Optical Conductivity of Clean Metals

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Key words  optical conductivity, Drude, Fermi liquid, Umklapp scattering
PACS  72.10.-d,72.15.Eb,74.25.Gz

Dedicated to the memory of Paul Drude.

We briefly review some basic aspects of transport in clean metals focusing on the role of electron-electron interactions and neglecting the effects of impurities, phonons and interband transitions. Both for small Fermi surfaces of two and three-dimensional metals and open Fermi surfaces of quasi one-dimensional metals the dc conductivity \( \sigma \) is largely dominated by momentum and pseudo-momentum conservation, respectively. In general, the frequency and temperature dependencies of \( \sigma(\omega, T) \) have very little in common. For small Fermi surfaces in three dimensions we find for example that the scattering rate is quadratic in frequency, \( \Gamma \propto \omega^2 \), even in the absence of a \( T^2 \) contribution.

1 Introduction

More than a century after Drude [1] published his pioneering work ‘Zur Ionentheorie der Metalle’ (‘On the ion theory of metals’), the optical conductivity is one of the most useful tools [2] to investigate the basic properties of metals. Both the excitation spectrum of metals (gaps, phonons, magnons, interband transitions, ...) and the scattering mechanisms leave their distinct traces in transport. It is surprising that the Drude model, which was developed before even the concepts of electrons (and Fermions in general) had been established, describes much of the phenomenology of low-frequency transport in metals correctly.

In this paper we will review some basic (but apparently not very well known) facts about the conductivity and low-frequency optical properties of clean metals. We will concentrate our discussion on the lowest frequencies, i.e. on the Drude peak at low but finite temperature, and on the \( T = 0 \) optical conductivity at the lowest frequencies. We will not consider the effects of disorder or phonons but study only the effects of electron-electron interactions in clean Fermi liquids. Parts of results discussed here have already been published in Ref. [3] (\( T = 0 \) optical conductivity of d-wave superconductors and metals) and Ref. [4] (clean quasi one-dimensional Fermi liquids at low \( T \)).

In a Galilean invariant system, the electrical conductivity is infinite as the current \( J = P/m \) is conserved, where \( P \) is the momentum and \( m \) the electron mass. In a crystal, translational invariance is partially broken which has two consequences. First, momentum is not conserved anymore as it can be transferred to the underlying lattice in units of the reciprocal lattice vectors \( G_i \) (using \( \hbar = 1 \)) via so-called ”Umklapp scattering processes”. A macroscopic momentum will therefore decay. Second, due to band-structure effects, current is not anymore proportional to the momentum. Even processes which conserve momentum can induce a partial decay of the current. As we elaborate below, non-Umklapp processes will lead to a decay of current components perpendicular to the momentum while the parallel component is not affected. The rather different role of current and momentum is reflected in the disparate frequency and temperature dependence of the optical conductivity. While the \( T \) dependence of the dc conductivity is often determined

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Fig. 1  a) Sketch of a scattering process using two different Brillouin zones. Two particles with state 1 and 2 are scattered into the states 1' and 2' (or equivalently 2''). While in hole-like Fermi surface (solid lines) no momentum is transferred to the lattice, the same scattering event in the dashed unit cell (replacing 2'' by the equivalent 2'') is an Umklapp process. b) Fermi surface of an anisotropic metal close to half filling. Both “forward” (I) and “Umklapp” (II) scattering processes do not lead to a decay of the pseudo-momentum $\tilde{P}_{21}$ as long as the momenta are within the shaded area. The scattering event III leads to a decay of $\tilde{P}_{21}$. c) Also in a d-wave superconductor with point nodes close to $(\pi/2, \pi/2)$, the pseudo momentum $\tilde{P}_{21}$ is the slowest mode in the systems as two-particle scattering events cannot relax $\tilde{P}_{21}$ if all momenta remain within the shaded regions.

by the decay rate of the momentum (or of the pseudo-momenta introduced in Sec.2), the finite frequency properties reflect that $J \neq P/m$, as we will explain in detail below.

In the following section we will first explore the role of momentum conservation. We emphasize that momentum has no unique definition in lattice and that the concept of momentum can and should be generalized e.g. in quasi one-dimensional systems or d-wave superconductors. We will first investigate temperature and then the frequency dependence of the optical conductivity.

