POINTWISE BOUNDED ASYMPTOTIC MORPHISMS AND THOMSEN’S NON-STABLE k-THEORY

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Abstract

In this paper I show that pointwise bounded asymptotic morphisms between separable metrisable locally convex *-algebras induce continuous maps between the quasi-unitary groups of the algebras, provided that the algebras support a certain amount of functional calculus. This links the asymptotic morphisms directly to Thomsen’s non-stable definition of $k$-theory in the $C^*$ algebra case. A result on composition of asymptotic morphisms is also given.

1 Introduction

Thomsen defined a non-stable version of $K$-theory, called $k$-theory, in terms of the topology of the unitary group of a $C^*$ algebra [6]. Here we link the pointwise bounded asymptotic morphisms (PBAMs) on separable metrisable locally convex (SMLC) algebras to the topology of their unitary groups (for *-algebras). For other algebras, it may be more appropriate to consider the group of invertible elements, for which a similar approach could be taken, but in this paper we concentrate on *-algebras and the unitary group.

To avoid problems with algebras which do not contain a unit, Thomsen used the following procedure, following Palmer [4, 5]. On an algebra $A$ we define an associative binary operation $a \bullet b = a + b + a b$, which has 2-sided unit $0 \in A$. Note that if we had a unit 1 for algebra multiplication, we would have $(1 + a)(1 + b) = 1 + a \bullet b$. By this means it can be seen that this
construction is really standard, though phrased in slightly unusual terms. We define \( gl(A) \), the quasi-invertibles in an algebra \( A \), and \( \mathcal{U}(A) \), the quasi-unitaries, by

\[
\begin{align*}
gl(A) &= \{ a \in A : \exists a' \in A \ a \cdot a' = a' \cdot a = 0 \} \\
\mathcal{U}(A) &= \{ u \in A : u \cdot u^* = u^* \cdot u = 0 \}.
\end{align*}
\]

It is easily seen that both of these are groups under the \( \cdot \) operation, with identity 0. Thomsen’s non-stable \( k \)-groups for a \( C^* \) algebra \( E \) are defined as \( k_n(E) = \pi_{n+1}(\mathcal{U}(E)) \).

The idea of asymptotic morphisms for \( C^* \) algebras was introduced by Connes and Higson \[2\] for \( E \)-theory, and later considered in the non-stable \( C^* \) case by Dădărlat \[3\]. In \[1\] PBAMs were introduced to generalise the idea of asymptotic morphism to SMLC algebras, and examples were given. The definition of a PBAM will be given later in this paper. The general idea of all these constructions is to have a family of maps \( f_t : A \to B \) between algebras indexed by \( t \in [0, \infty) \), where \( f_t \) becomes more like an algebra map as \( t \to \infty \).

Under a certain assumption about functional calculus on SMLC \(*\)-algebras \( A \) and \( B \), we show that a PBAM \( f : A \times [0, \infty) \to B \) induces a unique topological homotopy class \( \mathcal{U}(f) \in [\mathcal{U}(A), \mathcal{U}(B)] \). In the \( C^* \) algebra case, this gives a map \( f_* : k_*(A) \to k_*(B) \) by composition, \( f_*[h] = [\mathcal{U}(f) \circ h] \). The question naturally arises of whether \( \mathcal{U} \) is a functor from SMLC \(*\)-algebras and PBAMs to topological spaces. The answer is no, as SMLC \(*\)-algebras and PBAMs themselves do not form a category. However there is a compatibility condition between PBAMs \( f : A \times [0, \infty) \to B \) and \( g : B \times [0, \infty) \to C \) which ensures that there is a composition \( g \circ f : A \times [0, \infty) \to C \), defined up to homotopy of PBAMs. We shall see that this same condition also implies that \( \mathcal{U}(g \circ f) = \mathcal{U}(g) \circ \mathcal{U}(f) \in [\mathcal{U}(A), \mathcal{U}(C)] \). I choose to retain the compatibility condition rather than restrict to a category where composition is automatic (and there are such categories) in order not to unnecessarily restrict the results.

2 Asymptotic results and definitions on metric spaces

**Lemma 2.1** Let \( X \) be a separable metric space. Suppose that there is a function \( \Psi : X \times X \times [0, \infty) \times [0, \infty) \to \{ \text{true, false} \} \) with the following property: For all \( x \in X \) there is a \( \delta(x) > 0 \) and a \( Q(x) \geq 0 \) so that for all \( q \geq Q \) there is a \( R(x, q) \geq 0 \) and an \( \epsilon(x, q) > 0 \) for which

\[
\forall y \in X \ \forall t \geq Q \ \forall s \geq R \ \ [ t \in (q - \epsilon, q + \epsilon) \text{ and } d(x, y) < \delta ] \Rightarrow \Psi(x, y, s, t).
\]
Then there are continuous functions $\alpha: X \to [0, \infty)$ and $\phi: [0, \infty) \to [0, \infty)$ (increasing) so that
\[
\forall y \in X \ \exists x \in X \ \forall t \geq \alpha(y) \ \forall s \geq \phi(t) \ [d(x, y) < \delta(x) \text{ and } \Psi(x, y, s, t)] .
\]

**Proof**

**Part 1:** Given $x \in X$ and $r \geq Q(x)$, there is an open cover $(q - \varepsilon(x,q), q + \varepsilon(x,q))$ of the interval $[Q, r]$ indexed by $q \in [Q, r]$. By compactness of $[Q, r]$ there is a finite subcover indexed by $q_1, \ldots, q_n$. Let $S(x, r)$ be the maximum of the $R(x, q_i)$. Then
\[
\forall y \in X \ \forall t \in [Q, r] \ \forall s \geq S \ d(x, y) < \delta \Rightarrow \Psi(x, y, s, t) .
\]

