The Björling problem for prescribed mean curvature surfaces in $\mathbb{R}^3$

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Abstract

In this paper we solve the Björling problem for the class of immersed surfaces in $\mathbb{R}^3$ whose mean curvature is given as an analytic function depending on its Gauss map. As an application, we prove the existence of surfaces with the topology of a Möbius strip for an arbitrary large class of prescribed functions. In particular, we use the Björling problem to construct the first known examples of self-translating solitons of the mean curvature flow with the topology of a Möbius strip in $\mathbb{R}^3$.

1 Introduction

A classical problem in minimal surface theory in $\mathbb{R}^3$ is the so called Björling problem [Nit]. This problem was posed in 1844 by the very Björling and solved in 1890 by Schwarz [Sch], and asks the following:

Given a regular analytic curve $\beta(s)$ in $\mathbb{R}^3$ and an analytic distribution of oriented planes $\Pi(s)$ along $\beta(s)$ such that $\beta'(s) \in \Pi(s)$, find all minimal surfaces in $\mathbb{R}^3$ containing $\beta(s)$ and such that the tangent plane distribution along $\beta(s)$ is given by $\Pi(s)$.

Schwarz solved this problem via an integral representation formula, using holomorphic data. Indeed, a Weierstrass representation formula can be given for this problem, and can be applied in other generic situations: for instance, to study surfaces with certain symmetries [DHKW], and to solve global problems in the minimal surface theory.

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[AlMi, GaMi1]; see also [ACG, GaMi2, GaMi3] and references therein for an outline on the development of the geometric Cauchy problem. The Björling problem has been also studied when the mean curvature is a non-vanishing constant, see [BrDo].

Our objective in this paper is to solve the Björling problem for a certain class of prescribed mean curvature surfaces immersed in $\mathbb{R}^3$. Specifically, let $\Sigma$ be an immersed surface in $\mathbb{R}^3$ and let $\mathcal{H} \in C^1(\mathbb{S}^2)$. We say that $\Sigma$ has prescribed mean curvature $\mathcal{H}$ (in short, $\Sigma$ is an $\mathcal{H}$-surface) if the mean curvature $H_\Sigma$ of $\Sigma$ satisfies

$$H_\Sigma(p) = \mathcal{H}(\eta_p), \quad \forall p \in \Sigma,$$

where $\eta : \Sigma \to \mathbb{S}^2$ is the Gauss map of $\Sigma$.

In general, the study of hypersurfaces in $\mathbb{R}^{n+1}$ defined by a prescribed curvature function in terms of the Gauss map goes back to the famous Christoffel and Minkowski problems for ovaloids, see e.g. [Chr]. In particular, the existence and uniqueness of ovaloids with prescribed mean curvature (1.1) was studied among others by Alexandrov and Pogorelov [Ale, Pog] but the global geometry of these surfaces remained largely unexplored. In [GaMi4] the authors studied uniqueness of immersed $\mathcal{H}$-spheres proving a Hopf-type theorem for this class of immersed surfaces, and in the making they proved a standing conjecture by Alexandrov. The global properties of $\mathcal{H}$-hypersurfaces immersed in $\mathbb{R}^{n+1}$ have been recently developed in [BGM], where the authors studied several topics such as classification of rotational $\mathcal{H}$-hypersurfaces, a priori height estimates and a structure theorem for properly embedded $\mathcal{H}$-surfaces in $\mathbb{R}^3$, and curvature estimates for stable $\mathcal{H}$-surfaces immersed in $\mathbb{R}^3$.

This paper is organized as follows: in Section 2 we solve the Björling problem for the class of analytic functions $\mathcal{H} \in C^\omega(\mathbb{S}^2)$, giving rise to a large amount of examples of $\mathcal{H}$-surfaces besides the ones exposed in [BGM].

