Abstract. This note briefly reviews the Mirror Principle as developed in the series of papers [19][20][21][22][23]. We illustrate this theory with a few new examples. One of them gives an intriguing connection to a problem of counting holomorphic disks and annuli. This note has been submitted for the proceedings of the Workshop on Strings, Duality and Geometry the C.R.M. in Montreal of March 2000.
1. Some Background

In the aforementioned series of papers we develop the mirror principle in increasing generality and breadth. Given a projective manifold $X$, mirror principle is a theory that yields relationships for and often computes the intersection numbers of cohomology classes of the form $b(V_D)$ on stable moduli spaces $\bar{M}_{g,k}(d, X)$. Here $V_D$ is a certain induced vector bundles on $\bar{M}_{g,k}(d, X)$ and $b$ is any given multiplicative cohomology class. In the first paper [19], we consider this problem in the genus zero $g = 0$ case when $X = \mathbb{P}^n$ and $V_D$ is a bundle induced by any convex and/or concave bundle $V$ on $\mathbb{P}^n$. As a consequence, we have proved a mirror formula which computes the intersection numbers via a generating function. When $X = \mathbb{P}^n$, $V$ is a direct sum of positive line bundles on $\mathbb{P}^n$, and $b$ is the Euler class, a second proof of this special case has been given in [24][5] following an approach proposed in [9]. Other proofs in this case has also been given in [3][8], and when $V$ includes negative line bundles, in [6]. In [20], we develop mirror principle when $X$ is a projective manifold with $TX$ convex. In [21], we consider the $g = 0$ case when $X$ is an arbitrary projective manifold. Here emphasis has been put on a class of $T$-manifolds (which we call balloon manifolds) because in this case mirror principle yields a (linear!) reconstruction algorithm which computes in principle all the intersection numbers above for any convex/concave equivariant bundle $V$ on $X$ and any equivariant multiplicative class $b$. Moreover, specializing this theory to the case of line bundles on toric manifolds and $b$ to Euler class, we give a proof of the mirror formula for toric manifolds. In both [21] and [22], we develop mirror principle for higher genus. We also extend the theory to include the intersection numbers for cohomology classes of the form $ev^*(\phi)b(V_D)$. Here $ev : \bar{M}_{g,k}(d, X) \to X^k$ is the usual evaluation map into the product $X^k$ of $k$ copies of $X$, and $\phi$ is any cohomology class on $X^k$.

For motivations and some historical background of the mirror principle, we refer the reader to the introduction of [19][20].

In section 2, we outline the main ideas of the mirror principle, and explain one of our main theorems. In section 3, we discuss a few examples.

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2. Mirror Principle

For simplicity, we restrict our discussions to the genus zero theory, and refer the interested reader to [22] for a theory of higher genus. Let $X$ be a projective $n$-fold, and $d \in H^2(X, \mathbb{Z})$. Let $M_{0,k}(d, X)$ denote the moduli stack of $k$-pointed, genus 0, degree $d$, stable maps $(C, f, x_1, .., x_k)$ on $X$ [16]. (Note that our notation is without the bar.) By [18] (cf. [4]), each nonempty $M_{0,k}(d, X)$ admits a homology cycle $LT_{0,k}(d, X)$ of degree $\dim X + \langle c_1(X), d \rangle + k - 3$. This cycle plays the role of the fundamental class in topology, hence $LT_{0,k}(d, X)$ is called the virtual fundamental class.

Let $V$ be a convex vector bundle on $X$. (ie. $H^1(P^1, f^*V) = 0$ for every holomorphic map $f : P^1 \to X$.) Then $V$ induces on each $M_{0,k}(d, X)$ a vector bundle $V_d$, with fiber at $(C, f, x_1, .., x_k)$ given by the section space $H^0(C, f^*V)$. Let $b$ be any multiplicative characteristic class [12]. (ie. if $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles, then $b(E) = b(E')b(E'')$.) The problem we study here is to compute the characteristic numbers

$$K_d := \int_{LT_{0,0}(d, X)} b(V_d)$$

and their generating function:

$$\Phi(t) := \sum K_d e^{d\cdot t}.$$ 

There is a similar and equally important problem if one starts from a concave vector bundle $V$ [19]. (ie. $H^0(P^1, f^*V) = 0$ for every holomorphic map $f : P^1 \to X$.) More generally, $V$ can be a direct sum of a convex and a concave bundle.

