COXETER COVERS OF THE SYMMETRIC GROUPS

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Abstract. We study Coxeter groups from which there is a natural map onto a symmetric group. Such groups have natural quotient groups related to presentations of the symmetric group on an arbitrary set $T$ of transpositions. These quotients, denoted here by $C^Y(T)$, are a special type of the generalized Coxeter groups defined in [4], and also arise in the computation of certain invariants of surfaces.

We use a surprising action of $S_n$ on the kernel of the surjection $C^Y(T) \to S_n$ to show that this kernel embeds in the direct product of $n$ copies of the free group $\pi_1(T)$ (with the exception of $T$ being the full set of transpositions in $S_4$). As a result, we show that the groups $C^Y(T)$ are either virtually Abelian or contain a non-Abelian free subgroup.

1. Introduction

The symmetric group on $n$ letters is generated by the transpositions $s_i = (i \ i + 1), i = 1, \ldots, n - 1$. These generators satisfy the well known relations $s_i^2 = 1$, $s_is_j = s_js_i$ ($|i - j| \geq 2$) and $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$. Moreover, the abstract group defined by the $s_i$ with the given relations is a Coxeter group, isomorphic to $S_n$.

This set $\{s_i\}$ can be presented by a graph on the vertices $1, \ldots, n$, where $s_i$ is the edge connecting $i$ and $i + 1$. More generally, one can use any connected graph $T$ on $n$ vertices to define a Coxeter group $C(T)$, from which there is a natural projection onto the corresponding symmetric group.

The kernel of this projection is generated by elements coming from two families; one corresponding to triples of vertices in $T$ which meet in a common vertex, and one to the cycles of $T$. Let $C_Y(T)$ denote the quotient of $C(T)$ obtained by assuming the first family of relations to hold.

One motivation to study the groups $C_Y(T)$ comes from algebraic geometry, where these groups are a key ingredient in studying certain...
invariants of surfaces (see [10] for a discussion on the computation of those invariants). From another direction, signed graphs are used in [4] to define generalized Coxeter groups, which are quotients of ordinary Coxeter groups. Our groups \( C_Y(T) \) belong to this class. The generalized Coxeter groups which our results enable us to compute are discussed in Subsection 7.4. These include the group \( D_2 \) whose computation occupies a large portion of [5], and a certain family of Tsaranov groups. For example, the Tsaranov group of a hexagon (which attracted much attention, see [4, Example 8.6]), is identified in Corollary 7.16.

There are other indications that \( C_Y(T) \) is a natural quotient of the Coxeter group \( C(T) \). For example, their parabolic subgroups are well behaved: if \( T' \subseteq T \) is a subgraph, then the subgroup of \( C_Y(T) \) generated by the elements of \( T' \) is isomorphic to the abstract group defined on \( T' \). This is shown in Subsection 7.2.

We prove that (with the exception of \( T \) equals \( K_4 \), the complete graph on 4 vertices), \( C_Y(T) \) is contained in the semidirect product \( S_n \ltimes \pi_1(T)^n \), thus solving the word problem for these groups. On the other hand \( C_Y(T) \) contains copies of \( \pi_1(T) \), showing that it is virtually solvable (that is, has a solvable subgroup of finite index) iff \( T \) has at most one cycle. Moreover, if \( T \) has one cycle then \( C_Y(T) \) is virtually Abelian. This supports Teicher’s conjecture that the invariants mentioned above are either virtually solvable, or contain a free subgroup [9].

Recently, Margulis and Vinberg [7, Cor. 2] proved that infinite non-affine Coxeter groups are large (i.e. virtually have a free quotient). In particular the group \( C(T) \) is large for every graph \( T \) other than a line, a \( Y \)-shaped graph (on four vertices), or a cycle. Our results provide more information on \( C(T) \), proving that already \( C_Y(T) \) is large if \( T \) has at least two cycles, while the kernel of \( C(T) \rightarrow C_Y(T) \) is large otherwise.

In Section 2 we give the basic definitions and properties, and briefly describe an application for our results to algebraic geometry. Spanning subtrees of \( T \) are an important tool throughout, and in Section 3 we prove that the subgroup generated by a spanning subtree is the symmetric group. We then describe an action of \( S_n \) on the kernel of the projection \( C_Y(T) \rightarrow S_n \), which uses two different embeddings of \( S_n \) to \( C_Y(T) \). In Section 6 we prove the main result, that this kernel is isomorphic to a given abstract group, given by generators and relations. This group is studied in Section 5, where we show it embeds in a direct product of free groups.

The applications to Theorem 6.1 are given in Section 7: in Corollary 7.1 we give the criterion for \( C_Y(T) \) to be virtually solvable. Another
immediate result is that $C_Y(T)$ depends only on the number of vertices and cycles of $T$. In Subsection 7.3 we discuss the Coxeter graph of $C(T)$ and some special cases.

2. Presentations of $S_n$ on transpositions

Let $T$ be a graph on $n$ vertices. Consider the group generated by the transpositions $(ab) \in S_n$ for the edges $(a, b)$ in $T$; obviously this is the full symmetric group $S_n$ iff $T$ is connected. Throughout the paper, all our graphs are simple (i.e. no repeated edges or loops).

Recall that a Coxeter group is a group with generators $s_1, \ldots, s_k$, and defining relations $s_i^2 = 1$ and $(s_is_j)^{m_{ij}} = 1$, where $m_{ij} \in \{2, 3, \ldots, \infty\}$. The finite Coxeter groups are completely classified (see [3]), and they are the finite (real) reflection groups.

We use the graph $T$ to define a Coxeter group $C(T)$, as follows.

Definition 2.1. The group $C(T)$ is generated by the edges $u \in T$, subject to the following relations:

1. $u^2 = 1$ for all $u \in T$,
2. $uv = vu$ if $u, v$ are disjoint, and
3. $uvu = vuv$ if $u, v$ intersect.

Note that the last relation is equivalent to $(uv)^3 = 1$, so $C = C(T)$ is indeed a Coxeter group.

Definition 2.2. Let $T$ be a graph. The map

$$\phi : C(T) \to S_n$$

is defined by sending $u = (a, b)$ to the transposition $(ab)$.

This map is easily seen to be well defined. It is natural to ask what relations we need to add to $C(T)$ in order to obtain a presentation of $S_n$, or in other words what elements generate the kernel of $\phi$. Let $u, v, w \in T$ be three edges meeting in the vertex $a$, and let $a_u, a_v, a_w$ denote the other vertices of the respective edges. The transposition $(a, a_u)$ commutes with $(a, a_v)(a, a_w)(a, a_v) = (a_u, a_w)$. This motivates the following relation:

$$[u, vwu] = 1 \quad \text{for } u, v, w \in T \text{ which meet in a vertex.}$$

Let $u_1, \ldots, u_m \in T$ be a cycle. In that we mean that each $u_i$ shares a common vertex with $u_{i-1}$ and $u_{i+1}$ (and $u_1$ with $u_m$), and there are no
other intersections. One easily checks that the following relation holds in $S_n$:

\[(5) \quad u_1 \ldots u_{m-1} = u_2 \ldots u_m.\]

It turns out that the relations listed above are enough:

**Theorem 2.3** ([8, Prop. 3.4]). Let $T$ be a connected graph on $n$ vertices. The symmetric group $S_n$ has a presentation with the edges of $T$ as generators, and the relations (1)–(3), (4) and (5).

The object we study in this paper is the group $C_Y(T)$, which we now define. The subscript $Y$ symbolizes the three edges meeting in a vertex.

**Definition 2.4.** Let $T$ be a graph on $n$ vertices. $C_Y(T)$ is the group generated by the edges of $T$, with the relations (1)–(4).

The map of Definition 2.2 induces a map $\phi: C_Y(T) \to S_n$. We let $K(T) \subseteq C_Y(T)$ denote the kernel of this map.

If $T$ is a tree (i.e. a connected graph with no cycles), then (5) is vacuously satisfied, so from Theorem 2.3 we obtain

**Corollary 2.5.** If $T$ is a tree then $C_Y(T) \cong S_n$.

The main result of this paper is Theorem 6.1, which presents $K(T)$ (defined above) as a certain subgroup of the direct product $\pi_1(T)^n$. It follows (Corollary 7.1) that $C_Y(T)$ is virtually solvable iff $T$ has at most one cycle, in which case it is virtually Abelian.

One application of the study of $C_Y(T)$ is in algebraic geometry, specifically to the computation of fundamental groups of complements of plane curves in algebraic surfaces. This problem goes back to Zariski, and was, in part, the motivation behind van Kampen’s celebrated theorem. In recent years Moishezon and Teicher have made systematic attempts to compute the fundamental groups of certain canonical configurations explicitly in terms of generators and relations, using this to obtain information on the structure of these groups, see [10].

In these computations, the first step is to use van Kampen’s theorem to obtain a presentation of the fundamental group. This group is naturally generated by pairs of elements $\Gamma_j, \Gamma_j'$ ($j = 1, \ldots, N$ for some $N$), and usually has many defining relations. Let $G$ denote the fundamental group, modulo the relations $\Gamma_j^2 = \Gamma_j'^2 = 1$. There is a natural map $\phi_G$ from $G$ onto a symmetric group $S_n$, sending the generators $\Gamma_j, \Gamma_j'$ to transpositions (where $\phi_G(\Gamma_j) = \phi_G(\Gamma_j')$). It is known that the fundamental group of a Galois cover of the surface, with respect to a generic projection to $\mathbb{C}P^2$, is the kernel of $\phi_G$. This important invariant of surfaces is useful for classification of moduli spaces.
One can formally define a quotient group $C = \langle u_j \rangle$ of $G$ by applying
\( \theta : \Gamma_j, \Gamma_{j'} \mapsto u_j \) to the relations of $G$. Then the map $\phi_G$ splits as
$\phi_G = \phi \circ \theta$, where $\phi : C \to S_n$ is defined by $\phi(u_j) = \phi_G(\Gamma_j)$. It turns out
that $C$ is a Coxeter group of the form $C(T)$ (for a certain graph $T$ on $n$ vertices), with some extra relations: relation (5) holds for
some of the cycles in $T$. As the next remarks show, in the presence of enough
cyclic relations, $C$ actually becomes a quotient of $C_Y(T)$, and so our
computation of $C_Y(T)$ allows to compute $C$.

**Remark 2.6.** If $u, v, w \in T$ meet in a vertex, then the relations
$[u, vvw] = 1$, $[v, wuw] = 1$ and $[w, uvu] = 1$ are all equivalent (since
$[v, wuw] = w[vww, u]w$).

**Remark 2.7.** If $u_1, \ldots, u_m$ form a cycle in $T$, then all the relations of
the form of (5) corresponding to that cycle are equivalent: it does not
matter which edge is labelled $u_1$, nor in which direction the edges are
labelled.

**Proof.** Assume $u_1 \ldots u_{m-1} = u_2 \ldots u_m$. Multiplying by $u_2$ from the left
we obtain $u_1u_2u_1u_3 \ldots u_{m-1} = u_2u_1u_2 \ldots u_{m-1} = u_3 \ldots u_m$, but since
$u_1$ commutes with $u_3, \ldots, u_{m-1}$, we get $u_1u_2 \ldots u_{m-1} = u_3 \ldots u_mu_1$. Thus
rotation of the labels preserves the equality. Taking inverses in
(5) we see that this is also the case with changing the direction. \( \square \)

These two observations justify calling (4) the relation induced by the
triple $u, v, w$, and (5) the equation induced by the cycle. We conclude
with the following remark, which explains how cyclic relations imply 'triple' relations.

**Remark 2.8.** Let $u_1, \ldots, u_m$ be a cycle in $T$, and $w$ an edge meeting
the cycle at the vertex $u_i \cap u_{i+1}$. Then the relation (4) corresponding
to $u_i, u_{i+1}, w$ follows from the relation (5) associated to the cycle.

**Proof.** We may assume $i = 1$. Then by assumption we have $u_2u_1u_2 = u_3 \ldots u_{m-1}u_mu_{m-1} \ldots u_3$, and the $u_i$ ($i \geq 3$) commute with $w$, showing
that $[w, u_2u_1u_2] = 1$. \( \square \)

### 3. The subgroup of $C_Y(T)$ generated by a tree

In later sections, it will be useful to know that the subgroup of $C_Y(T)$
generated by the edges of a spanning subtree $T_0$ of $T$, is isomorphic to
the abstract group defined on $T_0$, i.e. to $C_Y(T_0) = S_n$.

