 23, 11\], we investigated the negation-free fragment of bilateral natural deduction.

Dual intuitionistic logic, which is symmetric to intuitionistic logic, is well known in structural proof theory \[19, 37, 34\]. A combined logic of intuitionistic and dual intuitionistic logics is classical logic. Whereas most of the logics are based on sequent calculi, Wansing constructed a natural deduction that can perform verification and falsification that corresponds to proving $+ A$ and $\neg A$, respectively, in our calculus \[42\]. However, a series of his works analyzed refutation, which is a proof for falsification in the context of studying various negations as seen in structural proof theory \[10, 41, 42\]. This is different from the objective in the present paper. He also neither provided a $\lambda$-calculus based on bilateralism nor described computational aspects, such as continuation controls. Tranchini also constructed a natural deduction of dual intuitionistic logic \[35\]. Our calculus seems to correspond to a negation-free fragment of his natural deduction.

7 Conclusion and Future Work

In this paper, we proposed a symmetric $\lambda$-calculus called the bilateral $\lambda$-calculus with the but-not connective based on bilateralism in proof-theoretic semantics. The formulae-as-types notion was extended to consider Rumfitt’s reductio, which corresponds to reduction ad absurdum as a $\mu$-abstraction of a first-class command in our calculus. Its call-by-value calculus can be defined as a sub-calculus of Wadler’s call-by-value dual calculus. We showed that the duality of functions is derived from definability of the mutual transformations between expressions and continuations in the bilateral $\lambda$-calculus. We also showed that every typable function has dual types.

In this paper, we have provided a method to justify a few notions in the theory of $\lambda$-calculi on bilateralism. The bilateral analysis in this paper targets the duality of functions
in Filinski’s symmetric λ-calculus. Bilateral analyses of asymmetric calculi constitute our future work.

The call-by-value variant of BLC corresponds to a sub-calculus of the call-by-value dual calculus with the but-not connective. It is also future work to clarify what practical uses are derived from the difference between BLC and the dual calculus.
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Heinrich Wansing. Falsification, natural deduction and bi-intuitionistic logic. *Journal of Logic and Computation*, 26(1):425–450, 2016.
A. The Call-by-Value Calculus of The Bilateral Lambda-Calculus

We introduce a call-by-value strategy to BLC and define a computationally consistent call-by-value calculus, which is equivalent to a sub-calculus of the call-by-value dual calculus by Wadler \[39\] without negation. Consequently, the consistency of our call-by-value calculus is obtained from the consistency of the call-by-value dual calculus.

A.1 The Call-by-Value Dual Calculus \(\text{CbV-Dc}\)

This subsection compares \(\text{CbV-Blc}\) with dual calculus invented by Wadler \[38, 39\], which corresponds to the classical sequent calculus on the notion of formulae-as-types. The call-by-value calculus of the dual calculus is known as a well established and computationally consistent system because it has the so-called CPS-semantics \[38\] and is equivalent to the call-by-value \(\lambda\mu\)-calculus \[39\]. We will show that our \(\text{CbV-Blc}\) is equivalent to a sub-calculus of the call-by-value dual calculus by giving an isomorphism between them.

We first recall the dual calculus. Suppose that countable sets of type variables, term variables, and coterm variables are given. Let \(\chi\), \(x\), and \(\alpha\) range over type variables, term variables, and coterm variables, respectively. Types, terms, coterms, statements, and syntactical objects are summarized in Figure 10. Substitution \([M/x]O\) of \(x\) in an expression \(O\) for \(M\) is defined in a standard component-wise and capture-avoiding manner. Similarly, substitution \([K/\alpha]O\) is also defined.

A judgment has the form of \(\Gamma \vdash \Delta | M : A, \Gamma | S \vdash \Delta\), or \(K : A | \Gamma \vdash \Delta\), where \(\Gamma\) is a type environment for terms that is a finite set of the form \(x : A\) and \(\Delta\) is a type environment for coterms that is a finite set of the form \(\alpha : A\). Figure 11 shows the inference rules.

We then recall the call-by-value calculus of the dual calculus. The values and the call-by-value evaluation contexts are defined as follows:

(values) \(W \::=\ x | \langle W, W \rangle | \langle W \rangle \text{inl} | \langle W \rangle \text{inr} | [K] \text{not} \)
\(| (W \bullet \text{fst}[\alpha]).\alpha | (W \bullet \text{snd}[\alpha]).\alpha\)

(contexts) \(F \::=\ \{ \} | \langle F, M \rangle | \langle F \rangle \text{inl} | \langle F \rangle \text{inr}\)

Figure 12 presents the call-by-value equation \(=\text{dcv}\) of the dual calculus. In the call-by-value dual calculus, the implication type \(A_0 \to A_1\) with its term \(\lambda x.M\), coterm \(M @ K\) can be defined as

\[
A_0 \to A_1 \equiv \neg (A_0 \land \neg A_1) \quad \lambda x.M \equiv [x'.(x' \bullet \text{fst}[x.(x' \bullet \text{snd}[\text{not}(M)]))]\text{not} \]
\[M @ K \equiv \text{not}([M, [K] \text{not}]\}

