A $^4$He shadow wavefunction with an inverse seventh power particle-particle correlation function

Orion Ciftja and Siu A. Chin

Department of Physics, Texas A&M University, College Station, Texas 77843

Francesco Pederiva

Dipartimento di Fisica, Università di Trento, Povo, Trento, Italy

(March 21, 2022)

Abstract

Many ground state studies of $^4$He using a shadow wave function with an inverse fifth power McMillan particle-particle correlation function have yielded radial distribution functions with misplaced peaks. It has been conjectured that this is due to the specific choice of the McMillan correlation function. However, beyond the use of fully optimized two-particle correlation functions, there has been little study of simple alternatives that can correct this defect. In this work we show that the remedy is surprisingly simple. When a shadow wavefunction with an inverse seventh power particle-particle correlation function is used to study $^4$He, it gives a correctly peaked radial distribution function, lowers the energy at all liquid and solid densities, and produces an excellent equation of state.

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The ground state properties of liquid and solid $^4$He have been studied extensively over the years by a variety of many-body techniques, ranging from hypernetted-chain (HNC) theories\cite{1,2} and variational Monte Carlo (VMC) methods\cite{3,4} to “exact” Monte Carlo methods (EMC) such as Green’s Function Monte Carlo (GFMC) and Diffusion Monte Carlo (DMC). The advance of EMC have seemingly obliterated the need for the “brutish” and bias-laden method of VMC. However, the introduction of the shadow wavefunction by Vitiello et al\cite{8} has added new subtlety and refinement to this approach. It has been shown that shadow wavefunctions can describe both the liquid and solid phase of $^4$He with excellent energy, while simultaneously maintaining translational and Bose symmetry. Since any VMC calculation is an order of magnitude less computationally demanding than corresponding EMC calculations, the method of shadow wavefunctions remained economically and conceptionally appealing.

However, it has been noted for some time that shadow wavefunctions with the McMillan inverse fifth power particle-particle correlation function do not give a correct radial distribution function for bulk liquid $^4$He. All the peaks are misplaced as in the original McMillan calculation. While there have been continued improvements on the form of two-particle correlation, leading to fully optimized correlations expressible in terms of a basis state, there has been no explorations of simpler alternatives to cure this defect. In this work, we shown that a simple replacement of the inverse fifth power by that of an inverse seventh power significantly improves simultaneously, the ground state energy, the equation of state and the radial distribution function.

The first VMC calculation of the groundstate properties of liquid and solid $^4$He was carried out by McMillan\cite{3} who employed a trial wave function with an inverse fifth power of the particle separation as the two-body correlation function. In this early study the potential between $^4$He atoms is taken to be the two-body Lennard-Jones (LJ) potential, $v(r) = 4\epsilon[(\sigma/r)^{12} - (\sigma/r)^6]$, with the DeBoer-Michels parameters, $\sigma = 2.556 \text{Å}$ and $\epsilon = 10.22 K^o$. Since then, the two-body HFDHE2 potential of Aziz et al\cite{9} has superceded the LJ potential as the potential of choice for the Helium studies. More recent minor revisions
of this potential have added, but have not greatly altered the quality of description of the interaction between Helium atoms at low pressure. In this work, we will continue to use the HFDHE2 potential to facilitate comparison with existing calculations in the literature.

The original shadow wavefunction of Vitiello et al can be written as:

\[ \Psi(R) = \Phi(R) \int dS \Theta(R, S) \Phi_S(S), \] (1)

with a Gaussian particle-shadow correlation function

\[ \Theta(R, S) = \exp \left[ -\sum_{i=1}^{N} C(r_i - s_i)^2 \right], \] (2)

where \( R \equiv \{r_1, r_2, \ldots, r_N\} \) and \( S \equiv \{s_1, s_2, \ldots, s_N\} \) represent the set of particle and shadow coordinates. The particle-particle correlation function \( \Phi(R) \) and the shadow-shadow correlation function \( \Phi_S(S) \) are both of the pair-wise product form, \( \Phi(R) = \exp \left[ -\sum_{i>j}^N u(r_{ij}) \right] \) and \( \Phi_S(S) = \exp \left[ -\sum_{i>j}^N u_S(s_{ij}) \right] \), where \( r_{ij} = |r_i - r_j| \) and \( s_{ij} = |s_i - s_j| \). Both \( \Phi(R) \) and \( \Phi_S(S) \) are taken as McMillan form of inverse fifth powers (\( m=5 \)) with \( u(r) = \frac{1}{2} (\frac{b}{r})^m \) and \( u_S(s) = (\frac{b'}{s})^m \), where \( b \) and \( b' \) are two variational parameters. The one-to-one coupling constant \( C \) between particles and shadows is treated as a variational parameter. We refer to this wavefunction as M+MS.

