No smooth Julia sets for polynomial
diffeomorphisms of $\mathbb{C}^2$ with positive entropy

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§0. Introduction. There are several reasons why the polynomial diffeomorphisms of $\mathbb{C}^2$ form an interesting family of dynamical systems. One of these is the fact that there are connections with two other areas of dynamics: polynomial maps of $\mathbb{C}$ and diffeomorphisms of $\mathbb{R}^2$, which have each received a great deal of attention. The question arises whether, among the polynomial diffeomorphisms of $\mathbb{C}^2$, are there maps with the special status of having smooth Julia sets? Here we show that is not the case.

More generally, we consider a holomorphic mapping $f : X \to X$ of a complex manifold $X$. The Fatou set of $f$ is defined as the set of points $x \in X$ where the iterates $f^n := f \circ \cdots \circ f$ are locally equicontinuous. If $X$ is not compact, then in the definition of equicontinuity, we consider the one point compactification of $X$; in this case, a sequence which diverges uniformly to infinity is equicontinuous. By the nature of equicontinuity, the dynamics of $f$ is regular on the Fatou set. The Julia set is defined as the complement of the Fatou set, and this is where any chaotic dynamics of $f$ will take place. The first nontrivial case is where $X = \mathbb{P}^1$ is the Riemann sphere, and in this case Fatou (see [M1]) showed that if the Julia set $J$ is a smooth curve, then either $J$ is the unit circle, or $J$ is a real interval. If $J$ is the circle, then $f$ is equivalent to $z \mapsto z^d$, where $d$ is an integer with $|d| \geq 2$; if $J$ is the interval, then $f$ is equivalent to a Chebyshev polynomial. These maps with smooth $J$ play special roles, and this sparked our interest to look for smooth Julia sets in other cases.

Here we address the case where $X = \mathbb{C}^2$, and $f$ is a polynomial automorphism, which means that $f$ is biholomorphic, and the coordinates are polynomials. Since $f$ is invertible, there are two Julia sets: $J^+$ for iterates in forward time, and $J^-$ for iterates in backward time. Polynomial automorphisms have been classified by Friedland and Milnor [FM]; every such automorphism is conjugate to a map which is either affine or elementary, or it belongs to the family $H$. The affine and elementary maps have simple dynamics, and $J^\pm$ are (possibly empty) algebraic sets (see [FM]).

Thus we will restrict our attention to the maps in $H$, which are finite compositions $f := f_k \circ \cdots \circ f_1$, where each $f_j$ is a generalized Hénon map, which by definition has the form $f_j(x, y) = (y, p_j(y) - \delta_j x)$, where $\delta_j \in \mathbb{C}$ is nonzero, and $p_j(y)$ is a monic polynomial of degree $d_j \geq 2$. The degree of $f$ is $d := d_1 \cdots d_k$, and the complex Jacobian of $f$ is $\delta := \delta_1 \cdots \delta_k$. In [FM] and [Sm] it is shown that the topological entropy of $f$ is $\log d > 0$. The dynamics of such maps is complicated and has received much study, starting with the papers [H], [HO1], [BS1] and [FS].

For maps in $H$, we can ask whether $J^+$ can be a manifold. For any saddle point $q$, the stable manifold $W^s(q)$ is a Riemann surface contained in $J^+$. Thus $J^+$ would have to have real dimension at least two. However, $J^+$ is also the support of a positive, closed current $\mu^+$ with continuous potential, and such potentials cannot be supported on a Riemann surface (see [BS1, FS]). On the other hand, since $J^+ = \partial K^+$ is a boundary, it cannot have interior. Thus dimension 3 is the only possibility for $J^+$ to be a manifold. In fact, there are examples of $f$ for which $J^+$ has been shown to be a topological 3-manifold (see [FS], [HO2], [Bo], [RT]).
The purpose of this paper is to prove the following:

**Theorem.** For any polynomial automorphism of $\mathbb{C}^2$ of positive entropy, neither $J^+$ nor $J^-$ is smooth of class $C^1$, in the sense of manifold-with-boundary.

We may interchange the roles of $J^+$ and $J^-$ by replacing $f$ by $f^{-1}$, so there is no loss of generality if we consider only $J^+$.

In an Appendix, we discuss the non-smoothness of the related sets $J$, $J^*$, and $K$.

**Acknowledgment.** We wish to thank Yutaka Ishii and Paolo Aluffi for helpful conversations on this material.

§1. **No boundary.** Let us start by showing that if $J^+$ is a $C^1$ manifold-with-boundary, then the boundary is empty. Recall that if $J^+$ is $C^1$, then for each $q_0 \in J^+$ there is a neighborhood $U \ni q_0$ and $r, \rho \in C^1(U)$ with $d r \wedge d \rho \neq 0$ on $U$, such that $U \cap J^+ = \{ r = 0, \rho \leq 0 \}$. If $J^+$ has boundary, it is given locally by $\{ r = \rho = 0 \}$. For $q \in J^+$, the tangent space $T_q J^+$ consists of the vectors that annihilate $d r$. This contains the subspace $H_q \subseteq T_q J^+$, consisting of the vectors that annihilate $d r$. $H_q$ is the unique complex subspace inside $T_q J^+$, so if $M \subset J^+$ is a complex submanifold, then $T_q M = H_q$.

We start by showing that if $J^+$ is $C^1$, then it carries a Riemann surface lamination.

**Lemma 1.1.** If $J^+$ is $C^1$ smooth, then $J^+$ carries a Riemann surface foliation $\mathcal{R}$ with the property that if $W^s(q)$ is the stable manifold of a saddle point $q$, then $W^s(q)$ is a leaf of $\mathcal{R}$. If $J^+$ is a $C^1$ smooth manifold-with-boundary, then $\mathcal{R}$ extends to a Riemann surface lamination of $J^+$. In particular, any boundary component is a leaf of $\mathcal{R}$.