by using \(\land\), \(\lor\), and \(\neg\) as follows:
\[ \Gamma \vdash \Delta | M : A \quad K : A | \Gamma \vdash \Delta \]
\[ \frac{\Gamma, x : A \vdash \Delta | x : A} {\Gamma \vdash \Delta | M \cdot K \vdash \Delta} \]
\[ \frac{\alpha : A | \Gamma \vdash \Delta, \alpha : A} {\Gamma \vdash \Delta | M_1 : A_1 \quad \Gamma \vdash \Delta | M_2 : A_2} {\Gamma \vdash \Delta | (M_1, M_2) : A_1 \land A_2} \]
\[ K : A_1 | \Gamma \vdash \Delta \quad K : A_2 | \Gamma \vdash \Delta \]
\[ \frac{\text{fst}[K] : A_1 \land A_2 | \Gamma \vdash \Delta} {\Gamma \vdash \Delta | M : A_1} \quad \frac{\text{snd}[K] : A_1 \land A_2 | \Gamma \vdash \Delta} {\Gamma \vdash \Delta | M : A_2} \]
\[ \frac{\Gamma \vdash \Delta | M_1 : A_1}{\Gamma \vdash \Delta | (\langle M \rangle \text{in1}) : A_1 \lor A_2} \quad \frac{\Gamma \vdash \Delta | M_2 : A_2}{\Gamma \vdash \Delta | (\langle M \rangle \text{inr}) : A_1 \lor A_2} \]
\[ K_1 : A_1 | \Gamma \vdash \Delta \quad K_2 : A_2 | \Gamma \vdash \Delta \]
\[ \frac{\langle K_1, K_2 \rangle : A_1 \land A_2 | \Gamma \vdash \Delta} {\Gamma \vdash \Delta | M : A} \]
\[ \frac{K : A | \Gamma \vdash \Delta}{\Gamma \vdash \Delta | \langle K \rangle \text{not} : \neg A} \quad \frac{\Gamma \vdash \Delta | M : A}{\Gamma \vdash \Delta | \langle M \rangle \text{not} : \neg A} \]
\[ \frac{\Gamma \vdash \Delta, \alpha : A}{\Gamma \vdash \Delta | (S) \cdot \alpha : A} \quad \frac{x : A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta | (S) : A} \]

\[ \begin{array}{c}
\text{Figure 11} \\
\text{The inference rules of the dual calculus.}
\end{array} \]

**Proposition A.1.** The following inferences are derivable:

\[ \frac{\Gamma, x : A_0 \vdash \Delta | M : A_1}{\Gamma \vdash \Delta | \lambda x.M : A_0 \rightarrow A_1} \quad \frac{\Gamma \vdash \Delta | M : A_0 \quad K : A_1 | \Gamma \vdash \Delta}{M \cdot K : A_0 \rightarrow A_1 | \Gamma \vdash \Delta} \]

Also, \( \lambda x.M \) is a value and the following equations hold:

\[
\begin{align*}
\beta \rightarrow & \quad \lambda x.M_0 \cdot (M_1 \cdot K) = \text{dcv} M_1 \cdot x.(M_0 \cdot K) \\
\eta \rightarrow & \quad W = \text{dcv} \lambda x.((W \cdot (x \cdot \alpha)).\alpha).
\end{align*}
\]

**Proof.** The inference part is shown immediately by the definition of \( \lambda x.M \) and \( M \cdot K \). The term \( \lambda x.M \) is a value, since it has the form \( [K]\text{not} \). The equation \( (\beta \rightarrow) \) is shown by the case analysis of \( M_1 \).

(a) If \( M_1 \) is not a value, then the claim is obtained by using \( (\nu \wedge_0) \):

\[
\begin{align*}
\lambda x.M_0 \cdot (M_1 \cdot K) & \equiv [x'.(x' \cdot \text{fst}[x.(x' \cdot \text{snd}[\neg(M_0)])])]) \cdot \neg \cdot \neg((M_1, [K]\text{not})) \\
& = \text{dcv} (M_1, [K]\text{not}) \cdot x'.(x' \cdot \text{fst}[x.(x' \cdot \text{snd}[\neg(M_0)])]) \\
& = \text{dcv} M_1 \cdot x''.([x''.([K]\text{not}) \cdot x'.(x' \cdot \text{fst}[x.(x' \cdot \text{snd}[\neg(M_0)])])]) \\
& = \text{dcv} M_1 \cdot x''.([x''.([K]\text{not}) \cdot \text{fst}[x.(x''.([K]\text{not}) \cdot \text{snd}[\neg(M_0)]))]) \\
& = \text{dcv} M_1 \cdot x''.(x.([K]\text{not} \cdot \neg(M_0))) \\
& = \text{dcv} M_1 \cdot x.([K]\text{not} \cdot \neg(M_0)) \\
& = \text{dcv} M_1 \cdot x.(M_0 \cdot K).
\end{align*}
\]
\[(\beta \land 0) \quad (W_0, W_1) \cdot \text{fst}[K] =_{dcv} W_0 \cdot K\]
\[(\beta \lor 1) \quad (W_0, W_1) \cdot \text{snd}[K] =_{dcv} W_1 \cdot K\]
\[(\beta \lor 0) \quad (W) \cdot \text{inl}[K_0, K_1] =_{dcv} W \cdot K_0\]
\[(\beta \lor 1) \quad (W) \cdot \text{inr}[K_0, K_1] =_{dcv} W \cdot K_1\]
\[(\beta \neg) \quad [K] \cdot \text{not}(M) =_{dcv} M \cdot K\]
\[(\beta R) \quad W \cdot x.(S) =_{dcv} W/x]S\]
\[(\beta L) \quad (S) \cdot \alpha \cdot K =_{dcv} [K/\alpha]S\]
\[(\eta R) \quad M =_{dcv} (M \cdot \alpha) \cdot \alpha\] if \(\alpha\) is fresh
\[(\eta L) \quad K =_{dcv} x.(x \cdot K)\] if \(x\) is fresh
\[(\eta \wedge) \quad W =_{dcv} ([W \cdot \text{fst}[\alpha]], \alpha, (W \cdot \text{snd}[\alpha]), \alpha)\] if \(\alpha\) is fresh
\[(\eta \lor) \quad K =_{dcv} [x.(x \cdot \text{inl}[K]), x.(x \cdot \text{inr}[K])]\] if \(x\) is fresh
\[(\eta \neg) \quad W =_{dcv} x.(W \cdot \text{not}(x)) \cdot \text{not}\] if \(x\) is fresh
\[(\zeta) \quad \mathcal{F}(M) \cdot K =_{dcv} M \cdot x.(\mathcal{F}(x) \cdot K)\] if \(x\) is fresh

Figure 12 The equations of the call-by-value dual calculus.

