FINITE GORENSTEIN REPRESENTATION TYPE IMPLIES SIMPLE SINGULARITY

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To Lucho Avramov on his sixtieth birthday

ABSTRACT. Let $R$ be a commutative noetherian local ring and consider the set of isomorphism classes of indecomposable totally reflexive $R$-modules. We prove that if this set is finite, then either it has exactly one element, represented by the rank 1 free module, or $R$ is Gorenstein and an isolated singularity (if $R$ is complete, then it is even a simple hypersurface singularity). The crux of our proof is to argue that if the residue field has a totally reflexive cover, then $R$ is Gorenstein or every totally reflexive $R$-module is free.

INTRODUCTION

Remarkable connections between the module theory of a local ring and the character of its singularity emerged in the 1980s. They show how finiteness conditions on the category of maximal Cohen–Macaulay modules\footnote{The finitely generated modules whose depth equals the Krull dimension of the ring.} characterize particular isolated singularities. We develop these connections in several directions.

A local ring with only finitely many isomorphism classes of indecomposable maximal Cohen–Macaulay modules is said to be of finite Cohen–Macaulay (CM) representation type. By work of Auslander \[5\], every complete Cohen–Macaulay local ring of finite CM representation type is an isolated singularity.

Specialization to Gorenstein rings opens to a finer description of the singularities; it centers on the simple hypersurface singularities identified in Arnol’d’s work on germs of holomorphic functions \[1\]. By work of Buchweitz, Greuel, and Schreyer \[12\], Herzog \[18\], and Yoshino \[32\], a complete Gorenstein ring of finite CM representation type is a simple singularity in the generalized sense of \[32\]. Under extra assumptions on the ring, the converse holds by work of Knörrer \[21\] and Solberg \[25\].

In this introduction, $R$ is a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k$. To avoid the a priori condition in \[12, 18, 32\] that $R$ is Gorenstein, we replace finite CM representation type with a finiteness condition on the category $\mathcal{G}(R)$ of modules of Gorenstein dimension 0. Over a Gorenstein ring, these modules are precisely the maximal Cohen–Macaulay modules, but they are known to exist over any ring, unlike maximal Cohen–Macaulay modules.

**Theorem A.** Let $R$ be complete. If the set of isomorphism classes of non-free indecomposable modules in $\mathcal{G}(R)$ is finite and not empty, then $R$ is a simple singularity.
The category $\mathcal{G}(R)$ was introduced by Auslander and Bridger [4, 6]. An $R$-module $G$ is in $\mathcal{G}(R)$ if there is an exact complex of finitely generated free $R$-modules

$$F = \cdots \to F_{n+1} \xrightarrow{\partial_{n+1}} F_n \xrightarrow{\partial_n} F_{n-1} \to \cdots,$$

such that $G$ is isomorphic to $\text{Coker} \partial_0$ and the complex $\text{Hom}_R(F, R)$ is exact. Every finitely generated free $R$-module is in $\mathcal{G}(R)$, and the modules in this category have Gorenstein dimension 0 as in [4, 6]; following [11] we call them totally reflexive.

The aforementioned works [12, 18, 32] show that Theorem A follows from the next result, which is proved as (4.3).

**Theorem B.** If the set of isomorphism classes of indecomposable modules in $\mathcal{G}(R)$ is finite, then $R$ is Gorenstein or every module in $\mathcal{G}(R)$ is free.

As this theorem does not require $R$ to be complete, we considerably strengthen Theorem A using work of Huneke, Leuschke, and R. Wiegand [19, 22, 30]; this occurs in (4.5). Theorem B was conjectured by R. Takahashi [29], who proved it for henselian rings of depth at most two [27, 28, 29]. The class of rings over which all totally reflexive modules are free is poorly understood, but it is known to include all Golod rings [11], in particular, all Cohen–Macaulay rings of minimal multiplicity.

To prove Theorem B we use a notion of $\mathcal{G}(R)$-approximations, which is close kin to the CM-approximations of Auslander and Buchweitz [7]. When $R$ is Gorenstein, a $\mathcal{G}(R)$-approximation is exactly a CM-approximation. By [7], every module over a Gorenstein ring has a CM-approximation. Our proof of Theorem B goes via the following strong converse, proved as (3.4).

**Theorem C.** Let $R$ be a local ring and assume there is a non-free module in $\mathcal{G}(R)$. If the residue field $k$ has a $\mathcal{G}(R)$-approximation, then $R$ is Gorenstein.

This theorem complements recent developments in relative homological algebra. The notion of totally reflexive modules has two extensions to non-finitely generated modules; see [13] for details. One is Gorenstein projective modules, which allows arbitrary free modules in the definition above. By recent work of Jørgensen [20], every module over a complete local ring has a Gorenstein projective precovers. The other extension is Gorenstein flat modules. By a result of Enochs and López-Ramos [15], every module has a Gorenstein flat precovers.

Theorem C counterposes these developments; it shows that for finitely generated modules, the precovers found in [20] and [15] cannot, in general, be finitely generated. Assume that $R$ is complete. Then a finitely generated $R$-module has a $\mathcal{G}(R)$-approximation if and only if it has a $\mathcal{G}(R)$-precovers. Assume further that $R$ is not Gorenstein. Theorem C shows that if $X \to k$ is a Gorenstein projective/flat precovers and $X$ is not free, then $X$ is not finitely generated.

1. **Categories and covers**

In this paper, rings are commutative and noetherian; modules are finitely generated (unless otherwise specified). We write $\text{mod}(R)$ for the category of finitely generated modules over a ring $R$.

For an $R$-module $M$, we denote by $M_i$ the $i$th syzygy in a free resolution. When $R$ is local, we denote by $\Omega^i_R(M)$ the $i$th syzygy in the minimal free resolution of $M$. For an $R$-module $M$, set $M^* = \text{Hom}_R(M, R)$; we refer to this module as the algebraic dual of $M$. 


We only consider full subcategories of \( \text{mod}(R) \); this allows us to define a subcategory by specifying its objects. In the following, \( \mathcal{B} \) is a subcategory of \( \text{mod}(R) \).

