THE STABLE MODULE CATEGORY OF A GENERAL RING

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Abstract. For any ring \( R \) we construct two triangulated categories, each admitting a functor from \( R \)-modules that sends projective and injective modules to 0. When \( R \) is a quasi-Frobenius or Gorenstein ring, these triangulated categories agree with each other and with the usual stable module category. Our stable module categories are homotopy categories of Quillen model structures on the category of \( R \)-modules. These model categories involve generalizations of Gorenstein projective and injective modules that we derive by replacing finitely presented modules by modules of type \( FP_\infty \).

1. Introduction

This paper is about the generalization of Gorenstein homological algebra to arbitrary rings. Gorenstein homological algebra first arose in modular representation theory, where one looks at representations of a finite group \( G \) over a field \( k \) whose characteristic divides the order of \( G \). In this case, the group ring \( k[G] \) is quasi-Frobenius, which means that projective and injective modules coincide. It is natural, then, to ignore them and form the stable module category \( \text{Stmod}(k[G]) \) by identifying two \( kG \)-module maps \( f, g : M \to N \) when \( g \circ f \) factors through a projective module. The stable module category is the main object of study in modular representation theory. For example, the Tate cohomology of \( G \) is just \( \text{Stmod}(k[G])(k,k)_* \).

One could perform this construction for any ring \( R \), but the resulting category only has good properties when the ring is quasi-Frobenius. In this case, the resulting category \( \text{Stmod}(R) \) is not abelian, but it is triangulated. The shift functor \( \Sigma M \) is given by taking the cokernel of a monomorphism from \( M \) into an injective module; this is unique up to isomorphism in \( \text{Stmod}(R) \). There is an exact coproduct-preserving functor

\[
\gamma : R\text{-Mod} \to \text{Stmod}(R),
\]

exact in the sense that if

\[
0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0
\]

is a short exact sequence in \( R\text{-Mod} \), then there is an exact triangle

\[
M' \to M \xrightarrow{g} M'' \xrightarrow{h} \Sigma M'
\]

in the triangulated category \( \text{Stmod}(R) \). Note that \( \gamma \) takes projective (and injective, since they coincide) modules to 0, and also that \( \gamma \) is universal with respect to this
property. That is, given any other exact functor \( \delta : R \text{-Mod} \to C \) into a triangulated category that sends projectives to 0, there is a unique induced exact functor \( \overline{\delta} : \text{Stmod}(R) \to C \) such that \( \overline{\delta} \gamma = \delta \). In fact, \( \overline{\delta} \) is just \( \delta \), since by assumption \( \delta \) sends projective modules to 0 and therefore identifies \( f \) and \( g \) when their difference factors through a projective module.

We have often wondered whether it would be possible to give a construction of \( \text{Stmod}(R) \) for a general ring \( R \), and it is the goal of this paper to make such a construction. We would like \( \text{Stmod}(R) \) to have similar properties; it should be triangulated with an exact functor \( \gamma : R \text{-Mod} \to \text{Stmod}(R) \) that sends projectives and injectives to 0, and it should be somehow universal with respect to this property. It should also be natural with respect to at least some ring homomorphisms, and of course it should give the same answer in cases where the stable module category has already been defined (this includes the case when \( R \) is Gorenstein [Hov02]).

We began this paper by considering the stable derived category of Krause [Kra05]. This is the chain homotopy category of exact (unbounded) complexes of injective modules. Krause only considers this for Noetherian commutative rings \( R \) (actually for Noetherian schemes), but most of his construction works more generally. The stable derived category is triangulated, and there is an exact functor from \( R \text{-Mod} \) into it that sends projectives to 0, but it seems unlikely that it sends injectives to 0. (Explicit examples are hard to understand except in the easiest cases). There is also no hint that it is universal. However, the stable derived category does coincide with the stable module category when \( R \) is Gorenstein.

The stable derived category suggests approaching the stable module category through unbounded complexes, rather than modules. We give two general constructions of Quillen model structures on unbounded chain complexes, one where everything is cofibrant and the fibrant objects are certain complexes of injectives, and one where everything is fibrant and the cofibrant objects are certain complexes of projectives. We recover the stable derived category as the homotopy category of one such model structure. However, the key idea comes from Gorenstein homological algebra, where one considers totally acyclic complexes. In the injective case, these are exact complexes of injectives that remain exact after applying \( \text{Hom}(I, -) \) for any injective module \( I \). This has always seemed a strange idea, but this is the key to constructing a homotopy category where the injectives go to 0 as well as the projectives.

The definition of totally acyclic is really only appropriate for Noetherian rings \( R \); perhaps the main contribution of this paper is describing how Gorenstein homological algebra should work for general rings \( R \). The idea is as follows. An injective module is a module \( I \) such that \( \text{Ext}^1(M, I) = 0 \) for all modules \( M \), but in fact it is equivalent to assume this for all finitely generated modules \( M \). But finitely generated modules are only well-behaved for Noetherian rings. One might try considering absolutely pure modules \( I \) instead; these are modules such that \( \text{Ext}^1(M, I) = 0 \) for all finitely presented \( M \). But finitely presented modules are only well-behaved over coherent rings. So what we really need are modules which have not only finite generators and relations, but finite relations between the relations, and so on. A module \( M \) is said to have type \( FP_\infty \) if it has a projective resolution by finitely generated projectives. We can then define a module \( I \) to be absolutely clean, thinking of clean as a little weaker than pure, if \( \text{Ext}^1(M, I) = 0 \) for all \( M \) of type \( FP_\infty \). Finally, we define a complex \( X \) of injectives to be exact.
AC-acyclic if it is exact and $\text{Hom}(I, X)$ remains exact for any absolutely clean module $I$. This is our generalization of totally acyclic.

We then get a Quillen model structure on chain complexes where everything is cofibrant and the fibrant objects are the exact AC-acyclic complexes of injectives, for any ring $R$. The homotopy category of this model structure is our candidate for the stable module category; it is the chain homotopy category of exact AC-acyclic complexes of injectives. Further evidence that this is the right thing is the fact that it is indeed a homotopy category of modules, rather than just complexes. That is, there is a model structure on $R$-Mod whose homotopy category is this stable module category. In this model structure, everything is cofibrant and the fibrant objects are the Gorenstein AC-injective modules; these are the zero-cycles of exact AC-acyclic complexes of injectives. Our stable module category is then equivalent to the quotient category of Gorenstein AC-injective modules where two maps are identified when their difference factors through an injective module. Note, however, that we need the model structure on complexes to approach the one on modules; we cannot work directly with modules.

We have concentrated here on the injective case, but we also construct analogous model structures using exact complexes of projectives. There is a similar notion of an exact AC-acyclic complex of projectives, and we get a likely different notion of a stable module category. This is the chain homotopy category of exact AC-acyclic complexes of projectives, or equivalently, the quotient category of Gorenstein AC-projective modules obtained by identifying maps that factor through a projective.

We have had to leave many loose ends in this paper that we hope to address in future work. We need more explicit examples. It seems likely to us that our two notions of stable module category likely agree with some conditions on the ring; perhaps when the ring has a dualizing module. And our construction of the stable module category includes a generalization of Tate cohomology to all algebras over $k$, and in particular to all groups, finite or not. Such generalizations of Tate cohomology have been considered by many authors; in particular, Benson uses modules of type $FP_{\infty}$ in his generalization [Ben97], but in a different way. We would like to understand the precise relationship between these approaches. This paper grew out of the Ph. D. thesis of the first author [Bra11]. We also mention the work of Hanno Becker [Bec12], who independently constructed some of our model categories in the Noetherian case and has a very nice interpretation of the recollement constructed by Krause in terms of model categories.

Throughout this paper, $R$ will denote a ring with unity, and $R$-Mod will denote the category of left $R$-modules.

2. Finiteness, flatness, and injectivity

Homological algebra of any sort is about approximating objects by projective, injective, and flat objects. In this section, we point out that the notions of flatness and injectivity depend on a choice of which modules are considered to be finite, and that the usual choices (finitely generated or finitely presented modules) may not be appropriate for general rings. Instead, we use the modules of type $FP_{\infty}$, as our finite objects. The “injective” modules for this choice are the absolutely clean modules and the “flat” modules are the level modules. Over any ring $R$, these modules have properties that injective modules only have over Noetherian rings and flat modules only have over coherent rings. They are also dual to each other.
under taking character modules. The drawback of this approach is that for some non-coherent rings, the only modules of type \( FP_\infty \) are free, so that every module is absolutely clean and level.

More precisely, let us recall that a nonempty full subcategory \( D \) of an abelian category is called thick if it is closed under summands and whenever we have a short exact sequence

\[
0 \to M' \to M \to M'' \to 0
\]

where two of the three entries are in \( D \), then the third entry is also in \( D \).

The following proposition is well-known.

**Proposition 2.1.** Let \( R \) be a ring.

1. The category of finitely generated left \( R \)-modules is thick if and only if \( R \) is left Noetherian.
2. The category of finitely presented left \( R \)-modules is thick if and only if \( R \) is left coherent.

**Proof.** Suppose the category of finitely generated left \( R \)-modules is thick and \( a \) is a left ideal. Then the short exact sequence

\[
0 \to a \to R \to R/a \to 0
\]

shows that \( a \) is finitely generated, so \( R \) is left Noetherian. The converse is standard.

Similarly, suppose the category of finitely presented left \( R \)-modules is thick, and \( a \) is a finitely generated ideal. Then \( R \) and \( R/a \) are finitely presented, so the same short exact sequence shows that \( a \) is finitely presented. Thus \( R \) is left coherent. The converse is also standard, but perhaps less well-known; see [Gla89, Theorem 2.5.1].

The hardest part is to prove that if we have a short exact sequence

\[
0 \to M' \xrightarrow{j} M \xrightarrow{q} M'' \to 0
\]

and \( M \) and \( M'' \) are finitely presented, then \( M' \) is as well. It suffices to prove that \( M' \) is finitely generated [Lam99, Corollary 4.52], and for this we take a finite presentation

\[
P_1'' \xrightarrow{t''} P_0'' \xrightarrow{q''} M'' \to 0
\]

of \( M'' \) and a finitely generated projective module \( P \) equipped with a surjection \( P \twoheadrightarrow M \). There is then a map \( \alpha \colon P \to P_0'' \) such that \( q'' \alpha = gq \). Then

\[
(\alpha, r'') \colon P \oplus P_1'' \to P_0''
\]

is a (necessarily split) surjection, and thus \( \ker(\alpha, r'') \) is a finitely generated projective mapping onto \( M \).

We interpret Proposition 2.1 as saying that finitely presented is only the correct notion of finiteness when the ring \( R \) is left coherent. In general, we need not just the relations, but all the syzygies of a finite module to be finitely generated. That is, we need the following definition.

**Definition 2.2.** A (left) module \( M \) over a ring \( R \) is said to be of type \( FP_\infty \) if \( M \) has a projective resolution by finitely generated projectives.

So if \( R \) is left Noetherian, the modules of type \( FP_\infty \) are precisely the finitely generated modules, and if \( R \) is left coherent, the modules of type \( FP_\infty \) are precisely the finitely presented modules.
Modules of type $FP_\infty$ have been studied before, but mostly in the context of geometric group theory. The standard reference for them is [Bie81]. They have been used in representation theory of certain infinite groups by Benson in [Ben97] and Kropholler [Kro99], and their homological algebra has been studied more generally in the Ph. D. thesis of Livia Miller (now Livia Hummel) [Mil08].

In particular, Bieri proves the following proposition.

**Proposition 2.3.** For any ring $R$, the modules of type $FP_\infty$ form a thick subcategory.

So the modules of type $FP_\infty$ are a reasonable candidate for the “finite” modules over any ring $R$. However, there is no guarantee that there are very many modules of type $FP_\infty$.

**Proposition 2.4.** Suppose $R$ is a ring. Every left $R$-module is a direct limit of modules of type $FP_\infty$ if and only if $R$ is left coherent.

**Proof.** Suppose that every module $M$ can be written as a direct limit $\lim_{\to} M_i$ of modules of type $FP_\infty$. Then if $M$ is finitely presented we have

$$\text{Hom}_R(M,M) \cong \text{Hom}_R(M,\lim_{\to} M_i) \cong \lim_{\to} \text{Hom}_R(M,M_i).$$

In particular, this means that the identity map of $M$ factors through some $M_i$, so that $M$ is a summand of $M_i$ and is therefore itself of type $FP_\infty$. Thus the finitely presented modules coincide with the modules of type $FP_\infty$ and so form a thick subcategory, and so $R$ is left coherent. The converse is immediate, since every module is always a direct limit of finitely presented modules. □

To illustrate this proposition, we consider the following example.

**Proposition 2.5.** Let $k$ be a field and let $R = k[x_1, x_2, \ldots]/m^2$ denote the quotient of the polynomial ring over $k$ on the generators $x_1, x_2, \ldots$ by the square of the maximal ideal $m = (x_1, x_2, \ldots)$. Then the only $R$-modules of type $FP_\infty$ are the finitely generated free modules.

**Proof.** Suppose $M$ is finitely presented but not free, and write $M$ as the quotient of two free modules

$$R^k \xrightarrow{g} R^n \to M \to 0$$

where $n$ is minimal and $k$ is minimal for that value of $n$ (and positive since $M$ is not free). Let us denote the standard generators of $R^k$ by $e_1, \ldots, e_k$ and the standard generators of $R^n$ by $e'_1, e'_2, \ldots, e'_n$. Because $n$ is minimal, we must have $\text{Im } g \subseteq mR^n$ for all $i$. Indeed, if not, then if we write

$$g(e_i) = \sum_j a_{ij}e'_j$$

then there must be an $i$ and a $j$ with $a_{ij}$ not in $m$, so that $a_{ij}$ is a unit. We can then solve for $e_j$. The effect of this is to write $M$ as the quotient

$$R^{k-1} \xrightarrow{h} R^{n-1} \to M \to 0$$

where $h$ is obtained by eliminating $e_i$ from $R^k$ and $e'_j$ from $R^n$ and rewriting $g$.

Now because $k$ is also minimal, we claim that $\ker g \subseteq mR^k$. Indeed, otherwise there is some $\sum a_i e_i$ in the kernel of $g$ and some $j$ with $a_j$ not in $m$, so a unit. We can then write this as

$$\sum a_i g(e_i) = 0$$
and use this to solve for $g(e_j)$ in terms of the other $gf(e_i)$. This means that we can delete $e_j$ and rewrite $M$ as the quotient
\[ R^{k-1} \xrightarrow{h} R^n \to M \to 0 \]
where $h$ is the restriction of $g$, violating the minimality of $k$.

Now, since $m^2 = 0$ and $\text{Im } g \subseteq mR^n$, we also have $mR^k \subseteq \ker g$. Therefore,
\[ \ker g = mR^k. \]
But $mR^k$ is not finitely generated, from which we conclude that $M$ is not even of type $FP_2$, let alone of type $FP_\infty$. $\square$

Now that we have a general notion of finiteness, there are corresponding notions of flatness and injectivity. More precisely, we have the following definitions.

**Definition 2.6.** Let $R$ be a ring. A left $R$-module $N$ is called $FP_\infty$-injective or absolutely clean if $\text{Ext}^1_R(M,N) = 0$ for all modules $M$ of type $FP_\infty$. Similarly, $N$ is called level if $\text{Tor}^1_R(M,N) = 0$ for all right $R$-modules $M$ of type $FP_\infty$.

Absolutely clean modules are analogous to absolutely pure modules, and coincide with them when $R$ is left coherent. More precisely, a short exact sequence $E$ is pure if and only if $K \otimes_R E$ is exact for any right $R$-module $K$, and this is equivalent to $\text{Hom}_R(M,E)$ being exact for any finitely presented left $R$-module $M$. It follows that a module $N$ is FP-injective ($\text{Ext}^1_R(M,N) = 0$ for all finitely presented $M$) if and only if $N$ is absolutely pure (every short exact sequence beginning with $N$ is pure). In analogy with this, we define a short exact sequence $E$ to be clean if $\text{Hom}_R(M,E)$ is exact for all $M$ of type $FP_\infty$. Then $N$ is absolutely clean if and only if every short exact sequence beginning with $N$ is clean.

