Degree Growth, Linear Independence and Periods of a Class of Rational Dynamical Systems

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Abstract

We introduce and study algebraic dynamical systems generated by triangular systems of rational functions. We obtain several results about the degree growth and linear independence of iterates as well as about possible lengths of trajectories generated by such dynamical systems over finite fields. Some of these results are generalisations of those known in the polynomial case, some are new even in this case.

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1 Introduction

Let $\mathbb{F}$ be an arbitrary field $\mathbb{F}$ and let $F_1, \ldots, F_m \in \mathbb{F}(X_1, \ldots, X_m)$ be $m$ rational functions in $m$ variables over $\mathbb{F}$. For each $i = 1, \ldots, m$ we define the
$k$-th iteration of the rational function $F_i$ by the recurrence relation

$$F_i^{(0)} = X_i, \quad F_i^{(k)} = F_i \left( F_1^{(k-1)}, \ldots, F_m^{(k-1)} \right), \quad k = 1, 2, \ldots \quad (1)$$

In this paper we consider dynamical systems generated by multivariate rational functions, we refer to [1, 21, 22] for a background on algebraic dynamical systems.

More precisely, we define the vectors $u_n = (u_{n,1}, \ldots, u_{n,m}) \in \mathbb{F}^m$ by the recurrence relation

$$u_{n+1,i} = F_i(u_{n,1}, \ldots, u_{n,m}), \quad n = 0, 1, \ldots, \quad i = 1, \ldots, m \quad (2)$$

with some initial vector $u_0 = (u_{0,1}, \ldots, u_{0,m}) \in \mathbb{F}^m$.

As we work with rational functions, we make the standard convention (see [3, 9, 10]) that

$$0^{-1} = 0 \quad (3)$$

Using the following vector notation

$$\mathbf{F} = (F_1(X_1, \ldots, X_m), \ldots, F_m(X_1, \ldots, X_m)),$$

we have the recurrence relation

$$u_{n+1} = \mathbf{F}(u_n), \quad n = 0, 1, \ldots \quad (4)$$

In particular, for any $n \geq 0$ and $i = 1, \ldots, m$ we have

$$u_{n,i} = F_i^{(n)}(u_0) = F_i^{(n)}(u_{0,1}, \ldots, u_{0,m})$$

or

$$u_n = \mathbf{F}^{(n)}(u_0),$$

provided that $u_n$ has been generated by (4) without using the convention (3) (that is, no poles have been encountered).

Clearly, if we work over a finite field of $q$ elements, the above sequence (4) of vectors $\{u_n\}$ is eventually periodic with some period $\tau \leq q^m$.

One of the important characteristics of the dynamical system generated by $F_1, \ldots, F_m \in \mathbb{F}(X_1, \ldots, X_m)$ is the degree growth of the functions (1). It is of great interest for the theory of dynamical systems and has been studied in a number of works, see, for example, [2, 25] and references therein. It is also important for applications to pseudorandom number generators [24].
More precisely, although for a “typical” system an exponential degree growth is expected, there are several examples of systems where the degree grows much slower (which is highly beneficial for their applications), and such systems are of special interest.

For example, in [14, 17] several types of multivariate polynomial systems \( \mathcal{F} = \{F_1, \ldots, F_m\} \) of \( m \) polynomials in \( m \) variables over a finite field \( \mathbb{F}_q \) have been constructed and studied, having the “triangular” form

\[
F_1(X_1, \ldots, X_m) = X_1 G_1(X_2, \ldots, X_m) + H_1(X_2, \ldots, X_m),
\]

\[
\ldots
\]

\[
F_{m-1}(X_1, \ldots, X_m) = X_{m-1} G_{m-1}(X_m) + H_{m-1}(X_m),
\]

\[
F_m(X_1, \ldots, X_m) = g_m X_m + h_m,
\]

with \( G_i, H_i \in \mathbb{F}_q[X_{i+1}, \ldots, X_m], \ i = 1, \ldots, m - 1, \) and \( g_m, h_m \in \mathbb{F}_q, \ g_m \neq 0. \) These systems have been further investigated in [13, 18, 19, 20].

For the systems (5), in the case of constant polynomials \( G_i \in \mathbb{F}_q^\ast \) in [14] and polynomials \( G_i \) with leading terms of special form in [17, 18], a series of results have been obtained about the distribution of the corresponding sequences given by (2) that are much stronger than those known for generic systems. Moreover, for these classes of polynomials, it has been shown in [17] that the degrees of the iterations of the polynomials \( F_i, \ i = 1, \ldots, m, \) grow significantly slower than the exponential growth expected for the iterations of a “generic” system of \( m \) polynomials in \( m \) variables. In turn, this leads (see [18]) to much better estimates of exponential sums, and thus of discrepancy, for vectors generated by (5) than for those originated from arbitrary polynomial systems (see [4, 5, 16]).

We also note that the results obtained in [17, 18] regarding the degree growth of the iterations of the polynomials in (5) hold over any field \( \mathbb{F} \).

