To the question of the existence and uniqueness of the scattering problem solution. The case of three one-dimensional quantum particles interacting by finite repulsive pair potentials.

A.M. Budylin, S.B. Levin

Introduction

In the work [4] asymptotic formulas at infinity in configuration space were offered for the first time for absolutely continuous spectrum eigenfunctions uniform at angle variables. We considered the case of three one-dimensional short-range quantum particles interacting by the repulsive pair potentials. The mentioned asymptotic formulas were obtained in terms of formal asymptotic decompositions within the framework of a rather subtle heuristic analysis.

The present work aims at announcing a proof of the existence and uniqueness theorem of the scattering problem solution in the finite repulsive pair potentials case. It is worth emphasizing that the limitation of treatments for the finite potentials case does not lead to a problem simplification in its essence as the interaction potential of all three particles remains non-decreasing at infinity but allows us to switch off from a certain number of technical details.

1 Preliminaries

Initial Setting and Reducing of the Problem. In the initial setting the non-relativistic Hamiltonian is considered

\[ H\psi = -\Delta\psi + \frac{1}{2} \sum_{1 \leq i \neq j \leq 3} v(z_i - z_j)\psi, \quad z_i \in \mathbb{R}, \quad z = (z_1, z_2, z_3) \in \mathbb{R}^3, \quad \psi = \psi(z) \in \mathbb{C}, \]

\( \Delta \) is a Laplace operator in \( \mathbb{R}^3 \), \( v \) is an even finite integrable function \( \mathbb{R}^3 \to [0, +\infty) \), defining two-particle interaction. In this case the essential self-adjointness of the operator \( H \) in the square-integrable functions space is well known.

To separate the center-of-mass motion, we restrict the Hamiltonian on the surface \( \Pi \) which is denoted by the equation \( \sum z_i = 0 \). With a certain negligence, we will hereafter denote the Laplace-Beltrami operator on the plane \( \Pi \subset \mathbb{R}^3 \) by \( \Delta \). Here at \( \Pi \) it is convenient to use any pair of \( (x_i, y_i), \; i = 1, 2, 3, \) the so-called Jacobi coordinates, uniquely denoted by the equations \( x_i = \frac{1}{\sqrt{2}}(z_k - z_j), \; y_i = \sqrt{2}z_i \), the indices \( (i, j, k) \) here form even permutations.

In view of the orthonormality of the Jacobi coordinates and the Laplace-Beltrami operator invariance we have \( \Delta = \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} \) and our operator takes on the form

\[ H = -\Delta + V, \quad V = \sum_{i=1}^{3} v_i(x_i, y_i) = v(x_i). \]  

(1.1)

Note that the support of the potential \( V \) lies in an infinite cross domain.

Define the resolvent of the operator \( H \): \( R(\lambda) = (H - \lambda I)^{-1}, \; \lambda \notin [0, +\infty) \). Here and hereafter \( I \) is an identity operator. The resolvent \( (H_0 - \lambda I)^{-1} \) of the unperturbed operator \( H_0 = -\Delta \) will be denoted by \( R_0(\lambda) \).
Saturable Absorption Principle. We consider the scattering problem within the framework of a so-called stationary approach. In this case a wave operators treatment is replaced by the study of limit values $R(E \pm i\varepsilon)$ of the considered resolvent, when a spectral parameter $\lambda$ approaches the real axis ($\lambda \sim E \pm i\varepsilon$, $E \in (0, +\infty)$).

The proof of the existence of such limit values in the appropriate topology is the very essence of the saturable absorption principle. Once the existence of the resolvent limit values is determined, the study of the wave operators takes the known standard form, see, for example, \cite{2,9}.

The existence of the resolvent limit values $R(E \pm i\varepsilon)$ is most often considered in the following weak sense (a so-called rigged Hilbert space method, see \cite{7}). In this case a certain Banach space $\mathcal{B}$ is continuously included into the main Hilbert space $\mathcal{H}$, and this, in its turn, allows to include $\mathcal{H}$ into the conjugate to $\mathcal{B}$ space $\mathcal{B}^*$ with a further proof that $R(E \pm i\varepsilon) : \mathcal{B} \to \mathcal{B}^*$ has a continuous extension at $\varepsilon \downarrow 0$.

Thus the generalized eigenfunctions in this case are treated as the elements of the space $\mathcal{B}^*$, while the main object of study becomes a scalar product $(R(E \pm i\varepsilon)\varphi, \varphi)$, $\varphi \in \mathcal{B}$, with $\varepsilon \downarrow 0$.

Hereafter, to be definite, we restrict ourselves to a consideration of the case $\text{Im}\lambda \downarrow 0$.

