How to extract information from Green’s functions in Landau gauge

Attilio Cucchieri\textsuperscript{a} and Tereza Mendes\textsuperscript{a,b}
\textsuperscript{a} Instituto de Física de São Carlos, Universidade de São Paulo, Caixa Postal 369, 13560-970 São Carlos, SP, Brazil
\textsuperscript{b} DESY, Platanenallee 6, 15738 Zeuthen, Germany

Abstract

The infrared behavior of gluon and ghost propagators offers a crucial test of confinement scenarios in Yang-Mills theories. A nonperturbative study of these propagators from first principles is possible in lattice simulations, but one must consider significantly large lattice sizes in order to approach the infrared limit. We propose constraints based on general properties of the propagators to gain control over the extrapolation of data to the infinite-volume limit. These bounds also provide a way to relate the propagators to simpler, more intuitive quantities. We apply our analysis to the case of pure $SU(2)$ gauge theory in Landau gauge, using the largest lattice sizes to date. Our results seem to contradict commonly accepted confinement scenarios. We argue that it is not so.

1 The Gribov-Zwanziger Confinement Scenario

About thirty years ago, Gribov proposed an interesting confinement mechanism for color charges in Landau (and Coulomb) gauge [1]. His idea was based on the restriction of the physical configuration space to the region $\Omega$ of transverse configurations, delimited by the so-called first Gribov horizon, where the smallest (non-trivial) eigenvalue of the Faddeev-Popov (FP) operator $M = -D_\mu \partial_\mu$ is zero. The limitation of the functional integration to the (first Gribov) region $\Omega$ was an attempt to fix the gauge completely, getting rid of spurious gauge copies, known thereafter as Gribov copies. Since the ghost propagator $G(p)$ is given by $\langle p | M^{-1} | p \rangle$ and $M$ is semi-positive definite for gluon fields $A \in \Omega$, one cannot have a singularity for $G(p)$ at a finite momentum $p$. Using perturbation theory up to second order, Gribov wrote (for Landau gauge) the no-pole condition [1, 2]

$$\sigma(0) = \frac{N_c}{4 (N_c^2 - 1)} \int \frac{d^4q}{(2\pi)^4} \frac{\langle A_\lambda^a(q) A_\lambda^b(-q) \rangle}{q^2} < 1 . \quad (1)$$

Here $N_c$ refers to the gauge group $SU(N_c)$, an average over the Lorentz indices $\lambda$ has been considered and the quantity $\sigma(p)$ enters the ghost propagator as $G(p) \approx p^{-2} [1 - \sigma(p)]^{-1}$. The above inequality tells us that, in the infrared (IR) limit, the Landau gluon propagator $D_{\mu\nu}(p) = \langle A_\mu^a(p) A_\nu^b(-p) \rangle$ is less singular than $1/p^2$ (in the $4d$ case). Moreover, by using the above no-pole condition as a characterization of the first Gribov region, one can show that the tree-level gluon propagator becomes

$$D(p) = g_0^2 p^2/(p^4 + \gamma^4) , \quad (2)$$
where the (Gribov) mass parameter $\gamma$ is fixed by the gap equation $\int d^4 q (2\pi)^{-4} (q^4 + \gamma^4)^{-1} = 4/(3N_c g^2)$. Then, by using (2) and the gap equation, one can study the IR limit of $\sigma(p)$, obtaining an IR-enhanced ghost propagator $G(p) \propto 1/p^4$. As stressed by Gribov [1], this enhancement is an indication of a long-range effect in the theory that may explain color confinement. In Coulomb gauge, this enhancement of $G(p)$ can be directly related to a color-Coulomb potential linearly rising with distance.

It is interesting that a bound similar to the one above has also been obtained in Ref. [3] by considering a variational method applied to the FP operator $M$. This bound, known as the ellipsoidal bound, can be written (in the continuum) as

$$\int \frac{d^d q}{(2\pi)^d} \frac{\langle A^a_\mu(q) A^b_\mu(-q) \rangle}{q^2} \leq C,$$

where $C$ depends on the dimensionality $d$ of the space-time and on the gauge group. One should stress that the ellipsoidal $E$, defined by the ellipsoidal bound, is a region of transverse configurations that includes the first Gribov region $\Omega$. A similar bound can also be obtained on the lattice [4]. Moreover, it is convenient to define on the lattice a region $\Theta$, included in the ellipsoid $E$ and including the first Gribov region $\Omega$, i.e. $\Omega \subset \Theta \subset E$. For all configurations belonging to $\Theta$, one can prove [5]