(b) If \(M_1\) is a value (say \(W\)), then
\[(\lambda x. M_0) \cdot (W \otimes K)\]
\[\equiv [x'.(x' \cdot \text{fst}[x'.(x' \cdot \text{snd}[\text{not}(M_0)])]) \cdot \text{not} \cdot \text{not} ([W', K \cdot \text{not}])\]
\[=_{dcv} (W', [K] \cdot \text{not}) \cdot x'.(x' \cdot \text{fst}[x'.(x' \cdot \text{snd}[\text{not}(M_0)])])\]
\[=_{dcv} (W', [K] \cdot \text{not}) \cdot \text{fst}[x'.([W', K \cdot \text{not}]), x \cdot \text{snd}[\text{not}(M_0)])\]
\[=_{dcv} W \cdot x.(M \cdot \text{not} \cdot \text{not}(M_0))\]
\[=_{dcv} W \cdot x.(M \cdot K)\]

The but-not type \(A_0 \leftrightarrow A_1\) with its term \(K \otimes M\), coterm \(\lambda \alpha. K\) are also defined as
\[A_0 \leftrightarrow A_1 \equiv A_0 \land \neg A_1\]
\[K \otimes M \equiv (M, [K] \cdot \text{not})\]
\[\lambda \alpha. K \equiv x.(x \cdot \text{snd}[\text{not}(x \cdot \text{fst}[K] \cdot \alpha)])\]

**Proposition A.2.** The following inferences are derivable:
\[
\begin{array}{c}
\Gamma \vdash \Delta \mid M : A_0 \quad K : A_1 \mid \Gamma \vdash \Delta \\
\hline
\Gamma \vdash \Delta \mid K \otimes M : A_0 \leftarrow A_1 \\
\end{array}
\]
\[
\begin{array}{c}
K : A_0 \mid \Gamma \vdash \Delta, \alpha : A_1 \\
\hline
\lambda \alpha. K : A_0 \leftarrow A_1 \mid \Gamma \vdash \Delta \\
\end{array}
\]

Also, \(K \otimes W\) is a value and the following equations hold:
\[(\beta \leftarrow) \quad (K_0 \otimes W) \cdot (\lambda \alpha. K_1) =_{dcv} (W \cdot K_1) \cdot \alpha \cdot K_0\]
\[(\eta \leftarrow) \quad K =_{dcv} \lambda \alpha. (x.((\alpha \otimes x) \cdot K))\]
\[(\zeta \leftarrow) \quad (K_0 \otimes M) \cdot K_1 =_{dcv} M \cdot x.((K_0 \otimes x) \cdot K_1)\]

if \(x\) is fresh.

**Proof.** The inference part is shown immediately by the definition of \(K \otimes M\) and \(\lambda \alpha. K\). By the definition, it is immediately checked that a term of the form \(K \otimes W\) is a value.
The first equation $(\beta\leftarrow)$ is shown as follows:

$$(K_0 \vdash W) \cdot (\lambda x. K_1) \equiv (W, [K_0 \not\alpha] \cdot x \cdot \text{snd}((x \cdot \text{fst}[K_1], \alpha)))$$

$$_{dcv} = (W, [K_0 \not\alpha] \cdot \text{snd}((W, [K_0 \not\alpha] \cdot \text{fst}[K_1], \alpha)))$$

$$_{dcv} = (K_0 \not\alpha \cdot \not\alpha((W \cdot K_1), \alpha))$$

$$_{dcv} = (W \cdot K_1)، \alpha \cdot K_0$$

The second equation $(\zeta\leftarrow)$ is shown with $(\nu \land_0)$ as follows:

$$(K_0 \vdash M) \cdot K_1 \equiv (M, [K_0 \not\alpha] \cdot K_1)$$

$$_{dcv} = M \cdot x.((x, [K_0 \not\alpha] \cdot K_1))$$

$$\equiv M \cdot x.((K_0 \vdash x) \cdot K_1)$$

We define a sub-calculus CbV-DC$\rightarrow$ of the call-by-value dual calculus that is obtained by removing the negation type $\neg A$ and adding the implication and but-not types with their syntactical objects, typing rules, and equations as primitives. The calculus CbV-DC$\rightarrow$ can be roughly understood as a sub-calculus of the call-by-value dual calculus that forbids free occurrences of the negation connective and allows only the occurrences necessary to define the implication and the but-not connectives.

A.2 Equivalence between CbV-BLC and CbV-DC$\rightarrow$

We define a translation $(\beta)\rightleftharpoons$ from CbV-BLC into CbV-DC$\rightarrow$. The translation is designed so that judgments in the bilateral natural deduction are mapped to sequents in the sequent calculus including proofs.

We assume that there exist variables $x_{cat}^\alpha$ and covariables $\alpha_{\bullet\bullet}$ of CbV-DC$\rightarrow$ for any constant expressions $cst^\alpha$ and constant continuations $\bullet^\alpha$ of CbV-BLC, respectively.

