EIGENVECTORS OF Z-TENSORS ASSOCIATED WITH LEAST H-EIGENVALUE WITH APPLICATION TO HYPERGRAPHS

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ABSTRACT. Unlike an irreducible Z-matrices, a weakly irreducible Z-tensor $A$ can have more than one eigenvector associated with the least H-eigenvalue. We show that there are finitely many eigenvectors of $A$ associated with the least H-eigenvalue. If $A$ is further combinatorial symmetric, the number of such eigenvectors can be obtained explicitly by the Smith normal form of the incidence matrix of $A$. When applying to a connected uniform hypergraph $G$, we prove that the number of Laplacian eigenvectors of $G$ associated with the zero eigenvalue is equal to the number of adjacency eigenvectors of $G$ associated with the zero eigenvalue if zero is a signless Laplacian eigenvalue.

1. INTRODUCTION

A real tensor (also called hypermatrix) $A = (a_{i_1i_2...i_m})$ of order $m$ and dimension $n$ refers to a multiarray of entries $a_{i_1i_2...i_m} \in \mathbb{R}$ for all $i_j \in [n] := \{1, 2, \ldots, n\}$ and $j \in [m]$, which can be viewed to be the coordinates of the classical tensor under an orthonormal basis. For a vector $x = (x_1, \cdots, x_n) \in \mathbb{C}^n$, $Ax^{m-1} \in \mathbb{C}^n$, which is defined as

$$(Ax^{m-1})_i = \sum_{i_2, \cdots, i_m \in [n]} a_{i_2...i_m}x_{i_2} \cdots x_{i_m}, \ i \in [n].$$

Let $I = (i_{1}, i_{2}, \ldots, i_{m})$ be the identity tensor of order $m$ and dimension $n$, that is, $i_{1}, i_{2}, \ldots, i_{m} = 1$ if $i_1 = i_2 = \cdots = i_m$ and $i_{1}, i_{2}, \ldots, i_{m} = 0$ otherwise.

Definition 1.1 ([13] [17] [2]). Let $A$ be an $m$-th order $n$-dimensional tensor. For some $\lambda \in \mathbb{C}$, if the polynomial system $(\lambda I - A)x^{m-1} = 0$, or equivalently $Ax^{m-1} = \lambda x^{(m-1)}$, has a solution $x \in \mathbb{C}^n \setminus \{0\}$, then $\lambda$ is called an eigenvalue of $A$ and $x$ is an eigenvector of $A$ associated with $\lambda$, where $x^{(m-1)} := (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})$.

For a real tensor $A$, an eigenvalue $\lambda$ of $A$ is called an $H$-eigenvalue if there is a real eigenvector associated with $\lambda$, implying that $\lambda$ is real. Denote by $\lambda_{\text{min}}(A)$ the least H-eigenvalue of $A$, and $\rho(A)$ the spectral radius of $A$ (i.e. the largest modulus of the eigenvalues of $A$). Let $\mathbb{P}^{n-1}$ be the complex projective spaces of dimension $n - 1$. Consider the projective variety

$$V_\lambda = V_\lambda(A) = \{ x \in \mathbb{P}^{n-1} : Ax^{m-1} = \lambda x^{(m-1)} \},$$

which is called the projective eigenvariety of $A$ associated with $\lambda$ [7]. In this paper the number of eigenvectors of $A$ is considered in $V_\lambda(A)$.

By the Perron-Frobenius theorem of nonnegative tensors [1] [10] [21] [22] [23], for an irreducible (or weakly irreducible) nonnegative tensor $A$, the spectral radius $\rho(A)$ is an eigenvalue of $A$ associated with a unique nonnegative eigenvector (or positive.
eigenvector) up to a scalar, which is called the Perron vector of $\mathcal{A}$. In [6] the authors investigate the spectral symmetry of $\mathcal{A}$ by using the eigenvalues with modulus $\rho(\mathcal{A})$, which generalizes some spectral properties of nonnegative irreducible matrices. But, different from the matrices case, $\mathcal{A}$ can have more than one eigenvector associated with $\rho(\mathcal{A})$, including the Perron vector.