2 Momenta and pseudo-momenta in lattices

In a one-band model, the electrical current is given by $J = \sum_{\sigma, k \in BZ} v_k c^\dagger_k \sigma c_k \sigma$ where the momentum sum extends over the first Brillouin zone (BZ) and $v_k = d\epsilon_k / dk$ is the velocity of electrons with creation operator $c^\dagger_k \sigma$ and band energy $\epsilon_k$. While in single band models the current can simply be obtained from a continuity equation, this is not the case in more complex multi-band models where inter-band matrix elements of the current depend on the details of the underlying microscopic model. As we are only interested in the conductivity at low frequencies and temperatures, we will not discuss such inter-band transitions in the following.

As the velocity is a periodic function of momentum, $v_{k+G_i} = v_k$, the definition of $J$ is obviously independent of the choice of the unit-cell in reciprocal space. This is, however, not the case for the momentum $P = \sum_{\sigma, k \in BZ} k c^\dagger_k \sigma c_k \sigma$ which depends on the chosen (first) Brillouin zone. For a given definition of $P$, one can separate all scattering events in 'non-Umklapp' and 'Umklapp' processes, where the latter are defined by a change of the momentum by a reciprocal lattice vector. The choice of the BZ will consequently decide whether a given scattering event is an Umklapp process as is shown in Fig. 1c. As it is common knowledge that in clean systems only Umklapp processes contribute to the dc conductivity and as the physics should not depend on the completely arbitrary choice of the BZ, one has to ask which definition of the momentum operator is the 'correct' one. Momentum conservation is only important in situations where $P$ decays very slowly. Therefore the relevant momentum is the mode with the slowest decay rate. This can for example be seen within a simple Boltzmann equation (essentially equivalent argument hold also in situations, where a Boltzmann description is not possible, see e.g. [5]). After linearization, the
Boltzmann equation in the presence of an electric field $E$ takes (in units where $e=1$) the form

$$\partial_t \Phi_k + v_k E = \sum_{k'} M_{kk'} \Phi_{k'}$$

(1)

$$\sigma_{ij}(\omega) = -\sum_{k,k'} v_k^i [M - i\omega]^{-1} v_{k'}^j \frac{\partial f_0}{\partial \epsilon_k}$$

(2)

where $\sigma_{ij}(\omega)$ is the (diagonal) optical conductivity tensor, $f_k = f_0(\epsilon_k) - \Phi_k \partial f_0/\partial \epsilon_k$ the distribution function linearized around the Fermi function $f_0$, and $M_{kk'}$ the well-known 3 scattering matrix. For a diagonal scattering matrix $M = \frac{1}{2} \delta_{kk'}$ one recovers the celebrated Drude formula. For the models considered in this paper, such a relaxation-time approximation cannot be applied as certain eigenvectors of $M$ have very small eigenvalues. Physically, this is a consequence of the approximate conservation of momentum or pseudo momentum (see below).

Let us first consider the well-known case of a Fermi liquid with a small Fermi surface, where Umklapp processes are strongly suppressed. In this case the inverse of the matrix $M$ is dominated by a single eigenvector with a very small eigenvalue. This eigenvector is with high precision the momentum parallel to the current. Introducing the scalar product $(a|b) = -\sum_k a_k b_k \partial f_0/\partial \epsilon_k$, one finds in such a situation 6

$$\sigma_{xx}(\omega = 0) = \langle v^x | M^{-1} | v^x \rangle = \max_{\Phi} \frac{\langle v^x | \Phi | M^{-1} | \Phi \rangle}{\langle \Phi | M | \Phi \rangle} \approx \frac{\langle v^x | p^x | p^x | v^x \rangle}{\langle p^x | M | p^x \rangle}$$

(3)

where the variational vector $\Phi$ takes at the maximum a value proportional to $M^{-1} |v^x\rangle$.

We now use 4 to generalize the concept of momentum and to resolve the above discussed ambiguities related to the choice of the BZ. First of all, momentum conservation (modulo reciprocal lattice vectors) is only relevant if the momentum decays much slower than other eigenvectors of $M$. For example, for a complex three-dimensional Fermi surface both Umklapp and non-Umklapp scattering rates are proportional to $T^2$ and momentum conservation and its generalizations are therefore not important. Second, when defining the momentum, $P_x = \sum_{\sigma,k \in \text{BZ}} k_x c^\dagger_k c_k$, one should use the BZ which results in the slowest decay rate of $P_x$ or, equivalently, in the largest value on the RHS of Eq. 5. Third, in many situations it is necessary to generalize the concept of momentum by introducing so-called ‘pseudo momenta’ 5 as the relevant eigenvector of $M$ is not proportional to momentum defined for any choice of the BZ.