In this paper we extend the class of rational dynamical systems with slow degree growth and present an analogue of the construction (5), but with rational functions defined by

\[
F_1(X_1, \ldots, X_m) = X_1^{e_1} G_1(X_2, \ldots, X_m) + H_1(X_2, \ldots, X_m),
\]

\[
\ldots
\]

\[
F_{m-1}(X_1, \ldots, X_m) = X_{m-1}^{e_{m-1}} G_{m-1}(X_m) + H_{m-1}(X_m),
\]

\[
F_m(X_1, \ldots, X_m) = g_m X_m^{e_m} + h_m,
\]

with \( e_1, \ldots, e_m \in \{-1, 1\}, \ G_i, H_i \in \mathbb{F}[X_{i+1}, \ldots, X_m], \ i = 1, \ldots, m - 1, \) and \( g_m, h_m \in \mathbb{F}, \ g_m \neq 0. \)
We note that for \( m = 1 \) and \( e = 1 \) we obtain the classical linear congruential generator which have been successfully used for decades in the theory of Quasi Monte Carlo methods, see [7, 8], and for \( m = 1 \) and \( e = -1 \), the classical inversive generator, see [9, 10, 11, 12].

For the above class of multivariate rational functions, we study the degree growth under iterations and, using an approach similar to that of [17, Lemma 1], we show in Section 3 that under certain additional conditions imposed on the systems of rational functions (6), the degree grows polynomially.

Moreover, for applications to pseudorandom number generators, following the standard technique almost identical to that of [17], one almost immediately obtains bounds on the exponential sums with elements of the sequence (4) generated by the system (6) (satisfying the conditions outlined in Section 3), that in turn leads to estimates on the uniformity of distribution of the vectors (4). However, one has also to prove that for any \( k \neq l \) and nonzero vector \( \mathbf{a} = (a_1, \ldots, a_{m-1}) \in \mathbb{F}_{m-1}^m \), the linear combination

\[
Q_{k,l,a} = \sum_{i=1}^{m-1} a_i (F_i^{(k)} - F_i^{(l)})
\]

is a non-constant rational function. We note that in the case of rational functions this does not follow directly from the degree argument as in the case of the polynomial systems (5) in [17], but we give such a result in Section 4.

Since the derivation of such bounds of exponential sums for our systems does not bring anything new to the area, we do not do this here but rather concentrate on the study of the degree and linear independence of iterates, which is also of interest for the general area of algebraic dynamics.

Furthermore, we consider a related question about the length of trajectories generated by iterations (4) over a finite field \( \mathbb{F}_q \). We remark that in this case a trajectory falls into a cycle if \( \mathbf{u}_t = \mathbf{u}_s \) for some integers \( t > s \geq 0 \). In particular, we show that under some rather broad conditions for any fixed \( \varepsilon > 0 \), for all but \( o(q^m) \) initial vectors \( \mathbf{u}_0 \in \mathbb{F}_q^m \), the trajectory length \( t \) of the iterations (4) is at least \( q^{1/3-\varepsilon} \).

We note that Silverman [23] has considered a question about periods of general polynomial systems but in somewhat dual situation when the initial value is fixed and the iterations are considered over a family of finite fields.
The results of [23] apply to very general systems, however the estimates are only logarithmic rather than a power of the field size.

Moreover, we give necessary and sufficient conditions for the systems (6) to generate sequences of maximal period. We note that for the case $c_i = 1$ for all $i = 1, \ldots, m$, the maximal period length of the sequence generated by the system (5) is achieved whenever the conditions of [15, Theorem 6] are satisfied. Our result is a generalisation of that of [15].

## 2 Structure of the Iterations

As in [17], we can describe explicitly the iterations of the rational functions $F_i$ as follows.

Let us define the sets

$$I_+ = \{1 \leq i \leq m : e_i = 1\} \quad \text{and} \quad I_- = \{1 \leq i \leq m : e_i = -1\}. \quad (8)$$

We also define

$$G_i^{(\ell)}(X_{i+1}, \ldots, X_m) = G_i\left(F_{i+1}^{(\ell-1)}, \ldots, F_m^{(\ell-1)}\right),$$
$$H_i^{(\ell)}(X_{i+1}, \ldots, X_m) = H_i\left(F_{i+1}^{(\ell-1)}, \ldots, F_m^{(\ell-1)}\right).$$

**Lemma 1.** Let $F_1, \ldots, F_m$ be rational functions defined by (6). Then for $i = 1, \ldots, m - 1$ and $k = 0, 1, \ldots$, for the rational functions $F_i^{(k)}$ given by (1),

1. for every $i \in I_+$, $i < m$, we have

$$F_i^{(k)} = X_iG_i^{(k)} + H_i^{(k)}, \quad (9)$$

where $G_i^{(k)}, H_i^{(k)} \in \mathbb{F}(X_{i+1}, \ldots, X_m)$ are defined by

$$G_i^{(k)} = G_iG_i^{(2)} \ldots G_i^{(k)},$$
$$H_i^{(k)} = H_iG_i^{(2)} \ldots G_i^{(k)} + H_i^{(2)}G_i^{(3)} \ldots G_i^{(k)} + \ldots + H_i^{(k-1)}G_i^{(k)} + H_i^{(k)}; \quad (10)$$