We start with the following easy observation.

**Remark 3.1.** Let $T' \subseteq T$ be any subgraph. Then the subgroup $\langle T' \rangle$ of
$C_Y(T)$ generated by the edges of $T'$ is a quotient of $C_Y(T')$. 
Proof. The map $C_Y(T') \to \langle T' \rangle$ defined by $u \mapsto u$ is well defined, since every relation in $C_Y(T')$ is assumed to hold in $C_Y(T)$. Its image is obviously $\langle T' \rangle$, so we are done. 

Though it is possible to prove directly that the map $\pi : C_Y(T') \to \langle T' \rangle$ is an isomorphism, we postpone further treatment of the subject to Section 7 (Proposition 7.8), where it will be derived as an easy consequence of the main results.

**Proposition 3.2.** Let $T$ be a graph, and $T_0 \subseteq T$ a subtree. Let $\langle T_0 \rangle$ be the subgroup of $C_Y(T)$ which is generated by the vertices $u \in T_0$. Then $\langle T_0 \rangle \cong S_n$.

Proof. By the remark, $\langle T_0 \rangle$ is the image of the map $\pi : C_Y(T_0) \to C_Y(T)$ defined by $u \mapsto u$. In the diagram below, $\phi, \phi_0$ are the maps of Definition 2.2 for $T, T_0$, respectively. The diagram commutes by definition of $\pi$.

$$
\begin{array}{ccc}
C_Y(T_0) & \xrightarrow{\pi} & C_Y(T) \\
\downarrow & & \downarrow \\
\phi_0 & \xrightarrow{\phi} & S_n
\end{array}
$$

By Corollary 2.5, $\phi_0$ is an isomorphism, so the composition $(\phi_0^{-1} \phi) \circ \pi$ is the identity on $C_Y(T_0)$. On the other hand if $u \in T_0$ then $\phi_0^{-1} \phi(u)$ is the generator $u \in C_Y(T)$, so that $\pi \circ (\phi_0^{-1} \phi|_{\langle T_0 \rangle}) = 1_{\langle T_0 \rangle}$. This proves that $\pi : C_Y(T_0) \to \langle T_0 \rangle$ is an isomorphism. 

\square

4. $S_n$-action on cycles

In this section we focus on a single cycle in $T$. We study the cyclic relation (5) corresponding to this cycle, and describe an action of the symmetric group on the normal subgroup it generates. Recall that $n$ is the number of vertices in $T$. Throughout, we multiply permutations by $(\sigma \tau)(a) = \tau(\sigma(a))$.

Let $u_1, \ldots, u_m$ denote the edges of the cycle ($m \leq n$ is the length of the cycle), and renumber the vertices so that $u_i = (i-1,i)$ ($i = 2, \ldots, m$), with $u_1 = (1,m)$. Let

$$
\gamma_i = u_{i+1} \ldots u_m u_1 \ldots u_i, \quad i = 1, \ldots, m-2,
$$

and also set $\gamma_{m-1} = u_1 \ldots u_{m-1}$ and $\gamma_m = u_2 \ldots u_m$.

According to Theorem 2.3, the kernel of $\phi : C_Y(T) \to S_n$ is the normal subgroup generated by elements of the form $\gamma_{m-1}^{-1} \gamma_m$ (ranging over all the cycles of $T$). In Remark 2.7 we have seen that $\gamma_j^{-1} \gamma_i$ belongs to this kernel for all $i, j = 1, \ldots, m$. Choose a spanning subtree $T_0$ of $T$. 


It would be nice to have a natural action of the subgroup $S_n = \langle T_0 \rangle$ on the set \{\gamma_i^{-1}\gamma_j\}; such an action can take the form

\begin{equation}
\sigma^{-1}\gamma_j^{-1}\gamma_i \sigma = \gamma_{\sigma j}^{-1}\gamma_{\sigma i}, \quad \sigma \in \langle T_0 \rangle.
\end{equation}

For (7) to even make sense, we need $\gamma_i$ to be defined for every $i = 1, \ldots, n$, not just for vertices on the cycle. We will show that assuming (7) to hold, we are led to a unique definition of the $\gamma_i$. We will later show that using this definition, Equation (7) is indeed satisfied. Since $\gamma_j^{-1}\gamma_i$ is mapped to the unit element of $S_n$, all the $\gamma_i$ map to the same permutation $\phi(\gamma_1) = (23) \ldots (m - 1 \, m)(1m) = (m \ldots 321)$.

A path in a graph is an ordered list of edges, in which every two consecutive edges have one vertex in common, and there are no other intersections.

**Proposition 4.1.** Suppose that elements $\gamma_i \in C_Y(T)$ ($i = 1, \ldots, n$) are defined for every cycle in $T$, such that Equation (7) holds, and for the vertices $i$ on the cycle, $\gamma_i$ is defined by Equation (6).

Then $K(T) = \text{Ker}(\phi : C_Y(T) \to S_n)$ is generated by the elements $\gamma_j^{-1}\gamma_i$ as a subgroup (rather than a normal subgroup) of $C_Y(T)$.

**Proof.** Let $N$ denote the subgroup of $C_Y(T)$ generated by the elements $\gamma_j^{-1}\gamma_i$ ($i, j = 1, \ldots, n$) for all cycles in $T$. We already know that $K(T)$ is generated by the $\gamma_{m - 1}^{-1}\gamma_m$ as a normal subgroup. Equation (7) indicates that the $\gamma_j^{-1}\gamma_i$ are all conjugate to each other, so $N \subseteq K(T)$.

We need to show that $N$ is normal. Let $\gamma_a^{-1}\gamma_b$ be one of the generators, and let $x \in T$. If $x \in T_0$, then by Equation (7), $x\gamma_b^{-1}\gamma_ax^{-1}$ is of the same form. Otherwise, there is a unique path in $T_0$ connecting the two vertices of $x$, and together with $x$ this is a cycle in $T$. Denote by $\delta_t$ the elements defined in (6) for that cycle (on the vertices $1', \ldots, m'$), where $x$ is the edge connecting $1'$ and $2'$. Then $x\delta_t^{-1}\delta_{m'} \in \langle T_0 \rangle$. Let $\sigma = \phi(x\delta_t^{-1}\delta_{m'})$. Then $x\gamma_b^{-1}\gamma_ax^{-1} = \sigma(\delta_{m'}^{-1}\delta_t)(\gamma_b^{-1}\gamma_a)(\delta_t^{-1}\delta_{m'})\sigma^{-1}$, which is in the subgroup generated by the $\gamma_j^{-1}\gamma_i$ and the $\delta_t^{-1}\delta_i$. $\square$

From Equation (7) alone it is not clear how a conjugate of $\gamma_i$ by $\sigma$ looks like. We cannot expect $\sigma^{-1}\gamma_i \sigma$ to be of the form $\gamma_j$, since $\phi(\gamma_1)$ is not a central element of $S_n$. Still, from Equation (7) it follows that $\gamma_i \sigma \gamma_{\sigma i}^{-1}$ is independent of $i$, so for every $\sigma \in \langle T_0 \rangle$ there is some $\tilde{\sigma} \in C_Y(T)$ such that

\begin{equation}
\tilde{\sigma}^{-1}\gamma_a \sigma = \gamma_{\sigma a}, \quad a = 1, \ldots, n.
\end{equation}

From this it would follow that $\sigma \mapsto \tilde{\sigma}$ is a homomorphism. We will show that this is the case after the $\tilde{\sigma}$ are specified.
Equation (8) can serve as the definition of \( \tilde{\sigma} = \gamma_a \sigma \gamma_a^{-1} \) (ignoring for the moment the fact that \( \gamma_a \) is not always defined, and that we need to show the definition is independent of \( a \)). Fixing the cycle, let us compute \( \gamma_a u \gamma_a^{-1} \) for arbitrary \( u \in T_0 \) (and \( a \) of our choice). There are four cases to consider.

1. \( u \) does not touch the cycle. Then choosing \( a \) on the cycle we have that \( \gamma_a \) commutes with \( u \), so that \( \gamma_a u \gamma_a^{-1} = u \).

2. \( u \) is part of the cycle, say \( u = u_i \) (\( i = 1, \ldots, m \)). Taking \( a = i-2 \) (with \( a = m \) if \( i = 2 \) and \( a = m-1 \) if \( i = 1 \)), we see that
   \[
   \gamma_{i-2} u_{i-2} \gamma_{i-2}^{-1} = \gamma_{i-2} u_{i-2} \gamma_{i-2}^{-1} = u_1 \ldots u_m u_1 \ldots u_i u_{i-2} \ldots u_1 u_m \ldots u_i = u_{i+1} u_{i+1} u_i u_{i+1} u_i = u_{i+1},
   \]
   where for \( i = m \) the calculation gives \( u_1 \).

3. \( u \) touches the cycle at one vertex, \( i \) (\( 1 \leq i \leq m \)). Taking \( a = i-1 \), we have
   \[
   \gamma_{i-1} u_{i-1} \gamma_{i-1}^{-1} = \gamma_{i-1} u_{i-1} \gamma_{i-1}^{-1} = u_{i+1} \ldots u_m u_1 \ldots u_{i-1} u_{i-1} \ldots u_1 u_m \ldots u_{i+1} = u_{i+1} u_{i+1} u_{i+1} u_{i+1} u_{i+1} = u_{i+1} u_{i+1} u_{i+1} u_{i+1} = u_{i+1} u_{i+1} u_{i+1} u_{i+1}.
   \]
   Notice that if \( \phi(u) = (i, k) \), then \( \phi(u_{i+1} u_{i+1}) = (i+1, k) \).

4. \( u \) touches the cycle at two vertices, \( i, j \), with \( j > i + 1 \). Then \( u(i-1) = i - 1 \) and
   \[
   \gamma_{i-1} u_{i-1} \gamma_{i-1}^{-1} = u_{i+1} \ldots u_m u_1 \ldots u_{i-1} u_{i-1} \ldots u_1 u_m \ldots u_{i+1} = u_{i+1} \ldots u_{j+1} u_{j+1} \ldots u_{i+1} = u_{i+1} u_{j+1} u_{j+1} u_{i+1}.
   \]
   Here \( \phi(u) = (i, j) \) by assumption, and \( \phi(u_{i+1} u_{j+1} u_{j+1} u_{i+1}) = (i + 1, j + 1) \).

We take the results of this computation as our definition.

**Definition 4.2.** Fixing a (directed) cycle in \( T \), for every \( u \in T \) we define \( \tilde{u} \in C_Y(T) \), as follows:

\[
\tilde{u} = \begin{cases} 
  u & \text{if } u \text{ does not touch the cycle} \\
  u_{i+1} & \text{if } u = u_i \text{ on the cycle} \\
  u_{i+1} u_{i+1} & \text{if } u \text{ touches the cycle at vertex } i \text{ only} \\
  u_{i+1} u_{j+1} u_{j+1} u_{i+1} & \text{if } u \text{ touches the cycle at the vertices } i, j.
\end{cases}
\]
Remark 4.3. Let \( \tau = (m \ldots 321) \). From the definition it follows that for every \( u \in T \), \( \phi(\tilde{u}) = \tau \phi(u) \tau^{-1} \).

From now and to the rest of the paper, whenever we enumerate the edges of a cycle \( u_1, \ldots, u_m \), we will choose the enumeration so that \( u_2, \ldots, u_m \in T_0 \) (so that necessarily \( u_1 \notin T_0 \)). Under this assumption, an edge \( u \in T_0 \) cannot touch the cycle more than once. As we shall see in Section 6, this makes our cycles "basic cycles" (with respect to \( T_0 \)).

Corollary 4.4. The map \( u \mapsto \tilde{u} \) extends to a homomorphism \( \sigma \mapsto \tilde{\sigma} \) from \( \langle T_0 \rangle \) to \( C_Y(T) \).

