Frederick W. Gehring (7 August 1925–29 May 2012)

Elected to NAS 1989

Gaven J Martin

Fred Gehring
Abstract

Frederick William Gehring was a hugely influential mathematician who spent most of his career at the University of Michigan – appointed in 1955 and as the T H Hildebrandt Distinguished University Professor from 1987. Gehring’s major research contributions were to Geometric Function Theory, particularly in higher dimensions $\mathbb{R}^n$, $n \geq 3$. This field he developed in close coordination with colleagues, primarily in Finland, over three decades 1960 – 1990. Gehring’s seminal work drove this field forward initiating important connections with geometry and nonlinear partial differential equations, while addressing and solving major problems. During his career Gehring received many honours from the international mathematical community. He was invited three times to address the International Congress of Mathematicians, at Moscow in 1966, at Vancouver in 1974, and at Berkeley (a plenary lecture) in 1986. He was awarded honorary degrees from the University of Helsinki (1979), the University of Jyväskylä (1990), and the Norwegian University of Science and Technology (1997). In 1989, he was elected to the American Academy of Arts and Sciences and the National Academy of Sciences. Other honors include an Alexander von Humboldt Stiftung (1981–84), Commander of the Order of Finland’s White Rose (Commander class, Finland’s highest scientific honour for foreigners), and a Lars Onsager Professorship at the University of Trondheim (1995–96). He served for 19 years on various Committees of the American Mathematical Society. In 2006 he received the American Mathematical Society’s Steele Prize for Lifetime Achievement. He served three terms as Chairman of the Mathematics Department at the University of Michigan and played a leading role in shaping that department through the latter part of the 20th century.

Life. Frederick William Gehring was born in Ann Arbor, Michigan on August 7, 1925. His family was of German origin; his great-grandfather Karl Ernst Gehring (1829–1893) had emigrated from Germany in 1847 and settled in Cleveland, Ohio, where he founded the Gehring Brewery. Gehring’s grandfather Frederick William Gehring (1859–1925) was treasurer of the brewery and co-founder of a bank in Cleveland. Gehring’s father Carl Ernest Gehring (1897–1966) loved music and was an amateur composer. He came to Ann Arbor to study engineering at the University of Michigan (UM), but soon switched to journalism and later worked for the Ann Arbor News as state news editor and music critic. Gehring’s mother Hester Reed Gehring (1898–1972) was the daughter of John Oren Reed (1856–1916), a physics professor
at UM who later became Dean of the College. She and Carl met as undergraduates at UM. After the birth of their three children, she went on to complete a PhD in German and served as a foreign language examiner for the UM Graduate School.

Gehring grew up in Ann Arbor and graduated from University High School in 1943. He was then admitted to MIT to study physics or engineering, but he chose instead to enlist in the US Navy V-12 program, not knowing where he would be sent. By coincidence, the Navy sent him to Ann Arbor for a special program in electrical engineering at UM. He later graduated with a double major in electrical engineering and mathematics. By that time the war was over, but the Navy sent him to sea for 4 months.

Upon his return, he re-enrolled at UM and decided, at the suggestion of Ruel Churchill, to concentrate on mathematics. After receiving an MA degree from UM in 1949, he went to the University of Cambridge on a Fulbright Scholarship earning a PhD in mathematics in 1952, taking courses from such famous analysts such as J E Littlewood and A S Besicovitch while writing a thesis under the direction of J C Burkill.

At Cambridge, Gehring met Lois Bigger, who had come from Iowa, also on a Fulbright Scholarship, to continue her study of microbiology. She also received her Ph.D. from Cambridge in 1952. Gehring went to Harvard as a Benjamin Peirce Instructor, Bigger to Yale on a Research Fellowship. They were married on August 29, 1953 in Bigger’s hometown of Mt Vernon, Iowa.

Gehring at graduation and wedding to Lois Bigger
Gehring’s acquaintance with Lars Ahlfors during his time at Harvard drew him closer to the area of complex analysis. When the position at Harvard ended in 1955, T H Hildebrandt hired Gehring back to Ann Arbor, where he remained for the rest of his life. A turning point in Gehring’s mathematical career occurred in 1958. Hoping to spend a year abroad, Gehring applied for Fulbright, Guggenheim, and NSF fellowships and was awarded all three. His colleague Jack Lohwater had spent the year 1956–57 in Helsinki, where he saw that Olli Lehto and Kalle Virtanen, among others, were working on the theory of normal functions. Encouraged also by Ahlfors, Gehring decided to go to Helsinki. Lohwater put him in touch with Lehto, and Gehring arranged to spend the year 1958–59 there.

Lehto [29] tells the story of Gehring’s year in Helsinki. When Gehring arrived, he learned to his dismay that his Finnish hosts were no longer interested in meromorphic functions, but had been working on quasiconformal mappings. As the story goes, Gehring said “Fine, I like quasiconformal mappings – what are they?” He was told he would learn, as they were about to start a seminar on the topic, in Finnish! Gehring responded to the challenge and learned to speak Finnish that year. Fred and Lois Gehring’s first son Kalle was born in Helsinki that year, while their son Peter was born two years later in Ann Arbor.

In the year 1959–60, Gehring moved to Zürich, where Albert Pfluger had also been working on quasiconformal mappings. Stimulated by discussions with Pfluger and a paper by Charles Loewner, Gehring began to further develop the higher-dimensional theory of quasiconformal mappings. In those years abroad Gehring made professional contacts which would strongly influence the entire course of his career. When he returned to Michigan in 1960, Gehring began training students in quasiconformal mappings, and his first student graduated from UM in 1963. In the course of his career, Gehring directed 29 PhD students, most of whom have had active careers in teaching and research at academic institutions.

An important aspect of Gehring’s professional work was his extensive editorial service. At various times in his career, he worked on the Editorial Boards of 9 different research journals. He also served as Editor of book series for Van Nostrand (1963–70), North Holland (1970–94), and most famously for Springer-Verlag (1974–2003) with the Undergraduate Texts in Mathematics series.

At the University of Michigan, Gehring was promoted to Professor in 1962. He served three terms as Chairman of the Mathematics Department,
in 1973–75, 1977–80, and 1981–84. He was named to a collegiate chair in 1984, and became the TH Hildebrandt Distinguished University Professor in 1987. The University honoured him with a Distinguished Faculty Achievement Award in 1981, the Henry Russel Lectureship in 1990, and a Sokol Faculty Award in 1994. An international conference on “Quasiconformal Mappings and Analysis” was held in Ann Arbor in August 1995 on the occasion of his 70th birthday. He retired in 1996.

Gehring was invited three times to address the International Congress of Mathematicians, at Moscow in 1966, at Vancouver in 1974, and at Berkeley (a plenary lecture) in 1986. He was awarded honorary degrees from the University of Helsinki (1979), the University of Jyväskylä (1990), the Norwegian University of Science and Technology (1997) and an Sc.D from Cambridge in 1976. In 1989, he was elected to the American Academy of Arts and Sciences and the National Academy of Sciences (USA). Other honours include an Alexander von Humboldt Stiftung (1981–84), Commander of the Order of Finland’s White Rose (1988–2012), and a Lars Onsager Professorship at the University of Trondheim (1995–96). He served for 19 years on various Committees of the American Mathematical Society, including 3 years on the Council, 4 years on the Executive Committee, and 10 years on the Board of Trustees. He served on most of the AMS’ important committees.

In 2006, the AMS honored Gehring with a Steele Prize for Lifetime Achievement. The citation (Notices Amer. Math. Soc. 53 (2006), 468–469) says in part, “Largely because of Gehring’s work, the theory of quasiconformal mappings has influenced many other parts of mathematics, including complex dynamics, function theory, partial differential equations, and topology. Higher dimensional quasiconformality is an essential ingredient of the Mostow rigidity theorem and of recent work of Donaldson and Sullivan on gauge theory and four-manifolds... Gehring’s mathematics is characterized by its elegance and simplicity and by its emphasis on deceptively elementary questions which later become surprisingly significant.”

Fred Gehring died in Ann Arbor on May 29, 2012 at age 86, after a long illness.

Mathematics. In his lifetime Fred Gehring wrote around 130 mathematical papers, starting with his first article in 1951. Gehring had wide and varied interests, real and complex analysis, geometric function theory, partial differential equations (pde), discrete groups and hyperbolic geometry, and geometric group theory just to name a few. He collaborated widely,
with nearly 40 different coauthors. As noted above, Gehring’s PhD thesis at Cambridge was ostensibly under the supervision of J C Burkill, but he certainly felt that J E Littlewood and A S Besicovitch were equally involved in his research directions and mentorship. Indeed, he thanks Besicovitch for the problem which gave rise to his first published paper.

Any brief survey on Gehring’s mathematical achievements is going to miss a lot of important work. There are obviously important contributions where he solved problems that had been open for decades, but there are other papers which introduced key concepts and initiated other work by subsequent authors. This of course does not even begin to address the many ideas and mathematically fruitful routes which he freely gave away to others, and in particular his many students. Gehring’s many and varied projects were often running simultaneously so it is impossible to order his work in time. There are threads that run through his entire career and the study of quasidisks is one-such. Gehring’s paper *Quasiconformal mappings in Euclidean spaces* in the Handbook of complex analysis, 2, 1–29, 2005 gives a fairly concise summary of his perspective of the main results achieved in the theory of quasiconformal mappings since the 1930’s and his broad vision of how they connect with other areas of mathematics. Gehring was certainly leading many of these developments.

Gehring was a meticulous writer, carefully crafting the statement of a theorem so as to reflect the contents of the proof. He worked tirelessly to ensure that an outline and an estimate became a sharp proof and he never used a result unless he understood its proof. Thus each paper was typically accompanied by an “Idiots Guide” carefully explaining all the details, many ran to several hundred pages to accompany a twelve page paper.

