We investigate the connection between the time-evolution of averages of stochastic quantities and the Fisher information and its induced statistical length. As a consequence of the Crámer-Rao bound, we find that the rate of change of the average of any observable is bounded from above by its variance times the temporal Fisher information. As a consequence of this bound, we obtain a speed limit on the evolution of stochastic observables: Changing the average of an observable requires a minimum amount of time given by the change in the average squared, divided by the fluctuations of the observable times the thermodynamic cost of the transformation. In particular for relaxational dynamics, which do not depend on time explicitly, we show that the Fisher information is a monotonically decreasing function of time and that this minimal required time is determined by the initial preparation of the system. We further show that the monotonicity of the Fisher information can be used to detect hidden variables in the system and demonstrate our findings for simple examples of continuous and discrete random processes.

I. INTRODUCTION

Information geometry [1] is a branch of information theory which describes information in terms of differential geometry. This can be motivated by a question central to any physical experiment: Given a system described by a set of parameters, how much information about the system can we gain from a slight variation of the parameters? Under certain regularity conditions, how the system changes under such a small parameter variation defines a metric, the so-called Fisher information metric [2–5]. This metric encodes the maximum amount of information that can be gained by measuring the change of any observable due to the parameter change.

The relation between the measurement of (macroscopic) observables and information gained about the physical system is also central to thermodynamics. Deciding which observables to measure and which parameters to vary in doing so is essential for reconstructing the thermodynamic potentials and thus obtaining a complete information about the macroscopic state of the system. Thus, it is not surprising that there exists a strong connection between thermodynamics and information theory, which, despite dating back all the way to Gibbs and Boltzmann [6], has recently received much attention [7–10]. This is in part motivated by improved experimental techniques, allowing to probe the relation between information and thermodynamic quantities on a more detailed and microscopic level [17, 18], but also by new theoretical proposals based on understanding information as a quantity that is just as physical as matter or energy.

Recently, a connection between information geometry and stochastic thermodynamics was established in Ref. [19]. Stochastic thermodynamics describes the behavior of thermodynamic quantities like heat, work and entropy in small systems, where these quantities fluctuate due to the presence of noise [20, 21]. In particular, Ref. [19] found an intimate connection between Fisher information and stochastic entropy. In this case, the parameter, whose change is described by the Fisher information, is time. Thus, the temporal Fisher information quantifies how much information can be gained from the time evolution of the system.

In this work, our aim is to expand on the idea of describing the time evolution of a Markovian stochastic system in terms of information, and to elucidate the consequences for the behavior of measurable observables. Our first result is a speed limit on the time evolution of any observable, which is related to the Crámer-Rao bound [22, 23]. Specifically, the rate of change of an observable is bounded from above by its fluctuations times the temporal Fisher information. This provides a measurable consequence of the Fisher information as the maximum obtainable information. Interpreting the Fisher information as a thermodynamic cost, this result complements a class of recently derived steady-state thermodynamic uncertainty relations [24–28]. As our second main result, we show that if the stochastic system describes a relaxation process without time-dependent driving, then the temporal Fisher information is a monotonically decreasing function of time. Thus, the amount of information that can be gained by observing a relaxation process gradually decreases. Together with our first result, this provides an explicit quantification of the physical intuition that the time evolution of a system should gradually slow down during a relaxation process. The monotonicity of the Fisher information for relaxation processes has two profound consequences: First, it results in a lower bound on the time required to evolve a stochastic system from an initial to a final configuration, extending previously obtained speed limits for stochastic dynamics [19, 29]. Second, it can serve as an indicator for the presence of hidden variables in the system: If we observe an increase of the Fisher information during a relaxation process, this necessarily implies that we are missing some infor-
mation about the system. We show that this discrepancy between observed and total information can be used to detect hidden degrees of freedom.

The present paper is organized as follows: In Section II, we introduce the Fisher information and some of its basic properties. In Section III we show how, in the case of a time-dependent stochastic system, the Cr´amer-Rao bound leads to a speed limit for arbitrary observables. We briefly discuss the relation between this speed limit and previously obtained bounds. Section IV contains the explicit proof of the monotonicity of the Fisher information in Markovian dynamics without explicit time-dependence, followed by a more general argument based on the relation between Fisher information and the Kullback-Leibler divergence. In Section V we use the monotonicity to derive a generalization of a previously obtained speed limit for stochastic dynamics. As a second consequence, we show in Section VI how a non-monotonic behavior of the Fisher information can be used to detect hidden degrees of freedom in the system. Section VII is dedicated to examples that provide an explicit demonstration of the general results of the previous sections; starting from the explicit expression for the Fisher information for an arbitrary normal distribution, we discuss the paradigmatic examples of diffusion and a particle in a parabolic potential. We go on to construct a simple jump process and show that the behavior of the Fisher information behaves qualitatively differently depending on whether hidden states are present in the system or not. We finish with some concluding remarks and outlook in Section VIII.

II. LENGTH AND FISHER INFORMATION

We study a general stochastic system that is described by a probability density \( P(\mathbf{X} = \mathbf{x} | \theta) \equiv P(\mathbf{x}, \theta) \), where \( \mathbf{X} \) is a vector of \( M \) continuous random variables \( \mathbf{X} = (X_1, \ldots, X_M) \in \mathbb{R}^M \) and \( \theta \in \mathbb{R} \) is a parameter. If \( \theta \) is equal to the observation time \( t \in [0, T] \) then \( P(\mathbf{x}, t) \) describes the time evolution of the probability density. However, \( \theta \) may also be some other, more general parameter, e. g. \( P(\mathbf{x}, \theta) \) could be the steady state probability density of the system and \( \theta \) some externally tunable field. In the following, we assume that \( P(\mathbf{x}, \theta) \) depends smoothly on \( \theta \), such that, in particular, the derivative \( \partial_\theta P(\mathbf{x}, \theta) \) exists and is a continuous function and the second derivative \( \partial_\theta^2 P(\mathbf{x}, \theta) \) exists. The Fisher information \( I(\theta) \) is defined by [30]

\[
I(\theta) = \int d\mathbf{x} \frac{\left( \partial_\theta P(\mathbf{x}, \theta) \right)^2}{P(\mathbf{x}, \theta)} \quad (1)
\]

\[
= \left\langle (\partial_\theta \ln P)^2 \right\rangle_\theta - \left\langle (\partial_\theta^2 \ln P) \right\rangle_\theta,
\]

where \( \langle \ldots \rangle_\theta \) denotes an average with respect to \( P(\mathbf{x}, \theta) \). Here, and in the following, we use the shorthand \( \partial_\theta f = \partial f / \partial \theta \) for partial and \( d_\theta f = df / d\theta \) for total derivatives.

The last equality follows from the normalization of the probability density \( \partial_\theta \left[ \int d\mathbf{x} P(\mathbf{x}, \theta) = \partial\theta \right] = 0 \). We note that, by definition, the Fisher information is positive and vanishes only if the probability density is independent of \( \theta \). The Fisher information is related to the Kullback-Leibler divergence or relative entropy between two distributions \( P(\mathbf{x}) \) and \( Q(\mathbf{x}) \),

\[
D_{\text{KL}}(Q \shortmid P) = \int d\mathbf{x} \ Q(\mathbf{x}) \ln \left( \frac{Q(\mathbf{x})}{P(\mathbf{x})} \right). \quad (2)
\]

Choosing \( Q(\mathbf{x}) = P(\mathbf{x}, \theta + d\theta) \), i. e. the probability distribution at an infinitesimally different value of \( \theta \), the corresponding Kullback-Leibler divergence is to leading order in \( d\theta \) given by

\[
D_{\text{KL}}(P(\theta + d\theta) \shortmid P(\theta)) = \frac{1}{2} I(\theta) d\theta^2 + O(d\theta^3), \quad (3)
\]

and the Fisher information thus is the curvature of the Kullback-Leibler divergence. Similar to the Kullback-Leibler divergence, the Fisher information is additive in the following sense: Suppose that we subdivide the random variables into two sets \( \mathbf{X} = (\mathbf{Y}, \Psi) \). Introducing the conditional probability density \( P_{\Psi} | \mathbf{Y} (\psi, \theta | \mathbf{y}) = P(\Psi = \psi | \mathbf{Y} = \mathbf{y}, \theta) \), we can then write

\[
P(\mathbf{x}, \theta) = P_{\Psi} | \mathbf{Y} (\psi, \theta | \mathbf{y}) P_{\mathbf{Y}} (\mathbf{y}, \theta), \quad (4)
\]

where \( P_{\mathbf{Y}} (\mathbf{y}, \theta) = P(\mathbf{Y} = \mathbf{y} | \theta) \) is the marginal density of the random variables \( \mathbf{Y} \). Then a straightforward calculation shows that

\[
I(\theta) = I_{\Psi} | \mathbf{Y} (\theta) + I_{\mathbf{Y}} (\theta) \quad \text{with}
\]

\[
I_{\Psi} | \mathbf{Y} (\theta) = \int d\psi \int d\mathbf{y} \ \left( \frac{\partial_\theta P(\psi, \theta | \mathbf{y})}{P_{\Psi} | \mathbf{Y} (\psi, \theta | \mathbf{y})} \right)^2 P_{\mathbf{Y}} (\mathbf{y}, \theta) \quad (5)
\]

\[
= \left\langle (\partial_\theta P_{\Psi} | \mathbf{Y} (\psi, \theta | \mathbf{y}))^2 \right\rangle_{\theta}
\]

\[
I_{\mathbf{Y}} (\theta) = \int d\mathbf{y} \ \left( \frac{\partial_\theta P_{\mathbf{Y}} (\mathbf{y}, \theta)}{P_{\mathbf{Y}} (\mathbf{y}, \theta)} \right)^2 = \left\langle (\partial_\theta P_{\mathbf{Y}})^2 \right\rangle_{\theta}.
\]

The Fisher information can thus be decomposed into two positive terms, depending on the conditioned statistics of the random variables \( \Psi \) and the statistics of the random variables \( \mathbf{Y} \), respectively. In particular, we have \( I(\theta) \geq I_{\mathbf{Y}} (\theta) \), i. e. eliminating variables decreases the Fisher information. If the random variables \( \Psi \) and \( \mathbf{Y} \) are further independent, then we have \( I(\theta) = I_{\Psi} (\theta) + I_{\mathbf{Y}} (\theta) \). The geometric interpretation of the Fisher information follows from defining a statistical line element \( ds \) by

\[
ds^2 = I(\theta) d\theta^2. \quad (6)
\]

The quantity \( ds \) may be thought of as a dimensionless distance between the probability densities at two infinitesimally different values of \( \theta \), i. e. between \( P(\mathbf{x}, \theta) \) and \( P(\mathbf{x}, \theta + d\theta) \). The infinitesimal statistical line element in a natural way defines a statistical length,

\[
L(\theta_2, \theta_1) = \int_{\theta_1}^{\theta_2} ds = \int_{\theta_1}^{\theta_2} |d\theta| \sqrt{I(\theta)}. \quad (7)
\]
The parameterization that realizes this minimal length is the geodesic curve,

\[ P^*(x, \theta) = \frac{1}{\sin \left( \frac{\Lambda}{2} \right)} \left( \sin \left( \frac{\Lambda}{2} \theta_2 - \theta_1 \right) \sqrt{P(x, \theta_1)} + \sin \left( \frac{\Lambda}{2} \theta_1 - \theta_2 \right) \sqrt{P(x, \theta_2)} \right)^2, \tag{9} \]

which simultaneously minimizes the action integral

\[ C(\theta_2, \theta_1) = \frac{1}{2} \int_{\theta_1}^{\theta_2} d\theta \; I(\theta). \tag{10} \]

For the geodesic curve, we thus have

\[ L^*(\theta_2, \theta_1) = \Lambda, \quad C^*(\theta_2, \theta_1) = \frac{\Lambda^2}{2(\theta_2 - \theta_1)}, \tag{11} \]

while for any other parameterization \( P(x, \theta) \), we have the inequalities

\[ C(\theta_2, \theta_1) \geq \frac{L^2}{2(\theta_2 - \theta_1)} \geq \frac{\Lambda^2}{2(\theta_2 - \theta_1)}, \tag{12} \]

where the first inequality follows from applying the Cauchy-Schwarz inequality to \( L^2 \) and the second one is a consequence of \( L \geq \Lambda \).

