One-loop helicity amplitudes for all \(2 \rightarrow 2\) processes in QCD and \(N=1\) supersymmetric Yang-Mills theory

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Abstract

One-loop corrections to the helicity amplitudes of all \(2 \rightarrow 2\) subprocesses are calculated in QCD and in \(N=1\) supersymmetric Yang-Mills theory using two versions of dimensional regularization: the ‘t Hooft-Veltman scheme and dimensional reduction. Studying the structure of the soft and collinear singularities, we found universal transition rules for the squared matrix element which can be used to translate the results obtained in these schemes to the results valid in the conventional dimensional regularization scheme. With explicit calculation it is demonstrated that the one-loop helicity amplitudes of the \(2 \rightarrow 2\) subprocesses calculated using dimensional reduction in the \(N=1\) supersymmetric \(SU(N)\) gauge theory respect the supersymmetry Ward identities. Our transition rules can also be used to calculate the next-to-leading order Altarelli-Parisi kernels in the dimensional reduction scheme when they satisfy supersymmetry Ward identities as well.

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1 Introduction

This paper discusses the consistency of various versions of the dimensional regularization schemes for regulating both ultraviolet and infrared divergences in calculations of infrared safe quantities for hadron-hadron collisions at next-to-leading order in QCD and in N=1 supersymmetric Yang-Mills theory. The discussion is based upon the calculation of the one-loop radiative corrections to the helicity amplitudes of all $2 \rightarrow 2$ parton scattering processes. The motivation for this study is a recent paper by Bern and Kosower [1] in which the authors worked out string-theory-based technique for evaluating multi-gluon amplitudes in one-loop order. The power of the method is demonstrated clearly by the recent calculation of the one-loop corrections to five gluon helicity amplitudes using the string-theory-based technology [2]. The field theory interpretation of the new results suggested that the use of background Feynman gauge, the helicity method and a new version of dimensional regularization, the so called four-dimensional helicity scheme, results in great technical advantages also in the standard field theory calculations.

In perturbative QCD, with the evaluation of the loop corrections one does not obtain yet physical cross-sections. The Bremsstrahlung contributions have to be added as well. In particular, finite hard scattering cross-sections are obtained only (after trivial ultraviolet renormalization) by cancelling soft and collinear singular terms between loop corrections and Bremsstrahlung contributions, and by subtracting the initial-state collinear singularities. In ref. [3], the singular terms appearing in this procedure have been worked out analytically using the conventional dimensional regularization scheme. The analytic expression obtained in ref. [3] for the soft and collinear contributions have universal, process independent character, but they depend on the regularization scheme used. The sensitivity to the regularization scheme, however, are also exhibited analytically, therefore, the result of ref. [3] can easily be transformed to the various versions of the dimensional regularization schemes.

Physical cross sections of infrared safe quantities are obtained by folding the finite hard-scattering cross section with parton densities. In a next-to-leading order calculation the $Q^2$-evolution of the parton densities has to be carried out with the next-to-leading order Altarelli-Parisi kernels which also depend on the regularization, factorization and ultraviolet-subtraction schemes. In the conventional $\overline{\text{MS}}$ factorization scheme the Altarelli-Parisi kernels have been calculated by Curci, Furmanski and Petronzio [4]. Changing the regularization scheme will generate changes in the next-to-leading order Altarelli-Parisi kernels, in the hard-scattering cross sections and in the value of $\Lambda_{\text{QCD}}$, but the physical cross section has to remain the same. Therefore, calculating the hard-scattering cross sections in different schemes, we can identify the universal transition terms which define the transformation of the next-to-leading order Altarelli-Parisi kernel from one scheme to another.

Recently, it has become clear that with appropriately modified dimensional regularization schemes the advantages of the helicity method can be maintained also for loop calculations [1, 5]. Three modifications have been proposed: the ‘t Hooft-Veltman scheme, regularization by dimensional reduction and the so called four-dimensional helicity scheme [4]. Changing the regularization scheme, the soft and collinear singular contributions will change but this

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1The four-dimensional helicity scheme was proposed in the context of the string-theory derivation of the one-loop corrections to the four-gluon helicity amplitudes [3]. According to our knowledge, no operational definition of this regularization has been defined in the standard field theory framework.
change has universal character. Therefore, it will be sufficient to find the changes generated in the hard scattering cross sections only for few processes. Then one knows the transition functions for all other processes. Deriving the transition terms has great practical importance because the phenomenological fit of the parton densities and their $Q^2$-evolution is worked out in the conventional regularization scheme. Therefore, if we calculate the next-to-leading order corrections in some other schemes, we must be able to transform the result into the conventional scheme.

In this paper, with the explicit calculation of the next-to-leading order corrections to all $2 \to 2$ parton-parton scattering amplitudes, we determine the transition rules necessary to translate the results obtained in the 't Hooft-Veltman and dimensional reduction schemes into those in the conventional scheme. The rules are process independent, therefore, they can be applied to other processes. For example, calculating the next-to-leading order corrections to the $2 \to 3$ scattering amplitudes, one can use such a regularization scheme that is consistent with the application of the helicity method and supersymmetry (dimensional reduction) and then one can easily translate the result into the phenomenologically relevant conventional scheme.

It has been noted long time ago that in calculating scattering amplitudes of massless partons in perturbative QCD, supersymmetry can be used as a technical tool to check and simplify the calculation. The reason for this is that if we change the color representation of quarks to the adjoint representation of a Majorana fermion we obtain the N=1 supersymmetric Yang-Mills theory in the Wess-Zumino gauge. If we use dimensional reduction which respects supersymmetry, then we can test supersymmetry Ward identities between the amplitudes of various subprocesses with different number of external fermions and bosons. These relations, which are valid in all order in perturbation theory, provide us a very significant check on the correctness of the calculation. One can also turn the argument around and use the supersymmetry Ward identities to obtain gluonic amplitudes from the fermionic ones correcting for the differences in the contribution of fermion loops.

We carried out all the calculations presented in this paper using standard Feynman diagram method, but using all the advantages of the helicity method, the background Feynman gauge and the 't Hooft-Veltman and dimensional reduction regularization schemes. The string theory derivation of the gluonic amplitudes in ref. [1] suggests that significant technical advantages can be achieved in addition with cleverly combining class of Feynman diagrams and reducing tensor integrals to scalar integrals. We did not make attempt to simplify our calculation in these directions. However, we used color subamplitudes [2] and the method of ref. [3] to reduce the tensor loop integrals to scalar integrals which is easily adaptable to calculations in massless QCD. This part of our field theory calculation still cannot match the simplicity achieved by the string-based technique.

The organization of our paper is as follows. In sect. 2, we describe the various versions of dimensional regularization briefly. In sects. 3 and 4, we give an overview of the techniques which enter our calculations: the helicity method and color decomposition of the amplitudes, the reduction of tensor integrals and use of background field method. These technicalities have been discussed in much details elsewhere in the literature, therefore, our presentation is constrained to setting the notation. Sect. 5 contains the one-loop color subamplitudes in QCD which constitutes one of our main results. We present the results in the unphysical region where all kinematic variables are negative which may become useful in the future. We
also calculate the shift in the strong coupling constant caused by using different regularization schemes. In sect. 6, we calculate the next-to-leading order loop-correction to the square of the matrix element in order that we can compare our results to that of Ellis and Sexton \[8\]. We establish transition rules among the one-loop amplitudes obtained using different versions of dimensional regularization. Sect. 7 presents the one-loop color subamplitudes in the supersymmetric limit of QCD, N=1 supersymmetric Yang-Mills theory, and elaborates on the usefulness of supersymmetry in QCD calculations. In order to find the complete transition rules among the various regularization schemes, in sect. 8, we discuss the difference among the Bremsstrahlung contributions obtained using different regulators. In sect. 9, we shall spell out how the hard-scattering cross section changes with changing the regularization scheme. In sect. 10, we shall discuss how the change in the hard-scattering cross section due to the use of different regularization schemes can be compensated by a change in the parton density functions. In this way, we can present some consistency checks on the correctness and process independent property of our transition rules. Finally, sect. 11 will summarize the main results.

2 Regularization schemes within dimensional regularization

In massless QCD, the conventional dimensional regularization is an attractive scheme because it simultaneously regulates infrared and ultraviolet divergences, is manifestly gauge invariant, consistent with unitarity and simple to implement.

When defining the different versions of dimensional regularization both for loop and phase space integrals, it is useful to distinguish two class of particles: observed and unobserved ones. Unobserved particles are those virtual ones which circulate in internal loops as well as those which are external but soft or collinear with other external particles. All the rest are observed particles. Unitarity demands that unobserved particles are treated uniformly.

The most important ingredient of dimensional regularization is the continuation of the momenta of the unobserved particles into \(d \neq 4\) dimensions, thus rendering the integrals over these momenta finite. Having this done, one is left with a lot of freedom how to treat the momenta of observed particles and the polarization vectors of all particles. The different choices lead to different versions of dimensional regularization.

The original choice made by ‘t Hooft-Veltman \[9\] was to continue both the momenta and the helicities of unobserved particles into \(d \neq 4\) dimensions, while leaving the momenta and helicities of observed particles in four dimensions.\[3\]

There is a third version of dimensional regularization, the dimensional reduction \[11\], widely used in connection with supersymmetric theories. This technique consist of continuing the virtual momenta of loop integrals into dimensions \(d < 4\), but keeping all polarization vectors in four dimensions. Because of \(d < 4\), the method is not directly applicable for regulating infrared divergent phase space integrals. It is essential that \(d < 4\) because the

\[2\]As a matter of fact, the original proposal was meant for ultraviolet regularization of loop integrals only. The possibility of using it also for regulating infrared singularities was initiated in ref. \[10\]. For a more detailed recent application see ref. \[5\].
four-dimensional vector field is split into a $d$-dimensional vector plus a field which transforms as $(4-d)$-dimensional scalars under gauge transformation [12]. It can be shown, however, that the use of this Lagrangian is equivalent to using the original Lagrangian without the splitting of the gauge field and perform the algebra in four dimensions. The only subtle point which is a reminiscent of the splitting is that one has to distinguish between four-dimensional metric tensors coming from the Feynman rules and $d$-dimensional ones emerging from momentum integrals with more than one loop momentum in the numerator. In this way, gauge invariance is maintained which has been checked explicitly up to two loops [12]. We remark that the operational definition of dimensional reduction is not complete in complicated cases (see e.g., refs. [13, 14]). However, this incompleteness does not affect our calculation.

In supersymmetric theories, there is a different formulation of dimensional reduction when supergraph technique is applied. In this approach all algebra is performed in four dimensions until only final scalar integrals are left in which the continuation of the loop momenta is performed. In ref. [12], for relatively simple calculations the equivalence of the two approaches has been pointed out.

Inspired by string-based rules for calculating loop amplitudes in gauge theories, in ref. [1] a new version of dimensional regularization, the four-dimensional helicity scheme has been proposed. The operational definition in the string calculation is such that only the momenta of unobserved particles are continued into $d$ dimensions, all polarization vectors and the momenta of observed particles are kept in four dimensions. The meaning of such a definition in field theory is obvious in most of the algebra except for the reduction of tensor loop integrals. In the string-based calculation, the loop momentum is integrated out at the string level which implies that the reduction of tensor integrals has been implicitly achieved in four dimensions. Therefore, in the field theory version of the four-dimensional helicity scheme, one may try to set $d = 4$ in the algebraic part of tensor loop integrals and treat the remaining scalar loop integral in $d$ dimensions. In the present paper, we do not pursue this possibility any further.

The requirements that a certain regularization scheme is a consistent regulator in a gauge field theory are that it has to respect gauge invariance and unitarity. Gauge invariance is known to be maintained by the conventional, ‘t Hooft-Veltman and dimensional reduction schemes. Unitarity is obviously maintained if the momenta and helicities of all particles are continued into $d$ dimensions, therefore, the conventional scheme is considered a consistent scheme in massless QCD. The other three versions of dimensional regularization do not treat momenta and helicities equally, therefore, it is not obvious that unitarity is respected by these schemes.

In axial gauge, the collinear singularities come from self-energy contributions on external lines only [1], therefore, it is expected that unitarity is preserved in the ‘t Hooft-Veltman scheme. In the case of dimensional reduction and four-dimensional helicity scheme, we do not know such a general argument.

3 The helicity method and color partial amplitudes

We use the helicity method in its simplest [15], crossing symmetric version [16]. In order to fix our notation, we present the definitions for the spinor algebra here. For more details, the
reader is referred to Appendix A of ref. [18].

Let $\psi(p)$ be a four-dimensional spinor satisfying the massless Dirac equation:

$$p\psi(p) = 0, \quad p^2 = 0. \tag{3.1}$$

We define the two helicity states of $\psi(p)$ by the two chiral projections

$$\psi^\pm(p) = \frac{1}{2}(1 \pm \gamma_5)\psi(p) = (\psi^\top(p))^c. \tag{3.2}$$

The second equation is a conventional choice of relative phase between opposite helicity spinors fixed by the properties under charge conjugation $C$,

$$(\psi(p))^c = C(\psi(p))^*, \quad C\gamma^\mu C^{-1} = \gamma^\mu. \tag{3.3}$$

Following ref. [1, 15, 16], we introduce a new notation

$$|p\pm\rangle = \psi_{\pm}(p), \quad \langle p\pm| = \psi^\pm(p), \tag{3.4}$$

$$\langle pq \rangle = \langle p - |p+\rangle, \quad [pq] = \langle p + |p-\rangle. \tag{3.5}$$

With this notation the normalization of the spinor is expressed as

$$\langle p \pm |\gamma^\mu|p\pm\rangle = 2p^\mu, \tag{3.6}$$

and we have the useful property

$$|p\pm\rangle\langle p\pm| = \frac{1}{2}(1 \pm \gamma_5)p. \tag{3.7}$$

The polarization vector of an outgoing massless vector of momentum $p$ is defined as

$$\epsilon^{\pm}(p, k) = \pm \frac{\langle p \pm |\gamma^\mu|k\pm\rangle}{\sqrt{2}(k \mp |p\pm\rangle} \tag{3.8}$$

where $k$ is an arbitrary (reference) momentum. Then we find that the usual requirements for the polarization vectors,

$$\epsilon^+(p, k) \cdot \epsilon^+(p, k) = 0, \tag{3.9}$$

$$\epsilon^+(p, k) \cdot \epsilon^-(p, k) = -1 \tag{3.10}$$

are satisfied.

As mentioned in the introduction, we also make use of the color subamplitudes which give gauge invariant decomposition in color space. For the tree and one-loop four-point amplitudes we find

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3 We use the same notation as in ref. [8] to distinguish the different processes and take all particles outgoing. We note that in the case of gluon-gluon scattering, our decomposition differs formally from that given in ref. [9]. The difference is the omission of the term proportional to $\text{Tr} T^a = 0$. 

