Quantum Control: discovered, repeated and reformulated ideas mark the progress

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Abstract

The dynamical manipulation of physical systems is necessary to explain the structure of physical theories. However, the manipulation techniques in quantum theories progress slowly, giving still no certainty that all theoretically described states and all unitary transformations can be indeed achieved. Below, we report some elements of idealized control of the evolution processes in time dependent quadratic potentials, with attention focused on the abstract $\delta$-pulses of the elastic forces. Some of them, if empirically approximated, could become essential for a sequence of difficult interpretation problems.

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1 Introduction

One of nontrivial challenges of the present day quantum theories are the dynamical manipulations of particle systems by variable external conditions. While multiple experiences already exist, they are still incomplete. However, if the physical theories were at all created it is only because the physical systems can be actively controlled by flexible external forces making possible to perform experiments. Indeed, without the interaction with variable surrounding, not even the theory of freely propagating particles in an empty space could be formulated.

In spite of the limited scope of the dynamical operations, the modern quantum theories show spectacular achievements (the description of the anomalous magnetic moments, the probable detection of the Higgs boson, etc.). However, almost all methods applied are borrowed from the theory of stationary oscillations or just static fields. The mathematical concepts too are repetitive, with pure states described always by vectors in the linear (Hilbert) spaces and the time evolution (in absence of dissipation) given by linear, unitary
operations. The picture persists in the theories of coupled systems and quantum field theories (QFT) where the states are represented in tensor product spaces - in an obvious intent to conserve the linearity of basic laws at the cost of multiplying the number of variables.

Yet, some doubts remain. Can all theoretically predicted states be indeed created \[1\]? Can all unitary operations be indeed achieved (or at least approached) by physical evolution? Can all measurements described by self-adjoint operators be truly performed? (The close interrelation between the last two questions can be noticed \[2\]).

One of the main obstacles in checking the 'obligatory beliefs' of quantum theories, are the perturbative complications and the difficulties of extrapolating to 'the little' and to 'the great' \[3\]. What could help are the exact algorithms, though they still require exceptional effort and patience \[4, 5, 6, 7, 8, 9, 10\]. The purpose of this talk is to show that even the simplest cases of the exactly soluble evolution, for particles in time dependent quadratic potentials, have some consequences which are not generally known. To them belong the possibility manipulating or even inverting the free evolution (i.e. the 'quantum time machine'), modeling the repulsive oscillator pulses by the attractive ones, and programming the operational schemes which challenge the time-energy uncertainty \[11\].

2 Quadratic Hamiltonians: the simplest approach

The quantum theory obeys the correspondence principle, turning classical for \(\hbar \to 0\). However, in some dynamical theories the evolution laws are shared by classical and quantum systems without the need of any limiting transition. This occurs for the non-relativistic quadratic Hamiltonians - where the motion equations for canonical variables are exactly the same in classical and quantum levels. While the fundamental problems were carefully revised by Barry Simon \[12\], what is still worth attention are the algebraic aspects of the evolution generated by the quadratic Hamiltonians:

\[
H(t) = \frac{1}{2} \sum \beta_{k,l}(t) Q_k Q_l
\]

where \((Q_1...Q_{2n}) = (q_1, q_2, ..., q_n, p_1, p_2, ..., p_n)\) are the canonical position and momentum variables and \(\beta_{k,l}(t) = \beta_{l,k}(t) \in \mathbb{R}\) the time dependent coefficients defining the external forces. While the most general dynamical operations of these Hamiltonians remain of significant interest, the subject of the present report will be some exact solutions of the evolution problem (1) for \(n \leq 3\) (i.e., taking place in \(\mathbb{R}^n, n \leq 3\)), with the special attention dedicated to the quadratic Hamiltonians in one space dimension:

\[
H(t) = \frac{p^2}{2} + \beta(t) \frac{q^2}{2}
\]

typically representing the separable parts of the evolution process in the traditional, quadrupole ion traps. The only challenge we shall attend is strictly combinatorial: how should one
program the oscillations of the c-number amplitude $\beta(t)$, to generate some useful quantum operations? On purely mathematical level, the problem is elementary, though not always simple.

In the classical theory and for nonsingular $\beta(t)$ the canonical equations lead to the linear evolution of the canonical variables, represented by the $2 \times 2$ symplectic evolution matrices $u(t, t_0)$:

$$ \begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = u(t, t_0) \begin{pmatrix} q(t_0) \\ p(t_0) \end{pmatrix} $$

(3)

given by the matrix equations:

$$ \frac{d}{dt} u(t, t_0) = \Lambda(t) u(t, t_0) \quad \frac{d}{dt_0} u(t, t_0) = -u(t, t_0) \Lambda(t_0) $$

(4)

with:

$$ \Lambda(t) = \begin{pmatrix} 0 & 1 \\ -\beta(t) & 0 \end{pmatrix} $$

(5)

and:

$$ u(t, \theta) u(\theta, t_0) = u(t, t_0) \quad u(t_0, t_0) = 1 $$

(6)

In quantum case, the corresponding evolution operators $U(t, t_0)$, are given by:

$$ i \frac{d}{dt} U(t, t_0) = H(t) U(t, t_0), \quad i \frac{d}{dt_0} U(t, t_0) = -U(t, t_0) H(t_0) $$

(7)

with:

$$ U(t, \theta) U(\theta, t_0) = U(t, t_0) \quad U(t_0, t_0) = 1 $$

(8)

defining the evolution of the observables $q$ and $p$ in Heisenberg’s picture (the “Heisenberg’s trajectory”), generating the same symplectic matrices (3-5):

$$ U(t, t_0) \begin{pmatrix} q \\ p \end{pmatrix} U(t, t_0) = u(t, t_0) \begin{pmatrix} q \\ p \end{pmatrix} $$

(9)

though now $q, p$ are operators with $[q, p] = i$ (we put for simplicity $\hbar = 1$). The exact formal correspondence of the classical and quantum cases was explored by multiple research groups, quite frequently without knowing of each other. In all these approaches, a useful observation is

**Proposition 1.** In absence of spin or any additional degrees of freedom, each unitary evolution operator $U(t, t_0)$ in $L^2(\mathbb{R})$ generated by the quadratic, time dependent Hamiltonians (2) is determined up to a phase factor by the classical motion trajectories.
Proof. Instead of any sophisticated arguments, notice that if two unitary operators $U_1$ and $U_2$ produce the same transformation of the canonical variables i.e., $U_1^* q U_1 = U_2^* q U_2$ and $U_1^* p U_1 = U_2^* p U_2$, then $U_1 U_2^*$ commutes with both $q$ and $p$. Hence, it commutes also with any function of $q$ and $p$. Since in $L^2(\mathbb{R})$ the functions of $q$ and $p$ generate an irreducible algebra, then $U_1 U_2^*$ must be a c-number and since it is unitary, it can be only a phase factor, $U_1 U_2^* = e^{i\varphi} \Rightarrow U_1 = e^{i\varphi} U_2$ where $\varphi \in \mathbb{R}$ \cite{13, 12, 14, 15}. Note that the same statement holds for the systems (1) with any number of canonical variables. □

Any two unitary operators which differ only by a c-number phase can generate different transformations of the state vectors, but the same transformation of quantum states, so we shall call them equivalent and write $U_1 \equiv U_2$. The fact that they might wear different phase factors $e^{i\varphi}$ can be of interest for the linear representation theory studying the metaplectic groups \cite{16, 17, 18} but does not affect the operations on physical states, the only subject of our interest. From now on, our report will thus concern the manipulations of quantum states, skipping almost all contributions to the group representation as well as the pictures of “passive” unitary transformations (meaning the change of the observation frame). Moreover, when describing the non-relativistic, time dependent Hamiltonians our review (with little exceptions) will consider only the evolution generated by physical fields, which, at least in principle can be created by some controllable external conditions. We thus omit the reports on hypothetical media with the time dependent dielectric or magnetic constants, or (with one exception) the time dependent mass. The so defined subject might seem narrow, but precisely, most interesting for the problem of quantum manipulation.

