Interlace polynomials for multimatroids and delta-matroids

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1. Introduction

The discovery of the interlace polynomial by Arratia, Bollobás, and Sorkin [3,5] can be traced from 4-regular graphs, via circle graphs, to general graphs. In this paper we generalize this graph polynomial to Δ-matroids and, even more generally, to multimatroids. In fact, we introduce generic polynomials for both Δ-matroids and multimatroids that allow for a unified presentation of the
interlace polynomial and various related graph polynomials. This introduction provides a brief sketch of a number of these known polynomials and their relationships, see, e.g., [21,22] for a survey.

The Martin polynomial [27], which is defined for both directed and undirected graphs, computes the number of circuit partitions of that graph. More precisely, the number \( a_k \) of \( k \)-component circuit partitions in a graph \( G \) is equal to the coefficient of \( x^k \) of the Martin polynomial of \( G \). In case \( G \) is a 2-in, 2-out digraph, an Eulerian circuit \( C \) of \( G \) defines a circle graph \( G \). The interlace polynomial of \( G \) turns out to coincide (up to a trivial transformation) with the Martin polynomial of (the underlying 2-in, 2-out digraph) \( G \)– the interlace polynomial is however defined for graphs in general (i.e., not only for circle graphs) [3,5]. The interlace polynomial is known to be invariant under the graph operations of local and edge complementation (where local complementation is defined here only for looped vertices) and moreover the interlace polynomial fulfills a recursive reduction relation. As demonstrated in [2,4], the interlace polynomial \( q(G) \) of a graph \( G \) may be explicitly defined through the nullity values of the set of subgraphs of \( G \). The interlace polynomial moreover coincides with the Tutte–Martin polynomial defined for isotropic systems [5,14] in the case where graph \( G \) does not have loops. Restricting to such simple graphs is an essential loss of generality as, e.g., the Tutte–Martin polynomial does not explain the invariance of the interlace polynomial under local complementation.

Related polynomials have been defined such as the different “interlace polynomial” \( Q(G) \) of [2], which is invariant under local, loop, and edge complementation. In case \( G \) is a circle graph (with possibly loops), then \( Q(G) \) coincides with the Martin polynomial of a 4-regular graph corresponding to \( G \). Moreover, \( Q(G) \) coincides with the “global” Tutte–Martin polynomial defined for isotropic systems [14] in the case where graph \( G \) is simple. As \( Q(G) \) is invariant under loop complementation, this is not an essential loss of generality. Also, the bracket polynomial for graphs [32] has recursive relations similar to the recursive relations of the interlace polynomial.

The notion of delta-matroid (or \( \Delta \)-matroid), introduced by Bouchet [8], is a generalization of the notion of matroid. In addition, binary \( \Delta \)-matroids may be viewed as a generalization of graphs. Moreover, as pointed out in [24], the graph operations of local and edge complementation have particularly simple interpretations in terms of \( \Delta \)-matroids. In [17] a suitable generalization of the notion of loop complementation for graphs has been defined for a subclass of the \( \Delta \)-matroids called vf-safe \( \Delta \)-matroids. Moreover, in [18], the notion of nullity for the adjacency matrix of a graph (or skew-symmetric matrix in general) is shown to correspond to a natural distance measure within \( \Delta \)-matroids.

Multimatroids were introduced by Bouchet [11] as a common generalization of both \( \Delta \)-matroids and isotropic systems. It turns out that \( \Delta \)-matroids precisely correspond to multimatroids with two “directions”, called 2-matroids. A particularly interesting class of multimatroids are the tight multimatroids, defined in [13]. The so-called even \( \Delta \)-matroids precisely correspond to the tight 2-matroids, while the class of isotropic systems corresponds to a subclass of tight 3-matroids (i.e., tight multimatroids with three directions). We show in Section 5 that the class of vf-safe \( \Delta \)-matroids (those \( \Delta \)-matroids that “allow” loop complementation in conjunction with twist) precisely corresponds to the class of tight 3-matroids. As a consequence, vf-safe \( \Delta \)-matroids strictly generalize isotropic systems. The equivalence of vf-safe \( \Delta \)-matroids and tight 3-matroids is of independent interest and confirms that the vf-safe \( \Delta \)-matroids form a natural family even beyond their convenient closure properties.

In Section 3 we define the (weighted) transition polynomial \( Q(Z) \) for multimatroids \( Z \) in a natural way as a “nullity generating” polynomial, and moreover prove a number of evaluations of \( Q(Z) \) in case \( Z \) is a tight \( k \)-matroid (i.e., with \( k \) directions). Then, in Section 6, we define the (weighted) transition polynomial \( Q(M) \) for set systems \( M \) using the generalizations of the notions of nullity, and of local, loop, and edge complementation mentioned above. In Section 7 we show that the transition polynomial \( Q(Z) \) for tight 3-matroids \( Z \) corresponds to the transition polynomial \( Q(M) \) for vf-safe \( \Delta \)-matroids \( M \). It turns out that the various graph polynomials recalled above are all special cases of the transition polynomials for (vf-safe) \( \Delta \)-matroids and multimatroids. In this way, these polynomials provide unified views in which invariance properties, recursive relations, and evaluations of these polynomials may be investigated.

The proofs of these results use combinatorial properties of multimatroids, and are not only more general and unified, but also much more concise compared to the proofs which rely on
graph-theoretical arguments. In particular, the evaluation on $-1$ of the interface polynomial for graphs (and also of the Tutte polynomial for binary matroids) is strikingly simple when viewed as a special case of an evaluation of the transition polynomial for tight multimatroids. Moreover, due to the highly symmetrical nature of multimatroids, we naturally obtain new interrelationships and evaluations for the corresponding polynomials on graphs (Section 10).

Since the minor-closed class of vf-safe $\Delta$-matroids strictly contains the class of quaternary matroids (this is recalled in Section 4), the main results of this paper are not likely obtainable using isotropic systems which fundamentally deal with binary matroids.

Inspired by the connection between the Martin and the Tutte polynomial in [27] (and its relation to isotropic systems described in [9]) we find that the Tutte polynomial $t_M(x,y)$ for matroids $M$ can, for the case $x = y$, be seen as a special case of the transition polynomial for $\Delta$-matroids. In addition, the recursive relations of the transition polynomial and $t_M(y,y)$ coincide when restricting to matroids $M$. The obtained evaluations of the transition polynomial are then straightforwardly carried over to $t_M(y,y)$.

Table 1 summarizes the relations of the above recalled polynomials from the literature with the transition polynomial $Q(M)$ for (vf-safe) $\Delta$-matroids $M$.

### 2. Multimatroids

We assume the reader is familiar with the basic notions concerning matroids, which can be found, e.g., in [35,28].

We take the terminology of multimatroids as developed by Bouchet [11–13]. A carrier is a tuple $(U, \Omega)$ where $\Omega$ is a partition of a finite set $U$, called the ground set. Every $\omega \in \Omega$ is called a skew class, and a $p \subseteq \omega$ with $|p| = 2$ is called a skew pair of $\omega$. A transversal (subtransversal, resp.) $T$ of $\Omega$ is a subset of $U$ such that $|T \cap \omega| = 1$ ($|T \cap \omega| \leq 1$, resp.) for all $\omega \in \Omega$. We denote the set of transversals of $\Omega$ by $T(\Omega)$, and the set of subtransversals of $\Omega$ by $\delta(\Omega)$. The power set of a set $X$ is denoted by $2^X$.

We recall now the notion of multimatroid. Like matroids, multimatroids can be defined in terms of rank, circuits, independent sets, etc. We define multimatroids here in terms of independent sets.

**Definition 1** ([11]). A multimatroid $Z$ (described by its independent sets) is a triple $(U, \Omega, \mathcal{I})$, where $(U, \Omega)$ is a carrier and $\mathcal{I} \subseteq \delta(\Omega)$ such that:

1. for each $T \in T(\Omega)$, $(T, T \cap 2^I)$ is a matroid (described by its independent sets) and
2. for any $I \in \mathcal{I}$ and any skew pair $p = \{x, y\}$ of some $\omega \in \Omega$ with $\omega \cap I = \emptyset$, $I \cup \{x\} \in \mathcal{I}$ or $I \cup \{y\} \in \mathcal{I}$.

Terminology concerning $\Omega$ carries over to $Z$: hence we may, e.g., speak of a transversal of $Z$. We often simply write $U$ and $\Omega$ to denote the ground set and the partition of the multimatroid $Z$ under consideration, respectively. Each $I \in \mathcal{I}$ in the definition of multimatroid $Z$ is called an independent set of $Z$. The family $\mathcal{I}$ of independent sets of $Z$ is denoted by $I_Z$. The family $\text{max}(I_Z)$ of sets of $I_Z$ that are maximal with respect to inclusion is denoted by $B_Z$. The elements of $B_Z$ are called the bases.

**Table 1**

| Special cases | Description |
|---------------|-------------|
| $Q_{1,0}(M)(y)$ | Martin polynomial for 2-in, 2-out digraphs [27]. Single-variable interlace polynomial [5]. Tutte–Martin polynomial for isotropic systems [9]. Tutte polynomial for matroids restricted to the diagonal |
| $Q_{1,1}(M)(y)$ | Two-variable interlace polynomial [4] |
| $Q_{0,1}(M)(y)$ | Tutte polynomial for matroids restricted to part of the plane |
| $Q_{1,1}(M)(y)$ | Bracket polynomial for graphs [32] |
| $Q_{1,1,1}(M)(y)$ | Martin polynomial for 4-regular graphs [27], “Global” Tutte–Martin polynomial for isotropic systems [9]. Polynomial $Q(G, x)$ for simple graphs $G$ as defined in [2] |

Special cases of multimatroids include:

- for any $I \subseteq \mathcal{I}$ and any skew pair $p = \{x, y\}$ of some $\omega \in \Omega$ with $\omega \cap I = \emptyset$, $I \cup \{x\} \in \mathcal{I}$ or $I \cup \{y\} \in \mathcal{I}$.
of $Z$. Note that the bases uniquely determine $Z$ (given its carrier). For any $X \subseteq U$, the restriction of $Z$ to $X$, denoted by $Z[X]$, is the multimatroid $(X, \mathcal{O}', I \cap 2^X)$ with $\mathcal{O}' = \{\omega \cap X \mid \omega \in \mathcal{O} \}$. If $X$ is a subtransversal, then we identify $Z[X]$ with the matroid $(X, I \cap 2^X)$ since $\mathcal{O}' = \{\{u\} \mid u \in X\}$ captures no additional information. The rank of $S \in \mathcal{O}(\mathcal{O})$ in $Z$, denoted by $r_Z(S)$, is the rank $r(Z[S])$ of the multimatroid $Z[S]$. The nullity of $S \in \mathcal{O}(\mathcal{O})$ in $Z$, denoted by $n_Z(S)$, is $n_Z(S) = |S| - r_Z(S)$, i.e., the nullity $n(Z[S])$ of the matroid $Z[S]$. The second condition of Definition 1 can equivalently be formulated as follows: For all $S \in \mathcal{O}(\mathcal{O})$ and any skew pair $p = \{x, y\}$ of $\omega \in \mathcal{O}$ with $w \cap S = \emptyset$, $n_Z(S \cup \{x\}) = n_Z(S) + n_Z(S \cup \{y\}) - n_Z(S) \leq 1$ (see the definition of multimatroid in terms of rank in [11]).

The minor of $Z$ induced by $X$, denoted by $Z[X]$, is the multimatroid $(U', \mathcal{O}', I')$, where $\mathcal{O}' = \{\omega \in \mathcal{O} \mid \omega \cap X = \emptyset\}$, $U' = \cup_{\omega \in \mathcal{O}'} \omega$, and $r_{Z[X]}(S) = r_Z(S \cup X) - r_Z(X)$ for all $S \in \mathcal{O}(\mathcal{O})$. By a standard property of matroid contraction (see, e.g., [28, Proposition 3.1.6]), for each $T \subseteq T(\mathcal{O})$, the multimatroid $Z[X[T]]$ is equal to $Z[T \cup X]/X$, where $/$ denotes matroid contraction. In case $X = \{u\}$ is a singleton, we also write $Z[U]$ to denote $Z[\{u\}]$. An element $u \in U$ is called singular in $Z$ if $n_Z(\{u\}) = 1$. Thus if $u \in U$ is nonsingular in $Z$, then $n_Z(\{u\}) = 0$. Note that, by the second condition of Definition 1, each skew class contains at most one singular element. A skew class that contains a singular element is called singular. We recall the following result of [13].

Proposition 2 (Proposition 5.5 of [13]). If a skew class $\omega$ of a multimatroid $Z$ is singular, then $Z[U] = Z[\{U \setminus \omega\}]$ for all $u \in \omega$.

For any positive integer $q$, a $k$-matroid is a multimatroid with carrier $(U, \mathcal{O})$ such that $|\omega| = k$ for all $\omega \in \mathcal{O}$. Note that a (regular) matroid corresponds to a 1-matroid.

A multimatroid $Z$ is called nondegenerate if $|\omega| > 1$ for all $\omega \in \mathcal{O}$. It is shown in [11], that if $Z$ is nondegenerate, then $2^Z = I_Z \cap T(\mathcal{O})$.

A multimatroid $Z$ is called tight if both $Z$ is nondegenerate and for all $S \in \mathcal{O}(\mathcal{O})$ with $|S| = |\mathcal{O}| - 1$, there is a $x \in \omega$ such that $n_Z(S \cup \{x\}) = n_Z(S) + 1$, where $\omega \in \mathcal{O}$ is the unique skew class such that $S \cap \omega = \emptyset$. Consequently, by the second condition in Definition 1, the values $n_S(S \cup \{y\})$ for all $y \in \omega$ are equal, except for exactly one $x \in \omega$ (for which the value $n_S(S \cup \{x\})$ is one larger than the others). It is shown in [13, Proposition 4.1] that tightness is preserved under minors. Moreover, the notion of tightness is characterized in terms of bases.

Proposition 3 (Theorem 4.2c of [13]). Let $Z$ be a nondegenerate multimatroid. Then $Z$ is tight if and only if for every basis $X$ and every skew class $\omega$ of $Z$, exactly one of the transversals $(X \setminus \omega) \cup \{u\}$, $u \in \omega$, is not a basis of $Z$.

As an example, we briefly recall from [11] that the Eulerian circuits of a connected 4-regular graph form the bases of a tight 3-matroid.

Example 4. Given a connected 4-regular graph $G$, any Eulerian circuit $C$ of $G$ visits every vertex of $G$ exactly twice. The Eulerian circuit $C$ defines a transition at each vertex $v$, i.e., a set of two (disjoint) unordered pairs of half-edges incident to $v$, that coincides with the trajectory of $C$ at $v$. Note that each vertex has three possible transitions. Let $(U, \mathcal{O})$ be a carrier, where each skew class of $\mathcal{O}$ consists of the three possible transitions at a vertex of $G$. The transversals of $(U, \mathcal{O})$ are in one-to-one correspondence with the circuit partitions of $G$, where a circuit partition is a set of mutually edge-disjoint circuits (circuits are considered unoriented) of $G$ covering each edge of $G$. It turns out that the transversals of $(U, \mathcal{O})$ that correspond to the Eulerian circuits of $G$ are the bases of a 3-matroid $Z$, called the Eulerian multimatroid of $G$ [11]. It is easily seen by Proposition 3 that $Z$ is tight: for every Eulerian circuit $C$ of $G$ and every vertex $v$ of $G$, it is possible to split $C$ in two by changing the transition of $C$ at $v$ in a suitable way.

3. Polynomials for multimatroids

It is often worthwhile to study weighted/multivariate variants of enumerating polynomials, see, e.g., the weighted variants of the Tutte polynomial studied in [7,23,30,36]. We now define a weighted
polynomial enumerating the nullity values of transversals of a multimatroid. It is called the transition polynomial, motivated by the eponymous polynomial for 4-regular graphs studied by Jaeger [26].