**Part 2:** Take a cover of $X$ by open balls center $x \in X$ radius $\delta(x)$. There is a countable subcover of $X$ by open balls $B_i$ (integer $i \geq 0$) center $x_i$ radius $\delta(x_i)$. Now define an increasing sequence $\phi_i \in [0, \infty)$ by using the recursive inequalities
\[
\forall j \in \{0, \ldots, i\} \text{ if } i \geq Q(x_j) \text{ then } \phi_i \geq S(x_j, i + 1) .
\]

Construct the graph of the continuous increasing function $\phi: [0, \infty) \to [0, \infty)$ by joining the dots $(0, \phi_0), (1, \phi_1), \ldots$ with straight lines. Take a continuous partition of unity $\theta_j: X \to [0, 1]$ with locally finite supports, where $\theta_j$ has support in $B_j$, and define
\[
\alpha(x) = \sum_j (1 + \max\{Q(x_j), j\}) \theta_j(x) .
\]

**Part 3:** For a given $y \in X$ we choose the minimum value of $\max\{Q(x_j), j\}$ over the finite number of indices $j$ for which $\theta_j(y) \neq 0$. If this minimum is $\max\{Q(x_k), k\}$ where $\theta_k(y) \neq 0$, it follows that $\alpha(y) \geq 1 + \max\{Q(x_k), k\}$ and $y \in B_k$. Now if $t \geq 1 + \max\{Q(x_k), k\}$ then there is an $i \in \mathbb{N}$ with $i + 1 \geq t \geq i \geq Q(x_k)$ and $i \geq k$. Now if $s \geq \phi(t)$ then $s \geq \phi_i$, so $s \geq S(x_k, i + 1)$. It follows that $\Psi(x_k, y, s, t)$.

**Lemma 2.2** For $n \in \mathbb{N}$, let $X_n$ be a separable metric space. Suppose that there are functions $\Psi_n: X_n \times X_n \times [0, \infty) \times [0, \infty) \to \{\text{true, false}\}$ (for $n \in \mathbb{N}$) which satisfy the property in the statement of lemma 2.1, using functions $\delta_n, Q_n, R_n$ and $\epsilon_n$. Then there is a continuous increasing function $\psi: [0, \infty) \to [0, \infty)$ and continuous functions $\beta_n: X_n \to [0, \infty)$ so that
\[
\forall n \in \mathbb{N} \ \forall y \in X_n \ \exists x \in X_n \ \forall t \geq \beta_n(y) \ \forall s \geq \psi(t) \ [d(x, y) < \delta_n(x) \text{ and } \Psi_n(x, y, s, t)] .
\]
**Proof** By using lemma 2.1 on each of the \( \Psi_n \), we get continuous functions \( \alpha_n : X_n \to [0, \infty) \) and \( \phi_n : [0, \infty) \to [0, \infty) \). Now we make a new continuous increasing function \( \psi : [0, \infty) \to [0, \infty) \) which satisfies the condition that if \( t \geq n \) then \( \psi(t) \geq \phi_n(t) \). Now set \( \beta_n(x) = \max\{n, \alpha_n(x)\} \). \( \square \)

**Definition 2.3** For metric spaces \( X \) and \( Y \), the function \( F : X \times [0, \infty) \to Y \) is said to be strongly asymptotically continuous if, for all \( \epsilon > 0 \) and all \( x \in X \) there is an \( \eta(x, \epsilon) > 0 \) and a \( P(x, \epsilon) \geq 0 \) so that for all \( x' \in X \) with \( d(x, x') < \eta \) we have \( d(F(x), F(x')) < \epsilon \) for all \( t \geq P \).

**Lemma 2.4** For metric spaces \( X \) and \( Y \), suppose that the function \( F : X \times [0, \infty) \to Y \) is continuous and strongly asymptotically continuous. Then, given \( \epsilon > 0 \) and \( x \in X \), there are \( \eta(x, \epsilon) > 0 \) and \( P(x, \epsilon) \geq 0 \) so that, given \( q \geq P \) there is a \( \delta(x, \epsilon, q) > 0 \) so that for all \( x' \in X \) and all \( t \geq 0 \),

\[
[|t - q| < \delta \text{ and } d(x, x') < \eta] \Rightarrow d(F_q(x), F_t(x')) < \epsilon.
\]

**Proof** From definition 2.3 we take \( \eta = \eta(x, \epsilon/2) > 0 \) and \( P(x, \epsilon/2) \geq 0 \), and set \( P = P + 1 \). By continuity of \( F \), given \( q \geq P \) there is a \( 1 > \delta(x, \epsilon, q) > 0 \) so that \( |t - q| < \delta \) implies \( d(F_q(x), F_t(x)) < \epsilon/2 \). As \( |t - q| < \delta \) implies \( t \geq P \), we have \( d(F_t(x), F_t(x')) < \epsilon/2 \) for all \( x' \in X \) with \( d(x, x') < \eta \). \( \square \)

3 Some functional calculus

**Definition 3.1** Let \( A \) be a topological *-algebra, and take \( A_{sad} \) to be the subset of self adjoint elements, i.e. those \( a \in A \) for which \( a^* = a \). \( A \) is said to have the inverse square root property if there is a convex open set \( V_A \subset A_{sad} \) containing \( 0 \in A \) and a continuous function \( \vartheta : V_A \to A_{sad} \) so that

1) \( a \bullet \vartheta(a) = \vartheta(a) \bullet a \) for all \( a \in V_A \),

2) \( a \bullet (\vartheta(a) \bullet \vartheta(a)) = 0 \) for all \( a \in V_A \),

3) for all \( a \in V_A \), \( a \) is \( \bullet \)-invertible,

4) \( \vartheta(0) = 0 \).