Below, our attention will be focused on the evolution caused by quadratic, time dependent Hamiltonians (2). The possibility of deducing the quantum state evolution from the transformations of the canonical variables (9) permits one to program ample families of the classical/quantum control operations. To classify them, the algebraic types of matrices (3-9) are quite essential. Since every evolution matrix $u = u(t, t_0)$ is symplectic (Det $u = 1$) its algebraic type is defined just by one invariant Tr$u$. The characteristic equation:

$$D(\lambda) = \text{Det}(\lambda - u) = \lambda^2 - \text{Tr}(u)\lambda + 1 = 0$$

has two roots $\lambda_{\pm} = \frac{1}{2} \text{Tr} u \pm i\sqrt{\Delta}$, where $\Delta = 1 - \frac{1}{4} (\text{Tr} u)^2$, permitting to distinguish three types of evolution matrices:

(I) If $|\text{Tr}(u)| < 2$, then $u$ has two complex eigenvalues $\lambda_{\pm} = e^{\pm i\sigma}$ where $0 \neq \sigma \in \mathbb{R}$

(II) If $|\text{Tr}(u)| = 2$ then $u$ is on the threshold: $\lambda_{+} = \lambda_{-} = \pm 1$

(III) If $|\text{Tr}(u)| > 2$, then $u$ has two real eigenvalues $\lambda_{\pm} = e^{\pm \sigma}$ where $0 \neq \sigma \in \mathbb{R}$

The classification turns specially relevant if the function $\beta(t)$ in (2) is periodic, $\beta(t + T) = \beta(t)$, defining a Floquet process. The (crucial) Floquet matrices $u(t_0 + T, t_0)$ describe
then the repeated evolution incidents. One easily shows that their types do not depend on \( t_0 \). Choosing \( t_0 = 0 \) and denoting for simplicity \( u(t) = u(t, 0) \) one sees that the results of the evolution in the sequence of expanding intervals \([0, nT]\) are given by repetitions of \( u(T) \), i.e., \( u(nT) = u(T)^n \). Now, if \( u(T) \) is in the class (I) the evolution is oscillatory. The eigenvectors of \( u(T) \) define a pair of variables \( A_\pm \) (the global creation and annihilation operators for the entire periodicity intervals) which for \( t = nT \) perform just the phase rotation \( U(t)^\dagger A_\pm U(t) = e^{\pm i\sigma t} A_\pm \). However, if \( u(T) \) is in the class (III) then the equilibrium is lost: the eigenvectors of \( u(T) \) define now two real canonical variables \( A_\pm \) which are multiplied by \( e^{\pm i\sigma t} \), \((0 \neq \sigma \in \mathbb{R}) \) i.e., endlessly squeezed or endlessly amplified as \( t = nT \to \infty \). In turn, the threshold cases (II) offer some exceptional manipulation techniques which might deserve attention.

3 The early branches of quantum control

We still comment the evolution caused by the Hamiltonians (2). Even if elementary, the problem has a considerable history. The subject was extensively studied already in 1967-1969 by Lewis and Riesenfeld [19, 20, 21] via the concept of invariants, then by Malkin, Manko, Trifonov [22, 23] leading later to an important techniques of quantum tomography [7, 8, 10]. The subsequent steps are no less interesting; they form a colorful story of achievements and obstacles. The concepts of invariants in [19, 20], though suggestive, were not immediate to apply. As the matter of fact, the problem seemed to obey some law of ‘difficulty conservation’. In order to solve the initial 2-nd order differential eq. for the canonical coordinate \( q \), i.e.: \( \ddot{q} + \beta(t)q(t) = 0 \), the techniques of invariants required the solution of an equivalent 2-nd order equation (see eqs. (45) or (101) in [20])

\[
\frac{d^2}{dt^2} \rho + \beta(t) \rho + \frac{1}{\rho^3} = 0
\]

(11)

not simpler than the original one.\(^1\) A stumble was also, that the model used by Lewis and Riesenfeld [20] while mathematically correct, was unphysical. It described the free motion of a charged particle along the \( z \)-axis, while on the orthogonal plane \( x, y \) the evolution was obeying the homogeneous time dependent magnetic field \( B(t) \) parallel to \( z \) and the time dependent scalar potential \( \phi = e\eta(t)(x^2 + y^2) \). It was enough to apply the Laplace operator \( \Delta \) to see, that the electromagnetic model in [20] involved a homogeneous, but time dependent charge density all over the space (acknowledged by the authors), awakening some uneasy thoughts about the charge conservation.\(^2\) The same scenario was adopted later by Baseia et al [24], apparently without perceiving any problem. Of course, one can

\(^1\)By consulting [20] one can even get an impression that the ‘invariants’ complicate rather than facilitate the solution.

\(^2\)Indeed, it is not an accident that the time dependent quadratic potentials in the ion traps in 3D have the form \( \eta(t)(x^2 + y^2 - 2z^2) \) but not \( \eta(t)(x^2 + y^2) \)
agree that the creative achievement in [19, 20] consists not so much in the realistic control models but rather in understanding better the mathematical structure. Moreover, their works initiated the whole manipulation trend. (One can notice though, that the authors of subsequent papers, while developing the method, abstain from assuming the unphysical scenarios [22, 23]).

A historical anecdote is also that Lewis and Riesenfeld in their eq. (11) arrived (without knowing) at the transformed Schrödinger equation discovered already by Milne in 1930 [25], used to determine the energy spectra of the Schrödinger electron [26, 27] the works widely known (with 269, 462 and 651 citations respectively) in an epoch when the computer techniques were not yet available. In 1950 the same equation was again discovered in a short (1/2 page) note by Pinney [28], without presenting any application, but since that time (perhaps in honor of Pinney) the eq. (11) was known as the Milne-Pinney equation. Yet, the *quid pro quo* does not end up here.