$Z$-matrices and $M$-matrices are the generalization of nonnegative matrices. Recently they were generalized to $Z$-tensors and $M$-tensors respectively [24, 4].

**Definition 1.2** ([24, 4]). A real tensor $\mathcal{A}$ is called a $Z$-tensor if all of its off-diagonal entries are non-positive, or equivalently it can be written as

$$
\mathcal{A} = s\mathcal{I} - \mathcal{B},
$$

where $s > 0$ and $\mathcal{B}$ is nonnegative. If $s \geq \rho(\mathcal{B})$, then $\mathcal{A}$ is called an $M$-tensor; if $s > \rho(\mathcal{B})$, then $\mathcal{A}$ is called a nonsingular $M$-tensor or strong $M$-tensor.

Zhang et al. [24] showed that the minimum real part of all eigenvalues of a $Z$-tensor is the least H-eigenvalue. Some characterization of a $Z$-tensor can be referred to [1, 10]. The generalization of the minimum real part of all eigenvalues of a $Z$-tensor as in (1.1) is different from the case of irreducible $Z$-matrices.

**2. Preliminaries**

Let $\mathcal{A}$ be an $m$-th order $n$-dimensional real tensor. $\mathcal{A}$ is called symmetric if its entries are invariant under any permutation of their indices. The irreducibility or weakly irreducibility of a tensor can be referred to [1, 10]. The support of $\mathcal{A}$, denoted by $\text{supp}(\mathcal{A}) = \{s_{i_1i_2\ldots i_m}\}$, is defined as a tensor with same order and dimension as $\mathcal{A}$, such that $s_{1\ldots i_m} = 1$ if $a_{i_1\ldots i_m} \neq 0$, and $s_{i_1\ldots i_m} = 0$ otherwise. $\mathcal{A}$ is called combinatorial symmetric if $\text{supp}(\mathcal{A})$ is symmetric. Let $\mathcal{A}$ be a combinatorial symmetric tensor of order $m$ and dimension $n$. Set

$$
E(\mathcal{A}) = \{(i_1, i_2, \ldots, i_m) \in [n]^m : a_{i_1i_2\ldots i_m} \neq 0, 1 \leq i_1 \leq \cdots \leq i_m \leq n\}.
$$

Define

$$
b_{e,j} = |\{k : i_k = j, e = (i_1, i_2, \cdots, i_m) \in E(\mathcal{A}), k \in [m]\}|$$

and obtain an $|E(\mathcal{A})| \times n$ matrix $\mathcal{I}_A = (b_{e,j})$, called the incidence matrix of $\mathcal{A}$.

A hypergraph $G = (V(G), E(G))$ consists of a vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and an edge set $E(G) = \{e_1, e_2, \ldots, e_l\}$, where $e_j \subseteq V(G)$ for $j \in [l]$. If $|e_j| = m$ for each $j \in [l]$, then $G$ is called an $m$-uniform hypergraph. The adjacency tensor of an $m$-uniform hypergraph $G$ is defined to be $A(G) = (a_{i_1i_2\ldots i_m})$, where $a_{i_1i_2\ldots i_m} = \frac{1}{(m-1)!}$ if $\{v_{i_1}, v_{i_2}, \ldots, v_{i_m}\} \in E(G)$, and is 0 otherwise [13]. Let $\mathcal{D}(G)$ be an $m$-th order $n$-dimensional diagonal tensor, where $d_{v_i} = d_{v_i}$, the degree of the vertex $v_i$, for each $i \in [n]$. Then $\mathcal{L}(G) = \mathcal{D}(G) - A(G)$ and $\mathcal{Q}(G) = \mathcal{D}(G) + A(G)$ are called the Laplacian tensor and the signless Laplacian tensor of $G$, respectively [13]. The adjacency, Laplacian or signless Laplacian tensor of $G$ is symmetric, and it is weakly irreducible if and only if $G$ is connected [16, 23]. The incidence matrix of
where $r \geq 0$, $1 \leq d_i \leq m-1$, $d_i | d_{i+1}$ for $i \in [r-1]$, and $d_i | m$ for all $i \in [r]$. The matrix in (2.1) is called the Smith normal form of $B$ over $\mathbb{Z}_m$.

