Revisiting the Softmax Bellman Operator: Theoretical Properties and Practical Benefits

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Abstract

The softmax function has been primarily employed in reinforcement learning (RL) to improve exploration and provide a differentiable approximation to the max function, as also observed in the mellowmax paper by Asadi and Littman. This paper instead focuses on using the softmax function in the Bellman updates, independent of the exploration strategy. Our main theory provides a performance bound for the softmax Bellman operator, and shows it converges to the standard Bellman operator exponentially fast in the inverse temperature parameter. We also prove that, under certain conditions, the softmax operator can reduce the overestimation error and the gradient noise. A detailed comparison among different Bellman operators is then presented to show the trade-off when selecting them. We apply the softmax operator to deep RL by combining it with the deep Q-network (DQN) and double DQN algorithms in an off-policy fashion, and demonstrate that these variants can often achieve better performance in several Atari games, and compare favorably to their mellowmax counterparts.

1 Introduction

The Bellman equation (Bellman 1957) has been a fundamental tool in reinforcement learning (RL), as it provides a sufficient condition for the optimal policy in dynamic programming. The use of the max function in the Bellman equation further suggests that the optimal policy should be greedy w.r.t. the $Q$-values. On the other hand, the trade-off between exploration and exploitation (Tarin 1992) motivates the use of exploratory and potentially sub-optimal actions during learning, and one commonly-used strategy is to add randomness by replacing the max function with the softmax function, as in Boltzmann exploration (Sutton and Barto 1998). Furthermore, the softmax function is a differentiable approximation to the max function, and hence can facilitate analysis (Reverdy and Leonard 2016).

Despite its wide deployment in RL, little has been known in terms of the theoretical properties for the softmax Bellman operator. It has also been demonstrated that the softmax Bellman operator is not a contraction, for certain temperature parameters (Littman 1996). In Asadi and Littman (2017), an alternative mellowmax operator was proposed, and shown as a contraction and also differentiable, with experimental results suggesting that it can improve exploration. Note that the form of the mellowmax operator lacks an explicit policy representation, and needs to be transformed into a softmax policy first before use, where additional computation is necessary to determine the temperature parameter.

In an effort to better understand the softmax Bellman operator, we first investigate its properties other than exploration and being differentiable. Starting from the same initial $Q$-values, we bound how far the $Q$-functions computed with the softmax operator can deviate from those computed with the regular Bellman operator, with both lower and upper bounds presented. We further show that the softmax Bellman operator converges to the optimal Bellman operator in an exponential rate w.r.t. the inverse temperature parameter, a desirable property. As also motivated by recent work (van Hasselt, Guez, and Silver 2016, Anschel, Baram, and Shimkin 2017) targeting bias and instability of the original deep Q-network (DQN) (Mnih et al. 2015), we further investigate whether the softmax Bellman operator can alleviate these issues. As discussed in van Hasselt, Guez, and Silver (2016), one possible explanation for the poor performance of the vanilla DQN on some Atari games was the overestimation bias when computing the target network, due to the max operator therein. We prove that given the same assumptions as van Hasselt, Guez, and Silver (2016), the softmax Bellman operator can reduce the overestimation bias, for any inverse temperature parameters. We also quantify the overestimation reduction by providing its lower and upper bounds.

To validate the softmax Bellman operator in practice, we combine it with the DQN and double DQN (DDQN) algorithms by replacing the max function therein with the softmax function, in the target network. We then test the variants on several games in the Arcade Learning Environment (ALE) (Bellemare et al. 2013), a standard large-scale deep RL testbed. The results show that the variants using the softmax Bellman operator can achieve higher test scores, and reduce the $Q$-value overestimation as well as the gradient noise on most of them. When compared with its max and mellowmax counterpart, the softmax operator often stays on the favorable side of the trade-off between the Bellman optimality and the overestimation reduction.
2. Background and Notation

A Markov decision process (MDP) can be represented as a 5-tuple \( \langle S, \mathcal{A}, P, R, \gamma \rangle \), where \( S \) is the state space, \( \mathcal{A} \) is the action space, \( P \) is the transition kernel whose element \( P(s'|s, a) \) denotes the transition probability from state \( s \) to state \( s' \) under action \( a \), \( R \) is a reward function whose element \( R(s, a) \) denotes the expected reward for executing action \( a \) in state \( s \), and \( \gamma \in (0, 1) \) is the discount factor. The policy \( \pi \) in an MDP can be represented in terms of a probability mass function (PMF), where \( \pi(s, a) \in [0, 1] \) denotes the probability of selecting action \( a \) in state \( s \), and \( \sum_{a \in \mathcal{A}} \pi(s, a) = 1 \).

For a given policy \( \pi \), its state-action value function \( Q^\pi(s, a) \) is defined as the accumulated, expected, discount reward, when taking action \( a \) in state \( s \), and following policy \( \pi \) afterwards, i.e., \( Q^\pi(s, a) = \mathbb{E}_{\tau \sim \pi} \sum_{t=0}^{\infty} \gamma^t R_t | s_0 = s, a_0 = a \). For the optimal policy \( \pi^* \), its corresponding \( Q \)-function satisfies the following Bellman equation:

\[
Q^*(s, a) = R(s, a) + \gamma \max_{a'} \sum_{s'} P(s'|s, a) Q^*(s', a').
\]

In DQN (Mnih et al. 2015), the \( Q \)-function is parameterized with a neural network as \( Q_\theta(s, a) \), which takes the state \( s \) as input and outputs the corresponding \( Q \)-value in the final fully-connected linear layer, for every action \( a \). The training objective for the DQN can be represented as:

\[
\min_{\theta} \frac{1}{2} \left\| Q_\theta(s, a) - [R(s, a) + \gamma \max_{a'} Q_{\theta^-}(s', a')] \right\|^2,
\]

(1)

where \( \theta^- \) corresponds to the frozen weights in the target network, and is updated at fixed intervals. The optimization of Eq. (1) is performed via RMSProp (Tieleman and Hinton 2012), with mini-batches sampled from a replay buffer.

To reduce the overestimation bias, the double DQN (DDQN) algorithm modified the target that \( Q_\theta(s, a) \) aims to fit in Eq. (1) as:

\[
R(s, a) + \gamma Q_{\theta^-}(s', \arg\max_a Q_{\theta}(s', a)).
\]

Note that a separate network based on the latest estimate \( \theta_t \) is employed for action selection, and the evaluation of this policy is due to the frozen network.

