A new product on $2 \times 2$ matrices

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Abstract

We study a bilinear multiplication rule on $2 \times 2$ matrices which is intermediate between the ordinary matrix product and the Hadamard matrix product, and we relate this to the hyperbolic motion group of the plane.

The observables of many systems in physics are treated by sets of matrices. The elements of these matrices are taken from the field $F$ of real or complex numbers. A multiplicative and bilinear composition rule $(a, b) \mapsto a \ast b$ of matrices provides the system with the structure of an algebra $A$. The standard composition rule for two matrices is matrix multiplication. This composition is given by (row by column) multiplication. By contrast, the Hadamard product of matrices $[1]$ is defined by the entry-wise multiplication of elements. We shall study and interpret a composition rule intermediate between the standard and the Hadamard case.

1 Properties of the matrix algebra and the $\ast$-product.

Suppose that $F$ is a field. We will be mainly interested in the case where $F$ is the field of real numbers $\mathbb{R}$, but most our results are valid in general (for example, $F$ could also denote the field of complex numbers). The multiplicative group of the field will be denoted by $F^\times$. We recall that an algebra is a vector space $A$ over $F$ with a bilinear product $\ast$ defined on pairs of vectors. Hence we require for all vectors $u, v, w \in A$ and all scalars $s \in F$ that

\[
(u + v) \ast w = u \ast w + v \ast w \\
(w \ast u) \ast v = w \ast (u \ast v) \\
(su) \ast v = u \ast (sv) = s(u \ast v).
\]

The algebra $A$ is called associative if $u \ast (v \ast w) = (u \ast v) \ast w$ holds for all vectors $u, v, w$. For example, the $n \times n$-matrices with the usual matrix product form an associative algebra $\text{Mat}(n, F)$, whose unit element is the identity matrix $1$. But other products on matrices have also been studied.

In this note we propose a new product $\ast$ on $2 \times 2$ matrices and study some of its properties. The product $\ast$ is defined as

\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix} \ast 
\begin{pmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{pmatrix} = 
\begin{pmatrix}
a_{11}b_{11} & a_{11}b_{12} + a_{12}b_{21} \\
a_{21}b_{11} + a_{22}b_{21} & a_{22}b_{22}
\end{pmatrix}.
\]

In order to study this product in a systematic way we set up the following notation. We denote $n \times n$-matrices by capital letters $A, B, C$. The identity matrix is denoted by $1$ and the zero matrix is denoted by $0$. Every $n \times n$ square matrix $A$ can be decomposed uniquely as a sum

\[
A = A_0 + A_1
\]
of a diagonal matrix $A_0$ and a matrix $A_1$ with zeros on the diagonal. We define a new product on $n \times n$-matrices by putting

$$A \star B = A_0 B_0 + (AB)_1.$$  

(2)

For the products on the right-hand side we use ordinary matrix multiplication. Hence entries on the diagonal of $A \star B$ are computed by multiplying diagonal entries, while the off-diagonal entries are computed in the ordinary matrix multiplication way. The star product thus mixes the ordinary matrix product with the Hadamard product. It is clear from the definition that the product $\star$ is bilinear,

$$A \star (B + C) = A \star B + A \star C \text{ and } (B + C) \star A = B \star A + C \star A.$$  

It is also clear that the product $\star$ is for $n = 1$ the ordinary multiplication of scalars. For $n = 2$, this product given in (2) coincides with the one given in equation (1).