We obtain in this section a recursive relation for our transition polynomial as well as evaluations of the polynomial for tight multimatroids, following the lead of [26].

**Definition 5.** Let $Z$ be a multimatroid with carrier $(U, \Omega)$. We define the (weighted) transition polynomial of $Z$ as

$$Q(Z)(\vec{x}, y) = \sum_{T \in \mathcal{T}(U)} x_T y^{r_Z(T)},$$

where $\vec{x}$ is a vector indexed by $U$ with entries $x_u$ for all $u \in U$, and $x_T = \prod_{u \in T} x_u$.

We now provide recursive relations for $Q(Z)$.

**Theorem 6.** Let $Z$ be a multimatroid and let $\omega \in \Omega$.

If $\omega$ is nonsingular in $Z$, then

$$Q(Z)(\vec{x}, y) = \sum_{v \in \omega} x_v Q(Z|v)(\vec{x}', y),$$

and if $\omega$ is singular in $Z$, then for all $w \in \omega$

$$Q(Z)(\vec{x}, y) = \left( x_u y + \sum_{v \in \omega \setminus \{u\}} x_v \right) Q(Z|w)(\vec{x}', y),$$

where $u \in \omega$ is singular in $Z$, and $\vec{x}'$ is obtained from $\vec{x}$ by removing the entries indexed by $\omega$.

**Proof.** Let $\Omega' = \Omega \setminus \{\omega\}$. We have

$$Q(Z)(\vec{x}, y) = \sum_{u \in \omega} \sum_{T \in \mathcal{T}(U)} x_T y^{r_Z(T)} = \sum_{u \in \omega} x_u \sum_{T \in \mathcal{T}(Z) \setminus \{u\}} x_T^{r_Z(T)}$$

$$= \sum_{u \in \omega} \sum_{T \in \mathcal{T}(\Omega')} x_T y^{r_Z(T \cup \{u\})}.$$

Let $u \in \omega$. We have, for all $S \in \mathcal{S}(\Omega')$, $r_Z|u|(S) = r_Z(S \cup \{u\}) - r_Z(\{u\})$ or, equivalently, $n_Z|u|(S) = n_Z(S \cup \{u\}) - n_Z(\{u\})$.

Assume first that $u$ is nonsingular in $Z$. Then $n_Z|u|(S) = n_Z(S \cup \{u\})$ for all $S \in \mathcal{S}(\Omega')$, and thus

$$\sum_{T \in \mathcal{T}(\Omega') \setminus \{u\}} x_T y^{r_Z(T \cup \{u\})} = Q(Z|u)(\vec{x}', y).$$

Assume now that $u$ is singular in $Z$. Then $n_Z|u|(S) = n_Z(S \cup \{u\}) - 1$ for all $S \in \mathcal{S}(\Omega')$, and thus

$$\sum_{T \in \mathcal{T}(\Omega') \setminus \{u\}} x_T y^{r_Z(T \cup \{u\})} = yQ(Z|u)(\vec{x}', y).$$

The result follows by Proposition 2. □

Note that the recursive relations of Theorem 6, along with the fact that $Q(Z)(\vec{x}, y) = 1$ when $Z$ is the unique multimatroid with empty ground set, characterize the transition polynomial $Q(Z)(\vec{x}, y)$.

The following result is inspired by [26, Proposition 10] from the context of 4-regular graphs.

**Theorem 7.** Let $Z$ be a tight multimatroid, let $\omega \in \Omega$, and let $\vec{x}'$ be a vector indexed by $U$ with $x'_v = x'_w$ for all $v, w \in \omega$ and $x'_v = 0$ for all $v \in U \setminus \omega$. Then $Q(Z)(\vec{x}, 1 - k) = Q(Z)(\vec{x} + \vec{x}', 1 - k)$ with $k = |\omega|$.

**Proof.** Let $c = x'_v$ for all $v \in \omega$. We prove by induction.

Assume that $|\Omega| = 1$. Since $Z$ is tight, there is a $u \in \omega$ singular in $Z$. By Theorem 6, $Q(Z)(\vec{x} + \vec{x}', 1 - k) = (x_u + c)(1 - k) + \sum_{v \in \omega \setminus \{u\}} x_v = Q(Z)(\vec{x}, 1 - k) + c(1 - k) + \sum_{v \in \omega \setminus \{u\}} x_v = Q(Z)(\vec{x}, 1 - k)$.

Assume that $|\Omega| > 1$. Let $\omega' \in \Omega$ with $\omega' \neq \omega$. Note that $\vec{x}$ and $\vec{x} + \vec{x}'$ coincide for the elements of $\omega'$. The induction hypothesis ensures now (recall that tightness is preserved under minors) that the right-hand sides of the equalities in Theorem 6 coincide for $Q(Z)(\vec{x}, 1 - k)$ and $Q(Z)(\vec{x} + \vec{x}', 1 - k)$. □
The polynomial \( Q(Z)(\vec{x}, y) \), where \( x_u = 1 \) for all \( u \in U \), is denoted by \( Q_1(Z)(y) = \sum_{T \in \mathcal{T}(\Omega)} y^{n_2(T)} \). Note that \( Q_1(Z)(1) = |\mathcal{T}(\Omega)| = \prod_{\omega \in \Omega} |\omega| \) and \( Q_1(Z)(0) = |\{|T \in \mathcal{T}(\Omega) \mid n_2(T) = 0\}| = |\mathcal{B}_2| \).

We now obtain the following two corollaries to Theorem 7.

**Theorem 8.** Let \( Z \) be a tight multimatroid with \( U \neq \emptyset \). If \( \omega \) is a skew class of \( Z \) with \( |\omega| = k \), then \( Q_1(Z)(1 - k) = 0 \).

**Proof.** By Theorem 7, \( Q_1(Z)(1 - k) = Q(Z)(\vec{x}, 1 - k) \) with \( \vec{x} \) the zero vector indexed by \( U \). Since \( U \neq \emptyset \), \( Q(Z)(\vec{x}, 1 - k) = 0 \). \( \square \)

The following result is inspired by [26, Proposition 11] from the context of 4-regular graphs.

**Theorem 9.** Let \( Z \) be a tight \( k \)-matroid for some \( k > 1 \) and let \( T \in \mathcal{T}(\Omega) \). Then \( Q_1(Z[U \setminus T])(1 - k) = (-1)^{|\Omega|}(1 - k)^{n_2(T)} \).

**Proof.** Note that \( Q_1(Z[U \setminus T])(y) \) is equal to the polynomial \( Q(Z)(\vec{x}, y) \), where \( x_u = 1 \) if \( u \in U \setminus T \) and \( x_u = 0 \) if \( u \in T \). Moreover, \( Q(Z)(\vec{x}', 1 - k) = (-1)^{|\Omega|}(1 - k)^{n_2(T)}\) where \( x'_u = 0 \) if \( u \in U \setminus T \) and \( x'_u = -1 \) if \( u \in T \). Since \( \vec{x}' \) is obtained from \( \vec{x} \) by subtracting 1 for each component, the result follows from Theorem 7 (by iteration on every skew class). \( \square \)

It is interesting to remark that, by Theorem 9, \( Q_1(Z')(1 - k) \) for a particular \( (k - 1) \)-matroid \( Z' \) depends on the nullity of a set \( T \) in a larger tight \( k \)-matroid \( Z \), where \( T \) is disjoint from the ground set of \( Z' \). We show in Theorem 13 that \( Z \) is essentially unique given \( Z' \).

### 4. Pivot and loop complementation for set systems and delta-matroids

In this section we recall the algebra of set systems generated by the operations of pivot and loop complementation [17]. Additionally we recall from [18] the notion of distance in set systems and its main properties with respect to \( \Delta \)-matroids.

A **set system** (over \( V \)) is a tuple \( M = (V, D) \) with \( V \) a finite set called the **ground set** and \( D \subseteq 2^V \) a family of subsets of \( V \). Let \( X \subseteq V \). We define \( M[X] = (X, D') \) where \( D' = \{Y \in D \mid Y \subseteq X\} \), and define \( M \setminus X = M[V \setminus X] \). In case \( X = \{u\} \) is a singleton, we also write \( M \setminus u \) to denote \( M \setminus \{u\} \). Set system \( M \) is called **nonempty** if \( D \neq \emptyset \). We write simply \( Y \in M \) to denote \( Y \in D \). A set system \( M \) with \( \emptyset \in M \) is called **normal** (note that every normal set system is nonempty). In particular the set system \( (\emptyset, \{\emptyset\}) \) is normal. A set system \( M \) is called **equicardinal** if for all \( X_1, X_2 \in M \), \( |X_1| = |X_2| \). For convenience we will often simply denote the ground set of the set system under consideration by \( V \).

Let \( M = (V, D) \) be a set system. We define, for \( X \subseteq V \), **pivot** (also called **twist** in the literature) of \( M \) on \( X \), denoted by \( M * X \), as \( (V, D') \), where \( D' = \{Y \in D \mid Y \subseteq X\} \), and \( \Delta \) denotes symmetric difference. In case \( X = \{u\} \) is a singleton, we also write simply \( M * u \). Moreover, we define, for \( u \in V \), **loop complementation of** \( M \) on \( u \) (the motivation for this name is from graphs, see Section 9), denoted by \( M + u \), as \( (V, D') \), where \( D' = D \Delta \{X \cup \{u\} \mid X \in D, u \notin X\} \). We assume left associativity of set system operations. Therefore, e.g., \( M + u \setminus u * v \) denotes \( ((M + u) \setminus u) * v \).

It has been shown in [17] that pivot \(*u \) and loop complementation \(+u \) on a common element \( u \in V \) are involutions (i.e., of order 2) that generate a group \( F_u \) isomorphic to \( S_3 \), the group of permutations on 3 elements. In particular, we have \(+u * u + u = u * u + u \), which is the third involution (in addition to pivot and loop complementation), and is called **dual pivot**, denoted by \( * \). Explicitly, if \( M = (V, D) \) and \( u \in V \), then \( M * u = (V, D') \), where \( D' = D \Delta \{X \setminus \{u\} \mid X \in D, u \notin X\} \). The elements of \( F_u \) are called **vertex flips**: \( \forall u \in V \), \( *u * u = u * u * u \). We have, e.g., \(+u * u + u = *u * u * u \) and \(*u * u + u = u * u + u = *u * u + u \) for \( u \in V \) (which are the two vertex flips in \( F_u \) of order 3).

While on a single element the vertex flips behave as the group \( S_3 \), they commute when applied on different elements. Hence, e.g., \( M * u + v = M + v * u \) and \( M * u + v = M + v * u \) when \( u \neq v \).

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\footnote{The notion of vertex flip as defined in this paper corresponds to the notion of invertible vertex flip in [17]—as we consider only invertible vertex flips in this paper for notational convenience we omit the adjective “invertible”}
Fig. 1. The orbit of a set system under vertex flips on the ground set, cf. Example 10.

$M + u + v = M + v + u$ and thus we (may) write, for $X = \{u_1, u_2, \ldots, u_n\} \subseteq V, M + X$ to denote $M + u_1 \cdot \cdot \cdot + u_n$ (as the result is independent of the order in which the operations $+u_i$ are applied). Similarly, we define $M \bar{\ast} X$ for $X \subseteq V$. We will often use the above equalities without explicit mention.

One may explicitly define the sets in $M \ast V$, $M + V$, and $M \bar{\ast} V$ as follows: $X \in M \ast V$ iff $V - X \in M$, and $X \in M + V$ iff $|\{Z \in M \mid Z \subseteq X\}|$ is odd. Dually, $X \in M \bar{\ast} V$ iff $|\{Z \in M \mid X \subseteq Z\}|$ is odd. In particular, $\emptyset \in M \bar{\ast} V$ iff the number of sets in $M$ is odd.

Example 10. Let $V = \{p, q, r\}$. Fig. 1 shows the set systems in the orbit under vertex flips on $V$ of the leftmost set system $M = (V, \{\{p, q\}, \{q, r\}, \{p\}, \emptyset\})$. The set systems are depicted as a Hasse diagram where the sets that belong to the set system are indicated by a circle (where, e.g., “qr” denotes the set $\{q, r\}$). In Section 9 we learn that the topmost four set systems represent graphs, and these graphs are also indicated in the figure.

It is shown in [17] that vertex flips and the removal of elements from the ground set commute when applied on different elements. Moreover, $M + u \setminus u = M \setminus u$. This is explicitly stated as a lemma.

Lemma 11 ([17]). Let $M$ be a set system and $u, v \in V$ with $u \neq v$. Then $M + u \setminus v = M \setminus v + u$, $M \ast u \setminus v = M \setminus v \ast u$, and $M + u \setminus u = M \setminus u$.

Assume that $M$ is nonempty. For $X \subseteq V$, we define $d_M(X) = \min(|X \Delta Y| \mid Y \in M)$. Hence, $d_M(X)$ is the smallest distance between $X$ and the sets of $M$, where the distance between two sets is defined as the number of elements in the symmetric difference. We set $d_M = d_M(\emptyset)$, the cardinality of a smallest set in $M$.

Lemma 12 ([18]). Let $M$ be a nonempty set system. Then $d_{M+Z}(X) = d_M(X \Delta Z)$ for all $X, Z \subseteq V$, and $d_{M+Y} = d_M$ for all $Y \subseteq V$.

In particular, we have by Lemma 12, $d_{M+Z} = d_M(Z)$ for all $Z \subseteq V$. 

$M + u + v = M + v + u$ and thus we (may) write, for $X = \{u_1, u_2, \ldots, u_n\} \subseteq V, M + X$ to denote $M + u_1 \cdot \cdot \cdot + u_n$ (as the result is independent of the order in which the operations $+u_i$ are applied). Similarly, we define $M \bar{\ast} X$ for $X \subseteq V$. We will often use the above equalities without explicit mention.
A $\Delta$-matroid is a nonempty set system $M$ that satisfies the symmetric exchange axiom: For all $X, Y \subseteq M$ and all $u \in X \setminus Y$, either $X \setminus \{u\} \in M$ or there is a $v \in X \setminus Y$ with $v \neq u$ such that $X \setminus \{u, v\} \in M$ [8]. Note that $\Delta$-matroids are closed under pivot, i.e., $M \ast X$ for $X \subseteq V$ is a $\Delta$-matroid when $M$ is a $\Delta$-matroid. Also, if $M \setminus u$ with $u \in V$ is nonempty, then $M \setminus u$ is a $\Delta$-matroid when $M$ is a $\Delta$-matroid. A $\Delta$-matroid $M$ is called even if the cardinalities of all sets in $M$ are of equal parity.

It turns out that a set system is an equicardinal $\Delta$-matroid iff it is a matroid described by its family of bases [10, Proposition 3]. In this way, the notion of $\Delta$-matroid is a generalization of the notion of matroid. Note that if $M$ is a matroid (described by its bases), then $d_M$ and $d_{M^*}$ are the rank and nullity of $M$. We implicitly assume that a matroid $M$ is described by its bases when viewing $M$ as a $\Delta$-matroid.

The result of applying loop complementation to a $\Delta$-matroid is not necessarily a $\Delta$-matroid [17, Example 10]. We say that a $\Delta$-matroid $M$ is $\psi$-safe if for any sequence $\psi$ of vertex flips (equivalently, pivots and loop complementations) over $V$ we have that $M \psi$ is a $\Delta$-matroid. We say that a matroid $M$ is $\psi$-safe if $M$ is $\psi$-safe as a $\Delta$-matroid. The class of $\psi$-safe $\Delta$-matroids strictly contains the class of quaternary matroids, i.e., matroids representable over GF(4) [19]. Recall that the class of quaternary matroids in turn strictly contains the class of binary matroids. It is conjectured that the class of $\psi$-safe $\Delta$-matroids is equal to the class $\mathcal{N}$ from [25], consisting of matroids that have no minors isomorphic to $U_{2,6}, U_{4,6}, P_6, F_7^-$, or $(F_7^-)^*$. For a description of these matroids, see [28].

