WEAK AND STRONG TYPES ESTIMATES FOR SQUARE FUNCTIONS ASSOCIATED WITH OPERATORS

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Abstract. Let $L$ be a linear operator in $L^2(\mathbb{R}^n)$ which generates a semigroup $e^{-tL}$ whose kernels $p_t(x,y)$ satisfy the Gaussian upper bound. In this paper, we investigate several kinds of weighted norm inequalities for the conical square function $S_{\alpha,L}$ associated with an abstract operator $L$. We first establish two-weight inequalities including bump estimates, and Fefferman-Stein inequalities with arbitrary weights. We also present the local decay estimates using the extrapolation techniques, and the mixed weak type estimates corresponding Sawyer’s conjecture by means of a Coifman-Fefferman inequality. Beyond that, we consider other weak type estimates including the restricted weak-type $(p, p)$ for $S_{\alpha,L}$ and the endpoint estimate for commutators of $S_{\alpha,L}$. Finally, all the conclusions aforementioned can be applied to a number of square functions associated to $L$.

1. Introduction

Given an operator $L$, the conical square function $S_{\alpha,L}$ associated with $L$ is defined by

$$S_{\alpha,L}(f)(x) := \left( \int_{\Gamma_\alpha(x)} |t^m Le^{-t^m L} f(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}},$$

(1.1)

where $\Gamma_\alpha(x) = \{(x,t) \in \mathbb{R}^n \times (0,\infty) : |x-y| < \alpha t\}$. In particular, if $m = 2$ and $L = -\Delta$, $S_{\alpha,L}$ is the classical area integral function. The conical square functions associated with abstract operators played an important role in harmonic analysis. For example, by means of $S_{\alpha,L}$, Auscher et al. [2] introduced the Hardy space $H^1_L$ associated with an operator $L$. Soon after, Duong and Yan [15] showed that $\text{BMO}_L^*$ is the dual space of the Hardy space $H^1_L$, which can be seen a generalization of Fefferman and Stein’s result on the duality between $H^1$ and BMO spaces. Later, the theory of function...
spaces associated with operators has been developed and generalized to many other different settings, see for example [13, 16, 17, 24]. Recently, Martell and Prisuelos-Arribas [27] studied the weighted norm inequalities for conical square functions. More specifically, they established boundedness and comparability in weighted Lebesgue spaces of different square functions using the Heat and Poisson semigroups. Using these square functions, they [28] define several weighted Hardy spaces $H^1_L(w)$ and showed that they are one and the same in view of the fact that the square functions are comparable in the corresponding weighted spaces. Very recently, Bui and Duong [3] introduced several types of square functions associated with operators and established the sharp weighted estimates.

In this paper, we continue to investigate several kinds of weighted norm inequalities for such operators, including bump estimates, Fefferman-Stein inequalities with arbitrary weights, the local decay estimates, the mixed weak type estimates corresponding Sawyer’s conjecture. Beyond that, we consider other weak type estimates including the restricted weak-type $(p,p)$ estimates and the endpoint estimate for the corresponding commutators. For more information about the progress of these estimates, see [6, 14, 30, 31, 29, 32] and the reference therein.

Suppose that $L$ is an operator which satisfies the following properties:

(A1) $L$ is a closed densely defined operators of type $\omega$ in $L^2(\mathbb{R}^n)$ with $0 \leq \omega < \pi/2$, and it has a bounded $H_\infty$-functional calculus in $L^2(\mathbb{R}^n)$.

(A2) The kernel $p_t(x, y)$ of $e^{-tL}$ admits a Gaussian upper bound. That is, there exists $m \geq 1$ and $C, c > 0$ so that for all $x, y \in \mathbb{R}^n$ and $t > 0$,

$$|p_t(x, y)| \leq \frac{C}{t^{n/m}} \exp\left(-\frac{|x - y|^{m/(m-1)}}{c t^{1/(m-1)}}\right).$$

Examples of the operator $L$ which satisfies condition (A1) and (A2) include: Laplacian $-\Delta$ on $\mathbb{R}^n$, or the Laplace operator on an open connected domain with Dirichlet boundary conditions, or the homogeneous sub-Laplacian on a homogeneous group; Schrödinger operator $L = -\Delta + V$ with a nonnegative potential $0 \leq V \in L^{1}_{\text{loc}}(\mathbb{R}^n)$.

The main results of this paper can be stated as follows. We begin with the bump estimates for $S_{\alpha,L}$.

**Theorem 1.1.** Let $1 < p < \infty$, and let $S_{\alpha,L}$ be defined in (1.1) with $\alpha \geq 1$ and $L$ satisfying (A1) and (A2). Given Young functions $A$ and $B$, we denote

$$\|(u, v)\|_{A,B,p} := \begin{cases} \sup_Q \|u^\frac{1}{p}\|_{p,Q}\|v^{-\frac{1}{p}}\|_{B,Q}, & \text{if } 1 < p \leq 2, \\ \sup_Q \|u^\frac{2}{p}\|_{A,Q}\|v^{-\frac{1}{p}}\|_{B,Q}, & \text{if } 2 < p < \infty. \end{cases}$$

If the pair $(u, v)$ satisfies $\|(u, v)\|_{A,B,p} < \infty$ with $\tilde{A} \in B_{(p/2)'}$ and $\tilde{B} \in B_p$, then

$$\|S_{\alpha,L}(f)\|_{L^p(u)} \lesssim \alpha^{n/\tilde{A}'_p} \|f\|_{L^p(v)},$$

(1.2)
where
\[
\mathcal{M}_p := \begin{cases} 
\|(u, v)\|_{A,B,p,\delta} \mathcal{B}^{1/p}_{B_p}, & \text{if } 1 < p \leq 2, \\
\|(u, v)\|_{A,B,p,\delta} \mathcal{B}^{1/p}_{B_p}, & \text{if } 2 < p < \infty.
\end{cases}
\]

**Theorem 1.2.** Let \(1 < p < \infty\), and let \(S_{\alpha,L}\) be defined in (1.1) with \(\alpha \geq 1\) and \(L\) satisfying (A1) and (A2). Let \(A\) be a Young function. If the pair \((u, v)\) satisfies \([u, v]_{A, p'} < \infty\) with \(\overline{A} \in B_{p'}\), then
\[
\|S_{\alpha,L}(f)\|_{L^p(w)} \lesssim [u, v]_{A, p'} \mathcal{B}^{1/p}_{B_{p'}} \|f\|_{L^p(v)}. \tag{1.3}
\]

We next present the Fefferman-Stein inequalities with arbitrary weights.

**Theorem 1.3.** Let \(1 < p < \infty\), and let \(S_{\alpha,L}\) be defined in (1.1) with \(\alpha \geq 1\) and \(L\) satisfying (A1) and (A2). Then for every weight \(w\),
\[
\|S_{\alpha,L}(f)\|_{L^p(Mw)} \lesssim \alpha^n \|f\|_{L^p(Mw)}, \quad 1 < p \leq 2, \tag{1.4}
\]
\[
\|S_{\alpha,L}(f)\|_{L^p(Mw)} \lesssim \alpha^n \|f(Mw/w)^{1/2}\|_{L^p(w)}, \quad 2 < p < \infty, \tag{1.5}
\]
where the implicit constants are independent of \(w\) and \(f\).

We turn to some weak type estimates for \(S_{\alpha,L}\).

**Theorem 1.4.** Let \(S_{\alpha,L}\) be defined in (1.1) with \(\alpha \geq 1\) and \(L\) satisfying (A1) and (A2). Let \(B \subset X\) be a ball and every function \(f \in L^\infty_c(\mathbb{R}^n)\) with \(\supp(f) \subset B\). Then there exist constants \(c_1 > 0\) and \(c_2 > 0\) such that
\[
|\{x \in B : S_{\alpha,L}(f)(x) > tM(f)(x)\}| \leq c_1 e^{-c_2 t^2} |B|, \quad \forall t > 0. \tag{1.6}
\]

**Theorem 1.5.** Let \(S_{\alpha,L}\) be defined in (1.1) with \(\alpha \geq 1\) and \(L\) satisfying (A1) and (A2). If \(u\) and \(v\) satisfy
\[
(1) \quad u \in A_1 \text{ and } uv \in A_\infty, \quad \text{or} \quad (2) \quad u \in A_1 \text{ and } v \in A_\infty,
\]
then we have
\[
\left\| \frac{S_{\alpha,L}(f)}{v} \right\|_{L^1(\infty)(uv)} \lesssim \|f\|_{L^1(u)}. \tag{1.7}
\]
In particular, \(S_{\alpha,L}\) is bounded from \(L^1(u)\) to \(L^1(\infty)(u)\) for every \(u \in A_1\).

Given \(1 \leq p < \infty\), \(A^R_p\) denotes the class of weights \(w\) such that
\[
[w]_{A^R_p} := \sup_{E \subset Q} \left| \frac{E}{Q} \left( \frac{w(Q)}{w(E)} \right) \right|^{1/p} < \infty,
\]
where the supremum is taken over all cubes \(Q\) and all measurable sets \(E \subset Q\). This \(A^R_p\) class was introduced in [21] to characterize the restricted weak-type \((p, p)\) of the Hardy-Littlewood maximal operator \(M\) as follows:
\[
\|M1_E\|_{L^p(\infty)(w)} \lesssim [w]_{A^R_p} w(E)^{1/p}. \tag{1.8}
\]
We should mention that \(A_p \subsetneq A^R_p\) for any \(1 < p < \infty\).
Theorem 1.6. Let $S_{\alpha,L}$ be defined in (1.1) with $\alpha \geq 1$ and $L$ satisfying (A1) and (A2). Then for every $2 < p < \infty$, for every $w \in A^p_R$, and for every measurable set $E \subset \mathbb{R}^n$,

$$\|S_{\alpha,L}(1_E)\|_{L^{p,\infty}(w)} \lesssim [w]_{A^p_R}^{1+\frac{2}{p}} w(E)^{\frac{1}{p}},$$

where the implicit constants are independent of $w$ and $E$.

Finally, we obtain the endpoint estimate for commutators of $S_{\alpha,L}$ as follows. Given an operator $T$ and measurable functions $b$, we define, whenever it makes sense, the commutator by

$$C_b(T)(f)(x) := T((b(x) - b(\cdot))f(\cdot))(x).$$

Theorem 1.7. Let $S_{\alpha,L}$ be defined in (1.1) with $\alpha \geq 1$ and $L$ satisfying (A1) and (A2). Then for every $w \in A_1$,

$$w(\{x \in \mathbb{R}^n : C_b(S_{\alpha,L})f(x) > t\} \lesssim \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{t}\right) w(x) dx, \quad \forall t > 0,$$

where $\Phi(t) = t(1 + \log^+ t)$.

2. Applications

The goal of this section is to give some applications of Theorems 1.1–1.7. To this end, we introduce some new operators. Associated with $L$ introduced in Section 1, we can also define the square functions $g_L$ and $g^*_\lambda,L$ ($\lambda > 0$) as follows:

$$g_L(f)(x) := \left(\int_0^\infty |t^m Le^{-t^{\lambda} L} f(x)|^2 \frac{dt}{t}\right)^{\frac{1}{2}},$$

$$g^*_\lambda,L(f)(x) := \left(\int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t + |x - y|}\right)^{n\lambda} \left|t^m Le^{-t^{\lambda} L} f(y)\right|^2 \frac{dy dt}{t^{n+1}}\right)^{\frac{1}{2}}.$$

If $L$ satisfies (A1) and (A2), we have the following estimates (cf. [3, p. 891]):

$$g_L(f)(x) \lesssim g^*_\lambda,L(f)(x), \quad x \in \mathbb{R}^n, \quad (2.1)$$

$$g^*_\lambda,L(f)(x) \lesssim \sum_{k=0}^\infty 2^{-k\lambda n/2} S_{2k,L} f(x), \quad x \in \mathbb{R}^n, \quad (2.2)$$

whenever $\lambda > 2$. By (2.1), (2.2) and Theorems 1.1–1.7, we conclude the following:

Theorem 2.1. Let $L$ satisfy (A1) and (A2). Then Theorems 1.1–1.7 are also true for $g_L$ and $g^*_\lambda,L$, whenever $\lambda > 2$.

