The NLS limit for bosons in a quantum waveguide∗

Johannes von Keler and Stefan Teufel

Mathematisches Institut, Universität Tübingen, Germany.

stefan.teufel@uni-tuebingen.de

March 10, 2016

Abstract

We consider a system of $N$ bosons confined to a thin waveguide, i.e. to a region of space within an $\varepsilon$-tube around a curve in $\mathbb{R}^3$. We show that when taking simultaneously the NLS limit $N \to \infty$ and the limit of strong confinement $\varepsilon \to 0$, the time-evolution of such a system starting in a state close to a Bose-Einstein condensate is approximately captured by a non-linear Schrödinger equation in one dimension. The strength of the non-linearity in this Gross-Pitaevskii type equation depends on the shape of the cross-section of the waveguide, while the “bending” and the “twisting” of the waveguide contribute potential terms. Our analysis is based on an approach to mean-field limits developed by Pickl [24].

1 Introduction

We consider a system of $N$ identical weakly interacting bosons confined to a thin waveguide, i.e. to a region $\mathcal{T}_\varepsilon \subset \mathbb{R}^3$ contained in an $\varepsilon$-neighborhood of a curve $c : \mathbb{R} \to \mathbb{R}^3$. The Hamiltonian of the system is

$$H_{\mathcal{T}_\varepsilon}(t) = \sum_{i=1}^{N} (-\Delta z_i + V(t, z_i)) + \sum_{i \leq j} a \mu^3 w \left( \frac{z_i - z_j}{\mu} \right),$$

where $z_j \in \mathbb{R}^3$ is the coordinate of the $j$th particle, $\Delta z_j$ the Laplacian on $\mathcal{T}_\varepsilon$ with Dirichlet boundary conditions, $V$ a possibly time-dependent external potential and $w$ a positive pair interaction potential. The coupling $a := \varepsilon^2 / N$ is chosen such that for $N$-particle states supported along a fixed part of the curve the interaction energy per particle remains of order one for all $N \in \mathbb{N}$ and $\varepsilon > 0$. For $\beta > 0$ the effective range of the interaction $\mu := (\varepsilon^2 / N)^{\beta}$ goes to zero for $N \to \infty$ and $\varepsilon \to 0$ and $\mu^{-3} w(\cdot / \mu)$ converges to a point interaction. We consider in the following only $\beta \in (0, 1/3)$, the so called mean-field regime where $a/\mu^3$ still goes to zero. For recent papers containing concise reviews of the mean-field and NLS limit for bose gases we refer to [18, 22]. For a detailed discussion of bose condensation in general and also the problem of dimensional reduction we refer to [19].

∗This work was supported by the German Science Foundation within the GRK 1838 “Spectral theory and dynamics of quantum systems”.

arXiv:1510.03243v2 [math-ph] 9 Mar 2016.
Let us give a somewhat informal account of our result before we discuss the details. Assume that the initial state $\psi_{N,\varepsilon} \in L^2_+ (\mathcal{T}_\varepsilon^N) := \bigotimes_{\text{sym}}^N L^2 (\mathcal{T}_\varepsilon)$ has a one-particle density matrix $\gamma_1$, i.e., the operator with kernel

$$\gamma_1(z, z') := \int \psi_{N,\varepsilon}^*(z, z_2, \ldots, z_N) \overline{\psi}_{N,\varepsilon}^*(z', z_2, \ldots, z_N) \, dz_2 \ldots dz_N,$$

(2)

that is asymptotically close to a projection $p = |\varphi\rangle\langle\varphi|$ onto a single particle state $\varphi = \Phi_0 \chi \in L^2(\mathcal{T}_\varepsilon)$, where $\Phi_0$ is the wavefunction along the curve and $\chi$ is the “ground state” in the confined direction. Then we show that all $M$-particle density matrices $\gamma_M(t)$ of the solution $\psi_{N,\varepsilon}(t)$ of the Schrödinger equation

$$i \frac{d}{dt} \psi_{N,\varepsilon}(t) = H_{T_\varepsilon}(t) \psi_{N,\varepsilon}(t)$$

are asymptotically close to $|\varphi(t)\rangle\langle\varphi(t)|^{\otimes M}$, where $\varphi(t) = \Phi(t) \chi$ with $\Phi(t)$ the solution of the one-dimensional non-linear Schrödinger equation

$$i \partial_t \Phi(t, x) = \left(-\frac{\partial^2}{\partial x^2} + V_{\text{geom}}(x) + V(t, x, 0) + b|\Phi(t, x)|^2\right) \Phi(t, x) \quad \text{with} \quad \Phi(0) = \Phi_0. \quad (3)$$

The strength $b$ of the nonlinearity depends on the details of the asymptotic limit. We distinguish two regimes: In the case of moderate confinement the width $\varepsilon$ of the waveguide shrinks slower than the range $\mu$ of the interaction and $b = \int_{\Omega_t} \chi(y)^4 \, d^2y \cdot \int_{\mathbb{R}^3} w(r) \, dr$, where $\Omega_t$ is the cross section of the waveguide and $\chi$ the ground state of the $2d$-Dirichlet Laplacian on $\Omega_t$. In the case of strong confinement the width $\varepsilon$ of the waveguide shrinks faster than the range $\mu$ of the interaction and $b = 0$. The geometric potential $V_{\text{geom}}(x)$ depends on the geometry of the waveguide and is the sum of two parts. The curvature $\kappa(x)$ of the curve contributes a negative potential $-\kappa(x)^2/4$, while the twisting of the cross-section relative to the curve contributes a positive potential. Note that quasi one-dimensional Bose-Einstein condensates in non-trivial geometric structures have been realized experimentally [11, 14] and that the transport and manipulation of condensates in waveguides is a highly promising experimental technique, see e.g. the review [9].

The rigorous derivation of the non-linear Gross-Pitaevskii equation from the underlying linear many-body Schrödinger equation has been a very active topic in mathematics during the last decade, however, almost exclusively without the confinement to a waveguide. Then the Gross-Pitaevskii equation [8] is still an equation on $\mathbb{R}^3$. The first rigorous and complete derivation in $\mathbb{R}^3$ is due to Erdős, Schlein and Yau [7]. Their proof is based on the BBGKY hierarchy, a system of coupled equations for all $M$-particle density matrices $\gamma_M(t), \, M = 1, \ldots, N$. Independently Adami, Golse and Teta solved the problem in one dimension [1]. Shortly after, Pickl developed an alternative approach [24] that turned out very flexible concerning time-dependent external potentials [25], non-positive interactions [26], and singular interactions [15]. Yet another approach based on Bogoliubov transformations and coherent states on Fock space was developed for the most difficult case $\beta = 1$ in [4]. Recently also corrections to the mean-field dynamics were established in [12, 23]. There are also several lecture notes reviewing the different approaches to the NLS-limit, e.g. [31, 10, 4, 29]. For our purpose the approach of Pickl turned out fruitful and our proof follows his general strategy and uses his formalism. However, since the NLS limit in a geometrically nontrivial waveguide required also crucial modifications, our paper is fully self-contained.
Also the problem of deriving lower dimensional effective equations for strongly confined bose gases has been considered before. In [2] the authors start with the Gross-Pitaevskii equation in dimension \( n + d \) confined to a \( n \)-dimensional plane by a strong harmonic potential and derive an effective NLS in dimension \( n \). In [21] the reduction of the Gross-Pitaevskii equation in dimension two to an \( \varepsilon \)-neighbourhood of a curve is considered. In both cases this corresponds to first taking the mean field limit and then the limit of strong confinement. However, we will see that the two limits do not commute and thus, that a direct derivation of the Gross-Pitaevskii equation in lower dimension from the \( N \)-particle Schrödinger evolution in higher dimension is of interest. This was done for a gas confined to a plane in \( \mathbb{R}^3 \) in [5], and for a gas confined to a straight line in [6] using the BBGKY-approach of [7].

2 Main result

In order to explain our result in full detail we need to start with the construction of the waveguide \( T_\varepsilon \). Consider a smooth curve \( c : \mathbb{R} \to \mathbb{R}^3 \) parametrized by arc-length, i.e. with \( \|c'(x)\|_{\mathbb{R}^3} = 1 \). Along the curve we define a frame by picking an orthonormal frame \((\tau(0), e_1(0), e_2(0))\) at \( c(0) \) with \( \tau(0) = c'(0) \) tangent to the curve and then defining \( (\tau(x), e_1(x), e_2(x)) \) by parallel transport along the curve, i.e. by solving

\[
\begin{pmatrix}
\tau' \\
e_1' \\
e_2'
\end{pmatrix} =
\begin{pmatrix}
0 & \kappa_1 & \kappa_2 \\
-\kappa_1 & 0 & 0 \\
-\kappa_2 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\tau \\
e_1 \\
e_2
\end{pmatrix}
\]

with the components of the mean curvature vector \( \kappa_j : \mathbb{R} \to \mathbb{R} \) \((j = 1, 2)\) given by

\[
\kappa_j(x) := \langle \tau'(x), e_j(x) \rangle_{\mathbb{R}^3} = \langle c''(x), e_j(x) \rangle_{\mathbb{R}^3}.
\]

Let the cross-section \( \Omega_\ell \subset \mathbb{R}^2 \) of the waveguide be open and bounded and let \( \theta : \mathbb{R} \to \mathbb{R} \) be a smooth function that defines the twisting of the cross-section relative to the parallel frame. In order to define the thin waveguide it is convenient to introduce the following maps separately. Denote the scaling map by

\[
D_\varepsilon : \mathbb{R}^3 \to \mathbb{R}^3, \quad r = (x, y) \mapsto (x, \varepsilon y) =: r^\varepsilon,
\]

the twisting map by

\[
T_\theta : \mathbb{R}^3 \to \mathbb{R}^3, \quad (x, y) \mapsto (x, T_\theta(x)y), \quad \text{where } T_\theta(x) = \begin{pmatrix}
\cos \theta(x) & -\sin \theta(x) \\
\sin \theta(x) & \cos \theta(x)
\end{pmatrix},
\]

and the embedding map by

\[
f : \mathbb{R}^3 \to \mathbb{R}^3, \quad r = (x, y_1, y_2) \mapsto f(r) = c(x) + y_1 e_1(x) + y_2 e_2(x).
\]

The waveguide is now defined by first scaling, then twisting and finally embedding the set \( \Omega := \mathbb{R} \times \Omega_\ell \subset \mathbb{R}^3 \) into a neighbourhood of \( c(\mathbb{R}) \). For \( \varepsilon \) small enough, the map

\[
f_\varepsilon : \Omega := \mathbb{R} \times \Omega_\ell \to \mathbb{R}^3, \quad r \mapsto f_\varepsilon(r) := f \circ T_\theta \circ D_\varepsilon(r)
\]

is, by Assumption A1, a diffeomorphism onto its range

\[
T_\varepsilon := f_\varepsilon(\Omega) \subset \mathbb{R}^3,
\]
which defines the region in space accessible to the particles, i.e. the waveguide. Now the evolution of an $N$-particle system in a waveguide is given by the Hamiltonian $\mathbf{1}$, which acts on $L^2(\mathcal{T}_\varepsilon)^{\otimes N} \cong L^2(\mathcal{T}_\varepsilon^N)$ with Dirichlet boundary conditions.

However, for the formulation and the derivation of our result it is more convenient to always work on the fixed, $\varepsilon$-independent product-domain $\Omega = \mathbb{R} \times \Omega_t$ instead of the tube $\mathcal{T}_\varepsilon$. This is achieved by the natural unitary transformation. For $\varepsilon$ small enough the map $f_\varepsilon$ is a diffeomorphism and therefore the map

$$U_\varepsilon : L^2(\mathcal{T}_\varepsilon) \to L^2(\Omega), \quad \psi \mapsto (U_\varepsilon \psi)(r) := \sqrt{\det Df_\varepsilon(r)} \psi(f_\varepsilon(r)) =: \sqrt{\rho_\varepsilon(r)} \psi(f_\varepsilon(r))$$

is unitary. Using $(U_\varepsilon)^{\otimes N}$ we can unitarily map the waveguide Hamiltonian $H_{\mathcal{T}_\varepsilon}(t)$ in $\mathbf{1}$ to

$$H(t) := (U_\varepsilon)^{\otimes N} H_{\mathcal{T}_\varepsilon}(t)(U_\varepsilon^*)^{\otimes N} + \sum_{i=1}^N \frac{1}{2} V^\perp(y_i)$$

where

$$= \sum_{i=1}^N \left( -(U_\varepsilon \Delta U_\varepsilon^*)_{z_i} + \frac{1}{\varepsilon^2} V^\perp(y_i) + V(t, f_\varepsilon(r_i)) \right) + \frac{1}{\mu^3} w \left( \frac{f_\varepsilon(r_i) - f_\varepsilon(r_j)}{\mu} \right),$$

with $\mu := \frac{\varepsilon^2}{N}$ and $\varepsilon$-dependent coupling $a$ is given by $a = \varepsilon^2/N$ and the effective range of the interaction by $\mu = (\varepsilon^2/N)^\beta$.

We will consider simultaneously the mean-field limit $N \to \infty$ and the limit of strong confinement $\varepsilon \to 0$ for the time-dependent Schrödinger equation with Hamiltonian $H(t)$ on the Dirichlet domain $D \subset H^2(\Omega^N) \cap H_0^1(\Omega^N)$. Recall that the effective coupling $a$ is given by $a = \varepsilon^2/N$ and the effective range of the interaction by $\mu = (\varepsilon^2/N)^\beta$.

Compared to the standard $N$-particle Schrödinger operator we thus have in $\mathbf{1}$ the shrinking domain and the strongly confining potential $V^\perp$, a pair interaction that is no longer exactly a function of the separation $r_i - r_j$ of two particles, and a modified kinetic energy operator.

**Lemma 2.1.** The Laplacian in the new coordinates has the form

$$U_\varepsilon \Delta U_\varepsilon^* = - (\partial_x + \theta'(x) L)^2 - \frac{1}{\varepsilon^2} \Delta_y - V_{\text{bend}}(r) - \varepsilon S^\varepsilon,$$

where

$$L = y_1 \partial_{y_2} - y_2 \partial_{y_1},$$

$$V_{\text{bend}}(r) = \frac{-\kappa(x)^2}{4 \rho_\varepsilon(r)^2} - \varepsilon \frac{T_{\theta(x)} y \cdot \kappa(x)}{2 \rho_\varepsilon(r)^3} - \varepsilon^2 \frac{5(T_{\theta(x)} y \cdot \kappa'(x))^2}{4 \rho_\varepsilon(r)^4},$$

$$S^\varepsilon = (\partial_x + \theta'(x) L) s^\varepsilon(r) (\partial_x + \theta'(x) L),$$

$$\rho_\varepsilon(r) = 1 - \varepsilon T_{\theta(x)} y \cdot \kappa(x), \quad \text{and} \quad s^\varepsilon(r) = \frac{\rho_\varepsilon^2(r) - 1}{\varepsilon \rho_\varepsilon^2(r)}.$$
corresponding expression with \( \partial_x \) instead of \( \partial_x + \theta L \). Now the rotation by the angle \( \theta(x) \) in the \( y \)-plane is implemented on \( L^2(\mathbb{R}^3) \) by the operator \( R(\theta(x)) = e^{\partial(y_1 \partial y_2 - y_2 \partial y_1)} \), such that
\[
R(\theta(x))^* \partial_x R(\theta(x)) = \partial_x + \theta' L.
\]

Before stating our main result we give a list of assumptions.

**A1 Waveguide**: Let \( \Omega_f \subset \mathbb{R}^2 \) be open and bounded. Let \( c : \mathbb{R} \to \mathbb{R}^3 \) be injective and six times continuously differentiable with all derivatives bounded, i.e. \( c \in C^6_b(\mathbb{R}, \mathbb{R}^3) \), and such that \( \|c'(x)\|_{\mathbb{R}^3} \equiv 1 \). To avoid overlap of different parts of the waveguide injectivity is not sufficient and we assume that there are constants \( c_1, c_2 > 0 \) such that
\[
\|c(x_1) - c(x_2)\|_{\mathbb{R}^3} \geq \min\{c_1 |x_1 - x_2|, c_2\}.
\]

Finally let \( \theta : \mathbb{R} \to \mathbb{R} \) satisfy \( \theta \in C_0^5(\mathbb{R}) \).

**A2 Interaction**: Let the interaction potential \( w \) be a non-negative, radially symmetric function such that \( w(r) = \tilde{w}(|r|^2) \) for a function \( \tilde{w} \in C^2(\mathbb{R}) \) with support in \((-1,1)\).

If the waveguide is straight and untwisted, i.e. if \( f = T_b = \text{id} \), then we only assume that \( w \) is a non-negative function in \( L^2(\mathbb{R}^3; d^3r) \cap L^1(\mathbb{R}^3; (1 + |r|) d^3r) \).

**A3 External potentials**: Let the external single particle potential \( V : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \) for each fixed \( t \in \mathbb{R} \) be bounded and four times continuously differentiable with bounded derivatives, \( V(t, \cdot) \in C^4_b(\mathbb{R}^3) \). Moreover assume that the map \( \mathbb{R} \to L^\infty(\Omega), t \mapsto V(t, \cdot) \) is differentiable and that \( \tilde{V}(t, \cdot) \in C^4_b(\mathbb{R}^3) \).

Let the confining potential \( V^\perp : \Omega_f \to \mathbb{R} \) be relatively bounded with respect to the Dirichlet Laplacian on \( \Omega_f \) with relative bound smaller than one.

**Remark 1.** (a) Note that for geometrically non-trivial waveguides we will have to Taylor expand the interaction \( w \) up to second order, hence condition A2. Otherwise the much weaker condition formulated for straight and untwisted waveguides suffices. Note also that any radially symmetric function can be written uniquely in the form \( w(r) = \tilde{w}(|r|^2) \) and that the regularity we need for the Taylor expansion is most conveniently formulated in terms of \( \tilde{w} \).

(b) The high regularity requirements for the waveguide in A1 and the external potential in A3 are only needed to ensure the existence of global solutions of the NLS equation \eqref{eq:nlse} that remain bounded in \( H^2(\mathbb{R}) \).

Let \( \psi^{N, \varepsilon}(t) \) be the solution to the time-dependent \( N \)-particle Schrödinger equation
\[
i \frac{d}{dt} \psi^{N, \varepsilon}(t) = H(t) \psi^{N, \varepsilon}(t)
\] (5)

with the Hamiltonian \( H(t) \) defined in \( \mathbb{H} \) and \( \psi^{N, \varepsilon}(0) \in D(H(t)) \equiv H^2(\Omega^N) \cap H^1_0(\Omega^N) \).