**Lemma 1.** The identity matrix $1$ is a unit element, $1 \star A = A = A \star 1$. The product $\star$ is associative for $n = 2$, but not associative for $n \geq 3$. For $n = 2$ we have

$$A \star B = A_0 B_0 + A_0 B_1 + A_1 B_0.$$  

(3)

Proof. The fact that $1$ is a unit is clear from formula (2). We always have

$$(AB)_1 = A_1 B_0 + A_0 B_1 + (A_1 B_1)_1,$$

because $(A_0 B_0)_1 = 0$. If $n = 2$, then the product of two matrices with zeros on the diagonal is a diagonal matrix, whence $(A_1 B_1)_1 = 0$ in this case. Therefore we have for $n = 2$ the formula

$$A \star B = A_0 B_0 + A_0 B_1 + A_1 B_0.$$

and thus

$$(A \star B) \star C = (A_0 B_0 + A_0 B_1 + A_1 B_0) \star C$$

$$= A_0 B_0 C_0 + A_0 B_0 C_1 + A_0 B_1 C_0 + A_1 B_0 C_0 = A \star (B \star C).$$

For $n = 3$ we have, however,

$$\left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \star \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right) \star \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) = \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

and

$$\left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \star \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right) \star \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right),$$

which shows that $\star$ is not associative. The same example works for all $n \geq 3$ by extending these three $3 \times 3$ matrices with zeros to $n \times n$ matrices.

2 The group of $\star$-invertible matrices

Now we study the $\star$-invertible matrices, for $n = 2$.

**Lemma 2.** Suppose that $n = 2$. Then $A = A_0 + A_1$ is $\star$-invertible if and only if $A_0$ is invertible in the ordinary sense. The $\star$-inverse $B$ of $A = A_0 + A_1$ is then

$$B = A_0^{-1} - A_0^{-1}A_1A_0^{-1}.$$
Proof. If $A_0$ is invertible, we have

$$ (A_0 + A_1) \star (A_0^{-1} - A_0^{-1}A_1^{-1}) = 1 + A_1A_0^{-1} - A_1A_0^{-1} = 1. $$

and similarly $(A_0^{-1} - A_0^{-1}A_1A_0^{-1}) \star (A_0 + A_1) = 1$. Hence $A = A_0 + A_1$ is $\star$-invertible with $\star$-inverse $B = A_0^{-1} - A_0^{-1}A_1A_0^{-1}$.

If $A_0$ is not invertible, let $B_0$ denote the diagonal matrix where the two diagonal entries of $A_0$ are exchanged. Then $A_0B_0 = 0$ and the matrix $B = B_0 - A_1$ satisfies

$$ A \star B = A_0B_0 - A_0A_1 + A_1B_0 = 0, $$

hence $A$ cannot be $\star$-invertible if $A_0$ is not invertible. \hfill \Box

We let $G$ denote the group of all $\star$-invertible matrices of our algebra, for $n = 2$. Every $\star$-invertible element is of the form

$$ A = A_0 + A_1 = A_0(1 + A_0^{-1}A_1) = A_0 \star (1 + B_1), \text{ where } B_1 = A_0^{-1}A_1. \quad (3) $$

Let $D$ denote the set of all $2 \times 2$ diagonal matrices. On this set $D$ of diagonal matrices, the $\star$-product and the usual matrix product coincide. The $\star$-invertible matrices in $D$ thus form a commutative subgroup $H$ of $G$. Let $N$ denote the set of all matrices of the form $B = 1 + B_1$. These matrices are $\star$-invertible and they form a group, with group law

$$ (1 + B_1) \star (1 + C_1) = 1 + B_1 + C_1. $$

Hence the group $N$ is also commutative and isomorphic to the additive group $E = F \times F$.

Lemma 3. The group $G$ is the semidirect product of $H$ and the invariant subgroup $N$,

$$ G = HN = H \rtimes N. $$

Proof. The subgroups $H, N \subseteq G$ have obviously trivial intersection $H \cap N = \{1\}$ and we noted above in equation (3) that $G = HN$. For $A = A_0$ in $H$ and $B = 1 + B_1$ in $N$ we have

$$ A \star B = A_0 + A_0B_1 = (1 + A_0B_1A_0^{-1}) \star A_0 = \tilde{B} \star A, $$

which shows that $N \subseteq G$ is an invariant subgroup in $G$. Hence $G = H \rtimes N$. \hfill \Box