Proof. We only need to show that \( u \mapsto \tilde{u} \) preserves the defining relations of \( \langle T_0 \rangle \cong C_Y(T_0) \). Let \( T'_0 \) be an arbitrary spanning subtree of \( T \). If \( R(u) \) is a relator on the generators of \( C_Y(T_0) \), then \( \phi(R(\tilde{u})) = R(\phi(\tilde{u})) = R(\tau \phi(u) \tau^{-1}) = \tau \phi(R(u)) \tau^{-1} = 1 \), showing that \( R(\tilde{u}) \in \text{Ker}(\phi) = K(T) \). Thus, if for all the generators \( u \) involved in a relator we have \( \tilde{u} \in \langle T'_0 \rangle \), then \( R(\tilde{u}) \in \langle T'_0 \rangle \cap K(T) = 1 \).

Let \( u, v \in T_0 \); then the subgraph of \( T \) built of the generators participating in \( \tilde{u}, \tilde{v} \) never contains a cycle. So it can be completed to a spanning subtree \( T'_0 \), showing that the relations in which \( u, v \) are involved are satisfied by \( \tilde{u}, \tilde{v} \). The same argument applies to \( u, v, w \in T_0 \) which meet in a common vertex, showing that relation (4) is also preserved, so we are done.

In Subsection 6.2 we will improve this result, and show that \( u \mapsto \tilde{u} \) actually extends to an automorphism of \( C_Y(T) \). However it should be emphasized that in general Equation (8) does not hold for every \( u \in T \).

Now that \( \tilde{\sigma} \) is defined, we can use (8) to define \( \gamma_a \) for arbitrary \( a \): let \( v_1, \ldots, v_s \) denote the path in \( T_0 \) from the vertex 1 to \( a \), and define

\[
\gamma_a = \tilde{v}_s \ldots \tilde{v}_1 \gamma_1 v_1 \ldots v_s.
\]

We first need to check that this new definition coincides with (6). Indeed, if \( a \) is on the cycle then the path from 1 to \( a \) is \( u_2, \ldots, u_a \), so

\[
\tilde{u}_a \ldots \tilde{u}_2 \gamma_1 u_2 \ldots u_a = u_{a+1} \ldots u_3 \cdot u_3 \ldots u_m u_1 \cdot u_2 \ldots u_a = u_{a+2} \ldots u_m u_1 u_2 \ldots u_a
\]

which is the previous definition. Secondly, if \( b \) is a vertex on the path from 1 to \( a \), say \( b = v_{k-1} \cap v_k \), then by definition of \( \gamma_b \), \( \gamma_a = \tilde{v}_s \ldots \tilde{v}_k \gamma_b v_k \ldots v_s \).

Remark 4.5. Recall that \( \tau = \phi(\gamma_1) = (m \ldots 321) \). For every \( a \) we have by Remark 4.3 that \( \phi(\gamma_a) = \tau \phi(v_s \ldots v_1) \tau^{-1} \phi(\gamma_1) \phi(v_1 \ldots v_s) = \tau \).

Proposition 4.6. With the above definitions of the \( \gamma_i \) and \( \tilde{u} \), Equation (8) holds for every \( \sigma \in \langle T_0 \rangle \) and \( a = 1, \ldots, n \).
Proof. It is enough to assume $\sigma = u \in T_0$, and we need to check that $\tilde{u}^{-1}\gamma_a u = \gamma_u(a)$. We use induction on the distance $s$ of $a$ from the vertex 1. First assume that $a = 1$. There are four cases to consider.

1. $u$ touches the vertex 1. Then by definition $\gamma_u(1) = \tilde{u}\gamma_1 u$.

2. $u$ is on the cycle, $u = u_i$ with $2 < i \leq m$. Then

   $\gamma_1 u\gamma_u^{-1} = u_3 \ldots u_m u_1 u_1 u_m \ldots u_3$
   $= u_3 \ldots u_i u_{i+1} u_{i+1} u_i \ldots u_3$
   $= u_3 \ldots u_{i-1} u_{i+1} u_{i-1} \ldots u_3$
   $= u_{i+1} = \tilde{u}$.

3. $u$ touches the cycle at a vertex $i$ other than 1. Then $\tilde{u} = u_{i+1} uu_{i+1}$, and

   $\gamma_1 u\gamma_u^{-1} = u_3 \ldots u_m u_1 u_1 u_m \ldots u_3$
   $= u_3 \ldots u_{i+1} u_{i+1} \ldots u_3$
   $\gamma \equiv u_{i+1} uu_{i+1} = \tilde{u}$.

4. $u$ does not touch the cycle. Then $\tilde{u} = u$ commutes with every $u_i$ on the cycle, so $\tilde{u}^{-1}\gamma_1 u = \gamma_1$.

Now suppose $a \neq 1$, and let $v_1, \ldots, v_s$ denote the path from 1 to $a$. Let $b = v_s(a)$; then $v_1, \ldots, v_{s-1}$ is the path from 1 to $b$. By the definition of $\gamma_b$ and $\gamma_a$, and the induction hypothesis, we have that

$\tilde{u}\gamma_a u = \tilde{w}\gamma_b v_s u$

$= \tilde{w}\tilde{v}_s \cdot \tilde{v}\gamma_b u \cdot uv_s u$

$= \tilde{w}\tilde{v}_s u \cdot \gamma_{u(b)} \cdot uv_s u$.

In particular, if $u$ is disjoint from $v_s$ then $\tilde{w}\tilde{v}_s \tilde{u} = \tilde{v}_s$ by Corollary 4.4, $u(b) = b$, and $\tilde{w}\gamma_a u = \tilde{v}_s\gamma_b v_s = \gamma_a$. Thus, we may assume that $u$ touches $a$ or $b$. There are two cases to check if $u$ touches $a$: either $u = v_s$, in which case $u(a) = b$ and $\tilde{u}^{-1}\gamma_a u = \tilde{v}_s\gamma_v v_s = \gamma_b = \gamma_u(a)$; or $u(a) \neq b$, where $\gamma_u(a) = \tilde{w}\gamma_a u$ by definition of $\gamma_u(a)$. Next, assume $u(a) = a$, but $c = u(b) \neq b$.

If $u = v_{s-1}$, let $b' = v_{s-1}(b)$. Then by the induction hypothesis $\tilde{w}\gamma_a u = \tilde{w}\gamma_{b'} v_s u = \tilde{w}\tilde{v}_s \gamma_{b'} v_s = \gamma_{b'} v_s = \gamma_b$. So we may assume $u$ is not part of the path $v_1, \ldots, v_s$; see Figure 1. We need to show that $\tilde{w}\gamma_a u = \gamma_a$; writing $\gamma_a = \tilde{v}_s\gamma_b v_s$, this is equivalent to $\tilde{v}_s \tilde{w}\gamma_{b'} v_s = \gamma_b$, which is symmetric under replacing $a$ and $v_s$ by $c$ and $u$.

There are three cases to consider.
(1) $a$ is on the cycle. Then $b$ is also on the cycle, but $c$ is not. Compute:
\[
\gamma_a u \gamma_{u(a)}^{-1} = u_{a+2} \ldots u_m u_1 \ldots u_a uu_a \ldots u_1 u_m \ldots u_{a+2}
\]
\[
\gamma_{u(a)}^{-1} = u_a uu_a = \tilde{u}.
\]

(2) $a, c$ are not on the cycle but $b$ is. Then
\[
\gamma_a u \gamma_{u(a)}^{-1} = \tilde{v}_s \gamma_{b} u v_s u v_s \gamma_{b}^{-1} \tilde{v}_s
\]
\[
\gamma_{b}^{-1} = \tilde{v}_s u v_s u v_s \gamma_{b}^{-1} \tilde{v}_s
\]
\[
\gamma_{b}^{-1} = u_{b+1} v_s u_{b+1} v_s u v_s \cdot u_{b+1} v_s u_{b+1}
\]
\[
\gamma_{b}^{-1} = u_{b+1} v_s u v_s u v_s u_{b+1} = u_{b+1} uu_{b+1} = \tilde{u}.
\]

(3) $b$ is not on the cycle. Let $w_1, \ldots, w_p$ be the path from a vertex $j$ on the cycle to $b$ (so that $w_p = v_{s-1}$).
\[
\gamma_a u \gamma_{u(a)}^{-1} = \tilde{v}_s \tilde{w}_p \ldots \tilde{w}_1 \gamma_j w_1 \ldots w_p v_s u v_s \gamma_{j-1} \tilde{w}_1 \ldots \tilde{w}_p \tilde{v}_s
\]
\[
\gamma_{j-1} = \tilde{v}_s \tilde{w}_p \ldots \tilde{w}_1 \gamma_j v_s u v_s \gamma_{j-1} \tilde{w}_1 \ldots \tilde{w}_p \tilde{v}_s
\]
\[
\gamma_{j-1} = \tilde{v}_s \tilde{w}_p v_s u v_s \tilde{w}_p \tilde{v}_s
\]
\[
\gamma_{j-1} = v_s \tilde{w}_p v_s u v_s \tilde{w}_p v_s
\]
\[
\gamma_{j-1} = v_s v_s u v_s v_s = u = \tilde{u}.
\]

Multiplying $\tilde{\sigma}^{-1} \gamma_i \sigma = \gamma_{\sigma i}$ and the inverse $\sigma^{-1} \gamma_{j}^{-1} \tilde{\sigma} = \gamma_{\sigma j}^{-1}$, we now get
Corollary 4.7. Equation (7) holds for every $\sigma \in \langle T_0 \rangle$.

We conclude with the following properties of the $\gamma_i$.

Proposition 4.8. Let $\tau = \phi(\gamma_1)$. For every $j = 1, \ldots, n$, we have that

$$
\gamma_j \gamma_m = \gamma_1 \gamma_{\tau(j)}.
$$

Proof. Since $\gamma_m = u_2 \ldots u_m \in \langle T_0 \rangle$, we have that $\tilde{\gamma}_m = \tilde{u}_2 \ldots \tilde{u}_m = u_3 \ldots u_m u_1 = \gamma_1$. Set $\sigma = \gamma_m$ in Equation (8), to get $\gamma_1^{-1} \gamma_j \gamma_m = \gamma_{\tau(j)}$ for arbitrary $j$. \hfill \Box

Similarly, we have

Proposition 4.9. For every $i, j$ on the cycle, we have

$$
\gamma_j \gamma_{i-1} = \gamma_i \gamma_{j-1}.
$$

Proof. The equality holds for $i = 1$ by the last proposition. But this is a relation on the generators $u_1, \ldots, u_m$, so we are done by renaming the edges (it can also be proved by direct computation). \hfill \Box

From Proposition 6.8 it follows that $\gamma_j \gamma_{\tau(i)} = \gamma_i \gamma_{\tau(j)}$ for any $i, j$, but we will not need this here.

5. Two abstract groups

In this section we describe two abstract groups, $A_{t,n}$ and $F_{t,n}$, and show that for all the interesting pairs $(t, n)$ (with one exception), they are isomorphic. In the next section we will prove that the kernel $\text{Ker}(\phi : C_Y(T) \to S_n)$, denoted by $K(T)$, is isomorphic to $A_{t,n}$ (for a certain $t$ depending on $T$). The isomorphism from $K(T)$ to $A_{t,n}$ is rather natural, but $A_{t,n}$ is better understood through its identification with $F_{t,n}$ (which is an explicit subgroup of the direct power of a free group). For reasons that will later become clear, if $n = 4$ we are only interested in the case $t \leq 3$, and if $n < 4$ in the case $t \leq 1$. The definitions are, however, general.

5.1. The group $A_{t,n}$. Let $t \geq 0$, $n \geq 1$. Fix a set $X = \{x, y, z, \ldots\}$ of size $t$.

Definition 5.1. The group $A_{t,n}$ is generated by the $n^2 |X|$ elements $x_{ij}$ $(x \in X, i, j = 1, \ldots, n)$, with the defining relations

$$
x_{ii} = 1, \tag{12}
x_{ij} x_{jk} = x_{ik}, \tag{13}
x_{jk} x_{ij} = x_{ik}, \tag{14}
$$
and
\[(15) \quad [x_{ij}, y_{kl}] = 1 \quad \text{if } i, j, k, l \text{ are different.}\]

We will sometimes use \(A_{X,n}\) to specify the set of generators of \(A\); this is the same group as \(A_{t,n}\) for \(t = |X|\).

