In our paper [1], a minor misprint occurs in equations (3) and (4). In the element in the fourth row, second column of the determinants in these equations, $k_T$ inside the parenthesis should be replaced by $k$, and in the element in the fourth row, fourth column, $k_c T$ inside the parenthesis should be replaced by $k_0 T$. The correct expressions for the determinants are:

$$U_{n}^{\text{TM}} = \begin{vmatrix} J_0(k^2 a) & J_0(k_c^2 a) & Y_0(k_c^2 a) & 0 \\ \frac{k}{\eta k^2} J'_0(k^2 a) & \frac{k_c}{\eta_c k_c^2} J'_0(k_c^2 a) & \frac{k_c}{\eta_c k_c^2} Y'_0(k_c^2 a) & 0 \\ 0 & J_0(k_c^2 a_c) & Y_0(k_c^2 a_c) & J_0(k_0^2 a_c) \\ 0 & \frac{k_c}{\eta_c k_c} J'_0(k_c^2 a_c) & \frac{k_c}{\eta_c k_c} Y'_0(k_c^2 a_c) & \frac{k_0}{\eta_0 k_0} J'_0(k_0^2 a_c) \end{vmatrix}$$

(3)

$$V_{n}^{\text{TM}} = \begin{vmatrix} J_0(k^2 a) & J_0(k_c^2 a) & Y_0(k_c^2 a) & 0 \\ \frac{k}{\eta k^2} J'_0(k^2 a) & \frac{k_c}{\eta_c k_c^2} J'_0(k_c^2 a) & \frac{k_c}{\eta_c k_c^2} Y'_0(k_c^2 a) & 0 \\ 0 & J_0(k_c^2 a_c) & Y_0(k_c^2 a_c) & J_0(k_0^2 a_c) \\ 0 & \frac{k_c}{\eta_c k_c} J'_0(k_c^2 a_c) & \frac{k_c}{\eta_c k_c} Y'_0(k_c^2 a_c) & \frac{k_0}{\eta_0 k_0} Y'_0(k_0^2 a_c) \end{vmatrix},$$

(4)

This misprint does not affect any result and conclusion in the paper, since all the reported numerical results were calculated using the correct expressions.

We would like to thank A Rezaei and F Mohajeri, Shiraz University, Iran, to bring these two typos to our attention.

Reference

[1] Alù A., Kerkhoff A. and Rainwater D. 2010 New J. Phys. 12 103028
Plasmonic cloaking of cylinders: finite length, oblique illumination and cross-polarization coupling

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Abstract. Metamaterial cloaking has been proposed and studied in recent years by following several interesting approaches. One of them, the scattering-cancelation technique or plasmonic cloaking, exploits the plasmonic effects of suitably designed thin homogeneous metamaterial covers to drastically suppress the scattering of moderately sized objects within specific frequency ranges of interest. In addition to its inherent simplicity, this technique also holds the promise of an isotropic response and weak polarization dependence. Its theory has been applied extensively to symmetrical geometries and canonical three-dimensional (3D) shapes, but the application of it to elongated objects has not been explored with the same level of detail. We derive here closed-form theoretical formulae for infinitely long cylinders under arbitrary wave incidence, and validate their performance with full-wave numerical simulations, also considering the effects of finite lengths and truncation effects in cylindrical objects. In particular, we find that a single isotropic (idealized) cloaking layer may successfully suppress the dominant scattering coefficients of moderately thin elongated objects, even for finite lengths comparable with the incident wavelength, providing weak dependence on the incidence angle. These results may pave the way for application of plasmonic cloaking in a variety of practical scenarios of interest.

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1. Introduction

In recent years, the applications of metamaterials to cloaking and invisibility have been explored in several exciting papers [1]–[15]. In these contributions, several alternative realizations and techniques have been discussed, and the exotic properties of several classes of metamaterials have been shown to be possibly tailored in order to provide much reduced scattering from finite-size objects in different configurations and schemes, for a wide range of frequencies of interest. Recent reviews of the various possibilities are available to the interested reader (see e.g. [16, 17]).

One such possibility takes advantage of the anomalous scattering response of thin plasmonic layers. As shown in [15], artificial plasmonic materials with low or negative effective permittivity may provide scattering cancelation via their local negative polarizability. This technique, called plasmonic cloaking, is consistent with earlier works that have speculated how a composite particle combining positive and negative permittivity may provide identically zero scattering in the static limit [18]. A recent study of the dynamic case has shown that plasmonic cloaks may suppress not only the dominant dipolar scattering from moderately sized objects, but also higher-order multipolar orders for larger scatterers [19]. In this vein, it is worth stressing that by 'cloaking' we mean strong, or maximized, scattering reduction over a finite frequency band—not necessarily complete invisibility, since residual scattering orders may always make the scatterer detectable to a certain extent. Still, significant reduction of visibility is achievable with proper plasmonic cloak design, as shown in several recent papers [20]–[28] and as discussed in the following.
Plasmonic cloaking has been shown to offer several intriguing properties in a variety of setups. Examples include intrinsic robustness to frequency and design variations [20]–[22], straightforward extension to arbitrary collections of objects [23] and multi-frequency operation [24, 25]. Moreover, the admission of fields inside the cloaked region, peculiar to this technique, may be used to suppress the inherent scattering from receiving antennas and sensing devices, which may open several interesting venues in non-invasive probing and sensing applications [26]. Extensions to ultrathin surface cloaks have also been put forward in the same context [27]. All these studies were conducted, for simplicity, for a canonical spherical object, illuminated by a generic polarization of the impinging field. Recent studies [28] have shown that analogous concepts may be applied to more complex three-dimensional (3D) geometries, based on the overall robustness of the integral cancelation effect.

However, in several practical applications, in particular for the radar community, elongated objects may become of specific interest for these applications. Alù and Engheta [15] provide quasi-static formulae for infinite circular dielectric cylinders under normal incidence, which was later extended to two-dimensional (2D) infinite conducting cylinders, also illuminated at normal incidence in [29]. These results were also preliminarily extended to oblique incidence in [30]. Moreover, in [31], the results of [24] for multi-frequency operation of spherical cloaks were extended to dielectric infinite cylinders. Also, recent theoretical and experimental efforts [32, 33] reported the practical realization of plasmonic cloaking in 2D cylindrical geometries, proposing metamaterial designs for specific polarization of interest.

The literature on plasmonic cloaking applied to cylinders, however, has often dealt with idealized 2D geometries: infinite cylinders, and incident waves normal to the cylinder axis, with specific polarization properties. This assumption, common to several other cloaking techniques applied to cylinders, makes the resulting calculations quite limited from a practical standpoint. It may be argued, in particular, that once the angle of incidence is modified, and the end effects of finite-length cylinders are considered, such cloaking effects may be severely limited, if not completely lost. In [34, 35], the effects of truncation were preliminarily considered for normal incidence.

Here we analyze all these issues in great detail, first deriving a general cloaking theory for arbitrary illumination of an infinite cylinder. We show that it is indeed possible to find a suitable, robust plasmonic cloaking layer in this scenario, which under suitable conditions may operate over a broad range of incidence angles. We corroborate these findings with an extensive numerical analysis of finite-length and truncation effects in practical cylindrical geometries, studying the overall scattering reduction for different angles of incidence and several design parameters. The present analysis applies to idealized metamaterial cloaks with isotropic properties. We leave for future work the practical limitations introduced by the specific realization of engineered metamaterial cloaks.

