GAUSSIAN HEAT KERNEL ESTIMATES OF BAMLER-ZHANG TYPE
ALONG SUPER RICCI FLOW
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Abstract. Bamler-Zhang have developed geometric analysis on Ricci flow with scalar curvature bound. The aim of this paper is to extend their work to various geometric flows. We generalize some of their results to super Ricci flow whose Müller quantity is non-negative, and obtain Gaussian heat kernel estimates.

1. Introduction

1.1. Backgrounds. Let \((M, g(t))_{t \in [0, T)}\) be a solution to the Ricci flow

\[
\partial_t g = -2 \text{Ric}
\]

on an \(n\)-dimensional compact smooth manifold with \(T < +\infty\). One of the common problems in the Ricci flow theory is to understand the blow-up behavior of curvature when \(t \nearrow T\). In a pioneering work by Hamilton [14], he has proved that the supremum of the norm of the Riemannian curvature tensor blows up at \(T\); namely,

\[
\lim_{t \nearrow T} \sup_M |Rm| = +\infty.
\]

After that under the same setting, Šešum [29] has generalized (1.2) to the Ricci curvature; namely, she has proved

\[
\lim_{t \nearrow T} \sup_M |\text{Ric}| = +\infty.
\]

It is natural to ask whether the same phenomenon occurs for the scalar curvature; namely,

\[
\lim_{t \nearrow T} \sup_M R = +\infty.
\]

This is known to occur in the case of \(n = 2, 3\) or in the Kähler case (see [15], [19], [37]), but it seems to be still open in the higher dimensional and non-Kähler case. From this point of view, it is important to investigate the structure of Ricci flow with scalar curvature bound.

Based on the background, Bamler-Zhang [7] have vastly developed geometric analysis on Ricci flow with scalar curvature bound, and obtained useful tools such as distance distortion estimates, a nice cutoff function, and mean value inequalities. Furthermore, by using them, they have concluded Gaussian heat kernel estimates, backward pseudolocality theorem, and strong \(\varepsilon\)-regularity theorem. Recently, their estimates have yielded numerous results in the sequel [8], and a series of works by Bamler [3], [4], [5], [6] while being improved.
1.2. Main results. The purpose of this paper is to extend the results by Bamler-Zhang [7] to more general geometric flows. We focus on a super solution to the Ricci flow (1.1). An $n$-dimensional compact manifold $(M, g(t))_{t \in [0, T)}$ equipped with a time-dependent metric is called super Ricci flow when
\[ \partial_t g \geq -2 \mathrm{Ric}. \]
McCann-Topping [25] have introduced this notion, and studied the relation between the Ricci flow theory and optimal transport theory. Recently, this notion has begun to be investigated not only from the viewpoint of the Ricci flow theory but also from that of metric measure geometry (see e.g., [20], [21], [22], [30]).

We examine super Ricci flow under the assumption that its Müller quantity is non-negative. For a (time-dependent) vector field $V$, the Müller quantity is defined by the following formula (see [26, Definition 1.3]):
\[ \mathcal{D}(V) := \partial_t H - \Delta H - 2|h|^2 + 4 \operatorname{div} h(V) - 2\langle \nabla H, V \rangle + 2 \operatorname{Ric}(V, V) - 2h(V, V), \]
where
\[ h := -\frac{1}{2} \partial_t g, \quad H := \operatorname{tr} h. \]
It is well-known that several results for Ricci flow can be extended to more general geometric flow under the assumption that its Müller quantity is non-negative; for instance, the monotonicity of $W$-functional and the reduced geometry (more precisely, see Subsections 2.2, 2.3). Also, there are some examples of super Ricci flow whose Müller quantity is non-negative (see [26, Section 2], [12, Section 7]):

(1) Ricci flow;
(2) Ricci flow coupled with the heat equation, called List flow ([23]);
(3) Ricci flow coupled with the harmonic map heat flow, called Müller flow ([27]);
(4) mean curvature flow for spacelike hypersurfaces in Lorentzian manifold of non-negative sectional curvature;
(5) (scaled) twisted Kähler-Ricci flow.

For $x, y \in M$ and $s, t \in [0, T)$ with $s < t$, we denote by $G(x, t; y, s)$ the heat kernel for the heat equation (more precisely, see Subsection 2.2). One of our main results is the following Gaussian heat kernel estimate, which has been formulated by Bamler-Zhang [7] for Ricci flow (see [7, Theorem 1.4]):

**Theorem 1.1.** Let $(M, g(t))_{t \in [0, T)}$ be an $n$-dimensional compact super Ricci flow with $T < +\infty$ satisfying $\mathcal{D}(V) \geq 0$ for all vector fields $V$. Then for any $A > 0$, there exist positive constants $C_1, C_2, C_3, C_4, C_5, C_6 > 0$ depending only on $n, T, g(0)$ and $A$ such that the following holds: We assume $H \leq H_1$ for $H_1 > 0$. For $s, t \in [0, T)$ with $s < t$, we suppose $t-s \leq AH_1^{-1}$ and $s \geq (t-s)/A$. Then we have

\[ G(x, t; y, s) \geq \frac{C_1}{(t-s)^{n/2}} \exp \left( - \frac{C_2 d_s(x, y)^2}{t-s} \right), \tag{1.3} \]
\[ G(x, t; y, s) \leq \frac{C_3}{(t-s)^{n/2}} \exp \left( - \frac{C_4 d_s(x, y)^2}{t-s} \right), \tag{1.4} \]
\[ |\nabla_x G|(x, t; y, s) \leq \frac{C_5}{(t-s)^{(n+1)/2}} \exp \left( - \frac{C_6 d_s(x, y)^2}{t-s} \right). \tag{1.5} \]

**Remark 1.2.** We give a remark for the dependence of the constants on $g(0)$. More precisely, they depend on the volume and Sobolev constant of $M$ with respect to $g(0)$, and the minimum of $H$ at $t = 0$, which is derived from the logarithmic Sobolev inequality of Fang-Zheng [12] (cf. [12, Theorem 1.1], Subsection 2.2). This remark can be applied throughout this paper.
2. Preliminaries

We recall some previous works for general geometric flow. Let \((M, g(t))_{t \in [0, T)}\) denote an \(n\)-dimensional compact manifold with a time-dependent metric with \(T < +\infty\).

2.1. Notations and basics. In the present subsection, we introduce some notations, and elementary facts that will be used frequently throughout this article.

We denote by \(d_t\) and \(m_t\) the Riemannian distance and Riemannian volume measure with respect to \(g(t)\), respectively. When \(t\) is clear from the context, we omit it, and write them as \(d\) and \(m\). Notice that the time derivative of \(dm\) is equal to \(-H dm\). For \(x_0 \in M, r_0 > 0\) and \(t_0 \in [0, T)\), we denote by \(B(x_0, r_0, t_0)\) the open ball of radius \(r_0\) centered at \(x_0\) with respect to \(g(t_0)\). Let \(Q^+(x_0, r_0, t_0)\) stand for the forward parabolic cube defined as

\[
Q^+(x_0, r_0, t_0) := \{(x, t) \in M \times [0, T) \mid x \in B(x_0, r_0, t), t \in [t_0, t_0 + r_0^2]\}.
\]

We begin with the following elementary fact (see \cite{12} Lemma 3.2):

**Proposition 2.1** \((\cite{12})\). Assume that \((M, g(t))_{t \in [0, T)}\) satisfies \(\mathcal{D}(V) \geq 0\) for all vector fields \(V\). Then we have

\[
H \geq -\frac{n}{2t}.
\]

**Remark 2.2.** We also mention the rescaling argument. For a positive \(r_0 > 0\), we will consider the parabolic rescaling

\[
\bar{g}(\tau) := r_0^{-2}g(r_0^2\tau) = r_0^{-2}g(t)
\]

with \(\tau := r_0^{-2}t\). We see that \(\bar{h} = h, \bar{H} = r_0^2H\) and \(\bar{\mathcal{D}}(V) = r_0^2\mathcal{D}(r_0^{-2}V)\), where \(\bar{h}, \bar{H}, \bar{\mathcal{D}}(V)\) are corresponding objects for \(\bar{g}(\tau)\). In particular, the super Ricci flow and the non-negativity of the Müller quantity are preserved under this rescaling.

2.2. Monotonicity of entropy. The monotonicity of Perelman’s \(\mathcal{W}\)-functional is one of the powerful tools in the Ricci flow theory \((\cite{28})\). It is well-known that such a monotonicity can be extended to general geometric flow under the non-negativity of the Müller quantity (see \cite{18} Theorem 3.1, \cite{13} Theorem 5.2, \cite{12} Lemma 3.1). In this subsection, we collect some results that can be derived from the monotonicity.

Fang-Zheng \cite{12} obtained a logarithmic Sobolev inequality by using the monotonicity of \(\mathcal{W}\)-functional, and derived the following Sobolev inequality (see \cite{12} Theorems 1.1, 1.3, and cf. \cite{31}, \cite{35}):

**Proposition 2.3** \((\cite{12})\). Assume that \((M, g(t))_{t \in [0, T)}\) satisfies \(\mathcal{D}(V) \geq 0\) for all vector fields \(V\). Then for all \(t \in [0, T)\) and \(u \in W^{1,2}(M)\) we have

\[
\left(\int_M |u|^{\frac{2n}{n-2}} \, dm\right)^{\frac{n-2}{n}} \leq C_1 \int_M \left(\nabla u^2 + \frac{H}{4} u^2\right) \, dm + C_2 \int_M u^2 \, dm,
\]

where \(C_1, C_2 > 0\) are positive constants depending only on \(n, T\) and \(g(0)\).

