ISOMETRIC EMBEDDING OF A WEIGHTED FERMAT-FRECHET MULTITREE FOR ISOPERIMETRIC DEFORMATIONS OF THE BOUNDARY OF A SIMPLEX TO A FRECHET MULTISIMPLEX IN THE $K$-SPACE

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Abstract. In this paper, we study the weighted Fermat-Frechet problem for a $\frac{N(N+1)}{2}$-tuple of positive real numbers determining $N$-simplexes in the $N$-dimensional $K$-Space ($N$-dimensional Euclidean space $\mathbb{R}^N$ if $K = 0$, the $N$-dimensional open hemisphere of radius $\frac{1}{\sqrt{K}}$ ($S^N_{\sqrt{K}}$) if $K > 0$ and the Lobachevsky space $\mathbb{H}^N_K$ of constant curvature $K$ if $K < 0$). The (weighted) Fermat-Frechet problem is a new generalization of the (weighted) Fermat problem for $N$-simplexes.

We control the number of solutions (weighted Fermat trees) with respect to the weighted Fermat-Frechet problem that we call a weighted Fermat-Frechet multitree, by using some conditions for the edge lengths discovered by Dekster-Wilker. In order to construct an isometric immersion of a weighted Fermat-Frechet multitree in the $K$-Space, we use the isometric immersion of Godel-Schoenberg for $N$-simplexes in the $N$-sphere and the isometric immersion of Gromov (up to an additive constant) for weighted Fermat (Steiner) trees in the $N$-hyperbolic space $\mathbb{H}^N_K$.

Finally, we create a new variational method, which differs from Schaflii’s, Luo’s and Milnor’s techniques to differentiate the length of a geodesic arc with respect to a variable geodesic arc, in the $3K$-Space. By applying this method, we eliminate one variable geodesic arc from a system of equations, which give the weighted Fermat-Frechet solution for a sextuple of edge lengths determining (Frechet) tetrahedra.

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1. Introduction

In [20], Frechet posed the following problem in distance geometry (see also in [41]):

Let $a_{ik} = a_{ki}$, $i \neq k; i, k = 0, 1, 2, \cdots, N$ ($a_{ik} = 0$, for $i = k$) be $\frac{N(N+1)}{2}$ given positive quantities. What are the necessary and sufficient conditions that they be the lengths of an $N$-simplex $A_0A_1A_2\cdots A_N$ in $\mathbb{R}^N$?

Schoenberg restates and solves Frechet problem in a more general form ([41, Theorem 1]):

A necessary and sufficient condition that the $a_{ik}$ be the lengths of the edges of an $r$-simplex $A_0A_1A_2\cdots A_N$ lying in a Euclidean space $\mathbb{R}^r$ ($1 \leq r \leq N$) but not in a $\mathbb{R}^{r-1}$ ($a_{ik} = 0$ if and only if $i = k$) is

$$F(x_1, x_2, \ldots, x_N) = \frac{1}{2} \sum_{i,k=1}^{N} (a_{0i}^2 + a_{0k}^2 - 2a_{ik}x_i x_k) \geq 0$$

and

$$\text{rank}(F(x_1, x_2, \ldots, x_N)) = r.$$ 

An extension of the Frechet problem for the construction of simplexes in the $(r-1)$-dimensional spherical space $S^r_{\rho}$ of radius $\rho$ is given in [41, Theorem 2].

Necessary and sufficient conditions that the $a_{ik}$ be the $\frac{N(N-1)}{2}$ lengths (mutual spherical distances) of the edges of an $r-1$-simplex $A_1A_2\cdots A_N$ lying in $S^r_{\rho}$ ($1 < r \leq N$) but not in a $S^r_{\rho-2}$ for $i, k = 1, 2, \cdots, N$ are:

$$\varphi(x_1, x_2, \ldots, x_N) = \frac{1}{2} \sum_{i,k=1}^{n} \cos \left( \frac{a_{ik}}{\rho} \right) x_i x_k \geq 0,$$

$$a_{ik} \leq \pi \rho,$$

$$\text{rank}(\varphi(x_1, x_2, \cdots, x_N)) = r.$$ 

Menger independently solved the Euclidean Frechet problem obtaining equations and inequalities by using determinants ([36, Third Fundamental theorem, pp. 738-743]) and Blumenthal-Garrett, Klanfer extends Menger’s method of determinants for spherical simplexes in $S^r_{\rho}$ of radius $\rho$ ([4], [34]).

A difficult problem is to determine $\frac{1}{2} \frac{N(N+1)!}{(N+1)!}$ pairwise incongruent $N$-simplexes with the same $\frac{1}{2}N(N+1)$ edges $N > 1$ in $\mathbb{R}^N$. 

In [5, 21], Blumenthal and Hertog, respectively, studied the Euclidean case for \( N = 3 \) and obtained conditions for six edges forming thirty incongruent tetrahedra in \( \mathbb{R}^3 \).

Dekster and Wilker discovered necessary and sufficient conditions for incongruent simplexes on spaces with constant curvature ([12], [14]), by applying ideas and techniques taken from the theory of distance matrices first developed by Seidel ([42]) and Neumaier ([34]). These conditions offer a generalization of the triangle inequality for simplexes.

The \( N \)-dimensional \( K \)-space (\( \mathbb{E}^N_K \)) is the \( N \)-dimensional Euclidean space \( \mathbb{R}^N \) if \( K = 0 \), the \( N \)-dimensional open hemisphere of radius \( \frac{1}{\sqrt{K}} \) \( (S^N_{\sqrt{K}}) \) if \( K > 0 \) and the Lobachevsky space \( \mathbb{H}^N_K \) of constant curvature \( K \) if \( K < 0 \).

The weighted Fermat problem in the \( \mathbb{E}^N_K \) refers to finding the unique point minimizing the sum of geodesic distances from this point to each point from a finite set of fixed non-collinear points multiplied by a positive real number (weight), which corresponds to each distance. The solution is referred as the weighted Fermat point. If the weighted Fermat point does not belong to this finite set of fixed points (vertices), the solution is called a weighted (floating) Fermat-Torricelli point, otherwise it is called a weighted (absorbing) Fermat-Torricelli point. The branching solution, which consists of the weighted branches that connect the weighted Fermat point with each fixed vertex is called a weighted Fermat tree.

We focus on the connection of the weighted Fermat problem with the Frechet problem (weighted Fermat-Frechet problem) in the \( \mathbb{E}^N_K \).

**Problem 1** (Weighted Fermat-Frechet problem). *Find the discrete set of weighted Fermat trees, which correspond to a given \( N \)-tuple of positive real numbers determining \( \frac{1}{2}N(N+1) \) edges of incongruent \( N \)-simplexes in \( \mathbb{E}^N_K \).*

In [5], Blumenthal claimed that the maximum number of incongruent \((N + 1)\)-simplexes in \( \mathbb{R}^N \), which may be obtained by a given \( N \)-tuple of positive real numbers determining \( \frac{1}{2}N(N+1) \) edges is \( \frac{1}{2}(N+1)! \frac{(N+1)!}{(N+1)!} \).

In [50] and [51], we studied the weighted Fermat-Frechet problem for \( N = 3 \) (geodesic triangles) in the two dimensional \( K \)-plane (\( \mathbb{E}^2_K \)) and we found the position of the weighted Fermat-Torricelli trees, which yields the weighted Fermat-Frechet (multitree) solution.

In [53], we found the position of the weighted Fermat-Torricelli trees for the weighted Fermat-Frechet problem for a given sextuple of positive real numbers determining the edge lengths of tetrahedra in \( \mathbb{R}^3 \),
by substituting Caley-Menger determinants in some weighted volume entropy equalities for tetrahedra derived in [19].

In the present paper, we introduce a class of weighted Fermat-Torricelli trees (Fermat-Frechet multitree) for a given \( \frac{N(N+1)}{2} \) tuple of positive real numbers determining \( N \)-simplexes in \( \mathbb{E}^N \). We apply ideas and methods by Godel, Schoenberg and Gromov, in order to construct isometric immersions of a weighted Fermat-Frechet multitree for isoperimetric deformations of the boundary of incongruent simplexes in simplexes inside \( \mathbb{E}^N \).

The main results of the paper are the following:

First, we study the conditions for the solution of the weighted Fermat-Frechet problem (Fermat-Frechet multitree), which corresponds to a union of weighted Fermat-Torricelli trees for all incongruent \( N \)-simplexes. The family of incongruent \( N \)-simplexes, whose edge lengths satisfy the conditions given by Dekster-Wilker ([12], [13], [14], [15]) form a Frechet multisimplex in \( \mathbb{E}^N_K \) (Theorems 1, 3, 6, 8).

Next we use the ideas of Godel-Schoenberg ([41]), which focus on Godel’s observation of thinking of the edges of an \( N \)-simplex to made of flexible strings and on placing in the interior a small sphere, which was gradually inflated. When the sphere reaches a definite size, it will be tightly packed within the edges (strings) of the \( N \)-simplex. Our idea is to consider the case of tightly packed a weighted Fermat-Torricelli tree with respect to a boundary \( (N-1) \)-simplex in a proper \( (N-1) \) dimensional sphere. Thus, following Godel-Shoenberg methods reversely we derive some controlled isometric immersions of a weighted Fermat-Frechet multitree to a Frechet simplex in a larger spherical space (Theorems 11, 14). The conditions of Dekster-Wilker need to be satisfied, in order to control the isometric immersions of Frechet multisimplex enriched with the corresponding weighted Fermat trees, which contains all incongruent simplexes.

We apply the theory of Gromov’s isometric embedding of geodesic trees in the \( N \) dimensional hyperbolic space of constant curvature \( K \mathbb{H}_K^N \) ([24]), in order to construct an embedding (inclusion map) of a Fermat-Steiner-Frechet multitree solution for a given \( \frac{N(N+1)}{2} \) tuple of edge lengths determining incongruent boundary \( N \)-simplexes to an associated family of Gromov isometries up to an additive constant for \( N \)-simplexes and ideal \( N \)-simplexes in \( \mathbb{H}_K^N \). The Fermat-Steiner-Frechet multitree is a union of (intermediate) weighted Fermat-Steiner trees having upto \( N-2 \) nodes (Fermat-Steiner points) inside the \( N \)-simplex and may be considered as a generalization of weighted minimal binary
trees studied by Ivanov and Tuzhilin in \[28\] (Theorems 15, 16). Furthermore, we apply Gromov’s theory of \(\delta\) thin metric trees in hyperbolic spaces, in order to measure the reduction of intelligence of intermediate Fermat-Steiner trees (Number of Fermat Steiner nodes), by applying Gromov’s convergence of ideal simplexes to two subsimplexes (Theorem 17).

We continue, by giving a new variational method, which focus on the generalization of cosine law in the 3-\(K\)-Space (Theorem 18). We note that a generalization of the cosine law in \(\mathbb{R}^3\) has been given in \[49\]. This variational method differs from Schafli (\[40\]), Luo’s (\[35\]) and Milnor’s (\[37\]) techniques to differentiate the length of a geodesic arc with respect to a variable geodesic arc in the 3-\(K\)-Space. By applying this method to a system of equations that deal with the weighted Fermat-Frechet solution in \(\mathbb{E}_N^K\), for \(N = 4\), we eliminate one variable geodesic arc from the equations and we determine the weighted Fermat-Frechet solution for a sextuple of edge lengths for Frechet multitetrahedra (Theorem 19).

Finally, we conclude with some calculations for the determination of weighted Fermat trees for tetrahedra having one or three vertices at infinity in \(\mathbb{R}^3\) (Theorem 21).

The paper is organized as follows: In Section 2, we study the conditions to solve the weighted Fermat-Frechet problem in the \(K\)-Space (Theorems 1- 9, Proposition 1, 2, 3)

In Section 3, we construct a controlled Godel-Schoenberg isometric immersion of the weighted Fermat Frechet multitee for \(N – 1\) boundary Frechet multisimplex in \(\mathbb{S}_{\rho_1}^{N-2}\) to a class of weighted simplexes in \(\mathbb{S}_{\rho_0}^{N-1}\) for \(\rho_0 > \rho_1\) (Theorems 9-14).

In Section 4, we construct a Gromov isometric immersion up to an additive constant for weighted Fermat-Steiner Frechet multitee in \(\mathbb{H}_N^K\) (Theorem 15- 17)

In Section 5, we give a new variational method to derive the weighted Fermat-Frechet multitee for Frechet multitetrahedra in the 3–\(K\)-Space (Theorem 18-19, Proposition 4).

In the last section, we present a computational method to obtain weighted Fermat-Torricelli trees with respect to a tetrahedron having one or three (ideal) vertices at infinity (Theorem 20-21, Propositions 7-8).
2. THE WEIGHTED FERMAT-FRECHET PROBLEM IN THE $N$ DIMENSIONAL $K$-SPACE

In this section, we introduce the weighted Fermat-Frechet problem for a given $\frac{N(N+1)}{2}$-tuple of positive real numbers determining $N$-simplexes in $E^N_K$ and we study the conditions to obtain a class of weighted Fermat-Torricelli trees, which form the weighted Fermat-Frechet multitree solution.

We start by describing the Dekster-Wilker domain given in [12], [13], [14], [15], which gives the maximum number of mutually incongruent simplexes for the same $\frac{N(N+1)}{2}$-tuple of edges in $E^N_K$.

We denote by $\{A_1, A_2, \cdots, A_{N}\}$ an $(N-1)$-simplex, by $A_0$ an interior point and by $\{A_1, A_2, \cdots, A_{N}\} \subset \{A_1, A_2, \cdots, A_{N}\}$ an $(N-1)$-simplex, which is derived by replacing the vertex $A_j \in \{A_1, A_2, \cdots, A_{N}\}$ with $A_0$ in $E^N_K$.

A. The weighted Fermat-Frechet problem in $\mathbb{R}^N$

Dekster and Wilker use the notations $\ell = \max_{i,j} a_{ij}$, $s = \min_{i,j} a_{ij}$ and

$$\lambda_N(\ell) = \begin{cases} \\
\ell \sqrt{1 - \frac{2(N+1)}{N(N+2)}} & \text{for even } N \geq 2, \\
\ell \sqrt{1 - \frac{2}{(N+1)}} & \text{for odd } N \geq 3
\end{cases}$$

**Definition 1.** The Dekster-Wilker Euclidean domain $DW_{\mathbb{R}^N}(\ell, s)$ is a closed domain in $\mathbb{R}^2$ between the ray $s = \ell$, and the graph of a function $\lambda_N(\ell)$, $\ell \geq 0$, which is less than $\ell$ for $\ell \neq 0$.

**Lemma 1.** Dekster-Wilker incongruent simplexes in $\mathbb{R}^N$, [13] (1.3), (1.4)

If $s \leq a_{ij} \leq \ell$, for $\ell, s \in DW_{\mathbb{R}^N}(\ell, s)$, then $a_{ij}$ determine the edge lengths of incongruent simplexes in $\mathbb{R}^N$.

The characterization of solutions (absorbing/floating Fermat-Torricelli trees) of the Fermat-problem in $\mathbb{R}^N$ was first given by Sturm in [44] and it was extended for the weighted case by Kupitz and Martini ([6, Chapter II, Theorem 18.37]) using calculus. These two cases of weighted Fermat trees are used to derive the weighted Fermat-Frechet solution for a given $\frac{N(N+1)}{2}$-tuple of positive real numbers $a_{ij}$ determining the edge lengths of incongruent $N$-simplexes in $\mathbb{R}^N$.