After the work of Maamache [29] it was suddenly discovered that the Ukrainian mathematician Ermakov from Kiev University, has written around 1880 a handbook for students presenting the transformed versions the 2-nd order ordinary differential equation \( \frac{d^2}{dt^2} \rho + \beta(t) \rho = 0 \). One of them was precisely the Milne-Pinney eq. (11) used later by Lewis and Riesenfeld. Of course, Ermakov could not know that he made a work in the Schrödinger’s quantum mechanics; neither Milne, Pinney, Lewis and Riesenfeld could know that in some distant past, behind the curtain of the Cyrillic alphabet, somebody had already written the eq. (11). Soon, a sequence of papers started to "rewrite the past" (*not* in the sense of Orwell [30], since Ermakov indeed existed!). Now, however, the name of Milne completely disappeared, though Pinney curiously survived: in a recent trend of publications the authors tell about "Ermakov-Pinney" equation. Moya and Leach [31] tell delicately that the Ermakov invariant was "revisited" by Lewis in 1967 and hence, they call it the "Ermakov-Lewis invariant". Yet, in all this labyrinth of results, the inspirations of Lewis and Riesenfeld still occupy a key position. Meanwhile, the dynamical control theory climbed already on different branches.

4 The evolution controlled by sharp pulses

In spite of the development of invariants [19, 20, 31] and the abundant works on group representations, the exact solutions of (2) were still absent. In their 1973 paper [23] Malkin, Man’ko and Trifonov modified the method of [20], developing the step-by-step approximations in terms of the adiabatic invariants. Yet, they admit: "Attractive as these results may seem at first sight, there is, however, one difficulty: The point is that the exact invariants are expressed in terms of the solutions of linear differential equations (5) or (38), but neither (5) nor (38) can be solved exactly for every \( R(t) \) or \( \Omega(t) \)" ([23], p.580, l. 4-9 of Sec.III). It was henceforth obvious that the problem of explicit solutions was still open.
An alternative possibility of driving the quantum states by $\delta(t)$-pulses of the external fields was first considered by Lamb Jr. [32], but the idea was not applied for some years. Yet, an extremely simple class of exact though formal "pulse solutions" of (2) in $L^2(\mathbb{R})$ (though ignoring the paper of Lamb [32]), was obtained in a sequence of our papers [14, 33, 34, 35, 36] by superposing two types of elementary operations: the incidents of free evolution and the effects of the sharp pulses of oscillator potentials. Since it seems that they brought some operational possibilities, let me describe them shortly.

Each free evolution incident in any interval $[t_0, t_1]$ produces the unitary evolution operator $e^{-i\tau p^2/2}$, where $\tau = t_1 - t_0$. In turn, the result of each sudden $\delta$-kick of the quadratic potential $V(q, t) = a\delta(t - t_0)q^2$ (where $a$ is the pulse amplitude) is most easily described by adopting the rectangular $\delta$-model defined by $\delta_\epsilon(t) = 1/\epsilon$ in a narrow interval $[t_0, t_0 + \epsilon]$ and vanishing outside. The evolution in $[t_0, t_0 + \epsilon]$ is then generated by the constant Hamiltonian $H_\epsilon = p^2/2 + a\epsilon q^2$ and the corresponding unitary operator $U_\epsilon = e^{-i\epsilon(p^2/2 + a\epsilon q^2)} = e^{-ia\epsilon q^2/2}$ for $\epsilon \to 0$. In agreement with the Baker formula [37]:

$$e^{\lambda A}Be^{-\lambda A} = B + \lambda[A, B] + \frac{\lambda^2}{2!}[A, [A, B]] + \frac{\lambda^3}{3!}[A, [A, [A, B]]] + \ldots,$$

(12)

which for $[A, [A, B]] = 0$ reduces just to the two first terms, both operations lead to extremely simple transformations of the canonical pair $q, p$. The free evolution incidents generate:

$$e^{i\tau p^2/2} e^{-i\frac{\tau p}{2}} = \left( \begin{array}{cc} q + \frac{\tau p}{2} & \frac{\tau}{2} \\ p & 0 \end{array} \right) \left( \begin{array}{c} q \\ p \end{array} \right)$$

(13)

while the potential shocks:

$$e^{-ia\frac{q^2}{2}} e^{-ia\frac{p^2}{2}} = \left( \begin{array}{cc} q & 0 \\ p - aq & 1 \end{array} \right) \left( \begin{array}{c} q \\ p \end{array} \right)$$

(14)

Within this scheme, an interesting operation is performed by a pair of free evolution steps separated by an oscillator pulse:

$$e^{-i\frac{\tau p^2}{2}} e^{-i\frac{\tau q^2}{2}} e^{-i\tau \frac{p^2}{2}} \equiv F_\tau$$

(15)

It generates:

$$q \to \tau p$$

$$p \to -\frac{1}{\tau}q$$

(16)

which might be called the squeezed Fourier transformation. Curiously, an equivalent operation is performed by:
\[ e^{-i\frac{q}{\tau}} e^{-i\frac{p}{\tau}} e^{-i\frac{1}{\tau}} \equiv F_\tau \]  

(17)

Henceforth the following product of the 6 unitary operations yields the transformation \( q \rightarrow -q \) and \( p \rightarrow -p \) (the parity operator)

\[ e^{-i\tau} e^{-i\frac{q}{\tau}} \cdots e^{-i\tau} e^{-i\frac{q}{\tau}} \equiv P, \]

(18)

whereas the sequence of 12 unitary terms produces an evolution loop [14]:

\[ \underbrace{e^{-i\tau} e^{-i\frac{q}{\tau}} \cdots e^{-i\tau} e^{-i\frac{q}{\tau}}} \equiv 1 \]

(19)

An intriguing property of (19) is that all 6 free evolution exponents arise with the same signs and so do the exponents of the kick operations (a kind of non-perturbative Baker-Campbell-Hausdorff effect [37]). More important aspect of (19) is that it contains the free evolution intervals (see Fig.1). It means that the remaining eleven unitary operations must cause the free evolution inversion:

\[ \underbrace{e^{-i\frac{q}{\tau}} e^{-i\tau} e^{-i\frac{q}{\tau}} \cdots e^{-i\tau} e^{-i\frac{q}{\tau}}} \equiv e^{+i\tau} \]

(20)

Figure 1: The evolution loop formed by 12 elementary evolution operators. The \( \delta \)-pulses of the attractive oscillator potential of amplitudes \( 1/\tau \) are represented by the hexagon vertices’s, while the 6 sides symbolize the \( \tau \)-intervals of the free evolution. Each 3 consecutive operators yield the squeezed Fourier operation (15-17). Each 11 operations (6 consecutive oscillator kicks separated by 5 free evolution intervals) invert the free evolution.
If the idealized pulses could be indeed applied, the effect would be generated for every wave packet independently of its initial shape [14]. (The effect seemed strange and was objected by Marcos Moshinski though accepted by Bernardo Wolf as a case of the Fourier engineering [38]). Notice also that in this way, the loop mechanism (19, 20) predicted a part of the 1990 hypothesis about the quantum time machine [39].

