Stability of the Parisi solution for the Sherrington–Kirkpatrick model near $T = 0$

A Crisanti$^1$ and C De Dominicis$^2$

$^1$ Dipartimento di Fisica, Università di Roma La Sapienza and ISC-CNR, P.le Aldo Moro 2, I-00185 Roma, Italy
$^2$ Institut de Physique Théorique, CEA-Saclay-Orme des Merisiers, 91191 Gif sur Yvette, France

E-mail: andrea.crisanti@phys.uniroma1.it and cirano.de-dominicis@cea.fr

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Abstract
To test the stability of the Parisi solution near $T = 0$, we study the spectrum of the Hessian of the Sherrington–Kirkpatrick model near $T = 0$, whose eigenvalues are the masses of the bare propagators in the expansion around the mean-field solution. In the limit $T \ll 1$, two regions can be identified. In the first region, for $x$ close to 0, where $x$ is the Parisi replica symmetry breaking scheme parameter, the spectrum of the Hessian is not trivial and maintains the structure of the full replica symmetry breaking state found at higher temperatures. In the second region $T \ll x \leq 1$, as $T \to 0$, the components of the Hessian become insensitive to changes of the overlaps and the bands typical of the full replica symmetry breaking state collapse. In this region only two eigenvalues are found: a null one and a positive one, ensuring stability for $T \ll 1$. In the limit $T \to 0$, the width of the first region shrinks to 0 and only the positive and null eigenvalues survive. As byproduct we enlighten the close analogy between the static Parisi replica symmetry breaking scheme and the multiple time-scales approach of dynamics, and compute the static susceptibility showing that it equals the static limit of the dynamic susceptibility computed via the modified fluctuation dissipation theorem.

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1. Introduction
The physics of spin glasses is still an active field of research because the methods and techniques developed to analyze the static and dynamic properties have found application in a variety of others fields of the complex system world, such as neural networks, combinatorial optimization or glass physics. In the study of spin glasses, a central role is played by the Sherrington–Kirkpatrick (SK) model [1], introduced in the mid-1970s as a mean-field model.
for spin glasses. Despite that its solution, known as the ‘Parisi solution’ [2–4], was found 30 years ago, some aspects are still far from being completely understood. In this work, we discuss the spectrum of the Hessian of the fluctuations for the Parisi solution in the limit of vanishing temperature, a problem still not fully explored.

The Hessian spectrum plays a central role not only for the stability of the Parisi solution of the mean-field SK model, but also for the study of finite-dimensional systems. Its eigenvalues are indeed the masses of the ‘bare’ propagators in the loop expansion about the mean-field limit. Thus, knowledge of the Hessian spectrum of the SK model is a prerequisite for any theory obtained from the development about the mean-field limit.

The stability of the Parisi solution for the SK model near its critical temperature $T_c$, has been established long ago [5, 6] by exhibiting the eigenvalues of the Hessian matrix. Briefly, one has a Replicon band whose lowest eigenvalues are zero modes, and a Longitudinal–Anomalous (LA) band, sitting at $(T_c - T)$, of positive eigenvalues (both with) a band width of order $(T_c - T)^2$. The analysis was partially extended later [7] via the derivation of Ward–Takahashi identities, showing that the zero Replicon modes would remain null in the whole low-temperature phase, and hence would not ruin the stability under loop corrections to the mean-field solution.

Despite these efforts, a complete analysis of the stability in the zero temperature limit is still missing. Near $T_c$ one can take advantage of the vanishing of the order parameter for $T = T_c$ and expand the free energy, a simplification clearly missing close to zero temperature, where the order parameter stays finite. Moreover the $T = 0$ limit is highly non-trivial. All these make the derivation of ‘effective’ approximations valid for $T \to 0$ a rather difficult task [8, 9].

In this work, anticipating the main results, we show that in the limit $T \ll 1$ the spectrum of the Hessian can be divided into two regions. The first region where the spectrum maintains a structure similar to that found close to $T_c$, and the second region where only two eigenvalues, one null and the other positive, are found. In the limit $T \to 0$, the width of the first region shrinks to 0, and only the second region survives.

The outline of the paper is as follows. In section 2, we describe how the Hessian of fluctuations associated with the SK model is obtained. In section 3, we discuss the properties of the Parisi solution in the low-limit $T \ll 1$ and how these affect the Hessian spectrum by considering three simple cases. In section 4, we show how spin averages, and response functions involving any number of spins can be computed within the Parisi replica symmetry breaking scheme with a finite number $R$ of replica symmetry breaking steps. In sections 5 and 6, using the results of section 4, we derive the Hessian spectrum in the $T \to 0$ limit for both the Replicon and LA sectors. Finally section 7 contains some discussions and conclusions. The two appendices contain details on the calculation of spin averages in the continuous $R \to \infty$ limit (appendix A) and the $T \ll 1$ limit (appendix B). For completeness, in appendix C we report the approach in terms of frozen fields probability distribution functions.

2. Free energy functional, fluctuations and propagator masses

The model is defined by the Hamiltonian [10]

$$H = -\frac{1}{2} \sum_{i,j} J_{ij} s_i s_j,$$

where $s_i = \pm 1$ are $N$ Ising spins located on a regular $d$-dimensional lattice and the symmetric bonds $J_{ij}$, which couple nearest-neighbor spins only, are random quenched Gaussian variables of the zero mean. The variance is properly normalized to ensure a well-defined thermodynamic
limit \( N \to \infty \). To average over the disorder one introduces replicas. After standard manipulations the free-energy density functional \( f \) in the thermodynamic limit is written as a function of the symmetric \( n \times n \) site-dependent replica overlap matrix \( Q_{ab} \) as \([11]\)

\[
e^{-Nf/T} = \int \prod_{(ab)} \prod_{i} dQ_{ab}^i \exp \mathcal{L}\{Q_{ab}^i\}
\]

\[
\mathcal{L}\{Q_{ab}^i\} = -\frac{\beta^2}{2} \sum_{p} (p^2 + 1) \sum_{(ab)} (Q_{ab}^p)^2 + \sum_{i} \ln \text{Tr}_{s} \exp \left( \frac{\beta^2}{2} \sum_{(ab)} Q_{ab}^i s^a s^b \right),
\]

where \( Q_{ab}^p \) is the spatial Fourier transform of \( Q_{ab}^i \) with respect to the site index \( i \) and \( \beta = 1/T \).

The notation ‘\((ab)\)’ means that the sum is over distinct ordered pairs \( a < b \) of replicas. Equations (2) and (3) are the starting points of the perturbative expansion around the mean-field theory. One then writes

\[
Q_{ab}^i = Q_{ab}^0 + \delta Q_{ab}^i.
\]

where \( Q_{ab}^0 \) is the mean-field order parameter, and expands \( \mathcal{L} \) in powers of \( \delta Q_{ab}^i \),

\[
\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \cdots.
\]

The first term

\[
\mathcal{L}^{(0)} = N \left[ -\frac{\beta^2}{2} \sum_{(ab)} (Q_{ab}^0)^2 + \ln \text{Tr}_{s} \cdot \exp \left( \frac{\beta^2}{2} \sum_{(ab)} Q_{ab}^0 s^a s^b \right) \right]
\]


gives the free energy density \( f \) in the mean-field limit, and equals that of the SK model. The second term reads

\[
\mathcal{L}^{(1)} = -\beta^2 \sum_{i} \sum_{(ab)} \delta Q_{ab}^i [Q_{ab}^0 - \langle s^a s^b \rangle],
\]

where \n
\[
\langle s^a s^b \rangle = \frac{\text{Tr}_{s} s^a s^b \exp(\beta^2 \sum_{(ab)} Q_{ab}^0 s^a s^b) / \text{Tr}_{s} \exp(\beta^2 \sum_{(ab)} Q_{ab}^0 s^a s^b)}. \]

The vanishing of \( \mathcal{L}^{(1)} \) yields the stationary condition that determines the mean-field value of the order parameter \( Q_{ab}^0 = \langle s^a s^b \rangle \), and ensures that tadpoles do not show up in the loop expansion. Below the critical temperature \( T_c \), the phase of the SK model is characterized by a large, yet not extensive, number of degenerate locally stable states in which the system freezes. The symmetry under replica exchange is broken and the overlap matrix \( Q_{ab} \) becomes a non-trivial function of replica indices. In the Parisi parameterization \([12]\), the matrix \( Q_{ab} \) for \( R \) steps of replica exchange symmetry breaking is divided into successive boxes of decreasing size \( p_r \), with \( p_0 = n \) and \( p_{R+1} = 1 \), and elements given by

\[
Q_{ab} = Q_r, \quad r = 0, \ldots, R + 1,
\]

where \( r = a \cap b \) denotes the overlap between the replica \( a \) and \( b \), and means that \( a \) and \( b \) belong to the same box of size \( p_r \), but to two distinct boxes of size \( p_{r+1} < p_r \). The solution of the SK model is obtained by letting \( R \to \infty \). In this limit the matrix \( Q_{ab} \) is described by a continuous non-decreasing function \( Q(x) \) parameterized by a variable \( x \), which in the Parisi

\[
Q_{ab} = Q_{R+1} = 1.
\]

The equality \( Q_{ab} = \langle s^a s^b \rangle \) that follows from the stationarity condition is valid only for \( a \neq b \). For consistency one defines \( Q_{aa} = Q_{R+1} = 1. \)
scheme is \( x \in [0, 1] \) and measures the probability for a pair of replicas to have an overlap not larger than \( Q(x) \).