**Theorem 1** (The weighted Fermat-Frechet solution for $N$-simplexes in $\mathbb{R}^N$). The weighted Fermat-Frechet solution for a given $\frac{N(N+1)}{2}$-tuple $a_{ij}$, such that $s \leq a_{ij} \leq \ell$, for $\ell, s \in DW_{\mathbb{R}^N}(\ell, s)$ consists of a maximum number of $\frac{N(N+1)!}{(N+1)!}$ weighted Fermat trees, which belong to one of the following two cases:
(I) If for each index \( k \in \{1, 2, 3, \ldots, N\} \),
\[
\sum_{i=1, i<j}^{N} B_i B_j \frac{a_{ik}^2 + a_{jk}^2 - a_{ij}^2}{2a_{ik}a_{jk}} > \frac{B_k^2 - \sum_{i=1, i<j}^{N} B_i^2}{2},
\]  
we obtain the weighted floating Fermat-Torricelli tree \( \{a_{01}, a_{02}, \ldots, a_{0N}\} \).

(II) If there is an index \( k \in \{1, 2, \ldots, N\} \), such that:
\[
\sum_{i=1, i<j}^{N} B_i B_j \frac{a_{ik}^2 + a_{jk}^2 - a_{ij}^2}{2a_{ik}a_{jk}} \leq \frac{B_k^2 - \sum_{i=1, i<j}^{N} B_i^2}{2},
\]  
we obtain the weighted absorbing Fermat-Torricelli tree \( \{a_{k1}, a_{k2}, \ldots, a_{kN}\} \).

**Proof.** The edge lengths \( a_{ij} \) yield a simplex \( A_1A_2\ldots A_{N+1} \) in \( \mathbb{R}^N \). Let \( X \) be a point in \( \mathbb{R}^N \). We assume that a positive number \( B_i \) (weight) correspond to each length of the segment \([X, A_i]\), for \( i = 1, 2, \ldots, N \).

By applying the method of directional derivatives by Kupitz and Martini [6, Chapter II, Theorem 18.16] for the weighted case and by applying the cosine law in \( \Delta A_iA_jA_k \), we get (2.1) and (2.2).

The existence and uniqueness of the weighted Fermat-Torricelli point \( X \equiv A_0 \) proved by convexity and compactness arguments gives the existence and uniqueness of the weighted Fermat-Torricelli trees for each simplex, which belongs to the class of incongruent boundary simplexes. Thus, the maximum number of arrangements of \( \frac{N(N+1)}{2} \) edges gives \( \frac{1}{2} \frac{N(N+1)!}{(N+1)!} \) mutually incongruent simplexes, which correspond to \( \frac{1}{2} \frac{N(N+1)!}{(N+1)!} \) weighted floating and absorbing Fermat-Torricelli trees.

**Theorem 2** (The weighted Euclidean Fermat-Frechet problem for large \( N \)). The weighted Fermat-Frechet solution for a given large number \( N \) and \( \frac{N(N+1)}{2} \)-tuple of edge lengths determining simplexes, consists of a maximum number of
\[
\left(\frac{N(N+1)}{2}\right)^{\frac{N(N+1)+1}{N(N+2)}} \exp\left(-\frac{1}{2} + \frac{\theta_1}{6N(N+1)} - \frac{\theta_2}{12(N+1)}\right)
\]
weighted Fermat trees, for \( \theta_1, \theta_2 \in (0, 1) \).

**Proof.** It is a direct consequence of Theorem II by using Stirling’s formula (II 6.1.38, p. 257):
\[
x! = \sqrt{2\pi x} x^{x+\frac{1}{2}} \exp(-x+\frac{\theta}{12x}),
\]  
for \( x > 0 \) and \( \theta \in [0, 1] \).

By substituting in (2.3) \( x \equiv \frac{N(N+1)}{2} \) and \( x \equiv N+1 \) for \( \theta_1, \theta_2 \in (0, 1) \), and dividing the two derived relations gives an upper bound of
\[
\frac{(N(N+1))^{N(N+1)+1}}{N(N+1)^{N+1}+2} \exp(-(N+1)(\frac{N}{2}-1) + \frac{\theta_1}{6N(N+1)} - \frac{\theta_2}{12(N+1)}) \text{ weighted Fermat trees that correspond to the mutually incogruent simplexes in } \mathbb{R}^N.
\]

\[\square\]

The volume of an \(N\)-simplex \(A_1A_2\cdots A_N\) in \(\mathbb{R}^N\) is given by the Caley-Menger determinant in terms of edge lengths (\cite{43}, (5.1), p. 125):

\[
\text{Vol}(A_1A_2\cdots A_N)^2 = \frac{1}{(-1)^N2^{N-1}(((N-1)!))^2}
\]

\[
\begin{vmatrix}
0 & 1 & 1 & \cdots & 1 & 1 \\
1 & 0 & a_{12}^2 & \cdots & a_{1(N-1)}^2 & a_{1N}^2 \\
1 & a_{21}^2 & 0 & \cdots & a_{2(N-1)}^2 & a_{2N}^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & a_{N1}^2 & a_{N2}^2 & \cdots & a_{N(N-1)}^2 & 0
\end{vmatrix}
\]

**Theorem 3** (The Euclidean Fermat-Torricelli Frechet solution). The following equations depending on the variable lengths \(a_{01}, a_{02}, \cdots, a_{0N}\) and the segments \(\{a_{ij}\}\) provide a necessary condition for the determination of the weighted floating Fermat-Torricelli trees \(\{a_{01}, a_{02}, \cdots, a_{0N}\}\), which belong to the weighted Fermat-Frechet solution:

\[
\sum_{i=2}^{N} B_i \left( a_{01}^2 + a_{0i}^2 - a_{1i}^2 \right) - \sum_{i=1, i \neq j}^{N} B_i \left( a_{0j}^2 + a_{0i}^2 - a_{ji}^2 \right) = -2(B_1 a_{01} - B_j a_{0j}),
\]

\[
(2.4)
\]

\[
\text{Vol}(A_1, A_2, \cdots, A_N) = \sum \text{Vol}(A_1, A_2, \cdots, A_0, \cdots, A_N),
\]

\[
(2.5)
\]

for \(j = 2, \cdots, N\).

**Proof.** It is well known that the first variational formula of a line segment \(A_0A_i\) with respect to a physical parameter \(A_0A_j\) (line segment), which meet at a point \(A_0\) in \(\mathbb{R}^N\) is given by:

\[
\frac{da_{0i}}{da_{0j}} = \cos \angle A_i A_0 A_j,
\]

\[
(2.6)
\]

for \(i, j = 1, 2, \cdots, N\). Differentiating the objective function

\[
f(a_{01}, a_{02}, \cdots, a_{0N}) = \sum_{i=1}^{N} B_i a_{0i} \text{ with respect to } a_{0i}, \text{ for } i = 1, 2, \cdots N,
\]

we get:

\[
\sum_{i=1}^{N} B_i \cos(\alpha_{10i}) = 0,
\]

\[
(2.7)
\]
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\[
\sum_{i=1}^{N} B_i \cos(\alpha_{20i}) = 0, \quad (2.8)
\]

\[
\ldots
\]

\[
\sum_{i=1}^{N} B_i \cos(\alpha_{N0i}) = 0. \quad (2.9)
\]

By subtracting (2.8)-(2.9) from (2.7) and by applying the cosine law in $\triangle A_j A_0 A_i$ for $j = 1, 2, \cdots, N$, we derive (2.4).

The interior point $A_0$ of $A_1 A_2 \cdots A_N = \bigcup A_1 A_2 \cdots A_0 \cdots A_N$ gives the property of addition of Euclidean volumes of simplexes in $\mathbb{R}^N$. By substituting in $\text{Vol}(A_1, A_2, \cdots, A_N)$, $\text{Vol}(A_1, A_2, \cdots, A_0, \cdots, A_N)$ the corresponding Caley Menger determinant depending on the edge lengths $a_{ij}$, for $i, j = 0, 1, 2, \cdots, N$, we obtain (2.5).

\[\square\]

B. The weighted Fermat-Frechet problem in the hyperbolic space $\mathbb{H}_K^N$, for $K < 0$.

Dekster and Wilker ([13], (1.5),(1.6),[15]) describe a class of incongruent hyperbolic simplexes in terms of $\ell, s$, such that: $\ell = \max_{i,j} a_{ij}$, $s = \min_{i,j} a_{ij}$ and

\[
\lambda_N(\ell) =
\]
\[
\begin{cases}
\frac{1}{K} \cosh^{-1}\left(\sqrt{1 + 2\left(1 - \frac{2}{N}\right) \sinh^2 K \ell \sqrt{1 + 2\left(1 - \frac{2}{N+2}\right) \sinh^2 K \frac{\ell}{2}}}\right) & \text{for even } N \geq 2, \\
\frac{2}{K} \sinh^{-1}\left(\sqrt{1 - \frac{2}{(N+1)\sinh^2 K \ell \sqrt{1 + 2\left(1 - \frac{2}{N+2}\right) \sinh^2 K \frac{\ell}{2}}}\right) & \text{for odd } N \geq 3
\end{cases}
\]

**Definition 2.** The Dekster-Wilker hyperbolic domain \(\text{DW}_{\mathbb{H}_K^N}(K\ell, Ks)\) is a closed domain in \(\mathbb{R}^2\) between the ray \(s = \ell\), and the graph of a function \(\lambda_N(\ell), \ell \geq 0\), which is less than \(K\ell\) for \(K\ell \neq 0\).

**Lemma 3. Dekster-Wilker incongruent simplexes in \(\mathbb{H}_K^N\), \([13, (1.5),(1.6)]\)**

If \(s \leq a_{ij} \leq \ell\), for \(K\ell, Ks \in \text{DW}_{\mathbb{H}_K^N}(K\ell, Ks)\), then \(a_{ij}\) determine the edge lengths of incongruent simplexes in \(\mathbb{H}_K^N\).

In \([39, \text{Theorem 1, Proposition 2}, \text{p. 97}]\), Noda, Sakai, Morimoto derive a characterization of solutions (absorbing/floating Fermat-Torricelli trees) for the Fermat-problem in \(\mathbb{H}_K^N\) and more general for simply connected smooth Riemannian manifolds with non-positive sectional curvature (Hadamard manifolds). An extension of characterizations of solutions of Fermat trees for the weighted case leads to the weighted Fermat-Frechet solution for a given \(\frac{N(N+1)}{2}\)-tuple of positive real numbers \(a_{ij}\) determining the edge lengths of incongruent \(N\)-simplexes in \(\mathbb{H}_K^N\).

We cannot reformulate Theorem 3 for the hyperbolic case, by using directly the addition property of Volumes of hyperbolic simplexes. In \([37, \text{Theorem}, \text{p. 200}]\), Milnor shows that the addition property of the Lobachevsky function, which is connected with hyperbolic volume holds for ideal (having their vertices at infinity) hyperbolic 3-simplexes and computes by using power series the volume of an ideal regular \(N\)-simplex \([37, \text{pp. 206-207}]\). In \([26]\), Haagerup and Munkholm proved an isoperimetric inequality of hyperbolic simplexes in \(\mathbb{H}^N\).

**Lemma 4. Isoperimetric inequality of a hyperbolic regular simplex in \(\mathbb{H}_K^N\), \([26]\)**

A hyperbolic \(N\)-simplex is of maximal volume if it is ideal and regular.

**Theorem 4 (The weighted Fermat-Frechet problem in \(\mathbb{H}_K^N\)).**
The weighted Fermat-Frechet solution for a given \(\frac{N(N+1)}{2}\)-tuple \(a_{ij}\), such that \(s \leq a_{ij} \leq \ell\), for \(\ell, s \in \text{DW}_{\mathbb{H}_K^N}(\ell, s)\) consists of a maximum number of \(\frac{N(N+1)!}{(N+1)!}\) weighted Fermat trees, which belong to one of the following two cases:
(I) If for each index \( k \in \{1, 2, 3, \ldots, N\} \),
\[
\sum_{i=1, i<j}^{N} B_i B_j \frac{\cosh K a_{ik} \cosh K a_{jk} - \cosh K a_{ij}}{\sinh K a_{ik} \sinh K a_{jk}} > \frac{B_k^2 - \sum_{i=1,i<j}^{N} B_i^2}{2},
\]
we obtain the weighted floating Fermat-Torricelli tree \( \{a_{01}, a_{02}, \ldots, a_{0N}\} \).

(II) If there is an index \( k \in \{1, 2, 3, \ldots, N\} \), such that:
\[
\sum_{i=1, i<j}^{N} B_i B_j \frac{\cosh K a_{ik} \cosh K a_{jk} - \cosh K a_{ij}}{\sinh K a_{ik} \sinh K a_{jk}} \leq \frac{B_k^2 - \sum_{i=1,i<j}^{N} B_i^2}{2},
\]
we obtain the weighted absorbing Fermat-Torricelli tree \( \{a_{k1}, a_{k2}, \ldots, a_{kN}\} \).

**Proof.** The existence and uniqueness of the weighted Fermat-Torricelli point \( A_0 \) in \( \mathbb{H}_K^N \) is given by compactness and convexity arguments of the distance function \( a_0 \) following Thurston [46, Theorem 2.5.8] under the weighted conditions of Noda-Sakai-Morimoto gives a maximum number of weighted Fermat trees for \( s \leq a_{ij} \leq \ell : (K s, K \ell) \in DW_{\mathbb{H}_K^N}(\ell, s) \). By substituting the hyperbolic law of cosines in \( \triangle A_i A_j A_k \) in the weighted norm inequalities of unit tangent vectors, we obtain (2.10) and (2.11).

**Theorem 5** (The hyperbolic Fermat-Torricelli Frechet solution). The following equations depending on the variable hyperbolic edge lengths \( a_{01}, a_{02}, \ldots, a_{0N} \) and the hyperbolic edge lengths \( \{a_{ij}\} \) provide a necessary condition for the location of the weighted floating Fermat-Torricelli trees \( \{a_{01}, a_{02}, \ldots, a_{0N}\} \), which belong to the weighted Fermat-Frechet multitree solution:
\[
\sum_{i=2}^{N} \frac{B_i}{\sinh K a_{0i} \sinh K a_{0i}} (\cosh K a_{01} \cosh K a_{0i} - \cosh K a_{1i}) -
\sum_{i=1, i \neq j}^{N} \frac{B_i}{\sinh K a_{0j} \sinh K a_{0i}} (\cosh K a_{0j} \cosh K a_{0i} - \cosh K a_{ji}) = 0,
\]

\[
\text{Vol}(A_1, A_2, \ldots, A_0, \ldots, A_N) < \sqrt{N} \sum_{k=0}^{\infty} \frac{\beta(\beta + 1) \cdots (\beta + k - 1)}{(N + 2k)!} A_{N,k}
\]

with \( \beta = \frac{N+1}{2} \), \( A_{N,k} = \sum_{i_0 + \cdots + i_N = k} \frac{(2i_0)! \cdots (2i_N)!}{i_0! \cdots i_N!} \), for \( j = 2, \ldots, N \).

**Proof.** By differentiating the objective function \( f(a_{01}, a_{02}, \cdots, a_{0N}) = \sum_{i=1}^{N} B_i \) with respect to arc length and by using the first variation
formula of a variable arc length \( a_{0i} \) with respect to arc length \( a_{01} \), and with respect to \( a_{0j} \) and by subtracting the two derived relations and then substituting \( \cos \angle A_iA_0A_j \) by the hyperbolic law of cosines in \( \triangle A_iA_0A_j \), we get (2.12). By substituting in Lemma 4 Milnor’s computation of a hyperbolic regular \( N \) simplex (upper bound), we derive (2.13).

