HOPF ALGEBRA ACTIONS AND RATIONAL IDEALS

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ABSTRACT. This note discusses a framework for the investigation of the prime spectrum of an associative algebra $A$ that is equipped with an action of a Hopf algebra $H$. In particular, we study a notion of $H$-rationality for ideals of $A$ and comment on a possible Dixmier-Moeglin equivalence for $H$-prime ideals of $A$.

INTRODUCTION

0.1. Actions of a group or Lie algebra on a given algebra $A$ have proven a useful tool in analyzing its prime spectrum, Spec $A$; see, e.g., [9], [12]. The present article explores the more general situation where a Hopf algebra $H$ acts on $A$. Our particular focus will be on a notion of rationality for prime ideals of $A$ that takes the $H$-action into account. Rational ideals were introduced into non-commutative algebra by Dixmier in connection with his investigation of primitive ideals in enveloping algebras of finite-dimensional Lie algebras. A highlight of this work is the celebrated Dixmier-Moeglin equivalence [8], [19]: rationality is equivalent to primitivity and also to local closedness for primes of enveloping algebras. In the context of arbitrary associative algebras, a notion of rationality was defined in [11].

0.2. Let $A$ be an associative algebra (with 1) over a field $k$ and let $H$ be a Hopf $k$-algebra. A (left) $H$-action on $A$ is a $k$-linear map $H \otimes_k A \rightarrow A$, $h \otimes a \mapsto h.a$, that makes $A$ into a left $H$-module and satisfies $h.(ab) = (h_1.a)(h_2.b)$ and $h.1 = (\varepsilon, h)1$ for all $h \in H$ and $a, b \in A$. Here, $\Delta h = h_1 \otimes h_2$ is the comultiplication $\Delta : H \otimes H \rightarrow H$ and $\varepsilon : H \rightarrow k$ is the counit. We will write $H \subseteq A$ to indicate such an action. An algebra $A$ that is equipped with an $H$-action is called an $H$-module algebra. With algebra maps that are also $H$-module maps as morphisms, $H$-module algebras form a category, $H_{\text{Alg}}$.

0.3. Let $A \in H_{\text{Alg}}$. An ideal $I$ of $A$ that is also an $H$-submodule will be referred to as an $H$-ideal. In this case, $A/I \in H_{\text{Alg}}$. The sum of all $H$-ideals of $A$ that are contained in a given arbitrary ideal $I$, clearly the unique maximal $H$-ideal of $A$ that is contained in $I$, will be called the $H$-core of $I$ and denoted by $I:H$. Explicitly,

\[ I:H = \{ a \in A \mid H.a \subseteq I \}. \]

If $A \neq 0$ and the product of any two nonzero $H$-ideals of $A$ is again nonzero, then $A$ is said to be $H$-prime. An $H$-ideal $I$ of $A$ is called $H$-prime if $A/I$ is $H$-prime. We will denote the

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collection of all $H$-primes of $A$ by $H$-$\text{Spec} \ A$. It is easy to see that $H$-cores of prime ideals are $H$-prime. Thus, we have a map,$$
abla H = \cdot : H \rightarrow H$-$\text{Spec} \ A$.

A routine application of Zorn’s Lemma shows that $\nabla H$ is surjective if the (two-sided) ideal of $A$ that is generated by $H.a$ is finitely generated for each $a \in A$ [14, Exercise 10.4.4]. This certainly holds if $A$ is noetherian or the action $H \subset A$ is locally finite. The fibers of $\nabla H$ are called the $H$-$\text{strata}$ of $\text{Spec} \ A$:

$$\text{Spec}_I A := \nabla H^{-1}(I) = \{ P \in \text{Spec} \ A \mid P \cdot H = I \} \quad (I \in H$-$\text{Spec} \ A).$$

0.4. Rational and $H$-rational ideals are special prime and $H$-prime ideals, respectively, that are of particular interest. They are defined in Section 3 below in terms of the symmetric ring of quotients rather than the right Amitsur-Martindale quotient ring that was employed in [11]. While this makes no essential difference, because the centers of these quotient rings are isomorphic for semiprime rings [14, E.3], symmetric quotient rings have some advantages such as their apparent right-left symmetry. Sections 1 and 2 deploy the requisite background material on symmetric quotient rings. In Section 3, we show that the core map $\nabla H$ sends rational ideals to $H$-rational ideals. Thus, denoting the collections of rational and $H$-rational ideals of $A$ by $\text{Rat} \ A$ and $H$-$\text{Rat} \ A$, respectively, we have a diagram of maps,

$$\begin{array}{ccc}
\text{Spec} \ A & \xrightarrow{\nabla H} & H$-$\text{Spec} \ A \\
\cup & & \cup \\
\text{Rat} \ A & \xrightarrow{\nabla H} & H$-$\text{Rat} \ A
\end{array} \quad \text{(1)}$$

The notion of $H$-rationality and the diagram (1) have previously been explored in some special cases including the following.

