Determine Arbitrary Feynman Integrals by Vacuum Integrals

Xiao Liu\textsuperscript{1} and Yan-Qing Ma\textsuperscript{1,2,3}

\textsuperscript{1} School of Physics and State Key Laboratory of Nuclear Physics and Technology, Peking University, Beijing 100871, China
\textsuperscript{2} Center for High Energy Physics, Peking University, Beijing 100871, China
\textsuperscript{3} Collaborative Innovation Center of Quantum Matter, Beijing 100871, China

(Dated: February 1, 2018)

By introducing an auxiliary parameter, we find a series representation for Feynman integral, which is defined as analytical continuation of a calculable series. To obtain the series representation, one only needs to deal with some much simpler vacuum integrals. The series representation therefore translates the problem of computing Feynman integrals to the problem of performing analytical continuations. As a Feynman integral is fully determined by its series representation, its reduction relation to master integrals can be achieved. Furthermore, differential equations of master integrals with respective to the auxiliary parameter can also be set up. By solving the differential equations, the desired analytical continuations are realized. The series representation thus provides a novel method to reduce and compute Feynman integrals.

\textbf{Introduction.} — Computation of Feynman loop integrals is in the heart of modern physics, which is important both for testing the particle physics Standard Model and for discovering new physics. A good method to compute one-loop integrals was proposed as early as in 1970s, the strategy of which is to first express scattering amplitudes in terms of linear combinations of master integrals (MIs) and then compute these MIs\textsuperscript{1–3}. Based on this method, one can compute one-loop scattering amplitudes systematically and efficiently if the number of external legs is no more than 4. With further improvement of the traditional tensor reduction\textsuperscript{4} and the development of unitarity-based reduction\textsuperscript{5–7}, computation of multi-leg one-loop scattering amplitudes is also a solved problem right now.

Yet, about 40 years later, it is still a challenge to compute multi-loop integrals, even for two-loop integrals with 4 external legs. The mainstream approach to calculate multi-loop integrals in literature is similar to that at one-loop level, by first reducing Feynman integrals to MIs and then calculating these MIs. However, both of the two steps are much harder to achieve than one-loop case. Although compact and explicit expressions for one-loop MIs can be easily obtained\textsuperscript{2,3}, the computation of multi-loop MIs is very challenging. There are many methods in literature to compute multi-loop MIs, such as the sector decomposition\textsuperscript{8}, Mellin-Barnes representation\textsuperscript{9,10}, and the differential equation method\textsuperscript{11–14}, but none of them provides a satisfactory solution. In Ref.\textsuperscript{15}, we proposed a systematic and efficient method to calculate multi-loop MIs by constructing and numerically solving a system of ordinary differential equations (ODEs). The differential variable, say $\eta$, is an auxiliary parameter introduced to all Feynman propagators. With the ODEs, physical results at $\eta = 0^+$ are fully determined by boundary conditions chosen at $\eta = \infty$, which can be obtained almost trivially. Therefore, MIs can be treated as special functions as far as there is a good reduction method to set up ODEs.

Reduction of multi-loop integrals is an even harder problem. Significantly different from one-loop case, propagators in a multi-loop integral are usually not enough to form a complete set to expand all independent scalar products, either between a loop momentum and an external momentum or between two loop momenta. As a consequence, the unitarity-based multi-loop reduction\textsuperscript{16–20} has difficulty to fully reduce scattering amplitudes. Although the integration-by-parts (IBP) reduction\textsuperscript{27–31} is general enough to reduce any scattering amplitude to MIs, the incompleteness of multi-loop propagators makes it hard to generate efficient reduction relations. Currently, IBP reduction is mainly based on Laporta’s algorithm\textsuperscript{25}, which is a brute force algorithm that becomes extremely inefficient for slightly complicated problems. E.g., it cannot give a complete reduction for Higgs pair hadroproduction at two-loop order\textsuperscript{32}. Improvements for IBP reduction method can be found in\textsuperscript{33–34} and references therein.