2. for every $i \in I_-$, $i < m$, we have:

$$F_i^{(k)} = \frac{X_iR_{i,k} + S_{i,k}}{X_iR_{i,k-1} + S_{i,k-1}}, \quad (11)$$
where $R_{i,k}, S_{i,k}$ are defined by the recurrence relations

$$
R_{i,k} = G_i^{(k)} R_{i,k-2} + H_i^{(k)} R_{i,k-1} \\
S_{i,k} = G_i^{(k)} S_{i,k-2} + H_i^{(k)} S_{i,k-1}
$$

for $k \geq 1$, with the initial rational functions

$$
R_{i,0} = 1, \quad S_{i,0} = 0, \quad R_{i,1} = H_i, \quad S_{i,1} = G_i;
$$

3. if $m \in I_+$, then

$$
F_m^{(k)} = g_m^k X_m + (g_m^{k-1} + \ldots + g_m + 1) h_m;
$$

4. if $m \in I_-$, then

$$
F_m^{(k)} = \frac{(A^k)_{1,1} X_m + (A^k)_{1,2}}{(A^k)_{2,1} X_m + (A^k)_{2,2}},
$$

where

$$
A^k = \begin{pmatrix} h_m & g_m \\ 1 & 0 \end{pmatrix}^k = \begin{pmatrix} (A^k)_{1,1} & (A^k)_{1,2} \\ (A^k)_{2,1} & (A^k)_{2,2} \end{pmatrix}.
$$

Proof. The case $e_i = 1$, $i = 1, \ldots, m$, is given by [18, Lemma 1]. We consider now that $e_i = -1$ and prove the result by induction on the number of iterations $k$. For $k = 1$ it is clear from the definition of the system, so we consider the statement true for the first $k - 1$ iterations and we prove it for the $k$-th iteration. For $i = 1, \ldots, m - 1$, we have

$$
F_i^{(k)} = F_i(F_i^{(k-1)}, F_{i+1}^{(k-1)}, \ldots, F_m^{(k-1)})
$$

$$
= \frac{F_i^{(k-1)} H_i^{(k)} + G_i^{(k)}}{F_i^{(k-1)}} = \frac{X_i R_{i,k-1} + S_{i,k-1} H_i^{(k)} + G_i^{(k)}}{X_i R_{i,k-2} + S_{i,k-2}}
$$

$$
= \frac{X_i (G_i^{(k)} R_{i,k-2} + H_i^{(k)} R_{i,k-1}) + G_i^{(k)} S_{i,k-2} + H_i^{(k)} S_{i,k-1}}{X_i R_{i,k-1} + S_{i,k-1}}
$$

and thus we conclude this case. When $e_m = -1$, it is also clear as the $k$-th iteration of

$$
F_m = \frac{h_m X_m + g_m}{X_m}
$$

is given by $A^k$ as simple calculations show. $\square$
We want to describe the degree growth of the iterations of the rational functions defined by (6), and in particular to prove that we have the same effect of slow degree growth as for the polynomial systems (5) described in [17, Lemma 1]. To be able to give an explicit formula for the degree growth we need to impose some further conditions on the degrees of the polynomials $G_i$ and $H_i$, $i = 1, \ldots, m - 1$.

Let $F_1, \ldots, F_m$ be rational functions defined by (6). From now on we consider the system (6) satisfying the following conditions for $F_i$ for any $i = 1, \ldots, m$:

1. if $e_i = 1$, as in [17, 18], we assume that the polynomial $G_i$ has a unique leading monomial $X_{i+1}^{s_{i,i+1}} \cdots X_{m}^{s_{i,m}}$, that is

$$G_i = g_i X_{i+1}^{s_{i,i+1}} \cdots X_m^{s_{i,m}} + \tilde{G}_i,$$

where $g_i \in \mathbb{F}^*$ and $\tilde{G}_i \in \mathbb{F}[X_{i+1}, \ldots, X_m]$ with

$$\deg_{X_j} \tilde{G}_i < s_{i,j}, \quad \deg_{X_j} H_i < s_{i,j}, \quad j = i + 1, \ldots, m; \quad (13)$$

2. if $e_i = -1$, we assume that the polynomial $H_i$ has a unique leading monomial $X_{i+1}^{s_{i,i+1}} \cdots X_{m}^{s_{i,m}}$, that is

$$H_i = h_i X_{i+1}^{s_{i,i+1}} \cdots X_m^{s_{i,m}} + \tilde{H}_i,$$

where $h_i \in \mathbb{F}^*$ and $\tilde{H}_i \in \mathbb{F}[X_{i+1}, \ldots, X_m]$, and

$$\deg_{X_j} \tilde{H}_i < s_{i,j}, \quad \deg_{X_j} G_i < 2s_{i,j}, \quad j = i + 1, \ldots, m. \quad (14)$$

We note that having these conditions also allows us to consider the rational function system with constant multipliers $G_i$, $i = 1, \ldots, m - 1$. We remark that in [14], the case of constant polynomials $G_i$, $i = 1, \ldots, m - 1$, in the system (5) was considered, but this case is different from the case of rational functions as the conditions on the degrees also differ, see (13) and (14). Having this, we prove the following formula for the degree growth which coincides with [17, Lemma 1].
3 Degree Growth

**Theorem 2.** Let $F_1, \ldots, F_m$ be rational functions defined by (6) satisfying the conditions (13) and (14) and such that $s_{i,i+1} \neq 0$, $i = 1, \ldots, m-1$. Then the degrees of the iterations of $F_1, \ldots, F_m$ grow as follows

$$\deg F_i^{(k)} = \frac{1}{(m-i)!}k^{m-i}s_{i,i+1} \cdots s_{m-1,m} + \psi_i(k), \quad i = 0, \ldots, m-1,$$

$$\deg F_m^{(k)} = 1,$$

where $\psi_i(T) \in \mathbb{Q}[T]$ is a polynomial of degree $\deg \psi_i < m-i$.

**Proof.** The proof is based on Lemma 1. The case $e_i = 1$, using (9) and (11), follows exactly the same as in [17, Lemma 1].

