Higher Order Corrections to Positronium Energy Levels

G S Adkins
Franklin & Marshall College, Lancaster, Pennsylvania, 17604, USA
E-mail: gadkins@fandm.edu

Abstract. Positronium spectroscopy is of continuing interest as a high-precision test of our understanding of binding in QFT. Spectroscopic studies of low-lying states (n = 1 hyperfine splitting, n = 2 fine structure, and the 1S − 2S interval) have reached a precision of order 1MHz, and ongoing experimental efforts give the promise of improved results. Theoretical calculations of positronium energies at order $m\alpha^6 \sim 18.7$MHz are complete, but only partial results are known at order $m\alpha^7 \sim 0.14$MHz. We report on the status of the positronium energy calculations and give some details of the methods employed.

1. Introduction

Positronium, the electron-positron bound state, has a number of characteristics that make it both interesting and useful for testing the bound state formalism of quantum field theory and for exploring the limits of fundamental theory. Positronium represents the purest example of binding in QFT as the constituents are structureless and their interactions are dominated by QED with only negligible contributions from strong and weak effects. Positronium differs from other Coulombic bound systems such as hydrogen or muonium in having maximal recoil—the constituent mass ratio m/M is one. The states of positronium are eigenstates of parity and charge conjugation allowing for tests of fundamental symmetries. Annihilation into photons, both real and virtual, limits the lifetime of positronium and contributes to predicted energies of its bound states.

Positronium has been extensively studied since its first production by Deutsch in 1951 [1]. Precision measurements have focused on energy levels and decay rates of low-lying states, the decay rate of the positronium negative ion, tests of discrete symmetries, and searches for exotic decay modes [2, 3, 4, 5]. Our interest here is on measurements of energy levels, specifically the precisely measured ground state hyperfine splitting (hfs, $1^3S_1 − 1^1S_0$ in $n(2s+1)\ell_j$ spectroscopic notation), n = 2 fine structure intervals, and the $1^3S_1 − 2^3S_1$ interval. A sustained effort to measure the ground state hyperfine interval culminated in the early 1980s with results by Mills and Bearman [6, 7] and by Hughes and collaborators [8] that gave a combined value of

$$\Delta E(\text{hfs}) = 203\,388.82(67)\text{MHz} \quad (3.3 \text{ ppm}).$$

A very recent result for the same interval is [9]

$$\Delta E(\text{hfs}) = 203\,394.2(1.6\text{stat})(1.3\text{sys})\text{MHz} \quad (8.0 \text{ ppm})(6.4 \text{ ppm}).$$
The new measurement was made with a careful focus on systematics such as thermalization and is in tension with the earlier results. Many of the $n=2$ fine structure intervals were measured between 1975 and 1993 with uncertainties ranging from 1.5MHz to 6.4MHz[10, 11, 12, 13, 14]. The $^{3}S_1-^{3}S_1$ interval was the subject of a decade-long program of measurement by Chu, Mills, and collaborators in the 1980s and early 1990s [15, 16, 17] leading to the final result [18]

$$\Delta E(^{3}S_1-^{3}S_1) = 1 233 607 216.4(3.2)\text{MHz (2.6 ppb)}. \quad (3)$$

A look at the natural line widths: $\sim 1300\text{MHz}$ for the hyperfine interval, $\sim 50\text{MHz}$ for $n=2$ fine structure transitions, and $\sim 1.3\text{MHz}$ for the $^{3}S_1-^{3}S_1$ interval, leads one to expect that there is room for improvement in measurements of the fine structure and especially the $^{3}S_1-^{3}S_1$ interval. The potential for significantly improved results has been discussed in a number of places [19, 3, 20, 21, 22].