0.4.1. If $H = U \mathfrak{g}$ is the enveloping algebra of a Lie $\mathbb{k}$-algebra $\mathfrak{g}$, then an $H$-module algebra structure on $A$ amounts to a Lie homomorphism $\mathfrak{g} \rightarrow \text{Der} \ A$, the Lie algebra of derivations of $A$. A special case of this setup arises from a Poisson structure on $A$, that is, a $\mathbb{k}$-bilinear map $\{ \cdot, \cdot \} : A \times A \rightarrow A$ such that $\mathfrak{g} := (A, \{ \cdot, \cdot \})$ is a Lie $\mathbb{k}$-algebra and $\{ a, \cdot \} \in \text{Der} \ A$ for all $a \in A$. We then have an action of $H = U \mathfrak{g}$ on $A$ that is determined by $x.a = \{ x, a \}$ for $x \in \mathfrak{g}, a \in A$. In this setting, $H$-cores and $H$-primes are called Poisson cores and Poisson primes, respectively, $H$-$\text{Spec} \ A$ is denoted by $\mathcal{P}$-$\text{Spec} \ A$ or similar, and $H$-rational ideals are called Poisson rational. See, for example, [6], [9], [2].

0.4.2. References [11], [12] consider (1) for a group algebra $H = \mathbb{k}G$, with particular emphasis on the case where $\mathbb{k}$ is algebraically closed and $G$ is an affine algebraic $\mathbb{k}$-group that acts rationally on $A$. In this setting, both maps in (1) are surjective and their fibers are known. The fibers of $\nabla H$ are exactly the $G$-orbits in $\text{Rat} \ A$. When $G$ is connected, the strata $\text{Spec}_I A$ can be described in terms of the spectra of certain commutative algebras; more generally, this holds for the $H$-strata of any “integral” action $H \subset A$ with $H$ cocommutative [15]. If $G$ is an algebraic $\mathbb{k}$-torus, for example, then each stratum is homeomorphic to the prime spectrum of a suitable commutative Laurent polynomial algebra over some $\mathbb{k}$-field [13].
0.5. The Dixmier-Moeglin equivalence ties, under suitable hypotheses, the representation-theoretic notion of “primitivity” to the the field-theoretic notion of “rationality” and to the topological notion of “local closedness.” Section 4 describes a topology on $H$-Spec $A$, a straightforward generalization of the familiar Jacobson-Zariski topology on Spec $A$. This topology was considered earlier for group actions, differential algebras, and in other settings; see, e.g., [12], [9], [10]. Aside from establishing a general framework, this section and the rest of this note offer little in the way of substantive results. Hopefully, the setup described here will lead to deeper investigations into the topological aspects of the Dixmier-Moeglin equivalence, perhaps along the lines of the interesting work in [3], or into some other possible avenues for future work that are pointed out below.

**Notations and conventions.** We will work over an arbitrary base field $k$ and write $\otimes = \otimes_k$. The notations and hypotheses introduced in the foregoing will remain in effect for the remainder of this paper. In particular, $H$ will always be a Hopf $k$-algebra with antipode $S$ and counit $\varepsilon$, and $A$ will be a left $H$-module $k$-algebra with action $H \subseteq A$ written as $h \otimes a \mapsto h.a$. We assume throughout that the antipode $S$ is bijective.

1. **Background on Quotient Rings**

In this section, $R$ is an arbitrary ring (with 1). We recall some basics concerning the symmetric ring of quotients, $QR$. For details, see [14, Appendix E]. Throughout, we let

$$\mathcal{E} = \mathcal{E}(R)$$

denote the collection of all (two-sided) ideals of $R$ having zero left and right annihilator in $R$.

1.1. **Symmetric Quotient Rings.** The ring $R$ is a subring of $QR$. Moreover, the following hold for any $q \in QR$:

$$D_q := \{ r \in R \mid qRr \subseteq R \text{ and } rRq \subseteq R \} \in \mathcal{E}. \quad (2)$$

$$I \in \mathcal{E}, q \neq 0 \implies qI \neq 0 \text{ and } Iq \neq 0. \quad (3)$$

For any $I \in \mathcal{E}$, let $\text{Hom}(RI, R) \times \text{Hom}(IR, R)$ denote the collections of all left and right $R$-module maps $I \mapsto R$, respectively, and define

$$\mathcal{H}_I := \{ (f, g) \in \text{Hom}(RI, R) \times \text{Hom}(IR, R) \mid f(a)b = ag(b) \forall a, b \in I \}.$$  

Writing $f(a) = af$ and $g(b) = gb$, the condition $f(a)b = ag(b)$ above becomes a variant of associativity: $(af)b = a(gb)$. In fact,

$$I \in \mathcal{E}, (f, g) \in \mathcal{H}_I \implies \exists q \in QR : \forall a, b \in I.$$  

By (3), the above $q$ is unique; we will write $q = [f, g] \in QR$.

1.2. **The Extended Center.** The center $CR := Z(QR)$ is called the extended center of $R$; it coincides with the centralizer of $R$ in $QR$:

$$CR = \{ q \in QR \mid qr = rq \forall r \in R \} = \{ q \in QR \mid \exists I \in \mathcal{E} : qa = aq \forall a \in I \}. \quad (5)$$

In particular, $ZR \subseteq CR$. If $ZR = CR$, then $R$ is called centrally closed. In general, the following subring of $QR$ has center $CR$ and may be strictly larger than $R$:

$$\overline{R} := R(CR) \subseteq QR.$$
If $R$ is semiprime, then $\tilde{R}$ is a centrally closed ring [11, Theorem 3.2], called the central closure of $R$. If $R$ is a $k$-algebra, then so are $QR$ and $\tilde{R}$, because $ZR \subseteq CR = Z\tilde{R}$.

1.3. An extension lemma. We will need a version of [11, Lemma 4] for symmetric rings of quotients. The proof is essentially identical to the one in [11], but we include it here in full detail because of its somewhat technical nature.