**Early years.** Gehring’s first paper is *Images of convergent sequences in sets*, published in J. London Math. Soc. 26, (1951), 249–256. It builds on earlier work of H. Hadwiger (well known for his work on convex bodies) and from H. Kestelman both in 1947. Hadwiger showed that if a set $E \subseteq \mathbb{R}^n$ has positive interior measure, then it contains a self similar image of any finite subset $S \subseteq \mathbb{R}^n$. Gehring considered this for infinite sets $S$, and in particular when $S$ is a convergent sequence - somewhat closer to Kestelman’s problem. Gehring’s first significant paper was that he wrote with Lehto on his first visit to Finland. That paper is *On the total differentiability of functions of a complex variable*, appearing in Ann. Acad. Sci. Fenn. Ser. A I, 1959 9 pp. The paper is still a classic and is often used by researchers, see [5 §3.3]. The
result is false in higher dimensions and it is the interplay between topology and analysis in the proof that was one of Gehring’s favourite things. The main theorem of the paper is the following:

**Theorem 1** Let \( f : \Omega \to \mathbb{C} \) be a continuous open mapping of a planar domain \( \Omega \). Then \( f \) is differentiable almost everywhere in \( \Omega \) if and only if \( f \) has finite first partials almost everywhere.

For planar homeomorphisms this result had been earlier established by D. Menchoff in 1931. The proof that Gehring and Lehto give is a model of exposition and has not changed much over the years - it is almost basic measure theory, but with a wonderful insight. The topology comes in as open mappings satisfy a maximum principle. As a consequence of this theorem every discrete open Sobolev mapping \( f \in W_{loc}^{1,1}(\Omega, \mathbb{C}) \) is differentiable almost everywhere. Using this result one can connect the volume derivatives and the pointwise Jacobian of a mapping and thereby obtain a first versions of the change-of-variables formula and further observe that for discrete open mappings the Jacobian cannot change sign on the domain of definition, so analytic degree is defined. Clearly this result is central, underpinning a lot of planar geometric function theory.

**Higher dimensional quasiconformal mappings.** From early in 1960 Gehring began the task of laying down some of the (now) basic tools of
quasiconformal mappings in higher dimensions. There were others working closely in this area, most notably Jussi Väisälä (in Finland) and Yu. G. Reshetnyak (in Russia). Gehring wrote quite a long review of Väisälä’s papers, “On quasiconformal mappings in space” and “On quasiconformal mappings of a ball” (see [46, 47]). The major tool in Väisälä’s work was the extremal length method - that is via the geometric definition of quasiconformality. Gehring and Väisälä had already written together on this method in “On the geometric definition for quasiconformal mappings,” Comment. Math. Helv., 36, (1961) 19–32. Of course these ideas for planar quasiconformal mappings go back to Lars V. Ahlfors [1], Pfluger [42] and even before to Ahlfors’ work with Arne Beurling in 1946 [2].

Initially both Gehring and Väisälä were seeking to show the equivalence of the various natural generalisations to higher dimensions of the many equivalent definitions for planar quasiconformal mappings - see two below. They also sought extensions of other well known planar results to higher dimensions. For instance Väisälä established the Caratheodory Theorem in this context in [47]. Let us give two possible definitions for a quasiconformal mapping of a domain $\Omega \subset \mathbb{R}^n$.

Here then are two apparently independent definitions for a quasiconformal mapping. First the analytic definition. Suppose $f$ is a homeomorphism which lies in the Sobolev space $W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ of functions whose first derivatives are integrable with exponent $n = \text{dimension}$. Then $f$ is $K$-quasiconformal, $1 \leq K < \infty$, if

$$|Df(x)|^n \leq K J(x, f), \quad \text{for almost all } x \in \Omega. \quad (1)$$

Here $Df$ is the $n \times n$ differential of $f$, $J(x, f)$ is its determinant (Jacobian) and $|\cdot|$ is the operator norm.

Next a geometric definition. Let $\Gamma$ be a family of curves in $\Omega$. For instance the family of all curves connecting two sets $E, F \subset \Omega$. A non-negative Borel integrable function $\rho : \Omega \to \mathbb{R}_{\geq 0}$ is admissible for $\Gamma$ if for each $\gamma \in \Gamma$ we have

$$\int_{\gamma} \rho(s) \, ds \geq 1$$

The modulus of $\Gamma$ is

$$M(\Gamma) = \inf \left\{ \int_{\Omega} \rho^n(x) \, dx : \rho \text{ is admissible for } \Gamma \right\}$$

8
We say that a homeomorphism $f : \Omega \to f(\Omega)$ is $\tilde{K}$-quasiconformal, $1 \leq \tilde{K} < \infty$, if for every curve family $\Gamma \subset \Omega$

$$\frac{1}{K} M(\Gamma) \leq M(f\Gamma) \leq \tilde{K} M(\Gamma) \quad (2)$$

Here $f\Gamma = \{f \circ \gamma : \gamma \in \Gamma\}$ is a curve family in $f(\Omega)$.

Hadamard’s inequality gives $|Df(x)|^n \geq J(x,f)$ and so equality holds in (1) for $K = 1$ and if also the dimension $n = 2$, a little calculation reveals the Cauchy-Riemann equations. Further, and relevant to our later discussion, from the Looman-Menchoff theorem we see the weaker hypothesis that $f \in W^{1,1}\text{loc}(\Omega,\mathbb{R}^2)$ would suffice to conclude $f$ is conformal. For (2), when $\tilde{K} = 1$, the argument is a little longer to deduce that $f$ is conformal but the relationship to the length-area inequality for conformal mappings is clear. Next, in dimension $n \geq 3$ the constants $K$ and $\tilde{K}$ may differ for the same mapping, but both are simultaneously equal to 1 (not hard) and simultaneously finite or infinite (a hard theorem). Thus the space of quasiconformal mappings is well defined, but when specifying constants one must refer to a particular definition.

Gehring’s two major papers here both appeared in the Transactions of the American Math. Soc., Symmetrization of rings in space, TAMS, 101, (1961) 499 – 519 and Rings and quasiconformal mappings in space, TAMS, 103, (1962), 353–393. There was an earlier announcement of the main results in the Proceedings of the National Academy, 47, (1961), 98–105. Gehring starts with results on the spherical symmetrisation of rings in space. This is quite difficult, but motivated by earlier two dimensional results of G Bol, T Carleman and G Pólya and G Szegö among others. This symmetrisation basically identifies extremal configurations for rings (read conformal invariants) in terms of geometric information such as diameters of components and distance between them - generalising the “length-area” method from complex analysis. From this one can give estimates on arbitrary rings. This is a key fact and leads to direct proofs of such things as the Hölder continuity of higher dimensional quasiconformal mappings previously established by E D Callender [11], following arguments of C B Morrey and L Nirenberg). Naturally, this modulus of continuity will give compactness via the Arzela-Ascoli theorem. Next, Gehring proved the existence and uniqueness of an extremal function realising the Loewner capacity of a ring (doubly connected domain) in space. This extremal function $u$ solves the nonlinear pde $\text{div}(|\nabla u|\nabla u) = 0.$
Gehring then used these results to establish equivalences between the analytic definition and the geometric definition. These proofs establish deep connections between the use of the Rademacher-Stepanoff theorem and the absolute continuity and geometric properties of mappings.

The most recognisable thing that Gehring proved using these two papers is the Liouville theorem. In 1850, Joseph Liouville added a short note to a new edition of Gaspard Monge’s classic work Application de l’Analyse à la Géométrie, whose publication Liouville was overseeing. The note was prompted by a series of three letters that Liouville had received in 1845/6 from the British physicist William Thomson (better known as Lord Kelvin) who had studied in Paris under Liouville’s in the mid-1840s. In his letters, Thomson asked Liouville a number of questions concerning inversions in spheres, questions that had arisen in conjunction with Thomson’s research in electrostatics, in particular, with the so-called principle of electrical images (note that reflection in the unit sphere $S^2$ of $\mathbb{R}^3$ is often referred to in physics as the “Kelvin transform.”) In Liouville’s time conformal mappings were certainly understood to be many times differentiable and following his motivation for writing the article, Liouville framed his discussion in the language of differential forms so his original formulation bears little resemblance to the modern-day theorem. However Liouville’s title, “Extension au cas de trois dimensions de la question du tracé géographique” gives no hint whatsoever as to the results. It was only later that Liouville published his theorem in a form approximating the usual statement which we now discuss.

In higher dimensions the group of all Möbius or Conformal transformations of $\mathbb{R}^n$ consists of all finite compositions of reflections in spheres and hyperplanes. Thus it contains the similarities $x \mapsto \lambda Ox + b$, $\lambda > 0$, $O \in O(n)$, the orthogonal group, and $b \in \mathbb{R}^n$ and the inversions $x \mapsto \frac{x-a}{|x-a|^2}$. It is easy to see that these mappings provide examples of conformal transformations - their derivative at any point is a scalar multiple of an orthogonal transformation and therefore the map infinitesimally preserves angles (i.e. is conformal). These mappings are all infinitely differentiable and the space of all Möbius transformations of $\mathbb{R}^n$ forms a finite dimensional Lie group.

Liouville proved in 1850 that if $f : \Omega \to \mathbb{R}^3$ is a 3 times continuously differentiable conformal mapping, then there is a Möbius transformation $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Phi|\Omega = f$. This is a very surprising rigidity theorem as in two dimensions the space of smooth conformal mappings is infinite dimensional and there is no local-to-global extension theorem. One significant corollary of Liouville’s theorem is that the only subdomains $\Omega$ in $\mathbb{R}^n$
with \( n \geq 3 \) that are conformally equivalent to the unit ball \( \mathbb{B}^n \) are Euclidean balls and half-spaces. This stands in stark contrast to the famous Riemann mapping theorem, announced in 1851 a year after Liouville’s note was published: any simply connected proper subdomain \( \Omega \subset \mathbb{C} \) of the complex plane is conformally equivalent to the unit disk \( \mathbb{D} \).