The rough idea of the Mirror Principle is that the classes the induced bundles $V_d$ on the stable moduli inherit a number of universal structures (ie. exist in all stable map moduli of any projective manifold). These structures combined with the multiplicative properties of the classes $b(V_d)$ give rise to some remarkable quadratic identities. It is often the case (when sufficient symmetry is present on $X$) that these identities are strong enough for a complete reconstruction of the intersection numbers $K_d$. We explain this idea further below without proofs. For details, see [21].

Step 1. Localization on the linear sigma model. Consider the moduli spaces $M_d(X) := M_{0,0}((1, d), P^1 \times X)$. The projection $P^1 \times X \to X$ induces a map $\pi : M_d(X) \to M_{0,0}(d, X)$. Moreover, the standard action of $S^1$ on $P^1$ induces an $S^1$ action on $M_d(X)$. We first study a slightly different problem. Namely consider the classes $\pi^* b(V_d)$ on $M_d(X)$, instead of $b(V_d)$ on $M_{0,0}(d, X)$. First, there is a canonical way to embed fiber products

$$F_r = M_{0,1}(r, X) \times_X M_{0,1}(d - r, X)$$
each as an $S^1$ fixed point component into $M_d(X)$. Let $i_r : F_r \to M_d(X)$ be the inclusion map. Second, there is an evaluation map $e : F_r \to X$ for each $r$. Third, there is a (product of) projective space $W_d$ equipped with an $S^1$ action, and there is an equivariant map $\varphi : M_d(X) \to W_d$, and embeddings $j_r : X \to W_d$, such that the diagram

$$
\begin{array}{ccc}
F_r & \xrightarrow{i_r} & M_d(X) \\
e | & & \downarrow \varphi \\
X & \xrightarrow{j_r} & W_d
\end{array}
$$

commutes. Let $\alpha$ denotes the weight of the standard $S^1$ action on $\mathbb{P}^1$. Applying the localization formula [2][14], this diagram allows us to recast our problem to one of studying the $S^1$-equivariant classes

$$Q_d := \varphi_* \pi^* b(V_d)$$

defined on $W_d$.

The projective space $W_d$ in the commutative diagram above is called a linear sigma model of $X$. They have been introduced in [17] following [25].

**Step 2. Gluing identity.** Consider the vector bundle $U_d := \pi^* V_d \to M_d(X)$, restricted to the fixed point components $F_r$. A point in $(C, f)$ in $F_r$ is a pair $(C_1, f_1, x_1) \times (C_2, f_2, x_2)$ of 1-pointed stable maps glued together at the marked points, i.e. $f_1(x_1) = f_2(x_2)$. From this, we get an exact sequence of bundles on $F_r$:

$$0 \to i_r^* U_d \to U'_r \oplus U'_{d-r} \to e^* V \to 0.$$

Here $i_r^* U_d$ is the restriction to $F_r$ of the bundle $U_d \to M_d(X)$. And $U'_r$ is the pullback of the bundle $U_r \to M_{0,1}(d, X)$ induced by $V$, and similarly for $U'_{d-r}$. Taking the multiplicative characteristic class $b$, we get the identity on $F_r$:

$$e^* b(V) b(i_r^* U_d) = b(U'_r) b(U'_{d-r}).$$

This is what we call the **gluing identity**. This may be translated to a similar quadratic identity, via Step 1, for $Q_d$ in the equivariant Chow groups of $W_d$. The new identity is called the Euler data identity.

