A NOTE ON THE CONVERGENCE OF THE SOLUTION OF THE NOVIKOV EQUATION

GIUSEPPE MARIA COCLITE
Dipartimento di Meccanica, Matematica e Management
Politecnico di Bari
via E. Orabona 4, 70125 Bari, Italy

LORENZO DI RUVO
Dipartimento di Matematica
Università di Bari
via E. Orabona 4, 70125 Bari, Italy

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Abstract. We consider the Novikov and Camassa-Holm equations, which contain nonlinear dispersive effects. We prove that as the diffusion parameter tends to zero, the solution of the dispersive equation converges to the unique entropy solution of a scalar conservation law. The proof relies on deriving suitable a priori estimates together with an application of the compensated compactness method in the $L^p$ setting.

1. Introduction. Consider the Camassa Holm type equations (see [14])

$$\partial_t u - \alpha \partial_x^3 u + \partial_x f(u) = \alpha n u^m \partial_x u \partial_x^2 u + \alpha u^{m+1} \partial_x^3 u, \quad n, m \in \mathbb{N}. \quad (1)$$

If $f(u) = 3u^2/2$, $n = 2$, $m = 0$, (1) becomes the Camassa-Holm equation (see [1])

$$\partial_t u - \alpha \partial_x^3 u + \frac{3}{2} \partial_x u^2 = 2\alpha \partial_x u \partial_x^2 u + \alpha u \partial_x^3 u, \quad (2)$$

it models the propagation of unidirectional shallow water waves on a flat bottom. The unknown $u(t, x)$ represents the fluid velocity at time $t$ in the horizontal direction $x$ (see [1, 23]). In [15], the authors derived (2) (in a more general form), as an equation describing finite length, small amplitude radial deformation waves in cylindrical compressible hyperelastic rods. The Camassa-Holm equation goes beyond the Korteweg-de Vries (KdV) and the Benjamin-Bona-Mahony (BBM) ones in the sense that (2) appears as a water-wave equation at quadratic order in an asymptotic expansion for unidirectional shallow water waves modeled by the incompressible Euler equations, whereas the KdV and BBM equations appear at first order in this asymptotic expansion (see [1, 23]). From a mathematical point of view, the Camassa-Holm equation is by now well studied. Local well-posedness results are proved in [11, 18, 25, 29] It is also known that there exist global solutions for a...
certain class of initial data and also solutions that blow up in finite time for a large class of initial data (see [10, 11, 12]). Existence and uniqueness results for global weak solutions of (2) are proven in [8, 12, 13, 31, 32].

If \( f(u) = 4u^3/3 \), \( n = 3 \), \( m = 1 \), (1) becomes the Novikov equation (see [28])

\[
\partial_t u - \alpha \partial_{xxx}^3 u + \frac{4}{3} \partial_x u^3 = 3\alpha u \partial_x u \partial_x^2 u + \alpha u^2 \partial_{xxx}^3 u.
\]  

(3)

In [17, 18], the Cauchy problem for (3), well–posedness and dependence on initial data in Sobolev spaces are studied. Local well–posedness in the Besov spaces for the Cauchy problem for (3) was proved in [27]. Existence and uniqueness of global weak solution of (3) with initial data under some conditions was proved in [33]. Finally, peakon solutions were studied in [18, 19, 20].

If we send \( \alpha \to 0 \) in (2), we pass from (2) to the scalar conservation law

\[
\begin{align*}
\begin{cases}
\partial_t u + \frac{3}{2} \partial_x u^2 = 0, & t > 0, \ x \in \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R}.
\end{cases}
\end{align*}
\]  

(4)

Instead, we send \( \alpha \to 0 \) in (3), we pass from (3) to the scalar conservation law

\[
\begin{align*}
\begin{cases}
\partial_t u + 4 \partial_x u^3 = 0, & t > 0, \ x \in \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R}.
\end{cases}
\end{align*}
\]  

(5)

To study the dispersion-diffusion for (2) and (3), we fix \( \varepsilon, \alpha \) and consider the following fourth order approximation

\[
\begin{align*}
\begin{cases}
\partial_t u_{\varepsilon, \alpha} - \alpha \partial_{xxx}^3 u_{\varepsilon, \alpha} + \partial_x f(u) - \alpha u_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} \partial_x^2 u_{\varepsilon, \alpha} - \alpha \partial_{xxx}^3 u_{\varepsilon, \alpha} \\
= \varepsilon \partial_{xxx}^3 u_{\varepsilon, \alpha} - \alpha \varepsilon \partial_{xxx}^3 u_{\varepsilon, \alpha} + m \alpha^2 \varepsilon (\partial_x u_{\varepsilon, \alpha})^2 \partial_x u_{\varepsilon, \alpha}, & t > 0, \ x \in \mathbb{R}, \\
u_{\varepsilon, \alpha}(0, x) = u_{\varepsilon, \alpha, 0}(x), & x \in \mathbb{R},
\end{cases}
\end{align*}
\]  

(6)

where \( f(u) = 3u^2/2, n = 2, m = 0, \) or \( f(u) = 4u^3/3, n = 3, m = 1. \)

Observe that, when \( f(u) = 3u^2/2, n = 2, m = 0, \) (6) reads

\[
\begin{align*}
\begin{cases}
\partial_t u_{\varepsilon, \alpha} - \alpha \partial_{xxx}^3 u_{\varepsilon, \alpha} + \frac{3}{2} \partial_x u^2 - 2\alpha \partial_x u_{\varepsilon, \alpha} \partial_x^2 u_{\varepsilon, \alpha} \\
= \varepsilon \partial_{xxx}^3 u_{\varepsilon, \alpha} - \alpha \varepsilon \partial_{xxx}^3 u_{\varepsilon, \alpha} + \varepsilon (\partial_x u_{\varepsilon, \alpha})^2 \partial_x u_{\varepsilon, \alpha}, & t > 0, \ x \in \mathbb{R}, \\
u_{\varepsilon, \alpha}(0, x) = u_{\varepsilon, \alpha, 0}(x), & x \in \mathbb{R},
\end{cases}
\end{align*}
\]  

(7)

which is the same approximation of [9].

In the paper, using (6), we prove the solution of (2) converges to the unique entropy one of (4) (see Theorem 2.1), under the assumption

\[
u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}), \quad \alpha = \mathcal{O}(\varepsilon^4).\]

(8)

For (3), using the approximation (7) with \( m = 1 \), we prove the convergence of the solution of (3) to the distributional one of (5), under the assumption

\[
u_0 \in L^2(\mathbb{R}) \cap L^6(\mathbb{R}), \quad \alpha = \mathcal{O}(\varepsilon^8).\]

(9)

Instead, under the assumption

\[
u_0 \in L^2(\mathbb{R}) \cap L^6(\mathbb{R}), \quad \alpha = o(\varepsilon^8),
\]

(10)

we prove the convergence of the solution of (3) to the unique entropy solution of (5) (see Theorem 3.1).
Observe that if in (1) the flux verifies the following condition
\[ |f'(u)| \leq k_0 |u|, \quad |f(u)| \leq k_1 u^2, \quad u \in \mathbb{R}, \] (11)
for some constants \( k_0, k_1 > 0 \), and the genuinely nonlinear condition
\[ \text{meas}\{u \in \mathbb{R} : f''(u) = 0\} = 0, \] (12)
using the approximation of [9], under the assumption (9) (see [4, 9]), we have the
convergence of the solution of (1) to the distributional solution of
\[
\begin{cases}
\partial_t u + \partial_x f(u) = 0, & t > 0, x \in \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R}.
\end{cases}
\] (13)
Moreover, following [16], under the assumption
\[ u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}), \quad \alpha = o(\varepsilon^4), \] (14)
the dissipation of energy is proven.

In what follows we say that a pair of functions \((\eta, q)\) is an entropy–entropy flux
pair if \(\eta : \mathbb{R} \to \mathbb{R}\) is a \(C^2\) function and \(q : \mathbb{R} \to \mathbb{R}\) is defined by
\[ q' = f' \eta'. \]
Moreover, an entropy-entropy flux pair \((\eta, q)\) is called convex/compactly supported
if, in addition, \(\eta\) is convex/compactly supported.

In [21], under the assumption (14), the convergence of the solution of (1) to the
unique entropy solution of (13) is proven. In order to do this, the author used
kinetic methods, which were introduced in [22].

The paper is organized as follows. In Section 2, we prove the convergence of the
solution of (2) to the unique entropy solution of (4), under the assumption (8). In
the Section 3, under the assumption (10), the convergence of the solution of (3) to
the unique entropy one of (5) is proven.

2. The Camassa-Holm equation. In this section we consider (2). We augment
(2) with the initial condition
\[ u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}). \] (15)
We study the dispersion-diffusion for (2). Therefore, following [9], we fix \(0 < \varepsilon, \alpha < 1\) and consider the following fourth order approximation
\[
\begin{cases}
\partial_t u_{\varepsilon, \alpha} - \alpha \partial_{xxx} u_{\varepsilon, \alpha} + \frac{3}{2} \partial_x u^2 - 2 \alpha \partial_x u_{\varepsilon, \alpha} \partial_{xx} u_{\varepsilon, \alpha} \\
- \alpha u_{\varepsilon, \alpha} \partial_{xxx} u_{\varepsilon, \alpha} + \varepsilon \partial_{xx}^2 u_{\varepsilon, \alpha} - \alpha \varepsilon \partial_{xxxx} u_{\varepsilon, \alpha},
\end{cases}
\] (16)
t > 0, x \in \mathbb{R},
where \(u_{\varepsilon, \alpha, 0}\) is a \(C^\infty\) approximation of \(u_0\) such that
\[
\begin{align*}
&u_{\varepsilon, \alpha, 0} \to u_0 \quad \text{in } L_{\text{loc}}^p(\mathbb{R}), \ 1 \leq p < 4, \ \text{as } \varepsilon, \alpha \to 0, \\
&|u_{\varepsilon, \alpha, 0}|_{L^2(\mathbb{R})} + |u_{\varepsilon, \alpha, 0}|_{L^4(\mathbb{R})} \leq C_0, \ \varepsilon, \alpha > 0, \\
&\left(\sqrt{\alpha} + \varepsilon\right) |\partial_x u_{\varepsilon, \alpha, 0}|_{L^2(\mathbb{R})} + \left(\sqrt{\alpha \varepsilon} + \sqrt{\alpha \varepsilon}\right) |\partial_x u_{\varepsilon, \alpha, 0}|_{L^2(\mathbb{R})} \leq C_0, \ \varepsilon, \alpha > 0, \\
&\alpha \int_{\mathbb{R}} u_{\varepsilon, \alpha, 0}^3 \partial_{xx}^2 u_{\varepsilon, \alpha, 0} dx = C_0, \ \varepsilon, \alpha > 0,
\end{align*}
\] (17)
for some constant \(C_0\) independent on \(\varepsilon, \alpha\).
The main result of this section is the following theorem.

**Theorem 2.1.** Assume (15) and (17). Fix $T > 0$, if
\[
\alpha = O(\varepsilon^4) \tag{18}
\]
then there exist two sequences $\{\varepsilon_k\}_{k \in \mathbb{N}}$, $\{\alpha_k\}_{k \in \mathbb{N}}$, with $\varepsilon_k, \alpha_k \to 0$, and a limit function
\[
u \in L^\infty((0, T); L^2(\mathbb{R}) \cap L^4(\mathbb{R}))
\]
such that
\[
u_{\varepsilon_k, \alpha_k} \to u, \quad \text{strongly in } L^p_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}), \text{ for each } 1 \leq p < 4, \text{ and a.e.} \quad \tag{19}
\]
$u$ is an entropy solution of (4). \( \tag{20} \)

The assumption (18) consists in
\[
\alpha \leq D^4 \varepsilon^4, \tag{21}
\]
where $D$ is a positive constant such that
\[
D < \min \left\{ \frac{C_1}{4C_0}, \sqrt{\frac{5}{64C_0^2}}, \frac{1}{2C_0} \right\}, \tag{22}
\]
where $C_1$ is a positive constant which is defined in (40).

Let us prove some a priori estimates on $\nu_{\varepsilon, \alpha}$, denoting with $C_0$ the constants which depend only on the initial data.

Arguing as in [9, Lemma 4.1], we have the following result

**Lemma 2.2.** For each $t > 0$,
\[
\|\nu_{\varepsilon, \alpha}(t, \cdot)\|^2_{L^2(\mathbb{R})} + \alpha \\|\partial_x \nu_{\varepsilon, \alpha}(t, \cdot)\|^2_{L^2(\mathbb{R})}
\]
\[
+ 2 \varepsilon \int_0^t \|\partial_x \nu_{\varepsilon, \alpha}(s, \cdot)\|^2_{L^2(\mathbb{R})} \, ds + 2 \alpha \varepsilon \int_0^t \|\partial_{xx}^2 \nu_{\varepsilon, \alpha}(s, \cdot)\|^2_{L^2(\mathbb{R})} \, ds \leq C_0. \tag{23}
\]
Moreover,
\[
\|\nu_{\varepsilon, \alpha}\|_{L^\infty([0, T] \times \mathbb{R})} \leq C_0 \alpha^{-\frac{3}{4}}. \tag{24}
\]

Following [2, Lemma 2.5] and [3, Lemma 2.2], we prove the following result.

**Lemma 2.3.** Fix $T > 0$. Assume (21) and (22). There exists $C_0 > 0$, independent on $\varepsilon, \alpha$, such that
\[
\|\partial_x \nu_{\varepsilon, \alpha}\|_{L^\infty((0, T) \times \mathbb{R})} \leq C_0 \alpha^{-\frac{3}{4}}. \tag{25}
\]
Moreover, for every $0 \leq t \leq T$,
\[
\alpha^\frac{3}{4} \|\partial_x \nu_{\varepsilon, \alpha}(t, \cdot)\|^2_{L^2(\mathbb{R})} + \alpha^\frac{3}{2} \|\partial_{xx}^2 \nu_{\varepsilon, \alpha}(t, \cdot)\|^2_{L^2(\mathbb{R})}
\]
\[
+ 2 \alpha \varepsilon \int_0^t \|\partial_{xx}^2 \nu_{\varepsilon, \alpha}(s, \cdot)\|^2_{L^2(\mathbb{R})} \, ds + 2 \alpha \varepsilon^\frac{3}{4} \int_0^t \|\partial_{xxx} \nu_{\varepsilon, \alpha}(s, \cdot)\|^2_{L^2(\mathbb{R})} \, ds \leq C_0 \alpha^{-\frac{3}{4}}. \tag{26}
\]
Proof. Let $0 \leq t \leq T$. Multiplying (16) by $-2\alpha^{\frac{4}{3}} \partial^2_{x,z}u_{\varepsilon,\alpha}$ and integrating on $\mathbb{R}$, we have
\[
-2\alpha^{\frac{4}{3}} \int_{\mathbb{R}} \partial^2_{x,z}u_{\varepsilon,\alpha} \partial_{t}u_{\varepsilon,\alpha} dx + 2\alpha^{\frac{4}{3}} \int_{\mathbb{R}} \partial^2_{x,z}u_{\varepsilon,\alpha} \partial^3_{x,z}u_{\varepsilon,\alpha} dx \\
- 6\alpha^{\frac{4}{3}} \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial_{t}u_{\varepsilon,\alpha} \partial^2_{x,z}u_{\varepsilon,\alpha} dx + 4\alpha^{\frac{4}{3}} \int_{\mathbb{R}} \partial_{x}u_{\varepsilon,\alpha}(\partial^2_{x,z}u_{\varepsilon,\alpha})^2 dx \\
+ 2\alpha^{\frac{4}{3}} \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial^2_{x,z}u_{\varepsilon,\alpha} \partial^3_{x,z}u_{\varepsilon,\alpha} dx
\]
(27)
\[
= -2\alpha^{\frac{4}{3}} \varepsilon \left\| \partial^2_{x,z}u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + 2\alpha^{\frac{4}{3}} \varepsilon \int_{\mathbb{R}} \partial^2_{x,z}u_{\varepsilon,\alpha} \partial^4_{x,z,z,z}u_{\varepsilon,\alpha} dx.
\]
Observe that
\[
-2\alpha^{\frac{4}{3}} \int_{\mathbb{R}} \partial^2_{x,z}u_{\varepsilon,\alpha} \partial_{t}u_{\varepsilon,\alpha} dx + 2\alpha^{\frac{4}{3}} \int_{\mathbb{R}} \partial^2_{x,z}u_{\varepsilon,\alpha} \partial^3_{x,z}u_{\varepsilon,\alpha} dx
\]
\[
= \frac{d}{dt} \left( \alpha^{\frac{4}{3}} \left\| \partial_{x}u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \alpha^{\frac{4}{3}} \left\| \partial^2_{x,z}u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \right) \\
+ 2\alpha^{\frac{4}{3}} \varepsilon \left\| \partial^2_{x,z}u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + 2\alpha^{\frac{4}{3}} \varepsilon \left\| \partial^3_{x,z,z,z}u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})}
\]
(28)
\[
= 6\alpha^{\frac{4}{3}} \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial_{x}u_{\varepsilon,\alpha} \partial^2_{x,z}u_{\varepsilon,\alpha} dx - 3\alpha^{\frac{4}{3}} \int_{\mathbb{R}} \partial_{x}u_{\varepsilon,\alpha}(\partial^2_{x,z}u_{\varepsilon,\alpha})^2 dx.
\]
Since
\[
6\alpha^{\frac{4}{3}} \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial_{x}u_{\varepsilon,\alpha} \partial^2_{x,z}u_{\varepsilon,\alpha} dx = -3\alpha^{\frac{4}{3}} \int_{\mathbb{R}} (\partial_{x}u_{\varepsilon,\alpha})^3 dx,
\]
by (21) and (28), we have
\[
\frac{d}{dt} \left( \alpha^{\frac{4}{3}} \left\| \partial_{x}u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \alpha^{\frac{4}{3}} \left\| \partial^2_{x,z}u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \right)
\]
\[
+ 2\alpha^{\frac{4}{3}} \varepsilon \left\| \partial^2_{x,z}u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + 2\alpha^{\frac{4}{3}} \varepsilon \left\| \partial^3_{x,z,z,z}u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})}
\]
\[
= -3\alpha^{\frac{4}{3}} \int_{\mathbb{R}} (\partial_{x}u_{\varepsilon,\alpha})^3 dx - 3\alpha^{\frac{4}{3}} \int_{\mathbb{R}} \partial_{x}u_{\varepsilon,\alpha}(\partial^2_{x,z}u_{\varepsilon,\alpha})^2 dx
\]
\[
\leq 3D \varepsilon \int_{\mathbb{R}} \left| \partial_{x}u_{\varepsilon,\alpha} \right|^3 dx + 3D \alpha \varepsilon \int_{\mathbb{R}} \left| \partial_{x}u_{\varepsilon,\alpha} \right| (\partial^2_{x,z}u_{\varepsilon,\alpha})^2 dx
\]
\[
\leq 3D \varepsilon \left\| \partial_{x}u_{\varepsilon,\alpha} \right\|_{L^\infty((0,T) \times \mathbb{R})} \left\| \partial_{x}u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})}
\]
\[
+ 3D \alpha \varepsilon \left\| \partial_{x}u_{\varepsilon,\alpha} \right\|_{L^\infty((0,T) \times \mathbb{R})} \left\| \partial^2_{x,z}u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})}.
\]
It follows from an integration on $(0,t)$, (17) and (23) that
\[
\alpha^{\frac{4}{3}} \left\| \partial_{x}u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \alpha^{\frac{4}{3}} \left\| \partial^2_{x,z}u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})}
\]
\[
+ 2\alpha^{\frac{4}{3}} \varepsilon \int_{0}^{t} \left\| \partial^2_{x,z}u_{\varepsilon,\alpha}(s, \cdot) \right\|^2_{L^2(\mathbb{R})} ds + 2\alpha^{\frac{4}{3}} \varepsilon \int_{0}^{t} \left\| \partial^3_{x,z,z,z}u_{\varepsilon,\alpha}(s, \cdot) \right\|^2_{L^2(\mathbb{R})} ds
\]
We prove (25). Due to (23), (29) and the H"older inequality,

\begin{align}
\intertext{Introducing the notation}
\text{Lemma 2.4.}
\intertext{Fix}
\intertext{(25) follows from (31) and (32). Arguing as in [2, Lemma 2.5], we have that}
\intertext{Finally, (25) and (29) give (26).}
\end{align}

We prove (25). Due to (23), (29) and the H"older inequality,

\begin{align}
(\partial_x u_{\varepsilon, \alpha}(t, x))^2 = & 2\int_{-\infty}^{x} \partial_x u_{\varepsilon, \alpha} \partial_x^2 u_{\varepsilon, \alpha} dy \\
\leq & 2 \int_{\mathbb{R}} |\partial_x u_{\varepsilon, \alpha}| \partial_x^2 u_{\varepsilon, \alpha} dx \\
\leq & ||\partial_x u_{\varepsilon, \alpha}(t, \cdot)||_{L^2(\mathbb{R})} \||\partial_x^2 u_{\varepsilon, \alpha}(t, \cdot)||_{L^2(\mathbb{R})} \\
\leq & \frac{C_0}{\alpha^2} \left(1 + ||\partial_x u_{\varepsilon, \alpha}||_{L^\infty((0,T) \times \mathbb{R})}\right).
\end{align}

Hence,

\begin{align}
||\partial_x u_{\varepsilon, \alpha}||_{L^\infty((0,T) \times \mathbb{R})}^4 \leq & \frac{C_0}{\alpha^2} \left(1 + ||\partial_x u_{\varepsilon, \alpha}||_{L^\infty((0,T) \times \mathbb{R})}\right) .
\end{align}

Introducing the notation

\begin{align}
y := ||\partial_x u_{\varepsilon, \alpha}||_{L^\infty((0,T) \times \mathbb{R})}, \quad \delta := \alpha^2 ,
\end{align}

(30) reads.

\begin{align}
y^4 \leq \frac{C_0}{\delta} (1 + y) .
\end{align}

Arguing as in [2, Lemma 2.5], we have that

\begin{align}
y \leq C_0 \delta^{-\frac{1}{4}} .
\end{align}

(25) follows from (31) and (32).