They further concluded that the following \(\kappa\)-noncollapsing estimate holds (see \cite{31} Theorem 6.1, \cite{12} Theorem 5.1):

**Theorem 2.4** \((\cite{12})\). Assume that \((M, g(t))_{t \in [0, T)}\) satisfies \(\mathcal{D}(V) \geq 0\) for all vector fields \(V\). For \(x \in M, r \in (0, \sqrt{T})\) and \(t \in [0, T)\), we assume \(H \leq r^{-2}\) on \(B(x, r, t)\). Then we have

\[
m(B(x, r, t)) \geq \kappa r^n,
\]

where \(\kappa > 0\) is a positive constant depending only on \(n, T\) and \(g(0)\).
For \( x, y \in M \) and \( s, t \in [0, T] \) with \( s < t \), we denote by \( G(x, t; y, s) \) the heat kernel for the heat equation; namely, for a fixed \((y, s) \in M \times [0, T]\), it solves
\[
(\partial_t - \Delta_x)G(\cdot, \cdot; y, s) = 0, \quad \lim_{t \searrow s} G(\cdot, t; y, s) = \delta_y.
\]
Notice that \( G(x, t; \cdot, \cdot) \) is the kernel for the conjugate heat equation; namely, for any \((x, t) \in M \times [0, T]\),
\[
(-\partial_s - \Delta_y + H)G(x, t; \cdot, \cdot) = 0, \quad \lim_{s \nearrow t} G(x, t; \cdot, s) = \delta_x.
\]
They also showed the following upper bound of Zhang type (see \cite{36} (1.5), \cite{12} Lemma 6.3):

**Proposition 2.5** (\cite{12}). Assume that \((M, g(t))_{t \in [0, T]}\) satisfies \(D(V) \geq 0\) for all vector fields \(V\). Then for all \(x, y \in M\) and \(s, t \in [0, T]\) with \(s < t\), we have
\[
G(x, t; y, s) \leq \frac{C}{(t - s)^{n/2}},
\]
where \(C > 0\) is a positive constant depending only on \(n, T\) and \(g(0)\).

### 2.3. Reduced geometry.

Perelman’s reduced geometry is also a crucial concept in the Ricci flow theory (\cite{28}). It is also well-known that the reduced geometry can be extended to general geometric flow under the assumption that the Müller quantity is non-negative (see e.g., \cite{26}, \cite{18}, \cite{32}, \cite{33}). Here we present some results concerning the reduced geometry.

Let us recall the notion of the reduced distance. Let \(x, y \in M\) and \(s, t \in [0, T]\) with \(s < t\). The \(L\)-distance \(L_{(x, t)}(y, s)\) from a space-time base point \((x, t)\) to \((y, s)\) is defined by
\[
L_{(x, t)}(y, s) := \inf_{\gamma} \int_s^t \sqrt{t - \xi} \left( H + \left| \frac{d\gamma}{d\xi} \right|^2 \right) d\xi,
\]
where the infimum is taken over all curves \(\gamma: [s, t] \rightarrow M\) with \(\gamma(s) = y\) and \(\gamma(t) = x\). The reduced distance \(\ell_{(x, t)}(y, s)\), and its modified one \(\overline{L}_{(x, t)}(y, s)\) are defined as
\[
\ell_{(x, t)}(y, s) := \frac{1}{2\sqrt{t - s}} L_{(x, t)}(y, s), \quad \overline{L}_{(x, t)}(y, s) := 4(t - s) \ell_{(x, t)}(y, s).
\]

It is well-known that the following fundamental inequality holds under the non-negativity of the Müller quantity (see \cite{26} Lemmas 5.2, 5.3, and also \cite{18} Theorem 3.4, \cite{32} Subsection 2.3, \cite{33}):

**Proposition 2.6** (\cite{26}, \cite{18}, \cite{32}, \cite{33}). Assume that \((M, g(t))_{t \in [0, T]}\) satisfies \(D(V) \geq 0\) for all vector fields \(V\). Then we have
\[
(-\partial_s + \Delta)\overline{L}_{(x, t)} \leq 2n
\]
in the barrier sense.

For the Ricci flow, this estimate (2.1) together with the maximum principle leads us to the uniform upper bound of the minimum of the reduced distance (see \cite{28}, \cite{10} Lemmas 7.48, 7.50). By the same argument, Proposition 2.6 gives us the following:

**Proposition 2.7.** Assume that \((M, g(t))_{t \in [0, T]}\) satisfies \(D(V) \geq 0\) for all vector fields \(V\). Then for all \(x \in M\) and \(s, t \in [0, T]\) with \(s < t\) we have
\[
\min_{y \in M} \ell_{(x, t)}(y, s) \leq \frac{n}{2}.
\]

We also mention the following relation between the reduced distance and heat kernel (see \cite{9} Lemma 2.4):
Proposition 2.8 ([2]). Assume that \((M, g(t))_{t \in [0,T]}\) satisfies \(\mathcal{D}(V) \geq 0\) for all vector fields \(V\). Then for all \(x, y \in M\) and \(s, t \in [0, T)\) with \(s < t\) we have

\[
G(x, t; y, s) \geq \frac{1}{(4\pi(t-s))^{n/2}} \exp \left(-\ell(x,t)(y,s)\right).
\]

2.4. Zhang type gradient estimates. For the super Ricci flow, one can derive the following gradient estimate of Zhang type (see [131] Theorem 3.1, [12] Lemma 6.5):

Theorem 2.9 ([12]). Let \((M, g(t))_{t \in [0,T]}\) be a super Ricci flow. For \(t_0 \in [0, T)\), let \(u \in C^\infty(M \times (t_0, T))\) be a positive solution to the heat equation. For \(t_1, t_2 \in (t_0, T)\) with \(t_1 < t_2\), we assume \(u \leq A \) for \(A > 0\) on \(M \times [t_1, t_2]\). Then for all \(x, y \in M\) we have

\[
\frac{|\nabla u|(x, t)}{u(x, t)} \leq \frac{1}{\sqrt{t-t_1}} \sqrt{\log \frac{A}{u(x,t)}}.
\]

Due to Theorem 2.9 we have the following Harnack inequality (see [36] [12] Lemma 6.5):

Corollary 2.10 ([12]). Let \((M, g(t))_{t \in [0,T]}\) be a super Ricci flow. For \(t_0 \in [0, T)\), let \(u \in C^\infty(M \times (t_0, T))\) be a positive solution to the heat equation. For \(t_1, t_2 \in (t_0, T)\) with \(t_1 < t_2\), we assume \(u \leq A \) for \(A > 0\) on \(M \times [t_1, t_2]\). Then for all \(x, y \in M\) we have

\[
u(y, t_2) \leq A^{1/2} u(x, t_2)^{1/2} \exp \left(\frac{d_t^2(x, y)^2}{4(t_2 - t_1)}\right).
\]

Using Propositions 2.5, 2.8 and Corollary 2.10 Fang-Zheng [12] have obtained the following lower Gaussian heat kernel estimate (see [36], [12] Lemma 6.6):

Theorem 2.11 ([12]). Let \((M, g(t))_{t \in [0,T]}\) be a super Ricci flow satisfying \(\mathcal{D}(V) \geq 0\) for all vector fields \(V\). For \(s, t \in [0, T)\) with \(s < t\), and for \(H_1 > 0\), we assume \(H \leq H_1\) on \(M \times [s,t]\). Then for all \(x, y \in M\) we have

\[
G(x, t; y, s) \geq \frac{C}{(t-s)^{n/2}} \exp \left(-\frac{d_t^2(x, y)^2}{t-s}\right),
\]

where \(C > 0\) is a positive constant depending only on \(n, T, g(0)\) and \((t-s)H_1\).

From Propositions 2.5, 2.8 and Corollary 2.10 they also concluded the following \(\kappa\)-noninflating theorem of Zhang type (see [36] Theorem 1.1, [12] Theorem 6.1):

Theorem 2.12 ([12]). Let \((M, g(t))_{t \in [0,T]}\) be a super Ricci flow satisfying \(\mathcal{D}(V) \geq 0\) for all vector fields \(V\). For \(x_0 \in M\), \(t_0 \in (0, T)\), \(r_0 \in (0, \sqrt{t_0})\) and \(A > 0\), we assume \(H \leq A/(t_0 - t)\) on \(B(x_0, r_0, t_0) \times [t_0 - \sqrt{r_0^2}, t_0]\). Then we have

\[
m(B(x_0, r_0, t_0)) \leq \kappa r_0^n,
\]

where \(\kappa > 0\) is a positive constant depending only on \(n, T, g(0)\) and \(A\).

2.5. Characterization of super Ricci flow. It is well-known that super Ricci flow can be characterized by a gradient estimate, logarithmic Sobolev inequality, and Poincaré inequality for the heat semigroup, which can be regarded as a time-dependent version of the equivalence of a lower Ricci curvature bound established by Bakry-Émery [11], Bakry-Ledoux [2] (see e.g., [17], [24], [16], [22], and also [25], [30] for other characterizations). In this subsection, we present some results that can be deduced from such a characterization.

Let \(x, y \in M\), and let \(s, t \in [0, T)\) with \(s < t\). The heat kernel measure \(\nu_{(x,t)}(y, s)\) is defined by

\[
d\nu_{(x,t)}(y, s) := G(x, t; y, s) \, dm_s(y).
\]

We possess the following logarithmic Sobolev inequality (see [24] Theorem 1.1, [16] Theorem 1.5], [22] Theorem 1.3], and cf. [17] Theorem 1.10 (2)):
Proposition 2.15. Let \((M, g(t))_{t \in [0,T]}\) be a super Ricci flow. Then for all \(u \in C^\infty(M)\) with \(\int_M u^2(y) \, d\nu(x,t)(y,s) = 1\) we have
\[
\int_M u^2 \log u^2(y) \, d\nu(x,t)(y,s) \leq 4(t-s) \int_M |\nabla u|^2(y) \, d\nu(x,t)(y,s).
\]

Theorem 2.13 together with the method of the proof of [17, Theorem 1.13] implies the following Gaussian concentration inequality (cf. [17, Theorem 1.13]):

Corollary 2.14. Let \((M, g(t))_{t \in [0,T]}\) be a super Ricci flow. Let \(\nu := \nu(x,t)(\cdot, s)\). Then for all \(\Omega_1, \Omega_2 \subset M\) we have
\[
\nu(\Omega_1) \nu(\Omega_2) \leq \exp\left(-\frac{d_s(\Omega_1, \Omega_2)^2}{8(t-s)}\right).
\]

Corollary 2.14 together with the method of the proof of [17, (1.20)] tells us the following average bound for the heat kernel (cf. [17, (1.20)]):

Proposition 2.15. Let \((M, g(t))_{t \in [0,T]}\) be a super Ricci flow. Let \(x, y \in M\), and let \(s, t \in [0, T]\) with \(s < t\). For \(c > 0\) we assume \(m(B(y, c\sqrt{t-s}, s)) \geq \kappa(t-s)^{n/2}\). Then
\[
\frac{1}{m(B(y, c\sqrt{t-s}, s))} \int_{B(y, c\sqrt{t-s}, s)} G(x, t; z, s) \, dm_{s}(z) \leq \left(\int_{B(x, c\sqrt{t-s}, s)} G(x, t; z, s) \, dm_{s}(z)\right)^{-1} \kappa^{-1} e^{c^2/2} \exp\left(-\frac{d_s(x, y)^2}{16(t-s)}\right).
\]

In [17], a similar inequality has been stated only for \(c = 1\). For the sake of our argument, we present a slightly general form. The method is completely same as that of [17, (1.20)].

