Leading logarithms of the $b$ quark mass
in inclusive $B \rightarrow X_s \gamma$ decay*

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Abstract

Part of the order $\alpha_s$ correction to the inclusive $B \rightarrow X_s \gamma$ photon spectrum is enhanced by $\log(m_b^2/m_s^2)$. We discuss its origin and sum the corrections proportional to $[\alpha_s \log(m_b^2/\mu^2)]^n$ to all orders. These are the calculable leading logarithms in the parton fragmentation functions of a quark or gluon into a photon. Although the gluon fragmentation into a photon starts only at order $\alpha_s^2$, its contribution is of the same order as the $s$ quark’s in the leading log sum. For not too small values of the photon energy, the resummation yields a moderate suppression. In the standard model, the coefficient of the operator whose matrix element gives rise to such terms is small. A measurement of the photon spectrum around 1 GeV would provide a theoretically clean determination of $C_8$, the Wilson coefficient of the $b \rightarrow s g$ operator.

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I. INTRODUCTION

Inclusive $B \to X_s \gamma$ decay is very sensitive to physics beyond the standard model (SM) [1–6]. As any flavor changing neutral current process, it can only arise at one-loop level in the SM, and therefore possible new physics can yield comparable contributions. As the recent CLEO measurement [7] excludes large deviations from the SM, a next-to-leading order calculation is essential. The absence of such a calculation will soon become the major obstacle in improving the bounds on new physics from this process. Certain parts of this computation have already been carried out [4,8], but the technically most challenging parts are yet to be calculated.

It was observed recently [9] that moments of the photon spectrum can be predicted to order $\alpha_s$ accuracy by knowing the coefficients of the effective Hamiltonian only to the presently available leading logarithmic accuracy. This will provide a model independent determination of the $b$ quark pole mass (see also [10]), that is, the $\bar{\Lambda}$ and $\lambda_1$ matrix elements of the heavy quark effective theory.

In this paper we point out that part of the next-to-leading order correction to the photon spectrum induced by the $b \to s g$ operator is enhanced by $\log(m_b^2/m_s^2)$. We describe these contributions in section II. In section III we resum the series of leading logarithms of the form $[\alpha_s \log(m_b^2/\mu^2)]^n$. We discuss the meaning of the infrared scale $\mu$ and how nonperturbative phenomena enter the results. The resummation of leading logarithms does not eliminate the logarithmic dependence on the $b$ quark mass, which formally cancels the $\alpha_s$ suppression of this term. In section IV we discuss possible phenomenological implications.

II. $\log(m_b^2/m_s^2)$ TERMS IN THE ORDER $\alpha_{em} \alpha_s$ RESULT

In the standard model, $B \to X_s \gamma$ decay is mediated by penguin diagrams. QCD corrections to this process form a power series in the parameter $\alpha_s \log(M_W^2/m_b^2)$, which is too large to provide a reliable expansion. Therefore, it is convenient to integrate out the virtual
top quark and \( W \) boson effects (and possible new physics) at the \( W \) scale, and sum up the large logarithms using the operator product expansion and the renormalization group. We work with the operator basis and effective Hamiltonian of Ref. [2]

\[
H_{\text{eff}} = -\frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_{i=1}^{8} C_i(\mu) O_i(\mu) .
\] (2.1)

For the present paper only two of these operators are directly relevant

\[
O_7 = \frac{e}{16\pi^2} \bar{s}_\alpha \sigma^{\mu\nu} F_{\mu\nu} (m_b P_R + m_s P_L) b_\alpha ,
\]

\[
O_8 = \frac{g}{16\pi^2} \bar{s}_\alpha \sigma^{\mu\nu} G_{\mu\nu} T^a_{\alpha\beta} (m_b P_R + m_s P_L) b_\beta ,
\] (2.2)

where \( \alpha, \beta \) are color indices, and \( P_{R,L} = \frac{1}{2} (1 \pm \gamma_5) \). Throughout our discussion \( C_7 \) and \( C_8 \) will refer to the “effective” Wilson coefficients [3] that include the leading order contribution of the four quark operators.