**Remark 5.2.** For every fixed \(x \in X\), the subgroup \(<x_{ij}>\) < \(A_{t,n}\) is commutative.

**Proof.** First, \(x_{ji} = x_{ij}^{-1}\) by (12) and (13). For every \(i, j\) we have \(x_{1i}x_{1j} = x_{ji}x_{ij}\) by (13),(14), so \(x_{1j}x_{1i} = x_{1i}x_{1j}\) and \(<x_{1i}>\) is commutative. Now \(x_{ij} = x_{1i}^{-1}x_{1j}\), and we are done. \(\square\)

**Example 5.3.** For \(t = 1\), \(A_{1,n}\) is generated by a single set \(\{x_{ij}\}\) and is commutative by Remark 5.2. Relation (15) is then redundant, so \(x_{i,i+1}\) are free generators, and \(A_{1,n} \cong \mathbb{Z}^{n-1}\).

**Proposition 5.4.** If \(n \geq 5\), or if \(t \leq 2\), then \(A_{t,n}\) satisfies the following relation:
\[(16) \quad [x_{ij}, z_{ik}y_{kl}z_{ki}] = 1 \quad \text{if } i, j, k, l \text{ are different.}\]

**Proof.** First assume \(n \geq 5\), so we can choose \(r \neq i, j, k, l\). Using the decomposition \(z_{ik} = z_{ir}z_{rk}\) and relation (15), we compute
\[
[x_{ij}, z_{ik}y_{kl}z_{ki}] = x_{ij}z_{ik}y_{kl}z_{ki}x_{ji}z_{ik}y_{lk}z_{ki} = x_{ij}z_{ir}z_{rk}y_{kl}z_{ki} = z_{rk}x_{ij}y_{kl}z_{ir}z_{ki} = z_{rk}x_{ij}y_{kl}z_{ir}z_{kr} = 1
\]

Now suppose \(t = |X| \leq 2\). If in (16) we set \(z = y\) or \(z = x\) the assertion is trivial, so assume \(x = y\), and compute:
\[
[x_{ij}, z_{ik}x_{kl}z_{ki}] = x_{ij}z_{ik}x_{kl}z_{ki}x_{ji}z_{ik}x_{lk}z_{ki} = x_{ij}z_{ik}x_{kj}x_{ji}z_{kl}z_{ki} = x_{ij}x_{ji}z_{ik}x_{kj}x_{ji}z_{kl}z_{ik}x_{lk}z_{ki} = x_{il}z_{lk}x_{kj}x_{ji}z_{il}x_{lk}z_{ki} = x_{il}x_{ji}z_{lk}z_{il}x_{kj}x_{lk}z_{ki} = x_{ji}z_{ik}x_{lj}z_{ki} = 1
\]

Essentially the same proof shows that under the same assumptions,
\[(17) \quad [x_{ij}, z_{ki}y_{kl}z_{ik}] = 1 \quad \text{if } i, j, k, l \text{ are different.}\]

also holds in \(A_{t,n}\).

The next proposition will be useful in Section 7.
Proposition 5.5. $A_{t-1,n}$ is a subgroup (in fact a retract) of $A_{t,n}$.

Proof. Let $X' \subset X$ be sets of sizes $t - 1$ and $t$, respectively, and let $w$ denote the element in $X - X'$. Define $\rho' : A_{X',n} \to A_{X,n}$ by $x_{ij} \mapsto x_{ij}^w$ for $x \in X'$, and $\rho : A_{X,n} \to A_{X',n}$ by $\rho(x_{ij}) = x_{ij}$ for $x \in X'$, and $\rho(w_{ij}) = 1$. Since all the relations are preserved, both maps are well defined.

For every generator $x_{ij}$ of $A_{X',n}$ ($x \in X'$, $1 \leq i, j \leq n$), the maps satisfy $\rho'(x_{ij}) = \rho(x_{ij}) = x_{ij}$, so $\rho'$ is the identity on $1_{A_{X',n}}$. In particular $\rho'$ is one-to-one, injecting $A_{X',n}$ onto its image, the subgroup $\langle x_{ij} : x \in X' \rangle$ of $A_{X,n}$. \hfill \square

5.2. The group $F_{t,n}$. Again let $X$ be a set of size $t \geq 0$. For every $i = 1, \ldots, n$, let $F^{(i)}$ denote the free group on the $t$ letters $x_i$ ($x \in X$). Set $F_{t,n}^* = F^{(1)} \times \cdots \times F^{(n)}$. Then $F_{t,n}^*$ has generators $x_i$ ($x \in X$, $i = 1, \ldots, n$) and defining relations

$$[x_i, y_j] = 1 \quad (x, y \in X, i \neq j).$$

The Abelianization map $ab : F^{(i)} \to \mathbb{Z}^t$ defined on every component by $ab(x_i) = x$ (where $\mathbb{Z}^t$ is thought of as the free Abelian group generated by $X$), can be extended to a map $F_{t,n}^* \to \mathbb{Z}^t$ by summing the entries:

$$ab(p_1, \ldots, p_n) = ab(p_1) + \cdots + ab(p_n).$$

Definition 5.6. $F_{t,n}$ is the kernel of $ab : F_{t,n}^* \to \mathbb{Z}^t$.

Theorem 5.7. $A_{t,n} \cong F_{t,n}$ in the following cases: $t \leq 1$, or $n \geq 5$, or $n = 4$ and $t = 2$.

Moreover, if $n = 4$, then $F_{t,n}$ is a quotient of $A_{t,n}$.

Proof. If $t = 0$ then both groups are trivial. If $t = 1$ then $A_{t,n} = \mathbb{Z}^{n-1}$, and $F_{t,n}^* = \mathbb{Z}^n$ by definition, so $F_{t,n} = \mathbb{Z}^{n-1}$. We may thus assume $n \geq 4$.

Define a group $A_{t,n}^*$ with generators $x_{ij}$ and $s_x$ ($x \in X$, $i, j = 1, \ldots, n$), and defining relations (12)–(15) and the following:

$$[s_x, y_{ij}] = [x_{nk}, y_{ij}] \quad (x, y \in X, k \neq i, j),$$

$$[s_x, s_y] = [x_{ni}, y_{nj}] \quad (i \neq j, i, j \neq n).$$

Define a map $\mu : A_{t,n}^* \to F_{t,n}^*$ as follows:

$$\mu(x_{ij}) = x_j^{-1}x_i,$$

$$\mu(s_x) = x_n.$$

A routine check shows that $\mu$ respects all the relations (12), (13), (14), (15), (20) and (21), so is well defined.
Also, \( \mu(s_xx) = x_i \), so \( \mu \) is onto \( F_{t,n}^* \). Since all the relations of \( A_{t,n} \) are assumed to hold in \( A_{t,n}^* \), \( x_{ij} \mapsto x_{ij} \) defines a map \( \rho : A_{t,n} \to A_{t,n}^* \). Also define \( ab : A_{t,n}^* \to \mathbb{Z}^l \) by \( ab(x_{ij}) = 1 \) and \( ab(s_x) = x \).

Now define \( \mu' : F_{t,n}^* \to A_{t,n}^* \) by

\[
\mu'(x_i) = s_xx_i
\]

(in particular we have \( \mu'(x_n) = s_xx_{nn} = s_x \)). The fact that (18) is preserved by \( \mu' \) for \( x = y \), follows from Remark 5.2. If \( x \neq y \), choose \( k \neq j, n \) and \( l \neq i, k, n \) (this requires \( n \geq 4 \)) and compute:

\[
[s_xx_i, s_yy_j] = s_xx_is_yy_js_xy_is_x^{-1}y_{nj}y_s^{-1} = x_is_yy_js_xy_is_x^{-1}y_{nj}
\]

\[
\overset{(20)}{=} x_inx_ky_jn_x_kn_s_xy_is_x^{-1}y_{ni}y_is_y^{-1}y_{nj}
\]

\[
\overset{(20)}{=} x_inx_ky_jn_x_kn_s_xy_is_x^{-1}s_y^{-1}y_{ni}y_{ni}y_{nj}y_{nj}
\]

\[
\overset{(21)}{=} x_inx_ky_jn_x_kn_x_kn_s_xy_is_x^{-1}s_y^{-1}y_{ni}y_{ni}y_{nj}y_{nj}y_{nj}y_{nj}
\]

\[
\overset{(12)-(14)}{=} x_{ii}y_{ij}x_{ik}y_{jl}
\]

\[
\overset{(15)}{=} 1.
\]

This proves that \( \mu' \) is well defined. Moreover, it is easy to check that \( \mu\mu' = 1_{F_{t,n}}\) and \( \mu'\mu = 1_{A_{t,n}^*} \). We also use \( \mu \) to denote the map \( A_{t,n} \to F_{t,n} \) defined by \( \mu(x_{ij}) = x_j^{-1}x_i \). Then the following diagram commutes:

\[
\begin{array}{ccc}
A_{t,n} & \xrightarrow{\rho} & A_{t,n}^* \\
\downarrow{\mu} & & \downarrow{\mu'} \\
1 & \xrightarrow{ab} & \mathbb{Z}^l & \xrightarrow{1}
\end{array}
\]

It is easy to see that as a normal subgroup of \( F_{t,n}^* \), \( F_{t,n} \) is generated by the elements \( x_i^{-1}x_n \) and \([x_n, y_n] (x, y \in X) \). Thus, applying \( \mu' \), we see that \( \mu'(F_{t,n}) \) is generated by the elements \( x_{ni} \) and \([s_x, s_y] = [x_{ni}, y_{nj}] \); but these are the generators of \( \text{Im}(\rho) \), showing that \( F_{t,n} \) and \( \text{Im}(\rho) \) are isomorphic.

From this it follows that \( F_{t,n} \) is a quotient group of \( A_{t,n} \). In the next proposition we show that \( \rho : A_{t,n} \to A_{t,n}^* \) is injective if \( n \geq 5 \) or \( t \leq 2 \), thus completing the proof.

We now show that if \( n \geq 5 \) or \( t \leq 2 \), then the mapping \( \rho : A_{t,n} \to A_{t,n}^* \) defined by \( \rho(x_{ij}) = x_{ij} \) is an embedding. Fix an arbitrary order on \( X \). For \( z \in X \), let \( A_{z}^* \) denote the group generated by \( A_{t,n} \) and the elements \( s_x (x \leq z) \), subject to the relations (20)–(21). Of course,
when defining $A^*_z$, the range of the variables $x$ and $y$ in these relations is $x, y < z$ (rather than all $X$). Thus, if $z$ is the maximal element of $X$, we have that $A^*_z = A^*_{t,n}$.

**Proposition 5.8.** Assume $n \geq 5$ or $t \leq 2$.

If $u < z$ are consecutive elements of $X$, then $A^*_z$ is a semidirect product of $A^*_u$ and $\mathbb{Z} = \langle s_z \rangle$. In particular, $A_{t,n}$ is the subgroup of $A^*_{t,n}$ generated by the $x_i$.

**Proof.** Let us define an automorphism of $A^*_u$ by

\[
\begin{align*}
\tilde{s}_z(x_{ij}) &= z_{nk}x_{ij}z_{nk}^{-1} \quad (x \in X, \ k \neq i, j), \\
\tilde{s}_z(s_x) &= [z_{ni}, x_{nj}]s_x \quad (x < z, \ i \neq j, \ i, j \neq n).
\end{align*}
\]

We must first verify that the definition does not depend on the choice of $k$ (in the first case), or $i, j$ (in the second). The equality $z_{nk}x_{ij}z_{nk}^{-1} = z_{nk'}x_{ij}z_{nk'}^{-1}$ (for $k, k' \neq i, j$) follows from (15). Now suppose that $i, j, n$ are different, and that $i', j', n$ are different. If $i' \neq j$ then it is easy to check that $[z_{ni}, x_{nj}] = [z_{ni'}, x_{nj'}] = [z_{ni'}, x_{nj'}]$; likewise the equality follows if $i \neq j'$. In order to show that $[z_{ni}, x_{nj}] = [z_{nj}, x_{ni}]$, choose $k \neq i, j, n$; then $[z_{ni}, x_{nj}] = [z_{ni}, x_{nk}] = [z_{nj}, x_{nk}] = [z_{nj}, x_{ni}]$.

