Strong consistency of MLE for finite mixtures of location-scale distributions when the ratios of the scale parameters are exponentially small

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Abstract

In finite mixtures of location-scale distributions, if there is no constraint on the parameters then the maximum likelihood estimate does not exist. But when the ratios of the scale parameters are restricted appropriately, the maximum likelihood estimate exists. We prove that the maximum likelihood estimator (MLE) is strongly consistent, if the ratios of the scale parameters are restricted from below by \( \exp(-n^d) \), \( 0 < d < 1 \), where \( n \) is the sample size.

Key words and phrases: Mixture distribution, maximum likelihood estimator, consistency.

1 Introduction

In this paper we consider mixtures of \( M \) location-scale densities which is defined by

\[
f(x; \theta) \equiv \sum_{m=1}^{M} \alpha_m f_m(x; \mu_m, \sigma_m),
\]

where \( \alpha_m \), called the mixing weights, are nonnegative real numbers that sum to one and \( f_m(x; \mu_m, \sigma_m) \), called the components of the mixture, are location-scale density functions with the location parameter \( \mu_m \in \mathbb{R} \) and the scale parameter \( \sigma_m > 0 \). \( \theta \) contains all the parameters in the mixture and can be written as \( \theta = (\alpha_1, \mu_1, \sigma_1, \ldots, \alpha_M, \mu_M, \sigma_M) \).

The location-scale densities satisfy

\[
f_m(x; \mu_m, \sigma_m) = \frac{1}{\sigma_m} f_m \left( \frac{x - \mu_m}{\sigma_m}; 0, 1 \right).
\]
We allow \( f_m(x; \mu_m, \sigma_m) \) to belong to different families. For example, \( f_1(x; \mu_1, \sigma_1) \) may be a normal density, \( f_2(x; \mu_2, \sigma_2) \) may be a uniform density, etc.

In finite mixtures of location-scale distributions, the maximum likelihood estimate over the whole parameter space does not exist. For a given data \( x_1, \ldots, x_n \), the likelihood function for a mixture is unbounded. For example, if we set \( \mu_1 = x_1 \) and let \( \sigma_1 \to 0 \) in a mixture of two normal densities with means \( \mu_1, \mu_2 \) and standard deviations \( \sigma_1, \sigma_2 \), then the likelihood will tend to infinity.

Let us consider the constrained parameter space

\[
\Theta_b \equiv \{ \theta \in \Theta \mid \min_{1 \leq m, m' \leq M} \sigma_m / \sigma_{m'} \geq b \} \quad , \quad 0 < b < 1,
\]

where \( \Theta \) denote the unconstrained parameter space. Hathaway (1985) showed that the global maximizer \( \hat{\theta}_b \) of the likelihood function over \( \Theta_b \) exists and if true value of parameter belongs to \( \Theta_b \) then \( \hat{\theta}_b \) is strongly consistent. But there is the problem how small we have to choose \( b \) to ensure strong consistency. An interesting question here is whether we can decrease the bound \( b \) to zero with the sample size yet guarantee the strong consistency of the maximum likelihood estimator. This question is mentioned in Hathaway (1985), McLachlan and Peel (2000) and treated as an unsolved problem.

Meanwhile in Tanaka and Takemura (2006), we consider mixtures of location-scale distributions with constraint imposed on the scale parameters themselves and showed that the maximum likelihood estimator is strongly consistent if the scale parameters are restricted from below by \( \exp(n^{-d}) \), \( 0 < d < 1 \). Tanaka and Takemura (2006) implies that the rate \( \exp(n^{-d}) \) obtained in Tanaka and Takemura (2006) is almost the lower bound to ensure strong consistency. The method used in Tanaka and Takemura (2006) is useful for solving the problem stated in Hathaway (1985) in which the constraints are imposed on the ratios of the scale parameters.

In this paper, we solve the problem stated in Hathaway (1985). We prove that the maximum likelihood estimator is strongly consistent, if the ratios of the scale parameters are restricted from below by \( \exp(-n^d) \), \( 0 < d < 1 \).

The organization of the paper is as follows. In section 2 we prepare notation and summarize some preliminary results. In section 3 we state our main result in theorem 2. Section 4 is devoted to the proof of theorem 2. The last section is conclusion and future work.

