Considerations on elliptical failure envelope associated to Mohr-Coulomb criterion

A M Comanici¹, P D Barsanescu²
¹,² “Gheorghe Asachi” Technical University of Iasi-Romania, Department of Mechanical Engineering, Mechatronics and Robotics, Blvd. Mangeron, No. 63, 700050, Iasi, Romania
E-mail: ana.comanici@yahoo.com

Abstract. Mohr-Coulomb theory is mostly used in civil engineering as it is suitable for soils, rock, concretes, etc., meaning that the theory is generally used for brittle facture of the materials, but there are cases when it matches ductile behaviour also. The failure envelope described by the Mohr-Coulomb criterion is not completely accurate to the real yield envelope. The ductile or brittle behaviour of materials could not be incorporated in a linear envelope suggested by classic stress state theories and so, there have been a number of authors who have refined the notion of yield envelope so that it would fit better to the actual behaviour of materials. The need of a realistic yield envelope comes from the demand that the failure state should be able to be predicted in a fair manner and with as little errors as possible. Of course, certain criteria will be closer to the actual situation, but there is a constant need to unify and refine the limit stress theories in order to avoid problems as defining boundaries of application areas on numerical programs. Mohr-Coulomb’s yield envelope is the most used one on programs, can be reduced to Tresca theory when the materials are conducting a ductile behaviour and has a linear simplified form. The paper presents some considerations with respect to the elliptical failure envelope correlated to the Mohr-Coulomb theory. The equations have been rewritten for triaxial situation to describe a more accurate state of stress that is encountered under real conditions in materials. Using the Mohr’s circles to define the yield envelope, the calculus has been made in order to determine the yield stress at tensile tests.

Keywords: failure envelope, limit stress state theories, analytic functions.

1. Introduction
Metallic glasses are solid metallic alloys that have an atomic-scale structure. These types of materials are also known as amorphous metals, non-crystalline, and they have a structure similar to common glasses, but they differentiate because metallic glasses have a very good electrical conductivity [1]. Amorphous metals can be produced by very rapid cooling, physical vapour deposition, ion irradiation and mechanical alloying [2]. Caltech was the first producer of metallic glass in 1957 and the material (an alloy - Au₇₅Si₂₅) was first discovered by Duwez [3].

At the beginning, the manufacturing of these alloys was based on extremely rapid cooling in order to avoid crystallization of the melt mixed metal. An important disadvantage to these types of materials due to the fast cooling process is the fact that the metallic glasses are very thin [4]. Due to the non-crystalline structure, the alloy presents no grain boundaries, which subsequently means that the metallic glasses have higher tensile yield strengths and higher elastic strain limits than polycrystalline metal alloys [5,6].
The fracture of MG materials usually happens in a shear mode along a very narrow shear band in which severe plastic deformation localizes [7,8]. Besides the shear stress which is the driving force of shear plastic deformation, the normal stress on the shear band plane also influences the fracture behaviors [9].

The applications of metallic glasses vary from sports equipment [10], medical devices [11,12] till electronic equipment [13,14].

2. Classical limit stress state theories
For biaxial or triaxial state of stress, it is determined an equivalent dangerous stress using a stress state theory, which has to meet the condition:

\[ \sigma_{eq} \leq \sigma_a \]  \hspace{1cm} (1)

where \( \sigma_a \) is the tolerable stress (the admissible stress).

The equivalent stresses determined with these theories are presented in the following, depending on the principal stresses.

**Rankine** theory:

\[ \sigma_{eq} = \left| \sigma_1 \right| \]  \hspace{1cm} (2)

**Tresca** theory:

\[ \sigma_{eq} = \sigma_1 - \sigma_3 \]  \hspace{1cm} (3)

**von Mises** theory:

\[ \sigma_{eq} = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1 \sigma_2 - \sigma_2 \sigma_3 - \sigma_3 \sigma_1} \]  \hspace{1cm} (4)

For a biaxial state of stress \( \sigma_2=0 \) and equation (4) becomes:

\[ \sigma_{eq} = \sqrt{\sigma_1^2 + \sigma_3^2 - \sigma_3 \sigma_1} \]  \hspace{1cm} (5)

**Mohr-Coulomb** theory:

\[ \sigma_{eq} = \sigma_1 - K \sigma_3 \]  \hspace{1cm} (6)

where:

\[ K = \frac{\sigma_{LT}}{\sigma_{LC}} \]  \hspace{1cm} (7)