In order to study simultaneously the mean-field limit \( N \to \infty \) and the limit of strong confinement \( \varepsilon \to 0 \), we consider families of initial data \( \psi^{N, \varepsilon}(0) \) along sequences \( (N_n, \varepsilon_n) \to (\infty, 0) \).
Definition 2.1. For $\beta \in (0, \frac{1}{3}]$ we call a sequence $(N_n, \varepsilon_n)$ in $\mathbb{N} \times (0, 1]$ admissible, if
\[
\lim_{n \to \infty} (N_n, \varepsilon_n) = (\infty, 0) \quad \text{and} \quad \lim_{n \to \infty} (\varepsilon_n)_{N_n}^{\beta} = 0 \quad \text{for} \quad \mu_n := \left(\frac{\varepsilon_n}{N_n}\right)^{\beta}. \tag{6}
\]
We say that the sequence $(N_n, \varepsilon_n)$ is moderately confining, if, in addition,
\[
\lim_{n \to \infty} \frac{\mu_n}{\varepsilon_n} = 0,
\]
i.e. if the effective range $\mu$ of the interaction shrinks faster than the width $\varepsilon$ of the waveguide.
We say that the sequence $(N_n, \varepsilon_n)$ is strongly confining, if instead
\[
\lim_{n \to \infty} \frac{\varepsilon_n}{\mu_n} = 0,
\]
i.e. if the width of the waveguide is small even on the scale of the interaction.

Note that the admissibility condition in (6) requires that the width $\varepsilon$ of the waveguide cannot shrink too slowly compared to the range of the interaction $\mu$. This is a technical requirement that simplifies the proof considerably. It assures that the energy gap between $E_0$ and the first excited state in the normal direction, which is of order $\frac{1}{\varepsilon^2}$, grows sufficiently quickly so that transitions into excited states in the normal direction become negligible at leading order. In the following we will be concerned almost exclusively with the case of moderate confinement, where the effective one dimensional equation is nonlinear. The analysis of the strongly confining case turns out to be much simpler.

Before we can formulate our precise assumptions on the family of initial states, we need to introduce the one-particle energies. For $\psi \in D(H)$ the “renormalised energy per particle” is
\[
E^\psi(t) := \frac{1}{N} \langle \psi, H(t) \psi \rangle_{L^2(\Omega^N)} - \frac{E_0}{\varepsilon^2},
\]
and for $\Phi \in H^2(\mathbb{R})$ let the “effective energy per particle” be
\[
E^\Phi(t) := \left\langle \Phi, \left( -\frac{\partial^2}{\partial x^2} - \frac{\kappa(x)^2}{4} + |\theta'(x)|^2 \|L\chi\|_2^2 + V(t, x, 0) + \frac{b}{2} |\Phi|_2^2 \right) \Phi \right\rangle_{L^2(\mathbb{R})}. \tag{7}
\]
Recall that $\chi$ is the ground state wave function of $-\Delta_y + V^\perp(y)$ on $\Omega_f$ with Dirichlet boundary conditions and $E_0$ the corresponding ground state eigenvalue. As with $L^2_+(\mathcal{T}_\varepsilon)$, we also denote the symmetric subspace of $L^2(\Omega^N)$ by $L^2_+(\Omega^N) := \bigotimes_{\text{sym}}^N L^2(\Omega)$.

**A4 Initial data:** Let the family of initial data $\psi^{N,\varepsilon}(0) \in D(H) \cap L^2_+(\Omega^N)$, $\|\psi^{N,\varepsilon}(0)\|_2 = 1$, be close to a condensate with single particle wave function $\varphi_0 = \Phi_0 \chi$ for some $\Phi_0 \in H^2(\mathbb{R})$ in the following sense: for some admissible sequence $(N, \varepsilon) \to (\infty, 0)$ it holds that
\[
\lim_{(N, \varepsilon) \to (\infty, 0)} \text{Tr} L^2(\Omega) \left| \gamma^{N,\varepsilon}(0) - |\varphi_0\rangle \langle \varphi_0| \right| = 0,
\]
where $\gamma^{N,\varepsilon}(0)$ is the one particle density matrix of $\psi^{N,\varepsilon}(0)$, cf. (2). In addition we assume that also the energy per particle converges,
\[
\lim_{(N, \varepsilon) \to (\infty, 0)} \left| E^{\psi^{N,\varepsilon}}(0) - E^{\Phi_0}(0) \right| = 0.
\]
Finally, let $\Phi(t)$ be the corresponding solution of the effective nonlinear Schrödinger equation

$$\begin{align*}
\frac{\partial}{\partial t} \Phi(t) &= \left( -\frac{\partial^2}{\partial x^2} - \frac{\kappa(x)^2}{4} + |\theta'(x)|^2 \|L\chi\|^2 + V(t, x, 0) + b|\Phi(t)|^2 \right) \Phi(t) \\
&=: \dot{h} \Phi(t)
\end{align*}$$

where

$$b := \begin{cases} 
\int_{\Omega} |\chi(y)|^4 \, d^2 y \cdot \int_{\mathbb{R}^3} w(r) \, d^3 r & \text{in the case of moderate confinement,} \\
0 & \text{in the case of strong confinement.}
\end{cases}$$

The unique existence and properties of solutions to (5) and (8) are well known and briefly discussed in Appendix A.

**Theorem 1.** Let the waveguide satisfy assumption A1 and let the potentials satisfy assumptions A2 and A3. For $\beta \in (0, 1)$ let $\psi^{N, \varepsilon}(0)$ be a family of initial data satisfying A4. Let $\psi^{N, \varepsilon}(t)$ be the solution of the N-particle Schrödinger equation (5) with initial datum $\psi^{N, \varepsilon}(0)$ and $\gamma^{N, \varepsilon}_M(t)$ its M-particle reduced density matrix. Let $\Phi(t)$ be the solution of the effective equation (8) with initial datum $\Phi(0)$.

Then for any $t \in \mathbb{R}$ and any $M \in \mathbb{N}$

$$\lim_{(N, \varepsilon) \to (\infty, 0)} \text{Tr} \left| \gamma^{N, \varepsilon}_M(t) - |\Phi(t)\rangle\langle\Phi(t)|^{\otimes M} \right| = 0,$$

and

$$\lim_{(N, \varepsilon) \to (\infty, 0)} \left| E^{\psi^{N, \varepsilon}}(t) - E^{\Phi(t)}(t) \right| = 0$$

where the limits are taken along the sequence from A4.

**Remark 2.** (a) In Assumption A4 we assume that the initial state is close to a complete Bose-Einstein condensate. To show that the ground state of a bose gas is actually of this form is in itself an important and difficult problem. For a straight wave guide and the case $\beta = 1$ this was shown in [20], see also [19] for a detailed review and [30]. The analysis of ground states in geometrically non-trivial wave guides is, as far as we know, an open problem. For the latest results for $\beta \in (0, 1)$, but without strong confinement, we refer to [18].

(b) The assumption in A2 that the interaction $w$ is non-negative seems to be crucial to our proof, although it is used only once in the proof of the energy estimate of Lemma 4.7. The results of [6] suggest, however, that also our result should hold for interactions with a certain negative part.

(c) The negative part $-\kappa(x)^2/4$ of the geometric potential stemming from the curvature $\kappa(x)$ of the curve is often called the bending potential, while the positive part $|\theta'(x)|^2\|L\chi\|^2$ is called the twisting potential. Both appear in exactly the same form also for non-interacting particles in a waveguide, as they originate just from the transformation of the Laplacian in Lemma 2.1. See also [16] for a review in the one-particle case.
(d) One could also consider a waveguide with a cross-section that varies along the curve, e.g. having constrictions or thickenings. But then \( E_0 = E_0(x) \) would be a function of \( x \) and an effective potential of size \( \frac{E_0(x)}{\varepsilon^2} \) would appear in the effective equation. As a consequence also the kinetic energy in the \( x \)-direction would be of order \( \frac{1}{\varepsilon^2} \), i.e. \( \| \Phi \|_{H^1(\mathbb{R})}^2 = O(\frac{1}{\varepsilon^2}) \).

It is conceivable that a similar result to Theorem 1 holds also in this setting of large tangential energies. However, this is a much more difficult problem, since transitions into excited normal modes will be energetically possible. Using adiabatic theory, this problem is treated in the single-particle case in \[34, 17, 13\].

(e) Another interesting modification of the setup is the confinement only by potentials, without the Dirichlet boundary. Also this would introduce additional technical complications, since in this case the map \( f \) is no longer a global diffeomorphism and has to be cut off, c.f. \[34\].

(f) Let us briefly comment on the main differences of our result compared to the work of Chen and Holmer [6]. While our focus is on geometrically non-trivial wave guides, the authors of [6] consider confinement by a harmonic potential of constant shape to a straight line. However, their main focus are attractive pair interactions, more precisely pair potentials with \( \int_{\mathbb{R}^3} w(r) dr \leq 0 \), a situation which is excluded in our result. On the other hand, at least in the case of a straight wave guide, our approach needs much less regularity for \( w \) and can incorporate external time-dependent potentials. Finally, our proof yields also convergence rates, which, as far as we understand, is not the case for [6]. As explained below, we refrain from stating these rates because they are quite complicated and most likely far from optimal.

(g) In [20] the authors exhibit five different scaling regimes with different effective energy functionals for the ground state energy. Note that a direct comparison with our two regimes is not sensible for two reasons: First we assume \( \beta \in (0, \frac{1}{3}) \) while in [20] the Gross-Pitaevskii scaling \( \beta = 1 \) is considered. As a consequence, in [20] the scattering length, i.e. the range of the interaction \( w \), is always small compared to the small diameter \( \varepsilon \) of the wave guide. The situation \( \varepsilon/\mu \to 0 \) (what we called strong confinement) does not occur for \( \beta = 1 \). Secondly, [20] is specifically concerned with the ground state energy, where some terms in the energy functional can become negligible or can take a specific form depending on details of the ground state.

Acknowledgements

We thank Steffen Gilg, Stefan Haag, Christian Hainzl, Jonas Lampart, Sören Petrat, Peter Pickl, Guido Schneider, and Christof Sparber for helpful discussions. The support by the German Science Foundation (DFG) within the GRK 1838 “Spectral theory and dynamics of quantum systems” is gratefully acknowledged.

Ethical Statement

Funding: This work was funded by the German Science Foundation (DFG) within the GRK 1838. Conflict of Interest: The authors declare that they have no conflict of interest.
3 Structure of the proof and the main argument

In the proof we will not directly control the difference $\text{Tr} \left| \gamma_M^{N,\varepsilon}(t) - |\varphi(t)\rangle\langle\varphi(t)| \otimes M \right|$, but use a functional $\alpha(\psi^{N,\varepsilon}(t), \varphi(t))$ introduced by Pickl \cite{24, 15, 27} to measure the “distance” between $\psi^{N,\varepsilon}$ and $\varphi$. For this measure of distance our proof yields also rates of convergence, which could be translated into rates of convergence also for $\text{Tr} \left| \gamma_M^{N,\varepsilon}(t) - |\varphi(t)\rangle\langle\varphi(t)| \otimes M \right|$. However, since these rates are presumably far from optimal, we refrain from stating them explicitly.

The functional $\alpha$ is constructed from the following projections in the $N$-particle Hilbert space.

**Definition 3.1.** Let $p$ be an orthogonal projection in the one-particle space $L^2(\Omega)$.

For $i \in \{1, \ldots, N\}$ define on $L^2(\Omega) \otimes N$ the projection operators

$$p_i := \underbrace{1 \otimes \cdots \otimes 1}_\text{i-1 times} \otimes p \otimes \underbrace{1 \otimes \cdots \otimes 1}_\text{N-i times} \quad \text{and} \quad q_i := 1 - p_i.$$  

For $0 \leq k \leq N$ let

$$P_k := \left( q_1 \cdots q_k p_{k+1} \cdots p_N \right)_{\text{sym}} := \sum_{J \subseteq \{1, \ldots, N\}} \prod_{j \in J} q_j \prod_{j \notin J} p_j.$$  

For $k < 0$ and $k > N$ we set $P_k = 0$.

We will use the many-body projections $P_k$ exclusively for $p = |\varphi\rangle\langle\varphi|$, the orthogonal projection onto the subspace spanned by the condensate state $\varphi \in L^2(\Omega)$ with $\|\varphi\|_{L^2(\Omega)} = 1$. However, a number of simple algebraic relations, like

$$\sum_{k=0}^N P_k = 1, \quad \sum_{i=1}^N q_i P_k = k P_k,$$  

hold independently of the special choice for $p$ and will turn out very useful in the analysis of the mean field limit. The first identity in \ref{9} follows from the fact that $q_i + p_i = 1$. For the second identity note that together with the first identity we have

$$\sum_{i=1}^N q_i = \sum_{i=1}^N q_i \sum_{k'=0}^N P_{k'} = \sum_{k'=0}^N \sum_{i=1}^N q_i P_{k'} = \sum_{k'=0}^N k' P_{k'}.$$  

Projecting with $P_k$ yields the second identity, since $P_k P_{k'} = \delta_{k,k'} P_k$.

**Definition 3.2.** For any function $f : \mathbb{N}_0 \to \mathbb{R}$ define the bounded linear operator

$$\hat{f} : L^2(\Omega^N) \to L^2(\Omega^N), \quad \psi \mapsto \hat{f}\psi := \sum_{k=0}^N f(k) P_k \psi$$  

and the functional $\alpha_f : L^2(\Omega^N) \times L^2(\Omega) \to \mathbb{R}$

$$\alpha_f(\psi, \varphi) := \left\langle \psi, \hat{f}\psi \right\rangle_{L^2(\Omega^N)} = \sum_{k=0}^N f(k) \left\langle \psi, P_k \psi \right\rangle_{L^2(\Omega^N)}.$$  

9
The heuristic idea behind this definition is the following. The operator $P_k$ projects onto the subspace of $L^2(\Omega^N)$ of those states, where exactly $k$ out of the $N$ particles are not condensed into $\varphi$. Components of $\psi \in L^2(\Omega^N)$ with $k$ particles outside the condensate are weighted by $f(k)$ in $\alpha_f(\psi, \varphi)$. In order to obtain a useful measure of distance between $\psi$ and the condensate $\varphi^\otimes_N$, the function $f$ should thus be increasing and $f(0)$ should be (close to) zero. For $n(k) := \sqrt{k/N}$ it is easily seen that the functional $\alpha_{n^2}$ is a good measure for condensation: Using the shorthand

$$\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L^2(\Omega^N)},$$

we find for any symmetric $\psi \in L^2(\Omega^N)$

$$\alpha_{n^2}(\psi, \varphi) = \sum_{k=0}^N \frac{k}{N} \langle \psi, P_k \psi \rangle \overset{\text{symmetry}}{=} \sum_{k=0}^N \frac{1}{N} \sum_{i=1}^{N} \langle \psi, q_i P_k \psi \rangle$$

$$= \sum_{k=0}^N \langle \psi, q_1 P_k \psi \rangle = \langle \psi, q_1 \psi \rangle = \|q_1 \psi\|^2.$$ 

(10)

And in general we have the following equivalences.

Lemma 3.1. Let $\psi^N \in L^2_+ (\Omega^N)$ be a sequence of normalised $N$-particle wave functions and let $\gamma^N_M$ be the sequence of corresponding $M$-particle density matrices for some fixed $M \in \mathbb{N}$. Let $\varphi \in L^2(\Omega)$ be normalised. Then the following statements are equivalent:

(i) $\lim_{N \to \infty} \alpha_n(\psi^N, \varphi) = 0$ for some $a > 0$
(ii) $\lim_{N \to \infty} \alpha_n(\psi^N, \varphi) = 0$ for any $a > 0$
(iii) $\lim_{N \to \infty} \|\gamma^N_M - |\varphi\rangle \langle \varphi|^\otimes_M\| = 0$ for all $M \in \mathbb{N}$
(iv) $\lim_{N \to \infty} \text{Tr} [\gamma^N_M - |\varphi\rangle \langle \varphi|^\otimes_M] = 0$ for all $M \in \mathbb{N}$
(v) $\lim_{N \to \infty} \text{Tr} [\gamma^N_1 - |\varphi\rangle \langle \varphi|] = 0$

The proof of this lemma collects different statements somewhat scattered in the literature, c.f. [27, 15]. Since the claim is at the basis of our result and since the proof is short and simple, we give it at the end of Subsection 4.1 for the convenience of the reader.

