SYMMETRIC \((2,3,5)\) DISTRIBUTIONS, AN INTERESTING ODE
OF \(7\)th ORDER AND PLEBAŃSKI METRIC

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Abstract. We show that the unique \(7\)th order ODE having 10 contact symmetries appears naturally in the theory of generic 2-distributions in dimension five.

1. Introduction

Recently Dunajski and Sokolov have found [4] a general solution to an interesting \(7\)th order ODE:

\[
10y^{(3)}y^{(7)} - 70y^{(3)}y^{(4)}y^{(6)} - 49y^{(3)}y^{(5)}y^{(5)} - 280y^{(3)}y^{(4)}y^{(4)}y^{(5)} - 175y^{(4)}y^{(5)} = 0.
\]

This equation can be characterized as a unique (modulo contact transformations of variables) \(7\)th order ODE which has a 10-dimensional group of local contact symmetries [7, 9, 10]. In this short note we show that this equation also turns out to be a natural geometric condition for a certain class of generic 2-distributions in dimension five.

We say that a 2-distribution \(D = \text{Span}(X_4, X_5)\), where \(X_4\) and \(X_5\) are two vector fields on a 5-dimensional manifold \(M\), is generic, or \((2,3,5)\) on \(M\), if the system of five vector fields

\[
(X_1, X_2, X_3, X_4, X_5) = ([X_5, [X_4, X_5]], [X_4, [X_4, X_5]], [X_4, X_5], X_4, X_5)
\]

forms a frame on \(M\). Locally, a generic 2-distribution \(D\) on \(M\) can be defined as the annihilator of three 1-forms \((\omega_1, \omega_2, \omega_3)\) on \(M\), defined in terms of a single real function \(f = f(x, y, p, q, z)\), such that \(f_{qq} \neq 0\), via:

\[
\omega_1 = dy - pdx, \quad \omega_2 = dp - qdx, \quad \omega_3 = dz - f(x, y, p, q, z)dx.
\]

Here \((x, y, p, q, z)\) is a local coordinate system in \(M\).

The local geometry of \((2,3,5)\) distributions is nontrivial: there exist generic distributions \(D_1\) and \(D_2\) on \(M\) which do not admit a local diffeomorphism \(\varphi : M \rightarrow M\) such that \(\varphi_*D_1 = D_2\). For example, distributions corresponding to a function \(f = q^3\) and \(f = q^4\) in (1.1) do not admit such a diffeomorphism. In such case we say that they are locally nonequivalent. The full set of differential invariants of \((2,3,5)\) distributions considered modulo local diffeomorphism was given by Cartan in [3]. For each \((2,3,5)\) distribution he associated a Cartan connection, with values in the split real form of the exceptional Lie algebra \(g_2\), whose curvature provided the invariants. These invariants can be also understood in terms of a certain conformal class of metrics [8], defined on \(M\) by \(D\). This conformal class is defined as follows:

Let \(D\) be defined as the annihilator of 1-forms \((\omega_1, \omega_2, \omega_3)\), as e.g. in (1.1). We supplement them by the 1-forms \(\omega_4\) and \(\omega_5\), in such a way that \(\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 \wedge \omega_5\)
with some functions $\omega_4 \wedge \omega_5 \neq 0$. In case of (1.1) we take $\omega_4 = dq$ and $\omega_5 = dx$. Consider the forms $(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5)$ defined on $M$ via:

$$
\begin{pmatrix}
\theta^1 \\
\theta^2 \\
\theta^3 \\
\theta^4 \\
\theta^5
\end{pmatrix}
= 
\begin{pmatrix}
b_{11} & b_{12} & b_{13} & 0 & 0 \\
b_{21} & b_{22} & b_{23} & 0 & 0 \\
b_{31} & b_{32} & b_{33} & 0 & 0 \\
b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\
b_{51} & b_{52} & b_{53} & b_{54} & b_{55}
\end{pmatrix}
\begin{pmatrix}
\omega^1 \\
\omega^2 \\
\omega^3 \\
\omega^4 \\
\omega^5
\end{pmatrix},
$$

with some functions $b_{ij}$, $i, j = 1, 2, \ldots, 5$, on $M$ such that $\theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 \wedge \theta^5 \neq 0$. It follows that for a $(2,3,5)$ distribution $\mathcal{D}$ one can always find functions $b_{ij}$ and 1-forms $\Omega_\mu$, $\mu = 1, 2, \ldots, 7$, on $M$ such that