In Section 3 we restrict ourselves to the class of analytic functions $\mathcal{H} \in C^\omega(\mathbb{S}^2)$ satisfying the symmetry condition $\mathcal{H}(-x) = -\mathcal{H}(x), \forall x \in \mathbb{S}^2$, and we use the solution of the Björling problem for adequate Björling data to construct non-orientable $\mathcal{H}$-surfaces with the topology of a Möbius strip. A particular analytic function with this symmetry condition is the one given by $\mathcal{H}(x) = \langle x, e_3 \rangle, \forall x \in \mathbb{S}^2$. The $\mathcal{H}$-surfaces arising for this choice are the self-translating solitons of the mean curvature flow in $\mathbb{R}^3$, a well studied class of surfaces in the past decades. See e.g. [CSS, Hui, MSHS] for relevant works regarding this topic. As an application, we construct self-translating solitons of the mean curvature flow in $\mathbb{R}^3$ with the topology of a Möbius strip. Up to our knowledge, these translating Möbius strips are the first known examples of self-translating solitons with non-orientable topology.

Finally, in Section 4 we use the solution of the Björling problem to construct further examples of $\mathcal{H}$-surfaces for the prescribed function $\mathcal{H}(x) = \langle x, e_3 \rangle$. In particular, we construct helicoidal $\mathcal{H}$-surfaces, and $\mathcal{H}$-surfaces based on the Enneper core curve.

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2 Resolution of the Björling problem for $\mathcal{H}$-surfaces

Definition 1 Let $\mathcal{H} \in C^\omega(S^2)$. A pair of Björling data for $\mathcal{H}$-surfaces in $\mathbb{R}^3$ is a regular analytic curve $\beta : I \rightarrow \mathbb{R}^3$ and an analytic vector field $B : I \rightarrow \mathbb{R}^3$ along $\beta(s)$ such that $|\beta'(s)| - |B(s)| = \langle \beta'(s), B(s) \rangle = 0$, $\forall s \in I$.

From this definition we get two obvious consequences. First, given $\mathcal{H} \in C^\omega(S^2)$, a regular analytic curve $\beta(s)$ and an oriented distribution of planes $\Pi(s)$ along $\beta(s)$, we get a pair of Björling data by just defining $B(s) = J\beta'(s)$, where $J$ denotes the $\pi/2$-rotation in the tangent plane. And second, if $\nabla$ denotes the Riemannian connection of $\Sigma$ and $\beta(s)$ is parametrized by arc-length, if we define $\beta''(s) := \nabla_{\beta'(s)}\beta'(s)$ then $B(s) = \beta''(s)/|\beta''(s)|$ is an analytic unit vector field along $\beta(s)$ such that $\langle \beta'(s), B(s) \rangle = 0$. Thus, $B(s)$ determines an oriented distribution of planes $\Pi(s)$ along $\beta(s)$ by defining $\Pi(s) = \beta(s) + B(s)^\perp$. In particular, the Björling problem generalizes the problem of finding a surface which contains a given curve as a geodesic.

Although we do not have a Weierstrass representation for $\mathcal{H}$-surfaces, and thus we cannot solve the Björling problem with an integral representation formula just as in the case $\mathcal{H} = 0$ (see e.g. [Mir]), we can solve this problem by applying different methods.

Let $\Sigma$ be an orientable Riemannian surface and let $\psi : \Sigma \rightarrow \mathbb{R}^3$ be an isometric immersion of $\Sigma$ in $\mathbb{R}^3$. Then, it is well known that the coordinates of $\psi$ satisfy the elliptic PDE

$$\Delta_\Sigma \psi = 2H_\Sigma \eta,$$

(2.1)

where $\Delta_\Sigma$ stands for the Laplace-Beltrami operator on $\Sigma$, $\eta$ denotes the Gauss map of $\Sigma$ and $H_\Sigma$ is the mean curvature of $\Sigma$ computed with respect to $\eta$.

