Every finite group is the group of self-homotopy equivalences of an elliptic space

by

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1. Introduction

For simply connected CW-complexes $X$ of finite type, we are interested in the group of homotopy classes of self-homotopy equivalences, $E(X)$, and the realizability problem for groups. Namely, if a given group $G$ can appear as the group $E(X)$ for some space $X$. This problem has been placed as the first to solve in [3], being around for over fifty years and recurrently appearing in surveys and lists of open problems about self-homotopy equivalences [2], [14], [20], [21], [26]. The difficulty of this question relies on the fact that techniques used so far are specific to certain groups [6], [7], [12], [22], [24], and have not proved fruitful when addressing this problem in general.

Apart from the group of automorphisms of a group $\pi$, $\text{Aut}(\pi)$, which is isomorphic to $E(K(\pi, n))$ for an Eilenberg–MacLane space $K(\pi, n)$, there is no global picture in this context. A special mention deserves the cyclic group of order 2, which is the group of automorphisms of the cyclic group of order 3, and hence it can be realized as $E(K(\mathbb{Z}_3, n))$. Arkowitz and Lupton show that, moreover, it is the group of self-homotopy equivalences of a rational space, pointing out the surprising appearance of a finite group in rational homotopy theory, and raising the question of when finite groups can be realized by rational spaces [4].

In this paper, we give a complete answer to the realizability problem for finite groups.

Theorem 1.1. Every finite group $G$ can be realized as the group of self-homotopy equivalences of infinitely many (non-homotopy-equivalent) rational elliptic spaces $X$. 

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To build up those spaces, we introduce a general method which we hope can be useful for obtaining examples with interesting properties in subjects of different nature. For instance, it appears to produce differential manifolds related to a question of Gromov, as mentioned below (see also §3). Indeed, we construct a contravariant functor from a subcategory of finite graphs to the homotopy category of differential graded commutative algebras whose cohomology is 1-connected and of finite type. Then, the geometric realization functor of Sullivan gives the equivalence of categories between the homotopy category of minimal Sullivan algebras and the homotopy category of rational simply connected spaces of finite type.

We remark that by dropping the requirement on the finiteness type of the differential graded algebras, our method can be extended to infinite, locally finite graphs. This is a subtle and technical point that is handled in [10], where this extended version of our techniques is used to obtain an isomorphism criteria for a large class of groups, having thus consequences in representation theory.

In this paper, we prove the following theorem.

**Theorem 1.2.** Let \( G \) be a finite connected graph with more than one vertex. Then, there exists an elliptic minimal Sullivan algebra \( M_G \) such that the group of automorphisms of \( G \) is realizable by the group of self-homotopy equivalences of \( M_G \).

Our idea of using graphs has its origin on the following classical result ([16], [17]).

**Theorem 1.3.** (Frucht, 1939) Given a finite group \( G \), there exist infinitely many non-isomorphic connected (finite) graphs \( G \) whose automorphism group is isomorphic to \( G \).

Because of the equivalence given by the geometric realization functor of Sullivan, Theorem 1.1 follows directly from Theorems 1.2 and 1.3 (see Proposition 2.7). Applying Theorem 1.1 to the trivial group, we supply a partial answer to Problem 3 in [21]. This problem consists on determining spaces, which were thought to be quite rare [20], with a trivial group of self-homotopy equivalences, the so-called homotopically rigid spaces.

**Corollary 1.4.** There exist infinitely many rational spaces that are homotopically rigid.

Recall that in homotopy theory, naive dichotomy [13] classifies spaces in either elliptic or hyperbolic. Ellipticity is a very severe restriction on a space \( X \), and it is remarkable that many of the spaces which play an important role in geometry are rationally elliptic. In particular the rational cohomology of \( X \) satisfies Poincaré duality [19] and, with extra hypothesis on the dimension of the fundamental class, \( X \) has the rational homotopy type of a simply connected manifold ([5], [27]). Indeed, the spaces in Theorem 1.1 can be