Mean-field forward and backward SDEs with jumps. Associated nonlocal quasi-linear integral-PDEs

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Abstract

In this paper we consider a mean-field backward stochastic differential equation (BSDE) driven by a Brownian motion and an independent Poisson random measure. Translating the splitting method introduced by Buckdahn, Li, Peng and Rainer [6] to BSDEs, the existence and the uniqueness of the solution \((Y^{t,\xi}, Z^{t,\xi}, H^{t,\xi})\), \((Y^{t,x,P_\xi}, Z^{t,x,P_\xi}, H^{t,x,P_\xi})\) of the split equations are proved. The first and the second order derivatives of the process \((Y^{t,x,P_\xi}, Z^{t,x,P_\xi}, H^{t,x,P_\xi})\) with respect to \(x\), the derivative of the process \((Y^{t,x,P_\xi}, Z^{t,x,P_\xi}, H^{t,x,P_\xi})\) with respect to the measure \(P_\xi\), and the derivative of the process \((\partial_\mu Y^{t,x,P_\xi}(y), \partial_\mu Z^{t,x,P_\xi}(y), \partial_\mu H^{t,x,P_\xi}(y))\) with respect to \(y\) are studied under appropriate regularity assumptions on the coefficients, respectively. These derivatives turn out to be bounded and continuous in \(L^2\). The proof of the continuity of the second order derivatives is particularly involved and requires subtle estimates. This regularity ensures that the value function \(V(t,x,P_\xi) := Y^{t,x,P_\xi}_t\) is regular and allows to show with the help of a new Itô formula that it is the unique classical solution of the related nonlocal quasi-linear integral-partial differential equation (PDE) of mean-field type.

Keyword. BSDEs with jump, mean-field BSDEs with jump, integral-PDE of mean-field type, Itô’s formula, value function

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1 Introduction

Mean-field stochastic differential equations, also called McKean-Vlasov equations, can be dated back to the works of Kac [14], [15] in the 1950s. Nonlinear mean-field backward stochastic differential equations had not been investigated before the work of Buckdahn, Djehiche Li and Peng [3] in 2009. Since their work the theory of mean-field forward-backward stochastic differential equations (FBSDEs), as well as that of the associated partial differential equations (PDEs) of mean-field type has been intensively investigated. For example, Buckdahn, Li and Peng [5] obtained for mean-field BSDEs an existence and uniqueness theorem, but also a comparison theorem. Using a BSDE approach, first introduced by Peng [26] in 1997, the authors also gave a probabilistic interpretation.*

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to related nonlocal partial differential equations. Min, Peng and Qin [24] proved through a con-
tinuation method that fully coupled mean-field FBSDEs have a unique square integrable adapted
solution. On the other hand, with the development of the theory of mean-field FBSDEs, many
stochastic control problems in the mean-field framework have also been considered. For instance,
Li [19] studied a stochastic maximum principle for the mean-field controls. A stochastic optimal
control problem with delay and of mean-field type was considered by Shen, Meng and Shi [27].
With the help of the theory of FBSDEs involving the value function, but with frozen partial initial
values, Hao and Li investigated an optimal control problem with systems of decoupled controlled
mean-field FBSDEs [11], as well as fully coupled controlled mean-field FBSDEs [12]. We remark
that, generally speaking, the dynamic programming principle for mean-field FBSDEs does not hold
true anymore because of the presence of expectation terms in coefficients. For this reason, in [5], [11]
and [12], the authors adopted a new method: They fixed partially the initial values, to overcome
this difficulty. Besides, there are also many other works in the mean-field area, see, e.g., Kloeden
and Lorenz [16], Kotelenez and Kurtz [17], Yong [29] and references therein. In particular, Lasry
and Lions [18] extended the application areas for mean-field problems to Economics, Finance and
game theory.

The lectures given by P.L. Lions [23] at Coll`ege de France and the notes edited by Cardaliguet
[7] give the definition of the derivative for a function $\varphi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ with respect to measure.
Many works adopt this definition, for example, R. Carmona and F. Delarue [9], Cardaliaguet
[5]. Among all these works, we refer in particular to that of Buckdahn, Li, Peng and Rainer [6].
The authors considered general mean-field SDEs and related nonlocal PDEs, and proved that the
solution $(X^{t,\xi}, X^{t,x,P_\xi})$ of such a couple of forward SDEs satisfies the flow property. This allowed
to prove that the associated nonlocal PDE has a unique classical solution. This approach overcame
the drawback of partial freezing of initial data, see [5]. Recently, Chassagneux, Crisan and Delarue
[10] considered fully coupled mean-field FBSDEs driven by Brownian motion, and proved that the
fully coupled mean-field FBSDEs have unique solutions.

We are interested here in more general mean-field FBSDE with jumps. The theory of FBSDEs
with jumps has developed very dynamically in the recent years because of its variable applications.
There are many works on FBSDEs with jumps, see, e.g., Bass [1], Barles, Buckdahn and Pardoux
[2], Tang and Li [28], Li and Peng [20], Buckdahn, Li and Hu [4], Li and Wei [21], [22]. On the other
hand, Hao and Li [13] studied mean-field SDEs with jumps. They showed that the unique solution
$(X^{t,\xi}, X^{t,x,P_\xi})$ of the split mean-field SDE with jumps satisfies the flow property, and using a new
approach the authors succeeded in proving the existence and the uniqueness of classical solutions
for the related nonlocal linear integral-PDEs. Inspired by the works of Hao, Li [13] and Pardoux,
Peng [25], the objective of our present work is to associate the mean-field (forward) SDE with
jumps with a mean-field BSDE driven by a Brownian motion and an independent Poisson random
measure, and to describe the associated nonlocal integral-PDE of mean-field type which unlike [13]
and [6] is quasi-linear. We emphasize that this generalization is far from being trivial and related
with very subtle BSDE estimates.

More precisely, given the solution of the split forward SDE $(X^{t,\xi}, X^{t,x,P_\xi})$ (see the equations
(3.1) and (3.2)), we consider the split BSDEs with jumps (see the equations (4.1) and (4.2)), driven
by the Brownian motion $B$ and the independent compensated Poisson random measure $N_\lambda$ (with
associated Lévy measure $\lambda$ defined over $K \subset \mathbb{R}^\ell \setminus \{0\}$). From Theorem 10.1 in the Appendix
it follows equation (4.1) has a unique solution \((Y_{t}^{t,\xi},Z_{t}^{t,\xi},H_{t}^{t,\xi})\). Once knowing \((Y_{t}^{t,\xi},Z_{t}^{t,\xi},H_{t}^{t,\xi})\), equation (4.2) can be treated as a classical BSDE with jumps, and it possesses a unique solution \((Y_{t}^{t,x,\xi},Z_{t}^{t,x,\xi},H_{t}^{t,x,\xi})\). We show that this solution of (4.2) depends on \(\xi\) only through its law, but not on \(\xi\) itself (see Proposition 4.1), which allows to define \((Y_{t}^{t,x,P_{\xi}},Z_{t}^{t,x,P_{\xi}},H_{t}^{t,x,P_{\xi}}) = (Y_{t}^{t,x,\xi},Z_{t}^{t,x,\xi},H_{t}^{t,x,\xi})\). The flow property of \((X_{t}^{t,x,P_{\xi}},Y_{t}^{t,x,P_{\xi}})\) (see (3.5)) leads to a corresponding property for \((Y_{t}^{t,x,P_{\xi}},Y_{t}^{t,x,P_{\xi}})\) (see (4.9)), which is crucial to study the related nonlocal quasi-linear integral-PDE of mean-field type. As we are interested in classical solutions of the related PDEs, we have to study the regularity of \((Y_{t}^{t,x,P_{\xi}},Z_{t}^{t,x,P_{\xi}},H_{t}^{t,x,P_{\xi}})\), i.e., its twice continuous differentiability with respect to \(x\), its continuous differentiability with respect to the law and the continuous differentiability of this latter derivative with respect to the variable which is generated by the derivative with respect to the law. The study of these second order derivatives for a BSDE leads to new BSDEs whose driver depends, in particular, on non-linear functions of \((Y_{t}^{t,x,P_{\xi}},Z_{t}^{t,x,P_{\xi}},H_{t}^{t,x,P_{\xi}})\) multiplied with the square of the first order derivatives of the processes \(Z_{t}^{t,x,P_{\xi}}\) and \(H_{t}^{t,x,P_{\xi}}\), which are only square integrable with respect to the time parameter. This makes the proof of the continuity of the second order derivatives of these processes very subtle and is related with very technical estimates (see, in particular, Section 8, Section 10.3), a point which in their study of classical BSDEs and classical solutions of associated PDEs in [25] was not developed there. The regularity of \((Y_{t}^{t,x,P_{\xi}},Z_{t}^{t,x,P_{\xi}},H_{t}^{t,x,P_{\xi}})\) yields that of the value function \(V\) defined by \(V(t,x,P_{\xi}) = Y_{t}^{t,x,P_{\xi}}\). We prove that this value function \(V(t,x,P_{\xi})\) is the unique classical solution of the new nonlocal quasi-linear integral-PDE of mean-field type (9.1) (see Theorem 9.2). For this we first prove a new more general Itô’s formula \(F(t,U_{t},P_{X_{t}})\), where \(U\) and \(X\) are Itô processes with jumps, respectively. In particular, unlike [6] and [13] we don’t need the existence of the second order mixed derivatives \(\partial_{x}\partial_{\mu}F, \partial_{\mu}\partial_{x}F, \partial_{\mu}^{2}F\) for the Itô formula, see Theorem 2.1. This new Itô formula simplifies the proof of Theorem 9.2, even for the more special case studied in [6] and [13]. We also get the representation formulas for the solutions of (4.1) and (4.2), see (9.2) and (9.3).

This paper is organized as follows. In Section 2 we recall the definition of the derivative of a function \(\varphi\) defined on \(P_{2}(\mathbb{R}^{d})\) with respect to the measure. We also prove a new general Itô formula. Section 3 studies mean-field SDEs with jumps. The properties of the solution for our split mean-field BSDEs with jumps are proved in Section 4. Section 5 shows that the first order derivatives of the process \(X_{t}^{t,x,P_{\xi}}\) with respect to \(x\) and the measure \(P_{\xi}\) exist, and the corresponding estimates are obtained. Section 6 is devoted to study the first order derivatives of \((Y_{t}^{t,x,P_{\xi}},Z_{t}^{t,x,P_{\xi}},H_{t}^{t,x,P_{\xi}})\) with respect to \(x\) and the measure \(P_{\xi}\), respectively, which are bounded and Lipschitz continuous in \(L^{2}\). In Section 7 the second order derivatives of \(X_{t}^{t,x,P_{\xi}}\) are discussed. The second order derivatives of \((Y_{t}^{t,x,P_{\xi}},Z_{t}^{t,x,P_{\xi}},H_{t}^{t,x,P_{\xi}})\) are investigated in Section 8. In Section 9 we prove by using our new Itô’s formula that our associated integral-PDEs of mean-field type has a unique classical solution. Section 10 (the Appendix) gives the proof of Theorem 2.1 (Subsection 10.1), that of an auxiliary result for Proposition 9.1 (Subsection 10.3), and recalls some basic results on mean-field BSDEs with jumps (Subsection 10.2).

### 2 Preliminaries

Let us consider a complete probability space \((\Omega, \mathcal{F}, P)\) on which is defined a \(d\)-dimensional Brownian motion \(B(= (B_{t}^{1}, \ldots, B_{t}^{d})) = (B_{t})_{t \in [0,T]}\); and an independent Poisson random measure \(N\) on \(\mathbb{R}_{+} \times K\).
Here $K \subset \mathbb{R}^d \setminus \{0\}$ is a nonempty open set equipped with its Borel field $\mathcal{K}$. The compensator 
$\nu(de, dt) = \lambda(de)dt$ of $N$ is such that $\left\{N_{\lambda}(0, t] \times E = (N - \nu)(0, t] \times E)\right\}_{t \geq 0}$ is a martingale for all $E \in \mathcal{K}$ satisfying $\lambda(E) < \infty$, and $\lambda$ is a given $\sigma$-finite Lévy measure on $(K, \mathcal{K})$, i.e., a measure on $(K, K)$ with the property that $\int_K (1 \wedge |e|^2)\lambda(de) < \infty$. Let $T > 0$ denote an arbitrarily fixed time horizon. We suppose that there is a sub-$\sigma$-field $\mathcal{F}_0 \subset \mathcal{F}$ such that

i) the Brownian motion $B$ and the Poisson random measure $N$ are independent of $\mathcal{F}_0$,

ii) $\mathcal{F}_0$ is “rich enough”, i.e., $\mathcal{P}_2(\mathbb{R}^k) = \{P_\vartheta, \vartheta \in L^2(\mathcal{F}_0; \mathbb{R}^k), k \geq 1\}$,

iii) $\mathcal{F}_0 \supset \mathcal{N}_P$, where $\mathcal{N}_P$ is the set of all $P$-null subsets of $\mathcal{F}$.

By $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ we denote the filtration generated by this Brownian motion $B$ and the Poisson random measure $N$, augmented by $\mathcal{F}_0$, i.e.,

$$\mathcal{F}_t^0 = \sigma\{B_s, N([0, s] \times E) | s \leq t, E \in \mathcal{K}\},$$

$$\mathcal{F}_t := \mathcal{F}_{t-} \vee \mathcal{F}_0 = (\bigcap_{s < t} \mathcal{F}_s^0) \vee \mathcal{F}_0, \quad t \in [0, T].$$

Note that $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ satisfies the standard assumption of right-continuity and completeness. Let us introduce the following spaces which are needed in what follows.

- $\mathcal{H}_2^2(t, T; \mathbb{R}^d) := \{\psi : \Omega \times [t, T] \rightarrow \mathbb{R}^d \text{ is an } \mathbb{F} \text{-predictable process with } \mathbb{E}\left[\int_t^T |\psi_s|^2 ds\right] < +\infty\}$;
- $\mathcal{S}_2^2(t, T; \mathbb{R}^d) := \{\varphi : \Omega \times [t, T] \rightarrow \mathbb{R}^d \text{ is an } \mathbb{F} \text{-adapted càdlàg process with } \mathbb{E}\left[\sup_{0 \leq s \leq T} |\varphi_s|^2 ds\right] < +\infty\}$;
- $\mathcal{K}_2^2(t, T; \mathbb{R}^d) := \{H : H : \Omega \times [t, T] \times K \rightarrow \mathbb{R}^d \text{ is } \mathcal{P}_0 \otimes \mathcal{B}(K) \text{-measurable and } \mathbb{E}\left[\int_t^T \int_K |H_s(e)|^2 \lambda(de) ds\right] < +\infty\}$.

Here $t \in [0, T]$ and $\mathcal{P}_0$ denotes the $\sigma$-field of $\mathbb{F}$-predictable subsets of $\Omega \times [0, T]$. Note that we may omit $\mathbb{R}^d$ and just write $\mathcal{H}_2^2(t, T)$ when $d = 1$, similar to other notations.

Let us introduce some notations and concepts, which are used frequently in what follows. By $\mathcal{P}(\mathbb{R}^d)$ we denote the set of probability measures over $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$; $\mathcal{P}_2(\mathbb{R}^d)$ denotes the set of probability measures $\mu$ from $\mathcal{P}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} |x|^2 \mu(dx) < +\infty$. Let $\mathcal{P}_2(\mathbb{R}^d)$ be endowed with the $2$-Wasserstein metric: For $\nu, \tilde{\nu} \in \mathcal{P}_2(\mathbb{R}^d)$,

$$W_2(\nu, \tilde{\nu}) := \inf \left\{ \left( \int_{\mathbb{R}^d} |x - y|^2 \rho(dx, dy) \right)^{\frac{1}{2}}, \rho \in \mathcal{P}_2(\mathbb{R}^{2d}) \right\},$$

such that $\rho(A_1 \times \mathbb{R}^d) = \nu(A_1)$, $A_1 \in \mathcal{B}(\mathbb{R}^d)$, $\rho(\mathbb{R}^d \times A_2) = \tilde{\nu}(A_2)$, $A_2 \in \mathcal{B}(\mathbb{R}^d)$.

We now introduce the notion of differentiability of a function defined on $\mathcal{P}_2(\mathbb{R}^d)$ with respect to probability measure. Here we adopt the approach introduced by Lions in his course at Collège de France (23) and later edited in the notes by Cardaliaguet (7). Given a function $\phi : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$, we consider the lifted function $\tilde{\phi}(\xi) := \phi(P_\xi)$, $\xi \in L^2(F; \mathbb{R}^d)(:= L^2(\Omega, \mathcal{F}, P; \mathbb{R}^d))$. If for a given $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ there exists a random variable $\xi_0 \in L^2(F; \mathbb{R}^d)$ satisfying $P_{\xi_0} = \mu_0$, such that $\tilde{\phi} : L^2(F; \mathbb{R}^d) \rightarrow \mathbb{R}$ is Fréchet differentiable at this point $\xi_0$, then we called that $\phi$ is differentiable with respect to $\mu_0$.

This is equivalent with the existence of a continuous linear mapping $D\tilde{\phi}(\xi_0) : L^2(F; \mathbb{R}^d) \rightarrow \mathbb{R}$ (i.e., $D\tilde{\phi}(\xi_0) \in L(L^2(F; \mathbb{R}^d); \mathbb{R})$) such that

$$\tilde{\phi}(\xi_0 + \zeta) - \tilde{\phi}(\xi_0) = D\tilde{\phi}(\xi_0)(\zeta) + o(|\zeta|_{L^2}),$$

for $\zeta \in L^2(F; \mathbb{R}^d)$ with $|\zeta|_{L^2} \rightarrow 0$. Riesz’s Representation Theorem allows to show that there exists a unique $\eta \in L^2(F; \mathbb{R}^d)$ such that $D\tilde{\phi}(\xi_0)(\zeta) = \mathbb{E}[\eta \cdot \zeta]$, $\zeta \in L^2(F; \mathbb{R}^d)$. But this random variable
η is a Borel measurable function of ξ₀, refer to Cardaliaguet [7]. This means that η is of the form η = ψ(ξ₀), where ψ is a Borel measurable function depending on ξ₀ only through its law. Hence, combining (2.1) and the above argument, we have

\[ \varphi(P_{ξ₀+ζ}) - \varphi(P_{ξ₀}) = E[ψ(ξ₀) \cdot ζ] + o(\|ζ\|₂). \]

In the spirit of Lions and Cardaliaguet, the derivative of \( \varphi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) with respect to the measure \( P_{ξ₀} \) is denoted by \( \partial_μ \varphi(P_{ξ₀}, y) := ψ(y), \ y ∈ \mathbb{R}^d \). Observe that \( \partial_μ \varphi(P_{ξ₀}, y) \) is only \( P_{ξ₀}(dy) \)-a.e. uniquely determined; see also Definition 2.1 in Buckdahan, Li, Peng and Rainer [6].

The following two spaces are used frequently. For more details the reader may refer to [6].

**Definition 2.1.** 1) We say that \( \varphi \) belongs to \( C^{1,1}_b(\mathcal{P}_2(\mathbb{R}^d)) \), if \( \varphi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) is differentiable on \( \mathcal{P}_2(\mathbb{R}^d) \) and \( \partial_μ \varphi(·, ·) : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d \) is bounded and Lipschitz continuous, i.e., there exists some positive constant \( L \) such that

(i) \( |\partial_μ \varphi(μ, y)| ≤ L, \ μ ∈ \mathcal{P}_2(\mathbb{R}^d), \ y ∈ \mathbb{R}^d \),

(ii) \( |\partial_μ \varphi(μ, y) - \partial_μ \varphi(μ', y')| ≤ L(W_2(μ, μ') + |y - y'|), \ μ, \ μ' ∈ \mathcal{P}_2(\mathbb{R}^d), \ y, y' ∈ \mathbb{R}^d \).

2) By \( C^2_b(\mathcal{P}_2(\mathbb{R}^d)) \) we denote the space of all functions \( \varphi ∈ C^{1,1}_b(\mathcal{P}_2(\mathbb{R}^d)) \) with \( (\partial_μ \varphi)_j(μ, ·) : \mathbb{R}^d \to \mathbb{R} \) is differentiable, for every \( μ ∈ \mathcal{P}_2(\mathbb{R}^d) \), and the derivative \( \partial_y \partial_μ \varphi : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d \) is bounded and continuous.

Here we use the notation \( \partial_μ \varphi(μ, y) := ((\partial_μ \varphi)_j(μ, y))_{1 ≤ j ≤ d'}(μ, y) ∈ \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \).

**Definition 2.2.** We say that \( \varphi \) belongs to \( C^{1,2,2}_b([0, T] \times \mathbb{R}^d × \mathcal{P}_2(\mathbb{R}^d)) \), if \( \varphi : [0, T] \times \mathbb{R}^d × \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) satisfies

(i) \( \varphi(·, ·, μ) ∈ C^{1,2}_b([0, T] \times \mathbb{R}^d), \) for all \( μ ∈ \mathcal{P}_2(\mathbb{R}^d) \);

(ii) \( \varphi(t, x, ·) ∈ C^2_b(\mathcal{P}_2(\mathbb{R}^d)), \) for all \( (t, x) ∈ [0, T] × \mathbb{R}^d \);

(iii) All derivatives of order 1 and 2 are continuous on \( [0, T] × \mathbb{R}^d × \mathcal{P}_2(\mathbb{R}^d) × \mathbb{R}^d \), \( \partial_μ \varphi \) and \( \partial_y(\partial_μ \varphi) \) are bounded over \( [0, T] × \mathbb{R}^d × \mathcal{P}_2(\mathbb{R}^d) × \mathbb{R}^d \).

Now we give a general Itô’s formula for the jump case which generalizes that in [6] and [13].

**Theorem 2.1.** (Itô’s formula)

Let \( F ∈ C^{1,2,2}_b([0, T] × \mathbb{R}^d × \mathcal{P}_2(\mathbb{R}^d)) \). We consider the following two Itô processes:

\[ X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s + \int_0^t \int K_s(e) N_λ(ds, de), \ t ∈ [0, T], \quad (2.2) \]

where \( b ∈ \mathcal{H}^2(0, T; \mathbb{R}^d), \ σ ∈ \mathcal{H}^2(0, T; \mathbb{R}^{d×d}), \ β ∈ \mathcal{K}^2(0, T; \mathbb{R}_+), \ X_0 ∈ L^2(\mathcal{F}_0; \mathbb{R}^d) \), and

\[ U_t = U_0 + \int_0^t u_s ds + \int_0^t v_s dB_s + \int_0^t \int K_s(e) N_λ(ds, de), \ t ∈ [0, T], \quad (2.3) \]

where \( u ∈ L^2_0(\Omega; L^1([0, T]; \mathbb{R}^d)), \ v ∈ L^2_0(\Omega; L^2([0, T]; \mathbb{R}^{d×d})), \ γ ∈ \mathcal{K}^0_λ(0, T; \mathbb{R}^d) \) is such that \( |γ_ε(e)| ≤ ζ(1 + |e|), \) P-a.s., \( (s, e) ∈ [0, T] × K, \) with \( ζ ≥ 0, ζ ∈ L^p(\mathcal{F}_0), \) and \( U_0 ∈ L^p(\mathcal{F}_0; \mathbb{R}^d) \).

Then, for all \( t ∈ [0, T] \) we have

\[ *1) \ L^2(\Omega; L^1([0, T]; \mathbb{R}^d)) \) is the set of \( F \)-adapted processes \( u : [0, T] × Ω → \mathbb{R}^d \) with \( \int_0^T |u(s)| ds < +∞ \), P-a.s.;

\[ 2) \ L^2(\Omega; L^2([0, T]; \mathbb{R}^{d×d})) \) is the set of \( F \)-predictable processes \( v : [0, T] × Ω → \mathbb{R}^{d×d} \) with \( \int_0^T |v(s)|^2 ds < +∞ \), P-a.s.;

\[ 3) \ \mathcal{K}^0_λ(0, T; \mathbb{R}^d) \) is the set of \( \mathcal{P}^0 ⊗ \mathcal{B}(K) \)-measurable processes \( γ : [0, T] × Ω × K → \mathbb{R}^d \) with \( \int_0^T \int K_s(e)^2 λ(de) ds < +∞ \), P-a.s.
\[
F(t, U_t, P_{X_t}) - F(0, U_0, P_{X_0})
= \int_0^t \left\{ (\partial_s F)(s, U_s, P_{X_s}) + \sum_{i=1}^d (\partial_{x_i} F)(s, U_s, P_{X_s})v_s^i + \frac{1}{2} \sum_{i,j,k=1}^d (\partial^2_{x_i x_j} F)(s, U_s, P_{X_s})v_s^i v_s^j v_s^k \right\} \, ds 
+ \int_K \left( F(s, U_s + \gamma_s(e), P_{X_s}) - F(s, U_s, P_{X_s}) - \sum_{i=1}^d (\partial_{x_i} F)(s, U_s, P_{X_s}) \gamma_s^i(e) \right) \lambda(de) \, ds 
+ \int_0^t \hat{E} \left[ \sum_{i=1}^d (\partial_s F)_i(s, U_s, P_{X_s}, \bar{X}_s) \hat{b}_s^i + \frac{1}{2} \sum_{i,j,k=1}^d \partial_{y_i} (\partial_{x_j} F)_i(s, U_s, P_{X_s}, \bar{X}_s) \sigma_s^i \sigma_s^j \sigma_s^k \right] \, ds 
+ \int_0^t \int_0^1 \sum_{i=1}^d \left\{ (\partial_s F)_i(s, U_s, P_{X_s}, \bar{X}_s + \rho \beta_s(e)) - (\partial_{x_i} F)_i(s, U_s, P_{X_s}, \bar{X}_s) \right\} \beta(s(e)) \, \rho \lambda(de) \, ds 
+ \int_0^t \sum_{i,j=1}^d (\partial_{x_i} F)(s, U_s, P_{X_s}) v_s^i v_s^j \, dB_s^i + \int_0^t \int_K \left( F(s, U_s - + \gamma_s(e), P_{X_s}) - F(s, U_s - , P_{X_s}) \right) N_\lambda(ds, de). \tag{2.4}
\]

Here \((\bar{X}, \hat{b}, \sigma, \beta)\) denotes an independent copy of \((X, b, \sigma, \beta)\), defined on a probability space \((\Omega, \hat{F}, \hat{P})\). The expectation \(\hat{E}[-]\) on \((\Omega, \hat{F}, \hat{P})\) concerns only random variables endowed with the superscript \(\hat{\cdot}\). The proof of this theorem is given in the Appendix for the convenience.

**Remark 2.1.** Observe that unlike \([12]\) and \([13]\) we don’t need the existence of the second order mixed derivatives \(\partial_x \partial_t F, \partial_t \partial_x F, \partial_t^2 F\) for the Itô formula. This is why they are neither introduced in the definition of the space \(C^{1,2}(\mathbb{R}^d \times \mathbb{P}_2(\mathbb{R}^d))\).

### 3 Mean-field stochastic differential equations with jumps

From now on let us be given deterministic Lipschitz functions \(\sigma : \mathbb{R}^d \times \mathbb{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d \times d}\), \(b : \mathbb{R}^d \times \mathbb{P}_2(\mathbb{R}^d) \to \mathbb{R}^d\), and \(\beta : \mathbb{R}^d \times \mathbb{P}_2(\mathbb{R}^d) \times K \to \mathbb{R}^d\) satisfying

**Assumption (H3.1)**

(i) \(b\) and \(\sigma\) are bounded and Lipschitz continuous on \(\mathbb{R}^d \times \mathbb{P}_2(\mathbb{R}^d)\);

(ii) There exists a positive constant \(L\) such that, for all \(e \in K\), \(x, \bar{x} \in \mathbb{R}^d\), \(\nu, \bar{\nu} \in \mathbb{P}_2(\mathbb{R}^d)\),

\(|\beta(x, \nu, e)| \leq L(1 \wedge |e|), |\beta(x, \nu, e) - \beta(\bar{x}, \bar{\nu}, e)| \leq L(1 \wedge |e|)(|x - \bar{x}| + W_2(\nu, \bar{\nu})).\]

We consider for the initial data \((t, x) \in [0, T] \times \mathbb{R}^d\) and \(\xi \in L^2(\mathcal{F}_t; \mathbb{R}^d)\) the following both stochastic differential equations (SDEs) with jumps:

\[
X^{t, \xi}_s = x + \int_t^s b(X^{t, \xi}_r, P_{X^{t, \xi}_r}) \, dr + \int_t^s \sigma(X^{t, \xi}_r, P_{X^{t, \xi}_r}) \, dB_r + \int_t^s \int_K \beta(X^{t, \xi}_r, P_{X^{t, \xi}_r}, e) N_\lambda(ds, de), \tag{3.1}
\]

and

\[
X^{t, x, \xi}_s = x + \int_t^s b(X^{t, x, \xi}_r, P_{X^{t, x, \xi}_r}) \, dr + \int_t^s \sigma(X^{t, x, \xi}_r, P_{X^{t, x, \xi}_r}) \, dB_r + \int_t^s \int_K \beta(X^{t, x, \xi}_r, P_{X^{t, x, \xi}_r}, e) N_\lambda(ds, de). \tag{3.2}
\]

where \(s \in [t, T]\).

We recall that under the assumption (H3.1) the both SDEs have a unique solution in \(\mathcal{S}_B^2(t, T; \mathbb{R}^d)\) (see, e.g., Hao and Li \([13]\)). In particular, the solution \(X^{t, \xi}\) of the equation \((3.1)\)
allows to determine that of (3.2), and \(X^{t,x,\xi} \in S^2(t, T; \mathbb{R}^d)\) is independent of \(\mathcal{F}_t\). As SDE standard estimates show, we have for some \(C \in \mathbb{R}_+\) depending only on the Lipschitz constants of \(\sigma, b\) and \(\beta\),

\[
E\left[ \sup_{s \in [t,T]} |X^{s,x,\xi}_s - X^{s',x',\xi}_s|^2 \right] \leq C|x - x'|^2,
\]

for all \(t \in [0,T]\), \(x, x' \in \mathbb{R}^d\), \(\xi \in L^2(\mathcal{F}_t; \mathbb{R}^d)\). This allows to substitute in (3.2) for \(x\) the random variable \(\xi\) and shows that \(X^{t,x,\xi}|_{x=\xi}\) solves the same SDE as \(X^{t,\xi}\). From the uniqueness of the solution we conclude

\[
X^{t,\xi}_s = X^{s,x,\xi}_{x=\xi} = X^{s,\xi}_s, \quad s \in [t, T].
\]  

Moreover, we deduce the following flow property

\[
(X^{s,x,\xi}_r, p, \xi) = (X^{s,t,\xi}_r, p, \xi), \quad r \in [s, T], \quad \text{for all } 0 \leq t \leq s \leq T, \ x \in \mathbb{R}^d, \ \xi \in L^2(\mathcal{F}_t; \mathbb{R}^d).
\]

In fact, putting \(\eta = X^{s,\xi}_s \in L^2(\mathcal{F}_s; \mathbb{R}^d)\), and considering the SDEs (3.1) and (3.2) with the initial data \((s, \eta)\) and \((s, y)\), respectively,

\[
X^{s,\eta}_r = \eta + \int_s^r b(x^{s,\eta}_u, P_{X^{s,\eta}_u}) du + \int_s^r \sigma(x^{s,\eta}_u, P_{X^{s,\eta}_u}) dB_u + \int_s^r \beta(x^{s,\eta}_u, P_{X^{s,\eta}_u}, e) N_\lambda(du, de),
\]

and

\[
X^{s,y}_r = y + \int_s^r b(x^{s,y}_u, P_{X^{s,y}_u}) du + \int_s^r \sigma(x^{s,y}_u, P_{X^{s,y}_u}) dB_u + \int_s^r \beta(x^{s,y}_u, P_{X^{s,y}_u}, e) N_\lambda(du, de),
\]

\(r \in [s, T]\), we get from the uniqueness of the solution of (3.1) that \(X^{s,\eta}_r = X^{t,\xi}_r\), \(r \in [s, T]\), and, consequently, from the uniqueness of the solution of (3.2) \(X^{s,x,\xi}_{r, \xi} = X^{t,x,\xi}_r\), \(r \in [t, T]\), i.e., we have (3.5).

We have to show that the solution \(X^{t,x,\xi}\) does not depend on \(\xi\) itself but only on its law \(P_\xi\). For this, the following lemma is very useful; please refer to Hao and Li [13].

