Quantum Effects in Neural Networks

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We develop the statistical mechanics of the Hopfield model in a transverse field to investigate how quantum fluctuations affect the macroscopic behavior of neural networks. When the number of embedded patterns is finite, the Trotter decomposition reduces the problem to that of a random Ising model. It turns out that the effects of quantum fluctuations on macroscopic variables play the same roles as those of thermal fluctuations. For an extensive number of embedded patterns, we apply the replica method to the Trotter-decomposed system. The result is summarized as a ground-state phase diagram drawn in terms of the number of patterns per site, \( \alpha \), and the strength of the transverse field, \( \Delta \). The phase diagram coincides very accurately with that of the conventional classical Hopfield model if we replace the temperature \( T \) in the latter model by \( \Delta \). Quantum fluctuations are thus concluded to be quite similar to thermal fluctuations in determination of the macroscopic behavior of the present model.

KEYWORDS: neural networks, Hopfield model, quantum effects, macrovariables, phase diagram

§1. Introduction

Statistical mechanics has been applied successfully to the analysis of various problems in neural networks. In particular, the Hopfield model, a prototype of associative memory, was solved explicitly by combining well-established techniques in the mean-field theory of random spin systems. In the case of a finite number of embedded patterns, the application of the mean-field method for the Ising ferromagnet to the neural network was shown to prove the existence of a memory-retrieval phase as well as mixed (confused) states. When the number of embedded patterns is extensive, it is necessary to introduce the replica method in order to investigate the properties of the network at finite temperatures, similarly to the case of the Sherrington-Kirkpatrick model of spin glasses. The resulting phase diagram is characterized by three macroscopic phases, namely, the retrieval, spin glass and paramagnetic phases.

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The introduction of temperature in these studies was motivated by the apparent randomness in signal transmission at a synapse. A pulse reaching the terminal bulb of an axon does not always cause the release of neuro-transmitters contained in vesicles. The probability of release is generally small. This randomness in signal transmission is usually taken into account as thermal fluctuations, which leads to the stochastic formulation of the problem in terms of the kinetic Ising model. However, detailed considerations of the origin of this randomness suggest that quantum effects may be the major driving force to cause uncertainty in the release of neurotransmitters from vesicles into the synaptic cleft. For example, Stapp pointed out that the migration of calcium ions in the bulb is a quantum diffusion process and thus the uncertainty in the ion positions leads to quantum fluctuations in the signal transmission at a synapse. Beck and Eccles argued that the uncertainty in the positions of hydrogen atoms in the vesicular grid is the origin of quantum fluctuations of comparable order of magnitude as thermal fluctuations in the brain at room temperature. These investigations indicate the necessity to treat randomness in the signal transmission in terms of quantum mechanics, not simply as thermal fluctuations as has been the case conventionally.

The relation between quantum mechanics and the brain functioning has been discussed also in the context to clarify the fundamental significance of wave functions and observations in quantum mechanics (see Refs. 3 and 4 and references therein). However, few of the previous investigations in this area have paid attention to the behavior of macroscopic observables. Discussions have been given mostly in terms of microscopic wave functions, though experiments are often carried out on macroscopically observable variables. We should point out that the superposition of various microscopic state vectors does not always lead to the uncertainty in macroscopic observables. Quantum fluctuations may work in a manner similar to thermal fluctuations, leading only to weak stochastic deteriorations of observed values of macrovariables. We will show this effect explicitly in the present paper.

We therefore have sufficient reasons to introduce quantum fluctuations into neural networks. It is in general difficult to reflect directly microscopic quantum processes in a simple model amenable to analytical investigations. We thus adopt the Hopfield model in a transverse field,

\[ \mathcal{H} = - \sum_{\langle ij \rangle} J_{ij} \sigma_i^x \sigma_j^x - \Delta \sum_i \sigma_i^z, \]

(1.1)

where \( \sigma_i^x \) and \( \sigma_i^z \) are the components of a Pauli matrix at site \( i \). The interactions \( J_{ij} \) are given by the Hebb rule as specified explicitly in §2. Admittedly this model is not a faithful reproduction of real processes in the brain. For example, the state of a neuron has quantum uncertainty in an eigenstate of the Hamiltonian (1.1), quite an improbable situation in reality. However, our purpose is not to explain the brain itself in detail. (One may argue in this regard that the Hopfield model without the transverse-field term is already inadequate as a model of the brain.) We rather aim
to clarify the role of quantum fluctuations in large-scale networks at a phenomenological level. We believe that the present system (1.1) serves as a first step toward this goal.

This paper is organized as follows. In §2 the transverse-field Hopfield model (1.1) is solved for a finite number of embedded patterns. It is shown that quantum fluctuations have almost the same effects as thermal fluctuations on macroscopic properties of the network. The case with an extensive number of embedded patterns is studied in §3. The resulting phase diagram turns out to be almost the same as that of the classical Hopfield model qualitatively and even quantitatively. (We call the model with \( \Delta = 0 \) in (1.1) the classical Hopfield model in this paper.) The last section is devoted to discussions. A preliminary report of a part of the present work has already been given elsewhere.

§2. Finite Number of Patterns Embedded

We first consider the case in which the number of embedded patterns \( p \) remains finite in the thermodynamic limit.

2.1 Formulation

The Hamiltonian of the Hopfield model in a transverse field has already been given in (1.1) as

\[
\mathcal{H} = -\sum_{(ij)} J_{ij} \sigma^z_i \sigma^z_j - \Delta \sum_i \sigma^x_i \equiv \mathcal{H}_0 + \mathcal{H}_1.
\]

The interactions \( J_{ij} \) obey the Hebb rule,

\[
J_{ij} = \frac{1}{N} \sum_{\mu=1}^p \xi_i^\mu \xi_j^\mu,
\]

with \( \xi_i^\mu = 1 \) or \( -1 \) randomly. The summation over the indices \( (ij) \) in (2.1) runs over all combinations of pairs of sites. The partition function of this quantum system can be represented in terms of simple Ising variables by the Trotter decomposition

\[
Z = \lim_{M \to \infty} \text{Tr} \left( e^{-\beta \mathcal{H}_0/M} e^{-\beta \mathcal{H}_1/M} \right)^M = \lim_{M \to \infty} Z_M,
\]

where

\[
Z_M = \sum_{\{\sigma = \pm 1\}} \exp \left( \frac{\beta}{MN} \sum_{K=1}^M \sum_{(ij)} \sum_{\mu=1}^p \xi_i^\mu \xi_j^\mu \sigma_i \sigma_j + B \sum_{K=1}^M \sum_{i=1}^N \sigma_i \sigma_{i,K+1} \right).
\]

The coupling constant \( B \) in the Trotter direction is related to the coefficient \( \Delta \) of the transverse field term in (2.1) by

\[
B = \frac{1}{2} \log \csc \frac{\beta \Delta}{M}.
\]

We follow the standard procedure to decompose the double summation over \( (ij) \) using a Gaussian integral,
\[ Z_M = \int \prod_{K\mu} dm_{K\mu} \sum_{\sigma} \exp \left( -\frac{N\beta}{2M} \sum_{K\mu} m_{K\mu}^2 + \frac{\beta}{M} \sum_{K\mu} \sum_i m_{K\mu} \xi_i \sigma_{iK} + B \sum_{K_i} \sigma_{iK} \sigma_{i,K+1} \right), \quad (2.6) \]

where we have ignored the overall constant which is irrelevant for the following arguments.

In the thermodynamic limit \( N \to \infty \) with \( p \) kept finite, the saddle point of the integrand of (2.6) yields the equilibrium free energy per spin as

\[ f = \frac{1}{2M} \sum_{K\mu} m_{K\mu}^2 - T \ll \log \sum_{\sigma} \exp \left( \frac{\beta}{M} \sum_{K\mu} m_{K\mu} \xi_{\mu} \sigma_K + B \sum_K \sigma_K \sigma_{K+1} \right) \gg , \quad (2.7) \]

where the double brackets \( \ll \cdots \gg \) denote the average over the randomness of embedded patterns \( \{\xi_{\mu}\} \). We have assumed the self-averaging property of the free energy to derive the above expression.

The saddle-point condition leads to the equation of state

\[ m_{K\mu} = \ll \xi_{\mu} \langle \sigma_K \rangle \gg , \quad (2.8) \]

where the brackets \( \langle \cdots \rangle \) stand for the average by the weight

\[ \exp \left( \frac{\beta}{M} \sum_{K\mu} m_{K\mu} \xi_{\mu} \sigma_K + B \sum_K \sigma_K \sigma_{K+1} \right) . \]

Equation (2.8) shows that the parameter \( m_{K\mu} \) represents the overlap of the spin configuration in the \( K \)th Trotter slice with the \( \mu \)th embedded pattern. The solutions of (2.8) describe the thermodynamic properties of the system.

2.2 Symmetric solution near the critical point

Let us first discuss the symmetric solutions of the equation of state (2.8) in the form \( m_{K\mu} = m \) for all \( K \) and \( \mu \leq l \), \( l \) being a given integer. If \( \mu \) exceeds \( l \), \( m_{K\mu} = 0 \). The stability of this type of solutions will be considered later. The free energy (2.7) is now written as

\[ f = \frac{1}{2} lm^2 - T \ll \log \sum_{\sigma} \exp \left( \frac{\beta}{M} \sum_K \sum_{\mu=1}^l \xi_{\mu} \sigma_K + B \sum_K \sigma_K \sigma_{K+1} \right) \gg . \quad (2.9) \]

To determine the critical temperature, we expand (2.9) to \( O(m^4) \) as

\[ f = \frac{1}{2} lm^2 - T \log Z_0 - \frac{\beta m^2}{2M^2} \ll z_0^2 \gg \left( \sum_K \sigma_K \right)_{0c}^2 \]

\[ - \frac{\beta^3 m^4}{24M^4} \ll z_0^4 \gg \left( \sum_K \sigma_K \right)_{0c}^4 , \quad (2.10) \]
where
\[ Z_0 = \sum_{\sigma} e^{B \sum_K \sigma_K \sigma_{K+1}} \]
and
\[ z_l = \sum_{\mu=1}^{l} \xi^\mu. \]

In (2.10) the brackets \( \langle \cdots \rangle_{0c} \) represent the cumulants calculated from the average \( \langle \cdots \rangle_0 \) defined by
\[ \langle Q \rangle_0 \equiv 1 \sum_{\sigma} Q e^{B \sum_K \sigma_K \sigma_{K+1}}. \]  
(2.11)

In order to evaluate the cumulants appearing in (2.10), we calculate the generating function
\[ Z_{h,M} = \sum_{\sigma} \exp \left( B \sum_K \sigma_K \sigma_{K+1} + \frac{h}{M} \sum_K \sigma_K \right). \]  
(2.12)

In the limit of large \( M \), we find, using the Trotter decomposition formula:
\[ Z_h \equiv \lim_{M \to \infty} Z_{h,M} = \text{Tr} e^{h \sigma_x + \beta \Delta \sigma_x} = 2 \cosh \sqrt{h^2 + \beta^2 \Delta^2}. \]  
(2.13)