**Notation**

The softmax function is defined as

\[
f_\tau(x) = \frac{\exp(\tau x_1), \exp(\tau x_2), \ldots, \exp(\tau x_m)}{\sum_{i=1}^m \exp(\tau x_i)},
\]

where the superscript \( T \) denotes the vector transpose. Subsequently, the softmax-weighted function is represented as \( g_\tau(x) = f_\tau^T(x) x \), as a function of \( \tau \). Also, we define the vector \( Q(s, \cdot) = [Q(s, a_1), Q(s, a_2), \ldots, Q(s, a_m)]^T \). We further set \( m \) to be the size of the action set \( \mathcal{A} \) in the MDP. Finally, \( R_{\text{min}} \) and \( R_{\text{max}} \) denote the minimum and the maximum immediate rewards, respectively.

3. The Softmax Bellman Operator

We start by providing the following standard Bellman operator:

\[
\mathcal{T} Q(s, a) = R(s, a) + \gamma \sum_{s'} P(s'|s, a) \max_{a'} Q(s', a').
\]

(2)

We propose to use the following softmax Bellman operator, defined as

\[
\mathcal{T}_{\text{soft}} Q(s, a) = R(s, a) + \gamma \sum_{s'} P(s'|s, a) \max_{a'} Q(s', a') \frac{\exp[\tau Q(s', a')] \sum_{a''} \exp[\tau Q(s', a'')]}{\sum_{a''} \exp[\tau Q(s', a'')]} Q(s', a'),
\]

(3)

where \( \tau \geq 0 \) denotes the inverse temperature parameter. Note that \( \mathcal{T}_{\text{soft}} \) will reduce to \( \mathcal{T} \) and the mean operator, when \( \tau \to \infty \) and \( \tau = 0 \), respectively. In contrast to its typical use to improve exploration, the softmax Bellman operator is combined in this paper with the DQN and DDQN algorithms in an off-policy fashion.

Even though the softmax Bellman operator was shown in Littman (1996) not to be a contraction, with a counterexample, we are not aware of any prior work showing the performance bound by using the softmax Bellman operator in Q-iteration. Furthermore, it is interesting to investigate how fast \( \mathcal{T}_{\text{soft}} \) approaches \( \mathcal{T} \), given the fact that \( \mathcal{T}_{\text{soft}} \to \mathcal{T} \) when \( \tau \to \infty \). We propose to answer these questions in Section 3.1. We also show in Section 3.2 how the softmax Bellman operator can reduce the overestimation bias and gradient noise, given certain assumptions.

Before presenting our main theoretical results, we first show the following useful lemma to bound the distance between the softmax-weighted and max operators.

**Lemma 1.** By defining \( \delta = \sup_Q \max_{s,i,j} |Q(s, a_i) - Q(s, a_j)| \) and assuming \( \delta > 0 \), we have \( \forall Q \) and \( \forall s \),

\[
\frac{\delta}{m \exp(\tau \delta)} \leq \max_a Q(s, a) - f_\tau^T(Q(s, \cdot)) \leq (m - 1) \max \left\{ \frac{1}{\tau + 2}, \frac{2Q_{\text{max}}}{\tau + \exp(\tau)} \right\},
\]

where \( Q_{\text{max}} = \frac{R_{\text{max}}}{1 - \gamma} \) represents the maximum \( Q \)-value.

**Proof.** We first sort the sequence \( \{Q(s, a_i)\} \) such that \( Q(s, a_{[1]}) \geq \cdots \geq Q(s, a_{[m]}) \). Then, \( \forall Q \) and \( \forall s \), we have

\[
\max_a Q(s, a) - f_\tau^T(Q(s, \cdot)) = Q(s, a_{[1]}) - \frac{m}{\sum_{i=1}^m \exp[\tau Q(s, a_{[i]})]} \sum_{i=1}^m \exp[\tau Q(s, a_{[i]})] (Q(s, a_{[1]}) - Q(s, a_{[i]})),
\]

(4)

By introducing \( \delta_i = Q(s, a_{[i]}) - Q(s, a_{[i+1]}) \), and noting \( \delta_i \geq 0 \) and \( \delta_1 = 0 \), we can proceed from Eq. (4) as

\[
\frac{m}{\sum_{i=1}^m \exp(-\tau \delta_i)} \delta_i = \frac{m}{1 + \sum_{i=2}^m \exp(-\tau \delta_i)} \delta_i.
\]

(5)

Now, we can proceed from Eq. (5) to prove each direction separately as follows.

\[\text{The sup}_{Q} \text{ is over all } Q \text{-functions that occur during } Q \text{-iteration, starting from } Q_0 \text{ until the iteration terminates.}\]
(i) Upper bound: First note that for any two non-negative sequences \( \{x_i\} \) and \( \{y_i\} \),
\[
\frac{\sum_i x_i}{1 + \sum_i y_i} \leq \sum_i \frac{x_i}{1 + y_i}.
\]
We then apply Eq. (6) to Eq. (5) as
\[
\frac{\sum_{i=2}^{m} \exp(-\tau \delta_i)}{1 + \sum_{i=2}^{m} \exp(-\tau \delta_i)} \leq \frac{\sum_{i=2}^{m} \exp(-\tau \delta_i) \delta_i}{1 + \exp(-\tau \delta_i)} = \sum_{i=2}^{m} \frac{\delta_i}{1 + \exp(\tau \delta_i)}.
\]
Next, we bound each term in Eq. (7), by considering the following two cases:
1) \( \delta_i > 1 \): \( \frac{\delta_i}{1 + \exp(\tau \delta_i)} \leq \frac{\delta_i}{1 + \exp(\tau \delta_i)} \leq \frac{2Q_{\text{max}}}{1 + \exp(\tau \delta_i)} \cdot \frac{1}{\tau \delta_i} \cdot \frac{1}{\tau \delta_i} \leq \frac{1}{\tau \delta_i} \cdot \frac{1}{\tau \delta_i} \cdot \frac{1}{\tau \delta_i} 
\]
where we first expand the denominator using Taylor series for the exponential function.
By combining these two cases with Eq. (7), we achieve the upper bound.
(ii) Lower bound:
\[
\frac{\sum_{i=2}^{m} \exp(-\tau \delta_i)}{1 + \sum_{i=2}^{m} \exp(-\tau \delta_i)} \geq \sum_{i=2}^{m} \frac{\exp(-\tau \delta_i) \delta_i}{m} \geq \sum_{i=2}^{m} \frac{\delta_i}{m \exp(\tau \delta_i)} \geq \frac{\delta_i}{m \exp(\tau \delta_i)}.
\]
Note that another upper bound for this gap was provided in [O’Donoghue et al., 2017]. Our proof here uses a different strategy, by considering possible values for the difference between \( Q \)-values with different actions. We further derive a lower bound for this gap.