$$
d\theta^1 = \theta^1 \wedge (2\Omega_1 + \Omega_4) + \theta^2 \wedge \Omega_2 + \theta^3 \wedge \theta^4 \\
d\theta^2 = \theta^1 \wedge \Omega_3 + \theta^2 \wedge (\Omega_1 + 2\Omega_4) + \theta^3 \wedge \theta^5 \\
d\theta^3 = \theta^1 \wedge \Omega_5 + \theta^2 \wedge \Omega_6 + \theta^3 \wedge (\Omega_1 + \Omega_4) + \theta^4 \wedge \theta^5 \\
d\theta^4 = \theta^1 \wedge \Omega_7 + \frac{4}{3} \theta^3 \wedge \Omega_6 + \theta^4 \wedge \Omega_1 + \theta^5 \wedge \Omega_2 \\
d\theta^5 = \theta^2 \wedge \Omega_7 - \frac{4}{3} \theta^3 \wedge \Omega_5 + \theta^4 \wedge \Omega_3 + \theta^5 \wedge \Omega_4.
$$

And now, it turns out that the $(3,2)$-signature conformal class $[g_\mathcal{D}]$ represented on $M$ by the metric

$$(1.2) \quad g_\mathcal{D} = g_{ij} \theta^i \otimes \theta^j = \theta^1 \otimes \theta^5 + \theta^5 \otimes \theta^1 - \theta^2 \otimes \theta^4 - \theta^4 \otimes \theta^2 + \frac{4}{3} \theta^3 \otimes \theta^3$$

is well defined, and that its Weyl tensor can be used to get all the basic differential invariants of the distribution $\mathcal{D}$. The simplest of these invariants, the so called Cartan’s quartic $C(\zeta)$ of $\mathcal{D}$, can be obtained in terms of the Weyl tensor of the conformal class $[g_\mathcal{D}]$ as follows [16]:

Calculate the Weyl tensor $W = W_{ijkl} \theta^i \otimes \theta^j \otimes \theta^k \otimes \theta^l$ for the metric $g_\mathcal{D}$ in the coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5)$. (Use the metric $g_\mathcal{D}$ to lower the index $i$ from the natural placement $W^i_{jkl}$ to $W_{ijkl}$, $W_{ijkl} = g_{ip}W^p_{jkl}$). Then Cartan’s quartic for $\mathcal{D}$ is

$$C(\zeta) := A_1 + 4A_2 \zeta + 6A_3 \zeta^2 + 4A_4 \zeta^3 + A_5 \zeta^4$$

with the functions $A_i$, $i = 1, 2, \ldots, 5$, given by:

$$A_1 = W_{4114}, \quad A_2 = W_{4124}, \quad A_3 = W_{4125}, \quad A_4 = W_{4225}, \quad A_5 = W_{5225}.$$ 

The simplest equivalence class of $(2,3,5)$ distributions corresponds to the vanishing of Cartan’s quartic, $C(\zeta) \equiv 0$, or equivalently, $A_1 \equiv 0$ for all $i = 1, 2, \ldots, 5$. Modulo local diffeomorphisms there is only one such distribution $\mathcal{D}$. It may be represented by (1.1) with $f = q^2$. This distribution has maximal group of local symmetries. This group is isomorphic to the split real form of the exceptional group $G_2$ [3].

This provokes a problem: find all functions $f = f(x,y,p,q,z)$, which via (1.1), define a generic distribution $\mathcal{D}$, which is locally diffeomorphically equivalent to the most symmetric one, the one with $f = q^2$.

The general solution to this problem requires rather elaborate calculations, and it follows that the PDEs required for $f$ to correspond to vanishing $A_i$s are quite ugly. However in the restricted case when the function $f$ depends only on a single variable $q$ the solution is quite nice, see [5], equation (57). For completeness we recall this solution in the next section.
2. Cartan quartic for the distribution with \( f = f(q) \)