To get a satisfactory solution for multi-loop calculation, new ideas seem to be indispensable. Inspired by our previous work\textsuperscript{15}, in this Letter we construct a novel method to compute Feynman loop integrals. The most important observation is that, with the introduction of the auxiliary parameter $\eta$, any Feynman integral can be defined as the analytical continuation of a calculable asymptotic series, which we call series representation of the Feynman integral. As the series contains only vacuum integrals, it can be easily computed order by order. This new representation therefore translates the problem of computing Feynman integrals to the problem of performing analytical continuations. The series representation can also be used to set up reduction relation between any Feynman integral and MIs, as while as to set up ODEs for the MIs. Setting up reduction relations using series representation has an advantage that the incompleteness of multi-loop propagators does not introduce any difficulty. With the reduction relations and ODEs, the desired analytical continuations can be achieved eas-
ily. With a two-loop example, we show that our method can be much more efficient than existing ones.

A series representation for Feynman integral.
— Following Ref. [15], we introduce a dimensionally regularized $L$-loop Feynman integral with an auxiliary parameter $\eta$,

$$\mathcal{M}(D, \vec{s}, \eta) \equiv \int \frac{d^D q_i}{(2\pi)^D/2} \prod_{\alpha=1}^{N(L)} \left( \frac{1}{(D_\alpha + \eta)^{s_\alpha}} \right),$$

(1)

where $D$ is the spacetime dimension, $D_\alpha \equiv q_\alpha^2 - m_\alpha^2$ are usual inverse Feynman propagators with $m_\alpha$ being corresponding masses and $q_\alpha$ being linear combinations of loop momenta $\ell_i$ and external momenta $p_i$. The expansion can be interpreted as vacuum integrals with present in denominators anymore, thus each term of the expansion, all external momenta and masses do not depend on $\eta$. Therefore, all propagators can be expanded like

$$\mathcal{M}(D, \vec{s}, 0) \equiv \lim_{\eta \to 0^+} \mathcal{M}(D, \vec{s}, \eta),$$

(2)

with $0^+$ defining the causality of Feynman amplitudes.

The study in Ref. [15] shows that, as $\eta \to \infty$, there is only one integration region for $\mathcal{M}(D, \vec{s}, \eta)$, where all components of loop momenta are at the order of $|\eta|^{1/2}$. Therefore, all propagators can be expanded like

$$\frac{1}{(\ell + p)^2 - m^2 + i\eta} = \sum_{j=0}^{\infty} \left( \frac{-2\ell \cdot p + p^2 - m^2}{\ell^2 + i\eta} \right)^j,$$

(3)

where $\ell$ is a linear combination of loop momenta $\ell_i$, and $p$ is a linear combination of external momenta $p_i$. After the expansion, all external momenta and masses do not present in denominators anymore, thus each term of the expansion can be interpreted as vacuum integrals with equal internal squared masses $-i\eta$. Inserting Eq. (3) into Eq. (1) and rescaling all loop momenta by $|\eta|^{1/2}$, we get an asymptotic expansion around $\eta \to \infty$,

$$\mathcal{M}(D, \vec{s}, \eta) = \eta^{L/2 - \sum_m^\infty \eta^{-\mu_m} \mathcal{M}_{\mu_m}^{\text{bub}}(D, \vec{s})},$$

(4)

where $\mathcal{M}_{\mu_0}^{\text{bub}}(D, \vec{s})$ consist of vacuum integrals with equal internal squared masses $-i$, which can be easily expressed as linear combinations of vacuum MIs, $\{I_k^{\text{bub}}(D), \ldots, I_k^{\text{bub}}(D)\}$. Here $B_L$ is the total number of $L$-loop equal-mass vacuum MIs, with $B_1 = 1$, $B_2 = 2$, $B_3 = 6$ and so on. Thus, we have the decomposition

$$\mathcal{M}_{\mu_0}^{\text{bub}}(D, \vec{s}) = \sum_{k=1}^{B_L} I_k^{\text{bub}}(D) \sum_{\vec{\mu} \in \Omega_{\mu_0}^L} C_{k}^{\mu_0 \cdots \mu_r}(D) s^{\mu_1_1} \cdots s^{\mu_r_r},$$

(5)

where $\vec{\mu}$ is a $r$-dimensional vector in $\Omega_{\mu_0}^L \equiv \{\vec{\mu} \in \mathbb{N}^r | \mu_1 + \cdots + \mu_r = \mu_0\}$, $C_{k}^{\mu_0 \cdots \mu_r}(D)$ are fractional polynomials of $D$.

