INTERSECTION SHEAVES FOR ABEL MAPS

JASON MICHAEL STARR

Abstract. Intersection sheaves, i.e., the Deligne pairings, were first introduced by Deligne in the setting of Poincaré duality for étale cohomology, and later in his work on the determinant of cohomology. Intersection sheaves were generalized from smooth schemes to Cohen-Macaulay schemes by Elkik, and then beyond Cohen-Macaulay schemes by Munoz-Garcia. To define the Abel maps arising in rational simple connectedness on the natural parameter spaces of rational sections, we need a variant of the construction of Munoz-Garcia that has the good properties of the construction using det and Div. In addition, we prove basic properties of the classifying stacks that arise in the definition of Abel maps.

1. Introduction

The Deligne pairing, or intersection sheaf, was introduced for families of smooth, proper curves in [SGA73, Exposé XVIII] as part of the proof of Poincaré duality in étale cohomology. It was developed for more general smooth morphisms in [Del87] where Deligne also enriched the determinant of cohomology in many ways. In contrast to the Chow-theoretic pushforward of an intersection product of divisor classes, which it closely mirrors, the intersection sheaf respects nilpotence of the target (some versions of Chow theory do not), it is integral in the sense that there are no denominators, and it is defined for a robust class of targets including mixed characteristic schemes. It is well-adapted to refined versions such as in Arakelov theory. Finally, and this is crucial here, via its additivity property it extends from \(\mathbb{G}_m\)-torsors, i.e., invertible sheaves, to torsors for more general group schemes of multiplicative type.

Intersection sheaves satisfying the axioms of the original construction were extended to families of (not-necessarily-smooth) Cohen-Macaulay schemes in [Elk89], and finally extended to non-Cohen-Macaulay schemes in [MnGa00]. The construction of the Abel map in [dJHS11] is a special case of an intersection sheaf in relative dimension 0. The intersection sheaf in relative dimension 0 extends the usual norm of invertible sheaves and Cartier divisors for a finite, flat morphism to morphisms that are not necessarily finite and flat. The construction in [dJHS11] in terms of det and Div, cf. [KM76], is essentially the original construction. In addition to satisfying the axioms of intersection sheaves, the det-Div construction gives a formula for the contribution to the intersection sheaf from the locus where the morphism is not finite and flat. This is crucial in [dJHS11]: that formula implies that the Abel maps are compatible with the inductive structure on the moduli spaces of stable sections coming from “boundary” correspondences between these moduli spaces.

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In positive characteristic, the stacks of stable sections are not Deligne-Mumford stacks. Even in positive characteristic, the Hilbert scheme is a projective scheme that parameterizes sections and their specializations. The Hilbert scheme parameterizes closed subschemes that are not necessarily Cohen-Macaulay. Thus, the appropriate version of intersection sheaves is \([\text{MnGa00}]\). However, the formula for the contribution to the intersection sheaf from the non-flat locus is not part of that construction. Our main result is that the construction of \([\text{MnGa00}]\) gives the same intersection sheaves as the det-Div construction in our setting. The result, Proposition 5.3, shows that the det-Div construction of the intersection sheaf extends to the non-Cohen-Macaulay setting and satisfies many of the usual axioms: enough axioms so that it agrees with every other construction satisfying these axioms, including the construction of \([\text{MnGa00}]\).

Since the intersection sheaves are multiadditive, they naturally extend to torsors for \(G_{\rho}^m\) for \(\rho \geq 1\), i.e., ordered \(\rho\)-tuples of invertible sheaves. Combined with fppf descent, which we quickly review in Section 6 this leads to an extension of intersection sheaves to intersection torsors for more general group schemes than \(G_{\rho}^m\). Of course this was part of Deligne’s original construction, allowing him to define the intersection pairing on torsors for finite, flat, commutative group schemes.

To define Abel maps for fibrations of higher Picard rank \(\rho \geq 1\), the relevant group schemes are tori that are not necessarily split. Corollary 7.2 gives the extension of intersection sheaves to intersection torsors for tori.

The strategy of the proof of existence of sections in \([dJHS11]\) and \([Zhu]\) is to ascend up the tower of moduli spaces of stable sections via the boundary correspondences and to prove that “eventually” the fibers of the Abel map are rationally connected. The terms in this tower are indexed by the degree of a torsor on a curve together with its natural partial ordering, so degrees of torsors are also important. Proposition 8.4 gives a degree map satisfying some natural properties.

### 2. Axioms and the Basic Construction

For every morphism of schemes \(\pi : C \to T\) that is proper, that is fppf of pure relative dimension \(d\), and that is cohomologically flat in degree 0, the relative Picard functor is an algebraic space over \(T\), \(\text{Pic}_{C/T}\).

**Definition 2.1.** Let \(\pi\) be a morphism that is proper, fppf of pure relative dimension \(d\), and cohomologically flat in degree 0. An intersection datum for \(\pi\) is a tuple

\[
(n,T_0 \to T,p_0 : Y_0 \to T_0,g_0 : Y_0 \to C_0,(L_0,\ldots,L_n))
\]

of an integer \(n \geq 0\), a \(T\)-scheme \(T_0\), a proper, fppf morphism \(p_0\) of relative dimension \(\leq d + n\) that is projective fppf locally over \(T_0\), a proper, perfect morphism \(g_0\) of \(T_0\)-schemes, and an \((n+1)\)-tuple of invertible sheaves on \(Y_0\).

For fixed \((n,T_0,p_0,g_0)\), an isomorphism between \((n+1)\)-tuples \((L_0,\ldots,L_n)\) and \((L'_0,\ldots,L'_n)\) is an \((n+1)\)-tuple of isomorphisms of invertible sheaves \(L_i \to L'_i\). For a datum as above and for a morphism of \(T\)-schemes, \(T_1 \to T_0\), the pullback datum is

\[
(n,T_1,Y_0 \times_{T_0} T_1 \to T_1,Y_0 \times_{T_0} T_1 \xrightarrow{g_0 \times \text{Id}} C_0 \times_{T_0} T_1, (\text{pr}_{Y_0}^* L_0,\ldots,\text{pr}_{Y_0}^* L_n)).
\]

A relative intersection sheaf for \(\pi\) is an assignment to every intersection datum of a section \(I_{g_0}(L_0,\ldots,L_n)\) over \(T_0\) of \(\text{Pic}_{C/T}\) that satisfies the following axioms.
Definition 2.3. For every Noetherian scheme $Y$, denote by $K(Y)$ the Grothendieck group of bounded, perfect complexes of $\mathcal{O}_Y$-modules with its usual ring structure and lambda operations. For every locally constant function $r : \mathcal{L} \to \mathcal{O}_Y$ that is regular when restricted to every fiber of $p_0$, for the associated Cartier divisor $\iota_0 : Z_0 \hookrightarrow Y_0$, $I_{g_0 \circ \iota_0}(t^0_0 \mathcal{L}_0, \ldots, t^0_{n-1} \mathcal{L}_{n-1})$ equals $I_{g_0}(\mathcal{L}_0, \ldots, \mathcal{L}_n)$.

Remark 2.4. The element $\langle a_0, \ldots, a_n \rangle$ is in the gamma filtration $\gamma^{n+1}$ on $K(Y)$. The gamma filtration is the filtration by ideals that is smallest among those such that every element $\langle a_0, \ldots, a_n \rangle$ is in $\gamma^{n+1}$ and that is “strictly” compatible with pullbacks to all projective space bundles (so that we can use the “splitting principle”) in the following sense. For every projective bundle $f : \mathbb{P}(E) \to Y$ of relative dimension $r$ and with relative Serre twisting sheaf $\mathcal{O}(1)$, an element $a \in K(Y)$ is in $\gamma^{n+1}$ if and only if $[\mathcal{O}(1)] - [\mathcal{O}(E)] = f^* a$ is in $\gamma^{n+1}$. For invertible sheaves $(\mathcal{L}_0, \ldots, \mathcal{L}_n)$, the operation $\langle a_0, \ldots, a_n \rangle$ is not multiaadditive for tensor product of invertible sheaves. However, it is multiaadditive for tensor products modulo the next piece of the gamma filtration, $\gamma^{n+2}$. The intersection sheaf below extends to an additive map $\det(Rg_*(\mathcal{O}_Y))$ defined on $\gamma^{n+1}$. If that map annihilates $\gamma^{n+2}$, then the ideal sheaf $\langle \mathcal{L}_0, \ldots, \mathcal{L}_n \rangle \mapsto \det(Rg_*(\mathcal{L}_0, \ldots, \mathcal{L}_n))$ is
multiadditive for tensor products. It seems difficult to prove directly that any one of the constructions of the intersection sheaf annihilates $\gamma^{n+2}$.

**Hypothesis 2.5.** Let $T$ be a Noetherian scheme. Let $p : Y \to T$, $\pi : C \to T$ be morphisms of algebraic spaces.

**Lemma 2.6.** If $\pi$ is smooth, and if $p$ is locally fppf, then every $T$-morphism $g : Y \to C$ is a perfect morphism in the sense of [BGI71, Définition III.4.1]. If $\pi$ is separated and $p$ is proper, then also $g$ is proper.

**Proof.** Since $\pi$ is smooth, the immersion $\Gamma_g : Y \to Y \times_T C$ is a regular immersion, hence an LCI morphism. By [BGI71, Proposition VIII.1.7], $\Gamma_g$ is a perfect morphism. If $\pi$ is separated, then $\Gamma_g$ is a closed immersion, hence proper. Since $p$ is locally fppf, also $p \times \text{Id}_C : Y \times_T C \to C$ is locally fppf. Hence by [BGI71 Corollaire III.4.3.1], also $p \times \text{Id}_C$ is perfect. If $p$ is proper, then $p \times \text{Id}_C$ is proper. Finally by [BGI71 Proposition III.4.5], the composition $(p \times \text{Id}_C) \circ \Gamma_g$ is perfect, i.e., $g$ is perfect. Also, if $p$ is proper and if $\pi$ is separated, then both $p \times \text{Id}_C$ and $\Gamma_g$ are proper. Then the composition $g \circ \Gamma_g$ is proper. □

**Hypothesis 2.7.** Assume that $\pi : C \to T$ is flat of constant relative dimension $d$. Assume that $p : Y \to T$ is flat of constant relative dimension $d + n$ for an integer $n \geq 0$. Assume that $g : C \to Y$ is a proper, perfect $T$-morphism.

By [BGI71 Proposition III.4.8], for every perfect complex $E$ of bounded amplitude on $Y$, also $Rg_*(E)$ is a perfect complex of bounded amplitude on $C$. Thus, by [KM76], there is an associated invertible sheaf $\det(Rg_*(E))$ on $C$. This is additive for direct sum, and even for distinguished triangles of perfect complexes. By this additivity property, $\det(Rg_*(-))$ extends to an additive homomorphism from the Grothendieck group of virtual classes of perfect complexes on $Y$ to the Picard group of $C$.

**Definition 2.8.** Assuming Hypothesis 2.7, for every ordered $(n+1)$-tuple $(\mathcal{L}_0, \ldots, \mathcal{L}_n)$ of invertible sheaves on $Y$, the intersection sheaf or Deligne pairing relative to $g$, $I_g(\mathcal{L}_0, \ldots, \mathcal{L}_n)$, is the invertible sheaf on $C$ obtained by applying $\det(Rg_*(-))$ to the virtual class $\langle \mathcal{L}_0, \ldots, \mathcal{L}_n \rangle := ([\mathcal{L}_0] - [\mathcal{O}_Y]) \otimes \cdots \otimes ([\mathcal{L}_n] - [\mathcal{O}_Y])$.

The virtual class $\langle \mathcal{L}_0, \ldots, \mathcal{L}_n \rangle$ is basically the cup product of the K-theoretic first-Chern-classes (up to twisting by the K-theory class of an invertible sheaf). This is an explicit alternating sum of K-theory classes of invertible $\mathcal{O}_Y$-modules, and $\det(Rg_*(-))$ of this class is the alternating tensor product of the corresponding terms. The intersection sheaf is functorial for the category of ordered $(n+1)$-tuples of invertible $\mathcal{O}_Y$-modules with isomorphisms as the morphisms. This is also $\mathfrak{S}_{n+1}$-equivariant for permutation of the terms of $(\mathcal{L}_0, \ldots, \mathcal{L}_n)$. If any $\mathcal{L}_i$ is isomorphic to $\mathcal{O}_Y$, then the intersection sheaf is isomorphic to $\mathcal{O}_C$.

### 3. Regular Sequences

The intuition of intersection sheaves is stated most simply in terms of regular sequences, and this is the basis of the construction for many authors. Whether or
Let $(s_i)_i$ be an $\mathcal{O}_Y$-module homomorphism,

$$(s_i)_{0 \leq i \leq n} : \bigoplus_{i=0}^{n} \mathcal{L}_i^g \rightarrow \mathcal{O}_Y.$$ 

**Hypothesis 3.1.** Assume that $d \geq 1$, and assume that every fiber of $\pi$ satisfies Serre’s condition $S_2$ (if $d$ equals 1, then conditions $S_1$ and $S_2$ are equivalent). Assume that the restriction of $(s_i)_i$ to every fiber of $p$ is a regular sequence.

**Proposition 3.2.** Under the above hypotheses, there exists $s = I_g(s_0, \ldots, s_n)$, an $\mathcal{O}_C$-module homomorphism $\mathcal{O}_C \rightarrow I_g(\mathcal{L}_0, \ldots, \mathcal{L}_n)$ whose restriction to every fiber of $\pi$ is injective. The closed subset $g(\text{Supp}(\text{Coker}(s_i)_i))$ contains $\text{Supp}(\text{Coker}(s))$. Finally, for every virtual linear combination $\mathcal{H}$ of locally free $\mathcal{O}_Y$-modules of virtual dimension $r \in \mathbb{Z}$, $\det(Rg_*(\mathcal{H} \otimes (\mathcal{L}_0, \ldots, \mathcal{L}_n)))$ is isomorphic to $I_g(\mathcal{L}_0, \ldots, \mathcal{L}_n)^{\otimes r}$.

**Proof.** Denote by $K(s_i)_i$ the Koszul complex of $(s_i)_i$. This is the free differential graded $\mathcal{O}_Y$-algebra on $(s_i)_i : \bigoplus_i \mathcal{L}_i^g \rightarrow \mathcal{O}_Y$. It is a complex of locally free sheaves concentrated in degrees $[-n-1, 0]$. In degree $-d$, the associated locally free sheaf is

$$K_d(s_i)_i = \bigwedge_{\mathcal{O}_Y} \left( \bigoplus_{i=0}^{n} \mathcal{L}_i^g \right).$$

Thus, the $K$-theory class of $K(s_i)_i$ is $\mathcal{L} \otimes (\mathcal{L}_0, \ldots, \mathcal{L}_n)$, for the invertible sheaf

$$\mathcal{L} = \mathcal{L}_0 \otimes \cdots \otimes \mathcal{L}_n.$$ 

By the hypothesis, the only nonzero homology sheaf of this complex is in degree 0, and that homology sheaf is $\text{Coker}(s_i)_i$. By hypothesis, this is flat over $T$ of relative dimension $d-1$.