**Lemma 3.1.** For all \(p \geq 2\) there is a constant \(C_p > 0\) only depending on the Lipschitz constants of \(\sigma, b\) and \(\beta\), such that we have the following estimates

\[
\begin{align*}
\text{i)} \quad & E\left[ \sup_{s \in [t,T]} |X^{s,x,\xi}_s - \hat{X}^{s,x,\xi}_s|^p |\mathcal{F}_t| \right] \leq C_p \left( |\xi - \hat{\xi}|^p + W_2(P_\xi, \hat{P}_\xi)^p \right), \\
\text{ii)} \quad & E\left[ \sup_{s \in [t,T]} |X^{s,x,\xi}_s - \hat{X}^{s,x,\xi}_s|^p |\mathcal{F}_t| \right] \leq C_p \left( |x - \hat{x}|^p + W_2(P_\xi, \hat{P}_\xi)^p \right), \\
\text{iii)} \quad & E\left[ \sup_{s \in [t,T]} |X^{s,x,\xi}_s|^p |\mathcal{F}_t| \right] \leq C_p (1 + |x|^p), \\
\text{iv)} \quad & E\left[ \sup_{s \in [t,T]} |X^{s,\xi}_s|^p |\mathcal{F}_t| \right] \leq C_p (1 + |\xi|^p), \\
\text{v)} \quad & \sup_{s \in [t,T]} W_2(P_{X^{s,t,\xi}}, P_{X^{s,\xi}}) \leq C_2 W_2(P_\xi, \hat{P}_\xi), \\
\text{vi)} \quad & E\left[ \sup_{|t,t+h|} |X^{s,\xi}_s - \xi|^p + |X^{s,x,\xi}_s - x|^p |\mathcal{F}_t| \right] \leq C_p h,
\end{align*}
\]

for all \(t \in [0,T]\), \(x, \hat{x} \in \mathbb{R}^d\), \(\xi, \hat{\xi} \in L^2(\mathcal{F}_t; \mathbb{R}^d)\).
Assumption (H4.1) and satisfy:

\[ \tilde{\xi} \quad \text{and, extending the notation introduced in the preceding section for functions to random variables} \]

\[ \tilde{\xi} \in L^2(\mathcal{F}_t; \mathbb{R}^d) \]

Remark 3.1. An immediate consequence of the above Lemma 3.1(ii) is that, given \((t, x) \in [0, T] \times \mathbb{R}^d\), the processes \(X^{t,x,\xi_1}\) and \(X^{t,x,\xi_2}\) are indistinguishable, whenever the laws of \(\xi_1 \in L^2(\mathcal{F}_t; \mathbb{R}^d)\) and \(\xi_2 \in L^2(\mathcal{F}_t; \mathbb{R}^d)\) are the same. But this means that we can define

\[ X^{t,x,F^k} := X^{t,x,\xi}, \quad (t, x) \in [0, T] \times \mathbb{R}^d, \quad \xi \in L^2(\mathcal{F}_t; \mathbb{R}^d), \]  

(3.9)

and, extending the notation introduced in the preceding section for functions to random variables and processes, we shall consider the lifted process

\[ \tilde{X}^{t,x,F^k} := X^{t,x,F^k} = X^{t,x,\xi}, \quad s \in [t, T], \quad (t, x) \in [0, T] \times \mathbb{R}^d, \quad \xi \in L^2(\mathcal{F}_t; \mathbb{R}^d). \]

However, we prefer to continue to write \(X^{t,x,\xi}\) and reserve the notation \(\tilde{X}^{t,x,F^k}\) for an independent copy of \(X^{t,x,F^k}\), which we will introduce later.

4 Mean-field BSDEs with jumps

In this section we consider mean-field BSDEs driven by a Brownian motion and an independent compensated Poisson random measure. The existence and the uniqueness of the solution for this type of BSDEs is proved; for more details please refer to Section 10.2 in the Appendix.

Let \(f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}) \to \mathbb{R}\) and \(\Phi : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}\) be deterministic and satisfy:

**Assumption (H4.1)** The functions \(f\) and \(\Phi\) are bounded and Lipschitz, i.e., there exists a constant \(C > 0\) such that, for all \(x, x' \in \mathbb{R}^d\), \(y, y' \in \mathbb{R}\), \(z, z' \in \mathbb{R}^d\), \(h, h' \in \mathbb{R}\), \(\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R})\),

\[
|f(x, y, z, h, \mu) - f(x', y', z', h', \mu')| = |\Phi(x, \mu) - \Phi(x', \mu')| 
\leq C(|x - x'| + |y - y'| + |z - z'| + |h - h'| + W_2(\mu, \mu')).
\]

Given \(x \in \mathbb{R}^d\) and \(\xi \in L^2(\mathcal{F}_t; \mathbb{R}^d)\) we consider the following both BSDEs with jumps:

\[
\begin{align*}
\{ & dY_s^{t,x,\xi} = -f(\Pi_s^{t,x,\xi}, P_{\Pi_s^{t,x,\xi}})ds + Z_s^{t,x,\xi}dB_s + \int_K H_s^{t,x,\xi}(e)N(\lambda(de, ds), \quad s \in [t, T], \\
& Y_T^{t,x,\xi} = \Phi(X_T^{t,x,\xi}, P_{X_T^{t,x,\xi}}), \}
\end{align*}
\]

(4.1)

and

\[
\begin{align*}
\{ & dY_s^{t,x,\xi} = -f(\Pi_s^{t,x,\xi}, P_{\Pi_s^{t,x,\xi}})ds + Z_s^{t,x,\xi}dB_s + \int_K H_s^{t,x,\xi}(e)N(\lambda(de, ds), \quad s \in [t, T], \\
& Y_T^{t,x,\xi} = \Phi(X_T^{t,x,\xi}, P_{X_T^{t,x,\xi}}), \}
\end{align*}
\]

(4.2)

where

\[
\begin{align*}
\Pi_s^{t,x,\xi} := (X_s^{t,x,\xi}, Y_s^{t,x,\xi}, Z_s^{t,x,\xi}, \int_K H_s^{t,x,\xi}(e)l(e)(\lambda(de))), \quad \Pi_s^{t,x,\xi} := (X_s^{t,x,\xi}, Y_s^{t,x,\xi}, Z_s^{t,x,\xi}, \int_K H_s^{t,x,\xi}(e)l(e)(\lambda(de))),
\end{align*}
\]

and \(l : K \to \mathbb{R}\) is a Borel function with growth condition \(|l(e)| \leq C(1 + |e|), \ e \in K\). Recall that the processes \(X^{t,x,\xi}\) and \(X^{t,x,\xi}\) are the solution of SDEs (3.1) and (3.2), respectively.

Under Assumption (H4.1) we know that from Theorem 10.1 in the Appendix the equation (4.1) has a unique solution \((Y^{t,x,\xi}, Z^{t,x,\xi}, H^{t,x,\xi}) \in S^2_\mathcal{F}(t, T; \mathbb{R}) \times H^2_\mathcal{F}(t, T; \mathbb{R}^d) \times \mathcal{K}^2_\mathcal{F}(t, T; \mathbb{R})\). On the other hand, once having the solution of (4.1), under Assumption (H4.1) the BSDE (4.2) becomes classical and possesses a unique solution \((Y^{t,x,\xi}, Z^{t,x,\xi}, H^{t,x,\xi}) \in S^2_\mathcal{F}(t, T; \mathbb{R}) \times H^2_\mathcal{F}(t, T; \mathbb{R}^d) \times \mathcal{K}^2_\mathcal{F}(t, T; \mathbb{R})\).

Indeed, once we have got \(\Pi_s^{t,x,\xi} = (X_s^{t,x,\xi}, Y_s^{t,x,\xi}, Z_s^{t,x,\xi}, \int_K H_s^{t,x,\xi}(e)l(e)(\lambda(de)))\), we define

\[ \tilde{f}(s, y, z, h) = f(X_s^{t,x,\xi}, y, z, h, P_{\Pi_s^{t,x,\xi}}), \quad \tilde{\xi} = \Phi(X_T^{t,x,\xi}, P_{X_T^{t,x,\xi}}). \]
Proposition 4.1. Suppose the Assumption (H4.1) holds true. Then, for all $p \geq 2$, there exists a constant $C_p > 0$ only depending on the Lipschitz constants of $\sigma$, $b$, $\beta$, $f$ and $\Phi$, such that, for $t \in [0,T]$, $x$, $\tilde{x} \in \mathbb{R}^d$, $\tilde{\xi} \in L^2(\mathcal{F}_t; \mathbb{R}^d)$,

\begin{align}
(\text{i}) & \quad E \left[ \sup_{s \in [t,T]} |Y^t,x,\xi^s|_s^p + \left( \int_t^T |Z^t,x,\xi^s|^2_2 ds \right)^{\frac{p}{2}} + \left( \int_t^T \int_K |H^t,x,\xi^s(e)|^2 \lambda(de)ds \right)^{\frac{p}{2}} |\mathcal{F}_t \right] \leq C_p; \\
(\text{ii}) & \quad E \left[ \sup_{s \in [t,T]} \left| Y^t,x,\xi^s - Y^t,x,\xi^\tilde{x}^s \right|_s^p + \left( \int_t^T |Z^t,x,\xi^s - Z^t,x,\xi^\tilde{x}^s|^2_2 ds \right)^{\frac{p}{2}} + \left( \int_t^T \int_K |H^t,x,\xi^s(e) - H^t,x,\xi^\tilde{x}^s(e)|^2 \lambda(de)ds \right)^{\frac{p}{2}} |\mathcal{F}_t \right] \\
& \quad \leq C_p(|x - \tilde{x}|^p + W_2(P_x, P_{\tilde{x}})^p); \\
(\text{iii}) & \quad \int_t^T W_2(P_{I^{t,x,\xi}}, P_{I^{t,x,\xi}})^2 ds \leq CW_2(P_x, P_{\tilde{x}})^2.
\end{align}

Proof. From Lemma 10.1-2 we get (i) directly. Now we prove (ii) and (iii).

Notice that $\Pi^{t,x,\xi}$ is independent of $\mathcal{F}_t$ and, hence, of $\xi \in L^2(\mathcal{F}_t; \mathbb{R}^d)$. This allows to consider $\Pi^{t,x,\xi}|_{x=\xi}$, and from the uniqueness of the solution of (4.1) and (4.2), it follows from (3.4) that $\Pi^{t,\xi} = \Pi^{t,x,\xi}|_{x=\xi}$. On the other hand, it also follows that, if $\xi' \in L^2(\mathcal{F}_t; \mathbb{R}^d)$ has the same law as $\xi$, then also $\Pi^{t,\xi}|_{x=\xi'}$ and $\Pi^{t,\xi}$ are of the same law. Hence, $P_{\Pi^{t,\xi}} = P_{\Pi^{t,x,\xi}, \xi'}$, $d$-a.e. Then, for given $\xi_i \in L^2(\mathcal{F}_t; \mathbb{R}^d)$ and $\xi_i' \in L^2(\mathcal{F}_t; \mathbb{R}^d)$ of the same law as $\xi_i$, we consider the following BSDE:

\begin{align}
\begin{cases}
\quad dY^{t,\xi_i,\xi_i'}_s = -f(\Pi^{t,\xi_i,\xi_i'}, ds) + Z^{t,\xi_i,\xi_i'}_s dB_s + \int_K H^{t,\xi_i,\xi_i'}_s(e) N_\lambda(ds,de), \quad s \in [t,T], \\
\quad Y^{t,\xi_i,\xi_i'}_T = \Phi(X^{t,\xi_i,\xi_i'}_{T^-}, P_{X^{t,\xi_i,\xi_i'}}).
\end{cases}
\end{align}

From Lemma 10.1-1) and (H4.1) we have, for all $\delta > 0$ (to be specified later) there exists $\beta > 0$, such that

\begin{align}
E[\int_t^T e^{\beta(s-t)} (|Y^{t,\xi_i,\xi_i'}_s - Y^{t,\xi_i,\xi_i'}_s|^2 + |Z^{t,\xi_i,\xi_i'}_s|^2 + |H^{t,\xi_i,\xi_i'}_s(e)|^2 \lambda(de) ds)] \\
\quad \leq Ce^{\beta(T-t)} E[|X^{t,\xi_i,\xi_i'}_T - X^{t,\xi_i,\xi_i'}_T|^2 + W_2(P_{X^{t,\xi_i,\xi_i'}}, P_{X^{t,\xi_i,\xi_i'}}^2)] + C^1 \delta E[\int_t^T e^{\beta(s-t)} (|X^{t,\xi_i,\xi_i'}_s - X^{t,\xi_i,\xi_i'}_s|^2 + W_2(P_{X^{t,\xi_i,\xi_i'}}, P_{X^{t,\xi_i,\xi_i'}}^2)^2) ds] \\
\quad \leq C_{\beta, \delta} E[\sup_{s \in [t,T]} |X^{t,\xi_i,\xi_i'}_s - X^{t,\xi_i,\xi_i'}_s|^2 + W_2(P_{X^{t,\xi_i,\xi_i'}}, P_{X^{t,\xi_i,\xi_i'}}^2)] + C^1 \delta \int_t^T e^{\beta(s-t)} W_2(P_{\Pi^{t,\xi_i,\xi_i'}}, P_{\Pi^{t,\xi_i,\xi_i'}})^2 ds,
\end{align}

(4.4)
where $C^1$ depends only on the Lipschitz constants of $f$ and $\Phi$, while $C_{\beta, \delta}$ depends also on $\beta$ and $\delta$.

From Lemma 3.1 we get

\begin{itemize}
  \item[i)] $W_2(P_{X_T^{t,\xi_1}}, P_{X_T^{t,\xi_2}}) \leq CW_2(P_{\xi_1}, P_{\xi_2})$;
  \item[ii)] $E[\sup_{s \in [t,T]} |X^{t,\xi_1}_s - X^{t,\xi_2}_s|^2] = E[\sup_{s \in [t,T]} |X^{t,x_1}_s - X^{t,x_2}_s|^2] |F_t|_{x_1=\xi_1}^{x_2=\xi_2}$
\end{itemize}

\[ \leq CE[|\xi_1' - \xi_2'|^2 + W_2(P_{\xi_1}, P_{\xi_2})^2]. \]

Therefore, from the above (4.4) and the definition of 2-Wasserstein metric we get

\[ E[\int_t^T e^{\beta(s-t)} (|Y^{t,\xi_1}_s - Y^{t,\xi_2}_s|^2 + |Z^{t,\xi_1}_s - Z^{t,\xi_2}_s|^2)^2 + \int_K |H^{t,\xi_1}_s(e) - H^{t,\xi_2}_s(e)|^2 \lambda (de) ds] \]
\[ \leq C_{\beta, \delta}E[|\xi_1' - \xi_2'|^2 + W_2(P_{\xi_1}, P_{\xi_2})^2] + C \delta \int_t^T e^{\beta(s-t)} W_2(P_{\Xi^{t,\xi_1}_s}, P_{\Xi^{t,\xi_2}_s})^2 ds \]
\[ \leq C_{\beta, \delta}E[|\xi_1' - \xi_2'|^2 + W_2(P_{\xi_1}, P_{\xi_2})^2] + C^1 \delta E[\int_t^T e^{\beta(s-t)} (|Y^{t,\xi_1}_s - Y^{t,\xi_2}_s|^2 + |Z^{t,\xi_1}_s - Z^{t,\xi_2}_s|^2)^2 + \int_K |H^{t,\xi_1}_s(e) - H^{t,\xi_2}_s(e)|^2 \lambda (de) ds]. \]

(4.5)

Now we take $\delta > 0$ small enough such that $C^1 \delta \leq \frac{1}{2}$ we get

\[ E[\int_t^T e^{\beta(s-t)} (|Y^{t,\xi_1}_s - Y^{t,\xi_2}_s|^2 + |Z^{t,\xi_1}_s - Z^{t,\xi_2}_s|^2)^2 + \int_K |H^{t,\xi_1}_s(e) - H^{t,\xi_2}_s(e)|^2 \lambda (de) ds] \]
\[ \leq CE[|\xi_1' - \xi_2'|^2 + W_2(P_{\xi_1}, P_{\xi_2})^2]. \]

(4.6)

Furthermore, from the properties of $W_2$, ii) and (4.6) we get

\[ \int_t^T W_2(P_{\Xi^{t,\xi_1}_s}, P_{\Xi^{t,\xi_2}_s})^2 ds = \int_t^T W_2(P_{\Xi^{t,\xi_1}_s}, P_{\Xi^{t,\xi_2}_s})^2 ds \]
\[ \leq E[\int_t^T (|X^{t,\xi_1}_s - X^{t,\xi_2}_s|^2 + |Y^{t,\xi_1}_s - Y^{t,\xi_2}_s|^2 + |Z^{t,\xi_1}_s - Z^{t,\xi_2}_s|^2)^2 \lambda (de) ds]
\[ + \int_K |H^{t,\xi_1}_s(e) - H^{t,\xi_2}_s(e)|^2 \lambda (de) ds] \]
\[ \leq CE[|\xi_1' - \xi_2'|^2 + W_2(P_{\xi_1}, P_{\xi_2})^2]. \]

Hence, taking the infimum over all $\xi_1', \xi_2' \in L^2(F_t; \mathbb{R}^d)$ with $P_{\xi_i} = P_{\xi_i}, i = 1, 2$, we get

\[ \int_t^T W_2(P_{\Xi^{t,\xi_1}_s}, P_{\Xi^{t,\xi_2}_s})^2 ds \leq CW_2(P_{\xi_1}, P_{\xi_2})^2, \xi_1, \xi_2 \in L^2(F_t; \mathbb{R}^d). \]

(4.7)

This allows now to apply Lemma 10.1.2 to BSDE (4.2) with $g_i(s,y,z,h) := f(X^{t,x_i}_s, y, z, h, P_{\Xi^{t,\xi_i}})$, $\theta_i := \Phi(X^{t,x_i}_T, P_{\xi_i}, P_{X^{t,\xi_i}})$. Then, thanks to Lemma 3.1 and (4.7), for any $p \geq 2$, there exists some $C_p > 0$ only depending on the Lipschitz constants of $b$, $\sigma$, $\beta$, $f$ and $\Phi$, such that for any $\xi_1, \xi_2 \in \mathbb{R}^d$. A.2.9.
\( \in L^2(\mathcal{F}_t; \mathbb{R}^d), \quad x_1, x_2 \in \mathbb{R}^d, \)
\[
E \left[ \sup_{s \in [t,T]} |Y^{t,x_1,\xi}_s - Y^{t,x_2,\xi}_s|^p + \left( \int_t^T |Z^{t,x_1,\xi}_s - Z^{t,x_2,\xi}_s|^2 ds \right)^{\frac{p}{2}} \right] \\
\quad + \left( \int_t^T \int_{\mathcal{K}} |H^{t,x_1,\xi}_s(e) - H^{t,x_2,\xi}_s(e)|^2 \lambda(de)ds\right)^{\frac{p}{2}} |\mathcal{F}_t| \\
\leq C_p \left( E[|X^{t,x_1,\xi}_s - X^{t,x_2,\xi}_s|^p] + W_2(P_{X^{t,x_1,\xi}_s}, P_{X^{t,x_2,\xi}_s})^p \right) \\
\quad + \left( \int_t^T \left(|X^{t,x_1,\xi}_s - X^{t,x_2,\xi}_s|^2 + W_2(P_{\Pi^t_\xi}, P_{\Pi^t_\xi})^2|ds\right)\right)^{\frac{p}{2}} |\mathcal{F}_t| \\
\leq C_p (|x_1 - x_2|^p + W_2(P_{\xi_1}, P_{\xi_2})^p).
\]

The proof is complete. \( \square \)

Recalling that \( (Y^{t,\xi}, Z^{t,\xi}, H^{t,\xi}) = (Y^{t,x,\xi}, Z^{t,x,\xi}, H^{t,x,\xi})|_{x=\xi} \), we have the following result.

**Corollary 4.1.** Suppose the Assumption (H4.1) holds true. Then, for all \( p \geq 2 \), there exists a constant \( C_p > 0 \) only depending on the Lipschitz constants of the coefficients, such that, for \( t \in [0,T], \xi_1, \xi_2 \in L^2(\mathcal{F}_t; \mathbb{R}^d), \)
\[
E \left[ \sup_{s \in [t,T]} |Y^{t,\xi}_s - Y^{t,\xi}_s|^p + \left( \int_t^T |Z^{t,\xi}_s - Z^{t,\xi}_s|^2 ds \right)^{\frac{p}{2}} \right] \\
\quad + \left( \int_t^T \int_{\mathcal{K}} |H^{t,\xi}_s(e) - H^{t,\xi}_s(e)|^2 \lambda(de)ds\right)^{\frac{p}{2}} |\mathcal{F}_t| \\
\leq C_p (E[|\xi_1 - \xi_2|^p] + W_2(P_{\xi_1}, P_{\xi_2})^p) \leq C_p E[|\xi_1 - \xi_2|^p].
\]

From Proposition 4.1 the processes \( Y^{t,x,\xi} = \{Y^{t,x,\xi}_s\}_{s \in [t,T]}, Z^{t,x,\xi} = \{Z^{t,x,\xi}_s\}_{s \in [t,T]} \) and \( H^{t,x,\xi} = \{H^{t,x,\xi}_s\}_{s \in [t,T]} \) depend on \( \xi \) only through its distribution, which means \( (Y^{t,x,\xi}, Z^{t,x,\xi}, H^{t,x,\xi}) \) and \( (Y^{t,\xi}, Z^{t,\xi}, H^{t,\xi}) \) are indistinguishable as long as \( \xi \) and \( \tilde{\xi} \) have the same distribution. Hence we can define \( Y^{t,x,\xi}, Z^{t,x,\xi} \) and \( H^{t,x,\xi} \) by
\[
Y^{t,x,\xi} := Y^{t,x,\xi}, \quad Z^{t,x,\xi} := Z^{t,x,\xi}, \quad H^{t,x,\xi} := H^{t,x,\xi}.
\]

And it follows from the uniqueness of the solution of BSDEs (4.1) and (4.2) that
\[
Y^{t,\xi} = Y^{t,x,\xi}|_{x=\xi}, \quad Z^{t,\xi} = Z^{t,x,\xi}|_{x=\xi}, \quad H^{t,\xi} = H^{t,x,\xi}|_{x=\xi}.
\]

In particular, from (4.3) for \( 0 \leq t \leq s \leq T, x \in \mathbb{R}^d, \xi \in L^2(\mathcal{F}_t; \mathbb{R}^d), \) it holds
\[
Y^{t,\xi} = Y^{t,x,\xi}|_{x=\xi}, \quad Z^{t,\xi} = Z^{t,x,\xi}|_{x=\xi}, \quad H^{t,\xi} = H^{t,x,\xi}|_{x=\xi}.
\]

Now we introduce the value function
\[
V(t, x, \Pi) := Y^{t,x,\Pi}_t.
\]

Notice that \( V(t, x, \Pi) \) is deterministic because we are in the Markovian case. On the other hand, from Proposition 4.1, we can get
\[
V(s, X^{t,x,\Pi}_s, P_{X^{t,x,\Pi}_s}) = Y^{s,X^{t,x,\Pi}_s,\Pi}_{s} = Y^{s,T,\Pi}_{t}, \quad s \in [t,T].
\]

An immediate consequence of Proposition 4.1 is
Proposition 4.2. For $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^d$, $P_x, P_{\bar{x}} \in \mathcal{P}_2(\mathbb{R}^d)$, $$|V(t, x, P_x) - V(t, \bar{x}, P_{\bar{x}})| \leq C(|x - \bar{x}| + W_2(P_x, P_{\bar{x}})).$$

In fact, the value function $V(t, x, P_x)$ is also $\frac{1}{2}$-Hölder continuous with respect to $t$.

Proposition 4.3. There exists some constant $C > 0$ such that, for all $t$, $t' \in [0, T]$, $x \in \mathbb{R}^d$, $\xi \in L^2(F_t; \mathbb{R}^d)$, $$|V(t, x, P_x) - V(t', x, P_x)| \leq C|t-t'|^{\frac{1}{2}}.$$

Proof. Without loss of generality let $0 \leq t < t'$. Then we have

$$|V(t, x, P_x) - V(t', x, P_x)| \leq |E[Y_t^{t,x,P_x} - Y_{t'}^{t',x,P_x}] + E[|Y_{t'}^{t,x,P_x} - Y_{t'}^{t',x,P_x}|]|.$$  \hfill (4.12)

We begin with estimating $|E[Y_t^{t,x,P_x} - Y_{t'}^{t',x,P_x}]|$, as $f$ is bounded we have

$$|E[Y_t^{t,x,P_x} - Y_{t'}^{t',x,P_x}]| \leq |E[\int_t^{t'} |f(\Pi_s^{t,x,P_x}, P_H^{t,x,P_x})|ds] \leq C(t-t').$$  \hfill (4.13)

On the other hand, from (4.9), Proposition 4.1 and Lemma 3.1(v) and vi) we have

$$E[|Y_t^{t,x,P_x} - Y_{t'}^{t',x,P_x}|^2] \leq \left( E[|Y_t^{t,x,P_x} - Y_{t'}^{t',x,P_x}|^2 F_t] \right)^{\frac{1}{2}} \leq C \left( E[|X_t^{t,x,P_x} - X_{t'}^{t',x,P_x}|^2 + |X_{t'}^{t',x,P_x} - \xi|] \right)^{\frac{1}{2}} \leq C|t-t'|^{\frac{1}{2}}.$$  \hfill (4.14)

Hence, from (4.12), (4.13) and (4.14), we get $|V(t, x, P_x) - V(t', x, P_x)| \leq C|t-t'|^{\frac{1}{2}}$. \hfill $\square$

5 First order derivatives of $X^{t,x,P_x}$

In this section we revisit the first order derivatives of $X^{t,x,P_x}$ with respect to $x$ and the measure $P_x$, studied by Hao and Li [13]. For the reader's convenience we give the main results here, for more details the reader is referred to [13], or [6] for the case without jumps.

Assumption (H5.1) For each $e \in K$, the triple of coefficients $(b, \sigma, \beta(\cdot, \cdot, e))$ belongs to $C^{1,1}_{b, \sigma, \beta}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \times \mathbb{R}^{d \times d} \times \mathbb{R}^d)$, i.e., the components $b_j, \sigma_{i,j}, \beta_{j}(\cdot, \cdot, e), 1 \leq i, j \leq d$, satisfy the following properties:

(i) For all $x \in \mathbb{R}^d, e \in K, \sigma_{ij}(x, \cdot), b_j(x, \cdot), \beta_j(x, \cdot, e) \in C^{1,1}_b(\mathcal{P}_2(\mathbb{R}^d))$;

(ii) For all $\nu \in \mathcal{P}_2(\mathbb{R}^d), e \in K, \sigma_{ij}(\cdot, \nu), b_j(\cdot, \nu), \beta_j(\cdot, \nu, e) \in C^{1,1}_b(\mathcal{P}_2(\mathbb{R}^d))$;

(iii) The derivatives $\partial_{x} \sigma_{i,j}, \partial_{x} b_j : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ and $\partial_{\nu} \sigma_{i,j}, \partial_{\nu} b_j : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$ are Lipschitz continuous and bounded;

(iv) There is a constant $C \in \mathbb{R}_+$ such that $\partial_{x} \beta_{j}(\cdot, \cdot, \cdot) : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ and $\partial_{\nu} \beta_{j}(\cdot, \cdot, \cdot, \cdot) : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$ have $C(1 \wedge |e|)$ as bound and as Lipschitz constant, i.e., for all $x, x', y, y' \in \mathbb{R}^d, \nu, \nu' \in \mathcal{P}_2(\mathbb{R}^d), e \in K, 1 \leq j \leq d$;

(v) $|\partial_{\nu} \beta_{j}(x, \nu, e, y)| \leq C(1 \wedge |e|)$, $|\partial_{x} \beta_{j}(x, \nu, e)| \leq C(1 \wedge |e|)$;

(vi) $|\partial_{x} \beta_{j}(x, \nu, e) - \partial_{x} \beta_{j}(x', \nu', e)| \leq C(1 \wedge |e|)(|x - x'| + W_2(\nu, \nu'))$, $|\partial_{\nu} \beta_{j}(x, \nu, e, y) - \partial_{\nu} \beta_{j}(x', \nu', e, y')| \leq C(1 \wedge |e|)(|x - x'| + |y - y'| + W_2(\nu, \nu'))$.

Now we give the first order derivative of $X^{t,x,P_x}$ with respect to $x$. 

Theorem 5.1. Suppose Assumption (H5.1) holds true. Then the $L^2$-derivative of $X^{t,x,P}_k$ with respect to $x$ exists, which is denoted by $\partial_x X^{t,x,P}_k = (\partial_x X^{t,x,P}_k)_{1 \leq j \leq d}$, and it satisfies the following SDE with jumps: $s \in [t, T]$, $1 \leq i, j \leq d$,

$$\partial_x X^{t,x,P}_{s,i,j} = \delta_{ij} + \sum_{k=1}^{d} \int_t^s \partial_x b_j(X^{t,x,P}_r, P_{X^{t,x}_r}) \partial_x X^{t,x,P}_{r,k} \, dr$$

$$+ \sum_{k,l=1}^{d} \int_t^s \partial_x \sigma_{j,l}(X^{t,x,P}_r, P_{X^{t,x}_r}) \partial_x X^{t,x,P}_{r,k} \, dB_r$$

$$+ \sum_{k=1}^{d} \int_t^s \partial_x \beta_j(X^{t,x,P}_r, P_{X^{t,x}_r}, e) \partial_x X^{t,x,P}_{r,k} N_k(dr, de).$$  (5.1)

For the proof the reader is referred to Theorem 4.1 in [13], and for the case without jumps also to Theorem 3.1 in [6]. From the standard estimates of classical SDEs with jumps we have

Proposition 5.1. For all $p \geq 2$, there exists a constant $C_p > 0$ only depending on the Lipschitz constants of $\partial_x \sigma$, $\partial_x b$ and $\partial_x \beta$, such that, for all $t \in [0, T]$, $x, x' \in \mathbb{R}^d$, $\xi, \xi' \in L^2(\mathcal{F}_t; \mathbb{R}^d)$, $P$-a.s.,

(i) $E\left[ \sup_{s \in [t,T]} |\partial_x X^{t,x,P}_s|^p |\mathcal{F}_t \right] \leq C_p$,

(ii) $E\left[ \sup_{s \in [t,T]} |\partial_x X^{t,x,P}_s - \partial_x X^{t',x',P'}_s|^p |\mathcal{F}_t \right] \leq C_p(|x - x'|^p + W_2(P_\xi, P_{\xi'})^p)$,

(iii) $E\left[ \sup_{s \in [t,t+h]} |\partial_x X^{t,x,P}_s - I_{d \times d}|^p |\mathcal{F}_t \right] \leq C_p h, \ 0 \leq t \leq t + h \leq T$.

The following theorem shows that the unique solution $X^{t,x,\xi}_s$ of equation (3.2) interpreted as a functional of $\xi \in L^2(\mathcal{F}_t; \mathbb{R}^d)$ is Fréchet differentiable.

Theorem 5.2. Let $(\sigma, b, \beta)$ satisfy Assumption (H5.1). Then for all $0 \leq t \leq s \leq T$, $x \in \mathbb{R}^d$, the lifted process $L^2(\mathcal{F}_t; \mathbb{R}^d) \ni \xi \rightarrow X^{t,x,\xi}_s := X^{t,x,P}_s \in L^2(\mathcal{F}_s; \mathbb{R}^d)$ is Fréchet differentiable, and the Fréchet derivative is characterized by

$$DX^{t,x,\xi}_s(\eta) = \dot{E}[U^{t,x,P}_s(\xi) \cdot \eta] = \left( \dot{E}\left[ \sum_{j=1}^{d} U^{t,x,P}_s(\xi) \cdot \eta_j \right] \right)_{1 \leq i \leq d},$$

for all $\eta = (\eta_1, \eta_2, \ldots, \eta_d) \in L^2(\mathcal{F}_t; \mathbb{R}^d)$, where for all $y \in \mathbb{R}^d$, $U^{t,x,P}_s(y) = ((U^{t,x,P}_{s,i,j}(y))_{s \in [t,T]})_{1 \leq i \leq d \leq d} \in S^2_t(t,T; \mathbb{R}^{d \times d})$ is the unique solution of the following SDE:

$$U^{t,x,P}_{s,i,j}(y) = \sum_{k=1}^{d} \int_t^s \partial_x b_k(X^{t,x,P}_r, P_{X^{t,x}_r}) U^{t,x,P}_{r,k,j}(y) \, dr$$

$$+ \sum_{k,l=1}^{d} \int_t^s \partial_x \sigma_{i,l}(X^{t,x,P}_r, P_{X^{t,x}_r}, e) U^{t,x,P}_{r,k,j} N_k(dr, de)$$

$$+ \sum_{k=1}^{d} \int_t^s E[(\partial_x \sigma_{i,k})_s(z, P_{X^{t,x}_r}, X^{t,y,P}_r) \partial_x X^{t,y,P}_r, P_{X^{t,x}_r}, e) U^{t,x,P}_{r,k,j}(y) N_k(dr, de)$$

$$+ \sum_{k=1}^{d} \int_t^s E[(\partial_x b_k)_s(z, P_{X^{t,x}_r}, X^{t,y,P}_r) \partial_x X^{t,y,P}_r, P_{X^{t,x}_r}, e) U^{t,x,P}_{r,k,j}(y) N_k(dr, de)$$

$$+ \sum_{k=1}^{d} \int_t^s (\partial_x b_i)_s(z, P_{X^{t,x}_r}, X^{t,y,P}_r) \partial_x X^{t,y,P}_r, P_{X^{t,x}_r}, e) U^{t,x,P}_{r,k,j}(y) N_k(dr, de).$$
We recall from Proposition 4.1 that \((Y_x)\)

\[ \frac{d}{dt} \int_{K} E[(\partial_{\mu} \beta_i)_k(z, P_{X_t^\xi}, X_{r}^{t,y,P,E}, e) \partial_{x_i} X_{r}^{t,y,P,E,k}}

+ (\partial_{\mu} \beta_i)_k(z, P_{X_t^\xi}, X_{r}^{t,y,E}, e) \cdot U_{r,k,j}(y) \right|_{z=X_{r}^{t,y,P,E,k}} N_\lambda(dr, de), \quad s \in [t, T], \quad 1 \leq i, j \leq d, \]

where \(U^{t,y}_{s} = (U^{t,y}_{s,i,j}(y))_{s \in [t, T]} \) satifies (5.2) with \(x\) replaced by \(\xi\).

**Proposition 5.2.** For every \(p \geq 2\), we know that there exists a constant \(C_p > 0\) only depending on the Lipschitz constants of \(b\) and \(\sigma\), such that, for all \(t \in [0, T]\), \(x, x', y, y' \in \mathbb{R}^d\) and \(\xi, \xi' \in L^2(F_T; \mathbb{R}^d)\),

\[
\begin{align*}
(i) & \quad E \left[ \sup_{s \in [t, T]} \left( |U^{t,y}_{s}(y)|^p + |U^{t,y}_{s}(\xi)|^p \right) \right] \leq C_p, \\
(ii) & \quad E \left[ \sup_{s \in [t, T]} \left( |U^{t,y}_{s}(y) - U^{t,y}_{s}(\xi)|^p + |U^{t,y}_{s}(y) - U^{t,y}_{s}(\xi')|^p \right) \right] \\
& \quad \leq C_p \left( |x - x'|^p + |y - y'|^p + W_2(\xi', \xi')^p \right), \\
(iii) & \quad E \left[ \sup_{s \in [t, t+h]} |U^{t,y}_{s}(y)|^p \right] \leq C_p h, \quad 0 \leq h \leq T - t.
\end{align*}
\]

For the proof of Theorems 5.2 and Proposition 5.2, we refer the reader to Section 4 in [13].

