A sufficient condition for the existence of plane spanning trees on geometric graphs

Eduardo Rivera-Campo† and Virginia Urrutia-Galicia‡

Abstract
Let \( P \) be a set of \( n \geq 3 \) points in general position in the plane and let \( G \) be a geometric graph with vertex set \( P \). If the number of empty triangles \( \triangle uvw \) in \( P \) for which the subgraph of \( G \) induced by \( \{u, v, w\} \) is not connected is at most \( n-3 \), then \( G \) contains a non-self intersecting spanning tree.

Keywords. Geometric Graph. Plane Tree. Empty Triangle.

1 Introduction
Throughout this article \( P \) denotes a set of \( n \geq 3 \) points in general position in the Euclidean plane. A geometric graph with vertex set \( P \) is a graph \( G \) drawn in such a way that each edge is a straight line segment with both ends in \( P \). A plane spanning tree of \( G \) is a non-self intersecting subtree of \( G \) that contains every vertex of \( G \). Plane spanning trees with or without specific conditions have been studied by various authors.

A well known result of Károlyi et al. [3] asserts that if the edges of a finite complete geometric graph \( GK_n \) are coloured by two colours, then there exists a plane spanning tree of \( GK_n \) all of whose edges are of the same colour. Keller et al. [4] characterized those plane spanning trees \( T \) of \( GK_n \) such that the complement graph \( T^c \) contains no plane spanning trees.

A plane spanning tree \( T \) is a geometric independency tree if for each pair \( \{u, v\} \) of leaves of \( T \), there is an edge \( xy \) of \( T \) such that the segments \( uv \) and \( xy \) cross each other. Kaneko et al. [2] proved that every complete geometric graph with \( n \geq 5 \) vertices contains a geometric independency tree with at least \( \frac{n}{5} \) leaves.

Let \( k \) be an integer with \( 2 \leq k \leq 5 \) and \( G \) be a geometric graph with \( n \geq k \) vertices such that all geometric subgraphs of \( G \) induced by \( k \) vertices have a plane spanning tree. Rivera-Campo [6] proved that \( G \) has a plane spanning tree.

Three points \( u, v \) and \( w \) in \( P \) form an empty triangle if no point of \( P \) lies in the interior of the triangle \( \triangle uvw \). For any geometric graph \( G \) with vertex set \( P \) we say that an empty triangle \( \triangle uvw \) of \( P \) is disconnected in \( G \) if the subgraph of \( G \) induced by \( \{u, v, w\} \) is not connected.

Let \( s(G) \) denote the number of disconnected empty triangles of \( G \). Our result is the following:

---

*Partially supported by Conacyt, México.
†Corresponding author.
‡Departamento de Matemáticas, Universidad Autónoma Metropolitana-Iztapalapa, Av. San Rafael Atlixco 186, México D.F., C.P. 09340, {erc, vug}@xanum.uam.mx.
Theorem 1. If $G$ is a geometric graph with $n \geq 3$ vertices such that $s(G) \leq n - 3$, then $G$ has a plane spanning tree.

For each $n \geq 3$, let $u_1, u_2, \ldots, u_n$ be the vertices of a regular $n$-gon and denote by $T_n$ and $T_n^c$ the plane path $v_1, u_2, \ldots, u_n$ and its complement, respectively. The geometric graph $T_n^c$ contains no plane spanning tree and is such that $s(T_n^c) = n - 2$. This shows that the condition in Theorem 1 is tight.

2 Proof of Theorem 1

For every oriented straight line $L$ we denote by $L^-$ the set of points in $P$ which are on or to the left of $L$ and by $L^+$ the points which are on or to the right of $L$.

A $k$-set of $P$ is a subset $X$ of $P$ with $k$ elements that can be obtained by intersecting $P$ with an open half plane. The main tool in the proof of Theorem 1 is the following procedure of Erdős et al [3], used to generate all $k$-sets of $P$: Let $L = L_1$ be an oriented line passing through precisely one point $v_1$ of $P$ with $|L_1^-| = k + 1$. Rotate $L$ clockwise around the axis $v_1$ by an angle $\theta$ until a point $v_2$ in $P$ is reached. Now rotate $L$ in the same direction but around $v_2$ until a point $v_3$ in $P$ is reached, and continue rotating $L$ in a similar fashion obtaining a set of oriented lines $C(L)$ and a sequence of points $v_1, v_2, \ldots, v_s$, not necessarily distinct, where $v_s = v_1$ when the angle of rotation $\theta$ reaches $2\pi$.