5. Equivalence of $\psi$-safe delta-matroids and tight 3-matroids

In this section we show that $\psi$-safe $\Delta$-matroids may be viewed as “fragments” of tight 3-matroids. Moreover, we show that given a $\psi$-safe $\Delta$-matroid, it is possible to reconstruct the entire tight 3-matroid (up to renaming elements from the ground set). In this way, $\psi$-safe $\Delta$-matroids and tight 3-matroids turn out to be essentially equivalent.

5.1. Unique tight extension

First we consider a general result that says that there is at most one way to extend a $k$-matroid to a tight $(k + 1)$-matroid.

**Theorem 13.** Let $(U, \Omega)$ be a carrier. Also, let $T \in \mathcal{T}(\Omega)$ and $\Omega' = \{\omega \setminus T \mid \omega \in \Omega\}$. If $Z'$ is a nondegenerate multimatroid with carrier $(U \setminus T, \Omega')$, then there is at most one tight multimatroid $Z$ with carrier $(U, \Omega)$ such that $Z[U \setminus T] = Z'$.

**Proof.** Let $Z_1$ and $Z_2$ be tight multimatroids such that $Z_1[U \setminus T] = Z' = Z_2[U \setminus T]$ is nondegenerate. Let $X \in \mathcal{T}(\Omega)$. We show by induction on $|X \setminus T|$ that $n_{Z_1}(X) = n_{Z_2}(X)$ (this is sufficient as $Z'$ is nondegenerate). If $|X \setminus T| = 0$, then $X \in \mathcal{T}(\Omega')$ and thus $n_{Z_1}(X) = n_{Z_2}(X)$. Assume that the assertion holds for all $X \in \mathcal{T}(\Omega)$ with $|X \setminus T| = n$. Let $|X \setminus T| = n + 1$. Let $x \in X \setminus T$ belong to the skew class $\omega \in \Omega$. Let $p$ be any skew pair with $x \in p$. Then $|(X \setminus p) \setminus T| = n$, and thus, by the induction hypothesis, $n_{Z_1}(X \setminus p) = n_{Z_2}(X \setminus p)$. Hence, if we consider $S = X \setminus \{x\}$, then $n_{Z_1}(S \cup \{y\}) = n_{Z_2}(S \cup \{y\})$ for all $y \in \omega \setminus \{x\}$. Since $Z_1$ and $Z_2$ are tight, the value of $n_{Z_1}(S \cup \{x\})$ (and $n_{Z_2}(S \cup \{x\})$) is uniquely determined given $n_{Z_1}(S \cup \{y\}) = n_{Z_2}(S \cup \{y\})$ for all $y \in \omega \setminus \{x\}$. Thus, $n_{Z_1}(X) = n_{Z_1}(S \cup \{x\}) = n_{Z_2}(S \cup \{x\}) = n_{Z_2}(X)$. \qed

**Theorem 13** raises the problem of characterizing those $k$-matroids $Z$ that are extendible to a tight $(k + 1)$-matroid. We consider the case $k = 2$ in Sections 5.3 and 5.4.

5.2. Tight 2-matroids and even $\Delta$-matroids

In this subsection we recall from [11,13] that 2-matroids correspond to $\Delta$-matroids, and tight 2-matroids correspond to even $\Delta$-matroids.

An $(\ell, k)$-carrier, for positive integers $\ell$ and $k$, is a carrier $(U, \Omega)$ such that $|\Omega| = \ell$ and for all $\omega \in \Omega$, $|\omega| = k$. A transversal $k$-tuple is a sequence $(T_1, \ldots, T_k)$ of mutually disjoint transversals of $\Omega$.
A projection of \((U, \Omega)\) is a surjective function \(\pi : U \to V\) such that \(\pi(x) = \pi(y)\) iff \(x\) and \(y\) are in the same skew class \(\omega \in \Omega\). Thus each skew class is assigned by \(\pi\) to a unique element of \(V\).

Let \(Z\) be a 2-matroid with carrier \((U, \Omega)\), and fix a projection \(\pi : U \to V\) of \((U, \Omega)\). Let \(\tau = (T_1, T_2)\) be a transversal 2-tuple of \(\Omega\). Then the section of \(Z\) by \(\tau\), denoted by \(M_{Z,\tau,\pi}\), is the set system \((V, D)\) with \(D = \{\pi(X \cap T_2) \mid X \in B_2\}\). If \(\pi\) is clear from the context, then we often write simply \(M_{Z,\tau}\) to denote \(M_{Z,\tau,\pi}\). A section of a 2-matroid is a \(\Delta\)-matroid, see [11, Proposition 4.2]. We define, for all \(v \in V\), \(\tau \ast v = (T_1 \Delta p, T_2 \Delta p)\) where \(p = \pi^{-1}(v)\). It is easy to verify that \(M_{Z,\tau \ast v} \ast v = M_{Z,\tau}\) (or see part of the proof of Theorem 14).

Conversely, given a \(\Delta\)-matroid \(M\) over \(V\), a \((|V|, 2)\)-carrier \((U, \Omega)\), a transversal 2-tuple \(\tau\) of \(\Omega\), and a projection \(\pi : U \to V\) we may construct the 2-matroid \(Z\), denoted by \(Z_{M,\tau}\), such that \(M = M_{Z,\tau}\). We have that \(B_2 = \{X \in \mathcal{T}(\Omega) \mid \pi(X \cap T_2) \in M\}\). The 2-matroid \(Z_{M,\tau}\) is called the lift of \(M\) with respect to \(\tau\) [13, Construction 3.5]. Hence a section of a 2-matroid \(Z\) retains all essential information of \(Z\).

It is shown in [13, Theorem 5.3] that \(Z\) is tight iff there is a transversal 2-tuple \(\tau\) of \(Z\) such that \(\Delta\)-matroid \(M_{Z,\tau}\) is even iff for all transversal 2-tuples \(\tau\) of \(Z\), \(M_{Z,\tau}\) is even.

5.3. \(vf\)-safe delta-matroids from tight 3-matroids

In this subsection and the next we characterize the family of \(\Delta\)-matroids for which the corresponding 2-matroid is \(vf\)-extendible (in the sense of Theorem 13) to a tight 3-matroid. The approach is to extend Section 5.2 by including both \(\mathbb{H}\) and \(+\) and an additional transversal \(T_3\).

In this subsection we fix a \((|V|, 3)\)-carrier \((U, \Omega)\), a transversal 3-tuple \(\tau\) of \((U, \Omega)\), and a projection \(\pi : U \to V\) of \((U, \Omega)\).

Let \(v \in V\), \(\omega = \pi^{-1}(v) \in \Omega\), and \(\tau = (T_1, T_2, T_3)\). Let, for \(i \in \{1, 2, 3\}\), \(p_i \subseteq \omega\) be the (unique) skew pair of \(\omega\) with \(p_i \cap T_i = \emptyset\). Then we define

\[
\begin{align*}
\tau \ast v &= (T_1 \Delta p_3, T_2 \Delta p_3, T_3), \\
\tau \ast v &= (T_1, T_2 \Delta p_1, T_3 \Delta p_1), \\
\tau \ast v &= (T_1 \Delta p_2, T_2, T_3 \Delta p_2).
\end{align*}
\]

Note that \(\ast, \ast, \ast v\) generates a group isomorphic to \(S_3\). In particular, \(\tau \ast v = ((\tau + v) \ast v) \ast v\). Moreover, these operations commute on distinct elements of \(V\). Again, we assume left associativity of these operations. We extend this notation to sets, and write, for all \(L \subseteq V\), e.g., \(\tau + L\) to denote applying \(\tau + v\) for all \(v \in L\) (in arbitrary order).

Let \(Z\) be a 3-matroid over \((U, \Omega)\). Since \(Z[T_1 \cup T_2]\) is a 2-matroid, we have by Section 5.2 that the section \(M_{Z[T_1 \cup T_2],(T_1, T_2),\pi}\) is a \(\Delta\)-matroid. We denote \(M_{Z[T_1 \cup T_2],(T_1, T_2),\pi}\) simply by \(M_{Z,\tau,\pi}\). Again we drop the subscript \(\pi\) when \(\pi\) is clear from the context. We now show that \(M_{Z,\tau}\) is \(vf\)-safe when \(Z\) is tight.

**Theorem 14.** Let \(Z\) be a tight 3-matroid over \((U, \Omega)\). For \(v \in V\), \(M_{Z,\tau} = M_{Z,\tau + v} + v = M_{Z,\tau + v} \ast v = M_{Z,\tau + v} \ast v\). In particular, \(M_{Z,\tau}\) is \(vf\)-safe.

**Proof.** Let \(\tau = (T_1, T_2, T_3)\).

For all \(Y \subseteq V\), there is a unique “lift” \(X \in \mathcal{T}(\Omega)\) such that \(\pi(X \cap T_2) = Y\) and \(X \cap T_3 = \emptyset\), which we denote by \(Y^{(t)}\). Thus, \(Y \in M_{Z,\tau}\) iff \(Y^{(t)} \in B_3\).

Let, for \(i \in \{1, 2, 3\}\), \(p_i\) be the (unique) skew pair of \(\pi^{-1}(v)\) with \(p_i \cap T_i = \emptyset\).

We first consider \(\ast\). Note that \(\tau \ast v = (T_1 \Delta p_3, T_2 \Delta p_3, T_3)\). Hence \(Y^{(t \ast v)} = (Y \Delta \{v\})^{(t)}\). We have \(Y \in M_{Z,\tau} \ast v\) iff \(Y \Delta \{v\} \in M_{Z,\tau}\) iff \((Y \Delta \{v\})^{(t)} \in B_3\) iff \(Y \in M_{Z,\tau \ast v}\).

We now consider \(+\). Note that \(\tau + v = (T_1, T_2 \Delta p_1, T_3 \Delta p_1)\). Assume first that \(v \notin Y\). Then \(Y^{(t)} = Y^{(t + v)}\) as \(Y \cap \omega \subseteq T_1\). We have \(Y \in M_{Z,\tau} + v\) iff \(Y \in M_{Z,\tau}\) iff \(Y^{(t + v)} \in B_3\) iff \(Y \in M_{Z,\tau + v}\).

Assume now that \(v \in Y\). Then \(Y^{(t)} = (Y \setminus \{v\})^{(t + v)}\). We have \(Y \in M_{Z,\tau} + v\) iff either \(Y \in M_{Z,\tau}\) or \(Y \setminus \{v\} \in M_{Z,\tau}\) but not both iff either \(Y^{(t)} \in B_3\) or \((Y \setminus \{v\})^{(t)} \in B_3\) but not both iff \(Y^{(t)} \Delta p_1 \in B_3\) (since \(Z\) is tight) iff \(Y^{(t + v)} \in B_3\) iff \(Y \in M_{Z,\tau + v}\).

We finally consider \(\ast\), which is now easy. We have \(M_{Z,\tau} \ast v = M_{Z,\tau} + v \ast v + v = M_{Z,\tau + v} \ast v + v = M_{Z,\tau + v} \ast v\). □
5.4. Tight 3-matroids from v-safe delta-matroids

A pre-multimatroid $Z$ is a triple $(U, \Omega, I)$, where $(U, \Omega)$ is a carrier and $I \subseteq \delta(\Omega)$. Note that we may define, e.g., the restriction $Z[X]$ for $X \in \delta(\Omega)$ exactly as we did for multimatroids.

In this subsection we fix a nonempty set system $M$ over $V$, a $(|V|, 3)$-carrier $(U, \Omega)$, a transversal 3-tuple $\tau = (T_1, T_2, T_3)$ of $\Omega$, and a projection $\pi : U \rightarrow V$ of $(U, \Omega)$.

We define

$$B_{M, \tau, \pi} = \{ X \in \mathcal{T}(\Omega) \mid \pi(X \cap T_2) \in M \Rightarrow \pi(X \cap T_3) \}.$$  

For symmetry's sake, note that the condition in the definition of $B_{M, \tau, \pi}$ is equivalent to $\varnothing \in M \Rightarrow \pi(X \cap T_3) \in M$ by Lemma 12. As the sets $\pi(X \cap T_1)$, $i \in \{1, 2, 3\}$, are disjoint, we can perform these operations in any order.

We denote by $Z_{M, \tau, \pi}$ the pre-multimatroid $(U, \Omega, B_{M, \tau, \pi})$.

Note that if $M$ is a $\Delta$-matroid, then, by Section 5.2, $Z_{M, \tau}[T_1 \cup T_2]$ is a 2-matroid and the lift of $M$ with respect to $(T_1, T_2)$. Thus, $Z_{M, \tau}[T_1 \cup T_2] = Z_{M_{\Delta}(T_1, T_2)}$.

Lemma 15. Let $v \in V$. Then $Z_{M, \tau} = Z_{M+v, \tau+v} = Z_{M*\varepsilon, \tau*\varepsilon} = Z_{M*\varepsilon, \tau*\varepsilon}$.

Proof. For later use, we first show a general property of vertex flip operations on a set system $M$. Let $B, C \subseteq V$ be disjoint, and let $Y \subseteq V$. Then

$$M + Y * B \bar{*} C = M * (B \Delta Y') \bar{*} (C \Delta Y') + Y \quad \text{with} \quad Y' = Y \cap (B \cup C),$$

$$M * Y * B \bar{*} C = M * (B \Delta Y') \bar{*} C + (Y \cap C) \quad \text{with} \quad Y' = Y \setminus C,$$

$$M \bar{*} Y * B \bar{*} C = M \bar{*} B \bar{*} (C \Delta Y') + (Y \cap B) \quad \text{with} \quad Y' = Y - B. \tag{3}$$

As the vertex flips on different vertices commute, it suffices to consider only a single element $v \in V$. In other words, we may assume without loss of generality that $Y, B,$ and $C$ are all subsets of $\{v\}$. If $v \notin Y$, then there is nothing to prove. Assume $v \in Y$. Then, depending on whether $v$ is in $B$ or $C$, we observe that $M + v \bar{*} v = M \bar{*} v + v$ (for $v \in B$), $M + v \bar{*} v = M \bar{*} v + v$ (for $v \in C$), and $M + v = M + v$ (for $v \notin B \cup C$). This proves Equality (1).

Similarly, $M \bar{*} v \bar{*} v = M \bar{*} v + v$ (for $v \in C$), and $M \bar{*} v = M \bar{*} v$ (for $v \notin B \cup C$), which leads to Equality (2). Equality (3) is proved in an analogous way.

By Lemma 12,

$$\varnothing \in M + Y * B \bar{*} C \quad \text{iff} \quad \varnothing \in M * (B \Delta Y') \bar{*} (C \Delta Y') \quad \text{with} \quad Y' = Y \cap (B \cup C),$$

$$\varnothing \in M * Y * B \bar{*} C \quad \text{iff} \quad \varnothing \in M * (B \Delta Y') \bar{*} C \quad \text{with} \quad Y' = Y \setminus C,$$

$$\varnothing \in M \bar{*} Y * B \bar{*} C \quad \text{iff} \quad \varnothing \in M \bar{*} B \bar{*} (C \Delta Y') \quad \text{with} \quad Y' = Y \setminus B.$$  

For $i \in \{1, 2, 3\}$, $p_i \subseteq \omega$ be the (unique) skew pair of $\omega$ with $p_i \cap T_j = \varnothing$. Let $X \in \mathcal{T}(\Omega)$. We thus have $X \in B_{M+i, \tau}$ if and only if $\varnothing \in M + v \bar{*} \pi(X \cap T_2) \bar{*} \pi(X \cap T_2) \pi(X \cap T_2) \Delta(Y')$ if and only if $\varnothing \in M * (\pi(X \cap T_2) \Delta Y') \bar{*} (\pi(X \cap T_2) \Delta Y')$ if and only if $\varnothing \in M \bar{*} v \bar{*} (\pi(X \cap T_2) \Delta Y')$ if and only if $\varnothing \in M * \pi(X \cap T_2 \Delta \psi) \bar{*} \pi(X \cap T_2 \Delta \psi)$ if and only if $\varnothing \in M \bar{*} \pi(X \cap T_2 \Delta \psi) \bar{*} \pi(X \cap T_2 \Delta \psi)$. Hence $X \in B_{M+i, \tau}$.