3. Distance distortion estimate

In what follows, we denote by \((M, g(t))_{t \in [0,T]}\) an \(n\)-dimensional compact super Ricci flow with \(T < +\infty\) satisfying \(D(V) \geq 0\) for all vector fields \(V\). Unless otherwise stated, positive constants which appear in the proof will depend only on \(n, T\) and \(g(0)\) (cf. Remark [17, 2]).

The aim of this section is to prove a distance distortion estimate. We start with the following key lemma, which has been obtained by Bamler-Zhang [7] for Ricci flow (see [7, Lemma 3.1]):

Lemma 3.1. There exists a positive constant \(C_n > 0\) depending only on \(n\) such that the following holds: For \(t_0 \in [0, T]\), let \(u \in C^\infty(M \times (t_0, T))\) be a positive solution to the heat equation. For \(t_1, t_2 \in (t_0, T)\) with \(t_1 < t_2\), we assume \(u \leq A\) for \(A > 0\) on \(M \times [t_1, t_2]\). Then for all \(x \in M\) and \(t \in (t_1, t_2]\) we have
\[
\left(|\Delta u| + \frac{|
abla u|^2}{u} - AH\right)(x, t) \leq \frac{C_n A}{t - t_1}.
\]

Proof. We may assume \(A = 1\). We first estimate
\[
L_1 := -\Delta u + \frac{|
abla u|^2}{u} - H.
\]
Using the Bochner formula, we deduce
\[
(\partial_t - \Delta)\Delta u = \partial_t \Delta u - \Delta \partial_t u = 2\langle \nabla^2 u, h \rangle + 2 \div h(\nabla u) - \langle \nabla H, \nabla u \rangle,
\]
\[
(\partial_t - \Delta)\frac{|
abla u|^2}{u} = -\frac{2}{u} \left|\nabla^2 u - \frac{du \otimes du}{u}\right|^2 + \Ric(\nabla u, \nabla u) - h(\nabla u, \nabla u),
\]
\[
(\partial_t - \Delta)H = 2|h|^2 + D(0).
\]
Therefore, by \( u \leq 1, \text{Ric} \geq h \) and \( \mathcal{D}(V) \geq 0 \) we obtain

\[
(\partial_t - \Delta) L_1 = -\frac{2}{u} \left| \nabla^2 u - \frac{du \otimes du}{u} \right|^2 - 2\langle \nabla^2 u, h \rangle - 2|h|^2 \\
- 2 \text{div } h(\nabla u) + \langle \nabla H, \nabla u \rangle - \frac{2}{u} (\text{Ric}(\nabla u, \nabla u) - h(\nabla u, \nabla u)) - \mathcal{D}(0)
\]

\[
\leq -\left| \nabla^2 u - \frac{du \otimes du}{u} \right|^2 - 2\langle \nabla^2 u, h \rangle - 2|h|^2 \\
- 2 \text{div } h(\nabla u) + \langle \nabla H, \nabla u \rangle - (\text{Ric}(\nabla u, \nabla u) - h(\nabla u, \nabla u)) - \mathcal{D}(0)
\]

\[
= -\left| \nabla^2 u - \frac{du \otimes du}{u} + h \right|^2 - \left| \frac{du \otimes du}{u} + h \right|^2 + \frac{\langle \nabla u \rangle_4}{u^2} - \frac{1}{2}(\mathcal{D}(\nabla u) + \mathcal{D}(0))
\]

\[
\leq -\frac{1}{n} \left( \Delta u - \frac{\langle \nabla u \rangle_2}{u} + H \right)^2 + \frac{\langle \nabla u \rangle_4}{u^2} = -\frac{1}{n} L_1^2 + \frac{\langle \nabla u \rangle_4}{u^2}.
\]

Theorem 2.9 tells us that

\[
(3.2) \quad \frac{\langle \nabla u \rangle_2}{u} \leq \frac{u}{t - t_1} \log \frac{1}{u} \leq \frac{1}{e(t - t_1)}
\]

on \( M \times (t_1, t_2] \), and hence

\[
(\partial_t - \Delta) L_1 \leq -\frac{1}{n} L_1^2 + \frac{1}{e^2(t - t_1)^2}.
\]

Let \( C_{n,1} > 0 \) be a positive constant determined by

\[
\frac{C_{n,1} + e^{-2}}{C_{n,1}^2} = \frac{1}{n}
\]

depending only on \( n \). Note that

\[
(\partial_t - \Delta) \left( \frac{C_{n,1}}{t - t_1} \right) = -\frac{1}{n} \left( \frac{C_{n,1}}{t - t_1} \right)^2 + \frac{1}{e^2(t - t_1)^2}.
\]

It follows that

\[
(\partial_t - \Delta) \left( L_1 - \frac{C_{n,1}}{t - t_1} \right) \leq -\frac{1}{n} \left( L_1 + \frac{C_{n,1}}{t - t_1} \right) \left( L_1 - \frac{C_{n,1}}{t - t_1} \right).
\]

The maximum principle leads us to \( L_1 \leq C_{n,1}/(t - t_1) \).

We next estimate

\[
L_2 := \Delta u + \frac{\langle \nabla u \rangle_2}{u} - H.
\]
Using (3.1), $u \leq 1$, Ric $\geq h$ and $D(V) \geq 0$ again, one can calculate

\[
(\partial_t - \Delta) L_2 = -\frac{2}{u} \left| \nabla^2 u - \frac{du \otimes du}{u} \right|^2 + 2 \langle \nabla^2 u, h \rangle - 2|h|^2 \\
+ 2 \text{div } h(\nabla u) - \langle \nabla H, \nabla u \rangle - \frac{2}{u} \left( \text{Ric}(\nabla u, \nabla u) - h(\nabla u, \nabla u) \right) - D(0)
\]

\[
\leq - \left| \nabla^2 u - \frac{du \otimes du}{u} \right|^2 + 2 \langle \nabla^2 u, h \rangle - 2|h|^2 \\
- 2 \text{div } h(-\nabla u) + \langle \nabla H, (-\nabla u) \rangle - \left( \text{Ric}(\nabla u, \nabla u) - h(\nabla u, \nabla u) \right) - D(0)
\]

\[
= - \left| \nabla^2 u - \frac{du \otimes du}{u} \right|^2 - \left| \frac{du \otimes du}{u} - h \right|^2 + \frac{|\nabla u|^4}{u^2} + \frac{1}{2} \left( D(-\nabla u) + D(0) \right)
\]

\[
\leq - \frac{1}{n} \left( \Delta u - \frac{|\nabla u|^2}{u} - H \right)^2 + \frac{|\nabla u|^4}{u^2}
\]

\[
= - \frac{1}{2n} \left( L_2 - 4 \frac{|\nabla u|^2}{u} \right)^2 - \frac{1}{2n} L_2^2 + \left( 1 + \frac{4}{n} \right) \frac{|\nabla u|^4}{u^2} \leq - \frac{1}{2n} L_2^2 + \left( 1 + \frac{4}{n} \right) \frac{|\nabla u|^4}{u^2}.
\]

The estimate (3.2) implies

\[
(\partial_t - \Delta) L_2 \leq - \frac{1}{n} L_2^2 + \left( 1 + \frac{4}{n} \right) \frac{1}{e^2(t-t_1)^2}
\]

over $M \times (t_1, t_2]$. Let $C_{n,2} > 0$ be a constant determined by

\[
C_{n,2}^{-1} + \left( 1 + \frac{4}{n} \right) \frac{1}{e^2} C_{n,2}^{-2} = \frac{1}{2n},
\]

which depends only on $n$. We notice that

\[
(\partial_t - \Delta) \left( \frac{C_{n,2}}{t-t_1} \right) = - \frac{1}{2n} \left( \frac{C_{n,2}}{t-t_1} \right)^2 + \left( 1 + \frac{4}{n} \right) \frac{1}{e^2(t-t_1)^2},
\]

in particular,

\[
(\partial_t - \Delta) \left( L_2 - \frac{C_{n,2}}{t-t_1} \right) \leq - \frac{1}{2n} \left( L_2 + \frac{C_{n,2}}{t-t_1} \right) \left( L_2 - \frac{C_{n,2}}{t-t_1} \right).
\]

Due to the maximum principle, we obtain $L_2 \leq C_{n,2}/(t-t_1)$. This completes the proof. \(\square\)

We are now in a position to prove the following distance distortion estimate, which has been established by Bamler-Zhang \cite{7} for Ricci flow (see \cite[Theorem 1.1]{7}):

**Theorem 3.2.** There exists a positive constant $\alpha \in (0, 1)$ depending only on $n, T$ and $g(0)$ such that the following holds: For $t_0 \in (0, T)$ and $r_0 \in (0, \sqrt{t_0}]$, we assume $H \leq r_0^{-2}$. Let $x_0, y_0 \in M$ satisfy $d_{t_0}(x_0, y_0) \geq r_0$. Then for all $t \in [0, T]$ with $|t - t_0| \leq \alpha r_0^2$,

\[
ad_{t_0}(x_0, y_0) \leq d_t(x_0, y_0) \leq \alpha^{-1} d_{t_0}(x_0, y_0).
\]

**Proof.** By the argument of \cite[Theorem 1.1]{7}, it suffices to prove the upper bound for $d_t(x_0, y_0)$ only in the case of $d_{t_0}(x_0, y_0) \leq 2r_0$. By the parabolic rescaling, it is enough to show the claim when $r_0 = 1$ (see Remark 2.2). Then we have $t_0 \geq 1$ and $H \leq 1$. We further possess $H \geq -n$ on $M \times [t_0 - 1/2, t_0 + 1/2]$ by virtue of Proposition 2.1. Let $\gamma : [0, 1] \to M$ be a minimal geodesic from $x_0$ to $y_0$ with respect to $g(t_0)$. By Proposition 2.7

\[
(3.3) \quad \ell_{(x_0, t_0)} \left( z, t_0 - \frac{1}{2} \right) \leq \frac{n}{2}
\]

for some $z \in M$. We put $G(x, t) := G(x, t; z, t_0 - 1/2)$. 