For $i = 1, 2, \ldots, s - 1$, let $L(v_i, v_{i+1})$ be the line in $C(L)$ that passes through points $v_i$ and $v_{i+1}$ and for $i = 2, 3, \ldots, s - 1$, let $L_i$ be any line in $C(L)$ between $L(v_{i-1}, v_i)$ and $L(v_i, v_{i+1})$.

It is well known that for each line $L_j$ either $L_{j+1}^+ = L_j^+$ and $L_{j+1}^- = (L_j^- \setminus \{v_j\}) \cup \{v_j+1\}$, or $L_{j+1}^+ = (L_j^+ \setminus \{v_j\}) \cup \{v_{j+1}\}$ and $L_{j+1}^- = L_j^-$. In both cases $|L_{j+1}^-| = |L_j^-| = k + 1$ and $|L_{j+1}^+| = |L_j^+| = n - k$. It is also easy to see that if $v_{j+1} \in L_j^+$, then $L^-(v_j, v_{j+1}) = L_j^- \cup \{v_{j+1}\}$ and $L^+(v_j, v_{j+1}) = L_j^+$, and if $v_{j+1} \in L_j^-$, then $L^-(v_j, v_{j+1}) = L_j^-$ and $L^+(v_j, v_{j+1}) = (L_j^- \setminus \{v_j\}) \cup \{v_{j+1}\}$.

The following lemma will be used in the proof of Theorem 1.

Lemma 2. Let $L_i, L_j \in C(L)$ with $i \neq j$. If $x, y$ and $z$ are points of $P$ lying in $L_i^+ \cap L_j^-$, then there are integers $k$ and $l$ with $i \leq k < l < j$ such that $v_k \in \{x, y, z\}$, $x, y, z \in L_k^+ \cap L_j^-$ and such that $L_i$ crosses the triangle $\triangle xyz$.

Proof. Consider the lines $L_i, L_{i+1}, \ldots, L_j$. The result follows from the fact that at each step $t$, at most one of the points $x, y, z$ switches from $L_i^+$ to $L_{i+1}^-$. See Fig. 1.

Let $G$ be a geometric graph with $n \geq 3$ vertices such that $s(G) \leq n - 3$ and let $P$ denote the vertex set of $G$. If $n = 3$ or $n = 4$, it is not difficult to verify by inspection that $G$ has a plane spanning tree. Let us proceed with the proof of Theorem 1 by induction and assume $n \geq 5$ and that the result is valid for each geometric subgraph of $G$ with $k$ vertices, where $3 \leq k \leq n - 1$.

Let $v_1$ be a point in $P$ and $L_1$ be an oriented line through $v_1$ such that $|L_1^-| = \lceil \frac{n+1}{2} \rceil$ and $|L_1^+| = \lfloor \frac{n+1}{2} \rfloor$. Let $C(L)$ be the set of oriented lines obtained from $L = L_1$ as above.

For every $i \geq 1$, define $G_i^-$ and $G_i^+$ as the geometric subgraphs of $G$ induced by $L_i^-$ and $L_i^+$ respectively, and $G^-\left(v_i, v_{i+1}\right)$ and $G^+\left(v_i, v_{i+1}\right)$ as the geometric subgraphs of $G$ induced by $L^-\left(v_i, v_{i+1}\right)$ and $L^+\left(v_i, v_{i+1}\right)$, respectively.
We show there is a line in $C(L)$ for which induction applies to the corresponding graphs $G^-$ and $G^+$, giving plane spanning trees $T^-$ of $G^-$ and $T^+$ of $G^+$. As $T^-$ and $T^+$ lie in opposite sides of $L$, their union contains a plane spanning tree of $G$. We analyse several cases.

**Case 1.** $s(G^-_1) \leq |L^-_1| - 3$ and $s(G^+_1) \leq |L^+_1| - 3$.

By induction there exist plane spanning trees $T^-_1$ of $G^-_1$ and $T^+_1$ of $G^+_1$. Since $T^-_1$ and $T^+_1$ lie in opposite sides of $L_1$ and contain exactly one point in common, the graph $T^-_1 \cup T^+_1$ is a plane spanning tree of $G$.

