On the quasi-Hopf structure of deformed double Yangians

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Abstract

We construct universal twists connecting the centrally extended double Yangian $\mathcal{DY}(\mathfrak{sl}(2))_c$ with deformed double Yangians $\mathcal{DY}_r(\mathfrak{sl}(2))_c$, thereby establishing the quasi-Hopf structures of the latter.

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# 1 Introduction

Universal twists connecting (affine) quantum groups to (elliptic) (dynamical) (affine) algebras have been constructed in [1, 2, 3]. They show in particular the quasi-Hopf structure of elliptic and dynamical algebras. These twists transform the universal $R$-matrix $R$ of the first object into the universal $R$-matrix $R^F$ of the second one as follows:

$$R^F_{12} = F_{21}R_{12}F^{-1}_{12}.$$  (1.1)

The double degeneracy limits of elliptic $R$-matrices, whether vertex-type [4, 5, 6] or face-type [7], give rise to algebraic structures which have been variously characterised as scaled elliptic algebras [6, 3, 2], or double Yangian algebras [6, 3, 2]. As pointed out earlier [4, 6], although represented by formally identical Yang–Baxter relations $RLL = LLR$ [10], these two classes of objects differ fundamentally in their structures (as is reflected in the very different mode expansions of $L$ defining their individual generators) and must be considered separately.

In our previous paper [7] we have defined, in the quantum inverse scattering or RLL formulation, various algebraic structures of double Yangian type connected by twist-like operators, i.e. such that their evaluated $R$-matrices were related as:

$$R^F_{12} = F_{21}R_{12}F^{-1}_{12}$$  (1.2)

for a particular matrix $F$. It was conjectured that these twist-like operators were indeed evaluation representations of universal twists obeying a shifted cocycle condition thereby raising the relation (1.2) to the status of a genuine twist connection (1.1) between quasi-Hopf algebras.

We shall be concerned here only with algebraic structures related to the algebra $\hat{sl}(2)_c$, and henceforth dispense with indicating it explicitly: for instance $DY$ is thus to be understood as $DY(\hat{sl}(2)_c)$.

It is our purpose here to establish such connections, at the level of universal $R$-matrices, between the double Yangian structures respectively known as $DY$, $DY_r^{V6}$, $DY_r^{V8}$ and $DY_r^F$. $DY$ is the double Yangian defined in [4, 1]. $DY_r^{V6}$ is characterised by a scaled “elliptic” $R$-matrix defined in [4], $DY_r^{V8}$ is characterised by a scaled $R$-matrix defined in [4, 3]. In connection with our previous caveat, note that these $R$-matrices are also used to describe respectively the scaled elliptic algebras $A_{h,0}$, $A_{h,η}$ [3, 2, 4]. $DY_r^F$ is the deformed double Yangian obtained by a particular limit of the dynamical $R$-matrix characterising elliptic $B_{q,p,λ}$ algebra [4].

A crucial ingredient for our procedure is a linear (difference) equation obeyed by the twist. This type of equation for twist operators was first written in [12]. It is also present in [2, 3]. Our method consists in $i)$ finding a twist-like action in representation $ii)$ interpreting this representation as an infinite product $iii)$ defining a linear equation obeyed by this infinite product $iv)$ promoting this linear equation for the representation to the level of linear equation for universal twist. $v)$ The solution of this linear equation is obtained as a infinite product as in [2] which $vi)$ is then proved to obey the shifted cocycle condition as in [2, 3] and $vii)$ has an evaluation representation identical to the twist-like action found in $i$).

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1We wish to thank S. Pakuliak for clarifying this point to us.
This provides us with the universal $R$-matrix and quasi-Hopf structure of the twisted algebras $\mathcal{DY}_{rV^6,V^8,F}$, thereby realising a fully consistent description of these algebraic structures.

The universal $R$-matrix and Hopf algebra structure for $\mathcal{DY}$ were described in [9, 11]. We construct a universal twist between $\mathcal{DY}$ and $\mathcal{DY}_{rV^6}$. We then show the existence of a universal coboundary (trivial) twist, the evaluation of which realises the connection between the evaluated $R$-matrices of $\mathcal{DY}_{rV^6}$ and $\mathcal{DY}_{rV^8}$, leading to identification of these two as quasi-Hopf algebras. Finally another universal coboundary-like twist realises, when evaluated, the connection between the $R$-matrices of $\mathcal{DY}_{rV^6}$ and $\mathcal{DY}_{rF}$.