In the proof of our main theorem we will work with the functional $\alpha_m$, where

$$m(k) := \begin{cases} n(k) & \text{for } k \geq N^{1-2\xi} \\ \frac{1}{2} (N^{-1+\xi} k + N^{-\xi}) & \text{else} \end{cases}$$

(11)

for some $0 < \xi < \frac{1}{2}$ to be specified below. Since $n(k) \leq m(k) \leq \max(n(k), N^{-\xi})$ holds for all $k \in \mathbb{N}_0$, convergence of $\alpha_m$ to zero is equivalent to convergence of $\alpha_n$ to zero and thus to all cases in Lemma 3.1. We will use the shorthand

$$\alpha_m(t) := \alpha_m(\psi^{N,\xi}(t), \varphi(t))$$

when we evaluate the functional $\alpha_m$ on the solutions to the time-dependent equations. Finally, the quantity that we can actually control in the proof is

$$\alpha_\xi(t) := \alpha_m(t) + \left| E^{\psi^{N,\xi}(t)}(t) - E^{\varphi(t)}(t) \right|.$$ 

(12)
We will now state two key propositions and then give the proof of Theorem \ref{thm:main}. The simple strategy is to show bounds for the time-derivative of $\alpha_\varepsilon$ and then use Grönwall’s inequality. With the expression from Lemma \ref{lem:Laplacian} for the Laplacian in the adapted coordinates we find that

$$\begin{align*}
H(t) &= \sum_{i=1}^{N} \left( -(\varepsilon \Delta U^-)_{r_i} + \frac{1}{\varepsilon^2} V_{\perp}(y_i) + V(t, f_\varepsilon(r_i)) \right) + a \sum_{i<j} \frac{1}{\mu^3} w \left( \frac{f_\varepsilon(r_i) - f_\varepsilon(r_j)}{\mu} \right) \\
&= \sum_{i=1}^{N} \left( -\frac{\partial^2}{\partial x_i^2} - \left( \theta'(x_i)L_i \right)^2 - \frac{1}{\varepsilon^2} \Delta y_i + \frac{1}{\varepsilon^2} V_{\perp}(y_i) + V(t, r_i^\varepsilon) - \frac{\kappa(x_i)^2}{4} + R_i^{(1)} \right) \\
&\quad + \frac{1}{N-1} \sum_{i<j} w_{ij}^{\varepsilon,\beta,N}
\end{align*}$$

with

$$R_i^{(1)} := -\partial_x \theta'(x_i) L_i - \theta'(x_i)L_i \partial_{x_i} + \left( V_{\text{bend}}(r_i) + \frac{\kappa(x_i)^2}{4} \right) - \varepsilon S_i^\varepsilon$$

and

$$w_{ij}^{\varepsilon,\beta,N}(r_1, \ldots, r_N) := (N-1) \frac{a}{\mu^3} w \left( \frac{f_\varepsilon(r_i) - f_\varepsilon(r_j)}{\mu} \right).$$

**Proposition 3.2.** Let the assumptions of Theorem \ref{thm:main} hold and let $\alpha_\varepsilon(t)$ be given by \eqref{eq:alpha}. Then the time-derivative of $\alpha_\varepsilon(t)$ is bounded by

$$\left| \frac{d}{dt} \alpha_\varepsilon(t) \right| \leq 2|I(t)| + |II(t)| + 2|III(t)| + |IV(t)|$$

with

\begin{align*}
I(t) &:= N \left\langle \psi_{\varepsilon,N}(t), p_1 p_2 \left[ w_{12}^{\varepsilon,\beta,N} - b |\Phi(t, x_2)|^2, \tilde{m} \right] q_1 q_2 \psi_{\varepsilon,N}(t) \right\rangle \\
II(t) &:= N \left\langle \psi_{\varepsilon,N}(t), p_1 p_2 \left[ w_{12}^{\varepsilon,\beta,N}, \tilde{m} \right] q_1 q_2 \psi_{\varepsilon,N}(t) \right\rangle \\
III(t) &:= N \left\langle \psi_{\varepsilon,N}(t), p_1 q_2 \left[ w_{12}^{\varepsilon,\beta,N} - b |\Phi(t, x_1)|^2, \tilde{m} \right] q_1 q_2 \psi_{\varepsilon,N}(t) \right\rangle \\
IV(t) &:= \left\langle \left\langle \psi_{\varepsilon,N}(t), \tilde{V}(t, x_1, \varepsilon y_1) \psi_{\varepsilon,N}(t) \right\rangle - \langle \Phi, \tilde{V}(t, x_1, 0) \rangle_{L^2(\mathbb{R})} \right\rangle + 2N \left\langle \psi_{\varepsilon,N}(t), p_1 \left[ V(t, x_1, \varepsilon y_1) - V(t, x_1, 0), \tilde{m} \right] q_1 \psi_{\varepsilon,N}(t) \right\rangle + 2N \left\langle \psi_{\varepsilon,N}(t), p_1 \left[ |\theta'(x_1)L_i|^2 + |\theta'(x)|^2 \|L\|^2, \tilde{m} \right] q_1 \psi_{\varepsilon,N}(t) \right\rangle + 2N \left\langle \psi_{\varepsilon,N}(t), p_1 \left[ R_i^{(1)}, \tilde{m} \right] q_1 \psi_{\varepsilon,N}(t) \right\rangle
\end{align*}

The three terms I–III contain the two-body interaction and are delicate to bound because of the factor $N$ in front. Very roughly speaking, Term I is small because in between the projection $p_1$ onto the state $\varphi$ in the first variable the full interaction and the mean-field interaction cancel each other at leading order. In Term II and Term III the full interaction $w_{12}^{\varepsilon,\beta,N}$ acting on the range of $q_1 q_2$ becomes singular as $(N, \varepsilon) \to (\infty, 0)$, but both can still be bounded in terms of $\alpha_\varepsilon$, however, with considerable effort. The one-particle contributions in Term IV are rather easy to handle, as all potentials appearing remain bounded also on the range of $q_1$. However, the first line of IV is only small if $\psi$ is close to the condensate.
In the following estimates we use the function \( g(t) > 0 \) given in terms of the a priori bound on the energy per particle,

\[
|E^{\psi_N}(t)| \leq |E^{\psi_N}(0)| + \int_0^t \|V(s, \cdot)\|_{L^\infty(\Omega)} ds =: g^2(t) - 1.
\]

If the external potential is time-independent, then \( g^2(t) \equiv 1 + |E^{\psi_N}(0)| \).

We defer the proof of Proposition 3.2 and also of the following one to Section 4.

**Proposition 3.3.** For moderate confinement we have the bounds

\[
|I(t)| \lesssim g(t) \|\Phi(t)\|_{H^2(\mathbb{R})}^3 \left( \frac{\mu}{\varepsilon} + N^\xi \varepsilon + \frac{\varepsilon^2}{\mu^2} \right), \quad |\Pi(t)| \lesssim \|\Phi(t)\|_{H^2(\mathbb{R})}^2 \alpha_\xi(t) + N^\xi \frac{\alpha}{\mu^3}, \quad |\Pi(t)| \lesssim g(t) \|\Phi(t)\|_{H^2(\mathbb{R})}^3 \left( \alpha_\xi(t) + \frac{\mu}{\varepsilon} + \frac{\varepsilon^2}{\mu^2} + \frac{\varepsilon^3}{\mu^3} \right), \quad |IV(t)| \lesssim \alpha_\xi(t) + \varepsilon \|\Phi(t)\|_{H^2(\mathbb{R})} + g(t)N^\xi \varepsilon.
\]

For strong confinement we have the bounds

\[
|I(t)| \lesssim \mu \|\Phi(t)\|_{H^2(\mathbb{R})}^2, \quad |\Pi(t)| \lesssim (\alpha_\xi(t) + \mu) \|\Phi(t)\|_{H^2(\mathbb{R})}, \quad |\Pi(t)| \lesssim (\alpha_\xi(t) + \mu) \|\Phi(t)\|_{H^2(\mathbb{R})}^3, \quad |IV(t)| \lesssim \alpha_\xi(t) + \varepsilon \|\Phi(t)\|_{H^2(\mathbb{R})} + g(t)N^\xi \varepsilon.
\]

Here and in the remainder of the paper we use the notation \( A \lesssim B \) to indicate that there exists a constant \( C \in \mathbb{R} \) independent of all “variable quantities” \( \varepsilon, N, t, \xi, \Psi^{N,0}, \Phi \) such that \( A \leq CB \). Note that \( C \) can depend on “fixed quantities” like the shape of the waveguide determined by \( c, \theta, \Omega_t \), and also on the potentials \( V, w, V^\perp \) and on \( \beta \).

**Proof of Theorem 1.** Combining Propositions 3.2 and 3.3 we obtain for the case of moderate confinement that

\[
\frac{d}{dt} \alpha_\xi(t) \leq CG(t) \|\Phi(t)\|_{H^2(\mathbb{R})}^3 \left( \alpha_\xi(t) + \frac{\mu}{\varepsilon} + \frac{\varepsilon^2}{\mu^2} + N^\xi \varepsilon + N^\xi \frac{\alpha}{\mu^3} \right)
\]

for a constant \( C < \infty \) independent of \( t, \varepsilon, N, \beta, \xi \) and \( \psi^{N,0} \). Thus Grönwall’s lemma proves Theorem 1 once we show that for some \( \xi > 0 \) all terms in the bracket besides \( \alpha_\xi(t) \) vanish in the limit \((N, \varepsilon) \to (\infty, 0)\) along any admissible and moderately confining sequence \((N, \varepsilon)\). This is true for \( \mu / \varepsilon \) and \( \varepsilon^2 / \mu^3 = \sqrt{\varepsilon^4 / \mu^3} \) by assumption. Since

\[
\frac{\varepsilon^4}{\mu^3} = \varepsilon^{4-\beta} N^{3\beta} \to 0 \quad \text{implies} \quad \varepsilon \frac{\alpha^{3\beta}}{N^{3\beta}} \to 0,
\]

we have that

\[
N^\xi \varepsilon = \left( N^\xi N^{-\frac{3\beta}{4-6\beta}} \frac{\alpha^{3\beta}}{N^{3\beta}} \right) \to 0 \quad \text{for} \quad 0 < \xi \leq \frac{3\beta}{4-6\beta}
\]

and

\[
N^\xi \frac{\alpha}{\mu^3} = N^\xi \varepsilon^{2-\beta} N^{3\beta-1} = \left( N^\xi N^{-\frac{3\beta(2-6\beta)}{4-6\beta}} \right) \left( \varepsilon N^{\frac{3\beta}{4-6\beta}} \right)^{2-\beta} N^{3\beta-1} \to 0 \quad \text{for} \quad 0 < \xi \leq \frac{2-6\beta}{2-3\beta}.
\]

Thus in the case of moderate confinement

\[
\lim_{(N, \varepsilon) \to (\infty, 0)} \alpha_\xi(t) = 0
\]

follows by Grönwall’s lemma for \( 0 < \xi \leq \min \left\{ \frac{3\beta}{4-6\beta}, \frac{2-6\beta}{2-3\beta} \right\} \) and thus with Lemma 3.1 also Theorem 1. Analogously the statement for strong confinement follows for \( 0 < \xi \leq \frac{3\beta}{4-6\beta} \). \( \square \)
4 Proofs of the Propositions

4.1 Preliminaries

In this section we prove several lemmata that will be used repeatedly in the proofs of the propositions. The first ones are concerned with properties of the operators \( \hat{f} \) that are at the basis of the condensation-measures \( \alpha_f \) (see Definition 3.2). One should keep in mind, that they are defined with respect to some orthogonal projection \( p \) in the one-particle space \( L^2(\Omega) \). While the first lemma is purely algebraic and holds for general \( p \), later on \( p = |\varphi\rangle\langle\varphi| \) will always be the projection onto the one-dimensional subspace spanned by the condensate vector \( \varphi \in L^2(\Omega) \).

**Definition 4.1.** For \( j \in \mathbb{Z} \) we define the shift operator on a function \( f : \{0, \cdots, N\} \rightarrow \mathbb{R} \) by

\[
(\tau_j f)(k) = f(k + j),
\]

where we set \( (\tau_j f)(k) = 0 \) for \( k + j \notin \{0, \ldots, N\} \).

**Lemma 4.1.** Let \( f, g : \{0, \cdots, N\} \rightarrow \mathbb{R}, j \in \{1, \ldots, N\} \), and \( k \in \{0, \ldots, N\} \).

(a) It holds that

\[
\hat{f} \hat{g} = \hat{g} \hat{f} = \hat{f}, \quad \hat{p}_j = \hat{p}_j \hat{f}, \quad \hat{q}_j = \hat{q}_j \hat{f}, \quad \text{and} \quad \hat{P}_k = P_k \hat{f}.
\]

(b) Let \( \phi, \psi \in L^2_1(\Omega^N) \) be symmetric and \( n(k) = \sqrt{k/N} \), then

\[
\left\langle \phi, \hat{f} q_j \psi \right\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \left\langle \phi, \hat{f} \n^n \psi \right\rangle.
\]

If, in addition, \( f \) is non-negative, then for \( i \in \{1, \ldots, N\} \), \( i \neq j \), it holds that

\[
\left\langle \psi, \hat{f} q_i q_j \psi \right\rangle \leq \frac{N}{N-1} \left\langle \psi, \hat{f} \n^n \psi \right\rangle.
\]

(c) Let \( T : L^2(\Omega)^{\otimes 2} \rightarrow L^2(\Omega)^{\otimes 2} \) be a bounded operator that acts only on the factors \( i \) and \( j \) in the tensor product, e.g. the two-body potential \( w_{ij} \). Then for \( Q_0 := p_i p_j \), \( Q_1 \in \{p_i q_j, q_i p_j\} \), and \( Q_2 := q_i q_j \) we have

\[
\hat{f} Q_\nu T Q_\mu = Q_\nu T Q_\mu \tau_{\nu - \mu} f
\]

\[
Q_\nu T Q_\mu \hat{f} = \tau_{\mu - \nu} f Q_\nu T Q_\mu.
\]

**Proof.** (a) All commutation relations follow immediately from the definitions. E.g.

\[
\hat{f} \hat{g} = \sum_{k,l} f(k)g(l)P_k P_l = \sum_k f(k)g(k)P_k = \hat{f} \hat{g} = \hat{g} \hat{f}.
\]

(b) For the equality we find using the symmetry of \( \psi \) and \( \phi \) and \( [9] \) that

\[
\left\langle \phi, \hat{f} q_j \psi \right\rangle = \frac{1}{N} \sum_{j=1}^{N} \left\langle \phi, \hat{f} q_j \psi \right\rangle = \sum_{k=0}^{N} \sum_{j=1}^{N} \frac{f(k)}{N} \left\langle \phi, q_j P_k \psi \right\rangle = \sum_{k=0}^{N} f(k) \frac{k}{N} \left\langle \phi, P_k \psi \right\rangle = \left\langle \phi, \hat{f} \n^n \psi \right\rangle.
\]
For the proof of the inequality let without loss of generality \( i = 1, j = 2 \). Then

\[
\langle \psi, \hat{f} q_1 q_2 \psi \rangle = \frac{1}{N(N-1)} \sum_{i \neq j} \langle \psi, \hat{f} q_i q_j \psi \rangle \leq \frac{1}{N(N-1)} \sum_{i,j} \langle \psi, \hat{f} q_i q_j \psi \rangle \]

\[
= \frac{1}{N(N-1)} \sum_{k=0}^{N} \sum_{i,j=1}^{N} f(k) \langle \psi, q_i q_j P_k \psi \rangle \]

\[
\leq \frac{N^2}{N(N-1)} \sum_{k=0}^{N} \sum_{i,j=1}^{N} f(k) k^2 \langle \psi, P_k \psi \rangle \]

\[
= \frac{N}{(N-1)} \langle \psi, \hat{f} n^4 \psi \rangle.
\]

(c) Let without loss of generality \( i = 1 \) and \( j = 2 \), and let \( P_{k, N-2}^{12} \) be the operator \( P_{k, N-2} \), where \( P_{k, N-2} \) is the operator \( P_{k} \) defined on \( L^2(\Omega) \). Then

\[
\hat{f} Q_{\nu} TQ_{\mu} = \sum_{k=0}^{N} f(k) P_k Q_{\nu} TQ_{\mu} = \sum_{k=\nu}^{N-2+\nu} f(k) P_{k-2} Q_{\nu} TQ_{\mu} = \sum_{k=\nu}^{N-2+\nu} f(k) Q_{\nu} TQ_{\mu} P_{k-2}^{12}
\]

\[
= \sum_{k=\nu}^{N-2+\nu} f(k) Q_{\nu} TQ_{\mu} P_{k-\nu+\mu} = \sum_{k'=\mu}^{N-2+\mu} f(k') (\nu - \mu) Q_{\nu} TQ_{\mu} P_{k'}
\]

and the converse direction follows in the same way.

From now on \( P_k \) and the derived operations \( \hat{\sim} \) and \( \alpha \) refer to the projection \( p = |\varphi\rangle\langle\varphi| \) onto the one-dimensional subspace spanned by the one-particle wave function \( \varphi \in L^2(\Omega) \). We make this explicit only within the following lemma.

**Lemma 4.2.** Let \( \varphi(t) = \Phi(t)\chi \), where \( \chi \) is an eigenfunction of \(-\Delta_y + V_\perp\) on \( \Omega \) and \( \Phi(t) \) a solution to (8) with \( \Phi_0 \in H^2(\mathbb{R}) \). Then for all \( f : \{0, \ldots, N\} \rightarrow \mathbb{R} \)

(a) \( P^\varphi_k \subseteq C^1(\mathbb{R}, \mathcal{L}(L^2(\Omega))) \) for all \( k \in \{0, \ldots, N\} \) and thus also \( \hat{f}\varphi(t) \subseteq C^1(\mathbb{R}, \mathcal{L}(L^2(\Omega))) \),

(b) \( [-\Delta_y + V_\perp(y_i), \hat{f}\varphi(t)] = 0 \) for all \( i \in \{1, \ldots, N\} \),

(c) Let \( H^\Phi(t) := \sum_{i=1}^{N} h_i^\Phi(t) \) where \( h_i^\Phi(t) \) denotes the one-particle operator \( h_i^\Phi(t) \) (c.f. (8)) acting on the \( i \)th factor in \( L^2(\Omega) \). Then

\[
\frac{d}{dt} \hat{f}\varphi(t) = \left[ H^\Phi(t), \hat{f}\varphi(t) \right].
\]

**Proof.** (a) This follows immediately from \( \varphi \in C^1(\mathbb{R}, L^2(\Omega)) \).

(b) This is the fact that a self-adjoint operator commutes with its spectral projections.
(c) Because of (8) the projection $|\Phi(t)\rangle\langle\Phi(t)|$ satisfies the differential equation $i\partial_t |\Phi(t)\rangle\langle\Phi(t)| = [\hat{H}^\Phi(t), |\Phi(t)\rangle\langle\Phi(t)|]$ and thus $i\partial_t p_i(t) = [\hat{h}_i^\Phi(t), p_i(t)]$ and $i\partial_t q_i(t) = [\hat{h}_i^\Phi(t), q_i(t)]$. The product rule then implies for any $J \subset \{1, \ldots, N\}$ that
\[
i\partial_t \prod_{j \in J} q_j(t) \prod_{j \notin J} p_j(t) = \sum_{j=1}^N [\hat{h}_{ij}^\Phi(t), \prod_{j \in J} q_j(t) \prod_{j \notin J} p_j(t)] = [\hat{H}^\Phi(t), \prod_{j \in J} q_j(t) \prod_{j \notin J} p_j(t)].\]

As $\hat{f}^\varphi$ is a linear combination of operators of the above form, the claim follows.

For the next lemma recall the definition (11) of the function $m(k)$ defining our weight $\alpha_m$. Because of Lemma 4.1 (c) and the form of the terms I–IV in Proposition 3.2 the difference $m_\ell(k)$ defined below will appear many times in our estimates.

**Lemma 4.3.** Let $\psi \in L^2_+(\Omega^N)$ be symmetric, $\ell \in \mathbb{N}$ and
\[
m_\ell(k) := N(m(k) - \tau_{-\ell}m(k)) = N(m(k) - m(k - \ell)), \tag{13}\]
where the function $m(k)$ was defined in (11).

(a) It holds that
\[
0 \leq m_\ell(k) \leq \begin{cases} \ell \sqrt{\frac{N}{\kappa}} & k \geq N^{1-2\xi} + \ell \\ \frac{\ell}{2} N^\xi & k < N^{1-2\xi} + \ell \end{cases},
\]
and
\[
\|\hat{m}_1 q_1 \psi\| \leq \ell \|\psi\| \quad \text{and} \quad \|N\hat{n} - \tau_{-\ell} \hat{n}\| q_1 \psi\| \leq \ell \|\psi\|.
\]

(b) Let $q^\chi := 1_{L^2(\mathbb{R})} \otimes (1_{L^2(\Omega^k)} - |\chi\rangle\langle\chi|)$ be the projection onto the orthogonal complement of the ground state in the confined direction. Then
\[
\langle \psi, q^\chi_1 \psi \rangle \lesssim \varepsilon^2 \left(1 + |E^\psi(t)|\right)
\]
and
\[
\|\hat{m}_1 q_1 \psi\| \lesssim N^\xi \varepsilon \left(1 + |E^\psi(t)|\right)^{\frac{1}{2}}.
\]

**Proof.** First recall that $n$ and $m$ are monotonically increasing functions, c.f. (11). Moreover
\[
(n(k) - n(k - \ell))^2 = \left(\frac{\sqrt{k - \ell} - \sqrt{k}}{\sqrt{N}}\right)^2 = \frac{\ell^2}{(\sqrt{k} + \sqrt{k-\ell})^2 N} \leq \frac{\ell^2}{2 N}
\]
and thus also $m_\ell(k) \leq \ell \sqrt{N \frac{k}{\kappa}}$ for $k \geq N^{1-2\xi} + \ell$ follows. The bound $m_\ell(k) \leq \frac{\ell}{2} N^\xi$ is obvious for $k < N^{1-2\xi}$ and holds also for $N^{1-2\xi} \leq k < N^{1-2\xi} + \ell$, since $\sqrt{\frac{k}{N}} \leq \frac{\ell}{2} (N^{-1+\xi} k + N^{-\xi})$ for such $k$.