Because the fibers of $\pi$ satisfy $S_2$, every point of $C$ of depth $\leq 0$ is a generic point of its $\pi$-fiber, and every point of $C$ of depth $\leq 1$ is either a generic point or a codimension 1 point of its $\pi$-fiber. By hypothesis, after restricting to the fiber over each point $t$ of $T$, the support of $\text{Coker}(s_i)_i$ has dimension $\leq d-1$. Since the $\pi$-fiber $C_t$ has dimension $d$, the fiber of $g$ over every codimension 0 point of $C_t$ is disjoint from the support of $\text{Coker}(s_i)_i$. Similarly, the fiber of $g$ over every codimension 1 point of $C_t$ intersects the support of $\text{Coker}(s_i)_i$ in a zero-dimensional scheme. Altogether, $K(s_i)_i$ satisfies the transversality hypothesis $Q_{-1}$ relative to $g : Y \rightarrow C$. [KM76, Definition, p. 50]; in fact it even satisfies the hypothesis after restricting to the fiber over every point of $T$.

By [KM76, Proposition 9], there is an associated Div section

$$s : \mathcal{O}_C \rightarrow \det(Rg_*(K(s_i)_i)),$$

and for every locally free sheaf $\mathcal{H}$ of rank $r$, $\det(Rg_*(\mathcal{H} \otimes K(s_i)_i))$ is canonically isomorphic to $\det(Rg_*(K(s_i)_i))^{\otimes r}$. (Please note: the statement of the proposition is only for $r = 1$, but the proof of the proposition works for all $r \geq 1$.)

In particular, for $\mathcal{H}$ equal to $\mathcal{L}$, $\det(Rg_*(K(s_i)_i))$ is canonically isomorphic to $I_g(\mathcal{L}_0, \ldots, \mathcal{L}_n)$. Using additivity of $\det(Rg_*(-))$, for every virtual linear combination $\mathcal{H}$ of locally free $\mathcal{O}_Y$-modules, $\det(Rg_*(\mathcal{H} \otimes (\mathcal{L}_0, \ldots, \mathcal{L}_n)))$ is isomorphic to $I_g(\mathcal{L}_0, \ldots, \mathcal{L}_n)^{\otimes r}$. □
Deligne's study of the functoriality of $I_g(s_0, \ldots, s_n)$ in $(s_0, \ldots, s_n)$ is used to prove important properties of $I_g(L_0, \ldots, L_n)$ such as additivity. Let $\mathcal{A}$ and $\mathcal{B}$ be invertible sheaves on $Y$. Let

$$\alpha : \mathcal{A}^\vee \to \mathcal{O}_Y, \quad \beta : \mathcal{B}^\vee \to \mathcal{O}_Y$$

be $\mathcal{O}_Y$-module homomorphisms. Then $\mathcal{A} \otimes_{\mathcal{O}_Y} \mathcal{B}$ is an invertible sheaf on $Y$, and there is a tensor product $\mathcal{O}_Y$-module homomorphism,

$$\alpha \otimes \beta : \mathcal{A}^\vee \otimes_{\mathcal{O}_Y} \mathcal{B}^\vee \to \mathcal{O}_Y.$$

There is a useful compatibility between the Koszul complexes $K(\alpha)$, $K(\beta)$, $K(\alpha \otimes \beta)$, and $K(\alpha, \beta)$.

There are natural morphisms of differential graded $\mathcal{O}_Y$-algebras, $K(\alpha \otimes \beta) \to K(\alpha)$ and $K(\alpha \otimes \beta) \to K(\beta)$. There are morphisms of complexes of $\mathcal{O}_Y$-modules,

$$L_\alpha : \mathcal{A}^\vee \otimes_{\mathcal{O}_Y} K(\beta) \to K(\alpha \otimes \beta),$$

$$L_\beta : (\mathcal{A}^\vee \otimes_{\mathcal{O}_Y} \mathcal{B}^\vee) \to K(\alpha \otimes \beta),$$

$$L_{\alpha \beta} : (\mathcal{A}^\vee \otimes_{\mathcal{O}_Y} \mathcal{B}^\vee) \to K(\alpha \otimes \beta).$$

Since $\mathcal{A}$ and $\mathcal{B}$ are invertible sheaves, these are even homomorphisms of differential graded modules over $K(\alpha \otimes \beta)$. The direct sum is a morphism of complexes of $\mathcal{O}_Y$-modules,

$$L_\alpha \oplus L_\beta : (\mathcal{A}^\vee \otimes_{\mathcal{O}_Y} K(\beta)) \oplus (K(\alpha) \otimes \mathcal{B}^\vee) \to K(\alpha \otimes \beta).$$

Define $C(\alpha, \beta)$ to be the mapping cone of $L_\alpha \oplus L_\beta$,

$$\begin{align*}
(A^\vee \otimes_{\mathcal{O}_Y} B^\vee) \oplus (A^\vee \otimes_{\mathcal{O}_Y} B^\vee) &\xrightarrow{d_{C, 2}} A^\vee \oplus B^\vee \oplus (A^\vee \otimes_{\mathcal{O}_Y} B^\vee) \xrightarrow{d_{C, 1}} \mathcal{O}_Y, \\
&= \frac{d_{C, 1} = \left[ \begin{array}{ccc} -\alpha & -\beta & \alpha \otimes \beta \\ -\text{Id} & 0 & 0 \\ 0 & -\alpha \otimes \text{Id} & -\text{Id} \end{array} \right]}.
\end{align*}$$

This is a differential graded $\mathcal{O}_Y$-algebra. There is a mapping cone short exact sequence,

$$0 \to K(\alpha \otimes \beta) \xrightarrow{u} C(\alpha, \beta) \xrightarrow{v} (A^\vee \otimes_{\mathcal{O}_Y} K(\beta)) \oplus (K(\alpha) \otimes_{\mathcal{O}_Y} B^\vee) \xrightarrow{[1]} 0. \quad (1)$$

The morphism $u$ is a homomorphism of differential graded $\mathcal{O}_Y$-algebras, and $v$ is a homomorphism of differential graded $K(\alpha \otimes \beta)$-modules.

There is a natural morphism of differential graded $\mathcal{O}_Y$-algebras, $K(\alpha, \beta) \to K(\text{Id})$. The zero map,

$$0 : \mathcal{A}^\vee \otimes_{\mathcal{O}_Y} \mathcal{B} \otimes_{\mathcal{O}_Y} K(\text{Id}) \to K(\alpha, \beta),$$
is a homomorphism of differential graded $K(\alpha, \beta)$-modules. The mapping cone $C'(\alpha, \beta)$ is just $(A^\vee \otimes_{O_Y} B^\vee \otimes_{O_Y} K(\operatorname{Id})) [1] \oplus K(\alpha, \beta)$,

$$(A^\vee \otimes_{O_Y} B^\vee) \oplus (A^\vee \otimes_{O_Y} B^\vee) \xrightarrow{d_{C',2}} (A^\vee \otimes_{O_Y} B^\vee) \oplus A^\vee \oplus B^\vee \xrightarrow{d_{C',1}} O_Y,$$

$$d_{C',1} = \begin{bmatrix} 0 & \alpha & \beta \\ \operatorname{Id} & 0 & 0 \end{bmatrix},$$

$$d_{C',2} = \begin{bmatrix} \operatorname{Id} & 0 & 0 \\ 0 & -\operatorname{Id} \otimes \beta \\ 0 & \alpha \otimes \operatorname{Id} \end{bmatrix}.$$ This is a differential graded $O_Y$-algebra. There is a mapping cone short exact sequence,

$$0 \to K(\alpha, \beta) \xrightarrow{u'} C'(\alpha, \beta) \xrightarrow{v'} A^\vee \otimes_{O_Y} B^\vee \otimes_{O_Y} K(\operatorname{Id}) \to 0.$$ (2)

The morphism $u'$ is a homomorphism of differential graded $O_Y$-algebras, and $v'$ is a homomorphism of differential graded $K(\alpha, \beta)$-modules.

**Lemma 3.3.** The complexes of $O_Y$-modules, $C(\alpha, \beta)$ and $C'(\alpha, \beta)$, are isomorphic.

**Proof.** Consider the diagram

\[
\begin{array}{c}
(A^\vee \otimes_{O_Y} B^\vee) \oplus (A^\vee \otimes_{O_Y} B^\vee) \xrightarrow{d_{C,2}} A^\vee \otimes B^\vee \oplus (A^\vee \otimes_{O_Y} B^\vee) \xrightarrow{d_{C,1}} O_Y \\
\phi_2 \downarrow \quad \phi_1 \downarrow \quad \downarrow \operatorname{Id} \\
(A^\vee \otimes_{O_Y} B^\vee) \oplus (A^\vee \otimes_{O_Y} B^\vee) \xrightarrow{d_{C',2}} (A^\vee \otimes_{O_Y} B^\vee) \oplus A^\vee \oplus B^\vee \xrightarrow{d_{C',1}} O_Y,
\end{array}
\]

\[
\phi_1 = \begin{bmatrix} 0 & 0 & \operatorname{Id} \\ -\operatorname{Id} & 0 & \operatorname{Id} \otimes \beta \\ 0 & -\operatorname{Id} & 0 \end{bmatrix},
\]

\[
\phi_2 = \begin{bmatrix} -\operatorname{Id} & -\operatorname{Id} \\ 0 & \operatorname{Id} \end{bmatrix}.
\]

It is straightforward to check that $\phi$ is a morphism of complexes. Also, each of $\phi_1$ and $\phi_2$ is an isomorphism. Thus, $\phi$ is an isomorphism of complexes of $O_Y$-modules. \hfill \Box

In what follows, all we need is existence of this isomorphism; we need no other compatibilities. That is fortunate: the isomorphism above is not unique, and $\phi$ is not an isomorphism of differential graded $O_Y$-algebras. Nonetheless, it is a lifting to perfect differential graded $O_Y$-algebras of an elementary homomorphism of $O_Y$-algebras. Since $A^\vee \otimes_{O_Y} B^\vee \otimes_{O_Y} K(\operatorname{Id})$ is an acyclic complex, $u'$ is a quasi-isomorphism. Thus, in the derived category of complexes of $O_Y$-modules, there is a distinguished triangle,

\[
(A^\vee \otimes_{O_Y} K(\beta)) \oplus (K(\alpha) \otimes_{O_Y} B^\vee) \xrightarrow{-L_{\alpha} \otimes L_{\beta}} K(\alpha \otimes \beta) \xrightarrow{(u')^{-1} \circ \phi_{\operatorname{un}}} K(\alpha, \beta)
\]

\[
\xrightarrow{\circ \phi_{\operatorname{un}}^{-1} \circ u'} (A^\vee \otimes_{O_Y} K(\beta)) \oplus (K(\alpha) \otimes_{O_Y} B^\vee) [1]
\]

When $(\alpha, \beta)$ is a regular sequence, this distinguished triangle is a lifting to complexes of locally free sheaves of the short exact sequence of $O_Y$-modules,

$$0 \to (A^\vee \otimes_{O_Y} (O_Y/\beta)) \oplus ((O_Y/\alpha) \otimes_{O_Y} B^\vee) \to O_Y/(\alpha \otimes \beta) \to O_Y/(\alpha + \beta) \to 0.$$
Hypothesis 3.4. Assume that $d \geq 1$, and assume that every fiber of $\pi$ satisfies Serre’s condition $S_2$ (if $d$ equals 1, then conditions $S_1$ and $S_2$ are equivalent). Assume that the restriction of $(s_0, \ldots, s_{n-1}, \alpha, \beta)$ to every fiber of $p$ is a regular sequence.

Proposition 3.5. Under the above hypotheses, there is an equality of effective Cartier divisors, $I_g(s_0, \ldots, s_{n-1}, \alpha \otimes \beta) = I_g(s_0, \ldots, s_{n-1}, \alpha) \otimes I_g(s_0, \ldots, s_{n-1}, \beta)$.

In particular, there is an isomorphism of intersection sheaves, $I_g(\mathcal{L}_0, \ldots, \mathcal{L}_{n-1}, \mathcal{A} \otimes \mathcal{O}_Y, \mathcal{B}) \cong I_g(\mathcal{L}_0, \ldots, \mathcal{L}_{n-1}, \mathcal{A}) \otimes \mathcal{O}_Y, I_g(\mathcal{L}_0, \ldots, \mathcal{L}_{n-1}, \mathcal{A})$.

Proof. Because the sequence $(s_0, \ldots, s_{n-1}, \alpha, \beta)$ is regular on every fiber, so are the subsequences $(s_0, \ldots, s_{n-1}, \alpha)$ and $(s_0, \ldots, s_{n-1}, \beta)$. Also, for every fiber of $p$, since both $\overline{\mathfrak{p}}$ and $\overline{\mathfrak{b}}$ are regular on the quotient by $(s_0, \ldots, s_{n-1})$, also $\alpha \otimes \beta$ is regular on the quotient by $(s_0, \ldots, s_{n-1})$. Thus, also the sequence $(s_0, \ldots, s_{n-1}, \alpha \otimes \beta)$ is regular on every fiber of $p$. Thus all of the Cartier divisors above are defined.

Tensoring the Koszul complex $K(s_0, \ldots, s_{n-1})$ with the short exact sequence in Equation 8 gives a mapping complex short exact sequence,

$$0 \to K(s_0, \ldots, s_{n-1}, \alpha, \beta) \xrightarrow{\text{Id} \otimes u'} K(s_0, \ldots, s_{n-1}) \otimes \mathcal{O}_Y, C'((\alpha, \beta) \xrightarrow{\text{Id} \otimes v'} \mathcal{A}' \otimes \mathcal{O}_Y, B' \otimes \mathcal{O}_Y, K(s_0, \ldots, s_{n-1}, \text{Id}) \to 0.$$ 

The derived functor $Rg_*$ preserves mapping cones. Thus there is a mapping cone short exact sequence,

$$0 \to Rg_*(K(s_0, \ldots, s_{n-1}, \alpha, \beta)) \xrightarrow{Rg_*(\text{Id} \otimes u')} Rg_*(K(s_0, \ldots, s_{n-1}) \otimes \mathcal{O}_Y, C'((\alpha, \beta) \xrightarrow{Rg_*(\text{Id} \otimes v')} \mathcal{A}' \otimes \mathcal{O}_Y, B' \otimes \mathcal{O}_Y, K(s_0, \ldots, s_{n-1}, \text{Id}) \to 0.$$ 

The third complex is acyclic. By hypothesis, $K(s_0, \ldots, s_{n-1}, \alpha, \beta)$ satisfies hypothesis $Q_{-2}$ relative to $g$, [KM76, Definition, p. 50]. In other words, the open complement $U$ of $g(\text{Supp}(\text{Coker}(s_0, \ldots, s_{n-1}, \alpha, \beta)))$ contains all depth 0 and depth 1 points of $C$, and $Rg_*(K(s_0, \ldots, s_{n-1}))$ is acyclic on $U$. By [KM76, Proposition 9], there is a Div Cartier divisor of $Rg_*(K(s_0, \ldots, s_{n-1}, \alpha, \beta))$, and this Cartier divisor is acyclic on $U$. By Krull’s Hauptidealsatz, every minimal prime over a principal ideal (assuming that there are any such primes) has height 0 or 1. Thus, the Div Cartier divisor is trivial, $O_C \xrightarrow{\text{Id}} O_C$. Finally, by [KM76, Theorem 3], also $Rg_*$ of the middle complex is good, and the associated Div Cartier divisor is trivial.