Recall that a ring homomorphism $\varphi: R \to S$ is called centralizing if the ring $S$ is generated by the image $\varphi R$ and the centralizer $C_S(\varphi R) = \{ s \in S \mid s\varphi(r) = \varphi(r)s \ \forall r \in R \}$. Any such $\varphi$ maps the center $ZR$ to $ZS$ and, for any ideal $I$ of $R$, the ideal of $S$ that is generated by $\varphi I$ given by $\langle \varphi I \rangle := (\varphi I)C_S(\varphi R) = C_S(\varphi R)(\varphi I)$.

**Lemma 1.** Let $\varphi: R \to S$ be a centralizing ring homomorphism and let

$$C_\varphi = \{ q \in CR \mid \langle \varphi D_q \rangle \in \mathcal{E}(S) \}.$$ 

Then $RC_\varphi$ is a subring of $\tilde{R}$ containing $R$. The map $\varphi$ extends uniquely to a homomorphism $\tilde{\varphi}: RC_\varphi \to \tilde{S}$ which is centralizing. In particular, $\tilde{\varphi}C_\varphi \subseteq CS$.

**Proof.** Below, we put $C := C_S(\varphi R)$ and $I_q := \langle \varphi D_q \rangle = (\varphi D_q)C$ for $q \in QR$. Let

$$R_\varphi = \{ q \in QR \mid I_q \in \mathcal{E}(S) \}.$$ 

Since $1 \in D_q$ for $q \in R$, we certainly have $R \subseteq R_\varphi$. For any $q, q' \in QR$, one checks that $D_q \cap D_{q'} \subseteq D_{q+q'}$ and $D_q D_{q'} \subseteq D_{q+q'}$. Hence $I_q I_{q'} \subseteq I_{q+q'}$. If $I_q$ and $I_{q'}$ both belong to $\mathcal{E}(S)$, then $I_q I_{q'} \in \mathcal{E}(S)$ and hence also $I_{q+q'}, I_{q+q'} \in \mathcal{E}(S)$. Thus, $R_{\varphi}$ is a subring of $QR$.

Note that $C_\varphi = CR \cap R_{\varphi} = ZR_{\varphi}$. So $RC_\varphi$ is also a subring of $QR$, containing $R$.

It remains to construct a ring homomorphism $\tilde{\varphi}: RC_\varphi \to \tilde{S}$ that extends $\varphi$. Such an extension will necessarily be unique by (5), because $\varphi(qd) = \tilde{\varphi}(q)\varphi(d)$ for $q \in RC_\varphi$, $d \in D_q$ and $I_q \in \mathcal{E}(S)$.

First, we define a ring homomorphism $\tilde{\varphi}: C_\varphi \to CS$. To this end, let $q \in C_\varphi$ be given. Define $f_q: I_q \to S$ by

$$f_q \left( \sum_i \varphi(d_i) c_i \right) := \sum_i \varphi(qd_i) c_i \quad (d_i \in D_q, c_i \in C).$$

To see that $f_q$ is well defined, let $d \in D_q$. Then $\varphi(dq) \sum_i \varphi(d_i)c_i = \sum_i \varphi(dgd_i)c_i = \varphi(d) \sum_i \varphi(gdq_i)c_i$. Thus, if $\sum_i \varphi(d_i)c_i = \sum_j \varphi(d'_j)c'_j$ with $d_i, d'_j \in D_q$ and $c_i, c'_j \in C$, then

$$0 = \varphi(D_qq) \left( \sum_i \varphi(d_i)c_i - \sum_j \varphi(d'_j)c'_j \right) = \varphi D_q \left( \sum_i \varphi(qd_i)c_i - \sum_j \varphi(qd'_j)c'_j \right)$$

and so $\sum_i \varphi(gdq_i)c_i = \sum_j \varphi(qd'_j)c'_j$ because $I_q \in \mathcal{E}(S)$. Therefore, $f_q$ is well defined. Next, we show that $f_q$ is an $(S, S)$-bimodule map. For $s = \varphi(r)c \in S$ with $r \in R$, $c \in C$ and $x = \sum_i \varphi(d_i)c_i \in I_q$, we compute using the fact that $qr = rq$,

$$f_q(sx) = f_q \left( \sum_i \varphi(rd_i)cc_i \right) = \sum_i \varphi(qrd_i)cc_i = \sum_i \varphi(qd_i)c_i = sf_q(x).$$

Similarly, $f_q(xs) = f_q(x)s$. So $f_q$ is indeed an $(S, S)$-bimodule map; equivalently, the pair $(f_q, f_q)$ belongs to the set $H_{I_q}$ in (4). The element $t = [f_q, f_q] \in QS$ satisfies $xt = f_q(x) =$
tx for all \(x \in I_q\). By (5), it follows that \(t \in CS\). Defining \(\tilde{\varphi}(q) := t\) we obtain a map \(\tilde{\varphi} : C_\varphi \to CS\). The definition of \(\tilde{\varphi}\) gives

\[
\tilde{\varphi}(q)\varphi(d) = f_q(\varphi(d)) = \varphi(qd) \quad (q \in C_\varphi, d \in D_q).
\]

In particular, for \(q, q' \in C_\varphi\) and \(d \in D_q, d' \in D_{q'}\), we obtain \(\tilde{\varphi}(q'q')\varphi(d'd) = \varphi(qq'd'd) = \tilde{\varphi}(q)\varphi(q'\varphi(d'd)) = \tilde{\varphi}(q)\varphi(q'\varphi(d')\varphi(d'))\), which implies \(\tilde{\varphi}(qq') = \tilde{\varphi}(q)\tilde{\varphi}(q')\) because \(I_qI_q \subset \mathcal{E}(S)\).