Expansion of this equation in powers of \( h \) gives
\[ \log Z_h = \log Z_0 \bigg|_{M \to \infty} + \frac{h^2 \tanh a}{2a} + \frac{h^4 (a - \tanh a - a \tanh^2 a)}{12a^3}, \]  
(2.14)

with \( a = \beta \Delta \). Comparison of (2.14) with (2.10) gives the cumulants in the limit \( M \to \infty \) as
\[ \frac{1}{M^2} \langle \sum_K \sigma_K \rangle^2_{0c} = \frac{\tanh a}{a}, \]
\[ \frac{1}{M^4} \langle \sum_K \sigma_K \rangle^4_{0c} = \frac{3}{a^3} (a - \tanh a - a \tanh^2 a). \]

Using the relations
\[ \ll z_l^2 \gg = l, \]  
(2.15)
\[ \ll z_l^4 \gg = l(3l - 2), \]  
(2.16)

we finally obtain the explicit expression of the free energy (2.10) as
\[ \frac{f}{l} = \frac{1}{2} m^2 - \frac{\beta m^2}{2} \cdot \frac{\tanh a}{a} \]
\[ \quad - (3l - 2) \cdot \frac{\beta^2 m^4}{8} \cdot \frac{a - \tanh a - a \tanh^2 a}{a^3}. \]  
(2.17)
The critical condition is thus expressed as

$$\beta_c \tanh a = 1$$

or

$$\tanh \beta_c \Delta = \Delta.$$  \hfill (2.18)

This formula shows that the critical temperature does not depend on the parameter $l$ similarly to the case of the classical model.\(^1\) The critical line (2.18) is drawn in Fig. 1. The asymptotic form of the overlap order parameter around the critical temperature is derived from (2.17) as

$$m^2 = \frac{2 \left( 1 - \Delta^{-1} \tanh \beta \Delta \right)}{\beta_c^3 (3l - 2) g(a_c)},$$  \hfill (2.19)

where $a_c = \beta_c \Delta$ and

$$g(a) = \frac{1}{a^3} (a - \tanh a - a \tanh^2 a).$$

2.3 Symmetric solution in the ground state

In the limit of large $M$, the free energy (2.9) is written as

$$f = \frac{1}{2} \beta m^2 - T \ll \log \text{Tr} e^{\beta m^2 \sigma_z + \beta \Delta \sigma_x} \gg$$

according to the Trotter decomposition formula. This expression is further evaluated in the limit $T \to 0$ as

$$f = \frac{1}{2} \beta m^2 - T \ll 2 \beta \text{cosh} \beta \sqrt{m^2 z_l^2 + \Delta^2} \gg$$

$$\longrightarrow \frac{1}{2} \beta m^2 - \ll \sqrt{m^2 z_l^2 + \Delta^2} \gg.$$  \hfill (2.20)
The equation of state is obtained from (2.20) as
\[ lm = \ll \frac{m z^2}{\sqrt{m^2 z^2 + \Delta^2}} \rr. \] (2.21)

For a finite value of the overlap order parameter \( m \), (2.21) reads
\[ l = \ll \frac{z^2}{\sqrt{m^2 z^2 + \Delta^2}} \rr. \]

In consideration of (2.15), this equation implies that \( m \) decreases to 0 as \( \Delta \) approaches 1. Thus the critical value of \( \Delta \) in the ground state is 1. The free energy (or the energy) at this critical point is −1 independent of \( l \) as is apparent from (2.20).

It is straightforward to derive the explicit forms of the order parameter and the free energy for small values of \( l \) from (2.21) and (2.20). The results are given by
\[ m = \sqrt{1 - \Delta^2}, \quad f = -\frac{1}{2}(1 + \Delta^2) \] (2.22)
and
\[ m = \frac{1}{2}\sqrt{1 - \Delta^2}, \quad f = -\frac{1}{4}(1 + \Delta)^2 \] (2.23)
for \( l = 1 \) and \( l = 2 \), respectively, and
\[ f = \frac{3}{2}m^2 - \frac{1}{4}\left(\sqrt{9m^2 + \Delta^2} + 3\sqrt{m^2 + \Delta^2}\right) \] (2.24)
for \( l = 3 \). These results are plotted in Fig. 2 for the order parameter and in Fig. 3 for the free energy. These figures show that in the ground state quantum effects represented by the parameter \( \Delta \) play very similar roles to thermal fluctuations represented by the temperature \( T \) in the classical model in which the order parameter and the free energy behave in almost the same way as functions of \( T \) as in Figs. 2 and 3 if we replace \( \Delta \) by \( T \).

2.4 Stability of the symmetric solutions (I)

It is necessary to check the stability of the symmetric solutions given in the previous subsection. The overlap order parameter \( m_{K\mu} \) should depend upon \( K \) and \( \mu \) in general. The possible dependence of \( m_{K\mu} \) on the Trotter number \( K \) is investigated in the present subsection. The pattern number dependence will be treated in the next subsection.

Let us assume that \( m_{K\mu} = m_K \) if \( \mu \leq l \) and \( m_{K\mu} = 0 \) otherwise. Then, the free energy (2.7) is expanded to second order of \( m_K \) as
\[ f = -T \log Z_0 + \frac{l}{2M} \sum_K m_K^2 - \frac{\beta l}{2M^2} \sum_{KK'K''} m_K m_{K'} m_{K''} (\sigma_K \sigma_{K''})_0 = -T \log Z_0 + \frac{l}{2M} \sum_{KK'} A_{KK'} m_K m_{K'}, \] (2.25)
Fig. 2. The ground-state magnetization as a function of $\Delta$ for $l = 1, 2$ and 3.

Fig. 3. The ground-state free energy (or the energy) as a function of $\Delta$ for $l = 1, 2$ and 3.

where

$$A_{KK'} = \delta_{KK'} - \frac{\beta}{M} \langle \sigma_K \sigma_{K'} \rangle_0 .$$

The average $\langle \cdots \rangle_0$ was defined in (2.11). The expansion of the free energy (2.25) corresponds to the approach to the critical point from the paramagnetic phase because the order parameter vanishes in equilibrium in the high-temperature paramagnetic phase. Correspondingly, from (2.24), all eigenvalues of the matrix $A_{KK'}$ are seen to be positive for small $\beta$ or large $T$, implying the stability
of the paramagnetic solution $m_K = 0$. The critical temperature is determined by the vanishing point of the minimum eigenvalue $\lambda_0$ of the matrix $A_{KK'}$,

$$
\lambda_0 = 1 - \frac{\beta}{M^2} \sum_{KK'} \langle \sigma_K \sigma_{K'} \rangle_0 \\
= 1 - \frac{\beta}{M^2} \langle \left( \sum_K \sigma_K \right)^2 \rangle_0 \\
\rightarrow 1 - \frac{\tanh \beta \Delta}{\Delta}.
$$

(2.27)

The critical condition $\tanh \beta \Delta = \Delta$ of (2.18) is thus recovered. An important point here is that the eigenstate corresponding to the minimum eigenvalue $\lambda_0$ is a uniform mode $m_K = m$. This means that an ordered state uniform in the Trotter number $K$ is formed slightly below the critical temperature. Therefore, we conclude that the symmetric solution $m_{K\mu} = m$ is stable against the Trotter-number dependence slightly below the critical temperature.

At an arbitrary temperature the free energy is written as

$$
f = \frac{l}{2M} \sum_K m_K^2 - T \ll \log \sum_{\sigma} \exp \left( \frac{\beta z_l}{M} \sum_K m_K \sigma_K + B \sum_K \sigma_K \sigma_{K+1} \right) \gg.
$$

(2.28)

To investigate the stability of the symmetric solution, we check the eigenvalues of the Hessian

$$
A_{KL} = \left. \frac{\partial^2 f}{\partial m_K \partial m_L} \right|_{m_K = m} = \frac{l}{M} \delta_{KL} - \frac{\beta}{M^2} \ll z_l^2 \left( \langle \sigma_K \sigma_L \rangle - \langle \sigma_K \rangle \langle \sigma_L \rangle \right) \gg,
$$

(2.29)

where the brackets $\langle \cdots \rangle$ denote the average with respect to the weight

$$
\exp \left( \frac{\beta z_l m}{M} \sum_K \sigma_K + B \sum_K \sigma_K \sigma_{K+1} \right).
$$

Since the value of the matrix element $A_{KL}$ depends only on the difference $|K - L|$ and the quantity in the double brackets $\ll \cdots \gg$ in (2.29) is positive, the lowest eigenvalue of this Hessian is given by

$$
\lambda_0 = \sum_{L=1}^{M} A_{KL}.
$$

The symmetric solution is stable if $\lambda_0$ is positive. Explicitly,

$$
M \lambda_0 = l - \beta \frac{\partial^2}{\partial (\beta m^2)} \ll \log \sum_{\sigma} \exp \left( \frac{\beta z_l m}{M} \sum_K \sigma_K + B \sum_K \sigma_K \sigma_{K+1} \right) \gg \\
\rightarrow l - T \frac{\partial^2}{\partial m^2} \ll \log \text{Tr} e^{\beta z_l m \sigma_z + \beta \Delta \sigma_x} \gg \\
= l - \Delta^2 \ll \frac{z_l^2}{(m^2 z_l^2 + \Delta^2)^{3/2}} \tanh \beta \sqrt{m^2 z_l^2 + \Delta^2} \gg
$$

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\[-\beta m^2 < \frac{z_i^4}{m^2 z_i^2 + \Delta^2 \text{sech}^2 \beta \sqrt{m^2 z_i^2 + \Delta^2}} \]. \quad (2.30)\]

At \(T = 0\), the above expression reduces to
\[M\lambda_0 = l - \Delta^2 < \frac{z_i^2}{(m^2 z_i^2 + \Delta^2)^{3/2}} \].

For \(\Delta \to 0\), \(M\lambda_0\) tends to \(l\), a positive value. If, on the other hand, \(\Delta\) is close to 1 (\(\Delta = 1 - \epsilon\) with \(\epsilon \ll 1\)), \(M\lambda_0\) approaches \(2\epsilon l\), again positive. We thus expect that the eigenvalue \(\lambda_0\) remains positive between these two limiting values of \(\Delta\) when \(T = 0\).

In the case of general finite temperature, we have numerically confirmed the positivity of \(\lambda_0\) for \(l = 3\). It is expected that the same property holds for other values of \(l\), though it is difficult to prove it explicitly.