As $I_{L,k}^{\text{bub}}(D)$ can be easily calculated [35–40], the series (4) defines an analytical function around $\eta = \infty$, which therefore determines $\mathcal{M}(D, \vec{s}, \eta)$ for any value of $\eta$ based on analytical continuation. Especially, the desired physical value at $\eta = 0^+$ is fully determined. As a result, the expression (4) can be thought as a series representation of $\mathcal{M}(D, \vec{s}, 0)$. We note that the series representation can be applied not only for individual scalar Feynman integrals, but also for tensor integrals and scattering amplitudes. As far as we know, this is the only series representation of Feynman integral in literature. All other representations, such as Feynman (or Schwinger) parametric representation [41], the Baikov representation [42] and Mellin-Barnes representation [43], which have played important roles in the study of Feynman integrals, are integral representations.

Comparing with integral representations, there is a nice feature of the series representation. As values of the series representation in the $\eta \to \infty$ region can be easily computed order by order in $1/\eta$ expansion, by analytical continuation one can obtain physical value at $\eta = 0^+$. Thus the problem of computing Feynman integrals is translated to the problem of performing analytical continuations. This conceptual change of interpretation of Feynman integrals may both deepen our understanding of scattering amplitudes and result in powerful methods to compute scattering amplitudes.

However, evaluation of $\mathcal{M}(D, \vec{s}, 0)$ using this series representation is a highly nontrivial problem. The reason is that, in practice, the Eq. (4) must be truncated to some orders in $1/\eta$ expansion, which makes the analytical continuations to $\eta = 0^+$ be very hard both numerically and analytically. Fortunately, the analytical continuations can be achieved thanks to the fact that any given family of Feynman integrals can always form a finite-dimensional linear space, as we will explain in the following.

Reduction relations from series representation.
— An important property of Feynman loop integrals is that the number of MIs is finite [43]. More precisely, for loop integrals constructed from any given set of propagators, there exists a finite set of loop integrals (called MIs, which can be found easily [44]) so that all other loop integrals can be expressed as linear combinations of them, with coefficients being fractional polynomials of kinematic variables and spacetime dimension. In other words, loop integrals with given set of propagators form a finite-dimensional linear space. In the following discussion, we will suppress the dependence on $D$, $\vec{s}$ and $\eta$ in loop integrals whenever it does not introduce any confusion.

For a given problem, let us assume that the dimension of the linear space is $n$. Then any given set of
\( n + 1 \) integrals \( \{ \mathcal{M}_1, \ldots, \mathcal{M}_{n+1} \} \) must be linearly dependent, which means that there exists \( n + 1 \) polynomials \( \{ \mathcal{Q}_1, \ldots, \mathcal{Q}_{n+1} \} \) so that
\[
\sum_{i=1}^{n+1} \mathcal{Q}_i(D, \mathbf{s}, \eta) \mathcal{M}_i(D, \mathbf{s}, \eta) = 0. \tag{6}
\]
We denote the mass dimension of \( \mathcal{M}_i \) by \( \text{Dim}(\mathcal{M}_i) \) and the mass dimension of \( \mathcal{Q}_i \) by \( 2d_i \), then they are constrained by
\[
2d_i + \text{Dim}(\mathcal{M}_i) = \cdots = 2d_{n+1} + \text{Dim}(\mathcal{M}_{n+1}). \tag{7}
\]
Therefore, for each choice of \( d_i \), all other \( d_j \) would be fixed. For fixed \( d_i \), we can expand \( \mathcal{Q}_i(D, \mathbf{s}, \eta) \) as
\[
\mathcal{Q}_i(D, \mathbf{s}, \eta) = \sum_{(\lambda_0, \lambda) \in \Omega_{d_i}^n} \mathcal{Q}_i^{\lambda_0 \cdots \lambda_r}(D) \eta^{\lambda_0} \mathbf{s}_1^{\lambda_1} \cdots \mathbf{s}_r^{\lambda_r}, \tag{8}
\]
where \( \mathcal{Q}_i^{\lambda_0 \cdots \lambda_r}(D) \) are fractional polynomials of \( D \) to be determined.