2 Notation and definitions

2.1 Notation

Let \( \Omega_m = \mathbb{R} \times (0, \infty) \) denote the parameter space of the \( m \)-th component \( (\mu_m, \sigma_m) \). Then the entire parameter space \( \Theta \) can be represented as follows.

\[
\Theta = \{ (\alpha_1, \ldots, \alpha_M) \in \mathbb{R}^M \mid \sum_{m=1}^{M} \alpha_m = 1, \ \alpha_m \geq 0 \} \times \prod_{m=1}^{M} \Omega_m.
\]
Let $\mathcal{X}$ be a subset of $\{1, 2, \ldots, M\}$ and let $|\mathcal{X}|$ denote the number of elements in $\mathcal{X}$. Denote by $\theta_{\mathcal{X}}$ a subvector of $\theta \in \Theta$ consisting of the components in $\mathcal{X}$. Then the parameter space of subprobability measures consisting of the components in $\mathcal{X}$ is

$$\bar{\Theta}_{\mathcal{X}} \equiv \{ \theta_{\mathcal{X}} \mid \theta \in \Theta, \sum_{m \in \mathcal{X}} \alpha_m \leq 1 \}.$$ 

Corresponding density and the set of subprobability densities are denoted by

$$f_{\mathcal{X}}(x; \theta_{\mathcal{X}}) \equiv \sum_{m \in \mathcal{X}} \alpha_m f_m(x; \mu_m, \sigma_m),$$

$$\mathcal{G}_{\mathcal{X}} \equiv \{ f_{\mathcal{X}}(x; \theta_{\mathcal{X}}) \mid \theta_{\mathcal{X}} \in \bar{\Theta}_{\mathcal{X}} \}.$$

Furthermore denote the set of subprobability densities with no more than $K$ components by

$$\mathcal{G}_K \equiv \bigcup_{|\mathcal{X}| \leq K} \mathcal{G}_{\mathcal{X}} \quad (1 \leq K \leq M).$$

### 2.2 Identifiability and strong consistency of estimators

In general, a parametric family of distributions is identifiable if different values of parameter designate different distributions. In mixtures of distributions, different parameters may designate the same distribution. For example, if $\alpha_1 = 0$, then for all parameters which differ only in $\mu_1$ or $\sigma_1$, we have the same distribution. Thus mixtures of distributions are not identifiable. Therefore we have to carefully define strong consistency of an estimator. The following definition is essentially the same as Redner (1981). We assume that the parameter space $\Theta$ is a subset of Euclidean space and $\text{dist}(\theta, \theta')$ denotes the Euclidean distance between $\theta, \theta' \in \Theta$. Furthermore we define

$$\text{dist}(U, V) \equiv \inf_{\theta \in U} \inf_{\theta' \in V} \text{dist}(\theta, \theta')$$

for $U, V \subset \Theta$.

**Definition 1.** (a strongly consistent estimator)

Let $\Theta_0$ denote the set of true parameters

$$\Theta_0 \equiv \{ \theta \in \Theta \mid f(x; \theta) = f(x; \theta_0) \quad a.e. \: x \},$$

where $\theta_0$ is one of parameters designating the true distribution, and let

$$\Theta(\hat{\theta}) \equiv \{ \theta \in \Theta \mid f(x; \theta) = f(x; \hat{\theta}) \quad a.e. \: x \}.$$

An estimator $\hat{\theta}_n$ is strongly consistent if

$$\text{Prob} \left( \lim_{n \to \infty} \text{dist}(\Theta(\hat{\theta}_n), \Theta_0) = 0 \right) = 1.$$
2.3 Preliminaries

In Tanaka and Takemura (2006), we assume the following regularity conditions for strong consistency of the constrained maximum likelihood estimator.

Assumption 1. There exist real constants $v_0, v_1 > 0$ and $\beta > 1$ such that

$$f_m(x; \mu_m = 0, \sigma_m = 1) \leq \min\{v_0, v_1 \cdot |x|^{-\beta}\}$$

for all $m$.

Let $\Gamma$ denote any compact subset of $\Theta$.

Assumption 2. For $\theta \in \Theta$ and any positive real number $\rho$, let

$$f(x; \theta, \rho) \equiv \sup_{\text{dist}(\theta', \theta) \leq \rho} f(x; \theta').$$

For each $\theta \in \Gamma$ and sufficiently small $\rho$, $f(x; \theta, \rho)$ is measurable.

Assumption 3. For each $\theta \in \Gamma$, if $\lim_{j \to \infty} \theta^{(j)} = \theta, (\theta^{(j)} \in \Gamma)$ then $\lim_{j \to \infty} f(x; \theta^{(j)}) = f(x; \theta)$ except on a set which is a null set and does not depend on the sequence $\{\theta^{(j)}\}_{j=1}^\infty$.