It follows that the three deformed structures are in fact one single quasi-Hopf algebra described by three different choices of generators, more precisely given in three different gauges.

We shall denote throughout this paper $\mathcal{F}[\mathcal{A};\mathcal{B}]$ the universal or represented twist connecting $R$-matrices as $R_{B} = \mathcal{F}_{21}[\mathcal{A};\mathcal{B}] \mathcal{R}_{A} \mathcal{F}_{12}^{-1}[\mathcal{A};\mathcal{B}]$.

### 2 Presentation of the double Yangians $\mathcal{DY}$ and $\mathcal{DY}_{r}$

#### 2.1 Double Yangian $\mathcal{DY}$

The double Yangian $\mathcal{DY}$ is defined by the $R$-matrix

\[
R(\beta) = \rho(\beta) \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & i\beta & \pi & 0 \\
0 & i\beta + \pi & i\beta & 0 \\
0 & \pi & 0 & 1
\end{pmatrix},
\]

with the normalisation factor

\[
\rho(\beta) = \frac{\Gamma(\frac{\beta}{\pi} | 2) \Gamma(2 + \frac{\beta}{\pi} | 2)}{\Gamma(1 + \frac{\beta}{\pi} | 2)^2},
\]

and the mode expansions

\[
L^{+}(\beta) = \sum_{k \geq 0} L^{+}_{k} \beta^{-k} \quad \text{and} \quad L^{-}(\beta) = \sum_{k \leq 0} L^{-}_{k} \beta^{-k}.
\]

It is important to point out that $L^{+}$ and $L^{-}$ are independent. There exists in this case a Gauss decomposition of the Lax matrices allowing for an alternative Drinfeld presentation [11]. Indeed, $L^{\pm}$ are decomposed as

\[
L^{\pm}(x) = \begin{pmatrix}
1 & f^{\pm}(x^{\mp}) \\
0 & 1
\end{pmatrix} \begin{pmatrix}
k^{\pm}_{1}(x) & 0 \\
0 & k^{\pm}_{2}(x)
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
e^{\pm}(x) & 1
\end{pmatrix}
\]

(2.6)
with \( x^+ \equiv x \equiv \frac{i\beta}{\pi} \) and \( x^- \equiv x - c \). Furthermore, \( k_1^+(x)k_2^+(x-1) = 1 \) and one defines \( h^\pm(x) \equiv k_2^\pm(x)^{-1}k_1^\pm(x) \).

The evaluation representation \( \pi_x \) is then easily defined by its action on a two-dimensional vector space by
\[
\pi_x(e_k) = x^k \sigma^+, \quad \pi_x(f_k) = x^k \sigma^-, \quad \pi_x(h_k) = x^k \sigma^3, \quad (2.7)
\]

where
\[
e^{\pm}(u) = \pm \sum_{k \geq 0 \atop k < 0} e_k u^{-k-1}, \quad f^{\pm}(u) = \pm \sum_{k \geq 0 \atop k < 0} f_k u^{-k-1}, \quad h^{\pm}(u) = 1 \pm \sum_{k \geq 0 \atop k < 0} h_k u^{-k-1}. \quad (2.8)
\]

### 2.2 Deformed double Yangian \( \mathcal{D}Y_r^{V6} \)

The \( R \)-matrix of the deformed double Yangian \( \mathcal{D}Y_r^{V6} \) is related to the two-body \( S \) matrix of the sine–Gordon theory \( S_{SG}(\beta, r) \) and is given by
\[
R_{V6}(\beta, r) = \cotg(\frac{i\beta}{2}) S_{SG}(\beta, r) = \rho_r(\beta) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin \frac{i\beta}{r} & \sin \frac{\pi}{r} & 0 \\ 0 & \sin \frac{\pi}{r} & \sin \frac{i\beta}{r} & 0 \\ 0 & \sin \frac{\pi+i\beta}{r} & \sin \frac{\pi+i\beta}{r} & 1 \end{pmatrix}, \quad (2.9)
\]

where the normalisation factor is
\[
\rho_r(\beta) = \frac{S_2^2(1 + \frac{i\beta}{r} \mid r, 2)}{S_2^2(\frac{i\beta}{r} \mid r, 2) S_2^2(2 + \frac{i\beta}{r} \mid r, 2)}. \quad (2.10)
\]

