On Gauge–Invariant Boundary Conditions for 2d Gravity with Dynamical Torsion

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Abstract

In the example of $R^2 + T^2$ gravity on the unit two dimensional disk we demonstrate that in the presence of an independent spin connection it is possible to define local gauge invariant boundary conditions even on boundaries which are not totally geodesic. One-loop partition function and the corresponding heat kernel are calculated.

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1. An important problem in quantum gravity and quantum cosmology is the formulation of gauge invariant boundary conditions (see monograph [1] and references therein). This problem is especially complicated on manifolds with non totally geodesic boundaries as exemplified e.g. by the Euclidean disk. The boundary conditions used so far in actual computations [2,3,4] either involve only part of the degrees of freedom [2], or are only ”partially” invariant [3]. The Barvinsky boundary conditions [5], while having all necessary invariance properties are non-local, which makes computations very complicated.

Two dimensional quantum gravity is frequently used as a laboratory for studying various theoretical ideas. As a particular model we choose here the $R^2 + T^2$ gravity [6,7], which is both classically and quantum integrable [6-8] on manifolds without boundaries. The aim of this work is twofold. First, we explore the possibility to define local gauge invariant boundary conditions in the presence of an independent spin connection. Second, we study the ultraviolet divergencies of $R^2 + T^2$ gravity due to the presence of boundaries and comment on the quantum equivalence to a model with only finite number of quantized modes.

For the sake of simplicity we restrict ourselves to perturbative one-loop path integral on the background represented by the unit Euclidean disk. To keep unified notations with the Minkowski signature models we call the $O(2)$ rotations local Lorentz transformations. Our central aim is the definition of local gauge invariant boundary conditions. Locality means that the boundary conditions for a field $\Phi$ can be represented in the form $P_D \Phi|_{\partial M} = 0,$ $(\partial_0 + c) P_N \Phi|_{\partial M} = 0,$ where $\partial_0$ is the normal derivative, $P_D$ and $P_N$ are complementary local projectors. In this context, gauge invariance means that the gauge transformations map the functional space defined by the boundary conditions onto itself. The requirement of locality seems to be technical. However, it is needed to obtain a controllable quantum theory. Unlike recent works [9] on boundary dynamics in dilaton gravity we impose boundary conditions on all the components of zweibein fluctuations, i.e. the boundary conditions are imposed before fixing a gauge. This ensures gauge-independence of results. We discover that local gauge invariant boundary conditions exist only if the spin connection is independent of the zweibein. This is exactly the case whenever torsion becomes dynamical. By computing the heat kernel expansion we demonstrate that the divergencies proportional to the length of the boundary are not cancelled. This means that to obtain a model contain-
ing only a finite number of quantized modes (such models do not give rise to 
surface terms with $t^{-\frac{1}{2}}$ in the heat kernel expansion) one should either aban-
don the locality requirement, or manifolds with totally geodesic boundaries 
should be considered.

2. Consider a 2-dimensional diffeomorphism covariant theory described 
by zweibein $e_{\mu}^a$ and spin-connection $\omega_{\mu}^{ab} = \omega_{\mu}^{eab}$, $\epsilon^{01} = 1$. For the sake of 
simplicity let us restrict ourselves to small fluctuations around the flat two-
dimensional Euclidean disk, then the background values of zweibein and 
connection are $e_0^0 = 1$, $e_1^1 = r$, $\omega_1 = 1$, where the polar coordinate system is 
adopted, $x^1 = \theta$, $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$. $r = 1$ corresponds to the boundary 
of the unit disk.

Let us try to define local boundary conditions for the perturbations $h_{\mu}^a$ 
of the zweibein $e_{\mu}^a$ possessing both diffeomorphism and Lorentz invariance. 
The diffeomorphism transformations with infinitesimal parameter $\xi^\mu$ look as 
follows:

$$
\begin{align*}
\delta h_0^0 &= \partial_0 \xi^0, \quad \delta h_0^1 = r \partial_0 \xi^1, \\
\delta h_1^0 &= \partial_1 \xi^0, \quad \delta h_1^1 = r \partial_1 \xi^1 + \xi^0.
\end{align*}
$$

(1)

The boundary conditions for $\xi$ will later become the boundary conditions for 
the ghost fields. This is why we require them to be local too. Let us also 
adopt the quantum cosmology boundary conditions for $h_1^1$:

$$
h_1^1|_{\partial M} = 0 \quad (2)
$$

From the last of the equations (1) it is clear that $\xi^1$ and $\xi^0$ should satisfy 
Dirichlet boundary conditions too:

$$
\xi^\mu|_{\partial M} = 0 \quad (3)
$$

From (3) we immediately conclude that $h_1^0$ also satisfy the Dirichlet boundary 
conditions, while $h_0^0$ and $h_1^0$ obey Neumann boundary conditions with some 
constants $c_0$ and $c_1$, whose precise value is irrelevant:

$$
\begin{align*}
&h_1^0|_{\partial M} = 0, \quad (\partial_0 + c_0)h_0^0|_{\partial M} = 0, \quad (\partial_0 + c_1)h_1^0|_{\partial M} = 0
\end{align*}
$$