The loop effects can be also caused by elastic pulses with alternating signs. Their basic fragment might be the sequence of 4 operators $S = e^{-i\tau^2/2}e^{-i\alpha^2/2}e^{-i\tau^2/2}e^{i\alpha^2/2}$, represented by the nilpotent matrices $Q = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $Q^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$:

$$s = e^{\tau Q^\dagger}e^{-\alpha Q}e^{\tau Q^\dagger}e^{\alpha Q} = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$$

(21)

The algebraic properties of $S$ depend again on the $\text{Tr} s = 2 - \tau^2\alpha^2$. If $\tau^2\alpha^2 > 4 \Rightarrow \text{Tr} s < -2$, then the matrix $s$ is of the type (III) and the multiple pulse repetitions generate an unstable motion of squeezing type. However, if $0 \neq \tau^2\alpha^2 < 4 \Rightarrow -2 < \text{Tr} s < 2$ the matrix $s$ is of type (I) and the repeated pulses produce a confined motion, including the possibility of the evolution loops [33, 15]. One of simplest cases occurs for $\tau^2\alpha^2 = 2 \Rightarrow \text{Tr} s = 0$; the Hamilton Cayley eq. for $s$ then implies $s^2 = -\mathbb{1}$ $\Rightarrow S^2 \equiv -1$ and so, the pulse pattern $S$, if repeated, creates an octagonal evolution loop [33], in which the sum of the oscillator pulses cancels, but the effects don’t (see Fig.2):

$$[e^{-i\sqrt{2}\tau^2/2}e^{-i\alpha^2/2}e^{i\sqrt{2}\tau^2/2}e^{-i\tau^2/2}]^4 \equiv 1$$

(22)

The possibility of non-adiabatic loops generated by time dependent oscillator forces was noticed already in 1970 by Malkin, Man’ko and Trifonov [40], though without elaborating the dynamical consequences. Yet, as shown subsequently, the existence of such phenomenon leads to a certain operational hypothesis. It states that one of the simplest ways to control the dynamical evolution consists in generating a closed evolution pattern (evolution loop) and then considering their perturbed or deformed versions [14, 33, 34]. If the unperturbed evolution is represented by a certain family of unitary operators $U_0(t)$, then the perturbed evolution operators split into $U(t) = U_0(t)W(t)$, where $U_0(t)$ represent the basic dynamical process and $W(t)$ is the correction (the evolution in the interaction frame). If now at some moment $T$ the basic evolution closes to a loop $U_0(T) \equiv 1$, then the full evolution reduces just to the pure deformation, $U(T) \equiv W(T)$, see Fig.3, in general, much easier to manipulate by the external fields [14, 33, 34, 15]. The most elementary evolution loops occur in the time independent oscillator potentials, leading e.g., to the non-demolishing quantum measurements [41], but do not exhaust the manipulation techniques.

In fact, the same mechanism predicts an ample family of elementary effects such as the 'time squeezing', i.e., accelerating or slowing down the free evolution [33, 34]. Moreover,
Figure 2: The 8 intervals of free evolution interrupted by 4 attractive and 4 repulsive oscillator pulses close the dynamical process to an octagonal evolution loop. Similarly as for Fig.1, the incomplete fragments of the process generate the free evolution inversions.

Figure 3: The applications of an evolution loop. The basic and perturbed evolution processes are represented by $U_0(t)$ and $U(t) = U_0(t)W(t)$, where the $W(t)$ is the *evolution in the interaction frame*. If for some $T$, the $U_0(T)$ closes to the evolution loop, $U_0(T) \equiv 1$, then the whole process reduces to $W(T)$ alone, the *precession operator*, sensitive to manipulation programs.
some simple, asymmetric sequences of the oscillator pulses can produce the squeezing and/or magnification of canonical variables. The most elementary such effects can be produced by two different *squeezed Fourier* operations, \( F_\alpha \) and \( F_\beta \) [15]:

\[
F_\alpha F_\beta \rightarrow \begin{pmatrix}
0 & \alpha \\
-\frac{1}{\alpha} & 0
\end{pmatrix} \begin{pmatrix}
0 & \beta \\
-\frac{1}{\beta} & 0
\end{pmatrix} = \begin{pmatrix}
-\frac{\alpha}{\beta} & 0 \\
0 & -\frac{\beta}{\alpha}
\end{pmatrix} = \begin{pmatrix}
-\sigma & 0 \\
0 & -\frac{1}{\sigma}
\end{pmatrix}
\]  

(23)

where \( \sigma = \frac{\alpha}{\beta} \). Some of these phenomena were independently observed in [42, 43]. An ample collection of most general squeezing effects was described by Dodonov [44]; the manipulations by complex Hamiltonians were also considered [45, 46, 47, 48, 49]. Some “exotic” effects are described in [15]. All this does not yet exhaust the early history.

5 The Optical Prehistory

While the oscillator kicks have some aspects of Platonic ideas, it was discovered by Wolf [17, 50] that their empirical equivalents were known since long time in geometrical optics describing images on the optical bench. In particular, the \( 2 \times 2 \) matrix transformations (13, 14, 23) of the canonical variables \( q, p \) correspond to the application of some typical optical instruments (microscopes, telescopes etc). Here, \( q \) is the distance of a light ray from the bench axis \( z \) and \( p = c \sin \theta \), where \( \theta \) is an angle between the light ray and the bench axis, \( c \) meaning the light velocity (or simply \( p = \sin \theta \), in dimensionless variables where \( c = 1 \)). The matrix (13) then describes the propagation of images formed by the light rays on the planes orthogonal to the bench axis. In turn, (14) describes the action of a thin optical lens placed on the bench, the amplitude \( a \) meaning the (positive or negative) lens curvature [51]). A slightly different optical representation of symplectic matrices is proposed by Hiley [52], who adopts (the dimensionless) \( p = \theta \), perhaps thinking about the spherical screens. The most detailed and complete report on the optical interpretation of the symplectic matrices is given by Collins [53].

Both interpretations have some advantages. Thus, what in the dynamical language is an *evolution loop* (useful for the manipulation algorithms), in optical terms is the simple reproduction of an optical image. The squeezing/amplification mechanisms produced by two or more oscillator shocks in quantum mechanics are easily interpretable as the applications of the microscope (or telescope) in geometrical optics. (So, in a sense, the effects of the oscillator kicks were known already to Galileo). Moreover, what was not so easy to predict in the dynamical language, i.e., the asymmetry of \( \beta(t) \) needed to produce the squeezing, is immediately obvious at the optical level. (In fact, the symmetric apparatus could not produce amplified or reduced images).

Both have also some limits. In the area of optics, the applications of too close lenses with too big \( |a| \) would mean that a part of one lens (if not the whole) must overlap with the interior of the other. Some more exact equivalents of the elastic pulses (14) appeared...
later in works on optical signals in dispersive fibers [54] (more recently, c.f. also [55]). All this does not yet explain how to create the elastic pulses in the ion traps. Of course, an ideal application of the δ kicks is practicably impossible. Forgetting even about the finite resistance of the trap walls, no infinite energy shocks can be engineered. Yet, despite all their imperfections, the idealized dynamical operations might be of interest for some unfinished fundamental discussions.