Assume that the Frechet class of hyperbolic simplexes contains an orthosimplex \( A^o_{N+1}A^o_1A^o_2 \cdots A^o_N \) with edge lengths \( a^o_{ij} : s \leq a^o_{ij} \leq \ell : (Ks, K\ell) \in DW_{\mathbb{H}_K^N}(K\ell, Ks) \). An \( N \)-dimensional orthosimplex (Euclidean or non-Euclidean) is an \( N \)-simplex whose faces \( F_0, \cdots F_N \) satisfy: \( F_i \perp F_j \) for \( |i - j| \geq 2 \) ([37, p. 207]). By using Milnor’s elegant computation of volume of a hyperbolic orthosimplex, we get a reformulation of Theorem 6 using the following lemma:

**Lemma 5. Computation of the Volume of an orthosimplex in \( \mathbb{H}_K^N \), [37, p. 208]**

\[
\text{Vol}((A^o_{N+1}A^o_1, A^o_2, \cdots, A^o_N)) = \sum_{i_1,i_2,\cdots,i_N=0}^{\infty} \frac{\beta(\beta + 1)\cdots(\beta + k - 1) a^2_{i_1} \cdots a^2_{i_N}}{i_1!i_2!\cdots i_N!(2i_1 + \cdots + 2i_N + N) \cdots (2i_N + 1)}
\]

(2.14)

where

\[
k = i_1 + i_2 + \cdots + i_N, \quad \beta = \frac{N+1}{2}
\]

\[
\tanh Ka_{(N+1)1} = a_1, \quad \tanh Ka_{12} = \frac{a^2_2}{\sqrt{1-a^2_1}}, \cdots, \\
\tanh Ka_{(N-1)N} = \frac{a^2_N}{\sqrt{1-a^2_{N-1}}}
\]

**Theorem 6** (The hyperbolic Fermat-Torricelli Frechet solution). The following equations depending on the variable hyperbolic edge lengths \( a_{01}, a_{02}, \cdots, a_{0N} \) and the hyperbolic edge lengths \( \{a_{ij}\} \) provide a necessary condition for the location of the weighted floating Fermat-Torricelli trees \( \{a_{01}, a_{02}, \cdots, a_{0N}\} \), which belong to the weighted Fermat-Frechet
(orthosimplex) solution:

\[
\sum_{i=2}^{N} \frac{B_i}{\sinh K a_{01} \sinh K a_{0i}} (\cosh K a_{01} \cosh K a_{0i} - \cosh K a_{1i}) - \\
\sum_{i=1, i \neq j}^{N} \frac{B_i}{\sinh K a_{0j} \sinh K a_{0i}} (\cosh K a_{0j} \cosh K a_{0i} - \cosh K a_{ji}) = 0,
\]

\[
\text{Vol}(A_1, A_2, \cdots, A_0 \cdots, A_N, A_{N+1}) < \lambda \text{Vol}(A_{N+1}^\circ A_1^\circ, A_2^\circ \cdots A_N^\circ) + m.
\]

(2.15)

for some \( \lambda, m > 0 \).

Given \( \{a_0, a_2, \cdots, a_N\} \) a weighted (floating) Fermat-Torricelli tree with respect to the hyperbolic simplex \( A_1 A_2 \cdots A_N \) in \( \mathbb{H}_K^N \) with unknown constant curvature \( K \), which belongs to the hyperbolic weighted Fermat-Frechet multitree solution, we may compute the curvature \( K \) and the radius \( R \) of the sphere of the Klein model, by using the hyperbolic Caley-Menger determinant. The hyperbolic Caley Menger determinant for \( \{A_0, A_1, \cdots, A_N\} \) is given by:

\[
\text{det}(A_0, A_1, A_2, \cdots, A_N) = \\
\begin{vmatrix}
1 & \cosh K a_{01} & \cosh K a_{02} & \cdots & \cosh K a_{0N} \\
\cosh K a_{10} & 1 & \cosh K a_{12} & \cdots & \cosh K a_{1N} \\
\cosh K a_{20} & \cosh K a_{21} & 1 & \cdots & \cosh K a_{2N} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\cosh K a_{N0} & \cosh K a_{N1} & \cosh K a_{N2} & \cdots & 1
\end{vmatrix}.
\]

Thus, by setting \( \text{det}(A_0, A_1, A_2, \cdots, A_N) = 0 \), we may derive an estimate of \( K \). In [45], Terence Tao used a spherical Caley-Menger determinant to obtain estimate of the radius of the Earth.

Proposition 2. If a weighted Fermat-Frechet multitree \( \{a_0, a_2, \cdots, a_N\} \) is given in \( \mathbb{H}_K^N \) then we get the same sphere on the Klein model.

Proof. Solving \( \text{det}(A_0, A_1, A_2, \cdots, A_N) = 0 \), for each weighted Fermat-Torricelli tree that consists the weighted Fermat-Torricelli multitree yields an estimate for the constant negative curvature \( K \) and the corresponding radius \( R \) in the Klein model.

C. The weighted Fermat-Frechet problem in the spherical space \( S_K^N \), for \( K > 0 \) and \( R = \frac{1}{\sqrt{K}} \).

Dekster and Wilker ([43] (3.1),(3.6),(1.3),(1.5],[45] pp. 9–11)) introduced a class of incongruent spherical simplexes in terms of \( \ell, s, \) such
that: \( \ell = \max_{i,j} a_{ij}, \ s = \min_{i,j} a_{ij} \) and the following notations are used:

\[
\lambda_N(\ell) =
\begin{cases}
\frac{1}{K} \arccos\left(\sqrt{1 + 2(1 - \frac{2}{N}) \sin^2 K \frac{\ell}{2}}\right) & \text{for even } N \geq 2, \\
\frac{2}{K} \arcsin\left(\sqrt{1 - \frac{2}{(N+1)} \sin K \frac{\ell}{2}}\right) & \text{for odd } N \geq 3
\end{cases}
\]

\[
m_N(K\ell) = \frac{2}{K} \arcsin K \sqrt{\frac{2 + 2(N-1) \cos K \ell}{1 + (N-2) \cos K \ell} \sin K \frac{\ell}{2}}
\]

\[
\ell^*_N = \frac{2}{K} \arcsin K \sqrt{\frac{N + 1}{2N}},
\]

\[
a_N = \frac{2}{K} \arcsin \frac{\sqrt{7N - 4 + \sqrt{N^2 + 8N}}}{N - 1},
\]

\( K \ OC : K s = K \ell, K \ell \in [0, K \ell^*_N], \)

\( K \ CF : K s = K m_N(K\ell), K \ell \in [K \ell^*_N, Ka_N], \)

\( K \ OE : K s = K \lambda_N(K \ell), K \ell \in [0, K \ell^*_N], \)

\( EF : K s = K \lambda_N(K \ell), K \ell \in [K \ell^*_N, Ka_N]. \)

**Definition 3.** [13, 15] The Dekster-Wilker spherical domain \((K\ell, Ks) \in DW_{\mathbb{S}^N}(K\ell, Ks)\) is a closed domain in \(\mathbb{R}^2\) bounded by the arcs \(K \ OC, K \ CF, K \ OE, K \ EF.\)

**Definition 4.** [11, p. 333] A set \(D\) in \(\mathbb{S}^N_K\), is convex if for any \(X, Y \in D\) there exists a geodesic arc \(XY\) in \(D\) such that \(XY\) is the unique minimizer in \(\mathbb{S}^N_K\), connecting \(X\) to \(Y.\)

**Definition 5.** [11, p. 335] For any \(X\) in \(\mathbb{S}^N_K\), the convexity radius is given by: \(\text{ConvB}(X, r) = \sup\{\rho : B(X, r)\text{ is convex for all } r < \rho, \text{ where } B(X, r)\text{ is a disk with center at } X\text{ and radius } r.\)

**Lemma 6.** Convexity Radius for the spherical space \(\mathbb{S}^N_K, [33, p. 510], [19]\) Appendix, Lemma 1] The convexity radius \(\text{ConvB}(X, r)\) for each \(X \in \mathbb{S}^N_K\), is \(\frac{\pi}{4\sqrt{K}}.\)

**Theorem 7** (The weighted Fermat-Frechet problem in \(\mathbb{S}^N_K\)). The weighted Fermat-Frechet solution for a given \(\frac{N(N+1)}{2}\)-tuple \(a_{ij}\), such that \(s \leq a_{ij} \leq \ell\), for \((K\ell, Ks) \in DW_{\mathbb{S}^N}(K\ell, Ks)\) and \(\ell \leq \frac{\pi}{4\sqrt{K}}\) consists of a
maximum number of \( \frac{\binom{N}{2}}{(N-1)!} \) weighted Fermat trees, which belong to one of the following two cases:

(I) If for each index \( k \in \{1, 2, 3, \cdots, N\} \),

\[
\sum_{i=1, i<j}^{N} B_i B_j \frac{-\cos Ka_{ik} \cos Ka_{jk} + \cos Ka_{ij}}{\sin Ka_{ik} \sin Ka_{jk}} > \frac{B_k^2 - \sum_{i=1, i<j}^{N} B_i^2}{2},
\]

we obtain the weighted floating Fermat-Torricelli tree \( \{a_{01}, a_{02}, \cdots, a_{0N}\} \).

(II) If there is an index \( k \in \{1, 2, 3, \cdots, N\} \), such that:

\[
\sum_{i=1, i<j}^{N} B_i B_j \frac{-\cos Ka_{ik} \cos Ka_{jk} + \cos Ka_{ij}}{\sin Ka_{ik} \sin Ka_{jk}} \leq \frac{B_k^2 - \sum_{i=1, i<j}^{N} B_i^2}{2},
\]

we obtain the weighted absorbing Fermat-Torricelli tree \( \{a_{k1}, a_{k2}, \cdots, a_{kN}\} \).

Proof. The existence and uniqueness of the weighted Fermat-Torricelli point \( A_0 \) in \( S^N_K \) is given by compactness and the convexity of the distance function \( a_{0i} \) in \( S^N_K \) (1.2.4), by selecting incongruent \( N \)-simplexes \( A_1 A_2 \cdots A_N \) in \( B_{A_i}, \frac{\sqrt{K}}{N} \). Therefore, the maximum number of weighted Fermat trees for \( s \leq a_{ij} \leq \ell : (Ks, K\ell) \in DW_{\mathbb{S}^N_K}(K\ell, Ks) \) is \( \frac{\binom{N}{2}}{(N-1)!} \).

The gradient of the objective function \( f(a_{01}, a_{02}, \cdots, a_{N0}) = \sum_{i=1}^{N} B_i a_{0i} \) gives a weighted sum of outward pointing unit tangent vectors at \( A_0 \)
\[
\nabla f(a_{10}, a_{20}, \cdots, a_{N0}) = \sum_{i=1}^{N} B_i \exp_{A_0}^{-1} \frac{X(A_0, A_i)}{a_{0i}}.
\]

Taking the norm of the inner product \( \nabla f(a_{10}, a_{20}, \cdots, a_{N0}) \exp_{A_0}^{-1} \frac{X(A_0, A_i)}{a_{0i}} \) yields an extension of weighted norm conditions of Kupitz-Martini for the spherical case. By substituting the spherical law of cosines in in \( \triangle A_i A_j A_k \) in the weighted norm inequalities of unit tangent vectors, we get (2.16) and (2.17). \( \square \)

In [7], Boroczky proved an isoperimetric inequality of spherical simplexes in \( S^N_K \).

Lemma 7. Isoperimetric inequality of a spherical regular simplex in \( S^N_K \). A spherical \( N \) simplex is of maximal volume if it is regular.

Theorem 8 (The spherical Fermat-Torricelli Frechet solution). The following equations depending on the variable spherical edge lengths \( a_{01}, a_{02}, \cdots, a_{0N} \) and the spherical edge lengths \( \{a_{ij}\} \) provide a necessary condition for the location of the weighted floating Fermat-Torricelli trees \( \{a_{01}, a_{02}, \cdots, a_{0N}\} \), which belong to the weighted Fermat-Frechet
multitree solution:

\[
\sum_{i=2}^{N} \frac{B_i}{\sin K a_{01} \sin K a_{0i}} (- \cos K a_{01} \cos K a_{0i} + \cos K a_{1i}) - \\
\sum_{i=1, i \neq j}^{N} \frac{B_i}{\sin K a_{0j} \sin K a_{0i}} (- \cos K a_{0j} \cos K a_{0i} + \cos K a_{ji}) = 0, \quad (2.18)
\]

where \(C_1, C_2, \cdots, C_N\) is a regular spherical \(N\)-simplex with edge length \(c_{ij} = \sum_{i,j} a_{ij} (N - 1) / (N - 1) N \).

**Proof.** By differentiating the objective function \(f(a_{01}, a_{02}, \cdots, a_{0N}) = \sum_{i=1}^{N} B_i\) with respect to arc length and by using the first variation formula of a variable arc length \(a_{0i}\) with respect to arc length \(a_{01}\), and with respect to \(a_{0j}\) and by subtracting the two derived relations and then substituting \(\cos \angle A_i A_0 A_j\) by the spherical law of cosines in \(\triangle A_i A_0 A_j\), we get (2.18). Taking into account Lemma 7 we obtain:

\[
\text{Vol}(A_1, A_2, \cdots, A_0 \cdots, A_N) < \text{Vol}(C_1, C_2, \cdots, C_N) < \\
\text{Vol}(A_1, A_2, \cdots, A_N)
\]

which gives (2.19).

Given \(\{a_{01}, a_{02}, \cdots, a_{0N}\}\) a weighted (floating) Fermat-Torricelli tree with respect to the spherical simplex \(A_1 A_2 \cdots A_N\) in \(S_K^N\) with unknown constant curvature \(K\), which belongs to the spherical weighted Fermat-Frechet multitree, we may compute the curvature \(K\) and the radius \(R\) of \(S_K^N\), by using the spherical Caley-Menger determinant. The spherical Caley Menger determinant for \(\{A_0, A_1, \cdots, A_N\}\) is given by:

\[
\text{det}(A_0, A_1, A_2, \cdots, A_N) = \\
\begin{vmatrix}
1 & \cos K a_{01} & \cos K a_{02} & \cdots & \cos K a_{0N} \\
\cos K a_{10} & 1 & \cos K a_{12} & \cdots & \cos K a_{1N} \\
\cos K a_{20} & \cos K a_{21} & 1 & \cdots & \cos K a_{2N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\cos K a_{N0} & \cos K a_{N1} & \cos K a_{N2} & \cdots & 1
\end{vmatrix}
\]

By setting \(\text{det}(A_0, A_1, A_2, \cdots, A_N) = 0\), we derive proposition 8 which gives an estimate of \(K\), (see in [45] for \(N = 3\)).
Proposition 3. If a weighted Fermat-Frechet multitree \( \{a_{01}, a_{02}, \cdots, a_{0N}\} \) is given in \( S^N_K \) then the intersection of the solution set
\[
\det(A_0, A_1, A_2, \cdots, A_N) = 0,
\]
for all incongruent spherical simplexes with respect to \( K \) gives the same sphere with radius \( \frac{1}{\sqrt{K}} \).

3. A controlled Godel-Schoenberg’s isometric immersion of weighted Fermat trees in \( S^{N-2}_{\rho_1} \) to a weighted simplex in \( S^{N-1}_{\rho_0} \)

In this section, we use Godel Schoenberg’s techniques, in order to construct isometric immersions of weighted Fermat trees in \( S^{N-2}_{\rho_1} \) to a weighted simplex in \( S^{N-1}_{\rho_0} \).

In \cite{23} (see also in \cite{41}, Footnote 5, p. 730), Godel considered the edges of a tetrahedron in \( \mathbb{R}^3 \) to be made of flexible strings and placed in the interior of the tetrahedron a small sphere, which was gradually inflated. After time \( t \), this sphere will become tightly packed within the six edges of the tetrahedron. We are interested in the case where the three edges of the tetrahedron correspond to a boundary spherical triangle and the other edges are the three branches of the weighted Fermat (geodesic) tree, which meet at an interior Fermat point.

Let \( \triangle A_1A_2A_3 \) be a spherical triangle in the open hemisphere \( S^K_2 \) of radius \( R = \frac{1}{\sqrt{K}} \equiv \frac{1}{\kappa} \). We denote by \( A_0 \) the weighted Fermat point inside \( \triangle A_1A_2A_3 \) \( \alpha_{ijk} \equiv \angle A_iA_jA_k \) and by \( \{A_0A_1, A_0A_2, A_0A_3\} \) the weighted Fermat tree, such that a positive real number (weight) \( B_r \) corresponds to each geodesic branch \( A_0A_r \), for \( i, j, k = 0, 1, 2, 3, r = 1, 2, 3 \).

