APPROXIMATE SPECTRAL GAPS FOR MARKOV CHAINS
MIXING TIMES IN HIGH-DIMENSION*

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This paper introduces a concept of approximate spectral gap to analyze the mixing time of Markov Chain Monte Carlo (MCMC) algorithms for which the usual spectral gap is degenerate or almost degenerate. We use the idea to analyze a class of MCMC algorithms to sample from mixtures of densities. As an application we study the mixing time of a popular Gibbs sampler for variable selection in linear regression models. Under some regularity conditions on the signal and the design matrix of the regression problem, we show that for well-chosen initial distributions the mixing time of the Gibbs sampler is polynomial in the dimensional of the space.

1. Introduction. Markov Chain Monte Carlo (MCMC) is the gold standard for Bayesian computation, and understanding the type of problems for which fast MCMC sampling is possible is a question of practical interest. The study of the size of the spectral gap is a widely used approach to gain insight into the behavior of MCMC algorithms. However there are many Markov chains with zero (degenerate) spectral gap. This is the case for instance for the so-called sub-geometrically ergodic Markov chains commonly encountered in MCMC ([14, 6]). Many other Markov chains have near-degenerate spectral gaps, as for instance with MCMC algorithms designed to sample from multimodal distributions with isolated modes. In these latter cases the Markov chain can get trapped for long times in isolated local modes. Obviously when dealing with small isolated modes, a Markov chain that avoids such modes can still sample approximately well from the target distribution. We are interested in this work in measuring the mixing times of Markov chains as they avoid small sets.

Building on the s-conductance of L. Lovasz and M. Simonovits ([19, 20]), we develop

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an idea of approximate spectral gap (that we call $\zeta$-spectral gap, for some $\zeta \in [0, 1]$) which allows us to measure the mixing time of a Markov chain as it avoids small (potentially problematic) sets. We show that the $\zeta$-spectral gap can be used to bound the convergence rate of the Markov chain towards stationarity up to a – typically small – additive term that depends on $\zeta$ (see Lemma 1).

We use the idea to analyze a class of MCMC algorithms to sample from mixtures of densities. Much is known on the computational complexity of various MCMC algorithms for log-concave densities (see e.g. [19, 20, 9, 18, 21, 7] and the references therein). However these results cannot be directly applied to mixtures, since for instance a mixture of log-concave densities is not log-concave in general. In fact, sampling from mixtures is a more challenging problem. For instance it is shown in [10] that no polynomial-time MCMC algorithm exists to sample from mixtures of densities with inequal covariance matrix, if the algorithm uses only the marginal density of the mixture and its derivative. For mixtures of strongly log-concave densities (or close to be) with equal covariance matrices, [10] then proposes a polynomial-time algorithm based on Langevin dynamics and tempering. One major shortcoming of [10] is that their algorithm is impractical when the number of mixture components is very large. In such settings, a Gibbs sampler is commonly employed. A very nice lower bound on the spectral gap of such Gibbs samplers (and generalizations thereof) is developed in [22]. However the analysis of [22] does not exploit any structure of the target distribution and its direct application in large-scale problems typically leads to exponential mixing times. We re-examine [22]'s argument using the concept of $\zeta$-spectral gap, leading to Theorem 3.

Our initial motivation into this work is in large-scale Bayesian variable selection problems. The Bayesian posterior distributions that arise from these problems are typically mixtures of log-concave densities with very large numbers of components, and the aforementioned Gibbs sampler is commonly used for sampling (see e.g. [11, 25]). The proposed concept of $\zeta$-spectral gap and Theorem 3, combined with Bayesian posterior contraction principles are used to show that the algorithm – with a good initialization – has a mixing time that is polynomial in the number of regressors in the model (see Theorem 9 and Corollary 10).

The paper is organized as follows. We develop the concept of $\zeta$-spectral gap in Section 2. The main result there is Lemma 1. In Section 3 we study the mixing time of mixtures of Markov kernels, and derive (Theorem 3) a generalization of Theorem
1.2 of [22]. We put these two results together to analysis the linear regression model in Section 4, leading to Theorem 9 and Corollary 10. Some numerical simulations are detailed in Section 4.3, and some closing remarks are gathered in Section 5.

2. Approximate spectral gaps for Markov chains. Let $\pi$ be a probability measure on some Polish space $(X, B)$ (where $B$ is its Borel sigma-algebra), equipped with a reference sigma-finite measure denoted $dx$. In the applications that we have in mind, $X$ is the Euclidean space $\mathbb{R}^p$ equipped with its Lebesgue measure. We assume that $\pi$ is absolutely continuous with respect to $dx$, and we will abuse notation and use $\pi$ to denote both $\pi$ and its density: $\pi(dx) = \pi(x)dx$. We let $L^2(\pi)$ denote the Hilbert space of all real-valued measurable functions on $X$, equipped with the inner product $\langle f, g \rangle_\pi \coloneqq \int_X f(x)g(x)\pi(dx)$ with associated norm $\| \cdot \|_{2, \pi}$. More generally, for $s \geq 1$, we set $\| f \|_{s, \pi} \coloneqq \left( \int_X |f(x)|^s \pi(dx) \right)^{1/s}$. For $s = +\infty$, we set $\| f \|_{\infty, \pi} = \sup_{x \in X} |f(x)|$.

If $P$ is a Markov kernel on $X$, and $n \geq 1$ an integer, $P^n$ denotes the $n$-th iterate of $P$, defined recursively as $P^n(x, A) \coloneqq \int_X P^{n-1}(x, dz)P(z, A)$, $x \in X$, $A$ measurable. If $f : X \to \mathbb{R}$ is a measurable function, then $Pf : X \to \mathbb{R}$ is the function defined as $Pf(x) \coloneqq \int_X P(x, dz)f(z)$, $x \in X$, assuming that the integral is well defined. And if $\mu$ is a probability measure on $X$, then $\mu P$ is the probability on $X$ defined as $\mu P(A) \coloneqq \int_X \mu(dz)P(z, A)$, $A \in B$. The total variation distance between two probability measures $\mu, \nu$ is defined as

$$\|\mu - \nu\|_{\text{tv}} \coloneqq \sup_{A \in B} (\mu(A) - \nu(A)).$$

Let $K$ be a Markov kernel on $X$ that is reversible with respect to $\pi$. That is for all $A, B \in B$,

$$\int_A \pi(dx) \int_B K(x, dy) = \int_B \pi(dx) \int_A K(x, dy).$$

We will also assume throughout that $K$ is lazy in the sense that $K(x, \{x\}) \geq \frac{1}{2}$. The concept of spectral gap and the related Poincaré’s inequalities are commonly used to quantify Markov chains mixing times. For $f \in L^2(\pi)$, we set $\pi(f) \coloneqq \int_X f(x)\pi(dx)$, $\text{Var}_\pi(f) \coloneqq \| f - \pi(f) \|_{2, \pi}^2$, and $\mathcal{E}(f, f) \coloneqq \frac{1}{2} \int \int (f(y) - f(x))^2 \pi(dx)K(x, dy)$. The spectral gap of $K$ is then defined as

$$\text{SpecGap}(K) \coloneqq \inf \left\{ \frac{\mathcal{E}(f, f)}{\text{Var}_\pi(f)}, f \in L^2(\pi), \text{ s.t. } \text{Var}_\pi(f) > 0 \right\}.$$
It is well-known and easy to establish (see for instance [24] Corollary 2.15) that if 
\[ \pi_0(dx) = f_0(x)\pi(dx), \quad \text{and} \quad f_0 \in L^2(\pi), \]
then
\[ \|\pi_0 K^n - \pi\|^2_{tv} \leq \text{Var}_\pi(f_0) (1 - \text{SpecGap}(K))^n. \]  

Therefore, lower-bounds on the spectral gap can be used to derive upper-bounds on 
the mixing time of \( K \). We refer the reader to ([29, 28, 5, 24]) for more details, and for 
various strategies to lower-bound \( \text{SpecGap}(K) \). In many examples, the conductance 
of \( K \), defined as
\[ \Phi(K) \overset{\text{def}}{=} \inf \left\{ \int_A \pi(dx)K(x, A^c) \right\}, \quad A \in \mathcal{B} : 0 < \pi(A) < 1 \]
is easier to control than the spectral gap. Cheeger’s inequality for Markov chains 
([16, 29]) can then be used to translate a lower-bound on \( \Phi(K) \) into a lower-bound 
on the spectral gap:
\[ \frac{1}{8} \Phi(K)^2 \leq \text{SpecGap}(K) \leq \Phi(K). \]

As mentioned in the introduction, there are many Markov chains for which the 
spectral gap and the conductance are zero, making the bound in (2.1) useless. The 
concept of \( s \)-conductance introduced by L. Lovacze and M. Simonivits ([19, 20], see also 
[21]) as a generalization of the conductance has proven very useful in these settings.

For \( \zeta \in [0, 1/2) \) – using a definition slightly different from [19, 20] – we define the 
\( \zeta \)-conductance of the Markov kernel \( K \) as
\[ \Phi_\zeta(K) \overset{\text{def}}{=} \inf \left\{ \int_A \pi(dx)K(x, A^c) \right\}, \quad \zeta < \pi(A) < \frac{1}{2} \]
where the infimum above is taken over measurable subsets of \( \mathcal{X} \). Note that \( \Phi_0(K) = \Phi(K) \). Plainly put, \( \Phi_\zeta(K) \) captures the same concept as \( \Phi(K) \), except that in \( \Phi_\zeta(K) \) we disregard sets that are either too small or too large under \( \pi \). It turns out that 
\( \Phi_\zeta(K) \) still controls the mixing time of \( K \) up to an additive constant that depends on 
\( \zeta \) (see [20] Corollary 1.5). One important drawback of the \( \zeta \)-conductance is that the arguments that relate \( \Phi_\zeta(K) \) to the mixing time of \( K \) (Theorem 1.4 of [20]) is rather 
involved, and this has limited the scope and the usefulness of the concept. Motivated 
by the \( \zeta \)-conductance, we introduce a similar concept of \( \zeta \)-spectral gap that directly 
approximates the spectral gap. And we show that the proposed \( \zeta \)-spectral gap still 
controls the mixing time of the Markov chains.
Let \( \| \cdot \|_* : L^2(\pi) \to [0, \infty] \) denote a norm-like function on \( L^2(\pi) \) with the following properties: \( \| \alpha f \|_* = |\alpha| \| f \|_* \), if \( \| f \|_* = 0 \) then \( \text{Var}_\pi(f) = 0 \), and
\[
\| Kf \|_* \leq \| f \|_*, \quad f \in L^2(\pi).
\]

For \( \zeta \in (0, 1/2) \), we define the \( \zeta \)-spectral gap of \( K \) as
\[
\text{SpecGap}_\zeta(K) \overset{\text{def}}{=} \inf \left\{ \frac{\mathcal{E}(f,f)}{\text{Var}_\pi(f)} - \zeta, \; f \in L^2(\pi), \; \text{Var}_\pi(f) > \zeta, \; \text{and} \; \| f \|_* = 1 \right\}.
\]

We note that \( \text{SpecGap}_\zeta(K) \) depends on the choice of \( \| \cdot \|_* \), although we will not make that dependence explicit. We note also that if \( \zeta = 0 \) and \( \| f \|_* = \| f \|_{2,\pi} \), then we recover \( \text{SpecGap}_0(K) = \text{SpecGap}(K) \).

The idea of \( \zeta \)-spectral gap is somewhat similar to the concept of weak Poincaré inequality developed for continuous-time Markov semigroups with zero spectral gap ([17, 27, 4]). One key difference is that weak Poincaré inequalities lead to subgeometric rates of convergence of the semi-group, whereas the idea of \( \zeta \)-spectral gap as introduced here leads to a geometric convergence rate, plus an additive remainder that depends on \( \zeta \). More precisely, we have the following analog of (2.1). The proof is similar to the proof of (2.1), and is based on an argument from [23] (see also [8, 24]).