Due to Proposition 2.5 on $M \times [t_0 - 1/4, t_0 + 1/4]$ it holds that
\begin{equation}
G(x, t) \leq \frac{C_1}{(t - (t_0 - 1/2))^{n/2}} \leq C_2.
\end{equation}

Thanks to Proposition 2.8 and (3.3),
\begin{equation}
G(x_0, t_0) \geq \frac{1}{(4\pi(t_0 - (t_0 - 1/2)))^{n/2}} \exp \left(-\ell_{(x_0, t_0)}(z, t_0 - 1/2)\right) \geq C_3.
\end{equation}

By (3.4), Corollary 2.10 (3.5) and the assumption $d_t(x_0, y_0) \leq 2$, for all $x \in \gamma([0, 1])$,
\begin{equation}
G(x, t) \geq C_2^{-1}G(x, t_0)^2 \exp \left(-\frac{d_t(x_0, x)^2}{2(t_0 - (t_0 - 1/4))}\right)
\end{equation}
\begin{equation}
\geq C_2^{-1}G^2_{\gamma} \exp (-2d_t(x_0, y_0)^2) \geq e^{-8}C_2^{-1}C_3^2 =: C_4.
\end{equation}

For the dimensional constant $C_n > 0$ obtained in Lemma 3.1 we set
\[\alpha_0 := \min \left\{ \frac{1}{8}, \frac{C_4}{4C_2(1 + 8C_n)} \right\}.\]

Using (3.6), (3.4), Lemma 3.1 $H \leq 1$, for all $x \in \gamma([0, 1])$ and $t \in [t_0 - \alpha_0, t_0 + \alpha_0]$, we see
\begin{equation}
G(x, t) \geq G(x, t_0) - \int_{t_0 - \alpha_0}^{t_0 + \alpha_0} \left| \partial_t G\right|(x, t) dt = G(x, t_0) - \int_{t_0 - \alpha_0}^{t_0 + \alpha_0} \left| \Delta G\right|(x, t) dt
\end{equation}
\begin{equation}
\geq C_4 - C_2 \int_{t_0 - \alpha_0}^{t_0 + \alpha_0} \left( H(x, t) + \frac{C_n}{t - (t_0 - 1/4)} \right) dt
\end{equation}
\begin{equation}
\geq C_4 - 2\alpha_0 C_2 \left( 1 + \frac{C_n}{1/4 - \alpha_0} \right) \geq C_4 - 2\alpha_0 C_2 (1 + 8C_n) \geq \frac{C_4}{2}.
\end{equation}

By (3.4), Corollary 2.10 (3.7), for all $s \in [0, 1], t \in [t_0 - \alpha_0/2, t_0 + \alpha_0/2]$ and $x \in B(\gamma(s), \sqrt{\alpha_0}, t)$ we have
\begin{equation}
G(x, t) \geq C_2^{-1}G(\gamma(s), t)^2 \exp \left(-\frac{d_t(x, \gamma(s))^2}{2(t - (t_0 - \alpha_0))}\right) \geq \frac{C_2^{-1}C_3^2}{4} e^{-1} =: C_5.
\end{equation}

Let $\{x_i\}_{i=1}^N$ denote a maximal $(2\sqrt{\alpha_0})$-separated set in $\gamma([0, 1])$ with respect to $g(t)$. The argument in the proof of [7, Theorem 1.1] tells us that $d_t(x_0, y_0) \leq 4N\sqrt{\alpha_0}$. On $(t_0 - 1/2, t_0 + 1/2]$, we see
\[\frac{d}{dt} \int_M G \, dm = \int_M \left( \Delta G - H \, G \right) \, dm \leq n \int_M G \, dm\]
since $H \geq -n$ on $M \times [t_0 - 1/2, t_0 + 1/2]$, and hence
\begin{equation}
\int_M G \, dm \leq \exp (n(t - (t_0 - 1/2))) \leq e^n.
\end{equation}

Combining (3.8) and Theorem 2.4 we see that if $t \in [t_0 - \alpha_0/2, t_0 + \alpha_0/2]$, then
\[e^n \geq \int_M G(\cdot, t) \, dm \geq \sum_{i=1}^N \int_{B(x_i, \sqrt{\alpha_0}, t)} G(\cdot, t) \, dm \geq N(k\alpha_0^{n/2})C_5.
\]

We conclude that for all $t \in [t_0 - \alpha_0/2, t_0 + \alpha_0/2]$, 
\[d_t(x_0, y_0) \leq \frac{4e^n}{\kappa\alpha_0^{(n-1)/2}C_5} < \frac{8e^n}{\kappa\alpha_0^{(n-1)/2}C_5} d_t(x_0, y_0).
\]

Setting
\[\alpha := \min \left\{ \frac{\alpha_0}{2}, \frac{\kappa\alpha_0^{(n-1)/2}C_5}{8e^n} \right\},\]
we arrive at the desired upper bound. Thus we complete the proof. □

4. Cutoﬀ Function

Based on Theorem 3.2, we construct the following cutoﬀ function, which has been obtained by Bamler-Zhang [7] for Ricci ﬂow (see [7] Theorem 1.3):

**Theorem 4.1.** There exists a constant ρ ∈ (0, 1) depending only on n, T and g(0) such that the following holds: Let t_0 ∈ (0, T) and r_0 ∈ (0, √t_0]. We assume H ≤ r_0^{-2}. Then for any x_0 ∈ M and τ ∈ (0, ρ^2r_0^2], there is φ ∈ C^∞(M × [t_0 − τ, t_0]) satisfying the following properties:

1. 0 ≤ φ < 1;
2. φ ≥ ρ on B(x_0, ρr_0, t_0) × [t_0 − τ, t_0];
3. φ = 0 on (M \ B(x_0, r_0, t_0)) × [t_0 − τ, t_0];
4. |∇φ| ≤ r_0^{-1} and |∂_t φ| + |Δφ| ≤ r_0^{-2}.

**Proof.** By the rescaling, we may assume r_0 = 1 (see Remark 2.2). We possess t_0 ≥ 1 and H ≤ 1. Moreover, H ≥ −n on M × [t_0 − 1/2, t_0] by Proposition 2.1. Let θ ∈ (0, 1/2), τ ∈ (0, θ/4), which will be determined later.

Due to Proposition 2.2, there exists z ∈ M such that

\[ \ell(x_0, t_0)(z, t_0 − \theta) \leq \frac{n}{2}. \]

Set \( G(x, t) := G(x, t; z, t_0 − \theta) \). Thanks to Proposition 2.8,

\[
G(x_0, t_0) \geq \frac{1}{(4\pi(t_0 − (t_0 − \theta)))^{n/2}} \exp\left(-\ell(x_0, t_0)(z, t_0 − \theta)\right)
\]

\[
\geq \frac{1}{(4\pi\theta)^{n/2}}e^{-n/2} =: C_n\theta^{-n/2},
\]

where \( C_n > 0 \) depends only on \( n \). We deﬁne

\[ Ω := \{ x ∈ M \mid G(x, t_0) > C_n\theta^{-n/2}/2 \}. \]

From (4.1) we derive \( x_0 ∈ Ω \). Let \( Ω_0 ⊂ Ω \) stand for the connected component of \( Ω \) containing \( x_0 \). We prove that for any suﬃciently small \( θ \), we have \( Ω_0 ⊂ B(x_0, 1, t_0) \) by contradiction. Let us assume that \( Ω_0 \) is not contained in \( B(x_0, 1, t_0) \). Then we can take a curve \( γ : [0, 1] → Ω_0 \) from \( x_0 \) to a point in \( ∂B(x_0, 1, t_0) \cap Ω_0 \). Let \( \{x_i\}_{i=1}^N \) be a maximal (2√θ)-separated set in \( γ([0, 1]) \) with respect to \( g(t_0) \). The argument in the proof of [7] Theorem 1.3 tells us that

\[
N ≥ \frac{1}{4\sqrt{θ}}.
\]

Proposition 2.3 tells us that on \( M × [t_0 − θ/2, t_0] \) we see

\[
G(x, t) ≤ \frac{C_1}{(t − (t_0 − θ))^{n/2}} ≤ C_2θ^{-n/2}.
\]

Due to (4.3), Corollary 2.10 for all \( y ∈ Ω_0 \) and \( z ∈ B(y, \sqrt{θ}, t_0) \) we have

\[
G(z, t_0) ≥ C_2^{-1}θ^{n/2}G(y, t_0)^2 \exp\left(-\frac{d_i^2(y, z)^2}{2θ}\right) ≥ \frac{e^{-1/2}}{2}C_2^{-1}C_nθ^{-n/2} =: C_3θ^{-n/2}.
\]

On the other hand, on \( (t_0 − θ, t_0] \) we see

\[
\frac{d}{dt} \int_M G \, dm = \int_M (ΔG − H \, G) \, dm ≤ n \int_M G \, dm
\]
since $H \geq -n$ on $M \times [t_0 - \theta, t_0]$, and hence
\begin{equation}
(4.5) \quad \int_M G \, dm \leq \exp \left( n(t - (t_0 - \theta)) \right) \leq e^{n\theta} \leq e^{n/2}.
\end{equation}
Combining (4.2), (4.4), (4.5) and Theorem 2.4, we obtain
\[ e^{n/2} \geq \int_M G(\cdot, t_0) \, dm \geq \sum_{i=1}^N \int_{B(x_i, \sqrt{\theta} \, t_0)} G(\cdot, t_0) \, dm \geq N(C_3 \theta^{-n/2}) \kappa \theta^{n/2} = N \kappa C_3 \geq \frac{\kappa C_3}{4\sqrt{\theta}}. \]
If $\theta < 16^{-1}e^{-n} \kappa^2 C_3^2$, then this is a contradiction. Hereafter, we fix $\theta \in (0, 16^{-1}e^{-n} \kappa^2 C_3^2)$ such that $\Omega_0 \subset B(x_0, 1, t_0)$.