**Step 3. Linking theorem.** The construction above is functorial, so that if $X$ comes equipped with a torus $T$ action, then the entire construction becomes $G = S^1 \times T$ equivariant and not just $S^1$ equivariant. In particular, the Euler data identity is an identity of $G$-equivariant classes on $W_d$. Our problem is to first compute the $G$-equivariant classes
$Q_d$ on $W_d$ satisfying the Euler data identity. Note that the restrictions $Q_d|_p$ to the $T$ fixed points $p$ in $X_0 \subset W_d$ are polynomials functions on the Lie algebra of $G$. Suppose that $X$ is a balloon manifold. This is a complex projective $T$-manifold satisfying the following conditions [13]:

(i) The $T$ fixed points are isolated.

(ii) Let $p$ be a $T$ fixed point. Then the $T$ weights $\lambda_1, ..., \lambda_n$ of the isotropic representation on the tangent space $T_pX$ are pairwise linearly independent. We further assume that the moment map is 1-1 on the fixed point set.

In this case, the classes $Q_d$ are uniquely determined by the values of the $Q_d|_p$, when $\alpha$ is some scalar multiple of a weight $\lambda_i$. These values of $Q_d|_p$, which we call the linking values (see [21] for precise definition), can be computed explicitly by exploiting the moment map [1][11] as well as certain structure of a balloon manifold.

**Theorem 2.1.** [21] The equivariant classes $Q_d = \varphi_*\pi^*b(V_d)$, as a solution to the Euler data identity, can be completely recovered from the linking values.

Once the linking values are known, it is often easy to manufacture explicitly the $G$-equivariant classes $Q_d$ using the linking values as a guide. Many explicit examples are discussed in [21].

**Step 4. Computing the $K_d$.** Once the classes $Q_d = \varphi_*\pi^*b(V_d)$ are determined, one has to carefully unwind the commutative diagram and maps in Step 1. This allows us to establish a crucial integral identity between $K_d$ and the classes $Q_d$, which in turn allows us to compute the $K_d$.

3. Some Examples

We now give some examples of our main theorem.

**Line bundles on a balloon manifold.** Let $X$ be balloon manifold, and set $b$ to be the Chern polynomial. We fix a base $H_1, ..., H_m$ of $H^2(X)$, and let $t_1, ..., t_m$ be formal variables. We denote by $e_{S^1}(X_0/W_d)$ the equivariant Euler class of the normal bundle of $X_0 := j_0(X) \subset W_d$. Let

$$V = V^+ \oplus V^-,$$

$$V^+ := \oplus L^+_i, \quad V^- := \oplus L^-_j$$
satisfying $c_1(V^+) - c_1(V^-) = c_1(X)$, and $rk V^+ - rk V^- - (n - 3) \geq 0$. where the $L_i^\pm$ are respectively convex/concave line bundles on $X$. Let

$$\Omega = B_0 := c(V^+)/c(V^-) = \prod_i (x + c_1(L_i^+))/\prod_j (x + c_1(L_j^-))$$

$$B_d := \frac{1}{e_{S^1}(X/W)} \times \prod_i \prod_{k=0}^{\langle c_1(L^+_i), d \rangle} (x + c_1(L_i^+) - k\alpha) \times \prod_j \prod_{k=1}^{\langle c_1(L^-_j), d \rangle - 1} (x + c_1(L_j^-) + k\alpha).$$

$B(t) := e^{-H \cdot \tilde{t}} \sum B_d e^{d \cdot t}$

$\Phi(t) := \sum K_d e^{d \cdot t}$.

**Theorem 3.1.** There exist unique power series $f(t), g(t)$ such that the following formula holds:

$$\frac{1}{s!} \left( \frac{d}{dx} \right)^s \int_X \left( e^{f/\alpha} B(t) - e^{-H \cdot \tilde{t}/\alpha} \Omega \right) = \alpha^{-3} x^{-s} \left( 2\Phi(\tilde{t}) - \sum_i \tilde{t}_i \frac{\partial \Phi(\tilde{t})}{\partial \tilde{t}_i} \right).$$

where $s := rk V^+ - rk V^- - (n - 3)$, $\tilde{t} := t + g$. Moreover, $f, g$ are determined by the condition that the integrand on the left hand side is of order $O(\alpha^{-2})$.