For any $f : \{0, \ldots, N\} \rightarrow \mathbb{R}$ we find with Lemma 4.1 (b) that
\[
\left\|\left(\hat{f} - \tau_{-l}\hat{f}\right) q_1 \psi\right\|^2 = \left\langle \psi, \left(\hat{f} - \tau_{-l}\hat{f}\right) q_1 \psi\right\rangle = \sum_{k=1}^N \left(f(k) - f(k - l)\right)^2 \frac{k}{N} \left\langle \psi, P_k \psi\right\rangle.
\]
Hence part (a) of the lemma follows with the above estimates on $m_\ell$ and the identity \([9]\). From
\[
E^\psi(t) = \frac{1}{N} \left\langle \psi, (H(t) - N \frac{E^\psi}{\varepsilon}) \psi \right\rangle
\]
\[
= \left\langle \psi, \frac{1}{N} \left( \sum_{i=1}^{N} \left( (\partial_{x_i} + \theta'(x_i) L_i) \rho_\varepsilon^{-2}(r_i) (\partial_{x_i} + \theta'(x_i) L_i) - V_{\text{bend}}(r_i) + V(r_i) \right) \right) \right\rangle
\]
\[
+ \frac{1}{\varepsilon^2} \left(-\Delta y_i + V^\perp(y_i) - E_0\right) + \frac{a}{\mu^2} \sum_{j<i} w \left( \frac{f^x_{\phi}(r_i) - f^x_{\phi}(r_j)}{\mu} \right) \right\rangle \right\rangle
\]
\[
A^2 \geq \left\langle \psi, \left(-V_{\text{bend}}(r_1) + \frac{1}{\varepsilon^2} \left(-\Delta y_1 + V^\perp(y_1) - E_0\right) \right) \psi \right\rangle
\]
\[
= - \left\langle \psi, V_{\text{bend}}(r_1) \psi \right\rangle + \left\langle \psi, \frac{1}{\varepsilon^2} \left(-\Delta y_1 + V^\perp(y_1) - E_0\right) q_1^x \psi \right\rangle
\]
\[
\geq - \|V_{\text{bend}}\|_{L^\infty(\Omega)} + \frac{C}{\varepsilon^2} \left\langle \psi, q_1^x \psi \right\rangle
\]
we infer that $\left\langle \psi, q_1^x \psi \right\rangle \lesssim \varepsilon^2 \left(1 + |E^\psi(t)|\right)$. For the proof of the remaining estimate we use that $q_1^x$ commutes with $q_1$ and thus also with $P_k$. Hence
\[
\|\hat{m}_1 q_1^x \psi\|^2 \leq N^2 \sum_{k=1}^{N_{1-2\xi}} m_1^2(k) \left\langle \psi, P_k q_1^x \psi \right\rangle
\]
\[
\leq \frac{1}{4} \sum_{k=1}^{[N_{1-2\xi}]} N^2 \varepsilon \left\langle \psi, P_k q_1^x \psi \right\rangle + \frac{1}{4} \sum_{k=[N_{1-2\xi}]}^{N} \frac{N}{k} \left\langle \psi, P_k q_1^x \psi \right\rangle
\]
\[
\leq \frac{1}{2} N^2 \varepsilon \sum_{k=1}^{N} \left\langle \psi, P_k q_1^x \psi \right\rangle = \frac{1}{2} N^2 \varepsilon \sum_{k=1}^{N} \left\langle \psi, q_1^x \psi \right\rangle \lesssim N^2 \varepsilon^2 \left(1 + |E^\psi(t)|\right).
\]

\[\square\]

The meaning of Lemma \([4.3]\) (b) is the following. Due to symmetry of the wave function, the “probability” that one specific particle in a many-body state gets excited in the confined direction can be controlled by the renormalised energy per particle $E^\psi(t)$ for any $t \in \mathbb{R}$.

Next we Taylor expand the scaled two-body interaction $w_{12}^{\varepsilon,\beta,N}$.

**Lemma 4.4.** Assuming $A^2$ for the two-body potential $w$ and $A^1$ for the geometry of the waveguide, it holds that
\[
\frac{\varepsilon^2}{\mu^3} \frac{\rho_\varepsilon^{-2}(r_1) - \rho_\varepsilon^{-2}(r_2)}{\mu} = \frac{\varepsilon^2}{\mu^3} w \left( \frac{r_1^x - r_2^x}{\mu} \right) + R(r_1, r_2) \frac{\varepsilon^2}{\mu^3} w' \left( \frac{\|r_2^x - r_1^x\|}{\mu^2} \right) + \frac{\varepsilon^2}{\mu^3} O(R(r_1, r_2)^2)
\]
\[
= \frac{\varepsilon^2}{\mu^3} w_1^0(r_1, r_2) + T_1(r_1, r_2) + T_2(r_1, r_2)
\]
with
\[
\overline{R} := \sup_{r_1, r_2 \in \Omega} |R(r_1, r_2)| \lesssim \varepsilon + \mu.
\]
Proof. The proof is in essence a Taylor expansion, but we need to be careful with the different scalings. First recall the maps \( f, T_\theta \) and \( D_\varepsilon \) defined in the introduction. \( D_\varepsilon \) is linear and for the differentials of \( f \) and \( T_\theta \) one easily computes

\[
DT_\theta(x, y) = \begin{pmatrix} 1 & 0 \\ T_{\theta(x)}y & T_{\theta(x)} \end{pmatrix}
\]

\[
Df(r)h = (c'(x), e_1(x), e_2(x))h + (y^1e_1'(x) + y^2e_2'(x))h_x
\]

\[=: A(x)h - y \cdot \kappa(x)c'(x)h_x.\]

For \( f_\theta := f \circ T_\theta \) we thus obtain

\[
Df_\theta(r) = D(f \circ T_\theta)(r)h = Df \circ T_\theta(r) DT_\theta(r)h
\]

\[
= (A \circ T_\theta)(x) + (b \circ T_\theta)(r)(1, 0, 0)^T \begin{pmatrix} 1 & 0 \\ T_{\theta(x)}y & T_{\theta(x)} \end{pmatrix} h
\]

\[
= A(x)T_\theta(x)h + A(x) \begin{pmatrix} 0 & 0 \\ T'_{\theta(x)}y & 0 \end{pmatrix} h - (T_{\theta(x)}y \cdot \kappa(x)c'(x)h_x
\]

\[
= A(x)T_\theta(x)h + (e_1(x), e_2(x))T'_{\theta(x)}y - (T_{\theta(x)}y \cdot \kappa(x)c'(x))h_x
\]

\[=: A_\theta(x)h + b_\theta(r)h_x.\]

Note that \( A_\theta(x) \) is an orthogonal matrix for all \( x \in \mathbb{R} \) and \( \|b(x, y)\|_{\mathbb{R}^3} \lesssim \|y\|_{\mathbb{R}^2} \). Hence for \( \|y\|_{\mathbb{R}^2} \) small enough, \( Df_\theta(r) \) is invertible and

\[
\|f_\theta(r_2) - f_\theta(r_1)\|_{\mathbb{R}^3} - \|r_2 - r_1\|_{\mathbb{R}^3} \lesssim \|y\|_{\mathbb{R}^2} \|f_\theta(r_2) - f_\theta(r_1)\|_{\mathbb{R}^3}. \tag{14}
\]

Since \( f \in C^\infty(\mathbb{R}^3) \), Taylor expansion gives

\[
f_\theta(r_2) - f_\theta(r_1) = A_\theta \left( \frac{r_1 + r_2}{2} \right) (r_2 - r_1) + b_\theta \left( \frac{r_1 + r_2}{2} \right) (x_2 - x_1) + O(|r_2 - r_1|^3)
\]

and thus

\[
\|f_\theta(r_2) - f_\theta(r_1)\|^2 = \langle A_\theta \left( \frac{r_1 + r_2}{2} \right) (r_2 - r_1), A_\theta \left( \frac{r_1 + r_2}{2} \right) (r_2 - r_1) \rangle
\]

\[
+ 2 \langle A_\theta \left( \frac{r_1 + r_2}{2} \right) (r_2 - r_1), b_\theta \left( \frac{r_1 + r_2}{2} \right) \rangle (x_2 - x_1)
\]

\[
+ \langle b_\theta \left( \frac{r_1 + r_2}{2} \right), b_\theta \left( \frac{r_1 + r_2}{2} \right) \rangle (x_2 - x_1)^2 + O(|r_2 - r_1|^3)
\]

\[
= \|r_2 - r_1\|^2 + \bar{R}(r_1, r_2)
\]

with

\[
|\bar{R}(r_1, r_2)| = O(|r_2 - r_1| |x_2 - x_1| |y|_{\mathbb{R}^2} + |x_2 - x_1|^2 |y|_{\mathbb{R}^2}^2 + \|r_2 - r_1\|^3).
\]

Now recall that

\[
f_\varepsilon := f \circ (T_\theta \circ D_\varepsilon \ i.e. \ f_\varepsilon(r) = f_\theta(r^\varepsilon).
\]

Since \( w \left( \frac{1}{\varepsilon} (f_\varepsilon(r_2) - f_\varepsilon(r_1)) \right) \neq 0 \) only for \( \|f_\varepsilon(r_2) - f_\varepsilon(r_1)\| < \mu \) and thus according to \( \|f_\varepsilon(r_2) - f_\varepsilon(r_1)\| \lesssim \|y\|_{\mathbb{R}^2} \|r_2 - r_1\|^3 \) also \( \|r_2 - r_1\| < \mu(1 + \varepsilon) \), we have that

\[
|\bar{R}(r_1^\varepsilon, r_2^\varepsilon)| = O(\mu^2 \varepsilon + \mu^2 \varepsilon^2 + \mu^3).
\]
Taylor expansion of \( w(r) = \tilde{w}(|r|^2) \) finally gives the desired result with \( R := \tilde{R}/\mu^2 \),
\[
\begin{align*}
w \left( \frac{1}{\mu^2} (f_{\varepsilon}(r_2) - f_{\varepsilon}(r_1)) \right) &= \tilde{w} \left( \frac{1}{\mu^2} \| f_{\varepsilon}(r_2) - f_{\varepsilon}(r_1) \|^2 \right) \\
&= \tilde{w} \left( \frac{1}{\mu^2} \| r_2^\varepsilon - r_1^\varepsilon \|^2 \right) + \frac{\tilde{R}(r_1^\varepsilon, r_2^\varepsilon)}{\mu^2} \tilde{w}' \left( \frac{1}{\mu^2} \| r_2^\varepsilon - r_1^\varepsilon \|^2 \right) + O \left( \frac{(\tilde{R}(r_1^\varepsilon, r_2^\varepsilon))^2}{\mu^4} \right).
\end{align*}
\]

The next lemma collects elementary facts that will allow us to estimate one- and two-body potentials in different situations.

**Lemma 4.5.** Let \( g : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R} \) be a measurable function such that \( |g(r_1, r_2)| \leq v(r_2 - r_1) \) almost everywhere for some measurable function \( v : \mathbb{R}^3 \rightarrow \mathbb{R} \), and let \( \varphi \in L^2(\Omega) \cap L^\infty(\Omega) \) with \( \| \varphi \|_2 = 1 \), and \( p = |\varphi\rangle \langle \varphi| \).

(a) For \( v \in L^2(\mathbb{R}^3) \) we have
\[
\| v(r)p \|_{L^2(\Omega)} \leq \| v \|_{L^2(\mathbb{R}^3)} \| \varphi \|_{L^\infty(\Omega)} \quad \text{and} \quad \| g(r_1, r_2)p_1 \|_{L^2(\Omega)} \leq \| v \|_{L^1(\mathbb{R}^3)} \| \varphi \|_{L^\infty(\Omega)}
\]

(b) For \( v \in L^1(\mathbb{R}^3) \) we have
\[
\| p_1 g(r_1, r_2)p_1 \|_{L^2(\Omega)} \leq \| v \|_{L^1(\mathbb{R}^3)} \| \varphi \|_{L^\infty(\Omega)}^2.
\]

(c) For \( \Phi : H^2(\mathbb{R}) \)
\[
\| \Phi \|_{L^\infty}^2 \leq \| \Phi \|_{H^1}^2 \leq \| \Phi \|_{H^2}^2 \quad \text{and} \quad \| \nabla \Phi \|_{L^2}^2 \leq 2 \| \Phi \|_{L^\infty} \| \Phi \|_{H^1}.
\]

**Proof.** All three estimates in (a) and (b) are elementary:
\[
\begin{align*}
\| v(r)p \|_{L^2(\Omega)}^2 &= \sup_{\| \psi \| = 1} \langle \psi, p v(r)^2 p \psi \rangle_{L^2(\Omega)} = \langle \varphi(r), |v(r)|^2 \varphi(r) \rangle_{L^2(\Omega)} \sup_{\| \psi \| = 1} \| p \psi \|_{L^2(\Omega)}^2 \\
&\leq \langle v(r), |\varphi|^2 v(r) \rangle_{L^2(\Omega)} \leq \| v \|_{L^2(\mathbb{R}^3)}^2 \| \varphi \|_{L^\infty(\Omega)}^2,
\end{align*}
\]
\[
\begin{align*}
\| g(r_1^\varepsilon, r_2^\varepsilon)p_1 \|_{L^2(\Omega)}^2 &= \sup_{\| \psi \| = 1} \langle p_1 \psi, |g(r_1^\varepsilon, r_2^\varepsilon)|^2 p_1 \psi \rangle_{L^2(\Omega)} \\
&\leq \sup_{\| \psi \| = 1} \| p_1 \psi \|_{L^2(\Omega)} \sup_{r_2 \in \Omega} \int_{\Omega} |\varphi(r_1)|^2 |v(r_2^\varepsilon - r_1^\varepsilon)|^2 \, dr_1 \\
&\leq \| \varphi \|_{L^\infty(\Omega)}^2 \sup_{r_2 \in \Omega} \int_{\Omega} |v(r_2^\varepsilon - r_1^\varepsilon)|^2 \, dr_1 \\
&\leq \| \varphi \|_{L^\infty(\Omega)}^2 \frac{1}{|\mathbf{2}|} \| v \|_{L^2(\mathbb{R}^3)}^2,
\end{align*}
\]
\[
\| p_1 g(r_1^\varepsilon, r_2^\varepsilon)p_1 \|_{L^2(\Omega)} \leq \sup_{r_2 \in \Omega} \int_{\Omega} |\varphi(r_1)|^2 |g(r_1^\varepsilon, r_2^\varepsilon)| \, dr_1 \leq \sup_{r_2 \in \Omega} \int_{\Omega} |\varphi(r_1)|^2 |v(r_2^\varepsilon - r_1^\varepsilon)| \, dr_1 \\
&\leq \| \varphi \|_{L^\infty(\Omega)}^2 \frac{1}{|\mathbf{2}|} \| v \|_{L^1(\mathbb{R}^3)}.
\]

For (c) note that for \( \Phi : H^2(\mathbb{R}) \subset C^1(\mathbb{R}) \) we have with Cauchy-Schwarz
\[
\Phi(x)\Phi(x) = \int_{-\infty}^x \Phi'(s)\Phi(s) + \Phi(s)\Phi'(s) \, ds \leq 2 \| \Phi' \|_{L^2(\mathbb{R})} \| \Phi \|_{L^2(\mathbb{R})} \leq \| \Phi' \|_{L^2}^2 + \| \Phi \|_{L^2}^2 = \| \Phi \|_{H^1}^2.
\]
Using the bound and we obtain analogously

\[
|\nabla |\Phi|^2|_{L^2}^2 \leq \int 4|\Phi'(x)|^2|\Phi(x)|^2 \, dx \leq 4\|\Phi\|_{L^\infty}^2 \|\Phi'\|_{L^2}^2 \leq 4\|\Phi\|_{L^\infty}^2 \|\Phi\|_{H^1}.
\]

In the following corollary we collect bounds on the two-body interaction that will be used repeatedly.

**Corollary 4.6.** For the scaled two-body interaction \(w_{12}^{\varepsilon, N}\) we have that

\[
\|w_{12}^{\varepsilon, N}p_1\|_{L^2(\Omega^2)} \lesssim \|\Phi\|_{H^2(\mathbb{R})} \cdot \begin{cases} \frac{\varepsilon^2}{\mu^2} & \text{for moderate confinement} \\ \frac{\varepsilon^4}{\mu^4} & \text{for strong confinement} \end{cases}
\]

and

\[
\|w_{12}^{\varepsilon, N}p_1\|_{L^2(\Omega^2)} \lesssim \|\|\Phi\|_{H^2(\mathbb{R})} \cdot \begin{cases} \frac{\varepsilon^2}{\mu^2} & \text{for moderate confinement} \\ \frac{\varepsilon^4}{\mu^4} & \text{for strong confinement} \end{cases}
\]

In the following corollary we collect bounds on the two-body interaction that will be used repeatedly.