Tensoring the Koszul complex $K(s_0, \ldots, s_{n-1})$ with the mapping cone short exact sequence in Equation 8 gives another mapping cone short exact sequence,

$$0 \to K(s_0, \ldots, s_{n-1}, \alpha \otimes \beta) \xrightarrow{\text{Id} \otimes u_{L_0} \otimes L_3} K(s_0, \ldots, s_{n-1}) \otimes C(\alpha, \beta) \xrightarrow{\text{Id} \otimes v_{L_0} \otimes L_3} (A' \otimes \mathcal{O}_Y, K(s_0, \ldots, s_{n-1}, \beta)) \oplus (K(s_0, \ldots, s_{n-1}, \alpha) \otimes \mathcal{O}_Y, K(\alpha)) \to 0.$$ 

The derived functor $Rg_*$ preserves mapping cones. Thus there is a mapping cone short exact sequence,

$$0 \to Rg_*(K(s_0, \ldots, s_{n-1}, \alpha \otimes \beta)) \xrightarrow{Rg_*(\text{Id} \otimes u_{L_0} \otimes L_3)} Rg_*(K(s_0, \ldots, s_{n-1}) \otimes C(\alpha, \beta)) \xrightarrow{Rg_*(\text{Id} \otimes v_{L_0} \otimes L_3)} Rg_*(A' \otimes \mathcal{O}_Y, K(s_0, \ldots, s_{n-1}, \beta)) \oplus (K(s_0, \ldots, s_{n-1}, \alpha) \otimes \mathcal{O}_Y, K(\alpha)) \to 0.$$ 

The first and third complexes on $Y$ satisfy hypothesis $Q_{-1}$ relative to $g$. Thus, the first and third complexes on $C$ are good, [KM76, p. 47]. By [KM76, Theorem 3(ii)],
also the middle complex on $C$ is good, and the “sum” of the Div Cartier divisors of the first and third complex equals the Div Cartier divisor of the middle complex. By Lemma 3.3, the middle complex above is isomorphic to the middle complex from the previous paragraph. That complex had trivial Div Cartier divisor. Thus, the middle complex above has trivial Div Cartier divisor. Therefore, the Div Cartier divisor of the first complex is the inverse Cartier divisor of the Div Cartier divisor of the third complex. Combined with Proposition 3.2, this precisely gives

$$I_g(s_0, \ldots, s_{n-1}, \alpha \otimes \beta) = I_g(s_0, \ldots, s_{n-1}, \beta) \otimes I_g(s_0, \ldots, s_{n-1}, \alpha).$$

□

This is the basic additivity of intersection sheaves under a regularity hypothesis. Via the $\mathcal{S}_{n+1}$-equivariance and usual methods of multilinear algebra, this implies other properties. The following property is helpful in proving additivity with no regularity hypothesis.

For $i = 0, \ldots, n-1$, let $L_i'$ and $L_n''$ be invertible $O_Y$-modules with $O_Y$-module homomorphisms,

$$s_i' : (L_i')^\vee \to O_Y, \quad s_n'' : (L_n'')^\vee \to O_Y.$$

Let $A', A'', B', \text{ and } B''$ be invertible $O_Y$-modules with $O_Y$-module homomorphisms,

$$\alpha' : (A')^\vee \to O_Y, \quad \alpha'' : (A'')^\vee \to O_Y,$$

$$\beta' : (B')^\vee \to O_Y, \quad \beta'' : (B'')^\vee \to O_Y.$$

Define $L_n' = A' \otimes_{O_Y} B'$, resp. $L_n'' = A'' \otimes_{O_Y} B''$. Define $s_n' = \alpha' \otimes \beta'$, resp. $s_n'' = \alpha'' \otimes \beta''$.

Denote $J = \{0, \ldots, n-1, n\}$, a set with $n + 1$ elements. For every partition $J = J' \sqcup J''$, one of $J'$ or $J''$ contains $n$. Define $L_{J', J''}$, resp. $L_{J', J''}, A, \text{ and } L_{J', J''}, B$, to be the length-$(n+1)$ sequence of invertible sheaves $(L_0, \ldots, L_{n-1}, L_n)$ where for $i = 0, \ldots, n-1$, $L_i$ equals $L_i'$ or $L_n'$ depending on whether $i \in J'$ or $i \in J''$, and where $L_n$ equals $L_n'$, resp. $A_n'$, $B_n'$, or $L_n''$, resp. $A_n''$, $B_n''$, depending on whether $n \in J'$ or $n \in J''$. Similarly, define $L_{J', J''}, A, B$ to be the length-$(n+1)$ sequence of invertible sheaves $(L_0, \ldots, L_{n-1}, A, B)$ as above, where $A$, resp. $B$, equals $A'$ or $A''$, resp. $B'$ or $B''$, depending on whether $n \in J'$ or $n \in J''$. For each of these sequences, there is a corresponding sequence $s_{J', J''}$, resp. $s_{J', J''}, A, s_{J', J'', B}, s_{J', J''}, A, B$ of $O_Y$-module homomorphisms $s_j : L_{J'}^j \to O_Y$, resp. $\alpha : A^\vee \to O_Y,$ $\beta : B^\vee \to O_Y,$ using the $O_Y$-module homomorphisms from the previous paragraph. Finally, let $(r_{J', J''})_{J', J''}$ be a sequence of integers $r_{J', J''}$ indexed by all partitions $(J', J'')$ of $J$.

**Hypothesis 3.6.** Assume that $d \geq 1$, and assume that every fiber of $\pi$ satisfies Serre’s condition $S_2$ (if $d$ equals 1, then conditions $S_1$ and $S_2$ are equivalent). Assume that for every partition $(J', J'')$ of $J = \{0, \ldots, n\}$, the restriction of the sequence $s_{J', J'', A, B}$ to every fiber of $p$ is a regular sequence.

**Corollary 3.7.** Under the above hypotheses, there is an equality of Cartier divisors (written additively),

$$\sum_{J', J''} r_{J', J''} \Div(I_g(s_{J', J''})) = \sum_{J', J''} r_{J', J''}(\Div(I_g(s_{J', J''}, A)) + \Div(I_g(s_{J', J'', B}))).$$
In particular, there is an isomorphism of intersection sheaves,
\[ \bigotimes_{J', J''} I_g(\mathcal{L}_{J', J''}) \cong \bigotimes_{J', J''} (I_g(\mathcal{L}_{J', J''}, A) \otimes_{\mathcal{O}_C} I_g(\mathcal{L}_{J', J''}, B)) \otimes_{\mathcal{O}_C} I_g(\mathcal{L}_{J', J''}, C). \]

Proof. It suffices to prove for every partition \( (J', J'') \) that the Cartier divisor \( I_g(s_{J', J''}) \) equals \( I_g(s_{J', J''}, A) \otimes I_g(s_{J', J''}, B) \), using multiplicative notation. This follows from Proposition 3.5. \( \square \)

4. Properties

There are a few straightforward properties of the intersection sheaf. For every morphism \( a : C_0 \rightarrow C \), denote by \( Y_0 \) the fiber product \( Y \times_C C_0 \), and denote by \( g \times Id_{C_0} \) the projection \( Y \times_C C_0 \rightarrow C_0 \).

Lemma 4.1. Assuming Hypothesis 2.7, assuming that \( a \) is fppf of pure relative dimension \( e \), then \( g \times Id_{C_0} \) is proper and perfect. For every ordered \((n + 1)\)-tuple \((\mathcal{L}_0, \ldots, \mathcal{L}_n)\) of invertible \( \mathcal{O}_Y \)-modules, there is an isomorphism of invertible sheaves on \( C_0 \),
\[ I_g(\mathcal{L}_0, \ldots, \mathcal{L}_n) : a^* I_g(\mathcal{L}_0, \ldots, \mathcal{L}_n) \rightarrow I_{g \times Id_{C_0}}(pr_Y^* \mathcal{L}_0, \ldots, pr_Y^* \mathcal{L}_n). \]
This isomorphism is natural in \((\mathcal{L}_0, \ldots, \mathcal{L}_n)\) and in \( a \).

Proof. The morphism \( g \times Id_{C_0} \) is perfect by [BCI71 Corollaire III.4.7.1]. It is proper since it is a base change of a proper morphism. Pullback under \( pr_Y^* \) is a ring homomorphism from the \( K \)-ring of virtual perfect complexes on \( Y \) to the \( K \)-ring of virtual perfect complexes on \( Y \times_C C_0 \). Similarly, \( \text{det} (Rg_*(-)) \) is compatible with \( a \) [KM76 p. 46]. The lemma follows from these compatibilities. \( \square \)

Similarly, for a morphism of Noetherian schemes \( b : T_0 \rightarrow T \), denote \( C_0 = C \times_T T_0 \), denote \( Y_0 = Y \times_T T_0 \), and denote \( g \times Id_{T_0} : Y_0 \rightarrow C_0 \) the projection. The same proof proves the following.

Lemma 4.2. Assuming Hypothesis 2.7, the morphism \( g \times Id_{T_0} \) is proper and perfect. For every ordered \((n + 1)\)-tuple \((\mathcal{L}_0, \ldots, \mathcal{L}_n)\) of invertible \( \mathcal{O}_Y \)-modules, there is an isomorphism of invertible sheaves on \( C_0 \),
\[ I_g(\mathcal{L}_0, \ldots, \mathcal{L}_n) : pr_C^* I_g(\mathcal{L}_0, \ldots, \mathcal{L}_n) \rightarrow I_{g \times Id_{T_0}}(pr_Y^* \mathcal{L}_0, \ldots, pr_Y^* \mathcal{L}_n). \]
This isomorphism is natural in \((\mathcal{L}_0, \ldots, \mathcal{L}_n)\) and in \( b \).

Hypothesis 4.3. Let \( p : Y \rightarrow T \) be an fppf morphism. Let \((Z_i \hookrightarrow Y)\), be a nonempty finite set of closed subschemes that are each \( T \)-flat of constant relative dimension \( d_i \geq 0 \). Let \( \mathcal{L} \) be an invertible sheaf on \( Y \). Let \( O(1) \) be a \( p \)-relatively ample invertible sheaf on \( Y \).

Lemma 4.4. Under the above hypotheses, there exists an integer \( \bar{m} \) such that for every \( m \geq \bar{m} \), after base change from \( T \) to a Zariski cover, there exists a homomorphism of coherent sheaves \( s : \mathcal{O}(-m) \mathcal{L} \) such that \( s \), resp. \( s|_{Z_i} \), is injective after restriction to every fiber of \( Y \rightarrow T \), resp. after restriction to every fiber of \( Z_i \rightarrow T \). Thus, the support of \( \text{Coker}(s) \), resp. \( \text{Coker}(s|_{Z_i}) \), is a \( T \)-flat Cartier divisor in \( Y \), resp. in \( Z_i \).
Proof. By semicontinuity, there exists $\tilde{m}$ such that for all $m \geq \tilde{m}$, $p_* \mathcal{L}(m)$ surjects onto the sections of $\mathcal{L}(m)$ on each fiber of $p$. Thus, every homomorphism $s$ defined on a fiber of $p$ extends to a Zariski neighborhood of that point. For a homomorphism $s$, resp. $s|_{Z_i}$, if the restriction to a fiber of $p$ is injective, then $s$, resp. $s|_{Z_i}$, is injective with flat cokernel on a Zariski open neighborhood of that fiber, \cite[Theorem 23.7]{Mat}. Thus, to construct a Zariski neighborhood of a fiber of $p$ and $s$ as above, it suffices to prove the result for the fiber. Thus, assume that $T$ is Spec $k$ for a field $k$.

By primary decomposition, there are finitely many associated points of $Y$, resp. of the finitely many closed subschemes $Z_i$. By ampieness, up to increasing $\tilde{m}$, for every $m \geq \tilde{m}$, there exists $s : \mathcal{O}(-m) \to \mathcal{L}$ that is an isomorphism on the stalks at each of the finitely many associated points. Since the set of zero divisors in a Noetherian ring is precisely the union of the associated primes, $s$ is injective, resp. each $s|_{Z_i}$ is injective.

Hypothesis 4.5. Let $p : Y \to T$ be an fpf morphism. Let $(Z_i \hookrightarrow Y)_i$ be a nonempty finite set of closed subschemes that are each $T$-flat of constant relative dimension $d_i \geq 0$. Let $\mathcal{O}(1)$ be a $p$-relatively ample invertible sheaf on $Y$. Let $e \geq 1$ be an integer, and let $J$ be $\{0, \ldots, e\}$. Let $(\mathcal{L}_j)_{j \in J}$ be a finite collection of $e$ invertible sheaves on $Y$.

Lemma 4.6. Under the above hypotheses, for every integer $\tilde{m}$, after base change from $T$ to a Zariski cover of $T$, there exists a sequence $(m_j)_{j \in J}$ of integers $m_j \geq \tilde{m}$ and there exists a collection $(s_j)_{j \in J}$ of homomorphisms of coherent sheaves $s_j : \mathcal{O}(-m_j) \to \mathcal{L}_j$ such that for every $i$ and for every finite subset $J' \subset J$ of size $e_i \leq d_i + 1$, the sequence $(s_j|_{Z_i})_{j \in J'}$ is regular on every fiber of $Z_i \to T$. Thus the closed subscheme $Z_{i,J'}$ of $Z_i$ cut out by this regular sequence is $T$-flat of constant relative dimension $d_i - e_i$; $Z_i$ is empty when $e_i$ equals $d_i + 1$. Moreover, the set $\mathcal{M} \subset \mathbb{Z}_{\geq 0}^{n+1}$ of sequences $(m_j)_{j \in J}$ for which there exists such a datum has the following property: for every $(m_0, \ldots, m_n) \in \mathcal{M}$, for every $r = 1, \ldots, n$, there exists an integer $\tilde{m}_r$ such that for every $m \geq \tilde{m}_r$, there exists $(m_{r+1}, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^{n-r}$ with $(m_0, \ldots, m_{r-1}, m, m_{r+1}, \ldots, m_n)$ in $\mathcal{M}$.

Proof. This is proved by induction on $e$. When $e$ equals 0, this follows from Lemma 4.4. Thus, by way of induction, assume that $e > 0$, and assume the result is proved for smaller values of $e$. Partition $J$ as $\{e\} \sqcup J_e$. By the induction hypothesis, after base change to a Zariski cover of $T$, there exists $(m_j)_{j \in J_e}$ and there exists $(s_j)_{j \in J_e}$ such that for every $i$ and for every subset $J' \subset J_e$ of size $e_i \leq d_i$, the sequence $(s_j|_{Z_i})_{j \in J'}$ is regular on all fibers of $Z_i \to T$. Thus the closed subscheme $Z_{i,J'}$ of $Z_i$ cut out by this regular sequence is $T$-flat of relative dimension $d_i - e_i$.

For every $(m_j)_{j \in J_e}$ and $(s_j)_{j \in J_e}$ as above, consider the collection $(Z_{i,J'})_{i,J'}$ of $T$-flat closed subschemes of $Y$ where for every $i$, $J'$ varies over all subsets $J' \subset J_{e-1}$ of size $e_i \leq d_i$. When $J'$ is the empty set, interpret $Z_{i,\emptyset}$ as $Z_i$.