If the distribution is given as the annihilator of
\[
\omega_1 = dy - pdx \\
\omega_2 = dp - qdx \\
\omega_3 = dz - f(q)dx
\]
(2.1)
the conformal class \([g_D]\) can be represented by (1.2), with the forms \((\theta^1, \theta^2, \theta^3, \theta^4, \theta^5)\) given by:
\[
\begin{align*}
\theta^1 &= \omega_1 - \frac{1}{f''}(f'\omega_2 - \omega_3) \\
\theta^2 &= \frac{1}{f''}(f'\omega_2 - \omega_3) \\
\theta^3 &= \frac{4f''^2 - f'(3)}{4f''^2}\omega_2 + \frac{f(3)}{4f''^2}\omega_3 \\
\theta^4 &= \frac{(7f(3)^2 - 4f''f(4))}{40f''^3}(f'\omega_2 - \omega_3) + \omega_4 - \omega_5 \\
\theta^5 &= -\omega_4,
\end{align*}
\]
where
\[
\omega_4 = dq, \quad \omega_5 = dx.
\]
With this choice of \(\theta^i\)'s the Cartan quartic is
\[
C(\zeta) = \frac{a_5}{100f''^4}\zeta^4
\]
with
\[
a_5 = 10f(6)f''^3 - 80f''^2f(3)f(5) - 51f''^2f(4)^2 + 336f''^2f(3)^2f(4) - 224f(3)^4.
\]
We see, in particular, that the only nonvanishing component of Cartan’s quartic is \(A_5\), and that the quartic has a quadruple root, which makes it of type IV, in the terminology of [11].

We have the following corollary.

**Corollary 2.1.** Necessary and sufficient conditions for the distribution
\[
D = \text{Span}(\partial_q, \partial_x + p\partial_y + q\partial_p + f(q)\partial_z)
\]
to have split real form of the exceptional Lie group \(G_2\) as a group of its local symmetries are:
(2.2) \(f'' \neq 0\)
and
(2.3) \(10f(6)f''^3 - 80f''^2f(3)f(5) - 51f''^2f(4)^2 + 336f''^2f(3)^2f(4) - 224f(3)^4 = 0\).

Thus apart from the genericity condition (2.2) the function \(f\) must satisfy quite a complicated 6th order ODE (2.3).

Strangely enough, this ODE is closely related to the equation studied by Dunajski and Sokolov mentioned in the Introduction. We have the following proposition.
Proposition 2.2. Suppose that a real, sufficiently many times differentiable, function \( f = f(q) \) satisfies equation (2.3). Let \( \Theta = \Theta(x_5) \) be another real, sufficiently many times differentiable, function of a real variable \( x_5 \), whose second and third derivatives with respect to \( x_5 \) are related to \( f \) via an equation:

\[
(2.4) \quad f(-\Theta^{(3)}) + x_5 \Theta^{(3)} - \Theta'' = 0.
\]

Assume in addition that \( \Theta^{(4)} \neq 0 \). Then the function \( \Theta = \Theta(x_5) \) satisfies the following 8th order ODE:

\[
10\Theta^{(4)} \Theta^{(8)} - 70\Theta^{(4)} \Theta^{(6)} + 49\Theta^{(4)} \Theta^{(6)} + 280\Theta^{(4)} \Theta^{(5)}^2 \Theta^{(5)} - 175\Theta^{(5)}^4 = 0.
\]

Proof. The proof consists in a successive differentiation of the equation \( f(-\Theta^{(3)}) = -x_5 \Theta^{(3)} + \Theta'' \) using the chain rule. We have: \(-\Theta^{(4)} f' = -\Theta^{(3)} - x_5 \Theta^{(4)} + \Theta^{(3)}\), i.e. \( f' = x_5 \). Then, in the same way:

\[
f^{(p)} = -\frac{1}{\Theta^{(4)}} \frac{d}{dx_5} f^{(p-1)} \quad \text{for} \quad p = 2, 3, \ldots,
\]

i.e. \( f'' = -\frac{1}{\Theta^{(4)}} f^{(3)} = \Theta^{(5)} \Theta^{(7)} + \Theta^{(4)} \Theta^{(6)} - 3\Theta^{(5)} \Theta^{(5)} \Theta^{(4)} \Theta^{(4)} \), etc. Inserting these derivatives of \( f \) into the definition of \( a_5 \) we get

\[
a_5 = \frac{-10\Theta^{(4)} \Theta^{(8)} - 70\Theta^{(4)} \Theta^{(6)} + 49\Theta^{(4)} \Theta^{(6)} + 280\Theta^{(4)} \Theta^{(5)}^2 \Theta^{(5)} - 175\Theta^{(5)}^4}{\Theta^{(4)}}.
\]

Thus if the equation \( a_5 = 0 \) for \( f \) is satisfied, i.e. if (2.3) holds, then the function \( \Theta = \Theta(x_5) \) satisfies the 8th order ODE from the proposition, as claimed. \( \Box \)

Magically, the equation (2.3), when transformed via (2.4) into the 8th order ODE from Proposition 2.2 and then, when reduced by one order via \( y = \Theta' \), becomes the 7th order ODE considered by Dunajski and Sokolov. The magic is in a peculiar form of the transformation (2.4) relating \( f \) and \( \Theta \). The geometric reason for this transformation is explained in the next section.