\( K \) constant is the ratio of tensile limit stress and compression limit stress, and it represents a feature of material. Limit shear stress predicted with **Tresca** and, respectively, **von Mises** theories is:

\[ \tau_L = \frac{\sigma_{LT}}{2} \]  \hspace{1cm} (8)

\[ \tau_L = \frac{\sigma_{LT}}{\sqrt{2}} \]  \hspace{1cm} (9)

In the case of ductile materials the limit stresses are equal:

\[ \sigma_{LT} = \sigma_{LC} \]  \hspace{1cm} (10)

which means that **Mohr-Coulomb** theory is reduced to **Tresca** theory, \( \sigma_{LT} = \sigma_{LC} \).
3. Modelling the failure envelope with a segment of the ellipse

The version where the curve failure envelope is approximated with a line of equation (6), in Mohr-Coulomb theory, it was also used for modeling of failure in metallic glasses. R. Tao and Z.F. Zhang [15] have shown that Mohr-Coulomb classical theory is not in good accordance with the experimental results. For this reason, they proposed to approximate the limit state curve with an arc of ellipse. In the coordinate system $\sigma$-$\tau$, the ellipse equation is:

$$\frac{\sigma^2}{\sigma_L^2} + \frac{\tau^2}{\tau_L^2} = 1$$

(11)

where $\sigma_L$ and $\tau_L$ are the limit stresses, that represent the semi-major and semi-minor axis of the ellipse (figure 1).

![Figure 1. Stress states represented by Mohr's circles.](image)

In the following will write equation (11) in principal stresses domain $\sigma_1$-$\sigma_3$ ($\sigma_2=0$), which is often used in engineering practice. From equation (11) issue:

$$\tau^2 = \left(1 - \frac{\sigma^2}{\sigma_L^2}\right)\tau_L^2 = \left(\sigma_L^2 - \sigma^2\right)k_{ST}^2$$

(12)

where is noted with $k_{ST}$ the ratio of limit stresses (failure or yielding) tension and, respectively, shear:

$$k_{ST} = \frac{\tau_L}{\sigma_L}$$

(13)

In other words, $k_{ST}$ represents the ratio between semi-minor and semi-major axis of the proposed ellipse (figure 1).

Mohr's circle written in $\sigma$-$\tau$ domain (figure 2) is:

$$\left(\sigma - \frac{\sigma_1 + \sigma_3}{2}\right)^2 + \tau^2 = \left(\frac{\sigma_1 - \sigma_3}{2}\right)^2$$

(14)

where $\sigma_1$ and $\sigma_3$ are the principal stresses for the biaxial state of stress ($\sigma_2=0$).
Figure 2. The failure envelope for biaxial state of stress according to Mohr-Coulomb theory.

From equation (12) and (14) successively results:

\[
\sigma^2 - (\sigma_1 + \sigma_3)\sigma + \frac{1}{4}(\sigma_1 + \sigma_3)^2 + k_{ST}^2(\sigma_L^2 - \sigma^2) = \frac{1}{4}(\sigma_1 - \sigma_3)^2
\]  
(15)

\[
\sigma^2 - (\sigma_1 + \sigma_3)\sigma + \frac{1}{4}(\sigma_1^2 + 2\sigma_1\sigma_3 + \sigma_3^2) + k_{ST}^2\sigma_L^2 - k^2\sigma^2 = \frac{1}{4}(\sigma_1^2 - 2\sigma_1\sigma_3 + \sigma_3^2)
\]  
(16)

\[
\sigma^2(1-k_{ST}^2) - (\sigma_1 + \sigma_3)\sigma + \sigma_1\sigma_3 + k_{ST}^2\sigma_L^2 = 0
\]  
(17)

The roots of the equation (17) are:

\[
\sigma_{1,2} = \frac{1}{2(1-k_{ST}^2)}\left[\sigma_1 + \sigma_3 \pm \sqrt{(\sigma_1 + \sigma_3)^2 - 4(1-k_{ST}^2)(\sigma_1\sigma_3 + k_{ST}^2\sigma_L^2)}\right]
\]  
(18)

The equation (18) is rewritten for tensile stress. Substituting:

\[
\sigma_3 = 0
\]  
(19)

\[
\sigma_1 = \sigma > 0
\]  
(20)

\[
\sigma_L = \sigma_{LT} = \sigma
\]  
(21)

is obtained:

\[
\sigma_{1,2} = \frac{1}{2(1-k_{ST}^2)}\left[\sigma \pm \sqrt{1-4(1-k_{ST}^2)k_{ST}^2}\right]
\]  
(22)

\[
\sigma_{1,2} = \frac{\sigma}{2(1-k_{ST}^2)}\left[1 \pm \sqrt{1-4(1-k_{ST}^2)k_{ST}^2}\right]
\]  
(23)
\[ \sigma_{1,2} = \frac{\sigma}{2(1-k_{ST}^2)} \left[ 1 \pm \sqrt{\left(1 - 2k_{ST}^2\right)^2} \right] = \frac{\sigma}{2(1-k_{ST}^2)} \left[ 1 \pm \left(1 - 2k_{ST}^2\right) \right] \] (24)