By Lemma 4.4 there exists an integer $\tilde{m}_e \geq \tilde{m}$ such that for every integer $m_e \geq \tilde{m}_e$, after replacing $T$ by a Zariski cover once more, there exists $s_e : \mathcal{O}(-m_e) \to \mathcal{L}_e$ that is regular after restriction to every fiber of $Z_{i,J'} \to T$. Thus, since the restriction to each fiber of $Z_i \to T$ of $(s_j)_{j \in J'}$ is regular, and also $s_e$ is regular on each fiber of $Z_{i,J'} \to T$, the entire sequence $(s_j)_{j \in J'} \cup (s_e)$ is regular on each fiber of $Z_i \to T$. The result follows by induction on $e$. 

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Moreover, since for every \((m_0, \ldots, m_{e-1})\) as above, for every \(m \geq \tilde{m}_e\), the element \((m_0, \ldots, m_{e-1}, m)\) is in \(\mathcal{M}\), also the observation about \(\mathcal{M}\) follows by induction on \(e\). □

A first application of this is in the special case that every \(L_j\) is \(O_Y\).

**Lemma 4.7.** In Lemma 4.6, if every \(L_j\) is \(O_Y\), then for every integer \(m_0\), after base change from \(T\) to a Zariski cover of \(T\), there exists \((m_j)_{j \in J}\) and \((s_j)_{j \in J}\) as in the lemma and satisfying the extra hypothesis that \(m_1 = \cdots = m_e\).

**Proof.** By Lemma 4.6, there exists some sequence of integers \((n_j)\), not necessarily all equal, and there exists a sequence \((t_j)_{j \in J}\) of homomorphisms \(t_j : O(-n_j) \to O_Y\) as in the lemma. Now let \(m\) be the least common multiple of all \(n_j\), i.e., for each \(j\) there exists a positive integer \(r_j\) such that \(m = n_jr_j\). Set \(s_j\) equal to \(t_j^{r_j}\). Then every \(s_j\) is a homomorphism \(O(-m) \to O_Y\). Since \((t_j)_{j \in J'}\) is regular, \((s_j)_{j \in J'}\) is also regular. [Mat89, Theorem 16.1]. □

### 5. The Main Result

**Definition 5.1.** A **Stein factorization** of \(\pi\) is a pair of finitely presented morphisms, \[ C \xrightarrow{\pi'} T' \xrightarrow{\rho} T, \] such that \(\rho \circ \pi'\) equals \(\pi\), such that \(\rho\) is quasi-finite, and such that the natural homomorphism of \(O_{T'}\)-algebras, \[ O_{T'} \to \pi'_* O_C, \] is an isomorphism.

If \(\pi\) is proper, then there exists a Stein factorization of \(\pi\). Assuming that a Stein factorization exists, for every invertible sheaf \(L\) on \(T'\), the natural homomorphism of \(O_{T'}\)-modules, \[ L \to \pi'_* (\pi')^* L, \] is an isomorphism. Moreover, Stein factorizations are compatible with fpqc base change (they are compatible with arbitrary base change if \(\pi\) is proper and cohomologically flat in degree 0).

**Hypothesis 5.2.** Assume that \(\pi : C \to T\) is flat of constant relative dimension \(d \geq 1\). Assume that every fiber of \(p\) satisfies Serre’s condition \(S_2\) (if \(d\) equals 1, then conditions \(S_1\) and \(S_2\) are equivalent). Assume that \(p : Y \to T\) is flat of constant relative dimension \(d + n\) for an integer \(n \geq 0\). Assume that \(g : C \to Y\) is a proper, perfect \(T\)-morphism.

**Proposition 5.3.** As above, let \(T\) be a Noetherian scheme. Let \(\pi : C \to T\) be a flat morphism of constant relative dimension \(d \geq 1\), and assume that all fibers are \(S_2\). Let \(p : Y \to T\) be a finite type, flat morphism of constant relative dimension \(d + n\) for \(n \geq 0\). Let \(g : Y \to C\) be a proper, perfect \(T\)-morphism. Assume that there exists an invertible sheaf \(O(1)\) on \(Y\) that is \(p\)-ample. All of the following hold after base change of \(T\) by a Zariski cover, setting \(T' = T\); resp. if there exists a Stein factorization of \(\pi\), the following hold without base change for \(T'\) as in the Stein factorization.
(i) For every \((n + 2)\)-tuple of invertible sheaves, \((\mathcal{L}_0, \ldots, \mathcal{L}_{n-1}, \mathcal{A}, \mathcal{B})\), there exists an invertible sheaf \(I_g, \pi(\mathcal{L}_0, \ldots, \mathcal{L}_{n-1}, \mathcal{A}, \mathcal{B})\) on \(T\) and an isomorphism of \(O_C\)-modules,
\[
I_g(\mathcal{L}_0, \ldots, \mathcal{L}_{n-1}, \mathcal{A}) \otimes I_g(\mathcal{L}_0, \ldots, \mathcal{L}_{n-1}, \mathcal{B}) \cong
I_g(\mathcal{L}_0, \ldots, \mathcal{L}_{n-1}, \mathcal{A} \otimes \mathcal{B}) \otimes_{O_T} I_g(\mathcal{L}_0, \ldots, \mathcal{L}_{n-1}, \mathcal{A}, \mathcal{B}).
\]

(ii) For every virtual perfect complex \(H\) on \(Y\) of virtual rank \(r\), there exists an invertible sheaf \(I_g(\mathcal{L}_0, \ldots, \mathcal{L}_n, \mathcal{H})\) on \(T\) and an isomorphism of \(O_C\)-modules,
\[
\det(Rg_*(\mathcal{H} \otimes (\mathcal{L}_0, \ldots, \mathcal{L}_n))) \cong I_g(\mathcal{L}_0, \ldots, \mathcal{L}_n)^{\otimes r} \otimes_{O_T} I_g(\mathcal{L}_0, \ldots, \mathcal{L}_n, \mathcal{H}).
\]

(iii) For every \(O_Y\)-module homomorphism \(s_n : \mathcal{L}_n^r \to \mathcal{O}_Y\) whose restriction to every fiber of \(p\) is injective, for the closed subscheme \(v : Z \to Y\) that is the effective Cartier divisor of \(s\), there exists an invertible sheaf \(I_g(\mathcal{L}_0, \ldots, \mathcal{L}_n, s_n)\) on \(T\) and an isomorphism of \(O_C\)-modules,
\[
I_g(\mathcal{L}_0, \ldots, \mathcal{L}_n, s_n) \cong I_g(\mathcal{L}_0, \ldots, \mathcal{L}_n, \mathcal{L}_n)^{\otimes r} \otimes_{O_T} I_g(\mathcal{L}_0, \ldots, \mathcal{L}_n, \mathcal{H}).
\]

Proof. For a Stein factorization, since \(\pi^*(\pi')^* \mathcal{L}\) equals \(\mathcal{L}\) for every invertible sheaf on \(T\), the invertible sheaves \(I_g(\mathcal{L}_0, \ldots, \mathcal{L}_{n-1}, \mathcal{A}, \mathcal{B}), I_g(\mathcal{L}_0, \ldots, \mathcal{L}_n, \mathcal{H})\), and \(I_g(\mathcal{L}_0, \ldots, \mathcal{L}_n, s_n)\) are uniquely determined and can be constructed after base change from \(T\) to a Zariski cover of \(T\). Thus, in what follows, we perform such base changes freely.

(i) Denote \(A \otimes_{O_T} B = \mathcal{L}_n\). Denote by \(\mathcal{T}\) a set of size \(n + 3\), \(\{0, \ldots, n-1, A, B\}\). Denote by \(\mathcal{J}\), resp. \(J, J_n\), the subset \(\{0, \ldots, n-1, A, B\}\), resp. \(\{0, \ldots, n\}, \{0, \ldots, n-1\}\). For every subset \(J' \subset \mathcal{J}\), define \(J'_n\) to be the subset \(J' \cap J_n\) of \(J_n\).

Define \(P\) to be the subset of the power set of \(\mathcal{J}\) consisting of a subset \(J' \subset \mathcal{J}\) of type \(P\) if for the subset \(J'_n = J' \cap J_n\), one of the following hold:

\[
J' = J'_n \cup \{A\}, \quad J' = J'_n \cup \{B\}, \quad J' = J'_n \cup \{A, B\}, \quad \text{or} \quad J' = J'_n \cup \{n\}.
\]

Define \(P\) to be the collection of all subsets \(J' \subset \mathcal{J}\) of this type. Said differently \(J'\) fails to be in \(P\) if and only if \(J'\) contains the subset \(\{A, n\}\) or contains the subset \(\{B, n\}\). Notice, since \(J'_n\) has size \(\leq n\), every \(J'\) in \(P\) has size \(\leq n + 2\).

By Lemma \ref{lem:1.7} after base change of \(T\) to a Zariski cover, there exists an integer \(m \geq 1\) and sections \((t_0, \ldots, t_{n-1}, t_A, t_B)\) in \(J\),
\[
t_j : \mathcal{O}(-m) \to \mathcal{O}_Y,
\]

such that for every subset \(J' \subset J\), the restriction of \((t_j)_{j \in J'}\) to every fiber of \(p\) is a regular sequence. Up to replacing \(\mathcal{O}(1)\) by \(\mathcal{O}(m)\), assume that \(m = 1\).

For every \(J'\) in \(P\) of size \(e'\), we next define a length-\(e'\) sequence \(t_{J'}\) of sections of ample invertible sheaves whose restriction to every fiber of \(p\) is a regular sequence. Thus the zero scheme \(Z_{J'}\) of this regular sequence is flat over \(T\). If \(e' \leq n + 1\), by considering the intersection with fibers of \(g\), every fiber of \(Z_{J'} \to T\) is nonempty, and hence has pure dimension \(n + d - e'\). If \(e'\) equals \(n + 2\), then over every connected open scheme of \(T\) where \(Z_{J'} \to T\) has some nonempty fiber, then every fiber is nonempty of pure dimension \(d - 2\), but there may well be connected components of \(T\) over which \(Z_{J'}\) is empty.

For \(J' \subset \mathcal{J}\), define \(t_{J'}\) to be \((t_j)_{j \in J'}\). By construction, this is a regular sequence on every fiber of \(p\). For every subset \(J'_n \subset J_n\), whose size \(e\) automatically satisfies \(e \leq n\), on every fiber of \(p\), both of the sequences \((t_j)_{j \in J' \cup \{A\}}\) and \((t_j)_{j \in J' \cup \{B\}}\)
are regular. Since both the images \( T_A \) and \( T_B \) are regular modulo the sequence \( (t_j)_{j \in J'} \), also \( t_A t_B \) is regular modulo \( (t_j)_{j \in J'} \), i.e., \( (t_j)_{j \in J'} \cup (t_A t_B) \) is a regular sequence on every fiber of \( p \). For the set \( J' = J'_n \cup \{ n \} \), define \( t_{J'} \) to be the sequence \( (t_j)_{j \in J'_n} \cup (t_A t_B) \) of length \( e + 1 \). Define \( Z_{J'} \) to be the zero scheme of this sequence.

By Lemma 4.6 applied to the closed subschemes \( (Z_{J'})_{J' \in P} \) and the sequence of invertible sheaves \( (L_0, \ldots, L_{n-1}, A, B) \), there exists a sequence of positive integers \( (m_0, \ldots, m_{n-1}, m_A, m_B) \) and a sequence \( (s_0, \ldots, s_{n-1}, s_A, s_B) \) of \( \mathcal{O}_Y \)-module homomorphisms,

\[
s_j : \mathcal{O}(-m_j) \to L_j, \quad s_A : \mathcal{O}(-m_A) \to A, \quad s_B : \mathcal{O}(-m_B) \to B
\]

such that for every \( J' \) in \( P \) with size \( e' \) and for every finite subset \( J'' \subseteq J \) whose size \( e'' \) satisfies \( e'' \leq n + d + 1 - e' \), the sequence \( (s_j)_{j \in J'} \) is regular when restricted to every fiber of \( Z_{J'} \to \mathbb{T} \).

Define \( m_n = m_A + m_B \) and define \( s_n \) to be \( s_A \otimes s_B \),

\[
\mathcal{O}(-m_A - m_B) \xrightarrow{s_A \otimes s_B} A \otimes \mathcal{O}_Y B = L_n.
\]

For every \( J' \) in \( P \) with size \( e' \), for every subset \( J'' \subseteq J_n \) of size \( e'' \leq n + d - e' \), both of the following sequences are regular when restricted to fibers of \( Z_{J'} \to \mathbb{T} \):

\[
(s_j)_{j \in J'_n} \cup (s_A) \quad \text{and} \quad (s_j)_{j \in J'_n} \cup (s_B).
\]

Thus, also the sequence \( (s_j)_{j \in J'_n} \cup (s_n) \) is regular.

For every \( i = 0, \ldots, n - 1 \), define \( L'_i = \mathcal{O}(m_i) \) and \( L''_i = L_i(m_i) \). Similarly, define \( A' = \mathcal{O}(m_A), \quad A'' = A(m_A), \quad B' = \mathcal{O}(m_B), \) and \( B'' = B(m_B) \). For every \( i = 0, \ldots, n - 1 \), define \( s'_i \) to be \( t_A^{m_i} \), and define \( s''_i \) to be \( s_i \). Define \( \alpha', \beta' \), to be \( t_A^{m_i}, \) resp. \( t_B^{m_i}, \) resp. \( \alpha'', \beta'' \), to be \( s_A, \) resp. \( s_B \).

Let \( (J', J'') \) be a partition of \( J \). If \( n \in J' \), resp. if \( n \in J'' \), by construction the sequence

\[
(t_j)_{j \in J'_n} \cup (s_j)_{j \in J''_n} \cup (t_A t_B),
\]

respectively the sequence

\[
(t_j)_{j \in J'_n} \cup (s_j)_{j \in J''_n} \cup (s_A, s_B),
\]

is regular on every fiber of \( p \). Thus, also the sequence

\[
(t^m_j)_{j \in J'_n} \cup (s_j)_{j \in J''_n} \cup (t_A^{m_i} t_B^{m_i}),
\]

respectively the sequence

\[
(t^{m''}_j)_{j \in J'_n} \cup (s_j)_{j \in J''_n} \cup (s_A, s_B),
\]

is regular on every fiber of \( p \). Thus, the hypotheses of Corollary 5.7 are satisfied.