**Proof.** Given $q_0 \in J^+$, let us choose holomorphic coordinates $(z, w)$ such that $d r(q_0) = d w$. We work in a small neighborhood which is a bidisk $\Delta_\eta \times \Delta_\eta$. We may choose $\eta$ small enough that $|r_z/r_w| < 1$. In the $(z, w)$-coordinates, the tangent space $H_q$ has slope less than 1 at every point $\{ |z|, |w| < \eta \}$. Now let $\hat{q}$ be a saddle point, and let $W^s(\hat{q})$ be the stable manifold, which is a complex submanifold of $\mathbb{C}^2$, contained in $J^+$. Let $M$ denote a connected component of $W^s(\hat{q}) \cap (\Delta_\eta \times \Delta_\eta/2)$. Since the slope is $< 1$, it follows that there is an analytic function $\varphi : \Delta_\eta \rightarrow \Delta_\eta$ such that $M \subset \Gamma_\varphi := \{(z, \varphi(z)) : z \in \Delta_\eta \}$. Let $\Phi$ denote the set of all such functions $\varphi$. Since a stable manifold can have no self-intersections, it follows that if $\varphi_1, \varphi_2 \in \Phi$, then either $\Gamma_{\varphi_1} = \Gamma_{\varphi_2}$ or $\Gamma_{\varphi_1} \cap \Gamma_{\varphi_2} = \emptyset$. Now let $\hat{\Phi}$ denote the set of all normal limits (uniform on compact subsets of $\Delta_\eta$) of elements of $\Phi$. We note that by Hurwitz’s Theorem, the graphs $\Gamma_{\varphi}, \varphi \in \hat{\Phi}$ have the same pairwise disjointness property. Finally, by [BS2], $W^s(q_0)$ is dense in $J^+$, so the graphs $\Gamma_{\varphi}, \varphi \in \hat{\Phi}$ give the local Riemann surface lamination.

If $q_1$ is another saddle point, we may follow the same procedure and obtain a Riemann surface lamination whose graphs are given locally by $\varphi \in \hat{\Phi}_1$. However, we have seen that the tangent space to the foliation at a point $q$ is given by $H_q$. Since these two foliations have the same tangent spaces everywhere, they must coincide.

We have seen that all the graphs are contained in $J^+$, so if $J^+$ has boundary, then the boundary must coincide locally with one of the graphs. \hfill $\Box$

We will use the observation that $K^+ \subset \{(x, y) \in \mathbb{C}^2 : |y| > \max(|x|, R)\}$. Further, we will use the Green function $G^-$ which has many properties, including:
(i) $G^-$ is pluri-harmonic on $\{G^- > 0\}$,
(ii) $\{G^- = 0\} = K^-$,
(iii) $G^- \circ f = d^{-1}G^-$.

Further, the restriction of $G^-$ to $\{|y| \leq \max(|x|, R)\}$ is a proper exhaustion.

**Lemma 1.2.** Suppose that $J^+$ is a $C^1$ smooth manifold-with-boundary, and $M$ is a component of the boundary of $J^+$. Then $M$ is a closed Riemann surface, and $M \cap K \neq \emptyset$.

**Proof.** We consider the restriction $g := G^-|_M$. If $M \cap K = \emptyset$, then $g$ is harmonic on $M$. On the other hand, $g$ is a proper exhaustion of $M$, which means that $g(z) \to \infty$ as $z \in M$ leaves every compact subset of $M$. This means that $g$ must assume a minimum value at some point of $M$, which would violate the minimum principle for harmonic functions. \qed

**Lemma 1.3.** Suppose that $J^+$ is a $C^1$ smooth manifold-with-boundary, then the boundary is empty.

**Proof.** Let $M$ be a component of the boundary of $J^+$. By Lemma 1.2, $M$ must intersect $\Delta_{2}^\pm$. Since $J^+$ is $C^1$, there can only finitely many boundary components of $J^+ \cap \Delta_{2}^\pm$. Thus there can be only finitely many components $M$, which must be permuted by $f$. If we take a sufficiently high iterate $f^N$, we may assume that $M$ is invariant. Now let $h := f^N|_M$ denote the restriction to $M$. We see that $h$ is an automorphism of the Riemann surface $M$, and the iterates of all points of $M$ approach $K \cap M$ in forward time. It follows that $M$ must have a fixed point $q \in M$, and $|h'(q)| < 1$. The other multiplier of $Df$ at $q$ is $\delta/h'(q)$.

We consider three cases. First, if $|\delta/h'(q)| > 1$, then $q$ is a saddle point, and $M = W^s(q)$. On the other hand, by [BS2], the stable manifold of a saddle points is dense in $J^+$, which makes it impossible for $M$ to be the boundary of $J^+$. This contradiction means that there can be no boundary component $M$.

The second case is $|\delta/h'(q)| < 1$. This case cannot occur because the multipliers are less than 1, so $q$ is a sink, which means that $q$ is contained in the interior of $K^+$ and not in $J^+$.

The last case is where $|\delta/h'(q)| = 1$. In this case, we know that $f$ preserves $J^+$, so $Df$ must preserve $T_q(J^+)$. This means that the outward normal to $M$ inside $J^+$ is preserved, and thus the second multiplier must be $+1$. It follows that $q$ is a semi-parabolic/semi-attracting fixed point. It follows that $J^+$ must have a cusp at $q$ and cannot be $C^1$ (see Ueda [U] and Hakim [Ha]). \qed

§2. Maps that do not decrease volume. We note the following topological result (see Samelson [S] for an elegant proof): If $M$ is a smooth 3-manifold (without boundary) of class $C^1$ in $\mathbb{R}^4$, then it is orientable. This gives:

**Proposition 2.1.** For any $q \in M$, there is a neighborhood $U$ about $q$ so that $U - M$ consists of two components $\mathcal{O}_1$ and $\mathcal{O}_2$, which belong to different components of $\mathbb{R}^4 - M$.

