Constraints on Regge models from perturbation theory

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Our aim

- We will consider the vector-vector correlator

\[
\Pi_{V}^{\mu\nu}(q) \equiv (q^\mu q^\nu - g^{\mu\nu} q^2) \Pi_V(q) \equiv i \int d^4xe^{iqx}\langle vac|J_{V}^{\mu}(x)J_{V}^{\nu}(0)|vac\rangle
\]

where

\[
J_{V}^{\mu} = \sum_{f} Q_f \bar{\psi}_{f}\gamma^{\mu}\psi_{f}
\]

- In particular we will focus on the Adler function

\[
A(Q^2) \equiv -Q^2 \frac{d}{dQ^2} \Pi_{V}(Q^2) = Q^2 \int_{0}^{\infty} \frac{1}{(Q^2 + t)^2} \frac{1}{\pi} \text{Im} \Pi_{V}(t)
\]
Our aim

\[ A(Q^2) \equiv -Q^2 \frac{d}{dQ^2} \Pi_V(Q^2) = Q^2 \int_0^\infty \frac{1}{(Q^2 + t)^2} \frac{1}{\pi} \text{Im}\Pi_V(t) \]

**Perturbative calculation**

**Non-perturbative calculation**

- We want to address the constraints that the knowledge of the perturbative calculation (in particular the perturbative expansion in \( \alpha_s(Q^2) \)) imposes on the hadronic spectrum.
Large $N_c$ limit + Regge theory

- In the large $N_c$ limit the meson spectrum consists of infinitely narrow resonances

$$\text{Im} \Pi(t) = \sum_{n=0}^{\infty} F_V^2(n) \pi \delta(t - M_V^2(n))$$
Large $N_c$ limit + Regge theory

- In the large $N_c$ limit the meson spectrum consists of infinitely narrow resonances

$$
\Pi(Q^2) = \sum_{n=0}^{\infty} \frac{F_V^2(n)}{M_V^2(n) + Q^2}
$$
Large $N_c$ limit + Regge theory

- In the large $N_c$ limit the meson spectrum consists of infinitely narrow resonances

\[
A(Q^2) = Q^2 \sum_{n=0}^{\infty} \frac{F_V^2(n)}{(Q^2 + M_V^2(n))^2}
\]
Large $N_c$ limit + Regge theory

- In the large $N_c$ limit the meson spectrum consists of infinitely narrow resonances

$$A(Q^2) = Q^2 \sum_{n=0}^{\infty} \frac{F_V^2(n)}{(Q^2 + M_V^2(n))^2}$$

- High excitations of the QCD spectrum are believed to follow Regge trajectories

$$\lim_{n \to \infty} \frac{M_V^2(n)}{n} = \text{constant}$$
Large $N_c$ limit + Regge theory

- In the large $N_c$ limit the meson spectrum consists of infinitely narrow resonances

\[ A(Q^2) = Q^2 \sum_{n=0}^{\infty} \frac{F_{V}^2(n)}{(Q^2 + M_{V}^2(n))^2} \]

- At leading order

\[ M_{V}^2(n) = B_{V} n \]

This is consistent with perturbation theory in the Euclidean region at leading order in $\alpha_s$ if

\[ F_{V}^2(n) = \text{Const.} \]
Large $N_c$ limit + Regge theory

- In the large $N_c$ limit the meson spectrum consists of infinitely narrow resonances

\[ A(Q^2) = Q^2 \sum_{n=0}^{\infty} \frac{F_V^2(n)}{(Q^2 + M_V^2(n))^2} \]

- Going further...

\[ M_V^2(n) = \sum_{s=-1}^{\infty} B_V(-s) n(-s) = B_V n + A_V + \frac{C_V}{n} + \cdots \]
In the large $N_c$ limit the meson spectrum consists of infinitely narrow resonances

$$A(Q^2) = Q^2 \sum_{n=0}^{\infty} \frac{F^2_V(n)}{(Q^2 + M^2_V(n))^2}$$

Going further...