Of course, for the ring of Proposition 2.5, the only modules of type $FP_\infty$ are free, so every module is both absolutely clean and level.

Recall that injective $R$-modules are closed under direct limits if and only if $R$ is left Noetherian, and absolutely pure $R$-modules are closed under direct limits if and only if $R$ is left coherent $\text{[Ste70]}$. Absolutely clean modules are always closed under direct limits, and share many other properties of absolutely pure modules.

**Proposition 2.7.** For any ring $R$, the following hold:

1. If $N$ is an absolutely clean $R$-module, then $\text{Ext}^n(M,N) = 0$ if $n > 0$ and $M$ is of type $FP_\infty$.
2. The class of absolutely clean modules is closed under pure submodules and pure quotients.
3. The class of absolutely clean modules is coresolving; that is, it contains the injective modules and is closed under extensions and cokernels of monomorphisms.
4. The class of absolutely clean modules is closed under products and direct limits, and so also under transfinite extensions.
5. There is a set $S$ of absolutely clean modules such that every absolutely clean module is a transfinite extension of modules in $S$.

For the last two statements, given a collection of modules $\mathcal{D}$, we say that $N$ is a transfinite extension of objects in $\mathcal{D}$ if there is an ordinal $\lambda$ and a colimit-preserving functor $X : \lambda \to R$-$\text{Mod}$ with $X_0 \in \mathcal{D}$ such that each map $X_i \to X_{i+1}$ is a monomorphism whose cokernel is in $\mathcal{D}$, and such that $\text{colim}_{i<\lambda} X_i \cong N$. 

Proof. Given a module $M$ of type $FP_\infty$, let $P_\ast$ be a resolution of $M$ by finitely generated projectives. Define $M_0 = M$ and $M_i = B_{i-1}P_\ast$ for $i > 0$. Then if we truncate $P_\ast$ by setting everything below $i$ to 0, we get a projective resolution of $M_i$ by finitely generated projectives, so $M_i$ is also of type $FP_\infty$. Since we have short exact sequences

$$0 \rightarrow M_i \rightarrow P_{i-1} \rightarrow M_{i-1} \rightarrow 0,$$

we see that

$$\text{Ext}^n(M, N) = \text{Ext}^1(M_{n-1}, N),$$

from which the first statement follows.

For the second statement, if the short exact sequence

$$E : 0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$$

is pure and $N$ is absolutely clean, then $\text{Hom}(M, E)$ is still exact for any module $M$ of type $FP_\infty$. Hence $\text{Ext}^1(M, N')$ is a submodule of the zero module $\text{Ext}^1(M, N)$, and so $N'$ is absolutely clean. Then $\text{Ext}^1(M, N'')$ is a submodule of $\text{Ext}^1(M, N')$, which is also zero by the first part of the proposition. Hence $N''$ is also absolutely clean.

Now suppose

$$0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$$

is a short exact sequence where $N', N$ are absolutely clean. By applying $\text{Hom}(M, -)$ to this short exact sequence, where $M$ is $FP_\infty$, we see that $\text{Ext}^1(M, N'')$ is trapped between the two zero groups $\text{Ext}^1(M, N)$ and $\text{Ext}^2(M, N')$. Hence it is zero, and so $N''$ is absolutely clean. A similar argument shows that absolutely clean modules are closed under extensions, giving us the third statement.

It is clear that absolutely clean modules are closed under products. Now suppose $N_\alpha$ is a directed family of absolutely clean modules. Then

$$\text{Ext}^1(M, \varinjlim N_\alpha) \cong H^1(\text{Hom}(P_\ast, \varinjlim N_\alpha)) \cong H^1(\varinjlim \text{Hom}(P_\ast, N_\alpha))$$

$$\cong \varinjlim H^1 \text{Hom}(P_\ast, N_\alpha) \cong \varinjlim \text{Ext}^1(M, N_\alpha),$$

giving us the fourth statement. The last statement is an immediate consequence of Proposition 2.8 below. \hfill $\Box$

We owe the reader the following useful proposition.

**Proposition 2.8.** Suppose $\mathcal{A}$ is a class of $R$-modules that is closed under taking pure submodules and quotients by pure submodules. Then there is a cardinal $\kappa$ such that every module in $\mathcal{A}$ is a transfinite extension of modules in $\mathcal{A}$ with cardinality less than $\kappa$.

The proof below does not require $\mathcal{A}$ to be closed under transfinite extensions, but in practice one usually wants this as well, so that a module is in $\mathcal{A}$ if and only if it is a transfinite extension of modules in $\mathcal{A}$ with cardinality less than $\kappa$.

**Proof.** Take $\kappa$ to be any cardinal larger than $|R|$. Given $M \in \mathcal{A}$, we define a strictly increasing chain $M_i \subseteq M$ with $M_i \in \mathcal{A}$ by transfinite induction on $i$. For $i = 0$, we let $M_0$ be a nonzero pure submodule of $M$ of cardinality less than $\kappa$ (using [EJ00, Lemma 5.3.12]). Having defined the pure submodule $M_i$ of $M$ and assuming that $M_i \neq M$, we let $N_i$ be a nonzero pure submodule of $M/M_i$ of cardinality less than $\kappa$. Since $M_i$ is a pure submodule of $M$, $M/M_i$ is also in $\mathcal{A}$, so $N_i$ is as well. We then let $M_{i+1}$ be the preimage in $M$ of $N_i$, so that $M_{i+1}$ is a pure submodule of $M$
so also in $\mathcal{A}$. For the limit ordinal step, we define $M_\beta = \bigcup_{\alpha < \beta} M_\alpha$: this is a colimit of pure submodules of $M$ so is also a pure submodule of $M$. This process will eventually stop when $M_i = M$, at which point we have written $M$ as a transfinite extension of modules in $\mathcal{A}$ with cardinality less than $\kappa$. \qed

We then have the following corollary to Proposition 2.7.

**Corollary 2.9.** A ring $R$ is left coherent if and only if absolutely clean left $R$-modules and absolutely pure left $R$-modules coincide.

**Proof.** If absolutely clean and absolutely pure modules coincide, then absolutely pure modules are closed under direct limits, so $R$ is left coherent. \qed

We will see below that level modules are dual to absolutely clean modules, so should expect dual properties.

**Proposition 2.10.** For any ring $R$, the following hold:

1. If $N$ is a level $R$-module, then $\text{Tor}_n^R(M, N) = 0$ if $n > 0$ and $M$ is of type $\text{FP}_\infty$.
2. The class of level modules is closed under pure submodules and pure quotients.
3. The class of level modules is resolving; that is, it contains the projective modules and is closed under extensions and cokernels of epimorphisms.
4. The class of level modules is closed under products and direct limits, and so also under transfinite extensions.
5. There is a set $S$ of level modules such that every level module is a transfinite extension of modules in $S$.

**Proof.** For part (a), suppose $M$ is of type $\text{FP}_\infty$ and take a projective resolution $P_\ast$ of $M$ by finitely generated projectives. As we have seen before, this gives us short exact sequences

$$0 \to M_{i+1} \to P_i \to M_i \to 0$$

for all $i$, with $M_0 = M$ and $M_{i+1} = Z_i P$. Each $M_i$ is then of type $\text{FP}_\infty$, because we can truncate $P_\ast$ to get a resolution of $M_i$. Then

$$\text{Tor}_n^R(M, F) = \text{Tor}_1^R(M_{n-1}, F) = 0$$

for $n > 1$.

For the second statement, if the short exact sequence

$$E : 0 \to N' \to N \to N'' \to 0$$

is pure and $N$ is level, then $M \otimes_R E$ is exact for any right $R$-module $M$ of type $\text{FP}_\infty$. Hence $\text{Tor}_1(M, N'')$ is a quotient of the zero module $\text{Tor}_1(M, N)$, and so $N''$ is level. Then $\text{Tor}_1(M, N')$ is a quotient of $\text{Tor}_2(M, N')$, which is also zero by the first part of the proposition. Hence $N'$ is also level.

Now suppose

$$0 \to N' \to N \to N'' \to 0$$

is a short exact sequence where $N, N''$ are level. By applying $M \otimes_R -$ to this short exact sequence, where $M$ is $\text{FP}_\infty$, we see that $\text{Tor}_1(M, N')$ is trapped between the two zero groups $\text{Tor}_1(M, N)$ and $\text{Tor}_2(M, N'')$. Hence it is zero, and so $N'$ is level. A similar argument shows that level modules are closed under extensions, giving us the third statement.
Because Tor commutes with colimits, it is clear that level modules are closed under direct limits. Now let \( N_i \) be level for all \( i \), and let \( M \) be a right module \( M \) of type \( \text{FP}_\infty \). Take a projective resolution \( P_* \) of \( M \) by finitely generated projectives. Then
\[
\text{Tor}^R_1(M, \prod N_i) = H_1(P_* \otimes \prod N_i)
\]
\[
\cong H_1(\prod (P_* \otimes N_i)) \cong \prod H_1(P_* \otimes N_i) = 0,
\]
where we have used the fact that each \( P_n \) is finitely presented to move it inside the product \[\text{Lam99\ Proposition 4.44}\]. Finally, the last statement is immediate from Proposition \[\text{Lam99\ Proposition 2.10}\]. \( \square \)

**Corollary 2.11.** A ring \( R \) is right coherent if and only if level (left)\( R \)-modules and flat \( R \)-modules coincide.

**Proof.** If \( R \) is right coherent, then every finitely presented right \( R \)-module has type \( \text{FP}_\infty \), so a level module \( N \) has \( \text{Tor}^R_1(M, N) = 0 \) for all finitely presented \( M \). Since every module is a direct limit of finitely presented modules, and \( \text{Tor}^R_1(-, N) \) preserves direct limits, \( N \) is flat. Conversely, if every level module is flat, then Proposition \[\text{Lam99\ Proposition 2.10}\] shows that products of flat left \( R \)-modules are flat. This forces \( R \) to be right coherent by Chase’s theorem \[\text{Lam99\ Theorem 4.47}\]. \( \square \)

Now recall the well-known fact that a left \( R \)-module \( N \) is flat if and only if its character module \( N^+ = \text{Hom}_R(M, \mathbb{Q}/\mathbb{Z}) \) is injective (or, equivalently, absolutely pure) as a right \( R \)-module \[\text{Lam99\ Theorem 4.9}\], and the less well-known fact that if \( R \) is left Noetherian then \( N \) is injective if and only if \( N^+ \) is flat as a right \( R \)-module \[\text{EJ00\ Corollary 3.2.17}\]. This partial duality between flat and injective modules is a reflection of a perfect duality between level and absolutely clean modules.

**Theorem 2.12.** For any ring \( R \), a module \( N \) is level if and only if \( N^+ \) is absolutely clean, and \( N \) is absolutely clean if and only if \( N^+ \) is level.

**Proof.** Suppose \( N \) is level and \( M \) is a right \( R \)-module of type \( \text{FP}_\infty \). Consider a short exact sequence
\[
E: 0 \to M_1 \to P \to M \to 0
\]
where \( P \) is projective. Because \( N \) is level, \( E \otimes_R N \) is exact. Hence \( (E \otimes_R N)^+ \) is also exact, but by adjointness this is the same as \( \text{Hom}_R(E, N^+) \). It then follows that \( \text{Ext}^1_R(M, N^+) = 0 \), so \( N^+ \) is absolutely clean. Conversely, if \( N^+ \) is absolutely clean, then \( \text{Hom}_R(E, N^+) \cong (E \otimes_R N)^+ \) is exact. Since \( \mathbb{Q}/\mathbb{Z} \) is an injective cogenerator of the category of abelian groups, \( E \otimes_R N \) must be exact, and so \( \text{Tor}^R_1(M, N) = 0 \) and \( N \) is level.

Now suppose that \( N \) is absolutely clean and \( M \) is a left \( R \)-module of type \( \text{FP}_\infty \). We take a short exact sequence
\[
E: 0 \to M_1 \to P \to M \to 0
\]
where \( P \) is finitely generated projective and \( M_1 \) is also of type \( \text{FP}_\infty \), and in particular finitely presented. Since \( N \) is absolutely clean, \( \text{Hom}_R(E, N) \) is exact, and so \( (\text{Hom}_R(E, N))^+ \) is also exact. Since \( E \) consists of finitely presented modules and \( \mathbb{Q}/\mathbb{Z} \) is an injective abelian group, we can apply Theorem 3.2.11 of \[\text{EJ00}\] to conclude that
\[
(\text{Hom}_R(E, N))^+ \cong N^+ \otimes_R E.
\]
It then follows immediately that $\text{Tor}_1^R(N^+, M) = 0$, so $N^+$ is level. As before, the converse is a matter of reversing the steps. Indeed, if $N^+$ is level, then $N^+ \otimes_R E$ is exact, so $(\text{Hom}_R(E, N))^+$ is exact, so $\text{Hom}_R(E, N)$ is exact. We conclude that $\text{Ext}_R^1(M, N) = 0$ and so $N$ is absolutely clean. □

Note that this theorem implies that if $R$ is left coherent and $N$ is an injective left $R$-module, then $N^+$ is level, so must be flat. On the other hand, if $R$ is left coherent but not left Noetherian, then there is an absolutely pure module $N$ that is not injective, and it too will have $N^+$ flat.

Now we recall the notion of a complete cotorsion pair. Given an abelian category $\mathcal{A}$, a cotorsion pair is a pair of classes of objects $(\mathcal{F}, \mathcal{C})$ of $\mathcal{A}$ such that $\mathcal{F}^\perp = \mathcal{C}$ and $\mathcal{F} = \perp \mathcal{C}$. Here $\mathcal{F}^\perp$ is the class of objects $Y \in \mathcal{A}$ such that $\text{Ext}^1(F, Y) = 0$ for all $F \in \mathcal{F}$, and similarly $\perp \mathcal{C}$ is the class of objects $X \in \mathcal{A}$ such that $\text{Ext}^1(X, C) = 0$ for all $C \in \mathcal{C}$. Two simple examples of cotorsion pairs in $R$-Mod are $(\mathcal{P}, \mathcal{A})$ and $(\mathcal{A}, \mathcal{I})$, where $\mathcal{P}$ is the class of projectives, $\mathcal{I}$ is the class of injectives and $\mathcal{A}$ is the class of all $R$-modules. The canonical example of a cotorsion pair is $(\mathcal{F}, \mathcal{C})$ where $\mathcal{F}$ is the class of flat $R$-modules and $\mathcal{C}$ is the class of cotorsion $R$-modules $\mathcal{EJ00}$.

The cotorsion pair is said to have enough injectives if for any $A \in \mathcal{A}$ there is a short exact sequence $0 \to C \to F \to A \to 0$ where $C \in \mathcal{C}$ and $F \in \mathcal{F}$. We say it has enough projectives if it satisfies the dual statement. These two statements are in fact equivalent for a cotorsion pair as long as the category $\mathcal{A}$ has enough projectives and injectives $\mathcal{EJ00}$ Proposition 7.1.7. We say that the cotorsion pair is complete if it has enough projectives and injectives. The book $\mathcal{EJ00}$ is a standard reference for cotorsion pairs.

Since level modules are analogous to flat modules, we should expect them to be the left half of a complete cotorsion pair.

**Definition 2.13.** A module $N$ is called cospiral if $\text{Ext}^1(F, N) = 0$ for all level modules $F$.

**Theorem 2.14.** For any ring $R$, the pair (level modules, cospiral modules) forms a complete cotorsion pair that is cogenerated by a set. In particular, the level modules form a covering class.