3. Distribution with \( f = f(q) \) as a twistor distribution

A particular class of \((2, 3, 5)\) distributions is associated with 4-dimensional split signature metrics. This is carefully explained in [1], see Section 2, for every split signature metric. Here we concentrate on a special case, when the metric is given in terms of a one real function of four variables, called Plebański second heavenly function.

Let \( \Theta = \Theta(x, y, z, w) \) be a real, sufficiently smooth, function of four real variables \( (x, y, z, w) \). Such a function enables us to define a 4-metric \( g \), on a manifold \( \mathcal{U} \) parametrized by \( (x, y, z, w) \), via:

\[
(3.1) \quad g = dw dx + dz dy - \Theta_{xx} dx^2 - \Theta_{yy} dy^2 + 2 \Theta_{xy} dw dz.
\]

This is Plebański’s second heavenly metric \(^1\) on \( \mathcal{U} \). It can be written in the form

\[
g = \tau^1 \otimes \tau^2 + \tau^2 \otimes \tau^1 + \tau^3 \otimes \tau^4 + \tau^4 \otimes \tau^3,
\]

\(^1\)Note that we do not assume that the heavenly function \( \Theta \) satisfies any kind of equations. It is a free differentiable function of four variables. In particular it is not required to satisfy Plebański’s heavenly equation.
This inevitably leads to where (2.1) and (3.2), we see that the relation between the function \( \Theta \) variables \((x,y,z,w)\)

\[
\begin{align*}
\tau^1 &= dx - \Theta_{yy} dw + \Theta_{xy} dz \\
\tau^2 &= dw \\
\tau^3 &= dy - \Theta_{x} dz + \Theta_{xy} dw \\
\tau^4 &= dz.
\end{align*}
\]

Since \( g \) has split signature on \( \mathcal{U} \), there is a natural circle bundle \( S^1 \to T(\mathcal{U}) \to \mathcal{U} \) over \( \mathcal{U} \). In this bundle, which we call as the circle twistor bundle for \( (\mathcal{U}, g) \), every point in the fiber over \( x \in \mathcal{U} \) is a certain real totally null selfdual 2-plane at \( x \). There is an entire circle of such planes at \( x \). The bundle \( T(\mathcal{U}) \to \mathcal{U} \) is naturally equipped with a 2-dimensional distribution \( D \). Its plane \( D_p \) at a point \( p \in T(\mathcal{U}) \), which as we know can be identified with a certain real totally null 2-plane \( N(p) \) at \( \pi(p) \), is the tautological horizontal lift of \( N(p) \) from \( \pi(p) \) to \( p \). Horizonality in \( T(\mathcal{U}) \) is induced by the Levi-Civita connection of \( g \) from \( \mathcal{U} \). (See [1], Section 2, for details). In case of the Plebański metric (3.1), given in terms of the heavenly 1-forms \( \omega \), the circle twistor bundle can be locally parametrized by \((x,y,z,w,\xi)\) and the twistor distribution can be defined as the annihilator of the 1-forms

\[
\tilde{\omega}_1 = d\xi - ( (\partial_x + \xi \partial_y)^3 \Theta ) dz
\]
\[
\tilde{\omega}_2 = dw + \xi dz
\]
\[
\tilde{\omega}_3 = dy - \xi dx - ( (\partial_x + \xi \partial_y)^2 \Theta ) dz
\]

A little tweak (see [11], Thm 3.3.5), which we have learned from Ian Anderson [2], and which he attributes to Goursat [5], consists in introducing new coordinates \((x_1,x_2,x_3,x_4,x_5)\) on \( T(\mathcal{U}) \):

\[
x_1 = z, \quad x_2 = w, \quad x_3 = -\xi, \quad x_4 = y - \xi x, \quad x_5 = x,
\]

and enables us to conclude that the twistor distribution for the Plebański metric (3.1) can equivalently be defined by the annihilator of the forms

\[
\begin{align*}
\omega_1 &= dx_2 - x_3 dx_1 \\
\omega_2 &= dx_3 + \Theta_{555} dx_1 \\
\omega_3 &= dx_4 - (\Theta_{55} - x_5 \Theta_{555}) dx_1.
\end{align*}
\]

Here \( \Theta_{55} = \frac{\partial^2 \Theta}{\partial x_1^2}, \Theta_{555} = \frac{\partial^3 \Theta}{\partial x_1^3} \), and because \( \Theta \) is originally function of only four variables \((x,y,z,w)\), we have \( \Theta_3 + x_5 \Theta_4 = 0 \).

