Nash equilibrium and the Legendre transform in optimal stopping games with one dimensional diffusions

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Abstract

We show that the value function of an optimal stopping game driven by a one-dimensional diffusion can be characterised using a modification of the Legendre transformation if and only if the optimal stopping game exhibits a Nash equilibrium (i.e. a saddle point of the optimal stopping game exists). This result is an analytical complement to the results in [19] where the ‘duality’ between a concave-biconjugate which is modified to remain below an upper barrier and a convex-biconjugate which is modified to remain above a lower barrier is proven by appealing to the probabilistic result in [18]. The main contribution of this paper is to show that, in this special case, the semi-harmonic characterisation of the value function may be proven using only results from convex analysis.

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1 Introduction

This paper examines the connection between convex analysis and optimal stopping of one-dimensional diffusions. The connection between convex analysis and optimal stopping dates back to Dynkin [7] where the solution to an optimal stopping problem of the type

\[ V(x) = \sup_{\tau} E_x \{ G(X_{\tau}) \} \]

where \( X \) is a Markov process was first characterised the smallest superharmonic majorant of the gains function \( G \). When \( X \) is a one-dimensional diffusion, the corresponding superharmonic functions can been characterised in terms of a generalised type of concavity (see [10] pp. 115). This so called ‘superharmonic characterisation’ of the value function is discussed in further detail in [20] Chapter IV Section 9, to illustrate the properties of the solution to certain free-boundary problems associated with optimal stopping problems. More recently, the change of time and scale technique underpinning the superharmonic characterisation was ‘re-introduced’ in [6].

The minimax version of this problem is referred to as an optimal stopping (or Dynkin) game. The sup-player selects a stopping time \( \tau \) with the aim of maximising the functional

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Consider a stochastic basis \((\Omega, \mathcal{F}, \mathbb{F}, \{P_x\}_{x \in I})\) supporting a one dimensional diffusion \(X = (X_t)_{t \geq 0}\) with \(X_0 = x \in I \subseteq \mathbb{R}\) under \(P_x\). The level passage times of \(X\) will be denoted by \(T_y = \inf \{ t \geq 0 \mid X_t = y \}\). The state space \(I\) of \(X\) is an open interval on \(\mathbb{R}\) i.e. \(I = (a, b)\) and the boundaries of \(I\) are natural, i.e. \(P_x(T_a < \infty) = P_x(T_b < \infty) = 0\) for all \(x \in I\). The diffusion \(X\) is also assumed to be regular in the sense that for all \(x \in I\), \(P_x(T_y < \infty) > 0\) for some \(y > x\).

A Borel measurable function \(U : I \rightarrow \mathbb{R}\) is \(r\)-superharmonic with respect to \(X\) if

\[
U(x) \geq E_x \left[ U(X_\tau) e^{-r\tau} 1_{\tau \leq \sigma} \right]
\]

for all \(x \in I\) and all stopping times \(\tau\). The function \(U\) is \(r\)-subharmonic with respect to \(X\) if

\[
U(x) \leq E_x \left[ U(X_\tau) e^{-r\tau} 1_{\tau < \sigma} \right]
\]

for all \(x \in I\) and all stopping times \(\tau\). Moreover, \(U\) is referred to as \(r\)-harmonic, if it is both \(r\)-superharmonic and \(r\)-subharmonic.

The generator of \(X\) is denoted \(L_X\) and under some additional regularity conditions (see [4] Section 4.6) can be expressed as

\[
L_X f(x) = \mu(x) \frac{d}{dx} f(x) + D(x) \frac{d^2}{dx^2} f(x)
\]

for \(x \in I\) where \(\mu\) is the drift and \(D > 0\) is the diffusion coefficient of \(X\). Take a constant \(r > 0\) and consider the ODE

\[
L_X f(x) = r f(x)
\]  

(2.1)
which has two linearly independent positive solutions, denoted \( \varphi \) and \( \psi \). As the boundaries of \( I \) are natural, \( \varphi \) and \( \psi \) may be taken such that \( \varphi \) is increasing and \( \psi \) is decreasing with

\[
\varphi(a+) = \psi(b-) = 0 \quad ; \quad \varphi(b-) = \psi(a+) = +\infty.
\]