3.1 Performance Bound for \( T_{\text{soft}} \)

The optimal state-action value function \( Q^* \) is known to be a fixed point for the standard Bellman operator \( T \) (Williams and Baird [1993]), i.e., \( TQ^* = Q^* \). Since \( T \) is a contraction with rate \( \gamma \), we also know that \( \lim_{k \to \infty} T^k Q_0 = Q^* \), for arbitrary \( Q_0 \). Given these facts, one may wonder in the limit, how far iteratively applying \( T_{\text{soft}} \), in lieu of \( T \), over \( Q_0 \) will be away from \( Q^* \), as a function of \( \tau \).

**Theorem 2.** Let \( T^k Q_0 \) and \( T_{\text{soft}}^k Q_0 \) denote that the operators \( T \) and \( T_{\text{soft}} \) are iteratively applied over an initial state-action value function \( Q_0 \) for \( k \) times. Then,
\[
(I) \forall (s, a), Q^*(s, a) \geq \lim_{k \to \infty} T_{\text{soft}}^k Q_0(s, a) \geq Q^*(s, a) - \frac{\gamma}{(1 - \gamma)} \max \left\{ \frac{1}{1 + \exp(\tau \delta)}, \frac{2Q_{\text{max}}}{1 + \exp(\tau \delta)} \right\}.
\]
\[
(II) T_{\text{soft}} \text{ converges to } T \text{ with an exponential rate, in terms of } \tau, \text{ the proof of which does not depend on the bound in part (I)}.
\]
A noteworthy point about Theorem 2 is that it does not contradict the negative convergence results for \( T_{\text{soft}} \) since its proof and result do not need to assume the convergence of \( T_{\text{soft}} \). Part (I) implies that the approximation error for \( T_{\text{soft}} \) gradually disappears when increasing \( \tau \), which is consistent with the fact that \( T_{\text{soft}} \) converges to \( T \) when \( \tau \to \infty \).

Note that although the bound for mellowmax under the entropy-regularized MDP framework was shown in [Lee, Choi, and Oh, 2018] to have a better scalar term \( \log(m) \) instead of \( (m - 1) \), its convergence rate is linear w.r.t. \( \tau \). In contrast, our softmax Bellman operator result has an exponential rate, as shown in Part (II). The error bound in the sparse Bellman operator (Lee, Choi, and Oh, 2018) improves upon \( \log(m) \), but is still linear w.r.t. \( \tau \). We further empirically illustrate the faster convergence for the softmax Bellman operator in Figure 3 of Section 3.3.

3.2 Overestimation and Gradient Noise Reduction

One may wonder why the softmax Bellman operator should be employed, as the greedy policy from the Bellman equation suggests that the max operator should be optimal, and this paper is not focused on exploration, nor is it motivated by regularizing the policy, as in the entropy regularized MDP (Schulman, Chen, and Abbeel, 2017; Nachum et al., 2017; Asadi and Littman, 2017; Lee, Choi, and Oh, 2018). As discussed in earlier work (van Hasselt, Guez, and Silver, 2016; Anschel, Baram, and Shimkin, 2017), max leads to the significant issue of overestimation for the \( Q \)-function. Furthermore, max tends to have higher gradient noise. Here we aim to provide analysis of how the softmax Bellman operator can overcome these issues. Although our analysis is focused on the softmax Bellman operator, this work could potentially give further insight into practical benefits of entropy regularization as well.

**Overestimation Bias Reduction** \( Q \)-learning’s overestimation bias, due to the max operator, was first discussed in [Thrun and Schwartz, 1994]. It was later shown in [van Hasselt, Guez, and Silver, 2016] and [Anschel, Baram, and Shimkin, 2017] that overestimation leads to the poor performance of DQN in some Atari games. Following the same assumptions as [van Hasselt, Guez, and Silver, 2016], we can show the softmax operator reduces the overestimation bias.

**Theorem 3.** Given the same assumptions as [van Hasselt, Guez, and Silver, 2016], where (A1) there exists some \( V^*(s) \) such that the true state-action value function satisfies \( Q^*(s, a) = V^*(s) \), for different actions. (A2) the estimation error is modeled as \( Q_0(s, a) = V^*(s) + \epsilon_a \), then (i) the overestimation errors from \( T_{\text{soft}} \) are smaller or equal to those of \( T \) using the max operator, for any \( \epsilon \geq 0 \); (ii) the overestimation reduction by using \( T_{\text{soft}} \) in lieu of \( T \) is within \( \left| \frac{\delta}{m \exp(\tau \delta)} \right| (m - 1) \max \left\{ \frac{1}{1 + \exp(\tau \delta)}, \frac{2Q_{\text{max}}}{1 + \exp(\tau \delta)} \right\} \); (iii) the overestimation error for \( T_{\text{soft}} \) monotonically increases w.r.t. \( \tau \in [0, \infty) \).

An observation about Theorem 3 is that for any positive value of \( \tau \), there will still be some potential for overestimation bias because noise can also influence the softmax oper-
ator. Depending upon the amount of noise, it is possible that, unlike double DQN, the reduction caused by softmax could exceed the bias introduced by max. This can be seen in our experimental results below, which show that it is possible to have negative error (overcompensation) from the use of softmax. However, when combined with double Q-learning, this effect becomes very small and decreases with the number of actions.

To elucidate Theorem 3, we simulate standard normal variables $\epsilon_a$ with 100 independent trials for each action $a$, using the same setup as van Hasselt, Guez, and Silver (2016). Figure 1 shows the mean and one standard deviation of the overestimation bias, for different values of $\tau$ in the softmax operator. For both single and double implementations of the softmax operator, they achieve smaller overestimation errors than their max counterparts, thus validating our theoretical results. Note that the gap becomes smaller when $\tau$ increases, which is intuitive as $\tau \rightarrow \infty$, and also consistent with the monotonicity result in Theorem 5.