**Lemma 3.2** If \( a \in A \) has \( a^* \bullet a \in V_A \) and \( a \bullet a^* \in V_A \), then \( a \in \mathfrak{gl}(A) \) and \( a \bullet \vartheta(a^* \bullet a) \in \mathcal{U}(A) \).
Proof  Set \( u = a \bullet \vartheta(a^* \bullet a) \). Then
\[
 u^* \bullet u = \vartheta(a^* \bullet a) \bullet (a^* \bullet a) \bullet \vartheta(a^* \bullet a) = (a^* \bullet a) \bullet (\vartheta(a^* \bullet a) \bullet \vartheta(a^* \bullet a)) = 0 .
\]
Since \( a^* \bullet a \) and \( a \bullet a^* \) are \( \bullet \)-invertible, there are \( b \in A \) and \( c \in A \) so that
\[
 a^* \bullet a \bullet b = b \bullet a^* \bullet a = 0 \quad \text{and} \quad a \bullet a^* \bullet c = c \bullet a \bullet a^* = 0 .
\]
Now
\[
 (a^* \bullet a) \bullet (b \bullet a^*) = 0 \bullet a^* = a^* = a^* \bullet 0 = a^* \bullet (a \bullet a^* \bullet c) = (a^* \bullet a) \bullet (a^* \bullet c) ,
\]
so since \( a^* \bullet a \) is \( \bullet \)-invertible we deduce that \( b \bullet a^* = a^* \bullet c \) is the \( \bullet \)-inverse of \( a \). The formula
\[
 ((a^* \bullet a) \bullet \vartheta(a^* \bullet a)) \bullet \vartheta(a^* \bullet a) = 0
\]
shows that \( \vartheta(a^* \bullet a) \) is \( \bullet \)-invertible, so \( u \) is \( \bullet \)-invertible, and so \( u \bullet u^* = 0 \), i.e. \( u \in \mathcal{U}(A) \). \( \square \)

Example 3.3  Any Banach \( * \)-algebra \( A \) has the inverse square root property. This can be seen by taking \( V_A = \{ a \in A_{sad} : |a| < \frac{1}{2} \} \), and using functional calculus with the function \( \vartheta(x) = (1 + x)^{-1/2} - 1 \). The easiest way to see that this works is to observe that \( \vartheta \) has a Taylor series with radius of convergence 1, and the fact that there is no constant term in the series means that we do not require \( A \) to have a unit.

Example 3.4  The algebra \( C^\infty(M, \mathbb{C}) \) of smooth complex valued functions on a compact manifold \( M \) (with the smooth topology) has the inverse square root property. We set \( V = \{ f \in C^\infty(M, \mathbb{R}) : \forall m \in M \ |f(m)| < \frac{1}{2} \} \), and \( \vartheta(f)(m) = (1 + f(m))^{-1/2} - 1 \).

4  Pointwise bounded asymptotic morphisms

Suppose that both \( A \) and \( B \) are separable metrisable locally convex \( * \)-algebras (SMLC \( * \)-algebras for short). Recall that a metrisable locally convex topological vector space has topology defined by a countable number of seminorms, which we write \( |.|_0, |.|_1, \) etc. We can change any of the seminorms to a stronger continuous seminorm without altering the topology. It will be convenient to use this fact to assume that the seminorms are non-decreasing, i.e. that \( |b|_n \leq |b|_{n+1} \) for all \( n \in \mathbb{N} \) and \( b \in B \). In a SMLC \( * \)-algebra \( B \) we assume that the multiplication and
star operations are continuous. It will also be convenient to assume, by strengthening seminorms
where appropriate, that for all \( b, b' \in B \) and \( n \in \mathbb{N} \), \( |b^*|_n \leq |b|_{n+1} \) and \( |bb'|_n \leq |b|_{n+1}|b'|_{n+1} \). This
implies the following result for \( b, c \in B \), which is used later:

\[
|b^* \cdot b - c^* \cdot c|_n \leq 2 |b - c|_{n+2} (1 + |b|_{n+2} + |b - c|_{n+2}) .
\] (1)

**Definition 4.1** The map \( f : A \times [0, \infty) \to B \) is called pointwise asymptotically bounded if, for
all \( a \in A \), the set \( \{ f_t(a) : t \geq 0 \} \) is bounded in \( B \). In effect this means that for every \( n \in \mathbb{N} \) the
subset of the reals \( \{|f_t(a)|_n : t \geq 0\} \) is bounded above.

**Definition 4.2** If \( A \) and \( B \) are SMLC *-algebras, call a continuous function \( f : A \times [0, \infty) \to B \)
a pointwise bounded asymptotic morphism (PBAM for short) if

a) \( f : A \times [0, \infty) \to B \) is pointwise asymptotically bounded (see 4.1),

b) \( f \) is strongly asymptotically continuous (see 2.3),

c) \( f_t(a)^* - f_t(a^*) \to 0 \) as \( t \to \infty \) for all \( a \in A \),

d) \( \lambda f_t(a) - f_t(\lambda a) \to 0 \) as \( t \to \infty \) for all \( a \in A \) and all \( \lambda \in \mathbb{C} \),

e) \( f_t(a) + f_t(a') - f_t(a + a') \to 0 \) as \( t \to \infty \) for all \( a, a' \in A \),

f) \( f_t(a) f_t(a') - f_t(aa') \to 0 \) as \( t \to \infty \) for all \( a, a' \in A \).

Note that the additive condition (e) implies that \( f_t(0) \to 0 \) as \( t \to \infty \).

5 **Pointwise bounded asymptotic morphisms and quasi-unitary groups**

In this section we suppose that \( A \) and \( B \) are SMLC *-algebras, that \( B \) has the inverse square
root property (see 3.1), and that \( f : A \times [0, \infty) \to B \) is a PBAM.