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\[ A_4^{\text{tree}}(\bar{q}, \bar{q}; Q, q) = g^2 \left( \delta_{i_1 i_3} \delta_{i_2 i_4} - \frac{1}{N_c} \delta_{i_1 i_4} \delta_{i_2 i_3} \right) a_{4;0}(1, 2; 3, 4). \]  (3.11)

\[ A_4^{\text{tree}}(\bar{q}, q; q, q) = g^2 \sum_{\sigma \in S_2} \left( \delta_{i_1 \sigma(3)} \delta_{i_2 \sigma(4)} - \frac{1}{N_c} \delta_{i_1 \sigma(4)} \delta_{i_2 \sigma(3)} \right) \times b_{4;0}(1, 2; \sigma(3), \sigma(4)). \]  (3.12)

\[ A_4^{\text{tree}}(g, q; q, q) = g^2 \sum_{\sigma \in S_2} (T^{a_\sigma(1)} T^{a_\sigma(2)})_{i_3 i_4} c_{4;0}(\sigma(1), \sigma(2); 3, 4). \]  (3.13)

\[ A_4^{\text{tree}}(g, g, g) = g^2 \sum_{\sigma \in S_4 / Z_4} \text{Tr}(T^{a_\sigma(1)} T^{a_\sigma(2)} T^{a_\sigma(3)} T^{a_\sigma(4)}) \times d_{4;0}(\sigma(1), \sigma(2), \sigma(3), \sigma(4)). \]  (3.14)

\[ A_4^{1-\text{loop}}(\bar{q}, \bar{q}; Q, q) = g^4 \left[ \left( \delta_{i_1 i_3} \delta_{i_2 i_4} - \frac{1}{N_c} \delta_{i_1 i_4} \delta_{i_2 i_3} \right) a_{4;1}(1, 2; 3, 4) \right. \]
\[ \left. +\delta_{i_1 i_3} \delta_{i_2 i_4} a_{4;2}(1, 2; 3, 4) \right]. \]  (3.15)

\[ A_4^{1-\text{loop}}(\bar{q}, \bar{q}; q, q) = g^4 \sum_{\sigma \in S_2} \left[ \left( \delta_{i_1 \sigma(3)} \delta_{i_2 \sigma(4)} - \frac{1}{N_c} \delta_{i_1 \sigma(4)} \delta_{i_2 \sigma(3)} \right) \times b_{4;1}(1, 2; \sigma(1), \sigma(2)) \right. \]
\[ \left. +\delta_{i_1 \sigma(3)} \delta_{i_2 \sigma(4)} b_{4;2}(1, 2; \sigma(3), \sigma(4)) \right]. \]  (3.16)

\[ A_4^{1-\text{loop}}(g, q; q, q) = g^4 \left[ \sum_{\sigma \in S_2} (T^{a_\sigma(1)} T^{a_\sigma(2)})_{i_3 i_4} c_{4;1}(\sigma(1), \sigma(2); 3, 4) \right. \]
\[ \left. +\delta_{i_1 i_3} \delta_{i_2 i_4} c_{4;2}(1, 2; 3, 4) \right]. \]  (3.17)

\[ A_4^{1-\text{loop}}(g, g, g) = g^4 \left[ \sum_{\sigma \in S_4 / Z_4} \text{Tr}(T^{a_\sigma(1)} T^{a_\sigma(2)} T^{a_\sigma(3)} T^{a_\sigma(4)}) \times d_{4;1}(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) \right. \]
\[ \left. +\sum_{\sigma \in S_4 / Z_4^2} \text{Tr}(T^{a_\sigma(1)} T^{a_\sigma(2)}) \text{Tr}(T^{a_\sigma(3)} T^{a_\sigma(4)}) \times d_{4;2}(\sigma(1), \sigma(2); \sigma(3), \sigma(4)) \right]. \]  (3.18)

In these equations, \( S_n \) is the permutation group of \( n \) elements, \( Z_n \) is the cyclic group of \( n \) elements, i.e., \( S_n / Z_n \) means the group of permutations of \( n \) elements with cyclic permutations removed.

The helicity subamplitudes for \( n \)-gluon processes enjoy a number of useful properties \[ \square \] which simplify the actual calculation (or alternatively, can be used as checks of the results). The amplitudes of the other processes possess much less symmetry. However, useful relations can be discovered in the supersymmetric limit (see sect. 7).

The Feynman diagrams for quark-quark, quark-gluon and gluon-gluon scattering have been presented in ref. \[ \square \]; we do not repeat them in our paper. The diagrams that enter
the calculation of a given partial amplitude can be found by imposing the condition that they have the proper color structure. In the sect. 4, we describe how to compute a one-loop Feynman diagram. The external legs of the diagrams to be computed are on shell, i.e., for massless partons the momentum squared of the external legs is zero. In dimensional regularization a loop insertion on a particle line is proportional to the momentum squared of the line. This means that those diagrams which have a loop insertion on an external line are zero in dimensional regularization. Motivated by the results of ref. [19], we shall use background Feynman gauge. The necessary Feynman rules can be found in the standard reference [20].

4 Calculation of one-loop Feynman diagrams

One of the methods we used for calculating one-particle irreducible Feynman diagrams is as described in ref. [7]. In this section, we recall the basic ingredients of the method and give all necessary formulæ for calculating four-parton processes explicitly.

The general algebraic expression for a one-loop one-particle irreducible Feynman diagram with $N$ propagators in $d = 2\omega$ ($\omega = 2 - \varepsilon$) dimensions is

$$M = \mu^{2\varepsilon} \int \frac{d^{2\omega} \ell}{(2\pi)^{2\omega}} \frac{N(q,p)}{\prod_{r=1}^{N}(q_r^2 - m_r^2 + i\eta)},$$

(4.1)

where $\ell$ is the loop momentum, $p_i$ are the external outgoing momenta, $q_r$ are the momenta of the propagators and $N(q,p)$ is the numerator which receives its contributions from vertices and numerators of propagators and contains the dimensionless coupling. After Feynman parametrization the loop integral can be performed to give

$$M = (-1)^N \frac{i}{(4\pi)^2 (4\pi \mu^2)^\varepsilon} \sum_{k=0}^{\varepsilon} \Gamma(N - \omega - k) \int \prod_{r=1}^{N} dx_r \frac{N_k(J,p)}{D(x,p)^{N-\omega-k}} \delta \left( \sum_{r=1}^{N} x_r - 1 \right).$$

(4.2)

Cutting two of the propagators in the loop produces two tree diagrams — a two-tree. A double cut can be performed \( \binom{N}{2} \) different ways. The set of two-trees will be denoted by $T_2$. With this notation and for massless particles in the propagators

$$D(x,p) = -\sum_{T_2} (x_{c_i}, x_{c_j}) \left( \sum_c p_c \right)^2,$$

(4.3)

where $x_{c_i}, x_{c_j}$ are the Feynman parameters of the two cut lines and the sum over $c$ is the sum of the external momenta belonging to one of the trees. In the numerator in eq. (4.2), $N_0(J,p) = N(J,p)$ and

$$J_r = \sum_{T_2(r)} x_{c_i} \left( \sum_c p_c \right).$$

(4.4)

\textsuperscript{4}A factor of $\mu^{(N-2)\varepsilon}$ is omitted. When a cross section is calculated this factor appears in the corresponding tree diagrams, as well. Therefore, for the cross section it means an irrelevant overall factor which disappears at the end of the calculation when physical dimension is considered.
In this equation one of the cut lines is \( r \), while \( x_{c_i} \) is the Feynman parameter of the other cut line. In the sum over \( c \), the sum of the external momenta belonging to the tree to which \( q_r \) flows is to be taken. \( N_k(J,p) \) for \( k > 0 \) is obtained from \( N_0(J,p) \) by contracting \( k \) pairs of \( J \)'s in all possible ways and summing over all such contractions. If no contraction is possible then \( N_k = 0 \). Contraction of \( J \)'s means

\[
J^\mu J_s^\nu \rightarrow -\frac{1}{2} g^{\mu\nu}_{(2\omega)}. \tag{4.5}
\]

This rule is valid in this simple form only if the propagators in the loop constitute a continuous flow of the momentum which for a one-loop diagram can always be arranged.

We give formula (4.2) for the cases that enter our calculation (\( N = 2, 3, 4 \)) together with the necessary integrals explicitly in Appendix A.

As a second method, we calculated the one-loop tensor integrals using the standard Passarino-Veltman type scheme \[17\].

We note that the tree level diagrams with one-loop renormalization insertions have also to be taken into account. The \( \overline{\text{MS}} \) ultraviolet counterterm for the \( n \)-point scattering amplitude at the one-loop level is well known:

\[
(n - 2) \left[ -\frac{1}{2} \beta_0 \left( \frac{g}{4\pi} \right)^2 \frac{(4\pi)^\epsilon}{\epsilon \Gamma(1 - \epsilon)} A_n^{\text{tree}}(1, \ldots, n) \right], \tag{4.6}
\]

where \( \beta_0 = (11N_c - 2N_f)/3 \) and \( A_n^{\text{tree}} \) is the four-dimensional tree-level amplitude.

## 5 One-loop color subamplitudes in QCD

In this section, we present the results obtained for the QCD color subamplitudes. First, we list the amplitudes in the ‘t Hooft-Veltman scheme. Next, we give the differences of the amplitudes obtained in the ‘t Hooft-Veltman and dimensional reduction schemes. Finally, we calculate \( \Lambda^{\overline{\text{DR}}}_{\overline{\text{MS}}} \) for dimensional reduction in terms of \( \Lambda_{\overline{\text{MS}}} \) of conventional dimensional regularization.\[6\]

In order to check the following results, we calculated the nontrivial integrals in two different ways (one is described in Appendix A) and checked gauge invariance in all external gluon legs using longitudinal gluons. The amplitudes for gluon-gluon scattering without the fermionic contribution have already been calculated in ref. \[1\]. Our results are in complete agreement with the published ones.

### 5.1 Results in the ‘t Hooft-Veltman scheme

In the following, we constrain the presentation of the results to the minimal information from which after trivial algebra — i.e., using parity transformation or the cyclic property of four-gluon amplitudes — any helicity amplitude can be obtained. To calculate the square of the matrix element for process B, the same partial amplitudes are to be used as for process

\[5\] This contraction is done in \( 2\omega \) dimensions both in ‘t Hooft-Veltman and dimensional reduction schemes.

\[6\] \( \Lambda_{\overline{\text{MS}}} \) in the ‘t Hooft-Veltman and conventional schemes are identical.
A, only the color and helicity sums differ. Therefore, the subamplitudes for processes A and B are not written separately. The \( d_{4;2} \) subamplitudes will not play any role in the rest of this paper. They can be obtained from the from the \( N_f \) independent part of the \( d_{4;1} \) amplitudes using the decoupling equations of ref. [1], where they were published, therefore, we do not repeat them here. We remind the reader that according to our convention all particles are outgoing.

We start our description of the results with those amplitudes which are finite in four dimensions (the corresponding tree amplitudes are vanishing).

Process A:

The amplitudes which vanish at tree level vanish at one-loop level as well.

Process C:

\[
\begin{align*}
  c_{4;1}^{HV}(+,+;+,-) &= -\frac{i}{48\pi^2} \frac{\langle 34\rangle[12][13][23]}{s_{12}s_{14}} \left[ \frac{3}{2} \frac{N_c^2 + 1}{N_c} + (N_c - N_f) \frac{s_{14}}{s_{12}} \right]. \\
  c_{4;1}^{HV}(-,-;+,-) &= \frac{i}{48\pi^2} \frac{\langle 12\rangle\langle 14\rangle\langle 24\rangle\langle 34\rangle}{s_{12}s_{14}} \left[ \frac{3}{2} \frac{N_c^2 + 1}{N_c} + (N_c - N_f) \frac{s_{14}}{s_{12}} \right].
\end{align*}
\]

(5.1) (5.2)

Process D:

\[
\begin{align*}
  d_{4;1}^{HV}(+,+,+,+) &= \frac{i}{48\pi^2} (N_c - N_f) \frac{s_{12}s_{14}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}, \\
  d_{4;1}^{HV}(-,+,+,-) &= \frac{i}{48\pi^2} (N_c - N_f) \frac{[24]^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} (s_{12} + s_{14}).
\end{align*}
\]

(5.4) (5.5)

All the other partial amplitudes are divergent in four dimensions. We present the results in the unphysical region, where all kinematic variables are negative, therefore, the arguments of the logarithms are positive. The analytic continuation to the physical region can easily be performed keeping in mind that the branch cut of the logarithm is obtained by inserting the \( i\eta \) associated with each kinematic variable:

\[
\begin{align*}
  (-s_{ij})^{-\epsilon} &\to |s_{ij}| e^{-i\pi\epsilon\Theta(s_{ij})}, \\
  \log(-s_{ij}) &\to \log |s_{ij}| + i\pi\Theta(s_{ij}),
\end{align*}
\]

(5.7) (5.8)

where \( \Theta(x) \) is the usual step function. The definition of \( \beta_0 \), appearing in the following results, is given after eq. (4.6). One finds:

\footnote{We have checked the results for Processes A and C in the ‘t Hooft-Veltman scheme using Passarino-Veltman type reduction of tensor integrals.}
Process A:

\[ a_{4:1}^{HV}(-,-;+,+) = c_\tau a_{4:0}(-,-;+,+) F_{a:1}^{--}(\varepsilon, s_{12}, s_{13}, s_{14}), \quad (5.9) \]

\[ F_{a:1}^{--}(\varepsilon, s_{12}, s_{13}, s_{14}) = \]

\[ \left( -\frac{\mu^2}{s_{14}} \right)^\varepsilon \left\{ N_e \left[ -\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} + \frac{11}{3\varepsilon} - \frac{2}{\varepsilon} \log \frac{s_{14}}{s_{12}} + \frac{13}{9} + \pi^2 \right] + N_f \left[ -\frac{2}{3\varepsilon} - \frac{10}{9} \right] \right. \]

\[- \frac{1}{N_e} \left[ -\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - \frac{2}{\varepsilon} \log \frac{s_{12}}{s_{13}} - 8 + \frac{1}{s_{14}} \left( 1 - \frac{s_{13}}{s_{12}} \right) \left( \log^2 \frac{s_{14}}{s_{12}} + \pi^2 \right) \frac{s_{14}}{s_{12}} \log \frac{s_{14}}{s_{13}} \right] \left\} - \frac{1}{\varepsilon} \beta_0. \]

\[ a_{4:1}^{HV}(-,+;-,+)= c_\tau a_{4:0}(-,+-,-+) F_{a:1}^{-+}(\varepsilon, s_{12}, s_{13}, s_{14}), \quad (5.11) \]

\[ F_{a:1}^{-+}(\varepsilon, s_{12}, s_{13}, s_{14}) = \]

\[ \left( -\frac{\mu^2}{s_{14}} \right)^\varepsilon \left\{ N_e \left[ -\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} + \frac{11}{3\varepsilon} - \frac{2}{\varepsilon} \log \frac{s_{14}}{s_{12}} + \frac{13}{9} + \pi^2 \right] + N_f \left[ -\frac{2}{3\varepsilon} - \frac{10}{9} \right] \right. \]

\[- \frac{1}{N_e} \left[ -\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - \frac{2}{\varepsilon} \log \frac{s_{12}}{s_{13}} - 8 \right] \]

\[ \left( N_e + \frac{1}{N_e} \right) \left[ \frac{1}{s_{14}} \left( 1 - \frac{s_{12}}{s_{13}} \right) \left( \log^2 \frac{s_{14}}{s_{12}} + \pi^2 \right) + \frac{s_{14}}{s_{13}} \log \frac{s_{14}}{s_{12}} \right] \left\} - \frac{1}{\varepsilon} \beta_0. \]

\[ a_{4:2}^{HV}(-,-;+,+) = c_\tau a_{4:0}(-,+-,+)+ F_{a:2}^{--}(\varepsilon, s_{12}, s_{13}, s_{14}), \quad (5.13) \]

\[ F_{a:2}^{--}(\varepsilon, s_{12}, s_{13}, s_{14}) = \]

\[ \left( -\frac{\mu^2}{s_{14}} \right)^{\varepsilon} \left\{ V \left[ -\frac{2}{\varepsilon} \log \frac{s_{12}}{s_{13}} + \frac{1}{s_{14}} \left( 1 - \frac{s_{13}}{s_{12}} \right) \left( \log^2 \frac{s_{14}}{s_{12}} + \pi^2 \right) + \frac{s_{14}}{s_{12}} \log \frac{s_{14}}{s_{13}} \right] \cdot \right. \]

\[ a_{4:2}^{HV}(-,+;-,+)= c_\tau a_{4:0}(-,+-,-+) F_{a:2}^{-+}(\varepsilon, s_{12}, s_{13}, s_{14}), \quad (5.15) \]

\[ F_{a:2}^{-+}(\varepsilon, s_{12}, s_{13}, s_{14}) = \]

\[ \left( -\frac{\mu^2}{s_{14}} \right)^{\varepsilon} \left\{ V \left[ -\frac{2}{\varepsilon} \log \frac{s_{12}}{s_{13}} - \frac{1}{s_{14}} \left( 1 - \frac{s_{12}}{s_{13}} \right) \left( \log^2 \frac{s_{14}}{s_{12}} + \pi^2 \right) - \frac{s_{14}}{s_{13}} \log \frac{s_{14}}{s_{12}} \right] \right. \]
Process C:

\[
F_{c_1}^{++}(\varepsilon, s_{12}, s_{13}, s_{14}) = 
\left(-\frac{\mu^2}{s_{12}}\right)^\varepsilon \left\{ N_e \left[ -\frac{3}{\varepsilon^2} - \frac{3}{\varepsilon} \log \frac{s_{13}}{s_{12}} - \frac{7}{2} + \frac{1}{2} \pi^2 - \frac{1}{2} \log^2 \frac{s_{13}}{s_{12}} + \frac{3}{2} \log \frac{s_{13}}{s_{12}} \\
- \frac{1}{N_e} \left[ -\frac{3}{\varepsilon^2} - \frac{3}{2\varepsilon} - 4 \right] \left( \frac{s_{13}}{s_{14}} \log \frac{s_{13}}{s_{12}} \right)^2 + \log \frac{s_{13}}{s_{12}} + \left( \frac{s_{13}}{s_{14}} \right)^2 \pi^2 \right]\right\} - \frac{1}{\varepsilon} \beta_0.
\]