(4)

Intuitively it is clear that the normal derivative $\partial_0$ changes the type of bound-
dary conditions. We shall demonstrate this below in a more rigorous way.
Consider now local Lorentz transformations with infinitesimal parameter \( \sigma \), \( \delta h^a_\mu = \epsilon^{ab} \epsilon^b_\mu \sigma(x) \). On the disk we obtain
\[
\delta h^0_0 = \delta h^1_1 = 0, \quad \delta h^1_0 = -\sigma, \quad \delta h^0_1 = r \sigma.
\] (5)

By comparing the transformation rule (5) for \( h^0_1 \) with the boundary condition (4) we are forced to a Dirichlet boundary condition for \( \sigma \)
\[
\sigma|_{\partial M} = 0.
\] (6)

However, eq. (6) is in contradiction with the boundary conditions and the transformation law for \( h^1_0 \). We conclude that it is impossible to define local gauge invariant boundary conditions for the zweibein fluctuations. A careful analysis shows that this is true for any manifold with non-vanishing second fundamental form of the boundary regardless of precise form of the boundary conditions for \( h^1_1 \).

Only on a totally geodesic boundary the situation is different. Then the last of the equations (1) would contain only the \( \xi^1 \). This makes it possible to define local gauge invariant boundary conditions for the zweibein.

3. To consider the spin-connection \( \omega_\mu \) as an independent field opens completely new possibilities. Indeed, we may define then a new Lorentz invariant field
\[
\tilde{e}^a_\mu = \exp(-\phi \epsilon^{ab}) \epsilon^b_\mu,
\] (7)
where \( \phi \) is the longitudinal part of the connection fluctuation \( \rho_\mu \) of \( \omega_\mu \)
\[
\rho_\mu = \partial_\mu \phi + \rho^T_\mu, \quad \nabla^\mu \rho^T_\mu = 0.
\] (8)

\( \nabla_\mu \) denotes background covariant derivative. The diffeomorphism transformations of the connection field \( \omega \) are
\[
\delta \omega_\mu = \xi^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu \xi^\nu.
\] (9)
Truncated to the linear order in fluctuations over the disk and transformation parameter they become
\[
\delta \phi = \xi^1, \quad \delta \rho^T_\mu = 0.
\] (10)
Thus we see that only the longitudinal part of \( \rho \) transforms. One can easily obtain the linearized transformation rules for the fluctuations \( \tilde{h}^a_\mu \) of \( \tilde{e}^a_\mu \)
\[
\delta \tilde{h}^a_\mu = e^{a\nu} \nabla_\mu \xi_\nu.
\] (11)
where $e$ and $\nabla$ are the background zweibein and covariant derivative, respectively. Repeating step by step our previous calculations we obtain Dirichlet boundary conditions (3) for $\xi_\mu$ and mixed boundary conditions for $\tilde{h}$

$$\tilde{h}_1^a|_{\partial M} = 0, \quad (\partial_0 + \frac{1}{r})\tilde{h}_0^a, \quad a = 0, 1.$$  (12)

Now the precise form of the Neumann boundary condition is essential. Consider the derivation of eq. (12) in more detail. We shall use the observation [10] that it is enough to define boundary conditions for eigenfunctions of the Laplace operator $\Delta$. For example,

$$ (\partial_0 + \frac{1}{r})\delta \tilde{h}_0^0 = (\partial_0 + \frac{1}{r})\partial_0 \xi_0 = (\Delta - \frac{1}{r^2}\partial_1^2 + \frac{1}{r^2})\xi_0 + \frac{2}{r^3}\partial_1 \xi_1, $$  (13)

where we used an explicit expression for the vector Laplace operator on the disk (see e.g. ref. [1]). The $\xi_\mu$ can be choosen as eigenvector of both $\Delta$ and $\partial_1$. In this case the r.h.s. of (13) vanishes term by term on the boundary provided $\xi$ satisfies the Dirichlet boundary conditions (3).

Using the transformation rule (10) we can now define the boundary condition for $\phi$

$$ \phi|_{\partial M} = 0. $$  (14)

The equation (14) fixes the boundary condition for the rotation parameter $\sigma$ and for $\rho_\mu$ since we require locality of the boundary conditions for the latter field as well:

$$ \sigma|_{\partial M} = 0, \quad \rho_1|_{\partial M} = 0, \quad \partial_0 \rho_0|_{\partial M} = 0. $$  (15)

In the theory of the de Rham complex the conditions for $\rho_\mu$ (15) are known as relative boundary conditions [11].

Hence in the case of an independent spin-connection we are indeed able to define local gauge invariant boundary conditions for the fluctuations $\rho$ and $\tilde{h}$.

4. As an example, we consider the Euclidean $R^2 + T^2$ action with zero cosmological constant.

$$ S = \int d^2x e(4R^2 + \alpha T^a T^a), $$

$$ eR = e^{\mu\nu} \partial_\mu \omega_\nu, \quad eT^a = e^{\mu\nu}(\partial_\mu e^a_\nu - \omega_\mu e^{ab}_\nu) $$  (16)
The background in question is a stationary point of the action (16).

