Coulomb-gas approach for boundary conformal field theory

Shinsuke Kawai

Theoretical Physics, Department of Physics, University of Oxford,
1 Keble Road, Oxford OX1 3NP, United Kingdom

(January 14, 2022)

We present a construction of boundary states based on the Coulomb-gas formalism of Dotsenko and Fateev. It is shown that Neumann-like coherent states on the charged bosonic Fock space provide a set of boundary states with consistent modular properties. Such coherent states are characterised by the boundary charges, which are related to the number of bulk screening operators through the charge neutrality condition. We illustrate this using the Ising model as an example, and show that all of its known consistent boundary states are reproduced in our formalism. This method applies to \( c < 1 \) minimal conformal theories and provides an unified computational tool for studying boundary states of such theories.

PACS number(s): 98.80.Bp, 98.80.Cq, 04.20.Dw

I. INTRODUCTION

The basic concepts and techniques of boundary conformal field theory (BCFT) were introduced in the eighties. Much of the seminal work was done by Cardy, who discussed surface critical behaviour and invented a method to calculate boundary correlation functions \([1]\), studied the restriction on the operator content imposed by boundary conditions \([2]\), developed a systematic classification of boundary states based on the modular transformation and introduced the concept of boundary operators \([3,4]\). After the relatively dormant era of the middle nineties, this field is now enjoying its second stage of development. The growing interest in BCFT is largely motivated by noticing the importance of boundary states of open strings after the discovery of D-branes. Classification of boundary states is now recognised as of prime importance since D-branes are an essential element of non-perturbative string theory.

Our understanding of the underlying algebraic structure of BCFT has recently been improved enormously. Importance of complete sets of boundary conditions \([5]\) has been recognised widely. An extra boundary condition was discovered in the simplest non-diagonal minimal model \([6]\) and the resulting set of boundary conditions was shown to be complete \([7]\). The sewing relations originated by Cardy and Lewellen \([8]\) were explicitly solved for some cases \([9]\), and it is now understood that solving Cardy’s consistency condition reduces to finding non-negative integer-valued matrix representations of the Verlinde algebra \([10]\). A rational conformal theory is rational with respect to a symmetry which is in general larger than the Virasoro symmetry. Classification of boundary states from such a symmetry-breaking viewpoint is also being done \([11,12]\). The algebraic construction of boundary states (or D-branes) is now being extended to various rational conformal theories far beyond the minimal models.

In this paper we would like to consider another approach for BCFT, namely, the construction of boundary states from free fields. This is particularly important from the practical point of view since any correlation function should be calculable ab initio from the operator algebra. Such BCFTs for free bosons and fermions have been well established for a long time, and they are indeed essential building blocks of the open string theory. Aside from bosons and fermions, the boundary states of symplectic fermions at \( c = -2 \) were constructed recently \([13]\). However, to the author’s knowledge, such an approach for other CFTs seems to be absent. In order to generalise the free-field construction of boundary states we re-formulate the Coulomb-gas picture of Dotsenko and Fateev \([14]\) in the presence of a boundary. Work in this direction was done by Schulze \([15]\), who discussed Coulomb-gas system on the half plane and calculated boundary correlation functions using contour integrals. In the present paper we shall consider the system on an annulus which is a suitable arena for discussing modular integrals. In the present paper we shall consider the system on an annulus which is a suitable arena for discussing modular integrals. In the present paper we shall consider the system on an annulus which is a suitable arena for discussing modular integrals.

The plan of this paper is as follows. We start in the next section by reviewing the Coulomb-gas formalism on the Riemann surfaces and fix our notation. In Sec. III the charged bosonic Fock space (CBFS) is defined for the theory on an annulus. We also construct the boundary coherent states on CBFS and find conditions for the conformal invariance of such states. The charge-neutrality conditions for the boundary Coulomb-gas are considered

*E-mail address: s.kawai@physics.ox.ac.uk
and the closed-string channel amplitudes are calculated in Sec. IV. We illustrate our method in Sec V using the Ising model as an example. In Sec. VI we summarise and conclude.

II. COULOMB-GAS AND THE CHARGED BOSONIC FOCK SPACE

The essential ingredient of the Coulomb-gas formalism is the non-minimal coupling of the free scalar field to the background curvature. This makes the $U(1)$ symmetry anomalous, modifying the central charge and the conformal dimensions of $c = 1$ theory to generate the minimal models. In this section we collect the basic components of the Coulomb-gas formalism without the boundary \[13\] \[16\] \[18\]. Variation of the action

$$S = \frac{1}{8\pi} \int d^2x \sqrt{g}(\partial_{\mu} \Phi \partial_{\mu} \Phi + 2\sqrt{2} \alpha_0 i \Phi R),$$

with respect to the metric gives the energy-momentum tensor

$$T(z) = -2\pi T_{zz} = -\frac{1}{2} : \partial \varphi \partial \varphi : + i \sqrt{2} \alpha_0 \partial^2 \varphi,$$

where $\varphi$ is the holomorphic part of the boson, $\Phi(z, \bar{z}) = \varphi(z) + \bar{\varphi}(\bar{z})$. The antiholomorphic part is similar. From $T(z)$ the central charge is read off as

$$c = 1 - 24\alpha_0^2.$$

The chiral vertex operator defined as

$$V_\alpha(z) = e^{i\sqrt{2} \alpha \varphi(z)} :,$$

then has the conformal dimension $h_\alpha = \alpha^2 - 2\alpha_0 \alpha$, which is easily verified by computing the OPE with $T(z)$. Among these vertex operators, $V_\pm(z) \equiv V_{\alpha_\pm}(z)$ with $\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + 1}$ play a special role. They have conformal dimensions 1 and the closed contour integrals,

$$Q_\pm \equiv \oint dz V_\pm(z),$$

are the screening operators which do not change the conformal properties but carry charges. The condition that the fields must be screened by such screening operators leads to the quantisation of the spectrum,

$$\alpha_{r,s} = \frac{1}{2}(1 - r)\alpha_+ + \frac{1}{2}(1 - s)\alpha_-,$$

where $r$ and $s$ are positive integers. The vertex operators $V_{\alpha_{r,s}}(z)$ then have conformal dimensions

$$h_{r,s} = \frac{1}{4}(r\alpha_+ + s\alpha_-)^2 - \alpha_0^2,$$

and are identified with the operators appearing in the Kac formula.