Note that when $x \to 0$, the formula above reduces to the case when $b$ is the Euler class.

*The tangent bundle on $\mathbb{P}^n$.** The example above deals, of course, with direct sum of line bundles only. We now give an example starting from the tangent bundle $V = T_X$ on $X = \mathbb{P}^n$, which is nonsplit. Consider the case where $b_T$ the $T$-equivariant Chern polynomial. Let $\lambda_i$ be the weights of the standard $T$ action on $\mathbb{P}^n$, and $p_i$ be the $i$th fixed point. Recall that $\Omega := b_T(V) = \frac{1}{x} \prod_i (x + H - \lambda_i)$, where $H$ is the equivariant class on $\mathbb{P}^n$ with $H|_{p_i} = \lambda_i$.

Using the $T$ equivariant Euler sequence

$$0 \to \mathcal{O} \to \oplus_{i=0}^n \mathcal{O}(H - \lambda_i) \to T_X \to 0$$

One can compute the linking values in this case. There are given by

$$\prod_i \prod_{k=0}^d (x + \lambda_j - \lambda_i - k\lambda/d).$$
Here \( p, q \) are the \( j \)th and the \( l \)th fixed points in \( \mathbb{P}^n \), and \( \lambda = \lambda_j - \lambda_l \). We can use this to set up a system of linear equations to solve for \( A(t) \) inductively. However, there is an easier way to compute \( A(t) \) in this case. Using the linking values above as a guide, we set

\[
B_d := \frac{1}{x} \prod_i \prod_{k=0}^d (x + H - \lambda_i - k\alpha)
\]

and let

\[
B(t) := e^{-H \cdot t/\alpha} \sum \frac{B_d}{\prod_i \prod_{k=1}^d (H - \lambda_i - k\alpha)}.
\]

Then it can be shown that the series

\[
A(t) := e^{-H \cdot t/\alpha} \sum \frac{j^*_0 Q_d}{\prod_i \prod_{k=1}^d (H - \lambda_i - k\alpha)} e^{d \cdot t}
\]

is related to \( B(t) \) by

\[
A(t + g) = e^{f/\alpha} B(t)
\]

where \( f, g \) are explicitly computable functions, similar to those in the previous example. This relation, once again, allows us to compute all the \( K_d \) simultaneously.

\( V := \mathcal{O}(-1) \oplus \mathcal{O}(-1) \) on \( \mathbb{P}^1 \). In this case, we let \( b \) be the Euler class \( c_{top} \), and we would like to compute the one-pointed intersection numbers

\[
\int_{M_{0,1}(d, \mathbb{P}^1)} e^*(H) b(V'_d).
\]

Here \( V'_d \) is the bundle induced on \( M_{0,1}(d, \mathbb{P}^1) \) by \( V \), \( H \) is the hyperplane class on \( \mathbb{P}^1 \), and \( e: M_{0,1}(d, \mathbb{P}^1) \to \mathbb{P}^1 \) is the evaluation map. We can easily specialize the first example above to the case of \( X = \mathbb{P}^1 \) and \( V = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \). In this case, \( f = g = 0 \), and we get the formula

\[
\int_{\mathbb{P}^1} e^{-H t/\alpha} \frac{j^*_0 Q_d}{\prod_{m=1}^d (H - m\alpha)^2} = \alpha^{-3} (2 - dt) K_d.
\]

Apply \( \frac{d}{dt} \) to both sides, and combine the result with Theorem 3.2 in [19] (see the first eqn. on p36 there). What we get is

\[
\int_{M_{0,1}(d, \mathbb{P}^1)} e^*(H) b(V'_d) = d K_d = d^{-2}.
\]