**Proof.** Suppose that $\mathcal{O}_1$ and $\mathcal{O}_2$ belong to the same component of $\mathbb{R}^4 - M$. Then we can construct a simple closed curve $\gamma \subset \mathbb{R}^4$ which crosses $M$ transversally at $q$ and has no other intersection with $M$. It follows that the (oriented) intersection is $\gamma \cdot M = 1$ (modulo 2). But the oriented intersection modulo 2 is a homotopy invariant (see [M2]), and $\gamma$ is contractible in $\mathbb{R}^4$, so we must have $\gamma \cdot M = 0$ (modulo 2). \qed
Corollary 2.2. If $J^+$ is $C^1$ smooth, then $f$ is an orientation preserving map of $J^+$.

Proof. $U^+ := C^2 - K^+$ is a connected (see [HO1]) and thus it is a component of $C^2 - J^+$. Since $f$ preserves $U^+$, it also preserves the orientation of $J^+$, which is $\pm \partial U^+$.

We recall the following result of Friedland and Milnor:

**Theorem [FM]**. If $|\delta| > 1$, then $K^+$ has zero Lebesgue volume, and thus $J^+ = K^+$. If $|\delta| = 1$, then $\text{int}(K^+) = \text{int}(K^-) = \text{int}(K)$. In particular, there exists $R$ such that $J^+ = K^+$ outside $\Delta^2_R$.

**Proof of Theorem in the case $|\delta| \geq 1$**. Let $q \in J^+$ be a point outside $\Delta^2_R$, as in the Theorem above. Then near $q$ there must be a component $\mathcal{O}$, which is distinct from $U^+ = C^2 - K^+$. Thus $\mathcal{O}$ must belong to the interior of $K^+$. But by the Theorem above, the interior of $K^+$ is not near $q$.

§3. Volume decreasing maps. Throughout this section, we continue to suppose that $J^+$ is $C^1$ smooth, and in addition we suppose that $|\delta| < 1$. For a point $q \in J^+$, we let $T_q := T_q(J^+)$ denote the real tangent space to $J^+$. We let $H_q := T_q \cap iT_q$ denote the unique (one-dimensional) complex subspace inside $T_q$. Since $J^+$ is invariant under $f$, so is $H_q$, and we let $\alpha_q$ denote the multiplier of $D_qf|_{H_q}$.

**Lemma 3.1.** Let $q \in J^+$ be a fixed point. There is a $D_qf$-invariant subspace $E_q \subset T_q(C^2)$ such that $H_q$ and $E_q$ generate $T_q$. We denote the multiplier of $D_qf|_{E_q}$ by $\beta_q$. Thus $D_qf$ is linearly conjugate to the diagonal matrix with diagonal elements $\alpha_q$ and $\beta_q$. Further, $\beta_q \in \mathbb{R}$, and $\beta_q > 0$.

**Proof.** We have identified an eigenvalue $\alpha_q$ of $D_qf$. If $D_qf$ is not diagonalizable, then it must have a Jordan canonical form \(\begin{pmatrix} \alpha_q & 1 \\ 0 & \alpha_q \end{pmatrix}\). The determinant is $\alpha_q^2 = \delta$, which has modulus less than 1. Thus $|\alpha_q| < 1$, which means that $q$ is an attracting fixed point and thus in the interior of $K^+$, not in $J^+$. Thus $D_qf$ must be diagonalizable, which means that $H_q$ has a complementary invariant subspace $E_q$. Since $E_q$ and $T_q$ are invariant under $D_qf$, the real subspace $E_q \cap T_q \subset E_q$ is invariant, too. Thus $\beta_q \in \mathbb{R}$. By Corollary 2.2, $D_qf$ will preserve the orientation of $T_q$, and so $\beta_q > 0$.

Let us recall the Riemann surface foliation of $J^+$ which was obtained in Lemma 1.1. For $q \in J^+$, we let $R_q$ denote the leaf of $\mathcal{R}$ containing $q$. If $q$ is a fixed point, then $f$ defines an automorphism $g := f|_{R_q}$ of the Riemann surface $R_q$. Since $R_q \subset K^+$, we know that the iterates of $g^n$ are bounded in a complex disk $q \in \Delta_q \subset R_q$. Thus the derivatives $(Dg)^n = D(g^n)$ are bounded at $q$. We conclude that $|\alpha_q| = |D_qg(q)| \leq 1$. If $|\alpha_q| = 1$, then $\alpha_q$ is not a root of unity. Otherwise $g$ is an automorphism of $R_q$ fixing $q$, and $Dg^n(q) = 1$ for some $n$. It follows that $g^n$ must be the identity on $R_q$. This means that $R_q$ would be a curve of fixed points for $f^n$, but by [FM] all periodic points of $f$ are isolated, so this cannot happen.

**Lemma 3.3.** If $q \in J^+$ is a fixed point, then $q$ is a saddle point, and $\alpha_q = \delta/d$, and $\beta_q = d$.
Proof. First we claim that $|\alpha_p| < 1$. Otherwise, we have $|\alpha_q| = 1$, and by the discussion above, this means that $\alpha_q$ is not a root of unity. Thus the restriction $g = f|_{R_q}$ is an irrational rotation. Let $\Delta \subset R_q$ denote a $g$-invariant disk containing $q$. Since $|\delta| = |\alpha_q \beta_q| = |\beta_q|$ has modulus less than 1, we conclude that $f$ is normally attracting to $\Delta$, and thus $q$ must be in the interior of $K^+$, which contradicts the assumption that $q \in J^+$. Now we have $|\alpha_q| < 1$, so if $|\beta_q| = 1$, we have $\beta_q = 1$, since $\beta_q$ is real and positive. This means that $q$ is a semi-parabolic, semi-attracting fixed point for $f$. We conclude by Ueda [U] and Hakim [Ha] that $J^+$ has a cusp at $q$ and thus is not smooth. Thus we conclude that $|\beta_q| > 1$, which means that $q$ is a saddle point.