Theoretical studies of positronium properties commenced before its discovery. Positronium energies can be expressed as an expansion in powers of the fine structure constant $\alpha$:

$$E = m\left\{ A_{20}\alpha^2 + A_{40}\alpha^4 + A_{51}\alpha^5 L + A_{50}\alpha^5 + A_{61}\alpha^6 + A_{72}\alpha^7 L^2 + A_{71}\alpha^7 L + A_{70}\alpha^7 + \cdots \right\} \quad (4)$$

where $m$ is the electron mass, $L \equiv \ln(1/\alpha)$, $c = 1$ in natural units, and the $A$’s are constants. The lowest approximation is that of Bohr at $O(\alpha^2)$, the same as for hydrogen except that the reduced mass becomes exactly half of the electron mass. Results for energy levels at $O(\alpha^4)$ were in hand by 1951 [23, 24, 25, 26]. These results included effects of relativity, the spin-orbit, Darwin, magnetic, and tensor spin-spin interactions, and one-photon annihilation. Terms of $O(\alpha^6)$ were quickly worked out [27, 28, 29, 30]. A new feature appeared at this order—the presence of logarithmic factors coming from terms dependent on more than one energy scale [31, 32]. Many people contributed to the evaluation of the $O(\alpha^6)$ corrections—the final results (with references) are contained in [33, 34, 35]. The contribution of the $A_{60}$ term is 8.5MHz for the $^{1}S_0$ state and 1.2MHz for the $^{3}S_1$ state, so clearly higher corrections will be required if there is any significant improvement in the experimental situation. Indeed, the $O(\alpha^7L^2)$ [36, 37, 38] and some of the $O(\alpha^7L)$ [39, 40, 41] logarithmic terms had already been found by 2001. More recently, many of the pure $\alpha^7$ contributions have been obtained [42, 43, 44, 45, 46, 47, 48, 49, 50], as reviewed in [50], but many remain to be done. The calculation of positronium energies at $O(\alpha^8)$ is the subject of this work.

2. Method of calculation

The main challenge facing bound state QED is the presence of multiple energy scales in Feynman integrals. A typical integral for an energy level correction contains a contribution at some leading order in the fine structure constant $\alpha$ as well as an infinite string of contributions of higher order in $\alpha$, often accompanied by associated logarithms. A procedure to isolate the various orders is required. Our approach is based on the effective non-relativistic quantum field theory “Non-relativistic quantum electrodynamics” (NRQED) [51]. However, even in the context of NRQED there are still a number of possible options about how to proceed. The choices we make are summarized here:

(i) Our calculations are ultimately based on QED, but using the effective quantum field theory
NRQED instead of QED directly. The NRQED Lagrangian has the form [51, 52]

\[ \mathcal{L} = \psi_t^\dagger \left\{ i D_t + \frac{\vec{D} \cdot \vec{D}}{2m} + \frac{\vec{D} \cdot \vec{D}}{8m^4} + c_F \frac{q}{2m} \vec{\sigma} \cdot \vec{B} + c_D \frac{q}{2m^2} \left( \vec{B} \cdot \vec{E} - \vec{E} \cdot \vec{B} \right) \right. \\
+ \left. c_S \frac{iq}{8m^2} \vec{\sigma} \cdot \left( \vec{D} \times \vec{E} - \vec{E} \times \vec{D} \right) + \ldots \right\} \psi_e \\
+ \text{positron terms} + \text{four–fermion contact terms} + \text{photon terms}, \tag{5} \]

where \( q \) and \( m \) are the electron charge and mass. The covariant derivatives are given in terms of vector potential as \( D_t = \frac{\partial}{\partial t} + iqA^0 \) and \( \vec{D} = \vec{\nabla} - iq\vec{A}, \) and the electric and magnetic fields are defined in the usual way: \( \vec{E} = -\vec{\nabla}A^0 - \frac{\partial \vec{A}}{\partial t}, \ \vec{B} = \vec{\nabla} \times \vec{A}. \) The electron quantum field \( \psi_e \) is a two component Pauli spinor—there is no built-in unification of particles with their corresponding antiparticles due to the NR nature of the theory. There is another contribution to the Lagrange density—the “positron term”—that mirrors the electron term shown, but involves an independent two-component positron field \( \psi_p. \) The “matching coefficients” \( c_F, c_D, c_S, \ldots \) are determined by the requirement that the full theory QED and the effective theory NRQED agree for processes for which both are valid, such as low energy scattering. The “four–fermion contact terms” are point-like interactions between an incoming electron-positron pair and an outgoing electron-positron pair that plays the role in the low-energy theory of all high-energy electron-positron interactions, including all processes involving virtual annihilation of the electron-positron pair to photons. The “photon terms” include the usual \(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\) Maxwell density plus corrections arising from vacuum polarization. The NRQED Lagrange density is represented as an expansion in powers of \( 1/m. \) This is appropriate, as higher powers of \( 1/m \) are associated with additional derivatives in the numerator in order to keep the units right, and additional derivatives translate to more momentum factors in the Feynman rules. Because \( \langle p \rangle/m \sim \alpha \) for Coulombic bound states, terms with more factors of \( 1/m \) give smaller energy contributions. The Lagrange density shown includes all terms of order \( 1/m^2 \) (and one of the \( 1/m^3 \) terms), and is adequate to calculate all contributions to positronium energies of order \( ma^5. \) The expansion through terms of order \( 1/m^4 \) (plus the \( \vec{D}^6/m^5 \) term) will be needed in order to obtain energies at \( O(ma^7). \) The form of the Lagrange density of order \( 1/m^4 \) is known [53]. It involves a number of additional interactions and matching coefficients.