We now use (4.3) and Lemma 3.1. On $M \times [t_0 - \tau, t_0]$, we possess
\[ G(x, t) < C_0 \theta^{-n/2} = 0.5C_2 \theta^{-n/2} + 0.1C_0 \theta^{-n/2} = 6C_0 \theta^{-n/2}. \]
Now, we define a function $\psi \in C^\infty(M \times [t_0 - \tau, t_0])$ by
\[ \hat{\psi}(x, t) := \begin{cases} \max\{G(x, t) - 0.6C_0 \theta^{-n/2} & \text{on } \Omega_0 \times [t_0 - \tau, t_0], \\ 0 & \text{on } (M \setminus \Omega_0) \times [t_0 - \tau, t_0], \end{cases} \]
which is compactly supported on $\Omega_0 \times [t_0 - \tau, t_0]$. This function enjoys the following properties:
\begin{enumerate}
\item $0 \leq \hat{\psi} < C_5 \theta^{-n/2}$;
\item $\hat{\psi}(x_0, t_0) \geq 0.4C_0 \theta^{-n/2}$;
\item $|\nabla \hat{\psi}| \leq 2C_5 \theta^{-n/2-1/2}$;
\item $|\partial_t \hat{\psi} + |\Delta \hat{\psi}| \leq 2C_5 \theta^{-n/2-1}.$
\end{enumerate}
The first inequality follows from (4.3) and $C_2 < C_5$. The second one is a consequence of $x_0 \in \Omega$. By (4.3) and Theorem 2.9 on $M \times [t_0 - \tau, t_0],$
\[ |\nabla G(x, t)| \leq \frac{1}{\sqrt{t - (t_0 - \theta/2)}} \sqrt{\log \frac{C_2 \theta^{-n/2}}{G(x, t)} G(x, t) \leq 2C_2 \theta^{-n/2-1/2} < 2C_5 \theta^{-n/2-1/2}}, \]
and this implies the third one. The fourth one can be derived from (4.6). If we further define $\tilde{\phi} := 32^{-1}C_5^{-1} \theta^{n/2} + \hat{\psi}$, then it satisfies the following properties:
\begin{enumerate}
\item $0 \leq \tilde{\phi} < 1$;
\item $\tilde{\phi}(x_0, t_0) \geq C_6 \theta$;
\item $|\nabla \tilde{\phi}| \leq 1/8$;
\item $|\partial_t \tilde{\phi} + |\Delta \tilde{\phi}| \leq 1/8$.
\end{enumerate}
Once we obtain this function, we can conclude the desired assertion by the same argument as in the proof of [7, Theorem 1.3]. We complete the proof. \hfill \Box
5. Mean value inequality

In the present section, we produce a mean value inequality for conjugate heat equation. To do so, we first yield the following integral estimate based on the Moser iteration argument, which has been obtained by Bamler-Zhang [7] for Ricci flow (see [7, Lemma 4.1]):

**Lemma 5.1.** Let \( p \geq 2 \). Then there are positive constants \( \beta \in (0,1) \) and \( C > 0 \) depending only on \( n, T, g(0) \) and \( p \) such that the following holds: For \( t_0 \in (0, T) \) and \( r_0 \in (0, \sqrt{t_0}) \), we assume \( H \leq r_0^{-2} \). Let \( u \in C^\infty(M \times [t_0, t_0 + r_0^2]) \) denote a positive solution to the conjugate heat equation. Then for all \( x_0 \in M \) we have

\[
\left( \int_{Q^+(x_0, r_0, t_0)} u^p \, dm \, dt \right)^{1/p} \leq C \int_{t_0}^{(n+2)(p-2)/2p} \left( \int_{Q^+(x_0, r_0, t_0)} u^2 \, dm \, dt \right)^{1/2}.
\]

**Proof.** By the rescaling, we may assume \( r_0 = 1 \) (see Remark 2.2). It holds that \( t_0 \geq 1 \) and \( H \leq 1 \). Moreover, \( H \geq -n \) on \( M \times (t_0 - 1/2, t_0 + 1) \) in view of Proposition 2.1. Let \( \alpha, \rho \in (0, 1) \) be the constants obtained in Theorems 3.2 and 4.1 respectively. We define

\[ \theta := \alpha \in (0, 1). \]

Note that \((\rho \theta)^2 \in (0, \alpha)\). By Theorem 3.2 we have

\[ B(x_0, \theta, t_1) \subset B(x_0, 1, t_2) \]

for all \( t_1, t_2 \in [t_0, t_0 + (\rho \theta)^2] \). We further set

\[ \sigma := \frac{\alpha \rho \theta}{\sqrt{2}} \in (0, \rho \theta). \]

Notice that \( 2\sigma^2 \in (0, \alpha(\rho \theta)^2) \). Using Theorem 3.2 again, we obtain

\[ B(x_0, \sigma, t_1) \subset B(x_0, \rho \theta, t_2) \]

for all \( t_1, t_2 \in [t_0, t_0 + 2\sigma^2] \). With the help of Theorem 4.1, there is \( \phi \in C^\infty(M \times [t_0, t_0 + 2\sigma^2]) \) such that the following holds:

1. \( 0 \leq \phi < 1; \)
2. \( \phi \geq \rho \) on \( B' \times [t_0, t_0 + 2\sigma^2]; \)
3. \( \phi = 0 \) on \( (M \setminus B) \times [t_0, t_0 + 2\sigma^2]; \)
4. \( |\nabla \phi| \leq \theta^{-1} \) and \( |\partial_t \phi| \leq \theta^{-2}, \)

where \( B := B(x_0, \theta, t_0 + 2\sigma^2) \) and \( B' := B(x_0, \rho \theta, t_0 + 2\sigma^2) \). By (5.1) and (5.2), we possess

\[ Q^+(x_0, \sigma, t_0) \subset B' \times [t_0, t_0 + 2\sigma^2] \subset B \times [t_0, t_0 + 2\sigma^2] \subset Q^+(x_0, 1, t_0). \]

Let \( \eta \in C^\infty([t_0, t_0 + 2\sigma^2]) \) be a function satisfying the following properties:

1. \( 0 \leq \eta \leq 1; \)
2. \( \eta \equiv 1 \) on \( [t_0, t_0 + \sigma^2]; \)
3. \( \eta(t_0 + 2\sigma^2) = 0; \)
4. \( |\eta'| \leq C_1. \)

Then we define \( \psi(x, t) := \eta(t)\phi(x, t) \). Note that \( \psi \geq \rho \) on \( Q^+(x_0, \sigma, t_0) \).

Let \( p \geq 2 \). Since \( u \) is a solution to the conjugate heat equation, we see

\[
\partial_t u^{p/2} + \Delta u^{p/2} - \frac{p}{2} Hu^{p/2} = \frac{p}{2} \left( \frac{p}{2} - 1 \right) u^{p/2 - 2} |\nabla u|^2 \geq 0.
\]
On \([t_0, t_0 + 2\sigma^2]\), it holds that
\[
\frac{d}{dt} \int_M u^p \psi^2 \, dm = 2 \int_M (\partial_t u^{p/2}) u^{p/2} \psi^2 \, dm + 2 \int_M u^p (\partial_t \psi) \psi \, dm - \int_M u^p \psi^2 H \, dm \\
\geq 2 \int_M \left( -\Delta u^{p/2} + \frac{p}{2} H u^{p/2} \right) u^{p/2} \psi^2 \, dm - C_2 \int_B u^p \, dm \\
\geq 2 \int_M \langle \nabla u^{p/2}, \psi \nabla (u^{p/2} \psi) + (u^{p/2} \psi) \nabla \psi \rangle \, dm - C_3 \int_B u^p \, dm \\
= 2 \int_M \langle \nabla (u^{p/2} \psi) - u^{p/2} \nabla \psi, \nabla (u^{p/2} \psi) + u^{p/2} \nabla \psi \rangle \, dm - C_3 \int_B u^p \, dm \\
= 2 \int_M |\nabla (u^{p/2} \psi)|^2 \, dm - 2 \int_M u^p |\nabla \psi|^2 \, dm - C_3 \int_B u^p \, dm \\
\geq 2 \int_M |\nabla (u^{p/2} \psi)|^2 \, dm - C_5 \int_B u^p \, dm.
\]
From Proposition 2.3, we conclude
\[
\frac{d}{dt} \int_M u^p \psi^2 \, dm \geq C_4 \left( \int_M u^{\frac{pm}{n+2}} \psi^{\frac{2n}{n+2}} \, dm \right)^{\frac{n-2}{n}} - C_5 \int_B u^p \, dm.
\]
Let us integrate this inequality from a fixed \(t_1 \in [t_0, t_0 + 2\sigma^2]\) to \(t_0 + 2\sigma^2\). We deduce
\[
- \int_{B'} u^p(\cdot, t_1) \psi^2(\cdot, t_1) \, dm_{t_1} \\
\geq - \int_M u^p(\cdot, t_1) \psi^2(\cdot, t_1) \, dm_{t_1} \\
\geq C_4 \int_{t_1}^{t_0+2\sigma^2} \left( \int_M u^{\frac{pm}{n+2}} \psi^{\frac{2n}{n+2}} \, dm \right)^{\frac{n-2}{n}} \, dt - C_5 \int_{t_1}^{t_0+2\sigma^2} \int_B u^p \, dm \, dt.
\]
This leads us to
\[
\sup_{t \in [t_0, t_0 + 2\sigma^2]} \int_{B'} u^p(\cdot, t) \psi^2(\cdot, t) \, dm_t \leq C_6 \int_{t_0}^{t_0+2\sigma^2} \int_B u^p \, dm \, dt, \\
\int_{t_0}^{t_0+2\sigma^2} \left( \int_{B'} u^{\frac{pm}{n+2}} \psi^{\frac{2n}{n+2}} \, dm \right)^{\frac{n-2}{n}} \, dt \leq C_7 \int_{t_0}^{t_0+2\sigma^2} \int_B u^p \, dm \, dt.
\]
By the Hölder inequality,
\[
\int_{t_0}^{t_0+2\sigma^2} \int_{B'} u^p(1+\frac{2}{n}) \psi^2(1+\frac{2}{n}) \, dm \, dt \\
= \int_{t_0}^{t_0+2\sigma^2} \int_{B'} \left( u^p \psi^2 \right) \left( u^{\frac{2}{n}} \psi^{\frac{2}{n}} \right) \, dm \, dt \\
\leq \int_{t_0}^{t_0+2\sigma^2} \left( \int_{B'} u^{\frac{pm}{n+2}} \psi^{\frac{2n}{n+2}} \, dm \right)^{\frac{n-2}{n}} \left( \int_{B'} u^p \psi^2 \, dm \right)^{\frac{2}{n}} \, dt \\
\leq \left[ \int_{t_0}^{t_0+2\sigma^2} \left( \int_{B'} u^{\frac{pm}{n+2}} \psi^{\frac{2n}{n+2}} \, dm \right)^{\frac{n-2}{n}} \, dt \right]^{\frac{n-2}{n}} \left[ C_6 \int_{t_0}^{t_0+2\sigma^2} \int_B u^p \, dm \, dt \right]^{\frac{2}{n}} \\
\leq \left( C_8 \int_{t_0}^{t_0+2\sigma^2} \int_B u^p \, dm \, dt \right)^{1+\frac{2}{n}}.
\]
In view of (5.3) and \( \psi \geq \rho \) on \( Q^+(x_0, \sigma, t_0) \), we arrive at
\[
\left( \int_{Q^+(x_0, \sigma, t_0)} u^{p(1+\frac{2}{\theta})} \, dm \, dt \right)^{1/p(1+\frac{2}{\theta})} \leq C_9 \left( \int_{Q^+(x_0, 1, t_0)} u^p \, dm \, dt \right)^{1/p}.
\]
Once we obtain this estimate, we can conclude the desired one by the same argument as in the proof of [7, Lemma 4.2].