Finally, (25) and (29) give (26).

Following [5, Lemma 2.2], we prove the following result

\textbf{Lemma 2.4.} \textit{Fix } T > 0. \textit{Assume (21) and (22). Then,}

i) \textit{the family } \{u_{\varepsilon, \alpha}\} \textit{is bounded in } L^\infty((0,T); L^4(\mathbb{R}));

ii) \textit{the families}

\begin{align}
\{\varepsilon \partial_x u_{\varepsilon, \alpha}\} \varepsilon, \alpha, \{\alpha^{\frac{1}{2}} \varepsilon \partial_x^2 u_{\varepsilon, \alpha}\} \varepsilon, \alpha, \{\alpha^{\frac{1}{2}} u_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha}\} \varepsilon, \alpha,
\end{align}

\textit{are bounded in } L^\infty((0,T); L^2(\mathbb{R}));

iii) \textit{the families}

\begin{align}
\{\varepsilon \partial_t u_{\varepsilon, \alpha}\} \varepsilon, \alpha, \{\alpha^{\frac{1}{2}} \varepsilon \partial_x^2 u_{\varepsilon, \alpha}\} \varepsilon, \alpha, \{\varepsilon \partial_x u_{\varepsilon, \alpha}\} \varepsilon, \alpha,
\end{align}

\textit{are bounded in } L^2((0,T); \times \mathbb{R}).
Proof. We begin by observing that

\[
- \alpha \int R u_{\epsilon,\alpha}^3 \partial_{xx}^3 u_{\epsilon,\alpha} \, dx = 3\alpha \int R u_{\epsilon,\alpha}^2 \partial_x u_{\epsilon,\alpha} \partial_{xx}^2 u_{\epsilon,\alpha} \, dx = -6\alpha \int R u_{\epsilon,\alpha} (\partial_x u_{\epsilon,\alpha})^2 \partial_t u_{\epsilon,\alpha} \, dx - 3\alpha \int R u_{\epsilon,\alpha} \partial_{xx}^2 u_{\epsilon,\alpha} \partial_t u_{\epsilon,\alpha} \, dx.
\]

(33)

Since

\[
\partial_t (u_{\epsilon,\alpha}^3 \partial_{xx}^2 u_{\epsilon,\alpha}) = 3u_{\epsilon,\alpha}^2 \partial_t u_{\epsilon,\alpha} \partial_{xx}^2 u_{\epsilon,\alpha} + u_{\epsilon,\alpha} \partial_{txx}^3 u_{\epsilon,\alpha},
\]

it follows from (33) that

\[
-2\alpha \int R u_{\epsilon,\alpha}^3 \partial_{txx}^3 u_{\epsilon,\alpha} \, dx = -6\alpha \int R u_{\epsilon,\alpha} (\partial_x u_{\epsilon,\alpha})^2 \partial_t u_{\epsilon,\alpha} \, dx - \frac{d}{dt} \int R u_{\epsilon,\alpha}^3 \partial_{xx}^2 u_{\epsilon,\alpha} \, dx.
\]

(34)

that is

\[
-\alpha \int R u_{\epsilon,\alpha}^3 \partial_{xx}^3 u_{\epsilon,\alpha} \, dx = -3\alpha \int R u_{\epsilon,\alpha} (\partial_x u_{\epsilon,\alpha})^2 \partial_t u_{\epsilon,\alpha} \, dx - \frac{\alpha}{2} d \int R u_{\epsilon,\alpha}^3 \partial_{xx}^2 u_{\epsilon,\alpha} \, dx.
\]

Fix 0 ≤ t ≤ T. Let A, B be two positive constants which will be specified later. Multiplying (16) by

\[
u_{\epsilon,\alpha}^3 - A\varepsilon^2 \partial_{xx}^2 u_{\epsilon,\alpha} + B\varepsilon \partial_t u_{\epsilon,\alpha},
\]

we have

\[
(u_{\epsilon,\alpha}^3 - A\varepsilon^2 \partial_{xx}^2 u_{\epsilon,\alpha} + B\varepsilon \partial_t u_{\epsilon,\alpha}) \partial_t u_{\epsilon,\alpha} + 3(u_{\epsilon,\alpha}^3 - A\varepsilon^2 \partial_{xx}^2 u_{\epsilon,\alpha} + B\varepsilon \partial_t u_{\epsilon,\alpha}) \partial_x u_{\epsilon,\alpha} \\
- \alpha (u_{\epsilon,\alpha}^3 - A\varepsilon^2 \partial_{xx}^2 u_{\epsilon,\alpha} + B\varepsilon \partial_t u_{\epsilon,\alpha}) \partial_{xx}^3 u_{\epsilon,\alpha} \\
- 2\alpha (u_{\epsilon,\alpha}^3 - A\varepsilon^2 \partial_{xx}^2 u_{\epsilon,\alpha} + B\varepsilon \partial_t u_{\epsilon,\alpha}) \partial_x u_{\epsilon,\alpha} \partial_{xx}^2 u_{\epsilon,\alpha} \\
- \alpha (u_{\epsilon,\alpha}^3 - A\varepsilon^2 \partial_{xx}^2 u_{\epsilon,\alpha} + B\varepsilon \partial_t u_{\epsilon,\alpha}) u_{\epsilon,\alpha} \partial_{xx}^3 u_{\epsilon,\alpha},
\]

(36)

Observe that

\[
\int R (u_{\epsilon,\alpha}^3 - A\varepsilon^2 \partial_{xx}^2 u_{\epsilon,\alpha} + B\varepsilon \partial_t u_{\epsilon,\alpha}) \partial_t u_{\epsilon,\alpha} \, dx
\]

\[
= \frac{d}{dt} \left( \frac{1}{4} \|u_{\epsilon,\alpha}(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \frac{A\varepsilon^2}{2} \|\partial_x u_{\epsilon,\alpha}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right)
\]

\[
+ B\varepsilon \|\partial_t u_{\epsilon,\alpha}(t, \cdot)\|_{L^2(\mathbb{R})}^2,
\]

3 \int R (u_{\epsilon,\alpha}^3 - A\varepsilon^2 \partial_{xx}^2 u_{\epsilon,\alpha} + B\varepsilon \partial_t u_{\epsilon,\alpha}) u_{\epsilon,\alpha} \partial_x u_{\epsilon,\alpha} \, dx
\]

\[
= -3A\varepsilon^2 \int R u_{\epsilon,\alpha} \partial_x u_{\epsilon,\alpha} \partial_{xx}^2 u_{\epsilon,\alpha} \, dx + B\varepsilon \int R u_{\epsilon,\alpha} \partial_x u_{\epsilon,\alpha} \partial_t u_{\epsilon,\alpha} \, dx,
\]

\[
- \alpha \int R (u_{\epsilon,\alpha}^3 - A\varepsilon^2 \partial_{xx}^2 u_{\epsilon,\alpha} + B\varepsilon \partial_t u_{\epsilon,\alpha}) \partial_{xx}^3 u_{\epsilon,\alpha} \, dx
\]

\[
= -\alpha \int R u_{\epsilon,\alpha} \partial_{xx}^3 u_{\epsilon,\alpha} \, dx + \frac{A\varepsilon^2}{2} \frac{d}{dt} \|\partial_{xx}^2 u_{\epsilon,\alpha}(t, \cdot)\|_{L^2(\mathbb{R})}^2
\]

\[
+ B\alpha \varepsilon \|\partial_{xx}^2 u_{\epsilon,\alpha}(t, \cdot)\|_{L^2(\mathbb{R})}^2,
\]
\[-2\alpha \int_{\mathbb{R}} (u_{\varepsilon, \alpha}^3 - A\varepsilon^2 \partial_{xx}^2 u_{\varepsilon, \alpha} + B\varepsilon \partial_t u_{\varepsilon, \alpha}) \partial_x u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha} \, dx \]

\[= 3\alpha \int_{\mathbb{R}} u_{\varepsilon, \alpha}^2 (\partial_x u_{\varepsilon, \alpha})^3 \, dx + 2A\alpha \varepsilon^2 \int_{\mathbb{R}} \partial_x u_{\varepsilon, \alpha} (\partial_{xx}^2 u_{\varepsilon, \alpha})^2 \, dx \]

\[ - 2B\alpha \varepsilon \int_{\mathbb{R}} \partial_x u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha} \partial_t u_{\varepsilon, \alpha} \, dx, \]

\[-\alpha \int_{\mathbb{R}} (u_{\varepsilon, \alpha}^3 - A\varepsilon^2 \partial_{xx}^2 u_{\varepsilon, \alpha} + B\varepsilon \partial_t u_{\varepsilon, \alpha}) u_{\varepsilon, \alpha} \partial_{xxx}^3 u_{\varepsilon, \alpha} \, dx \]

\[= -4\alpha \int_{\mathbb{R}} u_{\varepsilon, \alpha}^3 \partial_x u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha} \, dx - \frac{A\alpha \varepsilon^2}{2} \int_{\mathbb{R}} \partial_x u_{\varepsilon, \alpha} (\partial_{xx}^2 u_{\varepsilon, \alpha})^2 \, dx \]

\[+ B\alpha \varepsilon \int_{\mathbb{R}} \partial_x u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha} \partial_t u_{\varepsilon, \alpha} \, dx + B\alpha \varepsilon \int_{\mathbb{R}} u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha} \partial_t u_{\varepsilon, \alpha} \, dx \]

\[= 6\alpha \int_{\mathbb{R}} u_{\varepsilon, \alpha}^2 (\partial_x u_{\varepsilon, \alpha})^3 \, dx - \frac{A\alpha \varepsilon^2}{2} \int_{\mathbb{R}} \partial_x u_{\varepsilon, \alpha} (\partial_{xx}^2 u_{\varepsilon, \alpha})^2 \, dx \]

\[+ B\alpha \varepsilon \int_{\mathbb{R}} \partial_x u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha} \partial_t u_{\varepsilon, \alpha} \, dx + B\alpha \varepsilon \int_{\mathbb{R}} u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha} \partial_t u_{\varepsilon, \alpha} \, dx, \]

\[\varepsilon \int_{\mathbb{R}} (u_{\varepsilon, \alpha}^3 - A\varepsilon^2 \partial_{xx}^2 u_{\varepsilon, \alpha} + B\varepsilon \partial_t u_{\varepsilon, \alpha}) \partial_{xx}^2 u_{\varepsilon, \alpha} \, dx \]

\[= -3\varepsilon \| u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \| \partial_{xx}^2 u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \]

\[+ \frac{B\varepsilon^2}{2} \frac{d}{dt} \| \partial_{xx} u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2, \]

\[-\alpha \varepsilon \int_{\mathbb{R}} (u_{\varepsilon, \alpha}^3 - A\varepsilon^2 \partial_{xx}^2 u_{\varepsilon, \alpha} + B\varepsilon \partial_t u_{\varepsilon, \alpha}) \partial_{xxx}^4 u_{\varepsilon, \alpha} \, dx \]

\[= 3\alpha \varepsilon \int_{\mathbb{R}} u_{\varepsilon, \alpha}^2 \partial_x u_{\varepsilon, \alpha} \partial_{xxx}^3 u_{\varepsilon, \alpha} \, dx - A\alpha \varepsilon^3 \| \partial_{xxx}^3 u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \]

\[+ \frac{B\alpha \varepsilon^3}{2} \frac{d}{dt} \| \partial_{xxx} u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \]

\[= -6\alpha \varepsilon \int_{\mathbb{R}} u_{\varepsilon, \alpha} (\partial_x u_{\varepsilon, \alpha}) \partial_{xx}^2 u_{\varepsilon, \alpha} \, dx - 3\alpha \varepsilon \| u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})} \| \partial_{xx}^2 u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \]

\[- A\alpha \varepsilon^3 \| \partial_{xx}^2 u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{B\alpha \varepsilon^3}{2} \frac{d}{dt} \| \partial_{xx} u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2. \]

An integration of (36) on \( \mathbb{R} \) and (34) give

\[
\frac{d}{dt} \left( \frac{1}{4} \| u_{\varepsilon, \alpha}(t, \cdot) \|_{L^4(\mathbb{R})}^4 + \frac{(A + B)\varepsilon^2}{2} \| \partial_x u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \right) \]

\[+ B\varepsilon \| \partial_t u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + B\alpha \varepsilon \| \partial_{xx}^2 u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \]

\[+ 3\varepsilon \| u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})} \| \partial_{xx}^2 u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \]

\[+ 3A\alpha \| \partial_{xx}^2 u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + A\alpha \varepsilon^3 \| \partial_{xxx}^3 u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \]

\[= 3A\alpha \varepsilon^2 \int_{\mathbb{R}} u_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha} \, dx - B\varepsilon \int_{\mathbb{R}} u_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} \partial_t u_{\varepsilon, \alpha} \, dx \]
Due to (21), (24), (25) and the Young inequality,

\[ 3A\varepsilon^2 \int \mu_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha} \, dx \leq 3A\varepsilon^2 \int |\mu_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha}| \partial_{xx}^2 u_{\varepsilon, \alpha} \, dx \]

\[ = \int |\varepsilon^2 \mu_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha}| \partial_{xx}^2 u_{\varepsilon, \alpha} \, dx \]

\[ \leq \frac{\varepsilon^2}{2} \| \mu_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} \|^2_{L^2(\mathbb{R})} + \frac{3A^2 \varepsilon^3}{2} \left\| \partial_{xx}^2 u_{\varepsilon, \alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})}, \]

\[ - B\varepsilon \int u_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} \partial_{xx} u_{\varepsilon, \alpha} \, dx \leq B\varepsilon \int |u_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha}| |\partial_t u_{\varepsilon, \alpha}| \, dx \]

\[ = \varepsilon \int |u_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha}| |B \partial_t u_{\varepsilon, \alpha}| \, dx \]

\[ \leq \frac{\varepsilon}{2} \| \mu_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} \|^2_{L^2(\mathbb{R})} + \frac{B^2 \varepsilon}{2} \left\| \partial_t u_{\varepsilon, \alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})}, \]

\[ 3\alpha \int \mu_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} \partial_{xx}^3 u_{\varepsilon, \alpha} \, dx \leq 3\alpha \| \mu_{\varepsilon, \alpha} \|^2_{L^\infty((0, T) \times \mathbb{R})} \int \| \mu_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} \| |\partial_t u_{\varepsilon, \alpha}| \, dx \]

\[ \leq C_0 \alpha \frac{1}{2} \int \| \mu_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} \| |\partial_t u_{\varepsilon, \alpha}| \, dx \leq C_0 D\varepsilon \int \| \mu_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} \| |\partial_t u_{\varepsilon, \alpha}| \, dx \]

\[ \leq C_0 D\varepsilon \| \mu_{\varepsilon, \alpha}(t, \cdot) \|^2_{L^2(\mathbb{R})} + C_0 \varepsilon \| \partial_t u_{\varepsilon, \alpha}(t, \cdot) \|^2_{L^2(\mathbb{R})}, \]

\[ - 9\alpha \int u_{\varepsilon, \alpha}^2 \partial_x u_{\varepsilon, \alpha} \partial_t u_{\varepsilon, \alpha} \, dx \leq 9\alpha \| \mu_{\varepsilon, \alpha} \|^2_{L^\infty((0, T) \times \mathbb{R})} \| \mu_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} \|^2_{L^2(\mathbb{R})} \]

\[ \leq C_0 \alpha \frac{1}{2} \| \mu_{\varepsilon, \alpha} \|^2_{L^\infty((0, T) \times \mathbb{R})} \| \partial_x u_{\varepsilon, \alpha} \|^2_{L^2(\mathbb{R})} \]

\[ \leq C_0 \alpha \frac{1}{2} \| \partial_x u_{\varepsilon, \alpha} \|^2_{L^2(\mathbb{R})} \leq C_0 D\alpha \varepsilon^3 \| \partial_{xx}^2 u_{\varepsilon, \alpha}(t, \cdot) \|^2_{L^2(\mathbb{R})}, \]

\[ - 3A\alpha\varepsilon^2 \int \partial_x u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha} \partial_{xx} u_{\varepsilon, \alpha} \, dx \leq \frac{3A\alpha\varepsilon^2}{2} \| \partial_x u_{\varepsilon, \alpha} \|^2_{L^\infty((0, T) \times \mathbb{R})} \| \partial_{xx}^2 u_{\varepsilon, \alpha}(t, \cdot) \|^2_{L^2(\mathbb{R})} \]

\[ \leq C_0 \alpha \frac{1}{2} \| \partial_t u_{\varepsilon, \alpha} \|^2_{L^2(\mathbb{R})} \leq C_0 D\alpha \varepsilon^3 \| \partial_{xx}^2 u_{\varepsilon, \alpha}(t, \cdot) \|^2_{L^2(\mathbb{R})}, \]

\[ - B\alpha \varepsilon \int \partial_x u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha} \partial_{xx} u_{\varepsilon, \alpha} \, dx \leq B\alpha \varepsilon \int |u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha}| |\partial_{xx} u_{\varepsilon, \alpha}| \, dx \]

\[ = B\alpha \varepsilon \int |u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha}| |\partial_{xx} u_{\varepsilon, \alpha}| \, dx \leq C_0 D\alpha \varepsilon^2 \int |\partial_{xx}^2 u_{\varepsilon, \alpha}| |\partial_{xx} u_{\varepsilon, \alpha}| \, dx \]

\[ = C_0 \int |D^2 u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha}| |\partial_{xx} u_{\varepsilon, \alpha}| \, dx \]

\[ \leq C_0 D\alpha \varepsilon^3 \| \partial_{xx}^2 u_{\varepsilon, \alpha}(t, \cdot) \|^2_{L^2(\mathbb{R})} + C_0 B\alpha \varepsilon \| \partial_t u_{\varepsilon, \alpha}(t, \cdot) \|^2_{L^2(\mathbb{R})}, \]

\[ - B\alpha \varepsilon \int u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha} \partial_{x}^2 u_{\varepsilon, \alpha} \leq B\alpha \varepsilon \int |u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha}| |\partial_{x}^2 u_{\varepsilon, \alpha}| \, dx \]

\[ = \alpha \varepsilon \int |u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha}| |B \partial_{x}^2 u_{\varepsilon, \alpha}| \, dx \]
We choose \( C_0 \) such that
\[
B, \quad C_0 > 0, \quad 1 - \frac{5B}{2} > 0, \quad \frac{A}{2} - \frac{3A^2}{2} > 0, \quad \frac{A}{2} - C_0D > 0, \quad A - 
\]
Consequently, by the second equation of (39), we have that

\[ D < \frac{C_1}{4C_0}. \]  

(42)

Thanks to the sixth inequality of (39), we take

\[ A = \frac{1}{4}. \]  

(43)

It follows from the seventh inequality of (39) that

\[ D < \sqrt{\frac{5}{64C_0}}. \]  

(44)

Consequently, from the fourth and fifth inequalities of (39), (40) and (44), we have (22).