\( S_2(x \mid \omega_1, \omega_2) \) is Barnes’ double sine function of periods \( \omega_1 \) and \( \omega_2 \) defined by:
\[
S_2(x \mid \omega_1, \omega_2) = \frac{\Gamma_2(\omega_1 + \omega_2 - x \mid \omega_1, \omega_2)}{\Gamma_2(x \mid \omega_1, \omega_2)}, \quad (2.11)
\]

where \( \Gamma_r \) is the multiple Gamma function of order \( r \) given by
\[
\Gamma_r(x \mid \omega_1, \ldots, \omega_r) = \exp \left( \frac{\partial}{\partial s} \sum_{n_1, \ldots, n_r \geq 0} (x + n_1 \omega_1 + \cdots + n_r \omega_r)^{-s} \right) \bigg|_{s=0}. \quad (2.12)
\]

The algebra \( \mathcal{D}Y_r^{V6} \) is defined by
\[
R_{12}(\beta_1 - \beta_2) L_1(\beta_1) L_2(\beta_2) = L_2(\beta_2) L_1(\beta_1) R_{12}^*(\beta_1 - \beta_2), \quad (2.13)
\]

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where \( R_{12}^*(\beta, r) \equiv R_{12}(\beta, r - c) \).

The Lax matrix \( L \) must now be expanded on both positive and negative powers as
\[
L(\beta) = \sum_{k \in \mathbb{Z}} L_k \beta^{-k}.
\] (2.14)

A presentation similar to the double Yangian case is achieved by introducing the following two matrices:
\[
L^+(\beta) \equiv L(\beta - i\pi c),
\] (2.15)
\[
L^-(\beta) \equiv \sigma_3 L(\beta - i\pi r) \sigma_3.
\] (2.16)

They obey coupled exchange relations following from (2.13) and periodicity/unitarity properties of the matrices \( R_{12} \) and \( R_{12}^* \):
\[
R_{12}(\beta_1 - \beta_2) L_1^+(\beta_1) L_2^+(\beta_2) = L_2^+(\beta_2) L_1^+(\beta_1) R_{12}^*(\beta_1 - \beta_2),
\] (2.17)
\[
R_{12}(\beta_1 - \beta_2 - i\pi c) L_1^+(\beta_1) L_2^-(\beta_2) = L_2^-(\beta_2) L_1^+(\beta_1) R_{12}(\beta_1 - \beta_2).
\] (2.18)

Contrary to the case of the double Yangian, the matrices \( L^+ \) and \( L^- \) are not independent. Note also that, due to conflicting conventions, the \( r \to \infty \) limit of \( L^\pm \) in \( DY^V_{r^6} \) corresponds to \( L^\mp \) in \( DY \).

### 2.3 Deformed double Yangian \( DY^V_{r^8} \)

The \( R \)-matrix of the deformed double Yangian \( DY^V_{r^8} \) is obtained as the scaling limit of the \( R \)-matrix of the elliptic algebra \( A_{q,p} \) \[4,5\]. It reads
\[
R_{V8}(\beta, r) = \rho_r(\beta) \begin{pmatrix}
\cos \frac{\beta i}{2r} & \cos \frac{\pi}{2r} & 0 & 0 & -\sin \frac{\beta i}{2r} & \sin \frac{\pi}{2r} \\
-\sin \frac{\beta i}{2r} & \sin \frac{\pi + \beta i}{2r} & \cos \frac{\beta i}{2r} & \cos \frac{\pi i}{2r} & 0 & 0 \\
0 & -\sin \frac{\beta i}{2r} & \sin \frac{\pi}{2r} & \cos \frac{\beta i}{2r} & \cos \frac{\pi + \beta i}{2r} & 0 \\
0 & 0 & \sin \frac{\beta i}{2r} & \sin \frac{\pi + \beta i}{2r} & \cos \frac{\beta i}{2r} & \cos \frac{\pi}{2r} \\
-\sin \frac{\beta i}{2r} & \sin \frac{\pi}{2r} & 0 & 0 & \cos \frac{\beta i}{2r} & \cos \frac{\pi i}{2r} \\
\cos \frac{\beta i}{2r} & \cos \frac{\pi}{2r} & 0 & 0 & -\sin \frac{\beta i}{2r} & \sin \frac{\pi i}{2r}
\end{pmatrix}.
\] (2.19)