**Lemma 5.1** For every \( u \in \mathcal{U}(A) \) there is a \( T(u) \geq 0 \) and an open neighbourhood \( O(u) \) of \( u \) in
\( A \) so that

\[
\forall t \geq T(u) \quad \forall v \in O(u) \quad f_t(v) \cdot f_t(v)^* \in V_B \text{ and } f_t(v)^* \cdot f_t(v) \in V_B .
\]
Proof First note that, for the given \( u \in \mathcal{U}(A) \), \( f_t(u) \cdot f_t(u)^* \to 0 \) and \( f_t(u)^* \cdot f_t(u) \to 0 \) as \( t \to \infty \). To see this, apply \( f_t \) to \( u^* \cdot u = 0 \) to get \( f_t(u) + f_t(u^*) + f_t(u^*) f_t(u) \to 0 \) as \( t \to \infty \). Now we use \( f_t(u)^* - f_t(u^*) \to 0 \) as \( t \to \infty \) together with the fact that \( f_t(u) \) is bounded for \( t \geq 0 \) to show that \( f_t(u)^* \cdot f_t(u) \to 0 \). Similarly for \( f_t(u) \cdot f_t(u)^* \to 0 \).

As \( V_B \) is an open neighbourhood of 0 in \( B_{sad} \), there is an \( n \geq 0 \) and an \( \epsilon > 0 \) so that \( \{ b \in B_{sad} : |b|^n < 2\epsilon \} \subset V_B \). As \( f \) is pointwise bounded, there is a \( K \geq 0 \) so that \( |f_t(u)|_{n+2} \leq K \) for all \( t \). Choose \( \delta > 0 \) so that \( 2\delta(1 + K + \delta) < \epsilon \). By strong asymptotic continuity there is a neighbourhood \( O(u) \) of \( u \) in \( A \) and a \( P \geq 0 \) so that \( |f_t(v) - f_t(u)|_{n+2} \leq \delta \) for all \( t \geq P(u) \) and all \( v \in O(u) \). Then by (\[ \[ \] \]), \( |f_t(v)^* \cdot f_t(v) - f_t(u)^* \cdot f_t(u)|_n < \epsilon \). Likewise we get \( |f_t(v) \cdot f_t(v)^* - f_t(u) \cdot f_t(u)^*|_n < \epsilon \). Finally choose \( T \geq P \) so that \( |f_t(u)^* \cdot f_t(u)|_n < \epsilon \) and \( |f_t(u) \cdot f_t(u)^*|_n < \epsilon \) for all \( t \geq T \). \( \square \)

Lemma 5.2 There is a continuous function \( \alpha : \mathcal{U}(A) \to [0, \infty) \) so that

\[
\forall \nu \in \mathcal{U}(A) \quad \forall \tau \geq \alpha(\nu) \quad f_t(\nu) \cdot f_t(\nu)^* \in V_B \text{ and } f_t(\nu)^* \cdot f_t(\nu) \in V_B.
\]

Proof From lemma [5.1] take a countable subcover \( O(u_n) (n \in \mathbb{N}) \) of \( \mathcal{U}(A) \), and a locally finite continuous partition of unity \( \theta_n \) on \( \mathcal{U}(A) \), with the support of \( \theta_n \) contained in \( O(u_n) \). Now set \( \alpha = \sum T(u_n) \theta_n \). \( \square \)

Proposition 5.3 There is a well defined homotopy class \( \mathcal{U}(f) \in [\mathcal{U}(A), \mathcal{U}(B)] \) with representative function \( \tilde{f}_\alpha(\nu) = f(\nu)^* \cdot \vartheta(f(\nu)^* \cdot f(\nu)) \), where \( \alpha \) is the function in lemma 5.2.

Here well defined means independent of the choice of \( \alpha \).

Proof Note that the function \( \tilde{f}_\alpha \) takes values in \( \mathcal{U}(B) \) by lemma 3.2. Suppose that we have \( \gamma : \mathcal{U}(A) \to [0, \infty) \) with \( \gamma(u) \geq \alpha(u) \) for all \( u \in \mathcal{U}(A) \). Then there is a homotopy \( H : \mathcal{U}(A) \times [0,1] \to \mathcal{U}(B) \) connecting \( \tilde{f}_\alpha \) and \( \tilde{f}_\gamma \) defined by \( H(\nu, p) = \tilde{f}_{p\alpha+(1-p)\gamma}(\nu) \). Now if \( \beta \) is another choice of function in lemma 5.2, we see that \( \tilde{f}_\alpha \) and \( \tilde{f}_\beta \) are homotopic, as they are both homotopic to \( \tilde{f}_{\max\{\alpha, \beta\}} \). \( \square \)

6 Homotopy invariance

In this section we suppose that \( A \) and \( B \) are SMLC \(*\)-algebras, and that \( f \) and \( g : A \times [0, \infty) \to B \) are two PBAMs.
Definition 6.1 The PBAMs \( f \) and \( g \) are said to be pointwise bounded asymptotically homotopy equivalent (PBA homotopic for short) if there is a PBAM \( h : A \times [0, \infty) \to C([0, 1], B) \) so that \( h_t(a)(0) = f_t(a) \) and \( h_t(a)(1) = g_t(a) \). (Use the supremum metric on \( C([0, 1], B) \), the continuous functions from \([0, 1]\) to \( B \).)

Proposition 6.2 If \( f \) and \( g \) are PBA homotopic then \( \mathcal{U}(f) = \mathcal{U}(g) \in [\mathcal{U}(A), \mathcal{U}(B)] \).