The functions \( \varphi \) and \( \psi \) are shown in [23] V.50 to be continuous, strictly monotone and strictly convex. Furthermore, for \( y \in I \), the process \((e^{-r(t\land T_y)}\psi(X_{t\land T_y}))_{t \geq 0}\) is a \((P_x, \mathcal{F}_{t\land T_y})\)-martingale for all \( y > x \) while \((e^{-r(t\land T_y)}\psi(X_{t\land T_y}))_{t \geq 0}\) is a \((P_x, \mathcal{F}_{t\land T_y})\)-martingale for all \( y \leq x \). For \( x, y \in I \) the Laplace transforms of the first passage times may be expressed as

\[
E_x \left[ e^{-rT_y}\mathbb{1}_{T_y < \infty} \right] = \begin{cases} \frac{\psi(x)}{\psi(y)} & \text{for } x > y \\ \frac{\varphi(x)}{\varphi(y)} & \text{for } x \leq y \end{cases}.
\]

**Definition 1.** Let \( J : I \rightarrow \mathbb{R} \) be a monotone function. A Borel measurable function \( f : I \rightarrow \mathbb{R} \) is \( J \)-concave if

\[
f(x) \geq f(y) \frac{J(z) - J(x)}{J(z) - J(y)} + f(z) \frac{J(x) - J(y)}{J(z) - J(y)}
\]

for \( x \in [y, z] \subseteq I \). The \( J \)-derivative of \( f \) is denoted

\[
\frac{d}{dJ} f(x) := \lim_{y \uparrow z} \frac{f(y) - f(x)}{J(y) - J(x)}.
\]

Define a pair of strictly increasing functions

\[
F(x) = \frac{\varphi(x)}{\psi(x)} \quad ; \quad \tilde{F}(x) = \frac{\psi(x)}{\varphi(x)} = -\frac{1}{F(x)}.
\]

it is well-known (see [8] Theorem 16.4) that a Borel measurable function \( U \) is \( r \)-superharmonic if and only if \( U/\psi \) is \( F \)-concave, or equivalently, \( U/\varphi \) is \( \tilde{F} \)-concave. For \( a < y \leq x \leq z < b \) the Laplace transforms of the exit times from the open set \((y, z) \subset I \) are

\[
\begin{align*}
E_x \left[ e^{-rT_y}\mathbb{1}_{T_y < T_z} \right] &= \frac{\psi(x) F(z) - F(x)}{\psi(y) F(z) - F(y)} = \frac{\varphi(x) \tilde{F}(z) - \tilde{F}(x)}{\varphi(y) \tilde{F}(z) - \tilde{F}(y)} \\
E_x \left[ e^{-rT_z}\mathbb{1}_{T_z > T_y} \right] &= \frac{\psi(x) F(x) - F(y)}{\psi(z) F(x) - F(y)} = \frac{\varphi(x) \tilde{F}(x) - \tilde{F}(y)}{\varphi(z) \tilde{F}(x) - \tilde{F}(y)}.
\end{align*}
\]

Take a payoff function \( G : I \rightarrow \mathbb{R} \) which is upper semi-continuous and such that for all \( x \in I \),

\[
P_x \left( \lim_{t \rightarrow \infty} G(X_t) e^{-rt} = c \right) = 1 \quad , \quad E_x \left[ \sup_{t \geq 0} |G(X_t) e^{-rt}| \right] < +\infty
\]

for some \( c \in \mathbb{R} \). Section 3 focuses on an alternative characterisation of the solution to the discounted optimal stopping problem

\[
V(x) = \sup_{r} E_x \left[ G(X_r) e^{-rt} \right].
\]

The problem will be approached via the concave biconjugate of a transformed objective function as in [19] but without reference to previous results on the superharmonic characterisation of the
value function (see [20] Chapter 1 and the references therein for further details). Theorem 3 shows that the function $x \mapsto V(x)$ defined in (2.5) solves the ‘dual problem’

$$V = \inf_{F \in \text{Sup}(G)} F$$

(2.6)

where

$$\text{Sup}(G) := \{ F : I \to [G, +\infty) \mid F \text{ is continuous and } r\text{-superharmonic} \}.$$

Hence Theorem 3 provides a proof of the superharmonic characterisation of the value function in this specific case using only analytical tools from convex analysis.