Friedrichs - Faddeev Model In the Friedrichs - Faddeev model, see \cite{2,6}, within the framework of the saturable absorption principle, an unperturbed operator is treated as an operator of multiplication by an auxiliary space the limit in a weak sense values of the resolvent will be considered.

Our choice of an auxiliary space will be also defined by this circumstance. In the topology of the auxiliary space the limit in a weak sense values of the resolvent will be considered.

Thus, the existence of the limit values of resolvent $R(\lambda)$ is determined, the study of the wave operators takes the known standard form, see, for example, \cite{6,9}.

Alternating Schwartz Method. In the problem considered there stands out first of all a possibility of separating of variables for a partial Hamiltonian $H_i = -\Delta + v_i$.

Thus, the existence of the limit values of resolvent $R_i(\lambda) = (H_i - \lambda I)^{-1}$ is easily controlled. As a consequence, there arises a question about accounting such contributions into resolvent $R(\lambda)$ of a complete Hamiltonian $H$ with a total potential $V = \sum v_i$.

The scheme of such accounting is known in literature under the name of the alternating Schwartz method. Denote by $\{G_i\}_{i=1}^n$ a certain set of linear operators in a complex vector space $\mathcal{X}$. Define the operator $G = \sum_{i=1}^n G_i$. Assuming that all operators $I - G_i$ are bijective, let $I - \Gamma_i = (I - G_i)^{-1}$. Operator $\Gamma_i$ is called an inversion operator relative to the operator $G_i$.

The essence of the alternating Schwartz method resolves itself into the following. Bijectivity of the operator matrix $L = \begin{pmatrix} I & \Gamma_1 & \ldots & \Gamma_1 \\ \Gamma_2 & I & \ldots & \Gamma_2 \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_n & \Gamma_n & \ldots & I \end{pmatrix}$ in space $\mathcal{X}^n$ is equivalent to the bijection of the operator $I - G$ in space $\mathcal{X}$.

Moreover, if the operator matrix $\gamma$ is a solution of the equation $L \cdot \gamma = \text{diag}(\Gamma_1, \ldots, \Gamma_n)$, where $\text{diag}(\Gamma_1, \ldots, \Gamma_n)$ a diagonal matrix is denoted, then the operator $\Gamma = \sum \gamma_{ij}$, where summation extends to all matrix elements of operator matrix $\gamma = (\gamma_{ij})$, defines an inverse to $I - G$ operator by the equality $(I - G)^{-1} = I - \Gamma$.  

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Note that if the operator matrix $\Gamma = L - I$ in an adequate Banach space is restricted and its norm is less than unity, then

$$\Gamma = \sum \Gamma_i - \sum_{i \neq j} \Gamma_i \Gamma_j + \sum \Gamma_i \Gamma_j \Gamma_k - \ldots$$

where a series of sums is taken as convergent in norm. Exactly this formula explains the name of the method.

Finally, note that the equality $(I + \Gamma)^{-1} = (I - \Gamma^2)^{-1}(I - \Gamma)$, reduces the inversion of the operator matrix $L$ to the inversion of the operator matrix $I - \Gamma^2$.

But the matrix $\Gamma^2$ as its components has the sums of the operators as $\Gamma_i \Gamma_j$ at $i \neq j$.

Applied to the considered problem it is the analysis of such products $\Gamma_i \Gamma_j$ that has brought the desired result. To embed our problem into this scheme all we need is to separate a free resolvent $R_0(\lambda)$. As this takes place $R(\lambda) = R_0(\lambda)(I - G(\lambda))^{-1}, G(\lambda) = \sum G_i(\lambda), G_i(\lambda) = v_i R_0(\lambda)$. The reflection operators with respect to $G(\lambda)$ or $G_i(\lambda)$ will be denoted, accordingly, by $\Gamma(\lambda) \Gamma_i(\lambda)$. Thus,

$$R(\lambda) = R_0(\lambda)(I - \Gamma(\lambda)) \cdot (1.3)$$

2 Short Summary of the Results

Further we assume that the Fourier transform of the finite function $\psi$, defining a two-particle interaction, belongs to $H^{\mu, \theta}_0(\mathbb{R})$ at certain $\mu > 0, \theta > 0$.