Alternatives to the use of NRQED exist, particularly “potential NRQED”: pNRQED [54, 55, 56, 57, 58]. A decision on the merits of pNRQED vs. strict NRQED for bound states such as positronium awaits the comparison of calculations at high orders using both approaches.

(ii) Dimensional regularization is used to control all ultraviolet (UV) and infrared (IR) divergences. Other regularization methods have been used with NRQED, such as the imposition of an explicit cut-off, but dimensional regularization is by far the most efficient for use in calculations and, in conjunction with the “method of regions” (see point (iv)), allows for a convenient separation of the contributions of the various momentum space regions that contribute to a given Feynman integral.

(iii) The gauge freedom of QED and NRQED allows one to choose the most convenient gauge for a given calculation. Specifically, we use Coulomb gauge for bound state calculations, and covariant gauges such as Feynman gauge for scattering calculations.

(iv) We use the method of regions to segregate the various momentum space scales that contribute to NRQED integrals. Four distinct regions have been identified [59] that are important when analyzing the loop-momentum integrals that occur in the study of Coulomb two-body bound states: the hard region with \( q_0 \sim |q| \sim m; \) the soft region with...
We employ a lowest-order approximation to the full NRQED Bethe-Salpeter equation that

\[ q_0 \sim |\vec{q}| \sim m\alpha; \] the potential region with \( q_0 \sim m\alpha^2, |\vec{q}| \sim m\alpha; \] and the ultrasoft region with \( q_0 \sim |\vec{q}| \sim m^2. \] For each integral, the contributing regions are identified, and region by region, expansions in small parameters are made. The expanded terms are integrated over the full \( d = 4 - 2\epsilon \) dimensional space. The individual terms that result have a single power of the fine structure constant \( \alpha. \) For divergent contributions that power might be modified by a logarithm after expansion in the small parameter \( \epsilon, \) as in \( \Gamma(\alpha\epsilon) = \frac{1}{\epsilon} + \ln \alpha - \gamma_E + O(\epsilon) \) where \( \gamma_E \) is the Euler-Mascheroni constant.

(v) Some expression derived from the fundamental theory must be available that gives the energies

\[ E_n \] where \( n \) different masses. For electron-positron systems specifically we have

\[ \epsilon \] that power might be modified by a logarithm after expansion in the small parameter \( \epsilon, \) as in \( \Gamma(\alpha\epsilon) = \frac{1}{\epsilon} + \ln \alpha - \gamma_E + O(\epsilon) \) where \( \gamma_E \) is the Euler-Mascheroni constant.

\[ G' = S + SKG' \] (6)

where \( G' \) is the fully-corrected 2-to-2 Green function of NRQED, \( S \) is the product of uncorrected single-particle propagators–one for the electron and one for the positron, and \( K \) is the two-particle-irreducible interaction kernel of NRQED. We have chosen to include all electron and positron self-energy effects in \( K \) instead of in \( S \) in order to keep the full Bethe-Salpeter equation as simple as possible. Implicit in (6) are a set of \( d \)-dimensional integrals over the relative momenta of the particles. Making these integrals explicit, (6) can be written as

\[ G'(E;p_2,p_1) = S(E;p_2,p_1) + S(E;p_2) \int d^d q K(E;p_2,q)G'(E,q,p_1). \] (7)