2. Analytical results for infinite cylinders

2.1. General derivation

Consider the geometry of figure 1, depicting a circular cylinder of finite length \( L \), radius \( a \), permittivity \( \epsilon \) and permeability \( \mu \), covered by a thin conformal cylindrical cloak shell of thickness \((a_c - a)\), permittivity \( \epsilon_c \) and permeability \( \mu_c \). In this section, we examine the limiting case of an infinite circular cylinder \((L \to \infty)\) illuminated by an arbitrary plane wave at oblique
Figure 1. Schematic diagram of the cloaked cylindrical object problem and its geometry. A dielectric core of length $L$ and diameter $2a$ is covered uniformly, except the ends, by a metamaterial shell of radial thickness $(a_c - a)$. The object is illuminated by an arbitrarily polarized plane wave, incident at an angle $\alpha$ with respect to the cylinder axis.

angle. The general scattering problem may be solved by expanding the impinging and scattered fields in cylindrical harmonics (see e.g. [36]–[38]), which we apply to derive the scattering response as a function of incidence angle $\alpha$.

Without loss of generality, the incident wave may be decomposed into its transverse-magnetic (TM$_z$) and transverse-electric (TE$_z$) components, with respect to the cylinder ($z$) axis. By matching boundary conditions at the two radial interfaces, the problem may be solved exactly for each cylindrical harmonic [36]–[38]. Analytical results for normal incidence, specifically applied to the cloaking problem at hand, were derived in [32]. In that special case ($\alpha = 90^\circ$), only two tangential field components are non-zero at each interface: $E_z$, $H_\phi$ for TM polarization and $E_\phi$, $H_z$ for TE. This ensures that the scattering problem is easily solved as a set of four equations, and the TE and TM Mie scattering coefficients are easily expressed as a function of $4 \times 4$ determinants, $U_{n\text{TM,TE}}$ and $V_{n\text{TM,TE}}$ [32]:

$$c_n = \frac{-U_n}{U_n + iV_n}$$

(1)

for either TM or TE polarization. We have assumed here and in the following an $e^{-i\omega t}$ time dependence. The overall scattering efficiency, defined as the ratio of the total scattering cross-section normalized to the cylinder’s physical cross-section, is given by [37]

$$C_s = \frac{2}{k_0a_c} \sum_{n=-\infty}^{\infty} \left(|c_{n\text{TM}}|^2 + |c_{n\text{TE}}|^2\right).$$

(2)

As discussed in [32], by tuning the cloak’s electrical and geometrical properties, the cylinder’s visibility can be minimized by canceling the dominant scattering orders, which is possible under the condition $U_n = 0$. In the quasi-static limit, i.e. for very thin cylinders, approximate closed-form expressions for these cloaking conditions for TE and TM incidence were reported in [29, 39].
Scattering for oblique incidence is a more challenging problem, since polarization coupling is usually involved [38], as discussed in the appendix. This implies that for oblique incidence purely TE or TM cylindrical waves may excite also TM and TE scattered waves, respectively. This feature will play an important role when analyzing the response of the cloak for different incidence angles. In the special circumstances for which this coupling may be absent, or minimized, e.g., as discussed in the appendix, for azimuthally symmetric modes \((n = 0)\), for conducting or high-contrast cylinders or for small cross-sections, the scattering coefficients may be still expressed as in equation (1), where \(U_n\) and \(V_n\) have generalized expressions:

\[
U_n^{TM} = \begin{bmatrix}
J_n(k^T a) & J_n(k^T a) & Y_n(k^T a) & 0 \\
\frac{k}{n^T a} J'_n(k^T a) & \frac{k}{n^T a} J'_n(k^T a) & \frac{k}{n^T a} Y'_n(k^T a) & 0 \\
0 & \frac{k}{n^T a} J'_n(k^T a) & \frac{k}{n^T a} J'_n(k^T a) & J_n(k^T a) \\
0 & 0 & \frac{k}{n^T a} J'_n(k^T a) & \frac{k}{n^T a} J'_n(k^T a)
\end{bmatrix}
\]

and

\[
V_n^{TM} = \begin{bmatrix}
J_n(k^T a) & J_n(k^T a) & Y_n(k^T a) & 0 \\
\frac{k}{n^T a} J'_n(k^T a) & \frac{k}{n^T a} J'_n(k^T a) & \frac{k}{n^T a} Y'_n(k^T a) & 0 \\
0 & J_n(k^T a_c) & Y_n(k^T a_c) & Y_n(k^T a_c) \\
0 & 0 & \frac{k}{n^T a} J'_n(k^T a_c) & \frac{k}{n^T a} J'_n(k^T a_c)
\end{bmatrix}
\]

where \(J_n\) and \(Y_n\) are the cylindrical Bessel functions of the first and second kind of order \(n\), \(J'\) and \(Y'\) are their derivatives with respect to the argument, and \(\eta_i = \sqrt{\mu_i/\epsilon_i}\) are the characteristic impedances of each region. The TE expressions may be found by electromagnetic duality: \(\epsilon_i \leftrightarrow \mu_i\).

These expressions coincide with those in [32] for normal incidence \((\alpha = 90°)\), as expected. The dependence on the angle of incidence \(\alpha\) is encoded in the transverse wavenumbers, \(k^T_i\). Specifically, \(k^T_i = \sqrt{k_i^2 - \beta^2}\), where \(\beta = k_0 \cos \alpha\) is the wavenumber component along the cylinder axis. In particular, \(k^T_i = k_0 \sin \alpha\), as expected. In the general case, where polarization coupling will occur, equation (1) should be modified to include the TM–TE coupling coefficients, consistent with the derivation in the appendix and the results reported in [36].

### 2.2. Quasi-static analysis

As in the spherical scattering scenario [15] and cylindrical scenario at normal incidence [32], it is instructive to first consider the case of electrically small cylinders, i.e. \(k_i a_c \ll 1\). In this limit, as we discuss in the following, the analysis is made simpler by the fact that: (i) the dependence of equations (3) and (4) on the incidence angle is negligible; consequently, so is the cross-polarization effect on the cloaking conditions; (ii) the scattered wave is dominated by a limited number of multipolar orders. The combination of these features makes the plasmonic cloaking easier to achieve, more effective and robust, compared with larger cylinders. This is as expected, since satisfying the condition \(U_n = 0\) for only a few dominant terms in equation (2) is easier to achieve, and such conditions are less dependent on the angle of incidence in this regime. In particular, it is easy to show that for regular dielectric cylinders \((\epsilon > \epsilon_0, \mu = \mu_0)\), the dominant terms in equation (2) are \(c_0^{TM}\) and \(c_0^{TE}\), respectively, for TM and TE excitations. For arbitrary polarization, \(c_0^{TM}\) scattering dominates. This also applies to a conducting cylinder, whose limiting case corresponds to \(\epsilon \rightarrow -j\infty\) and \(\mu \rightarrow 0\) in equations (3) and (4). For a magnetic cylinder \((\mu > \mu_0, \epsilon = \epsilon_0)\), dual considerations apply, so \(c_0^{TE}\) dominates.
2.2.1. Quasi-static cloaking conditions. As discussed in the appendix, for electrically small cylinders \((k_i a_c \ll 1)\), the cross-coupling terms vanish for any angular order \(n\). We may thus derive the quasi-static cloaking condition by simply taking the first-order Taylor expansion of the coefficients of equation (1). This leads to simple closed-form solutions for the appropriate ratio of cloak to core radii to achieve cloaking in the quasi-static limit for each scattering coefficient, consistent with analogous formulae available in the spherical [15] and cylindrical normal-incidence [32] scenarios:

\[
\begin{align*}
c_{TE}^0 &= \frac{a_c}{a} \sqrt{\frac{\mu_c - \mu}{\mu_c - \mu_0}}, \\
c_{n \neq 0}^{TE} &= \frac{a_c}{a} \sqrt{\frac{(\epsilon_c - \epsilon)(\epsilon_c + \epsilon_0)}{(\epsilon_c - \epsilon_0)(\epsilon_c + \epsilon)}}, \\
c_{TM}^0 &= \frac{a_c}{a} \sqrt{\frac{\epsilon_c - \epsilon}{\epsilon_c - \epsilon_0}}, \\
c_{n \neq 0}^{TM} &= \frac{a_c}{a} \sqrt{\frac{(\mu_c - \mu)(\mu_c + \mu_0)}{(\mu_c - \mu_0)(\mu_c + \mu)}}.
\end{align*}
\] (5)

For perfectly electric conducting (PEC) cylinders, a case of particular interest for cloaking applications at radio frequencies, we find that

\[
\begin{align*}
c_{TE}^0 &= \frac{a_c}{a} \sqrt{\frac{\mu_c}{\mu_c - \mu_0}}, \\
c_{n \neq 0}^{TE} &= \frac{a_c}{a} \sqrt{\frac{\epsilon_c + \epsilon_0}{\epsilon_c - \epsilon}}, \\
c_{n \neq 0}^{TM} &= \frac{a_c}{a} \sqrt{\frac{\mu_c + \mu_0}{\mu_c - \mu_0}}.
\end{align*}
\] (6)

We note that Irci and Ertürk [29] misreported that there is no possible quasi-static condition for \(c_{0}^{TE}\). The correct formula, shown above, results from taking the proper limits in our previous analysis for \(\epsilon \to -j\infty\) and \(\mu \to 0\) in \(U_n^{TE}\). This is particularly relevant for oblique incidence, since enforcing only \(\epsilon \to -j\infty\) to model a PEC boundary, as suggested in [29], would not ensure zero normal component of the magnetic field on the boundary, as required in the PEC limit [40]. As correctly reported in [29], there is no quasi-static condition for the \(c_{0}^{TM}\) coefficient in the PEC scenario.

A few important points should be highlighted regarding equations (5) and (6). First of all, equation (5) for TE polarization formally coincides with the derivation of [15]. The formulae here have been obtained for completely generic oblique incidence, generalizing the previously published result. They show that in the long wavelength limit, cloaking conditions are unaffected by an arbitrary variation of the angle of incidence. In this regime, the TM–TE polarization cross-terms are of second-order (cf the appendix); thus equation (5) ensures that an electrically small magnetodielectric cylinder may always be cloaked using its dominant scattering order in the Taylor expansion, regardless of the angle of incidence. As in the spherical scenario, these quasi-static formulae split the role of permittivity and permeability between TM and TE polarizations. Special attention should be paid to the azimuthally symmetric modes, for
which the role of permittivity or permeability is reversed compared with higher-order modes. Moreover, again as in the spherical case, the cloaking conditions depend only on the ratio $a_c/a$, implying that a thin homogeneous shell may be employed to suppress the relevant scattering orders in this quasi-static scenario. It should be stressed that equations (5) and (6) may not be met by arbitrary values of the constituent parameters. Rather, they should lie in specific ranges of permittivity, due to the simple physical constraint on geometry $a_c/a > 1$.

From a practical standpoint, an electrically thin dielectric cylinder, for which the $c_{0\text{TM}}$ coefficient is dominant, may always be cloaked by a thin shell with $\epsilon_c < \epsilon_0$. Under the first of the conditions in equation (5), most of the scattering may be suppressed, although significantly negative $\epsilon_c$ may be required for achieving a very thin shell. By duality, an electrically thin magnetic cylinder will require $\mu_c < \mu_0$.

For such infinite cylinders, the cloaking conditions may in general become trickier than in the spherical case, since the azimuthally symmetric cylindrical waves follow special cloaking conditions: they are permittivity based for zeroth order when higher orders are permeability based, and vice versa. This is particularly relevant for the conducting scenario, for which the dominant $c_{0\text{TM}}$ coefficient may not be canceled at all in the long wavelength limit (cf equation (6)). It should be emphasized that it is still possible to suppress scattering in the fully dynamic scenario, since it may always be canceled with its dynamic expression $U_{0\text{TM}}$. However, in the long wavelength limit, the required wavenumber in the cloak considerably grows, implying that quasi-static considerations such as those used in equation (6) may not be applied. However, it is evident that conducting objects are significantly more challenging to cloak for cylinders than spheres [20]. The physical reason for this difference is evidently related to the fact that electrically thin cylinders are not electrically small objects. These systems are still infinite in the axial direction (for this analytical treatment), which implies that their scattering may not necessarily be small—above all when conduction currents may be induced in the $z$-direction, as with the TM$_0$ cylindrical wave.

2.2.2. Validation of the quasi-static conditions: electrically small dielectric cylinder. To assess the validity of the previous approximate cloaking conditions in the fully dynamic scattering problem, we present some numerical examples of interest in this section, solved using the exact analytical theory reported above. Consider first the case of a dielectric cylinder with $\epsilon = 3\epsilon_0$ and normalized diameter $k_0a_c = 0.1$, covered by a thin uniform cloaking shell with $a_c/a = 1.1$. Figure 2(a) shows the variation of the total scattering efficiency $C_s$ as a function of the cloak permittivity and of the angle of incidence of a TM illuminating plane wave. It is immediately apparent that the use of negative-permittivity thin cloaks may significantly reduce the overall scattering efficiency of the cylinder, and in particular, minimum scattering is obtained very close to the corresponding quasi-static cloaking condition for $c_{0\text{TM}}$ in equation (5), which would yield $\epsilon_c = -8.524\epsilon_0$. Moreover, minimum scattering is achieved at the same value of negative permittivity, independent of the angle of incidence, consistent with the previous theoretical results, despite some polarization coupling for oblique angles. Overall scattering reduction, compared with no cloak ($\epsilon_c = \epsilon_0$), is especially significant in the normal incidence case, which provides over 50 dB reduction. For oblique incidence, the coupling with TE coefficients affects the cloaking performance and generates residual scattering at the cloaking condition, but it is seen that the same (negative) value of cloak permittivity may provide significant scattering reduction (over 10 dB) for a wide angular range without drastic changes.

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Figure 2. Total scattering efficiency for electrically thin infinite-length circular cylinders with $k_0a_c = 0.1$ and $a_c = 1.1a$: (a) dielectric ($\epsilon = 3\epsilon_0$, $\mu = \mu_0$) and (b) magnetic ($\epsilon = \epsilon_0$, $\mu = 3\mu_0$); for three different angles of incidence of TM$_z$-polarized radiation. In the electrically thin limit, the cloaking conditions have weak dependence on the angle of incidence.