### III. CRÁMER-RAO BOUND AND DIFFERENTIAL SPEED LIMIT

We define an observable \( R(X) \) as a function of the random variables; its average corresponding to a parameter value \( \theta \) is given by

\[ \langle R \rangle_\theta = \int dx \; R(x) P(x, \theta). \tag{13} \]

Since \( P(x, \theta) \) is a normalized probability density, we have \( \int dx \; \partial_\theta P(x, \theta) = 0 \), or \( \langle \partial_\theta \ln P \rangle_\theta = 0 \). Then, using the covariance inequality, it is straightforward to obtain the following inequality

\[ \left( \langle \partial_\theta \langle R \rangle_\theta \rangle_\theta \right)^2 = \langle R \; \partial_\theta \ln P \rangle_\theta^2 = \langle \Delta R \; \partial_\theta \ln P \rangle_\theta \leq \langle \Delta R^2 \rangle_\theta I(\theta), \tag{14} \]

where we defined \( \Delta R(x) = R(x) - \langle R \rangle_\theta \). Taking the square root of the above, and using the definition Eq. [6] of the statistical line-element,

\[ \left| \frac{d\langle R \rangle_\theta}{\sqrt{\langle \Delta R^2 \rangle_\theta}} \right| \leq ds. \tag{15} \]

This inequality states that the change in some observable \( R \) due to a change in \( \theta \) relative to its fluctuations is bounded by the line element \( ds \). Interpreting the left-hand side as a distance in the average of \( R \), the intuitive interpretation of this inequality is that the distance
elapsed in the space of probability densities is always larger than the distance in the projection of this space onto any observable. Note that if the observable \( \hat{R} = \hat{\theta} \) is an unbiased estimator of \( \theta \), i.e., \( \langle \hat{\theta} \rangle_{\theta} = \theta \), then this is equivalent to the Crémé-Rao bound \([22, 23]\),

\[
\langle \Delta \hat{\theta}^2 \rangle_{\theta} \geq \frac{1}{I(\theta)}. \tag{16}
\]

Written in terms of the Fisher-information, the inequality \([15]\) is thus equivalent to the generalized Crémé-Rao bound \([30]\),

\[
\langle \Delta \hat{R}^2 \rangle_{\theta} \geq \frac{(d\theta\langle R \rangle_{\theta})^2}{I(\theta)}, \tag{17}
\]

where \( \hat{R} \) is any unbiased estimator of \( \langle R \rangle_{\theta} \). The variance of an estimator of \( R \) is thus always larger than the sensitivity of the expectation of \( R \) with respect to changes in the parameter \( \theta \) divided by the Fisher information. The Crémé-Rao bound is widely used in estimation theory and statistics. However, the inequality \([15]\) and thus the Crémé-Rao bound have another intriguing physical interpretation if \( \theta = t \) is equal to the physical time. In this case, we have the bound

\[
(d_t \langle R \rangle_t)^2 \leq \langle \Delta R^2 \rangle_t I(t) = \langle \Delta R^2 \rangle_t \left( \frac{ds}{dt} \right)^2, \tag{18}
\]

where we defined the temporal Fisher information

\[
I(t) = \int dx \frac{(\partial_t P(x,t))^2}{P(x,t)}. \tag{19}
\]

This bound provides a differential speed limit for the time evolution of any observable without explicit time dependence: the rate of change of any such observable is bounded by its fluctuations times the speed \( ds/dt \) of the evolution of the probability density. Alternatively, we can interpret the temporal Fisher information as the maximum information that one can obtain by observing the time evolution of the system. The speed limit Eq. \([18]\) then implies that measuring the time evolution of any observable can only yield less information.

The appearance of the fluctuations of the observable appear in the bound \([18]\) reflects that, if the fluctuations of an observable are large, then we also have to observe a large change in the average value in order to make a meaningful statement about the system. Occasionally, it may be useful to consider vector-valued observables \( \mathbf{R} (\mathbf{X}) = \{ R_1 (\mathbf{X}), \ldots, R_N (\mathbf{X}) \} \). In this case, the speed-limit Eq. \([18]\) generalizes to

\[
(d_t \langle \mathbf{R} \rangle_t)^T \Xi_R(t)^{-1} (d_t \langle \mathbf{R} \rangle_t) \leq I(t), \tag{20}
\]

where \( \Xi_R \) is the covariance matrix of \( \mathbf{R} \),

\[
(\Xi_R(t))_{ij} = \langle R_i R_j \rangle_t - \langle R_i \rangle_t \langle R_j \rangle_t. \tag{21}
\]

In Ref. \([19]\), a thermodynamic interpretation of the action Eq. \([10]\) defined by \( (ds/dt)^2 \),

\[
\mathcal{C} = \frac{1}{2} \int_0^T dt \left( \frac{ds}{dt} \right)^2, \tag{22}
\]

was proposed as the thermodynamic cost associated with the time evolution of the system during the time interval \([0, T]\). This thermodynamic cost measures the rate of local entropy entropy production; see Appendix \([A]\) for a more detailed discussion for the case of Fokker-Planck dynamics. At this point, we briefly recall the thermodynamic uncertainty relation for currents in steady state systems \([24, 28]\),

\[
\langle \hat{X} \rangle_{\text{st}}^2 \leq D_X \langle \sigma^\text{bath} \rangle_{\text{st}}, \tag{23}
\]

where \( \langle \hat{X} \rangle_{\text{st}} \) is the steady state current of some observable \( X \), \( D_X = \lim_{T \to \infty} \langle (\Delta X^2)_{\text{st}} / (2 T) \rangle \) the corresponding diffusion coefficient (quantifying fluctuations) and \( \langle \sigma^\text{bath} \rangle_{\text{st}} \) the average rate of entropy production in the heat bath (quantifying the thermodynamic cost of maintaining the current). In analogy to Eq. \([23]\), also Eq. \([18]\) can be understood as a uncertainty relation, since it relates the rate of a change in the system to the fluctuations and the thermodynamic cost of the time evolution. In this sense Eqs. \([15]\) and \([23]\) are dual to each other: The uncertainty relation provides a bound on the rate at which a quantity is transported in a steady state situation, while the speed limit bounds the rate of change of a quantity due to a transient dynamics. We remark that a speed limit similar to Eq. \([18]\) can also be obtained for higher order moments of \( R \), e.g. for the variance

\[
\left( d_t \langle \Delta R^2 \rangle_t \right)^2 \leq 2 \left( \kappa_R(t) - \frac{1}{2} \gamma_R(t)^2 \right) \left( \frac{ds}{dt} \right)^2, \tag{24}
\]

where \( \kappa_R(t) = \langle \Delta R^4 \rangle_t / \langle \Delta R^2 \rangle_t^2 \) and \( \gamma_R(t) = \langle \Delta R^3 \rangle_t / (\langle \Delta R^2 \rangle_t)^{3/2} \) denote the kurtosis and skewness of the distribution with respect to \( R \). We provide a derivation of this bound in Appendix \([3]\).

In Ref. \([19]\) also an integral speed limit for the total evolution time \( T \) was derived

\[
T \geq \frac{\mathcal{L}^2}{2 \mathcal{C}}, \tag{25}
\]

where \( \mathcal{L} = \mathcal{L}(T, 0) \) is the length of the path traced by the probability density during the time evolution from \( 0 \) to \( T \) and \( \mathcal{C} \) is the corresponding thermodynamic cost. From the definition of the statistical length \( \mathcal{L} \), Eq. \([7]\) and Eq. \([18]\) we have

\[
\mathcal{L} = \int_0^T ds(t) \geq \int_0^T dt \frac{|d_t \langle R \rangle_t|}{\sqrt{(\langle \Delta R^2 \rangle_t)}} \geq \frac{|\langle R \rangle_T - \langle R \rangle_0|}{\sqrt{\langle \Delta R^2 \rangle_{\text{max}}}}, \tag{26}
\]

where \( \langle \Delta R^2 \rangle_{\text{max}} \) denotes the maximum variance of \( R \) in the interval \([0, T]\). Then we can get an integral speed
limit in terms of the observable $R$ from Eq. (25),
\[
T \geq \frac{C^2}{2C} \geq \frac{1}{2C} \left( \langle R \rangle_T - \langle R \rangle_0 \right)^2 / (\Delta R^2)_{\text{max}}.
\] (27)

Thus the time for needed the system to evolve from one state to another is bounded from below by the change of any observable between the two states relative to its fluctuations, divided by the cost $C$. Note that the speed limit Eq. (25) constitutes a tighter bound on the evolution time than the observable-dependent bound Eq. (27). However, if we are not interested in the precise state of the system but only in the value of the observable $R$, the latter bound may be the more relevant one. Indeed, we can also read it as a bound on the required thermodynamic cost to change the value of the observable from $\langle R \rangle_0$ to $\langle R \rangle_T$ within time $T$,
\[
C \geq \frac{1}{2} \left( \frac{\langle R \rangle_T - \langle R \rangle_0}{\Delta R^2} \right)^2.
\] (28)

Read in this way, the bound states that a fast (small $T$) and precise (small fluctuations) change of an observable necessarily incurs a large thermodynamic cost. Again, this is similar to the uncertainty relation Eq. (24), which states that fast transport with small fluctuations likewise requires a large investment in terms of entropy production [24] [31].