Similarly, we obtain $X \in B_{M*\varepsilon, \psi} \bar{*} v$ if $X \in B_{M+\varepsilon, \tau}$ and $X \in B_{M*\varepsilon, \tau*\varepsilon}$ if $X \in B_{M, \tau*\varepsilon}$. \hfill \Box

We now show that $Z_{M, \tau}$ is a tight 3-matroid when $M$ is a v-safe $\Delta$-matroid.

Theorem 16. Let $M$ be a v-safe $\Delta$-matroid. Then $Z_{M, \tau}$ is a tight 3-matroid described by its bases.

Proof. Let $T \in \mathcal{T}(\Omega)$. We first argue that $Z' = Z_{M, \tau}[T]$ is a matroid. Let $\psi$ be a sequence of $+, \bar{*}$, and $\bar{*}$ operations such that $\psi = (T_1', T_2', T_3')$ with $T \subseteq T_1' \cup T_2'$. Then by Lemma 15, $Z_{M, \tau} = Z_{M*\nu, \psi}$. By Section 5.2, $Z_{M*\nu, \psi}[T_1 \cup T_2'] = Z_{M*\nu,(T_1', T_2')}$. Consequently, $Z' = Z_{M*\nu,(T_1', T_2')}[T]$ is a matroid.

We now show the second defining property of multimatroids (in Definition 1). Let $I = \{i \subseteq X \mid X \in B_{M, \tau}\}$. Let $I \in \mathcal{I}$ and $\omega \in \Omega$ with $\omega \cap I \neq \varnothing$, and let $X \in B_{M, \tau}$ with $I \subseteq X$. Consider now
Let $M$ be a $\Delta$-matroid, or $\Delta$-matroid, over $V$. Let $M'$ be a $\Delta$-matroid over $V$. Let $\tau$ be a transversal $\Delta$-matroid over $V$. Let $\pi : U \to V$ be a projection of $(U, \Omega)$. Then for all $T \in \mathcal{T}(\Omega)$, $n_{Z_{M',\tau}}(T) = d_{M\pi(T \cap T_2)\pi^{-1}(T \cap T_3)}$. 

**Proof.** Let $\tau = (T_1, T_2, T_3)$ and $T \in \mathcal{T}(\Omega)$. Let $M' = M * \pi (T \cap T_2) \pi (T \cap T_3)$. By Theorem 16, $Z_{M',\tau}$ is a $\Delta$-matroid. By Lemma 15, $n_{Z_{M',\tau}}(T) = n_{Z_{M,\tau}}(T')$ with $T' = \pi (T \cap T_2) \pi (T \cap T_3)$. Now, $T' = (T_1', T_2', T_3')$ with $T_1' = T$. Let $Z = Z_{M',\tau}$.

We have that $r_2(T) = r_2[\pi^{-1}(T)](T) = r_2(T[T \cap T_2])$ which is equal to the largest independent set $I$ of $T[T \cap T_2]$ with $I \subseteq T$. Hence, $r_2(T)$ is the maximum value of $|X \cap T|$ with $X \in \mathcal{B}_{Z[T[T \cap T_2]]}$. Consequently, $n_2(T)$ is the minimum value of $|\Omega| - |T \cap T_1| - |\pi (X \cap T)| + |\pi (X \cap T)|$ with $X \in \mathcal{B}_{Z[T[T \cap T_2]]}$. Hence $d_{M',\tau}(T) = d_{M}$. By change of variables $M := M'$ we obtain the desired result. 

Let $M$ be a $\Delta$-matroid. Since $Z_{M,\tau}$ is tight, we have by Lemma 18 that the values $d_{M,\tau}$, $d_{M+\tau}$, and $d_{M+\tau}$ are such that precisely two of the three are equal, to say $m$, and the third is equal to $m + 1$. This has also been shown in a direct way (without the use of multimatroids) in [18].

Obviously, we may formulate the "2-matroid version" of Lemma 18: if $Z$ is a 2-matroid with transversal 2-tuple $\tau = (T_1, T_2)$, then $n_2(T) = d_{M_{2,\tau}}$ for all $T \in \mathcal{T}(\Omega)$. Note that the 2-matroid version of Lemma 18 is not strictly a special case of Lemma 18 as not all 2-matroids allow for an extension to tight 3-matroids (or, equivalently, not every $\Delta$-matroid is $\Delta$-safe).

We now turn to minors.

**Lemma 19.** Let $M$ be a $\Delta$-matroid over $V$. Let $Z = Z_{M,\tau}$ with $\tau = (T_1, T_2, T_3)$ a transversal 3-tuple of $\Omega$, $(U, \Omega)$ a $(|V|, 3)$-carrier, and $\pi : U \to V$ a projection of $(U, \Omega)$. Moreover, let $v \in V$, $M_1 = M \setminus v$, $M_2 = M * v \setminus v$, $M_3 = M \setminus v \setminus v$, and $v = \pi^{-1}(v)$. 

Recall from Section 5.2 that the lift $M \mapsto Z_{M(T_1, T_2)}$ is a one-to-one correspondence from the family of $\Delta$-matroids over $V$ to the family of $\Delta$-matroids with carrier $(T_1 \cup T_2, \omega \cap T_3, \omega \in \Omega))$, with inverse mapping $Z \mapsto Z_{M(T_1, T_2)}$.

By Theorems 14 and 16, for a $\Delta$-matroid $M$ over $V$, the 3-matroid $Z_{M,T}$ is the (unique) extension of the 2-matroid $Z_{M(T_1, T_2)}$ to a tight 3-matroid over $(U, \Omega)$ (as in Theorem 13). The 2-matroids involved are precisely the 2-matroids that allow for such a (unique) extension.

Let $\Theta$ be the mapping from 3-matroids over $(U, \Omega)$ to 2-matroids which sends each $Z$ to the restriction $Z[T_1 \cup T_2]$. Then, for tight 3-matroids over $(U, \Omega)$, $Z \mapsto Z_{M,T} = Z_{\Theta(Z),T}$ is the inverse of $M \mapsto Z_{M,T}$.

This leads to the following result.

**Theorem 17.** The mapping $M \mapsto Z_{M,T}$ is a one-to-one correspondence from the family of $\Delta$-matroids over $V$ to the family of tight 3-matroids over $(U, \Omega)$, with inverse $Z \mapsto Z_{M,T}$. 

5.5. Carrying over nullity and minor

In this subsection we tie the notion of nullity in a tight 3-matroid to that of distance in the corresponding $\Delta$-matroid. We also carry over the concept of minor for tight 3-matroids to $\Delta$-matroids (and similarly for 2-matroids to $\Delta$-matroids).

**Lemma 18.** Let $M$ be a $\Delta$-matroid over $V$. Let $(U, \Omega)$ be a $(|V|, 3)$-carrier, $\tau$ be a transversal $\Delta$-matroid over $V$, and $\pi : U \to V$ be a projection of $(U, \Omega)$. Then for all $T \in \mathcal{T}(\Omega)$, $n_{Z_{M,\tau}}(T) = d_{M\pi(T \cap T_2)\pi^{-1}(T \cap T_3)}$.
For all $i \in \{1, 2, 3\}$, $M_i$ is nonempty iff the (unique) element $t_i \in T_i \cap \omega$ is nonsingular in $Z$. Moreover, if $t_i$ is nonsingular in $Z$, then $Z|t_i = Z_{M_i,\tau'}$ with $\tau' = (T_1 \setminus \omega, T_2 \setminus \omega, T_3 \setminus \omega)$ and so $M_i$ is a $v\setminus w$-safe $\Delta$-matroid.

**Proof.** First, $t_1$ is nonsingular in $Z$ iff $Z[T_1 \cup T_2] = Z_{M,(T_1,T_2)}$ has a basis containing $t_1$ iff $M$ contains a set without $v$ iff $M \setminus v$ is nonempty. By Lemma 15, $t_2$ is nonsingular in $Z$ iff $t_1$ is nonsingular in $Z_{M,v+} = Z_{M,v,\tau}$ iff $M \setminus v \setminus \bar{v}$ is nonempty. In the same way, we obtain that $t_3$ is nonsingular in $Z$ iff $M \setminus v \setminus \bar{v}$ is nonempty.

Let $t_i$ be nonsingular in $Z$ for some $i \in \{1, 2, 3\}$, and let $X \in \mathcal{T}(\Omega \setminus \{\omega\})$. Then, by the “2-matroid version” of Lemma 19, a $v$ is nonsingular in a $\Delta$-matroid $M$ iff all $u \in \pi^{-1}(v)$ are nonsingular in $Z_{M,\tau}$ for suitable transversal 2-tuple $\tau$ iff $\pi^{-1}(v)$ is nonsingular in $Z_{M,\tau}$.

We call $v \in V$ nonsingular in a nonempty set system $M$ if both $M \setminus v$ and $M \setminus v \setminus \bar{v}$ are nonempty. Thus, by the “2-matroid version” of Lemma 19, $v$ is nonsingular in a $\Delta$-matroid $M$ iff all $u \in \pi^{-1}(v)$ are nonsingular in $Z_{M,\tau}$.

Let note that $v \in V$ is nonsingular in a nonempty set system $M$ iff there are $X_1, X_2 \in M$ with $v \in X_1 \Delta X_2$. Clearly, for all $w \in V$, $v$ is nonsingular in $M$ iff $v$ is nonsingular in $M \setminus w$. Note that every $v \in V$ is singular in $M$ if $M$ contains only one subset. Note that if $M$ is a matroid, then $M \setminus v$ is nonempty iff $v$ is not a loop or a coloop of $M$. Hence $v$ is nonsingular in $M \setminus v$ if and only if $v$ is neither a loop nor a coloop.

We call $v \in V$ strongly nonsingular in a nonempty set system $M$ if $M \setminus v$, $M \setminus v \setminus \bar{v}$ are all nonempty. Thus, by Lemma 19, $v$ is strongly nonsingular in a $\setminus w$-safe $\Delta$-matroid $M$ iff $\pi^{-1}(v)$ is nonsingular in $Z_{M,\tau}$ for suitable transversal 3-tuple $\tau$.

**Remark 20.** In line with matroid theory, one may define an element $v$ of a $\Delta$-matroid $M$ to be a coloop of $M$ if $M \setminus v$ is empty, and a loop of $M$ if $M \setminus v \setminus \bar{v}$ is empty. Then one may define the deletion of $v$ in $M$, here denoted by $M \setminus v$, to be $M \setminus v$ if $v$ is not a loop of $M$, and $M \setminus v \setminus \bar{v}$ otherwise. Similarly, one may define the contraction of $v$ in $M$, denoted by $M/v$, to be $M \setminus v \setminus \bar{v}$ if $v$ is not a loop of $M$ and $M \setminus v \setminus \bar{v}$ otherwise. Note that if $M$ is a matroid described by its bases, then these notions of coloop, loop, deletion and contraction coincide with the usual matroid-theoretical definitions (note, e.g., that if $v$ is not a loop, then $M/v = (V \setminus \{v\}, \{X \setminus \{v\} \mid v \in X \in M\}) = M \setminus v \setminus \bar{v}$). Now, by the 2-matroid version of Lemma 19, Proposition 2, and the fact that minors on distinct elements in a multimatroid commute, we have that deletion and contraction on $\Delta$-matroids $M$ commute on distinct elements. Note that these operations do not commute for set systems $M$ in general: for example, if $M = (\{u, v, w\}, \{\{u\}, \{v, w\}\})$, then $M \setminus u \setminus v = (\{w\}, \{\emptyset\})$ and $M \setminus v \setminus u = (\{w\}, \{\emptyset\})$. A similar remark can be made for $\setminus w$-safe $\Delta$-matroids and their three types of minors using Lemma 19.

For notational convenience, we stick with the set system operation $\setminus$ in this paper, but the reader may trivially reformulate the results in this paper in terms of the minor definitions given here.

### 6. Transition polynomial for set systems

#### 6.1. The transition polynomial

We move from multimatroids to set systems, and consider a, rather generic, weighted polynomial for set systems. We first obtain a technical result that shows how the variables of the polynomial change when one of the vertex flip operations $+Y$, $\ast Y$ and $\ast Y$ is applied. In the next subsections more interesting polynomials appear as specializations.

Let $V$ be a finite set. We define $\mathcal{P}_3(V)$ to be the set of triples $(V_1, V_2, V_3)$ where $V_1$, $V_2$, and $V_3$ are pairwise disjoint subsets of $V$ such that $V_1 \cup V_2 \cup V_3 = V$. Therefore $V_1$, $V_2$, and $V_3$ form an “ordered partition” of $V$ where $V_i = \emptyset$ for some $i \in \{1, 2, 3\}$ is allowed.
We now define a weighted polynomial for set systems.

**Definition 21.** Let \( M \) be a nonempty set system over \( V \). We define the (weighted) transition polynomial of \( M \) as follows:

\[
Q(M)(\vec{a}, \vec{b}, \vec{c}, y) = \sum_{(A, B, C) \in \mathcal{P}_2(V)} a_A b_B c_C y^{d_{M + \vec{a} \cup \vec{b} \cup \vec{c}}},
\]

where \( \vec{a} \) is a vector indexed by \( V \) with entries \( a_v \) for all \( v \in V \), \( a_A = \prod_{v \in A} a_v \), and similarly for \( \vec{b}, \vec{c}, \) and \( c_C \).

For notational convenience, we often omit the variables \( y \) in \( Q(M) \). In fact, we often also omit the variable \( y \) in \( Q(M)(y) \). In the proof of Lemma 15 we have

**Proof.** By Equality (1) in the proof of Lemma 15 we have \( d_{M + \vec{a} \cup \vec{b} \cup \vec{c}} = d_{M + (\vec{a} \cup \vec{b} \cup \vec{c})} \). The equality for \( Q(M + Y) \) is obtained by changing variables \( B := B \Delta Y' \) and \( C := C \Delta Y' \). The equalities for \( Q(M \nabla Y) \) and \( Q(M \ast Y) \) are obtained similarly. \( \square \)

### 6.2. Specializations of the transition polynomial

In this subsection we consider four interesting special cases of the transition polynomial \( Q(M) \), which generalizes the two types of interlace polynomials [2,5] and the bracket polynomial [32] for graphs. These special cases each fulfill a particular invariance result with respect to pivot, loop complementation, or dual pivot. These invariance results do not hold for \( Q(M) \) in general.

In this subsection we again let \( M \) be a nonempty set system. We let \( Q_{(a, b, c)}(M) \) be \( Q(M) \) where \( a_u = a, b_u = b, \) and \( c_u = c \) for all \( u \in V \). Thus

\[
Q_{(a, b, c)}(M)(y) = \sum_{(A, B, C) \in \mathcal{P}_2(V)} a^{|A|} b^{|B|} c^{|C|} y^{d_{M + \vec{a} \cup \vec{b} \cup \vec{c}}}.\]

Note that by Theorem 22, \( Q_{(a, b, c)}(M) = Q_{(a, c, b)}(M + V) = Q_{(c, b, a)}(M \nabla V) = Q_{(b, a, c)}(M \ast V) \).

We consider specific values for \( a, b, \) and \( c \) as specializations of \( Q_{(a, b, c)}(M) \).