Assumption 4.

$$\int |\log f(x; \theta_0)| f(x; \theta_0) dx < \infty.$$

Let $E_0[\cdot]$ denote the expectation under the true parameter $\theta_0$. In Tanaka and Takemura (2006), we showed the following theorem.

Theorem 1. (Tanaka and Takemura (2006)) Suppose that assumptions [1-4] are satisfied and $f(x; \theta_0) \in \mathcal{G}_M \setminus \mathcal{G}_{M-1}$. Let $c_0 > 0$ and $0 < d < 1$. If $c_n = c_0 \cdot \exp(-n^d)$ and

$$\Theta_{c_n} \equiv \{\theta \in \Theta \mid \min_{1 \leq m \leq M} \sigma_m \geq c_n\},$$

then

$$\text{Prob}\left(\lim_{n \to \infty} \text{dist}(\hat{\theta}_{c_n}, \Theta_0) = 0\right) = 1,$$

where $\hat{\theta}_{c_n}$ is the maximum likelihood estimator restricted to $\Theta_{c_n}$.

3 Main result

To show the strong consistency of the constrained maximum likelihood estimator in the problem stated in Hathaway (1985), we replace the assumption [1] with the following assumption.
Assumption 5. There exist real constants $v_0, v_1 > 0$ and $\beta > 2$ such that

$$f_m(x; \mu_m = 0, \sigma_m = 1) \leq \min\{v_0, v_1 \cdot |x|^{-\beta}\}$$

for all $m$.

Now we state the main theorem of this paper.

Theorem 2. Suppose that the assumptions 2–5 are satisfied and $f(x; \theta_0) \in \mathcal{G}_M \setminus \mathcal{G}_{M-1}$. Let $b_0 > 0$ and $0 < d < 1$. If $b_n = b_0 \cdot \exp(-n^d)$ and

$$\Theta_{b_n} \equiv \{\theta \in \Theta \mid \min_{1 \leq m \neq m', \leq M} \frac{\sigma_m}{\sigma_{m'}} \geq b_n\},$$

then

$$\operatorname{Prob}\left(\lim_{n \to \infty} \operatorname{dist}(\hat{\Theta}_{b_n}, \Theta_0) = 0\right) = 1,$$

where $\hat{\Theta}_{b_n}$ is the maximum likelihood estimator restricted to $\Theta_{b_n}$.

4 Proof

In this section, we prove theorem 2 by using theorem 1. The organization of this section is as follows. First in subsection 4.1 we partition the parameter space $\Theta_{b_n}$ into two sets. Then the proof for strong consistency of the maximum likelihood estimator restricted to $\Theta_{b_n}$ is also partitioned and the proof for one set is shown by applying the result of theorem 1. The proof for another set is shown in section 4.2.

4.1 Partitioning the parameter space

Let $0 < d < 1$ be any constant and define $b_n \equiv \exp(-n^d)$. We choose $d'$ such that $d < d' < 1$ and define $c_n \equiv \exp(-n^{d'})$. Notice that the following arguments also hold even when we define $b_n \equiv b_0 \cdot \exp(-n^d)$ with a positive constant $b_0$. Define $\Theta_{b_n} \equiv \{\theta \in \Theta \mid \min_{1 \leq m \neq m', \leq M} \frac{\sigma_m}{\sigma_{m'}} \geq b_n\}$ and $\Theta_{c_n} \equiv \{\theta \in \Theta \mid \min_{1 \leq m \leq M} \sigma_m \geq c_n\}$. The constrained parameter space $\Theta_{b_n}$ can be divided into two sets.

$$\Theta_{b_n} = (\Theta_{b_n} \cap \Theta_{c_n}) \cup (\Theta_{b_n} \cap \Theta_{c_n}^C),$$

where $\Theta_{c_n}^C$ is the complement of $\Theta_{c_n}$.

From theorem 1, the maximum likelihood estimator over $\Theta_{b_n} \cap \Theta_{c_n}$ is strongly consistent. If the maximum of likelihood function over $\Theta_{b_n} \cap \Theta_{c_n}^C$ is very small, then the maximum likelihood estimator over $\Theta_{b_n}$ is strongly consistent. By the argument used in [Wald, 1949], in order to prove theorem 2 it suffices to prove the following lemma.