As a particularly interesting case of Eq. (25), we note that the time derivative of the Shannon entropy $\Sigma^\text{sys}(t) = -\int dx \ln(P(x,t)) P(x,t)$ is given by
\[
d_x \Sigma^\text{sys}(t) = -\int dx \ln(P(x,t)) \partial_t P(x,t).
\] (29)

We thus have the bound
\[
\sigma^\text{sys}(t)^2 \leq \left( \langle \ln(P)^2 \rangle_t - \langle \ln(P) \rangle_t^2 \right) I(t).
\] (30)

This provides an inequality between two central quantities of information theory, the Fisher information and the rate of change of Shannon entropy. If we consider the Shannon entropy as the average of a stochastic Shannon entropy $\Phi(x,t) = -\ln(P(x,t))$, then the first factor on the right-hand side can be interpreted as the fluctuations of this stochastic Shannon entropy. The inequality (30) then states that the average rate of Shannon entropy change is always less than the fluctuations of the Shannon entropy times the Fisher information.

IV. MONOTONICITY OF FISHER INFORMATION

Up to this point, the origin of the probability density $P(x,t)$, i.e., the precise stochastic system that is described by the latter, has not been specified. We now assume that $P(x,t)$ describes the time-evolution of a diffusive dynamics, i.e., is the solution of the Fokker-Planck equation [32]
\[
\partial_t P(x,t) = \mathcal{G}(x,t) P(x,t) \quad \text{with} \quad \mathcal{G}(x,t) = -\partial_x \left( a_i(x,t) - \frac{1}{2} \partial_x B_{ij}(x,t) \right),
\] (31)

where a sum over repeated indices is implied. Here $a(x,t)$ is a drift vector and $B(x,t)$ is a symmetric and positive semidefinite diffusion matrix, i.e.
\[
v_i B_{ij}(x,t) v_j \geq 0 \quad (32)
\]

for an arbitrary vector $v$ and for all $x$ and $t$. The Fokker-Planck operator $\mathcal{G}$ is the generator of the dynamics. We further introduce the adjoint of the generator,
\[
\mathcal{G}^\dagger(x,t) = \left( a_i(x,t) + \frac{1}{2} B_{ij}(x,t) \partial_{x_j} \right) \partial_{x_i},
\] (33)

which satisfies

\[\begin{align}
\int dx \ f \mathcal{G} g &= \int dx \ g \mathcal{G}^\dagger f \\
\mathcal{G}^\dagger f^2 &= 2 f \mathcal{G}^\dagger f + \left[ \partial_{x_i} f \right] B_{ij} \left[ \partial_{x_j} f \right]
\end{align}\]

for suitable (smooth and integrable) functions $f(x,t)$ and $g(x,t)$. For such a dynamics, we consider the time-derivative of the Fisher information
\[
d_t I(t) = \int dx \frac{2 P (\partial_t P) \left[ \partial_t^2 P \right] - \left( \partial_t P \right)^2}{P^2},
\] (35)

with the convention that derivatives inside square brackets do not act on terms outside the brackets. Here and in the following, we omit the arguments of the respective functions for brevity. We write the second time-derivative of the probability density as $\partial_t^2 P = \partial_t \mathcal{G} P = \mathcal{G}^\dagger P + \mathcal{G} \partial_t P$, where we introduced the time-derivative of the Fokker-Planck operator
\[
\mathcal{G}(x,t) = -\partial_x \left( a_i(x,t) - \frac{1}{2} \partial_{x_j} B_{ij}(x,t) \right). \quad (36)
\]

Defining the generalized potential $\Phi(x,t) = -\ln(P(x,t))$, which can be identified as a stochastic Shannon entropy in the sense that the Shannon entropy is the average of $\Phi$, $\Sigma^\text{sys} = -\int dx \ln(P) P = \langle \Phi \rangle_t$, we can write for the time-derivative of the Fisher
information

\[ d_t I(t) + 2 \int dx \left[ \partial_t \phi \right] \mathcal{G} P \]

\[ = - \int dx \left( 2 \left[ \partial_t \phi \right] \mathcal{G}^2 P + \left[ \partial_t \phi \right]^2 \mathcal{G} P \right) \]

\[ = - \int dx \left( 2 \left[ \mathcal{G} P \left[ \mathcal{G}^\dagger \partial_t \phi \right] + P \left[ \mathcal{G}^\dagger \left( \partial_t \phi \right)^2 \right] \right) \]

\[ = - \int dx \left( 2 \left[ \mathcal{G} P \left[ \mathcal{G}^\dagger \partial_t \phi \right] + 2P \left[ \partial_t \phi \right] \left[ \mathcal{G}^\dagger \partial_t \phi \right] \right) \]

\[ + P \left[ \partial_x \partial_t \phi \right] B_{ij} \left[ \partial_{x_j} \partial_t \phi \right] P \]

\[ = - \int dx \left[ \partial_x \partial_t \phi \right] B_{ij} \left[ \partial_{x_j} \partial_t \phi \right] P \]

\[ - 2 \int dx \left( \left[ \mathcal{G} P \right] + P \left[ \partial_t \phi \right] \right) \left[ \mathcal{G}^\dagger \partial_t \phi \right]. \]

From the definition of \( \phi \) we have \( P \partial_t \phi = - \partial_t P = - \mathcal{G} P \), such that the last term vanishes. We thus arrive at

\[ d_t I(t) = -2 \left( \left[ \mathcal{G}^\dagger \partial_t \phi \right] \right)_t - \left( \left[ \partial_x \partial_t \phi \right] B_{ij} \left[ \partial_{x_j} \partial_t \phi \right] \right)_t \]  

(38)

with the operator

\[ \mathcal{G}^\dagger \left( x, t \right) = \left( \left[ \partial_t \partial_{x_i} \left( x, t \right) \right] + \frac{1}{2} \left[ \partial_{x_i} B_{ij} \left( x, t \right) \right] \partial_{x_j} \right) \partial_{x_i}. \]  

(39)

The rate of change of the Fisher information thus decomposes into two terms. We can think of the two terms as the dynamical and statistical contribution to the change in Fisher information: The former contribution is proportional to the explicit time-dependence of the dynamics via the drift vector \( a \) and diffusion matrix \( B \) and can be either positive or negative. The latter contribution, on the other hand, is always less or equal zero, since the diffusion matrix is positive semidefinite. It characterizes the relaxation of the system towards the instantaneous steady state and the loss of information due to this relaxation process. In particular, if the dynamics do not have any explicit time-dependence (i.e. \( a \) and \( B \) do not depend on time), the Fisher information is a non-increasing function of time,

\[ d_t I(t) \leq 0. \]  

(40)

For systems possessing a steady state \( P^\text{st}(x) = \lim_{t \to \infty} P(x, t) \), this guarantees that the approach towards the steady state is always monotonic with respect to the Fisher information, independent of the initial state. This is obviously not the case for arbitrary observables, which need not approach their steady state value in a monotonic fashion, e.g. for an underdamped particle in a confining potential, whose position may exhibit oscillations. Note that the monotonic behavior of the Fisher information is a consequence of the time-translation invariance of the generator.

We remark that the same result holds for e.g. Markov jump processes (see Appendix D). It is in fact a consequence of the monotonicity of the Kullback-Leibler divergence: For both Fokker-Planck and Markov jump dynamics, given two solutions for the probability (density) \( P(t) \) and \( Q(t) \),

\[ \partial_t P(t) = \mathcal{G}(t) P(t) \quad \text{and} \quad \partial_t Q(t) = \mathcal{G}(t) Q(t), \]

(41)

their Kullback-Leibler divergence is a non-increasing function \[ d_t D_{KL}(Q(t) || P(t)) \leq 0. \]

(42)

If the generator of the dynamics is independent of time, then both \( P(t) \) and \( Q(t) = P(t + \tau) \) are valid solutions. To leading order in \( \tau \), their Kullback-Leibler divergence is given by

\[ D_{KL}(P(t + \tau) || P(t)) = \frac{1}{2} \tau^2 + O(\tau^3) \]

(43)

and we thus have

\[ d_t D_{KL}(Q(t) || P(t)) = \frac{1}{2} d_t I(t) \tau^2 + O(\tau^3) \leq 0 \quad \Rightarrow \quad d_t I(t) \leq 0. \]

(44)

If the system possesses a steady state, then obviously also the Kullback-Leibler divergence between the instantaneous probability and the steady state is a non-increasing quantity. We thus have two related measures for the approach of a system to the steady state: Both the Fisher information \( I(t) = \langle (\partial_t P/P)^2 \rangle_t \) and the Kullback-Leibler divergence \( D_{KL}(P(t) || P^\text{st}) = \langle \ln(P/P^\text{st}) \rangle_t \), are positive, non-increasing functions of time and zero only in the steady state. However, the Fisher information has the advantage that it is local in time, i.e. it depends only on the instantaneous state of the system and does not require knowledge about (or even the existence of) the steady state.

V. INTEGRAL SPEED LIMIT

Using the properties of the Fisher information discussed above, specifically the minimal statistical length Eq. (11) and the monotonic behavior of the Fisher information for dynamics without explicit time-dependence Eq. (40), we can conclude from the speed limit Eq. (25)

\[ \mathcal{T} \geq \frac{L^2}{2\mathcal{C}} \geq \frac{\Lambda^2}{2\mathcal{C}} \geq \frac{\mathcal{L}^2}{I(0)}. \]

(45)

Here, we used that \( \mathcal{L} \geq \Lambda \) and \( 2\mathcal{C} = \int_0^\mathcal{T} dt \ I(t) \leq \int_0^\mathcal{T} dt \ I(0) = \mathcal{T} I(0) \). We thus have the following speed limit for dynamics without explicit time-dependence

\[ \mathcal{T} \geq 2 \arccos \left( \frac{\int dx \sqrt{P^1(x)P^3(x)}}{\int dx \sqrt{\mathcal{G}^\dagger(x)\mathcal{G}(x)}} \right). \]

(46)
importantly, this speed limit depends only on the initial and final state and on the generator $G$ of the dynamics. We remark that such speed limits have been extensively discussed in quantum-mechanical systems (see e.g. Ref. [29]), however, it has recently been found that similar bounds also apply to classical and stochastic dynamics [29]. We note that in contrast to the Margolus-Levitin-type bound derived in Ref. [29] (Eq. (23) therein), this result does not require any particular spectral properties of the generator or existence of a steady state; the only requirement is that the generator does not depend explicitly on time. It is further tighter than the Mandelstam-Tamm-type bound derived in Ref. [29] (Eq. (26) therein) for a particle relaxing in a binding potential, since we have $2 \arccos(x) \geq \pi(1-x)$ for $x > 0$.

Using the monotonicity of the Fisher information, we also have from Eq. (18)

$$\frac{(d_t \langle R \rangle_t)^2}{\langle \Delta R^2 \rangle_t} \leq I(t) \leq \int dx \frac{(GP_i(x))^2}{P_i(x)} \tag{47}$$

Thus, in the absence of explicit time dependence, the initial state limits how fast any observable may evolve at any later time. As mentioned before, $\langle R \rangle_t$ is not necessarily a monotonic function of time, however, the magnitude of $d_t \langle R \rangle_t$ relative to the fluctuations of $R$ is bounded from above by a decreasing function. Thus, if $\langle R \rangle_t$ exhibits oscillations, this result implies that the amplitude of the oscillations necessarily decreases over time, provided $\langle \Delta R^2 \rangle_t$ is bounded.