6 The uncertainties about the Time-Energy uncertainty

Indeed, it is enough to remind the polemics about the "time-energy uncertainty principle". The first objections against the (too narrow) interpretations of Landau and Peierls [56] supporting this principle, appeared already in the study of Aharonov and Bohm in 1961 [57] (see also [58]). Further counter-arguments were collected by Aharonov, Massar and Popescu [11] who argue that an arbitrarily exact measurement of the energy of a quantum system can be performed in an arbitrarily short time, provided that the Hamiltonian is known and "the measurement is brutal". Their example was the spin measurement. As it seems, the pulse patterns of Sec.4 bring the next illustration to the same idea for the continuous canonical variables.

Indeed, suppose a δ-pulse of the oscillator potential surrounded by two short intervals of the free evolution performs the "squeezed Fourier transformation" (15) of a free particle with an energy \( \frac{p^2}{2m} \). After the operation, the unknown (classical or quantum) momentum \( p \) is converted into the new particle position \( \tilde{q} = \tau p \). Could such transformation be produced, it would be no longer necessary to detect directly the particle energy, e.g., by observing its collision with a heavier microobject [56]. It would be enough to measure its new position \( \tilde{q} \). Whatever the technical difficulties, there is no fundamental law which would forbid to determine \( \tilde{q} \) in an arbitrarily short time.

An objection can still arise, that our prescription (15) offers an inconvenient relation between the final \( \tilde{q} \) and the initial momentum \( p \). The \( \tilde{q} = \tau p \) implies \( \Delta \tilde{q} = \tau \Delta p \), so if the operation time \( \tau \) is very short, then little errors in \( \tilde{q} \) will correspond to much greater errors in \( p \). It looks almost as a revenge of the time-energy uncertainty. Yet, it is not. The parameter \( \tau \in \mathbb{R} \) marks the time of an external operation (i.e., an external time of Aharonov and Bohm [57]), while \( \Delta \tilde{q} \) is not limited by any universal constant. Moreover, (still accepting the δ-pulses), the dynamics of (2) can offer also much better measurement methods which we proposed to call the "Fourier microscopes" [15]. A simple option can be to apply three elastic shocks divided by two free evolution incidents (i.e., an incomplete case of the triangular evolution loop [33]):
\[
\begin{equation}
\begin{aligned}
u &= e^{-aQ}e^{\tau Q}e^{-bQ}e^{\tau Q}e^{-cQ} = \begin{pmatrix}
1 & 0 \\
-a & 1
\end{pmatrix}
\begin{pmatrix}
1 & \tau \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-b & 1
\end{pmatrix}
\begin{pmatrix}
1 & \tau \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-c & 1
\end{pmatrix} \\
&= \frac{1}{2} \left( a\tau + b\tau + c\tau + (a\tau)(b\tau) + 2(a\tau)(c\tau) + (b\tau)(c\tau) + (a\tau)(b\tau)(c\tau) \right) \begin{pmatrix}
1 + b\tau + 2c\tau + (b\tau)(c\tau) & \tau(2 - b\tau) \\
(1 + 2a\tau + b\tau + (a\tau)(b\tau)) & 1 + 2a\tau + b\tau + (a\tau)(b\tau)
\end{pmatrix}
\end{aligned}
\end{equation}
\]

Figure 4: Depending on the values of the elastic pulses \(a, b, c\) the incomplete version of the triangular evolution loop can generate the distorted cases of the inverted free evolution or else, the cases of Fourier microscope with an arbitrary magnification coefficient achieved in an arbitrarily short time.

The above simple pattern offers a variety of dynamical operations. By asking \(u_{11} = u_{22} = \pm 1\), one can obtain the free evolution with ‘squeezed’ or even inverted time. In turn, by asking that \(u_{11} = u_{22} = 0\), one obtains the ‘squeezed Fourier’ with \(\Gamma = |u_{12}| = |\tau(2 - b\tau)|\) arbitrarily great, i.e., a “Fourier microscope” yielding the final position \(\hat{q} = \Gamma p\) large enough to assure very little errors of \(\Delta p = \frac{1}{\Gamma} \Delta \hat{q}\) even if the measurement of \(\hat{q}\) is far from perfect. While technical difficulties still exist, they do not seem a fundamental obstacle. This appears to confirm the idea of the “brutal measurement” in an arbitrarily short time [58].

**Is there the "time operator"?** The whole discussion could not avoid an interrelated question, as to, whether the quantum theories should adopt the *time of event* as a legitimate quantum observable, represented by a certain hermitian operator \(t\) in a Hilbert space, a canonical counterpart of the energy \(E\) with \([t, E] = i\hbar\). The idea, though, was not so easy
to accept. As many other controversies it produced an enormous sequence of papers, but no final solution. Apart of difficulties with defining consistently the wave function on the time axis, the main obstacle was the theorem of Pauli, stating that then \( E \) must have a continuous, displacement invariant spectrum extending over the entire \( \mathbb{R} \).

Trying to avoid the obstacle, Kijowski [59] proposed in 1974 a simplified 1D model replacing the free energy operator \( E = \frac{p^2}{2} \) by \( \tilde{E} = \frac{|p|^2}{2} \) with an infinite spectrum covering \( \mathbb{R} \), and then he proved a theorem finding the canonically conjugated 'time' \( \tilde{t} \). The further development has again some aspects of a historical anecdote. The construction of Kijowski remained almost unperceived for more than 2 decades (Not so surprising. The “publish or perish” culture produces such mountains of publications, that neither the XXII century archeologists will have patience to read all of them!). Yet, around 2000, a group of authors noticed that Kijowski results may be useful to determine the time of arrival into a waiting detector (or screen) for the wave packets which belong only to the positive (or only to the negative) subspace of the momentum operator. The observation produced a trend of papers describing either the wave packets localized initially to the left of the detector and moving to the right (the 'right movers'), or vice versa [60, 61]. Yet, to restrict the problem to the 'left component' is not the same as to consider the 'right movers' (and vice versa). Moreover, the method did not describe the interference of both, left and right components (if they both exist), converging from opposite sides to the detection point (the difficulty, as far as I know, discovered first by Leavens [62, 63, 64, 65, 66]).

In spite of subsequent works on POV measures and some new but still partial results [67, 68, 69], it looks that the idea of time-operator has too many gaps to grant the time-energy uncertainty (except if some additional limitations of quantum measurements proved real [70]).

Here again a historical anecdote. In the summer of 2001 we have finished with Gabino Torres Vega an article containing those and other critical remarks about the time operator. Our text was ended up with a little story which (we thought) exclude completely the existence of an orthodox quantum mechanical time operator at least for a class of physical phenomena, such as the radioactive decay of unstable atoms. We argued:

"Note that this negative result might imply some good news. Indeed, imagine a simple experiment which consists in registering the time moment in which an unstable particle decays. The result is a real number (the decay time); yet, an attempt to describe it in terms of the orthodox scheme (i.e. as an eigenvalue of a self-adjoint time operator) would lead to a wrong conclusion: about the existence of initial states for which the moment of the decay can be predicted with certainty! Should this be true, the consequences would be quite dramatic. They would include a suspense story about a suitcase full of radioactive atoms, smuggled safely through the airport security, with all atoms programmed to decay tomorrow! So, perhaps we should not regret that the time of events does
After submitting the work to Phys.Rev.A, we thought that the truth of our little story must conquer the referees. However, it was not so. Our article circulated until November, meeting finally some counter-anecdotes. Since the discussion is fundamental, some of them deserve to be cited. The First Referee has written:

“... As a general comment, I also have problems to be sympathetic with the general negative tone of this article. While criticism and finding difficulties are inherent (...) for the progress of science, as much or even more effort should be put in finding the solution to the problems. This positive component (and effort?) is essentially absent in this article (...). The article would much improve if it could balance negative aspects with something positive. As a minor point, the final anecdote in the paragraph on the “good news” is possibly not a very happy idea considering the present situation of international affairs. I admit of course that these are largely matters of taste”.