Recall that $\Sigma$ inherits a Riemann surface structure induced by the first fundamental form, and thus we can consider a conformal coordinate $z = s + it$ in a simply connected domain $\Omega \subset \mathbb{C}$. For this conformal coordinate, we define the usual Weingarten operators $\partial_z = 1/2(\partial_s - i\partial_t)$, $\overline{\partial}_z = 1/2(\partial_s + i\partial_t)$. Then, the induced metric on $\Sigma$ expresses as $\langle \cdot, \cdot \rangle = \lambda^2|dz|^2$, where $|dz|^2$ is the flat metric on $\Omega$ and $\lambda^2 = \langle \partial_s, \partial_s \rangle = \langle \partial_t, \partial_t \rangle > 0$ is the conformal factor. For such a conformal coordinate, the operator $\Delta_\Sigma$ expresses as

$$\Delta_\Sigma = \frac{1}{\lambda^2} \Delta_0 = \frac{1}{\lambda^2}(\partial_{ss} + \partial_{tt}) = \frac{4}{\lambda^2} \partial_{\overline{z}z},$$

(2.2)

where $\Delta_0$ denotes the usual flat Laplacian, and we used the relation of the Laplace-Beltrami operator between two conformal metrics. On the other hand, the Gauss map $\eta$ of $\Sigma$ has the following expression with respect to $z$

$$\eta = \frac{2}{i} \frac{\psi_z \wedge \psi_{\overline{z}}}{\sqrt{\langle \psi_z \wedge \psi_{\overline{z}}, \psi_z \wedge \psi_{\overline{z}} \rangle}} = \frac{2}{i\lambda^2} \psi_z \wedge \psi_{\overline{z}}.$$

(2.3)

Let us denote by $C^\omega(S^2)$ the class of analytic functions defined on the sphere, and suppose that the immersion $\psi : \Sigma \rightarrow \mathbb{R}^3$ defines an $\mathcal{H}$-surface for some $\mathcal{H} \in C^\omega(S^2)$. 

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By Equations (2.2) and (2.3), Equation (2.1) writes as

$$\psi_{z\bar{z}} = -i\mathcal{H}(\eta)\psi_z \wedge \psi_{\bar{z}}. \quad (2.4)$$

Viewing the immersion in coordinates \(\psi = (\psi_1, \psi_2, \psi_3)\), then Equation (2.4) can be seen as a system of partial differential equations. In this setting, we can solve the Björling problem for the class of \(\mathcal{H}\)-surfaces, as stated next:

**Theorem 2 (Björling problem for \(\mathcal{H}\)-surfaces)** Let \(\mathcal{H} \in C^\omega(\mathbb{S}^2)\) and let \(\beta(s), B(s)\) be a pair of Björling data defined on a real interval \(I\). Then, there exists an open domain \(\Omega \subset \mathbb{C}\) containing \(I \times \{0\} \subset \Omega\), and there exists a conformal immersion \(\psi : \Omega \to \mathbb{R}^3\) that solves the system of partial differential equations with initial data

$$\begin{cases}
\psi_{z\bar{z}} = -i\mathcal{H}(\eta)\psi_z \wedge \psi_{\bar{z}} \\
\psi(s,0) = \beta(s) \\
\psi_1(s,0) = B(s)
\end{cases} \quad (2.5)$$

As a consequence, \(\psi(\Omega)\) defines an \(\mathcal{H}\)-surface that contains the curve \(\beta(s)\), and the tangent plane distribution \(T_{\beta(s)\Sigma}\) at each point \(\beta(s) \in \psi(\Omega)\) is spanned by the vectors \(\beta'(s), B(s)\).

**Proof:** The system (2.5) is of Cauchy-Kovalevskaya type, and thus it has local existence and uniqueness. As the system (2.4) is elliptic without characteristic points, we have that the existence and uniqueness extends to the whole interval \(I\) where \(\beta(s)\) and \(B(s)\) are defined. Thus, there exists \(\delta > 0\) and functions \(\psi_1, \psi_2, \psi_3\) defined on \(I \times (-\delta, \delta) \subset \mathbb{C}\) such that \(\psi = (\psi_1, \psi_2, \psi_3)\) is a solution of (2.4) satisfying \(\psi(s,0) = \beta(s)\), \(\psi_1(s,0) = B(s)\), \forall s \in I\).