2.5 Stability of the symmetric solution (II)

We next check the pattern number (\(\mu\)) dependence of the solution of the equation of state. If we assume \(m_K\mu = m_\mu\) for \(\mu \leq l\) and \(m_K\mu = 0\) otherwise, the free energy (2.7) reads
\[f = \frac{1}{2} \sum_\mu m_\mu^2 - T \ll \log \sum_\sigma \exp \left( \frac{\beta}{M} \sum_\mu m_\mu \xi_\mu \sum_K \sigma_K + B \sum_K \sigma_K \sigma_{K+1} \right) \] \[\stackrel{M \to \infty}{\longrightarrow} \frac{1}{2} m^2 - T \ll 2 \cosh \beta \sqrt{(m \cdot \xi)^2 + \Delta^2} \]. \quad (2.31)\]

The vector \(m\) has components \((m_1, m_2, \cdots, m_l, 0, 0, \cdots)\), and similarly for \(\xi\). The Hessian around the symmetric solution is given by
\[A_{\mu\nu} = \frac{\partial^2 f}{\partial m_\mu \partial m_\nu} \bigg|_{m_\mu = m} \]
\[= \delta_{\mu\nu} - \Delta^2 < \frac{\xi_\mu \xi_\nu}{(m^2 z_i^2 + \Delta^2)^{3/2}} \text{tanh} \beta \sqrt{m^2 z_i^2 + \Delta^2} \]
\[-\beta m^2 < \frac{z_i^2 \xi_\mu \xi_\nu}{m^2 z_i^2 + \Delta^2 \text{sech}^2 \beta \sqrt{m^2 z_i^2 + \Delta^2}} \]. \quad (2.32)\]

The diagonal element of \(\{A_{\mu\nu}\}\) is written in the following form
\[e_1 \equiv A_{\mu\mu} = 1 - \Delta^2 < \frac{\text{tanh} \beta \sqrt{m^2 z_i^2 + \Delta^2}}{(m^2 z_i^2 + \Delta^2)^{3/2}} \]
\[+ \beta q - \beta m^2 < \frac{z_i^2}{m^2 z_i^2 + \Delta^2} >. \quad (2.33)\]
where
\[ q = \ll \frac{m^2 z_l^2}{m^2 z_l^2 + \Delta^2} \tanh^2 \beta \sqrt{m^2 z_l^2 + \Delta^2} \gg \]
\[ = \ll \langle \sigma_K \rangle^2 \gg . \tag{2.34} \]

All off-diagonal elements have the same value:
\[ e_2 \equiv A_{\mu \nu} \quad (\mu \neq \nu) \]
\[ = -\Delta^2 \ll \frac{\xi^\mu \xi^\nu}{(m^2 z_l^2 + \Delta^2)^{3/2}} \tanh \beta \sqrt{m^2 z_l^2 + \Delta^2} \gg \]
\[ -\beta m^2 \ll \frac{z_l^2 \xi^\mu \xi^\nu}{m^2 z_l^2 + \Delta^2} \text{sech}^2 \beta \sqrt{m^2 z_l^2 + \Delta^2} \gg . \tag{2.35} \]

From (2.32), (2.33) and (2.35), it is easy to see that there are three eigenvalues of the Hessian matrix \( \{ A_{\mu \nu} \} \),
\[ \lambda_1 = e_1 + (l - 1)e_2 , \tag{2.36a} \]
\[ \lambda_2 = e_2 , \tag{2.36b} \]
\[ \lambda_3 = e_1 - e_2 , \tag{2.36c} \]
with the degeneracy \( p - l, 1 \) and \( l - 1 \), respectively.

To investigate the stability of the symmetric solution near the critical temperature, we expand various terms in (2.33) and (2.35) to second order of \( m \) using (2.34):
\[ m^2 \ll \frac{z_l^2}{m^2 z_l^2 + \Delta^2} \gg \approx \frac{m^2 l}{\Delta} , \]
\[ q \approx m^2 l \cdot \frac{\tanh^2 \beta \Delta}{\Delta^2} \]

and
\[ \ll \frac{\tanh \beta \sqrt{m^2 z_l^2 + \Delta^2}}{(m^2 z_l^2 + \Delta^2)^{3/2}} \gg \approx \frac{1}{\Delta^3} \left( \tanh \beta \Delta - \frac{3m^2 l}{2\Delta^2} \tanh \beta \Delta + \frac{\beta m^2 l}{2\Delta \cosh^2 \beta \Delta} \right) . \tag{2.37} \]

These equations and (2.19) yield
\[ e_1 \approx \frac{2}{3l - 2} \left( \frac{\tanh \beta \Delta}{\Delta} - 1 \right) \approx \frac{2(1 - \Delta^2)}{3l - 2} \epsilon , \]
\[ e_2 \approx \frac{6}{3l - 2} \frac{(\Delta^{-1} \tanh \beta \Delta - 1)}{3l - 2} \approx \frac{3(1 - \Delta^2)}{3l - 2} \epsilon , \]
where \( \epsilon \) represents the deviation of the temperature from the critical value,
\[ \epsilon = T_c(\Delta) - T . \]
The signs of the eigenvalues in (2.36) are then found as \( \lambda_1, \lambda_2 > 0 \) and \( \lambda_3 < 0 \). Since the degeneracy of the last eigenvalue \( \lambda_3 \) is \( l - 1 \), the symmetric solution with \( l \geq 2 \) is unstable against fluctuations in the direction of the eigenvector corresponding to \( \lambda_3 \) slightly below the critical temperature. This property is exactly the same as in the case of the classical Hopfield model.

When \( T = 0 \), the matrix elements (2.33) and (2.35) reduce to

\[
e_1 = 1 - \Delta^2 \ll \frac{1}{(m^2 z_l^2 + \Delta^2)^{3/2}} \, ,
\]

\[
e_2 = -\Delta^2 \ll \frac{\xi \mu \xi \nu}{(m^2 z_l^2 + \Delta^2)^{3/2}} \, .
\]

For \( l = 1 \), \( \lambda_1 = e_1 = 1 - \Delta^2 \geq 0 \), implying the stability as expected trivially for the one-component solution. When \( l = 2 \), the eigenvalues are given by

\[
\begin{align*}
\lambda_1 &= 1 - \Delta^2 , \quad (2.38a) \\
\lambda_2 &= 1 - \frac{\Delta^2}{2} - \frac{1}{2\Delta} , \quad (2.38b) \\
\lambda_3 &= 1 - \frac{1}{\Delta} . \quad (2.38c)
\end{align*}
\]

The first eigenvalue \( \lambda_1 \) is always positive in the range \( 0 \leq \Delta < 1 \), while the second one \( \lambda_2 \) is positive for \( \Delta > 0.618 \) and is negative for \( \Delta < 0.618 \). The third eigenvalue \( \lambda_3 \) is always negative. Thus, this \( l = 2 \) solution is unstable at \( T = 0 \) for any \( \Delta \) between 0 and 1.

The symmetric solution with \( l = 3 \) has the following eigenvalues for \( T = 0 \):

\[
\begin{align*}
\lambda_1 &= 1 - \frac{\Delta^2}{4} \left[ \frac{3}{(9m^2 + \Delta^2)^{3/2}} + \frac{1}{(m^2 + \Delta^2)^{3/2}} \right] , \\
\lambda_2 &= 1 - \frac{\Delta^2}{4} \left[ \frac{1}{(9m^2 + \Delta^2)^{3/2}} + \frac{3}{(m^2 + \Delta^2)^{3/2}} \right] , \\
\lambda_3 &= 1 - \frac{\Delta^2}{(m^2 + \Delta^2)^{3/2}} .
\end{align*}
\]

These eigenvalues are plotted as functions of \( \Delta \) in Fig. 4. The first and second eigenvalues \( \lambda_1 \) and \( \lambda_2 \) are both positive, but \( \lambda_3 \) changes its sign at \( \Delta = 0.494 \). Similarly, the behavior of the eigenvalues for \( l = 4 \) and \( l = 5 \) is shown in Figs. 5 and 6. This analysis for \( l \) up to 5 thus suggests that the even-\( l \) solution is always unstable while the odd-\( l \) solution is stable in a finite range \( 0 \leq \Delta < \Delta_c \) when \( T = 0 \).

To confirm this conjecture, we have expanded the eigenvalues \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) as functions of \( \epsilon = 1 - \Delta \) for small \( \epsilon \). The equation of state (2.21) has the solution \( m = c\sqrt{\epsilon} \) with \( c = \sqrt{2/(3l - 1)} \). This gives

\[
\begin{align*}
e_1 &= \frac{1}{2}(3lc^2 - 2)\epsilon , \\
e_2 &= 3c^2\epsilon .
\end{align*}
\]
Fig. 4. Three eigenvalues of the Hessian for $l = 3$ and $T = 0$. The third eigenvalue $\lambda_3$ crosses 0 at $\Delta = 0.494$.

Fig. 5. Three Hessian eigenvalues for $l = 4$ and $T = 0$.

Then, the eigenvalues (2.36) are

\[
\begin{align*}
\lambda_1 &= 2\epsilon, \\
\lambda_2 &= \frac{2\epsilon}{3l - 2}, \\
\lambda_3 &= -\frac{3l + 2}{3l - 2}\epsilon.
\end{align*}
\]

Since the third eigenvalue $\lambda_3$ has degeneracy $l - 1$ and is negative, we conclude that all symmetric solutions with $l \geq 2$ are unstable near $\Delta = 1$. This, together with the analysis of eigenvalues for
$l = 2$ to $5$ explained before, suggests that the coefficient $\Delta$ of the transverse-field term of the present model in the ground state has effects very similar to the temperature of the classical Hopfield model ($\Delta = 0$) in which the even-$l$ solutions are unstable for any $T$ while the odd-$l$ solutions are stable below certain temperatures.

The instability of the even-$l$ solutions suggests the existence of asymmetric solutions of the equation of state (2.8) as was the case in the classical model. We actually have found many asymmetric solutions. An example of a solution of the type $(m, m, u, u, u, 0, 0, \cdots)$ is shown in Fig. 7. Although we were not able to find general rules for the existence and behavior of asymmetric solutions, we observed that, when $T = 0$, $\Delta$ has effects very similar to $T$ in the classical model.

§3. Extensive Number of Patterns Embedded

In this section the number of patterns embedded, $p$, is assumed to be proportional to the system size $N$. We closely follow Chap. 10 of Ref. 12 in our presentation.

3.1 General form of the free energy

According to (2.6), the replicated partition function averaged over the quenched randomness is written for a finite Trotter number $M$ as

$$
\ll \langle Z^n \rangle = \ll \int \prod_{K_{\mu \rho}} dm_{K_{\mu \rho}} \sum_{\sigma} \exp \left( -\frac{N \beta}{2M} \sum_{K_{\mu \rho}} m_{K_{\mu \rho}}^2 + \frac{\beta}{M} \sum_{K_{\mu \rho}} \sum_{i} m_{K_{\mu \rho}} \xi_{i}^{\mu} \sigma_{i \rho K} + B \sum_{K_{3 \rho}} \sigma_{i \rho K} \sigma_{i \rho, K+1} \right) \gg , \quad (3.1)
$$
where $\rho (=1,\ldots,n)$ represents the replica index. The overall constant is irrelevant and is ignored here.