For smaller incidence angles, we also observe excitation of a plasmonic resonance for slightly negative values of $\epsilon_c$. This is associated with the condition $V^{\text{TE}}_1 = 0$, obtained here in the quasi-static limit for $\epsilon_c = -0.07\epsilon_0$, consistent with findings on quasi-static plasmonic resonances in layered spheres and cylinders [39]. It is evident that the resonance is not excited at normal incidence, but arises as soon as oblique incidence is considered, due to cross-polarization coupling. This explains the sharp peak for small negative values of $\epsilon_c$ in figure 2(a).

Analogously, for low positive values of $\epsilon_c$, a second small dip in the scattering efficiency appears at $\alpha = \pi/4$, associated with the cloaking condition for $c_1^{\text{TE}} = 0$ in equation (5). Here, this provides the solution $\epsilon_c = 0.14\epsilon_0$. Once again, this dip does not appear at normal incidence, due to a lack of coupling between polarizations for $\alpha = \pi/2$. To conclude this discussion, it should be underlined that, in absolute value, scattering is larger for normal incidence in the uncloaked scenario ($\epsilon_c = \epsilon_0$ in the figure), as expected, due to the larger $E_z$ component. On the other hand, the proper cloak design may maximally suppress scattering in this worst-case condition, which is an inherent property of this cloaking approach.

2.2.3. Validation of the quasi-static conditions: the electrically small magnetic cylinder. As a second example, consider the same cylindrical geometry, but with magnetic properties ($\mu = 3\mu_0$), again excited by a TM$_z$ wave. All other parameters are kept the same. The cylinder’s scattering efficiency is shown in figure 2(b). As anticipated above, this configuration gives much weaker scattering in the uncloaked scenario ($\mu_c = \mu_0$), due to its small size and a lack of dielectric contrast with the background. In fact, the $c_1^{\text{TM}}$ coefficient is negligible in this case. Residual scattering is dominated by the higher-order $c_1^{\text{TM}}$ coefficient, which may be canceled with $\mu_c = -21.2\mu_0$ or $0.14\mu_0$ (note the dual behavior compared with $c_1^{\text{TE}}$ in figure 2(a)). Two clear scattering dips are visible around these two values, which also in this scenario do not depend on the angle of incidence, consistent with equation (5). The absence of scattering from higher-order modes and of coupling for the $n = 0$ modes ensures that the overall cloak performance remain unchanged by variations of the incidence angle $\alpha$, as evidenced by the different curves in figure 2(b).
2.2.4. Quasi-static conditions: conclusions. It should be stressed that the above numerical analyses neglect ohmic absorption losses in the cloak and/or core materials. We include these effects in the following sections. Nevertheless, the plasmonic cloaking technique has been shown to be inherently robust against moderate absorption [20], since it is not based on a resonant effect. This implies that the curves in figure 2 would remain practically unchanged near the cloaking regions when moderate losses are considered. On the other hand, the large scattering peaks associated with plasmonic resonances would be significantly dampened by realistic losses. This and the previous considerations imply that

1. The plasmonic cloaking technique may be successfully applied to infinite cylinders.
2. The quasi-static conditions, equations (5) and (6), hold to a very good approximation for thin cylinders, $k_0a_c \lesssim 0.1$, and ensure dramatic scattering reduction in this limit.
3. Cloaking conditions for electrically thin cylinders are very weakly dependent on $\alpha$.

Thus, cloak designs for electrically thin cylinders are robust against the angle of incidence for any scattering order $n$ and polarization. We explore numerically in the following sections how these properties are affected by considering thicker geometries and truncation effects.

3. Plasmonic cloaking design and optimization

For a given infinite cylinder of radius $a$ and permittivity $\epsilon$, a single-layer plasmonic cloaking shell may be optimized by varying its two design parameters $a_c$ (or $a_c/a$) and $\epsilon_c$, using the analytical results derived in section 2. In general, it is preferable to choose thin shell thicknesses, since drastically increased cross-sections imply reduced bandwidths and larger sensitivities to the design parameters [27].

This simple cloaking design inherently limits the overall total thickness of cylinders that we may cloak, since only a few scattering orders may be drastically suppressed independently. Magnetic properties of the cloak [19] as well as multi-layer designs [24] may be considered for increasing the available degrees of freedom and size of objects to be cloaked. In this work, however, for the sake of simplicity, we limit our interest to non-magnetic cloaks, and we consider moderate cross-sections of the objects to be cloaked. In particular, in the following we consider cylinders with diameters $2a = (\lambda_0/2, \lambda_0/4, \lambda_0/8)$ and relative permittivities of 3 and 10 in the dielectric scenario, in addition to a PEC cylinder. Since the previous theoretical discussion shows that infinite cylinders may be effectively cloaked, as anticipated, we consider overall lengths $L$ of several times the diameter, which may be comparable with or larger than the wavelength of operation.

Because of the TM–TE polarization coupling described in the previous section (cf also the appendix), completely general cloak optimization would be quite complicated and in general depend on the incidence angle and polarization of excitation. We choose instead to optimize cloak response at normal incidence—which is usually the angle for which larger scattering is produced—and then examine the response at oblique angles with numerical simulations. A more extensive cloak optimization would involve the choice of a cost function and the application of global (e.g. genetic algorithms [41]) and local (e.g. quasi-Newton) optimization techniques across all angles and polarizations, also as a function of the observer’s position. In general, further optimization may be achieved by considering magnetic cloaks, as discussed above. We will analyze these aspects in the near future.
3.1. Design parameter space: the view from above

To provide insight into the effects of the various parameters involved in cloak design, we examine the variation of scattering efficiency with $\frac{a_c}{a}$ and $\epsilon_c$ for TE or TM plane wave normal incidence in the infinite (2D) scenario, using the formulation developed in the previous section. To that end we write the scattering gain as

$$ Q_s\left(\frac{a_c}{a}, \epsilon_c\right) = \frac{C_s}{C_0^s}. $$

$C_s$ is the scattering efficiency from equation (2), and the superscript 0 represents the uncloaked case, calculated with equations (1)–(4), in the limit $a_c \to a$ and $\epsilon_c \to \epsilon$.

In general, the above discussions show that it is not possible to achieve large scattering reduction for both TM and TE illumination simultaneously with a single-layer permittivity cloak at the same frequency. For moderate cross-sections, however, such as those considered here, scattering is generally dominated by one of the two responses. It is thus best to design for maximal suppression of that polarization. Hence, we focus on TM waves, which dominate scattering from infinite dielectric and conducting cylinders in the long wavelength limit (cf quasi-static analysis in the previous section). Ideally, the cloak is designed to provide minimized scattering gain for normal incidence.

For our calculations we used Mathematica 7 (http://www.wolfram.com/mathematica) and truncated the Mie summations at order $n = 5$, which ensured convergence for the cross-sections considered here. Figure 3 shows scattering gain contour plots for infinite cylinders of diameter $2a = \lambda_0/4$, relative permittivities $\epsilon = 3$ (figure 3(a)) and $\epsilon = 10$ (figure 3(b)), and TM normal incidence. The plots show alternating loci of resonant peaks (large $Q_s$, light color) and cloaking regions (near-zero $Q_s$, dark color). As expected, thinner cloaks (smaller $a_c/a$) yield larger bandwidths, reflected by wider cloaking regions. This implies that the ideal cloak design would utilize $\epsilon_c < 0$, as expected from the results in section 2, from simple physical considerations and analogous results in the spherical scenario [15]. The cloaking loci for large positive values of $\epsilon_c$ are associated with anti-resonances that arise when the shell thickness is comparable with the wavelength in the shell material. These anti-resonances are narrow-bandwidth and highly sensitive to the design parameters, and they necessarily lie in close proximity to scattering resonant peaks, which makes them unsuitable for a robust cloaking design. Figure 4 shows constant-$a_c/a$ slices of figure 3 to better illustrate these characteristics. These slices illustrate how $\epsilon_c < 0$ choices typically produce lower scattering gain, i.e. better cloaking, away from dangerous resonant enhancements. This is especially true when cloaking low-density dielectric cylinders.