Certainly conformal mappings are \( 1 \)-quasiconformal mappings - that is conformal invariants such as moduli are preserved. What of the converse. Certainly this is implied when Liouville’s assumptions on smoothness hold. Gehring had by now developed the tools to identify extremal rings and their configurations and an understanding of how modulus behaved under continuous deformations and how this gave regularity for quasiconformal mappings. It was a short step (but the one he had been aiming at) to prove the following theorem.

**Theorem 2** Let \( \Omega \subset \mathbb{R}^n \), \( n \geq 3 \) and let \( f: \Omega \rightarrow f(\Omega) \) be a \( 1 \)-quasiconformal mapping. Then \( f \) is the restriction to \( \Omega \) of a Möbius transformation of \( \mathbb{R}^n \).

There are now refinements of this result, see [25, 26] and the curious discrepancy between what is known in even and odd dimensions remains one of the most interesting problems in the higher dimensional theory.

Lars Ahlfors’ reviews of these papers of Gehring begins “This is an announcement of important results in the theory of 3-dimensional quasiconformal mappings” and ends “In a final section he [Gehring] proves that a 1-quasiconformal mapping is a Möbius transformation. This is Liouville’s theorem without any regularity assumptions, a remarkable achievement.”

This last note was professionally important to Gehring as certainly Lars Ahlfors was a personal and mathematical hero to him, perhaps above all others.
Soon after these papers were published Gehring and Väisälä worked together on the paper *The coefficients of quasiconformality of domains in space*, in Acta Math., 114 (1965) 1–70. The problems they consider are typically of the following form: Suppose $\Omega \subset \mathbb{R}^n$ is a domain. Is there a quasiconformal homeomorphism $f : \mathbb{B}^n \to \Omega$? and if so, bound its distortion. They had worked together on these sort of problems before, but this paper produces the most definitive results (and led to a number of students extending the results as well, G Anderson, M.K. Vamanamurthy and K. Hag wrote on this subject, see [3, 21, 22]). As noted, a problem here is that there are different measures of distortion depending on the definition of quasiconformality used. Gehring and Väisälä carefully define the different distortion functions (which are all quantitatively, but not functionally related). Precise results depend on the distortion chosen.

In two dimensions all distortion functions coincide and a considerable literature on extremal mappings and various existence criteria due to the important relationship with Teichmüller theory. In higher dimensions these issues are still not resolved (not even close actually). The calculation of the coefficients of the dihedral wedge, cylinder, and cone by Gehring and Väisälä were the first examples of the calculation of such distortion quantities in the literature. Perhaps the most well known results of the paper, and certainly one of the most useful, is the following.

Gehring and Lars V Ahlfors
Theorem 3 Suppose that $0 < a < b$, that $D$ is a domain in $\mathbb{R}^3$, and that $C(D) \cap \{|x| < b\}$ has at least two components which meet $\{|x| = a\}$. Then the inner distortion, $K_I(D) \geq A \log(b/a)$. $A \geq 0.129$ is an absolute constant.

An interesting case they consider is the infinite cylinder $D = \{x = (r, \theta, x_3): 0 \leq r < 1\}$. They prove the outer distortion $K_0(D) = (q/2)^{1/2}$, where $q$ is the EllipticK functional value

$$q = \int_0^{\pi/2} \frac{1}{\sqrt{\sin u}} \, du$$

The extremal map is obtained as follows. $D$ is mapped onto the half-space $D' = \{x_3 > 0\}$ by $f_1(r, \theta, x_3) = (t, \theta, \varphi)$, where $(t, \theta, \varphi)$ are spherical coordinates,

$$r = \left(\frac{1}{q} \int_0^\varphi (\sin u)^{-1/2} \, du\right)^2, \quad x_3 = \frac{2}{q} \log t.$$ 

Then $D'$ is mapped onto the unit 3-ball by a Möbius transformation. Of course the hard part is explicitly computing certain conformal invariants in the domain and range to show this estimate is sharp.

Next they prove the following:

**Theorem 4** If $\partial D$ contains an inward directed spire or an outward directed ridge, then $K(D) = \infty$. For each $\varepsilon > 0$ there exists a domain whose boundary contains an outward directed spire and whose coefficients are within $\varepsilon$ of the corresponding coefficients of an infinite cylinder.

They use this result to construct a quasiconformal ball whose set of non-accessible boundary points has positive 3-dimensional measure.

There is important recent work related to this discussion, but obtaining nice geometric criteria on a topological ball to guarantee it is a quasiball is probably never going to happen. However many necessary criteria for a domain to be quasiconformally equivalent to a ball have found wide applications. The notion of uniform and John domains, quantitative versions of local connectivity and so forth have become important ideas in the theory of analysis on metric spaces as laid down by J. Heinonen and P. Koskela [23], and elsewhere.
Area distortion. Until Kari Astala’s solution in 1994, the most famous open problem about planar quasiconformal mappings was the area distortion conjecture raised in Gehring’s paper with Edgar Reich from 1966, *Area distortion under quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I (1966) 15 pp. Suppose that $f(D) = D$ and $f(0) = 0$. Given $E \subset D$, denote it’s measure by $|E|$. It follows from Bojaski’s improved regularity result for planar quasiconformal mappings (see the next section, and also [7]) that for each $K \geq 1$ there are numbers $a_K$ and $b_K$, such that if $f$ is $K$-quasiconformal, then

$$\frac{|f(E)|}{\pi} \leq b_K \left( \frac{|E|}{\pi} \right)^{a_K}, \quad \text{for each measurable set } E \subset D$$  \hspace{1cm} (3)

Gehring and Reich prove $a(K) = K^{-a}$ for some $1 \leq a \leq 40$, and that $b(K) = 1 + O(K-1)$ as $K \to 1$. They deduce this as a consequence of the inequality for the Beurling transform of characteristic functions

$$\int \int_D |S_{X_E}(z)| \, dx \, dy \leq a |E| \log(\pi/|E|) + b |E|$$  \hspace{1cm} (4)

for some $1 \leq a \leq 40$, and $0 \leq b \leq 2$ which they achieve using the Calderón-Zygmund theory of singular integral operators. They conjectured that $a = 1$ in both the inequalities (3) and (4) and showed that the best bound in either (3) or (4) gives the best bound in the other. Astala showed, among other things, that this first area distortion conjecture is true. But the other problem suggested by this paper concerns the $L^p$-norms of the Beurling transform,
and that still stands as one of the most challenging problems in the modern theory.

**Higher integrability.** We now come to the result that is undoubtedly Gehring’s most well-known contribution to mathematics. Though the results have been improved, many of the ideas are still used in mathematics today.

In a remarkable paper in 1973, actually a landmark in modern analysis and pde, *The $L^p$-integrability of the partial derivatives of a quasiconformal mapping*, *Acta Math.*, 130 (1973), 265–277, Gehring established that the Jacobian determinant of a $K$-quasiconformal mapping is integrable above the natural exponent ($n = \text{dimension}$). That is the usual assumption $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$, together with a bound as per (1) implies that $f \in W^{1,n+\epsilon}_{\text{loc}}(\Omega, \mathbb{R}^n)$ for some $\epsilon > 0$ depending on $n$ and $K$ and for which Gehring gave explicit estimates.

While this result was already known in the plane due to the work of B Bojarski [7], and perhaps anticipated in higher dimensions, it is impossible to overstate how important this result has proven to be in the theory of quasiconformal mappings and more generally Sobolev spaces and in the theory of non-linear pdes. The techniques developed to solve this problem, for instance the well-known reverse Hölder inequalities, are still one of the main tools used in non-linear potential theory, non-linear elasticity, pdes and harmonic analysis. In the introduction to his paper, Gehring says the results follow “using a quite elementary proof”. P Caraman repeats this claim in his review, however time has shown Gehring’s ideas here to be deep and highly innovative. It is perhaps a mark of the clarity of Gehring’s writing (and his own modesty) that one can follow the arguments easily and overlook how hard these ideas are to come by.

We state the following version of Gehring’s result as proved in [27] which also gives the result for quasiregular mappings.

**Theorem 5 (Higher Integrability)** Let $f : \Omega \to \mathbb{R}^n$ be a mapping of Sobolev class $W^{1,q}_{\text{loc}}(\Omega)$ satisfying the differential inequality

$$|Df(x)|^n \leq K \, J(x, f).$$

(5)

Then there are $\epsilon_K, \epsilon_K > 0$ such that if $q > n - \epsilon_K$, then $f \in W^{1,p}_{\text{loc}}(\Omega)$ for all $p < n + \epsilon_K$.

As an immediate corollary we have the following:
Theorem 6 Let $f : \Omega \to \mathbb{R}^n$ be a $K$ quasiconformal mapping. Then there is $p_K > n$ such that $f \in W^{1,p_K}_{loc}(\Omega)$.

The higher-dimensional integrability conjecture, perhaps the most important outstanding problem in the area, would assert that if $f$ satisfies (5) and lies in $W^{1,q}_{loc}(\Omega)$ for some $q > nK/(K + 1)$, then $f$ actually lies in the Sobolev space $W^{1,p}_{loc}(\Omega)$ for all $p < nK/(K - 1)$. In two-dimensions this conjecture was proven by Astala as mentioned earlier.

A key step in Gehring’s proof is the following

Lemma 1 (Reverse Hölder inequality) Let $f : \Omega \subset \mathbb{R}^n \to \Omega'$ be a $K$-quasiconformal mapping. Then there is $p = p(n, K) > 1$ and $C = C(n, K)$ such that

$$
\left( \frac{1}{|Q|} \int_Q J(x, f)^p \, dx \right)^{1/p} \leq \frac{C}{|Q|} \int_Q J(x, f) \, dx
$$

for all cubes $Q$ such that $2Q \subset \Omega$.

This result is often referred to as the Gehring Lemma. It is worth reading T. Iwaniec’s survey on this lemma [24]. That paper contains many recent related results, with numerous generalizations and some of the far-reaching applications in mathematical analysis.