$$|\tilde{A}^b_\mu(0)| \leq 2 \tan \left( \frac{\pi}{V^{1/d}} \right),$$

where $V$ is the lattice volume and $\tilde{A}^b_\mu(0) = V^{-1} \sum_x A^b_\mu(x)$ is the gluon field at zero momentum. This quantity may be viewed as a magnetization $M$. Thus, in the infinite-volume limit one can show that $|M|$ is zero. By adding an external color “magnetic” field $H$ coupled to $A$ in the action and using the above inequality, one also obtains that the free energy per unit volume is null when $V$ goes to infinity [5]. If $H$ is spatially modulated, then the susceptibility $\chi$ at zero external field coincides with the usual gluon propagator $D(p)$. The inequalities valid in the region $\Theta$ would then suggest that $\lim_{p \to 0} \chi(H = 0, p) = \lim_{p \to 0} D(p) = 0$ [4].

In order to restrict the functional integration to the first Gribov region $\Omega$, Zwanziger added to the usual Yang-Mills action a non-local term proportional to $M^{-1}$ [6]. This term clearly suppresses the probability of configurations near the boundary $\partial \Omega$ of the region $\Omega$. After localizing the action, zeroth-order perturbation theory allows one to obtain again the propagator in Eq. (2). The purely imaginary poles $p^2 = \pm i\gamma^2$ of this propagator make it incompatible with a Kallen-Lehmann representation [6]. These singularities at an unphysical location suggest that gluons are indeed not physical excitations. One should also note [4, 5] that $D(0) = \int d^d x D(x) = 0$ means a gluon propagator in position space $D(x)$ that is positive and negative in equal measure. This represents a maximal violation of reflection positivity. In Refs. [4, 5] the violation of reflection positivity for the gluon propagator has been proposed as a confinement mechanism for gluons.

The restriction of the functional integration to $\Omega$ has also been discussed in Ref. [7]. The non-local term $-\gamma^4 H$ is added to the Yang-Mills action, where $H$ is the so-called horizon function, containing an $M^{-1}$ factor. At the same time, the Gribov mass $\gamma$ is fixed implicitly by the horizon condition $\langle h \rangle = (N_c^2 - 1)d$, where $h$ is the horizon function per unit volume. It is interesting that the horizon condition implies a ghost propagator enhanced in the IR limit [8], i.e. $\lim_{p \to 0} [p^2 G(p)]^{-1} = 0$. Clearly, the enhancement of the
ghost propagator at \( p = 0 \) should indicate that \( G(p) \) feels the singularity of \( M^{-1} \) on \( \partial \Omega \). Indeed, since the configuration space has very large dimensionality one expects that in the infinite-volume limit, due to entropy considerations, the Boltzmann weight be concentrated on \( \partial \Omega \) \cite{7}. This implies that the smallest nonzero eigenvalue \( \lambda_{\text{min}} \) of \( M \) should go to zero in the infinite-volume limit. (This has been verified numerically in Landau gauge \cite{9}.)

The IR behavior of propagators and vertices in Landau gauge has also been studied in Ref. \cite{10} by considering the sets of coupled Dyson-Schwinger equations (DSE) for the basic propagators and vertices of Yang-Mills theory (in the 4d case). By using a simple power counting for the solutions in the IR limit and constraints obtained using a skeleton expansion, the authors found a consistent solution characterized by an IR-enhanced ghost propagator \( G(p) \sim p^{-2(1+\kappa)} \) and by an IR-finite gluon propagator \( D(p) \sim p^{2(2\kappa-1)} \) with \( \kappa \in [1/2, 3/4] \). Note that \( D(0) = 0 \) for \( \kappa > 1/2 \). On the other hand, the analysis carried out in \cite{10} allows also for a solution with a tree-level-like ghost propagator at small momenta \( G(p) \sim p^{-2} \) and a finite nonzero gluon propagator \( D(0) > 0 \). These two consistent solutions have also been obtained by several studies of DSE using specific approximations \cite{11}.