We also considered optimization of plasmonic cloaks for PEC cylinders of the same size. However, as noted in the quasi-static limit, here the dominant scattering contribution (via the $c_0^{TM}$ coefficient) may not be canceled in the long wavelength regime. This implies that in the corresponding contour plots there would not be cloaking regions for thin negative-permittivity shells, as in the dielectric scenario. Moderate scattering reduction may be achieved with large-$\epsilon_c$ thick shells, but as outlined above this implies large sensitivity to frequency, design variations and closely spaced resonant peaks, which may be excited at different incidence angles. It is thus evident that a simple permittivity cloak may not be sufficient to adequately cloak a PEC cylinder and, as in the spherical scenario [15], it may require a magnetic permeability $\mu_c$ different from the background for robust scattering reduction. In the following, therefore, we mainly focus our design efforts on dielectric cylinders.
3.2. Cloak design

The previous results show that optimal cloak configurations for dielectric cylinders are based on negative permittivity metamaterials and thin cloak shells. We should point out that negative values of effective permittivity may indeed be achieved in the microwave or THz frequency ranges using various metamaterial geometries, such as wire media or parallel-plate implants \[32\], and they are naturally available at larger frequencies. In particular, the parallel-plate implant technology may be particularly well suited for cloaking incident TMz waves, as has been already demonstrated theoretically and verified experimentally for normal incidence at microwave frequencies \[32, 33\]. We assume here, for the sake of simplicity, that the required value of effective permittivity is available for the shell geometry of interest and that the cloak material is isotropic. Possible anisotropy for a specific metamaterial realization, inherent in some proposed realizations \[32, 33\], may affect cloak performance for different
Table 1. Parameters from the cloak optimization procedure described in section 3.2 for different objects and cloak thicknesses. $Q_s$ corresponds to the scattering gain for normal-incidence TM$_z$ illumination.

| Core diameter ($2a$) | Core $\epsilon_0$ | Cloak $\epsilon_c$ | Cloak $\epsilon_{\lambda}$ | $Q_s$ |
|----------------------|-------------------|--------------------|------------------------|-------|
| 3                    | 1.10              | -8.16              | 0.26                   |
| 3                    | 1.40              | 22.45              | 0.13                   |
| 10                   | 1.05              | 13.37              | 0.22                   |
| 10                   | 1.10              | 6.91               | 0.22                   |
| PEC                  | 1.10              | 95.48              | 0.47                   |
| 3                    | 1.05              | -27.88             | 0.031                  |
| 3                    | 1.10              | -13.55             | 0.038                  |
| 10                   | 1.10              | -35.00             | 0.36                   |
| 10                   | 1.20              | 74.57              | 0.16                   |
| PEC                  | 1.50              | 14.01              | 0.37                   |
| 3                    | 1.05              | -20.26             | 0.00076                |
| 3                    | 1.10              | -9.45              | 0.00092                |
| 10                   | 1.10              | -56.25             | 0.0017                 |
| 10                   | 1.30              | -17.87             | 0.0034                 |
| PEC                  | 1.40              | 88.92              | 0.096                  |

Incidence angles, but our preliminary results show that for thin cloaks such effects may be minor. Thus, for simplicity we always assume idealized isotropic metamaterials in the following. The requirement that any passive metamaterial has a frequency dispersion ensuring that $\partial(\omega\epsilon_c)/\partial\omega > 0$ is met here by assuming a Drude dispersion model of the form

$$\epsilon_c(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega - j\gamma)}.$$ 

(8)

Here and in the numerical simulations in the next section, we calculated the plasma frequency $\omega_p$ to ensure that at the design frequency $f_0$ the real part of $\epsilon_c$ yields the required value. The damping frequency $\gamma$, associated with the level of losses in the metamaterial, has been always assumed in the following to be $\gamma = 10^{-2}\omega_p$. This value provides a moderate amount of loss, comparable with practically realizable metamaterial geometries at these frequencies.

Table 1 summarizes an extensive campaign of optimizations that we performed to cloak different cylindrical geometries, as outlined above, considering several cloaking designs. Inspecting table 1, we see that thin cylinders ($2a = \lambda/8$), consistent with the previous quasi-static analysis, require optimized cloaks with negative permittivity in the case of dielectric objects, and a large positive permittivity for conducting materials. The corresponding scattering gain may become extremely low, since only a few Mie coefficients effectively contribute to the total scattering, and they are properly canceled by the optimized cloak design. As a general rule of thumb, consistent with the previous considerations, thinner cloaks require larger values of permittivity, either positive or negative. Relatively thicker cloaks relax the requirements on very negative permittivity of the cloak, but, as highlighted above, larger thickness also tends to produce additional scattering terms, which limits overall performance. However, lower absolute values of negative (in the case of dielectric) or positive (for PEC) permittivity may
be easier to achieve and less sensitive to loss. A trade-off among cloak thickness, material reliability and overall cloaking performance should be considered. Larger dielectric objects may be cloaked by thicker cloaks with positive values of permittivity, since the dynamic nature of the wave in the cloak may produce a negative polarizability even with positive materials in larger shells. However, the overall scattering reduction is less dramatic; residual scattering is indeed expected for larger objects. In particular, as discussed above, conducting cylinders are most challenging to be cloaked against TM waves, due to the axial conduction currents induced on their surface. Scattering reduction of about 3 dB, however, may still be achieved even for conducting cylinders. Moreover, large values of core permittivity $\epsilon$ are harder to cloak, and the residual scattering is larger than that for a lower-permittivity core of the same size.

Our cloak designs in table 1 were optimized for TM normal incidence, the scenario with maximum scattering for the considered objects. At oblique incidence, however, TM–TE coupling excites additional scattering modes, which may affect the overall response, in particular for thicker objects and moving toward grazing incidence. We discuss these features in detail in the next section.

Figure 5 considers the typical frequency dependence of cloak performance on wave polarization. The left panel shows the uncloaked scattering widths for an infinite dielectric cylinder with $2a = \lambda_0/4$ and $\epsilon = 3$ for TM and TE normally incident plane waves. As expected, the TM polarization is more strongly scattered; thus it is of more relevance in designing an optimized cloak. However, despite the much smaller response to TE waves, it is interesting to analyze the behavior of a TM-optimized cloak to TE polarization, since a portion of the TM scattered power may be coupled to TE waves for TM oblique incidence, in particular for smaller incidence angles. The right panel shows the normalized scattering gain for TM and TE incidence, considering a conformal shell cloak with $\alpha_c = 1.1a$ and $\epsilon_c = -13.55$ at design frequency $f_0$ with Drude dispersion, which coincides with the optimized design at $f_0$ for TM normal incidence. We see how the cloak is very effective over a relatively broad range of frequencies around $f_0$ for TM incidence, despite the natural dispersion of the cloak material. However, this same cloak does not provide scattering reduction at the same frequency for impinging TE waves, supporting instead a small cloaking dip at lower frequencies. Although this per se is not a major issue, since the absolute value of the TE scattering is much smaller (left panel) and the quantities in the right panel are normalized to the corresponding uncloaked scattering, this issue may cause an overall dependence of the cloaking response on the incidence angle for TM excitation, due to the TM–TE polarization coupling. We verify these issues in the next section, which considers the performance of finite-length cylinders illuminated at oblique incidence.