Theorem 9 (Godel’s isometric immersion of a weighted Fermat tree with respect to a boundary spherical triangle to a weighted tetrahedron in \( \mathbb{R}^3 \)). If we select a triad of weights \( \{B_1, B_2, B_3\} \), which satisfy
\[
1 - \left( \frac{\sin(\kappa a_{13})}{\sin(\arccos(\frac{B_2^2 - B_1^2 - B_3^2}{2B_1B_3}))} \sin(\alpha_{213} - \arccos(\frac{\cos(\kappa a_{02}) - \cos(\kappa a_{01}) \cos(\kappa a_{12})}{\sin(\kappa a_{01}) \sin(\kappa a_{12})})) \right)^2 = 
\]
\[
(\cos(\kappa a_{01}) \cos(\kappa a_{13}) + \sin(\kappa a_{13}) \cos(\alpha_{213}) \frac{\cos(\kappa a_{02}) - \cos(\kappa a_{01}) \cos(\kappa a_{12})}{\sin(\kappa a_{12})} + 
\]
\[
+ \sin(\kappa a_{01}) \sin(\kappa a_{13}) \sin(\alpha_{213}) \cdot 
\]
\[
\sqrt{1 - \left( \frac{\cos(\kappa a_{02}) - \cos(\kappa a_{01}) \cos(\kappa a_{12})}{\sin(\kappa a_{01}) \sin(\kappa a_{12})} \right)^2} \right)^2,
\]
(3.1)

\[
(\cos(\kappa a_{12}) - \cos(\kappa a_{01}) \cos(\kappa a_{02}))^2 = (\sin(\kappa a_{01}) \sin(\kappa a_{02}) \frac{B_3^2 - B_1^2 - B_2^2}{2B_1B_2})^2
\]
(3.2)
such that \( \{a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23}\} \in DW_{R^2}(\ell, s) \) then the weighted Fermat tree \( \{a_{01}, a_{02}, a_{03}\} \) and the edges of the boundary spherical triangle are isometrically immersed to a weighted tetrahedron in \( \mathbb{R}^3 \) with corresponding edges \( \{a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23}\} \) and corresponding weights \( \{B_1, B_2, B_3, 1, 1, 1\} \).

Proof. We construct an isometric immersion of a Fermat tree with respect to \( \Delta A_1A_2A_3 \) following Godel’s observation of a tetrahedron with six flexible edges tightly pack to a sphere of given radius \( R = \frac{1}{\sqrt{K}} = \frac{1}{k} \) and taking into account two equations derived by [50, Theorem 2.4, (2.23),(2.24)], which determine the location of the weighted Fermat-tree. By restricting the edge lengths \( \{a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23}\} \in DW_{R^2}(\ell, s) \), we obtain a weighted tetrahedron \( A_0A_1A_2A_3 \) in \( \mathbb{R}^3 \).

Remark 1. We note that the two equations (3.1) and (3.2), which give the location of the weighted Fermat point \( \Delta A_1A_2A_3 \) on the K-plane (two dimensional sphere \( S^2_K \), and two dimensional hyperbolic plane \( H^2_K \)) can be merged to a rational algebraic equation, which depends only on the variable \( z \equiv \sin \angle A_0A_1A_3 \) (see in [51, Theorem 2.4, (2.8)])

In [41, Theorem 3,p. 728], Schoenberg obtained an isometric immersion of an \( (N - 1) \) spherical simplex in \( S_{\rho_0}^{N-1} \) to \( S_{\rho_1}^{N-2} \) by proving the existence of a radius \( \rho_1 \leq \rho_0 \). Thus, by applying Schoenberg’s isometric immersion, we derive the following reformulation of Theorem 9 for \( N = 4 \).

Theorem 10 (Schoenberg’s isometric immersion of a weighted Fermat tree with respect to a boundary spherical triangle in \( S_{\rho_0}^2 \) to a weighted tetrahedron in \( S_{\rho_0}^3 \)). If we select a triad of weights \( \{B_1, B_2, B_3\} \), which satisfy

\[
1 - \left( \frac{\sin(\kappa_1a_{13})}{\sin(\arccos(\frac{B_1^2-B_2^2-B_3^2}{2B_1B_3}))} \sin(\alpha_{213} - \arccos\left( \frac{\cos(\kappa_1a_{02}) - \cos(\kappa_1a_{01}) \cos(\kappa_1a_{12})}{\sin(\kappa_1a_{01}) \sin(\kappa_1a_{12})} \right) \right)^2 = \\
(\cos(\kappa_1a_{01}) \cos(\kappa_1a_{13}) + \sin(\kappa_1a_{13}) \cos(\alpha_{213}) \cos(\kappa_1a_{02}) - \cos(\kappa_1a_{01}) \cos(\kappa_1a_{12})) + \\
\sin(\kappa_1a_{01}) \sin(\kappa_1a_{13}) \cdot \sin(\alpha_{213}) \cdot \\
\sqrt{1 - \left( \frac{\cos(\kappa_1a_{02}) - \cos(\kappa_1a_{01}) \cos(\kappa_1a_{12})}{\sin(\kappa_1a_{01}) \sin(\kappa_1a_{12})} \right)^2 \right)^2, \tag{3.3}
\]

\[
(\cos(\kappa_1a_{12}) - \cos(\kappa_1a_{01}) \cos(\kappa_1a_{02}))^2 = (\sin(\kappa_1a_{01}) \sin(\kappa_1a_{02}) \frac{B_3^2 - B_2^2 - B_1^2}{2B_1B_2})^2 \tag{3.4}
\]
then there exists a \( \rho_0 > \rho_1 \), such that the weighted Fermat tree \( \{a_{01}, a_{02}, a_{03}\} \) and the edges of the boundary spherical triangle in \( S_{\rho_1}^2 \) are isometrically immersed to a weighted spherical tetrahedron in \( S_{\rho_0}^3 \) with corresponding edges \( \{a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23}\} \) and corresponding weights \( \{B_1, B_2, B_3, 1, 1, 1\} \),
for \( \{a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23}\} \in DW_{S_{\rho_0}^3}(\ell, s) \).

We note that we may use the weighted Fermat-Torricelli tree for an equilateral triangle having equal edge lengths \( \frac{\pi}{2} \) in \( S_{1}^2 \), which was given explicitly in [52], in order to construct a weighted isometric immersion in \( S_{\rho_0}^3 \) and in \( \mathbb{R}^3 \). These isometric immersions need to be controlled by the conditions for the edge lengths described by the Dekster-Wilker spherical and Euclidean domain.

Let \( A_1 A_2 \cdots A_{N-1} \) be an \((N-2)\)-simplex in \( S_{\rho_1}^{N-2} \), \( A_0 \) be the weighted Fermat point of the \((N-2)\)-simplex having degree \( N-1 \) and \( \{A_1 A_0, A_2 A_0, \cdots, A_{N-1} A_0\} \) the corresponding weighted Fermat tree. We denote by \( B_i \) the weight, which corresponds to each geodesic branch \( A_i A_0 \) and by \( a_{jm} \) the length of the geodesic arc \( A_j A_m \), for \( i = 0, 1, 2, \cdots N-1 \) and \( j, m = 0, 1, 2, \cdots, N-1 \).

We denote by \( \{A_1, A_2, \cdots, A_0, \cdots, A_{N-1}\} \subset \{A_1, A_2, \cdots, A_{N-1}\} \) an \((N-2)\)-simplex, which is derived by replacing the vertex \( A_j \in \{A_1, A_2, \cdots, A_{N-1}\} \) with \( A_0 \).

**Theorem 11** (Godel-Schoenberg’s isometric immersion of weighted Fermat trees in \( S_{\rho_1}^{N-2} \) to \( S_{\rho_0}^{N-1} \)). If we select an \((N-1)\)-tuple of weights \( \{B_1, \cdots, B_{N-1}\} \), which satisfy

\[
\sum_{i=2}^{N-1} \frac{B_i}{\sin Ka_{01} \sin Ka_{0i}} (-\cos Ka_{01} \cos Ka_{0i} + \cos Ka_{1i}) - \sum_{i=1, i\neq j}^{N-1} \frac{B_i}{\sin Ka_{0j} \sin Ka_{0i}} (-\cos Ka_{0j} \cos Ka_{0i} + \cos Ka_{ji}) = 0, \quad (3.5)
\]

\[
\text{Vol}(A_1, A_2, \cdots, A_0, \cdots, A_{N-1})) < \text{Vol}(C_1, C_2, \cdots, C_{N-1})) \quad (3.6)
\]

where \( C_1, C_2, \cdots C_{N-1} \) is a regular spherical \((N-2)\)-simplex with edge length \( c_{ij} = \frac{\sum_{a, j} a_{ij}}{N-2} \).

then there exists a \( \rho_0 > \rho_1 \), such that the weighted Fermat tree \( \{a_{01}, \cdots, a_{0N-1}\} \) and the edges of the boundary \((N-1)\)-simplex \( \{a_{ij}\} \) in \( S_{\rho_1}^{N-2} \) are isometrically immersed to a weighted spherical \((N-1)\)-simplex in \( S_{\rho_0}^{N-1} \) with corresponding edges \( \{a_{01}, \cdots, a_{0N-1}, \{a_{ij}\}\} \) and corresponding weights \( \{B_1, B_2, \cdots, B_{N-1}, 1, \cdots, 1\} \).
for the \(\frac{(N-1)N}{2}\) tuple of edge lengths \(\{a_0, \cdots, a_{N-2}, \{a_{ij}\}\} \in DW_{S_{\rho_0}}(\ell, s)\).

**Proof.** By applying Schoenberg isometric immersion and by taking into account the conditions of Theorem 8 we can construct an isometric immersion of weighted Fermat trees in \(S_{\rho_1}^N \to S_{\rho_0}^{N-1}\), the \(\frac{(N-1)N}{2}\) tuple of edge lengths \(\{a_0, \cdots, a_{N-2}, \{a_{ij}\}\} \in DW_{S_{\rho_0}}(\ell, s)\).

By substituting \(\rho = \infty\) in Schoenberg’s isometric immersion, we get Godel-Schoenberg’s isometric immersion (\cite{41}, Theorem 3’, p. 730), we derive Godel’s isometric immersion of weighted Fermat trees for a boundary \((N-1)\)-simplex in \(S_{\rho}^{N-1}\) to a weighted \(N\)-simplex \(\mathbb{R}^N\).

We denote by \(\{A_1, A_2, \cdots, A_0, \cdots, A_N\} \subset \{A_1, A_2, \cdots, A_N\}\) an \((N-1)\)-simplex, which is derived by replacing the vertex \(A_j \in \{A_1, A_2, \cdots, A_N\}\) with \(A_0\).

**Theorem 12** (Godel’s isometric immersion of weighted Fermat trees in \(S_{\rho}^{N-1} \to \mathbb{R}^N\)). If we select an \(N\)-tuple of weights \(\{B_1, \cdots, B_{N-1}, B_N\}\), which satisfy

\[
\sum_{i=2}^{N} \frac{B_i}{\sin K a_{01} \sin K a_{0i}} ( - \cos K a_{01} \cos K a_{0i} + \cos K a_{1i} ) - \sum_{i=1, i \neq j}^{N} \frac{B_i}{\sin K a_{0j} \sin K a_{0i}} ( - \cos K a_{0j} \cos K a_{0i} + \cos K a_{ji} ) = 0, \quad (3.7)
\]

where \(\{C_1, C_2, \cdots, C_{N-1}, C_N\}\) is a regular spherical \((N-1)\) simplex with edge length \(c_{ij} = \frac{\sum a_{ij}}{(N-1)N}\).

then there exists an isometric immersion of the weighted Fermat tree \(\{a_0, \cdots, a_{N}\}\) and the edges of the boundary \(N-1\) simplex \(\{a_{ij}\}\) in \(S_{\rho_1}^N\) to a weighted \(N\)-simplex in \(\mathbb{R}^N\) with corresponding edges \(\{a_{01}, \cdots, a_{0N}, \{a_{ij}\}\}\) and corresponding weights \(\{B_1, B_2, \cdots, B_{N}, 1, \cdots, 1\}\).

for the \(\frac{(N(N+1)}{2}\) tuple of edge lengths \(\{a_0, \cdots, a_{N}, \{a_{ij}\}\} \in DW_{\mathbb{R}^N}(\ell, s)\).

We consider a \(\frac{(N-1)N}{2}\) -tuple of positive real numbers \(a_{ij}\), determining the edge lengths of \((N-1)\) incongruent spherical simplexes, such that: \(a_{ij}\) belong to the spherical domain of Dekster-Wilker \(DW_{S_{\rho_0}}(\ell, s)\) and \(s \leq a_{ij} \leq \ell\). Thus, all incongruent pairwise spherical \((N-1)\)-simplexes may yield up to \(\frac{(N-1)N}{N!}\) weighted Fermat-Torricelli trees for a given \(N\)-tuple of weights \(\{B_1, B_2, \cdots, B_N\}\). The union of these weighted Fermat-Torricelli trees determine the weighted Fermat-Frechet multtree (solution).
**The Weighted Fermat-Frechet Multitree in the K-space**

**Definition 6** (Stable isometric immersion of a variable weighted Fermat-Frechet multitree). We call a stable isometric immersion of a variable weighted Fermat-Frechet multitree for a \((N-1)\)-tuple of positive real numbers \(a_{ij}\), determining \((N-1)\)-incongruent spherical simplexes in \(S^{N-1}_{\rho}\) to \(\mathbb{R}^{N}\) the solution set of the variable weights \(\{B_1, B_2, \cdots, B_N\}\), such that each \(N-1\) spherical simplex with the corresponding variable weighted Fermat tree in \(S^{N-1}_{\rho}\) is isometrically immersed to an \(N\)-simplex in \(\mathbb{R}^{N}\).

**Definition 7** (Godel-Frechet multisimplex). We call Godel-Frechet multisimplex a union of \(N\)-simplexes in \(\mathbb{R}^{N}\), which is derived by a stable isometric immersion of a variable weighted Fermat-Frechet multitree in \(S^{N-1}_{\rho}\).

We denote by \(B_i(t)\) a variable weight, which correspond to the geodesic branch \(A_0A_i\) of each weighted Fermat tree \(\{A_1A_0, c, \ldots, A_NA_0\}\), which forms a weighted Fermat-Frechet multitree, for \(t > 0, i = 1, 2, \cdots, N\).

**Theorem 13** (Godel’s isometric immersions of a weighted Fermat-Frechet multitree in \(S^{N-1}_{\rho}\) to \(\mathbb{R}^{N}\)). The following conditions for \(B_i(t)\) provide a stable isometric immersion of a weighted Fermat-Frechet multisimplex in \(S^{N-1}_{\rho}\) to a Godel-Frechet simplex in \(\mathbb{R}^{N}\):

\[
\sum_{i=2}^{N} \frac{B_i(t)}{\sin Ka_{01} \sin Ka_{0i}} (-\cos Ka_{01} \cos Ka_{0i} + \cos Ka_{1i}) - \\
\sum_{i=1, i\neq j}^{N} \frac{B_i(t)}{\sin Ka_{0j} \sin Ka_{0i}} (-\cos Ka_{0j} \cos Ka_{0i} + \cos Ka_{ji}) = 0, \quad (3.9)
\]

\[
\sum_{i=1}^{N} B_i(t) = 1, \quad (3.10)
\]

where \(C_1, C_2, \cdots, C_N\) is a regular spherical \((N-1)\)-simplex with edge length \(c_{ij} = \sum_{k \neq i,j} a_{0k}^{\frac{1}{N}}\).