(1.1) **Closures.** Recall that the category \( \mathcal{B} \) is said to be closed under extensions if for every short exact sequence \( 0 \to B \to X \to B' \to 0 \) with \( B \) and \( B' \) in \( \mathcal{B} \) also \( X \) is in \( \mathcal{B} \). The closure of \( \mathcal{B} \) under extensions is by definition the smallest subcategory containing \( \mathcal{B} \) and closed under extensions. Recall also that \( \mathcal{B} \) is closed under direct sums and direct summands when a direct sum \( M \oplus N \) is in \( \mathcal{B} \) if and only if both summands are in \( \mathcal{B} \). The closure of \( \mathcal{B} \) under addition is by definition the smallest subcategory containing \( \mathcal{B} \) and closed under direct sums and direct summands; we denote it by \( \text{add}(\mathcal{B}) \).

We define the closure \( \langle \mathcal{B} \rangle \) to be the smallest subcategory containing \( \mathcal{B} \) and closed under direct summands and extensions. It is straightforward to verify that the closure \( \langle \mathcal{B} \rangle \) is reached by countable alternating iteration, starting with \( \mathcal{B} \), between closure under addition and closure under extensions.

We say that \( \mathcal{B} \) is **closed under algebraic duality** if for every module \( B \in \mathcal{B} \) the module \( B^* \) is also in \( \mathcal{B} \). Similarly, we say that \( \mathcal{B} \) is **closed under syzygies** if for every module \( B \in \mathcal{B} \) every first syzygy \( B_1 \) is in \( \mathcal{B} \); then every syzygy \( B_i \) is in \( \mathcal{B} \).

(1.2) **Precovers and covers.** Let \( M \) be an \( R \)-module. A \( \mathcal{B} \)-precover of \( M \) is a homomorphism \( \varphi: B \to M \), with \( B \in \mathcal{B} \), such that every homomorphism \( X \to M \) with \( X \in \mathcal{B} \), factors through \( \varphi \); i.e., the homomorphism

\[
\text{Hom}_R(X, \varphi): \text{Hom}_R(X, B) \to \text{Hom}_R(X, M)
\]

is surjective for each module \( X \in \mathcal{B} \). A \( \mathcal{B} \)-precover \( \varphi: B \to M \) is a \( \mathcal{B} \)-cover if every \( \gamma \in \text{Hom}_R(B, B) \) with \( \varphi \gamma = \varphi \) is an automorphism.

Note that if the category \( \mathcal{B} \) contains \( R \), then every \( \mathcal{B} \)-precover is surjective.

(1.3) If there are only finitely many isomorphism classes of indecomposable modules in \( \mathcal{B} \), then every finitely generated \( R \)-module has a \( \mathcal{B} \)-precover; see [2, Prop. 4.2].

(1.4) Consider a diagram \( B \xrightarrow{\varphi} M \oplus N \xrightarrow{\pi} M \), where \( \pi_1 \) is the identity on \( M \). If \( \varphi \) is a \( \mathcal{B} \)-precover, then so is \( \pi \varphi: B \to M \).

The next two lemmas appear in Xu’s book [31, 2.1.1 and 1.2.8]. We include a proof of the second one since Xu left it to the reader.

(1.5) **Wakamatsu’s lemma.** Let \( \mathcal{B} \) be a subcategory of \( \text{mod}(R) \), and let \( \varphi \) be a \( \mathcal{B} \)-cover of an \( R \)-module \( M \). If \( \mathcal{B} \) is closed under extensions, then \( \text{Ext}^1_R(X, \ker \varphi) = 0 \) for all \( X \in \mathcal{B} \).

(1.6) **Lemma.** Let \( \mathcal{B} \) be a subcategory of \( \text{mod}(R) \), and let \( M \) be an \( R \)-module. If \( M \) has a \( \mathcal{B} \)-cover, then a \( \mathcal{B} \)-precover \( \varphi: X \to M \) is a cover if and only if \( \ker \varphi \) contains no non-zero direct summand of \( X \).

**Proof.** Let \( \psi: Y \to M \) be a \( \mathcal{B} \)-cover. For the “if” part, consider the commutative diagram below, where \( \alpha \) and \( \beta \) are given by the precovering properties of \( \varphi \) and \( \psi \).
Since $\psi\beta\alpha = \psi$ and $\psi$ is a cover, the composite $\beta\alpha$ is an automorphism, so $\beta$ is surjective. It also follows that $X$ is isomorphic to $\ker \beta \oplus \text{Im} \alpha$. As $\ker \varphi$ contains no non-zero summand of $X$, the inclusion $\ker \beta \subseteq \ker \varphi$ implies that $\beta$ is also injective. Consequently, $\varphi$ is a $B$-cover.

For the “only if” part, consider a decomposition $X = Y \oplus Z$, and assume there is an inclusion $Z \subseteq \ker \varphi$. Let $\pi$ be the endomorphism of $X$ projecting onto $Y$, then $\varphi\pi = \varphi$. Since $\varphi$ is a cover, $\pi$ is an automorphism, whence $Z = 0$. \hfill $\square$

2. Approximations and reflexive subcategories

Stability of (pre-)covers under base change is delicate to track. To avoid this task, we develop a notion between precover and cover. The next definition is in line with that of CM-approximations [7]; for $G(R)$ it broadens the notion used in [11].

(2.1) **Definitions.** Let $\mathcal{B}$ be a subcategory of $\text{mod}(R)$ and set

$$\mathcal{B}^\perp = \{ L \in \text{mod}(R) \mid \text{Ext}^i_R(B, L) = 0 \text{ for all } B \in \mathcal{B} \text{ and all } i > 0 \}.$$  

Let $M$ be an $R$-module. A **$\mathcal{B}$-approximation** of $M$ is a short exact sequence

$$0 \to L \to B \to M \to 0,$$

where $B$ is in $\mathcal{B}$ and $L$ is in $\mathcal{B}^\perp$.

(2.2) Let $\mathcal{B}$ be a subcategory of $\text{mod}(R)$ and $M$ be an $R$-module.

(a) If $0 \to \ker \varphi \to B \xrightarrow{\varphi} M \to 0$ is a $\mathcal{B}$-approximation of $M$, then $\varphi$ is a special $\mathcal{B}$-precover of $M$; see [31, Prop. 2.1.3].

(b) If $B \xrightarrow{\varphi} M$ is a surjective $\mathcal{B}$-cover, and $\mathcal{B}$ is closed under syzygies and extensions, then the sequence $0 \to \ker \varphi \to B \xrightarrow{\varphi} M \to 0$ is a $\mathcal{B}$-approximation of $M$ by Wakamatsu’s lemma.