Consider an arbitrary one-particle operator $\hat{P} = \sum_k \hat{p}_k c^\dagger_k c_k$ with the time derivative $\partial_t \hat{P} = i \sum_{kk',\sigma} U_{\sigma'}(\hat{p}_{k'} + \hat{p}_k - \hat{p}_{k+q} - \hat{p}_{k'-q}) c^\dagger_k c_{k',\sigma'} c_{k+q,\sigma'} c_{k'-q,\sigma'}$ in the presence of an arbitrary density–density interaction $U_q$. We want to construct an operator $\hat{P}$ which (i) does not decay by 2-particle processes close to the Fermi surface and (ii) has a sizeable overlap [defined by $\langle \hat{p} | v_x \rangle$] with the current in $x$ direction. We therefore require that $\hat{p}_k$ is finite close to the Fermi surface while $\hat{p}_k + \hat{p}_{k'} - \hat{p}_{k+q} - \hat{p}_{k'-q}$ should vanish for low-energy excitations. For a small Fermi surface, the usual momentum has these properties. If we are able to construct such a quantity in more complicated situations, we will call $\hat{P}$ ‘pseudo momentum’.

Such a construction is indeed possible when the low-energy excitations cover only a small part of the BZ as it is the case e.g. for quasi one-dimensional metals and unconventional superconductors with point or line nodes. For illustration, let us define a set of pseudo momenta labeled by the two integers $n$ and $m$ (generalizations are possible)

$$\tilde{P}_{nm} = \sum_k \delta_{k_{nm}} c^\dagger_k c_k = \sum_k (k_x - \frac{m G_x}{n} \text{sgn } k_x) c^\dagger_k c_k$$

(4)

where $G_x$ is the $x$ component of a reciprocal lattice vector. For $m = 0$ one recovers the standard momentum. The sign-function $\text{sgn } k_x$ implies that for positive/negative $k_x$ the momentum is measured with respect to the line $k_x = \pm \frac{m G_x}{n}$, i.e. the dashed lines in Figs 1 and c. These pseudo momenta dominate
A finite current \( \langle J \rangle \) typically decays initially very rapidly. However, if a (pseudo-) momentum \( \tilde{P} \) is present, which is approximately conserved, the component of the current parallel to \( \tilde{P} \) will decay much slower, giving rise to a low-frequency peak in the optical conductivity with a width which is determined by the slow decay rate \( \Gamma_{\text{slow}} \ll T^2 \).

The interplay of slow and fast decay, \( \Gamma_{\text{fast}} \sim T^2 \), results in a non-Drude like frequency dependence of the low-\( \omega \) peak. For \( \omega \gg \Gamma_{\text{slow}} \), a constant incoherent background should show up in generic three-dimensional materials. Higher frequencies are usually dominated by intraband transitions (and phonons etc.).

For the quasi one-dimensional Fermi surface shown in Fig. 1b, two types of scattering processes dominate for low \( T \). 'Forward' processes of type I conserve momentum and therefore also the pseudo-momentum \( \tilde{P}_{21} \) with \( \delta k_{21} = k_x - \text{sgn}(k_x)G_x/4 \). Umklapp processes of type II transfer the momentum \( G_x \) to the lattice: momentum decays rapidly. But as two particles move from the right to the left Fermi surface the pseudo momentum gets an extra contribution \( G_x/4 \) and remains conserved. More precisely, \( \tilde{P} \) does not decay by two-particle processes, if all four momenta are in the shaded region of Fig. 1, i.e. \( |\delta k_{21}| < G_{1x}/4 \). While the momentum is conserved modulo reciprocal lattice vectors, \( G \), the pseudo momentum \( \tilde{P}_{nm} \) can decay in quanta of size \( G_x/n \), for example by process III shown in Fig. 1. Such scattering processes are exponentially suppressed even at moderate temperatures. We conclude that in cases where the low-energy excitations are located in the shaded areas of Fig. 1b and c, the Boltzmann equation predicts for finite \( (\nu^x|\tilde{p}) \) an exponentially large dc-conductivity which has its origin in the approximate conservation of the pseudo-momentum \( \tilde{P}_{21} \).