4. Numerical results for finite-length cylinders and oblique incidence

After having considered the optimization of cloaks for dielectric and conducting infinite cylinders, we consider in this section the effects of finite length and varying angle of incidence on the performance of cylindrical plasmonic cloaks, using full-wave numerical simulations CST Design Studio 2009, http://www.cst.com. To keep the design simple, in our numerical simulations we simply truncated the cloaked cylinder, as optimized in the previous section, consistent with the geometry in figure 1, leaving the ends uncovered. This is expected to cause additional scattering, especially for more oblique incidence (smaller $\alpha$). However, since this cloaking technique is based on an integral effect [20], for moderate cross-sections these effects
Figure 5. Uncloaked scattering efficiency (a) and expected scattering gain (b), $C_s/C_0$, as a function of relative frequency, for an infinite cylinder with $2a = \lambda_0/4$, $\epsilon = 3$ and for $a_c = 1.1a$ (see table 1). Optimized for TM waves at normal incidence, the cloak will perform differently for TE illumination.

are not expected to significantly deteriorate the overall cloaking performance. This intuition is confirmed by the following numerical results. Improved performance may be obtained by locally tailoring the cloak around the edges, and covering the ends, as described for conical geometries in [28]. The small improvements achievable with this fine-tuning of the cloak design are in any case not relevant to the following general discussion.

4.1. Dielectric cylinder at normal incidence

Consider first the dielectric cylinder of figure 2(a) ($\epsilon = 3\epsilon_0$, $\mu = \mu_0$, diameter $2a = 2.5$ cm), but with finite length $L = 10$ cm. Our objective is to reduce its scattering cross-section at the design frequency $f_0 = 3$ GHz, for which this object has cross-sectional width $2a = \lambda_0/4$ and length $L = \lambda_0$; $\lambda_0 = 10$ cm is the free-space wavelength at a frequency $f_0$. In this frequency range, the object may produce significant scattering, since its length and electrical size are comparable with the incident wavelength. Scattering at normal incidence is dominated by the TM$_z$ polarization (cf section 2), so we excite the cylinder with an impinging TM wave, and use a cloak with permittivity $\epsilon = -13.55\epsilon_0$ ($\omega_p = 11.44$ GHz) at $f_0$ and thickness $a_c = 1.1a$, obtained in section 3. Note the relatively large negative permittivity required by such a thin layer. For a thicker cloak, $|\epsilon_c|$ may be reduced, but at the price of possible excitation of higher-order cylindrical harmonics, which may slightly deteriorate the cloak performance.

Figure 6(a) plots the total 3D radar cross-section (RCS) [36], obtained by integrating the scattering cross-section over all angles, calculated using commercial software based on the finite-integration technique (CST Design Studio). It compares the cloaked (solid green curve) and uncloaked (red dashed curve) scenarios for normal incidence ($\alpha = \pi/2$). Despite the cylinder’s finite electrical length and its non-negligible cross-section, scattering at the design frequency may be significantly suppressed with a proper cloak design. As expected, the scattering is minimum at $f_0$, and the bandwidth of operation is moderately large, as is common for plasmonic cloaks [20]. Significant scattering reduction is achieved over a fractional bandwidth of over 30%. Figure 6(b) plots the corresponding scattering gain, defined as the
Fig. 6. (a) Frequency dispersion of the total RCS for a dielectric cylinder with $\epsilon = 3\epsilon_0$, $\mu = \mu_0$, $2a = 2.5$ cm and finite length $L = 10$ cm, comparing uncloaked versus cloaked (as optimized in section 3) with $a_c = 1.1a$ and $\text{Re}[\epsilon_c] = -13.55\epsilon_0$ using a Drude model at $f_0 = 3$ GHz for TM illumination at normal incidence ($\omega_p = 11.44$ GHz). (b) The corresponding scattering gain (ratio of cloaked to uncloaked RCS), compared with analytical results obtained for an infinite cylinder with otherwise identical geometry. The full-wave simulations were performed using CST Design Studio.

Table 2 summarizes the calculated overall suppression at $f_0$ for the dielectric-core $a_c = 1.1a$ cloak designs of table 1 illuminated by TM waves at normal incidence. In particular, the table compares the simulated scattering gain for finite lengths $C_{s,\text{sym}}$ with the theoretical value predicted analytically for infinite cylinders $C_{s,\text{th}}$. We observe excellent agreement between predicted results from the analytical formulae for the infinite cylinder, and full-wave numerical simulations for the truncated geometries. This implies that truncation effects are not significant in several realistic setups when cloaking dielectric cylinders with the plasmonic cloaking technique.

4.2. Varying angle of incidence

Fig. 7 shows the variation of scattering gain for the cylinder of figure 6 varying the incidence angle $\alpha$, as depicted in figure 1. Even for small (near grazing) incidence angles, the cloaking
Table 2. Scattering gain at a frequency $f_0$ for the cloak designs of table 1 with 10% thickness ($a_c = 1.1a$) and TM$_z$ normal incidence. We compare the theoretical results for infinite cylinders ($Q_{s,th}$) with the numerical full-wave simulations for truncated cylinders with $L = \lambda_0$ ($Q_{s,sim}$).

| Core diameter ($2a$) | $L/2a$ | $\varepsilon_\xi/\varepsilon_0$ | Cloak $\varepsilon_c$ | $Q_{s,th}^{TM}$ | $Q_{s,sim}^{TM}$ |
|----------------------|--------|-----------------|-------------------|----------------|----------------|
| $\frac{1}{2}$        | 2      | 3               | -8.16             | 0.17           | 0.27           |
| $\frac{1}{3}$        | 4      | 3               | -13.55            | 0.038          | 0.046          |
| $\frac{1}{8}$        | 4      | 10              | -35.00            | 0.17           | 0.38           |

![Figure 7](http://www.njp.org/) Scattering gain for the cylinder of figure 6, for various angles of incidence $\alpha$. (a) The results of numerical simulation. (b) Numerical simulation (solid) versus analytical infinite-cylinder results (dashed) for select angles.

effects are barely different from the normal-incidence case. For smaller values of $\alpha$, cloaking is somewhat reduced at $f_0$, due to the excitation of TE scattered waves via cross-polarization coupling, as discussed in section 2. These results demonstrate strong agreement between the analytical calculations for infinite cylinders and the numerical simulations for finite $L$, although some expected minor deviation does appear at small angles, due to end effects. We emphasize that, although the normalized scattering gain decrease is less for smaller angles, the overall RCS is significantly smaller in these cases, since the electric field component along the cylinder is comparatively shorter, consistent with the discussion in the previous section. The results of figure 7 demonstrate convincingly that plasmonic cloaking may be applied to finite-length dielectric cylinders at oblique excitation, although at small $\alpha$ one may be required to cloak the TE contribution separately for improved performance. This is consistent with our findings in figure 5.