**Lemma 1.** Fix \( \zeta \in (0, 1/2) \). Suppose that \( \pi_0(dx) = f_0(x)\pi(dx) \) for a function \( f_0 \in L^2(\pi) \) such that \( \| f_0 \|_* < \infty \). Then for all integer \( n \geq 1 \), we have
\[
\| \pi_0 K^n - \pi \|_{tv}^2 \leq \max \left\{ \text{Var}_\pi(f_0), \zeta \| f_0 \|_*^2 \right\} \left( 1 - \text{SpecGap}_\zeta(K) \right)^n + \zeta \| f_0 \|_*^2.
\]

**Proof.** See Section 6.1.

Using Lemma 1 requires a lower bound on \( \text{SpecGap}_\zeta(K) \). We highlight a general approach to doing this. First, we introduce the related concept of restricted spectral gap. If \( X_0 \subseteq X \) is a non-empty measurable subset such that \( \pi(X_0) > 0 \), the \( X_0 \)-restricted spectral gap of \( K \) is defined as
\[
\text{SpecGap}_{X_0}(K) \overset{\text{def}}{=} \inf \left\{ \frac{\int_{X_0} \int_{X_0} \pi(dx)K(x,dy)(f(y) - f(x))^2}{\int_{X_0} \int_{X_0} \pi(dx)\pi(dy)(f(y) - f(x))^2}, \; f : X \to \mathbb{R} \right\},
\]
where the infimum is taken over all measurable functions \( f \) such that \( \int_{X_0} \int_{X_0} \pi(dx)\pi(dy)(f(y) - f(x))^2 > 0 \).
Lemma 2. Given $\zeta \in (0, 1/2)$, and taking $\| \cdot \| = \| \cdot \|_{m, \pi}$, for some real number $m \in (2, +\infty)$, let $X_\zeta$ be a measurable subset of $X$ such that $\pi(X_\zeta) \geq 1 - \left(\frac{\zeta}{5}\right)^{1 + \frac{2}{m-2}}$. We have

$$\text{SpecGap}_\zeta(K) \geq \text{SpecGap}_{X_\zeta}(K).$$

Proof. See Section 6.2.

Standard techniques to establish Poincare inequalities can be applied to lower bound $\text{SpecGap}_{X_\zeta}(K)$, particularly if $X_\zeta$ is a compact set. Such inequalities can be used to lower bound $\text{SpecGap}_\zeta(K)$ via Lemma 2. We illustrate the idea more specifically in the next section with mixtures of Markov kernels.

3. Mixing times of mixtures of Markov kernels. In this section we consider the case where the probability measure $\pi$ is a discrete mixture of the form

$$\pi(dx) \propto \sum_{i \in I} \pi(i, x)dx,$$

for nonnegative measurable functions $\{\pi(i, \cdot), i \in I\}$, where $I$ is a nonempty finite set. To avoid confusion we will write $\bar{\pi}$ to denote the joint distribution on $I \times X$ defined as

$$\bar{\pi}(D \times B) = \frac{\sum_{i \in D} \int_B \pi(i, x)dx}{\sum_{i \in I} \int_X \pi(i, x)dx}, \quad D \subseteq I, \quad B \in B.$$

Let $\pi(i|x) \propto \pi(i, x)$ (resp. $\pi(i) \propto \int_X \pi(i, x)dx$) denote the implied conditional (resp. marginal) distribution on $I$, and let $\pi_i(dx) \propto \pi(i, x)dx$ be the implied conditional distribution on $X$. For each $i \in I$, let $K_i$ be a transition kernel on $X$ with invariant distribution $\pi_i$. We assume that $K_i$ is reversible with respect to $\pi_i$, and ergodic (that is phi-irreducible and aperiodic). We then consider the Markov kernel $K$ defined as

$$K(x, dy) \overset{\text{def}}{=} \sum_{i \in I} \pi(i|x)K_i(x, dy),$$

that obviously has invariant distribution $\pi$ as in (3.1). Note that $K$ is also reversible with respect to $\pi$.

In [22] the authors developed a very nice lower bound on the spectral gap of $K$ knowing the spectral gaps of the $K_i$’s. Fix $\kappa > 0$, and construct a graph on $I$ such that there is an edge between $i, j \in I$ if and only if

$$\int_X \min(\pi_i(x), \pi_j(x)) \, dx \geq \kappa.$$
If $D(l)$ denotes the diameter of the graph thus defined\(^1\), Theorem 1.2 of [22] says that

\[
\text{SpecGap}(K) \geq \frac{\kappa}{2D(l)} \min_{i \in l} \{\pi(i)\text{SpecGap}(K_i)\}.
\]

The spectral gap and the lower bound in (3.3) describe the computational cost for obtaining an asymptotically exact sampling from $\pi$. The lower bound in (3.3) can be extremely small. For instance $\kappa$ (which measures the ability of the sampler to move from one component of the mixture to the other) can be very small if $l$ is large and/or some components of the mixtures are hard to move out from. The lower bound in (3.3) can also be very small in problems for which $\pi(i)$ is exponentially small for some components $i$. Now suppose that the isolated modes $i$ are precisely those for which the probabilities $\pi(i)$ are small. In that case we expect the $\zeta$-spectral gap to scale much better than the spectral gap. We have the following lower bound on the $\zeta$-spectral gap.

**Theorem 3.** Let $\pi$ as in (3.1), and $K$ as in (3.2) for some family $\{K_i, i \in l\}$ of Markov kernels on $\mathcal{X}$. Choose $\|\cdot\|_* = \|\cdot\|_{m,\pi}$, for some real number $m \in (2, +\infty]$. Fix $l_0 \subseteq l$, and $\{B_i, i \in l_0\}$ a family of nonempty measurable subsets of $\mathcal{X}$, and set $\bar{B} \overset{\text{def}}{=} \bigcup_{i \in l_0} \{i\} \times B_i$. Fix $\kappa > 0$, and let a graph on $l_0$ be such that

\[
\int_{B_i \cap B_j} \min\left(\frac{\pi_i(x)}{\pi_i(B_i)}, \frac{\pi_j(x)}{\pi_j(B_j)}\right) dx \geq \kappa,
\]

whenever there is an edge between $i, j \in l_0$. Let $D(l_0)$ denote the diameter of the graph. Given $\zeta \in (0, 1/2)$, if $\bar{\pi}(\bar{B}) \geq 1 - \left(\frac{\zeta}{2}\right)^{1+\frac{2}{m-2}}$, then

\[
\text{SpecGap}_\zeta(K) \geq \frac{\kappa}{2D(l_0)} \min_{i \in l_0} \{\pi_i(B_i)^2\} \min_{i \in l_0} \{\pi(i)\text{SpecGap}_{B_i}(K_i)\}.
\]

**Proof.** See Section 6.3. \qed

Note the similarity with (3.3). However Theorem 3 allows us to restrict the analysis of the chain to the set $\bar{B}$. Hence the computational cost of approximate sampling from $\pi$ can scale better if $\min_{i \in l_0} \pi(i)$ and the connectivity $\kappa$ of the sub-graph on $l_0$ scale better. Another aspect of Theorem 3 is that it shows that we may replace the

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\(^1\)We recall that the diameter of a graph is the length (the number of edges) of the longest among all the shortest paths between all pairs of vertices.
spectral gaps \text{SpecGap}(K_i) over the entire \( \mathcal{X} \) (which is difficult to control when \( \mathcal{X} \) is unbounded) by their restricted versions \text{SpecGap}_{B_i}(K_i), which are sometimes easier to bound. However in the interest of space we will not explore here how to bound these restricted spectral gaps.

In the important special case where \( K_i(x, dy) = \pi_i(dy) \), that is in the case where we do an exact Gibbs sampler, and one chooses \( B_i = \mathcal{X} \), Theorem 3 shows that

\[
\text{SpecGap}_\mathcal{X}(K) \geq \frac{\kappa}{2D(I_0)} \min_{i \in I_0} \{\pi(i)\}.
\]

4. Analysis of a Gibbs sampler. We use the results above to analysis a popular Gibbs sampler commonly used for Bayesian variable selection. We consider the Bayesian treatment of a linear regression problem with response variable \( z \in \mathbb{R}^n \), and covariate matrix \( X \in \mathbb{R}^{n \times p} \). The regression parameter is denoted \( \theta \in \mathbb{R}^p \). In settings where the number of regressors \( p \) is very large, and one is interested in selecting the most significant regressors and the corresponding coefficients, it is common practice to introduce an additional variable selection parameter \( \delta \in \Delta \)\( \overset{\text{def}}{=} \{0, 1\}^p \), and to use a spike-and-slab prior distribution on \( \theta \). More precisely, given hyper-parameters \( q \in (0, 1), \rho, \gamma \in (0, +\infty) \), and integer \( \bar{s} \geq 1 \), we assume that the prior distribution of \( \delta \) is given by

\[
\omega_\delta \propto q^{\|\delta\|_0} (1 - q)^{p - \|\delta\|_0} 1_{\Delta_{\bar{s}}}(\delta), \quad \delta \in \Delta,
\]

where \( \Delta_{\bar{s}} \overset{\text{def}}{=} \{\delta \in \Delta : \|\delta\|_0 \leq \bar{s}\} \). We then assume that the components of \( \theta \) are conditionally independent given \( \delta \), and we assume that \( \theta_j|\delta \) has density \( \mathcal{N}(0, \frac{1}{\rho}) \) if \( \delta_j = 1 \), and density \( \mathcal{N}(0, \gamma) \) otherwise, where \( \mathcal{N}(\mu, v^2) \) denotes the univariate Gaussian distribution with mean \( \mu \) and variance \( v^2 \). The resulting posterior distribution on \( \Delta \times \mathbb{R}^p \) is

\[
\Pi(\delta, d\theta|z) \propto \omega_\delta e^{-\frac{1}{2} \theta^T D(\delta)^{-1} \theta} e^{-\frac{1}{2\sigma^2} \|z - X\theta\|^2_2} d\theta,
\]

where \( D(\delta) \in \mathbb{R}^{p \times p} \) is a diagonal matrix with \( j \)-th diagonal element equal to \( 1/\rho \) if \( \delta_j = 1 \), and \( \gamma \) if \( \delta_j = 0 \). The regression error \( \sigma \) is assumed known. This model is very popular in the application ([11, 13, 25]), mainly because it is straightforward to sample from (4.1). Indeed, the posterior conditional distribution \( \Pi(\delta|\theta, z) \) is a product of independent Bernoulli distributions conditioned to be sparse, with closed
form probabilities:

\[(4.2) \quad \Pi(\delta|\theta, z) \propto \prod_{j=1}^{p} [q_j \delta_j [1 - q_j]^{1 - \delta_j} \mathbf{1}_{\Delta}(\delta), \]

where \( q_j \text{ def } = \frac{1 - \frac{1}{q}}{q} \sqrt{\frac{1}{\gamma \rho} e^{\frac{1}{2} (\rho - \frac{1}{2}) \|\theta\|^2}}, \quad j = 1, \ldots, p. \)

One can sample from (4.2) by generating vectors \( \delta = (\delta_1, \ldots, \delta_p) \) of independent Bernoulli random variables \( \delta_j \sim \text{Ber}(q_j) \), until \( \|\delta\|_0 \leq s \), where \( \text{Ber}(\alpha) \) denotes the Bernoulli distribution with success probability \( \alpha \). This method works very well provided that \( q \) is taken small (which is usually the case when \( p \) is large, see H1), and \( s \) is sufficiently large.

Given \( \delta \), the conditional distribution of \( \theta \) given \( \delta \) is \( N_p(m_\delta, \sigma^2 \Sigma_\delta) \), with \( m_\delta \) and \( \Sigma_\delta \) given by

\[(4.3) \quad m_\delta \text{ def } = \Sigma_\delta X' z \quad \text{and} \quad \Sigma_\delta \text{ def } = \left( X' X + \sigma^2 D^{-1}\delta \right)^{-1}. \]

Put together these two conditional distributions yield a simple Gibbs sampling algorithm for (4.1). We consider the following version.