(c) Assume $\text{mod}(R)$ has the Krull–Schmidt property (e.g., $R$ is henselian) and $\mathcal{B}$ is closed under direct summands. The module $M$ has a $\mathcal{B}$-cover if and only if it has a $\mathcal{B}$-precover; see [29, Cor. 2.5].

The next two results study the behavior of approximations under base change.

Let $\vartheta : R \to S$ be a ring homomorphism. We say that $\vartheta$ is of finite flat dimension if $S$, viewed as an $R$-module through $\vartheta$, has a bounded resolution by flat $R$-modules. We write $\text{Tor}^R_{i > 0}(S, B) = 0$ if for all $B \in \mathcal{B}$, and for all $i > 0$, the modules $\text{Tor}^R_i(S, B)$ vanish. We denote by $S \otimes \mathcal{B}$ the subcategory of $S$-modules $S \otimes_R B$ with $B \in \mathcal{B}$.

(2.3) **Lemma.** Let $R \to S$ be a ring homomorphism of finite flat dimension. Let $\mathcal{B}$ be a subcategory of $\text{mod}(R)$ such that $\text{Tor}^R_{i > 0}(S, \mathcal{B}) = 0$. If $L \in \mathcal{B}^\perp$ and $\text{Tor}^R_{i > 0}(S, L) = 0$, then for every $m \in \mathbb{Z}$ and every $B \in \mathcal{B}$ there is an isomorphism

$$\text{Ext}^m_S(S \otimes_R B, S \otimes_R L) \cong \text{Tor}^R_m(S, \text{Hom}_R(B, L)).$$

In particular, there are isomorphisms $\text{Hom}_S(S \otimes_R B, S \otimes_R L) \cong S \otimes_R \text{Hom}_R(B, L)$, and $S \otimes_R L$ is in $(S \otimes \mathcal{B})^\perp$.

**Proof.** Fix $B \in \mathcal{B}$. Take a free resolution $E \to B$ and a bounded flat resolution $F \to S$ over $R$. By the vanishing of (co)homology, the induced morphisms

$$S \otimes_R E \to S \otimes_R B, \quad F \otimes_R L \to S \otimes_R L, \quad \text{and } \text{Hom}_R(B, L) \to \text{Hom}_R(E, L)$$

provide the required isomorphisms.
are homology isomorphisms. In particular, the first one is a free resolution of the $S$-module $S \otimes_R B$. The functors $\text{Hom}_R(E, -)$ and $F \otimes_R -$ preserves homology isomorphisms. This explains the first, third, and fifth isomorphisms below.

$$\text{Ext}_S^m(S \otimes_R B, S \otimes_R L) \cong H^m(\text{Hom}_S(S \otimes_R E, S \otimes_R L))$$

$$\cong H^m(\text{Hom}_R(E, S \otimes_R L))$$

$$\cong H^m(\text{Hom}_R(E, F \otimes_R L))$$

$$\cong H^m(F \otimes_R \text{Hom}_R(E, L))$$

$$\cong H^m(F \otimes_R \text{Hom}_R(B, L))$$

$$\cong \text{Tor}_m^R(S, \text{Hom}_R(B, L))$$

The second isomorphism follows from Hom-tensor adjointness, and the fourth is tensor evaluation; see [17, Prop. II.5.14]. For $m = 0$ the composite isomorphism reads $\text{Hom}_S(S \otimes_R B, S \otimes_R L) \cong S \otimes_R \text{Hom}_R(B, L)$. That $S \otimes_R L$ is in $\langle S \otimes B \rangle^\perp$ follows as $\text{Tor}_i^R$ is zero for $i < 0$. □

(2.4) **Proposition.** Let $R \to S$ be a ring homomorphism and $\mathcal{B}$ be a subcategory of mod($R$). Let $M$ be an $R$-module with a $\mathcal{B}$-approximation $0 \to L \to B \to M \to 0$. If $\text{Tor}_{i>0}^R(S, \mathcal{B}) = 0$ and $\text{Tor}_{i>0}^R(S, M) = 0$, then

$$0 \to S \otimes_R L \to S \otimes_R B \to S \otimes_R M \to 0$$

is an $(S \otimes \mathcal{B})$-approximation.

**Proof.** By the assumptions on $\mathcal{B}$ and $M$, application of the functor $S \otimes_R -$ to the $\mathcal{B}$-approximation of $M$ yields the desired short exact sequence and also equalities $\text{Tor}_{i>0}^R(S, L) = 0$. Now Lemma (2.3) gives that $S \otimes_R L$ is in $\langle S \otimes \mathcal{B} \rangle^\perp$. □

(2.5) Let $\mathcal{B}$ be a subcategory of mod($R$) with $R \in \mathcal{B}^\perp$. For every $B \in \mathcal{B}$ and every $R$-module $N$, dimension shifting yields

$$\text{Ext}_R^i(B, N) \cong \text{Ext}_R^{i+h}(B, N_h) \quad \text{for } i > 0 \text{ and } h \geq 0.$$ 

Moreover, for $h \geq 0$ the algebraic dual $B^*$ is a $h$th syzygy of $(B_h)^*$, so

$$\text{Ext}_R^i(B^*, N) \cong \text{Ext}_R^{i+h}((B_h)^*, N) \quad \text{for } i > 0 \text{ and } h \geq 0.$$ 

If, furthermore, $\mathcal{B}$ is closed under syzygies and algebraic duality, then these isomorphisms combine to yield

$$\text{Ext}_R^i(B^*, N_j) \cong \text{Ext}_R^i((B_h)^*, N_{j-h}) \quad \text{for } i > 0 \text{ and } j \geq h \geq 0.$$ 

In particular, (2.5.1) holds when $\mathcal{B}$ is a category satisfying the next definition.

(2.6) **Definition.** A subcategory $\mathcal{B}$ of mod($R$) is reflexive if $R$ is in $\mathcal{B} \cap \mathcal{B}^\perp$ and $\mathcal{B}$ is closed under

1. direct sums and direct summands,
2. syzygies, and
3. algebraic duality.

