Abstract. Motivated by Barría-Halmos’s [6, Question 19] and Halmos’s [22, Problem 237], we explore projections in Toeplitz algebra on the Hardy space. We show that the product of two Toeplitz (Hankel) operators is a projection if and only if it is the projection onto one of the invariant subspaces of the shift (backward shift) operator. As a consequence one obtains new proofs of criterion for Toeplitz operators and Hankel operators to be partial isometries. Furthermore, we completely characterize when the self-commutator of a Toeplitz operator is a projection. This provides a class of nontrivial projections in Toeplitz algebra.

1. Introduction

Let $\mathbb{D}$ be the open disk in the complex plane and $\mathbb{T}$ its boundary. The Hardy space $H^2$ is the subspace of $L^2 = L^2(\mathbb{T})$ consisting of functions whose Fourier coefficients corresponding to negative integers vanish. A function $\vartheta \in H^2$ is called an inner function if $|\vartheta(e^{i\theta})| = 1$ a.e.

For $\varphi$ in $L^\infty = L^\infty(\mathbb{T})$, the Toeplitz operator $T_\varphi$ with symbol $\varphi$ and the Hankel operator $H_\varphi$ with symbol $\varphi$ are defined on $H^2$ as the following:

$$T_\varphi f = P(\varphi f),$$

$$H_\varphi f = (I - P)(\varphi f), \quad f \in H^2,$$

where $P$ is the orthogonal projection of $L^2$ onto $H^2$. The Toeplitz algebra $\mathfrak{T}_{L^\infty}$ is the $C^*$-algebra generated by $\{T_\varphi, \varphi \in L^\infty\}$. We say that a bounded operator $Q$ on a Hilbert space is a projection if $Q$ satisfies

$$Q = Q^* = Q^2.$$

The study of projections, and applications of such study to illuminate structure of $C^*$-algebras, have been an enduring theme in operator algebra. In particular, progresses on projections in Toeplitz algebra will shed new light on the structure of $\mathfrak{T}_{L^\infty}$, for instance, compact perturbation or essential commutant problem [4, 13, 19, 11, 30], when a Hankel operator is in $\mathfrak{T}_{L^\infty}[5, 10]$, is Cesàro operator in $\mathfrak{T}_{L^\infty}[23]$?, etc.

In [6, Question 19], J. Barría and P. R. Halmos raised a problem:

“Which projections belong to $\mathfrak{T}_{L^\infty}$?”
They remarked that although the question is vague, it “might give a hint to a suitably general context in which Toeplitz theory can be embedded”, and in which problems in Toeplitz theory become “more manageable”. To better understand this problem, we first observe that if $T$ is a finite rank diagonal operator with diagonal entries equal to 0 or 1, then $T \in T\ell_\infty$, by the formula
\[ I - Tz_{n+1}T\bar{z}_{n+1} = z^n \otimes z^n (n \geq 0). \]
Are there any other projections in $T\ell_\infty$? For a unital $C^*$-algebra, the projections 0 and $I$ are trivial. The purpose of the current paper is to find more nontrivial projections in $T\ell_\infty$, and classify them in some sense.

It is easy to see that all finite sums of finite products of Toeplitz operators form a dense set in $T\ell_\infty$. For J. Barría and P. R. Halmos’ problem, we should find a condition for the operator
\[ \sum_{i=1}^{m} \prod_{j=1}^{n} T_{\varphi_{ij}} \]
A projection in $T\ell_\infty$. A. Brown and R. Douglas in [8, Theorem 2.6].

The central role in this work is played by the following theorem (see[14, 7.11] or[15, Theorem 2]):

**Symbol mapping.** Every operator in $T\ell_\infty$ is of the form
\[ T = Tf + S, \quad f \in L^\infty, S \in \mathcal{S} \]
where $\mathcal{S}$ is the semicommutator ideal generated by all semicommutators $Tfg - TfTg$, $f, g \in L^\infty$.

Since a Toeplitz operator is a projection if and only if it is 0 or $I$ [9, Corollary 5]. In the view of the symbol mapping theorem and the following important formula
\[ Tfg - TfTg = H_f^*H_g, \quad f, g \in L^\infty, \quad (1.1) \]
in what follows we shall consider that for which functions $f$ and $g$, $H_f^*H_g$ is a projection?

Let $\vartheta$ be a nonconstant inner function, the corresponding model space $K_\vartheta^2$ is defined to be
\[ K_\vartheta^2 = H^2 \ominus \vartheta H^2. \]
Moreover, $K_\vartheta^2$ is a nontrivial invariant subspace of $T^*_z$. In section 4, we show that if $H_f^*H_g$ is a projection, then it must be a projection onto a model space. This result covers the description of the partially isometric Hankel operators [25, Theorem 2.6].

For an operator $T$ on a separable Hilbert space $\mathcal{H}$, the self-commutator of $T$ is defined by $T^*T - TT^*$. The study of self-commutator has attracted much interest. For example, every self-adjoint operator on an infinite dimensional Hilbert space is the sum of two self-commutators [21] and Berger-Shaw’s theorem [7], etc. P. R. Halmos [22, Problem 237] asked that can $T^*T - TT^*$ be a projection, and, if so, how? He also proved that if $T$ is an abnormal operator (i.e., operators that have
no normal direct summands) and $\|T\| = 1$, such that self-commutator of $T$ is a projection, then $T$ is an isometry. It is still an interesting question for Toeplitz operator. Note that the self-commutator of $T_f$ is in $\mathcal{Z}_{L^\infty}$.

In section 5, we give the necessary and sufficient condition for the self-commutator of $T_f$ to be a projection when $T_f$ remains unrestricted. There are several difficulties in proving this result. One is that the symbol mapping theorem is fail to get the information of symbol $f$, since the corresponding symbol of $T_f^*T_f - T_fT_f^*$ is zero. Another is to obtain the range of $T_f^*T_f - T_fT_f^*$. We overcome these obstacles by linking hyponormal Toeplitz operators and truncated Toeplitz operators.

In section 6, we describe the $C^*$—algebra generated by $T_uT_{\bar{u}}$ for all inner functions $u$. We can now state our main results.

**Theorem 3.4** If $f, g \in L^\infty(\mathbb{T})$, then the following statements are equivalent.

1. $T_fT_g$ is a nontrivial projection;
2. $T_fT_g$ is a projection, and its range is a nontrivial invariant subspace of the shift operator $T_z$;
3. There exist a nonconstant inner function $\theta$ and a nonzero constant $a$ such that $f = a\theta$ and $g = \frac{\theta}{\bar{a}}$.

**Theorem 4.1** If $f, g \in L^\infty(\mathbb{T})$, then the following statements are equivalent.

1. $H^2_fH_g$ is a nontrivial projection operator;
2. The range of $H^2_fH_g$ is a model space $K^2_\theta$, where $\theta$ is an inner function;
3. $\bar{f} + \bar{\mu}\bar{\theta}, g + \frac{\bar{\mu}}{\mu} \in H^2$, where $\mu \in \mathbb{C} \setminus \{0\}$.

**Theorem 5.8** If $\varphi \in L^\infty(\mathbb{T})$, then $T_\varphi T_\varphi^* - T_\varphi T_\varphi^*$ is a nontrivial projection operator if and only if one of following conditions holds

1. The range of $T_\varphi^*T_\varphi - T_\varphi T_\varphi^*$ is a model space, and $\varphi = a\theta + b\bar{\theta} + c$, where $\theta$ is an inner function, $a, b$ and $c$ are constant with $|a|^2 - |b|^2 = 1$;
2. The range of $T_\varphi^*T_\varphi - T_\varphi T_\varphi^*$ is not a model space, and $\varphi = uv + \bar{v} + c$, where $u$ is inner, $c$ is constant, $v \in H^2$ with $|v|^2 = Re(uh + 1)(h \in H^2)$.

2. **Self-adjointness of $T_fT_g + T_gT_f$**

As a preparation, we obtain a necessary and sufficient condition for self-adjointness of $T_fT_g + T_gT_f$. The main tool is finite rank operators.

Given vectors $f$ and $g$ in a separable Hilbert space $\mathcal{H}$, we define the rank-one operator $f \otimes g$ mapping $\mathcal{H}$ into itself by

$$(f \otimes g)h = \langle h, g \rangle f. \quad (2.1)$$

The following properties of rank-one operators are well known.

**Lemma 2.1.** Given vectors $f$ and $g$ in a separable Hilbert space $\mathcal{H}$,

1. If $f \otimes g = 0$ if and only if either $f = 0$ or $g = 0$;
2. $(f \otimes g)^* = g \otimes f$;
3. For bounded operators $A$ and $B$, $A(f \otimes g)B = (Af) \otimes (B^*g)$.