Proof. By substituting \(B_i(t) \rightarrow B_i\) in (3.7) taking into account the isoperimetric condition for the variable weights (3.10) and the volume inequality (3.8) taken from Theorem 12, we derive (3.9), which yields
the desired stable isometric immersion of the variable weighted Fermat-Frechet spherical multitree to the variable weighted Godel-Frechet multisimplex in $\mathbb{R}^N$. □

We focus on small perturbations $\epsilon_{ij}$ of a $\frac{(N-1)N}{2}$-tuple of positive real numbers $a_{ij}$, determining the edge lengths of $(N - 1)$ incongruent spherical simplexes, such that: $a_{ij}, a_{ij} + \epsilon_{ij} \in DW_{S^{N-1}}(\ell, s)$ and $s \leq a_{ij} \leq \ell$, under the condition:

$$\sum_{i,j=1,i<j}^{N} a_{ij} = \sum_{i,j=1,i<j}^{N} a_{ij} + \epsilon_{ij},$$

for $\|\epsilon_{ij}\| << 1$ and $\sum_{i,j=1,i<j}^{N} \epsilon_{ij} = 0$.

By substituting $a_{ij} + \epsilon_{ij} \rightarrow a_{ij}$ in Theorem 13, we derive the conditions to create a family of stable isometric immersions of spherical variable weighted Fermat-Frechet multitrees to Godel-Frechet multisimplexes depending on the isoperimetric perturbations $\epsilon_{ij}$ of the initial boundary simplex in $S^{N-1}$.

**Theorem 14** (Stable isometric immersions of weighted Fermat-Frechet multitrees for isoperimetric deformations of the boundary $N-$simplex in $S^{N-1}$ to $\mathbb{R}^N$). The following conditions for $B_i(t)$ provide a family of stable isometric immersions of variable weighted Fermat-Frechet multisimplexes in $S^{N-1}$ to $\epsilon_{ij}$-Godel-Frechet simplexes in $\mathbb{R}^N$:

$$\sum_{i=2}^{N} \frac{B_i(t)}{\sin Ka_{01} \sin Ka_{0i}} (-\cos Ka_{01} \cos Ka_{0i} + \cos K(a_{1i} + \epsilon_{1i}) -$$

$$\sum_{i=1,i\neq j}^{N} \frac{B_i(t)}{\sin Ka_{0j} \sin Ka_{0i}} (-\cos Ka_{0j} \cos Ka_{0i} + \cos K(a_{ji} + \epsilon_{ji})) = 0,$$

(3.12)

$$\sum_{i=1}^{N} B_i(t) = 1,$$

(3.13)

$$\text{Vol}(A_1, A_2, \cdots, A_0 \cdots, A_N)(\{\epsilon_{ij}\}) < \text{Vol}(C_1, C_2, \cdots, C_N)(\{\epsilon_{ij}\})$$

(3.14)

where $C_1, C_2, \cdots, C_N$ is a regular spherical $(N - 1)$-simplex with edge length $c_{ij} = \frac{\sum_{i,j} a_{ij} + \epsilon_{ij}}{N-1}$, for the $\frac{(N(N+1)}{2}$-tuple of edge lengths $\{a_{01}, \cdots, a_{0N}, a_{ij}\} \in DW_{\mathbb{R}^N}(\ell, s)$. 
4. Gromov isometries for the Fermat-Steiner-Frechet solution in $\mathbb{H}^N$

In this section, we introduce an embedding (inclusion map) of a Fermat-Steiner-Frechet multitree solution for a given $\frac{N(N+1)}{2}$-tuple of edge lengths determining incongruent boundary $N$-simplexes to an associated family of Gromov isometries up to an additive constant for $N$-simplexes and ideal $N$-simplexes in $\mathbb{H}_K^N$.

We proceed by giving the definitions of an ideal $k$-simplex $\Delta_k^i \equiv A_1^i A_2^i \cdots A_{k+1}^i$ in $\mathbb{H}_K^N$ with constant curvature $K = -\epsilon^2 < 0$, a class of geodesic trees enriched by some properties given by Gromov (intermediate geodesic Fermat-Steiner trees) and Gromov’s (+) isometry (up to an additive constant).

The hyperbolic $N$ space $\mathbb{H}_N^k$ may be represented as the Poincare disc model or the half space model or the projective (Klein) model.

If we consider that $\mathbb{H}_N$ is represented by the Poincare disc model $\mathbb{H}_N \approx D^N = \{x \in \mathbb{R}^N ||x|| < 1\}$ with the Riemannian metric

$$ds^2 = \frac{4}{1 - r^2} \sum_{i=1}^{N} (dx_i)^2, \quad r^2 \equiv \sum_{i=1}^{N} x_i^2$$

(see in [26]).

**Definition 8.** [26, p. 1] The hyperbolic boundary $\partial \mathbb{H}^N$ is the "sphere at infinity"

$$\partial \mathbb{H}^N = \{x \in \mathbb{R}^N ||x|| = 1\} = S^{N-1}.$$  

**Definition 9.** An ideal $k$-simplex $\Delta_k^i$, [24, p. 233] An ideal $N$-simplex in $\mathbb{H}_{-\epsilon^2}^N$ is the convex hull of $N + 1$ distinct points (vertices) $\{A_1^d, A_2^d, \cdots, A_N^d, A_{N+1}^d\}$ in the hyperbolic boundary $\partial \mathbb{H}_{-\epsilon^2}^N$ ("sphere at infinity").

In [46], Thurston showed that all ideal triangles in $\mathbb{H}_{-1}^2$ are isometric. In [47], Milnor derived that ideal hyperbolic tetrahedra are not mutually isometric. In [24, pp. 233-234] Gromov assigned properties in a geodesic tree $S \subset \Delta_N^N \subset \mathbb{H}^N$ and used in a canonical way an inclusion map $S \to \Delta_N^N$, to confirm that ideal $N$–simplexes are not mutually isometric for $N \geq 3$.

We extend the definitions of Steiner tree topologies given in [22] and intermediate Fermat-Steiner tree topologies, which have been introduced in [54] in $\mathbb{R}^3$, for geodesic trees in $\mathbb{H}_{-\epsilon^2}^N$. 
Definition 10 (Geodesic tree topology). A geodesic tree topology is a connection matrix specifying which pairs of points from the list \( \{A_1, A_2, \ldots A_N, X_1, X_2, \ldots X_{N-2}\} \in \mathbb{H}_N^{N-2} \) have a connecting geodesic segment (edge).

Definition 11 (Degree of a vertex). The degree of a vertex corresponds to the number of connections of the vertex with geodesic segments.

Definition 12 (Degree of an intermediate Fermat-Steiner point). The degree of a (weighted) intermediate Fermat-Steiner point (vertex) \( X \) with respect to a boundary simplex \( A_1, A_2, \ldots A_N \in \mathbb{H}_N^{N-2} \) is greater or equal than 3 and less or equal than \( N \).

Definition 13 (Fermat (geodesic) tree topology). A Fermat tree topology is a tree topology with a connection matrix \( A_1, A_2, \ldots A_N, X_1, X_2, \ldots X_{N-1} \) such that the boundary vertices \( A_1, A_2, \ldots A_N \) have degree 1 and the Fermat-point \( X_1 \) has degree \( N \).

Definition 14 (Fermat-Steiner (geodesic)) tree topology). A (full) Fermat-Steiner tree topology is a tree topology with a connection matrix \( A_1, A_2, \ldots A_N, X_1, X_2, \ldots X_{N-1} \) such that the boundary vertices of the simplex \( A_1, A_2, \ldots A_N \) have degree 1 and the Fermat-Steiner points \( X_1, X_2, \ldots X_{N-1} \) have degree 3.

Definition 15 (Intermediate Fermat-Steiner (geodesic) tree topology). An intermediate Fermat-Steiner tree topology is a tree topology, which has \( m \leq N-2 \) \( X_1, X_2, \ldots X_m \) vertices inside the simplex \( A_1, A_2, \ldots A_N \in \mathbb{H}_N^{N-2} \), such that the boundary vertices of the simplex \( A_1, A_2, \ldots A_N \) have degree 1 and the intermediate Fermat-Steiner points \( X_1, X_2, \ldots X_m \) have degree less than \( N \).

The intermediate Fermat-Steiner tree \( S \) is a union of some geodesic segments between the intermediate Fermat-Steiner vertices \( X_j \) and some geodesic segments connecting each \( A_i \) with some \( X_j \). In [24], Gromov assigned in a canonical way an intermediate Fermat-Steiner geodesic tree for an ideal simplex \( A_1, A_2, \ldots A_N \) some geometric properties [24] (a),(b),(c),(d),(e), p. 233).

Example 1. An intermediate Fermat-Steiner tree \( S \) of a 4-simplex \( A_1, A_2, A_3, A_4, A_5 \) associated with an intermediate Fermat-Steiner tree topology is a collection of the geodesic segments \( \{A_1X_1, A_4X_1, X_1X_2, A_5X_2, A_2X_2, A_3X_2\} \). The intermediate Fermat Steiner points \( X_1, X_2 \) have degrees 3 and 4, respectively and the boundary vertices \( \{A_1, A_2, A_3, A_4, A_5\} \) have degree 1.
Definition 16 (Intermediate Fermat-Steiner Frechet multitree). An intermediate Fermat-Steiner Frechet multitree is a collection of intermediate Fermat-Steiner trees, which correspond to incongruent \( N \)-simplexes derived by a \( \frac{N(N+1)}{2} \)-tuple of positive real numbers determining edge lengths.

Remark 2. An intermediate Fermat-Steiner Frechet multitree \( MS \) coincides with a Fermat-Frechet multitree, by setting only one vertex \( X_1 \) inside each derived simplex (Frechet multisimplex \( F\Delta^N \)) having degree \( N \).

Definition 17. A Fermat-Frechet multispanning tree is a Fermat-Frechet multitree with zero interior vertices \( (m = 0) \) at each simplex, which belongs to the Frechet multisimplex.

In [24, 6.2, pp. 157-158], Gromov constructed a geodesic tree (log \( 2 \) \( N \) spanning geodesic tree) \( \tau \) subset \( X \), for a finite subset \( \{V = V_1, V_2, \cdots, V_N\} \) of a geodesic \( \delta \) hyperbolic space \( X \) with following properties:

(1) \( \tau \) is a union of at most \( N - 1 \) geodesic segments in \( X \),

(2) the set of extremal points of \( \tau \) equals \( V \),

(3) Every two points \( V_i, V_j \) in \( V \) can be joined by a broken geodesic \( g \) having \( k \leq 1 + 2\log_2 N \) segments, such that:

\[
\text{length} g \leq |V_i - V_j| + C\delta(\log_2(\text{log} N))^2, \\
\]

for \( C \leq 100 \).

In [8], [9], Bowditch obtained another tree-like structure of a \( \delta \) hyperbolic space \( X \) in the sense of Gromov, by determining an isometry of a spanning tree up to an additive constant, which depend on \( N \) and a constant \( \delta \), which lead to an isometric embedding of "logarithmic \( \log N \) spanning trees".

Lemma 8. Isometric embedding of a spanning tree up to an additive constant. [9, Proposition 6.7, pp. 49-51] There is a function \( h : N \rightarrow [0, \infty) \) such that if \( F \subset X \) with \( |F| = N \), then there is a (spanning) tree \( \tau \), such that for all \( X, Y \in F \), \( d_\tau(X, Y) \leq XY + \delta h(N) \), \( (d_\tau \) is distance measured in \( \tau \) and all the edges of \( \tau \) are geodesic segments).

Lemma 9. Isometric embedding of a spanning tree up to an additive logarithmic constant. [8, Proposition 7.3.1, Theorem 7.6.1] There is a function \( f : N \rightarrow \mathbb{R} \), such that the following holds. Suppose \( (X, d) \) is a \( \delta \) hyperbolic geodesic space and the vertex set \( V \subset X \) is a set of
$(N + 1)$ points. Then there is an immersed spanning tree $τ$ for $V$ in $X$, such that for any $X, Y \in V$, we have

$$\rho_τ(X, Y) \leq d(X, Y) + hF(N),$$

where $F(N) = O(\log N)$.

We proceed by obtaining an embedding of Fermat-Steiner Frechet multitree to $\Delta^N$, taking into consideration Gromov’s observation that the regular ideal simplex having the maximal volume corresponds to a tree $S$ having a maximal length metric among the Fermat-Steiner trees, which consists of the union of $N + 1$ rays joining the ideal points $A_i$ with only a single point $X_1 \in H^N$ [24 Property (d), p. 233].

We consider an example of a tree $S$ for a boundary tetrahedron in $\mathbb{R}^3$. We show that we can perturb the maximal length (metric) of the unweighted Fermat-Steiner trees, by assigning properly some weights at each vertex of the tetrahedron, which yields a maximal weighted length metric of the Fermat tree greater than the maximal unweighted length metric of the weighted Fermat tree having the same geometric structure (tree topology).

**Example 2.** Let $A_1A_2A_3A_4$ be a tetrahedron and $X_1$ be the corresponding Fermat point in $\mathbb{R}^3$. One can rotate by a suitable angle (twist angle), such that $A_1A_2A_3A_4$ and $X_1$ lie on the same plane. Thus, $X_1$ is the intersection of the diagonals $A_1A_3$ and $A_2A_4$. One can assign the following weights $B_i$ (positive real numbers) at each vertex $A_i$ (Fig. 1). The length of the unweighted ($B_i = 1$) Fermat tree is given by

$$\text{length}_g = A_1A_3 + A_2A_4.$$

We can perturb the length metric of the Fermat tree, by assigning the following weights at each $A_i$, such that the weighted Fermat point $X_1$ remains the same:

$$B_1 = B_3 = 1 + \epsilon, \quad B_2 = B_4 = 1 - \epsilon.$$

$$\text{length}_{g_p} = (1 + \epsilon)A_1A_3 + (1 - \epsilon)A_2A_4.$$

Hence, we obtain the inequality for $\epsilon > 0$ and $a_{13} > a_{24}$ or $\epsilon < 0$ and $a_{13} < a_{24}$:

$$\text{length}_{g_p} - \text{length}_g = \epsilon(a_{13} - a_{24}) > 0.$$

**Theorem 15.** The embedding of an intermediate Fermat-Steiner Frechet multitree $MS \to F\Delta^N$ is a collection of isometries up to an additive constant, where the implied constants only depends on $N$ and $\epsilon$. 


Proof. A. A Proof using weights

We can assign weights \( B_1, B_2, \cdots B_{N+1} \) to each vertex of the simplex \( A_1A_2\cdots A_{N+1} \in \mathbb{H}_{\epsilon,2}^N \), to perturb the corresponding maximal length metric of the Fermat-tree, which gives an isometry up to an additive \( \log_2 N \) or \( \log N \) constant due to Gromov, Bowditch for a given \( \frac{N(N+1)}{2} \) tuple of edge lengths determining a simplex by using the solution domain of Dekster-Wilker \( \text{DW}(\mathbb{H}_{\epsilon,2}^N) \). Thus, the collection of the isometries up to an additive constant depend on \( \epsilon \) and \( N \) and gives a "multi-"inclusion map of intermediate Fermat-Steiner-Frechet multi-trees \( MS \) to a Frechet multisimplex \( F\Delta^N \).

B. Proof without using weights

We use the following result for the length metric of minimal Steiner trees and associated minimal spanning trees, which has been proved by Ivanov Tuzhilin and Cieslik for manifolds ([31], [29]). The length of the Fermat-Steiner tree with intermediate vertices is less than length of the spanning tree with zero intermediate vertices. Therefore, by applying
Gromov-Bowditch constructions, we get

\[ \text{length}(\text{Fermat Steiner tree}) < \text{length Spanning tree} \leq |V_i - V_j| + C\delta(\log_2(N))^2, \]

for \( C \leq 100 \).

We define a dynamic intermediate Fermat-Steiner Frechet multitree for some perturbed hyperbolic Frechet multisimplexes and we consider embeddings of dynamic intermediate Fermat-Steiner Frechet multitrees to almost regular Frechet multisimplex in terms of Dekster’s lower bound estimate for the generalized Santalo’s thickness of simplexes in \( \mathbb{H}^N_{-\epsilon} \) ([16 Theorem 2 (2.3), pp. 51-52]).