**Proof.** By Proposition 2.10 there is a set $S$ of level modules such that the class of level modules is precisely the transfinite extensions of $S$. Then $S$ cogenerated a complete cotorsion theory $(\mathcal{D}, \mathcal{E})$, where $\mathcal{D}$ is the class of all summands of transfinite extensions of elements of $S$; that is, the level modules. Then $\mathcal{E}$ is necessarily the class of cospiral modules. Completeness of the cotorsion theory proves that the class of level modules is precovering. Since it is also closed under directed colimits, it is covering $\mathcal{EJ00}$ Corollary 5.2.7. □

3. Chain complexes

We recall some basics about chain complexes and model categories in this section. Recall that $R$ is a ring with unity, and $R$-Mod is the category of left $R$-modules. We will denote the category of unbounded chain complexes of (left) $R$-modules by $\text{Ch}(R)$. A chain complex $\cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \to \cdots$ will be denoted by $(X, d)$ or simply $X$. We say $X$ is bounded below (above) if $X_n = 0$ for $n < k$ ($n > k$) for some $k \in \mathbb{Z}$. We say it is bounded if it is bounded above and below. The $n$th cycle module is defined as $\ker d_n$ and is denoted $Z_n X$. The $n$th
boundary module is $\text{Im} d_{n+1}$ and is denoted $B_n X$. The nth homology module
is defined to be $Z_n X/B_n X$ and is denoted $H_n X$. Given an $R$-module $M$, we let
$S^n(M)$ denote the chain complex with all entries 0 except $M$ in degree $n$. We let
$D^n(M)$ denote the chain complex $X$ with $X_n = X_{n-1} = M$ and all other entries
0. All maps are 0 except $d_n = 1_M$. Given $X$, the suspension of $X$, denoted $\Sigma X$,
is the complex given by $(\Sigma X)_n = X_{n-1}$ and $(d_{\Sigma X})_n = -d_n$. The complex $\Sigma(\Sigma X)$
is denoted $\Sigma^2 X$ and inductively we define $\Sigma^n X$ for all $n \in \mathbb{Z}$.

Given two chain complexes $X$ and $Y$ we define $\text{Hom}(X,Y)$ to be the complex
of abelian groups $\cdots \to \prod_{k \in \mathbb{Z}} \text{Hom}(X_k,Y_{k+n}) \xrightarrow{\delta_n} \prod_{k \in \mathbb{Z}} \text{Hom}(X_k,Y_{k+n-1}) \to \cdots$, where $(\delta_n f)_k = d_{k+n} f_k - (-1)^n f_{k-1} d_k$. This gives a functor $\text{Hom}(X,\_): \text{Ch}(R) \to \text{Ch}(\mathbb{Z})$ which is left exact, and exact if $X_n$ is projective for all $n$. Similarly the
contravariant functor $\text{Hom}(\_, Y)$ sends right exact sequences to left exact sequences
and is exact if $Y_n$ is injective for all $n$.

Recall that $\text{Ext}^1_{\text{Ch}(R)}(X,Y)$ is the group of (equivalence classes) of short exact
sequences $0 \to Y \to Z \to X \to 0$ under the Baer sum. We let $\text{Ext}^1_{\text{dw}}(X,Y)$ be
the subgroup of $\text{Ext}^1_{\text{Ch}(R)}(X,Y)$ consisting of those short exact sequences which are
split in each degree. The next lemma is very useful. It is standard and we will not
prove it.

Lemma 3.1. For chain complexes $X$ and $Y$, we have
$$\text{Ext}^1_{\text{dw}}(X,\Sigma^{(-n-1)} Y) \cong H_n \text{Hom}(X,Y) = \text{Ch}(R)(X,\Sigma^{-n} Y)/\sim,$$
where $\sim$ is chain homotopy.

In particular, for chain complexes $X$ and $Y$, $\text{Hom}(X,Y)$ is exact iff for any
$n \in \mathbb{Z}$, any $f: \Sigma^n X \to Y$ is homotopic to $0$ (or iff any $f: X \to \Sigma^n Y$ is homotopic
to $0$).

Next recall that a model category is a category $\mathcal{M}$ with all small limits and colimits
equipped with three classes of maps called cofibrations, fibrations, and weak
 equivalences, all subject to a list of axioms allowing one to formally introduce homotopy theory on $\mathcal{M}$. We assume the reader has a basic understanding or interest
in model categories. Standard references include [Hov99] and [DS95]. In [Hov02],
the third author found the following 1-1 correspondence between complete cotorsion pairs (discussed above just after Theorem 2.12) and abelian model category
structures on $\mathcal{A}$.

Theorem 3.2. An abelian model structure on a bicomplete abelian category $\mathcal{A}$ is equivalent to a thick subcategory $\mathcal{W}$ and two classes $\mathcal{Q}$ and $\mathcal{R}$ for which $(\mathcal{Q}, \mathcal{R} \cap \mathcal{W})$ and $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ are complete cotorsion pairs. In this case $\mathcal{W}$ is the class of
trivial objects, $\mathcal{Q}$ the cofibrant objects and $\mathcal{R}$ the fibrant objects. Here an abelian
model structure is one which a map is a (trivial) cofibration if and only if it is a
monomorphism with (trivially) cofibrant cokernel. Equivalently, a map is a (trivial)
fibration if and only if it is a surjection with (trivially) fibrant kernel.

The following two Propositions are really just Corollaries of Theorem 3.2. We
will use them to construct many different model structures on $\text{Ch}(R)$.

Proposition 3.3 ((Construction of an injective model structure)). Let $\mathcal{A}$ be a
bicomplete abelian category with enough injectives and denote the class of injectives
by $\mathcal{I}$. Let $\mathcal{F}$ be any class of objects and set $\mathcal{W} = \perp \mathcal{F}$. Suppose the following
conditions hold:
(1) \((W, F)\) is a complete cotorsion pair.
(2) \(W\) is thick.
(3) \(I \subseteq W\).

Then there is an abelian model structure on \(A\) where every object is cofibrant, \(F\) is the class of fibrant objects, \(W\) is the class of trivial objects, and \(I = F \cap W\) is the class of trivially fibrant objects.

Proof. Let \(A\) also denote the class of all objects in the category. Then \((A, I)\) is a complete cotorsion pair since the category has enough injectives. By Theorem 3.2 we only still need (i) \(A \cap W = W\), and (ii) \(F \cap W = I\). But of course (i) is true so we just need to prove (ii).

First we see \(F \cap W \supseteq I\), because \((W, F)\) being a cotorsion pair automatically implies \(I \subseteq F\), and also by assumption we have \(I \subseteq W\). Next we show \(F \cap W \subseteq I\). So suppose \(X \in F \cap W\). Then find a short exact sequence \(0 \to X \to I \to I/X \to 0\) where \(I\) is injective. By hypothesis \(I \in W\). But \(W\) is assumed to be thick, which means \(X/I \in W\). But now since \((W, F)\) is a cotorsion pair the short exact sequence splits. Therefore \(X\) is a direct summand of \(I\), proving \(X \in I\).

□

We also list the dual for easy reference.

**Proposition 3.4** ((Construction of a projective model structure)). Let \(A\) be a bicomplete abelian category with enough projectives and denote the class of projectives by \(P\). Let \(C\) be any class of objects and set \(W = C^\perp\). Suppose the following conditions hold:

(1) \((C, W)\) is a complete cotorsion pair.
(2) \(W\) is thick.
(3) \(P \subseteq W\).

Then there is an abelian model structure on \(A\) where every object is fibrant, \(C\) are the cofibrant objects, \(W\) are the trivial objects, and \(P = C \cap W\) are the trivially cofibrant objects.

4. **Injective model structures on \(\text{Ch}(R)\)**

In this section, we give a general construction of some abelian model structures on \(\text{Ch}(R)\) for which every object is cofibrant and the fibrant objects are contained in complexes of injectives. We then use this to build many different model structures on \(\text{Ch}(R)\), and in particular, to build one whose homotopy category is our injective stable module category of \(R\).

**Theorem 4.1.** Given a ring \(R\), let \(A\) be a fixed left \(R\)-module. Let \(F\) be the class of \(A\)-acyclic complexes of injectives; that is, chain complexes \(F\) that are degreewise injective and such that \(\text{Hom}_R(A, F) = \text{Hom}(S^0(A), F)\) is exact. Then there is a cofibrantly generated abelian model structure on \(\text{Ch}(R)\) where every object is cofibrant, \(F\) is the class of fibrant objects, \(W = \perp F\) is the class of trivial objects, and the injective complexes \(I = F \cap W\) are the trivially fibrant objects. Furthermore, \(W\) contains all contractible complexes. We call this model structure the \(A\)-acyclic injective model structure. The homotopy category of the \(A\)-acyclic injective model structure is equivalent to the chain homotopy category of \(A\)-acyclic complexes of injectives.
Proof. We use Proposition 3.3. So we must show $(\mathcal{W}, \mathcal{F})$ is a complete cotorsion pair, that $\mathcal{W}$ is thick, and that $\mathcal{I} \subseteq \mathcal{W}$. Let

$$S = \{ D^n(R/a)| n \in \mathbb{Z}, a \text{ a left ideal of } R \} \cup \{ S^n(A)| n \in \mathbb{Z} \}.$$ 

Then

$$\mathcal{F} = S^\perp.$$ 

Indeed, one can readily check that

$$\text{Ext}^1_{\text{Ch}(R)}(D^n(R/a), F) \cong \text{Ext}_R^1(R/a, F).$$ 

Baer’s criterion for injectivity then implies that $S^\perp$ consists of the complexes of injectives $F$ such that

$$\text{Ext}^1_{\text{Ch}(R)}(S^n(A), F) = 0$$

for all $n$. But because $F$ is a complex of injectives,

$$\text{Ext}^1_{\text{Ch}(R)}(S^n(A), F) = \text{Ext}^1_{dw}(S^n(A), F) = \text{H}_n \text{Hom}(S^0(A), F),$$

and so $\mathcal{F} = S^\perp$ as claimed. Anytime we have a set $S$ in a Grothendieck category with enough projectives, then $(\perp(S^\perp), S^\perp)$ is always a complete cotorsion pair by [Hov02, Theorem 2.4], so $(\mathcal{W}, \mathcal{F})$ is so.

To see that $\mathcal{W}$ is thick, first note that, because $\mathcal{F}$ consists of complexes of injectives, Lemma 3.1 implies that $X \in \mathcal{W}$ if and only if $\text{Hom}(X, F)$ is acyclic for all $F \in \mathcal{F}$. Now suppose we have a short exact sequence

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0,$$

where two out of three of the entries are in $\mathcal{W}$. Now suppose $F \in \mathcal{F}$. Since $F$ is a complex of injectives, the resulting sequence

$$0 \rightarrow \text{Hom}(Z, F) \rightarrow \text{Hom}(Y, F) \rightarrow \text{Hom}(X, F) \rightarrow 0$$

is still short exact. Since two out of three of these complexes are acyclic, so is the third.

Note that if $X$ is contractible, then $\text{Hom}(X, F)$ is obviously acyclic for any $F$, so $X \in \mathcal{W}$. This model structure is cofibrantly generated by the results of [Hov02, Section 6]. Explicitly, the generating trivial cofibrations can be taken to be the set of all $D^n(a) \rightarrow D^n(R)$ for $a$ a left ideal of $R$ and $n \in \mathbb{Z}$ together with the set of all $S^n(K) \rightarrow S^n(P)$ for $n \in \mathbb{Z}$, where

$$0 \rightarrow K \rightarrow P \rightarrow A \rightarrow 0$$

is exact and $P$ is projective. The generating cofibrations can be taken to be the $S^n(a) \rightarrow S^n(R)$ for $a$ a left ideal of $R$ and $n \in \mathbb{Z}$. We leave to the reader the statement about the homotopy category.

Note that the proof of this theorem shows that the complete cotorsion pair is cogenerated by the $D^n(R/a)$ and the $S^n(A)$. This means that $\mathcal{W}$ consists of all summands of transfinite extensions of chain complexes of the form $D^n(R/a)$ and $S^nA$.

Of course, $\mathcal{W}$ is also a thick subcategory. So, for simplicity, if an object $B$ is in the smallest thick subcategory that is closed under transfinite extensions and contains some set $S$ of objects, we will say that $B$ is built from $S$.

In particular, if a module $B$ is built from the given module $A$, then the complex $S^0(B)$ is built from $S^0(A)$, so $S^0B$ is in $\mathcal{W}$. So if $F$ is fibrant in the $A$-acyclic injective model category, then it is also fibrant in the $B$-acyclic model structure.
Therefore, the identity functor from the \( A \)-acyclic injective model category to the \( B \)-acyclic injective model category preserves fibrations, and the trivial fibrations are the same in the two model structures. We therefore get the following proposition.

**Lemma 4.2.** Suppose \( A \) and \( B \) are left \( R \)-modules and \( B \) is built from \( A \). Then the identity functor is a left Quillen functor from the \( B \)-acyclic injective model structure to the \( A \)-acyclic injective model structure; in fact the \( A \)-acyclic injective model structure is a left Bousfield localization of the \( B \)-acyclic injective model structure. In particular, if \( A \) is also built from \( B \), then the \( A \)-acyclic injective model structure coincides with the \( B \)-acyclic injective model structure.

Before considering examples, we point out some basic properties of the map from \( R\text{-Mod} \) to the homotopy category of the \( A \)-acyclic injective model structure.

**Proposition 4.3.** Let \( R \) be a ring and \( A \) a left \( R \)-module. Consider the composite functor

\[
\gamma: R\text{-Mod} \xrightarrow{S^0} \text{Ch}(R) \to \text{Ho Ch}(R)
\]

from \( R \)-modules to the homotopy category of the \( A \)-acyclic injective model structure. Then \( \gamma \) is an exact coproduct-preserving functor to the triangulated category \( \text{Ho Ch}(R) \). The kernel of \( \gamma \) contains all modules built from \( A \).

**Proof.** The homotopy category \( \text{Ho Ch}(R) \) is triangulated because the shift is an equivalence of categories and is also equivalent to the suspension functor that exists in any model category (see Section 7.1 of [Hov99] for a general discussion of when the homotopy category of a pointed model category is triangulated).

Any monomorphism in \( \text{Ch}(R) \) is a cofibration in the \( A \)-acyclic injective model structure, and so short exact sequences in \( \text{Ch}(R) \) give rise to exact triangles in \( \text{Ho Ch}(R) \). The functor \( S^0 \) is exact, so we conclude that \( \gamma \) is exact.

The kernel of \( \gamma \) consists of all modules \( M \) such that \( S^0 M \) is trivial in the \( A \)-acyclic injective model structure. In view of Lemma 3.1 this is all modules \( M \) such that \( \text{Hom}(M, X) \) is exact for all \( A \)-acyclic complexes of injectives \( X \). In view of the proof of Lemma 4.2 this includes all modules built from \( A \). \( \square \)

The simplest case of the \( A \)-acyclic injective model structure is of course when \( A = 0 \).

**Corollary 4.4.** For any ring \( R \) there is a cofibrantly generated abelian model structure on \( \text{Ch}(R) \), the **Inj model structure**, in which every object is cofibrant and the fibrant objects are the complexes of injectives. The trivially fibrant objects coincide with the injective complexes, and the homotopy category is the chain homotopy category of complexes of injectives.

At the other extreme, we could take \( A \) to be the direct sum of all the finitely generated modules. Since every module is a transfinite extension of finitely generated modules, the fibrant objects in this case would be exact complexes of injectives \( X \) such that \( \text{Hom}(M, X) \) is exact for all left \( R \)-modules \( M \). In particular, by taking \( M = Z_n X \), we see that the inclusion \( Z_n X \to X_n \) factors through \( d_{n+1}: X_{n+1} \to X_n \). This means that \( X_{n+1} \cong Z_{n+1} X \oplus Z_n X \), from which it follows easily that \( X \) is an injective complex. So this is the model structure in which every map is a weak equivalence, the cofibrations are the monomorphisms, and the fibrations are the split epimorphisms with injective kernel.
It is more interesting to take $A = R$, when we get the following corollary. Note that any projective module is built from $R$, since it is a summand of a direct sum of copies of $R$.

**Corollary 4.5.** For any ring $R$ there is a cofibrantly generated abelian model structure on $\text{Ch}(R)$, the **exact injective model structure**, in which every object is cofibrant and the fibrant objects are the exact complexes of injectives. The trivially fibrant objects coincide with the injective complexes. For this model structure, all projective modules are sent to 0 by the functor $\gamma$ of Proposition 4.3. The homotopy category is the chain homotopy category of exact complexes of injectives, the stable derived category of [Kra05], and is denoted $S(R)$.

Note that, in general, if we replace $A$ by $A \oplus R$, we change the fibrant objects from the $A$-acyclic complexes of injectives to the exact $A$-acyclic complexes of injectives. We therefore sometimes refer to the $A \oplus R$-acyclic injective model structure as the **exact $A$-acyclic injective model structure**.