We consider the situation in which a finite number of patterns have nonvanishing overlaps. That is, $m_{K\mu\rho}$ is of order unity for $\mu = 1,2,\ldots,s$ and is of order $1/\sqrt{N}$ for other $\mu$’s with $s$ being a finite number. Then, the configurational average $\langle \cdots \rangle$ in (3.1) can be evaluated explicitly for patterns $\mu > s$. For the $\mu$th pattern with $\mu > s$,

$$\prod_i \ll \exp \left( \frac{\beta}{M} \sum_{K\rho} m_{K\mu\rho} \sigma_{i\rho K} \right) \gg = \prod_i \cosh \left( \frac{\beta}{M} \sum_{K\rho} m_{K\mu\rho} \sigma_{i\rho K} \right) \approx \exp \left( \frac{\beta^2}{2M^2} \sum_i \sum_{\rho\sigma} \sum_{KL} m_{K\mu\rho} m_{L\mu\sigma} \sigma_{i\rho K} \sigma_{i\sigma L} \right). \quad (3.2)$$

The terms containing $\{m_{K\mu\rho}\}_{\mu>s}$ in (3.1) and (3.2) are collected in a single formula,

$$E_\mu \equiv \exp \left( -\frac{\beta N}{2M} \sum_{K\rho} \sum_{L\sigma} \tilde{\Lambda}_{K\rho,L\sigma} m_{K\mu\rho} m_{L\mu\sigma} \right),$$

where

$$\tilde{\Lambda}_{K\rho,L\sigma} = \delta_{K\rho,L\sigma} - \frac{\beta}{NM} \sum_i \sigma_{i\rho K} \sigma_{i\sigma L}. \quad (3.3)$$

The integration of $E_\mu$ over $\{m_{K\mu\rho}\}$ gives

$$\int \prod_{K\rho} dm_{K\mu\rho} E_\mu = \text{const.} \times (\det \tilde{\Lambda})^{-1/2}.$$
The product of this result over \( \mu = s + 1, \cdots, p \) is
\[
(\det \hat{\Lambda})^{-(p-s)/2} \approx (\det \tilde{\Lambda})^{-p/2} = \exp \left( -\frac{p}{2} \sum_{Kp} \log \tilde{\lambda}_{Kp} \right).
\]

Here we have neglected the finite number \( s \) in the exponent since it is small compared to the extensive number \( p \). The eigenvalues of the matrix \( \tilde{\Lambda} \) are denoted as \( \tilde{\lambda}_{Kp} \).

The eigenvalue \( \tilde{\lambda}_{Kp} \) depends on the spin configuration as is apparent from (3.3). To avoid this complication, we introduce another matrix \( \Lambda \) without explicit dependence upon the spin configurations:
\[
\Lambda_{Kp,L\sigma} = \delta_{Kp,L\sigma} - \frac{\beta}{M} q_{p\sigma}(KL) - \delta_{p\sigma} \frac{\beta}{M} S_{p}(KL).
\]

This matrix \( \Lambda \) is equal to \( \tilde{\Lambda} \), if
\[
q_{p\sigma}(KL) = \begin{cases} 
\frac{1}{N} \sum_{i} \sigma_{ipK} \sigma_{i\sigma L} & (\rho \neq \sigma) \\
0 & (\rho = \sigma)
\end{cases}
\]

and
\[
S_{p}(KL) = \frac{1}{N} \sum_{i} \sigma_{ipK} \sigma_{i\rho L}
\]
are satisfied. Physically, \( q_{p\sigma}(KL) \) is the spin glass order parameter for spins at the Trotter slices \( K \) and \( L \), and \( S_{p}(KL) \) is a measure of quantum fluctuations. If there are no quantum fluctuations, the spin configuration \( \sigma_{ipK} \) does not depend on the Trotter number \( K \) and \( S_{p}(KL) = 1 \). Quantum fluctuations reduce \( S_{p}(KL) \) from unity. Hence \( 1 - S_{p}(KL) \) gives the scale of quantum fluctuations.

It is convenient to rewrite a function of \( \tilde{\lambda}_{Kp} \) as
\[
G\{\tilde{\lambda}_{Kp}\} = \int \prod_{(Kp,L\sigma)} dq_{p\sigma}(KL) \prod_{\rho} dS_{p}(KL) 
\]
\[
\times \delta \left( q_{p\sigma}(KL) - \frac{1}{N} \sum_{i} \sigma_{ipK} \sigma_{i\sigma L} \right) 
\]
\[
\times \delta \left( S_{p}(KL) - \frac{1}{N} \sum_{i} \sigma_{ipK} \sigma_{i\rho L} \right) G\{\lambda_{Kp}\},
\]
where \((KL)\) denotes an arbitrary combination of \( K \) and \( L \) including \( K = L \), and \((Kp, L\sigma)\) expresses an arbitrary pair except for \( \rho = \sigma \). Using the Fourier representations of the delta functions, (3.7) is written as
\[
G\{\tilde{\lambda}_{Kp}\} = \int \prod_{(Kp,L\sigma)} dq_{p\sigma}(KL) dr_{p\sigma}(KL) \prod_{\rho} dS_{p}(KL) d\ell_{\rho}(KL) 
\]
\[
\times \exp \left[ -\frac{N\alpha\beta^2}{M^2} r_{p\sigma}(KL) q_{p\sigma}(KL) + \frac{\alpha\beta^2}{M^2} r_{p\sigma}(KL) \sum_{i} \sigma_{ipK} \sigma_{i\sigma L} \right]
\]
\[- \frac{N\alpha\beta^2}{M^2} t_\rho(KL) S_\rho(KL) + \frac{\alpha\beta^2}{M^2} t_\rho(KL) \sum_i \sigma_i \rho K \sigma_i \rho L \Bigg] G\{\lambda_{K\rho}\}, \quad (3.8)\]

where we have ignored the overall constant.

The average of the replicated $Z$ is thus

\[
\ll Z^n \gg = \int \prod_{K\mu\rho} dm_{K\mu\rho} \prod_{(K\rho, L\sigma)} dq_{\rho\sigma}(KL)dr_{\rho\sigma}(KL) \prod_{\rho} \prod_{(KL)} dS_\rho(KL)dt_\rho(KL)
\times \exp \left[ - \frac{N\beta}{2M} \sum_{K\mu\rho} m^2_{K\mu\rho} - \frac{N\alpha}{2} \sum_{K\rho} \log \lambda_{K\rho} \right]
\times \exp \left[ - \frac{N\alpha\beta^2}{2M^2} \sum_{(K\rho, L\sigma)} r_{\rho\sigma}(KL) q_{\rho\sigma}(KL) - \frac{N\alpha^2}{2M^2} \sum_{\rho} \sum_{(KL)} t_\rho(KL) S_\rho(KL) \right]
\times \sum_\sigma \exp \left[ \frac{\beta}{M} \sum_{K\rho \mu \leq s} m_{K\mu\rho} \sum_i \xi^\mu_i \sigma_i \rho K + B \sum_{i} \sum_{K\rho} \sigma_i \rho K \sigma_i \rho, K+1 \right.
\left. + \frac{\alpha\beta^2}{2M^2} \sum_{i} \sum_{(K\rho, L\sigma)} r_{\rho\sigma}(KL) \sigma_i \rho K \sigma_i \sigma L + \frac{\alpha^2}{2M^2} \sum_{i} \sum_{\rho} \sum_{(KL)} t_\rho(KL) \sigma_i \rho K \sigma_i \rho L \right] \gg (3.9)\]

If we write the above integrand as $\exp(-N\beta f)$, the saddle point of the integral yields the free energy per spin $f$ as

\[
f = \frac{1}{2M} \sum_{K\rho \mu \leq s} m^2_{K\mu\rho} + \frac{\alpha}{2\beta} \sum_{K\rho} \log \lambda_{K\rho}
+ \frac{\alpha\beta}{2M^2} \sum_{(K\rho, L\sigma)} r_{\rho\sigma}(KL) q_{\rho\sigma}(KL) + \frac{\alpha^2}{2M^2} \sum_{\rho} \sum_{(KL)} t_\rho(KL) S_\rho(KL)
- T \ll \log \sum_\sigma \exp \left[ \frac{\beta}{M} \sum_{K\rho \mu \leq s} m_{K\mu\rho} \xi^\mu_i \sigma_i \rho K + B \sum_{K\rho} \sigma_i \rho K \sigma_i \rho, K+1 \right.
\left. + \frac{\alpha\beta^2}{2M^2} \sum_{(K\rho, L\sigma)} r_{\rho\sigma}(KL) \sigma_i \rho K \sigma_i \sigma L + \frac{\alpha^2}{2M^2} \sum_{\rho} \sum_{(KL)} t_\rho(KL) \sigma_i \rho K \sigma_i \rho L \right] \gg . (3.10)\]

We have used the self-averaging property of the free energy in dropping the site index $i$ in the above equation. The stationarity conditions of the free energy give the equations of state:

\[
m_{K\mu\rho} = \ll \xi^\mu_i \langle \sigma_i \rho K \rangle \gg , \quad (3.11)
q_{\rho\sigma}(KL) = \ll \langle \sigma_\rho K \rangle \langle \sigma_\sigma L \rangle \gg , \quad (3.12)
r_{\rho\sigma}(KL) = \frac{1}{\alpha} \sum_{\mu > s} \ll m_{K\mu\rho} m_{L\mu\sigma} \gg , \quad (3.13)\]
\[ S_{\rho}(KL) = \langle \sigma_{\rho K} \sigma_{\rho L} \rangle \gg , \quad (3.14) \]
\[ t_{\rho}(KL) = \frac{1}{\alpha} \sum_{\mu \geq s} \langle m_{K\mu \rho} m_{L\mu \rho} \rangle \gg . \quad (3.15) \]

The physical meaning of the parameters \( m_{K\mu \rho} \), \( q_{\rho \sigma}(KL) \) and \( r_{\rho \sigma}(KL) \) is the same as in the classical Hopfield model: \( m_{K\mu \rho} \) is the overlap, \( q_{\rho \sigma}(KL) \) denotes the spin glass order parameter and \( r_{\rho \sigma}(KL) \) represents the effects of uncondensed patterns. The deviation of \( S_{\rho}(KL) \) from unity reflects quantum fluctuations as mentioned before. The last quantity \( t_{\rho}(KL) \) is for effects of uncondensed patterns in the same replica, or in other words, the diagonal element of \( r_{\rho \sigma}(KL) \).

### 3.2 RS solution in the static approximation

It is very difficult to solve the equations of state in their general forms (3.11)-(3.15). We instead look for the solutions in the replica symmetric (RS) subspace under the static approximation.\(^{10,11}\)

We thus neglect the dependence of the order parameters on the replica index (the RS approximation) and the Trotter number (the static approximation):

\[ m_{K\mu \rho} = m_{\mu} , \quad (3.16) \]
\[ q_{\rho \sigma}(KL) = q , \quad (3.17) \]
\[ r_{\rho \sigma}(KL) = r , \quad (3.18) \]
\[ t_{\rho}(KL) = t , \quad (3.19) \]
\[ S_{\rho}(KL) = \begin{cases} S & (K \neq L) \\ 1 & (K = L) \end{cases} . \quad (3.20) \]

The stability of the replica symmetry will be considered later. The static approximation is expected to give at least qualitatively reliable results as long as the parameter \( \Delta \) representing quantum effects is not too large as was the case of the SK model in a transverse field.\(^{10,11}\) The consistency of the RS and static approximations can be checked from the viewpoint of another approximate method as explained in §3.3 and Appendix A.