Next, we introduce a class of square functions associated to $L$ and $D$, where $D$ is an operator which plays the role of the directional derivative or gradient operator. Assume that $m$ is a positive even integer. Let $D$ be a densely defined linear operator on $L^2(\mathbb{R}^n)$ which possess the following properties:

(D1) $D^{m/2} L^{-1/2}$ is bounded on $L^2(\mathbb{R}^n)$;
Given \( \alpha \geq 1 \) and \( \lambda > 2 \), we define the following square functions associated to \( L \) and \( D \):

\[
g_{D,L}(f)(x) := \left( \int_0^\infty |t^{\frac{m}{2}} D^{\frac{m}{2}} e^{-t^m L} f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}},
\]

\[
S_{\alpha,D,L}(f)(x) := \left( \int_{\Gamma_{\alpha}(x)} \int_0^\infty |t^{\frac{m}{2}} D^{\frac{m}{2}} e^{-t^m L} f(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}},
\]

\[
g_{\lambda,D,L}^*(f)(x) := \left( \int_0^\infty \int_{\mathbb{R}^n} \left( \frac{t}{t + |x - y|} \right)^{n\lambda} |t^{\frac{m}{2}} D^{\frac{m}{2}} e^{-t^m L} f(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}}.
\]

It was proved in [3, p. 895] that

\[
g_{D,L}(f)(x) \lesssim g_{\lambda,D,L}^*(f)(x), \quad x \in \mathbb{R}^n \text{ and } \lambda > 2. \tag{2.3}
\]

On the other hand, we note that \( S_{\alpha,D,L} \) has the same properties as \( S_{\alpha,L} \) and

\[
g_{\lambda,D,L}^*(f)(x) \lesssim \sum_{k=0}^{\infty} 2^{-k\lambda n/2} S_{2k,D,L}(f)(x), \quad x \in \mathbb{R}^n \text{ and } \lambda > 2. \tag{2.4}
\]

Then, (2.3), (2.4) and Theorems 1.1–1.7 give the following.

**Theorem 2.2.** Let \( L \) satisfy (A1) and (A2) and \( D \) satisfy (D1) and (D2). Let \( \alpha \geq 1 \) and \( \lambda > 2 \). Then Theorems 1.1–1.7 also hold for \( g_{D,L}, S_{\alpha,D,L}, \) and \( g_{\lambda,D,L}^* \).

Finally, we defined a class of more general square functions. Assume that \( L \) is a nonnegative self-adjoint operator in \( L^2(\mathbb{R}^n) \) and satisfies (A2). Denote by \( E_L(\lambda) \) the spectral decomposition of \( L \). Then by spectral theory, for any bounded Borel function \( F : [0, \infty) \to C \) we can define

\[
F(L) = \int_0^\infty F(\lambda) dE_L(\lambda)
\]

as a bounded operator on \( L^2(\mathbb{R}^n) \).

Let \( \psi \) be an even real-valued function in the Schwartz space \( \mathcal{S}(\mathbb{R}) \) such that \( \int_0^\infty \psi^2(s) \frac{ds}{s} < \infty \). Given \( \alpha \geq 1 \) and \( \lambda > 2 \), we now consider the following square functions:

\[
g_{\psi,L}(f)(x) := \left( \int_0^\infty |\psi(t^{\frac{m}{2}} \sqrt{L}) f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}},
\]

\[
S_{\alpha,\psi,L}(f)(x) := \left( \int_{\Gamma_{\alpha}(x)} \int_0^\infty |\psi(t^{\frac{m}{2}} \sqrt{L}) f(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}},
\]

\[
g_{\lambda,\psi,L}^*(f)(x) := \left( \int_0^\infty \int_{\mathbb{R}^n} \left( \frac{t}{t + |x - y|} \right)^{n\lambda} |\psi(t^{\frac{m}{2}} \sqrt{L}) f(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}}.
\]
Observe that for any $N > 0$,
\[ |\psi(t^{m/2}\sqrt{t})(x,y)| \leq C_N \frac{1}{t^n} \left(1 + \frac{|x-y|}{t}\right)^{-N}, \quad t > 0, \ x, y \in \mathbb{R}^n. \]  
(2.5)

Using (2.5) and the argument for $S_{\alpha,L}$, we obtain that the estimates in Section 1 is true for $S_{\alpha,D,L}$. Additionally, for any $\lambda > 2$,
\[ g_{\lambda,\psi,L}(f)(x) \lesssim g_{\lambda,\eta,L}(f)(x) + g_{\lambda,\psi,L}^*(f)(x), \quad x \in \mathbb{R}^n, \]  
(2.6)
\[ g_{\lambda,\psi,L}^*(f)(x) \lesssim \sum_{k=0}^{\infty} 2^{-k\lambda n/2} S_{2^k,\psi,L}(f)(x), \quad x \in \mathbb{R}^n, \]  
(2.7)
where $\varphi \in \mathcal{S}(\mathbb{R})$ is a fixed function supported in $[2^{-m/2},2^{m/2}]$. The proof of (2.6) is given in [3], while the proof of (2.7) is as before. Together with Theorems 1.1–1.7, these estimates imply the conclusions as follows.

**Theorem 2.3.** Let $L$ be a nonnegative self-adjoint operator in $L^2(\mathbb{R}^n)$ and satisfy (A2). Let $\alpha \geq 1$ and $\lambda > 2$. Then Theorems 1.1–1.7 are true for $g_{\psi,L}$, $S_{\alpha,\psi,L}$ and $g_{\lambda,\psi,L}^*$.

### 3. Preliminaries

#### 3.1. Muckenhoupt class.

By a weight $w$, we mean that $w$ is a nonnegative locally integrable function on $\mathbb{R}^n$. The weight $w$ is said to belong to the Muckenhoupt class $A_p$, $1 < p < \infty$, if
\[ [w]_{A_p} := \sup_Q \left( \frac{1}{Q} \int_Q w \, dx \right) \left( \frac{1}{Q} \int_Q w^{-1/p} \, dx \right)^{p-1} < \infty, \]
where the supremum is taken over all cubes in $\mathbb{R}^n$.

#### 3.2. Dyadic cubes.

Denote by $\ell(Q)$ the sidelength of the cube $Q$. Given a cube $Q_0 \subset \mathbb{R}^n$, let $\mathcal{D}(Q_0)$ denote the set of all dyadic cubes with respect to $Q_0$, that is, the cubes obtained by repeated subdivision of $Q_0$ and each of its descendants into $2^n$ congruent subcubes.

**Definition 3.1.** A collection $\mathcal{D}$ of cubes is said to be a dyadic grid if it satisfies
\begin{enumerate}
  \item For any $Q \in \mathcal{D}$, $\ell(Q) = 2^k$ for some $k \in \mathbb{Z}$.
  \item For any $Q, Q' \in \mathcal{D}$, $Q \cap Q' = \{Q, Q', \emptyset\}$.
  \item The family $\mathcal{D}_k = \{Q \in \mathcal{D} : \ell(Q) = 2^k\}$ forms a partition of $\mathbb{R}^n$ for any $k \in \mathbb{Z}$.
\end{enumerate}

**Definition 3.2.** A subset $\mathcal{S}$ of a dyadic grid is said to be $\eta$-sparse, $0 < \eta < 1$, if for every $Q \in \mathcal{S}$, there exists a measurable set $E_Q \subset Q$ such that $|E_Q| \geq \eta|Q|$, and the sets $\{E_Q\}_{Q \in \mathcal{S}}$ are pairwise disjoint.

By a median value of a measurable function $f$ on a cube $Q$ we mean a possibly non-unique, real number $m_f(Q)$ such that
\[ \max \{|\{x \in Q : f(x) > m_f(Q)\}|, |\{x \in Q : f(x) < m_f(Q)\}|\} \leq |Q|/2. \]

The decreasing rearrangement of a measurable function $f$ on $\mathbb{R}^n$ is defined by
\[ f^*(t) = \inf\{\alpha > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}| < t\}, \quad 0 < t < \infty. \]
The local mean oscillation of \( f \) is
\[
\omega_\lambda(f; Q) = \inf_{c \in \mathbb{R}} \left( (f - c)1_Q \right)^* (\lambda |Q|), \quad 0 < \lambda < 1.
\]
Given a cube \( Q_0 \), the local sharp maximal function is defined by
\[
M_{\lambda,Q_0}^f(x) = \sup_{x \in Q \subset Q_0} \omega_\lambda(f; Q).
\]
Observe that for any \( \delta > 0 \) and \( 0 < \lambda < 1 \)
\[
|m_f(Q)| \leq (f1_Q)^* (|Q|/2) \text{ and } (f1_Q)^* (\lambda |Q|) \leq \left( \frac{1}{\lambda} \int_Q |f|^\delta \, dx \right)^{1/\delta}.
\]

(3.1)

The following theorem was proved by Hytönen [18, Theorem 2.3] in order to improve Lerner’s formula given in [22] by getting rid of the local sharp maximal function.

**Lemma 3.3.** Let \( f \) be a measurable function on \( \mathbb{R}^n \) and let \( Q_0 \) be a fixed cube. Then there exists a (possibly empty) sparse family \( S(Q_0) \subset D(Q_0) \) such that
\[
|f(x) - m_f(Q_0)| \leq 2 \sum_{Q \in S(Q_0)} \omega_{2^{-n/2}}(f; Q)1_Q(x), \quad a.e. \ x \in Q_0.
\]

(3.2)

3.3. Orlicz maximal operators. A function \( \Phi : [0, \infty) \to [0, \infty) \) is called a Young function if it is continuous, convex, strictly increasing, and satisfies
\[
\lim_{t \to 0^+} \frac{\Phi(t)}{t} = 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty.
\]

Given \( p \in [1, \infty) \), we say that a Young function \( \Phi \) is a \( p \)-Young function, if \( \Psi(t) = \Phi(t^{1/p}) \) is a Young function.

If \( A \) and \( B \) are Young functions, we write \( A(t) \asymp B(t) \) if there are constants \( c_1, c_2 > 0 \) such that \( c_1A(t) \leq B(t) \leq c_2A(t) \) for all \( t \geq t_0 > 0 \). Also, we denote \( A(t) \preceq B(t) \) if there exists \( c > 0 \) such that \( A(t) \leq cB(t) \) for all \( t \geq t_0 > 0 \). Note that for all Young functions \( \phi, t \leq \phi(t) \). Further, if \( A(t) \leq cB(t) \) for some \( c > 1 \), then by convexity, \( A(t) \leq B(ct) \).