If $R$ is left Noetherian, we can take $A$ to be the direct sum of all the indecomposable injectives. This will give us a model structure where the fibrant objects are complexes of injectives $X$ such that $\text{Hom}_R(I, X)$ is exact for all injective modules $I$. Such complexes are called **Inj-acyclic**, and if they are also exact they are called **totally acyclic complexes of injectives**. We then get the following corollary.

**Corollary 4.6.** For any left Noetherian ring $R$, there is a cofibrantly generated abelian model structure on $\text{Ch}(R)$, the **Inj-acyclic injective model structure**, in which every object is cofibrant and the fibrant objects are the Inj-acyclic complexes of injectives. There is also a cofibrantly generated abelian model structure on $\text{Ch}(R)$, the **totally acyclic injective model structure**, in which every object is cofibrant and the fibrant objects are the totally acyclic complexes of injectives. In both of these model structures, the trivially fibrant objects coincide with the injective complexes. In the Inj-acyclic injective model structure, all injective modules are sent to 0 by the functor $\gamma$ of Proposition 4.3; in the totally acyclic injective model structure, both projective modules and injective modules are sent to 0 by $\gamma$. In both cases, the homotopy category is the chain homotopy category of the fibrant objects.

We now consider the simplest case, when $R$ is Gorenstein. Here we are using Gorenstein in the non-commutative sense, so that $R$ is left and right Noetherian and has finite self-injective dimension on the left and the right. The main property of Gorenstein rings that we need is that the modules of finite projective dimension coincide with the modules of finite injective dimension; see [EJ00, Chapter 9].

**Proposition 4.7.** Suppose $R$ is Gorenstein. Then the exact injective model structure, the totally acyclic injective model structure and the Inj-acyclic injective model structure all coincide.

**Proof.** Consider a general complex of injectives $X$, and let $\mathcal{E}$ denote the collection of all $R$-modules $M$ such that $\text{Hom}_R(M, X)$ is acyclic. Note that $\mathcal{E}$ is obviously closed under direct sums and retracts. We claim that $\mathcal{E}$ is thick. Indeed, if we have a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

there is an induced short exact sequence of complexes

$$0 \to \text{Hom}(M'', X) \to \text{Hom}(M, X) \to \text{Hom}(M', X) \to 0,$$
because $X$ is a complex of injectives. The long exact sequence in homology now proves that $\mathcal{E}$ is thick.

It follows that, if $X$ is an exact complex of injectives, then $\text{Hom}(M,X)$ is exact for all $M$ of finite projective dimension. In particular, if $R$ is Gorenstein, then every injective module has finite projective dimension, so $X$ is totally acyclic. Thus the exact injective and totally acyclic injective model structures coincide.

Similarly, if $X$ is Inj-acyclic, it follows that $\text{Hom}(M,X)$ is exact for all $M$ of finite injective dimension. If $R$ is Gorenstein, then $R$ itself has finite injective dimension, and so $X$ is acyclic. Thus the Inj acyclic and totally acyclic Inj model structure coincide as well. $\square$

We now give an example to show that the exact injective model structure may differ from the totally acyclic injective model structure if the ring $R$ is not Gorenstein. Let $R = k[x,y]/(x^2, xy, y^2)$ where $k$ is a field, so that $R$ is an Artinian local ring of Krull dimension zero with nilpotent maximal ideal $m = (x, y)$. There is only one indecomposable injective $J$, the injective hull of $R/m \cong k$. One can easily check that $J = R \oplus R/K$, where $K$ is generated by $(x,0)$, $(y,-x)$, and $(0,y)$.

**Proposition 4.8.** For $R = k[x,y]/(x^2, xy, y^2)$ as above, every module is built from $R$ and $J$. Therefore every totally acyclic complex of injectives is actually an injective complex, and so the homotopy category of the totally acyclic injective model structure is trivial. On the other hand, there is an exact complex of injectives that is not totally acyclic, so the homotopy category of the exact injective model structure is non-trivial. Similarly, there is a complex of injectives that is Inj-acyclic but not totally acyclic.

**Proof.** The injective envelope of $R$ is $J \oplus J$. Indeed, we can write $J = k\langle \alpha, \beta, \gamma \rangle$, where $x\alpha = y\beta = 0$ and $y\alpha = \gamma = x\beta$. The map $R \to J \oplus J$ that takes $1$ to $(\alpha, \beta)$ is then a monomorphism. The cokernel of this map is $k \oplus k \oplus k$. Therefore, $k$ is built from $R$ and $J$. Since $k$ is the only simple $R$-module and $R$ is Artinian, every finitely generated module is built from $k$. But then every module is built from $k$.

It follows that if $X$ is a totally acyclic complex of injectives, then $\text{Hom}(M,X)$ is acyclic for any $M$. We have seen in the discussion following Corollary 4.13 that this means that $X$ is injective as a complex.

To construct an example of an exact complex of injectives that is not an injective complex, so not totally acyclic, let $X_n = \bigoplus_{i=1}^{\infty} J$ for each $n$, and define $d$: $X_n \to X_{n-1}$ by $d_{\alpha_i} = \gamma_{2i-1}$ (that is, send the $\alpha$ in $J_1$ to the $\gamma$ in $J_{2i-1}$) and $d(\beta_i) = \gamma_{2i}$. One can easily check then that

$$\ker d = \text{Im} d = \bigoplus_{i=1}^{\infty} k$$

generated by the $\gamma_i$. So the complex is exact and cannot be an injective complex, since its cycles are not injective.

To find a complex of injectives $Y$ that is Inj-acyclic but not exact, we define $Y_n = X_n$ but this time we define the differential by

$$d(\alpha_{2i-1}) = \gamma_i, d(\beta_{2i-1}) = 0 = d(\alpha_{2i}), d(\beta_{2i}) = \gamma_i.$$  

This complex is obviously not exact, since $\alpha_1 - \beta_2$ is a cycle that is not a boundary. On the other hand, $\text{Hom}(J,Y)$ is a countable sum of copies of $R$ in each dimension,
with differential $d(1_{2i-1}) = y_i$ and $d(1_{2i}) = x_i$. One can then check easily that $\text{Hom}(J,Y)$ is exact, and therefore $Y$ is Inj-acyclic.

In view of Section 2, we can extend the Inj-acyclic and totally acyclic injective model structures to general rings by taking $A$ to be the direct sum of a set of absolutely clean modules as in Proposition 2.7(e) that generate all others by transfinite extensions. This will give a model structure where the fibrant objects are AC-acyclic, in the sense that $\text{Hom}(I,X)$ is exact for all absolutely clean modules $I$ (so in particular for all injective modules $I$).

**Theorem 4.9.** For any ring $R$, there is a cofibrantly generated abelian model structure on $\text{Ch}(R)$, the AC-acyclic model structure, in which every object is cofibrant and the fibrant objects are the AC-acyclic complexes of injectives. There is also a cofibrantly generated abelian model structure on $\text{Ch}(R)$, the exact AC-acyclic model structure, in which every object is cofibrant and the fibrant objects are the exact AC-acyclic complexes of injectives. The homotopy category of the (exact) AC-acyclic model structure is the chain homotopy category of (exact) AC-acyclic complexes of injectives. For these model structures, all absolutely clean modules are sent to 0 by the functor $\gamma$ of Proposition 4.3, and all projective modules are sent to 0 for the exact AC-acyclic model structure. When $R$ is left Noetherian, the AC-acyclic model structure coincides with the Inj-acyclic injective model structure, and the exact AC-acyclic model structure coincides with the totally acyclic injective model structure.

Note that for the ring $R$ of Proposition 2.5, every module is absolutely clean, so the fibrant objects in the AC-acyclic model structure are the injective complexes, and every map is a weak equivalence.

We would like to acknowledge the work of Pinzon [Pin08], whose theorems about absolutely pure modules over left coherent rings led us to the more general notion of absolutely clean modules, which of course agree with absolutely pure modules over left coherent rings. Note that Pinzon’s theorems generalize; for example, the class of absolutely clean modules is covering for any ring $R$.

When $R$ is left Noetherian, Krause constructs a recollement involving $K(\text{Inj})$, the stable derived category $S(R)$, and the derived category $D(R)$ in [Kra05]. Half of this recollement arises because $S(R)$ is a Bousfield localization of $K(\text{Inj})$. The other half can also be interpreted in terms of model structures; this has been done very nicely in [Bec12]. Inspired by Becker’s work, the second author has given a general study of when these recollements happen in [Gil12].

5. The Gorenstein AC-injective model structure on modules

Recall that the usual stable module category can be obtained directly from a model category structure on modules. This is explained for quasi-Frobenius rings in [Hov99, Section 2.2] and for Gorenstein rings in [Hov02]. In this section, we prove that this can be done for the exact AC-acyclic model structure of the previous section.

Following the method used for Gorenstein rings in [Hov02], we define an $R$-module $M$ to be **Gorenstein AC-injective** if $M = Z_0X$ for some exact AC-acyclic complex of injectives. If $R$ is left Noetherian, the Gorenstein AC-injectives are the usual Gorenstein injectives; if $R$ is left coherent, the Gorenstein AC-injectives are the Ding injectives discussed in [Gil10]. We would like to put an abelian model
structure on $R$-Mod in which everything is cofibrant and the fibrant objects coincide with the class $\mathcal{F}$ of Gorenstein AC-injectives. This forces us to define

$$W = \perp \mathcal{F}.$$  

We need to understand how $W$ relates to the class of trivial objects in the exact $A$-injective model structure.

**Lemma 5.1.** Let $A$ be an $R$-module, and suppose $X$ is a complex with $H_iX = 0$ for $i < 0$ and such that $X_i$ is absolutely clean for $i > 0$. Then $X$ is trivial in the exact AC-acyclic model structure if and only if $Z_0X \in W$.

**Proof.** Suppose $M$ is Gorenstein AC-injective, so that $M = Z_0Y$ for some exact AC-acyclic complex of injectives $Y$. We claim that there is an isomorphism

$$\text{Ext}^1(X, Y) \rightarrow \text{Ext}^1(Z_0X, M);$$

this isomorphism would prove the lemma. Indeed, because $Y$ is a complex of injectives, Lemma 3.1 gives us an isomorphism

$$\text{Ext}^1(X, Y) \rightarrow \text{Ch}(R)(X, \Sigma Y)/\sim,$$

where $\sim$ denotes chain homotopy. A chain map $\phi: X \rightarrow \Sigma Y$ induces a map $Z_0X \rightarrow Z_{-1}Y$. A chain homotopy between $\phi$ and 0 gives us maps $D_n: X_n \rightarrow Y_n$ with $-dD_n + D_{n-1}d = \phi_n$. In particular $\phi_0 = -dD_0$ on $Z_0X$. Thus, there is a natural map

$$\tau: \text{Ch}(R)(X, \Sigma Y)/\sim \rightarrow \text{Hom}(Z_0X, Z_{-1}Y)/\text{Hom}(Z_0X, Y_0) \cong \text{Ext}^1(Z_0X, M).$$

To see that $\tau$ is surjective, suppose we have a map $f: Z_0X \rightarrow Z_{-1}Y$. Because $Y_{-1}$ is injective, there is a map $f_0: X_0 \rightarrow Y_{-1}$ extending $f$. We therefore get an induced map $Z_{-1}X = B_{-1}X \rightarrow B_2Y$ using the fact that $H_{-1}X = 0$. Because $Y_{-2}$ is injective, this extends to a map $f_{-1}: X_{-1} \rightarrow Y_{-2}$. Continuing in this fashion we can define maps $f_n$ for all $n \leq 0$. To define $f_n$ in positive degrees, we use the fact that $Y$ is exact AC-acyclic. Indeed, our given map $f$ induces the composite

$$X_1 \xrightarrow{d} Z_0X \xrightarrow{f_0} Y_0.$$  

Since $X_1$ is absolutely clean and $Y$ is AC-acyclic, there is a map $f_1: X_1 \rightarrow Y_0$ such that $df_1$ is this composite. This means that $f_1d: X_2 \rightarrow Y_0$ lands in $Z_0Y$, and so, because $X_2$ is absolutely clean and $Y$ is AC-acyclic, there is a map $f_2: X_2 \rightarrow Y_1$. Continuing in this fashion, we complete $f$ to a chain map as required.

To see that $\tau$ is injective, suppose we have a chain map $\phi: X \rightarrow \Sigma Y$ such that $Z_0\phi$ factors through $Y_0$ via a map $g_0: Z_0X \rightarrow Y_0$. We need to construct a chain homotopy $D$ between $\phi$ and 0, and we do this following the same plan. As before, we extend $Z_0\phi$ to a map $D_0: X_0 \rightarrow Y_0$. Then $dD_0 - \phi_0$ induces a map $g_{-1}: Z_{-1}X = B_{-1}X \rightarrow Z_{-1}Y$, which we extend to a map $D_{-1}: X_{-1} \rightarrow Y_{-1}$, so that $dD_{-1} - D_{-1}d = \phi_0$. We continue in this fashion to construct $D_n$ for all negative $n$. We then define $D_n$ for positive $n$ using the fact that $X_i$ is absolutely clean for $i > 0$ and $Y$ is AC-acyclic, as we did above.

As a scholium of the above proof, we note the following proposition.

**Proposition 5.2.** For any ring $R$, suppose $M$ and $N$ are Gorenstein AC-injective $R$-modules, with $M = Z_0X$ and $N = Z_0Y$ for $X$ and $Y$ exact AC-acyclic complexes of injectives. Given a map $f: M \rightarrow N$, there is a chain map $\phi: X \rightarrow Y$ with $Z_0\phi = f$. 

We now use this proposition to prove the following lemma.

**Lemma 5.3.** For any ring $R$, the collection of Gorenstein AC-injective $R$-modules is closed under retracts.

The proof we give here is inspired by the proof for the corresponding fact for Gorenstein injectives over a Noetherian ring given in [Hol04].

**Proof.** For any ring $R$, the exact AC-acyclic complexes of injectives are closed under products, and so Gorenstein AC-injectives are as well. The Eilenberg swindle then implies that if $N$ is a retract of a Gorenstein AC-injective, there is a Gorenstein AC-injective module $M$ with $N \oplus M \cong M$. Indeed, if $N \oplus N'$ is Gorenstein AC-injective, we take $M = \prod_{i=1}^\infty (N \oplus N')$, so that

\[
M \cong N \oplus \prod_{i=1}^\infty (N' \oplus N) \cong N \oplus M.
\]

Now choose an exact AC-acyclic complex of injectives $X$ with $M = Z_0 X$. By Proposition 5.2, the monomorphism $f: M \to M \oplus N \cong M$ is induced by a chain map $\phi: X \to X$, so that $Z_0 \phi = f$. We would like to take the cokernel of $\phi$ and claim that it is an exact AC-acyclic complex of injectives with zero cycles equal to $N$. Unfortunately, $\phi$ need not be a monomorphism. So instead we use $\psi: X \to X' = X \oplus \prod_{i \neq 0} D^{i+1} X_i$, where $\psi$ has components $\phi$ and the maps $\rho_i: X \to D^{i+1} X_i$ that are $d$ in degree $i + 1$ and the identity in degree $i$. Note that $\psi$ is clearly a monomorphism in every degree except possibly 0, but also in degree 0 because $Z_0 \phi$ is a monomorphism. Furthermore, $Z_0 \psi = f$.

We then get a short exact sequence

\[
0 \to X \xrightarrow{\psi} X' \to Y \to 0
\]

of complexes, necessarily degreewise split since $X$ is a complex of injectives. Both $X$ and $X'$ are exact AC-acyclic complexes of injectives (since the disks we add are contractible), and so $Y$ is as well. Furthermore, because $X$ is exact, the induced sequence

\[
0 \to Z_0 X \xrightarrow{Z_0 \psi} Z_0 X' \to Z_0 Y \to 0
\]

is still exact, and so $Z_0 Y \cong N$ and so $N$ is Gorenstein AC-injective. \hfill \Box

**Lemma 5.4.** A module $M$ over a ring $R$ is in $\mathcal{W}$ if and only if $S^0 M$ is trivial in the exact AC-acyclic model structure.

