Integral curvatures of Finsler manifolds and applications

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Abstract In this paper, we study the integral curvatures of Finsler manifolds. Some Bishop-Gromov relative volume comparisons and several Myers type theorems are obtained. We also establish a Gromov type precompactness theorem and a Yamaguchi type finiteness theorem. Furthermore, the isoperimetric and Sobolev constants of a closed Finsler manifold are estimated by integral curvature bounds.

Keywords integral curvature · Finsler manifold · comparison theorem · Myers theorem · Gromov-Hausdorff distance · isoperimetric constant · Sobolev constant

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1 Introduction

In Riemannian geometry, the structure of manifolds with a lower Ricci curvature bound has been studied for a long time and numerous significant results are obtained (see [13],[19],[29], etc. for a survey). There have been many efforts to generalize such results to situations with weaker curvature bounds. One direction has been to assume integral curvature bounds, which are introduced in [21]. More precisely, given a Riemannian manifold $\langle M, g \rangle$, let $\text{Ric}_-$ denote
smallest eigenvalue for $\text{Ric} : T_xM \to T_xM$ and set

$$K(k,q) := \frac{1}{\text{vol}(M)} \int_M ((n-1)k - \text{Ric}_-)^q_+ d\text{vol}, \text{ if } M \text{ is compact},$$

$$K(k,q,R) := \sup_{x \in M} \frac{1}{\text{vol}(B_x(R))} \int_{B_x(R)} ((n-1)k - \text{Ric}_-)^q_+ d\text{vol}, \text{ if } M \text{ is noncompact},$$

where $R > 0$, $k \in \mathbb{R}$, $q \in \mathbb{N}$ and $(\cdot)_+ := \max\{\cdot, 0\}$. It is easy to see that $K(k,q) = 0$ iff $\text{Ric} \geq (n-1)k$. Moreover, if $K(k,q,R)$ is small for $q > n/2$, then interesting phenomena emerge. Refer to [20,21,22], etc. for topological and analytical results under such conditions.

Roughly speaking, a Finsler metric is a Riemannian metric without quadratic restriction. Almost every geometric quantity of a non-Riemannian Finsler manifold $(M,F)$ is defined on the projective tangent bundle $PM$ instead of the underlying manifold $M$. In particular, the Ricci curvature here is not a $(1,1)$-tensor but a scalar function on $PM$, which implies that the assumptions on the Ricci curvature in the Finsler setting are not as strong as the ones in the Riemannian case.

On the other hand, many important results concerned with a lower Ricci curvature bound in Riemannian geometry such as the (Bonnet-)Myers theorem, the Bishop-Gromov relative volume comparison and Gromov’s precompactness theorem, remain valid in the Finsler case ([4,25,30,40]), although one usually has to assume additional bounds on some non-Riemannian quantities. Also refer to [4,24] for more details.

Thus, it is natural as well as meaningful to study whether these results hold under weaker curvature bounds. Many efforts have been made in this direction recently. For example, Ohta introduced the weighted curvatures $\text{Ric}_N$ and $\text{Ric}_\infty$ in [15], which are combinations of the Ricci curvature, the S-curvature and the dimension, and he proved the Myers theorem and the Bishop-Gromov volume comparison by the lower $\text{Ric}_N$ bounds only. Also refer to [15,16,17] for more results.

Note that both the Ricci curvature and the weighted Ricci curvatures are pointwise defined. In order to find weaker assumptions on the Ricci curvature, it is worthwhile to study its $L^q$ norm. Since the Ricci curvature is a scalar function, one needs to consider $\text{Ric}(x) := \min_{y \in S_xM} \text{Ric}(y)$ instead of the lowest eigenvalue. Inspired by [21,12], Wu ([32]) introduced the following quantity to study the structure of a compact Finsler manifold

$$\epsilon(n,k,q) := \frac{1}{\text{vol}_{\min}(M)} \int_M ((n-1)k - \text{Ric}(x))^q_+ d\text{vol}_{\max}(x).$$

Here, $\text{vol}_{\min}(M)$ and $\text{vol}_{\max}(M)$ are two measures defined by Wu, which are called extreme measures (cf. [31,34]).

It is remarkable that $d\text{vol}_{\min}(x) \leq d\text{vol}_{\max}(x)$ with equality iff $F|_x$ is Riemannian. Thus, $\epsilon(n,k,q)$ is not normalized unless the metric is Riemannian. On the other hand, compared with extreme measures, the Busemann-Hausdorff measure and the Holmes-Thompson measure are used more frequently and relatively easy to compute (cf. [1,2,8,24]). Therefore, in this paper,
we investigate the properties of Finsler manifolds by the following geometric quantities

\[ K_{d^m}(q,k) := \frac{1}{m(M)} \int_M ((n-1)k - R)^q d^m, \quad \text{if } M \text{ is compact}, \]

\[ K_{d^m}(q,k,R) := \sup_{p \in M} \left( \frac{1}{m(B^+_p(R))} \int_{B^+_p(R)} ((n-1)k - R)^q d^m \right), \quad \text{if } M \text{ is noncompact}, \]

where we use \( d^m \) to denote both the Busemann-Hausdorff measure and the Holmes-Thompson measure. Note that \( d^m \) is exactly the canonical measure when \( F \) is Riemannian and hence, these quantities are the natural extensions of integral curvatures in Riemannian geometry.

We first consider the Myers theorem, because it is not only a beautiful result in differential geometry, but also has a close connection with general relativity (cf. [5,11]), which says that a forward complete Finsler \( n \)-manifold \( M \) is compact with \( \text{diam}(M) \leq \pi\sqrt{K} \) if \( \text{Ric} \geq (n-1)K > 0 \) (cf. [4,24]). In [15], Ohta proved \( \text{diam}(M) \leq \pi\sqrt{(N-1)/K} \) if \( \text{Ric}_N \geq K > 0 \). Yin [37] obtained similar compactness results under \( \text{Ric}_\infty \geq (n-1)K > 0 \) and additional assumptions on non-Romanian quantities.

All the results above rely on uniformly positive (lower) curvature bounds. Thus, it is natural to ask whether a Finsler manifold is still compact if the Ricci curvature is negative in some small places. In spirited by [20,22,23], we give an affirmative answer. In the following, \( A_F \) (resp., \( \lambda_F \)) denotes the uniformity constant [10] (resp., the reversibility [18]) of a Finsler manifold \( (M,F) \). Then we have

**Theorem 1** Given any \( n > 1, q \geq 1, k \in \mathbb{R}, K > 0, R > 0 \) and \( \delta \geq 1, \) for each \( \rho > 0 \), there exists an \( \varepsilon = \varepsilon(n,q,k,K,\delta,R,\rho) > 0 \) such that every forward complete Finsler \( n \)-manifold \( (M,F) \) with

\[ \text{Ric} \geq -(n-1)k^2, \quad A_F \leq \delta^2, \quad K_{d^m}(q,K,R) < \varepsilon \]

satisfies

\[ \text{diam}(M) \leq \frac{\pi}{\sqrt{K}} + \rho. \]

In particular, the universal covering \( \tilde{M} \) is compact and hence, \( \pi_1(M) \) is finite. Moreover, if \( F \) is Berwald and \( q > n/2 \), then one still has

\[ \text{diam}(M) \leq \delta^2 \left( \frac{\pi}{\sqrt{K}} + \rho \right) \]

without the assumption \( \text{Ric} \geq -(n-1)k^2 \).

We remark that the non-positive lower bound \( \text{Ric} \geq -(n-1)k^2 \) in Theorem [1] is to establish a Finslerian version of the Cheeger-Colding segment inequality [6 Theorem 2.1] and it can be removed in the Berwaldian case. Thus, it is natural to ask whether Theorem [1] remains valid without this assumption in the general case. In Riemannian geometry, Petersen and Sprouse [20] solved
this problem by using analytic techniques. Unfortunately, their method does not work in the Finsler setting as the Finslerian Laplacian is nonlinear, and we will discuss it somewhere else. It is also remarkable that Wu in \cite{Wu2004} obtained a compactness theorem by assuming positivity for the integral of the Ricci curvature along all geodesics from a point. Clearly, this condition is weaker than the uniform curvature bound but stronger than ours.

In this paper, we also study the properties of compact Finsler manifolds by $\K_{\text{dm}}(q,k)$. More precisely, we consider the precompactness and the finiteness of topological types of such manifolds. Since the distance of a Finsler manifold is usually asymmetric, the Gromov-Hausdorff topology on the family of Finsler manifolds is different from the original one in \cite{Gromov1984}. See Appendix \cite{Chen2019} and \cite{Hirsch1976} for details. Then we have the following result.

\textbf{Theorem 2} Given $n > 1$, $q > n/2$, $k \leq 0$, $\delta \in [1, \infty)$, $D < \infty$ and $R < \infty$, one can find $\epsilon = \epsilon(n,q,k,\delta,D) > 0$ such that the class of compact Finsler $n$-manifolds $(M,F)$ with

$$A_F \leq \delta^2, \quad \text{diam}(M) \leq D, \quad \K_{\text{dm}}(k,q) \leq \epsilon$$

is precompact in the $\delta$-Gromov-Hausdorff topology.

Moreover, there exists an $\epsilon = \epsilon(n,q,k,\delta,D,R) > 0$ such that the class of closed Finsler $n$-manifolds $(M,F)$ with

$$A_F \leq \delta^2, \quad \text{diam}(M) \leq D, \quad \K_{\text{dm}}(k,q) \leq \epsilon, \quad c_M \geq R$$

contains at most finitely many homotopy types and only finitely many diffeomorphism types if in addition $n \neq 4$, where $c_M$ denotes the contractibility radius of $(M,F)$.

In particular, if $F$ is Berwaldian, then the consequences above remain valid under the weaker condition $\lambda_F \leq \delta$ instead of $A_F \leq \delta^2$.

It is not hard to see that the first part of Theorem 2 implies Gromov’s precompactness theorem \cite{Gromov1984} while the second part is a generalization of Yamaguchi’s finiteness theorem \cite{Yamaguchi1997}. Also refer to \cite{Cheeger2000,Morgan2004,Nicolini2005} for other versions of Gromov’s precompactness theorem.

Besides, we estimate Cheeger’s constant $h(M,\text{dm})$, the (Neumann $n$-) isoperimetric constant $\text{IN}(M,\text{dm})$ and (Neumann $n$-)Sobolev constant $\text{SN}(M,\text{dm})$ of a closed Finsler manifold. The definitions of these constants can be found in Section 4.2.

\textbf{Theorem 3} Given $n > 1$, $q > n/2$, $k \leq 0$, $\delta \in [1, \infty)$, $D < \infty$ and $V > 0$, one can find $\epsilon = \epsilon(n,q,k,\delta,D,V) > 0$ such that a closed Finsler $n$-manifolds $(M,F)$ with

$$A_F \leq \delta^2, \quad \text{diam}(M) \leq D, \quad m(M) \geq V, \quad \K_{\text{dm}}(k,q) \leq \epsilon,$$
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satisfies

\[ h(M, d) \geq \frac{(n-1)c_{n-1}V}{2^{2q+1}c_{n-2}D\delta^{n+1}v(n, k, D)}, \]

\[ SN(M, d) \geq IN(M, d) \geq \frac{c_{n-1}}{(c_n/2)^{1-\frac{1}{2}}\delta^{4n+5}} \left( \frac{V}{2^{2q+1}\delta^{4n+5}v(n, k, D)} \right)^{1+\frac{1}{n}}, \]

where \( c_n := \text{vol}(\mathbb{S}^n) \) and \( v(n, k, r) \) is the volume of \( r \)-ball in the Riemannian space form \( \mathbb{M}^n_k \).

This result together with the Cheeger inequality \( \lambda_1 \geq h^2(M, d)/4\lambda_2 \) (cf. [7]) yields an estimate of the first eigenvalue. Furthermore, in the Riemannian case (i.e., \( A_F = 1 \) and \( d = d\text{vol} \)), one can obtain Croke’s estimate [9] by supposing \( K_{dm}(k, q) = 0 \) in Theorem 3.

The arrangement of contents of this paper is as follows. In Section 2, we brief some necessary definitions and properties concerned with Finsler geometry. Section 3 consists of two parts. A Bishop-Gromov type comparison is establish by \( K_{dm}(q, k, R) \) in Subsection 3.1 and Theorem 1 is proved in Subsection 3.2. Section 4 is divided into 2 parts. We obtain a relative comparison theorem concerned with \( K_{dm}(q, k) \) and prove Theorem 2 in Subsection 4.1. Subsection 4.2 is devoted to the proof of Theorem 3. In Appendix A, we give an estimate of distortion by uniformity constants. In Appendix B, we introduce the generalized Gromov-Hausdorff topology briefly.

2 Preliminaries

In this section, we recall some definitions and properties about Finsler manifolds. See [4,24] for more details.

A Finsler \( n \)-manifold \((M, F)\) is an \( n \)-dimensional differential manifold \( M \) equipped with a Finsler metric \( F \) which is a nonnegative function on \( TM \) satisfying the following two conditions:

1. \( F \) is positively homogeneous, i.e., \( F(\lambda y) = \lambda F(y) \), for any \( \lambda > 0 \) and \( y \in TM \);
2. \( F \) is smooth on \( TM\setminus\{0\} \) and the Hessian \( \frac{1}{2}[F^2]_{y'y'}(x, y) \) is positive definite, where \( F(x, y) := F(y \frac{\partial}{\partial x}) \).