In section 4 we examine the smallest $F$-concave function lying above the function $G$ yet below the function $H$. The functions $G, H$ are continuous and for all $x \in I$ satisfy:

$$G(x) \leq H(x) \quad ; \quad E_x \left[ \sup_{t \geq 0} |G(X_t) e^{-rt}| \right] < +\infty \quad ; \quad E_x \left[ \sup_{t \geq 0} |H(X_t) e^{-rt}| \right] < +\infty$$

(2.7)

and

$$P_x \left( \lim_{t \to \infty} G(X_t) e^{-rt} = \lim_{t \to \infty} H(X_t) e^{-rt} \right) = 1.$$  

(2.8)

Under these assumptions on $G$ and $H$ it is shown in Theorem 17 that the smallest $F$-concave function dominating $G$ which is below $H$ coincides with the largest $F$-convex function minorising $H$ which is above $G$.

Section 5 studies the solution to the optimal stopping game with lower value $V_-$ defined as

$$V_- (x) = \sup_{\tau, \sigma} E_x \left[ (G(X_\tau) \mathbb{1}_{[\tau \leq \sigma]} + H(X_\sigma) \mathbb{1}_{[\tau > \sigma]}) e^{-r(\tau \land \sigma)} \right],$$  

(2.9)

and upper value $V_+$ defined as

$$V_+ (x) = \inf_{\tau, \sigma} E_x \left[ (G(X_\tau) \mathbb{1}_{[\tau \leq \sigma]} + H(X_\sigma) \mathbb{1}_{[\tau > \sigma]}) e^{-r(\tau \land \sigma)} \right].$$  

(2.10)

where $G$ and $H$ satisfy the assumptions outlined above. For ease of notation, the objective function is denoted

$$R_x (\tau, \sigma) := E_x \left[ (G(X_\tau) \mathbb{1}_{[\tau \leq \sigma]} + H(X_\sigma) \mathbb{1}_{[\tau > \sigma]}) e^{-r(\tau \land \sigma)} \right].$$  

(2.11)

The optimal stopping game has a Stackelberg equilibrium if the upper and lower values coincide, i.e. $V_- (x) = V_+ (x)$ for all $x \in I$. In [11] and [12] it is shown via probabilistic means that this game exhibits a Stackelberg equilibrium when both (2.7) and (2.8) hold. The assumptions in [12] are slightly more general, in which case the Stackelberg equilibrium is determined by how the objective function is specified at the natural boundaries. A saddle point is a pair of stopping times $(\tau^*, \sigma^*)$ such that for any other stopping times $\tau, \sigma$

$$R_x (\tau, \sigma^*) \leq R_x (\tau^*, \sigma^*) \leq R_x (\tau^*, \sigma) \quad \forall x \in I.$$

The optimal stopping game exhibits a Nash equilibrium if the game has saddle point. In particular, existence of a Nash equilibrium implies a Stackelberg equilibrium exists but the converse is not true. The result in [11] (which applies to more general processes) shows that, under the assumptions (2.7) and (2.8), the optimal stopping game described above has a Nash equilibrium.
Under the weaker assumptions used in [12] a saddle point need not exist, the conditions under which Nash equilibrium fail are discussed in Section 5. Introduce a pair of dual problems

\[ \hat{V} := \inf_{F \in \text{Sup}(G,H)} F \quad ; \quad \check{V} := \sup_{F \in \text{Sub}(G,H)} F \]  

where the admissible sets of functions are

\[ \text{Sup}(G,H) = \{ F : I \rightarrow [G,H] \mid F \text{ is finely continuous and } r\text{-superharmonic on } \{ F > G \} \text{ and } \{ F > V \} \}, \]

\[ \text{Sub}(G,H) = \{ F : I \rightarrow [G,H] \mid F \text{ is finely continuous and } r\text{-subharmonic on } \{ F < H \} \text{ and } \{ F < V \} \}. \]

It has been shown in [18] Theorem 2.1 that when (2.7) and (2.8) hold \( \hat{V} = \check{V} \) and that the value of the dual problems coincides with the value of the optimal stopping game with upper and lower values (2.9)-(2.10) if and only if the optimal stopping game has a Nash equilibrium. In [19] the dual formulation (2.12) was used to show that the value functions of optimal stopping games based on one dimensional diffusions absorbed upon exit from a compact set can be represented using a modification of the Legendre transform. Section 4 provides converse results to [19] by deriving the duality between these generalised Legendre transforms without appealing to results from optimal stopping. Section 5 relates the modified Legendre transformations introduced in [19] to the dual problems (2.12) and the solution of the optimal stopping game. In Theorem 18 the optimal stopping game (2.9)-(2.10) is shown to have a Stackelberg equilibrium. Theorem 22 shows that when (2.7) and (2.8) are assumed to hold the value of the dual problems (2.12) coincide, i.e. \( \hat{V} = \check{V} \), and the the optimal stopping game exhibits a Nash equilibrium.