4.3. Varying diameter

The left panel of figure 8 shows the scattering gain under normal-incidence TM$_z$ illumination for the cylinder of section 4.1 (cf figure 6) next to that for a cylinder with twice that diameter,
2a = \lambda_0/2 (cf table 1 for cloak parameters). The figure also compares these results with the infinite-cylinder analytical result of section 2. End effects are negligible except at low f, and the analytical results match very well the numerical simulations—especially at frequencies where plasmonic resonances are not excited. Cloaking is effective over a relatively broad bandwidth, even for the thicker cylinder. This implies that the designed cloaks may be effective over a relatively broad range of object sizes. The excellent agreement between the analytical curves for infinite cylinders and the simulation results for truncated geometries indicates that truncation effects are negligible for cloaking performance and suggests the use of the previously derived analytical formulae for fast cloak optimization. A simple permittivity cloak, as considered here, is effective in achieving significant scattering reduction for cylinders with diameter of the order of the wavelength. Thicker cylinders may require the use of multilayer and/or magnetic cloaks, as discussed in [19] for spherical geometries.

4.4. Varying object permittivity

The right panel of figure 8 shows the scattering gain for the cylinder of section 4.1 (cf figure 6), for the same dielectric and also with a denser core, \( \epsilon = 10\epsilon_0 \) (see table 1). In the large \( \epsilon \) limit, this comparison underscores some of the possible challenges involved in cloaking a conducting object, which would coincide with a dielectric core in the limit of very large permittivity. There is significant scattering reduction around the design frequency \( f_0 \) also in the denser scenario, but this example simultaneously supports even stronger cloaking at lower frequencies, \( f \sim 2 \) GHz. This effect, predicted by the analytical results for infinite cylinders, is associated with the frequency dispersion of the cloak, which matches the cloaking condition for the coefficient at lower frequencies (for more negative values). Effectively, this scenario presents
a coincidental cloaking effect at another frequency. This does not imply that one could tune the second suppression arbitrarily, since its position depends on the natural metamaterial dispersion. However, one could tune its separation from $f_0$ to a limited degree if the cloak thickness were not fixed at $a_c = 1.1a$.

4.5. Varying cylinder length

Figure 9 analyzes in more detail the effects of truncation for the case of figure 6 in section 4.1, by considering different truncation lengths for several incidence angles: $\alpha = \pi/2, \pi/3, \pi/6$ for the three panels, respectively. Each compares different lengths, including the analytical result for infinite cylinders. The cloaking effect is indeed robust against variations in incidence angle, and truncation effects are quite moderate, as the scattering gain follows the line calculated analytically in the infinite cylinder geometry. Even for the shortest length ($L = 5$ cm, a 2:1 aspect ratio), the full-wave simulation still follows the infinite-length case with surprising accuracy, above all near $f_0$. In this case, scattering is characterized by small resonant peaks at lower frequencies, associated with longitudinal resonances due to the finite length. The results are nevertheless extremely encouraging, in particular for shorter cylinders, for which the cloaking design works extremely well in the frequency range of interest.

4.6. Near- and far-field plots

Figures 10 and 11 illustrate the full-wave near- and far-field numerical results for the cylinder of section 4.1 (figure 6), illuminated by a TM wave impinging at an angle $\alpha = \pi/3$. Figures 10 and 11 compare the cloaked configuration at the design frequency $f_0$ (b) with the uncloaked case (a). Consistent with figure 2(a), the overall calculated RCS reduction at the central frequency is $-14.2$ dB, and the following figures will depict the effective functionality of the cloak in its near- and far-field regions. We have chosen $\alpha = \pi/3$ to indicate the performance of the cloak to oblique excitation; similar considerations apply for any other angle of incidence, which we have verified via detailed study.

Figure 10 shows the (normal) magnetic field distribution on the $E$-plane (snapshot in time), comparing the uncloaked scenario (a) with the cloaked one (b). Several interesting features are noteworthy. In the uncloaked case, the wave penetrates the dielectric rod and experiences a wavelength shortening that effectively distorts the planar wavefronts on the back of the cylinder, producing significant shadow and scattered fields all around the object. The thin plasmonic metamaterial shell is able to re-establish the proper planar fronts just outside the cloak (b), ensuring reduced scattering and suppressed visibility for an outside observer positioned anywhere in the near- or far-field of the object. It should be noted that this effect is obtained for oblique incidence, and the cylinder ends are uncloaked. Additional improvement may be achieved by proper cloaking of these terminations. One may also easily observe how the plasmonic layer supports surface-plasmon waves traveling along the shell, as expected because of its negative permittivity. It is interesting to observe how these waves effectively cancel the residual scattering and restore the phase fronts to almost exactly match those of the original plane wave, had it traveled through free space instead.

Figure 11 illustrates the far-field radiation patterns for this same cylinder, comparing on the same scale the scattering from the uncloaked (a) and cloaked (b) objects, showing drastic
Figure 9. Scattering gain for the cylinder of figure 6 and variations in length by a factor of two, for three different angles of incidence under TM excitation. The cloaking design is robust even for very short lengths, almost up to $L \sim 2a$.

suppression of the bistatic RCS at all angles. Uncloaked scattering manifests itself mainly as a shadow on the back of the cylinder; various higher-order scattering harmonics contribute to this residual scattering pattern. Most of the scattering is suppressed by the cloak. Figure 11(c) shows the cloaked residual scattering on a much smaller scale: as expected, scattering is not identically zero, and small lobes, associated with higher-order (and more directive) cylindrical harmonics not completely suppressed, are still present. Their relevance, however, is very limited compared with the original scattering levels.

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Figure 10. Contour plots (snapshot in time) of the magnetic field on the $E$-plane of polarization for a TM$_z$ wave at $\alpha = \pi/3$ (the origin is in the top right corner) for the object of figure 6: (a) uncloaked; (b) the $a_c = 1.1a$ cloak. Severe uncloaked wavefront distortions are almost totally restored by the thin cloak. Cloak interface nodes (right) are associated with plasmonic surface waves.

Figure 11. (a, b) Far-field scattering patterns in (a) the uncloaked and (b) the cloaked ($a_c = 1.1a$) cylinders of figure 6, plotted on the same scale. (c) An enlargement of (b) for showing the residual detail of the scattering pattern, dominated by higher-order scattering modes. In (a, b), a dramatic scattering reduction from a properly tuned plasmonic cloak is demonstrated.
5. Conclusions

We have presented an extensive investigation of the application of the plasmonic cloaking technique to circular cylinders illuminated by plane waves of arbitrary polarization and angle of incidence. We have derived analytical formulae for the general oblique-angle scenario and showed that in the electrically thin limit there is no angular dependence on the cloaking response, i.e. the design formulae are independent of the angle of incidence.

To study the characteristics of cylindrical cloaks, we have designed a set of TM\textsubscript{z}-optimized cloaks for a wide range of parameters and cylinders of interest, using the normal-incidence analytical formulae with realistic losses and metamaterial frequency dispersion implemented in a Drude model. For the cloak design, we have focused on TM polarization because it dominates the scattering of moderately thick dielectric and conducting cylinders, which is of interest for several applications within the radar community. The optimized cloaks’ drastic scattering reduction was corroborated with full-wave numerical simulation, taking into account variations of the angle of incidence, core permittivity, cylinder diameter and most importantly the truncation effects due to finite length.