$$F^2_V(n) = \sum_{s=0}^{\infty} F^2_{V,s}(n) \frac{1}{n^s} = F^2_{V,0}(n) + \frac{F^2_{V,1}(n)}{n} + \frac{F^2_{V,2}(n)}{n^2} + \cdots$$

where

$$F^2_{V,s}(n) = \sum_{r=0}^{\infty} C^{(r)}_{V,s}(n) \frac{1}{\ln^r n}$$
Large $N_c$ limit + Regge theory

- In the large $N_c$ limit the meson spectrum consists of infinitely narrow resonances

\[ A(Q^2) = Q^2 \sum_{n=0}^{\infty} \frac{F^2_V(n)}{(Q^2 + M^2_V(n))^2} \]

- Going further...

\[ F^2_V(n) = \sum_{s=0}^{\infty} F^2_{V,s}(n) \frac{1}{n^s} = F^2_{V,0}(n) + \frac{F^2_{V,1}(n)}{n} + \frac{F^2_{V,2}(n)}{n^2} + \cdots \]

where

\[ F^2_{V,s}(n) = \sum_{r=0}^{\infty} C_{V,s}^{(r)}(n) \frac{1}{\ln^r n} \]

\[ \frac{1}{n^s} \sim \frac{1}{(Q^2)^s} \]

\[ \frac{1}{\ln^r n} \sim \alpha_s(Q^2)^r \]
Matching with the OPE

\[ A(Q^2) = Q^2 \sum_{n=0}^{\infty} \frac{F^2_V(n)}{(Q^2 + M^2_V(n))^2} \]

\[ A_{OPE}(Q^2) = \sum_f Q^2_f \left[ \frac{4}{3} \frac{N_c}{16\pi^2} \left( 1 + \frac{3}{8} N_c \frac{\alpha_s(Q^2)}{\pi} + \ldots \right) \right. \]

\[ + \frac{C(\alpha_s(Q^2))}{Q^4} \beta(\alpha_s(\nu)) \langle vac|G^2(\nu)|vac \rangle + \mathcal{O} \left( \frac{1}{Q^6} \right) \]
Matching with the OPE

- Use the Euler-MacLaurin formula

\[ A(Q^2) = Q^2 \left( \sum_{n=0}^{n^*-1} \frac{F_V^2(n)}{(Q^2 + M_V^2(n))^2} + \sum_{n=n^*}^{\infty} \frac{F_V^2(n)}{(Q^2 + M_V^2(n))^2} \right) \]

\( n^* \) arbitrary but formally large, \( \Lambda_{QCD} n^* \ll Q \)
Matching with the OPE

- Use the Euler-MacLaurin formula

\[
A(Q^2) = Q^2 \int_0^\infty dn \frac{F_V^2(n)}{(Q^2 + M_V^2(n))^2} + Q^2 \left[ \sum_{n=0}^{n^*-1} \frac{F_V^2(n)}{(Q^2 + M_V^2(n))^2} - \int_0^{n^*} \frac{F_V^2(n)}{(Q^2 + M_V^2(n))^2} \right] \\
+ \frac{Q^2}{2} \frac{F_V^2(n^*)}{(Q^2 + M_V^2(n^*))^2} + Q^2 \sum_{k=1}^{\infty} (-1)^k \left| B_{2k} \right| \frac{d^{(2k-1)}}{(2k)!} \left. \int_0^{n^*} \frac{F_V^2(n)}{(Q^2 + M_V^2(n))^2} \right|_{n=n^*}
\]

\( n^* \) arbitrary but formally large, \( \Lambda_{QCD} n^* \ll Q \)
Matching with the OPE

- Use the Euler-MacLaurin formula

\[ A(Q^2) = Q^2 \int_0^\infty dn \frac{F_V^2(n)}{(Q^2 + M_V^2(n))^2} + Q^2 \left[ \sum_{n=0}^{n^*-1} \frac{F_V^2(n)}{(Q^2 + M_V^2(n))^2} - \int_0^{n^*} \frac{F_V^2(n)}{(Q^2 + M_V^2(n))^2} \right] + \frac{Q^2}{2} \frac{F_V^2(n^*)}{(Q^2 + M_V^2(n^*))^2} + Q^2 \sum_{k=1}^{\infty} (-1)^k \left| B_{2k} \right| \frac{d^{(2k-1)}}{(2k)!} \left| \frac{F_V^2(n)}{(Q^2 + M_V^2(n))^2} \right|_{n=n^*} \]