We prove now the case when $e_i = -1$. Using the conditions (13) and (14) and the recurrence relation (12), it is easy to see that

$$\deg F_i^{(k)} = \deg R_i^{(k)} + 1 = \deg H_i^{(k)} R_{i,k-1} + 1$$

$$= \deg H_i H_i^{(2)} \cdots H_i^{(k)} + 1 = \sum_{j=1}^{k} \deg H_i^{(j)} + 1. \quad (15)$$

As in [17, Lemma 1], we use induction on the number of variables $m$. For $m = 2$ we easily see that $\deg H_1^{(j)} = \deg H_1 = s_{1,2}$, and thus, by (15), we have that $\deg F_1^{(k)} = ks_{1,2} + 1$. We assume now that the theorem is true for $m - 1$ variables and we prove it for $m$. For any $i = 1, \ldots, m - 1$, by the induction hypothesis, we have

$$\deg F_i^{(k)} = \sum_{j=1}^{k} \deg H_i^{(j)} + 1 = \sum_{j=1}^{k} \deg H_i(F_i^{(j-1)}, \ldots, F_m^{(j-1)}) + 1$$

$$= \sum_{j=1}^{k} \deg ((F_i^{(j-1)})^{s_{i,i+1}} \cdots (F_m^{(j-1)})^{s_{i,m}}) + 1$$

$$= \sum_{j=1}^{k} \left( \frac{1}{(m-i-1)!} (j-1)^{m-i-1} s_{i,i+1} s_{i+1,i+2} \cdots s_{m-1,m} + \right.$$  

$$\left. \cdots + (j-1) s_{i,m} s_{m-1,m} \right) + 1.$$

As

$$\sum_{j=1}^{k} j^{m-1-i} = \frac{1}{m-i}(B_{m-i}(k+1) - B_{m-i}(0)),$$
where $B_{m-i}$ is the Bernoulli polynomial of degree $m-i$ (which has the leading coefficient equal to 1), we finally obtain the desired result. □

4 Linear Independence

Theorem 3. Let $F_1, \ldots, F_m$ be rational functions defined by (6) satisfying the conditions (13) and (14) and such that $s_{i,i+1} \neq 0$, $i = 1, \ldots, m-1$. Then, for $k \neq l$ and a nonzero vector $a \in \mathbb{F}^{m-1}$, $Q_{k,l,a}$ is a non-constant rational function.

Proof. The proof reduces to proving that $\deg(F_s^{(k)} - F_s^{(l)}) > 1$, where $s \leq m-1$ is the smallest index such that $a_s \neq 0$, as the variable $X_s$ does not appear in the polynomial

$$Q_{k,l,a} - a_s(F_s^{(k)} - F_s^{(l)}) = \sum_{i=s+1}^{m-1} a_i(F_i^{(k)} - F_i^{(l)}).$$

If $e_s = 1$, it is clear, as from Theorem 2, for $k > l$ we have $\deg G_{s,k} > \deg G_{s,l}$.

If $e_s = -1$, by (11), we have

$$F_s^{(k)} - F_s^{(l)} = \frac{X_sR_{s,k} + S_{s,k}}{X_sR_{s,k-1} + S_{s,k-1}} - \frac{X_sR_{s,l} + S_{s,l}}{X_sR_{s,l-1} + S_{s,l-1}} = \frac{U_{k,l,s}}{V_{k,l,s}},$$

where

$$U_{k,l,s} = X_s^2(R_{s,k}R_{s,l-1} - R_{s,k-1}R_{s,l}) + X_s(R_{s,k}S_{s,l-1} + S_{s,k}R_{s,l-1} - R_{s,k-1}S_{s,l} - S_{s,k-1}R_{s,l}) + S_{s,k}S_{s,l-1} - S_{s,k-1}S_{s,l}.$$

and

$$V_{k,l,s} = X_sR_{s,k-1}R_{s,k-1} + X_s(R_{s,k-1}S_{s,l-1} + R_{s,l-1}S_{s,k-1}) + S_{s,k-1}S_{s,l-1}.$$

Without loss of generality we may assume that $k > l$. Using (12), we obtain

$$R_{s,k}R_{s,l-1} - R_{s,k-1}R_{s,l} = (G_s^{(k)}R_{s,k-2} + H_s^{(k)}R_{s,k-1})R_{s,l-1} - R_{s,k-1}(G_s^{(l)}R_{s,l-2} + H_s^{(l)}R_{s,l-1}),$$

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and thus, using Lemma 1 and Theorem 2, we derive
\[
\deg(R_{s,k}R_{s,l-1} - R_{s,k-1}R_{s,l}) = \deg(H^{(k)}_s - H^{(l)}_s)R_{s,k-1}R_{s,l-1} \\
> \deg H^{(k)}_s R_{s,k-1}R_{s,l-1} > \deg R_{s,k-1}R_{s,l-1} > 1
\]  \hspace{1cm} (16)
for \( k > l \), which concludes the proof. \( \square \)

Note that, as in [18], we can include \( m \)-term linear combinations
\[
\overline{Q}_{k,l,a} = \sum_{i=1}^{m} a_i(F^{(k)}_i - F^{(l)}_i)
\]
with \( a \in \mathbb{F}_m \), but in the case of \( a_1 = \ldots = a_{m-1} = 0 \), \( a_m \neq 0 \), the nontriviality also depends on the divisibility of \( k - l \) by the multiplicative order of \( g_m \).

## 5 Trajectory Lengths

In this section we work over a finite field \( \mathbb{F}_q \).