VI. Fisher Information as an Indicator for Hidden Degrees of Freedom

For Fokker-Planck and Markov jump dynamics without explicit time-dependence, we have shown in Section IV respectively Appendix D that the Fisher information has to decrease monotonically with time. Combining this with the additivity of Fisher information (see Eq. 5), we can make a statement about the behavior of the Fisher information in the presence of hidden degrees of freedom. Suppose that, as in Eq. 5, the system of interest is composed two sets of degrees of freedom $Y$ and $\Psi$. Physically, we assume that $Y$ contains the observable degrees of freedom, that are accessible to direct observation, and $\Psi$ is composed of hidden degrees of freedom, which are not directly observable. If the system is time-independent, we then have for the Fisher information of the total system from Eqs. 5 and 40

$$d_t I(t) = d_t I_{\Psi|Y}(t) + d_t I_Y(t) \leq 0 \tag{48}$$

While each individual term may be positive or negative, the sum of the terms has to be negative. This means that if we measure $I_Y(t)$ from the probability distribution of the observable degrees of freedom and find $d_t I_Y(t) > 0$ at any time, than this is a clear indicator that hidden degrees of freedom are present in the system. We can make this precise in the form of the following statement: If, for some stochastic process $Y(t)$, we observe $d_t I_Y(t) > 0$ at any time $t$, the process cannot be described in terms of a diffusion process with time-independent drift and diffusion coefficient. Thus, either the drift and/or diffusion coefficient depend explicitly on time, or there are hidden degrees of freedom in the system which effectively render the process $Y(t)$ non-Markovian. Similarly, for a Markov jump dynamics with states labeled by the set $X = \{1, \ldots, M\}$ and occupation probabilities $p_i, i \in X$, we can divide the states into observable states $Y \subset X$ and hidden states $\Psi = X \setminus Y$. For a state $i \in B$, we define the occupation probability restricted on the set of observable states as $p_{i|Y} = p_i / p_Y$, where $p_Y = \sum_{i \in Y} p_i$ is the probability to find the system in an observable state. Then the Fisher information can be decomposed as

$$I = \sum_{i \in X} \frac{(d_t p_i)^2}{p_i} = I_{|Y} p_Y + I_{Y|\Psi} p_\Psi + \frac{(d_t p_\Psi)^2}{p_\Psi (1-p_\Psi)} \tag{49}$$

with $I_{|Y} = \sum_{i \in Y} \frac{(d_t p_{i|Y})^2}{p_{i|Y}}, I_{Y|\Psi} = \sum_{i \in C} \frac{(d_t p_{i|\Psi})^2}{p_{i|\Psi}}$.

Here we used that the probability of the system being in a hidden state satisfies $p_\Psi = 1 - p_Y$. The total Fisher information thus consists of three terms: The first two contain the Fisher information $I_{|Y}, I_{Y|\Psi}$ due to changes in the occupation probabilities within the observable and hidden states, respectively. These correspond to the dynamics within the subsets $Y$ and $\Psi$. The third term describes exchange of probability between the subsets $Y$ and $\Psi$. If only the set $Y$ of states is accessible to observation, then only the restricted occupation probabilities $p_{i|Y}$ and thus $I_{|Y}$ can be measured. Assuming that the transition rates are time-independent, we have for time-derivative of the total Fisher information $d_t I(t) \leq 0$ (see Eq. (10)). If we now measure $I_{|Y}(t)$ as a function of time and find $d_t I_{|Y}(t) > 0$ at any time, then this is a clear indication for the presence of hidden states. Thus, the existence of hidden states can potentially be determined from the behavior of the Fisher information of the observable states.

VII. Examples

A. General normal distributions

A particularly succinct and widely applicable example for the relation between statistical length, Fisher information and observables is for a normal distribution in $M$
variables,

\[ P(x, t) = \frac{1}{\sqrt{(2\pi)^M \det(\Xi(t))}} \times \exp \left[ -\frac{1}{2}(x - \mu(t))^T \Xi(t)^{-1} (x - \mu(t)) \right] \]  

(50)

with the average \( \langle x \rangle_t = \mu(t) \) and the (symmetric and positive definite) covariance matrix \( \Xi(t) \) defined by

\[ \Xi_{ij}(t) = \langle (x_i - \mu_i(t))(x_j - \mu_j(t)) \rangle_t. \]  

(51)

Here the subscript \( T \) denotes transposition and det the determinant. In this case, we can compute the rate of Shannon entropy change \( \sigma^{\text{sys}}(t) = d_t \Sigma^{\text{sys}}(t) \) and the Fisher information explicitly [31],

\[ \sigma^{\text{sys}}(t) = \frac{1}{2} d_i \ln (\det(\Xi)) = \frac{1}{2} \text{tr}(\Xi(t)^{-1} \Xi(t)) \]  

(52a)

\[ I(t) = (\dot{\mu}(t))^T \Xi(t)^{-1} \dot{\mu}(t) + \frac{1}{2} \text{tr}(\Xi(t)^{-1} \dot{\Xi}(t) \Xi(t)^{-1} \Xi(t)), \]  

(52b)

where \( \dot{\mu}(t) \) and \( \dot{\Xi}(t) \) are the component-wise time derivatives of the respective quantities and \( \text{tr} \) is the trace. A normal distribution can arise from the solution of a Fokker-Planck equation with linear drift coefficients

\[ \partial_t P(x, t) = -\partial_{x_i} \left( a_i(x, t) - \frac{1}{2} B_{ij}(t) \partial_{x_j} \right) P(x, t) \]  

(53)

with \( a_i(x, t) = K_{ij}(t)x_j + k_i(t) \)

with a symmetric, positive semidefinite matrix \( B \), provided that the initial distribution is normal

\[ P_0(x) = \frac{1}{\sqrt{(2\pi)^M \det(\Xi_0)}} \times \exp \left[ -\frac{1}{2}(x - \mu_0)^T \Xi_0^{-1} (x - \mu_0) \right]. \]

(54)

The mean and covariance matrix then are determined by the differential equations

\[ d_t \mu_i(t) = K_{ij}(t) \mu_j(t) + k_i(t) \]  

(55a)

\[ d_t \Xi_{ij}(t) = K_{ii}(t) \Xi_{ij}(t) + K_{ji}(t) \Xi_{ij}(t) + \frac{1}{2} (B_{ij}(t) + B_{ji}(t)), \]  

(55b)

or in matrix notation (using that \( B \) is symmetric)

\[ \dot{\mu}(t) = K(t) \mu(t) + k(t) \]  

(56a)

\[ \dot{\Xi}(t) = K(t) \Xi(t) + \Xi(t) K^T(t) + B(t), \]  

(56b)

with initial condition \( \mu(0) = \mu_0 \) and \( \Xi(0) = \Xi_0 \). These equations allow us to write the Fisher information without relying on time-derivatives,

\[ I(t) = (K \mu + k)^T \Xi^{-1} (K \mu + k) + \frac{1}{2} \text{tr} \left[ (\Xi^{-1} K \Xi + K^T)^2 + 2B(\Xi^{-1} K + K^T \Xi^{-1}) + B \Xi^{-1} B \Xi^{-1} \right]. \]  

(57)

Obviously, any normal distribution is uniquely determined by its mean and covariance matrix and thus the latter two quantities also specify the average of any observable \( R(x) \) and its time evolution. However, how precisely the time evolution of the mean and covariance matrix impact the time evolution of \( \langle R \rangle_t \), i.e. the explicit expression of \( \langle R \rangle_t \) in terms of \( \mu \) and \( \Xi \) is not obvious except in simple cases. Nevertheless, from Eq. (20), we always have the bound

\[ (\dot{R})^T \Xi^{-1}_R (\dot{R}) \leq \mu^T \Xi^{-1} \mu + \frac{1}{2} \text{tr}(\Xi^{-1} \dot{\Xi} \Xi^{-1} \dot{\Xi}). \]  

(58)

This bound is particularly instructive for a time-independent covariance matrix \( \Xi = 0 \), where it states that the change in the average of any observable, relative to its covariance matrix, is always less than the respective quantity for the mean of the distribution. In this sense, no observable can change faster than the mean of the distribution. We further note a result valid for any probability distribution which depends on time only via its mean \( \mu \), and can thus be written as \( P(x, t) = \tilde{P}(x - \mu(t)). \) For such a probability distribution, the Fisher information is always larger than for a normal distribution with the same mean and variance,

\[ I(t) \geq I_{\text{normal}}(t) = (\dot{\mu}(t))^T \Xi(t)^{-1} \dot{\mu}(t). \]  

(59)

Thus, a normal distribution minimizes the Fisher information for pure translations. We give the proof of this result in Appendix D. Note that the inequality [59] breaks down if the variance or some higher cumulants depend on time.

For a normal distribution, the relation between Fisher information and Shannon entropy (5) change takes a particularly simple form, since, as we show in Appendix E we have

\[ \langle (\ln P)^2 \rangle_t - \langle \ln P \rangle_t^2 = \frac{M}{2}, \]  

(60)

independent of the shape of the covariance matrix. For a normal distribution, we thus have the relation between Shannon entropy and Fisher information

\[ (\sigma^{\text{sys}}(t))^2 \leq \frac{M}{2} I(t). \]  

(61)

Using \( \Sigma^{\text{sys}}(T) - \Sigma^{\text{sys}}(0) = \int_0^T dt \sigma^{\text{sys}}(t) \) and applying the Cauchy-Schwarz inequality, this yields

\[ T \geq \frac{(\Sigma^{\text{sys}}(T) - \Sigma^{\text{sys}}(0))^2}{MC}. \]  

(62)

Since we generally expect both \( C \) and \( \Sigma^{\text{sys}} \) to scale linearly with the number \( M \) of degrees of freedom, we can write this in terms of the following speed limit for normal distributions

\[ T \geq \frac{(\Sigma^{\text{sys}}(T) - \Sigma^{\text{sys}}(0))^2}{C}. \]  

(63)
where we \( \tilde{\Sigma}^\text{sys} = \Sigma^\text{sys}/M \) and \( \tilde{\xi} = C/M \) are the Shannon entropy and thermodynamic cost per degree of freedom. This result has two interesting consequences: First, it provides a speed limit in terms of the Shannon entropy difference between initial and final state. Second, it explicitly demonstrates that, at least in the case of a normal distribution, this speed limit remains useful in the limit of a macroscopic number of degrees of freedom \( M \gg 1 \).

We stress that the latter statement is not self-evident: For the case of the speed limit Eq. (66), the numerator is obviously bounded from above by \( \pi \), the largest possible arc length on the unit sphere. On the other hand, the denominator scales as \( \sqrt{M} \) for independent degrees of freedom, since the Fisher information is additive in this case. Thus the right-hand side of Eq. (66) is typically of order \( 1/\sqrt{M} \) and the bound becomes meaningless in the macroscopic limit.