The Hilbert space of the theory defined on a Riemann surface is a direct sum of charged bosonic Fock spaces (CBFSs) with BRST projection [13]. The chiral CBFS $F_{\alpha,\alpha_0}$ with vacuum charge $\alpha$ and background charge $\alpha_0$ is built on the highest-weight vector $|\alpha; \alpha_0\rangle$ as a representation of the Heisenberg algebra

$$[a_m, a_n] = m\delta_{m+n,0},$$

where $a_n$ are the mode operators defined by

$$\varphi(z) = \varphi_0 - i\alpha_0 \ln z + i \sum_{n\neq 0} \alpha_n z^{-n}.$$

The zero-mode operators satisfy the commutation relation $[\varphi_0, \alpha_0] = i$. The highest-weight vector is constructed from the vacuum $|0; \alpha_0\rangle$ by operating with $e^{i\sqrt{2}\alpha_0 \varphi_0}$,

$$|\alpha; \alpha_0\rangle = e^{i\sqrt{2}\alpha_0 \varphi_0}|0; \alpha_0\rangle,$$

and is annihilated by the action of $a_{n>0}$. The charge $\alpha$ is related to the eigenvalue of $\alpha_0$ by

$$\alpha_0|\alpha; \alpha_0\rangle = \sqrt{2}\alpha|\alpha; \alpha_0\rangle.$$

The Virasoro generators are written in terms of the mode operators as

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} a_{n-k} a_k - \sqrt{2}\alpha_0(n+1)a_n,$$

$$L_0 = \sum_{k \geq 1} a_{-k} a_k + \frac{1}{2} a_0^2 - \sqrt{2}\alpha_0 a_0.$$

With these generators the CBFS $F_{\alpha,\alpha_0}$ has the structure of a Virasoro module. It is easy to check that

$$L_0|\alpha; \alpha_0\rangle = (\alpha^2 - 2\alpha_0 \alpha)|\alpha; \alpha_0\rangle,$$

that is, the conformal dimension of $|\alpha; \alpha_0\rangle$ is $\alpha^2 - 2\alpha_0 \alpha$. Because of $[L_0, a_{-n}] = na_{-n}$ ($\forall n \geq 0$), $F_{\alpha,\alpha_0}$ is graded by $L_0$ and written as

$$F_{\alpha,\alpha_0} = \bigoplus_{n=0}^\infty F_{\alpha,\alpha_0}(n),$$

where $(F_{\alpha,\alpha_0})_n$ is the subspace with conformal dimension $\alpha^2 - 2\alpha_0 \alpha + n$. Counting the number of states the character of $F_{\alpha,\alpha_0}$ is found to be

$$\chi_{\alpha,\alpha_0}(q) \equiv \sum_{L_0, a_{-n}} q^{L_0 - c/24} = \sum_{n=0}^\infty \frac{(\alpha-\alpha_0)^2}{\eta(\tau)},$$

where $q = e^{2\pi i \tau}$ is the modular parameter and $\eta(\tau) \equiv q^{1/24} \prod_{n \geq 1} (1 - q^n)$ is the Dedekind eta function.
The dual space $F^*_{α,α_0}$ of $F_{α,α_0}$ is built on a contravariant highest-weight vector $⟨α; α_0|$ satisfying the condition

$$⟨α; α_0|α; α_0⟩ = κ,$$  \hspace{1cm} (17)

where $κ$ is a normalisation factor which is usually set to 1 in unitary models. The modules are endowed with a dual Virasoro structure

$$⟨ω|L_{−n}ξ⟩ = ⟨ωL_n|ξ⟩$$  \hspace{1cm} (18)

for any $⟨ω⟩ ∈ F^*_{α,α_0}$, $|ξ⟩ ∈ F_{α,α_0}$. This dual structure naturally incorporates the transpose $A^t$ of an operator $A$ through the relation

$$⟨ω|Aξ⟩ = ⟨ωA^t|ξ⟩.$$  \hspace{1cm} (19)

In particular, $L_{−n}^{t} = L_n$, $a_{−n}^{t} = 2√2α_0δ_{n,0} − a_n$. With this definition of transpose, $F^*_{α,α_0}$ is shown to be a Fock space isomorphic to $F_{2α_0−a,α_0}$. The contravariant highest-weight vector $⟨α; α_0|$ is annihilated by the action of $a_n$ for $n < 0$ (or $a_n^t$ for $n > 0$),

$$⟨α; α_0|a_n|α; α_0⟩ = 0.$$  \hspace{1cm} (20)

From the uniqueness of the expression $⟨α; α_0|a_n|α; α_0⟩$ and the right operation of the zero mode (11) we immediately have

$$⟨α; α_0|a_0⟩ = √2α(α; α_0).$$  \hspace{1cm} (21)

Analogously to (10) we find

$$⟨α; α_0⟩ = (0; α_0)e^{−i√2αφ},$$  \hspace{1cm} (22)

where the contravariant vector $⟨0; α_0⟩$ is the vacuum with the normalisation $⟨0; α_0|0; α_0⟩ = κ$. From (11) and (22), the in-state $|α; α_0⟩$ and the out-state $⟨α; α_0|$ are interpreted as possessing charges $α$ and $−α$, respectively. The non-vanishing inner product (17) is consistent with the neutrality of the total charge, $−α + α = 0$. Since the inner product must vanish when the total charge is not zero, we have in general

$$⟨α; α_0|β; α_0⟩ = κδ_{α, β}.$$  \hspace{1cm} (23)

On the plane the minimal conformal theory is realized through the usual radial quantisation scheme, by sending the in-state to zero and the out-state to infinity. Expectation values are usually taken between $⟨2α_0; α_0⟩$ and $⟨0; α_0⟩$, which is interpreted as placing a charge $−2α_0$ at infinity. Correlation functions of vertex operators are calculated with suitable insertion of the screening operators to realise the charge neutrality, leading in general to integral representations. The Coulomb-gas formalism also applies to Riemann surfaces of higher genus and such theories have been studied by many authors [16,19–21]. On the torus it is shown that taking the trace over the BRST cohomology space is equivalent to the alternated summation [16]. For example, the zero-point function on the torus for the conformal block corresponding to the representation $(r, s)$ of the minimal models is calculated in Coulomb-gas method as [16]

$$Tr_{(r,s)}q^{L_{−c/24}} = \frac{1}{η(τ)}(θ_{pr−p's, pp'}(τ) − θ_{pr+p's, pp'}(τ)),$$  \hspace{1cm} (24)

which is nothing but the Rocha-Caridi character formula [22] as it should be. Here, we have defined the Jacobi theta function as $θ_{λ,μ}(τ) = \sum_{k∈Z} q^{2(k+λ)^2/4iμ}$. Formulas for Jacobi theta and Dedekind eta functions are summarised in App. A.

### III. CBFS WITH BOUNDARY

In this section we discuss the Fock space representation of BCFT where the interplay between holomorphic and antiholomorphic sectors is important. Let us start with the geometry of the upper half-plane. We define $ζ = x + iy$, $x, y ∈ R$ and consider a CFT defined on the region $Imζ ≥ 0$. The boundary is $y = 0$, or $ζ = ̅ζ$. The antiholomorphic dependence of the correlators on the upper half plane may be mapped into the holomorphic dependence on the lower half plane [6]. This introduces a mirror image on the lower half plane, and the boundary condition tells how the images on the upper and lower half-planes are glued on the mirror, $ζ = ̅ζ$. The energy-momentum tensor on the lower half plane is obtained by the mapping from the upper half plane, $T(ζ^*) = ̅T(̅ζ)$. The condition on the boundary

$$[T(ζ) − ̅T(̅ζ)]_{ζ=̅ζ} = 0,$$  \hspace{1cm} (25)

indicates the absence of the energy-momentum flow across the boundary. Since the energy-momentum tensor is the generator of conformal transformations, (25) also means the conformal invariance of the boundary. Going from the upper half plane (or holomorphic part) to the lower half plane (antiholomorphic part) is generally accompanied by a parity transformation $P$. The free boson transforms under $P$ as $φ(ζ) → Ωφ(Ωζ)$, $Ω = ±1$. This leads to the condition on the boundary

$$[φ(ζ) − Ωφ(Ωζ)]_{ζ=̅ζ} = 0.$$  \hspace{1cm} (26)

When $Ω = 1$, the non-chiral free boson $Φ(ζ, ̅ζ)$ is a scalar and the boundary condition is called Neumann, whereas when $Ω = −1$, $Φ(ζ, ̅ζ)$ is a pseudo-scalar and such boundary condition is called Dirichlet. Under the parity transformation the chiral vertex operators $V_α(ζ) = e^{i√2αφ(ζ)}$ are mapped into $̅V_α(̅ζ) = e^{i√2αφ(Ωζ)}$. When $Ω = −1$ (Dirichlet) the mirror image has a charge $Ωα = −α$ which has the opposite sign from the original one. In the Neumann case
\( \Omega = 1 \), the mirror and the original vertex operators have the same charge \( \alpha \). Coulomb-gas system on the half plane was studied in \([3]\) where the boundary correlation functions of the Ising model are calculated using the mirroring technique of \([1]\).