Let \( \pi : PM \to M \) and \( \pi^*TM \) be the projective sphere bundle and the pullback bundle, respectively. Then a Finsler metric \( F \) induces a natural Riemannian metric \( g = g_{ij}(x, [y]) \, dx^i \otimes dx^j \), which is the so-called fundamental tensor, on \( \pi^*TM \), where

\[ g_{ij}(x, [y]) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}, \quad dx^i = \pi^*dx^i. \]

Note that \( F \) may be irreversible, that is, \( F(x, y) \neq F(x, -y) \). Hence, Rademacher [18] and Egloff [10] introduced the reversibility \( \lambda_F \) and the uni-
formity constant $A_F$ to describe its asymmetry. More precisely, set
\[
\lambda_F := \sup_{y \in S_M} F(-y), \quad A_F := \sup_{X,Y,Z \in SM} g_Z(Y,Y),
\]
where $S_xM := \{ y \in T_xM : F(x,y) = 1 \}$ and $SM := \cup_{x \in \Omega} S_xM$. Clearly, $A_F \geq \lambda_F^2 \geq 1$. In particular, $\lambda_F = 1$ iff $F$ is reversible, while $A_F = 1$ iff $F$ is Riemannian.

On the other hand, $F$ also induces the average Riemannian metric $\hat{g}$ on $M$, which is defined by
\[
\hat{g}(X,Y) := \frac{1}{\nu(S_xM)} \int_{S_xM} g_y(X,Y) d\nu_x(y), \quad \forall X,Y \in T_xM,
\]
where $\nu(S_xM) = \int_{S_xM} d\nu_x(y)$, and $d\nu_x$ is the Riemannian volume form of $S_xM$ induced by $F$. It is noticeable that
\[
A_F^{-1} \cdot F^2(X) \leq \hat{g}(X,X) \leq A_F \cdot F^2(X),
\]
with equality iff $F$ is Riemannian.

The geodesic coefficient is defined by
\[
G^i(y) := \frac{1}{4} g^{ii}(y) \left\{ 2 \frac{\partial g_{ij}}{\partial x^k}(y) - \frac{\partial g_{ik}}{\partial x^j}(y) \right\} y^j y^k.
\]

The Riemannian curvature $R_g$ of $F$ is a family of linear transformations on tangent spaces. More precisely, set $R_g := R^i_k(y) \frac{\partial}{\partial x^k} \otimes dx^i$, where
\[
R^i_k(y) := 2 \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.
\]

The flag curvature is defined by
\[
K(y,v) := \frac{g_y(R_y(v),v)}{g_y(y,y)g_y(v,v) - g_y^2(y,v)},
\]
and the Ricci curvature of $y$ is defined by
\[
\text{Ric}(y) := \sum_i K(y,e_i) = \frac{R^i_i(y)}{F^2(y)},
\]
where $e_1, \ldots, e_n$ is a $g_y$-orthonormal base on $(x,y) \in TM \setminus 0$. We also use the notation
\[
\text{Ric}(x) := \min_{y \in S_xM} \text{Ric}(y).
\]

Let $\gamma : [0,1] \to M$ be a Lipschitz continuity path from $p,q$. The length of $\gamma$ is defined by
\[
L_F(\gamma) := \int_0^1 F(\gamma(t)) dt.
\]
Define the distance function $d : M \times M :\to [0, +\infty)$ by $d(p, q) := \inf L_F(\gamma)$, where the infimum is taken over all Lipshitz continuous paths $\gamma : [a, b] \to M$ with $\gamma(a) = p$ and $\gamma(b) = q$. Generally speaking, $d_F$ satisfies all axioms for a metric except symmetry, i.e., $d(p, q) \neq d(q, p)$, unless $F$ is reversible.

A smooth curve $\gamma(t)$ is called a (constant speed) geodesic if it satisfies

$$\frac{d^2\gamma^i}{dt^2} + 2C^{ij}_{kl} \left( \frac{d\gamma^k}{dt} \right) \frac{d\gamma^l}{dt} = 0.$$  

Every short geodesic minimizes the arc length functional $\frac{1}{2}$. That is, if there exists a Lipschitz curve $\gamma$ from $p$ to $q$ satisfying $L_F(\gamma) = d(p, q)$, then $\gamma$ is a geodesic. However, for a nonreversible Finsler manifold, the reverse of a geodesic is usually not a geodesic.

Given $y \in S_p M$, let $\gamma_y(t), t \geq 0$ denote the geodesic with $\dot{\gamma}_y(0) = y$. The cut value $i_y$ of $y$ is defined by

$$i_y := \sup \{ r : \text{the segment } \gamma_y(t), 0 \leq t \leq r \text{ is globally minimizing} \}.$$  

The injectivity radius at $p$ is defined as $i_p := \inf_{y \in S_p M} i_y$, whereas the cut locus of $p$ is

$$\text{Cut}_p := \{ \exp_p(i_y \cdot y) : y \in S_p M \text{ with } i_y < \infty \}.$$  

It should be remarked that $\text{Cut}_p$ is closed and null Lebesgue measure.

Given $R > 0$, the forward and backward metric balls $B^+_p(R)$ and $B^-_p(R)$ are defined by

$$B^+_p(R) := \{ x \in M : d(p, x) < R \}, \quad B^-_p(R) := \{ x \in M : d(x, p) < R \}.$$  

If $F$ is reversible, forward metric balls coincide with backward ones.

There is only one reasonable notion of the measure for Riemannian manifolds. However, the situation is different in Finsler geometry, because the determinant of the fundamental tensor depends on the direction of $y$. Thus, measures on a Finsler manifold can be defined in various ways and essentially different results may be obtained.

Let $dm$ be a measure on $M$. In a local coordinate system $(x^i)$, express $dm = \sigma(x)dx^1 \wedge \cdots \wedge dx^n$. Define the distortion of $(M, F; dm)$ as

$$\tau(y) := \log \sqrt{\frac{\det g_{ij}(x, y)}{\sigma(x)}}, \text{ for } y \in T_x M \setminus \{0\}.$$  

And the $S$-curvature $S$ is defined by

$$S(y) := \frac{d}{dt} [\tau(\gamma_y(t))]|_{t=0},$$  

where $\gamma_y(t)$ is the geodesic with $\dot{\gamma}_y(0) = y$. It is easy to see both the distortion and the $S$-curvature vanish in the Riemannian case.
There are two measures used frequently in Finsler geometry, which are the so-called Busemann-Hausdorff measure \( d_m^{BH} \) and Holmes-Thompson measure \( d_m^{HT} \). They are defined by

\[
d_m^{BH} := \frac{\text{vol}(\mathbb{B}^n)}{\text{vol}(B_x M)} \int_{B_x M} dx^1 \wedge \cdots \wedge dx^n,
\]

\[
d_m^{HT} := \left( \frac{1}{\text{vol}(\mathbb{B}^n)} \int_{B_x M} \det g_{ij}(x, y) dy^1 \wedge \cdots \wedge dy^n \right) dx^1 \wedge \cdots \wedge dx^n,
\]

where \( B_x M := \{ y \in T_x M : F(x, y) < 1 \} \). Each of them becomes the canonical Riemannian measure if \( F \) is Riemannian. However, their properties are different. Even in the reversible case, \( d_m^{BH} \leq d_m^{HT} \) with equality iff \( F \) is Riemannian (cf. [1]). According to [24,41], the S-curvatures of \( d_m^{BH} \) and \( d_m^{HT} \) always vanish when \( F \) is Berwaldian.

### 3 Results concerned with \( \overline{K}_{dm}(q, k, R) \)

#### 3.1 A Bishop-Gromov type comparison theorem

In this paper, we always use \( dm \) to denote the Busemann-Hausdorff measure and the Holmes-Thompson measure. Set

\[
\overline{K}_{dm}(q, k, R) := \sup_{p \in M} \left( \frac{1}{m(B^+_p(R))} \int_{B^+_p(R)} ((n-1)k - \text{Ric})^q_{+} dm \right).
\]

In this section, we will establish a relative volume comparison by this quantity. In order to do this, we need to recall the polar coordinate system of a Finsler manifold first. Also refer to [31,40] for more details.

Set \( D_p := \{ t y \in T_p M : y \in S^p M, 0 \leq t < i_y \} \), and \( D^p := \exp_p(D_p) \). Thus, \( M = D_p \sqcup \text{Cut}_p \) (cf. [4, Prop. 8.5.2]). Now, let \( r(x) := d(p, x) \). Then the polar coordinates at \( p \) actually describe a diffeomorphism of \( D_p - \{ 0 \} \) onto \( D^p - \{ p \} \), given by

\[
(r, y) \mapsto \exp_p ry.
\]

Now write

\[
dm := \tilde{\sigma}_p(r, y) dr \wedge d\nu_p(y),
\]

where \( d\nu_p(y) \) is the Riemannian volume measure induced by \( F \) on \( S^p M \). A direction calculation yields

\[
\nabla r = \frac{\partial}{\partial r}, \quad \Delta r = \frac{\partial}{\partial r} \log \tilde{\sigma}_p(r, y), \quad 0 < r < i_y,
\]

and

\[
\lim_{r \to 0^+} e^{-r(\gamma_p(r))} \tilde{\sigma}_p^{-1}(r) = 1,
\]

where \( \gamma_p(r) \) is the focal radius of \( p \).
where \( \gamma_y(t) \) is a geodesic with \( \dot{\gamma}_y(0) = y \), and \( s_k(t) \) is the unique solution to \( f'' + kf = 0 \) with \( f(0) = 0 \) and \( f'(0) = 1 \). It is easy to check that (also see [32, (4.5)])

\[
\frac{\partial}{\partial r} H \leq -\text{Ric}(\nabla r) - \frac{H^2}{n-1}, \quad (r, y) \in D_p,
\]

where

\[
H(r, y) := \frac{\partial}{\partial r} \log \left[ \tilde{\sigma}_p(r, y) e^{\tau(\dot{\gamma}_y(r))} \right] = \frac{\partial}{\partial r} \log \sqrt{\det g},
\]

and \( \det g \) is the determinant of \( g \) in \( (r, y) \). Given \( k \leq 0 \), set

\[
H_k(r) := \frac{\partial}{\partial r} \log s_k^{n-1}(r).
\]

Thus, we have

\[
\frac{\partial}{\partial r} H_k = -(n-1)k - \frac{H^2_k}{n-1}.
\]

**Lemma 1**

\[
\lim_{r \to 0^+} [H(r, y) - H_k(r)] = 0. \tag{3.3}
\]

**Proof** Choose a small \( r_0 > 0 \) such that \( \overline{B^+_p(r_0)} \subset D_p \). Since \( \overline{B^+_p(r_0)} \) is compact, we can suppose that \( k_1 \leq K \frac{B^+_p(r_0)}{|B^+_p(r_0)|} \leq k_2 \). Then using the Laplacian comparison theorem and (3.2), one can obtain (also see [40, (3.8)])

\[
H(r, y) - H_{k_1}(r) \leq 0, \quad H(r, y) - H_{k_2}(r) \geq 0, \quad 0 < r < r_0.
\]

Now the lemma follows from \( H - H_k = (H - H_{k_1}) + (H_{k_1} - H_k) \), \( i = 1, 2 \). \( \Box \)

Let \( v(n, k, r) \) denote the volume of \( r \)-ball in space form \( \mathbb{M}^n_k \), i.e.,

\[
v(n, k, r) := \text{vol}(\mathbb{S}^{n-1}) \int_0^r s_k^{n-1}(t) dt.
\]

Then we have following theorem, which is inspired by [21].