\( n^* \) arbitrary but formally large, \( \Lambda_{QCD} n^* \ll Q \)

All the terms involving Logs of \( Q^2 \) come from this integral. This greatly simplifies the matching with the logarithmic terms within the OPE. We cannot find a closed expression for the finite (Log(\( Q^2 \))-independent) terms.
Matching at LO

- The only term in $A(Q^2)$ that can produce Logs of $Q^2$ that are not suppressed by $1/Q^2$ is

$$Q^2 \int_0^\infty dn \frac{F_{V,0}^2(n)}{(Q^2 + B_V n)^2}$$

- This must match the leading term in the OPE (the perturbative QCD calculation)

$$Q^2 \int_0^\infty \frac{1}{(Q^2 + t)^2} \frac{1}{\pi} \text{Im} \Pi_{V}^{\text{pert.}}(t)$$
Matching at LO

The condition to be fulfilled is

\[ \frac{F_{V,0}^2(n)}{B_V} = \frac{1}{\pi} \text{Im} \Pi_{V}^{\text{pert.}}(B_V n) \]
Matching at LO

- The condition to be fulfilled is

$$\frac{F_{V,0}^2(n)}{B_V} = \frac{1}{\pi} \text{Im} \Pi_{pert.}^V (B_V n)$$

- By using the 3-loop expression of $\text{Im} \Pi^V_{pert.}$ we obtain

$$F_{V,0}^2(n) = B_V \frac{4}{3} \frac{N_c}{16\pi^2} \sum_f Q_f^2 \left\{ 1 + \frac{3}{8\pi} N_c \alpha_s(nB_V) + \frac{243 - 176 \zeta(3)}{128\pi^2} N_c^2 \alpha_s^2(nB_V) \\
+ \frac{346201 - 2904\pi^2 - 324528 \zeta(3) + 63360 \zeta(5)}{27648\pi^3} N_c^3 \alpha_s^3(nB_V) + O(\alpha_s^4(nB_V)) \right\}$$
Matching at LO

- If we use

\[ \alpha_s(nB_V) = \frac{1}{\beta_0 \ln \left( \frac{nB_V}{\Lambda^2_{\text{MS}}} \right)} - \frac{1}{\beta_0^3 \ln^2 \left( \frac{nB_V}{\Lambda^2_{\text{MS}}} \right)} \beta_1 \ln \left( \ln \left( \frac{nB_V}{\Lambda^2_{\text{MS}}} \right) \right) \]

\[ + \frac{1}{\beta_0^3 \ln^3 \left( \frac{nB_V}{\Lambda^2_{\text{MS}}} \right)} \left\{ \frac{\beta_2}{\beta_0} \left[ \ln^2 \left( \ln \left( \frac{nB_V}{\Lambda^2_{\text{MS}}} \right) \right) - \ln \left( \ln \left( \frac{nB_V}{\Lambda^2_{\text{MS}}} \right) \right) - 1 \right] + \frac{\beta_2}{\beta_0} \right\} \]
Matching at LO

- If we use

\[
\alpha_s(n_B V) = \frac{1}{\beta_0 \ln \left( \frac{n_B V}{\Lambda^2_{\overline{MS}}} \right)} - \frac{1}{\beta_0^3 \ln^2 \left( \frac{n_B V}{\Lambda^2_{\overline{MS}}} \right)} \beta_1 \ln \left( \ln \left( \frac{n_B V}{\Lambda^2_{\overline{MS}}} \right) \right) \\
+ \frac{1}{\beta_0^3 \ln^3 \left( \frac{n_B V}{\Lambda^2_{\overline{MS}}} \right)} \left\{ \frac{\beta_1^2}{\beta_0^2} \left[ \ln^2 \left( \ln \left( \frac{n_B V}{\Lambda^2_{\overline{MS}}} \right) \right) - \ln \left( \ln \left( \frac{n_B V}{\Lambda^2_{\overline{MS}}} \right) \right) - 1 \right] + \frac{\beta_2}{\beta_0} \right\}
\]