**Case 2.** $s(G^-_1) \geq |L^-_1| - 2$ and $s(G^+_1) \geq |L^+_1| - 2$.

Clearly $s(G^-_1) + s(G^+_1) \geq (|L^-_1| - 2) + (|L^+_1| - 2) = n - 3 \geq s(G) \geq s(G^-_1) + s(G^+_1)$. This implies $s(G^-_1) = |L^-_1| - 2$, $s(G^+_1) = |L^+_1| - 2$ and that $L_1$ does not cross any disconnected empty triangle of $G$.

Consider the line $L_m$ in $C(L)$ parallel to $L_1$ with opposite orientation. As $L^+_1 \subset L^-_m$, any disconnected empty triangle of $G^+_1$ is also a disconnected empty triangle of $G^-_m$. By Lemma 2 there exists a line in $C(L)$ that crosses a disconnected empty triangle of $G$. Let $j$ be the smallest integer such that $L_{j+1}$ crosses a disconnect empty triangle $\triangle xyz$ of $G$.

Since $L_1, L_2, \ldots, L_j$ do not cross any disconnected empty triangle of $G$, it follows that $s(G^-_j) = s(G^-_1) = |L^-_1| - 2 = |L^-_j| - 2$ and that $s(G^+_j) = s(G^+_1) = |L^+_1| - 2 = |L^+_j| - 2$. Moreover, also by Lemma 2 the axis vertex $v_j$ of $L_j$ must be one of the vertices $x$, $y$, or $z$, since $L_{j+1}$ crosses $\triangle xyz$ while $L_1, L_2, \ldots, L_j$ do not. Without loss of generality we assume $z = v_j$. See Fig. 2.

**Case 2.1.** $v_{j+1} \in L^+_j$.

In this case $\triangle xyz$ is a disconnected empty triangle of $G^-_j$, see Fig. 2 (left) and Fig. 3. Let $i \geq j + 1$ be the smallest integer such that the axis vertex $v_{i+1}$ of $L_{i+1}$ lies in $L^-_j$. By the choice of $i$, all points $v_{j+1}, v_{j+2}, \ldots, v_i$ lie in $L^+_j$ and therefore $L^+_i = L^+_i = \cdots = L^+_j$. It follows that $G^+_i = G^+_i = \cdots = G^+_j$ and that $s(G^+_i) = s(G^+_i) = \cdots = s(G^+_j)$.

Again by the choice of $i$, $L^-_{k+1} = (L^-_k \setminus \{v_k\}) \cup \{v_{k+1}\}$ for $k = j, j + 1, \ldots, i - 1$ and therefore $L^-_i = (L^-_j \setminus \{v_j\}) \cup \{v_i\}$. Moreover, all lines $L_i, L_{i-1}, \ldots, L_{j+1}$ cross $\triangle xyz$. This implies $s(G^-_i) \leq s(G^-_j) - 1$ since $\triangle xyz$ is a disconnected empty triangle of $G^-_j$.
Figure 2: $L_1, L_2, \ldots, L_j$ do not cross any disconnected empty triangle of $G$ and $L_{j+1}$ crosses a disconnected empty triangle $\triangle xyv_j$ of $G$.

Figure 3: $v_{j+1}, v_{j+2}, \ldots, v_i \in L_i^+, L_{i+1} \in L_i^-$. 

Now consider the line $L(v_i, v_{i+1})$ and notice that $L^-(v_i, v_{i+1}) = L_i^- = L_i^+ \cup \{v_{i+1}\}$ because $v_{i+1} \in L_i^-$. Therefore

$$|L^-(v_i, v_{i+1})| = |L_i^-| = |L_j^-| \quad \text{and} \quad |L^+(v_i, v_{i+1})| = |L_i^+| + 1 = |L_j^+| + 1$$

Also notice that $s(G^-(v_i, v_{i+1})) = s(G_i^-)$ and $s(G^+(v_i, v_{i+1})) = s(G_i^+)$ because no empty triangle of $G$ contained in $L^+(v_i, v_{i+1})$ has $v_{i+1}$ as one of its vertices since $L_j$ does not cross any empty triangle of $G$. Therefore

$$s(G^-(v_i, v_{i+1})) = s(G_i^-) = s(G_j^-) - 1 = (|L_j^-| - 2) - 1 = |L_j^-| - 3 = |L^-(v_i, v_{i+1})| - 3$$

and

$$s(G^+(v_i, v_{i+1})) = s(G_i^+) = |L_i^-| - 2 = (|L_i^+(v_i, v_{i+1})| - 1) - 2 = |L^+(v_i, v_{i+1})| - 3.$$

By induction, there exist plane spanning trees $T^-$ of $G^-(v_i, v_{i+1})$ and $T^+$ of $G^+(v_i, v_{i+1})$. The theorem follows since $T^- \cup T^+$ contains a plane spanning tree of $G$. 