A function \( \Phi \) is said to be doubling, or \( \Phi \in \Delta_2 \), if there is a constant \( C > 0 \) such that \( \Phi(2t) \leq C\Phi(t) \) for any \( t > 0 \). Given a Young function \( \Phi \), its complementary function \( \bar{\Phi} : [0, \infty) \to [0, \infty) \) is defined by
\[
\bar{\Phi}(t) := \sup_{s > 0} \{st - \Phi(s)\}, \quad t > 0,
\]
which clearly implies that
\[
st \leq \Phi(s) + \bar{\Phi}(t), \quad s, t > 0.
\]

(3.3)

Moreover, one can check that \( \Phi \) is also a Young function and
\[
t \leq \Phi^{-1}(t)\bar{\Phi}^{-1}(t) \leq 2t, \quad t > 0.
\]

(3.4)

In turn, by replacing \( t \) by \( \Phi(t) \) in first inequality of (3.4), we obtain
\[
\bar{\Phi}\left(\frac{\Phi(t)}{t}\right) \leq \Phi(t), \quad t > 0.
\]

(3.5)
Given a Young function \( \Phi \), we define the Orlicz space \( L^\Phi(\Omega, \mu) \) to be the function space with Luxemburg norm

\[
\|f\|_{L^\Phi(\Omega, \mu)} := \inf \left\{ \lambda > 0 : \int_\Omega \Phi \left( \frac{|f(x)|}{\lambda} \right) d\mu(x) \leq 1 \right\}.
\]  

(3.6)

Now we define the Orlicz maximal operator

\[
M_\Phi f(x) := \sup_{Q \ni x} \|f\|_{\Phi,Q} := \sup_{Q \ni x} \|f\|_{L^\Phi(Q, \frac{dx}{|Q|})},
\]

where the supremum is taken over all cubes \( Q \) in \( \mathbb{R}^n \). When \( \Phi(t) = t^p \), \( 1 \leq p < \infty \),

\[
\|f\|_{\Phi,Q} = \left( \int_Q |f(x)|^p dx \right)^{\frac{1}{p}} =: \|f\|_{p,Q}.
\]

In this case, if \( p = 1 \), \( M_\Phi \) agrees with the classical Hardy-Littlewood maximal operator \( M \); if \( p > 1 \), \( M_\Phi f = M_p f := M(|f|^p)^{1/p} \). If \( \Phi(t) \preceq \Psi(t) \), then \( M_\Phi f(x) \leq c M_\Psi f(x) \) for all \( x \in \mathbb{R}^n \).

The Hölder inequality can be generalized to the scale of Orlicz spaces [11, Lemma 5.2].

**Lemma 3.4.** Given a Young function \( A \), then for all cubes \( Q \),

\[
\int_Q |fg| dx \leq 2\|f\|_{A,Q}\|g\|_{A,Q}.
\]

(3.7)

More generally, if \( A, B \) and \( C \) are Young functions such that \( A^{-1}(t)B^{-1}(t) \leq c_1 C^{-1}(t) \), for all \( t \geq t_0 > 0 \), then

\[
\|fg\|_{C,Q} \leq c_2\|f\|_{A,Q}\|g\|_{B,Q}.
\]

(3.8)

The following result is an extension of the well-known Coifman-Rochberg theorem. The proof can be found in [19, Lemma 4.2].

**Lemma 3.5.** Let \( \Phi \) be a Young function and \( w \) be a nonnegative function such that \( M_\Phi w(x) < \infty \ a.e. \). Then

\[
[(M_\Phi w)^\delta]_{A_1} \leq c_{n, \delta}, \quad \forall \delta \in (0, 1),
\]

(3.9)

\[
[(M_\Phi w)^{-\lambda}]_{RH_{\infty}} \leq c_{n, \lambda}, \quad \forall \lambda > 0.
\]

(3.10)

Given \( p \in (1, \infty) \), a Young function \( \Phi \) is said to satisfy the \( B_p \) condition (or, \( \Phi \in B_p \)) if for some \( c > 0 \),

\[
\int_c^\infty \frac{\Phi(t)}{t^p} dt < \infty.
\]

(3.11)

Observe that if (3.11) is finite for some \( c > 0 \), then it is finite for every \( c > 0 \). Let \([\Phi]_{B_p} \) denote the value if \( c = 1 \) in (3.11). It was shown in [11, Proposition 5.10] that if \( \Phi \) and \( \Phi \) are doubling Young functions, then \( \Phi \in B_p \) if and only if

\[
\int_c^\infty \left( \frac{t^{p'}}{\Phi(t)} \right)^{p-1} dt < \infty.
\]
Let us present two types of $B_p$ bumps. An important special case is the “log-bumps” of the form
\[ A(t) = t^p \log(e + t)^{p-1+\delta}, \quad B(t) = t^{p'} \log(e + t)^{p'-1+\delta}, \quad \delta > 0. \]
(3.12)

Another interesting example is the “loglog-bumps” as follows:
\[ A(t) = t^p \log(e + t)^{p-1} \log(e + t)^{p-1+\delta}, \quad \delta > 0 \]
(3.13)
\[ B(t) = t^{p'} \log(e + t)^{p'-1} \log(e + t)^{p'-1+\delta}, \quad \delta > 0. \]
(3.14)

Then one can verify that in both cases above, $\mathcal{A} \in B_{p'}$ and $\mathcal{B} \in B_p$ for any $1 < p < \infty$.

The $B_p$ condition can be also characterized by the boundedness of the Orlicz maximal operator $M_\Phi$. Indeed, the following result was given in [11, Theorem 5.13] and [19, eq. (25)].

**Lemma 3.6.** Let $1 < p < \infty$. Then $M_\Phi$ is bounded on $L^p(\mathbb{R}^n)$ if and only if $\Phi \in B_p$.
Moreover, $\|M_\Phi\|_{L^p(\mathbb{R}^n)} \leq C_{n,p} [\Phi]_{B_p}$. In particular, if the Young function $A$ is the same as the first one in (3.12) or (3.13), then
\[ \|M_\Phi\|_{L^p(\mathbb{R}^n)} \leq C p^2 \delta^{-1/p}, \quad \forall \delta \in (0, 1]. \]
(3.15)

**Definition 3.7.** Given $p \in (1, \infty)$, let $A$ and $B$ be Young functions such that $\mathcal{A} \in B_p$ and $\mathcal{B} \in B_p$. We say that the pair of weights $(u, v)$ satisfies the double bump condition with respect to $A$ and $B$ if
\[ [u, v]_{A,B,p} := \sup_Q \|u^{1/p}\|_{A,Q} \|v^{-1/p'}\|_{B,Q} < \infty. \]
(3.16)
where the supremum is taken over all cubes $Q$ in $\mathbb{R}^n$. Also, $(u, v)$ is said to satisfy the separated bump condition if
\[ [u, v]_{A,B,p'} := \sup_Q \|u^{1/p}\|_{A,Q} \|v^{-1/p'}\|_{B,Q} < \infty, \]
(3.17)
\[ [u, v]_{p,B} := \sup_Q \|u^{1/p}\|_{p,Q} \|v^{-1/p'}\|_{B,Q} < \infty. \]
(3.18)

Note that if $A(t) = t^p$ in (3.17) or $B(t) = t^p$ in (3.18), each of them actually is two-weight $A_p$ condition and we denote them by $[u, v]_{A_p} := [u, v]_{p,p'}$. Also, the separated bump condition is weaker than the double bump condition. Indeed, (3.16) implies (3.17) and (3.18), but the reverse direction is incorrect. The first fact holds since $\mathcal{A} \in B_{p'}$ and $\mathcal{B} \in B_p$ respectively indicate $A$ is a $p$-Young function and $B$ is a $p'$-Young function. The second fact was shown in [1, Section 7] by constructing log-bumps.

**Lemma 3.8.** Let $1 < p < \infty$, let $A$, $B$ and $\Phi$ be Young functions such that $A \in B_p$ and $A^{-1}(t)B^{-1}(t) \lesssim \Phi^{-1}(t)$ for any $t > t_0 > 0$. If a pair of weights $(u, v)$ satisfies $[u, v]_{p,B} < \infty$, then
\[ \|M_\Phi f\|_{L^p(u)} \leq C[u, v]_{p,B} [A]_{B_p} \|f\|_{L^p(v)}. \]
(3.19)
Moreover, (3.19) holds for $\Phi(t) = t$ and $B = A$ satisfying the same hypotheses. In this case, $\mathcal{A} \in B_p$ is necessary.
The two-weight inequality above was established in [11, Theorem 5.14] and [12, Theorem 3.1]. The weak type inequality for $M_\Phi$ was also obtained in [11, Proposition 5.16] as follows.

**Lemma 3.9.** Let $1 < p < \infty$, let $B$ and $\Phi$ be Young functions such that \( t^{\frac{1}{p}} B^{-1}(t) \lesssim \Phi^{-1}(t) \) for any $t > t_0 > 0$. If a pair of weights $(u, v)$ satisfies $[u, v]_{p, B} < \infty$, then
\[
\|M_\Phi f\|_{L^p, \infty (u)} \leq C\|f\|_{L^p(v)}.
\] (3.20)

Moreover, (3.20) holds for $M$ if and only if $[u, v]_{A_p} < \infty$.

4. **Proof of main results**

4.1. **Sparse domination.** Let $\Phi$ be a radial Schwartz function such that $1_{B(0,1)} \leq \Phi \leq 1_{B(0,2)}$. We define
\[
\tilde{S}_{\alpha, L}(f)(x) := \left( \int_0^\infty \int_{\mathbb{R}^n} \Phi \left( \frac{|x-y|}{\alpha t} \right) |Q_{t,L} f(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2},
\]
where $Q_{t,L} f := t^m L^{-m} f$. It is easy to verify that
\[
S_{\alpha, L}(f)(x) \leq \tilde{S}_{\alpha, L}(f)(x) \leq S_{2\alpha, L}(f)(x), \quad x \in \mathbb{R}^n. \quad (4.1)
\]
Additionally, it was proved in [2] that $S_{1, L}$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{1, \infty}(\mathbb{R}^n)$. Then, this and (4.1) give that
\[
\|\tilde{S}_{\alpha, L}(f)\|_{L^{1, \infty}(\mathbb{R}^n)} \lesssim \alpha^n \|f\|_{L^1(\mathbb{R}^n)}. \quad (4.2)
\]

Using these facts, we can establish the sparse domination for $S_{\alpha, L}$ as follows.