Proof Take the homotopy to be \( h : A \times [0, \infty) \to C([0, 1], B) \). By lemma 5.2 there are functions \( \alpha, \beta \) and \( \gamma : \mathcal{U}(A) \to [0, \infty) \) corresponding to \( f, g \) and \( h \) respectively. Define \( \eta : \mathcal{U}(A) \to [0, \infty) \) by \( \eta(u) = \max\{\alpha(u), \beta(u), \gamma(u)\} \). Now \( f_\alpha \) is homotopic to \( f_\eta \), and \( g_\beta \) is homotopic to \( g_\eta \). Finally \( f_\eta \) and \( g_\eta \) are homotopic by the homotopy \( H : \mathcal{U}(A) \times [0, 1] \to \mathcal{U}(B) \) defined by \( H(v, p) = h_\eta(v)(p) \), using the fact that \( \mathcal{U}(C([0, 1], B)) = C([0, 1], \mathcal{U}(B)) \).  

7 Composition of pointwise bounded asymptotic morphisms

In this section we suppose that \( A, B \) and \( C \) are SMLC *-algebras, and that \( f : A \times [0, \infty) \to B \) and \( g : B \times [0, \infty) \to C \) are PBAMs. We will show that under certain conditions it is possible to compose \( f \) and \( g \). To save a lot of time we will use the results on composing strongly asymptotic morphisms in [4].

Definition 7.1 For PBAMs \( f \) and \( g \), and a continuous increasing function \( \phi : [0, \infty) \to [0, \infty) \) we define \( g \circ_\phi f : A \times [0, \infty) \to C \) by \( (g \circ_\phi f)_t(a) = g_{\phi(t)}(f_t(a)) \). The function \( \phi \) is called a valid reparameterisation for \( f \) and \( g \) if

a) \( g \circ_\phi f \) is a PBAM, and

b) for every continuous increasing function \( \theta : [0, \infty) \to [0, \infty) \) with \( \theta(t) \geq \phi(t) \) \( \forall t \in [0, \infty) \), \( g \circ_\theta f \) is also a PBAM, and is PBA homotopic to \( g \circ_\phi f \), using \( h : A \times [0, \infty) \to C([0, 1], B) \) given by \( h_t(a)(p) = g_{p\phi(t)+(1-p)\theta(t)}(f_t(a)) \).

Definition 7.2 The PBAMs \( f \) and \( g \) are said to satisfy conditions C1 and C3 if:

C1: For all \( a \in A \) and all \( \nu > 0 \) there is a \( \xi(a, \nu) > 0 \) and a \( Q'(a, \nu) \geq 0 \) so that, for all \( t \geq Q' \) there is a \( S'(t, a, \nu) \geq 0 \) so that

\[
[ \ d(f_t(a), b) < \xi \ 	ext{and} \ s \geq S' \ ] \ \Rightarrow \ d(g_s(f_t(a)), g_s(b)) < \nu .
\]
For all \(a \in A\) and all \(n \in \mathbb{N}\) there is a \(Q_n(a) \geq 0\) and an \(M_n(a) \geq 0\) so that, for all \(t \geq Q_n(a)\) there is a \(S_n(a,t) \geq 0\) so that for \(s \geq S_n\) we have \(|g_s(f_t(a))|_n \leq M_n\).

**Lemma 7.3** If the PBAMs \(f\) and \(g\) satisfy conditions \(C1\) and \(C3\), then the property \(\Psi_n : A \times A \times [0, \infty) \times [0, \infty) \to \{\text{true, false}\}\) defined by \(\Psi_n(x, y, s, t)\) being the logical value of \(|g_s(f_t(y))|_n \leq M_n(x) + 1\) satisfies the conditions of lemma 2.4.

**Proof** Choose \(\nu > 0\) so that for the translation invariant metric \(d\) on \(C\), \(d(c, c') < \nu\) implies \(|c - c'| < 1\). By applying \(C3\) to \(x \in A\), we get \(Q_n(x) \geq 0\) and \(M_n(x) \geq 0\) so that, for all \(q \geq Q_n(x)\) there is a \(S_n(x, q) \geq 0\) so that for \(s \geq S_n\) we have \(|g_s(f_q(x))|_n \leq M_n\). By applying \(C1\), we get \(\xi(x, \nu) > 0\) and a \(Q'(x, \nu) \geq 0\) so that, for all \(q \geq Q'\) there is a \(S'(q, x, \nu) \geq 0\) so that

\[
[d(f_q(x), b) < \xi \text{ and } s \geq S'] \Rightarrow |g_s(f_q(x)) - g_s(b)|_n < 1 .
\]

By lemma 2.4 there are \(\overline{\eta}(x, \xi) > 0\) and \(\overline{Q}(x, \xi) \geq 0\) so that, for \(q \geq \overline{Q}\) there is a \(\overline{\delta}(x, \xi, q) > 0\) so that for all \(y \in A\) and all \(t \geq 0\),

\[
[d(x, y) < \overline{\eta} \text{ and } |t - q| < \overline{\delta}] \Rightarrow d(f_q(x), f_t(y)) < \xi .
\]

Now set \(\delta(x) = \overline{\eta}(x, \nu)\) and \(Q(x) = \max\{Q_n(x), Q'(x, \nu), \overline{Q}(x, \xi)\}\). Given \(q \geq Q\) set \(\epsilon(x, q) = \overline{\delta}(x, \xi, q)\) and \(R(x, q) = \max\{S'(q, x, \nu), S_n(x, q)\}\). \(\square\)

**Theorem 7.4** If the PBAMs \(f\) and \(g\) satisfy conditions \(C1\) and \(C3\), then there is a continuous increasing function \(\phi : [0, \infty) \to [0, \infty)\) which is a valid reparameterisation for \(f\) and \(g\).

