An Oka principle for equivariant isomorphisms

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Stein spaces and Oka principle

- With F. Kutzschebauch and F. Lárusson.
- Let $X$ be a complex manifold. Then $X$ is Stein iff $X$ is biholomorphic to a closed complex submanifold of some $\mathbb{C}^n$.
- Holomorphic analogue of smooth complex affine variety.
- Can also define when a complex space is Stein. Analogue of complex affine variety.

Oka Principle

On reduced Stein spaces, there are only topological obstructions to solving holomorphic problems that can be formulated cohomologically.
Let $G$ be a complex Lie group and $X$ a reduced Stein space.

**Theorem (Grauert)**

Inclusion induces an isomorphism between isomorphism classes of holomorphic principal $G$-bundles on $X$ and topological principal $G$-bundles on $X$.

Note that isomorphism classes of principal $G$-bundles are given by a certain cohomology set $H^1(X, \mathcal{G})$ where $\mathcal{G}$ is maps of open sets of $X$ to $G$.

Theorem of Grauert is an Oka principle.

Equivariant version due to Heinzner and Kutzschebauch.
Want an Oka principle for equivariant maps.

Let $X$ be a connected Stein manifold with holomorphic action of the complex reductive Lie group $G$.

We have the quotient space $Z = X \sslash G$, a reduced Stein space.

The space $Z$ has points corresponding to the closed $G$-orbits in $X$ and the pull-back of the structure sheaf on $Z$ is the sheaf of $G$-invariant holomorphic functions on $X$.

Let $x \in X$ such that $Gx$ is closed. Then $G_x$ is reductive and the representation of $G_x$ on $T_x(X)/T_x(Gx)$ is called the slice representation at $x$.

$Z$ has a stratification $Z(H)$ where the points in $Z(H)$ correspond to the closed orbits with isotropy group conjugate to the reductive subgroup $H$ of $G$. 
The stratification $Z_{(H)}$ is a locally finite stratification of $Z$ by locally closed smooth subvarieties of $Z$.

Example. Let $G = \mathbb{C}^*$ and $V = \mathbb{C}^2$ where $t(a, b) = (ta, t^{-1}b)$, $t \in \mathbb{C}^*$, $(a, b) \in V$. Let $x$ and $y$ be the coordinate functions. Then $O(V)^G$ is generated by $xy$.

Let $\pi = xy : V \rightarrow Z = \mathbb{C}$. Then $\pi^*\mathcal{H}(Z) = \mathcal{H}(V)^G$.

Nonzero closed orbits $Gx$ have $G_x = \{e\}$. The origin has isotropy group $G$. Then the strata of $Z = \mathbb{C}$ are $\mathbb{C} \setminus \{0\}$ and $\{0\}$. 
• Let \( Y \) be another Stein \( G \)-manifold with quotient mapping \( \pi_Y : Y \to Z \). Same quotient space as \( X \).
• We say that \( X \) and \( Y \) are **locally isomorphic over \( Z \)** if there are \( G \)-biholomorphisms \( \psi_i : \pi_X^{-1}(U_i) \cong \pi_Y^{-1}(U_i) \) which induce the identity on \( U_i \) for an open cover \( \{U_i\} \) of \( Z \).
• Hoped for Oka principle: \( X \) and \( Y \) are \( G \)-biholomorphic (over \( \text{Id} : Z \to Z \)) iff a topological condition is satisfied.
• For \( U \subset Z \) let \( F(U) \) denote the \( G \)-equivariant biholomorphisms of \( \pi_X^{-1}(U) \) inducing \( \text{Id} : U \to U \). Sheaf of groups.
• Then \( \psi_{ij} := \psi_i^{-1} \circ \psi_j \) is in \( F(U_i \cap U_j) \) and \( \{\psi_{ij}\} \in H^1(Z, F) \).

There is an equivariant biholomorphism \( \varphi : X \to Y \) over the identity of \( Z \) iff \( \{\psi_{ij}\} \) is a coboundary.
• Example. Let $X$ and $Y$ be holomorphic principal $G$-bundles over the Stein manifold $Z$

• Then $X//G = Y//G = Z$ and $X$ and $Y$, as Stein $G$-manifolds, are locally isomorphic over $Z$.