### B. Brownian motion

The most basic example of a continuous-valued random process is Brownian motion. Let us first consider the classical case of an overdamped particle in a environment at temperature \( T \), described by the diffusion equation

\[
\partial_t P(x, t) = -\frac{1}{\gamma} \partial_x \left( F_0 - T \partial_x \right) P(x, t),
\]

or, equivalently the overdamped Langevin equation

\[
\gamma \dot{x}(t) = F_0 + \sqrt{2\gamma T} \xi(t)
\]

where \( \gamma \) is the friction coefficient, \( F_0 \) is a constant bias force, \( T \) is the temperature and \( \xi(t) \) is Gaussian white noise. The solution of the diffusion equation is straightforward,

\[
P(x, t) = \frac{1}{\sqrt{2\pi(2D_x t + \langle \Delta x^2 \rangle_0)}} \exp \left( -\frac{(x - \langle x \rangle_0)^2}{2(2D_x t + \langle \Delta x^2 \rangle_0)} \right),
\]

where \( \langle x \rangle_0 \) and \( \langle \Delta x^2 \rangle_0 \) are the initial average and variance of the particle’s position at time \( t = 0 \). Here, we introduced the diffusion coefficient \( D_x \) given by the Einstein relation \( D_x = T/\gamma \). As we only have one degree of freedom, the expression for the Fisher information Eq. (52b) simplifies to

\[
I(t) = \frac{F_0^2}{2\gamma T (t + \langle \Delta x^2 \rangle_0/2D_x)} + \frac{1}{2 (t + \langle \Delta x^2 \rangle_0/2D_x)^2}.
\]

Both with and without bias, the Fisher information for Brownian motion is a monotonously decaying function and thus (biased) Brownian motion is a generalized relaxation process. Note that even though the Fisher information decreases, the time-derivative of the average position \( d_t \langle x \rangle_t = F_0/\gamma \) does not decay to zero but remains constant. This is not in contradiction with the speed limit Eq. (18), which only demands that the time derivative of \( \langle x \rangle_t \) relative to the fluctuations of \( x \)—which in this case increase with time—should decrease along with the Fisher information.

### C. Particle in a parabolic trap

As a second paradigmatic example, we consider a single overdamped particle with position \( x(t) \) in a parabolic trap \( U(x) = \kappa(t) (x - r(t))^2/2 \),

\[
\partial_t P(x, t) = -\frac{1}{\gamma} \partial_x \left( \kappa(t)(x - r(t)) + T(t) \partial_x \right) P(x, t),
\]

or, equivalently, the Langevin equation

\[
\gamma \dot{x}(t) = \kappa(t)(x - r(t)) + \sqrt{2\gamma T(t)} \xi(t),
\]

where \( \gamma \) is the friction coefficient, \( \kappa \) the spring constant and \( T \) the temperature. We allow the spring constant, temperature and equilibrium position \( r(t) \) of the trap to change as a function of time. Provided that the initial state is given by a normal distribution with average \( \langle x \rangle_0 \) and variance \( \langle \Delta x^2 \rangle_0 \), the solution to this problem is the normal distribution

\[
P(x, t) = \frac{1}{\sqrt{2\pi(\Delta x^2)_t}} \exp \left[ -\frac{(x - \langle x \rangle_t)^2}{2(\Delta x^2)_t} \right],
\]

where the average and variance of the position obey the following differential equations,

\[
d_t \langle x \rangle_t = -\frac{\kappa(t)}{\gamma} (\langle x \rangle_t - r(t)) \quad \left( 71a \right)
\]

\[
d_t (\Delta x^2)_t = -2\frac{\kappa(t)}{\gamma} (\Delta x^2)_t + \frac{2T(t)}{\gamma}. \quad \left( 71b \right)
\]

Again, for a single degree of freedom, the expression for the Fisher information is immediate from Eq. (52b)

\[
I(t) = \left( \frac{ds}{dt} \right)^2 = \frac{1}{2} \left( \frac{d_t(\Delta x^2)_t}{\langle \Delta x^2 \rangle_t} \right)^2 + \frac{(d_t \langle x \rangle_t)^2}{\langle \Delta x^2 \rangle_t} \quad \left( 72 \right)
\]

Here, the Fisher information (and thus the thermodynamic cost \( C \)) consists of two positive terms: The first one is non-zero if the variance changes as a function of time, the second one if the average position changes. The average rates of change of Shannon \( \sigma^\text{sys}(t) = d_t \Sigma^\text{sys}(t) \) and total entropy \( \sigma^{\text{tot}}(t) = d_t \Sigma^{\text{tot}}(t) \) (see Appendix A) are given by

\[
\sigma^\text{sys}(t) = \frac{1}{2} \frac{d_t(\Delta x^2)_t}{\langle \Delta x^2 \rangle_t} \quad \left( 73 \right)
\]

\[
\sigma^{\text{tot}}(t) = \frac{\gamma(\Delta x^2)_t}{T(t)} \left( \frac{1}{4} \left( \frac{d_t(\Delta x^2)_t}{\langle \Delta x^2 \rangle_t} \right)^2 + \left( \frac{d_t \langle x \rangle_t}{\langle \Delta x^2 \rangle_t} \right)^2 \right) \quad \left( 74 \right)
\]
In this case, the bound Eq. \((61)\) on the rate of change of the Shannon entropy is obvious, since we have \((M = 1)\)

\[
I(t) = \frac{1}{2} \left( \frac{d_t(\Delta x^2)_t}{(\Delta x^2)_t} \right)^2 + \frac{d_t(x)_t}{(\Delta x^2)_t} \geq 1 \left( \frac{d_t(\Delta x^2)_t}{(\Delta x^2)_t} \right)^2 = 2(\sigma_{xyz}(t))^2.
\]

Then the time-derivative of the Fisher information can be calculated as

\[
d_t I(t) = \frac{2 d_t(\Delta x^2)_t}{\gamma(\Delta x^2)_t^2} T(t) - \frac{2 \kappa(t) d_t(x)_t}{\gamma(\Delta x^2)_t^2} r(t)
+ \left( \frac{2 \kappa(t)(x)_t^2 - 2 d_t(\Delta x^2)_t}{\gamma(\Delta x^2)_t^2} \right) \dot{\kappa}(t)
- \frac{2 T(t)}{\gamma} \left( \frac{(d_t(\Delta x^2)_t)^2}{(\Delta x^2)_t^2} + \frac{(d_t(x)_t)^2}{(\Delta x^2)_t^2} \right).
\]

The first three terms depend explicitly on the time-derivative of \(T\), \(r\) and \(\kappa\), respectively, while the last term is negative. In particular, if the parameters \(T\), \(r\) and \(\kappa\) are independent of time, then we have

\[
d_t I(t) = -\frac{2 T(t)}{\gamma} \left( \frac{(d_t(\Delta x^2)_t)^2}{(\Delta x^2)_t^2} + \frac{(d_t(x)_t)^2}{(\Delta x^2)_t^2} \right) \leq 0,
\]

and the Fisher information decreases monotonically, as predicted by Eq. \((40)\).
solution of the equations is already quite involved and we refrain from writing down the cumbersome expression for the Fisher information, which can be obtained from Eq. (52b). However, in this case, since we have two degrees of freedom, already the case where do not depend on time offers some interesting insights. In this case, we observe a relaxation from the initial state to the equilibrium state with \( \langle x \rangle_{eq} = r \), \( \langle v \rangle_{eq} = 0 \), \( \langle \Delta x^2 \rangle_{eq} = T/\kappa \), \( \langle \Delta x \Delta v \rangle_{eq} = 0 \) and \( \langle \Delta v^2 \rangle_{eq} = T/m \). For a non-equilibrium initial condition corresponding to potential with \( \kappa > \kappa^* \) and \( \dot{r} \neq r \), the Fisher information of the relaxation process is shown in Fig. 2. While the Fisher information of the joint distribution decays monotonically, as predicted by Eq. (40), the Fisher information restricted to the observable states, \( I_Y(t) = \sum_{j=1}^{3} \frac{(d_j p_j(t))^2}{p_j(t)} \), is a monotone function of time \( d_j I(t) \leq 0 \), this is not necessarily true for the Fisher information restricted to the observable states, \( I_Y(t) = \sum_{j=1}^{3} \frac{(d_j p_j(t))^2}{p_j(t)} \).

We can thus use this to distinguish between the system with the hidden state 4 present and a system without this hidden state by examining the time-dependence of the Fisher information. In the following we choose \( \beta = 1 \), \( E_1 = 0.79 \), \( E_2 = 1.19 \), \( E_3 = 1.14 \) and \( E_4 = 3 \) and use the following transition rate matrix
\[
W = \begin{pmatrix}
0 & 1.28 & 1.31 & 10.5 \\
0.86 & 0 & 1.06 & 6.50 \\
0.93 & 1.12 & 0 & 5.36 \\
1.15 & 1.06 & 0.83 & 0
\end{pmatrix},
\]
which is obtained by randomly assigning values between 0.8 and 1.2 to the lower-left half of the matrix and then enforcing the detailed balance condition on the upper-right half. Note that the entries in the last column are larger, reflecting the large transition rates out of the short-lived state 4. We initialize the system with equal probability in each of the observable states, \( p_1(0) = p_2(0) = p_3(0) = 1/3 \) and then evolve it with the above transition rate matrix. Figure 3 shows the resulting time-evolution of the occupation probabilities and the Fisher information. Clearly, the Fisher information of the observable degrees of freedom, Eq. (40), shows a non-monotone behavior in the presence of the hidden state. Thus, observing only the occupation probabilities of the observable states, we can conclude that a three-state model with time-independent transition rates cannot possibly describe the dynamics correctly.

**VIII. DISCUSSION**

The speed limit Eq. (18) on the time-evolution of the average of a fluctuating observable shows that the behavior of measurable observables (averages and fluctuations) is governed by the information-theoretic concept...
FIG. 3. Occupation probabilities of the observable states (left) and the resulting Fisher information (right) for the four-state Markov-jump model. The solid lines are the probabilities $p_i(t)$ of the observable states, Eq. (84), in the system including the hidden state; the dashed lines correspond to a three-state model with the same transition rates but without the hidden state, obtained by deleting the last row and column from the transition matrix $W$. Note that while the presence of the hidden state modifies the time-evolution of the occupation probabilities (left), there is no qualitative difference discernible. On the other hand, the Fisher information (right) displays a markedly different behavior: Without the hidden state, the Fisher information $I_Y(t)$, Eq. (86), becomes non-monotonic (solid line), giving a clear indication that the system cannot be described by a three-state Markov model any longer.

of Fisher information. A similar connection between the Fisher information and the family of thermodynamic uncertainty relations was recently obtained in Refs. [35, 36]. Such a connection can potentially be exploited in several ways. If the underlying probability distribution and the corresponding Fisher information is not known, then we can obtain a lower bound in terms of measurable quantities. Since the lower bound is guaranteed to hold for all observables, we may also compare the bounds obtained by measuring different observables in order to find the observable that contains the most information about the time-evolution of the probability density.