**Proof.** A calculation using Lemma 3.1 shows that

\[
\text{Ext}^1(S^0 M, X) = \text{Ext}^1(M, Z_1 X).
\]

for any exact complex of injectives $X$. \hfill \Box

**Theorem 5.5.** For any ring $R$, there is an abelian model structure on $R$-Mod, the Gorenstein AC-injective model structure, in which every object is cofibrant and the fibrant objects are the Gorenstein AC-injective modules.
This model structure generalizes the Gorenstein injective model structure defined in [Hov02] and also its generalization in [Gil10]. Note, though, that we do not have an explicit cogenerating set for the cotorsion pair \((\mathcal{W}, \mathcal{F})\) when \(R\) is not Gorenstein, though we will prove below that a cogenerating set does exist. When \(R\) is Gorenstein, the syzygies of the indecomposable injective modules cogenerate.

One aspect of this model structure is that the class of Gorenstein AC-injectives is special pre-enveloping for any ring \(R\).

**Proof.** As above, we take \(\mathcal{F}\) to be the Gorenstein AC-injective modules, and define \(\mathcal{W} = \perp \mathcal{F}\). Then Lemma 5.4 shows that \(\mathcal{W}\) is thick and contains the injective modules. Now for any module \(M\), we have a short exact sequence

\[
0 \to S^0 M \to X \to Y \to 0
\]

in which \(X\) is an exact AC-acyclic complex of injectives and \(Y\) is trivial in the exact AC-acyclic model structure. Applying \(Z_0 = \text{Hom}_{\text{Ch}(R)}(S^0R, -)\), we get an exact sequence

\[
0 \to M \to Z_0 X \to Z_0 Y \to \text{Ext}^1_{\text{Ch}(R)}(S^0R, S^0 M) = 0.
\]

Of course \(Z_0 X\) is Gorenstein AC-injective by definition, but \(Z_0 Y\) is in \(\mathcal{W}\) as well by Lemma 5.1 since \(Y_i\) is injective for all \(i \neq 0\) and \(H_i Y = 0\) for all \(i \neq 1\). So the purported cotorsion pair \((\mathcal{W}, \mathcal{F})\) has enough injectives, and hence enough projectives as well if it is a cotorsion pair.

We can then use this to show that \(\mathcal{F} = \mathcal{W}^\perp\), so that \((\mathcal{W}, \mathcal{F})\) is in fact a cotorsion pair. Indeed, suppose \(M \in \mathcal{W}^\perp\). Find a short exact sequence

\[
0 \to M \to J \to N \to 0
\]

where \(J\) is Gorenstein AC-injective and \(N \in \mathcal{W}\). By assumption, this must split, and so \(M\) is a retract of \(J\) and hence is Gorenstein AC-injective by Lemma 5.3. Proposition 5.3 now completes the proof. \(\square\)

It is enlightening to examine the homotopy category of the Gorenstein AC-injective model structure, but to do so we need a lemma.

**Lemma 5.6.** For any ring \(R\), the cotorsion pair \((\mathcal{W}, \mathcal{F})\), where \(\mathcal{F}\) is the Gorenstein AC-injective modules, is hereditary, so that \(\text{Ext}^n(\mathcal{W}, \mathcal{F}) = 0\) for all \(n > 0\), \(W \in \mathcal{W}\), and \(F \in \mathcal{F}\). Hence the collection of Gorenstein AC-injective modules is coresolving (closed under cokernels of monomorphisms).

**Proof.** Suppose \(F\) is Gorenstein AC-injective, so that \(F = Z_0 X\), where \(X\) is an exact AC-acyclic complex of injectives. We have short exact sequences

\[
0 \to Z_i X \to X_i \to Z_{i-1} X \to 0
\]

for all \(i\), and each \(Z_i X\) is also Gorenstein AC-injective. A simple computation then shows that

\[
\text{Ext}^n(M, F) = \text{Ext}^1(M, Z_{n+1} X),
\]

so \((\mathcal{W}, \mathcal{F})\) is hereditary. It follows that \(\mathcal{F}\) is coresolving, just as in the proof of the third part of Proposition 2.7. \(\square\)

**Theorem 5.7.** For any ring \(R\), the homotopy category of the Gorenstein AC-injective model structure is the quotient category of the category of Gorenstein AC-injective modules obtained by identifying two maps when their difference factors through an injective module.
Proof. Since the Gorenstein AC-injectives are the cofibrant and fibrant objects, the homotopy category is the quotient category of the category of Gorenstein injectives by the homotopy relation. Since injective modules are trivial, any map that factors through an injective module gets sent to 0 in the homotopy category. So if \( f - g \) factors through an injective, then \( f = g \) in the homotopy category, and so \( f \) and \( g \) are homotopic. Conversely, suppose \( f, g: M \to N \) are homotopic, where \( M \) and \( N \) are Gorenstein AC-injective. Then there is a homotopy \( H: M \to N' \) between them for any path object \( N' \) of \( N \). We can obtain a path object by factoring the diagonal map \( N \to N \times N \) into a trivial cofibration \( N \xrightarrow{i} N' \) followed by a fibration \( N' \xrightarrow{p} N \times N \). In particular, since \( p \) is a fibration and the Gorenstein AC-injectives are closed under extensions (like the right half of any cotorsion pair), we see that \( N' \) is Gorenstein AC-injective. The cokernel of \( i \) is therefore also Gorenstein AC-injective since the Gorenstein AC-injectives are coresolving. But this cokernel is in \( W \) as well, so the cokernel of \( i \) is in fact an injective module. If we let \( d: N \times N \to N \) denote the difference map, we then have

\[
f - g = d \circ (f, g) = d \circ p \circ H.
\]

But \( d \circ p \circ i = 0 \), so \( d \circ p \) factors through the injective module \( \text{cok} \ i \), and so \( f - g \) does as well.

The Gorenstein AC-injective model structure and the exact AC-acyclic model structure are Quillen equivalent.

**Theorem 5.8.** For any ring \( R \), the functor \( S^0: R\text{-Mod} \to \text{Ch}(R) \) is a Quillen equivalence from the Gorenstein AC-injective model structure to the exact AC-acyclic model structure.

**Proof.** Since \( S^0 \) is exact, it preserves cofibrations. Lemma \([5.1]\) shows that it preserves trivial cofibrations, so it is a left Quillen functor with right adjoint \( Z_0 \). We will show that \( Z_0 \) reflects weak equivalences between fibrant objects and that the map \( M \to Z_0RS^0M \) is a weak equivalence, where \( R \) denotes fibrant replacement in the exact AC-acyclic model structure. In view of Corollary 1.3.16 of \([Hov99]\), this will complete the proof.

We first show that \( Z_0 \) reflects weak equivalences between fibrant objects. As with any right Quillen functor, it suffices to show that if \( f: X \to Y \) is a cofibration of fibrant objects such that \( Z_0f \) is a weak equivalence, then \( f \) is a weak equivalence. So we are given a short exact sequence

\[
0 \to X \xrightarrow{i} Y \to C \to 0
\]

with \( X \) and \( Y \) exact AC-acyclic complexes of injectives. This sequence is necessarily degreewise split, from which it follows that \( C \) is also an exact AC-acyclic complex of injectives. The functor \( Z_0 \) is left exact but not exact, but since \( X \) is exact we do get a short exact sequence

\[
0 \to Z_0X \to Z_0Y \to Z_0C \to 0.
\]

Since \( Z_0f \) is a weak equivalence, \( Z_0C \) is in \( W \). Lemma \([5.1]\) then implies that \( C \) is trivial in the exact AC-acyclic model structure, and so \( f \) is a weak equivalence.

Now take any module \( M \), and let \( RS^0M \) be a fibrant replacement for \( S^0M \), so that we have a short exact sequence

\[
0 \to S^0M \to RS^0M \to Y \to 0
\]
with $Y$ trivial in the exact AC-acyclic model category. Applying the functor $Z_0 = \text{Hom}_{\text{Ch}(R)}(S^0 R, -)$, we get an exact sequence

$$0 \to M \to Z_0 RS^0 M \to Z_0 Y \to \text{Ext}_{\text{Ch}(R)}^1(S^0 R, S^0 M) = 0.$$ 

Furthermore, $Y_i$ is injective for all $i \neq 0$ and $H_i Y = 0$ for all $i \neq 1$, so Lemma 5.4 implies that $Z_0 Y \in W$. Hence $M \to Z_0 RS^0 M$ is a weak equivalence. \hfill \Box

In view of Theorem 5.8, the functor

$$\gamma : R\text{-Mod} \to \text{Ho} R\text{-Mod}$$

to the homotopy category of the Gorenstein AC-injective model structure is an exact functor to a triangulated category that preserves coproducts and sends all absolutely clean and all projective modules to 0.

In fact, it is also initial in a sense.

**Proposition 5.9.** The homotopy category of the Gorenstein AC-injective model structure is initial among all triangulated categories with an exact functor from $R\text{-Mod}$ that preserves coproducts and sends all elements of $W$ to zero.

**Proof.** Suppose we have an exact functor $F : R\text{-Mod} \to C$ that sends all elements of $W$ to 0. Then $F$ sends all trivial cofibrations and fibrations to isomorphisms, since these are each part of exact sequences where the other term is in $W$, so map to exact triangles where one term is 0. Since every weak equivalence is a composition of a trivial cofibration followed by a trivial fibration, $F$ sends all weak equivalences to isomorphisms. Hence $F$ extends uniquely to the homotopy category. \hfill \Box

To truly understand the Gorenstein AC-injective model structure, then, we need to know what $W$ is. All we know in general is that $W$ contains all absolutely clean and all projective modules, and that $W$ is a thick subcategory closed under direct limits (as the kernel of any $F$ as in Proposition 5.9 must be). So we would like to know, for example, that $W$ is the smallest thick subcategory containing all absolutely clean and projective modules and closed under direct limits. But we do not know this.

We can at least prove that $(W, F)$ is cogenerated by a set.

**Proposition 5.10.** For any ring $R$, the cotorsion pair $(W, F)$, where $F$ is the class of Gorenstein AC-injectives, is cogenerated by a set. Thus the Gorenstein AC-injective model structure is cofibrantly generated.

We use the work of Šťovíček [Št’13] on deconstuctibility. The following lemma is not stated explicitly in [Št’13], so we prove it here, but it is implicit there.

**Lemma 5.11.** If $(A, B)$ is a cotorsion pair in a Grothendieck category that is cogenerated by a set, then there exists a set $S \subseteq A$ such that every element of $A$ is a transfinite extension of objects of $S$.

Recall that by definition there is a set $T$ of elements of $A$ such that every element of $A$ is a summand in a transfinite extension of objects of $T$. This lemma removes the summand condition at the expense of making $T$ larger. We should also note that although this lemma has a similar conclusion as Proposition 2.8 its hypotheses are quite different.
Proof. Let $T$ be a cogenerating set as above, and let $D$ be the class of all transfinite extensions of objects of $T$. By definition, this is a deconstructible class in the sense of Šťovíček [Št'13]. Now $A$ is the class of all summands of objects of $D$, but Šťovíček [Št'13, Proposition 2.9] proves that this means that $A$ is also deconstructible. Hence there is a set $S$ such that $A$ is the collection of all transfinite extensions of elements of $A$. □

Proof of Proposition 5.10. Take a set $S'$ of complexes as in Lemma 5.11 for the cotorsion pair $(W', F')$, where $F'$ is the class of exact AC-acyclic complexes of injectives. Now let $S$ be the collection of all modules $M$ such that $S^0 M \in S'$. Then $S \subseteq W$ by Lemma 5.11 and if $N \in W$, then $S^0 N \in W'$ by the same lemma, so $N$ must be a transfinite extension of objects of $S'$. However, each term $X_\alpha$ in this transfinite composition is a subobject of $S^0 N$, so must be $S^0 M_\alpha$ for some module $M_\alpha$. It follows that $M$ is a transfinite extension of objects in $S$. □

6. Projective model structures on $\text{Ch}(R)$

Having constructed model structures based on complexes of injective modules, it is natural to try to make similar constructions with complexes of projective modules. The projective case is harder to deal with because we must still prove that our cotorsion pairs are cogenerated by a set, and there is no natural choice for such a set even in the simplest case, as there is no dual version of Baer’s criterion for injectivity. We follow the standard idea of using all objects in the left half of the cotorsion pair that are not too big as a cogenerating set.

The goal of this section, then, is to state the following theorem, analogous to Theorem 4.1, and derive some analogous corollaries. The proof of Theorem 6.1 is technical and will be postponed until the next section.

**Theorem 6.1.** Given a ring $R$, let $A$ be a given right $R$-module. Let $C$ be the class of $A$-acyclic complexes of projectives; that is, chain complexes $C$ that are degreewise projective and such that $A \otimes_R C$ is exact. Then there is a cofibrantly generated abelian model structure on $\text{Ch}(R)$ where every object is fibrant, $C$ is the class of cofibrant objects, $W = C^\perp$ is the class of trivial objects, and the projective complexes $P = C \cap W$ are the trivially cofibrant objects. Furthermore, $W$ contains all contractible complexes. We call this model structure the $A$-acyclic projective model structure. Its homotopy category is equivalent to the chain homotopy category of $A$-acyclic complexes of projectives.

We have the dual dependence on $A$ that we had with the $A$-acyclic injective model structure.

**Lemma 6.2.** Suppose $A$ and $B$ are left $R$-modules and $B$ is built from $A$. Then the identity functor is a left Quillen functor from the $A$-acyclic projective model structure to the $B$-acyclic projective model structure; in fact the $A$-acyclic projective model structure is a right Bousfield localization of the $B$-acyclic projective model structure. In particular, if $A$ is also built from $B$, then the $A$-acyclic projective model structure coincides with the $B$-acyclic projective model structure.

**Proof.** Given a complex of projectives $C$, the collection of all right modules $M$ such that $M \otimes_R C$ is exact is a thick subcategory that is closed under transfinite extensions. Hence if $C$ is $A$-acyclic it is also $B$-acyclic. It follows that the identity
functor from the $A$-acyclic projective model structure to the $B$-acyclic one preserves cofibrations, and they have the same trivial cofibrations. 

The homotopy category of the $A$-acyclic projective model structure has similar properties to that of the $A$-acyclic injective model structure.

**Proposition 6.3.** Let $R$ be a ring and $A$ a right $R$-module. Consider the composite functor

$$\gamma: R\text{-Mod} \xrightarrow{S^0} \text{Ch}(R) \rightarrow \text{Ho Ch}(R)$$

from $R$-modules to the homotopy category of the $A$-acyclic projective model structure. Then $\gamma$ is an exact product-preserving functor to the triangulated category $\text{Ho Ch}(R)$. The kernel of $\gamma$ consists of all modules $M$ such that $\text{Hom}_R(X,M)$ is exact for all $A$-acyclic complexes of projectives $X$.

**Proof.** The homotopy category $\text{Ho Ch}(R)$ is triangulated because the inverse shift is an equivalence of categories and is also equivalent to the loop functor that exists in any model category (see Section 7.1 of [Hov99] for a general discussion of when the homotopy category of a pointed model category is triangulated).

Any epimorphism in $\text{Ch}(R)$ is a fibration in the $A$-acyclic projective model structure, and so short exact sequences in $\text{Ch}(R)$ give rise to exact triangles in $\text{Ho Ch}(R)$. The functor $S^0$ is exact, so we conclude that $\gamma$ is exact.

The kernel of $\gamma$ consists of all modules $M$ such that $S^0 M$ is trivial in the $A$-acyclic projective model structure. In view of Lemma 3.1 this is all modules $M$ such that $\text{Hom}_R(X,M)$ is exact for all $A$-acyclic complexes of projectives $X$. 

The simplest case of the $A$-acyclic projective model structure is when $A = 0$.

**Corollary 6.4.** For any ring $R$, there is a cofibrantly generated abelian model structure on $\text{Ch}(R)$, the Proj model structure, in which the cofibrant objects are the complexes of projectives and every object is fibrant. The trivially cofibrant objects coincide with the projective complexes, and the homotopy category is the chain homotopy category of complexes of projectives.

At the other extreme, we could take $A$ to be the direct sum of all finitely generated right $R$-modules. At first glance this appears to be more interesting than in the injective case. Here the cofibrant objects are the complexes of projectives $C$ such that $M \otimes_R C$ is exact for all right $R$-modules $M$. So these are the pure exact complexes of projectives. However, as we show in Theorem A.1 any pure exact complex is trivial in the Proj model structure. So a pure exact complex of projectives is trivially cofibrant in the Proj model structure, and hence is a projective complex. Thus, this is the model structure in which every map is a weak equivalence, the fibrations are the epimorphisms, and the cofibrations are the split monomorphisms with projective cokernel.