The meaning of \( x \) depends on the parameterization used for the matrix \( Q^{ab} \). In the dynamical approach [13], \( x \) labels the relaxation time scale \( t_x \), so that \( Q(x) = \langle s(t_x) s(0) \rangle \). Here the angular brackets denote time (and disorder) averaging. The smaller the \( x \), the longer \( t_x \) will be. All time scales diverge in the thermodynamic limit but \( t_x / t_x' \rightarrow \infty \) if \( x > x' \). To make contact with the static Parisi solution one takes \( x \in [0, 1] \), with \( x = 0 \) corresponding to the largest possible relaxation time and \( x = 1 - \) to the shortest one. With this assumption one recovers \( Q(0) = 0 \) and \( Q(1 -) = q_c(T) \), the largest overlap. In both cases \( Q(1) = 1 \), since it gives the self or equal-time overlap. Other choices are possible, e.g., those used in [14–18] to tackle the \( T \rightarrow 0 \) limit. We stress however that different choices just give a different parameterization of the function \( Q(x) \), but do not change the physics, since this is given by the possible values \( q \) that the function \( Q(x) \) can take and by their probability distribution \( P(q) \).

This property is called gauge invariance [13, 14, 19]. In what follows, unless explicitly stated, we take for \( x \) the Parisi parameterization.

The quadratic term

\[
\mathcal{L}^{(2)} = -\frac{\beta^2}{2} \sum_p (p^2 + 1) \sum_{(ab)} (\delta Q^{ab}_p)^2 + \frac{\beta^4}{2} \sum_{(ab),(cd)} \delta Q^{ab}_i \delta M^{ab,cd} \delta Q^{cd}_i
\]

defines the ‘bare’ propagators of the theory. This quadratic form in \( \delta Q^{ab}_i \) contains the Hessian matrix

\[
M^{ab,cd} = \delta_{(ab);(cd)} - \beta^2 \delta M^{ab,cd}
\]

of the SK model whose eigenvalues rule the stability of the mean-field solution, and give the masses of the ‘bare’ propagators. Terms with higher powers of \( \delta Q^{ab}_i \) in expansion (5) define the interaction vertices of the theory.

In the reminder of this paper we shall consider the eigenvalue spectrum of the Hessian matrix \( M^{ab,cd} \) of the SK model for the Parisi solution in the very low temperature limit \( T \ll 1 \).

2.1. The Hessian \( M^{ab,cd} \): the Replicon and LA sectors

With four replicas the Hessian is characterized by three overlaps. We can distinguish two different geometries.

(i) The LA sector. This is characterized by the two overlaps \( r = a \cap b \) and \( s = c \cap d \) and, if \( r \neq s \), the single cross-overlap \( t = \max[a \cap c, a \cap d, b \cap c, b \cap d] \). Then we denote the matrix element in the LA sector as

\[
M^{ab,cd} = M^{rs}_{t,s}, \quad r, s = 0, 1, \ldots, R; \quad t = 0, 1, \ldots, R.
\]

Note that \( t = R + 1 \) if \( a = c \) or \( a = d \) or \( b = c \) or \( b = d \).

(ii) The Replicon sector. In this case \( a \cap b = c \cap d = r \), and the geometry is characterized by the two cross-overlaps

\[
\begin{align*}
u &= \max[a \cap c, a \cap d] \\
u &= \max[b \cap c, b \cap d]
\end{align*}
\]

For the Replicon sector the matrix elements are denoted as

\[
M^{ab,cd} = M^{uv}_{uv}, \quad u, v \geq r + 1.
\]
The element $M^{r^r}_{u,v}$, however, contains contribution from both the Replicon and LA sectors, and one has [20]

$$M^{r^r}_{u,v} = R M^{r^r}_{u,u} + M^{r^r}_v + M^{r^r}_v - M^{r^r}_v,$$

where the first is the Replicon contribution while the others come from the LA sector. The latter can be projected out by taking the double replica Fourier transform (RFT) on the cross-overlaps $u, v$:

$$M^r_{k,l} = \sum_{u=1}^{R+1} \sum_{v=1}^{R+1} p_u p_v [M^{r^r}_{u,v} - M^{r^r}_{u-1,v} - M^{r^r}_{u,v-1} + M^{r^r}_{u,v-1-1}].$$

The LA terms indeed cancel in this expression and one can replace $M^{r^r}_{u,v}$ in the double RFT by $R M^{r^r}_{u,v}$. This in turn implies that the inverse double RFT of $M^r_{k,l}$ yields the Replicon contribution $R M^{r^r}_{u,v}$ and not $M^{r^r}_{u,v}$.

3. How things work near $T = 0$: simplest cases

The equation for $Q(x)$ is rather difficult to solve by analytical and/or numerical methods for $T \to 0$. The origin of this difficulty can be traced back to the fact that, as the temperature decreases toward $T = 0$, the probability of finding overlaps $Q^{ab}$ sensibly smaller than $q_c(T) = 1 - \alpha T^2 + O(T^3)$, with $\alpha = 1.575 \ldots$, vanishes with $T^{\frac{1}{2}}$[21, 22]. There is however a finite probability $x_c \simeq 0.524 \ldots$ that $Q^{ab} \leq q_c(T)$. As consequence of this, the order parameter function $Q(x)$ in the Parisi parameterization develops for $T \ll 1$ a boundary layer of thickness $\delta \sim T$ close to $x = 0$, as shown in figure 1. From the figure we see that for very small $T$ the function $Q(x)$ is slowly varying for $\delta \ll x \ll x_c$. However, in the boundary layer $0 < x \leq \delta$, it undergoes an abrupt and rapid change. In the limit $T \to 0$, the thickness $\delta \sim T \to 0$ and the order parameter function becomes discontinuous at $x = 0$.

Uniform approximate solutions valid for $T \ll 1$ can be constructed using the boundary layer theory, that is by studying the problem separately inside (inner region) and outside (outer region).
region) the boundary layer [23]. One then introduces the notion of the inner and outer limit of the solution. The outer limit is obtained by choosing a fixed $x$ outside the boundary layer, that is in $\delta \ll x \ll 1$, and allowing $T \to 0$. Similarly, the inner limit is obtained by taking $T \to 0$ with $x \leq \delta$. This limit is conveniently expressed introducing an inner variable $a$, such as $a = x/\delta$, in terms of which the solution is slowly varying inside the boundary layer as $T \to 0$. The inner and outer solutions are then combined together by matching them in the intermediate limit $x \to 0$, $x/\delta \to \infty$ and $T \to 0$. The inner solution $Q(a)$ is a smooth function of $a$ for $T \to 0$ varying between 0 and $q_c \simeq 1$ [14, 16, 17, 24], similar to $Q(x)$ at finite temperature. In the rest of this paper we concentrate on the outer solution since as $T \to 0$ it covers the overwhelming part of the interval $[0, 1]$.