Therefore, it follows from (22), (41) and (43)

\[
\begin{align*}
\frac{d}{dt} \left( \frac{1}{4} \left\| u_{\varepsilon, \alpha}(t, \cdot) \right\|_{L^4(\mathbb{R})}^4 + \frac{(A + B)\varepsilon^2}{2} \left\| \partial_\varepsilon u_{\varepsilon, \alpha}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\
+ \frac{d}{dt} \left( \frac{(A + B)\varepsilon^2}{2} \left\| \partial_{xx} u_{\varepsilon, \alpha}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) - \frac{\alpha}{2} \int_{\mathbb{R}} u_{\varepsilon, \beta}^3 \partial_{xx} u_{\varepsilon, \beta} \, dx \\
+ K_1 \varepsilon \left\| \partial_\varepsilon u_{\varepsilon, \alpha}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + K_2 \alpha \varepsilon \left\| \partial_{xx} u_{\varepsilon, \alpha}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
+ K_3 \varepsilon \left\| u_{\varepsilon, \alpha}(t, \cdot) \partial_\varepsilon u_{\varepsilon, \alpha}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + K_4 \varepsilon^3 \left\| \partial_{xx}^2 u_{\varepsilon, \alpha}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
+ \frac{5\alpha\varepsilon}{2} \left\| u_{\varepsilon, \alpha}(t, \cdot) \partial_{xx}^2 u_{\varepsilon, \alpha}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + K_5 \alpha \varepsilon^3 \left\| \partial_{xx}^3 u_{\varepsilon, \alpha}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq 0,
\end{align*}
\]

for some \( K_1, K_2, K_3, K_4, K_5 > 0 \). An integration on \((0, t)\) and (17) give

\[
\begin{align*}
\frac{1}{4} \left\| u_{\varepsilon, \alpha}(t, \cdot) \right\|_{L^4(\mathbb{R})}^4 \\
+ \left( \frac{A + B)\varepsilon^2}{2} \left\| \partial_\varepsilon u_{\varepsilon, \alpha}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\
+ \left( \frac{(A + B)\varepsilon^2}{2} \left\| \partial_{xx} u_{\varepsilon, \alpha}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) - \frac{\beta}{2} \int_{\mathbb{R}} u_{\varepsilon, \beta}^3 \partial_{xx} u_{\varepsilon, \beta} \, dx \\
+ K_1 \varepsilon \int_{0}^{t} \left\| \partial_\varepsilon u_{\varepsilon, \alpha}(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds + K_2 \alpha \varepsilon \int_{0}^{t} \left\| \partial_{xx} u_{\varepsilon, \alpha}(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds \\
+ K_3 \varepsilon \int_{0}^{t} \left\| u_{\varepsilon, \alpha}(s, \cdot) \partial_\varepsilon u_{\varepsilon, \alpha}(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds + K_4 \varepsilon^3 \int_{0}^{t} \left\| \partial_{xx}^2 u_{\varepsilon, \alpha}(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds \\
+ \frac{5\alpha\varepsilon}{2} \int_{0}^{t} \left\| u_{\varepsilon, \alpha}(s, \cdot) \partial_{xx}^2 u_{\varepsilon, \alpha}(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds + K_5 \alpha \varepsilon^3 \int_{0}^{t} \left\| \partial_{xx}^3 u_{\varepsilon, \alpha}(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds \\
\leq \frac{3\varepsilon}{4} \int_{0}^{t} \left\| \partial_\varepsilon u_{\varepsilon, \alpha}(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds + C_0 \alpha \varepsilon \int_{0}^{t} \left\| \partial_{xx} u_{\varepsilon, \alpha}(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds \\
+ C_0 \alpha^2 \varepsilon \int_{0}^{t} \left\| \partial_{xx}^2 u_{\varepsilon, \alpha}(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds \leq C_0.
\end{align*}
\]

Since

\[
- \frac{\alpha}{2} \int_{\mathbb{R}} u_{\varepsilon, \alpha}^3 \partial_{xx}^2 u_{\varepsilon, \alpha} \, dx = \frac{3\alpha}{2} \left\| u_{\varepsilon, \alpha}(t, \cdot) \partial_\varepsilon u_{\varepsilon, \alpha}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,
\]

we have

\[
\frac{1}{4} \left\| u_{\varepsilon, \alpha}(t, \cdot) \right\|_{L^4(\mathbb{R})}^4 \\
+ \left( \frac{A + B)\varepsilon^2}{2} \left\| \partial_\varepsilon u_{\varepsilon, \alpha}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \]
Let us consider a compactly supported entropy–entropy flux
Proof of Theorem 2.1. 
{L_2 \leq n} \eta \in K \Omega \leq 3 \alpha \eta (x) \partial_{xx} u_{\alpha}(s, \cdot) 
\int \Omega \partial_{tt} u_{\alpha}(s, \cdot) \geq \int \Omega \partial_{tt} u_{\alpha}(s, \cdot) \geq C_0.

Hence,
\|u_{\alpha}(t, \cdot)\|_{L^1(\Omega)} \leq C_0,
\|u_{t,\alpha}(t, \cdot)\|_{L^2(\Omega)} \leq C_0,
\alpha^2 \|u_{\alpha}(t, \cdot)\|_{L^2(\Omega)} \leq C_0,
\alpha^2 \|u_{\alpha}(t, \cdot)\|_{L^2(\Omega)} \leq C_0,
\varepsilon \int\int \|u_{\alpha}(s, \cdot)\|_{L^2(\Omega)} \leq C_0,
\varepsilon \int\int \|u_{\alpha}(s, \cdot)\|_{L^2(\Omega)} \leq C_0,
\varepsilon \int\int \|u_{\alpha}(s, \cdot)\|_{L^2(\Omega)} \leq C_0,
\varepsilon \int\int \|u_{\alpha}(s, \cdot)\|_{L^2(\Omega)} \leq C_0,
\varepsilon \int\int \|u_{\alpha}(s, \cdot)\|_{L^2(\Omega)} \leq C_0,
\alpha \varepsilon \int\int \|u_{\alpha}(s, \cdot)\|_{L^2(\Omega)} \leq C_0,
\alpha \varepsilon \int\int \|u_{\alpha}(s, \cdot)\|_{L^2(\Omega)} \leq C_0,
\alpha \varepsilon \int\int \|u_{\alpha}(s, \cdot)\|_{L^2(\Omega)} \leq C_0,
\alpha \varepsilon \int\int \|u_{\alpha}(s, \cdot)\|_{L^2(\Omega)} \leq C_0,
for every 0 \leq t \leq T.

To prove Theorem 2.1, the following technical lemma is needed [26].

Lemma 2.5. Let \Omega be a bounded open subset of \mathbb{R}^2. Suppose that the sequence \{\mathcal{L}_n\} \in \mathbb{N} of distributions is bounded in \mathcal{W}^{-1, \infty}(\Omega). Suppose also that
\mathcal{L}_n = \mathcal{L}_{1,n} + \mathcal{L}_{2,n},
where \{\mathcal{L}_{1,n}\} \in \mathbb{N} lies in a compact subset of \mathcal{H}^{-1}_{\text{loc}}(\Omega) and \{\mathcal{L}_{2,n}\} \in \mathbb{N} lies in a bounded subset of \mathcal{M}_{\text{loc}}(\Omega). Then \{\mathcal{L}_n\} \in \mathbb{N} lies in a compact subset of \mathcal{H}^{-1}_{\text{loc}}(\Omega).

Proof of Theorem 2.1. Let us consider a compactly supported entropy–entropy flux pair \((\eta, q)\) for (4). Multiplying (16) by \eta'(u_{\alpha}), we have
\partial_t \eta(u_{\alpha}) + \partial_{xx} q(u_{\alpha}) = \varepsilon \eta'(u_{\alpha}) \partial_{xx} u_{\alpha} - \alpha \eta'(u_{\alpha}) \partial_{xxx} u_{\alpha} + \alpha \eta'(u_{\alpha}) \partial_{xx} u_{\alpha} + \alpha \eta'(u_{\alpha}) u_{\alpha} \partial_{xxx} u_{\alpha} = I_1, \partial_{xx} u_{\alpha} + I_2, \partial_{xx} u_{\alpha} + I_3, \partial_{xx} u_{\alpha} + I_4, \partial_{xx} u_{\alpha} + I_5, \partial_{xx} u_{\alpha} + I_6, \partial_{xx} u_{\alpha} + I_7, \partial_{xx} u_{\alpha} + I_8, \partial_{xx} u_{\alpha} + I_9, \partial_{xx} u_{\alpha} + I_{10}, \partial_{xx} u_{\alpha} + I_{11}, \partial_{xx} u_{\alpha}.
where

\[ I_1, \varepsilon, \alpha = \partial_x (\varepsilon \eta' (u_{\varepsilon, \alpha}) \partial_x u_{\varepsilon, \alpha}), \]
\[ I_2, \varepsilon, \alpha = -\varepsilon \eta'' (u_{\varepsilon, \alpha}) (\partial_x u_{\varepsilon, \alpha})^2, \]
\[ I_3, \varepsilon, \alpha = -\partial_x (\alpha \varepsilon \eta' (u_{\varepsilon, \alpha}) \partial_{x x} u_{\varepsilon, \alpha}), \]
\[ I_4, \varepsilon, \alpha = \alpha \varepsilon \eta'' (u_{\varepsilon, \alpha}) \partial_x u_{\varepsilon, \alpha} \partial_{x x} u_{\varepsilon, \alpha}, \]
\[ I_5, \varepsilon, \alpha = \partial_x (\alpha \eta' (u_{\varepsilon, \alpha}) \partial_{x x} u_{\varepsilon, \alpha}), \]
\[ I_6, \varepsilon, \alpha = -\alpha \eta'' (u_{\varepsilon, \alpha}) \partial_x u_{\varepsilon, \alpha} \partial_{x x}^2 u_{\varepsilon, \alpha}, \]
\[ I_7, \varepsilon, \alpha = \partial_x (\alpha \eta' (u_{\varepsilon, \alpha}) (\partial_x u_{\varepsilon, \alpha})^2), \]
\[ I_8, \varepsilon, \alpha = -\alpha \eta'' (u_{\varepsilon, \alpha}) (\partial_x u_{\varepsilon, \alpha})^3, \]
\[ I_9, \varepsilon, \alpha = \partial_t (\alpha \eta' (u_{\varepsilon, \alpha}) u_{\varepsilon, \alpha} \partial_{x x}^2 u_{\varepsilon, \alpha}), \]
\[ I_{10}, \varepsilon, \alpha = -\alpha \eta'' (u_{\varepsilon, \alpha}) u_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} \partial_{x x}^2 u_{\varepsilon, \alpha}, \]
\[ I_{11}, \varepsilon, \alpha = -\alpha \eta'' (u_{\varepsilon, \alpha}) \partial_x u_{\varepsilon, \alpha} \partial_{x x}^2 u_{\varepsilon, \alpha}. \]

Fix \( T > 0 \). Arguing as in [9, Lemma 5.1], we have that

\[ I_{1, \varepsilon, \alpha} \to 0 \text{ in } H^{-1}((0, T) \times \mathbb{R}), \text{ as } \alpha, \varepsilon \to 0, \]
\[ \{I_{2, \varepsilon, \alpha}\}_{\varepsilon, \alpha > 0} \text{ is bounded in } L^1((0, T) \times \mathbb{R}), \]
\[ I_{3, \varepsilon, \alpha} \to 0 \text{ in } H^{-1}((0, T) \times \mathbb{R}), \text{ as } \alpha, \varepsilon \to 0, \]
\[ I_{5, \varepsilon, \alpha} \to 0 \text{ in } H^{-1}((0, T) \times \mathbb{R}), \text{ as } \alpha, \varepsilon \to 0, \]
\[ I_{7, \varepsilon, \alpha} \to 0 \text{ in } H^{-1}((0, T) \times \mathbb{R}), \text{ as } \alpha, \varepsilon \to 0, \]
\[ I_{9, \varepsilon, \alpha} \to 0 \text{ in } H^{-1}((0, T) \times \mathbb{R}), \text{ as } \alpha, \varepsilon \to 0. \]

Instead, arguing as in [4, Theorem 1.1]

\[ I_6, \varepsilon, \alpha \to 0 \text{ in } L^1((0, T) \times \mathbb{R}), \text{ as } \alpha, \varepsilon \to 0, \]
\[ I_{10}, \varepsilon, \alpha \to 0 \text{ in } L^1((0, T) \times \mathbb{R}), \text{ as } \alpha, \varepsilon \to 0, \]
\[ I_{11}, \varepsilon, \alpha \to 0 \text{ in } L^1((0, T) \times \mathbb{R}), \text{ as } \varepsilon \to 0. \]

We have

\[ \{I_{8, \varepsilon, \alpha}\}_{\varepsilon, \alpha > 0} \text{ is bounded in } L^1((0, T) \times \mathbb{R}), \]

Thanks to (21), Lemma 2.2 and (25),

\[
\|\alpha \eta'' (u_{\varepsilon, \alpha}) (\partial_x u_{\varepsilon, \alpha})^3\|_{L^1((0, T) \times \mathbb{R})} \\
\leq \alpha \|\eta''\|_{L^\infty(\mathbb{R})} \int_0^T \int_{\mathbb{R}} |\partial_x u_{\varepsilon, \alpha}|^3 dt dx \\
\leq \alpha \|\eta''\|_{L^\infty(\mathbb{R})} \|\partial_x u_{\varepsilon, \alpha}\|_{L^\infty((0, T) \times \mathbb{R})} \|\partial_x u_{\varepsilon, \alpha}\|_{L^2((0, T) \times \mathbb{R})}^2 \\
\leq C_0 \|\eta''\|_{L^\infty(\mathbb{R})} \alpha^{-\frac{3}{2}} \|\partial_x u_{\varepsilon, \alpha}\|_{L^2((0, T) \times \mathbb{R})}^2 \leq C_0.
\]

Therefore, (19) follows from Lemmas 2.2, 2.4, 2.5 and the \( L^p \) compactness of [30].

Arguing as in [9, Lemma 5.1], we have that \( u \) is a a distributional solution of (4). Instead, arguing as in [4, Theorem 1.1], we have that \( u \) is the entropy solution of (4). \( \square \)
3. The Novikov equation. In this section, we consider (3). We augment (3) with the initial condition

\[ u_0 \in L^2(\mathbb{R}) \cap L^6(\mathbb{R}). \tag{47} \]

We study the dispersion-diffusion for (3). Therefore, we fix \( 0 < \varepsilon, \alpha < 1 \), and consider the following fourth order approximation

\[
\begin{aligned}
\partial_t u_{\varepsilon, \alpha} &- \alpha \partial_{xxx}^3 u_{\varepsilon, \alpha} + \frac{4}{3} \partial_x u^3 \\
-3\alpha u_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha} - \alpha u_{\varepsilon, \alpha}^2 \partial_{xxx}^3 u_{\varepsilon, \alpha} \\
= \varepsilon \partial_{xx}^2 u_{\varepsilon, \alpha} - \alpha \varepsilon \partial_{xxxx}^4 u_{\varepsilon, \alpha} + \alpha \varepsilon (\partial_x u_{\varepsilon, \alpha})^2 \partial_{xx}^3 u_{\varepsilon, \alpha}, & \quad t > 0, x \in \mathbb{R}, \\
u_{\varepsilon, \alpha}(0, x) = u_{\varepsilon, \alpha, 0}(x), \quad x \in \mathbb{R},
\end{aligned}
\tag{48}
\]

where \( u_{\varepsilon, \alpha, 0} \) is a \( C^\infty \) approximation of \( u_0 \) such that

\[
\|u_{\varepsilon, \alpha, 0}\|_{L^2(\mathbb{R})} + \|u_{\varepsilon, \alpha, 0}\|_{L^6(\mathbb{R})} \leq C_0, \quad \varepsilon, \alpha > 0,
\]

\[
\left( \sqrt[\varepsilon]{\alpha} + \varepsilon \right) \|\partial_x u_{\varepsilon, \alpha, 0}\|_{L^2(\mathbb{R})} + \left( \sqrt[\varepsilon]{\alpha} + \sqrt{\alpha \varepsilon} \right) \|\partial_{xx}^2 u_{\varepsilon, \alpha, 0}\|_{L^2(\mathbb{R})} \leq C_0, \quad \varepsilon, \alpha > 0,
\]

\[
\sqrt[\varepsilon]{\alpha} \sqrt[\varepsilon]{\alpha} \|\partial_x u_{\varepsilon, \alpha, 0}\|_{L^4(\mathbb{R})} \leq C_0, \quad \varepsilon, \alpha > 0,
\]

for some constant \( C_0 \) independent on \( \varepsilon, \alpha \).

The main result of this section is the following theorem.