**Lemma 4.1.** For any $\alpha \geq 1$, we have
\[
S_{\alpha, L}(f)(x) \lesssim \alpha^n \sum_{j=1}^{2^n} A^2_S \tilde{S} j(\alpha f)(x), \quad a.e. \ x \in \mathbb{R}^n, \quad (4.3)
\]
where
\[
A^2_S(f)(x) := \left( \sum_{Q \in S} |f|_Q^2 1_Q(x) \right)^{\frac{1}{2}}.
\]

**Proof.** Fix $Q_0 \in D$. By (3.1), Kolmogorov’s inequality and (4.2), we have
\[
|m_{\tilde{S}_{\alpha, L} f}(Q_0)| \lesssim \left( \int_{Q_0} |\tilde{S}_{\alpha, L}(f 1_{Q_0})|^2 dx \right)^4 \lesssim \|	ilde{S}_{\alpha, L}(f 1_{Q_0})\|_{L^{1, \infty}(\mathbb{R}^n)}^2 \lesssim \alpha^{2n} \left( \int_{Q_0} |f| dx \right)^2. \quad (4.4)
\]
From [3, Proposition 3.2], we obtain that for any dyadic cube $Q \subset \mathbb{R}^n$, $\alpha \geq 1$ and $\lambda \in (0, 1)$,
\[
\omega_\lambda(\tilde{S}_{\alpha, L} f)^2; Q \lesssim \alpha^{2n} \sum_{j=0}^{\infty} 2^{-j\delta} \left( \int_{2^j Q} |f| dx \right)^2, \quad (4.5)
\]
where $\delta \in (0, 1)$ is some constant. Invoking Lemma 3.3, (4.4) and (4.5), one can pick a sparse family $S(Q_0) \subset D(Q_0)$ so that
\[
\tilde{S}_{\alpha,L}(f)(x)^2 \lesssim |m_{\tilde{S}_{\alpha,L}(f)^2}(Q_0)| + \sum_{Q \in S(Q_0)} \omega_\varepsilon(\tilde{S}_{\alpha,L}(f)^2; Q)1_Q(x)
\]
\[
\lesssim \alpha^{2n} \sum_{Q \in S(Q_0)} \sum_{j=0}^{\infty} 2^{-j\delta} \|f\|_{2j+2}^2 1_Q(x)
\]
\[
=: \alpha^{2n} \sum_{j=0}^{\infty} 2^{-j\delta} T_{S(Q_0), j}^2(f)(x)^2, \text{ a.e. } x \in Q_0. \tag{4.6}
\]
where
\[
T_{S,j}^2(f)(x) := \left( \sum_{Q \in S} \|f\|_{2j+2}^2 1_Q(x) \right)^{\frac{1}{2}}.
\]
Denote
\[
T_{S,j}(f, g)(x) := \sum_{Q \in S} \|f\|_{2j+2} \|g\|_{2j+2} 1_Q(x),
\]
\[
A_{S,j}(f, g)(x) := \sum_{Q \in S} \|f\|_Q \|g\|_{2Q} 1_Q(x).
\]
Then, $T_{S,j}^2(f)(x) = T_{S,j}(f, f)(x)^{\frac{1}{2}}$. On the other hand, the arguments in [23, Sections 11-13] shows that there exist $3^n$ dyadic grids $S_j \in \mathcal{D}_j, j = 1, \ldots, 3^n$, such that
\[
\sum_{j=0}^{\infty} 2^{-j\delta} T_{S(Q_0), j}^1(f, f)(x) \lesssim \sum_{j=1}^{3^n} A_{S,j}(f, f)(x) = \sum_{j=1}^{3^n} A_{S,j}^2(f)(x)^2. \tag{4.8}
\]
Gathering (4.7) and (4.8), we deduce that
\[
\tilde{S}_{\alpha,L}(f)(x) \lesssim \alpha^n \sum_{j=1}^{3^n} A_{S,j}^2(f)(x), \text{ a.e. } x \in Q_0.
\]
Since $\mathbb{R}^n = \bigcup_{Q \in \mathcal{D}} Q$, it leads that
\[
S_{\alpha,L}(f)(x) \leq \tilde{S}_{\alpha,L}(f)(x) \lesssim \alpha^n \sum_{j=1}^{3^n} A_{S,j}^2(f)(x), \text{ a.e. } x \in \mathbb{R}^n.
\]
This completes our proof. \qed

4.2. Bump conjectures. In this subsection, we are going to show two-weight inequalities invoking bump conjectures.

**Proof of Theorem 1.1.** By Lemma 4.1, the inequality (1.2) follows from the following
\[
\|A_S^2(f)\|_{L^p(u)} \lesssim \mathcal{M}_p \|f\|_{L^p(v)}, \tag{4.9}
\]
for every sparse family $S$, where the implicit constant does not depend on $S$. 


To prove (4.9), we begin with the case $1 < p \leq 2$. Actually, the H"{o}lder’s inequality (3.7) gives that
\[
\|A_S^2(f)\|_{L^p(u)}^p = \int_X \left( \sum_{Q \in S} |f|^2_Q 1_Q(x) \right)^{\frac{p}{2}} u(x) dx \leq \sum_{Q \in S} (|f|^p_Q)^{\frac{p}{2}} \int_Q u(x) dx
\]
\[
\leq \sum_{Q \in S} \|fv^\frac{1}{p}\|_{B,Q}^p \|v^{-\frac{1}{p}}\|_{B,Q}^p \|u^\frac{1}{p}\|_{p,Q}^p |Q|
\]
\[
\leq ||(u,v)||_{A,B,p}^p \sum_{Q \in S} \left( \inf_{Q} M_B(fv^\frac{1}{p}) \right)^p |E_Q|
\]
\[
\leq ||(u,v)||_{A,B,p}^p \int_X M_B(fv^\frac{1}{p})(x)^p dx
\]
\[
\leq ||(u,v)||_{A,B,p}^p \|M_B\|_{L^p(v)}^p \|f\|_{L^p(u)}^p,
\]
(4.10)
where Lemma 3.6 is used in the last step.

Next let us deal with the case $2 < p < \infty$. By duality, one has
\[
\|A_S^2(f)\|_{L^p(u)} = \|A_S^2(f)^2\|_{L^{p/2}(u)} = \sup_{0 \leq h \in L^{(p/2)'}(u), \|h\|_{L^{(p/2)'}(u)} = 1} \int_{\mathbb{R}^n} A_S^2(f)^2 h u dx.
\]
(4.11)
Fix a nonnegative function $h \in L^{(p/2)'}(u)$ with $\|h\|_{L^{(p/2)'}(u)} = 1$. Then using H"{o}lder’s inequality (3.7) three times and Lemma 3.6, we obtain
\[
\int_X A_S^2(f)(x)^2 h(x) u(x) dx \leq \sum_{Q \in S} (|f|^2_Q h u(x) 1_Q) |Q|
\]
\[
\leq \sum_{Q \in S} \|fv^\frac{1}{p}\|_{B,Q}^2 \|v^{-\frac{1}{p}}\|_{B,Q}^2 \|hu^{1-\frac{2}{p}}\|_{A,Q}^2 \|u^\frac{2}{p}\|_{A,Q} |Q|
\]
\[
\leq ||(u,v)||_{A,B,p}^2 \sum_{Q \in S} \left( \inf_{Q} M_B(fv^\frac{1}{p}) \right)^2 \left( \inf_{Q} M_A(hu^{1-\frac{2}{p}}) \right) |E_Q|
\]
\[
\leq ||(u,v)||_{A,B,p}^2 \int_X M_B(fv^\frac{1}{p})(x)^2 M_A(hu^{1-\frac{2}{p}})(x) dx
\]
\[
\leq ||(u,v)||_{A,B,p}^2 \|M_B(fv^\frac{1}{p})\|_{L^{p/2}}^2 \|M_A(hu^{1-\frac{2}{p}})\|_{L^{(p/2)'}(u)}^2
\]
\[
\leq ||(u,v)||_{A,B,p}^2 \|M_B\|_{L^{p/2}}^2 \|M_A\|_{L^{(p/2)'}(u)}^2 \|f\|_{L^p(u)}^2 \|h\|_{L^{(p/2)'}(u)}^2.
\]
(4.12)
Therefore, (4.9) immediately follows from (4.10), (4.11) and (4.12). \qed

Let us present an endpoint extrapolation theorem from [11, Corollary 8.4].

**Lemma 4.2.** Let $\mathcal{F}$ be a collection of pairs $(f,g)$ of nonnegative measurable functions. If for every weight $w$,
\[
\|f\|_{L^{1,\infty}(w)} \leq C \|g\|_{L^1(Mw)}, \quad (f,g) \in \mathcal{F},
\]
then for all \( p \in (1, \infty) \),
\[
\|f\|_{L^p, \infty(u)} \leq C \|g\|_{L^p(u)}, \quad (f, g) \in \mathcal{F},
\]
whenever \( \sup_B \|u^\frac{2}{p}\|_{A,B} \|v^{-\frac{1}{p'}}\|_{p',B} < \infty \), where \( \bar{A} \in B_{p'} \).

**Proof of Theorem 1.2.** In view of, it suffices to prove that for every weight \( w \),
\[
\|S_{\alpha,L}(f)\|_{L^{1,\infty}(w)} \leq C\|f\|_{L^1(\mathcal{M}w)},
\]
where the constant \( C \) is independent of \( w \) and \( f \). We should mention that although the norm of weights does not appear in [11, Corollary 8.4], one can check its proof to obtain the norm constant in (1.3). Invoking Lemma 4.1, we are reduced to showing that there exists a constant \( C \) such that for every sparse family \( S \) and for every weight \( w \),
\[
\|A^2_S(f)\|_{L^{1,\infty}(w)} \leq C\|f\|_{L^1(\mathcal{M}w)}, \quad (4.13)
\]
Without loss of generality, we may assume that \( f \) is bounded and has compact support. Fix \( \lambda > 0 \) and denote \( \Omega := \{x \in \mathbb{R}^n : M_D f(x) > \lambda\} \). By the Calderón-Zygmund decomposition, there exists a pairwise disjoint family \( \{Q_j\} \subset \mathcal{D} \) such that \( \Omega = \bigcup_j Q_j \) and
1. \( f = g + b \),
2. \( g = f1_\Omega + \sum_j \langle f \rangle_{Q_j} 1_{Q_j} \),
3. \( b = \sum_j b_j \) with \( b_j = (f - \langle f \rangle_{Q_j}) 1_{Q_j} \),
4. \( \|f\|_{Q_j} > \lambda \) and \( |g(x)| \leq 2^n \lambda \), a.e. \( x \in \mathbb{R}^n \),
5. \( \text{supp}(b_j) \subset Q_j \) and \( \int_{Q_j} b_j \, dx = 0 \).

Then by (1), we split
\[
w(\{x \in \mathbb{R}^n : A^2_S(f)(x) > \lambda\}) \leq w(\Omega) + I_g + I_b, \quad (4.14)
\]
where
\[
I_g = w(\{x \in \Omega^c : A^2_S(g)(x) > \lambda/2\}) \quad \text{and} \quad I_b = w(\{x \in \Omega^c : A^2_S(b)(x) > \lambda/2\})
\]
For the first term, we by (4) have
\[
w(\Omega) \leq \sum_j w(Q_j) \leq \frac{1}{\lambda} \sum_j \frac{w(Q_j)}{|Q_j|} \int_{Q_j} |f(x)| \, dx
\]
\[
\leq \frac{1}{\lambda} \sum_j \int_{Q_j} |f(x)| M_D w(x) \, dx \leq \frac{1}{\lambda} \int_{\mathbb{R}^n} |f(x)| M_D w(x) \, dx. \quad (4.15)
\]
To estimate \( I_b \), we claim that \( A^2_S(b_j)(x) = 0 \) for all \( x \in \Omega^c \) and for all \( j \). In fact, if there exist \( x_0 \in \Omega^c \) and \( j_0 \) such that \( A^2_S(b_{j_0})(x_0) \neq 0 \), then there is a dyadic cube \( Q_0 \in \mathcal{S} \) such that \( x_0 \in Q_0 \) and \( \langle b_{j_0} \rangle_{Q_0} \neq 0 \). The latter implies \( Q_0 \nsubseteq Q_{j_0} \) because of the support and the vanishing property of \( b_{j_0} \). This in turn gives that \( x_0 \in Q_{j_0} \), which contradicts \( x_0 \in \Omega^c \). This shows our claim. As a consequence, the set \( \{x \in \Omega^c : A^2_S(b)(x) > \lambda/2\} \) is empty, and hence \( I_b = 0 \).
In order to control $I_\kappa$, we first present a Fefferman-Stein inequality for $A_S^2$. Note that $v(x) := M_D w(x) \geq \langle w \rangle_Q$ for any dyadic cube $Q \in S$ containing $x$. Then for any Young function $A$ such that $A \in B_2$,

\[
\|A_S^2(f)\|_{L^2(w)}^2 = \sum_{Q \in S} \langle |f| \rangle_Q^2 w(Q) \leq \sum_{Q \in S} \|f v^{\frac{1}{2}}\|_{A,Q}^2 \|v^{-\frac{1}{2}}\|_{A,Q}^2 w(Q)
\]

\[
\leq \sum_{Q \in S} \|f v^{\frac{1}{2}}\|_{A,Q}^2 (\inf_{Q \in S} M_A(f v^{\frac{1}{2}}))^2 |E_Q| \leq \int_{\mathbb{R}^n} M_A(f v^{\frac{1}{2}}) (x)^2 \, dx \lesssim \|f\|_{L^2(w)}^2 = \|f\|_{L^2(M_D w)}^2,
\]

where we used Lemma 3.6 in the last inequality. Note that for any $x \in Q_j$,

\[
M_D(w 1_{\Omega^c})(x) \leq M_D(w 1_{Q^c_j})(x) = \sup_{x \in Q \in D} \frac{1}{|Q|} \int_{Q \cap Q^c_j} w(y) \, dy \leq \inf_{Q \subseteq \Omega \cap Q^c_j} \frac{1}{|Q|} \int_{Q \cap Q^c_j} w(y) \, dy \leq \inf_{Q \subseteq \Omega \cap Q^c_j} M_D w.
\]