Now since $E_q$ is transverse to $H_q$, it follows that $W^u(q)$ intersects $J^+$ transversally, and thus $J^+ \cap W^u(q)$ is $C^1$ smooth. Let us consider the uniformization

$$\phi : \mathbb{C} \to W^u(q) \subset \mathbb{C}^2, \quad \phi(0) = q, \quad f \circ \phi(\zeta) = \phi(\lambda^u \zeta)$$

The pre-image $\tau := \phi^{-1}(W^u(q) \cap J^+) \subset \mathbb{C}$ is a $C^1$ curve passing through the origin and invariant under $\zeta \mapsto \lambda^u \zeta$. It follows that $\lambda^u \in \mathbb{R}$, and $\tau$ is a straight line containing the origin. Further, $g^+ := G^+ \circ \phi$ is harmonic on $\mathbb{C} - \tau$, vanishing on $\tau$, and satisfying $g^+(\lambda^u \zeta) = d \cdot g^+(\zeta)$. Since $\tau$ is a line, it follows that $g^+$ is piecewise linear, so we must have $\lambda^u = \pm d$. Finally, since $f$ preserves orientation, we have $\lambda^u = d$.

**Lemma 3.4.** There can be at most one fixed point in the interior of $K^+$. There are at least $d - 1$ fixed points are contained in $J^+$, and at each of these fixed points, the differential $Df$ has multiplier of $d$.

**Proof.** Suppose that $q$ is a fixed point in the interior of $K^+$. Then $q$ is contained in a recurrent Fatou domain $\Omega$, and by [BS2], $\partial \Omega = J^+$. If there is more than one fixed point in the interior of $K^+$, we would have $J^+$ simultaneously being the boundary of more than one domain, in addition to being the boundary of $U^+ = \mathbb{C}^2 - K^+$. This is not possible if $J^+$ is a topological submanifold of $\mathbb{C}^2$.

By [FM] there are exactly $d$ fixed points, counted with multiplicity. By Lemma 3.3, the fixed points in $J^+$ are of saddle type, so they have multiplicity 1. Thus there are at least $d - 1$ of them.

§4. **Fixed points with given multipliers.** If $q = (x, y)$ is a fixed point for $f = f_n \circ \cdots \circ f_1$, then we may represent it as a finite sequence $(x_j, y_j)$ with $j \in \mathbb{Z}/n\mathbb{Z}$, subject to the conditions $(x, y) = (x_1, y_1) = (x_{n+1}, y_{n+1})$ and $f_j(x_j, y_j) = (x_{j+1}, y_{j+1})$. Given the form of $f_j$, we have $x_{j+1} = y_j$, so we may drop the $x_j$’s from our notation and write $q = (y_n, y_1)$. We identify this point with the sequence $\hat{q} = (y_1, \ldots, y_n) \in \mathbb{C}^n$, and we define the polynomials

$$\varphi_1 := p_1(y_1) - \delta_1 y_n - y_2$$
$$\varphi_2 := p_2(y_2) - \delta_2 y_1 - y_3$$
$$\ldots \ldots$$
$$\varphi_n := p_n(y_n) - \delta_n y_{n-1} - y_1$$

The condition to be a fixed point is that $\hat{q} = (y_1, \ldots, y_n)$ belongs to the zero locus $Z(\varphi_1, \ldots, \varphi_n)$ of the $\varphi_i$’s. We define $q_i(y_i) := p_i(y_i) - y_i^{d_i}$ and $Q_i := q_i(y_i) - y_{i+1} - \delta_i y_{i-1}$,
The formula for the determinant gives

\[ \Phi = y_i^{d_i} + q_i(y_i) - y_{i+1} - \delta_i y_{i-1} = y_i^{d_i} + Q_i \]  

(\star)

Since \( p_j \) is monic, the degrees of \( q_i \) and \( Q_i \) are \( \leq d_i - 1 \).

By the Chain Rule, the differential of \( f \) at \( q = (y_n, y_1) \) is given by

\[ Df(q) = \begin{pmatrix} 0 & 1 \\ -\delta_n & p_n'(y_n) \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -\delta_1 & p_1'(y_1) \end{pmatrix} \]

We will denote this by \( M_n = M_n(y_1, \ldots, y_n) := \begin{pmatrix} m^{(n)}_{11} & m^{(n)}_{12} \\ m^{(n)}_{21} & m^{(n)}_{22} \end{pmatrix} \).

We consider special monomials in \( p_j' = p_j'(y_j) \) which have the form \( (p')^L := p_{\ell_1}' \cdots p_{\ell_s}' \), with \( L = \{\ell_1, \ldots, \ell_s\} \subset \{1, \ldots, n\} \). Note that the factors \( p_{\ell_i}' \) in \( (p')^L \) are distinct. Let us use the notation \( |L| \) for the number of elements in \( L \), and \( H_{m} \) for the linear span of \( \{(p')^L : |L| = m - 2k, 0 \leq k \leq n/2\} \). With this notation, \( m \) indicates the maximum number of factors of \( p_j' \) in any monomial, and in every case the number of factors differs from \( m \) by an even number.

**Lemma 4.1.** The entries of \( M_n \):

1. \( m^{(n)}_{11} \) and \( m^{(n)}_{22} - p_1'(y_1) \cdots p_n'(y_n) \) both belong to \( H_{n-2} \).
2. \( m^{(n)}_{12}, m^{(n)}_{21} \in H_{n-1} \).