**Theorem 4** Given \( \delta \geq 1 \), \( q > n/2 \) and \( k \leq 0 \), for any \( \alpha < \delta^{-4n} \), there exists an \( \varepsilon = \varepsilon(n, q, k, \delta, R, \alpha) > 0 \) such that for any forward complete Finsler \( n \)-manifold \((M, F)\) with

\[
\Lambda_F \leq \delta^2, \quad \overline{K}_{da}(q, k, R) < \varepsilon,
\]

satisfies

\[
\alpha \cdot \frac{v(n, k, r_1)}{v(n, k, r_2)} \leq \frac{m(B^+_p(r_1))}{m(B^+_p(r_2))},
\]

for all \( x \in M \) and \( 0 < r_1 \leq r_2 \leq R \). In particular, if \( F \) is Berwaldian, then the result holds without the assumption \( \Lambda_F \leq \delta^2 \), in which case \( \alpha < 1 \).
Proof Step 1. Define two functions on \([0, +\infty) \times S_p M\) by

\[
\Psi(r, y) := \begin{cases} 
(H(r, y) - H_k(r))_+, \quad & \text{if } 0 \leq r < i_y, \ y \in S_p M \\
0, \quad & \text{if } r \geq i_y.
\end{cases}
\]

\[
\tau(r, y) := \begin{cases} 
\tau((\exp_p)_{*ry} y), \quad & \text{if } 0 \leq r < i_y, \ y \in S_p M, \\
-\infty, \quad & \text{if } r \geq i_y.
\end{cases}
\]

Given \(y \in S_p M\), one gets

\[
\frac{d}{dr} \left[ e^{\tau(r,y)} \hat{\sigma}_p(r,y) \right] \leq \Psi(r,y) \frac{e^{\tau(r,y)} \hat{\sigma}_p(r,y)}{s_k^{n-1}(r)}.
\]

Hence, for all \(0 \leq t < r < +\infty\) (even if \(0 \leq t < i_y < r < \infty\)), (3.4) yields

\[
g_k^{n-1}(t) \cdot e^{\tau(r,y)} \hat{\sigma}_p(r,y) - g_k^{n-1}(r) \cdot e^{\tau(t,y)} \hat{\sigma}_p(t,y) \leq g_k^{n-1}(r) \int_t^r \Psi(s,y)e^{\tau(s,y)} \hat{\sigma}_p(s,y)ds \leq g_k^{n-1}(r) \int_0^r \Psi(s,y)e^{\tau(s,y)} \hat{\sigma}_p(s,y)ds,
\]

which implies

\[
\frac{d}{dr} \left[ \int_{B_p^+(r)} e^{r} dm \right]_{v(n,k,r)} \leq \frac{\text{vol}(S^{n-1}) \int_{S_p M} \left[ g_k^{n-1}(t) e^{\tau(r,y)} \hat{\sigma}_p(r,y) - g_k^{n-1}(r) e^{\tau(t,y)} \hat{\sigma}_p(t,y) \right] dt d\nu_p(y)}{v(n,k,r)^2} \leq \frac{\text{vol}(S^{n-1}) \cdot r \cdot g_k^{n-1}(r) \int_{S_p M} \Psi(s,y)e^{\tau(s,y)} \hat{\sigma}_p(s,y)ds d\nu_p(y)}{v(n,k,r)^2} \leq \frac{\text{vol}(S^{n-1}) \cdot r \cdot g_k^{n-1}(r) \cdot \left( \int_{B_p^+(r)} \Psi^{1/2 q} e^{r} dm \right)^{1/2 q} \left( \int_{B_p^+(r)} e^{r} dm \right)^{1-1/2 q}}{v(n,k,r)^2}.
\]

Set

\[
C_4(n,k,r) := \max_{t \in [0,r]} \frac{\text{vol}(S^{n-1}) \cdot t \cdot g_k^{n-1}(t)}{v(n,k,t)} < \infty.
\]
It is not hard to see that $C_1(n, k, r)$ is nondecreasing in $r$ and $C_1(n, k, r) = n$ as $r = 0$. Since $k < 0$, $C_1 \to \infty$ as $r \to \infty$. In particular, $C_1 = n$ if $k = 0$. Thus, one has

$$\frac{d}{dr} \left( \frac{\int_{B^+_r(n,k,r)} e^\tau dm}{v(n, k, r)} \right) \leq C_1(n, k, r) \cdot \left( \int_{\mathcal{B}^+_r(n,k,r)} e^\tau dm \right)^{\frac{1}{q}} \cdot \left( \int_{\mathcal{B}^+_r(n,k,r)} \psi^{2q} e^\tau dm \right)^{\frac{1}{2q}} \cdot v(n, k, r)^{-\frac{1}{2q}}.$$

**Step 2.** Now we claim that there exists $C_2(n, q) > 0$ such that if $q > n/2$, then

$$\int_0^r \psi^{2q} e^\tau(t, y) \hat{\sigma}_p(t, y) dt \leq C_2(n, q) \int_0^r \rho^q e^\tau(t, y) \hat{\sigma}_p(t, y) dt, \quad \forall r > 0,$$

where

$$\rho(r, y) := \begin{cases} ((n-1)k - \text{Ric} (\nabla r))_+, & \text{if } 0 < r < i_y, \ y \in S_p M, \\ 0, & \text{if } r \geq i_y \text{ or } r = 0. \end{cases}$$

In fact, the definition yields that for almost every $r > 0$, we have

$$\frac{\partial}{\partial r} \psi + \frac{\psi^2}{n-1} + 2 \frac{\psi \cdot H_k}{n-1} \leq \rho,$$

which implies

$$\int_0^r \frac{\partial}{\partial t} \psi \cdot \psi^{2q-2} e^\tau \hat{\sigma}_p dt \leq \int_0^r \psi^{2q} e^\tau \hat{\sigma}_p dt + \frac{1}{n-1} \int_0^r \psi^{2q} e^\tau \hat{\sigma}_p dt + \frac{2}{n-1} \int_0^r H_k \psi^{2q-1} e^\tau \hat{\sigma}_p dt \leq \int_0^r \rho \cdot \psi^{2q-2} e^\tau \hat{\sigma}_p dt. \quad (3.5)$$

Note that (3.3) implies $\Psi(0) = 0$ and $\Psi \geq 0$. Thus, the first term of (3.5) satisfies

$$\int_0^r \frac{\partial}{\partial t} \psi \cdot \psi^{2q-2} e^\tau \hat{\sigma}_p dt \geq -\frac{1}{2q-1} \int_0^r \psi^{2q-1} (H_k + \psi) e^\tau \hat{\sigma}_p dt,$$

which together with (3.5) yields

$$\left( \frac{1}{n-1} - \frac{1}{2q-1} \right) \int_0^r \psi^{2q} e^\tau \hat{\sigma}_p dt + \left( \frac{2}{n-1} - \frac{1}{2q-1} \right) \int_0^r \psi^{2q-1} H_k e^\tau \hat{\sigma}_p dt \leq \int_0^r \rho \psi^{2q-2} e^\tau \hat{\sigma}_p dt.$$
Since \( q > n/2 \), \( \frac{1}{n-1} - \frac{1}{2q-1} > 0 \). Hence,
\[
\left( \frac{1}{n-1} - \frac{1}{2q-1} \right) \int_0^r \psi^{2q} e^{\tau} \hat{\sigma}_p dt \leq \int_0^r \rho \psi^{2q-2} e^{\tau} \hat{\sigma}_p dt
\]
\[
\leq \left( \int_0^r \rho^q e^{\tau} \hat{\sigma}_p dt \right)^{1/q} \cdot \left( \int_0^r \psi^{2q} e^{\tau} \hat{\sigma}_p dt \right)^{1-1/q}.
\]
That is,
\[
\int_0^r \psi^{2q} e^{\tau} \hat{\sigma}_p dt \leq \left( \int_0^r \rho^q e^{\tau} \hat{\sigma}_p dt \right)^{1/q} \cdot \left( \int_0^r \psi^{2q} e^{\tau} \hat{\sigma}_p dt \right)^{1-1/q}.
\]
Hence, the claim is true, which together with Step 1 yields
\[
\int_0^r \psi^{2q} e^{\tau} \hat{\sigma}_p dt \leq \frac{1}{n-1} - \frac{1}{2q-1} \cdot \rho^q e^{\tau} \hat{\sigma}_p dt.
\]

Hence, the claim is true, which together with Step 1 yields
\[
\frac{d}{dr} \left( \int_{B_p^+(r)} e^{\tau} dm \right) \leq C_3(n, q, k, r) \cdot \left( \int_{B_p^+(r)} e^{\tau} dm \right)^{1-\frac{1}{2q}} \cdot (k_p(q, k, r))^{\frac{1}{2q}} \cdot v(n, k, r)^{-\frac{1}{2q}},
\]
where \( C_3(n, q, k, r) \) is nondecreasing in \( r \) and
\[
k_p(q, k, r) := \int_{B_p^+(r)} \rho^q e^{\tau} \hat{\sigma}_p = \int_{S_p M} d\nu_p(y) \int_0^r \rho^q(t, y) e^{\tau(t, y)} \hat{\sigma}_p(t, y) dt.
\]

**Step 3.** Now set
\[
h(r) := \int_{B_p^+(r)} e^{\tau} dm \cdot f(r) := C_3(n, q, k, r) \cdot (k_p(q, k, r))^{\frac{1}{2q}} \cdot v(n, k, r)^{-\frac{1}{2q}}.
\]
Using (3.2), one gets
\[
h(0) = \frac{\int_{S_p M} d\nu_p}{\text{vol}(S^{n-1})} = \frac{\nu(S_p M)}{\text{vol}(S^{n-1})}.
\]
Then we have following equation:
\[
h' \leq h^{1-\frac{1}{2q}} \cdot f(r), \quad h(0) = \frac{\nu(S_p M)}{\text{vol}(S^{n-1})}, \quad h > 0.
\]
Thus, for any \( 0 < r_1 \leq r_2 \leq R \),
\[
2q \cdot h^{\frac{1}{2q}}(r_2) - 2q \cdot h^{\frac{1}{2q}}(r_1) \leq \int_{r_1}^{r_2} f(s) ds.
\]
Note that \( C_3(n, q, k, r) \) and \( k_p(q, k, r) \) are nondecreasing in \( r \). Hence,
\[
\int_{r_1}^{r_2} f(s) ds \leq C_3(n, q, k, R) \cdot \int_0^R v(n, k, s)^{-\frac{1}{2q}} ds \cdot (k_p(q, k, R))^{\frac{1}{2q}}.
\]
Set
\[
C_4(n, q, k, R) := \frac{1}{2q} C_3(n, q, k, R) \cdot \int_0^R v(n, k, s)^{-\frac{1}{2q}} ds.
\]
$C_4(n, q, k, R)$ is well-defined if $q > n/2$. Then we obtain

\[
h \frac{1}{v(n, k, r_2)} - h \frac{1}{v(n, k, r_1)} \leq C_4(n, q, k, R) \cdot \left( k_p(q, k, R) \right)^{\frac{1}{p}}
\]

which together with Proposition A1 yields

\[
h \frac{1}{v(n, k, r_2)} - h \frac{1}{v(n, k, r_1)} \leq C_4(n, q, k, R) \cdot \left( \delta^{2n} \cdot K_{p, \alpha}(k, q, R) \right)^{\frac{1}{p}},
\]

(3.7)

where

\[
K_{p, \alpha}(k, q, R) := \int_{B^+_{\delta}(R)} ((n-1)k - \text{Ric}(x))^p \delta^\alpha dm(x)
\]

**Step 4.** It is not hard to see that (3.7) implies

\[
\left( \frac{v(n, k, r_1)}{v(n, k, r_2)} \right)^{\frac{1}{\gamma}} = \left( \frac{\int_{B^+_{\delta}(r_1)} e^\tau dm}{\int_{B^+_{\delta}(r_2)} e^\tau dm} \right)^{\frac{1}{\gamma}} \leq C_4(n, q, k, R) \cdot \delta^{n/q} \cdot K_{p, \alpha}(k, q, R)^{\frac{1}{\gamma}} \cdot \left( \frac{v(n, k, r_1)}{v(n, k, r_2)} \right)^{\frac{1}{\gamma}},
\]

(3.8)

which together with Proposition A1 implies

\[
(1-c) \left( \frac{v(n, k, r_1)}{v(n, k, r_2)} \right)^{\frac{1}{\gamma}} \leq \left( \frac{\int_{B^+_{\delta}(r_1)} e^\tau dm}{\int_{B^+_{\delta}(r_2)} e^\tau dm} \right)^{\frac{1}{\gamma}} \Rightarrow \delta^{-4n} (1-c)^{2q} \frac{v(n, k, r_1)}{v(n, k, r_2)} \leq \frac{m(B^+_{\delta}(r_1))}{m(B^+_{\delta}(r_2))},
\]

where

\[
c = C_4(n, q, k, R) \cdot \delta^{n/q} \cdot K_{p, \alpha}(k, q, R)^{\frac{1}{\gamma}} \cdot \left( \frac{v(n, k, r_2)}{\int_{B^+_{\delta}(r_2)} e^\tau dm} \right)^{\frac{1}{\gamma}}.
\]

In order to estimate $c$, we use (3.7) again ($r_2 \to R$, $r_1 \to r_2$) and obtain

\[
\left( \frac{v(n, k, r_2)}{\int_{B^+_{\delta}(r_2)} e^\tau dm} \right)^{\frac{1}{\gamma}} \leq \left( \frac{\int_{B^+_{\delta}(R)} e^\tau dm}{v(n, k, R)} \right)^{\frac{1}{\gamma}} - C_4(n, q, k, R) \cdot \delta^{n/q} \cdot K_{p, \alpha}(k, q, R)^{\frac{1}{\gamma}}
\]

\[
\leq \left( \frac{v(n, k, R)}{\int_{B^+_{\delta}(R)} e^\tau dm} \right)^{\frac{1}{\gamma}} \left( 1 - C_4(n, q, k, R) \cdot v(n, k, R)^{\frac{1}{\gamma}} \cdot \delta^{2n/q} \cdot K_{p, \alpha}^{\frac{1}{\gamma}}(k, q, R) \right)^{-1}.
\]

Hence, there exists an $\epsilon_1 = \epsilon_1(n, q, k, \delta, R) > 0$ such that if $K_{\alpha}(k, q, R) < \epsilon_1$, then

\[
\left( \frac{v(n, k, r_2)}{\int_{B^+_{\delta}(r_2)} e^\tau dm} \right)^{\frac{1}{\gamma}} \leq 2 \left( \frac{v(n, k, R)}{\int_{B^+_{\delta}(R)} e^\tau dm} \right)^{\frac{1}{\gamma}}.
\]

On the other hand, we can choose $\epsilon_2 = \epsilon_2(n, q, k, \delta, R, \alpha) > 0$ such that if $K_{\alpha}(k, q, R) < \epsilon_2$, then the above inequality together with Proposition A1 yields

\[
c \leq 2C_5(n, q, k, R) \cdot \delta^{2n/q} K_{\alpha}^{\frac{1}{\gamma}}(k, q, R) \leq 2C_5(n, q, k, R) \cdot \delta^{2n/q} \cdot \epsilon_2 \leq 1 - \alpha^{\frac{1}{\gamma}} \cdot \delta^{\frac{2n}{\gamma}},
\]
where $C_5(n, q, k, R) := C_4(n, q, k, R) \cdot v(n, k, R)^{1/(2q)}$. We are done by setting $\varepsilon := \min \{\varepsilon_1, \varepsilon_2\}$.