In the spirit of Lions and Cardaliaguet (refer to [23], [7]), the derivative of \(X^{t,x,P,E}_s\) with respect to the probability measure can be defined as follows

\[ \partial_x X^{t,x,P,E}_s(y) := U^{t,x,P,E}_s(y), \quad s \in [t, T], \quad t \in [0, T], \quad x \in \mathbb{R}^d, \quad \xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^d), \quad y \in \mathbb{R}^d. \]

With this definition we have \(DX^{t,x,P,E}_s(\xi) = E[\partial_x X^{t,x,P,E}_s(\xi)], \) for all \(\eta \in L^2(\mathcal{F}_T; \mathbb{R}^d).\)

As an immediate result of Proposition 5.2, we have

**Proposition 5.3.** For all \(p \geq 2\), there exists a constant \(C_p > 0\) only depending on the Lipschitz constants of \(b\) and \(\sigma\), such that, for \(t \in [0, T]\), \(x, x', y, y' \in \mathbb{R}^d, \xi, \xi' \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^d), \)

\[
\begin{align*}
(i) & \quad E \left[ \sup_{s \in [t, T]} \left| \partial_x X^{t,x,P,E}_s(y) \right|^p \right] \leq C_p; \\
(ii) & \quad E \left[ \sup_{s \in [t, T]} \left| \partial_x X^{t,x,P,E}_s(y') - \partial_x X^{t,x,P,E}_s(y) \right|^p \right] \leq C_p \left( |x - x'|^p + |y - y'|^p + W_2(\xi', \xi')^p \right); \\
(iii) & \quad E \left[ \sup_{s \in [t, t+h]} \left| \partial_x X^{t,x,P,E}_s(y) \right|^p \right] \leq C_p h, \quad 0 \leq h \leq T - t.
\end{align*}
\]

**6** First order derivatives of \((Y^{t,x,P,E}, Z^{t,x,P,E}, H^{t,x,P,E})\)

We recall from Proposition 4.1 that \((Y^{t,x,P,E}, Z^{t,x,P,E}, H^{t,x,P,E})\) depends on \(\xi\) only through its law, which allows to define \((Y^{t,x,P,E}, Z^{t,x,P,E}, H^{t,x,P,E}) := (Y^{t,x,P,E}, Z^{t,x,P,E}, H^{t,x,P,E})\). This section is devoted to study the first order derivatives of \((Y^{t,x,P,E}, Z^{t,x,P,E}, H^{t,x,P,E})\) with respect to \(x\) and \(P_{\xi}\), respectively.

**Assumption (H6.1)** Let \(\Phi \in C^{1,1}_b(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))\) and \(f \in C^{1,1}_b(\mathbb{R}^{d+1+d+1} \times \mathcal{P}_2(\mathbb{R}^{d+1+d+1})), \) i.e., \(\Phi\) and \(f\) satisfy:

\[
\begin{align*}
i) & \quad \text{For all } x \in \mathbb{R}^d, y \in \mathbb{R}, z \in \mathbb{R}^d, h \in \mathbb{R}, \Phi(x, \cdot) \in C^{1,1}_b(\mathbb{P}_2(\mathbb{R}^d)), f(x, y, z, h, \cdot) \in C^{1,1}_b(\mathbb{P}_2(\mathbb{R}^{d+1+d+1}));
\end{align*}
\]
Under the Assumptions (H5.1) and (H6.1) the equation (4.2) with respect to $x$ has a solution $u(t,x,P)$, and
the following BSDE with jumps:

$$\int_{t}^{T} \partial_t \Phi(X_t, x, P) dt + \int_{t}^{T} \partial_x \Phi(X_t, x, P) \partial_x X_t dt + \int_{t}^{T} \{ \sum_{\ell=1}^{d} \partial_{x \ell} f(\Pi^{t,x,P}_{t}, P_{t}) \partial_{x \ell} X_t \}\ dt + \int_{t}^{T} \{ \sum_{\ell=1}^{d} \partial_{x \ell} f(\Pi^{t,x,P}_{t}, P_{t}) \partial_{x \ell} Z_t \}\ dt$$

$$+ \int_{t}^{T} \partial_{g}(\Pi^{t,x,P}_{t}, P_{t}) \partial_{x} Y_t \ dt + \int_{t}^{T} \partial_{h}(\Pi^{t,x,P}_{t}, P_{t}) \partial_{x} \int_{t}^{T} \Phi(s, x, P) ds \ dt$$

$$- \int_{t}^{T} \int_{K} \partial_{x} H^{t,x,P}_{r}(e) \lambda(ds, de), \ s \in [t, T], \ 1 \leq i \leq d,$$

where $\Pi^{t,x,P}_{t} = (X^{t,x,P}_{t}, Y^{t,x,P}_{t}, Z^{t,x,P}_{t}, \int_{K} H^{t,x,P}_{r}(e) \lambda(ds, de))$, $\Pi^{t,x,P}_{t} |_{x = \xi} = (X^{t,x,P}_{t}, Y^{t,x,P}_{t}, Z^{t,x,P}_{t}, \int_{K} H^{t,x,P}_{r}(e) \lambda(ds, de))$.

As the $L^2$-derivative of the driving coefficient $f(\Pi^{t,x,P}_{t}, P_{t})$ concerns only $\Pi^{t,x,P}_{t}$ but not the law $P_{t}$, the arguments of the proof are standard; the reader is referred, for instance, to [25].

From Lemma 10.1 the standard estimates for classical BSDEs with jumps, combining with Proposition 4.1, Corollary 4.1 and Proposition 5.1, we have that, for every $p \geq 2$, there exists a constant $C_p > 0$ only depending on the Lipschitz constants of the coefficients such that, for all $t \in [0, T]$, $x$, $x' \in \mathbb{R}^d$, $P_t, P_t' \in \mathcal{P}_2(\mathbb{R}^d), d$.

$$E[\sup_{s \in [t, T]} |\partial_x Y^{t,x,P}_{s} - \partial_x Y^{t,x',P}_{s}|^p + \left( \int_{t}^{T} |\partial_x Z^{t,x,P}_{s}|^2 ds \right)^{p/2} + \left( \int_{t}^{T} |\partial_x H^{t,x,P}_{s}|^2 \lambda(ds) ds \right)^{p/2} + \left( \int_{t}^{T} |\partial_x Z^{t,x,P}_{s} - \partial_x Z^{t,x,P}_{s}'|^2 ds \right)^{p/2} + \left( \int_{t}^{T} |\partial_x H^{t,x,P}_{s} - \partial_x H^{t,x,P}_{s}'|^2 \lambda(ds, de) \right)^{p/2} \right] \leq C_p |x - x'|^p + W_2(P_t, P_t').$$

Theorem 6.2. Assume the Assumptions (H5.1) and (H6.1) hold. Then, for all $0 \leq t \leq s \leq T$, $x \in \mathbb{R}^d$, the lifted processes $L^2(\mathcal{F}_t, \mathbb{R}^d) \ni \xi \mapsto \Pi^{t,x,P}_{s}(\xi):= Y^{t,x,P}_{s}(\xi), Z^{t,x,P}_{s}(\xi), H^{t,x,P}_{s}(\xi)$ are Fréchet differentiable, with Fréchet derivatives

$$D\Pi^{t,x,P}_{s}(\xi) = \mathbb{E}[O^{t,x,P}_{s}(\xi)], s \in [t, T], \text{P. a.s.}, \quad D\Pi^{t,x,P}_{s}(\xi) = \mathbb{E}[Q^{t,x,P}_{s}(\xi)], dsdP \text{-a.e.},$$

$$DH^{t,x,P}_{s}(\xi) = \mathbb{E}[R^{t,x,P}_{s}(\xi)], dsd\lambda dP \text{-a.e.},$$

(6.3)
for all $\eta = (\eta_1, \ldots, \eta_d) \in L^2(\mathcal{F}_t; \mathbb{R}^d)$, where, for all $y \in \mathbb{R}^d$, $(O^{t,x,P_\xi}(y), Q^{t,x,P_\xi}(y), R^{t,x,P_\xi}(y)) = \bigg( ((O^{t,x,P_\xi}_{s,j}(y))_{s \in [t,T]} \bigg)_{1 \leq j \leq d} \bigg) \in S^2_{\mathcal{F}}(t,T; \mathbb{R}^d) \times \mathcal{H}^2_{\mathcal{F}}(t,T; \mathbb{R}^{d \times d}) \times \mathcal{K}^2_{\mathcal{F}}(t,T; \mathbb{R}^d)$ is the unique solution of the following BSDE:

$$O^{t,x,P_\xi}_{s,j}(y) = \sum_{k=1}^d \partial_{x_k} \Phi(X^{t,x,P_\xi}_{T}, P_{X^{t,x,P_\xi}_{T}}) \partial_{\mu} X^{t,x,P_\xi}_{T,j} \bigg| \sum_{s=1}^T \int_{\xi} \bigg( \sum_{k=1}^d \partial_{x_k} f(P^{t,x,P_\xi}_{r}, P^{t,x,P_\xi}_{r}) \partial_{\mu} \xi_2 \bigg) \bigg|_{z=\xi}^T \bigg)$$

$$+ \sum_{k=1}^T \int_{s}^T \bigg[ \sum_{k=1}^d \partial_{x_k} f(P^{t,x,P_\xi}_{r}, P^{t,x,P_\xi}_{r}) \partial_{\mu} \xi_2 \bigg] \bigg|_{z=\xi}^T \bigg) \bigg)$$

$$- \sum_{k=1}^d \int_{s}^T Q^{t,x,P_\xi}_{r,k,j}(y) dB^k_r - \sum_{k=1}^T \int_{s}^T R^{t,x,P_\xi}_{r,j} (y,e) \lambda (de), \; s \in [t,T], \; 1 \leq j \leq d,$$

where $(O^{t,\xi}, Q^{t,\xi}, R^{t,\xi}) = (O^{t,\xi,P_\xi}, Q^{t,\xi,P_\xi}, R^{t,\xi,P_\xi})$ is the unique solution of the above BSDE with $x$ replaced by $\xi$.

In order to prove Theorem 6.2, we need the following three lemmas. For simplicity of redaction but w.l.o.g., let us restrict to the dimension $d = 1$ and to $f(x, y, z, h, \gamma) = f(z, h, \gamma(\mathbb{R} \times \mathbb{R} \times \cdot), (x, y, z, h) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $\gamma \in \mathcal{P}_2(\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$) and $\Phi(x, \gamma) = \Phi(x), (x, \gamma) \in \mathbb{R} \times \mathcal{P}_2(\mathbb{R}).$ We first consider the following BSDE with jumps, which is obtained by formal differentiation of the lifted solution $(Y^{t,x,\xi+h,\eta}, Z^{t,x,\xi+h,\eta}, H^{t,x,\xi+h,\eta})$ of BSDE (4.2) (with $\xi+h, \eta$ instead of $\xi, \xi, \xi, \eta \in L^2(\mathcal{F}_t)$) with respect to $h$ at $h = 0$. This formal $L^2$-differentiation (which will be made rigorous later) leads to a triple of processes $(O^{t,\xi}(\eta), Q^{t,\xi}(\eta), R^{t,\xi}(\eta))$ solving the BSDE:

$$O^{t,\xi}(\eta) = \partial_{x} \Phi(X^{t,x}_T, \mathcal{M}_{X^{t,x}_T}(\eta))$$

$$+ \int_{s}^T \bigg[ \partial_{x} f(P^{t,\xi}_r, P^{t,\xi}_r) \mathcal{Q}^{t,\xi}_r(\eta) + \partial_{\mu} f(P^{t,\xi}_r, P^{t,\xi}_r) \int_K \mathcal{R}^{t,\xi}_r (\eta, e) \lambda (de) \bigg] \bigg)$$

$$+ \int_{s}^T \mathcal{E} \bigg[ \partial_{\mu} f + \mathcal{E} \bigg] \bigg)$$

$$+ \int_{s}^T \mathcal{E} \bigg[ \partial_{\mu} f + \mathcal{E} \bigg] \bigg)$$

$$16$$
where \((O(t,\xi), Q(t,\xi), R(t,\xi)) = (O(x,t,\xi), Q(x,t,\xi), R(x,t,\xi))|_{x=\xi}\) is the solution of \((6.5)\) for \(x\) replaced by \(\xi\), and \(U_{s}^{t,x,\xi}(\eta) := DX^{s}_{t,x,\xi}(\eta) = E[\partial_{u}X^{s}_{t,x,\xi}(\tilde{\eta})]\) and \(U_{s}^{t,\xi,\xi}(\eta) = U_{r}^{t,\xi,\xi}(\eta)|_{x=\xi}, r \in [t,T].\)

Of course, in the above BSDE we still use the notations

\[
\Pi^{t,x,\xi}_{s} = (X^{t,x,\xi}_{s}, Y^{t,x,\xi}_{s}, Z^{t,x,\xi}_{s}, \int_{s}^{T} H^{t,x,\xi}_{s}(e)l(e)\lambda(de)), \quad \Pi^{t,\xi}_{s} = (X^{t,\xi}_{s}, Y^{t,\xi}_{s}, Z^{t,\xi}_{s}, \int_{s}^{T} H^{t,\xi}_{s}(e)l(e)\lambda(de)),
\]

and \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\) is a probability space carrying with \((\xi, \tilde{\eta}, \tilde{B}, \tilde{N}_{\lambda})\) an (independent) copy of \((\xi, \eta, B, N_{\lambda})\) (defined on \((\Omega, \mathcal{F}, P)\)); \((\tilde{X}^{t,x,\xi}_{t}, \tilde{Y}^{t,x,\xi}_{t}, \tilde{Z}^{t,x,\xi}_{t}, \tilde{H}^{t,x,\xi}_{t})\) (resp., \((\tilde{X}^{t,\xi}_{t}, \tilde{Y}^{t,\xi}_{t}, \tilde{Z}^{t,\xi}_{t}, \tilde{H}^{t,\xi}_{t})\)) is the solution of the same equation as that for \((X^{t,x,\xi}_{t}, Y^{t,x,\xi}_{t}, Z^{t,x,\xi}_{t}, H^{t,x,\xi}_{t})\) (resp., \((X^{t,\xi}_{t}, Y^{t,\xi}_{t}, Z^{t,\xi}_{t}, H^{t,\xi}_{t})\)), but with the data \((\tilde{\xi}, \tilde{\eta}, \tilde{B}, \tilde{N}_{\lambda})\) instead of \((\xi, B, N_{\lambda})\).

From Theorem 10.1 the equation \((6.5)\) with \(x\) replaced by \(\xi\) has a unique solution \((O^{t,\xi}(\eta), Q^{t,\xi}(\eta), R^{t,\xi}(\eta)) \in S_{P}^{2}(t, T) \times \mathcal{H}_{P}^{2}(t, T) \times K_{\lambda}^{2}(t, T)\). Moreover, from Theorem 10.3 we have that, for all \(p \geq 2\), there exists a constant \(C_{p} > 0\) depending only on \(p\) and the bounds of the coefficients, such that

\[
E[\sup_{s \in [t,T]} |O^{t,\xi}_{s}(\eta)|^{p} + (\int_{t}^{T} |Q^{t,\xi}_{s}(\eta)|^{2}ds)^{p/2} + (\int_{t}^{T} \int_{K} |R^{t,\xi}_{s}(\eta,e)|^{2}\lambda(de)ds)^{p/2}] \leq C_{p}. \tag{6.6}
\]

Once having \((O^{t,\xi}(\eta), Q^{t,\xi}(\eta), R^{t,\xi}(\eta))\), from Theorem 10.1 and Theorem 10.3 again that \((6.5)\) possesses a unique solution \((O^{t,x,\xi}(\eta), Q^{t,x,\xi}(\eta), R^{t,x,\xi}(\eta)) \in S_{P}^{2}(t, T) \times \mathcal{H}_{P}^{2}(t, T) \times K_{\lambda}^{2}(t, T)\), and that for all \(p \geq 2\), there is a constant \(C_{p} > 0\) only depending on the bounds of the coefficients, such that

\[
E[\sup_{s \in [t,T]} |O^{t,x,\xi}_{s}(\eta)|^{p} + (\int_{t}^{T} |Q^{t,x,\xi}_{s}(\eta)|^{2}dr)^{p/2} + (\int_{t}^{T} \int_{K} |R^{t,x,\xi}_{s}(\eta,e)|^{2}\lambda(de)dr)^{p/2}] \leq C_{p}. \tag{6.7}
\]

Lemma 6.1. Suppose \((H5.1)\) and \((H6.1)\) hold true. Then, for all \((t,x) \in [0,T] \times \mathbb{R}, \xi \in L^{2}(\mathcal{F}_{t})\), there exist three stochastic processes \(O^{t,x,\xi}_{s}(y) \in S_{P}^{2}(t, T), Q^{t,x,\xi}_{s}(y) \in \mathcal{H}_{P}^{2}(t, T), R^{t,x,\xi}_{s}(y) \in K_{\lambda}^{2}(t, T)\), depending measurably on \(y \in \mathbb{R}\), such that

\[
O^{t,x,\xi}_{s}(\eta) = E[O^{t,x,\xi}_{s}(\xi) \cdot \tilde{\eta}], \quad P\text{-a.s.}, \quad s \in [t,T], \quad Q^{t,x,\xi}_{s}(\eta) = E[Q^{t,x,\xi}_{s}(\xi) \cdot \tilde{\eta}], \quad dsdP\text{-a.e.},
\]

\[
R^{t,x,\xi}_{s}(\eta) = E[R^{t,x,\xi}_{s}(\xi) \cdot \tilde{\eta}], \quad dsd\lambda dP\text{-a.e.}
\]

In particular, for all \(x \in \mathbb{R}, 0 \leq t \leq s \leq T, \xi \in L^{2}(\mathcal{F}_{t})\), the mappings

\[
O^{t,x,\xi}(\cdot) : L^{2}(\mathcal{F}_{t}) \mapsto L^{2}(\mathcal{F}_{s}), \quad Q^{t,x,\xi}(\cdot) : L^{2}(\mathcal{F}_{t}) \mapsto \mathcal{H}_{P}^{2}(t, T), \quad R^{t,x,\xi}(\cdot) : L^{2}(\mathcal{F}_{t}) \mapsto K_{\lambda}^{2}(t, T),
\]

are linear and continuous.

Remark 6.1. For \((O^{t,\xi}_{s}(y), Q^{t,\xi}_{s}(y), R^{t,\xi}_{s}(y)) := (O^{t,x,\xi}_{s}(y), Q^{t,x,\xi}_{s}(y), R^{t,x,\xi}_{s}(y))|_{x=\xi}, s \in [t,T], \xi \in L^{2}(\mathcal{F}_{t}), y \in \mathbb{R}\), we see directly from Lemma 6.1 that

\[
O^{t,\xi}_{s}(\eta) = E[O^{t,\xi}_{s}(\xi) \cdot \tilde{\eta}], \quad P\text{-a.s.}, \quad s \in [t,T], \quad Q^{t,\xi}_{s}(\eta) = E[Q^{t,\xi}_{s}(\xi) \cdot \tilde{\eta}], \quad dsdP\text{-a.e.},
\]

\[
R^{t,\xi}_{s}(\eta) = E[R^{t,\xi}_{s}(\xi) \cdot \tilde{\eta}], \quad dsd\lambda dP\text{-a.e.}, \quad \eta \in L^{2}(\mathcal{F}_{t}).
\]
Proof. For $y \in \mathbb{R}$, let $(O_t^{t,x,P_t^y}(y), Q_t^{t,x,P_t^y}(y), R_t^{t,x,P_t^y}(y)) \in S^2_{\mathbb{P}}(t,T) \times \mathcal{H}^2_{\mathbb{F}}(t,T) \times \mathcal{K}^2_{\mathbb{F}}(t,T)$ be the unique solution of BSDE (6.4), which, for our special case $(d = 1$ and $f = f(z,h,\gamma(\mathbb{R} \times \mathbb{R} \times \cdot)))$, writes as follows

$$
O_t^{t,x,P_t^y}(y) = \partial_x \Phi(X_t^{t,x,P_t^y}(y)) \partial_x X_t^{t,x,P_t^y}(y)
$$

$$
+ \int_t^T \left[ \partial_x f(\Pi^{t,x,P_t^y}_r, P_{\Pi^{t,x,P_t^y}_r})Q_r^{t,x,P_t^y}(y) + \partial_h f(\Pi^{t,x,P_t^y}_r, P_{\Pi^{t,x,P_t^y}_r}) \int_K R_r^{t,x,P_t^y}(y,e)l(e)\lambda(de) \right] dr
$$

$$
+ \int_s^T \tilde{E} \left[ (\partial_x f_1(\Pi^{t,x,P_t^y}_r, P_{\Pi^{t,x,P_t^y}_r}, \Pi^{t,y,P_t^y}_r) \partial_x \tilde{Z}_r^{t,y,P_t^y} + (\partial_x f_1(\Pi^{t,x,P_t^y}_r, P_{\Pi^{t,x,P_t^y}_r}, \Pi^{t,y,P_t^y}_r) \tilde{Q}_r^{t,y,P_t^y}(y) \right] dr
$$

$$
+ \int_s^T \tilde{E} \left[ (\partial_x f_2(\Pi^{t,x,P_t^y}_r, P_{\Pi^{t,x,P_t^y}_r}, \Pi^{t,y,P_t^y}_r) \int_K \partial_x \tilde{H}_r^{t,y,P_t^y}(e)l(e)\lambda(de) \right] dr
$$

$$
- \int_s^T Q_r^{t,x,P_t^y}(y)dB_r - \int_s^T \int_K R_r^{t,x,P_t^y}(y,e)N_\lambda(dr,de), \ s \in [t,T],
$$

where $(O_t^{t,x,P_t^y}(y), Q_t^{t,x,P_t^y}(y), R_t^{t,x,P_t^y}(y)) := (O_t^{t,x,P_t^y}(y), Q_t^{t,x,P_t^y}(y), R_t^{t,x,P_t^y}(y))_{|x=\xi} \in S^2_{\mathbb{P}}(t,T) \times \mathcal{H}^2_{\mathbb{F}}(t,T) \times \mathcal{K}^2_{\mathbb{F}}(t,T)$ is the unique solution of (6.8) with $x$ replaced by $\xi$, $\Pi^{t,x,P_t^y}_s = (Z^{t,x,P}_s, \int_K H^{t,x,P}_s(e)l(e)\lambda(de))$, and $\Pi^{t,x,P_t^y}_s = \Pi^{t,x,P_t^y}_{t-s}|_{x=\xi}$. It follows from Theorem 10.3 that, for any $p \geq 2$, there is some constant $C_p > 0$ only depending on the bounds of the coefficients such that, for all $t \in [0,T]$, $x \in \mathbb{R}$, $\xi \in L^2(\mathcal{F}_t;\mathbb{R})$, $y \in \mathbb{R}$,

$$
E\left[ \sup_{s \in [t,T]} |O_t^{t,x,P_t^y}(y)|^p + \int_t^T |Q_t^{t,x,P_t^y}(y)|^2 ds \right]^{p/2} + \int_t^T \int_K |R_t^{t,x,P_t^y}(y,e)|^2 \lambda(de)ds \right]^{p/2} \leq C_p,
$$

then again from Theorem 10.3 we get

$$
E\left[ \sup_{s \in [t,T]} |O_t^{t,x,P_t^y}(y)|^p + \int_t^T |Q_t^{t,x,P_t^y}(y)|^2 ds \right]^{p/2} + \int_t^T \int_K |R_t^{t,x,P_t^y}(y,e)|^2 \lambda(de)ds \right]^{p/2} \leq C_p.
$$

Let the couple $(\tilde{\xi}, \tilde{\eta})$ defined on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ be an independent copy of $(\xi, \eta)$ on $(\Omega, \mathcal{F}, P)$ and, in particular, also an independent copy of $(\xi, \tilde{\eta})$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. Substituting in (6.8) for $x$ the random variable $\xi$ and for $y$ the random variable $\tilde{\xi}$, and then multiplying $\tilde{\eta}$ on both sides of the such obtained equation and taking expectation $\tilde{E}[\cdot]$, we obtain

$$
\tilde{E}[O_t^{t,x,P_t^y}(\tilde{\xi}) \cdot \tilde{\eta}] = \partial_x \Phi(X_t^{t,x,P_t^y}(\tilde{\xi})) \partial_x X_t^{t,x,P_t^y}(\tilde{\xi}) \cdot \tilde{\eta}
$$

$$
+ \tilde{E} \left[ \int_t^T \left[ \partial_x f(\Pi^{t,x,P_t^y}_r, P_{\Pi^{t,x,P_t^y}_r})Q_r^{t,x,P_t^y}(\tilde{\xi}) \cdot \tilde{\eta} + \partial_h f(\Pi^{t,x,P_t^y}_r, P_{\Pi^{t,x,P_t^y}_r}) \int_K R_r^{t,x,P_t^y}(\tilde{\xi},e) \cdot \tilde{\eta}l(e)\lambda(de) \right] dr \right]
$$

$$
+ \tilde{E} \left[ \int_s^T \tilde{E} \left[ (\partial_x f_1(\Pi^{t,x,P_t^y}_r, P_{\Pi^{t,x,P_t^y}_r}, \Pi^{t,y,P_t^y}_r) \partial_x \tilde{Z}_r^{t,y,P_t^y} \cdot \tilde{\eta} + (\partial_x f_1(\Pi^{t,x,P_t^y}_r, P_{\Pi^{t,x,P_t^y}_r}, \Pi^{t,y,P_t^y}_r) \tilde{Q}_r^{t,y,P_t^y}(\tilde{\xi}) \cdot \tilde{\eta} \right) dr
$$

$$
+ \int_s^T \tilde{E} \left[ (\partial_x f_2(\Pi^{t,x,P_t^y}_r, P_{\Pi^{t,x,P_t^y}_r}, \Pi^{t,y,P_t^y}_r) \int_K \partial_x \tilde{H}_r^{t,y,P_t^y}(e) \cdot \tilde{\eta}l(e)\lambda(de) \right] dr \right]
$$

$$
- \tilde{E} \left[ \int_s^T Q_r^{t,x,P_t^y}(\xi) \cdot \tilde{\eta}dB_r \right] - \tilde{E} \left[ \int_s^T \int_K R_r^{t,x,P_t^y}(\xi,e) \cdot \tilde{\eta}N_\lambda(dr,de), \ s \in [t,T]. \right.
$$

(6.11)
Since \((\xi, \eta)\) is independent of \((\xi, \eta, \Pi_{t,x}^\xi)\) and \((\xi, \eta, \Pi_{t,x}^\xi, P_{t,x}^\xi)\), and of the same law as \((\xi, \eta)\), we have

\begin{align}
i & E[\tilde{E}[((\partial_\mu f)_1(\Pi_{t,x}^\xi, P_{t,x}^\xi, \Pi_{t,x}^\xi, P_{t,x}^\xi, \Pi_{t,x}^\xi, P_{t,x}^\xi, \Pi_{t,x}^\xi, P_{t,x}^\xi, \Pi_{t,x}^\xi, P_{t,x}^\xi)\tilde{\partial}_x \tilde{Z}_{t,x}^\xi P_{t,x}^\xi \cdot \tilde{\eta}] = E[\tilde{E}[(\partial_\mu f)_1(\Pi_{t,x}^\xi, P_{t,x}^\xi, \Pi_{t,x}^\xi, P_{t,x}^\xi, \Pi_{t,x}^\xi, P_{t,x}^\xi, \Pi_{t,x}^\xi, P_{t,x}^\xi, \Pi_{t,x}^\xi, P_{t,x}^\xi)\tilde{\partial}_x \tilde{Z}_{t,x}^\xi P_{t,x}^\xi \cdot \tilde{\eta}]] \\
ii & E[\tilde{E}[(\partial_\mu f)_2(\Pi_{t,x}^\xi, P_{t,x}^\xi, \Pi_{t,x}^\xi, P_{t,x}^\xi, \Pi_{t,x}^\xi, P_{t,x}^\xi, \Pi_{t,x}^\xi, P_{t,x}^\xi, \Pi_{t,x}^\xi, P_{t,x}^\xi)\int_K \tilde{\partial}_x \tilde{R}_{t,x}^\xi P_{t,x}^\xi(e) \cdot \tilde{\eta}l(e)l(\eta)] \\
& = E[\tilde{E}[(\partial_\mu f)_2(\Pi_{t,x}^\xi, P_{t,x}^\xi, \Pi_{t,x}^\xi, P_{t,x}^\xi, \Pi_{t,x}^\xi, P_{t,x}^\xi, \Pi_{t,x}^\xi, P_{t,x}^\xi, \Pi_{t,x}^\xi, P_{t,x}^\xi)\int_K \tilde{\partial}_x \tilde{R}_{t,x}^\xi P_{t,x}^\xi(e) \cdot \tilde{\eta}l(e)l(\eta)],
\end{align}

similar to other terms. From the above equalities and the uniqueness of the solution of equation (6.3) with \(x\) replaced by \(\xi\) it follows

\begin{equation}
\begin{align}
O_{t,x}^{t,x}(\eta) &= E[O_{t,x}^{t,x}(\xi) \cdot \tilde{\eta}], \quad \text{P-a.s., } s \in [t, T], \quad Q_{t,x}^{t,x}(\eta) = E[Q_{t,x}^{t,x}(\xi) \cdot \tilde{\eta}], \quad \text{d}sdP\text{-a.e.}, \\
R_s^{t,x}(\eta) &= E[R_s^{t,x}(\xi) \cdot \tilde{\eta}], \quad \text{d}sd\lambda dP\text{-a.e.} \quad (6.12)
\end{align}
\end{equation}

Furthermore, from (6.9) we get

\begin{equation}
E[|O_{t,x}^{t,x}(\eta)|^2] = E[|E[O_{t,x}^{t,x}(\xi) \cdot \tilde{\eta}]|^2] \leq E[E[|O_{t,x}^{t,x}(\xi)|^2] \cdot |\tilde{\eta}|^2] = CE[|\tilde{\eta}|^2]. \quad (6.13)
\end{equation}

That means \(|O_{t,x}^{t,x}(\eta)|_{L^2} \leq C|\eta|_{L^2}\), for every \(\eta \in L^2(F_t)\). Hence, \(O_{t,x}^{t,x}(\cdot) : L^2(F_t) \to L^2(F_t)\) is a linear and continuous mapping, for all \(s \in [t, T]\), and \(|O_{t,x}^{t,x}(\cdot)|_{L(L^2, L^2)} \leq C\).

Furthermore, also

\begin{equation}
\begin{align}
E[\int_t^T |Q_{t,x}^{t,x}(\eta)|^2 ds] = E[\int_t^T |E[Q_{t,x}^{t,x}(\xi) \cdot \tilde{\eta}]|^2 ds] \leq E[E[\int_t^T |Q_{t,x}^{t,x}(\xi)|^2 ds \cdot |\tilde{\eta}|^2]] \\
= E[E[\int_t^T |Q_{t,x}^{t,x}(\eta)|^2 ds] | y = \xi \cdot |\tilde{\eta}|^2] \leq CE[|\tilde{\eta}|^2],
\end{align}
\end{equation}

and

\begin{equation}
\begin{align}
E[\int_t^T \int_K |R_s^{t,x}(\eta, e)|^2 \lambda(\eta, e) ds] = E[\int_t^T \int_K |E[R_s^{t,x}(\xi, e) \cdot \tilde{\eta}]|^2 \lambda(\eta, e) ds] \\
\leq E[\int_t^T \int_K E[|R_s^{t,x}(\xi, e)|^2 \cdot |\tilde{\eta}|^2] \lambda(\eta, e) ds] = E[E[\int_t^T \int_K |R_s^{t,x}(\eta, e)|^2 \lambda(\eta, e) ds] | y = \xi \cdot |\tilde{\eta}|^2] \\
\leq CE[|\tilde{\eta}|^2].
\end{align}
\end{equation}

Therefore, \(Q_{t,x}^{t,x}(\cdot) : L^2(F_t) \to H_{t,x}(t, T)\) and \(R_s^{t,x}(\cdot) : L^2(F_t) \to K_{t,x}(t, T)\) are continuous linear mappings. Making use of the above argument, but for \((O_{t,x}^{t,x}, P_{t,x}^\xi(\eta), Q_{t,x}^{t,x}, P_{t,x}^\xi(\eta), R_s^{t,x}, P_{t,x}^\xi(\eta), \Pi_{t,x}^\xi(\eta))\) instead of \((O_{t,x}^{t,x}(\eta), Q_{t,x}^{t,x}(\eta), R_s^{t,x}(\eta), \Pi_{t,x}^\xi(\eta))\), we also have, for all \(\eta \in L^2(F_t)\),

\begin{equation}
\begin{align}
O_{t,x}^{t,x}(\eta) &= E[O_{t,x}^{t,x}(\xi) \cdot \tilde{\eta}], \quad \text{P-a.s., } s \in [t, T], \quad Q_{t,x}^{t,x}(\eta) = E[Q_{t,x}^{t,x}(\xi) \cdot \tilde{\eta}], \quad \text{d}sdP\text{-a.e.}, \\
R_s^{t,x}(\eta) &= E[R_s^{t,x}(\xi) \cdot \tilde{\eta}], \quad \text{d}sd\lambda dP\text{-a.e.} \quad (6.16)
\end{align}
\end{equation}

Moreover, by using (6.10), similar to (6.13), (6.14) and (6.15), we obtain that \(O_{t,x}^{t,x}(\cdot), Q_{t,x}^{t,x}(\cdot), R_s^{t,x}(\cdot)\) are also linear and continuous mappings (over the same spaces as \(O_{t,x}^{t,x}, Q_{t,x}^{t,x}, R_s^{t,x}\)).