First, observe that the equation \(\langle \psi_{z\bar{z}}, \psi_z \rangle = 0\) holds by substituting \(\psi_{z\bar{z}}\) for its expression (2.4), and thus the function \(\langle \psi_z, \psi_{\bar{z}} \rangle\) is holomorphic. As \(\psi\) satisfy the described initial conditions, the function \(\langle \psi_z, \psi_{\bar{z}} \rangle\) evaluated at \((s,0)\) is equal to \(|\beta'(s)|^2 - |B(s)|^2 - 2i\langle \beta'(s), B(s)\rangle\). This expression vanishes identically, as \(\beta'(s)\) and \(B(s)\) are orthogonal vector fields with the same length. In this conditions, \(\langle \psi_z, \psi_{\bar{z}} \rangle\) is a holomorphic function vanishing at the real axis and by the identity principle of holomorphic functions, \(\langle \psi_z, \psi_{\bar{z}} \rangle\) is identically zero in \(I \times (-\delta, \delta)\). We conclude that the map \(\psi : I \times (-\delta, \delta) \to \mathbb{R}^3\) is conformal.

Now we prove that the map \(\psi\) defines an immersion. This property is given by the condition \(\langle \psi_z, \psi_{\bar{z}} \rangle > 0\). If we evaluate again the function \(\langle \psi_z, \psi_{\bar{z}} \rangle\) at the real axis \((s,0)\), we get \(1/4(|\beta'(s)|^2 + |B(s)|^2) > 0\). Consequently, \(\langle \psi_z, \psi_{\bar{z}} \rangle > 0\) in the real axis, and thus there exists \(\varepsilon > 0\) (which can be chosen smaller than \(\delta\)) such that if we define the open set \(\Omega := I \times (-\varepsilon, \varepsilon)\), the map \(\psi : \Omega \to \mathbb{R}\) is a conformal immersion. Lastly, as \(\psi\) is a solution of Equation (2.4), \(\psi\) defines a conformal immersion of an \(\mathcal{H}\)-surface. This concludes the proof of Theorem 2.

\(\Box\)
For example, consider the analytic function $\mathcal{H}(x) = \langle x, e_3 \rangle$, $\forall x \in S^2$. The $\mathcal{H}$-surfaces arising for this particular choice are the self-translating solitons of the mean curvature flow.

The rotational self-translating solitons of the mean curvature flow are fully classified as follows: an entire, strictly convex vertical graph that intersects the axis of rotation orthogonally, called the bowl soliton; and a one parameter family of properly embedded annuli, with both ends pointing towards the $e_3$ direction, called the wing-like solitons or translating catenoids. The family of wing-like solitons are parametrized by the necksize, i.e. the minimum distance to the axis of rotation, attained at a circumference of radius equal to the waist, see [CSS] for details.

Bearing this in mind, we can recover this family by choosing adequate Björling data. Indeed, consider the one parameter family of Björling data $\beta_\tau(s) = \tau \langle \cos s, \sin s, 0 \rangle$ and $B_\tau(s) = (0, 0, \tau)$, $\forall s \in \mathbb{R}$, $\tau > 0$. Then, for each fixed $\tau > 0$, the translating soliton $\Sigma_\tau$ generated by this Björling data corresponds to the wing-like soliton with necksize equal to $\tau$. When $\tau \to 0$, the sequence $\Sigma_\tau$ converges smoothly to a double cover of the bowl soliton minus the vertex. See Figure 1 for a plot of the wing-like soliton with necksize equal to 1.

![Figure 1: A wing-like example for the Björling data $\mathcal{H}(x) = \langle x, e_3 \rangle$, $\beta(s) = \langle \cos s, \sin s, 0 \rangle$ and $B(s) = (0, 0, 1)$.
]
3 \( \mathcal{H} \)-Möbius strips in \( \mathbb{R}^3 \)

The resolution of the Björling problem adds a large amount of \( \mathcal{H} \)-surfaces to the ones studied in [BGM], for any given \( \mathcal{H} \in C^\omega(\mathbb{S}^2) \). By choosing an adequate Björling data, the examples arising may have some prescribed symmetries, as well as some topological properties. In this context, the resolution of the Björling problem motivates us for the search of non-orientable surfaces. The most recurrent examples are the surfaces with the topology of a Möbius strip. See [Mee, Mir] for relevant works regarding minimal Möbius strips in \( \mathbb{R}^n \) given as the solution of the Björling problem for adequate Björling data.