Thus, combining (4.16) with (4.17), we have

\[
I_\kappa \leq \frac{4}{\lambda^2} \int_{\Omega^c} A_S^2(g)(x) w(x) \, dx 
\]

\[
\lesssim \frac{1}{\lambda^2} \int_{\mathbb{R}^n} |g(x)| M_D(w 1_{\Omega^c})(x) \, dx \lesssim \frac{1}{\lambda} \int_{\mathbb{R}^n} |g(x)| M_D(w 1_{\Omega^c})(x) \, dx 
\]

\[
\leq \frac{1}{\lambda} \int_{\Omega^c} |f(x)| M_D(w 1_{\Omega^c})(x) \, dx + \frac{1}{\lambda} \sum_j |f(x)| \int_{Q_j} M_D(w 1_{\Omega^c})(x) \, dx
\]

\[
\leq \frac{1}{\lambda} \int_{\mathbb{R}^n} |f(x)| M_D w(x) \, dx + \frac{1}{\lambda} \sum_j \int_{Q_j} |f(y)| M_D w(y) \, dy \lesssim \frac{1}{\lambda} \int_{\mathbb{R}^n} |f(x)| M_D w(x) \, dx.
\]

Consequently, gathering (4.14), (4.15) and (4.18), we conclude that (4.13) holds. \quad \square

4.3. Fefferman-Stein inequalities. In order to show Theorem 1.3, we recall an extrapolation theorem for arbitrary weights in [9, Theorem 1.3], or [5, Theorem 4.11] in the general Banach function spaces.

Lemma 4.3. Let $F$ be a collection of pairs $(f, g)$ of nonnegative measurable functions. If for some $p_0 \in (0, \infty)$ and for every weight $w$,

\[
\|f\|_{L^{p_0}(w)} \leq C \|g\|_{L^{p_0}(M_D w)}, \quad (f, g) \in F,
\]

then for every $p \in (p_0, \infty)$ and for every weight $w$,

\[
\|f\|_{L^p(w)} \leq C \|g(M_D w / w) \|_{L^p(w)}, \quad (f, g) \in F.
\]
Proof of Theorem 1.3. Note that \( v(x) := M_w(x) \geq \langle w \rangle_Q \) for any dyadic cube \( Q \in S \) containing \( x \). Let \( A \) be a Young function such that \( \bar{A} \in B_p \). By Lemma 3.6, we have

\[
\| M_{\bar{A}}(fv^{1/2}) \|_{L^p} \lesssim \| f \|_{L^p(w)}.
\]

(4.19)

Thus, using Lemma 4.1, Hölder’s inequality and (4.19), we deduce that

\[
\| S_{\alpha,L}(f) \|_{L^p(w)} \lesssim \alpha^{pn} \sum_{j=1}^{K_0} \sum_{Q \in S_j} \| f \|_{L^p(w)}^{1/2} \| \nu^{-1/2} \|_{A,Q}^{1/2} \int_Q w(x) dx
\]

\[
\lesssim \alpha^{pn} \sum_{j=1}^{K_0} \sum_{Q \in S_j} \| f \|_{L^p(w)}^{1/2} \| \nu^{-1/2} \|_{A,Q}^{1/2} |Q|
\]

\[
\lesssim \alpha^{pn} \sum_{j=1}^{K_0} \sum_{Q \in S_j} \left( \inf_Q M_{\bar{A}}(fv^{1/2}) \right)^p \| f \|_{L^p(w)}^{1/2} \| \nu^{-1/2} \|_{A,Q}^{1/2} |Q|
\]

\[
\lesssim \alpha^{pn} \sum_{j=1}^{K_0} \sum_{Q \in S_j} \| f \|_{L^p(w)}^{1/2} \| \nu^{-1/2} \|_{A,Q}^{1/2} E_Q
\]

This shows (1.4). Finally, (1.5) is a consequence of (1.4) and Lemma 4.3.

□

4.4. Local decay estimates. To show Theorem 1.4, we need the following Carleson embedding theorem from [19, Theorem 4.5].

Lemma 4.4. Suppose that the sequence \( \{a_Q\}_{Q \in D} \) of nonnegative numbers satisfies the Carleson packing condition

\[
\sum_{Q \in D : Q \subset Q_0} a_Q \leq A w(Q_0), \quad \forall Q_0 \in D.
\]

Then for all \( p \in (1, \infty) \) and \( f \in L^p(w) \),

\[
\left( \sum_{Q \in D} a_Q \left( \frac{1}{w(Q)} \int_Q f(x) w(x) \, du(x) \right)^{\frac{1}{p}} \right)^{\frac{1}{p}} \leq \alpha p' \| f \|_{L^p(w)}.
\]

(4.20)

Lemma 4.5. For every \( 1 < p < \infty \) and \( w \in A_p \), we have

\[
\| S_{\alpha,L} f \|_{L^2(B,w)} \leq c_{n,p} \alpha^n \| w \|_{A_p}^{1/2} \| M f \|_{L^2(B,w)},
\]

for every ball \( B \subset \mathbb{R}^n \) and \( f \in L^\infty(\mathbb{R}^n) \) with \( \text{supp}(f) \subset B \).

Proof. Let \( w \in A_p \) with \( 1 < p < \infty \). Fix a ball \( B \subset \mathbb{R}^n \). From (4.6), there exists a sparse family \( S(Q) \subset D(Q) \) so that

\[
\tilde{S}_{\alpha,L}(f)(x)^2 \lesssim \alpha^{2n} \sum_{Q \in S(Q)} \sum_{j=0}^{\infty} 2^{-j\delta} \| |f| \|_{2^{j+1}Q'}^2 \mathbf{1}_{Q'}(x)
\]
\[ \lesssim \alpha^{2n} \sum_{Q' \in \mathcal{S}(Q)} \inf_{Q'} M(f)^2 1_{Q'}(x), \quad \text{a.e. } x \in Q. \]

This implies that
\[ \| \tilde{S}_{a,L}(f) \|_{L^2(Q,w)}^2 \lesssim \alpha^{2n} \sum_{Q' \in \mathcal{S}(Q)} \inf_{Q'} M(f)^2 w(Q'). \]

From this and (4.1), we see that to obtain (4.20), it suffices to prove
\[ \sum_{Q' \in \mathcal{S}(Q)} \inf_{Q'} M(f)^2 w(Q') \lesssim [w]_{A_p} \| M(f) \|_{L^2(Q,w)}. \quad (4.21) \]

Recall that a new version of \( A_\infty \) was introduced by Hytönen and Pérez [19]:
\[ [w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w1_Q)(x) \, dx. \]

By [19, Proposition 2.2], there holds
\[ c_n[w]_{A_\infty} \leq [w]_{A_\infty} \leq [w]_{A_p}. \quad (4.22) \]

Observe that for every \( Q'' \in \mathcal{D} \),
\[ \sum_{Q' \in \mathcal{S}(Q): Q' \subset Q''} w(Q') = \sum_{Q' \in \mathcal{S}(Q): Q' \subset Q''} \langle w \rangle_{Q'} |Q'| \lesssim \sum_{Q' \in \mathcal{S}(Q): Q' \subset Q''} \inf_{Q'} M(w1_{Q'}) |E_{Q'}| \lesssim \int_{Q''} M(w1_{Q''})(x) \, dx \leq [w]_{A_\infty} w(Q'') \lesssim [w]_{A_p} w(Q''), \]

where we used the disjointness of \( \{ E_{Q'} \}_{Q' \in \mathcal{S}(Q)} \) and (4.22). This shows that the collection \( \{ w(Q') \}_{Q' \in \mathcal{S}(Q)} \) satisfies the Carleson packing condition with the constant \( c_n[w]_{A_p} \). As a consequence, this and Lemma 4.4 give that
\[ \sum_{Q' \in \mathcal{S}(Q)} \inf_{Q'} M(f)^2 w(Q') \lesssim \sum_{Q' \in \mathcal{S}(Q)} \left( \frac{1}{w(Q')} \int_{Q'} M(f) 1_{Q'} w \, dx \right)^2 w(Q') \\lesssim [w]_{A_p} \| M(f) 1_Q \|_{L^2(w)}^2 = [w]_{A_p} \| M(f) 1_Q \|_{L^2(Q,w)}, \]

where the above implicit constants are independent of \( [w]_{A_p} \) and \( Q \). This shows (4.21) and completes the proof of (4.20). \( \square \)

Next, let us see how Lemma 4.5 implies Theorem 1.4.

**Proof of Theorem 1.4.** Let \( p > 1 \) and \( r > 1 \) be chosen later. Define the Rubio de Francia algorithm:
\[ \mathcal{R}h = \sum_{k=0}^{\infty} \frac{M^k h}{2^k \| M \|_{L^r(\mathbb{R}^n)}^{k)}. \]

Then it is obvious that
\[ h \leq \mathcal{R}h \quad \text{and} \quad \| \mathcal{R}h \|_{L^{r'}(\mathbb{R}^n)} \leq 2 \| h \|_{L^{r'}(\mathbb{R}^n)}. \quad (4.23) \]
Moreover, for any nonnegative $h \in L^{r'}(\mathbb{R}^n)$, we have that $\mathcal{R}h \in A_1$ with

$$[\mathcal{R}h]_{A_1} \leq 2\|M\|_{L^{r'} ightarrow L^{r'}} \leq c_n r. \quad (4.24)$$

By Riesz theorem and the first inequality in (4.23), there exists some nonnegative function $h \in L^{r'}(Q)$ with $\|h\|_{L^{r'}(Q)} = 1$ such that

$$\mathcal{F}_Q^r := \left\{|x \in Q : S_{\alpha,L}(f)(x) > tM(f)(x)\}\right\|^{\frac{1}{r}} = \left\{|x \in Q : S_{\alpha,L}(f)(x)^2 > t^2M(f)(x)^2\}\right\|^{\frac{1}{r}} \leq \frac{1}{t^2} \left\|\left(S_{\alpha,L}(f)\right)^2\right\|_{L^{r'}(Q)} \leq \frac{1}{t^2} \int_Q S_{\alpha,L}(f)^2 h M(f)^{-2} dx \leq t^{-2}\|S_{\alpha,L}(f)\|_{L^2(Q,w)}^2, \quad (4.25)$$

where $w = w_1 w_2^{1-p}$, $w_1 = \mathcal{R}h$ and $w_2 = M(f)^{2(p'-1)}$. Recall that Coifmann-Rochberg theorem [20, Theorem 3.4] asserts that

$$[(M(f))^\delta]_{A_1} \leq \frac{c_n}{1 - \delta}, \quad \forall \delta \in (0, 1). \quad (4.26)$$