Similarly, one checks that \(\tilde{\varphi}(q + q') = \tilde{\varphi}(q) + \tilde{\varphi}(q')\); so \(\tilde{\varphi}\) is a ring homomorphism.

Finally, define \(\tilde{\varphi} : RC_\varphi \to S\) by

\[
\tilde{\varphi}(\sum_i r_i q_i) = \sum_i \varphi(r_i)\tilde{\varphi}(q_i) \quad (r_i \in R, q_i \in C_\varphi).
\]

For well definedness, assume \(\sum_i r_i q_i = \sum_j r'_j q'_j\) and let \(d \in D = \bigcap_{i,j} (D_{q_i} \cap D_{q'_j})\). Then

\[
\left(\sum_i \varphi(r_i)\tilde{\varphi}(q_i) - \sum_j \varphi(r'_j)\tilde{\varphi}(q'_j)\right)(d) = \sum_i \varphi(r_i)\varphi(q_id) - \sum_j \varphi(r'_j)\varphi(q'_jd) = \sum_i \varphi(r_iq_id) - \sum_j \varphi(r'_jq'_jd) = \varphi\left((\sum_i r_iq_i - \sum_j r'_jq'_j)d\right) = 0.
\]

Therefore, \(\sum_i \varphi(r_i)\tilde{\varphi}(q_i) = \sum_j \varphi(r'_j)\tilde{\varphi}(q'_j)\) by (3), because \(\langle \varphi(D) \rangle \subset \mathcal{E}(S)\). This proves well definedness of \(\tilde{\varphi}\). The fact that \(\tilde{\varphi}\) is a centralizing ring homomorphism follows readily from the corresponding properties of \(\varphi\) and \(\tilde{\varphi}|_{C_\varphi}\). This completes the proof. \qed

2. The extended \(H\)-center

In this section, we return to \(H\)-actions. We fix \(A \in H\text{-Alg}\) and put \(\mathcal{E} = \mathcal{E}(A)\). We also put

\[
\mathcal{E}^H = \{I \in \mathcal{E} \mid I \text{ is an } H\text{-ideal}\}.
\]

2.1. Extended \(H\)-invariants. It may not always be possible to extend the \(H\)-action on \(A\) to an action \(H \subset QA\). Thus, the familiar definition of \(H\)-invariants,

\[
A^H = \{a \in A \mid h.a = \langle \varepsilon, h \rangle a \text{ for all } h \in H\},
\]

is not directly applicable to \(QA\) in this form. Instead, using the ideal \(D_q \in \mathcal{E}\) in (2), we define

\[
Q^HA := \{q \in QA \mid h.(dq) = (h.d)q, h.(qd) = q(h.d) \forall h \in H, d \in D_q\}.
\]

This makes sense, since the above equations only involve the \(H\)-action on elements of \(A\).

Lemma 2. (a) \(D_q \in \mathcal{E}^H\) for any \(q \in Q^HA\).

(b) Let \(q \in QA\). If there is some \(I \in \mathcal{E}^H\) such that \(I \subseteq D_q\) and \(h.(aq) = (h.a)q, h.(qa) = q(h.a) \forall h \in H, a \in I\), then \(q \in Q^HA\).

(c) \(Q^HA\) is a subalgebra of \(QA\) with \(A^H \subseteq Q^HA\).

Proof. (a) We will use the following identities in \(A\); see \cite{14 Exercise 10.4.1}:

\[
(h.a)b = h_1.(a(S(h_2).b)), \quad a(h.b) = h_2.((S^{-1}(h_1).a)b) \quad (a, b \in A, h \in H).
\]
Now let \( q \in Q^H A \) and \( d \in D_q \) be given. Then, for any \( a \in A \) and \( h \in H \),
\[
(h.d)aq = (h_1.(d(S(h_2).a)))q = h_1.(d(S(h_2).a)q),
\]
where the second equality follows from \( (h.d')q = h.(d'q) \) for all \( h \in H, d' \in D_q \), because \( d(S(h_2).a) \in D_q \). The last expression belongs to \( A \); so \( (H.D_q)Aq \subseteq A \). The inclusion \( qA(H.D_q) \subseteq A \) follows similarly from the computation
\[
qa(h.d) = qh_2.((S^{-1}(h_1).a)d) = h_2.(q(S^{-1}(h_1).a)d).
\]
This shows that \( H.D_q \subseteq D_q \).

(b) Let \( q \in QA \) and \( I \in \mathcal{E}^H \) be as in the statement of (b). We need to check that \( h.(dq) = (h.d)q \) and \( h.(qd) = q(h.d) \) for \( h \in H, d \in D_q \). For any \( a \in I \), the first identity in (6) gives
\[
(h.(dq))a = h_1.(dq(S(h_2).a)) = h_1.(d(S(h_2).qa)) = (h.d)qa.
\]
Thus, \( (h.(dq) - (h.d)q)I = 0 \) and so \( h.(dq) = (h.d)q \) by (3). The equality \( h.(qd) = q(h.d) \) is proved similarly.