- We can express the result as

\[
F_{V,0}^2(n) = B_V \frac{4}{3} \frac{N_c}{16\pi^2} \sum_f Q_f^2 \left\{ 1 + \frac{9}{22} \ln \tilde{n} + \frac{1}{\ln^2 \tilde{n}} \left[ - \frac{459}{1331} \ln \ln \tilde{n} + \frac{144}{121} \left( 243 \frac{243}{128} - \frac{11}{8} \zeta(3) \right) \right] \\
+ \frac{1}{\ln^3 \tilde{n}} \left[ \frac{46818}{161051} \ln^2 \ln \tilde{n} + \frac{459}{322102} (-2877 + 1936\zeta(3)) \ln \ln \tilde{n} + \frac{42272605}{2576816} \\
- \frac{3 \pi^2}{22} - \frac{20283 \zeta(3)}{1331} + \frac{360 \zeta(5)}{121} \right] + \mathcal{O} \left( \frac{1}{\ln^4 n} \right) \right\} \quad \text{with} \quad \tilde{n} = n_B V / \Lambda_{\overline{MS}}
\]
Matching at NLO

- We expand the integral from the Euler-Maclaurin expression of $A(Q^2)$ to NLO

$$Q^2 \int_0^\infty \frac{F_{V,0}^2(n) + F_{V,1}^2(n)}{(Q^2 + A_V + B_V n)^2} = Q^2 \int_0^\infty \frac{F_{V,0}^2(n)}{(Q^2 + B_V n)^2} + Q^2 \int_0^\infty \frac{-2A_V F_{V,0}^2(n)}{(Q^2 + B_V n)^3} + Q^2 \int_0^\infty \frac{F_{V,1}^2(n)}{n (Q^2 + B_V n)^2} + \ldots$$
Matching at NLO

We expand the integral from the Euler-Maclaurin expression of $A(Q^2)$ to NLO

\[ Q^2 \int_0^\infty dn \frac{F_{V,0}^2(n) + \frac{F_{V,1}^2(n)}{n}}{(Q^2 + A_V + B_V n)^2} = Q^2 \int_0^\infty dn \frac{F_{V,0}^2(n)}{(Q^2 + B_V n)^2} \]

\[ + Q^2 \int_0^\infty dn \frac{-2A_V F_{V,0}^2(n)}{(Q^2 + B_V n)^3} + Q^2 \int_0^\infty dn \frac{F_{V,1}^2(n)}{n (Q^2 + B_V n)^2} + \ldots \]

There is no $1/Q^2$ term in the OPE, so all logarithmic (and finite) terms at this order must vanish

\[ Q^2 \int_0^\infty dn \frac{-2A_V F_{V,0}^2(n)}{(Q^2 + B_V n)^3} + Q^2 \int_0^\infty dn \frac{F_{V,1}^2(n)}{n (Q^2 + B_V n)^2} = 0 \]
Matching at NLO

We find

\[
\frac{F_{V,1}^2(n)}{n} = \frac{4}{3} \frac{N_c}{16 \pi^2} \sum_f Q_f^2 \frac{1}{n} \left\{ -\frac{11}{32 \pi^2} N_c^2 \alpha_s^2(nBV) - \frac{2877 - 1936 \zeta(3)}{768 \pi^3} N_c^3 \alpha_s^3(nBV) \right. \\
\left. - \frac{11 (376357 - 2904 \pi^2 - 344112 \zeta(3) + 63360 \zeta(5))}{110592 \pi^4} N_c^4 \alpha_s^4(nBV) + O(\alpha_s^5(nBV)) \right\}
\]
Matching at NNLO

$$\frac{35}{121} \frac{\alpha_s(Q^2)}{4\pi} \frac{\beta(\alpha_s(\nu))}{Q^4} \langle \text{vac} | G^2(\nu) | \text{vac} \rangle$$

$$\Rightarrow Q^2 \int_{n^*}^{\infty} \frac{dn}{(Q^2 + B_V n)^2} \left[ \frac{F_{V,2}^2(n)}{n^2} - \frac{1}{B_V} \frac{d}{dn} \left( \frac{C_V F_{V,0}^2(n)}{n} + \frac{A_VF_{V,1}^2(n)}{2n} \right) \right]$$