In view of (4.24) and (4.26), we see that $w_1, w_2 \in A_1$ provided $p > 3$. Then the reverse $A_p$ factorization theorem gives that $w = w_1 w_2^{1-p} \in A_p$ with

$$[w]_{A_p} \leq [w_1]_{A_1} [w_2]_{A_1}^{p-1} \leq c_n r. \quad (4.27)$$

Thus, gathering (4.20), (4.25) and (4.27), we obtain

$$\mathcal{F}_Q^r \leq c_n t^{-2} \alpha^{2n} [w]_{A_p} \|M(f)\|_{L^2(Q,w)}^2 = c_n t^{-2} \alpha^{2n} [w]_{A_p} \|\mathcal{R}h\|_{L^1(Q)} \leq c_n t^{-2} \alpha^{2n} [w]_{A_p} \|\mathcal{R}h\|_{L^{r'}(Q)} |Q|^\frac{1}{r} \leq c_n t^{-2} \alpha^{2n} [w]_{A_p} \|h\|_{L^{r'}(Q)} |Q|^\frac{1}{r} \leq c_n t^{-2} \alpha^{2n} |Q|^\frac{1}{r}. \quad (4.28)$$

Consequently, if $t > \sqrt{c_n e^{\alpha n}}$, choosing $r > 1$ so that $t^2/e = c_n \alpha^{2n} r$, we have

$$\mathcal{F}_Q \leq (c_n \alpha^{2n} t^2 r^{-2})^\frac{1}{2} |Q| = e^{-\gamma} |Q| = e^{-\frac{t^2}{c_n e^{\alpha n}}} |Q|. \quad (4.29)$$

If $0 < t \leq \sqrt{c_n e^{\alpha n}}$, it is easy to see that

$$\mathcal{F}_Q \leq |Q| \leq e \cdot e^{-\frac{t^2}{c_n e^{\alpha n}}} |Q|. \quad (4.29)$$

Summing (4.28) and (4.29) up, we deduce that

$$\mathcal{F}_Q = \mu\{x \in Q : S_{\alpha,L}(f)(x) > tM(f)(x)\} \leq c_1 e^{-c_2 t^2/\alpha^{2n}} |Q|, \quad \forall t > 0.$$

This proves (1.6).
4.5. Mixed weak type estimates. To proceed the proof of Theorem 1.5, we establish a Coifman-Fefferman inequality.

**Lemma 4.6.** For every $0 < p < \infty$ and $w \in A_{\infty}$, we have
\[
\|S_{\alpha,L}f\|_{L^p(w)} \lesssim \|Mf\|_{L^p(w)},
\] (4.30)

**Proof.** Let $w \in A_{\infty}$. Then, it is well known that for any $\alpha \in (0, 1)$ there exists $\beta \in (0, 1)$ such that for any cube $Q$ and any measurable subset $A \subset Q$
\[
|A| \leq \alpha|Q| \implies w(A) \leq \beta w(Q).
\]
Thus, for the sparsity constant $\eta_j$ of $S_j$ there exists $\beta_j \in (0, 1)$ such that for $Q \in S_j$, we have
\[
w(E_Q) = w(Q) - w(Q \setminus E_Q) \geq (1 - \beta_j) w(Q),
\] (4.31)
since $w(Q \setminus E_Q) \leq (1 - \eta_j) w(Q)$. It follows from (4.3) and (4.31) that
\[
\int_{\mathbb{R}^n} S_{\alpha,L}(\vec{f})(x)^2 w(x) \, dx \lesssim \sum_{j=1}^{3^n} \sum_{Q \in S_j} \langle |f| \rangle^2_Q w(Q)
\lesssim \sum_{j=1}^{3^n} \sum_{Q \in S_j} \left( \inf_{Q} M(f) \right)^2 w(E_Q)
\lesssim \sum_{j=1}^{3^n} \sum_{Q \in S_j} \int_{E_Q} M(f)(x)^2 w(x) \, dx
\lesssim \int_{\mathbb{R}^n} M(f)(x)^2 w(x) \, dx.
\] (4.32)
This shows the inequality (4.30) holds for $p = 2$.

To obtain the result (4.30) for every $p \in (0, \infty)$, we apply the $A_{\infty}$ extrapolation theorem from [11, Corollary 3.15] in the Lebesgue spaces or [8, Theorem 3.36] for the general measure spaces. Let $\mathcal{F}$ be a family of pairs of functions. Suppose that for some $p_0 \in (0, \infty)$ and for every weight $v_0 \in A_{\infty},$
\[
\|f\|_{L^{p_0}(v_0)} \leq C_1 \|g\|_{L^{p_0}(v_0)}, \quad \forall (f,g) \in \mathcal{F}.
\] (4.33)
Then for every $p \in (0, \infty)$ and every weight $v \in A_{\infty},$
\[
\|f\|_{L^p(v)} \leq C_2 \|g\|_{L^p(v)}, \quad \forall (f,g) \in \mathcal{F}.
\] (4.34)
From (4.32), we see that (4.33) holds for the exponent $p_0 = 2$ and the pair $(S_{\alpha,L}f, Mf)$. Therefore, (4.34) implies that (4.30) holds for the general case $0 < p < \infty$. \hfill \Box

**Proof of Theorem 1.5.** In view of Lemma 4.6, we just present the proof for $S_{\alpha,L}$. We use a hybrid of the arguments in [10] and [25]. Define
\[
\mathcal{R}h(x) = \sum_{j=0}^{\infty} \frac{T_j^2 h(x)}{2^j K_0^j}.
\]
where $K_0 > 0$ will be chosen later and $T_u f(x) := M(fu)(x)/u(x)$ if $u(x) \neq 0$, $T_u f(x) = 0$ otherwise. It immediately yields that

$$h \leq \mathcal{R}h \quad \text{and} \quad T_u (\mathcal{R}h) \leq 2K_0 \mathcal{R}h.$$ (4.35)

Moreover, follow the same scheme of that in [10], we get for some $r > 1$,

$$\mathcal{R}h \cdot uv^\frac{1}{r} \in A_\infty \quad \text{and} \quad \|\mathcal{R}h\|_{L^{r',1}(uv)} \leq 2\|h\|_{L^{r',1}(uv)}. \quad \text{(4.36)}$$

Note that

$$\|f^q\|_{L^{p,\infty}(w)} = \|f\|_{L^{p,\infty}(w)^q}, \quad 0 < p, q < \infty. \quad \text{(4.37)}$$

This implies that

$$\left\| \frac{S_{\alpha,L}(f)}{v} \right\|_{L^{1,\infty}(uv)}^{\frac{1}{p}} = \left\| \left( \frac{S_{\alpha,L}(f)}{v} \right)^{\frac{1}{r'}} \right\|_{L^{r',\infty}(uv)}$$

$$= \sup_{\|h\|_{L^{r',1}(uv)} = 1} \left\| \int_{\mathbb{R}^n} |S_{\alpha,L}(f)(x)|^{\frac{1}{r'}} h(x) u(x) v(x) \frac{dx}{x} \right\|$$

$$\leq \sup_{\|h\|_{L^{r',1}(uv)} = 1} \int_{\mathbb{R}^n} |S_{\alpha,L}(f)(x)|^{\frac{1}{r'}} \mathcal{R}h(x) u(x) v(x) \frac{dx}{x}.$$ (4.38)

Invoking Lemma 4.6 and Hölder’s inequality, we obtain

$$\int_{\mathbb{R}^n} |S_{\alpha,L}(f)(x)|^{\frac{1}{r'}} \mathcal{R}h(x) u(x) v(x) \frac{dx}{x}$$

$$\leq \int_{\mathbb{R}^n} M(f)(x)^{\frac{1}{r'}} \mathcal{R}h(x) u(x) v(x) \frac{dx}{x}$$

$$= \int_{\mathbb{R}^n} \left( \frac{M(f)(x)}{v(x)} \right)^{\frac{1}{r'}} \mathcal{R}h(x) u(x) v(x) dx$$

$$\leq \left\| \left( \frac{M(f)}{v} \right)^{\frac{1}{r'}} \right\|_{L^{r',\infty}(uv)} \|\mathcal{R}h\|_{L^{r',1}(uv)}$$

$$\leq \left\| \frac{M(f)}{v} \right\|_{L^{1,\infty}(uv)}^{\frac{1}{r'}} \|h\|_{L^{r',1}(uv)};$$

where we used (4.37) and (4.36) in the last inequality. Here we need to apply the weighted mixed weak type estimates for $M$ proved in Theorems 1.1 in [26]. Consequently, collecting the above estimates, we get the desired result

$$\left\| \frac{S_{\alpha,L}(f)}{v} \right\|_{L^{1,\infty}(uv)} \lesssim \left\| \frac{M(f)}{v} \right\|_{L^{1,\infty}(uv)} \lesssim \|f\|_{L^1(u)}.$$ (4.39)

The proof is complete. \qed
4.6. **Restricted weak type estimates.**

**Proof of Theorem 1.6.** In view of (4.3), it suffices to show that

$$
\|A_S^2(1_E)\|_{L^p,\infty(w)} \lesssim [w]_{A^p}^{p+1} w(E)^{\frac{2}{p}}.
$$

(4.38)

Since $S$ is sparse, for every $Q \in S$, there exists $E_Q \subset Q$ such that $|E_Q| \simeq |Q|$ and $\{E_Q\}_{Q \in S}$ is a disjoint family. Note that $Q \subset Q(x, 2\ell(Q)) \subset 3Q$ for any $x \in Q$, where $Q(x, 2\ell(Q))$ denotes the cube centered at $x$ and with sidelength $2\ell(Q)$. Hence, for all $Q \in S$ and for all $x \in Q$,

$$
\frac{w(Q(x, 2\ell(Q)))}{w(E_Q)} \simeq \frac{w(Q)}{w(E_Q)} \leq [w]_{A^p}^{p} \left( \frac{|Q|}{|E_Q|} \right)^{p} \lesssim [w]_{A^p}^{p}. \quad (4.39)
$$

By duality, one has

$$
\|A_S^2(1_E)\|^2_{L^p,\infty(w)} = \|A_S^2(1_E)^2\|_{L^{p/2,\infty}(w)} = \sup_{0 \leq h \in L^{(p/2)'\frac{n}{2}}(w)} \int_{\mathbb{R}^n} A_S^2(1_E)^2 h w \, dx. \quad (4.40)
$$