Now we prove the following estimate for the solution of equation (6.8).
Proposition 6.1. For all $p \geq 1$, there exists a constant $C_p > 0$ only depending on the Lipschitz constant of the coefficients, such that, for all $t \in [0, T]$, $x$, $x'$, $y$, $y' \in \mathbb{R}^d$, and $\xi, \xi' \in L^2(\mathcal{F}_t; \mathbb{R}^d)$,

$$E[\sup_{s \leq t \leq T} |O^{t,x,P}_s(y) - O^{t,x',P'}_s(y')|^{2p} + (\int_t^T |Q^{t,x,P}_s(y) - Q^{t,x',P'}_s(y')|^2 ds)^p] + (\int_t^T \int_K |R^{t,x,P}_s(y, e) - R^{t,x',P'}_s(y', e)|^2 (de)(ds)) \leq C_p |x - x'|^{2p} + |y - y'|^{2p} + W(\mathcal{P}, \mathcal{P}', \mathcal{P}^2).$$

(6.17)

Proof. Recall that for simplicity of redaction, $d = 1$ and $f(\Pi^{t,x,\xi}_{\mathcal{F}_t}, P^{t,\xi}_{\mathcal{F}_t})$ depends only on $\Pi^{t,x,\xi}_{\mathcal{F}_t} = (Z^{t,x,\xi}_{\mathcal{F}_t}, \int K H^{t,x,\xi}(e)I(e)\lambda(de))$ and $P^{t,\xi}_{\mathcal{F}_t} = \Pi^{t,x,\xi}_{\mathcal{F}_t}$. Let $\xi, \xi', \partial, \vartheta' \in L^2(\mathcal{F}_t)$ be such that $P_\vartheta = P_\xi$, $P_\vartheta' = P_{\xi'}$. Notice that $\Pi^{t,x,P}_s(y)$ and $(O^{t,x,P}_s(y), Q^{t,x,P}_s(y), R^{t,x,P}_s(y))$, $t \leq s \leq T$, are independent of $\mathcal{F}_t$. Hence, from [GS], we get the following BSDE:

$$O^{t,x,P}_s(y) - O^{t,x',P'}_s(y') = \Xi(x, x') + \int_s^T R(r, x, x')dr$$

$$+ \int_s^T \left( (\partial f)(\Pi^{t,x,\xi}_{\mathcal{F}_t}, P^{t,\xi}_{\mathcal{F}_t})(Q^{t,x,P}_r(y) - Q^{t,x',P'}_r(y')) \right)$$

$$+ (\partial h)(\Pi^{t,x,\xi}_{\mathcal{F}_t}, P^{t,\xi}_{\mathcal{F}_t})(\int_K (R^{t,x,P}_r(y, e) - R^{t,x',P'}_r(y', e))I(e)\lambda(de))$$

$$+ \hat{E}[(\partial \mu)f_1(\Pi^{t,x,\xi}_{\mathcal{F}_t}, P^{t,\xi}_{\mathcal{F}_t}, \hat{\Pi}^{\partial,\vartheta}_r)(\lambda^{t,\vartheta,\xi'}_r(y) - \lambda^{t,\vartheta,\xi'}_r(y'))]$$

$$+ \hat{E}[(\partial \mu)f_2(\Pi^{t,x,\xi}_{\mathcal{F}_t}, P^{t,\xi}_{\mathcal{F}_t}, \hat{\Pi}^{\partial,\vartheta}_r)(\int_K \hat{R}_r\lambda^{t,\vartheta,\xi'}_r(y, e)I(e)\lambda(de))]$$

$$- \int_s^T (Q^{t,x,P}_r(y) - Q^{t,x',P'}_r(y'))dB_r - \int_s^T \int_K (R^{t,x,P}_r(y, e) - R^{t,x',P'}_r(y', e))N_\lambda(dr, de),$$

where

$$R(r, x, x') = ((\partial f)(\Pi^{t,x,\xi}_{\mathcal{F}_t}, P^{t,\xi}_{\mathcal{F}_t}) - (\partial f)(\Pi^{t,x',\xi'}_{\mathcal{F}_t}, P^{t,\xi'}_{\mathcal{F}_t}))(Q^{t,x',P'}_r(y'))$$

$$+ ((\partial h)(\Pi^{t,x,\xi}_{\mathcal{F}_t}, P^{t,\xi}_{\mathcal{F}_t}) - (\partial h)(\Pi^{t,x',\xi'}_{\mathcal{F}_t}, P^{t,\xi'}_{\mathcal{F}_t}))(\int_K R^{t,x',P'}_r(y', e)I(e)\lambda(de))$$

$$+ \hat{E}[(\partial \mu)f_1(\Pi^{t,x,\xi}_{\mathcal{F}_t}, \hat{\Pi}^{\partial,\vartheta}_r, \hat{\Pi}^{\partial,\vartheta,\xi'}_r)(\lambda^{t,\vartheta,\xi'}_r(y) - \lambda^{t,\vartheta,\xi'}_r(y'))]$$

$$+ \hat{E}[(\partial \mu)f_2(\Pi^{t,x,\xi}_{\mathcal{F}_t}, \hat{\Pi}^{\partial,\vartheta}_r, \hat{\Pi}^{\partial,\vartheta,\xi'}_r)(\int_K \hat{R}_r\lambda^{t,\vartheta,\xi'}_r(y, e)I(e)\lambda(de))]$$

$$+ \hat{E}[(\partial \mu)f_1(\Pi^{t,x,\xi}_{\mathcal{F}_t}, P^{t,\xi}_{\mathcal{F}_t}, \hat{\Pi}^{\partial,\vartheta,\xi'}_r)(\partial \vartheta Z^{t\vartheta,\xi'}_r - (\partial \mu)f_1(\Pi^{t,x,\xi}_{\mathcal{F}_t}, P^{t,\xi}_{\mathcal{F}_t}, \hat{\Pi}^{\partial,\vartheta,\xi'}_r)]$$

$$+ \hat{E}[(\partial \mu)f_2(\Pi^{t,x,\xi}_{\mathcal{F}_t}, P^{t,\xi}_{\mathcal{F}_t}, \hat{\Pi}^{\partial,\vartheta,\xi'}_r)(\int_K \hat{R}_r\lambda^{t,\vartheta,\xi'}_r(e)I(e)\lambda(de))]$$

$$- (\partial \mu)f_2(\Pi^{t,x,\xi}_{\mathcal{F}_t}, P^{t,\xi}_{\mathcal{F}_t}, \hat{\Pi}^{\partial,\vartheta,\xi'}_r)(\int_K \hat{R}_r\lambda^{t,\vartheta,\xi'}_r(e)I(e)\lambda(de))]$$

and

$$\Xi(x, x') = (\partial_x \Phi)(X^{t,x,P}_T)\partial_{\mu}X^{t,x,P}_T(y) - (\partial_x \Phi)(X^{t,x',P'}_T)\partial_{\mu}X^{t,x',P'}_T(y').$$
From Propositions [5.1 and 5.3] for all \( p \geq 1 \), it holds
\[
E[(\Xi(x,x'))^{2p}] \leq C_p(|x-x'|^{2p} + W_2(P_\xi, P_{\xi'})^{2p} + |y-y'|^{2p}).
\] (6.19)

We now give the estimate for \( R(r,x,x') \). From (6.10) and Proposition 4.1 we notice that
\[
i) \quad E\left[\left(\int_0^T ((\partial_\mu f)(T_r^{1,x',\xi}, P_{t',\xi'}) - (\partial_\mu f)(T_r^{1,x',\xi}, P_{t',\xi'})) \cdot |Q_r^{t,x',\xi}(y')| dr \right)^{2p}\right]
\leq C_p\left(E\left[\left(\int_0^T |Q_r^{t,x',\xi}(y')| dr \right)^{2p}\right]\right)^{\frac{1}{2}} \left(E\left[\left(\int_0^T (\Pi_r^{2,x',\xi} - \Pi_r^{2,x',\xi'})^2 + W_2(P_{t',\xi}, P_{t',\xi'})^2\right) dr \right)^{2p}\right)^{\frac{1}{2}}
\leq C_p(|x-x'|^{2p} + W_2(P_\xi, P_{\xi'})^{2p} + \left(E[|\hat{\vartheta} - \vartheta'|]^p\right)^{p});
\]
\[
ii) \quad E\left[\left(\int_0^T |\tilde{E}((\partial_\mu f)(T_r^{1,x',\xi}, P_{t',\xi}), \hat{T}_r^{1,x',\xi}, \hat{P}_r)\right) dr \right)^{2p}\right]
\leq C_p\left(E\left[\left(\int_0^T |\tilde{E}(\hat{T}_r^{1,x',\xi}, \hat{P}_r)\right) dr \right)^{2p}\right)^{\frac{1}{2}} \left(E\left[\left(\int_0^T (\Pi_r^{1,x',\xi} - \Pi_r^{1,x',\xi'})^2 + W_2(P_{t',\xi}, P_{t',\xi'})^2\right) dr \right)^{2p}\right)^{\frac{1}{2}}
\leq C_p(|x-x'|^{2p} + W_2(P_\xi, P_{\xi'})^{2p} + \left(E[|\hat{\vartheta} - \vartheta'|]^p\right)^{p});
\]
\[
iii) \quad E\left[\left(\int_0^T |\partial_\vartheta Z_r^{1,y,\xi} - \partial_\vartheta Z_r^{1,y,\xi'}| dr \right)^{2p}\right]
\leq C_p\left(E\left[\left(\int_0^T |\partial_\vartheta Z_r^{1,y,\xi} - \partial_\vartheta Z_r^{1,y,\xi'}| dr \right)^{2p}\right]\right)^{\frac{1}{2}} \left(E\left[\left(\int_0^T (\Pi_r^{2,x',\xi} - \Pi_r^{2,x',\xi'})^2 + W_2(P_{t',\xi}, P_{t',\xi'})^2\right) dr \right)^{2p}\right)^{\frac{1}{2}}
\leq C_p(|x-x'|^{2p} + W_2(P_\xi, P_{\xi'})^{2p} + |y-y'|^{2p}).
\]
The other terms of \( R(r,x,x') \) are estimated in a similar way. Consequently, we get
\[
E[\left(\int_0^T |R(r,x,x')| dr \right)^{2p}] \leq C_p(|x-x'|^{2p} + W_2(P_\xi, P_{\xi'})^{2p} + \left(E[|\vartheta - \vartheta'|]^p\right)^{p}) + |y-y'|^{2p}).
\] (6.20)

Substituting in BSDE (6.18) \( x = \vartheta \) and \( x' = \vartheta' \), since \( E[(\Xi(\vartheta, \vartheta')^2) = E[\Xi(x,x')] = \leq C(\Xi(\vartheta - \vartheta')^2 + W_2(P_\xi, P_{\xi'})^2 + |y-y'|^2) \), and
\[
E[\left(\int_0^T |R(r,x,x')| dr \right)^{2p} = E[E\left[\left(\int_0^T |R(r,x,x')| dr \right)^{2p}\right] \leq C(W_2(P_\xi, P_{\xi'})^2 + E[|\vartheta - \vartheta'|^2] + |y-y'|^2),
\]
it follows from Corollary 10.1 that
\[
E[\sup_{s \in [0,T]} |Q_s^{t,x',\xi}(y) - Q_s^{t,x',\xi}(y')|^2 + \int_0^T (\|Q_r^{t,x',\xi}(y) - Q_r^{t,x',\xi}(y')\|^2)
\leq C(W_2(P_\xi, P_{\xi'})^2 + E[|\vartheta - \vartheta'|^2] + |y-y'|^2).
\] (6.21)
This estimate allows to return to BSDE (6.18). Note that

\[
E\left[\left(\int_t^T \tilde{E}(\partial_u f)_1(\Pi_{1,t}^{t,x,\xi}, \tilde{P}_{t,x,\xi}, \tilde{P}_{t,x,\xi}) (\tilde{Q}_{r}^{t,x,\xi}(y) - \tilde{Q}_{r}^{t,x,\xi}(y')) |dr\right)^{2p}\right]
\leq C_p\left(\tilde{E}\left[\left(\int_t^T \tilde{Q}_{r}^{t,x,\xi}(y) - \tilde{Q}_{r}^{t,x,\xi}(y') |dr\right)^{p}\right] \right. \\
\left. \leq C_p(W_2(P_\xi, P_{\xi'})^{2p} + (E[|\vartheta - \vartheta'|^2])^{p} + |y-y'|^{2p})\right),
\]

and analogously,

\[
E\left[\left(\int_t^T |\tilde{E}(\partial_u f)_2(\Pi_{1,t}^{t,x,\xi}, \tilde{P}_{t,x,\xi}, \tilde{P}_{t,x,\xi}) (\tilde{R}_{r}^{t,x,\xi}(y, e) - \tilde{R}_{r}^{t,x,\xi}(y', e)|l(e)\lambda(de)) |dr\right)^{2p}\right]
\leq C_p(W_2(P_\xi, P_{\xi'})^{2p} + (E[|\vartheta - \vartheta'|^2])^{p} + |y-y'|^{2p}).
\]

Consequently, with the help of Theorem [10.3] recalling (6.19), (6.20), (6.22) and (6.23), we get for the solution of BSDE (6.18)

\[
E\left[\sup_{s \in [t,T]} |\tilde{O}_s^{t,x,P_\xi}(y) - \tilde{O}_s^{t,x,P_{\xi'}}(y')|^{2p} + (\int_t^T |Q_t^{t,x,P_\xi}(y) - Q_t^{t,x,P_{\xi'}}(y')|^{2r}dr)^{p}\right]
+ (\int_t^T \int_K |R_t^{t,x,P_\xi}(y, e) - R_t^{t,x,P_{\xi'}}(y', e)|^{2\lambda(de)ds})^{p}
\leq C_p(|x-x'|^{2p} + W_2(P_\xi, P_{\xi'})^{2p} + |y-y'|^{2p} + (E[|\vartheta - \vartheta'|^2])^{p}),
\]

for all \(\vartheta, \vartheta' \in L^2(F_t)\) with \(P_\vartheta = P_\xi\) and \(P_{\vartheta'} = P_{\xi'}\).

Then from the definition of the 2-Wasserstein metric we get

\[
E\left[\sup_{s \in [t,T]} |\tilde{O}_s^{t,x,P_\xi}(y) - \tilde{O}_s^{t,x,P_{\xi'}}(y')|^{2p} + (\int_t^T |Q_t^{t,x,P_\xi}(y) - Q_t^{t,x,P_{\xi'}}(y')|^{2r}dr)^{p}\right]
+ (\int_t^T \int_K |R_s^{t,x,P_\xi}(y, e) - R_s^{t,x,P_{\xi'}}(y', e)|^{2\lambda(de)ds})^{p}
\leq C_p(|x-x'|^{2p} + W_2(P_\xi, P_{\xi'})^{2p} + |y-y'|^{2p}).
\]

Lemma 6.2. Suppose \((H5.1)\) and \((H6.1)\) hold true. Then, for \(0 \leq t \leq s \leq T\) and \(x \in \mathbb{R}\), the lifted processes \(L^2(F_t) \ni \xi \mapsto Y_s^{t,x,\xi} := Y_s^{t,x,\xi} \in L^2(F_s), L^2(F_t) \ni \xi \mapsto Z_s^{t,x,\xi} := Z_s^{t,x,\xi} \in H^2_F(t, T), L^2(F_t) \ni \xi \mapsto H_s^{t,x,\xi} := H_s^{t,x,\xi} \in K^2(\lambda, \lambda)(t, T)\) as functionals of \(\xi\) are Gâteaux differentiable, and the Gâteaux derivatives in direction \(\eta \in L^2(F_t)\) are just \(O_s^{t,x,\xi}(\eta), Q_s^{t,x,\xi}(\eta)\) and \(R_s^{t,x,\xi}(\eta)\), respectively, i.e.,

\[
\partial_\xi Y_s^{t,x,\xi}(\eta) = O_s^{t,x,\xi}(\eta) = \tilde{E}[O_s^{t,x,P_\xi}(\xi) \cdot \eta], \text{ P.a.s., } s \in [t,T],
\]

\[
\partial_\xi Z_s^{t,x,\xi}(\eta) = Q_s^{t,x,\xi}(\eta) = \tilde{E}[Q_s^{t,x,P_\xi}(\xi) \cdot \eta], \text{ dsdP-a.e.,}
\]

\[
\partial_\xi H_s^{t,x,\xi}(\eta) = R_s^{t,x,\xi}(\eta) = \tilde{E}[R_s^{t,x,P_\xi}(\xi) \cdot \eta], \text{ dsd}\lambda dP-a.e.,
\]

where \(O_s^{t,x,\xi}(\eta), Q_s^{t,x,\xi}(\eta), R_s^{t,x,\xi}(\eta)\) are defined in Lemma 6.1.

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Proof. The proof is split into two steps.

**Step 1.** We prove that the directional derivatives of \(Y^{t,x,\xi}, Z^{t,x,\xi}, H^{t,x,\xi}\) in all direction \(\eta \in L^2(\mathcal{F}_t)\) exist, and

\[
\begin{align*}
\mathcal{O}^{t,x,\xi}(\eta) - \frac{1}{h}(Y^{t,x,\xi+h\eta} - Y^{t,x,\xi}) \xrightarrow{h \to 0} 0, & \quad \mathcal{Q}^{t,x,\xi}(\eta) - \frac{1}{h}(Z^{t,x,\xi+h\eta} - Z^{t,x,\xi}) \xrightarrow{h \to 0} 0, \\
\mathcal{R}^{t,x,\xi}(\eta) - \frac{1}{h}(H^{t,x,\xi+h\eta} - H^{t,x,\xi}) \xrightarrow{h \to 0} 0.
\end{align*}
\]

In fact, for all \(s \in [t,T]\),

\[
\begin{align*}
\frac{1}{h}(Y^{s,x,\xi+h\eta} - Y^{s,x,\xi}) - \mathcal{O}^{s,x,\xi}(\eta) &= \frac{1}{h} \left( \Phi(X^{s,x,\xi+h\eta}) - \Phi(X^{s,x,\xi}) \right) - \partial_x \Phi(X^{s,x,\xi})U^{s,x,\xi}(\eta) \\
&+ \frac{1}{h} \int_s^T \left( f(\Pi^{s,x,\xi,h\eta}, P_{\Pi^{s,x,\xi}}) - f(\Pi^{s,x,\xi}, P_{\Pi^{s,x,\xi}}) \right) \lambda(\eta) \lambda(de) \\
&- \left\{ \int_s^T \partial_x f(\Pi^{s,x,\xi}, P_{\Pi^{s,x,\xi}}) \mathcal{Q}^{s,x,\xi}(\eta) \right\} d\mathcal{K} \\
&+ \int_s^T \mathcal{E}[(\partial_x f)(\Pi^{s,x,\xi}, P_{\Pi^{s,x,\xi}}) + \partial_x \Phi(X^{s,x,\xi})U^{s,x,\xi}(\eta)] d\mathcal{K} \\
&+ \int_s^T \mathcal{E}[(\partial_x f)(\Pi^{s,x,\xi}, P_{\Pi^{s,x,\xi}}) + \partial_x \Phi(X^{s,x,\xi})U^{s,x,\xi}(\eta)] d\mathcal{K} \\
&= I_1(x) + I_2(x) - \int_s^T \left( Z^{s,x,\xi+h\eta} - Z^{s,x,\xi} \right) \lambda(\eta) \lambda(de) \\
&- \int_s^T \left( H^{s,x,\xi+h\eta} - H^{s,x,\xi} \right) \lambda(\eta) \lambda(de),
\end{align*}
\]

where

\[
\begin{align*}
I_1(x) &= \frac{1}{h} \left( \Phi(X^{t,x,\xi+h\eta}) - \Phi(X^{t,x,\xi}) \right) - \partial_x \Phi(X^{t,x,\xi})U^{t,x,\xi}(\eta) \\
&= \int_0^1 \partial_x \Phi(X^{t,x,\xi} + \rho(X^{t,x,\xi+h\eta} - X^{t,x,\xi})) d\rho \left( X^{t,x,\xi+h\eta} - X^{t,x,\xi} \right) - \partial_x \Phi(X^{t,x,\xi})U^{t,x,\xi}(\eta),
\end{align*}
\]

and \(I_2(x)\) is then defined by (6.26). Notice that

\[
\begin{align*}
I_1(x) &= \frac{1}{h} \left( \Phi(X^{t,x,\xi+h\eta}) - \Phi(X^{t,x,\xi}) \right) - \partial_x \Phi(X^{t,x,\xi})U^{t,x,\xi}(\eta) \\
&= \int_0^1 \partial_x \Phi(X^{t,x,\xi} + \rho(X^{t,x,\xi+h\eta} - X^{t,x,\xi})) d\rho \left( X^{t,x,\xi+h\eta} - X^{t,x,\xi} \right) - \partial_x \Phi(X^{t,x,\xi})U^{t,x,\xi}(\eta) \\
&= \int_0^1 \partial_x \Phi \left( X^{t,x,\xi} + \rho(X^{t,x,\xi+h\eta} - X^{t,x,\xi}) \right) d\rho \left( X^{t,x,\xi+h\eta} - X^{t,x,\xi} \right) + \partial_x \Phi(X^{t,x,\xi}) \left( X^{t,x,\xi+h\eta} - X^{t,x,\xi} \right).
\end{align*}
\]
Consequently, as $\partial_x \Phi$ is Lipschitz and bounded, and as $I_1(x)$ is independent of $\mathcal{F}_t$,

$$E[I_1(x)^2|\mathcal{F}_t] = E[I_1(x)^2] \leq C \frac{1}{h^2} E[|X_T^{t,x,\xi + h\eta} - X_T^{t,x,\xi}|^4] + CE[|\int_0^1 (\partial_y X_T^{t,x,\xi} (\xi + \rho h\eta) - \partial_y X_T^{t,x,\xi} (\xi))\,d\rho\cdot\eta|].$$

From Lemma 5.1, we have

$$E[|X_T^{t,x,\xi + h\eta} - X_T^{t,x,\xi}|^4] \leq CW_2(P_{\xi + h\eta}, P_\xi)^4 \leq C h^4 (E[\eta^2])^2.$$

On the other hand, from Proposition 5.3, as

$$\frac{1}{h^2} (X_T^{t,x,\xi + h\eta} - X_T^{t,x,\xi}) - U_T^{t,x,\xi}(\eta) = \frac{1}{h^2} \int_0^1 \left( \partial_y X_T^{t,x,\xi, \rho h\eta} (\xi + \rho h\eta) - \partial_y X_T^{t,x,\xi} (\xi) \right)\,d\rho\cdot\eta,$$

we have

$$E\left[ \frac{1}{h^2} (X_T^{t,x,\xi + h\eta} - X_T^{t,x,\xi}) - U_T^{t,x,\xi}(\eta) \right]^2 \leq E\left[ \left( \int_0^1 \left( \partial_y X_T^{t,x,\xi, \rho h\eta} (\xi + \rho h\eta) - \partial_y X_T^{t,x,\xi} (\xi) \right)\,d\rho \right)^2 \right] \leq E[\eta^2] \int_0^1 \int_0^1 \left( \partial_y X_T^{t,x,\xi, \rho h\eta} (y) - \partial_y X_T^{t,x,\xi} (y') \right)^2\,d\rho\,dy' \leq CE[\eta^2] \int_0^1 \int_0^1 \left( W_2(P_{\xi + h\eta}, P_\xi)^2 + |y - y'|^2 \right)\,d\rho\,dy' \leq C h^2 (E[\eta^2])^2.$$

This shows that

$$E[I_1(x)^2|\mathcal{F}_t] = E[I_1(x)^2] \leq C h^2 (E[\eta^2])^2. \quad (6.27)$$

We now consider $I_2(x) = \int_*^r I_2(r)\,dr$ with $I_2(r) = I_{2,1}(r) - I_{2,2}(r)$, where

$$I_{2,1}(r) = \frac{1}{h} (f_1 \Pi^{t,x,\xi + h\eta}, P_{\Pi^{t,x,\xi + h\eta}}) - f_1 \Pi^{t,x,\xi}, P_{\Pi^{t,x,\xi}});$$

$$I_{2,2}(r) = (\partial_y f)(\Pi^{t,x,\xi}, P_{\Pi^{t,x,\xi}}) Q_{t,x}^{t,x,\xi} (\eta) + (\partial_y f)(\Pi^{t,x,\xi}, P_{\Pi^{t,x,\xi}}) \int_K R_{t,x}^{t,x,\xi} (\eta, e)(l(e)\lambda(de))$$

$$+ \int_0^1 \left( \partial_y (\Pi^{t,x,\xi}, P_{\Pi^{t,x,\xi}}, \hat{\Pi}^{t,x,\xi} \hat{\eta} + \hat{Q}_{t,x}^{t,x,\xi} (\hat{\eta})) \right) \left( \partial_x \hat{\Pi}^{t,x,\xi} (e) \hat{l}(e)\lambda(de) + \int_K \hat{R}_{t,x}^{t,x,\xi} (\hat{\eta}, e)(l(e)\lambda(de)) \right). \quad (6.28)$$

We put

$$\Pi^{t,x,\xi}(\eta, \rho) := \Pi^{t,x,\xi} + \rho (\Pi^{t,x,\xi + h\eta} - \Pi^{t,x,\xi}), \quad \Pi^{t,\xi}(\eta, \rho) := \Pi^{t,\xi} + \rho (\Pi^{t,\xi + h\eta} - \Pi^{t,\xi}).$$

Then, using the fact that $f \in C^{1,1}_b(\mathbb{R}^2 \times P_2(\mathbb{R}^2))$, we have

$$I_{2,1}(r) = \frac{1}{h} (f_1 \Pi^{t,x,\xi + h\eta}, P_{\Pi^{t,x,\xi + h\eta}}) - f_1 \Pi^{t,x,\xi}, P_{\Pi^{t,x,\xi}})$$

$$= \frac{1}{h} \int_0^1 \partial_y f_1 (\Pi^{t,x,\xi}(\eta, \rho), P_{\Pi^{t,x,\xi}(\eta, \rho)})\,d\rho$$

$$= \int_0^1 \left\{ (\partial_y f)(\Pi^{t,x,\xi}(\eta, \rho), P_{\Pi^{t,x,\xi}(\eta, \rho)}) \left( \frac{1}{h} (Z^{t,x,\xi + h\eta} - Z^{t,x,\xi}) \right) + (\partial_y f)(\Pi^{t,x,\xi}(\eta, \rho), P_{\Pi^{t,x,\xi}(\eta, \rho)}) \left( \int_K \frac{1}{h} (H^{t,x,\xi + h\eta}(e) - H^{t,x,\xi}(e))\lambda(de) \right) \right\} \,d\rho$$

$$+ \int_0^1 \left\{ \hat{E} \left[ (\partial_y f)(\Pi^{t,x,\xi}(\eta, \rho), P_{\Pi^{t,x,\xi}(\eta, \rho)}), \hat{\Pi}^{t,x,\xi}(\hat{\eta}, \rho) \right] \left( \frac{1}{h} (\hat{Z}^{t,x,\xi + h\eta} - \hat{Z}^{t,x,\xi}) \right) \right\} \,d\rho$$

$$+ \hat{E} \left[ (\partial_y f)(\Pi^{t,x,\xi}(\eta, \rho), P_{\Pi^{t,x,\xi}(\eta, \rho)}), \hat{\Pi}^{t,x,\xi}(\hat{\eta}, \rho) \right] \left( \int_K \frac{1}{h} (\hat{H}^{t,x,\xi + h\eta}(e) - \hat{H}^{t,x,\xi}(e))\lambda(de) \right) \right\} \,d\rho.$$
From the Lipschitz property of the derivative of \( f \) we get

\[
I_{2,1}(r) = \langle \partial_x f \rangle (\Pi_{r}^{t,x,\xi}, P_{\Pi_{r}^{t,x,\xi}}(\frac{1}{h}(Z_{r}^{t,x,\xi+h\eta} - Z_{r}^{t,x,\xi})))
+ \langle \partial_x f \rangle (\Pi_{r}^{t,x,\xi}, P_{\Pi_{r}^{t,x,\xi}}(\int_{K} \frac{1}{h}(H_{r}^{t,x,\xi+h\eta}(e) - H_{r}^{t,x,\xi}(e))l(e)\lambda(de)))
+ \hat{E}[(\partial_x f)_1(\Pi_{r}^{t,x,\xi}, P_{\Pi_{r}^{t,x,\xi}}, \hat{\Pi}_{r}^{t,\xi})(\frac{1}{h}(\hat{H}_{r}^{t,\xi+h\eta}(e) - \hat{H}_{r}^{t,\xi}(e))(l(e)\lambda(de))) + R_1(x,h)(r),

where \( R_1(x,h)(r) \) is defined in an obvious way. Also recall that \( \Pi_{r}^{t,x,\xi} = (Z_{r}^{t,x,\xi}, \int_{K} H_{r}^{t,x,\xi}(e)l(e)\lambda(de)), \Pi_{r}^{t,\xi} = (Z_{r}^{t,\xi}, \int_{K} H_{r}^{t,\xi}(e)l(e)\lambda(de)) \). Let us put \( R_{s}^{1}(x,h) = \int_{s}^{T} R_1(x,h)(r)dr \), and \( \| R_{s}^{1}(x,h) \| := \int_{s}^{T} |R_1(x,h)(r)|dr \). Then

\[
E[\| R_{s}^{1}(x,h) \|] \leq CE\left[\left(\int_{t}^{T} \int_{0}^{1} (|\Pi_{r}^{t,x,\xi}(\eta,\rho) - \Pi_{r}^{t,x,\xi}| + W_2(P_{\Pi_{r}^{t,x,\xi}(\eta,\rho)}, P_{\Pi_{r}^{t,x,\xi}}))dr\right)^{2}\right]
+ CE\left[\left(\int_{t}^{T} \int_{0}^{1} \hat{E}[|\Pi_{r}^{t,x,\xi}(\eta,\rho) - \Pi_{r}^{t,x,\xi}| + W_2(P_{\Pi_{r}^{t,x,\xi}(\eta,\rho)}, P_{\Pi_{r}^{t,x,\xi}}) + |\hat{\Pi}_{r}^{t,\xi}(\eta,\rho) - \hat{\Pi}_{r}^{t,\xi}|)
\right)^{2}\right] = I_{3,1} + I_{3,2},
\]

where \( I_{3,1} := CE\left(\int_{t}^{T} \int_{0}^{1} (|\Pi_{r}^{t,x,\xi}(\eta,\rho) - \Pi_{r}^{t,x,\xi}| + W_2(P_{\Pi_{r}^{t,x,\xi}(\eta,\rho)}, P_{\Pi_{r}^{t,x,\xi}})(\frac{1}{h}(Z_{r}^{t,x,\xi+h\eta} - Z_{r}^{t,x,\xi})) + |\Pi_{r}^{t,x,\xi}(\eta,\rho) - \Pi_{r}^{t,x,\xi}|(\frac{1}{h}(H_{r}^{t,x,\xi+h\eta}(e) - H_{r}^{t,x,\xi}(e))l(e)\lambda(de)))dr\right)^{2} \), and

\( I_{3,2} := CE\left(\int_{t}^{T} \int_{0}^{1} \hat{E}[|\Pi_{r}^{t,x,\xi}(\eta,\rho) - \Pi_{r}^{t,x,\xi}| + W_2(P_{\Pi_{r}^{t,x,\xi}(\eta,\rho)}, P_{\Pi_{r}^{t,x,\xi}}) + |\hat{\Pi}_{r}^{t,\xi}(\eta,\rho) - \hat{\Pi}_{r}^{t,\xi}|)
\right)^{2}\).