In this paper we mainly study BCFT defined on a finite cylinder, or an annulus. We consider a finite cylinder of length \( T \) and circumference \( L \), or an annulus on the \( z \)-plane with \( 1 \leq |z| \leq \exp(2\pi T/L) \). We also introduce a modular parameter as \( \bar{q} = e^{2\pi i T}, \bar{r} = 2iT/L \). With this the annulus is \( 1 \leq |z| \leq \bar{q}^{-1/2} \). We regard this cylinder as a propagating closed string, and call the direction along it as \( \text{time} \). A merit of considering such a geometry is that the familiar energy-momentum tensor for the full-plane may be used without modification. We conformally map a semi-annular domain in the upper-half \( \zeta \)-plane onto a full-annulus in the \( z \)-plane by \( z = \exp(2\pi i w/L) \) and \( w = (T/\pi) \ln \zeta \). The boundary \( \zeta = \zeta_0 \) is then mapped on the \( z \)-plane to \( |z| = 1 \), \( \exp(2\pi T/L) \). Since the \( z \)-plane allows radial quantization, the conformal invariance \([24]\) on the \( |z| = 1 \) boundary becomes the conditions on the quantum states \( |B\rangle \) \([23]\),

\[
(L_k - \bar{L}_{-k})|B\rangle = 0.
\] (27)

As \( \varphi(\zeta) \) and \( \bar{\varphi}(\bar{\zeta}) \) are not primary, the condition \([20]\) cannot be mapped to the annulus. However, the derivative of \( \text{uncharged} \) bosons is primary and

\[
\left[ \partial \varphi(\zeta) - \Omega \bar{\partial} \bar{\varphi}(\bar{\zeta}) \right]_{\zeta = \zeta_0} = 0
\]

on the \( \zeta \)-plane is mapped on the \( z \)-plane as

\[
(a_n + \Omega \bar{a}_{-n})|B\rangle = 0.
\] (29)

This expression no longer makes sense for the \( \text{charged} \) bosons since \( \partial \varphi \) and \( \bar{\partial} \bar{\varphi} \) cease to be primary when they are coupled to the background curvature. However, \([27]\) is still valid and is indeed a necessary condition for the conformally invariant boundary states. The vertex operators are safely mapped to \( z \)-plane since they remain primary. In the rest of this section we construct a Fock space representation of boundary states which satisfy the conformal invariance condition \([27]\).

Our starting point is recalling that a BCFT consists of a pair of chiral CFTs whose holomorphic and anti-holomorphic sectors are glued together on the boundary. The construction of the boundary states then requires a Fock space which is common to both holomorphic and anti-holomorphic sectors. As we have the same central charge \( c \) for both holomorphic and anti-holomorphic sectors, \( a_0 \), which is related to \( c \) by \([3]\), is common to both sectors, although we are free to choose different vacuum charges for each sector. Hence let us define the highest-weight vectors at the two boundaries of the annulus as \( |\alpha, \bar{\alpha}; a_0\rangle \) and \( \langle \alpha, \bar{\alpha}; a_0| \), satisfying

\[
|\alpha, \bar{\alpha}; a_0\rangle = \sqrt{2} \alpha |\alpha, \bar{\alpha}; a_0\rangle,
\]

\[
\bar{a}_0 |\alpha, \bar{\alpha}; a_0\rangle = \sqrt{2} \bar{\alpha} |\alpha, \bar{\alpha}; a_0\rangle,
\]

\[
\langle \alpha, \bar{\alpha}; a_0| |\alpha, \bar{\alpha}; a_0\rangle = (\alpha, \bar{\alpha}; a_0) \sqrt{2} \alpha,
\]

\[
\langle \alpha, \bar{\alpha}; a_0| \bar{a}_0 = (\alpha, \bar{\alpha}; a_0) \sqrt{2} \bar{\alpha},
\]

which are essentially the direct products of holomorphic and antiholomorphic parts of \([11]\), \([21]\). The state \( |\alpha, \bar{\alpha}; a_0\rangle \) has holomorphic charge \( \alpha \) and anti-holomorphic charge \( \bar{\alpha} \), and \( \langle \alpha, \bar{\alpha}; a_0| \) has holomorphic charge \( -\alpha \) and anti-holomorphic charge \( -\bar{\alpha} \). The mode operators of the antiholomorphic sector are defined, similarly to the holomorphic part \([3]\), by the mode expansion of \( \bar{\varphi}(\bar{z}) \) as

\[
\bar{\varphi}(\bar{z}) = \bar{\varphi}_0 - i\bar{a}_0 \ln \bar{z} + i \sum_{n \neq 0} \frac{\bar{a}_n}{n} \bar{z}^{-n}.
\]

The antiholomorphic mode operators satisfy the same Heisenberg algebra as their holomorphic counterpart:

\[
|\bar{a}_m, \bar{a}_n\rangle = m \delta_{m+n, 0},
\]

\[
|\bar{\varphi}_0, \bar{a}_0\rangle = i.
\]

There is a subtlety in the treatment of \( \bar{\varphi}_0 \) and \( \bar{a}_0 \) since the zero mode of the boson \( \Phi(z, \bar{z}) \) does not naturally decouple into left and right. We split them into two identical and independent copies such that \( |\bar{\varphi}_0, \bar{a}_0\rangle = |\bar{\varphi}_0, \bar{a}_0\rangle = 0 \). In such decomposition the existence of the dual field is implicit \([24]\). The highest-weight vector \( |\alpha, \bar{\alpha}; a_0\rangle \) is annihilated by the action of \( a_{n>0} \) and \( \bar{a}_{n<0} \), and the contravariant highest-weight vector \( \langle \alpha, \bar{\alpha}; a_0| \) is annihilated by \( a_{n<0} \) and \( \bar{a}_{n<0} \). Following \([23]\) we assume the highest-weight vectors are normalised as

\[
\langle \alpha, \bar{\alpha}; a_0| |\beta, \bar{\beta}; a_0\rangle = k' \delta_{\alpha, \bar{\beta}} \delta_{\bar{\alpha}, \bar{\beta}},
\]

where \( k' \) is a normalisation factor, which may be set to 1 if the sector is unitary. If \( k' \) is negative we set it to \(-1\).