The behavior of $Q(x)$ for $T \ll 1$ has strong consequences on other relevant quantities, such as, e.g., the four-spin correlation entering into the Hessian matrix. We shall make this more quantitative in the following sections. Here the only feature we wish to retain is that in such as, e.g., the four-spin correlation entering into the Hessian matrix. We shall make this insensitivity of the $Q(x)$, where $\alpha = \lim_{T \to 0} (1 - q(x_c))/T^2$. We note that the breakpoint $x_c$ depends on $T$. The dependence is however very weak for low temperatures [22] and the approximation $x_c(T) \simeq x_c = 0.524 \ldots$ is rather good for $T \sim 0$. From this expression we see that the variation of $Q(x)$ in the outer region is

$$Q(x) = 1 - c (\beta x)^2 + \left(\frac{c}{x_c} - \alpha\right) T^2 + \mathcal{O}((\beta x)^3, T^3),$$

where $c = 0.4108 \ldots$ and $\alpha = \lim_{T \to 0} (1 - q(x_c))/T^2$. We note that the breakpoint $x_c$ depends on $T$. The dependence is however very weak for low temperatures [22] and the approximation $x_c(T) \simeq x_c = 0.524 \ldots$ is rather good for $T \sim 0$. From this expression we see that the variation of $Q(x)$ in the outer region is

$$\frac{Q(x_c) - Q(x)}{Q(x)} \simeq c \left(\frac{T}{x}\right)^2 \left[1 - \left(\frac{x}{x_c}\right)^2\right]$$

so that one can safely take the approximation $Q(x) \sim Q(x_c) = q_c(T)$ as $T \to 0$, the error being $\mathcal{O}(T^2)$ at least. Going back to $R$ steps of replica symmetry breaking this approximation translates into

$$Q_r \sim Q_R = q_c(T) = 1 - a T^2 + \mathcal{O}(T^3), \quad T \to 0$$

for all $r$ in the outer region, that is, such that $T \ll x(Q_r) = p$, and $T \to 0$, or, equivalently, for fixed $r \neq 0$ and $T \to 0$. We shall make this insensitivity with respect to the overlaps $r$ in the $T \ll 1$ limit more precise in the following sections. Here we just discuss the consequence of the insensitivity on the elements of the Hessian by considering some simple cases.

Suppose the two pairs of replicas are equal: $(a, b) = (c, d)$. In this case, from equation (11) one constructs the simplest Hessian component

$$M^{ab, ab} = 1 - \beta^2 [\langle s^a s^h \rangle^2] - \langle s^a s^h \rangle \langle s^a s^b \rangle]$$

$$= 1 - \beta^2 [1 - \langle Q^{ab} \rangle^2]$$

(20)

that for the overlap $a \cap b = r$ gives

$$M^{R^r}_{R^r + 1, R^r + 1} = 1 - \beta^2 (1 - Q^2_r).$$

(21)

Insensitivity implies that for fixed $r$ and $T \to 0$ we have

$$M^{R^r}_{R^r + 1, R^r + 1} \sim M^{R^R}_{R^R + 1, R^R + 1} = 1 - 2a + \mathcal{O}(T^2), \quad T \to 0.$$ 

(22)

The next simple case is when only three replicas are different, in which case we have

$$M^{ab, ac} = -\beta^2 [(s^b s^c) - \langle s^a s^h \rangle \langle s^a s^c \rangle], \quad b \neq c.$$ 

(23)
Ultrametricity imposes that the three replicas \(a, b, c\) with \(a \cap b = r\) can be only disposed as shown in figure 2. The LA geometries (a) and (b) lead for \(T \to 0\) and fixed \(r\) and \(s\) to

\[
M_{R+1}^{s,r} = -\frac{Q_r - Q_s}{T^2} \simeq -\frac{Q_r(1 - Q_r)}{T^2} = -\alpha + O(T^2),
\]

while the Replicon geometry (c) yields

\[
M_{R+1}^{s,r} = -\frac{Q_s - Q_r^2}{T^2} \simeq -\frac{Q_r(1 - Q_r)}{T^2} = -\alpha + O(T^2).
\]

We shall see below that insensitivity implies that \(M_{s,r}^{s,r} \sim 0\), and that all Replicon components vanish. Then from equations (15) and (25) it follows

\[
M_{R+1}^{R;R} = -\alpha + O(T^2), \quad T \to 0.
\]

Similarly from (15) and (22) one obtains

\[
2M_{R+1}^{R;R} - M_{R}^{R;R} = 1 - 2\alpha + O(T^2),
\]

which combined with (26) gives

\[
M_{R}^{R;R} = -1 + O(T^2), \quad T \to 0.
\]

The general case with four different replicas cannot be reduced to simple forms and the expression of the four-spin averages is required. This will be derived in the next section.

4. Spin averages

The evaluation of the Hessian components requires the computation of the four-spin averages \(\langle s_a s_b s_c s_d \rangle\) for a generic geometry of the four replicas. This can be done by introducing the generating function

\[
\mathcal{Z}(b) = \exp[nG_{-1}(b)] = \text{Tr}_r \exp\left(\frac{1}{2} \sum_{ab} \Lambda_{ab} s_a s_b + \sum_a b_a s_a^a\right),
\]

where \(\Lambda_{ab}\), equal to \(\beta^2 Q_{ab}\) with \(\beta = 1/T\) for the SK model, is a generic \(n \times n\) symmetric matrix with Parisi’s block structure:

\[
\Lambda_{ab} \big|_{a \cap b = r} = \lambda_r, \quad r = 0, \ldots, R + 1.
\]
Spin averages follow from the differentiation
\[ \langle s^a s^b \cdots \rangle = \lim_{n \to 0} \frac{1}{Z(b)} \left[ \frac{\partial}{\partial b_a} \frac{\partial}{\partial b_b} \cdots \right] Z(b) \bigg|_{b_1 = \cdots = b_n = 0}. \]  
(31)

Introducing the 'block indices' \( a_k \)
\[ a = (a_0, a_1, \ldots, a_R), \quad a_k = 0, \ldots, \frac{p_k}{p_{k+1}} - 1, \]  
(32)

where \( p_k \), with \( n = p_0 > p_1 > \cdots > p_R > p_{R+1} = 1 \), are the block sizes, the generating function can be written as multiple integrals over independent Gaussian variables:
\[ Z(b) = \int D_R(\alpha) \prod R \exp G_R(b^R_a + b_a), \]  
(33)

where \( D_R(\alpha) \) is the short-hand notation for
\[ \int D_R(\alpha) \equiv \prod R \int Dz^I \equiv R \int \prod_{t=0}^{R} Dz_{a_0,\ldots,a_{t-1}} \]  
(34)

and \( z_{a_0,\ldots,a_{t-1}}^I \) are independent Gaussian random variables of zero mean and variance one:
\[ Dz \equiv \frac{dz}{\sqrt{2\pi}} e^{-z^2/2}. \]  
(35)

The function \( G_R(b) \) is the ‘free energy’ of a single spin in a field \( b \),
\[ \exp G_R(b) = \text{Tr}_b \exp(b s) = 2 \cosh b, \]  
(36)

and the frozen (random) field \( b^R_a \), given by
\[ b^R_a = \sum_{t=0}^{R} \sqrt{\Delta \lambda_t z_{a_0,\ldots,a_{t-1}}}, \]  
(37)

where \( \Delta \lambda_t = \lambda_t - \lambda_{t-1} \), keeps track of contributions from the various blocks.

Inserting the form (33) of \( Z(b) \) into equation (31), and noting that the differentiation with respect to \( b_a \) can be replaced by the differentiation with respect to \( b^R_a \), we obtain
\[ \langle s^a s^b \cdots \rangle = \lim_{n \to 0} \frac{1}{Z(0)} \left[ \frac{\partial}{\partial b^R_a} \frac{\partial}{\partial b^R_b} \cdots \right] \int D_R(\alpha) \prod R \exp G_R(b^R_a)
\[ = \lim_{n \to 0} \frac{1}{Z(0)} \int D_R(\alpha) \prod R \exp G_R(b^R_a) \left[ \frac{\partial}{\partial b^R_a} G_R(b^R_a) \right] \left[ \frac{\partial}{\partial b^R_b} G_R(b^R_b) \right] \cdots \]  
(38)

For any given geometry of the replicas \( a, b, c \ldots \) the integrals can now be performed recursively from scale \( R \) up to scale 0. To illustrate the procedure let us consider
\[ Z(0) = \int D_R(\alpha) \prod R \exp G_R(b^R_a). \]  
(39)

4 We use Greek letters for summed replica indices.
The field $b_{\alpha}^R$ can be written as

$$b_{\alpha}^R = \sqrt{\Delta \lambda_{R}^R} z_{\alpha}^R + \sum_{t=0}^{R-1} \sqrt{\Delta \lambda_{t}} z_{t}^R$$

Then splitting out the $z_{\alpha}^R$-integrals, and recalling that $z_{\alpha}^R$ depends only upon indices $\alpha_0, \ldots, \alpha_{R-1}$, one has

$$Z(0) = \int D_{R-1}(\alpha) \prod_{0}^{R-1} Dz_{\alpha}^R \exp \{ p_{R} G_{R}(\sqrt{\lambda_{R}^R} z_{\alpha}^R + b_{\alpha}^{R-1}) \}. \tag{41}$$