**Proof.** We proceed by induction. The case \( n = 1 \) is clear. If \( n = 2 \),

\[ M_2 = \begin{pmatrix} 0 & 1 \\ -\delta_2 & p_2' \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\delta_1 & p_1' \end{pmatrix} = \begin{pmatrix} -\delta_1 & p_1' \\ -\delta_1 p_2' & p_1' p_2' - \delta_2 \end{pmatrix} \]

which satisfies (1) and (2). For \( n > 2 \), we have

\[ M_n = \begin{pmatrix} 0 & 1 \\ -\delta_n & p_n' \end{pmatrix} M_{n-1} = \begin{pmatrix} m^{(n-1)}_{22} & m^{(n-1)}_{21} \\ -\delta_n m^{(n-1)}_{11} & m^{(n-1)}_{12} \end{pmatrix} \begin{pmatrix} m^{(n-1)}_{21} & m^{(n-1)}_{22} \\ -\delta_n m^{(n-1)}_{11} + m^{(n-1)}_{21} p_n' & -\delta_n m^{(n-1)}_{12} + p_n' m^{(n-1)}_{22} \end{pmatrix} \]

which gives (1) and (2) for all \( n \). \( \square \)

The condition for \( Df \) to have a multiplier \( \lambda \) at \( q \) is \( \Phi(\tilde{q}) = 0 \), where

\[ \Phi = \det \left( M_n - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) \]

**Lemma 4.2.** \( \Phi - p_1'(y_1) \cdots p_n'(y_n) \in H_{n-2} \).

**Proof.** The formula for the determinant gives

\[ \Phi = \lambda^2 - \lambda \text{Tr}(M_n) + \det(M_n) = \lambda^2 - \lambda (m^{(n)}_{11} + m^{(n)}_{22}) + \delta \]

since \( \delta \) is the Jacobian determinant of \( Df \). The Lemma now follows from Lemma 4.1. \( \square \)
The degree of the monomial \( y^a := y_1^{a_1} \cdots y_n^{a_n} \) is \( \deg(y^a) = a_1 + \cdots + a_n \). We will use the graded lexicographical order on the monomials in \( \{y_1, \ldots, y_n\} \). That is, \( y^a > y^b \) if either \( \deg(y^a) > \deg(y^b) \), or if \( \deg(y^a) = \deg(y^b) \) and \( a_i > b_i \), where \( i = \min\{1 \leq j \leq n : a_j \neq b_j\} \). If \( f \in \mathbb{C}[y_1, \ldots, y_n] \), we denote \( \text{LT}(f) \) for the leading term of \( f \), \( \text{LC}(f) \) for the leading coefficient, and \( \text{LM}(f) \) for the leading monomial.

**Lemma 4.3.** With the graded lexicographical order, \( G := \{\varphi_1, \ldots, \varphi_n\} \) is a Gröbner basis.

**Proof.** We will use Buchberger’s Algorithm (see [CLO, Chapter 2]). For each \( i = 1, \ldots, n \), \( \text{LT}(\varphi_i) = \text{LM}(\varphi_i) = y_i^{d_i} \), so for \( i \neq j \), the least common multiple of the leading terms is \( \text{L.C.M.} = y_i^{d_i} y_j^{d_j} \). The \( S \)-polynomial is

\[
S(\varphi_i, \varphi_j) := \frac{\text{L.C.M.}}{\text{LM}(\varphi_j)} \varphi_i - \frac{\text{L.C.M.}}{\text{LM}(\varphi_i)} \varphi_j = y_j^{d_j} Q_i - y_i^{d_i} Q_j = \varphi_j Q_i - Q_j \varphi_i
\]

where we use the \( Q_j \) from (4.1) and cancel terms. Now let \( \mu_i := \deg(Q_i) \). Since \( \mu_i < d_i \) for all \( i \), the monomials \( \text{LM}(\varphi_j Q_i) = y_j^{d_j} y_i^{d_i} \) and \( \text{LM}(\varphi_i Q_j) = y_i^{d_i} y_j^{d_j} \) are not equal in our monomial ordering. Thus \( \text{LM}(S(\varphi_i, \varphi_j)) \geq \max(\text{LM}(\varphi_j Q_i), \text{LM}(\varphi_i Q_j)) \). It follows from Buchberger’s Algorithm that \( \{\varphi_1, \ldots, \varphi_n\} \) is a Gröbner basis.

We will use the Multivariable Division Algorithm, by which any polynomial \( g \in \mathbb{C}[y_1, \ldots, y_n] \) may be written \( g = A_1 \varphi_1 + \cdots + A_n \varphi_n + R \) where \( \text{LM}(g) \geq \text{LM}(A_j \varphi_j) \) for all \( 1 \leq j \leq n \), and \( R \) contains no terms divisible by any \( \text{LM}(\varphi_j) \). An important property of a Gröbner basis, is that \( g \) belongs to the ideal \( \langle \varphi_1, \ldots, \varphi_n \rangle \) if and only if \( R = 0 \) (see, for instance, [CLO] or [BW]).

If all fixed points have the same value of \( \lambda \) as multiplier, then it follows that \( \Phi \) must vanish on the whole zero set \( Z(\varphi_1, \ldots, \varphi_n) \). Since we have a Gröbner basis, we easily determine the following:

**Corollary 4.4.** \( \Phi \notin \langle \varphi_1, \ldots, \varphi_n \rangle \).

**Proof.** The leading monomial of \( \Phi \) is \( y_1^{d_1-1} \cdots y_n^{d_n-1} \), but this is not divisible by any of the leading monomials \( \text{LM}(\varphi_j) = y_j^{d_j} \). Since \( \{\varphi_1, \ldots, \varphi_n\} \) is a Gröbner basis, it follows that \( \Phi \) does not belong to the ideal \( \langle \varphi_1, \ldots, \varphi_n \rangle \).

§5. **Proof of the Theorem.** In this section we prove the Theorem, which will follow from Lemma 3.4, in combination with:

**Proposition 5.1.** Suppose \( F = f_n \circ \cdots \circ f_1, \ n \geq 3, \) is a composition of generalized Hénon maps with \( |\delta| < 1 \). Suppose that \( F \) has \( d = d_1 \cdots d_n \) distinct fixed points. It is not possible that \( d - 1 \) of these points have the same multipliers.