Thanks to Proposition 4.1 we get that, here using the notation \( |\Pi_{r}^{t,x,\xi'} - \Pi_{r}^{t,x,\xi}| := |Z_{r}^{t,x,\xi'} - Z_{r}^{t,x,\xi}| \), \( \int_{K} |H_{r}^{t,x,\xi'}(e) - H_{r}^{t,x,\xi}(e)|l(e)\lambda(de) \) (similar to \( |\Pi_{r}^{t,x,\xi'} - \Pi_{r}^{t,x,\xi}| \)),

\[
E(\int_{t}^{T} |\Pi_{r}^{t,x,\xi+h\eta} - \Pi_{r}^{t,x,\xi}|^2 dr)^{p} \leq C_{p}(|x - x'|^{2p} + W_2(P_{\xi+h\eta}, P_{\xi})^{2p})
\leq C_{p}(|x - x'|^{2p} + (|h|^2E[\eta^2])^{p}), \quad p \geq 1,
\]
i.e.,
\[\begin{align*}
&i) \quad E(\int_{t}^{T} |\Pi_{r}^{t,x,\xi+h\eta} - \Pi_{r}^{t,x,\xi}|^2 dr)^{2} \leq C_{h}^{2}(E[\eta^2])^{2},
&i) \quad \int_{t}^{T} W_2(P_{\Pi_{r}^{t,x,\xi}(\eta,\rho)}, P_{\Pi_{r}^{t,x,\xi}})^2 dr
\leq E(\int_{t}^{T} |\Pi_{r}^{t,x,\xi+h\eta} - \Pi_{r}^{t,x,\xi}|^2 dr) = E[E(\int_{t}^{T} |\Pi_{r}^{t,x,\xi+h\eta} - \Pi_{r}^{t,x,\xi}|^2 dr)_{|x'=x+h\eta, x=x}]
\leq CE(|x' - x|^2 + W_2(P_{\xi+h\eta}, P_{\xi})^{2})_{|x'=x+h\eta, x=x} \leq C_{h}E[\eta^2];
\end{align*}\]
and, thus,
\[
I_{3,1} \leq \frac{C}{h^2} E[\left( \int_t^T |\Pi_r^{t,x,\xi+h\eta} - \Pi_r^{t,x,\xi}|^2 dr \right)^2] \\
+ \frac{C}{h^2} E[\left( \int_t^T \int_0^1 W_2(P_{\Pi_r^{t,x,\xi+h\eta}}, P_{\Pi_r^{t,x,\xi}}) d\rho \cdot |\Pi_r^{t,x,\xi+h\eta} - \Pi_r^{t,x,\xi}| dr \right)^2] \leq Ch^2(E[|\eta|^2])^2.
\]

On the other hand, we have
\[
I_{3,2} \leq \frac{C}{h^2} E\left[\left( \int_t^T (|\Pi_r^{t,x,\xi+h\eta} - \Pi_r^{t,x,\xi}| + h(E[|\eta|^2]) \right)^2 + (h^2 E[|\eta|^2]) E[\left( \int_t^T |\Pi_r^{t,x,\xi+h\eta} - \Pi_r^{t,x,\xi}| dr \right)^2] \\
\leq \frac{C}{h^2} \left( E\left[\left( \int_t^T |\Pi_r^{t,x,\xi+h\eta} - \Pi_r^{t,x,\xi}|^2 dr \right)^2 \right] + (h^2 E[|\eta|^2]) E[\left( \int_t^T |\Pi_r^{t,x,\xi+h\eta} - \Pi_r^{t,x,\xi}|^2 dr \right)^2] \right) \leq Ch^2(E[|\eta|^2])^2.
\]

From above we get that
\[
E[|| R_t^{(1)}(x, h)||^2 | F_t] \leq Ch^2(E[|\eta|^2])^2. \tag{6.30}
\]

We remark that from (6.22)
\[
\int_t^T (E[\frac{1}{h} (\hat{Z}_r^{t,x,\xi+h\eta} - \hat{Z}_r^{t,x,\xi}) - (\partial_x \hat{Z}_r^{t,x,P_k} \cdot \hat{\eta}) + \frac{1}{h} (\hat{Z}_r^{t,x,P_k} - \hat{Z}_r^{t,x,P_k})])^2 dr \\
= \int_t^T (E[\int_0^1 (\partial_x \hat{Z}_r^{t,x,h\rho \eta,P_k} - \partial_x \hat{Z}_r^{t,x,P_k}) d\rho \cdot \hat{\eta}])^2 dr \\
\leq 1 \int_0^1 \int_t^T (|\partial_x \hat{Z}_r^{t,x,h\rho \eta,P_k} - \partial_x \hat{Z}_r^{t,x,P_k}|^2 dr d\rho \cdot \hat{\eta}) \\
\leq CE[|\eta|^2](h^2 E[|\eta|^2]) + W_2(P_{\Pi_r^{t,x,\xi+h\eta}}, P_{\Pi_r^{t,x,\xi}})^2 \leq C(E[|\eta|^2])^2 \cdot h^2,
\]

and, analogously,
\[
\int_t^T (E[\int_K \left\{ \frac{1}{h} (\hat{H}_r^{t,x,h\rho \eta,P_k} - \hat{H}_r^{t,x,P_k}) \right\} l(\xi) \lambda(\eta, e)])^2 dr \leq C(E[|\eta|^2])^2 \cdot h^2.
\]

Summarizing our above estimates we have from (6.26), (6.27), (6.28) and (6.29)
\[
\frac{1}{h}(Y_r^{t,x,\xi+h\eta} - Y_r^{t,x,\xi}) - Q_r^{t,x,\xi}(\eta) \\
= I_1(x) + I_2(x) - \int_t^T \left( \frac{1}{h} (Z_r^{t,x,\xi+h\eta} - Z_r^{t,x,\xi}) - Q_r^{t,x,\xi}(\eta) \right) dB_r \\
- \int_t^T \int_K \left( \frac{1}{h} (H_r^{t,x,\xi+h\eta} - H_r^{t,x,\xi}) - \tau_r^{t,x,\xi}(\eta, e) \right) N_\lambda(ds, de) \\
= I_1(x) + \int_t^T \left\{ (\partial_x f)(\Pi_r^{t,x,\xi}, P_{\Pi_r^{t,x,\xi}}) \frac{1}{h} (Z_r^{t,x,\xi+h\eta} - Z_r^{t,x,\xi}) - Q_r^{t,x,\xi}(\eta) \\
+ (\partial_t f)(\Pi_r^{t,x,\xi}, P_{\Pi_r^{t,x,\xi}}) \left( \int_K \frac{1}{h} (H_r^{t,x,\xi+h\eta} - H_r^{t,x,\xi}) - \tau_r^{t,x,\xi}(\eta, e) \right) l(\xi) \lambda(\eta, e) \right\} dB_r \\
+ \hat{E}[\int_0^T (\Pi_r^{t,x,\xi}, P_{\Pi_r^{t,x,\xi}}, \hat{P}_r^{t,x,\xi}) (\frac{1}{h} (\hat{Z}_r^{t,x,P_k} - \hat{Z}_r^{t,x,P_k}) - \hat{Q}_r^{t,x,\xi}(\eta))] \\
+ \hat{E}[\left( \int_t^T |\Pi_r^{t,x,\xi+h\eta} - \Pi_r^{t,x,\xi}|^2 dr \right)^2] \leq Ch^2(E[|\eta|^2])^2.
\]
This latter estimate now allows to deduce from (6.33) by using Corollary 10.1 that

\[ E \sum_k (\eta_k e_k) \left( \int_{t_k}^{t_{k+1}} \frac{1}{h} (\hat{H}^{t,x}_r P_{t,x} \eta e - \hat{H}^{t,x}_r P_{t,x} \eta) (\eta e) (\eta e) \lambda (de) \right) \]  

\[ + \int_{s}^{T} \frac{1}{h} (Z^{t,x}_r P_{t,x} \eta - Z^{t,x}_r P_{t,x} \eta) dR_r \]

\[ - \int_{s}^{T} \int_{K} \frac{1}{h} (H^{t,x}_r(x, \eta) - H^{t,x}_r(x, \eta)) N_x (dr, de), \quad s \in [t, T]. \]

Substituting in (6.33) for \( x \) the variable \( \xi \) we get

\[ I_1 (\xi) = \int_{s}^{T} \left\{ (\nabla f)(\Pi_{t,x}^{t,y}, P_{t,x}^{t,y}) \left( \frac{1}{h} (Z^{t,x}_r P_{t,x} \eta - Z^{t,x}_r P_{t,x} \eta) - Q^{t,x}_r \eta \right) \right\} dr \]

\[ + \int_{s}^{T} \frac{1}{h} (H^{t,x}_r P_{t,x} \eta - H^{t,x}_r P_{t,x} \eta) dR_r \]

\[ - \int_{s}^{T} \int_{K} \frac{1}{h} (H^{t,x}_r P_{t,x} \eta - H^{t,x}_r P_{t,x} \eta) N_x (dr, de), \quad s \in [t, T]. \]

Notice that we have \( E[\left| \int_{t}^{T} |R_r (\xi, h)|^2 dr \right|] = E[\left| \int_{t}^{T} |R_r (x, h)|^2 dr \right|] \leq C h^2 (E[|h|^2])^2 \) from (6.30), (6.31), (6.32); and \( E[|I_1 (\xi)|^2] \leq C h^2 (E[|h|^2])^2 \) from (6.27).

Therefore, applying Corollary 10.1 to BSDE (6.34) we get that

\[ E[ \sup_{s \in [t, T]} \left| \frac{1}{h} (Y^{t,x}_s P_{t,x} \eta - Y^{t,x}_s P_{t,x}) - O^{t,x}_s \eta \right|^2] + E\left\{ \int_{t}^{T} \left( \frac{1}{h} (Z^{t,x}_r P_{t,x} \eta - Z^{t,x}_r P_{t,x} \eta) - Q^{t,x}_r \eta \right) \right\} \]

\[ + \int_{s}^{T} \frac{1}{h} (H^{t,x}_r P_{t,x} \eta - H^{t,x}_r P_{t,x} \eta) dR_r \]

\[ \leq C (E[|h|^2])^2 \cdot h^2. \]

This latter estimate now allows to deduce from (6.33) by using Corollary 10.1 that

\[ E[ \sup_{s \in [t, T]} \left| \frac{1}{h} (Y^{t,x}_s P_{t,x} \eta - Y^{t,x}_s P_{t,x}) - O^{t,x}_s \eta \right|^2] + E\left\{ \int_{t}^{T} \left( \frac{1}{h} (Z^{t,x}_r P_{t,x} \eta - Z^{t,x}_r P_{t,x} \eta) - Q^{t,x}_r \eta \right) \right\} \]

\[ + \int_{s}^{T} \frac{1}{h} (H^{t,x}_r P_{t,x} \eta - H^{t,x}_r P_{t,x} \eta) dR_r \]

\[ \leq C (E[|h|^2])^2 \cdot h^2. \]

(6.35)

**Step 2.** In Step 1 we have proved that the directional derivatives of \( Y^{t,x}_s, Z^{t,x}_s, H^{t,x}_s \) in all direction \( \eta \in L^2 (F_t) \) exist and the directional directives \( \partial_\eta Y^{t,x}_s, \partial_\eta Z^{t,x}_s, \partial_\eta H^{t,x}_s \) coincide with \( O^{t,x}_s (\eta), Q^{t,x}_s (\eta), R^{t,x}_s (\eta) \). Recall that \( O^{t,x}_s (\cdot), Q^{t,x}_s (\cdot), R^{t,x}_s (\cdot) \) are linear and continuous mappings. Consequently, \( Y^{t,x}_s, Z^{t,x}_s, H^{t,x}_s \) as functionals of \( \xi \) are Gâteaux differentiable, and furthermore, from Lemma 6.1 the Gâteaux derivatives can be characterized by
The proof is complete.

In order to prove \(Y^{t,x,\xi}, Z^{t,x,\xi}, H^{t,x,\xi}\) are Fréchet differentiable, we want to show
\[
L^2(F_t) \ni \xi \mapsto O_s^{t,x,\xi} \in L(L^2(F_t), L^2(F_s)), \quad L^2(F_t) \ni \xi \mapsto Q_s^{t,x,\xi} \in L(L^2(F_t), \mathcal{H}_2^2(t,T)), \quad L^2(F_t) \ni \xi \mapsto R_s^{t,x,\xi} \in L(L^2(F_t), \mathcal{K}_2^2(t,T))
\]
are continuous.

**Lemma 6.3.** Under the Assumptions (H5.1) and (H6.1), for all \(t \in [0,T], x \in \mathbb{R}\), the mappings
\[
\partial_\xi Y_s^{t,x,\xi} = O_s^{t,x,\xi}, \quad \partial_\xi Z_s^{t,x,\xi} = (Q_s^{t,x,\xi}), \quad \partial_\xi H_s^{t,x,\xi} = (R_s^{t,x,\xi})
\]
onlyx{\text{as the functionals of } \xi \text{ are continuous.}

**Proof.** We only prove that \(\partial_\xi Y_s^{t,x,\xi} = O_s^{t,x,\xi}, \quad \partial_\xi Z_s^{t,x,\xi} = (Q_s^{t,x,\xi})\), and \(\partial_\xi H_s^{t,x,\xi} = (R_s^{t,x,\xi})\) can be proved with a similar argument. From (6.17) we have
\[
|\partial_\xi Y_s^{t,x,\xi} - \partial_\xi Y_s^{t,x,\xi'}|^2_{L^2(F_t),L^2(F_s)} = \sup_{\eta \in L^2(F_t), \|\eta\|_{L^2} \leq 1} |\partial_\xi Y_s^{t,x,\xi} - \partial_\xi Y_s^{t,x,\xi'}(\eta)|^2_{L^2} = \sup_{\eta \in L^2(F_t), \|\eta\|_{L^2} \leq 1} E[|\partial_\xi Y_s^{t,x,\xi} - \partial_\xi Y_s^{t,x,\xi'}(\eta)|^2_{L^2}] \leq \sup_{\eta \in L^2(F_t), \|\eta\|_{L^2} \leq 1} E[E[|O_s^{t,x,\xi'}(\eta) - O_s^{t,x,\xi'}(\eta')|^2_{L^2}] \leq C(E[|\xi - \xi'|^2 + W_2(P_s, P_{s'})^2) \leq 2CE|\xi - \xi'|^2, \quad t \leq s \leq T.
\]

So far, combining the Lemmas 6.1, 6.2 and 6.3 Theorem 6.2 has been proved. As shown in Section 5, \((O^{t,x,P_s}, Q^{t,x,P_s}, P^{t,x,P_s})\) are the derivatives of \((Y^{t,x,P_s}, Z^{t,x,P_s}, H^{t,x,P_s})\) with respect to the measure \(P_s\), i.e., \(\partial_\mu Y_s^{t,x,P_s}(y) := O_s^{t,x,P_s}(y), \partial_\mu Z_s^{t,x,P_s}(y) := Q_s^{t,x,P_s}(y), \partial_\mu H_s^{t,x,P_s}(y) := R_s^{t,x,P_s}(y), \quad s \in [t,T]\). As a direct result of (6.10) and Proposition 6.1 we have

**Proposition 6.2.** For \(p \geq 2\), there exists a constant \(C_p > 0\) only depending on the bounds and Lipschitz constants of the coefficients, such that for all \(t \in [0,T], x, \bar{x} \in \mathbb{R}^d, y, \bar{y} \in \mathbb{R}^d, P_s\), \(P_s \in \mathcal{P}_2(\mathbb{P}^d)\),

\[\begin{align*}
i) E[\sup_{s \in [t,T]} |\partial_\mu Y_s^{t,x,P_s}(y)|^p + (\int_t^T |\partial_\mu Z_s^{t,x,P_s}(y)|^2 ds)^{p/2} + (\int_t^T |\partial_\mu H_s^{t,x,P_s}(y)|^2 \lambda(de)ds)^{p/2}] \leq C_p, \\
ii) E[\sup_{s \in [t,T]} |\partial_\mu Y_s^{t,x,P_s}(y) - \partial_\mu Y_s^{t,x,P_s}(\bar{y})|^p + (\int_t^T |\partial_\mu Z_s^{t,x,P_s}(y) - \partial_\mu Z_s^{t,x,P_s}(\bar{y})| ds)^{p/2} + (\int_t^T |\partial_\mu H_s^{t,x,P_s}(y) - \partial_\mu H_s^{t,x,P_s}(\bar{y})| ds)^{p/2}] \leq C_p(|x - \bar{x}|^p + |y - \bar{y}|^p + W_2(P_s, P_{s'})^p).
\end{align*}\]
7 Second order derivatives of $X^{t,x,P_t}$

In this section we investigate the second order derivatives of $X^{t,x,P_t}$. For this we first give the following definition.

**Definition 7.1.** We say that $g \in C^{2,1}_b(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$, if $g \in C^{1,1}_b(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ is such that

i) For all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $\partial_x g(\cdot, \mu) \in C^{1,1}(\mathbb{R}^d \rightarrow \mathbb{R})$;

ii) For all $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $\partial_x g(x, \mu, \cdot) \in C^{1}(\mathbb{R}^d \rightarrow \mathbb{R}^d)$;

iii) The derivatives $\partial_2^2 g : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}^{d \times d}$, $\partial_y(\partial_\mu g) : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d \times d}$ are bounded and Lipschitz.

(Recall the notation $g \in C^{1,1}_b(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ contains that $\partial_x g$ and $\partial_\mu g$ are bounded and Lipschitz.)

**Assumption (H7.1)** $(b, \sigma) \in C^{2,1}_b(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}^d \times \mathbb{R}^{d \times d})$, $\beta(\cdot, e) \in C^{2,1}_b(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}^d)$ with bounds of the form $C(1 \wedge |e|)$ for all its derivatives of first and second order, and with a Lipschitz constant of the form $C(1 \wedge |e|)$ for $\partial_2^2 \beta(\cdot, e)$ and $\partial_y(\partial_\mu \beta)(\cdot, e)$.

**Theorem 7.1.** Under the assumption (H7.1) the first order derivatives $\partial_x X^{t,x,P_t}_s$ and $\partial_\mu X^{t,x,P_t}_s(y)$ are in $L^2$-differentiable w.r.t. $x$ and $y$, respectively, and interpreted as a functional of $\xi \in L^2(\mathcal{F}_t; \mathbb{R}^d)$, and for

$$M^{t,x,P_t}_{s,i,j}(y) := (\partial^2_{x,i,j}X^{t,x,P_t}_s, \partial_y(\partial_\mu X^{t,x,P_t}_s(y)))$$

we have that, for all $p \geq 2$, there exists a constant $C_p \in \mathbb{R}_+$ such that, for all $t \in [0, T]$, $x$, $x'$, $y$, $y' \in \mathbb{R}^d$, and $\xi$, $\xi' \in L^2(\mathcal{F}_t; \mathbb{R}^d)$, $1 \leq i$, $j \leq d$,

i) $E[\sup_{s \in [t,T]} |M^{t,x,P_t}_{s,i,j}(y)|^p] \leq C_p$;

ii) $E[\sup_{s \in [t,T]} |M^{t,x,P_t}_{s,i,j}(y) - M^{t,x,P_t}_{s,i,j}(y')|^p] \leq C_p(|x - x'|^p + |y - y'|^p + W(\mathbb{P}_t, \mathbb{P}_{t'}))^p$; (7.1)

iii) $E[\sup_{s \in [t,T]} |M^{t,x,P_t}_{s,i,j}(y)|^p] \leq C_p h$, $0 \leq h \leq T - t$.

For the proof we can refer to [13]; here the situation is even more simple since we don’t need to consider the mixed derivatives $\partial_x \partial_\mu, \partial_\mu \partial_x$.

8 Second order derivatives of $(Y^{t,x,P_t}, Z^{t,x,P_t}, H^{t,x,P_t})$

This section is devoted to the study of second order derivatives of $(Y^{t,x,P_t}, Z^{t,x,P_t}, H^{t,x,P_t})$.

**Assumption (H8.1)** Let $\Phi \in C^{2,1}_b(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$, $f \in C^{2,1}_b(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}^{d+1+d+1}))$ (Recall Definition 7.1).

**Theorem 8.1.** Assuming (H7.1) and (H8.1) we have, for all $t \in [0, T]$, $x$, $y \in \mathbb{R}^d$, $\xi \in L^2(\mathcal{F}_t; \mathbb{R}^d)$,

i) The differentiability (in $L^2$) of the mappings

$$\mathbb{R}^d \ni x \rightarrow (\partial_y Y^{t,x,P_t}(x), \partial_x Z^{t,x,P_t}(x), \partial_x H^{t,x,P_t}(x)) \in S^2_\mathbb{R}(t, T; \mathbb{R}^d) \times \mathcal{H}^2(t, T; \mathbb{R}^{d \times d}) \times \mathcal{K}^2(t, T; \mathbb{R}^d)$$

and

$$\mathbb{R}^d \ni y \rightarrow (\partial_y Y^{t,x,P_t}(y), \partial_y Z^{t,x,P_t}(y), \partial_y H^{t,x,P_t}(y)) \in S^2_\mathbb{R}(t, T; \mathbb{R}^d) \times \mathcal{H}^2(t, T; \mathbb{R}^{d \times d}) \times \mathcal{K}^2(t, T; \mathbb{R}^d)$$

Moreover, for all $p \geq 2$, there is some constant $C_p > 0$ only depending on the bounds and the Lipschitz constants of the coefficients $\sigma, b, f, \Phi$ and their first and second order derivatives, such that, for both $(\zeta^{t,x,P_t}_s(y), \xi^{t,x,P_t}_s(y), \eta^{t,x,P_t}_s(y, \cdot)) \in \{ (\partial^2_{y} Y^{t,x,P_t}_s, \partial^2_{x} Z^{t,x,P_t}_s, \partial^2_{x} H^{t,x,P_t}_s(\cdot)) \}, (\partial_y \partial_\mu Y^{t,x,P_t}_s(y), \partial_y \partial_\mu Z^{t,x,P_t}_s(y), \partial_y \partial_\mu H^{t,x,P_t}_s(y, \cdot))$,
Proof. Similar to Theorem 6.1, as the $L^2$-derivative of $(\partial_x Y^{t,x,P}, \partial_z Z^{t,x,P}, \partial_y H^{t,x,P})$ with respect to $x$ concerns only $\Pi^{t,x,P}$ but not the law $P_{y,t}^{\Pi^x}$ of the coefficients of BSDE (6.1), the proof is standard, the reader may refer to, for instance, [25]. Then applying Lemma 10.1 to BSDE satisfied by $(\partial^2_x Y^{t,x,P}, \partial^2_z Z^{t,x,P}, \partial^2_y H^{t,x,P})$ we get directly the estimate a) for $(\partial_x Y^{t,x,P}(y), \partial_z Z^{t,x,P}(y), \partial_y H^{t,x,P}(y))$.

Now let us prove i) for $(\partial_y Y^{t,x,P}(y), \partial_z Z^{t,x,P}(y), \partial_y H^{t,x,P}(y))$ and the associated estimates in ii). Large parts of the proof are standard or similar to the proofs of Theorem 6.2 and Proposition 6.1.

Before in order to consider the main difficulties here but w.l.o.g. let us study the case of dimension $d = 1$, with $\Phi(X_T^{t,x,P}, P_{X_T^{t,x,P}}) = \Phi(X_T^{t,x,P}, P_{X_T^{t,x,P}})$, and $f(\Pi^{t,x,P}, P_{\Pi^{t,x,P}})$ with $\Pi_T^{t,x,P} = \int K H^{t,x,P}(e)\lambda(\mu) \lambda(\mu), \Pi_T^{y,z} = \Pi_T^{y,z}$, and $f(\Pi^{t,x,P}, P_{\Pi^{t,x,P}})$ in our case now $(\partial_y Y^{t,x,P}(y), \partial_z Z^{t,x,P}(y), \partial_y H^{t,x,P}(y))$ is a solution of the following BSDE:

$$a) \ E[ \sup_{t \leq s \leq T} (\ell^{t,x,P}(\xi)(y))^p + (\int_t^T |\ell^{t,x,P}(\xi)(y)|^2 ds)^\frac{p}{2} + (\int_t^T \int K \theta^{t,x,P}(\xi)(y)^2 \lambda(\mu) ds)^\frac{p}{2}) \leq C_p;$$

$$b) \ E[ \sup_{t \leq s \leq T} (\ell^{t,x,P}(\xi)(y))^p - (\ell^{t,x,P}(\xi)(y))^p + (\int_t^T \int K \theta^{t,x,P}(\xi)(y)^2 \lambda(\mu) ds)^\frac{p}{2}) \leq C_p;$$

for all $t \in [0, T], x, x' \in \mathbb{R}^d$, $y, y' \in \mathbb{R}^d$, $\xi, \xi' \in L^2(\mathcal{F}; \mathbb{R}^d)$, $M > 0$, with $\rho_M(t, y, P_{\xi}) \rightarrow 0$, and $E[\rho_M(t, y, P_{\xi})] \rightarrow 0$, as $M \rightarrow \infty$.

Notice that all second order derivatives of $\Phi$ and $f$ are bounded and $\widehat{E}[\int_t^T |\ell^{t,x,P}(\xi)|^2 ds] \leq C_p$ (see (6.2)). We first consider the above equation (8.1) with $x$ replaced by $\xi$, then from Theorem 10.1
this equation has a unique solution \((\partial_y(\partial_x Y_{t,x,p}^{t,x,p}(y)), \partial_y(\partial_x Z_{t,x,p}^{t,x,p}(y)), \partial_y(\partial_x H_{t,x,p}^{t,x,p}(y))) \in S^2(\xi, \xi) \times K^2(t, T) \times K^2(t, T)\), and, furthermore, from Theorem 10.3, for all \(p \geq 2\), there is some \(C_p > 0\) only depending on the bounds and the Lipschitz constants of the coefficients and its derivatives of order 1 and 2 such that

\[
E \left[ \sup_{s \in [t,T]} |\partial_y(\partial_x Y_{t,x,p}^{t,x,p}(y))|^p + \left( \int_t^T |\partial_y(\partial_x Z_{t,x,p}^{t,x,p}(y))|^2 ds \right)^{\frac{p}{2}} \right] \leq C_p, \quad \text{for all } t \in [0,T], \ y \in \mathbb{R}, \ \xi \in L^2(\mathcal{F}_t).
\]

Then return to the equation (8.1) again from Theorem 10.1 it has a unique solution \((\partial_y(\partial_x Y_{t,x,p}^{t,x,p}(y)), \partial_y(\partial_x Z_{t,x,p}^{t,x,p}(y)), \partial_y(\partial_x H_{t,x,p}^{t,x,p}(y))) \in S^2(\xi, \xi) \times K^2(t, T) \times K^2(t, T)\), and from Theorem 10.3,

\[
E \left[ \sup_{s \in [t,T]} |\partial_y(\partial_x Y_{t,x,p}^{t,x,p}(y))|^p + \left( \int_t^T |\partial_y(\partial_x Z_{t,x,p}^{t,x,p}(y))|^2 ds \right)^{\frac{p}{2}} \right] \leq C_p, \quad \text{for all } t \in [0,T], \ y \in \mathbb{R}, \ \xi \in L^2(\mathcal{F}_t).
\]

Let \(\xi, \xi', \vartheta, \vartheta' \in L^2(\mathcal{F}_T)\) be such that \(P_{\vartheta} = P_{\xi}, \ P_{\vartheta'} = P_{\xi'}\). Notice that \(\Pi_{r,x,p}^{t,x,p}(y), \ Q_{r,x,p}^{t,x,p}(y), \ R_{r,x,p}^{t,x,p}(y)\), \(t \leq s \leq T\), are independent of \(\mathcal{F}_t\). Hence, from (8.1) we get the following BSDE:

\[
\partial_y(\partial_x Y_{t,x,p}^{t,x,p}(y)) - \partial_y(\partial_x Y_{t,x',p'}(y')) = I_1(x, y, P_{\xi}) - I_1(x', y', P_{\xi'}) + \int_s^T R(r, x, x')dr
\]

\[
+ \int_s^T (\vartheta f)(\Pi_{r,x,p}^{t,x,p}, P_{\Pi_{r}^{t,x,p}})(\partial_y(\partial_x \Pi_{r,x,p}^{t,x,p}(y)) - \partial_y(\partial_x \Pi_{r,x',p'}(y'))) dr
\]

\[
+ \int_s^T \hat{E}\left[ (\partial_y f)(\Pi_{r,x,p}^{t,x,p}, P_{\Pi_{r}^{t,x,p}})(\partial_y(\partial_x \Pi_{r,x,p}^{t,x,p}(y)) - \partial_y(\partial_x \Pi_{r,x',p'}(y'))) \right] dr
\]

\[
- \int_s^T (\partial_y(\partial_x Z_{r,x,p}^{t,x,p}(y)) - \partial_y(\partial_x Z_{r,x',p'}(y'))) dB_r
\]

\[
- \int_s^T \int_K (\partial_y(\partial_x H_{r,x,p}^{t,x,p}(y, e)) - \partial_y(\partial_x H_{r,x',p'}(y', e))) N_{\lambda}(dr, de),
\]

where \(I_1(x, y, P_{\xi}) := (\partial_x \Phi)(X_{r,T}^{t,x,p}(x))\partial_y(\partial_x X_{r,T}^{t,x,p}(y));\)

\[
R(r, x, x') := I_2(r, x, y, P_{\xi}) - I_2(r, x', y', P_{\xi'}) + I_3(r, x, y, P_{\xi}) - I_3(r, x', y', P_{\xi'});
\]

\[
I_2(r, x, y, P_{\xi}) := \hat{E}[((\partial_y f)(\Pi_{r,x,p}^{t,x,p}, P_{\Pi_{r}^{t,x,p}})(\partial_x^2 \Pi_{r,x,p}^{t,x,p}(y))];\]

\[
I_3(r, x, y, P_{\xi}) := \hat{E}[\partial_y(\partial_y f)(\Pi_{r,x,p}^{t,x,p}, P_{\Pi_{r}^{t,x,p}})(\partial_x \Pi_{r,x,p}^{t,x,p}(y))\partial_y \Pi_{r,x,p}^{t,x,p}(y)];\]

\[
R_1(r, x, y, P_{\xi}; x', y', P_{\xi'}) := ((\partial_y f)(\Pi_{r,x,p}^{t,x,p}, P_{\Pi_{r}^{t,x,p}}) - (\partial_y f)(\Pi_{r,x',p'}^{t,x',p'}(y'))\partial_y(\partial_x \Pi_{r,x',p'}(y')));
\]

\[
R_2(r, x, y, P_{\xi}; x', y', P_{\xi'}) := \hat{E}[(\partial_y f)(\Pi_{r,x,p}^{t,x,p}, P_{\Pi_{r}^{t,x,p}})(\partial_x \Pi_{r,x,p}^{t,x,p}(y))\partial_y(\partial_x \Pi_{r,x',p'}(y'))] - (\partial_y f)(\Pi_{r,x',p'}^{t,x',p'}(y'), P_{\Pi_{r}^{t,x',p'}}(y'))\partial_y(\partial_x \Pi_{r,x',p'}(y'));\]

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Observe that
i) It follows from Lemma 3.1 and Theorem 7.1 that
\[ E[|I_1(x, y, P_\xi) - I_1(x', y', P_\xi')|^p] \leq C_p(|x - x'|^p + |y - y'|^p + W_2(P_\xi, P_\xi')^p). \]

ii) As \( \partial_y f \) is bounded and Lipschitz, we have from (8.3) and Proposition 4.1
\[
E\left[ \left( \int_t^T \left| R_1(r, x, y, P_\xi; x', y', P_\xi') \right| dr \right)^p \right] 
\leq C_p \left( \left( \int_t^T \left( |\Pi_r^{t,x,y} - \Pi_r^{t,x',y'}|^2 + W_2(P_{\Pi_r^{t}}; P_{\Pi_r^{t'}})^2 \right) dr \right)^p \right) \frac{2}{p} 
\leq C_p \left( |x - x'|^p + W_2(P_\xi, P_\xi')^p \right).
\]

iii) Similar arguments to ii) from (8.2) and Proposition 4.1, we see that also
\[
E\left[ \left( \int_t^T \left| R_2(r, x, y, P_\xi; x', y', P_\xi') \right| dr \right)^p \right] 
\leq C_p \left( |x - x'|^p + W_2(P_\xi, P_\xi')^p + \left( E[|\vartheta - \vartheta'|^2] \right)^{\frac{p}{2}} \right).
\]

iv) Similar arguments to ii) we get first:
\[
E\left[ \left( \int_t^T \left| I_3(r, x, y, P_\xi) - I_3(r, x', y', P_\xi') \right| dr \right)^p \right] 
\leq I_{3,1}(y, P_\xi; y', P_\xi') + I_{3,2}(x, y, P_\xi; x', y', P_\xi'),
\]
with
\[
I_{3,1}(y, P_\xi; y', P_\xi') := C_p E\left[ \left( \int_t^T \left| (\partial_x \Pi_r^{t,y,y,P_\xi})^2 - (\partial_x \Pi_r^{t,y',y',P_\xi'})^2 \right| dr \right)^p \right]
\] and
\[
I_{3,2}(x, y, P_\xi; x', y', P_\xi') := C_p E\left[ \left( \tilde{E}\left( \int_t^T \left| \partial_x \Pi_r^{t,y,y,P_\xi} \right|^2 \right) \cdot \min\{C, |\Pi_r^{t,x,y,P_\xi} - \Pi_r^{t,x',y',P_\xi'}| + W_2(P_{\Pi_r^{t}}; P_{\Pi_r^{t'}}) + |\tilde{\Pi}_r^{t,y,y,P_\xi} - \tilde{\Pi}_r^{t,y',y',P_\xi'} | \} dr \right)^p \right].
\]
(Recall that \( \partial_y (\partial_\mu f) \) is bounded and Lipschitz). Obviously, from (6.2),
\[
I_{3,1}(y, P_\xi; y', P_\xi') \leq C_p \left( E\left[ \left( \int_t^T \left| \partial_x \Pi_r^{t,y,y,P_\xi} - \partial_x \Pi_r^{t,y',y',P_\xi'} \right|^2 \right| dr \right]^p \right) \frac{1}{p}
\leq C_p \left( |y - y'|^p + W_2(P_\xi, P_\xi')^p \right).
\]

On the other hand, from Proposition 4.1
\[
I_{3,2}(x, y, P_\xi; x', y', P_\xi') 
\leq C_p M_p \left( E\left[ \left( \int_t^T \left( |\Pi_r^{t,x,y,P_\xi} - \Pi_r^{t,x',y',P_\xi'}|^2 + W_2(P_{\Pi_r^{t}}; P_{\Pi_r^{t'}})^2 \right) \right)^p \right] \right)^{\frac{1}{p}}
\leq C_p \left( |x - x'|^p + W_2(P_\xi, P_\xi')^p + |y - y'|^p \right) + \rho_{M,p}(t, y, P_\xi),
\]
where \( \rho_{M,p}(t, y, P_\xi) \to 0 \) \( M \to \infty \) and \( E[\rho_{M,p}(t, \xi, P_\xi)] \to 0 \) \( M \to \infty \) thanks to the Dominated Convergence Theorem (indeed, \( \sup_{y \in \mathbb{R}} E\left[ \left( \int_t^T |\partial_x \Pi_r^{t,y,y,P_\xi}|^2 \right)^p \right] \leq C_p < \infty \).
v) Remarking that, in analogy to [33], from Lemma 10.1 with more classical arguments not involving the derivative with respect to the measure, we can show on one hand $E\left[\left(\int_t^T |\partial_x^2 \Pi^{t,x,P_t}|^2 dr\right)^{\frac{p}{2}}\right] \leq C_p, \ (t,x,P_t) \in [0,T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R})$, and on the other hand (meeting for the estimate the same difficulty as in iv)-the difficulty is already inherent to the classical case (see, Pardoux and Peng [25]) although not developed there

$$E\left[\left(\int_t^T |\partial_x^2 \Pi^{t,x,P_t} - \partial_x^2 \Pi^{t,x',P_t'}|^2 dr\right)^{\frac{p}{2}}\right] \leq C_p M^p \left(|x-x'|^p + W_2(P_t,P_t')^p\right) + \rho_{M,p}(t,x,P_t),$$

where $\rho_{M,p}(t,x,P_t) \to 0$ and $E[\rho_{M,p}(t,\xi,P_t)] \to 0 \ (M \to \infty)$. Therefore,

$$E\left[\int_t^T |I_2(r, x, y, P_t) - I_2(r, x', y', P_t')| dr\right]^{\frac{p}{2}} \leq \rho_{M,p}(t,y,P_t) + C_p M^p \left(|x-x'|^p + |y-y'|^p + W_2(P_t,P_t')^p + (E[|\vartheta - \vartheta'|^2])^{\frac{p}{2}}\right).$$

Now substituting in BSDE $(8.4)$ $x = \vartheta$ and $x' = \vartheta'$, similar to $(6.21)$ we get

$$E\left[\sup_{s \in [t,T]} |\partial_y (\partial_\mu Y_s^{t,\vartheta,P_t}(y)) - \partial_y (\partial_\mu Y_s^{t,\vartheta',P_t'}(y'))|^2 + \int_t^T |\partial_y (\partial_\mu Z_r^{t,\vartheta,P_t}(y)) - \partial_y (\partial_\mu Z_r^{t,\vartheta',P_t'}(y'))|^2 dr + \int_t^T \int_K |\partial_y (\partial_\mu H_r^{t,\vartheta,P_t}(y,e)) - \partial_y (\partial_\mu H_r^{t,\vartheta',P_t'}(y',e))|^2 \lambda(de) dr\right] \leq \rho_{M}(t,y,P_t) + CM^2(|y-y'|^2 + W_2(P_t,P_t')^2 + E[|\vartheta - \vartheta'|^2]).$$

$(8.7)$

This estimate allows to study BSDE $(8.4)$, following the same arguments for $(6.23)$, $(6.24)$ and $(6.25)$, we obtain that, for all $t \in [0,T], \ x, x', y, y' \in \mathbb{R}, \ \xi, \xi' \in L^2(\mathcal{F}_t; \mathbb{R})$,

$$E\left[\sup_{s \in [t,T]} |\partial_y (\partial_\mu Y_s^{t,x,P_t}(y)) - \partial_y (\partial_\mu Y_s^{t,x',P_t'}(y'))|^p + \int_t^T |\partial_y (\partial_\mu Z_r^{t,x,P_t}(y)) - \partial_y (\partial_\mu Z_r^{t,x',P_t'}(y'))|^2 dr\right]^{\frac{p}{2}} \leq C_p M^p \left(|x-x'|^p + |y-y'|^p + W_2(P_t,P_t')^p + \rho_{M,p}(t,y,P_t)\right),$$

$(8.8)$

where $\rho_{M,p}(t,y,P_t) \to 0, \ M \to \infty$, and $E[\rho_{M,p}(t,\hat{\xi},P_t)] \to 0, \ M \to \infty$.  