Here \( m_1 + m_2 + E \) is the center of mass energy (including rest masses), \( p_2 \) (\( p_1 \)) is the relative momentum of the outgoing (incoming) electron-positron pair, and \( d^d q = d^d q/(2\pi)^d \) is the integration measure of \( d \)-dimensional momentum space. The uncorrected NRQED electron-positron propagator is

\[ S(E;p_2,p_1) = S(E;p_2)(2\pi)^d \delta^d(p_2-p_1) \] with

\[ S(E;p) = \frac{i}{\xi_1 E + p_0 - \vec{p}^2/(2m_1) + ie \xi_2 E - p_0 - \vec{p}^2/(2m_2) + ie} \] (8)

where \( \xi_1 + \xi_2 = 1. \) (We have written \( S(E;p) \) for the more general case of fermions with different masses. For electron-positron systems specifically we have \( m_1 = m_2 = m \). Bound state energies are contained in \( G'(E,p_2,p_1) \) as the positions of poles:

\[ G'(E;p_2,p_1) \rightarrow i \sum_k \frac{\Psi_{nak}(p_2)\Psi_{nak}(p_1)}{E-E_{na}} \text{ for } E \rightarrow E_{na}, \] (9)

where \( n \) is the principal quantum number, \( a \) stands for additional quantum numbers that the energies \( E_{na} \) might depend on, and \( k \) labels the degeneracy.

(vi) We employ a lowest-order approximation to the full NRQED Bethe-Salpeter equation that can be solved exactly, at least in the limit that \( D \rightarrow 3 \), and a bound state perturbation theory built on this exact solution. This was an important approach when dealing with the Bethe-Salpeter equation in QED and continues to be useful in NRQED. For the lowest order interaction kernel we use the exchange of a Coulomb photon: \( K_0(E;p_2,p_1) = -iV(\vec{p}_2 - \vec{p}_1), \)
where \( V(\vec{q}) = -4\pi\alpha\vec{\mu}^2/\vec{q}^2 \) is the usual Coulomb potential. (A mass scale \( \vec{\mu} \) must be included in the momentum space potential so that the configuration space potential has the conventional units. We use dimensional regularization in the modified minimal subtraction (\( \overline{\text{MS}} \)) variant—hence the use of \( \vec{\mu} \) instead of simply \( \mu \).) The reference propagator satisfies the crucial property that

\[
\int d\vec{p}_0 S(E;\vec{p}) = \int d\vec{p}_0 \frac{i}{\xi_1 E + p_0 - \vec{p}^2_{\text{2}m_e} + i\epsilon} \frac{i}{\xi_2 E - p_0 - \vec{p}^2_{\text{2}m_e} + i\epsilon} = is(E;\vec{p}),
\]

(10)

where \( s(E;\vec{p}) \) is the non-relativistic propagator

\[
s(E;\vec{p}) = \frac{1}{E - \vec{p}^2_{\text{2}m_e} + i\epsilon}
\]

(11)

with reduced mass \( m_r \). It follows that the reference lowest order Bethe-Salpeter equation

\[
G_0 = S + SK_0 G_0
\]

(12)

can be solved exactly. The reference 2-to-2 Green function takes the form

\[
G_0 = (1 - SK_0)^{-1} S = S + SK_0 S + SK_0 SK_0 S + \cdots,
\]

(13)

where the series can be summed. The \( d = 4 - 2\epsilon \) integrals over the relative momenta can be reduced to \( D = 3 - 2\epsilon \) dimensional spatial integrals by use of (10):

\[
[K_0SK_0](E;p_2,p_1) = \int d^Dq K_0(E;p_2,q_2)S(E,q_2,q_1)K_0(E;q_1,p_1)
\]

\[
= \int d^Dq (-i)V(\vec{p}_2 - \vec{q})s(E;\vec{q})V(\vec{q} - \vec{p}_1)
\]

\[
= -i[VsV](E;\vec{p}_2,\vec{p}_1),
\]

(14)

where in the last form the integral over the relative spatial momentum is implicit. Higher terms in the series for \( G_0 \) can also be reduced to \( D \)-dimensional form

\[
[K_0SK_0]S(E;p_2,p_1) = -i[VsV](E;\vec{p}_2,\vec{p}_1), \text{ and so on, so that}
\]

\[
G_0 = S + SK_0 S + S(-i)hS,
\]