**Proof that Proposition 5.1 implies the Theorem.** To prove the Theorem, it remains to deal with the case \( |\delta| < 1 \). If \( f = f_1 \) is a single generalized Hénon map, we consider \( F = f_1 \circ f_1 \circ f_1 \) with \( n = 3 \) and the same Julia set. Lemma 3.4 asserts that if \( J^+ \) is \( C^1 \), there are \( d - 1 \) saddle points with unstable multiplier \( \lambda = d \). So by Proposition 4.1 we conclude that \( J^+ \) cannot be \( C^1 \) smooth.
We give the proof of Proposition 5.1 at the end of this Section. For \( J \subset \{1, \ldots, n\} \), we write
\[
\Lambda_J := \{(p')^L : L \subset J, |L| = |J| - 2k, \text{ for some } 1 \leq k \leq |J|/2\},
\]
We let \( H_J \) denote the linear span of \( \Lambda_J \). To compare with our earlier notation, we note that \( H_J \subset H_{|J|-2} \) and that \( (p')^j \notin H_J \). The elements of \( H_J \) depend only on the variables \( y_j \) for \( j \in J \). Now we formulate a result for dividing certain terms by \( \varphi_j \):

**Lemma 5.2.** Suppose that \( J \subset \{1, \ldots, n\} \) and \( h \in H_J \). Then for each \( j \in J \) and \( \alpha \in \mathbb{C} \), we have
\[
(y_j - \alpha) \left((p')^j + h\right) = A(y)\varphi_j + B(y) \left((p')^{j-\{j\}} + \rho_1\right) + (y_j - \alpha) \cdot \rho_2, \tag{†}
\]
where \( \rho_1, \rho_2 \in H_{J - \{j\}} \), and \( B = \eta_j(y_j) + d_j y_{j+1} + d_j \delta_j y_{j-1} \) with
\[
\eta_j(y_j) = y_j q_j'(y_j) - \alpha p_j'(y_j) - d_j q_j(y_j). \tag{‡}
\]

The leading monomials satisfy:
\[
LM \left(\left((y_j - \alpha) \left((p')^j + h\right)\right)\right) = LM(A(y)\varphi_j)
\]

**Proof.** Let us start with the case \( J = \{1, \ldots, m\}, m \leq n, \) and \( j = 1 \), so \( J - \{j\} = J_1 = \{2, \ldots, n\} \). We divide by \( p'_1 \) and remove any factor of \( p'_1 \) in \( h \). This gives
\[
(p')^j + h = p'_1(y_1)\mu_1 + \rho_2
\]
where \( \mu_1 = (p')^{j_1} + \rho_1 \), and \( \rho_1, \rho_2 \in H_{\{2, \ldots, m\}} \), and \( \mu_1, \rho_1, \rho_2 \) are independent of the variable \( y_1 \). Thus
\[
(y_1 - \alpha) \left((p')^j + h\right) = (y_1 - \alpha)(d_1 y_1^{d_1-1} + q'_1(y_1))\mu_1 + (y_1 - \alpha)\rho_2
\]
\[
= d_1 y_1^{d_1-1} \mu_1 + (y_1 q'_1(y_1) - \alpha p'_1(y_1))\mu_1 + (y_1 - \alpha)\rho_2
\]
\[
= (d_1 \mu_1)\varphi_1 + (\eta_1(y_1) + d_1 y_2 + d_1 \delta_1 y_1)\mu_1 + (y_1 - \alpha)\rho_2
\]
where in the last line we substitute \( \eta_1 \) defined by (‡). Using (§), we see that this gives (†).

It remains to look at the leading terms of \( T_1 := (y_1 - \alpha) \left((p')^j + h\right) \) and \( T_2 := d_1 \mu_1 \varphi_1 \). We see that \( T_1 \) and \( T_2 \) both contain nonzero multiples of \( y_j \prod_{i=1}^m y_i^{d_i-1} \), and all other monomials in \( T_1 \) and \( T_2 \) have lower degree. Thus we have \( LM(T_1) = LM(T_2) \) for the graded ordering, independent of any ordering on the variables \( y_1, \ldots, y_n \). The choices of \( J = \{1, \ldots, m\} \) and \( j = 1 \) just correspond to a permutation of variables, and this does not affect the conclusion that \( LM(T_1) = LM(T_2) \).

**Lemma 5.3.** For any \( \alpha \in \mathbb{C} \), \( (y_1 - \alpha)\Phi \notin \langle \varphi_1, \ldots, \varphi_n \rangle \).

**Proof.** By [FM], we may assume that \( p_j(y_j) = y_j^{d_j} + q_j(y_j) \), and \( \deg(q_j) \leq d_j - 2 \). We consider two cases. The first case is that there is at least one \( j \) such that \( \eta_j \) is not the zero polynomial. If we conjugate by \( f_{j-1} \circ \cdots \circ f_1 \), we may “rotate” the maps in \( f \) so that the
factor $f_j$ becomes the first factor. If there exists a $j$ for which $\eta_j(y_j)$ is non constant, we choose this for $f_1$. Otherwise, if all the $\eta_j$ are constant, we choose $f_1$ to be any factor such that $\eta_1 \neq 0$.

We will apply the Multivariate Division Algorithm on $(y_1 - \alpha)\Phi$ with respect to the set $\{\varphi_1, \ldots, \varphi_n\}$. We will find that there is a nonzero remainder, and since $\{\varphi_1, \ldots, \varphi_n\}$ is a Gröbner basis, it will follow that $(y_1 - \alpha)\Phi$ does not belong to the ideal $\langle \varphi_1, \ldots, \varphi_n \rangle$.