Next, one deduces from the improved regularity bounds on the distortion of Hausdorff dimension $\dim_H(E)$ of a set $E \subset \mathbb{R}^n$.

Theorem 7 If $f : \Omega \to \mathbb{R}^n$ is $K$-quasiconformal, then there is $a \geq 1$ such that for each $E \subset \mathbb{R}^n$,

$$
K^{-a} \left( \frac{1}{\dim_H(f(E))} - \frac{1}{n} \right) \leq \frac{1}{\dim_H(f(E))} - \frac{1}{n} \leq K^a \left( \frac{1}{\dim_H(f(E))} - \frac{1}{n} \right)
$$

This result shows in particular that sets of 0 and $n$ dimensional Hausdorff measure are preserved, proving an earlier conjecture Gehring made in his work with Väisälä. The optimal conjecture would be that $a = 1$ here.

Gehring was always keen to give a lot of credit to E De Giorgi’s results [18] to support his proof. While De Giorgi’s work (and also that of J F Nash) was hugely important, it was substantially earlier and well known in the pde community by that time. Gehring found considerable insight and gave many new ideas to solve these problems.
Quasiconformal Homogeneity. Another theme that continues to generate a lot of interest is Gehring’s paper with former student Bruce Palka, *Quasiconformally homogeneous domains*, which appears in J. Analyse Math. 30 (1976), 172–199. As mentioned earlier, one of Gehring’s lifelong tasks was to find a characterisation of those domains $\Omega \subset \mathbb{R}^n$ which are of the form $\Omega = f(B^n)$ where $f$ is a quasiconformal mapping and $B^n$ is the unit ball - *quasiballs* (quasidisks in two dimensions). This paper was written “in the hope that this may eventually lead to a new characterisation for domains quasiconformally equivalent to $B^n$”. There appears to be no simple characterisation such as Ahlfors’ criterion in higher dimensions. So some other ideas need to be advanced. There are clear characterisations in terms of quasiconformal generalizations of the local collaring theorems of M Brown and others from geometric topology. Here one assumes that the boundary $\partial \Omega$ has a neighborhood $U$ such that $U \cap \Omega$ can be mapped quasiconformally into $\mathbb{B}^n$ so that $\partial \Omega$ corresponds to $\partial B^n$ (Gehring first did this in 1967. It directly implies that smoothly bounded Jordan domains are quasiballs in higher dimensions). Another criterion, unfortunately very hard to apply, is that of L.G. Lewis [30]: the corresponding $n$-dimensional Royden algebras $A_n(\Omega)$ and $A_n(B^n)$ should be isomorphic.

In this work mentioned above, Gehring and Palka study domains $\Omega$ which are homogeneous with respect to a quasiconformal family, a quasiconformal group, or a conformal family, where homogeneous with respect to a family $\Gamma$ means that for each pair $a, b \in D$, there exists an $f \in \Gamma$ such that $f(a) = b$.

Gehring and Palka also prove that if $D$ is homogeneous with respect to a quasiconformal family and has a $k$-tangent ($1 \leq k \leq n - 1$) at some finite boundary point, then $D$ is quasiconformally equivalent to (a) $B^n$ if $k = n - 1$ and to (b) $\mathbb{R}^n - \mathbb{R}^k$ if $k = 1, \ldots, n-2$, $n > 2$. They obtain also, in the particular case $n = 2$, that $D$ is homogeneous with respect to a quasiconformal group if and only if $D$ can be mapped conformally onto $\mathbb{R}^2$, $\mathbb{R}^2 - \{0\}$ or $\mathbb{D}$. In the $n$-dimensional case they obtain similar results when $D$ is homogeneous with respect to a 1-quasiconformal family. There the results basically follow from Lie theory since the 1-quasiconformal mappings form a group. Actually it turns out that quasiconformal groups are Lie groups in all dimensions as well [34].

They give examples to show there can be no easy characterization for $\Omega$ in $\mathbb{R}^n$ quasiconformally equivalent to $B^n$ in terms of the tangential or smoothness properties of their boundaries.
Recent work on qc-homogeneity has connections with rigidity. The ques-
tion of the existence of a lower bound (strictly greater than one) on the
quasiconformal homogeneity constant of quasiconformally homogeneous sets
first appeared in this paper. Recently F. Kwakkel and V. Markovic prove
that here exists a constant $\epsilon$ such that, if $M$ is a $K$-quasiconformally homogeneity
planar domain which is not simply connected, then $K \geq 1 + \epsilon$.[41].
Earlier work looked at this problem on hyperbolic manifolds in dimension
$n \geq 3$ see [3] and also in Riemann surfaces where the problem is still open
though partial results can be found, see [9][10].

One of the reasons this paper is widely cited is a key tool identified and de-
veloped for the first time; namely the quasihyperbolic metric – defined below
– and estimates establishing how it distorts under quasiconformal mappings.

**Quasihyperbolic geometry** Subsequently Gehring wrote with his former
student Brad Osgood, *Uniform domains and the quasihyperbolic metric*, which
appeared in J. Analyse Math., 36 (1979), 50 – 74. Uniform domains were
introduced by O. Martio and J. Sarvas around 1979 in connection with the
injectivity properties of functions [37]. They found their way into P W Jones’
work characterising the domains $\Omega \subset \mathbb{R}^n$ for which each BMO function $u$ in
$\Omega$ has a BMO extension to $\mathbb{R}^n$. A proper subdomain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is said
to be uniform if there are $0 < a, b < \infty$ such that any pair of points $x_1, x_2 \in \Omega$
can be joined by a rectifiable curve $\gamma \subset \Omega$ for which $s(\gamma) \leq a|x_1 - x_2|$ and
$\min_{j=1,2} s(\gamma(x_j, x)) \leq bd(x, \partial D)$ for all $x \in \gamma$, where $s(\gamma)$ denotes the Eu-
dclidean length of $\gamma$, $\gamma(x_j, x)$ is the part of $\gamma$ between $x_j$ and $x$, and $d(x, \partial D)$
is the Euclidean distance from $x$ to $\partial D$.

Simply connected planar domains are uniform if and only if they are
quasidisks.

Again the quasihyperbolic metric, $k_D$ for a domain $D$, is key tool. This
metric is defined by

$$k_D(x_1, x_2) = \inf \int_\gamma d(x, \partial D)^{-1} ds$$

where the infimum is taken over all rectifiable curves $\gamma \subset D$ joining $x_1$ to $x_2$
in $D$. It is not difficult to see that

$$k_D(x_1, x_2) \geq j_D(x_1, x_2) = \frac{1}{2} \log \left[ \left( \frac{|x_1 - x_2|}{d(x_1, \partial D)} + 1 \right) \left( \frac{|x_1 - x_2|}{d(x_2, \partial D)} + 1 \right) \right]$$

for all $x_1, x_2 \in D$. Gehring and Osgood first prove using the geodesics
of $k_D$ (which turn out to be $C^{1,1}$ curves) that $D$ is uniform if and only if
there are constants $0 < c, d < \infty$ such that $k_D(x_1, x_2) \leq cj_D(x_1, x_2) + d$ for all $x_1, x_2 \in D$ and this was actually what P. Jones used in [38, 39]. They also proved that $k_D$ and $j_D$ are quasi-invariant under quasiconformal mappings of $\Omega$ and this gives another proof of the invariance of the family of uniform domains under quasiconformal mappings of $\mathbb{R}^n$; an earlier result of Olli Martio [35]. Gehring and Osgood then define a domain $D \subset \mathbb{R}^2$ to be quasiconformally decomposable if there is $K \in [1, \infty)$ such that for any $x_1, x_2 \in D$ there is a quasidisk $D_0 \subset D$ which contains $x_1$ and $x_2$. This gives a surprising new characterisation of uniform domains: $D \subset \mathbb{R}^2$ is uniform if and only if $D$ is quasiconformally decomposable, and via this characterization they give an alternative proof for the main injectivity properties of uniform domains in $\mathbb{R}^2$ that were discovered by Martio and J. Sarvas [37]. This decomposition theorem is false in higher dimensions $n \geq 3$, but is true if one redefines decomposability via the equivalent assumption in two dimensions, that there is $L \geq 1$ such that every pair of points lies in the $L$-bilipschitz image of a ball, [32].

**Discrete groups and hyperbolic geometry.** In the last couple of decades of Gehring’s research career he and G J Martin worked on the geometry of discrete groups, writing about 30 joint papers. This represented a new research direction for Gehring and was motivated by his attendance at Alan Beardon’s series of lectures on the geometry of discrete groups in Ann Arbor around 1980 - Beardon was just putting the finishing touches on his excellent book [6]. Gehring started thinking about how the general theory of Möbius (or 1-quasiconformal) groups would generalise to arbitrary discrete groups of quasiconformal mappings - typically with the assumption of a uniform bound on the distortion and touched on in his homogeneity paper with Palka. It became clear that a surprising amount of the general theory of discrete groups used only the compactness properties of quasiconformal mappings. These initial studies became the paper *Discrete quasiconformal groups. I.*, Proc. London Math. Soc., 55 (1987), 331–358. Part II never was finished. The theory of convergence groups, identified and developed in that paper, was a timely idea and meshed well with Mikhael Gromov’s theory of hyperbolic groups [20] published in the same year. In the hands of Pekka Tukia it was quickly used to characterise certain conjugates of Möbius groups among groups of homeomorphisms [43], a programme that was completed by David Gabai [19] and Andrew Casson & Douglas Jungreis [13] and, with earlier work of Geoffery Mess, led directly to the resolution of the
Seifert conjecture: a compact orientable irreducible 3-manifold with fundamental group having an infinite cyclic normal subgroup is Seifert fibered. Both results include a far reaching generalisation of the Nielsen Realisation problem, earlier solved by Steve Kerckhoff [40]. Mike Freedman even showed an equivalence between the four-dimensional surgery conjecture and an extension problem for convergence groups of \( \mathbb{R}^3 \), [14]. These new and surprising connections between quasiconformal mappings and low dimensional topology and so forth was enormously exciting to Gehring, and some of the most interesting unresolved questions in low dimensional topology - for instance the Cannon Conjecture, a group-theoretic generalization of the generic case of W. P. Thurston’s famous Geometrization Conjecture – can be framed within the theory of convergence groups, see [12] and [37].