The overlap of electrical current and pseudo momentum \( (\nu^x|\tilde{p}) \), see Eq. 5, is easily calculated within the approximation scheme underlying the Boltzmann approach. By partial integration one obtains

\[
(\nu^x|\tilde{p}_{nm}) = \sum_k f_k^0 \partial k_x \delta k_{nm} = \Delta \rho = \rho - \rho_{\text{max}}n/m
\]

where \( \rho \) is the total particle density, \( \rho_{\text{max}} \) the density of a filled band. For an exactly half-filled band one therefore has \( (\nu^x|\tilde{p}) = 0 \) and conservation of \( \tilde{P}_{21} \) is not important but dominates transport for small doping away from half-filling. In Ref. 4 it is shown that these statements also hold beyond perturbation theory even if a simple Fermi liquid description is not possible.

How does the presence of an approximately conserved pseudo-momentum affect the optical conductivity (a more quantitative version of the following arguments can be found in Ref. 4, a numerical test in Ref. 7)? Consider the following Gedanken experiment illustrated in Fig. 2: prepare a state with a finite current \( \langle J(t=0) \rangle \) > 0 and switch off the driving electric field at time \( t = 0 \). As the current is not conserved, it will decay rather fast with a typical rate which we denote by \( \Gamma_J \). The initial state with finite current will typically also have a finite pseudo momentum [assuming \( (\nu^x|\tilde{p}) \neq 0 \)] which will decay at a much slower rate than the current; \( \Gamma_{\tilde{P}} \ll \Gamma_J \). The slow decay of the pseudo-momentum will, as time goes...
on, induce a slow decay of the current since a state with finite pseudo-momentum will typically carry a finite current as \( \langle \hat{m} \hat{\rho} \rangle \neq 0 \) according to our assumptions. Thus, in the long time limit, a finite fraction, \( D/D_0 \), of \( \langle J \rangle \) will not decay with the fast rate \( \Gamma_J \) but with the much smaller rate \( \tilde{\Gamma} \), as is shown schematically in Fig. 2 (the effect of exact conservation laws has been studied a long time ago by Mazur and Suzuki [8, 9] and more recently by Zotos and coworkers [10]). The slow long-time decay of some fraction of the current leads to a corresponding long-time tail in the (equilibrium) current-current correlation function. Therefore, one expects a low-frequency peak in the optical conductivity which carries the fraction \( D/D_0 \) (a calculation of \( D \) within Fermi liquid theory is given in Ref. [4]). The width of the low-frequency peak is determined by the decay-rate of \( \tilde{\Gamma} \). As the weight of a peak is approximately given by its width multiplied with its height, we expect that the height and therefore the dc-conductivity is of the order of \( D/\tilde{\Gamma} \).

3 Temperature dependence of the dc conductivity: Small Fermi surfaces and quasi one-dimensional metals

The qualitative analysis of the Boltzmann equation sketched above suggests that the dc conductivity of a clean metal is exponentially large [11,12], \( \sigma \sim (\Delta \rho)^2 e^{\Delta E/T}/T^2 \), if the Fermi surface is either sufficiently small or sufficiently one-dimensional such that a momentum or pseudo-momentum can be found which does not decay by two-particle scattering processes close to the Fermi surface. Here, the so-called Umklapp gap \( \Delta E \) depends on the detailed shape of the Fermi surface.

However, it is not difficult to see that the exponential \( T \) dependence of \( \sigma \) (an often cited result going back to Peierls [11,12]) is an artifact of a Boltzmann equation which considers only two-particle scattering processes. A low-energy scattering process involving back to Peierls [11,12]) is an artifact of a Boltzmann equation which considers only two-particle scattering processes. A low-energy scattering process involving back to Peierls [11,12]) is an artifact of a Boltzmann equation which considers only two-particle scattering processes. A low-energy scattering process involving back to Peierls [11,12]), \( \langle J \rangle \) will not decay with the fast rate \( \Gamma_J \) but with the much smaller rate \( \tilde{\Gamma} \), as is shown schematically in Fig. 2 (the effect of exact conservation laws has been studied a long time ago by Mazur and Suzuki [8, 9] and more recently by Zotos and coworkers [10]). The slow long-time decay of some fraction of the current leads to a corresponding long-time tail in the (equilibrium) current-current correlation function. Therefore, one expects a low-frequency peak in the optical conductivity which carries the fraction \( D/D_0 \). The qualitative analysis of the Boltzmann equation sketched above suggests that the dc conductivity of a clean metal is exponentially large [11,12], \( \sigma \sim (\Delta \rho)^2 e^{\Delta E/T}/T^2 \), if the Fermi surface is either sufficiently small or sufficiently one-dimensional such that a momentum or pseudo-momentum can be found which does not decay by two-particle scattering processes close to the Fermi surface. Here, the so-called Umklapp gap \( \Delta E \) depends on the detailed shape of the Fermi surface.