(15)

where \( h \) is defined as in [60] to be

\[
h \equiv V + VsV + VsVsV + \cdots.
\]

(16)

This is the same \( h \) that forms the core of the solution to the inhomogeneous Schrödinger-Coulomb (SC) equation

\[
(s^{-1} - V)g = 1,
\]

(17)

which can be written in more explicit form as

\[
\left( E - \frac{\vec{p}_2^2}{2m_r} \right) g(E;\vec{p}_2,\vec{p}_1) - \int d^Dq V(\vec{p}_2,\vec{q})g(E;\vec{q},\vec{p}_1) = (2\pi)^D \delta^D(\vec{p}_2 - \vec{p}_1).
\]

(18)

Then the solution of (17) is

\[
g = (1 - sV)^{-1} s = s + Vs + shs.
\]

(19)
The explicit expression for $h$ in the limit $D \to 3$ is due to Hostler [61, 62] and Schwinger [63].

We use the perturbation theory of Kato [64, 65, 66] to calculate energies:

$$E = E^0 + (\delta K) + \left( \delta K \hat{G}_0 \delta K \right) + (\delta K)' + O(\delta K)^3$$

(20)

where $E^0$ is the Bohr energy level (with reduced mass), $\delta K = K - K_0$, and the expectation values are calculated according to

$$(M) = \left. i \bar{\Psi}^{0} M \Psi^{0} \right| = \int d^Dp_2 d^Dp_1 \bar{\Psi}^{0}(p_2)M(E^0:p_2,p_1)\Psi^{0}(p_1),$$

$$(M)' = \left. i \bar{\Psi}^{0} \frac{dM}{dE} \right|_{E^0} \Psi^{0} = \int d^Dp_2 d^Dp_1 \bar{\Psi}^{0}(p_2)\frac{dM(E^0:p_2,p_1)}{dE} \Psi^{0}(p_1),$$

(21)

The reference wave functions $\bar{\Psi}^0, \Psi^0$ are found by factoring the residues of the bound state poles of $G_0$, in analogy to (9), and $\hat{G}_0$ is the reduced Green function obtained from $G_0$ by subtracting the appropriate pole term.

(vii) The lowest order 2-to-2 Green function of NRQED contains within it, through the function $h$ of (11) and (13), the nonrelativistic SC Green function of standard quantum mechanics (in $D$ dimensions). A detailed understanding of the $D$-dimensional SC problem [67] is needed for the successful application of bound state perturbation theory for the evaluation of the divergent expectation values that appear in perturbation theory when using dimensional regularization. Experience has shown that appropriate subtractions can be made on the reduced Green function $\hat{G}_0$ to render its expectation values finite in the limit $D \to 3$. Evaluation of the typically divergent subtraction terms depend on the Coulomb wave functions near $r = 0$ in $D$ dimensions. These wave function satisfy the non-relativistic $D$-dimensional Schrödinger equation

$$\frac{1}{2m_r} \nabla^2 \psi(x) + V(r) \psi(x) = E \psi(x)$$

(22)

with a Coulomb potential obtained from $V(q) = -4\pi\alpha\hat{\mu}^{2r}/q^2$ by Fourier transformation:

$$V(r) = -\frac{\Gamma(D/2 - 1)\hat{\mu}^{2r}}{\pi^{D/2 - 1}r^{D - 2}}.$$  

(23)

After separation of the angular from the radial variables, the radial function is found to satisfy

$$\frac{1}{2m_r} \left\{ -\partial_r^2 - \frac{D-1}{r} \partial_r + \frac{\ell(\ell + D - 2)}{r^2} \right\} R_{n\ell}(r) + V(r) R_{n\ell}(r) = E_{n\ell} R_{n\ell}(r).$$

(24)

Analysis of the short and long distance behavior suggests a solution of the form

$$R_{n\ell}(r) = \phi_{n\ell} \Omega^{1/2}_{D-1} \left( \frac{(n + \ell)!}{n(n - \ell - 1)!} \right)^{1/2} \frac{\rho^\ell e^{-\rho/2}}{(2\ell + 1)!} L_{n\ell}(\rho)$$