**Definition 18** (A dynamic intermediate Fermat-Steiner Frechet multitree). A dynamic intermediate Fermat-Steiner-Frechet multitree is a union of Fermat-Steiner-Frechet multitrees, which are derived by a given \( \frac{N(N+1)}{2} \)-tuple of positive real numbers determined by the edge lengths \( \ell_i \) of a Frechet multisimplex in \( \mathbb{H}^N_k \) perturbed by a real number \( \epsilon_i : \ell'_i = \ell_i + \epsilon_i \):

\[
\frac{N(N+1)}{2} \sum_{i=1}^{N(N+1)} \ell'_i = \sum_{i=1}^{N(N+1)} \ell'_i, 
\]

where \( \sum_{i=1}^{N(N+1)} \epsilon_i = 0 \) and \( \ell'_i \) are the edge lengths of a Frechet simplex obtained by the Dekster Wilker solution domain \( \text{DW}(\mathbb{H}^N_k) \).

We note that Dekster ([16 p 52, (1.14), Example, pp. 53-54]) extended Gromov’s thinness of geodesic triangles in \( \delta = \frac{1}{2} \log 3 \) hyperbolic spaces ([24 6.3, Lemma 6.3.A], [9 Example 4, p. 54]), by obtaining a generalization of Santalo’s thickness from \( \mathbb{H}^2 \) to \( \mathbb{H}^N \) and distinguish the thickness of simplexes of a compact convex set \( C \) (simplex) \( t_g(C) \) to the general thickness \( t_g(C) \) and the normal thickness \( t_N \), which yields \( t_N(C) \geq t_g(C) \).

**Lemma 10.** [16 Theorem 2, (2.3), pp. 54-55] Let \( \Delta^N \) be an \( N \)-simplex in \( \mathbb{H}^N \) with edges \( a_{ij} \in [s, \ell] \subset [\lambda_N(\ell), \ell] \) where \( \ell > 0 \) and \( \lambda_N(\ell) \) is given by

\[
\cosh \lambda_N(\ell) = \sqrt{\cosh^2 \ell - 2f_N(\cosh \ell - 1) \left[ 1 + \frac{N}{N+1} (\cosh \ell - 1) \right],}
\]

where

\[
f_N = \begin{cases} 
\frac{2}{N+1} & \text{for odd } N, \\
\frac{2}{N+1} & \text{for even } N 
\end{cases}
\]
Then the general thickness $t_g$ of $\Delta^N$ satisfies:

$$t_g \geq \cosh^{-1}(1 + \frac{\cosh s - \cosh \lambda_N(\ell)}{\cosh^2 \ell}).$$

**Theorem 16** (Isometric embedding of a dynamic Fermat-Steiner-Frechet multitree to a Frechet almost regular multisimplex). The embedding of a dynamic intermediate Fermat-Steiner Frechet multitree $MS \rightarrow F\Delta^N$, $F\Delta^N_\ell$ is a family of almost regular multiFrechet simplexes for $s = \ell - \max\{|\epsilon_1|, \cdots |\epsilon_i|\}$ very close to $\ell$, which gives a collection of isometries up to an additive constant, where the implied constants only depends on $N$ and $\epsilon$, such that $t_g(F\Delta^N) \geq \frac{1}{2\epsilon} \log 3$ for $s \leq a_{ij} \leq \ell$ and $(s, \ell) \in DW_{H^N,2}$.

**Proof.** Let $A_1A_2\cdots A_{N+1}$ be an almost regular simplex for $s$ very close to $\ell$ and $X_1$ is the corresponding Fermat point. By taking $X_1$ as a reference point, the rays $X_1A_i$ issuing from $X_1$ intersect $\partial H_{N,2}$ at $A_i^\circ$, for $i = 1, 2, \cdots, N$, which yields that $A_1^\circ A_2^\circ \cdots A_N^\circ$ is an ideal almost regular $N$-simplex. By using Gromov’s observation that we can assign a Fermat tree having a maximal length metric to each $A_1^\circ A_2^\circ \cdots A_N^\circ$, which has an almost maximal volume and by applying Theorem 15, we can embed isometrically up to an additive constant to each member of multiFrechetsimplexes $F\Delta^N_\ell$, for $s = \ell - \max\{|\epsilon_1|, \cdots |\epsilon_i|\}$ very close to $\ell : s \leq a_{ij} \leq \ell$.

The Fermat point of $A_1A_2\cdots A_{N+1}$ and $A_1^\circ A_2^\circ \cdots A_N^\circ$ remains the same, because

$$\sum_{i=1}^{N+1} \frac{\exp^{-1}_{X_1} X_1 A_i}{|X_1 A_i|} = \sum_{i=1}^{N+1} \frac{\exp^{-1}_{X_1} X_1 A_i^\circ}{|X_1 A_i^\circ|} = 0.$$

□

Let $S$ be an intermediate Fermat-Steiner tree with two intermediate Fermat-Steiner points $X_1$ and $X_2$ inside the ideal $N$ simplex $A_1^\circ \cdots A_{N+1}^\circ$ in $\mathbb{H}_k^N$ having degree $p$ and $(N+2) - p$ and the boundary ideal vertices $A_i^\circ$ having degree one, for $3 \leq p \leq N - 2$.

**Lemma 11.** Convergence of $\Delta^3$ [24, Example, p. 234] If $X_1X_2 \rightarrow \infty$, for $N = 3$, $p = 3$, then $\Delta^3$ converges exponentially fast converges to the union of two triangles spanned by the triples $\{A_1^\circ, A_2^\circ, \frac{X_1+X_2}{2}\}$ and $\{A_3^\circ, A_4^\circ, \frac{X_1+X_2}{2}\}$.

**Lemma 12** (Extension of convergence of an $N$-simplex to a $p$-simplex and $N + 3 - p$ simplex). If $X_1X_2 \rightarrow \infty$, then $\Delta^N$ exponentially fast
converges to the union of two hyperbolic simplexes spanned by the p-tuple \( \{ A_1^0, A_2^0, \ldots, A_{p-1}^0, \frac{X_1 + X_2}{2} \} \) and the \((N + 3 - p)\)-tuple \( \{ A_p^0, A_{p+1}^0, \ldots, A_{N+2-p}^0, \frac{X_1 + X_2}{2} \} \), respectively.

In [24, Example, p. 234], Gromov also observed that if the basic invariant of \( \Delta^3 X_1 X_2 < \infty \), then the geometry of the ideal simplex \( \Delta^3 \) is close to that of the regular ideal simplex.

In Nature, an evolutionary tree with respect to a boundary closed polyhedron in \( \mathbb{R}^3 \) tends to maximize the volume of the closed polyhedron formed by the boundary vertices with a minimum communication via minimum transfer of mass along the branches. The evolutionary tree reduces its length by placing intermediate Fermat-Steiner points inside the boundary polyhedron.

**Definition 19 (Degree of intelligence).** We call degree of intelligence of an evolutionary tree in \( \mathbb{H}^N_K \) the number of intermediate Fermat Steiner points, which correspond to an intermediate Fermat Steiner tree of an \( N \)-simplex in \( \mathbb{H}^N_K \).

**Definition 20 (Maximum degree of intelligence of an evolutionary tree).** The maximum degree of intelligence of an evolutionary tree, which correspond to an intermediate Fermat Steiner tree of an \( N \)-simplex in \( \mathbb{H}^N_K \) is \( N - 2 \).

Consider two trees \( T \) and \( T' \) having the same set of boundary vertices in \( \mathbb{H}^2_k \).

**Definition 21 (Intelligent trees).** The tree \( T \) is more intelligent than then tree \( T' \) if the degree of intelligence of \( T \) is greater than the degree of intelligence of \( T' \).

We continue with Gromov’s definition of thin triangles in \( \delta \) hyperbolic metric spaces, which help us to view \( \delta \) hyperbolic metric spaces as metric trees with a prescribed thickness ([24], [10, 1.1 Definition, Fig. H.1, p. 399]).

**Definition 22.** Thin triangle. [24, p. 183] [10, 1.1 Definition, Fig. H.1, p. 399] A geodesic triangle in a metric space is said to be \( \delta \) thin if each of its sides is contained in the \( \delta \) neighborhood of the union of the other two sides.

A metric (geodesic) tree with zero thickness is an \( \mathbb{R} \) tree (Zero hyperbolic geodesic space)

**Definition 23 (Degree of intelligence of a \( \delta \) metric tree).** The degree of intelligence of a \( \delta \) metric tree embedded in an \( N \) simplex is the degree of
intelligence of the corresponding zero (δ = 0) metric tree (intermediate Fermat-Steiner tree) with respect to the same boundary N-simplex.

Consider an intermediate Fermat-Steiner tree having two degrees of intelligence $X_1, X_2$ with corresponding degrees $p$ and $N + 2 - p$, respectively inside the ideal simplex $A_1^o A_2^o \cdots A_N^o \in \mathbb{H}_c^N$.

**Theorem 17** (Reduction of the degree of intelligence of a metric δ tree). *If the hyperbolic distance $X_1^o X_2^o \to \infty$, then the degree of intelligence of the $\frac{1}{2\pi} \log 3$ metric tree is reduced by one degree from two to one.*

*Proof.* By applying Lemma [12] and taking into account $X_1^o X_2^o \to \infty$, $\Delta^N$ exponentially fast converges to the union of two hyperbolic simplexes spanned by the $p$-tuple $\{A_1^o, A_2^o, \ldots, A_{p-1}^o, \frac{X_1+X_2}{2}\}$ and the $N + 3 - p$-tuple $\{A_p^o, A_{p+1}^o, \ldots, A_{N+2-p}^o, \frac{X_1+X_2}{2}\}$, respectively.

Hence, one may consider the vertex $\frac{X_1+X_2}{2}$ as the weighted Fermat point with respect to the union of the two hyperbolic simplexes, which minimize the objective function (zero Fermat (metric) geodesic tree)

$$w_1 A_1^o \frac{X_1+X_2}{2} + w_2 A_2^o \frac{X_1+X_2}{2} + \cdots + w_{N+2-p} A_{N+2-p}^o \frac{X_1+X_2}{2}.$$

Taking into account the corresponding $\delta = \frac{1}{2\pi} \log 3$ we may consider a $k\delta$ hyperbolic metric tree ([24, Lemma, p. 183]) as a union of the $\delta$ branches $A_i^o \frac{X_1+X_2}{2}$, for $i = 1, 2, \ldots, N + 1$, for a proper $N + 1$ tuple of weights (positive real numbers), we obtain a metric tree structure having one degree of intelligence (one Fermat point). □

5. A variational approach to the weighted Fermat-Frechet problem for a given sextuple of edge lengths determining a 3-simplex in the threedimensional $K$-Space

In this section, we obtain a method to differentiate the length of a geodesic arc with respect to a geodesic arc in the 3 $K$-Space, which is different than the method derived by Schlaffi ([10] and Luo ([35]). This new variational technique can be applied to solve the weighted Fermat-Frechet problem for boundary incongruent tetrahedra in the 3 $K$-Space and determine the corresponding weighted Fermat trees for each derived tetrahedron, such that their edges belongs to the Dekster-Wilker spherical domain $DW_{S^3}$ $(\ell, s)$ or hyperbolic domain $DW_{H^3}$ $(\ell, s)$.

Let $A_1 A_2 A_3 A_4$ be a tetrahedron in the 3$K$-Space and $A_0$ be the corresponding weighted Fermat point inside the tetrahedron.
We denote by $\alpha$ the dihedral angle formed by the planes $\triangle A_1A_2A_3$ and $\triangle A_1A_2A_0$ and by $\alpha_{g_4}$ the dihedral angle formed by the planes $\triangle A_1A_2A_3$ and $\triangle A_1A_2A_4$, by $h_{0,12}$ the height of $\triangle A_1A_2A_3$ from $A_0$ to $A_1A_2$.

by $A_{0,12}$ the projection of $A_0$ to $A_1A_2$ by $A_{0,123}$ the projection of $A_0$ to the $K$-plane defined by $\triangle A_1A_2A_3$ by $h_{0,123}$ the length of $\triangle A_0A_{0,12,13}$ and by $l$ the length of $A_{0,12}A_2$, for $i = 1, 2, 3$.

We set $\alpha_{ijk} \equiv \angle A_iA_jA_k$, $\beta \equiv \angle A_1A_2A_{0,123}$, $h_{0,12} \equiv A_0A_{0,12}$, $h_{0,123} \equiv A_0A_{0,123}$ and $l \equiv A_2A_{0,12}$.

for $i, j, k = 0, 1, 2, 3$.

\textbf{Theorem 18.} The geodesic edges $a_{03}$ and $a_{04}$ can be expressed as functions of $a_{01}, a_{02}, \alpha$:

$$\cos k\alpha_{03} = \cos k\alpha_{02} \cos k\alpha_{23} +$$
$$+ \sin k\alpha_{23} \cos k\alpha_{01} \cos k\alpha_{02} \sin k\alpha_{123} \cos kl(a_{01}, a_{02}) +$$
$$+ \sin k\alpha_{23} \sin k\alpha_{123} \cos \alpha$$ \quad (5.1)

$$\cos k\alpha_{04} = \cos k\alpha_{02} \cos k\alpha_{24} +$$
$$+ \sin k\alpha_{24} \cos k\alpha_{01} \cos k\alpha_{02} \sin k\alpha_{124} \cos kl(a_{01}, a_{02}) +$$
$$+ \sin k\alpha_{24} \sin k\alpha_{124} \cos (\alpha_{g_4} - \alpha)$$ \quad (5.2)

\textbf{Proof.} From the cosine law in $\triangle A_1A_2A_3 \in K$-plane, we get:

$$\cos \alpha_{102} = \frac{\cos k\alpha_{12} - \cos k\alpha_{01} \cos k\alpha_{02}}{\sin k\alpha_{01} \sin k\alpha_{02}}$$ \quad (5.3)

From the sine law in $\triangle A_2A_{0,12}A_0$, we get:

$$\sin \alpha_{120} = \frac{\sin k\alpha_{h_{0,12}}}{\sin k\alpha_{02}}$$ \quad (5.4)

By substituting (5.23) in the sine law of $\triangle A_1A_2A_0$, we derive:

$$\frac{\sin k\alpha_{12}}{\sin \alpha_{102}} = \frac{\sin k\alpha_{1}}{\sin \alpha_{120}} = \frac{\sin k\alpha_{01} \sin k\alpha_{02}}{\sin k\alpha_{h_{0,12}}}$$ \quad (5.5)

or

$$\sin k\alpha_{h_{0,12}} = \frac{\sin k\alpha_{01} \sin k\alpha_{02}}{\sin k\alpha_{12}} \sin \alpha_{102}.$$ \quad (5.6)

By substituting (5.3) in (5.6), we obtain:
\[
\sin \kappa h_{0,12} = \frac{\sin \kappa a_{01} \sin \kappa a_{02}}{\sin \kappa a_{12}} \sqrt{1 - \left( \frac{\cos \kappa a_{12} - \cos \kappa a_{01} \cos \kappa a_{02}}{\sin \kappa a_{01} \sin \kappa a_{02}} \right)^2} \tag{5.7}
\]

From the cosine law in \(\triangle A_0 A_{0,123} A_3\), we get:

\[
\cos \kappa a_{03} = \cos \kappa h_{0,123} \cos \kappa x_3 \tag{5.8}
\]

From the cosine law in \(\triangle A_{0,123} A_3 A_2\), we have:

\[
\cos \kappa x_3 = \cos \kappa x_2 \cos \kappa a_{23} + \sin \kappa x_2 \sin \kappa a_{23} \cos (\alpha_{123} - \beta) \tag{5.9}
\]

By substituting (5.9) in (5.8), we get:

\[
\cos \kappa a_{03} = \cos \kappa x_2 \cos \kappa h_{0,123} \cos \kappa a_{23} + \\
+ \sin \kappa x_2 \cos \kappa h_{0,123} \sin \kappa a_{23} \cos (\alpha_{123} - \beta) \tag{5.10}
\]