As series representation fully determines all analytical functions \( \mathcal{M}_i \), it certainly also determines the reduction relation Eq. (6). With a choice of \( d_i \), to determine the unknown polynomials \( \mathcal{Q}_i^{\lambda_0 \cdots \lambda_r}(D) \) using series representation, we substitute Eq. (4), (5) and (8) into Eq. (6) and then expand it in terms of monomials of \( \eta \) and \( \mathbf{s}_i \), which gives
\[
\sum_{k, \rho_0, \rho_r} f_k^{\rho_0 \cdots \rho_r}(D) \mathcal{I}_{L,k}^{\rho_0} \mathbf{s}_1^{\rho_1} \cdots \mathbf{s}_r^{\rho_r} = 0, \tag{9}
\]
where \( f_k^{\rho_0 \cdots \rho_r}(D) \) are linear combinations of \( \mathcal{Q}_i^{\lambda_0 \cdots \lambda_r}(D) \). As \( \mathcal{I}_{L,k}^{\rho_0} \mathbf{s}_1^{\rho_1} \cdots \mathbf{s}_r^{\rho_r} \) are independent of each other, their coefficients must vanish, which results in a system of linear equations
\[
f_k^{\rho_0 \cdots \rho_r}(D) = 0, \quad \text{for each } k, \rho_0, \rho_r. \tag{10}
\]
By calculating the series representation to sufficiently high order in \( 1/\eta \), we can generate enough linear equations to pin down all unknown \( \mathcal{Q}_i^{\lambda_0 \cdots \lambda_r}(D) \).

It is worth mentioning that, if the value of \( d_1 \) is chosen too small, there will be no solution for \( \mathcal{Q}_i^{\lambda_0 \cdots \lambda_r}(D) \); while if \( d_1 \) is chosen too large, there will be more than one solution, which is easy to understand because the Eq. (6) is unchanged if we multiplied it by any polynomial. Therefore, to find out the minimal value of \( d_1 \) so that Eq. (6) holds, we generate and solve the linear equations (10) for each choice of \( d_1 \), running from a small enough value to larger values. We stop once solutions for \( \mathcal{Q}_i^{\lambda_0 \cdots \lambda_r}(D) \) are found.

In practice, we can choose a few special values of \( D \) and determine \( \mathcal{Q}_i^{\lambda_0 \cdots \lambda_r}(D) \) for each special value. As \( \mathcal{Q}_i^{\lambda_0 \cdots \lambda_r}(D) \) are fractional polynomials of \( D \), their results with different choice of \( D \) can be used to fit their exact expressions.

It is needed to emphasize that, as \( \mathcal{Q}_i^{\lambda_0 \cdots \lambda_r}(D) \) are independent of \( \eta \), their values determined by series representation in the \( \eta \to \infty \) region are the same as their values in other regions. Therefore, we get an analytic relation (0) valid for any value of \( \eta \). Especially, by taking \( \eta \to 0^+ \) limit in Eq. (9), we get a correct reduction relation between \( \mathcal{M}_i(D, \mathbf{s}, 0) \).

**Analytical continuation.** — As series representation can relate any loop integral to MIs, the problem of performing analytical continuations for arbitrary loop integrals is reduced to the problem of performing analytical continuations for MIs.

Let us denote \( \mathcal{I}(D, \mathbf{s}, \eta) \) as the vector of a complete set of \( n \) MIs. As \( \frac{\partial}{\partial \eta} \mathcal{I}(D, \mathbf{s}, \eta) \) are special loop integrals, the reduction method described above can also express them in terms of linear combinations of \( \mathcal{I}(D, \mathbf{s}, \eta) \). Therefore, we obtain a system of ODEs,
\[
\frac{\partial}{\partial \eta} \mathcal{I}(D, \mathbf{s}, \eta) = A(D, \mathbf{s}, \eta) \mathcal{I}(D, \mathbf{s}, \eta), \tag{11}
\]
where \( A(D, \mathbf{s}, \eta) \) is the calculable \( n \times n \) coefficient matrix. With the ODEs, analytical continuation of MIs from \( \eta = \infty \) to \( \eta = 0^+ \) can be obtained straightforwardly by numerically solving the ODEs with BCs chosen at \( \eta = \infty \). The process is well-studied mathematically, and final results can be obtained efficiently to high precision [15].