It is standard that the category $\mathcal{G}(R)$ of totally reflexive $R$-modules is a reflexive subcategory of mod($R$). Moreover, using the characterization of $\mathcal{G}(R)$ provided by [13, (1.1.2) and (4.1.4)], it is straightforward to verify that every reflexive subcategory of mod($R$) is, in fact, a subcategory of $\mathcal{G}(R)$. 

[17] Proposition II.5.14
(2.7) In the rest of the paper, $\mathcal{F}(R)$ denotes the category of finitely generated free $R$-modules. Let $\mathcal{B}$ be a reflexive subcategory of $\text{mod}(R)$. There are containments

$$\mathcal{F}(R) \subseteq \mathcal{B} \subseteq \mathcal{G}(R).$$

Further, let $R \to S$ be a ring homomorphism of finite flat dimension, then

$$\text{Tor}^R_{i>0}(S, \mathcal{B}) = 0,$$

as every module in $\mathcal{B}$ is an infinite syzygy.

The next observation is crucial for our proofs of the main theorems.

(2.8) Assume $\text{mod}(R)$ has the Krull–Schmidt property (e.g., $R$ is henselian) and let $\mathcal{B}$ be a reflexive subcategory of $\text{mod}(R)$ closed under extensions. We claim that an $R$-module $M$ has a $\mathcal{B}$-precover if and only if it has a $\mathcal{B}$-approximation. Indeed, let $\varphi: B \to M$ be a $\mathcal{B}$-precover; by (2.2)(c) the module $M$ also has a $\mathcal{B}$-cover. Decompose $B$ as $B' \oplus B''$, where $B''$ is the largest direct summand of $B$ contained in $\ker \varphi$. By Lemma (1.6) the factorization $\varphi': B' \to M$ is a cover, and by (2.2)(b) the sequence $0 \to \ker \varphi' \to B' \to M \to 0$ is a $\mathcal{B}$-approximation.

(2.9) Lemma. Let $\mathcal{B}$ be a reflexive subcategory of $\text{mod}(R)$ and $M$ be an $R$-module. If $M$ has a $\mathcal{B}$-approximation, then every syzygy of $M$ has a $\mathcal{B}$-approximation.

Proof. Let $0 \to L \to B \to M \to 0$ be a $\mathcal{B}$-approximation. It is sufficient to prove that every first syzygy $M_1$ has a $\mathcal{B}$-approximation. By the horseshoe construction, there is a short exact sequence $0 \to L_1 \to B_1 \to M_1 \to 0$, and the syzygy $B_1$ is in $\mathcal{B}$ by assumption. Let $X$ be in $\mathcal{B}$. Since $\mathcal{B}$ is reflexive, there is an isomorphism $X \cong X^{**}$, and also the module $((X^*)_1)^*$ is in $\mathcal{B}$. Now (2.5.1) yields the second isomorphism in the chain

$$\text{Ext}^i_R(X, L_1) \cong \text{Ext}^i_R(X^{**}, L_1) \cong \text{Ext}^i_R(((X^*)_1)^*, L) = 0. \quad \square$$

(2.10) Proposition. Let $R \to S$ be a ring homomorphism of finite flat dimension. If $\mathcal{B}$ is a reflexive subcategory of $\text{mod}(R)$, then $(\mathcal{S} \otimes \mathcal{B})$ is a reflexive subcategory of $\text{mod}(S)$. In particular, $(\mathcal{S} \otimes \mathcal{G}(R))$ is reflexive.

Proof. The ring $S$ is in $(\mathcal{S} \otimes \mathcal{B})$. As $R \in \mathcal{B}^\perp$, it follows from (2.7) and Lemma (2.3) that $S$ is in $(\mathcal{S} \otimes \mathcal{B})^\perp$. By definition, $(\mathcal{S} \otimes \mathcal{B})$ is closed under direct sums and direct summands; this leaves (2) and (3) in Definition (2.6) to verify.

First we prove closure under syzygies. Take $B \in \mathcal{B}$ and consider a short exact sequence $0 \to B_1 \to F \to B \to 0$, where $F$ is a free $R$-module. By assumption, the syzygy $B_1$ is in $\mathcal{B}$. By (2.7) the sequence

$$0 \to S \otimes_R B_1 \to S \otimes_R F \to S \otimes_R B \to 0$$

is exact. It shows that the syzygy $S \otimes_R B_1$ of $S \otimes_R B$ is in $S \otimes \mathcal{B}$. Moreover, it follows that any summand of $S \otimes_R B$ has a first syzygy in $\text{add}(S \otimes \mathcal{B})$, in particular, in $(S \otimes \mathcal{B})$. By Schanuel’s lemma, a module in $(S \otimes \mathcal{B})$ with some first syzygy in $\mathcal{S} \otimes \mathcal{B}$ has every first syzygy in $(\mathcal{S} \otimes \mathcal{B})$. Finally, given a short exact sequence $0 \to M \to X \to N \to 0$, where $M$, $N$, and their first syzygies are in $(S \otimes \mathcal{B})$, we claim that also a first syzygy of $X$ is in $(S \otimes \mathcal{B})$. Indeed, take presentations of $M$ and $N$. Since $(S \otimes \mathcal{B})$ is closed under extensions, it follows from the horseshoe construction that a first syzygy of $X$ is in $(S \otimes \mathcal{B})$. 

Next we prove closure under algebraic duality. Take $B \in \mathcal{B}$ and note that by (2.7), Lemma (2.3) applies (with $L = R$) to yield the isomorphism
$$\text{Hom}_S(S \otimes_R B, S) \cong S \otimes_R \text{Hom}_R(B, R).$$
Thus, the algebraic dual of $S \otimes_R B$ is in $S \otimes \mathcal{B}$. Moreover, the algebraic dual of any summand of $S \otimes_R B$ is in $\text{add}(S \otimes \mathcal{B})$, in particular, in $(S \otimes \mathcal{B})$. It is now sufficient to prove that for every short exact sequence $0 \to M \to X \to N \to 0$, where $M$, $N$, and the duals $M^*$ and $N^*$ are in $(S \otimes \mathcal{B})$, also the dual $X^*$ is in $(S \otimes \mathcal{B})$. Since $(S \otimes \mathcal{B})$ is closed under extensions, this is immediate from the exact sequence
$$0 \to N^* \to X^* \to M^* \to \text{Ext}^1_S(N, S),$$
where $\text{Ext}_S(N, S) = 0$ as $S$ is in $(S \otimes \mathcal{B})^\perp$.