(c) The defining conditions of \( Q^H A \) are readily checked for any \( q \in A^H \), with \( D_q = A \). Thus, \( A^H \subseteq Q^H A \). To show that \( Q^H A \) is a subalgebra of QA, let \( q, q' \in Q^H A \) be given. Then the ideal \( I = D_q \cap D_{q'} \) belongs to \( \mathcal{E}^H \) and \( I \subseteq D_{q+q'} \). Moreover, for any \( a \in I \) and \( h \in H \), one has
\[
h.(a(q + q')) = h.(aq) + h.(aq') = (h.a)q + (h.a)q' = (h.a)(q + q').
\]
Similarly, \( h.((q + q')a) = (q + q')(h.a) \). In view of (b), it follows that \( q + q' \in Q^H A \). The proof of \( qq' \in Q^H A \) is analogous, using \( I = D_{q'}D_q \).

\( \square \)

Remark. Lemma 2(a) shows that \( Q^H A \) is contained in a certain subring of QA, the symmetric ring of quotients for the ideal filter \( \mathcal{E}^H \); see [7, 20]. Denoting this ring of quotients by \( Q_{\mathcal{E}^H} A \), it has been shown in these references that \( Q_{\mathcal{E}^H} A \subseteq HAlg \) via a unique extension of the action \( H \subseteq A \) to \( Q_{\mathcal{E}^H} A \). Hence, the algebra of \( H \)-invariants, \( (Q_{\mathcal{E}^H} A)^H \), is defined as usual. In fact,
\[
(Q_{\mathcal{E}^H} A)^H = Q^H A.
\]
To see this, let \( q \in (Q_{\mathcal{E}^H} A)^H \). Then \( h.(dq) = (h_1.d)(h_2.q) = (h_1.d)(\varepsilon, h_2)q = (h.d)q \) for all \( h \in H \) and \( d \in D_q \). Similarly, \( h.(qd) = q(h.d) \); so \( q \in Q^H A \). Conversely, let \( q \in Q^H A \). Then, for any \( h \in H \) and \( d \in D_q \), the first identity in (6) gives
\[
(h.q) d = h_1.(q(S(h_2).d)) = h_1 S(h_2).(qd) = \langle \varepsilon, h \rangle qd,
\]
where the second equality uses that \( q \in Q^H A \) and \( S(h_2).d \in D_q \) by Lemma 2(a). Therefore, \( h.q = \langle \varepsilon, h \rangle q \) by (3) and so \( q \in (Q_{\mathcal{E}^H} A)^H \).

Example 3. If \( H \) is pointed, then \( QA \subseteq HAlg \) via a unique extension of the action \( H \subseteq A \) to \( QA \) [20, 2.3]. Hence \( (QA)^H \) is defined. As in the Remark, it follows that \( Q^H A = (QA)^H \).
2.2. Extended $H$-centers. We define the extended $H$-center of $A$ by
\[ C^H A = CA \cap Q^H A. \]

**Example 4.** If $H$ is cocommutative, then the center of any $H$-module algebra is $H$-stable [7 Proposition 4]. Thus, if $H$ is pointed cocommutative (e.g., a group algebra or an enveloping algebra), then Remark (2) above gives that $C^H A = (CA)^H$, the subalgebra of $H$-invariants in the extended center of $A$.

**Example 5.** Let $A$ be a Poisson algebra. With $H = \mathfrak{g}$ as in §0.4.1, the algebra of $H$-invariants in $A$ is called the *Poisson center* of $A$ and usually denoted by $Z_P(A)$; so
\[ Z_P(A) = \{ a \in A \mid \{ a, \cdot \} = 0 \}. \]
If $A$ is a commutative domain, then $QA = CA = \text{Fract } A$, the field of fractions of $A$, and $C^H A = Z_P(\text{Fract } A)$.

An important general property of $C^H A$ is stated in the following proposition the first part of which is due to Matczuk [17].

**Proposition 6.** If $A$ is $H$-prime then $C^H A$ is a $k$-field. Conversely, if $A$ is semiprime and $C^H A$ is a field, then $A$ is $H$-prime.

*Proof.* Assume that $A$ is $H$-prime and let $0 \neq q \in C^H A$ be given. Recall from Lemma 2(a) that $D_q \in \mathfrak{g}^H$. Consequently, $I := qD_q$ is a nonzero ideal of $A$ by (3) and it is and $H$-ideal by definition of $Q^H A$. Since $A$ is $H$-prime, it follows that $I \in \mathfrak{g}$ [7 Corollary 3]. Therefore, the map $f : D_q \to I, d \mapsto qd$ is an isomorphism of $(A, A)$-bimodules. Thus, the pair $(f, f)$ belongs to the set $\mathcal{H}_I$ in (4) and $q = [f, f]$. The desired inverse of $q$ is given by $[f^{-1}, f^{-1}]$.

Next, assume that $A$ is semiprime but not $H$-prime. Then there exists a nonzero $H$-ideal $I$ of $A$ such that $J = \text{ann } A I \neq 0$. By [7 Corollary 2], $J$ is an $H$-ideal. Since $A$ is semiprime, the sum $I + J$ is direct and $I + J$ has zero annihilator; so $I + J \in \mathfrak{g}^H$. Define $(A, A)$-bimodule maps $f, f' : I + J \to A$ by $f(i + j) = i$ and $f'(i + j) = j$ and put $q = [f, f], q' = [f', f'] \in CA$. Since $f$ and $f'$ are $H$-equivariant, we also have $q, q' \in Q^H A$. Thus, $0 \neq q, q' \in C^H A$ but $qq' = 0$, whence $C^H A$ is not a field. \[\square\]

3. Rationality and $H$-rationality

We continue to assume that $A \in \mathcal{H}_\text{Alg}$ throughout this section.