We give a proof of the following mean value inequality, which has been formulated by Bamler-Zhang [7] for Ricci flow (see [7, Lemma 4.2]):

**Theorem 5.2.** There exist positive constants \( \gamma \in (0, 1) \) and \( C > 0 \) depending only on \( n, T \) and \( q(0) \) such that the following holds: For \( t_0 \in (0, T) \) and \( r_0 \in (0, \sqrt{T_0}] \), we assume \( H \leq r_0^{-2} \).

Let \( u \in C^\infty(M \times [t_0, t_0 + r_0^2]) \) be a positive solution to the conjugate heat equation. Then for all \( x_0 \in M \) we have
\[
\sup_{Q^+(x_0, r_0 \gamma, t_0)} u^2 \leq \frac{C}{\gamma^{2n+2}} \int_{Q^+(x_0, r_0, t_0)} u^2 \, dm \, dt.
\]

**Proof.** By the rescaling, we may assume \( r_0 = 1 \) (see Remark 2.2). We have \( t_0 \geq 1 \) and \( H \leq 1 \). Furthermore, we have \( H \geq -n \) on \( M \times [t_0 - 1/2, t_0 + 1] \) by Proposition 2.1. Let \( \alpha, \rho \in (0, 1) \) be the constants obtained in Theorems 3.2 and 1.1 respectively. Let \( \beta \in (0, 1) \) be the constant obtained in Lemma 5.1 for a fixed \( p > n + 2 \). We define
\[
\theta := \alpha \beta \in (0, 1).
\]

Note that \((\rho \theta)^2 \in (0, \alpha \beta^2)\). By Theorem 3.2, it holds that
\[
(5.4) \quad B(x_0, \theta, t_1) \subset B(x_0, \beta, t_2)
\]
for all \( t_1, t_2 \in [t_0, t_0 + (\rho \theta)^2] \). We further define
\[
\gamma := \frac{\alpha \rho \theta}{\sqrt{2}} \in (0, \rho \theta).
\]

Notice that \( 2 \gamma^2 \in (0, \alpha (\rho \theta)^2) \). In virtue of Theorem 3.2, we possess
\[
(5.5) \quad B(x_0, \gamma, t_1) \subset B(x_0, \rho \theta, t_2)
\]
for all \( t_1, t_2 \in [t_0, t_0 + 2 \gamma^2] \). By Theorem 4.1, there is \( \phi \in C^\infty(M \times [t_0, t_0 + 2 \gamma^2]) \) such that the following holds:

(1) \( 0 \leq \phi < 1 \);
(2) \( \phi \geq \rho \) on \( B' \times [t_0, t_0 + 2 \gamma^2] \);
(3) \( \phi = 0 \) on \( (M \setminus B) \times [t_0, t_0 + 2 \gamma^2] \);
(4) \( |\nabla \phi| \leq \theta^{-1} \) and \( |\partial_t \phi| + |\Delta \phi| \leq \theta^{-2} \),

where \( B := B(x_0, \theta, t_0 + 2 \gamma^2) \) and \( B' := B(x_0, \rho \theta, t_0 + 2 \gamma^2) \). From (5.4) and (5.5), we deduce
\[
(5.6) \quad Q^+(x_0, \gamma, t_0) \subset B' \times [t_0, t_0 + 2 \gamma^2] \subset B \times [t_0, t_0 + 2 \gamma^2] \subset Q^+(x_0, \beta, t_0).
\]

Let \( \eta \in C^\infty([t_0, t_0 + 2 \gamma^2]) \) be a function satisfying the following properties:

(1) \( 0 \leq \eta \leq 1 \);
(2) \( \eta \equiv 1 \) on \( [t_0, t_0 + \gamma^2] \);
(3) \( \eta(t_0 + 2 \gamma^2) = 0 \);
(4) \( |\eta'| \leq C_1 \).
Then we define \( \psi(x, t) := \eta(t)\phi(x, t) \). In view of (5.6), we notice \( \psi \geq \rho \) on \( Q^+(x_0, \gamma, t_0) \). For a fixed \( (x, t) \in Q^+(x_0, \gamma, t_0) \), we set \( G(z, t) := G(z, t; x, t) \). Then

\[
(u\psi)(x, t) = -\int_{t}^{t_0 + 2\gamma^2} \int_M G(u\Delta\psi + u\partial_t\psi + 2\langle \nabla u, \nabla\psi \rangle) \, dm \, dl.
\]

Here we used

\[
\partial_t(u\psi) + \Delta(u\psi) - H(u\psi) = u\Delta\psi + u\partial_t\psi + 2\langle \nabla u, \nabla\psi \rangle,
\]

which is a consequence of the fact that \( u \) is a solution to the conjugate heat equation. Integration by parts yields

\[
(u\psi)(x, t) = -\int_{t}^{t_0 + 2\gamma^2} \int_M G(-\Delta\psi + \partial_t\psi) + 2u\langle \nabla G, \nabla\psi \rangle \, dm \, dl.
\]

From the properties of \( \psi \) and (5.6), it follows that

\[
(u\psi)(x, t) \leq C_2 \int_{t}^{t_0 + 2\gamma^2} \int_B (G \psi + u|\nabla G|) \, dm \, dl =: C_2(I_1 + I_2).
\]

We first estimate \( I_1 \). By the Hölder inequality, (5.6) and Lemma 5.1

\[
I_1 \leq \left( \int_{t}^{t_0 + 2\gamma^2} \int_B G^q \, dm \, dl \right)^{1/q} \left( \int_{t}^{t_0 + 2\gamma^2} \int_B u^p \, dm \, dl \right)^{1/p}
\]

\[
\leq \left( \int_{t}^{t_0 + 2\gamma^2} \int_B G^q \, dm \, dl \right)^{1/q} \left( \int_{Q^+(x_0, \beta, t_0)} u^p \, dm \, dl \right)^{1/p}
\]

\[
\leq C_3 \left( \int_{t}^{t_0 + 2\gamma^2} \int_B G^q \, dm \, dl \right)^{1/q} \| u \|_{L^2(Q^+(x_0, 1, t_0))}
\]

for \( q = p/(p - 1) \). By Proposition 2.5 for all \( l \in (t, t_0 + 2\gamma^2] \) we have

\[
G(z, l) \leq \frac{C_4}{(l - t)^{n/2}}.
\]

We also see

\[
\frac{d}{dl} \int_M G \, dm = -\int_M H \, G \, dm \leq n \int_M G \, dm
\]

since \( H \geq -n \) on \( M \times [t_0 - 1/2, t_0 + 1] \); in particular, that for all \( l \in (t, t_0 + 2\gamma^2] \),

\[
\int_M G \, dm \leq \exp(n(l - t)) \leq C_5.
\]

Combining (5.8), (5.9) and (5.10), we obtain

\[
I_1 \leq C_6 \left( \int_{t}^{t_0 + 2\gamma^2} \int_B \frac{1}{(l - t)^{(q - 1)n/2}} \, dm \, dl \right)^{1/q} \| u \|_{L^2(Q^+(x_0, 1, t_0))}
\]

\[
\leq C_7 \left( \int_{t}^{t_0 + 2\gamma^2} \int_B \frac{1}{(l - t)^{(q - 1)n/2}} \, dl \right)^{1/q} \| u \|_{L^2(Q^+(x_0, 1, t_0))}.
\]

Since \( p > n + 2 \), we have \( (q - 1)n/2 < 1 \); in particular,

\[
I_1 \leq C_8 \| u \|_{L^2(Q^+(x_0, 1, t_0))}.
\]

We next derive a bound of \( I_2 \). The estimate (5.9) implies that for each \( l \in (t, t_0 + 2\gamma^2] \),

\[
A_l := \sup_{M \times [t_0 + 2\gamma^2, l]} G \leq \frac{C_9}{(l - t)^{n/2}}.
\]
Using Theorem 2.9, we possess
\[ \frac{\|\nabla G\|^2}{G^2}(z, l) \leq \frac{2}{l - t} \log \frac{A_l}{G(z, l)}. \]
Since \( G(z, l)/A_l \leq 1 \), we have
\[ |\nabla G|^q \leq G \frac{C_{10}A_l^{q-1}}{(l - t)^{q/2}} \left( \frac{G}{A_l} \right)^{q-1} \left( \log \frac{A_l}{G} \right)^{q/2} \leq G \frac{C_{11}A_l^{q-1}}{(l - t)^{q/2}} \leq C_{12} \frac{(l - t)^{(n+1)q-n/2}}{G} \]
at \((z, l)\), where we used a fact that a function \( \varphi(\xi) := \xi^{q-1}(1/\xi)^{q/2} \) takes its maximum value \( e^{-q/2}(q/2(q-1))^{q/2} \) at \( \xi = e^{-q/2(q-1)} \) on \((0, 1]\). Using the Hölder inequality, this gradient bound, (5.10), Lemma 5.1 with (5.6), we conclude
\[ I_2 \leq \left( \int_t^{\rho + 2} \left( \int_B |\nabla G|^q \, dm \, dt \right)^{1/q} \left( \int_B u \, dm \, dt \right)^{1/p} \right)^{1/p} \leq C_{13} \left( \int_t^{\rho + 2} \frac{1}{(l - t)^{(n+1)q-n/2}} \, dt \right)^{1/q} \|u\|_{L^2(Q+(x_0, t_0))}. \]
From \( p > n + 2 \), we deduce ((n + 1)q - n)/2 < 1, and hence
\[ (u\psi)(x, t) \leq C_{15} \|u\|_{L^2(Q+(x_0, t_0))}. \]
Combining (5.7), (5.11), (5.12), we arrive at
\[ \theta := \alpha^{-1}A \in (0, 1], \quad B := \alpha^{-1}A^{-1/2}, \quad r_0 := A^{-1/2}\sqrt{l - s}. \]
It holds that
\[ r_0 \leq \sqrt{s} \leq \sqrt{t}, \quad H \leq H_1 \leq \frac{A}{l - s} = r_0^2, \quad t - s \leq \theta^{-1}(t - s) \leq \alpha r_0^2, \quad d_l(x, y) \geq \alpha^{-1}r_0 \geq r_0, \]
and hence one can apply Theorem 3.2. It follows that
\[ d_l(x, y) \geq \alpha d_l(x, y) \geq \alpha A^{1/2}d_l(x, y) = B^{-1}d_l(x, y), \]
which is a desired estimate.