**Lemma 2.2.** Given vectors $f$ and $g$ in a separable Hilbert space. If nonzero operator $f \otimes g$ is self-adjoint if and only if there is a nonzero real constant $\lambda$ such that $f = \lambda g$. 

Proof. Assume that $f \otimes g$ is self-adjoint, we have $f \otimes g = g \otimes f$, and therefore
\[
(f \otimes g)g = (g \otimes f)g,
\]
\[
\langle g, g \rangle f = \langle g, f \rangle g,
\]
\[
f = \frac{\langle g, f \rangle}{\langle g, g \rangle} g.
\]
If $\langle g, f \rangle = 0$, then $f$ is the zero vector. By Lemma 2.1(1), this contradict that $f \otimes g$ is a nonzero operator. Let $\lambda = \frac{\langle g, f \rangle}{\langle g, g \rangle} \neq 0$. Substituting $f = \lambda g$ into $f \otimes g = g \otimes f$,
\[
\lambda g \otimes g = \bar{\lambda} g \otimes g.
\]
(2.2)
Hence, $\lambda$ is a nonzero real number. The converse follows easily from (2.2). □

Lemma 2.3. Given vectors $f, g, \phi$ and $\psi$ in a separable Hilbert space. If operator $f \otimes g + \phi \otimes \psi$ is zero if and only if one of following statement hold
(1) either $f$ or $g$ is the zero vector and either $\phi$ or $\psi$ is the zero vector;
(2) $f, g, \phi$ and $\psi$ are all nonzero vectors, $f = \lambda \phi$ and $\psi = -\bar{\lambda} g$, $\lambda$ is a nonzero constant.

Proof. If one of four vectors $f, g, \phi$ and $\psi$ is zero, it is easy to see condition (1) hold, by Lemma 2.1(1).

Suppose that $f, g, \phi$ and $\psi$ are all nonzero vectors and $f \otimes g = -\phi \otimes \psi$, we have
\[
(f \otimes g)g = -(\phi \otimes \psi)g
\]
\[
\langle g, g \rangle f = -\langle g, \psi \rangle \phi
\]
\[
f = -\frac{\langle g, \psi \rangle}{\langle g, g \rangle} \phi.
\]
Let $\lambda = -\frac{\langle g, \psi \rangle}{\langle g, g \rangle}$, since $f$ is a nonzero vector, $\lambda \neq 0$. Write $f = \lambda \phi$, we have,
\[
f \otimes g + \phi \otimes \psi = \lambda \phi \otimes g + \phi \otimes \psi
\]
\[
= \phi \otimes (\bar{\lambda} g + \psi) = 0.
\]
Since $\phi$ is a nonzero vector and Lemma 2.1, $\bar{\lambda} g + \psi = 0$. It is easy to check that the converse is true. □

Lemma 2.4. Given vectors $f, g, \phi$ and $\psi$ in a separable Hilbert space $\mathcal{H}$. If $f \otimes g + \phi \otimes \psi$ is self-adjoint, then $\{f, g\}$ is linearly dependent if and only if $\{\phi, \psi\}$ is linearly dependent.

Proof. If one of $\{f, g, \phi, \psi\}$ is a nonzero vector, by Lemma 2.2, $\{f, g\}$ and $\{\phi, \psi\}$ are both linearly dependent.

Assume that $f, g, \phi$ and $\psi$ are four nonzero vectors and $\{f, g\}$ is linearly dependent, then there exist a nonzero constant $\lambda$, such that
\[
f = \lambda g.
\]
(2.3)
Since $f \otimes g + \phi \otimes \psi$ is self-adjoint,
\[
f \otimes g + \phi \otimes \psi = g \otimes f + \psi \otimes \phi,
\]
(2.4)
Substituting (2.3) into (2.4), we have
\[(\lambda - \bar{\lambda})g \otimes g = \psi \otimes \phi - \phi \otimes \psi.\]  \hspace{1cm} (2.5)

If \(\lambda\) is real, Lemma 2.1(2) implies \(\psi \otimes \phi\) is self-adjoint, by Lemma 2.2, we have \(\{\phi, \psi\}\) is linearly dependent.

When \(\lambda \neq \bar{\lambda}\), assume that \(\{\phi, \psi\}\) is linearly independent, by Gram-Schmidt procedure, there exist two nonzero vectors \(x\) and \(y\) such that
\[\langle x, \phi \rangle = 1, \langle x, \psi \rangle = 0,\]
\[\langle y, \psi \rangle = 1, \langle y, \phi \rangle = 0.\]

Applying operator equation (2.5) to \(x\) and \(y\) give
\[(\lambda - \bar{\lambda})\langle x, g \rangle g = \psi,\]
\[(\lambda - \bar{\lambda})\langle y, g \rangle g = -\phi.\]

Since \(\phi\) and \(\psi\) are nonzero vectors,
\[(\lambda - \bar{\lambda})\langle x, g \rangle \neq 0,\]
\[(\lambda - \bar{\lambda})\langle y, g \rangle \neq 0.\]

This contradicts our assumption (\(\{\phi, \psi\}\) is linearly independent). The rest of proof is the same as the above reasoning. \(\square\)

**Lemma 2.5.** Given nonzero vectors \(f, g, \phi, \psi\) in a separable Hilbert space. \(f \otimes g + \phi \otimes \psi\) is a nonzero self-adjoint operator if and only if one of following statement holds

1. \(f = \lambda g\) and \(\phi = \mu \psi\), where \(\lambda, \mu \in \mathbb{R} \setminus \{0\}\);
2. \(f = \lambda g, \phi = \mu \psi, \) and \(\psi = -ag\), where \(\lambda, \mu, a \in \mathbb{C} \setminus \{0\}, Im(\lambda) \neq 0, Im(\mu) \neq 0, |a|^2 \frac{Im(\mu)}{Im(\lambda)} = -1.\)
3. Both \(\{f, g\}\) and \(\{\phi, \psi\}\) are linearly independent,

   \[\phi = a_{11}f + a_{12}g\]
   \[\psi = a_{21}f + a_{22}g,\]

   where \(a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{C}.\ a_{11}a_{21}, a_{12}a_{22} \in \mathbb{R}, a_{12}a_{21} - a_{11}a_{22} = 1.\)

**Proof.** By Lemma 2.4, there are two cases to consider.

**Case I**

Assume that \(\{f, g\}\) and \(\{\phi, \psi\}\) are both linearly dependent, there are two nonzero constants \(\lambda\) and \(\mu\) such that
\[f = \lambda g, \quad \phi = \mu \psi.\]  \hspace{1cm} (2.6)

Since \(f \otimes g + \phi \otimes \psi\) is self-adjoint,
\[f \otimes g + \phi \otimes \psi = g \otimes f + \psi \otimes \phi\]  \hspace{1cm} (2.7)

Substituting (2.6) into (2.7), we have
\[(\lambda - \bar{\lambda})g \otimes g = (\bar{\mu} - \mu)\psi \otimes \psi.\]  \hspace{1cm} (2.8)
This means that $\lambda = \bar{\lambda}$ if and only if $\bar{\mu} = \mu$. If $Im(\lambda) = Im(\mu) = 0$, then

$$f \otimes g + \phi \otimes \psi = \lambda g \otimes g + \mu g \otimes \phi,$$

and $f \otimes g + \phi \otimes \psi$ is a self-adjoint operator. If $Im(\lambda)$ and $Im(\mu)$ both are nonzero, (2.8) becomes

$$g \otimes g + \frac{Im(\mu)}{Im(\lambda)} \psi \otimes \psi = 0.$$

By Lemma 2.3, we have

$$g = a \frac{Im(\mu)}{Im(\lambda)} \psi, \psi = -ag, a \in \mathbb{C} \setminus \{0\},$$

and $|a| \left( \frac{Im(\mu)}{Im(\lambda)} \right) = -1$.