The Second Referee wrote:

“... It is problematic that the authors limit themselves to a sharp criticism of the Kijowski proposal without delivering any ideas on how to actually construct a feasible operator of time. However, this referee liked the fact that the authors bothered to give numerical examples illustrating the action of Kijowski’s operator (this down-to-earth approach is certainly required to achieve some progress in the field). But I also strongly oppose the aggressive language used throughout this contribution”.

The first objection could be answered on logical grounds. Our paper was dedicated to show that the “time operator” does not exist. So, the request that we should complete our argument by constructing the time operator (both referees) was indeed demanding too much! Besides, the first referee objection against our anecdote seemed incomprehensible: our argument was correct! But suddenly, we understood the chronological complication. Our paper was send to Phys.Rev.A at the beginning of August 2001, then it started to circulate, and the referee must have obtained it in October or November 2001. He thought that we were joking of September 11 terrorist attack! We understood that the defense might be difficult and we sent the paper to the open access Journal in Lodz University in Poland, see [71].

Another slight surprise in this discussion was, however, the voice of the 3-rd referee, which was Kijowski himself. He defended his result, as a mathematical theorem, but not necessarily as a physical fact. He quoted the following fragment of his 1999 paper [72]:

not obey the axioms of Dirac and v. Neumann?”
“Although there is no room to modify the mathematically unique definition of probability (...) within the standard mathematical framework of Quantum Mechanics (spectral measures, self-adjoint operators etc) the problem is still open from the physical point of view.”. This may be taken as a hint, that the ‘standard framework’ is not an absolute truth, but still requires careful studies. But then, how much we really know?

7 Are the operations of $\delta$-pulses real?

The operations considered until now concern the states in 1 space dimension and are singular. Their generalizations in 2D and 3D, employing the harmonic $\delta$-pulses (without an unphysical charge density of [20]), were designed by Fernandez, who discussed the equivalents of all previously predicted effects, including the comments on the time-energy uncertainty. The paper was ‘softly’ rejected by a referee in Phys.Rev.A who did not question the mathematical results but decided that the $\delta$-pulses are not acceptable, as they do not reflect the empirical reality required in Phys.Rev.A (yet, the papers on kicked rotators were accepted without any problems, perhaps by referees impressed by the luggage of special functions?). Distasted by the suspicious attitude in Phys.Rev.A, Fernadez made a quick decision to publish the results in Nuovo Cimento [73], where they are now waiting for some new followers of the control problem.

But, what’s about the reality of $\delta$-pulses? They don’t exactly exist in nature. Yet, their effects could be idealized forms of some real phenomena. In fact, the strong, but bounded fields (e.g. in form of rectangular steps) allowed already to design a family of dynamical operations generating the “distorted free evolution”, rigid displacement of the wave packet, etc. The possibility of obtaining still more general operations for softly varying, differentiable fields (without any steps!) was considered in [74, 34], then confirmed by computer studies [75, 76, 77, 33, 34, 35, 78]).

Here, let me also mention some manifestly unphysical results, but distinguished by mathematical elegance. I am referring to papers Caldirolla [79], Kanai [80] and specially Baseia et al. [81, 82] dedicated to the time dependent mass. Note an essential difference: while the effective mass depending on space points can describe the average particle propagation in some inhomogeneous media [83, 84, 85], the Schrödinger’s particles with the time dependent mass do not seem to have any empirical equivalent. Yet, the Caldirolla-Kanai Hamiltonian with an exponentially varying mass:

$$H(t) = \frac{1}{2m(t)} p^2 + \frac{m(t) \omega^2}{2} q^2, \quad m(t) = m_0 e^{\lambda t}$$  \hspace{1cm} (25)$$

was persistently used as a description of a quantum system affected by friction: a conviction hardly acceptable in spite of the work of Bateman [86].

Indeed, the evolution operations defined by any Hamiltonian with a time dependent mass (in classical or quantum case) cannot describe the friction, since they yield always
the invertible transformations of pure into pure states. For quadratic Hamiltonians of two canonical variables, this is typically either the trapped motion, or the squeezing of one canonical variable at the cost of another [81, 87], but not a dissipative process, which needs a more general evolution laws for mixed ensembles (see Lindblad [88], Gisin and Percival [89]). This fact found a nice illustration in [82], for the Hamiltonian (25) with $m(t)$ forming a sequence of sharp, rectangular steps. Even if operationally impossible, it offers a clean mathematical image of a squeezing which affects just a pair of canonical variables $q,p$. Moreover, the authors are able to design a programme of steps which invert the operation. The composition of both closes up to a clean evolution loop, illustrating nicely the absence of any dissipative process for Hamiltonians of type (25). The story does not end up here.

Some surprise was, that the merits of the evolution loops were finally noticed in Phys.Rev.Lett., in 1999 work of Harel and Akulin on the control problem in finite dimensional spaces [45]. They wrote:

Our approach is based on the following idea: We should first find an “identity map $T_1,T_2,...,T_{N^2}$ for the particular case $\hat{H}_{\text{eff}} = 0$. If we succeed, the operator of the evolution over the period $T = \sum_{n=1}^{N^2} T_n$ is the identity transformation

$$\hat{U}(T) = e^{-i\hat{B}T_{N^2}}e^{-i\hat{A}T_{N^2-1}}...e^{-i\hat{B}T_2}e^{-i\hat{A}T_1} = 1$$

For small variations (...) the expansion of the exponential factors (...) yields (...) This repeats almost (though not completely) our early evolution loops typically composed of two alternative operations. However, since Harel and Akulin work in finite dimensional Hilbert spaces and their Hamiltonians are not quadratic, it looks that their inspiration was independent. Quite simply, in the Platonic heaven some predatory ideas are circulating looking all the time for some new victims! Anyhow, our loop doctrine was recognized by a group of quantum control in Texas [47]. In further development, the quantum control by sharp pulses have found new forms and new leaders (or victims?). The pulses have now the form of general Hamiltonians interrupting the continuous evolution, called decouplers [46]; the evolution affected by sequences of their sharp kicks are studied for the spin systems, as one of the most promising quantum control methods [45, 46] (obeying an equally dangerous Platonic idea, known as the “bam-bam-control”), though less violent methods are also considered [48, 90]. The question arises, what is the future of their and our approaches? The ‘pulse operations’ of course must have some natural limits. Hence, the soft alternatives of the potential kicks can be no less interesting. This brings us back to the manipulation problems for the quadratic Hamiltonians (2).
8 The soft operations