Recall that the topologic construction of a Möbius strip is by identifying antipodally two opposite sides of a square. For an arbitrary \( \mathcal{H} \in C^\omega(\mathbb{S}^2) \) we cannot guarantee the existence of these examples with non-orientable topology; even though in the case when \( \mathcal{H} \) is a positive constant, such surfaces do not exist. The main problem comes when trying to close the surface by identifying the opposite sides of the square, as in these points the orientation is changed and thus the value of the mean curvature will not agree. The advantage in the minimal case is the possibility of reverse the orientation of the surface without changing the value of the mean curvature. Bearing this in mind, we restrict ourselves to prescribed functions \( \mathcal{H} \in C^\omega(\mathbb{S}^2) \) satisfying the antipodal antisymmetry condition \( \mathcal{H}(-x) = -\mathcal{H}(x), \forall x \in \mathbb{S}^2 \). Note that the particular case \( \mathcal{H} \equiv 0 \) is included in this family, as well as \( \mathcal{H}(x) = \langle x, e_3 \rangle \).

The following result is inspired in the ideas developed in [Mir].

**Proposition 3** Let \( \mathcal{H} \in C^\omega(\mathbb{S}^2) \) such that \( \mathcal{H}(-x) = -\mathcal{H}(x), \forall x \in \mathbb{S}^2 \), and let \( \beta(s), B(s) \) be Björling data such that \( \beta(s) \) is \( T \)-periodic and \( B(s) \) is \( T \)-antiperiodic, for some \( T > 0 \).

Then, the \( \mathcal{H} \)-surface generated by means of Theorem 2 for this Björling data has the topology of a Möbius strip, with fundamental group spanned by \( \beta(s) \).

Conversely, every \( \mathcal{H} \)-Möbius strip is generated in this way.

**Proof:** Consider an immersion of an \( \mathcal{H} \)-Möbius strip \( \psi : \mathcal{M} \to \mathbb{R}^3 \), and let \( \Gamma \) be a regular analytic closed curve in \( \mathcal{M} \) that spans its fundamental group.

Let be \( \tilde{\mathcal{M}} \) the two sheeted cover of \( \mathcal{M} \) and denote by \( \pi : \tilde{\mathcal{M}} \to \mathcal{M} \) the canonical projection. In this setting, \( \tilde{\mathcal{M}} \) inherits a Riemann surface structure, endowed with an antiholomorphic involution without fixed points, i.e. a map \( I : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}} \) such that \( I(z_1) = z_2 \) if and only if \( z_1 \neq z_2 \) and \( \pi(z_1) = \pi(z_2) \). Note that \( \mathcal{M} \) is topologically a cylinder, and so there exists an analytic closed curve \( \tilde{\Gamma} \) that generates the fundamental group of \( \tilde{\mathcal{M}} \), defined as \( \pi(\tilde{\Gamma}) = \Gamma \). As \( \tilde{\Gamma} \) is a closed curve, we can parametrize \( \tilde{\Gamma} \) as \( \tilde{\Gamma}(s) : \mathbb{R} \to \tilde{\Gamma} \), where \( \tilde{\Gamma}(s) \) is \( 2T \)-periodic for some \( T > 0 \), and verifies \( I(\tilde{\Gamma}(s)) = \tilde{\Gamma}(s + T), \forall s \in \mathbb{R} \). In particular, \( \Gamma(s) := \pi(\tilde{\Gamma}(s)) \) is \( T \)-periodic.