A group of homeomorphisms acting on \( \mathbb{R}^n \) is called a \textit{quasiconformal group} if there is some finite \( K \) such that each \( g \in G \) is \( K \)-quasiconformal. In two dimensions every quasiconformal group acting on \( \mathbb{R}^2 = \hat{\mathbb{C}} \), the Riemann sphere, is the quasiconformal conjugate of a conformal group or a Möbius group as result of Dennis Sullivan and proved by Tukia [44], that is there is a quasiconformal \( f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \) and a group of Möbius transformations \( \Gamma \) such that \( G = f \Gamma f^{-1} \). On the other hand, if \( n \geq 3 \), then there is a discrete quasiconformal group not quasiconformally conjugate to a Möbius group [45, 33].

A \textit{convergence group} \( G \) is a group of self homeomorphisms of \( \mathbb{R}^n \) which has the following property: every infinite subfamily of \( G \) contains a sequence \( \{g_j\} \) such that one of the following holds:

1. there is a self-homeomorphism \( g \) of \( \mathbb{R}^n \) such that \( g_j \rightarrow g \) and \( g_j^{-1} \rightarrow g^{-1} \) uniformly in \( \mathbb{R}^n \) as \( j \rightarrow \infty \),

2. there are points \( x_0 \) and \( y_0 \) in \( \mathbb{R}^n \) such that \( g_j \rightarrow y_0 \) and \( g_j^{-1} \rightarrow x_0 \) locally uniformly in \( \mathbb{R}^n - \{x_0\} \) and \( \mathbb{R}^n - \{y_0\} \), respectively, as \( j \rightarrow \infty \).

Convergence groups are similarly defined on other spaces. Quasiconformal groups are convergence groups. A convergence group is \textit{discrete} if (1) above never holds. The limit set \( L(G) \) of a discrete convergence group \( G \) is defined as for discrete Möbius groups - the points of accumulation of a generic orbit. It has quite similar structure to that of a discrete Möbius group: \( L(G) \) is either nowhere dense or is equal to \( \mathbb{R}^n \). If \( L(G) \) contains three points, then \( L(G) \) is a perfect set. The elements of a discrete convergence group fall into
three kinds, elliptic, parabolic and loxodromic. Examples are obtained from the following: if \( E \) is a totally disconnected closed set in \( \mathbb{R}^n \) and if a group \( G \) of self-homeomorphisms of \( \mathbb{R}^n \) is properly discontinuous in \( \mathbb{R}^n \setminus E \), then \( G \) is a discrete convergence group whose limit set lies in \( E \).

This initial foray into the geometry of discrete groups led to further research in Kleinian groups motivated by attempts to generalize to higher dimensions the universal geometric constraints on Fuchsian groups that Beardon gave an exposition of [6]. They observed a very interesting connection between criteria for discrete groups and holomorphic dynamics - the iteration of polynomials of one complex variable on the Riemann sphere, [15]. This connection was already inherent in Jørgensen’s proof of his remarkable inequality for discrete groups [28]. This polynomial came from a trace identity in \( SL(2, \mathbb{C}) \), basically the Fricke identity. Gehring and Martin had come to Mittag-Leffler in 1990 to work on this project as part of a six month thematic programme. They discovered entirely new classes of polynomial trace identities and thus an entirely new tool to study the geometry of Kleinian groups which led to a number of new developments. Among the more important papers is Commutators, collars and the geometry of Möbius groups, J. Anal. Math., 63 (1994), 175 – 219. Jane Gilman wrote “This paper is the seventh in a series of ten remarkable papers that Gehring and Martin have written on the geometry of discrete Möbius groups . . . The papers use a combination of hyperbolic geometry and complex iteration theory to obtain discreteness criteria for Möbius groups. The results obtained in the current paper include a sharp analogue (for subgroups of \( PSL(2, \mathbb{C}) \) with an elliptic element) to the Shimizu-Leutbecher inequality . . . and the elimination of a large region of possible values for the commutator parameter (trace\([g,h]-2\) with \(g,h\in G\)) of a discrete group \( G \) with two elliptic generators. From this the authors are able to obtain a sharp lower bound for the distance between the axes of elliptics of order \( n \) and \( m \) in any discrete groups (for many values of \( m \) and \( n \)). The results have as a corollary substantial improvements ... on volumes of hyperbolic orbifolds ... .” Even with these new tools at hand it took quite some time to claim the real prize. This came in the paper Minimal co-volume hyperbolic lattices. I. The spherical points of a Kleinian group, Ann. of Math., 170 (2009), 123–161. Gehring’s last research article. This paper, along with a sequel [31] completes the identification of the hyperbolic 3-orbifold of smallest volume. This solves a problem of Siegel from 1942 in three dimensions, also Problem 3.60 (F) in the Kirby problem list.
Quasidisks. Finally we should talk about the ubiquitous quasidisk. A quasidisk $\Omega$ is the image of the unit disk $\mathbb{D}$ under a quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$, so $f(\mathbb{D}) = \Omega$. Thus a quasidisk is a simply connected planar domain with reasonable geometric control on the structure of its boundary by way of such things as Hausdorff dimension, and “turning conditions”. The importance of the concept is reflected in the many remarkable and diverse applications in planar geometric function theory and also in low-dimensional geometry and topology where quasidisks form the components of ordinary set (where the action is properly discontinuous) quasifuchsian groups. In dynamics, quasidisks form the components of the filled in Julia set of a hyperbolic rational map. Further, the theory of holomorphic motions shows that when suitably interpreted a holomorphic perturbation of the unit disk in the space of injections gives a quasidisk. It was another lifelong task of Gehring to collect characterisations of quasidisks, to give new and different proofs for these, to find new applications of the theory and to generally spread the word about these wonderful objects.

The nearest thing to a book that Gehring wrote himself and in his lifetime is the monograph Characteristic properties of quasidisks which appears in Séminaire de Mathématiques Supérieures, 84. Presses de l’Université de Montréal, 1982, 97 pp. This book was subsequently expanded into the long awaited book The ubiquitous quasidisk, in Mathematical Surveys and Monographs, 184. AMS, 2012 with Kari Hag.
theory and analysis. In the first set of notes, Gehring gave 17 different characterisations, this was subsequently extended to 30 (!). Ahlfors’ criterion has been known for rather a long time; the Jordan domain $\Omega$ is a quasidisk if and only if there exists a constant $c$ such that, for all pairs of points $z_1, z_2 \in \partial \Omega$, 

$$\min_{j=1,2} \{ \text{diam}(\gamma_j) \leq dc \left| z_1 - z_2 \right| \}$$

where $\gamma_1, \gamma_2$ are the components of $\partial \Omega \setminus \{z_1, z_2\}$. This is typical of the geometric characterisations found, others include the notions of linear local connectedness and so forth. Other geometric characterisations are based on the estimates on the hyperbolic or quasihyperbolic metric and similar sorts of things.

Function-theoretic characterisation include the Schwarzian derivative property. For an analytic $\varphi$

$$S \varphi = \left( \frac{\varphi''}{\varphi'} \right)' - \frac{1}{2} \left( \frac{\varphi''}{\varphi'} \right)^2,$$

and $\Omega$ is a quasidisk if and only if there exists a constant $c > 0$ such that $f$ is injective whenever $f$ is analytic in $\Omega$ with $|S \varphi| \leq c \rho^2_\Omega$, $\varphi' \neq 0$ in $\Omega$. Of course the Schwarzian derivative is invariant under Möbius transformations and has wide application in modular forms and hypergeometric functions and Teichmüller theory. Further criteria found by Gehring and Hag concern the injectivity properties of quasi-isometries, extension properties of BMO or $W^{1,2}$ spaces (and their near relatives). A lot of the discussion concerns the many characterizations involving the hyperbolic and the quasihyperbolic metric. We recommend the reader to thumb through the book to see the vast and diverse collection of results which cover a broad spectrum of planar function theory.

References

[1] L. V. Ahlfors, On quasiconformal mappings, J. Analyse Math., 3, (1954), 1–58.

[2] L. V. Ahlfors and A. Beurling, Invariants conformes et problèmes extrémaux, C. R. Dixime Congrés Math. Scandinaves 1946, pp. 341–351. Jul. Gjellerups Forlag, Copenhagen, 1947.
[3] G.D. Anderson, *The coefficients of quasiconformality of ellipsoids*, Ann. Acad. Sci. Fenn. Ser. A I No. 411, (1967) 14 pp.

[4] K. Astala, *Area distortion of quasiconformal mappings*, Acta Math., **173**, (1994), 37–60.

[5] K. Astala, T. Iwaniec and G.J. Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane*, Princeton Mathematical Series, 48. Princeton University Press, Princeton, NJ, 2009.

[6] A. F. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, 91. Springer-Verlag, New York, 1983.

[7] B. V. Bojarski, *Generalized solutions of a system of differential equations of first order and of elliptic type with discontinuous coefficients*, Mat. Sb. N.S., **43**, (1957), 451–503.

[8] P. Bonfert-Taylor, R. D. Canary, G.J. Martin and E. Taylor, *Quasiconformal homogeneity of hyperbolic manifolds*, Math. Ann., **331**, (2005), 281–295.

[9] P. Bonfert-Taylor, G.J. Martin, A.W. Reid and E. Taylor, *Teichmüller mappings, quasiconformal homogeneity, and non-amenable covers of Riemann surfaces*, Pure Appl. Math. Q. **7** (2011), Special Issue: In honor of Frederick W. Gehring, Part 2, 455 –468.

[10] P. Bonfert-Taylor, Petra; R.D. Canary, J. Souto, and E.C. Taylor, *Exotic quasi-conformally homogeneous surfaces*, Bull. Lond. Math. Soc., **43**, (2011), 57–62.