3. Approximations detect the Gorenstein property

The main result of this section is Theorem C from the introduction. Lemma (3.2) furnishes the base case; for that we study a standard homomorphism.

(3.1) For modules $X$ and $N$ over a ring $S$ there is a natural map
$$\theta_{X,N} : X \otimes_S N \to \text{Hom}_S(X^*, N),$$
given by evaluation $\theta(x \otimes n)(\zeta) = \zeta(x)n$. Auslander computed the kernel and cokernel of this map in [3, Prop. 6.3]. Because the map is pivotal for our proof of the next lemma, we include a computation for the case where $X$ is totally reflexive.

Consider a short exact sequence $0 \to N_1 \to F \to N \to 0$, where $F$ is a free $S$-module. For any totally reflexive $S$-module $X$, the evaluation homomorphism $\theta_{X,F}$ is an isomorphism, and the commutative diagram
\[
\begin{array}{ccc}
X \otimes_S N_1 & \longrightarrow & X \otimes_S F \\
\downarrow \theta_{X,N_1} & \cong & \downarrow \theta_{X,F} & \downarrow \theta_{X,N} \\
0 & \longrightarrow & \text{Hom}_S(X^*, N_1) & \longrightarrow & \text{Hom}_S(X^*, F) & \longrightarrow & \text{Hom}_S(X^*, N) & \longrightarrow & \text{Ext}^1_S(X^*, N_1) & \longrightarrow & 0
\end{array}
\]
shows that there is an isomorphism $\text{Coker} \theta_{X,N} \cong \text{Ext}^1_S(X^*, N_1)$. The snake lemma applies to yield $\text{Ker} \theta_{X,N} \cong \text{Coker} \theta_{X,N_1} \cong \text{Ext}^1_S(X^*, N_2)$, and then (2.5.1) gives
\[(3.1.1) \quad \text{Ker} \theta_{X,N} \cong \text{Ext}^1_S((X_2)^*, N) \quad \text{and} \quad \text{Coker} \theta_{X,N} \cong \text{Ext}^1_S((X_1)^*, N).
\]

(3.2) **Lemma.** Let $(S, n, \ell)$ be a complete local ring of depth $0$. Let $\mathcal{C}$ be a reflexive subcategory of $\text{mod}(S)$. If $\ell$ has a $\mathcal{C}$-approximation and $\ell$ is not in $\mathcal{C}$, then $\mathcal{C} = \mathcal{F}(S)$.

**Proof.** Consider a $\mathcal{C}$-approximation $0 \to L \xrightarrow{\alpha} C \to \ell \to 0$, and dualize to get $0 \to \ell^* \to C^* \xrightarrow{\alpha^*} L^*$. Let $I$ be the image of $\alpha^*$, and let $\varphi$ be the factorization of $\alpha^*$ through the inclusion $I \to L^*$.

First we prove that the surjection $\varphi$ is a $\langle \mathcal{C} \rangle$-cover of $I$. Let $X$ be a module in $\langle \mathcal{C} \rangle$. If $X$ is a free $S$-module, then any homomorphism $X \to I$ lifts through $\varphi$. We may now assume that $X$ is indecomposable and not free. Because $\text{Hom}_S(X, I)$ is a submodule of $\text{Hom}_S(X, L^*)$, it suffices to prove surjectivity of
$$\text{Hom}_S(X, \alpha^*) : \text{Hom}_S(X, C^*) \to \text{Hom}_S(X, L^*),$$
which we do next.
The vertical maps in the commutative diagram below are evaluation homomorphisms, see (3.1).

\[
\begin{array}{c}
X \otimes_R L \xrightarrow{\theta_{XL}} X \otimes_R C \xrightarrow{\theta_{XC}} X \otimes_R \ell \xrightarrow{\theta_{XL}} 0 \\
0 \xrightarrow{\theta_{XL}} \Hom_S(X^*, L) \xrightarrow{\theta_{XC}} \Hom_S(X^*, C) \xrightarrow{\theta_{XL}} \Hom_S(X^*, \ell) \xrightarrow{\Ext^1_S(X^*, L)} 0
\end{array}
\]

First we argue that the rows of this diagram are short exact sequences. The module \(X\) is in \(\langle C \rangle\) and hence in \(\mathcal{G}(S)\), see (2.7), so \(\Ext^1_S(X^*, L) = 0\). Moreover, \(\theta_{XL}\) is an isomorphism by (3.1.1), hence \(\ell\) is injective. Next note that for every \(\zeta \in X^*\) the image of \(\zeta : X \to S\) is in \(n\) as \(X\) is indecomposable and not free. Thus, for all \(x \in X\) and \(u \in \ell\), we have \(\theta_{XL}(x \otimes u)(\zeta) = \zeta(x)u = 0\). Finally, apply \(\Hom_S(-, S)\) to the diagram above and use \(\Hom\)-tensor adjointness to get

\[
\begin{array}{c}
0 \xrightarrow{\theta_{XL}} \Hom_S(X^*, L) \xrightarrow{\theta_{XC}} \Hom_S(X^*, C) \xrightarrow{\theta_{XL}} \Hom_S(X^*, \ell) \xrightarrow{\Ext^1_S(X^*, L)} 0 \\
\Hom_S(X, \ell^*) \xrightarrow{\Hom_S(X, \alpha^*)} \Hom_S(X, C^*) \xrightarrow{\Hom_S(X, \ell^*)} \Ext^1_S(X, \ell^*, L) \xrightarrow{\Ext^1_S(X \otimes_R \ell, S)} 0
\end{array}
\]

The diagram shows that \(\Hom_S(X, \alpha^*)\) is surjective, as desired.

Now \(\varphi : C^* \to I\) is a \(\langle C \rangle\)-precover, so by completeness of \(S\), the module \(I\) has a \(\langle C \rangle\)-cover; see (2.2)(c). The ring has depth 0, so \(\ell^*\) is a non-zero \(\ell\)-vector space. By the assumptions on \(C\), the residue field \(\ell\) cannot be a direct summand of \(C^*\). As \(\Ker \varphi = \ell^*\), it follows from Lemma (1.6) that \(\varphi\) is a \(\langle C \rangle\)-cover. For every \(X \in \langle C \rangle\) Wakamatsu’s lemma gives \(\Ext^1_S(X, \ell^*) = 0\). Consequently, every module in \(C\) is projective and hence free, since \(S\) is local.