Now we can demystify transformation (2.4). Indeed, comparing the formulae (2.1) and (3.2), we see that the relation between the function \( f \) in (2.1) and the function \( \Theta \) in (3.2) is

\[
q = -\Theta_{555}, \quad f(q) = \Theta_{55} - x_5 \Theta_{555}.
\]

This inevitably leads to

\[
f(-\Theta_{555}) = \Theta_{55} - x_5 \Theta_{555},
\]

i.e., when \( \Theta \) is a function of \( x_5 \) only, \( \Theta = \Theta(x_5) \), to the relation (2.4).

For an explicit derivation of the ODE discussed by Dunajski and Sokolov in terms of the Plebański second heavenly metric, we find explicit formulae for the conformal class \([g_\mathcal{D}]\) associated with the distribution \( D \) defined by (3.2) with \( \Theta = \Theta(x_5) \). Since for this the function \( \Theta \) is a function of one variable only, we will denote the
derivatives w.r.t. $x_5$ by primes, double primes, etc. First we extend the forms (3.2) by
\[ \omega_4 = dx_1, \quad \omega_5 = dx_5 \]
to a coframe on $\mathbb{T}(U)$, and then find a suitable representatives of the forms ($\theta^1, \theta^2, \theta^3, \theta^4, \theta^5$) defining, via [1,2], the conformal class $[g_D]$. These forms can be taken to be:
\[
\begin{align*}
\theta^1 &= \omega_1 - \Theta^{(4)}(x_5 \omega_2 - \omega_3) \\
\theta^2 &= \Theta^{(4)}(x_5 \omega_2 - \omega_3) \\
\theta^3 &= -\frac{4\Theta^{(4)} + x_5 \Theta^{(5)}}{4 \Theta^{(4)}} \omega_2 + \frac{\Theta^{(5)}}{4 \Theta^{(4)}} \omega_3 \\
\theta^4 &= -\frac{5\Theta^{(5)} - 4\Theta^{(4)} \Theta^{(6)}}{40 \Theta^{(4)}^3} (x_5 \omega_2 - \omega_3) + \omega_4 - \Theta^{(4)} \omega_5 \\
\theta^5 &= \Theta^{(4)} \omega_5,
\end{align*}
\]

with
\[
\begin{align*}
\omega_1 &= dx_2 - x_3 dx_1 \\
\omega_2 &= dx_3 + \Theta^{(3)} dx_1 \\
\omega_3 &= dx_4 - (\Theta'' - x_3 \Theta^{(3)}) dx_1 \\
\omega_4 &= dx_1 \\
\omega_5 &= dx_5.
\end{align*}
\]

It is straightforward now to calculate Cartan’s quartic for $g_D$ with these forms $\theta^1$. It reads:
\[
C(\zeta) = -\frac{\alpha_5 \zeta^4}{100 \Theta^{(4)}},
\]
where $\alpha_5$ is given by
\[
\alpha_5 = 10\Theta^{(4)}^3 \Theta^{(8)} - 70 \Theta^{(4)}^2 \Theta^{(5)} \Theta^{(7)} - 49 \Theta^{(4)}^2 \Theta^{(6)}^2 + 280 \Theta^{(4)} \Theta^{(5)}^2 \Theta^{(6)} - 175 \Theta^{(5)}^4.
\]

Thus under the condition $\Theta^{(4)} \neq 0$, Cartan’s quartic identically vanishes if and only if $y = \Theta'$ satisfies the 7th order ODE considered by Dunajski and Sokolov.

We have just proved the following theorem.

**Theorem 3.1.** The twistor distribution $\mathcal{D}$ on the circle twistor bundle $S^1 \to \mathbb{T}(M) \to M$ of the Plebański second heavenly manifold $(M, g)$ with the metric
\[ g = dw dx + dz dy - \Theta''dz^2 \]
and the second heavenly function $\Theta = \Theta(x)$ has the split real form of the exceptional group $G_2$ as a group of its local symmetries if and only if the heavenly function $\Theta$ satisfies $\Theta^{(4)} \neq 0$ and the celebrated ODE:
\[
10 \Theta^{(4)}^3 \Theta^{(8)} - 70 \Theta^{(4)}^2 \Theta^{(5)} \Theta^{(7)} - 49 \Theta^{(4)}^2 \Theta^{(6)}^2 + 280 \Theta^{(4)} \Theta^{(5)}^2 \Theta^{(6)} - 175 \Theta^{(5)}^4 = 0.
\]

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