**Step 5.** Now additionally suppose that $F$ is Berwaldian. It follows from [24,26] that the S-curvature of $d_m$ always vanishes, which implies that the distortion $\tau(r, y)$ only depends on $y$. In particular, we have

$$H = \frac{\partial}{\partial r} \log \left[ \hat{\sigma}_p(r, y) e^{\tau(r)} \right] = \frac{\partial}{\partial r} \log \hat{\sigma}_p(r, y).$$

Now we extend $\hat{\sigma}_p$ on $M$ by $\hat{\sigma}_p(r, y) := 0$ if $r \geq i_y$. Thus, (3.4) yields that, given $y \in S_p M$, on almost every $r > 0$

$$\frac{d}{dr} \left[ \hat{\sigma}_p(r, y) \right] \leq \Psi(r, y) \frac{\hat{\sigma}_p(r, y)}{\hat{\sigma}_k^{-1} \left( \frac{1}{2} \right) \left( r \right)}.$$

The same argument yields the result. \hfill \square

**Remark 1.** Theorem 4 can be extended to any measure if the assumption $\Lambda_F \leq \delta^2$ is replaced by $a \leq \tau \leq b$.

### 3.2 Compactness theorems

In order to prove Theorem 4 we need to generalize the so-called segment inequality [6, Theorem 2.1] to the Finsler setting.

**Theorem 5.** Let $(M, F, d_m)$ be a forward complete Finsler $n$-manifold with

$$A_F \leq \delta^2, \quad \text{Ric} \geq (n - 1)k.$$

Let $A_i$, $i = 1, 2$ be two bounded open subsets and let $W$ be an open subset such that for each two $x_i \in A_i$, the normal minimal geodesic $\gamma_{x_1 x_2}$ from $x_1$ to $x_2$ is contained in $W$. Thus, for any non-negative integrable function $f$ on $W$, we have

$$\int_{A_1 \times A_2} \int_0^{d(x_1, x_2)} f(\gamma_{x_1 x_2}(s)) ds \, d\mu_x \leq C(n, \delta, k, D) \left[ m(A_2) \text{diam}(A_2) + m(A_1) \text{diam}(A_1) \right] \int_W f \, d\mu,$$

where $d\mu_x$ is the product measure induced by $d\mu, D := \sup_{x_1 \in A_1, x_2 \in A_2} d(x_1, x_2)$ and

$$C(n, k, \delta, D) = \delta^{4n} \sup_{0 \leq \frac{1}{2} \leq t \leq D} \left( \frac{\sigma_k(t)}{\sigma_k(\frac{1}{2})} \right)^n.$$
Proof Step 1. Set

\[ B := \{(x_1, x_2) \in A_1 \times A_2 : \text{there exists a unique minimal geodesic from } x_1 \text{ to } x_2\}. \]

Since \( \overline{A}_i, i = 1, 2 \) are compact, \( m_\times(B) = m_\times(A_1 \times A_2) \). Let

\[ E(x_1, x_2) := \int_0^{d(x_1, x_2)} f(\gamma_{x_1 x_2}(s))ds = \int_0^{d(x_1, x_2)} f(\gamma_{x_1 x_2}(s))ds + \int_0^{1/2} d(x_1, x_2) \]

\[ =: E_1(x_1, x_2) + E_2(x_1, x_2). \]

Given \( y \in S_{x_1} M \), set

\[ I(x_1, y) := \{ t : \gamma_y(t) \in A_2, \gamma_y|[0, t] \text{ is minimal} \}. \]

Clearly, the length of \( I(x_1, y) \) is not larger than \( \text{diam}(A_2) \). Set

\[ T(y) := \sup\{t : t \in I(x_1, y)\}. \]

Let \( (r, y) \) denote the polar coordinate system at \( x_1 \). Since \( \text{Ric} \geq (n - 1)k^{1/2} \), according to [10], we have

\[ \frac{e^r(s, y)\hat{\sigma}_{x_1}(s, y)}{g_k^{n-1}(s)} \geq \frac{e^r(r, y)\hat{\sigma}_{x_1}(r, y)}{g_k^{n-1}(r)}, \quad 0 < r < i_y, \]

which together with Proposition A1 yields

\[ \int_{x_2 \in A_2} E_1(x_1, x_2)dm_\times(x_2) = \int_{S_{x_1} M} \int_{I(x_1, y)} E_1(x_1, \exp_{x_1}(ry)) \cdot \hat{\sigma}_{x_1}(r, y) d\nu_{x_1}(y) \]

\[ = \int_{S_{x_1} M} d\nu_{x_1}(y) \int_{I(x_1, y)} \hat{\sigma}_{x_1}(r, y)dr \int_0^r f(\gamma_{x_1 \exp_{x_1}(ry)}(s))ds \]

\[ \leq C(n, k, \delta, D) \int_{S_{x_1} M} d\nu_{x_1}(y) \int_{I(x_1, y)} dr \int_0^r f(\gamma_{x_1 \exp_{x_1}(ry)}(s)) \hat{\sigma}_{x_1}(s, y)ds \]

\[ \leq C(n, k, \delta, D) \text{diam}(A_2) \int_{S_{x_1} M} d\nu_{x_1}(y) \int_0^{T(y)} f(\gamma_{x_1 \exp_{x_1}(ry)}(s)) \hat{\sigma}_{x_1}(s, y)ds \]

\[ \leq C(n, k, \delta, D) \text{diam}(A_2) \int_W f dm. \]

Hence, we have

\[ \int_{A_1 \times A_2} E_1(x_1, x_2)dm_\times \leq C(n, k, \delta, D) \cdot \text{diam}(A_2) \cdot m(A_1) \int_W f dm. \]

Step 2. In this step, we estimate

\[ E_2(x_1, x_2) = \int_0^{1/2} d(x_1, x_2) f(\gamma_{x_1 x_2}(s))ds. \]

Let \( \overline{F}(x, y) := F(x, -y) \) be the reverse metric (cf. [13,15]). Let \( c_{x_2 x_1}(s) \) be the reverse of \( \gamma_{x_1 x_2} \). Thus, \( c_{x_2 x_1}(s) \) is a normal minimal geodesic from \( x_2 \) to \( x_1 \).
in \((M, \tilde{F})\) with the length \(d(x_1, x_2)\). Thus, 
\[ \gamma_{x_1x_2}(s) = c_{x_1x_2}(d(x_1, x_2) - s), \]
\[ \tilde{d}(x_2, x_1) = d(x_1, x_2) \]
and
\[ \gamma_{x_1x_2}(s) = c_{x_1x_2}(\tilde{d}(x_2, x_1)) = c_{x_2x_1}(-\tilde{d}(x_2, x_1)) \]
Thus, 
\[ E_2(x_1, x_2) = \int_0^{\frac{1}{2}\tilde{d}(x_1, x_2)} f(\gamma_{x_1x_2}(s))ds = \int_0^{\frac{1}{2}\tilde{d}(x_2, x_1)} f(c_{x_2x_1}(s))ds : = \tilde{E}_1(x_2, x_1). \]

Note that \(\tilde{\text{Ric}} \geq (n-1)k\). The same argument as above yields
\[ \int_{A_2 \times A_1} \tilde{E}_1(x_2, x_1)d\tilde{m} \leq C(n, k, \delta, D) \cdot \tilde{\text{diam}}(A_1) \cdot \tilde{m}(A_2) \int_W fd\tilde{m}. \]
It is not hard to check that \(d\tilde{m} = dm\), \(\tilde{m}(A_2) = m(A_2)\) and \(\tilde{\text{diam}}(A_1) = \text{diam}(A_1)\), which implies
\[ \int_{A_1 \times A_2} E_2(x_1, x_2)dm \leq C(n, k, \delta, D) \cdot \text{diam}(A_1) \cdot m(A_2) \int_W fdm. \]
Step 1 together with Step 2 yields the result. \(\square\)

Remark 2 By the comparison theorem in [15], one can see that the theorem above remains valid under a simpler assumption \(\text{Ric}_N \geq (n-1)k\) instead of \(\Lambda F \leq \delta^2\), \(\text{Ric} \geq (n-1)k\).

In [23], Sprouse proved some compactness theorems by \(L^1\)-Ricci curvature bounds in the Riemannian case. Inspired by his work, we show the following result by the \(L^q\)-norm (\(\forall q \geq 1\)).

**Lemma 2** Given any \(n > 1\), \(q \geq 1\), \(k \leq 0\), \(K > 0\) and \(\delta \geq 1\), for each \(R > \pi/\sqrt{K}\), there exists \(\varepsilon = \varepsilon(n, q, k, \delta, R, \rho) > 0\) such that every forward complete Finsler \(n\)-manifold \((M, F)\) with 

\[ \text{Ric} \geq (n-1)k, \ A_F \leq \delta^2, \ K_{dm}(q, K, R) < \varepsilon \]

satisfies 
\[ \text{diam}(M) \leq \frac{\pi}{\sqrt{K}} + \rho, \]
where \(\rho\) is any positive number with 
\[ \rho < \frac{1}{1+\delta} \left( R - \frac{\pi}{\sqrt{K}} \right). \]

**Proof** Step 1. Fix a point \(p \in M\) and set \(W = B^+_p(R)\). Choose any point \(p' \in W\) with 
\[ \frac{\pi}{\sqrt{K}} + \delta(3 + 2\delta)r < d(p, p') < R - (2 + \delta)r, \]
where \(r = \frac{\rho}{\pi(1+\delta)}\) is a fixed number.

Set \(A_1 := B^+_p(r)\) and \(A_2 := B^+_p(r')\). We now show that \(A_1, A_2, W\) satisfy the conditions in Theorem [3].
Thus, (3.8) implies that position A yields

\[ d(x_1, x_2) \leq d(x_1, p) + d(p, p') + d(p', x_2) < (1 + \delta)r + R - (2 + \delta)r = R - r, \]

which implies

\[ d(p, \gamma_{x_1x_2}(s)) \leq d(p, x_1) + d(x_1, \gamma_{x_1x_2}(s)) < r + R - r = R. \]

That is, \( \gamma_{x_1x_2} \) is contained in \( W \). Now Theorem [5] yields

\[
\int_{A_1 \times A_2} \left( \int_0^{d(x_1, x_2)} ((n-1)K - \text{Ric})_+^q (\gamma_{x_1x_2}(s)) ds \right) dm_x \leq C(n, k, \delta, R) \cdot (1 + \delta)r \cdot [m(A_1) + m(A_2)] \int_W ((n-1)K - \text{Ric})_+^q dm. \tag{3.8}
\]

Clearly, \( m(B_p^+(1, \delta)) \geq m(W) \) and [10] Remark 3.5 together with Proposition [4] yields

\[
\frac{m(A_1)}{m(W)} \geq \delta^{-4n} \frac{v(n, k, r)}{v(n, k, R)}, \quad \frac{m(A_2)}{m(B_p^+(1, \delta))} \geq \delta^{-4n} \frac{v(n, k, r)}{v(n, k, (1 + \delta)R)}.
\]

Thus, (3.8) implies

\[
\inf_{(x_1, x_2) \in A_1 \times A_2} \int_0^{d(x_1, x_2)} ((n-1)K - \text{Ric})_+^q (\gamma_{x_1x_2}(s)) ds \leq C(n, k, \delta, R) \cdot (1 + \delta)r \cdot \left( \frac{1}{m(A_1)} + \frac{1}{m(A_2)} \right) \int_W ((n-1)K - \text{Ric})_+^q dm \\
\leq 2\delta^{4n} \cdot C(n, k, \delta, R) \cdot (1 + \delta)r \cdot \frac{v(n, k, (1 + \delta)R)}{v(n, k, R)} \cdot \frac{1}{m(W)} \int_W ((n-1)K - \text{Ric})_+^q dm
\]

\[ := C'(n, k, \delta, R, \rho) \cdot \bar{K}_{dm}(q, K, R). \]

Since \( A_1 \times A_2 \) is compact, there exist two points \( x_i \in A_i, i = 1, 2 \) such that the normal minimal geodesic \( \gamma = \gamma_{x_1x_2} \) satisfies

\[
\int_0^{d(x_1, x_2)} ((n-1)K - \text{Ric})_+^q (\gamma(s)) ds \leq C'(n, k, \delta, R, \rho) \cdot \bar{K}_{dm}(q, K, R) < C'(n, k, \delta, \rho, R) \cdot \varepsilon. \tag{3.9}
\]