**Case II**

If both \{f, g\} and \{\phi, \psi\} are linearly independent, by Gram-Schmidt procedure, there exist two nonzero vectors $x$ and $y$ such that

$$\langle y, \psi \rangle = 1, \langle y, \phi \rangle = 0,$$

$$\langle x, \phi \rangle = 1, \langle x, \psi \rangle = 0.$$

Applying operator equation (2.7) to $x$ and $y$ give

$$\phi = -\langle y, g \rangle f + \langle y, f \rangle g,$$

$$\psi = \langle x, g \rangle f - \langle x, f \rangle g.$$

Let $a_{11} = -\langle y, g \rangle, a_{12} = \langle y, f \rangle, a_{21} = \langle x, g \rangle$ and $a_{22} = -\langle x, f \rangle$. Write

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}. \tag{2.9}$$

Substituting (2.9) into (2.7), we have

$$f \otimes g + (a_{11} f + a_{12} g) \otimes (a_{21} f + a_{22} g)$$

$$= g \otimes f + (a_{21} f + a_{22} g) \otimes (a_{11} f + a_{12} g).$$

After simplifying we get

$$(a_{11} \bar{a}_{21} - a_{11} a_{21}) f + (a_{12} \bar{a}_{21} - a_{11} a_{22} - 1) g) \otimes f$$

$$= (a_{12} \bar{a}_{22} - a_{12} \bar{a}_{22} + (a_{12} a_{21} - a_{11} a_{22} - 1) f) \otimes g.$$

Since \{f, g\} is linearly independent and Lemma 2.3,

$$a_{11} \bar{a}_{21} - a_{11} a_{21} = 0,$$

$$a_{12} \bar{a}_{22} - a_{12} \bar{a}_{22} = 0,$$

$$\bar{a}_{12} a_{21} - a_{11} \bar{a}_{22} = 1.$$

The converse follows immediately from the above reasoning. □

Define an operator $V$ on $L^2$ by

$$V f(w) = \overline{w} f(w)$$
for \( f \in L^2 \). It is easy to check that \( V \) is anti-unitary. The operator \( V \) satisfies the following properties [29, Lemma 2.1]:

\[
V^2 = I, \\
VPV = (I - P), \\
VH_f V = H_f^*. 
\]

**Lemma 2.6.** If \( f \) and \( g \) are in \( L^\infty \), then

\[
T_z T_f T_g T_z - T_f T_g = (V H_f 1) \otimes (V H_g 1). 
\]

**Proof.** By the following identity:

\[
I - T_z T_f T_g T_z = 1 \otimes 1, 
\]

we have

\[
T_z T_f T_g T_z = T_z T_f (1 \otimes 1 + T_z T_z) T_g T_z \\
= T_z T_f (1 \otimes 1) T_g T_z + T_z T_f T_z T_g T_z \\
= T_z T_f (1 \otimes 1) T_g T_z + T_f T_g \\
= (T_z f 1) \otimes (T_z g 1) + T_f T_g. 
\]

On the other hand, one easily verifies that

\[
T_z f 1 = P z f 1 = PV \bar{f} = VP_{-} \bar{f} = VH_f 1, 
\]

Thus,

\[
T_z T_f T_g T_z - T_f T_g = (V H_f 1) \otimes (V H_g 1). 
\]

\( \square \)

Next, we present a proof of the result of K. Stroethoff [28, Theorem 4.4].

**Lemma 2.7.** If \( f, g, \phi \) and \( \psi \) are in \( L^\infty (\mathbb{T}) \), then \( T_f T_g + T_\phi T_\psi \) is a Toeplitz operator if and only if

\[
(V H_f 1) \otimes (V H_g 1) + (V H_\phi 1) \otimes (V H_\psi 1) = 0 
\]

if and only if one of the following cases holds:

1. either \( \bar{f} \) or \( g \) is analytic and either \( \bar{\phi} \) or \( \psi \) is analytic;
2. \( f - \lambda \phi \in \overline{H^2}, \psi + \lambda g \in H^2 \), where \( \lambda \in \mathbb{C} \setminus \{0\} \).

In this case, \( T_f T_g + T_\phi T_\psi = T_{fg + \phi \psi} \).

**Proof.** By [9, Theorem 6] and Lemma 2.6 we get that \( T_f T_g + T_\phi T_\psi \) is a Toeplitz operator if and only if

\[
T_z (T_f T_g + T_\phi T_\psi) T_z = T_f T_g + T_\phi T_\psi. 
\]

if and only if

\[
(V H_f 1) \otimes (V H_g 1) + (V H_\phi 1) \otimes (V H_\psi 1) = 0. 
\]

(2.10)

If (2.10) holds, Lemma 2.3 yields

1. either \( \bar{f} \) or \( g \) is analytic and either \( \bar{\phi} \) or \( \psi \) is analytic; or
2. \( f - \lambda \phi \in \overline{H^2}, \psi + \lambda g \in H^2 \), where \( \lambda \) is a constant.
Conversely, if either $\tilde{f}$ or $g$ is analytic and either $\tilde{\phi}$ or $\psi$ is analytic, by [9, Theorem 8], we have
\[ T_f T_g + T_{\phi} T_{\psi} = T_{fg + \phi \psi}. \]

An easy computation gives
\begin{equation}
T_f T_g + T_{\phi} T_{\psi} = T_f T_g - T_f g + T_f g + T_{\phi} \psi + T_{\phi} T_{\psi}
= -H_f^* H_g - H_{\phi}^* H_{\psi} + T_{fg + \phi \psi}.
\end{equation}

If $f - \lambda \phi \in \overline{H^2}$, $\psi + \lambda g \in H^2$, where $\lambda$ is a constant, then
\begin{align*}
- H_f^* H_g - H_{\phi}^* H_{\psi} \\
= - H_{\phi}^* H_{\phi} - H_{\phi}^* H_{\lambda g} \\
= - \lambda H_{\phi}^* H_g + \lambda H_{\phi}^* H_g = 0.
\end{align*}

\begin{lemma}
If $f, g, \phi$ and $\psi$ are in $L^\infty$, then $T_f T_g + T_{\phi} T_{\psi}$ is not a Toeplitz operator and is self-adjoint if and only if one of the following cases holds:
\begin{enumerate}
\item either $f$ or $g \in H^2$, $\tilde{\phi} \notin H^2$ and $\psi \notin H^2$, $\tilde{\phi} - a \psi \in H^2$, $a \in \mathbb{R} \setminus \{0\}$, $fg + \phi \psi$ is real-valued.
\item either $\phi$ or $\psi \in H^2$, $f \notin H^2$ and $g \notin H^2$, $\tilde{f} - bg \in H^2$, $b \in \mathbb{R} \setminus \{0\}$, $fg + \phi \psi$ is real-valued.
\item $\tilde{f}, g, \tilde{\phi}$ and $\psi$ are not in $H^2$, $fg + \phi \psi$ is real-valued.
\begin{enumerate}
\item $\tilde{f} - \lambda g \in H^2$ and $\tilde{\phi} - \mu \psi \in H^2$. where $\lambda, \mu \in \mathbb{C} \setminus \{0\}$;
\item $\text{Im}(\lambda) = \text{Im}(\mu) = 0$;
\item $\text{Im}(\lambda) \neq 0$ and $\text{Im}(\mu) \neq 0$, $\psi + cg \in H^2$, $c \in \mathbb{C} \setminus \{0\}$, $|c|^2 \frac{\text{Im}(\mu)}{\text{Im}(\lambda)} = -1$.
\item $\tilde{\phi} - a_{11} \tilde{f} - a_{12} g \in H^2$, and $\psi - a_{21} \tilde{f} - a_{22} g \in H^2$, where $a_{11}, a_{12}, a_{21}$ and $a_{22}$ are constant, $a_{11} a_{21}$ and $a_{12} a_{22}$ are real numbers, $a_{12} a_{21} - a_{11} a_{22} = 1$.
\end{enumerate}
\end{enumerate}
\end{lemma}

\begin{proof}
Assume that $T_f T_g + T_{\phi} T_{\psi}$ is not a Toeplitz operator and is self-adjoint, we have
\begin{align*}
T_f T_g + T_{\phi} T_{\psi} &= T_f T_g + T_{\phi} T_{\psi} \\
= T_f T_g + T_{\phi} T_{\psi}. \\
T_f T_g + T_{\phi} T_{\psi} &= T_f T_g + T_{\phi} T_{\psi}.
\end{align*}
By symbol map [14, 7.11] and [15, Theorem 2], we have $fg + \phi \psi$ is real-valued. By Lemma 2.6 and [9, Theorem 6] we get
\begin{align*}
(VH_f 1) \otimes (VH_g 1) + (VH_{\phi} 1) \otimes (VH_{\psi} 1)
= (VH_g 1) \otimes (VH_f 1) + (VH_{\psi} 1) \otimes (VH_{\phi} 1),
\end{align*}
and
\begin{align*}
T_f T_g + T_{\phi} T_{\psi} &= T_f T_g + T_{\phi} T_{\psi} \\
T_f T_g + T_{\phi} T_{\psi} &= T_f T_g + T_{\phi} T_{\psi}.
\end{align*}
Hence,
\begin{align*}
(VH_f 1) \otimes (VH_g 1) + (VH_{\phi} 1) \otimes (VH_{\psi} 1)
\end{align*}
is a nonzero self-adjoint operator.
If either $VH_f 1$ or $VH_g 1$ is the zero vector, and $VH_{\phi} 1$ and $VH_{\psi} 1$ are both nonzero vectors, then either $\tilde{f} \in H^2$ or $g \in H^2$, and $\tilde{\phi} \notin H^2$ and $\psi \notin H^2$. Thus
\((VH_{\bar{\phi}}1) \otimes (VH_{\psi}1)\) is a nonzero self-adjoint operator. By Lemma 2.2, we have \(\bar{\phi} - a\psi \in H^2, a \in \mathbb{R} \setminus \{0\}\).