Term $(E)^\beta$, coterm $(C)^\beta$, and statement $(N)^\beta$ are defined inductively as shown in Figure 13. We show that the translation preserves typability. Let Cons$^+$ and Cons$^-$ be the sets of constants of the form $cst^\alpha$ and $\bullet^\alpha$, respectively. For any $X \subseteq_{\text{fin}} \text{Cons}^+$ and $Y \subseteq_{\text{fin}} \text{Cons}^-$, we respectively define

$$(\Pi)^\beta_X = \Pi \cup \{x_{cat}^\alpha : \alpha_{\bullet\bullet} \in X\} \quad \text{and} \quad (\Sigma)^\beta_Y = \Sigma \cup \{\alpha_{\bullet\bullet} = \bullet^\alpha \in Y\}$$

- **Proposition A.3.** 1. $\Pi; \Sigma \vdash_+ E$: A implies $(\Pi)^\beta_X \vdash (\Sigma)^\beta_Y \vdash (E)^\beta$: A for any Cons$^+$ $(E) \subseteq X$ and Cons$^-$ $(E) \subseteq Y$,
2. $\Pi; \Sigma \vdash_-$ C: A implies $(C)^\beta$: A $\vdash (\Pi)^\beta_X \vdash (\Sigma)^\beta_Y$ for any Cons$^+$ $(C) \subseteq X$ and Cons$^-$ $(C) \subseteq Y$, and
3. $\Pi; \Sigma \vdash_\circ N$ implies $(\Pi)^\beta_X \vdash (N)^\beta_D \vdash (\Sigma)^\beta_Y$ for any Cons$^+$ $(N) \subseteq X$ and Cons$^-$ $(N) \subseteq Y$.

**Proof.** The claims are shown by simultaneous induction on the derivation of the bilateral $\lambda$-calculus.

- **Lemma A.4.** 1. $E$ is a value of CbV-BLC if and only if $(E)^\beta$ is a value of CbV-DC$\rightarrow$,
2. $(V/x)D)^\beta \equiv [(V)^\beta/x][D)^\beta \text{ and } ([C/o]D)^\beta \equiv ([C)^\beta/o](D)^\beta$. 

The claim (1) can be shown immediately.

\[(\text{cst}^α)^2 \equiv x_{\text{cst}}\]
\[(x^A)^2 \equiv x\]
\[(\lambda x^A.E)^2 \equiv \lambda x.(E)^2\]
\[(E_0,E_1)^2 \equiv ((E_0)^2 \bullet ((E_1)^2 \otimes α))\cdot α\]
\[(E_0,E_1)^2 \equiv ((E_0)^2 \cdot (E_1)^2)\]
\[(\pi_0(E))^2 \equiv ((E)^2 \cdot \text{fst}[α])\cdot α\]
\[(\pi_1(E))^2 \equiv ((E)^2 \cdot \text{snd}[α])\cdot α\]
\[(\mu x^A.N)^2 \equiv ((N)^2)\cdot α\]
\[
\begin{align*}
\langle E \mid C \rangle^2 & \equiv (E)^2 \cdot (C)^2 \\
\end{align*}
\]

**Figure 13** A translation from CbV-BLC into CbV-DC++

**Proof.** The claim (1) can be shown immediately.

The former claim of (2) is shown by induction on \(S\). Note that, by (1), \((E)^1\) is a value if and only if \([(V/x]\ E)^1\) is a value.

The case of \(S \equiv \text{cst}^α\):

\[([V/x]\ \text{cst}^α)^1 \equiv ([V/x]\ x_{\text{cst}})^1 \equiv [(V)^1/x]\ x_{\text{cst}} \equiv [(V)^1/x]\ (\text{cst}^α)^1\]

The case of \(S \equiv x^A\):

\[([V/x]\ x^A)^1 \equiv ([V)^1/x]\ x \equiv [(V)^1/x]\ (x^A)^1\]

The case of \(S \equiv x_0^A\), where \(x^A \not\equiv x_0^A\):

\[([V/x]\ x_0^A)^1 \equiv x_0^A \equiv [(V)^1/x]\ x_0 \equiv [(V)^1/x]\ (x_0^A)^1\]

The case of \(S \equiv \pi_0(E)\) is shown by using induction hypothesis.

\[([V/x]\ \pi_0(E))^1 \equiv ([V/x]\ E)^1\]
\[\equiv (((V/x/E)^1 \cdot \text{fst}[α])) \cdot \pi_0)\]
\[\equiv (((V)^1/x/E)^1 \cdot \text{fst}[α])) \cdot \pi_0)\]
\[\equiv [(V)^1/x]((E)^1 \cdot \text{fst}[α])) \cdot \pi_0)\]
\[\equiv [(V)^1/x]((E)^1 \cdot \text{fst}[α])) \cdot \pi_0)\]

The other cases \(S \equiv \pi_1(E), (E_0,E_1), \lambda x.E, E_0E_1, \mu α.N, \bullet α, a^A, \pi_0(C), \pi_1(C), (C_0, C_1), \lambda α.C, C_0C_1, μx.N, \text{ and } (E \mid C)\) are also shown straightforwardly by using induction hypothesis.

The latter claim of (2) is shown by induction on \(S\).

The case of \(S \equiv \bullet α\):

\[([C/α]\bullet α)^1 \equiv (\bullet α)^1 \equiv ((C)^1/α]\bullet α \equiv [(C)^1/α]\ (\bullet α)^1\]

The case of \(S \equiv a^A\):

\[([C/α]\ a^A)^1 \equiv (C)^1 \equiv ([(C)^1/α]\ a \equiv [(C)^1/α]\ (a^A)^1\]
The case of $S \equiv a_0^{A_0}$, where $a^A \not\equiv a_0^{A_0}$:

$$\begin{align*}
([C/\alpha]a_0^{A_0})^\dagger & \equiv (a_0^{A_0})^\dagger \equiv a_0 \equiv [(C)^\dagger/\alpha]a_0 \equiv [(C)^\dagger/\alpha](a_0^{A_0})^\dagger
\end{align*}$$

The case of $S \equiv \pi_0(C_0)$ is shown by using induction hypothesis:

$$
\begin{align*}
([C/\alpha]\pi_0(C_0))^\dagger & \equiv (\pi_0([C/\alpha]C_0))^\dagger \\
\equiv (x.(x)\text{inl} \bullet (x)\pi_0(C_0))^\dagger & \equiv (x.(x)\text{inl} \bullet (C)^\dagger/\alpha)(C_0)^\dagger \\
\equiv [(C)^\dagger/\alpha](x.(x)\text{inl} \bullet (C_0)^\dagger) & \equiv [(C)^\dagger/\alpha](\pi_0(C_0))^\dagger \\
\end{align*}
$$

by induction hypothesis.