**Proof** First note that \((C1\) and \(C3)\) implies \(C2\) from 1 for the \(*\)-operation, scalar multiplication, addition and algebra multiplication. To see this we use the inequalities

\[
|g_s(f_t(a))^* - g_s(b)^*|_n \leq |g_s(f_t(a)) - g_s(b)|_{n+1} ,
\]

\[
|\lambda g_s(f_t(a)) - \mu g_s(b)|_n \leq |\lambda - \mu| |g_s(f_t(a))|_n + (|\lambda| + |\mu|) |g_s(f_t(a)) - g_s(b)|_n ,
\]

\[
|g_s(f_t(a)) + g_s(f_t(a')) - g_s(b) - g_s(b')|_n \leq |g_s(f_t(a)) - g_s(b)|_n + |g_s(f_t(a')) - g_s(b')|_n ,
\]

\[
|g_s(f_t(a)) g_s(f_t(a')) - g_s(b) g_s(b')|_n \leq |g_s(f_t(a))|_{n+1} |g_s(f_t(a')) - g_s(b')|_{n+1} + |g_s(f_t(a)) - g_s(b)|_{n+1} \times
\]

\[
|g_s(f_t(a')) - g_s(b')|_{n+1} \times
\]
\[(|g_s(f_t(a'))|_{n+1} + |g_s(f_t(a')) - g_s(b')|_{n+1}) . \quad (2)\]

From \[\ref{1}\] we now know that there is a valid reparameterisation (for strongly asymptotic morphisms) \(\psi : [0, \infty) \to [0, \infty)\) so that \(g \circ \psi f\) is a \textbf{strongly} asymptotic morphism. From \[7.3\] and \[7.2\] there are functions \(\beta_n : A \to [0, \infty) (n \in \mathbb{N})\) and \(\theta : [0, \infty) \to [0, \infty)\) so that, given \(n \in \mathbb{N}\) and \(y \in A\), there is an \(x \in A\) so that

\[t \geq \beta_n(y)\text{ and } s \geq \theta(t) \Rightarrow |g_s(f_t(y))|_n \leq M_n(x) + 1 .\]

Let \(\phi(y) = \max\{\psi(y), \theta(y)\}\). Then \(g \circ \phi f\) is a PBAM, and if \(\chi : [0, \infty) \to [0, \infty)\) has \(\chi(t) \geq \phi(t)\) \(\forall t \in [0, \infty)\), then the map \(h : A \times [0, \infty) \to C([0, 1], B)\) given by \(h_t(a)(p) = g_p \cdot \phi(t)(1-p) \chi(t)(f_t(a))\) is a PBA homotopy between \(g \circ \phi f\) and \(g \circ \chi f\). \(\Box\)

\section{A conditional functoriality}

Suppose that \(A, B\) and \(C\) are SMLC *-algebras, and that \(B\) and \(C\) have the inverse square root property. We are given PBAMs \(f : A \times [0, \infty) \to B\) and \(g : B \times [0, \infty) \to C\) which satisfy \(C1\) and \(C3\).

\textbf{Lemma 8.1} \textit{Given} \(x \in \mathcal{U}(A)\), \textit{there is an} \(N(x) \geq 0\) \textit{and a} \(\nu(x) > 0\) \textit{so that, for} \(q \geq N\) \textit{there is an} \(L(x, q) \geq 0\) \textit{so that}

\[\forall z \in B \quad [d(f_q(x), z) < \nu \text{ and } s \geq L] \Rightarrow \left[|g_s(z)^* \cdot g_s(z) \in V_C \text{ and } g_s(z) \cdot g_s(z)^* \in V_C\right].\]

\textbf{Proof} \textbf{Part 1}: We shall only consider the \(g_s(z)^* \cdot g_s(z) \in V_C\) part, the other is almost identical. Choose \(\epsilon > 0\) and \(n \geq 0\) so that for \(c \in C\), \(|c|_n < \epsilon\) implies \(c \in V_C\). Since \(g_s(0) \to 0\) as \(s \to \infty\), and using the strong asymptotic continuity of \(g\), there is a \(\eta > 0\) and \(P \geq 0\) so that

\[\left[\text{\(d(b, 0) < \eta\text{ and } s \geq P\)}\right] \Rightarrow |g_s(b)|_n < \epsilon/4 . \quad (3)\]

Given \(x \in \mathcal{U}(A)\), by \(C3\) and \(\ref{1}\) there is a \(Q_{n+2}(x) \geq 0\) and \(M_{n+2}(x) \geq 0\) so that for any \(q \geq Q_{n+2}\) there is a \(S_{n+2}(x, q)\) so that for \(s \geq S_{n+2}\),

\[|g_s(z)^* \cdot g_s(z) - g_s(f_q(x))^* \cdot g_s(f_q(x))|_n \leq 2 |g_s(z) - g_s(f_q(x))|_{n+2} \times (1 + M_{n+2} + |g_s(z) - g_s(f_q(x))|_{n+2}). \quad (4)\]
Now choose $\delta(x) > 0$ so that $2 \delta (1 + M_{n+2} + \delta) < \epsilon/2$. By CI there is a $\xi(x)$ and a $Q'(x)$ so that, for all $q \geq Q'$ there is a $S'(x, q)$ so that

$$[\ d(f_q(x), z) < \xi \text{ and } s \geq S' \ ] \Rightarrow |g_s(z) - g_s(f_q(x))|_{n+2} < \delta. \tag{5}$$