In particular, for every invertible sheaf \( H \) on \( Y \),

\[
det(Rg_* (H \otimes (L_{J'}, B))) \cong I_d(L_{J', B} \otimes \mathcal{O}_Y)) \cong I_d(L_{J', B}) \otimes \det(Rg_* (H \otimes (L_{J'}, B))) \otimes \det(Rg_* (H \otimes (L_{J'}, B))).
\]

In the K-group of locally free \( \mathcal{O}_Y \)-modules, there is an identity,

\[
[L_j] - [\mathcal{O}_Y] = [\mathcal{O}(-m_j)] \otimes (([L'_j] - [\mathcal{O}_Y]) - ([\mathcal{O}(m_j)] - [\mathcal{O}])) =

[\mathcal{O}(-m_j)] \otimes (([L''_j] - [\mathcal{O}_Y]) - ([L'_j] - [\mathcal{O}])).
\]
Denote \( m = m_0 + \cdots + m_n \). Via the multiadditivity of the operation \( (\mathcal{L}_0, \ldots, \mathcal{L}_n) \mapsto \langle \mathcal{L}_0, \ldots, \mathcal{L}_n \rangle \), in the K-group there is an identity,

\[
\langle \mathcal{L}_0, \ldots, \mathcal{L}_n \rangle = [\mathcal{O}(m)] \otimes \left( \sum_{r=0}^{n} (-1)^r \sum_{(j', j'')} \langle \mathcal{L}_{j', j''} \rangle \right).
\]

Since \( Rg_* \) and \( \det \) are additive, using Proposition 3.5 it follows that \( I_g(\mathcal{L}_1, \ldots, \mathcal{L}_n) \) equals an alternating tensor product of invertible sheaves \( I \mathcal{L}_n \mathcal{L}_n \mathcal{I} \mathcal{O} \mathcal{O}_Y \mathcal{B} \mathcal{A} \mathcal{L} \mathcal{A} \mathcal{O}_C \mathcal{I}_g(\mathcal{L}_0, \ldots, \mathcal{L}_{n-1}, \mathcal{B}) \).

(ii) Via the same strategy as in the proof (i), this follows from the corresponding statement in Proposition 3.2.

(iii) Since \( R(g \circ \iota)_* \) equals \( Rg_* \circ R\iota_* \), it suffices to prove an identity

\[
\langle \mathcal{L}_0, \ldots, \mathcal{L}_{n-1}, \mathcal{L}_n \rangle = [\mathcal{H}] \otimes R\iota_*(\iota^* \mathcal{L}_0, \ldots, \iota^* \mathcal{L}_{n-1})
\]

for an invertible \( \mathcal{O}_Y \)-module \( \mathcal{H} \). Since \( \iota^* \) is a ring homomorphism of K-rings,

\[
\langle \iota^* \mathcal{L}_0, \ldots, \iota^* \mathcal{L}_{n-1} \rangle \cong \iota^* \langle \mathcal{L}_0, \ldots, \mathcal{L}_{n-1} \rangle.
\]

Thus, by the projection formula,

\[
R\iota_*(\iota^* \mathcal{L}_0, \ldots, \iota^* \mathcal{L}_{n-1}) \cong \langle \mathcal{L}_0, \ldots, \mathcal{L}_{n-1} \rangle \otimes [\iota_* \mathcal{O}_Z].
\]

Finally, the resolution \( s_n : \mathcal{L}_n \twoheadrightarrow \mathcal{O}_Y \) of \( \iota_* \mathcal{O}_Z \) gives an identity,

\[
[\mathcal{H}] \otimes [\iota_* \mathcal{O}_Z] = ([\mathcal{L}_n] - [\mathcal{O}_Y]),
\]

for \( \mathcal{H} = \mathcal{L}_n \). □

The projectivity hypothesis on \( p \) is only necessary fpqc locally.

**Corollary 5.4.** In the previous proposition, replace the hypothesis that there exists a \( p \)-ample invertible sheaf on \( C \) with the hypothesis that for some fpqc morphism \( T_0 \to T \) there exists a \( T_0 \)-ample invertible sheaf on \( C \times_T T_0 \). Also assume that \( \pi \) has a Stein factorization. Then the proposition still holds.

**Proof.** Because of compatibility of pushforward and flat base change, the base change of the Stein factorization,

\[
C \times_T T_0 \xrightarrow{\pi \times \id} T' \times_T T_0 \xrightarrow{\rho \times \id} T_0,
\]

is a Stein factorization of \( C \times_T T_0 \to T_0 \). Now use the same observation as in the previous proof: because \( \pi'_*(\pi^*)^* \mathcal{L} \) equals \( \mathcal{L} \) for every invertible sheaf on \( T' \), the invertible sheaves \( I_g(\mathcal{L}_0, \ldots, \mathcal{L}_{n-1}, \mathcal{A}, \mathcal{B}) \), \( I_g(\mathcal{L}_0, \ldots, \mathcal{L}_n, \mathcal{H}) \), and \( I_g(\mathcal{L}_0, \ldots, \mathcal{L}_n, s_n) \) are uniquely determined. Thus, the invertible sheaves on \( T' \times_T T_0 \) constructed using Proposition 5.4 satisfy the fpqc descent condition. □

**Corollary 5.5.** Let \( \pi : C \to T \) be a morphism that is proper, fppf of pure relative dimension \( d \geq 1 \), and cohomologically flat in degree 0. Also assume that every geometric fiber of \( \pi \) satisfies Serre’s condition \( S_2 \). Then there exists a relative intersection sheaf for \( \pi \) as in Definition 2.1. Moreover, this intersection sheaf is unique.
Proof. By the proposition, the intersection sheaves from Definition 2.8 satisfy the axioms from Definition 2.1. Moreover, the proof of the proposition proves that, via Axioms (ii) and (iii), the intersection sheaf for an arbitrary \((n+1)\)-tuple of invertible sheaves can be reconstructed from those \((n+1)\)-tuples of invertible sheaves that admit sections forming a regular sequence of length \(n+1\). For such \((n+1)\)-tuples, Axiom (iv) and induction on \(n\) reduces to the case that \(n = 0\). For \(n = 0\), Axiom (v) uniquely determines the intersection sheaf. Thus, the relative intersection sheaf is unique. \(\square\)

6. Tori and Torsors

The intersection sheaves above are sufficient to construct Abel maps in case the target has Picard rank one. To deal with higher Picard rank, it is necessary to generalize the intersection sheaf from \(\mathbb{G}_m\)-torsors to torsors for a more general group scheme \(Q\) over \(\mathbb{C}\), as in [SGA73, Expos´e XVIII, Formulaire 1.3.8]. The group scheme \(Q\) will be isomorphic to a product of copies of \(\mathbb{G}_m\), i.e., a split torus, after pullback by a finite, étale morphism \(C_0 \to C\). Unfortunately, cohomological flatness in degree 0 is not preserved by finite, étale morphisms.

Example 6.1. Let \(k\) be a field. Let \(\overline{C}\) be a smooth, projective, geometrically connected curve over \(k\) of genus \(g \geq 2\). For simplicity assume that \(\overline{C}\) has a \(k\)-point, so that there exists an invertible sheaf \(P\) on \(\overline{C} \times_{\text{Spec } k} \text{Pic}^{2g-2}_{\overline{C}/k}\) representing the relative Picard functor. The pushforward \(\text{pr}_2_* P\) is flat and of formation compatible with arbitrary base change when restricted over the open complement of the unique \(k\)-point parameterizing the dualizing sheaf \(\omega\). On every open neighborhood of \([\omega]\), this sheaf is neither flat nor of formation compatible with arbitrary base change.

Let \(T \subset \text{Pic}^{2g-2}_{\overline{C}/k}\) be a dense open subset. On \(\overline{C} \times_{\text{Spec } k} T\), define \(A\) to be the commutative, coherent sheaf of algebras

\[ A = \mathcal{O} \oplus P\epsilon, \]

where \(\epsilon^2 = 0\). Define \(\nu : C \to \overline{C} \times_{\text{Spec } k} T\) to be the relative Spec, \(C = \text{Spec } A\).

The projection \(\pi : C \to T\) is projective, flat, and even LCI. Since \(\pi_* \mathcal{O}_C\) equals \(\mathcal{O}_T \oplus \text{pr}_T^* \mathcal{P}\), \(\pi\) is cohomologically flat in degree 0 if and only if \(T\) does not contain the distinguished \(k\)-point \([\omega]\). Now let \(\overline{C}_0 \to \overline{C}\) be the finite, étale morphism of degree 2 associated to a nontrivial 2-torsion invertible sheaf \(T\) on \(\overline{C}\). This extends uniquely to a finite, étale morphism of degree 2, \(C_0 \to C\). The morphism \(C_0 \to T\) is cohomologically flat in degree 0 if and only if \(T\) contains neither \([\omega]\) nor \([\omega \otimes \mathcal{O}_{C_0} T]\).

Thus, it can happen that \(C \to T\) is cohomologically flat in degree 0, yet \(C_0 \to T\) is not cohomologically flat in degree 0.

Hypothesis 6.2. Let \(T\) be quasi-compact. Let \(\pi : C \to T\) be a morphism that is proper, fppf of pure relative dimension \(d \geq 1\), and whose geometric fibers \(C_t\) are reduced, are \(S_2\), and have only finite geometric covers. That last hypothesis says that for every étale morphism \(b : C_0 \to C_t\) satisfying the valuative criterion of properness (but \(b\) need not be quasi-compact) and with \(C_0\) having only finitely many connected components, then \(b\) is a finite morphism. These hypotheses holds if \(C_t\) is normal or if \(C_t\) is an at-worst-nodal curve of “compact type”.

Definition 6.3. A torus over \(C\) is a smooth group scheme \(Q\) over \(C\) that is étale locally isomorphic to \(\mathbb{G}_m^\rho\) over \(C\) for an integer \(\rho \geq 0\), the rank of the torus. The Cartier
dual of \( Q \) is the étale group scheme \( Q^D \) over \( C \) whose associated étale sheaf of Abelian groups is the sheaf \( \text{Hom}_{C-\text{gr}}(Q,G_m,C) \). This sheaf is locally constant with fiber \( \mathbb{Z}^ρ \).

The group scheme \( Q^D \to C \) is never quasi-compact if \( ρ > 0 \). However, since \( C \) is quasi-compact, \( Q^D \) is a countable increasing union of open subschemes that are quasi-compact. Because \( C \) is quasi-compact, and because of the hypothesis on the geometric fibers of \( π \), for the smallest open and closed subscheme \( Q^D \) of \( Q^D \) containing a specified quasi-compact open, \( Q^D \) is finite over \( C \). For every geometric point of \( C \), some \( Q^D \) contains a (finite) set of generators for the geometric fiber \( Q^D \) (as a group). Again using that \( C \) is quasi-compact, there exists a \( Q^D \) that generates \( Q^D \) as a group scheme.

**Example 6.4.** If we drop the hypothesis on the fibers of \( π \), this can easily fail. For instance, let \( C \) be a nodal plane cubic, let \( b : C_0 \to C \) be the unique finite, étale morphism of degree 2 with connected domain, and let \( Q^D \) be the étale group scheme that is isomorphic to \( \mathbb{Z}^2 \) on each of the two irreducible components of \( C_0 \), yet where the glueing isomorphisms at the two nodes differ by an infinite order automorphism of \( \mathbb{Z}^2 \), e.g., \((m,n) \mapsto (m,m+n)\). Presumably there is a hypothesis weaker than unibranch that works and that allows the fibers to be nonreduced (yet \( S_2 \)). Reducedness of fibers is useful not only here, but also because it implies cohomological flatness in degree 0 of \( C_0 \to T \) for all finite, étale covers of \( C \).

For every set \( Σ \), denote by \( L_{C,Σ} \) the étale \( C \)-scheme together with a set map \( Σ \to L_{C,Σ}(C) \) that represents the functor associating to every étale \( C \)-scheme \( L \) the collection of all set maps \( \text{Hom}_{\text{set}}(Σ,L(C)) \). Thus the sheaf of \( L_{C,Σ} \) is locally constant with fiber \( Σ \). locally \( λ \) defines an isomorphism of the constant sheaf \( \mathbb{Z}^ρ \). In particular, \( L_{C,\mathbb{Z}^ρ} \) is the étale group scheme over \( C \) with fiber \( \mathbb{Z}^ρ \).

**Lemma 6.5.** There exists a finite, étale, surjective morphism \( C_0 \to C \) representing the functor that associates to every \( C \)-scheme \( S \) the set of all isomorphisms of étale group \( C \)-schemes, \( φ : L_{S,\mathbb{Z}^ρ} \to S \times C Q^D, \) mapping the basis elements of \( \mathbb{Z}^ρ \) into the subscheme \( S \times C Q^D \).

**Proof.** This is straightforward. Since both \( L_{C,\mathbb{Z}^ρ} \) and \( Q^D \) are finite, étale over \( C \), the Hom scheme \( H = \text{Hom}_C(L_{C,\mathbb{Z}^ρ},Q^D) \) is finite, étale over \( C \). The universal morphism over \( H \) extends to a morphism of group schemes \( φ : L_{H,\mathbb{Z}^ρ} \to H \times C Q^D \). The target is étale locally isomorphic to \( L_{H,\mathbb{Z}^ρ} \), thus the determinant of \( φ \) is étale locally well-defined up to a sign. This determinant is an étale locally constant function. Thus, there is an open and closed subscheme \( C_0 \) of \( H \) that is the locus on which this determinant is \( +1 \) or \( −1 \). As an open and closed subscheme of a finite, étale scheme over \( C \), also \( C_0 \) is a finite, étale scheme over \( C \). Since \( Q^D \) is étale locally isomorphic to \( L_{C,\mathbb{Z}^ρ} \) and since \( Q^D \) generates the group scheme on all geometric fibers, \( C_0 \to C \) is surjective.

**Remark 6.6.** By the lemma, \( Q \times C C_0 \) is a split multiplicative group on \( C_0 \). By adjointness of pullback and restriction of scalars, there is a natural morphism of group schemes \( Q \to R_{C_0/C}(Q \times C C_0) \). Checking étale locally, this morphism is unramified and locally split. Thus, the lemma is just a global version of the “standard” argument that every torus embeds in the torus of a “permutation module”.

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For a morphism $a : C \to C$, define $C_1 = C_0 \times_C C_0$, resp. $C_2 = C_0 \times_C C_0 \times_C C_0$. Denote the projections by

\[ \text{pr}_1, \text{pr}_2 : C_1 \to C_0, \]

\[ \text{pr}_{1,2}, \text{pr}_{1,3}, \text{pr}_{2,3} : C_2 \to C_1. \]

A descent datum of an affine $C$-scheme relative to $C_0$ is a pair $(f_0 : E_0 \to C_0, \phi)$ of an affine morphism $f_0$ and an isomorphism of $C_1$-schemes, $\phi : \text{pr}_1^* E_0 \to \text{pr}_2^* E_0$ such that the following cocycle condition or descent condition is satisfied,

\[ \text{pr}_{2,3}^* \phi \circ \text{pr}_{1,2}^* \phi = \text{pr}_{1,3}^* \phi. \]

For descent data $(f_0 : E_0 \to C_0, \phi)$ and $(f'_0 : E'_0 \to C_0, \phi')$, an morphism between these descent data is an morphism of $C_0$-schemes, $g_0 : E_0 \to E'_0$ such that $\text{pr}_2^* g_0 \circ \phi$ equals $\phi' \circ \text{pr}_2^* g_0$. The identity is a morphism, and the composition of two morphisms is an morphism. Thus, descent data form a category.