Using

$$\frac{1}{Q^4} \alpha_s(Q^2) = \frac{\beta_0}{8\pi} \alpha_s^2(nB_V) ,$$

we find
Matching at NNLO

\[ F_{V,2}^2(n) = -C_V \frac{4}{3} \frac{N_c}{16\pi^2} \sum_f Q_f^2 \left\{ 1 + \frac{3}{8\pi} N_c \alpha_s(nB_V) \right\} \]

\[ + \left[ \frac{287 - 176 \zeta(3)}{128\pi^2} - \frac{11A_V^2}{64\pi^2B_VC_V} - \frac{35}{88} \frac{\beta(\alpha_s(\nu)) \langle \text{vac}|G^2(\nu)|\text{vac}\rangle}{B_VC_VN_c^2} \right] N_c^2 \alpha_s^2(nB_V) \]

+ \mathcal{O}(\alpha_s^3(nB_V)) \}
Some numerics...

|       | \( n = 1 \)          | \( n = 2 \)          | \( n = 3 \)          | \( n = 4 \)          |
|-------|----------------------|----------------------|----------------------|----------------------|
| \( M_\rho \) | 781.3 \((775.5 \pm 0.4)\) | 1440.2 \((1459 \pm 11)\) | 1891.8 \((1870 \pm 20)\) | 2257 \((2265 \pm 40)\) |
| \( M_{al} \) | 1235.6 \((1230 \pm 40)\) | 1621.7 \((1647 \pm 22)\) | 1962 \((1930^{+30}_{-70})\) | 2257.8 \((2270^{+55}_{-40})\) |
| \( F_V \)  | 152 \((156 \pm 1)\)  | 153                  | 153                  | 152                  |
| \( F_A \)  | 121 \((122 \pm 24)\) | 136                  | 138                  | 138                  |

\[
\text{taking } \alpha_s(1\text{GeV}) = 0.33 \text{ and } \beta\langle G^2 \rangle = -(352\text{MeV})^4
\]

\[
M_{V,A}^2(n) = B_{V,A}n + A_{V,A} + C_{V,A} \frac{1}{n} + \ldots
\]

\[
B_V = 1.525 \times 10^6 \text{MeV}^2 \quad A_V = -1.038 \times 10^6 \text{MeV}^2 \quad C_V = 0.123 \times 10^6 \text{MeV}^2
\]

\[
B_A = 1.278 \times 10^6 \text{MeV}^2 \quad A_A = -0.100 \times 10^6 \text{MeV}^2 \quad C_A = 0.349 \times 10^6 \text{MeV}^2
\]
Some numerics...

\[ F_{X,0}^2(n) = \sum_{r=0}^{\infty} C_{X,0}^{(r)}(n) \frac{1}{\ln^r n} \]
Some numerics...

\[ F_{V,LO}(n) = F_{V,0}^2(n) \]

\[ F_{V,NLO}(n) = F_{V,0}^2(n) + \frac{F_{V,1}^2(n)}{n} \]

\[ F_{V,NNLO}(n) = F_{V,0}^2(n) + \frac{F_{V,1}^2(n)}{n} + \frac{F_{V,2}^2(n)}{n^2} \]
Conclusions

- We have studied the constraints that perturbation theory imposes on a model for the meson spectrum inspired on the large $N_c$ limit plus Regge theory, focusing on current-current correlators.

- Introducing $1/n$ corrections to the mass spectrum and the decay constants, we can match the hadronic calculation with the perturbative calculation at 3-loop order in $\alpha_s$.

- A numerical analysis shows that the corrections are small, but significant (more so in the case of the axial-vector mesons). Although numerically unstable and with considerable uncertainties, our results compare favorably with existing experimental data.

- We can have a different large $n$ behavior for the vector and the axial-vector mesons complying with the OPE.