**Theorem 4.** Let \( F_1, \ldots, F_m \) be rational functions defined by (6) satisfying the conditions (13) and (14) and such that \( s_{i,i+1} \neq 0 \), \( i = 1, \ldots, m-1 \). Then, for any \( T \geq 1 \), for all but \( O(T^3q^{m-1}) \) initial vectors \( u_0 \in \mathbb{F}_q^m \), the trajectory length of the iterations (4) exceeds \( T \).

**Proof.** Let \( \mathcal{U} \) be the set of \( u = (u_1, \ldots, u_m) \in \mathbb{F}_q^m \) such that
\[
G_i(u_{i+1}, \ldots, u_m) = 0
\]
for some \( i = 1, \ldots, m-1 \). Clearly \( \#\mathcal{U} = O(q^{m-1}) \).

We now build a sequence of sets \( \mathcal{U}_k \), \( k = 0, 1, \ldots, \), recursively.

We put \( \mathcal{U}_0 = \mathcal{U} \).

Assume that \( \mathcal{U}_0, \ldots, \mathcal{U}_k \) are defined and let
\[
\mathcal{W}_k = \mathcal{U}_0 \cup \ldots \cup \mathcal{U}_k.
\]

Then we let \( \mathcal{U}_{k+1} \) be the set of the initial values
\[
u_0 \in \mathbb{F}_q^m \setminus \mathcal{W}_k
\]
such that for the corresponding sequence of vectors (4) we have \( \nu_{k+1} \in \mathcal{U} \).

Inspecting (6), we now see that, by our assumption, for any \( \nu \in \mathbb{F}_q^m \), there
is a unique preimage $u \in \mathbb{F}^m_q \setminus U$ under the map given by (6) (that is, with $v = F(u)$). In turn, we see that for any $v \in U$ there is a unique corresponding initial value $u_0 \in \mathbb{F}^m_q \setminus W_k$ with $u_{k+1} = v$. Thus $\#U_k = \#U$.

Since there are $O(q^{m-1})$ vectors $u \in \mathbb{F}^m_q$ that contain a zero in at least one component, we see that the set $E_T$ of initial values for which, for some integer $t \leq T$, the vector $u_t$ has a zero component, satisfies

$$E_T = O(q^{m-1} + T \#U) = O(T q^{m-1}).$$  

(17)

We see that if a vector $u_0 \in \mathbb{F}^m_q \setminus E_T$ generates a trajectory of lengths $t \leq T$ then $u_t = u_s$ for some nonnegative integer $s < t$.

Now, if $e_{m-1} = 1$, then we remove the set $F_T$ of initial vectors $u_0 \in \mathbb{F}^m_q$ such that

$$G_{m-1,t}(u_0) = G_{m-1,s}(u_0)$$

for some integers $s$ and $t$ with $T \geq t > s \geq 0$. By Lemma 1 we see that $G_{m-1,t} - G_{m-1,s}$ is a nontrivial polynomial of degree $O(t)$. Hence,

$$\#F_T = O\left( \sum_{0 \leq s < t \leq T} t q^{m-1} \right) = O\left( T^3 q^{m-1} \right).$$  

(18)

Furthermore, if $e_{m-1} = -1$, then we remove the set $F_T$ of initial vectors $u_0 \in \mathbb{F}^m_q$ such that

$$R_{m-1,t}(u_0)R_{m-1,s-1}(u_0) = R_{m-1,t-1}(u_0)R_{m-1,s}(u_0)$$

for some integers $s$ and $t$ with $T \geq t > s \geq 0$. As in the proof of Theorem 3 (in particular, see (16)) we note that Lemma 1 implies that $R_{m-1,t}R_{m-1,s-1} - R_{m-1,t-1}R_{m-1,s}$ is a nontrivial polynomial of degree $O(t)$. Hence, again we obtain the bound (18).

We remark that for $T \geq t > s \geq 0$, for any solution $u_0 = (u_{0,1}, \ldots, u_{0,m}) \in \mathbb{F}^m_q \setminus F_T$ to the equation

$$F(t)(u_0) = F(s)(u_0),$$

the component $u_{0,m-1}$ is uniquely defined by $u_{0,m}$. So there at most $q^{m-1}$ such solutions for every fixed $t$ and $s$ with $T \geq t > s \geq 0$ and thus at most $T^2 q^{m-1}$ for such $t$ and $s$. Combining this bound with (17) and (18) we conclude the proof.  

$\square$
Clearly if the map $u \mapsto F(u)$ is a permutation, as for example, in [13], then all trajectories are purely periodic. So we always have $s = 0$ in the argument of the proof of Theorem 4. This leads to a better estimate $O(T^2 q^{m-1})$ on the number of initial values generating trajectories of length at most $T$.

6 Maximal Periods

In this section we show that the periods of the rational function systems over $\mathbb{F}_q$ defined by (6) with $e_i = -1$ for all $i = 1, \ldots, m$ are given by the orbit lengths of certain linear fractional transformations, also called Möbius transformations. In particular, we describe the case when the systems (6) achieve maximal periods in their orbits. We also note that in [15, Theorem 6] there are given necessary and sufficient conditions for the system (6) to achieve maximal period in the case $e_i = 1$ for all $i = 1, \ldots, m$.