### 3.3 Gradient Variance Reduction

When the Q-function cannot be exactly represented, we approximate it with a functional form as $Q_{\theta}$, parameterized by $\theta$. In the DQN, $\theta$ corresponds to parameters of the convolutional neural network. Subsequently, the task of learning a policy in the DQN is converted to estimating $\theta$ in the neural network, which is performed via stochastic gradient descent (SGD). Given the loss function $L$ in Eq. 1 for DQN, its gradient is then represented as

$$\nabla_\theta L = \mathbb{E}_{s,a,r,s'} \left\{ \left[ Q_{\theta}(s,a) - \tau Q_{\theta}(s,a) \right] \nabla_\theta Q_{\theta}(s,a) \right\},$$

and the gradient under the softmax operator can be obtained accordingly by replacing $T$ with $T_{\text{soft}}$.

As shown in van Hasselt et al. (2016), one source for the gradient noise is the magnitude for the $Q$-function itself, and scaling the $Q$-function may dramatically change its magnitude. We show in Section 3.2 that the softmax Bellman operator can reduce the overestimation bias, and also observe in experiments that it generates smaller $Q$-value estimates.

Another source for the variance of the gradient is the scaling term $[Q_{\theta}(s,a) - T Q_{\theta}(s,a)]$ (a similar analysis for the deep generative model is provided in Mnih and Gregor, 2014). The following proposition demonstrates for the case without function approximation that, starting from the same initial $Q$-function, the update for $T_{\text{soft}}$ will always be less than the update for $T$. This result easily generalizes to the Q-learning case if we assume that both updates use the same training data.

**Proposition 4.** Let $Q_i$ and $Q_{i,\text{soft}}$ denote the $Q$-values obtained at the $i$th step during $Q$-iteration, by using $T$ and $T_{\text{soft}}$ respectively. When the initial $Q$-value is set as $Q_0(s,a) = Q_{i,\text{soft}}(s,a), \forall (s,a)$, then for any index $i \geq 0$ during $Q$-iteration, $T Q_i(s,a) - Q_i(s,a) \geq T_{\text{soft}} Q_i^{\text{soft}}(s,a) - Q_i^{\text{soft}}(s,a), \forall (s,a)$.

This proposition comes with two caveats. First, in the function approximation case, proving that $T_{\text{soft}}$ has a smaller scaling term may require knowledge of the optimization surface of the DQN, and may not hold in all cases. Second, this result bounds the difference in just one direction and does not preclude the magnitude of the update in $T_{\text{soft}}$ being larger due to negative values. It is possible, however, to ensure that $T_{\text{soft}} Q_i^{\text{soft}}(s,a) - Q_i^{\text{soft}}(s,a) \geq 0$ for sufficiently small $\tau$, discussion of which is deferred to Supplemental Material.

### 3.4 Comparison for Different Bellman Operators

Similar to the softmax Bellman operator, the mellowmax operator (Asadi and Littman, 2017) used the log-sum-exp function in lieu of the max function. A thorough comparison among these Bellman operators will not only exhibit their differences, but also reveal the trade-offs when selecting them in practice.

Table 1 shows the comparison from different criteria. We also illustrate in Figure 2 the trade-off between converging to the Bellman optimality and overestimation reduction in terms of the inverse temperature parameter, where the softmax-max operator approaches the max operator in a faster speed while the mellowmax operator can further reduce the overestimation error. Given the facts that both softmax and mellowmax operators (i) converge to the mean and max operators when $\tau \rightarrow 0$ and $\tau \rightarrow \infty$, respectively; (ii) are continuous on $\tau \in [0, \infty)$, it is also worth noting the equivalence between the two in Q-iteration with an off-policy setting: For the softmax-weighted $Q$-function in the Bellman update, i.e., $g_{Q}(s') = \sum_a \sum_{a'} \frac{\exp[\tau Q(s',a')]}{\sum_a \exp[\tau Q(s',a)]} Q(s',a')$ in Eq. (3), there exists another $\tau^*$ such that the mellowmax-weighted $Q$-function achieves the same value.
Furthermore, note that the mellowmax operator cannot directly represent a policy, and needs to be transformed into a softmax policy, where numerical methods are necessary to determine the corresponding state-dependent temperature parameters. The lack of an explicit policy representation also prevents the mellowmax operator from being directly applied in double $Q$-learning. Finally, although the softmax operator is not guaranteed to converge for any inverse temperature parameters, our experiments in Section 5 demonstrate it is not sensitive to this parameter and always achieves competitive performance in Atari games.

## 4 Related Work

[Littman (1996)](https://doi.org/10.1162/cogv.1996.2.3.413) first showed that the Boltzmann operator could be expansive in a specific MDP by tuning the temperature parameter. However, we are not aware of any previous work providing the performance bound for the softmax Bellman operator, in a general MDP. The mellowmax operator proposed in [Asadi and Littman (2017)](https://arxiv.org/abs/1712.01500) was shown to be a non-expansion, whose log-sum-exp component on the $Q$-functions was also employed in the Boltzmann backup operator [Schulman, Chen, and Abbeel (2017)](https://arxiv.org/abs/1707.06887) [Nachum et al. (2017)](https://arxiv.org/abs/1706.01109), derived from the Shannon-entropy regularized soft $Q$-learning [Neu, Jonsson, and Gómez (2017)](https://arxiv.org/abs/1704.04359) and actor-critic learning. Most recently, the sparsemax operator based on Tsallis entropy was proposed in [Lee, Choi, and Oh (2018)](https://arxiv.org/abs/1806.01589) to improve the error bound for mellowmax. However, these operators were not shown to address the instability issues in the DQN, and their corresponding policies were used to improve the exploration performance, which is different from our focus here.