This structure suggests introducing quantities $G_{r}(b)$ as

$$\exp \{ p_{r} G_{r-1}(b_{r}^{r-1}) \} = \int Dz_{r}^R \exp \{ p_{r} G_{r}(\sqrt{\lambda_{r}^R} z_{r}^R + b_{r}^{r-1}) \} \tag{42}$$

so that equation (39) can be written as

$$Z(0) = \int D_{R-1}(\alpha) \prod_{0}^{R-1} \exp \{ p_{R} G_{R-1}(b_{\alpha}^{R-1}) \}, \tag{43}$$

which has the same form of (39) provided $R \rightarrow R - 1$. The entire process can be iterated up to level 0 and leads to

$$Z(0) = \exp[p_{0} G_{-1}(0)] = \int Dz \exp[p_{0} G_{0}(\sqrt{\lambda_{0}} z)]. \tag{44}$$

In the limit $p_{0} = n \rightarrow 0$ one recovers the usual expression [12]

$$G_{-1}(0) = \int Dz G_{0}(\sqrt{\lambda_{0}} z). \tag{45}$$

Equation (42) has an interesting ‘physical’ interpretation. The quantity $G_{R}(b^R)$ is the free energy of a system of one spin, i.e. of size 1, in the replica space in the presence of the frozen field $b^R$, that is with all random (Gaussian) $z^R$ held fixed. To move one level up, $R \rightarrow R - 1$, we have to un freeze and integrate over $z^R$, while keeping all other fields $z^t$ with $t < R$ frozen. The fields $z^t$ with $t < R$ give the effective action, under the form of a (random) field, of the spins $s^t$ on the spin $s^R$ with $a \cap b = t < R$. Then integration over the field $z^R$ means that only the spins $s^a$ and $s^b$ such that $a \cap b = R$ are summed in the trace. All others are kept frozen. Thus, the quantity $G_{R-1}(b^{R-1})$ can be seen as the free energy (density) of a system in the replica space of size $p_R$ in the presence of an external field $b^{R-1}$, which gives the interaction with the frozen spins, that is the frozen degrees of freedom. Extension to the successive $z^t$-integration is straightforward. The quantity $G_{r-1}(b^{r-1})$ is obtained by integrating out in turn the random fields $z^t$ with $t \geq r$, while keeping all $z^t$ with $t < r$ frozen. This means that the trace is restricted to spins $s^a$ and $s^b$ such that $a \cap b = t \geq r$. The contribution from the spins not included into the trace, and hence frozen, is taken into account by the frozen field $b^{r-1}$. The quantity $G_{r-1}(b^{r-1})$ is then the free energy (density) of a system of size $p_r$ in the replica space in the presence of the external field $b^{r-1}$, which accounts for the degrees of freedom still frozen at scale $r$. 

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The free energy $G_{r-1}(0)$ is a part of the total free energy density of the system, see equations (6) and (29), and thus it is itself an intensive quantity in the real space. This implies that $p_r G_{r-1}$ are intensive quantities, and hence as $n \to 0$ the $p_r$ become densities in the real space: $0 < p_r < 1$. The $p_r$ give a measure of the density of the frozen degrees of freedom at scale $r - 1$ as measured from the overlap. Consider indeed the function

$$x(q) = n + \sum_{r=0}^{R} (p_{r+1} - p_r) \theta(q - Q_r),$$

which equals the number of pairs of replicas with overlap $Q_{ab}$ less or equal to $q$: $x(q) = p_{r+1}$ if $Q_r < q < Q_{r+1}$. The function $x(q)$ is not decreasing with $q$; thus, $p_r < p_{r'}$ if $r < r'$ as $n \to 0$. Indeed while moving from level $r$ to level $r - 1$, the number of unfrozen degrees of freedom, that is, the number of spins in the replica space over which the trace is done, increases, and hence the number of frozen degrees of freedom decreases, as signaled by the decrease of the value of the overlap. This picture is fully consistent with the dynamical formulation of CHS [28, 29] in terms of time-scales and density of frozen/unfrozen degrees of freedom.

We can now turn to the problem of calculating spin averages. This differs from that of $Z(0)$ by the presence of terms that depend on the fields $b^a_r$, cf equations (38) and (39). The recursion relation (42) is the usual rule to compute the free energy when some frozen degrees of freedom become unfrozen, and hence must be summed up in the trace. In the specific case those frozen at scale $r$ but unfrozen at scale $r - 1$, are represented by the fields $z^r$. The presence of $p_r$ instead of $p_{r+1}$ in the integrand follows because at scale $r$ there are $p_r/p_{r+1}$ disjoint systems in the replica space, all with the same free energy, that merge at scale $r - 1$. This suggests the following recursion relation for the calculation of spin averages. Let $F_s(b^a_s)$ be a generic function of the field $b^a_s$ at scale $s$. We then define the quantity $F_{s-1}(b^a_{s-1})$ at scale $s - 1$ as the average of $F_s(b^a_s)$ over the random field $z^s$ weighted with the statistical weight of the state, that is,

$$F_{s-1}(b^a_{s-1}) = Z_{s-1}(a)^{-1} \int Dz^s_a Z_s(a)^{p_s/p_{s+1}} F_s(\sqrt{\Delta \lambda_r} z^s_a + b^a_{s-1}),$$

where

$$Z_r(a) = Z_r(b^a_r) = \exp \left\{ p_{r+1} G_r(b^a_r) \right\} = \exp \left\{ p_{r+1} G_r(\sqrt{\Delta \lambda_r} z^s_a + b^a_{s-1}) \right\}.$$  

This recursion relation is supplemented by the boundary condition

$$F_r(b^a_r) \big|_{s=r} = F_r(b^a_r),$$

where $F_r(b_r)$ is a known expression. Assume for example that

$$F_r(b^a_r) = \frac{\partial}{\partial b^a_r} G_r(b^a_r),$$

then a simple calculation shows that

$$F_{r-1}(b^a_{r-1}) = \frac{\partial}{\partial b^a_{r-1}} G_{r-1}(b^a_{r-1}).$$

This result is not unexpected since it just states that the magnetization $m_r$ at any scale $r$ is given by the derivative of the free energy of that scale with respect to the applied field at that scale:

$$m_r(b^a_r) = \frac{\partial}{\partial b^a_r} G_r(b^a_r).$$
Figure 3. $(s^a s^b) \cap b = r$, tree configuration.

From this result it immediately follows that

$$
\langle s^a \rangle = m_{-1}(h) = \frac{\partial}{\partial h} G_{-1}(h) = \frac{\partial}{\partial h} \int Dz \, G_0(\sqrt{\lambda_0} \, z + h) = \int Dz \, m_0(\sqrt{\lambda_0} \, z + h), \quad (53)
$$

where $h$ is an external field$^5$. Clearly $\langle s^a \rangle = 0$ if $h = 0$.

To compute the two-spin correlation $\langle s^a s^b \rangle$ with $a \cap b = r$, i.e. the overlap $Q_r$, we have to evaluate the integral

$$
\langle s^a s^b \rangle \rightarrow \int \mathcal{D} \mathcal{R}(\alpha) \prod_0^R Z_\mathcal{R}(\alpha) \, m_{\mathcal{R}}(b^R_{a}) \, m_{\mathcal{R}}(b^R_{b}), \quad (54)
$$

where we used (52). On a branch-tree diagram the two replicas $a$ and $b$ with $a \cap b = r$ are on different branches for scales $s > r$, see figure 3, i.e., they belong to different systems. The two local fields $b^R_{a}$ and $b^R_{b}$, and hence the magnetizations, are independent from each other and the integration factorizes. Then

$$
\langle s^a s^b \rangle \rightarrow \int \mathcal{D} \mathcal{R}(\alpha) \prod_0^r Z_\mathcal{R}(\alpha) \, m_{\mathcal{R}}(b^R_{a})^2, \quad (55)
$$

since at scale $r$ the two replicas end up in the same system of size $p_r$ and the two fields $b^R_{a}$ and $b^R_{b}$ become equal. This expression suggests introducing the quantity

$$
F_r(b^R_{a}) = m_{(2)}^{(2)}(b^R_{a}) \quad (56)
$$

with the boundary condition

$$
m_{(2), 0}^{(2)}(b^R_{a}) = m_{(2)}(b^R_{a})^2 \quad (57)
$$

in terms of which we have

$$
\langle s^a s^b \rangle \cap a \cap b = r = \int \mathcal{D} \mathcal{R}(\alpha) \prod_0^{r-1} Z_\mathcal{R}(\alpha) \, m_{\mathcal{R}}(b^R_{a}) \, m_{\mathcal{R}}(b^R_{b}), \quad (58)
$$

where $m_{(2), 0}^{(2)}$ is computed from the recursion relation (47) with (56) and (57).