4
Case 2.2. \( v_{j+1} \in L_j^- \).
In this case \( \triangle xyz \) is a disconnected empty triangle of \( G_j^+ \), see Fig. 2 (right). The proof is analogous to that of Case 2.1.

Case 3. \( s(G_1^-) \geq |L_1^-| - 2 \) and \( s(G_1^+) \leq |L_1^+| - 3 \).
If for every \( L_j \in C(L) \),
\[
s(G_j^-) \geq |L_j^-| - 2 \quad \text{and} \quad s(G_j^+) \leq |L_j^+| - 3,
\]
then for \( L_m \) in particular, the line in \( C(L) \) parallel to \( L_1 \) with the opposite orientation, we have that
\[
s(G_m^-) \geq |L_m^-| - 2 \quad \text{and} \quad s(G_m^+) \leq |L_m^+| - 3.
\]

If \( n \) is odd, then \( L_1 \) and \( L_m \) are the same line but with opposite orientations, in which case \( L_m^- = L_1^+ \) and \( L_m^+ = L_1^- \). It follows that
\[
|L_m^-| - 2 = |L_1^-| - 2 \leq s(G_1^-) = s(G_m^+) \leq |L_m^+| - 3,
\]
which is not possible.

If \( n \) is even, then \( L_1 \) and \( L_m \) are parallel lines with opposite orientations, with \( L_m \) to the left of \( L_1 \) and with \( |L_1^+| + |L_m^+| = n \). This implies that there are no points between \( L_1 \) and \( L_m \). Therefore every empty triangle of \( G_1^- \) contains points in \( L_m^+ \) and every empty triangle of \( G_m^- \) contains points in \( L_1^+ \). Thus no empty triangle of \( G_1^- \) is also an empty triangle of \( G_m^- \), see Fig 4.

![Diagram](image)

Figure 4: No empty triangle of \( G \) is contained in \( L_1^- \cap L_m^- \).

It follows that \( s(G_1^-) + s(G_m^-) \leq s(G) \) which is also a contradiction since
\[
s(G) \leq n - 3 < n - 2 = |L_1^-| - 2 + |L_m^-| - 2 \leq s(G_1^-) + s(G_m^-).
\]
Therefore, there exists \( L_k \in C(L) \) such that
\[
s(G_k^-) \geq |L_k^-| - 2 \quad \text{and} \quad s(G_k^+) \leq |L_k^+| - 3,
\]
while
\[ s(G_{k+1}^-) \leq |L_{k+1}^-| - 3 \quad \text{or} \quad s(G_{k+1}^+) \geq |L_{k+1}^+| - 2. \]

Since \( L_{k+1}^- = L_k^- \) or \( L_{k+1}^+ = L_k^+ \), it must happen that either
\[ s(G_{k+1}^-) \leq |L_{k+1}^-| - 2 \quad \text{and} \quad s(G_{k+1}^+) \leq |L_{k+1}^+| - 3 \]
\[ \text{or} \]
\[ s(G_{k+1}^-) > |L_{k+1}^-| - 3 \quad \text{and} \quad s(G_{k+1}^+) > |L_{k+1}^+| - 3 \]

which are Case 1 and Case 2, respectively.

**Case 4.** \( s(G_1^-) \leq |L_1^-| - 3 \) and \( s(G_1^+) \geq |L_1^+| - 2 \)

As above, let \( L_m \) be the line in \( C(L) \) parallel to \( L_1 \) with opposite orientation. If \( n \) is odd, then \( L_m = L_1^+ \) and \( L_m = L_1^- \). Therefore \( s(G_m^-) \geq |L_1^-| - 2 \) and \( s(G_m^+) \leq |L_1^+| - 3 \) which is Case 3.

Since \( s(G_m^-) \geq |L_1^-| - 2 \) and \( s(G_m^+) \leq |L_1^+| - 3 \), we have,
\[ |L_1^+| - 2 \leq s(G_1^+) \leq s(G_m^-) \leq |L_m^-| - 3 = |L_m^+| - 2 \]
and
\[ |L_m^+| - 2 \leq s(G_m^+) \leq s(G_1^-) \leq |L_1^-| - 3 = |L_1^+| - 2 \]

which implies \( s(G_1^+) = s(G_1^-) = s(G_m^-) = s(G_m^+) \), since \( |L_1^+| = |L_m^-| \).