Lemma 1.

$$\lim_{n \to \infty} \sup_{\theta \in \Theta_{b_n} \cap \Theta_{c_n}^C} \frac{\prod_{i=1}^{n} f(x_i; \theta)}{\prod_{i=1}^{n} f(x_i; \theta_0)} = 0, \quad \text{a.e.}$$
4.2  Proof of lemma 1

Let
\begin{align*}
\sigma(1) &= \min_{1 \leq m \leq M} \sigma_m, \quad \sigma(M) = \max_{1 \leq m \leq M} \sigma_m. \\
\end{align*}
(2)

Then for \( \theta \in \Theta_{\text{C}n} \),
\[ \sigma(1) \leq c_n. \]  \hfill (3)

Furthermore for \( \theta \in \Theta_{\text{b}n} \),
\[ \frac{\sigma(1)}{\sigma_m} \geq b_n, \quad 1 \leq m \leq M. \]

Therefore for \( \theta \in \Theta_{\text{b}n} \cap \Theta_{\text{C}n} \),
\[ \sigma_m \leq c_n / b_n = \exp(n^d - n^{d'}) , \quad 1 \leq m \leq M. \]  \hfill (4)

This means that all the scale parameters \( \sigma_m \) of \( \theta \in \Theta_{\text{b}n} \cap \Theta_{\text{C}n} \) are very small for large \( n \). Hence the maximum of likelihood function over \( \Theta_{\text{b}n} \cap \Theta_{\text{C}n} \) seems to be small.

Let \( E_0[ \cdot ] \) denote the expectation under the true parameter \( \theta_0 \). By law of large numbers [1] is implied by
\[
\limsup_{n \to \infty} \sup_{\theta \in \Theta_{\text{b}n} \cap \Theta_{\text{C}n}} \frac{1}{n} \sum_{i=1}^{n} \log f(x_i; \theta) < E_0[\log f(x; \theta_0)] \quad \text{a.e.} \]  \hfill (5)

Therefore, in order to prove [1] it suffices to prove [5].

4.2.1  Step 1 : Bounding the components by step functions

![Diagram of step function bounding](image)

Figure 1: Each component is bounded by a step function.

Define
\[ \nu(\sigma) = \left( \frac{v_1}{v_0} \right)^{\frac{1}{2}} \cdot \sigma^{1-\frac{2}{\mu}}. \]  \hfill (6)

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From assumption 5 each component is bounded from above as
\[ f_m(x; \mu_m, \sigma_m) \leq \max \{1_{[\mu_m - \nu(\sigma_m), \mu_m + \nu(\sigma_m)]]}(x) \cdot \frac{v_0}{\sigma_m}, v_0 \sigma_m\} . \]

See figure 1. From this and \(2\) we obtain the following lemma.

**Lemma 2.**
\[ f_m(x; \mu_m, \sigma_m) \leq \max \{1_{[\mu_m - \nu(\sigma_m), \mu_m + \nu(\sigma_m)]]}(x) \cdot \frac{v_0}{\sigma_1}, v_0 \sigma_M\} , \quad 1 \leq m \leq M. \]

### 4.2.2 Step 2: Bounding the likelihood function by two terms

Let \(R_n(V)\) denote the number of observation which belong to a set \(V \subset \mathbb{R}\). Define
\[ J(\theta) \equiv \bigcup_{m=1}^{M} [\mu_m - \nu(\sigma_m), \mu_m + \nu(\sigma_m)] . \]

**Lemma 3.**
\[ \forall \theta \in \Theta_{b_n} \cap \Theta_{c_n}^C , \quad \sum_{i=1}^{n} \log f(x_i; \theta) \leq R_n(J(\theta)) \cdot \log \frac{v_0}{\sigma_1} + R_n(J(\theta)^C) \cdot \log (v_0 \sigma_M) \]

**Proof:** From lemma 2 we obtain
\[
\sum_{i=1}^{n} \log f(x_i; \theta) = \sum_{i=1}^{n} \log \left\{ \sum_{m=1}^{M} \alpha_m f_m(x_i; \mu_m, \sigma_m) \right\} \\
\leq \sum_{i=1}^{n} \left\{ \max_{m=1,\ldots,M} \log f_m(x_i; \mu_m, \sigma_m) \right\} \\
\leq \sum_{i=1}^{n} \max_{m=1,\ldots,M} \max \{1_{[\mu_m - \nu(\sigma_m), \mu_m + \nu(\sigma_m)]]}(x) \cdot \frac{v_0}{\sigma_1}, v_0 \sigma_M\} \\
= R_n(J(\theta)) \cdot \log \frac{v_0}{\sigma_1} + R_n(J(\theta)^C) \cdot \log (v_0 \sigma_M). \]

In the following we bound the right hand side of (7) from above.