We can take $A = R$ to get the following corollary.

**Corollary 6.5.** For any ring $R$ there is a cofibrantly generated abelian model structure on $\text{Ch}(R)$, the exact projective model structure, in which the cofibrant objects are the exact complexes of projectives and every object is fibrant. The trivially cofibrant objects coincide with the projective complexes and the homotopy category is the chain homotopy category of exact complexes of projectives. For this model structure, all injective modules are sent to 0 by the functor $\gamma$ of Proposition 6.3.
The last sentence follows from Lemma [3.1] since $\text{Hom}_R(C, I)$ is always exact if $C$ is exact and $I$ is injective.

Just as in the injective case, replacing $A$ by $A \oplus R$ changes the cofibrant objects from $A$-acyclic complexes of projectives to exact $A$-acyclic complexes of projectives, so we sometimes call the $A \oplus R$-acyclic projective model structure the exact $A$-acyclic projective model structure.

Now, if $R$ is right Noetherian, we can take $A$ to be the direct sum of all indecomposable injectives and $R$ itself. One might expect this to give us the totally acyclic projective model structure. But an exact complex of projectives $C$ is defined to be totally acyclic if $\text{Hom}(C, P)$ is exact for all projective modules $P$. We get instead exact complexes of projectives such that $\text{Hom}(C, F)$ is exact for all flat modules $F$. Furthermore, this result extends to right coherent rings, and has a natural extension to all rings.

**Definition 6.6.** Let $R$ be a ring and $C$ a complex of projective $R$-modules. We say that $C$ is AC-acyclic if $I \otimes_R C$ is exact for all absolutely clean right $R$-modules $I$. We say that $C$ is firmly acyclic if $\text{Hom}(C, F)$ is exact for all level left $R$-modules $F$. If $C$ is itself exact, we will call $C$ exact AC-acyclic or exact firmly acyclic as the case may be.

We will then prove the following theorem in the appendix, as Corollary A.7.

**Theorem 6.7.** For any ring $R$, a complex of projectives $C$ is AC-acyclic if and only if it is firmly acyclic. If level $R$-modules all have finite projective dimension, these conditions are equivalent to $\text{Hom}_R(C, P)$ being exact for all projective $R$-modules $P$.

If $R$ is right Noetherian, this theorem says that a complex of projectives $C$ has $I \otimes_R C$ exact for all injective right $R$-modules $I$ if and only if $\text{Hom}(C, F)$ is exact for all flat left $R$-modules $F$. This was proved by Murfet and Salarian in [MS11]. If $R$ is right coherent, Theorem 6.7 says that a complex of projectives $C$ has $I \otimes_R C$ exact for all absolutely pure right $R$-modules $I$ if and only if $\text{Hom}_R(C, F)$ is exact for all flat left $R$-modules $F$, extending the Murfet-Salarian theorem to the coherent case.

For a general ring $R$, as in the injective case, we can take $A$ to be the direct sum of all the absolutely clean modules of small cardinality to get the following theorem.

**Theorem 6.8.** For any ring $R$, there is a cofibrantly generated abelian model structure on $\text{Ch}(R)$, the firmly acyclic model structure, in which every object is fibrant and the cofibrant objects are the firmly acyclic complexes of projectives. There is also a cofibrantly generated abelian model structure on $\text{Ch}(R)$, the **exact firmly acyclic model structure**, in which every object is fibrant and the cofibrant objects are the exact firmly acyclic complexes of projectives. The homotopy category of the (exact) firmly acyclic projective model structure is the chain homotopy category of (exact) firmly acyclic complexes of projectives. The functor $\gamma$ of Proposition 6.3 sends all injective modules to 0 in both the firmly acyclic and exact firmly acyclic model structures, and also sends level modules to 0 in the exact firmly acyclic model structure.

To get a totally acyclic projective model structure, we need to know slightly more about $R$. 
Corollary 6.9. Suppose $R$ is a ring in which all level modules have finite projective dimension. Then a complex of projectives $C$ is cofibrant in the (exact) firmly acyclic model structure if and only if $C$ is (exact) totally acyclic. If $R$ is right coherent, it suffices for all flat modules to have finite projective dimension.

Note that there are many rings in which every flat module has finite projective dimension. This is true if $R$ is right Noetherian and has a dualizing complex [Jør05].

Just as in the injective case, most of these model structures coincide when $R$ is Gorenstein.

Proposition 6.10. Suppose $R$ is Gorenstein. Then the exact projective model structure, the firmly acyclic model structure, and the exact firmly acyclic model structure all coincide.

Proof. The proof is very similar to the injective case. So consider a general complex of projectives $X$, and let $\mathcal{E}$ denote the collection of all right $R$-modules $M$ such that $M \otimes_R X$ is acyclic. As in the injective case, it is easy to check that $\mathcal{E}$ is thick. It follows that, if $X$ is an exact complex of projectives, then $M \otimes_R X$ is exact for all $M$ of finite projective dimension. In particular, if $R$ is Gorenstein, then every injective module has finite projective dimension, so $X$ is firmly acyclic. Thus the exact projective and the exact firmly acyclic model structures coincide.

Similarly, if $X$ is firmly acyclic, it follows that $M \otimes_R X$ is exact for all $M$ of finite injective dimension. If $R$ is Gorenstein, then $R$ itself has finite injective dimension, and so $X$ is exact. Thus the firmly acyclic and exact firmly acyclic model structures coincide as well. □

The same example as we used in the injective case shows that these model structures do not coincide in general if $R$ is not Gorenstein.

Proposition 6.11. Suppose $R = k[x,y]/(x^2, xy, y^2)$ where $k$ is a field. Then every totally acyclic complex of projectives is a projective complex, so the homotopy category of the firmly acyclic projective model structure is trivial. On the other hand, there is an exact complex of projectives that is not firmly acyclic, so the homotopy category of the exact projective model structure is not trivial. Similarly, there is a firmly acyclic complex of projectives that is not exact.

Note that this ring $R$ has a dualizing module $k$, so in particular a dualizing complex, so a complex of projectives is exact firmly acyclic if and only if it is totally acyclic.

Proof. If $X$ is a totally acyclic complex of projectives, then it is exact and firmly acyclic, so $I \otimes_R X$ is exact for any injective module $I$. In particular, both $R \otimes_R X$ and $J \otimes_R X$ are exact, where $J$ is the unique indecomposable injective. Since every $R$-module is built from $R$ and $J$ by Proposition 4.8, we see that every cycle group $Z_n X$ is projective.

To find an exact complex of projectives that is not totally acyclic, let $X_n = \bigoplus_{i=1}^{\infty} R$, and define $d: X_n \to X_{n-1}$ by $d(1_{2i-1}) = x_i$ and $d(1_{2i}) = y_i$. Then one
can easily that
\[ \ker d = \text{Im} d = \bigoplus_{i=1}^{\infty} k \]
generated by the \( x_i \) and the \( y_i \), so \( X \) is an exact complex of projectives. It is not a projective complex since the cycles are not projective.

To find a firmly acyclic complex \( Y \) of projectives that is not exact, we let \( Y_n = X_n \) with differential \( d(1_i) = x_{2i} + y_{2i-1} \). Then \( x_1 \) is a cycle that is not a boundary, so \( Y \) is not exact. But \( J \otimes_R Y \) is a countable direct sum of copies of \( J \), with \( d(\alpha_i) = \gamma_{2i-1} \) and \( d(\beta_i) = \gamma_{2i} \). So \( J \otimes_R Y \) is exact, and this \( Y \) is firmly acyclic. \( \square \)

7. Proof of the Projective Model Structures

The proof of Theorem 6.1 is technical. The basic plan is to find well-behaved small subcomplexes inside a complex \( C \) for which \( A \otimes_R C \) is exact. The key fact that makes this work is the result of Kaplansky [Kap58] that every projective module is a direct sum of countably generated projective modules. We first consider how to cope with complexes, each of whose degrees is a direct sum.

**Lemma 7.1** ((Covering Lemma)). Let \( \kappa \) be an infinite cardinal and suppose \( X \) is a nonzero complex in which each \( X_n \) has a direct sum decomposition \( X_n = \bigoplus_{i \in I_n} M_{n,i} \) where \( |M_{n,i}| < \kappa \) for all \( i \in I_n \). Then for any choice of subcollections \( J_n \subseteq I_n \) (at least one of which is nonempty), with \( |J_n| < \kappa \), we can find a nonzero subcomplex \( S \subseteq X \) with each \( S_n = \bigoplus_{i \in K_n} M_{n,i} \) for some subcollections \( K_n \subseteq I_n \) satisfying \( J_n \subseteq K_n \) and \( |K_n| < \kappa \).

**Proof.** Suppose we are given such subcollections \( J_n \subseteq I_n \). First, for each \( n \), we may build a subcomplex \( X^n \) of \( X \) as follows: In degree \( n \) the complex will consist of \( \bigoplus_{i \in J_n} M_{n,i} \). Then noting \( d(\bigoplus_{i \in J_n} M_{n,i}) \subseteq \bigoplus_{i \in I_{n-1}} M_{n-1,i} \) we define \( L_n-1 = \{ i \in I_{n-1} \mid d(\bigoplus_{i \in J_n} M_{n,i}) \cap M_{n-1,i} \neq 0 \} \). This essentially “covers” \( d(\bigoplus_{i \in J_n} M_{n,i}) \) with summands in the sense that \( d(\bigoplus_{i \in J_n} M_{n,i}) \subseteq \bigoplus_{i \in L_{n-1}} M_{n-1,i} \) and yet \( |L_{n-1}| < \kappa \) because \( |d(\bigoplus_{i \in J_n} M_{n,i})| < \kappa \). Now the subcomplex of \( X \) we are constructing will consist of \( \bigoplus_{i \in L_{n-1}} M_{n-1,i} \) in degree \( n-1 \). We continue down in the same way finding \( L_{n-2} \subseteq I_{n-2} \) with \( |L_{n-2}| < \kappa \) and with \( d(\bigoplus_{i \in L_{n-1}} M_{n-1,i}) \subseteq \bigoplus_{i \in L_{n-2}} M_{n-2,i} \). In this way we get a subcomplex of \( X \):

\[ X^n = \cdots \rightarrow 0 \rightarrow \bigoplus_{i \in J_n} M_{n,i} \rightarrow \bigoplus_{i \in L_{n-1}} M_{n-1,i} \rightarrow \bigoplus_{i \in L_{n-2}} M_{n-2,i} \rightarrow \cdots \]

Finally set \( X = \bigcup_{n \in \mathbb{N}} X^n \) and note that this complex, obviously nonzero because at least one \( I_n \neq \emptyset \), will work. (The sets \( K_n \) we claim to exist are the union of all the \( J_n \)’s and all the various \( L_n \) in sight. We still have \( |K_n| < \kappa \).) \( \square \)

Now that we find small subcomplexes of complexes with degreewise direct sum decompositions, we need to find small exact subcomplexes of exact complexes with degreewise direct sum decompositions.

**Lemma 7.2** ((Exact Covering Lemma)). Let \( \kappa \) be an infinite cardinal and suppose \( Y \) is an exact complex in which each \( Y_n \) has a direct sum decomposition \( Y_n = \bigoplus_{i \in I_n} M_{n,i} \) where \( |M_{n,i}| < \kappa \) for all \( i \in I_n \). Then for any choice of subcollections \( K_n \subseteq I_n \) with \( |K_n| < \kappa \), we can find an exact subcomplex \( T \subseteq Y \) with each \( T_n = \bigoplus_{i \in J_n} M_{n,i} \) for some subcollections \( J_n \subseteq I_n \) satisfying \( K_n \subseteq J_n \) and \( |J_n| < \kappa \).
Proof. We prove this in two steps.

(Step 1) We first show the following: If $X \subseteq Y$ is any exact subcomplex with $|X| < \kappa$, then for any single one of the given $K_n$, we can find an exact subcomplex $T \subseteq Y$ containing $X$ and so that for this given $n$, $T_n = \oplus_{i \in L_n} M_{n,i}$ for some $L_n \subseteq I_n$ with $K_n \subseteq L_n$ and $|L_n| < \kappa$.

For this given $n$, first set $D_n = \{ i \in I_n \mid X_n \cap M_{n,i} \neq 0 \}$. Since $|X_n| < \kappa$, we have $|D_n| < \kappa$. Now define $L_n = D_n \cup K_n$ and set $T_n = \oplus_{i \in L_n} M_{n,i}$. Of course $|L_n| < \kappa$ and $X_n \subseteq T_n$.

So all we need to do is extend $T_n$ into an exact subcomplex containing $X$ and with cardinality less than $\kappa$. We build down by setting $T_{n-1} = S_{n-1} + d(T_n)$ and $T_i = S_i$ for all $i < n - 1$. One can check that

$$T_n \to S_{n-1} + d(T_n) \to S_{n-2} \to \cdots$$

is exact. In particular, we have exactness in degree $n-1$ since $d(S_n) \subseteq d(T_n)$.

Next we build up from $T_n$. To start, take the kernel of $T_n \to T_{n-1}$ and find a $T_{n+1} \subseteq Y_{n+1}$ such that $|T_{n+1}| < \kappa$ and $T_{n+1}$ maps surjectively onto this kernel. Then take $T_{n+1} = S_{n+1} + T_{n+1}$. Now $T_{n+1}$ also maps surjectively onto this kernel. We continue upward to build $T_{n+2}, T_{n+3}, \cdots$ in the same way and we are done.

We now finish the proof. From Step 1, taking $X = 0$ and the subcollection to be $K_0$ we can find an exact subcomplex $T^0 \subseteq Y$ such that $(T^0)_{i_0} = \oplus_{i \in L_0} M_{0,i}$ for some $L_0 \subseteq I_0$ with $K_0 \subseteq L_0$ and $|L_0| < \kappa$. Now using Step 1 again, with $X = T^0$ and using $K_{-1}$, we get another exact subcomplex $T^1$ containing $T^0$ and such that $(T^1)_{-1} = \oplus_{i \in L_{-1}} M_{-1,i}$ for some $L_{-1} \subseteq I_{-1}$ with $K_{-1} \subseteq L_{-1}$ and $|L_{-1}| < \kappa$. Lets say that $T^0$ was constructed using a “degree 0 operation” and $T^1$ was constructed using a “degree -1 operation”. Then we can continue to use “degree k operations” with the following back and forth pattern on $k$:

$$0, -1, 0, 1, -2, -1, 0, 1, 2, -3, -2, -1, 0, 1, 2, 3 \cdots$$

to build an increasing union of exact subcomplexes, $\{ T^l \}$. Finally set $T = \cup_{l \in \mathbb{N}} T^l$. Then by a cofinality argument we see that for each $n$ we have $T_n = \oplus_{i \in J_n} M_{n,i}$ for some subsets $J_n \subseteq I_n$ (the $J_n$’s are each a countable union of the newly constructed $L_n$’s obtained in each “pass”, and so $|J_n| < \kappa$). Clearly each $K_n \subseteq J_n$ and $T$ is exact.

With these lemmas in hand, we turn to our specific situation. Let $P$ be any complex of projective modules. As we have discussed, Kaplansky [Kap58] proved that we can write $P_n = \oplus_{i \in L_n} P_{n,i}$ for each $n$. Note that if $\kappa > \max\{ |R|, \omega \}$ is a regular cardinal then $|P_{n,i}| < \kappa$.

Next let $A$ be a given $R$-module. Using the natural isomorphism

$$M \otimes_R (\oplus_{i \in S} N_i) \cong \oplus_{i \in S} M \otimes_R N_i$$

we may identify $A \otimes_R P$ with the complex whose degree $n$ is $\oplus_{i \in I_n} A \otimes_R P_{n,i}$. Moreover, for any subcomplex $S \subseteq P$ of the form $S_n = \oplus_{i \in K_n} P_{n,i}$ for some $K_n \subseteq I_n$ we can and will identify $A \otimes_R S$ with the subcomplex of $A \otimes_R P$ whose degree $n$ is $\oplus_{i \in K_n} A \otimes_R P_{n,i} \subseteq \oplus_{i \in I_n} A \otimes_R P_{n,i}$. We note that if $\kappa > \max\{ |R|, \omega \}$ is a regular cardinal, then such a subcomplex $S$ satisfies $|S| < \kappa$ whenever $|K_n| < \kappa$. Similarly, if $\kappa > \max\{ |A|, \omega \}$ is a regular cardinal, note that $|A \otimes_R S| < \kappa$ whenever $|K_n| < \kappa$.