For higher order spin correlations we proceed in a similar way. Consider for example the four-spin correlation

$$
\langle s^a s^b s^c s^d \rangle \cap a \cap b = r \cap c \cap d \rightarrow \int \mathcal{D} \mathcal{R}(\alpha) \prod_0^R Z_\mathcal{R}(\alpha) \, m_{\mathcal{R}}(b^R_{a}) \, m_{\mathcal{R}}(b^R_{b}) \, m_{\mathcal{R}}(b^R_{c}) \, m_{\mathcal{R}}(b^R_{d}). \quad (59)
$$

$^5$ The field $h$ includes a factor $\beta$ from the statistical weight $\exp(-\beta H)$, see also equation (29). Thus the correct expression in terms of the real external field would have $\beta h$. We prefer to leave the factor $\beta$ hidden into the field to have no factors $\beta$ in the definition of the spin averages via equation (31).
where the replicas $a, b, c, d$ have the LA sector configuration shown in figure 4. The four replicas are independent from scale $R$ to $t$, where the replicas $b$ and $c$ end up in the same system in the replica space and the fields $b'_b$ and $b'_c$ become equal. Thus,

$$\langle s^a s^b s^c s^d \rangle_{t,s} \rightarrow \int D_1(\alpha) \prod_0^t Z_t(\alpha) m_r(b'_b) m_r(b'_c) m_r(b'_d).$$

(60)

Moving up along the tree, the surviving three replicas remain in different systems up to scale $r$, where the replicas $a$ and $b$ eventually find themselves in the same system. Then, using the quantity $m_{t,r}(b')$ introduced for the two-spin correlation, we can write

$$\langle s^a s^b s^c s^d \rangle_{t,s} \rightarrow \int D_r(\alpha) \prod_0^r Z_r(\alpha) m_r(b'_a) m_{r,s}(b'_c) m_r(b'_d).$$

(61)

where $m_{t,r}(b')$ is obtained from the recursion relation (47) with the initial condition $m_{r,t}(b') = m_t(b')^2$. For the next step we observe that replicas $a$ and $d$ remain in different subspaces up to scale $s$. Thus, by introducing the quantity

$$F_k(b'_a) = m_{r,s,k}^{2,1}(b'_a)$$

(62)

with the boundary condition

$$m_{r,s}^{2,1}(b'_a) = m_r(b'_a) m_{r,s}^{2,1}(b'_c),$$

(63)

we can move up along the tree up to scale $s$, and

$$\langle s^a s^b s^c s^d \rangle_{t,s} \rightarrow \int D_s(\alpha) \prod_0^s Z_s(\alpha) m_{r,s,k}^{2,1}(b'_c) m_s(b'_d).$$

(64)

The last step from scale $s$ to scale $0$ is now straightforward. We introduce the quantity

$$F_k(b'_a) = m_{r,s,k}^{2,1,1}(b'_a)$$

(65)

with the boundary condition

$$m_{r,s,k}^{2,1,1}(b'_a) = m_{r,s,k}^{2,1}(b'_a) m_s(b'_d).$$

(66)

so that the final result reads

$$\langle s^a s^b s^c s^d \rangle_{t,s} = m_{r,s}^{2,1,1} \langle h \rangle \rightarrow 0 \Rightarrow \int D_k m_{r,s,k}^{2,1,1}(h) \sqrt{k_0} (\sqrt{k_0} z + h).$$

(67)

In a similar way, once the replica geometry is specified, one can compute spin averages involving any number of spins.

The above results, valid for any finite $R$, are easily extended to the continuous case $R \rightarrow \infty$. In this case, since the values of $Q^0$, and hence those of $\lambda_r$, are bounded in a finite interval, the differences $\Delta \lambda_r \rightarrow 0$ as $R \rightarrow \infty$ to account for an infinite number of values.
in a finite interval\(^6\). As a consequence the recursion relations are replaced by differential equations. In particular the recursion relation (42) becomes the Parisi equation [12], while equation (47) is replaced by the partial differential equation

\[ \dot{F}(x, b) = -\frac{1}{2} \left[ F''(x, b) + 2 \lambda(x) \right] \]

where \(m(x, b) = G'(x, b)\), with the initial condition

\[ F(x, b) \big|_{x = t} = F(t, b) \]

where \(F(t, b)\) is some known expression at scale \(t\). As usual the ‘dot’ and the ‘prime’ denote partial derivative with respect to \(x\) and \(b\), respectively. Details are in appendix A.

We conclude this section by noting that this formalism can be easily extended to calculate the (static) response of the system to external perturbations that act at given scales. Let us denote by \(\epsilon_r^a\) a small external perturbation acting only on scale \(r\). We then define the static response as

\[ \chi_r = \frac{\partial}{\partial \epsilon_r^a} \langle s^a \rangle \bigg|_{\epsilon_r = 0} \]

where \(\langle s^a \rangle\) is the spin average in the presence of \(\epsilon_r\). The perturbation \(\epsilon_r^a\) can be seen as an extra contribution to the frozen field \(b_r^a\). Then, since \(\epsilon_r^a\) acts only on scale \(r\), in the recursion relation we end up with

\[ G_{r+1} \left( \sqrt{\Delta \lambda_{r+1}} \epsilon_r^a + b_r^a + \epsilon_r^a \right) \]

By expanding to the first order in \(\epsilon_r^a\), we finally have

\[ \langle s^a \rangle \rightarrow \int Dz \exp \left\{ p_{r+1} G_{r+1} (b_r^a) \right\} \]

and a similar expression for the normalization \(Z_r(0)\). By neglecting all unnecessary indices, the recursion relation (47) is then replaced by

\[ m_{r+1}^\epsilon (b) = \exp \left\{ -p_{r+1} G_{r+1}^\epsilon (b_r^a) \right\} \]

with

\[ \exp \left\{ p_{r+1} G_{r+1}^\epsilon (b_r^a) \right\} = \int Dz \exp \left\{ p_{r+1} G_{r+1} (\sqrt{\Delta \lambda_{r+1}} z + b) \right\} \]

Taking the derivative with respect to \(\epsilon_r^a\), and setting \(\epsilon_r^a \rightarrow 0\), leads to

\[ \frac{\partial}{\partial \epsilon_r^a} m_{r+1}^\epsilon (b_r^a) \bigg|_{\epsilon_r = 0} = p_{r+1} \left( m_r^{(2)} (b_r^a) - m_r (b_r^a)^2 \right) \]

Then from equations (70) and (71) we have\(^7\)

\[ \chi_r = p_{r+1} (Q_{r+1} - Q_r) \]

The derivative of the normalization factor \(Z_r(0)\) gives a term proportional to \(\langle s^a \rangle^2\), which vanishes in the absence of an external field. Expression (75) is the static limit of the modified fluctuation dissipation theorem introduced by CHS [28, 29] in dynamics.

\(^6\) One can allow for a finite number of ‘jumps’, that is points where \(\Delta_r\) does not vanish as \(R \rightarrow \infty\). One then obtains mixed-type solutions as those found, e.g., in spherical \(p\)-spin models [26, 27].

\(^7\) The perturbation \(\epsilon_r^a\) includes a factor \(\beta\). This removes the factor \(\beta\) on the rhs.
5. Replicon sector

The Hessian is a \( n(n-1)/2 \times n(n-1)/2 \) symmetric matrix that after block-diagonalization \([30, 31]\) becomes a string of \((R+1) \times (R+1)\) blocks along the diagonal for the LA sector, followed by \(1 \times 1\) fully diagonalized blocks, for the Replicon sector. The diagonal elements in the Replicon sector are given by

\[
\lambda_r^{r; \hat{k} \hat{l}} = M^{r; r}_{\hat{k} \hat{l}}.
\]

To evaluate \(M^{r; r}_{\hat{k} \hat{l}}\) we need the matrix elements \(M^{2; 2}_{u, v}\), equation (11), that is, the four-spin average \(\langle s^a s^b s^c s^d \rangle\) for the Replicon geometry shown in figure 5. From the results of section 4 and appendix A this is given, in the absence of an applied external field and for \(R \to \infty\), by

\[
\langle s^a s^b s^c s^d \rangle_{u, v}^{2; 2} = m^{(2,2)}_{u, v}(0, 0).
\]

The function \(m^{(2,2)}_{u, v}(x, b)\) is the solution of a chain of the partial differential equation of the form

\[
\dot{F}(x, b) = -\beta^2 \frac{\dot{Q}(x)}{2} [F''(x, b) + 2x m(x, b) F'(x, b)],
\]

where \(m(x, b)\) is the local magnetization at scale \(x\) in the presence of the field \(b\).