The integral kernel of the operator $\Gamma_j(\lambda)$ looks as $\Gamma_j(z, z'|\lambda) = v(x_j) \int_{\mathbb{R}^2} \frac{\psi(z, q)\psi(z', q)}{-(\lambda - q^2)},$ here $dq$ is a Lebesgue measure at the momentum variable plane $q = (k, p)$, dual to $z = (x_j, y_j)$ and $z' = (x'_j, y'_j)$, while $\psi_j$ is a continuous spectrum generalized eigenfunction of the operator $H_j$. It looks as $\psi_j(z, q) = \phi_j(x, k)e^{i|x||p|}$, where $\phi_j(x, k)$ is a solution of a one-dimensional Schroedinger equation $\left(-\frac{d^2}{dx^2} + v(x)\right)\phi(x, k) = k^2\phi(x, k)$ and is defined by the scattering data of a corresponding one-dimensional scattering problem. In a certain sense a key to the proof of the announced result is the computation of the asymptotic of the kernel $\Gamma_j(\lambda)\Gamma_k(\lambda)$ with $|z| \to \infty$. It permitted to obtain a partition of the operator $\Gamma_j(\lambda)\Gamma_k(\lambda)$ into a sum $\Gamma_j(\lambda)\Gamma_k(\lambda) = A_{jk}(\lambda) + B_{jk}(\lambda)$, where $A_{jk}(\lambda)$ is the 2nd rank operator including all the bad part of this product, i.e. the one which goes beyond the space $L_2(\mathbb{R}^2)$ with $\lambda < 0$, while the operator $B_{jk}(\lambda)$ is a compact operator in $H^{\mu, \theta},$ strongly continuous in $\lambda$ with $Im \lambda \geq 0$ and $0 < c_1 \leq Re \lambda \leq c_2 < \infty$, if $\mu$ and $\theta$ are small enough. The limit operator $A_{jk}(E + i0)$ act on functions from $H^{\mu, \theta}$ into a two-dimensional space of the functions of the type $v(x_j)\phi_j(x_j, 0)|y_j|^{-1/2}e^{iy_j|\sqrt{E}(C_1\chi(y_j) + C_2\chi(-y_j))}$, where $\chi$ can be defined as a smoothed characteristic function of the semi-axis $(T, +\infty)$ with $T \gg 1$.

The space of such functions has square root analitical singularities on the axis in a momentum representation, exactly - the singularities of the type $(p \pm \sqrt{E})^{-1/2}$. We will denote the described above range of the operator $A_{jk}(\lambda)$ by the space of functions of the type $A_j$.

It is worth emphasizing that it is the necessity of an extraction of such an operator as $A_{jk}$ that is a distinguishing feature of this problem - the three one-dimensional particles scattering problem - in comparison with the case of the three three-dimensional particles scattering problem considered in the work [2].

The consequence of the partition $\Gamma_j(\lambda)\Gamma_k(\lambda)$ is the representation $I - \Gamma^2(\lambda) = I - A(\lambda) - B(\lambda)$, where the operator matrices $A(\lambda)$ and $B(\lambda)$ inherit the properties of the corresponding scalar operators $A_{jk}(\lambda)$ and $B_{jk}(\lambda)$. The following assertion is valid.

Lemma 2.1. With $Im \lambda > 0$ the operator $(I - \Gamma^2)^{-1}$ allow a representation $(I - \Gamma^2)^{-1} = I - A(\lambda) - B(\lambda)$ where the matrix components $A(\lambda)$ are the finite rank operators acting into the algebraic sum of the spaces of the type $A_j, j = 1, 2, 3$ while the matrix components $B$ are compact operators in $H^{\mu, \theta}$ strongly continuous in $\lambda$ with $Im \lambda \geq 0$ and $0 < c_1 \leq Re \lambda \leq c_2 < \infty$, if $\mu$ and $\theta$ are small enough.

It is easy to see if $\phi_\lambda$ is a function of the type $A_j$ and $\psi \in H^{\mu, \theta}$ then $(R_0(\lambda)\phi_\lambda, \psi)$ has a limit at $Im \lambda \downarrow 0$. As a consequence for the desired scalar operator $\Gamma(\lambda)$ the constructions of the alternating Schwartz method described above lead to the representation: $\Gamma(\lambda) = \sum \Gamma_i(\lambda) + N(\lambda)$ where the operator $N(\lambda)$ possess the
property that the product $R_0(\lambda)N(\lambda)$ has a weak limit in $\hat{H}^{\mu,\theta}$ with $\text{Im}\lambda \downarrow 0$ and $0 < c_1 \leq \text{Re}\lambda \leq c_2 < \infty$, if $\mu$ and $\theta$ are small enough. It means, see (1.3) that the following is valid

**Theorem 2.2.** $R(\lambda)$ has a weak limit in $\hat{H}^{\mu,\theta}$ with $\text{Im}\lambda \downarrow 0$ and $0 < c_1 \leq \text{Re}\lambda \leq c_2 < \infty$, if $\mu$ and $\theta$ are small enough.

Note that this theorem guarantees a validity of the asymptotic constructions of the works [4],[3].

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