As $f$ is a PBAM we know that $f_t(x)^* \cdot f_t(x) - f_t(x^* \cdot x) \to 0$ as $t \to \infty$, and as $x \in U(A)$ this means $f_t(x)^* \cdot f_t(x) \to 0$. Choose $N'(x) \geq 0$ so that $d(f_t(x)^* \cdot f_t(x), 0) < \eta$ for all $t \geq N'$. Given $q \geq 0$, as $g$ is a PBAM we know that $g_s(f_q(x))^* \cdot g_s(f_q(x)) - g_s(f_q(x)^* \cdot f_q(x)) \to 0$ as $s \to \infty$. Choose $S''(x, q) \geq 0$ so that

$$s \geq S'' \Rightarrow |g_s(f_q(x))^* \cdot g_s(f_q(x)) - g_s(f_q(x)^* \cdot f_q(x))|_n < \epsilon/4. \tag{6}$$

Part 2: Set $N = \max\{Q_{n+2}(x), Q'(x), N'(x)\}$ and $\nu = \xi(x)$. Given $q \geq N$ we set $L = \max\{P, S'(x, q), S''(x, q), S_{n+2}(x, q)\}$. Combining (3) and (4), we have

$$d(f_q(x), z) < \nu \text{ and } s \geq L \Rightarrow |g_s(z)^* \cdot g_s(z) - g_s(f_q(x))^* \cdot g_s(f_q(x))|_n < \epsilon/2. \tag{7}$$

As $q \geq N'$, we have $d(f_q(x)^* \cdot f_q(x), 0) < \eta$, and putting this into (3) we get

$$s \geq L \Rightarrow |g_s(f_q(x))^* \cdot f_q(x))|_n < \epsilon/4. \tag{8}$$

Combining this with (3) gives

$$s \geq L \Rightarrow |g_s(f_q(x))^* \cdot g_s(f_q(x))|_n < \epsilon/2, \tag{9}$$

which together with (4) gives the result. \(\square\)

**Lemma 8.2** For all $x \in U(A)$, all $n \in \mathbb{N}$ and all $\epsilon > 0$, there is an $\eta_n(x, \epsilon) > 0$ and a $T_n(x, \epsilon) \geq 0$ so that for all $y \in A$ and all $p \in [0, 1]$

$$t \geq T_n \text{ and } d(y, x) < \eta_n \Rightarrow |\vartheta_B(p(f_t(y)^* \cdot f_t(y)))|_n < \epsilon.$$

**Proof** The function $\vartheta_B : V_B \to B$ is continuous, so there is a $m(\epsilon, n) \in \mathbb{N}$ and a $\delta(\epsilon, n)$ so that

$$\forall b \in A_{sad} \quad |b|_m < \delta \Rightarrow [b \in V_B \text{ and } |\vartheta_B(b)|_n < \epsilon].$$

The problem then reduces to finding conditions under which $|f_t(y)^* \cdot f_t(y)|_m < \delta$, and this is essentially contained in the proof of lemma 5.1. \(\square\)
Definition 8.3 For values $t \in [0, \infty)$, $u \in \mathcal{U}(A)$ and $p \in [0,1]$ for which $f_t(u)^* \cdot f_t(u) \in V_B$ and $f_t(u) \cdot f_t(u)^* \in V_B$, we define $\hat{f}_{t,p}(u) = f_t(u) \cdot \vartheta(p(f_t(u)^* \cdot f_t(u)))$.

Lemma 8.4 The property $\Psi(x,y,s,t)$ defined as follows satisfies the conditions of lemma 2.4. For $x,y \in X = \mathcal{U}(A)$ and $s,t \geq 0$, $\Psi(x,y,s,t)$ is the logical value of the statement

$$f_t(y)^* \cdot f_t(y) \in V_B \quad \text{and} \quad f_t(y)^* \cdot f_t(y) \in V_B$$

$$\forall p \in [0,1] \quad [g_s(\hat{f}_{t,p}(y))^* \cdot g_s(\hat{f}_{t,p}(y)) \in V_C \quad \text{and} \quad g_s(\hat{f}_{t,p}(y))^* \cdot g_s(\hat{f}_{t,p}(y)) \in V_C ] \tag{10}$$

Proof Part 1: By 2.4, for every $x \in \mathcal{U}(A)$ there is a $T(x) \geq 0$ and an open neighbourhood $O(x)$ of $x$ in $A$ so that

$$\forall t \geq T(x) \quad \forall y \in O(x) \quad f_t(y) \cdot f_t(y)^* \in V_B \quad \text{and} \quad f_t(y)^* \cdot f_t(y) \in V_B .$$

By lemma 8.1, given $x \in \mathcal{U}(A)$, there is an $N(x) \geq 0$ and a $\nu(x) > 0$ so that, for $q \geq N$ there is an $L(x,q) \geq 0$ so that

$$\forall z \in B \quad [d(f_q(x),z) < \nu \quad \text{and} \quad s \geq L ] \Rightarrow [g_s(z)^* \cdot g_s(z) \in V_C \quad \text{and} \quad g_s(z) \cdot g_s(z)^* \in V_C ] .$$

We take $q \geq \max\{T,N\} + 1$, $y \in O(x) \cap \mathcal{U}(A)$ and $|t - q| < 1$ in what follows, so $\hat{f}_{t,p}(y)$ (see 8.3) is defined for all $p \in [0,1]$.

Part 2: Take $n \in \mathbb{N}$ and $1 > \chi > 0$ so that in $B$, $|z - f_q(x)|_n < \chi$ implies $d(f_q(x),z) < \nu$. Now consider the inequality

$$|f_t(y) \cdot b - f_q(x)|_n \leq |f_t(y) - f_q(x)|_{n+1} + |b|_{n+1} \left(1 + |f_q(x)|_{n+1} + |f_t(y) - f_q(x)|_{n+1}\right), \tag{11}$$

where $b = \vartheta(p(f_t(y)^* \cdot f_t(y)))$, so that $f_t(y) \cdot b = \hat{f}_{t,p}(y)$. As $f$ is pointwise bounded there is an $M(x,n+1)$ so that $|f_q(x)|_{n+1} \leq M$ for all $q \geq 0$. The conditions

$$|f_t(y) - f_q(x)|_{n+1} < \chi/2 \quad \text{and} \quad |\vartheta(p(f_t(y)^* \cdot f_t(y)))|_{n+1} < \chi/(4 + 2M) \tag{12}$$

and $s \geq L(x,q)$ then imply $\Psi(x,y,s,t)$.