**Algorithm 1.** For some initial distribution \( \nu_0 \) on \( \mathbb{R}^p \), draw \( u_0 \sim \nu_0 \). Given \( u_0, \ldots, u_n \) for some \( n \geq 0 \), draw independently \( I_{n+1} \sim \text{Ber}(0.5) \).

1. If \( I_{n+1} = 0 \), set \( u_{n+1} = u_n \).
2. If \( I_{n+1} = 1 \),
   
   (a) Draw \( \delta \sim \Pi(\cdot|u_n, z) \) as given in (4.2), and
   
   (b) draw \( u_{n+1} \sim N_p(m_\delta, \sigma^2 \Sigma_\delta) \) as given in (4.3).

**Remark 4.** The introduction of the indicator variable \( I_n \) implies that half of the time the chain does not move: we have a lazy Markov chain, as needed in our theory. This trick is not used in practice, and for the numerical illustrations presented below we only implemented the plain Gibbs sampler.

The indicator variables \( \delta \) discarded in Algorithm 1 are important in practice for the variable selection problem, and are usually collected along the iterations. Here we focus the analysis on the continuous variables \( u_n \in \mathbb{R}^p \). Obviously we do not lose anything, since given \( u_n \) exact sampling of \( \delta \) is possible as discussed above.
Remark 5. The computational cost per iteration of Algorithm 1 is dominated by the cost of sampling from the Gaussian distribution $N(m_\delta, \Sigma_\delta)$, which itself is dominated by the Cholesky decomposition of $\Sigma_\delta$. Hence each iteration of Algorithm 1 in general has a cost that scales with $p$ as $O(p^3)$. However a faster implementation that exploits the structure of $\Sigma_\delta$ is possible as in [2]. The per-iteration computational cost of this approach is $O(n^3 + p^2 n)$. Hence the per-iteration computational cost of the algorithm is $O(p^2 \min(n, p))$, which matches other state of the art algorithms for high-dimensional regression ([2]).

Our objective is to analyze the mixing time of the marginal chain $\{u_n, n \geq 0\}$ from Algorithm 1. As easily seen, $\{u_n, n \geq 0\}$ is a Markov chain with invariant distribution

\begin{equation}
\Pi(d\theta|z) \propto \sum_{\delta \in \Delta} \omega_\delta e^{-\frac{1}{2} \theta' D_\delta^{-1} \theta} e^{-\frac{1}{2} \pi^2 \|z - X\theta\|_2^2} d\theta,
\end{equation}

which is of the form (3.1), and with transition kernel

\begin{equation}
K(u, d\theta) \overset{\text{def}}{=} \sum_{\omega \in \Delta} \Pi(\omega|u, z) \left[ \frac{1}{2} \delta_u(d\theta) + \frac{1}{2} \Pi(d\theta|\omega, z) \right],
\end{equation}

which is of the form (3.2). One important remark to make is that, consistently with the viewpoint in Bayesian asymptotics, we will view the data $z$ as the realization of a random variable $Z$ on $\mathbb{R}^n$. In that sense, $\Pi(\cdot|z)$ is a realization of a random probability measure $\Pi(\cdot|Z)$, and therefore the Markov kernel $K$ is also a realization of a random Markov kernel. We are then able to make probabilistic statements about $K$ and $\Pi(\cdot|Z)$ under the distribution of $Z$ (that is, under the data generating mechanism). To avoid notation overload, we will omit the dependence of the Markov kernel $K$ on $z$ and $Z$.

The next assumption introduces, among other things, the random variable $Z$ and its distribution.

H1. 1. The data $z \in \mathbb{R}^n$ is the realization of a random variable $Z \overset{\text{def}}{=} (Z_1, \ldots, Z_n) \sim N(X\theta_*, \sigma^2 I_n)$, for some unknown parameter $\theta_* \in \mathbb{R}^p$, and a known absolute constant $\sigma^2 > 0$.

\footnote{We are grateful to Anirban Bhattacharya for pointing out this paper to us.}
2. The matrix $X$ is non-random and normalized such that

\[(4.6)\quad \|X_j\|_2^2 = n, \quad j = 1, \ldots, p,\]

where $X_j \in \mathbb{R}^n$ denotes the $j$-th column of $X$.

3. The prior parameter $q$ is chosen such that

\[(4.7)\quad \frac{q}{1-q} = \frac{1}{p^{u+1}},\]

for some absolute constant $u > 0$.

4. The prior parameters $\rho$ and $\gamma$ are taken such that

\[(4.8)\quad 0 < \gamma < \frac{1}{2\rho}.\]

Remark 6. Overall these are very basic assumptions. We assume in H1-(1) that the statistical model is well specified. The assumption that the regression errors are Gaussian is imposed mostly for simplicity, and can be replaced by a sub-Gaussian assumption, with minimal change to what follows. The prior assumption in H1-(3) is fairly standard, and follows [3, 25, 1]. We insist to point out that the hyper-parameter $u > 0$ is an absolute constant that is not adjusted to the dimension $p$. H1-(4) simply says that the variance of the slab prior density should be sufficiently larger than the variance of the spike prior density.

We will write $P_*$ (resp. $E_*$) to denote the probability distribution (resp. expectation operator) of the random variable $Z$ assumed in H1.

4.1. Contraction behavior. We first establish that the posterior distribution $\Pi$ puts most of its probability mass on a small number of components of $\Delta$. But first for convenience, we introduce some notations. For $\theta, \theta' \in \mathbb{R}^p$, we write $\theta \cdot \theta' \in \mathbb{R}^p$ to represent the component-wise product of $\theta$ and $\theta'$. For $\delta \in \Delta$, and $\theta \in \mathbb{R}^p$, we write $\theta_\delta$ as a short for $\theta \cdot \delta$, and we define $\delta^c \stackrel{\text{def}}{=} 1 - \delta$, that is $\delta^c_j = 1 - \delta_j$, $1 \leq j \leq p$. For a matrix $A \in \mathbb{R}^{q \times p}$, $A_\delta$ (resp. $A_{\delta^c}$) denotes the matrix of $\mathbb{R}^{q \times \|\delta\|_0}$ (resp. $\mathbb{R}^{q \times (p-\|\delta\|_0)}$) obtained by keeping only the columns of $A$ for which $\delta_j = 1$ (resp. $\delta_j = 0$). The support of a vector $u \in \mathbb{R}^p$ is the vector $\text{supp}(u) \in \Delta$ such that $\text{supp}(u)_j = 1$ if and only if $|u_j| > 0$. 
For an integer $s \geq 1$, we define
\[ \bar{v}(s) \overset{\text{def}}{=} \sup \left\{ \frac{\theta'(X'X)\theta}{n\|\theta\|_2^2}, \theta \neq 0, \|\theta\|_0 \leq s \right\}. \]

It is not hard to see that under (4.6), $\bar{v}(s) \leq s$. An important role is played in the analysis by the matrices
\[ L_\delta \overset{\text{def}}{=} I_n + \frac{1}{\sigma^2} XD(\delta)X' = I_n + \frac{1}{\sigma^2\rho} \sum_{j: \delta_j = 1} X_j X_j' + \frac{\gamma}{\sigma^2} \sum_{j: \delta_j = 0} X_j X_j', \quad \delta \in \Delta. \]

The following quantity can be interpreted as a coherence of the design matrix $X$:
\[ C_X \overset{\text{def}}{=} \max_{\delta \in \Delta: \|\delta\|_0 \leq s} \max_{j \neq \ell, \delta_j = 0} |X_j' L_\delta^{-1} X_\ell|. \]

We make the following important assumption.

**H2.** There exist $\varrho > 0$ and an integer $k_0 \geq 0$, $k_0 \leq s$, such that for all $\delta \in \Delta$ satisfying $\|\delta\|_0 \leq s$, for all vector $u \in \mathbb{R}^p$ such that $\delta^c \supseteq \text{supp}(u)$, and $\text{supp}(u) \leq k_0 + 1$, we have
\[ u' \left( X'L_\delta^{-1} X \right) u \geq n\varrho \|u\|_2^2. \]

**Remark 7.** For $\gamma$ small and $\rho$ small, the matrix $L_\delta^{-1}$ can be loosely interpreted as the projector on the orthogonal of the space spanned by the columns of $X_\delta$. Therefore, H2 is precluding the situation where a small number of columns of $X$ has the same linear span as all the columns of $X$. We show in Lemma 14 in the appendix that if $X$ is a random matrix with i.i.d. standard normal entries (Gaussian ensemble) and $\gamma$ is taken small enough, then H2 holds with high probability, and
\[ C_X \leq c_0 \sqrt{n \log(p)}, \]
for some universal constant $c_0$. \qed

We need few more quantities in order to state the theorem. We define
\[ \epsilon \overset{\text{def}}{=} \sigma \sqrt{\frac{\log(p)}{n}}, \quad (4.9) \]
that we view as the signal detectability threshold. Let $J_*$ be the set of $j \in \{1, \ldots, p\}$ such that $|\theta_{*,j}| > \epsilon$. Let $\tilde{\delta}_*$ be the element of $\Delta$ that indicates which components of $\theta_*$ are greater than $\epsilon$ in absolute value: $\tilde{\delta}_{*,j} = 1$ if and only if $|\theta_{*,j}| > \epsilon$. Finally, set

$$\theta_* = \inf \{|\theta_{*,i}| : i \in J_*\},$$

and we convene that $\theta_* = +\infty$ if $J_* = \emptyset$. Note that by definition $\theta_* \geq \epsilon$. Given $k \geq 0$, we define

$$D_k \overset{\text{def}}{=} \left\{ \delta \in \Delta : \delta \supseteq \tilde{\delta}_*, \|\delta\|_0 \leq \|\tilde{\delta}_*\|_0 + k \right\}.

**Theorem 8.** Assume H1-H2. Suppose also that

$$\frac{10}{\sigma} \left(1 + \frac{\|\theta_*\|_1 C_X}{\sqrt{n \log(p)}}\right)^2 \leq u, \tag{4.10}$$

and

$$n \sigma \theta_*^2 \geq 64 \sigma^2 (1 + u)(\bar{s} + 1)\bar{v}(s_\star) \log(p). \tag{4.11}$$

Then with probability at least $1 - 6p^{-1}$, and for all $0 \leq k \leq k_0$, we have

$$\Pi(D_k | Z) \geq 1 - \frac{4}{p^{(u+1)/2}}. \tag{4.11}$$

**Proof.** See Section 6.4.

The contraction properties of $\Pi$ has been studied in [25]. However Theorem 8 is more precise, in the sense that here we identify separately how the signal and the features of design matrix impact the contraction properties of the posterior distribution. In particular the result shows that the components of $\theta_*$ that are below the threshold value $\epsilon$ cannot be recovered in general. But with enough sample size, the remaining parameters are recovered with high probability. The sample size condition (4.11) implies that one needs some minimal sample size or order $(s_\star \bar{s} \log(p))$ for the contraction properties to kick in. Condition (4.10) highlights the need of the coherence $C_X$ to be well-behaved. In the case of the Gaussian ensemble, $C_X \leq c_0 \sqrt{n \log(p)}$ (see Lemma 14), which implies that in this case, if $\|\theta_*\|_1$ does not increase with $p$, (4.10) holds if $u$ is taken large enough. However we noted in our simulations with the Gaussian ensemble that the posterior distribution behaves well even with $u = 1$. 


4.2. Mixing time analysis. Theorem 3 gives the following simple analysis of Algorithm 1. We recall that for $\delta \in \Delta$, $\Pi(\cdot | \delta, z)$ denotes the conditional distribution of $\theta$ given $\delta$ under the posterior distribution (see (4.3)).