The free energy (3.10) divided by \( n \) (the number of replicas) is now

\[
f = \frac{1}{2} m^2 + \frac{\alpha}{2\beta n} \sum_{K,\rho} \log \lambda_{K\rho} + \frac{\alpha \beta}{2M^2} M^2 (n - 1) r q + \frac{\alpha \beta}{2M^2} M^2 t S - \frac{T}{n} \ll \log \sum_{\sigma} \exp \left[ \frac{\beta}{M} \sum_{K,\rho} \sum_{\mu \geq s} m_{\mu} \xi_{\mu} + B \sum_{K,\rho} \sigma_{\rho K} \sigma_{\rho, K+1} \right. \\
+ \left. \frac{\alpha \beta^2}{2M^2} r \left\{ \left( \sum_{\rho} \sum_{K} \sigma_{\rho K} \right)^2 - \sum_{\rho} \left( \sum_{K} \sigma_{\rho K} \right)^2 \right\} + \frac{\alpha \beta^2}{2M^2} t \sum_{\rho} \left( \sum_{K} \sigma_{\rho K} \right)^2 \right]\gg . (3.21)
\]

The summation of \( \log \lambda_{K\rho} \) appearing above is carried out as follows. There are three values of the
matrix elements of \( \{ \Lambda_{K, \rho, L, \sigma} \} \) defined in (3.4),
\[
-\frac{\beta}{M} q \quad (\rho \neq \sigma), \\
-\frac{\beta}{M} S \quad (\rho = \sigma, K \neq L), \\
1 - \frac{\beta}{M} \quad (\rho = \sigma, K = L),
\]
under the static approximation. The eigenvalues of this matrix are easily found to be
\[
\lambda_1 = 1 - \frac{\beta}{M} - \frac{M-1}{M} \beta S - (n-1)\beta q, \\
\lambda_2 = 1 - \frac{\beta}{M} - \frac{M-1}{M} \beta S + \beta q, \\
\lambda_3 = 1 - \frac{\beta}{M} + \frac{\beta}{M} S,
\]
with degeneracies 1, \( n-1 \) and \( (M-1)n \), respectively. Thus, we have
\[
\lim_{n \to 0} \frac{1}{n} \sum_{K, \rho} \log \lambda_{K, \rho} = \log \left( 1 - \frac{\beta}{M} - \frac{M-1}{M} \beta S + \beta q \right) \\
- \frac{\beta q}{1 - \frac{\beta}{M} - \frac{M-1}{M} \beta S + \beta q} + (M-1) \log \left( 1 - \frac{\beta}{M} + \frac{\beta}{M} S \right) \\
\to_{M \to \infty} \log (1 - \beta S + \beta q) - \frac{\beta q}{1 - \beta S + \beta q} - \beta + \beta S .
\] (3.22)

The Gaussian integral enables us to decompose the last term in the exponential appearing in (3.21) into independent replicas:
\[
\exp \frac{\alpha \beta^2}{2M^2} \left[ r \left( \sum_\rho \sum_K \sigma_{\rho K} \right)^2 + (t-r) \sum_\rho \left( \sum_K \sigma_{\rho K} \right)^2 \right] \\
= \int \mathcal{D}z \exp \left[ \frac{\beta}{M} \sqrt{\alpha r} z \sum_{\rho K} \sigma_{\rho K} + \frac{\alpha \beta^2 (t-r)}{2M^2} \sum_\rho \left( \sum_K \sigma_{\rho K} \right)^2 \right] ,
\] (3.23)
where \( \mathcal{D}z \) denotes the Gaussian measure \( e^{-z^2/2 d\sqrt{2\pi}} \). Then, the summation over \( \sigma_\rho \) in (3.21) can be carried out independently for each \( \rho \) as
\[
\log \sum_\sigma \exp \left[ \frac{\beta}{M} \sum_K \sum_\rho \sigma_{\rho K} \sum_\rho m_\mu \xi_\mu + B \sum_K \sum_\rho \sigma_{\rho K} \sigma_{\rho, K+1} \right. \\
+ \frac{\alpha \beta^2}{2M^2} r \left( \sum_\rho \sum_K \sigma_{\rho K} \right)^2 + \frac{\alpha \beta^2}{2M^2} (t-r) \sum_\rho \left( \sum_K \sigma_{\rho K} \right)^2 \right]
\]
\[
\log \int Dz \left\{ \sum_{\sigma} \exp \left[ \frac{\beta}{M} \sum_{K} \sigma_{\rho K} + B \sum_{K} \sigma_{\rho K} \sigma_{\rho, K+1} \right. \right. \\
\left. \left. + \frac{\beta}{M} \sqrt{\alpha r z} \sum_{K} \sigma_{\rho K} + \frac{\alpha \beta^2}{2M^2}(t-r) \left( \sum_{K} \sigma_{\rho K} \right)^2 \right] \right\}^n
\]

\[
\approx n \int Dz \log \sum_{\sigma} \exp \left[ \frac{\beta}{M} \sum_{K} \sigma_{K} + B \sum_{K} \sigma_{K} \sigma_{K+1} \right. \\
\left. + \frac{\beta}{M} \sqrt{\alpha r z} \sum_{K} \sigma_{K} + \frac{\alpha \beta^2}{2M^2}(t-r) \left( \sum_{K} \sigma_{K} \right)^2 \right]
\]

\[
= n \int Dz log \sum_{\sigma} \int Dw \exp \left[ \frac{\beta}{M} \sum_{K} \sigma_{K} + B \sum_{K} \sigma_{K} \sigma_{K+1} \right. \\
\left. + \frac{\beta}{M} \sqrt{\alpha r z} \sum_{K} \sigma_{K} + \frac{\beta}{M} \sqrt{\alpha (t-r) w} \sum_{K} \sigma_{K} \right]
\]

\[
\xrightarrow{M \to \infty} n \int Dz \log \int Dw \text{Tr} \exp \left[ \beta \left\{ m \cdot \xi + \sqrt{\alpha r z} + \sqrt{\alpha (t-r) w} \right\} \sigma_z + \beta \Delta \sigma_x \right]
\]

\[
= n \int Dz \log \int Dw 2 \cosh \beta \left[ \sqrt{m \cdot \xi + \sqrt{\alpha r z} + \sqrt{\alpha (t-r) w}^2} + \Delta^2 \right]. \quad (3.24)
\]

The total free energy is then obtained from (3.21), (3.22) and (3.24) as

\[
f = \frac{1}{2} m^2 + \frac{\alpha}{2} \beta \left[ \log(1 - \beta S + \beta q) - \frac{\beta q}{1 - \beta S + \beta q} - \beta(1 - S) \right] + \frac{\alpha \beta}{2} (tS - rq) \\
- T \ll \int Dz \log \int Dw 2 \cosh \beta \left[ \sqrt{m \cdot \xi + \sqrt{\alpha r z} + \sqrt{\alpha (t-r) w}^2} + \Delta^2 \right] \gg . \quad (3.25)
\]

### 3.3 Equations of state at \( T = 0 \)

Variation of the free energy (3.25) gives the equations of state for the order parameters. Let us consider the case in which only one of the patterns is retrieved, \( m_\mu = \delta_{\mu 1} \cdot m \). For brevity of expressions, we introduce the following notations,

\[
g = m + \sqrt{\alpha rz} + \sqrt{\alpha (t-r) w} , \\
u = \sqrt{g^2 + \Delta^2} , \\
Y = \int Dw \cosh \beta u .
\]

The equations of state obtained by the variation of the free energy (3.25) with respect to \( m, q, s, r \) and \( t \) are, respectively,

\[
m = \int Dz Y^{-1} \int Dw gu^{-1} \sinh \beta u , \quad (3.26)
\]
Quantum effects play most important roles in the ground state where thermal fluctuations are absent. We therefore only consider the case \( T = 0 \) hereafter. The equations of state simplify considerably in the \( T = 0 \) limit because \( q = S \) and \( t = r \) when \( T = 0 \) as proved below. For very small \( T \), we find from (3.29) and (3.30)

\[
S = \int Dz \left( Y^{-1} \int Dw g^{-1} u \sinh \beta u \right)^2, \quad (3.29)
\]

\[
S = \int Dz Y^{-1} \left( \int Dw g^2 u^{-2} \cosh \beta u \right. + T\Delta^2 \int Dw u^{-3} \sinh \beta u \Big), \quad (3.30)
\]

If we assume that \( S \) is strictly larger than \( q \) at \( T = 0 \), (3.28) immediately leads to \( t = r \) and also (3.27) to \( r = 0 \) because \(-\beta S + \beta q \) diverges. Then, \( u \) is a constant \((m^2 + \Delta^2)^{1/2}\), and hence (3.29) and (3.30) imply \( S = q \), a contradiction to the assumption \( S > q \). Therefore, the equality \( S = q \) holds in the ground state.

To derive the ground-state equation of state for \( r \) from (3.27), we have to estimate the \( T \to 0 \) limit of \( \beta(S - q) \). Comparison of (3.29) and (3.30) leads to

\[
\lim_{T \to 0} \beta(S - q) = \Delta^2 \int Dz \frac{1}{(m + \sqrt{\alpha} rz + \Delta^2)^{3/2}} \equiv C. \quad (3.31)
\]

The equations of state at \( T = 0 \) are thus written as

\[
m = \int Dz \frac{m + \sqrt{\alpha} rz}{\sqrt{(m + \sqrt{\alpha} rz)^2 + \Delta^2}}, \quad (3.32)
\]

\[
q = \int Dz \frac{(m + \sqrt{\alpha} rz)^2}{(m + \sqrt{\alpha} rz)^2 + \Delta^2}, \quad (3.33)
\]

\[
r = \frac{q}{(1 - C)^2}. \quad (3.34)
\]

The same equations of state can also be derived by a direct mean-field analysis as explained in detail in Appendix A. The reason why we have presented the replica analysis in the present section is three fold. First, the replica method gives the free energy by means of which we can distinguish two different retrieval phases as explained in the next subsection. Second, the AT line, namely the
stability limit of the RS solution, can be determined by the replica formalism as discussed in §3.5. Lastly, it is useful to confirm that two different methods lead to the same results so that we acquire confidence in the appropriateness of the present approximations.