However, it is not difficult to see that the exponential \( T \) dependence of \( \sigma \) (an often cited result going back to Peierls [11,12]) is an artifact of a Boltzmann equation which considers only two-particle scattering processes. A low-energy scattering process involving \( N \) particles can – for sufficiently large \( N \) – transfer momentum to the lattice even for small Fermi surfaces. We therefore have to generalize our discussion to \( N \) particle scattering events. Due to conservation of lattice momentum and particle number, \( \tilde{P}_{nm} \) can be altered only by an amount in quanta of size \( \frac{1}{m} G_1^{1} \) (for small Fermi surfaces use \( m = 0 \) and \( n = 1 \) such that \( \delta k_{nm} = k \)). Therefore, a relaxation of \( \tilde{P} \) is not possible if all \( 2N \) pseudo momenta \( \delta k_{ix} \) involved in an \( N \)-particle scattering process are smaller than \( \frac{1}{2N} \frac{1}{m} G_1^{1} \). For a given Fermi surface, the decay of \( \tilde{P} \) by \( N \)-particle collision at low \( T \) is possible only for

\[
N > \frac{G_{1x}/(2n)}{\max \left| \delta k_{nm}^P \right|},
\]  

(6)

where \( \max \left| \delta k_{nm}^P \right| \) is the maximal distance of the Fermi surface from the plane \( k_x = \pm m G_{1x} \) (dashed line in Fig. 1b). In the case of a small Fermi surface, \( \delta k_{nm}^P \) is just the Fermi momentum.

The \( T \) dependence of the decay rate of the (pseudo-) momentum arising from an \( N \) particle scattering event can be obtained from the usual phase space arguments: when a particle of energy \( \sim T \) decays into \( 2N - 1 \) particle and hole excitations, one of the energies is fixed by energy conservation, and the remaining \( 2N - 2 \) energies each have a phase-space of order \( T \). Therefore, \( \tilde{\Gamma} \propto T^{2N-2} \)

\[
\sigma(\omega = 0) \sim \frac{(\Delta \rho)^2}{T^{2N-2}}
\]  

(7)

where the integer \( N \) is the smallest value consistent with [6,13,14]. The prefactor of \( \tilde{\Gamma} \) and therefore also the prefactor in (7) depends in a rather delicate way on the strength and range of the interaction, the screening, and the band-curvature, and the prefactor in Eq. (7) will be very large for a weakly interacting system and large \( N \). The crossover from a high-temperature regime where \( 2 \)-particle scattering dominates to low temperatures can crudely be mimicked by an interpolating formula of the type \( \tilde{\Gamma} \sim U^2 T^2 e^{-\Delta E/T} + U^N T^{2N-2} \), where \( U \) is an effective interaction and \( \Delta E \) the Umklapp gap for \( 2 \)-particle collisions.
We want to stress that the analysis given above is valid both for small Fermi surfaces and quasi one-dimensional systems with moderate curvature of the Fermi surface. It holds for interactions of arbitrary strength as long as a Fermi liquid description is possible. For practical purposes, a very clean system is needed to observe a large exponent $N$ as otherwise the (pseudo) momentum will decay by impurity scattering. In this case, one recovers the usual $T^2$ correction to the resistivity for the lowest $T$, however with a prefactor which depends on the amount of disorder in the system.