3 A geometric interpretation of the group $G$

For this last section we assume that

$$ 1 + 1 \neq 0, $$

which is certainly true for the case that $F$ is the field of real numbers $\mathbb{R}$. We denote by $\mathrm{SO}(1, 1)$ the group of all $2 \times 2$ matrices of determinant 1 which leave the bilinear form

$$ b(u, v) = u_1v_1 - u_2v_2 $$

on the 2-dimensional vector space $E = F \times F$ invariant. This group is abelian and consists of all matrices of the form

$$ R = \begin{pmatrix} c & s \\ s & c \end{pmatrix} \quad \text{with} \quad c^2 - s^2 = 1. \quad (4) $$
For the case $F = \mathbb{R}$ the group $\text{SO}(1, 1)$ has a subgroup $\text{SO}^+(1, 1)$ of index 2 consisting of all matrices of the form

$$\varphi(t) = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}, \quad t \in \mathbb{R},$$

and the one parameter group $t \mapsto \varphi(t)$ is a Lie group isomorphism

$$\varphi : \mathbb{R} \xrightarrow{\cong} \text{SO}^+(1, 1).$$

In general, the map $\psi : F^* \to \text{SO}(1, 1)$ that maps the nonzero scalar $x$ to the matrix

$$\psi(x) = \begin{pmatrix} \frac{1}{2}(x + x^{-1}) & \frac{1}{2}(x - x^{-1}) \\ \frac{1}{2}(x - x^{-1}) & \frac{1}{2}(x + x^{-1}) \end{pmatrix}$$

is a group isomorphism $\psi : F^* \xrightarrow{\cong} \text{SO}(1, 1)$.

The hyperbolic motion group $\text{ISO}(1, 1)$ consists of all affine transformations $T_{[R,u]}$ of the 2-dimensional plane $E = F \times F$ of the form

$$T_{[R,u]}(x) = Rx + u,$$

where $R$ is a $2 \times 2$ matrix in $\text{SO}(1, 1)$ and $u$ is a vector in $E$. Such a transformation consists thus of a hyperbolic rotation $R$ followed by a translation by a vector $u$. The group law on $\text{ISO}(1, 1)$ is thus

$$T_{[R,u]} * T_{[S,v]} = T_{[RS,u+Rv]}.$$

We define three auxiliary maps $\alpha, \beta, \gamma$ as follows. We put

$$\alpha \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(x + y) & \frac{1}{2}(x - y) \\ \frac{1}{2}(x - y) & \frac{1}{2}(x + y) \end{pmatrix},$$

Then $\alpha : H \to \text{SO}(1, 1)$ is a surjective group homomorphism whose kernel consists of all diagonal matrices of the form $s1$, with $s \neq 0$. We also put

$$\beta \begin{pmatrix} 1 & p \\ q & 1 \end{pmatrix} = \begin{pmatrix} p + q \\ p - q \end{pmatrix},$$

and we note that $\beta$ is a group isomorphism $N \xrightarrow{\cong} E$. Finally, we put

$$\gamma \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = x$$

and we note that $\gamma : H \to F^*$ is a surjective group homomorphism whose kernel consists of all diagonal matrices of the form $(1, 0)$. We define a map $\Phi : G \to \text{ISO}(1, 1)$ by

$$\Phi(A) = \Phi(A_0 + A_1) = T_{[\alpha(A_0), \beta(1 + A_1 A_0^{-1})]} = T_{[\alpha(A_0), \beta(AA_0^{-1})]}.$$

**Proposition 4.** The map

$$\Phi : G \to \text{ISO}(1, 1)$$

is a surjective group homomorphism whose kernel consists of all matrices of the form $s1$ with $s \neq 0$. 
Proof. We have to verify that $\Phi(A \star B) = \Phi(A) \star \Phi(B)$ holds for all $A, B \in G$. If $A = A_0$ and $B = B_0$, then

$$\Phi(A) \star \Phi(B) = T_{[\alpha(A_0),0]} \star T_{[\alpha(B_0),0]} = T_{[\alpha(A_0)\alpha(B_0),0]} = T_{[\alpha(A_0B_0),0]} = \Phi(A \star B).$$