The values \( d^{-2} \) remind us of a result that Vafa obtains via the physics of local mirror symmetry. He considers the problem of counting holomorphic disks in a Calabi-Yau 3-fold.
equipped with a choice of Lagrangian submanifold. The boundary of the disks are required to lie in the Lagrangian submanifold. A counting problem is heuristically formulated into a problem of determining the Euler classes of certain yet-to-be-defined moduli spaces. In this case the physics of local mirror symmetry indicates that the Euler classes should be given by a “multiple-cover” contribution \( d^{-2} \), where \( d \) is the winding number of the disk’s boundary circle along the Lagrangian submanifold.

Vafa’s result suggests the following interpretation. The Lagrangian submanifold plays the role of a vanishing cycle in a certain limit. A holomorphic disk with boundary landing on the Lagrangian submanifold would look like a \( \mathbb{P}^1 \) with one marked point in this limit. Local mirror symmetry suggests that we should use the stable map moduli spaces of \( \mathbb{P}^1 \) as a model for this problem. The requirement that the marked point lands on the vanishing cycle may be thought of as the incidence condition on the map \( \mathbb{P}^1 \to \mathbb{P}^1 \) with one point mapped to the cycle \( H \). The appropriate moduli spaces in this model should then be \( M_{0,1}(d, \mathbb{P}^1) \), and the Euler classes should correspond to the “multiple-cover” formula for the bundle \( V'_d \) induced by \( V = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \). So a good candidate for the intersection numbers are

\[
\int_{M_{0,1}(d, \mathbb{P}^1)} e_2^*(H) b(V'_d) = d^{-2}.
\]

Two Lagrangian \( S^3 \) in a CY 3-fold? Another interesting situation considered by physicists is the problem of counting annuli in a CY 3-fold equipped with two Lagrangian 3-spheres \( S^3 \), subject to the incidence condition that each of the boundary circles of the annulus lands inside one of the 3-spheres. Again the 3-spheres plays the role of two vanishing cycle, and are allowed to contract to points \( x, y \). The annulus looks like a \( \mathbb{P}^1 \) with two marked points anchored to \( x, y \). By analogy with the previous example as in local mirror symmetry, the corresponding stable map moduli in this case should be the two-pointed moduli \( M_{0,2}(d, \mathbb{P}^1) \), and the corresponding intersection numbers should be

\[
\int_{M_{0,2}(d, \mathbb{P}^1)} e_1^*(H) e_2^*(H) b(V''_d).
\]

Here \( V''_d \) is the induced bundle \( \rho_2^* V'_d \) on \( M_{0,2}(d, \mathbb{P}^1) \), where \( \rho_2 : M_{0,2}(d, \mathbb{P}^1) \to M_{0,1}(d, \mathbb{P}^1) \) is the map that forgets the second marked point, and the \( e_i \) are the usual evaluation maps on \( M_{0,2}(d, \mathbb{P}^1) \).

The intersection number can be easily computed in a way analogous to the previous example. By writing \( b(V''_d) = \rho_2^* b(V'_d) \), we get

\[
\int_{M_{0,2}(d, \mathbb{P}^1)} e_1^*(H) e_2^*(H) b(V''_d) = \int_{M_{0,1}(d, \mathbb{P}^1)} e^*(H) b(V'_d) \rho_2^* e_2^*(H).
\]
By integrating along a fiber of the map $\rho_2$, we see that the last factor in the integrand contributes an overall factor $d$. Thus we get the answer

$$\int_{M_{0,2}(d,\mathbb{P}^1)} e_1^*(H) e_2^*(H) b(V''_d) = d^{-1}.$$

The last two example suggests the very interesting possibility that one may be able to use stable map moduli spaces as models for some of the moduli spaces in the problem of counting holomorphic disks and annuli with suitable incidence conditions. Moreover, the appropriate intersection numbers should come from Euler classes of induced bundles, which is exactly what the mirror principle is designed to study. This possibility deserves further investigations.
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