We will use all of the above observations in the proof of our theorem below.
Theorem 7.3. Let $A$ be a given $R$-module and take $\kappa > \max\{ |R|, |A|, \omega \}$ to be a regular cardinal. Let $P$ be any nonzero complex of projectives in which $A \otimes_R P$ is exact. Then we can write $P$ as a continuous union $P = \bigcup_{\alpha < \lambda} Q_\alpha$ where each $Q_\alpha, Q_{\alpha+1}/Q_\alpha$ are also $A \otimes_R$-exact complexes of projectives and $|Q_\alpha|, |Q_{\alpha+1}/Q_\alpha| < \kappa$.

Proof. Write each $P_n = \bigoplus_{i \in I_n} P_{n,i}$ where $P_{n,i}$ are countably generated. We prove the theorem in two steps.

(Step 1). We first show the following: We can find a nonzero subcomplex $Q \subseteq P$ of the form $Q_n = \bigoplus_{i \in I_n} P_{n,i}$ for some subcollections $I_n \subseteq I_n$ having $|I_n| < \kappa$ and such that $A \otimes_R Q$ is exact.

Since $P$ is nonzero at least one $P_n \neq 0$. For this $n$, take any nonempty $J_n \subseteq I_n$ having $|J_n| < \kappa$. Apply the Covering Lemma with $P$ in the place of $X$ and taking the subcollections to consist of this $J_n$ and all the other $J_n$ may be empty. This gives us a nonzero subcomplex with $S^1_n = \bigoplus_{i \in K^1_n} P_{n,i}$ for some subcollections $K^1_n \subseteq I_n$ satisfying $J_n \subseteq K^1_n$ and $|K^1_n| < \kappa$ for each $n$.

Now $A \otimes_R S^1$ is the subcomplex of $A \otimes_R P$ having $(A \otimes_R S^1)_n = \bigoplus_{i \in K^1_n} A \otimes_R P_{n,i}$. That is, the subcollections $K^1_n \subseteq I_n$ determine $A \otimes_R S^1$. We now apply the Exact Covering Lemma with $A \otimes_R P$ in the place of $Y$ and taking the subcollections to be the $K^1_n$. This gives us an exact subcomplex $T^1 \subseteq A \otimes_R P$ with each $T^1_n = \bigoplus_{i \in J^1_n} A \otimes_R P_{n,i}$ for some subcollections $J^1_n \subseteq I_n$ satisfying $K^1_n \subseteq J^1_n$ and $|J^1_n| < \kappa$.

But perhaps now the direct sums $\bigoplus_{i \in J^1_n} P_{n,i}$ don’t even form a subcomplex of $P$ (because the tensor product with $A$ may send some maps to 0). So we apply the Covering Lemma to $P$ with the $J^1_n$ as the subcollections to find a subcomplex $S^2 \subseteq P$ with each $S^2_n = \bigoplus_{i \in K^2_n} P_{n,i}$ for some subcollections $K^2_n \subseteq I_n$ satisfying $J^1_n \subseteq K^2_n$ and $|K^2_n| < \kappa$. Of course $S^1 \subseteq S^2$ because $K^1_n \subseteq K^2_n$ for each $n$.

But now certainly $A \otimes_R S^2$ need not be exact, so we again apply the Exact Covering Lemma to $A \otimes_R P$ taking the subcollections to be the $K^2_n$. This gives us an exact subcomplex $T^2 \subseteq A \otimes_R P$ with each $T^2_n = \bigoplus_{i \in J^2_n} A \otimes_R P_{n,i}$ for some subcollections $J^2_n \subseteq I_n$ satisfying $K^2_n \subseteq J^2_n$ and $|J^2_n| < \kappa$. Notice that we have $A \otimes_R S^1 \subseteq T^1 \subseteq A \otimes_R S^2 \subseteq T^2$ because $K^1_n \subseteq J^1_n \subseteq K^2_n \subseteq J^2_n$.

And so it goes. The $\bigoplus_{i \in J^2_n} P_{n,i}$ need not form a subcomplex of $P$. So we continue this back and forth, applying the Covering Lemma to $P$ and the newly obtained subcollections $J^l_n$, and then applying the Exact Covering Lemma to $A \otimes_R P$ and the newly found subcollections $K^l_n$. We obtain an increasing sequence of subcomplexes of $P$

$$0 \neq S^1 \subseteq S^2 \subseteq S^3 \subseteq \cdots$$

corresponding to the subcollections $J^1_n \subseteq J^2_n \subseteq J^3_n \subseteq \cdots$. We also get an increasing sequence of subcomplexes of $A \otimes_R P$

$$A \otimes_R S^1 \subseteq T^1 \subseteq A \otimes_R S^2 \subseteq T^2 \subseteq A \otimes_R S^3 \subseteq T^3 \subseteq \cdots$$

with each $T^l$ exact.

So we set $Q = \bigcup_{l \in \mathbb{N}} S^l$ and claim that $Q$ satisfies the properties we sought. Indeed notice each $Q_n = \bigoplus_{i \in I_n} P_{n,i}$ where $L_n = \bigcup_{l \in \mathbb{N}} J^l_n$. Also we still have $|L_n| < \kappa$. Finally, since $A \otimes_R -$ commutes with direct limits we get $A \otimes_R Q = \bigcup_{l \in \mathbb{N}} A \otimes_R S^l = \bigcup_{l \in \mathbb{N}} T^l$. This is exact because each $T^l$ is exact.

(Step 2). We now can easily finish to obtain the desired continuous union. Start by finding a nonzero $Q^0 \subseteq P$ of the form $Q^0_n = \bigoplus_{i \in L^0_n} P_{n,i}$ for some subcollections $L^0_n \subseteq
$L_n$ having $|L_n^0| < \kappa$ and such that $A \otimes_R Q^0$ is exact. Note that $Q^0$ and $P/Q^0$ are also complexes of projectives and since $0 \to Q^0 \to P \to P/Q^0 \to 0$ is a degreewise split short exact sequence, so is $0 \to A \otimes_R Q^0 \to A \otimes_R P \to A \otimes_R P/Q^0 \to 0$. It follows that $A \otimes_R P/Q^0$ must also be exact. So if it happens that $P/Q^0$ is nonzero we can in turn find a nonzero subcomplex $Q^1/Q^0 \subseteq P/Q^0$ with $Q^1/Q^0$ and $(P/Q^0)/(Q^1/Q^0) \cong P/Q^1$ both $A \otimes_R$ - exact complexes of projectives with cardinality less than $\kappa$. Note that we can identify these quotients such as $P/Q^0$ as complexes whose degree $n$ entry is $\bigoplus_{i \in I_n} L_n^i P_{n,i}$ and in doing so we may continue to find an increasing union $0 \neq Q^0 \subseteq Q^1 \subseteq Q^2 \subseteq \cdots$ corresponding to a nested union of subsets $L_n^0 \subseteq L_n^1 \subseteq L_n^2 \subseteq \cdots$ for each $n$. Assuming this process doesn’t terminate we set $Q^\omega = \bigcup_{n < \omega} Q_n$ and note that $Q_n^\omega = \bigoplus_{i \in L_n^\omega} P_{n,i}$ where $L_n^\omega = \bigcup_{n < \omega} L_n^\alpha$. So still, $Q^\omega$ and $P/Q^\omega$ are complexes of projectives and are $A \otimes_R$ - exact since $A \otimes_R$ - commutes with direct limits. Therefore we can continue this process with $P/Q^\omega$ to obtain $Q^{\omega + 1}$ with all the properties we desire. Using this process we can obtain an ordinal $\lambda$ and a continuous union $P = \bigcup_{\alpha < \lambda} Q^\alpha$ with each $Q_\alpha, Q_{\alpha + 1}/Q_\alpha$ $A \otimes_R$ - exact complexes of projectives having $|Q_\alpha|, |Q_{\alpha + 1}/Q_\alpha| < \kappa$. \hfill \Box

We can now prove Theorem 6.1

**Proof.** The plan is to apply Proposition 5.4. First let $\kappa > \max\{|R|, |A|, \omega\}$ be a regular cardinal and let $S$ be the set of all complexes $P \in \mathcal{C}$ such that $|P| \leq \kappa$. (We really need to take a representative for each isomorphism class so that we actually get a set as opposed to a proper class). Since any set $S$ cogenerates a complete cotorsion pair $\langle (S^-, S^+) \rangle$ it is enough to show $S^+ = \mathcal{C}^-$. But this follows right away from the chain complex version of Theorem 7.3.4 of [EH00]. The remaining two properties of Proposition 6.2 to check hold by straight duality of the proofs for the injective models in Theorem 4.1. \hfill \Box

## 8. The Gorenstein AC-projective model structure on modules

We saw in Section 5 that the exact AC-acyclic model structure on $\operatorname{Ch}(R)$ gives rise to a Quillen equivalent model structure on $R$-$\operatorname{Mod}$ in which the fibrant objects are the Gorenstein AC-injectives. One would then expect the exact firmly acyclic model structure to give rise to a similar model structure on $R$-$\operatorname{Mod}$. We construct this Gorenstein AC-projective model structure in this section.

Recall that a module $M$ is Gorenstein projective if $M = Z_0 X$ for some totally acyclic complex of projectives; that is, $X$ is exact and $\operatorname{Hom}(X, P)$ is exact for all projective modules $P$. In view of Theorem 6.7 we define $M$ to be Gorenstein AC-projective if $M = Z_0 X$ for some exact firmly acyclic complex of projectives; that is, $X$ is exact and $\operatorname{Hom}(X, F)$ is exact for all level (left) modules $F$, equivalently, $I \otimes_R X$ is exact for all AC-modules $I$. Note that every Gorenstein AC-projective is Gorenstein projective, and the two concepts agree if every level module has finite projective dimension. On the other hand, over coherent rings the Gorenstein AC-projectives are exactly the Ding projectives from [GH10].

Our model structure on $R$-$\operatorname{Mod}$ will then have $\mathcal{C}$ consist of the Gorenstein AC-projective modules, $\mathcal{F}$ consist of all modules, and $\mathcal{W} = \mathcal{C}^\perp$. We now proceed as in Section 5.

**Lemma 8.1.** Let $R$ be a ring and suppose $Y$ is a complex of $R$-modules with $H_i Y = 0$ for $i > 0$ and $Y_i$ level for $i < 0$. Then $Y$ is trivial in the exact firmly acyclic model structure if and only if $Y_0/B_0 Y \in \mathcal{W}$. 


The proof of this lemma is very similar to the proof of Lemma 5.1, suitably dualized. We will set up the outline, then leave the rest of the proof to the reader.

Proof. Suppose $M$ is Gorenstein AC-projective, so that $M = Z_{-1}X = B_{-1}X = X_0/B_0X$ for some exact firmly acyclic complex of projectives $X$. We claim that there is an isomorphism

$$\operatorname{Ext}^1(X,Y) \rightarrow \operatorname{Ext}^1(M,Y_0/B_0Y);$$

this isomorphism would prove the lemma. Indeed, because $X$ is a complex of projectives, Lemma 3.1 gives us an isomorphism

$$\operatorname{Ext}^1(X,Y) \rightarrow \operatorname{Ch}(R)(X,\Sigma Y)/\sim,$$

where $\sim$ denotes chain homotopy. A chain map $\phi: X \rightarrow \Sigma Y$ induces a map $B_0X \cong X_1/B_1X \rightarrow Y_0/B_0Y$. A chain homotopy between $\phi$ and 0 gives us maps $D_n: X_n \rightarrow Y_n$ with $-dD_n + D_{n-1}d = \phi_n$. In particular, $D_0d = \phi_1$ as a map from $X_1$ to $Y_0/B_0Y$. Thus, there is a natural map

$$\operatorname{Ch}(R)(X,\Sigma Y)/\sim \rightarrow \operatorname{Hom}(B_0X,Y_0/B_0Y)/\operatorname{Hom}(X_0,Y_0/B_0Y) \cong \operatorname{Ext}^1(M,Y_0/B_0Y).$$

We now show this map is an isomorphism in analogous fashion to the proof of Lemma 5.1.

Just as in the injective case, we then get the following scholium.

**Proposition 8.2.** For any ring $R$, suppose $M$ and $N$ are Gorenstein AC-projective modules, with $M = Z_0X$ and $N = Z_0Y$ for $X$ and $Y$ exact firmly acyclic complexes of projectives. Given a map $f: M \rightarrow N$, there is a chain map $\phi: X \rightarrow Y$ with $Z_0\phi = f$.

This proposition then leads to the following lemma, analogous to Lemma 5.3 with the dual proof.

**Lemma 8.3.** For any ring $R$, the collection of Gorenstein AC-projective modules is closed under retracts.

**Lemma 8.4.** A module $M$ over a ring $R$ is in $W$ if and only if $S^0M$ is trivial in the exact firmly acyclic model structure.

Proof. A calculation using Lemma 3.1 shows that

$$\operatorname{Ext}^1(X,S^0M) = \operatorname{Ext}^1(Z_{-1}X,M),$$

for any exact complex of projectives $X$. \qed

**Theorem 8.5.** For any ring $R$, there is an abelian model structure on $R\text{-Mod}$, the Gorenstein AC-projective model structure, in which every object is fibrant and the cofibrant objects are the Gorenstein AC-projective modules.

This model structure generalizes the Gorenstein projective model structure for Gorenstein rings constructed in [Hov02], as well as its generalization constructed in [Gil10].

The proof of Theorem 8.5 is dual to the proof of Theorem 5.5 and so we leave it to the reader.

Just as in the injective case, we have the following lemma.
Lemma 8.6. For any ring R, the cotorsion pair (C, W), where C is the Gorenstein AC-projective modules, is hereditary, so that Ext^n(C, W) = 0 for all n > 0, W ∈ W, and C ∈ C. Hence the collection of Gorenstein AC-projective modules is resolving (closed under kernels of epimorphisms).

Proof. Suppose C is Gorenstein AC-projective, so that F = Z_0X, where X is an exact firmly acyclic complex of projectives. We have short exact sequences

$$0 \to Z_iX \to X_i \to Z_{i-1}X \to 0$$

for all i, and each Z_iX is also Gorenstein AC-projective. A simple computation then shows that

$$\text{Ext}^n(C, M) = \text{Ext}^1(Z_{n-1}C, M),$$

so (C, W) is hereditary. It follows that C is resolving. □

We then have the analogue of Theorem 5.7, whose proof is dual.

Theorem 8.7. For any ring R, the homotopy category of the Gorenstein AC-projective model structure is the quotient category of the category of Gorenstein AC-projective modules obtained by identifying two maps when their difference factors through a projective module.

Just as in the injective case, the Gorenstein AC-projective model structure and the exact firmly acyclic model structure are Quillen equivalent.

Theorem 8.8. For any ring R, the functor F: Ch(R) → R-Mod defined by F(X) = X_0/B_0X is a Quillen equivalence from the exact firmly acyclic model structure to the the Gorenstein AC-projective model structure.

Proof. The right adjoint of F is S^0, which is exact and so obviously preserves fibrations. In fact, S^0 also preserves trivial fibrations by Lemma 8.4. So S^0 is a right Quillen functor. To complete the proof, we will show that F reflects weak equivalences between cofibrant objects and that the natural map FCS^0M → S^0M is a weak equivalence for all modules M, where C denotes cofibrant replacement in the exact firmly acyclic model structure. In view of Corollary 1.3.16 of [Hov99], this will complete the proof.