**Theorem 3.1.** Assume (47) and (49). If

\[ \alpha = O(\varepsilon^8), \tag{50} \]

then there exist two sequences \( \{\varepsilon_k\}_{k \in \mathbb{N}}, \{\alpha_k\}_{k \in \mathbb{N}} \), with \( \varepsilon_k, \alpha_k \to 0 \), and a limit function

\[ u \in L^\infty((0, T); L^2(\mathbb{R}) \cap L^6(\mathbb{R})), \tag{51} \]

such that

\[ u_{\varepsilon_k, \alpha_k} \to u, \quad \text{strongly in } L^p_{loc}(\mathbb{R}^+ \times \mathbb{R}), \text{ for each } 1 \leq p < 6, \text{ and a.e.} \tag{52} \]

\[ u \text{ is a distributional solution of } (5). \tag{53} \]

Moreover, if

\[ \alpha = o(\varepsilon^8), \tag{54} \]

(51) holds, and

\[ u \text{ is the unique entropy solution of } (5). \tag{55} \]

The assumptions (50), or (53), consists in

\[ \alpha \leq D^8 \varepsilon^8, \tag{56} \]

where \( D \) is a positive constant such that

\[ D < \min \left\{ \sqrt[3]{\frac{3}{26C_0}}, \sqrt[12]{\frac{1}{10C_0}} \right\}. \tag{57} \]

Let us prove some priori estimates on \( u_{\varepsilon, \alpha} \), denoting with \( C_0 \) the constants which depend only on the initial data.
Lemma 3.2. For each $t > 0$,
\[
\|u_{\varepsilon,\alpha}(t, \cdot)\|^2_{L^2(\mathbb{R})} + \alpha \|\partial_x u_{\varepsilon,\alpha}(t, \cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon \int_0^t \|\partial_x u_{\varepsilon,\alpha}(s, \cdot)\|^2_{L^2(\mathbb{R})} \, ds + 2\alpha \varepsilon \int_0^t \|\partial_{xx}^2 u_{\varepsilon,\alpha}(t, \cdot)\|^2_{L^2(\mathbb{R})} \, ds
\]
\[
+ \frac{2\alpha^4 \varepsilon}{3} \int_0^t \|\partial_x u_{\varepsilon,\alpha}(s, \cdot)\|^4_{L^4(\mathbb{R})} \, ds \leq C_0.
\]
Moreover, (24) holds.

Proof. Multiplying (48) by $2u_{\varepsilon,\alpha}$ and integrating on $\mathbb{R}$, we have
\[
2 \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial_t u_{\varepsilon,\alpha} \, dx - 2\alpha \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial_{xx}^3 u_{\varepsilon,\alpha} \, dx + 8 \int_{\mathbb{R}} u_{\varepsilon,\alpha}^3 \partial_x u_{\varepsilon,\alpha} \, dx
\]
\[
- 6\alpha \int_{\mathbb{R}} u_{\varepsilon,\alpha}^2 \partial_x u_{\varepsilon,\alpha} \partial_{xx}^2 u_{\varepsilon,\alpha} \, dx - 2\alpha \int_{\mathbb{R}} u_{\varepsilon,\alpha}^3 \partial_{xxx}^3 u_{\varepsilon,\alpha} \, dx
\]
\[
= 2\varepsilon \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial_{xx}^2 u_{\varepsilon,\alpha} \, dx - 2\alpha \varepsilon \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial_{xxx}^3 u_{\varepsilon,\alpha} \, dx
\]
\[
+ \alpha^4 \varepsilon \int_{\mathbb{R}} u_{\varepsilon,\alpha} (\partial_x u_{\varepsilon,\alpha})^2 \partial_{xx}^2 u_{\varepsilon,\alpha} \, dx.
\]
Observe that
\[
2 \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial_t u_{\varepsilon,\alpha} \, dx - 2\alpha \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial_{xx}^3 u_{\varepsilon,\alpha} \, dx
\]
\[
= \frac{d}{dt} \left( \|u_{\varepsilon,\alpha}(t, \cdot)\|^2_{L^2(\mathbb{R})} + \alpha \|\partial_x u_{\varepsilon,\alpha}(t, \cdot)\|^2_{L^2(\mathbb{R})} \right),
\]
\[
8 \int_{\mathbb{R}} u_{\varepsilon,\alpha}^3 \partial_x u_{\varepsilon,\alpha} \, dx - 6\alpha \int_{\mathbb{R}} u_{\varepsilon,\alpha}^2 \partial_x u_{\varepsilon,\alpha} \partial_{xx}^2 u_{\varepsilon,\alpha} \, dx - 2\alpha \int_{\mathbb{R}} u_{\varepsilon,\alpha}^3 \partial_{xxx}^3 u_{\varepsilon,\alpha} \, dx
\]
\[
= 8 \int_{\mathbb{R}} u_{\varepsilon,\alpha}^3 \partial_x u_{\varepsilon,\alpha} \, dx - 6\alpha \int_{\mathbb{R}} u_{\varepsilon,\alpha}^2 \partial_x u_{\varepsilon,\alpha} \partial_{xx}^2 u_{\varepsilon,\alpha} \, dx
\]
\[
+ 6\alpha \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial_x u_{\varepsilon,\alpha} \partial_{xx}^2 u_{\varepsilon,\alpha} \, dx = 0,
\]
\[
2\varepsilon \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial_{xx}^2 u_{\varepsilon,\alpha} \, dx - 2\alpha \varepsilon \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial_{xxx}^3 u_{\varepsilon,\alpha} \, dx + \alpha^4 \varepsilon \int_{\mathbb{R}} u_{\varepsilon,\alpha} (\partial_x u_{\varepsilon,\alpha})^2 \partial_{xx}^2 u_{\varepsilon,\alpha} \, dx
\]
\[
= -2\varepsilon \|\partial_x u_{\varepsilon,\alpha}(t, \cdot)\|^2_{L^2(\mathbb{R})} - 2\alpha \varepsilon \|\partial_{xx}^2 u_{\varepsilon,\alpha}(t, \cdot)\|^2_{L^2(\mathbb{R})} - \frac{2\alpha^4 \varepsilon}{3} \|\partial_x u_{\varepsilon,\alpha}(t, \cdot)\|^4_{L^4(\mathbb{R})}.
\]
Hence, we can rewrite (58) in the following way:
\[
\frac{d}{dt} \left( \|u_{\varepsilon,\alpha}(t, \cdot)\|^2_{L^2(\mathbb{R})} + \alpha \|\partial_x u_{\varepsilon,\alpha}(t, \cdot)\|^2_{L^2(\mathbb{R})} \right) + 2\varepsilon \|\partial_x u_{\varepsilon,\alpha}(t, \cdot)\|^2_{L^2(\mathbb{R})}
\]
\[
+ 2\alpha \varepsilon \|\partial_{xx}^2 u_{\varepsilon,\alpha}(t, \cdot)\|^2_{L^2(\mathbb{R})} + \frac{2\alpha^4 \varepsilon}{3} \|\partial_x u_{\varepsilon,\alpha}(t, \cdot)\|^4_{L^4(\mathbb{R})} = 0.
\]
Integrating on $(0, t)$, from (49), we have (23). Arguing as in Lemma 2.2, we obtain (24).

Since the $L^\infty$ estimate (24) does not guarantee the $L^6$ conservation, following [6, Lemma 2.2], we prove the following one.
Lemma 3.3. Fix $T > 0$. Assume (50), or (53), (55) and (56). There exists $C_0 > 0$, such that
\[ \| u_\varepsilon \|_{L^\infty(0,T) \times \mathbb{R}} \leq C_0 \alpha^{-\frac{1}{4}}. \]  

In particular, we have
\[ \alpha^\frac{1}{4} \| \partial_x u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \alpha^\frac{5}{4} \| \partial_{xx}^2 u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \]
\[ + 2 \alpha^\frac{3}{4} \int_0^t \| \partial_{xx}^3 u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 dt + 2 \alpha^\frac{1}{4} \int_0^t \| \partial_{xxx}^4 u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \]  

Proof. Let $0 \leq t \leq T$. Multiplying (48) by $-2 \alpha^\frac{1}{4} \partial_{xx} u_{\varepsilon, \alpha}$ and integrating on $\mathbb{R}$ we have
\[ -2 \alpha^\frac{1}{4} \int_{\mathbb{R}} \partial_{xx}^2 u_{\varepsilon, \alpha} \partial_t u_{\varepsilon, \alpha} dx + 2 \alpha^\frac{3}{4} \int_{\mathbb{R}} \partial_{xx}^3 u_{\varepsilon, \alpha} \partial_{xxx}^3 u_{\varepsilon, \alpha} dx \]
\[ -8 \alpha^\frac{1}{4} \int_{\mathbb{R}} \partial_{xx}^2 u_{\varepsilon, \alpha} \partial_{xx} \partial_{xx}^2 u_{\varepsilon, \alpha} dx + 6 \alpha^\frac{1}{4} \int_{\mathbb{R}} \partial_{xx} u_{\varepsilon, \alpha} \partial_{xx} \partial_{xx}^2 u_{\varepsilon, \alpha} \]  

Observe that
\[ -2 \alpha^\frac{1}{4} \int_{\mathbb{R}} \partial_{xx}^2 u_{\varepsilon, \alpha} \partial_t u_{\varepsilon, \alpha} dx + 2 \alpha^\frac{3}{4} \int_{\mathbb{R}} \partial_{xx}^3 u_{\varepsilon, \alpha} \partial_{xxx}^3 u_{\varepsilon, \alpha} dx \]
\[ = \frac{d}{dt} \left( \alpha^\frac{1}{4} \| \partial_x u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \alpha^\frac{5}{4} \| \partial_{xx} u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \right). \]

Moreover,
\[ 2 \alpha^\frac{1}{4} \int_{\mathbb{R}} \partial_{xx}^2 u_{\varepsilon, \alpha} \partial_{xxx}^3 u_{\varepsilon, \alpha} dx = -2 \alpha^\frac{1}{4} \int_{\mathbb{R}} u_{\varepsilon, \alpha} \partial_{xx} \partial_{xx}^2 u_{\varepsilon, \alpha} \]  

Hence, by (61), we get
\[ \frac{d}{dt} \left( \alpha^\frac{1}{4} \| \partial_x u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \alpha^\frac{5}{4} \| \partial_{xx} u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \right) \]
\[ + 2 \alpha^\frac{1}{4} \varepsilon \| \partial_{xx} u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + 2 \alpha^\frac{1}{4} \varepsilon \| \partial_{xxx} u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \]
\[ + 2 \alpha^\frac{1}{4} \varepsilon \| \partial_{x} u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \]  

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Since $0 < \varepsilon < 1$, due to (24), (55) and the Young inequality,

$$ - 8\alpha^{\frac{3}{2}} \int_{\mathbb{R}} u_{\varepsilon, \alpha} (\partial_x u_{\varepsilon, \alpha})^3 dx \leq 8\alpha^{\frac{3}{2}} \int_{\mathbb{R}} |u_{\varepsilon, \alpha} (\partial_x u_{\varepsilon, \alpha})^3| dx $$

$$ = 8\alpha^{\frac{3}{2}} \int_{\mathbb{R}} \left| \frac{u_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha}}{\varepsilon^{\frac{1}{2}}} \right| \left| \varepsilon^{\frac{1}{2}} (\partial_x u_{\varepsilon, \alpha})^2 dx \right| $$

$$ \leq \frac{4\alpha^{\frac{3}{2}}}{\varepsilon} \int_{\mathbb{R}} u_{\varepsilon, \alpha} (\partial_x u_{\varepsilon, \alpha})^2 dx + 4\alpha^{\frac{3}{2}} \varepsilon \| \partial_x u_{\varepsilon, \alpha} (t, \cdot) \|^4_{L^4(\mathbb{R})} $$

$$ \leq 4D\varepsilon \| u_{\varepsilon, \alpha} \|^2_{L^\infty((0,T) \times \mathbb{R})} \| \partial_x u_{\varepsilon, \alpha} (t, \cdot) \|^2_{L^2(\mathbb{R})} + 4\alpha^{\frac{3}{2}} \varepsilon \| \partial_x u_{\varepsilon, \alpha} (t, \cdot) \|^4_{L^4(\mathbb{R})} , $$

$$ - 4\alpha^{\frac{3}{2}} \int_{\mathbb{R}} u_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} (\partial_{xx}^2 u_{\varepsilon, \alpha})^2 dx \leq 4\alpha^{\frac{3}{2}} \int_{\mathbb{R}} \left| u_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} (\partial_{xx}^2 u_{\varepsilon, \alpha})^2 dx \right| $$

$$ = 2\alpha^{\frac{3}{2}} \int_{\mathbb{R}} \left| \frac{2\alpha^{\frac{1}{2}} u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha}}{\varepsilon^{\frac{1}{2}}} \right| \left| \varepsilon^{\frac{1}{2}} \partial_x u_{\varepsilon, \alpha} \partial_{xx}^2 u_{\varepsilon, \alpha} \right| dx $$

$$ \leq \frac{4\alpha^2}{\varepsilon} \int_{\mathbb{R}} u_{\varepsilon, \alpha} (\partial_{xx}^2 u_{\varepsilon, \alpha})^2 dx + \alpha^{\frac{3}{2}} \| \partial_x u_{\varepsilon, \alpha} (t, \cdot) \|^2_{L^2(\mathbb{R})} $$

$$ \leq \frac{4\alpha^2}{\varepsilon} \| u_{\varepsilon, \alpha} \|^2_{L^\infty((0,T) \times \mathbb{R})} \| \partial_{xx}^2 u_{\varepsilon, \alpha} (t, \cdot) \|^2_{L^2(\mathbb{R})} + \alpha^{\frac{3}{2}} \| \partial_x u_{\varepsilon, \alpha} (t, \cdot) \|^2_{L^2(\mathbb{R})} $$

$$ \leq C_0 \alpha^{\frac{3}{2}} \| u_{\varepsilon, \alpha} \|^2_{L^\infty((0,T) \times \mathbb{R})} \| \partial_{xx} u_{\varepsilon, \alpha} (t, \cdot) \|^2_{L^2(\mathbb{R})} + \alpha^{\frac{3}{2}} \| \partial_x u_{\varepsilon, \alpha} (t, \cdot) \|^2_{L^2(\mathbb{R})} $$

$$ \leq C_0 D^{\alpha} \varepsilon \| \partial_{xx} u_{\varepsilon, \alpha} (t, \cdot) \|^2_{L^2(\mathbb{R})} + \alpha^{\frac{3}{2}} \| \partial_x u_{\varepsilon, \alpha} (t, \cdot) \|^2_{L^2(\mathbb{R})} . $$

Hence, by (62), we get

$$ \frac{d}{dt} \left( \alpha^{\frac{1}{2}} \| \partial_x u_{\varepsilon, \alpha} (t, \cdot) \|^2_{L^2(\mathbb{R})} + \alpha^{\frac{3}{2}} \| \partial_{xx} u_{\varepsilon, \alpha} (t, \cdot) \|^2_{L^2(\mathbb{R})} \right) $$

$$ + 2\alpha^{\frac{1}{2}} \varepsilon \| \partial_{xx} u_{\varepsilon, \alpha} (t, \cdot) \|^2_{L^2(\mathbb{R})} + 2\alpha^2 \varepsilon \| \partial_{xxx} u_{\varepsilon, \alpha} (t, \cdot) \|^2_{L^2(\mathbb{R})} $$

$$ + \alpha^{\frac{3}{2}} \| \partial_x u_{\varepsilon, \alpha} (t, \cdot) \|^2_{L^2(\mathbb{R})} $$

$$ \leq 4D\varepsilon \| u_{\varepsilon, \alpha} \|^2_{L^\infty((0,T) \times \mathbb{R})} \| \partial_x u_{\varepsilon, \alpha} (t, \cdot) \|^2_{L^2(\mathbb{R})} + 4\alpha^{\frac{3}{2}} \varepsilon \| \partial_x u_{\varepsilon, \alpha} (t, \cdot) \|^4_{L^4(\mathbb{R})} + C_0 D^{\alpha} \varepsilon \| \partial_{xx} u_{\varepsilon, \alpha} (t, \cdot) \|^2_{L^2(\mathbb{R})} . $$

Integrating on $(0, t)$, by (49) and (57), we have

$$ \alpha^{\frac{1}{2}} \| \partial_x u_{\varepsilon, \alpha} (t, \cdot) \|^2_{L^2(\mathbb{R})} + \alpha^{\frac{3}{2}} \| \partial_{xx} u_{\varepsilon, \alpha} (t, \cdot) \|^2_{L^2(\mathbb{R})} $$

$$ + 2\alpha^{\frac{1}{2}} \varepsilon \int_0^t \| \partial_{xx} u_{\varepsilon, \alpha} (s, \cdot) \|^2_{L^2(\mathbb{R})} ds + 2\alpha^2 \varepsilon \int_0^t \| \partial_{xxx} u_{\varepsilon, \alpha} (s, \cdot) \|^2_{L^2(\mathbb{R})} ds $$

$$ + \alpha^{\frac{3}{2}} \int_0^t \| \partial_x u_{\varepsilon, \alpha} (s, \cdot) \|^2_{L^2(\mathbb{R})} ds $$

$$\leq C_0 + 4D\varepsilon \| u_{\varepsilon, \alpha} \|^2_{L^\infty((0,T) \times \mathbb{R})} \int_0^t \| \partial_x u_{\varepsilon, \alpha} (s, \cdot) \|^2_{L^2(\mathbb{R})} ds $$

$$ + 4\alpha^{\frac{3}{2}} \varepsilon \int_0^t \| \partial_x u_{\varepsilon, \alpha} (s, \cdot) \|^2_{L^2(\mathbb{R})} ds + \alpha \varepsilon \int_0^t \| \partial_{xx} u_{\varepsilon, \alpha} (s, \cdot) \|^2_{L^2(\mathbb{R})} ds $$

$$\leq C_0 \left( 1 + \| u_{\varepsilon, \alpha} \|^2_{L^\infty((0,T) \times \mathbb{R})} \right) .$$
We prove (59). Due to (57), (63) and the Hölder inequality,
\begin{align*}
    u_{x}^{2}(t,x) & = 2 \int_{-\infty}^{x} u_{x,\alpha} \partial_{x} u_{x,\alpha} dy \\
    & \leq 2 \int_{\mathbb{R}} |u_{x,\alpha}| |\partial_{x} u_{x,\alpha}| dx \\
    & \leq \|u_{x,\alpha}(t,\cdot)\|_{L^{2}(\mathbb{R})} \|\partial_{x} u_{x,\alpha}(t,\cdot)\|_{L^{2}(\mathbb{R})} \\
    & \leq \frac{C_{0}}{\alpha^{\frac{3}{4}}} \sqrt{\left(1 + \|u_{x,\alpha}\|_{L^{\infty}((0,T) \times \mathbb{R})}^{2}\right)}.
\end{align*}

Hence,
\begin{equation}
    \|u_{x}\|_{L^{\infty}((0,T) \times \mathbb{R})}^{4} \leq \frac{C_{0}}{\alpha^{\frac{3}{4}}} \left(1 + \|u_{x,\alpha}\|_{L^{\infty}((0,T) \times \mathbb{R})}^{2}\right). \tag{64}
\end{equation}

Introducing the notation
\begin{equation}
    y := \|u_{x}\|_{L^{\infty}((0,T) \times \mathbb{R})}, \quad \alpha^{\frac{4}{3}} = \delta, \tag{65}
\end{equation}
(64) reads
\begin{equation}
    y^{4} \leq \frac{C_{0}}{\delta} (1 + y^{2}). \tag{66}
\end{equation}

Arguing as in [7, Lemma 2.3], we have that
\begin{equation}
    y \leq C_{0}\delta^{-\frac{1}{4}}. \tag{67}
\end{equation}

(59) follows from (65) and (67).