**Theorem 9.** Suppose that we initialize Algorithm 1 with $\nu_0 = \Pi(\cdot | \delta^{(i)}, z)$, for some initial selection $\delta^{(i)} \in \Delta$. For some arbitrary integer $k \geq 1$, and $\zeta_0 \in (0, 1/2)$, set

$$E(k, \zeta_0) \overset{\text{def}}{=} \{ z \in \mathbb{R}^n : \frac{1 - \Pi(D_k | z)}{\Pi(\delta^{(i)} | z)^2} \leq \frac{\zeta_0^2}{10} \}.$$  

For any $z \in E(k, \zeta_0)$, and any integer $N$ that satisfies

$$N \geq 2 \left[ \log \left( \frac{1}{\Pi(\delta^{(i)} | z)} \right) + \log \left( \frac{1}{\zeta_0} \right) \right] + \frac{4k}{\gamma} \left( \frac{1}{\gamma} \right)^{\frac{5}{2}} \left( 1 + \frac{C_X \| \theta \|_1}{\pi} \right)^2 \frac{1}{\min_{\delta \in D_k} \pi(\delta | z)},$$

we have

$$\| \nu_0 K^N - \Pi(\cdot | z) \|_{tv} \leq \zeta_0.$$

**Proof.** See Section 6.5. \qed

Theorem 9 would imply fast mixing if the right-hand side of (4.12) is polynomial in $p$, and the set $E(k, \zeta_0)$ is large. To show this, let us consider the case where the initial selection $\delta^{(i)}$ satisfies $\delta^{(i)} \supseteq \tilde{\delta}_*$, and let $FP \overset{\text{def}}{=} \| \delta^{(i)} \|_0 - \| \tilde{\delta}_* \|_0$ be the number of false positive of $\delta^{(i)}$. Suppose also that H1-H2, as well as (4.10) and (4.11) hold. Under these assumptions, Theorem 8 applies, and says that with probability at least $1 - 6p^{-1}$, $\Pi(\tilde{\delta}_* | Z) \geq 1 - \frac{4}{p^{u+1}} \geq 1/2$, and for $0 \leq k \leq k_0$,

$$\Pi(D_k | Z) \geq 1 - \frac{4}{p^{u+1} (k+1)}.$$  

Using $\Pi(\tilde{\delta}_* | Z) \geq 1/2$ we can write,

$$\frac{1}{\Pi(\delta^{(i)} | Z)} \leq \frac{2\Pi(\tilde{\delta}_* | Z)}{\Pi(\delta^{(i)} | Z)}.$$  

Using (6.11) with $B = \{ \tilde{\delta}_* \}$ and $\delta_0 = \delta^{(i)}$, and using (6.13) and (6.14), we deduce that

$$\frac{1}{\Pi(\delta^{(i)} | Z)} \leq 2p^{(u+1)\text{FP}} \sqrt{\det \left( I_{\text{FP}} + \tau X'_{(\delta^{(i)} - \tilde{\delta}_*)} \tilde{L}_{\tilde{\delta}_*}^{-1} X_{(\delta^{(i)} - \tilde{\delta}_*)} \right)},$$

$$\leq 2p^{(u+1)} \sqrt{\frac{n \bar{v}(\text{FP})}{\sigma^2 \rho}}.$$
We can combine the last inequality with (4.13) to conclude that
\[
\frac{(1 - \Pi(D_k|Z))}{\Pi(D(k)|Z)^2} \leq 16 \left(1 + \frac{nFP}{\sigma^2 \rho}\right)^{FP} p^{2(u+1)FP} \times \frac{1}{p^{\frac{n}{2} (k+1)}}.
\]

Hence, if the number of false-positive FP is not too large, and satisfies
\[
(4.14) \quad k \geq 4 \left(1 + \frac{1}{u}\right) \text{FP} + \frac{2FP \log \left(1 + \frac{nFP}{\sigma^2 \rho}\right)}{u \log(p)}
\]
then it follows that for all p large enough, \(\mathbb{P}_\star(Z \notin \mathcal{E}(k, \zeta_0)) \leq 6/p\). Furthermore, for \(\delta \in D_k\), using (6.11) with \(B = \{\delta\}\) and \(\delta_0 = \tilde{\delta}_\star\), together with (6.13) and (6.14), we have
\[
\pi(\delta|z) \geq \frac{1}{2} \pi(\delta|z) \geq \frac{1}{2} p^{k(u+1)} \left(1 + \frac{n\tilde{v}(k)}{\sigma^2 \rho}\right)^{-\frac{k}{2}}.
\]
This implies that the leading term on the right-hand side of (4.12) is
\[
\frac{1}{(\gamma \rho)} \times p^{\frac{1}{2} \left(1 + \frac{c_X \|\theta^{u+1}\|_1}{3 \sigma \sqrt{n \log(p)}}\right)^2} \times p^{k(u+1)},
\]
which is polynomial in \(p\) under (4.10). We summarize this discussion in the next corollary.

**Corollary 10.** Suppose that H1-H2, as well as (4.10) and (4.11) hold. Suppose also that the initial selection \(\delta^{(i)}\) satisfies \(\delta^{(i)} \supseteq \tilde{\delta}_\star\), with a number of false-positive \(FP \triangleq \|\delta^{(i)}\|_0 - \|\tilde{\delta}_\star\|_0\) that satisfies (4.14) for some \(k \leq k_0\). Then with probability at least \(1 - 6p^{-1}\), the following holds: given \(\zeta_0 \in (0, 1)\) there exists \(A\) that does not depend on \(p\) such that for
\[
(4.15) \quad N \geq \frac{A}{(\gamma \rho)} \times p^{\frac{1}{2} + k(u+1)},
\]
we have
\[
\|\nu_0 K^N - \Pi(\cdot|z)\|_{tv} \leq \zeta_0.
\]

**Remark 11.** We note from (4.14) that the power \(k\) that appears in (4.15) grows with \(FP\). This suggests that the mixing time of the algorithm can rapidly deteriorate if \(FP\) is large. It is unclear whether the precise dependence on \(p\) thus expressed in
(4.15) is tight. However we did observe in the simulations a sharp increase in the mixing time of the algorithm as \( \text{FP} \) increases, which seems consistent with (4.15).

With respect to the initialization, the natural question is how the mixing time behaves if \( \delta^{(i)} \) admits false-negatives. Our methods of proof are not adapted to provide an answer to this question (different techniques are often needed to establish fast mixing and slow mixing. See e.g. [30]). Nonetheless to gain some intuition, we perform some numerical simulations which seem to suggest that the polynomial mixing time obtained in Corollary 10 no longer hold if \( \delta^{(i)} \) has false-negatives.

Corollary 10 has important practical implications. It suggests that the commonly used initialization strategy where \( \delta^{(i)} \) is taken as the zero vector is sub-optimal, and might result in Markov chain with exponential mixing times. Instead, a more sensible strategy is to take \( \delta^{(i)} \) as the support of the lasso estimate – or some other similarly-behaved frequentist estimate.

\[ \square \]

4.3. Numerical illustrations. We illustrate some of the conclusions with the following simulation study. We consider a linear regression model with Gaussian noise \( \mathbf{N}(0, \sigma^2) \), where \( \sigma^2 \) is set to 1. We experiment with sample size \( n = p/10 \), and dimension \( p \in \{500, 1000, 2000, 3000, 4000\} \). We take \( X \in \mathbb{R}^{n \times p} \) as a random matrix with i.i.d. standard Gaussian entries. We fix the number of non-zero coefficients to \( s^{\star} = 10 \), and \( \delta^{\star} \) is given by

\[ \delta^{\star} = (1, \ldots, 1, 0, \ldots, 0)^\top. \]

The non-zero coefficients of \( \theta^{\star} \) are uniformly drawn from \((-a - 1, -a) \cup (a, a + 1)\), where

\[ a = 4 \sqrt{\frac{\log(p)}{n}}. \]

We use the following prior parameters values:

\[ u = 1, \quad \rho = \frac{1}{\sqrt{n}}, \quad \gamma = \frac{0.1\sigma^2}{\lambda_{\text{max}}(X^\top X)}. \]

We use an initial distribution \( \nu_0 = \Pi(\cdot|\delta^{(i)}, z) \), where we vary the number of false-positives of \( \delta^{(i)} \). To monitor the mixing, we compute the sensitivity and the precision at iteration \( k \) as

\[ \text{SEN}_k = \frac{1}{s^{\star}} \sum_{j=1}^{p} 1_{\{\delta_{k,j} > 0\}} 1_{\{\delta^{\star},j > 0\}}, \quad \text{PREC}_k = \frac{\sum_{j=1}^{p} 1_{\{\delta_{k,j} > 0\}} 1_{\{\delta^{\star},j > 0\}}}{\sum_{j=1}^{p} 1_{\{\delta_{k,j} > 0\}}}. \]
We empirically measure the mixing time of the algorithm as the first time $k$ where both $\text{SEN}_k$ and $\text{PREC}_k$ reach 1, truncated to $2 \times 10^4$ – that is we stop any run that has not mixed after 20000 iterations. The average empirical mixing time thus obtained (based on on 46 independent MCMC replications) are presented in Table 1 and Figure 1. These estimates are consistent with our results. They show only a modest increase in mixing time as $p$ increases, but a sharp increase in mixing time as the number of false-positives increases. We also explore the behavior of the sampler in the presence of false-negatives in the initialization. More specifically we consider the case where $\delta^{(i)}$ has 2 false-negatives, but no false-positive. In this setting, and for all 46 replications, the sampler fails to recover all 10 significant components within 20,000 iterations.

| FP = 1%  | $p = 500$ | $p = 1000$ | $p = 2000$ | $p = 3000$ | $p = 4000$ |
|----------|-----------|-----------|-----------|-----------|-----------|
|          | 15.7 (21.1) | 71.6 (280.3) | 43.5 (45.6) | 42.8 (47.5) | 65.4 (100.6) |
| FP = 5%  | 93.8 (247.8) | 93.5 (102.3) | 130.8 (164.9) | 186.9 (303.9) | 225.9 (239.5) |
| FP = 10% | >11325.6   | >7916.3    | >8955.0    | >10648.0   | >12113.4   |
| FP = 20% | >20000     | >20000     | >20000     | >20000     | >20000     |
| FP = 0, FN = 2 | >20000 | >20000 | >20000 | >20000 | >20000 |

**Table 1**

Table showing the average empirical mixing time of the sampler. Based on 46 simulation replications. The numbers in parenthesis are standard errors. The notation $>a$ means that some (or all) of the replicated mixing times have been truncated to 20,000.

**Fig 1.** Boxplots of the average empirical mixing times. Based on 46 simulation replications.
5. Concluding remarks. The paper introduced a concept of approximate spectral gap for Markov chains. The idea makes it easy to combine together results from Bayesian asymptotics and Markov chain theory in order to analyze more precisely the behavior of MCMC algorithms that are used in Bayesian data analysis. In the linear regression model considered our results suggest that one should initialize the Gibbs sampler using the support of a frequentist estimator. In which case the Gibbs sampler typically mixes in polynomial time.

The idea of $\zeta$-spectral gap can be extended in several directions. For instance one can define along similar lines a similar concept of approximate log-Sobolev constants. The idea of approximate spectral gap can also be adapted to study the mixing time of continuous-time Langevin diffusion processes for sampling from densities that are not log-concave.