The other cases $S \equiv \text{cst}^a$, $x^A$, $\pi_0(E)$, $\pi_1(E)$, $(E_0, E_1)$, $\lambda x.E$, $E_0 E_1$, $\mu \alpha.N$, $\pi_1(C)$, $(C_0, C_1)$, $\lambda a.C$, $C_0 C_1$, $\mu x.N$, and $(E \mid C)$ are also shown straightforwardly by using induction hypothesis.

\begin{flushright}
\textbf{Theorem A.5.} $D_0 =_v D_1$ implies $(D_0)^\sharp =_{dcv} (D_1)^\sharp$.
\end{flushright}

\textbf{Proof.} First, we define a coterm $K$-indexed translation $(-)^\sharp_K$ from contexts for expressions of CbV-BLC into contexts of CbV-DC$\rightarrow_{v}$ as follows:

$$
\begin{align*}
(\{\}_{K})^\sharp & \equiv \{\}_{K} \\
(V\{E\})^\sharp_K & \equiv (E)^\sharp_{x.\{x\}^{\sharp}(\sharp xK)} & (\pi_0(E))^\sharp_K & \equiv (\pi_1(E))^\sharp_{x.(\{x\}^{\sharp}(\sharp x)\pi_0(K))} & (\pi_0(E))^\sharp_{K} & \equiv (\pi_1(E))^\sharp_{K} \equiv (\pi_1(E))^\sharp_{x.\{x\}^{\sharp}(\sharp x)\pi_0(K)}.
\end{align*}
$$

Next, we can immediately confirm

$$
(\pi_0(E))^\sharp =_{dcv} (\pi_1(E))^\sharp \equiv x.((\pi_1(E))^\sharp_K \{x\}).
$$

by induction on $E$.

Finally, the theorem can be shown by the case analysis of $=_v$ and Lemma A4.

The case of $(\lambda x.E)V =_v [V/\alpha]E$. By using Lemma A4(1) and (2), we have

$$
\begin{align*}
((\lambda x.E)V)^\sharp & \equiv ((\pi_0(V))^\sharp \bullet (\pi_1(V))^\sharp \bullet (\pi_0(V))^\sharp) \alpha =_{dcv} ((\pi_0(V))^\sharp \bullet (\pi_1(V))^\sharp \bullet (\pi_0(V))^\sharp) \alpha \\
=_{dcv} ((\pi_0(V))^\sharp \bullet (\pi_1(V))^\sharp \bullet (\pi_0(V))^\sharp) \alpha & \equiv (\pi_0(V))^\sharp \bullet (\pi_1(V))^\sharp \bullet (\pi_0(V))^\sharp) \alpha =_{dcv} ([V/\alpha]E)^\sharp .
\end{align*}
$$

The case of $\lambda x.Vx =_v V$. By using Lemma A4(1), we have

$$
\begin{align*}
(\lambda x.Vx)^\sharp & \equiv \lambda x.((\pi_0(V))^\sharp \bullet (\pi_1(V))^\sharp \bullet (\pi_0(V))^\sharp) \alpha =_{dcv} (V)^\sharp .
\end{align*}
$$

The case of $\pi_0((V_0, V_1)) =_v (V_0, V_1)$. By using Lemma A4(1), we have

$$
\begin{align*}
(\pi_0((V_0, V_1)))^\sharp & \equiv (((V_0)^\sharp, (V_1)^\sharp) \bullet \text{fst}[\alpha]) \alpha =_{dcv} ((V_0)^\sharp) \bullet (V_0)^\sharp \alpha =_{dcv} (V_0)^\sharp .
\end{align*}
$$

The case of $\pi_1((V_0, V_1)) =_v V_1$ is shown similarly.

The case of $(\pi_0(V), \pi_1(V)) =_v V$. By using Lemma A4(1), we have

$$
\begin{align*}
((\pi_0(V), \pi_1(V)))^\sharp & \equiv (((V)^\sharp \bullet \text{fst}[\alpha]) \alpha =_{dcv} ((V)^\sharp \bullet \text{fst}[\alpha]) \alpha =_{dcv} (V)^\sharp .
\end{align*}
$$

The case of $\mu a.(E \mid a) =_v E$. We have

$$
\begin{align*}
(\mu a.(E \mid a))^\sharp & \equiv ((E)^\sharp \bullet \alpha) \alpha =_{dcv} (E)^\sharp .
\end{align*}
$$
The case of \(<V | \mu x.N>\) = \([V/x]N\): By using Lemma \([A.4]\) (1) and (2), we have
\[\langle\langle V | \mu x.N\rangle\rangle^\sharp = (V)^\sharp \cdot x.(\langle\langle N\rangle\rangle^\sharp) = _{dcv} [(V)^\sharp/x]\langle\langle N\rangle\rangle^\sharp = \langle(V|x)\langle\langle N\rangle\rangle^\sharp\rangle^\sharp.\]

The case of \(<\mathcal{E}\{E\} | C>\) = \(<E | \mu x.\mathcal{E}\{x\} | C>\). By using (\(*\)), we have
\[\langle\langle E \{E\} | C\rangle\rangle^\sharp = \langle\langle E\{E\}\rangle\rangle^\sharp \cdot \langle\langle C\rangle\rangle^\sharp = _{dcv} (E)^\sharp \cdot x.(\langle\langle E\{E\}\rangle\rangle^\sharp) = _{dcv} (E)^\sharp \cdot x.(\langle\langle C\rangle\rangle^\sharp) \equiv \langle\langle E | \mu x.\mathcal{E}\{x\} | C\rangle\rangle^\sharp.\]

The case of \((\lambda a. C_0) C_1 = _v [C_1/a] C_0\). By using Lemma \([A.4]\) (2), we have
\[\langle\lambda a. C_0 | C_1\rangle^\sharp = \lambda a.\langle\langle x.(\langle x \alpha \rangle \bullet (C_0)^\sharp)\rangle\rangle = _{dcv} (C)^\sharp.\]

The case of \(\pi_0((C_0, C_1)) = _v C_0\). We have
\[\pi_0((\langle C_0, C_1\rangle)^\sharp) = \pi_0(\langle\langle x.(\langle x \alpha \rangle \bullet (C_0)^\sharp)\rangle\rangle) = _{dcv} (C)^\sharp.\]

The case of \(\pi_1((\langle C_0, C_1\rangle) = _v C_1\) is also shown similarly.