In our case, starting from the bottom of the tree in figure 5, we first have to solve equation (78) for \(F(x, b) = m(x, b)\) and the boundary condition

\[
F(x, b) \big|_{x=1^-} = m(1, b) = \tanh b.
\]

The range of \(x\) is \(u \leq x \leq 1\) for the left branch of the tree, and \(v \leq x \leq 1\) for the right branch. To proceed toward scale 0 we have to solve the next equation (78) for \(F(x, b) = m^{(2)}_{u, v}(x, b)\) and the initial condition

\[
F(x, b) \big|_{x=t} = m^{(2)}_{u, v}(t, b) = m(t, b)^2,
\]

where \(t = u, v\) depending upon we are on the left or on the right branch of the tree. The range of \(x\) is either \(r \leq x \leq u\), left branch, or \(r \leq x \leq v\), right branch, see figure 5.

To accomplish the last step, \(0 \leq x \leq r\), we finally solve equation (78) for \(F(x, b) = m^{(2,2)}_{u, v}(x, b)\) and the initial condition

\[
F(x, b) \big|_{x=r} = m^{(2,2)}_{u, v}(r, b) = m^{(2)}_{u}(r, b) m^{(2)}_{v}(r, b).
\]

While these equations are valid for any temperature \(T\), we are interested into their solution in the limit \(T \to 0\). Following Pankov [25] one can show, see appendix B, that in the outer region \(x \gg T\) and \(T \to 0\) the solution of the partial differential equation (78) loses its explicit dependence on the scale variable \(x\). As a consequence for \(R \to \infty\), the matrix element \(M^{r; r}_{u, v}\) becomes independent of \(u\) and \(v\) for all \(r\) such that \(p_r = x(Q_r) \gg T\) as \(T \to 0\), and this
in turn implies that $M_{ki}^{r,r}$ is independent of $k$ and $l$. Thus, by exploiting this insensitivity we conclude that for all $r$ in the outer region

$$M_{ki}^{r,r} = M_{r+1,r+1}^{r,r} = O \left( \frac{1}{R^2} \right)^{R \to \infty} = 0. \quad (82)$$

The second equality follows from a Ward–Takahashi identity [7]. In the inner region the Replicon spectrum maintains its complexity. However its relevance becomes less and less important as $T$ approaches 0, and vanishes in the limit $T \to 0$ when the thickness of the boundary shrinks to 0. The Replicon spectrum, similarly to the order parameter function $Q(x)$, becomes then discontinuous at $x = 0$.

6. LA sector

The LA sector corresponds to the $(R + 1) \times (R + 1)$ diagonal blocks along the diagonal labeled by the index $k = 0, \ldots, R + 1$. The matrix element in each block turns out to be

$$\Lambda_k(r) = A_k \left( \delta_r^{kr} + \frac{1}{4} M_{k,r}^{r,r} \delta_s^{(k)} \right), \quad r, s = 0, \ldots, R \quad (83)$$

where $\Lambda_k(r)$ is a shorthand for

$$\Lambda_k(r) = \begin{cases} M_{k,r+1}^{r,r} & k > r + 1, \\ M_{r+1,r}^{r,r} & k \leq r + 1, \end{cases} \quad (84)$$

and $\delta_s^{(k)} = p_s^{(k)} - p_{s+1}^{(k)}$, $k = 0, 1, \ldots, R + 1$, with

$$p_s^{(k)} = \begin{cases} p_s & s \leq k, \\ 2p_s & s > k. \end{cases} \quad (85)$$

$M_{k}^{r,r}$ is the RFT of the matrix element $M_{i}^{r,r}$ with respect the cross-overlap $t$, see figure 6, that is,

$$M_{k}^{r,r} = \sum_{i=0}^{R+1} p_{s}^{(r,s)} \left( M_{i}^{r,s} - M_{i-1}^{r,s} \right) \quad (86)$$

with, if $r < s$,

$$p_{s}^{(r,s)} = \begin{cases} p_s & t \leq r, \\ 2p_s & r < t \leq s, \\ 4p_s & s < t. \end{cases} \quad (87)$$

For finite temperature, the matrix elements are different in each block; however, for scales $k$ in the outer region the RFT, $M_{k}^{r,r}$ and $\Lambda_k(r)$ become insensitive to the value of $k$ as $T \to 0$. All correspondent blocks are then diagonalized through the single eigenvalue equation$^8$:

$$\lambda_{LA} f' = M_{R+1,R+1}^{r,r} f' + \frac{1}{4} \sum_{s=0}^{R} M_{R+1, R+1}^{r,s} \delta_s f', \quad (88)$$

where $\delta_s = p_s - p_{s+1}$. In the outer region the eigenvectors $f'$ satisfy $f' \neq 0$ if $T \ll x(Q_r) \ll x_c$ as $T \to 0$, and zero otherwise. The eigenvalue equation then reduces to

$$\lambda_{LA} f' = \frac{1}{4} M_{R+1, R+1}^{r,r} \delta_s f', \quad r = 7, \ldots, R \quad (89)$$

$^8$ The boundary term $t = 0$ in the RFT is proportional to $p_0 = n$ and vanishes for $n \to 0$. The next term is proportional to $p_1 M_{1}^{r,r}$, since $M_{0}^{r,r} = 0$, and vanishes as $R \to \infty$, so the only term which survives is $k = R + 1$. 


where $\mathcal{T}$ is the lower bound of the outer region: $x(Q_{\mathcal{T}}) = \overline{\pi} \sim \delta$ as $T \to 0$. The diagonal Replicon contribution vanishes for $R \to \infty$, as ensured by the Ward–Takahashi identity, and does not contribute. This equation has two distinct solutions. The first

$$\lambda_{LA} = 0$$

(90)

for $\sum_{s=\mathcal{T}}^R \delta_s f^s = 0$, and

$$\lambda_L = \frac{1}{4} \left( \sum_{s=\mathcal{T}}^R \delta_s \right) M_{R,R}^{R,R+1}
= (\overline{\pi} - 1)(1 - \beta^2 (1 - q_c(T))
= (\alpha - 1) + O(T), \quad T \to 0$$

(91)

for $\sum_{s=\mathcal{T}}^R \delta_s f^s \neq 0$. The last equality follows from equation (19), and $\overline{\pi} \sim \delta \sim T$ as $T \to 0$. In the inner region, where the LA spectrum maintains the RSB structure, the solutions are smooth functions of the inner variable even for $T \to 0$. For $T \to 0$, the thickness of the boundary layer shrinks to 0, and eigenvalues (90) and (91) cover the whole LA spectrum, with a discontinuity at $x = 0$.

7. Summary and conclusions

In this work we have presented the analysis of the very low temperature limit $T \ll 1$ of the spectrum of the Hessian for the Parisi solution of the SK model. It has long been known that in this regime two distinct regions of the interval $x \in [0, 1]$ can be identified according to the variation of the order parameter function $Q(x)$ with $x$. We have shown that this has strong consequences on the structure of the Hessian spectrum. In the first region $x \leq \delta \sim T$, where $Q(x)$ varies rapidly from $Q(0) = 0$ up to $Q(\delta) \sim q_c(T) \sim 1$, the spectrum maintains the complex structure observed close to the critical temperature $T_c$ for the full RSB state. We can call this region the RSB-like regime. In the second region, $T \ll x \leq x_c$ with $x_c \sim 0.575 \ldots$, where $Q(x)$ is slowly varying, the Hessian spectrum has a completely different structure. Here the components of the Hessian matrix become insensitive to changes of the overlaps.
and the bands observed in the replica symmetry breaking regime collapse. In this region only two distinct eigenvalues survive: a null one and the positive one. This ensures that the Parisi solution of the SK model then remains stable as the temperature goes to 0. The occurrence of zero modes is remarkable in both the Replicon and the LA sectors. Null eigenvalues arise from the Replicon geometry, with Ward–Takahashi identities protecting them. Note, however, that the zero modes arise also from the LA geometry, that is without protection of the Ward–Takahashi identities.

We observe that for $T \ll 1$ the order parameter function is almost constant for $T \ll x \leq x_c$, the variation being indeed of order $[Q(x_c) - Q(x)]/Q(x) = O((T/x)^2)$. Thus, in this region we have a marginally stable (almost) replica symmetric solution, that becomes a genuine replica symmetric solution in the limit $T \rightarrow 0$, with self-averaging trivially restored.