Similarly, if either \(VH_{\bar{\phi}}1\) or \(VH_{\psi}1\) is the zero vector, and if both \(VH_f1\) and \(VH_g1\) are nonzero vectors, then either \(\bar{\phi}\) or \(\psi \in H^2, \bar{f} \notin H^2\) and \(g \notin H^2, \bar{f} - bg \in H^2, b \in \mathbb{R} \setminus \{0\}\).

If \(VH_f1, VH_g1, VH_{\bar{\phi}}1,\) and \(VH_{\psi}1\) are nonzero vectors, Lemma 2.5 now gives

(I) \(\bar{f} - \lambda g \in H^2\) and \(\bar{\phi} - \mu \psi \in H^2\). where \(\lambda, \mu \in \mathbb{C} \setminus \{0\}\);

(i) \(\lambda\) and \(\mu\) are real;

(ii) \(\text{Im}(\lambda) \neq 0\) and \(\text{Im}(\mu) \neq 0\), \(\psi + cg \in H^2, c \in \mathbb{C} \setminus \{0\}\), \(|c|\frac{2\text{Im}(\mu)}{\text{Im}(\lambda)} = -1\).

(II) \(\bar{\phi} - a_{11}\bar{f} - a_{12}g \in H^2, \psi - a_{21}\bar{f} - a_{22}g \in H^2\), where \(a_{11}, a_{12}, a_{21}\) and \(a_{22}\) are constant, \(a_{11}a_{21}\) and \(a_{12}a_{22}\) are real numbers, \(a_{11}a_{21} - a_{11}a_{22} = 1\).

To verify condition (1), an easy computation gives

\[
T_fT_g + T_{\bar{\phi}}T_{\psi} = T_{fg + \phi\psi} - T_{\phi\psi} + T_{\phi\psi} = T_{fg + \phi\psi} - H_{\phi}^*H_{\psi} = T_{fg + \phi\psi} - aH_{\psi}^*H_{\psi},
\]

\(T_{fg + \phi\psi} - aH_{\psi}^*H_{\psi}\) is self-adjoint, and condition (1) is verified.

Condition (2) is verified in the same way as condition (1).

To verify condition (3)(a)(i), using (5.3) we obtain

\[
T_fT_g + T_{\phi}T_{\psi} = T_{fg + \phi\psi} - H_f^*H_g - H_{\phi}^*H_{\psi} = T_{fg + \phi\psi} - \lambda H_g^*H_g - \mu H_{\psi}^*H_{\psi},
\]

therefore, \(T_{fg + \phi\psi} - \lambda H_g^*H_g - \mu H_{\psi}^*H_{\psi}\) is self-adjoint, and Condition (3)(a)(i) is verified.

To verify condition (3)(a)(ii): \(\bar{f}, g, \bar{\phi}\) and \(\psi\) are not in \(H^2, fg + \phi\psi\) is real-valued. \(\bar{f} - \lambda g \in H^2\) and \(\bar{\phi} - \mu \psi \in H^2\). where \(\lambda, \mu \in \mathbb{C} \setminus \{0\}\); \(\text{Im}(\lambda) \neq 0\) and \(\text{Im}(\mu) \neq 0\), \(\psi + cg \in H^2, c \in \mathbb{C} \setminus \{0\}\), \(|c|\frac{2\text{Im}(\mu)}{\text{Im}(\lambda)} = -1\). Again using (5.3) we obtain

\[
T_fT_g + T_{\phi}T_{\psi} = T_{fg + \phi\psi} - H_f^*H_g - H_{\phi}^*H_{\psi} = T_{fg + \phi\psi} - \lambda H_g^*H_g - \mu H_{\psi}^*H_{\psi} = T_{fg + \phi\psi} - (\bar{\lambda} + \bar{\mu}|c|^2)H_g^*H_g
\]

Since \(|c|^2\frac{2\text{Im}(\mu)}{\text{Im}(\lambda)} = -1, \bar{\lambda} + \bar{\mu}|c|^2\) is a real constant, \(T_{fg + \phi\psi} - (\bar{\lambda} + \bar{\mu}|c|^2)H_g^*H_g\) is self-adjoint, Condition (3)(a)(ii) is verified.

To verify condition (3)(b): \(\bar{\phi} - a_{11}\bar{f} - a_{12}g\) and \(\psi - a_{21}\bar{f} - a_{22}g\) are in \(H^2\), where \(a_{11}, a_{12}, a_{21}\) and \(a_{22}\) are constant, \(a_{11}a_{21}\) and \(a_{12}a_{22}\) are real numbers, \(a_{11}a_{21} -
that either subset \( E \) or \( T \) is unbounded.

Proof. Suppose \( T \) is a projection. Then \( T \) must be 0 and \( \phi = 0 \).

Lemma 3.1. Let \( R \) be a Hankel operator on \( H^2 \).

(1) \( \ker R \) is an invariant subspace of \( T_z \);

(2) \( R \) has nontrivial kernel if and only if the symbol of \( R \) has the form \( \bar{\theta} \phi \) where \( \theta \) is some inner function and \( \phi \in H^\infty \). Furthermore:

(a) \( \ker H_{\bar{\theta}\phi} = \theta H^2 \;
\)(b) \( H^*_{\bar{\theta}\phi} = z\theta H^2 \);

(c) \( \ker \left\{ \text{Range}(H^*_{\bar{\theta}\phi}) \right\} = (\ker H_{\bar{\theta}\phi})^\perp = H^2 \ominus \theta H^2 = K^2_\theta \).

Lemma 3.2. Let \( f, g \in L^\infty(\mathbb{T}) \). If \( T_fT_g \) is a nontrivial idempotent, then \( fg = 1 \) a.e. on \( \mathbb{T} \).

Proof. Suppose \( T_fT_g \) is a nontrivial idempotent, namely, \( (T_fT_g)^2 = T_fT_g \). By symbol map \( \{14, \text{Theorem 7.11}\} \), we have \( (fg)^2 = fg \). Then there exists a measurable subset \( E \) of \( \mathbb{T} \) such that

\[
(fg)(e^{i\theta}) = \begin{cases}
1, & e^{i\theta} \notin E, \\
0, & e^{i\theta} \in E.
\end{cases}
\]

If \( m(E) > 0 \), then there exists a subset \( E_1 \) of \( E \) with positive measure, such that either \( f|_{E_1} = 0 \) or \( g|_{E_1} = 0 \).

If \( f|_{E_1} = 0 \), by Guo’ Lemma \( \{20, \text{Lemma 1}\} \), then \( \ker T_f = \ker T_f = \{0\} \).

Since \( T_fT_g \) is a nontrivial idempotent, \( \ker T_fT_g \neq \{0\} \). For any nonzero vector \( x \in \ker T_fT_g \), we have \( T_gx = 0 \), hence \( \ker T_g \neq \{0\} \). By Coburn’ Lemma \( \{14, \text{7.24}\} \), \( \ker T_g^* = \ker T_g = \{0\} \).

Since \( T_fT_g \) is a nontrivial idempotent, \( (T_fT_g)^* = T_gT_f \) is also a nontrivial idempotent. Hence \( \ker T_gT_f \neq \{0\} \), it is a contradiction.

If \( g|_{E_1} = 0 \), same considerations apply to \( T_gT_f \), we can also get a contradiction. Hence \( m(E) = 0 \).

Lemma 3.3. If a Toeplitz operator is a projection, it must be 0 and 1.

Proof. By \( \{9, \text{Corollary 5}\} \), the only idempotent Toeplitz operators are 0 and 1.

Hence, a Toeplitz operator cannot be a nontrivial projection.
**Theorem 3.4.** If \( f, g \in L^\infty(\mathbb{T}) \), then the following statements are equivalent.

1. \( T_f T_g \) is a nontrivial projection;
2. \( T_f T_g \) is a projection, and its range is a nontrivial invariant subspace of the shift operator \( T_z \);
3. There exist a nonconstant inner function \( \theta \) and a nonzero constant \( a \) such that \( f = a\theta \) and \( g = \overline{\theta}a \).

**Proof.** (1) \( \Rightarrow \) (2): Suppose \( T_f T_g \) is a nontrivial projection, by Lemma 2.8 (2), we have \( \overline{f} = \lambda g + h, \lambda \in \mathbb{R} \setminus \{0\}, h \in H^2 \). \( T_f T_g \) is a nontrivial projection if and only if \( I - T_f T_g \) is a nontrivial projection. By Lemma 3.2, we have \( I = T_f T_g \). Hence

\[
I - T_f T_g = T_f g - T_f T_g
= H_f^* H_g
= (\lambda H_g^* + H_h) H_g
= \lambda H_g^* H_g,
\]

thus \( \lambda H_g^* H_g \) is a nontrivial projection.