Recently it was shown \cite{12,13} that using the Gribov-Zwanziger approach, i.e. by restricting the functional integration to the Gribov region \( \Omega \), one can also obtain in 3d and 4d a finite nonzero gluon propagator \( D(0) \) and a tree-level-like ghost propagator in the IR limit. Here the dynamical mechanism is related to a suitable mass term that may be added to the action while preserving its renormalizability. As a consequence, one can show that the restriction to \( \Omega \) induces a soft breaking of the BRST symmetry \cite{13}. It is interesting that the same approach cannot be extended to the 2d case \cite{14}, because the new mass term produces IR singularities that make the restriction to \( \Omega \) impossible.

An IR-enhanced Landau ghost propagator is also obtained as a consequence of the so-called confinement criterion of Kugo-Ojima \cite{15}. At the same time, this criterion suggests that the perturbative massless pole in the transverse gluon propagator should disappear \cite{16}. In this sense, an IR-suppressed gluon propagator (not necessarily vanishing) can be accommodated in this confinement scenario \cite{17}. Finally, even though the Gribov-Zwanziger and the Kugo-Ojima confinement scenarios seem to predict similar IR behavior for the propagators, it is not clear how to relate the (Euclidean) cutoff at the Gribov horizon to the (Minkowskian) approach of Kugo-Ojima \cite{18}.

## 2 Bounds for the Gluon and the Ghost Propagators

Recently we have introduced \cite{19} rigorous upper and lower bounds for the gluon propagator at zero momentum \( D(0) \) by considering the quantity

\[
M(0) = \frac{1}{d(N_c^2 - 1)} \sum_{b,\mu} |\tilde{A}_\mu^b(0)|,
\]

with \( \tilde{A}_\mu^b(0) \) defined in the previous section. Indeed, by straightforward calculations one finds that

\[
V \langle M(0) \rangle^2 \leq D(0) \leq V d(N_c^2 - 1) \langle M(0) \rangle^2.
\]

Thus, if \( M(0) \) goes to zero as \( V^{-\alpha} \) we obtain that \( D(0) \to 0 \), \( 0 < D(0) < +\infty \), or \( D(0) \to +\infty \), respectively if \( \alpha \) is larger than, equal to or smaller than 1/2. Recall that
$M(0)$ should go to zero at least as $V^{-1/d}$ in the $d$-dimensional case [see Eq. (1)]. At the same time, a necessary condition to find $D(0) = 0$ is that $M(0)^2$ goes to zero faster than $1/V$. We note that the above bounds apply to any gauge and that they can be immediately extended to the case $D(p)$ with $p \neq 0$.

We investigated the bounds [9] for pure $SU(2)$ gauge theory in Landau gauge considering several lattice volumes in 2$d$, in 3$d$, and in 4$d$ with the largest lattice corresponding (respectively) to $a^2V \approx (70 \text{ fm})^2$, $a^3V \approx (85 \text{ fm})^3$ and to $a^4V \approx (27 \text{ fm})^4$. By using the Ansatz $B_u/L^u$ for $a^2\langle M(0)^2 \rangle$ we obtain $u = 2.72(1)$ in the 2$d$ case, implying $D(0) = 0$. A similar analysis in 3$d$ and in 4$d$ for the lower and the upper bounds gives $0.4(1) \text{ GeV}^{-2} \leq a^2D(0) \leq 4(1) \text{ GeV}^{-2}$ in 3$d$ and $2.2(3) \text{ GeV}^{-2} \leq a^2D(0) \leq 29(5) \text{ GeV}^{-2}$ in 4$d$. Recently, a study for the 4$d$ $SU(3)$ case [20] also finds a value for $\alpha$ very close to $1/2$. Although the authors conclude that $D(0) = 0$ in the infinite-volume limit, one should observe that in this case the lattice volumes considered are relatively small and the statistics is rather low. Thus, a more detailed analysis in the $SU(3)$ case should be carried out in order to verify if the IR behavior of the gluon propagator agrees [21] or not with the $SU(2)$ case.

One can also obtain lower and upper bounds for the ghost propagator [22]. In Landau gauge, for any nonzero momentum $p$, one finds

$$\frac{1}{N_c^2 - 1} \frac{1}{\lambda_{\text{min}}} \sum_a |\bar{\psi}_{\text{min}}(a,p)|^2 \leq G(p) \leq \frac{1}{\lambda_{\text{min}}},$$