### 4.2.3 Step 3: Bounding the first term

Let \(x_1, \ldots, x_n\) denote a random sample of size \(n\) from \(f(x; \theta_0)\) and let
\[ x_{n,1} \equiv \min \{x_1, \ldots, x_n\} , \quad x_{n,n} \equiv \max \{x_1, \ldots, x_n\}. \]

In Tanaka and Takemura (2006) we showed the following lemma.
Lemma 4. (Tanaka and Takemura (2006)) For any real positive constants $A_0 > 0, \zeta > 0$, define

$$A_n = A_0 \cdot n^{\frac{2\zeta}{1}}. \quad (8)$$

Then

$$\text{Prob} \left( x_{n,1} < -A_n \text{ or } x_{n,n} > A_n \ i.o. \right) = 0.$$  

By this lemma we can bound the behavior of the minimum and the maximum of the sample with probability 1. In the following we ignore the event $\{x_{n,1} < -A_n \text{ or } x_{n,n} > A_n\}$.

Next we prove the following lemma for bounding the first term of (7).

Lemma 5.

$$\forall \theta \in \Theta_{b_n} \cap \Theta_{c_n}^C, \forall \epsilon > 0 \ , \ \text{Prob} \left( \max \{ R_n(J(\theta)) - 4M, 0 \} > \epsilon \ i.o. \right) = 0$$

Proof: Define

$$w_n \equiv \nu(c_n/b_n). \quad (9)$$

From lemma 4 we can ignore the event $\{x_{n,1} < -A_n \text{ or } x_{n,n} > A_n\}$. Now we divide $[-A_n, A_n]$ from $-A_n$ to $A_n$ by short intervals of length $2w_n$. Let $k(w_n)$ be the number of the short intervals and let $I_1(w_n), \ldots, I_{k(w_n)}(w_n)$ be the divided short intervals. The length of the rightmost short interval $I_{k(w_n)}(w_n)$ may be less than $2w_n$. See figure 2. Then we have

$$k(w_n) \leq \frac{2A_n}{2w_n} + 1 = \frac{A_n}{w_n} + 1. \quad (10)$$

From (4), (6) and (9) we have

$$\nu(\sigma_1), \nu(\sigma_2), \ldots, \nu(\sigma_M) \leq \nu(c_n/b_n) = w_n.$$  

Since $J(\theta) = \bigcup_{m=1}^{M} [\mu_m - \nu(\sigma_m) \mu_m + \nu(\sigma_m)]$ consists of M intervals of length at most $2w_n$, $J(\theta) \cap [-A_n, A_n]$ is covered by at most $2M$ short intervals of
I_1(w_n), \ldots, I_k(w_n)(w_n). \text{ Therefore the following relation holds.}

\{\max\{R_n(J(\theta)) - 4M, 0\} > \epsilon\} \iff \{\max\{R_n(J(\theta) \cap [-A_n, A_n]) - 4M, 0\} > \epsilon\}
\Rightarrow \{1 \leq k \leq k(w_n), R_n(I_k(w_n)) \geq 2\}.

Let u_0 \equiv \sup_x f(x; \theta_0). \text{ Let } P_0(V) \text{ denote the probability of } V \subset \mathbb{R} \text{ under the true density } P_0(V) \equiv \int_V f(x; \theta_0)dx.

From (10), R_n(I_k(w_n)) \sim \text{Bin}(n, P_0(I_k(w_n))) \text{ and } P_0(I_k(w_n)) \leq 2w_nu_0 \text{ we obtain}

\text{Prob}(\max\{R_n(J(\theta)) - 4M, 0\} > \epsilon) \leq \sum_{k=1}^{k(w_n)} \text{Prob}(R_n(I_k(w_n)) \geq 2)
\leq k(w_n) \cdot \left\{\max_{1 \leq k \leq k(w_n)} \text{Prob}(R_n(I_k(w_n)) \geq 2)\right\}
\leq \left(\frac{A_n}{w_n} + 1\right) \sum_{k=2}^{n} \binom{n}{k} (2w_nu_0)^k (1 - 2w_nu_0)^{n-k}
\leq \left(\frac{A_n}{w_n} + 1\right) \sum_{k=2}^{n} \frac{n^k}{k!} (2w_nu_0)^k
\leq \left(\frac{A_n}{w_n} + 1\right) (2nw_nu_0)^2 \exp(2nw_nu_0) . \quad (11)

From (6), (8) and (9), the order of the right hand side of (11) is evaluated as follows.