The boundary conditions (12), (15) have the following easily established properties.

(i) \( \tilde{h}_\mu^a = e \epsilon_{\mu \nu} \nabla^\nu v^a + e^{\nu a} \nabla^\mu \xi^\nu, \quad \rho_\mu = \partial_\mu \phi + e \epsilon_{\mu \nu} \nabla^\nu s \) (17)

with background covariant derivative \( \nabla \) and zweibein \( e^a_\mu \).

(ii) The decompositions (17) are orthogonal with respect to ordinary inner products without surface terms.

(iii) The fields \( v^a \) and \( s \) satisfy Neumann boundary conditions

\[ \partial_0 v^a|_{\partial M} = 0, \quad \partial_0 s|_{\partial M} = 0. \] (18)

(iv) The kernel of the map \( \{ v^a, \xi^\nu \} \rightarrow \tilde{h}_\mu^a \) consists of two covariantly constant vectors \( v^a \). The kernel of the map \( \{ \phi, s \} \rightarrow \rho_\mu \) consists of one constant scalar \( s \).

The natural gauge fixing conditions are

\[ \nabla^\mu \tilde{h}_\mu^a = 0, \quad \nabla^\mu \rho_\mu = 0 \] (19)

Due to the flatness of the background equations (19) are equivalent to \( \xi = 0 \) and \( \phi = 0 \).

Now we are able to write down the one loop path integral in the gauge (19)

\[ Z = \int Dv^a Ds J_v J_s \exp \left( - \int d^2 x e [\alpha (\epsilon^{ba} \Delta v^a + \nabla^b s)^2 + s \Delta^2 s] \right) \]

\[ J_v = \det (-\Delta)^{\frac{1}{2}}_{v,D} \text{det}'(-\Delta)^{\frac{1}{2}}_{v,N}, \quad J_s = \det (-\Delta)^{\frac{1}{2}}_{s,D} \text{det}'(-\Delta)^{\frac{1}{2}}_{s,N} \] (20)

where the subscripts \( v, s, D, N \) denote vectors, scalars, Dirichlet and Neumann boundary conditions, respectively. The prime indicates the exclusion of zero modes.

Let us change the variables in (20), \( \{ v^a, s \} \rightarrow \{ u^a, s \}, \; u^a = \epsilon^{ab} \Delta v^b - \nabla^b s, \)

\[ Dv^a Ds = \text{det}'(-\Delta)^{\frac{1}{2}}_{v,N} \text{det}'(-\Delta)^{\frac{1}{2}}_{s,N} \text{det}(-\Delta)^{\frac{1}{2}}_{v,D} \text{det}(-\Delta)^{\frac{1}{2}}_{s,D}. \] (21)

By performing the Gaussian integration in (20) with the help of (21) we obtain

\[ Z = \text{det}'(-\Delta)^{\frac{1}{2}}_{v,N} \text{det}'(-\Delta)^{\frac{1}{2}}_{s,N} \text{det}(-\Delta)^{\frac{1}{2}}_{v,D} \text{det}(-\Delta)^{\frac{1}{2}}_{s,D}. \] (22)
Let us make use of the proper time representation of the path integral

\[ \log Z = -\frac{1}{2} \int_0^\infty \frac{dt}{t} K(t) \]  

(23)

\[ K(t) = K_{v,N}(t) + K_{s,N}(t) - K_{v,D}(t) - K_{s,D}(t), \]

where \( K_{v,N}(t) = \text{tr}' \exp(t\Delta_{v,N}) \) etc. The corresponding heat kernels \( K(t) \) can be evaluated with the help of standard expressions [12]

\[ K(t) = \frac{3}{2} \left( \frac{\pi}{t} \right)^{1/2} - 3 + O(t^{-1/2}). \]  

(24)

The heat kernel (24) completely defines the ultra-violet divergencies of \( R^2 + T^2 \) gravity on the unit disk.

Now some comments are in order. We observe that the term with \( t^{-1} \), which is typical for two dimensions, is cancelled as it should be in a model with equal number of bosonic (zweibein and connection) and fermionic (ghosts) degrees of freedom. The surface divergence with \( t^{-1/2} \) is not cancelled. This is just a manifestation of the fact that the ghosts obey Dirichlet boundary conditions while the gauge-fixed fields satisfy the Neumann ones. Note, that if only a finite number of modes is quantized the heat kernel expansion starts with the zeroth power of the proper time \( t \).

In conclusion, let us formulate the main lesson to be drawn from our present study. First, in the presence of an independent connection field it is possible to define local boundary conditions for fluctuations of \( \bar{\theta} \) and \( \omega \) in a diffeomorphism and local Lorentz invariant way even in the case of a not totally geodesic boundary. Second, these boundary conditions do not correspond to a quantum integrable model with a finite number of modes. As a consequence, a model of the latter type can be constructed either for more sophisticated non-local boundary conditions or on a manifold with totally geodesic boundary.

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