(7)

where $\lambda_{\text{min}}$ is the smallest nonzero eigenvalue of the FP operator $\mathcal{M}$ and $\bar{\psi}_{\text{min}}(a,p)$ is the corresponding eigenvector. If we assume $\lambda_{\text{min}} \sim N^{-\delta}$ and $G(p) \sim p^{-2-2\kappa}$ at small $p$, we should find $2 + 2\kappa \leq \delta$, i.e. $\delta > 2$ is a necessary condition for the IR enhancement of $G(p)$. Note that a similar analysis can be carried out [23] for any generic gauge condition $\mathcal{F}[A] = 0$ imposed on the lattice by minimizing a functional $E[U]$, where $U$ is the (gauge) link variable. Indeed, from the second variation of $E[U]$ one can obtain the corresponding FP matrix $\mathcal{M}$ and the set of local minima of $E[U]$ defines the Gribov region $\Omega$, where all eigenvalues of $\mathcal{M}$ are positive. In the infinite-volume limit, entropy favors configurations near $\partial \Omega$ (where $\lambda_{\text{min}}$ goes to zero). Thus, inequalities of the type (7) can tell us if one should expect an enhancement of the ghost propagator $G(p)$ when the Boltzmann weight gets concentrated on $\partial \Omega$.

A study in the $SU(2)$ Landau case [22] suggests that $\delta > 2$ in 2$d$, implying IR enhancement of $G(p)$, while $\delta < 2$ in 4$d$. These results are confirmed if one considers the dressing function $p^2G(p^2)$ for very large lattice volumes [22]. Indeed, the data in the 2$d$ case can be fitted by $\sim p^{-2\kappa}$, with $\kappa$ between 0.1 and 0.2. On the contrary, in 3$d$ and in 4$d$ the data are well described by $a - b \log(1 + cp^2)$, supporting $\kappa = 0$.

Let us note that our data for the gluon and ghost propagators are in good agreement with results obtained by other groups using very large lattice volumes [23]. Of course, one should also recall that the region $\Omega$ is actually not free of Gribov copies [3, 5, 26] and that the configuration space should be identified with the so-called fundamental modular region (FMR) $\Lambda$. On the other hand, the restriction of the configuration space to the FMR should not make any difference on the numerical verification of the Gribov–Zwanziger scenario.

\footnote{For example, in 4$d$ Maximally Abelian gauge one sees that $\lambda_{\text{min}}$ goes to zero at large volume but the ghost propagator stays finite at zero momentum [24].}
Indeed, as we have seen in the previous section, this scenario is based on the restriction of the configuration space to the region $\Omega$, which includes $\Lambda$. Actually, the bounds obtained for the gluon fields (see again Section I) apply to regions, such as $\Theta$ and $\mathcal{E}$, that are even larger than the region $\Omega$. Finally, as explained in [27], the restriction to the FMR can only make the ghost propagator less singular, as confirmed by recent lattice data [28].

3 Conclusions

We have presented simple properties of gluon and ghost propagators that constrain (by upper and lower bounds) their IR behavior. For the gluon case we define a magnetization-like quantity, while for the ghost case we relate the propagator to $\lambda_{\text{min}}$ of the FP matrix. We propose the study of these quantities, as a function of the lattice volume, in order to gain better control of the infinite-volume limit for the propagators in the IR regime.

Our data support a Landau-gauge gluon propagator that is IR finite in $3d$ and $4d$. This result can be interpreted [19] as a consequence of “self-averaging” of a magnetization-like quantity, i.e. $M(0)$ without the absolute value. In particular, one may think of $D(0)$ as a response function (susceptibility) of this magnetization. In this case it is natural to expect $D(0) > 0$ in the infinite-volume limit. In the $2d$ case the magnetization is “over self-averaging” and the susceptibility is zero. These results are in agreement with the suppression of the IR components of the gluon field $A$ due to the limitation of the functional space to the first Gribov region $\Omega$. At the same time the gluon propagator displays a clear violation of reflection positivity in Landau gauge [31], i.e. the confinement mechanism for gluons proposed in [4, 5] is confirmed by lattice data.

For the ghost propagator we find that in $3d$ and $4d$ the behavior at small momenta is essentially tree-level like, while in $2d$ this propagator seems to be clearly enhanced compared to the perturbative behavior $p^{-2}$. As described in Section I these results are not necessarily in contradiction with the Gribov-Zwanziger approach [12, 13, 14].

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References

[1] V.N. Gribov, Nucl. Phys. B 139, 1 (1978).
[2] R.F. Sobreiro and S.P. Sorella, arXiv:hep-th/0504095.
[3] G. Dell’Antonio and D. Zwanziger, Nucl. Phys. B 326, 333 (1989).