It is also obtained from the \( R \)-matrix of \( DY^V_{r^6} \) by a gauge transformation \[4\]. The algebra \( DY^V_{r^8} \) is defined by the same relation as \( DY^V_{r^6} \), albeit with the matrix \( R_{V8} \), and the same type of Lax matrix with positive and negative modes.
2.4 Deformed double Yangian $\mathcal{DY}_F^r$

The $R$-matrix of $\mathcal{DY}_F^r$ is given by

$$R(\beta; r) = \rho_r(\beta) \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \sin \frac{i\beta}{r} & 0 & e^{\beta/r} \\
0 & 0 & \sin \frac{\pi + i\beta}{r} & 0 \\
0 & e^{-\beta/r} & \sin \frac{\pi + i\beta}{r} & 1
\end{pmatrix}.$$  \hspace{1cm} (2.20)

The normalisation factor is the same as for $\mathcal{DY}_r^{V6}$. The definition of the algebra and the Lax operator are again formally identical.

3 Twist from $\mathcal{DY}$ to $\mathcal{DY}_r$: representation formula

3.1 A notation for $P_{12}$ invariant matrices

Let us denote by $M(b^+, b^-)$ the $4 \times 4$ matrix given by

$$M(b^+, b^-) \equiv \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{2}(b^+ + b^-) & \frac{1}{2}(b^+ - b^-) & 0 \\
0 & \frac{1}{2}(b^+ - b^-) & \frac{1}{2}(b^+ + b^-) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$  \hspace{1cm} (3.1)

With this definition, we have $M(a, b)M(a', b') = M(aa', bb')$ and $M(a, b)^{-1} = M(a^{-1}, b^{-1})$.

Now,

$$R[\mathcal{DY}](\beta) = \rho(\beta)M \left(1, \frac{i\beta - \pi}{i\beta + \pi}\right).$$  \hspace{1cm} (3.2)

We have $R[\mathcal{DY}_r^{V6}](\beta) = \rho_r(\beta)M(b_r^+, b_r^-)$, with

$$b_r^+ = \frac{\cos \frac{i\beta - \pi}{2r}}{\cos \frac{i\beta + \pi}{2r}} = \frac{\Gamma_1(r + \frac{i\beta}{\pi} + 1|2r)\Gamma_1(r - \frac{i\beta}{\pi} - 1|2r)}{\Gamma_1(r + \frac{i\beta}{\pi} - 1|2r)\Gamma_1(r - \frac{i\beta}{\pi} + 1|2r)},$$ \hspace{1cm} (3.3)

$$b_r^- = \frac{\sin \frac{i\beta - \pi}{2r}}{\sin \frac{i\beta + \pi}{2r}} = \frac{\Gamma_1(\frac{i\beta}{\pi} + 1|2r)\Gamma_1(2r - \frac{i\beta}{\pi} - 1|2r)}{\Gamma_1(\frac{i\beta}{\pi} - 1|2r)\Gamma_1(2r - \frac{i\beta}{\pi} + 1|2r)},$$ \hspace{1cm} (3.4)

$$= \frac{\Gamma_1(2r + \frac{i\beta}{\pi} + 1|2r)\Gamma_1(2r - \frac{i\beta}{\pi} - 1|2r)}{\Gamma_1(2r + \frac{i\beta}{\pi} - 1|2r)\Gamma_1(2r - \frac{i\beta}{\pi} + 1|2r)} \cdot \frac{i\beta - \pi}{i\beta + \pi}. $$ \hspace{1cm} (3.5)
3.2 The linear equation in representation