By Lemma 3.1 (1), \( \ker H_g \) is an invariant subspace of shift operator \( T_z \). Moreover,

\[
\ker H_g = \ker H_g H_g = \ker \lambda H_g^* H_g = \ker (I - T_f T_g) = \text{Range}(T_f T_g).
\]

Therefore, the range of \( T_f T_g \) is a nontrivial invariant subspace of the shift operator \( T_z \).

(2) \( \Rightarrow \) (3): By Beurling’s theorem [14, 6.11], \( \text{Range}(T_f T_g) = \theta H^2 \) for some nonconstant inner function \( \theta \). \( T_\theta T_\theta \) is the orthogonal projection of \( L^2 \) onto \( \theta H^2 \). Hence

\[
T_f T_g = T_\theta T_\theta.
\]

By Lemma 2.7, we have

\[
f - a\theta \in \overline{H^2}, \overline{\theta} - ag \in H^2, a \in \mathbb{C} \setminus \{0\}.
\]

Note that

\[
T_{\overline{a}f} T_{ag} = T_f T_g,
\]

let

\[
F \triangleq \frac{1}{a} f = \theta + \varphi,
G \triangleq ag = \psi + \bar{\theta}, \tag{3.1}
\]

where \( \varphi \) and \( \psi \) are in \( H^\infty \). Since Lemma 3.2, \( FG = 1, \overline{\theta} F \theta G = 1 \) and \( \theta \bar{F} \theta G \) is an inner function.

If \( \theta \bar{F} \theta G \neq 1 \), then \( \text{Re}(1 - \theta \bar{F} \theta G) > 0 \), by [24, Part A. 4.2.2], we have \( 1 - \theta \bar{F} \theta G \) is outer. Using (3.1), then

\[
1 - \theta \bar{F} \theta G = 1 - \theta (\bar{\theta} + \varphi) \theta (\bar{\theta} + \psi)
= 1 - (1 + \theta \varphi)(1 + \theta \psi)
= -\theta (\varphi + \psi + \varphi \psi),
\]
it is a contradiction. Hence $\theta F \theta G = 1$. Note that $\theta F$ and $\theta G$ are in $H^\infty$, by [14, 6.20], $\theta F$ and $\theta G$ are outer functions. Since $\theta F \theta G = 1 = \bar{\theta} F \theta G$, $\theta F = \frac{1}{\bar{\theta} G}$, $\theta F = (\frac{1}{\bar{\theta} G})$, and $\theta F$ and $\theta G$ are real-valued functions in $H^\infty$, there exists a nonzero real constant $c$ such that

$$F = c\theta \quad \text{and} \quad G = \frac{\bar{\theta}}{c}.$$ 

Combining this with (3.1), we arrive at

$$(c - 1)\theta = \varphi \quad \text{and} \quad \frac{1}{c} - 1)\bar{\theta} = \psi.$$ 

Since $\theta$ is not a constant, $c = 1$, it follows that

$$F = \theta \quad \text{and} \quad G = \bar{\theta}.$$ 

From (3.1), we have

$$f = a\theta \quad \text{and} \quad g = \frac{\bar{\theta}}{a}.$$ 

(3) $\Rightarrow$ (1): Suppose $f = a\theta$ and $g = \frac{\theta}{a}$. Then

$$T_f T_g = T_\theta T_{\bar{\theta}}.$$ 

Hence $T_f T_g$ is a nontrivial projection operator. \hfill \Box

Remark 3.5. Widom [14, 7.46] proved that the spectrum of a Toeplitz operator is a connected subset of complex plane. It is natural to ask whether the spectrum of the product of two Toeplitz operator is connected? Since the spectrum of a projection operator is $\{0, 1\}$, by Theorem 3.3, the answer to the question is negative.

Lemma 3.6. [16, Theorem 7.22] Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and $A$ an operator in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ Then the following are equivalent:

1. $A$ is a partial isometry;
2. $A^*$ is a partial isometry;
3. $AA^*$ is an orthogonal projection, $AA^* = P_{(\ker A^*)^\perp}$;
4. $A^*A$ is an orthogonal projection, $A^*A = P_{(\ker A)^\perp}$.

Using Theorem 3.4, we present a new proof of the result of A. Brown and R. Douglas [8].

Corollary 3.7. If $f \in L^\infty$ then the following statements are equivalent.

1. $T_f$ is a partial isometry;
2. $T_f^*$ is a partial isometry;
3. either $f$ or $\bar{f}$ is inner.

Proof. Using Lemma 3.6 and Theorem 3.4. \hfill \Box
4. The product of two Hankel operators is a projection

**Theorem 4.1.** If \( f, g \in L^\infty \) then the following statements are equivalent.

1. \( H_f^*H_g \) is a nontrivial projection operator;
2. The range of \( H_f^*H_g \) is a model space \( K^2_\theta \), where \( \theta \) is an inner function;
3. \( \bar{f} + \bar{\theta} \bar{g} + \frac{\bar{\theta}}{\mu} g \in H^2 \), where \( \mu \in \mathbb{C} \setminus \{0\} \).

**Proof.** (1) \( \Rightarrow \) (2): We can suppose \( T_{fg} - T_f T_g \) is a nontrivial projection because \( H_f^*H_g = T_{fg} - T_f T_g \). By Lemma 2.8(2), we have

\[
\bar{f} = \lambda g + h, \quad \lambda \in \mathbb{R} \setminus \{0\}, \quad h \in H^2.
\]  

Moreover,

\[
T_{fg} - T_f T_g = H_f^*H_g = (\lambda H_g^* + H_h) H_g = \lambda H_g^* H_g.
\]

Hence,

\[
\ker(H_f^*H_g) = \ker(\lambda H_g^* H_g) = \ker H_g.
\]

By Lemma 3.1 (1), \( \ker H_g \) is an invariant subspace of shift operator \( T_z \), by Beurling’s theorem [14, 6.11], \( \ker(H_f^*H_g) = \theta H^2 \) for some nonconstant inner function \( \theta \). \( T_\theta T_\bar{\theta} \) is the orthogonal projection of \( L^2 \) onto \( \theta H^2 \). Hence

\[
\lambda H_g^* = I - T_\theta T_\bar{\theta},
\]

\[
\lambda(T_{\theta g} - T_\theta T_{\bar{\theta} g}) = I - T_\theta T_\bar{\theta},
\]

\[
T_{\lambda |\theta|^2 - 1} = T_{\theta g} - T_\theta T_{\bar{\theta} g}.
\]

Since projection operator is positive, \( \lambda > 0 \). By Lemma 2.7, we have

\[
\lambda g + \mu \theta \in \overline{H^2}, \quad \bar{\theta} + \mu g \in H^2, \quad \mu \in \mathbb{C} \setminus \{0\}.
\]  

(4.2)

Hence,

\[
(\lambda - |\mu|^2)g \in H^2.
\]

If \( \lambda \neq |\mu|^2 \), then \( g \in H^2 \) and \( H_g = 0 \). By assumption \( T_{fg} - T_f T_g = \lambda H_g^* H_g \) is a nontrivial projection, so \( \lambda = |\mu|^2 \). Using (4.2), we have

\[
g + \frac{\bar{\theta}}{\mu} \in H^2.
\]  

(4.3)

Combining (4.3) with (4.1) gives

\[
f + \mu \theta \in \overline{H^2}.
\]

(4) \( \Rightarrow \) (1): Suppose \( f + \mu \theta \in \overline{H^2} \), \( g + \frac{\bar{\theta}}{\mu} \in H^2 \), \( \mu \in \mathbb{C} \setminus \{0\} \). Then

\[
H_f^*H_g = H_f^* H_\mu H_{\bar{\mu}}
\]

\[
= H_\theta H_{\bar{\theta}}
\]

\[
= I - T_\theta T_{\bar{\theta}}.
\]
We next derive an alternative proof of [25, Theorem 2.6].

**Corollary 4.2.** If $f \in L^\infty$ then $H_f$ is a partial isometry if and only if $\bar{f}$ is inner.

**Proof.** Using Lemma 3.6 and Theorem 4.1. □

5. **Projection as self-commutators of Toeplitz operators**

The problem 237 in Paul R. Halmos’s famous text: A Hilbert space problem book [22] states: can $T^*T - TT^*$ be a projection and, if so, how? He discusses the following two cases.

(a) If $T$ is an abnormal operator of norm 1, such that $T^*T - TT^*$ is a projection, then $T$ is an isometry.

(b) Does the statement remain true if the norm condition is not assumed?

In particular, if $T$ is a Toeplitz operator. Let $f \in L^\infty(T)$, we consider that when is $T_f^*T_f - T_fT_f^*$ a projection?