**Step 2.** Set \( L/\sqrt{K} = d(x_1, x_2) \). Let \( T := \gamma \) and \( \{ E_\alpha, T \} \) be a \( g_T \)-orthnormal parallel frame field along \( \gamma \). Set

\[
Y_\alpha(t) := \sin \left( \frac{\pi \sqrt{K}}{L} t \right) \cdot E_\alpha(t), \quad \alpha = 1, \ldots, n - 1.
\]
Let $C_\alpha(t, s)$ be the fixed-endpoint variation of curves corresponding to $Y_\alpha$ (i.e., $Y_\alpha = (\partial_s C_\alpha)(t, 0)$) and $L_\alpha(s)$ be the length of $C_\alpha(\cdot, s)$. Then we have

$$
\sum_\alpha \left. \frac{d^2}{ds^2} \right|_{t=0} L_\alpha = \sum_\alpha \int_0^\infty g_T(\nabla_T Y_\alpha, \nabla_T Y_\alpha) + R_T(Y_\alpha, T, Y_\alpha) \, dt \\
= \frac{(n-1)L\sqrt{K}}{2} \left( 1 - \left( \frac{\pi}{L} \right)^2 \right) + \Delta,
$$

(3.10)

where

$$
\Delta := \int_0^\infty \sin^2 \left( \frac{\pi \sqrt{K}}{L} t \right) \left[ (n-1)K - \text{Ric}(T) \right] \, dt.
$$

Now using the Hölder inequality and (3.9), one gets

$$
\Delta \leq \int_0^\infty \sin^2 \left( \frac{\pi \sqrt{K}}{L} t \right) ((n-1)K - \text{Ric}(T))_+ \, dt \\
\leq \left( \int_0^\infty \sin^2 \left( \frac{\pi \sqrt{K}}{L} t \right) \, dt \right)^{\frac{1}{2}} \left( \int_0^\infty ((n-1)K - \text{Ric}(T))^q_+ \, dt \right)^{\frac{1}{2}} \\
= C(q, n, k, K, \delta, R, \rho) \cdot L^{1-1/q} \cdot \varepsilon^{\frac{1}{2}}.
$$

(3.11)

**Step 3.** Now we claim $L \leq \pi + \frac{\rho}{2} \sqrt{K}$, if $\varepsilon$ is small enough. If not, i.e., $L > \pi + \frac{\rho}{2} \sqrt{K}$, then we consider any $\varepsilon > 0$ with $\varepsilon \leq \left[ \frac{(n-1)\sqrt{K} \left( 1 - \left( \frac{\pi}{\pi + \frac{\rho}{2} \sqrt{K}} \right)^2 \right)^q}{2 \cdot C(q, n, k, \delta, R, \rho)} \right] \cdot \left( \pi + \frac{\rho}{2} \sqrt{K} \right) =: \varepsilon_1$.

It follows from (3.10) and (3.11) that

$$
\sum_\alpha \left. \frac{d^2}{ds^2} \right|_{t=0} L_\alpha = \frac{(n-1)L\sqrt{K}}{2} \left( 1 - \left( \frac{\pi}{L} \right)^2 \right) + \Delta < 0,
$$

which is a contradiction, since $\gamma$ is a minimal geodesic. Hence, the claim is true and then the triangle inequality implies that

$$
d(p, p') \leq d(p, x_1) + d(x_1, x_2) + d(x_2, p') \leq (1 + \delta)r + \frac{L}{\sqrt{K}} < \pi + \rho.
$$

Now recall that $p'$ is an arbitrary point in $B_p^+(R - (2 + \delta)r)$ and hence,

$$
B_p^+(R - (2 + \delta)r) \subset B_p^+ \left( \frac{\pi}{\sqrt{K}} + \rho \right).
$$

However, it is easy to check that

$$
R - (2 + \delta)r > \frac{\pi}{\sqrt{K}} + \rho \Rightarrow M = B_p^+ \left( \frac{\pi}{\sqrt{K}} + \rho \right).
$$
In particular, $M$ is compact.

**Step 4.** Now we estimate $\text{diam}(M)$. Since $M$ is compact, we can suppose that there exist two points $p, p' \in M$ such that

$$D := \text{diam}(M) = d(p, p') > \frac{1}{1 + \delta} \left( \frac{\pi}{\sqrt{K}} + \rho \right).$$

Otherwise, we are done. Fix a number $r$ with

$$0 < r < \min \left\{ \frac{1}{(1 + \delta)^2} \left( \frac{\pi}{\sqrt{K}} + \rho \right), \frac{\rho}{2(1 + \delta)} \right\}$$

and set

$$R_0 := \frac{\pi}{\sqrt{K}} + \rho, \ A_1 := B_p^+(r), \ A_2 := B_{p'}^+(r), \ W := M = B_p^+(R_0) = B_{p'}^+(R).$$

Since $D < (1 + \delta)R_0$, the same argument as above (see (3.9)) yields that there exist two points $x_i \in A_i$, $i = 1, 2$ with

$$\int_0^{d(x_1, x_2)} ((n - 1)K - \text{Ric})_q^q(\gamma(s))ds = \int_0^{d(x_1, x_2)} (n - 1)K - \text{Ric})_q^q(\gamma_{x_1, x_2}(s))ds \
\leq C'(n, k, \delta, R_0, \rho) \cdot \frac{1}{m(B_{p}^{+}(R))} \int_{B_{P}^{+}(R)} ((n - 1)K - \text{Ric})_q^q dm \
< C'(n, k, \delta, R_0, \rho) \cdot \varepsilon.$$ 

Set $d(x_1, x_2) =: L/\sqrt{K}$. Using the same arguments in Step 2-3, one can show that if

$$\varepsilon \leq \left[ \frac{(n - 1)\sqrt{K} \left(1 - \left(\frac{\pi}{\frac{\pi}{2}\sqrt{K}}\right)^2\right)^{q}}{2 \cdot C(q, n, k, K, \delta, R_0, \rho)} \right] \cdot \left( \frac{\rho}{2 \sqrt{K}} \right) =: \varepsilon_2,$$

then $L \leq \pi + \frac{\varepsilon}{\sqrt{K}}$. Now, one gets

$$D = d(p, p') \leq d(p, x_1) + d(x_1, x_2) + d(x_2, p') < L/\sqrt{K} + (1 + \delta)r \leq \frac{\pi}{\sqrt{K}} + \rho.$$ 

Now we are done by setting $\varepsilon := \min \{\varepsilon_1, \varepsilon_2\}$. $\square$

**Remark 3** In [23], for a Riemannian manifold, Sprouse showed that $\text{diam}(M) \leq \pi/\sqrt{K} + \rho$ when $\text{Ric} \geq (n - 1)k = 0$ and claimed this result still holds in the general case. Here, we improve his method and make it more efficient to deal with the case $k < 0$. Moreover, we show the result in any $L^q$-norm of Ricci curvature instead of $L^1$-norm.
We now recall the definition and properties of fundamental domain. See [13, 27, 28] for more details. Let \( f : \tilde{M} \to M \) be a covering with deck transformation group \( \Gamma \). \( \Omega \subset \tilde{M} \) is called a fundamental domain of \( \tilde{M} \) if \( f(\Omega) = M \) and \( \gamma(\Omega) \cap \Omega = \emptyset \), for all \( \gamma \in \Gamma - \{1\} \). If \( \Omega \) is a fundamental domain, then

\[
\bigcup_{\gamma \in \Gamma} \gamma(\Omega) = \tilde{M}, \quad f|_{\gamma(\Omega)} : \gamma(\Omega) \to f(\Omega) \text{ is a homeomorphism, } \forall \gamma \in \Gamma.
\]

If \((M, F)\) is forward complete, one can get a fundamental domain as follows. For any \( p \in M \),

\[
p \mapsto D_p \subset T_p M \mapsto f|_{\tilde{p}}^{-1}(D_p) \subset T_{\tilde{p}} \tilde{M} \mapsto \exp_{\tilde{p}}(f|_{\tilde{p}}^{-1}(D_p)) =: \Omega_p,
\]

where \( \tilde{p} \) is an arbitrary point in \( f^{-1}(p) \). Thus, \( \Omega_p \) is a fundamental domain.

On the other hand, if \( d_m \) is either the Busemann-Hausdorff measure or the Holmes-Thompson measure, then the pull-back measure \( f^* d_m \) is exactly the same kind of measure on \((\tilde{M}, f^* F)\). By abuse of notation, \( d_m \) also denotes the pull-back measure.

**Theorem 6** Given any \( n > 1, q \geq 1, k \leq 0, K > 0, R > 0 \) and \( \delta \geq 1 \), for each \( \rho > 0 \), there exists \( \varepsilon = \varepsilon(n, q, k, K, R, \delta, \rho) > 0 \) such that every forward complete Finsler \( n \)-manifold \((M, F)\) with \( \text{Ric} \geq (n-1)k, A_F \leq \delta^2, \ K \text{dm}(q, K, R) < \varepsilon \) satisfies

\[
\text{diam}(M) \leq \frac{\pi}{\sqrt{K}} + \rho.
\]

In particular, the universal covering \( \tilde{M} \) is compact and hence, \( \pi_1(M) \) is finite.

**Proof Step 1.** We first estimate the diameter of \( M \). If \( R > \pi/\sqrt{K} \), then the theorem follows from Lemma [2] directly. Hence, we just need to consider the case when \( R \leq \pi/\sqrt{K} \).

Fix \( R' > \pi/\sqrt{K} \) and choose any point \( x \in M \). Let \( \{B^{+}_{x_0}(R/(1+\delta))\} \) be a maximal disjoint family in \( B^{+}_{x_0}(R') \). Thus, \( \{B^{+}_{x_0}(R)\} \) is a covering of \( B^{+}_{x_0}(R') \) and hence,

\[
\frac{1}{\text{m}(B^{+}_{x_0}(R'))} \int_{B^{+}_{x_0}(R')} ((n-1)K - \text{Ric})_+^q \text{dm} \\
\leq \frac{1}{\text{m}(B^{+}_{x_0}(R'))} \sum_i \int_{B^{+}_{x_0}(R')} ((n-1)K - \text{Ric})_+^q \text{dm} \\
= \sum_i \frac{\text{m}(B^{+}_{x_0}(R))}{\text{m}(B^{+}_{x_0}(R'))} \frac{1}{\text{m}(B^{+}_{x_0}(R))} \int_{B^{+}_{x_0}(R)} ((n-1)K - \text{Ric})_+^q \text{dm} \\
\leq \sum_i \frac{\text{m}(B^{+}_{x_0}(R))}{\text{m}(B^{+}_{x_0}(R'))} \text{Kdm}(q, K, R)
\]
\[
\leq \sum_i \frac{m(B^+_x(R))}{m(B^+_x(R/(1+\delta)))} \frac{m(B^+_x(R/(1+\delta)))}{m(B^+_x(R'))} \mathcal{K}_{dm}(q, K, R)
\]
\[
\leq C(n, k, \delta, R) \sum_i \frac{m(B^+_x(R/(1+\delta)))}{m(B^+_x(R'))} \mathcal{K}_{dm}(q, K, R)
\]
\[
\leq C(n, k, \delta, R) \cdot \mathcal{K}_{dm}(q, K, R),
\]
where
\[
C(n, k, \delta, R) := \delta^{4n} \frac{v(n, k, R)}{v(n, k, R/(1+\delta))}.
\]
That is,
\[
\mathcal{K}_{dm}(q, K, R') \leq C(n, k, \delta, R) \cdot \mathcal{K}_{dm}(q, K, R).
\]
Choose \(\varepsilon = \varepsilon(n, q, k, K, \delta, R', \rho)\) as defined in Lemma 2 and set
\[
C_3(n, k, \delta, R) \cdot \mathcal{K}_{dm}(q, K, R) < \varepsilon.
\]
Then the estimate of the diameter follows from Lemma 2.

**Step 2.** Now we show that \(\tilde{M}\) is compact. Suppose that \(R > \pi/\sqrt{K}\). It follows from Lemma 2 that \(\text{diam}(M) < (1+\delta)R\). We show that \(\mathcal{K}_{dm}(q, K, R)\) is controlled by \(\mathcal{K}_{dm}(q, K, R)\), where \(\mathcal{K}_{dm}(q, K, R)\) is the integral curvature of \((\tilde{M}, f^*F)\).