The two roots of equation (18) become:

\[ \sigma_{(i)} = \frac{\sigma}{2(1-k_{ST}^2)} 2\left(1-k_{ST}^2\right) = \sigma \] (25)

\[ \sigma_{(2)} = \frac{k_{ST}^2 \sigma}{1-k_{ST}^2} \] (26)

It is observed that only the root \( \sigma_{(i)} \), corresponding to the (+) sign in front of the radical has physical meaning. Retaining only the (+) sign, the equation (18) can be rewritten for the equivalent dangerous stress given by this criterion:

\[ \sigma_{eq} = \frac{1}{2(1-k_{ST}^2)} \left[ \sigma_1 + \sigma_3 + \sqrt{\left(\sigma_1 + \sigma_3\right)^2 - 4\left(1-k_{ST}^2\right)\left(\sigma_1\sigma_3 + k_{ST}^2\sigma_L^2\right)} \right] \] (27)

In the above relationship, the \( \sigma_L \) stress represents the tensile limit (figure 1). For the studied metallic glasses, R. Tao and Z.F. Zhang [15] found the following report between the limit stress for compression and tension:

\[ \frac{\sigma_L}{\sigma_T} \leq 1.58 \] (28)

In quadrants I and IV \( \sigma_L = \sigma_T \). For quadrants III and IV it can be chosen another ellipse that has the semi-major axis \( \sigma_L = \sigma_C \) (figure 1). In the following we determine \( \sigma_{eq} \) for the particular case of equibiaxial tension. Substituting \( \sigma_1 = \sigma_3 = \sigma_T \) (29)

in equation (27) it issue:

\[ \sigma_{ech} = \left(1 + \sqrt{1-k_{ST}^2}\right) \sigma_T \] (30)

All the new criterions need a lot of experimental verification. According to [15], for the metallic glass Zr_{65}Fe_{5}Al_{10}Cu_{20} characteristics were determined:

\[ \sigma_{LC} = 1.66\text{GPa} \quad \sigma_{LT} = 1.62\text{GPa} \quad \tau_L = 0.92\text{GPa} \] (31)

With the above values it is calculated:

\[ K = 0.976 \quad k_{ST} = 0.568 \] (32)

For equibiaxial tension the equation (21) is used to determine:

\[ \sigma_{eq} = 2.691\sigma_L \] (33)

It is observed that for equibiaxial tension the Tresca theory can’t be used (\( \sigma_{eq} = 0 \)), and Rankine and von Mises theories give same value:

\[ \sigma_{eq} = \sigma \] (34)
With classical *Mohr-Coulomb* theory we calculate:

\[ \sigma_{eq} = (1 - K) \sigma \]  

(35)

\[ \sigma_{eq} = (1 - 0.976) \sigma = 0.024 \sigma \]  

(36)

In the situations when it appears a echi-triaxial state of stress, the values calculated with *Mohr-Coulomb* modified theory are closer to those obtained using *Rankine* and *von Mises* theories, then those obtained with *Mohr-Coulomb* classical theory. This last one gives values that do not fit the material's experimental data.

4. Experimental verification

Using the experimental data from equation (31) and one complex state of stress, studied by Tao for the metallic glass Zr\(_{65}\)Fe\(_{5}\)Al\(_{10}\)Cu\(_{20}\), we've determined the equivalent stress using different classic criteria in order to compare with the *Mohr-Coulomb* modified theory's values (figure 3).

![Figure 3](image)

**Figure 3.** Comparison between different equivalent stresses calculated with classic criteria for a metallic glass and one complex state of stress studied by [15], to highlight the lower errors when modified *Mohr-Coulomb* theory is used.

Of course, we need much more experimental data to validate a criterion. Unfortunately, very few experimental data for failure of metallic glasses under compound state of stress are presented in the literature.

5. Conclusions

It is noticed that there are big differences between the equivalent stresses calculated with these theories. Verification requires experimental data but, unfortunately, they are missing. The equi-biaxial state of stress it could be achieved simply by bulge test. However, the calculated value with classical Mohr-Coulomb theory it does not seem credible.

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