We denote

$$\tilde{u}_{0,i} = (u_{0,i+1}, \ldots, u_{0,m}) \in \mathbb{F}_q^{m-i}, \quad i = 1, \ldots, m - 1.$$ 

Lemma 5. Let $F_1, \ldots, F_m \in \mathbb{F}_q[X_1, \ldots, X_m]$ be as in (6) with $e_i = -1$ for all $i = 1, \ldots, m$. Assume that the sequence generated by the lower $m - i$ rational functions $F_{i+1}, \ldots, F_m$ in $\mathbb{F}_q^{m-i}$ is purely periodic with period $\tau_{i+1}$ for some $i = 1, \ldots, m - 1$. Then we have the following description for the $k\tau_{i+1}$-th iteration of $F_i$ on any initial vector $u_0 \in \mathbb{F}_q^m$:

$$F_i^{(k\tau_{i+1})}(u_0) = f_i^{(k)}(u_{0,i}), \quad k \geq 1,$$  \hspace{1cm} (19)

where $R_{i,\tau_{i+1}}$ and $S_{i,\tau_{i+1}}$ are defined by (12) and $f_i$ is the Möbius transformation in the variable $Y$,

$$f_i(Y) = \frac{Y R_{i,\tau_{i+1}}(\tilde{u}_{0,i}) + S_{i,\tau_{i+1}}(\tilde{u}_{0,i})}{Y R_{i,\tau_{i+1}-1}(\tilde{u}_{0,i}) + S_{i,\tau_{i+1}-1}(\tilde{u}_{0,i})}.$$ 

In particular, the orbit length of $F_i$ in $u_0$ is given by the orbit length of $f_i$ in $u_{0,i}$.

Proof. We first note that the orbit length of $F_i$ is a multiple of $\tau_{i+1}$. Indeed, let $\tau_i$ be the orbit length of the system $F_i, \ldots, F_m$ in the initial vector $u_0$. Then $\tau_i = \text{lcm}(\tau_{i+1}, \eta_i)$, where $\eta_i$ is the period of the sequence $\{u_{n,i}\}$ defined by the iterations of the polynomial $F_i$, and thus $\tau_i$ is a multiple of $\tau_{i+1}$. This
shows that, in order to describe the period of $F_i, \ldots, F_m$ on the initial vector $u_0$, it is enough to consider only $k\tau_{i+1}$-th iterations of $F_i$.

By (11) we have

$$F_{i}^{(k\tau_{i+1})}(u_0) = \frac{u_{0,i}R_{i,\tau_{i+1}}(\tilde{u}_{0,i}) + S_{i,\tau_{i+1}}(\tilde{u}_{0,i})}{u_{0,i}R_{i,\tau_{i+1}-1}(\tilde{u}_{0,i}) + S_{i,\tau_{i+1}-1}(\tilde{u}_{0,i})} = f_{i}(u_{0,i}).$$

(20)

Now,

$$F_{i}^{(k\tau_{i+1})}(u_0)$$

$$= F_{i}^{(k\tau_{i+1})}(F_{i}^{((k-1)\tau_{i+1})}(u_0), \ldots, F_{m}^{((k-1)\tau_{i+1})}(u_0))$$

$$= F_{i}^{((k-1)\tau_{i+1})}(u_0), u_{0,i+1}, \ldots, u_{0,m}(21)$$

To prove (19) we use induction over $k$. For $k = 1$ it is clear.

We now assume that the statement is true for $k - 1$ and we prove it for $k$. Using (20), (21) and the induction hypothesis, we derive

$$F_{i}^{(k\tau_{i+1})}(u_0) = \frac{f_{i}^{(k-1)}(u_{0,i})R_{i,\tau_{i+1}}(\tilde{u}_{0,i}) + S_{i,\tau_{i+1}}(\tilde{u}_{0,i})}{f_{i}^{(k-1)}(u_{0,i})R_{i,\tau_{i+1}-1}(\tilde{u}_{0,i}) + S_{i,\tau_{i+1}-1}(\tilde{u}_{0,i})} = f_{i}^{(k)}(u_{0,i}),$$

which concludes the proof. \qed

**Lemma 6.** Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a system of polynomials over $\mathbb{F}_q$ defined by (6). Let $i = 1, \ldots, m$ such that $e_i = -1$ in the system (6). Then we have

$$R_{i,k}S_{i,k-1} - R_{i,k-1}S_{i,k} = (-1)^k G_{i}(G_{i}^{(2)} \ldots G_{i}^{(k)}),$$

where $R_{i,k}, S_{i,k}$ are defined by (12).

**Proof.** We use induction on $k$. If $k = 1$, by (12), we get

$$R_{i,1}S_{i,0} - R_{i,0}S_{i,1} = -G_{i}.$$ We assume the statement true for $k$ and we prove it for $k + 1$. By (12) and the induction hypothesis we have

$$R_{i,k+1}S_{i,k} - R_{i,k}S_{i,k+1}$$

$$= (G_{i}^{(k+1)} R_{i,k-1} + H_{i}^{(k+1)} R_{i,k})S_{i,k} - R_{i,k}(G_{i}^{(k+1)} S_{i,k-1} + H_{i}^{(k+1)} S_{i,k})$$

$$= -G_{i}^{(k+1)} (R_{i,k}S_{i,k-1} - R_{i,k-1}S_{i,k})$$

$$= (-1)^{k+1} G_{i}G_{i}^{(2)} \ldots G_{i}^{(k)} G_{i}^{(k+1)}.$$
and thus we conclude the proof.