We have found that, as predicted by the analytical formulae presented here, for elongated objects with diameters up to one-half the wavelength (for which our cloaking technique is most effective), but with lengths comparable with the wavelength of operation, a simple one-layer permittivity cloak is very effective, providing significant scattering reduction highly robust to variations in the angle of incidence. Performance is slightly weakened at near-grazing angles, for which TM–TE polarization coupling partially affects the overall performance of a single-layer cloak optimized for TM polarization alone. We note, however, that scattering approaching grazing angles is the weakest in absolute value and is thus less important for achieving overall scattering reduction.

The analytical theory developed here for infinite cylinders has been strongly corroborated by our numerical simulations for finite lengths, implying that the truncation effects do not significantly perturb the cloaking effect. We are currently working on the extension of these concepts to multi-layered metamaterial cloaks for suppression of multiple scattering coefficients, and contemporary suppression of TE and TM scattered waves, which may increase the size of the cloaked objects and the angular range of operation. We are also currently pursuing an experimental realization of the plasmonic cloaking concept at radio frequencies for practical finite cylinders and cloaks.

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Appendix. Analytical solution of the scattering from a two-layer circular cylinder for oblique incidence

Consider the geometry of figure 1, consisting of a circular cylinder of radius $a$, permittivity $\epsilon$ and permeability $\mu$, covered by a thin conformal cylindrical cloak shell of thickness $(a_c - a)$, permittivity $\epsilon_c$ and permeability $\mu_c$. When excited by an impinging infinite plane wave of
arbitrary incidence angle and polarization, the general scattering problem may be solved in the limit of long cylinders \((L \rightarrow \infty)\) by expanding the impinging and scattered fields in terms of cylindrical harmonics \([36]-[38]\). The problem may be split into the orthogonal polarizations with \(E\)-field transverse to the cylinder axis \((\text{TE}_z)\) and transverse \(H\)-field \((\text{TM}_z)\).

For a TM, plane wave impinging at an angle \(\alpha\) from the cylinder axis \(\hat{z}\), the incoming magnetic field may be written as

\[
H_i = \frac{E_0}{\eta_0} e^{-i\beta z} e^{-ik_i^{TM} \hat{z}},
\]

where \(E_0\) is the electric field amplitude, \(\eta_0\) and \(k_0\) are the free-space characteristic impedance and wave number, \(\beta = k_0 \cos \alpha\) is the wavenumber component along the cylinder axis, \(k_i^{TM} = \sqrt{k_0^2 - \beta^2} = k_0 \sin \alpha\) is the transverse component of the wavenumber, and the corresponding impinging electric field may be calculated using the curl Maxwell’s equations. Expanding in cylindrical waves, we may write electric and magnetic fields as

\[
E_{TM} = \frac{1}{(k_0^2 - \beta^2) \rho \partial \phi} \frac{\partial^2 u_i^{TM}}{\partial \phi^2} \hat{\rho} + \frac{1}{\rho k_0^2 \partial \phi} \frac{\partial u_i^{TM}}{\partial \phi} \hat{\phi} + u_i^{TM} \hat{z},
\]

\[
H_{TM} = \frac{i k_0}{\rho \eta_0 k_0^2} \frac{\partial u_i^{TM}}{\partial \phi} \hat{\rho} - \frac{i k_0}{\rho k_0^2} \frac{\partial u_i^{TM}}{\partial \rho} \hat{\phi},
\]

where

\[
u_i^{TM} = E_0 i^{-n} (\sin \alpha) J_n(k_i^T \rho) e^{i\phi} e^{-i\beta z}
\]

and \(J_n\) is the cylindrical Bessel function of order \(n\). Analogous expressions for \(\text{TE}_z\) waves may be derived using duality.

Using the orthogonality of cylindrical waves, the boundary conditions at the radial interfaces may be met by assuming the existence of transmitted cylindrical waves in the two dielectric regions and a scattered wave in free space, which may be written consistently with equation \((A.2)\), using the scalar potentials:

\[
u_1^{TM} = i^{-n} E_0 e^{i\phi} e^{-i\beta z} \sin \alpha c_{1,n}^{TM} J_n(k_i^T \rho) \quad \rho < a,
\]

\[
u_2^{TM} = i^{-n} E_0 e^{i\phi} e^{-i\beta z} \sin \alpha \left[ c_{2,n}^{TM} J_n(k_i^T \rho) + c_{2,n}^{TM} Y_n(k_i^T \rho) \right] \quad a < \rho < a_c,
\]

\[
u_s^{TM} = i^{-n} E_0 e^{i\phi} e^{-i\beta z} \sin \alpha c_{s,n}^{TM} H_n^{(1)}(k_i^T \rho) \quad \rho > a_c,
\]

for the fields induced in the core region, shell and for the scattered field, respectively. The core and cloak regions are labeled ‘1’ and ‘2’, respectively, and ‘s’ represents the scattered wave outside the cloak; \(k_i\) are the relevant transverse wavenumbers for each region and \(k_i^T = \sqrt{k_i^2 - \beta^2}\). \(J_n\) and \(Y_n\) are the cylindrical Bessel functions of the first and second kind, for incoming and outgoing waves, whereas \(H_n^{(1)}\) is the cylindrical Hankel function. The complex scattering coefficients are the unknowns. Analogous equations may be written for \(\text{TE}_z\) waves.

For normal incidence \((\alpha = \pi/2)\), for PEC objects or for the azimuthally symmetric mode \(n = 0\), the problem is easily solved by matching the two nonzero tangential field components at each radial interface:

\[
E_{s,\phi} = E_{2,\phi}, \quad \rho = a_c,
\]

\[
E_{2,\phi} = E_{1,\phi}, \quad \rho = a,
\]

\[
H_{s,\phi} = H_{2,\phi}, \quad \rho = a_c,
\]

\[
H_{2,\phi} = H_{1,\phi}, \quad \rho = a.
\]

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for TM$_z$ (TE$_z$) polarization. This yields the familiar rank-4 determinant expressions reported in equations (3) and (4). Consistent expressions for the TE$_z$ coefficients may be obtained by applying duality. In the general case of oblique incidence on a dielectric cylinder, however, the higher-order $n > 0$ modes are characterized by all four independent tangential components of the fields at each interface, which may not be matched independently for each TE$_z$ or TM$_z$ harmonic. The boundary conditions may instead be met, as derived in [38], by linearly combining TE$_z$ and TM$_z$ harmonics of the same order $n > 0$. The corresponding solution of an eight-by-eight system of equations, derived in [38], provides the exact expression for the scattering coefficients in the general case, to be used in equation (2) to derive the total scattering width of the cylinder. This form of polarization coupling is inherently associated with the asymmetry introduced by the oblique excitation for higher-order modes, and it may be avoided only in the case of conducting objects or for $n = 0$. In the quasi-static limit, however, the dependence of the arguments of the Bessel functions in equations (A.4) on the transverse wavenumber $k_c$ is negligible as well; therefore this form of cross-polarization coupling is negligible. In this limit, equations (3) and (4) may be used for any angle of incidence, on the basis of the derivation of equations (5)–(6), which do not depend on $\alpha$.

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