We eventually find that all MIs, and thus arbitrary loop integrals, can be determined unambiguously by the series representation, which basically involves only vacuum integrals.

**Example.** — We take the sunrise diagram in Fig 1 as a simple but nontrivial example to illustrate how our method works. Let us consider a family of Feynman integrals
\[
\hat{\mathcal{I}}_{\nu_1 \nu_2 \nu_3} = \int \frac{D \ell_1}{\ell_1^{n_D/2}} \frac{D \ell_2}{\ell_2^{n_D/2}} \frac{D \ell_3}{\ell_3^{n_D/2}} \mathcal{I}_{\nu_1 \nu_2 \nu_3}, \tag{12}
\]
with inverse propagators
\[
D_1 = (\ell_1 + p)^2 - m^2, \quad D_2 = \ell_2^2, \quad D_3 = (\ell_1 + \ell_2)^2. \tag{13}
\]
This family forms a 2-dimensional linear space, with basis can be chosen as \( \{ \hat{\mathcal{I}}_{211}, \hat{\mathcal{I}}_{111} \} \). To compute \( \hat{\mathcal{I}}_{\nu_1 \nu_2 \nu_3} \), we introduce similar Feynman integrals \( I_{\nu_1 \nu_2 \nu_3} \) by changing \( D_1 \to D_1 + i\eta \). Note that, \( I_{\nu_1 \nu_2 \nu_3} \) form a 5-dimensional linear space, with basis can be chosen as \( \{ I_{2111}, I_{1211}, I_{1111}, I_{1110}, I_{0111} \} \). By taking \( \eta \to 0^+ \) limit to the 5 basis, the obtained \( I_{1110} \) and \( I_{0111} \) vanish in dimension regularization, and \( I_{1211} \) can be further reduced to \( I_{2111} \) and \( I_{1111} \). Suppose that we are now interested in the computation of \( I_{n_11} \) with large \( \nu_1 \).

We should first expand \( I_{n_11} \) and the corresponding 5 basis in large \( \eta \) region to obtain series representations. E.g., we have
\[
I_{1111} = \eta^{D-3} \left\{ \left[ 1 - \frac{D - 3}{3} \frac{m^2}{\eta} + \frac{(D + 4)(D - 3)}{9D} \frac{p^2}{\eta} \right] \mathcal{I}_{21/2} \right\}.
\]
we can reduce them even easily because they are essentially one-loop integrals. In this way, we can eventually get the desired reduction relation.

The next step is to set up a reduction relation between $I_{\nu11}$ and the 5 basis using the series representations. However, setting up the relation directly through Eq. (10) is almost impossible for large $\nu$ because the difference between the mass dimension of $I_{\nu11}$ and that of the 5 basis is very large, which results in too many integrals up to $O(\eta^{-2})$ in Eq. (6) to determine. This difficulty can be easily passed. Instead of generating the relation directly, we can generate it indirectly by first relating $I_{\nu11}$ to a complete set of intermediate integrals which are “close” to $I_{\nu11}$, and then relating these intermediate integrals to the 5 basis, again indirectly. For example, we can obtain the following intermediate reduction relation:

$$I_{\nu11} \rightarrow \{I_{\nu10}, I_{(\nu-1)11}, I_{(\nu-1)21}, I_{(\nu-2)11}, I_{(\nu-2)21}\}. \quad (15)$$

To set up this relation, we need to expand all relevant integrals up to $O(\eta^{-4})$ and solve a system of 30 linear equations with 22 unknown variables. The new type of integral $I_{\nu21}$ introduced in Eq. (15) can be similarly reduced as

$$I_{\nu21} \rightarrow \{I_{\nu20}, I_{\nu11}, I_{(\nu-1)21}, I_{(\nu-1)11}, I_{(\nu-2)21}\}, \quad (16)$$

which can be set up by solving a system of 30 linear equations with 17 unknown variables. For the integrals $I_{\nu20}$ and $I_{\nu10}$ introduced in the above two reduction relations, we can reduce them even easily because they are essentially one-loop integrals. In this way, we can eventually get the desired reduction relation.