We are looking for conformally invariant boundary states built on the highest-weight vectors \( |\alpha, \bar{\alpha}; a_0\rangle \) and \( \langle \alpha, \bar{\alpha}; a_0| \). Since we know that such states for (uncharged) bosonic strings are found in the form of coherent states in string theory, let us start with an ansatz

\[
|B_{\alpha, \bar{\alpha}; a_0}\rangle_{\Omega} = \prod_{k>0} \exp \left( -\frac{\Omega}{k} a_{-k} \bar{a}_{-k} \right) |\alpha, \bar{\alpha}; a_0\rangle,
\]

\[
\Omega \langle B_{\alpha, \bar{\alpha}; a_0}| = \langle \alpha, \bar{\alpha}; a_0| \prod_{k>0} \exp \left( -\frac{1}{k^2} a_{-k} \bar{a}_{-k} \right).
\]

These states satisfy

\[
(a_n + \Omega \bar{a}_{-n}) |B_{\alpha, \bar{\alpha}; a_0}\rangle_{\Omega} = 0 \quad (n \neq 0),
\]

\[
\Omega \langle B_{\alpha, \bar{\alpha}; a_0}|(a_n + \Omega \bar{a}_{-n}) = 0 \quad (n \neq 0).
\]

Using the expression of Virasoro operators \([12]\), \([13]\) we see that \( |B_{\alpha, \bar{\alpha}; a_0}\rangle_{\Omega} \) does not satisfy the condition (27) straightaway. For example, we have
indeed a sufficient condition for the conformal invariance.

[54x299]α

conformal invariance for non-zero α and satisfying the conditions (44) and (45). Since the anti-

shall consider the conformally invariant boundary states even for n > 0, and

for n > 0, and

\[ (L_0 - \tilde{L}_0)|B_{\alpha, \bar{\alpha}; a_0}\rangle_{\Omega} = \prod_{k>0} \exp \left( -\frac{\Omega}{k} a_k \bar{a}_k \right) \times \left\{ \sqrt{2} a_{-n} [(\Omega - 1)n a_0 + (\Omega + 1)a_0 - \Omega \alpha - \bar{\alpha}] + \frac{1}{2} \sum_{0 < j < n} a_{-j} \bar{a}_{j-n} (\Omega^2 - 1) \right\} |\alpha, \bar{\alpha}; a_0\rangle \]  

which are in general not zero. However, it can be easily seen that the expressions (42) and (43) do vanish when

\[ \Omega = 1, \]  

and

\[ \alpha + \bar{\alpha} - 2a_0 = 0, \]  

even for \( a_0 \neq 0 \). It is easily verified that these conditions also lead to \((L_n - \tilde{L}_n)|B_{\alpha, \bar{\alpha}; a_0}\rangle_{\Omega} = 0\) for \( n < 0 \) and are indeed a sufficient condition for the conformal invariance. Similarly it can be checked that \( 1|B_{\alpha, \bar{\alpha}; a_0}|(L_n - \tilde{L}_n) = 0 \) as long as \( \Omega = 1 \) and \( \alpha + \bar{\alpha} - 2a_0 = 0 \). Note that the “Dirichlet” condition \( \Omega = -1 \) is not compatible with the conformal invariance for non-zero \( a_0 \) because of the term proportional to \( n \) in (12). In the rest of this paper we shall consider the conformally invariant boundary states satisfying the conditions (44) and (45). Since the anti-holomorphic charge is determined by the condition (45), such boundary states are characterised by only one parameter \( \alpha \), apart from the value of the background charge \( a_0 \) which is fixed by the central charge. For simplicity we shall denote these boundary states as

\[ |B(\alpha)\rangle = |B_{\alpha, 2a_0 - \alpha; a_0}\rangle_{\Omega=1}, \]

and

\[ \{B(\alpha)\} = \Omega=1|B_{\alpha, 2a_0 - \alpha; a_0}| . \]

The background charge \( a_0 \) is suppressed since no confusion arises.

**IV. COHERENT AND CONSISTENT BOUNDARY STATES**

Identifying boundary states which may be realised in a physical system is one of the main goals in BCFT. Such boundary states are not only conformally invariant, but must satisfy some extra conditions. Indeed, any linear combination of conformally invariant boundary states is conformally invariant, whereas the number of physical boundary states are usually finite. One of the most powerful and systematic method for finding such physical boundary states is Cardy’s fusion method [3], which we shall review briefly.

The extra condition used in Cardy’s method is the duality in boundary partition functions. The partition function calculated in the open-string channel and the closed string channel leads to different expressions, and their equivalence gives a constraint on the boundary states. In the open string channel, the partition function is a sum of the chiral characters, \( Z_{\alpha \beta}(q) = \sum_j n_{\alpha \beta}^j \chi_j(q) \), where \( \alpha \) and \( \beta \) stand for boundary conditions on the two ends of an open string. \( n_{\alpha \beta}^j \) is a non-negative integer representing the multiplicity, and \( \chi_j(q) \) is the character for the representation \( j \). This means \( n_{\alpha \beta}^j \) copies of the representation \( j \) appear in the bulk when the conditions of two boundaries are \( \alpha \) and \( \beta \). We have introduced the modular parameter \( q \) as \( q = e^{-\pi L/T} \). In the closed string channel, the partition function is a tree-level amplitude of a closed string propagating from one boundary \( \alpha \) to the other \( \beta \), which is written as \( \langle \alpha|e^{\pi L/T}|\beta \rangle \). Here, \( H \) is the Hamiltonian \( H = (2\pi/L)(L_0 + \tilde{L}_0 - c/12) \). Using the modular parameter \( \hat{q} = e^{-4\pi T/L} \), the amplitude becomes \( \langle \alpha|(q^{1/2})^{L_0 + \tilde{L}_0 - c/12}|\beta \rangle \). The duality of the partition function now demands \( Z_{\alpha \beta}(q) = \langle \alpha|(q^{1/2})^{L_0 + \tilde{L}_0 - c/12}|\beta \rangle \), which is called Cardy’s consistency condition. Boundary states satisfying the above condition, which we call consistent boundary states and denote with tilde (‘), may be expanded with a complete set of the space of boundary states \( \{|a\} \) and \( \{|a\} \)\). Cardy’s condition is now written as

\[ \sum_j n_{\alpha \beta}^j \chi_j(q) = \sum_{a, b} \langle \tilde{\alpha}|\langle a|(q^{1/2})^{L_0 + \tilde{L}_0 - c/12}|b\rangle \rangle \langle b|\tilde{\beta} \rangle . \]

By solving this equation, the consistent boundaries are expressed as linear sums of the basis states \( \{|a\} \) and \( \{|a\} \). A convenient set of such basis states is the Ishibashi states \( |j\rangle \) [2], which diagonalise the above closed string amplitudes and give characters [1].

Although in some literature the term ‘Ishibashi state’ is used to mean any boundary state satisfying the condition [2], we use this term in a narrower sense meaning the particular solution found by Ishibashi [2]. In this paper we call the states including coherent, Ishibashi and consistent boundary states collectively as ‘boundary states,’ whereas some authors use this term for what we call ‘consistent boundary states’ here.
\[
\langle i | \left( q^{1/2} L_0 + L_0 - c/12 \right) | j \rangle = \delta_{ij} \chi_j(q). \tag{49}
\]
We may, however, choose any set of boundary states for the basis as long as they are complete.

In order to use the above machinery and express the consistent boundary states in terms of the coherent states we found in the last section, we need to calculate the closed string amplitudes between \( |B(\alpha)\rangle \) and \(|B(\beta)\rangle \).