Part 3: From 2.4 and 8.2 there are $T_{n+1}, \overline{P} \geq 0$ and $\eta_{n+1}, \overline{\eta} > 0$ (all depending on $x$) and, given $q \geq \overline{P}$, a $\overline{\delta} > 0$ (depending on $x$ and $q$) so that the conditions in (12) are true if $t \geq T_{n+1}$, $|t - q| < \overline{\delta}$ and $d(x,y) < \min\{\eta_{n+1}, \overline{\eta}\}$. Now define $Q(x) = \max\{T_{n+1}, \overline{P}, T, N\} + 1$, and choose $\min\{\eta_{n+1}, \overline{\eta}\} > \delta(x) > 0$ so that $d(x,y) < \delta$ implies $y \in O(x)$. Given $q \geq Q$, set $R(x,q) = L(x,q)$ and $\epsilon(x,q) = \min\{\overline{\delta}, 1\}$. □
Corollary 8.5 There are continuous functions $\gamma : \mathcal{U} \to [0, \infty)$ and $\theta : [0, \infty) \to [0, \infty)$ (increasing) so that the formula

$$r_{s,t,p}(u) = g_\phi(\hat{f}_{t,p}(u)) \cdot \partial_C( g_\phi(\hat{f}_{t,p}(u))^* \cdot g_\phi(\hat{f}_{t,p}(u)))$$

defines a continuous function

$$r : \{(u, s, t, p) \in \mathcal{U}(A) \times [0, \infty) \times [0, \infty) \times [0, 1] : t \geq \gamma(u) \text{ and } s \geq \theta(t) \} \to \mathcal{U}(C).$$

**Proof** Use 2.1 on 8.4. □

Theorem 8.6 $\mathcal{U}(g \circ f) = \mathcal{U}(g \circ \mathcal{U}(f)) \in [\mathcal{U}(A), \mathcal{U}(C)]$.

**Proof** By [f] there is a valid reparameterisation $\phi : [0, \infty) \to [0, \infty)$ for $g$ and $f$. Take the functions $\gamma : \mathcal{U}(A) \to [0, \infty)$ and $\theta : [0, \infty) \to [0, \infty)$ from lemma 8.5, and define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \max\{t, \phi(t), \theta(t)\}$. As $f$, $g$ and $g \circ \phi$ are PBAMs, we can generate respective functions $\alpha : \mathcal{U}(A) \to [0, \infty)$, $\beta : \mathcal{U}(B) \to [0, \infty)$ and $\mu : \mathcal{U}(A) \to [0, \infty)$ from [f]. Define $\lambda : \mathcal{U}(A) \to [0, \infty)$ by $\lambda(u) = \max\{\alpha(u), \gamma(u)\}$ and $\omega : \mathcal{U}(A) \to [0, \infty)$ by $\omega(u) = \max\{\alpha(u), \beta(\hat{f}_\lambda(u)), \gamma(u), \mu(u)\}$.

Now the function $h : \mathcal{U}(A) \times [0, 1] \to \mathcal{U}(C)$ defined by $h_p(u) = r_{\psi(\omega(u)), \omega(u), p}(u)$ gives a homotopy between $h_0(u) = (g \circ \phi)^\omega(u)$ and

$$h_1(u) = g_{\psi(\omega(u))}((\hat{f}_\omega(u))) \cdot \partial_C( g_{\psi(\omega(u))}(\hat{f}_\omega(u))^* \cdot g_{\psi(\omega(u))}(\hat{f}_\omega(u))) .$$

By using $\tau, \zeta : [0, 1] \times \mathcal{U}(A) \to [0, \infty)$ defined by $\tau(p)(u) = (1 - p)\omega(u) + p\lambda(u)$ and $\zeta(p)(u) = \max\{\psi(\tau(p)(u)), \beta(\hat{f}_\lambda(u))\}$ we define another homotopy $h' : \mathcal{U}(A) \times [0, 1] \to \mathcal{U}(C)$ by

$$h'_p(u) = g_{\zeta(p)(u)}(\hat{f}_{\tau(p)}(u)) \cdot \partial_C( g_{\zeta(p)(u)}(\hat{f}_{\tau(p)}(u))^* \cdot g_{\zeta(p)(u)}(\hat{f}_{\tau(p)}(u))) .$$

This time $h'_0 = h_1$ as $\psi(\omega(u)) \geq \omega(u) \geq \beta(\hat{f}_\lambda(u))$, and

$$h'_1(u) = g_{\zeta(1)(u)}(\hat{f}_{\lambda(u)}) \cdot \partial_C( g_{\zeta(1)(u)}(\hat{f}_{\lambda(u)})^* \cdot g_{\zeta(1)(u)}(\hat{f}_{\lambda(u)})).$$

Define another homotopy $h'' : \mathcal{U}(A) \times [0, 1] \to \mathcal{U}(C)$ by

$$h''_p(u) = g_{\eta(p)(u)}(\hat{f}_{\lambda(u)}) \cdot \partial_C( g_{\eta(p)(u)}(\hat{f}_{\lambda(u)})^* \cdot g_{\eta(p)(u)}(\hat{f}_{\lambda(u)})) ,$$

where $\eta(p)(u) = (1 - p)\zeta(1)(u) + p\beta(\hat{f}_\lambda(u))$. Now $h''_0 = h'_1$ and $h''_1 = \bar{g}_\beta \circ \hat{f}_\lambda$. □
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