From the cosine law in \(\triangle A_{0,123} A_2 A_0, \triangle A_0 A_{0,12} A_2\), we get respectively:

\[
\cos \kappa a_2 = \cos \kappa x_2 \cos \kappa h_{0,123} \tag{5.11}
\]

and

\[
\cos \kappa h_{0,123} = \cos \kappa \frac{\cos \kappa h_{0,12}}{\cos \kappa d}. \tag{5.12}
\]

By substituting (5.11) and (5.12) in (5.10), we have:

\[
\cos \kappa a_{03} = \cos \kappa a_{02} \cos \kappa a_{23} + \\
+ \sin \kappa a_{23} \sin \kappa x_2 \cos \kappa h_{0,12} \cos \alpha_{123} \cos \beta + \\
+ \sin \kappa a_{23} \sin \kappa x_2 \frac{\cos \kappa h_{0,12}}{\cos \kappa d} \sin \alpha_{123} \sin \beta \tag{5.13}
\]

From the sine law and cosine law in \(\triangle A_{0,123} A_{0,12} A_2\), we get:

\[
\sin \kappa x_2 = \frac{\sin \kappa d}{\sin \beta} \tag{5.14}
\]

\[
\cos \kappa x_2 = \cos \kappa d \cos \kappa l. \tag{5.15}
\]

By substituting (5.14) and (5.15) in (5.13), we have:
\[
\begin{align*}
\cos \kappa a_03 &= \cos \kappa a_{02} \cos \kappa a_{23} + \\
&+ \sin \kappa a_{23} \tan \kappa d \cos \kappa h_{0,12} \cos \alpha_{123} \cos \beta \sin \kappa x_2 + \\
&+ \sin \kappa a_{23} \tan \kappa d \cos \kappa h_{0,12} \sin \alpha_{123} \sin \beta \sin \kappa x_2 
\end{align*}
\] (5.16)

From the cosine law and sine law in \( \triangle A_{0,123}A_{0,12}A_2 \), we get:
\[
\begin{align*}
\cos \beta &= \frac{\cos \kappa d - \cos \kappa l \cos \kappa x_2}{\sin \kappa l \sin \kappa x_2} \\
\sin \beta &= \frac{\sin \kappa d}{\sin \kappa x_2} \\
\cos \kappa l &= \frac{\cos \kappa a_{02}}{\cos \kappa h_{0,12}}
\end{align*}
\] (5.17, 5.18, 5.19)

Taking into account that \( h_{012} = h_{012}(a_{01}, a_{02}) \), (5.19) yields:
\[
l = l(a_{01}, a_{02})
\]

By substituting (5.17) and (5.18) in (5.16), we have:
\[
\begin{align*}
\cos \kappa a_03 &= \cos \kappa a_{02} \cos \kappa a_{23} + \\
&+ \sin \kappa a_{23} \cos \kappa h_{0,12} \cos \alpha_{123} \sin \kappa l + \\
&+ \sin \kappa a_{23} \cos \kappa h_{0,12} \sin \alpha_{123} \tan \kappa d
\end{align*}
\] (5.20)

From the cosine law and sine law in \( \triangle A_0A_{0,12}A_{0,123} \), we get:
\[
\begin{align*}
\cos \kappa h_{0,123} &= \frac{\cos \kappa h_{0,12}}{\cos \kappa d} \\
\sin \kappa h_{0,123} &= \frac{\sin \kappa h_{0,12}}{\sin \alpha}
\end{align*}
\] (5.21, 5.22)

\[
\begin{align*}
\cos \kappa h_{0,123} &= \cos \kappa d \cos \kappa h_{0,12} + \sin \kappa d \sin \kappa h_{0,12} \cos \alpha
\end{align*}
\] (5.23)

Substituting (5.21), (5.22) in (5.23) yields:
\[
\tan \kappa d = \tan \kappa h_{0,12} \cos \alpha
\] (5.24)

By substituting (5.24) in (5.20), we obtain (5.1).

By working cyclically and changing the index 3 \( \rightarrow \) 4, we derive (5.2). \( \square \)
Proposition 4.
\[
\kappa a_{04} = \kappa a_{04}(a_{01}, a_{02}, a_{03}; a_{12}, a_{13}, a_{14}, a_{21}, a_{23}, a_{24}, a_{34}, \kappa).
\]
(5.25)

Proof. By solving (5.1) with respect to \(\cos \alpha\) and substituting this variable in (5.2) and by replacing \(\alpha_{123}, \alpha_{124}\), from the cosine law and the sine law in \(\Delta A_1 A_2 A_4\) and \(A_1 A_2 A_3\) in (5.2) we get (5.25). \(\square\)

Theorem 19 (The weighted Fermat-Torricelli Frechet solution). The following equations depending on the variable edge lengths \(a_{01}, a_{02}, a_{03}\) the constant sectional curvature \(K\) and the edge lengths \(\{ a_{ij}\}\) provide a necessary condition for the location of the weighted floating Fermat-Torricelli trees \(\{ a_{01}, a_{02}, a_{03}, a_{04}\}\), which belong to the weighted Fermat-Frechet (multitree solution in the 3K-space:

\[
\sum_{i=2}^{4} \frac{B_i}{\sinh Ka_{01} \sinh Ka_{0i}} (\cosh Ka_{01} \cosh Ka_{0i} - \cosh Ka_{1i}) - \sum_{i=1, i \neq j}^{4} \frac{B_i}{\sinh Ka_{0j} \sinh Ka_{0i}} (\cosh Ka_{0j} \cosh Ka_{0i} - \cosh Ka_{ji}) = 0,
\]

\[
\text{Vol}(A_0, A_1, A_2, A_3) < \text{Vol}(C_1 C_2, C_3, C_4).
\]
(5.26)

for \(i, j, k = 1, 2, 3, 4, i \neq j \neq k\).

Proof. It is a direct consequence of Proposition 4 and Theorems 6, 8 for \(N = 4\). \(\square\)

In [55, Problem 5, Theorem 3], we study the weighted Fermat-Steiner problem for a boundary tetrahedron \(A_1 A_2 A_3 A_4\) in \(\mathbb{R}^3\). The weighted Fermat-Steiner solution is a tree having two weighted Fermat (Steiner) points \(A_0\) and \(A'_0\) inside \(A_1 A_2 A_3 A_4\). The weighted Fermat Steiner tree topology consists of the branches (line segments) \(\{A_1 A_0, A_2 A_0, A_0 A'_0, A'_0 A_3, A'_0 A_4\}\), with corresponding weights \(\{B_1, B_2, \frac{B_5+B_6}{2}, B_3, B_4\}\). We call degree of intelligence of a a weighted Fermat-Steiner network with respect to a boundary \(N\)-simplex in \(\mathbb{R}^N\) the number of weighted Fermat-Steiner points it possesses inside the simplex. Thus, a full weighted Fermat-Steiner network for boundary tetrahedra having two weighted Fermat-Steiner points has two degrees of intelligence.

In [2, Figures 8, 9, pp. 18-19], P. Alexandrov makes an elegant exposition of algebraic complexes by considering them as higher dimensional generalization of ordinary directed polygonal paths, taking into account that a line which is traversed twice in opposite directions does not count. By applying this technique to a weighted Fermat-Steiner...
network for boundary tetrahedra, we may isolate the two degrees of intelligence in two non-intersecting triangles $\triangle A_1A_2A_0$ and $\triangle A_1A_2A'_0$.

We consider the following two networks:

- a weighted Fermat-Steiner tree for $A_1A_2A_3A_4$, such that:
  \[ f(A_0, A'_0) = B_1A_1A_0 + B_2A_2A_0 + B_3A_3A'_0 + B_4A_4A'_0 + \frac{B_0 + B'_0}{2} A_0A'_0 \rightarrow \min \]

- the two broken lines formed by the boundaries $\triangle A_1A_2A_0$ and $A_3A_4A'_0$ connected by their minimum distance $A_0A'_0$.

By using the orientation $A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_0 \rightarrow A'_0 \rightarrow A_3 \rightarrow A_4 \rightarrow A'_0 \rightarrow A_0$,

we obtain the following proposition, which deals with isolated intelligence of the boundaries $\triangle A_1A_2A_0$ and $A_3A_4A'_0$.

**Proposition 5** (A directed weighted Fermat-Steiner Frechet isolated multitree for Frechet tetrahedra in $\mathbb{R}^3$ in the sense of P. Alexandrov).

If the orientation

$A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_0 \rightarrow A'_0 \rightarrow A_3 \rightarrow A_4 \rightarrow A'_0 \rightarrow A_0$,

for the boundary triangles $\triangle A_1A_2A_0$ and $A_3A_4A'_0$ occurs for incongruent boundary tetrahedra derived by a sextuple of positive real numbers under the conditions of Blumenthal, Herzog, Dekster-Wilker, such that \{A_1A_0, A_2A_0, A_0A'_0, A'_0A_3, A_0A_4\} is a union of weighted Fermat-Steiner trees (Fermat-Steiner Frechet multitree) with corresponding weights \{B_1, B_2, \frac{B_0 + B'_0}{2}, B_3, B_4\} for the boundary tetrahedron $A_1A_2A_3A_4$, then we derive two boundary triangles with two isolated degrees of intelligence for all incongruent boundary tetrahedra (Frechet multitetrahedron).

*Proof.* The orientation

$A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_0 \rightarrow A'_0 \rightarrow A_3 \rightarrow A_4 \rightarrow A'_0 \rightarrow A_0$

in the sense of P. AlExandrov cancels the line segment $A_0$ and $A'_0$ from the weighted Fermat-Steiner tree \{A_1A_0, A_2A_0, A_0A'_0, A'_0A_3, A_0A_4\}, which leads to two boundaries $\triangle A_1A_2A_0$ and $A_3A_4A'_0$, with two isolated degrees of intelligence at the vertices $A_0$ and $A'_0$. Therefore, taking into account the conditions for a sextuple of positive real numbers determining the edge lengths of a maximum of thirty incongruent tetrahedra (Frechet multitetrahedron) studied by Blumenthal, Herzog and Dekster Wilker, we may get a union of isolated boundary triangles, which correspond to the weighted Fermat-Steiner Frechet tree (multitree) for the Frechet multitetrahedron. \qed
In [17] and [18], Edelstein, Schwartz and Eremenko proved Gehrink’s problem on linked curves in $\mathbb{R}^3$.

**Lemma 13.** [17], [18] If $C_1$ and $C_2$ are two linked closed curves (any continuous deformation of $C_1$ to a point intersect $C_2$) in $\mathbb{R}^3$ and the distance between $C_1$ and $C_2$ is 1, then the length of $C_1$ or $C_2$ is at least $2\pi$.

**Proposition 6** (Gehring’s linked curved inequality associated with the weighted Fermat-Steiner Frechet multitree in $\mathbb{R}^3$). If $A_1A_2A_3A_4$ belongs to a union of incongruent tetrahedra, which give the Frechet multitetrahedron, formed by a given sextuple of positive real numbers, determining edge lengths, $C_1 \equiv \text{boundary}(\Delta A_1A_0A_0')$, $C_2 \equiv \text{boundary}(\Delta A_2A_3A_4)$, $A_0$, $A_0'$ the corresponding Fermat-Steiner points and $\text{dist}(\Delta A_1A_0A_0', \Delta A_2A_3A_4)) = 1$, then the length of $\text{boundary}(\Delta A_1A_0A_0')$ or $\text{boundary}(\Delta A_2A_3A_4)$ is at least $2\pi$.

**Proof.** By applying lemma [13] for $C_1 \equiv \text{boundary}(\Delta A_1A_0A_0')$, $C_2 \equiv \text{boundary}(\Delta A_2A_3A_4)$, we derive that the length of $\text{boundary}(\Delta A_1A_0A_0')$ or $\text{boundary}(\Delta A_2A_3A_4)$ is at least $2\pi$. □

**Remark 3.** A generalization of Proposition 2 for Steiner trees in $\mathbb{R}^3$ may give a new perspective to the Steiner ratio conjecture, which was proved for manifolds by Ivanov-Tuzhlin-Cieslik in [31].

6. SOME COMPUTATIONS ON THE VARIATION OF FERMAT TREES FOR TETRAHEDRA HAVING ONE OR THREE VERTICES AT INFINITY IN $\mathbb{R}^3$

In this section, we present a new class of weighted Fermat trees for a tetrahedron $A_1A_2A_3A_4$ having the vertex $A_4$ at infinity or the three vertices $A_1$, $A_2$, $A_3$ at infinity, such that $A_4$ belongs to the perpendicular line w.r.t to the plane defined by $\Delta A_1A_2A_3$ at the corresponding weighted Fermat point $A_{0,123}$ of $\Delta A_1A_2A_3$, which is derived by setting $B_4 = 0$ in the weighted Problem for $A_1A_2A_3A_4$ and the length of $A_4A_{0,123}$ is a large positive real number $M$, for $B_4 = 1$, and $B_i = B_i(M)$, are positive linear function w.r.t to $M$, for $i = 1, 2, 3$.

Let $A_1A_2A_3A_4$ be a tetrahedron having the vertex $A_4$ at infinity in $\mathbb{R}^3$, with corresponding weights $B_1(M) = b_1M + c_1 > 0$, $B_2(M) = b_2M + c_2 > 0$, and $B_3(M) = b_3M + c_3 > 0$, $B_4 = 1$ having positive real values for a given large number $M$ and given real numbers $b_i$, $c_i$ for $i = 1, 2, 3$.

We denote by $A_{0,123}$ the corresponding weighted Fermat-Torricelli point of $\Delta A_1A_2A_3$ for given weights $B_1(M)$, $B_2(M)$ and $B_3(M)$, which
satisfy the inequalities (2.1) of the weighted floating case of Theorem [1] by setting \( B_4 = 0 \).

We set \( \varphi \equiv \angle A_{0,123}A_1A_3 \), \( a_{i,0123} \equiv \| A_{0,123}A_i \| \) and \( B_i \equiv B_i(M) \), for \( i = 1, 2, 3 \).

**Lemma 14.** The exact position of the weighted Fermat-Torricelli tree w.r to \( \triangle A_1A_2A_3 \) is given by:

\[
\varphi = \arccot \left( \frac{\sin(\alpha_{213}) - \cos(\alpha_{213}) \cot(\arccos \frac{B_2^2 - B_1^2 - B_3^2}{2B_1B_2}) - \frac{a_{11}}{a_{12}} \cot(\arccos \frac{B_2^2 - B_1^2 - B_3^2}{2B_1B_2})}{\cos(\alpha_{213}) - \sin(\alpha_{213}) \cot(\arccos \frac{B_2^2 - B_1^2 - B_3^2}{2B_1B_2}) + \frac{a_{11}}{a_{12}}} \right) \tag{6.1}
\]

and

\[
a_{1,0123} = \frac{\sin \left( \varphi + \arccos \frac{B_2^2 - B_1^2 - B_3^2}{2B_1B_3} \right) a_{13}}{\sin \left( \arccos \frac{B_2^2 - B_1^2 - B_3^2}{2B_1B_3} \right)} \tag{6.2}
\]

From Lemma 14 we derive that:

**Lemma 15.** The line segments \( a_{2,0123} \) and \( a_{3,0123} \) depend on \( B_1, B_2, B_3, a_{12}, a_{13}, a_{23} \) and \( \varphi \):

\[
a_{2,0123} = \sqrt{a_{1,0123}^2 + a_{12}^2 - 2a_{1,0123}a_{12} \cos(\angle A_2A_1A_3 - \varphi)} \tag{6.3}
\]

and

\[
a_{3,0123} = \sqrt{a_{1,0123}^2 + a_{13}^2 - 2a_{1,0123}a_{13} \cos(\varphi)}. \tag{6.4}
\]

We assume that \( A_4 \) lies on the normal line w.r. to the plane defined by \( \triangle A_1A_2A_3 \), for \( B_4 = 1 \) and the the length of \( A_{0,123}A_4 \) is \( M \).