Next, we need to show that $\tilde{s}_z$ respects the defining relations of $A^*_u$: (12)–(15), and the restricted versions of (20)–(21). Relation (12) holds trivially. For (13) and (14), choose $l \neq i, j, k$. Then $\tilde{s}_z$ acts on $x_{ij}, x_{jk}, x_{ik}$ as conjugation by $z_{nl}$, so the equality is preserved. In order to prove that $\tilde{s}_z$ preserves (15) (for certain $i, j, k, l$), we choose to write $\tilde{s}_z(x_{ij}) = z_{nk}x_{ij}z_{nk}^{-1}$ and $\tilde{s}_z(y_{kl}) = z_{ni}y_{kl}z_{ni}^{-1}$. Then we have $[\tilde{s}_z(x_{ij}), \tilde{s}_z(y_{kl})] = [z_{nk}x_{ij}z_{nk}^{-1}, z_{ni}y_{kl}z_{ni}^{-1}] = z_{nk}[x_{ij}, z_{nk}y_{kl}z_{ki}]z_{nk}^{-1} = 1$ by (16) (which holds under our assumptions on $t, n$). In order to check that

\[
\tilde{s}_z(s_x)\tilde{s}_z(y_{ij})\tilde{s}_z(s_x)^{-1} = \tilde{s}_z(x_{nk})\tilde{s}_z(y_{ij})\tilde{s}_z(x_{nk})^{-1}
\]

(where $x < z, y \in X$ is arbitrary and $k \neq i, j$), we need to choose parameters for the action of $\tilde{s}_z$ on various generators. We may assume that $i \neq j$, so for example $j \neq n$. Then, write $\tilde{s}_z(s_x) = [z_{nk}, x_{nj}]s_x$, $\tilde{s}_z(y_{ij}) = z_{nk}y_{ij}z_{kn}$, and $\tilde{s}_z(x_{nk}) = z_{nj}x_{nk}z_{jn}$. Then the equality becomes

\[
[z_{nk}, x_{nj}]s_xz_{nk}y_{ij}z_{kn}s_x^{-1} = x_{nj}z_{nk}x_{jn} \quad \text{and} \quad s_xy_{ij}s_x^{-1} = x_{nk}y_{ij}x_{kn},
\]

so the equation becomes

\[
z_{jk}x_{nk}y_{ij}x_{kn}z_{kj} = x_{nk}z_{jk}y_{ij}z_{kj}x_{kn}.
\]
There are two cases to consider. If $i = n$ then choose $r \neq j, k, n$, and compute that

$$z_{jk}x_{nk}y_{ij}x_{kn}z_{kj} = \frac{z_{jr}z_{rk}x_{nk}y_{ij}x_{kn}z_{kr}z_{rj}}{z_{jr}x_{nk}y_{ij}x_{kn}z_{jr}} = \frac{x_{nk}z_{jr}y_{ij}z_{rj}x_{kn}}{x_{nk}z_{jr}y_{ij}z_{rj}x_{kn}} = \frac{x_{nk}z_{jr}y_{ij}z_{kr}z_{rj}x_{kn}}{x_{nk}z_{jr}y_{ij}z_{kr}z_{rj}x_{kn}}. \tag{17}$$

If $i \neq n$ then $x_{kn}$ commute with $y_{ij}$, and the equality is equivalent to $z_{jk}[y_{ij}, z_{kj}x_{nk}]z_{kj} = 1$, which again holds by (17).

Finally, we need to check (21), where $x, y < z$, that is, to show that $[s_z(s_x), s_z(s_y)] = [s_z(x_{ni}), s_z(y_{nj})]$. Let $k \neq i, j, n$. We choose to write $s_z(s_x) = [z_{nk}, x_{ni}]s_x, s_z(s_y) = [z_{nk}, y_{nj}]s_y, s_z(x_{ni}) = z_{nk}x_{ni}z_{kn}$, and $s_z(y_{nj}) = z_{nk}y_{nj}z_{kn}$, so that $s_z([x_{ni}, y_{nj}]) = z_{nk}[x_{ni}, y_{nj}]z_{kn}$.

Now the equation becomes

$$[z_{nk}, x_{ni}]s_x[z_{nk}, y_{nj}]s_ys_x^{-1}[x_{ni}, z_{nk}]s_y^{-1}[y_{nj}, z_{nk}] = z_{nk}[x_{ni}, y_{nj}]z_{kn}.\tag{18}$$

Letting $s_x$ act as conjugation by $x_{ni}$ on $z_{nk}, y_{nj}$, and $s_y$ as conjugation by $y_{nj}$ on $z_{nk}, x_{ni}$, we get

$$z_{nk}x_{ni}[z_{nk}, y_{nj}][x_{ni}, z_{nk}]y_{nj}[y_{nj}, z_{nk}] = z_{nk}[x_{ni}, y_{nj}]z_{kn},\tag{19}$$

which is equivalent (using only (12)–(14)) to

$$y_{nj}x_{ni}[s_x, s_y]y_{nj}x_{ni} = 1.\tag{20}$$

This is relation (21), so we are done.

Now that $s_z$ is well defined, we are done by noting that $A^*_u$ is generated from $A^*_u$ by adding one generator, $s_z$, and the relevant portion of relations (20)–(21), which can be rewritten in the form $s_zws_z^{-1} = s_z(w)$ for $w = x_{ij}$ ($x \in X$) or $w = s_x$ ($x < z$). But these are precisely the relations defining the semidirect product of $\mathbb{Z} = \langle s_z \rangle$ acting on $A^*_u$ via the automorphism $s_z$.

5.3. Action of $S_n$. The symmetric group $S_n$ naturally acts on $F_{t,n}^* = (\mathbb{F}_t)^n$ by the action on indices:

$$\sigma^{-1}x_i\sigma = x_{\sigma(i)}. \tag{22}$$

Likewise, it acts on $A_{t,n}$ by

$$\sigma^{-1}x_{ij}\sigma = x_{\sigma_{i},\sigma_{j}}. \tag{23}$$

(recall that $(\sigma\tau)(i) = \tau(\sigma(i))$).
Remark 5.9. Under the assumptions of Theorem 5.7, the isomorphism $F_{t,n} \cong A_{t,n}$ carries $x_{ij}$ to $x_{j}^{-1}x_{i} \in F_{t,n}$, so it agrees with the actions of $S_{n}$ on both groups. In particular the resulting semidirect products $S_{n} \ltimes F_{t,n}$ and $S_{n} \ltimes A_{t,n}$ are isomorphic.

5.4. Some identities in $A_{t,n}$. The identities we prove here are most easily derived from the isomorphism $A_{t,n} \cong F_{t,n}$, when it holds. However, since we also want to cover the case $n = 4$ and $t = 3$, our proofs use direct computation in $A_{t,n}$.

Remark 5.10. Choose any $x \in X$. The map fixing $y_{ij}$ for every $y \neq x$ and sending $x_{ij} \mapsto x_{ji}$ is an automorphism of $A_{t,n}$.

Proof. This map preserves the relations (12) and (15), and switches (13) and (14). □

Proposition 5.11. Suppose $n \geq 4$. For every $x, y \in X$ and distinct $i, j, k$, the following relations hold in $A_{t,n}$.

\begin{align*}
(24) \quad x_{jk}y_{ki}x_{ij} &= y_{ji}x_{ik}y_{kj} \\
(25) \quad x_{jk}y_{jk}x_{ij} &= y_{ij}x_{ik}y_{jk} \\
(26) \quad x_{kj}y_{ik}x_{ji} &= y_{ij}x_{ki}y_{jk}
\end{align*}

Proof. Let $r \neq i, j, k$, and compute the quotient of the left hand and right hand sides in (24):

$\frac{x_{jk}y_{ki}x_{ij}}{y_{ji}x_{ik}y_{kj}} - 1 = x_{jk}y_{ki}x_{ij}y_{jk}x_{ki}y_{ij}$

$= x_{jk}y_{ki}x_{ir}x_{rj}y_{jk}x_{ki}y_{ij}$

$= x_{jk}x_{rj}y_{ki}y_{jk}x_{ji}x_{ki}y_{ij}$

$= x_{rk}y_{ji}x_{kr}y_{ij} = 1$.

The other two relations are obtained by inverting $x_{rs}$, or $x_{rs}$ and $y_{rs}$, using Remark 5.10. □

Proposition 5.12. Suppose $n \geq 5$ or $t \leq 2$.

If $u, v, w \in X$ and $i, j, k, s$ are distinct, we have that

\begin{equation}
(27) \quad u_{si}v_{ij}u_{js}w_{sk} = w_{sk}u_{ki}v_{ij}u_{jk}.
\end{equation}

Proof. This identity is equivalent to Equation (16), as the following computation shows.

\begin{align*}
1 = u_{si}v_{ij}u_{js}w_{sk}u_{kj}v_{ji}u_{ik}w_{ks} &= u_{si}v_{ij}u_{js}w_{sk}(u_{kj}v_{ji}u_{jk})u_{ij}w_{ks} \\
&= u_{si}v_{ij}u_{js}w_{sk}u_{kj}v_{ji}u_{jk}w_{sk}u_{ij}w_{ks} \\
&= u_{si}v_{ij}u_{ks}v_{ji}u_{jk}w_{sk}u_{ij}w_{ks} \\
&= u_{si}u_{ks}u_{jk}u_{ij} = 1.
\end{align*}

□
6. The group $C_Y(T)$

6.1. The main result. In this subsection we prove the following

**Theorem 6.1.** Let $T$ be a connected graph on $n$ vertices. Then $C_Y(T)$ is isomorphic to the semidirect product $G = S_n \ltimes A_{t,n}$, where $t$ is the rank of the free group $\pi_1(T)$.

Recall that $C_Y(T)$ was defined in Definition 2.4, the group $A_{t,n}$ in Subsection 5.1, and the action of $S_n$ on $A_{t,n}$ in Subsection 5.3.

The proof is rather direct: we define maps $\Phi : C_Y(T) \rightarrow S_n \ltimes A_{t,n}$ and $\Psi : S_n \ltimes A_{t,n} \rightarrow C_Y(T)$, show that they are well defined, and check that they invert each other. Showing that $\Psi$ is well defined is the longest part of the proof, relying on Corollary 4.7.

Choose a spanning tree $T_0 \subseteq T$, and let $X = T - T_0$. Arbitrarily fix a direction for every edge $x \in X$, so $x$ has a starting point $a$ and an ending point $b$. There is a unique path in $T_0$ from $a$ to $b$, and together with $x$ this path forms a cycle in $T$, called a basic cycle (this is the unique cycle in $T_0 \cup \{x\}$). The set of basic cycles is a free set of generators for the fundamental group $\pi_1(T)$. Let $t = |X|$ be the number of basic cycles. We will always use $X$ as the set of size $t$ indexing the generators of $A_{t,n}$.

Note that the number $t$ of basic cycles in a graph on $n = 4$ vertices is bounded by $t \leq 3$, and if $n = 3$ there is at most one cycle. If $t = 0$ Theorem 6.1 is vacuous (since $A_{0,n} = 1$ and $C_Y(T) = S_n$ by Corollary 2.5), so we may assume $t \geq 1$ and $n \geq 3$. It follows that the only case where Theorem 5.7 does not apply is where $n = 4$ and $t = 3$, which is the case iff $T$ is the complete graph $K_4$ on four vertices (this is the only graph on 4 vertices with 3 independent cycles).

**Definition 6.2** (Definition of $\Phi : C_Y(T) \rightarrow S_n \ltimes A_{t,n}$). Let $u \in T$ be an edge from $a$ to $b$. We set

$$
\Phi(u) = \begin{cases} 
(ab) & \text{if } u \in T_0, \\
(ab)u_{ab} & \text{if } u \in X.
\end{cases}
$$

**Proposition 6.3.** The map $\Phi$ is well defined on $C_Y(T)$.

*Proof.* We need to check the relations (1)–(4). If $u \in T_0$ then $\Phi(u)^2 = (ab)^2 = 1$, and for $u \in X$ we have $\Phi(u) = (ab)u_{ab}(ab)u_{ab} = (ab)^2u_{ba}u_{ab} = u_{aa} = 1$ by (12), so Equation (1) is preserved.