For every affine morphism $f : E \to C$, the associated descent datum is $E_0 = E \times_C C_0$ with projection $f_0 : E_0 \to C_0$ and with $\phi$ equal to the natural isomorphism

\[ (E \times_{C,a} C_0) \times_{C_0,\text{pr}_1} C_1 \cong E \times_{C,a,\text{pr}_2} C_1 = E \times_{C,a} C_0 \times_{C_0,\text{pr}_2} C_1 \cong (E \times_{C,a} C_0) \times_{C_0,\text{pr}_2} C_1. \]

For an morphism of affine $C$-schemes, $g : E \to E'$, the associated morphism of descent data is $g_0 : E_0 \to E'_0$,

\[ g \times \text{Id}_{C_0} : E \times_C C_0 \to E' \times_C C_0. \]

Every descent datum isomorphic to the descent datum associated to an affine $C$-scheme is an effective descent datum. The basic result of effective fpqc descent is the following.

**Theorem 6.7.** [Gro62, Théorème 2, p. 190-19] Assume that $C_0 \to C$ is faithfully flat and quasi-compact. For every pair of affine $C$-schemes, $f : E \to C$ and $f' : E' \to C$, every morphism of the associated descent datum is associated to a unique morphism $g : E \to E'$ of $C$-schemes. Every descent datum of affine schemes relative to $C_0$ is effective.

This is relevant here because $Q$-torsors are affine and fpqc. Thus, they can be used both as the affine scheme and the fpqc cover in the previous theorem. In particular, this leads quickly to an existence result for "induced torsors". Let $Q$ and $Q'$ be faithfully flat, finitely presented group schemes over a scheme $C$. Let $\psi : Q \to Q'$ be a morphism of $C$-group schemes. Denote by $m_\psi : Q \times_C Q' \to Q'$ the left $Q$-action by multiplication on the right by the inverse, $m_\psi(q, q') = q' \psi(q)^{-1}$. For every left $Q$-torsor $E$, say $m_E : Q \times_C E \to E$, $C$, there is an induced "diagonal" left action of $Q$ on $Q' \times_C E$,

\[ m_{\psi,E} : Q \times_C Q' \times_C E \to Q' \times_C E, \text{ pr}_2 \circ m_{\psi,E} = m_E \circ \text{pr}_{1,3}, \text{ pr}_2 \circ m_{\psi,E} = m_\psi \circ \text{pr}_{1,2}. \]

Altogether, this makes $Q' \times_C E$ into a $Q' \times_C Q$-torsor over $C$. (This is the reason for using the inverse of the right $Q$-action on $Q'$; so that it commutes with the left regular action of $Q'$ on itself.) The goal is to construct a left $Q'$-torsor $\psi_* E$ and a $Q$-invariant, left $Q'$-equivariant morphism $p_{\psi,E} : Q' \times_C E \to \psi_* E$ realizing $Q' \times_C E$ as a $Q$-torsor over $\psi_* E$. In particular, $p_{\psi,E}$ is a categorical quotient of the diagonal action. Thus, if $p_{\psi,E}$ exists, then it is unique up to unique isomorphism. This strong uniqueness insures the cocycle condition for a descent datum.
Corollary 6.8. There exists a left \( Q' \)-torsor \( \psi_*E \) and a morphism \( p_{\psi,E} : Q' \times_C E \to \psi_*E \) that is \( Q' \)-equivariant, that is invariant for the diagonal \( Q \)-action, and that realizes \( Q' \times_C E \) as a \( Q \)-torsor over \( \psi_*E \) for its diagonal \( Q \)-action.

Proof. Since \( Q \) is fpqc, and since \( E \) is isomorphic to \( Q \) as a \( Q \)-torsor after passing to an fpqc cover of \( C \), also the projection morphism \( a : E \to C \) is fpqc. By Theorem 3.7 Gro62 [Théorème 2, p. 190-191], it suffices to construct an \( \alpha \)-descent datum for \( \psi_*E \) and \( p_{\psi,E} \) satisfying the conditions. Because of the strong uniqueness, both the morphism \( \phi \) in the descent datum and the cocycle conditions are automatic. Thus, it suffices to prove the existence of a \( Q' \)-torsor \( \psi_*E_a \) over \( E \) and a morphism \( p_{\psi,E} : (Q' \times_C E) \times_{C,a} E \to \psi_*E_a \) from the \( \alpha \)-pullback of \( Q' \times_C E \) to \( \psi_*E_a \) that is left \( Q' \)-equivariant, that is \( Q \)-invariant, and that realizes \( (Q' \times_C E) \times_{C,a} E \) as a \( Q \)-torsor over \( \psi_*E_a \).

Since \( E \) is a \( Q \)-torsor, the following morphism is an isomorphism,
\[
\lambda_E : Q \times_C E \to E \times_C E, \quad \text{pr}_1 \circ \lambda_E = m_E, \quad \text{pr}_2 \circ \lambda_E = \text{pr}_2.
\]
Thus, there is an induced isomorphism,
\[
\text{Id}_{Q'} \times \lambda_E : Q' \times_C Q \times_C E \to Q' \times_C E \times_C E.
\]
This is an isomorphism from the trivial \( Q' \times_C Q \)-torsor over \( E \) to the pullback by \( a \) of the \( Q' \times_C Q \)-torsor \( Q' \times_C E \). Consider the following morphism,
\[
r : Q' \times_C Q \times_C E \to Q' \times_C E, \quad (q', q, e) \mapsto (q' \psi(q), e).
\]
This is left \( Q' \)-equivariant, it is \( Q \)-invariant, and it realizes \( Q' \times_C Q \times_C E \) as a \( Q \)-torsor over \( Q' \times_C E \). Since \( \text{Id}_{Q'} \times \lambda_E \) is an isomorphism, there is a unique morphism
\[
p_{\psi,E,a} : Q' \times_C E \times_C E \to Q' \times_C E
\]
such that \( p_{\psi,E,a} \circ (\text{Id}_{Q'} \times \lambda_E) \) equals \( r \). Thus, \( p_{\psi,E,a} \) is also left \( Q' \)-equivariant, it is \( Q \)-invariant, and it realizes \( Q' \times_C E \times_C E \) as a \( Q \)-torsor over \( Q' \times_C E \). 

The operation \( E \mapsto \psi_*E \) induces a \( 1 \)-morphism of classifying stacks.

Definition 6.9. For \( C \) as in Hypothesis 6.2 and for \( Q \) a torus over \( C \), denote by \( BQ \) the \( C \)-stack classifying \( Q \)-torsors. This is a quasi-compact, quasi-separated algebraic \( C \)-stack with affine diagonal. Denote by \( BQ_{C/T} \) the \( T \)-stack \( \text{Hom}_T(C, BQ) \). For every morphism \( \psi : Q \to Q' \) of tori over \( C \), denote by \( B\psi : BQ \to BQ' \), resp. by \( B\psi_{C/T} : BQ_{C/T} \to BQ'_{C/T} \), the \( 1 \)-morphism defined by \( E \mapsto \psi_*E \).

Proposition 6.10. If \( \pi : C \to T \) is proper and fppf, then \( BQ_{C/T} \) is an algebraic \( T \)-stack that is locally finitely presented and whose diagonal is quasi-compact and separated. If \( \pi \) satisfies Hypothesis 6.2, then \( BQ_{C/T} \) has a coarse moduli space \( |BQ_{C/T}| \) that is locally finitely presented and quasi-separated (typically not separated). Assuming, moreover, that \( \pi \) has geometrically reduced fibers and that the finite part \( T'/T \) of the Stein factorization of \( C/T \) is étale, fppf locally \( BQ_{C/T} \to |BQ_{C/T}| \) is a gerbe for a commutative, fppf group scheme that is the kernel of a morphism of tori. Finally, assuming further that a \( Q \)-trivializing, finite, étale, surjective cover \( b : C_0 \to C \) of \( Q \) is geometrically integral over the Stein factorization \( T' \) of \( C/T \), and assuming there exists a \( T' \)-section of \( C_0 \), then this gerbe is split, i.e., there is a section \( |BQ_{C/T}| \to BQ_{C/T} \).
Proof. By [Lie06, Proposition 2.3.4] and [Ols06, Theorem 1.5], $BQ_{C/T}$ is a locally finitely presented algebraic $T$-stack whose diagonal is quasi-compact and separated. The proof that the diagonal is quasi-separated is roughly the same as the proof that $BQ_{C/T}$ has a coarse moduli space, so here it is quickly. By Lemma 6.3, there exists a finite, étale morphism $a : C_0 \to C$ such that $a^*Q$ is isomorphic to $G^p_{m,C_0}$. Quasi-separatedness of this stack is equivalent to quasi-compactness of the Isom scheme $\text{Isom}_{Q/T}(E, E')$ for $Q$-torsors $E$ and $E'$ over $C$, not only for $Q$-torsors on $C/T$, but also for $Q$-torsors on every base change $C \times_T S/S$. By fpqc descent, Theorem 6.7, [Gro62, Théorème 2, p. 190-19], $Q$-torsors on $C$ are equivalent to $C_0/C$ descent data of $Q$-torsors. On $C_0$, the $Q$-torsors are equivalent to $\rho$-tuples of invertible sheaves. So the Isom scheme is an $\rho$-fold fiber product over $T$ of the Isom schemes of the component invertible sheaves. These Isom schemes are quasi-compact by [Gro63, Corollaire 7.7.8]. The scheme parameterizing the isomorphism $\phi$ on $C_1$ is again described in terms of Isom schemes of invertible sheaves on $C_1$. This Isom scheme is again quasi-compact by [Gro63, Corollaire 7.7.8]. Of course to descend, this data must satisfy the cocycle condition. These are closed conditions: the subscheme parametrizing data satisfying the cocycle condition is a closed subscheme. A closed subscheme of a quasi-compact scheme is again quasi-compact. Thus, altogether, the stack is quasi-separated.

In the same way, the coarse moduli functor of $Q$-torsors over $C$ is relatively a finitely presented, affine scheme over the coarse moduli functor of $Q$-torsors on $C_0$. Since the pullback of $Q$ to $C_0$ is $G^p_{m,C_0}$, this functor is a quasi-separated, locally finitely presented algebraic $T$-space by Hypothesis 5.2 and [Art69, Theorem 7.3].

Next, assume that $\pi$ has geometrically reduced fibers and that $T'/T$ is étale. Then for every étale cover $C_0 \to C$, the finite part of the Stein factorization, $T_0 \to T$, is étale. In particular, after a finite, étale base change of $T$, assume that $T'$ is a disjoint union of $m$ copies of $T$. After a further fppf base change, essentially by $C \to T$, assume that there is an $m$-tuple $s = (s_1, \ldots, s_m)$ of sections $s_i : T \to C$ splitting the morphism $C \to T'$. After a further finite, étale base change of $T$, there exist finite, étale morphism $b : C_0 \to C$ and a trivialization $\psi : Q_0 \to G^p_{m,C_0}$, $Q_0 = Q \times_C C_0$, such that $C_0 \to T'$ is the Stein factorization and such that there exists a lift, $s_{i,0} : T \to C_0$, of each section $s_i$.

Thus, assume now that $C_0 \to T'$ has geometrically integral fibers, and assume that there exists a section $s_0 : T' \to C_0$ of the Stein factorization. This induces a section $s = b \circ s_0 : T' \to C$. Since $C_0 \to T'$ is projective and flat with integral geometric fibers, every automorphism of a $G^p_{m,C_0}$-torsor $E$ is just multiplication by a $T'$-section of $G^p_{m,T}$. Denote by $BQ_{C/T,s}$ the rigidified stack of $Q$-torsors $E$ together with a trivialization $\tau : s^*E \cong s^*Q$ as $s^*Q$-torsors. By Theorem 6.7, [Gro62, Théorème 2, p. 190-19], every automorphism of such a pair $(E, \tau)$ is uniquely determined by the induced automorphism of the corresponding object of $(BQ_0)_{C_0/T,s_0}$. As noted above, these automorphisms are trivial. Thus, the stack $BQ_{C/T,s}$ is an algebraic space. There is a forgetful morphism $\Phi : BQ_{C/T,s} \to BQ_{C/T}$. This is a torsor for the pullback to $BQ_{C/T}$ of the group scheme $s^*Q$: any two trivializations $\tau$ differ by post-composing by multiplication by a section of $s^*Q$. Thus, the induced map of coarse moduli spaces is an isomorphism, i.e., $BQ_{C/T,s}$ is equivalent, as a $T$-stack and thus also as a $T$-algebraic space, to the coarse moduli space $|BQ_{C/T}|$.

Therefore, the morphism $BQ_{C/T} \to |BQ_{C/T}|$ admits a section.
There is a commutative group scheme $R$ over $BQ_{C/T,s}$ defined as the pushforward via $\pi$ of the Isom scheme of the torsor $E$. For the finite, étale covers $C_0 \to C$, resp. $C_1 \to C$, there are associated group schemes $R^0$ and $R^1$. Again using Theorem 6.7 [Gro62 Théorème 2, p. 190-19], there is an exact sequence of commutative group schemes,

$$0 \to R \to R^0 \xrightarrow{d^0} R^1,$$

where $d^0$ sends $g_0$ to $\phi \circ \text{pr}_1^*g_0 - \text{pr}_2^*g_0 \circ \phi$. By construction, both $R^0$ and $R^1$ are tori: $R^0$ is isomorphic to the Weil restriction of $G_{m,T'}^d$ for the finite, étale cover $T'/T$, and $R^1$ is the Weil restriction of a split torus for the finite, étale cover arising from its Stein factorization over $T$. Thus, étale locally on $T$, the morphism $d^0$ is a morphism between split tori. Choosing splittings of these tori, $d^0$ is equivalent to an integer valued matrix. Considering the Smith normal form of this matrix, $R$ is equivalent to a product of copies of $G_{m,T}$ and copies of finite, flat group scheme $\mu_\ell$ for various integers $\ell$ (possibly divisible by the characteristic, thus not necessarily étale). The 1-morphism $BQ_{C/T} \to BQ_{C/T,s}$ is a gerbe for this group scheme $R$. □

7. Intersection Torsors

There is a category $\text{Torus}_C$ whose objects are pairs $(a : C_0 \to C, Q_0)$ of a finite, étale morphism $a$ and a torsor $Q_0$ over $C_0$. For objects $(a' : C'_0 \to C, Q'_0)$ and $(a : C_0 \to C, Q_0)$, a morphism from the first to the second is a pair $(b : C'_0 \to C_0, \psi)$ of a $C$-morphism $b$ (automatically finite and étale) and a morphism of group schemes over $C'_0$, $\psi : Q_0 \times_{C_0} C'_0 \to Q'_0$ (note, this is contravariant in the second argument). Composition and identities are defined in the evident manner. This category is fibered over the category of finite, étale $C$-schemes, and each fiber category is an additive category (it is not an Abelian category, because there are no cokernels satisfying the usual axioms for an Abelian category). The cline associates to $(a : C_0 \to C, Q_0)$ and $b : C'_0 \to C_0$ the object $(a' : C'_0 \to C_0, Q_0 \times_{C_0} C'_0)$.

For a $Q_0$-torsor $E_0$ on $C_0$, there is an associated torsor $b^*E_0$ on $C'_0$ for $Q_0 \times_{C_0} C'_0$. By Corollary 6.8 there is an associated $Q'_0$-torsor $\psi, b^*E_0$ on $C'_0$. This pullback torsor $(b, \psi)^*E_0$ is contravariant in $(b, \psi)$ and covariant in $E_0$.