Optimizing the shadow-shadow pseudopotential in the form of \( u_S(s) = (\frac{b'}{s})^n \) with two variational parameters \( b' \) and \( n \), produces no significant improvements. A better choice was found following a suggestion by Reatto et al.: \( u_S(s) = \tau v(\alpha s) \), where \( v(r) \) is the Aziz HFDHE2 potential and \( \tau \) and \( \alpha \) are variational parameters. This scaled Aziz shadow correlation, introduced by MacFarland et al, which uses a McMillan inverse fifth power particle-particle pseudopotential (\( m=5 \)), will be denoted as M+AS. The M+AS shadow wavefunction substantially lowered the variational energies and improved the description of liquid and solid \(^4\text{He}\) at all densities. For example, at the GFMC equilibrium density \( \rho\sigma^3 = 0.365 \) and at freezing density of \( \rho\sigma^3 = 0.438 \), the M+AS energy is about 0.5 \( K^o \) lower than the M+MS energy.

Both wavefunctions, however, produce a radial distribution function at equilibrium density whose main peak is shifted outward by about 0.1 Å as compared with the experimental
value. The same misplacement is also observed at the GFMC freezing density. Such a misplacement can be corrected by optimizing the two-body correlations through the method of basis state expansion\textsuperscript{[3]} (The peak height is still underestimated, however.)

In this work, we show that this crucial defect can be simply corrected by a better choice of the inverse power (from 5 to 7) in McMillan’s form of the particle-particle correlation function.

Without reoptimizing the M+AS wavefunction’s shadow parameters, but only varying the variational parameter $b$ with $m = 7$, we obtained lower energies than those of M+AS at all liquid and solid densities, in an amount ranging from 0.1 to $0.3K^\circ$. This choice of the wavefunction in our work, referred to as M7+AS, allows us to improve the quality of the shadow wavefunction while retaining the same level of simplicity as before.

For a system of $N$ Helium atoms interacting via two-body forces only, the Hamiltonian has the form

$$\hat{H} = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \nabla_i^2 + \sum_{i>j}^{N} v(r_{ij}) \text{.}$$

where $v(r_{ij})$ is the Aziz HFDHE2 potential. The expectation value of the Hamiltonian can be expressed as

$$E = \frac{\int d\mathbf{R} \hat{\Psi}^*(\mathbf{R}) \hat{H} \hat{\Psi}(\mathbf{R})}{\int d\mathbf{R} |\Psi(\mathbf{R})|^2} \text{.}$$

$$= \int d\mathbf{R} d\mathbf{S} d\mathbf{S}' \ p(\mathbf{R}, \mathbf{S}, \mathbf{S}') \ E_L(\mathbf{R}, \mathbf{S}, \mathbf{S}') \text{.}$$

The local energy is written as

$$E_L(\mathbf{R}, \mathbf{S}, \mathbf{S}') = \frac{\hat{H} \Phi(\mathbf{R})\Theta(\mathbf{R}, \mathbf{S})}{\Phi(\mathbf{R})\Theta(\mathbf{R}, \mathbf{S})} \text{,}$$

and does not depend on $\Phi_S(\mathbf{S})$ since $\hat{H}$ acts only upon the variables describing the system of real particles.