3.1. Hearts of $H$-primes. For any $I \in \text{H-Spec } A$, we define
\[ \mathcal{H}_H(I) = C^H(A/I); \]
this is always a $k$-field by Proposition 6. The $H$-prime $I$ will be called $H$-rational if the field extension $\mathcal{H}_H(I)/k$ is algebraic. For a trivial $H$-action, we obtain the usual definitions: the heart $\mathcal{H}(P) = C(A/P)$ of a prime $P \in \text{Spec } A$, which is called rational when $\mathcal{H}(P)/k$ is algebraic. As in the Introduction, we will denote the collections of rational and $H$-rational ideals of $A$ by $\text{Rat } A$ and $\text{H-Rat } A$, respectively.

**Example 7.** Let $A$ be a commutative noetherian Poisson algebra and assume that $\text{char } k = 0$. Then each Poisson prime $I \in \text{P-Spec } A$ is prime [9]. Instead of of $\mathcal{H}_H(I)$ with $H = \mathfrak{g}$ (§0.4.1), one generally writes $\mathcal{H}_P(I)$; so $\mathcal{H}_P(I) = Z_P(\text{Fract } A/I)$. Moreover, $H$-rational
ideals of $A$ are called Poisson rational: $I \in \mathcal{P}$-$\text{Spec } A$ is Poisson rational iff the field extension $Z_P(\text{Fract } A/I)/k$ is algebraic.

For the special case of a group algebra $H = kG$, the following result is [11 Proposition 12] and for commutative differential algebras, it was proved in [9 Proposition 1.2]; see also [16 Lemma 3.4].

**Proposition 8.** Let $P \in \text{Spec } A$. There is an embedding of $k$-fields \( \mathcal{H}_H(P;H) \hookrightarrow \mathcal{H}(P) \). In particular, if $P \in \text{Rat } A$ then $P;H \in H$-$\text{Rat } A$.

**Proof.** We may assume that $P : H = 0$. Thus $C^H A$ is a field and it suffices to construct a $k$-algebra map $C^H A \to C(A/P)$. For a given $q \in C^H A$, we know that $D_q \subseteq \mathcal{E} H$ by Lemma 2(a). Therefore, $D_q \not\subseteq P$. Letting $\varphi : A \to A/P$ denote the canonical epimorphism, we have $\varphi(D_q) \subseteq \mathcal{E}(A/P)$. Thus, we may apply Lemma 1 with $C^H A \subseteq C$, and we obtain an extension $\tilde{\varphi}$ of $\varphi$ such that $\tilde{\varphi}(C^H A) \subseteq C(A/P)$. This is the desired homomorphism. \( \square \)

Thus, we have a well-defined map $\kappa^\text{Rat}_H = \kappa_H|_{\text{Rat } A} : \text{Rat } A \to H$-$\text{Rat } A$. In contrast with the map $\kappa_H : \text{Spec } A \to H$-$\text{Spec } A$, which is often surjective, surjectivity of $\kappa^\text{Rat}_H$ seems to require stronger hypotheses.

**Example 9.** Assume that $\text{char } k = 0$ and let $A = k(x, y)$ be the rational function field over $k$, equipped with the Poisson bracket that is determined by $\{x, y\} = x$. Of course, $A$ has no rational primes at all, yet it is not hard to see that $Z_P(A) = k$; so the zero ideal is Poisson rational. On the other hand, considering the polynomial algebra $k[x, y]$ with the above Poisson bracket, the zero ideal is still Poisson rational, but now it is also the Poisson kernel of any maximal ideal of $k[x, y]$ that does not contain $x$. Indeed, it is easy to see that all nonzero Poisson primes of $k[x, y]$ contain $x$.

3.2. Rational strata. It would be interesting to have a description of the $\kappa^\text{Rat}_H$-fiber $\text{Rat}_I A := \text{Spec } A \cap \text{Rat } A = \{ P \in \text{Rat } A \mid P : H = I \}$ over a given $I \in H$-$\text{Rat } A$. The following result may serve as a first approximation. For any ideal $I$ of $A$, we may consider the convolution algebra $\text{Hom}_k(H, A/I)$ and the “hit” action $H \subseteq \text{Hom}_k(H, A/I)$ that is defined by $(h \cdot f)(k) = f(kh)$ for $h, k \in H$ and $f \in \text{Hom}_k(H, A/I)$ [14 10.4.2]. The map

$$\alpha_I : A \to \text{Hom}_k(H, A/I), \quad a \mapsto (h \mapsto h.a + I)$$

is a map in $H$-$\text{Alg}$ and $I;H = \text{Ker } \alpha_I$.