Definition 7.1. Let $\pi : C \to T$ be a morphism satisfying Hypothesis 6.2. For every object $(a : C_0 \to C, Q_0)$ of $\text{Torus}_C$, an intersection datum for $\pi$ is a tuple

$$(n, T_0 \to T, p_0 : Y_0 \to T, g_0 : Y_0 \to T_0 \times_T C_0, (\mathcal{L}_0, \ldots, \mathcal{L}_{n-1}, E))$$

of an integer $n \geq 0$, a $T$-scheme $T_0$, a proper, fppf morphism $p_0$ of relative dimension $\leq d + n$ that is projective fppf locally over $T_0$, a proper, perfect morphism of $T_0$-schemes, an $n$-tuple of invertible sheaves $(\mathcal{L}_0, \ldots, \mathcal{L}_{n-1})$ on $Y_0$, and a torsor over $Y_0$ for $Q_0 \times_{C_0} Y_0$. For fixed $(n, T_0, p_0, g_0)$, an isomorphism between $(n + 1)$-tuples $(\mathcal{L}_0, \ldots, \mathcal{L}_{n-1}, E)$ and $(\mathcal{L}_0', \ldots, \mathcal{L}_{n-1}', E')$ is an $n$-tuple of isomorphisms between the component invertible sheaves $\mathcal{L}_i \to \mathcal{L}_i'$, and an isomorphism of torsors $E \to E'$ under $Q_0 \times_{C_0} Y_0$. For a datum as above, and for a morphism in $C$, $(b, \psi) : (C'_0, Q'_0) \to (C_0, Q_0)$, the $(b, \psi)$-pullback of the datum is the datum

$$(n, T_0 \to T, p_0 \times \text{Id} : Y_0 \times_{C_0} C'_0 \to T, g_0 \times \text{Id} : Y_0 \times_{C_0} C'_0 \to T_0 \times_T C'_0, (\text{pr}_{Y_0}^*\mathcal{L}_0, \ldots, \text{pr}_{Y_0}^*\mathcal{L}_{n-1}, \psi, \text{pr}_{Y_0}^*E)).$$
Similarly, for a datum as above and for a morphism of \( T \)-schemes, \( T_1 \to T_0 \), the pullback datum is
\[
(n, T_1, Y_0 \times_{T_0} T_1 \to T_1, Y_0 \times_{T_0} T_1 \to \mathbb{A} \times_{\mathbb{A}} C_0 \times_{T_0} T_1, (pr^*_{Y_0} L_0, \ldots, pr^*_{Y_0} L_{n-1}, pr^*_{Y_0} E)).
\]

A relative intersection torsor for \( \pi \) is an assignment to every intersection datum of a section \( I_{g_0}(L_0, \ldots, L_{n-1}, E) \) over \( T_0 \) of \( |(BQ_0)_{C_0/T}| \) that satisfies all of the following axioms.

(i) The assignment is invariant under isomorphism of \( (n + 1) \)-tuples.
(ii) The assignment is \( \mathcal{S}_n \)-invariant in \( (L_0, \ldots, L_{n-1}) \).
(iii) The assignment is additive in every argument of \( (L_0, \ldots, L_{n-1}, E) \).
(iv) For every \( i = 0, \ldots, n-1 \), for every section \( s_i : L^i_i \to O_{Y_0} \) that is regular on every fiber of \( p_0 \), for the associated Cartier divisor \( \iota_0 : Z_0 \to Y_0 \),
\[
I_{g_0}(t_0^* L_0, \ldots, t_0^* L_{i-1}, t_0^* L_{i+1}, \ldots, t_0^* L_{n-1}, t^* E) = I_{g_0}(L_0, \ldots, L_{n-1}, E).
\]
(v) If \( n \) equals 0, and if \( Q_0 \) equals \( \mathbb{G}_{m,C_0} \), then for the \( \mathbb{G}_{m,C_0} \)-torsor \( E \) of an invertible sheaf \( L_0 \), \( I_{g_0}(E) \) equals \( \det(Rg_*(L_0)) \otimes_{O_{C_0}} \det(Rg_*(\mathcal{O}_{Y_0})) \).

**Corollary 7.2.** Let \( \pi : C \to T \) be a morphism satisfying Hypothesis 8.1. Also assume that every geometric fiber of \( \pi \) satisfies Serre’s condition \( S_2 \). Then there exists a relative intersection torsor for \( \pi \), and this intersection torsor is unique.

**Proof.** This is a consequence of Corollary 5.5 and fpqc descent, Theorem 6.7 [Gro62, Théorème 2, p. 190-19]. Corollary 5.5 gives a relative intersection torsor whenever \( Q_0 \) is the split torus \( \mathbb{G}_{m,C_0}^n \); for a \( \mathbb{G}_{m,C_0}^n \)-torsor \( E = (E_1, \ldots, E_n) \) with \( E_j \) associated to an invertible sheaf \( \mathcal{N}_j \), the intersection \( \mathbb{G}_{m,C_0}^n \)-torsor \( (I_1, \ldots, I_n) \) has \( I_j \) associated to the invertible sheaf \( I_{g_0}(L_0, \ldots, L_{n-1}, \mathcal{N}_j) \). Additivity in the last argument implies that this is compatible with \( (b, \psi) \)-pullbacks for morphisms \( \psi : \mathbb{G}_{m,C_0}^n \to \mathbb{G}_{m,C_0}^n \).

Having constructed the relative intersection torsor whenever \( Q_0 \) is a split torus, now Lemma 6.5 and fpqc descent, Theorem 6.7 [Gro62, Théorème 2, p. 190-19], extends the relative intersection torsor to general torsors \( Q_0 \). The cocycle condition for fpqc descent is precisely compatibility with \( (b, \psi) \)-pullbacks.

### 8. Degrees

The last aspect of Abel maps has to do with degrees of torsors. For this, we make a strong hypothesis that is satisfied in the case of Abel maps.

**Hypothesis 8.1.** Let \( T \) be an excellent, Noetherian scheme. Let \( \pi : C \to T \) be a projective, smooth morphism of pure relative dimension \( d \).

**Definition 8.2.** For every finite, flat morphism \( b : C_0 \to C \) with \( \pi_0 = \pi \circ b \) smooth, for every torsor \( Q_0 \) on \( C_0 \), define \( \Lambda(Q_0/C_0/T) \) to be the pushforward with respect to \( \pi_0 \) of the cocharacter lattice \( \text{Hom}_{\mathbb{G}_{m,C_0}^d}(\mathbb{G}_{m,C_0}, Q_0) \) (as an étale sheaf). This is contravariant in \( C_0 \), it is covariant in \( Q_0 \), and it is compatible with étale base change of \( T \).
The étale group scheme $\Lambda(Q/C/T)$ is the natural target for the degree map. In particular, when $T$ is Spec of a finite field, this degree is a logarithmic height, and the map to $\Lambda(Q/C/T)$ is a "multithéight" as in [BGS94, Gub97].

**Definition 8.3.** A degree datum for $\pi$ is a tuple 

$$(b : C_0 \to C, Q_0, (L_1, \ldots, L_{d-1}, E))$$

of a finite, flat morphism $b$ such that $\pi_0 = \pi \circ b$ is smooth, a torus $Q_0$ on $C_0$, a $(d-1)$-tuple of invertible sheaves $(L_1, \ldots, L_{d-1})$ on $C_0$, and a $Q_0$-torsor $E$ on $C_0$. Isomorphisms and pullbacks are as in Definition 7.4. A relative degree map for $\pi$ is an assignment to every degree datum of a section $\deg_\pi(L_1, \ldots, L_{d-1}, E)$ of $\Lambda(Q_0/C_0/T)$ over $T_0$ that satisfies all of the following axioms.

(i) The assignment is $\mathcal{G}_{d-1}$-invariant in $(L_1, \ldots, L_{d-1})$.
(ii) The assignment is additive in every argument of $(L_1, \ldots, L_{d-1}, E)$.
(iii) The assignment is compatible with base change of $T$.
(iv) For every finite, flat morphism $g : C_1 \to C_0$ such that $\pi_1 = \pi_0 \circ g$ is smooth, for every $(d-1)$-tuple $(L_0, \ldots, L_{d-1})$ of invertible sheaves on $C_0$, for every $Q_0 \times_{C_0} C_1$-torsor $F$ on $C_1$,

$$\deg_\pi(g^*L_1, \ldots, g^*L_{d-1}, F) = g^*\deg_\pi(L_1, \ldots, L_{d-1}, L_g(F)).$$

(v) For every morphism of tori $\psi : Q_0 \to Q_0'$,

$$\deg_\pi(L_1, \ldots, L_{d-1}, \psi_*E) = \psi_*\deg_\pi(L_1, \ldots, L_{d-1}, E).$$

(vi) When $pr_0$ has connected geometric fibers, and when $Q_0$ equals $\mathbb{G}_{m,C}$ for the associated invertible sheaf $L_0$ of $E$, $\deg_\pi(L_1, \ldots, L_{d-1}, E)$ equals the virtual rank of $R\pi_0_*([L_1, \ldots, L_{d-1}, L_d]).$

**Proposition 8.4.** Under Hypothesis [7.1], there exists a relative degree map. The relative degree map is unique. Denoting by $\Lambda(Q/C/T)$ also the étale group scheme over $T$ whose étale sheaf is as above, the relative degree map defines a morphism of $T$-schemes, $Pic_{C/T} \times_T \cdots \times_T Pic_{C/T} \times_T [BQ/C/T] \to \Lambda(Q/C/T)$.

**Proof.** This is proved by the same method as in the proof of Corollary [7.2]. The definition of the degree in the general case reduces to the case that $Q_0$ equals $\mathbb{G}_{m,C}$. The main point is that for every element $a \in \gamma^{d+1}$, the virtual rank of $R\pi_*(a)$ is zero. Since the virtual rank is locally constant, this can be checked after base change to the residue field of a point of $T$. Now the claim follows by Riemann-Roch, cf. [Man69, Corollary 8.10, Proposition 16.12]. Modulo $\gamma^{d+1}$, the rule $(L_1, \ldots, L_{d-1}, L_d) \mapsto ([L_1, \ldots, L_{d-1}, L_d])$ is multiadditive. Since $R\pi_*(-)$ is also additive for the K-group, it follows that $\deg_\pi(L_1, \ldots, L_{d-1}, L_d)$ is multiadditive.

For every $(d-1)$-tuple $(L_1, \ldots, L_{d-1}), (L_1, \ldots, L_{d-1})$ in $\gamma^{d-1}$. For every virtual object $F$ in the K-ring having virtual rank $r$ and determinant $det(F)$, the class $F - r[O_C]$ is congruent to $det(F) - [O_C]$ modulo $\gamma^2$. Since the gamma filtration is multiplicative, the virtual rank of $R\pi_*(-)$ on $(L_1, \ldots, L_{d-1}) \otimes (F - r[O_C])$ is the same as $\deg_\pi(L_1, \ldots, L_{d-1}, det(F))$. In particular, for $g : C_1 \to C$ as above, for an invertible sheaf $\mathcal{M}$ on $C_1$, the virtual object $g_*[\mathcal{M}] - g_*[O_{C_1}]$ has virtual rank 0 and determinant equal to $I_g([\mathcal{M}])$. Thus, via the projection formula,

$$\deg_\pi(g^*L_1, \ldots, g^*L_{d-1}, \mathcal{M}) = R\pi_*(Rg_*((g^*L_1, \ldots, g^*L_{d-1}) \otimes ([\mathcal{M}] - [O_{C_0}]))) = R\pi_*([L_1, \ldots, L_{d-1}] \otimes (g_*[\mathcal{M}] - g_*[O_{C_1}])) = \deg_\pi(L_1, \ldots, L_{d-1}, I_g([\mathcal{M}])).$$
Since \( \pi \) is smooth, the geometric fibers are reduced, and the same holds for all finite, étale covers. Thus, for every finite, étale morphism \( C_0 \to C \), the morphism \( C_0 \to T \) is cohomologically flat in degree 0. Since the geometric fibers of \( \pi \) are smooth curves, they satisfy Serre’s condition \( S_2 \). Thus, the hypothesis above implies Hypothesis 6.2.

**Definition 8.5.** For every algebraic stack \( \mathcal{X} \) over \( C \), define \( \text{CurveMaps}(\mathcal{X}/T) \) to be the \( T \)-stack whose objects are commutative diagrams

\[
\begin{array}{ccc}
Y_0 & \xrightarrow{g_0} & C \\
p \downarrow & & \downarrow \pi \\
T_0 & \xrightarrow{h} & T
\end{array}
\]

Together with a 1-morphism over \( C \), \( \zeta : Y_0 \to \mathcal{X} \). Here \( h : T_0 \to T \) is a morphism of schemes, and \( p \) is a flat, proper morphism of relative dimension \( \leq 1 \). When \( \mathcal{X} \) is the classifying stack \( BQ \), the \( T \)-stack above is denoted \( \text{CurveMaps}(BQ/T) \).

By Lemma 2.6, \( g_0 : Y_0 \to T_0 \times_T C \) is perfect and proper. Every flat, proper morphism of relative dimension \( \leq 1 \) is projective fppf locally over \( T_0 \), cf. [dJHS11, Proposition 3.3]. Thus, by Corollary 7.2, there is an intersection torsor \( I_{g_0}(E) \) that is a section of \( |BQ_{C/T}| \) over \( T_0 \).

**Proposition 8.6.** The \( T_0 \)-section \( I_{g_0}(E) \) of \( |BQ_{C/T}| \) defines a 1-morphism \( \text{CurveMaps}(BQ/T) \to |BQ_{C/T}| \). When the diagram above is a family of stable sections over \( C \), this 1-morphism agrees with the 1-morphisms constructed in [dJHS11] and [Zhu].

**Proof.** By the axioms, the intersection torsor is functorial for pullback in \( T_0 \) and for isomorphism of \( Q \)-torsors over \( Y_0 \). Thus, this is a 1-morphism. The construction of an intersection torsor via det and Div is precisely how the Abel maps are constructed in [dJHS11] and [Zhu]. □

**Hypothesis 8.7.** Let \( T \) be an excellent, Noetherian scheme. Let \( \pi : C \to T \) be a projective, smooth morphism of pure relative dimension 1, i.e., a family of projective, smooth curves over \( T \) (not necessarily geometrically connected).

Denote by \( T' \to T \) the finite part of the Stein factorization of \( \pi \). Since \( \pi \) is smooth, \( T'/T \) is finite and étale. Since \( \pi \) has relative dimension 1, the relative degree map defines a morphism of \( T \)-schemes,

\[
\deg_{Q/C/T} : |BQ_{C/T}| \to \Lambda(Q/C/T).
\]

**Proposition 8.8.** Assuming Hypothesis 8.7, resp. assuming both Hypothesis 8.7 and that \( \mathcal{O}_T(T) \) contains a characteristic 0 field, the stack \( BQ_{C/T} \) is smooth and the coarse moduli space \( |BQ_{C/T}| \) is fppf, resp. \( BQ_{C/T} \) is smooth and the coarse moduli space is quasi-compact and smooth. Also the degree morphism is fppf, resp. the morphism is surjective and smooth.