(25)

where $\rho = 2\gamma_{n\ell} r$ is dimensionless with $\gamma_{n\ell} = (-2m_r E_{n\ell})^{1/2}$, and $\Omega_N \equiv 2\pi^{N/2}/\Gamma(N/2)$ is the surface area of a unit $N$-sphere. The constant $\phi_{n\ell}$ is, for $S$ states, the value of the wave function $\psi_{n,\ell=0}$ at the origin. For $D \neq 3$, a power series solution for $L(\rho)$ of the usual type does not work, and we require a series having the form

$$L_{n\ell}(\rho) = \sum_{j=0}^{\infty} \sum_{k=0}^{j} a_{jk} (\hat{n}_{n\ell})^k \rho^{j+2k},$$

(26)

6
Figure 1. NRQED kernels that contribute to bound state energy levels through order $m\alpha^5$. Terms (a)-(f) and (k) contribute at order $m\alpha^4$, and the rest are also required at order $m\alpha^5$. The first term—(a) fourth-order relativistic kinetic energy—come from the self-energy corrections. The next two—(b) spin-orbit, and (c) Darwin—involve the exchange of a Coulomb photon with a Coulomb vertex on one end. The next three involve the exchange of a transverse photon: (d) the magnetic interaction, (e) Fermi spin-orbit, and (f) Fermi spin-spin. Part (g) represents a Coulomb exchange with a vacuum polarization correction. Part (h) contains the full SC propagator spanned by an ultrasoft transverse photon. Part (i), the Salpeter correction, has a full SC propagator crossed by an ultrasoft transverse photon. The single transverse exchange term must be subtracted as it was included in (d) above. Part (j) is a transverse seagull graph, where the inner loop is purely soft. Part (k) is a contact term containing the hard two-photon exchange contribution and the effect of virtual annihilation to one or two photons.

where $\bar{n}_{nl}$ is a numerical parameter determined in the course of solving for $L_{nl}(\rho)$. The coefficients $a_{jk}$ satisfy a recursion relation that allows them all to be computed starting with the initial conditions $a_{00} = 1$ and $a_{jk} = 0$ unless $0 \leq k \leq j$. Expansions of the various quantities in the parameter $\epsilon$ near $\epsilon = 0$ are given in [67].

3. Progress towards order $m\alpha^7$

All corrections to positronium energy levels at orders $m\alpha^4$ and $m\alpha^5$ are contained in the first-order of perturbation theory—that is, they are contained in the expression $\delta K$. Of course, the kernel $\delta K$ has a lot of terms—those required for orders $m\alpha^4$ and $m\alpha^5$ are shown in figure 1.

Contributions at order $m\alpha^4$ come from a relativistic kinetic energy correction and the spin-orbit, Darwin, magnetic, Fermi spin-orbit, and spin-spin tensor interactions (figure 1a-f) along with a four fermion contact interaction representing the effect of virtual annihilation to one photon (figure 1k). At this order it is adequate to replace $1/k^2$ by $1/(−\vec{k}^2)$ in the propagators for the transverse exchange photons since the only contributing region is potential.

All kernels displayed in figure 1 are required at order $m\alpha^5$, but still only in the first order of perturbation theory. Now the regions analysis is much more interesting. The contributions of figures 1b,c are calculated as before, but now the order $\alpha$ terms in the matching coefficients $c_S$ and $c_D$ are required. The magnetic correction figure 1d, involving the exchange of a transverse photon with convection vertices on each end, has a contribution of order $m\alpha^3$ from the region.
where the outgoing (incoming) 4-momentum $p_2$ ($p_1$) and also the exchanged 4-momentum $k = p_2 - p_1$ are all potential, and contributions of order $ma^5$ from the region where $p_2$, $p_1$ are potential with $k$ ultrasoft and the region where only one of the $p$'s is potential while $k$ and the other $p$ are soft. Figure 1h with an ultrasoft spanning photon, in conjunction with the order $ma^5$ part of the Darwin correction figure 1c give rise to the traditional Lamb shift. The Salpeter correction of figure 1i contributes at order $ma^5$ when the exchanged transverse photon is ultrasoft, and the seagull term figure 1j contributes at order $ma^5$ when both exchanged photons are soft while the outgoing and incoming fermions are potential.

For order $ma^6$ a number of additional terms in the $(\delta K)$ come into play, as well as terms from the second order of perturbation theory. We are now working on a complete calculation of the order $ma^6$ energy levels using the formalism described here [68]. This exercise is giving us confidence that the method is working correctly and provides the basis for higher order work.