Define

$$Q = T_f^*T_f - T_fT_f^*.$$ 

**Example 5.1.** Corresponding case (a), we next show that if there is a constant $\lambda$ such that $\|T_{f+\lambda}\| \leq 1$ and $Q$ is a nontrivial projection, then $T_{f+\lambda}$ is an isometry. Note that

$$T_{f+\lambda}T_{f+\lambda} - T_{f+\lambda}T_{f+\lambda}^* = T_f^*T_f - T_fT_f^*, \quad \lambda \in \mathbb{C}.$$ 

Using the idea of [22, Solution 237] and $\|T_f\| = \|f\|_\infty$ we have

$$\|h\|^2 \geq \|T_{f+\lambda}h\|^2 = \langle T_{f+\lambda}^*T_{f+\lambda}h, h \rangle = \langle T_{f+\lambda}^*T_{f+\lambda}h, h \rangle + \langle Qh, h \rangle$$

$$= \|T_{f+\lambda}^*h\|^2 + \|Qh\|^2.$$ 

Replace $h$ by $Qx(x \in H^2)$ in the above formula, we have $T_{f+\lambda}^*Q = 0$ and $T_{f+\lambda}$ is quasinormal. A Theorem in [3] tells us that a quasinormal Toeplitz operator is either normal or analytic and $f + \lambda = c\theta$, where $c$ is a constant and $\theta$ is an inner function. $Q$ is a nontrivial projection, we have $f + \lambda = c\theta$. Hence,

$$Q = T_f^*T_f - T_fT_f^*$$

$$= T_f^2 - T_fT_f$$

$$= H_f^*H_f$$

$$= |c|^2 H_{\bar{\theta}}^*H_{\bar{\theta}}.$$ 

Since $Q$ is an idempotent, $|c| = 1$. By [9, Corollary 3], $T_{f+\lambda}$ is an isometry if and only if $f + \lambda$ is an inner function. In this case, $Q = H_{\bar{\theta}}^*H_{\bar{\theta}}$ is the projection onto model space $K_{\theta}^2$.

**Example 5.2.** Let us recall Abrahamse’s theorem [1]. If

1. $f$ or $\bar{f}$ is of bounded type;
2. $T_f$ is hyponormal;
3. $\ker Q$ is invariant for $T_f$.
then $T_f$ is normal or analytic.

Using the above theorem, if

1. $f$ or $\bar{f}$ is of bounded type;
2. $Q$ is a nontrivial projection;
3. $\ker Q$ is invariant for $T_f$,

then $f$ is analytic. Hence,

$$Q = T_f^*T_f - T_fT_f^* = T_{|f|^2} - T_f T_{\bar{f}} = H_f^*H_{\bar{f}}.$$

By Theorem 4.1, there is a constant $c$ such that $f = \theta + c$, where $\theta$ is an inner function. In this case, $Q = H_{\bar{\theta}}^*H_{\bar{\theta}}$ is the projection onto model space $K_{\theta}^2$.

From the above two examples, we need to consider two things: if $Q$ is a non-trivial projection,

1. when is the range of $Q$ a model space?
2. is the range of $Q$ necessarily a model space?

**Lemma 5.3.** If $\varphi \in L^\infty$ then $T_{\varphi}^*T_{\varphi} - T_{\varphi}T_{\varphi}^*$ is the projection on to a model space $K_{\theta}^2$ if and only if $\varphi = a\theta + b\bar{\theta} + c$, where $a, b$ and $c$ are constant with $|a|^2 - |b|^2 = 1$.

**Proof.** If $\varphi = a\theta + b\bar{\theta} + c$, where $a, b$ and $c$ are constant with $|a|^2 - |b|^2 = 1$, then

$$T_{\varphi}^*T_{\varphi} - T_{\varphi}T_{\varphi}^* = T_{\varphi}^*T_{\varphi} - T_{\varphi}\bar{\varphi} + T_{\varphi}\bar{\varphi} - T_{\varphi}T_{\varphi}^* = H_{\varphi}^*H_{\varphi} - H_{\varphi}^*H_{\varphi} = (|a|^2 - |b|^2)H_{\bar{\theta}}^*H_{\bar{\theta}} = H_{\bar{\theta}}^*H_{\bar{\theta}}.$$

Conversely, suppose $T_{\varphi}^*T_{\varphi} - T_{\varphi}T_{\varphi}^*$ is the projection on to a model space $K_{\bar{\theta}}^2$, then

$$T_{\varphi}^*T_{\varphi} - T_{\varphi}T_{\varphi}^* = I - T_{\theta}T_{\bar{\theta}}.$$

Write $\varphi = f + \bar{g}$, $f$ and $g$ in $H^2$, using Lemma 2.6, we have

$$(VH_{\theta}1) \otimes (VH_{\bar{\theta}}1) - (VH_f1) \otimes (VH_{\bar{f}}1) = -(VH_{\bar{\theta}}1) \otimes (VH_{\bar{\theta}}1).$$

**Case 1.**

Assume that $\{H_{\bar{\theta}}1, H_f1\}$ is linearly dependent, there are two constants $k_1$ and $k_2$ such that

$$k_1H_{\bar{\theta}}1 + k_2H_f1 = 0.$$

If $k_1$ is not zero, let $\lambda = \frac{k_2}{k_1}$, then $\bar{g} - \lambda\bar{f} \in H^2$ and

$$T_{\varphi}^*T_{\varphi} - T_{\varphi}T_{\varphi}^* = H_{\varphi}^*H_{\varphi} - H_{\varphi}^*H_{\varphi} = H_f^*H_f - H_{\bar{\theta}}^*H_{\bar{\theta}} = (1 - |\lambda|^2)H_f^*H_f.$$
Then \((1 - |\lambda|^2)H_f^*H_f = H_\theta^*H_\theta\) is a projection, and \(1 - |\lambda|^2 > 0\). By Theorem 4.1, we have \(f + \frac{\mu}{\sqrt{1 - |\lambda|^2}} \theta \in \mathcal{H}^2, \mu\) is unimodular constant. Therefore, \(\varphi = -\frac{\mu}{\sqrt{1 - |\lambda|^2}} \theta - \frac{\lambda \mu}{\sqrt{1 - |\lambda|^2}} \theta + c\), where \(c\) is a constant. Let \(a = -\frac{\mu}{\sqrt{1 - |\lambda|^2}}\) and \(b = -\frac{\lambda \mu}{\sqrt{1 - |\lambda|^2}}\), we have
\[
\varphi = a\theta + b\bar{\theta} + c, \tag{5.2}
\]
where \(|a|^2 - |b|^2 = 1\). If \(k_2\) is not zero, repeating the previous reasoning, we can prove the same equality (5.2) hold.

**Case 2.**

Assume that \(\{H_{\bar{g}}, H_f\}\) is linearly independent. Since \(V\) is anti-unitary, \(\{VH_{\bar{g}}, VH_f\}\) is linearly independent, by Gram-Schmidt procedure, there exist a nonzero function \(x_0\) in span \(\{VH_{\bar{g}}, VH_f\}\) such that
\[
\langle VH_{\bar{g}}, x_0 \rangle = 1, \quad \langle VH_f, x_0 \rangle = 0.
\]
Applying operator equation (5.1) to \(x_0\) gives
\[
VH_{\bar{g}} = - \langle x_0, VH_{\bar{g}} \rangle VH_{\bar{g}}, \quad VH_f = - \langle VH_{\bar{g}}, x_0 \rangle H_{\bar{g}}. 
\]
Let \(b = -\langle x_0, VH_{\bar{g}} \rangle\), thus \(g - b\theta \in \mathcal{H}^2\), and \(g - b\theta\) is a constant.

Similarly, there exists a constant \(a\) such \(f - a\theta\) is a constant. Therefore,
\[
T_\varphi^*T_\varphi - T_\varphi T_\varphi^* = H_f^*H_f - H_{\bar{g}}^*H_{\bar{g}} = (|a|^2 - |b|^2)H_{\bar{g}}^*H_{\bar{g}},
\]
and \(|a|^2 - |b|^2 = 1\). \(\square\)

Recall the definition of truncated Toeplitz operator. For \(\varphi\) in \(L^2(\mathbb{T})\), the truncated Toeplitz operator \(A_\varphi^\theta\) is densely defined on \(K_\varphi^\theta\) by
\[
A_\varphi^\theta f = (P - T_\theta T_\varphi)(\varphi f).
\]
The algebraic properties of truncated Toeplitz operator will play key role in the following Lemma.

**Lemma 5.4.** If \(\varphi \in L^\infty\) then \(T_\varphi^*T_\varphi - T_\varphi T_\varphi^*\) is a nontrivial projection operator and its range is not a Model space if and only if \(\varphi = uv + \bar{v} + a\), where \(u\) is inner, \(v \in \mathcal{H}^2\) with \(|v|^2 - 1 \in u\mathcal{H}^2 + u\mathcal{H}^2\) and \(a\) is constant.