3.4 Phase diagram at \( T = 0 \)

To draw the phase diagram, the equations of state (3.32), (3.33) and (3.34) have been solved numerically. The result is drawn in Fig. 8. The asymptotic form of the overlap order parameter around the The three phases are characterized by the relations \( m = q = 0 \) (paramagnetic, P), \( m = 0, q > 0 \) (spin glass, SG) and \( m \neq 0, q > 0 \) (retrieval, R-I and R-II), respectively. The retrieval phase is separated into two parts, one with \( f_R < f_{SG} \) (R-I) and the other with \( f_{SG} < f_R \) (R-II), where \( f_R \) is the free energy of the retrieval state and \( f_{SG} \) is that of the spin glass state.

The transition between the paramagnetic and the spin glass phases is of second order, and so the shape of the boundary can be determined analytically by expansion of the equation of state (3.33). The result is

\[
\Delta = 1 + \sqrt{\alpha}.
\]

(3.35)

This is exactly the same relation as the corresponding classical phase boundary if we replace \( \Delta \) by \( T \). The other phase transitions (between SG and R-II and between R-I and R-II) are both of first order. Therefore, it is in general impossible to obtain the analytic expressions of these phase transitions...
boundaries. However, when \( \alpha \) is very small, we can determine the asymptotic forms of the phase boundaries by expansions of the equations of state and the free energy as in the case of the classical model. Details are described in Appendix 1. The result is

\[
\Delta \simeq 1 - 1.95\sqrt{\alpha}
\]

(3.36)

for the transition between the SG and R-II phases, and

\[
\Delta \simeq 1 - \frac{33}{16}\sqrt{\alpha} = 1 - 2.0625\sqrt{\alpha}
\]

(3.37)

between the R-I and R-II regions. The relation (3.36) exactly agrees with that of the classical case if we replace \( \Delta \) by \( T \). The relation (3.37) shows that the R-I region of this model in the vicinity of \( (\alpha = 0, \Delta = 1) \) is wider than that of the classical case, \( T \simeq 1 - 2.6\sqrt{\alpha} \) though the deviation from unity is also proportional to \( \sqrt{\alpha} \).

When \( \Delta = 0 \), the R-II phase changes into the SG phase at \( \alpha = 0.1379 \) as it should. The boundary between the SG and R-II phases is slightly reentrant in the low-temperature region. That is, for a fixed \( \alpha \) slightly larger than 0.1379, the SG phase once changes to the R-II phase as \( \Delta \) is decreased but the SG phase once again becomes stable for very small \( \Delta \). The same is true for the boundary between the R-I and R-II regions. The reentrance in these two cases is observed also in the classical model. It should be noted, however, that the effect of replica symmetry breaking treated in the next subsection obscures the significance of reentrance within the RS solution near \( \alpha = 0.1379 \), again the same situation as in the classical model.

3.5 AT line

The stability of the replica symmetric solution against replica symmetry breaking can be checked following the standard procedure. We keep the static approximation intact and investigate only the effects of replica symmetry breaking. Calculations are somewhat involved but straightforward. Details are given in Appendix C. The stability limit of the replica symmetric solution (the AT line) is found to be given by

\[
q = \alpha r \Delta^4 \int Dz \left[ (m + \sqrt{\alpha}rz)^2 + \Delta^2 \right]^{-3}
\]

(3.38)

This AT line is also drawn in the phase diagram in Fig. 8.

The region near \( \alpha = 0.1379 \) and \( \Delta = 0 \) is drawn enlarged in Fig. 9. The AT line merges with the boundary between the R-I and SG phases at \( \alpha \simeq 0.1378 \) and \( \Delta \simeq 0.022 \). Since the replica-symmetric solution is unstable below this line, the reentrant behavior mentioned in the previous subsection may be an artifact of the replica symmetric solution.

§4. Summary and Discussion

The Hopfield model in a transverse field has been introduced and solved. For a finite number of embedded patterns, we have used the method of the Trotter decomposition to reduce the quantum
Fig. 9. The phase diagram near $\alpha = 0.1379$ and $\Delta = 0$.

problem to a classical form. We found that the solutions of the equations of state in the ground state have quite similar properties to those of the classical model at finite temperatures.

If the number of embedded patterns is proportional to the system size, it is necessary to employ the replica technique, in addition to the Trotter decomposition, to trace out the quenched randomness. We cannot obtain the full exact solution and have to appeal to the replica symmetric (RS) and static approximations. The stability analysis of the RS solution leads to the AT line, above which the RS solution is at least locally stable. On the other hand, the static approximation probably does not give the exact solutions for all values of the parameters except for the limits of $\Delta \to 0$ (the classical model) and $\alpha \to 0$ (the finite-$p$ model). Nevertheless, the experience in the transverse SK model suggests that the static approximation is expected to capture the qualitative features of spin systems in transverse fields. The resulting phase diagram has three phases, the retrieval, spin glass and the paramagnetic phases. The shapes of phase boundaries turn out to be almost the same as those of the classical model if we replace $\Delta$ by $T$.

We have found that the quantum fluctuations have almost the same effects as thermal fluctuations in the Hopfield model. The uncertainties in signal transmission at a synapse have conventionally been treated in model analyses in terms of thermal fluctuations. The results of the present study show that we may instead consider quantum fluctuations without changing conclusions on the macroscopic behavior of the network.

The above-mentioned fact means that quantum uncertainties do not lead to many possible alternative values of macroscopic variables but rather cause only quantitative deteriorations as thermal
fluctuations do. Quantum uncertainties at microscopic levels are far from the source of the simultaneous existence of macroscopically distinguishable states. This point seems to be disregarded in some of the arguments concerning the significance of quantum mechanics in the functioning of the brain.\footnote{fn:1}

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**Appendix A: Equation of State by the Mean-Field Theory**

In this Appendix we derive the equations of state (3.32), (3.33) and (3.34) by a direct mean-field analysis.\footnote{fn:2} The Hamiltonian of the Hopfield model (2.1) with the Hebb rule (2.2) is approximated by the following single-site effective Hamiltonian:

$$\mathcal{H}_i = -\sigma_i^z \sum_{\mu} \xi_{\mu}^i m_{\mu} - \Delta \sigma_i^x ,$$

(A.1)

where $m_{\mu}$ stands for the overlap

$$m_{\mu} = \frac{1}{N} \sum_j \xi_{\mu j}^}\langle \sigma_j^z \rangle .$$

(A.2)

The thermal expectation value of $\sigma_j^z$ appearing in (A.2) can be calculated easily under the Hamiltonian (A.1) to give

$$\langle \sigma_j^z \rangle = \frac{u_i}{\sqrt{u_i^2 + \Delta^2}} \tanh \beta \sqrt{u_i^2 + \Delta^2} ,$$

(A.3)

where

$$u_i = \sum_{\mu} m_{\mu} \xi_{\mu}^i .$$

(A.4)

Let us now assume that a single pattern, the first one ($\mu = 1$) for instance, is retrieved. Then, $m_1 \equiv m$ is of order unity and all other $m_{\mu}$’s are of order $1/\sqrt{N}$. The equation of state (A.2) for $\mu = 1$ is then written as

$$m = \frac{1}{N} \sum_i \frac{m + \sqrt{\alpha rz_i}}{\sqrt{(m + \sqrt{\alpha rz_i})^2 + \Delta^2}} \times \tanh \beta \sqrt{(m + \sqrt{\alpha rz_i})^2 + \Delta^2} ,$$

(A.5)

where

$$\sqrt{\alpha rz_i} = \sum_{\mu \geq 2} m_{\mu} \xi_{\mu i}^1 \xi_{i}^1 .$$

(A.6)

Since $z_i$ is proportional to the sum of extensively many random variables, it is not unreasonable to assume that $z_i$ is a Gaussian random variable according to the central limit theorem. Strictly speaking, the terms appearing in the definition (A.6) are mutually correlated through the $\xi$-dependence
of $m_\mu$ and the central limit theorem is not applicable. Nevertheless, the results obtained under this approximation of Gaussian distribution agree with those from the replica method in the case of the classical Hopfield model.\[^{[5]}\] We thus adopt the same approach in the present quantum case.

The average value of $z_i$ is expected to be vanishing because $\xi_\mu \xi_1$ is $\pm 1$ with equal probability. The variance of $z_i$ is unity if we define $r$ in the following way:

\[ \ll \left( \sum_{\mu \geq 2} m_\mu \xi_\mu \xi_1 \right)^2 \gg = \ll \sum_{\mu \geq 2} m_\mu^2 \gg = \alpha r . \tag{A.7} \]

Then, in the thermodynamic limit, the summation over $i$ in (A.5) reduces to the average over the Gaussian distribution as

\[ m = \int Dz \frac{m + \sqrt{\alpha rz}}{\sqrt{(m + \sqrt{\alpha rz})^2 + \Delta^2}} \times \tanh \beta \sqrt{(m + \sqrt{\alpha rz})^2 + \Delta^2} , \tag{A.8} \]

where $Dz$ denotes the Gaussian measure $e^{-z^2/2}dz/\sqrt{2\pi}$.

To derive the equation of state for $r$, we need to evaluate $m_\nu$ for $\nu \geq 2$ according to (A.7). For this purpose we decompose $u_i$ defined in (A.4) into three parts by extracting contributions from $\mu = 1$ and $\mu = \nu$:

\[ u_i \xi_1 = m + \sqrt{\alpha rz}_i + \eta_\nu m_\nu , \tag{A.9} \]

with $\eta_\nu = \xi_\nu \xi_1$. Although the definition of $z_i$ here does not include the contribution from $\mu = \nu$ in contrast to (A.6), this difference does not affect the results because the only property of $z_i$ we use is its Gaussian distribution, which remains the same if we drop a single term from (A.6). It should be noted here that $m_\nu$ is of order $1/\sqrt{N}$ and is quite small compared to the other two terms in (A.9).