Among variants of the DQN in [Mnih et al. (2015)](https://doi.org/10.5555/2969744.2969885), DDQN ([van Hasselt, Guez, and Silver (2016)](https://arxiv.org/abs/1509.06578)) first identified the overestimation bias issue caused by the max operator, and mitigated it via the double $Q$-learning algorithm [van Hasselt (2010)](https://arxiv.org/abs/1006.0007) [Anschel, Baram, and Shimkin (2017)](https://arxiv.org/abs/1706.01109) later demonstrated that averaging over the previous $Q$-value estimates in the learning target can also reduce the bias, while the analysis for the bias reduction is restricted to a small MDP with a special structure. Furthermore, an adaptive normalization scheme was developed in [van Hasselt et al. (2016)](https://arxiv.org/abs/1611.05654), and demonstrated to reduce the gradient norm. However, unlike the softmax Bellman operator, neither [van Hasselt, Guez, and Silver (2016)](https://arxiv.org/abs/1509.06578) nor [van Hasselt et al. (2016)](https://arxiv.org/abs/1611.05654) was proposed to simultaneously reduce the overestimation bias and the gradient noise. The softmax function was also employed in the categorical DQN [Bellemare, Dabney, and Munos (2017)](https://arxiv.org/abs/1702.06938), but with a different purpose of generating distributions in its distributional Bellman operator. Finally, we notice that other variants [Wang et al. (2016)](https://arxiv.org/abs/1611.05654) [Mnih et al. (2016)](https://arxiv.org/abs/1511.06561) [Schaul et al. (2016)](https://arxiv.org/abs/1601.03141) [He et al. (2017)](https://arxiv.org/abs/1611.05654) [Hessel et al. (2018)](https://arxiv.org/abs/1706.01109) have also been proposed to improve the vanilla DQN, though they were not explicitly designed to tackle the issues of overestimation error and gradient noise.

## 5 Experiments

Our theoretical results apply to the case of a tabular value function representation and known next-state distributions. To assess the practical impact of the softmax Bellman operator when combined with sampled next states and function approximation, we combine the operator with DQN and DDQN, by replacing the max function therein with the softmax function, with all other steps being the same. The corresponding new algorithms are termed as S-DQN and S-DDQN, respectively. Their exploration strategies are set to be $\epsilon$-greedy, same as DQN and DDQN. We also implemented the mellowmax operator and combined it with DQN in the same way as the softmax operator. We tried the mel-
Table 2: Mean of test scores for different values of $\tau$ in S-DDQN (standard deviation in parenthesis). Each game is denoted with its initial. Note that S-DDQN reduces to DDQN when $\tau = \infty$.

| $\tau$ | 1      | 5      | 10     | $\infty$ |
|-------|--------|--------|--------|----------|
| Q     | 12068.66 (1085.65) | 11049.28 (1565.57) | 11191.31 (1336.35) | 10577.76 (1508.27) |
| M     | 2492.40 (183.71) | 2566.44 (227.24) | 2546.18 (259.82) | 2293.73 (160.50) |
| B     | 313.08 (20.13) | 350.88 (35.58) | 303.71 (65.59) | 284.64 (60.83) |
| C     | 10740.91 (4617.90) | 11111.07 (5047.19) | 104049.46 (6686.84) | 96373.08 (9244.27) |
| A     | 3476.91 (460.27) | 10266.12 (2682.00) | 6588.13 (1183.10) | 5523.80 (694.72) |
| S     | 272.20 (49.75) | 2701.05 (10.06) | 6254.01 (697.12) | 5695.35 (1862.59) |

We tested on six Atari games: Q*Bert, Ms. Pacman, Crazy Climber, Breakout, Asterix, and Seaquest. Our code is built on the Theano+Lasagne implementation from https://github.com/spragunr/deep_q_rl/. The training contains 200 epochs in total. The test procedures and all the hyperparameters are set the same as DQN, with details described in Mnih et al. (2015). We set the inverse temperature parameter $\tau$ to be 1, unless otherwise noted. We also implemented the logarithmic cooling scheme (Mitra, Romeo, and Sangiovanni-Vincentelli, 1986) in simulated annealing to gradually increase $\tau$, but did not observe a better policy, compared with the constant temperature. The results statistics are obtained by running with five independent random seeds.

Figure 3 shows the mean and one standard deviation for the average test score on the Atari games, as a function of the training epoch: S-DQN and S-DDQN achieve higher scores on most of them, illustrating the promise of replacing max with softmax. For Asterix and Seaquest, S-DQN scores on most of them, illustrating the promise of replacing max with softmax. For Asterix, often used as an example of the overestimation issue in DQN (van Hasselt et al., 2016), increasing $\tau$ makes the Bellman optimality, since the max operator (corresponding to $\tau = \infty$) is employed in Bellman equation. On the other hand, using a larger $\tau$ will lead to the issues of overestimation and high gradient noise.

We further check the estimated $Q$-values and gradient noise on Atari games. To do this, we report the $\ell_2$ norm of the gradient in the final fully-connected linear layer, by averaging over 50 independent inputs.

Figure 4 shows that increasing $\tau$ in S-DDQN will lead to higher gradient variance and larger $Q$-values, matching the fact that $T_\text{soft}$ approaches $T$ when $\tau$ becomes larger. Figure 5 shows the estimated Q-values and gradient norm.

\footnote{Due to the space limit, we plot fewer games from here, and provide the full plots in Supplemental Material.}
Figure 4: Q-value and gradient norm on the Atari games for S-DDQN, with different values of \( \tau \).

Table 3: Mean and standard deviation of test scores for different Bellman operators.

| Bellman Operator | Max Mean | Softmax Mean | Mellowmax Mean |
|------------------|----------|--------------|----------------|
| Q*Bert           | 8331.72  | 11307.10     | 11775.93       |
|                  | (1597.67)| (1332.80)    | (1173.51)      |
| Ms. Pacman       | 2368.79  | 2856.82      | 2458.76        |
|                  | (219.17) | (369.75)     | (130.34)       |
| C. Climber       | 90923.40 | 106422.27    | 99601.47       |
|                  | (11059.39)| (4821.40)   | (19271.53)     |
| Breakout         | 255.32   | 345.56       | 355.94         |
|                  | (64.69)  | (34.19)      | (25.85)        |
| Asterix          | 196.91   | 8868.00      | 11203.75       |
|                  | (135.16) | (2167.35)    | (3818.40)      |
| Seaquest         | 4090.36  | 8066.78      | 6476.20        |
|                  | (1455.73)| (1646.51)    | (1952.12)      |

demonstrate that DQNs and DDQNs can be improved by replacing the standard Bellman operator with its softmax counterpart, and that the softmax Bellman operator achieves competitive performance, compared with the recently proposed mellowmax operator.

An interesting direction for future work is to provide more theoretical analysis for the performance trade-off when selecting \( \tau \) in \( T_{\text{soft}} \), and to design an efficient cooling scheme.
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Supplemental Material

A1 Proof for Performance Bound

Proof of Theorem[2] We first prove the upper bound by induction as follows.