The next step is to complete the calculations at $O(ma^7)$. This will involve many new pieces, including the evaluation of some new contributions to the four-fermion matching coefficients (coming from four photon exchange, four photon annihilation, and other places), two-loop Bethe logarithms [69], and the careful analysis using the method of regions of all the terms that contribute at orders $ma^4$, $ma^5$, and $ma^6$ to uncover the new contributions they contain at $O(ma^7)$. Also necessary is a careful accounting of the higher order contributions contained in the matching coefficients, such as $c_D$, $c_S$, $c_F$, which contributed initially at $O(ma^4)$ but will contribute at $O(ma^7)$ as well because the matching coefficients themselves are power series in $a$. A great deal of extremely interesting work remains to be done.

Acknowledgments

This work was supported by the U.S. National Science Foundation through Grant No. PHY-1707489.

References

[1] Deutsch M 1951 Phys. Rev. 82 455–6
[2] Rich A 1981 Rev. Mod. Phys. 53 127–65
[3] Mills A P, Jr and Chu S 1990 Precision measurements in positronium Quantum Electrodynamics ed Kinoshita T (World Scientific) pp 774–821
[4] Karshenboim S G 2004 Int. J. Mod. Phys. A 19 3879–96
[5] Cassidy D B 2018 Eur. Phys. J. D 72 53
[6] Mills A P, Jr and Bearman G H 1975 Phys. Rev. Lett. 34 246–50
[7] Mills A P, Jr 1983 Phys. Rev. A 27 262–7
[8] Ritter M W, Egan P O, Hughes V W and Woodle K A 1984 Phys. Rev. A 30 1331–8
[9] Ishida A, Namba T, Asai S, Kobayashi T, Saito H, Yoshida M, Tanaka K and Yamamoto A 2014 Phys. Lett. B 734 338–44
[10] Mills A P, Jr, Berko S and Canter K F 1975 Phys. Rev. Lett. 34 1541–4
[11] Hatamian S, Conti R S and Rich A 1987 Phys. Rev. Lett. 58 1833–6
[12] Conti R S, Hatamian S, Lapidus L, Rich A and Skalsey M 1993 Phys. Lett. A 177 43–8
[13] Hagena D, Ley R, Weil D, Werth G, Arnold W and Schneider H 1993 Phys. Rev. Lett. 71 2887–90
[14] Ley R, Hagena D, Weil D, Werth G, Arnold W and Schneider H 1994 Hyperfine Int. 89 327–41
[15] Chu S, Mills A P, Jr and Hall J L 1984 Phys. Rev. Lett. 52 1689–92
[16] Danzmann K, Fee M S and Chu S 1989 Phys. Rev. A 39 6072–3
[17] Fee M S, Mills A P, Jr, Chu S, Shaw E D, Danzmann K, Chichester R J and Zuckerman D M 1993 Phys. Rev. Lett. 70 1397–400
[18] Fee M S, Chu S, Mills A P, Jr, Chichester R J, Zuckerman D M, Shaw E D and Danzmann K 1993 Phys. Rev. A 48 192–219
[19] Ley R 2002 Appl. Surf. Sci. 194 301–6
[20] Cassidy D B, Tom H W K and Mills A P, Jr 2008 AIP Conf. Proc. 1037 66–83
[21] Crivelli P, Cooke D A and Friedrich S 2014 Int. J. Mod. Phys.: Conf. Series 30 1460257
[22] Cooke D A, Crivelli P, Ahinis J, Antognini A, Brown B, Friedrich S, Gabard A, Haensch T W, Kirch K, Rubbia A and Vrankovic V 2015 Hyperfine Int. 233 67–73
[23] Pirenne J 1947 Archives des sciences physiques et naturelles 29 265–300
[24] Berestetski V B and Landau L D 1949 Zhurnal eksperimental’noi i teoreticheskoi fiziki 19 673–9
[25] Berestetski V B 1949 Zhurnal eksperimental’noi i teoreticheskoi fiziki 19 1130–5