The probability $p(\mathbf{R}, \mathbf{S}, \mathbf{S}')$ is given by

$$p(\mathbf{R}, \mathbf{S}, \mathbf{S}') = \frac{\Phi(\mathbf{R})^2 \Theta(\mathbf{R}, \mathbf{S}) \Phi_S(\mathbf{S}) \Theta(\mathbf{R}, \mathbf{S}') \Phi_S(\mathbf{S}')}{\int d\mathbf{R} d\mathbf{S} d\mathbf{S}' \Phi(\mathbf{R})^2 \Theta(\mathbf{R}, \mathbf{S}) \Phi_S(\mathbf{S}) \Theta(\mathbf{R}, \mathbf{S}') \Phi_S(\mathbf{S}')} \text{.}$$
To evaluate the expectation value of the Hamiltonian we use the Metropolis Monte Carlo algorithm to sample the probability density \( p(\mathbf{R}, \mathbf{S}, \mathbf{S}') \) from the \( 9N \) dimensional configuration space of the particles and two sets of shadow coordinates. In these computations, the Metropolis steps are subdivided in two parts. In the first, one attempts to move real particle coordinates at random inside cubical boxes of side length \( \Delta \). In the second, analogous attempts to move shadow coordinates are made inside cubical boxes of side length \( \Delta' \). After we attempt to move all the shadow coordinates of set \( \{\mathbf{S}\} \), the same is done for those in set \( \{\mathbf{S}'\} \). The parameters \( \Delta \) and \( \Delta' \) were adjusted so that the acceptance ratio for both particle and shadow moves was nearly 50%.

We compute the ground state variational energy, the radial distribution function \( g(r) \), and the static structure factor \( S(k) \). These quantities are spherical averages and have been computed for both the real particles and the shadow coordinates. The radial distribution function is defined by

\[
  g(r) = \frac{1}{N\rho} \sum_{i \neq j}^{N} \langle \delta(|\mathbf{r}_i - \mathbf{r}_j - \mathbf{r}|) \rangle ,
\]

(8)

where the angular brackets denote an average with respect to \( |\Psi(\mathbf{R})|^2 \) and \( \rho \) is the particle density. The static structure factor is obtained from the average

\[
  S(k) = \frac{1}{N} \langle \rho_{-k}\rho_k \rangle ,
\]

(9)

where \( \rho_k \) is given by \( \rho_k = \sum_{j=1}^{N} \exp(-i\mathbf{k}\mathbf{r}_j) \). By using this procedure \( S(k) \) is computed for a discrete set of \( \mathbf{k} \) values where the smaller wave vector compatible with the periodic boundary condition of the system is \( k = 2\pi/L \) ( \( L \) is the side of the simulation box).

All simulations presented in this work have been done with \( N = 108 \) atoms of \(^4\)He in a cubic box with periodic boundary conditions. To enforce periodicity, the two-body interaction potential \( v(r) \) smoothly goes to zero at a cutoff distance, \( r_c = L/2 \), equal half the side of the simulation box. We actually use a slightly modified two-body interaction potential \( v'(r) = v(r) - \Delta v(r) \) according to the replacement
\[ v'(r) = \begin{cases} 
  v(r) + v(2r_c - r) - 2v(r_c), & r \leq r_c \\
  0, & r > r_c 
\end{cases} \tag{10} \]

A correction \( \Delta V = (\rho/2) \int d^3 r g(r) \Delta v(r) \) was then added to the computed potential energy, where the radial distribution function \( g(r) \) comes from the simulation and is taken equal to 1 for \( r > r_c \). The shadow-shadow pseudopotential \( u_S(s) \) was modified according to the same prescription as \( v(r) \), while the particle-particle inverse power McMillan pseudopotential \( u(r) \) and its first two derivatives were slightly modified near the edge of the simulation box in order to go smoothly to zero, by using a third degree polynomial fit to the pseudopotential near the edge of the simulation box.

All calculations start from a perfect fcc crystal. Our runs consisted of a total of about 5.5 \( \cdot \) 10^5 passes during each of which an attempt was made to move particles and shadows. We allowed about 50 \( \cdot \) 10^3 passes for equilibration followed by about 5 \( \cdot \) 10^5 passes which comprise the equilibrated random walk.

In Table I we show the energy per particle obtained from the M7+AS shadow wavefunction after simulations with \( N = 108 \) particles at several densities of liquid \(^4\)He. Also included in the table are several results from the literature, GFMC refers to the results of Kalos et al.\(^6\).

In Table II we show the values of the optimum variational parameters \( b, C, \tau, \) and \( \alpha \) for the M7+AS shadow wavefunction at different densities \( \rho \) in the liquid phase.

The energy per particle for the M7+AS shadow wavefunction after simulations with \( N = 108 \) particles at some densities in the solid phase is shown in Table III. In the same table we show the VMC results obtained with the M+MS and M+AS shadow wavefunction, as well as the GFMC results.