Such amplitudes generally involve screening operators, or floating charges in the bulk. Let us consider the situation where \( m \) positive \((\alpha_+\)) and \( n \) negative \((\alpha_-)\) floating charges are present. The closed-string amplitude for such a process is
\[
A_{\alpha,\beta} = \langle B(\alpha) | e^{-TH} Q_+^m Q_-^m \hat{Q}_+^n \hat{Q}_-^n | B(\beta) \rangle = \langle B(\alpha) | \left( q^{1/2} L_0 + L_0 - c/12 \right) Q_+^m Q_-^m \hat{Q}_+^n \hat{Q}_-^n | B(\beta) \rangle, \tag{50}
\]
where \( Q_{\pm} \) is defined in (34) and
\[
Q_{\pm} = \int d\bar{z} \bar{V}_{\pm}(\bar{z}), \tag{51}
\]
\[
\bar{V}_{\pm}(\bar{z}) = e^{i \sqrt{2} \alpha_{\pm} \bar{q}(\bar{z})}. \tag{52}
\]

The integration contours must be non-self-intersecting closed curves with non-trivial homotopy. In our geometry such contours are the ones which simply go around the cylinder just once. A comment on the uniqueness of the amplitude (50) is in order. It is easy to show that \([Q_+, Q_-] = 0, [Q_+, \hat{Q}_+] = 0, [Q_-, \hat{Q}_-] = 0\) because the holomorphic and antiholomorphic mode operators commute. As the screening operators have trivial conformal dimension, they commute with the Virasoro operators: \([L_n, Q_{\pm}] = 0, [L_n, \hat{Q}_{\pm}] = 0\). In particular, \([L_0, Q_+] = 0\) and \([L_0, \hat{Q}_+] = 0\). Hence the order and the position of the screening operators do not matter and the amplitude with \( m \) positive and \( n \) negative floating charges may be always written in the form (50).

The numbers of the screening charges \( m \) and \( n \) are not arbitrary but they must satisfy the charge neutrality condition (otherwise the amplitude vanishes). Note that our formalism (see the normalisation (53)) demands charge neutrality in both holomorphic and antiholomorphic sectors. In the holomorphic sector, we have charges \(-\alpha\) and \(\beta\) on the boundaries, and \( m \) positive and \( n \) negative screening charges in the bulk. The total charge in the holomorphic part is then
\[
-\alpha + \beta + m\alpha_+ + n\alpha_-, \tag{53}
\]
which must be zero. Similarly, the total charge in the antiholomorphic part is \(-\bar{\alpha} + \bar{\beta} + m\alpha_+ + n\alpha_-\), or, using the condition (53),
\[
\alpha - \beta + m\alpha_+ + n\alpha_-, \tag{54}
\]
which is also zero. Since the sum of the holomorphic and antiholomorphic charges must also vanish, summing above two expressions we have \(ma_+ + na_- = 0\). Now let us recall that the screening charges of the minimal models are characterised by two co-prime integers \(p\) and \(p'\) \((p > p')\) as \(\alpha_+ = \sqrt{p/p'}, \alpha_- = -\sqrt{p'/p}\). Then we have
\[
mp - p'n = 0. \tag{55}
\]
Since \(p\) and \(p'\) are co-prime, \(m\) and \(n\) are written using an integer \(l\) as \(m = lp', n = lp\). This means the net floating charges must vanish in both holomorphic and antiholomorphic sectors. The simplest charge configuration obeying this condition is \(m = n = 0\), or no screening operators. In this case the amplitude (50) is particularly easily evaluated. The oscillating part is calculated with the Heisenberg algebras (8) (35) and repeated use of Hausdorff formula, as
\[
\prod_{k=1}^{\infty} \frac{1}{1 - q^k} = q^{1/24} \bar{\eta}^{\bar{\eta}}. \tag{56}
\]
The zero-mode part,
\[
\langle \alpha, \bar{\alpha}; \alpha_0 | \left( q^{1/2} (a_0^2 + \bar{a}_0^2) / 2 - \sqrt{2} (a_0 + \bar{a}_0) - c/12 \right) | \beta, \bar{\beta}; \alpha_0 \rangle, \tag{57}
\]
is simplified with the central charge (3), the condition on boundary charges for conformal invariance (43) and the operation of zero-modes on the highest-weight vectors (24) (25), as
\[
\langle \alpha, 2\alpha_0 - \alpha; \alpha_0 | q^{a^2 - 2a\alpha + \alpha^2 - 1/24} | \beta, 2\alpha_0 - \beta; \alpha_0 \rangle. \tag{58}
\]
Using the normalisation of the highest weight vectors (37) we have
\[
A_{\alpha,\beta} = \langle B(\alpha) | \left( q^{1/2} L_0 + L_0 - c/12 \right) | B(\beta) \rangle = \frac{q^{(\alpha - \alpha_0)^2}}{\bar{\eta}^{\bar{\eta}}} \kappa^\prime \delta_{\alpha,\beta}. \tag{59}
\]
Note the similarity of these amplitudes to the characters (14) of CBFS. This is not a coincidence, but is understood as follows.

Just as a BCFT on the upper half plane can be viewed as a chiral CFT on the full-plane by the mirroring procedure we mentioned at the beginning of the last section, a BCFT on the annulus may be viewed in two ways (Fig.1).

We can see the system as the holomorphic \((H)\) and antiholomorphic \((\bar{H})\) sectors residing on a finite cylinder, and they are tied together on the two boundaries \((a)\). We can map the antiholomorphic sector \(\bar{H}\) to a continuation of the holomorphic sector, via a parity transformation \(P\). This introduces a torus with two cells separated by two boundaries \((b)\). We may also apply a time-reversal transformation \(T\) as well as \(P\), so that one can go from \(H\) through a boundary to the mapped antiholomorphic sector \(TP(H)\), and then through the other boundary and back to \(H\) along one direction of periodic time. In this
way the BCFT on the annulus can be seen as a chiral theory on the torus. The closed-string picture is based on the non-chiral picture (a), but the amplitude should also represent the chiral picture on the torus (b).

In order to describe the minimal models, it is convenient to introduce boundary states $|a_{r,s}\rangle$ and $|a_{r,-s}\rangle$ defined as

$$|a_{r,s}\rangle = \sum_{k \in Z} |B(k \sqrt{pp'} + \alpha_{r,s})\rangle,$$

$$|a_{r,-s}\rangle = \sum_{k \in Z} |B(k \sqrt{pp'} + \alpha_{r,-s})\rangle.$$  

(60)  

Similarly we define

$$\langle a_{r,s}| = \sum_{k \in Z} \langle B(k \sqrt{pp'} + \alpha_{r,s})|,$$

$$\langle a_{r,-s}| = \sum_{k \in Z} \langle B(k \sqrt{pp'} + \alpha_{r,-s})|.$$  

(61)

These are linear sums of countably many coherent states $|a\rangle$ and (17) defined in the previous section. Using (59) it is shown that

$$\langle a_{r,s} | \langle q^{1/2} L_{0} + L_{0} + c/12 | a_{r',s'}\rangle = \Theta_{pr-p'r',s'} \delta_{r,r'} \delta_{s,s'},$$

$$\langle a_{r,s} | \langle q^{1/2} L_{0} + L_{0} - c/12 | a_{r',s'}\rangle = \Theta_{pr-p'r',s'} \delta_{r,r'} \delta_{s,s'},$$

(64)

$$\langle a_{r,-s} | \langle q^{1/2} L_{0} + L_{0} + c/12 | a_{r',s'}\rangle = -\Theta_{pr-p'r',s'} \delta_{r,r'} \delta_{s,s'},$$

$$\langle a_{r,-s} | \langle q^{1/2} L_{0} + L_{0} - c/12 | a_{r',s'}\rangle = -\Theta_{pr-p'r',s'} \delta_{r,r'} \delta_{s,s'},$$

(65)

and

$$\langle a_{r,s} | \langle q^{1/2} L_{0} + L_{0} + c/12 | a_{r',s'}\rangle = 0.$$  