[11] E.D. Callender, *Hölder continuity of n-dimensional quasi-conformal mappings*, Pacific J. Math., **10**, 1960, 499–515.

[12] J.W. Cannon, *The combinatorial Riemann mapping theorem*, Acta Math., **173**, (1994), 155–234.

[13] A. Casson and D. Jungreis, *Convergence groups and Seifert fibered 3-manifolds*, Invent. Math., **118**, (1994), 441–456.

[14] M.H. Freedman, *A geometric reformulation of 4-dimensional surgery*, Special volume in honor of R. H. Bing (1914–1986). Topology Appl., **24** (1986), 133–141.
[15] F.W. Gehring and G.J. Martin, *Iteration theory and inequalities for Kleinian groups*, Bull. Amer. Math. Soc., 21, (1989), 57–63.

[16] F.W. Gehring and G.J. Martin, *Precisely invariant collars and the volume of hyperbolic 3-folds*, J. Differential Geom., 49, (1998), 411–435.

[17] F.W. Gehring, C. Maclachlan, G.J. Martin and A.W. Reid, *Arithmeticity, discreteness and volume*, Trans. Amer. Math. Soc., 349, (1997), 3611–3643.

[18] E. De Giorgi, *Sulla differenziabilit e l’analiticit delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat., 3, 1957 25–43.

[19] D. Gabai, *Convergence groups are Fuchsian groups*, Ann. of Math., 136, (1992), 447–510.

[20] M. Gromov, *Hyperbolic groups*, Essays in group theory, 75–263, Math. Sci. Res. Inst. Publ., 8, Springer, New York, 1987.

[21] K. Hag and M. Näätänen, *On the outer coefficient of quasiconformality of a cylindrical map of a convex dihedral wedge*, Ann. Acad. Sci. Fenn. Ser. A I Math., 5, (1980), 125 – 130.

[22] K. Hag and M.K. Vamanamurthy, *The coefficients of quasiconformality of cones in n-space*, Ann. Acad. Sci. Fenn. Ser. A I Math., 3, (1977), 267–275.

[23] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math., 181, (1998), 1–61.

[24] T. Iwaniec, *The Gehring lemma*, Quasiconformal mappings and analysis (Ann Arbor, MI, 1995), 181204, Springer, New York, 1998.

[25] T. Iwaniec, *p-harmonic tensors and quasiregular mappings*, Ann. of Math., 136, (1992), 589–624.

[26] T. Iwaniec and G. Martin, *Quasiregular mappings in even dimensions*, Acta Math., 170, (1993), 29–81.
[27] T. Iwaniec and G. Martin, *Geometric function theory and non-linear analysis*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2001.

[28] T. Jørgensen, *On discrete groups of M"obius transformations*, Amer. J. Math., 98, (1976), 739–749.

[29] O. Lehto, *Gehring Gehring and Finnish mathematics*, Quasiconformal mappings and analysis. A collection of papers honoring F. W. Gehring. Papers from the International Symposium held in Ann Arbor, MI, August 1995. Edited by Peter Duren, Juha Heinonen, Brad Osgood and Bruce Palka. Springer-Verlag, New York, 1998.

[30] L.G. Lewis, *Quasiconformal mappings and Royden algebras in space*, Trans. Amer. Math. Soc., 158, 1971 481–492.

[31] T.H. Marshall and G.J. Martin, *Minimal co-volume hyperbolic lattices, II: Simple torsion in a Kleinian group*, Ann. of Math., 176, (2012), 261–301.

[32] G.J. Martin, *Quasiconformal and bi-Lipschitz homeomorphisms, uniform domains and the quasihyperbolic metric*, Trans. Amer. Math. Soc., 292, (1985), 169–191.

[33] G.J. Martin, *Discrete quasiconformal groups that are not the quasiconformal conjugates of M"obius groups*, Ann. Acad. Sci. Fenn. Ser. A I Math., 11, (1986), 179–202.

[34] G.J. Martin, *The Hilbert-Smith conjecture for quasiconformal actions*, Electron. Res. Announc. Amer. Math. Soc., 5, (1999), 66–70.

[35] G.J. Martin and R. Skora, *Group actions on $S^2$*, Amer. J. Math., 111, (1989), 387–402.

[36] O. Martio, *Definitions for uniform domains*, Ann. Acad. Sci. Fenn. Ser. A I Math., 5, (1980), 197–205.

[37] O. Martio and J. Sarvas, *Injectivity theorems in plane and space*, Ann. Acad. Sci. Fenn. Ser. A I Math., 4, (1979), 383 – 401.

[38] P. W. Jones, *Extension theorems for BMO*, Indiana Univ. Math. J., 29, (1980), 1–66.
[39] P. W. Jones, *Quasiconformal mappings and extendability of functions in Sobolev spaces*, Acta Math., **147**, (1981), 71–88.

[40] S.P. Kerckhoff, *The Nielsen realization problem*, Ann. of Math., **117**, (1983), 235–265.

[41] F. Kwakkel and V. Markovic, *Quasiconformal homogeneity of genus zero surfaces*, J. Anal. Math., **113**, (2011), 173–195.

[42] A. Pfluger, *Über die Äquivalenz der geometrischen und der analytischen Definition quasikonformer Abbildungen*, Comment. Math. Helv., **33**, (1959), 23–33.

[43] P. Tukia, *Homeomorphic conjugates of Fuchsian groups*, J. Reine Angew. Math., **391**, (1988), 1–54.

[44] P. Tukia, *On two-dimensional quasiconformal groups*, Ann. Acad. Sci. Fenn. Ser. A I Math., **5**, (1980), 73–78.

[45] P. Tukia, *A quasiconformal group not isomorphic to a Mbius group*, Ann. Acad. Sci. Fenn. Ser. A I Math., **6**, (1981), 149–160.

[46] J. Väisälä, *On quasiconformal mappings in space*, Ann. Acad. Sci. Fenn. Ser. A I, (1961), 36 pp.

[47] J. Väisälä, *On quasiconformal mappings of a ball*, Ann. Acad. Sci. Fenn. Ser. A I, (1961), 7 pp.
Frederick W. Gehring: Published Works.

Gehring, F. W., *Images of convergent sequences in sets*, J. London Math. Soc., 26, (1951), 249–256.
Burkill, J. C.; Gehring, F. W., *A scale of integrals from Lebesgue’s to Denjoy’s*, Quart. J. Math., Oxford Ser., 4, (1953), 210–220.

Gehring, F. W., *A study of α-variation. I*, Trans. Amer. Math. Soc., 76, (1954), 420–443.

Gehring, F. W., *A note on a paper by L. C. Young*, Pacific J. Math., 5, (1955), 67–72.

Gehring, F. W., *On the Dirichlet problem*, Michigan Math. J., 3, (1955–1956), 201.

Gehring, F. W., *On the radial order of subharmonic functions*, J. Math. Soc. Japan, 9, (1957), 77–79.

Gehring, F. W., *The Fatou theorem and its converse*, Trans. Amer. Math. Soc., 85, (1957), 106–121.

Gehring, F. W., *The Fatou theorem for functions harmonic in a half-space*, Proc. London Math. Soc., 8, (1958), 149–160.

Gehring, F. W., *The asymptotic values for analytic functions with bounded characteristic*, Quart. J. Math. Oxford Ser., 9, (1958), 282–289.

Gehring, F. W.; Lohwater, A. J., *On the Lindelöf theorem*, Math. Nachr., 19, (1958), 165–170.

Gehring, F. W., *On solutions of the equation of heat conduction*, Michigan Math. J., 5, (1958), 191–202.

Gehring, F. W.; Lehto, O., *On the total differentiability of functions of a complex variable*, Ann. Acad. Sci. Fenn. Ser. A I, 272, (1959), 9 pp.
Gehring, F. W., *The boundary behavior and uniqueness of solutions of the heat equation*, Trans. Amer. Math. Soc., **94**, (1960), 337–364.

Gehring, F. W., *Harmonic functions and Tauberian theorems*, Proc. London Math. Soc., **10**, (1960), 88–106.

Gehring, F. W., *The definitions and exceptional sets for quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I, **281**, (1960), 28 pp.

Gehring, F. W.; Haahiti, H., *The transformations which preserve the harmonic functions*, Ann. Acad. Sci. Fenn. Ser. A I, **293**, (1960), 12 pp.

Gehring, F. W., *Rings and quasiconformal mappings in space*, Proc. Nat. Acad. Sci. U.S.A., **47**, (1961), 98–105.

Gehring, F. W., *Symmetrization of rings in space*, Trans. Amer. Math. Soc., **101**, (1961), 499–519.

Gehring, F. W., *A remark on the moduli of rings*, Comment. Math. Helv., **36**, (1961), 42–46.

Gehring, F. W.; Väisälä, J., *On the geometric definition for quasiconformal mappings*, Comment. Math. Helv., **36**, (1961), 19–32.

Gehring, F. W., *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc., **103**, (1962), 353–393.

Gehring, F. W., *Extremal length definitions for the conformal capacity of rings in space*, Michigan Math. J., **9**, (1962), 137–150.

Gehring, F. W.; Hayman, W. K., *An inequality in the theory of conformal mapping*, J. Math. Pures Appl., **41**, (1962), 353–361.

Gehring, F. W., *Quasiconformal mappings in space*, Bull. Amer. Math. Soc., **69**, (1963), 146–164.

Gehring, F. W.; af Hällström, G., *A distortion theorem for functions univalent in an annulus*, Ann. Acad. Sci. Fenn. Ser. A I , **325**, (1963), 16 pp.
Gehring, F. W., *The Carathéodory convergence theorem for quasiconformal mappings in space*, Ann. Acad. Sci. Fenn. Ser. A I, 336, (1963), 21 pp.

Agard, S. B.; Gehring, F. W., *Angles and quasiconformal mappings*, Proc. London Math. Soc., 14, (1965), 1–21.