\[
c_{4_1}^{HV}(+; +; +; -) = c_T c_{4_0}(-; +; +; -) F_{c_1}^{++}(\varepsilon, s_{12}, s_{13}, s_{14}),
\]

\[
F_{c_1}^{--}(\varepsilon, s_{12}, s_{13}, s_{14}) = 
\left(-\frac{\mu^2}{s_{12}}\right)^\varepsilon \left\{ N_e \left[ -\frac{3}{\varepsilon^2} - \frac{3}{\varepsilon} \log \frac{s_{13}}{s_{12}} - \frac{7}{2} + \frac{1}{2} \pi^2 - \frac{1}{2} \log^2 \frac{s_{13}}{s_{12}} + \frac{3}{2} \log \frac{s_{13}}{s_{12}} \\
- \frac{1}{N_e} \left[ -\frac{3}{\varepsilon^2} - \frac{3}{2\varepsilon} - 4 \right] \left( \frac{s_{13}}{s_{14}} \log \frac{s_{13}}{s_{12}} \right)^2 + \log \frac{s_{13}}{s_{12}} + \left( \frac{s_{13}}{s_{14}} \right)^2 \pi^2 \right]\right\} - \frac{1}{\varepsilon} \beta_0.
\]

\[
c_{4_1}^{HV}(-; +; +; -) = c_T c_{4_0}(+; -; +; +) F_{c_1}^{--}(\varepsilon, s_{12}, s_{13}, s_{14}),
\]

\[
F_{c_2}^{++}(\varepsilon, s_{12}, s_{13}, s_{14}) = 
-2 \left[ \frac{1}{s_{12}} \left( -\frac{\mu^2}{s_{12}} \right)^\varepsilon \log \frac{s_{14}}{s_{13}} + \frac{1}{s_{13}} \left( -\frac{\mu^2}{s_{13}} \right)^\varepsilon \log \frac{s_{14}}{s_{12}} \right] + \frac{3}{2s_{14}} \left[ \log^2 \frac{s_{13}}{s_{12}} + \pi^2 \right].
\]

\[
c_{4_2}^{HV}(+, +; +; -) = c_T \left\{ \frac{24}{14} \right\} F_{c_2}^{++}(\varepsilon, s_{12}, s_{13}, s_{14}) F_{c_2}^{--}(\varepsilon, s_{12}, s_{13}, s_{14}),
\]

\[
F_{c_2}^{--}(\varepsilon, s_{12}, s_{13}, s_{14}) = 
-2 \left[ \frac{1}{s_{12}} \left( -\frac{\mu^2}{s_{12}} \right)^\varepsilon \log \frac{s_{14}}{s_{13}} + \frac{1}{s_{13}} \left( -\frac{\mu^2}{s_{13}} \right)^\varepsilon \log \frac{s_{14}}{s_{12}} \right] - \frac{3}{2s_{13}} \left[ \log^2 \frac{s_{14}}{s_{12}} + \pi^2 \right].
\]
Process D:

\[ d_{4i}^{HV}(-, -, +, +) = c_G d_{4,0}(-, -, +, +) F_{4i}^{--}(\varepsilon, s_{12}, s_{13}, s_{14}), \quad (5.25) \]

\[ F_{4i}^{--}(\varepsilon, s_{12}, s_{13}, s_{14}) = \left( -\frac{\mu^2}{s_{14}} \right) \varepsilon \left\{ N_c \left[ -\frac{4}{\varepsilon^2} - \frac{11}{3\varepsilon} - \frac{2}{\varepsilon} \log \frac{s_{14}}{s_{12}} - \frac{67}{9} + \pi^2 \right] + N_f \left[ \frac{2}{3\varepsilon} + \frac{10}{9} \right] \right\} - \frac{1}{\varepsilon} \beta_0. \quad (5.26) \]

\[ d_{4i}^{HV}(-, +, -, +) = c_G d_{4,0}(-, +, -, +) F_{4i}^{-+}(\varepsilon, s_{12}, s_{13}, s_{14}), \quad (5.27) \]

\[ F_{4i}^{-+}(\varepsilon, s_{12}, s_{13}, s_{14}) = \left( -\frac{\mu^2}{s_{14}} \right) \varepsilon \left\{ N_c \left[ -\frac{4}{\varepsilon^2} - \frac{11}{3\varepsilon} - \frac{2}{\varepsilon} \log \frac{s_{14}}{s_{12}} - \frac{67}{9} + \pi^2 \right] + N_f \left[ \frac{2}{3\varepsilon} + \frac{10}{9} \right] \right\} - \frac{1}{\varepsilon} \beta_0. \quad (5.28) \]

In these equations

\[ c_G = \frac{1}{(4\pi)^{2-\varepsilon}} \frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)} \quad (5.29) \]

is a ubiquitous prefactor, \( V = N_c^2 - 1 \). For the sake of completeness, we list the tree level expressions here:

\[ a_{4,0}(-, -; +, +) = i \frac{\langle 12 | 34 \rangle}{s_{14}}, \quad (5.30) \]

\[ a_{4,0}(-, +; -, +) = i \frac{\langle 13 | 24 \rangle}{s_{14}}, \quad (5.31) \]

\[ c_{4,0}(g, g; q^+, \bar{q}^-) = -i \frac{\langle I q \rangle \langle I \bar{q} \rangle^3}{\langle 12 | 2q \rangle \langle \bar{q} \bar{q} \rangle \langle q1 \rangle}, \quad (5.32) \]

\[ d_{4,0}(g, g, g, g) = i \frac{\langle I J \rangle^4}{\langle 12 | 23 \rangle \langle 34 \rangle \langle 41 \rangle}, \quad (5.33) \]

For process C, \( c_{4,0} \) is vanishing if the gluons have the same helicity. In eqs. (5.32) \( I \) denotes the negative helicity gluon. For process D, \( d_{4,0} \) is non-vanishing only if two gluons have positive and two of them have negative helicity. In eq. (5.33), \( I \) and \( J \) denote the negative helicity gluons.
5.2 Results obtained using dimensional reduction

The one-loop color subamplitudes obtained using dimensional reduction can most conveniently be written with the help of the results given in the previous subsection. Obviously, to $\mathcal{O}(\varepsilon)$, only the divergent amplitudes may differ in the various schemes, therefore, we do not repeat the finite ones.

The difference in the algebra when 't Hooft-Veltman or dimensional reduction schemes are used together with the tensor reduction of sect. 4, can be described by a single parameter which is chosen to be $d$-dimensional in the 't Hooft-Veltman scheme and has four-dimensional value in dimensional reduction. Changing the value of this parameter, one finds the following differences:

$$
a^\text{DR}_{4;1}(-,\pm;\mp,+) - a^\text{HV}_{4;1}(-,\pm;\mp,+) = c_\Gamma a_{4;0}^\text{(2)} (-,\pm;\mp,+) \left( \frac{2}{3} N_c - \frac{1}{N_c} \right).
$$

$$
c^\text{DR}_{4;1}(-,\pm;\mp,+) - c^\text{HV}_{4;1}(-,\pm;\mp,+) = c_\Gamma c_{4;0}^\text{(2)} (-,\pm;\mp,+) \frac{1}{2} \left( N_c - \frac{1}{N_c} \right).
$$

$$
d^\text{DR}_{4;1}(-,\pm;\mp,+) - d^\text{HV}_{4;1}(-,\pm;\mp,+) = c_\Gamma d_{4;0}^\text{(2)} (-,\pm;\mp,+) \frac{1}{3} N_c,
$$

while the $m^\text{a;2}_4$, ($m = a, c$ or $d$) subamplitudes are identical in the two schemes. We remark that our results for the gluon-gluon scattering without the fermionic contribution, obtained using dimensional reduction is identical with the corresponding expression obtained using string-based rules and the four-dimensional helicity scheme in ref. [1].

These differences may have two origins. They can arise from the different regularization of both the ultraviolet and the infrared singularities. The difference in the ultraviolet regularization can be absorbed into the change of $\Lambda_{\text{QCD}}$. This difference can be understood as a finite renormalization of the gauge coupling and can be calculated separately. In the background field technique we obtain the finite renormalization from calculating the gluon self energy in two different regularization schemes and use the background field Ward identity [20],

$$
Z^f_g = (Z^f_A)^{-1/2}.
$$

The self energy has the well-known form,

$$
\Pi^{ab}_{\mu\nu}(k) = i\delta^{ab}(k_\mu k_\nu - k^2 g_{\mu\nu})\Pi(k^2),
$$

where

$$
\Pi^{\text{HV}}(k^2) = \left( \frac{g_\pi}{4\pi} \right)^2 \left( -\frac{4\pi\mu^2}{k^2} \right)^\varepsilon \Gamma(1 + \varepsilon) \left( -\frac{1}{\varepsilon} \beta_0 - \frac{67 N_c - 10 N_f}{9} \right) Z^f_A - 1 + \mathcal{O}(\varepsilon)
$$

in the 't Hooft-Veltman scheme and

$$
\Pi^{\text{DR}}(k^2) = \left( \frac{g_\pi}{4\pi} \right)^2 \left( -\frac{4\pi\mu^2}{k^2} \right)^\varepsilon \Gamma(1 + \varepsilon) \left( -\frac{1}{\varepsilon} \beta_0 - \frac{64 N_c - 10 N_f}{9} \right) + \mathcal{O}(\varepsilon),
$$

in dimensional reduction. As a result of the finite renormalization, the difference between the two results vanish, therefore,

$$
Z^f_A = 1 + \frac{1}{3} N_c \left( \frac{g_\pi}{4\pi} \right)^2
$$
and
\[ Z_g^f = 1 - \frac{1}{6} N_c \left( \frac{g_r}{4\pi} \right)^2. \] (5.42)

After this renormalization, the difference between the two schemes for the amplitudes becomes
\[ a^\text{DR}_{4;1}(-, \pm; \mp, +) - a^\text{HV}_{4;1}(-, \pm; \mp, +) = \Gamma a_{4;0}(-, \pm; \mp, +) \left( N_c - \frac{1}{N_c} \right), \] (5.43)
\[ c^\text{DR}_{4;1}(-, \pm; \mp, +) - c^\text{HV}_{4;1}(-, \pm; \mp, +) = \Gamma c_{4;0}(-, \pm; \mp, +) \left( \frac{5}{6} N_c - \frac{1}{2N_c} \right), \] (5.44)
\[ d^\text{DR}_{4;1}(-, \pm; \mp, +) - d^\text{HV}_{4;1}(-, \pm; \mp, +) = \Gamma d_{4;0}(-, \pm; \mp, +) \left( \frac{5}{6} N_c - \frac{1}{2N_c} \right). \] (5.45)

The tilde in eqs. (5.43–5.45) reminds us that the expansion parameter is the same \( \alpha^\text{MS} \) in both schemes. In sect. 10, we shall discuss how these differences can be incorporated into the definition of the parton densities.

As a final step in this section, following ref. [21], we calculate the change in \( \Lambda_{QCD} \) caused by the finite renormalization of the gauge coupling according to eq. (5.42):
\[ \Lambda_{\text{DR}} = \Lambda_{\text{MS}} \exp \frac{c}{\beta_0}. \] (5.46)

The constant \( c \) is defined by
\[ \alpha_s^{\text{DR}} = \alpha_s^{\text{MS}} \left( 1 + c \frac{\alpha_s^{\text{MS}}}{2\pi} \right), \] (5.47)
i.e., \( c = N_c/6 \) in agreement with the result of ref. [22] obtained in Lorentz-Feynman gauge.

6 Loop contribution to the next-to-leading order matrix element

In this section, first we give explicit formulæ how to obtain the loop contribution to the next-to-leading order matrix element summed over color and helicities and then compare the obtained results with the corresponding expressions given in ref. [8]. In this way, we can establish the transition rules for loop amplitudes which connect our results to those obtained in the conventional scheme.

6.1 Squared matrix element in the ‘t Hooft-Veltman scheme

It is useful to rewrite the \( F \) functions in terms of the physical variables \( s, t \) and \( u \) in a form which will be more convenient when the square of the matrix element is calculated. In the next-to-leading order amplitudes of sect. 5, we factored out the Born terms. Therefore, it is sufficient to present only the real part of the \( F \) functions since the imaginary part will not play any role in a next-to-leading order calculation of the squared matrix elements \( \langle \text{Re} F \rangle \equiv F \). To keep the crossing symmetry of the matrix element manifest, we leave the
To keep the real parts of the logarithms, we use the following substitutions:  
\[ \log \left( \frac{s_{12}}{s_{14}} \right) \to \ell(s) - \ell(t), \quad \log^2 \left( \frac{s_{12}}{s_{14}} \right) \to \ell_2(s) + \ell_2(t) - 2\ell(s)\ell(t), \]  
(6.1)  
\[ \log \left( \frac{s_{12}}{s_{13}} \right) \to \ell(s) - \ell(u), \quad \log^2 \left( \frac{s_{12}}{s_{13}} \right) \to \ell_2(s) + \ell_2(u) - 2\ell(s)\ell(u), \]  
(6.2)  
\[ \log \left( \frac{s_{13}}{s_{14}} \right) \to \ell(u) - \ell(t), \quad \log^2 \left( \frac{s_{13}}{s_{14}} \right) \to \ell_2(u) + \ell_2(t) - 2\ell(u)\ell(t), \]  
(6.3)  
where  
\[ \ell(x) = \log \left| \frac{x}{Q^2} \right| \]  
and  
\[ \ell_2(x) = \ell^2(x) - \pi^2 \Theta(x). \]  
(6.4)  

One finds the following results.

Process A:

For the square of the matrix element only the sum of the \( F_{a:1} \) and \( F_{a:2} \) functions is needed.

\[ F_{a}^{--}(\varepsilon, s, t, u) \equiv \Re \{ F_{a:1}^{--}(\varepsilon, s, t, u) + F_{a:2}^{--}(\varepsilon, s, t, u) \} = \]  
(6.5)  
\[ \left( \frac{\mu^2}{Q^2} \right)^\varepsilon \left\{ \frac{V}{N_c} \left( -\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} \right) - \frac{10}{9} N_f + \frac{V}{N_c} \left( \frac{2}{\varepsilon} \ell(t) - \ell_2(t) + 3\ell(t) \right) + \frac{2}{3} N_f \ell(t) \right. \]  
\[ \left. + N_c \left[ \frac{2}{\varepsilon} (\ell(u) - \ell(t)) + \frac{13}{9} + \pi^2 + 2\ell_2(t) - 2\ell(t)\ell(u) - \frac{11}{3} \ell(t) \right] \right. \]  
\[ + \frac{t}{2s} \left( 1 - \frac{u}{s} \right) \left( \ell_2(t) + \ell_2(u) + \pi^2 - 2\ell(t)\ell(u) \right) + \frac{t}{s} (\ell(t) - \ell(u)) \right] \]  
\[ + \frac{1}{N_c} \left[ \frac{4}{\varepsilon} (\ell(s) - \ell(u)) + 8 + 2\frac{t}{s} (\ell(u) - \ell(t)) \right] \]  
\[ - \frac{t}{s} \left( 1 - \frac{u}{s} \right) \left( \ell_2(t) + \ell_2(u) + \pi^2 - 2\ell(t)\ell(u) - 4\ell(s)\ell(t) + 4\ell(t)\ell(u) \right) \right\} \]  
\[ - \frac{1}{\varepsilon} \beta_0. \]  
(6.6)  

\[ F_{a}^{--}(\varepsilon, s, t, u) \equiv \Re \{ F_{a:1}^{--}(\varepsilon, s, t, u) + F_{a:2}^{--}(\varepsilon, s, t, u) \} = \]  
(6.7)  
\[ \left( \frac{\mu^2}{Q^2} \right)^\varepsilon \left\{ \frac{V}{N_c} \left( -\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} \right) - \frac{10}{9} N_f + \frac{V}{N_c} \left( \frac{2}{\varepsilon} \ell(t) - \ell_2(t) + 3\ell(t) \right) + \frac{2}{3} N_f \ell(t) \right. \]  
\[ \left. + N_c \left[ \frac{2}{\varepsilon} (\ell(u) - \ell(t)) + \frac{13}{9} + \pi^2 + 2\ell_2(t) - 2\ell(t)\ell(u) - \frac{11}{3} \ell(t) \right] \right. \]  
\[ + \frac{t}{u} \left( 1 - \frac{s}{u} \right) \left( \ell_2(t) + \ell_2(s) + \pi^2 - 2\ell(t)\ell(s) + 2\frac{t}{u} (\ell(t) - \ell(s)) \right) \right\} \]  
\[ - \frac{1}{\varepsilon} \beta_0. \]  
(6.8)
Process B:

For the square of the matrix element, we shall need $\mathcal{F}_{a;1}^{-}$ in addition to the $\mathcal{F}_a$ functions.

\[
\mathcal{F}_{a;1}^{-} (\varepsilon, s, t, u) = \left( \frac{\mu^2}{Q^2} \right)^{\varepsilon} \left\{ \frac{V}{N_c} \left( - \frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - \frac{10}{9} N_f + \frac{V}{N_c} \left( \frac{2}{\varepsilon} \ell(t) - \ell_2(t) + 3\ell(t) \right) + \frac{2}{3} N_f \ell(t) \right. \\
+ N_c \left[ \frac{2}{\varepsilon} (\ell(s) - \ell(t)) + \frac{13}{9} + \pi^2 + 2\ell_2(t) - 2\ell(t)\ell(s) - \frac{11}{3} \ell(t) \right] \\
+ \frac{1}{N_c} \left[ \frac{2}{\varepsilon} (\ell(s) - \ell(u)) + 8 - 2\ell(s)\ell(t) + 2\ell(t)\ell(u) \right. \\
- \frac{t}{2s} \left( 1 - \frac{u}{s} \right) (\ell_2(t) + \ell_2(u) + \pi^2 - 2\ell(t)\ell(u)) + \frac{t}{s} (\ell(u) - \ell(t)) \left. \right] \right\} \\
- \frac{1}{\varepsilon} \beta_0.
\]

Process C:

\[
\mathcal{F}_{c;1}^{+} (\varepsilon, s, t, u) = \left( \frac{\mu^2}{Q^2} \right)^{\varepsilon} \left\{ \frac{V}{N_c} \left( - \frac{1}{\varepsilon^2} - \frac{3}{2\varepsilon} - \frac{7}{2} \right) - N_c \frac{2}{\varepsilon} \right. \\
+ \frac{V}{N_c} \left[ \frac{1}{\varepsilon} \ell(s) + \frac{3}{2} \ell(s) - \frac{1}{2} \ell_2(s) \right] + N_c \left[ \frac{2}{\varepsilon} \ell(u) + \ell_2(s) + \pi^2 \right] \\
- 2N_c \ell(s)\ell(u) + \frac{N_c^2 + 1}{2N_c} \left( 1 + \frac{s}{t} \left( \ell_2(s) + \ell_2(u) - 2\ell(s)\ell(u) + \pi^2 \right) \right) \left. \right\} \\
- \frac{1}{\varepsilon} \beta_0.
\]

\[
\mathcal{F}_{c;1}^{-} (\varepsilon, s, t, u) = \left( \frac{\mu^2}{Q^2} \right)^{\varepsilon} \left\{ \frac{V}{N_c} \left( - \frac{1}{\varepsilon^2} - \frac{3}{2\varepsilon} - \frac{7}{2} \right) - N_c \frac{2}{\varepsilon} \right. \\
+ \frac{V}{N_c} \left[ \frac{1}{\varepsilon} \ell(s) + \frac{3}{2} \ell(s) - \frac{1}{2} \ell_2(s) \right] + N_c \left[ \frac{2}{\varepsilon} \ell(u) + \ell_2(s) + \pi^2 \right] \\
- \frac{1}{2} N_c \left[ \ell_2(s) + \ell_2(u) + \pi^2 + 2\ell(s)\ell(u) + 3\ell(s) - 3\ell(u) \right. \\
+ \frac{u}{t} \left( 1 + \left( 1 - \frac{u}{t} \right) (\ell(s) - \ell(u)) \right. \\
\left. \left. + \left( \frac{1}{t} \right)^2 \left( \ell_2(s) + \ell_2(u) - 2\ell(s)\ell(u) + \pi^2 \right) \right) \right] \right. \\
+ \frac{1}{2N_c} \left[ 1 + \frac{s}{t} \left( 1 - \left( 1 - \frac{s}{t} \right) (\ell(s) - \ell(u)) \right. \\
\left. \left. \left. + \left( \frac{1}{t} \right)^2 \left( \ell_2(s) + \ell_2(u) - 2\ell(s)\ell(u) + \pi^2 \right) \right) \right] \right\} \\
- \frac{1}{\varepsilon} \beta_0.
\]
It turns out that only the sum and difference of the $F_{c,2}$ functions are needed.

\[ F^+_c(\varepsilon, s, t, u) \equiv \Re e(F^{-+}_{c,2}(\varepsilon, s, t, u) + F^{+-}_{c,2}(\varepsilon, s, t, u)) = \]
\[
\left( \frac{\mu^2}{Q^2} \right)^\varepsilon \left\{ \begin{array}{l}
\frac{2}{s} \left( \ell(u) - \ell(t) \right) + \frac{1}{t} \left( \ell(u) - \ell(s) \right) + \frac{1}{u} \left( \ell(s) - \ell(t) \right) \\
+ \frac{4}{s} \left( \ell(s) \ell(t) - \ell(s) \ell(u) \right) \\
+ \frac{1}{t} \left[ \frac{3}{2} \left( \ell_2(s) + \ell_2(u) + \pi^2 \right) - 3 \ell(s) \ell(u) + 2 \ell(s) \ell(t) - 2 \ell(t) \ell(u) \right] \\
- \frac{1}{u} \left[ \frac{3}{2} \left( \ell_2(s) + \ell_2(t) + \pi^2 \right) - 3 \ell(s) \ell(t) + 2 \ell(s) \ell(u) - 2 \ell(t) \ell(u) \right]
\end{array} \right\} \tag{6.10}
\]

\[ F^-_c(\varepsilon, s, t, u) \equiv \Re e(F^{-+}_{c,2}(\varepsilon, s, t, u) - F^{+-}_{c,2}(\varepsilon, s, t, u)) = \]
\[
\left( \frac{\mu^2}{Q^2} \right)^\varepsilon \left\{ \begin{array}{l}
\frac{2}{\varepsilon} \left[ \frac{1}{t} \left( \ell(s) - \ell(t) \right) + \frac{1}{u} \left( \ell(s) - \ell(t) \right) \right] \\
+ \frac{1}{t} \left[ \frac{3}{2} \left( \ell_2(s) + \ell_2(u) + \pi^2 \right) - 3 \ell(s) \ell(u) - 2 \ell(s) \ell(t) + 2 \ell(t) \ell(u) \right] \\
+ \frac{1}{u} \left[ \frac{3}{2} \left( \ell_2(s) + \ell_2(t) + \pi^2 \right) - 3 \ell(s) \ell(t) - 2 \ell(s) \ell(u) + 2 \ell(t) \ell(u) \right]
\end{array} \right\} \tag{6.11}
\]

Process D:

\[ F^{-}_{d,1}(\varepsilon, s, t, u) = \]
\[
\left( \frac{\mu^2}{Q^2} \right)^\varepsilon \left\{ \begin{array}{l}
N_c \left[ - \frac{4}{\varepsilon^2} - \frac{11}{3\varepsilon} + \frac{2}{\varepsilon} (\ell_2(s) + \ell(t)) - \frac{67}{9} + \pi^2 + \frac{11}{3} \ell(t) - 2 \ell(s) \ell(t) \right] \\
+ N_f \left[ \frac{2}{3\varepsilon} + \frac{10}{9} - \frac{2}{3} \ell(t) \right] \right\} - \frac{1}{\varepsilon} \beta_0. \tag{6.12}
\]

\[ F^{+}_{d,1}(\varepsilon, s, t, u) = \]
\[
\left( \frac{\mu^2}{Q^2} \right)^\varepsilon \left\{ \begin{array}{l}
N_c \left[ - \frac{4}{\varepsilon^2} - \frac{11}{3\varepsilon} + \frac{2}{\varepsilon} (\ell_2(s) + \ell(t)) - \frac{67}{9} + \pi^2 + \frac{11}{3} \ell(t) - 2 \ell(s) \ell(t) \right] \\
+ N_f \left[ \frac{2}{3\varepsilon} + \frac{10}{9} - \frac{2}{3} \ell(t) \right] \\
- (N_c - N_f) \frac{st}{u^2} \left[ 1 - \left( \frac{s}{u} - \frac{t}{u} \right) (\ell(t) - \ell(s)) \right] \\
- \left( \frac{st}{u^2} - 2 \right) \left( \ell_2(s) + \ell_2(t) + \pi^2 - 2 \ell(s) \ell(t) \right) \\
+ \beta_0 \frac{t}{u} (\ell(t) - \ell(s)) - \frac{3}{2} N_f \frac{st}{u^2} \left( \ell_2(s) + \ell_2(t) + \pi^2 - 2 \ell(s) \ell(t) \right) \right\} - \frac{1}{\varepsilon} \beta_0. \tag{6.13}
\]
The square of the matrix element summed over helicities can be expressed in terms of the $F$ functions.

Process A:

$$\sum_{\text{hel}} \sum_{\text{col}} [\mathcal{M}^* \mathcal{M}]_{\text{NLO}} = 2g_6^6 c_T 2V \left( \frac{s^2}{t^2} F_a^{--}(s, t, u) + \frac{u^2}{t^2} F_a^{--}(s, t, u) \right). \quad (6.14)$$

Process B:

$$\sum_{\text{hel}} \sum_{\text{col}} [\mathcal{M}^* \mathcal{M}]_{\text{NLO}} = 2g_6^6 c_T 2V \left( \frac{s^2}{t^2} F_a^{--}(s, t, u) + \frac{u^2}{t^2} F_a^{--}(s, t, u) + \frac{s^2}{u^2} F_a^{--}(s, u, t) + \frac{t^2}{u^2} F_a^{--}(s, u, t) \right. \left. - \frac{1}{N_c} \frac{s^2}{tu} (F_a^{--}(s, t, u) + F_a^{--}(s, u, t)) \right). \quad (6.15)$$

Process C:

$$\sum_{\text{hel}} \sum_{\text{col}} [\mathcal{M}^* \mathcal{M}]_{\text{NLO}} = 2g_6^6 c_T 2V \frac{V}{N_c} \left\{ 2V \frac{1}{s} \left[ tu \left( F_{c_1}^{--}(s, t, u) + F_{c_1}^{--}(s, u, t) \right) + \frac{t^3}{u} F_{c_1}^{--}(s, t, u) + \frac{u^3}{t} F_{c_1}^{--}(s, u, t) \right] \right. \left. - \frac{2}{s^2} \left[ t^2 \left( F_{c_1}^{--}(s, t, u) + F_{c_1}^{--}(s, u, t) \right) + u^2 \left( F_{c_1}^{--}(s, t, u) + F_{c_1}^{--}(s, u, t) \right) \right] \right. \left. - N_c \frac{1}{s} \left[ s^2 F_c^{--}(s, t, u) + t^2 F_c^{--}(s, t, u) - u^2 F_c^{++}(s, t, u) \right] \right\}. \quad (6.16)$$

Process D:

$$\sum_{\text{hel}} \sum_{\text{col}} [\mathcal{M}^* \mathcal{M}]_{\text{NLO}} = 2g_6^6 c_T 2N_c^2 V (F_d(s, t, u) + F_d(t, u, s) + F_d(u, s, t)), \quad (6.17)$$

where

$$F_d(s, t, u) = 2\frac{s^2}{t^2} F_{d_1}^{--}(s, t, u) + 2\frac{s^2}{u^2} F_{d_1}^{--}(s, u, t) + \frac{s^4}{t^2u^2} \left( F_{d_1}^{--}(u, t, s) + F_{d_1}^{--}(t, u, s) \right). \quad (6.18)$$
6.2 Comparison with existing results

The matrix elements given in the previous subsection differ in finite terms from those calculated in the conventional regularization scheme by Ellis and Sexton \[8\]. The difference can easily be described if one rewrites the Ellis-Sexton matrix elements in the form given in ref. \[3\].

Following ref. \[3\], we denote a certain cross section (e.g., inclusive one-jet cross section) by $I$. At next-to-leading order, $I$ is a sum of two terms,

$$I = I[2 \to 2] + I[2 \to 3],$$

(6.19)

where $I[2 \to n]$ is the $[2 \to n]$ part of the cross section. According to the factorization theorem, the physical cross section in the QCD improved parton model for hadron-hadron scattering is a folding between the parton densities and the hard-scattering cross section:

$$I[2 \to n] = \sum_{a,b} \int_0^1 dx_a \int_0^1 dx_b f_{a/A}(x_a, \mu) f_{b/B}(x_b, \mu) d\hat{\sigma}_{a,b}(x_a p_A, x_b p_B, \mu, \alpha_s(\mu)).$$

(6.20)

In eq. (6.20), $d\hat{\sigma}_{a,b}$ is the hard-scattering cross section for the process $a + b \to j_1 + \ldots + j_n$. It is defined as a product of the flux factor and the integral of the squared matrix element over the phase space of the final state particles

$$d\hat{\sigma}_{a,b} = \frac{1}{n!} \sum_{j_1, \ldots, j_n} \frac{1}{2x_a x_b s} \int dP S(n)(\vec{p}_{j_1}) S_n(p_{j_i}^\mu) \langle |M(a + b \to j_1 + \ldots + j_n)|^2 \rangle (2\pi)^d \delta^d \left( p_{j_1}^\mu + \ldots + p_{j_n}^\mu - \sum_{i=1}^n p_{j_i}^\mu \right),$$

(6.21)

where $S_n(p_{j_i}^\mu)$ is the so called measurement function that defines the infrared-safe physical quantity. The counting factors $1/n!$ are present when all partons are treated indistinguishable and we sum over the possible parton types. For the $[2 \to 2]$ process, the square of the matrix element — summed over final spins and colors and averaged over initial spins and colors — has the following perturbative expansion

$$\langle |M(a + b \to j_1 + j_2)|^2 \rangle = \frac{g^4}{\omega(a) \omega(b)} \left\{ \psi^{(4)}(\vec{a}, \vec{p}) + 2g^2 \left( \frac{\mu^2}{Q^2} \right)^\varepsilon c_T \psi^{(6)}(\vec{a}, \vec{p}) + O(g^4) \right\},$$

(6.22)

where we denote $\vec{a} = (a, b, j_1, j_2)$, $\vec{p} = (p_{j_1}^a, p_{j_2}^b, p_{j_1}^a, p_{j_2}^b)$ and $\omega(a)$ represents the number of spin and color states of a parton type $a$. Comparing our notation to the one just defined, we see that

$$2g^6 \left( \frac{\mu^2}{Q^2} \right)^\varepsilon c_T \psi^{(6)}(\vec{a}, \vec{p}) \equiv \sum_{\text{hel}} \sum_{\text{col}} [M^* M]^{\text{NLO}}.$$ 

(6.23)

In ref. \[3\], using the results of Ellis and Sexton, the following structure has been found for the next-to-leading order term:

$$\psi^{(6)}(\vec{a}, \vec{p}) = \psi^{(4)}(\vec{a}, \vec{p}) \left\{ -\frac{1}{\varepsilon^2} \sum_n C(a_n) - \frac{1}{\varepsilon} \sum_n \gamma(a_n) \right\}$$

$$+ \frac{1}{\varepsilon} \sum_{m<n} \log \left( \frac{2p_n \cdot p_m}{Q^2} \right) \psi^{(4,c)}_{mn}(\vec{a}, \vec{p})$$

$$+ \psi^{(6)}_{\text{NS}}(\vec{a}, \vec{p}),$$

(6.24)
where $\psi(4)(\vec{a}, \vec{p})$ is the $d$-dimensional Born term, $\psi_{mn}^{(4,c)}(\vec{a}, \vec{p})$ are the color-linked Born squared matrix elements in $d$ dimensions as defined in Appendix A of ref. [3] and $\psi_{NS}^{(6)}(\vec{a}, \vec{p})$ represents the remaining finite terms. The sum over $m$ and $n$ runs from one to four. In eq. (6.24), $C(a)$ is the color charge of parton $a$ and the constant $\gamma(a)$ represents the contribution from virtual diagrams to the Altarelli-Parisi kernel. Specifically,

$$C(g) = N_c, \quad \gamma(g) = \frac{1}{2} \beta_0,$$

$$C(q) = \frac{V}{2N_c}, \quad \gamma(q) = \frac{3V}{4N_c}.$$  

Using this notation, the transition from the conventional scheme to the 't Hooft-Veltman scheme can be achieved by substituting the four-dimensional expressions for the $\psi^{(4)}$ and $\psi^{(4,c)}_{mn}$ functions and leaving the $\psi^{(6)}_{NS}$ functions unchanged. In sect. 8, we shall analyze the structure of the squared matrix element for the Bremsstrahlung part of the cross section and shall point out that part of the cross section undergoes analogous change when passing from the conventional to the 't Hooft-Veltman scheme.