6. Proofs.

6.1. Proof Lemma 1. We first note that if a probability measure $\nu$ is absolutely continuous with respect to $\pi$ with Radon-Nikodym derivative $f_\nu$, then for any $A \in \mathcal{B}$,

$$\nu K(A) = \int \nu(dx) K(x, A) = \int \int f_\nu(x) 1_A(y) \pi(dx) K(x, dy)$$

$$= \int \int 1_A(x) f_\nu(y) \pi(dx) K(x, dy) = \int_A \pi(dx) \int K(x, dy) f_\nu(y),$$

where the third equality uses the reversibility of $K$. This calculation says that $\nu K$ is also absolutely continuous with respect to $\pi$ with Radon-Nikodym derivative $x \mapsto K f_\nu(x)$. More generally

$$\|\nu K^n - \pi\|_{tv}^2 = \left( \int \left| \frac{d(\nu K^n)}{d\pi}(x) - 1 \right| \pi(dx) \right)^2$$

$$= \left( \int |K^n f_\nu(x) - 1| \pi(dx) \right)^2$$

$$\leq \|K^n f_\nu - 1\|_{L_2,\pi}^2$$

$$= \text{Var}(K^n f_\nu).$$

(6.1)

Take $f \in L^2(\pi)$. Since $\pi(f) = \pi(K f)$, we have

$$\text{Var}(K f) - \text{Var}(f) = \langle K f, K f \rangle_\pi - \langle f, f \rangle_\pi$$

$$= -\frac{1}{2} \int \int (f(y) - f(x))^2 \pi(dx) K^2(x, dy),$$

(6.2)
where the last equality exploits the reversibility of $K$. The lazyness of the chain gives that for any pair of measurable sets $A, B$,

$$
\int_A \int_B \pi(dx) K^2(x, dy) = \int_A \pi(dx) \int_X K(x, dy_1) K(y_1, B) \geq \frac{1}{2} \int_A \pi(dx) K(x, B) + \int_A \pi(dx) \int_B K(x, dy_1) K(y_1, B) \geq \int_A \pi(dx) K(x, B).
$$

It follows that for all $f \in L^2(\pi)$,

$$
\int \int (f(y) - f(x))^2 \pi(dx) K^2(x, dy) \geq \int \int (f(y) - f(x))^2 \pi(dx) K(x, dy).
$$

Using the last display together with (6.2), and the definition of $E(f, f)$, we conclude that for all $f \in L^2(\pi)$,

$$
\text{Var}(K f) \leq \text{Var}(f) - E(f, f).
$$

(6.3)

But if $\text{Var}(f) > \|f\|_\|^2 > 0$, then by (6.3),

$$
\text{Var}(K f) = \|f\|_\|^2 \text{Var} \left( K \left( \frac{f}{\|f\|_\|} \right) \right) \\
\leq \|f\|_\|^2 \left( \text{Var} \left( \frac{f}{\|f\|_\|} \right) - E \left( \frac{f}{\|f\|_\|}, \frac{f}{\|f\|_\|} \right) \right) \\
\leq \text{Var}(f) - \|f\|_\|^2 \text{SpecGap}_\|, \text{SpecGap}(K) \left( \text{Var} \left( \frac{f}{\|f\|_\|} \right) - \zeta \right), \\
= \text{Var}(f) \left( 1 - \text{SpecGap}_\|, \text{SpecGap}(K) \right) + \zeta \|f\|_\|^2 \text{SpecGap}_\|, \text{SpecGap}(K).
$$

Clearly the last display (which is derived assuming that $\|f\|_\| > 0$) continues to hold if $\|f\|_\| = 0$. We conclude that for all $f \in L^2(\pi)$,

$$
\text{Var}(K f) \leq \max(\text{Var}(f), \zeta \|f\|_\|^2) \left( 1 - \text{SpecGap}_\|, \text{SpecGap}(K) \right) + \zeta \|f\|_\|^2 \text{SpecGap}_\|, \text{SpecGap}(K).
$$
Since \( \|Kf\|_* \leq \|f\|_* \), it follows that for all \( f \in L^2(\pi) \)

\[
(6.4) \quad \max (\text{Var}(Kf), \zeta \|Kf\|_*^2) \leq \max (\text{Var}(f), \zeta \|f\|_*^2) \left(1 - \text{SpecGap}_\zeta(K)\right) + \zeta \|f\|_*^2 \cdot \text{SpecGap}_\zeta(K).
\]

We can iterate the above inequality to deduce that for all \( f \in L^2(\pi) \), such that \( \|f\|_* < \infty \), and for all \( n \geq 1 \),

\[
\max (\text{Var}(K^n f), \zeta \|K^n f\|_*^2) \leq \max (\text{Var}(f), \zeta \|f\|_*^2) \left(1 - \text{SpecGap}_\zeta(K)\right)^n + \zeta \|f\|_*^2.
\]

Now, if \( \pi_0 = f_0 \pi \), the last display combined with (6.1) implies that

\[
\|\pi_0 K^n - \pi\|_{TV}^2 \leq \max (\text{Var}(K^n f_0), \zeta \|K^n f_0\|_*^2) \leq \max (\text{Var}(f_0), \zeta \|f_0\|_*^2) \left(1 - \text{SpecGap}_\zeta(K)\right)^n + \zeta \|f_0\|_*^2.
\]

This ends the proof.

\[
\square
\]

6.2. **Proof Lemma 2.** Take \( f: \mathcal{X} \to \mathbb{R} \) such that \( \text{Var}_\pi(f) > \zeta \), and \( \|f\|_* = \|f\|_{m,\pi} = 1 \). We have

\[
2\text{Var}_\pi(f) = \int_{\mathcal{X}_\zeta} \int_{\mathcal{X}_\zeta} (f(y) - f(x))^2 \pi(dx)\pi(dy) + 2 \int_{\mathcal{X}_\zeta} \int_{\mathcal{X}\setminus\mathcal{X}_\zeta} (f(y) - f(x))^2 \pi(dx)\pi(dy).
\]

Using the convexity inequality \((a + b)^2 \leq 2a^2 + 2b^2\), and Holder’s inequality,

\[
\int_{\mathcal{X}_\zeta} \int_{\mathcal{X}\setminus\mathcal{X}_\zeta} (f(y) - f(x))^2 \pi(dx)\pi(dy) \\
\leq 2\pi(\mathcal{X}_\zeta) \int_{\mathcal{X}\setminus\mathcal{X}_\zeta} f(x)^2 \pi(dx) + 2\pi(\mathcal{X}\setminus\mathcal{X}_\zeta) \int_{\mathcal{X}_\zeta} f(x)^2 \pi(dx) \\
\leq 2\pi(\mathcal{X}_\zeta) \pi(\mathcal{X}\setminus\mathcal{X}_\zeta)^{1 - \frac{2}{m}} \|f\|_{m,\pi}^2 + 2\pi(\mathcal{X}\setminus\mathcal{X}_\zeta) \|f\|_{m,\pi}^2 \\
\leq 4\pi(\mathcal{X}\setminus\mathcal{X}_\zeta)^{1 - \frac{2}{m}}.
\]
With similar calculation, 
\[ \int_{X \setminus X} \int_{X \setminus X} (f(y) - f(x))^2 \pi(dx) \pi(dy) \leq 4\pi(A \setminus A) \pi(A \setminus A)^{1 - \frac{m}{2}} \leq 2\pi(A \setminus A)^{1 - \frac{m}{2}}. \]

Using \( \pi(A) \geq (\zeta/5)^{1+2/(m-2)} \), we get
\[ 2(\text{Var}_\pi(f) - \zeta) \geq \int_{A} \int_{A} \pi(dx) \pi(dy) (f(y) - f(x))^2. \]

Hence
\[ \mathcal{E}(f, f) - \zeta = \int_{A} \int_{A} \pi(dx) \pi(dy) (f(y) - f(x))^2 \geq \text{SpecGap}_{A}. \]

This ends the proof. \( \square \)

6.3. **Proof Theorem 3.** The proof of the theorem is similar to the proof of Lemma 2. But first, we need the following lemma.

**Lemma 12.** Let \( \nu(dx) = f_\nu(x)dx, \mu(dx) = f_\mu(x)dx \) be two probability measures on some measurable space with reference measure \( dx \), such that \( \int \min(f_\mu(x), f_\nu(x))dx > \epsilon \) for some \( \epsilon > 0 \). Then for any measurable function \( h \) such that \( \int h^2(x)\nu(dx) < \infty \) and \( \int h^2(x)\mu(dx) < \infty \), we have
\[
\int (h(y) - h(x))^2 \mu(dy) \nu(dx) \leq \frac{2 - \epsilon}{2\epsilon} \left[ \int (h(y) - h(x))^2 \mu(dy) \mu(dx) + \int (h(y) - h(x))^2 \nu(dy) \nu(dx) \right].
\]

**Proof.** This result is established as part of the proof of Theorem 1.2 of [22] (see inequality (47)). \( \square \)

Choose \( f \in L^2(\pi) \) such that \( \|f\|_{m, \pi} = 1 \). Given \( i \in I \), we set
\[ \mathcal{E}_i(f, f) = \frac{1}{2} \int_{B_i} \int_{B_i} (f(y) - f(x))^2 \pi_i(dx) K_i(x, dy). \]

By the definition of \( \text{SpecGap}_{B_i}(K_i) \), we have
\[ \mathcal{E}_i(f, f) \geq \frac{1}{2} \text{SpecGap}_{B_i}(K_i) \int_{B_i} \int_{B_i} (f(y) - f(x))^2 \pi_i(dx) \pi_i(dy). \]
By Fubini’s theorem, and using (6.5), we have

\[
2\mathcal{E}(f, f) = \int_{\mathcal{X}} \pi(dx) \left\{ \sum_{i \in I} \pi_x(i) \int_{\mathcal{X}} K_i(x, dy) \right\} (f(y) - f(x))^2
\]

\[
= \sum_{i \in I} \pi(i) \int_{\mathcal{X}} \int_{\mathcal{X}} (f(y) - f(x))^2 \pi_i(dx) K_i(x, dy)
\]

\[
\geq 2 \sum_{i \in I_0} \pi(i) \mathcal{E}_i(f, f)
\]

\[
\geq \sum_{i \in I_0} \pi(i) \text{SpecGap}_{B_i}(K_i) \int_{B_i} \int_{B_i} (f(y) - f(x))^2 \pi_i(dx) \pi_i(dy)
\]

\[
\geq \min_{i \in I_0} \left\{ \pi(i) \pi_i(B_i) \text{SpecGap}_{B_i}(K_i) \right\}
\]

\[
\times \sum_{i \in I_0} \frac{1}{\pi_i(B_i)} \int_{B_i} \int_{B_i} (f(y) - f(x))^2 \pi_i(dx) \pi_i(dy).
\]

(6.6)

Using \(\bar{B} = \cup_{i \in I_0} \{i\} \times B_i\), and \(\bar{B}^c \overset{\text{def}}{=} (1 \times \mathcal{X}) \setminus \bar{B}\), we write

\[
2\text{Var}_\pi(f) = \int_{1 \times \mathcal{X}} \int_{1 \times \mathcal{X}} (f(y) - f(x))^2 \bar{\pi}(i, dx) \bar{\pi}(j, dy)
\]

\[
\leq \int_B \int_{\bar{B}} (f(y) - f(x))^2 \bar{\pi}(i, dx) \bar{\pi}(j, dy) + 10\pi(\bar{B}^c)^{1 - \frac{2}{m}},
\]

by using similar calculations as in Lemma 2. And since \(\bar{B}\) is such that \(5\pi(\bar{B}^c)^{1 - \frac{2}{m}} \leq \zeta\), we conclude that

\[
2(\text{Var}_\pi(f) - \zeta) \leq \int_{\bar{B}} \int_{\bar{B}} (f(y) - f(x))^2 \bar{\pi}(i, dx) \bar{\pi}(j, dy),
\]

\[
= \sum_{i \in I_0} \sum_{j \in I_0} \pi(i) \pi(j) \int_{B_i} \int_{B_j} (f(y) - f(x))^2 \pi_i(dx) \pi_j(dy).
\]

\[
= \sum_{i \in I_0} \sum_{j \in I_0} \pi(i) \pi_i(B_i) \pi(j) \pi_j(B_j) \int_{B_i} \int_{B_j} (f(y) - f(x))^2 \frac{\pi_i(dx) \pi_j(dy)}{\pi_i(B_i) \pi_j(B_j)}
\]

(6.7)

For \(i, j \in I_0\), let us write \((i, j)\) to denote the path from \(i\) to \(j\), and given an edge \(e\), let us write \(e\) as \((e_1, e_2)\) where \(e_1\) and \(e_2\) denote the incident nodes of \(e\). By the
Cauchy-Schwarz inequality,
\[
\int_{B_i} \int_{B_j} (f(y) - f(x))^2 \frac{\pi_i(dx) \pi_j(dy)}{\pi_i(B_i) \pi_j(B_j)} \leq \sum_{e \in \{i,j\}} \frac{1}{\min(\pi_e(B_e_1), \pi_e(B_e_2))} \times \sum_{e \in \{i,j\}} \min(\pi_e(B_e_1), \pi_e(B_e_2)) \int_{B_e_1} \int_{B_e_2} (f(y) - f(x))^2 \frac{\pi_e_1(dx) \pi_e_2(dy)}{\pi_e_1(B_e_1) \pi_e_2(B_e_2)}.
\]