We recall that the Cayley-Menger determinant \( D(S) \) is given by:

\[
D(S) = \det \begin{pmatrix}
0 & a_{12}^2 & a_{13}^2 & a_{14}^2 & 1 \\
a_{12}^2 & 0 & a_{23}^2 & a_{24}^2 & 1 \\
a_{13}^2 & a_{23}^2 & 0 & a_{34}^2 & 1 \\
a_{14}^2 & a_{24}^2 & a_{34}^2 & 0 & 1 \\
1 & 1 & 1 & 1 & 0
\end{pmatrix}. \tag{6.5}
\]

We recall that \( a_{ij} \) is length of the line segment \( A_iA_j \), \( a_{ijk} \equiv \angle A_iA_jA_k \), the dihedral angle \( \alpha \) is defined by the planes formed by \( \triangle A_0A_1A_2 \) and \( \triangle A_1A_2A_3 \), the dihedral angle \( \alpha_{ijk} \) is defined by the planes formed by \( \triangle A_1A_2A_3 \) and \( \triangle A_1A_2A_4 \), \( h_{0,12} \) is the height of \( \triangle A_0A_1A_2 \) from \( A_0 \), by \( h_{0,12,m} \) the distance of \( A_0 \) from the plane defined by \( \triangle A_1A_2A_m \), for \( i, j, k = 0, 1, 2, 3, 4 \) and \( m = 3, 4 \).

The variable line segments \( a_{03}(a_{01}, a_{02}, \alpha), a_{04}(a_{01}, a_{02}, \alpha) \), are derived in [49] Formulas (2.14), (2.20) p. 116].
Proposition 7. The variable lengths $a_{03}(a_{01}, a_{02}, \alpha), a_{04}(a_{01}, a_{02}, \alpha; M)$, are given by:

$$a_{03}(a_{01}, a_{02}, \alpha) = \sqrt{a_{02}^2 + a_{23}^2 - 2a_{23}\left[\sqrt{a_{02}^2 - h_{0,12}^2} \cos \alpha_{123} + h_{0,12} \sin \alpha_{123} \cos \alpha\right]}$$

(6.6)

and

$$a_{04}(a_{01}, a_{02}, \alpha; M) = \sqrt{a_{02}^2 + a_{24}^2 - 2a_{24}\left[\sqrt{a_{02}^2 - h_{0,12}^2} \cos \alpha_{124} + h_{0,12} \sin \alpha_{124} \cos (\alpha_{94} - \alpha)\right]}$$

(6.7)

where

$$h_{0,12} = h_{0,12}(a_{01}, a_{02}, a_{12}) = \frac{a_{01}a_{02}}{a_{12}} - \sqrt{1 - \left(\frac{a_{01}^2 + a_{02}^2 - a_{12}^2}{2a_{01}a_{02}}\right)^2}.$$  

(6.8)

such that:

$$\lim_{M \to +\infty} a_{04}(a_{01}, a_{02}, \alpha; M) = +\infty.$$  

(6.9)

Proof. The angle $\alpha_{123}$ does not depend on $M$:

$$\cos \alpha_{123} = \frac{a_{12}^2 + a_{23}^2 - a_{13}^2}{2a_{12}a_{23}},$$

(6.10)

and

$$\sin \alpha_{123} = \frac{\sqrt{(a_{12} + a_{23} + a_{13})(a_{23} + a_{13} - a_{12})(a_{12} + a_{13} - a_{23})(a_{12} + a_{23} - a_{13})}}{2a_{12}a_{23}}.$$  

(6.11)

By replacing (6.10), (6.11) and (6.8) in (6.6) we derive that $a_{03} = a_{03}(a_{01}, a_{02}, \alpha)$.

We shall show that $a_{41}, a_{42}, \alpha_{124}$ depend on $M$.

From the right triangles $\triangle A_1A_0_{123}A_4, \triangle A_2A_0_{123}A_4$ and taking into account (6.2) and (6.3), we get:

$$a_{41} = a_{41}(M) = \sqrt{M^2 + a_{1,0123}^2}$$

(6.12)

and

$$a_{42} = a_{42}(M) = \sqrt{M^2 + a_{2,0123}^2}.$$  

(6.13)
\[ a_{4}^2 = a_{2}^2 + a_{24}^2 - 2a_{24}\sqrt{a_{2}^2 - h_{0,12}^2 \cos \alpha_{124} + h_{0,12} \sin \alpha_{124}(\cos \alpha_{g_{4}} \cos \alpha + \sin \alpha_{g_{4}} \sin \alpha)} \]  

(6.14) 

where 

\[ \cos \alpha_{124} = \frac{a_{12}^2 + a_{24}^2 - a_{14}^2}{2a_{12}a_{24}}, \]  

(6.15) 

\[ \sin \alpha_{124} = \frac{\sqrt{(a_{12} + a_{24} + a_{14})(a_{24} + a_{14} - a_{12})(a_{12} + a_{14} - a_{24})(a_{12} + a_{24} - a_{14})}}{2a_{12}a_{24}}, \]  

(6.16) 

\[ \alpha_{g_{4}} = \arccos \left( \frac{\frac{a_{41}^2 + a_{42}^2 - a_{43}^2}{2a_{23}} - \sqrt{a_{42}^2 - h_{4,12}^2 \cos \alpha_{123}}}{h_{4,12} \sin \alpha_{123}} \right) \]  

(6.17) 

and 

\[ h_{4,12} = h_{4,12}(a_{41}, a_{42}, a_{12}) = \frac{a_{41}a_{42}}{a_{12}} \sqrt{1 - \left( \frac{a_{11}^2 + a_{12}^2 - a_{13}^2}{2a_{41}a_{42}} \right)^2}. \]  

(6.18) 

The dihedral angle \( \alpha_{g_{4}} \) is derived by setting in (6.6) and (6.8) the index from 0 \( \rightarrow \) 4. 

By replacing (6.12), (6.13), (6.15), (6.16), (6.17), (6.8) in (6.7) we derive that: \( a_{04} = a_{04}(a_{01}, a_{02}, \alpha; M) \). 

Theorem 20. The following four equations provide a necessary condition to determine the position of the weighted Fermat-Torricelli tree at the interior of \( A_{1}A_{2}A_{3}A_{4} \):

\[ \left( \frac{B_{3}}{a_{3} \text{Vol}(A_{0}A_{1}A_{2}A_{4})} \right)^2 = \left( \frac{B_{4}}{a_{4} \text{Vol}(A_{0}A_{1}A_{2}A_{3})} \right)^2 = C^2, \]  

(6.19)
THE WEIGHTED FERMAT-FRECHET MULTITREE IN THE $k$-SPACE

\[
\left( \frac{B_3}{a_3 \text{Vol}(A_0A_1A_2A_4)} \right)^2 = \left( \frac{B_1}{a_1 \text{Vol}(A_0A_2A_3A_4)} \right)^2 = C^2, \quad (6.20)
\]

\[
\left( \frac{B_3}{a_3 \text{Vol}(A_0A_1A_2A_4)} \right)^2 = \left( \frac{B_2}{a_2 \text{Vol}(A_0A_1A_3A_4)} \right)^2 = C^2, \quad (6.21)
\]

and

\[
\left( \frac{B_1}{a_1 \text{Vol}(A_0A_2A_3A_4)} \right)^2 = \left( \frac{B_2}{a_2 \text{Vol}(A_0A_1A_3A_4)} \right)^2 = C^2. \quad (6.22)
\]

where

\[
B_i(M) = b_iM + c_i > 0, \quad (6.23)
\]

for $i = 1, 2, 3$

and

\[
B_4 = 1.
\]

Proof. The objective function is given by:

\[
f(a_1, a_2, \alpha; M) = B_1a_1 + B_2a_2 + B_3a_3(a_1, a_2, \alpha) + B_4a_4(a_1, a_2, \alpha; M). \quad (6.24)
\]

or

\[
f(a_1, a_4, \alpha'; M) = B_1a_1 + B_4a_4 + B_2a_2(a_1, a_4, \alpha') + B_3a_3(a_1, a_4, \alpha'; M). \quad (6.25)
\]

or

\[
f(a_2, a_3, \alpha''; M) = B_2a_2 + B_3a_3 + B_1a_1(a_2, a_3, \alpha'') + B_4a_4(a_2, a_3, \alpha''; M). \quad (6.26)
\]

where $\alpha'$ is the dihedral angle formed by $\triangle A_1A_0A_4$ and $\triangle A_1A_4A_2$ and $\alpha''$ is the dihedral angle formed by $\triangle A_2A_0A_3$ and $\triangle A_2A_1A_3$.

By differentiating (6.24) w.r to $\alpha$, (6.25) w.r. to $\alpha'$ and (6.26) w.r. to $\alpha''$ we derive (see also in [49, Formula (2.25), pp. 117]):

\[
\frac{B_3}{a_3 \text{Vol}(A_0A_1A_2A_4)} = \frac{B_4}{a_4 \text{Vol}(A_0A_1A_2A_3)} = \frac{B_1}{a_1 \text{Vol}(A_0A_2A_3A_4)} = \frac{B_2}{a_2 \text{Vol}(A_0A_1A_3A_4)} = C, \quad (6.27)
\]

The volume $\text{Vol}(A_0A_iA_jA_k)$, for $i, j, k = 1, 2, 3, 4$ is given by([48, pp. 249-255]):
\[288 \text{Vol}(A_0 A_1 A_2 A_3)^2 = D(\{a_1, a_2, a_3, a_{23}, a_{13}, a_{12}\}) \] \hspace{1cm} (6.28)

\[288 \text{Vol}(A_0 A_2 A_3 A_4)^2 = D(\{a_4, a_2, a_3, a_{24}, a_{43}, a_{23}, a_{12}\}) \] \hspace{1cm} (6.29)

\[288 \text{Vol}(A_0 A_1 A_3 A_4)^2 = D(\{a_1, a_4, a_3, a_{43}, a_{13}, a_{14}\}) \] \hspace{1cm} (6.30)

and

\[288 \text{Vol}(A_0 A_1 A_2 A_4)^2 = D(\{a_1, a_2, a_4, a_{14}, a_{12}\}). \] \hspace{1cm} (6.31)

By squaring both parts of the equations in (6.27) and then by replacing (6.28), (6.31), (6.30), (6.29) in the derived equations, we deduce (6.19), (6.20), (6.21) and (6.22) which depend on \(a_1, a_2, \alpha\) and \(M\).

For \(M \to +\infty\), the solution of the weighted Fermat-Torricelli problem is a weighted Fermat-Torricelli tree with branches \(A_1 A_{0,123}, A_2 A_{0,123}, A_3 A_{0,123}\) and \(A_4 A_{0,123}\). We call the weighted Fermat-Torricelli tree for a tetrahedron having one vertex at infinity a large tree because one of the four branches \(A_1 A_{0,123} \to \infty\).

The unique solution of the inverse problem for tetrahedra in \(\mathbb{R}^3\) has been established in [49].

**Problem 2. Inverse weighted Fermat problem for tetrahedra in \(\mathbb{R}^3\), [49]**

Given a point \(A_0\) and a positive real number \(C\) which belongs to the interior of \(A_1 A_2 A_3 A_4\) in \(\mathbb{R}^3\), does there exist a unique set of positive weights \(B_i\), such that

\[B_1 + B_2 + B_3 + B_4 = C,\]

for which \(A_0\) minimizes

\[f(A_0) = \sum_{i=1}^{4} B_i a_{0i}.\]

We denote by \(\alpha_{i,j,0,k}\) the angle that is formulated by the line segment \(A_0 A_i\) and the line segment that connects \(A_0\) with the trace of the orthogonal projection of \(A_i\) to the plane defined by \(\triangle A_j A_0 A_k\). A positive answer w.r. to the inverse problem for \(A_1 A_2 A_3 A_4\) is given in [49, Proposition 1]):

**Lemma 16.** [49, Proposition 1, Solution of Problem 2] The weight \(B_i\) are uniquely determined by the formula:

\[B_i = \frac{C}{1 + \left| \frac{\sin \alpha_{i,j,0,k}}{\sin \alpha_{j,k,0,l}} \right| + \left| \frac{\sin \alpha_{i,j,0,l}}{\sin \alpha_{k,j,0,l}} \right| + \left| \frac{\sin \alpha_{i,j,0,l}}{\sin \alpha_{k,j,0,l}} \right|}, \] \hspace{1cm} (6.32)
where

\[
\frac{B_j}{B_i} = \frac{\sin \alpha_{i,k0l}}{\sin \alpha_{j,k0l}} \tag{6.33}
\]

\[
\sin \alpha_{i,k0l} = \sin \alpha_{j,k0l} \sqrt{\frac{\sin^2 \alpha_{k0m} - \cos^2 \alpha_{m0i} - \cos^2 \alpha_{k0i} + 2 \cos \alpha_{k0m} \cos \alpha_{m0i} \cos \alpha_{k0i}}{\sin^2 \alpha_{k0m} - \cos^2 \alpha_{m0j} - \cos^2 \alpha_{k0j} + 2 \cos \alpha_{k0m} \cos \alpha_{m0j} \cos \alpha_{k0j}}} \tag{6.34}
\]

for \(i, j, k, l = 1, 2, 3, 4\) and \(i \neq j \neq k \neq l\).

For \(B_4 = 0\), we obtain the inverse weighted Fermat-Torricelli problem for \(\triangle A_1A_2A_3\).

Lemma 17. \[25\] The weight \(B_i\) are uniquely determined by the formula:

\[
B_i = \frac{C}{1 + \frac{\sin \alpha_{i0k}}{\sin \alpha_{j0k}} + \frac{\sin \alpha_{i0l}}{\sin \alpha_{j0l}}}, \tag{6.35}
\]

where

\[
\frac{B_j}{B_i} = \frac{\sin \alpha_{i0k}}{\sin \alpha_{j0k}}. \tag{6.36}
\]

for \(i, j, k = 1, 2, 3\).

Theorem 21. If \(M \rightarrow +\infty\), for \(B_4 = 1\) the solution of the inverse problem for the tetrahedron \(A_1A_2A_3A_4\) having the vertex \(A_4\) at infinity in \(\mathbb{R}^3\) coincides with the solution of the inverse problem for \(\triangle A_1A_2A_3\).

Proof. For \(M \rightarrow +\infty\), \(A_0 \rightarrow A_{0,123}\) and \(\angle A_4A_0A_i = 90^\circ\), for \(i = 1, 2, 3\). By replacing \(\angle A_4A_0A_1 = \angle A_4A_0A_2 = \angle A_4A_0A_3 = 90^\circ\) in (6.34) and (6.36) for \(i, j = 1, 2, 3, k, l, m = 1, 2, 3, 4\) and \(i \neq j \neq k \neq l\), we derive

\[
\frac{B_j}{B_i} = \frac{\sin \alpha_{i0k}}{\sin \alpha_{j0k}},
\]

for \(i, j, k = 1, 2, 3\) and \(B_4 = 1\). \(\sqcup\)

A direct consequence of Theorem 4, for

\[
B_1(M) + B_2(M) + B_3(M) + B_4(M) = C
\]

\((C\) is a constant real number independent of \(M\))
gives the following Proposition (the weights \(B_i\) are independent of \(M\))
Proposition 8. For $M \to +\infty$ and 

$$B_i(M) = b_i M + c_i > 0,$$

$$B_4 = 1,$$

for $i = 1, 2, 3$ such that 

$$b_1 + b_2 + b_3 = 0,$$

$B_i$ are uniquely determined by the formula:

$$B_i = \frac{C}{1 + \frac{\sin \alpha_{i0k}}{\sin \alpha_{j0k}} + \frac{\sin \alpha_{i0j}}{\sin \alpha_{i0k}}}$$ (6.37)

for $C = 1 + c_1 + c_2 + c_3$ and $i, j, k = 1, 2, 3$.

As a future work, we may focus on studying properties of the Betti group of a Frechet multisimplex.

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