On the other hand, if we have a theoretical model for a particular physical system, then the speed limit can serve as a test for the validity of the model: If we find that the observed time evolution of any observable exceeds the Fisher information bound predicted by the theoretical model, then this is a sure indication that crucial information about the system is missing in the model. For systems without explicit time-dependence the monotonic decay of the Fisher information provides even stricter restrictions on the type of models that can describe a given system. Finally, if the Fisher information itself is known, then the speed limit imposes a regularity condition on the system in the sense that it limits the rate of change of any conceivable observable.

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Appendix A: Thermodynamic cost for Fokker-Planck dynamics

In Ref. [19], the justification for referring to the quantity $C$ as a thermodynamic cost was provided by relating it to the entropy change upon the system transitioning between two discrete states $x$ and $x'$. To provide the analog for the case when the system is described by a set of continuous variables, we first note that the Fokker-Planck equation [31] for the probability density is equivalent to the stochastic evolution of the state $x(t)$ of the system described by the Langevin equation [32]

$$\dot{x}(t) = a(x(t), t) + \sqrt{B(x(t), t)} \cdot dW(t),$$

(A1)

where $\sqrt{B}$ refers to the unique positive semidefinite principal square root of the symmetric and positive semidefinite matrix $B$. $W$ is a vector of mutually uncorrelated Wiener processes and $\cdot$ denotes the Itô product. We want to describe the stochastic Shannon entropy (or generalized potential)

$$\Phi_{\text{sys}}(t) = -\ln P(x(t), t),$$

(A2)
with \( \langle \Phi^{\text{sys}} \rangle_t = S^{\text{sys}}(t) \). We rewrite Fokker-Planck equation \(^{(31)}\) as a continuity equation in terms of the probability current \( j(x, t) \),

\[
\partial_t P(x, t) = -\nabla j(x, t) \quad \text{with} \quad j(x, t) = \left( a(x, t) - \frac{1}{2} \nabla B(x, t) \right) P(x, t),
\]

where we define the operator \( \nabla B = \partial_x B_{ij} \). By Itô’s Lemma, we have for the differential of \( \Phi^{\text{sys}} \),

\[
d\Phi^{\text{sys}}(t) = -\partial_t \ln P(x(t), t) dt - \nabla \ln P(x(t), t) \cdot dx(t) - \frac{1}{2} B(x(t)) \nabla \nabla \ln P(x(t), t),
\]

where by \( B \nabla \nabla \) we mean the operator \( B_{ij} \partial_x \partial_x \). We can equivalently write this using the Stratonovich product \( \circ \),

\[
d\Phi^{\text{sys}}(t) = -\partial_t \ln P(x(t), t) dt - \nabla \ln P(x(t), t) \circ dx(t).
\]

The first term describes the change in Shannon entropy in a fixed state \( x \) due to the change in the ensemble probability \( P(x, t) \) to be in state \( x \). We interpret this term as a global (in the sense of ensemble) contribution; note that due to conservation of probability, this term always vanishes on average. On the other hand, the second, local, contribution describes the change in Shannon entropy due to a change in state from \( x \) to \( x' = x + dx \); this change in Shannon entropy in a transition \( \Delta \sigma_{x' \to x}^{\text{sys}} \) of a Markov jump process, as defined in Ref. \(^{19}\). In analogy to Ref. \(^{19}\), we thus interpret

\[
\Delta \Sigma^{\text{sys}}_{\text{loc}}(x, t) \equiv -\nabla \ln P(x, t)
\]

as the local change in Shannon entropy, which is related to the change in average Shannon entropy via

\[
d\Sigma^{\text{sys}} = \langle \Delta \Sigma^{\text{sys}}_{\text{loc}} \circ dx \rangle = \int dx \Delta \Sigma^{\text{sys}}_{\text{loc}}(x, t) j(x, t) dt
\]

Using this definition and integrating by parts, it is then easy to show that

\[
-\langle \partial_t \Delta \Sigma^{\text{sys}}_{\text{loc}} \circ dx \rangle_t \equiv -\int dx \ j(x, t) \partial_t \Delta \Sigma^{\text{sys}}_{\text{loc}}(x, t) = \int dx \ j(x, t) \nabla \partial_t \ln P(x, t) = -\int dx \ \partial_t \ln P(x, t) \nabla j(x, t)
\]

\[
= \int dx \ \partial_t \ln P(x, t) \partial_t P(x, t) = \left( \frac{ds}{dt} \right)^2,
\]

in analogy to Eq. (37) of Ref. \(^{19}\). For a diagonal diffusion matrix \( B_{ij} = B_i \delta_{ij} \) with \( B_i > 0 \), we can further write the change in total entropy as follows \(^{37} 38\),

\[
d\Phi^{\text{tot}}(t) = d\Phi^{\text{sys}}(t) + d\Phi^{\text{med}}(t) \quad \text{with} \quad d\Phi^{\text{med}}(t) = 2(B_i(x(t), t))^{-1}\left( a_i(x(t), t) - \frac{1}{2} \partial_x B_i(x(t), t) \right) \circ dx_i(t).
\]

Defining the local change in medium entropy and total entropy

\[
\Delta \Sigma^{\text{med}}_{\text{loc}}(x, t) \equiv 2\left( a(x, t) - \frac{1}{2} b_i(x, t) \right) B^{-1}(x, t)
\]

\[
\Delta \Sigma^{\text{tot}}_{\text{loc}}(x, t) \equiv \Delta \Sigma^{\text{med}}_{\text{loc}}(x, t) + \Delta \Sigma^{\text{sys}}_{\text{loc}}(x, t)
\]

with the vector \( b_i'(x, t) = \partial_x B_i(x, t) \), we thus have for the average change in total entropy \(^{38}\)

\[
d\Sigma^{\text{tot}} = \left( \Delta \Sigma^{\text{med}}_{\text{loc}} + \Delta \Sigma^{\text{sys}}_{\text{loc}} \right) \circ dx = 2 \int dx \ \frac{j(x, t) B^{-1}(x, t) j(x, t)}{P(x, t)} dt.
\]

This further allows us to write

\[
\left( \frac{ds}{dt} \right)^2 = \left( \partial_t \Delta \Sigma^{\text{med}}_{\text{loc}} - \partial_t \Delta \Sigma^{\text{tot}}_{\text{loc}} \right) \circ dx,
\]

again in analogy to the identification made in Ref. \(^{19}\).
Appendix B: Speed limit for the variance

We want to prove the bound Eq. \[(24)\]

\[
\frac{d_t \langle \Delta R^2 \rangle}{\langle \Delta R^2 \rangle} \leq \left( \frac{\dot{\kappa}_R(t)}{\dot{\gamma}_R(t)} - \frac{1}{2} \gamma_R(t)^2 \right) \frac{d_t}{dt},
\]

(B1)

where \(\kappa_R(t) = \langle \Delta R^4 \rangle / \langle \Delta R^2 \rangle^2\) and \(\gamma_R(t) = \langle \Delta R^3 \rangle / (\langle \Delta R^2 \rangle)^{3/2}\) denote the kurtosis and skewness of the distribution with respect to \(R\). We first note that \(d_t \langle \Delta R^2 \rangle\) cannot be written as the time-derivative of a time-independent observable \(\int dx \psi(x) \partial_t P(x,t)\). However, by taking the trivially extended probability distribution \(Q(x,y,t) = P(x,t)P(y,t)\) with \(\partial_t Q(x,y,t) = P(x,t)\partial_t P(y,t) + P(x,t)\partial_t P(y,t)\), we can write

\[
d_t \langle \Delta R^2 \rangle = \partial_t \int dx dy \left( R(x)^2 - R(x)R(y) \right) P(x,t)P(y,t) = \int dx dy \left( R(x)^2 - R(x)R(y) \right) \partial_t Q(x,y,t).\]

Now we can apply the covariance inequality

\[
\left( \frac{d_t \langle \Delta R^2 \rangle}{\langle \Delta R^2 \rangle} \right)^2 \leq \left( \int dx dy \left( R(x)^2 - R(x)R(y) \right)^2 Q(x,y,t) - \left( \int dx dy \left( R(x)^2 - R(x)R(y) \right) Q(x,y,t) \right)^2 \right) \times \int dx dy \frac{(\partial_t Q(x,y,t))^2}{Q(x,y,t)}.
\]

(B3)

For the second factor on the right-hand side, it is easy to see that

\[
\int dx dy \frac{(\partial_t Q(x,y,t))^2}{Q(x,y,t)} = \int dx dy \frac{P(x,t)^2(\partial_t P(y,t))^2 + 2P(x,t)P(y,t)\partial_t P(x,t)\partial_t P(y,t) + P(y,t)^2(\partial_t P(x,t))^2}{P(x,t)P(y,t)} = 2 \int dx \frac{(\partial_t P(x,t))^2}{P(x,t)},
\]

since the middle term vanishes. In the first factor, we replace \(R(x) = \Delta R(x,t) + \langle R \rangle_t\) with \(\Delta R(x,t) = R(x) - \langle R \rangle_t\) and, after some algebra, find

\[
\int dx dy \left( R(x)^4 - 2R(x)^3R(y) + R(x)^2R(y)^2 \right) P(x,t)P(y,t) - \left( \langle \Delta R^2 \rangle \right)^2 = \langle \Delta R^4 \rangle + 2 \langle \Delta R^3 \rangle \langle R \rangle_t + 2 \langle \Delta R^2 \rangle \langle R \rangle_t^2.
\]

(B5)

Finally, we use that \(R(x)\) and \(\dot{R}(x,t) = R(x) + r_0(t)\) with an arbitrary function \(r_0(t)\) have the same fluctuations, i. e. \(\Delta R(x,t) = \Delta \dot{R}(x,t)\) to write the bound as follows

\[
\left( \frac{d_t \langle \Delta R^2 \rangle}{\langle \Delta R^2 \rangle} \right)^2 \leq 2 \left( \langle \Delta R^4 \rangle + 2 \langle \Delta R^3 \rangle \langle R \rangle_t + \langle \Delta R^2 \rangle \langle R \rangle_t \right) \int dx \frac{(\partial_t P(x,t))^2}{P(x,t)}.
\]

(B6)

Since \(r_0(t)\) is arbitrary, we can minimize the right-hand side with respect to \(r_0(t)\) to obtain the tightest bound, which yields \(r_0(t) = -\langle R \rangle_t - \langle \Delta R^3 \rangle / (2 \langle \Delta R^2 \rangle_t)\) and thus

\[
\left( \frac{d_t \langle \Delta R^2 \rangle}{\langle \Delta R^2 \rangle} \right)^2 \leq 2 \left( \frac{\langle \Delta R^4 \rangle}{\langle \Delta R^2 \rangle_t} - \frac{1}{2} \frac{\langle \Delta R^3 \rangle^2}{\langle \Delta R^2 \rangle_t} \right) \int dx \frac{(\partial_t P(x,t))^2}{P(x,t)}.
\]