Next, we consider the squared matrix element in the 't Hooft-Veltman and dimensional reduction schemes in the form as given in eq. (6.24). The difference described at amplitude level by eqs. (5.43–5.45) means a difference in the $\psi^{(6)}_{NS}$ functions. We see, however, that the difference is proportional to the Born term. Consequently, we can use the same (properly modified) $\psi^{(6)}_{NS}$ functions in both schemes and include the difference into a scheme dependent modification of the $\gamma(a)$ functions. Instead of $\gamma(a)$, we use the following $\varepsilon$-dependent $\gamma(a, \varepsilon)$ functions:

$$\gamma(a, \varepsilon) = \gamma(a) + \varepsilon \tilde{\gamma}(a),$$

where

$$\tilde{\gamma}(g) = \frac{1}{6} N_c, \quad \tilde{\gamma}(q) = \frac{1}{2} C_F,$$

with $C_F = V/(2N_c)$. We obtained the above expressions for the $\tilde{\gamma}(a)$ terms from the differences among the squared matrix elements in the 't Hooft-Veltman and dimensional reduction schemes for all processes. These differences determine an overconstrained system of linear equations (four equations for two unknowns). We could solve this overconstrained system consistently, which indicates that using the $\gamma(a, \varepsilon)$ functions in eq. (6.24), universality will be maintained. We receive further indication of universality if we recall our observation that for the four-gluon amplitudes in pure gauge theory, one obtains identical results using the string-based rules and four-dimensional helicity scheme as in field theory using dimensional reduction. Extrapolating this rule to the five-gluon amplitudes, one can easily check, using the results of ref. [2], that universality is maintained at the five-point level. We have an additional confirmation that the $\gamma(a, \varepsilon)$ functions are process independent. We can obtain them if we extend the validity of the momentum sum-rule to the $\varepsilon$-dependent part of the Altarelli-Parisi kernels in the 't Hooft-Veltman scheme, as can easily be checked using the expressions of Appendix B.

When we write the results for the squared matrix element in the form of eq. (6.24), but in terms of $\gamma(a, \varepsilon)$ functions, then the transition from the 't Hooft-Veltman scheme to dimensional reduction is accomplished by setting $\tilde{\gamma}(a) = 0$. This observation will be
important when we discuss how to shift the difference in the loop amplitudes described by \( \tilde{\gamma}(a) \) into the parton density functions.

7 Scattering amplitudes in N=1 supersymmetric Yang-Mills theory at one loop

The classical Lagrangian of the N=1 supersymmetric Yang-Mills theory in Wess-Zumino gauge for the component fields reads

\[
\mathcal{L} = -\frac{1}{4} (F^a_{\mu\nu})^2 - \frac{1}{2} \bar{\lambda}^a \mathcal{D} \lambda_a, \tag{7.1}
\]

where \( F^a_{\mu\nu} \) is the usual Yang-Mills field strength of a vector field \( g \) in the adjoint representation, \( \mathcal{D} \) is the usual covariant derivative and \( \lambda \) is a Weyl spinor in the adjoint representation. From this Lagrangian one immediately sees that the kinematic structure of the gluon-gluino-gluino coupling in N=1 supersymmetric Yang-Mills theory is the same as the gluon-quark-antiquark coupling in QCD. The only difference between these two couplings is the color charge: in QCD the quarks are in the fundamental representation while in the supersymmetric theory the gluinos are in the adjoint representation. Therefore, changing the color matrices \( T^a \) in our previous calculation for the QCD amplitudes to \( F^a_{\mu\nu} \), we can obtain the gluino-gluino, gluon-gluino and gluon-gluon scattering amplitudes in N=1 supersymmetric Yang-Mills theory. Such a procedure has a direct application in tree-level QCD calculation: the subamplitudes for gluon-gluon scattering can be obtained from those of quark-gluon scattering \[24, 25\]. At one-loop level the color structure of the amplitudes in QCD does not allow for such a direct use of supersymmetry. One obvious application is going into the supersymmetry limit and use supersymmetry Ward identities as important checks on the calculation (see below). A more direct application is the following. Calculate the quark-gluon scattering amplitudes, change the color charges to obtain the gluino-gluon scattering amplitudes. Using supersymmetry Ward identities, obtain the gluon-gluon scattering amplitudes in N=1 supersymmetric Yang-Mills theory. Calculate that part of the gluon-gluon scattering amplitude in QCD which is proportional to the number of quark flavors, \( N_f \). This calculation is much simpler than the complete calculation because it involves the evaluation of only those Feynman diagrams which contain a closed fermion loop. Subtract this contribution from the result in the supersymmetric theory with color charges corresponding to the adjoint representation of the fermions (i.e, \( N_f \rightarrow N_c \)) and simultaneously add it with the QCD color charges. Thus we obtain the correct amplitudes for gluon-gluon scattering in QCD saving considerable amount of work in that part of the calculation which is the most difficult to carry out.

In order to obtain a meaningful result in a supersymmetric theory beyond tree level, one has to use a supersymmetric regulator or alternatively one has to restore the supersymmetry Ward identities if a supersymmetry breaking regularization has been used. At one-loop level, dimensional reduction is known to respect supersymmetry. It is, however, interesting to see, how supersymmetry can be restored when a non-supersymmetry-preserving regulator (such as the ’t Hooft-Veltman scheme) is used. Therefore, we shall elaborate on both approaches.
Let us now give the results of the calculation performed in the dimensional reduction scheme. We shall use a self-evident extension of naming the processes, i.e., process B will be the gluino-gluino scattering, process C will be the gluon-gluino scattering and process D will be the gluon-gluon scattering. The results will be given for one gluino flavor. In a supersymmetric theory, the amplitudes for those helicity configurations which are vanishing at tree level have to vanish at one loop as well. Explicit calculation shows that this property is fulfilled in the one-loop calculation. In the following, we give only those non-vanishing amplitudes from which one can obtain other amplitudes using parity transformation or cyclic property of the amplitudes. As usual, all particles are outgoing. The results can be given in a concise form:

\[
m_{4,1}^{\text{SUSY,DR}}(-,\mp,\pm,+) = c_{1} m_{4,0}^{\text{SUSY}}(-,\mp,\pm,+) N_{c} F_{1}^{-\mp}(\varepsilon,s_{12},s_{13},s_{14}),
\]

where \( m \) stands for \( b, c \) or \( d \). The \( F_{1}^{-\mp} \) functions are universal:

\[
F_{1}^{-\lambda}(\varepsilon,s_{12},s_{13},s_{14}) = \left(-\frac{\mu^{2}}{s_{14}}\right)^{\varepsilon} \left(-\frac{4}{\varepsilon^{2}} - \frac{3}{\varepsilon} - \frac{2}{\varepsilon} \log \frac{s_{14}}{s_{12}} - 6 + \pi^{2}\right) - \frac{3}{\varepsilon} \\
+ \delta^{\varepsilon} \left(3 \frac{s_{14}}{s_{13}} \log \frac{s_{14}}{s_{12}} - \frac{3}{2} \frac{s_{12} s_{14}}{s_{13}^{2}} \left(\log^{2} \frac{s_{14}}{s_{12}} + \pi^{2}\right)\right),
\]

It is easy to check that these results are indeed supersymmetric, i.e., they satisfy certain on-shell supersymmetry Ward identities. These Ward identities can easily be derived noting that the supersymmetry charge \( Q(\eta) \) with \( \eta \) being the fermionic parameter of the transformation annihilates the vacuum. Then it follows that the commutator of \( Q(\eta) \) with any string of operators creating or annihilating of a gluon or a gluino has a vanishing vacuum expectation value [26]. This statement is true in any order of perturbation theory. If \( a_{i} \) represent these operators, then we find the following supersymmetry Ward identities:

\[
0 = \left\langle \left[ Q, \prod_{i=1}^{n} a_{i} \right] \right\rangle_{0} = \sum_{i=1}^{n} \left\langle a_{1} \ldots [Q, a_{i}] \ldots a_{n} \right\rangle_{0}.
\]

In the \( N=1 \) supersymmetric Yang-Mills theory we considered above, the \( a_{i} \) stand for \( g^{\pm} \) and \( \lambda^{\pm} \). The superscripts \( \pm \) refer to the two possible helicity states of the vector and spinor. The action of the supersymmetry charge on the doublet \( (g, \lambda) \) is as follows [26]:

\[
[Q(\eta), g^{\pm}(p)] = \mp \Gamma^{\pm}(p,\eta) \lambda^{\pm}.
\]

\[
[Q(\eta), \lambda^{\pm}(p)] = \mp \Gamma^{\mp}(p,\eta) g^{\pm}.
\]

Substituting these commutation relations into eq. (7.4) we obtain a relation among various scattering amplitudes for particles with different spin. These relations are the on-shell supersymmetry Ward identities referred to above.

We remark that the supersymmetry Ward identities hold separately for each of the subamplitudes in which one can expand the full amplitude [18].

In order to find explicit relations, the \( \Gamma^{\pm}(p,\eta) \) functions are to be specified yet. \( \Gamma^{\pm}(p,\eta) \) is a complex function linear in the anticommuting \( c \)-number components of \( \eta \) and satisfies

\[
\Gamma^{+}(p,\eta) = (\Gamma^{-}(p,\eta))^{*} = \bar{\eta} u_{-}(p),
\]
where \( u_-(p) \) is a negative helicity spinor satisfying the massless Dirac equation with momentum \( p \). Using the freedom in choosing the supersymmetry parameter \( \eta \), we choose it to be a negative helicity spinor obeying the massless Dirac equation with arbitrary momentum \( k \) times a Grassmann variable \( \theta \). Then,

\[
\Gamma^+(p, k) \equiv \Gamma^+(p, \eta(k)) = \theta(k + |p|).
\] (7.8)

As an application, we show how to derive the \( b_{5,0}^{\text{SUSY}} \) amplitudes from the well-known results for \( d_{4,0} \) (see eq. (5.33)). To obtain \( \langle \lambda_1^- \lambda_2^- \lambda_3^+ \lambda_4^+ \rangle_0 \), consider \( \langle [Q, g_1^- g_2^- g_3^+ \lambda_4^+] \rangle_0 \) and \( \langle [Q, \lambda_1^- g_2^- \lambda_3^+ \lambda_4^+] \rangle_0 \):

\[
0 = \langle [Q, g_1^- g_2^- g_3^+ g_4^+] \rangle_0 = \Gamma^-(p_1, k)\langle \lambda_1^- g_2^- g_3^+ \lambda_4^+ \rangle_0
+ \Gamma^-(p_2, k)\langle \lambda_1^- g_2^- g_3^+ \lambda_4^+ \rangle_0 - \Gamma^-(p_4, k)\langle g_1^- g_2^- g_3^+ g_4^+ \rangle_0.
\] (7.9)

\[
0 = \langle [Q, \lambda_1^- g_2^- \lambda_3^+ \lambda_4^+] \rangle_0 = -\Gamma^-(p_2, l)\langle \lambda_1^- \lambda_2^- \lambda_3^+ \lambda_4^+ \rangle_0
+ \Gamma^-(p_3, l)\langle \lambda_1^- \lambda_2^- \lambda_3^+ \lambda_4^+ \rangle_0 - \Gamma^-(p_4, l)\langle \lambda_1^- \lambda_2^- \lambda_3^+ \lambda_4^+ \rangle_0.
\] (7.10)

Choose \( k = p_2 \) and \( l = p_4 \). Using the properties of the spinor products and momentum conservation, we find

\[
\langle \lambda_1^- \lambda_2^- \lambda_3^+ \lambda_4^+ \rangle_0 \equiv b_{5,0}^{\text{SUSY}} (-, -, +, +) = \frac{-i\langle 12 \rangle [34]}{s_{14}}.
\] (7.11)

In an exactly analogous fashion, from \( \langle [Q, g_1^- g_2^- g_3^- \lambda_4^+] \rangle_0 \) and \( \langle [Q, \lambda_1^- g_2^+ \lambda_3^- \lambda_4^+] \rangle_0 \) with \( k = p_3 \) and \( l = p_4 \), we obtain

\[
\langle \lambda_1^- \lambda_2^+ \lambda_3^- \lambda_4^+ \rangle_0 \equiv b_{5,0}^{\text{SUSY}} (-, +, -, +) = \frac{i\langle 13 \rangle [24]}{s_{14}} \frac{s_{13}}{s_{12}}.
\] (7.12)

We now see that our results given in eq. (7.2) are indeed supersymmetric due to the universality of the \( F_{1^-} \) functions. When applying a supersymmetry transformation, the only change one finds is such that the Born terms transform into one another.

From the results of sect. 5, we see that the color subamplitudes calculated in the ’t Hooft-Veltman and dimensional reduction schemes only slightly differ. Consequently, one expects that the ’t Hooft-Veltman scheme breaks supersymmetry only by a small amount at one-loop level. This is indeed what one finds in a direct calculation:

\[
m_{4,1}^{\text{SUSY, HV}} (-, \mp, \pm, +) = c_{t} m_{4,0}^{\text{SUSY}} (-, \mp, \pm, +) N_c \left( F_{1^-} (\varepsilon, s_{12}, s_{13}, s_{14}) - \frac{\delta_m}{3} \right).
\] (7.13)

with the same process independent \( F_{1^-} \) functions as in eq. (7.2), but with process dependent \( \delta_m \): \( \delta_b = 5, \delta_c = 3, \delta_d = 1 \).