• Then $X$ is $G$-biholomorphic to $Y$ over $Z$ if and only if the two holomorphic principal bundles are isomorphic if and only if the principal bundles are $G$-homeomorphic (Grauert) if and only if $X$ is $G$-homeomorphic to $Y$ over $Z$. 
Generic actions

- There is a unique open stratum $Z_{pr} \subset Z$, called the principal stratum. Let $X_{pr} = \pi_X^{-1}(Z_{pr})$.
- We say that $X$ is generic if $X_{pr}$ consists of closed orbits with trivial isotropy group and $\text{codim } X \setminus X_{pr} \geq 2$.
- $X$ is generic iff every slice representation is generic.
- $X_{pr} \to Z_{pr}$ is a principal $G$-bundle.
- For a fixed simple group $H$ and $H$-modules $W$ with $W^H = (0)$, up to isomorphism, only finitely many $W$ are not generic!
- Similar statement for $H$ semisimple. Thus “almost any” $X$ is generic.
Special automorphisms

- Let $\psi : X \to X$ be holomorphic, equivariant, induce identity on $Z$. Say $\psi$ is special if there is a holomorphic map $\gamma : X \to G$ such that $\psi(x) = \gamma(x) \cdot x$.

**Lemma**

If $X$ is generic, then every holomorphic $\psi$ is special. Moreover, we have that $\gamma(gx) = g\gamma(x)g^{-1}$.

- Let $\mathcal{G}$ be the sheaf on $Z$ corresponding to equivariant holomorphic $\gamma : \pi_X^{-1}(U) \to G$, $U$ open in $Z$.
- If $X$ is generic, then $\mathcal{F} \simeq \mathcal{G}$, by the Lemma.
• Let $\mathcal{G}_c$ be the sheaf of groups corresponding to continuous equivariant maps to $G$.

**Theorem (HK)**

The natural map $H^1(Z, \mathcal{G}) \to H^1(Z, \mathcal{G}_c)$ is an isomorphism.

**Corollary**

$X \simeq Y$ over $Z$, equivariantly, iff a topological condition is satisfied.
G-finite functions

- Now we see what a topological condition should be.
- $G$ acts on $\mathcal{H}(X)$, $f \mapsto g \cdot f$ where $(g \cdot f)(x) = f(g^{-1}x)$, $x \in X$.

**Definition**

$f \in \mathcal{H}(X)$ is $G$-finite if $\{g \cdot f \mid g \in G\}$ spans a finite-dimensional $G$-module.

- The $G$-finite functions are an $\mathcal{H}(X)^G$-module.
- Let $V_i$ be a finite-dimensional $G$-module and let $\mathcal{H}(X)_{V_i}$ denote the sum of the subspaces of $G$-finite functions that transform by $V_i$. Covariants.
- Assume $\exists$ collection of irreducible representations $V_i$ such that the $\mathcal{H}(X)_{V_i}$ generate the algebra of $G$-finite functions on $X$ and that the $\mathcal{H}(X)_{V_i}$ are finitely generated $\mathcal{H}(X)^G$-modules. (True locally over $\mathbb{Z}$).
Strongly continuous maps

- Let \( \psi : X \to X \) be equivariant biholomorphic over \( Z \). Let \( f_1, \ldots, f_n \) generate the \( \mathcal{H}(X)_{\mathcal{V}_i} \). Then

\[
\psi^* f_i = \sum a_{ij}(z)f_j \quad \text{where the } a_{ij}(z) \in \mathcal{H}(Z).
\]

- \( \psi \) is determined by the \( a_{ij} \).
- Let \( \varphi : X \to X \) be a \( G \)-equivariant homeomorphism.

**Definition**

We say that \( \varphi \) is strongly continuous if \( \varphi^* f_i = \sum a_{ij}(z)f_j \) where the \( a_{ij}(z) \) are continuous.
\[ \varphi^* f_i = \sum_{ij} a_{ij}(z)f_j. \]

- The fibers of \( \pi \) are affine \( G \)-varieties and the \( \mathcal{H}(X)_V \) generate \( \mathcal{O}(\pi_X^{-1}(z)) \).
- Hence \( \varphi \) induces a \( G \)-automorphism of \( \pi_X^{-1}(z) \). So \( \varphi \) is a continuous family of \( G \)-isomorphisms of the fibers of \( \pi_X \).
- Strongly continuous maps are the natural kinds of topological maps one should consider.
- Suppose that \( \mathcal{F} \) is represented by a group scheme \( \tilde{\mathcal{F}} \) over \( Z \), i.e., the fibers of \( \tilde{\mathcal{F}} \to Z \) are groups and \( \mathcal{F}(U) \cong \Gamma(U, \tilde{\mathcal{F}}) \). Then the continuous sections of \( \tilde{\mathcal{F}} \) are the strongly continuous homeomorphisms.
Let \( \varphi: X \to Y \) be a \( G \)-homeomorphism over \( Z \). Then \( \varphi \) is strongly continuous if \( \psi_i^{-1} \circ \varphi: \pi^{-1}_X(U_i) \to \pi^{-1}_X(U_i) \) is strongly continuous for all \( i \). Recall \( \psi_i: \pi^{-1}_X(U_i) \cong \pi^{-1}_Y(U_i) \) over \( U_i \).