(3.3) Let \((R, m, k)\) be a local ring and denote by \(\mathcal{M}(R)\) the category of maximal Cohen–Macaulay \(R\)-modules.

(a) If \(R\) is Cohen–Macaulay, then \(\mathcal{G}(R) \subseteq \mathcal{M}(R)\) by the Auslander–Bridger formula [4, §3.2 Prop. 3]. Conversely, if \(\mathcal{G}(R) \subseteq \mathcal{M}(R)\), then \(R\) is Cohen–Macaulay.

(b) If \(R\) is Gorenstein, then the categories \(\mathcal{G}(R)\) and \(\mathcal{M}(R)\) coincide by [4, §3.2 Thm. 3] and the Auslander–Bridger formula. Conversely, if \(\mathcal{G}(R) = \mathcal{M}(R)\), then \(R\) is Gorenstein. Indeed, \(R\) is Cohen–Macaulay by (a), so \(\Omega^R_{\dim R}(k)\) is in \(\mathcal{M}(R)\), hence in \(\mathcal{G}(R)\), and therefore \(R\) is Gorenstein by [4, §3.2, Rmk. after Thm. 3].

(c) If \(R\) is Gorenstein, then a short exact sequence \(0 \to L \to G \to M \to 0\) is a CM-approximation if and only if it is a \(\mathcal{G}(R)\)-approximation. This follows from (b) and the fact that \(L\) is in \(\mathcal{M}(R)\) if and only if \(L\) has finite injective dimension.

If \(R\) is Gorenstein, then every \(R\)-module has a CM-approximation by [7, Thm. A]. In view of (3.3)(c) the next result contains a converse, cf. Theorem C.

(3.4) Theorem. Let \((R, m, k)\) be a local ring and \(\mathcal{B}\) be a reflexive subcategory of \(\mod(R)\). If \(k\) has a \(\mathcal{B}\)-approximation, then \(R\) is Gorenstein or \(\mathcal{B} = \mathcal{F}(R)\).

In our proof of this theorem we use the next lemma. We do not know a reference giving a direct argument, so one is supplied here.

(3.5) Lemma. Let \((R, m, k)\) be a local ring, and let \(x = x_1, \ldots, x_n\) be a sequence in \(m \setminus m^2\). If \(x\) is linearly independent modulo \(m^2\), then \(k\) is a direct summand of the module \(\Omega^R_n(k)/x\Omega^R_n(k)\).
Proof.} Let $(K(x), d)$ be the Koszul complex on $x$. If necessary, supplement $x$ to a minimal generating sequence $x, y$ for $m$. Let $(F, \partial)$ be a minimal free resolution of $k$. The identification $R/(x, y) = k$ lifts to a morphism of complexes $\sigma: K(x, y) \to F$. Serre proves in [24, Appendix I.2] that $\sigma$ is injective and degreewise split. The natural inclusion $\iota: K(x) \hookrightarrow K(x, y)$ is also degreewise split, so the composite $\rho = \sigma \iota$ is an injective morphism of complexes and degreewise split.

From the short exact sequence $0 \to \Omega^R_n(k) \to F_{n-1} \to \Omega^R_{n-1}(k) \to 0$, we get an exact sequence in homology that reads in part

\[(\ast) \quad \text{Tor}_R^n(R/(x), \Omega^R_{n-1}(k)) \to R/(x) \otimes_R \Omega^R_n(k) \to \frac{R/(x) \otimes_R F_{n-1}}{R/(x) \otimes_R \Omega^R_{n-1}(k)}.
\]

The module $\text{Tor}_R^n(R/(x), \Omega^R_{n-1}(k)) \cong \text{Tor}_R^n(R/(x), k)$ is annihilated by $m$.

Let $e$ be a generator of $K(x)_n$. The image $\rho_n(e)$ in $F_n$ is a minimal generator as $\rho_n$ is split. Set $\varepsilon = \partial_n \rho_n(e) \in \Omega^R_n(k)$; since $F$ is minimal, $\varepsilon$ is a minimal generator of the syzygy $\Omega^n_R(k)$. The minimal generator $1 \otimes \varepsilon$ of $R/(x) \otimes_R \Omega^R_n(k)$ is in the kernel of $(R/(x) \otimes_R \iota)$, as the element $\varepsilon = \partial_n \rho_n(e) = \rho_{n-1} \partial_0(e)$ is in $x F_{n-1}$. By exactness of $(\ast)$ the element $1 \otimes \varepsilon$ is annihilated by $m$, hence it generates a 1-dimensional $k$-vector space that is a direct summand of $\Omega^R_n(k)/x \Omega^R_n(k)$.

Proof of (3.4).} We aim to apply Lemma (3.2). By Propositions (2.4) and (2.10), and by faithful flatness of $\hat{R}$, we may assume $R$ is complete. Set $d = \text{depth } R$; by Lemma (2.9) the $d$th syzygy $\Omega^R_d(k)$ has a $B$-approximation:

\[
0 \to L \to B \to \Omega^R_d(k) \to 0.
\]

Let $x = x_1, \ldots, x_d$ be an $R$-regular sequence in $m \setminus m^2$ linearly independent modulo $m^2$. The Koszul homology modules

\[
H_i(K(x) \otimes_R \Omega^R_d(k)) \cong \text{Tor}_R^i(R/(x), \Omega^R_d(k)) \cong \text{Tor}_R^i(R/(x), k)
\]

vanish for $i > 0$, so $x$ is also $\Omega^R_d(k)$-regular.

Set $S = R/(x)$; by (2.7) and Proposition (2.4) the sequence

\[
0 \to S \otimes_R L \to S \otimes_R B \to \Omega^R_d(k) \to 0
\]

is a $(S \otimes B)$-approximation. Moreover, the category $(S \otimes B)$ is reflexive by Proposition (2.10). By Lemma (3.5) the residue field $k$ is a direct summand of $S \otimes_R \Omega^R_d(k)$, so by (1.4) there is an $(S \otimes B)$-precover of $k$. Since $S$ is complete, it follows from (2.8) that $k$ has a $(S \otimes B)$-approximation.