Let $A$ be a tensor of order $m$ and dimension $n$. Define

$$D^{(0)} = D^{(0)}(A) = \{ D : A = D^{-(m-1)}AD, d_1 = 1 \}$$

where $D$ is an $n \times n$ invertible diagonal matrix and the product is defined as in [19].

**Definition 2.1 ([7]).** For a general tensor $A$, the cardinality of $D^{(0)}(A)$, denoted by $s(A)$, is called the stabilizing index of $A$.

By [7] Lemma 2.5(1), $D^{(0)}(A)$ is an abelian group under the usual matrix multiplication, which is determined by the support of $A$ by [7] Lemma 2.6. Suppose that $A$ is a nonnegative weakly irreducible tensor. By [23] Theorem 3.7, for each $y \in V_{\rho(A)}$, $|y| =: v_p$ is the unique positive Perron vector of $A$. Therefore, we can assume each $y \in V_{\rho(A)}$ satisfies $y_1 = 1$. Define

$$D_y = \text{diag}(y_1/|y_1|, \ldots, y_n/|y_n|),$$

and a quasi-Hadamard product $\circ$ in $V_{\rho(A)}$ as follows:

$$y \circ \hat{y} := D_y D_{\hat{y}} v_p.$$

**Lemma 2.2 ([7], Lemma 2.5, Lemma 3.1).** Let $A$ be a nonnegative weakly irreducible tensor. Then the following results hold.

1. $D^{(0)}(A) = \{ D_y : y \in V_{\rho(A)}, y_1 = 1 \}$, and hence $s(A) = |V_{\rho(A)}|$.

2. $(V_{\rho(A)} \circ)$ is an abelian group isomorphism to $D^{(0)}(A)$.

Further assume $A$ is also combinatorial symmetric of order $m$. By [7] Lemma 2.5(3), $D^m = I$ for each $D \in D^{(0)}$. Hence for each $y \in V_{\rho(A)}$, $y^m = D^m y = v_p$ (the identity), which implies that $(V_{\rho(A)} \circ)$ admits a $\mathbb{Z}_m$-module. Define

$$S_0(A) = \{ x \in Z_m^n : I_A x = 0 \text{ over } \mathbb{Z}_m, x_1 = 0 \},$$

where $I_A$ is the incidence matrix of $A$.

**Theorem 2.3 ([7]).** Let $A$ be a nonnegative combinatorial symmetric weakly irreducible tensor of order $m$. Then $V_{\rho(A)}$ is $\mathbb{Z}_m$-module isomorphic to $S_0(A)$.

**Theorem 2.4 ([7]).** Let $A$ be a nonnegative combinatorial symmetric weakly irreducible tensor of order $m$ and dimension $n$. Suppose that $I_A$ has a Smith normal form over $\mathbb{Z}_m$ as in (2.7). Then $1 \leq r \leq n-1$, and

$$S_0(A) \cong \oplus_{i,d_i \neq 1} Z_{d_i} \oplus (n-1-r)Z_m.$$

**Theorem 2.5 ([8]).** Let $A$ be a weakly irreducible nonnegative tensor. Then $V_{\rho(A)}$ has dimension zero, i.e. there are finite many eigenvectors of $A$ corresponding to $\rho(A)$ up to a scalar.
3. Z-tensors, Laplacian tensors and signless Laplacian tensors

Lemma 3.1. Let $A = sI - B$ be a $Z$-tensor, where $s > 0$ and $B \geq 0$. Then

1. $\lambda_{\min}(A) = s - \rho(B)$, which is the eigenvalue of $A$ with the least real part.
2. $V_{\lambda_{\min}(A)} = V_{\rho(B)}$.
3. $D^{(0)}(A) = D^{(0)}(B)$, $s(A) = s(B)$.
4. For any diagonal tensor $D$, $s(A) = s(D + A)$.