Before continuing, it is interesting to compare our reduction of $I_{\nu11}$ to the corresponding 5 basis with the IBP reduction of $I_{\nu11}$ to the corresponding 2 basis. We list the time consumed by our reduction and that by the IBP reduction using FIRE5 in Tab. I for different values of $\nu$. Although that our method is realized in Mathematica while FIRE5 is written in C++, and that the reduction of $I_{\nu11}$ seems to be much harder than the reduction of $I_{\nu11}$, the time consumed by our method is significantly shorter than that by FIRE5, especially for large $\nu$. There are mainly two reasons why our method is more efficient. The first reason is that we can generate reduction relation for any set of integrals, as far as they are linearly dependent. So we always generate relations to reduce an integral to some “simpler” integrals. While in IBP method, one does not know how to generate these good relations, and thus one needs to generate plenty of relations to eventually get the desired reduction, as shown in Tab. I for the number of relations generated by the two methods. The second reason is that, although reduction relations we obtain are analytical, to get them we do not need to manipulate analytical expressions, but only rational numbers. We further note that the reduction of $I_{\nu11}$ can be used not only for sunrise integrals with one massive propagators, but also for two or three massive propagators depending on the choice of the value of $\eta$.

By taking $\eta \to 0^+$ limit for the reduction relation between $I_{\nu11}$ and the 5 basis, we obtain a relation between $I_{\nu11}$ and $\{\tilde{I}_{211}, \tilde{I}_{121}, \tilde{I}_{111}\}$. Thus the value of the former can be got once values of the later are known. To compute the later integrals, we use series representations to set up ODEs for the 5 basis w.r.t. $\eta$, the procedure of which is similar to the reduction of $I_{\nu11}$ described above. By solving the ODEs [15], we get the values of $\tilde{I}_{211}$, $\tilde{I}_{121}$, and $\tilde{I}_{111}$, and thus the value of $\tilde{I}_{\nu11}$ can be obtained. For example, with $m^2 = 1$ and $p^2 = 3.3$ we get

$$10^4 \tilde{I}_{(100)11} = 1.0307153 \times 10^4 + (4.9596399 + 3.2380877i) + (8.0586259 + 15.581168i)e + O(\epsilon^2). \quad (17)$$

As far as the best knowledge of us, there is no other method that can compute $\tilde{I}_{(100)11}$ to the same precision that we quoted. FIESTA4 [15] can only compute $\tilde{I}_{\nu11}$ with $\nu < 7$, in which cases the obtained results agree.
with ours.

**Summary and outlook.** — In this Letter, we find a novel representation for Feynman integrals, which is defined as analytical continuation of a calculable asymptotic series. Distinguished from all existed representations, it is a series representation but not integral representation. The new representation translates the problem of computing Feynman integrals to the problem of performing analytical continuations. This new perspective of Feynman integrals may be helpful to deepen our understanding of Feynman integrals and scattering amplitudes.

To realize analytical continuations, we first use the series representation to set up reduction relations between Feynman integrals and MIs, as well as to set up ODEs for the MIs. Then the desired analytical continuations can be achieved easily by solving the ODEs. As series representation can generate reduction relations freely, we can always choose to generate more efficient relations comparing with IBP reduction. With a two-loop example, we show that our method to compute Feynman integrals can be much more efficient than all existed methods.

Our method can be further improved from many aspects. One possible improvement is to take advantage of finite fields technique (see Ref. [46] for an introduction) to avoid large numbers in the middle of calculation process. Another possible improvement is to find out simpler reduction relations, which may only involve Feynman integrals in the subspace of the entire linear space. For example, we indeed find that, if \( \nu_2 > 2 \), there is a very simple relation between \( I_{\nu_1,\nu_2,\nu_3} \) and the following three integrals:

\[
\{I_{(\nu_1+2)(\nu_2-2)\nu_3}, I_{(\nu_1+1)(\nu_2-2)\nu_3}, I_{\nu_1(\nu_2-1)\nu_3}\}.
\]

Besides, the number of Feynman integrals involved in dimensional recurrence relations is usually also less than the dimension of the entire linear space.

**Acknowledgments.** — We thank Kuang-Ta Chao, Feng Feng, Yu Jia, Zhao Li, Xiaohui Liu, Ce Meng and Chen-Yu Wang for helpful discussions. This work is in part supported by the Recruitment Program of Global Youth Experts of China.