As usual, we say that a polynomial \( f \in \mathbb{F}_q[X] \) of degree \( d \geq 1 \) is primitive if it is the minimal polynomial over \( \mathbb{F}_q \) of a primitive element of \( \mathbb{F}_{q^d} \) (that is, an element of multiplicative order \( q^d - 1 \)), see [6].

Next, we present necessary and sufficient conditions for the system (6) to achieve maximal period over the prime field \( \mathbb{F}_p \).

Using [15, Lemma 2] which holds for the functions \( F_i \) in the system (6) for which \( e_i = 1 \), we have the following analogue of [15, Lemma 5] (with an almost identical proof which we do not present here).

**Lemma 7.** Let \( \mathcal{F} = \{F_1, \ldots, F_m\} \) be a system of polynomials over \( \mathbb{F}_p \) defined by (6). Let the index \( 1 \leq i \leq m \) such that \( e_i = 1 \) and assume that the period of the sequence generated by the lower \( m - i \) polynomials \( F_{i+1}, \ldots, F_m \) in \( \mathbb{F}_p^{m-i} \) is \( p^{m-i} \) and that \( G_{i,p^{m-i}}(\tilde{u}_{0,i}) = 1 \). Then, for the rational functions \( H_{i,p^{m-i}} \) defined by (10), we have

\[
H_{i,p^{m-i}}(\tilde{u}_{0,i}) = \sum_{v \in \mathbb{F}_p^{m-i}} R_i(v),
\]

where

\[
R_i \equiv H_i G_i^{(2)} \cdots G_i^{(p^{m-i})}.
\]

Now, using Lemma 7 and [15, Theorem 6] for \( F_i \) with \( e_i = 1 \) in the system (6), we have the following result.

We recall that the sets \( I_+ \) and \( I_- \) are given by (8).

**Theorem 8.** Let \( \mathcal{F} = \{F_1, \ldots, F_m\} \) be a system of polynomials over \( \mathbb{F}_p \) defined by (6). Then the sequence \( \{u_n\} \) generated by (4) is purely periodic with period \( \tau = p^m \) if and only if the following conditions are satisfied

1. for every \( i \in I_+ \), \( i < m \), we have

\[
\prod_{v \in \mathbb{F}_p^{m-i}} G_i(v) = 1 \quad \text{and} \quad \sum_{v \in \mathbb{F}_p^{m-i}} R_i(v) \neq 0;
\]

2. for every \( i \in I_- \), \( i < m \), we have:

   (a) if \( R_{i,p^{m-i-1}}(u_0) = 0 \), then

   \[
   R_{i,p^{m-i}}(u_0) = S_{i,p^{m-i-1}}(u_0) \quad \text{and} \quad S_{i,p^{m-i}}(u_0)S_{i,p^{m-i-1}}(u_0) \neq 0;
   \]
(b) if $R_{i,p^{m-i-1}}(u_0) \neq 0$, then

$$X^2 - \frac{R_{i,p^{m-i}}(u_0)}{R_{i,p^{m-i-1}}(u_0)} X - \frac{\prod_{v \in F_p^{m-1}} G_1(v)}{R_{i,p^{m-i-1}}(u_0)}$$

is a primitive polynomial over $F_p$;

3. if $m \in I_+$, then $g_m = 1$;

4. if $m \in I_-$, then $X^2 - h_m X - g_m$ is a primitive polynomial over $F_p$.

**Proof.** We prove the result by induction on $m$. For $m = 1$, if $m \in I_+$, then it’s clear that the period $p$ is achieved if and only if $g_m = 1$ and $h_m \in F_p^*$. If $m \in I_-$, we have

$$F_1 = g_1 X_1^{-1} + h_1,$$

which, by [3, Theorem 1], has maximal period $p$ if and only if the polynomial $X^2 - h_1 X - g_1$ is a primitive polynomial over $F_p$.

We assume the statement true for the first $m - 1$ variables and we want to prove it for $m$. Let the sequence $\{\tilde{u}_{n,1}\} = \{(u_{n,2}, \ldots, u_{n,m})\}$ be defined by the last $m - 1$ components of the vectors in the sequence $\{u_n\}$. By the induction hypothesis we know that the period $\tilde{\tau}$ of the sequence $\{\tilde{u}_{n,1}\}$ is $p^{m-1}$, and taking into account the first remark in the proof of Lemma 5, we see that the period of $\{\tilde{u}_{n,1}\}$ is of the form $kp^{m-1}$, for some $1 \leq k \leq q$. Thus, proving the maximality of the period of $\{u_n\}$ reduces to proving that $k = q$. We note that the representation of $F_1^{(k)}$ given by Lemma 1 does not depend how we choose $e_2, \ldots, e_m$ in the functions $F_2, \ldots, F_m$, but only the functions $G_{1,k}, H_{1,k}$ or $G_{1}^{(k)}, H_{1}^{(k)}$ if $1 \in I_+$ or $1 \in I_-$, respectively. Thus, the representation of $F_1^{(k)}$ given by Lemma 1 is the same regardless if $i_j \in I_+$ or $i_j \in I_-$ for $2 \leq j \leq m$.

Thus, the case $1 \in I_+$ follows identically as in the proof of [15, Theorem], and we do not repeat it here.