It follows that no disconnected empty triangle \( \triangle xyz \) of \( G_1^- \) has \( v_1 \) as one of its vertices, otherwise \( L_m \) must cross \( \{x, y, z\} \) in which case \( s(G_m^-) < s(G_1^-) \) because \( G_m^+ \) is a subgraph of \( G_1^- \).

By our assumption, the same argument can be applied to every line \( L_j \) in \( C(L) \) and therefore for each graph \( G_j^- \), no disconnected empty triangle of \( G_j^- \) has \( v_j \) as one of its vertices.

To reach a contradiction consider any disconnected empty triangle \( \triangle xyz \) of \( G_1^- \). As \( L_m \) is parallel to \( L_1 \) and to the left of \( L_1 \), then \( \triangle xyz \) is also a disconnected empty triangle of \( G_m^- \) and therefore \( \triangle xyz \) lies to the right of \( L_1 \) and to the left of \( L_m \). By Lemma[2] there is a line \( L_k \) in \( C(L) \) with \( 1 < t < m \) such that \( \triangle xyz \) is a disconnected empty triangle of \( G_t^- \) and one of its vertices is precisely \( v_t \), which is the contradiction, see Fig. [5]

As in Case 3, there is a line \( L_k \) in \( C(L) \) such that
\[ s(G_k^-) \leq |L_k^-| - 3 \quad \text{and} \quad s(G_k^+) \geq |L_k^+| - 2, \]
while
\[ s(G_k^+) \geq |L_k^+| - 2 \quad \text{or} \quad s(G_k^+) \leq |L_k^+| - 3 \]

Again, since \( L_k^- = L_k^- \) or \( L_k^+ = L_k^+ \), it must happen that either
\[ s(G_k^-) \leq |L_k^-| - 3 \quad \text{and} \quad s(G_k^+) \leq |L_k^+| - 3 \]
\[ \text{or} \]
\[ s(G_k^-) > |L_k^-| - 3 \quad \text{and} \quad s(G_k^+) > |L_k^+| - 3 \]

which are Case 1 and Case 2, respectively. This ends the proof of Theorem[1]
3 Final Remark

For $n \geq 5$, let $v_1, v_2, \ldots, v_{n-1}$ be the vertices of a regular $(n - 1)$-gon and let $w$ be a point closed to $v_{n-1}$ and in the interior of the triangle $\triangle v_{n-3}v_{n-2}v_{n-1}$. Denote by $R_n$ and $R^c_n$ the plane path $v_1, v_2, \ldots, v_{n-1}, w$ and its complement, respectively. The geometric graph $R^c_n$ is such that $s(R^c_n) = n - 3$ and both graphs $R_n$ and $R^c_n$ contain plane spanning trees. This shows that Theorem 1 is not (at least not an immediate) consequence of the result by Károlyi et al mentioned above.

4 Acknowledgments

We thank the anonymous referees for their suggestions that help us to improve the organisation and readability of the paper.

References

[1] Erdős P., Lovász, L., Simmons, A., Straus, E.G., Dissection graphs of planar point sets, in: G. Srivastava (Ed.), A Survey of Combinatorial Theory, North-Holland, Amsterdam, 1973, 139–149.

[2] Kaneko, A., Oda, Y., Yoshimoto, K., On geometric independency trees for point sets in the plane, Discrete Math. 258, 2002, 93–104.

[3] Károlyi, G., Pach, J., Tóth, G., Ramsey-type results for geometric graphs I. ACM Symposium on Computational Geometry (Philadelphia, PA, 1996), Discrete Comput. Geom. 18 No. 3, 1997, 247–255.
[4] Keller, C., Perles, M. A., Rivera-Campo, E., Urrutia-Galicia, V., Blockers for non-crossing spanning trees in complete geometric graphs, to appear in: J. Pach (Ed.), *Thirty essays in geometric graph theory*, Springer (2012).

[5] Lovász, L., On the number of halving lines. *Annal. Univ. Sci. Budapest. de Rolando Eötvös Nominatae, Sectio Math.* **14**, 1971, 107-108.

[6] Rivera-Campo, E., A note on the existence of plane spanning trees of geometric graphs, *Discrete and Computational Geometry* (Tokyo, 1998), *Lecture Notes in Comput. Sci.* **1763**, 2000, 274–277.