According to the supersymmetry Ward identities (7.4), we have used a supersymmetry-preserving regulator, the constants \( \delta_m \) would have been process independent in eq. (7.13). In the followings, we shall argue that the supersymmetry breaking in the ’t Hooft-Veltman scheme can consistently be restored applying a special form of finite renormalization. We obtain this renormalization via calculating the gluon and gluino self energies to \( O(\varepsilon) \) order.
in ’t Hooft-Veltman scheme and dimensional reduction scheme and require that the finite difference between the results can be compensated by a finite renormalization. For the self energies, we have the well-known expressions

\[ \Pi^{ab}(k^2) = i\delta^{ab}(k_\mu k_\nu - k^2g_{\mu\nu})\Pi(k^2), \] (7.14)

\[ i\Sigma^j_i(k) = \delta^j_i\Sigma(k^2), \] (7.15)

The functions \(\Pi(k^2)\) and \(\Sigma(k^2)\) are regularization-scheme dependent. Explicit calculation for space-like \(k\) in the background-field gauge gives

\[ \Pi^\text{HV}(k^2) = N_c \left( \frac{g_r}{4\pi} \right)^2 \left( -\frac{4\pi\mu^2}{k^2} \right)^\varepsilon \Gamma(1 + \varepsilon) \left( -\frac{3}{\varepsilon} - \frac{57}{9} \right) + Z^f_A - 1 + \mathcal{O}(\varepsilon), \] (7.16)

\[ \Pi^\text{DR}(k^2) = N_c \left( \frac{g_r}{4\pi} \right)^2 \left( -\frac{4\pi\mu^2}{k^2} \right)^\varepsilon \Gamma(1 + \varepsilon) \left( -\frac{3}{\varepsilon} - \frac{54}{9} \right) + \mathcal{O}(\varepsilon), \] (7.17)

\[ \Sigma^\text{HV}(k^2) = N_c \left( \frac{g_r}{4\pi} \right)^2 \left( -\frac{4\pi\mu^2}{k^2} \right)^\varepsilon \Gamma(1 + \varepsilon) \left( \frac{1}{\varepsilon} + 1 \right) + Z^f_\lambda - 1 + \mathcal{O}(\varepsilon), \] (7.18)

\[ \Sigma^\text{DR}(k^2) = N_c \left( \frac{g_r}{4\pi} \right)^2 \left( -\frac{4\pi\mu^2}{k^2} \right)^\varepsilon \Gamma(1 + \varepsilon) \left( \frac{1}{\varepsilon} + 2 \right) + \mathcal{O}(\varepsilon). \] (7.19)

Requiring that the results are equal in the two schemes, we obtain the desired finite renormalizations:

\[ Z^f_A = 1 + \frac{1}{3}N_c \left( \frac{g_r}{4\pi} \right)^2 \] (7.20)

and

\[ Z^f_\lambda = 1 + N_c \left( \frac{g_r}{4\pi} \right)^2 \] (7.21)

So far, this renormalization is very similar to that of sect. 6, where we discussed the difference between \(\Lambda_{\text{QCD}}\) in the ’t Hooft-Veltman and dimensional reduction schemes. There is, however, an important difference. The renormalizations discussed in the present section have to vanish as \(k^2 \to 0\). The reason is the following: In the spirit of dimensional regularization, the ultraviolet and infrared divergencies cancel exactly for on-shell propagation independently whether ’t Hooft-Veltman scheme or dimensional reduction has been used. Therefore, the only breaking of supersymmetry could occur when calculating the loop correction for off-shell lines. This implies that the renormalization insertions occur on off-shell propagator lines only. Of course, in order not to break gauge invariance, additional finite renormalization of the vertices is necessary according to the Ward identities

\[ Z^f_{\text{AAA}} = Z^f_g(Z^f_A)^{3/2}, \quad Z^f_{\text{AAA}} = Z^f_g(Z^f_A)^{1/2}Z^f_\lambda, \] (7.22)

where \(Z^f_g\) is the gauge-coupling renormalization. From these equations we obtain the following condition among the renormalization constants

\[ Z^f_{\text{AAA}}(Z^f_A)^{-1} = Z^f_{\text{AAA}}(Z^f_\lambda)^{-1}. \] (7.23)

\[ ^8\text{We note that in the helicity formalism the renormalization of the four-gluon vertex at the one-loop level is irrelevant.} \]
This condition is sufficient to ensure that if the renormalization procedure is consistent for the four-point amplitudes then it is consistent for any \( n \)-point one-loop amplitude. The proof goes by induction. We assume that the finite renormalization consistently restores supersymmetry for the \((n-1)\)-point amplitudes. Attaching an external gluon leg on an \((n-1)\)-point tree amplitude on a gluon line (either propagator or external leg) we introduce a finite renormalization of \( Z_{AAA}^f(Z_A^f)^{-1} \), while attaching the external gluon on a fermion line we introduce a renormalization of \( Z_{A\lambda\lambda}^f(Z_A^f)^{-1} \). According to eq. (7.13), these renormalizations are equal, therefore, if the renormalization was consistent at \((n-1)\)-point level, it remains consistent at \( n \)-point level. Using this argument, and the results of ref. [2] for the five-gluon amplitudes in the pure gauge sector presented in the 't Hooft-Veltman scheme, we can deduce the field theory result in dimensional reduction which turns out to be identical to the result in the four-dimensional helicity scheme. This is a further indication that dimensional reduction in field theory gives the same result as the four-dimensional helicity scheme when the string based rules are used.

The consistency of our renormalization procedure at the four-point level can be seen by performing the finite renormalization on formula (7.13) explicitly. We find that the constants \( \delta_m \) in eq. (7.13) become universal, hence the required supersymmetry Ward identities are fulfilled. However, the absolute normalization is not set by condition (7.23), consequently, the actual value of \( \delta_m \) is unknown. In order to set the absolute normalization, we require the background-gauge Ward identity between the gauge-field and gauge-coupling renormalizations, namely

\[
Z_g^f = (Z_A^f)^{-1/2} = 1 - \frac{1}{6} N_c \left( \frac{g_r}{4\pi} \right)^2. \tag{7.24}
\]

Having the absolute normalization set, we obtain \( \delta_m = 0 \), i.e., we recover the result obtained using dimensional reduction exactly together with the correct shift in \( \Lambda_{QCD} \) between the two schemes.

8 Bremsstrahlung contributions in ‘t Hooft-Veltman and dimensional reduction schemes

In the first part of this paper, we have seen that one-loop amplitudes of massless QCD can be calculated using the ‘t Hooft-Veltman version of dimensional regularization and a simple rule for the transition to the conventional scheme has been found. The advantage of using the ‘t Hooft-Veltman scheme is obvious: both the calculation and the final result is much simpler than in the conventional scheme. We have also seen the usefulness of performing the loop calculation using a supersymmetry preserving or weakly violating regulator. On the other hand, the most widely accepted regularization scheme for the regularization of infrared divergencies which is also used in the factorization procedure is the conventional scheme. These facts motivate a thorough study of the various dimensional regularization schemes to establish the transition rules among them. We already know, how to make the

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\[9\] We use the background field formalism to the gluon sector only, therefore, we have to use the non-supersymmetric condition.
transition for the loop corrections. In order to establish the transition for physical quantities, we have to investigate the Bremsstrahlung contributions as well.

To start with, we recall the form of the Bremsstrahlung contribution found in ref. [3]. In order to keep the discussion reasonably concise, we give the integrals in a schematic form and spell out those factors explicitly which depend on the regularization scheme. We admit that it is difficult to follow our discussion without consulting ref. [3] for more details and precise definitions. However, we find it pointless to recapitulate every details which can be found in ref. [3].

The structure of the real corrections after integrating over the phase space is same as eq. (6.24) for the virtual corrections, although the actual expression is far more complicated. Schematically we write

\[ I_{[2 \to 3]} = \sum_{i=1}^{4} I_{[2 \to 3]_i} = I_{[\text{soft}]_i} + I_{[\text{coll}]_i} + I_{[\text{NS}]_i}, \quad (8.1) \]

where \( I_{[2 \to 3]} \) is the \([2 \to 3]\) cross section. To obtain \( I_{[2 \to 3]} \), we have to integrate over the momenta of the three final state partons. We can always arrange the calculation such that we label by 3 the parton with the smallest transverse momentum. The squared matrix element can be decomposed into a sum of four terms, where one term \( \langle \mathcal{A}_5^{\text{tree}} \rangle_i \) (\( i = 1, 2, 3, 4 \)) has singularities only when parton 3 is soft or collinear to parton \( i \). Then \( I_{[2 \to 3]_i} \) is the part of the cross section obtained by integrating \( \langle \mathcal{A}_5^{\text{tree}} \rangle_i \); \( I_{[\text{soft}]_i} \), \( I_{[\text{coll}]_i} \) and \( I_{[\text{NS}]_i} \) are its soft, collinear and finite terms respectively. The actual form of the \( I_{[\text{soft}]_i} \), \( I_{[\text{coll}]_i} \) and \( I_{[\text{NS}]_i} \) integrals depends whether parton \( i \) is in the initial or final state.

### 8.1 Soft contributions

In the soft limit \( p_3 \to 0 \), the variables of parton 3 can be integrated analytically. One is left with an integral over the variables of partons 1 and 2:

\[ I_{[\text{soft}]_i} = \sum_{a,b,j_1,j_2} \int \! D_i L(a, b) S_2(p_{j_1}^\mu, p_{j_2}^\mu) \psi_{i}^{\text{soft}}(\vec{a}, \vec{p}), \quad (8.2) \]

where the sum runs over all possible flavors of the initial (\( a \) and \( b \)) and final (\( j_1 \) and \( j_2 \)) state partons, \( D_i \) is a ubiquitous prefactor which does not depend on the regularization scheme. The function \( L(a, b) \) describes the parton luminosity, \( S_2(p_{j_1}^\mu, p_{j_2}^\mu) \) is the measurement function and the function \( \psi_{i}^{\text{soft}}(\vec{a}, \vec{p}) \) is the soft limit of the squared matrix element integrated over the variables of parton 3. They all depend on the regularization scheme. However, the dependence in the parton luminosity and the measurement function is irrelevant as far as our considerations are concerned because the same type of integral occurs when the \([2 \to 2]\) matrix elements are integrated over the phase space (see eqs. (29) and (33) in ref. [3]).

When \( i \) is a final state parton, the function \( \psi_{i}^{\text{soft}}(\vec{a}, \vec{p}) \) has the form

\[ \psi_{i}^{\text{soft}}(\vec{a}, \vec{p}) = \psi_{(4)}^{(4)}(\vec{a}, \vec{p}) \left\{ \frac{1}{\varepsilon} C(a_i) - \frac{1}{\varepsilon} 2C(a_i) \log(x) \right\} \]

\[ + \sum_{m \neq i} \psi_{(4,c)}^{(4,c)}(\vec{a}, \vec{p}) \left\{ -\frac{1}{2\varepsilon} \log \left( \frac{2p_i \cdot p_m}{Q^2} \right) + \bar{\mathcal{I}}_{im}(x) + \mathcal{O}(\varepsilon) \right\}, \quad (i = 1, 2), \quad (8.3) \]
while for the case when $i$ is an initial state parton, we have

$$
\psi_{i}^{\text{soft}}(\vec{a},\vec{p}) = \psi^{(4)}(\vec{a},\vec{p}) \left\{ \frac{1}{\varepsilon^2} C(a_i) + \frac{1}{\varepsilon} 2C(a_i) \log \left( \frac{x_i}{1-x_i} \right) \right\} + \sum_{m \neq i} \psi^{(4,c)}_{im}(\vec{a},\vec{p}) \left\{ -\frac{1}{2\varepsilon} \log \left( \frac{2p_i \cdot p_m}{Q^2} \right) + \tilde{I}_{im}(x) + \mathcal{O}(\varepsilon) \right\}, \quad (i = 3, 4).
$$

(8.4)

There is a term proportional to $-\frac{\log(x)}{\varepsilon}$ in eq. (8.3) (the same $x$ appears in the argument of the $\tilde{I}_{im}$ functions too). In this term, $x$ is an arbitrary positive number, less than one and it represents the fraction of $p_i$ that sets the upper limit of the integration in $p_3$. It is used to separate the soft and collinear contributions in the soft-collinear region. The limit $x \to 1$ means that in the collinear region all momentum configuration is considered collinear except when $p_3 \simeq 0$. In ref. [3], $x = 1/2$ has been chosen for the sake of convenience. For our purpose of establishing the transition rules among the regularization schemes, it is more suitable to keep $x$ arbitrary. The reason will be clear when the collinear integrals are investigated.

Using the derivations in Appendix A of ref. [3], we find that in eqs. (8.3) and (8.4) only the $\psi^{(4)}$ and $\psi^{(4,c)}_{im}$ functions depend on the regularization scheme: for the conventional scheme they are the $d$-dimensional expressions, while for the 't Hooft-Veltman or dimensional reduction schemes they are to be taken in four dimensions.

8.2 Collinear contributions

The integral for the final state collinear singularities has an analogous form to that of the soft terms in eq. (8.2):

$$
\mathcal{I}[\text{coll}]_i = \sum_{a,b,j_1,j_2} \int D_{1,2} L(a,b) S_{2}(p_{j_1}^\mu, p_{j_2}^\mu) \psi^{\text{coll}}_{i}(\vec{a},\vec{p}), \quad (i = 1, 2).
$$

(8.5)

In eq. (8.5),

$$
\psi^{\text{coll}}_{i}(\vec{a},\vec{p}) = -\frac{1}{\varepsilon} \left( \frac{Q^2}{16p_i^2} \right)^\varepsilon \frac{\Gamma(1-\varepsilon)}{\Gamma(1+\varepsilon)\Gamma(1-2\varepsilon)} \mathcal{Z}(a_i, x) \psi^{(4)}(\vec{a},\vec{p})
$$

(8.6)

is the collinear limit of the squared matrix element integrated over the variables of parton 3. The function $\mathcal{Z}(a, x)$ is an integral of a finite expression:

$$
\mathcal{Z}(a, x) = \int_x^1 dz (1-z)^{-2\varepsilon} \left[ z^{-2\varepsilon} \sum_b \tilde{P}_{b/a}(z, \varepsilon) - \frac{2C(a)}{1-z} \right].
$$

(8.7)

The lower limit of the integration is the same $x$ we have met in eq. (8.3). $\tilde{P}_{b/a}(z, \varepsilon)$ is the usual Altarelli-Parisi splitting function in $(4 - 2\varepsilon)$ dimensions without $z \to 1$ regulation. We shall term them $\varepsilon$-dependent Altarelli-Parisi kernels. In Appendix B, we collected these $\varepsilon$-dependent Altarelli-Parisi kernels in all three regularization schemes. Using those expressions, one can easily verify that $\mathcal{Z}(a, x)$ does not depend on the scheme in the limit $x \to 1$. We are allowed to take that limit, because the sum of the soft and collinear terms is independent of $x$ (as can be checked using the precise formulæ of ref. [3]). Therefore, we find
again that in the final state collinear term, eq. (8.3), apart from the irrelevant dependence in $L$ and $S_2$, it is only the Born term, $\psi_i^{(4)}(\vec{a}, \vec{p})$ which is regularization-scheme dependent.

The structure of $\mathcal{I}[\text{coll}]_i$ for initial state parton is more complicated. It can be split into two terms. The first has the structure encountered at the final state terms, while the second contains the the initial state collinear pole that is involved in the factorization theorem.

$$\mathcal{I}[\text{coll}]_i = \sum_{a,b,j_1,j_2} \int \left[ D_i L(a,b) S_2(p^\mu_{j_1}, p^\mu_{j_2}) \psi_i^{\text{coll}}(\vec{a}, \vec{p}) + G_i^{\text{coll}}(\vec{a}, \vec{p}) \right], \quad (i = 3, 4). \quad (8.8)$$

In eq. (8.8), $\psi_i^{\text{coll}}$ has the form

$$\psi_i^{\text{coll}}(\vec{a}, \vec{p}) = \psi_i^{(4)}(\vec{a}, \vec{p}) \left\{ \frac{1}{\varepsilon} \left[ \gamma(a_i) - 2C(a_i) \log \left( \frac{x_i}{1 - x_i} \right) \right] + \log \left( \frac{Q^2}{\mu^2} \right) \left[ \gamma(a_i) - 2C(a_i) \log \left( \frac{x_i}{1 - x_i} \right) \right] \right\}. \quad (8.9)$$

Again, the whole regularization-scheme dependence in $\psi_i^{\text{coll}}$ is contained in the Born term. The other term, $G_i^{\text{coll}}$ has a more complicated structure. As always, we emphasize only that part which depends on the regularization scheme,

$$G_i^{\text{coll}}(\vec{a}, \vec{p}) = -G_{CT,i}^{(2 \to 2)} - \alpha_s^2 p_2^2 \left( \frac{p_2}{2 \pi \mu} \right)^{-2\varepsilon} \psi_i^{(4)}(\vec{a}, \vec{p}) S_2(p^\mu_{j_1}, p^\mu_{j_2}) \sum_a \frac{\omega(\vec{a})}{\omega(a_i)} \int \frac{dz}{z^2} L \left( \vec{a}, b, \frac{x_i}{z} \right) \frac{\alpha_s}{2\pi} \tilde{P}_{a/\vec{a}}(z)$$

$+$ finite terms.

In eq. (8.10), $G_{CT,i}^{(2 \to 2)}$ is the part of the $\overline{\text{MS}}$ factorization counter term that is associated with particle $i$,

$$G_{CT,i}^{(2 \to 2)} = \frac{\alpha_s^2 p_2^2}{2s^2} \left( \frac{p_2}{2 \pi \mu} \right)^{-2\varepsilon} \psi_i^{(4)}(\vec{a}, \vec{p}) S_2(p^\mu_{j_1}, p^\mu_{j_2}) \sum_a \frac{\omega(\vec{a})}{\omega(a_i)} \int \frac{dz}{z^2} L \left( \vec{a}, b, \frac{x_i}{z} \right) \frac{(4\pi)^{\varepsilon}}{\varepsilon \Gamma(1 - \varepsilon)} \frac{\alpha_s}{2\pi} \tilde{P}_{a/\vec{a}}(z). \quad (8.11)$$

It cancels exactly, when the factorization counter term is added. The Born term and the factors $\omega(a)$ which represent the number of spin and color states of a parton type $a$ are four-dimensional expressions in the second line of eq. (8.10). $\tilde{P}^\varepsilon$ represents the $\varepsilon$-dependent part of the $\varepsilon$-dependent Altarelli-Parisi kernels,

$$\tilde{P}_{a/b}^\varepsilon(z) = \frac{\partial}{\partial \varepsilon} \tilde{P}_{a/b}(z, \varepsilon) \bigg|_{\varepsilon = 0}. \quad (8.12)$$

The functions $\tilde{P}^\varepsilon$ are given explicitly in Appendix B. The scheme dependence in eq. (8.10) occurs in the $\tilde{P}^\varepsilon$ functions only.