**Proof.** According to Lemma 4.5 (b) and (c) we have

\[
\|w_{12}^{0}p_1\|_{L^2(\Omega^2)} \lesssim \|\varphi\|_{L^\infty(\Omega)} \|w_{12}^{0}\|_{L^2(\Omega)} \lesssim \|\Phi\|_{L^\infty(\mathbb{R})}^2 \cdot \begin{cases} \frac{\varepsilon^2}{\mu^2} & \text{for moderate confinement} \\ \frac{\varepsilon^4}{\mu^4} & \text{for strong confinement} \end{cases}
\]

and

\[
\|p_1w_{12}^{0}p_1\|_{L^2(\Omega^2)} \lesssim \|\varphi\|_{L^\infty(\Omega)} \|w_{12}^{0}\|_{L^1(\Omega)} \lesssim \|\Phi\|_{L^\infty(\mathbb{R})}^2 \cdot \begin{cases} \frac{\varepsilon^2}{\mu^2} & \text{for moderate confinement} \\ \frac{\varepsilon^4}{\mu^4} & \text{for strong confinement} \end{cases}
\]

Using the bound

\[
|T_1(r_1^2, r_2^\varepsilon)| \leq \frac{\varepsilon^2}{\mu^3} \tilde{R} \tilde{w}'(\frac{|r_2^\varepsilon-r_1^2|}{\mu^2}) =: \psi(r_2^\varepsilon-r_1^2)
\]

we obtain analogously

\[
\|T_1p_1\|_{L^2(\Omega^2)} \lesssim \|\varphi\|_{L^\infty(\Omega)} \|v\|_{L^2(\Omega)} \lesssim \|\Phi\|_{L^\infty(\mathbb{R})}^2 \|\mu^2(\varepsilon + \mu)^2
\]

and

\[
\|p_1T_1p_1\|_{L^2(\Omega^2)} \lesssim \|\varphi\|_{L^\infty(\Omega)} \|v\|_{L^1(\Omega)} \lesssim \|\Phi\|_{L^\infty(\mathbb{R})}^2 (\varepsilon + \mu)
\]

for moderate confinement and an additional factor \(\varepsilon^2\) for strong confinement. With \(\|T_2\| \leq \frac{\varepsilon^2(\varepsilon + \mu)^2}{\mu^4} \) and \(\|\Phi\|_{L^\infty(\mathbb{R})} \leq \|\Phi\|_{H^2(\mathbb{R})} \geq 1\) we arrive at

\[
\|w_{12}^{\varepsilon, N}p_1\|_{L^2(\Omega^2)} \lesssim \frac{\varepsilon}{\mu^2} \|\Phi\|_{H^2(\mathbb{R})} \quad \text{and} \quad \|p_1w_{12}^{\varepsilon, N}p_1\|_{L^2(\Omega^2)} \lesssim \|\Phi\|_{H^2(\mathbb{R})}^2
\]

for moderate confinement, and

\[
\|w_{12}^{\varepsilon, N}p_1\|_{L^2(\Omega^2)} \lesssim \sqrt{\mu} \|\Phi\|_{H^2(\mathbb{R})} \quad \text{and} \quad \|p_1w_{12}^{\varepsilon, N}p_1\|_{L^2(\Omega^2)} \lesssim \mu \|\Phi\|_{H^2(\mathbb{R})}^2
\]

19
for strong confinement. Finally

$$\left\| \sqrt{\frac{\varepsilon}{2} \beta N} p_1 \right\|_{L^2(L^2(\Omega^2))}^2 = \sup_{\|\psi\|=1} \left\langle \psi, p_1 u_{12}^{\varepsilon, \beta N} p_1 \psi \right\rangle \leq \left\| p_1 u_{12}^{\varepsilon, \beta N} p_1 \right\|_{L^2(L^2(\Omega^2))}. \quad \square$$

Finally we need also the following lemma that shows how to bound the “kinetic energy” of $q_1 \psi$ in terms of $\alpha_\xi$.

**Lemma 4.7.** Let the assumptions of Theorem 4 hold. Then in the moderately confining case

$$\left\| \left( \frac{\partial}{\partial x_1} + \theta'(x_1) L_1 \right) q_1 \psi_{N, \xi}(t) \right\|^2 \lesssim \|\Phi(t)\|_{H^2(\mathbb{R})}^3 \left( \alpha_\xi(t) + \frac{\mu}{\varepsilon} + \frac{a}{\mu^2} \right).$$

This energy estimate is actually quite subtle and we postpone its proof to Subsection 4.4. Note that it is the only place in our argument where the positivity of the interaction is crucial. We end this subsection with the proof of Lemma 3.1

**Proof of Lemma 3.1.** (i) $\Leftrightarrow$ (ii): Let $\lim_{N \to \infty} \|q_1 \psi_N\| = 0$ for some $a > 0$. Then $\alpha_{a,b}(\psi_N, \varphi) \leq \alpha_{a,b}(\psi_N, \varphi)$ for all $b > a$ since $n^b \leq n^a$. If $\frac{a}{2} \leq b < a$, then

$$\alpha_{a,b}(\psi_N, \varphi) = \left\langle \psi_N, \tilde{n}^b \psi_N \right\rangle = \left\langle \tilde{n}^{b-\frac{a}{2}} \psi_N, \tilde{n}^{\frac{a}{2}} \psi_N \right\rangle \leq \|\tilde{n}^{b-\frac{a}{2}} \psi_N\| \|\tilde{n}^{\frac{a}{2}} \psi_N\| \leq \sqrt{\alpha_{a,b}(\psi_N, \varphi)}.$$

(i) $\Rightarrow$ (iii): For $a = 2$ we have $\lim_{N \to \infty} \|q_1 \psi_N\| = 0$ according to (10). Let $P_k^M$ be the projector from Definition 3.1 acting on $L^2(\Omega^M)$ with $p = |\varphi\rangle \langle \varphi|$. Then in

$$\gamma_M^N = \sum_{k=0}^{M} \sum_{k'=0}^{M} P_k^M \gamma_M^N P_{k'}^M,$$

all terms but the one with $k = k' = 0$ go to zero in norm for $N \to \infty$. Hence,

$$\lim_{N \to \infty} \gamma_M^N = \lim_{N \to \infty} P_{\gamma_M^N} = p^\otimes M,$$

where the last equality follows from $\text{Tr} \gamma_M^N = 1$ and the fact that $p^\otimes M$ has rank one.

(iii) $\Rightarrow$ (iv): We learned the following argument from [28]. Since $p^\otimes M$ has rank one, the operator $\gamma_M^N - p^\otimes M$ can have at most one negative eigenvalue $\lambda_- < 0$. Since $\text{Tr} (\gamma_M^N - p^\otimes M) = 0$, $|\lambda_-|$ equals the sum of all positive eigenvalues. Hence

$$\text{Tr} |\gamma_M^N - p^\otimes M| = 2 |\lambda_-| = 2 \|\gamma_M^N - p^\otimes M\|.$$

(iv) $\Rightarrow$ (v) is obvious and (v) $\Rightarrow$ (i) follows for $a = 2$ from

$$\alpha_{a^2}(\psi_N, \varphi) = \left\langle \psi, (1 - p_1) \psi \right\rangle = \text{Tr} (p - p \gamma_1^N p) = \text{Tr} |p - p \gamma_1^N p| = \text{Tr} |p (p - \gamma_1^N)| \leq \|p\| \text{Tr} |p - \gamma_1^N| \quad \square$$

20
4.2 Proof of Proposition 3.2

Recalling the definition (12) we need to estimate

$$|\frac{d}{dt}\alpha_\xi(t)| \leq |\frac{d}{dt}\alpha_m(\psi^{N,\varepsilon}(t), \varphi(t))| + |\frac{d}{dt}|E^{\psi^{N,\varepsilon}(t)}(t) - E^\Phi(t)|.$$

For better readability we abbreviate $\psi = \psi^{N,\varepsilon}(t)$ and $\Phi = \Phi(t)$ in the remainder of this proof. The derivative of the second term yields

$$|\frac{d}{dt}|E^\Psi(t) - E^\Phi(t)|| = \|\langle \psi, \hat{\nabla}(t, x_1, \varepsilon y_1)\psi \rangle - \langle \Phi, \hat{\nabla}(t, x_1, 0)\Phi \rangle_{L^2(\mathbb{R})}||.$$

As of Lemma 4.2, the map $t \mapsto \alpha_m(\psi, \varphi) \in C^1(\mathbb{R})$ and we find

$$\frac{d}{dt}\alpha_m = \frac{d}{dt}\langle \psi, \hat{\nabla}\rangle = i\langle \psi, [H_N^\varepsilon - H^\Phi, \hat{\nabla}] \psi \rangle$$

$$+ i\langle \psi, \left[\sum_{i<j}^N w_{i,j}^{\varepsilon,\beta,N} - \sum_{i=1}^N b|\Phi(x_i)|^2, \hat{m}\right] \psi \rangle + i\langle \psi, \left[\sum_{i=1}^N V(x_i, \varepsilon y_i) - \sum_{i=1}^N V(x_i, 0), \hat{m}\right] \psi \rangle$$

$$- iN \langle \psi, [(\theta'(x_1)L_1)^2 + |\theta'(x_1)|^2 \|L\chi\|^2, \hat{m}] \psi \rangle - iN \langle \psi, R^{(1)}_{1}, \hat{m} \rangle \hat{m} \psi \rangle + iN \langle \psi, [R^{(1)}_{1}, \hat{m}] \psi \rangle$$

$$= \frac{iN}{2} \langle \psi, (p_1 + q_1)(p_2 + q_2) \left[w_{12}^{\varepsilon,\beta,N} - b|\Phi(x_1)|^2 - b|\Phi(x_2)|^2, \hat{m}\right] (p_1 + q_1)(p_2 + q_2) \psi \rangle$$

$$+ iN \langle \psi, (p_1 + q_1) \left[V(t, x_1, \varepsilon y_1) - V(t, x_1, 0), \hat{m}\right] (p_1 + q_1) \psi \rangle$$

$$- iN \langle \psi, (p_1 + q_1) \left[(\theta'(x_1)L_1)^2 + |\theta'(x_1)|^2 \|L\chi\|^2, \hat{m}\right] (p_1 + q_1) \psi \rangle$$

$$+ iN \langle \psi, (p_1 + q_1) \left[R^{(1)}_{1}, \hat{m}\right] (p_1 + q_1) \psi \rangle.$$

According to Lemma 4.1 (c) all terms with the same number of $p$’s and $q$’s on each side of the commutator vanish. Therefore we find that (16)–(18) are bounded by

$$|16| + |17| + |18| \leq 2N \|\psi, p_1 \left[V(t, x_1, \varepsilon y_1) - V(t, x_1, 0), \hat{m}\right] q_1 \psi\|$$

$$+ 2N \|\psi, p_1 \left[(\theta'(x_1)L_1)^2 + |\theta'(x_1)|^2 \|L\chi\|^2, \hat{m}\right] q_1 \psi\|$$

$$+ 2N \|\psi, p_1 \left[R^{(1)}_{1}, \hat{m}\right] q_1 \psi\|.$$
The crucial step (c.f. [24]) is to split (15) according to

\[
\frac{1}{2} N \left\langle \psi, (p_1 + q_1)(p_2 + q_2) \left[ w_{i1}^{\epsilon, \beta, N} - b|\Phi(x_1)|^2 - b|\Phi(x_2)|^2, \hat{m} \right] (p_1 + q_1)(p_2 + q_2) \psi \rightangle
\]

\[
= \frac{1}{2} N \left\langle \psi, p_1 p_2 \left[ w_{i1}^{\epsilon, \beta, N} - b|\Phi(x_1)|^2 - b|\Phi(x_2)|^2, \hat{m} \right] p_1 p_2 \psi \rightangle
\]

\[
+ \frac{1}{2} N \left\langle \psi, (p_1 q_2 + q_1 p_2) \left[ w_{i1}^{\epsilon, \beta, N} - b|\Phi(x_1)|^2 - b|\Phi(x_2)|^2, \hat{m} \right] (p_1 q_2 + q_1 p_2) \psi \rightangle
\]

\[
+ \frac{1}{2} N \left\langle \psi, (p_1 q_2 + q_1 p_2) \left[ w_{i1}^{\epsilon, \beta, N} - b|\Phi(x_1)|^2 - b|\Phi(x_2)|^2, \hat{m} \right] (p_1 q_2 + q_1 p_2) \psi \rightangle
\]

\[
= \frac{1}{2} N \left\langle \psi, p_1 p_2 \left[ w_{i1}^{\epsilon, \beta, N} - b|\Phi(x_1)|^2 - b|\Phi(x_2)|^2, \hat{m} \right] \right\rangle_p q_1 q_2 \psi \rightangle + c.c.
\]

\[
+ \frac{1}{2} N \left\langle \psi, p_1 p_2 \left[ w_{i1}^{\epsilon, \beta, N} - b|\Phi(x_1)|^2 - b|\Phi(x_2)|^2, \hat{m} \right] q_1 q_2 \psi \rightangle + c.c.
\]

\[
+ iN \left\langle \psi, p_1 q_2 \left[ w_{i1}^{\epsilon, \beta, N} - b|\Phi(x_1)|^2 - b|\Phi(x_2)|^2, \hat{m} \right] q_1 q_2 \psi \rightangle + c.c.
\]

\[
= -2\mathcal{I} - \mathcal{I} - 2\mathcal{III}.
\]

The term with \( p_1 p_2 \) on both sides of the commutator vanishes again because of Lemma 4.1 [23]. For the second to last equality we used that for \( i \neq j \) the projection \( p_j \) commutes with \( \hat{m} \) and with \( |\Phi(x_i)|^2 \) and that \( p_j q_j = 0 \).

### 4.3 Proof of Proposition 3.3

**Proof of the bound for I.** In the case of moderate confinement, the term \( I \) is small due to the cancellation of the mean field and the full interaction. Since \( b|\Phi|^2 \) is the mean field for a condensate in the state \( \Phi \chi \), i.e. a condensate that is in the ground state with respect to the confined directions, this cancellation works only for the part of \( \psi \) that is in this confined ground state. We thus need to split \( \psi \) accordingly and introduce the following projections on \( L^2(\Omega) \),

\[
p^x := 1 \otimes |\chi \rangle \langle \chi|, \quad q^x = 1 - p^x, \quad p^\Phi = |\Phi \rangle \langle \Phi| \otimes 1, \quad q^\Phi = 1 - p^\Phi.
\]

As in Definition 3.1 we also introduce the projections \( p_j^x, q_j^x, p_j^\Phi \), and \( q_j^\Phi \) on \( L^2(\Omega^N) \). With these projections we can rewrite

\[
q_j = 1 - p_j = 1 - p_j^\Phi p_j^x = (1 - p_j^x) + (1 - p_j^\Phi) p_j^x = q_j^x + q_j^\Phi p_j^x,
\]

(19)
where we recall that $p_j := p_{j}^{0}$. Now with Lemma 4.1, 19 and 13 we find

$$
|I| = N \left| \left\langle \psi, p_1 p_2 \left[ w_{12}^{\varepsilon,N} - b \Phi^2(x_2), \hat{m} \right] p_1 q_2 \psi \right\rangle \right|
$$

(4.1)

$$
\leq N \left| \left\langle \psi, p_1 p_2 \left[ w_{12}^{\varepsilon,N} - b \Phi^2(x_2), \hat{m} \right] p_1 q_2 \psi \right\rangle \right|
$$

(4.3)

$$
\leq \left| \left\langle \psi, p_1 p_2 \left[ \left( \left\| w_{12}^{\varepsilon,N} - b \Phi^2(x_2) \right\| \right) p_1 q_2 \psi \right\rangle \right| + \left| \left\langle \psi, p_1 p_2 \left[ \left( \left\| w_{12}^{\varepsilon,N} \right\| \right) p_1 q_2 \psi \right\rangle \right| \right|
$$

(20)

In the first term (20) the interaction $w_{12}^{0}$ acts between states that are fixed in the $r_1$ and the $y_2$ variable, so only a $x_2$-dependence remains that approximately cancels the mean field $b|\Phi(x_2)|^2$. More precisely, between $p_1 p_2$ and $p_1 p_2^0$ the leading part $w_{12}^{0}$ of the interaction can be replaced by the effective potential

$$
\left\langle \varphi \otimes \chi, \frac{\varepsilon^2}{\mu^3} w \left( \frac{r_1 - r_2}{\mu} \right) \varphi \otimes \chi \right\rangle_{L^2(\Omega \times \Omega)} =
$$

$$
= \frac{\varepsilon^2}{\mu^3} \int |\Phi(x_1)|^2 |\chi(y_1)|^2 w \left( \mu^{-1}(x_1 - x_2), \varepsilon(y_1 - y_2) \right) |\chi(y_2)|^2 \, dx_1 \, dy_1 \, dy_2
$$

$$
= \int |\chi(y_2)|^2 \left( \frac{\varepsilon^2}{\mu^3} \int |\Phi(x_2 - x)|^2 |\chi(y_2 - y)|^2 w \left( \mu^{-1}(x, \varepsilon y) \right) \, dx \, dy \right) dy_2.
$$

(23)

To see that this is close to $b|\Phi(x_2)|^2$, first note that for $f \in C^\infty_0(\Omega)$ we have with $z := (x, y)$ that

$$
\frac{\varepsilon^2}{\mu^3} \int f(z_2 - z) \, w \left( \mu^{-1}(x, \varepsilon y) \right) \, dx \, dy
$$

$$
\left. \left. = \right. \right. f(z_2) \|w\|_{L^1(\mathbb{R}^3)} - \frac{\varepsilon^2}{\mu^3} \int \int_0^1 \nabla f(z_2 - sz) \cdot z \, w \left( \mu^{-1}(x, \varepsilon y) \right) \, ds \, dx \, dy
$$

$$
\left. \left. = \right. \right. f(z_2) \|w\|_{L^1(\mathbb{R}^3)} + R(z_2),
$$

where the $L^2$-norm of the remainder is bounded by

$$
\|R\|_{L^2(\Omega)}^2 \leq \|\nabla f\|_{L^2(\Omega)}^2 \left( \frac{\varepsilon^2}{\mu^3} \int \left| \frac{x, \varepsilon y}{\mu} \right| w \left( \frac{x, \varepsilon y}{\mu} \right) \, dx \, dy \right)^2
$$

$$
= \|\nabla f\|_{L^2(\Omega)}^2 \left( \frac{\varepsilon^2}{\mu^2} \int \left| \frac{x, \varepsilon y}{\mu} \right| w \left( \frac{x, \varepsilon y}{\mu} \right) \, dx \, dy \right)^2
$$

$$
= \|\nabla f\|_{L^2(\Omega)}^2 \left( \mu \varepsilon^2 \int \left| \frac{x, \varepsilon y}{\varepsilon^2} \right| w \left( \frac{x, \varepsilon y}{\varepsilon^2} \right) \, dx \, dy \right)^2
$$

$$
\leq \|\nabla f\|_{L^2(\Omega)}^2 \left( \mu \varepsilon^2 \int \left| \frac{x, \varepsilon y}{\varepsilon^2} \right| w \left( \frac{x, \varepsilon y}{\varepsilon^2} \right) \, dx \, dy \right)^2 \leq \frac{\mu^2}{\varepsilon^2} \|\nabla f\|_{L^2(\Omega)}^2 \|w(z)\|_{L^1(\mathbb{R}^3)}.
$$