We start with Lemma 4.2, according to which $\Phi = p_1 y_1 \cdots y_n + h$, where $h \in H_{n - 2} = H_{\{1, \ldots, n\}}$. The leading monomial of $(y_1 - \alpha)\Phi$ is $y_1^{d_1} \prod_{i=2}^{n} y_i^{d_i - 1}$, and $\varphi_1$ is the only element of the basis whose leading monomial divides this. Thus we apply Lemma 5.2, with $J = \{1, \ldots, n\}$, $j = 1$, and $J_1 := J \setminus \{j\} = \{2, \ldots, n\}$. This gives

$$(y_1 - \alpha)\Phi = A_1 \varphi_1 + (\eta_1(y_1) + d_1 y_2 + d_1 \delta_1 y_n) \left( \prod_{i=2}^{n} p_i'(y_i) + \rho_1 \right) + (y_1 - \alpha)\rho_2$$

$$= A_1 \varphi_1 + [d_1 y_2 ((p')^{J_1} + \rho_1)] + [d_1 \delta_1 y_n ((p')^{J_1} + \rho_1)] + [\eta_1 ((p')^{J_1} + \rho_1)] + \ell.o.t$$

$$= A_1 \varphi_1 + T_2 + T_n + R_1 + \ell.o.t$$

where $\rho_1, \rho_2 \in H_{\{2, \ldots, n\}}$. In particular, $T_2$ and $T_n$ depend on $y_2, \ldots, y_n$ but not on $y_1$. We note that $T_2$ (respectively, $T_n$) contains a term divisible by $LM(\varphi_2)$ (respectively, $LM(\varphi_n)$). We view $R_1$ as a remainder term, and note that $LM(R_1)$ is divisible by $y_2^{d_2} \cdots y_n^{d_n - 1}$, as well as the largest power of $y_1$ in $\eta_1(y_1)$. By “$\ell.o.t.$” we mean that none of its monomials is divisible by $LM(R_1)$ or by any of the $LM(\varphi_i)$.

Now we apply Lemma 5.2 to $T_2$, this time with $J = \{2, \ldots, n\}$ and $j = 2$, with $J \setminus \{2\} = J_{12} = \{3, \ldots, n\}$. We have

$$T_2 = A_2 \varphi_2 + d_2 y_3 ((p')^{J_{12}} + \rho_2^{(2)}) + d_2 \delta_2 y_1 ((p')^{J_{12}} + \rho_2^{(2)}) + \eta_2(y_2)(p')^{J_{12}} + \ell.o.t.$$

$$= A_2 \varphi_2 + T_2^{(2)} + R_2^{(2)} + \ell.o.t.$$

We see that $T_2^{(2)}$ contains terms that are divisible by $LM(\varphi_3)$, but the monomials in $R_1^{(2)}$ and $R_2^{(2)}$ are not divisible by $LM(\varphi_i)$ for any $i$. The remainder term here is $R_1^{(2)} + R_2^{(2)}$, and we observe that this cannot cancel the largest term in $R_1$. This is because $LM(R_1^{(2)})$ lacks a factor of $y_2$, and $LM(R_2^{(2)})$ is equal to $y_3^{d_3 - 1} \cdots y_n^{d_n - 1}$ times the largest power of $y_2$ in $\eta_2(y_2)$, and by (†), this power is no bigger than $d_2 - 1$. If $\eta_1$ is not constant, then we see that $LM(R_1) > LM(R_2^{(2)})$. If $\eta_1$ is constant, then $\eta_2$ must be constant, too, and again we have $LM(R_1) > LM(R_2^{(2)})$. Thus, with our earlier notation, $R_1^{(2)} + R_2^{(2)} = \ell.o.t.$

We do a similar procedure with $T_n, T_2^{(2)}$, etc., and again find that the remainder term does not contain a multiple of the leading monomial of $R_1$. We see that each time we do this process, the size of the exponent $L$ decreases in the term $(p')^L$. When we have $L = 0$, there are no terms that can be divided by any $LM(\varphi_j)$. Thus we end up with

$$(y_1 - \alpha)\Phi = A_1 \varphi_1 + \cdots + A_n \varphi_n + R_1 + \ell.o.t.$$

and $LT((y_1 - \alpha)\Phi) \geq LT(A_j \varphi_j)$ for all $1 \leq j \leq n$, and none of the remaining terms is divisible by any of the leading monomials of $\varphi_j$. Thus we have now finished the Multivariate
Division Algorithm, and we have a nonzero remainder. Thus \((y_1 - \alpha)\Phi\) does not belong to the ideal of the \(\varphi_j\)’s.

Now we turn to the second case, in which \(\eta_j = 0\) for all \(j\). By [FM], we may assume that \(\deg(q_j) \leq d_j - 2\). It follows that \(\alpha = 0\) and \(q_j = 0\). Thus \(p_j = y_j^{d_j}\) for all \(1 \leq j \leq n\), so \(p_j' = d_j y_j^{d_j-1}\), and \(H_j\) consists of linear combinations of products \((p')^I = y_i_1^{d_1-1} \cdots y_i_k^{d_k-1}\) for \(I = \{i_1, \ldots, i_k\} \subseteq J\), for even \(k \leq |J| - 2\). We will go through the multivariate division algorithm again. The principle is the same as before, but the details are different; in the first case we needed \(n \geq 2\), and now we will need \(n \geq 3\).

Again, it is only \(\varphi_1\) which has a leading monomial which can divide some terms in \((y_1 - \alpha)\Phi\). As before, we apply Lemma 5.2 with \(J = \{1, \ldots, n\}\), \(j = 1\), and \(J - \{1\} = J_1 = \{2, \ldots, n\}\). The polynomial in \((\hat{\varphi})\) becomes \(B = d_j y_{j+1} + d_j \delta_j y_{j-1}\), and we have:

\[
y_1 \Phi = A_1 \varphi_1 + d_1 y_2 ((p')^{J_1} + \rho_1) + d_1 \delta_1 y_n ((p')^{J_1} + \rho_1) + y_1 \rho_2
\]

\[
= A_1 \varphi_1 + T_2 + T_n + \ell.o.t.
\]

where \(\rho_1, \rho_2 \in H_{\{2, \ldots, n\}}\). Now we apply Lemma 5.2 to divide \(T_2\) (respectively \(T_n\)) by \(\varphi_2\) (respectively \(\varphi_n\)). This yields:

\[
y_1 \Phi = A_1 \varphi_1 + A_2 \varphi_2 + A_n \varphi_n + T_3 + T_n + R + \ell.o.t.
\]

where

\[
T_3 = d_1 d_2 y_3 ((p')^{J_2} + \tilde{\rho}_3), \quad T_n = d_1 d_n \delta_1 \delta_n y_{n-1} ((p')^{J_1} + \tilde{\rho}_n)
\]

with \(\tilde{\rho}_3 \in H_{\{3, \ldots, n\}}\) and \(\tilde{\rho}_n \in H_{\{2, \ldots, n-1\}}\), and

\[
R = \left(d_1 d_2 \delta_2 y_1 y_n^{d_n-1} + d_1 d_n \delta_1 y_1 y_2^{d_2-1}\right) \prod_{i=3}^{n-1} y_i^{d_i-1}
\]