(i) When \( i = 1 \), we start from the definitions for \( T \) and \( T_{\text{soft}} \) in Eq. (2) and Eq. (3), and proceed as

\[
TQ_0(s,a) - T_{\text{soft}}Q_0(s,a) = \sum_{s'} P(s'|s,a) \left[ \max_{a'} Q_0(s',a') - f_T^Q(Q_0(s',s))Q_0(s',s) \right] \geq 0.
\]

(ii) Suppose this claim holds when \( i = l \), i.e.,

\[
T^lQ_0(s,a) = T_{\text{soft}}^lQ_0(s,a) = \sum_{s'} P(s'|s,a) \left[ \max_{a'} Q_0(s',a') - f_T^Q(Q_0(s',s))Q_0(s',s) \right] \geq 0.
\]

Since \( Q^* \) is the fixed point for \( T \), we know

\[
\lim_{k \to \infty} T^kQ_0(s,a) = Q^*(s,a).\]

Therefore, from the definition of \( T_{\text{soft}} \), we know

\[
\lim_{k \to \infty} T_{\text{soft}}^kQ_0(s,a) \leq Q^*(s,a).
\]

To prove the lower bound, we first conjecture that

\[
T^kQ_0(s,a) - T_{\text{soft}}^kQ_0(s,a) \leq \sum_{j=1}^k \gamma^j \zeta, \tag{A1}
\]

where \( \zeta = \sup_{s,a} \max_{s'} \max_{a'} Q_0(s,a) - f_T^Q(Q_0(s',s))Q_0(s',s) \)

denotes the supremum of the difference between the maximum and softmax operators, over all \( Q \)-functions that occur during \( Q \)-iteration, and state \( s \). Eq. (A1) is proven using induction as follows.

(i) When \( i = 1 \), we start from the definitions for \( T \) and \( T_{\text{soft}} \) in Eq. (2) and Eq. (3), and proceed as

\[
TQ_0(s,a) - T_{\text{soft}}Q_0(s,a) = \sum_{s'} P(s'|s,a) \left[ \max_{a'} Q_0(s',a') - f_T^Q(Q_0(s',s))Q_0(s',s) \right] \leq \sum_{s'} P(s'|s,a) \zeta = \gamma \zeta.
\]

(ii) Suppose the conjecture holds when \( i = l \), i.e.,

\[
T^lQ_0(s,a) - T_{\text{soft}}^lQ_0(s,a) \leq \sum_{j=1}^l \gamma^j \zeta, \text{ then }
\]

\[
T^{l+1}Q_0(s,a) - T_{\text{soft}}^{l+1}Q_0(s,a) = T^{l}T^1Q_0(s,a) - T_{\text{soft}}^{l+1}Q_0(s,a) \leq T \left[ T_{\text{soft}}^lQ_0(s,a) + \sum_{j=1}^l \gamma^j \zeta \right] - T_{\text{soft}}^{l+1}Q_0(s,a) \leq \sum_{j=1}^l \gamma^{j+1} \zeta + (T - T_{\text{soft}}) T_{\text{soft}}^lQ_0(s,a) \leq \sum_{j=1}^{l+1} \gamma^j \zeta + \gamma \zeta = \sum_{j=1}^{l+1} \gamma^j \zeta,
\]

where the last inequality follows from the definition of \( \zeta \). By using the fact that \( \lim_{k \to \infty} T^kQ_0(s,a) = Q^*(s,a) \) and applying Lemma 1 to bound \( \zeta \), we finish the proof for Part (I).

To prove part (II), note that as a byproduct of Eq. (7), Eq. (A1) can be bounded as

\[
\lim_{k \to \infty} T^kQ_0(s,a) - T_{\text{soft}}^kQ_0(s,a) \leq \frac{\gamma}{1 - \gamma} \sum_{i=2}^m \delta_i \frac{\exp(\gamma \delta_i)}{1 + \exp(\tau \delta_i)}.
\]

(A2)

From the definition of \( \delta_i \) in the main text, we know \( \delta_m \geq \delta_{m-1} \geq \ldots \geq \delta_2 \geq 0 \). Furthermore, there must exist an index \( i^* \) such that \( \delta_i > 0, \forall i < i^* \) (otherwise the upper bound becomes zero). Subsequently, we can proceed from Eq. (A2) as

\[
\frac{\gamma}{1 - \gamma} \sum_{i=2}^m \delta_i \frac{\exp(\gamma \delta_i)}{1 + \exp(\tau \delta_i)} \leq \frac{\gamma}{1 - \gamma} \sum_{i=1}^m \delta_i \frac{\exp(\tau \delta_i)}{1 + \exp(\tau \delta_i)} \leq \frac{\gamma}{1 - \gamma} \sum_{i=1}^m \delta_i \frac{\exp(\tau \delta_i)}{1 + \exp(\tau \delta_i)} = \frac{\gamma}{1 - \gamma} \sum_{i=1}^m \delta_i,
\]

which implies an exponential convergence rate in terms of \( \tau \) and hence proves part (II).

\( \square \)

A2 Proofs for Overestimation and Gradient Variance Reduction

Lemma A5. \( g_\tau(x) = \sum_{i=1}^m \frac{\sum_{j=1}^m \exp(x_{ij} \tau x_i) \sum_{i=1}^m \exp(x_{ij} \tau x_i)}{\sum_{i=1}^m \sum_{j=1}^m \exp(x_{ij} \tau x_i)} \) is a monotonically increasing function for \( \tau \in [0, \infty) \).

Proof. The gradient of \( g_\tau(x) \) can be computed as

\[
\frac{\partial g_\tau(x)}{\partial \tau} = \left\{ \sum_{i=1}^m \exp(x_{ij} \tau x_i)^2 \right\} \left\{ \sum_{j=1}^m \exp(x_{ij} \tau x_i) \right\} - \left\{ \sum_{i=1}^m \exp(x_{ij} \tau x_i) \right\}^2 \geq 0,
\]

where the last step holds because of the Cauchy-Schwarz inequality.

\( \square \)

The overestimation bias due to the max operator can be observed by plugging assumption (A2) in Theorem 3 into Eq. (2) as

\[
\mathbb{E} \left[ \max_a (Q_t(s,a)) - \max_a (Q^*_t(s,a)) \right] = \mathbb{E} \left[ \max_a (Q_t(s,a) - V_*(s)) \right] = \mathbb{E} \left[ \max_a (\epsilon_a) \right],
\]
and $\max_a (\epsilon_a)$ is typically positive for a large action set and the noise satisfying a normal distribution, or a uniform distribution with the symmetric support.