In the limit $T \rightarrow 0$, the region where the RSB structure of the solution is found shrinks to zero, and only the RS part survives. This feature, in a sense, brings about some perfume of conciliation between aspects of the Parisi mean-field approach and the droplet approach [32, 33]. We stress, however, that in order to identify a genuine droplet behavior, corrections to the mean-field have to be studied in more detail [34, 35].

Concerning the multiplicity of the eigenvalues we observe that in each sector, Replicon and LA, one has to separate the contribution from the RSB-like and the droplet-like regions. The former is proportional to the width $\delta$ of the region. Therefore, in the limit $T \rightarrow 0$ the contribution from the RSB-like region vanishes, and one has the usual Replicon and LA multiplicities for the droplet-like region.

Finally, we have developed a method to compute spin averages in replica space involving any number of replicas, both for a finite number $R$ of the replica symmetry breaking steps and for the continuous limit $R \rightarrow \infty$. This generalizes some special cases known for the continuous limit $R \rightarrow \infty$, and to our knowledge it is new. Moreover it sheds light on the interpretation of the replica symmetry breaking method and its relation with the dynamical approach. For example, we were able to compute the static susceptibility and show that it equals the static limit of the dynamic susceptibility computed via the modified fluctuation dissipation theorem.

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**Appendix A. Spin averages: continuous case**

The expressions derived in section 4 are valid for any finite number $R$ of replica symmetry breaking steps. Here we shall address the limit $R \rightarrow \infty$, where the replica symmetry breaking becomes continuous. If the values of $\lambda_r$ are bounded in a finite interval, as is the case of the SK model, in the limit $R \rightarrow \infty$ they must be dense and

$$\Delta \lambda_r \rightarrow 0 \text{ as } R \rightarrow \infty, \quad r = 0, 1, \ldots, R$$

so that the integral in the recursion relations can then be evaluated by expanding in powers of $\Delta \lambda_r$. We observe that a finite number of `jumps’, that is values of $r$ where $\Delta \lambda_r$ remains finite.
as $R \to \infty$, are possible. One then obtains a mixed continuous-discrete phase described by a piecewise-order parameter function, as those observed, e.g., in spherical $p$ spin-glass models [26, 27]. We shall not discuss this case here.

Let us first consider the recursion relation for the free energy $G_r$. In the following we drop all unnecessary indices. By expanding $G_r(\sqrt{\Delta \lambda_r} z + b)$ in equation (42) in powers of $\Delta \lambda_r$ and integrating over the Gaussian variable $z$, a straightforward algebra leads to

$$G_{r-1}(b) = G_r(b) + \frac{\Delta \lambda_r}{2}[G''_r(b) + p_r(G'_r(b))^2] + O(\Delta \lambda^2_r),$$  \tag{A.2}

where the 'prime' denotes differentiation with respect to the argument, i.e. the field $b$:

$$G'_r(b) = \frac{\partial}{\partial b} G_r(b).$$  \tag{A.3}

Next we observe that $G_r(b)$ is a function of $p_r$, thus to extract the non-trivial part of equation (A.2) as $\Delta \lambda_r \to 0$ we have to specify what happens to $p_r - p_{r-1}$ as $R \to \infty$. Suppose $p_r - p_{r-1}$ does not vanish in the limit $R \to \infty$. In this case equation (A.2) implies that

$$G_{r-1}(b) = G_r(b) \quad \text{as} \Delta \lambda_r \to 0 \quad \text{but} \quad p_r - p_{r-1} \neq 0,$$  \tag{A.4}

that is, $G_r(b)$ does not depend on the scale. This is what happens, for example, in the SK model for $x_c < r < 1$, the so called plateau. If $p_r - p_{r-1} \to 0$ as $\Delta \lambda_r \to 0$ then, by defining

$$p_r \equiv x \quad \Rightarrow \quad p_r - p_{r-1} = \Delta x$$  \tag{A.5}

and changing the notation to $G_r(b) = G(x, b)$, equation (A.2) leads for $\Delta \lambda_r \to 0$ to the Parisi equation [12, 36]

$$\dot{G}(x, b) = -\frac{\dot{\lambda}(x)}{2}[G''(x, b) + x(G'(x, b))^2],$$  \tag{A.6}

where the 'dot' denotes the (left) derivative with respect to the scale $x$, e.g.,

$$\dot{G}(x, b) = \frac{\partial}{\partial x} G(x, b) \equiv \lim_{\Delta x \to 0} \frac{G(x, b) - G(x - \Delta x, b)}{\Delta x}.\tag{A.7}

We note that with this definition the validity of the partial differential equation (A.6) can be extended to include case (A.4) since the derivative is always well defined. As a consequence the initial condition simply reads, see equation (36),

$$G(x = 1^-, b) = \ln(2 \cosh b)$$  \tag{A.8}

while (45) becomes

$$G_{-1}(b) = \int Dz \, G(0, \sqrt{\lambda(0)} z + b).$$  \tag{A.9}

For the recursion relation (47) we follow a similar procedure, that is, we expand the rhs of (47) in powers of $\Delta \lambda_r$ and integrate over the Gaussian variable $z$. This leads to

$$F_{r-1}(b) = F_r(b) + \frac{\Delta \lambda_r}{2}[F''_r(b) + 2 p_r F'_r(b)] + O(\Delta \lambda^2_r).$$  \tag{A.10}

As before if $p_r - p_{r-1}$ does not vanish as $\Delta \lambda_r \to 0$, the recursion relation reduces to

$$F_{r-1}(b) = F_r(b).$$  \tag{A.11}

If, however, $p_r - p_{r-1} = \Delta x \to 0$ as $\Delta \lambda_r \to 0$ we end up with the partial differential equation

$$\dot{F}(x, b) = -\frac{\dot{\lambda}(x)}{2}[F''(x, b) + 2 x m(x, b) F'(x, b)], \quad x < t$$  \tag{A.12}
where \( m(x, b) = G'(x, b) \) is the magnetization at scale \( x \) in the presence of field \( b \), with the initial condition

\[
F(x, b)|_{x=t} = F(t, b), \tag{A.13}
\]

where \( F(t, b) \) is a known expression at scale \( t \).

As an example consider the two-spin correlation \( \langle s^a s^b \rangle \) with \( a \cap b = r \). From equation (58) it readily follows that

\[
\langle s^a s^b \rangle|_{a \cap b=r} = \int Dz \ m_r(0, \sqrt{\lambda(0)} z + h), \tag{A.14}
\]

where \( m_r(0, b) \) is the solution of the the partial differential equation

\[
\dot{m}_r(x, b) = -\frac{\dot{\lambda}(x)}{2} \left[ m_r''(x, b) + 2x \ m(x, b) m_r'(x, b) \right], \tag{A.15}
\]

for \( 0 < x < r \) and initial condition

\[
m_r(0, b) = m(r, b)^2. \tag{A.16}
\]

To the best of our knowledge, this equation was first derived by Goltsev [37].

For the four-spin correlation \( \langle s^a s^b s^c s^d \rangle_{t,s} \) of figure 4 we have a similar expression

\[
\langle s^a s^b s^c s^d \rangle_{t,s} = \int Dz \ m_{t,r,s}(0, \sqrt{\lambda(0)} z + h), \tag{A.17}
\]

where \( m_{t,r,s}(0, b) \) is the solution of the partial differential equation

\[
\dot{m}_{t,r,s}(x, b) = -\frac{\dot{\lambda}(x)}{2} \left[ m_{t,r,s}''(x, b) + 2x \ m(x, b) m_{t,r,s}'(x, b) \right], \tag{A.18}
\]

for \( 0 < x < s \) and the initial condition

\[
m_{t,r,s}(0, b) = m_{t,s}(s, b) m(s, b). \tag{A.19}
\]

The function \( m_{t,r,s}(x, b) \) is itself the solution of the partial differential equation

\[
\dot{m}_{t,r}(x, b) = -\frac{\dot{\lambda}(x)}{2} \left[ m_{t,r}''(x, b) + 2x \ m(x, b) m_{t,r}'(x, b) \right], \tag{A.20}
\]

for \( s < x < r \) and the initial condition

\[
m_{t,r}(0, b) = m_{t,r}(r, b) m(r, b). \tag{A.21}
\]

Finally \( m_r(0, b) \) is the solution of equation (A.15) for \( r < x < t \) and the initial condition \( m_r(0, b) = m(t, b)^2 \).