Let $u, v$ be two disjoint edges. If both belong to $T_0$ then obviously $\Phi(u)$ and $\Phi(v)$ commute. If $u = (a, b) \in T_0$ and $v = (c, d) \in X$, then $\Phi(u) = (ab)$ commutes with $\Phi(v) = (cd)x_{cd}$ since $(ab)$ acts trivially on
$x_{cd}$. And if $u, v \in X$, $(ab)u_{ab}$ commutes with $(cd)v_{cd}$ by (15). Thus (2) is preserved.

Let $u, v$ be two intersecting edges, e.g. $u = (a, b)$ and $v = (b, c)$. If $u, v \in T_0$ then Equation (3) is satisfied by the transpositions $(ab), (bc)$.

If $u \in T_0$ and $v \in X$ then $\Phi(u)\Phi(v) = (ab)(bc)v_{bc} = (cba)v_{bc}$ and
\[
\Phi(u)\Phi(v) = (cbac)v_{bc} = (cba)^2v_{bc} = v_{cb}v_{bc} = 1.
\]

The last case to check is when $u, v \in X$ (this can only happen if $t \geq 2$, so we assume $n \geq 4$). Then $\Phi(u) = (ab)u_{ab}$ and $\Phi(v) = (bc)v_{bc}$ so we have $\Phi(u)\Phi(v)\Phi(u) = (ab)u_{ab}(bc)v_{bc}(ab)u_{ab} = (ac)u_{ac}u_{ab}$ and $\Phi(v)\Phi(u)\Phi(v) = (bc)v_{bc}(ab)u_{ab}(bc)v_{bc} = (ac)v_{ac}u_{ac}v_{bc}$, and the equality follows from (25). Here we assumed that the ending point of $u$ is the starting point of $v$. The other possible ways to direct $u, v$ are checked similarly using (24) and (26), so Equation (3) is verified.

Finally, we need to check that $\Phi$ respects (4). Suppose $u, v, w \in T$ meet in the same vertex $s$, which $u$ connects to $i$, $v$ to $j$ and $w$ to $k$. If $u, v \in T_0$ then we are done since $(is)(js)(is) = (ji)$ commutes with $(ks)$ and $(ks)w_{ks}$. If $u \in T_0$ and $v \in X$ then $\Phi(u)\Phi(v)\Phi(u) = (is)(js)v_{js}(is) = (ji)v_{ji}$ which commutes with $(ks)$ and $(ks)w_{ks}$ by Relations (15) and (23). The last case to consider is when $u, v, w \in X$. This cannot happen if $n \leq 3$, or if $n = 4$ and $t = 3$, for in this case $T = K_4$ and the complement of $u, v, w$ would be a triangle, not containing a spanning subtree. So we may assume that $n \geq 5$, and then Proposition 5.4 applies. Then $\Phi(u)\Phi(v)\Phi(u) = (ij)u_{sj}v_{ji}u_{is}$, and
\[
[\Phi(u)\Phi(v)\Phi(u), \Phi(w)] = (ij)u_{sj}v_{ji}u_{is}(ks)w_{ks}(ij)u_{sj}v_{ji}u_{is}(ks)w_{ks} = u_{si}u_{sj}v_{js}w_{ks}u_{jk}v_{ji}u_{ik}w_{ks} = 1
\]
by Equation (27).

\[
\square
\]

Remark 6.4. By definition, the restriction of $\Phi : C_Y(T) \to S_n \rtimes A_{t,n}$ to the subgroup $(T_0)$ is $\Phi(u) = (ij)$ (where $i, j$ are the vertices of $u$), so $\Phi$ is an extension of the isomorphism $\phi_0 : (T_0) \to S_n$ of Proposition 3.2.

Definition 6.5 (Definition of $\Psi : S_n \rtimes A_{t,n} \to C_Y(T)$). We define $\Psi$ on $S_n$ to invert $\Phi : \Psi(\sigma)$ is the unique element of $(T_0)$ which $\Phi$ takes to $\sigma$.

If $x \in X$, let $u_2, \ldots, u_m \in T_0$ be the path connecting the vertices of $x$; label the vertices on the cycle so that $u_i$ connects $i - 1, i$ (and $x$ connects 1, $m$), and define $\gamma_i, i = 1, \ldots, n$, as in (9) (extending the definition (6)). Then define (for $i, j = 1, \ldots, n$)
\[
\Psi(x_{ij}) = \gamma_j^{-1}\gamma_i.
\]

Proposition 6.6. The map $\Psi$ is well defined on $S_n \rtimes A_{t,n}$.
Proof. As \( \Psi \) is well defined on \( S_n \) (see Remark 6.4), we only need to check the defining relations: (12)–(15) and (23). Observe that (12) and (14) are trivially preserved. Next, we already proved (23) in Corollary 4.7. This equation is very useful, as it allows us to carry 'local' proofs (for indices of our choice) to the general case.

In order to check Equation (13), we first prove that \( \Psi(x_{1m}) = \gamma_m^{-1}\gamma_1 \) and \( \Psi(x_{2m}) = \gamma_m^{-1}\gamma_2 \) commute. Recall from Equation (11) that if \( i,j \) are on the cycle, we have \( \gamma_i \gamma_j^{-1} = \gamma_{j+1}^{-1}\gamma_{i+1} \). Now compute:

\[
[\gamma_m^{-1}\gamma_1, \gamma_m^{-1}\gamma_2] = \gamma_m^{-1}\gamma_1\gamma_m^{-1}\gamma_2\gamma_1^{-1}\gamma_m\gamma_2^{-1}\gamma_m \\
= \gamma_m^{-1}\gamma_1\gamma_2\gamma_3\gamma_1^{-1}\gamma_3^{-1}\gamma_1\gamma_m \\
= \gamma_m^{-1}\gamma_1\gamma_2\gamma_3^{-1}\gamma_3^{-1}\gamma_1\gamma_m = 1.
\]

Now, given any distinct \( i,j,k \), choose a permutation \( \sigma \) which carries \( 1, m, 2 \) to \( i, j, k \); then using (14) we have

\[
\Psi(x_{ij})\Psi(x_{jk}) = \sigma^{-1}\Psi(x_{1m})\Psi(x_{2m})\sigma \\
= \sigma^{-1}\Psi(x_{2m})\Psi(x_{1m})\sigma = \sigma^{-1}\Psi(x_{12})\sigma = \Psi(x_{ik}).
\]

If \( t = 1 \) then Equation (15) is trivial, so we are done in this case. Using Proposition 6.7 and Corollary 6.9 (which are independent of the current proposition), the proof of Theorem 6.1 itself is complete if \( t = 1 \). This will be used instead of a lengthy case analysis below.

We will now show that \( \Psi \) respects Equation (15). Let \( x \neq y \in X \). Let \( \gamma_i \) be defined as usual with respect to \( x \), and similarly define \( \gamma_i' \) for \( y \) (we label the vertices of the basic cycle of \( x \) as \( 1, \ldots, m \) for \( x \), and the vertices of the basic cycle of \( y \) as \( 1', \ldots, m' \)). Let \( g_x = \gamma_m^{-1}\gamma_1x \) and \( g_y = \gamma_m^{-1}\gamma_1'y \); by definition of the \( \gamma_i, \gamma_i' \), we have that \( g_x, g_y \in \langle T_0 \rangle \). First assume \( x, y \) are disjoint, so the endpoints \( 1, m \) of \( x \) and \( 1', m' \) of \( y \) are distinct. Let \( T_1 = T_0 \cup \{ x \} \). Working in the abstract group \( C_X(T_1) \), we see that \( x \) commutes with \( g_y \) (since their images under the appropriate \( \Psi \) are \( (1m)x_{1m} \) and \( (1'm') \)). By Remark 3.1, \( x \) commutes with \( g_y \) in \( \langle T_0, x \rangle \leq C_X(T) \). Similarly, \( y \) commutes with \( g_x \). Moreover, \( g_x \) commutes with \( g_y \), as they correspond to disjoint transpositions in \( \langle T_0 \rangle = S_n \), and finally \( x, y \) commute by assumption. Since \( g_x, x \) commute with \( g_y, y \), we have that \( \Psi(x_{1m}) = \gamma_m^{-1}\gamma_1 = g_xx \) commutes with \( \Psi(y_{1'm'}) = g_yy \). Conjugating by arbitrary \( \sigma \in S_n \), we have that \( \Psi(x_{ij}) \) commutes with \( \Psi(y_{ki}) \) for arbitrary disjoint \( i, j, k, l \), which is what we need.

Now assume that \( x, y \) touch each other; let \( 1, m \) denote the endpoints of \( x \), and \( 1, m' \) the endpoints of \( y \). Consider the paths in \( T_0 \) connecting \( 1 \) to \( m \) and \( m' \). Since \( T_0 \) is a tree, their intersection is connected; we denote by \( u_2, \ldots, u_m \) the path from \( 1 \) to \( m \), and by \( v_2, \ldots, v_{m'} \) the path
from 1 to $m'$ (the two lists coincide at the beginning). See Figure 2.

As always, denote by $m - 1$ the other vertex of $u_m$. Exchanging $x, y$ if necessary, we may assume that $u_m$ is not in $\{v_2, \ldots, v_{m'}\}$. In particular $m - 1 \neq m'$.

![Figure 2](image)

Let $u = u_2 \ldots u_{m-1}$, and $T_2 = \{u_2, \ldots, u_m, x, y\} \subseteq T$. This subgraph has one cycle, and in $C_\gamma(T_2)$ (which is isomorphic to $S_{m+1} \ltimes \mathbb{Z}^m$ by the case $t = 1$ of Theorem 6.1), one checks that $y$ commutes with $u_1^{-1}g_xu$, as they correspond to $(1m')$ and $x_{m-1}$ (note that in this subgraph, $y$ is part of the spanning subtree). Similarly let $T_3 = \{u_2, \ldots, u_m, x, v_2, \ldots, v_{m'}\}$; this subgraph too has one cycle, and in $C_\gamma(T_3)$ we have that $g_y = v_2 \ldots v_{m'} \ldots v_2$ commutes with $u_1^{-1}g_xu$ (as they again correspond to $(1m')$ and $x_{m-1}$). Finally $u_1^{-1}g_xu = u_1^{-1}\gamma_m^{-1}\gamma_1 u = \gamma_m u_1^{-1}\gamma_1 = \gamma_m^{-1}\gamma_1 = \Psi(x_{m-1})$ by Equation (7), so $\Psi(x_{m-1})$ commutes with $\Psi(y_{m'}) = \gamma_m^{-1}\gamma_1 = gy$. Conjugating by arbitrary $\sigma \in S_n$ we see that $\Psi$ respects (15).

Proposition 6.7. The composition $\Psi \circ \Phi$ is the identity on $C_\gamma(T)$.

Proof. Let $u = (i, j) \in T_0$. Then $\Phi(u) = (ij)$, and by definition $\Psi\Phi(u) = \Psi(ij) = u$. Now let $x \in X$, and set $u_1 = x$. As usual we label the vertices along the basic cycle attached to $x$, so that $x$ connects 1 and $m$, and by definition $\Phi(x) = (1m)x_{1m}$. Let $\gamma_i$ be defined as in (9), and compute that $\gamma_{m}^{-1}\gamma_1 u_1 = u_2 \ldots u_m \ldots u_2$ is an element of $(T_0)$ which maps to $(1m)$, so $\Psi((1m)) = u_1^{-1}\gamma_1^{-1}\gamma_m$. By definition $\Psi(x_{1m}) = \gamma_m^{-1}\gamma_1$, so that $\Psi\Phi(x) = \Psi((1m)x_{1m}) = u_1 = x$, as required.

For the other direction, we first compute $\Phi(\gamma_a)$.

Proposition 6.8. Let $\gamma_a$ be defined as in (9), and set $\tau = (m \ldots 321)$. Then for every $a = 1, \ldots, n$ we have that $\Phi(\gamma_a) = \tau x_{am}$.
Proof. First compute $\Phi(\gamma_j)$ where $j$ is on the cycle (that is, $1 \leq j \leq m$). We may use (6) as the definition of $\gamma_j$. If $j < m$ we have

$$\Phi(\gamma_j) = \Phi(u_{j+2} \ldots u_m u_1 \ldots u_j) = (j + 1) j + 2 \ldots (m - 1)m)(1m)x_{1m}(12) \ldots (j - 1)j = \tau x_{jm},$$

and for $j = m, \Phi(\gamma_j) = \Phi(u_2 \ldots u_m) = (12) \ldots (m - 1)m) = \tau = \tau x_{mm}.$

In general, $\gamma_a$ is defined in (9). Let $v \in T_0$ be an edge connecting a vertex $j$ on the cycle to a vertex $b$ (not on the cycle). By definition, $\tilde{v} = u_{j+1}v u_{j+1}$. So if $j < m$ we have $\Phi(\tilde{v}) = (j + 1)(jb)(jj + 1) = (j + 1)b$, while if $j = m$ we have $\Phi(\tilde{v}) = (1m)x_{1m}(mb)(1m)x_{1m} = (1b)x_{1b}$.