Assume $R$ is not Gorenstein. Then $S$ is not Gorenstein, so the residue field $k$ is not in $G(S)$ and hence not in $(S \otimes B)$; see [4, §3.2, Rmk. after Thm. 3] or [13, Thm. (1.4.9)]. By Lemma (3.2) every module in $(S \otimes B)$ is now free, so for every $B \in B$ the module $S \otimes_R B$ is free over $S$. By (2.7) the sequence $x$ is $B$-regular; therefore, $B$ is a free $R$-module by Nakayama’s lemma.

An approximation of a module $M$ is minimal if the map onto $M$ is a cover. When $R$ is Gorenstein, every $R$-module has a minimal CM-approximation by unpublished work of Auslander; see [8, Sec. 4] and [14, Thm. 5.5]. Hence we have

(3.6) Corollary.} Let $(R, m, k)$ be a local ring and assume there is a non-free module in $G(R)$. The following are then equivalent:

(i) $R$ is Gorenstein.

(ii) $k$ has a $G(R)$-approximation.

(iii) Every finitely generated $R$-module has a minimal $G(R)$-approximation. □
(3.7) If \( R \) has a dualizing complex, cf. [17, V.§2], then \( k \) has a Gorenstein projective precover \( X \to k \) by [20, Thm. 2.11]. Assume \( X \) is finitely generated, i.e., \( X \) is in \( \mathcal{G}(R) \) and, further, that \( R \) is henselian. If \( X \) is free, then it follows from (2.8) that \( k \) has a \( \mathcal{G}(R) \)-approximation \( 0 \to L \to X' \to k \to 0 \), where \( X' \) is free. Hence, \( k \) is in \( \mathcal{G}(R)^- \) and then \( \mathcal{G}(R) = \mathcal{F}(R) \). If \( X \) is not free, then \( R \) is Gorenstein by (3.6).

(3.8) **Questions.** Let \( (R, m, k) \) be a local ring. If \( k \) has a \( \mathcal{G}(R) \)-precover, is then \( \mathcal{G}(R) \) precovering? If \( \mathcal{G}(R) \) is precovering and contains a non-free module, is then \( R \) Gorenstein?

4. ON THE NUMBER OF TOTALLY REFLEXIVE MODULES

In this section we prove Theorems A and B. Note that by (1.3) the latter would follow immediately from a positive answer to the second question in (3.8).

(4.1) **Lemma.** Let \( R \) be a local ring and \( M \) and \( N \) be finitely generated \( R \)-modules. If only finitely many isomorphism classes of \( R \)-modules \( X \) can fit in a short exact sequence \( 0 \to N \to X \to M \to 0 \), then the \( R \)-module \( \text{Ext}_R^1(M, N) \) has finite length.

**Proof.** Given an \( R \)-module \( X \), we denote by \([X]\) the subset of \( \text{Ext}_R^1(M, N) \) whose elements have representatives of the form \( 0 \to N \to Y \to M \to 0 \), where \( Y \cong X \). By assumption, there exist non-isomorphic \( R \)-modules \( X_0, \ldots, X_n \) such that \( \text{Ext}_R^1(M, N) \) is the disjoint union of the sets \([X_i]\). We may take \( X_0 = M \oplus N \), so \([X_0]\) is the zero submodule of \( \text{Ext}_R^1(M, N) \). We must prove that there is an integer \( q > 0 \) such that \( m^q \text{Ext}_R^1(M, N) \) is contained in \([X_0]\).

By [16, Cor. 1] there are integers \( p_i \) such that if \( M/m^p M \oplus N/m^p N \cong X_i/m^p X_i \) for some \( p \geq p_i \), then \( X_i \cong M \oplus N \). Set \( q = \max\{p_1, \ldots, p_n\} \). Take a short exact sequence \( \xi \) in \( m^q \text{Ext}_R^1(M, N) \); it belongs to some set \([X_i]\). By [26, Thm. 1.1] the sequence \( \xi \otimes_R m^q \) splits, so \( M/m^q M \oplus N/m^q N \cong X_i/m^q X_i \). By the choice of \( q \) this implies \( X_i \cong M \oplus N \), so \( i = 0 \), i.e. \( \xi \) is in the zero submodule \([X_0]\). \( \square \)

Let \( R \to S \) be a flat ring homomorphism. It does not follow from the natural isomorphism \( S \otimes_R \text{Ext}_R^1(M, N) \cong \text{Ext}_S^1(S \otimes_R M, S \otimes_R N) \) that every extension of the \( S \)-modules \( S \otimes_R N \) and \( S \otimes_R M \) has the form \( S \otimes_R X \) for some \( R \)-module \( X \). In a seminar, Roger Wiegand alerted us to the next result.

(4.2) **Lemma.** Let \( (R, m) \to (S, n) \) be a flat ring homomorphism with \( mS = n \) and \( R/m \cong S/n \). Let \( M \) and \( N \) be finitely generated \( R \)-modules and \( \xi \) be an element of the \( S \)-module \( \text{Ext}_S^1(S \otimes_R M, S \otimes_R N) \). If the \( R \)-module \( \text{Ext}_R^1(M, N) \) has finite length, then there is an element \( \chi \) in \( \text{Ext}_R^1(M, N) \) such that \( \xi = S \otimes_R \chi \).

**Proof.** The functor \( S \otimes_R - \) from the category mod(\( R \)) to itself induces a natural isomorphism \( K \to S \otimes_R K \) on \( R \)-modules of finite length. Applied to \( \text{Ext}_R^1(M, N) \) this yields the first isomorphism below

\[
\text{Ext}_R^1(M, N) \overset{\cong}{\longrightarrow} S \otimes_R \text{Ext}_R^1(M, N) \overset{\cong}{\longrightarrow} \text{Ext}_S^1(S \otimes_R M, S \otimes_R N).
\]

The composite sends an exact sequence \( \chi \) to \( S \otimes_R \chi \). \( \square \)

The next result is Theorem B from the introduction.

(4.3) **Theorem.** Let \( R \) be a local ring. If the set of isomorphism classes of indecomposable modules in \( \mathcal{G}(R) \) is finite, then \( R \) is Gorenstein or \( \mathcal{G}(R) = \mathcal{F}(R) \).
Then a sequence $\otimes G$ is exact, so by what has already been proved, the middle term By Theorem (4.6) the ring $R$ is Gorenstein and hence an isolated singularity by (2.8), and the desired conclusion follows from Theorem (3.4) and faithful flatness of $R$.