Summarizing this section, we conclude that the dependence on the regularization scheme in $\mathcal{I}[2 \to 3]$ appears only in the $\psi_i^{(4)}$ and $\psi_i^{(4,c)}$ functions and in a folding of the parton

\footnote{In ref. [3], the whole contribution has been called $G_i^{\text{coll}}$. For our purposes this decomposition is useful.}
luminosities with the $\varepsilon$-dependent part of the $\varepsilon$-dependent Altarelli-Parisi kernels, $\tilde{P}'$. The transition rule from the conventional scheme to the ‘t Hooft-Veltman or dimensional reduction schemes is very simple: the $\psi^{(4)}$ and $\psi^{(4,c)}_{im}$ functions has to be taken in four dimensions and $L \otimes \tilde{P}'$ has to be chosen in the proper scheme according to the formulas in Appendix B. This transition rule is in accordance with the transition rule found in sect. 6 for the virtual contributions.

9 Transition rules for the hard-scattering cross section among dimensional regularization schemes

According to the factorization theorem, the physical cross section in the QCD improved parton model for hadron-hadron scattering is a folding between the parton densities and the hard-scattering cross section:

$$d\sigma(p_A, p_B) = \sum_{a,b} \int_0^1 dx_a \int_0^1 dx_b f_{a/A}(x_a, \mu) f_{b/B}(x_b, \mu) d\tilde{\sigma}_{a,b}(x_ap_A, x_bp_B, \mu, \alpha_s(\mu)).$$  \hspace{1cm} (9.1)

In this equation, the $f_{a/A}$ parton densities are process independent, their $Q^2$-evolution is determined by the Altarelli-Parisi equations. At next-to-leading order the regularization and factorization scheme dependence of the Altarelli-Parisi kernels is cancelled by that of the hard-scattering cross section, $d\tilde{\sigma}_{a,b}$. Therefore, in principle all schemes are equally acceptable. In practice, however, one is forced to use the existing parton density functions which are obtained by fitting a large set of data in a rather complicated phenomenological procedure. This fitting and the $Q^2$-dependence of the fitted function is always worked out using the conventional regularization and $\overline{\text{MS}}$ (or DIS) factorization schemes. As a result, the existing parton density functions can be used only with those hard-scattering cross sections which are calculated in the conventional dimensional regularization scheme. On the other hand, as we have seen above, convenient applications of the helicity method to loop calculations require that regularization is performed in the ‘t Hooft-Veltman or dimensional reduction schemes. Therefore, it is vital to find the transition rules which tell us how to transform the expressions for the hard-scattering cross sections obtained in the ‘t Hooft-Veltman or dimensional reduction schemes into the corresponding expressions in the conventional scheme. That goal is easily achievable using the results of sections 6 and 8.

The cancellation of infrared divergencies can be seen manifestly when the analytic structure of the infrared singularities are exhibited according to eqs. (6.24) and (8.11). In this cancellation the $\psi^{(4)}(\vec{a}, \vec{p})$ and $\psi^{(4,c)}_{im}(\vec{a}, \vec{p})$ functions appear as formal objects, the only requirement is that they are the same in the real and virtual contributions and in the factorization counter term. We have seen that it is indeed the case in all three versions of dimensional regularization: in the conventional scheme, they are $d$-dimensional expressions, while in the ‘t Hooft-Veltman and dimensional reduction schemes, they are taken in four dimensions.

After cancellation of infrared divergencies and subtraction of initial-state collinear singularities, we find still process independent differences in the hard-scattering cross sections in the different regularization schemes. If we compare $d\tilde{\sigma}_{a,b}$ in the ‘t Hooft-Veltman scheme to
that in the conventional scheme, then we find that only the $\tilde{P}'$ functions are different. If we compare $d\tilde{\sigma}_{a,b}$ in the dimensional reduction scheme to that in the conventional scheme, then we find that the difference is contained in part in the redefinition of $\Lambda_{\text{QCD}}$, in part in the $\tilde{P}'$ functions and in part in $\gamma(a,\varepsilon)$: in the conventional scheme $\gamma$ takes its $\varepsilon$-dependent value, while in dimensional reduction, its $\varepsilon$-dependent part is set to zero. From these considerations we see that the only modification one has to perform on $d\tilde{\sigma}_{a,b}$ obtained in non-conventional schemes in order to recover $d\tilde{\sigma}_{a,b}$ in the conventional scheme is simply changing $\tilde{P}'$ to the value in the conventional scheme, use $\varepsilon$-dependent $\gamma$ parameters and the standard $\Lambda_{\overline{\text{MS}}}$ QCD parameter.

In order to find the proper form of the loop contributions in the 't Hooft-Veltman or dimensional reduction schemes which is useful for the cancellation mechanism (i.e., eq. (6.24)), we have to calculate the $\psi^{(4)}(\vec{a},\vec{p})$ and $\psi^{(4,c)}_{mn}(\vec{a},\vec{p})$ functions in four dimensions. But these are tree level calculations which can easily be done using the helicity method. For the definition of the $\psi^{(4,c)}_{mn}(\vec{a},\vec{p})$ functions, we refer to ref. [3]. Knowing the $\psi^{(4)}(\vec{a},\vec{p})$ and $\psi^{(4,c)}(\vec{a},\vec{p})$ functions in four dimensions and the complete loop contributions in $d$ dimensions, we can deduce the form of the $\psi_{\text{NS}}^{(6)}(\vec{a},\vec{p})$ functions which constitute the finite part of the loop corrections in the conventional scheme. As far as the Bremsstrahlung contributions are concerned, in the 't Hooft-Veltman and dimensional reduction schemes, both the soft and the hard part of the cross section have to be evaluated in four dimensions. It is only the calculation of the $\tilde{P}'$ functions in the 't Hooft-Veltman scheme which requires $d$-dimensional treatment (see Appendix B).

We have given the transition terms and functions needed to recover the hard-scattering cross section in the conventional regularization scheme from calculations performed in the 't Hooft-Veltman or dimensional reduction schemes. This is the main result of our paper, which was based upon the assumption that the analytic structure of the singularities found in ref. [3] is process independent even after the minor modification obtained by the introduction of $\varepsilon$-dependent $\gamma$ parameters. One expects that this structure is indeed universal because the singularities are linked to the external particles of the subprocess. To give more support to the universality, in sect. 10, we present further consistency check on the transition rules via going into the supersymmetric limit of QCD once again.

## 10 Next-to-leading order Altarelli-Parisi kernels in the 't Hooft-Veltman and dimensional reduction schemes

We have seen that the only dependence on the regularization scheme in the hard-scattering cross section, $d\tilde{\sigma}_{a,b}$ — beyond the redefinition of $\Lambda_{\text{QCD}}$ — is contained in $\tilde{P}'$ and $\tilde{\gamma}(a)$. We have established the transition rules to recover the hard-scattering cross section in the conventional $\overline{\text{MS}}$ scheme from a calculation performed in non-conventional schemes. Although, all phenomenological studies for the parton densities are performed using the conventional next-to-leading order Altarelli-Parisi kernels, it may still be useful to know these kernels also for dimensional reduction and 't Hooft-Veltman schemes. In particular, knowing the next-to-leading order Altarelli-Parisi kernels for a supersymmetry preserving regularization, such as dimensional reduction, has got its own interest.
The next-to-leading order Altarelli-Parisi kernels in dimensional reduction can be obtained in two different ways. On one hand, they can be determined in a direct calculation using dimensional reduction. On the other, requiring that the physical cross section should not depend on the regularization scheme, the regularization-scheme dependence of the next-to-leading order Altarelli-Parisi kernels can be uniquely determined from that of the hard-scattering cross section. If the two approaches lead to the same result, then it would prove that unitarity is not violated as well as it would ensure the universality of our transition rules. A direct calculation of the next-to-leading order Altarelli-Parisi kernels in non-conventional schemes has not been carried out yet. Nevertheless, we can still perform for the dimensional reduction scheme a consistency test: we can test explicitly whether the next-to-leading order Altarelli-Parisi kernels satisfy the supersymmetry Ward identities for N=1 supersymmetric Yang-Mills theory.

For the sake of completeness, we review the derivation of the transformation rule for the next-to-leading order Altarelli-Parisi kernels using the example of deep-inelastic lepton-hadron scattering. The $F_2$ structure function at next-to-leading order is

$$\frac{1}{x} F_2(x, Q) = \sum_i e_i^2 \left( f_{i/A}(x, \mu) + \frac{\alpha_s}{2\pi} \sum_a f_{a/A}(z, \mu) \otimes_x C_{ia}(z, Q/\mu) \right) + \mathcal{O}(\alpha_s^2),$$

(10.1)

where the following notation has been introduced:

$$f \otimes_x g = \int_0^1 dy \, dz \, f(z) g(y) \delta(x - yz)$$

(10.2)

and $i = q, \bar{q}, a = q, g$. In eq. (10.1), $C_{ia}$ denotes the hard-scattering cross section — their actual form is irrelevant for our purposes. We can freely change $C_{ia}$ to $C^\prime_{ia} = C_{ia} + \Delta C_{ia}$ if we simultaneously change the parton densities to

$$f_{a/A}(x, \mu) = f_{a/A}(x, \mu) - \frac{\alpha_s}{2\pi} \sum_b f_{b/A}(\mu) \otimes_x \Delta C_{ab}(Q/\mu)|_{Q=\mu}. $$

(10.3)

Indeed, writing eq. (10.1) in terms of $C^\prime_{ia}$ and $f^\prime_{a/A}$, we find that the change in $F_2$ is of the order $\mathcal{O}(\alpha_s^2)$ which is neglected. The $\mu$ dependence of both $f_{a/A}$ and $f^\prime_{a/A}$ is determined by the Altarelli-Parisi evolution equations, but with different kernels $P$ and $P^\prime$ respectively:

$$\mu \frac{\partial}{\partial \mu} f_{a/A}(x, \mu) = \frac{\alpha_s}{\pi} \sum_b P_{a/b}^{(\prime)} \otimes_x f_{b/A}^{(\prime)}(\mu) $$

(10.4)

In the equation for $f^\prime_{a/A}$, we substitute $f^\prime_{a/A}$ according to eq. (10.3) on the left hand side. Performing the differentiation and using

$$\mu \frac{\partial \alpha_s}{\partial \mu} = -\frac{1}{2} \beta_0 \frac{\alpha_s}{\pi} \alpha_s, $$

(10.5)

we obtain

$$\sum_b P_{a/b}^{\prime} \otimes_x f_{b/A}^{\prime}(\mu) = $$

$$\sum_b \left[ P_{a/b} \otimes_x f_{b/A}(\mu) + \frac{\alpha_s}{2\pi} \left( \frac{\beta_0}{2} f_{a/A} \otimes_x \Delta C_{ab} - \sum_c (P_{b/c} \otimes f_{c/A}) \otimes_x \Delta C_{ab} \right) \right] + \mathcal{O}(\alpha_s^2).$$

\footnote{In the language of deep-inelastic scattering, $C_{ia}$ is usually called coefficient function.}

\footnote{The DIS factorization scheme at next-to-leading order is defined by $C^\prime_{ia} = 0$.}
We can rewrite the right hand side in terms of $f'$ and read off the change in the next-to-leading order Altarelli-Parisi kernel induced by a change in the hard-scattering cross section:

$$P_{a/b}'(z) = P_{a/b}(z) + \frac{\alpha_s}{2\pi} \left[ \sum_c \left( P_{a/c} \otimes z \Delta C_{cb} - \Delta C_{ac} \otimes z P_{c/b} \right) + \frac{\beta_0}{2} \Delta C_{ab} \right] + O(\alpha_s^2).$$ (10.7)

Table 1 contains the change in the hard-scattering cross section for making the transition from the conventional scheme to the ‘t Hooft-Veltman and dimensional reduction regularization schemes in the $\overline{\text{MS}}$ factorization scheme. These values can be obtained from the $\tilde{D}$ and $\gamma(a)$ functions as described in Appendix B. Substituting these values for $\Delta C_{ab}$ into eq. (10.7), we obtain the next-to-leading order Altarelli-Parisi kernels in the $\overline{\text{MS}}$ scheme. We list below only the difference in comparison to the standard $\overline{\text{MS}}$ expressions obtained in the conventional scheme [27]:

$$P_{g/q}^{HV}(z) - P_{g/q}^{MS}(z) = C_F T_R N_f (2 - \frac{2}{3} x^{-1} - \frac{4}{3} x^2 + 2x \log x) + C_G \beta_0 x (1 - x).$$ (10.8)

$$P_{q/g}^{HV}(z) - P_{q/g}^{MS}(z) = C_F T_R N_f (-1 + x + 2x \log x - 4x^2 \log x) + C_G T_R N_f (-1 + \frac{2}{3} x^{-1} - \frac{13}{3} x^2 - 4x \log x + 4x^2 \log x).$$ (10.9)

$$P_{g/q}^{HV}(z) - P_{g/q}^{MS}(z) = C_F C_G (2 - \frac{2}{3} x^{-1} - \frac{4}{3} x^2 + 2x \log x).$$ (10.10)

$$P_{q/g}^{HV}(z) - P_{q/g}^{MS}(z) = C_F T_R N_f (-2 + \frac{2}{3} x^{-1} + \frac{4}{3} x^2 - 2x \log x).$$ (10.11)

$$P_{g/q}^{DR}(z) - P_{g/q}^{MS}(z) = C_F T_R N_f (3 - \frac{2}{3} x^{-1} + x - \frac{10}{3} x^2 + 4x \log x).$$ (10.12)

$$P_{q/g}^{DR}(z) - P_{q/g}^{MS}(z) = C_F T_R N_f (-3 + 2x + x^2 - \log x - 4x^2 \log x) + C_G T_R N_f (-2 + \frac{2}{3} x^{-1} - 10x + \frac{34}{3} x^2 - 8x \log x) + T_R N_f (\tilde{\gamma}(g) - \tilde{\gamma}(q))(1 - 2x + x^2).$$ (10.13)

$$P_{g/q}^{DR}(z) - P_{g/q}^{MS}(z) = C_F^2 (-x^{-1} + \frac{5}{2} x - 2 \log x - 2x \log x) + C_F C_G (2 - x^{-1} + x - 2x^2 + 4x \log x) + C_F [-\beta_0 x + (\tilde{\gamma}(g) - \tilde{\gamma}(q))(2 - 2x^{-1} - x)].$$ (10.14)

$$P_{q/g}^{DR}(z) - P_{q/g}^{MS}(z) = C_F T_R N_f (-\frac{9}{4} + \frac{2}{3} x^{-1} - x + \frac{10}{3} x^2 - 4x \log x) + C_F \beta_0 \frac{1}{2} (-1 + x).$$ (10.15)

In these equations, we used the notation of ref. [27]: $C_F = V/(2N_c)$, $C_G = N_c$ and $T_R = 1/2$. 
Now we turn to the $N=1$ supersymmetric Yang-Mills theory discussed in sect. 7. In this case, if the regularization scheme is supersymmetric, the Altarelli-Parisi kernels are to satisfy the following identity to all orders in perturbation theory:

$$P_{g/g}(z) + P_{\tilde{g}/\tilde{g}}(z) = P_{g/\tilde{g}}(z) + P_{\tilde{g}/g}(z).$$

(10.16)

The validity of this relation follows simply from the physical meaning of the Altarelli-Parisi kernels: they give the number density of the partons in the infinite momentum frame. Therefore, if we sum over the final states, it should not matter whether the initial state is a gluon or a gluino. The Altarelli-Parisi kernels are known to satisfy identity (10.16) to leading [27] and in dimensional reduction to next-to-leading order [28].