Now let $a$ be an arbitrary vertex, and let $w_1, \ldots, w_s$ denote the path connecting $a$ to a vertex $j$ on the cycle. Write $w = w_2 \ldots w_s$, and let $b = w_1(j)$. Note that $w(b) = a$. Since $w$ does not touch the cycle we have that $\tilde{w} = w$, so by definition $\gamma_a = w^{-1}\tilde{w}_1\gamma_j w_1 w$. Now if $j < m$,

$$\Phi(\gamma_a) = \Phi(w^{-1}\tilde{w}_1\gamma_j w_1 w) = \Phi(w)^{-1}\Phi(w_1\gamma_j w_1)\Phi(w) = \Phi(w)^{-1}(j + 1)b \tau x_{jm}(jb)\Phi(w) = \Phi(w)^{-1}(j + 1)b \tau (jb)\Phi(w)x_{am} = \Phi(w)^{-1}\tau \Phi(w)x_{am} = \tau x_{am},$$

where the last equality follows since $w$ does not intersect the cycle.

Similarly if $j = m$, we have

$$\Phi(\gamma_a) = \Phi(w^{-1}\tilde{w}_1\gamma_m w_1 w) = \Phi(w)^{-1}\Phi(w_1\gamma_m w_1)\Phi(w) = \Phi(w)^{-1}(1b)x_{1b}\tau(mb)\Phi(w) = \Phi(w)^{-1}(1b)\tau x_{mb}(mb)\Phi(w) = \Phi(w)^{-1}(1b)\tau(mb)\Phi(w)x_{am} = \Phi(w)^{-1}\tau \Phi(w)x_{am} = \tau x_{am}.$$

Corollary 6.9. $\Phi \circ \Psi$ is the identity on $S_n \ltimes A_{t,n}$.

Proof. For $\sigma \in S_n$ we have that $\Psi(\sigma) \in \langle T_0 \rangle$ and by definition $\Psi \Phi(\sigma) = \sigma$. Let $x \in X$ and let $i, j = 1, \ldots, n$. Then by definition $\Psi(x_{ij}) = \gamma_j^{-1}\gamma_i$ where $\gamma_i, \gamma_j$ are constructed as usual.

By the last proposition and (14), $\Phi(\gamma_j^{-1}\gamma_i) = x_{mj}\tau^{-1}\tau x_{im} = x_{ij}$. □

This ends the proof of Theorem 6.1.
6.2. The map $u \mapsto \tilde{u}$. As always, let $T_0$ be a spanning subtree of $T$ and $x \in T - T_0$, and let $u_2, \ldots, u_m \in T_0$ be the completion of $x$ to a basic cycle in $T$. In Section 4 we defined a map $u \mapsto \tilde{u}$ ($u \in T$), and showed it induces a homomorphism from $(T_0)$ to $C_Y(T)$. This map has an interesting role in the proof of Theorem 6.1, so before moving to the applications, we will prove the following:

**Proposition 6.10.** Assume $T \neq K_4$. Then the map $u \mapsto \tilde{u}$ of Definition 4.2 extends to an automorphism of $C_Y(T)$.

**Proof.** We identify the groups $A_{t,n}^*$ and $F_{t,n}^*$ (using the proof of Theorem 5.7). From Theorem 6.1 we have the isomorphism $\Psi : S_n \times F_{t,n} \rightarrow C_Y(T)$, inverted by $\Phi$ of Definition 6.2. Set $\tau = (m \ldots 321)$. We claim that for every $u \in T$,

$$\tilde{u} = \Psi(\tau x_m^{-1} \cdot \Phi(u) \cdot x_m \tau^{-1}).$$

The right hand side is an inner automorphism of $S_n \times F_{t,n}$, induced by the element $\tau x_m^{-1}$. In order to prove this claim, we choose for every $u \in T$ a vertex $c$ not lying on $u$. Then $x_c$ commutes with $\Phi(u)$, and we need to prove

$$\tilde{u} = \Psi(\tau x_m^{-1} x_c \cdot \Phi(u) \cdot x_c^{-1} x_m \tau^{-1}),$$

which using $\Psi(x_m^{-1} x_c) = \gamma_m^{-1} \gamma_c$ and the fact that $\Psi(\tau) = \gamma_m$ (see Proposition 6.8), translates to

$$\tilde{u} = \gamma_c u \gamma_c^{-1}.$$ 

Note that apart from the assumption that $u_2, \ldots, u_m \in T_0$, the map $u \mapsto \tilde{u}$ is independent of $T_0$. This is not true for the $\gamma_i$ in general (see Equation (9)), but for $i = 1, \ldots, m$, we do have that $\gamma_i$ is independent of $T_0$ (see (6)). Now, for every $u \in T$ which does not touch the cycle twice, we can a priori choose $T_0$ to include $u$; then the result follows from Proposition 4.6. The remaining case is where $u \in T$ touches the cycle twice, but we have shown in Case (4) of the calculations preceding Definition 4.2 that $\tilde{u} = \gamma_{i-1} u \gamma_i^{-1}$ in this case. \hfill \Box

In contrast, let us consider the case $T = K_4$. We label the vertices and edges as in Figure 3.

We choose the spanning subtree $T_0 = \{u_2, u_3, v\}$, and consider the basic cycle of $x$ (oriented as $u_1 = x, u_2, u_3$). Then by definition of $u \mapsto \tilde{u}$ we have $\tilde{u}_2 = u_3$, $\tilde{u}_3 = x$, $\tilde{x} = u_2$, $\tilde{v} = u_3 v u_3$, $\tilde{y} = u_2 y u_2$, and $\tilde{z} = x z x$. It is easy to directly check that $\langle u_3, x, u_3 v u_3 \rangle \cong S_4$ (as Corollary 4.4 predicts), but we do not know if $[x, y z y] = [u_3 v u_3, u_2 y u_2 x z x u_2 y u_2]$ is the trivial element in $C_Y(T)$. It is interesting to note that under the identification of $C_Y(T)$ with $S_4 \times A_4$, this element evaluates to $y_{32} (x_{41} z_{12} x_{24} y_{43} x_{32} z_{21} x_{13} y_{34}) y_{23}$, which is trivial iff (27) holds.
7. Applications

7.1. The structure of \( C_Y(T) \). In this section we apply Theorem 6.1 to answer some natural questions on the structure of \( C_Y(T) \) and its normal subgroup \( K(T) = \text{Ker}(\phi: C_Y(T) \to S_n) \).

Let \( t \) denote the rank of \( \pi_1(T) \); \( t = 0 \) if \( T \) is a tree, and \( t = 1 \) if \( T \) has a unique cycle. Recall that \( K_4 \) is the complete graph on 4 vertices, the only exception to Theorem 5.7.

**Corollary 7.1.** Let \( T \) be a connected graph on \( n \) vertices.

- \( a. \) \( C_Y(T) \) is virtually solvable iff \( t \leq 1 \).
- \( b. \) If \( T \neq K_4 \), then \( K(T) \subseteq \pi_1(T)^n = (F_t)^n \).
- \( c. \) \( K(T) \) maps onto \( \pi_1(T)^{n-1} \).
- \( d. \) The group \( C_Y(T) \) contains a subgroup isomorphic to \( \pi_1(T) \).

**Proof.** By Theorem 6.1, \( C_Y(T) \cong S_n \rtimes A_{t,n} \). If \( t = 0 \) then the kernel \( K(T) = \text{Ker}(\phi: C_Y(T) \to S_n) \) is trivial (Corollary 2.5). If \( t = 1 \) we have that \( C_Y(T) = S_n \rtimes \mathbb{Z}^{n-1} \) and \( K(T) = \mathbb{Z}^{n-1} \) (Example 5.3). Here \( \mathbb{Z}^{n-1} \) is the large irreducible component of the standard representation of \( S_n \).

If \( t \geq 2 \) then \( C_Y(T) \) cannot be even virtually super-solvable, by Part \( c \).

Part \( b \) follows from Theorems 6.1 and 5.7. Recall the presentation of \( F_{t,n}^* = F_t^n \) from Subsection 5.2, and let \( F_0 \) denote the subgroup generated by the elements \( x_i \) for \( i = 1, \ldots, n-1 \) (which is isomorphic to \( (F_t)^{n-1} \)). The map defined by \( x_i \mapsto x_i \) for \( i < n \) and \( x_n \mapsto 1 \) (for all \( x \in X \)) is onto, since \( x_i x_n^{-1} \in F_{t,n} \) covers \( x_i \). If \( T \neq K_4 \) we are done (as \( K(T) \cong F_{t,n}^* \)), but even if \( T = K_4 \), \( F_{t,n} \) is a quotient of \( A_{t,n} \cong K(T) \) by Theorem 5.7.

Finally, \( x \mapsto x_1 x_2^{-1} \) defines a map \( \pi_1(T) \to F_{t,n} \) which is obviously injective, thus proving Part \( d \). for \( T \neq K_4 \) (since then \( F_{t,n} \subseteq C_Y(T) \)).

For \( T = K_4 \), \( F_3 \subseteq F_2 \subseteq F_{2,4} \cong A_{2,4} \subseteq A_{3,4} \) by Proposition 5.5, so that the same result holds. \( \square \)

Note that Part \( d \) above can be strengthened to state that \( C_Y(T) \) contains \( \frac{n-1}{2} \) commuting copies of \( \pi_1(T) \), using the free subgroups \( \langle x_{12} : x \in X \rangle, \langle x_{34} : x \in X \rangle, \ldots \).
Example 7.2. Consider the graph $T$ of Figure 4, where the spanning subgraph is $T_0 = \{a, b, c, d, e\}$ and $X = \{x, y, z\}$.

The isomorphism $K(T) \cong A_{3,6}$ gives $x_{14} = cecx$, $y_{30} = bdby$ and $z_{56} = cadacz$. As three commuting free subgroups we can take

$$\langle caexecac, dadbybad, adczcd \rangle,$$

$$\langle bacecxab, ecabdydace, bdecadacedb \rangle$$

and

$$\langle daexecad, cadbybac, cadacz \rangle.$$

The following interesting property is immediate from Theorem 6.1:

Remark 7.3. The group $C_Y(T)$ depends on the graph $T$ only through the number of basic cycles and the number of vertices in the graph.

Let $T$ be a connected graph. The Abelianization of $C(T)$ is easily seen to be $\mathbb{Z}/2\mathbb{Z}$ as every relation of the form $uvu = vuv$ becomes $u = v$. The isomorphism $C_Y(T) \cong S_n \rtimes A_{t,n}$ proves that the commutator subgroup of $C_Y(T)$ coincides with $A_n \rtimes A_{t,n}$ (where $A_n$ is the alternating group). On the other hand we have

Proposition 7.4. Let $T$ be a connected graph. The Abelianization of $K(T)$ is $\mathbb{Z}^{(n-1)}$.

Proof. As an Abelian group, $A_{t,n}$ is freely generated by the $x_{i,i+1}$ ($x \in X, i = 1, \ldots, n - 1$).

Another easy property of $F_{t,n}$ allows us to conclude the following

Proposition 7.5. Let $T \neq K_4$ be a connected graph. Then the kernel $K(T)$ of $\phi : C_Y(T) \rightarrow S_n$ is torsion free.

The following is also of some interest:

Proposition 7.6. If $T \neq K_4$ is a connected graph, then $C_Y(T)$ is residually finite.

Proof. Given $1 \neq w \in C_Y(T)$, we need to show that $w$ is outside a finite index normal subgroup of $C_Y(T)$. If $w \notin K(T)$, we are done; and if $w \in K(T) \cong F_{t,n}$ use the residually finiteness of the free group.  