23
Hence
\[
\left\| \frac{\varepsilon^2}{\mu^3} \int f(\cdot - z) w \left( \mu^{-1}(x, \varepsilon y) \right) \, dxdy - f \left\| w \right\|_{L^1(\mathbb{R}^3)} \right\|_{L^2(\Omega)} \lesssim \frac{\mu}{\varepsilon} \left\| \nabla f \right\|_{L^2(\Omega)}
\]  
(24)

and this bound extends to \( f \in H^1(\Omega) \) by density, in particular, to \( f = |\Phi|^2 \chi |^2 \). Inserting this bound with (23) into (24) yields, together with Lemma 4.5 (a+d) and Lemma 4.3 the bound
\[
(20) \lesssim \frac{\mu}{\varepsilon} \left\| \nabla |\Phi|^2 \right\|_{L^2(\mathbb{R})} \left\| \Phi \right\|_{L^\infty(\mathbb{R})} \lesssim \frac{\mu}{\varepsilon} \left\| |\Phi|^3 \right\|_{H^2(\mathbb{R})}.
\]
For the term (21) we have with Corollary 4.6 and Lemma 4.3 that
\[
(21) \quad \leq \left( \|T_1 p_1\| + \|T_2 p_1\| \right) \left\| \hat{m}_1 q_2 \psi \right\| \lesssim \frac{(\varepsilon + \mu) \varepsilon}{\mu^{3/2}} \left\| \Phi \right\|_{H^2(\mathbb{R})}.
\]
The term (22) is small due to energy conservation and the energy gap of order \( \varepsilon^{-2} \) between the ground state and the first excited state in the confined direction. With the help of Lemma 4.3 we get
\[
\left\| \left\langle \psi, p_1 p_2 w_{12}^{\varepsilon, \beta, N} \hat{m}_1 q_1 \psi \right\rangle \right\| \leq \left\| \left\langle \psi, p_1 w_{12}^{\varepsilon, \beta, N} \right\rangle \right\| \left\| \left\langle \psi, \hat{m}_1 q_2 \psi \right\rangle \right\| \lesssim \frac{(\varepsilon + \mu) \varepsilon}{\mu^{3/2}} \left\| \Phi \right\|_{H^2(\mathbb{R})}.
\]
(25)

**Proof of the bound for II.** We start with the case of moderate confinement. Using again Lemma 4.1 (c) we find that
\[
\left\| \left\langle \psi, p_1 p_2 w_{12}^{\varepsilon, \beta, N} \hat{m}_2 q_1 \psi \right\rangle \right\| = \left\| \left\langle \psi, p_1 p_2 (\hat{m}_2 q_1) \right\rangle \right\| \lesssim \frac{(\varepsilon + \mu) \varepsilon}{\mu^{3/2}} \left\| \Phi \right\|_{H^2(\mathbb{R})}.
\]
(26)

The second factor of (25) is easily estimated by
\[
\left\| \hat{m}_2^{1/2} q_1 \psi \right\|^2 = \left\langle \psi, \hat{m}_2 q_1 \psi \right\rangle \leq \left\langle \hat{m}_2 q_1 \psi, \hat{m}_2 q_1 \psi \right\rangle \lesssim \alpha \psi.
\]
The first factor of (25) we split into a “diagonal” and an “off-diagonal” term and find
\[
\left\| \left\langle \sum_{j=2}^N q_j w_{1j}^{\varepsilon, \beta, N} (\hat{m}_2 q_1) \right\rangle \right\| \leq \left\| \sum_{j=2}^N \left\langle \psi, q_j p_1 (\hat{m}_2 q_1) \right\rangle \right\| \lesssim \sum_{j=2}^N \left\| \left\langle \psi, q_j p_1 (\hat{m}_2 q_1) \right\rangle \right\|.
\]
(26)
The first summand of (26) is bounded by

\[
(N - 1)(N - 2) \left\| \psi, q_2 p_1 p_3 (\tilde{\tau}_2 m_2)^{\frac{1}{2}} w_{e, \beta, N}^{\psi} q_3 (\tilde{\tau}_2 m_2)^{\frac{1}{2}} p_1 p_2 \psi \right\|
\]
\[
\leq N^2 \left\| \sqrt{w_{e, \beta, N}^{\psi}} \sqrt{w_{e, \beta, N}^{\psi}} q_3 (\tilde{\tau}_2 m_2)^{\frac{1}{2}} p_1 p_2 \psi \right\|^2 
\]
\[
\leq N^2 \left\| \sqrt{w_{e, \beta, N}^{\psi}} p_2 \sqrt{w_{e, \beta, N}^{\psi}} p_1 (\tilde{\tau}_2 m_2)^{\frac{1}{2}} q_3 \psi \right\|^2 
\]
\[
\leq N^2 \left\| \sqrt{w_{e, \beta, N}^{\psi}} p_2 \psi \right\|^4 \left\| (\tilde{\tau}_2 m_2)^{\frac{1}{2}} q_3 \psi \right\|^2
\]
\[
\lesssim N^2 \left\| \Phi \right\|_{H^2(\mathbb{R})}^4 \left\| (\tilde{\tau}_2 m_2)^{\frac{1}{2}} \n \psi \right\|^2 = N^2 \left\| \Phi \right\|_{H^2(\mathbb{R})}^4 \left\| \tilde{\n} \psi, (\tilde{\tau}_2 m_2) \n \tilde{\n} \psi \right\|
\]
\[
\lesssim N^2 \left\| \Phi \right\|_{H^2(\mathbb{R})}^4 \alpha \xi , \tag{27}
\]

where we used that \( \tau_2 m_2 n \) is bounded. The second summand of (26) is bounded by

\[
N \left\| (\tilde{\tau}_2 m_2)^{\frac{1}{2}} \psi, p_1 p_2 (w_{e, \beta, N}^{\psi})^2 p_1 (\tilde{\tau}_2 m_2)^{\frac{1}{2}} \psi \right\| \leq N \left\| p_1 (w_{e, \beta, N}^{\psi})^2 p_1 \right\| \left\| (\tilde{\tau}_2 m_2)^{\frac{1}{2}} \psi \right\|^2
\]
\[
\lesssim N \xi^2 \left\| \Phi \right\|_{H^2(\mathbb{R})}^2 N \xi , \tag{28}
\]
since \( \sup_{1 \leq k \leq N} m_2(k) \leq N^\xi \). Inserting the bounds (27) and (28) into (26), we obtain in continuation of (25) the desired bound,

\[
\left\| \right\| \leq \left( \left\| \Phi \right\|_{H^2(\mathbb{R})}^2 \sqrt{\alpha \xi} + N \frac{1}{\mu^2} \left\| \Phi \right\|_{H^2(\mathbb{R})} N \xi \right) \sqrt{\alpha \xi}
\]
\[
= \left\| \Phi \right\|_{H^2(\mathbb{R})}^2 \alpha \xi + N \xi \sqrt{\frac{a}{\mu^2}} \left\| \Phi \right\|_{H^2(\mathbb{R})} \sqrt{\alpha \xi} \leq \frac{3}{4} \left\| \Phi \right\|_{H^2(\mathbb{R})}^2 \alpha \xi + N \xi \frac{a}{\mu^2} . \tag*{\Box}
\]

In the strongly confining case we can easily estimate

\[
\left\| \right\| \leq \left\| \psi, p_1 p_2 w_{e, \beta, N}^{\psi} \tilde{m}_2 q_1 q_2 \psi \right\| \leq \left\| w_{e, \beta, N}^{\psi} p_1 \right\| \left\| \tilde{m}_2 q_1 q_2 \psi \right\| \lesssim \sqrt{\pi} \left\| \Phi \right\|_{H^2(\mathbb{R})} \sqrt{\alpha \xi}
\]
\[
\leq \left\| \Phi \right\|_{H^2(\mathbb{R})} (\alpha \xi + \mu) .
\]

**Proof of the bound for III.** The same manipulations as before yield

\[
\left\| \right\| = \left| N \left\| \psi, p_1 q_2 \left( w_{e, \beta, N}^{\psi} - b |\Phi|^2(x_1), \tilde{m} \right) q_1 q_2 \psi \right\| \right|
\]
\[
\lesssim \left| \left\| \psi, p_1 q_2 \left( w_{e, \beta, N}^{\psi} - b |\Phi|^2(x_1) \right) \tilde{m} q_1 q_2 \psi \right\| \right|
\]
\[
\leq \left| \left\| \psi, p_1 q_2 w_{e, \beta, N}^{\psi} \tilde{m} q_1 q_2 \psi \right\| \right| + \left| \left\| \psi, p_1 q_2 b |\Phi|^2(x_1) \tilde{m} q_1 q_2 \psi \right\| \right| . \tag{29}
\]

The second summand of (29) is easily bounded by

\[
\left| \left\| \psi, p_1 q_2 b |\Phi|^2(x_1) \tilde{m} q_1 q_2 \psi \right\| \right| \lesssim |q_2 \psi| \left\| \tilde{m} q_1 q_2 \psi \right\| \lesssim \alpha \xi .
\]

For the first term of (29) we use \( q = q^x + p^y q^\phi \) to obtain four terms

\[
\left| \left\| \psi, p_1 q_2 w_{e, \beta, N}^{\psi} \tilde{m} q_1 q_2 \psi \right\| \right| \leq \left| \left| \psi, p_1 q_2 \chi^{\psi} w_{e, \beta, N}^{\psi} \tilde{m} q_1 q_2 \psi \right| \right| + \left| \left| \psi, p_1 p_2 q_2 w_{e, \beta, N}^{\psi} \tilde{m} q_1 q_2 \psi \right| \right|
\]
\[
+ \left| \left| \psi, p_1 p_2 q_2 \Phi w_{e, \beta, N}^{\psi} \tilde{m} q_1 q_2 \psi \right| \right| + \left| \left| \psi, p_1 p_2 q_2 \Phi w_{e, \beta, N}^{\psi} \tilde{m} q_1 q_2 \psi \right| \right| . \tag{30}
\]
All terms but the last are easy to handle. The first term of (30) can be estimated by
\[
|\langle \psi, p_1 p_2 q_2 \Phi w_{12}^{e, \beta, N} \tilde{m}_1 q_1 q_2 \psi \rangle| \leq \| q_2^\chi \psi \| \| w_{12}^{e, \beta, N} p_1 \| \| \tilde{m}_1 q_1 q_2 \psi \|
\]
\[
\lesssim \varepsilon g(t) \frac{\varepsilon^2}{\mu \tau^2} \| \Phi \|_{H^2(\mathbb{R})} \sqrt{\alpha \xi} \leq g(t) \| \Phi \|_{H^2(\mathbb{R})} \left( \alpha \xi + \frac{\varepsilon^4}{\mu^2} \right),
\]
where we used Lemmas 4.1 and 4.3 (b) and Corollary 4.6 in the second step. For the second (and completely analogous the third) term in (30) we find in the same way
\[
|\langle \psi, p_1 p_2 q_2 \Phi w_{12}^{e, \beta, N} \tilde{m}_1 q_1 q_2 \psi \rangle| = |\langle \psi, p_1 p_2 q_2 \Phi \tilde{m}_1^{1/2} q_1 q_2 \psi \rangle|
\]
\[
\lesssim \sqrt{\alpha \xi} \frac{\varepsilon^2}{\mu \tau^2} \| \Phi \|_{H^2(\mathbb{R})} \varepsilon g(t) \leq g(t) \| \Phi \|_{H^2(\mathbb{R})} \left( \alpha \xi + \frac{\varepsilon^4}{\mu^2} \right),
\]
where we used
\[
\left\| \tilde{m}_1^{1/2} q_1 q_2 \psi \right\|^2 = \langle q_1^\chi \psi, \tilde{m}_1 q_1 \psi \rangle = \frac{1}{N-1} \sum_{j=2}^N \langle q_j^\chi \psi, \tilde{m}_1 q_1 \psi \rangle
\]
\[
= \frac{1}{N-1} \left( \sum_{j=1}^N \langle q_j^\chi \psi, \tilde{m}_1 q_1 \psi \rangle - \frac{1}{N-1} \langle q_1^\chi \psi, \tilde{m}_1 q_1 \psi \rangle \right)
\]
\[
= \frac{1}{N-1} \left( \sum_{j=1}^N \langle q_j^\chi \psi, \tilde{m}_1 q_1 \psi \rangle - \langle q_1^\chi \psi, q_1 q_1 \psi \rangle \right)
\]
\[
\leq \frac{1}{N-1} \| q_1^\chi \psi \|^2 + \| q_1^\chi \psi \|^2 \lesssim \varepsilon^2 g(t)^2.
\]
In the last term of (30) we again split the interaction according to Lemma 4.1
\[
|\langle \psi, p_1 p_2 q_2 \Phi w_{12}^{e, \beta, N} \tilde{m}_1 p_1^N q_1 q_2 \psi \rangle| = |\langle \psi, p_1 p_2 q_2 \Phi \tilde{m}_1 q_1 p_2^N q_2 \psi \rangle|
\]
\[
+ |\langle \psi, p_1 p_2 q_2 \Phi (T_1 + T_2) \tilde{m}_1 p_1^N q_1 q_2 \psi \rangle|
\]
asd and bound the second term with the help of Corollary 4.6 and Lemmas 4.1 and 4.3 (b),
\[
|\langle \psi, p_1 p_2 q_2 \Phi (T_1 + T_2) \tilde{m}_1 p_1^N q_1 q_2 \psi \rangle| \lesssim \varepsilon^2 \| \Phi \|_{H^2(\mathbb{R})} \sqrt{\alpha \xi} \leq \| \Phi \|_{H^2(\mathbb{R})} \left( \alpha \xi + \frac{(\varepsilon^2)^2}{\mu^2} \right).
\]
For the leading term containing \( w_{12}^0 \) we have to use a different approach. Here we know that the potential only acts on the function \( \chi \) in the confined directions. Thus, we can replace
\[
p_1 p_2^\chi w_{12}^0 p_1^\chi p_2^\chi = p_1 p_2^\chi w_{12}^0 p_1 p_2^\chi
\]
with
\[
\overline{w}^0(x_1 - x_2) := \frac{1}{\mu} \int_{\mathbb{R}^2} \frac{\varepsilon^2}{\mu^2} w \left( \mu^{-1} (x_1 - x_2, \varepsilon (y_1 - y_2)) \right) |\chi(y_1)|^2 |\chi(y_2)|^2 dy_1 dy_2.
\]
By inspection of the above formula on checks that \( \| \overline{w}^0 \|_{L^1(\mathbb{R})} \lesssim 1 \) and thus its anti-derivative
\[
\overline{W}^0(x) := \int_{-\infty}^x \overline{w}^0(x') dx' \leq \| \overline{w}^0 \|_{L^1(\mathbb{R})}
\]
remains bounded. Integration by parts therefore yields

$$\left\langle \psi, p_1 p_2 q_2 q_2 \tilde{m}_1 \chi_1 \chi_2 \psi \right\rangle = \left\langle \psi, p_1 p_2 q_2 \left( \frac{\partial}{\partial x_1} \tilde{W}_{12} \right) \tilde{m}_1 \chi_1 \chi_2 \psi \right\rangle$$

$$= - \left\langle \psi, \left( \frac{\partial}{\partial x_1} p_1 \right) p_2 q_2 \tilde{W}_{12} \tilde{m}_1 p_1 \chi_1 \chi_2 \psi \right\rangle - \left\langle \psi, p_1 p_2 q_2 \tilde{W}_{12} \frac{\partial}{\partial x_1} \tilde{m}_1 p_1 \chi_1 \chi_2 \psi \right\rangle,$$

where the first term is easily bounded by

$$\left\| \left\langle \psi, \left( \frac{\partial}{\partial x_1} p_1 \right) p_2 q_2 \tilde{W}_{12} \tilde{m}_1 p_1 \chi_1 \chi_2 \psi \right\rangle \right\| \leq \left\| \frac{\partial}{\partial x_1} p_1 \right\| \left\| q_2 \psi \right\| \left\| \tilde{W}_{12} \right\| \left\| \tilde{m}_1 q_2 \psi \right\| \approx \left\| \Phi \right\|_{H^1(\mathbb{R})} \alpha \varepsilon.$$

The second term is

$$\left\| \left\langle \psi, p_1 p_2 q_2 \tilde{W}_{12} \frac{\partial}{\partial x_1} \tilde{m}_1 p_1 \chi_1 \chi_2 \psi \right\rangle \right\| =$$

$$= \left\| \left\langle \psi, \frac{\partial}{\partial x_1} \left( p_1 + q_1 \right) q_2 \frac{\partial}{\partial x_1} \tilde{m}_1 q_1 \chi_1 \chi_2 \psi \right\rangle \right\|$$

$$\leq \left\| q_2 \psi \right\| \left\| \tilde{W}_{12} \right\| \left\| \left( \tilde{m}_1 + \tilde{m}_1 \right) q_2 \frac{\partial}{\partial x_1} q_1 \psi \right\|$$

$$\lesssim \sqrt{\alpha \varepsilon} \left( \left\| \Phi \right\|_{H^2(\mathbb{R})} \left( \alpha \varepsilon + \frac{\mu}{\varepsilon} + \frac{a}{\mu^3} + \varepsilon g(t) \right) \right)$$

$$\lesssim \left\| \Phi \right\|_{H^2(\mathbb{R})} \left( \alpha \varepsilon + \frac{\mu}{\varepsilon} + \frac{a}{\mu^3} \right) + g(t)(\alpha \varepsilon + \varepsilon^2),$$

where we used Lemma 4.7 and for $\ell = 0, 1$

$$\left\| \tilde{t} \tilde{m}_1 q_2 \frac{\partial}{\partial x_1} p_1 \chi_1 \psi \right\|^2 = \left\| \left( q_1 \frac{\partial}{\partial x_1} p_1 \chi_1 \psi, q_2 \tilde{t} \tilde{m}_1 \frac{\partial}{\partial x_1} p_1 \chi_1 \psi \right) \right\|$$

$$\leq \left\| q_1 \frac{\partial}{\partial x_1} p_1 \chi_1 \psi, \sum_{j=1}^{N} q_j \frac{\tilde{t} \tilde{m}_1}{N-1} q_1 \frac{\partial}{\partial x_1} p_1 \chi_1 \psi \right\|$$

$$\leq \left\| q_1 \frac{\partial}{\partial x_1} p_1 \chi_1 \psi, \sum_{k=0}^{N} \frac{\tilde{t} \tilde{m}_1(k)^2}{N-1} \sum_{j=1}^{N} q_j P_k q_1 \frac{\partial}{\partial x_1} p_1 \chi_1 \psi \right\|$$

$$\leq \left\| q_1 \frac{\partial}{\partial x_1} p_1 \chi_1 \psi, \sum_{k=0}^{N} \frac{\tilde{t} \tilde{m}_1(k)^2}{N-1} k P_k q_1 \frac{\partial}{\partial x_1} p_1 \chi_1 \psi \right\|$$

and

$$\left\| q_1 \frac{\partial}{\partial x_1} p_1 \chi_1 \psi \right\| \leq \left\| \left( q_1 p_1 + q_1 p_1 \theta'(x_1) L_1 q_1 \psi \right) \left. \right\| + \left\| p_1 \theta'(x_1) L_1 q_1 \psi \right\|$$

$$= \left\| q_1 p_1 \left( \frac{\partial}{\partial x_1} + \theta'(x_1) L_1 \right) q_1 \psi \right\| + \left\| p_1 \theta'(x_1) L_1 q_1 \psi \right\|$$

$$\leq \left\| \left( \frac{\partial}{\partial x_1} + \theta'(x_1) L_1 \right) q_1 \psi \right\| + \left\| p_1 \theta'(x_1) L_1 \right\| \left\| q_1 \psi \right\|$$

$$\lesssim \left( \left\| \Phi \right\|_{H^2(\mathbb{R})} \left( \alpha \varepsilon + \frac{\mu}{\varepsilon} + \frac{a}{\mu^3} \right) \right)^{\frac{1}{2}} + \varepsilon g(t).$$
In the strongly confining case we find again
\[ |\text{III}| \leq \left| \langle \psi, p_1 q_2 w_{12}^{\xi,\beta,N} \tilde{m}_1 q_2 \psi \rangle \right| \leq \|w_{12}^{\xi,\beta,N} p_1\| \|\tilde{m}_1 q_2 \psi\| \lesssim \sqrt{\mu} \|\Phi\|_{H^2(\mathbb{R})} \sqrt{\alpha_x} \]
\[ \leq \|\Phi\|_{H^2(\mathbb{R})} (\alpha_x + \mu). \]