4 Frequency dependence of the optical conductivity

In the previous section, we have discussed the $T$ dependence of the conductivity for $\omega = 0$ or more precisely $\omega \ll \Gamma_p$, where $\Gamma_p$ is the decay rate of the momentum at $\omega = 0$. In this section, we consider the small-frequency dependence of the $T = 0$ conductivity, or more precisely the limit $\omega \gg \Gamma_J$, where $\Gamma_J$ is the $\omega = 0$ decay rate of the current. We do not investigate the intermediate regime $\Gamma_p \lesssim \omega \lesssim \Gamma_J$. For simplicity, we also concentrate our discussion on systems with small Fermi surfaces to avoid the complications arising from the numerous crossover scales in the quasi one-dimensional system. In the limit $\omega \to 0$, $T = 0$, the frequency dependence of the optical conductivity of a quasi one-dimensional system with finite curvatures of the Fermi sheets is characterized by the same power laws as a generic three-dimensional metal.

For $\omega > 0$, $T = 0$ and in the absence of disorder, it is possible to calculated the optical conductivity from a straightforward perturbative evaluation of the Kubo formula.

$$\text{Re} \sigma(\omega) = \frac{1}{\omega} \text{Im} \langle \langle J, J \rangle \rangle_{\omega} = \frac{-i}{\omega} \int_0^{\infty} dt \, e^{i(\omega+\delta)t} \langle \langle [J(t), J(0)] \rangle \rangle. \tag{8}$$

In contrast, the calculation of dc transport at finite $T$ to leading order in perturbation theory involves usually the solution of some integral equations (for example, the quantum Boltzmann equation or equivalently a vertex equation) or equivalently a matrix inversion, see Eq. (2). The simplifications arising at $T = 0$, $\omega > 0$ can be seen by inspecting the (generalized) Drude formula

$$\sigma(\omega) = \frac{\chi}{\Gamma(\omega) - i\omega} \tag{9}$$

where $\chi$ is identified with the total optical weight and $\text{Re} \Gamma(\omega)$ is the (frequency-dependent) scattering rate. For $\omega = 0$ and $T > 0$ the conductivity is a singular function of $\Gamma(\omega) = 0$ and therefore an infinite resummation of perturbation theory [e.g. by solving a Boltzmann equation or by inverting the matrix in Eq. (2)] is required to calculate $\sigma$. However, in the opposite limit, $\omega > 0$ and $T = 0$, the scattering rate is small compared to $\omega$ allowing for a straightforward perturbative expansion in the small parameter $\Gamma/\omega$. This can be done most conveniently [15] by first multiplying Eq. (9) with $\omega^2$ to cancel the $\delta$-function at $\omega = 0$.

$$\omega^2 \text{Re} \sigma(\omega) = \frac{\text{Im} \langle \langle \partial_t J, \partial_t J \rangle \rangle_{\omega}}{\omega} = \omega^2 \text{Re} \frac{\chi}{\Gamma(\omega) - i\omega} \approx \chi \text{Re} \Gamma(\omega) \tag{10}$$

As $\partial_t J$ is already linear in the interactions (see below), it is sufficient to leading order to evaluate the correlation function in (10) to zeroth order in the couplings. We will use this approximation only at $T = 0$. At any finite temperature, the scattering rate $\Gamma(\omega \to 0)$ is finite and therefore the method described above will break down for $\omega \to 0$ but remains valid at higher frequencies where $|\Gamma(\omega)| \ll \omega$. The drastic simplification arising for $|\Gamma(\omega)| \ll \omega$ is also obvious from Eq. (2): while one has to invert the infinite-dimensional matrix $M$ to obtain the dc-conductivity, one can approximate $\text{Re} (M - i\omega)^{-1} \approx M/\omega^2$ if $\omega$ is larger than the largest eigenvalue of $M$.

It is worthwhile to discuss the range of validity of Eq. (10). Quantitatively, an evaluation of $\langle \langle \partial_t J, \partial_t J \rangle \rangle_{\omega}$ to lowest order in the interactions will only be justified if interactions are weak. Qualitatively, the lowest
order expression does, however, correctly describe the dynamics of 2-particle collisions. In a Fermi liquid, Re $\Gamma(\omega)$ will be of order $\omega^2$ or smaller due to Pauli blocking (see below) and therefore smaller than $\omega$ in a wide frequency range even if interactions are strong. Im $\Gamma(\omega)$ is linear in $\omega$ for small frequencies where $c$ is of order 1 for strong interactions. While this factor can lead to strong renormalizations of the optical mass and therefore of prefactors, such a correction will not change the power-law dependence of Re $\sigma(\omega)$ for small frequencies discussed below.