We conclude this section by noting that if we take \( F(x, b) = m(x, b) \), then the differential equation (A.12) becomes the known differential equation [14]:

\[
\dot{m}(x, b) = -\frac{\dot{\lambda}(x)}{2} \left[ m''(x, b) + 2x \ m(x, b) m'(x, b) \right], \tag{A.22}
\]

with the initial condition

\[
m(1^-, b) = \tanh b \tag{A.23}
\]
Appendix B. The Pankov scaling regime

To discuss the Pankov regime $T \to 0$ and $T \ll x \ll 1$ we first perform the change of variable $b = \beta y$ into the partial differential equation (A.12) to make the temperature dependence explicit. In the new variable the equation reads

$$F(x, y) = -\frac{\dot{\lambda}(x)}{2\beta^2} \left[ F''(x, y) + 2\beta x m(x, y) F'(x, y) \right],$$  \hspace{1cm} (B.1)

where the ‘prime’ now denotes the differentiation with respect to $y$, and the local magnetization is $m(x, y) = \beta^{-1} G'(x, y)$, see footnote 6. Following Pankov [25] we assume that the dependence on the local fields $y$ is via the combination $z = \beta xy$, that is,

$$F(x, y) = \tilde{F}(x, z), \quad z = \beta xy$$  \hspace{1cm} (B.2)

and similarly for $m(x, y)$. The differential equation (B.1) then becomes

$$x \tilde{F}'(x, z) = -\frac{x^3 \dot{\lambda}(x)}{2} \left[ \tilde{F}''(x, z) + 2\tilde{m}(x, z) \tilde{F}'(x, z) \right] - z \tilde{F}'(x, z).$$  \hspace{1cm} (B.3)

Pankov has shown that in the outer region the ‘tilded’ functions do not depend explicitly on the scale variable $x$: $\tilde{F}(x, z) = \tilde{F}(z)$. All dependences on scale, field and temperature enter via the combination $\beta xy$. Pankov called this the scaling regime.

From equation (B.3) it is clear that the Pankov scaling regime is only possible iff

$$\frac{x^3 \dot{\lambda}(x)}{2} = c = \text{constant} \Rightarrow \lambda(x) = \text{const.} - \frac{c}{x^2},$$  \hspace{1cm} (B.4)

in which case the partial differential equation (B.3) reduces to the ordinary second-order differential equation

$$\tilde{F}''(z) = -\left[ \frac{z}{c} + 2\tilde{m}(z) \right] \tilde{F}'(z).$$  \hspace{1cm} (B.5)

In the SK model $\lambda(x) = \beta^2 Q(x)$, where $Q(x)$ is the order parameter function, then from (B.4) it follows that in the outer region $Q(x)$ has the form

$$Q(x) = \text{const.} - \frac{c}{(\beta x)^2}.$$  \hspace{1cm} (B.6)

Equation (17) now follows by imposing $Q(x_c) = q_c(T) = 1 - \alpha T^2 + O(T^3)$.

Appendix C. Descending the replica tree: the frozen field probability distribution functions

In section 4 we have shown how spin averages can be computed using a bottom-up approach, that is starting from level $R$ at the bottom of the tree and climbing up toward level 0 at the top of the tree. A top-down approach is also possible.

To illustrate the procedure suppose we have to compute the following average:

$$\langle g \rangle = \frac{\int \mathcal{D}_a(\alpha) \prod_0^R Z_r(\alpha) g_r(p_r)}{\int \mathcal{D}_a(\alpha) \prod_0^R Z_r(\alpha)},$$  \hspace{1cm} (C.1)
where $g_r(b'_a)$ is a generic function of the frozen field $b'_a$ at scale $r$. For example, $g_r(b'_a) = m_r(b'_a)^2$ for the two-spin correlation, see equation (55). The average (C.1) can be rewritten in the simple form

$$\langle g \rangle = \int dy \, P_r(y) \, g_r(y)$$

(C.2)

by introducing the frozen field probability distribution function at scale $r$

$$P_r(y) = \frac{1}{\int D_r(\alpha) \prod_0^1 Z_r(\alpha) } \frac{\delta(b'_r - y)}{\delta(b'_r - y)} = \frac{\delta(b'_r - y)}{\delta(b'_r - y)}.$$

(C.3)

Following the procedure outlined in section 4 we can integrate the Gaussian variables $z'_a$ in equation (C.3), ending with

$$P_r(y) = \frac{1}{\int D_{r-1}(\alpha) \prod_0^1 Z_{r-1}(\alpha) } \frac{\delta(b'_{r-1} - y)}{\delta(b'_{r-1} - y)}.$$

(C.4)

where

$$g_{r-1}(b; y) = \frac{1}{\sqrt{2\pi \Delta \lambda_r}} \exp \left\{ \frac{-(y - b)^2}{2\Delta \lambda_r} \right\} + p_r [G_r(y) - G_{r-1}(b)].$$

(C.5)

Equation (C.4) has the same structure as equation (C.1); thus, we can write

$$P_r(y) = \int dy' P_{r-1}(y') \, g_{r-1}(y'; y),$$

(C.6)

where $P_{r-1}(y)$ is given by equation (C.3) with $r \to r - 1$. Inserting expression (C.5) into (C.6) leads to the recursion relation

$$P_r(y) = \int D \exp(p_r [G_r(y) - G_{r-1}(\sqrt{\Delta \lambda_r} z + y)]) P_{r-1}(\sqrt{\Delta \lambda_r} z + y),$$

(C.7)

which gives the frozen field distribution at scale $r$ once it is known at scale $r - 1$. The initial condition is specified at level 0, at the top of the tree, and reads

$$P_0(y) = \frac{1}{\sqrt{2\pi \Delta \lambda_0}} \exp \left\{ \frac{-(y - b)^2}{2\Delta \lambda_0} \right\}.$$

(C.8)

as follows from equations (C.4) and (C.5).

In the limit $R \to \infty \, P_r(y) \to P(x, y)$ and the recursion relation (C.7) is replaced by the partial differential equation

$$P(x, y) = \frac{\dot{\lambda}(x)}{2} [P''(x, y) - 2x m(x, y) \, P(x, y)].$$

(C.9)

where $m(x, y) = G'(x, y)$, with the initial condition at $x = 0$:

$$P(0, y) = \frac{1}{\sqrt{2\pi \Delta \lambda(0)}} \exp \left\{ \frac{-(y - b)^2}{2\Delta \lambda(0)} \right\}.$$

(C.10)

Equations (C.9) and (C.10) were first derived by Sommers and Dupont [14] using a variational approach. We note that taking $g_r(b) = m_r(b)^2$ from equation (C.2) one recovers in the limit $R \to \infty$ the Sommers–Dupont expression

$$Q(x) = \int dy \, P(x, y) \, m(x, y)^2.$$

(C.11)

The approach in terms of frozen field distribution functions can be generalized to deal with averages of quantities that depend on more then one local field. Suppose for example that $g_r \to g_r(b'_a, b'_b)$ with $a \neq b$. In this case equation (C.2) is replaced by

$$\langle g \rangle = \int dy_1 dy_2 \, P_r(y_1, y_2) \, g_r(y_1, y_2).$$

(C.12)
where $P_r(y_1, y_2)$ is the probability distribution function of the frozen fields $y_1$ and $y_2$ lying on two different branches of the tree at scale $r$. This satisfies the top-down recursion relation

$$P_r(y_1, y_2) = \int Dz_1 Dz_2 \prod_{k=1,2} \exp[p_r(G_r(y_k) - G_{r-1}(\sqrt{\Delta \lambda_r} z_k + y_k))]$$

$$\times P_{r-1}(\sqrt{\Delta \lambda_r} z_1 + y_1, \sqrt{\Delta \lambda_r} z_2 + y_2).$$

(C.13)

The initial condition is specified at the branching point $s < r$ where the two branches meet, and read

$$P_r(y_1, y_2) |_{r=s} = \delta(y_1 - y_2) P_s(y_1).$$

(C.14)

It is easy to verify that the frozen field distribution functions obey the sum-rule

$$\int dy_2 P_r(y_1, y_2) = P_r(y_1), \quad \forall r.$$  

(C.15)

In the continuous limit the recursion relation (C.13) is replaced by the partial differential equation

$$\dot{P}(x, y_1, y_2) = \frac{\dot{\lambda}(x)}{2} \sum_{k=1,2} \left[ \partial^2_k P(x, y_1, y_2) - 2 x \partial_k [m(x, y_k) \partial_k P(x, y_1, y_2)] \right].$$

(C.16)

where $\partial_k = (\partial/\partial y_k)$. The generalization to frozen field distribution functions of any number of independent frozen fields is straightforward.

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