Table IV shows the values of the optimum variational parameters \( b, C, \tau, \) and \( \alpha \) for the M7+AS shadow wavefunction at different densities \( \rho \) in the solid phase. We fit our equation of state in the liquid phase to a cubic polynomial of the form

\[ E(\rho) = E_0 + B \left( \frac{\rho - \rho_0}{\rho_0} \right)^2 + C \left( \frac{\rho - \rho_0}{\rho_0} \right)^3, \tag{11} \]
where $\rho_0$ is the equilibrium density. A similar function has been used to fit the experimental equation of state\cite{16} and we use it to analyze our M7+AS results. The values of the parameters in the fit and their errors are shown in Table \ref{tab:fit_params}, together with other results from the literature. One notes that the values of the coefficients $B$, $C$, and $\rho_0\sigma^3$ for the M7+AS shadow wavefunction are in good agreement with both GFMC and experimental results\cite{16}. In the solid phase we used the same parametrization as reported for the M+AS case. We fitted the energy to a cubic polynomial of the form

$$E(\rho) = E_0 + B \left( \frac{\rho}{\rho_S} - 1 \right)^2 + C \left( \frac{\rho}{\rho_S} - 1 \right)^3,$$

where the specific density $\rho_S\sigma^3 = 0.4486$ is taken from the GFMC calculation. The values of the parameters to this fit are shown in Table \ref{tab:fit_params}, together with other results from the literature. Again the M7+AS shadow wavefunction shows a good agreement with GFMC with the exception of a discrepancy in the coefficient $C$. In Fig. 4 we plot the equation of state for $^4$He liquid as obtained by using the values of the fitting parameters reported in Table \ref{tab:fit_params}. As already known, one notes that both M+AS and M7+AS wavefunctions give a better equation of state than the shadow wavefunction with fully optimized Jastrow particle-particle correlations (OJ+AS), although the OJ+AS wavefunction gives somewhat lower energies. It has been argued\cite{13} that possible causes of such behavior are the incomplete determination of the coefficients in the basis-set expansion for the OJ+AS wavefunction, or the missing full reoptimization of the shadow parameters. Indeed the recent VMC calculations with a fully optimized shadow wavefunction\cite{17} confirm this latter possibility.

In Fig. 4, we show the radial distribution function $g(r)$ obtained at the GFMC equilibrium density $\rho\sigma^3 = 0.365$. Our maximum of $g(r)$ is obtained at the same position $r_{\text{max}}$ as the GFMC value and it is clear that there is no shifting of our curve to larger values of $r$. Our variational peak $g(r_{\text{max}})$ is a little smaller than the GFMC value. Fig. 5 shows our results for the radial distribution function $g(r)$ determined at the GFMC freezing density $\rho\sigma^3 = 0.438$. Statistical errors in the GFMC $g(r)$ near the maximum $g(r_{\text{max}})$ are large, so a detailed comparison with GFMC is not possible. It appears that the position of our maximum of
$g(r)$ compares very well with the GFMC value, but the variational peak $g(r_{max})$ is again smaller. The trend seen at the GFMC equilibrium and freezing densities is repeated at all other densities.

In Fig. 4 we show $S(k)$ at the equilibrium density $\rho \sigma^3 = 0.365$. The experimental $S(k)$ shown in this figure is the result reported by Svensson et al. obtained by neutron diffraction at saturated vapor pressure at $T = 1.0K^\circ$. The agreement of the variational structure factor with experiment is seen to be very good for all $k$-s, except for small $k$. This is to be expected since our M7+AS wavefunction does not contain the proper long-range correlations necessary for the linear behavior of $S(k)$ which is observed in $^4$He.

In this work we have demonstrated the utility of our M7+AS shadow wavefunction for studying the ground state properties of liquid and solid $^4$He. The use of an inverse seventh power as the particle-particle correlation function has significantly improved the ground state energy, the equation of state and the radial density distribution. Such an improvement was obtained with very little additional computational effort. The wavefunction remained simple, compact and portable.
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TABLE I. Energies in liquid $^4$He at several densities including the experimental equilibrium density ($\rho\sigma^3 = 0.365$) and the GFMC freezing density ($\rho\sigma^3 = 0.438$) with $\sigma = 2.556\,\text{Å}$. All simulations use the Aziz HFDHE2 potential and have been performed for systems of $N = 108$ particles. The energies are given in Kelvin per particle. M+MS refers to a shadow wavefunction with McMillan fifth power-law pseudopotential (m=5) for both particle-particle and shadow-shadow pseudopotentials. M+AS refers to a shadow wavefunction with a rescaled Aziz HFDHE2 shadow-shadow pseudopotential and a McMillan fifth power-law particle-particle pseudopotential (m=5). M7+AS refers to a shadow wavefunction with a rescaled Aziz HFDHE2 shadow-shadow pseudopotential and a McMillan seventh power-law particle-particle pseudopotential (m=7) as used in this work. GFMC refers to the Green’s Function Monte Carlo calculations with the Mcmillan fifth power-law form for the importance and starting function.