The case of \((\pi_0(C), \pi_1(C)) = _v C\). We have
\[\langle\langle\pi_0(C), \pi_1(C)\rangle\rangle^\sharp = \langle\langle x.(\langle x \alpha \rangle \bullet (C_0)^\sharp)\rangle\rangle \cdot \langle\langle x.(\langle x \alpha \rangle \bullet (C_1)^\sharp)\rangle\rangle = _{dcv} (C)^\sharp.\]

The case of \(\mu x.\langle\langle x | C\rangle\rangle^\sharp = _v C\). We have
\[\langle\langle\mu x.\langle\langle x | C\rangle\rangle\rangle^\sharp = \langle\langle x.(\langle x \alpha \rangle \bullet (C)^\sharp)\rangle\rangle = _{dcv} (C)^\sharp.\]

The case of \(\langle\langle\mu a. N | C\rangle\rangle^\sharp = _v [C/a] N\). By using Lemma \([A.4]\) (2), we have
\[\langle\langle\mu a. N | C\rangle\rangle^\sharp = \langle\langle (N)^\sharp \cdot \alpha \bullet (C)^\sharp \rangle\rangle = _{dcv} ([C/^\alpha](N)^\sharp \equiv ([C/a]N)^\sharp.\]

We next define a translation \((-)^\sharp\) from CbV-DC\(\rightarrow^\pm\) into CbV-BLC. Expression \((M)^\sharp\), continuation \((K)^\sharp\), and command \((S)^\sharp\) for any typable \(M\), \(K\), and \(S\) in CbV-DC\(\rightarrow^\pm\) are defined inductively as shown in Figure \([A.8]\).

We show that this translation preserves typability. Let \(\Gamma\) be \(x_{cat} : A^\prime, x : A\). Then we define \((\Gamma)^\sharp\) by \(x_{cat} : A^\prime, x : A\). Similarly, we also define \((\Delta)^\sharp\).

\textbf{Proposition A.6.} 1. \(\Gamma \vdash \Delta | M : A \implies (\Gamma)^\sharp; (\Delta)^\sharp \vdash_+ (M)^\sharp : A\), \(K : A | \Gamma \vdash \Delta \implies (\Gamma)^\sharp; (\Delta)^\sharp \vdash_- (K)^\sharp : A\), and \(3. \Gamma | S \vdash \Delta \implies (\Gamma)^\sharp; \Delta^\sharp \vdash_o (S)^\sharp\).

\textbf{Proof.} The claims can be shown by simultaneous induction on the derivation of judgments of CbV-DC\(\rightarrow^\pm\).

We show that the translation \((-)^\sharp\) is an inverse of \((-)^\sharp\) up to \(=_v\) as follows:

\textbf{Theorem A.7.} 1. \(((D)^\sharp)^\sharp =_v D\) holds, and
The claims (1) and (2) are shown by induction on $M$.

**Proof.**

1. The claim (1) is shown immediately. The claims of (2) are shown by induction on $M$.

2. If $\alpha$ has a type $A$.

3. If $\alpha. K$ has a type $A_1 \leftarrow A$.

4. $\langle F \rangle^\beta \equiv \alpha^\beta$ if $\alpha$ has a type $A$.

5. $\langle \lambda \alpha. K \rangle^\beta \equiv \alpha^\beta$.

6. $\langle \lambda x. M \rangle^\beta \equiv \lambda x^\beta. (M)^\beta$

7. If $\alpha. K$ has a type $A_1 \leftarrow A$.

8. $\langle K \& M \rangle^\beta \equiv \mu a. (\langle M \rangle^\beta | a(K)^\beta)$

9. $\langle (E_0, E_1)^\beta \equiv (\langle E_0 \rangle^\beta, (E_1)^\beta)$

10. $\langle (\langle F \& M \rangle)^\beta \equiv \langle (\langle F \rangle^\beta, (M)^\beta)$

11. $\langle (\langle F \rangle^\beta | \pi_1(a))^\beta \equiv \mu a. (\langle F \rangle^\beta | \pi_1(a))^\beta$

12. $\langle (\langle F \rangle^\beta | \pi_1(a))^\beta \equiv \mu a. (\langle F \rangle^\beta | \pi_1(a))^\beta$.

---

**Lemma A.8.**

1. $M$ is a value of CbV-DC$_{\Leftrightarrow}$ if and only if there exists a value $V$ of CbV-BLC such that $V = (M)^\beta$, and

2. $\langle W/x \rangle^\beta = \langle (W)^\beta | x\rangle^\beta$ and $\langle (K/\alpha)O \rangle^\beta \equiv \langle (K)^\beta / a \rangle^\beta$.

**Proof.**

The claim (1) is shown immediately. The claims of (2) are shown by induction on $O$.

For any $\mathcal{F}$ (contexts of CbV-DC$_{\Leftrightarrow}$), we define $\langle \mathcal{F} \rangle^\beta$ (contexts for expressions of CbV-BLC) as follows:

$\langle \{\} \rangle^\beta \equiv \{\}$

$\langle \langle W, \mathcal{F} \rangle \equiv \langle (W)^\beta, (\mathcal{F})^\beta \rangle$

$\langle \langle \mathcal{F}, \lambda x. \mathcal{M} \rangle \equiv \langle \langle \mathcal{F} \rangle^\beta | \pi_1(a) \rangle$

$\langle \{\mathcal{F} \} \equiv \mathcal{F}^\beta \{\}^\beta \}^\beta \rangle$ holds, and

2. $\langle (\mathcal{F})^\beta (E) | C \rangle^\beta \equiv \langle \langle (\mathcal{F})^\beta (E) | C \rangle \rangle$ holds.