**The polynomial \( Q_1(M) \).** Let \( Q_1(M) = Q_{(1, 1, 1)}(M) \). Explicitly, we have

\[
Q_1(M) = \sum_{X, Y \subseteq V, X \cap Y = \emptyset} y^{d_{M + \vec{X} \cup \vec{Y}}}.\]

The last equality can be seen as follows. For \( X \cap Y = \emptyset \), we have \( M + \vec{Y} \ast X = M \ast Y + X \ast X = M + X \ast X + X = M + X \ast (Y \cup Y) + X \). Moreover, \( d_{M + \vec{X} \cup \vec{Y} \ast X} = d_{M + \vec{X} \cup \vec{Y} \ast X} = d_{M + X \ast (Y \cup Y)} = d_{M + X \ast (Y \cup Y)} \). Finally, change variables \( [Z := X, X := X \cup Y] \).

By Theorem 22, \( Q_1(M) \) is invariant under pivot, loop complementation, and dual pivot. We will see in Section 10 that \( Q_1(M) \) generalizes an interlace polynomial for simple graphs defined in [2].
The polynomials $q_i(M)$. We now give polynomials $q_1(M)$, $q_2(M)$, and $q_3(M)$ which are invariant under pivot, loop complementation, and dual pivot, respectively. In Section 10 we find that $q_1(M)$ generalizes the (single-variable) interlace polynomial (from [5]), and that $q_2(M)$ generalizes the bracket polynomial for graphs defined in [32].

Let $q_1(M) = Q_{1,0,0}(M)$. Thus, by Lemma 12,
\[ q_1(M) = \sum_{X \subseteq V} y^{d_M+X} = \sum_{X \subseteq V} y^{d_M(X)}. \]

We remark that $q_1(M)$ is a very natural formulation as it counts the distances of each $X \subseteq V$ with $M$. By Theorem 22 (or by the explicit formulation above), $q_1(M)$ is invariant under pivot.

Let $q_2(M) = q_1(M \ast V) = Q_{0,1,1}(M)$. Recall from above (regarding $Q_1$) that for $X, Y \subseteq V$ with $X \cap Y = \emptyset$, we have $d_{M+Y}X = d_{M+X}(X \cup Y)$. Therefore, considering the case $Y = V \setminus X$, we have
\[ q_2(M) = \sum_{X \subseteq V} y^{d_{M+V}(X)+X} = \sum_{X \subseteq V} y^{d_{M+X}(V)}. \]

By Theorem 22 (or by the explicit formulation above), $q_2(M)$ is invariant under loop complementation.

Finally $q_3(M) = q_1(M + V) = Q_{1,0,1}(M)$ completes the family. Thus,
\[ q_3(M) = \sum_{X \subseteq V} y^{d_{M+X}} = \sum_{X \subseteq V} y^{d_{M+X}(X)}. \]

The latter equality holds, since $d_{M+X} = d_{M+X+X} = d_{M+X} = d_{M+X}(X)$.

By Theorem 22 (or by the explicit formulation above), $q_3(M)$ is invariant under dual pivot. Moreover, by Theorem 22, we find that $q_3(M) = q_2(M \ast V)$. Note that since pivot, loop complementation, and dual pivot are involutions, we also have, e.g., $q_2(M) = q_3(M \ast V)$.

**Example 23.** Let $M = (V, \{ p, q \}, \{ q, r \}, \{ p \}, \{ r \}, \emptyset)$, cf. Example 10. Then $Q_1(M) = 16 + 10y + y^2$, $q_1(M) = 5 + 3y$, $q_2(M) = q_1(M \ast V) = 3 + 4y + y^2$, and $q_3(M) = q_1(M + V) = 6 + 2y$.

Note that the invariance properties of $Q_1(M)$ and $q_i(M)$ ($i \in \{1, 2, 3\}$) do not hold in general for the transition polynomial $Q(M)$ itself. In the next sections we will focus on particular recursive relations for $Q_1(M)$ and $q_i(M)$ ($i \in \{1, 2, 3\}$), which also do not hold for $Q(M)$ in general.

### 6.3. Two-variable polynomials and the Tutte polynomial

We often consider single-variable polynomials $Q_i(M)$ and $q_i(M)$ by letting $a$, $b$, and $c$ be constants in $Q_{a,b,c}(M)$. However, at some points it is useful to consider some of $a$, $b$, and $c$ as variables.

Again, let $M$ be a nonempty set system. We have $Q_{a,b,c}(M)(y) = \sum_{X \subseteq V} a^{\vol(X)}b^{\chi(X)}c^{d_{M+X}}$. Let $\bar{q}(M)(x, y) = Q_{1,0,1}(M)(y)$. Then obviously $\bar{q}(M)(x, y) = \sum_{X \subseteq V} x^{\vol(X)} y^{d_{M+X}}$ and $\bar{q}(M)(1, y) = q_1(M)(y)$. Note however that, unlike the single-variable case $q_1(M)$, $\bar{q}(M)(x, y)$ is not invariant under pivot. Polynomial $\bar{q}(M)(x, y)$ will be of interest in Section 10. Another two-variable case of interest in Section 10 is $\bar{q}(M \ast V \ast V) = Q_{0,1,1}(M)$.

Let $M$ be a matroid over $V$ (described by its bases). The Tutte polynomial is defined by
\[ t_M(x, y) = \sum_{X \subseteq V} (x - 1)^{r(Y) - r(X)}(y - 1)^{n(X)} \]
where $n(X) = \min\{|X \setminus Y|\}$ and $r(X) = |X| - n(X)$ are the nullity and rank of $X \subseteq V$ in $M$, respectively.

The following result is an extension of known results for 4-regular graphs [26] and for binary matroids [1, Proposition 4] to matroids in general.

**Theorem 24.** Let $M$ be a matroid over $V$. Then $Q_{a,b,c}(M)(y) = a^{\vol(V)}b^{r(V)}t_M(1 + \frac{a}{b}y, 1 + \frac{b}{a}y)$. 
We have \( a^{n(V)}b^{r(V)}t_M(1 + \frac{b}{a}y, 1 + \frac{b}{a}y) = a^{n(V)}b^{r(V)} \sum_{X \subseteq V} (\frac{b}{a}y)^{r(X) - r(V)} (\frac{b}{a}y)^{n(X)} = \sum_{X \subseteq V} a^{r(X)}b^{n(X)} \). It suffices to show that \( r(V) - r(X) + n(X) = d_M(X) \). We have \( n(X) = \min(|X \setminus Y| \mid Y \in M) \). Hence \( r(V) - r(X) + n(X) = r(V) - |X| + 2n(X) = \min((|Y| - |X| + |X \setminus Y|) + |Y \setminus X| \mid Y \in M) \) as \( r(V) \) is the cardinality of the sets \( Y \in M \). Now \( |Y| - |X| + |X \setminus Y| = |Y \setminus X| \) and thus we obtain \( \min((|Y| - |X| + |X \setminus Y|) + |X \setminus Y| \mid Y \in M) = \min(|X \setminus Y| \mid Y \in M) = d_M(X) \). \( \Box \)

If we take \( a = b = 1 \) in Theorem 24, then we obtain the following corollary.

**Corollary 25.** Let \( M \) be a matroid. Then \( t_M(y, y) = q_1(M)(y - 1) \).

Hence, the Tutte polynomial on the diagonal is essentially \( q_1(M) \) (in the sense that they both represent the same information) for the case where \( M \) is a matroid described by its bases. The polynomial \( R_M(x, y) = t_M(x + 1, y + 1) \) is known as the Whitney rank generating function, as recalled in [35, Section 15.4]. Hence \( R_M(y, y) = q_1(M)(y) \).

7. Recursive relations and \( \Delta \)-matroids

In this section we show that each of the polynomials considered in Section 6 fulfill a specific recursive relation when restricting to (vf-safe) \( \Delta \)-matroids.

Section 5 shows that \( \Delta \)-matroids may be lifted to 2-matroids, and vf-safe \( \Delta \)-matroids to tight 3-matroids. This allows us to transfer properties of polynomials for multimatroids to similar properties for \( \Delta \)-matroids.

7.1. The transition polynomial

We first show that the polynomials \( Q(M) \) for vf-safe \( \Delta \)-matroids \( M \) and \( Q(Z) \) for tight 3-matroids \( Z \) are equivalent.

**Theorem 26.** Let \( M \) be a vf-safe \( \Delta \)-matroid over \( V \), \( (U, \Omega) \) be a \((|V|, 3)\)-carrier, \( \pi : U \to V \) be a projection of \( (U, \Omega) \), and \( \tau = (T_1, T_2, T_2) \) be a transversal 3-tuple of \( \Omega \). Then \( Q(M)(\tilde{a}, \tilde{b}, \tilde{c}, y) = Q(Z_{M, \pi})(\tilde{x}, y) \), where, for \( u \in U \), \( x_u = a_{\pi(u)} \) if \( u \in T_1 \), \( x_u = b_{\pi(u)} \) if \( u \in T_2 \), \( x_u = c_{\pi(u)} \) if \( u \in T_3 \).

**Proof.** Let \( Z = Z_{M, \pi} \). By Lemma 18, \( n_Z(T) = d_{M+\pi(T\cap T_2)\|\pi(T\cap T_3)} \). Thus we have

\[
Q(Z)(\tilde{x}, y) = \sum_{T \in \pi(\Omega)} x_T y^{d_T} = \sum_{T \in \pi(\Omega)} a_{\pi(T\cap T_1)}b_{\pi(T\cap T_2)c_{\pi(T\cap T_3)}y^{d_T}(T)} = \sum_{(A, B, C) \in \mathcal{P}_3(V)} a_A b_B c_C y^{d_{M+\tilde{A}, \tilde{B}, \tilde{C}}} = Q(M)(\tilde{a}, \tilde{b}, \tilde{c}, y). \ \Box
\]

The “2-matroid version” of Theorem 26 is obtained similarly.

**Corollary 27.** Let \( M \) be a \( \Delta \)-matroid over \( V \), \( (U, \Omega) \) be a \((|V|, 2)\)-carrier, \( \pi : U \to V \) be a projection of \( (U, \Omega) \), and \( \tau = (T_1, T_2) \) be a transversal 2-tuple of \( \Omega \). Then \( Q(M)(\tilde{a}, \tilde{b}, \tilde{c}, y) = Q(Z_{M, \pi})(\tilde{x}, y) \), where, for \( u \in U \), \( x_u = a_{\pi(u)} \) if \( u \in T_1 \), \( x_u = b_{\pi(u)} \) if \( u \in T_2 \), and \( c_v = 0 \) for all \( v \in V \).

**Proof.** Since \( c_v = 0 \) for all \( v \in V \), it is sufficient to restrict the summation in the definition of \( Q(M) \) to the triples \( (A, B, C) \in \mathcal{P}_3 \) with \( C = \emptyset \). Other than this, the proof of this result is identical to the proof of Theorem 26. \( \Box \)

We are now ready to state recursive relations for \( Q(M) \).

**Theorem 28.** Let \( M \) be a vf-safe \( \Delta \)-matroid and let \( v \in V \).

1. If \( v \) is strongly nonsingular in \( M \), then

\[
Q(M) = a_v Q(M \setminus v) + b_v Q(M \setminus v) + c_v Q(M \setminus v).\]

2. If \( v \) is strongly singular in \( M \), then

\[
Q(M) = a_v Q(M \setminus v) + b_v Q(M) + c_v Q(M \setminus v).\]
(2) Assume \( v \) is not strongly nonsingular in \( M \), and let \( \{ (z_1, M_1), (z_2, M_2), (z_3, M_3) \} = \{ (a_v, M \setminus v), (b_v, M \ast v \setminus v), (c_v, M \ast v \setminus v) \} \). If \( M_1 \) is empty, then \( M_2 = M_3 \) is nonempty and
\[
Q(M) = (z_2 + z_3 + z_1y)Q(M_2).
\]

**Proof.** Let \( Z = Z_{M, \tau} \) for some transversal 3-tuple \( \tau = (T_1, T_2, T_3) \) for \((|V|, 3)\)-carrier \((U, \Omega)\) and projection \( \pi : U \to V \). By Theorem 26, \( Q(M)(\bar{a}, \bar{b}, \bar{c}, y) = Q(Z)(\bar{x}, y) \) for some suitable \( U \)-indexed vector \( \bar{x} \). Let \( \omega = \pi^{-1}(v) \).

Assume first that \( v \) is strongly nonsingular in \( M \). By Lemma 19, \( \omega \) is nonsingular in \( Z \). By Theorem 6, \( Q(Z)(\bar{x}, y) = \sum_{w \in \omega} \alpha_{x_0}Q(Z|w)(\bar{x}, y) = a_vQ(Z|t_1)(\bar{x}, y) + b_vQ(Z|t_2)(\bar{x}, y) + c_vQ(Z|t_3)(\bar{x}, y) \) where \( t_i \in T_i \cap \omega \) for all \( i \in \{ 1, 2, 3 \} \) and \( \bar{x} \) is obtained from \( \bar{x} \) by removing the entries indexed by \( \omega \). By Lemma 19, for all \( i \in \{ 1, 2, 3 \} \), \( Q(Z|t_i)(\bar{x}, y) = Q(Z_{N_i, \tau'})(\bar{x}, y) \) with \( N_1 = M \setminus v, N_2 = M \ast v \setminus v, N_3 = M \setminus v \setminus v \), and \( \tau' = (T_1 \setminus \omega, T_2 \setminus \omega, T_3 \setminus \omega) \). By Theorem 26, \( Q(Z_{N_i, \tau'})(\bar{x}, y) = Q(N_i)(\bar{a}', \bar{b}', \bar{c}', y) \) where the vectors \( \bar{a}', \bar{b}', \bar{c}' \) are obtained from \( \bar{a}, \bar{b}, \bar{c} \) by removing the entries indexed by \( u \).

Assume now that \( v \) is not strongly nonsingular in \( M \). Assume that \( M_1 \) is empty. Let \( \omega = \{ t_1, t_2, t_3 \} \) where for all \( i \in \{ 1, 2, 3 \} \), \( t_i \in T_i \). By Lemma 19, \( \omega \) is singular in \( Z \). Let \( u \in \omega \) be singular in \( Z \). By Theorem 6, \( Q(Z)(\bar{x}, y) = (\alpha_{x_0} + \sum_{w \in \omega \setminus \{ u \}} \alpha_{x_0})Q(Z|w)(\bar{x}, y) \) for all \( w \in \omega \). Since \( Z|m = Z|n \) for all \( m, n \in \omega \), we have by Lemma 19, \( Z_{M_2, \tau'} = Z_{M_3, \tau'} \) and thus \( M_2 = M_3 \) by Theorem 17. Thus, for all \( w \in \omega \), \( Q(Z|w)(\bar{x}, y) = Q(Z_{M_2, \tau'})(\bar{x}, y) \). Finally, by Theorem 26, \( Q(Z_{N_i, \tau'})(\bar{x}, y) = Q(N_i)(\bar{a}', \bar{b}', \bar{c}', y) \) where the vectors \( \bar{a}', \bar{b}', \bar{c}' \) are obtained from \( \bar{a}, \bar{b}, \bar{c} \) by removing the entries indexed by \( v \). □

Note that, just like for Theorem 6, the recursive relations of Theorem 28 characterize \( Q(M)(\bar{a}, \bar{b}, \bar{c}, y) \).

We now state the obvious 2-matroid version of Theorem 28.

**Corollary 29.** Assume that \( \bar{c} \) is the zero vector in \( Q(M) \). Let \( M \) be a \( \Delta \)-matroid and let \( u \in V \).

1. If \( u \) is nonsingular in \( M \), then
   \[
   Q(M) = a_uQ(M \setminus u) + b_uQ(M \ast u \setminus u).
   \]
2. Assume \( u \) is singular in \( M \). If \( M \setminus u \) is empty, then \( M \ast u \setminus u \) is nonempty and
   \[
   Q(M) = (b_u + a_u)yQ(M \ast u \setminus u).
   \]
   A similar statement holds if \( M \ast u \setminus u \) is empty.