We insert (A.9) into the right hand side of (A.2) with (A.3) taken into account and expand the result to first order of $m_\nu$ to find

\[ m_\nu = \frac{1}{N} \sum_i \frac{v_i}{v_i^2 + \Delta^2} \tanh \beta \sqrt{v_i^2 + \Delta^2} \]

\[ + \frac{m_\nu}{N} \sum_i \left[ \frac{\Delta^2}{(v_i^2 + \Delta^2)^{3/2}} \tanh \beta \sqrt{v_i^2 + \Delta^2} + \frac{\beta v_i^2}{v_i^2 + \Delta^2} \text{sech}^2 \beta \sqrt{v_i^2 + \Delta^2} \right] , \tag{A.10} \]

with $v_i = \eta_\nu (m + \sqrt{\alpha rz}_i)$. We now define $q$ and $S$ as

\[ q = \frac{1}{N} \sum_i \frac{v_i^2}{v_i^2 + \Delta^2} \tanh^2 \beta \sqrt{v_i^2 + \Delta^2} \]

\[ = \int Dz \frac{v^2}{v^2 + \Delta^2} \tanh^2 \beta \sqrt{v^2 + \Delta^2} \tag{A.11} \]
\[
S = 1 - \int Dz \left[ \frac{\Delta^2}{v^2 + \Delta^2} \right.
- \frac{T \Delta^2}{(v^2 + \Delta^2)^{3/2}} \tanh \beta \sqrt{v^2 + \Delta^2} \left. \right], \quad (A.12)
\]

with \( v = m + \sqrt{\alpha r z} \). Equation (A.10) can then be solved for \( m_{\nu} \) as
\[
m_{\nu} = \frac{1}{1 - \beta S + \beta q} \times \frac{1}{N} \sum_i \frac{v_i}{v_i^2 + \Delta^2} \tanh \beta \sqrt{v_i^2 + \Delta^2}. \quad (A.13)
\]

From (A.10) and (A.11), we find
\[
(1 - \beta S + \beta q)^2 \ll \sum_{\nu \geq 2} m_{\nu}^2 \gg
\]
\[
= \frac{p - 1}{N^2} \sum_i \frac{v_i^2}{v_i^2 + \Delta^2} \tanh^2 \beta \sqrt{v_i^2 + \Delta^2}
= \alpha q. \quad (B.1)
\]

Equation (A.17) and the above relation lead to
\[
r = \frac{q}{(1 - \beta S + \beta q)^2}. \quad (A.14)
\]

It is straightforward to check that the zero-temperature limits of (A.8), (A.11), (A.12) and (A.14) agree with (3.31), (3.32), (3.33) and (3.34) derived by the replica method, respectively.

**Appendix B: Phase Boundaries Near \( \alpha = 0 \)**

In this Appendix we derive the asymptotic forms of phase boundaries around the point \((\alpha = 0, \Delta = 1)\). The first boundary is between the SG and R-II phases and the second is between the R-I and R-II phases.

To derive the boundary between the SG and R-II phases, we expand the right-hand side of the equation of state (3.32) assuming \( m \) and \( r \) are small:
\[
m = \frac{1}{\Delta} \left[ m - \frac{1}{2\Delta^2} (m^3 + 3m \alpha r) \right].
\]

Let us write \( 1 - \Delta = \epsilon \). Then, the above equation is approximated to order \( \epsilon \) as
\[
m^2 + 3m \alpha r = 2 \epsilon. \quad (B.1)
\]

Similarly, the equations of state for \( C \), \( q \) and \( r \) ((3.31), (3.33) and (3.34)) behave asymptotically as
\[
C = (1 + \epsilon) \left[ 1 - \frac{3}{2} (m^2 + \alpha r) \right], \quad (B.2)
q = m^2 + \alpha r, \quad (B.3)
r = \frac{4(m^2 + \alpha r)}{(2\epsilon - 3m^2 - 3m \alpha r)^2}. \quad (B.4)
\]
To erase $r$ from (B.1) and (B.4), we define $x = m^2 + \alpha r$ and write the above equations as

$$x = m^2 + \frac{4\alpha x}{(3x - 2\epsilon)^2}, \quad (B.5)$$

$$x = \frac{2}{3}(\epsilon + m^2). \quad (B.6)$$

From these equations we obtain

$$\frac{2}{3}(\epsilon + m^2) = m^2 + \frac{2\alpha(\epsilon + m^2)}{3m^4}. \quad (B.7)$$

Using new parameters $y$ and $\tau$ defined by

$$y = m^2/\sqrt{\alpha}, \quad (B.8)$$

$$\tau = \epsilon/\sqrt{\alpha}, \quad (B.9)$$

(B.7) is expressed as

$$\frac{1}{2}y^3 - \tau y^2 + y + \tau = 0, \quad (B.10)$$

which coincides with (5.12) of Ref. 2, though the definition of $y$ is not the same. Since bifurcation occurs on the SG–R-II phase boundary, the derivative of this equation is also vanishing on this boundary, that is,

$$\frac{3}{2}y^2 - 2\tau y + 1 = 0. \quad (B.11)$$

Eliminating $\tau$ from (B.10) and (B.11), we have

$$\frac{1}{2}y^3 + y = \frac{3}{2}y^2 + 1,$$

and therefore

$$y^2 = \frac{5 + \sqrt{33}}{2} \quad \text{and} \quad \tau = 1.95 \cdots,$$

or equivalently,

$$\Delta \approx 1 - 1.95\sqrt{\alpha}.$$

The boundary between the R-I and R-II phases is the line on which the free energy of the solution with $m \neq 0$ and that of the solution with $m = 0$ are equal to each other. We first derive the explicit form of the free energy at $T = 0$. All we have to do is to take the limit $\beta \to \infty$ in the expression of the free energy at finite temperatures (3.25). However, we should be very careful in taking this limit because both $(S - q)$ and $(t - r)$ are of order $T$, as shown in (3.28) and (3.31). In order to evaluate the last term of (3.25), $-T \ll \cdots$, we expand this term up to first order of $(t - r)$ for the single retrieval case to obtain

$$-T \ll \cdots$$

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\[
\begin{align*}
&= -T \int Dz \log \int Dw \cosh \left[ \beta \sqrt{(m + \sqrt{\alpha rz})^2 + \Delta^2} \left( 1 + \frac{(m + \sqrt{\alpha rz}) \sqrt{\alpha(t-r)w}}{(m + \sqrt{\alpha rz})^2 + \Delta^2} + \cdots \right) \right] \\
&= -T \left\{ \int Dz \log \left[ \exp \left( \beta \sqrt{(m + \sqrt{\alpha rz})^2 + \Delta^2} \right) \right] \int Dw \exp \left( \frac{\beta(m + \sqrt{\alpha rz}) \sqrt{\alpha(t-r)w}}{(m + \sqrt{\alpha rz})^2 + \Delta^2} \right) \\
&\quad + \exp \left( -\beta \sqrt{(m + \sqrt{\alpha rz})^2 + \Delta^2} \right) \int Dw \exp \left( -\beta \frac{(m + \sqrt{\alpha rz}) \sqrt{\alpha(t-r)w}}{(m + \sqrt{\alpha rz})^2 + \Delta^2} \right) \right\} \\
&\quad + \frac{1}{2} \beta^2 \alpha(t-r) \int Dz \frac{(m + \sqrt{\alpha rz})^2}{(m + \sqrt{\alpha rz})^2 + \Delta^2} \\
&\xrightarrow{T \to 0} - \int Dz \sqrt{(m + \sqrt{\alpha rz})^2 + \Delta^2} - \frac{1}{2} \alpha \beta(t-r)q . \quad (B.12)
\end{align*}
\]

Substituting this formula to (3.25) and taking the limit \( T \to 0 \), we have

\[
\begin{align*}
f &= \frac{1}{2} \alpha^2 \frac{m^2}{2} - \frac{\alpha}{2} \left( 1 + \frac{C}{1-C} q - rC \right) \\
&\quad - \int Dz \sqrt{(m + \sqrt{\alpha rz})^2 + \Delta^2} . \quad (B.13)
\end{align*}
\]

Next, we expand this expression with respect to \( \epsilon = 1 - \Delta \). Note that the last term of (B.13) is not multiplied by a smallness parameter \( \alpha/2 \) in contrast to the preceding term, and thus we have to calculate up to second order of \( \epsilon \) in this term. We aim to express the free energy only using the parameters \( y \) and \( \tau \) as (B.10). For this purpose, we write all the parameters in terms of \( m^2 \) and \( \epsilon \). In the solution with \( m \neq 0 \), (B.1), (B.3) and (B.2) reduce to

\[
\begin{align*}
q &= \frac{2}{3} (m^2 + \epsilon) , \quad (B.14) \\
\alpha r &= \frac{2}{3} \epsilon - \frac{1}{3} m^2 , \quad (B.15) \\
C &= 1 + \epsilon - (m^2 + \epsilon) . \quad (B.16)
\end{align*}
\]

On the other hand, in the solution with \( m = 0 \), we use (3.34) instead of (B.1). Substituting the relation \( q = \alpha r \) to (3.34), we have \( (1 - C)^2 = \alpha \). Because the term

\[
\frac{\alpha}{2\beta} \log(1 - \beta S + \beta q) \xrightarrow{T \to 0} \frac{\alpha}{2\beta} \log(1 - C)
\]

exists in the original expression of the free energy (3.25), the condition \( C < 1 \) should be satisfied, and we obtain

\[
\begin{align*}
q &= \alpha r = \frac{2}{3} (\epsilon + \sqrt{\alpha}) , \quad (B.17) \\
C &= 1 - \sqrt{\alpha} . \quad (B.18)
\end{align*}
\]

Using (B.8), (B.9) and (B.13)–(B.18), after long but straightforward calculations we have

\[
f(m) - f(0) = \frac{\alpha}{6} \left[ \frac{1}{2} y^2 - 2\tau \left( y + \frac{1}{y} \right) + 4\tau + 1 \right] = 0 . \quad (B.19)
\]

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Solving this equation together with (B.10), we obtain
\[ y = 3 \quad \text{and} \quad \tau = \frac{33}{16} = 2.0625 , \]
or equivalently,
\[ \Delta \simeq 1 - 2.0625\sqrt{\alpha} . \]

Appendix C: AT Line

We derive the expression of the AT line (3.38) at \( T = 0 \). We follow the Appendix B of Ref.2 in the first half of the argument. To obtain the second derivatives of the free energy by the variables \( q_{\rho \sigma} \) and \( r_{\rho \sigma} \) under the static approximation, we retain only the relevant part in (3.10):

\[
f = \frac{\alpha}{2\beta} \sum_{K \rho} \log \lambda_{K \rho} + \frac{\alpha \beta}{2} \sum_{\rho \sigma} r_{\rho \sigma} q_{\rho \sigma} - T \ll \log \sum_{\sigma} \exp \left[ \frac{\beta}{M} \sum_{\rho \mu} m_{\rho \mu} \xi_{\mu} \sum_{K} \sigma_{\rho K} + B \sum_{K \rho} \sigma_{\rho K} \sigma_{\rho, K+1} + \frac{\alpha \beta}{2M^2} \sum_{\rho \sigma} \sum_{KL} \sigma_{\rho K} \sigma_{\sigma L} \right] \gg . \]  

(C.1)

We have set \( r_{\rho \rho} = t_\rho \) in anticipation of the ground state relation \( r = t \) at the replica-symmetric point as explained in §3.3.