We are interested in the order \( \alpha_s \) correction proportional to \( C_8^2 \), which is the order \( \alpha_{em} \) corrections to \( B \to X_s g \). The finite part of the spectrum is given by

\[
\left. \frac{d\Gamma_{88}}{dx} \right|_{x>0} = \Gamma_0 C_8^2 \frac{\alpha_s C_F}{4\pi} \left\{ \left( \frac{4+4r}{x-rx} - 4 + 2x \right) \log \frac{1-x+rx}{r} \right. \\
- \left( 1-x \right) \left[ 8 - (1-r) x (16 - 9x + 7rx) + (1-r)^2 x^3 (1-2x) \right] \\
\left. \frac{x}{(1-x+rx)^2} \right\}.
\] (2.3)

Here \( C_F = 4/3 \) in \( SU(3) \),

\[
\Gamma_0 = \frac{G_F^2 |V_{tb} V_{ts}^*|^2 \alpha_{em} C_7^2 m_b^5}{32 \pi^4} (1-r)^3 (1+r) ,
\] (2.4)

is the leading order contribution to the \( B \to X_s \gamma \) decay rate given by the matrix element of \( O_7 \), and we introduced the dimensionless parameters

\[
x = \frac{2E_\gamma}{(1-r)m_b} , \quad r = \frac{m_s^2}{m_b^2} .
\] (2.5)

The variable \( x \) corresponds to \( E_\gamma/E_{\gamma}^{\text{max}} \) in the free quark decay model. The extra factor of 9 suppression in eq. (2.3) results from the square of the \( b \) or \( s \) quark’s electric charge.

This contribution to the total \( B \to X_s \gamma \) decay rate is infinite. The divergence is canceled by the virtual correction to the \( B \to X_s g \) decay mode (with no photon in the final state).
While soft photons could be summed to all orders in $\alpha_{em}$ to make the $E_\gamma \to 0$ limit smooth, in finite orders of perturbation theory, it only makes sense to talk about the total inclusive $B \to X_s \gamma$ decay rate with an explicit lower cut-off on the photon energy.

The more interesting feature of the spectrum in eq. (2.3) is that it is not finite in the $m_b/m_s \to \infty$ ($r \to 0$) limit. In particular, the photon spectrum calculated in dimensional regularization with vanishing light quark masses would contain a part proportional to $1/(D-4)$ for any value of $E_\gamma$. This is a consequence of the fact that a massless quark cannot be distinguished from the same massless quark and one (or more) collinear massless partons with the same total charge and energy. The singularity is eliminated once a nonzero quark mass is included, but it results in the above logarithmic dependence of the spectrum on $r$.

Pictorially, this is simplest to understand by looking at the Dalitz plot in the $r \to 0$ limit (in the $E_\gamma - E_g$ variables), shown in Fig. 1. The shaded area represents the available phase space, the thick line denotes where collinear singularities occur, and the black box indicates where the virtual corrections contribute. Fig. 1/a corresponds to the square of the invariant matrix element of $O_7$. In this case both the collinear singularity and the virtual correction occur at $E_\gamma = m_b/2$. For this contribution, there is no problem whatsoever carrying out the calculation in the $m_s = 0$ limit. The reason is that the kinematic variable $(1-x)$ regularizes all singularities, and one finds the usual picture of Sudakov exponentiation near $E_\gamma = m_b/2$. However, the situation is more complicated for the contribution of the square of the invariant matrix element of $O_8$, shown in Fig. 1/b. In this case, collinear singularities occur at any value of $E_\gamma$; these are only canceled by the singular part of the virtual correction in the total decay rate, but not in the photon spectrum.

If the hadronic final state $X_s$ formed a jet, by excluding photons within a certain angular cut around the jet axis, the above discussed collinear singularity would be eliminated. However, at CLEO $B$ meson decay products are distributed in angle. Thus, when the experimental analysis excludes events in which a photon and a charged particle overlap in the detector, this cut is not related to the direction of the decay product $s$ quark at short distances.
III. RESUMMATION

To identify the origin of the leading logarithms of the form \( \alpha_s \log(\frac{m_b^2}{\mu^2})^n \), and to resum them to all orders, we observe that the analytic structure and \( r \)-dependence of the \( O_8 \) contribution is very similar to the photon structure function, as discussed in \[11\]. In the latter case one is interested in a hard photon scattering off another (almost) on-shell photon, while in the present problem an on-shell photon is created in the decay of a heavy particle.

We work to fixed (first) order in \( \alpha_{em} \) and to all orders in \( \alpha_s \). It is simplest to consider the discontinuity of the forward scattering amplitude corresponding to intermediate states with an \( s \) quark, a photon, and any number of light quarks (\( u, d, \) and \( s \)) and gluons. In an axial gauge the leading logarithms come from those diagrams in which at order \( \alpha_s^n \) there are precisely \( n \) ways of separating the external photon legs from the other external legs by cutting two internal lines (see, \( e.g. \), \[12\]). This implies that the result is given by the sum of two sets of ladder diagrams. One corresponds to the decay function of the \( s \) quark into a photon, and the other to that of the gluon. This second set of ladder diagrams starts only at order \( \alpha_s^2 \), but also yields leading logs. From these arguments it also follows that only the square of the \( O_8 \) operator’s invariant matrix element can yield these type of leading logs.