7.2. The polynomials \( q_i(M) \)

In the rest of this section we turn to various specializations of \( Q(M) \). We immediately obtain the following recursive relation for \( q_1(M) \).

**Theorem 30.** Let \( M \) be a \( \Delta \)-matroid. If \( u \in V \) is nonsingular in \( M \), then
\[
q_1(M) = q_1(M \setminus u) + q_1(M \ast u \setminus u).
\]
If every \( v \in V \) is singular in \( M \), then \( q_1(M) = (y + 1)^n \) with \( n = |V| \).

**Proof.** By Corollary 29, it suffices to consider the case where every \( v \in V \) is singular in \( M \), i.e., \( M \) contains only one subset. Repeat the recursion \( q_1(M) = (1 + y)q_1(M') \) where nonempty \( M' \) is either \( M \setminus u \) or \( M \ast u \setminus u \), until we reach \( M_\emptyset = (\emptyset, \{ \emptyset \}) \) which has \( q_1(M_\emptyset) = 1 \). □

**Example 31.** We recursively compute \( q_1(M) \) using Theorem 30 for \( \Delta \)-matroid \( M = ([p, q, r], \{ \emptyset, \{ p \}, \{ p, q \}, \{ q, r \}, \{ r \}) \) from Example 10. The computation tree is given in Fig. 2. Recursion stops when the \( \Delta \)-matroid is singular for all elements of the ground set. We verify that \( q_1(M) = 3y + 5 \).

Obviously \( q_2(M) \) and \( q_3(M) \) fulfill similar recursive relations as \( q_1(M) \) by applying the previous result to \( q_1(M \ast V) \) and \( q_1(M + V) \), respectively. However, as the family of \( \Delta \)-matroids is not closed
under loop complementation, we require that $M$ is $vf$-safe. In this case we obtain, e.g., $q_2(M) = q_2(M \overset{u}{\rightarrow} u \setminus u) + q_2(M \overset{u}{\rightarrow} u \setminus u)$.

We now consider the case where $M$ is normal, i.e., $\emptyset \in M$. Note that $u \in V$ is nonsingular in a normal $M$ iff there is an $X \in M$ with $u \in X$. Also, by Lemma 12, $M$ is normal iff $M + V$ is normal, and thus in this case $u$ is nonsingular in $M + V$ iff there is an $Y \in M + V$ with $u \in Y$.

We modify now Theorem 30 such that each of the “components” of $M$ in the recursive equality are normal whenever $M$ is normal. In this way we prepare for a corresponding result on graphs.

**Corollary 32.** Let $M$ be a normal $vf$-safe $\Delta$-matroid.

If $X \in M$ with $u \in X$, then

$$q_1(M) = q_1(M \setminus u) + q_1(M \overset{u}{\rightarrow} u \setminus u).$$

If both $\{u, v\} \in M$ and $\{u\} \not\in M$, then

$$q_2(M) = q_2(M \overset{u}{\rightarrow} u \setminus \{u, v\}) + q_2(M \overset{u}{\rightarrow} u \setminus \{u\} \setminus \{v\}) + q_2(M \overset{u}{\rightarrow} u \setminus \{u\}).$$

Moreover, each of the given “components” is normal.

**Proof.** Since $q_1$ is invariant under pivot, $q_1(M \overset{u}{\rightarrow} X \setminus u) = q_1(M \overset{u}{\rightarrow} \{X \setminus \{u\}\} \setminus u) = q_1(M \overset{u}{\rightarrow} \{u\} \setminus u \setminus (X \setminus \{u\}))$ and the result follows by Theorem 30.

Now consider $q_2$. We have the recursive formula $q_2(M) = q_2(M \overset{u}{\rightarrow} u \setminus u) + q_2(M \overset{u}{\rightarrow} u \setminus u)$ (assuming $u$ is nonsingular in $M$).

One may easily verify that given $\emptyset, \{u, v\} \in M$ and $\{u\} \not\in M$, we have $\emptyset, \{v\}, \{u, v\} \in M \overset{u}{\rightarrow} u$ and $\{u\} \not\in M \overset{u}{\rightarrow} u$. Therefore, $u$ is nonsingular in $M \overset{u}{\rightarrow} u$. Hence $q_2(M) = q_2(M \overset{u}{\rightarrow} u \setminus u) + q_2(M \overset{u}{\rightarrow} u \setminus u)$.

As we have $\emptyset, \{v\} \in (M \overset{u}{\rightarrow} u \setminus u \overset{u}{\rightarrow} \{v\})$, $v$ is nonsingular in this set system and therefore $q_2(M) = q_2(M \overset{u}{\rightarrow} u \setminus u \overset{u}{\rightarrow} \{v\} \setminus \{u, v\}) + q_2(M \overset{u}{\rightarrow} u \setminus u \overset{u}{\rightarrow} \{v\} \setminus \{u\}, \{v\}) + q_2(M \overset{u}{\rightarrow} u \setminus u \overset{u}{\rightarrow} \{v\} \setminus \{u\}, \{v\}, \{u\} \overset{u}{\rightarrow} \{v\} \setminus \{u, v\})$, and $M \overset{u}{\rightarrow} u \setminus u$, hence they are all normal. \(\square\)

**7.3. The polynomial $Q_1(M)$**

In case $M$ is a normal $\Delta$-matroid, we find that $Q_1(M)$ may be computed given $q_2(N)$ for all “sub-set systems” $N = M[X]$ of $M$.

**Lemma 33.** Let $M$ be a normal $\Delta$-matroid. Then $Q_1(M) = \sum_{X \subseteq V} q_2(M[X])$.

**Proof.** We start by observing that for any $\Delta$-matroid $M$ and $X \subseteq V$, $d_{M[X]} = d_M$ provided that $M[X]$ is nonempty. This follows from the fact that each minimal set (w.r.t. inclusion) of $M[X]$ is a minimal set.
of $M$, and the observation from [15, Property 4.1] that the minimal sets of $\Delta$-matroid $M$ are of equal cardinality.

Since $M$ is normal, $M[X]$ is nonempty for all $X \subseteq V$. We have therefore $q_2(M[X]) = \sum_{Z \subseteq X} y^{d_M[X]+z(X)}$ for all $X \subseteq V$. As $M$ is a $\Delta$-matroid, we have by the observation above, for all $Z \subseteq X$, $d_M[X] + z(X) = d_{M-Z}(X)$. Hence $Q_1(M) = \sum_{X \subseteq V} q_2(M[X])$. □

Note that if $M$ is a $\Delta$-matroid (not necessarily normal), then Lemma 33 can be applied, for any $Z \in M$, to the normal $\Delta$-matroid $M \ast Z$ to obtain $Q_1(M \ast Z) = Q_1(M)$ (recall that $Q_1$ is invariant under pivot).

From Theorem 28 we know that $Q_1(M)$ itself fulfills the recursive relation

$$Q_1(M) = Q_1(M \setminus u) + Q_1(M \ast u \setminus u) + Q_1(M \bar{u} \setminus u),$$

where $M$ is $\ast$-safe, and $u$ is strongly nonsingular in $M$. If every $u \in V$ is not strongly nonsingular in $M$, then $Q_1(M) = (y + 2)^n$ with $n = |V|$. Indeed, we have $Q_1(M) = (2 + y)Q_1(M')$ where $M'$ is nonempty and equal to either $M \setminus u$ or $M \ast u \setminus u$. We have that every element of $M'$ is not strongly nonsingular in $M'$ (it is easy to see that if $v$ is strongly nonsingular in $M'$, then so is $v$ in $M$). By iteration, we reach $M_0 = (\varnothing, \{\varnothing\})$ for which $Q_1(M_0) = 1$.

The recursive relation given does not hold for $\Delta$-matroids in general. Indeed, consider $\Delta$-matroid $M = (V, \mathcal{P}(V) \setminus \{\varnothing\})$ with $V = \{1, 2, 3\}$. It is shown in [18] that $M$ is not a $\ast$-safe $\Delta$-matroid. We have $Q_1(M) = 13y + 14$, while $Q_1(M \setminus u) + Q_1(M \ast u \setminus u) + Q_1(M \bar{u} \setminus u) = (3y + 6) + (y + 2)^2 + (y + 2)^2 = 2y^2 + 11y + 14$.

Example 34. We recursively compute $Q_1(M)$ for $\ast$-safe $\Delta$-matroid $M = (\{p, q, r\}, \{\varnothing, \{p\}, \{p, q\}, \{q, r\}, \{r\})$ from Example 10. The computation tree is given in Fig. 3, where we have omitted the ground sets for visual clarity. The recursion stops when every element of ground set of the $\ast$-safe $\Delta$-matroid under consideration is singular. We verify that $Q_1(M) = y^2 + 10y + 16$.

7.4. The Tutte polynomial

Let us reformulate Theorem 30 to obtain a recursive characterization of $q_1(M)$ which can be directly compared to that of the Tutte polynomial (for matroids).

Corollary 35. Let $M$ be a $\Delta$-matroid. Then

$$q_1(M) = \begin{cases} (y + 1)q_1(M \setminus u) & \text{if } M \ast u \setminus u \text{ is empty} \\ (y + 1)q_1(M \ast u \setminus u) & \text{if } M \setminus u \text{ is empty} \\ q_1(M \setminus u) + q_1(M \ast u \setminus u) & \text{otherwise} \end{cases}$$

for all $u \in V$, and $q_1(M) = 1$ if $M = (\varnothing, \{\varnothing\})$. 

Fig. 3. Recursive computation of $Q_1(M)$. 

\[X\]

\[Y\]

\[Z\]

\[\Delta\]

\[\nabla\]

\[\gamma\]

\[\delta\]

\[\epsilon\]

\[\zeta\]

\[\eta\]

\[\theta\]

\[\iota\]

\[\kappa\]

\[\lambda\]

\[\mu\]

\[\nu\]

\[\xi\]

\[\pi\]

\[\rho\]

\[\sigma\]

\[\tau\]

\[\upsilon\]

\[\phi\]

\[\chi\]

\[\psi\]

\[\omega\]
We recall now the recursive relations of the Tutte polynomial \( t_M(x, y) \). Let \( M \) be a matroid over \( V \) (described by its bases). Recall that \( u \) is a coloop of \( M \) iff \( M \setminus u \) is empty, and \( u \) is a loop iff \( M \ast u \setminus u \) is empty. Moreover, if \( u \in V \) is not a coloop, then deletion of matroids coincides with deletion of set systems, and if \( u \in V \) is not a loop, then contraction \( M/u \) of \( M \) by \( u \) coincides with \( M \ast u \setminus u \). The notion of loop used here is not to be confused with loop complementation and loops in graphs as we use in the rest of the paper.

The Tutte polynomial \( t_M(x, y) \) fulfills the following characteristic relations:

\[
t_M(x, y) = \begin{cases} 
  y t_{M\setminus u}(x, y) & \text{if } u \text{ is a loop} \\
  x t_{M/u}(x, y) & \text{if } u \text{ is a coloop} \\
  t_{M\setminus u}(x, y) + t_M(x, y) & \text{otherwise}
\end{cases}
\]

for all \( u \in V \), and \( t_M(x, y) = 1 \) if \( M = (\emptyset, \{\emptyset\}) \). Hence we find that the recursive relations of \( q_1(M)(y - 1) \), restricted to matroids \( M \), and \( t_M(x, y) \), for the case \( x = y \), coincide.

Note that while matroids may be viewed as 1-matroids, here we view the class of matroids also as a particular subclass of the class of even \( \Delta \)-matroids and therefore (essentially) as a particular subclass of the class of tight 2-matroids. The class of \( v \)-safe matroids may be viewed as a particular subclass of the class of tight 3-matroids as well. Both the tight 2-matroid viewpoint and the tight 3-matroid viewpoint will be used in Section 8 when we consider evaluations of the Tutte polynomial.

We remark that it is not possible to use the recursive relation of the Tutte polynomial to obtain a full, two-variable generalization of the Tutte polynomial for \( \Delta \)-matroids, as the outcome of the recursive relation would then depend on the order of evaluations of the elements of \( V \). For example, for the \( \Delta \)-matroid \( (\{a, b\}, \{\emptyset, \{a\}, \{a, b\}\}) \), the outcome of the recursion might then be either \( x + 2 \) or \( y + 2 \). Of course, we may use the extended recursive relation of Corollary 29 to cover a bigger portion (instead of only the diagonal) of the two-variable Tutte polynomial (for matroids), cf. Theorem 24.

### 7.5. Two-variable polynomials

Using the generic recursive relation for \( \Delta \)-matroids, Corollary 29, we formulate the special case for the two-variable polynomial \( \tilde{q}(M)(x, y) \) defined in Section 6.3.

**Theorem 36.** Let \( M \) be a \( \Delta \)-matroid, and \( u \in V \). If \( u \) is nonsingular in \( M \), then

\[
\tilde{q}(M) = \tilde{q}(M \setminus u) + x \tilde{q}(M \ast u \setminus u).
\]

If \( u \) is singular in \( M \), then either \( M \setminus u \) is empty, and \( \tilde{q}(M) = (x + y) \tilde{q}(M \setminus u) \), or \( M \ast u \setminus u \) is empty, and \( \tilde{q}(M) = (1 + xy) \tilde{q}(M \setminus u) \).

If all \( v \in V \) are singular in \( M \), then \( M \) contains a single element \( X \), and we can iterate the recursion to obtain \( \tilde{q}(M) = (x + y)^{|X|} (1 + xy)^{|X| - k} \) with \( n = |V| \) and \( k = |X| \).

Using Theorem 36 we derive a recursive relation in the style of the two-variable interlace polynomial for graphs [4], see Section 9.

**Lemma 37.** Let \( M \) be a \( \Delta \)-matroid, with \( \emptyset, \{u, v\} \in M \), while \( \{u\}, \{v\} \notin M \).

Then \( \tilde{q}(M) = \tilde{q}(M \setminus u) + \tilde{q}(M \ast \{u, v\} \setminus u) + (x^2 - 1) \tilde{q}(M \ast \{u, v\} \setminus \{u, v\}) \).

**Proof.** Obviously, \( u \) is nonsingular in \( M \). Hence by Theorem 36, \( \tilde{q}(M) = \tilde{q}(M \setminus u) + x \tilde{q}(M \ast u \setminus u) \).

(1) First assume that \( v \) is nonsingular in \( M \ast u \setminus u \). We have by Theorem 36, \( \tilde{q}(M \setminus u) + x \tilde{q}(M \ast u \setminus u) = \tilde{q}(M \setminus u) + x \tilde{q}(M \setminus u \setminus \{u, v\}) + x^2 \tilde{q}(M \ast \{u, v\} \setminus \{u, v\}) \).

Since \( v \) is nonsingular in \( M \ast u \setminus u \), \( v \) is nonsingular in \( M \ast \{u, v\} \setminus u \). By Theorem 36 we find that \( \tilde{q}(M \ast \{u, v\} \setminus u) = \tilde{q}(M \ast \{u, v\} \setminus \{u, v\}) + x \tilde{q}(M \ast \{u, v\} \setminus \{u, v\}) \).

Thus \( x \tilde{q}(M \ast \{u, v\} \setminus \{u, v\}) = \tilde{q}(M \ast \{u, v\} \setminus u) - \tilde{q}(M \ast \{u, v\} \setminus \{u, v\}) \), and we obtain the required equality.

(2) Now assume that \( v \) is singular in \( M \ast u \setminus u \). Note that \( \emptyset \in M \ast \{u, v\} \setminus \{u, v\} \), which means that \( (M \ast u \setminus u) \ast v \setminus v = M \setminus \{u, v\} \setminus \{u, v\} \) is nonempty. We have by Theorem 36, \( \tilde{q}(M \setminus u) + x \tilde{q}(M \ast u \setminus u) = \tilde{q}(M \setminus u) + x(1 + y) \tilde{q}(M \ast \{u, v\} \setminus \{u, v\}) \).