**Proof of Theorem 3.** First, the overestimation error from $T_{\text{soft}}$ can be represented as

$$E \left\{ \sum_a \frac{\exp[\tau Q_l(s, a)]}{\sum_a \exp[\tau Q_l(s, a)]} Q_l(s, a) - V^*(s) \right\}$$

$$= E \left\{ \sum_a \frac{\exp[\tau V^*(s) + \epsilon_a]}{\sum_a \exp[\tau V^*(s) + \epsilon_a]} [V^*(s) + \epsilon_a] - V^*(s) \right\}$$

$$= E \left\{ \sum_a \frac{\exp[\epsilon_a]}{\sum_a \exp[\epsilon_a]} \epsilon_a \right\} \leq E \left[ \max_a (\epsilon_a) \right].$$

To prove Part (II), note that the overestimation reduction of $T_{\text{soft}}$ from $\hat{T}$ can then be represented as

$$E \left[ \max_a (\epsilon_a) - \sum_a \frac{\exp[\epsilon_a]}{\sum_a \exp[\epsilon_a]} \epsilon_a \right]$$

$$= E \left\{ \max_a [\epsilon_a + V^*(s)] - \sum_a \frac{\exp[\epsilon_a]}{\sum_a \exp[\epsilon_a]} [\epsilon_a + V^*(s)] \right\}$$

$$= E \left\{ \max_a [Q_l(s, a)] - \sum_a \frac{\exp[\epsilon_a]}{\sum_a \exp[\epsilon_a]} [Q_l(s, a)] \right\}$$

$$\leq E \left\{ \max_a [Q_l(s, a)] - \sum_a \frac{\exp[\epsilon_a + \tau V^*(s)]}{\sum_a \exp[\epsilon_a + \tau V^*(s)]} \tau V^*(s) \sum_a \frac{\exp[\epsilon_a + \tau V^*(s)]}{\sum_a \exp[\epsilon_a + \tau V^*(s)]} \right\}$$

Subsequently, we can employ Lemma 1 to obtain the range.

Finally, the monotonicity for the overestimation error in $E$ has all non-negative elements.

**Definition 7.** The largest difference between $Q^{i+1}$ and $Q^i$ for state $s$ is defined as

$$\hat{\Delta}^i(s) = \max_a \Delta^i(s, a).$$

**Theorem 8.** When the inverse temperature parameter satisfies

$$0 \leq \tau \leq \inf_{s, a} \left\{ \ln \left[ \frac{Q^i(s, a)}{Q^{i-1}(s, a)} \right] - \frac{\Delta^i(s) - \Delta^{i-1}(s)}{\Delta^i(s)} \right\}$$

$T_{\text{soft}} Q^i \geq T_{\text{soft}} Q^{i-1}$ elementwise, $\forall j > i^*.$

**Proof.** WLOG, let’s assume $Q^*(s, a) \geq 0, \forall s$ and $\forall a$. When $j = i^* + 1$, we have

$$T_{\text{soft}} Q^j(s, a) = R(s, a) + \gamma \sum_{s'} P(s' | s, a) \sum_{a'} \frac{\exp[\tau Q^j(s', a')]}{\sum_a \exp[\tau Q^j(s', a)]} Q^j(s', a')$$

$$\geq R(s, a) + \gamma \sum_{s'} P(s' | s, a) \sum_{a'} \frac{\exp[\tau Q^{i+1}(s', a') + \tau \Delta^{i+1}(s', a')]}{\sum_a \exp[\tau Q^{i+1}(s', a) + \tau \Delta^{i+1}(s', a)]}$$

$$\geq R(s, a) + \gamma \sum_{s'} P(s' | s, a) \sum_{a'} \frac{\exp[\tau Q^i(s', a') + \tau \Delta^i(s', a')]}{\sum_a \exp[\tau Q^i(s', a) + \tau \Delta^i(s', a)]}$$

where the last inequality uses the definition of $\hat{\Delta}^i(s)$ in Eq. (A4).

To let $T_{\text{soft}} Q^j(s, a) \geq T_{\text{soft}} Q^{i+1}(s, a)$, one condition can be $\forall a'$ and $\forall s'$,

$$[Q^i(s', a') + \Delta^i(s', a')] \sum_a \frac{\exp[\tau Q^i(s', a') + \tau \Delta^i(s', a')]}{\sum_a \exp[\tau Q^i(s', a) + \tau \Delta^i(s', a)]} \geq Q^i(s', a') \exp[\tau Q^i(s', a)] \sum_a \exp[\tau Q^i(s', a)].$$

**A3 Monotonicity for $T_{\text{soft}}$ in $Q$-iteration**

The purpose of this section is to derive a lower bound for $\tau$ such that $T_{\text{soft}}$ can monotonically improve the $Q$-function with $Q$-iteration.

Similar to the monotonicity result for the standard Bellman operator using the max function (e.g., [Bertsekas 2012 Lemma 1.1.1]), we first assume that there exists a time step such that the current $Q$-value is not smaller than the $Q$-value in the previous step.

**Assumption 6.** In $Q$-iteration, there exists an iteration index $i^*$ such that $T_{\text{soft}} Q^{i^*} \geq T_{\text{soft}} Q^{i^*-1}$ elementwise, i.e.,

$$Q^{i+1} = T_{\text{soft}} Q^{i} - T_{\text{soft}} Q^{i-1} + \Delta^i = Q^i + \Delta^i,$$

where $\Delta^i$ has all non-negative elements.