The current of a one-band model, $J = \sum_{k} c_k^\dagger v_k c_k$, decays in the presence of interactions

$$
\partial_t J = i \sum_{kk'q\sigma\sigma'} U_q (v_k + v_{k'} - v_{k+q} - v_{k'-q}) c_k^\dagger c_{k+q,\sigma} c_{k'-q,\sigma'} c_{k',\sigma'}
$$

(11)

where we assumed a density–density interaction with momentum dependence $U_q$ (we use $U_q \approx U = \text{const.}$ for convenience in the following). $\partial_t J$ is proportional to the difference of incoming and outgoing velocities. This difference vanishes in a Galilean invariant system where $v_k = k/m$ but is finite in a lattice model.

Using Eq. (10) we obtain for $T = 0$ and $\omega > 0$ to leading order in $U$

$$
\text{Re } \sigma(\omega > 0) \approx \frac{4\pi U^2}{\omega^3} \sum_{1234G} f_1 f_2 (1 - f_3) (1 - f_4)
$$

$$
\times (v_1^2 + v_2^2 - v_3^2 - v_4^2)^2 \delta_{1,2,3,4} G [\delta(\omega - (\varepsilon_4 + \varepsilon_3 - \varepsilon_2 - \varepsilon_1)) - (\omega \leftrightarrow -\omega)]
$$

(12)

where 1, ..., 4 denote the momenta $k_1, ..., k_4$ in the first Brillouin zone, $f_i = f(\varepsilon_{k_i})$ are Fermi functions and momentum and energy is conserved modulo reciprocal lattice vectors $G$.

To perform the momentum integrals it is useful to split $k_i$ into a component perpendicular to the Fermi surface and an angular integration parallel to it. For small $\omega$ only a thin shell of width $\omega/v_F$ contributes for each of the three relevant momentum integrations perpendicular to the Fermi surface (the 4th integral takes care of the $\delta$-function arising from energy conservation and the angular integrations are used to fulfill momentum conservation). If we neglect the velocity prefactors, this leads to an $\omega^3$ dependence of the integrals which cancels the $1/\omega^3$ prefactor.

The velocity prefactors $(v_1^2 + v_2^2 - v_3^2 - v_4^2)^2$ can be approximated by constants of the order of $v_F^2$, if the Fermi surface is sufficiently large such that Umklapp scattering processes can take place. In the presence of Umklapp, one therefore obtains the well-known result that

$$
\text{Re } \sigma(\omega > 0) \sim \frac{U^2 k_F^5}{v_F^4} \approx \text{const.,} \quad \Gamma(\omega) \propto \omega^2
$$

(13)

Note that a constant “incoherent background” corresponds according to Eq. (10) to a scattering rate $\Gamma(\omega) \propto \omega^2$ characteristic of a Fermi liquid with Umklapp scattering in 2 or 3 dimensions.

More interesting is the corresponding result for small Fermi surfaces ($k_F < G/4$) where Umklapp scattering at the Fermi surface is not possible and one obtains rather different results in two and three dimensions. We first consider two dimensions and study the size of $(v_1^2 + v_2^2 - v_3^2 - v_4^2)^2$ in the limit $\omega \to 0$. For a simple Fermi surface (without pockets etc.), momentum conservation in the limit $\omega \to 0$ can only be fulfilled by combining opposite momenta, i.e. by choosing $k_1 = -k_2$ and $k_3 = -k_4$ (or $k_1 = k_3/4$ and $k_2 = k_4/3$). As $v_{-k} = -v_k$ the prefactor $(v_1^2 + v_2^2 - v_3^2 - v_4^2)^2$ vanishes linearly in $\omega$ for $\omega \to 0$ and one obtains from power counting

$$
\text{Re } \sigma(\omega > 0) \propto \omega^2, \quad \Gamma(\omega) \propto \omega^4
$$

(14)

for a small Fermi surface in $d = 2$.