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In order to show that the formal derivative \( \partial_y(\partial_y Y_{t,x,P}(y)) \) is really the \( L^2 \)-derivative of \( \partial_y Y_{t,x,P}(y) \), \( \partial_y(\partial_y Z_{t,x,P}(y)) \) and \( \partial_y(\partial_y H_{t,x,P}(y)) \), we estimate

\[ I := \frac{1}{h} \partial_y Y_{t,x,P}(y+h) - \partial_y Y_{t,x,P}(y), \]

\[ II := \frac{1}{h} \partial_y Z_{t,x,P}(y+h) - \partial_y Z_{t,x,P}(y), \]

\[ III := \frac{1}{h} \partial_y H_{t,x,P}(y+h) - \partial_y H_{t,x,P}(y), \]

with the same tools as for (8.8). Indeed, for \( h \in \mathbb{R} \setminus \{0\} \), one has to study the BSDE satisfied by

\[ \Xi_t \in \mathbb{R} \setminus \{0\}, \]

In analogy to the proof of the estimate (8.8) we meet, for example, the term (see iv)

\[ I := E \left[ \frac{1}{h} \left( \int_0^T |\partial_x \Pi_r^{y,y-h} (P_t) - \partial_x \Pi_r^{y,y-h} | \, |dr|^2 \right) \right] \]

\[ = E \left[ \frac{1}{h} \left( \int_0^T |\partial_x \Pi_r^{y,y-h} (P_t) - \partial_x \Pi_r^{y,y-h} | \, |dr|^2 \right) \right]. \]

Using that \( \partial_y(\partial_y f) \) is bounded and Lipschitz, this yields

\[ I \leq CE \left[ \frac{1}{h} \left( \int_0^T |\partial_x \Pi_r^{y,y-h} | \, |dr|^2 \right) \right] + CE \left[ \frac{1}{h} \left( \int_0^T |\partial_x \Pi_r^{y,y-h} | \, |dr|^2 \right) \right], \]

and the argument developed to prove (8.6) allows to see that \( I \leq CM^2 |h| + \rho_M(t, y, P) \), with \( \rho_M(t, y, P) \to 0, E[\rho_M(t, y, P)] \to 0 \), as \( M \to \infty \). On the other hand, it is easy to show that

\[ E\left[ \left( \int_0^T \left| \partial_y X_{t,x,P}(y+h) - \partial_y X_{t,x,P}(y) \right| \, |dr| \right)^2 \right] \leq Ch^2. \]

Furthermore, when applying Theorem 10.3 in the Appendix it yields by using a similar discussion for other terms corresponding to i), ii) and iv)

\[ E \left[ \sup_{s \in [t,T]} \frac{1}{h} \left( \partial_y Y_{s,x,P}(y+h) - \partial_y Y_{s,x,P}(y) \right) - \partial_y(\partial_y Y_{s,x,P}(y)) \right]^2 \]

\[ + \int_t^T \left( \frac{1}{h} \left| \partial_y Z_{s,x,P}(y+h) - \partial_y Z_{s,x,P}(y) \right| - \partial_y(\partial_y Z_{s,x,P}(y)) \right)^2 \, ds \]

\[ + \int_t^T \int_K \left( \frac{1}{h} \left| \partial_y H_{s,x,P}(y+h, e) - \partial_y H_{s,x,P}(y, e) \right| - \partial_y(\partial_y H_{s,x,P}(y, e)) \right)^2 \lambda(ce) \, ds \]

\[ \leq CM^2 |h|^2 + \rho_M(t, y, P), \]

with \( \rho_M(t, y, P) \to 0 \) as \( M \to \infty \).

It follows the wished \( L^2 \)-differentiability in \( y \) of \( \partial_y Y_{t,x,P}(y), \partial_y Z_{t,x,P}(y), \partial_y H_{t,x,P}(y) \).
9 Related integral-PDEs of mean-field type

The objective of this section is to study the related integral-PDEs of mean-field type. We will prove that $V(t, x, P_x)$ defined by (4.11) is the unique classical solution of the following new nonlocal quasi-linear integral PDE of mean-field type:

\[
\partial_t V(t, x, P_x) = -\left\{ \sum_{i=1}^{d} \partial_{x_i} V(t, x, P_x) b_i(x, P_x) + \frac{1}{2} \sum_{i,j,k=1}^{d} \partial_{x_i x_j}^2 V(t, x, P_x) \sigma_{i,k} \sigma_{j,k} \right\}(x, P_x) \\
+ \int_{K} \left( V(t, x + \beta(x, P_x, e), P_x) - V(t, x, P_x) - \sum_{i=1}^{d} \partial_{x_i} V(t, x, P_x) \beta_i(x, P_x, e) \right) \lambda(de) + f(x, V(t, x, P_x), \\
\sum_{i=1}^{d} \partial_{x_i} V(t, x, P_x) \sigma_i(x, P_x) + \int_{K} \left( V(t, x + \beta(x, P_x, e), P_x) - V(t, x, P_x) \right) l(e) \lambda(de), P(\xi, \psi(t, x, P_x)) \} \\
+ E \left\{ \sum_{i=1}^{d} (\partial_{\mu} V)_i(t, x, P_x, \xi) b_i(\xi, P_x) + \frac{1}{2} \sum_{i,j,k=1}^{d} \partial_{y_i} (\partial_{\mu} V)_j(t, x, P_x, \xi) \sigma_{i,k} \sigma_{j,k} \right\}(\xi, P_x) \\
+ \int_{0}^{1} \int_{K} \sum_{i=1}^{d} \left[ (\partial_{\mu} V)_i(t, x, P_x, \xi + \rho \beta(\xi, P_x, e)) - (\partial_{\mu} V)_i(t, x, P_x, \xi) \right] \beta_i(\xi, P_x, e) \lambda(de) dp \right\},
\]

\[
V(T, x, P_x) = \Phi(x, P_x), \quad (x, P_x) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d), \tag{9.1}
\]

where $\psi(t, x, P_x) :=$

\[
( V(t, x, P_x), \sum_{i=1}^{d} \partial_{x_i} V(t, x, P_x) \sigma_i(x, P_x), \int_{K} V(t, x + \beta(x, P_x, e), P_x) - V(t, x, P_x) l(e) \lambda(de) )
\]

The following two propositions study the regularity properties of the value function $V(t, x, P_x)$.

**Proposition 9.1.** Under the assumptions (H7.1) and (H8.1) the value function $V$ has the following properties:

i) $V \in C^{4,2,2,2}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$, which means,

a) $V(t, x, \cdot) \in C^2(\mathbb{R}^d)$, for all $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$;

b) $V(t, x, \cdot) \in C^2(\mathcal{P}_2(\mathbb{R}^d))$, for all $(t, x) \in [0, T] \times \mathbb{R}^d$;

c) The derivatives $\partial_x V, \partial^2_x V$ are continuous on $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, $\partial_{\mu} V, \partial_y (\partial_{\mu} V)$ are continuous and bounded on $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$;

d) $V(\cdot, x, \mu)$ is $\frac{1}{2}$-Hölder continuous in $t$, uniformly with respect to $(x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$.

ii) For all $\varphi \in \{ \partial_x V, \partial^2_x V, \partial_{\mu} V, \partial_y (\partial_{\mu} V) \}$, there exists a constant $C > 0$,

\[
|\varphi(t, x, P_x, y) - \varphi(t', x, P_x, y)| \leq C|t - t'|^{1/2}, \quad t, t' \in [0, T], \quad x, \ y \in \mathbb{R}^d, \quad \xi \in L^2(\mathcal{F}).
\]

**Proof.** i) follows directly from the preceding results-Proposition 4.3, Theorems 6.1 and 6.2, Proposition 6.2, Theorem 8.1 on $Y^{t, x, P_x}$ and its derivatives;

ii) follows from Lemma 10.2 in the Appendix. The proof is long, we give it in the appendix. \(\square\)

From (4.11) and (10.31) in the proof of Lemma 10.2 we get the following results.
Corollary 9.1. (Representation Formulas) Under the assumptions (H7.1) and (H8.1) we have the following representation formulas:

\[ Y_{s}^{t,x,P_{t}} = V(s, X_{s}^{t,x,P_{t}}, P_{X_{s}^{t,x}}), \text{P-a.s.}, \ s \in [t,T]; \]

\[ Z_{s}^{t,x,P_{t}} = \partial_{x}V(s, X_{s}^{t,x,P_{t}}, P_{X_{s}^{t,x}})\sigma(X_{s}^{t,x,P_{t}}, P_{X_{s}^{t,x}}), \text{dsdP-a.e.}; \]

\[ H_{s}^{t,x,P_{t}}(e) = V(s, X_{s}^{t,x,P_{t}} + \beta(X_{s}^{t,x,P_{t}}, P_{X_{s}^{t,x}}), P_{X_{s}^{t,x}}) - V(s, X_{s}^{t,x,P_{t}}, P_{X_{s}^{t,x}}), \text{dsd\lambdaP-a.e.}. \] (9.2)

Remark 9.1. From (4.8) the solution of BSDE (4.1) has the following representation formulas:

\[ Y_{s}^{t,x,P_{t}} = V(s, X_{s}^{t,x,P_{t}}, P_{X_{s}^{t,x}}), \text{P-a.s.}, \ s \in [t,T]; \]

\[ Z_{s}^{t,x,P_{t}} = \partial_{x}V(s, X_{s}^{t,x,P_{t}}, P_{X_{s}^{t,x}})\sigma(X_{s}^{t,x,P_{t}}, P_{X_{s}^{t,x}}), \text{dsdP-a.e.}; \]

\[ H_{s}^{t,x,P_{t}}(e) = V(s, X_{s}^{t,x,P_{t}} + \beta(X_{s}^{t,x,P_{t}}, P_{X_{s}^{t,x}}), P_{X_{s}^{t,x}}) - V(s, X_{s}^{t,x,P_{t}}, P_{X_{s}^{t,x}}), \text{dsd\lambdaP-a.e.}. \] (9.3)

Theorem 9.1. The value function \( V \in C^{1,2,2}([0,T] \times \mathbb{R}^{d} \times \mathcal{P}_{2}(\mathbb{R}^{d})) \).

Proof. From Proposition 9.1 we see that we only need to prove continuous differentiability of \( V \) with respect to \( t \). For simplicity of notations we still give the proof when \( d = 1 \). Using the notations in Lemma [10.2] and [10.29] in the Appendix, we have

\[ V(t, x, P_{t}) - V(t + h, x, P_{t}) = E[\int_{t}^{t+h} (\theta(t, t + h, s) + \delta(t, t + h, s) + f(\Pi_{s}^{t,x,P_{t}}, \Pi_{s}^{t,x,P_{t}}))ds]; \] (9.4)

and

\[ E[\theta(t, t + h, s)] = E[\partial_{x}V(s, X_{s}^{t,x,P_{t}}, P_{X_{s}^{t,x}})b(X_{s}^{t,x,P_{t}}, P_{X_{s}^{t,x}}) + \frac{1}{2}(\partial^{2}_{x}V)(s, X_{s}^{t,x,P_{t}}, P_{X_{s}^{t,x}})\sigma(X_{s}^{t,x,P_{t}}, P_{X_{s}^{t,x}})^{2} \]

\[ + \int_{K} (V(s, X_{s}^{t,x,P_{t}} + \beta(X_{s}^{t,x,P_{t}}, P_{X_{s}^{t,x}}), e) - V(s, X_{s}^{t,x,P_{t}}, P_{X_{s}^{t,x}})) - \partial_{x}V(s, X_{s}^{t,x,P_{t}}, P_{X_{s}^{t,x}}) \beta(X_{s}^{t,x,P_{t}}, P_{X_{s}^{t,x}}) ](de) + R_{1}(t, t + h)(s), \]

with \( |R_{1}(t, t + h)(s)| \leq Ch^{\frac{1}{2}} \) which follows from Proposition 9.1. Furthermore, from the Bounded Convergence Theorem we get

\[ E[\theta(t, t + h, s)] \rightarrow \partial_{x}V(t, x, P_{t})b(x, P_{t}) + \frac{1}{2}(\partial^{2}_{x}V)(t, x, P_{t})\sigma(x, P_{t})^{2} \]

\[ + \int_{K} (V(t, x + \beta(x, P_{t}, e), P_{t}) - V(t, x, P_{t}) - (\partial_{x}V)(t, x, P_{t})\beta(x, P_{t}, e))\lambda(de), \text{as } s \rightarrow t, \ h \downarrow 0, \]

(bounded by some \( K \) only depending on \( \partial_{x}V, \partial^{2}_{x}V \)). Similarly, we also have

\[ E[\delta(t, t + h, s)] = E[\hat{E}[\partial_{x}V(s, X_{s}^{t,x,P_{t}}, \hat{X}_{s}^{t,x})b(\hat{X}_{s}^{t,x}, P_{X_{s}^{t,x}}) + \frac{1}{2}\partial_{y}(\partial_{x}V)(s, X_{s}^{t,x,P_{t}}, P_{X_{s}^{t,x}}, \hat{X}_{s}^{t,x}) \]

\[ \sigma(\hat{X}_{s}^{t,x}, P_{X_{s}^{t,x}})^{2} + \int_{K} \int_{0}^{1} (((\partial_{y}V)(s, X_{s}^{t,x,P_{t}}, P_{X_{s}^{t,x}}, \hat{X}_{s}^{t,x} + \rho\beta(\hat{X}_{s}^{t,x}, P_{X_{s}^{t,x}, e}) \]

\[ - (\partial_{y}V)(s, X_{s}^{t,x,P_{t}}, P_{X_{s}^{t,x}, e})\beta(\hat{X}_{s}^{t,x}, P_{X_{s}^{t,x}, e})d\rho\lambda(de))] + R_{2}(t, t + h)(s), \]

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with $|R_2(t, t + h)(s)| \leq C h^k$ from Proposition 9.1, and from the Bounded Convergence Theorem we have

$$E[\delta(t, t + h, s)] \to \hat{E}[(\partial_\mu V)(t, x, P_\xi, \xi) b(\xi, P_\xi) + \frac{1}{2} \partial_y(\partial_\mu V)(t, x, P_\xi, \xi) \sigma(\xi, P_\xi)^2$$

$$+ \int_K \int_0^1 ((\partial_\mu V)(t, x, P_\xi, \xi + \rho \beta(\xi, P_\xi, e)) - (\partial_\mu V)(t, x, P_\xi, \xi)) \beta(\xi, P_\xi, e) d\rho \lambda(de)],$$

as $s \to t$, $h \downarrow 0$. Moreover, recall that

$$E[f(\Pi^{t, x, P_\xi}_s, P_{\Pi_s^t, \xi})]$$

$$= E[f(X^{t, x, P_\xi}_s, Y^{t, x, P_\xi}_s, Z^{t, x, P_\xi}_s, \int_K H^{t, x, P_\xi}_s(e) \lambda(de), P_{(X^{t, x, P_\xi}_s, Y^{t, x, P_\xi}_s, Z^{t, x, P_\xi}_s, \int_K H^{t, x, P_\xi}_s(e) \lambda(de))}],$$

and using the representation formulas (9.2) we see that we have also the convergence

$$E[f(\Pi^{t, x, P_\xi}_s, P_{\Pi_s^t, \xi})] \to f(x, V(t, x, P_\xi), \partial_x V(t, x, P_\xi) \sigma(x, P_\xi), \int_K (V(t, x + \beta(x, P_\xi, e), P_\xi)$$

$$- V(t, x, P_\xi)) \lambda(de), P_{(\xi, \psi(t, \xi, P_\xi))}), s \downarrow t,$$

where $\psi(t, x, P_\xi)$

$$:= (V(t, x, P_\xi), \partial_x V(t, x, P_\xi) \sigma(x, P_\xi), \int_K (V(t, x + \beta(x, P_\xi, e), P_\xi) - V(t, x, P_\xi)) \lambda(de).$$

Consequently, from (9.4) we can obtain that $V(t, x, P_\xi)$ is differentiable in $t$, and

$$- \partial_t V(t, x, P_\xi) = (\partial_\mu V)(t, x, P_\xi) b(x, P_\xi) + \frac{1}{2} (\partial^2_\mu V)(t, x, P_\xi) \sigma(x, P_\xi)^2$$

$$+ \int_K (V(t, x + \beta(x, P_\xi, e), P_\xi) - V(t, x, P_\xi) - (\partial_x V)(t, x, P_\xi) \beta(x, P_\xi, e)) \lambda(de)$$

$$+ E\left[(\partial_\mu V)(t, x, P_\xi, \xi) b(\xi, P_\xi) + \frac{1}{2} \partial_y(\partial_\mu V)(t, x, P_\xi, \xi) \sigma(\xi, P_\xi)^2$$

$$+ \int_K \int_0^1 ((\partial_\mu V)(t, x, P_\xi, \xi + \rho \beta(\xi, P_\xi, e)) - (\partial_\mu V)(t, x, P_\xi, \xi)) \beta(\xi, P_\xi, e) d\rho \lambda(de)\right]$$

$$+ f(x, V(t, x, P_\xi), \partial_x V(t, x, P_\xi) \sigma(x, P_\xi), \int_K (V(t, x + \beta(x, P_\xi, e), P_\xi) - V(t, x, P_\xi)) \lambda(de), P_\eta),$$

(9.5)

where $\eta := (\xi, \psi(t, \xi, P_\xi))$. As the whole right-hand side of (9.5) is continuous in $(t, x, P_\xi)$, this proves $V \in C^{1, 2, 2}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)).$

Moreover, it also shows the following main result.

**Theorem 9.2.** $V \in C^{1, 2, 2}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ is the classical solution of PDE (9.7), and it is unique in $C^{1, 2, 2}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)).$

**Proof.** As before we still assume $d = 1$. From (9.3) and the definition of $V$ we see immediately that $V \in C^{1, 2, 2}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ is the classical solution of PDE (9.1). Now we only need to prove the uniqueness of solution of PDE (9.1) in $C^{1, 2, 2}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)).$

Suppose $U(t, x, P_\xi) \in C^{1, 2, 2}([0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}))$ is another solution of the integral-PDE of mean-field type (9.1). Then, applying Itô’s formula to $U(s, X^{t, x, P_\xi}_s, P_{X^{t, x, P_\xi}_s})$ (Recall Theorem 2.1, now with $U_s = X^{t, x, P_\xi}_s$ and $X_s = X^{t, x}_s$), we have

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\[ dU(s, X^t_{s}, P^t_{X_s}, P_{X^t_{s}}) = \left\{ \partial_s U(s, X^t_{s}, P^t_{X_s}, P_{X^t_{s}}) \right\} ds + \partial_x U(s, X^t_{s}, P^t_{X_s}, P_{X^t_{s}}) \cdot \sigma(X^t_{s}, P^t_{X_s}, P_{X^t_{s}}), \]
\[ \int_K \left\{ \partial_x U(s, X^t_{s}, P^t_{X_s}, P_{X^t_{s}}) \cdot \sigma(X^t_{s}, P^t_{X_s}, P_{X^t_{s}}) \right\} ds \]

But, as \( U(t, x, P_x) \) satisfies equation (9.1), this yields

\[ dU(s, X^t_{s}, P^t_{X_s}, P_{X^t_{s}}) = -f(X^t_{s}, U(s, X^t_{s}, P^t_{X_s}, P_{X^t_{s}}), \partial_x U(s, X^t_{s}, P^t_{X_s}, P_{X^t_{s}}) \cdot \sigma(X^t_{s}, P^t_{X_s}, P_{X^t_{s}}), \]
\[ \int_K \left\{ \partial_x U(s, X^t_{s}, P^t_{X_s}, P_{X^t_{s}}) \cdot \sigma(X^t_{s}, P^t_{X_s}, P_{X^t_{s}}) \right\} ds \]

where \( \eta := (X^t_{s}, U(s, X^t_{s}, P^t_{X_s}, P_{X^t_{s}}), \partial_x U(s, X^t_{s}, P^t_{X_s}, P_{X^t_{s}}) \cdot \sigma(X^t_{s}, P^t_{X_s}, P_{X^t_{s}}), \int_K \left\{ \partial_x U(s, X^t_{s}, P^t_{X_s}, P_{X^t_{s}}) \cdot \sigma(X^t_{s}, P^t_{X_s}, P_{X^t_{s}}) \right\} ds \]

Now we replace \( x \) by \( \xi \) in the above (9.7) (recall that \( X^t_{s,\xi} = X^t_{s,x,\xi} \mid_{x=\xi} \), or applying Itô's formula directly to \( U(s, X^t_{s,\xi}, P_{X^t_{s,\xi}}) \) we get

\[ dU(s, X^t_{s,\xi}, P_{X^t_{s,\xi}}) = -f(X^t_{s,\xi}, U(s, X^t_{s,\xi}, P^t_{X_s,\xi}, P_{X^t_{s,\xi}}), \partial_x U(s, X^t_{s,\xi}, P^t_{X_s,\xi}, P_{X^t_{s,\xi}}) \cdot \sigma(X^t_{s,\xi}, P^t_{X_s,\xi}, P_{X^t_{s,\xi}}), \]
\[ \int_K \left\{ \partial_x U(s, X^t_{s,\xi}, P^t_{X_s,\xi}, P_{X^t_{s,\xi}}) \cdot \sigma(X^t_{s,\xi}, P^t_{X_s,\xi}, P_{X^t_{s,\xi}}) \right\} ds \]

From the uniqueness of the solution of mean-field BSDEs with jumps (4.11) we can get that:

\[ Y^t_{s} = U(s, X^t_{s}, P_{X^t_{s}}), P-a.s., \quad s \in [t, T]; \quad Z^t_{s} = \partial_x U(s, X^t_{s}, P_{X^t_{s}}) \sigma(X^t_{s}, P_{X^t_{s}}), \quad ds dP-a.e.; \]
\[ H_s^{t,\xi}(e) = U(s, X_s^{t,\xi} + \beta(X_s^{t,\xi}, P_{X_s^{t,\xi}}, e), P_{X_s^{t,\xi}}) - U(s, X_s^{t,\xi}, P_{X_s^{t,\xi}}), \text{dsd} \lambda \text{dP-a.e.} \quad (9.9) \]

Furthermore, with the help of (9.7) and (9.9) it follows from the uniqueness of the solution of BSDEs with jumps (1.2) we can conclude that

\[ Y_s^{t,x,P_\xi} = U(s, X_s^{t,x,P_\xi}, P_{X_s^{t,x,P_\xi}}), \text{P-a.s., } s \in [t, T]; \]
\[ Z_s^{t,x,P_\xi} = \partial_s U(s, X_s^{t,x,P_\xi}, P_{X_s^{t,x,P_\xi}}) \sigma(X_s^{t,x,P_\xi}, P_{X_s^{t,x,P_\xi}}), \text{dsdP-a.e.;} \]
\[ H_s^{t,x,P_\xi}(e) = U(s, X_s^{t,x,P_\xi} + \beta(X_s^{t,x,P_\xi}, P_{X_s^{t,x,P_\xi}}, e), P_{X_s^{t,x,P_\xi}}) - U(s, X_s^{t,x,P_\xi}, P_{X_s^{t,x,P_\xi}}), \text{dsd} \lambda \text{dP-a.e.} \]

In particular, as \( s = t, V(t, x, P_\xi) = Y_t^{t,x,P_\xi} = U(t, x, P_\xi) \). The proof is complete. \( \Box \)

10 Appendix

10.1 The proof of Theorem 2.1

For simplicity we just consider the case of \( d = 1 \); using the same argument, the results can be easily extended to the case \( d > 1 \).

Now we give the proof of Theorem 2.1

**Proof.** Let us begin to consider the special case \( F(s, x, \mu) = f(\mu), (s, x, \mu) \in [0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \).

**Step 1.** We first consider \( X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s + \int_0^t \int_K \beta_s(e) N_\lambda(ds, de), t \in [0, T], \)
\( X_0 \in L^2(\mathcal{F}_0), \) where \( b \in L^\infty_v(0, T), \sigma \in L^\infty_v(0, T), \) and \( \beta \in \mathcal{K}_\lambda(0, T) \) are bounded step processes such that, there exists a partition \( \tau = \{0 = t_0 < t_1 < \cdots < t_N = T\} \) with:

i) \( \sigma_s = \sigma_{t_k}, b_s = b_{t_k}, \beta_s(e) = \beta_{t_k}(e), s \in (t_k, t_{k+1}], 0 \leq k \leq N - 1; \)

ii) \( |\sigma_{t_k}|, |b_{t_k}| \leq C, |\beta_{t_k}(e)| \leq C(1 + |e|^2), e \in K, 0 \leq k \leq N - 1. \)

Recall that \( f \in C^2_v(\mathcal{P}_2(\mathbb{R})) \) with \( \partial_\mu, \partial_y(\partial_\mu f) \) are bounded and continuous (but not necessarily Lipschitz continuous).

For \( 0 \leq k \leq N - 1, t_k \leq t < t + h \leq t_{k+1}, \) since \( f \in C^2_v(\mathcal{P}_2(\mathbb{R})) \) is continuously differentiable with a bounded derivative \( \partial_\mu f, |\partial_\mu f(\mu, y)| \leq K, \) for all \( (\mu, y) \in \mathcal{P}_2(\mathbb{R}) \times \mathbb{R}, \) we have

\[ f(P_{X_t+h}) - f(P_{X_t}) = \int_0^1 E[(\partial_\mu f)(P_{X_t+\rho(X_{t+h}-X_t)}, X_t + \rho(X_{t+h}-X_t))(X_{t+h}-X_t)]d\rho. \quad (10.1) \]

Now we define

\[ X_t(\rho, h) := X_t + \rho(X_{t+h} - X_t) \]
\[ = X_t + \rho(b_{t_k} h + \sigma_{t_k} (B_{t+k} - B_t)) + \int_t^{t+h} \int_K \beta_{t_k}(e) N_\lambda(ds, de). \quad (10.2) \]

Then,

\[ E[(\partial_\mu f)(P_{X_t(\rho, h)}, X_t(\rho, h))(X_{t+h} - X_t)] = E[(\partial_\mu f)(P_{X_t(\rho, h)}, X_t(\rho, h))b_{t_k} h] + E[(\partial_\mu f)(P_{X_t(\rho, h)}, X_t(\rho, h))\sigma_{t_k} (B_{t+k} - B_t)] \]
\[ + E[(\partial_\mu f)(P_{X_t(\rho, h)}, X_t(\rho, h))\int_t^{t+h} \int_K \beta_{t_k}(e) N_\lambda(ds, de)] \quad (10.3) \]
\[ = I_1(t, \rho, h) + I_2(t, \rho, h) + I_3(t, \rho, h), \]
\[ 39 \]
where
\[ I_1(t, \rho, h) := E[\partial_\mu f](P_{X_t(\rho, h)}, X_t(\rho, h))b_t, h]; \]
\[ I_2(t, \rho, h) := E[\partial_\mu f](P_{X_t(\rho, h)}, X_t(\rho, h))\sigma_t (B_{t+h} - B_t); \]
\[ I_3(t, \rho, h) := E[\partial_\mu f](P_{X_t(\rho, h)}, X_t(\rho, h)) \int_t^{t+h} \int_K \beta_t(e)N_\lambda(ds, de). \]

Now we deal with \( I_2 \) and \( I_3 \).

a) Recall that for \( \varphi \in C^1_b(\mathbb{R}) \), and \( \zeta \) standard normal random variable, we have by partial integration
\[ E[\varphi(\zeta)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(x)e^{-\frac{x^2}{2}} dx \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi'(x)e^{-\frac{x^2}{2}} dx = E[\varphi'(\zeta)]. \]

Hence, as \((B_{t+h} - B_t)\) is independent of \( b_t, \sigma_t \) and \( \int_t^{t+h} \int_K \beta_t(e)N_\lambda(ds, de) \), we have
\[ I_2(t, \rho, h) = E[\partial_\mu f](P_{X_t(\rho, h)}, X_t(\rho, h))|\sigma_t|^2 \rho h]. \]

b) We remark that, as \(|\beta_t(e)| \leq C(1 \land |e|^2)\) we have
\[ E[\int_t^{t+h} \int_K |\beta_t(e)|N(ds, de)] = E[\int_t^{t+h} \int_K |\beta_t(e)|\lambda(de)ds] \leq C \int_K (1 \land |e|^2)\lambda(de)h, \]
i.e., we can consider the decomposition
\[ \int_t^s \int_K \beta_t(e)N_\lambda(ds, de) = \int_t^s \int_K \beta_t(e)N(dr, de) - (\int_K \beta_t(e)\lambda(de)(s - t), s \in [t, t + h]. \]

We define \( \zeta(t, \rho, h)(s) := X_t + \rho(b_t - \int_t^K \beta_t(e)\lambda(de))h + \rho\sigma_t (B_{t+h} - B_t) + \rho \int_t^s \int_K \beta_t(e)N(dr, de), \]
\( s \in [t, t + h] \). Then we have \( \zeta(t, \rho, h)(t + h) = X_t(\rho, h) \), and
\[
(\partial_\mu f)(P_{X_t(\rho, h)}, X_t(\rho, h)) - (\partial_\mu f)(P_{X_t(\rho, h)}, \zeta(t, \rho, h)(t))
= \sum_{t < s \leq t+h} \left( (\partial_\mu f)(P_{X_t(\rho, h)}, \zeta(t, \rho, h)(s)) - (\partial_\mu f)(P_{X_t(\rho, h)}, \zeta(t, \rho, h)(s-)) \right)
\]
\[ = \int_t^{t+h} \int_K \left\{ (\partial_\mu f)(P_{X_t(\rho, h)}, \zeta(t, \rho, h)(s)) - (\partial_\mu f)(P_{X_t(\rho, h)}, \zeta(t, \rho, h)(s-)) \right\} N(ds, de)
= \int_t^{t+h} \int_K \left\{ (\partial_\mu f)(P_{X_t(\rho, h)}, \zeta(t, \rho, h)(s)) - (\partial_\mu f)(P_{X_t(\rho, h)}, \zeta(t, \rho, h)(s-)) \right\} N_\lambda(ds, de)
+ R_1(t, \rho, h), \]
where
\[ R_1(t, \rho, h) = \int_t^{t+h} \int_K \left\{ (\partial_\mu f)(P_{X_t(\rho, h)}, \zeta(t, \rho, h)(s-)) - (\partial_\mu f)(P_{X_t(\rho, h)}, \zeta(t, \rho, h)(s-)) \right\} \lambda(de)ds, \]
and as \( \partial_\mu \) is bounded:
\[ |R_1(t, \rho, h)| \leq C \int_t^{t+h} \int_K |\beta_t(e)|\lambda(de)ds \leq C \int_K (1 \land |e|^2)\lambda(de)h \leq Ch. \]
Noting that $(\partial_{\mu}f)(P_{X_t(\rho,h)}, \zeta(t,\rho,h)(t))$ and $N(dsde)$, $s \in [t, t+h]$, are independent, we have

$$I_3(t, \rho, h)$$

$$= E\left[\left(\partial_{\mu}f( P_{X_t(\rho,h)}, X_t(\rho,h)) \right) \int_t^{t+h} \int_K \beta_{tk}(e)N_\lambda(ds,de)\right]$$

$$= E\left[\left(\int_t^{t+h} \int_K \left\{ (\partial_{\mu}f)(P_{X_t(\rho,h)}, \zeta(t,\rho,h)(s)) + \rho \beta_{tk}(e) \right\} (\partial_{\mu}f)(P_{X_t(\rho,h)}, \zeta(t,\rho,h)(s)) \right\} N_\lambda(ds,de)\right] + E\left[R_1(t, \rho, h)\left(\int_t^{t+h} \int_K \beta_{tk}(e)N_\lambda(ds,de)\right)\right]$$

$$= E\left[\left(\int_t^{t+h} \int_K \left\{ (\partial_{\mu}f)(P_{X_t(\rho,h)}, \zeta(t,\rho,h)(s)) + \rho \beta_{tk}(e) \right\} (\partial_{\mu}f)(P_{X_t(\rho,h)}, \zeta(t,\rho,h)(s)) \right\} \beta_{tk}(e)\lambda(ds)\right] + R_2(t, \rho, h), \quad (10.7)$$

where $R_2(t, \rho, h) := E\left[R_1(t, \rho, h)\left(\int_t^{t+h} \int_K \beta_{tk}(e)N_\lambda(ds,de)\right)\right]$. Notice that

$$|R_2(t, \rho, h)| \leq E\left[|R_1(t, \rho, h)\int_t^{t+h} \int_K \beta_{tk}(e)N_\lambda(ds,de)|\right]$$

$$\leq C h (E\left[\int_t^{t+h} \int_K |\beta_{tk}(e)|^2 \lambda(ds)\right])^{\frac{1}{2}} \leq C h^\frac{2}{3}.$$