Finally, (60) follows from (59) and (63). \qed

**Lemma 3.4.** Fix $T > 0$. Assume (50), or (53), (55) and (56). Then,
\begin{itemize}
    \item[i)] the family $\{u_{x,\alpha}\}_{\varepsilon,\alpha}$ is bounded in $L^{\infty}((0,T); L^{0}(\mathbb{R}))$;
    \item[ii)] the families
        \begin{equation*}
            \{\varepsilon \partial_{x} u_{x,\alpha}\}_{\varepsilon,\alpha}, \{\alpha^{\frac{3}{2}} \varepsilon^{\frac{3}{4}} \partial_{xx}^{2} u_{x,\alpha}\}_{\varepsilon,\alpha}
        \end{equation*}
    are bounded in $L^{\infty}(\mathbb{R}^{+}; L^{2}(\mathbb{R}))$;
    \item[iii)] the family $\{\alpha^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \partial_{x} u_{x,\alpha}\}_{\varepsilon,\alpha}$ is bounded in $L^{\infty}((0,T); L^{4}(\mathbb{R}))$;
    \item[iv)] the families
        \begin{equation*}
            \{\varepsilon^{\frac{1}{2}} \partial_{x} u_{x,\alpha}\}_{\varepsilon,\alpha}, \{\alpha^{\frac{3}{4}} \varepsilon^{\frac{3}{4}} \partial_{xx} u_{x,\alpha}\}_{\varepsilon,\alpha}, \{\varepsilon^{\frac{3}{2}} \partial_{xx}^{2} u_{x,\alpha}\}_{\varepsilon,\alpha}, \{\alpha^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \partial_{xx}^{2} u_{x,\alpha}\}_{\varepsilon,\alpha}, \{\alpha^{\frac{3}{4}} \varepsilon^{\frac{3}{4}} \partial_{xx}^{3} u_{x,\alpha}\}_{\varepsilon,\alpha}, \{\varepsilon^{\frac{1}{2}} \partial_{x} u_{x,\alpha}\partial_{x} u_{x,\alpha}\}_{\varepsilon,\alpha}, \{\alpha^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \partial_{x} u_{x,\alpha}\partial_{x}^{2} u_{x,\alpha}\}_{\varepsilon,\alpha}
        \end{equation*}
    are bounded in $L^{2}(\mathbb{R}^{+} \times \mathbb{R})$.
\end{itemize}

**Proof.** Let $0 \leq t \leq T$. Let $A, B$ some positive constants which will be specified later. Multiplying (48) by
\begin{equation*}
    u_{x,\alpha}^{5} - A \varepsilon^{2} \partial_{xx}^{2} u_{x,\alpha} + B \varepsilon \partial_{x} u_{x,\alpha},
\end{equation*}
we have that
\begin{align*}
    (u_{x,\alpha}^{5} & - A \varepsilon^{2} \partial_{xx}^{2} u_{x,\alpha} + B \varepsilon \partial_{x} u_{x,\alpha}) \partial_{x} u_{x,\alpha} \\
    & + 4 (u_{x,\alpha}^{5} - A \varepsilon^{2} \partial_{xx}^{2} u_{x,\alpha} + B \varepsilon \partial_{x} u_{x,\alpha}) u_{x,\alpha}^{2} \partial_{x} u_{x,\alpha}
\end{align*}

...
\begin{align*}
&- \alpha \left( u_{\varepsilon,\alpha}^5 - A \varepsilon^2 \partial_{xx}^2 u_{\varepsilon,\alpha} + B \varepsilon \partial_t u_{\varepsilon,\alpha} \right) \partial_{tx} u_{\varepsilon,\alpha} \\
&- 3 \alpha \left( u_{\varepsilon,\alpha}^5 - A \varepsilon^2 \partial_{xx}^2 u_{\varepsilon,\alpha} + B \varepsilon \partial_t u_{\varepsilon,\alpha} \right) u_{\varepsilon,\alpha} \partial_x u_{\varepsilon,\alpha} \partial_{xx} u_{\varepsilon,\alpha} \\
&- \alpha \left( u_{\varepsilon,\alpha}^5 - A \varepsilon^2 \partial_{xx}^2 u_{\varepsilon,\alpha} + B \varepsilon \partial_t u_{\varepsilon,\alpha} \right) u_{\varepsilon,\alpha}^2 \partial_{xx}^3 u_{\varepsilon,\alpha} \\
&= \varepsilon \left( u_{\varepsilon,\alpha}^5 - A \varepsilon^2 \partial_{xx}^2 u_{\varepsilon,\alpha} + B \varepsilon \partial_t u_{\varepsilon,\alpha} \right) \partial_{xx}^2 u_{\varepsilon,\alpha} \\
&- \alpha \varepsilon \left( u_{\varepsilon,\alpha}^5 - A \varepsilon^2 \partial_{xx}^2 u_{\varepsilon,\alpha} + B \varepsilon \partial_t u_{\varepsilon,\alpha} \right) \partial_{xxxx}^4 u_{\varepsilon,\alpha} \\
&+ \alpha^4 \varepsilon \left( u_{\varepsilon,\alpha}^5 - A \varepsilon^2 \partial_{xx}^2 u_{\varepsilon,\alpha} + B \varepsilon \partial_t u_{\varepsilon,\alpha} \right) (\partial_x u_{\varepsilon,\alpha})^2 \partial_{xx}^2 u_{\varepsilon,\alpha}.
\end{align*}

Observe
\begin{align*}
\int_{\mathbb{R}} \left( u_{\varepsilon,\alpha}^5 - A \varepsilon^2 \partial_{xx}^2 u_{\varepsilon,\alpha} + B \varepsilon \partial_t u_{\varepsilon,\alpha} \right) \partial_t u_{\varepsilon,\alpha} \, dx \\
&= \frac{d}{dt} \left( \frac{1}{6} \| u_{\varepsilon,\alpha}(t, \cdot) \|_{L^6(\mathbb{R})}^6 + \frac{A \varepsilon^2}{2} \| \partial_t u_{\varepsilon,\alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \right) \\
&\quad + B \varepsilon \| \partial_t u_{\varepsilon,\alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2, \\
4 \int_{\mathbb{R}} \left( u_{\varepsilon,\alpha}^5 - A \varepsilon^2 \partial_{xx}^2 u_{\varepsilon,\alpha} + B \varepsilon \partial_t u_{\varepsilon,\alpha} \right) u_{\varepsilon,\alpha}^2 \partial_{xx} u_{\varepsilon,\alpha} \, dx \\
&= -4A \varepsilon^2 \int_{\mathbb{R}} u_{\varepsilon,\alpha}^2 \partial_x u_{\varepsilon,\alpha} \partial_{xx}^2 u_{\varepsilon,\alpha} \, dx + 4B \varepsilon \int_{\mathbb{R}} u_{\varepsilon,\alpha}^2 \partial_x u_{\varepsilon,\alpha} \partial_t u_{\varepsilon,\alpha} \, dx, \\
- \alpha \int_{\mathbb{R}} \left( u_{\varepsilon,\alpha}^5 - A \varepsilon^2 \partial_{xx}^2 u_{\varepsilon,\alpha} + B \varepsilon \partial_t u_{\varepsilon,\alpha} \right) \partial_{tx} u_{\varepsilon,\alpha} \, dx \\
&= 5 \alpha \int_{\mathbb{R}} u_{\varepsilon,\alpha}^4 \partial_x u_{\varepsilon,\alpha} \partial_{xx}^2 u_{\varepsilon,\alpha} \, dx + \frac{A \alpha \varepsilon^2}{2} \frac{d}{dt} \| \partial_{xx}^2 u_{\varepsilon,\alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \\
&\quad + B \alpha \varepsilon \| \partial_{xx}^2 u_{\varepsilon,\alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2, \\
- 3 \alpha \int_{\mathbb{R}} \left( u_{\varepsilon,\alpha}^5 - A \varepsilon^2 \partial_{xx}^2 u_{\varepsilon,\alpha} + B \varepsilon \partial_t u_{\varepsilon,\alpha} \right) u_{\varepsilon,\alpha} \partial_x u_{\varepsilon,\alpha} \partial_{xx}^2 u_{\varepsilon,\alpha} \\
&= -3 \alpha \int_{\mathbb{R}} u_{\varepsilon,\alpha}^6 \partial_x u_{\varepsilon,\alpha} \partial_{xx}^2 u_{\varepsilon,\alpha} \, dx + 3A \alpha \varepsilon^2 \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial_x u_{\varepsilon,\alpha} (\partial_{xx}^2 u_{\varepsilon,\alpha})^2 \, dx \\
&\quad - 3B \alpha \varepsilon \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial_x u_{\varepsilon,\alpha} \partial_{xx}^2 u_{\varepsilon,\alpha} \partial_t u_{\varepsilon,\alpha} \, dx, \\
- \alpha \int_{\mathbb{R}} \left( u_{\varepsilon,\alpha}^5 - A \varepsilon^2 \partial_{xx}^2 u_{\varepsilon,\alpha} + B \varepsilon \partial_t u_{\varepsilon,\alpha} \right) u_{\varepsilon,\alpha}^2 \partial_{xx}^3 u_{\varepsilon,\alpha} \, dx \\
&= 7 \alpha \int_{\mathbb{R}} u_{\varepsilon,\alpha}^6 \partial_x u_{\varepsilon,\alpha} \partial_{xx}^2 u_{\varepsilon,\alpha} \, dx - A \alpha \varepsilon^2 \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial_x u_{\varepsilon,\alpha} (\partial_{xx}^2 u_{\varepsilon,\alpha})^2 \, dx \\
&\quad - B \alpha \varepsilon \int_{\mathbb{R}} u_{\varepsilon,\alpha}^2 \partial_{xx}^3 u_{\varepsilon,\alpha} \partial_t u_{\varepsilon,\alpha} \, dx, \\
\varepsilon \int_{\mathbb{R}} \left( u_{\varepsilon,\alpha}^5 - A \varepsilon^2 \partial_{xx}^2 u_{\varepsilon,\alpha} + B \varepsilon \partial_t u_{\varepsilon,\alpha} \right) \partial_{xx}^2 u_{\varepsilon,\alpha} \, dx \\
&= -5 \varepsilon \| u_{\varepsilon,\alpha}(t, \cdot) \partial_x u_{\varepsilon,\alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 - A \varepsilon^3 \| \partial_{xx}^2 u_{\varepsilon,\alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^3 \\
&\quad - \frac{B \varepsilon^2}{2} \frac{d}{dt} \| \partial_t u_{\varepsilon,\alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2, \\
- \alpha \varepsilon \int_{\mathbb{R}} \left( u_{\varepsilon,\alpha}^5 - A \varepsilon^2 \partial_{xx}^2 u_{\varepsilon,\alpha} + B \varepsilon \partial_t u_{\varepsilon,\alpha} \right) \partial_{xxxx}^4 u_{\varepsilon,\alpha} \, dx \\
&= 5 \alpha \varepsilon \int_{\mathbb{R}} u_{\varepsilon,\alpha}^4 \partial_x u_{\varepsilon,\alpha} \partial_{xx}^3 u_{\varepsilon,\alpha} \, dx - A \alpha \varepsilon^3 \| \partial_{xxxx}^4 u_{\varepsilon,\alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2.
\end{align*}
\[
- \frac{B\alpha \varepsilon^2}{2} \frac{d}{dt} \left\| \partial_{xx}^2 u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
= -20 \alpha \varepsilon \int_{\mathbb{R}} u_{\varepsilon,\alpha}^4 (\partial_x u_{\varepsilon,\alpha})^2 \partial_{xx}^2 u_{\varepsilon,\alpha} dx - 5 \alpha \varepsilon \left\| u_{\varepsilon,\alpha}^2 (t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
- A\alpha \varepsilon^3 \left\| \partial_{xx}^3 u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} - \frac{B\alpha \varepsilon^2}{2} \frac{d}{dt} \left\| \partial_{xx}^2 u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
= 20 \alpha \varepsilon \int_{\mathbb{R}} u_{\varepsilon,\alpha}^2 (\partial_x u_{\varepsilon,\alpha})^4 dx - 5 \alpha \varepsilon \left\| u_{\varepsilon,\alpha}^2 (t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
- A\alpha \varepsilon^3 \left\| \partial_{xx}^3 u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} - \frac{B\alpha \varepsilon^2}{2} \frac{d}{dt} \left\| \partial_{xx}^2 u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})},
\]
\[
\alpha^\frac{1}{2} \varepsilon \int_{\mathbb{R}} \left( u_{\varepsilon,\alpha}^2 - A\varepsilon \partial_{xx}^2 u_{\varepsilon,\alpha} + B\varepsilon \partial_{xx} u_{\varepsilon,\alpha} \right) (\partial_x u_{\varepsilon,\alpha})^2 \partial_{xx}^2 u_{\varepsilon,\alpha} dx \\
= - \frac{5}{3} \alpha^\frac{1}{2} \varepsilon \left\| u_{\varepsilon,\alpha}^2 (t, \cdot) \partial_x u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
- A\alpha \varepsilon^3 \left\| \partial_{xx} u_{\varepsilon,\alpha}(t, \cdot) \partial_{xx}^2 u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
- \frac{B\alpha \varepsilon^2}{12} \frac{d}{dt} \left\| \partial_{xx} u_{\varepsilon,\alpha}(t, \cdot) \right\|^4_{L^1(\mathbb{R})}.
\]

Consequently, by (68), we get
\[
\frac{d}{dt} \left( \frac{1}{6} \left\| u_{\varepsilon,\alpha}(t, \cdot) \right\|^6_{L^6(\mathbb{R})} + \frac{(A + B)\varepsilon^2}{2} \left\| \partial_x u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \right) \\
= \frac{d}{dt} \left( \frac{(A + B)\alpha \varepsilon^2}{2} \left\| \partial_{xx}^2 u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{B\alpha \varepsilon^2}{12} \left\| \partial_{xx} u_{\varepsilon,\alpha}(t, \cdot) \right\|^4_{L^1(\mathbb{R})} \right) \\
+ B\varepsilon \left\| \partial_t u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + B\alpha \varepsilon \left\| \partial_{xx}^2 u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
+ 5\varepsilon \left\| u_{\varepsilon,\alpha}^2 (t, \cdot) \partial_x u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + A\varepsilon^3 \left\| \partial_{xx}^3 u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
+ 5\alpha \varepsilon \left\| u_{\varepsilon,\alpha}^2 (t, \cdot) \partial_x u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + A\alpha \varepsilon^3 \left\| \partial_{xx}^2 u_{\varepsilon,\alpha}(t, \cdot) \partial_{xx}^2 u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
+ \frac{5}{3} \alpha^\frac{1}{2} \varepsilon \left\| u_{\varepsilon,\alpha}^2 (t, \cdot) \partial_x u_{\varepsilon,\alpha}(t, \cdot) \partial_{xx}^2 u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + A\alpha \varepsilon^3 \left\| \partial_{xx} u_{\varepsilon,\alpha}(t, \cdot) \partial_{xx}^2 u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
= 4A\varepsilon^2 \int_{\mathbb{R}} u_{\varepsilon,\alpha}^2 \partial_x u_{\varepsilon,\alpha} \partial_{xx}^2 u_{\varepsilon,\alpha} dx - 4B\varepsilon \int_{\mathbb{R}} u_{\varepsilon,\alpha}^2 \partial_x u_{\varepsilon,\alpha} \partial_t u_{\varepsilon,\alpha} dx \\
- 5\alpha \int_{\mathbb{R}} u_{\varepsilon,\alpha}^4 \partial_x u_{\varepsilon,\alpha} \partial_{xx}^2 u_{\varepsilon,\alpha} dx - 4\alpha \int_{\mathbb{R}} u_{\varepsilon,\alpha}^4 \partial_x u_{\varepsilon,\alpha} \partial_{xx}^2 u_{\varepsilon,\alpha} dx \\
- 2A\varepsilon^2 \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial_x u_{\varepsilon,\alpha} (\partial_{xx}^2 u_{\varepsilon,\alpha})^2 dx - 3B\alpha \varepsilon \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial_x u_{\varepsilon,\alpha} \partial_{xx}^2 u_{\varepsilon,\alpha} \partial_t u_{\varepsilon,\alpha} dx \\
+ B\alpha \varepsilon \int_{\mathbb{R}} u_{\varepsilon,\alpha}^2 \partial_{xx}^3 u_{\varepsilon,\alpha} \partial_t u_{\varepsilon,\alpha} dx + 20\alpha \varepsilon \int_{\mathbb{R}} u_{\varepsilon,\alpha}^2 (\partial_x u_{\varepsilon,\alpha})^4 dx.
\]