Proof. The first result follows by a similar discussion to [24, Theorem 3.3]. Obviously, $e$ is an eigenvector of $A$ associated with $\lambda_{\min}(A)$ if and only if it is an eigenvector of $B$ associated with $\rho(B)$. So the second result follows. The last two results follow from the definition. □

Lemma 3.2. Let $A$ be a weakly irreducible $Z$-tensor. Then

1. $V_{\lambda_{\min}(A)}$ is finite, i.e. there are finitely many eigenvectors of $A$ associated with $\lambda_{\min}(A)$.
2. $V_{\lambda_{\min}(A)}$ is an abelian group isomorphic to $D^{(0)}(A)$.
3. $s(A) = |V_{\lambda_{\min}(A)}|$.

Proof. Suppose $A = sI - B$, where $s > 0$ and $B \geq 0$. Then $V_{\lambda_{\min}(A)} = V_{\rho(B)}$ by Lemma 3.1. As $B$ is nonnegative and weakly irreducible, the finiteness of $V_{\lambda_{\min}(A)}$ follows from Theorem 2.5. By Lemma 2.2, $V_{\rho(B)}$, as well as $V_{\lambda_{\min}(A)}$, is an abelian group isomorphic to $D^{(0)}(B)$, which is equal to $D^{(0)}(A)$ by Lemma 3.1. □

Theorem 3.3. Let $A$ be a combinatorial symmetric weakly irreducible $Z$-tensor of order $m$ and dimension $n$. Suppose that $I_A$ has a Smith normal form over $\mathbb{Z}_m$ as in (2.7). Then

1. $1 \leq r \leq n - 1$.
2. $V_{\lambda_{\min}(A)}$ is a $\mathbb{Z}_m$-module with the decomposition
   
   $V_{\lambda_{\min}(A)} \cong \oplus_{i \neq 1} \mathbb{Z}_d_i \oplus (n - 1 - r)\mathbb{Z}_m$.
3. $s(A) = |V_{\lambda_{\min}(A)}| = m^{n-1-r}\Pi_{i=1}^r d_i$.

Proof. Let $A = sI - B$, where $s > 0$ and $B \geq 0$. That $B$ is combinatorial symmetric and weakly irreducible, $A$ and $B$ have the same incidence matrices, i.e. $I_A = I_B$ over $\mathbb{Z}_m$. So, by Theorem 2.3, $1 \leq r \leq n - 1$. By Theorem 2.3, $V_{\rho(B)}$ is $\mathbb{Z}_m$-module isomorphic to $S_0(B)$, which has a decomposition as in (2.6). As $V_{\lambda_{\min}(A)} = V_{\rho(B)}$ by Lemma 3.1, the second result follows. The last result follows from (2) and Lemma 3.2. □

Lemma 3.4. The Laplacian tensor $L(G)$ of a uniform hypergraph $G$ is a singular $M$-tensor.

Proof. As $L(G)$ is diagonal dominant, $L(G)$ is an $M$-tensor by [24, Theorem 3.15]. Note that 0 is the least $H$-eigenvalue of $L(G)$ associated with an all-ones eigenvector. So $L(G)$ is singular. □

Theorem 3.5. Let $G$ be a connected $m$-uniform hypergraph. Suppose that $I_G$ has a Smith normal form over $\mathbb{Z}_m$ as in (2.7). Then

$|V_0(L(G))| = s(L(G)) = s(A(G)) = m^{n-1-r}\Pi_{i=1}^r d_i$.

Proof. As $G$ is connected, $L(G)$ is a combinatorial symmetric weakly irreducible Z-tensor. The incidence matrix of $L(G)$ is same as that of $A(G)$, i.e. $I_G$ over $\mathbb{Z}_m$. So, by Theorem 3.3, $s(L(G)) = |V_0(L(G))| = m^{n-1-r}\Pi_{i=1}^r d_i$. The result now follows as $s(L(G)) = s(A(G))$ by definition. □
Let $G$ be an $m$-uniform hypergraph on $n$ vertices, where $m$ is even. $G$ is called odd-colorable \([15]\) if there exists a map $f : V(G) \rightarrow [m]$ such that for each $e \in E(G)$, $\sum_{v \in e} f(v) \equiv m \mod m$.

**Lemma 3.6** \([9]\). Let $G$ be an $m$-uniform connected hypergraph on $n$ vertices. Then the following are equivalent.