Since $S_n \rtimes F_{t,n} \subseteq S_n \rtimes F^*_{t,n}$, we also have

**Proposition 7.7.** The word problem is solvable in $C_Y(T)$ for every connected $T \neq K_4$.

### 7.2. Parabolic subgroups

In Proposition 3.2 we saw that a subgroup of $C_Y(T)$ generated by a subtree $T_0$ is isomorphic to the abstract group $C_Y(T_0)$ defined on that tree. This can be generalized. We will show that parabolic subgroups are well behaved, even without assuming the smaller defining graph is connected.

**Proposition 7.8.** Let $T$ be a graph, and $T' \subseteq T$ a subgraph. Let $C = \langle T' \rangle$ be the subgroup of $C_Y(T)$, generated by the vertices $u \in T'$. If $T' \neq K_4$, then $C \cong C_Y(T')$.

This result should be compared to the situation for Coxeter groups. If $D = \langle s_1, \ldots, s_k \rangle$ is a Coxeter group, then it is well known (e.g. [6, Section 5.5]) that for $I \subseteq \{1, \ldots, k\}$, the subgroup of $D$ generated by $\{s_i : i \in I\}$ is isomorphic to the appropriate abstract Coxeter group.

Our result that $C_Y(T')$ naturally embeds into $C_Y(T)$ (as the subgroup generated by the edges of $T'$) is of the same nature; even better, if $T'$ is connected we show that $C_Y(T')$ is a retract of $C_Y(T)$.

**Proof of Proposition 7.8.** First assume that $T'$ is connected. By induction we may assume that $|T - T'| = 1$. Let $w$ denote the element in $T - T'$. Let $T_0 \subseteq T'$ be a spanning subtree (of $T$), $X' = T' - T_0$ and $X = T - T_0$. Inspecting the image of $\langle X' \rangle$ under the isomorphism $\Phi : C_Y(T) \rightarrow S_n \rtimes A_{X,n}$, we only need to show that the subgroup $\langle x_{ij} : x \in X' \rangle$ of $A_{X,n}$ is isomorphic to $A_{X,n}$, and this is Proposition 5.5.

Now assume that $T'$ is not connected. Again we apply induction but this time it is necessary to choose the chain $T' = T^0 \subseteq T^1 \subseteq \ldots \subseteq T^s = T$ of subgraphs (with $|T^{i+1} - T^i| = 1$), in a way that $T^i \neq K_4$. Indeed, assume $T^i = K_4$ for some $i$. Since $T' \neq K_4$, $i > 0$ so $T^{i-1}$ is $K_4$ with one edge removed, i.e. the graph (1) in Figure 5. If $T = T^i$ then $T^{i-1}$ is a connected subgraph, so using the first case we can replace $T$ by $T^{i-1}$ (thus avoiding $K_4$). Otherwise $T^{i+1}$ must be the graph (3) in Figure 5. Then we can replace $T^i$ by the graph (4) in that Figure, proving the claim.

We continue under the assumption that $T'$ is a disconnected subgraph of the connected graph $T$, with $|T - T'| = 1$. In order to show that $C_Y(T') \subseteq C_Y(T)$, we prove that $F_{t,n-1}$ embeds in $F_{t,n}$. Consider
the commutative diagram

\[
\begin{array}{c}
1 \longrightarrow F_{t,n} \xleftarrow{\rho} F_{t,n}^{*ab} \longrightarrow Z' \longrightarrow 1 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
1 \longrightarrow F_{t-1,n}^{*ab} \longrightarrow Z' \longrightarrow 1
\end{array}
\]

where \( \rho \) is defined by \( x_i \mapsto x_i \) (\( i = 1, \ldots, n \)). This induces an embedding \( F_{t-1,n} \hookrightarrow F_{t,n} \) which agrees with the \( S_n \) action.

Letting \( S_{n-1} \) denote the stabilizer in \( S_n \) of the isolated vertex in \( T' \), we have

\[
C_Y(T') \cong S_{n-1} \ltimes F_{t,n-1} \leftarrow S_{n-1} \ltimes F_{t,n} \leftarrow S_n \ltimes F_{t,n} \cong C_Y(T),
\]

and the image of \( C_Y(T') \) is easily computed to be the subgroup \( \langle T' \rangle \) of \( C_Y(T') \).

The proof of Proposition 5.5 shows that \( A_{t-1,n} \) is a quotient group of \( A_{t,n} \); so we have proved

**Remark 7.9.** If \( T' \subseteq T \) is connected, then \( C_Y(T') \) is a retract of \( C_Y(T) \).

If \( T = T' \cup \{x\} \), then the group \( C_Y(T') \) is obtained by adding to \( C_Y(T) \) the cyclic relation associated to the basic cycle of \( x \). In other words, assuming the cyclic relation of a basic cycle amounts to erasing the non-\( T_0 \) edge on that cycle from \( T \).

Of course Remark 7.9 does not hold if \( T' \) is not connected, as \( S_{n_1} \times S_{n_2} \) is not a quotient of \( S_{n_1+n_2} \).

Until now we always assumed that \( T \) is connected. The generators from distinct components commute, so we have

**Remark 7.10.** Let \( T = T_1 \cup \cdots \cup T_s \) be the decomposition to connected components of the graph \( T \).

Then \( C(T) = C(T_1) \times \cdots \times C(T_s) \), \( C_Y(T) = C_Y(T_1) \times \cdots \times C_Y(T_s) \), and \( K(T) = K(T_1) \times \cdots \times K(T_s) \). The map \( \phi : C(T) \rightarrow S_n \) covers \( S_{n_1} \times \cdots \times S_{n_s} \) where \( n_i \) is the number of vertices in \( T_i \).
7.3. The Coxeter graph. It is customary to define a Coxeter group by a graph (called the Coxeter graph of the group), defined on the set of generators, where generators \( u, v \) are connected in the graph iff they do not commute; if the order of \( uv \) is assumed to be a number \( p > 3 \), the edge connecting \( v, u \) is labelled by \( p \).

From our definition of \( C(T) \) it follows that the Coxeter graph of \( C(T) \) is the dual graph \( T^\# \), which is defined on the set of edges of \( T \), and two edges of \( T \) are connected in \( T^\# \) iff they intersect in \( T \).

Example 7.11. Let \( T \) be a cycle on \( n \) vertices: since all the vertices are of degree 2, \( C(T) = C_Y(T) \). The Coxeter graph is again a cycle on \( n \) vertices, being the dual graph of \( T \). We obtain the well known result, that the Coxeter group of a cycle on \( n \) vertices, is isomorphic to \( S_n \rtimes \mathbb{Z}^{n-1} \).

Example 7.12. Let \( Y \) be the graph on four vertices, consisting of three edges meeting in a vertex, and \( \Delta \) the graph of a triangle. Then the dual of both graphs is a triangle, showing that \( C(Y) = C(\Delta) \).

Moreover, since in \( \Delta \) only two edges meet in every vertex, \( C_Y(\Delta) = C(\Delta) \). So we have that \( C(Y) = C(\Delta) = C_Y(\Delta) = S_3 \rtimes \mathbb{Z}^2 \), while \( C_Y(Y) = S_4 \). Let \( u, v, w \) denote the edges of \( Y \). Then in the isomorphism \( C(Y) = S_3 \rtimes \mathbb{Z}^2 \) we can take \( u, v \) as generators of the \( S_3 \) piece, and \( \langle uvw, vwv \rangle \) as generators of \( \mathbb{Z}^2 \). The kernel of the map to \( C_Y(Y) = S_4 \) is \( \langle [u, vw], [v, uwu] \rangle \cong \mathbb{Z}^2 \).

The graphs \( T^\# \) tend to have many edges. In particular not every graph is of the form \( T^\# \). The easiest example is the following.

Remark 7.13. If a graph \( S \) has the graph of Figure 6 as a subgraph, then it is not of the form \( T^\# \) for any graph \( T \).

![Figure 6. A forbidden subgraph of \( T^\# \)](image)

Using this observation, one easily scans the lists of finite or hyperbolic Coxeter groups (which can be found in [6], for example), to check the following

Remark 7.14. The only hyperbolic Coxeter groups of the form \( C(T) \) are the symmetric groups \( S_n \), and the three infinite non-compact groups obtained for the graphs of Figure 7.
The hyperbolic $T$s

The corresponding (dual) Coxeter graphs are shown in Figure 8. The groups $C_Y(T)$ are $S_5$, $S_5$ and $S_4 \rtimes \mathbb{Z}^3$, respectively.

Figure 7. The hyperbolic $T$s

Figure 8. Hyperbolic $T^\#$s

7.4. Coxeter groups defined by signed graphs. Let $V$ be the vector space spanned by the generators of a Coxeter group. In the Tits’ representation of Coxeter groups, a generator $u$ acts as $r_u: V \to V$, where $r_u$ is defined by $r_u(u) = -u$, $r_u(v) = v$ if $u, v$ commute in the group, and $r_u(v) = v + u$ if $(uv)^3 = 1$. Changing this slightly, to $r_u(v) = v - u$ if $(uv)^3 = 1$, we obtain a representation of $C_Y(T)$.

This observation, in a more general context, is one of the motivations of [4] to introduce certain quotients of Coxeter groups, indexed by signed graphs. We direct the reader to [4] for the definitions and notation used here. Let $f$ be a signing of the graph $T^\#$. If for every $u, v, w \in T$ which meet in a vertex, the resulting triangle in $T^\#$ is odd (with respect to $f$), then $\text{Cox}(T^\#, f)$ is a quotient of $C_Y(T)$. It is fairly direct to evaluate the ‘cut elements’ (which define $\text{Cox}(T^\#, f)$ as a quotient of $C(T)$) in $S_n \rtimes A_{t,n}$, so these groups can be computed using our results. For example, the group $D_2$ computed in [5] is our $C_Y(T)$ for $T$ the graph of Figure 5.(1) (indeed $D_2 = S_4 \rtimes F_{2,4}$).

Tsaranov groups are generated by elements $t_1, \ldots, t_n$ of order 3 with relations $(t_it_j^3)^3 = 1$, where the signs are indexed by a graph (the exponent being $-1$ iff $(i, j)$ is in the graph). In [4] it is shown that $\text{Ts}^*(\Gamma)$ (which is the Tsaranov group of the graph $\Gamma$, adjoint with an involution whose conjugation inverts the generators) is a certain quotient of the Coxeter group defined by the graph $\Gamma + 0$ (the complement of a disjoint union of $\Gamma$ and a point 0).
Proposition 7.15. Let \( a, b \geq t \geq 0 \). Let \( K_{a,b} \) be the complete bipartite graph on \( a + b \) vertices, and let \( \Gamma \) be the graph obtained by deleting \( t \) disjoint edges from \( K_{a,b} \). Set \( n = a + b + 2 - t \), and let \( X = \{x, y, \ldots\} \) be a set of size \( t \) indexing the generators of \( F_{t,n} \).

Then \( TS^*(\Gamma) \) is the subgroup \( S_n \rtimes F_{t,n} \) of \( S_n \rtimes F_{t,n}^1 \), modulo its normal subgroup \( \langle x_{2i} x_{-2j} \rangle \).

Proof. Taking \( T \) to be the union of \( t \) triangles with a common vertex \( u_0 \), to which we glue a star of \( a - t \) edges at one end and a star of \( b - t \) edges at the other, we see that \( \Gamma + 0 = T^\# \).

From Theorem 8.3 of [4] it then follows that \( TS^*(T) \) is a quotient of \( C_Y(T) \). The presentation given in Theorem 8.1 of that paper translates to adding the relations \( x_{2i} x_{-2j} = 1 \) to \( C_Y(T) = S_n \rtimes A_{t,n} \). Since \( T \) can never be the graph \( K_4 \), Proposition 5.8 applies, and \( A_{t,n} \subseteq A_{t,n}^* \), with \( \langle x_{2i} x_{-2j} \rangle \) being a normal subgroup in \( A_{t,n}^* \). Mapping this presentation to \( F_{t,n}^* \), we get the desired result.

Of special interest is the Tsaranov group of a hexagon (see [5, p. 179]). Note that deleting three disjoint edges from \( K_{3,3} \) gives a hexagon, so we get

Corollary 7.16. The Tsaranov group of a hexagon is \( S_5 \rtimes F_{3,5} \) modulo the relations \( x_i x_{-2j} \) (\( x \in X \)).

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