**Proposition 10.** Given $I \in H$-$\text{Spec } A$, there is a bijection

$$\left\{ P \in \text{Spec } A \mid A/P \cong k \text{ and } P;H = I \right\} \longleftrightarrow \left\{ \text{embeddings } A/I \to H^* \right\}$$

Proof. The set of all primes $P$ with $A/P \cong k$ is in bijection with the set of all algebra maps $A \to k$. Consider the isomorphism $\sim : \text{Hom}_k(A, k) \sim \text{Hom}_H(A, H^*)$ that is the composite of the canonical $k$-$\text{linear}$ isomorphisms

$$\text{Hom}_k(A, k) \sim \text{Hom}_k(H \otimes_H A, k) \sim \text{Hom}_H(A, \text{Hom}_k(H, k)) = \text{Hom}_H(A, H^*)$$

\( \square \)
where the second isomorphism is $\text{Hom} \otimes$ adjunction: $f \mapsto f'$ with $f'(a)(h) = f(h \otimes a)$; see, e.g., [14, B.2.2]. Explicitly,

$$\tilde{\varphi}(a)(h) = \varphi(h.a) \quad (\varphi \in \text{Hom}_k(A, k), h \in H, a \in A).$$

If $\varphi$ is an algebra map, then $\tilde{\varphi}$ is an algebra map as well: for $a, a' \in A$ and $h \in H$,

$$\tilde{\varphi}(aa')(h) = \varphi(h.(aa')) = \sum \varphi(h_1.a)\varphi(h_2.a') = (\tilde{\varphi}(a)\tilde{\varphi}(a'))(h);$$

so $\tilde{\varphi}(aa') = \tilde{\varphi}(a)\tilde{\varphi}(a')$. Conversely, if $\tilde{\varphi}$ is an algebra map, then so is $\varphi$:

$$\varphi(aa') = \tilde{\varphi}(aa')(1) = (\tilde{\varphi}(a)\tilde{\varphi}(a'))(1) = \tilde{\varphi}(a)(1)\tilde{\varphi}(a')(1) = \varphi(a)\varphi(a'),$$

Finally, $\text{Ker} \tilde{\varphi} = (\text{Ker} \varphi):H$. Thus, $\tilde{\varphi}$ gives the desired bijection. \hfill $\square$

4. THE TOPOLOGY OF $H$-$\text{Spec} A$

We continue to assume that $A \in H\text{Alg}$.

4.1. The Jacobson-Zariski topology. The familiar Jacobson-Zariski topology on $\text{Spec} A$ is defined by declaring all subsets of the form $\mathcal{V}S = \{P \in \text{Spec} A \mid P \supseteq S\}$ for any subset $S \subseteq A$ to be closed (e.g., [14, 1.3.4]). In analogy with this definition, we define the closed subsets of $H$-$\text{Spec} A$ to be those of the form

$$\mathcal{V}^H S = \{Q \in H$-$\text{Spec} A \mid Q \supseteq S\}.$$

Evidently, $\mathcal{V}^H \emptyset = H$-$\text{Spec} A$, $\mathcal{V}^H \{1\} = \emptyset$, and $\mathcal{V}^H \bigcup_\alpha S_\alpha = \bigcap_\alpha \mathcal{V}^H S_\alpha$ for any family of subsets $S_\alpha \subseteq A$. Since we may replace the set $S$ by the $H$-ideal of $A$ that is generated by $S$ without changing $\mathcal{V}^H S$, the closed subsets of $H$-$\text{Spec} A$ can also be described as the sets of the form $\mathcal{V}^H I$, where $I$ is an $H$-ideal of $A$. The defining property of $H$-prime ideals implies that $\mathcal{V}^H I \cup \mathcal{V}^H J = \mathcal{V}^H IJ$ for $H$-ideals $I$ and $J$. Thus, finite unions of closed sets are again closed, thereby verifying the topology axioms. We list some of its basic properties in the following lemma.

**Lemma 11.**

(a) The map $\kappa_H : \text{Spec} A \to H$-$\text{Spec} A$ is continuous.

(b) If all $H$-primes of $A$ are prime, then the topology of $H$-$\text{Spec} A$ is the initial topology for the inclusion map $H$-$\text{Spec} A \hookrightarrow \text{Spec} A$.

(c) Assume that $\kappa_H$ is surjective and $\bigcap \text{Spec}_I A = I$ for all $I \in H$-$\text{Spec} A$. Then the topology of $H$-$\text{Spec} A$ is the final topology for $\kappa_H$.

**Proof.** (a) The preimage of the closed set $C = \mathcal{V}^H I$, for an $H$-ideal $I$ of $A$, is given by

$$\kappa^{-1}_H(C) = \{P \in \text{Spec} A \mid P : H \supseteq I\} = \{P \in \text{Spec} A \mid P \supseteq I\} = \mathcal{V}^H I,$$

which is closed in $\text{Spec} A$ for the Jacobson-Zariski topology. Therefore, $\kappa_H$ is continuous.

(b) By definition, the closed sets in the initial topology for the inclusion $H$-$\text{Spec} A \hookrightarrow \text{Spec} A$ are exactly the sets $H$-$\text{Spec} A \cap \mathcal{V} S = \mathcal{V}^H S$ [4, Chap. 1 §2.3].

(c) A subset $C \subseteq H$-$\text{Spec} A$ is closed in the final topology for $\kappa_H$, by definition, if and only if $\kappa^{-1}_H(C)$ is closed in $\text{Spec} A$ [4, Chap. 1 §2.4]. By (a), this includes all sets of the form $C = \mathcal{V}^H I$ for an $H$-ideal $I$ of $A$. Conversely, assume that $D := \kappa^{-1}_H(C) = \bigcup_{I \in C} \text{Spec}_I A$
is closed in $\text{Spec } A$; so $D = \forall J$, where $J = \cap D$. Note that $J = \bigcap_{I \in \mathcal{C}} \cap \text{Spec } A = \cap \mathcal{C}$, an $H$-ideal of $A$. Since $\kappa_H$ is assumed surjective, it follows that $C$ has the desired form:

$$C = \kappa_H(D) = \kappa_H(\forall J) = \{P: H \mid P \in \text{Spec } A, P \supseteq J\}$$

$$= \{P: H \mid P \in \text{Spec } A, P:H \supseteq J\} = \{Q \in H-\text{Spec } A \mid Q \supseteq J\} = \forall^H J. \quad \square$$

If $A$ has the maximum condition on ideals, then the condition $\bigcap \text{Spec } A = I$ for all $I \in H-\text{Spec } A$ in part (b) of the lemma is equivalent to the requirement that $H$-cores of prime ideals of $A$ are semiprime. This requirement is satisfied for all group actions and for all actions of cocommutative Hopf algebras in characteristic 0 \footnote{15}.

