Self-Similarity Investigation on 3D Pigeon-hole Model as 3D Fractal Rock Model

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Abstract. A self-similarity relationship to the fractal dimension of 3D pigeon-hole rock models has been obtained by applying the box-counting fractal dimension. Self-similarity is a characteristic of fractals which shows that fractals consist of similarly shaped parts at each scale and produced the same 3D fractal dimension. 3D pigeon-hole models with self-similarity principle in structure and substructure (level 0, 1, 2) were constructed. Nine 3D rock models were generated with the size of $210 \times 210 \times 210$ pixels and porosity of 30%. For each complexity levels (0, 1, 2), the grains were varied to have a size (radius) of 6-10 pixels, 11-15 pixels, and 16-20 pixels. The average correlation coefficient of 3D fractal dimension results obtained for 3D pigeon-hole model similar with 3 different range of radius for level 0, 1, 2 is 0.921. Based on the level of pigeon-hole produced on the model, the fractal dimension is increases as level increases, and it is found consistent in each radius range. The average correlation coefficient of 3D fractal dimension obtained for 3D pigeon-hole model for level 0, 1, 2 in three different radius range is 0.990. Based on the radius range on the model, the fractal dimension is decreases as radius range increases, and it is found consistent in each level of pigeon-hole. As a conclusion, based on these result 3D pigeon-hole model can be considered as fractal rock model based on its self-similarity both in similary structure and substructure and its fractal dimension and grain shape model that close to rock grains. Regardless of the random distribution of the grain position, and the grain overlap condition in the modeling scheme.

1. Introduction
Several studies in the last decade have shown that many natural porous rock formations have fractal properties, i.e., they are self-similar over several length scales [1]. The concept of fractals was first introduced by Mandelbrot et al [2]. This name was coined by Mandelbrot from the Latin *Fractus* which describes the appearance of a broken stone. Fractal is an object that has self-similarity in different scales. That means, parts of the object looks the same as the object itself when viewed as a whole.

Several studies regarding fraccals dimension of digital data of rocks have been developed such as by Daigle et al [3] where they have been determined the dimensions of fractal of the nuclear magnetic resonance data on rocks with an internal magnetic field gradient. It was also found that the influence of the internal magnetic field gradient is most prominent on the rocks with pore size and the contrast between the high magnetic susceptibility liquid pores and mineral grains. Bieganowski et al [4] calculated the fractal dimension using PSD (Particle Size Distribution) method and it is found that the
amount of fraction and division or interval affect the calculated fractal dimension thus it is necessary to standardize the procedures of calculation. Three dimensional fractal dimension in geological rock structure has also been investigated by Feranie et al [1] and they found that the measured 3D box counting fractal dimension of geological rock samples are not integers which implies that the structure of geological rock samples is fractal in nature. 2D fracture upscaling using Iterated Function System (IFS) fractal model on geothermal rock has been investigated by Tobing et al [5] and it is found that fractal upscaling on geothermal rock using IFS fractal model is potential to larger scale model of 2D fracture.

Several techniques have been developed to acquire digital representation of the microstructure of rocks. The techniques includes the reconstruction by means of imaging instruments as well as by computer modeling. The reconstruction of rock models to mimic original rocks is important because the characterization of pore structures is considered easier and cheaper by developing computer rock models rather than digital reconstruction of real rocks.

In this study porous rock models were generated by grain based method using pigeon-hole grain model. This modeling method has been previously developed by Pape et al [5] for the 2D porous rock. The complexity of the pore space can be generated by increasing the level of the pigeon-hole model. Study on 3D pigeon-hole has been developed by the Latief et al [6] where the characteristics of pore structure of 3D pigeon-hole rock model was investigated and it was found that qualitatively, the pigeon-hole 3D rock model has similar characteristics as compared to Fontainebleau’s low porosity sandstone and Berea sandstone. The purpose of this study is to analyse the influence of the pigeon-hole complexity level and the grain radius against fractal dimension of the porous rock for the same porosity.

2. Research Methodology

According to Latief [6], a 3D pigeon-hole model has several advantages compared to the 2D model where 3D models can describe more complex structure and is able to describe the pore space and connectivity between the pore spaces. The basic structure of the pigeon-hole model is a spherical object in which the spherical object was surrounded by a substructure of the same shape with smaller size. Example of 3D pigeon-hole model construction is shown in Figure 1.

![Figure 1. Grain model using 3D pigeon-hole with similarity level of (a) 0, (b) 1, and (c) 2.](image)

Figure 1.a is the construction of the 3D pigeon-hole model at the pigeon-hole level 0 (PH0). Figure 1.b is the construction of the 3D pigeon-hole model at the pigeon-hole level 1 (PH1). Figure 1.c shows the construction of the 3D pigeon-hole model at the pigeon-hole level 2 (PH2). The pigeon-hole level 0 is pure spherical. For the next level, the base sphere is surrounded by smaller spheres, as seen in Figure 1.b. Level 2 of the pigeon-hole model was generated with similar approach (see Figure 1.c).

Nine 3D rock models were generated with a porosity of 30% with the size of 210 × 210 × 210 pixels. The grains were deposited randomly, thus they might overlap with each other. For each complexity levels (0, 1, 2), the grains were varied to have a size (radius) of 6-10 pixels, 11-15 pixels, and 16-20 pixels.
The measurement of 3D fractal objects was done based on the notion of the fractal dimension of Hausdorff-Besicovitch which can be described as follows:

\[ V = N(r)r^{D_{3D}}, \]  

(1)

where, \( r \) is the side length of the cube, \( N(r) \) is the number of the sub-cubes that covers a fractal object, \( D_{3D} \) is the fractal dimension. The fractal dimension itself is a dimension of a fractal object that has values that do not include integers. Based on Equation (1), the fractal dimension can be calculated as follows:

\[ D_{3D} = -\frac{\Delta \ln N(r)}{\Delta \ln r}. \]  

(2)

### 3. Results and Discussion

After a series of modelling procedures were performed, the following results can be analysed. Visualization of 3D pigeon-hole models of porous rocks is shown in Figure 2 through 4. The fractal dimension were obtained and presented in Table 1.