We will now show that F reflects weak equivalences between cofibrant objects. By factoring any map between cofibrant objects into a trivial cofibration followed by a fibration, we see that it suffices to show that if f: X → Y is a fibration of cofibrant objects such that Ff is a weak equivalence, then f is a trivial fibration. So we are given a short exact sequence

$$0 \to K \to X \xrightarrow{f} Y \to 0$$

with X and Y exact firmly acyclic complexes of projectives. This sequence is necessarily degreewise split, so K is also an exact firmy acyclic complex of projectives. The functor F is right exact but not exact, but since Y is exact we do get a short exact sequence

$$0 \to K_0/B_0K \to X_0/B_0X \to Y_0/B_0Y \to 0.$$ 

Since Ff is a weak equivalence, K_0/B_0K is in W. Lemma 8.1 then implies that K is trivial in the exact firmly acyclic model structure, so f is a trivial fibration.

Now take any module M, and let CS^0M be a cofibrant replacement for S^0M, so that we have a short exact sequence

$$0 \to Y \to CS^0M \to S^0M \to 0.$$
with \( Y \) trivial in the exact firmly acyclic model category. Because \( M \) is concentrated in degree 0, one can check that we get an exact sequence
\[
0 \to FY \to FCS^0M \to M \to 0
\]
Furthermore, \( Y_i \) is projective for all \( i \neq 0 \) and \( H_iY = 0 \) for all \( i \neq 0 \), so Lemma 5.3 implies that \( FY \in W \). Hence \( FCS^0M \to M \) is a weak equivalence. \( \square \)

The functor
\[
\gamma : R\text{-Mod} \to \text{Ho } R\text{-Mod}
\]
to the homotopy category of the Gorenstein AC-projective model structure is an exact functor to a triangulated category that preserves products and sends all level modules and all injective modules to 0.

Just as in the injective case, it also is initial in the following sense.

**Proposition 8.9.** The homotopy category of the Gorenstein AC-projective model structure is initial among all triangulated categories with an exact functor from \( R\text{-Mod} \) that preserves products and sends all elements of \( W \) to zero.

The proof is exactly the same as in the injective case.

Unfortunately, as in the injective case, we do not what \( W \) is. Ideally, it would be the smallest thick subcategory closed under products that contains the level and injective modules, but we do not know if this is true.

We do know that \((\mathcal{C}, W)\) is cogenerated by a set though.

**Proposition 8.10.** For any ring \( R \), the cotorsion pair \((\mathcal{C}, W)\), where \( \mathcal{C} \) is the class of Gorenstein AC-projectives, is cogenerated by a set. Thus the Gorenstein AC-projective model structure is cofibrantly generated.

The proof of this proposition is very similar to the proof of Proposition 5.10 so we leave it to the reader.

There is a relationship between the Gorenstein AC-projective and the Gorenstein AC-injective model structures.

**Proposition 8.11.** For any ring \( R \), the identity functor is a left Quillen functor from the Gorenstein AC-projective model structure to the Gorenstein AC-injective model structure. It is a Quillen equivalence when \( R \) is Gorenstein.

**Proof.** It is clear that the identity functor takes cofibrations in the Gorenstein AC-projective model structure, which are certain monomorphisms, to cofibrations in the Gorenstein AC-injective model structure, which are all monomorphisms. Similarly, it takes fibrations in the Gorenstein AC-injective model structure to fibrations in the Gorenstein AC-projective model structure. Together, these make it a left Quillen functor as required.

When \( R \) is Gorenstein, these model structure coincide with the ones constructed in [Hov02], where it is shows that they are Quillen equivalent. \( \square \)

**Appendix A. Complexes of projectives**

The object of this section is to prove Theorem 6.7 which we will restate below as Corollary A.7 and some related theorems. To prove these results, we will use the following theorem of independent interest.
Theorem A.1. Let $R$ be a ring and let $C$ be a complex of projective $R$-modules. If $Y$ is a pure exact complex of $R$-modules, then $\text{Hom}(C,Y)$ is exact, or, equivalently, $\text{Ext}^1_{\text{Ch}(R)}(C,Y) = 0$. Similarly, if $Z$ is a pure exact complex of right $R$-modules, then $Z \otimes_R C$ is exact.

This theorem generalizes Neeman’s result [Nee08] that $\text{Ext}^1(C,Y) = 0$ if $C$ is a complex of projectives and $Y$ is a complex of flat modules with flat cycles, as such $Y$ are automatically pure. However, Neeman also proves that such $Y$ are the only complexes of flat modules with $\text{Ext}^1(C,Y) = 0$ for all complexes of projectives $X$. The generalization of this fact is untrue; it is easy to see by induction that every chain map from a complex of projectives to a bounded below exact complex is chain homotopic to 0, and not all bounded below exact complexes are pure exact.

Our first goal is to reduce the study of all complexes of projectives to a manageable set of them.

Lemma A.2. For any ring $R$, let $P$ be a complex of projective $R$-modules. Then $P$ is a retract of a complex $F$ of free modules. Furthermore, if $P$ is exact then $F$ can be taken to be exact.

Proof. Recall Eilenberg’s swindle (Corollary 2.7 of [Lam99]) allows one to construct, for any projective module $P$ a free module $F$ such that $P \oplus F \cong F$. Given a complex of projectives $P$, we use the swindle to find for each $P_n$ a free $F_n$ such that $P_n \oplus F_n \cong F_n$. Then $P \oplus (\oplus_{n \in \mathbb{Z}} D^n(F_n))$ is a complex of free modules. Indeed in degree $n$ the complex equals $P_n \oplus F_n \oplus F_{n+1} \cong F_n \oplus F_{n+1}$ which is free. Of course $P$ is a retract of $P \oplus (\oplus_{n \in \mathbb{Z}} D^n(F_n))$ by construction and also $P \oplus (\oplus_{n \in \mathbb{Z}} D^n(F_n))$ is exact whenever $P$ is exact.

Theorem A.3. For any ring $R$, the cotorsion pair $(\mathcal{C}, \mathcal{W})$, where $\mathcal{C}$ is the class of all complexes of projective modules, is cogenerated by the collection of all bounded above complexes of finitely generated free modules.

Proof. Let $\mathcal{S}$ be the set of bounded above complexes of finitely generated free modules. It cogenerated a cotorsion pair $(\perp(S^\perp), S^\perp)$ and we wish to show $\mathcal{C} = \perp(S^\perp)$. Since $\mathcal{S}$ contains a set of projective generators we know that $\perp(S^\perp)$ is precisely the class of all retracts of transfinite extensions of objects in $\mathcal{S}$. (Although complexes of projective modules are not closed under all direct limits, they are closed under transfinite compositions, just as projective modules are.) Since Lemma A.2 tells us that any complex of projectives is a retract of a complex of free modules we only need to show that a complex of free modules is a transfinite composition of bounded above complexes of finitely generated free modules. So let $F$ be a complex of free modules and write each $F_n = \oplus_{i \in I_n} R_i$ for some $I_n$ and each $R_i = R$. We do a simplified version of the argument in Lemma 7.1. Assuming $F$ is nonzero we can find a nonzero $F_n$ and we take just one summand $R_j$ for some $j \in I_n$. We start to build a bounded above subcomplex $X \subseteq F$ by setting $X_n = R_j$ and setting $X_i = 0$ for all $i > n$. Now note $d(R_j) \subseteq \oplus_{i \in I_{n-1}} R_i$ and set $L_{n-1} = \{ i \in I_{n-1} \mid d(R_j) \cap \oplus_{i \in I_{n-1}} R_i \neq 0 \}$. We set $X_{n-1} = \oplus_{i \in I_{n-1}} R_i$ and note that $|L_{n-1}|$ must be finite. We can continue down in the same way finding $L_{n-2} \subseteq I_{n-2}$ with $|L_{n-2}|$ finite and with $d(\oplus_{i \in L_{n-2}} R_i) \subseteq \oplus_{i \in L_{n-2}} R_i$. In this way we get a subcomplex of $X$:

$$X' = \cdots \to 0 \to R_j \to \oplus_{i \in L_{n-1}} R_i \to \oplus_{i \in L_{n-2}} R_i \to \cdots$$

So $X$ is a nonzero bounded above complex of finitely generated free modules.
Now following the method of Step 2 in the proof of Theorem 7.3 see that we can write any complex of free modules as a continuous union of bounded above complexes of finitely generated free modules.

We would now like to find complexes in \( W = C^+ \). Since \( C \) consists of complexes of projectives, this is equivalent to finding complexes \( Y \) such that every chain map from a complex of projectives \( C \) to \( Y \) is chain homotopic to 0. In view of the above lemma, we can assume that \( C \) is a bounded above complex of finitely generated free modules. Of course, \( Y \) must be exact, since \( C \) contains \( S^n(R) \) for all \( n \). One might guess that \( Y \) should be slightly better than exact; the content of Theorem A.1 says that any pure exact \( Y \) is in \( W \).

**Lemma A.4.** Given a ring \( R \), suppose \( X \) is a bounded complex of finitely presented modules and \( Y \) is a pure exact complex. Then every chain map \( f: X \to Y \) is chain homotopic to 0. Similarly, if \( Z \) is a pure exact complex of right \( R \)-modules, then \( Z \otimes_R X \) is exact.

The second statement is actually true for any bounded complex \( X \), even if the entries are not finitely presented.

**Proof.** We begin with the first statement. Let \( n \) be the number of \( i \) for which \( X_i \) is nonzero. If \( n = 1 \), the result follows by definition of pure exactness. In general, let \( m \) be the largest degree \( i \) for which \( X_i \) is nonzero, and let \( A \) be the subcomplex of \( X \) with \( A_i = X_i \) for \( i < m \) and \( A_m = 0 \). The short exact sequence

\[
0 \to A \to X \to S^mX_m \to 0
\]

is degreewise split, so the induced sequence

\[
0 \to \text{Hom}(S^mX_m, Y) \to \text{Hom}(X, Y) \to \text{Hom}(A, Y) \to 0
\]

is still exact. The long exact sequence in homology now gives us the result by induction.

The proof of the second statement is virtually identical. The base case of \( n = 1 \) again follows by definition of pure exactness, and the proof of the induction step is the same except for replacing \( \text{Hom}(-, Y) \) by \( Z \otimes_R - \).

We can now prove Theorem A.1.

**Proof of Theorem A.1.** We first show that \( \text{Hom}(C, Y) \) is exact for \( C \) a complex of projectives and \( Y \) pure exact. In view of Theorem A.3 we may assume that \( C \) is a bounded above complex of finitely generated free modules. It suffices to show that any chain map \( f: C \to Y \) is chain homotopic to 0. We construct a chain homotopy \( D_n: C_n \to Y_{n+1} \) with \( dD_n + D_{n-1}d = f_n \) by downwards induction on \( n \). Since \( C \) is bounded above, we can take \( D_n = 0 \) for large \( n \) to begin the induction. So we suppose that \( D_i \) has been defined for \( i \geq n \) and that \( dD_{n+1} + D_{n}d = f_{n+1} \).

We will first modify \( D_n \) to a new map \( \tilde{D}_n \) so that this identity still holds, and then construct \( D_{n-1} \) such that \( dD_n + D_{n-1}d = f_n \). Note first that

\[
(f_n - dD_n)d = d(f_{n+1} - D_nd) = d^2D_{n+1} = 0,
\]

so there is an induced map \( g_n: C_n/B_nC \to Y_n \). Now consider the bounded complex \( X \) of finitely presented modules with \( X_n = C_n/B_nC, X_{n-1} = C_{n-1}, X_{n-2} = C_{n-1}/B_{n-1}C, \) and \( X_i = 0 \) for all other \( i \). There is a chain map \( g: X \to Y \) that is \( g_n \) in degree \( n \), \( f_{n-1} \) in degree \( n - 1 \), and \( f_{n-2}d \) in degree \( n - 2 \). By the
Proof. In view of Proposition A.5, if $\text{Hom} D_n \subset D_n - 1 = f_n - dD_n$. Put another way, this means that
\[
d(D_n + D'_n) + D_{n-1}d = f_n
\]
as required. Furthermore, we still have the required relation
\[
dD_{n+1} + (D_n + D'_n)d = f_{n+1}
\]
because $D'_nd = 0$.

For the second half of the theorem, we need to show that $Z \otimes_R C$ is exact, where $Z$ is any pure exact complex of right $R$-modules. Since homology commutes with direct limits, we can again assume that $C$ is a bounded above complex of finitely generated free modules. But any bounded above complex is a direct limit of its truncations $C^{-n}$, where $(C^{-n})_k = C_k$ if $k > -n$, $(C^{-n})_{-n} = B_{-n}C$, and $(C^{-n})_k$ is 0 otherwise. Each truncation $C^{-n}$ is a bounded complex, so $Z \otimes_R C^{-n}$ is exact by Lemma A.3. Therefore the direct limit $Z \otimes_R C$ is exact.

Now, recall the character module of of an $R$-module $M$ is $\text{Hom}_Z(M, Q/Z)$.

**Proposition A.5.** Let $R$ be a ring, $C$ a chain complex, and $M$ a right $R$-module. Then $M \otimes_R C$ is exact if and only if $\text{Hom}_R(C, M^+)$ is exact.

**Proof.** Note that $M \otimes_R C$ is exact if and only if $(M \otimes_R C)^+$ is exact, and
\[
(M \otimes R C)^+ \cong \text{Hom}_R(C, M^+).
\]

If $R$ is a ring, $C$ is a collection of right $R$-modules, and $D$ is a collection of left $R$-modules, we say that $(C, D)$ is a **duality pair** if $M \in C$ if and only if $M^+$ is in $D$, and $N \in D$ if and only if $N^+ \in C$.

**Theorem A.6.** Let $R$ be a ring, and suppose $(C, D)$ is a duality pair such that $D$ is closed under pure quotients. Let $C$ be a complex of projectives. Then $M \otimes_R C$ is exact for all $M \in C$ if and only if $\text{Hom}_R(C, N)$ is exact for all $N \in D$.

**Proof.** In view of Proposition A.5, if $\text{Hom}_R(C, N)$ is exact for all $N \in D$, then $M \otimes_R C$ is exact for all $M \in C$. Conversely, suppose $M \otimes_R C$ is exact for all $m \in C$. Then if $N \in D, N^+ \otimes_R C$ is exact, and so Proposition A.5 tells us that $\text{Hom}(C, N^+)$ is exact. We conclude that $\text{Hom}_R(C, K)$ is exact for all $K \in D^+$, and we note that $D^+ \subset D$ since $(C, D)$ is a duality pair.

Now, for any $N$, the natural map $N \to N^+$ is a pure monomorphism [EJ00 Proposition 5.3.9]. So if $N \in D$, the quotient $N^+ / N$ is also in $D$ since $D$ is closed under pure quotients. We can therefore create a resolution of $N \in D$ by elements of $D^+$. That is, we can find a pure exact chain complex $X$ where $X_i = 0$ for $i > 0$, $X_0 = N$, and each of the $X_i$ for $i < 0$ is in $D^+$. This gives a short exact sequence
\[
0 \to S^0N \to X \to Y \to 0
\]
in which $X$ is pure exact and $Y$ is a bounded above complex with entries in $D^+$. Theorem A.1 tells us that $\text{Hom}(C, X)$ is exact. So to complete the proof it will suffice to show that $\text{Hom}(C, Y)$ is exact. If $Z$ is a bounded complex with entries in $D^+$, then we can prove $\text{Hom}(C, Z)$ is exact by induction on the number of nonzero entries in $Z$. In general, any bounded above complex $Y$ is the inverse limit of its truncations $Y^{-n}$ for $n \in \mathbb{N}$, where $(Y^{-n})_i = Y_i$ for $i \geq -n$ and is
0 otherwise. This is a very simple inverse limit, and so it is easy to check that $\text{Hom}(C, Y) = \lim \text{Hom}(C, Y^{-n})$. It follows that $\text{Hom}(C, Y)$ is exact, completing the proof. □

We now recover Theorem 6.7.

Corollary A.7. For any ring $R$, a complex of projectives $C$ is AC-acyclic if and only if it is firmly acyclic. If level $R$-modules all have finite projective dimension, these conditions are equivalent to $\text{Hom}_R(C, P)$ being exact for all projective $R$-modules $P$.

Proof. The classes of absolutely clean modules and level modules form a duality pair by Theorem 2.12. For the second statement, note that the collection of all modules $M$ such that $\text{Hom}_R(C, M)$ is exact is a thick subcategory, because $C$ is a complex of projectives. So if it contains projective modules, it contains all modules of finite projective dimension. □

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