6. GAUSSIAN HEAT KERNEL ESTIMATE

In this section, we give a proof of Theorem 1.1. First, we prove the lower bound (1.3). To do so, we prepare the following lemma:

**Lemma 6.1.** For any \( A > 0 \), there is a constant \( B > 1 \) depending only on \( n, T, g(0) \) and \( A \) such that the following holds: We assume \( H \leq H_1 \) for \( H_1 > 0 \). For \( s, t \in [0, T] \) with \( s < t \), we suppose \( t - s \leq A H_1^{-1} \) and \( s \geq (t - s)/A \). If \( d_l(x, y) \geq B\sqrt{t - s} \), then \( d_s(x, y) \geq B^{-1}d_l(x, y) \).

**Proof.** Let \( \alpha \in (0, 1) \) be the constant obtained in Theorem 3.2. We first consider the case of \( A \in (0, \alpha) \). We set
\[ \theta := \alpha^{-1}A \in (0, 1], \quad B := \alpha^{-1}A^{-1/2}, \quad r_0 := A^{-1/2}\sqrt{l - s}. \]
It holds that
\[ r_0 \leq \sqrt{s} \leq \sqrt{t}, \quad H \leq H_1 \leq \frac{A}{l - s} = r_0^2, \quad t - s \leq \theta^{-1}(t - s) \leq \alpha r_0^2, \quad d_l(x, y) \geq \alpha^{-1}r_0 \geq r_0, \]
and hence one can apply Theorem 3.2. It follows that
\[ d_s(x, y) \geq \alpha d_l(x, y) \geq \alpha A^{1/2}d_l(x, y) = B^{-1}d_l(x, y), \]
which is a desired estimate.
We next consider the remaining case of $A \in (\alpha, \infty)$. We put
\[ \theta := \alpha^{-1}A \in (1, \infty), \quad B := \alpha^{-\theta+2}, \quad r_0 := A^{-1/2}\sqrt{t-s}. \]
Let $N \in \mathbb{N}$ be the integer part of $\theta$, and let $s = \tau_{N+1} \leq \tau_N < \cdots < \tau_1 < \tau_0 = t$ be a division of $[s,t]$ such that $\tau_i - \tau_{i+1} = \theta^{-1}(t-s)$ for $i = 0, \ldots, N-1$. We see $\tau_N - \tau_{N+1} \in [0, \theta^{-1}(t-s))$.

We possess
\[ r_0 \leq \sqrt{s} \leq \sqrt{\tau_i} \leq \sqrt{t}, \quad H \leq H_1 \leq \frac{A}{t-s} = r_0^2, \quad \tau_i - \tau_{i+1} \leq \theta^{-1}(t-s) = \alpha r_0^2 \]
for all $i = 0, \ldots, N$. In view of
\[ B \geq \alpha^{-\theta-1}A^{-1/2} \geq \alpha^{-N-1}A^{-1/2}, \]
we see
\[ d_{r_0}(x,y) = d_t(x,y) \geq B\sqrt{t-s} \geq \alpha^{-N-1}A^{-1/2}\sqrt{t-s} = \alpha^{-N-1}r_0 \geq r_0, \]
and hence Theorem $3.2$ implies
\[ d_{\tau_i}(x,y) \geq \alpha d_{r_0}(x,y) \geq \alpha^{-N}r_0 \geq r_0. \]

Using Theorem $3.2$ again, we obtain
\[ d_{\tau_2}(x,y) \geq \alpha d_{\tau_1}(x,y) \geq \alpha^{-N+1}r_0 \geq r_0. \]

By repeating this procedure, we arrive at
\[ d_{\tau_i}(x,y) \geq \alpha d_{\tau_{i-1}}(x,y) \geq \alpha^{-N+(i-1)}r_0 \geq r_0 \]
for all $i = 1, \ldots, N+1$. Therefore,
\[
\begin{align*}
    d_s(x,y) &= d_{\tau_{N+1}}(x,y) \\
    &\geq \alpha d_{\tau_N}(x,y) \geq \alpha^2 d_{\tau_{N-1}}(x,y) \geq \cdots \geq \alpha^{N+1}d_t(x,y) \\
    &\geq \alpha^{\theta+2}d_t(x,y) \geq B^{-1}\sqrt{t-s}.
\end{align*}
\]
This proves the lemma.

Let us show the lower bound $(1.3)$.

**Proposition 6.2.** For any $A > 0$, there exist positive constants $C_1, C_2 > 0$ depending only on $n, T, g(0)$ and $A$ such that the following holds: We assume $H \leq H_1$ for $H_1 > 0$. For $s, t \in [0,T)$ with $s < t$, we suppose $t-s \leq AH_1^{-1}$ and $s \geq (t-s)/A$. Then we have
\[
    G(x, t; y, s) \geq \frac{C_1}{(t-s)^{n/2}} \exp\left(-\frac{C_2 d_s(x, y)^2}{t-s}\right).
\]

**Proof.** We possess
\[
    H \leq H_1 \leq \frac{A}{t-s}.
\]
Furthermore, in view of Lemma $2.1$
\[
    -\frac{nA}{2(t-s)} \leq -\frac{n}{2s} \leq H
\]
on $M \times [s,t]$.

Theorem $2.11$ together with $(6.1)$ tells us that
\[
    G(x, t; y, s) \geq \frac{C}{(t-s)^{n/2}} \exp\left(-\frac{d_t(x, y)^2}{t-s}\right).
\]
Let $B > 1$ be the constant obtained in Lemma 6.1. If $d_t(x, y) \geq B\sqrt{t - s}$, then $d_s(x, y) \geq B^{-1}d_t(x, y)$. From (6.3) it follows that

$$G(x, t; y, s) \geq \frac{C}{(t - s)^{n/2}} \exp \left( -\frac{d_t(x, y)^2}{t - s} \right) \geq \frac{C}{(t - s)^{n/2}} \exp \left( -\frac{B^2d_s(x, y)^2}{t - s} \right).$$

On the other hand, if $d_t(x, y) < B\sqrt{t - s}$, then (6.3) also leads to

$$G(x, t; y, s) \geq \frac{C}{(t - s)^{n/2}} \exp \left( -\frac{d_t(x, y)^2}{t - s} \right) \geq \frac{Ce^{-B^2}}{(t - s)^{n/2}} \geq \frac{Ce^{-B^2}}{(t - s)^{n/2}} \exp \left( -\frac{d_t(x, y)^2}{t - s} \right).$$

Thus, we complete the proof.

We next prove the upper bound (1.4). Similarly to Lemma 6.1 we prepare the following:

**Lemma 6.3.** For any $A > 0$, there is a constant $B > 1$ with $B^{-1}A^{1/2} < 1$ depending only on $n, T, g(0)$ and $A$ such that the following holds: Assume $H \leq H_1$ for $H_1 > 0$. For $s, t \in [0, T)$ with $s < t$, we suppose $t - s \leq AH^{-1}$ and $s \geq (t - s)/A$. Assume $d_s(x, y) \geq B\sqrt{t - s}$. Then for all $l_1, l_2 \in [s, t]$ we have

$$B^{-1}d_{l_1}(x, y) \leq d_{l_2}(x, y) \leq B d_{l_1}(x, y).$$

**Proof.** Let $\alpha \in (0, 1)$ be the constant obtained in Theorem 3.2. We begin with the case of $A \in (0, \alpha]$. Similarly to the proof of Lemma 6.1 we put

$$\theta := \alpha^{-1}A \in (0, 1], \quad B := \alpha^{-1}A^{-1/2}, \quad r_0 := A^{-1/2}\sqrt{t - s}.$$

Note that $B > 1$ and $B^{-1}A^{1/2} < 1$. For all $l_1, l_2 \in [s, t]$ we see

$$r_0 \leq \sqrt{s} \leq \sqrt{t}, \quad H \leq H_1 \leq \frac{A}{t - s} = r_0^{-2}, \quad |l_1 - l_2| \leq \theta^{-1}(t - s) \leq \alpha r_0^2, \quad d_s(x, y) \geq \alpha^{-1}r_0 \geq r_0.$$

In virtue of Theorem 3.2 we obtain

$$d_{l_1}(x, y) \geq \alpha d_s(x, y) \geq r_0, \quad d_{l_2}(x, y) \geq \alpha d_{l_1}(x, y) \geq \alpha A^{1/2}d_{l_1}(x, y) = B^{-1}d_{l_1}(x, y).$$

By switching the roles of $l_1$ and $l_2$, we arrive at the desired estimate.