Gehring, F. W., *Extension of quasiconformal mappings in three space*, J. Analyse Math., 14, (1965), 171–182.

Gehring, F. W.; Väisälä, J., *The coefficients of quasiconformality of domains in space*, Acta Math., 114, (1965), 1–70.

Gehring, F. W.; Reich, E., *Area distortion under quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I, 388, (1966), 15 pp.

Gehring, F. W., *Coefficients of quasiconformality of domains in three space*, 1966 Contemporary Problems in Theory Anal. Functions (Internat. Conf., Erevan, 1965) pp. 83–88 Izdat. “Nauka”, Moscow.

Gehring, F. W., *Definitions for a class of plane quasiconformal mappings*, Nagoya Math. J., 29, (1967), 175–184.

Gehring, F. W., *Extension theorems for quasiconformal mappings in n-space*, J. Analyse Math., 19, (1967), 149–169.

Gehring, F. W., *Extension theorems for quasiconformal mappings in n-space*, 1968 Proc. Internat. Congr. Math. (Moscow, 1966) pp. 313–318 Izdat. “Mir”, Moscow.

Gehring, F. W., *Quasiconformal mappings of slit domains in three space*, J. Math. Mech., 18, (1969), 689–703.

Gehring, F. W., *Quasiconformal mappings which hold the real axis pointwise fixed*, 1970, Mathematical Essays Dedicated to A. J. Macintyre, pp. 145–148, Ohio Univ. Press, Athens, Ohio.

Gehring, F. W., *Extremal mappings of tori*, Certain problems of mathematics
and mechanics (on the occasion of the seventieth birthday of M. A. Lavrentiev) pp. 146–152. Izdat. "Nauka", Leningrad, 1970.

Gehring, F. W., *Inequalities for condensers, hyperbolic capacity, and extremal lengths*, Michigan Math. J., **18**, (1971), 1–20.

Gehring, F. W., *Lipschitz mappings and the $p$-capacity of rings in $n$-space*, Advances in the theory of Riemann surfaces (Proc. Conf., Stony Brook, N.Y., 1969), pp.175–193. Ann. of Math. Studies, **66**, Princeton Univ. Press, Princeton, N.J., 1971.

Gehring, F. W.; Huckemann, F., *Quasiconformal mappings of a cylinder*, Proc. Romanian-Finnish Seminar on Teichmüller Spaces and Quasiconformal Mappings (Braşov, 1969), pp.171–186. (Publ. House of the Acad. of the Socialist Republic of Romania, Bucharest, 1971)

Gehring, F. W., *Dilatations of quasiconformal boundary correspondences*, Duke Math. J., **39**, (1972), 89–95.

Gehring, F. W., *The $L^p$-integrability of the partial derivatives of quasiconformal mapping*, Bull. Amer. Math. Soc., **79**, (1973), 465–466.

Gehring, F. W.; Väisälä, J., *Hausdorff dimension and quasiconformal mappings*, J. London Math. Soc., **6**, (1973), 504–512.

Gehring, F. W., *The $L^p$-integrability of the partial derivatives of a quasiconformal mapping*, Acta Math., **130**, (1973), 265–277.

Gehring, F. W.; Kelly, J. C., *Quasiconformal mappings and Lebesque density*, Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973), pp. 171–179. Ann. of Math. Studies, **79**, Princeton Univ. Press, Princeton, N.J., 1974.

Gehring, F. W., *The Hausdorff measure of sets which link in Euclidean space*, Contributions to analysis (a collection of papers dedicated to Lipman Bers), pp. 159–167. Academic Press, New York, 1974.

Gehring, F. W., *Inequalities for condensers, hyperbolic capacity, and extremal
lengths, Topics in analysis (Colloq. Math. Anal., Jyväskylä, 1970), pp.133–136. Lecture Notes in Math., Vol. 419, Springer, Berlin, 1974.

Gehring, F. W., *The $L^p$-integrability of the partial derivatives of a quasiconformal mapping*, Proceedings of the Symposium on Complex Analysis (Univ. Kent, Canterbury, 1973), pp. 73–74. London Math. Soc. Lecture Note Ser., 12, Cambridge Univ. Press, 1974.

Gehring, F. W., *Lower dimensional absolute continuity properties of quasiconformal mappings*, Math. Proc. Cambridge Philos. Soc., 78, (1975), 81–93.

Gehring, F. W., *Quasiconformal mappings in $\mathbb{R}^n$*, Lectures on quasiconformal mappings, 44–110. Dept. Math., Univ. Maryland, Lecture Note, 14, Dept. Math., Univ. Maryland, College Park, Md., 1975.

Gehring, F. W., *Some metric properties of quasiconformal mappings*, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 2, pp. 203–206. Canad. Math. Congress, Montreal, Que., 1975.

Gehring, F. W.; Palka, B. P., *Quasiconformally homogeneous domains*, J. Analyse Math., 30, (1976), 172–199.

Gehring, F. W., *Absolute continuity properties of quasiconformal mappings*, Symposia Mathematica, Vol. LXVIII (Convegno sulle Transformazioni Quasiconformi e Questioni Connesse, INDAM, Rome, 1974), pp. 551–559. Academic Press, London, 1976.

Gehring, F. W., *Quasiconformal mappings*, Complex analysis and its applications (Lectures, Internat. Sem., Trieste, 1975), Vol. II, pp. 213–268. Internat. Atomic Energy Agency, Vienna, 1976.

Gehring, F. W., *A remark on domains quasiconformally equivalent to a ball*, Ann. Acad. Sci. Fenn. Ser. A I Math., 2, (1976), 147–155.

Gehring, F. W., *Univalent functions and the Schwarzian derivative*, Comment. Math. Helv., 52, (1977), 561–572.
Gehring, F. W., *Spirals and the universal Teichmüller space*, Acta Math., **141**, (1978), 99–113.

Gehring, F. W., *Some problems in complex analysis*, Proceedings of the First Finnish-Polish Summer School in Complex Analysis (Podlesice, 1977), Part II, pp. 61–64, Univ. Łódź, 1978.

Gehring, F. W., *Remarks on the universal Teichmüller space*, Enseign. Math., **24**, (1978), 173–178.

Gehring, F. W., *Univalent functions and the Schwarzian derivative*, Proceedings of the Rolf Nevanlinna Symposium on Complex Analysis (Math. Res. Inst., Univ. Istanbul, Silivri, 1976), pp. 19–24, Publ. Math. Res. Inst. Istanbul, 7, Univ. Istanbul, Istanbul, 1978.

Gehring, F. W.; Osgood, B. G., *Uniform domains and the quasihyperbolic metric*, J. Analyse Math., **36**, (1979), 50–74 (1980).

Beardon, A. F.; Gehring, F. W., *Schwarzian derivatives, the Poincaré metric and the kernel function*, Comment. Math. Helv., **55**, (1980), 50–64.

Gehring, F. W., *Spirals and the universal Teichmüller space*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), pp. 145–148, Ann. of Math. Stud., **97**, Princeton Univ. Press, Princeton, N.J., 1981.

Gehring, F. W.; Hayman, W. K.; Hinkkanen, A., *Analytic functions satisfying Hölder conditions on the boundary*, J. Approx. Theory, **35**, (1982), 243–249.

Teichmüller, Oswald Gesammelte Abhandlungen. [Collected papers] Edited and with a preface by Lars V. Ahlfors and F W. Gehring. Springer-Verlag, Berlin-New York, 1982.

Gehring, F. W., *Characteristic properties of quasidisks*, Séminaire de Mathématiques Supérieures [Seminar on Higher Mathematics], 84. Presses de l’Université de Montréal, Montreal, Que., 1982. 97 pp.
Gehring, F. W. *Injectivity of local quasi-isometries*, Comment. Math. Helv., **57**, (1982), 202–220.

Garnett, J. B.; Gehring, F. W.; Jones, P. W., *Conformally invariant length sums*, Indiana Univ. Math. J., **32**, (1983), 809–829.

Gehring, F. W.; Martio, O., *Quasidisks and the Hardy-Littlewood property*, Complex Variables Theory Appl., **2**, (1983), 67–78.

Astala, K.; Gehring, F. W. *Injectivity criteria and the quasidisk*, Complex Variables Theory Appl., **3**, (1984), 45–54.

Gehring, F. W.; Pommerenke, Ch., *On the Nehari univalence criterion and quasicircles*, Comment. Math. Helv., **59**, (1984), 226–242.

Astala, K.; Gehring, F. W., *Quasiconformal analogues of theorems of Koebe and Hardy-Littlewood*, Michigan Math. J., **32**, (1985), 99–107.

Anderson, J. M.; Gehring, F. W.; Hinkkanen, A., *Polynomial approximation in quasidisks*, Differential geometry and complex analysis, 75–86, Springer, Berlin, 1985.

Gehring, F. W.; Martio, O., *Lipschitz classes and quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I Math., **10**, (1985), 203–219.

Gehring, F. W.; Martio, O., *Quasieextremal distance domains and extension of quasiconformal mappings*, J. Analyse Math., **45**, (1985), 181–206.

Gehring, F. W., *Extension of quasi-isometric embeddings of Jordan curves*, Complex Variables Theory Appl., **5**, (1986), 245–263.

Astala, K.; Gehring, F. W., *Injectivity, the BMO norm and the universal Teichmüller space*, J. Analyse Math., **46**, (1986), 16–57.

Gehring, F. W., *Uniform domains and the ubiquitous quasidisk*, Jahresber. Deutsch. Math.-Verein., **89**, (1987), 88–103.
Gehring, F. W.; Martin, G. J., *Discrete quasiconformal groups. I*, Proc. London Math. Soc., 55, (1987), 331–358.

Gehring, F. W.; Martin, G. J., *Discrete convergence groups*, Complex analysis, I (College Park, Md., 1985–86), 158–167, Lecture Notes in Math., 1275, Springer, Berlin, 1987.

Gehring, F. W.; Hag, K., *Remarks on uniform and quasiconformal extension domains*, Complex Variables Theory Appl., 9, (1987), 175–188.