In fact, given any point \(\tilde{x} \in \tilde{M}\), let \(N\) denote the minimal number of the fundamental domains \(\gamma_i(\Omega)\) covering \(B^+_x(R)\), i.e.,
\[
B^+_x(R) \subset \bigcup_{i=1}^N \gamma_i(\Omega) \subset B^+_x((2+\delta)R).
\]
And hence,
\[
\frac{m(B^+_x(2+\delta)R)}{m(B^+_x(R))} \leq \delta^{4n} \frac{v(n, k, (2+\delta)R)}{v(n, k, R)} =: C'(n, k, \delta, R).
\]
It follows from (3.13) and (3.14) that
\[
\frac{1}{m(B^+_x(R))} \int_{B^+_x(R)} ((n-1)K - \text{Ric})_+^q \ dm
\]
\[
\leq \frac{N}{m(B^+_x(R))} \int_M ((n-1)K - \text{Ric})_+^q \ dm
\]
\[
\leq C'(n, k, \delta, R) \frac{N}{m(B^+_x((2+\delta)R))} \int_M ((n-1)K - \text{Ric})_+^q \ dm
\]
\[
\leq C'(n, k, \delta, R) \frac{1}{m(M)} \int_M ((n-1)K - \text{Ric})_+^q \ dm = C'(n, k, \delta, R) \cdot \mathcal{K}_{dm}(q, K, R),
\]
which implies that
\[
\mathcal{K}_{dm}(q, K, R) \leq C'(n, k, \delta, R) \cdot \mathcal{K}_{dm}(q, K, R).
\]
Let $\varepsilon = \varepsilon(n, q, K, \delta, R, \rho)$ defined in Lemma 2 and let

$$C'(n, k, \delta, R) \cdot \mathcal{K}_{dm}(q, K, R) < \varepsilon.$$ 

Now (3.15) together with $\tilde{Ric} \geq (n - 1)k$, $A_F \leq \delta^2$ and Lemma 2 yields that $\tilde{M}$ is compact.

For $R \leq \pi/\sqrt{K}$, one can use the same argument as Step 1 to show that $\tilde{M}$ is compact and hence, $\pi_1(M)$ is finite. \(\square\)

In [20], Petersen-Sprouse gave a better result in the Riemannian case by removing the assumption $\text{Ric} \geq (n - 1)k$ ($k \leq 0$). Unfortunately, their method does not work on general Finsler manifolds as neither the Laplacian is linear nor Green’s formula holds. However, for a Berwald space, we can obtain the following result by the connection with its average Riemannian metric.

**Theorem 7** Given any $n > 1$, $q > n/2$, $K > 0$, $R > 0$ and $\delta \geq 1$, for each $\rho > 0$, there exists $\varepsilon = \varepsilon(n, q, K, R, \delta, \rho)$ such that if a (forward) complete Berwald $n$-manifold $(M, F)$ satisfies $\Lambda_F \leq \delta^2$,

$$K_{dm}(q, K, R) < \varepsilon,$$

then

$$\text{diam}(M) \leq \delta^2 \left( \frac{\pi}{\sqrt{K}} + \rho \right).$$

**Proof** Let $\tilde{g}$ denote the average Riemannian metric of $g$ and let $B_x(r)$ denote the geodesic ball centered at $x$ with radius $r$ of $(M, \tilde{g})$. For each $p \in B_x(r) - \{x\}$, there exists a minimal normal geodesic $\gamma$ from $x$ to $p$. In a local coordinate system $(x^i)$, set

$$dm = \sigma \cdot dx^1 \wedge \cdots \wedge dx^n = h \cdot d\text{vol},$$

where $d\text{vol}$ is the Riemannian measure of $\tilde{g}$ and

$$h(\gamma(t)) = \frac{\sigma(\gamma(t))}{\sqrt{\text{det} g(\gamma(t), \dot{\gamma}(t))}} \cdot \sqrt{\frac{\text{det} g(\gamma(t), \dot{\gamma}(t))}{\text{det} \tilde{g}(\gamma(t))}}.$$

In the following, we estimate $h$. Since the Levi-Civita connection of $\tilde{g}$ is exactly the Chern connection of $g$, one can choose a $g_i$-orthonormal parallel frame field $\{E_i\}$ such that each $E_i$ is the eigenvector of $\tilde{g}$. Note that $h(\gamma(t))$ is independent of the choice of coordinates. Denote by $\text{det} g$ and $\text{det} \tilde{g}$ be the determinants of $g$ and $\tilde{g}$ w.r.t. $\{E_i\}$, respectively. It is easy to see that

$$\text{det} g(\gamma(t), \dot{\gamma}(t)) = 1, \quad \delta^{-2n} \leq \text{det} \tilde{g}(\gamma(t)) \leq \delta^{2n}.$$ 

Then

$$h(\gamma(t)) = e^{-\tau(\gamma)} \sqrt{\frac{\text{det} g(\gamma(t), \dot{\gamma}(t))}{\text{det} \tilde{g}(\gamma(t))}},$$

together with Proposition [A1] implies

$$\delta^{-3n} \leq h(\gamma(t)) \leq \delta^{3n} \Rightarrow \delta^{-3n} d\text{vol} \leq dm \leq \delta^{3n} d\text{vol},$$
and hence,
\[
\frac{1}{\text{vol}(B_p^+(R))} \int_{B_p^+(R)} f \, d\text{vol} \leq \delta^{6n} \frac{1}{\text{m}(B_p^+(R))} \int_{B_p^+(R)} f \, d\text{m},
\]
(3.16)
\[
\frac{1}{\text{vol}(B_p^+(R))} \leq \delta^{6n} \frac{\text{m}(B_p^+(R))}{\text{m}(B_p^+(R/(1+\delta)))}.
\]
(3.17)

Note that
\[
\mathcal{K}_{dm}(q, 0, R) \leq \mathcal{K}_{dm}(q, K, R).
\]

Hence, it follows from Theorem 4 that there exists \( \varepsilon_1 = \varepsilon_1(n, p, \delta, R) \) such that if \( \mathcal{K}_{dm}(q, K, R) < \varepsilon_1 \), then for any \( x \in M \),
\[
\frac{\text{m}(B_p^+(R))}{\text{m}(B_p^+(R/(1+\delta)))} \leq 2 \delta^{4n}(1+\delta)^n.
\]
(3.18)

Given \( p \in M \), let \( \{B_p^+(R/(1+\delta))\} \) denote the maximal family of disjoint forward balls in \( B_p(\delta R) \). Thus, \( \{B_p^+(R)\} \) is a covering of \( B_p(\delta R) \). Let \( J(p) \) denote the minimal eigenvalue of the Ricci tensor of \( \hat{g} \) at \( p \in M \). Thus, one gets
\[
(n-1)K - J \delta^2 \leq (n-1)K - \text{Ric},
\]
since \((M, F)\) is Berwaldian. Now it follows from (3.16)-(3.18) that
\[
\frac{\delta^{2q}}{\text{vol}(B_p(\delta R))} \int_{B_p(\delta R)} \left( \frac{n-1}{\delta^2} - \mathcal{J} \right)^q _+ \, d\text{vol}
\]
\[
\leq \frac{1}{\text{vol}(B_p(\delta R))} \sum_i \int_{B_i^+(R)} \left( (n-1)K - \mathcal{J} \delta^2 \right)_+ ^q \, d\text{vol}
\]
\[
\leq \delta^{6n} \sum_i \frac{\text{vol}(B_i^+(R))}{\text{vol}(B_i^+(R/(1+\delta)))} \frac{\text{vol}(B_i^+(R/(1+\delta)))}{\text{vol}(B_i^+(R(\delta R)))} \mathcal{K}_{dm}(q, K, R)
\]
\[
\leq \delta^{6n} \sup_i \frac{\text{vol}(B_i^+(R))}{\text{vol}(B_i^+(R/(1+\delta)))} \cdot \mathcal{K}_{dm}(q, K, R) \leq 2 \delta^{16n}(1+\delta)^n \cdot \mathcal{K}_{dm}(q, K, R).
\]

According to [20, Theorem 1.1], there exists \( \varepsilon_2 = \varepsilon_2(n, p, K, \delta, R, \rho) \) such that if
\[
\delta^{-2q} \cdot 2 \delta^{12n}(1+\delta)^n \cdot \mathcal{K}_{dm}(q, K, R) \leq \varepsilon_2 \leq \varepsilon_1,
\]
then
\[
\text{diam}_{\hat{g}}(M) \leq \delta \left( \frac{\pi}{\sqrt{K}} + \rho \right).
\]
The result now follows. 

\[\square\]

The proof of Theorem 1 follows from Theorem 6 and Theorem 7 directly. 
\[\square\]
4 Results concerned with $\mathcal{K}_{dm}(k,q)$

In this section, $(M,F)$ is always a compact Finsler $n$-manifold. Now set

$$
\mathcal{K}_{dm}(k,q) := \frac{1}{m(M)} \int_M ((n-1)k - \text{Ric}(x))^q_+ \, dm(x).
$$

4.1 Precompactness and finiteness theorems

If $\text{diam}(M) \leq D$, then for each $B_p^+(D) = M$ for all $p \in M$. Thus, using the same argument as Theorem 4, one gets the following, which is an generalization of [21, Theorem 1.1].

**Theorem 8** Given $n > 1$, $q > n/2$ and $k \leq 0$, $\delta < \infty$ and $D < \infty$, for any $\alpha < \delta^{-4n}$, there exists an $\varepsilon = \varepsilon(n,q,k,\delta,D,\alpha) > 0$ such that for any compact Finsler $n$-manifold $(M,F,dm)$ with

$$
\Lambda_F \leq \delta^2, \quad \text{diam}(M) \leq D, \quad \mathcal{K}_{dm}(k,q) \leq \varepsilon
$$

satisfies

$$
\alpha \cdot \frac{v(n,k,r)}{v(n,k,D)} \leq \frac{m(B_p^+(r))}{m(M)}.
$$

for any $p \in M$ and $r \leq D$.

In particular, if $(M,F)$ is a compact Berwald manifold, then without any assumption, we always have

$$
\left( \frac{m(B_p^+(r))}{v(n,k,R)} \right)^{\frac{1}{n}} - \left( \frac{m(B_p^+(r))}{v(n,k,r)} \right)^{\frac{1}{n}} \leq C(n,q,k,D) \cdot (\mathcal{K}_{dm}(k,q))^{\frac{1}{q}}.
$$

for $0 < r \leq R$, where

$$
\mathcal{K}_{dm}(k,q) := \int_M ((n-1)k - \text{Ric}(x))^q_+ \, dm(x).
$$

Now we turn to study the precompactness of compact Finsler manifolds in the $\delta$-Gromov-Hausdorff topology. See Appendix [B] for the definitions and properties of the $\delta$-Gromov-Hausdorff distance. Also refer to [26,31,32,39] for more details.

Now Theorem 8 together with Proposition [B] yields the following result.

**Theorem 9** Given $n > 1$, $q > n/2$ and $k \leq 0$, $\delta \in [1,\infty)$, $D < \infty$, there is an $\varepsilon = \varepsilon(n,q,k,\delta,D) > 0$ such that the class of closed Finsler $n$-manifolds $(M,F)$ with

$$
\Lambda_F \leq \delta^2, \quad \text{diam}(M) \leq D, \quad \mathcal{K}_{dm}(k,q) \leq \varepsilon
$$

is precompact in $\delta$-Gromov-Hausdorff topology. In particular, the condition $\Lambda_F \leq \delta^2$ can be replaced by $\Lambda_F \leq \delta$ if $F$ is Berwaldian.
Proof Set \( \alpha = 1/(2\delta^4n) \) and let \( \varepsilon \) be defined by Theorem \ref{thm:existence}. Given any \( r > 0 \), since \( M \) is compact, there exists only finitely many disjoint forward \( r \)-balls inside \( M \). Denote them by \( B^+_x(r), \ldots, B^+_x(r) \) are disjoint. Suppose that \( B^+_x(r) \) is the forward ball with the smallest \( m \)-volume. Thus, it follows from Theorem \ref{thm:existence} that

\[
 k \leq \frac{m(M)}{m(B^+_x(r))} \leq \alpha^{-1} \cdot \frac{v(n, k, D)}{v(n, k, r)} := N(r).
\]

This implies that \( \text{Cap}_M(2r) \leq N(r) \). Now the conclusion follows from Proposition \ref{prop:cap} directly. \( \square \)

Remark 4 It follows from \cite[Theorem 5.3]{26} that the limit space of any convergent subsequence in Theorem \ref{thm:limit} is a general length space, say \((X, d)\). In particular, the metric \( d \) is strictly intrinsic.

Now we recall the definition of critical point. A point \( q \) in a forward complete Finsler manifold \((M, F)\) is regular for \( r_p(\cdot) := d(p, \cdot) \) if there exists a unit vector \( y \in S^1_q M \) and an \( \eta > 0 \) such that

\[
r_p(\gamma_y(t)) \geq r_p(\gamma_y(0)) + \eta \cdot t
\]

for all sufficiently small \( t > 0 \), where \( \gamma_y(t) := \exp_q(ty) \). If \( q \in M \) is not a regular point for \( r_p \), then it is called a critical point. The contractibility radius \( c_M \) is defined as the supremum of \( r \) such that every forward metric ball of radius \( r \) contains no critical points of the distance function from the center. Refer to \cite{24,39} for more details.