We remark that the normalisation factor of $\mathcal{D}Y_r^V$ can be rewritten as:

$$\rho_r(\beta) = \rho_F(-\beta; r)\rho(\beta)\rho_F(\beta; r)^{-1}$$  \hspace{1cm} (3.6)

with

$$\rho_F(\beta) = \frac{\Gamma_2(i\frac{\beta}{\pi} + 1 + r | 2, r)^2}{\Gamma_2(i\frac{\beta}{\pi} + r | 2, r)\Gamma_2(i\frac{\beta}{\pi} + 2 + r | 2, r)}.$$  \hspace{1cm} (3.7)

Equations (3.2-3.6) allow us to write:

$$R[\mathcal{D}Y_r^V] = F_{21}(-\beta)R[\mathcal{D}Y]F_{12}(\beta)^{-1}.$$  \hspace{1cm} (3.8)

Using the notation (3.1), $F_{12}(\beta)$ is given by

$$F_{12}(\beta) = \rho_F(\beta) \cdot M\left(\frac{\Gamma_1(i\frac{\beta}{\pi} + r - 1 | 2r)}{\Gamma_1(i\frac{\beta}{\pi} + r + 1 | 2r)}, \frac{\Gamma_1(i\frac{\beta}{\pi} + 2r - 1 | 2r)}{\Gamma_1(i\frac{\beta}{\pi} + 2r + 1 | 2r)}\right).$$  \hspace{1cm} (3.9)

This twist-like matrix reads

$$F_{12}(\beta) = \rho_F(\beta) \prod_{n=1}^{\infty} M\left(1, \frac{i\beta + \pi + 2n\pi r}{i\beta - \pi + 2n\pi r}\right) M\left(\frac{i\beta + \pi + (2n - 1)\pi r}{i\beta - \pi + (2n - 1)\pi r}, 1\right)$$  \hspace{1cm} (3.10)

$$= \prod_{n=1}^{\infty} R(\beta - i(2n)\pi r)^{-1} \tau(R(\beta - i(2n - 1)\pi r)^{-1})$$  \hspace{1cm} (3.11)

with

$$\tau(M(a, b)) = M(b, a)$$  \hspace{1cm} (3.12)

and where, unless differently specified, $R$ is the $R$-matrix of $\mathcal{D}Y$. One uses here the representation of $\rho_F(\beta)$ as an infinite product

$$\rho_F(\beta) = \prod_{n=1}^{\infty} \rho(\beta - in\pi r)^{-1}.$$  \hspace{1cm} (3.13)

The automorphism $\tau$ may be interpreted as the adjoint action of $(-1)^{\frac{1}{2}h^{(1)}_0}$, so that

$$F_{12}(\beta) = \prod_{n=1}^{\infty} R(\beta - i(2n)\pi r)^{-1} \text{Ad}\left((-1)^{\frac{1}{2}h^{(1)}_0}\right) R(\beta - i(2n - 1)\pi r)^{-1}$$  \hspace{1cm} (3.14)

$$= \prod_{n=1}^{\infty} \text{Ad}\left((-1)^{\frac{1}{2}h^{(1)}_0}\right) R(\beta - in\pi r)^{-1}.$$  \hspace{1cm} (3.15)

Hence $F$ is solution of the difference equation

$$F(\beta - i\pi r) = (-1)^{-\frac{1}{2}h^{(1)}_0} F(\beta)(-1)^{\frac{1}{2}h^{(1)}_0} \cdot R(\beta - i\pi r) .$$  \hspace{1cm} (3.16)

It would be tempting to relate the automorphism $\tau$ to the one used in [3], although the naive scaling of the latter does not give back the former. For instance our $\tau$ is inner not outer.

All the infinite products are logarithmically divergent. They are consistently regularised by the $\Gamma_1$ and $\Gamma_2$ functions. In particular, $\lim_{r \to \infty} F = M(1, 1) = \mathbb{I}_4$. 