Fix such $h$ above. Then, using (4.39), the disjointness of $\{E_Q\}_{Q \in S}$, Hölder’s inequality and (1.8), we conclude that

$$
\int_{\mathbb{R}^n} A_S^2(1_E)^2 h w \, dx = \sum_{Q \in S} \langle 1_E \rangle_Q^2 \int_{Q} h w \, dx
$$

$$
\leq \sum_{Q \in S} \langle 1_E \rangle_Q^2 \left( \int_{Q(x, 2\ell(Q))} h \, dw \right) w(E_Q) \left( \frac{w(Q(x, 2\ell(Q)))}{w(E_Q)} \right)
$$

$$
\lesssim [w]_{A^p}^{p} \sum_{Q \in S} \left( \inf_{Q} M1_E \right)^2 \left( \inf_{Q} M^c_w h \right) w(E_Q)
$$

$$
\leq [w]_{A^p}^{p} \int_{\mathbb{R}^n} M1_E(x)^2 M^c_w h(x) w \, dx
$$

$$
\leq [w]_{A^p}^{p} \| (M1_E)^2 \|_{L^{p/2,\infty}(w)} \| M^c_w h \|_{L^{(p/2)'\frac{n}{2}}(w)}
$$

$$
\lesssim [w]_{A^p}^{p+2} w(E)^{\frac{2}{p}}. \quad (4.41)
$$

Therefore, (4.40) and (4.41) immediately imply (1.9). \hfill \square

4.7. **Endpoint estimates for commutators.** Recall that the sharp maximal function of $f$ is defined by

$$
M^\sharp f(x) := \sup_{x \in Q} \inf_{c \in \mathbb{R}} \left( \int_Q |f - c| \, dx \right)^{\frac{1}{p}}.
$$

If we write $Q_{t,L} := t^m L e^{-t^n L}$, then

$$
C_b(S_{\alpha,L}) f(x) = \left( \int_{Q_{t,L}} |Q_{t,L}((b(x) - b(\cdot))f(\cdot))(y)|^2 dy dt \right)^{\frac{1}{2}}.
$$
Lemma 4.7. For any $0 < \delta < 1$,

\[
M_\delta^\#(\widetilde{S}_{n,L} f)(x_0) \lesssim \alpha^{2n} M f(x_0), \quad \forall x_0 \in \mathbb{R}^n. \tag{4.42}
\]

Proof. For any cube $Q \ni x_0$. The lemma will be proved if we can show that

\[
\left( \int_Q |\widetilde{S}_{n,L}(f)^2(x) - c_Q|^{\delta} dx \right)^{\frac{1}{\delta}} \lesssim \alpha^{2n} M f(x_0)^2,
\]

where $c_Q$ is a constant which will be determined later.

Denote $T(Q) = Q \times (0, \ell(Q))$. We write

\[
\widetilde{S}_{n,L}(f)^2(x) = E(f)(x) + F(f)(x),
\]

where

\[
E(f)(x) := \int_{T(2Q)} \Phi \left( \frac{|x - y|}{at} \right) |Q_{t,L} f(y)|^2 \frac{dydt}{t^{n+1}},
\]

\[
F(f)(x) := \int_{\mathbb{R}^n \setminus T(2Q)} \Phi \left( \frac{|x - y|}{at} \right) |Q_{t,L} f(y)|^2 \frac{dydt}{t^{n+1}}.
\]

Let us choose $c_Q = F(f)(x_Q)$ where $x_Q$ is the center of $Q$. Then

\[
\left( \int_Q |\widetilde{S}_{n,L}(\psi)^2 - c_Q|^{\delta} dx \right)^{\frac{1}{\delta}} = \left( \int_Q |E(f)(x) + F(f)(x) - c_Q|^{\delta} dx \right)^{\frac{1}{\delta}} \lesssim \left( \int_Q |E(f)(x)|^{\delta} dx \right)^{\frac{1}{\delta}} + \left( \int_Q |F(f)(x)|^{\delta} dx \right)^{\frac{1}{\delta}} : = I + II
\]

We estimate each term separately. For the first term $I$, we set $f = f_0 + f^\infty$, where

\[
f_0 = f\chi_{Q^*}, f^\infty = f\chi_{(Q^*)^c} \text{ and } Q^* = 8Q.
\]

Then we have

\[
E(f)(x) \lesssim E(f_0)(x) + E(f^\infty)(x). \tag{4.43}
\]

Therefore,

\[
\left( \int_Q |E(f)(x)|^{\delta} dx \right)^{\frac{1}{\delta}} \lesssim \left( \int_Q |E(f_0)(x)|^{\delta} dx \right)^{\frac{1}{\delta}} + \left( \int_Q |E(f^\infty)(x)|^{\delta} dx \right)^{\frac{1}{\delta}}.
\]

It was proved in [3, p. 884], that $\|\widetilde{S}_{n,L}(f)\|_{L^{1,\infty}} \lesssim \alpha^n \|S_{1,L}(f)\|_{L^{1,\infty}}$. Then, by (4.2) and Kolmogorov inequality we have

\[
\left( \int_Q |E(f_0)(x)|^{\delta} dx \right)^{\frac{1}{\delta}} \lesssim \left( \int_Q |\widetilde{S}_{n,L}(f_0)|^{2\delta} dx \right)^{\frac{1}{2\delta}} \lesssim \|\widetilde{S}_{n,L}(f_0)\|^2_{L^{1,\infty}(Q^*,\frac{\alpha^n}{\delta})} \lesssim \alpha^{2n} \left( \int_{Q^*} f_0(x)^2 dx \right)^2. \tag{4.44}
\]
On the other hand,
\[
\left( \frac{1}{|Q|} \int_{\mathbb{R}^n} \int_{T(2Q)} \phi(\frac{x-y}{\alpha t}) |Q_{t,L}f(y)|^2 \frac{dydt}{t^{n+1}} dx \right)^\frac{1}{2}
\]
\[
= \frac{\alpha^{2n}}{|Q|} \int_{T(2Q)} |Q_{t,L}(f^\infty)(y)|^2 \frac{dydt}{t},
\]
since \( \int_{\mathbb{R}^n} \phi(\frac{x-y}{\alpha t}) dx \leq c_n(\alpha t)^n \).

For any \( k \in \mathbb{N}_+ \), \( p_{k,t}(x,y) \) denote the kernel of operator \((tL)^k e^{-tL}\). Note that condition (A2) implies that for any \( \delta_0 > 0 \), there exist \( C, c > 0 \) such that
\[
|p_{k,t}(x,y)| \leq \frac{C}{t^{n/m}} \left( \frac{t^{1/m}}{t^{1/m} + |x-y|} \right)^{n+\delta_0}, \quad \text{for all } x, y \in \mathbb{R}^n.
\] (4.45)

Thus, (4.45) implies that
\[
\left( \int_{2Q} |Q_{t,L}(f^\infty)(y)|^2 \frac{dy}{2} \right)^{1/2}
\]
\[
\lesssim \sum_{j \geq 3} \left\{ \int_{2Q} \left[ \int_{2j+1Q \setminus 2jQ} \frac{1}{t^n} \left( \frac{t}{t^n + |y-z|} \right)^{n+\delta_0} |f(z)|^2 dz \right] \frac{dy}{2} \right\}^{1/2}
\]
\[
\lesssim \sum_{j \geq 3} \left\{ \int_{2Q} \left[ \int_{2j+1Q \setminus 2jQ} \frac{1}{2^{j\ell(Q)}} \left( \frac{t}{2^{j\ell(Q)}} \right)^{n+\delta_0} |f(z)|^2 dz \right] \frac{dy}{2} \right\}^{1/2}
\]
\[
\lesssim \left( \frac{t}{\ell(Q)} \right)^{\delta_0} |2Q|^{1/2} \sum_{j \geq 3} \frac{1}{2^{j\delta_0}} \left( \int_{2jQ} |f(z)|^2 dz \right).
\]

Then one has
\[
\left( \int_{Q} |E(f^\infty)(x)|^\delta dx \right)^{\frac{1}{\delta}}
\]
\[
\lesssim \left[ \sum_{l=0}^{\infty} \frac{1}{2^{\delta_0 l}} \left( \int_{2^lQ} |f_j| dx \right) \right]^{2} \frac{\alpha^{2n}}{|Q|} \int_{T(2Q)} |2Q|(t/\ell(Q))^{2\delta_0} \frac{dydt}{t}
\]
\[
\lesssim \alpha^{2n} \left[ \sum_{l=0}^{\infty} \frac{1}{2^{\delta_0 l}} \left( \int_{2^lQ} |f_j| dx \right) \right]^{2} \lesssim \alpha^{2n} \sum_{l=0}^{\infty} \frac{1}{2^{\delta_0 l}} \left( \int_{2^lQ} |f_j| dx \right)^2,
\]
where in the last inequality we used Hölder’s inequality.

Therefore, we obtain that
\[
\left( \int_{Q} |E(f^\infty)(x)|^\delta dx \right)^{\frac{1}{\delta}} \lesssim \alpha^{2n} \sum_{l=0}^{\infty} \frac{1}{2^{\delta_0 l}} \left( \int_{2^lQ} |f_j| dx \right)^2.
\] (4.46)

Gathering (4.43), (4.44) and (4.46), we deduce that
\[
I \lesssim \alpha^{2n} M(f)(x_0).
\]
To estimate the second term $II$, we shall use the following estimate [3, Eq (35)]:

$$|F(f)(x) - F(f)(x_Q)| \lesssim \alpha^{2n} \sum_{l=0}^{\infty} \frac{1}{2^{ls}} \left( \int_{2^lQ} |f| \, dx \right)^{\frac{1}{s}},$$

for some $\delta > 0$ and all $x \in Q$, where $x_Q$ is the center of $Q$. As a consequence, we have

$$II = \left( \int_Q |F(f)(x) - F(f)(x_Q)|^\delta \, dx \right)^{\frac{1}{\delta}} \lesssim \alpha^{2n} Mf(x_0).$$

This finish the proof. \hfill \lrcorner

**Lemma 4.8.** For any $0 < \delta < \varepsilon < 1$ and for any $b \in \text{BMO}$,

$$M^b_\delta(Cb(\tilde{S}_{\alpha,L})f)(x) \lesssim \|b\|_{\text{BMO}} (M_{L,logL}f(x) + M_\varepsilon(\tilde{S}_{\alpha,L}f)(x)).$$

**Proof.** Let $x \in \mathbb{R}^n$, and let $Q$ be any arbitrary cube containing $x$. It suffices to show that there exists $c_Q$ such that

$$\mathcal{A} := \left( \int_Q |b(z) - b_Q| \tilde{S}_{\alpha,L}(f)(z)^\delta \, dz \right)^{\frac{1}{\delta}} \lesssim \|b\|_{\text{BMO}} (M_{L,logL}f(x) + M_\varepsilon(\tilde{S}_{\alpha,L}f)(x)).$$

Split $f = f_1 + f_2$, where $f_1 = f 1_{ScQ}$. Then, we have

$$\mathcal{A} \lesssim \left( \int_Q |b(z) - b_Q| \tilde{S}_{\alpha,L}(f)(z)^\delta \, dz \right)^{\frac{1}{\delta}}$$

$$+ \left( \int_Q |S(\alpha,L)((b - b_Q)f_1)(z)|^\delta \, dz \right)^{\frac{1}{\delta}}$$

$$+ \left( \int_Q |S(\alpha,L)((b - b_Q)f_2)(z) - c_Q|^\delta \, dz \right)^{\frac{1}{\delta}}$$

$$:= \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3.$$  (4.50)

To bound $\mathcal{A}_1$, we choose $r \in (1, \varepsilon/\delta)$. The Hölder’s inequality gives that

$$\mathcal{A}_1 \leq \left( \int_Q |b(z) - b_Q|^{\delta r'} \, dz \right)^{\frac{1}{r'}} \left( \int_Q |\tilde{S}_{\alpha,L}(f)(z)|^{\delta r} \, dz \right)^{\frac{1}{r}}$$

$$\lesssim \|b\|_{\text{BMO}} M_{b^r}(\tilde{S}_{\alpha,L}f)(x) \leq \|b\|_{\text{BMO}} M_\varepsilon(\tilde{S}_{\alpha,L}f)(x).$$  (4.51)