We consider now the case $1 \in I_-$. By Lemma 5 we have

$$F_1^{(kp^{m-1})}(u_0) = f_1^{(k)}(u_{0,1}), \quad k \geq 1,$$

where

$$f_1(Y) = \frac{Y R_{1,p^{m-1}}(\tilde{u}_{0,1}) + S_{1,p^{m-1}}(\tilde{u}_{0,1})}{Y R_{1,p^{m-1-1}}(\tilde{u}_{0,1}) + S_{1,p^{m-1-1}}(\tilde{u}_{0,1})},$$

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and thus the maximal period of the sequence generated by the iterations of $F_1$ is given by the case when $f_1$ achieves maximal orbit length in $u_{0,1}$.

We distinguish now two cases. If $R_{1,p^{m-1}-1}(\tilde{u}_{0,1}) = 0$, then we note that $S_{1,p^{m-1}-1}(\tilde{u}_{0,1}) \neq 0$, as otherwise $f_1 = 0$. Thus, we have the case of linear generator

$$ f_1(Y) = \frac{R_{1,p^{m-1}}(\tilde{u}_{0,1})}{S_{1,p^{m-1}-1}(\tilde{u}_{0,1})} Y + \frac{S_{1,p^{m-1}}(\tilde{u}_{0,1})}{S_{1,p^{m-1}-1}(\tilde{u}_{0,1})}. $$

The maximal period is achieved in this case if and only if

$$ \frac{R_{1,p^{m-1}}(\tilde{u}_{0,1})}{S_{1,p^{m-1}-1}(\tilde{u}_{0,1})} = 1, \quad \frac{S_{1,p^{m-1}}(\tilde{u}_{0,1})}{S_{1,p^{m-1}-1}(\tilde{u}_{0,1})} \neq 0, $$

which is equivalent to

$$ R_{1,p^{m-1}}(\tilde{u}_{0,1}) = S_{1,p^{m-1}-1}(\tilde{u}_{0,1}), \quad S_{1,p^{m-1}}(\tilde{u}_{0,1}) \neq 0. $$

We consider now the case of $R_{1,p^{m-1}-1}(\tilde{u}_{0,1}) = 0$. We note that in this case $f_1$ achieves maximal period if and only if, under a linear transformation, it has the same property. Taking into account that $R_{1,p^{m-1}-1}(\tilde{u}_{0,1}) \neq 0$, we can make the linear transformation

$$ Y \rightarrow R_{1,p^{m-1}-1}(\tilde{u}_{0,1})^{-1} Y - R_{1,p^{m-1}-1}(\tilde{u}_{0,1})^{-1} S_{1,p^{m-1}-1}(\tilde{u}_{0,1}) $$

and obtain the following inversive generator which, by a slightly abuse of notation, we denote also by $f_1$,

$$ f_1(Y) = \frac{-R_{1,p^{m-1}}(\tilde{u}_{0,1})S_{1,p^{m-1}-1}(\tilde{u}_{0,1}) + R_{1,p^{m-1}-1}(\tilde{u}_{0,1})S_{1,p^{m-1}}(\tilde{u}_{0,1})}{R_{1,m-1}(\tilde{u}_{0,1})} Y^{-1} + \frac{R_{1,p^{m-1}}(\tilde{u}_{0,1})}{R_{1,p^{m-1}-1}(\tilde{u}_{0,1})}. $$

Applying now Lemma 6 we have

$$ R_{1,p^{m-1}}(\tilde{u}_{0,1})S_{1,p^{m-1}-1}(\tilde{u}_{0,1}) - R_{1,p^{m-1}-1}(\tilde{u}_{0,1})S_{1,p^{m-1}}(\tilde{u}_{0,1}) $$

$$ = -G_1(\tilde{u}_{0,1})G_1(2)(\tilde{u}_{0,1}) \cdots G_1(p^{m-1})(\tilde{u}_{0,1}). $$

Let $\mathcal{F} = \{F_2, \ldots, F_m\}$. Now, as the period induced by $\mathcal{F}$ is $p^{m-1}$, the elements

$$ \tilde{u}_{0,1}, \mathcal{F}(\tilde{u}_{0,1}), \ldots, \mathcal{F}(p^{m-1})(\tilde{u}_{0,1}) $$
are all the distinct elements of $\mathbb{F}_p^{m-i}$, and thus we obtain
\[
G_1(\tilde{u}_{0,1})G_1^{(2)}(\tilde{u}_{0,1}) \ldots G_1^{(p^{m-1})}(\tilde{u}_{0,1})
= G_1(\tilde{u}_{0,1})G_1(\tilde{F}(\tilde{u}_{0,1})) \ldots G_1(\tilde{F}^{(p^{m-1}-1)}(\tilde{u}_{0,1})) = \prod_{v \in \mathbb{F}_p^{m-1}} G_1(v).
\]
This concludes that
\[
f_1(Y) = \frac{\prod_{v \in \mathbb{F}_p^{m-1}} G_1(v)}{R_{1,p^{m-1}}(\tilde{u}_{0,1})} Y^{-1} + \frac{R_{1,p^{m-1}}(\tilde{u}_{0,1})}{R_{1,p^{m-1}-1}(\tilde{u}_{0,1})}.
\]
Applying now [3, Theorem 1], we know that $f_1$ achieves maximal period $q$ if and only if the polynomial
\[
X^2 - \frac{R_{1,p^{m-1}}(\tilde{u}_{0,1})}{R_{1,p^{m-1}-1}(\tilde{u}_{0,1})} X - \frac{\prod_{v \in \mathbb{F}_p^{m-1}} G_1(v)}{R_{1,p^{m-1}-1}(\tilde{u}_{0,1})}
\]
is a primitive polynomial over $\mathbb{F}_p$. \hfill \Box

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