Since \(n > 2\), \(R\) is not the zero polynomial. We will continue the Multivariate Division Algorithm by dividing \(T_3\) by \(\varphi_3\) and \(T_n\) by \(\varphi_n\), but we see that any terms created cannot cancel \(R\). Thus when we finish the Division Algorithm, we will have a nonzero remainder. As in the previous case, we conclude that \(y_1 \Phi\) is not in the ideal \(\langle \varphi_1, \ldots, \varphi_n \rangle\). \[\square\]

Proof of Proposition 5.1. The fixed points of \(f\) coincide with the elements of \(Z(\varphi_1, \ldots, \varphi_n)\), which is a variety of pure dimension zero. Saddle points have multiplicity 1, and since there are \(d - 1\) of these, and since the total multiplicity is \(d\), there must be one more fixed point, also of multiplicity 1. It follows that the ideal \(I := \langle \varphi_1, \ldots, \varphi_n \rangle\) is equal to its radical (see [BW]). Since the saddle points all have multiplier \(\lambda\), \(\Phi\) must vanish at all the saddle points. If \((\alpha, \beta)\) is the other fixed point, we conclude that \((y_1 - \alpha)\Phi\) vanishes at all the fixed points. Thus \((y_1 - \alpha)\Phi\) belongs to the radical of \(I\), and thus \(I\) itself. This contradicts Lemma 5.3, which completes the proof of Proposition 5.1. \[\square\]

Appendix: Non-smoothness of \(J\), \(J^*\), and \(K\)

Let us turn our attention to other dynamical sets for polynomial diffeomorphisms of positive entropy. These are \(J := J^+ \cap J^-\), \(K := K^+ \cap K^-\), and the set \(J^*\), which coincides
with the closure of the set of periodic points of saddle type. (See [BS1], [BS3], and [BLS] for other characterizations of $J^\ast$.) We have $J^* \subset J \subset K$. We note that none of these sets can be a smooth 3-manifold: otherwise, for any saddle point $p$, it would be a bounded set containing $W^s(p)$ or $W^u(p)$, which is the holomorphic image of $\mathbb{C}$. The following was suggested by Remark 5.9 of Cantat in [C]; we sketch his proof:

**Proposition A.1.** If $J = J^*$, then it is not a smooth 2-manifold.

**Proof.** Let $p$ be a saddle point, and let $W^u(p)$ be the unstable manifold. The slice $J \cap W^u(p)$ is smooth and invariant under multiplication by the multiplier of $Df$. This means that in fact, the multiplier must be real, and the restriction of $G^+$ to the slice must be linear on each (half-space) component of $W^u(p) - J$.

The identity $G^+ \circ f = d \cdot G^+$ means that the canonical metric (defined in [BS8]) is multiplied by $d$. Thus $f$ is quasi-expanding on $J^*$. Now, applying this argument to $f^{-1}$ we get that $f$ is quasi-hyperbolic. Further, $J^* = J$, so it is quasi-hyperbolic on $J$. If $f$ fails to be hyperbolic, then by [BSm] there will be a one-sided saddle point, which can not happen since $J$ is smooth.

Now that $f$ is hyperbolic on $J$, there is a splitting $E^s \oplus E^u$ of the tangent bundle, so we conclude that $J$ is a 2-torus. The dynamical degree must be the spectral radius of an invertible 2-by-2 integer matrix, but this means it is not an integer, which contradicts the fact the the dynamical degree of a Hénon map is its algebraic degree. □

**Proposition A.2.** Suppose that the complex jacobian is not equal to $\pm 1$. Then for each saddle (periodic) point $p$ and each neighborhood $U$ of $p$, neither $J \cap U$ nor $J^* \cap U$ nor $K \cap U$ is a $C^1$ smooth 2-manifold.

**Proof.** Let us write $M := J \cap U$ and $g := f|_M$. (The following argument works, too, if we take $M = J^* \cap U$ or $M = K \cap U$.) The tangent space $T_pM$ is invariant under $Df$. The stable/unstable spaces $E^{s/u} \subset T_p\mathbb{C}^2$ are invariant under $D_p f$. The space $E^s$ (or $E^u$) cannot coincide with $T_p M$, for otherwise the complex stable manifold $W^s(p)$ (or $W^u(p)$) would be locally contained in $M$, and thus globally contained in $J$. But the $W^{s/u}$ are uniformized by $\mathbb{C}$, whereas $J$ is bounded. We conclude that $p$ is a saddle point for $g$, and thus the local stable manifold $W^{s}_{loc}(p; g)$ is a $C^1$-curve inside the complex stable manifold $W^s(p)$. As in Lemma 3.3, we conclude that the multiplier for $D_p f|_{E^s_p}$ is $\pm d$ and the multiplier for $D_p f|_{E^u_p}$ is $\pm 1/d$. Thus the complex Jacobian is $\delta = \pm 1$. □

**Solenoids.** The two results above concern smoothness, but no example is known where $J$, $J^*$ or $K$ is even a topological 2-manifold. In the cases where $J^+$ has been shown to be a topological 3-manifold (see [FS], [HO2], [Bo] and [RT]) it also happens that $J$ is a (topological) real solenoid, and in these cases it is also the case that $J = J^*$. Further, for every saddle (periodic) point $p$, there is a real arc $\gamma_p = W^u_{loc}(p) \cap J$. If we apply the argument of Proposition A.2 to this case, we conclude that $\gamma_p$ is not $C^1$ smooth.
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