Define next \( \tilde{\psi} : \tilde{\mathcal{M}} \to \mathbb{R}^3 \) by the relation \( \tilde{\psi} = \psi \circ \pi \). Then, \( \beta(s) = \tilde{\psi}(\tilde{\Gamma}(s)) \) is a regular analytic \( T \)-periodic curve in \( \mathbb{R}^3 \). Besides, denoting by \( \Pi(s) \) the oriented tangent
planes distribution of \( \tilde{\psi} \) along \( \beta(s) \), we have that the planes \( \Pi(s + T) \) and \( \Pi(s) \) agree with opposite orientation. If we denote by \( J \) the \( \pi/2 \)-rotation in each oriented plane, then the field \( B(s) := J\beta'(s) \) satisfies \( B(s + T) = -B(s) \), \( \forall s \in \mathbb{R} \). That is, \( B(s) \) is \( T \)-antiperiodic. Thus, we have proved that every \( H \)- Möbius strip defines a pair of Björling data with the periodicity properties stated in Proposition 3.

Conversely, let \( \beta(s) \) and \( B(s) \) be Björling data such that \( \beta(s) \) is \( T \)-periodic and \( B(s) \) is \( T \)-antiperiodic. In particular, the parameter \( s \) is defined on the entire real line. If we solve the Björling problem for the Björling data \( \beta(s) \) and \( B(s) \) by means of Theorem 2, we ensure the existence of an open set \( \Omega \subset \mathbb{C} \) containing the real axis \( \{ \Im(z) = 0; \ z \in \mathbb{C} \} \), and there exists a conformal immersion \( \psi : \Omega \to \mathbb{R}^3 \) defining an \( H \)-surface that solves (2.5).

Given \( J \subset \mathbb{R} \) we define the set \( \Omega_J = \{ z \in \Omega; \ \Re(z) \in J \} \), and name \( \psi_J = \psi_{\Omega_J} \). Because \( \beta(s) \) and \( B(s) \) are \( 2T \)-periodic, the solutions \( \psi_J \) and \( \psi_{J+2T} \) have the same Björling data, and by the uniqueness of Theorem 2 \( \psi_J = \psi_{J+2T} \), for all \( J \subset \mathbb{R} \). In particular, \( \Omega \) is \( 2T \)-periodic in the real line direction, i.e. \( \Omega = \Omega + (2T, 0) \), and the immersion \( \psi \) is defined, up to periodicity, on the open set \( \Omega_{[0,2T]} \). For definiteness, we will name this domain again \( \Omega \). This discussion implies that the quotient map \( \psi : \Omega/(2TZ) \to \mathbb{R}^3 \) is well defined, and generates a topological cylinder.

Moreover, we can suppose that \( \Omega \) is symmetric with respect to the conjugation, i.e.
\[ z \in \Omega, \, \forall z \in \Omega, \text{ and thus we can define on } \Omega/(2TZ) \text{ the map} \]

\[
I : \Omega/(2TZ) \rightarrow \Omega/(2TZ) \quad \text{such that} \quad z \mapsto z + T \tag{3.1}
\]

It is clear then that the map \( I \) has no fixed point, that is an involution and that reverses the orientation of \( \Omega/(2TZ) \). Consequently, \( I \) is an anti-holomorphic map for the conformal structure of \( \Omega/(2TZ) \). Moreover, the cylinder \( \Omega/(2TZ) \) is the orientable two sheeted cover of the space \( (\Omega/(2TZ), I) \). In this setting, for each point \( p \) in the quotient space \( (\Omega/(2TZ), I) \) there exist two points \( q_1, q_2 \) in \( \Omega/(2TZ) \) such that \( I(q_1) = I(q_2) = p \).

Define on \( \{ \Im(z) = 0 \} \subset \Omega \) the vector field \( \eta(s) := \beta'(s) \wedge B(s) \), which has an extension in a domain contained in \( \Omega \); we will keep naming \( \Omega \) to this domain, which can be supposed also to be symmetric with respect the conjugation. As \( \beta(s) \) is \( T \)-periodic and \( B(s) \) is \( T \)-antiperiodic, it is clear that \( \eta(z) = -\eta(I(z)), \, \forall z \in \Omega \). On the one hand, this condition allows us to define \( \eta \) in the quotient \( \Omega/(2TZ) \). On the other hand, given \( z, w \in \Omega/(2TZ) \) such that \( w = I(z) \), and because \( \psi \) defines an \( \mathcal{H} \)-surface in \( \mathbb{R}^3 \), we have

\[
H_\psi(\psi(w)) = \mathcal{H}(\eta_\psi(w)) = \mathcal{H}(-\eta_\psi(I(z))) = -\mathcal{H}(\eta_\psi(I(z))) = -H_\psi(\psi(I(z))). \tag{3.2}
\]