Theorem 10 Given \( n > 1, q > n/2, k \leq 0, \delta \in [1, \infty), D < \infty \) and \( R < \infty \), there is an \( \varepsilon = \varepsilon(n, q, k, \delta, D, R) > 0 \) such that the class of closed Finsler \( n \)-manifolds \((M, F)\) with

\[
 \Lambda_F \leq \delta^2, \quad \text{diam}(M) \leq D, \quad \text{K} \leq \frac{1}{c_M} \leq \frac{1}{c_M} \quad \text{contains at most finitely many homotopy types and only finitely many diffeomorphism types if in addition} \ n \neq 4. \quad \text{In particular, the condition} \ \Lambda_F \leq \delta^2 \ \text{can be replaced by} \ \lambda_F \leq \delta \ \text{if} \ F \ \text{is Berwaldian}.
\]

Proof It follows from Theorem \ref{thm:comp} that \( M \) is LGC(\( \rho \)), where \( \rho = \text{Id} : [0, R) \rightarrow [0, R) \) (cf. Definition \ref{def:lgc}). Due to Proposition \ref{prop:contract} and Proposition \ref{prop:homotopy}, the homotopy finiteness follows from the same argument as that in Theorem \ref{thm:limit}. Since every closed topological \( n \)-manifold carries at most finitely many smooth structures up to diffeomorphism when \( n \neq 4 \) (cf. \cite{14}), we have the diffeomorphism finiteness. \( \square \)

It is easy to see that Theorem \ref{thm:limit} together with Theorem \ref{thm:existence} yields Theorem \ref{thm:main}.
4.2 Estimates of isoperimetric and Sobolev constants

Given a hypersurface $\Gamma$ in $M$, denote by $n_+$ and $n_-$ the unit inward and outward normal vector fields along $\Gamma$, respectively. According to [7,24], the measures on $\Gamma$ induced by $n_\pm$ are defined by $dA_\pm = i^*(n_\pm)dm$. Usually, $A_+(\Gamma)$ is not equal to $A_-(\Gamma)$ unless $F$ is reversible.

Example 1 ([7]) Let $M$ be a unit ball in $\mathbb{R}^n$ equipped with a Funk metric

$$F(y) = \sqrt{|y|^2 - (|x|^2|y|^2 - \langle x \cdot y \rangle^2)} + \langle x \cdot y \rangle,$$

where $|\cdot|$ and $\langle , \rangle$ denote the Euclidean norm and inner product in $\mathbb{R}^n$, respectively. Given $r \in (0,1)$, set $\Omega_r := \{x \in M : |x| < r\}$. Let $n_+$ and $n_-$ denote the unit outward and inward normal vector fields along $\partial \Omega_r$, respectively. Then

$$n_- = -\frac{(1 + |r|)x^i}{|r|} \frac{\partial}{\partial x^i}, \quad n_+ = \frac{(1 - |r|)x^i}{|r|} \frac{\partial}{\partial x^i}.$$

In particular, for any measure on $M$, we always have

$$\lim_{r \to 1} \frac{A_-(\partial \Omega_r)}{A_+(\partial \Omega_r)} = +\infty.$$

We now recall the definitions of the Sobolev constant, Cheeger’s constant and the isoperimetric constant of a closed Finsler manifold (cf. [7,24,38]).

**Definition 1** Let $(M,F,\text{d}m)$ be a closed Finsler $n$-manifold. The Neumann $n$-Sobolev constant $SN(M,\text{d}m)$ is defined as

$$SN(M,\text{d}m) := \inf_{f \in C^\infty(M)} \frac{\int_M F^*(df)\text{d}m}{\inf_{a \in \mathbb{R}} \{\int_M |f-a|^{\frac{n}{n-1}}\text{d}m\}^{\frac{n-1}{n}}}.$$

Cheeger’s constant $H(M,\text{d}m)$ and the Neumann $n$-isoperimetric constant $\Pi(M,\text{d}m)$ are defined by

$$H(M,\text{d}m) := \frac{\min \{A_+(\Gamma)\}}{\min \{\text{m}(M_1), \text{m}(M_2)\}}, \quad \Pi(M,\text{d}m) := \inf_{\Gamma} \frac{\min \{A_+(\Gamma)\}}{\min \{\text{m}(M_1), \text{m}(M_2)\}^{\frac{n-1}{n}}}.$$

where $\Gamma$ varies over compact $(n-1)$-dimensional submanifolds of $M$ which divide $M$ into disjoint open submanifolds $M_1, M_2$ of $M$ with common boundary $\partial M_1 = \partial M_2 = \Gamma$.

One can show the Federer-Fleming type inequality (see [38]), i.e.,

$$\Pi(M,\text{d}m) \leq SN(M,\text{d}m) \leq 2\Pi(M,\text{d}m).$$
Proof of Theorem 3. Step 1. Firstly, using the similar argument to Theorem 4.7 to estimate \( \det g \), one can easily show \( \nu(S_pM) \leq c_{n-1}\delta^{2n} \). It follows from (3.7) that

\[
\begin{align*}
    h^{\frac{1}{n}}(R) - h^{\frac{1}{n}}(r) &\leq C(n, q, k, D) \cdot \left( \delta^{2n} \cdot \mathcal{K}_{dm}(k, q) \right)^{\frac{1}{n}} \cdot \left( \frac{m(M)}{v(n, k, D)} \right)^{\frac{1}{n}} v(n, k, D)^{\frac{1}{n}} \\
    &\leq C(n, q, \delta, k, D) \cdot \mathcal{K}_{dm}(k, q) \cdot \left( \frac{m(M)}{v(n, k, D)} \right)^{\frac{1}{n}} v(n, k, D)^{\frac{1}{n}}, \quad 0 < r \leq R \leq D.
\end{align*}
\]

Letting \( R = D \) and \( r = 0 \), one gets

\[
\delta^{-\frac{n}{2}} \left( \frac{m(M)}{v(n, k, D)} \right)^{\frac{1}{n}} \leq \left( \frac{\nu(S_pM)}{c_{n-1}} \right)^{\frac{1}{n}} \Rightarrow m(M) \leq \alpha^{-2q} \cdot \delta^{4n} \cdot v(n, k, D).
\]

For any fixed number \( \alpha \in (0, 1) \), one can choose \( \varepsilon_1 = \varepsilon_1(n, q, \delta, k, D, \alpha) > 0 \) such that

\[
C(n, q, \delta, k, D) \cdot \mathcal{K}_{dm}(k, q) \leq C(n, q, \delta, k, D) \cdot \varepsilon_1^{\frac{1}{n}} \leq (1 - \alpha)\delta^{-\frac{n}{2}}.
\]

That is,

\[
\alpha \cdot \delta^{-\frac{n}{2}} \left( \frac{m(M)}{v(n, k, D)} \right)^{\frac{1}{n}} \leq \left( \frac{\nu(S_pM)}{c_{n-1}} \right)^{\frac{1}{n}} \Rightarrow m(M) \leq \alpha^{-2q} \cdot \delta^{4n} \cdot v(n, k, D).
\]

Step 2. Given any \( p \in M \), let \( \tilde{S}_p \) be a subset of \( S_pM \). For each \( r > 0 \), set

\[
\Gamma(\tilde{S}_p, r) := \{ \exp_p(ty) : 0 \leq t < \min\{r, i_y\}, \ y \in \tilde{S}_p \}.
\]

Now we estimate \( m(\Gamma(\tilde{S}_p, r)) \). From Step 1, we have

\[
\begin{align*}
    \int_{\Gamma(\tilde{S}_p, D)} \rho^q e^r \, dm &\leq \int_{\Gamma} \rho^q e^r \, dm \cdot m(M) \\
    &\leq \delta^{2n} \cdot \mathcal{K}_{dm}(k, q) \cdot m(M) \leq \alpha^{-2q} \cdot \delta^{4n} \cdot v(n, k, D) \cdot \mathcal{K}_{dm}(k, q).
\end{align*}
\]

Now consider

\[
h(r) := \frac{\int_{\Gamma(\tilde{S}_p, r)} e^r \, dm}{v(n, k, r)}.
\]

The same argument as (3.7) together with (4.1) yields

\[
\begin{align*}
    h^{\frac{1}{n}}(R) - h^{\frac{1}{n}}(r) &\leq C(n, q, k, \delta, D) \cdot \left( \int_{\Gamma(\tilde{S}_p, D)} \rho^q e^r \, dm \right)^{\frac{1}{n}} \\
    &\leq C_1(n, q, k, \delta, D) \cdot \mathcal{K}_{dm}(k, q).
\end{align*}
\]

Again letting \( R = D \) and \( r = 0 \), we have

\[
\begin{align*}
\delta^{-\frac{n}{2}} \left( \frac{m(\Gamma(\tilde{S}_p, D))}{v(n, q, D)} \right)^{\frac{1}{n}} &\leq \left( \frac{\nu(S_p)}{c_{n-1}} \right)^{\frac{1}{n}} + \alpha^{-1} \cdot C_1(n, q, k, \delta, D) \cdot \mathcal{K}_{dm}(k, q).
\end{align*}
\]
Now we choose \( \varepsilon_2 = \varepsilon_2(n, q, k, \delta, D, \alpha) \) such that \( K_{dm}(k, q) < \varepsilon_2 \leq \varepsilon_1 \) and

\[
\alpha^{-1} \cdot C_1(n, q, k, \delta, D) \cdot \varepsilon_2^{\frac{1}{n}} \leq \frac{1}{2} t^{-\frac{2}{3}} \left( \frac{1}{2} V \right)^{\frac{1}{n}} \cdot \varepsilon_2^{\frac{1}{n}}.
\]

Hence,

\[
m(\Gamma(\hat{\mathcal{L}}_p, D)) \leq \delta^{2n} \cdot v(n, q, D) \cdot \left[ \left( \frac{\nu(\hat{\mathcal{L}}_p)}{c_{n-1}} \right)^{\frac{1}{n}} + \frac{1}{2} \delta^{-\frac{2}{3}} \left( \frac{1}{2} V \right)^{\frac{1}{n}} \cdot \varepsilon_2^{\frac{1}{n}} \right]^{2q}.
\]

**Step 3.** Let \( \Gamma \) be a \((n-1)\)-dimensional compact submanifold of \( M \) dividing \( M \) into two open submanifolds \( M_1 \) and \( M_2 \), such that \( \partial M_1 = \partial M_2 = \Gamma \). Suppose \( m(M_1) \leq m(M_2) \) and hence \( m(M_2) \geq m(M)/2 \).

Given \( x \in M_1 \), for each \( y \in S_x M \), set \( t(y) := \sup \{ t : \gamma_y([0, t]) \subset M_1 \} \) and \( U^+_y := \{ y \in S_x M : t(y) \leq i_y \} \). Let \( \overline{T} (\cdot) \) and \( \overline{i} (\cdot) \) be the corresponding quantities defined in \((M, \overline{F})\) and set \( U^-_x := \{ y \in S_x M : \overline{T} (-y) \leq \overline{i} - y \} \).

Set

\[
\omega_x^\pm := c_{n-1}^{-1} \int_{U_x^\pm} e^{\sigma(y)} d\nu_x(y) \geq \delta^{-2n} \nu(U_x^\pm) / c_{n-1}.
\]

Given \( x \in M_1 \), set

\[
O^-_x := \{ q \in M : \exists y \in U^-_x \text{ such that } q = \overline{T} - y(t), t \in (0, \overline{i} - y] \},
\]

where \( \overline{T} - y(t) \) is the geodesic in \((M, \overline{F})\) with \( \overline{T} - y(0) = -y \).

For any \( q \in M_2 \), there exists a minimal unit speed geodesic, say \( \overline{\gamma}_X(t) \), from \( x \) to \( q \). Clearly, \( \overline{\gamma}_X(t) \) must hit the boundary and therefore, \( \overline{T}(X) \leq \overline{i} - X \). Since \( F(-X) = \overline{F}(X) = 1 \), \( q \in O^-_x \) and hence, \( M_2 \subset O^-_x \).

Note that \( d\overline{m} = dm, \overline{K}_{dm} = K_{dm} \) and \( A_{\overline{F}} = A_{\overline{F}} \). Hence, it follows from Step 1-2 that

\[
\frac{1}{2} V \leq m(M_2) = m(M_2) \leq m(O^-_x) = \int_{y \in U^-_x} d\overline{\nu}_x(-y) \int_0^{\overline{i} - y} \sigma_x(r, -y) dr
\]

\[
\leq \overline{m}(\Gamma(U^-_x, D)) = m(\Gamma(U^-_x, D)) \leq \left[ \left( \delta^{2n} \omega_x^\pm \right)^{\frac{1}{n}} + \frac{1}{2} \delta^{-\frac{2}{3}} \left( \frac{1}{2} V \right)^{\frac{1}{n}} \cdot \varepsilon_2^{\frac{1}{n}} \right]^{2q} \cdot v(n, q, D).
\]

That is,

\[
\omega^\pm - (x) \geq \frac{V}{2^{2q+1} \delta^{2n} v(n, k, D)}.
\]

Set \( O^+_x := \{ q \in M : \exists y \in U^+_x \text{ such that } q = \gamma_y(t), t \in (0, i_y] \} \). It is easy to see that \( M_2 \subset O^+_x \). A similar argument yields

\[
\omega^+_x \geq \frac{V}{2^{2q+1} \delta^{2n} v(n, k, D)}.
\]
Now, the results follow from the Croke type inequalities \([40,38]\), namely,

\[
\frac{A_{\pm}(\partial M_1)}{m(M_1)} \geq \frac{(n-1)c_{n-1}\omega}{c_{n-2}DA_F^{2n+2}+2}, \quad \frac{A_{\pm}(\partial M_1)}{m(M_1)^{1-\frac{1}{n}}} \geq \frac{c_{n-1}\omega^{1+\frac{1}{n}}}{(c_{n}/2)^{1-\frac{1}{n}}A_F^{2n+2}},
\]

where \(\omega := \inf_{x\in M_1} \min\{\omega^+(x), \omega^-(x)\}\).

\[\Box\]
B Gromov-Hausdorff topology for general metric spaces

In this section, we recall some definitions and properties about general metric spaces in [26]. For the reversible case please refer to [26][28][19], etc.