Since $\tilde{S}_{\alpha,L} : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ and $0 < \delta < 1$, there holds

$$\mathcal{A}_2 \lesssim \|\tilde{S}_{\alpha,L}((b - b_Q)f_1)\|_{L^{1,\infty}(Q, \frac{dz}{|Q|})} \lesssim \int_Q |(b - b_Q)f_1| \, dz$$

$$\lesssim \|b - b_Q\|_{\text{expL}} \|f\|_{L^{logL,Q}} \lesssim \|b\|_{\text{BMO}} M_{L,logL}(f)(x).$$  (4.52)

For the last term, we take $c_Q = \tilde{S}_{\alpha,L}((b - b_Q)f_2)(z_Q)$, where $z_Q$ is the center of $B$. We have

$$\mathcal{A}_3 \leq \int_Q |\tilde{S}_{\alpha,L}((b - b_Q)f_2)(z) - c_Q| \, dz =: \int_Q J_Q(z) \, dz \leq \left( \int_Q J_Q(z)^2 \, dz \right)^{\frac{1}{2}}.$$  (4.53)
For any cube $Q \subset \mathbb{R}^n$, set $T_Q = Q \times (0, \ell(Q))$. Thus, for any $z \in Q$,
\[
J_Q(z)^2 \leq |\tilde{S}_{a,L}((b - b_Q)f_2)(z)|^2 - |\tilde{S}_{a,L}((b - b_Q)f_2)(z_Q)|^2
\]
\[
\leq \int_{T(2Q)} \Phi\left(\frac{z - y}{\alpha t}\right)|Q_{t,L}((b - b_Q)f_2)(y)|^2 \frac{dydt}{t^{n+1}}
\]
\[
+ \int_{T(2Q)} \Phi\left(\frac{z' - y}{\alpha t}\right)|Q_{t,L}((b - b_Q)f_2)(y)|^2 \frac{dydt}{t^{n+1}}
\]
\[
+ \int_{\mathbb{R}^{n+1} \setminus T(2Q)} \Phi(z - y) - \Phi(z' - y) \left|Q_{t,L}((b - b_Q)f_2)(y)\right|^2 \frac{dydt}{t^{n+1}}
\]
\[
=: J_{Q,1}(z) + J_{Q,2}(z) + J_{Q,3}(z).
\]

In order to estimate $J_{Q,1}(z)$, we note that
\[
\int_Q \Phi\left(\frac{z - y}{\alpha t}\right)dz \leq \frac{1}{|Q|} \int_{\mathbb{R}^n} \Phi\left(\frac{z - y}{\alpha t}\right)dz \lesssim (\alpha t)^n.
\]
Furthermore, the kernel estimate (4.45) gives that
\[
\left(\int_{2Q} |Q_{t,L}((b - b_Q)f_2)(y)|^2 \right)^{\frac{1}{2}}
\]
\[
\lesssim \sum_{j \geq 3} \left\{ \int_{2Q} \int_{2^{j+1}Q \setminus 2^jQ} \frac{1}{t^n} \left(\frac{t}{t + |z - y|}\right)^{n+\delta_0} |b - b_Q||y|dz \right\}^{\frac{1}{2}}
\]
\[
\lesssim \sum_{j \geq 3} \left\{ \int_{2Q} \int_{2^{j+1}Q \setminus 2^jQ} \frac{1}{t^n} \left(\frac{t}{2^j\ell(Q)}\right)^{n+\delta_0} |b - b_Q||y|dz \right\}^{\frac{1}{2}}
\]
\[
\lesssim \sum_{j \geq 3} \left(\frac{t}{\ell(Q)}\right)^{\delta_0} |Q|^{\frac{1}{2}} \int_{2^{j+1}Q} |b - b_Q||y|dz
\]
\[
\lesssim \left(\frac{t}{\ell(Q)}\right)^{\delta_0} |Q|^{\frac{1}{2}} \sum_{j \geq 0} 2^{-j\delta_0} \|b - b_Q\|_{L^2(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)}
\]
\[
\lesssim \left(\frac{t}{\ell(Q)}\right)^{\delta_0} |Q|^{\frac{1}{2}} \sum_{j \geq 0} 2^{-j\delta_0} \|b\|_{\text{BMO}} M_{L^1(\mathbb{R}^n)}(x)
\]
\[
\lesssim \left(\frac{t}{\ell(Q)}\right)^{\delta_0} |Q|^{\frac{1}{2}} \|b\|_{\text{BMO}} M_{L^1(\mathbb{R}^n)}(x).
\]

Then, gathering (4.55) and (4.56), we obtain
\[
\int_Q J_{Q,1}(z)dz \leq \int T(2Q) \left( \int_Q \Phi\left(\frac{z - y}{\alpha t}\right)dz \right) |Q_{t,L}((b - b_Q)f_2)(y)|^2 \frac{dydt}{t^{n+1}}
\]
\[
\lesssim \frac{1}{|Q|} \int_{0}^{\ell(Q)} \int_{2Q} |Q_{t,L}((b - b_Q)f_2)(y)|^2 dy \frac{dt}{t}
\]
Similarly, one has
\[
\int_Q J_{Q,2}(z)dz \lesssim \|b\|_{\text{BMO}} M_{L \log L} f(x).
\]
(4.58)

To control $J_{Q,3}$, invoking [3, eq. (35)], we have
\[
J_{Q,3}(z) \lesssim \sum_{j \geq 0} 2^{-j\delta_0} \left( \int_{2^j Q} |b - b_Q||f|dz \right)^2
\]
\[
\lesssim \sum_{j \geq 0} 2^{-j\delta_0} \|b - b_Q\|_{\exp L, 2^j Q}^2 \|f\|_{L, 2^j Q}^2
\]
\[
\lesssim \sum_{j \geq 0} 2^{-j\delta_0} \|b\|_{\text{BMO}} M_{L \log L} f(x) \lesssim \|b\|_{\text{BMO}} M_{L \log L} f(x).
\]
(4.59)

Combining (4.53), (4.54), (4.57), (4.58) and (4.59), we conclude that
\[
\mathcal{B} \lesssim \|b\|_{\text{BMO}} M_{L \log L} f(x).
\]
(4.60)
Therefore, (4.49) immediately follows from (4.50), (4.51), (4.52) and (4.60).

**Lemma 4.9.** For any $w \in A_\infty$ and $b \in \text{BMO}$,
\[
\sup_{t > 0} \Phi(1/t)^{-1}w(\{x \in \mathbb{R}^n : |C_b(S_{a,L})f(x)| > t\})
\]
\[
\lesssim \sup_{t > 0} \Phi(1/t)^{-1}w(\{x \in \mathbb{R}^n : M_{L \log L} f(x) > t\}),
\]
(4.61)
for all $f \in L^\infty_c(\mathbb{R}^n)$.

**Proof.** Recall that the weak type Fefferman-Stein inequality:
\[
\sup_{\lambda > 0} \varphi(\lambda) \omega(\{x \in \mathbb{R}^n : M_\delta f(x) > \lambda\}) \leq \sup_{\lambda > 0} \varphi(\lambda) \omega(\{x \in \mathbb{R}^n : M_\delta^2 f(x) > \lambda\})
\]
(4.62)
for all function $f$ for which the left-hand side is finite, where $\varphi : (0, \infty) \to (0, \infty)$ is doubling. We may assume that the right-hand side of (4.61) is finite since otherwise there is nothing to be proved. Now by the Lebesgue differentiation theorem we have
\[
\mathcal{B} := \sup_{t > 0} \Phi(1/t)^{-1}w(\{x \in \mathbb{R}^n : |C_b(S_{a,L})f(x)| > t\})
\]
\[
= \sup_{t > 0} \Phi(1/t)^{-1}w(\{x \in \mathbb{R}^n : |C_b(\tilde{S}_{a,L})f(x)| > t\})
\]
\[
\leq \sup_{t > 0} \Phi(1/t)^{-1}w(\{x \in \mathbb{R}^n : M_\delta(C_b(\tilde{S}_{a,L}))f(x)| > t\}).
\]
Then Lemma 4.7, Lemma 4.8 and (4.62) give that
\[
\mathcal{B} \lesssim \sup_{t > 0} \Phi(1/t)^{-1}w(\{x \in \mathbb{R}^n : M_\delta^2(C_b(\tilde{S}_{a,L}))f(x) > t\})
\]
\[
\lesssim \sup_{t > 0} \Phi(1/t)^{-1}w(\{x \in \mathbb{R}^n : M_{L \log L} f(x) + M_\epsilon(\tilde{S}_{a,L})f(x) > c_0 t\})
\]
\[ \lesssim \sup_{t > 0} \Phi(1/t)^{-1} w(\{ x \in \mathbb{R}^n : M_{L \log L} f(x) > t \}) + \sup_{t > 0} \Phi(1/t)^{-1} w(\{ x \in \mathbb{R}^n : M_e(S_{a,L}) f(x) > t \}) \]
\[ \lesssim \sup_{t > 0} \Phi(1/t)^{-1} w(\{ x \in \mathbb{R}^n : M_{L \log L} f(x) > t \}) + \sup_{t > 0} \Phi(1/t)^{-1} w(\{ x \in \mathbb{R}^n : M^2_e(S_{a,L}) f(x) > t \}) \]
\[ \lesssim \sup_{t > 0} \Phi(1/t)^{-1} w(\{ x \in \mathbb{R}^n : M_{L \log L} f(x) > t \}) + \sup_{t > 0} \Phi(1/t)^{-1} w(\{ x \in \mathbb{R}^n : M(f)(x) > t \}) \]
\[ \lesssim \sup_{t > 0} \Phi(1/t)^{-1} w(\{ x \in \mathbb{R}^n : M_{L \log L} f(x) > t \}). \]

The proof is complete. \qed

**Proof of Theorem 1.7.** Let \( w \in A_1 \). By homogeneity, it is enough to prove
\[ w(\{ x \in \mathbb{R}^n : C_b(S_{a,L}) f(x) > 1 \}) \lesssim \int_{\mathbb{R}^n} \Phi(\frac{|f(x)|}{t}) w(x) dx. \quad (4.63) \]
Let us recall a result from [7, Lemma 2.11] for \( m = 1 \). For any \( w \in A_1 \),
\[ w(\{ x \in \mathbb{R}^n : M_{L \log L} f(x) > t \}) \lesssim \int_{\mathbb{R}^n} \Phi(\frac{|f(x)|}{t}) w(x) dx, \quad \forall t > 0. \quad (4.64) \]
Since \( \Phi \) is submultiplicative, Lemma 4.9 and (4.64) imply that
\[ w(\{ x \in \mathbb{R}^n : C_b(S_{a,L}) f(x) > 1 \}) \]
\[ \lesssim \sup_{t > 0} \Phi(1/t)^{-1} w(\{ x \in \mathbb{R}^n : |C_b(S_{a,L}) f(x)| > t \}) \]
\[ \lesssim \sup_{t > 0} \Phi(1/t)^{-1} w(\{ x \in \mathbb{R}^n : M_{L \log L} f(x) > t \}) \]
\[ \lesssim \sup_{t > 0} \Phi(1/t)^{-1} \int_{\mathbb{R}^n} \Phi(\frac{|f(x)|}{t}) w(x) dx \]
\[ \leq \sup_{t > 0} \Phi(1/t)^{-1} \int_{\mathbb{R}^n} \Phi(|f(x)|) \Phi(1/t) w(x) dx \]
\[ \leq \int_{\mathbb{R}^n} \Phi(|f(x)|) w(x) dx. \]
This shows (4.63) and hence Theorem 1.7. \qed

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