By Lemma 12, integral on the right-hand side of the last display is upper bounded by
\[
\left(\frac{2 - \kappa}{2\kappa}\right) \frac{1}{\pi_i(B_i)^2} \int_{B_i} \int_{B_i} (f(y) - f(x))^2 \pi_i(dx) \pi_i(dy) + \left(\frac{2 - \kappa}{2\kappa}\right) \frac{1}{\pi_i(B_i)^2} \int_{B_i} \int_{B_i} (f(y) - f(x))^2 \pi_i(dx) \pi_i(dy).
\]

Therefore the last inequality becomes
\[
\int_{B_i} \int_{B_j} (f(y) - f(x))^2 \frac{\pi_i(dx) \pi_j(dy)}{\pi_i(B_i) \pi_j(B_j)} \leq \frac{D(L_0)}{\min_{i \in I_0} \pi_i(B_i)} \cdot \frac{2}{\kappa} \sum_{i \in I_0} \frac{1}{\pi_i(B_i)} \int_{B_i} \int_{B_i} (f(y) - f(x))^2 \pi_i(dx) \pi_i(dy).
\]

This inequality together with (6.7) and (6.6) gives
\[
\frac{\mathcal{E}(f,f)}{\text{Var}(f)} - \zeta \geq \frac{\kappa}{2D(L_0)} \min_{i \in I_0} \{\pi_i(B_i)^2\} \min_{i \in I_0} \{\pi(i) \text{SpecGap}_{B_i}(K_i)\}.
\]

This concludes the proof. \(\square\)

6.4. Proof of Theorem 8. We introduce the set

\[
(6.8) \quad \mathcal{E} \stackrel{\text{def}}{=} \left\{ z \in \mathbb{R}^n : \sup_{1 \leq j \leq p} \frac{1}{\sigma} \left| \langle L_{\delta_*^{-1}}X_j, z - X\theta_* \rangle \right| \leq 2\sqrt{n \log(p)}, \right. \\
\text{and} \sup_{j: \delta_{s,j} = 1} \sup_{\delta \in \Delta : \|\delta\|_0 \leq s} \frac{1}{\sigma} \left| \langle L_{\delta_*^{-1}}X_j, z - X\theta_* \rangle \right| \leq 2\sqrt{(s + 1)n \log(p)} \}
\]

We note that by Lemma 13, and under H1, \(\mathbb{P}_*(Z \notin \mathcal{E}) \leq C_0/p\), for some absolute constant \(C_0\) that can be taken as \(C_0 = 6\). In the sequel we fix \(z \in \mathcal{E}\).
Let \( \tilde{\theta}_* = \theta_* \cdot \delta_* \), and let \( A_1 \) denote the set of all \( \delta \in \Delta \) such that \( \|\delta\|_0 \leq s \), and \( \delta_j = 0 \) for at least one \( j \in J_* \) (set \( A_1 = \emptyset \) if \( J_* = \emptyset \)), and let \( A_2 \) be the set of all \( \delta \in \Delta \) such that \( s_* + k < \|\delta\|_0 \leq \bar{s} \), where \( s_* \) is some arbitrary integer. \( A_1 \) (resp. \( A_2 \)) is the set of models with false-negatives (resp. false positives).

Given the definition of \( D_k \), we can then write \( \Delta = D_k \cup A_1 \cup A_2 \cup \{ \delta \in \Delta : \|\delta\|_0 > \bar{s} \} \). And since \( \Pi(\{\delta \in \Delta : \|\delta\|_0 > \bar{s}\}|z) = 0 \), we have

\[
(6.9) \quad \Pi(D_k|z) = 1 - \Pi(A_1|z) - \Pi(A_2|z).
\]

Let us assume that \( A_1 \neq \emptyset \), and \( A_2 \neq \emptyset \), otherwise the corresponding probability is simply zero. In order to bound these terms, we start with some general remarks. For any subset \( B \) of \( \Delta \), and \( \delta_0 \in \Delta \),

\[
\Pi(B|z) = \frac{\Pi(\delta_0|z)}{\Pi(\delta_0|z)} \sum_{\delta \in B} \frac{\Pi(\|\delta\|_0-z)}{\Pi(\delta_0|z)} \int_{\mathbb{R}^p} e^{-\frac{1}{2\sigma^2} \|z - Xu\|^2} - \frac{1}{2} u^T D^{-1}_0 u du.
\]

By the determinant lemma \( \det(A + UV') = \det(A) \det(I_m + V'A^{-1}U) \) valid for any invertible matrix \( A \in \mathbb{R}^{n \times n} \), and \( U, V \in \mathbb{R}^{n \times m} \) we have

\[
(\gamma \rho) \frac{\sqrt{\det(\sigma^2 D^{-1}_0 + X'X)}}{\sqrt{\det(\sigma^2 D^{-1}_0 + X'X)}} = \sqrt{\det(I_n + \frac{1}{\sigma^2} XD(\delta_0)X')} \det(I_n + \frac{1}{\sigma^2} XD(\delta_0)X').
\]

By the Woodbury identity ([12] Section 0.7.4) which states that for any set of matrices \( U, V, A, C \) with matching dimensions, \( (A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + \)
\[
VA^{-1}U)^{-1}VA^{-1}, \text{ we have }
\]
\[
X \left( \sigma^2 D_{(\delta)}^{-1} + X'X \right)^{-1} X' = \frac{1}{\sigma^2} XD(\delta) X' - \frac{1}{\sigma^4} XD(\delta) X' \left( I_n + \frac{1}{\sigma^2} XD(\delta) X' \right)^{-1} XD(\delta) X'
\]
\[
= I_n - \left( I_n + \frac{1}{\sigma^2} XD(\delta) X' \right)^{-1}.
\]

Hence,
\[
e^{-\frac{1}{2\sigma^2}z'X \left( \sigma^2 D_{(\delta)}^{-1} + X'X \right)^{-1} X'z} = \frac{e^{-\frac{1}{2\sigma^2}z' \left( I_n + \frac{1}{\sigma^2} XD(\delta) X' \right)^{-1} z}}{e^{-\frac{1}{2\sigma^2}z' \left( I_n + \frac{1}{\sigma^2} XD(\delta) X' \right)^{-1} z}}.
\]

It follows from the above and (6.10) that for all \( \delta_0 \in \Delta, \) and \( B \subseteq \Delta, \)
\[
(6.11) \quad \Pi(B|z) = \Pi(\delta_0|z) \sum_{\delta \in B} \frac{\omega_{\delta}}{\omega_{\delta_0}} \sqrt{\frac{\det(L_{\delta_0})}{\det(L_{\delta})}} \frac{1}{e^{\frac{1}{2\sigma^2}z' L_{\delta_0}^{-1} z}} \frac{1}{e^{\frac{1}{2\sigma^2}z' L_{\delta}^{-1} z}},
\]

where, for \( \delta \in \Delta, \) we recall the definition \( L_{\delta} \overset{\text{def}}{=} I_n + \frac{1}{\sigma^2} XD(\delta) X'. \) Suppose that we have \( \vartheta, \delta \in \Delta \) such that \( \vartheta \supseteq \delta. \) Setting \( \tau \overset{\text{def}}{=} \frac{1}{\sigma^2} \left( \frac{1}{\rho} - \gamma \right), \) it is easily seen that
\[
(6.12) \quad L_{\vartheta} = L_{\delta} + \tau \sum_{j: \vartheta_j = 0, \theta_j = 1} X_j X'_j.
\]

Therefore by the determinant lemma,
\[
(6.13) \quad \frac{\det(L_{\vartheta})}{\det(L_{\delta})} = \det \left( I_{\|\vartheta - \delta\|_0} + \tau X'_{(\vartheta - \delta)} L_{\delta}^{-1} X_{(\vartheta - \delta)} \right).
\]

And by the Woodbury identity,
\[
(6.14) \quad L_{\vartheta}^{-1} = L_{\delta}^{-1} - \tau L_{\delta}^{-1} X_{(\vartheta - \delta)} \left( I_{\|\vartheta - \delta\|_0} + \tau X'_{(\vartheta - \delta)} L_{\delta}^{-1} X_{(\vartheta - \delta)} \right)^{-1} X'_{(\vartheta - \delta)} L_{\delta}^{-1}.
\]

Control of the term \( \Pi(A_1|Z). \) If \( J_* = \emptyset, \) then \( A_1 = \emptyset, \) and \( \Pi(A_1|z) = 0. \) So we may assume that \( J_* \neq \emptyset. \) Given \( \delta \in A_1, \) let \( \tilde{\delta} \) be the element of \( \Delta \) obtained by adding to \( \delta \) the active component of \( \delta_* \) that are missing from \( \delta: \tilde{\delta}_j = 1 \) for \( j \in J_* \), and \( \tilde{\delta}_j = \delta_j \) for \( j \notin J_* \). We have
\[
\Pi(A_1|z) = \sum_{\delta \in A_1} \Pi(\tilde{\delta}|z) \frac{\Pi(\delta|z)}{\Pi(\tilde{\delta}|z)} \leq \sup_{\delta \in A_1} \frac{\Pi(\delta|z)}{\Pi(\tilde{\delta}|z)} \sum_{\delta \in A_1} \Pi(\tilde{\delta}|z)
\]
\[
\leq \sup_{\delta \in A_1} \frac{\Pi(\delta|z)}{\Pi(\tilde{\delta}|z)} 2^{\tilde{\delta}_* \Pi(\{\tilde{\delta}_*\} \cup A_2|z)} \leq 2^{\tilde{\delta}_*} \sup_{\delta \in A_1} \frac{\Pi(\delta|z)}{\Pi(\tilde{\delta}|z)},
\]
where $\tilde{s}_* \overset{\text{def}}{=} \|\tilde{s}_*\|_0$. We apply (6.11) with $\delta_0 = \tilde{\delta}$, and $B = \{\delta\}$, with $\|\delta\|_0 - \|\delta\|_0 = k$ say, to get

$$\Pi(\delta|z) \leq \frac{\omega_{\delta}}{\omega_{\tilde{\delta}}} \sqrt{\frac{\det(L_{\tilde{\delta}})}{\det(L_{\delta})}} e^{\frac{1}{2\sigma^2} z' L_{\tilde{\delta}}^{-1} z}.$$

$$\leq p^{(1+u)k} \left( 1 + \frac{n\bar{v}(s_*)}{\sigma^2 \rho} \right) \frac{1}{2} e^{-\frac{r}{2\sigma^2} z' L_{\tilde{\delta}}^{-1} z} \left( 1 + r X_{(\delta-\delta_0)} L_{\tilde{\delta}}^{-1} L_{(\delta-\delta)} \right) X_{(\delta-\delta)} L_{\tilde{\delta}}^{-1} z$$

$$\leq p^{(1+u)k} \left( 1 + \frac{n\bar{v}(s_*)}{\sigma^2 \rho} \right) \frac{1}{2} e^{-\frac{r}{2\sigma^2} (1 + n \rho(\delta_0))} \|X_{(\delta-\delta)} L_{\tilde{\delta}}^{-1} z\|_2^2.$$

We note that for all real numbers $a, b$ such that $|b| \leq |a|/2$, we have $(a+b)^2 \geq (a/2)^2$. We will use this to lower bound the term $(X_j' L_{\tilde{\delta}}^{-1} z)^2$. Given $j$ such that $\tilde{\delta}_j = 1$ and $\delta_j = 0$, with $v = (z - X\theta_*)/\sigma$, we have

$$X_j' L_{\tilde{\delta}}^{-1} z = \sigma X_j' L_{\tilde{\delta}}^{-1} v + \theta_{*,j} X_j' L_{\tilde{\delta}}^{-1} X_j + \sum_{\ell: \delta_{*,\ell} = 1, \ell \neq j} \theta_{*,\ell} X_j' L_{\tilde{\delta}}^{-1} X_{\ell}.$$