Since \(0 < \alpha, \varepsilon < 1\), due to (55), (59) and the Young inequality,
\[
4A\varepsilon^2 \int_{\mathbb{R}} u_{\varepsilon,\alpha}^2 \partial_x u_{\varepsilon,\alpha} \partial_{xx}^2 u_{\varepsilon,\alpha} dx \leq 4A\varepsilon^2 \int_{\mathbb{R}} u_{\varepsilon,\alpha}^2 |\partial_x u_{\varepsilon,\alpha}| |\partial_{xx}^2 u_{\varepsilon,\alpha}| dx \\
= 2 \int_{\mathbb{R}} |\varepsilon^\frac{1}{2} u_{\varepsilon,\alpha} \partial_x u_{\varepsilon,\alpha}| |2A\varepsilon^2 \partial_{xx}^2 u_{\varepsilon,\alpha}| dx \\
\leq \varepsilon \left\| u_{\varepsilon,\alpha}^2 (t, \cdot) \partial_x u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + 4A^2 \varepsilon^3 \left\| \partial_{xx}^2 u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})},
\]
\[-4B\varepsilon \int_{\mathbb{R}} u_{\varepsilon,\alpha}^2 \partial_x u_{\varepsilon,\alpha} \partial_t u_{\varepsilon,\alpha} \, dx \leq 4B\varepsilon \int_{\mathbb{R}} u_{\varepsilon,\alpha}^2 |\partial_x u_{\varepsilon,\alpha}| |\partial_t u_{\varepsilon,\alpha}| \, dx \]
\[= 2\varepsilon \int_{\mathbb{R}} \left| u_{\varepsilon,\alpha}^2 \partial_x u_{\varepsilon,\alpha} \right| 2B\partial_t u_{\varepsilon,\alpha} \, dx \]
\[\leq \varepsilon \left\| u_{\varepsilon,\alpha}(t,\cdot) \partial_x u_{\varepsilon,\alpha}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + 4B^2 \varepsilon \left\| \partial_t u_{\varepsilon,\alpha}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2, \]
\[-5\alpha \int_{\mathbb{R}} u_{\varepsilon,\alpha}^4 \partial_x u_{\varepsilon,\alpha} \partial_{x,x}^2 u_{\varepsilon,\alpha} \, dx \leq 5\alpha \int_{\mathbb{R}} u_{\varepsilon,\alpha}^4 |\partial_x u_{\varepsilon,\alpha}| |\partial_{x,x}^2 u_{\varepsilon,\alpha}| \, dx \]
\[\leq \alpha \int_{\mathbb{R}} \left| \frac{u_{\varepsilon,\alpha}^4 \partial_x u_{\varepsilon,\alpha}}{B\varepsilon^2} \right| 5B\varepsilon \frac{1}{B\varepsilon} |\partial_{x,x}^2 u_{\varepsilon,\alpha}| \, dx \]
\[\leq \frac{\alpha}{2B\varepsilon} \int_{\mathbb{R}} u_{\varepsilon,\alpha}^4 \left| \partial_x u_{\varepsilon,\alpha}(t,\cdot) \right|^2 + \frac{25B^2 \alpha \varepsilon}{2} \left\| \partial_{x,x}^2 u_{\varepsilon,\alpha}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \]
\[\leq \frac{\alpha}{2B\varepsilon} \int_{\mathbb{R}} u_{\varepsilon,\alpha}(t,\cdot) \partial_x u_{\varepsilon,\alpha}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{25B^2 \alpha \varepsilon}{2} \left\| \partial_{x,x}^2 u_{\varepsilon,\alpha}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \]
\[\leq C_0 B_\varepsilon \left\| u_{\varepsilon,\alpha}(t,\cdot) \partial_x u_{\varepsilon,\alpha}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{25B^2 \alpha \varepsilon}{2} \left\| \partial_{x,x}^2 u_{\varepsilon,\alpha}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \]
\[\leq C_0 D_\varepsilon \left\| u_{\varepsilon,\alpha}(t,\cdot) \partial_x u_{\varepsilon,\alpha}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{25B^2 \alpha \varepsilon}{2} \left\| \partial_{x,x}^2 u_{\varepsilon,\alpha}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \]
\[-4\alpha \int_{\mathbb{R}} u_{\varepsilon,\alpha}^6 \partial_x u_{\varepsilon,\alpha} \partial_{x,x}^2 u_{\varepsilon,\alpha} \, dx \leq 4\alpha \int_{\mathbb{R}} u_{\varepsilon,\alpha}^6 |\partial_x u_{\varepsilon,\alpha}| |\partial_{x,x}^2 u_{\varepsilon,\alpha}| \, dx \]
\[= 2\alpha \int_{\mathbb{R}} \left| \frac{2u_{\varepsilon,\alpha} \partial_x u_{\varepsilon,\alpha}}{\varepsilon^2} \right| \left| \varepsilon \frac{1}{2} u_{\varepsilon,\alpha} \partial_{x,x}^2 u_{\varepsilon,\alpha} \right| \, dx \]
\[\leq 4\alpha \int_{\mathbb{R}} \left( u_{\varepsilon,\alpha} \partial_x u_{\varepsilon,\alpha} \right)^2 + \alpha \varepsilon \left\| u_{\varepsilon,\alpha}(t,\cdot) \partial_{x,x}^2 u_{\varepsilon,\alpha}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \]
\[\leq 4\alpha \int_{\mathbb{R}} \left\| u_{\varepsilon,\alpha}(t,\cdot) \partial_x u_{\varepsilon,\alpha}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + \alpha \varepsilon \left\| u_{\varepsilon,\alpha}(t,\cdot) \partial_{x,x}^2 u_{\varepsilon,\alpha}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \]
\[-2A\alpha \varepsilon^2 \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial_x u_{\varepsilon,\alpha} \left( \partial_{x,x}^2 u_{\varepsilon,\alpha} \right)^2 \, dx \leq 2A\alpha \varepsilon^2 \int_{\mathbb{R}} |u_{\varepsilon,\alpha}| |\partial_x u_{\varepsilon,\alpha}| \left( \partial_{x,x}^2 u_{\varepsilon,\alpha} \right)^2 \, dx \]
\[= 2\alpha \int_{\mathbb{R}} \left| \frac{u_{\varepsilon,\alpha} \partial_x u_{\varepsilon,\alpha}}{\varepsilon^2} \right| A\varepsilon \partial_x u_{\varepsilon,\alpha} \partial_{x,x}^2 u_{\varepsilon,\alpha} \, dx \]
\[\leq A\alpha \varepsilon \int_{\mathbb{R}} \left( \partial_{x,x}^2 u_{\varepsilon,\alpha} \right)^2 + A^2 \alpha \varepsilon^3 \left\| \partial_x u_{\varepsilon,\alpha}(t,\cdot) \partial_{x,x}^2 u_{\varepsilon,\alpha}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \]
\[\leq \frac{\alpha \varepsilon}{2} \left\| u_{\varepsilon,\alpha}(t,\cdot) \partial_{x,x}^2 u_{\varepsilon,\alpha}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\alpha \varepsilon}{2} \left\| \partial_{x,x}^2 u_{\varepsilon,\alpha}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \]
\[+ A^2 \alpha \varepsilon \varepsilon \left\| \partial_x u_{\varepsilon,\alpha}(t,\cdot) \partial_{x,x}^2 u_{\varepsilon,\alpha}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \]
\[3B\alpha \varepsilon \int_{\mathbb{R}} u_{\varepsilon,\alpha} \partial_x u_{\varepsilon,\alpha} \partial_{x,x}^2 u_{\varepsilon,\alpha} \partial_t u_{\varepsilon,\alpha} \, dx \leq 3B\alpha \varepsilon \int_{\mathbb{R}} |u_{\varepsilon,\alpha}| \partial_x u_{\varepsilon,\alpha} \partial_{x,x}^2 u_{\varepsilon,\alpha} \partial_t u_{\varepsilon,\alpha} \, dx \]
It follows from (69) that

\[
\frac{d}{dt} \left( \frac{1}{6} \left\| u_{\varepsilon, a}(t, \cdot) \right\|_{L^6(\mathbb{R})}^6 + \frac{(A + B)\varepsilon^2}{2} \left\| \partial_{x} u_{\varepsilon, a}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\
\leq \frac{d}{dt} \left( \frac{(A + B)\alpha \varepsilon^2}{2} \left\| \partial_{x}^2 u_{\varepsilon, a}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\
+ \left( 10B \varepsilon \right) B \varepsilon \left\| \partial_{t} u_{\varepsilon, a}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + (2 - 25B) \frac{B \alpha \varepsilon}{2} \left\| \partial_{x}^2 u_{\varepsilon, a}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]

(70)
\[ + \left( 3 - \frac{C_0 D^4}{B} - C_0 D^4 \right) \varepsilon \left\| u_{\varepsilon,\alpha}^2(t, \cdot) \partial_{x} u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \]
\[ + (1 - 4A) A \varepsilon \left\| \partial_{xx}^2 u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{7 \alpha \varepsilon}{2} \left\| u_{\varepsilon,\alpha}(t, \cdot) \partial_{xx}^2 u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \]
\[ + (A - C_0 D^4) \alpha \varepsilon \left\| \partial_{xxx}^3 u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{5 \alpha \varepsilon}{3} \left\| u_{\varepsilon}^2(t, \cdot) \partial_{x} u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \]
\[ + (A - A^2 - C_0 D^{12}) \alpha \varepsilon \left\| \partial_{x} u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \]
\[ \leq \frac{\alpha \varepsilon}{2} \left\| \partial_{xx}^2 u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + C_0 \alpha \varepsilon \left\| \partial_{x} u_{\varepsilon,\alpha}(t, \cdot) \right\|^4_{L^4(\mathbb{R})}. \]

We search \( A, B, D \), such that
\[
1 - 10B > 0, \quad 2 - 25B > 0, \quad 3 - \frac{C_0 (1 + B) D^4}{B} > 0, \]
\[
1 - 4A > 0, \quad A - C_0 D^4 > 0, \quad \frac{A}{2} - A^2 > 0, \]
\[
\frac{A}{2} - C_0 D^{12} > 0,
\]
that is
\[
B < \frac{1}{10}, \quad B < \frac{2}{25}, \quad D < \sqrt[4]{\frac{3B}{C_0 (1 + B)}},
\]
\[
A < \frac{1}{4}, \quad D < \sqrt[4]{\frac{A}{C_0}}, \quad A < \frac{1}{2}, \quad (71)
\]
\[
D < \sqrt[12]{\frac{A}{2C_0}}.
\]

From the first inequality and the second one of (71), we have
\[
B < \frac{2}{25}, \quad (72)
\]
Hence, we choose
\[
B = \frac{1}{25}. \quad (73)
\]
It follows from (73) and the third inequality of (71) that
\[
D < \sqrt[4]{\frac{3}{26C_0}}. \quad (74)
\]
By the fourth inequality and the sixth one of (71), we have
\[
A < \frac{1}{4}. \quad (75)
\]
We choose
\[
A = \frac{1}{5}. \quad (76)
\]
Substituting (76) in the fifth inequality and in seventh one of (71), we obtain
\[
D < \sqrt[4]{\frac{1}{5C_0}}, \quad D < \sqrt[12]{\frac{1}{10C_0}}. \quad (77)
\]
Therefore, (74) and (77) give (56). Hence, by (56), (70) (73) and (76),
\[
\frac{d}{dt} \left( \frac{1}{6} \left\| u_{\varepsilon,\alpha}(t, \cdot) \right\|^6_{L^6(\mathbb{R})} + \frac{(A + B) \varepsilon^2}{2} \left\| \partial_{x} u_{\varepsilon,\alpha}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \right)
\]
\[
\frac{d}{dt} \left( \frac{(A + B)\alpha \varepsilon^2}{2} \| \partial^2_{xx} u_{\varepsilon, \alpha}(t, \cdot) \|^2_{L^2(\mathbb{R})} + \frac{B \alpha \varepsilon^2}{12} \| \partial_{x} u_{\varepsilon, \alpha}(t, \cdot) \|^4_{L^4(\mathbb{R})} \right) \\
+ K_1 \varepsilon \| \partial_{t} u_{\varepsilon, \alpha}(t, \cdot) \|^2_{L^2(\mathbb{R})} + K_2 \alpha \varepsilon \| \partial^2_{xx} u_{\varepsilon, \alpha}(t, \cdot) \|^2_{L^2(\mathbb{R})} \\
+ K_3 \varepsilon \| u^2_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + K_4 \varepsilon^3 \| \partial^2_{xx} u_{\varepsilon, \alpha}(t, \cdot) \|^2_{L^2(\mathbb{R})} \\
+ \frac{7 \alpha \varepsilon}{2} \| u^2_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + K_5 \alpha \varepsilon^3 \| \partial^3_{xxx} u_{\varepsilon, \alpha}(t, \cdot) \|^2_{L^2(\mathbb{R})} \\
+ \frac{5}{3} \alpha \varepsilon \| u^2_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + K_6 \alpha \varepsilon^3 \| \partial_{x} u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \\
\leq \frac{\alpha \varepsilon}{2} \| \partial^2_{xx} u_{\varepsilon, \alpha}(t, \cdot) \|^2_{L^2(\mathbb{R})} + C_0 \alpha \varepsilon \| \partial_{x} u_{\varepsilon, \alpha}(t, \cdot) \|^4_{L^4(\mathbb{R})}.
\]

for some \( K_1, K_2, K_3, K_4, K_5, K_6 > 0 \). An integration on (0, t), (49) and (57) give

\[
\frac{1}{6} \| u_{\varepsilon, \alpha}(t, \cdot) \|^6_{L^6(\mathbb{R})} + \frac{(A + B)\varepsilon^2}{2} \| \partial_{x} u_{\varepsilon, \alpha}(t, \cdot) \|^2_{L^2(\mathbb{R})} \\
+ \frac{(A + B)\alpha \varepsilon^2}{2} \| \partial^2_{xx} u_{\varepsilon, \alpha}(t, \cdot) \|^2_{L^2(\mathbb{R})} + \frac{B \alpha \varepsilon^2}{12} \| \partial_{x} u_{\varepsilon, \alpha}(t, \cdot) \|^4_{L^4(\mathbb{R})} \\
+ K_1 \varepsilon \int_0^t \| \partial_{t} u_{\varepsilon, \alpha}(s, \cdot) \|^2_{L^2(\mathbb{R})} ds + K_2 \alpha \varepsilon \int_0^t \| \partial^2_{xx} u_{\varepsilon, \alpha}(s, \cdot) \|^2_{L^2(\mathbb{R})} ds \\
+ K_3 \varepsilon \int_0^t \| u^2_{\varepsilon, \alpha}(s, \cdot) \|_{L^2(\mathbb{R})}^2 + K_4 \varepsilon^3 \int_0^t \| \partial^2_{xx} u_{\varepsilon, \alpha}(s, \cdot) \|^2_{L^2(\mathbb{R})} ds \\
+ \frac{7 \alpha \varepsilon}{2} \int_0^t \| u^2_{\varepsilon, \alpha}(s, \cdot) \|_{L^2(\mathbb{R})}^2 + K_5 \alpha \varepsilon^3 \int_0^t \| \partial^3_{xxx} u_{\varepsilon, \alpha}(s, \cdot) \|^2_{L^2(\mathbb{R})} ds \\
+ \frac{5}{3} \alpha \varepsilon \int_0^t \| u^2_{\varepsilon, \alpha}(s, \cdot) \|_{L^2(\mathbb{R})}^2 + K_6 \alpha \varepsilon^3 \int_0^t \| \partial_{x} u_{\varepsilon, \alpha}(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds \\
\leq C_0 + \frac{\alpha \varepsilon}{2} \int_0^t \| \partial^2_{xx} u_{\varepsilon, \alpha}(s, \cdot) \|^2_{L^2(\mathbb{R})} ds \\
+ C_0 \alpha \varepsilon \int_0^t \| \partial_{x} u_{\varepsilon, \alpha}(s, \cdot) \|^4_{L^4(\mathbb{R})} ds \leq C_0.
\]

As a consequence,

\[
\| u_{\varepsilon, \alpha}(t, \cdot) \|_{L^6(\mathbb{R})} \leq C_0, \\
\varepsilon \| \partial_{t} u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})} \leq C_0, \\
\alpha \varepsilon \| \partial^2_{xx} u_{\varepsilon, \alpha}(t, \cdot) \|_{L^2(\mathbb{R})} \leq C_0, \\
\alpha \varepsilon \| \partial_{x} u_{\varepsilon, \alpha}(t, \cdot) \|_{L^4(\mathbb{R})} \leq C_0, \\
\varepsilon \int_0^t \| \partial_{t} u_{\varepsilon, \alpha}(s, \cdot) \|^2_{L^2(\mathbb{R})} ds \leq C_0.
\]
Let us consider a compactly supported entropy–entropy flux pair \((\eta, q)\) for (5). Assume that (47), (49) and (50) hold. Then for any compactly supported entropy–entropy flux pair \((\eta, q)\) for (5), there exist two sequences \(\{\varepsilon_k\}_{k \in \mathbb{N}}, \{\alpha_k\}_{k \in \mathbb{N}}, \) with \(\varepsilon_k \to 0, \) and a limit function 

\[ u \in L^\infty((0,T); L^2(\mathbb{R}) \cap L^6(\mathbb{R})), \]

such that (51) and (52) hold.

**Proof.** Let us consider a compactly supported entropy–entropy flux pair \((\eta, q)\) for (5). Multiplying (48) by \(\eta'(u_{\varepsilon, \alpha})\), we have

\[
\begin{align*}
\partial_t \eta(u_{\varepsilon, \alpha}) + \partial_x q(u_{\varepsilon, \alpha}) &= \varepsilon \eta'(u_{\varepsilon, \alpha}) \partial_x^3 u_{\varepsilon, \alpha} - \alpha \varepsilon \eta'(u_{\varepsilon, \alpha}) \partial_x^3 u_{\varepsilon, \alpha} \\
& \quad + \alpha \eta'(u_{\varepsilon, \alpha}) \partial_x^3 u_{\varepsilon, \alpha} + 3 \alpha \eta'(u_{\varepsilon, \alpha}) u_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} \partial_x^3 u_{\varepsilon, \alpha} \\
& \quad + \alpha \eta'(u_{\varepsilon, \alpha}) u_{\varepsilon, \alpha} \partial_x^3 u_{\varepsilon, \alpha} + \alpha \partial_x^2 u_{\varepsilon, \alpha} (\partial_x u_{\varepsilon, \alpha})^2 \\
& = I_1, \varepsilon, \alpha + I_2, \varepsilon, \alpha + I_3, \varepsilon, \alpha + I_4, \varepsilon, \alpha + I_5, \varepsilon, \alpha \\
& \quad + I_6, \varepsilon, \alpha + I_7, \varepsilon, \alpha + I_8, \varepsilon, \alpha + I_9, \varepsilon, \alpha + I_{10}, \varepsilon, \alpha,
\end{align*}
\]

where

\[
\begin{align*}
I_1, \varepsilon, \alpha &= \partial_x (\varepsilon \eta'(u_{\varepsilon, \alpha}) \partial_x u_{\varepsilon, \alpha}), \\
I_2, \varepsilon, \alpha &= -\varepsilon \eta''(u_{\varepsilon, \alpha}) (\partial_x u_{\varepsilon, \alpha})^2, \\
I_3, \varepsilon, \alpha &= -\partial_x (\alpha \varepsilon \eta'(u_{\varepsilon, \alpha}) \partial_x^2 u_{\varepsilon, \alpha}), \\
I_4, \varepsilon, \alpha &= \alpha \varepsilon \eta''(u_{\varepsilon, \alpha}) \partial_x u_{\varepsilon, \alpha} \partial_x^3 u_{\varepsilon, \alpha}, \\
I_5, \varepsilon, \alpha &= \partial_x (\alpha \eta''(u_{\varepsilon, \alpha}) \partial_x^2 u_{\varepsilon, \alpha}), \\
I_6, \varepsilon, \alpha &= -\alpha \eta''(u_{\varepsilon, \alpha}) \partial_x u_{\varepsilon, \alpha} \partial_x^2 u_{\varepsilon, \alpha}, \\
I_7, \varepsilon, \alpha &= \partial_x (\alpha \eta''(u_{\varepsilon, \alpha}) u_{\varepsilon, \alpha}^2 \partial_x^2 u_{\varepsilon, \alpha}), \\
I_8, \varepsilon, \alpha &= -\alpha \eta''(u_{\varepsilon, \alpha}) u_{\varepsilon, \alpha}^2 \partial_x u_{\varepsilon, \alpha} \partial_x^2 u_{\varepsilon, \alpha},
\end{align*}
\]

for every \(0 \leq t \leq T. \) \(\square\)

To prove Theorem 3.1, Lemma 2.5 is needed.

We begin by proving the following result.

**Lemma 3.5.** Fix \(T > 0.\) Assume that (47), (49) and (50) hold. Then for any compactly supported entropy–entropy flux pair \((\eta, q)\) for (5), there exist two sequences \(\{\varepsilon_k\}_{k \in \mathbb{N}}, \{\alpha_k\}_{k \in \mathbb{N}}, \) with \(\varepsilon_k \to 0, \alpha_k \to 0,\) and a limit function 

\[ u \in L^\infty((0,T); L^2(\mathbb{R}) \cap L^6(\mathbb{R})), \]

such that (51) and (52) hold.