(66)

Here, we have assumed $1 \leq r, r' < p'$ and $1 \leq s, s' < p$. See App. A for our convention of Jacobi theta functions. We have set $\kappa' = 1$ in (44) and $\kappa' = -1$ in (45). This means the states $|a_{r,s}\rangle$, $\langle a_{r,s}|$ belong to a unitary sector whereas $|a_{r,-s}\rangle$, $\langle a_{r,-s}|$ belong to a non-unitary sector. The amplitudes include all the theta functions appearing in the characters of minimal models (24) and thus we have reproduced the necessary set of boundary states covering the right hand side of the Cardy’s consistency condition (45). We shall see this in detail for the Ising model in the next section. It can be easily checked by using (34) - (39) and the character formula (24) that the states defined as sums of the coherent states,

$$|\langle r, s\rangle\rangle = |a_{r,s}\rangle + |a_{r,-s}\rangle,$$

$$\langle\langle r, s\rangle| = \langle a_{r,s}| + \langle a_{r,-s}|,$$

(67)  

(68)

diagonalise the amplitude and reproduce the minimal characters. These states $|\langle r, s\rangle\rangle$ may then be regarded as the Ishibashi states.

Before discussing the Ising model, we have three points to make about the boundary states $\{|a_{r,s}\rangle, |a_{r,-s}\rangle\}$. Firstly, the amplitudes (44), (45), (46) are diagonal, i.e. the boundary states are all orthogonal to each other. This is a consequence of the diagonal amplitude (53). Indeed, since the boundary charges $k \sqrt{pp'} + \alpha_{r,\pm s}$ are all different for each set of $(r, \pm s, k)$ and the boundary states $\{|a_{r,s}\rangle, |a_{r,-s}\rangle\}$ contain no charges in common, the amplitudes (44), (45) must vanish unless $(r, s) = (r', s')$. The second point is that these boundary states are unique (besides the degeneracy $(r, s) \leftrightarrow (p'-r, p-s)$) as long as we want to reproduce the theta functions as amplitudes between such boundaries. The infinite sum expressions (43) for the theta functions are power series of $q$, and the power is related to the boundary charge through the expression (58). By superimposing the boundary charges appearing in the expression of theta functions, the boundary states are constructed without ambiguity. Third, the negative-norm states $|a_{r,-s}\rangle$ seem to be unavoidable even for the unitary minimal models. The highest-weight vector $|\alpha, \alpha; 0\rangle$ is built on the vacuum $|0, 0; 0\rangle$ by operating with $e^{i\sqrt{2}\alpha_0\phi_0}$ and $e^{i\sqrt{2}\alpha_0\phi_0}$, and its norm $\kappa'$ is due to the normalisation of the vacuum $|0, 0; 0\rangle$ as $|0, 0; 0\rangle = \kappa'$. This $\kappa'$ may be rescaled to an arbitrary real number as long as it is either positive or negative definite, but the sign cannot be changed by the rescaling. The states with $\kappa' = 1$ and $\kappa' = -1$ (that is, $|a_{r,s}\rangle$ and $|a_{r,-s}\rangle$ above) therefore belong to different sectors with no intersection.

FIG. 1. BCFT of holomorphic and antiholomorphic sectors glued together on the two boundaries of a finite cylinder (a) is equivalently described as a chiral theory on a torus (b), where the antiholomorphic sector is now regarded as continuation of the holomorphic sector, with parity and time-reversal transformations.
V. ISING MODEL BOUNDARY STATES

The Ising model is the simplest non-trivial minimal model, with the two characterising co-prime integers $p = 4$, $p' = 3$, and the central charge 1/2. Also it is clearly one of the most extensively studied critical systems described by two-dimensional CFT. For its detailed description we refer the readers to e.g. [3]. In this section we demonstrate that the boundary states constructed in the previous section are enough to reproduce the known physical boundary states of the Ising model, by a parallel discussion with Cardy’s original paper [4].

The critical Ising model is known to have three physical boundary states, corresponding to the two fixed (up and down) and one free boundary conditions. They are identified and expressed as particular linear combinations of the Ishibashi states by solving the consistency equation [4]. The characters of minimal models are linearly transformed under $\tau \to \tilde{\tau} = -1/\tau$ as $\chi_i(q) = \sum_i S_{ij} \chi_j(q)$. Substituting this and (48) into (49), and equating the coefficients of the character functions we have

$$\sum_i n^i_{\alpha\beta} S_{ij} = \langle \tilde{a}|j\rangle \langle j|\tilde{b}\rangle. \quad (69)$$

We assume a state $|0\rangle$ satisfies the condition $n^i_{\alpha\beta} = n^i_{\alpha\beta} = \delta^i_\alpha \delta^i_\beta$. Putting $\tilde{a} = \tilde{b} = 0$ in (49) we have $\langle 0|0\rangle = \sqrt{S_{00}}$. Similarly letting $\tilde{a} \neq 0$ and $\tilde{b} = 0$ we find $\langle \tilde{a}|j\rangle = S_{a_{\alpha}/\sqrt{S_{00}}}$. Substituting these back into (48) we have

$$\sum_i n^i_{\alpha\beta} S_{ij} = \frac{S_{a_{\alpha}S_{\beta}j}}{S_{0j}}. \quad (70)$$

This is identical to the Verlinde formula [20] and therefore $n^i_{\alpha\beta}$ is concluded to be the same as the fusion coefficients. The consistent boundary states are now expressed using the Ishibashi states as

$$\langle \tilde{a}| = \sum_j \langle j| \langle j|\tilde{a}\rangle = \sum_j \frac{S_{\alpha|j}}{\sqrt{S_{0j}}} \langle j| \rangle. \quad (71)$$

The operators appearing in the Kac table of the Ising model are $\phi_{(1,1)} = \phi_{(2,3)}$, $\phi_{(2,1)} = \phi_{(1,3)}$ and $\phi_{(1,2)} = \phi_{(3,2)}$, which are identified as the identity $I$, the energy $\epsilon$ and the spin $\sigma$ operators, having the conformal dimensions 0, 1/2, 1/16, respectively. Since we know the modular matrix $S_{ij}$ for these representations, from (41) we immediately have

$$|\tilde{I}| = |0\rangle = 2^{-1/2}|I\rangle + 2^{-1/2}|\epsilon\rangle + 2^{-1/4}|\sigma\rangle, \quad (72)$$

$$|\tilde{\epsilon}| = 2^{-1/2}|I\rangle + 2^{-1/2}|\epsilon\rangle - 2^{-1/4}|\sigma\rangle, \quad (73)$$

$$|\tilde{\sigma}| = |I\rangle - |\epsilon\rangle. \quad (74)$$

Since the first two lines differ only by the sign of the Ishibashi state $|\sigma\rangle$ associated to the spin operator, they are identified as the fixed (up or down) boundary states. The last line then corresponds to the free boundary state.