To prove the claim, we must show that

$$\text{Hom}_{\tilde{R}}(H', \tilde{R} \otimes_R \varphi): \text{Hom}_{\tilde{R}}(H', \tilde{R} \otimes_R B) \to \text{Hom}_{\tilde{R}}(H', k)$$

is surjective for every module $H' \in (\tilde{R} \otimes \mathcal{G}(R))$. By flatness of $\tilde{R}$, surjectivity holds for modules in $\tilde{R} \otimes \mathcal{G}(R)$ and hence for every module in $\text{add}(\tilde{R} \otimes \mathcal{G}(R))$. It is now sufficient to prove that the category $\text{add}(\tilde{R} \otimes \mathcal{G}(R))$ is closed under extensions, because then $(\tilde{R} \otimes \mathcal{G}(R))$ is add$(\tilde{R} \otimes \mathcal{G}(R))$.

First we show that $\tilde{R} \otimes \mathcal{G}(R)$ is closed under extensions. Fix modules $G$ and $K$ in $\mathcal{G}(R)$, and consider short exact sequences $0 \to G \to H \to K \to 0$. Each $H$ is in $\mathcal{G}(R)$, and the minimal number of generators of each $H$ is bounded by the sum of the numbers of minimal generators for $G$ and $K$. Since the number of indecomposable modules in $\mathcal{G}(R)$ is finite, there are, up to isomorphism, only finitely many such modules $H$. By Lemma (4.1) the module $\text{Ext}^1_{\tilde{R}}(K, G)$ has finite length, and by (4.2) every element of $\text{Ext}^1_{\tilde{R}}(\tilde{R} \otimes_R K, \tilde{R} \otimes_R G)$ is extended from $\text{Ext}^1_{\tilde{R}}(K, G)$.

To prove that $\text{add}(\tilde{R} \otimes \mathcal{G}(R))$ is closed under extensions, let $G'$ and $K'$ be summands of extended modules, i.e., $G' \oplus G'' \cong \tilde{R} \otimes_R G$ and $K' \oplus K'' \cong \tilde{R} \otimes_R K$ for modules $G, K \in \mathcal{G}(R)$. Consider a short exact sequence $0 \to G' \to H' \to K' \to 0$. Then a sequence

$$0 \to G' \oplus G'' \to H' \oplus G'' \oplus K'' \to K' \oplus K'' \to 0,$$

is exact, so by what has already been proved, the middle term $H' \oplus G'' \oplus K''$ is in $\tilde{R} \otimes \mathcal{G}(R)$; whence $H'$ is in $\text{add}(\tilde{R} \otimes \mathcal{G}(R))$. \hfill \Box

In view of (3.3)(a) we have

(4.4) Corollary. Let $R$ be a Cohen–Macaulay local ring. If $R$ is of finite CM representation type, then $R$ is Gorenstein or $\mathcal{G}(R) = \mathcal{F}(R)$. \hfill \Box

The next result contains Theorem A from the introduction.

(4.5) Theorem. Let $R$ be a local ring and assume the set of isomorphism classes of indecomposable modules in $\mathcal{G}(R) \setminus \mathcal{F}(R)$ is finite and not empty. Then $R$ is Gorenstein and an isolated singularity. Further, $\tilde{R}$ is a hypersurface singularity; if finite CM representation type ascends from $R$ to $\tilde{R}$, then $\tilde{R}$ is even a simple singularity.

Proof. By Theorem (4.3) the ring $R$ is Gorenstein. From (3.3)(b) it follows that $R$ is of finite CM representation type and hence an isolated singularity by [19, Cor. 2]. By [18, Satz 1.2] the completion $\tilde{R}$ is a hypersurface singularity and, assuming that also $\tilde{R}$ is of finite CM representation type, it follows from [32, Cor. (8.16)] that $\tilde{R}$ is a simple singularity. \hfill \Box

(4.6) Remark. In [23] Schreier conjectured that a Cohen–Macaulay local $k$-algebra $R$ is of finite CM representation type if and only if $\tilde{R}$ is of finite CM representation type. In [30] R. Wiegand proved descent of finite CM representation type from $\tilde{R}$ to $R$ for any local ring $R$. Ascent is verified in [30] when $R$ is Cohen–Macaulay.
and either \( \hat{R} \) is an isolated singularity or \( \dim R \leq 1 \). Ascent also holds for excellent Cohen–Macaulay local rings by work of Leuschke and R. Wiegand [22].

(4.7) **Remarks.** Constructing rings with infinitely many totally reflexive modules is easy using Theorem (4.3). Indeed, let \( Q \) be a local ring of positive dimension and set \( R = Q[[X]]/(X^2) \). As \( R \) is not reduced, it is not an isolated singularity. The \( R \)-module \( R/(X) \) is in \( \mathcal{G}(R) \) and is not free, cf. [13, exa. (4.1.5)], so by (4.3) there are infinitely many non-isomorphic indecomposable modules in \( \mathcal{G}(R) \).

More generally, Avramov, Gasharov, and Peeva [9] construct a non-free totally reflexive module\(^2 \) \( G \) over any ring of the form \( R \cong Q/(x) \), where \( (Q,q) \) is local and \( x \in q^2 \) is a \( Q \)-regular sequence. Such a ring \( R \) is said to have an embedded deformation of codimension \( c \), where \( c \) is the length of \( x \). Again (4.3) implies the existence of infinitely many non-isomorphic indecomposable modules in \( \mathcal{G}(R) \). If \( \hat{R} \) has an embedded deformation of codimension \( c > 2 \), a recent argument of Avramov and Iyengar builds from \( G \) an infinite family of non-isomorphic indecomposable modules in \( \mathcal{G}(R) \); see [10, Thm. 7.8 and proof of 7.4.(1)]. For such \( R \), this gives a constructive proof of the abundance of modules in \( \mathcal{G}(R) \).

(4.8) **Question.** Let \( R \) be a local ring that is not Gorenstein. Given an indecomposable totally reflexive \( R \)-module \( G \not\cong R \), are there constructions that produce infinite families of non-isomorphic indecomposable modules in \( \mathcal{G}(R) \)?

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