---

1. [xiao6@pku.edu.cn](mailto:xiao6@pku.edu.cn)
2. [yqma@pku.edu.cn](mailto:yqma@pku.edu.cn)
3. [G. Passarino and M. J. G. Veltman, One Loop Corrections for \( e^+ e^- \) Annihilation Into \( \mu^+ \mu^- \) in the Weinberg Model, Nucl. Phys. B160 (1979) 151–207](https://inspirehep.net/record/271306)
4. [G. T. Hooft and M. J. G. Veltman, Scalar One Loop Integrals, Nucl. Phys. B153 (1979) 365–401](https://inspirehep.net/record/271306)
5. [G. J. van Oldenborgh and J. A. M. Vermaseren, New Algorithms for One Loop Integrals, Z. Phys. C46 (1990) 425–438](https://inspirehep.net/record/271306)
6. [A. Denner and S. Dittmaier, Reduction schemes for one-loop tensor integrals, Nucl. Phys. B734 (2006) 62–115](https://inspirehep.net/record/271306)
7. [R. Britto, F. Cachazo, and B. Feng, Generalized unitarity and one-loop amplitudes in \( N=4 \) super-Yang-Mills, Nucl. Phys. B725 (2005) 275–305](https://inspirehep.net/record/271306)
8. [G. Ossola, C. G. Papadopoulos, and R. Pittau, Reducing full one-loop amplitudes to scalar integrals at the integrand level, Nucl. Phys. B763 (2007) 147–169](https://inspirehep.net/record/271306)
9. [W. T. Giele, Z. Kunszt, and K. Melnikov, Full one-loop amplitudes from tree amplitudes, JHEP 04 (2008) 049](https://inspirehep.net/record/271306)
10. [T. Binoth and G. Heinrich, An automated algorithm to compute infrared divergent multiloop integrals, Nucl. Phys. B585 (2000) 741–759](https://inspirehep.net/record/271306)
11. [N. T. Usyukina, On a Representation for Three Point Function, Theor. Mat. Fiz. 22 (1975) 300–306](https://inspirehep.net/record/271306)
12. [V. A. Smirnov, Analytical result for dimensionally regularized massless on shell double box, Phys. Lett. B460 (1999) 397–404](https://inspirehep.net/record/271306)
13. [A. V. Kotikov, Differential equations method: New technique for massive Feynman diagrams calculation, Phys. Lett. B254 (1991) 158–164](https://inspirehep.net/record/271306)
14. [Z. Bern, L. J. Dixon, and D. A. Kosower, Dimensionally regulated one loop integrals, Phys. Lett. B302 (1993) 299–308](https://inspirehep.net/record/271306)
15. [E. Remiddi, Differential equations for Feynman graph amplitudes, Nuovo Cim. A110 (1997) 1435–1452](https://inspirehep.net/record/271306)
16. [T. Gehrmann and E. Remiddi, Differential equations for two loop four point functions, Nucl. Phys. B580 (2000) 485–518](https://inspirehep.net/record/271306)
17. [X. Liu, Y.-Q. Ma, and C.-Y. Wang, A Systematic and Efficient Method to Compute Multi-loop Master Integrals, arXiv:1711.09572](https://arxiv.org/abs/1711.09572)
18. [J. Gluza, K. Kajda, and D. A. Kosower, Towards a Basis for Planar Two-Loop Integrals, Phys. Rev. D83 (2011) 045012](https://journals.aps.org/prd/abstract/10.1103/PhysRevD.83.045012)
19. [D. A. Kosower and K. J. Larsen, Maximal Unitarity at Two Loops, Phys. Rev. D85 (2012) 045017](https://journals.aps.org/prd/abstract/10.1103/PhysRevD.85.045017)
20. [P. Mastrolia, E. Mirabella, G. Ossola, and T. Peraro, Integrand-Reduction for Two-Loop Scattering Amplitudes through Multivariate Polynomial Division, Phys. Rev. D87 (2013) 085026](https://journals.aps.org/prd/abstract/10.1103/PhysRevD.87.085026)
21. [S. Badger, H. Frellesvig, and Y. Zhang, Hepta-Cuts of Two-Loop Scattering Amplitudes, JHEP 04 (2012) 055](https://journals.aps.org/prd/abstract/10.1103/PhysRevD.85.045017)
22. [P. Mastrolia, E. Mirabella, G. Ossola, and T. Peraro, Multiplanar Integrand Reduction for Dimensionally Regulated Amplitudes, Phys. Rev. B302 (2000) 485–508](https://inspirehep.net/record/271306)
23. [K. J. Larsen and Y. Zhang, Integration-by-parts reductions from unitarity cuts and algebraic geometry, Phys. Rev. D93 (2016) 041701](https://journals.aps.org/prd/abstract/10.1103/PhysRevD.93.041701)
24. [H. Ito, Two-loop Integrand Decomposition into Master Integrals and Surface Terms, Phys. Rev. D94 (2016) 045017](https://journals.aps.org/prd/abstract/10.1103/PhysRevD.94.045017)
A. von Manteuffel and R. M. Schabinger, J. Boehm, A. Georgoudis, K. J. Larsen, M. Schulze, and S. Borowka, N. Greiner, G. Heinrich, S. Jones, R. N. Lee, C. Studerus, S. Abreu, F. Febres Cordero, H. Ita, M. Jaquier, and A. V. Smirnov, 