Consequently, the mean curvature at points with opposite orientation has the opposite sign. Changing the orientation in one of the sheets, we have that the mean curvature at the points \( \psi(z) \) and \( \psi(I(z)) \) agree for every \( z \in \Omega/(2TZ) \), as well as the Björling data after the change of the orientation.

This allows us to define the map \( \tilde{\psi} : (\Omega/(2TZ), I) \rightarrow \mathbb{R}^3 \) as \( \tilde{\psi}(z) = \psi(I(z)), \, \forall z \in \Omega/(2TZ) \) as a well defined map. As a set of points, the surfaces given by the conformal immersions \( \psi(\Omega/(2TZ)) \) and \( \tilde{\psi}(\Omega/(2TZ), I) \) agree, and the surface \( \psi(\Omega/(2TZ)) \) passes two times for each point of this set: the first time having an orientation, and the second time with the opposite orientation.

As a matter of fact, the maps \( \psi : \Omega/(2TZ) \rightarrow \mathbb{R}^3 \) and \( \tilde{\psi} : (\Omega/(2TZ), I) \rightarrow \mathbb{R}^3 \) define the same \( \mathcal{H} \)-surface. By the uniqueness given by Theorem 2, the map \( \tilde{\psi} : (\Omega/(2TZ), I) \rightarrow \mathbb{R}^3 \) is a well defined conformal immersion of an \( \mathcal{H} \)-surface in \( \mathbb{R}^3 \), with the topology of a Möbius strip with fundamental group spanned by \( \beta(s) \), and consequently is non-orientable. This concludes the proof of Proposition 3.

\[ \square \]

As we have mentioned before, the choice \( \mathcal{H}(x) = \langle x, e_3 \rangle, \, \forall x \in S^2 \) corresponding to the case of the self-translating solitons of the mean curvature flow in \( \mathbb{R}^3 \), lies in the hypothesis of Proposition 3. Thus, we ensure the existence of self-translating solitons of the mean curvature flow with the topology of a Möbius strip, which we will refer to as translating Möbius strips, see Figure 2. Up to our knowledge, this construction gives the first examples of self-translating solitons of the mean curvature flow with non-orientable topology. In Figure 3 we show the construction of a non-orientable translating soliton constructed by half-rotating the vector field \( B(s) \) 7 times along the curve \( \beta(s) \) before it closes. This surface is homeomorphic to the Möbius strip showed in Figure 2.
Figure 3: An $\mathcal{H}$-surface with the topology of a Möbius strip for the analytic choice $\mathcal{H}(x) = \langle x, e_3 \rangle$, $x \in \mathbb{S}^2$.

4 Further examples of $\mathcal{H}$-surfaces via the Björling problem

In this last Section we show the existence of some examples of $\mathcal{H}$-surfaces immersed in $\mathbb{R}^3$ for $\mathcal{H}(-x) = -\mathcal{H}(x) \in C^\omega(\mathbb{S}^2)$, motivated by the analogous examples defined in the minimal surface theory. The main difference with the minimal case $\mathcal{H} = 0$ in $\mathbb{R}^3$ is that we fail to have a Weierstrass representation, and thus explicit parametrizations of these surfaces are not expected. Even though, we can solve the Björling problem for some adequate Björling data with some prescribed symmetries, and obtain $\mathcal{H}$-surfaces that are the analogous to quite famous examples in the minimal surface theory.

$\mathcal{H}$-Helicoids

We choose as Björling data the vertical curve $\beta(s) = (0, 0, s)$ and a $T$-antiperiodic vector field $B(s)$ along $\beta(s)$, for some $T > 0$, and let $\Sigma$ be the translating soliton given as the solution of the Björling problem for this Björling data.