**Proof.**

The claims (1) and (2) are shown by induction on $\mathcal{F}$.

**Theorem A.10.** $O_0 =_{\text{dcv}} O_1$ implies $(O_0)^\beta = (O_1)^\beta$.

**Proof.**

The claim is shown by the case analysis of $=_{\text{dcv}}$.

The case of $(\beta \rightarrow)$ is shown as follows.

$\langle \lambda x. M_0 \rangle \bullet (M_1 \& K)^\beta \equiv \langle \lambda x. (M_0)^\beta | \mu x_1. (x_1 (M_1)^\beta | (K)^\beta) \rangle$

$= \langle \langle \lambda x. (M_0)^\beta \rangle (M_1)^\beta | (K)^\beta \rangle$

$= \langle \langle M_1 \rangle^\beta | \mu x. ((\lambda x. (M_0)^\beta) x | (K)^\beta) \rangle$

$= \langle \langle M_1 \rangle^\beta | \mu x. ((M_0)^\beta | (K)^\beta) \rangle$

$\equiv \langle (M_1 \bullet x. (M_0 \bullet K))^\beta \rangle$.
The case of \( (\beta\leftarrow) \) is shown as follows.

\[
\{(K_0 \leq M) \bullet (\lambda a.K_1)\}^b = \langle (\mu a.((M)^b) | a_1(K_1)^b) | \lambda a.(K_1)^b \rangle \\
=_{v} \langle (M)^b | (\lambda a.(K_1)^b)^b \rangle(K_0)^b \\
=_{v} \langle (M)^b | ((K_0)^b \sbarr a(K_1)^b) \\
\equiv [(K_0)^b \sbarr a((M)^b) | (K_1)^b) \\
=_{v} \langle \mu a.((M)^b) | (K_1)^b \rangle | (K_0)^b \rangle \\
\equiv ((M \bullet K_1).a \bullet K_0)^b .
\]

The case of \( (\beta \land_0) \). By using Lemma \(\A.S.8\) (1), We have

\[
\langle(W_0, W_1) \bullet \text{\texttt{fst}[K]}\rangle^b \equiv \langle \langle(W_0)^b, (W_1)^b) | \mu x.\langle \pi_0(x) | (K)^b \rangle \rangle \\
=_{v} \langle \pi_0((W_0)^b, (W_1)^b) | (K)^b \rangle \\
=_{v} \langle (W_0)^b | (K)^b \rangle \equiv (W_0 \bullet K)^b .
\]

The case of \( (\beta \land_1) \) is shown similarly.

The case of \( (\beta \lor_0) \). We have

\[
\langle(W) \text{\texttt{inl} \bullet [K_0, K_1]}\rangle^b \equiv \langle \mu a.\langle(W)^b | \pi_0(a) \rangle | (K_0)^b, (K_1)^b \rangle \\
=_{v} \langle(W)^b | \pi_0((K_0)^b, (K_1)^b) \rangle \\
=_{v} \langle(W)^b | (K)^b \rangle \equiv (W \bullet K)^b .
\]

The case of \( (\beta \lor_1) \) is shown similarly.

The case of \( (\beta R) \). By using Lemma \(\A.S.8\) (1) and (2), We have

\[
(W \bullet x.(S))^b \equiv \langle(W)^b | \mu x.\langle S \rangle^b \rangle =_{v} \langle(W)^b / x \rangle(S)^b \equiv \langle(W/x)S\rangle^b .
\]

The case of \( (\beta L) \). By using Lemma \(\A.S.8\) (2), We have

\[
\langle(S).a \bullet K\rangle^b \equiv \langle \mu a.\langle(S)^b | (K)^b \rangle \rangle =_{v} \langle (K)^b / a(K)S \rangle^b \equiv \langle(K/a)S\rangle^b .
\]

The case of \( (\eta \rightarrow) \) is shown by using Lemma \(\A.S.8\) (1).

\[
\langle\lambda x.(W \bullet (x \oplus a)).a)\rangle^b \equiv \langle \lambda x.\mu a.\langle(W)^b | \mu x.\langle x_1x \rangle a \rangle \rangle \\
=_{v} \langle \lambda x.\mu a.\langle(W)^b | a \rangle \rangle =_{v} \lambda x.(W)^b x =_{v} (W)^b .
\]

The case of \( (\eta \land) \).

\[
\langle\lambda a.(x.(a \land x) \bullet K)\rangle^b \equiv \langle \lambda a.\mu x.\langle a_1a \rangle x | (K)^b \rangle \\
=_{v} \langle \lambda a.\mu x.\langle(K)^b | a \rangle x \rangle =_{v} \lambda a.(K)^b a =_{v} (K)^b .
\]

The case of \( (\eta \land) \). Note that \( (W \bullet \text{\texttt{fst}[a]}).a) \equiv \pi_0((W)^b) \) by Lemma \(\A.S.8\) (1). Hence we have

\[
\langle(W \bullet \text{\texttt{fst}[a]}).a, (W \bullet \text{\texttt{snd}[a]}).a)\rangle^b =_{v} (\pi_0((W)^b), \pi_1((W)^b)) =_{v} (W)^b .
\]