FIRE5: a C++ implementation of K. G. Chetyrkin and F. V. Tkachov, S. Abreu, F. Febres Cordero, H. Ita, M. Jaquier, P. Mastrolia, T. Peraro, and A. Primo, 

Adaptive Local integrands, High precision calculation of multiloop Feynman integrals by difference equations, Loops: The Algorithm to Calculate beta Functions in 4 Loops, Feynman Integral REduction, Amplitudes from Numerical Unitarity, 

Two-Loop Four-Gluon B. Page, and M. Zeng, Method at Two Loops, Subleading Poles in the Numerical Unitarity Method at Two Loops, 

Phys. Rev. D95 (2017) 096011 arXiv:1705.05315 [hep-th/9803091 [InSPIRE]. 

B. A. Kniehl, A. F. Pikelner, and O. L. Veretin, Three-loop massive tadpoles and polylogarithms through weight six, JHEP 08 (2017) 024 arXiv:1705.05136 [hep-ph/0503209 [InSPIRE]. 

Y. Schr"{o}der and A. Vuorinen, High-precision epsilon expansions of single-mass-scale four-loop vacuum bubbles, JHEP 06 (2005) 051 hep-ph/0406248 [hep-ph/0503209 [InSPIRE]. 

T. Luthe, Fully massive vacuum integrals at 5 loops. PhD thesis, Bielefeld U., 2015 InSPIRE. https://pub.uni-bielefeld.de/publication/2776013. 

T. Luthe, A. Maier, P. Marquard, and Y. Schroder, Complete renormalization of QCD at five loops, JHEP 03 (2017) 020 arXiv:1701.07063 [hep-ph/1607.07498 InSPIRE]. 

R. P. Feynman, Space - time approach to quantum electrodynamics, Phys. Rev. 76 (1949) 769–789 [hep-ph/0503209 [InSPIRE]. 

P. A. Baikov, Explicit solutions of the multiloop integral recurrence relations and its application, Nucl. Instrum. Meth. A389 (1997) 347–349 hep-ph/9611449 [hep-ph/9611449 [InSPIRE]. 

A. V. Smirnov and A. V. Petukhov, The Number of Master Integrals is Finite, Lett. Math. Phys. 97 (2011) 37–44 arXiv:1004.4199 [hep-ph/0503209 [InSPIRE]. 

A. Georgoudis, K. J. Larsen, and Y. Zhang, Azurite: An algebraic geometry based package for finding bases of loop integrals, Comput. Phys. Commun. 221 (2017) 203–215 arXiv:1612.04252 [hep-ph/0503209 [InSPIRE]. 

A. V. Smirnov, FIESTA4: Optimized Feynman integral calculations with GPU support Comput. Phys. Commun. 204 (2016) 189–199 arXiv:1511.03614 [hep-ph/0503209 [InSPIRE]. 

T. Peraro, Scattering amplitudes over finite fields and multivariate functional reconstruction, JHEP 12 (2016) 030 arXiv:1608.01902 [hep-ph/0503209 [InSPIRE].