Decay functions are typically not calculable from first principles; moreover, they are usually divergent in perturbation theory. As the photon only couples to massive partons (quarks), in a perturbative calculation the light quark masses provide an infrared cut-off. That is why eq. (2.3) in the previous section contains a logarithmic dependence on \( m_s \). Higher order corrections would also include terms proportional to \( \log m_u \) and \( \log m_d \). However, in real QCD there is no such dependence on the light quark masses. Non-perturbative decay functions are well-defined, measurable physical quantities.

The crucial observation of Ref. \[11\] was that for a parton interacting with an on-shell photon, the absolute normalization of the leading logarithms can be calculated in perturbative QCD. (In other cases only the \( Q^2 \) evolution is calculable.) This large \( Q^2 \) (or large \( m_b^2 \)) behavior is determined by the matrix element of the photon operators between the
external photon states. The contributions of quark and gluon operators are logarithmically suppressed. To sum the leading log ladders, one introduces an infrared scale $\mu$, which is appropriately chosen to be a few times $\Lambda_{\text{QCD}}$. The question of the $\mu$-dependence is then non-leading in $\log m_b$. Addressing it consistently involves a combination of perturbative and non-perturbative (i.e., decay functions deduced from other experiments) terms of comparable magnitudes. All strongly coupled long distance effects are absorbed into the definitions of the decay functions.

The resummation of these leading logarithms is important, since $m_b \gg \mu$ implies $\log(m_b^2/\mu^2) \sim 1/\alpha_s(m_b)$. Only in this limit can inclusive decay rates be calculated model-independently, based on an operator product expansion \[13\]. The result of the resummation of the leading logarithms is given by the sum of the decay functions of the $s$ quark and the gluon into a photon

$$d\Gamma_{s\rightarrow\gamma} \Big|_{\text{resummed}} = \Gamma(b \rightarrow s g) \times [D^{s\rightarrow\gamma}(x) + D^{g\rightarrow\gamma}(x)].$$

(3.1)

Here $\Gamma(b \rightarrow s g) = \Gamma_0(\alpha_s C_F C_8^2/\alpha_{\text{em}} C_7^2)$ is the tree-level $b \rightarrow s g$ decay rate. The calculable leading logarithmic part of the decay functions are simplest to write down in terms of their moments $n \geq 2$ \[14\]

$$\int_0^1 dx \, x^{n-1} D^{s\rightarrow\gamma}(x) = \frac{\alpha_{\text{em}} \log Q^2}{2\pi} V_n \left[ e_s^2 - \langle e_q^2 \rangle + \frac{\langle e_q^2 \rangle K_n}{1 + d_{n+}^g} \right],$$

(3.2a)

$$\int_0^1 dx \, x^{n-1} D^{g\rightarrow\gamma}(x) = \frac{\alpha_{\text{em}} \log Q^2}{2\pi} V_n \frac{\langle e_q^2 \rangle d_{n+}^g K_n}{K_n}.$$  

(3.2b)

Here $V_n = \int_0^1 dx \, x^{n-2} [1 + (1-x)^2]$, $e_s = -\frac{1}{3}$ is the $s$ quark’s electric charge, and $\langle e_q^2 \rangle$ denotes the average of the squares of the light quarks’ charges. The $K_n$ and the $d_{n+}$’s are related to the anomalous dimension matrix of the quark and gluon operators \[15\].

In the absence of QCD, $D^{s\rightarrow\gamma}(x) = (\alpha_{\text{em}} e_s^2/2\pi) \log Q^2 [1 + (1-x)^2]/x$ and $D^{g\rightarrow\gamma}(x) = 0$. Substituting this into eq. (3.1), we recover the singular (in the $r \rightarrow 0$ limit) part of the order $\alpha_{\text{em}} \alpha_s$ result discussed in section II. The relevant part of eq. (2.3) is, as promised,

$$d\Gamma_{s\rightarrow\gamma} \Big|_{\text{sing.}} = \Gamma_0 \frac{C_8^2}{9 C_7^2} \frac{\alpha_s C_F}{2\pi} \frac{1 + (1-x)^2}{x} \log \frac{1}{r}. $$

(3.3)
It is not important that the $x \to 1$ limit of eq. (2.3) is zero, while that of eq. (3.3) is not. The resummed photon spectrum vanishes at $x = 1$ anyway. Away from $x = 1$ the resummation of the leading logarithms yields finite, non-zero corrections. Although the gluon fragmentation into a photon starts only at order $\alpha_s^2$, its contribution for not too large photon energies is comparable to that of the $s$ quark.