| $\rho\sigma^3$ | Method | Trial function | Energy ($K^o$) |
|----------------|--------|----------------|---------------|
| 0.328          | VMC    | M+AS           | -6.561 ± 0.032|
|                | VMC    | M7+AS          | -6.571 ± 0.015|
|                | GFMC   | ...            | -7.034 ± 0.037|
| 0.365          | VMC    | M+MS           | -6.061 ± 0.025|
|                | VMC    | M+AS           | -6.599 ± 0.034|
|                | VMC    | M7+AS          | -6.664 ± 0.021|
|                | GFMC   | ...            | -7.120 ± 0.024|
| 0.401          | VMC    | M+AS           | -6.398 ± 0.019|
|                | VMC    | M7+AS          | -6.497 ± 0.012|
|                | GFMC   | ...            | -6.894 ± 0.048|
| 0.438          | VMC    | M+MS           | -5.360 ± 0.035|
|                | VMC    | M+AS           | -5.871 ± 0.016|
|                | VMC    | M7+AS          | -6.067 ± 0.010|
|                | GFMC   | ...            | -6.564 ± 0.058|
TABLE II. The variational parameters of the M7+AS shadow wavefunction used in the simulation of $^4$He liquid with $N = 108$ particles at different densities.

| $\rho\sigma^3$ | $b/\sigma$ | $C\sigma^3$ | $\tau(K^{-1})$ | $\alpha$ |
|----------------|------------|-------------|----------------|--------|
| 0.328          | 1.02       | 5.5         | 0.088          | 0.915  |
| 0.365          | 1.01       | 5.5         | 0.095          | 0.915  |
| 0.401          | 1.01       | 6.0         | 0.105          | 0.920  |
| 0.438          | 1.01       | 5.9         | 0.110          | 0.910  |
TABLE III. Energies in solid $^4$He at several densities including the GFMC melting density ($\rho\sigma^3 = 0.491$) with $\sigma = 2.556\text{Å}$. All simulations use the Aziz HFDHE2 potential and have been performed for systems of $N = 108$ particles. The energies are given in Kelvin per particle. The notation is the same as in Table III. The GFMC result at density $\rho\sigma^3 = 0.550$ was interpolated from the GFMC results at $\rho\sigma^3 = 0.526$ and $\rho\sigma^3 = 0.560$. The M7+AS results represent this work.

| $\rho\sigma^3$ | Method | Trial function | Energy ($K^o$) |
|---------------|--------|----------------|----------------|
| 0.491         | VMC    | M+MS           | -5.004 ± 0.055 |
|               | VMC    | M+AS           | -5.052 ± 0.014 |
|               | VMC    | M7+AS          | -5.324 ± 0.010 |
|               | GFMC   | ...            | -5.610 ± 0.030 |
| 0.550         | VMC    | M+MS           | -3.521 ± 0.032 |
|               | VMC    | M+AS           | -3.639 ± 0.012 |
|               | VMC    | M7+AS          | -3.724 ± 0.017 |
|               | GFMC   | ...            | -4.197 ± 0.030 |
| 0.589         | VMC    | M+AS           | -1.947 ± 0.012 |
|               | VMC    | M7+AS          | -2.097 ± 0.010 |
|               | GFMC   | ...            | -2.680 ± 0.060 |

TABLE IV. The variational parameters of the M7+AS shadow wavefunction used in the simulation of $^4$He solid with $N = 108$ particles at different densities.

| $\rho\sigma^3$ | $b/\sigma$ | $C\sigma^3$ | $\tau(K^{-1})$ | $\alpha$ |
|---------------|------------|-------------|----------------|----------|
| 0.491         | 1.00       | 5.7         | 0.110          | 0.875    |
| 0.550         | 1.00       | 5.9         | 0.100          | 0.890    |
| 0.589         | 1.00       | 6.5         | 0.110          | 0.900    |
TABLE V. Fit parameters of the equation of state for $^4$He in the liquid phase. The OJ+AS shadow wavefunction incorporates a fully optimized Jastrow particle-particle pseudopotential. The GFMC result is taken from Kalos at al. The experimental equation of state (Exp) is taken from Roach et al.