Proof of the bound for IV. For the first two summands in IV we expand the potential around \( y_1 = 0 \). The assumption A3 guarantees that in both cases the error is a bounded operator. Therefore, we can write
\[ \hat{V}(t, x_1, \varepsilon y_1) = \hat{V}(t, x_1, 0) + \varepsilon Q \]
\[ V(t, x_1, \varepsilon y_1) = V(t, x_1, 0) + \varepsilon \hat{Q} \]
with \( \|Q\|, \|\hat{Q}\| \leq C \). Thus we find
\[ |\langle \psi, p_1 N [V(x_1, \varepsilon y_1) - V(x_1, 0), \tilde{m}] q_1 \psi \rangle| = |\langle \psi, p_1 \varepsilon \hat{Q} \tilde{m}_1 q_1 \psi \rangle| \lesssim \varepsilon \|\tilde{m}_1 q_1 \psi\| \lesssim \varepsilon. \tag{33} \]
For the term containing \( \hat{V} \) we first note that for \( f \in L^\infty(\mathbb{R}) \)
\[ |\langle \psi, f(x_1) \psi \rangle - \langle \Phi, f(x) \Phi \rangle| \lesssim \|f\|_{L^\infty(\mathbb{R})} \alpha_x. \tag{34} \]
Thus we can estimate
\[ |\langle \psi, \hat{V}(x_1, \varepsilon y_1) \psi \rangle - \langle \Phi, \hat{V}(x_1, 0) \Phi \rangle| \lesssim |\langle \psi, \hat{V}(x_1, 0) \psi \rangle - \langle \Phi, \hat{V}(x_1, 0) \Phi \rangle| + \varepsilon \]
\[ \lesssim \|\hat{V}(\cdot, 0)\|_{L^\infty(\mathbb{R})} \alpha_x + \varepsilon. \tag{35} \]
Equation (35) holds since
\[ |\langle \psi, f(x_1) \psi \rangle - \langle \Phi, f(x) \Phi \rangle| \leq |\langle \psi, p_1 f(x_1) p_1 \psi \rangle - \langle \Phi, f(x) \Phi \rangle| + |\langle \psi, q_1 f(x_1) p_1 \psi \rangle| + |\langle \psi, q_1 f(x_1) q_1 \psi \rangle| \]
\[ \leq \alpha_x \langle \Phi, f(x) \Phi \rangle + 2 \|\psi, \tilde{m}_1^{-1/2} p_1 f(x_1) \tilde{m}_1^{-1/2} q_1 \psi\| + \|f\|_{L^\infty(\mathbb{R})} \alpha_x \]
\[ \lesssim \|f\|_{L^\infty(\mathbb{R})} \alpha_x. \tag{41} \]
For the “twisting” term we find
\[ \langle \psi, p_1 ((\theta'(x_1) L_1)^2 + |\theta'(x_1)|^2\|L\chi\|^2) \tilde{m}_1 q_1 \psi \rangle = \langle \psi, p_1 ((\theta'(x_1) L_1)^2 + |\theta'(x_1)|^2\|L\chi\|^2) q_1^\Phi \tilde{m}_1 \psi \rangle \]
\[ + \langle \psi, p_1 ((\theta'(x_1) L_1)^2 + |\theta'(x_1)|^2\|L\chi\|^2) q_1^\chi \tilde{m}_1 \psi \rangle. \]
With
\[ \langle \chi, (\theta'(x) L_1)^2 \psi \rangle_{L^2(\Omega_1)} = -|\theta'(x)|^2 \langle L\chi, L\chi \rangle \]
we see that the first term vanishes identically. For the second term we find with Lemma 4.3(b) that
\[ |\langle \psi, p_1 ((\theta'(x_1) L_1)^2 + |\theta'(x_1)|^2\|L\chi\|^2) q_1^\chi \tilde{m}_1 \psi \rangle| \]
\[ \leq \|((\theta'(x_1) L_1)^2 + |\theta'(x_1)|^2\|L\chi\|^2) p_1 \psi\| \|\tilde{m}_1 q_1^\chi \psi\| \lesssim g(t) N^\xi \varepsilon. \]
The remaining one-body terms are
\[ R^{(1)} = -\partial_x \theta'(x) L - \theta'(x) L \partial_x + \left( V_{\text{bend}}(r) + \frac{\kappa(x)^2}{4} \right) - \varepsilon S^\varepsilon. \]

With \( \langle \chi, L \chi \rangle = 0 \) it holds that
\[ \| \psi, p_1 (\partial_{x_1} \theta'(x_1) L_1 + \theta'(x_1) L_1 \partial_{x_1}) p_1 \hat{m}_1 \psi \| = 0 \]
and for the remaining term
\[ \| \langle \psi, p_1 (\partial_{x_1} \theta'(x_1) L_1 + \theta'(x_1) L_1 \partial_{x_1}) q_1 \hat{m}_1 \psi \| \| \lesssim g(t) N^\varepsilon \]
as before. With
\[ V_{\text{bend}}(r) + \frac{\kappa(x)^2}{4 \rho_\varepsilon(r)^2} = -\varepsilon \frac{T_{\theta(x)} y : \kappa(x)''}{2 \rho_\varepsilon(r)^3} - \varepsilon^2 \frac{5(T_{\theta(x)} y : \kappa'(x))^2}{4 \rho_\varepsilon(r)^4} = O(\varepsilon) \]
we can proceed as in \([33]\) for this part. For the \( S^\varepsilon \) term first note that
\[ s^\varepsilon(r) := \varepsilon^{-1}(\rho_\varepsilon^{-2}(r) - 1) = \frac{2T_{\theta(x)} y : \kappa(x) - \varepsilon(T_{\theta(x)} y : \kappa(x))^2}{(1 - \varepsilon T_{\theta(x)} y : \kappa(x))^2} \]
is uniformly bounded on \( \Omega \) with all its derivatives. Hence
\[ \varepsilon \| \langle \psi, p_1 S^\varepsilon \hat{m}_1 q_1 \psi \| = \varepsilon \| \langle \psi, p_1 (\partial_{x_1} + \theta'(x_1) L_1) s^\varepsilon(r_1) (\partial_{x_1} + \theta'(x_1) L_1) \hat{m}_1 q_1 \psi \| \]
\[ \leq \varepsilon \| (\partial_{x_1} + \theta'(x_1) L_1) s^\varepsilon(r_1) (\partial_{x_1} + \theta'(x_1) L_1) p_1 \psi \| \| \hat{m}_1 q_1 \psi \| \lesssim \varepsilon \| \Phi \|_{H^2(\mathbb{R})}, \]
concluding the bound for IV.

**4.4 Proof of Lemma 4.7**

The strategy is to control the expression in terms of the energy per particle. To this end we observe that
\[ \left\| \left( \frac{\partial}{\partial x_1} + \theta'(x_1) L_1 \right) q_1 \psi \right\|^2 = -\left\langle q_1 \psi, \left( \frac{\partial}{\partial x_1} + \theta'(x_1) L_1 \right)^2 q_1 \psi \right\rangle \]
\[ \leq \left\langle q_1 \psi, \left( \frac{\partial}{\partial x_1} + \theta'(x_1) L_1 \right)^2 - \frac{1}{\varepsilon^2} \Delta y_1 + \frac{1}{\varepsilon^2} V^\perp(y_1) - \frac{E_0}{\varepsilon^2} \right) q_1 \psi \right\rangle \]
\[ \leq 2 \left\langle q_1 \psi, \left( \frac{\partial}{\partial x_1} + \theta'(x_1) L_1 \right) (1 + \varepsilon s^\varepsilon(r_1)) \left( \frac{\partial}{\partial x_1} + \theta'(x_1) L_1 \right) \frac{1}{\varepsilon^2} \Delta y_1 + \frac{1}{\varepsilon^2} V^\perp(y_1) - \frac{E_0}{\varepsilon^2} \right) q_1 \psi \right\rangle \]
\[ =: 2 \left\langle q_1 \psi, \hat{h}_1 q_1 \psi \right\rangle. \]
Hence we have
\[ \left\| \left( \frac{\partial}{\partial x_1} + \theta'(x_1) L_1 \right) q_1 \psi \right\|^2 \leq 2 \left\| \sqrt{\hat{h}_1} q_1 \psi \right\|^2 \leq \left\| \sqrt{\hat{h}_1} (1 - p_1 p_2) \psi \right\|^2 + \left\| \sqrt{\hat{h}_1} p_1 q_2 \psi \right\|^2 \]
\[ \leq \left\langle \psi, (1 - p_1 p_2) \hat{h}_1 (1 - p_1 p_2) \psi \right\rangle + \langle \varphi, \hat{h}_1 \varphi \rangle_{\alpha_\varepsilon}. \]
Note that
\[
\langle \varphi, \tilde{h}_1 \varphi \rangle = -\langle \varphi, (\frac{\partial}{\partial x} + \theta'(x)L_1) (1 + \varepsilon s^x(r)) \left( \frac{\partial}{\partial x} + \theta'(x)L \right) \varphi \rangle \\
\leq -2 \left\langle \varphi, \left( \frac{\partial}{\partial x} + \theta'(x)L \right)^2 \varphi \right\rangle = 2 \left( \| \frac{\partial}{\partial x} \Phi \|^2 + \| \theta'(x) \|^2 \| L \|^2 \| \Phi \|^2 \right) \\
\lesssim \| \Phi \|^2_{H^1(\mathbb{R})}.
\]
Then, after expanding and rearranging the energy difference
\[
E^\psi - E^\Phi = \frac{1}{\varepsilon} \langle \psi, H(t) \psi \rangle - \frac{E_0}{\varepsilon} - \left\langle \Phi, E^\Phi(t) \Phi \right\rangle_{L^2(\mathbb{R})} \\
= \left\langle \psi, \left( \tilde{h}_1 + \frac{\varepsilon}{\varepsilon_1^2} w_{N} + V(x_1, \varepsilon y_1) + V_{\text{bend}}(r_1) \right) \psi \right\rangle \\
- \left\langle \Phi, \left( -\frac{\partial^2}{\partial x^2} - \kappa(x)^2 + |\theta'(x)|^2 \| L \|^2 + V(x, 0) + \frac{\varepsilon}{2} |\Phi|^2 \right) \Phi \right\rangle_{L^2(\mathbb{R})}
\]
we arrive at
\[
\left\langle \psi, \left( 1 - p_1 p_2 \right) \tilde{h}_1 \left( 1 - p_1 p_2 \right) \psi \right\rangle = E^\psi - E^\Phi \\
- \left( \left\langle \psi, p_1 p_2 \tilde{h}_1 p_1 p_2 \psi \right\rangle - \left\langle \varphi, -\frac{\partial^2}{\partial x^2} - \frac{1}{\varepsilon} (\Delta_y + E_0) + |\theta'(x)|^2 \| L \|^2 \varphi \right\rangle \right) \tag{35} (35) \\
- \left( \left\langle \psi, \left( 1 - p_1 p_2 \right) \tilde{h}_1 p_1 p_2 \psi \right\rangle - \left\langle \psi, p_1 p_2 \tilde{h}_1 \left( 1 - p_1 p_2 \right) \psi \right\rangle \right) \tag{36} \tag{36} \\
- \frac{1}{2} \left( \left\langle \psi, p_1 p_2 w_1^{\varepsilon, \beta, N} p_1 p_2 \psi \right\rangle - \left\langle \Phi, b |\Phi|^2 \Phi \right\rangle \right) \tag{37} \tag{37} \\
- \frac{1}{2} \left( \left\langle \psi, \left( 1 - p_1 p_2 \right) w_1^{\varepsilon, \beta, N} p_1 p_2 \psi \right\rangle + \left\langle \psi, p_1 p_2 w_1^{\varepsilon, \beta, N} \left( 1 - p_1 p_2 \right) \psi \right\rangle \right) \tag{38} \tag{38} \\
- \frac{1}{2} \left( \left\langle \psi, \left( 1 - p_1 p_2 \right) w_1^{\varepsilon, \beta, N} \left( 1 - p_1 p_2 \right) \psi \right\rangle \right) \tag{39} \tag{39} \\
- \left( \left\langle \psi, V(x_1, \varepsilon y_1) \psi \right\rangle - \left\langle \Phi, V(x, 0) \Phi \right\rangle \right) + \left( \left\langle \psi, \frac{\kappa(x)^2}{4} \psi \right\rangle - \left\langle \Phi, \frac{\kappa(x)^2}{4} \Phi \right\rangle \right) \tag{40} \tag{40} \\
- \left\langle \psi, \left( V_{\text{bend}}(r_1) + \frac{\kappa(x)^2}{4} \right) \psi \right\rangle. \tag{41} \tag{41}
\]
We will estimate each line separately. For (35) we find
\[
\begin{align*}
\left| \left\langle \psi, p_1 p_2 \tilde{h}_1 p_1 p_2 \psi \right\rangle - \left\langle \varphi, \tilde{h}_1 \varphi \right\rangle \right| + \varepsilon \left| \left\langle \varphi, \tilde{h}_1 \varphi \right\rangle \right| \\
= \left| \left\langle \varphi, \tilde{h}_1 \varphi \right\rangle \right| \left| \left\langle \psi, p_1 p_2 \psi \right\rangle - \left\langle \varphi, \tilde{h}_1 \varphi \right\rangle \right| + \varepsilon \left| \left\langle \varphi, \tilde{h}_1 \varphi \right\rangle \right| \\
= \left| \left\langle \varphi, \tilde{h}_1 \varphi \right\rangle \right| \left| \left\langle \psi, (1 - p_1 p_2) \psi \right\rangle \right| + \varepsilon \left| \left\langle \varphi, \tilde{h}_1 \varphi \right\rangle \right| \\
\overset{1.1}{\lesssim} \| \Phi \|^2_{H^1(\mathbb{R})} (\alpha_\xi + \varepsilon),
\end{align*}
\]
and (36) is bounded in absolute value by
\[
\left| \left( \left\langle \psi, (1 - p_1 p_2) \tilde{h}_1 p_1 p_2 \psi \right\rangle - \left\langle \psi, (1 - p_1 p_2) \tilde{h}_1 p_1 p_2 \psi \right\rangle \right) \right| \leq 2 \left| \left\langle \psi, p_1 p_2 \tilde{h}_1 p_1 p_2 \psi \right\rangle \right| = 2 \left| \left\langle \psi, q_1 \tilde{n}^{-\frac{1}{2}} \tilde{h}_1 \tau_1 \tilde{n}^{\frac{1}{2}} p_1 p_2 \psi \right\rangle \right| \\
\leq 2 \left| \tilde{n}^{-\frac{1}{2}} q_1 \right| \left| \tilde{n}^{\frac{1}{2}} \right| \left| \tilde{n}^{-\frac{1}{2}} \tilde{h}_1 \tau_1 \tilde{n}^{\frac{1}{2}} \right| \lesssim \sqrt{\alpha_\xi} \| \Phi \|_{H^2(\mathbb{R})} \sqrt{\alpha_\xi + \frac{1}{\sqrt{N}}} \lesssim \| \Phi \|_{H^2(\mathbb{R})} \left( \alpha_\xi + \frac{1}{\sqrt{N}} \right).
\]
For (37) we first note that
\[
\left| \left\langle \Phi, b |\Phi|^2 \Phi \right\rangle - \left\langle \psi, p_1 p_2 b |\Phi|^2 \psi \right\rangle \right| = \left| \left\langle \Phi, b |\Phi|^2 \Phi \right\rangle - \left\langle \psi, (1 - p_1 p_2) \psi \right\rangle \right| \lesssim \| \Phi \|^2_{L^\infty(\mathbb{R})} \alpha_\xi.
\]
30
Hence,

\[
\left|37\right| \leq \left|\psi, p_1 p_2 \left( b \Phi^2 - w_{12}^{\varepsilon, \beta, N} \right) p_1 p_2 \psi \right| + \left\| \Phi \right\|_{L^\infty(\mathbb{R})}^2 \alpha \xi \\
\leq \left|\psi, p_1 p_2 \left( b \Phi^2 - w_{12}^0 \right) p_1 p_2 \psi \right| + \left\| (T_1 + T_2) p_1 \right\| + \left\| \Phi \right\|_{L^\infty(\mathbb{R})}^2 \alpha \xi \\
\leq \frac{\mu}{\varepsilon} \left\| \nabla \Phi \right\|_{L^2(\mathbb{R})} \left\| \Phi \right\|_{L^8(\mathbb{R})} + \frac{\varepsilon(\varepsilon + \mu)}{\mu^{3/2}} \left\| \Phi \right\|_{H^2(\mathbb{R})} + \left\| \Phi \right\|_{L^\infty(\mathbb{R})}^2 \alpha \xi .
\]

For \((38)\) we have that

\[
\left|38\right| \leq 2 \left|\psi, p_1 p_2 w_{12}^{\varepsilon, \beta, N} (1 - p_1 p_2) \psi \right| = \left|\psi, p_1 p_2 w_{12}^{\varepsilon, \beta, N} (q_1 p_2 + p_1 q_2 + q_1 q_2) \psi \right| \\
\leq 2 \left|\psi, p_1 p_2 w_{12}^{\varepsilon, \beta, N} q_1 p_2 \psi \right| + \left|\psi, p_1 p_2 w_{12}^{\varepsilon, \beta, N} q_1 q_2 \psi \right| .
\]