For $z \in E$, we have

$$\left| \sigma X_j' L_{\tilde{\delta}}^{-1} v + \sum_{\ell: \delta_{*,\ell} = 1, \ell \neq j} \theta_{*,\ell} X_j' L_{\tilde{\delta}}^{-1} X_{\ell} \right| \leq 2\sigma \sqrt{(s_{k_0} + 1)n \log(p)} + \|\theta_*\|_1 C_X,$$

and using $\varrho$,

$$\left| \frac{\theta_{*,j}}{2} X_j' L_{\tilde{\delta}}^{-1} X_j \right| \geq \frac{n \varrho \theta_{*,j}}{2} \geq 2\sigma \sqrt{(s_{k_0} + 1)n \log(p)} + \|\theta_*\|_1 C_X,$$

where the second inequality uses (4.11). Therefore, for $z \in E$,

$$(X_j' L_{\tilde{\delta}}^{-1} z)^2 \geq \left( \frac{\theta_{*,j}}{2} X_j' L_{\tilde{\delta}}^{-1} X_j \right)^2 \geq \frac{n^2 \varrho^2 \theta_{*,j}^2}{4}.$$

Hence

$$\Pi(A_1|z) \leq 2^k p^{(1+u)} \left( 1 + \frac{n\bar{v}(s_*)}{\sigma^2 \rho} \right) \frac{1}{2} e^{\frac{1}{16\sigma^2 \rho^2}} \leq 2^k e^{-\frac{n \varrho^2 \theta_{*,j}^2}{32\sigma^2 \rho^2}} \leq 1 \leq p^{u(k+1)/2},$$

where the last two inequalities follow from (4.11).
Control of the term $\Pi(A_2|z)$. The argument is the same as in the case of $\Pi(||\delta||_0 > s_{k_0}|z)$. We apply (6.11) to get

$$\Pi(A_2|z) \leq \sum_{\delta \in A_2} \frac{\omega_\delta}{\omega_\delta} \sqrt{\frac{\det(L_{\tilde{\delta}_*})}{\det(L_{\delta})}} \frac{e^{2\sigma^2 z'|L_{\tilde{\delta}_*}^{-1}z'}}{e^{2\sigma^2 z'|L_{\delta}^{-1}z'}}.$$  

We note from (6.13) that since $\delta \supseteq \tilde{\delta}_*$,

$$\frac{\det(L_{\tilde{\delta}_*})}{\det(L_{\delta})} \leq 1.$$

Using (6.14),

$$\frac{e^{2\sigma^2 z'|L_{\tilde{\delta}_*}^{-1}z'}}{e^{2\sigma^2 z'|L_{\delta}^{-1}z'}} = e^{-\sigma^2 z'|L_{\delta}^{-1}X_{(\delta - \tilde{\delta}_*)}X_{(\delta - \tilde{\delta}_*)}^{-1}L_{\delta}^{-1}z'}.$$  

Writing $z = X\theta_* + \sigma v$, where $v = (z - X\theta_*)/\sigma$, we have

$$\left( I_{\|\delta - \tilde{\delta}_*\|_0} + \tau X'_{(\delta - \tilde{\delta}_*)}L_{\delta_*}^{-1}X_{(\delta - \tilde{\delta}_*)} \right)^{-1/2} X'_{(\delta - \tilde{\delta}_*)}L_{\delta_*}^{-1}z$$

$$= \sum_{\ell, \delta_* \epsilon = 1} \theta_* \ell \left( I_{\|\delta - \tilde{\delta}_*\|_0} + \tau X'_{(\delta - \tilde{\delta}_*)}L_{\delta_*}^{-1}X_{(\delta - \tilde{\delta}_*)} \right)^{-1/2} X'_{(\delta - \tilde{\delta}_*)}L_{\delta_*}^{-1}X_{\ell}$$

$$+ \sigma \left( I_{\|\delta - \tilde{\delta}_*\|_0} + \tau X'_{(\delta - \tilde{\delta}_*)}L_{\delta_*}^{-1}X_{(\delta - \tilde{\delta}_*)} \right)^{-1/2} X'_{(\delta - \tilde{\delta}_*)}L_{\delta_*}^{-1}v.$$  

For $z \in \mathcal{E}$,

$$\left\| \left( I_{\|\delta - \tilde{\delta}_*\|_0} + \tau X'_{(\delta - \tilde{\delta}_*)}L_{\delta_*}^{-1}X_{(\delta - \tilde{\delta}_*)} \right)^{-1/2} X'_{(\delta - \tilde{\delta}_*)}L_{\delta_*}^{-1}v \right\|_2 \leq \frac{1}{\sqrt{1 + \tau \eta n}} \left\| X'_{(\delta - \tilde{\delta}_*)}L_{\delta_*}^{-1}v \right\|_2$$

$$= 2\sqrt{\|\delta\|_0 - \tilde{\delta}_*} \sqrt{\frac{n \log(p)}{1 + \tau \eta n}}.$$  

With similar calculations we get

$$\left\| \sum_{\ell, \delta_* \epsilon = 1} \theta_* \ell \left( I_{\|\delta - \tilde{\delta}_*\|_0} + \tau X'_{(\delta - \tilde{\delta}_*)}L_{\delta_*}^{-1}X_{(\delta - \tilde{\delta}_*)} \right)^{-1/2} X'_{(\delta - \tilde{\delta}_*)}L_{\delta_*}^{-1}X_{\ell} \right\|_2$$

$$\leq \frac{\sqrt{\|\delta\|_0 - \tilde{\delta}_*}}{1 + \tau \eta n} (\|\theta_*\|_1 C_X + n\epsilon).$$
It follows from the last two inequalities that for \( z \in \mathcal{E} \),

\[
\left\| \left( I_{\| \delta - \delta_0 \|_0} + \tau X'_{(\delta - \tilde{\delta}_2)} L_{\tilde{\delta}_2}^{-1} X_{(\delta - \delta_0)} \right)^{-1/2} X'_{(\delta - \tilde{\delta}_2)} L_{\tilde{\delta}_2}^{-1} z \right\|_2 \\
\leq \sqrt{\| \delta \|_0 - \tilde{s}_* + n \epsilon + C_{\chi} \| \theta \|_1 + 2\sigma n \log(p)}.
\]

Hence, setting \( a_2 \overset{\text{def}}{=} \frac{5}{\theta} \left( 1 + \frac{\| \theta \|_1 C_{\chi}}{3\sigma \sqrt{n \log(p)}} \right)^2 \), and noting that \( a_2 \leq \frac{u}{2} \) by assumption, we can do the following calculations

\[
\Pi(A_2 | z) \leq \sum_{\delta : \delta \in A_2} \frac{\omega_{\delta}}{\omega_{\tilde{s}_*}} p_{\tilde{s}_*} (\| \delta \|_0 - \tilde{s}_*) \\
= \sum_{j = k+1}^{\tilde{s}_* - \tilde{s}_*} \sum_{\delta : \delta \supset \tilde{\delta}_2, \| \delta \|_0 = \tilde{s}_* + j} \frac{\omega_{\delta}}{\omega_{\tilde{\delta}_2}} p_{\tilde{s}_*} \left( \frac{1}{p_{u+1}} \right)^j p_{\tilde{s}_*} \\
\leq \frac{2}{p^{u(k+1)/2}}.
\]

The ends the proof. \( \square \)

6.5. **Proof Theorem 9.** We recall that the initial distribution is taken as \( \nu_0 = \Pi(\cdot | \delta^{(i)} z) \), for some initial choice \( \delta^{(i)} \). Let

\[
f_0(\theta) \overset{\text{def}}{=} \frac{\nu_0(\theta)}{\Pi(\theta | z)}, \theta \in \mathbb{R}^p,
\]

be the density of \( \nu_0 \) with respect to \( \Pi(\cdot | z) \). Since \( \Pi(\theta | z) \geq \Pi(\delta^{(i)} | z) \Pi(\theta | \delta^{(i)} z) \), we have

\[
f_0(\theta) = \frac{\Pi(\theta | \delta^{(i)} z)}{\Pi(\theta | z)} \leq \frac{1}{\Pi(\delta^{(i)} | z)}.
\]

Hence \( f_0 \) is bounded. Therefore by Lemma 1 (where we take \( \| \cdot \|_* = \| \cdot \|_\infty \), and \( \zeta = 5 (1 - \Pi(D_k | z)) \)), for all integer \( N \geq 1 \), and all \( z \in \mathbb{R}^n \), we have

\[
\| \nu_0 K^N - \Pi(\cdot | z) \|_{tv} \leq \left( \| f_0 \|_\infty^2 (1 - \text{SpecGap}_{\zeta}(K)) + \zeta \| f_0 \|_\infty^2 \right)^{1/2}.
\]
For $z \in \mathcal{E}(k, \zeta_0)$, we have

$$\zeta\|f_0\|_\infty^2 \leq \frac{5(1 - \Pi(D_k|z))}{\Pi(\delta^0|z)^2} \leq \frac{\zeta_0^2}{2}.$$  

It follows from (6.15) that for

$$N \geq \left(\log \left(\frac{2}{\zeta_0^2}\right) + 2\log \left(\frac{1}{\Pi(\delta^0|z)}\right)\right) \frac{1}{\text{SpecGap}_{K}(K)},$$

we have $\|\nu_0 K^N - \Pi(\cdot|z)\|_{tv} \leq \sqrt{\frac{\zeta_0^2}{2} + \frac{\zeta_0^2}{2}} = \zeta_0$. So it remains only to lower bound $\text{SpecGap}_{K}(K)$.

**Lower bound on $\text{SpecGap}_{K}(K)$.** We apply Theorem 3 with the obvious choices $l = \Delta$, $l_0 = D_k$, and $B_\delta = \mathbb{R}^p$, and $m = +\infty$. By choice of $\zeta$, we have $\Pi(D_k|z) = 1 - (\zeta/5)$. We consider the follow graph on $l_0$; we link $\delta_1$ and $\delta_2$ if $\delta_1 \supseteq \delta_2$, or $\delta_2 \supseteq \delta_1$, and $\|\delta_2 - \delta_1\|_0 = 1$. To apply Theorem 3 it remain only to find $\kappa > 0$ such that for all $\delta_1, \delta_2 \in D_k$, such that if $\delta_1 \supseteq \delta_2$, or $\delta_2 \supseteq \delta_1$, and $\|\delta_2 - \delta_1\|_0 = 1$ we have

(6.16) \[ \int_{\mathbb{R}^p} \min(\Pi(\theta|\delta_1, z), \Pi(\theta|\delta_2, z)) d\theta \geq \kappa. \]

Suppose that $\delta_2 \supseteq \delta_1$. Then

$$\frac{\Pi(\theta|\delta_2, z)}{\Pi(\theta|\delta_1, z)} = \frac{\int_{\mathbb{R}^p} e^{-\frac{1}{\sigma^2}||z - X\theta||^2 + \frac{1}{2}\theta^t D^{-1}_1 \theta} d\theta}{\int_{\mathbb{R}^p} e^{-\frac{1}{\sigma^2}||z - X\theta||^2 + \frac{1}{2}\theta^t D^{-1}_2 \theta} d\theta} \geq \frac{\int_{\mathbb{R}^p} e^{-\frac{1}{\sigma^2}||z - X\theta||^2 + \frac{1}{2}\theta^t D^{-1}_1 \theta} d\theta}{\int_{\mathbb{R}^p} e^{-\frac{1}{\sigma^2}||z - X\theta||^2 + \frac{1}{2}\theta^t D^{-1}_2 \theta} d\theta}.$$  