4.2. **Toward a Dixmier-Moeglin equivalence with $H$-action.** Let $I \in H-\text{Spec } A$. We will say that $I$ is $H$-locally closed if the one-point set $\{I\}$ is a locally closed subset in the topology of $H-\text{Spec } A$ (\footnote{11}), that is, $\{I\}$ is open in its closure $\{I\} = \forall^H I$ or, equivalently, $\forall^H I \setminus \{I\}$ is a closed subset of $H-\text{Spec } A$. Explicitly, this means that

$$I \subsetneq \cap \{J \in H-\text{Spec } A \mid J \supseteq I\}.$$ Further, $I$ will be called $H$-primitive if $I = P:H$ for some primitive ideal $P$ of $A$.

To summarize, the following three properties of $H$-prime ideals were considered in the foregoing: (i) $H$-local closedness, (ii) $H$-primitivity, and (iii) $H$-rationality. The following question has been studied in various settings before; see the examples below.

**Question** (Dixmier-Moeglin equivalence with $H$-action). When are (i)–(iii) equivalent?

It turns out that the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hold under fairly general circumstances. Indeed, a wide range of algebras $A$ shares the following two properties.

**Weak Nullstellensatz**: The Schur division algebra $\text{End}_A(V)$ of very irreducible left $A$-module $V$ is algebraic over $k$.

**Jacobson Property**: Every prime ideal of $A$ is an intersection of primitive ideals.

For example, both statements apply to any countably generated noetherian algebra $A$ over an uncountable base field $k$ and they also hold for many algebras over arbitrary fields; see \footnote{S II.7} or \footnote{13 Chapter 9}. The following lemma records some rather straightforward instances of the aforementioned implications.

**Lemma 12.** (a) If $\kappa_H$ is surjective and $A$ has the Jacobson property, then every $H$-locally closed $H$-prime ideal of $A$ is $H$-primitive.

(b) If $A$ satisfies the weak Nullstellensatz, then every $H$-primitive ideal of $A$ is $H$-rational.

**Proof.** (a) Let $I \in H-\text{Spec } A$ be $H$-locally closed. By our hypotheses, $I = P:H$ for some $P \in \text{Spec } A$ and $P = \bigcap \lambda Q_\lambda$ for some family $(Q_\lambda)$ of primitive ideals of $A$. Since the core operator $h:H$ evidently commutes with intersections, we obtain $I = \bigcap \lambda Q_\lambda:H$. Finally, all $Q_\lambda:H$ belong to $\forall^H I$ and $I$ is locally closed. So we must have $I = Q_\lambda:H$ for some $\lambda$ and, therefore, $I$ is $H$-primitive.

(b) Assume that $I = P:H$ for some primitive ideal $P$ of $A$; say $P$ is the annihilator of the irreducible $A$-module $V$. Then the heart $\mathcal{H}(P)$ embeds into the Schur division algebra $\text{End}_A(V)$ \footnote{14 Proposition E.2}. Our hypothesis on $A$ now implies that $\mathcal{H}(P)/k$ is algebraic and Proposition \footnote{8} further implies that $\mathcal{H}_H(I)/k$ is algebraic as well, showing that $I$ is $H$-rational.

$\square$
Stronger hypotheses are needed to ensure the validity of (iii) ⇒ (i).

**Example 13.** By classical results of Hilbert, the ordinary Dixmier-Moeglin equivalence, without $H$-action, holds for primes of any affine commutative $k$-algebra, with “primitive” being the same as “maximal.” For an affine commutative Poisson algebra $A$ and $H = U_g$ as in [0.4.1] the above Question was originally posed in [6] (for $k = C$); the equivalence is known as the Poisson Dixmier-Moeglin equivalence in this setting. Lemma [12] covers the easy implications, (i) ⇒ (ii) ⇒ (iii). It was shown in [2] that (iii) ⇒ (ii) also holds—so (ii) and (iii) are in fact equivalent—but (iii) ⇒ (i) can fail if the Krull dimension of $A$ is at least 4.

**Example 14.** Let $G$ be an affine algebraic group over an algebraically closed field $k$ that acts rationally on the $k$-algebra $A$ (0.4.2). Assume that $A$ has the Jacobson property and satisfies the weak Nullstellensatz. Then, again, Lemma [12] gives (i) ⇒ (ii) ⇒ (iii) with $H = kG$. It is also known in this setting, that every $G$-rational ideal of $A$ has the form $P : H$ for some $P \in \text{Rat} A$ and that $P$ is locally closed if and only if $P : H$ is $H$-locally closed [12]. Thus, if $A$ satisfies the ordinary Dixmier-Moeglin equivalence, and hence all rational primes are locally closed, then the Question above has a positive answer for $A$ and $H = kG$.

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