1. $0$ is an eigenvalue of $Q(G)$.
2. $m$ is even and $G$ is odd-colorable.
3. $Q(G) = D^{-(m-1)}L(G)D$ for some diagonal matrix $D$ with $|D| = I$.
4. $\rho(L(G)) = \rho(Q(G))$.

**Theorem 3.7.** Let $G$ be an $m$-uniform connected hypergraph which is odd-colorable. Suppose that $V(G)$ has a Smith normal form over $\mathbb{Z}_m$ as in \((2,4)\). Then

1. $s(Q(G)) = s(L(G)) = s(A(G)) = m^{n-1-r} \prod_{i=1}^{r} d_i$.
2. $s(Q(G)) = |V_{\rho(Q(G))}| = |V_{\rho(L(G))}| = |V_{0}(Q(G))| = |V_{0}(L(G))|$.

*Proof.* The first result follows by definition and Theorem 2.6. By Lemma 3.6 as $G$ is odd-colorable, $0$ is an eigenvalue of $Q(G)$. Also by Lemma 3.6, there exists an invertible diagonal matrix $D$ such that $y$ is an eigenvector of $Q(G)$ associated with $0$ if and only if $Dy$ is an eigenvector of $L(G)$ associated with $0$. So $|V_{0}(Q(G))| = |V_{0}(L(G))| = s(L(G))$, where the last equality follows from Lemma 3.2. By Lemma 2.2 $s(Q(G)) = |V_{\rho(Q(G))}|$ as $Q(G)$ is nonnegative and weakly irreducible. The result follows. $\square$

**Example 3.8.** Let $K_{n}^{[m]}$ be a complete $m$-uniform hypergraph on $n$ vertices, where $n \geq m+1$. By \[7\] Example 4.4, $s(A(K_{n}^{[m]})) = 1$, so $s(L(K_{n}^{[m]})) = 1$ by Theorem 3.5, which implies that $\mathcal{L}(K_{n}^{[m]})$ has only one eigenvector (the all-ones vector) associated with the zero eigenvalue.

An odd bipartition $\{V_1, V_2\}$ of $G$ is a bipartition of $V(G)$ such that each edge of $G$ intersects with $V_1$ or $V_2$ in an odd number of vertices. $G$ is called odd-bipartite if $G$ has an odd bipartition \([11]\). Shao et al. \[20\] proved that $0$ is an $H$-eigenvalue of $Q(G)$ if and only if $m$ is even and $G$ is odd-bipartite. An odd-bipartite hypergraph is odd-colorable, but the converse is not true \([15]\). A cored hypergraph \([12]\) is one such that each edge contains a vertex of degree one. Obviously a cored hypergraph of even uniformity is odd-bipartite.

**Example 3.9.** Let $G$ be a connected $m$-uniform cored hypergraph on $n$ vertices with $t$ edges. Then $s(A(G)) = m^{n-1-t}$ by \[7\] Theorem 4.1. So $L(G)$ has $m^{n-1-t}$ eigenvectors associated with the zero eigenvalue. If $m$ is even, then $G$ is odd-bipartite. So $Q(G)$ has $m^{n-1-t}$ eigenvectors associated with the zero eigenvalue.

**Example 3.10.** Let $G_{m,m/2}$ be a generalized power hypergraph \([13]\), which is obtained from a simple graph $G$ by blowing each vertex into an $m/2$-set and preserving the adjacency relation, where $m$ is even. It is known that $G_{m,m/2}$ is non-odd-bipartite if and only if $G$ is non-odd-bipartite \([13]\). Suppose that $G$ is non-odd-bipartite. Then $\rho(L(G_{m,m/2})) = \rho(Q(G_{m,m/2}))$ if and only if $m$ divides $4$. Particularly, let $G$ be a triangle or $C_{3}$ and $m = 4$. Then $C_{3}^{4,2}$ is non-odd-bipartite but odd-colorable by Lemma 3.5. The incidence matrix of $C_{3}^{4,2}$ has invariant divisors $1, 1, 2$ over $\mathbb{Z}_4$.

So $Q(C_{3}^{4,2})$ has 32 eigenvectors associated with the zero eigenvalue.

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