**Figure 2.** Models of 3D pore medium level 0 with grain radius of (a) 6-10 pixels, (b) 11-15 pixels, (c) 16-20 pixels.

**Figure 3.** Models of 3D pore medium level 1 with grain radius of (a) 6-10 pixels, (b) 11-15 pixels, (c) 16-20 pixels.
Figure 4. Models of 3D pore medium level 2 with grain radius of (a) 6-10 pixels, (b) 11-15 pixels, (c) 16-20 pixels.

Table 1. Calculation of fractal dimension.

|         | pigeon-hole level 0 | pigeon-hole level 1 | pigeon-hole level 2 | Mean  | Deviation |
|---------|---------------------|---------------------|---------------------|-------|-----------|
| R1      | 2.9959              | 2.9995              | 2.9997              | 2.9983| 0.0021    |
| R2      | 2.9716              | 2.9843              | 2.9974              | 2.9844| 0.0129    |
| R3      | 2.9434              | 2.9729              | 2.9923              | 2.9695| 0.0246    |
| Mean    | 2.9703              | 2.9855              | 2.9964              |       |           |
| Deviation| 0.0262              | 0.0133              | 0.0037              |       |           |

From the results, it can be analysed that the mean and deviation of the fractal dimensions obtained for the 3D pigeon-hole model for three ranges of radius of 6-10 pixels, 11-15 pixels, and 16-20 pixels at the level of 0, 1, 2 each is (2.970 ± 0.026), (2.985 ± 0.013), (2.996 ± 0.003). Based on the level of pigeon-hole generated on the model, the fractal dimension increases with the increase of the pigeon-hole level.

The mean and deviation of the fractal dimensions obtained for the 3D pigeon-hole model for levels 0, 1, 2 in three different radius range from 11 - 20 pixels, 11-15 pixels and 16-20 pixels respectively are (2.998 ± 0.002), (2.984 ± 0.012), (2.969 ± 0.024). Based on the range of radius of grains on the model, the fractal dimension decreases as the grain radius increases.

Figure 5.a shows the plots of the fractal dimension for three different levels of pigeon-hole (0, 1, 2) for the sphere of radius in the range 6-10 pixels, 11-15 pixels, and 16-20 pixels. From the data shown in Figure 5.a, it is found that the greater the level of pigeon-hole model, the fractal dimension is also greater. The increase of the fractal dimension is consistent across each range of radius of the grain, as shown by the blue straight curve for a model of grains of 6-10 pixels radius, orange dashed curve for model of grains of 11-15 pixels radius, and grey dotted curve for a model of grains of 16-20 pixels radius. The increase of fractal dimension occurs because higher pigeon-hole levels cause the structure and sub-structure of the model to be more similar, which makes the self-similarity of the model to be higher. Since the self-similarity is a fractal characteristic, thus the greater the self-similarity, the fractal dimension also increases. It can be observed that the relationship between the level of pigeon-hole with the porous medium fractal dimension is directly proportional.
Figure 5. a: Fractal dimension vs. Levels of pigeon-hole (0, 1, 2) with grains size of 6-10 pixels, 11-15 pixels, and 16-20 pixels. b: Fractal dimensions vs. grains size (6-10 pixels, 11-15 pixels, and 16-20 pixels) with levels of pigeon-hole 0, 1, 2.

Figure 5.b shows the plot of fractal dimension vs the grain radius in the range of 6-10 pixels, 11-15 pixels, and 16-20 pixels for the three levels of pigeon-hole. Figure 5.b shows that the larger the radius of the sphere, the smaller the fractal dimension. The decrease in fractal dimension is consistent on each level of the self-similarity in the model, shown on the blue straight curve for the grain model with the self-similarity level 0, the orange dashed curve for the grain model with the self-similarity level 1, and the grey dotted curve for the grain model with the self-similarity level 2. The decrease in fractal dimension occurs because in the medium with the same volume, the smaller the sphere radius the number of sphere grids that fill the medium will be greater. The more grains which are deposited in the medium, the grain structure and pore structure on the medium is more complex. Therefore the calculation of the fractal dimension using box-counting method, the grains (matrix) will be detected more in each box in the model. Thus the relationship between the radius of the grain with the porous medium fractal dimension is inversely proportional.

Among the models, the highest percolation rate was observed for the PH0 model and qualitatively the PH0 model is the most permeable model [6]. Percolation is inversely related to complexity, thus it can be analysed that based on the pigeon-hole level, higher pigeon-hole level causes the grain structure and pore structure to be more complex and it causes the fractal dimension to increase. It is found that smaller grain size is related to higher fractal dimension. This can be explained as follows: random deposition of overlapped small grains produces small and complex pore structure. Thus for the calculation of fractal dimension using the box-counting method, this will result in a larger volume of solid than the ---, which yields a higher fractal dimension number. By understands the influence of the level of pigeon-hole and the size of the grain to the fractal dimension, we can determine the characteristics of the existing rocks in nature, determine the fractal dimension, and apply the concept of upscaling for the structure of the rock.

4. Conclusion
The fractal dimension of the models increases as level increases and it is consistent in each radius range. The fractal dimension of the models decreases as the range of grain size (radius) increases, and it is also consistent in each level of pigeon-hole. As a conclusion, based on these results the 3D pigeon-hole model can be considered as fractal rock model based on its self-similarity both in similarity of the shape as well as the fractal dimension.

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