6
4 The universal form of $\mathcal{F}[DY; DY^V_r]$

We construct a universal twist $\mathcal{F}$ from $DY$ to $DY^V_r$, such that

$$F(\beta_1 - \beta_2) = \pi_{\beta_1} \otimes \pi_{\beta_2}(\mathcal{F}).$$

(4.1)

The form of the difference equation (3.16) obeyed by the conjectural representation of the twist, together with the known generic structures of linear equations obeyed by universal twists [12, 4, 5] lead us to postulate the following linear equation for $\mathcal{F}$:

$$\mathcal{F} \equiv \mathcal{F}(r) = \text{Ad}(\phi^{-1} \otimes \mathbb{I})(\mathcal{F}) \cdot C$$

(4.2)

with

$$\phi = (-1)^{\frac{1}{2}h_0}e^{(r+c)d},$$

(4.3)

$$C \equiv e^{-\alpha c \otimes d - \gamma d \otimes c}R.$$  

(4.4)

We now prove the consistency of these postulates. We will use the following preliminary properties:

- The operator $d$ in the double Yangian $DY$ is defined by $[d, e(u)] = \frac{d}{du}e(u)$ (see [11]). The evaluation representations are related through $\pi_{\beta + \beta'} = \pi_{\beta} \circ \text{Ad}(\exp(\frac{\beta'}{\pi}d))$.

- The operator $d$ satisfies $\Delta(d) = d \otimes 1 + 1 \otimes d$.

- The generator $h_0$ of $DY$ is such that

$$h_0 e(u) = e(u)(h_0 + 2), \quad h_0 f(u) = f(u)(h_0 - 2), \quad [h_0, h(u)] = 0,$$

(4.5)

and hence $\tau = \text{Ad} \left((-1)^{\frac{1}{2}h_0^{(1)}}\right)$ satisfies $\tau^2 = 1$.

The equation (4.2) can be solved by

$$\mathcal{F}(r) = \prod_k \mathcal{F}_k(r), \quad \mathcal{F}_k(r) = \phi_k C_{12}^{-1} \phi_{-k}^{-1}.$$  

(4.6)

It is easily seen that equation (3.13) is the evaluation representation of this universal formula. As in [3], $\mathcal{F}_k$ satisfy the following properties:

$$(\Delta \otimes \text{id})(\mathcal{F}_k(r)) = \mathcal{F}_k^{(23)}(r + c_1)\mathcal{F}_k^{(13)}(r + c_2 + \frac{\alpha}{k}c_2),$$

(4.7)

$$(\text{id} \otimes \Delta)(\mathcal{F}_k(r)) = \mathcal{F}_k^{(12)}(r)\mathcal{F}_k^{(13)}(r - \frac{\gamma}{k}c_2),$$

(4.8)

and

$$\mathcal{F}_{k+l}^{(12)}(r)\mathcal{F}_{k+l}^{(13)}(r + \frac{l - \gamma}{k + l}c_2)\mathcal{F}_l^{(23)}(r + c_1) = \mathcal{F}_l^{(23)}(r + c_1)\mathcal{F}_{k+l}^{(13)}(r + \frac{l + \alpha}{k + l}c_2)\mathcal{F}_k^{(12)}(r).$$

(4.9)
It is then straightforward to follow [3] to prove the shifted cocycle relation, provided that \( \alpha + \gamma = -1 \). We then have

\[
\mathcal{F}^{(12)}(r)(\Delta \otimes \text{id})(\mathcal{F}(r)) = \mathcal{F}^{(23)}(r + c^{(1)}) (\text{id} \otimes \Delta)(\mathcal{F}(r)). \tag{4.10}
\]

It follows that \( R_{DY^V 12} = F_{21} R_{12} F^{-1} \) satisfies a shifted Yang–Baxter equation

\[
R_{12}(r + c^{(3)}) R_{13}(r) R_{23}(r + c^{(1)}) = R_{23}(r) R_{13}(r + c^{(2)}) R_{12}(r), \tag{4.11}
\]

and that \( DY^V_r \) is a quasi-Hopf algebra with \( \Delta \mathcal{F}(x) = \mathcal{F}(x) \mathcal{F}^{-1} \) and \( \Phi_{123} = \mathcal{F}_{23}(r) \mathcal{F}_{23}(r + c^{(1)})^{-1} \).