**Proof.** Assume that \(T_\varphi^*T_\varphi - T_\varphi T_\varphi^*\) is a projection. Since projection is positive, \(T_\varphi^*T_\varphi - T_\varphi T_\varphi^*\) is positive and \(T_\varphi\) is hyponormal. We recall the characterization of Hyponormality of Toeplitz operators form Carl C. Conwen [12]. The theorem can be stated as follows:

If \(\varphi\) is in \(L^\infty(\mathbb{T})\), where \(\varphi = f + \bar{g}\) for \(f\) and \(g\) in \(\mathcal{H}^2\), then \(T_\varphi\) is hyponormal if and only if
\[
g = c + T_uf \tag{5.3}
\]
for some constant \(c\) and some function \(u\) in \(H^\infty\) with \(\|u\|_\infty \leq 1\).
According to Conwen’s Theorem, if $Q$ is a nontrivial projection, using (5.3), we have

$$Q = H^*_f(I - S_u S_u)^{1/2} H_f,$$

$$= H^*_f(I - S_u S_u)^{1/2} H_f,$$

$$= H^*_f(I - S_u S_u)^{1/2} H_f,$$

$$= H^*_f(I - S_u S_u) H_f,$$

$$= H^*_f(I - S_u S_u) H_f,$$

$$= H^*_f(I - S_u S_u) H_f,$$

where $S_u x = P_+(ux), x \in (H^2)^\perp$.

Since $\|u\|\infty \leq 1$ and $\|S_u\| = \|u\|\infty, S_u$ is a contraction, we have $I - S_u S_u$ is positive, and $\ker(I - S_u S_u) = \ker(I - S_u S_u)^{1/2}$.

We claim that if $I - S_u S_u$ is not injective, then $u$ is an inner function. To see this, let $x$ be a nonzero vector such that $(I - S_u S_u)x = 0$.

Hence,

$$\langle (I - S_u S_u)x, x \rangle = \langle x, x \rangle - \langle S_u x, S_u x \rangle$$

$$= \|x\|^2 - \|S_u x\|^2 = 0$$

and

$$\int_T |x|^2 dm = \|x\|^2 = \|S_u x\|^2 = \|P_- ux\|^2 \leq \|ux\|^2 = \int_T |ux|^2 dm.$$

Since $\|u\|\infty \leq 1$,

$$|ux|^2 - |x|^2 = (|u|^2 - 1)|x|^2 \leq 0.$$

But $\int_T (|u|^2 - 1)|x|^2 dm \geq 0$, thus $(|u|^2 - 1)|x|^2 = 0.a.e$ on $T$. Hence, $|u| = 1.a.e$ on $T$, and $u$ is an inner function.

Write

$$Q = H^*_f(I - S_u S_u)^{1/2} (I - S_u S_u)^{1/2} H_f,$$

$$= ((I - S_u S_u)^{1/2} H_f)^* (I - S_u S_u)^{1/2} H_f,$$

note that $\ker Q = \ker((I - S_u S_u)^{1/2} H_f)$. According to the above claim, we have that if $u$ is not an inner function, then $\ker((I - S_u S_u)^{1/2} = \ker(I - S_u S_u) = \{0\}$ and $\ker Q = \ker H_f$. By Lemma 3.1(1), $H_f$ is an invariant subspace of $T_z$. Hence, the range of $Q$ is a model space, it is a contradiction.

It remains to consider the case that $u$ be an inner function. Write

$$Q = H^*_f(I - S_u S_u) H_f$$

$$= H^*_f(S_uu - S_u S_u) H_f$$

$$= H^*_f H_uH^*_u H_f.$$

By Gu’s theorem [18, Theorem 1.1], for two Hankel operators $H_u$ and $H_f$, either $\ker H^*_u H_f = \ker H_f$ or $\ker H^*_f H_u = \ker H_u$. 
If \( \ker H_\alpha^* H_f = \ker H_f \), then \( \ker Q = \ker H_\alpha^* H_f = \ker H_f \). By Lemma 3.1 (1), \( \ker H_f \) is an invariant subspace of \( T_z \). Hence, the range of \( Q \) is a model space, it is a contradiction.

By Lemma 3.6, \( H_\alpha^* H_a H_\alpha^* H_f \) is an orthogonal projection, then \( H_\alpha^* H_f H_\alpha^* H_a \) is an orthogonal projection.

If \( \ker H_f H_a = \ker H_a = uH^2 \) (Lemma 3.1 (2)(a)), then
\[
H_\alpha^* H_f H_\alpha^* H_a f = H_\alpha^* H_a. \tag{5.4}
\]

Using the property \( V \), we have
\[
VH_\alpha^* H_f H_\alpha^* H_a V = H_\alpha H_f H_\alpha^* H_a,
\]
\[
VH_\alpha^* H_a V = H_\alpha H_a^*.
\]

Hence
\[
H_\alpha H_f H_\alpha^* H_a = H_\alpha H_a^*. \tag{5.5}
\]

Note that \( \ker H_a^* = \overline{zuH^2} \) (Lemma 3.1 (2)(a)) and \( \overline{zH^2} \oplus \overline{zuH^2} = \overline{zK^2_u} = \overline{\bar{u}K^2_u} \).

For every \( h \in K^2_u \), we have \( H_\alpha^* \bar{u}h = P(u\bar{u}h) = h \), and
\[
\langle H_\alpha^* H_f H_\alpha^* \bar{u}h, \bar{u}h \rangle = \langle H_\alpha H_\alpha^* \bar{u}h, \bar{u}h \rangle,
\]
\[
\langle H_\alpha^* H_f H_\alpha^* \bar{u}h, H_\alpha^* \bar{u}h \rangle = \langle H_\alpha^* \bar{u}h, H_\alpha^* \bar{u}h \rangle,
\]
\[
\langle H_f H_f h, h \rangle = \langle h, h \rangle.
\]

Hence
\[
P_{K^2_u}(H_\alpha^* H_f)|_{K^2_u} = I_{K^2_u},
\]
where \( P_{K^2_u} \) is the orthogonal projection onto \( K^2_u \) and \( I_{K^2_u} \) is the identity operator on \( K^2_u \).

An easy computation gives
\[
P_{K^2_u} H_f H_f h = P_{K^2_u} P f(I - P) \tilde{f} h
\]
\[
= P_{K^2_u} f(I - P) \tilde{f} h
\]
\[
= P_{K^2_u} f \tilde{f} h - P_{K^2_u} f P \tilde{f} h
\]
\[
= P_{K^2_u} f \tilde{f} h - P_{K^2_u} f(P - uP \bar{u} + uP \bar{u}) \tilde{f} h
\]
\[
= P_{K^2_u} f \tilde{f} h - P_{K^2_u} f(P_{K^2_u} + uP \bar{u}) \tilde{f} h
\]
\[
= P_{K^2_u} f \tilde{f} h - P_{K^2_u} f P_{K^2_u} \tilde{f} h
\]
\[
= A_{\tilde{f}h}^u - A^u_{\tilde{f}h}.
\]

Hence
\[
A_{\tilde{f}h}^u - A^u_{\tilde{f}h} = I_{K^2_u}, \tag{5.6}
\]
\[
A^u_{\tilde{f}h} = A^u_{\tilde{f}h - 1}.
\]

Since \( f \) is analytic, using N. A. Sedlock’ theorem [27, Theorem 5.2] leads to \( A^u_{\tilde{f}} = cI_{K^2_u} \), where \( c \) is a constant, and \( A^u_{\tilde{f} - c} \) is the zero operator, then \( f - c \in uH^2 \) [26, Theorem 3.1]. There is a function \( v \in H^2 \), such that \( f = c + uv \). Since (5.3), \( \varphi = uv + \bar{v} + a \), where \( a \) is a constant.
Substituting \( f = c + uv \) into (5.4), we have

\[
H_{\bar{a}}^*H_{\bar{u}\bar{v}}H_{\bar{a}}^*H_{\bar{a}} = H_{\bar{a}}^*H_{\bar{a}}. \tag{5.7}
\]

Repeating the above reasoning form (5.5) again, we have

\[
A_{uv}^uA_{uv}^u = A_{\|v\|^2-1}^u.
\]

Note that \( A_{uv}^u = A_{f-c}^u = 0 \), hence \( A_{\|v\|^2-1}^u \) is zero operator, using [26, Theorem 3.1] again, we have \( \|v\|^2 - 1 \in uH^2 + \overline{uH^2} \).