Using (6.10), (6.13), and (6.13), we have

$$\int_{\mathbb{R}^p} e^{-\frac{1}{2\sigma^2}||z - X\theta||^2 + \frac{1}{2}\theta^t D^{-1}_1 \theta} d\theta \geq (\gamma \rho) e^{-\frac{\tau X'_{(\delta_2 - \delta_1)} X_{(\delta_2 - \delta_1)} L_{\delta_1}^{-1} L_{\delta_1}^{-1} X'_{(\delta_2 - \delta_1)} L_{\delta_1}^{-1} L_{\delta_1}^{-1} X_{(\delta_2 - \delta_1)}} = \gamma \rho e^{-\frac{\tau X'_{(\delta_2 - \delta_1)} L_{\delta_1}^{-1} z \|2^2}{2\sigma^2(1 + n_\rho)}}.$$  

As seen before in the proof of Theorem 8,

$$\|X'_{(\delta_2 - \delta_1)} L_{\delta_1}^{-1} z \|_2^2 \leq \left(3\sigma \sqrt{n \log(p)} + C_X \|\theta_*\|_1\right)^2.$$  

It follows easily that

$$\int_{\mathbb{R}^p} e^{-\frac{1}{2\sigma^2}||z - X\theta||^2 + \frac{1}{2}\theta^t D^{-1}_1 \theta} d\theta \geq (\gamma \rho) e^{-\frac{3}{\sigma} \left(3\sigma \sqrt{n \log(p)} + C_X \|\theta_*\|_1\right)^2}.$$
Hence we can apply Theorem 3 with 

\[ \kappa = (\gamma \rho) p^{-\frac{2}{3} \left(1 + \frac{c_X \|\theta^*\|_1}{3e \sqrt{n \log(p)}}\right)^2}. \]

The diameter of the graph thus constructed is 2k. We conclude from the above and Theorem 3 that for \( z \in \mathcal{E} \),

\[ \text{SpecGap}_\xi(K) \geq \frac{(\gamma \rho)}{4k} p^{-\frac{2}{3} \left(1 + \frac{c_X \|\theta^*\|_1}{3e \sqrt{n \log(p)}}\right)^2} \min_{\delta \in D_k} \pi(\delta|z). \]

This completes the proof. \( \square \)

**APPENDIX A: SOME TECHNICAL RESULTS**

We make use of the following standard Gaussian deviation bound.

**Lemma 13.** Let \( Z \sim N(0, I_m) \), and \( u_1, \ldots, u_N \) be vectors of \( \mathbb{R}^m \). Then for all \( x \geq 0 \),

\[ \mathbb{P} \left[ \max_{1 \leq j \leq N} \left| \langle u_j, Z \rangle \right| > \max_{1 \leq j \leq N} \|u_j\|_2 \sqrt{2(x + \log(N))} \right] \leq \frac{2}{e^x}. \]

The next result gives a bound on \( C_X \), and shows that H2 holds with high probability in the case of a Gaussian ensemble.

**Lemma 14.** Suppose that \( X \in \mathbb{R}^{n \times p} \) is a random matrix with i.i.d. standard Normal entries. Given an integer \( s \), and positive constants \( \sigma, \gamma \) and \( \rho \), set

\[ C_X \overset{\text{def}}{=} \max_{\delta \in \Delta, \|\delta\|_0 \leq s} \max_{i \neq j, \delta_i = 0} \left| X_j' \left( I_n + \frac{1}{\sigma^2 \rho} X_{\delta} X_{\delta}^t + \frac{\gamma}{\sigma^2} X_{\delta}^c X_{\delta}^c \right) X_i \right|. \]

Then there exist some universal finite constants \( c_0, a, A \) such that for \( n \geq As^2 \log(p) \), the following two statements hold with probability at least \( 1 - \frac{a}{p} \):

(A.1) \( C_X \leq 2c_0 \sqrt{n \log(p)} \), and

\[ \min_{\delta: \|\delta\|_0 \leq s} \inf \left\{ \frac{u'(X_{\delta}^t L_{\delta} X_{\delta}^c) u}{n \|u\|_2^2}, \ u \in \mathbb{R}^p, \ \delta^c \supseteq \text{supp}(u), \ 0 < \|\text{supp}(u)\|_0 \leq s \right\} \geq \frac{1}{32}, \]

provided that

(A.2) \( \sigma^2 \rho \leq c_0 \sqrt{n \log(p)}, \ \gamma p \sqrt{n \log(p)} \leq \frac{\sigma^2}{2c_0}, \) and \( \frac{\gamma n^{3/2}}{\sqrt{\log(p)}} \leq \frac{c_0 \sigma^2}{8}. \)
Proof. For a matrix \( M \in \mathbb{R}^{n \times p} \) we set
\[
v(M, s) \overset{\text{def}}{=} \inf \left\{ \frac{u'(M'M)u}{n\|u\|_2^2} \mid u \neq 0, \|u\|_0 \leq s \right\},
\]
and for \( \kappa_0 = 1/64 \) and \( c_0 = 8 \), we define
\[
\mathcal{E} \overset{\text{def}}{=} \left\{ M \in \mathbb{R}^{n \times p} : v(M, s) \geq \kappa_0, \max_{1 \leq j \leq p} \|M_j\|_2 \leq 2\sqrt{n}, \min_{1 \leq j \leq p} \|M_j\|_2 \geq \sqrt{n}, \text{ and } \max_{j \neq k} |\langle M_j, M_k \rangle| \leq c_0 \sqrt{n \log(p)} \right\}.
\]
By Theorem 1 of [26], Lemma 1-(4.2) of [15], and standard Gaussian deviation bounds, we can find universal constants \( a, A \), such that for \( n \geq As \log(p) \), we have
\[
P(X \not\in \mathcal{E}) \leq a p.
\]
So to obtained the statement of the lemma, it suffices to consider some arbitrary element \( X \in \mathcal{E} \) and show that (A.1) holds.

Fix \( \delta \in \Delta \) such that \( \|\delta\|_0 \leq s \). We set \( M_{\delta} \overset{\text{def}}{=} I_n + \frac{1}{\sigma^2 \rho} X_\delta X'_\delta \), so that \( L_{\delta} = M_{\delta} + \frac{\gamma}{\sigma^2} X_\delta c' X'_\delta \). The Woodbury identity gives
\[
(A.3) \quad L_{\delta}^{-1} = M_{\delta}^{-1} - \frac{\gamma}{\sigma^2} M_{\delta}^{-1} X_\delta c \left( I_{\|\delta\|_0} + \frac{\gamma}{\sigma^2} X'_\delta M_{\delta}^{-1} X_\delta c \right)^{-1} X'_\delta M_{\delta}^{-1}.
\]
Hence, for any \( j, k \),
\[
(A.4) \quad X'_j L_{\delta}^{-1} X_k = X'_j M_{\delta}^{-1} X_k - \frac{\gamma}{\sigma^2} X'_j M_{\delta}^{-1} X_\delta c \left( I_{\|\delta\|_0} + \frac{\gamma}{\sigma^2} X'_\delta M_{\delta}^{-1} X_\delta c \right)^{-1} X'_\delta M_{\delta}^{-1} X_k.
\]
If \( C_1 = \max_{i} X'_i M_{\delta}^{-1} X_i \), and \( C_0 = \max_{i \neq j, \delta_j = 0} |X'_i M_{\delta}^{-1} X_j| \), then we deduce easily from (A.4) that for all \( j \neq k \) such that \( \delta_j = 0 \),
\[
(A.5) \quad |X'_j L_{\delta}^{-1} X_k| \leq C_0 + \frac{\gamma}{\sigma^2} \left( C_1^2 + p C_0^2 \right).
\]
In order to proceed, we need to bound the term \( X_j M_{\delta}^{-1} X_k \). Easily, for \( X \in \mathcal{E} \), we have
\[
X_j M_{\delta}^{-1} X_k \leq X_j^2 |X_k|^2 \leq 4n.
\]
Another application of the Woodbury identity gives
\[
(A.6) \quad M_{\delta}^{-1} = I_n - \frac{1}{\sigma^2 \rho} X_\delta \left( I_{\|\delta\|_0} + \frac{1}{\sigma^2 \rho} X'_\delta X_\delta \right)^{-1} X'_\delta.
\]
If \( X_\delta = U \Lambda V' \) is the singular value decomposition of \( X_\delta \), with positive singular values \( \lambda_1 \geq \lambda_2 \ldots \geq \lambda_{\| \delta \|_0} \), and if \( P_{\perp} \) denotes the projector on the space orthogonal to the span of \( X_\delta \), we have

\[
M_\delta^{-1} = P_{\perp} + \sum_{\ell=1}^{\| \delta \|_0} \frac{\sigma^2 \rho}{\sigma^2 \rho + \lambda_\ell} U_\ell U_\ell'.
\]

We note that for \( X \in \mathcal{E} \), \( \lambda_2 \| \delta \|_0 \geq \kappa_0 n \). Therefore, for \( k \neq j \), and using the above,

\[
|X_j^t M_\delta^{-1} X_k| \leq |\langle X_j, P_{\perp}(X_k) \rangle| + \frac{\sigma^2 s \rho}{\kappa_0} \leq 2c_0 \sqrt{n \log(p)},
\]

provided that \( \sigma^2 s \rho \leq c_0 \kappa_0 \sqrt{n \log(p)} \). We combine this with (A.5) to obtain that for \( j \neq k \) such that \( \delta_j = 0 \),

\[
\text{(A.7)} \quad |X_j^t L_\delta^{-1} X_k| \leq 3c_0 \sqrt{n \log(p)} \left( 1 + \frac{\gamma}{\sigma^2} pc_0 \sqrt{n \log(p)} \right) + 16 \frac{\gamma}{\sigma^2} n^2 \leq 8c_0 \sqrt{n \log(p)},
\]

using (A.2). (A.7) says that \( C_X \leq 8c_0 \sqrt{n \log(p)} \), for \( X \in \mathcal{E} \), as claimed.

For \( j \) such that \( \delta_j = 0 \), (A.6) gives

\[
X_j^t M_\delta^{-1} X_j = \|X_j\|^2 - \frac{1}{\sigma^2 \rho} X_j^t X_\delta \left( I_{\| \delta \|_0} + \frac{1}{\sigma^2 \rho} X_\delta^t X_\delta \right)^{-1} X_\delta^t X_j
\]

\[
\geq \|X_j\|^2 - \frac{1}{\sigma^2 \rho} \|X_\delta^t X_j\|_2^2 \frac{1 + \frac{\kappa_0}{\sigma^2 \rho}}{n \kappa_0}
\]

\[
\geq \|X_j\|^2 - \frac{\|X_\delta^t X_j\|_2^2}{n \kappa_0}
\]

\[
\geq \|X_j\|^2 - \frac{sc_0^2 \log(p)}{\kappa_0}
\]

\[
\geq \frac{n}{4},
\]

since \( n \geq A \log(p) \), and by taking \( A \) large enough (\( A \geq 4c_0^2 / \kappa_0 \)). Equation (A.3) then yields

\[
X_j^t L_\delta^{-1} X_j \geq X_j^t M_\delta^{-1} X_j - \frac{\gamma}{\sigma^2} \|X_\delta^t M_\delta^{-1} X_j\|_2^2
\]

\[
= X_j^t M_\delta^{-1} X_j - \frac{\gamma}{\sigma^2} \left[ (X_j^t M_\delta^{-1} X_j)^2 + \sum_{k: \delta_k = 0, k \neq j} (X_j^t M_\delta^{-1} X_k)^2 \right].
\]
For $2\gamma \leq \sigma^2$, it follows that

$$X_j' L_{\delta}^{-1} X_j \geq \frac{n}{8} - \frac{\gamma}{\sigma^2} (p - \|\delta\|_0) \left(4c_0^2 n \log(p)\right),$$

which together with (A.7) and (A.2) implies that for any $u \in \mathbb{R}^p$ such that $\delta^c \supseteq \text{supp}(u)$, and $\|\text{supp}(u)\|_0 \leq s$, we have

$$u' X_j' L_{\delta}^{-1} X_j^c u \geq \frac{n}{32} \|u\|_2^2,$$

as claimed. 

\[\square\]

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