Now let us show that the above procedure can be reproduced using the coherent states on CBFS instead of the Ishibashi states. The three characters for the three operators of the Ising model follow immediately from (24) as

$$\chi_{\ell}(q) = \chi_{1,1}(q) = \frac{1}{\eta(q)} \Theta_{1,12}(\tau) - \Theta_{7,12}(\tau), \quad (75)$$

$$\chi_{\ell}(q) = \chi_{2,1}(q) = \frac{1}{\eta(q)} \Theta_{2,12}(\tau) - \Theta_{11,12}(\tau), \quad (76)$$

$$\chi_{\sigma}(q) = \chi_{2,2}(q) = \frac{1}{\eta(q)} \Theta_{2,12}(\tau) - \Theta_{11,12}(\tau). \quad (77)$$

Using the modular transformation formula of the theta functions [43] they are written as

$$\chi_{\ell}(q) = \frac{\Theta_{1,12}(\tau) + \Theta_{5,12}(\tau) - \Theta_{7,12}(\tau) - \Theta_{11,12}(\tau)}{2\eta(\tau)} + \frac{\Theta_{2,12}(\tau) - \Theta_{10,12}(\tau)}{\sqrt{2}\eta(\tau)}, \quad (78)$$

$$\chi_{\ell}(q) = \frac{\Theta_{1,12}(\tau) + \Theta_{5,12}(\tau) - \Theta_{7,12}(\tau) - \Theta_{11,12}(\tau)}{2\eta(\tau)} - \frac{\Theta_{2,12}(\tau) - \Theta_{10,12}(\tau)}{\sqrt{2}\eta(\tau)}, \quad (79)$$

$$\chi_{\sigma}(q) = \frac{\Theta_{1,12}(\tau) - \Theta_{5,12}(\tau) - \Theta_{7,12}(\tau) + \Theta_{11,12}(\tau)}{\sqrt{2}\eta(\tau)}. \quad (80)$$

These are the character functions appearing in the open-string channel (left hand side of the consistency equation [43]. In the closed string channel of [33] we expand the states with $|\sigma, \pm k\rangle$ and $|\sigma, \pm k\rangle$ defined in (69) - (83), with $1 \leq \ell \leq 2$, $1 \leq s \leq 3$, and $3s < 4\tau$. In the Ising model the non-trivial amplitudes (45), (46) are

$$\langle a_{11}|q^{1/2}(L_0 + L_0 - c/12)|a_{11}\rangle = \Theta_{1,12}(\tau)/\eta(\tau), \quad (81)$$

$$\langle a_{22}|q^{1/2}(L_0 + L_0 - c/12)|a_{22}\rangle = \Theta_{2,12}(\tau)/\eta(\tau), \quad (82)$$

$$\langle a_{1,1}|q^{1/2}(L_0 + L_0 - c/12)|a_{1,1}\rangle = \Theta_{5,12}(\tau)/\eta(\tau), \quad (83)$$

$$\langle a_{1,-1}|q^{1/2}(L_0 + L_0 - c/12)|a_{1,-1}\rangle = -\Theta_{7,12}(\tau)/\eta(\tau), \quad (84)$$

$$\langle a_{2,-2}|q^{1/2}(L_0 + L_0 - c/12)|a_{2,-2}\rangle = -\Theta_{10,12}(\tau)/\eta(\tau), \quad (85)$$

$$\langle a_{2,-1}|q^{1/2}(L_0 + L_0 - c/12)|a_{2,-1}\rangle = -\Theta_{11,12}(\tau)/\eta(\tau). \quad (86)$$

Substituting these into the right hand side of (48) and equating the coefficients of $\Theta_{1,12}(\tau)/\eta(\tau)$, $\Theta_{2,12}(\tau)/\eta(\tau)$, $\Theta_{5,12}(\tau)/\eta(\tau)$, $\Theta_{7,12}(\tau)/\eta(\tau)$, $\Theta_{10,12}(\tau)/\eta(\tau)$ and $\Theta_{11,12}(\tau)/\eta(\tau)$ on both sides, we have

$$\frac{1}{2} n^{i}_{\alpha\beta} + \frac{1}{2} n^{i}_{\alpha\beta} = \langle \tilde{a}|a_{1,1}\rangle \langle a_{1,1}|\tilde{b}\rangle, \quad (87)$$

$$\frac{1}{2} n^{i}_{\alpha\beta} - \frac{1}{2} n^{i}_{\alpha\beta} = \langle \tilde{a}|a_{1,2}\rangle \langle a_{1,2}|\tilde{b}\rangle, \quad (88)$$
\[ \frac{1}{2} n_{\alpha \beta} + \frac{1}{2} n^\ast_{\alpha \beta} - \frac{1}{\sqrt{2}} n^\ast_{\alpha \beta} = \langle \tilde{\alpha}|a_{1,3}\rangle \langle a_{1,3}|\tilde{\beta} \rangle, \quad (89) \]

\[ \frac{1}{2} n_{\alpha \beta} + \frac{1}{2} n^\ast_{\alpha \beta} + \frac{1}{\sqrt{2}} n^\ast_{\alpha \beta} = \langle \tilde{\alpha}|a_{1,-1}\rangle \langle a_{1,-1}|\tilde{\beta} \rangle, \quad (90) \]

\[ \frac{1}{\sqrt{2}} n_{\alpha \beta} - \frac{1}{\sqrt{2}} n^\ast_{\alpha \beta} = \langle \tilde{\alpha}|a_{1,-2}\rangle \langle a_{1,-2}|\tilde{\beta} \rangle, \quad (91) \]

\[ \frac{1}{\sqrt{2}} n_{\alpha \beta} - \frac{1}{\sqrt{2}} n^\ast_{\alpha \beta} = \langle \tilde{\alpha}|a_{1,-3}\rangle \langle a_{1,-3}|\tilde{\beta} \rangle. \quad (92) \]

Let us find the coefficients assuming that they are real and \((\tilde{\alpha}|a_{r,+}\rangle = (a_{r,-}|\tilde{\alpha} \rangle).\) We start by letting \(\tilde{\alpha} = \tilde{\beta} = 0\).

The first equation \((87)\) gives 
\[ |\langle \tilde{\alpha}|a_{1,1}\rangle|^2 = 1/2 \]
and we can choose \(\langle \tilde{\alpha}|a_{1,1}\rangle = 1/\sqrt{2}.\) Likewise, from \((88)\) - \((92)\) we find \(\langle \tilde{\alpha}|a_{2,2}\rangle = \langle \tilde{\alpha}|a_{2,-2}\rangle = 2^{-1/4}, \langle \tilde{\alpha}|a_{2,1}\rangle = \langle \tilde{\alpha}|a_{1,-1}\rangle = 0,\) \(\langle \tilde{\alpha}|a_{2,-1}\rangle = 1/\sqrt{2}.\) Next, letting \(\tilde{\alpha} = \tilde{\beta}\) and \(\tilde{\beta} = 0\) we find \(\langle \tilde{\alpha}|a_{1,1}\rangle = \langle \tilde{\alpha}|a_{2,1}\rangle = \langle \tilde{\alpha}|a_{1,-1}\rangle = \langle \tilde{\alpha}|a_{2,-1}\rangle = 1/\sqrt{2},\) \(\langle \tilde{\alpha}|a_{2,2}\rangle = \langle \tilde{\alpha}|a_{2,-2}\rangle = 2^{-1/4}.\) Lastly, putting \(\tilde{\alpha} = \tilde{\sigma}\) and \(\tilde{\beta} = 0\) we find \(\langle \tilde{\sigma}|a_{1,1}\rangle = \langle \tilde{\sigma}|a_{1,-1}\rangle = 1, \langle \tilde{\sigma}|a_{2,2}\rangle = \langle \tilde{\sigma}|a_{2,-2}\rangle = 0\) and \(\langle \tilde{\sigma}|a_{2,1}\rangle = \langle \tilde{\sigma}|a_{2,-1}\rangle = -1.\) Then the consistent boundary states are expressed in terms of the coherent states as

\[ |\tilde{I} \rangle = |\tilde{0} \rangle = 2^{-1/2}(|a_{1,1}\rangle + |a_{1,-1}\rangle + |a_{2,1}\rangle + |a_{2,-1}\rangle) + 2^{-1/4}(|a_{2,2}\rangle + |a_{2,-2}\rangle), \quad (93) \]

\[ |\tilde{e} \rangle = 2^{-1/2}(|a_{1,1}\rangle + |a_{1,-1}\rangle + |a_{2,1}\rangle + |a_{2,-1}\rangle) - 2^{-1/4}(|a_{2,2}\rangle + |a_{2,-2}\rangle), \quad (94) \]

\[ |\tilde{\sigma} \rangle = |a_{1,1}\rangle + |a_{1,-1}\rangle - |a_{2,1}\rangle - |a_{2,-1}\rangle. \quad (95) \]