Since \( v \) is singular in \( M \ast u \setminus u \), \( v \) is singular in \( M \ast \{u, v\} \setminus u \). By Theorem 36 we find that \( \tilde{q}(M \ast \{u, v\} \setminus u) = (1 + xy) \tilde{q}(M \ast \{u, v\} \setminus \{u, v\}) \). Thus \( xy \tilde{q}(M \ast \{u, v\} \setminus \{u, v\}) = \tilde{q}(M \ast \{u, v\} \setminus u) - \tilde{q}(M \ast \{u, v\} \setminus \{u, v\}) \), and we obtain the required equality. \( \Box \)
8. Polynomial evaluations

We consider now the polynomials of Section 6 at particular values of \( y \). We show in Section 10 that various results in this section can be linked to (and are motivated by) known results corresponding to the case of graphs [2]. We also reconsider the evaluation of the Tutte polynomial \( t_M \) at \((-1, -1)\) for quaternary matroids \( M \) from [34].

8.1. The polynomials \( Q_1 \) and \( q_i \)

Note that we have \( q_i(M)(1) = 2^n \) with \( n = |V| \) and \( i \in \{1, 2, 3\} \). Moreover, \( Q_1(M)(1) = \sum_{X \subseteq V, X \cap \bar{Y} = \varnothing} 1 = 3^n \). Also, as \( X \in M \) iff \( d_M(X) = 0 \), we have that \( q_1(M)(0) \) is equal to the number of sets in \( M \).

**Theorem 38.** Let \( M \) be a \( \Delta \)-matroid.

1. If \( M \) is even and \( |V| > 0 \), then \( q_1(M)(-1) = 0 \).
2. If \( M \) is \( \perp \)-safe and \( |V| > 0 \), then \( Q_1(M)(-2) = 0 \).
3. If \( M \) is \( \perp \)-safe, then \( q_1(M)(-2) = (-1)^{|V|}(-2)^{d_M} \).

**Proof.** By Corollary 27, \( q_1(M) = Q_{1,1,0}(M) = Q_1(Z_{M,r}) \) for suitable transversal 2-tuple \( \tau \) of some \((|V|, 2)\)-carrier \((U, \Omega)\) and projection \( \pi \). Since \( M \) is even, \( Z_{M,r} \) is tight by Section 5.2. By Theorem 8, we obtain (1).

By Theorem 26, \( Q_1(M) = Q_1(Z_{M,r}) \) for suitable transversal 3-tuple \( \tau \) of some \((|V|, 3)\)-carrier \((U, \Omega)\) and projection \( \pi \). Since \( M \) is \( \perp \)-safe, \( Z_{M,r} \) is tight by Theorem 16. By Theorem 8, we obtain (2).

Similarly, by Theorem 26, \( q_1(M) = Q_{1,1,0}(M) = Q_1(Z_{M,r}(U \setminus T_3)) \) for suitable transversal 3-tuple \( \tau = (T_1, T_2, T_3) \) of some \((|V|, 3)\)-carrier \((U, \Omega)\) and projection \( \pi \). Since \( M \) is \( \perp \)-safe, \( Z = Z_{M,r} \) is a tight 3-matroid by Theorem 16. By Theorem 9, we obtain \( Q_1(Z[U \setminus T_3]) = (-1)^{|V|}(-2)^{d_M} \) since \( n_2(T_2) = d_{M,\pi} \) by Lemma 18. Thus we obtain (3).

Observe that similar properties hold for the other two polynomials \( q_i \), in particular \( q_2(M)(-2) = (-1)^{|V|}(-2)^{d_M} \) for \( \perp \)-safe \( \Delta \)-matroid \( M \).

Theorem 38(1) does not hold for \( \Delta \)-matroids in general: for the \( \Delta \)-matroid \( M = (V, 2^V) \) the polynomial \( q_1(M) \) equals \((\text{the constant}) 2^{|V|}\). Note that for the \( \perp \)-safe \( \Delta \)-matroid \( M = (\varnothing, \{\varnothing\}) \) we have \( Q_1(M)(-2) = 1 \). Also, note that for the (not \( \perp \)-safe) \( \Delta \)-matroid \( M \) considered before Example 34, we have \( Q_1(M)(-2) = -12 \).

8.2. The Tutte polynomial

It is a direct consequence of the recursive relation of the Tutte polynomial that \( t_M(0, 0) = 0 \) for matroid \( M \), except for \( M = (\varnothing, \{\varnothing\}) \). This property may now be seen as a special case of Theorem 38(1) (as every matroid is an even \( \Delta \)-matroid).

By Corollary 25 we may apply Theorem 38(3) to obtain evaluations for the Tutte polynomial.

**Corollary 39.** Let \( M \) be a \( \perp \)-safe matroid. Then \( t_M(-1, -1) = (-1)^{|V|}(-2)^{d_M} \).

The evaluation \( t_M(-1, -1) \) for quaternary matroids was studied by Vertigan [34]. Let \( A \) be a representation over GF(4) of a quaternary matroid \( M \), and let \( C \) be the nullspace of \( A \). Then the space \( C \cap C^\perp \) is called the bicycle space of \( C \). It turns out that [34, Proposition 3.5] the dimension \( d \) of the bicycle space is independent of the representation \( A \) over GF(4) of \( M \), and \( d \) is denoted by bd(\( M, 4 \)) (where 4 represents the field size). We have bd(\( M, 4 \) = bd(\( M, 2 \)) for binary matroids \( M \). For binary matroids \( M \), not only the dimension of the bicycle space, but the bicycle space \( C \cap C^\perp \) itself is independent of the representation over GF(2).

It is shown in [34, Theorem 3.4] (by extending a result of [29] for binary matroids) that for quaternary matroids \( M \), \( t_M(-1, -1) = (-1)^{|V|}(-2)^{\text{bd}(M, 4)}. \) Consequently, by Corollary 39 and the fact that quaternary matroids are \( \perp \)-safe, \( d_M = \text{bd}(M, 4) \) when \( M \) is quaternary. Note that \( d_M = \text{bd} \) provides a natural extension of bd(\( M, 4 \)) to \( \perp \)-safe matroids \( M \) that are not quaternary.
It is moreover shown in [34, Theorem 8.3] that the evaluation \( t_M(-1, -1) \) is in some sense “well behaved” for the larger class \( \mathcal{N} \) of matroids that have no minors isomorphic to \( U_{2,6}, U_{4,6}, P_6, F_7, \) or \((F_7)^*\). We now illustrate this observation by applying Corollary 39 to a non-quaternary matroid.

**Example 40.** The non-quaternary Steiner-system matroid \( S(5, 6, 12) \) (see, e.g., [28] for a description of this matroid) is \( \text{vf-safe} \) \([18]\). One may verify that \( S(5, 6, 12) \nsubseteq V = S(5, 6, 12) \). Moreover, \( S(5, 6, 12) \) has rank 6. Hence, by Corollary 39, \( t_{S(5,6,12)}(-1, -1) = 2^6 \).

### 8.3. Binary delta-matroids

We now turn to binary \( \Delta \)-matroids.

For a \( V \times V \)-matrix \( A \) (the columns and rows of \( A \) are indexed by finite set \( V \)) over some field and \( X \subseteq V \), \( A[X] \) denotes the principal submatrix of \( A \) with respect to \( X \), i.e., the \( X \times X \)-matrix obtained from \( A \) by restricting to rows and columns in \( X \). For a \( V \times V \)-matrix skew-symmetric matrix \( A \) over some field (we allow nonzero diagonal entries over fields of characteristic 2), the set system \( \mathcal{M}_A = (V, \mathcal{D}_A) \) with \( \mathcal{D}_A = \{ X \subseteq V \mid \det A[X] \neq 0 \} \) is a \( \Delta \)-matroid (see [8]). By convention, the nullity of the empty matrix is 0, i.e., \( n(A[\emptyset]) = 0 \).

A \( \Delta \)-matroid \( M \) is called binary if \( M = \mathcal{M}_A \times X \) for some symmetric matrix \( A \) over GF(2) and \( X \subseteq V \). It is shown in [8] that a matroid \( M \) is binary in the above \( \Delta \)-matroid sense iff \( M \) is binary in the usual matroid sense. It is shown in [18] that the family of binary \( \Delta \)-matroids are closed under \( \ast \) and +, and therefore every binary \( \Delta \)-matroid is \( \text{vf-safe} \). In particular, every binary matroid is \( \text{vf-safe} \).

The following is a reformulation of Corollary 4.2 of [9] (and inspired by Theorem 6 of [2]). The proof of this reformulation is an Appendix of Section 10 as not all required notions are recalled at this point.

**Proposition 41 (Corollary 4.2 of [9]).** Let \( M \) be a binary \( \Delta \)-matroid. Then \( q_1(M)(2) = kq_1(M)(-2) \) for some odd integer \( k \).

The following example illustrates that Proposition 41 does not hold for \( \text{vf-safe} \) \( \Delta \)-matroids in general.

**Example 42.** Consider the 5-point line \( U_{2,5} \), which is a nonbinary \( \text{vf-safe} \) matroid. We have \( q_1(U_{2,5})(y) = y^2 + 6y^2 + 15y + 10 \). Hence, \( q_1(U_{2,5})(2) = 9 \cdot 2^4 \), while \( q_1(U_{2,5})(-2) = -2^5 \).

### 9. Pivot and loop complementation on graphs

In order to interpret the above results for graphs we recall in this section the necessary notions and results from the literature. We recall that each graph has its characteristic \( \Delta \)-matroid, defined via its adjacency matrix, and we recall two operations on graphs that precisely match the \( \Delta \)-matroid operations pivot \( \ast X \) and loop complementation \( +X \).

We consider undirected graphs without parallel edges, but we do allow loops. For a graph \( G = (V, E) \) we use \( V(G) \) and \( E(G) \) to denote its set of vertices \( V \) and set of edges \( E \), respectively, and for \( x \in V \), \( \{x\} \in E \) iff \( x \) has a loop.

With a graph \( G \) one associates its adjacency matrix \( A(G) \), which is a \( V \times V \)-matrix \((a_{u,v}) \) over GF(2) with \( a_{u,v} = 1 \) iff \( \{u, v\} \in E \) (we have \( a_{u,u} = 1 \) iff \( \{u\} \in E \)). In this way, the family of graphs with vertex set \( V \) corresponds precisely to the family of symmetric \( V \times V \)-matrices over GF(2). Therefore we often make no distinction between a graph and its matrix, so, e.g., by the nullity of graph \( G \), denoted \( n(G) \), we mean the nullity \( n(A(G)) \) of its adjacency matrix (computed over GF(2)). Also, we (may) write, e.g., \( G[X] = A(G)[X] \) for \( X \subseteq V \), the subgraph of \( G \) induced by \( X \). The graph \( G \setminus X = G[V \setminus X] \) is obtained from \( G \) by deleting the vertices in \( X \) with their incident edges. In case \( X = \{u\} \) is a singleton, we also write \( G \setminus u \) to denote \( G \setminus \{u\} \).

Recall that we have defined the \( \Delta \)-matroid \( \mathcal{M}_A \) for a skew symmetric matrix \( A \) as \((V, \mathcal{D}_A)\) with \( \mathcal{D}_A = \{ X \subseteq V \mid \det A[X] \neq 0 \} \), so we set accordingly \( \mathcal{M}_G = \mathcal{M}_{A(G)} \). The construction adheres a correspondence between distance and nullity: \( d_{\mathcal{M}_G}(X) = n(G[X]) \) for \( X \subseteq V \) (see [18]). Given \( \mathcal{M}_G \) for some graph \( G \), one can (re)construct the graph \( G \), see [15, Property 3.1]. In this way, the family of
graphs (with set \( V \) of vertices) can be considered as a (strict) subset of the family of binary \( \Delta \)-matroids (over set \( V \)).

For a graph \( G \) and a set \( X \subseteq V \), the graph obtained after loop complementation for \( X \) on \( G \), denoted by \( G + X \), is obtained from \( G \) by adding loops to vertices \( v \in X \) when \( v \) does not have a loop in \( G \), and by removing loops from vertices \( v \in X \) when \( v \) has a loop in \( G \). Hence, if one considers a graph as a matrix, then \( G + X \) is obtained from \( G \) by adding the \( V \times V \)-matrix with elements \( x_{ij} \) such that \( x_{ii} = 1 \) if \( i \in X \) and 0 otherwise. Note that \( (G + X) + Y = G + (X \Delta Y) \). It has been shown in [17] that \( \mathcal{M}_{G+X} = \mathcal{M}_G + X \) for \( X \subseteq V \).

For a \( V \times V \)-matrix \( A \) (over a field) and \( X \subseteq V \) such that \( A[X] \) is nonsingular, i.e., \( \det A[X] \neq 0 \), the principal pivot transform (pivot for short) of \( A \) on \( X \), denoted by \( A \star X \), is defined as follows [33]. Let \( A = \left( \begin{array}{cc} P & Q \\ R & S \end{array} \right) \) with \( P = A[X] \). Then \( A \star X = \left( \begin{array}{cc} P^{-1} & -P^{-1}Q \\ R P^{-1} & -S^{-1}1 \end{array} \right) \). If \( A \) is nonsingular, then \( A \star V = A^{-1} \). The pivot operation is an involution (operation of order 2), and more generally, if \( (A \star X) \star Y \) is defined, then \( A \star (X \Delta Y) \) is defined and they are equal. If \( A \) is skew-symmetric, then so is \( A \star X \). Thus, this operation is defined on a graph \( G \) via its adjacency matrix \( A(G) \), and yields a graph \( G \star X \). Moreover, pivot operations on graphs and on their \( \Delta \)-matroids coincide: \( \mathcal{M}_{G \star X} = \mathcal{M}_G \star X \) for \( X \subseteq V \) if the left-hand side is defined [8].

The pivots \( G \star X \) where \( X \) is a minimal set of \( \mathcal{M}_G \setminus \{ \emptyset \} \) (the set system obtained from \( \mathcal{M}_G \) by removing \( \emptyset \)) with respect to inclusion are called elementary.

It is noted in [24] that any pivot on a graph can be decomposed into a sequence of so-called elementary pivots, which are operations \( \star X \) either on a loop, \( X = \{ u \} \in E(G) \), or an edge, \( X = \{ u, v \} \in E(G) \), where (distinct) vertices \( u \) and \( v \) are both non-loops. The operation is known as local complementation and edge complementation, respectively. We do not recall (or use) their explicit graph-theoretical definitions in this paper. It can be found in, e.g., [17]. Similar as for pivot we may define, e.g., “dual local complementation” \( G \star \{ u \} = G + \{ u \} \star \{ u \} + \{ u \} \) which is identical to “regular” local complementation, except that it is defined for a non-loop \( \{ u \} \) (instead of a loop).

For convenience, the next proposition summarizes the key known results of this section, which we will use frequently in the remaining part of this paper.

**Proposition 43.** Let \( G \) be a graph and \( X \subseteq V \). Then the normal binary \( \Delta \)-matroid \( \mathcal{M}_G \) uniquely determines \( G \) (and the other way around). Moreover, \( \mathcal{M}_{G \star X} = \mathcal{M}_G \star X \) (if the left-hand side is defined), \( \mathcal{M}_{G+X} = \mathcal{M}_G + X \), and \( d_{\mathcal{M}_G}(X) = n(G[X]) \).