Conversely, if \( \varphi = uv + \bar{v} + c \), where \( u \) is inner, \( v \in H^2 \) with \( \|v\|^2 - 1 \in uH^2 + \overline{uH^2} \) and \( c \) is constant, by Lemma 5.3, the range of \( Q \) is not a model space. An easy computation gives

\[
T_{\bar{v}}T_{\bar{v}} - T_{\bar{v}}T_{\bar{v}} = H_{\bar{u}}^*H_{\bar{u}} - H_{\bar{u}}^*H_{\bar{u}} = T_{uv\bar{u}} - T_{uv\bar{u}} = (T_{v\bar{u}} - T_{vT\bar{v}})
\]

\[
= T_{v\|v\|^2 - 1} - T_{v\|v\|^2 - 1} = T_{v} - T_{vT\bar{v}}
\]

\[
= T_{v}(I - T_{u\bar{u}}T_{\bar{v}}
\]

\[
= T_{v}H_{\bar{u}}^*H_{\bar{u}}T_{\bar{v}}.
\]

Note that \( T_{v}H_{\bar{u}}^*H_{\bar{u}}T_{\bar{v}} = (H_{\bar{u}}T_{\bar{v}})^*H_{\bar{u}}T_{\bar{v}} \) is positive, must be self-adjoint.

It remains to show that \( T_{v}H_{\bar{u}}^*H_{\bar{u}}T_{\bar{v}} \) is an idempotent. Since \( v \) is analytic,

\[
T_{v}H_{\bar{u}}^*H_{\bar{u}}T_{\bar{v}}T_{v}H_{\bar{u}}^*H_{\bar{u}}T_{\bar{v}} = T_{v}H_{\bar{u}}^*H_{\bar{u}}T_{\|v\|^2}H_{\bar{u}}^*H_{\bar{u}}T_{\bar{v}}.
\]

let \( \|v\|^2 = uh + \bar{u}h + 1 \), \( h, h_1 \in H^2 \), for every \( k \) in \( K_{\bar{u}} \), we have

\[
H_{\bar{u}}^*H_{\bar{u}}T_{\|v\|^2}k = H_{\bar{u}}^*H_{\bar{u}}P(uh + \bar{u}h + 1)k
\]

\[
= H_{\bar{u}}^*H_{\bar{u}}P(uhk + \bar{u}h_1k + k)
\]

\[
= H_{\bar{u}}^*H_{\bar{u}}(uhk + k)
\]

\[
= k.
\]

Since \( \text{Range}(H_{\bar{u}}^*H_{\bar{u}}) = K_{\bar{u}} \), \( T_{v}H_{\bar{u}}^*H_{\bar{u}}T_{\bar{v}}T_{v}H_{\bar{u}}^*H_{\bar{u}}T_{\bar{v}} = T_{v}H_{\bar{u}}^*H_{\bar{u}}T_{\bar{v}}. \)

**Remark 5.5.** In fact, \( \|h\|_{\infty} = 1 \). Since \( \ker Q \) is nontrivial, there is a nonzero vector \( x \) such that

\[
H_{\bar{f}}^*H_{\bar{f}}x = H_{\bar{f}}^*H_{\bar{f}}x \neq 0, \quad \|H_{\bar{f}}x\| = \|H_{\bar{f}}x\|
\]

and \( \|H_{\bar{f}}x\| = \|S_{\bar{h}}H_{\bar{f}}x\| \leq \|S_{\bar{h}}\|\|H_{\bar{f}}x\|. \) Hence \( \|S_{\bar{h}}\| = \|h\|_{\infty} \geq 1 \).

**Lemma 5.6.** If \( v \in H^2 \) and \( u \) is inner, \( \|v\|^2 - 1 \in uH^2 + \overline{uH^2} \) if and only if there is a function \( h \in H^2 \) such that \( \|v\|^2 = Re(uh + 1) \).
Proof. Since
\[ Re(uh + 1) = \frac{1}{2}(uh + 1 + \bar{u}h + 1) = u\left(\frac{1}{2}h\right) + \bar{u}\left(\frac{1}{2}h\right) + 1, \]

\[ |v|^2 = Re(uh + 1) \implies |v|^2 - 1 \in uH^2 + \bar{u}H^2. \]

Suppose \(|v|^2 - 1 \in uH^2 + \bar{u}H^2\), then there exist \(F, G \in H^2\) such that \(|v|^2 - 1 = uF + \bar{u}G\), and \(uF + \bar{u}G\) is real-valued, \(uF + \bar{u}G = \bar{u}F + uG\). Hence,
\[ u(F - G) = \bar{u}(\bar{F} - G). \]

The left-hand side of the above equation is analytic, the right-hand side is conjugate analytic, \(u(F - G)\) is equals to a constant \(\lambda\). If \(\lambda\) is not zero, then \(u\frac{1}{\lambda}(F - G) = 1\), and \(u\) is outer [14, 6.20], that is a contradiction. Thus \(\lambda = 0, F = G\), and \(|v|^2 = Re(u(2F) + 1)\). \(\square\)

Remark 5.7. The set \(\Theta = \{v : v \in H^2, |v|^2 - 1 \in uH^2 + \bar{u}H^2\}\) is not empty. It is easy to see that if \(v\) is inner, \(v \in \Theta\). Using (5.8), we have \(Q = T_vT_{\bar{v}} - T_{uv}T_{\bar{uv}}\), and the range of \(Q\) is \(vH^2 \oplus vuH^2 = vK^2\). Moreover, \(u \pm 1 \in \Theta\).

The following theorem summarizes Lemma 5.3, Lemma 5.4 and Lemma 5.6.

Theorem 5.8. If \(\varphi \in L^\infty\) then \(T^*_\varphi T_{\varphi} - T_{\varphi}T^*_\varphi\) is a nontrivial projection operator if and only if one of following conditions holds

1. The range of \(T^*_\varphi T_{\varphi} - T_{\varphi}T^*_\varphi\) is a model space, and \(\varphi = ab + b\bar{\theta} + c\), where \(\theta\) is an inner function, \(a, b\) and \(c\) are constant with \(|a|^2 - |b|^2 = 1\);

2. The range of \(T^*_\varphi T_{\varphi} - T_{\varphi}T^*_\varphi\) is not a model space, and \(\varphi = uv + \bar{v} + c\), where \(u\) is inner, \(c\) is constant, \(v \in H^2\) with \(|v|^2 = Re(uh + 1)(h \in H^2)\).

6. Further discussion

Now we study the \(C^*\)-algebra \(\mathcal{T}_u\) generated by \(\{T_uT_{\bar{u}} : u\) is an inner function\}. Since the symbol mapping of every element in \(\mathcal{T}_u\) is constant, \(\mathcal{T}_u\) is a proper subalgebra of \(\mathcal{S}_{L^\infty}\). The following theorem will give some information of the structure of \(\mathcal{T}_u\).

Theorem 6.1. \(\mathcal{T}_u\) is irreducible and contains all compact operators.

Proof. Suppose that \(\mathcal{T}_u\) is reducible. Then there exists a nontrivial projection \(E\) which commutes with each \(T_uT_{\bar{u}}\) for all inner function \(u\). If \(u\) is a Möbius transform
\[ u = \varphi_z(w) = \frac{z - w}{1 - \bar{z}w}, \]
and \(k_z\) denote the normalized reproducing kernel at \(z : k_z(w) = \frac{\sqrt{1-|z|^2}}{1-\bar{z}w}\). We have the following identity:
\[ I - k_z \otimes k_z = T_{\varphi_z}T_{\varphi_z}, \tag{6.1} \]
the identity can be found in [31, p.480]. Hence,

\[ E(k_z \otimes k_z) = (k_z \otimes k_z)E \]

\[ (Ek_z) \otimes k_z = k_z \otimes (Ek_z) \]

\[ (Ek_z, k_z)Ek_z = (Ek_z, Ek_z)k_z \]

\[ \|Ek_z\|^2 Ek_z = \|Ek_z\|^2 k_z. \]

If \( Ek_z \) is not a zero vector, we have \( Ek_z = k_z \). Thus unit disc \( \mathbb{D} \) is the disjoint union of two sets, say \( \mathbb{D} = \Sigma_1 \cup \Sigma_2 \), where \( \Sigma_1 = \{ z \in \mathbb{D} : Ek_z = 0 \} \) and \( \Sigma_2 = \{ z \in \mathbb{D} : Ek_z = k_z \} \). So, at least one of \( \Sigma_1 \) and \( \Sigma_2 \) is an uncountable set. at least of \( \{ k_z : z \in \Sigma_1 \} \) and \( \{ k_z : z \in \Sigma_2 \} \) is dense in \( H^2 \). Hence, \( E \) is zero operator or identical operator, which is a contradiction. Using (6.1), we have \( T_u \) contains at least one nonzero compact operator. By [14, 5,39], \( T_u \) contains all compact operators.

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