As recently found, the control operations are significantly simplified if the field amplitude \( \beta(t) \) is symmetric with respect to the center of the operation interval \([-T, T]\). We shall show now, that in the symmetric generation program the matrices (3-9), together with the corresponding 'driving amplitudes' \( \beta(t) \) can be expressed exactly (without perturbations!) in terms of a single real function which, apart of details, may be fixed at will [15]. Without pretending too much, it facilitates enormously the task of programing the dynamical operations. Indeed, one has:

**Proposition 2.** Consider a nontrivial operation interval \([-T, T]\) with the quadratic potential \( \beta(t) \frac{q^2}{2} \) and suppose, \( \beta(t) \) is bounded, piecewise continuous and symmetric, \( \beta(t) = \beta(-t) \). Then the evolution matrix \( u(t, -t) \) and the driving amplitude \( \beta(t) \) in the expanding family of intervals \([-t, t]\) (where \( t \leq T \)) can be written explicitly in terms of \( \theta(t) = u_{12}(t, -t) \), which may be choosen at will everywhere except its zero points. Moreover, whenever \( u = u(t, -t) \), for \( t \in [0, T] \), reaches the stability threshold with \( \text{Tr} u = \pm 2 \), then either \( u \) or \(-u\) adopts one of the forms (13) or (14), imitating the results of a simple or distorted free evolution, or else, of a sharp oscillator kick.

**Proof.** In what follows, whenever there will be no reasonable doubt, we shall simplify the notation, writing just \( u(t) \) instead of \( u(t, -t) \) and \( u_{kl}(t) \) instead of \( u_{kl}(t, -t) \). Due to the symmetry of the Hamiltonians \( H(t) = H(-t) \), the unitary evolution operators \( U(t, -t) \) for the expanding intervals \([-t, t]\) satisfy:

\[
i \frac{d}{dt}U(t, -t) = H(t)U(t, -t) + U(t, -t)H(t)
\]

(26)

Hence, the corresponding evolution matrix \( u(t) = u(t, -t) \) fulfills:

\[
\frac{du}{dt} = \Lambda(t)u + u\Lambda(t)
\]

(27)

Since \( \Lambda(t) \) is given by (5), this becomes

\[
\frac{du}{dt} = \begin{pmatrix}
    u_{21} - \beta u_{12} & Tr u \\
    -\beta Tr u & u_{21} - \beta u_{12}
\end{pmatrix}
\]

(28)

\[
= (u_{21} - \beta u_{12}) I + Tr u \begin{pmatrix}
    0 & 1 \\
    -\beta & 0
\end{pmatrix}
\]

Therefore,

\[
\frac{d}{dt}(u_{12} u_{21}) = Tr u (u_{21} - \beta u_{12}) = Tr u \frac{1}{2} \frac{d}{dt}Tr u = \frac{1}{4} \frac{d}{dt}(Tr u)^2
\]

(29)

and integrating:

\[
\frac{d}{dt} \left[ u_{12} u_{21} - \frac{1}{4} (Tr u)^2 \right] = 0 \Rightarrow u_{12} u_{21} - \frac{1}{4} (Tr u)^2 = C = \text{const.}
\]

(30)
To determine $C$ it is enough to take $t = 0$. The initial condition $u(0, 0) = I$ then tells that $C = -1$, and so:

$$u_{12}u_{21} = \frac{1}{4} (\text{Tr} u)^2 - 1 \quad t \in [-T, T] \quad (31)$$

The above eqs. (28-31) provide an elementary solution of the inverse evolution problem, permitting to reconstruct the entire evolution matrices $u(t, -t)$ together with the driving pulse $\beta(t)$ in terms of one function $\theta = u_{1,2}(t)$ in the expanding intervals $[-t, t]$.

Indeed, (28) implies $\frac{du_{11}}{dt} = \frac{du_{22}}{dt} = u_{21} - \beta u_{12}$ and since the initial condition at $t = 0$ is $u_{11}(0) = u_{22}(0) = 1$, then $u_{11} = u_{22}$ in all intervals $[-t, t]$ ($t \leq T$). In view of (28) this means that $u_{11}(t) = u_{22}(t) = \frac{1}{2} \text{Tr} u = \frac{1}{2} \theta'(t)$. In turn, since $u$ is symplectic, then $(\frac{1}{2} \theta')^2 - \theta u_{2,1} = 1$ and the remaining matrix element $\alpha = u_{2,1}$ is determined as:

$$\alpha = \left(\frac{\frac{1}{2} \theta'}{\theta}ight)^2 - 1 \quad (32)$$

with the pulse shape $\beta(t)$ defined in terms of $\theta$ as well:

$$\beta = -\frac{\theta''}{2\theta} + \left(\frac{\frac{1}{2} \theta'}{\theta}\right)^2 - 1 \quad (33)$$

If no singularity of $\beta$ occurs for $t = 0$ (or in other points where $\theta$ might vanish) these expressions grant (28), yielding an exact solution of the symmetric evolution problem for the family of the expanding intervals $[-t, t]$. □

As one can notice, whenever the symmetric evolution matrix $u(t) = u(t, -t)$ reaches the threshold values $\text{Tr} u = \pm 2$ (the case II of our classification), (31) implies that either $u_{1,2}$ or $u_{2,1}$ (or both) must vanish, leading to the canonical transformations (28) which simulate the oscillator kicks, the incidents of the free evolution with distorted, squeezed or negative time, or just one of the evolution loops (c.f. [87, 91]), all of them with or without the simultaneous parity transformation [87, 91]. The operations in the stability areas might provide also the softly achieved “squeezed Fourier” operations, including the “Fourier microscopes” - which can be generated by strong though differentiable amplitudes $\beta(t)$ in arbitrarily short time intervals. An essential consequence of the soft imitations is that they bring several ideas one step closer to the laboratory experience. This includes our symbolic results in [14, 33, 34], then Fernandez 3D proposals [73] and the critiques of the time-energy uncertainty [11].

To complete this report, let me describe a kind of NO-effect: a driving amplitude which does not vanish in $[-T, T]$, though vanishes outside, and moreover, it represents an “hidden pulse” completely undetectable out of the time interval in which it is acting. This happens always if the action of a (symmetric) $\beta(t)$ in an interval $[-T, T]$ produces a matrix $u = u(-T, T)$ with $u_{11} = u_{22} = \theta'(T) = 1$ and $u_{12} = \theta(T) = 2T$, i.e., the $u$ will mimic exactly the free evolution case, in spite of $\beta \neq 0$ in $[-T, T]$. 

Examples of several invisible pulses obtained for simple polynomial models $\theta(t) = 2t - \theta_3 t^3 - \theta_5 t^5 - \theta_7 t^7 - \theta_9 t^9 - \theta_{11} t^{11}$ are represented in Fig. 5. Our mathematical algorithm used to construct them is not identical, though interrelated with the invisibility studies employing the complex variables [92]. The importance of the subject may extend from the single particle behavior, to the particle tunneling in solid state wells, or even to the apparently unrelated phenomena. But what is the precision of our manipulation effects?