Let \( \varphi: X \to Y \) be strongly continuous where \( X \) and \( Y \) are generic. Then there is an equivariant biholomorphism \( \varphi': X \to Y \).
Proof of Theorem

- Let $x \in X$, $Gx$ closed and let $(W, H)$ be the slice representation. (So $H = Gx$.) There is an $H$-saturated open set $0 \in B \subset W$ such that $\sigma_X : \pi_X^{-1}(U) \simeq G \times^H B$ where $U$ is a neighborhood of $z = \pi_X(x)$. Slice theorem.

- We similarly have a $\sigma_Y : \pi_Y^{-1}(U) \simeq G \times^H B$.

- Then $\varphi_U := \sigma_Y \circ \varphi \circ \sigma_X^{-1} : G \times^H B \to G \times^H B$.

- For $t \in \mathbb{C}^*$ let $t \cdot [g, w] = [g, tw]$ for $[g, w] \in G \times^H B$. We can assume that $G \times^H B$ is stable under this action for $|t| \leq 1$.

**Lemma**

Let $\varphi_t([g, w]) = t^{-1}\varphi([g, tw])$. Then $\varphi_0 := \lim_{t \to 0} \varphi_t$ exists and is special, where the associated map $\gamma$ is continuous.

- Using induction and a partition of unity argument, one can show that there is a homotopy $\varphi_t$ with $\varphi_1 = \varphi$ and $\varphi_0$ special.

- Now $\{\psi_{ij}\} \in H^1(Z, \mathcal{F}) = H^1(Z, \mathcal{G}) \simeq H^1(Z, \mathcal{G}_c)$ where the existence of $\varphi_0$ shows that the class in $H^1(Z, \mathcal{G}_c)$ is trivial. QED.
• The proof does not actually show that $\varphi$ is homotopic to a $G$-biholomorphism of $X$ and $Y$ over $Z$.

• What about actions that are not generic?

Latest Theorem

Suppose that $\varphi : X \to Y$ is strongly continuous. Then there is a homotopy $\varphi_t$ with $\varphi_1 = \varphi$ and $\varphi_0$ a $G$-biholomorphism of $X$ and $Y$ over $Z$.

• Can’t reduce to HK. Go through Cartan’s version of Grauert’s original theorem and modify everything to fit our situation.
Preliminary step. Let $\varphi : X \to X$ be strongly continuous. Then, after a homotopy, we can arrange the following.

Let $z \in Z$. Then there is a neighborhood $U_z$ of $z$ and $\psi_z \in \mathcal{F}(U_z)$ which agrees with $\varphi$ on $\pi^{-1}_X(z)$. Moreover, the family $\Psi(x, x') = \psi_{\pi_X(x)}(x')$ is smooth in $x$ and $x'$.

We can apply the Grauert proof to the $\varphi$ which admit an extension $\Psi$. (They form a sheaf of groups.)
• How can $G$ act on $\mathbb{C}^n$? Can we holomorphically change coordinates such that the action of $G$ is linear? Say $G$-action is linearizable.
• Derksen-Kutzschebauch: For every $G \neq \{e\}$ there is a $d$ and a nonlinearizable action of $G$ on $\mathbb{C}^n$ for $n \geq d$. The quotients $\mathbb{C}^n//G$ are rather horrible.

**Theorem**

Suppose that $V$ is a $G$-module and that $X$ and $V$ are locally isomorphic over a common quotient. Then $X$ is equivariantly biholomorphic to $V$. 
Theorem

Suppose that $V$ is not too “small” and suppose that $X \parallel G$ and $V \parallel G$ are biholomorphic by a mapping which preserves the Luna strata. Then $X$ is $G$-biholomorphic to $V$.

- For any simple Lie group $G$, only finitely many $V$ with $V^G = (0)$ are too small. Similarly for $G$-semisimple.
- The Luna stratification is finer than the stratification by conjugacy class of the isotropy group. On each irreducible component of $Z(H)$, the slice representation $(W, H)$ is constant. The Luna stratification is by the isomorphism class of the slice representation.