These are exactly the same result as \((72)\) - \((74),\) as the relation between the Ishibashi states and the coherent states are given in \((77)\) and \((78).\) We have thus shown for the Ising model that the coherent states constructed on CBFS are not merely a subspace of the boundary states, but they cover the space spanned by Cardy’s consistent boundary states.

In the case of the Ising model, a similar construction of the boundary states from coherent states has been done using free Majorana fermions \([28, 29].\) In a sense the present analysis is a generalisation of such a construction to general minimal theories.

**VI. DISCUSSION**

In this paper we have constructed a set of coherent states on CBFS which preserve the conformal invariance, and argued that Cardy’s consistent boundary states for minimal models are expressed as linear combinations of such states. We have demonstrated this explicitly in the example of Ising model. Our approach provides a new intuitive picture of boundary states in CFT, in terms of the boundary charges which obey the charge neutrality conditions with bulk screening operators.

We would like to conclude this paper by stressing that, apart from giving a new interpretation of boundary states, this approach is quite powerful in at least two respects.

Firstly, once consistent boundary states are expressed in terms of the coherent states, it is in principle possible to compute boundary \(n\)-point functions on an annulus directly without resorting to extra information on the boundary. The \(n\)-point function on the upper half plane involving an operator \(\phi_{r,s}\) is found in the conventional method by solving the \((r \times s)\)-th order differential equations satisfied by the \(2n\)-point function on the full plane \([1].\) Solutions to such a differential equation are in the form \(A_1 F_1 + A_2 F_2 + \cdots\) where \(F_i\) represent the conformal blocks, and the coefficients \(A_i\) reflect boundary conditions and are determined by considering e.g. the asymptotic behaviour of the \(n\)-point function. In our Coulomb-gas approach, \(n\)-point functions on an annulus are obtained by inserting vertex operators between the boundary-to-boundary amplitudes, with appropriate inclusion of screening operators, leading to an integral representation of the correlation functions. In practice, however, such expressions involving multiple integrals of theta-functions are not always easy to evaluate.

The second advantage of our approach is its wide applicability. The Coulomb-gas approach is not constrained to the minimal models, but it also applies to WZNW models \([29]\) and CFTs involving \(W\)-algebras \([30, 31]\). Although we only presented the results of the simplest minimal model here, generalizations of our approach to these models also seem to be possible. We hope to come back to these issues in future publications.

**ACKNOWLEDGMENTS**

I am grateful to John Wheater for fruitful discussions and careful reading of the manuscript. I also appreciate helpful conversations with Ian Kogan and Alexei Tsvelik.

**APPENDIX A: JACOBI THETA FUNCTIONS**

In this appendix we list some formulae of elliptic functions used in the main text. The Jacobi theta function \(\Theta_{\lambda,\mu}(\tau)\) and the Dedekind eta function \(\eta(\tau)\) are defined as

\[ \Theta_{\lambda,\mu}(\tau) = \sum_{k \in \mathbb{Z}} q^{(2\mu k + \lambda)^2/4\mu}, \quad (A1) \]

\[ \eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n), \quad (A2) \]

where \(q = e^{2\pi i \tau}.\) From this definition it is obvious that \(\Theta_{\lambda,\mu}(\tau)\) has the following symmetries:

\[ \Theta_{\lambda,\mu}(\tau) = \Theta_{\lambda + 2\mu,\mu}(\tau) = \Theta_{-\lambda,\mu}(\tau). \quad (A3) \]
They transform under the modular S transformation \((\tau \to -1/\tau)\) and the modular T transformation \((\tau \to \tau + 1)\) as

\[
\Theta_{\lambda,\mu}(-1/\tau) = \sqrt{-\frac{i\tau}{2\mu}} \sum_{\nu=0}^{2\mu-1} e^{\lambda\nu/\mu} \Theta_{\lambda,\mu}(\tau),
\]

\[
\eta(-1/\tau) = \sqrt{-i\tau\eta(\tau)},
\]

(A4)

and

\[
\Theta_{\lambda,\mu}(\tau + 1) = e^{2\pi i/2\mu} \Theta_{\lambda,\mu}(\tau),
\]

\[
\eta(\tau + 1) = e^{\pi i/12} \eta(\tau).
\]

(A5)

In the main text we only used the modular S transformation.

The theta functions we used for the Ising model are related to another commonly used notation,

\[
\theta_2(\tau) = \sum_{k \in \mathbb{Z}} q^{(k+1/2)^2/2},
\]

(A6)

\[
\theta_3(\tau) = \sum_{k \in \mathbb{Z}} q^{k^2/2},
\]

(A7)

\[
\theta_4(\tau) = \sum_{k \in \mathbb{Z}} (-1)^k q^{k^2/2},
\]

(A8)

by

\[
\sqrt{\eta(\tau)\theta_2(\tau)} = \Theta_{2,12}(\tau) - \Theta_{10,12}(\tau),
\]

(A9)

\[
\sqrt{\eta(\tau)\theta_3(\tau)} = \Theta_{1,12}(\tau) + \Theta_{5,12}(\tau) - \Theta_{7,12}(\tau) - \Theta_{11,12}(\tau),
\]

(A10)

\[
\sqrt{\eta(\tau)\theta_4(\tau)} = \Theta_{1,12}(\tau) - \Theta_{5,12}(\tau) - \Theta_{7,12}(\tau) + \Theta_{11,12}(\tau).
\]

(A11)

\[\text{References:}\]

[1] J. L. Cardy, Nucl. Phys. B 240 (1984) 514.
[2] J. L. Cardy, Nucl. Phys. B 275 (1986) 200.
[3] J. L. Cardy, Nucl. Phys. B 324 (1989) 581.
[4] J. L. Cardy and D. C. Lewellen, Phys. Lett. B 259 (1991) 274.
[5] G. Pradisi, A. Sagnotti and Y. S. Stanev, Phys. Lett. B 381 (1996) 97.
[6] I. Affleck, M. Oshikawa and H. Saleur, J. Phys. A 31 (1998) 5827.
[7] J. Fuchs and C. Schweigert, Phys. Lett. B 441 (1998) 141.
[8] D. C. Lewellen, Nucl. Phys. B 372 (1992) 654.
[9] I. Runkel, Nucl. Phys. B 549 (1999) 563; ibid. 579 (2000) 561.
[10] R. E. Behrend, P. A. Pearce, V. B. Petkova and J. B. Zuber, Phys. Lett. B 444 (1998) 163; Nucl. Phys. B 570 (2000) 525 [Nucl. Phys. B 579 (2000) 707].