The first summand in \((42)\) is bounded by

\[
\left|\psi, p_1 p_2 w_{12}^{\varepsilon, \beta, N} q_1 p_2 \psi \right| = \left|\psi, p_1 p_2 \frac{1}{n} \frac{1}{n - \frac{1}{2}} q_1 p_2 \psi \right| \\
\leq \left\| p_2 w_{12}^{\varepsilon, \beta, N} p_2 \right\| \left\| \frac{1}{n} \frac{1}{n - \frac{1}{2}} q_1 \psi \right\| \lesssim \left\| \Phi \right\|_{H^2(\mathbb{R})} (\alpha \xi + \frac{1}{\sqrt{N}}) .
\]

For the second summand in \((42)\) we first use symmetry to write

\[
\left|\psi, p_1 p_2 w_{12}^{\varepsilon, \beta, N} q_1 q_2 \psi \right| = \frac{1}{N - 1} \left| \sum_{j=0}^{N} \psi, p_1 p_j w_{1j}^{\varepsilon, \beta, N} q_j q_j \psi \right| \\
\leq \frac{1}{N - 1} \left| \sum_{j=0}^{N} q_j w_{1j}^{\varepsilon, \beta, N} p_1 p_j \right| \leq \sqrt{\frac{\alpha \xi}{N - 1}} \left| \sum_{j=0}^{N} w_{1j}^{\varepsilon, \beta, N} p_1 p_j \right| .
\]

Now the second factor can be split into a “diagonal” and an “off-diagonal” term,

\[
\left| \sum_{j=0}^{N} q_j w_{1j}^{\varepsilon, \beta, N} p_1 p_j \right|^2 = \sum_{j,k=0}^{N} \left| \psi, p_1 p_j w_{1j}^{\varepsilon, \beta, N} q_j q_j \right| p_1 p_j \psi \right| \\
\leq \sum_{2 \leq j < k \leq N} \left| \psi, q_j p_1 p_j w_{1j}^{\varepsilon, \beta, N} q_j p_1 p_j \psi \right| (N - 1) \left| w_{1j}^{\varepsilon, \beta, N} p_1 p_j \psi \right|^2 .
\]

The “off-diagonal” term is bounded by

\[
(N - 1)(N - 2) \left| q_2 p_1 p_3 w_{13}^{\varepsilon, \beta, N} \right| w_{12}^{\varepsilon, \beta, N} q_3 p_1 p_2 \psi \right|^2 \leq N^2 \left| \sqrt{w_{12}^{\varepsilon, \beta, N} p_2} \sqrt{w_{13}^{\varepsilon, \beta, N} p_1 q_3} \psi \right|^2 \\
\leq N^2 \left| \sqrt{w_{12}^{\varepsilon, \beta, N} p_2} \right|^4 \left| q_3 \psi \right|^2 \lesssim N^2 \left\| \Phi \right\|_{H^2(\mathbb{R})}^4 \alpha \xi .
\]

The “diagonal” term is bounded by

\[
N \left| \psi, p_1 p_2 \left( w_{12}^{\varepsilon, \beta, N} \right)^2 p_1 p_2 \psi \right| \leq N \left| p_1 \left( w_{12}^{\varepsilon, \beta, N} \right) p_1 \right|^2 \leq \frac{\varepsilon_{\beta}^2}{\mu^3} \left\| \Phi \right\|_{H^2(\mathbb{R})}^2 .
\]

31
and we conclude that the second summand of \((12)\) is bounded by
\[
\left| \left\langle \psi, p_1 p_2 w_\varepsilon, \beta, N_{12}, q_1 q_2 \psi \right\rangle \right| \lesssim \frac{\sqrt{\alpha \varepsilon}}{N} \sqrt{N^2 \| \Phi \|^4_{H^2(\mathbb{R})} \alpha \varepsilon} + \frac{N \varepsilon^2}{\mu^3} \| \Phi \|^2_{H^2(\mathbb{R})}
\]
\[
\leq \| \Phi \|^2_{H^2(\mathbb{R})} \alpha \varepsilon + \frac{\sqrt{\alpha \varepsilon}}{N} \sqrt{\frac{\varepsilon^2}{\mu^3}} \| \Phi \|_{H^2(\mathbb{R})} \lesssim \| \Phi \|^2_{H^2(\mathbb{R})} \alpha \varepsilon + \frac{\varepsilon^2}{N \mu^3}.
\]

In summary we thus have that
\[
(39) \lesssim \| \Phi \|^2_{H^2(\mathbb{R})} \left( \alpha \varepsilon + \frac{1}{\sqrt{N}} \right) + \frac{\varepsilon^2}{N \mu^3}.
\]

Since the interaction is non-negative, we have \((39) \leq 0\). With the same arguments as used in the proof of Proposition 3.3 part IV we find
\[
(40) \lesssim \alpha \varepsilon + \varepsilon,
\]
and obviously \((41) \lesssim \varepsilon\). In summary we thus showed
\[
\left\| \left( \frac{\partial}{\partial x_1} + \theta'(x_1) L_1 \right) q_1 \right\|^2 \lesssim \| \Phi \|^3_{H^2(\mathbb{R})} \left( \alpha \varepsilon + \frac{1}{\sqrt{N}} + \varepsilon + \frac{\mu}{\varepsilon} + \frac{\sqrt{\mu}}{\mu^3} + \frac{a}{\mu^3} \right)
\]
and with
\[
\varepsilon \lesssim \frac{\mu}{\varepsilon}, \quad \frac{1}{\sqrt{N}} \lesssim \frac{\mu}{\varepsilon}, \quad \sqrt{\mu} \lesssim \frac{\mu}{\varepsilon},
\]
which holds for moderate confinement, the statement of the lemma follows.

### A Well-posedness of the dynamical equations

The Hamiltonian \(H_{T_\varepsilon}(t)\) given in \((1)\) is self-adjoint on \(H^2(T_\varepsilon^N) \cap H^1_0(T_\varepsilon^N)\) for every \(t \in \mathbb{R}\), since the potentials \(V\) and \(w\) are bounded by assumptions A2 and A3. Hence \((U_{T_\varepsilon})^N H_{T_\varepsilon}(t)(U_{T_\varepsilon}^*)^N + \sum_{i=1}^N \frac{1}{2} V_\varepsilon(y_i)\) is self-adjoint on \(U_{T_\varepsilon} H^2(T_\varepsilon^N) \cap U_{T_\varepsilon} H^1_0(T_\varepsilon^N) = H^2(\Omega^N) \cap H^1_0(\Omega^N)\), as \(\sum_{i=1}^N \frac{1}{2} V_\varepsilon(y_i)\) is relatively bounded with respect to \((U_{T_\varepsilon})^N H_{T_\varepsilon}(t)(U_{T_\varepsilon}^*)^N\) with relative bound smaller than one. Finally \(t \mapsto V(t) \in \mathcal{L}(L^2)\) is continuous, so \(H(t)\) generates an strongly continuous evolution family \(U(t, 0)\) such that for \(\psi_0 \in H^2(\Omega^N) \cap H^1_0(\Omega^N)\) the map \(t \mapsto U(t, 0)\psi_0\) satisfies the time-dependent Schrödinger equation.

Although the questions of well-posedness, global existence and conservation laws for the NLS equation in our setting are well understood, we couldn’t find a reference for global existence of \(H^2\)-solutions to (3) with time-dependent potential. We thus briefly comment on this point. The standard contraction argument (see e.g. Proposition 3.8 (33)) gives unique local existence of \(H^s\)-solutions \(\Phi(t)\) for all \(\frac{1}{2} < s \leq 4\), since under the hypotheses A1 and A3 on the external potential and the waveguide all potentials appearing in (3) are \(C^4\). Moreover, \(\|\Phi(t)\|_{L^2} = \|\Phi(0)\|_{L^2}\) and the solution map \(\Phi(0) \mapsto \Phi(t)\) is continuous in \(H^s\). See [32] for the details of this argument in the case of time-dependent potentials.

In order to show also global existence, assume without loss of generality (we can always add a real constant to the potential) that
\[
\inf_{t,x \in \mathbb{R}} \left( -\frac{\kappa^2(x)}{4} + |\theta'(x)|^2 \|L\chi\|^2 + V(t, x, 0) \right) \geq 0
\]
and recall the definition $E^{\Phi(t)}(t) := \langle \Phi(t), E^{\Phi(t)}(t)\Phi(t) \rangle$ in (7). Then for $\Phi(t) \in H^2$ the map $t \mapsto E^{\Phi(t)}(t)$ is differentiable and we have

$$
\| \Phi(t) \|^2_{H^1} \leq E^{\Phi(t)}(t) + 1 = E^{\Phi(0)}(0) + 1 + \int_0^t \frac{d}{ds} E^{\Phi(s)}(s) \, ds
$$

$$
= E^{\Phi(0)}(0) + 1 + \int_0^t \langle \Phi(s), \nabla s \rangle \, ds
$$

$$
\leq C \| \Phi(0) \|^2_{H^1} + \| \Phi(0) \|^2 \int_0^t \| \nabla s \|_{L^\infty} \, ds,
$$

which, by continuity of the solution map, extends to $\Phi(t) \in H^1$. Hence $\| \Phi(t) \|_{L^\infty} \leq \| \Phi(t) \|_{H^1}$ cannot blow up in finite time, which implies global existence of $H^1$-solutions.

To control also the $H^2$-norm, first note that with

$$
\| E^{\Phi(t)}(t) \Phi(t) \|^2 = \langle \Phi(t), - \frac{\partial^2}{\partial x^2} - \frac{\kappa^2}{4} + |\theta'|^2 \| L x \|^2 + V(t, x, 0) + \frac{b}{2} |\Phi(t)|^2 \rangle \Phi(t)
$$

$$
\geq \left\| \frac{\partial^2}{\partial x^2} \Phi(t) \right\|^2 + 2 \mathcal{R} \langle \Phi(t), - \frac{\partial^2}{\partial x^2} \left( - \frac{\kappa^2}{4} + |\theta'|^2 \| L x \|^2 + V(t, x, 0) + \frac{b}{2} |\Phi(t)|^2 \right) \Phi(t) \rangle
$$

and

$$
| \langle \Phi(t), \frac{\partial^2}{\partial x^2} (f + b |\Phi(t)|^2) \Phi(t) \rangle | \leq | \langle \Phi(t), (f + b |\Phi(t)|^2) \Phi(t) \rangle | + | \langle \Phi(t), \frac{b}{2} \Phi(t) \Phi(t) \rangle | + | \langle \Phi(t), f' \Phi(t) \rangle |
$$

$$
\leq \| \Phi(t) \|^2_{H^1} (C + b \| \Phi(t) \|^2_{L^\infty}) + \frac{b}{2} \| \Phi(t) \|^2_{H^1} \| \Phi(t) \|^2_{L^\infty} + C \| \Phi(t) \|_{H^1}
$$

$$
\leq C_1 \| \Phi(t) \|^4_{H^1}
$$

for some constant $C_1 \in \mathbb{R}$ we have

$$
\left\| \frac{\partial^2}{\partial x^2} \Phi(t) \right\|^2 \leq \| E^{\Phi(t)}(t) \Phi(t) \|^2 + 2 C_1 \| \Phi(t) \|^4_{H^1}.
$$

Moreover, for $\Phi(t) \in H^4(\mathbb{R})$ we have

$$
\| E^{\Phi(t)}(t) \Phi(t) \|^2 + 1 = \| E^{\Phi(0)}(0) \Phi(0) \|^2 + 1 + \int_0^t \frac{d}{ds} \left\langle E^{\Phi(s)}(s) \Phi(s), E^{\Phi(s)}(s) \Phi(s) \right\rangle \, ds
$$

$$
= \| E^{\Phi(0)}(0) \Phi(0) \|^2 + 1 + 2 \int_0^t \mathcal{R} \langle \nabla s \Phi(s), E^{\Phi(s)}(s) \Phi(s) \rangle \, ds
$$

$$
\leq C_2 \| \Phi(0) \|^2_{H^2} + C_3 \int_0^t \| E^{\Phi(s)}(s) \Phi(s) \| \, ds
$$

$$
\leq C_2 \| \Phi(0) \|^2_{H^2} + C_3 \int_0^t \left( \| E^{\Phi(s)}(s) \Phi(s) \|^2 + 1 \right) \, ds.
$$

An application of the Grönwall inequality yields a bound of $\| E^{\Phi(t)}(t) \Phi(t) \|^2$ in terms of $\| \Phi(0) \|^2_{H^2}$, which, again by continuity of the solution map, extends to $\Phi(t) \in H^2(\mathbb{R})$. Hence the $H^2$-norm of $\Phi(t)$ remains bounded on bounded intervals in time.
References

[1] R. Adami, F. Golse, and A. Teta. Rigorous derivation of the cubic NLS in dimension one. *J. Stat. Phys.* 127(6): 1193–1220, 2007.

[2] N. Ben Abdallah, F. Méhats, C. Schmeiser, and R. Weishäupl. The nonlinear Schrödinger equation with a strongly anisotropic harmonic potential. *SIAM Journal on Mathematical Analysis*, 37(1):189–199, 2005.

[3] N. Benedikter, G. De Oliveira, and B. Schlein. Quantitative derivation of the Gross-Pitaevskii equation. *Communications on Pure and Applied Mathematics*, 68(8):1399–1482, 2015.

[4] N. Benedikter, M. Porta, and B. Schlein. Effective Evolution Equations from Quantum Dynamics. arXiv:1502.02498, 2015.

[5] X. Chen and J. Holmer. On the rigorous derivation of the 2d cubic nonlinear Schrödinger equation from 3d quantum many-body dynamics. *Archive for Rational Mechanics and Analysis*, 210(3):909–954, 2013.

[6] X. Chen and J. Holmer. Focusing quantum many-body dynamics II: the rigorous derivation of the 1d focusing cubic nonlinear Schrödinger equation from 3d. arXiv:1407.8457, 2014.

[7] L. Erdös, B. Schlein, and H.-T. Yau. Derivation of the cubic non-linear Schrödinger equation from quantum dynamics of many-body systems. *Invent. Math.*, 167:515–614, 2007.

[8] L. Erdös and H.-T. Yau. Derivation of the nonlinear Schrödinger equation from a many body Coulomb system. *Adv. Theor. Math. Phys.*, 5(6):1169–1205, 2001.

[9] J. Fortágh and C. Zimmermann. Magnetic microtraps for ultracold atoms. *Rev. Mod. Phys.*, 79(1):235–289, 2007.

[10] F. Golse. On the Dynamics of Large Particle Systems in the Mean Field Limit. arXiv:1301.5494, 2013.

[11] A. Görlich, J. M. Vogels, A. E. Leanhardt, C. Raman, T. L. Gustavson, J. R. Abo-Shaeer, A. P. Chikkatur, S. Gupta, S. Inouye, T. Rosenband, and W. Ketterle. Realization of Bose-Einstein condensates in lower dimensions. *Phys. Rev. Lett.*, 87:130402, 2001.

[12] M. Grillakis and M. Machedon. Pair excitations and the mean field approximation of interacting Bosons. *Commun. Math. Phys.*, 324:601–636, 2013.

[13] S. Haag, J. Lampart, and S. Teufel. Generalised quantum waveguides. *Annales Henri Poincaré*, 16:2535–2568, 2015.

[14] K. Henderson, C. Ryu, C. MacCormick, and M. G. Boshier. Experimental demonstration of painting arbitrary and dynamic potentials for Bose-Einstein condensates. *New Journal of Physics*, 11(4):043030, 2009.
[15] A. Knowles and P. Pickl. Mean-field dynamics: singular potentials and rate of convergence. *Comm. Math. Phys.*, 298(1):101–138, 2010.

[16] D. Krejčiřík. Twisting versus bending in quantum waveguides. *Analysis on Graphs and its Applications: Proceedings of the Symposium on Pure Mathematics, American Mathematical Society*, 617–636, 2008.

[17] J. Lampart and S. Teufel. The adiabatic limit of Schrödinger operators on fibre bundles. To appear in *Mathematische Annalen* (see also arXiv:1402.0382, 2014).

[18] M. Lewin, P.T. Nam, and N. Rougerie. The mean-field approximation and the non-linear Schrödinger functional for trapped Bose gases. *Trans. Amer. Math. Soc.*, 368:6131-6157, 2016.

[19] E. H. Lieb, R. Seiringer, J. P. Solovej, and J. Yngvason. *The mathematics of the Bose gas and its condensation*, Volume 34 of Oberwolfach Seminars. Birkhäuser, 2005.

[20] E. H. Lieb, R. Seiringer, and J. Yngvason. One-dimensional behaviour of dilute, trapped Bose gases. *Comm. Math. Phys.*, 244(2):347–393, 2004.

[21] F. Méhats and N. Raymond. Strong confinement limit for the nonlinear Schrödinger equation constrained on a curve. arXiv:1412.1049, 2014.

[22] P.T. Nam, N. Rougerie, and R. Seiringer. Ground states of large bosonic systems: The Gross-Pitaevskii limit revisited. arXiv:1503.07061, 2015.

[23] P.T. Nam and M. Napiórkowski. Bogoliubov correction to the mean-field dynamics of interacting bosons, arXiv:1509.04631, 2015.

[24] P. Pickl. On the time dependent Gross-Pitaevskii- and Hartree equation. arXiv:0808.1178, 2008.

[25] P. Pickl. Derivation of the time dependent Gross-Pitaevskii equation with external fields. *Rev. Math. Phys.*, 27:1550003, 2015. (see also arXiv:1001.4894).

[26] P. Pickl. Derivation of the time dependent Gross-Pitaevskii equation without positivity condition on the interaction. *J. Stat. Phys.* 140(1):76–89, 2010.

[27] P. Pickl. A simple derivation of mean field limits for quantum systems. *Lett. Math. Phys.*, 97(2):151–164, 2011.

[28] I. Rodnianski and B. Schlein. Quantum fluctuations and rate of convergence towards mean field dynamics. *Comm. Math. Phys.*, 291(1):31–61, 2009.

[29] N. Rougerie. De finetti theorems, mean-field limits and bose-Einstein condensation. arXiv:1506.05263, 2015.

[30] K. Schnee and J. Yngvason. Bosons in Disc-Shaped Traps: From 3D to 2D. *Comm. Math. Phys.*, 269(3):659–691, 2006.

[31] B. Schlein. Derivation of Effective Evolution Equations from Microscopic Quantum Dynamics. arXiv:0807.4307, 2008.
[32] C. Sparber. Weakly nonlinear time-adiabatic theory. *Annales Henri Poincaré*, Online First, 2015 (see also arXiv:1411.0335).

[33] T. Tao. *Nonlinear dispersive equations. Local and global analysis*. CBMS Regional Conference Series in Mathematics, 106, Volume 200, 2006.

[34] J. Wachsmuth and S. Teufel. Effective Hamiltonians for constrained quantum systems. *Memoirs of the AMS* 1083 (2013).