The unit normal vector field at the $e_3$-axis, namely $\eta(s) := \beta'(s) \wedge B(s)$, is a horizontal vector field satisfying $\eta(s) = -\eta(s + T)$, $\forall s \in \mathbb{R}$. Thus, the mean curvature of $\Sigma$ satisfies
$H_\Sigma(\beta(s)) = H(\eta(s)) = 0, \forall s \in \mathbb{R}$, and the Björling data $\beta(s+T), B(s+T)$ agree with the Björling data $\beta(s) + Te_3, -B(s)$. Moreover, as the condition $\eta(s) = -\eta(s + T)$, $\forall s \in \mathbb{R}$ holds, at the points $\beta(s)$ and $\beta(s + T)$ we have the same Björling data but with the opposite orientation. Since in these points the mean curvature vanishes, we may change the orientation of one of the Björling datas in a way that they agree.

Bearing this in mind, if we denote by $\Theta(p) = p + Te_3$, $\forall p \in \mathbb{R}^3$, the uniqueness of Theorem 2 ensures us that $\Sigma$ is invariant by the discrete group of translations $T\mathbb{Z}\Theta$ in the $e_3$-direction. Moreover, starting at some $s_0 \in \mathbb{R}$, $\Sigma$ twists jointly with $\eta(s)$ around the $e_3$-axis until reaching the instant $\eta(s_0 + T) = -\eta(s_0)$, generating a simply connected fundamental part of $\Sigma$. Repeating this process, which is just translating this fundamental part of $\Sigma$ under the action of $\Theta$, we get the whole $H$-surface $\Sigma$.

These surfaces are called $H$-helicoids, since they generalize the usual minimal helicoids in $\mathbb{R}^3$. See Figure 4 for a plot of an $H$-helicoid for the particular choice $\mathcal{H}(x) = \langle x, e_3 \rangle$, $x \in S^2$, and the Björling data $\beta(s) = (0, 0, s)$, $B(s) = (\cos s, \sin s, 0)$.

![Figure 4: An $H$-helicoid for the analytic choice $\mathcal{H}(x) = \langle x, e_3 \rangle$, $x \in S^2$.](image)

**Enneper-type $H$-surfaces**

Consider the curve

$$\beta(s) = (-\cos s - 1/3 \cos(3s), \sin s - 1/3 \sin(3s), \sin(2s)),$$
and the vector field

\[ B(s) = (\cos s + \cos(3s), 2\cos(2s) \sin s, -2\sin(2s)). \]

Then, both \( \beta(s) \) and \( B(s) \) are analytic and satisfy \( |\beta'(s)| - |B(s)| = \langle \beta'(s), B(s) \rangle = 0, \forall s \in \mathbb{R} \), i.e. they can be chosen to be Björling data. If \( \mathcal{H} = 0 \), then the surface given by Theorem 2 is the well-known Enneper’s minimal surface [LoWe], and for that reason \( \beta(s) \) will be called the core curve of Enneper’s minimal surface. For the choice \( \mathcal{H}(x) = \langle x, e_3 \rangle, \forall x \in \mathbb{S}^2 \), the surface arising (see Figure 5, left) resembles indeed Enneper’s minimal surface. As we fail to have an explicit parametrization via a Weierstrass representation, we cannot guarantee that the hole in the middle will eventually close.

Figure 5: An \( \mathcal{H} \)-surface with the topology of a Möbius strip for the analytic choice \( \mathcal{H}(x) = \langle x, e_3 \rangle, x \in \mathbb{S}^2 \).

This construction can be generalized by choosing different vector fields along the core curve; in particular, making the vector \( B(s) \) twist along the curve. In Figure 5, right we can see a self-translating soliton of the mean curvature flow containing the core curve \( \beta(s) \), and whose tangent plane distribution twists along \( \beta(s) \) until it closes, where the corresponding planes agree but opposite direction, i.e. we get another translating soliton with the topology of a Möbius strip.
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