The case of \( (\eta \lor) \). Note that \( (x.(x \text{\texttt{inl} \bullet K})\rangle^b =_{v} \pi_0((K)^b) \). Hence we have

\[
\langle[x.(x \text{\texttt{inl} \bullet K}), x.(x \text{\texttt{inr} \bullet K})]\rangle^b =_{v} (\pi_0((K)^b), \pi_1((K)^b)) =_{v} (K)^b .
\]
The cases of \((\eta_R)\) and \((\eta_L)\) are shown immediately. The case of \((\zeta)\) is shown by using Lemma A.9:

\[
(\mathcal{F}\{M\} \bullet K)^b \equiv (\mathcal{F}\{M\})^b \mid (K)^b =_v (\mathcal{F})^b \{\{M\}\} \mid (K)^b =_v (M \bullet x.(\mathcal{F}\{x\}) \bullet K)^b.
\]

The equivalence between CbV-DC and CbV-BLC clarifies an essential difference between the full dual calculus and the bilateral \(\lambda\)-calculus. The negation of the dual calculus is not involutive, since \(\neg\neg A\) is not isomorphic to \(A\). The dual calculus actually contains the involutive duality not as the object-level negation but as the meta-level operation such as the antecedent and succedent duality of the sequent calculus. On the other hand, the negation is represented using inversions of polarities in the bilateral \(\lambda\)-calculus. By definition, the negation is involutive.

## B Proofs

### Proof of Proposition 2.2

The proposition holds as follows:

\[
\begin{array}{c}
+ A_0 \rightarrow A_1 \quad [+ A_0] \\
+ A_1 \quad [- A_1] \\
\hline
\bot
\end{array}
\quad
\begin{array}{c}
- A_0 \rightarrow A_1 \\
- A_0 \rightarrow A_1
\end{array}
\]

\[
\begin{array}{c}
- A_0 \leftrightarrow A_1 \\
- A_0 \leftrightarrow A_1
\end{array}
\]

\[
\begin{array}{c}
+ A_0 \leftrightarrow A_1 \\
+ A_0 \leftrightarrow A_1
\end{array}
\]

\[
\begin{array}{c}
+ A_0 \rightarrow A_1 \\
- A_0 \rightarrow A_1
\end{array}
\]

---

### Proof of Proposition 2.4

1) It suffices to use (Non-contradiction) and (Reductio) with \((\land\text{-E}_{+0}), (\land\text{-E}_{+1}), (\land\text{-I}_{+}), (\lor\text{-E}_{-}), (\lor\text{-I}_{-}), (\lor\text{-L}_{-})\), and \((\lor\text{-L}_{-1})\) as follows:

\[
\begin{array}{c}
[- A_0] \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
[- A_1] \\
\hline
\mathcal{A} \\
\mathcal{A}^* \\
\mathcal{A}^* \\
\mathcal{A}^* \\
\mathcal{A}^* \\
\mathcal{A}^*
\end{array}
\]

\[
\begin{array}{c}
[+ A_0 \land A_1] \\
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[+ A_1] \\
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---
where $i = 0, 1$.

2) It suffices to use (Non-contradiction) and (Reductio) with ($\rightarrow$-E), ($\rightarrow$I), ($\leftarrow$-E), and ($\leftarrow$I) as follows:

\[
\begin{align*}
[+A_0] & \quad +A_0 \\
\vdash & \quad A_0 \\
\vdash & \quad A_1 \\
\vdash & \quad [A^*] \\
\vdash & \quad [-A_0 \lor A_1] \\
\vdash & \quad [+A_i] \\
\vdash & \quad [-A_i] \\
\vdash & \quad A_i \\
\vdash & \quad A_0 \lor A_1 \\
\vdash & \quad [A^*] \\
\vdash & \quad [+A_0] \\
\vdash & \quad [-A_0] \\
\vdash & \quad A_0 \\
\vdash & \quad A_1 \\
\vdash & \quad [A^*] \\
\vdash & \quad [-A_0 \lor A_1] \\
\vdash & \quad [+A_i] \\
\vdash & \quad [-A_i] \\
\vdash & \quad A_i \\
\vdash & \quad A_0 \lor A_1.
\end{align*}
\]

**Proof of Theorem 5.1.** The proposition holds because the following:

\[
\begin{align*}
II' & ; \Sigma' \vdash E : A_0 \rightarrow A_1 & II' & ; \Sigma' \vdash x : A_0 \\
II' & ; \Sigma' \vdash Ex : A_1 & II' & ; \Sigma' \vdash a : A_1 \\
II' & ; x : A_0; \Sigma, a : A_1 \vdash \langle Ex \mid a \rangle : A_1 & II' & ; \Sigma' \vdash \mu a. \langle Ex \mid a \rangle : A_0 \\
II' & ; \Sigma', a : A_1 \vdash \lambda a. \mu a. \langle Ex \mid a \rangle : A_0 \leftarrow A_1 \\
II' & ; \Sigma' \vdash E : A_0 \rightarrow A_1 & II' & ; \Sigma' \vdash x : A_0 \\
II' & ; \Sigma' \vdash Ex : A_1 & II' & ; \Sigma' \vdash a : A_1 \\
II' & ; x : A_0; \Sigma, a : A_1 \vdash \langle Ex \mid a \rangle : A_0 \\
II' & ; \Sigma' \vdash \mu a. \langle Ex \mid a \rangle : A_0 \leftarrow A_1 \\
II' & ; \Sigma' \vdash \lambda a. \mu a. \langle Ex \mid a \rangle : A_0 \leftarrow A_1 \\
II' & ; \Sigma' \vdash C : A_0 \leftarrow A_1 & II' & ; \Sigma' \vdash a : A_1 \\
II' & ; \Sigma' \vdash Ca : A_0 & II' & ; \Sigma' \vdash a : A_1 \\
II' & ; x : A_0; \Sigma, a : A_1 \vdash \langle x \mid Ca \rangle & II' & ; \Sigma' \vdash \mu a. \langle x \mid Ca \rangle : A_1 \\
II' & ; \Sigma \vdash \lambda x. \mu a. \langle x \mid Ca \rangle : A_0 \rightarrow A_1 \\
\end{align*}
\]

are derived where $II'$ and $\Sigma'$ are $II, x : A_0; \Sigma, a : A_1$, respectively.