**Definition 7.** The largest difference between $Q^{i+1}$ and $Q^i$ for state $s$ is defined as

$$\hat{\Delta}^i(s) = \max_a \Delta^i(s, a).$$

**Theorem 8.** When the inverse temperature parameter satisfies

$$0 \leq \tau \leq \inf_{s, a} \left\{ \ln \left[ \frac{Q^i(s, a)}{Q^{i-1}(s, a)} \right] - \frac{\Delta^i(s) - \Delta^{i-1}(s)}{\Delta^i(s)} \right\}$$

$T_{\text{soft}} Q^i \geq T_{\text{soft}} Q^{i-1}$ elementwise, $\forall j > i^*.$

**Proof.** WLOG, let’s assume $Q^*(s, a) \geq 0, \forall s$ and $\forall a$. When $j = i^* + 1$, we have

$$T_{\text{soft}} Q^j(s, a) = R(s, a) + \gamma \sum_{s'} P(s' | s, a) \sum_{a'} \frac{\exp[\tau Q^j(s', a')]}{\sum_a \exp[\tau Q^j(s', a)]} Q^j(s', a')$$

$$\geq R(s, a) + \gamma \sum_{s'} P(s' | s, a) \sum_{a'} \frac{\exp[\tau Q^{i+1}(s', a') + \tau \Delta^{i+1}(s', a')]}{\sum_a \exp[\tau Q^{i+1}(s', a) + \tau \Delta^{i+1}(s', a)]}$$

$$\geq R(s, a) + \gamma \sum_{s'} P(s' | s, a) \sum_{a'} \frac{\exp[\tau Q^{i}(s', a') + \tau \Delta^i(s', a')]}{\sum_a \exp[\tau Q^{i}(s', a) + \tau \Delta^i(s', a)]}$$

where the last inequality uses the definition of $\hat{\Delta}^i(s)$ in Eq. (A4).

To let $T_{\text{soft}} Q^j(s, a) \geq T_{\text{soft}} Q^{i+1}(s, a)$, one condition can be $\forall a'$ and $\forall s'$,

$$[Q^{i-1}(s', a') + \Delta_{i-1}(s', a')] \sum_a \frac{\exp[\tau Q^{i-1}(s', a') + \tau \Delta_{i-1}(s', a')]}{\sum_a \exp[\tau Q^{i-1}(s', a) + \tau \Delta_{i-1}(s', a)]} \geq Q^{i-1}(s', a') \exp[\tau Q^{i-1}(s', a)] \sum_a \exp[\tau Q^{i-1}(s', a)].$$

(A6)
where we use the fact that the transition kernel $P(s' | s, a)$ is always non-negative. Subsequently, Eq. (A6) can be simplified as $\forall a'$ and $\forall s'$,

$$
\exp \left\{ \tau \left[ Q^{j-1}(s', a') + \Delta^{j-1}(s', a') - \Delta^{j-1}(s') \right] \right\}
\times [Q^{j-1}(s', a') + \Delta^{j-1}(s', a')]
\geq Q^{j-1}(s', a') \exp[\tau Q^{j-1}(s', a')]
$$

\[\iff\]

$$
0 \leq \tau \leq \frac{\ln[Q^{j}(s, a)] - \ln[Q^{j-1}(s, a)\big]}{\Delta^{j-1}(s') - \Delta^{j-1}(s', a')}.
$$

(A7)

To ensure Eq. (A7) is satisfied $\forall a', \forall s'$, and $\forall Q$, Eq. (A5) can be used as a sufficient condition.

Finally, applying $\mathcal{T}_{\text{soft}}$ recursively for $j > i^* + 1$ leads to

$$
\mathcal{T}_{\text{soft}} Q^{i^*} \leq \mathcal{T}_{\text{soft}} Q^{i^*+1} \leq \mathcal{T}_{\text{soft}} Q^{i^*+2} \leq \cdots,
$$

which completes our proof.

\[\square\]

### Discussion

When the $Q$-function is approaching convergence, the sufficient condition for $\tau$ in Eq. (A5) might be problematic, since both denominator and numerator become close to zero. In this case, we re-derive the sufficient condition for $\tau$, as shown in Corollary A9.

**Corollary A9.** When the iteration index $j$ is sufficiently large such that $|Q^{j}(s, a) - Q^{j-1}(s, a)|$ is close to zero, $\forall s$ and $\forall a$, a new sufficient condition for $\tau$ to guarantee $\mathcal{T}_{\text{soft}} Q^{j} \geq \mathcal{T}_{\text{soft}} Q^{j-1}$ elementwise in Theorem A8 is

$$
0 \leq \tau \leq \frac{1}{2 \sup_{Q} \max_{s, a} Q^{j-1}(s, a)}.
$$

(A8)

given the assumption that $Q^{j}(s, a)$ converges with the same rate for different actions.

**Proof.** We can start from Eq. (A7) in the proof of Theorem A8 and note a new upper bound for $\tau$ can be

$$
\tau \leq \frac{\ln[Q^{j}(s, a)] - \ln[Q^{j-1}(s, a)\big]}{\Delta^{j-1}(s) + \Delta^{j-1}(s, a)}.
$$

(A9)

We then perform the Taylor expansion on $\ln[Q^{j}(s, a)]$ at $Q^{j-1}(s, a)$ in Eq. (A9), and proceed as

$$
\tau \leq \frac{1}{Q^{j-1}(s, a)^2} \Delta^{j-1}(s, a) + O\left(\left[\Delta^{j-1}(s, a)^2\right]^2\right)
\Delta^{j-1}(s) + \Delta^{j-1}(s, a)
\approx \frac{1}{2Q^{j-1}(s, a)}.
$$

(A10)

where in the last step, we ignore the higher-order terms, and apply the assumption to set $\Delta^{j-1}(s) \approx \Delta^{j-1}(s, a)$. Again, we can take the maximum of $Q^{j-1}(s, a)$ to guarantee the condition in Eq. (A10) is satisfied, $\forall s$, $\forall a$, and $\forall Q$. \[\square\]

### A4 Additional Plots and Setup

**Setup for Figure 2.** The inverse temperature parameter $\tau$ is chosen from a linear grid from 0.01 to 100. We simulate standard normal random variables, and compute the weighted $Q$-functions for the cases with different number of actions. The error is measured in terms of the difference between the max function and the corresponding softmax and mellowmax functions. The result statistics are reported by averaging over 100 independent trials.

Figures A1 and A2 are the full version of the corresponding figures in the main text, by plotting all six games.

Figures A3, A4, and A5 show the scores, gradient norm, and $Q$-values, for different values of $\tau$, for S-DDQN.
Figure A1: Mean and one standard deviation of the estimated $Q$-values on the Atari games, for different methods.

Figure A2: Mean and one standard deviation of the gradient norm on the Atari games, for different methods.
Figure A3: Mean and one standard deviation of test scores on the Atari games, for different values of $\tau$ in S-DDQN.

Figure A3: Mean and one standard deviation of the gradient norm on the Atari games, for different values of $\tau$ in S-DDQN.
Figure A5: Mean and one standard deviation of the estimated $Q$-values on the Atari games, for different values of $\tau$ in S-DDQN.