![Figure 5: The four “invisible pulses” in the expanding [-T,T] intervals for T=1/2, 3/4, 1 and 5/4. While they can be arbitrarily strong in their action intervals, their consequences are undetectable outside.](image)

9 Are there natural precision limits?

The difficulties are still abundant. If accepting the validity of the present day quantum field theories, then the described unitary operations are merely approximate results which may be easily spoiled by “hostile” external environments. In fact, the use of the quantum mechanical quadratic Hamiltonians (1) depends on the semiclassical picture interpretable as the low frequency approximation in which the charged microparticle interacts with a huge ensemble of extremely tiny quanta. It suggests that the driving pulse $\beta(t)$ should be composed only of weak and long waves (of course if the role of the Fourier analysis was not exaggerated in quantum theories.). The semiclassical models actually used include

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3Indeed, the 2-nd order differential equations related to Hamiltonians (2) appear also in astronomy and cosmology; see [94, 95, 7, 97, 98, 15].
the microwave frequencies, but what about the infrared components? Our description would be obviously broken for microparticles interacting not with some coherent clouds of practically imperceptible quanta, but with one or more “high energy quanta”, leading to the scattering picture rather than the smooth unitary evolution. Some other aspects of our solutions may also need a critical revision.

The exact solutions (28-31) describe the particle evolution driven by time dependent $\beta(t)$, basically applicable in the ion traps, but they do not take into account the packet reflections from the trap walls. To neglect them, the trap should be ample enough. Yet, if too huge, the variable potentials on the trap walls will propagate to the trap interior with relativistic delays, causing the timing errors $\delta t = |\Delta x|/c$ of the field values, (where $|\Delta x|$ are the distances between the points in the trap interior). In the dimensionless time variable $\frac{t}{\tau}$ used in the expression (2) (where $\tau$ is a certain conventional time unit), the delays would be therefore $\delta\left(\frac{t}{\tau}\right) = \frac{|\Delta x|}{\tau c}$. Small indeed, perhaps allowing the traps of the size of few cm (or more?).

A difficult problem is, however, how to assure that the scalar potential on the trap surfaces will evolve exactly according to $\phi(x, t) = \beta(t)\phi_0(x)$ where $\phi_0(x)$ represents the static case. This would mean that the little potential changes $\delta\phi = \delta\beta(t)\phi_0(x)$ should arrive at the surface points in a perfect synchronization, the demand not so easy to satisfy [15]. In real experiments, the potential changes are typically implemented just at some particular surface points; they need some time, extremely short, but still non-vanishing, to propagate all over the trap walls. To get rid of this effect for wider ion traps, some method of introducing the simultaneous potential changes in a certain (dense?) net of the wall points would be needed, though the technology is still missing.

An interesting remark is, however, that if only such simultaneity could be achieved on the trap surfaces, then the relativistic (scalar) potential in the trap interior, obtained by substituting $t$ in $\beta(t)$ by the retarded time $t_r = t - \frac{|x-x'|}{c}$[99] would not produce the retarded corrections of the orders of magnitude $\approx \frac{|\Delta x|}{\tau c}$ (post-Newtonian), but paradoxically, only the corrections of the 2-nd order, $\delta^2 t_{r} \approx \frac{|\Delta x|^2}{(\tau c)^2}$, post-post-Newtonian in the EIH hierarchy (see [15, 93]). The problem, of how to approximate this situation is still open. Yet, it seems that the classical electrodynamics does not preclude the existence of such retarded fields. As an example, it might be interesting to consider a time dependent field with $\beta(t)$ generalising the traditional Paul’s formula [100] in the simplest, cylindrical trap. Assuming the absence of charge and current densities in the trap interior, a simple (though idealized) expressions for the softly changing electromagnetic fields can be constructed by applying the step by step approximations, independent of any particular gauge. In particular, for a cylindrical Paul’s trap with perfectly hyperbolic surfaces, one might define a sequence of vector-functions $E_1, E_3, E_5,...$ and $B_2, B_4, B_6, ...$ such that: $E_1 = \nabla \times B_2$, $B_2 = -\nabla \times E_4$, $E_3 = \nabla \times B_4$, $B_4 = -\nabla \times E_5$, $E_5 = \nabla \times B_6$, ...; and henceforth: $E_1 = \Delta E_3$, $E_3 = \Delta E_5$, ... and $B_2 = \Delta B_4$, $B_4 = \Delta B_6$, ... or explicitly:
\[
E_1 = \begin{pmatrix} x \\ -y \\ 0 \end{pmatrix} \quad E_3 = \begin{pmatrix} \frac{1}{2}xy^2 + \frac{1}{3}x^3 \\ + \frac{1}{4}x^2y - \frac{1}{3}y^3 \\ 0 \end{pmatrix} \quad E_5 = \begin{pmatrix} \frac{1}{8}x^3y^2 \\ -\frac{1}{8}y^3x^2 \\ 0 \end{pmatrix} \ldots
\]
\[
B_2 = \begin{pmatrix} 0 \\ 0 \\ xy \end{pmatrix} \quad B_4 = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{12}x^3y + xy^3 \end{pmatrix} \quad B_6 = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{8}x^3y^3 \end{pmatrix} \ldots
\]

which can be used to construct the electric and magnetic fields

\[
E(x, y, t) = \beta(t)E_1 + \frac{1}{c^2}\beta''(t)E_3 + \frac{1}{c^4}\beta'''(t)E_5 + \ldots \quad (34)
\]
\[
B(x, y, t) = \frac{1}{c}\beta'(t)B_2 + \frac{1}{c^3}\beta'''(t)B_4 + \frac{1}{c^5}\beta'''(t)B_6, \ldots \quad (35)
\]

satisfying the Maxwell-Faraday equations, \( \nabla \times B = \frac{1}{c}\frac{\partial E}{\partial t} \), \( \nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t} \), where the principal part would be \( \beta(t)E_1 \), i.e., an electric analogue of Paul’s field driven by an arbitrary \( \beta(t) \) (instead of the trigonometric \( \beta = \beta_0 + 2\beta_1 \cos \omega t \) leading to the Mathieu functions [100]). In turn, \( \frac{1}{c}\beta'(t)B_2 \) is the main magnetic part which must unavoidably appear in any cyldinic trap. It will generate the contributions \( \sim \frac{1}{c^2} \) (post-post-Newtonian) to the Lorentz force, acting on the charged particles. The whole rest, are just the magnetic and electric relativistic corrections of increasing powers in \( \frac{1}{c^k} \). Yet, the expressions (34) and (35) cannot represent the exact time dependent fields of the cyldinic ion trap. since they ignore the boundary conditions on the trap surfaces. If taken to the letter, they can be just a hint, that the simultaneity of the fields on the trap surfaces cannot be improved beyond the \( \frac{1}{c^2} \) order of EIH [93].

So, what is the sense of our constructions? Are they a kind of a daydream? Or perhaps, an incomplete prediction of the approaching steps of quantum control?

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