5 Twist to \( DY_r^V 8 \)

5.1 In representation

The \( R \)-matrix of \( DY_r^V 6 \) and \( DY_r^V 8 \) are related by

\[
R[DY_r^V 8] = K_{21} R[DY_r^V 6] K_{12}^{-1}, \tag{5.1}
\]

where

\[
K = V \otimes V \quad \text{with} \quad V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{5.2}
\]

This implies an isomorphism between \( DY_r^V 8 \) and \( DY_r^V 6 \) where the Lax operators are connected by \( \mathcal{L}_V 8 = \mathcal{L}_V 6 V^{-1} \).

5.2 Universal form

We identify \( V \) with an evaluation representation of an element \( g \)

\[
V \equiv \pi_x(g) \quad \text{with} \quad g = \exp \left( \frac{\pi}{2} (f_0 - e_0) \right). \tag{5.3}
\]

Since \( e_0 \) and \( f_0 \) lie in the undeformed Hopf subalgebra \( sl(2) \) of \( DY \), the coproduct of \( g \) reads

\[
\Delta(g) = g \otimes g \tag{5.4}
\]

so that

\[
g_1 g_2 \Delta^F(g^{-1}) F = g_1 g_2 F g_1^{-1} g_2^{-1}. \tag{5.5}
\]

The two-cocycle \( g_1 g_2 \Delta^F(g^{-1}) \) is a coboundary (with respect to the coproduct \( \Delta^F \)). In representation, \( \mathbf{5.3} \) is equal to the scaling limit of the represented twist from \( \mathcal{U}_q \) to \( \mathcal{A}_q \). \( \mathbf{5.1} \).

Note that this case is similar to the gauge transformation used in \( \mathbf{13} \) although \( g \) is not purely Cartan. It follows that

\[
R[DY_r^V 8] \equiv g_1 g_2 \Delta^F_{21}(g^{-1}) R[DY_r^V 6] \Delta^F_{12}(g) g_1^{-1} g_2^{-1} \tag{5.6}
\]

satisfies the shifted Yang–Baxter equation \( \mathbf{11.11} \).

To recover \( \mathbf{5.1} \), use \( \mathbf{5.3} \) and remark that \( \pi_x \otimes \pi_x(g \otimes g) \) commutes with \( R[DY] \).
6 Twist to $ DY^F_r $  

6.1 Twist in representation  

The $ R $-matrices of $ DY_r^V $ and $ DY_r^F $ are related by:  

$$ R[DY_r^F](\beta_1 - \beta_2) = K_{21}^{(6)}(\beta_2, \beta_1) R[DY_r^V](\beta_1 - \beta_2) (K_{12}^{(6)})^{-1}(\beta_1, \beta_2) , $$  

where  

$$ K^{(6)}(\beta_1, \beta_2) = V'(\beta_1) \otimes V'(\beta_2) \quad \text{with} \quad V'(\beta) = \begin{pmatrix} e^{\frac{\beta}{2r}} & 0 \\ 0 & e^{-\frac{\beta}{2r}} \end{pmatrix} . $$  

6.2 Universal twist  

Again, one identifies $ V'(\beta) $ as the evaluation representation of an algebra element  

$$ V'(\beta) = \pi_{\beta}(g') , $$  

where  

$$ g' = \exp \left( \frac{h_1}{2r} \right) . $$  

One then defines the following shifted coboundary  

$$ K_{12}(r) = g'(r) \otimes g'(r + c^{(1)}) \Delta^F(g'^{-1}) . $$  

It obeys a shifted cocycle condition  

$$ K_{12}(r) (\Delta^F \otimes \text{id}) K(r) = K_{23}(r + c^{(1)}) (\text{id} \otimes \Delta^F) K(r) , $$  

with $ F'_{23}(r) = F_{23}(r + c^{(1)}) $, as a consequence of  

$$ (\Delta^F \otimes \text{id}) \Delta^F(g'^{-1}) = (\text{id} \otimes \Delta^F) \Delta^F(g'^{-1}) , $$  

which is the coassociativity property for $ \Delta^F $.  

Finally  

$$ R[DY_r^F] \equiv K_{21}(r) R[DY_r^V] K^{-1}_{12}(r) $$  

satisfies the shifted Yang–Baxter equation (4.11). Moreover, (6.8) together with (6.3) show that $ DY_r^F $ and $ DY_r^V $ are the same quasi-Hopf algebra.  

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