The Hessian, or the second-derivative matrix of the free energy, is written as

\[
\begin{bmatrix}
A^{\alpha \beta, \gamma \delta} & \delta^{\alpha \beta, \gamma \delta} \\
\delta^{\alpha \beta, \gamma \delta} & B^{\alpha \beta, \gamma \delta} \\
\end{bmatrix},
\]

(C.2)

where

\[
A^{\alpha \beta, \gamma \delta} = \frac{\partial^2 f}{\partial q_{\alpha \beta} \partial q_{\gamma \delta}} , \quad (C.3)
\]

\[
B^{\alpha \beta, \gamma \delta} = \frac{\partial^2 f}{\partial r_{\alpha \beta} \partial r_{\gamma \delta}} , \quad (C.4)
\]

\[
\delta^{\alpha \beta, \gamma \delta} = \frac{\partial^2 f}{\partial q_{\alpha \beta} \partial r_{\gamma \delta}} . \quad (C.5)
\]

There is no difficulty in evaluating the derivatives written above by taking into account that \( \{ \lambda_{K \rho} \} \) is a set of eigenvalues of the \( Mn \times Mn \) matrix (3.4) under the static approximation. Note that the replica-index dependence of the matrix elements should be kept untouched until differentiations are carried out. The result is almost the same as that of the classical Hopfield model.\(^2\)

For the submatrix \( A \),

\[
A^{\rho \sigma, \rho \sigma} = -\alpha \beta \left( C_{\rho \rho}^2 + C_{\rho \sigma}^2 \right) , \quad (C.6)
\]
\[ A^{\rho\sigma,\rho\gamma} = -\alpha \beta \left( C^{\rho\rho} C^{\rho\sigma} + C^{\rho\sigma 2} \right) , \]  
(C.7)

\[ A^{\rho\sigma,\gamma\delta} = -\alpha \beta \left( 2C^{\rho\sigma 2} \right) , \]  
(C.8)

where

\[
C^{\rho\sigma} = \frac{\beta q}{[1 - \beta(S - q)]^2} , \]  
(C.9)

\[
C^{\rho\rho} = C^{\rho\sigma} + \frac{1}{1 - \beta(S - q)} . \]  
(C.10)

The elements of the submatrix \( B \) are expressed as

\[
B^{\rho\sigma,\rho\sigma} = -\alpha^2 \beta^3 \ll \langle \sigma_{\rho K} \sigma_{\rho L} \rangle^2 - \langle \sigma_{\rho K} \rangle^4 \gg , \]  
(C.11)

\[
B^{\rho\sigma,\rho\gamma} = -\alpha^2 \beta^3 \ll \langle \sigma_{\rho K} \sigma_{\rho L} \rangle \langle \sigma_{\rho K} \rangle^2 - \langle \sigma_{\rho K} \rangle^4 \gg , \]  
(C.12)

\[
B^{\rho\sigma,\gamma\delta} = -\alpha^2 \beta^3 \ll \langle \sigma_{\rho K} \rangle^4 - \langle \sigma_{\rho K} \rangle^4 \gg = 0 , \]  
(C.13)

where \( \langle \cdots \rangle \) denotes the average with respect to the weight appearing as the exponential function in (C.1). This average should be evaluated at the replica-symmetric point to check the stability of the replica-symmetric solution against replica-symmetry breaking. The off-diagonal element of the block matrix \( (C.2) \) is given as

\[ \delta^{\rho\sigma,\gamma\delta} = \alpha \beta \left( \delta^{\rho\gamma} \delta^{\sigma\delta} + \delta^{\rho\delta} \delta^{\sigma\gamma} \right) . \]  
(C.14)

We follow the usual procedure to find the eigenvalue of the replicon mode,

\[
qu_{\rho\sigma} = q + \eta_{\rho\sigma} , \]  
(C.15)

\[
ru_{\rho\sigma} = r + xu_{\rho\sigma} , \]  
(C.16)

where

\[
\eta_{\rho\sigma} = \eta \ (\rho, \sigma \neq 1, 2) , \]
\[
\eta_{1\rho} = \eta_{2\rho} = \frac{1}{2} (3 - n) \eta \ (\rho \neq 1, 2) , \]
\[
\eta_{12} = \frac{1}{2} (2 - n)(3 - n) \eta , \]
\[
\eta_{\rho\rho} = 0 . \]

The eigenvalue equations are then given by

\[
\sum_{\gamma\delta} \left( A^{\rho\sigma,\gamma\delta} + xu^{\rho\sigma,\gamma\delta} \right) \eta_{\gamma\delta} = \lambda \eta , \]  
(C.17)

\[
\sum_{\gamma\delta} \left( xuB^{\rho\sigma,\gamma\delta} + xu^{\rho\sigma,\gamma\delta} \right) \eta_{\gamma\delta} = \lambda x \eta , \]  
(C.18)
with $\rho, \sigma \neq 1, 2$. Substitution of the expressions of $A^{\rho\sigma,\gamma\delta}$, $B^{\rho\sigma,\gamma\delta}$, $\delta^{\rho\sigma,\gamma\delta}$ and $\eta_{\gamma\delta}$ into these equations leads to

$$x = \tilde{\lambda} + \frac{1}{1 - \beta(S - q)^2},$$

$$x \left[ \alpha \beta^2 \ll \left( \langle \sigma_{\rho K} \sigma_{\rho L} \rangle - \langle \sigma_{\rho K} \rangle^2 \right)^2 \gg + \tilde{\lambda} \right] = 1,$$

with $\tilde{\lambda} = \lambda / \alpha \beta$. This set of equations is solved for the eigenvalues as

$$\tilde{\lambda}_\pm = -\frac{1}{2} (u + v) \pm \sqrt{\frac{1}{4} (u + v)^2 + 1 - uv},$$

where

$$u = \alpha \beta^2 \ll \left( \langle \sigma_{\rho K} \sigma_{\rho L} \rangle - \langle \sigma_{\rho K} \rangle^2 \right)^2 \gg,$$

$$v = \frac{1}{1 - \beta(S - q)^2} = \frac{r}{q}.$$

Here we have used the equation of state (3.27) in the last equality. The AT line is determined by the vanishing point of the eigenvalue $\tilde{\lambda}_+, \tilde{\lambda}_-$ or equivalently, $uv = 1$, which has the following explicit form,

$$\alpha \beta^2 \ll \left( \langle \sigma_{\rho K} \sigma_{\rho L} \rangle - \langle \sigma_{\rho K} \rangle^2 \right)^2 \gg = \frac{q}{r}. \quad (C.19)$$

It is necessary to evaluate the expectation values of spin variables appearing in (C.19). We define the quantity in the brackets $\ll \cdots \gg$ in (C.19) as $V$:

$$V = \ll \left( \langle \sigma_{\rho K} \sigma_{\rho L} \rangle - \langle \sigma_{\rho K} \rangle^2 \right)^2 \gg = \ll \langle \sigma_{\rho K} \sigma_{\rho L} \rangle \langle \sigma_{\lambda K} \sigma_{\lambda L} \rangle \gg - 2 \ll \langle \sigma_{\rho K} \sigma_{\rho L} \rangle \langle \sigma_{\lambda K} \rangle \langle \sigma_{\sigma K} \rangle \gg + \ll \langle \sigma_{\rho K} \rangle \langle \sigma_{\lambda K} \rangle \langle \sigma_{\sigma K} \rangle \langle \sigma_{\kappa K} \rangle \gg.$$

According to the usual replica formalism [4] this equation is equivalent to the following expression in the limit $n \to 0$:

$$V = \ll \sum_{\sigma} (\sigma_{\rho K} \sigma_{\rho L} \sigma_{\lambda K} \sigma_{\lambda L} - 2 \sigma_{\rho K} \sigma_{\rho L} \sigma_{\lambda K} \sigma_{\sigma K} + \sigma_{\rho K} \sigma_{\lambda K} \sigma_{\nu K} \sigma_{\kappa L}) \times \exp \left[ \frac{\beta}{M} m \cdot \xi \sum_{K, \rho} \sigma_{\rho K} + B \sum_{K, \rho} \sigma_{\rho K} \sigma_{\rho, K + 1} + \frac{\alpha \beta^2}{2M^2} r \sum_{K, \rho, \rho} \sigma_{\rho K} \sigma_{\rho L} \right] \gg, \quad (C.20)$$

where different suffixes $\rho, \lambda, \nu$ and $\kappa$ correspond to different replicas.

Let us consider the single-retrieval case $m_\mu = \delta_{\mu 1} \cdot m$. The double summation over $K, L, \rho, \sigma$ in (C.20) can be decoupled using the Gaussian integral. In the limit $n \to 0$, we find

$$V = \int Dz \left( \langle \sigma_{\lambda K} \sigma_{\lambda L} \rangle_z - \langle \sigma_{\lambda K} \rangle_z^2 \right)^2, \quad (C.21)$$
where \( \langle \cdots \rangle_z \) stands for the average with respect to the weight

\[
\exp \left[ \frac{\beta m}{M} \sum_K \sigma_K + B \sum_K \sigma_K \sigma_{K+1} + \frac{\beta \sqrt{\alpha r}}{M} z \sum_K \sigma_K \right].
\] (C.22)

The averages appearing in the above equation can be calculated easily. The single-spin average is given by

\[
\langle \sigma \rangle_z = \frac{\text{Tr} \sigma e^{\beta h \sigma + \beta \Delta \sigma}}{\text{Tr} e^{\beta h \sigma + \beta \Delta \sigma}} = \frac{h}{\sqrt{h^2 + \Delta^2}} \tanh \beta \sqrt{h^2 + \Delta^2},
\] (C.23)

with \( h = m + \sqrt{\alpha r} z \), and the first equality holds in the limit \( M \to \infty \). The two-spin expectation value in (C.21) is the correlation function of the one-dimensional Ising model in a uniform magnetic field, and thus can be calculated by the transfer-matrix method. The result is, in the limit \( M \to \infty \),

\[
\langle \sigma \sigma \rangle_z = \frac{h^2}{v^2} + \frac{\Delta^2}{v^2} \cosh \beta v \left(1 - 2y\right)
\] (C.24)

with \( v = (h^2 + \Delta^2)^{1/2} \) and \( y = (K - L)/M \).

The last expression (C.24) of the correlation function has dependence on the Trotter numbers \( K \) and \( L \) through \( y \), though we have calculated it under the static approximation which ignores such dependence. This inconsistency is remedied if we average (C.24) over the interval \( 0 \leq y \leq 1 \). For example, the ground-state expression of the order parameter \( S (= q) \) in the static approximation (3.33) can be obtained by the integral:

\[
\lim_{T \to 0} \int_0^1 dy \langle \sigma \sigma \rangle_z = \int Dz \frac{(m + \sqrt{\alpha rz})^2}{(m + \sqrt{\alpha rz})^2 + \Delta^2}.
\] (C.25)

This result seems quite natural in consideration of the definition (3.14) of the order parameter \( S(KL) \). Therefore, we replace the quantity in the brackets in (C.21) by its average,

\[
\int_0^1 dy \left( \langle \sigma \sigma \rangle_z - \langle \sigma \rangle_z^2 \right) = \frac{h^2}{v^2} \frac{\Delta^2}{\beta v^3} \tanh \beta v \to \frac{T \Delta^2}{v^3} \quad (T \to 0).
\]

Equation (C.21) now can be written as

\[
V = T^2 \Delta^4 \int \frac{Dz}{v^3},
\]

and the AT line is finally obtained from this relation and (C.13) as

\[
q = \alpha r \Delta^4 \int \frac{Dz}{\left((m + \sqrt{\alpha rz})^2 + \Delta^2\right)^3}.
\]
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\[ l = 3 \]
$l=5$