Gehring, F. W., *Topics in quasiconformal mappings*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 62–80, Amer. Math. Soc., Providence, RI, 1987.

Holomorphic functions and moduli. I & II. Proceedings of the workshop held in Berkeley, California, March 13–19, 1986. Edited by D. Drasin, C. J. Earle, F. W. Gehring, I. Kra and A. Marden. Mathematical Sciences Research Institute Publications, 11. Springer-Verlag, New York, 1988.

Garnett, J. B.; Gehring, F. W.; Jones, P. W., *Quasiconformal groups and the conical limit set*, Holomorphic functions and moduli, Vol. II (Berkeley, CA, 1986), 59–67, Math. Sci. Res. Inst. Publ., 11, Springer, New York, 1988.

Gehring, F. W.; Martin, G. J., *The matrix and chordal norms of Möbius transformations*, Complex analysis, 51–59, Birkhäuser, Basel, 1988.

Gehring, F. W., *Quasiconformal mappings*, A plenary address presented at the International Congress of Mathematicians held in Berkeley, California, August 1986. Introduced by G. D. Mostow. ICM Series. American Mathematical Society, Providence, RI, 1988. 1 videocassette (NTSC; 1/2 inch; VHS) (60 min.)

Gehring, F. W.; Martin, G. J., *Iteration theory and inequalities for Kleinian groups*, Bull. Amer. Math. Soc. (N.S.), 21, (1989), 57–63.

Gehring, F. W.; Pommerenke, Ch., *Circular distortion of curves and quasicircles*, Ann. Acad. Sci. Fenn. Ser. A I Math., 14, (1989), 381–390.
Gehring, F. W.; Martin, G. J., Stability and extremality in Jørgensen’s inequality, Complex Variables Theory Appl., 12, (1989), 277–282.

Gehring, F. W.; Hag, K.; Martio, O., Quasihyperbolic geodesics in John domains, Math. Scand., 65, (1989), 75–92.

Astala, K.; Gehring, F. W., Crickets, zippers and the Bers universal Teichmüller space, Proc. Amer. Math. Soc., 110, (1990), 675–687.

Halmos, Paul R.; Gehring, F. W., Allen L. Shields, Math. Intelligencer, 12, (1990), 20.

Gehring, F. W.; Martin, G. J., Inequalities for Möbius transformations and discrete groups, J. Reine Angew. Math., 418, (1991), 31–76.

Paul Halmos., Celebrating 50 years of mathematics, Edited by John H. Ewing and F. W. Gehring. Springer-Verlag, New York, 1991.

Gehring, F. W.; Martin, G. J., Some universal constraints for discrete Möbius groups, Paul Halmos, 205–220, Springer, New York, 1991.

Gehring, F. W.; Martin, G. J., Axial distances in discrete Möbius groups, Proc. Nat. Acad. Sci. U.S.A., 89, (1992), 1999–2001.

Gehring, F. W., Topics in quasiconformal mappings: Quasiconformal space mappings, 20–38, Lecture Notes in Math., 1508, Springer, Berlin, 1992.

Gehring, F. W.; Martin, G. J., 6-torsion and hyperbolic volume, Proc. Amer. Math. Soc., 117, (1993), 727–735.

Gehring, F. W.; Martin, G. J., The iterated commutator in a Fuchsian group, Complex Variables Theory Appl., 21, (1993), 207–218.

Gehring, F. W.; Martin, G. J., On the minimal volume hyperbolic 3-orbifold, Math. Res. Lett., 1, (1994), 107–114.

Gehring, F. W.; Martin, G. J., Commutators, collars and the geometry of Möbius groups, J. Anal. Math., 63, (1994), 175–219.
Gehring, F. W.; Martin, G. J., *Chebyshev polynomials and discrete groups*, Proceedings of the Conference on Complex Analysis (Tianjin, 1992), 114–125, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA, 1994.

Gehring, F. W.; Martin, G. J., *Holomorphic motions, Schottky’s theorem and an inequality for discrete groups*, Computational methods and function theory 1994 (Penang), 173–181, Ser. Approx. Decompos., 5, World Sci. Publ., River Edge, NJ, 1995.

Gehring, F. W.; Martin, G. J., *On the Margulis constant for Kleinian groups. I*, Ann. Acad. Sci. Fenn. Math., 21, (1996), 439–462.

Gehring, F. W.; Maclachlan, C.; Martin, G. J., *On the discreteness of the free product of finite cyclic groups*, Mitt. Math. Sem. Giessen, 228, (1996), 9–15.

Gehring, F. W.; Maclachlan, C.; Martin, G. J.; Reid, A. W., *Arithmeticity, discreteness and volume*, Trans. Amer. Math. Soc., 349, (1997), 3611–3643.

Gehring, F. W.; Martin, G. J., *Geodesics in hyperbolic 3-folds*, Michigan Math. J., 44, (1997), 331–343.

Cao, C.; Gehring, F. W.; Martin, G. J., *Lattice constants and a lemma of Zagier*, Lipa’s legacy (New York, 1995), 107–120, Contemp. Math., 211, Amer. Math. Soc., Providence, RI, 1997.

Gehring, F. W.; Martin, G. J., *Hyperbolic axes and elliptic fixed points in a Fuchsian group*, Complex Variables Theory Appl., 33, (1997), 303–309.

Gehring, F; Kra, I.; Osserman, R., *The mathematics of Lars Valerian Ahlfors*, Edited by Steven G. Krantz. Notices Amer. Math. Soc., 45, (1998), 233–242.

Gehring, F. W.; Maclachlan, C.; Martin, G. J., *Two-generator arithmetic Kleinian groups. II*, Bull. London Math. Soc., 30, (1998), 258–266.

Gehring, F. W.; Marshall, T. H.; Martin, G. J., *The spectrum of elliptic axial distances in Kleinian groups*, Indiana Univ. Math. J., 47, (1998), 1–10.
Gehring, F. W.; Martin, G. J., Precisely invariant collars and the volume of hyperbolic 3-folds, J. Differential Geom., 49, (1998), 411–435.

Gehring, F. W.; Marshall, T. H.; Martin, G. J., Collaring theorems and the volumes of hyperbolic n-manifolds, New Zealand J. Math., 27, (1998), 207–225.

Gehring, F. W.; Martin, G. J., The volume of hyperbolic 3-folds with p-torsion, p ≥ 6, Quart. J. Math. Oxford Ser., 50, (1999), 1–12.

Gehring, F. W.; Iwaniec, T., The limit of mappings with finite distortion, Ann. Acad. Sci. Fenn. Math., 24, (1999), 253–264.

Gehring, F. W.; Hag, K., Hyperbolic geometry and disks, Continued fractions and geometric function theory (CONFUN) (Trondheim, 1997). J. Comput. Appl. Math., 105, (1999), 275–284.

Gehring, F. W.; Hag, K., A bound for hyperbolic distance in a quasidisk, Computational methods and function theory 1997 (Nicosia), 233–240, Ser. Approx. Decompos., 11, World Sci. Publ., River Edge, NJ, 1999.

Gehring, F. W., Characterizations of quasidisks, Quasiconformal geometry and dynamics (Lublin, 1996), 11–41, Banach Center Publ., 48, Polish Acad. Sci., Warsaw, 1999.

Gehring, F. W., Variations on a theorem of Fejér and Riesz, XII-th Conference on Analytic Functions (Lublin, 1998). Ann. Univ. Mariae Curie-Skłodowska Sect. A, 53, (1999), 57–66.

Gehring, F. W.; Hag, K., Bounds for the hyperbolic distance in a quasidisk, XII-th Conference on Analytic Functions (Lublin, 1998). Ann. Univ. Mariae Curie-Skłodowska Sect. A 53,(1999), 67–72.

Gehring, F. W.; Hag, K., The Apollonian metric and quasiconformal mappings, In the tradition of Ahlfors and Bers (Stony Brook, NY, 1998), 143–163, Contemp. Math., 256, Amer. Math. Soc., Providence, RI, 2000.

Gehring, F. W., The Apollonian metric, Travaux de la Conférence Interna-
Gehring, F. W.; Gilman, J. P.; Martin, G. J., *Kleinian groups with real parameters*, Commun. Contemp. Math., 3, (2001), 163–186.

Brown, E. H.; Cohen, F. R.; Gehring, F. W.; Miller, H. R.; Taylor, B. A., *Franklin P. Peterson (1930–2000)*, Notices Amer. Math. Soc., 48, (2001), 1161–1168.

Gehring, F. W.; Hag, K., *Reflections on reflections in quasidisks*, Papers on analysis, 81–90, Rep. Univ. Jyväskylä Dep. Math. Stat., 83, Jyväskylä, 2001.

Gehring, F. W.; Marshall, Timothy H.; Martin, G. J., *Recent results in the geometry of Kleinian groups*, Comput. Methods Funct. Theory 2, (2002), 249–256.

Gehring, F. W.; Hag, K., *Sewing homeomorphisms and quasidisks*, Comput. Methods Funct. Theory 3, (2003), 143–150.

Gehring, F. W., *Quasiconformal mappings in Euclidean spaces*, Handbook of complex analysis: geometric function theory. 2, 1–29, Elsevier, Amsterdam, 2005.

Gehring, F. W.; Martin, G. J., *(p,q,r)-Kleinian groups and the Margulis constant*, Complex analysis and dynamical systems II, 149–169, Contemp. Math., 382, Amer. Math. Soc., Providence, RI, 2005.

Gehring, F. W.; Martin, G J., *Minimal co-volume hyperbolic lattices. I. The spherical points of a Kleinian group*, Ann. of Math. (2) 170, (2009), 123–161.

Gehring, F. W.; Hag, K., The ubiquitous quasidisk. With contributions by Ole Jacob Broch. Mathematical Surveys and Monographs, 184. American Mathematical Society, Providence, RI, 2012.

G.J. Martin. Massey University, New Zealand