Consequently, from (10.1), (10.3), (10.4), (10.6) and (10.7) we get

$$f(P_{X_{t+h}}) - f(P_{X_t})$$

$$= \int_t^{t+h} \int_0^1 E\left[\left| (\partial_{\mu}f)(P_{X_t(\rho,h)}, X_t(\rho,h))b_s + \rho \beta_{tk}(e)\right| \sigma_s \right|^2$$

$$+ \int_K \left\{ (\partial_{\mu}f)(P_{X_t(\rho,h)}, \zeta(t,\rho,h)(s)) + \rho \beta_{ts}(e) \right\} \beta_{ts}(e)\lambda(ds)\right] d\rho ds + \int_0^1 R_2(t, \rho, h)d\rho,$$  

$$\quad (10.8)$$

for $t_k \leq t < t + h \leq t_{k+1}$ ($0 \leq k \leq N - 1$). Let now $n \geq 1$, $t^n_i := t + \frac{ih}{n}$, $0 \leq i \leq n$, then

$$f(P_{X_{t+h}}) - f(P_{X_t}) = \sum_{i=0}^{n-1} (f(P_{X^n_{t+i+1}}) - f(P_{X^n_{t+i}}))$$

$$= \int_t^{t+h} \int_0^1 E\left[\sum_{i=0}^{n-1} I_{[t^n_i, t^n_{i+1})}(s) \Xi_i^{(n)}(s, \rho)\right] d\rho ds + R_3^{(n)}(t, t+h), \quad (10.9)$$

with $R_3^{(n)}(t, t+h) = \sum_{i=0}^{n-1} \int_0^1 R_2(t^n_i, \rho, \frac{h}{n})d\rho, \left|R_3^{(n)}(t, t+h)\right| \leq C h^\frac{2}{3} \cdot n^{-\frac{1}{3}} \to 0$, as $n \to \infty$, and

$$\Xi_i^{(n)}(s, \rho) = (\partial_{\mu}f)(P_{X^n_{t^n_i}(\rho, \frac{h}{n})}, X^n_{t^n_i}(\rho, \frac{h}{n}))b_s + \rho \beta_{ts}(e)\left| (\partial_{\mu}f)(P_{X^n_{t^n_i}(\rho, \frac{h}{n})}, X^n_{t^n_i}(\rho, \frac{h}{n}))\right| \sigma_s \right|^2$$

$$+ \int_K \left\{ (\partial_{\mu}f)(P_{X^n_{t^n_i}(\rho, \frac{h}{n})}, \zeta(t^n_i, \rho, \frac{h}{n})(s)) + \rho \beta_{ts}(e) \right\} \beta_{ts}(e)\lambda(ds),$$  

$$\quad (10.10)$$
where \( s \in [t^n_i, t^n_{i+1}] \), \( 0 \leq i \leq n - 1 \). As \( \partial_{\mu} f \) and \( \partial_{\gamma}(\partial_{\mu} f) \) are bounded,

\[
\left| \Xi^n_i(s, \rho) \right| \leq C (|b_s| + |\sigma_s|^2 + \int_K |\beta_s(e)|^2 \lambda(de)) \leq C, \ s \in [t^n_i, t^n_{i+1}], \ 0 \leq i \leq n - 1, \ \rho \in [0, 1],
\]

and, as

\[
E \left[ \sup_{s \in [t^n_i, t^n_{i+1}]} |X^n_T(\rho, \frac{h_n}{n} - X_s|^2 \right] \leq C E \left[ \sup_{s \in [t^n_i, t^n_{i+1}]} |X_s - X^n_T|^2 \right] \leq C \frac{h_n}{n}, \quad (10.11)
\]

we also have

\[
\begin{align*}
(i) \quad & \sup_{s \in [t^n_i, t^n_{i+1}]} W_2(P_{X^n_t(\rho, \frac{h_n}{n})}, P_{X_s}) \leq C \left( \frac{h_n}{n} \right)^{\frac{1}{2}}, \\
(ii) \quad & E \left[ \sup_{s \in [t^n_i, t^n_{i+1}]} |(t^n_i - \rho, \frac{h_n}{n}) - X_s|^2 \right] \leq C \left( \frac{h_n}{n} \right).
\end{align*} \quad (10.12)
\]

It follows from the continuity of \( \partial_{\mu} f \) and \( \partial_{\gamma}(\partial_{\mu} f) \) on \( P_2(\mathbb{R}) \times \mathbb{R} \) and the Dominated Convergence Theorem that taking limit in (10.9) as \( n \to \infty \) it yields

\[
\begin{align*}
f(P_{X^{n+h}}) - f(P_{X^n}) &= \int_t^{t+h} \int_0^1 E \left[ (\partial_{\mu} f)(P_{X^n}, X_s)b_s + \rho \partial_{\gamma}(\partial_{\mu} f)(P_{X^n}, X_s)|\sigma_s|^2 \right] ds \\
&\quad + \int_K \left\{ (\partial_{\mu} f)(P_{X^n}, X_s + \rho \beta_s(e)) - (\partial_{\mu} f)(P_{X^n}, X_s) \right\} \beta_s(e) \lambda(de) \right] ds \\
&= \int_t^{t+h} E \left[ (\partial_{\mu} f)(P_{X^n}, X_s)b_s + \frac{1}{2} \partial_{\gamma}(\partial_{\mu} f)(P_{X^n}, X_s)|\sigma_s|^2 \right] ds \\
&\quad + \int_K \int_0^1 \left\{ (\partial_{\mu} f)(P_{X^n}, X_s + \rho \beta_s(e)) - (\partial_{\mu} f)(P_{X^n}, X_s) \right\} \beta_s(e) d\rho \lambda(de) \right] ds.
\end{align*} \quad (10.13)
\]

As this holds for all \( t_k \leq t < t + h \leq t_{k+1} \), \( 0 \leq k \leq N \), it holds for all \( 0 \leq t < t + h \leq T \).

**Step 2.** Let now \( b, \sigma \in H^2_F(0, T), \beta \in K^2(0, T), X_0 \in L^2(F_0) \). Then we can approximate \( b, \sigma, \beta \) by processes \( b^n, \sigma^n, \beta^n \) which satisfy, for each \( n \), the assumptions made in **Step 1** (with bounds depending on \( n \) and a partition depending on \( n \)): \( b^n \to b, \sigma^n \to \sigma \) in \( H^2_F(0, T), \beta^n \to \beta \) in \( K^2(0, T) \).

Let

\[
X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s + \int_0^t \int_K \beta_s(e) N(ds, de), \ t \in [0, T],
\]

\[
X^n_t = X_0 + \int_0^t b^n_s ds + \int_0^t \sigma^n_s dB_s + \int_0^t \int_K \beta^n_s(e) N(ds, de), \ t \in [0, T], \ n \geq 1.
\]

Then,

\[
E \left[ \sup_{t \in [0, T]} |X_t - X^n_t|^2 \right] \leq C \left( E \left[ \int_0^T |b_s - b^n_s|^2 ds \right] + E \left[ \int_0^T |\sigma_s - \sigma^n_s|^2 ds \right] \right).
\]

As \( n \to \infty \), it holds for all \( t \in [0, T] \).

**Step 1** we know that for all \( 0 \leq t \leq t + h \leq T \),

\[
\begin{align*}
f(P_{X^{n+h}}) - f(P_{X^n}) &= \int_t^{t+h} E \left[ (\partial_{\mu} f)(P_{X^n}, X^n_s)b^n_s + \frac{1}{2} \partial_{\gamma}(\partial_{\mu} f)(P_{X^n}, X^n_s)|\sigma^n_s|^2 \right] ds \\
&\quad + \int_K \int_0^1 \left\{ (\partial_{\mu} f)(P_{X^n}, X^n_s + \rho \beta^n_s(e)) - (\partial_{\mu} f)(P_{X^n}, X^n_s) \right\} \beta^n_s(e) d\rho \lambda(de) \right] ds.
\end{align*} \quad (10.16)
\]
We observe that, for
\[
V^n_s := (\partial_\mu f)(P_{X^n_s}, X^n_s)b^n_s + \frac{1}{2} \partial_\mu \partial_\mu f(P_{X^n_s}, X^n_s)|\sigma^n_s|^2 + \int_K \int_0^1 \left\{ (\partial_\mu f)(P_{X^n_s}, X^n_s + \rho \beta^n_s(e)) - (\partial_\mu f)(P_{X^n_s}, X^n_s) \right\} \beta^n_s(e) d\rho \lambda(de),
\]
i) \( V^n \to V \) in measure \( dsdP \), where \( V \) is defined in the same way as \( V^n \), but with \( (X,b,\sigma,\beta) \) instead of \( (X^n,b^n,\sigma^n,\beta^n) \).

ii) As \( \partial_\mu f \) and \( \partial_\mu \partial_\mu f \) are bounded,
\[
|V^n_s| \leq C \left| b^n_s \right| + |\sigma^n_s|^2 + \int_K \left| \beta^n_s(e) \right|^2 \lambda(de), \quad s \in [0,T], \ n \geq 1.
\]

But, as \( b^n \to b \), \( \sigma^n \to \sigma \) in \( \mathcal{H}_F^2(0,T) \) and \( \beta^n \to \beta \) in \( \mathcal{K}_F^2(0,T) \), the right hand side is a uniformly integrable sequence over \( [0,T] \times \Omega \). Consequently, we can apply Lebesgue’s convergence Theorem to (10.16) and we get
\[
f(P_{X^{t+h}}) - f(P_{X^t}) = \int_t^{t+h} E\left[ (\partial_\mu f)(P_{X_s}, X_s)b_s + \frac{1}{2} \partial_\mu \partial_\mu f(P_{X_s}, X_s)|\sigma_s|^2 + \int_K \int_0^1 \left\{ (\partial_\mu f)(P_{X_s}, X_s + \rho \beta_s(e)) - (\partial_\mu f)(P_{X_s}, X_s) \right\} \beta_s(e) d\rho \lambda(de) \right] ds, \quad 0 \leq t \leq t+h \leq T.
\]

**Step 3.** Let now \( b, \sigma, \beta \) be as in Step 2. For \( u \in L_F^0(\Omega, L^1(0,T)), \ v \in L_F^0(\Omega, L^2(0,T)) \), \( \gamma \in \mathcal{K}_F^2(0,T) \) with \( |\gamma_s(e)| \leq \zeta(1 \wedge |e|) \), \( \text{P-a.s.} \), \( (s,e) \in [0,T] \times K \), for some real-valued random variable \( \zeta \geq 0 \), \( \text{P-a.s.} \), and \( U_0 \in L^0(F_0) \), we consider the Itô process
\[
U_t = U_0 + \int_0^t u_s ds + \int_0^t v_s dB_s + \int_0^t \int_K \gamma_s(e) N_\lambda(ds,de), \quad t \in [0,T].
\]

Let \( F \in C^{1,2}([0,T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R})) \), we emphasize that we do not need the existence of the second order mixed derivatives \( \partial_s \partial_\mu F, \partial_\mu \partial_s F \), nor that of \( \partial_\mu (\partial_\mu F) \), unlike [6] and [13]. Under the above assumptions then we have the Itô formula.

Indeed, let us first suppose that \( |b| + |\sigma| \leq C, \ |\beta(e)| \leq C(1 \wedge |e|) \) and \( b_s, \sigma_s, \beta_s(e) \) are continuous with respect to \( s \). Then we see from (10.17) that
\[
\partial_t [F(t,x,P_{X_t})] = (\partial_t F)(t,x,P_{X_t}) + E\left[ (\partial_\mu F)(t,x,P_{X_t},X_t)b_t + \frac{1}{2} \partial_\mu \partial_\mu F(t,x,P_{X_t},X_t)|\sigma_t|^2 + \int_K \int_0^1 \left\{ (\partial_\mu F)(t,x,P_{X_t},X_t + \rho \beta_t(e)) - (\partial_\mu F)(t,x,P_{X_t},X_t) \right\} \beta_t(e) d\rho \lambda(de) \right]
\]
is continuous in \((t,x)\), i.e., \( G(t,x) := F(t,x,P_{X_t}), \ (t,x) \in [0,T] \times \mathbb{R} \), belongs to \( C^{1,2}([0,T] \times \mathbb{R}) \).

But this means that we can apply to \( F(t,U_t,P_{X_t}) = G(t,U_t) \) the classical Itô formula. This yields
\[
dF(t,U_t,P_{X_t}) = dG(t,U_t)
\]
\[
= \left\{ (\partial_t G)(t,U_t) + (\partial_x G)(t,U_t)u_t + \frac{1}{2} (\partial_x^2 G)(t,U_t)v_t^2 + \int_K \left[ G(t,U_t + \gamma_t(e)) - G(t,U_t) - (\partial_x G)(t,U_t) \right] \lambda(de) \right\} dt + (\partial_x G)(t,U_t)v_t dB_t + \int_K \left[ G(t,U_{t-} + \gamma_t(e)) - G(t,U_{t-}) \right] N_\lambda(dt,de)
\]

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\[
\begin{aligned}
&= \left\{ (\partial_t F)(t, U_t, P_{X_t}) + (\partial_x F)(t, U_t, P_{X_t}) u_t + \frac{1}{2} (\partial^2_x F)(t, U_t, P_{X_t}) v_t^2 + \int_K (F(t, U_t + \gamma_t(e), P_{X_t}) \\
&\quad - F(t, U_t, P_{X_t}) - (\partial_x F)(t, U_t, P_{X_t}) \gamma_t(e)) \lambda(de) \right\} dt + \hat{E} \left[ (\partial_\mu F)(t, U_t, P_{X_t}, \tilde{X}_t) b_t + \frac{1}{2} \partial_\gamma (\partial_\mu F)(t, U_t, P_{X_t}, \tilde{X}_t) \right] \hat{\gamma}_t(e) d\rho(\lambda(de)) dt \\
&\quad + (\partial_x F)(t, U_t, P_{X_t}) v_t dB_t + \int_K \left( F(t, U_{t-} + \gamma_t(e), P_{X_t}) - F(t, U_{t-}, P_{X_t}) \right) N_\lambda(dt, de), \quad t \in [0, T].
\end{aligned}
\]

(10.18)

Step 4. For the general case let now \( b, \sigma, \beta \in H^2_F(0, T) \) and \( \beta \in K^2_F(0, T) \), and suppose that \( b^n, \sigma^n, \beta^n \) satisfy the assumptions made on \( b, \sigma, \beta \) in Step 3 and \( b^n \rightarrow b, \sigma^n \rightarrow \sigma \) in \( H^2_F(0, T) \) and \( \beta^n \rightarrow \beta \) in \( K^2_F(0, T) \). From Step 3 we have the Itô formula (10.15) for \( b^n, \sigma^n, \beta^n \) and \( X^n \) (defined by (10.15)) instead of \( b, \sigma \) and \( X \), and we have to take the limit as \( n \rightarrow \infty \).

For this notice that, as \( E[\sup_{t \in [0, T]} |X^n_t - X_t|^2] \rightarrow 0, n \rightarrow 0 \) (see Step 2), we have in particular that

\[ W_2(P_{X^n}, P_X) \leq E[\sup_{t \in [0, T]} |X^n_t - X_t|^2] \rightarrow 0, n \rightarrow 0. \]

Hence, \( \{P_{X^n}, n \geq 1\} \) is relatively compact in \( P_2(D([0, T])) \). \( D([0, T]) \) is the space of càdlàg functions over \([0, T]\), endowed with the supremum norm. Combining this with the pathwise càdlàg property of the process \( U \) and the continuity of \( \partial^2_x F : [0, T] \times \mathbb{R} \times P_2(\mathbb{R}) \rightarrow \mathbb{R} \), we see that \( \sup_{n \geq 1, t \in [0, T]} |\partial^2_x F(t, U_t, P_{X^n_t})| < \infty, P \)-a.s., and

i) \[ \int_0^t \partial^2_x F(s, U_s, P_{X^n_s}) v_s^2 ds \rightarrow \int_0^t \partial^2_x F(s, U_s, P_{X_s}) v_s^2 ds, \quad P \text{-a.s., from the Dominated Convergence Theorem.} \]

Using the same arguments and the fact that \( |\gamma_t(e)| \leq \zeta(1 \wedge |e|) \), we see that

\[
\sup_{n \geq 1, t \in [0, T]} |F(t, U_{t-} + \gamma_t(e), P_{X^n_t}) - F(t, U_{t-}, P_{X^n_t})|^2 \leq \sup_{n \geq 1, t \in [0, T], \rho \in [0, 1]} |\partial_x F(t, U_{t-} + \rho \gamma_t(e), P_{X^n_t})|^2 |\zeta|^2 (1 \wedge |e|)^2,
\]

and

\[
\sup_{n \geq 1, t \in [0, T]} |F(t, U_t + \gamma_t(e), P_{X^n_t}) - F(t, U_t, P_{X^n_t}) - \partial_x F(t, U_t, P_{X^n_t}) \gamma_t(e)|^2 \leq \sup_{n \geq 1, t \in [0, T], \rho \in [0, 1]} |\partial^2_x F(t, U_t + \rho \gamma_t(e), P_{X^n_t})|^2 |\zeta|^2 (1 \wedge |e|)^2,
\]

with \( \sup_{n \geq 1, t \in [0, T], \rho \in [0, 1]} |\partial_x F(t, U_t + \rho \gamma_t(e), P_{X^n_t})| < +\infty \), and \( \sup_{n \geq 1, t \in [0, T], \rho \in [0, 1]} |\partial^2_x F(t, U_t + \rho \gamma_t(e), P_{X^n_t})| < +\infty \), \( P \)-a.s. This allows to show that, for all \( t \in [0, T] \), \( P \)-a.s., as \( n \rightarrow \infty \),

ii) \[ \int_0^t \int_K \left( F(s, U_{s-} + \gamma_s(e), P_{X^n_s}) - F(s, U_{s-}, P_{X^n_s}) \right) \lambda(ds, de) \rightarrow \int_0^t \int_K \left( F(s, U_{s-} + \gamma_s(e), P_{X_s}) - F(s, U_{s-}, P_{X_s}) \right) \lambda(ds, de), \]

and

iii) \[ \int_0^t \int_K \left( F(s, U_{s} + \gamma_s(e), P_{X^n_s}) - F(s, U_{s}, P_{X^n_s}) - \partial_x F(s, U_s, P_{X^n_s}) \gamma_s(e) \right) \lambda(ds, de) \rightarrow \int_0^t \int_K \left( F(s, U_{s} + \gamma_s(e), P_{X_s}) - F(s, U_{s}, P_{X_s}) - \partial_x F(s, U_s, P_{X_s}) \gamma_s(e) \right) \lambda(ds, de). \]
On the other hand, from the boundedness of $\partial_y(\partial_u F)$ (with some bound $C > 0$) we see that $|(\partial_u F)(t, U_t, P_{X^u_t}, \tilde{X}^u_t + \rho \tilde{\beta}^u_t(e)) - (\partial_u F)(t, U_t, P_{X^u_t}, \tilde{X}^u_t)| \leq C(\tilde{\beta}^u_t(e))^2, (t, e) \in [0, T] \times \mathcal{E}$, P-a.s. This allows to conclude from Lebesgue’s Convergence Theorem that

\[
iv) \quad \mathbb{E}\left[\int_0^T \int_{\mathcal{F}_t} \left((\partial_u F)(s, U_s, P_{X^u_s}, \tilde{X}^u_s + \rho \tilde{\beta}^u_s(e)) - (\partial_u F)(s, U_s, P_{X^u_s}, \tilde{X}^u_s)\right) \tilde{\beta}^u_s(e) d\rho(s) ds\right] \rightarrow 0,
\]

for all $t \in [0, T]$, P-a.s., $n \to \infty$.

An analogous discussion for the remaining terms in (10.18) shows that the we have also for them the convergence. The proof is complete.

\[\square\]

\section{10.2 Mean field BSDEs with jumps}

We first give two classical estimates for the solutions of BSDEs with jumps, the proof is standard, the readers may refer to, e.g., [2, 21] and [22].

\textbf{Lemma 10.1.} Suppose $(Y^i, Z^i, H^i)$ is the unique solution of the following BSDE with data $(g^i, \theta^i)$,

\[
\begin{cases}
  dY^i_s = -g_i(s, Y^i_s, Z^i_s, H^i_s)ds + Z^i_sdB + \int_K H^i_s(e)N_\lambda(ds, de), \\
  Y^i_T = \theta^i,
\end{cases}
\]

where $\theta^i \in L^2(\mathcal{F}_T)$, and $g_i : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times L^2(K, \mathcal{B}(K), \lambda; \mathbb{R}) \to \mathbb{R}$, $i = 1, 2$, respectively, are $\mathbb{F}$-predictable and satisfy:

\textbf{Assumption (H10.0)} i) $g_i(\cdot, \cdot, 0, 0, 0) \in \mathcal{H}_2^\delta(0, T)$; 

ii) There exists a constant $c^* > 0$ such that, P-a.s., for any $t \in [0, T]$, $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$, $h_1, h_2 \in L^2(K, \mathcal{B}(K), \lambda)$, $|g_i(t, y_1, z_1, h_1) - g_i(t, y_2, z_2, h_2)| \leq c^*|y_1 - y_2| + |z_1 - z_2| + |h_1 - h_2|$.

For $(Y, Z, T) := (Y^1, Z^1, H^1) - (Y^2, Z^2, H^2)$, $\gamma := g_1 - g_2$, $\theta := \theta^1 - \theta^2$, we have the following estimates:

1) For all $\delta > 0$, there exists a suitable $\beta(\geq \frac{1}{2} + 2c^*(1 + 2c^* + \frac{1}{\delta}))$ such that

\[
|Y_t|^2 + \frac{1}{2}E\left[\int_t^T e^{\beta(s-t)}(Y_s^2 + Z_s^2) + \int_K |Z_s(e)|^2 \lambda(de) ds\right]_{\mathcal{F}_t} \leq E\left[e^{\beta(T-t)}|Y_T|^2|\mathcal{F}_T\right] + c^*\beta E\left[\int_t^T e^{\beta(s-t)}|\gamma(s, Y_s^1, Z_s^1, H_s^1)|^2 ds\right]_{\mathcal{F}_T}, \quad \text{P-a.s., } t \in [0, T].
\]

2) For all $p \geq 2$, there exists $C_p > 0$ (only depending on $p$ and the Lipschitz constants) such that

\[
E\left[\sup_{s \in [t, T]} |Y_s|^p + \left(\int_t^T |Z_s|^2 ds\right)^\frac{p}{2} + \left(\int_t^T \int_K |Z_s(e)|^2 \lambda(de) ds\right)^\frac{p}{2}\right]_{\mathcal{F}_T} \leq C_p E\left[|\gamma|^p + \left(\int_t^T |\gamma(s, Y_s^1, Z_s^1, H_s^1)|ds\right)^p\right]_{\mathcal{F}_T}, \quad \text{P-a.s., } t \in [0, T].
\]

We now consider a more general case of BSDE (10.19). Let $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times L^2(K, \mathcal{B}(K), \lambda) \times \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times L^2(K, \mathcal{B}(K), \lambda)) \to \mathbb{R}$ be $\mathbb{F}$-predictable and satisfy the following assumptions.

\textbf{Assumption (H10.1)} i) $f(\cdot, \cdot, 0, 0, 0, \delta_0) \in \mathcal{H}_2^\delta(0, T)$;
We define the mapping $\Phi(\cdot)$, which means $(Y,Z,H)$ is the solution of (10.22).

**Theorem 10.1.** Under the Assumption (H10.1), the following mean-field BSDE with jumps

$$
\left\{
\begin{array}{l}
\ dY_s = -f(s,Y,Z,H)ds + Z_s dB_s + \int_K H_s(e)N_\lambda(ds,de), \\
\ Y_T = \xi,
\end{array}
\right.
$$

(10.22)

has a unique solution $(Y,Z,H) \in S^2(0,T;\mathbb{R}) \times \mathcal{H}^2_F(0,T;\mathbb{R}) \times \mathcal{K}^2(0,T;\mathbb{R})$.

**Proof.** Let $(U,V,W) \in \mathcal{M} := \mathcal{H}^2_F(0,T;\mathbb{R}) \times \mathcal{H}^2_F(0,T;\mathbb{R}) \times \mathcal{K}^2(0,T;\mathbb{R})$, then there is a unique solution $(Y,Z,H) \in S^2(0,T;\mathbb{R}) \times \mathcal{H}^2_F(0,T;\mathbb{R}) \times \mathcal{K}^2(0,T;\mathbb{R})$ of the BSDE

$$
\left\{
\begin{array}{l}
\ dY_s = -f(s,Y,Z,H)ds + Z_s dB_s + \int_K H_s(e)N_\lambda(ds,de), \\
\ Y_T = \xi,
\end{array}
\right.
$$

(10.22)

We define the mapping $\Phi(U,V,W) := (Y,Z,H) : \mathcal{M} \to \mathcal{M}$. Let $(U^i,V^i,W^i) \in \mathcal{M}$, $(Y^i,Z^i,H^i) := \Phi(U^i,V^i,W^i), i = 1,2$. Let $(\overline{Y},\overline{Z},\overline{H}) := (Y^1,Z^1,H^1) - (Y^2,Z^2,H^2)$, $(\overline{U},\overline{V},\overline{W}) := (U^1,V^1,W^1) - (U^2,V^2,W^2)$. Then from (10.20), for $C$ is the Lipschitz constant of $f$, we have

$$
\|\overline{Y},\overline{Z},\overline{H}\|_2^2 := \frac{1}{2}E[\int_0^T e^{\beta s}(|\overline{Y}_s|^2 + |\overline{Z}_s|^2 + \int_K |\overline{H}_s(e)|^2 \lambda(de))ds]
$$

$$
\leq \delta C \int_0^T e^{\beta s}W_2(P(X_s,U_s,\overline{Y}_s,\overline{Z}_s,\overline{H}_s))ds
$$

$$
\leq \delta CE\int_0^T e^{\beta s}(|\overline{U}_s|^2 + |\overline{V}_s|^2 + \int_K |\overline{W}_s(e)|^2 \lambda(de))ds
$$

$$
= 2\delta C\|\overline{U},\overline{V},\overline{W}\|_2^2.
$$

Consequently, for choosing $\delta > 0$ such that $2\delta C \leq \frac{1}{4}$, then $\Phi : (\mathcal{M},\|\cdot\|_\beta) \to (\mathcal{M},\|\cdot\|_\beta)$ is a contraction mapping, i.e., there exists a unique fixed point $(Y,Z,H) \in \mathcal{M}$ such that $(Y,Z,H) = \Phi(Y,Z,H)$, which means $(Y,Z,H)$ is the solution of (10.22). \hfill \square

Similar to Lemma 10.1, we also have the following estimate for mean-field BSDE with jumps.

**Theorem 10.2.** Let $(Y^i,Z^i,H^i)$ be the unique solution of the following BSDE with data $(f_i,\xi^i)$,

$$
\left\{
\begin{array}{l}
\ dY^i_s = -f^i(s,Y^i_s,Z^i_s,H^i_s)ds + Z^i_s dB_s + \int_K H^i_s(e)N_\lambda(ds,de), \\
\ Y^i_T = \xi^i,
\end{array}
\right.
$$

(10.23)

where $(X,f_i,\xi^i)$ satisfy (H10.1), $i = 1,2$, respectively.

We denote $(\overline{Y},\overline{Z},\overline{H}) := (Y^1,Z^1,H^1) - (Y^2,Z^2,H^2)$, $\overline{f} := f^1 - f^2$, $\overline{\xi} := \xi^1 - \xi^2$. Then there exists a constant $C > 0$ such that, $P$-a.s., $t \in [0,T]$,

$$
E\left[ \sup_{s \in [t,T]} |\overline{Y}_s|^2 + \int_t^T (|\overline{Z}_s|^2 + \int_K |\overline{H}_s(e)|^2 \lambda(de))ds \right] \leq CE[|\overline{\xi}|^2 + \left( \int_t^T |\overline{f}(s,Y^i_s,Z^i_s,H^i_s,P(X_s,Y^i_s,Z^i_s,H^i_s))|ds \right)^2] \mathcal{F}_t.
$$

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In particular, if \( f^i(s, y, z, h, P_{Y_s, Z_s, H_s}) = u^i(s, y, z, h) + \hat{E}[v^i(s, \hat{Y}_s, \hat{Z}_s, \hat{H}_s)] \), where \( u^i, v^i : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \) and both satisfy the assumption \((\text{H}10.0)\), with \((Y, Z, H) \in S^2_{\mathcal{P}}(0, T; \mathbb{R}) \times \mathcal{H}^2_{\mathbb{P}}(0, T; \mathbb{R}) \times \mathcal{K}^2_{\mathbb{P}}(0, T; \mathbb{R}) \), \( i = 1, 2 \), respectively. Then, we have \( \forall \alpha \), \( \alpha \) is the solution to BSDE \((10.24)\).

\[
E\left[ \sup_{s \in [t, T]} |Y_s|^2 + \int_t^T (|Z_s|^2 + \int_K |H_s(e)|^2 \lambda(de)) ds \right] \leq CE\left[ |\xi|^2 + \int_t^T (|u^1 - u^2|^2(s, Y^1_s, Z^1_s, H^1_s) + (\int_t^T |(v^1 - v^2)(s, \hat{Y}^1_s, \hat{Z}^1_s, \hat{H}^1_s)| ds)^2 \right] \quad \text{for all } \xi \in L^2(\mathcal{F}_t);
\]

Now we give the estimates for a special type of mean field BSDEs with jumps, which are used frequently in our work. We suppose that

**Assumption (H10.2)**

(i) \( \xi \in L^2(\mathcal{F}_T) \);

(ii) \( \alpha = (\alpha_s), \gamma = (\gamma_s) \) are boundedly \( \mathcal{F} \)-progressively measurable processes, and \( \beta = (\beta_s(e)) \) is \( \mathcal{F} \)-progressively measurable such that \( |\beta_s(e)| \leq C(1 + |e|) \);

(iii) \( \zeta = (\zeta_s), \theta = (\theta_s) \) are boundedly \( \mathcal{G} \)-progressively measurable processes with \( \mathcal{G}_t = \mathcal{F}_t \otimes \hat{\mathcal{F}}_t \), where \((\Omega, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{P}) \) is a copy of \((\Omega, \mathcal{F}, \mathbb{P}), \) and \( \delta = (\delta_s(e)) \) is boundedly \( \mathcal{G} \)-progressively measurable such that \( |\delta_s(e)| \leq C(1 + |e|) \);

(iv) \( R = (R(s)) \) is \( \mathbb{F} \)-progressively measurable with \( E[(\int_0^T |R(r)| dr)^2] < +\infty \).

From Theorem \( 10.2 \) we get the following corollary directly.

**Corollary 10.1.** Suppose Assumption \((\text{H}10.2)\) holds. Let \((Y, Z, H) \in S^2_{\mathcal{P}}(0, T) \times \mathcal{H}^2_{\mathbb{P}}(0, T; \mathbb{R}) \times \mathcal{K}^2_{\mathbb{P}}(0, T) \) and \((\hat{Y}, \hat{Z}, \hat{H}) \) be a copy of \((Y, Z, H) \) on \((\Omega, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{P}) \) \( \text{i.e., } P_{Y,Z,H} = \hat{P}_{\hat{Y},\hat{Z},\hat{H}} \) such that

\[
Y_t = \xi + \int_s^T (R(r) + \alpha_r Y_r + \hat{E}[\zeta_r \hat{Y}_r] + \gamma_r Z_r + \hat{E}[\theta_r \hat{Z}_r] + \int_K \beta_r(e) H_r(e) \lambda(de)) dr + \hat{E} \left[ \int_K \delta_r(e) \hat{H}_r(e) \lambda(de) \right] ds - \int_s^T Z_r dB_r - \int_s^T H_r(e) N_e(ds), \quad s \in [t, T].
\] (10.24)

Then there exists \( C \in \mathbb{R}_+ \) only depending on the bounds of the coefficients such that

\[
E\left[ \sup_{t \in [0, T]} |Y_t|^2 + \int_0^T (|Z_s|^2 + \int_K |H_s(e)|^2 \lambda(de)) ds \right] \leq CE\left[ |\xi|^2 + \int_0^T |R(s)| ds^2 \right] \left( \int_0^T |R(s)| ds^2 \right).
\]

**Theorem 10.3.** Suppose the Assumption \((\text{H}10.2)\) holds. For all \( p > 1 \), there exists \( C_p \in \mathbb{R}_+ \) only depending on the bounds of the coefficients, such that

\[
E\left[ \sup_{t \in [0, T]} |Y_t|^{2p} + \left( \int_0^T (|Z_s|^2 + \int_K |H_s(e)|^2 \lambda(de)) ds \right)^p \right] \leq CE\left[ |\xi|^{2p} + \int_0^T |R(s)| ds^2 \right],
\]

where \((Y, Z, H) \) is the solution of BSDE \((10.24)\).
\[E[|Y_t|^2] \leq C \beta t + \frac{1}{2} E\left[\int_t^T (|Y_s|^2 + |Z_s|^2 + \int_K |H_s(e)|^2 \lambda(de))ds\right].\]
Finally, let 0 = \( t_0^n < t_1^n < \cdots < t_n^n = T \) be the sequence of partitions of the interval \([0, T]\) of the form \( t_i^n := \frac{i}{n}T \), 0 ≤ i ≤ n, where n ≥ n₀ is such that \( C_p(T_{n_0})^p ≤ \frac{1}{2} \). Then we have

\[
E[(\int_{t_i^n}^{t_{i+1}^n} (|Z_s|^2 + \int_K |H_s(e)|^2 \lambda(de))ds)^p] ≤ C_p E[|\xi|^2]^p + C_p E[(\int_0^T |R(s)|ds)^{2p}].
\]

It follows that

\[
E[(\int_{t_i^n}^{t_{i+1}^n} (|Z_s|^2 + \int_K |H_s(e)|^2 \lambda(de))ds)^p] ≤ C_p E[|\xi|^2]^p + C_p E[(\int_0^T |R(s)|ds)^{2p}].
\]

\( \square \)

### 10.3 Lemma for the proof of Proposition 9.1.

Under the assumptions made for Proposition 9.1 we have

**Lemma 10.2.** There exists a constant \( C > 0 \), such that, for all \( t, t' \in [0, T], \; x, y \in \mathbb{R}, \; \xi \in L^2(\Omega, \mathcal{F}_t, P) \), it holds

1. \(|V(t, x, P_\xi) - V(t', x, P_\xi)| ≤ C|t' - t|\),
2. \(|\partial_x V(t, x, P_\xi) - \partial_x V(t', x, P_\xi)| ≤ C|t' - t|^{\frac{1}{2}}\),
3. \(|\partial^2_x V(t, x, P_\xi) - \partial^2_x V(t', x, P_\xi)| ≤ C|t' - t|^{\frac{1}{2}}\),
4. \(|\partial_y V(t, x, P_\xi, y) - \partial_y V(t', x, P_\xi, y)| ≤ C|t' - t|^{\frac{1}{2}}\),
5. \(|\partial_y(\partial_y V)(t, x, P_\xi, y) - \partial_y(\partial_y V)(t', x, P_\xi, y)| ≤ C|t' - t|^{\frac{1}{2}}\).