Next, we study the case of $A \in (\alpha, \infty)$. As in the proof of Lemma 6.1 we define

$$\theta := \alpha^{-1}A \in (1, \infty), \quad B := \alpha^{-\theta+2}, \quad r_0 := A^{-1/2}\sqrt{t - s},$$

where $B > 1$ and $B^{-1}A^{1/2} < 1$. Let $N \in \mathbb{N}$ be the integer part of $\theta$, and let $s = \tau_0 < \tau_1 < \cdots < \tau_N \leq \tau_{N+1} = t$ be a division of $[s, t]$ such that $\tau_{i+1} - \tau_i = \theta^{-1}(t - s)$ for $i = 0, \ldots, N - 1$. We see $\tau_{N+1} - \tau_N \in [0, \theta^{-1}(t - s))$. It holds that

$$r_0 \leq \sqrt{s} \leq \sqrt{t} \leq \sqrt{t}, \quad H \leq H_1 \leq \frac{A}{t - s} = r_0^{-2}, \quad \tau_{i+1} - \tau_i \leq \theta^{-1}(t - s) = \alpha r_0^2$$

for every $i = 0, \ldots, N$. Since

$$d_{\tau_0}(x, y) = d_s(x, y) \geq B\sqrt{t - s} \geq B^{-N-1}A^{1/2}\sqrt{t - s} = \alpha^{-N-1}r_0 \geq r_0,$$

Theorem 3.2 tells us that

$$d_{\tau_i}(x, y) \geq \alpha d_{\tau_0}(x, y) \geq \alpha^{-N}r_0 \geq r_0.$$ 

In the same manner as in the proof of Lemma 6.1 we use Theorem 3.2 repeatedly. We obtain

$$(6.4) \quad d_{l_i}(x, y) \geq \alpha d_{l_{i-1}}(x, y) \geq \alpha^{-N+i-1}r_0 \geq r_0$$

for all $i = 1, \ldots, N + 1$. For $l_1, l_2 \in [s, t]$, let $i_0 \in \{0, \ldots, N\}$ satisfy $l_1 \in [\tau_{i_0}, \tau_{i_0+1}]$. Theorem 3.2 together with (6.3) leads us to

$$(6.5) \quad d_{l_1}(x, y) \geq \alpha d_{l_0}(x, y) \geq \alpha^{-N+i_0}r_0 \geq r_0.$$
By using Theorem 3.2 together with (6.4), (6.5) at most \( N + 1 \) times, we conclude
\[
d_{l_1}(x, y) \geq \alpha^{N+1} d_{l_1}(x, y) \geq B^{-1} d_{l_1}(x, y).
\]

Switching the roles of \( l_1 \) and \( l_2 \), we complete the proof. \( \square \)

We now prove the upper bound (1.4).

**Proposition 6.4.** For any \( A > 0 \), there exist positive constants \( C_3, C_4 > 0 \) depending only on \( n, T, g(0) \) and \( A \) such that the following holds: We assume \( H \leq H_1 \) for \( H_1 > 0 \). For \( s, t \in [0, T) \) with \( s < t \), we also suppose \( t - s \leq A H_1^{-1} \) and \( s \geq (t - s)/A \). Then we have
\[
G(x, t; y, s) \leq \frac{C_3}{(t - s)^{n/2}} \exp \left( -\frac{C_4 d_s(x, y)^2}{t - s} \right).
\]

**Proof.** We again notice (6.1) and (6.2). By Proposition 2.15 together with (6.7) implies
\[
(6.6) \quad G(x, t; y, l) \leq \frac{C_1}{(t - l)^{n/2}}.
\]

Let \( B > 1 \) be the constant in Lemma 6.3 satisfying \( B^{-1} A^{-1/2} < 1 \). When \( d_s(x, y) < B \sqrt{t - s} \), (6.6) implies
\[
G(x, t; y, s) \leq \frac{C_1 e^{B^2}}{(t - s)^{n/2}} \leq \frac{C_1 e^{B^2}}{(t - s)^{n/2}} \exp \left( -\frac{d_s(x, y)^2}{t - s} \right).
\]

This is a desired one.

We consider the case of \( d_s(x, y) \geq B \sqrt{t - s} \). We set \( c := B^{-1} A^{-1/2} \in (0, 1) \).

For all \( l \in [s, t) \) we see
\[
c \sqrt{t - l} < \sqrt{T}, \quad H \leq \frac{A}{t - s} \leq \frac{AB^2}{t - l} = \frac{1}{c^2(t - l)},
\]
and hence
\[
(6.7) \quad m(B(x, c \sqrt{t - l}, l)) \geq \kappa_1 (t - l)^{n/2}
\]
by Theorem 2.4. Proposition 2.15 together with (6.7) implies
\[
(6.8) \quad \frac{1}{m(B(y, c \sqrt{t - l}, l))} \int_{B(y, c \sqrt{t - l}, l)} G(x, t; z, l) \, dm(z) \leq \left( \int_{B(x, c \sqrt{t - l}, l)} G(x, t; z, l) \, dm(z) \right)^{-1} \frac{C_2}{(t - l)^{n/2}} \exp \left( -\frac{d_l(x, y)^2}{16(t - l)} \right).
\]

By Proposition 6.2
\[
(6.9) \quad G(x, t; z, l) \geq \frac{C_3}{(t - l)^{n/2}} \exp \left( -\frac{C_4 d_l(x, z)^2}{t - l} \right) \geq \frac{C_5}{(t - l)^{n/2}}
\]
for all \( z \in B(x, c \sqrt{t - l}, l) \). Combining (6.7), (6.8), (6.9), we conclude
\[
(6.10) \quad \frac{1}{m(B(y, c \sqrt{t - l}, l))} \int_{B(y, c \sqrt{t - l}, l)} G(x, t; z, l) \, dm(z) \leq \frac{C_6}{(t - l)^{n/2}} \exp \left( -\frac{d_l(x, y)^2}{16(t - l)} \right).
\]

Now, we possess
\[
c \sqrt{t - l} < A^{-1/2} \sqrt{t - l} \leq A^{-1/2} \sqrt{t - s} \leq \sqrt{s} \leq \sqrt{l},
\]
\[
H(\cdot, \tau) \leq \frac{A}{t - s} \leq \frac{c^2(t - l)}{t - \tau} \frac{A}{t - s} \leq \frac{c^2A}{l - \tau}
\]
for \( \tau \in [l - c^2(t - l), l] \), and hence Theorem 2.12 tells us that
\[
(6.11) \quad m_l(B(y, cv\sqrt{t-l}), l) \leq \kappa_2(t-l)^{n/2}.
\]
From (6.10), (6.11) and Lemma 6.3 one can derive
\[
\int_{B(y, cv\sqrt{t-l}, l)} G(x, t; z, l) \, dm(z) \leq C_7 \exp \left(-\frac{d_l(x, y)^2}{16(t-l)}\right) \leq C_7 \exp \left(-\frac{B^{-2} d_s(x, y)^2}{16(t-s)}\right).
\]
This together with (6.6) leads us to
\[
\int_{B(y, cv\sqrt{t-l}, l)} G(x, t; z, l)^2 \, dm(z) \leq \frac{C_8}{(t-l)^{n/2}} \exp \left(-\frac{B^{-2} d_s(x, y)^2}{16(t-s)}\right).
\]
Integrating over \([s, (s + t)/2]\), we obtain
\[
\int_s^{(s+t)/2} \int_{B(y, cv\sqrt{t-l}, l)} G(x, t; z, l)^2 \, dm(z) \, dl \leq \frac{C_9}{(t-s)^{n/2-1}} \exp \left(-\frac{B^{-2} d_s(x, y)^2}{16(t-s)}\right).
\]
We now set
\[
r := c\sqrt{\frac{l-s}{2}}.
\]
We notice that \( t - l \geq (t - s)/2 \) for \( l \in [s, (s + t)/2] \), and hence
\[
Q^+(y, r, s) \subset \left\{(z, l) \in M \times [0, T] \mid z \in B(y, cv\sqrt{t-l}, l), l \in \left[s, \frac{s+t}{2}\right]\right\}.
\]
It follows that
\[
\int_{Q^+(y, r, s)} G(x, t; z, l)^2 \, dm(z) \, dl \leq \frac{C_9}{(t-s)^{n/2-1}} \exp \left(-\frac{B^{-2} d_s(x, y)^2}{16(t-s)}\right).
\]
By virtue of Theorem 5.2
\[
G(x, t; y, s)^2 \leq \frac{C_{10}}{r^{n+2}} \int_{Q^+(y, r, s)} G(x, t; z, l)^2 \, dm(z) \, dl \leq \frac{C_{11}}{(t-s)^n} \exp \left(-\frac{B^{-2} d_s(x, y)^2}{16(t-s)}\right).
\]
Thus, we complete the proof. \(\square\)

We finally prove the gradient estimate (1.5).

**Proposition 6.5.** For any \( A > 0 \), there exist positive constants \( C_5, C_6 > 0 \) depending only on \( n, T, g(0) \) and \( A \) such that the following holds: We assume \( H \leq H_1 \) for \( H_1 \geq 0 \). For \( s, t \in [0, T) \) with \( s < t \), we also suppose \( t - s \leq A H_1^{-1} \) and \( s \geq (t-s)/A \). Then we have
\[
|\nabla_x G|(x, t; y, s) \leq \frac{C_5}{(t-s)^{(n+1)/2}} \exp \left(-\frac{C_6 d_s(x, y)^2}{t-s}\right).
\]

**Proof.** We set \( G(x, t) := G(x, t; y, s) \). Due to Theorem 2.9
\[
\frac{|\nabla G|^2}{G^2}(x, t) \leq \frac{C_1}{t-s} \log \frac{A}{G(x, t)},
\]
where
\[
A := \sup_{M \times [s, t]} G.
\]
It follows that
\[
|\nabla G|^2(x, t) \leq \frac{C_1}{t-s} AG(x, t) \frac{G(x, t)}{A} \log \frac{A}{G(x, t)} \leq \frac{C_1}{t-s} AG(x, t).
\]
since $G(x, t)/A \leq 1$. By Propositions 2.5 and 6.4, we possess
\[
A \leq \frac{C_2}{(t-s)^{n/2}}, \quad G(x, t) \leq \frac{C_3}{(t-s)^{n/2}} \exp \left( -\frac{C_4 \, d_s(x, y)^2}{t-s} \right).
\]
Combining them, we arrive at
\[
|\nabla G|^2(x, t) \leq \frac{C_5}{(t-s)^{n+1}} \exp \left( -\frac{C_4 \, d_s(x, y)^2}{t-s} \right).
\]
This completes the proof. \qed

Proof of Theorem 1.1. By Propositions 6.2, 6.4, 6.5, we complete the proof. \qed

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