(B7)

Introducing the kurtosis and skewness, as above, this is precisely Eq. \[(24)\]. Note that since the kurtosis and the skewness satisfy the relation \(\kappa_R(t) \geq \gamma_R(t)^2 + 1\), we have \(\kappa_R(t) - \gamma_R(t)^2 / 2 \geq (\kappa_R(t) + 1) / 2\) and the right hand side of Eq. \[(24)\] is indeed always positive.
Appendix C: Minimal cost probability density

Let us consider two particular values \( \theta_1, \theta_2 \) of a parameter and the corresponding probability densities \( P^*(x) = P(x, \theta_1) \) and \( P_b(x) = P(x, \theta_2) \). Note that there is an infinite number of possible parameterized probability densities satisfying these conditions, e.g., we may have two probability densities \( P(x, \theta) \) and \( \tilde{P}(x, \theta) \) that coincide at \( \theta_1 \) and \( \theta_2 \) but are different otherwise. Each of these possible choices has an associated statistical length and cost defined by Eqs. (4) and (22)

\[
\mathcal{L}(\theta_2, \theta_1) = \int_{\theta_1}^{\theta_2} d\theta \int dx \frac{(\partial_q P(x, \theta))^2}{P(x, \theta)}, \quad \mathcal{C}(\theta_2, \theta_1) = \frac{1}{2} \int_{\theta_1}^{\theta_2} d\theta \int dx \frac{(\partial_q P(x, \theta))^2}{P(x, \theta)}
\]

where we assumed \( \theta_2 > \theta_1 \) without loss of generality. Note that for different \( P \) and \( \tilde{P} \), also the length and cost are generally different. However, there exists a unique choice \( P^*(x, \theta) \) which simultaneously minimizes the length and cost. To see this, we first minimize the cost \( \mathcal{C} \) with respect to \( P(x, \theta) \). In order to simplify the notation, we first reparameterize \( \theta(q) = \theta_2 q + \theta_1 (1 - q) \) with \( q \in [0, 1] \). Using this, we can write the length and cost as

\[
\mathcal{L}(\theta_2, \theta_1) = \int_{0}^{1} dq \int dx \frac{(\partial_q P(x, q))^2}{P(x, q)}
\]

\[
\mathcal{C}(\theta_2, \theta_1) = \frac{1}{2(\theta_2 - \theta_1)} \int_{0}^{1} dq \int dx \frac{(\partial_q P(x, q))^2}{P(x, q)},
\]

with \( P(x, q) \equiv P(x, \theta(q)) \). We now want to minimize \( \mathcal{C} \) with respect to \( P(x, q) \), under the condition that \( P(x, q) \) is a well-defined probability density, i.e., \( P(x, q) > 0 \) and \( \int dx \ P(x, q) = 1 \). Introducing the Lagrange multiplier \( \alpha \), we thus have to minimize the auxiliary functional

\[
F_c[P, \partial_q P] = \int_{0}^{1} dq \ f_c[P, \partial_q P](q) \equiv \int_{0}^{1} dq \left( \int dx \frac{(\partial_q P)^2}{P} - 4\alpha \left( \int dx \ P - 1 \right) \right)
\]

where the factor 4 in front of \( \alpha \) is included for later notational convenience. The corresponding Euler-Lagrange equation reads

\[
\partial_q f_c - \frac{\partial_q^2 P}{P^2} \frac{(\partial_q P)^2}{P} - 2 \frac{\partial_q^2 P}{P} - 4\alpha = 0
\]

Since \( P(x, q) > 0 \), we can write this as

\[
(\partial_q P)^2 - 2P \partial_q^2 P - 4\alpha P^2 = 0
\]

which has the general solution

\[
P(x, q) = f(x) \cos \left( \sqrt{\alpha}(q - g(x)) \right)^2.
\]

The functions \( f(x) \) and \( g(x) \), as well as the value of \( \alpha \) are fixed by the boundary conditions \( P(x, 0) = P^a(x) \) and \( P(x, 1) = P^b(x) \) and the normalization. The final result for \( P^*(x, q) \) minimizing the cost reads,

\[
P^*(x, q) = \frac{1}{1 - \cos \left( \frac{\Lambda q}{2} \right)} \left( \sec \left( \frac{\Lambda q}{2} \right) \sqrt{P^a(x)} + \tan \left( \frac{\Lambda q}{2} \right) \sqrt{P^b(x)} \right)^2
\]

with \( \Lambda = 2 \arccos \left( \int dx \sqrt{P^a(x) P^b(x)} \right) \).

For this choice, we have \( I^*(q) = \int dx \ (\partial_q P^*(x, q))^2/P^*(x, q) = \Lambda^2 \) and thus the minimal cost and statistical length

\[
\mathcal{C}^* = \frac{\Lambda^2}{2(\theta_2 - \theta_1)}, \quad \mathcal{L}^* = \Lambda
\]

In hindsight, it is obvious that \( \mathcal{C} \) is minimized by a probability density that yields constant Fisher information, since the former is defined as \( \mathcal{C} = \int_{\theta_1}^{\theta_2} d\theta \ I(\theta) \). The same is true for the length \( \mathcal{L} \), which is thus also minimized by \( P^* \).
note that, in analogy to the discussion in Ref. [19], the choice \( P^*(x, q) \) is the geodesic curve connecting \( P^a(x) \) and \( P^b(x) \), however, the geometric analogy is now less intuitive, since the underlying space is infinite-dimensional. Since \( P^*(x, q) \) yields the minimal length between \( P^a \) and \( P^b \) for a normalized probability density, we can interpret \( \mathcal{L}^* = \Lambda \) as the arc length between \( P^a \) and \( P^b \) on the infinite-dimensional unit sphere.

Since \( \mathcal{C}^* \) is the minimal cost, any other normalized probability density \( \tilde{P}(x, q) \) results in a larger cost \( \tilde{\mathcal{C}} \geq \mathcal{C}^* \). In particular, for a simple linear interpolation
\[
\tilde{P}(x, q) = P^b(x)q + P^a(x)(1 - q),
\]
which is positive and normalized, we obtain the cost
\[
\tilde{\mathcal{C}}(\theta_2, \theta_1) = \frac{1}{2(\theta_2 - \theta_1)} \int d x \left( P^b - P^a \right) \ln \left( \frac{P^b}{P^a} \right)
\]
\[
= \frac{1}{2(\theta_2 - \theta_1)} \left( D_{KL}(P^b || P^a) + D_{KL}(P^a || P^b) \right) = \frac{1}{\theta_2 - \theta_1} D_{KL}^{sym}(P^b, P^a),
\]
where we defined the symmetrized Kullback-Leibler divergence or relative entropy. We thus obtain the by no means obvious lower bound on the latter,
\[
D_{KL}^{sym}(P^b, P^a) \geq 2 \arccos \left( \int d x \sqrt{P^a(x)P^b(x)} \right)^2.
\]

Applying the above discussion to the time evolution of a stochastic dynamics \( \theta = t \), we fix the initial and final state of the system, \( P(x, 0) = P^a(x) \) and \( P(x, T) = P^b(x) \). The optimal time evolution between these two states is given by Eq. (C7) with \( s = t/T \). Since this results in \( \mathcal{L}^* = \Lambda \) and \( \mathcal{C}^* = \Lambda^2/(2T) \), we obtain a lower bound on the thermodynamic cost of the evolution from the initial to the final state [19],
\[
\mathcal{C} \geq \frac{\Lambda^2}{2T} \text{ with } \Lambda = 2 \arccos \left( \int d x \sqrt{P^a(x)P^b(x)} \right).
\]

Thus, the minimal thermodynamic cost is given by the square of the shortest distance between the initial and final state, divided by the evolution time. This shows that, in particular, a faster evolution is generally associated with a larger thermodynamic cost; further, zero cost is only realizable in the quasistatic limit where the time evolution is infinitely slow.

Appendix D: Monotonicity of Fisher information for Markov jump processes

Consider a Markov jump process on a set of \( M \) discrete states defined by the (generally time-dependent) transition rates \( W_{ij}(t) \geq 0 \) from state \( j \) to state \( i \) and occupation probabilities \( p_i(t) \) of state \( i \). The time-evolution of the occupation probabilities is governed by the Master equation [39]
\[
d_t p_i(t) = \sum_j \left( W_{ij}(t)p_j(t) - W_{ji}(t)p_i(t) \right) = \sum_j \mathcal{G}_{ij}(t)p_j(t),
\]
where we defined the matrix-valued generator \( \mathcal{G}(t) \)
\[
\mathcal{G}_{ij}(t) = W_{ij}(t) - \delta_{ij} \sum_k W_{ki}(t).
\]

In analogy to the continuous case, the (temporal) Fisher information is given in terms of the time-derivative of the occupation probability [19],
\[
I(t) = \sum_i \frac{\left( d_t p_i(t) \right)^2}{p_i(t)}.
\]

The time-derivative of the Fisher information is then
\[
d_t I(t) = \sum_i 2p_i(t)\left[ d_t p_i(t) \right] \left[ d^2_t p_i(t) \right] - \left[ d_t p_i(t) \right]^3 = \sum_i \left( 2 \left[ d_t \ln p_i(t) \right] \left[ d^2_t p_i(t) \right] - \left[ d_t \ln p_i(t) \right]^2 \left[ d_t p_i(t) \right] \right),
\]
or in terms of the generator
\[ d_t I(t) = \sum_i \left( 2 \left[ d_t \ln p_i(t) \right] \left[ d_t \sum_j G_{ij}(t) p_j(t) \right] - \left[ d_t \ln p_i(t) \right]^2 \left[ \sum_j G_{ij}(t) p_j(t) \right] \right) \quad (D5) \]
\[ = 2 \sum_{i,j} \left[ d_t \ln p_i(t) \right] \dot{G}_{ij}(t) p_j(t) + \sum_{i,j} \left( 2 \left[ d_t \ln p_i(t) \right] G_{ij}(t) \left[ d_t \ln p_j(t) \right] - \left[ d_t \ln p_i(t) \right]^2 G_{ij}(t) \right) p_j(t), \]

where we introduced the time-derivative of the generator \( \dot{G}(t) \). We define \( a_i \equiv d_t \ln p_i(t) \), in terms of which we can rewrite the above as
\[ d_t I(t) = 2a^T \dot{G} b + \sum_{i,j} \left( 2a_i G_{ij} a_j - a_i^2 G_{ij} p_j \right). \quad (D6) \]