For each field \( K \) and \( K \)-valued point \( e \in \Lambda(Q/C/T)(K) \), for \( b : C_0 \to C \) as in Lemma 6.7, for every \( K \)-point \( E \) of \( (BQ_0)_{C_0/T} \) with degree \( b^*e \in \Lambda(Q_0/C_0/T)(K) \), the norm \( I_b(E) \) is a \( K \)-point of \( \Lambda(Q/C/T) \) with degree \( e \). Thus \( \deg_{Q/C/T}(K) \) is surjective if \( C_0 \) has a \( K \)-point.
The degree morphism is a torsor for the kernel \(|BQ^r_{C/T}|\). Locally on \(T\) for the fppf topology, the kernel group scheme admits a finite morphism to an Abelian scheme. Thus \(|BQ^r_{C/T}|\) is a proper, fppf group scheme, resp. it is an extension of a (commutative) finite, étale group scheme over \(T\) by an Abelian scheme over \(T\).

**Proof.** Since \(\Lambda(Q/C/T)\) is étale over \(T\), to prove that \(\text{deg}_{Q_{C/T}}\) is flat and finitely presented, resp. smooth, it is equivalent to prove that \(|BQ_{C/T}|\) is flat and finitely presented over \(T\), resp. smooth over \(T\). This can be checked fppf locally on \(T\). By Proposition 6.10 after fppf base change of \(T\), the stack \(|BQ_{C/T}|\) is a gerbe over \(|BQ_{C/T}|\) for a commutative, fppf group scheme \(R\) that is the kernel of a morphism of tori. In characteristic 0, \(R\) is a smooth group scheme. Thus, to prove that \(|BQ_{C/T}|\) is fppf, resp. to prove that \(|BQ_{C/T}|\) is smooth in characteristic 0, it suffices to prove that \(|BQ_{C/T}|\) is smooth over \(T\).

The infinitesimal deformation theory of \(|BQ_{C/T}|\), and more general “restriction of scalars” stacks, is given in [Ols06, Section 5.7]. The pullback by the identity section of the relative sheaf of differentials of \(Q/C\) is a local free sheaf \(q!\) over \(C\). The cotangent complex of \(BQ \to C\) is isomorphic to the pullback of the complex concentrated in cohomological degree +1, i.e., \(q![-1]\). This is a pullback because the group scheme \(Q\) is commutative; for a non-commutative group scheme \(G\), twist by the universal \(G\)-torsor and the adjoint action on the sheaf of left-invariant differential \(g!\). For an Artinian scheme \(\text{Spec } A\), for a quotient by a square-zero ideal

\[0 \to I \to A' \to A \to 0,\]

for a morphism \(\text{Spec } A \to T\), for a \(Q\)-torsor \(E_A\) over \(C_A = C \times_T \text{Spec } A\), the obstruction to deforming \(E_A\) is an element in \(H^2(C_A, q \otimes_A I)\), and this vanishes because \(C_A\) has dimension 1. Thus, \(|BQ_{C/T}|\) is smooth over \(T\).

Therefore the kernel \(|BQ^r_{C/T}|\) of the degree morphism is an fppf group scheme, resp. a smooth group scheme in characteristic 0. Because the degree morphism is flat and finitely presented, resp. smooth in characteristic 0, this morphism of group schemes is a torsor under the kernel assuming that the morphism is surjective.

To prove surjectivity in the general case, it suffices to prove surjectivity in case \(T\) equals \(\text{Spec } K\) for a field \(K\). To prove surjectivity of points (not surjectivity of the induced map of \(K\)-points), it suffices to check after base change to a larger field. Thus, consider the special case that there exists a finite, étale, surjective morphism \(b : C_0 \to C\) and an isomorphism \(\psi : Q_0 \to G^\rho_{m,C_0}\). Assume, moreover, that for the Stein factorization \(T' = \text{Spec } O_C(C)\), also \(C_0\) is geometrically integral over \(T'\). Finally, assume that there exists a \(T'\)-point \(s_0\) of \(C_0\). By Lemma 6.5 this holds after a finite field extension of \(K\). By Proposition 6.10 under these hypotheses the stacks are split gerbes over the coarse moduli spaces, so they have the same (isomorphism classes of) \(K\)-points. For every \(K\)-point \(e\) of \(\Lambda(Q/C/T)\) and for every \(K\)-point \(E_0\) of \(\Lambda(Q_0/C_0/T)\) of degree \(b^*e\), by Definition 8.3(iv) and Proposition 8.3 the norm \(E = I_b(E_0)\) has degree \(e\). Thus, to prove that \(\text{deg}_{Q/C/T}\) is (geometrically) surjective, it suffices to prove the same for \(\text{deg}_{Q_0/C_0/T}\). Via \(\psi\), the degree morphism is

\[|(BG_{m,C_0})_{C_0/T}| \to \Lambda(G^\rho_{m,C_0}/C_0/T).\]

This is the \(T'/T\)-Weil restriction of the associated morphism of \(T'\)-schemes,

\[(\text{Pic}_{C_0/T'})^\rho \to Z^\rho.\]
Since the degree of $\mathcal{O}_{C_0}(s_0)$ equals 1, this degree morphism is surjective, and even $\deg_{Q_0/C_0}(T)$ is surjective. Thus also $\deg_{Q/C}(T)$ is surjective on $T$-points. Since this holds for all sufficiently large field extensions, the degree morphism is surjective on points (on geometric points always, and on rational points assuming that $C_0$ has a $T$'-point). Note, moreover, that the kernel $(\text{Pic}_0^0)^{C_0/T}$ of the degree morphism is an Abelian scheme over $T'$. Since $T' \to T$ is finite and étale, the $T'/T$-Weil restriction is also an Abelian scheme over $T$.

Denote by $R$ the quotient torus, $R = b_0Q_0/Q$, for the unramified morphism $Q \to b_0Q_0$ of tori on $C$. By the same proof as in Proposition 8.10, $\pi_*R$ is an fppf group scheme on $T$ that is, fppf locally, a product of a torus and a commutative, finite, flat group scheme. The quotient $M$ by $\pi_*b_0Q_0 \cong \mathbb{G}_{m,T}$ is also such an fppf group scheme, cf. Corollary 6.8. By the long exact sequence of fppf cohomology, there is an exact sequence of fppf group schemes,

$$0 \to M \to |BQ_{C/T}| \xrightarrow{b^*} (\text{Pic}_{C_0/T})^\rho \xrightarrow{u} |BR_{C/T}|.$$  

The scheme $M$ is affine, and every locally closed subscheme of $(\text{Pic}_{C_0/T})^\rho$ is separated. Thus, $|BQ_{C/T}|$ is separated. Then the same argument also implies that $|BR_{C/T}|$ is separated. So the kernel of $u$ is a closed subgroup scheme. Altogether, $|BQ_{C/T}|$ is an $M$-torsor over the closed image of $b^*$.

Since $M$ is a product of a torus and a finite, flat group scheme, in order to prove that $M$ is a finite, flat group scheme, it suffices to prove that $M$ has finite order $m$. Denote by $m$ the degree of the finite, étale morphism $b$. For a torsor $E$ on $C_K$, $\mathbb{I}_b(b^*E)$ is multiplication of $E$ by $m$. Since $M$ equals the kernel of $b^*$, $M$ has order $m$. Therefore $M$ is a commutative, finite, flat group scheme that is étale locally a product of group schemes $\mu_\ell$ over $T$.

Also, since $\Lambda(Q_0/C_0)$ is torsion-free, the degree of $b^*E$ equals 0 if and only if the degree of $E$ equals 0. Thus, the image under $b^*$ of $|BQ_{C/T}|$ is the intersection of the closed image of $b^*$ with $|(BQ_0)_{C_0/T}| = (\text{Pic}_{C_K_0/T})^\rho$. This intersection is a closed subgroup scheme of an Abelian scheme over $T$. Altogether, $|BQ_{C/T}|$ is a commutative, proper group scheme over $T$ that is fppf. In characteristic 0, such a group scheme is a direct product of an Abelian scheme and a commut is an extension of a finite, étale group scheme by an Abelian variety. In characteristic 0, such a group scheme is an extension of a finite, étale group scheme by an Abelian scheme (the Abelian scheme is the connected component of the identity). Of course over geometric points of $T$, this extension is split by Chevalley’s structure theorem. □

**Proposition 8.9.** With hypotheses as in Proposition 8.8 assume that $T$ equals $\text{Spec } K$ a field, and assume that $C \to T'$ is a geometrically integral curve of genus $g(C) \geq 2$. The exponent of the torsion group $\Lambda(Q/C/K)(K)/|BQ_{C/T}|(K)$ divides the degree $[K_0 : K']$ of every residue field $K_0$ of every closed point for every finite, étale, surjective morphism $b : C_0 \to C$ such that $b^*Q$ is isomorphic to $\mathbb{G}_{m,C_0}$.

**Proof.** For each such field extension $K_0/K$, replacing $C_0$ be the irreducible component that contains the $K_0$-point, then $C_0$ is a geometrically integral $K_0$-curve with a $K_0$-point. Thus, by Proposition 8.8 $\deg_{Q/C/K}(K_0)$ is surjective, i.e., $\deg_{Q_{K_0}/C_{K_0}/K_0}(K_0)$ is surjective. Since $\deg_{Q_{K_0}/C_{K_0}/K}$ is the $K_0/K$-Weil restriction of $\deg_{Q_{K_0}/C_{K_0}/K_0}$, for every $e \in \Lambda(Q/C/K)(K)$, the multiple $[K_0 : K]e$ is the Weil restriction of
$e \otimes_K K_0$. Thus, for $E_0$ a $K_0$-point of $BQ_{C/K}$ whose $Q_{K_0}/C_{K_0}/K_0$-degree equals $e \otimes_K K_0$, the $Q/C/K$-degree of the norm $I_{K_0/K}(E_0)$ equals $|K_0 : K| e$. So the order of $\tau$ in the cokernel of $\deg_{Q/C/K}(K)$ divides $|K_0 : K|$.

Finally, in order to make sense of the asymptotics of quantities depending on the degree, there is a notion of a “positive structure”, i.e., a monoid of positive degrees.

**Definition 8.10.** A **positive structure** on $\Lambda(Q/C/T)$ is an open and closed subscheme $\Lambda^+(Q/C/T)$ that is a submonoid scheme, that generates a finite index subgroup scheme of $\Lambda(Q/C/T)$, and is sharp, i.e., it contains no subgroup scheme other than the trivial group scheme. The positive structure is **saturated**, resp. **finitely generated**, if its fibers over geometric points of $T$ are saturated (for multiplication by positive integers), resp. finitely generated (as a commutative monoid).

**Lemma 8.11.** Every saturated positive structure generates $\Lambda(Q/C/T)$.

**Proof.** This can be checked after base change to geometric points of $T$. By hypothesis, for every $e \in \Lambda(Q/C/T)$, there exists an integer $m > 0$ such that $m \cdot e$ is contained in the subgroup generated by $\Lambda^+(Q/C/T)$. Every nonnegative linear combination of elements of $\Lambda^+(Q/C/T)$ is an element of $\Lambda^+(Q/C/T)$. Thus there exists $e_+, e_- \in \Lambda^+(Q/C/T)$ such that $m \cdot e$ equals $e_+ - e_-$. Thus $m \cdot e + e_-$ is in $\Lambda^+(Q/C/T)$. Since $m \geq 0$ and since $e_-$ is in $\Lambda^+(Q/C/T)$, also $(m - 1) \cdot e_-$ is in $\Lambda^+(Q/C/T)$. Thus $m \cdot e + e_- + (m - 1) \cdot e_-$ is in $\Lambda^+(Q/C/T)$, i.e., $m \cdot (e + e_-)$ is in $\Lambda^+(Q/C/T)$. Since this submonoid is saturated, $e + e_-$ is in $\Lambda^+(Q/C/T)$. Therefore $e = (e + e_-) - e_-$ is in the subgroup generated by $\Lambda^+(Q/C/T)$. □

**Lemma 8.12.** Let $b : C_0 \to C$ and $\psi : Q_0 \to \mathbb{Q}^\rho_{m,C_0}$ be a trivializing finite, étale cover as above. Assume, moreover, that $b$ is Galois with Galois group $\Gamma$. For every $\Gamma$-stabilized positive structure on $\Lambda(Q_0/C_0/T)$, the intersection of $\Lambda^+(Q_0/C_0/T)$ with $\Lambda(Q/C/T)$ is a positive structure $\Lambda^{\Gamma}(Q/C/T)$ on $\Lambda(Q/C/T)$. If $\Lambda^{\Gamma}(Q_0/C_0/T)$ is saturated, resp. finitely generated, then also $\Lambda^+(Q/C/T)$ is saturated, resp. finitely generated.

**Proof.** The intersection $\Lambda^+(Q/C/T)$ is an open and closed subscheme that is a subsemigroup scheme and contains only the trivial group scheme. It remains to check that $\Lambda^+(Q/C/T)$ generates a finite index subgroup scheme. This can be checked on geometric fibers over $T$. Denote by $m$ the index in $\Lambda(Q_0/C_0/T)$ of the subgroup generated by $\Lambda^+(Q/C/T)$. Denote by $n$ the order of $\Gamma$. For every geometric point $e \in \Lambda(Q/C/T)$, by hypothesis, the element $m \cdot e$ in $\Lambda(Q_0/C_0/T)$ is a $\mathbb{Z}$-linear combination of elements of $\Lambda^+(Q_0/C_0/T)$, say

$$m \cdot e = \sum_i c_i f_i, \quad c_i \in \mathbb{Z}, \quad f_i \in \Lambda^+(Q_0/C_0/T).$$

Then also

$$(n \cdot m) \cdot e = \sum_{\gamma \in \Gamma} \gamma \cdot (m \cdot e) = \sum_i c_i (\sum_{\gamma \in \Gamma} \gamma \cdot f_i).$$

By hypothesis, each $\gamma \cdot f_i$ is in $\Lambda^+(Q_0/C_0/T)$. Thus the sum $\sum_i \gamma \cdot f_i$ is an element of $\Lambda^+(Q_0/C_0/T)$ that is $\Gamma$-invariant. By descent, this is an element in $\Lambda(Q/C/T)$, thus it is in $\Lambda^+(Q/C/T)$. So the subgroup generated by $\Lambda^+(Q/C/T)$ has finite index dividing $n \cdot m$. 27
The intersection of a saturated submonoid of an Abelian group with a subgroup is also saturated. Thus, if \( \Lambda^+(Q_0/C_0/T) \) is saturated, then \( \Lambda^+(Q/C/T) \) is also saturated. Over geometric points of \( T \), \( \Lambda(Q_0/C_0/T) \) is isomorphic to \( \mathbb{Z}^n \). Assume that \( \Lambda^+(Q_0/C_0/T) \) is a finitely generated monoid. The intersection of any finite collection of finitely generated monoids in \( \mathbb{Z}^n \) is a finitely generated monoid (the problem of algorithmically finding a finite set of generators is known as the “vertex enumeration problem”). In particular, the intersection of the finitely generated monoid \( \Lambda^+(Q_0/C_0/T) \) and the finitely generated monoid (even subgroup) \( \Lambda(Q/C/T) \) is a finitely generated monoid \( \Lambda^+(Q_0/C_0/T) \). □

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Department of Mathematics, Stony Brook University, Stony Brook, NY 11794

E-mail address: jstarr@math.stonybrook.edu