The photon spectrum can be reconstructed from the moments in eq. (3.2) by inverse Mellin transformation. We use three flavors in the anomalous dimension matrix, as charm quarks are excluded from the final state in the experimental analysis. The sensitivity of the result to the number of flavors in the $\beta$-function (three or four) is numerically very small. The $D^{g\to \gamma}$ term, being proportional to the square of the quark charges, would be substantially bigger if charm were not excluded. We plot in Fig. 2. the ratio of the resummed $\Gamma_{88}$ contribution to the photon spectrum divided by eq. (3.3). For not too small values of the photon energy the resummation suppresses the photon spectrum. However, this suppression is not very strong; it only exceeds a factor of two for about $x > 0.85$. (The spectrum is enhanced for $x < 0.3$.)

IV. PHENOMENOLOGY AND CONCLUSIONS

While we think the observations above are interesting even purely from a theoretical point of view, we would like to discuss whether there are any experimental implications. Although enhanced by $\log(m_b^2/\mu^2)$, the $\Gamma_{88}$ term discussed in this paper cannot become arbitrarily large as compared to $\Gamma_0$. The reason is that in the large $m_b$ limit the running of $\alpha_s(m_b)$ cancels the logarithmic enhancement. For a fixed value of $m_b$ the $\mu \to 0$ limit is not allowed. On the one hand, it is cut off by nonperturbative effects; on the other hand, the uncalculable parts of the fragmentation functions can be absorbed in $\mu$, as long as $\mu$ is of order $\Lambda_{QCD}$. Beyond perturbation theory, the $\log(1/r)$ term of the order $\alpha_{em} \alpha_s$ result should be understood as $\log(m_b^2/\mu^2)$, where $\mu$ is some scale of order $\Lambda_{QCD}$. Thus, it is also obvious that a measurement of the $\Gamma_{88}$ contribution would not be sensitive to the strange
quark mass.

For our numerical estimates we use $C_7(m_b) \simeq -0.31$ and $C_8(m_b) \simeq -0.15$, corresponding to $m_t = 175$ GeV and $\alpha_s(M_Z) = 0.12$. We also adopt $m_b = 4.8$ GeV and $0.3 < \mu < 1$ GeV. The $\Gamma_{88}$ contribution to the energy region relevant for CLEO turns out to be negligible. The reason is mainly the overall $C_8^2/9C_7^2$ suppression. On the other hand, any process that contributes to $B$ decays also contributes to $B \to X \gamma$ via the fragmentation of any of the decay product partons into a photon. We estimate that the tree-level $W$-mediated decay $B \to \bar{u}u \bar{d} \to X \gamma$ constitutes of order 1% background to one of the CLEO analyses that does not reconstruct a kaon. This effect becomes more significant and needs to be taken into account more precisely for any future measurements of the $b \to d \gamma$ decay that aim at measuring $V_{td}$.

The $\Gamma_{88}$ contribution, however, dominates the photon spectrum below about 1.2 GeV. Measuring the inclusive $B \to X_s \gamma$ spectrum at such low photon energies will probably remain impossible at CLEO. The situation may be better at asymmetric $B$ factories [16]. But even if such a measurement is possible in a bin around 1 GeV (say, $0.3 < x < 0.5$, that corresponds to roughly $0.7 < E_\gamma < 1.2$ GeV), the $B \to X_s \gamma$ branching fraction into this bin is about $6 \times 10^{-7}$. Taking into account expected experimental efficiencies for such an analysis, probably only a handful of events can be detected (assuming $10^8 B$’s). The reward for this rather hard experimental analysis would be a theoretically clean determination of $C_8$.

In conclusion, we pointed out that a part of the next-to-leading order result for the inclusive $B \to X_s \gamma$ decay rate is proportional to $\alpha_s \log(m_b^2/m_s^2)$. Summing the leading logarithms of the form $[\alpha_s \log(m_b^2/\mu^2)]^n$ to all orders in perturbation theory suppresses (enhances) this contribution to the photon spectrum for large (small) values of the photon energy. In the large $m_b$ limit, formally, the logarithmic factor cancels the $\alpha_s$ suppression of the $\Gamma_{88}$ contribution, as compared to the leading order result $\Gamma_0$. A measurement of the photon spectrum about $E_\gamma \sim 1$ GeV would provide a theoretically clean determination of $C_8$, the Wilson coefficient of the $b \to s g$ magnetic moment type operator.
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FIG. 1. Dalitz plot for the contribution of the square of the invariant matrix element a) $O_7$, and b) $O_8$ to the decay $b \to s \gamma g$ in the $m_s \to 0$ limit. The shaded area is the available phase space, the thick lines denote where collinear singularities occur, and the black box indicates where the virtual corrections contribute.
FIG. 2. The resummed $\Gamma_{88}$ contribution to the photon spectrum divided by the leading log part of the Born term, eq. (3.3), as a function of $x = 2E_\gamma/(1 - r)m_b$ (solid curve). Dashed curve is the gluon decay contribution, eq. (3.2b).