**Proof.** Let us prove Lemma 10.2 in four steps.

*Step 1.* To prove i) of Lemma 10.2.

For any 0 ≤ t ≤ t + h ≤ T, from (4.10) and (4.11) we have

\[
V(t, x, P_\xi) - V(t + h, x, P_\xi) = Y_{t,x,P_\xi}^{t+h} - Y_{t,x,P_\xi}^t
\]

where

\[
Y_{t,x,P_\xi}^{t+h} = (Y_{t,x,P_\xi}^t - Y_{t+h,x,P_\xi}^t) + (Y_{t+h,x,P_\xi}^t - Y_{t+h,x,P_\xi}^{t+h})
\]

(10.27)

As \( V(t + h, \cdot, \cdot) \in C^{2,2}(\mathbb{R} \times \mathcal{P}_2(\mathbb{R})) \) with bounded continuous derivatives of 1st and 2nd order which are uniformly with respect to t, it follows from the Itô formula-Theorem 2.1 that

\[
V(t + h, X_{t+h}^{t,x,P_\xi}, P_{X_{t+h}^{t,x,P_\xi}}) - V(t + h, x, P_\xi)
\]

\[
= \int_t^{t+h} \left( (\partial_x V)(t + h, X_{s}^{t,x,P_\xi}, P_{X_{s}^{t,x,P_\xi}})b(X_{s}^{t,x,P_\xi}, P_{X_{s}^{t,x,P_\xi}}) + \frac{1}{2}(\partial^2_x V)(t + h, X_{s}^{t,x,P_\xi}, P_{X_{s}^{t,x,P_\xi}})\sigma(X_{s}^{t,x,P_\xi}, P_{X_{s}^{t,x,P_\xi}})^2 \right)ds
\]

\[
+ \frac{1}{2}(\partial_y V)(t + h, X_{s}^{t,x,P_\xi}, P_{X_{s}^{t,x,P_\xi}})\beta(X_{s}^{t,x,P_\xi}, P_{X_{s}^{t,x,P_\xi}, e})V(t + h, X_{s}^{t,x,P_\xi}, P_{X_{s}^{t,x,P_\xi}})
\]

\[
- (\partial_y V)(t + h, X_{s}^{t,x,P_\xi}, P_{X_{s}^{t,x,P_\xi}})\lambda(de)
\]

\[
+ \int_t^{t+h} \hat{E}\left[ (\partial_y V)(t + h, X_{s}^{t,x,P_\xi}, P_{X_{s}^{t,x,P_\xi}})\beta(X_{s}^{t,x,P_\xi}, P_{X_{s}^{t,x,P_\xi}, e})\lambda(de) + \frac{1}{2}(\partial_y V)(t + h, X_{s}^{t,x,P_\xi}, P_{X_{s}^{t,x,P_\xi}})\right]ds
\]

\[
+ \int_t^{t+h} \hat{E}\left[ (\partial_y V)(t + h, X_{s}^{t,x,P_\xi}, P_{X_{s}^{t,x,P_\xi}})\beta(X_{s}^{t,x,P_\xi}, P_{X_{s}^{t,x,P_\xi}, e})\lambda(de) + \frac{1}{2}(\partial_y V)(t + h, X_{s}^{t,x,P_\xi}, P_{X_{s}^{t,x,P_\xi}})\right]ds
\]

\[
+ \int_t^{t+h} \hat{E}\left[ (\partial_y V)(t + h, X_{s}^{t,x,P_\xi}, P_{X_{s}^{t,x,P_\xi}})\beta(X_{s}^{t,x,P_\xi}, P_{X_{s}^{t,x,P_\xi}, e})\lambda(de) + \frac{1}{2}(\partial_y V)(t + h, X_{s}^{t,x,P_\xi}, P_{X_{s}^{t,x,P_\xi}})\right]ds
\]

\[
+ \int_t^{t+h} \hat{E}\left[ (\partial_y V)(t + h, X_{s}^{t,x,P_\xi}, P_{X_{s}^{t,x,P_\xi}})\beta(X_{s}^{t,x,P_\xi}, P_{X_{s}^{t,x,P_\xi}, e})\lambda(de) + \frac{1}{2}(\partial_y V)(t + h, X_{s}^{t,x,P_\xi}, P_{X_{s}^{t,x,P_\xi}})\right]ds
\]

\[\cdots\]
\[ Y \equiv \int_0^1 \left( \partial_\mu V(t + h, X^{t,x,P}_s, P_{X_s^{t,x}}, \hat{X}_s^{t,x}) + \rho \beta(\hat{X}_s^{t,x}, P_{X_s^{t,x}}) \right) \, d\rho \lambda(ds) \, de \]
As $|\theta(t, t + h, s)|, |\delta(t, t + h, s)| \leq C$, $f$ is bounded, and $V(t, x, P_\xi) - V(t + h, x, P_\xi)$ is deterministic, we get by taking expectation in the preceding equality:

$$|V(t, x, P_\xi) - V(t + h, x, P_\xi)| \leq C|h|, \ t, t + h \in [0, T]. \quad (10.30)$$

**Step 2.** We have the following representation formulas for $Z^{t,x,P_\xi}_{s}$ and $H^{t,x,P_\xi}_{s}(\cdot)$:

$$Z^{t,x,P_\xi}_{s} = \partial_x V(s, X^{t,x,P_\xi}_{s}, P_{X^{t,x,E}_{s}}) \sigma(X^{t,x,P_\xi}_{s}, P_{X^{t,x,E}_{s}}), \ dsdP \text{-a.e.,}$$

$$H^{t,x,P_\xi}_{s}(e) = V(s, X^{t,x,P_\xi}_{s} + \beta(X^{t,x,P_\xi}_{s}, P_{X^{t,x,E}_{s}}, e), P_{X^{t,x,E}_{s}}) - V(s, X^{t,x,P_\xi}_{s}, P_{X^{t,x,E}_{s}}), \ dsdP \text{-a.e.} \quad (10.31)$$

Indeed, we get from (10.29) combined with (10.30)

$$E\int_{t}^{t+h} \left| \left( \partial_x V(s, X^{t,x,P_\xi}_{s}, P_{X^{t,x,E}_{s}}) \sigma(X^{t,x,P_\xi}_{s}, P_{X^{t,x,E}_{s}}) - Z^{t,x,P_\xi}_{s} \right) \right|^2 ds |\mathcal{F}_t| + \int_{K} |V(s, X^{t,x,P_\xi}_{s} + \beta(X^{t,x,P_\xi}_{s}, P_{X^{t,x,E}_{s}}, e), P_{X^{t,x,E}_{s}}) - V(s, X^{t,x,P_\xi}_{s}, P_{X^{t,x,E}_{s}}) - H^{t,x,P_\xi}_{s}(e)|^2 \lambda(de) ds |\mathcal{F}_t| \leq C|h|^2, \ 0 \leq t \leq t + h \leq T, \ x \in \mathbb{R}, \ \xi \in L^2(\mathcal{F}_t).$$

Consequently, considering a partition $t^n_i = t + hi2^{-n}, 0 \leq i \leq 2^n$, we have from the preceding estimate applied to $(t^n_i, t^n_{i+1})$ instead of $(t, t + h)$:

$$E\int_{t}^{t+h} \left| \left( \partial_x V(s, X^{t,x,P_\xi}_{s}, P_{X^{t,x,E}_{s}}) \sigma(X^{t,x,P_\xi}_{s}, P_{X^{t,x,E}_{s}}) - Z^{t,x,P_\xi}_{s} \right) \right|^2 ds |\mathcal{F}_t| + \int_{K} |V(s, X^{t,x,P_\xi}_{s} + \beta(X^{t,x,P_\xi}_{s}, P_{X^{t,x,E}_{s}}, e), P_{X^{t,x,E}_{s}}) - V(s, X^{t,x,P_\xi}_{s}, P_{X^{t,x,E}_{s}}) - H^{t,x,P_\xi}_{s}(e)|^2 \lambda(de) ds |\mathcal{F}_t|$$

$$= E\left[ \sum_{i=0}^{2^n-1} E\int_{t^n_i}^{t^n_{i+1}} \left| \left( \partial_x V(s, X^{t^n,x,y,P_{\theta_{t^n}}}_{s}, P_{X^{t^n,x,E}_{s}}) \sigma(X^{t^n,x,y,P_{\theta_{t^n}}}_{s}, P_{X^{t^n,x,E}_{s}}) - Z^{t^n,x,y,P_{\theta_{t^n}}}_{s} \right) \right|^2 ds |\mathcal{F}_{t^n_i}] |y=_{X^{t^n,x,y,P_{\theta_{t^n}}}_{s}}=_{X^{t^n,x,E}_{s}}\right|$$

$$\leq C \sum_{i=0}^{2^n-1} (h2^{-n})^2 = Ch^22^{-n} \to 0, \text{ as } n \to \infty.$$
hence we can get

\[
E[\text{esssup}_{s \in [t,T]} |\partial_x Z_s^{t,x,P_k}|] \leq CE[\sup_{s \in [t,T]} |\partial_x X_s^{t,x,P_k}|] \leq C_p,
\]

\[
E[\text{esssup}_{s \in [t,T]} \int_K |\partial_x H_s^{t,x,P_k}(e)|^2 \lambda(de)] \leq C_p E[\sup_{s \in [t,T]} |\partial_x X_s^{t,x,P_k}|] \leq C_p.
\]

For simplicity, in order to concentrate on the hard kernel of the proof, let \( \Phi(x, \mu) = \Phi(x), f = f(\Pi_r^{t,x,P_k}, P_{\Pi_r^{t,x,P_k}}) \) with \( \Pi_r^{t,x,P_k} = \int_K H_r^{t,x,P_k}(e) l(e) \lambda(de) \), and recall that \( \Pi_r^{t,x,P_k} = \Pi_r^{t,x,P_k} \). Then

\[
Y_s^{t,x,P_k} = \Phi(X_T^{t,x,P_k}) + \int_s^T f(\Pi_r^{t,x,P_k}, P_{\Pi_r^{t,x,P_k}}) dr - \int_s^T Z_r^{t,x,P_k} dB_r - \int_s^T \int_K H_r^{t,x,P_k}(e) N \lambda(dr, de), \ s \in [t,T].
\]

(10.33)

From Theorems 6.1 and 8.1 we have

\[
\partial_s Y_s^{t,x,P_k} = \partial_s \Phi(X_T^{t,x,P_k}) \partial_x X_T^{t,x,P_k} + \int_s^T (\partial_s f)(\Pi_r^{t,x,P_k}, P_{\Pi_r^{t,x,P_k}}) \partial_x \Pi_r^{t,x,P_k} dr - \int_s^T \partial_x Z_r^{t,x,P_k} dB_r
\]

\[
- \int_s^T \int_K \partial_s H_r^{t,x,P_k}(e) N \lambda(dr, de), \ s \in [t,T],
\]

and

\[
\partial_s^2 Y_s^{t,x,P_k} = (\partial_s^2 \Phi(X_T^{t,x,P_k}) (\partial_x X_T^{t,x,P_k})^2 + \partial_s \Phi(X_T^{t,x,P_k}) \partial_x^2 X_T^{t,x,P_k}) + \int_s^T (\partial_s f)(\Pi_r^{t,x,P_k}, P_{\Pi_r^{t,x,P_k}}) (\partial_s \Pi_r^{t,x,P_k})^2 dr
\]

\[
+ \int_s^T (\partial_s f)(\Pi_r^{t,x,P_k}, P_{\Pi_r^{t,x,P_k}}) \partial_x^2 \Pi_r^{t,x,P_k} dr - \int_s^T \partial_x^2 Z_r^{t,x,P_k} dB_r - \int_s^T \int_K \partial_x^2 H_r^{t,x,P_k}(e) N \lambda(dr, de)
\]

\[
= I_1(t) + I_2(t,r) dr + \int_s^T (\partial_s f)(\Pi_r^{t,x,P_k}, P_{\Pi_r^{t,x,P_k}}) \partial_x^2 \Pi_r^{t,x,P_k} dr - \int_s^T \partial_x^2 Z_r^{t,x,P_k} dB_r
\]

\[
- \int_s^T \int_K \partial_x^2 H_r^{t,x,P_k}(e) N \lambda(dr, de),
\]

(10.34)

where

\[
I_1(t) := \partial_s^2 \Phi(X_T^{t,x,P_k}) (\partial_x X_T^{t,x,P_k})^2 + \partial_s \Phi(X_T^{t,x,P_k}) \partial_x^2 X_T^{t,x,P_k},
\]

\[
I_2(t,r) := (\partial_s f)(\Pi_r^{t,x,P_k}, P_{\Pi_r^{t,x,P_k}})(\partial_x \Pi_r^{t,x,P_k})^2.
\]

It is obvious that for \( 0 \leq t < t' \leq T \),

\[
E[|I_1(t) - I_1(t')|^2] \leq C(E[|\partial_x X_T^{t,x,P_k}|^4]^\frac{1}{2} (E[|X_T^{t,x,P_k} - X_T^{t,x,P_k}|^4]^\frac{1}{2}
\]

\[
+ C(E[|\partial_x X_T^{t,x,P_k} - X_T^{t,x,P_k}|^4]^\frac{1}{2} (E[|X_T^{t,x,P_k} - \partial_x X_T^{t,x,P_k}|^4]^\frac{1}{2}
\]

\[
+ C(E[|\partial_x^2 X_T^{t,x,P_k}|^4]^\frac{1}{2} (E[|X_T^{t,x,P_k} - \partial_x X_T^{t,x,P_k}|^4]^\frac{1}{2} + C E[|\partial_x^2 X_T^{t,x,P_k} - \partial_x X_T^{t,x,P_k}|^2]
\]

\[
\leq C (E[|X_T^{t,x,P_k} - X_T^{t,x,P_k}|^4]^\frac{1}{2} + C E[|\partial_x X_T^{t,x,P_k} - \partial_x X_T^{t,x,P_k}|^4]^\frac{1}{2} + C E[|\partial_x^2 X_T^{t,x,P_k} - \partial_x^2 X_T^{t,x,P_k}|^2]
\]

\[
= \mathbb{I}_1(t, t') + \mathbb{I}_2(t,t') + \mathbb{I}_3(t,t'),
\]

(10.36)

where \( \mathbb{I}_1(t, t') := C (E[|X_T^{t,x,P_k} - X_T^{t,x,P_k}|^4]^\frac{1}{2} \), \( \mathbb{I}_2(t,t') := C (E[|\partial_x X_T^{t,x,P_k} - \partial_x X_T^{t,x,P_k}|^4]^\frac{1}{2} \), \( \mathbb{I}_2(t,t') := \).
\[ CE[|\partial_x^2 X_{t,x,P}^i - \partial_x^2 X_{t,x,P}^j|^2]. \] From Lemma 3.1 we get

\[ \mathbb{I}_1(t,t') = CE\left( E\left[ |E[X_T^i|X^{i',P} - X_T^{j',P}|^4|F_t]\right|_{x'=X_{t'},\eta=\mu_{t'}} \right)^{\frac{1}{2}} \]
\[ \leq CE\left( E\left[ |X_{t',P}^i - x|^4 + W_2(P_{X_{t',P}^i}, P_{X_{t',P}^j})^4 \right]\right)^{\frac{1}{2}} \leq C|t' - t|^{\frac{1}{4}}. \]

On the other hand we have

\[ \partial_x (X_T^{i,x,P}) = \partial_x (X_T^{j,x,P}) = \partial_x X_T^{i,x,P} \cdot \partial_x X_T^{j,x,P}, \]

and from Theorem 6.1

\[ \mathbb{I}_2(t,t') \leq CE\left[ |\partial_x X_T^{i,x,P}| \frac{8}{3} |\partial_x X_T^{i,x,P} - 1|^{\frac{8}{3}} \right] + CE\left[ |\partial_x X_T^{i,x,P} - x|^{\frac{8}{3}} + W_2(P_{X_{t',P}^i}, P_{X_{t',P}^j})^{\frac{8}{3}} \right] \leq C(t' - t)^{\frac{1}{4}}. \]

Considering that \( \partial_x^2 (X_T^{i,x,P}) = (\partial_x^2 X_T^{i,x,P}) + (\partial_x X_T^{i,x,P})^2 \partial_x X_T^{i,x,P} \cdot \partial_x^2 X_T^{i,x,P}, \) a straight-forward estimate using Theorem 7.1 and, in particular, \( (E[|\partial_x^2 X_T^{i,x,P}|^4])^{\frac{4}{3}} \leq C|t' - t|^{\frac{1}{4}}, \)
\( (E[(|\partial_x X_T^{i,x,P})^2 - 1|^4])^{\frac{3}{2}} \leq C(E[|\partial_x X_T^{i,x,P} - 1|^8])^{\frac{1}{2}} \leq C|t' - t|^{\frac{1}{4}}, \) yields now \( \mathbb{I}_3(t,t') \leq C|t' - t|^{\frac{1}{4}}. \)

Consequently, from (10.36) we get

\[ E[|I_1(t) - I_1(t')|^2] \leq C|t' - t|^{\frac{1}{4}}. \]

Let us estimate now \( |I_2(t, s) - I_2(t', s)| \) for \( t < t' \). Using that (10.32) yields

\[ \text{ess sup}_{s \in [t,T]} |\partial_x \Pi_s^{i,x,P}| \leq C \text{ess sup}_{s \in [t,T]} |\partial_x \Pi_s^{i,x,P}|_{L^2(\lambda)} \leq C \sup_{s \in [t,T]} |\partial_x X_s^{i,x,P}|. \]

(10.38)

As \( (\partial_x^2 f) \) is bounded and Lipschitz, we get

\[ E\left( \int_t^T |I_2(t, r) - I_2(t', r)|^2 dr \right)^2 \leq C E\left( \int_t^T \left| (\partial_x \Pi_s^{i,x,P})^2 - (\partial_x \Pi_{s'}^{i',x,P})^2 \right|^2 dr \right)^2 \]
\[ + C \sup_{x \in \mathbb{R}} E\left( \int_t^T |\Pi_s^{i,x,P} - \Pi_{s'}^{i',x,P}|^2 dr \right) \leq C \left( E\left( \int_t^T \left| (\partial_x \Pi_{s'}^{i',x,P})^2 - (\partial_x \Pi_{s'}^{i',x,P})^2 \right|^2 dr \right)^{\frac{1}{2}} \right) \]
\[ + C \sup_{x \in \mathbb{R}} E\left( \int_t^T \left| \Pi_s^{i',x,P} - \Pi_{s'}^{i',x,P} \right|^2 dx \right)^2 dr \].

Then from (10.38), Propositions 5.1 4.1 and (6.2) as well as Lemma 3.1, we obtain

\[ E\left( \int_t^T |I_2(t, r) - I_2(t', r)| dr \right)^2 \leq C|t' - t|^{\frac{1}{4}}. \]

(10.39)

Applying Lemma 10.1-2) to the equation (10.35) it follows from (10.37), (10.39), and \( \partial_x f \) is Lipschitz that
On the other hand, from (10.38) and Theorem 8.1, we get also

\[ E \left[ \sup_{s \in [t,T]} |\partial_x^2 Y^t,x,P_k - \partial_x^2 Y^t,x,P_{k'}|^2 \right] + \int_t^T (|\partial_x^2 Z^t,x,P_k - \partial_x^2 Z^t,x,P_{k'}|^2 \right) \]

\[ + \int_K |\partial_x^2 H^t,x,P_k(e) - \partial_x^2 H^t,x,P_{k'}(e)|^2 \lambda(de) \, dr \right] \leq CE[|I_1(t) - I_1(t')|^2] + CE\left( \int_t^T |I_2(t,r) - I_2(t',r)| \, dr \right)^2 \]

\[ + CE\left( \int_{t'}^T (|\partial_h f(\Pi^{t,x,P_k}, P^{T,x}_{\Pi^{t,x}}) - \partial_h f(\Pi^{t',x,P_{k'}}, P^{T,x}_{\Pi^{t',x}})| \partial_x^2 \Pi^{t,x,P_k} \, dr \right)^2 \]

\[ \leq C|t' - t|^\frac{1}{2} + CE\left[ \int_{t'}^T (|\Pi^{t,x,P_k} - \Pi^{t',x,P_{k'}}|^2 + W_2(P^{T,x}_{\Pi^{t,x}}, P^{T,x}_{\Pi^{t',x}})^2) \, dr \right] \]

\[ \leq C|t' - t|^\frac{1}{2}. \quad \text{(The proof of the last inequality is similar to that of (10.39).)} \]

On the other hand, from (10.38) and Theorem 8.1, we get also

\[ |E[|\partial_x^2 Y^t,x,P_k - \partial_x^2 Y^t,x,P_{k'}|] \]

\[ \leq \int_t^{t'} E[|(|\partial_h f(\Pi^{t,x,P_k}, P^{T,x}_{\Pi^{t,x}}) - \partial_h f(\Pi^{t',x,P_{k'}}, P^{T,x}_{\Pi^{t',x}})|^2 + (\partial_h f)(\Pi^{t,x,P_k}, P^{T,x}_{\Pi^{t,x}})|^2 \, dr \right] \]

\[ \leq C(t' - t) + CE\left[ \int_t^{t'} (|\partial_x^2 \Pi^{t,x,P_k}|^2 \, dr \right]) (t' - t)^\frac{1}{2} \leq C(t' - t)^\frac{1}{2}. \]

Consequently, from (10.40) and (10.41) we have

\[ |\partial_x^2 V(t,x,P_k) - \partial_x^2 V(t',x,P_{k'})| \]

\[ \leq |E[|\partial_x^2 Y^t,x,P_k - \partial_x^2 Y^t,x,P_{k'}|] + (E[\sup_{s \in [t',T]} |\partial_x^2 Y^s,y,P_k - \partial_x^2 Y^s,y,P_{k'}|^2])^\frac{1}{2} \]

\[ \leq C|t' - t|^\frac{1}{2}, \quad t, t' \in [0,T], \quad x \in \mathbb{R}, \quad \xi \in L^2(\mathcal{F}_t). \]

**Step 4. To prove iv) and v) of Lemma 10.2**

Let us prove v) which is more complicated, similar to prove iv). From (8.1) we have

\[ \partial_y(\partial_y Y^{t,x,P_k}(y)) = \partial_x \Phi(X^{t,x,P_k}_T) \partial_y(\partial_y X^{t,x,P_k}(y)) + \int_s^T (\partial_h f)(\Pi^{t,x,P_k}, P^{T,x}_{\Pi^{t,x}}) \partial_y(\partial_y \Pi^{t,x,P_k}(y)) \, dr \]

\[ + \int_s^T \tilde{E}[\partial_y(\partial_h f)(\Pi^{t,x,P_k}, P^{T,x}_{\Pi^{t,x}}) \tilde{\Pi}^{t,x,y,P_k}(y)] \partial_y(\partial_y \tilde{\Pi}^{t,x,y,P_k}(y)) \, dr \]

\[ + \int_s^T \tilde{E}[\partial_y(\partial_y Z^{t,x,P_k}(y)) \partial_y(\partial_y \Pi^{t,x,y,P_k}(y)) ] \, dr - \int_s^T \partial_y(\partial_y Z^{t,x,P_k}(y)) \, dB \]

\[ - \int_s^T \int_K \partial_y(\partial_x H^{t,x,P_k}(y,e)) N_\lambda(dr,de), \quad s \in [t,T]. \]

(10.42)

Now replace \( x \) by \( \xi \) in above equation (10.42) we get the solution \( \partial_y(\partial_y Y^{t,x,\xi}(y)), \partial_y(\partial_y Z^{t,x,\xi}(y)), \partial_y(\partial_y H^{t,x,\xi}(y))) \) of the following BSDE:

\[ \partial_y(\partial_h Y^{t,x,\xi}(y)) = \partial_x \Phi(X^{t,x,\xi}_T) \partial_y(\partial_y X^{t,x,\xi}(y)) + \int_s^T (\partial_h f)(\Pi^{t,x,\xi}, P^{T,x}_{\Pi^{t,x}}) \partial_y(\partial_y \Pi^{t,x,\xi}(y)) \, dr \]

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\begin{align*}
&+ \int_s^T \tilde{E}[\partial_y (\partial_\mu f)(\Pi_r^{t,\xi}, P_{r,\Pi_r^{t,\xi}}, \tilde{\Pi}_r^{t,x,y,P_\xi})(\partial_\mu \tilde{\Pi}_r^{t,x,y,P_\xi})(\partial_\mu \tilde{\Pi}_r^{t,x,y,P_\xi})^2 + (\partial_\mu f)(\Pi_r^{t,\xi}, P_{r,\Pi_r^{t,\xi}}, \tilde{\Pi}_r^{t,x,y,P_\xi})(\partial_\mu \tilde{\Pi}_r^{t,x,y,P_\xi})^2 dr \\
&+ \int_s^T \tilde{E}[(\partial_\mu f)(\Pi_r^{t,\xi}, P_{r,\Pi_r^{t,\xi}}, \tilde{\Pi}_r^{t,x,y,P_\xi})\partial_y (\partial_\mu \tilde{\Pi}_r^{t,x,y,P_\xi}(y))]dr - \int_s^T \partial_y (\partial_\mu Z_r^{t,\xi}(y))dB_r

\text{(10.43)}
\end{align*}

On the other hand, notice that we have for \(0 \leq t < s \leq T\), P-a.s.,

\begin{align*}
\partial_y (\partial_\mu X_T^{t,\xi}(y)) &= (\partial_\mu X_T^{s,\xi}(P_{s,\Pi_{r,s}}), \partial_y (\partial_\mu X_T^{s,\xi}(y)) + \tilde{E}[\partial_y ((\partial_\mu X_T^{s,\xi}(P_{s,\Pi_{r,s}}), \partial_x X_T^{s,\xi}(P_{s,\Pi_{r,s}}))^2] \\
&+ \tilde{E}[(\partial_\mu X_T^{s,\xi}(P_{s,\Pi_{r,s}}), \partial_y (\partial_\mu X_T^{s,\xi}(P_{s,\Pi_{r,s}})))]
\text{(10.44)}
\end{align*}

Using similar arguments as for \(\text{(10.40)}\), applying Theorem 10.2 to above equation \(\text{(10.43)}\) by using \(\text{(10.44)}, \text{(10.38)}, \text{(10.40)}\), Theorem 7.1, Propositions 5.1 and 11.1 (6.2), Lemma 3.1, Theorem 8.1, we obtain

\begin{align*}
E \left[ \sup_{s \in [t,T]} |\partial_y (\partial_\mu Y_s^{t,\xi}(y)) - \partial_y (\partial_\mu Y_s^{t,\xi}(y))|^2 + \int_t^T \left( |\partial_y (\partial_\mu Z_r^{t,\xi}(y)) - \partial_y (\partial_\mu Z_r^{t,\xi}(y))|^2 \\
+ \int_K |\partial_y (\partial_\mu H_r^{t,\xi}(y,e)) - \partial_y (\partial_\mu H_r^{t,\xi}(y,e))|^2 \lambda(de) dr \right) \right]
\leq CE \left[ |J_1(t) - J_1(t')|^2 \right] + CE \left[ \left( \int_t^T |J_2(t,r) - J_2(t',r)|dr \right)^2 \right] \\
+ CE \left[ \left( \int_t^T \left( |(\partial_\mu f)(\Pi_r^{t,\xi}, P_{r,\Pi_r^{t,\xi}}) - (\partial_\mu f)(\Pi_r^{t,\xi}, P_{r,\Pi_r^{t,\xi}})|dr \right)^2 \right] \\
+ CE \left[ \left( \int_t^T \left( \tilde{E}[((\partial_\mu f)(\Pi_r^{t,\xi}, P_{r,\Pi_r^{t,\xi}}, \tilde{\Pi}_r^{t,x,y,P_\xi}) - (\partial_\mu f)(\Pi_r^{t,\xi}, P_{r,\Pi_r^{t,\xi}}, \tilde{\Pi}_r^{t,x,y,P_\xi}))\partial_y (\partial_\mu \tilde{\Pi}_r^{t,x,y,P_\xi}(y))]dr \right)^2 \right]
\leq C|t - t'|^\frac{1}{2},
\end{align*}

where \(J_1(t) := \partial_\xi \Phi (X_T^{t,\xi})\partial_y (\partial_\mu X_T^{t,\xi}(y)), J_2(t,r) := \tilde{E}[\partial_y (\partial_\mu f)(\Pi_r^{t,\xi}, P_{r,\Pi_r^{t,\xi}}, \tilde{\Pi}_r^{t,x,y,P_\xi})(\partial_\mu \tilde{\Pi}_r^{t,x,y,P_\xi})^2 \\
+ (\partial_\mu f)(\Pi_r^{t,\xi}, P_{r,\Pi_r^{t,\xi}}, \tilde{\Pi}_r^{t,x,y,P_\xi})(\partial_\mu \tilde{\Pi}_r^{t,x,y,P_\xi}).

Furthermore, still applying similar arguments to the equation \(\text{(10.42)}\), but also using the estimate \(\text{(10.45)}\), we obtain

\begin{align*}
E \left[ \sup_{s \in [t,T]} |\partial_y (\partial_\mu Y_s^{t,x,P_\xi}(y)) - \partial_y (\partial_\mu Y_s^{t,x,P_\xi}(y))|^2 + \int_t^T \left( |\partial_y (\partial_\mu Z_r^{t,x,P_\xi}(y)) - \partial_y (\partial_\mu Z_r^{t,x,P_\xi}(y))|^2 \\
+ \int_K |\partial_y (\partial_\mu H_r^{t,x,P_\xi}(y,e)) - \partial_y (\partial_\mu H_r^{t,x,P_\xi}(y,e))|^2 \lambda(de) dr \right) \right]
\leq C|t - t'|^\frac{1}{2}.
\end{align*}

On the other hand, similar to \(\text{(10.41)}\) using \(\text{(10.38)}\) and Theorem 8.1 we get

\begin{align*}
|E[\partial_y (\partial_\mu X_T^{t,x,P_\xi}(y)) - \partial_y (\partial_\mu X_T^{t,x,P_\xi}(y))]| \\
\leq E[\int_t^T |(\partial_\mu f)(\Pi_r^{t,x,P_\xi}, P_{r,\Pi_r^{t,x,P_\xi}})\partial_y (\partial_\mu X_T^{t,x,P_\xi}(y)) + \tilde{E}[\partial_y (\partial_\mu f)(\Pi_r^{t,x,P_\xi}, P_{r,\Pi_r^{t,x,P_\xi}}, \tilde{\Pi}_r^{t,x,y,P_\xi})(\partial_\mu \tilde{\Pi}_r^{t,x,y,P_\xi})^2 \\
+ (\partial_\mu f)(\Pi_r^{t,x,P_\xi}, P_{r,\Pi_r^{t,x,P_\xi}}, \tilde{\Pi}_r^{t,x,y,P_\xi})(\partial_\mu \tilde{\Pi}_r^{t,x,y,P_\xi})\partial_y (\partial_\mu \tilde{\Pi}_r^{t,x,y,P_\xi}(y))]dr]
\leq C(t' - t)^\frac{1}{2}.
\end{align*}
Hence, from above two estimates we get
\[
|\partial_y(\partial_\mu V)(t, x, P_\xi, y) - \partial_y(\partial_\mu V)(t', x, P_\xi, y)|
\leq |E[\partial_y(\partial_\mu Y_t^{t,x,P_\xi}(y)) - \partial_y(\partial_\mu Y_{t'}^{t',x,P_\xi}(y))]| + (E[\sup_{s \in [t',T]} |\partial_y(\partial_\mu Y_s^{t,x,P_\xi}(y)) - \partial_y(\partial_\mu Y_s^{t',x,P_\xi}(y))|^2])^{\frac{1}{2}}
\leq C|t' - t|^\frac{1}{8}, \ t, t' \in [0, T], \ x \in \mathbb{R}, \ y \in \mathbb{R}, \ \xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}).
\]

\[
\square
\]

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