\left(\frac{A_n}{w_n} + 1\right) (2nw_nu_0)^2 = O\left(n^{2+\frac{2\beta-\delta}{1-2/\beta}} \cdot e^{(n^d-n^{d'})d(1-2/\beta)}\right)
\exp(2nw_nu_0) = O(1)

Recall that 0 < d < d' < 1 \text{ and } \beta > 2 \text{ by assumption 5. Then}

\sum_{n=1}^{\infty} n^{2+\frac{2\beta-\delta}{1-2/\beta}} \cdot e^{(n^d-n^{d'})d(1-2/\beta)} < \infty.

Therefore, when we sum the right hand side of (11) over \(n\), the resulting series converges. Hence by Borel-Cantelli lemma, we have

\text{Prob}(\max\{R_n(J(\theta)) - 4M, 0\} > \epsilon \ i.o.) = 0.
4.2.4 Step 4: Evaluating the likelihood function

From lemma 5, we can ignore the event \( R_n(J(\theta)) > 4M \). In the following we consider only the event \( \{ R_n(J(\theta)) \leq 4M \} \). Note that \( R_n(J(\theta)) \geq n - 4M \).

For sufficiently large \( n \), the scale parameters \( \sigma_1, \ldots, \sigma_M \) of \( \theta \in \Theta_{b_n} \cap \Theta_{c_n} \) are very small so that we can bound the right hand side of (7) as follows.

\[
R_n(J(\theta)) \cdot \log \frac{\nu_0}{\sigma(1)} + R_n(J(\theta)^C) \cdot \log (\nu_0 \sigma(M)) \leq 4M \cdot \log \frac{\nu_0}{\sigma(1)} + (n - 4M) \cdot \log (\nu_0 \sigma(M))
\]

where the last inequality holds by (4) i.e. \( \sigma(M) \leq \sigma(1)/b_n \). Recall that we set \( 0 < d < d' < 1 \). Then from (3), (7) and (12) we have

\[
\sup_{\theta \in \Theta_{b_n} \cap \Theta_{c_n}} \frac{1}{n} \sum_{i=1}^{n} \log f(x_i; \theta) \leq \frac{1}{n} \left\{ n \cdot \log \nu_0 + (n - 8M) \cdot (n - 4M) \cdot \log \frac{1}{b_n} \right\} \rightarrow -\infty \quad \text{a.e.}
\]

Therefore we obtain (5) and lemma 1 is proved.

This completes the proof of theorem 2.

5 Conclusion

In this paper we prove that if we set \( b_n \equiv \exp(-n^d) \), \( 0 < d < 1 \), then the maximum likelihood estimator restricted to \( \Theta_{b_n} \equiv \{ \theta \in \Theta \mid \min_{1 \leq m \neq m' \leq M} \frac{\sigma_m}{\sigma_{m'}} \geq b_n \} \) is strongly consistent under very mild regularity conditions. Mixtures of normal distributions satisfy the regularity conditions. This means that the problem stated in Hathaway (1985) is solved.

If we define \( b_n \equiv \exp(-n^r) \), \( r > 1 \), and set \( \theta \) as

\[
\mu_1 = x_1, \quad \sigma_1 = \exp(-n^r) \quad \mu_m = 0, \quad \sigma_m = 1 \quad (m \neq 1),
\]

then \( \theta \in \Theta_{b_n} \) and the mean log likelihood of this density tends to infinity \((Tanaka and Takemura (2005))\). This means that the mean log likelihood of the true model which converges to finite value almost everywhere is dominated by that of other models. Therefore if \( b_n \) decreases to zero faster than \( \exp(-n) \), then the consistency of the maximum likelihood estimator fails. This implies that the rate of \( b_n \equiv \exp(-n^d), 0 < d < 1 \) obtained in this paper is almost the lower bound of the order of \( b_n \) which maintains the consistency.

In theorem 2 we assume \( f(x; \theta_0) \in G_M \setminus G_{M-1} \) i.e. the number of components of true model is known. To discuss the case that the number of components of true model is unknown, more complicated mathematical techniques are needed.
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