**Proof.** Let us consider a compactly supported entropy–entropy flux pair \((\eta, q)\) for (5). Multiplying (48) by \(\eta'(u_{\varepsilon, \alpha})\), we have

\[
\begin{align*}
\partial_t \eta(u_{\varepsilon, \alpha}) + \partial_x q(u_{\varepsilon, \alpha}) &= \varepsilon \eta'(u_{\varepsilon, \alpha}) \partial_x^3 u_{\varepsilon, \alpha} - \alpha \varepsilon \eta'(u_{\varepsilon, \alpha}) \partial_x^3 u_{\varepsilon, \alpha} \\
& \quad + \alpha \eta'(u_{\varepsilon, \alpha}) \partial_x^3 u_{\varepsilon, \alpha} + 3 \alpha \eta'(u_{\varepsilon, \alpha}) u_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} \partial_x^3 u_{\varepsilon, \alpha} \\
& \quad + \alpha \eta'(u_{\varepsilon, \alpha}) u_{\varepsilon, \alpha} \partial_x^3 u_{\varepsilon, \alpha} + \alpha \partial_x^2 u_{\varepsilon, \alpha} (\partial_x u_{\varepsilon, \alpha})^2 \\
& = I_1, \varepsilon, \alpha + I_2, \varepsilon, \alpha + I_3, \varepsilon, \alpha + I_4, \varepsilon, \alpha + I_5, \varepsilon, \alpha \\
& \quad + I_6, \varepsilon, \alpha + I_7, \varepsilon, \alpha + I_8, \varepsilon, \alpha + I_9, \varepsilon, \alpha + I_{10}, \varepsilon, \alpha,
\end{align*}
\]

where

\[
\begin{align*}
I_1, \varepsilon, \alpha &= \partial_x (\varepsilon \eta'(u_{\varepsilon, \alpha}) \partial_x u_{\varepsilon, \alpha}), \\
I_2, \varepsilon, \alpha &= -\varepsilon \eta''(u_{\varepsilon, \alpha}) (\partial_x u_{\varepsilon, \alpha})^2, \\
I_3, \varepsilon, \alpha &= -\partial_x (\alpha \varepsilon \eta'(u_{\varepsilon, \alpha}) \partial_x^2 u_{\varepsilon, \alpha}), \\
I_4, \varepsilon, \alpha &= \alpha \varepsilon \eta''(u_{\varepsilon, \alpha}) \partial_x u_{\varepsilon, \alpha} \partial_x^3 u_{\varepsilon, \alpha}, \\
I_5, \varepsilon, \alpha &= \partial_x (\alpha \eta''(u_{\varepsilon, \alpha}) \partial_x^2 u_{\varepsilon, \alpha}), \\
I_6, \varepsilon, \alpha &= -\alpha \eta''(u_{\varepsilon, \alpha}) \partial_x u_{\varepsilon, \alpha} \partial_x^2 u_{\varepsilon, \alpha}, \\
I_7, \varepsilon, \alpha &= \partial_x (\alpha \eta''(u_{\varepsilon, \alpha}) u_{\varepsilon, \alpha}^2 \partial_x^2 u_{\varepsilon, \alpha}), \\
I_8, \varepsilon, \alpha &= -\alpha \eta''(u_{\varepsilon, \alpha}) u_{\varepsilon, \alpha}^2 \partial_x u_{\varepsilon, \alpha} \partial_x^2 u_{\varepsilon, \alpha},
\end{align*}
\]

(78)
Fix $T > 0$. Arguing as in [2, Lemma 3.2], we have that
$$I_{1, \varepsilon, \alpha} \to 0 \text{ in } H^{-1}((0, T) \times \mathbb{R}) \text{ as } \alpha, \varepsilon \to 0,$$
we have that
$$\{I_{2, \varepsilon, \alpha}\}_{\varepsilon, \alpha > 0} \text{ is bounded in } L^1((0, T) \times \mathbb{R}).$$
We claim that
$$I_{3, \varepsilon, \alpha} \to 0 \text{ in } H^{-1}((0, T) \times \mathbb{R}), \text{ as } \alpha, \varepsilon \to 0.$$ 
Due to (50) and Lemma 3.4,
$$\|\alpha \varepsilon \eta' (u_{\varepsilon, \alpha}) \partial_{xxx} u_{\varepsilon, \alpha}\|_{L^2((0, T) \times \mathbb{R})}^2 \leq \frac{\alpha^2}{\varepsilon} \|\eta'\|_{L^{\infty}(\mathbb{R})} \|\partial^3_{xxx} u_{\varepsilon, \alpha}\|_{L^2((0, T) \times \mathbb{R})}^2 \leq \frac{\alpha}{\varepsilon} C_0 \|\eta'\|_{L^{\infty}(\mathbb{R})} \leq C_0 \|\eta'\|_{L^{\infty}(\mathbb{R})} \varepsilon^3 \to 0.$$ 
We have that
$$I_{4, \varepsilon, \alpha} \to 0 \text{ in } L^1((0, T) \times \mathbb{R}), \text{ as } \alpha, \varepsilon \to 0.$$ 
Thanks to (50), Lemmas 3.2, 3.4 and the Hölder inequality,
$$\|\alpha \varepsilon \eta'' (u_{\varepsilon, \alpha}) \partial_x u_{\varepsilon, \alpha} \partial^2_{xxx} u_{\varepsilon, \alpha}\|_{L^1((0, T) \times \mathbb{R})} \leq \frac{\alpha^2}{\varepsilon} \|\eta''\|_{L^{\infty}(\mathbb{R})} \int_0^T \|\partial_x u_{\varepsilon, \alpha}\| \|\partial^3_{xxx} u_{\varepsilon, \alpha}\| dt dx \leq \frac{\alpha^2}{\varepsilon} C_0 \|\eta''\|_{L^{\infty}(\mathbb{R})} \leq C_0 \|\eta''\|_{L^{\infty}(\mathbb{R})} \varepsilon^3 \to 0.$$ 
We get
$$I_{5, \varepsilon, \alpha} \to 0 \text{ in } H^{-1}((0, T) \times \mathbb{R}) \text{ as } \alpha, \varepsilon \to 0.$$ 
By (50) and Lemma 3.4,
$$\|\alpha \eta' (u_{\varepsilon, \alpha}) \partial^2_{tx} u_{\varepsilon, \alpha}\|_{L^2((0, T) \times \mathbb{R})}^2 \leq \frac{\alpha^2}{\varepsilon} \|\eta'\|_{L^{\infty}(\mathbb{R})} \|\partial^2_{tx} u_{\varepsilon, \alpha}\|_{L^2((0, T) \times \mathbb{R})}^2 \leq \frac{\alpha}{\varepsilon} C_0 \|\eta'\|_{L^{\infty}(\mathbb{R})} \leq C_0 \|\eta'\|_{L^{\infty}(\mathbb{R})} \varepsilon^7 \to 0.$$ 
We claim that
$$I_{6, \varepsilon, \alpha} \to 0 \text{ in } L^1((0, T) \times \mathbb{R}), \text{ as } \alpha, \varepsilon \to 0.$$ 
Thanks to (50), Lemmas 3.2, 3.4 and the Hölder inequality,
$$\|\alpha \eta'' (u_{\varepsilon, \alpha}) \partial_x u_{\varepsilon, \alpha} \partial^2_{tx} u_{\varepsilon, \alpha}\|_{L^1((0, T) \times \mathbb{R})} \leq \frac{\alpha}{\varepsilon} \|\eta''\|_{L^{\infty}(\mathbb{R})} \int_0^T \|\partial_x u_{\varepsilon, \alpha}\| \|\partial^2_{tx} u_{\varepsilon, \alpha}\| dt dx \leq \frac{\alpha}{\varepsilon} \|\eta''\|_{L^{\infty}(\mathbb{R})} \leq C_0 \|\eta''\|_{L^{\infty}(\mathbb{R})} \varepsilon^3 \to 0.$$
We show that

\[ I_{\gamma, \varepsilon, \alpha} \rightarrow 0 \quad \text{in} \quad H^{-1}((0,T) \times \mathbb{R}), \quad \text{as} \quad \alpha, \varepsilon \rightarrow 0. \]

Due to (59), (50) and Lemma 3.4,

\[
\| \alpha \eta'(u_{\varepsilon, \alpha}) u_{\varepsilon, \alpha}^2 \partial_x u_{\varepsilon, \alpha} \|_{L^2((0,T) \times \mathbb{R})}^2 \\
\leq \frac{\alpha^2 \varepsilon}{\varepsilon} \| \eta' \|_{L^\infty(\mathbb{R})} \int_0^T \int_{\mathbb{R}} u_{\varepsilon, \alpha}^4 \partial_x^2 u_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} \, dt \, dx \\
\leq \frac{\alpha^2 \varepsilon}{\varepsilon} \| \eta' \|_{L^\infty(\mathbb{R})} \| u_{\varepsilon, \alpha} \|_{L^\infty(\mathbb{R} \times \mathbb{R})} \| \partial_x u_{\varepsilon, \alpha} \|_{L^2((0,T) \times \mathbb{R})} \| \partial_x^2 u_{\varepsilon, \alpha} \|_{L^2((0,T) \times \mathbb{R})} \\
\leq \frac{C_0}{\varepsilon} \| \eta' \|_{L^\infty(\mathbb{R})} \frac{\alpha^2 \varepsilon}{\varepsilon} \leq C_0 \| \eta' \|_{L^\infty(\mathbb{R})} \varepsilon^3 \rightarrow 0.
\]

We get

\[ I_{8, \varepsilon, \alpha} \rightarrow 0 \quad \text{in} \quad L^1((0,T) \times \mathbb{R}), \quad \text{as} \quad \alpha, \varepsilon \rightarrow 0. \]

Due (50), Lemma 3.4 and the Hölder inequality,

\[
\| \alpha \eta''(u_{\varepsilon, \alpha}) u_{\varepsilon, \alpha}^2 \partial_x u_{\varepsilon, \alpha} \|_{L^1((0,T) \times \mathbb{R})} \\
\leq \frac{\alpha^2 \varepsilon}{\varepsilon} \| \eta'' \|_{L^\infty(\mathbb{R})} \int_0^T \int_{\mathbb{R}} u_{\varepsilon, \alpha}^2 \partial_x u_{\varepsilon, \alpha} \partial_x^2 u_{\varepsilon, \alpha} \, dt \, dx \\
\leq \frac{\alpha^2 \varepsilon}{\varepsilon} \| \eta'' \|_{L^\infty(\mathbb{R})} \| u_{\varepsilon, \alpha} \|_{L^2((0,T) \times \mathbb{R})} \| \partial_x u_{\varepsilon, \alpha} \|_{L^2((0,T) \times \mathbb{R})} \| \partial_x^2 u_{\varepsilon, \alpha} \|_{L^2((0,T) \times \mathbb{R})} \\
\leq \frac{\alpha}{\varepsilon} C_0 \| \eta'' \|_{L^\infty(\mathbb{R})} \leq C_0 \| \eta'' \|_{L^\infty(\mathbb{R})} \varepsilon \rightarrow 0.
\]

We claim that

\[ I_{9, \varepsilon, \alpha} \rightarrow 0 \quad \text{in} \quad L^1((0,T) \times \mathbb{R}), \quad \text{as} \quad \alpha, \varepsilon \rightarrow 0. \]

Thanks to (50), Lemmas 2.2, 3.4 and the Hölder inequality,

\[
\| \alpha \eta''(u_{\varepsilon, \alpha}) u_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} \partial_x^2 u_{\varepsilon, \alpha} \|_{L^1((0,T) \times \mathbb{R})} \\
\leq \frac{\alpha \varepsilon}{\varepsilon} \| \eta'' \|_{L^\infty(\mathbb{R})} \int_0^T \int_{\mathbb{R}} \partial_x^2 u_{\varepsilon, \alpha} \partial_x u_{\varepsilon, \alpha} \, dt \, dx \\
\leq \frac{\alpha \varepsilon}{\varepsilon} \| \eta'' \|_{L^\infty(\mathbb{R})} \| \partial_x u_{\varepsilon, \alpha} \|_{L^2((0,T) \times \mathbb{R})} \| \partial_x^2 u_{\varepsilon, \alpha} \|_{L^2((0,T) \times \mathbb{R})} \\
\leq \frac{\alpha^4}{\varepsilon} C_0 \| \eta'' \|_{L^\infty(\mathbb{R})} \leq C_0 \| \eta'' \|_{L^\infty(\mathbb{R})} \varepsilon \rightarrow 0.
\]

We have that

\[ \{ I_{10, \varepsilon, \alpha} \}_{\varepsilon, \alpha > 0} \quad \text{is bounded in} \quad L^1((0,T) \times \mathbb{R}). \]

Due to (50), Lemmas 3.2, 3.4 and the Hölder inequality,

\[
\| \alpha \frac{\varepsilon}{\varepsilon} \eta'(u_{\varepsilon, \alpha}) (\partial_x u_{\varepsilon, \alpha})^2 \partial_x^2 u_{\varepsilon, \alpha} \|_{L^1((0,T) \times \mathbb{R})} \\
\leq \frac{\alpha \varepsilon^2}{\varepsilon} \| \eta' \|_{L^\infty(\mathbb{R})} \int_0^T \int_{\mathbb{R}} \partial_x u_{\varepsilon, \alpha} \partial_x^2 u_{\varepsilon, \alpha} \, dt \, dx \\
\leq \frac{\alpha \varepsilon^2}{\varepsilon} \| \eta' \|_{L^\infty(\mathbb{R})} \| \partial_x u_{\varepsilon, \alpha} \|_{L^2((0,T) \times \mathbb{R})} \| \partial_x^2 u_{\varepsilon, \alpha} \|_{L^2((0,T) \times \mathbb{R})} \\
\leq \frac{\alpha^4}{\varepsilon} C_0 \| \eta' \|_{L^\infty(\mathbb{R})} \leq C_0.
\]
Due (50), Lemmas 3.2, 3.4, and the Hölder inequality, compactness of \([30]\).

Finally, we prove (52). Let \(\phi \in C^\infty(\mathbb{R}^2)\) be a test function with compact support. We have to prove that

\[
\int_0^\infty \int_\mathbb{R} \left( u \partial_t \phi + \frac{4u^3}{3} \partial_x \phi \right) \, dt \, dx + \int_\mathbb{R} u_0(x) \phi(0, x) \, dx = 0. \tag{79}
\]

We define

\[
u_{\varepsilon, k} := u_k. \tag{80}
\]

We have that

\[
\int_0^\infty \int_\mathbb{R} \left( u_k \partial_t \phi + \frac{u^3}{3} \partial_x \phi \right) \, dt \, dx + \int_0^\infty \int_\mathbb{R} u_0(x) \phi(0, x) \, dx \\
+ \varepsilon_k \int_0^\infty \int_\mathbb{R} u_k \partial_{xx} \phi \, dt \, dx - \alpha_k \int_0^\infty \int_\mathbb{R} u_k \partial_{xxx} \phi \, dt \, dx \\
+ 3\alpha_k \int_0^\infty \int_\mathbb{R} u_k \partial_x u_k \partial_{xx}^2 u_k \phi \, dt \, dx + \alpha_k \int_0^\infty \int_\mathbb{R} u_k^2 \partial_{xxx}^3 u_k \phi \, dt \, dx \\
- \frac{\alpha_k^4}{3} \int_0^\infty \int_\mathbb{R} (\partial_x u_k)^3 \partial_x \phi = 0.
\]

Observe that

\[
3\alpha_k \int_0^\infty \int_\mathbb{R} u_k \partial_x u_k \partial_{xx}^2 u_k \phi \, dt \, dx + \alpha_k \int_0^\infty \int_\mathbb{R} u_k^2 \partial_{xxx}^3 u_k \phi \, dt \, dx \\
= \alpha_k \int_0^\infty \int_\mathbb{R} u_k \partial_x u_k \partial_{xx}^2 u_k \phi \, dt \, dx - \alpha_k \int_0^\infty \int_\mathbb{R} u_k^2 \partial_{xx}^2 u_k \partial_x \phi \, dt \, dx. \tag{81}
\]

We prove that

\[
\alpha_k \int_0^\infty \int_\mathbb{R} u_k \partial_x u_k \partial_{xx}^2 u_k \phi \, dt \, dx \to 0. \tag{82}
\]

Due (50), Lemmas 3.2, 3.4, and the Hölder inequality,

\[
\alpha_k \int_0^\infty \int_\mathbb{R} |u_k \partial_x u_k \partial_{xx}^2 u_k \phi| \, dt \, dx \\
\leq \frac{\alpha_k \varepsilon_k}{\varepsilon_k} \| \phi \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \partial_x u_k \|_{L^2(\text{supp}(\phi))} \| u_k \partial_{xx}^2 u_k \|_{L^2(\text{supp}(\phi))} \\
\leq \frac{\alpha_k \varepsilon_k}{\varepsilon_k} \| \phi \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \partial_x u_k \|_{L^2((0, T) \times \mathbb{R})} \| u_k \partial_{xx}^2 u_k \|_{L^2((0, T) \times \mathbb{R})} \\
\leq C_0 \| \phi \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \frac{\alpha_k^4}{\varepsilon_k} \leq C_0 \| \phi \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \varepsilon_k^3. \tag{83}
\]

We show that

\[
- \alpha_k \int_0^\infty \int_\mathbb{R} u_k^2 \partial_{xx}^2 u_k \partial_x \phi \, dt \, dx \to 0. \tag{84}
\]

Thanks to (59), (50), Lemma 3.4 and the Hölder inequality,

\[
- \alpha_k \int_0^\infty \int_\mathbb{R} u_k^2 \partial_{xx}^2 u_k \partial_x \phi \, dt \, dx \\
\leq \alpha_k \int_0^\infty \int_\mathbb{R} u_k^2 \| \partial_{xx}^2 u_k \|_{L^2(\text{supp}(\partial_x \phi))} \| \partial_x \phi \|_{L^2(\text{supp}(\partial_x \phi))} \\
\leq \alpha_k \| u_k \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \partial_x \phi \|_{L^\infty(\text{supp}(\partial_x \phi))} \| \partial_{xx}^2 u_k \|_{L^2(\text{supp}(\partial_x \phi))}. \tag{85}
\]
We have that
\[ \|\partial_x \phi\|_{L^\infty((0,T) \times \mathbb{R})} \leq C_0 \epsilon^{\frac{1}{k}} \alpha \|\partial_x^2 u_k\|_{L^2((0,T) \times \mathbb{R})} \]
\[ \leq C_0 \epsilon^{\frac{1}{k}} \alpha \|\partial_x^2 u_k\|_{L^2((0,T) \times \mathbb{R})} \leq C_0 \|\partial_x \phi\|_{L^\infty((0,T) \times \mathbb{R})} \epsilon \to 0. \]

We have that
\[ -\frac{\alpha \epsilon^2}{3} \int_0^\infty \int_{\mathbb{R}} (\partial_x u_k)^3 \partial_x \phi \, dx \, dt \]
\[ \leq \frac{\alpha \epsilon^2}{3} \int_0^\infty \int_{\mathbb{R}} |u_k|^3 |\partial_x \phi| \, dx \, dt \]
\[ \leq \frac{\alpha \epsilon^2}{3} \|\partial_x \phi\|_{L^\infty(\text{supp}(\partial_x \phi))} \|\partial_x u_k\|_{L^2(\text{supp}(\partial_x \phi))} \|\partial_x u_k\|_{L^4(\text{supp}(\partial_x \phi))} \]
\[ \leq \frac{\alpha \epsilon^2}{3} \|\partial_x \phi\|_{L^\infty((0,T) \times \mathbb{R})} \|\partial_x u_k\|_{L^2((0,T) \times \mathbb{R})} \|\partial_x u_k\|_{L^4((0,T) \times \mathbb{R})} \]
\[ \leq C_0 \epsilon^{\frac{1}{k}} \alpha \|\partial_x \phi\|_{L^\infty((0,T) \times \mathbb{R})} \leq C_0 \|\partial_x \phi\|_{L^\infty((0,T) \times \mathbb{R})} \epsilon \to 0. \]

(79) follows from (19), (49), (81), (82), (83) and (84).