Some care is to be taken when going to the supersymmetric limit of eqs. (10.12–10.15). A virtual gluino loop on a gluon propagator contains a combinatorial factor of $1/2$, while the gluon splitting into a gluino pair does not contain the same factor. Therefore, to obtain the supersymmetric limit, we take $T_R N_f \to N_c$ in the splitting functions and $2T_R N_f \to N_c$ in loop corrections. This rule can be written more consistently by using $T_R N_f \to N_c$, $\beta_0 \to 3 N_c$, $\tilde{\gamma}(g) \to N_c/6$ and $\tilde{\gamma}(g) \to N_c/2$ in addition to the usual rules, $C_F \to N_c$, $C_G \to N_c$ (this is the reason for not writing $\beta_0$, $\tilde{\gamma}(g)$ and $\tilde{\gamma}(q)$ in terms of $C_G$, $T_R$ and $N_f$ in eqs. (10.12–10.13)). From eqs. (10.12 10.13) for the combination

$$\Delta(z) \equiv \hat{P}_{g/g}(z) + \hat{P}_{\tilde{g}/\tilde{g}}(z) - \hat{P}_{g/\tilde{g}}(z) - \hat{P}_{\tilde{g}/g}(z),$$

(10.17)

we obtain

$$\Delta^{\text{DR}}(z) - \Delta^{\text{MS}}(z) = - \left( \frac{5}{6} - \frac{23}{3} x + 7 x^2 - (2 x + 4 x^2 - 1) \log x + \frac{2}{3} x^{-1} \right) N_c^2,$$

(10.18)

where $\hat{P}_{a/b}(z)$ is the Altarelli-Parisi kernel without the part that is proportional to $\delta(1-z)$. Comparing eq. (10.18) to formula (19) of ref. [27], we conclude that in the dimensional reduction scheme $\Delta(z) = 0$. This is the consistency check on our transition rules we referred to above. This test, however, still does not prove that unitarity is maintained in the ‘t Hooft-Veltman and dimensional reduction schemes. It may be violated by terms which are proportional to $(N_c - N_f)$.

Finally, we remark that our derivation of eq. (10.18) is different from that of Antoniadis and Floratos [28] and can be considered as an independent confirmation of the result that in dimensional reduction, relation (10.16) remains valid in next-to-leading order.

11 Summary

We calculated the one loop corrections to the helicity amplitudes of all $2 \to 2$ parton scattering processes in QCD using dimensional reduction as well as ‘t Hooft-Veltman regularization schemes for regularizing both the ultraviolet and infrared singularities. We explicitly demonstrated the cancellation of the soft and collinear singularities between the loop corrections and Bremsstrahlung contributions using the general structure of the Bremsstrahlung singularities described in ref. [4]. Using somewhat heuristic arguments concerning the differences between the results obtained in the ‘t Hooft-Veltman scheme and in the conventional dimensional regularization scheme, we could reproduce the results of Ellis and Sexton for the
one loop contributions to the spin and color summed cross-sections of all $2 \rightarrow 2$ parton subprocesses. Using the general form of the soft and collinear singularities, we have found universal (process independent) transition rules which can be used to transform the results obtained in the ‘t Hooft-Veltman scheme and in dimensional reduction into those of the conventional regularization scheme. These transformation rules have practical importance because the phenomenological parton density functions determined from fits to the data assume next-to-leading order $Q^2$-evolution as given in the conventional dimensional regularization scheme. Therefore, when a physical cross section is calculated, the hard scattering parton cross sections have to be given as obtained in the conventional scheme. Since we know how to translate the results between the above mentioned three schemes, one can do the actual calculation in the most convenient scheme. We note that in the ‘t Hooft-Veltman scheme and in dimensional reduction one can use the helicity method directly.

With explicit calculation we demonstrated that in dimensional reduction, as expected, the supersymmetry Ward identities of the scattering amplitudes of the $2 \rightarrow 2$ parton subprocesses of the $N=1$ supersymmetric Yang-Mills theory are satisfied also in next-to-leading order. QCD differs from the $N=1$ supersymmetric Yang-Mills theory only in the color representation of the quarks and gluinos, therefore, the supersymmetry Ward identities can be used to obtain a non-trivial test of the correctness of the calculation and to obtain the amplitudes of different subprocesses with the help of the supersymmetry Ward identities.

We have shown that the supersymmetry Ward identities can be derived also using the non-supersymmetric ‘t Hooft-Veltman regularization scheme by working out simple finite renormalization factors. Using this result we find that the string theory based four-dimensional helicity scheme of Bern and Kosower gives the same result for the four- and five-gluon amplitudes as the dimensional reduction scheme.

It is not obvious that in the ‘t Hooft-Veltman scheme and in the dimensional reduction unitarity is not violated by some finite terms when the soft and collinear singularities are cancelled between the loop corrections and the Bremsstrahlung contributions. This difficulty does not arise if we know the transition rules to the conventional regularization scheme. Using the transition rules we calculated the next-to-leading order Altarelli-Parisi kernels for dimensional reduction and for the ‘t Hooft-Veltman scheme. Unitarity is maintained in this case by construction. We have found that the next-to-leading order Altarelli-Parisi kernels of the $N=1$ supersymmetric Yang-Mills theory obtained in dimensional reduction satisfies the supersymmetry Ward identities [28].

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Appendix A

In this Appendix, we spell out eq. (4.2) explicitly for $N = 2, 3, 4$ in a form which is used in the actual calculation. We also calculate all the necessary integrals.

Case $N = 2$:

$$M = \frac{i}{(4\pi)^2} \left( \frac{4\pi \mu^2}{-p^2} \right) \epsilon \sum_{k=0}^{1} (-p^2)^k \Gamma(-k + \epsilon) \int_{[0,1]^2} dx_1 dx_2 \delta(1 - x_1 - x_2)(x_1 x_2)^{k-\epsilon} N_k(J, p),$$  \hspace{1cm} (A.1)

where

$$J_1 = x_2 p,$$  \hspace{1cm} (A.2)
$$J_2 = -x_1 p.$$  \hspace{1cm} (A.3)

$N_k(J, p)$ is a polynomial of $x_1$ and $x_2$. The general form of a term in the integral is

$$\int_{[0,1]^2} dx_1 dx_2 \delta(1 - x_1 - x_2)x_1^{n_1} x_2^{n_2} (x_1 x_2)^{k-\epsilon} = B(n_1 + k + 1 - \epsilon, n_2 + k + 1 - \epsilon).$$  \hspace{1cm} (A.4)

Case $N = 3$:

$$D(x, p) = -(x_1 x_2 p_2^2 + x_2 x_3 p_3^2 + x_3 x_1 p_1^2).$$  \hspace{1cm} (A.5)

For the $0 \rightarrow 4$ partons case, two of the external legs are on shell. If $i$ and $j$ are the indices of the propagators joining to the massive external leg ($p_j$), then eq. (4.2) becomes

$$M = -\frac{i}{(4\pi)^2} \left( \frac{4\pi \mu^2}{-p^2} \right) \epsilon \sum_{k=0}^{1} (-p_j^2)^{k-1} \Gamma(1 - k + \epsilon) \int_{[0,1]^3} dx_i dx_j dx_k \delta(1 - x_i - x_j - x_k)(x_i x_j x_k)^{k-1-\epsilon} N_k(J, p),$$  \hspace{1cm} (A.6)

where

$$J_i = x_j p_j + x_k (p_j + p_k),$$  \hspace{1cm} (A.7)
$$J_j = x_k p_k + x_i (p_k + p_i),$$  \hspace{1cm} (A.8)
$$J_k = x_i p_i + x_j (p_i + p_j).$$  \hspace{1cm} (A.9)

$N_k(J, p)$ is a polynomial of $x_i, x_j$ and $x_k$. The general form of a term in the integral is

$$\int_{[0,1]^3} dx_i dx_j dx_k \delta(1 - x_i - x_j - x_k)x_i^{n_i} x_j^{n_j} x_k^{n_k} (x_i x_j x_k)^{k-1-\epsilon}.$$  \hspace{1cm} (A.10)

After changing variables,

$$x_i = x,$$
$$x_j = y(1 - x),$$
$$x_k = (1 - x)(1 - y),$$
the integral becomes
\[
\int_0^1 dx \int_0^1 dy \, x^{n_i}(1 - x)^{n_i + n_k} y^{n_k}(1 - y)^{n_k}(x(1 - x)y)^{k-1-x}(1 - x) = B(n_i + k - \varepsilon, n_j + n_k + k + 1 - \varepsilon) B(n_j + k - \varepsilon, n_k + 1). \tag{A.11}
\]

Case \(N = 4\):
\[
D(x, p) = -(x_1 x_2 p_1^2 + x_2 x_3 p_2^2 + x_3 x_4 p_3^2 + x_4 x_1 p_1^2 + x_1 x_3 s_{12} + x_2 x_4 s_{14}), \tag{A.13}
\]
where
\[
s_{ij} = (p_i + p_j)^2. \tag{A.14}
\]
For the \(0 \rightarrow 4\)-parton case, the external legs are on shell. Therefore, eq. (1.2) becomes
\[
\mathcal{M} = \frac{i}{(4\pi)^2} (4\pi)^\varepsilon \sum_{k=0}^{2} \Gamma(2 - k + \varepsilon) \mu^{2\varepsilon} \int_{[0,1]^4} dx_1 \, dx_2 \, dx_3 \, dx_4 \, \delta(1 - x_1 - x_2 - x_3 - x_4) \frac{N_k(J, p)}{(-x_1 x_3 s_{12} - x_2 x_4 s_{14})^{2-k+\varepsilon}} \tag{A.15}
\]
where
\[
J_1 = x_2 p_1 + x_3 (p_1 + p_2) + x_4 (p_1 + p_2 + p_3), \tag{A.16}
J_2 = x_3 p_2 + x_4 (p_2 + p_3) + x_1 (p_2 + p_3 + p_4), \tag{A.17}
J_3 = x_4 p_3 + x_1 (p_3 + p_4) + x_2 (p_3 + p_4 + p_1), \tag{A.18}
J_4 = x_1 p_4 + x_2 (p_4 + p_1) + x_3 (p_4 + p_1 + p_2). \tag{A.19}
\]
\(N_k(J, p)\) is a polynomial of \(x_1, x_2, x_3\) and \(x_4\). The generic form of the integral is
\[
\mathcal{I}^k[P(x_i)] = (4\pi)^{-2+\varepsilon} \Gamma(2 - k + \varepsilon) \times \mu^{2\varepsilon} \int_{[0,1]^4} dx_1 \, dx_2 \, dx_3 \, dx_4 \, \delta(1 - x_1 - x_2 - x_3 - x_4) \frac{P(\{x_i\})}{(-x_1 x_3 s_{12} - x_2 x_4 s_{14})^{2-k+\varepsilon}}. \tag{A.20}
\]

It is remarkable that the same type of integrals, but only for \(k = 0\), appear in the calculation of one-loop amplitudes using the string-based approach. In ref. [29], a general method has been described how to calculate a large class of these one-loop Feynman parametric integrals. The main idea of the evaluation is derivation of a differential relation between the basic scalar integral \(\mathcal{I}^k[1]\) and the so called scaled integrals [29] which are related to \(\mathcal{I}^k[P(x_i)]\) according to
\[
\hat{\mathcal{I}}^k[P(x_i)] \equiv \left( \prod_{j=1}^{4} \alpha_j \right)^{-1} \mathcal{I}^k[P(x_i/\alpha_i)] \tag{A.21}
\]
where \(\alpha_i\) are positive, real parameters, satisfying \(\alpha_1 \alpha_3 = (-s_{12})^{-1}\) and \(\alpha_2 \alpha_4 = (-s_{14})^{-1}\).

The basic scalar integral can be evaluated either using the partial differential equation technique of ref. [29], or by direct integration. We have chosen the latter approach and
found agreement with the result of ref. [1]. For the cases \( k \neq 0 \) the same method with trivial modifications works.

**Appendix B**

This Appendix is devoted to the derivation of the changes in the coefficient functions appearing in Table 1.

The first step in the determination of the \( \Delta C_{ab} \) functions is the derivation of the \( \varepsilon \)-dependent Altarelli-Parisi kernels in all three regularization schemes. This accounts for finding the \( \varepsilon \)-dependent splitting functions without \( z \to 1 \) regulation. These functions in the dimensional reduction scheme coincide with the four-dimensional Altarelli-Parisi kernels first obtained by Altarelli and Parisi. We recall them to set the notation:

\[
P_{g/g}(z) = 2C_G \left[ \frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right],
\]

\[
P_{q/g}(z) = T_R N_f \left[ z^2 + (1-z)^2 \right],
\]

\[
P_{g/q}(z) = C_F \frac{1 + (1-z)^2}{z},
\]

\[
P_{q/q}(z) = C_F \frac{1 + z^2}{1-z}.
\]

We denote the \( \varepsilon \)-dependent Altarelli-Parisi kernels by the regularization scheme-dependent functions \( \tilde{P}_{a/b}(z, \varepsilon) \) and decompose them into a scheme-independent and a scheme-dependent part:

\[
\tilde{P}_{a/b}(z, \varepsilon) = P_{a/b}(z) + \varepsilon \tilde{P}_a'(z).
\]

Our goal is to find the scheme-dependent functions, \( \tilde{P}_a'(z) \).

Clearly, when the initial parton is a quark, the only dependence on the dimension of space-time enters in the integral over the loop momentum. As a result, the splitting functions for quark splitting are the same in the ’t Hooft-Veltman and conventional schemes. Repeating the calculation of Altarelli and Parisi in \( (4-2\varepsilon) \) dimensions, one finds

\[
\tilde{P}_{g/q}^{CDR}(z) = \tilde{P}_{g/q}^{HV}(z) = -C_F z,
\]

\[
\tilde{P}_{q/q}^{CDR}(z) = \tilde{P}_{q/q}^{HV}(z) = -C_F(1-z).
\]

We follow the procedure of Appendix A of ref. [3] to derive the splitting functions for gluon splitting. In that reference, the splitting for a polarized gluon were obtained in the form

\[
\tilde{P}_{q/g}^{ij}(z, \varepsilon) = \frac{1}{2} \left[ \delta^{ij} - 4z(1-z) \frac{q^i q^j}{q^2} \right],
\]

\[
\tilde{P}_{g/g}^{ij}(z, \varepsilon) = 2C_G \left[ \left( \frac{z}{1-z} + \frac{1-z}{z} \right) \delta^{ij} + 2(1-\varepsilon)z(1-z) \frac{q^i q^j}{q^2} \right].
\]
In eqs. (B.8), (B.9), \( q \) is the transverse momentum of the product particle, \( q^i \) and \( q^j \) are the two components of \( q \). The \( \varepsilon \)-dependent Altarelli-Parisi kernel is then (see ref. [3])

\[
\tilde{P}_{a/g}(z, \varepsilon) = \frac{V}{\omega(g)} \delta_{ij} \tilde{P}^{ij}_{a/g}(z, q).
\]  

(B.10)

Both \( \omega(g) \) and the contraction of the transverse indices are scheme dependent. In the \('t\) Hooft-Veltman scheme they are taken in four dimensions, while in conventional dimensional regularization they are in \( d \)-dimensions. Using eqs. (B.8)-(B.10), we obtain

\[
\tilde{P}^\text{CDR}_{g/g}(z) = 0, \\
\tilde{P}^\text{CDR}_{q/g}(z) = -2T_R N_f z(1 - z)
\]  

(B.11)

in the conventional scheme, in agreement with the results of ref. [31], while in the \('t\) Hooft-Veltman scheme, we find

\[
\tilde{P}^\text{HV}_{g/g}(z) = -2C_G z(1 - z), \\
\tilde{P}^\text{HV}_{q/g}(z) = 0,
\]  

(B.13)

in agreement with the results of ref. [3].

The difference between the hard scattering cross sections when changing the regularization scheme determines the \( \Delta C_{ab}(z) \) functions. These differences were analyzed in great details in the main text. The differences can entirely be described by the scheme dependence of the \( \tilde{P}' \) functions and by that of the \( \tilde{\gamma}(a) \) constants. The \( \gamma(a) \) constants represent the contribution from the virtual graphs to the Altarelli-Parisi kernel, therefore, we may interpret the \( \tilde{\gamma}(a) \) constants as the contribution from the virtual graphs to the \( \varepsilon \)-dependent part of the \( \varepsilon \)-dependent Altarelli-Parisi kernel. The \( \tilde{\gamma}(a) \) constants are scheme dependent:

\[
\tilde{\gamma}^\text{CDR}(g) = \tilde{\gamma}^\text{HV}(g) = \frac{1}{6} N_c, \\
\tilde{\gamma}^\text{CDR}(q) = \tilde{\gamma}^\text{HV}(q) = \frac{1}{2} C_F,
\]  

(B.15)

Using eqs. (B.6), (B.7), (B.11), (B.14), (B.15), it is now easy to fill the entries of Table 1.
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