A general metric space is a pair \((X,d)\), where \(X\) is a set and \(d : X \times X \to \mathbb{R}^+ \cup \{\infty\}\), called a metric, is a function, satisfying the following two conditions for all \(x,y,z \in X\):
- (i) \(d(x,y) \geq 0\), with equality iff \(x = y\);
- (ii) \(d(x,y) + d(y,z) \geq d(x,z)\).

Since all the spaces here are general metric spaces, we just call them metric spaces for simplicity. Given a metric space \((X,d)\), the forward \(\varepsilon\)-ball, \(\varepsilon > 0\), centered at \(x \in X\) is defined as \(B^+(\varepsilon) := \{y \in X | d(x,y) < \varepsilon\}\). A subset \(U \subset X\) is said to be open if, for each point \(x \in U\), there is an forward \(\varepsilon\)-ball centered at \(x\) contained in \(U\). Then we get the topology on \(X\).

Given a metric space \(X\), the reversibility \(\lambda_d\) is defined by
\[
\lambda_d := \sup_{x,y \in X} \frac{d(x,y)}{d(y,x)}.
\]
Clearly, \(\lambda_d \geq 1\) with equality iff \(d\) is reversible (i.e., symmetric). If \(\lambda_d\) is finite, then the metric \(d\) is naturally continuous with respect to the product topology on \(X \times X\). Thus, the metric space is a Hausdorff \((T_2)\) space.

Given a metric space \(X\) and two subsets \(A,B \subset X\), the Hausdorff distance between \(A\) and \(B\) is defined by
\[
d_H(A,B) := \inf\{\varepsilon : A \subset B^+(\varepsilon), B \subset B^+(A,\varepsilon)\}.
\]
where \(B^+(A,\varepsilon) := \{x \in X : d(A,x) < \varepsilon\}\) and \(d(A,B) := \inf\{d(a,b) : a \in A, b \in B\}\). In particular, \(d_H\) is symmetric, i.e., \(d_H(A,B) = d_H(B,A)\).

Given a number \(\delta \in [1,\infty)\), let \(M^\delta\) be the collection of compact metric spaces with reversibility \(\leq \delta\). Given two metric spaces \(X,Y\) in \(M^\delta\), a \(\delta\)-admissible metric on the disjoint union \(X \cup Y\) is a metric that extends the given metrics on \(X\) and \(Y\) with reversibility \(\leq \delta\). The \(\delta\)-Gromov-Hausdorff distance between \(X\) and \(Y\) is defined as
\[
d_{G,H}^\delta(X,Y) := \inf\{d_H(X,Y) : \delta\text{-admissible metrics on } X \cup Y\}.
\]
This definition is equivalent to the one induced by isometric immersion (cf. [26] Remark 3.5). Moreover, if \(\delta = 1\), \(d_{G,H}^1\) is exactly the original Gromov-Hausdorff distance. According to [26] Proposition 3.7, \((M^\delta,d_{G,H}^\delta)\) is a pseudometric space. Moreover, if we consider equivalence classes of isometric spaces, then it becomes a reversible metric space.

A sequence \(\{X_n\}_{n=1}^\infty \subset M^\delta\) converges to a compact metric space \(X \in M^\delta\) if \(d_{G,H}^\delta(X_n,X) \to 0\) as \(n \to \infty\). Thus, \((M^\delta,d_{G,H}^\delta)\) complete, that is, every Cauchy sequence in \(M^\delta\) is convergent (cf. [26] Theorem 2.4).

**Definition 2** For a compact metric space \(X\), define the capacity \(\text{Cap}_X\) and covering \(\text{Cov}_X\) as follows:
\[
\text{Cap}_X(\varepsilon) := \text{maximum number of disjoint forward } \frac{\varepsilon}{2}\text{-balls in } X,
\]
\[
\text{Cov}_X(\varepsilon) := \text{minimum number of forward } \varepsilon\text{-balls it takes to cover } X.
\]

**Proposition B1** ([26]) Let \((X,d)\) be a compact metric space with \(\lambda_d < \infty\). Then
\[
\text{Cap}_X\left(\frac{\varepsilon}{\lambda_d}\right) \geq \text{Cov}_X(\varepsilon),
\]
for all \(\varepsilon > 0\).

Given a compact metric space \(X\), let \(\text{diam } X := \sup_{x,y \in X} d(x,y)\). Then we have the following result.
Proposition B2 ([26]) Let \( C \subset (M^d, d_{GH}^\delta) \) be a class such that there exist a constant \( D > 0 \) and a function \( N = N(\epsilon) \) satisfying:

1. \( \text{diam } X \leq D, \forall X \in C \);
2. For each \( \epsilon > 0 \), \( \text{Cap}_X(\epsilon) \leq N(\epsilon), \forall X \in C \).

Then \( C \) is pre-compact in the \( \delta \)-Gromov-Hausdorff topology.

Now we recall the definition of LGC spaces. Also refer to [24,39] for more details.

Definition 3 A contractibility function \( \rho : [0,r) \to [0,\infty) \) is a function satisfying:

a. \( \rho(0) = 0 \),

b. \( \rho(\epsilon) \geq \epsilon \),

c. \( \rho(\epsilon) \to 0 \) as \( \epsilon \to 0 \),

d. \( \rho \) is nondecreasing.

A metric space \( X \) is LGC(\( \rho \)) for some contractibility function \( \rho \), if for every \( \epsilon \in [0,r) \) and \( x \in X \), the forward ball \( B^+_x(\epsilon) \) is contractible inside \( B^+_x(\rho(\epsilon)) \).

Theorem 11 ([39]) Let \( (M,F) \) be a forward complete Finsler manifold with \( c_M \geq R > 0 \).

Then \( (M,F) \) is LGC(\( \rho \)), where \( \rho : [0,R) \to [0,R) \) is the identity map and \( c_M \) is the contractibility radius.

Proposition B3 ([39]) Fix a function \( N : (0,\alpha) \to (0,\infty) \) with

\[ \lim_{\epsilon \to 0^+} N(\epsilon) < \infty \]

and a contractibility function \( \rho : [0,r) \to [0,\infty) \). The class

\[ \mathcal{C}(N,\rho) := \{ X \in M^d : X \text{ is LGC}(\rho), \text{ Cov}(X,\epsilon) \leq N(\epsilon) \text{ for all } \epsilon \in (0,\alpha) \} \]

contains only finitely many homotopy types.

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References
1. Alvarez-Paiva, J., Berck, G.: What is wrong with the Hausdorff measure in Finsler spaces. Adv. in Math. 204, 647-663 (2006)
2. Alvarez-Paiva, J., Thompson, A.C.: Volumes in normed and Finsler spaces. In: A Sampler of Riemann-Finsler geometry (Cambridge) (D. Bao, R. Bryant, S.S. Chern, and Z. Shen, eds.), pp. 1-49. Cambridge University Press, (2004)
3. Burago, D., Burago, Y., Ivanov, S.: A Course in Metric Geometry, American Mathematical Society (2001)
4. Bao, D., Chern, S.S., Shen, Z.: An introduction to Riemannian-Finsler geometry, GTM 200. Springer-Verlag (2000)
5. Beem, J.K., Ehrlich, P.E.: Cut points, conjugate points and Lorentzian comparison theorems. Math. Proc. Cambridge Philos. Soc. 86, 365-384 (1979)
6. Cheeger, J., Colding, T.H.: Lower bounds on Ricci curvature and the almost rigidity of warped products. Ann. of Math., 144, 189-237 (1996)
7. Chen, B.: Some geometric and analysis problems in Finsler geometry, Doctoral thesis. Zhejiang University (2010)
8. Cheng, X., Shen, Z.: Finsler Geometry: An Approach via Randers Spaces. Springer (2013).
9. Croke, C.: Some isoperimetric inequalities and eigenvalue estimates. Ann. Sci. École Norm. Sup. 13, 419-435 (1980)
10. Egloff, D.: Uniform Finsler Hadamard manifolds. Ann. Inst. Henri Poincaré. 66, 323-357 (1997)
11. Hawking, S., and Ellis, G.: The large scale structure of space-time. Cambridge University Press (1973)
12. Hu, Z., Xu, S.: Bounds on the fundamental groups with integral curvature bound. Geom. Dedicata 143, 1-16 (2008)
13. Gromov, M., Lafontaine, J., Pansu, P.: Structures métriques pour les variété riemanniennes, rédigé par J. Lafontaine et P. Pansu, Cedic/Fernand Nathan, Paris (1981)
14. Kirby, P., Siebenmann, L.: Foundational essays on topological manifolds, smoothings and triangulations. Ann. Math. Stud. 88. Princeton University Press (1977)
15. Ohta, S.: Finsler interpolation inequalities. Calc. Var. Partial Differ. Equ. 36, 211-249 (2009)
16. Ohta, S.: Nonlinear geometric analysis on Finsler manifolds. Eur. J. Math. (to appear). Available at arXiv:1704.01257
17. Otha, S.: Optimal transport and Ricci curvature in Finsler geometry. In: Advances in Pure Mathematics, pp. 1-20. (2009)
18. Rademacher, H.: Nonreversible Finsler metrics of positive ag curvature. In: A sampler of Riemann-Finsler geometry, pp. 261-302. Cambridge Univ. Press, Cambridge (2004)
19. Petersen, P.: Riemannian geometry. Graduate Texts in Mathematics 171. Springer-Verlag, New York, Heidelberg, Berlin (1997)
20. Petersen, P., Sprouse, S.: Integral curvature bounds, distance estimates, and applications. Jour. Diff. Geom., 50(2), 269-298 (1998)
21. Petersen, P., Wei, G.: Relative volume comparison with integral curvature bounds. Geom. Funct. Anal., 7, 1031-1045 (1997)
22. Petersen, P., Wei, G.: Analysis and geometry on manifolds with integral curvature bounds. II. Trans. AMS 353(2), 457-478 (2000)
23. Sprouse, C.: Integral curvature bounds and bounded diameter. Comm. Anal. Geom. 8, 531-543 (2000)
24. Shen, Z.: Lectures on Finsler geometry. World Sci., Singapore (2001)
25. Shen, Z.: Volume comparison and its applications in Riemannian-Finsler geometry. Adv. in Math., 128, 306-328 (1997)
26. Shen, Y., Zhao, W.: Gromov pre-compactness theorems for nonreversible Finsler manifolds. Diff. Geom. Appl., 28, 565-581 (2010)
27. Shen, Y., Zhao, W.: On Fundamental Groups of Finsler Manifolds. Sci. China Math. 54, 1951-1964 (2011)
28. Shen, Y., Zhao, W.: Some results on fundamental groups and Betti numbers of Finsler manifolds. Int. J. Math. 23(6)(2012), https://doi.org/10.1142/S0129167X12500632
29. Wei, G.: Manifolds with a lower Ricci curvature bound. In: Surveys in differential geometry, Vol. XI 203-227. Int. Press, Somerville, MA (2007)
30. Wu, B., Xin, Y.: Comparison theorems in Finsler geometry and their applications. Math. Ann., 337, 177-196 (2007)
31. Wu, B.: Volume form and its applications in Finsler geometry. Publ. Math. Debrecen, 78, 725-741 (2011)
32. Wu, B.: On integral Ricci curvature and topology of Finsler manifolds. Int. J. Math. 23(11)(2012), https://doi.org/10.1142/S0129167X1250111X
33. Wu, B.: A note on the generalized Myers theorem for Finsler manifolds. Bull. Korean Math. Soc. 50, 833-837 (2013)
34. Wu, B.: Relative volume comparison theorems in Finsler geometry and their applications. Bull. Iran. Math. Soc. 40, 217-234 (2014)
35. Yang, D.: Convergence of Riemannian manifolds with integral bounds on curvature I. Ann. Sci. Écol. Norm. Sup. 25, 77-105 (1992)
36. Yamaguchi, T.: Homotopy type finiteness theorems for certain precompact families of Riemannian manifolds. Proc. Am. Math. Soc. 102, 660-666 (1988)
37. Yin, S.: Two compactness theorems on Finsler manifolds with Positive weighted Ricci curvature. Results Math. 72, 319-327 (2017)
38. Yuan, L., Zhao, W.: Some formulas of Santaló type in Finsler geometry and its applications. Publ. Math. Debrecen, 87, 79-101 (2015)
39. Zhao, W.: Homotopy Finiteness Theorems for Finsler Manifolds. Publ. Math. Debrecen, 83, 329-358 (2013)
40. Zhao, W., Shen, Y.: A Universal Volume Comparison Theorem for Finsler Manifolds and Related Results. Can. J. Math., 65, 1401-1435 (2013)
41. Zhao, W., Shen, Y.: A Cheeger finiteness theorem for Finsler manifolds. J. Math. Anal. Appl., 433(2), 1690-1717 (2016)