We now plug the explicit definition (D2) of the generator into the second term,
\[ \sum_{i,j} \left( 2a_i G_{ij} a_j - a_i^2 G_{ij} p_j \right) = \sum_{i,j} \left( a_i (2G_{ij} a_j - a_i G_{ij}) p_j \right) \]
\[ = \sum_{i,j} \left( a_i (2W_{ij} - \delta_{ij} \sum_k W_{ki}(t)) a_j - a_i (W_{ij}(t) - \delta_{ij} \sum_k W_{ki}(t)) p_j \right) \]
\[ = \sum_{i,j} \left( 2a_i W_{ij} a_j p_j - a_i^2 W_{ij} p_j \right) - \sum_{i,k} \left( 2a_i^2 p_i W_{ki} - a_i^2 W_{ki} \right) \]
\[ = \sum_{i,j} \left( 2a_i W_{ij} a_j p_j - a_i^2 W_{ij} b_j - W_{ij} a_i^2 p_j \right) \]
\[ = -\sum_{i,j} \left( (a_i - a_j)^2 W_{ij} p_j \right), \]

where we renamed the summation indices in the last term from \((i, k)\) to \((j, i)\) in the second-to-last step. Since the both the transition rates and occupation probabilities are positive, \( W_{ij} \geq 0 \) and \( p_i \geq 0 \), this term is evidently negative. We thus arrive at
\[ d_t I(t) = -2 \left[ d_t \Phi(t) \right]^T \dot{G}(t) p(t) - \sum_{i,j} \left( d_t \Phi_i(t) - d_t \Phi_j(t) \right)^2 W_{ij}(t) p_j(t), \quad (D8) \]

where, in analogy to the continuous case, we introduced the vector of state-dependent Shannon entropy \( \Phi(t) \) defined by \( \Phi_i = -\ln p_i \). As in Eq. (35), the time-derivative of the Fisher information decomposes into a term involving the explicit time-dependence of the generator and a negative semidefinite term. If the transition rates do not depend explicitly on time, \( d_t W_{ij} = 0 \), then, just as in the case of Fokker-Planck dynamics, the Fisher information decreases monotonically in time
\[ d_t I(t) \leq 0, \quad (D9) \]
in complete analogy to Eq. (40). We remark that the same result holds for a mixed process,
\[ \partial_t P^k(x, t) = -\partial_x \left( a^k_i(x) - \partial_x B^k_{ij}(x) \right) P^k(x, t) + \sum_l \left( W^{kl}(x) P^l(x, t) - W^{lk}(x) P^k(x, t) \right), \quad (D10) \]
i.e. a Fokker-Planck dynamics with additional discrete states labeled by \( k \) and a state-dependent drift vector and diffusion matrix, since the generator is the sum of a diffusion and jump part, to which the arguments leading to Eqs. (40) and (D9) can be applied separately.

Appendix E: Shannon entropy and Fisher information for normal distributions

We consider a multivariate normal distribution
\[ P(x) = \frac{1}{\sqrt{(2\pi)^M \det(\Xi)\cdot M!}} \exp \left[ -\frac{1}{2} (x - \mu)^T \Xi^{-1} (x - \mu) \right], \quad (E1) \]
where $M$ denotes the dimension of $\mathbf{x}$, $\Xi$ is the (positive definite and symmetric) covariance matrix defined by

$$
\Xi_{ij} = \langle (x_i - \mu_i)(x_j - \mu_j) \rangle
$$

(E2)

and $\mu$ is the average of $\mathbf{x}$. We want to compute the variance of the logarithm of $P$,

$$
\Delta_{\ln} = \langle (\ln P)^2 \rangle - \langle \ln P \rangle^2.
$$

(E3)

By definition, we have

$$
\ln(P(\mathbf{x})) = -\frac{1}{2} \left( M \ln(2\pi) + \ln(\det \Xi^{-1}) + (\mathbf{x} - \mu)^T \Xi^{-1} (\mathbf{x} - \mu) \right).
$$

(E4)

Since the first two terms are independent of $\mathbf{x}$, they do not contribute to the variance and we thus have

$$
\Delta_{\ln} = \frac{1}{4} \left( \langle (\mathbf{x} - \mu)^T \Xi^{-1} (\mathbf{x} - \mu) \rangle^2 - \langle (\mathbf{x} - \mu)^T \Xi^{-1} (\mathbf{x} - \mu) \rangle^2 \right).
$$

(E5)

The average in the second term is readily computed,

$$
\langle (\mathbf{x} - \mu)^T \Xi^{-1} (\mathbf{x} - \mu) \rangle = \langle (x_i - \mu_i)(\Xi^{-1})_{ij}(x_j - \mu_j) \rangle = \langle \Xi^{-1} \rangle_{ij} \langle (x_i - \mu_i)(x_j - \mu_j) \rangle = \langle \Xi^{-1} \rangle_{ij} \Xi_{ij},
$$

(E6)

where summation over repeated indices is implied. Since the covariance matrix is symmetric, this is equal to

$$
\langle (\mathbf{x} - \mu)^T \Xi^{-1} (\mathbf{x} - \mu) \rangle = \langle \Xi^{-1} \rangle_{ij} \Xi_{ji} = \text{Tr}(\Xi^{-1} \Xi) = \text{Tr}(\mathbf{I}) = M.
$$

(E7)

For the first term, on the other hand, we have

$$
\langle (\mathbf{x} - \mu)^T \Xi^{-1} (\mathbf{x} - \mu) \rangle = \langle (x_i - \mu_i)(\Xi^{-1})_{ij}(x_j - \mu_j) (x_k - \mu_k)(\Xi^{-1})_{kl}(x_l - \mu_l) \rangle
$$

(E8)

$$
= \langle \Xi^{-1} \rangle_{ij} \langle (x_i - \mu_i)(x_j - \mu_j)(x_k - \mu_k)(x_l - \mu_l) \rangle.
$$

We now apply Isserli’s theorem for higher order moments of normal random variables,

$$
\langle (x_i - \mu_i)(x_j - \mu_j)(x_k - \mu_k)(x_l - \mu_l) \rangle = \Xi_{ij} \Xi_{kl} + \Xi_{ik} \Xi_{jl} + \Xi_{il} \Xi_{jk}
$$

(E9)

and again use the symmetry of the covariance matrix to write,

$$
\langle (\Xi^{-1})_{ij}(\Xi^{-1})_{kl} \rangle = \langle \Xi_{ij} (\Xi^{-1})_{kl} + \Xi_{ik} (\Xi^{-1})_{jl} + \Xi_{il} (\Xi^{-1})_{jk} \rangle.
$$

(E10)

We now recast the sum over $l$ in matrix notation,

$$
\Xi_{ij} (\Xi^{-1})_{kl} \Xi_{lk} + \Xi_{ik} (\Xi^{-1})_{kl} \Xi_{lj} + \Xi_{jl} (\Xi^{-1})_{kl} \Xi_{il} = \Xi_{ij} (\Xi^{-1})_{kk} + \Xi_{ik} (\Xi^{-1})_{kj} + \Xi_{jl} (\Xi^{-1})_{ij} \Xi_{kl}
$$

(E11)

$$
= \Xi_{ij} \delta_{kk} + \Xi_{ik} \delta_{kj} + \Xi_{jl} \delta_{kl} + \Xi_{ij} M + \Xi_{ij} + \Xi_{ji},
$$

where we performed the sum over $k$ in the last step. We thus have

$$
\langle (\mathbf{x} - \mu)^T \Xi^{-1} (\mathbf{x} - \mu) \rangle = \langle \Xi^{-1} \rangle_{ij} \langle \Xi_{ij} M + \Xi_{ij} + \Xi_{ji} \rangle = M^2 + 2M.
$$

(E12)

Plugging the results for the first and second term into Eq. (E5), we obtain the result,

$$
\Delta_{\ln} = \frac{M}{2},
$$

(E13)

independent of the form of the covariance matrix.
Next, for any distribution that depends on time only via its mean, \[ P(x, t) = \hat{P}(x - \mu(t)), \] (E14)
with a function \( \hat{P}(z) \) that does not explicitly depend on time, the Fisher information can be written as
\[ I(t) = \int dx \frac{(\partial_t P(x, t))^2}{P(x, t)} = \int dz \frac{(\hat{\mu}(t)^T \nabla_z \hat{P}(z))^2}{\hat{P}(z)}. \] (E15)
We now use the operator inequality,
\[ D - \Xi^{-1} \geq 0, \] (E16)
in the sense that the expression on the left-hand side is a positive semidefinite matrix. Here we defined
\[ (D)_{ij} = \int dz \frac{\partial_z \hat{P}(z) \partial_z \hat{P}(z)}{\hat{P}(z)}. \] (E17)
This inequality holds for arbitrary differentiable probability distributions and leads to
\[ I(t) = \hat{\mu}(t)^T D \hat{\mu}(t) \geq \hat{\mu}(t)^T \Xi^{-1} \hat{\mu}(t). \] (E18)
Since the rightmost expression is just the Fisher information for a normal distribution with time-independent covariance matrix, Eq. (52b), this proves the bound [50]. What is left to do is to prove the operator inequality Eq. (E16). To do so, consider the covariance \( \text{cov}(f, g) \equiv \langle fg \rangle - \langle f \rangle \langle g \rangle \) with respect to some differentiable probability distribution \( P(x), x \in \mathbb{R}^M \),
\[ \text{cov}(a^T x, b^T \nabla \ln(P)) = \int dx \ a_i x_i \partial_x P(x) - \int dx \ a_i x_i P(x) \int dy \ b_j \partial_x P(x) \]
\[ = -\int dx \ a_i b_j P(x) \partial_x x_i = -a_i b_j \delta_{ij}, \] where \( a, b \in \mathbb{R}^M \) are arbitrary vectors and we sum over repeated indices. Here, we integrated by parts in the second-to-last step. On the other hand, we have from the covariance inequality
\[ \text{cov}(a^T x, b^T \nabla \ln(P))^2 \leq \text{var}(a^T x) \text{var}(b^T \nabla \ln(P)), \] (E20)
where \text{var} denotes the variance with respect to \( P(x) \), \text{var}(f) \equiv \langle f^2 \rangle - \langle f \rangle^2 \) First, we note that \( \langle b^T \nabla \ln(P) \rangle = 0 \) and, consequently, the variance of \( b^T \nabla \ln(P) \) is given by
\[ \text{var}(b^T \nabla \ln(P)) = \int dx \ b_i \frac{\partial_x P(x) \partial_x P(x)}{P(x)} b_j = b^T Db, \] (E21)
Next, evaluate the variance
\[ \text{var}(a^T x) = \int dx \ a_i a_j x_i x_j P(x) - \int dx \int dy \ a_i a_j x_i y_j P(x) P(y) = a^T \Xi a \] (E22)
Then, the covariance inequality (E20) can be written as
\[ b^T Db \geq \frac{(b^T a)^2}{a^T \Xi a}. \] (E23)
Since this holds for arbitrary \( a \) and \( \Xi \) is positive definite and thus invertible, we may choose
\[ a = \Xi^{-1} b. \] (E24)
For this choice, we obtain
\[ b^T Db \geq \frac{(b^T \Xi^{-1} b)^2}{b^T \Xi^{-1} \Xi \Xi^{-1} b} = b^T \Xi^{-1} b, \] (E25)
where we used the symmetry of \( \Xi \) and that \( \Xi \Xi^{-1} = 1 \). Since \( b \) is arbitrary, this is equivalent to the inequality (E16).