10. Graph polynomials

We now turn to graphs and reinterpret the above results on transition polynomials for \( \Delta \)-matroids in this domain. We consider now \( q_i(M) \) for \( i \in \{ 1, 2, 3 \} \), \( Q_1(M) \), \( Q(M) \), and (two-variable) \( \bar{q}(M) \) for the case \( M = \mathcal{M}_G \) for some graph \( G \). For notational convenience we denote them by \( q_i(G) \), \( Q_1(G) \), \( Q(G) \), and \( \bar{q}(G) \).

We obtain by Section 6 and by **Proposition 43** the following graph polynomials.

\[
q_1(G) = \sum_{X \subseteq V} y^{n(G[X])} = \sum_{X \subseteq V} y^{n(G \star X)}, \quad (4)
\]

\[
q_2(G) = \sum_{X \subseteq V} y^{n(G + X)}, \quad (5)
\]

\[
q_3(G) = \sum_{X \subseteq V} y^{n(G + \bar{v}(X))}. \quad (6)
\]

Polynomial \( q_1(G)(y - 1) \) is the single-variable interlace polynomial \( q(G)(y) \), which is defined in [5]. More generally, the two-variable interlace polynomial [4] \( q(G)(x, y) = \sum_{X \subseteq V} (x - 1)^{n(G[X])}(y - 1)^{n(G[X])} = \sum_{X \subseteq V} (x - 1)^{|X|} \left( \frac{y - 1}{x - 1} \right)^{n(G[X])} \) is equal to \( \bar{q}(G)(x - 1, \frac{y - 1}{x - 1}) \). Moreover, the two-variable
polynomial $\bar{q}(M \ast V \ast \bar{y}) = \sum_{x \in V} x^{(1)} x^{d_{M+x}X+x} \ast \sum_{x \leq V} x^{(1)} y^{d_{M+x}(V)}$ for $M = M_G$ is equal to the bracket polynomial [32], defined by $b_1(G)(y) = \sum_{x \leq V} x^{(1)} y^{(M+X)\ast x}$.

By the above, the definitions of $q_1(M)(y)$ and $q_2(M)(y)$ for arbitrary set systems $M$ can be seen as generalizations of the single-variable interlace polynomial and the single-variable (i.e., the case $x = 1$) bracket polynomial. From this point of view we notice a close similarity between the single-variable interlace polynomial and the single-variable bracket polynomial, as shown in the next result (which follows directly from the definitions of $q_2(M)$ and $q_3(M)$).

**Theorem 44.** Let $G$ be a graph. We have (1) $q_1(G \ast V) = q_2(G)$, $q_1(G+V) = q_3(G)$, $q_2(G \ast V) = q_3(G)$, assuming $G \ast V$ and $G \ast V$ are defined, and (2) for $Y \subseteq V$, $q_1(G \ast Y) = q_1(G)$, $q_2(G + Y) = q_2(G)$, $q_3(G \ast Y) = q_3(G)$, assuming $G + Y$ and $G \ast Y$ are defined.

We now formulate the recursive relations from Section 7 concerning the $q_i(M)$ for normal set systems $M$ for the case that $M$ represents a graph $G$.

**Theorem 45.** Let $G$ be a graph.

- Let $X \subseteq V$ such that $u \in X$ and $G \ast X$ is defined, then
  
  $q_1(G) = q_1(G \setminus u) + q_1(G \ast X \setminus u)$.

- If $\{u, v\}$ is an edge of $G$ where both $u$ and $v$ do not have loops, then
  
  $q_2(G) = q_2(G \ast \{u, v\} \setminus \{u, v\}) + q_2(G \ast \{u\} \setminus \{u, v\}) + q_2(G \ast \{v\} \setminus \{u, v\})$
  
  and
  
  $\bar{q}(G) = \bar{q}(G \setminus u) + \bar{q}(G \ast \{u, v\} \setminus \{u, v\}) + (x - 1)^2 \bar{q}(G \ast \{u, v\} \setminus \{u, v\})$.

**Proof.** The recursive relation for $q_1(G)$ follows directly from Corollary 32. Likewise the relation for $\bar{q}(G)$ follows Lemma 37.

We now consider $q_2(G)$. It is easy to see that $G \ast \{u, v\} \ast \{u\}$, $G \ast \{v\} \ast \{u, v\}$, and $G \ast \{u, v\}$ are defined. Thus the equality $q_2(G) = q_2(G \ast \{u, v\} \setminus \{u, v\}) + q_2(G \ast \{u\} \setminus \{u, v\}) + q_2(G \ast \{v\} \setminus \{u, v\})$ follows now from Corollary 32. Moreover we have $M_G \ast \{u, v\} \ast \{u\} \setminus \{u, v\} = M_G \ast \{u\} \ast \{v\} \setminus \{u, v\} = M_G \ast \{u, v\} \ast \{v\} \setminus \{u, v\}$ and $M_G \ast \{u\} \ast \{v\} \setminus \{u, v\} = M_G \ast \{u\} \ast \{v\} \setminus \{u, v\}$. As it is easy to see that $G \ast \{u, v\} \ast \{u\}$ is defined, and so we have $G \ast \{u, v\} \ast \{u\} \setminus \{u, v\} = G \ast \{u\} \ast \{u\} \setminus \{u, v\}$. Consequently, we obtain the other equality $q_2(G) = q_2(G \ast \{u\} \setminus \{u, v\}) + q_2(G \ast \{u\} \setminus \{u, v\}) + q_2(G \ast \{u\} \setminus \{u, v\})$.

The recursive relation for the single-variable interlace polynomial $q(G)$ in [4, Section 4] is the special case of the recursive relation for $q_1(G)$ in Theorem 45 where the pivot $G \ast X$ is elementary. Note that the recursive relation for $q_1(M)$ where $M$ is a $\Delta$-matroid is a generalization of this result. Also, it is shown in [16] that the recursive relation for $q_1(G)$ in Theorem 45 can be generalized for arbitrary $V \times V$-matrices $A$. In this way we find that $q_1(M)$ for $\Delta$-matroids and $q_1(A)$ for matrices are "incomparable" generalizations (in the sense that one is not more general than the other) of the graph polynomial $q_1(G)$.

Of the two recursive relations for $q_2$ given in Theorem 45, the former seems novel, while the latter is from [32, Theorem 1(i)]. The recursive relation for $\bar{q}$ given in Theorem 45 corresponds to the recursive relation of the two-variable interlace polynomial [4, Theorem 6].

**Remark 46.** The marked-graph bracket polynomial [31] for a graph $G$ is defined as $mgb_2(G, C) = \sum_{X \subseteq V} b^{(1)} y^{(G+X\setminus(C \cup X))}$ with $C \subseteq V$ (the elements of $C$ are the vertices of $G$ that are not marked). For the case $B = 1$, we have that $mgb_2(G, C) = q_2(G \ast (V \setminus C)) = q_3(G \ast C)$. Indeed, $s_3(M \ast C) = \sum_{X \subseteq V} x^{(1)} x^{d_{M+x}C}$, and $d_{M+x} C \setminus X = d_{M+x} (C \setminus (X \setminus C)) \setminus X = d_{M+x} (C \setminus X) \setminus X = d_{M+x} (C \setminus X) \setminus X = d_{M+x} (X \cup C)$ and so $mgb_2(G, C)$ is $q_3(M \ast C)$ where set system $M$ represents graph $G$.

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3 Actually, the bracket polynomial contains another variable $z$. However, as pointed out in [32] we may assume without loss of information either $x = 1$ or $z = 1$ (not both)—for convenience we choose $z = 1$. 
(i.e., $M = \mathcal{M}_C$). Therefore $mgb_1(G, C)$ can be seen as a “hybrid” polynomial “between” $q_2$ and $q_3$. The recursive relations for $mgb_3(G, C)$ deduced in [31] are straightforwardly deduced from the recursive relations for $q_1(G)$. Of course, one may also consider the hybrid polynomials $q_1(G + C) = q_2(G + (V \setminus C))$ and $q_1(G + C) = q_3(G + V \setminus C)$ for $C \subseteq V$ between $q_1(G)$ and $q_2(G)$, and between $q_1(G)$ and $q_3(G)$, respectively.

We now consider polynomial $Q_1(G)$. We have

$$Q_1(G) = \sum_{X \subseteq V} \sum_{Z \subseteq X} y^{n(G + Z[X])}.$$  

(7)

By Lemma 33 we have $Q_1(G) = \sum_{X \subseteq V} q_2(G[X])$. By Theorem 28 and the fact that $Q_1(M)$ is invariant under pivot and loop complementation, we have the following result.

Theorem 47. Let $G$ be a graph, and $Y \subseteq V$. We have $Q_1(G) = Q_1(G * Y)$ when $G * Y$ is defined, and $Q_1(G) = Q_1(G + Y)$ if $\{u, v\}$ is an edge in $G$ with $u \neq v$ where both $u$ and $v$ are non-loop vertices, then

$$Q_1(G) = Q_1(G \setminus u) + Q_1(G * \{u\} \setminus u) + Q_1(G * \{u, v\} \setminus u).$$

(8)

Note that the recursive relation of Theorem 47 may be easily modified for the cases where either $u$ or $v$ (or both) have a loop. For example, if $u$ has a loop and $v$ does not have a loop in $G$, then, since $Q_1(G) = Q_1(G + \{u\})$, we have $Q_1(G) = Q_1(G \setminus u) + Q_1(G * \{u\} \setminus u) + Q_1(G + \{u\} * \{u, v\} \setminus u)$. As $Q_1(G)$ is invariant under loop complementation, we may, as a special case, consider this polynomial for simple graphs $F$. In this way, we obtain the recursive relation

$$Q_1(F) = Q_1(F \setminus u) + Q_1(loca(F) \setminus u) + Q_1(F * \{u, v\} \setminus u),$$

(9)

where loca($F$) is the operation that complements the neighborhood of $u$ in $F$ without introducing loops (as $F$ is simple, no loops are removed).

Polynomial $Q'(F) = \sum_{X \subseteq V} \sum_{Z \subseteq X} (y - 2)^{n(F + Z[X])}$ has been considered in [2] for simple graphs $F$. As the difference in variable $y := y - 2$ in $Q_1(G)$ with respect to $Q'(F)$ is irrelevant, this polynomial is essentially $Q_1(G)$ restricted to simple graphs. The Equality (9) is shown in [2] from a matrix point-of-view. Similarly, the single-variable case (case $u = 1$) of the multivariate interlace polynomial $C(F)$ of [20] is (essentially) $Q(G)$ restricted to simple graphs.

Finally, we may state the graph analogs of the results of Section 8.

Theorem 48. Let $G$ be a graph, and $n = |V|$. Then

$$Q_1(G)(-2) = 0 \quad \text{if } n > 0$$

(10)

$$q_1(G)(-2) = (-1)^n(-2)^{n(G + V)}$$

(11)

$$q_2(G)(-2) = (-1)^n$$

(12)

$$q_1(G)(-1) = 0 \quad \text{if } n > 0 \text{ and } G \text{ has no loops}$$

(13)

$$q_1(G)(2) = kq_1(G)(-2) \quad \text{for some odd integer } k.$$  

(14)

Proof. Equality (10) follows directly from Theorem 38(2). As $d_{M + V} = d_{M + V + V} = d_{M + V + V} = d_{M + V}(V)$, Equality (11) follows from Theorem 38(3) (and Proposition 43). By the paragraph below Theorem 38 we have $q_2(G)(-2) = q_2(\mathcal{M}_C)(-2) = (-1)^n(-2)^{d_{\mathcal{M}_C}} = (-1)^n$. Equality (14) for $q_1(G)(2)$ follows from Proposition 41. Finally, we have that $\mathcal{M}_C$ is an even $\Delta$-matroid iff $G$ has no loops. Hence Equality (13) follows by Theorem 38(1). □
Conversely, if \( Q \) holds essentially unchanged for graphs \( G \) apart from a particular isotropic system depending on \( q \) and \( r \), the last two of these proofs one may recognize a type of “\( m, m, m + 1 \)” nullity result for matrices which in the present paper is captured by the tightness property.

Equality (11) is proven in [2, Theorem 2.2], [6, Theorem 1], and in [14, Corollary 1] for the case where \( G \) does not have loops. In fact in the first two of these proofs one may recognize a type of “\( m, m, m + 1 \)” nullity result for matrices which in the present paper is captured by the tightness property.

Equality (13) is mentioned in [2, Remark before Lemma 2]. Note that Equality (13) does not in general hold for graphs with loops. Indeed, e.g., for the graph \( G \) having exactly one vertex \( u \), where \( u \) is looped, we have \( q_1(G)(-1) = 2 \).

Equality (14) is proven in [2, Theorem 6], and in [14, Corollary 1] for the case where \( G \) does not have loops. The proof of Theorem 6 of [2] does not seem to be easily extended from simple graphs to graphs with loops.

We remark that it is shown in [2, Theorem 2] that \( Q_1(G)(2) = k2^n \) where \( k \) is the number of induced Eulerian subgraphs. This result cannot be generalized to \( \Delta \)-matroids as for \( \Delta \)-matroids \( M = (\{q, r, s\}, \{\{q\}, \{r\}, \{s\}, \{\{q, r\}, \{q, s\}, \{r, s\}\}) \) we have \( Q_1(M) = 9 \cdot (y + 2) \). Consequently, \( Q_1(M)(2) = 9 \cdot 10^2 \) is not of the form \( k \cdot 2^n \).

Finally, a consequence of the proof of [8, Theorem 4.4] is that if \( M \) is a binary matroid, then \( M = M_{\mathcal{C}} \times X \) for some bipartite graph \( G \) (hence \( G \) has no loops) with color classes \( X \) and \( V \setminus X \). Conversely, if \( G \) is a bipartite graph with color classes \( X \) and \( V \setminus X \), then \( M = M_{\mathcal{C}} \times X \) is a binary matroid (and also \( \mathcal{C} \cdot (V \setminus X) \)). Hence we obtain the following consequence of Corollary 25, which is essentially shown in [2, Theorem 3].

**Corollary 49** (Theorem 3 of [2]). Let \( G \) be a bipartite graph with color classes \( X \) and \( V \setminus X \), and let \( M = M_{\mathcal{C}} \times X \) be a corresponding binary matroid. Then \( q_1(G)(y - 1) = tM(y, y) \).

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**Appendix. Proof of Proposition 41**

With the definition of the (single-variable) interlace polynomial for graphs in place, we prove now Proposition 41 using Corollary 4.2 of [9] and Section 6 of [2]. We assume in this appendix that the reader is familiar with isotropic systems.

**Proof of Proposition 41.** Let \( M \) be an binary \( \Delta \)-matroid, and let \( X \in M \). Then \( M \times X = M_{\mathcal{C}} \) for some graph \( G \), and we have \( q_1(M)(y) = q_1(M \times X)(y) = q_1(M_{\mathcal{C}})(y) = q_1(G)(y) = q(G)(y + 1) \), where \( q(G)(y) \) is the single-variable interlace polynomial. In Section 6 of [2], it is shown that, for graphs \( G \) without loops, \( q(G)(y) = m(\mathcal{C}, A, y) \) where \( m \) is the Martin polynomial for isotropic systems, \( \mathcal{C} \) is a particular isotropic system depending on \( G \) and \( A \) is a particular complete vector (we refer to [9] for definitions of these notions). It is straightforward to verify that the reasoning of Section 6 of [2] holds essentially unchanged for graphs \( G \) where loops are allowed. The only difference is that we may have either \( C_v = 0 \) or \( C_v = 1 \) in the Claim of Section 6 of [2] instead of having only the former possibility (and similarly for \( D \)), but this difference has no effect on the conclusion that \( C(D) = 1 + 1 = 0 \). By Corollary 4.2 of [9], for any isotropic system \( S \) and complete vector \( A \) over the ground set of \( S \), \( m(S, A, 3) = k \cdot m(S, A, -1) \) for some odd integer \( k \). We conclude that \( q_1(M)(2) = q(G)(3) = kq(G)(-1) = kq_1(M)(-2) \).

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