Momentum conservation is a much less restrictive condition in three dimensions. For small Fermi surface in $d = 3$, momentum conservation on the Fermi surface does not require that the relevant moments
are located opposite to each other. For any given momenta \( k_1 \) and \( k_2 \), there is a one-dimensional manifold of momenta \( k_3 \) and \( k_4 \) on the Fermi surface with \( k_1 + k_2 = k_3 + k_4 \). For a generic (i.e. non-spherical) Fermi surface, the prefactor \( (v_F^2 v_F^2 - v_k^2 - v_k^2)^2 \sim v_F^2 (k_F a)^4 \) will therefore be finite. The suppression by the factor \( (k_F a)^4 \), where \( a \) is the lattice spacing, describes that the Fermi surface becomes more and more spherical for small \( k_F \) [here we assumed \( \epsilon_{-k} = \epsilon_k \) such that leading corrections to the \( k^2/(2m^*) \) dispersion arise to order \( k^4/(a^2 m^*) \)]. The finite prefactor implies that

\[
\text{Re } \sigma(\omega > 0) \sim \frac{U^2 k_F^5}{v_F^2} (k_F a)^4 \approx \text{const.}, \quad \Gamma(\omega) \propto \omega^2
\]  

for a small Fermi surface in \( d = 3 \). Even in the absence of Umklapp processes the scattering rate varies as \( \Gamma(T = 0, \omega) \propto \omega^2 \). It is, however, suppressed by the small factor \( (k_F a)^4 \) compared to a single-particle relaxation rate. In a Galilean invariant system, this factor vanishes.

The frequency dependence of the scattering rate has to be contrasted with the temperature dependence discussed in section 3 where we found that \( \Gamma(\omega = 0, T) \propto T^{2N-2} \) is dominated by \( N \)-particle scattering processes with \( N > 2 \) for a small Fermi surface.

### 5 Conclusions

In a generic clean metal, the frequency and temperature dependence of the optical conductivity arise from rather different physical processes and have in general very little in common. This can be seen most drastically in systems where momentum conservation (or its generalization the pseudo-momentum conservation) dominates transport. In such a situation, we found \( \Gamma(\omega, T = 0) \propto \omega^2 \) while \( \Gamma(\omega = 0, T) \propto T^{2N-2} \) where \( N > 2 \) depends on the size of the Fermi surface (or its curvature in the case of a quasi one-dimensional metal). This disparity behavior can be traced back to the simple fact that momentum and current are two different operators in a lattice system. The response at finite frequencies depends on the relaxation rate of the current operator. We have shown that in a three-dimensional metal current can relax efficiently even in the absence of Umklapp scattering as a typical low-energy scattering process of two particles changes the current by a finite amount \( (v_F^2 v_F^2 - v_k^2 - v_k^2)^2 \sim v_F^2 (k_F a)^4 \). The same type of process, however, does not relax the momentum. We have argued that this implies that a finite fraction of the current – the fraction which is 'parallel' to the momentum – does not relax by this process. The consequence is that such a process is not effective in determining the \( \omega = 0 \) relaxation rate at finite temperatures. The \( T \) dependence of the dc conductivity is given by the relaxation rate of the (pseudo-) momentum and therefore a completely different set of processes.

As the \( \omega \) and \( T \) dependence are completely independent for small Fermi surfaces or quasi one-dimensional metals, it is obvious that also in the presence of Umklapp scattering, when \( \Gamma(\omega, T) \approx a(k_F T)^2 + b(\hbar \omega)^2 \) there is in general no simple relation between the constants \( a \) and \( b \). We emphasize this fact since in the experimental literature such a relation has sometimes been claimed to exist \cite{2,16,17} but is actually not observed \cite{17,18}.

When discussing the role of momentum conservation on the one hand and Umklapp scattering on the other hand it is important to keep in mind that in general the concept of momentum is not uniquely well defined in a lattice. We have emphasized that in general one has to search for a slow mode. In the case of strongly anisotropic Fermi liquids (or Luttinger liquids \cite{5}, spin chains and ladders \cite{19} or superconductors \cite{8} with point nodes) this has motivated us to consider so-called pseudo-momenta, see Eq. 14, which can be identified with the momenta of effective low-energy field theories \cite{5}.

More than a century after the pioneering work of Drude the measurement and calculation of the optical conductivity remains an interesting and challenging area of condensed matter physics. Open questions include for example the transport close to quantum-critical points where the role of momentum conservation and Umklapp scattering deserves further investigations. But even in simple Fermi liquids it is interesting – both experimentally and theoretically – to study the precise shape of the Drude peak in regimes where impurity scattering can be neglected, especially in quasi one-dimensional metals \cite{20}. 

Acknowledgements  I would like to thank N. Andrei and P. Wölfle for helpful discussions at various stages of this work and the DFG for financial support under SFB 608.

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