Following [24], we prove the following result.

**Lemma 3.6.** Fix $T > 0$. Assume that (47), (49) and (53) hold. Then for any compactly supported entropy–entropy flux pair $(\eta, q)$ for (5), there exist two sequences \( \{\epsilon_k\}_{k \in \mathbb{N}}, \{\beta_k\}_{k \in \mathbb{N}} \), with $\epsilon_k, \beta_k \to 0$, and a limit function
\[ u \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}) \cap L^6(\mathbb{R})), \]

such that (51) holds and
\[ u \text{ is the unique entropy solution of (5).} \]

**Proof.** Let $T > 0$ and consider a compactly supported entropy–entropy flux pair $(\eta, q)$ for (5). Multiplying (48) by $\eta'(u_{\epsilon, \alpha})$, we have
\[ \partial_t \eta(u_{\epsilon, \alpha}) + \partial_x q(u_{\epsilon, \alpha}) = \epsilon \eta'(u_{\epsilon, \alpha}) \partial^2_{xx} u_{\epsilon, \alpha} - \alpha \epsilon \eta'(u_{\epsilon, \alpha}) \partial^3_{xxx} u_{\epsilon, \alpha} \]
\[ + \alpha \eta'(u_{\epsilon, \alpha}) \partial^2_{xx} u_{\epsilon, \alpha} + 3 \alpha \eta'(u_{\epsilon, \alpha}) u_{\epsilon, \alpha} \partial u_{\epsilon, \alpha} \partial^2_{xx} u_{\epsilon, \alpha} \]
\[ + \alpha \eta'(u_{\epsilon, \alpha}) u_{\epsilon, \alpha}^2 \partial^2_{xx} u_{\epsilon, \alpha} \]
\[ = I_{1, \epsilon, \alpha} + I_{2, \epsilon, \alpha} + I_{3, \epsilon, \alpha} + I_{4, \epsilon, \alpha} + I_{5, \epsilon, \alpha} + I_{6, \epsilon, \alpha} + I_{7, \epsilon, \alpha} + I_{8, \epsilon, \alpha} + I_{9, \epsilon, \alpha} + I_{10, \epsilon, \alpha} \]

where $I_{1, \epsilon, \alpha} + I_{2, \epsilon, \alpha} + I_{3, \epsilon, \alpha} + I_{4, \epsilon, \alpha} + I_{5, \epsilon, \alpha} + I_{6, \epsilon, \alpha} + I_{7, \epsilon, \alpha} + I_{8, \epsilon, \alpha} + I_{9, \epsilon, \alpha} + I_{10, \epsilon, \alpha}$ are defined in (78). Arguing as in Lemma 3.5
\[ I_{1, \epsilon, \alpha} \to 0 \text{ in } H^{-1}((0,T) \times \mathbb{R}) \text{ as } \alpha, \epsilon \to 0, \]
\[ \{I_{2, \epsilon, \alpha}\}_{\epsilon, \alpha > 0} \text{ is bounded in } L^1((0,T) \times \mathbb{R}), \]
\[ I_{3, \epsilon, \alpha} \to 0 \text{ in } H^{-1}((0,T) \times \mathbb{R}) \text{ as } \alpha, \epsilon \to 0, \]
\[ I_{4, \epsilon, \alpha} \to 0 \text{ in } L^1((0,T) \times \mathbb{R}) \text{ as } \alpha, \epsilon \to 0, \]
\[ I_{5, \epsilon, \alpha} \to 0 \text{ in } H^{-1}((0,T) \times \mathbb{R}) \text{ as } \alpha, \epsilon \to 0, \]
\[ I_{6, \epsilon, \alpha} \to 0 \text{ in } L^1((0,T) \times \mathbb{R}) \text{ as } \alpha, \epsilon \to 0, \]
We conclude by proving that \( I_{7, \varepsilon, \alpha} \to 0 \) in \( H^{-1}((0, T) \times \mathbb{R}) \) as \( \alpha, \varepsilon \to 0 \),
\( I_{8, \varepsilon, \alpha} \to 0 \) in \( L^1((0, T) \times \mathbb{R}) \) as \( \alpha, \varepsilon \to 0 \),
\( I_{9, \varepsilon, \alpha} \to 0 \) in \( L^1((0, T) \times \mathbb{R}) \) as \( \alpha, \varepsilon \to 0 \).

We prove that
\[ I_{10, \varepsilon, \alpha} \to 0 \quad \text{in} \quad L^1((0, T) \times \mathbb{R}), \quad \text{as} \quad \alpha, \varepsilon \to 0. \]

Thanks to (50), Lemmas 3.2, 3.4 and the Hölder inequality,
\[
\left\| \int_{0}^{1} \frac{t}{\varepsilon} \eta'(u_{\varepsilon, \alpha})(\partial_x u_{\varepsilon, \alpha})^2 \partial_{xx} u_{\varepsilon, \alpha} \right\|_{L^1((0, T) \times \mathbb{R})} \\
\leq \frac{\alpha^2 \varepsilon^2}{\varepsilon} \left\| \eta' \right\|_{L^\infty(\mathbb{R})} \int_{0}^{T} \left\| \partial_x u_{\varepsilon, \alpha} \right\| \left\| \partial_x u_{\varepsilon, \alpha} \partial_{xx} u_{\varepsilon, \alpha} \right\| dtdx \\
\leq \frac{\alpha^2 \varepsilon^2}{\varepsilon} \left\| \eta' \right\|_{L^\infty(\mathbb{R})} \left\| \partial_x u_{\varepsilon, \alpha} \right\|_{L^2((0, T) \times \mathbb{R})} \left\| \partial_x u_{\varepsilon, \alpha} \partial_{xx} u_{\varepsilon, \alpha} \right\|_{L^2((0, T) \times \mathbb{R})} \\
\leq \frac{\alpha^2}{\varepsilon} C_0 \left\| \eta'' \right\|_{L^\infty(\mathbb{R})} \to 0.
\]

We conclude by proving that \( u \) is the entropy solution of (5). Let us consider a compactly supported entropy–entropy flux pair \((\eta, q)\), and \( \phi \in C_c^2((0, \infty) \times \mathbb{R}) \) a non-negative test function. Fix \( T > 0 \). We have to prove that
\[
\int_{0}^{\infty} \int_{\mathbb{R}} (\partial_t \eta(u) + \partial_x q(u)) \phi dtdx \leq 0.
\]
(87)

Thanks to (80), we have
\[
\int_{0}^{\infty} \int_{\mathbb{R}} (\partial_t \eta(u_k) + \partial_x q(u_k)) \phi dtdx \\
= \varepsilon_k \int_{0}^{\infty} \int_{\mathbb{R}} \partial_x (\eta'(u_k) \partial_x u_k) \phi dx - \varepsilon_k \int_{0}^{\infty} \int_{\mathbb{R}} \eta''(u_k) (\partial_x u_k)^2 \phi dtdx \\
- \varepsilon_k \alpha_k \int_{0}^{\infty} \int_{\mathbb{R}} \partial_x (\eta''(u_k) \partial_x u_k \partial_{xx} u_k) \phi dtdx \\
+ \varepsilon_k \alpha_k \int_{0}^{\infty} \int_{\mathbb{R}} \eta''(u_k) \partial_x u_k \partial_{xx} u_k \phi dtdx \\
+ \alpha_k \int_{0}^{\infty} \int_{\mathbb{R}} \partial_x (\eta'(u_k) \partial_x u_k \partial_{xx} u_k) \phi dtdx \\
- \alpha_k \int_{0}^{\infty} \int_{\mathbb{R}} \partial_x u_k \partial_x u_k \partial_{xx} u_k \phi dtdx \\
+ \frac{3\alpha_k}{2} \int_{0}^{\infty} \int_{\mathbb{R}} \partial_x (\eta'(u_k) \partial_x u_k \partial_{xx} u_k) \phi dtdx \\
- \frac{3\alpha_k}{2} \int_{0}^{\infty} \int_{\mathbb{R}} \eta''(u_k) \partial_x u_k \partial_{xx} u_k \phi dtdx \\
+ \alpha_k \int_{0}^{\infty} \int_{\mathbb{R}} \partial_x (\eta'(u_k) \partial_x u_k \partial_{xx} u_k) \phi dtdx \\
- \alpha_k \int_{0}^{\infty} \int_{\mathbb{R}} \eta''(u_k) \partial_x u_k \partial_{xx} u_k \phi dtdx \\
+ \alpha_k \int_{0}^{\infty} \int_{\mathbb{R}} \partial_x (\eta'(u_k) \partial_x u_k \partial_{xx} u_k) \phi dtdx.
\]
\[ \leq -\varepsilon_k \int_0^\infty \eta'(u_k) \partial_x u_k \partial_x \phi \, dx \\
+ \varepsilon_k \alpha_k \int_0^\infty \int_\mathbb{R} \eta'(u_k) \partial_{xxx} u_k \partial_x \phi \, dt \, dx \\
+ \varepsilon_k \alpha_k \int_0^\infty \int_\mathbb{R} \eta''(u_k) \partial_x u_k \partial_{xxx} u_k \partial_x \phi \, dt \, dx \\
- \alpha_k \int_0^\infty \int_\mathbb{R} \eta'(u_k) \partial_{xx}^2 u_k \partial_x \phi \, dt \, dx \\
- \alpha_k \int_0^\infty \int_\mathbb{R} \eta''(u_k) \partial_x u_k \partial_{xx}^2 \phi u_k \, dt \, dx \\
- \frac{3\alpha_k}{2} \int_0^\infty \int_\mathbb{R} \eta'(u_k) u_k (\partial_x u_k)^2 \partial_x \phi \, dt \, dx \\
- \frac{3\alpha_k}{2} \int_0^\infty \int_\mathbb{R} \eta''(u_k) u_k (\partial_x u_k)^3 \phi \, dt \, dx \\
- \frac{\alpha_k}{2} \int_0^\infty \int_\mathbb{R} \eta'(u_k) u_k^2 \partial_{xx}^2 u_k \partial_x \phi \, dt \, dx \\
- \frac{\alpha_k}{2} \int_0^\infty \int_\mathbb{R} \eta''(u_k) u_k^2 \partial_{xx}^2 u_k \phi \, dt \, dx \\
+ \frac{\alpha_k}{2} \int_0^\infty \int_\mathbb{R} \eta'(u_k) (\partial_x u_k)^2 \partial_{xx}^2 u_k \phi \, dt \, dx \]

\[ \leq \varepsilon_k \| \eta' \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \partial_x u_k \|_{L^2(\text{supp} (\partial_x \phi))} \| \partial_x \phi \|_{L^2(\text{supp} (\partial_x \phi))} \\
+ \varepsilon_k \alpha_k \| \eta' \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \partial_{xxx} u_k \|_{L^2(\text{supp} (\partial_x \phi))} \| \partial_x \phi \|_{L^2(\text{supp} (\partial_x \phi))} \\
+ \varepsilon_k \alpha_k \| \eta' \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \phi \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \partial_x u_k \partial_{xxx} u_k \|_{L^1(\text{supp} (\phi))} \\
+ \alpha_k \| \eta' \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \phi \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \partial_{xx}^2 u_k \partial_x \phi \|_{L^1(\text{supp} (\phi))} \\
+ \frac{3\alpha_k}{2} \| \eta' \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \partial_x \phi \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| u_k (\partial_x u_k)^2 \|_{L^1(\text{supp} (\partial_x \phi))} \\
+ \frac{3\alpha_k}{2} \| \eta'' \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \phi \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| u_k (\partial_x u_k)^3 \|_{L^2(\text{supp} (\phi))} \\
+ \alpha_k \| \eta'' \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \phi \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| u_k^2 \partial_{xx} u_k \partial_x \phi \|_{L^2(\text{supp} (\partial_x \phi))} \\
+ \frac{\alpha_k}{2} \| \eta'' \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \phi \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| (\partial_x u_k)^2 \partial_{xx} u_k \|_{L^1(\text{supp} (\phi))} \]

\[ \leq \varepsilon_k \| \eta' \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \partial_x u_k \|_{L^2((0,T) \times \mathbb{R})} \| \partial_x \phi \|_{L^2((0,T) \times \mathbb{R})} \\
+ \varepsilon_k \alpha_k \| \eta' \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \partial_{xxx} u_k \|_{L^2((0,T) \times \mathbb{R})} \| \partial_x \phi \|_{L^2((0,T) \times \mathbb{R})} \\
+ \varepsilon_k \alpha_k \| \eta' \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \phi \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \partial_x u_k \partial_{xxx} u_k \|_{L^1((0,T) \times \mathbb{R})} \\
+ \alpha_k \| \eta' \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \phi \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \partial_{xx}^2 u_k \partial_x \phi \|_{L^2((0,T) \times \mathbb{R})} \\
+ \alpha_k \| \eta'' \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \phi \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \partial_x u_k \partial_{xx}^2 u_k \|_{L^1((0,T) \times \mathbb{R})} \\
+ \frac{3\alpha_k}{2} \| \eta' \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| \partial_x \phi \|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \| u_k (\partial_x u_k)^2 \|_{L^1((0,T) \times \mathbb{R})} \]
Due to (53) and Lemmas 3.2, 3.3,

\[ + \frac{3\alpha_k}{2} \|\eta''\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|\phi\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|u_k(\partial_x u_k)^3\|_{L^1((0,T) \times \mathbb{R})} \]

\[ + \alpha_k \|\eta''\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|u_k^2 \partial_x^2 u_k\|_{L^2((0,T) \times \mathbb{R})} \|\partial_x \phi\|_{L^2((0,T) \times \mathbb{R})} \]

\[ + \alpha_k \|\eta''\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|\phi\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|u_k^2 \partial_x u_k \partial_x^2 u_k\|_{L^1((0,T) \times \mathbb{R})} \]

\[ + \alpha_k^3 \varepsilon_k \|\eta''\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|\phi\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|\phi\|_{L^\infty((0,T) \times \mathbb{R})} \|\partial_x u_k^2 \partial_x^2 u_k\|_{L^1((0,T) \times \mathbb{R})}. \]

Arguing as in [4, Theorem 1.1], we have

\[ \varepsilon_k \|\eta''\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|\partial_x u_k\|_{L^2((0,T) \times \mathbb{R})} \|\partial_x \phi\|_{L^2((0,T) \times \mathbb{R})} \rightarrow 0, \]

\[ \varepsilon L \alpha_k \|\eta''\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|\partial_x^2 u_k\|_{L^2((0,T) \times \mathbb{R})} \|\partial_x \phi\|_{L^2((0,T) \times \mathbb{R})} \rightarrow 0, \]

\[ \varepsilon L \alpha_k \|\eta''\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|\partial_x u_k \partial_x^2 u_k\|_{L^1((0,T) \times \mathbb{R})} \rightarrow 0, \] (88)

\[ \alpha_k \|\eta''\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|\partial_x u_k\|_{L^2((0,T) \times \mathbb{R})} \|\partial_x^2 u_k\|_{L^2((0,T) \times \mathbb{R})} \rightarrow 0, \]

\[ \alpha_k \|\eta''\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|\partial_x u_k \partial_x^2 u_k\|_{L^1((0,T) \times \mathbb{R})} \rightarrow 0. \]

We show that

\[ \frac{3\alpha_k}{2} \|\eta''\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|\partial_x \phi\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|u_k(\partial_x u_k)^2\|_{L^1((0,T) \times \mathbb{R})} \rightarrow 0. \] (89)

Due to (53) and Lemmas 3.2, 3.3,

\[ \frac{3\alpha_k}{2} \|\eta''\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|\partial_x \phi\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|u_k(\partial_x u_k)^2\|_{L^1((0,T) \times \mathbb{R})} \]

\[ \leq \frac{3\alpha_k}{2} \|\eta''\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|\partial_x \phi\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|u_k\|_{L^\infty((0,T) \times \mathbb{R})} \|\partial_x u_k\|^2_{L^2((0,T) \times \mathbb{R})} \]

\[ \leq C_0 \alpha_k^3 \|\eta''\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|\partial_x^2 u_k\|_{L^2((0,T) \times \mathbb{R})} \|\partial_x u_k\|^2_{L^1((0,T) \times \mathbb{R})} \]

\[ \leq C_0 \|\eta''\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|\partial_x \phi\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \varepsilon^6 \rightarrow 0. \]

We get

\[ \frac{3\alpha_k}{2} \|\eta''\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|\phi\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|u_k(\partial_x u_k)^3\|_{L^1((0,T) \times \mathbb{R})} \rightarrow 0. \] (90)

Due to (53), Lemmas 3.2, 3.3 and the Hölder inequality,

\[ \frac{3\alpha_k}{2} \|\eta''\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|\partial_x \phi\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|u_k(\partial_x u_k)^3\|_{L^1((0,T) \times \mathbb{R})} \]

\[ \leq \frac{3\alpha_k}{2} \|\eta''\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|\partial_x \phi\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \times \]

\[ \times \|u_k\|_{L^\infty((0,T) \times \mathbb{R})} \|\partial_x u_k\|_{L^2((0,T) \times \mathbb{R})} \|\partial_x u_k\|^2_{L^1((0,T) \times \mathbb{R})} \]

\[ \leq C_0 \alpha_k^3 \|\eta''\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \times \]

\[ \times \|\partial_x \phi\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|\partial_x u_k\|_{L^2((0,T) \times \mathbb{R})} \|\partial_x^2 u_k\|_{L^2((0,T) \times \mathbb{R})} \]

\[ \leq C_0 \|\eta''\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|\partial_x \phi\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \varepsilon^6 \rightarrow 0. \]

We have that

\[ \alpha_k \|\eta''\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|u_k^2 \partial_x^2 u_k\|_{L^2((0,T) \times \mathbb{R})} \|\partial_x \phi\|_{L^2((0,T) \times \mathbb{R})} \rightarrow 0. \] (91)

Thanks to (59), (53) and Lemma 3.4

\[ \alpha_k \|\eta''\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})} \|u_k^2 \partial_x^2 u_k\|_{L^2((0,T) \times \mathbb{R})} \|\partial_x \phi\|_{L^2((0,T) \times \mathbb{R})} \]

\[ \rightarrow 0. \]
Due to (53), Lemma 3.4 and the Hölder inequality,
\[
\alpha_k \eta'' \| \phi \|_{L^\infty((R^+ \times R)} \left\| \partial_x \phi \right\|_{L^2((0,T) \times R)} \leq C_0 \eta'' \| \phi \|_{L^\infty((R^+ \times R)} \left\| \partial_x \phi \right\|_{L^2((0,T) \times R)},
\]
We obtain that
\[
\alpha_k \eta'' \| \phi \|_{L^\infty((R^+ \times R)} \left\| \partial_x \phi \right\|_{L^2((0,T) \times R)} \leq C_0 \eta'' \| \phi \|_{L^\infty((R^+ \times R)} \leq \frac{\alpha_k}{\epsilon_k} \to 0.
\]

Proof Theorem 3.1. Theorem 3.1 follows from Lemmas 3.5 and 3.6.
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E-mail address: giuseppemaria.coclite@poliba.it
E-mail address: lorenzo.diruvo77@gmail.com