Similarly, if $A = 1 + A_1$ and $B = 1 + B_1$, then

$$\Phi(A) \star \Phi(B) = T_{[1,\beta(A)]} \star T_{[1,\beta(B)]} = T_{[1,\beta(A) + \beta(B)]} = T_{[1,\beta(AB)]} = \Phi(A \star B).$$

Hence the map $\Phi$ is a group homomorphism both on $H$ and on $N$.

Since $G = HN$ is a semidirect product, it remains to show that $\Phi(A \star B) = \Phi(A) \star \Phi(B)$ holds for all matrices $A, B$ of the form $A = A_0 \in H$ and $B = 1 + B_1 \in N$. We put

$$A = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & p \\ q & 1 \end{pmatrix}.$$

Then $\alpha(A) = (\xi, \eta)$, where $c = \frac{1}{2}(\xi x + \eta y)$ and $s = \frac{1}{2}(\xi x - \eta y)$. We also put $z = c + s = \xi_y$, and we compute

$$\Phi(A) \star \Phi(B) = T_{\left(\begin{array}{cc} c & s \\ s & c \end{array}\right)} \star T_{\left(\begin{array}{cc} 0 & p \\ q & 0 \end{array}\right)} = T_{\left(\begin{array}{cc} c & s \\ s & c \end{array}, \left(\begin{array}{cc} zp + z^{-1}q \\ zp - z^{-1}q \end{array}\right)\right)}.$$

On the other hand, $A \star B = \left(\begin{array}{cc} x & 0 \\ 0 & y \end{array}\right) + \left(\begin{array}{cc} 0 & xp \\ yq & 0 \end{array}\right)$. Thus

$$\Phi(A \star B) = T_{\left(\begin{array}{cc} c & s \\ s & c \end{array}, \left(\begin{array}{cc} zp + z^{-1}q \\ zp - z^{-1}q \end{array}\right)\right)}.$$

The kernel of $\Phi$ consists of the multiples $s1$ of the identity matrix. This finishes the proof of the proposition. \qed

By a well-known construction, see [2], one can describe the 2-dimensional affine transformations by $3 \times 3$ matrices. We introduce a third coordinate which is set to 1. Then $T_{[R,u]}$ corresponds to the $3 \times 3$ matrix

$$\tilde{T}_{[R,u]} = \begin{pmatrix} r_{11} & r_{12} & u_1 \\ r_{21} & r_{22} & u_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

From this we obtain a faithful 3-dimensional representation $\rho$ of $G$ via

$$(8) \quad \rho \left(\begin{array}{cc} x & p \\ q & y \end{array}\right) = x \left(\begin{array}{cc} c & s \\ s & c \end{array}, \begin{array}{cc} zp + z^{-1}q \\ zp - z^{-1}q \end{array}\right),$$

where again $c = \frac{1}{2}(\xi x + \eta y)$ and $s = \frac{1}{2}(\xi x - \eta y)$ and $z = c + s = \xi_y$.

4 Conclusion.

We have shown that the matrix group $G$, equipped with the $\star$ multiplication, is a semidirect product of two abelian groups.

By a 3-dimensional representation, the elements of the group $G$ can be interpreted as planar hyperbolic rotations, followed by translations. The $\star$ multiplication of matrices can be interpreted as the composition rule of these elements.
References

[1] CR Johnson. Hadamard products of matrices. Linear and Multilinear Algebra, 1(4), 295-307 (1974)

[2] N Ja Vilenkin and A U Klimyk, Representations of Groups of Motions and Special Functions, Springer (1991), Vol. 1: Representations of the Group of Motions of Euclidean and Pseudo-Euclidean Planes, and Cylindrical Functions, Chapter 4, pp 173-206.