[26] Ferrell R A 1951 Phys. Rev. 84 858–9
[27] Karplus R and Klein A 1952 Phys. Rev. 87 848–58
[28] Fulton T and Martin P C 1954 Phys. Rev. 95 811–22
[29] Fulton T 1982 Phys. Rev. A 26 1794–5
[30] Gupta S N, Repko W W and Suchyta C J III 1989 Phys. Rev. D 40 4100–4
[31] Khriplovich I B, Milstein A I and Yelkhovsky A S 1993 Physica Scripta T46 252–60
[32] Eides M I, Grotch H and Shelyuto V A 2007 Theory of light hydrogenic bound states Springer Tracts in Modern Physics, Volume 222 (Berlin: Heidelberg: Springer)
[33] Pachucki K and Karshenboim S G 1998 Phys. Rev. Lett. 80 2101–4
[34] Czarnecki A, Melnikov K and Yelkhovsky A 1999 Phys. Rev. A 59 4316–30
[35] Zatorski J 2008 Phys. Rev. A 78 032103
[36] Karshenboim S G 1993 Sov. Phys. JETP 76 541–6
[37] Melnikov K and Yelkhovsky A 1999 Phys. Lett. B 458 143–51
[38] Pachucki K and Karshenboim S G 1999 Phys. Rev. A 60 2792–8
[39] Kniehl B A and Penin A A 2000 Phys. Rev. Lett. 85 5094–7
[40] Melnikov K and Yelkhovsky A 2001 Phys. Rev. Lett. 86 1498–501
[41] Hill R J 2001 Phys. Rev. Lett. 86 3280–3
[42] Marcus S R 2001 Ultralow contribution to the positronium hyperfine splitting Master’s thesis, Univ. Alberta
[43] Baker M, Marquard P, Penin A A, Piclum J and Steinhauser M 2014 Phys. Rev. Lett. 112 120407
[44] Adkins G S and Fell R N 2014 Phys. Rev. A 89 052518
[45] Adkins G S, Parsons C, Salinger M D, Wang R and Fell R N 2014 Phys. Rev. A 90 042502
[46] Eides M I and Shelyuto V A 2015 Phys. Rev. D 92 013010
[47] Adkins G S, Kim M, Parsons C and Fell R N 2015 Phys. Rev. Lett. 115 233401
[48] Adkins G S, Parsons C, Salinger M D and Wang R 2015 Phys. Lett. B 747 551–5
[49] Adkins G S, Tran L M and Wang R 2016 Phys. Rev. A 93 052511
[50] Eides M I and Shelyuto V A 2017 Phys. Rev. D 96 011301(R)
[51] Caswell W E and Lepage G P 1986 Phys. Lett. B 167 437–42
[52] Kinoshita T and Nio M 1996 Physical Review D 53 4909–29
[53] Hill R J, Lee G, Paz G and Solon M P 2013 Phys. Rev. D 87 053017
[54] Pineda A and Soto J 1998 Nucl. Phys. B (Proc. Suppl.) 64 428–32
[55] Pineda A and Soto J 1998 Phys. Lett. B 420 391–6
[56] Pineda A and Soto J 1998 Phys. Rev. D 59 016005
[57] Hoang A H 2002 Heavy quarkonium dynamics At the Frontier of Particle Physics-Handbook of QCD vol 4 ed Shifman M (World Scientific)
[58] Brambilla N, Pineda A, Soto J and Vairo A 2005 Rev. Mod. Phys. 77 1423–96
[59] Beneke M and Smirnov V A 1999 Nucl. Phys. B 522 321–44
[60] Adkins G S and Fell R N 1999 Phys. Rev. A 60 4461–75
[61] Hostler L 1962 Bull. Am. Phys. Soc. 7 (Series II) 609
[62] Hostler L and Pratt R H 1963 Phys. Rev. Lett. 10 469–70
[63] Schwinger J 1964 J. Math. Phys. 5 1606–8
[64] Kato T 1940 Prog. Theor. Phys. IV 514–23
[65] Messiah A 1962 Quantum Mechanics vol II (New York: Wiley)
[66] Lepage G P 1977 Phys. Rev. A 16 863–76
[67] Adkins G S 2018 Phys. Lett. A 382 1545–9
[68] Adkins G S, Akers B, Alam F, Tran L M and Zhang X (unpublished manuscript)
[69] Czarnecki A, Jentschura U D and Pachucki K 2005 Phys. Rev. Lett. 95 180404