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Note that a faster approach to the infinite volume could be obtained by using an extended gauge-fixing as done in [29].

The massive behavior displayed by the gluon propagator in the IR limit has been recently criticized [30] due to the observation that different lattice discretizations yield different mass values in lattice units. On the other hand, such a comparison only makes sense when data are extrapolated to the continuum limit and that, of course, is not the case when the simulations are done at $\beta = 0$.
[4] D. Zwanziger, Nucl. Phys. B 364, 127 (1991).
[5] D. Zwanziger, Phys. Lett. B 257, 168 (1991).
[6] D. Zwanziger, Nucl. Phys. B 323, 513 (1989).
[7] D. Zwanziger, Nucl. Phys. B 378, 525 (1992).
[8] D. Zwanziger, Phys. Rev. D 65, 094039 (2002).
[9] A. Maas, Phys. Rev. D 75, 116004 (2007); A. Cucchieri, A. Maas and T. Mendes, Phys. Rev. D 74, 014503 (2006); A. Sternbeck, E.M. Ilgenfritz and M. Muller-Preussker, Phys. Rev. D 73, 014502 (2006).
[10] R. Alkofer, M.Q. Huber and K. Schwenzer, arXiv:0801.2762 [hep-th].
[11] L. von Smekal, A. Hauck and R. Alkofer, Annals Phys. 267, 1 (1998) [Erratum-ibid. 269, 182 (1998)]; J.M. Pawlowski et al., Phys. Rev. Lett. 93, 152002 (2004); A.C. Aguilar and A.A. Natale, JHEP 0408, 057 (2004); Ph. Boucaud et al., JHEP 0606, 001 (2006); C.S. Fischer, J. Phys. G 32, R253 (2006); Ph. Boucaud et al. JHEP 0806, 012 (2008); JHEP 0806, 099 (2008).
[12] D. Dudal et al., Phys. Rev. D 77, 071501 (2008).
[13] D. Dudal et al., arXiv:0806.4348 [hep-th]; arXiv:0808.0893 [hep-th].
[14] D. Dudal et al., arXiv:0808.3379 [hep-th].
[15] T. Kugo and I. Ojima, Prog. Theor. Phys. Suppl. 66, 1 (1979) [Erratum-ibid. 71, 1121 (1984)].
[16] T. Kugo, arXiv:hep-th/9511033.
[17] R. Alkofer and L. von Smekal, Phys. Rept. 353, 281 (2001).
[18] D. Zwanziger, Phys. Rev. D 69, 016002 (2004) arXiv:hep-ph/0303028.
[19] A. Cucchieri and T. Mendes, Phys. Rev. Lett. 100, 241601 (2008).
[20] O. Oliveira and P.J. Silva, arXiv:0809.0258 [hep-lat].
[21] A. Cucchieri et al., Phys. Rev. D 76, 114507 (2007).
[22] A. Cucchieri and T. Mendes, arXiv:0804.2371 [hep-lat].
[23] A. Cucchieri, AIP Conf. Proc. 892, 22 (2007).
[24] T. Mendes, A. Cucchieri and A. Mihara, AIP Conf. Proc. 892, 203 (2007); T. Mendes et al., these proceedings.
[25] I.L. Bogolubsky et al., PoS LATTICE 2007, 290 (2007); A. Sternbeck et al., PoS LATTICE 2007, 340 (2007).
[26] G. Dell’Antonio and D. Zwanziger, Commun. Math. Phys. 138, 291 (1991).
[27] A. Cucchieri, Nucl. Phys. B 508, 353 (1997).
[28] I.L. Bogolubsky et al., Phys. Rev. D 74, 034503 (2006) arXiv:hep-lat/0511056; P.J. Silva and O. Oliveira, PoS LAT2007, 333 (2007) arXiv:0710.0669 [hep-lat]; A. Maas, arXiv:0808.3047 [hep-lat].
[29] I.L. Bogolubsky et al., Phys. Rev. D 77, 014504 (2008) [Erratum-ibid. D 77, 039902 (2008)].
[30] See for example the talk by A. Sternbeck at the Confinement 8 Conference (Mainz, Germany, September 2008, http://wwwthep.physik.uni-mainz.de/~confinement8/site/).
[31] A. Cucchieri, T. Mendes and A.R. Taurines, Phys. Rev. D 71, 051902 (2005); A. Sternbeck et al., PoS LAT2006, 076 (2006); P.O. Bowman et al., Phys. Rev. D 76, 094505 (2007).