|        | $E_0(K^o)$  | $B(K^o)$  | $C(K^o)$  | $\rho_0\sigma^3$ |
|--------|-------------|-----------|-----------|-----------------|
| M+AS   | -6.610 ± 0.036 | 10.3 ± 5.5 | 11.3 ± 18.5 | 0.3535 ± 0.0043 |
| OJ+AS  | -6.796 ± 0.025 | 14.10 ± 4.18 | -18.7 ± 18.1 | 0.3567 ± 0.0032 |
| M7+AS  | -6.662 ± 0.020 | 14.08 ± 1.36 | 6.72 ± 8.02  | 0.361 ± 0.001  |
| GFMC   | -7.110 ± 0.023 | 10.08 ± 3.20 | 12.59 ± 8.50 | 0.3600 ± 0.0049 |
| Exp    | -7.14        | 13.65      | 7.67       | 0.365           |

TABLE VI. Fit parameters of the equation of state for $^4$He in the solid phase. The notation is the same as in Table V. The fitting curve is $E = E_0 + B[(\rho - \rho_s)/\rho_s]^2 + C[(\rho - \rho_s)/\rho_s]^3$, where $\rho_s\sigma^3 = 0.4486$ is taken from the GFMC result.

|        | $E_0(K^o)$  | $B(K^o)$  | $C(K^o)$  | $\rho_s\sigma^3$ |
|--------|-------------|-----------|-----------|-----------------|
| M+AS   | -5.340 ± 0.021 | 31.00 ± 1.50 | 9.92 ± 4.34 | 0.4486 ± 0.0097 |
| OJ+AS  | -5.81 ± 0.02  | 47.7 ± 1.4  | -33.89 ± 4.14 | 0.4486 ± 0.0097 |
| M7+AS  | -5.693 ± 0.013 | 32.92 ± 1.00 | -13.92 ± 1.45 | 0.4486 ± 0.0097 |
| GFMC   | -5.899 ± 0.121 | 31.95 ± 5.26 | 3.395 ± 80.0   | 0.4486 ± 0.0097 |
FIGURES

FIG. 1. Equation of state of $^4$He liquid. The lines are polynomial cubic fits. The solid line is the experimental results. The dotted line denotes the GFMC result. The dashed line denotes OJ+AS. The solid line with opaque circles is the M7+AS result. The solid line with crosses is the M+AS result.

FIG. 2. Radial distribution function $g(r)$ at the GFMC equilibrium density $\rho \sigma^3 = 0.365$ after a VMC simulation with N=108 particles. The solid line denotes $M7 + AS$. Filled circles are the GFMC results of Kalos et al.

FIG. 3. Radial distribution function $g(r)$ at the GFMC freezing density $\rho \sigma^3 = 0.438$ after a VMC simulation with N=108 particles. The solid line denotes $M7 + AS$. Filled circles are the GFMC results of Kalos et al.

FIG. 4. Static structure factor $S(k)$ of liquid $^4$He at the GFMC equilibrium density $\rho \sigma^3 = 0.365$. The filled circles show our results obtained from the formula $S(k) = \frac{1}{N} \langle \rho_{-k} \rho_k \rangle$ at a discrete set of $k$-points. The solid line denotes the experimental results reported by Svensson and co-workers obtained at saturated vapor pressure by means of neutron diffraction at temperature $T = 1.0K$. 
\[ \rho \sigma^3 \]

-7.3
-7.2
-7.1
-7.0
-6.9
-6.8
-6.7
-6.6
-6.5
-6.4
-6.3
-6.2
-6.1
-6.0
-5.9
-5.8

$E$ (K)

-5.9
-5.8
-5.7
-5.6
-5.5
-5.4
-5.3
-5.2
-5.1
-5.0
-4.9
-4.8
-4.7
-4.6
-4.5
-4.4
-4.3
-4.2
-4.1
-4.0
-3.9
-3.8
-3.7
-3.6
-3.5
-3.4
-3.3
-3.2
-3.1
-3.0
-2.9
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-2.6
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-2.4
-2.3
-2.2
-2.1
-2.0
-1.9
-1.8
-1.7
-1.6
-1.5
-1.4
-1.3
-1.2
-1.1
-1.0
-0.9
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

\[ \rho \sigma^3 \]

Exp
GFMC
M7+AS
M+AS
OJ+AS

\[ \rho \sigma^3 \]