### 3 Optimal stopping using the Legendre transformation

Before proceeding to solve the optimal stopping problem (2.5), we shall first recall the definition and some properties of the concave biconjugate. Let \( f : \text{dom}(f) \rightarrow \mathbb{R} \) be a proper, measurable function on the domain \( \text{dom}(f) \subseteq \mathbb{R} \). The concave conjugate of \( f \), denoted \( f^\ast \), is defined for \( c \in \mathbb{R} \) as

\[ f^\ast(c) = \inf_{x \in \text{dom}(f)} (cx - f(x)) . \]

The concave biconjugate of \( f \) is defined as

\[ f^{**}(x) = \inf_{y \in \text{dom}(f^\ast)} (xy - f^\ast(y)) = \inf_{y \in \text{dom}(f^\ast)} \sup_{c \in \text{dom}(f)} (y(x - c) + f(c)) . \]  

The epigraph of a function \( f \) is the set of all points above the graph of \( f \), that is

\[ \text{epi}(f) := \{ (x, \mu) \in \mathbb{R}^2 \mid \mu \geq f(x) \} . \]

The convex hull of the set \( \text{epi}(f) \) is the intersection of all convex sets containing \( \text{epi}(f) \) and is denoted \( \text{conv}(f) \). With a slight abuse of notation, let

\[ \text{conv}(f)(x) := \inf \{ y \in \mathbb{R} \mid (x, y) \in \text{conv}(f) \} , \]
then $-f_{**}$ is the upper semi-continuous modification of $\text{conv} (-f) (x)$, see [22] Theorem 12.2 and Corollary 12.1.1. A constant $c \in \mathbb{R}$ is a subgradient of the function $f$ at $x \in \mathbb{R}$ if
\[
f (z) \geq f (x) + c (z - x) \quad \forall z \in \mathbb{R}.
\]
The set of all such supergradients, denoted
\[
\partial f (x) := \{ c \in \mathbb{R} \mid f (y) - f (x) \leq c (y - x) \quad \forall y \in \mathbb{R} \}
\] (3.14)
is referred to as the superdifferential. The function $f$ is referred to as concave at $x \in \mathbb{R}$ when $\partial f (x) \neq \emptyset$. When $f$ is concave and differentiable at $x$ then $\partial f (x) = \{ f' (x) \}$.

The next lemma characterises the set upon which $f$ coincides with $f_{**}$. The proof illustrates how $x \mapsto f_{**} (x)$ can be constructed using a spike variation.

**Lemma 2.** Any function $f : \mathbb{R} \to \mathbb{R} \cup \{ +\infty \}$ coincides with its concave biconjugate on the set
\[
\{ x \in \mathbb{R} \mid f_{**} (x) = f (x) \} = \{ x \in \mathbb{R} \mid \partial f (x) \neq \emptyset \}.
\] (3.15)

**Proof.** Suppose that $\partial f (x) \neq \emptyset$ and take $c \in \partial f (x)$ then by definition $f (y) + c (x - y) \leq p$ for all $p \geq f (x)$ and all $y \in \mathbb{R}$. Hence
\[
f_*(c) := \inf_{y \in \mathbb{R}} (cy - f(y)) = cx - f(x).
\]
Consequently, $f_{**}$ can be written as
\[
f_{**} (x) = \inf_{c \in \mathbb{R}} (c (x - x) + f (x)) = f (x).
\]
so $\{ x \in \mathbb{R} \mid \partial f (x) \neq \emptyset \} \subseteq \{ x \in \mathbb{R} \mid f_{**} (x) = f (x) \}$. To show this inclusion holds with equality assume that $\partial f (x) = \emptyset$ and for a fixed $c \in \mathbb{R}$ let
\[
\varepsilon (x;c) := \inf \{ \varepsilon \geq 0 \mid f (y) + c (x - y) \leq f (x) + \varepsilon \quad \forall y \in \mathbb{R} \}.
\]
It follows that
\[
-f_*(c) = \sup_{y \in \mathbb{R}} (f(y) - cy) = f(x) + \varepsilon(x;c) - cx,
\]
and $f_{**}$ can be written as
\[
f_{**} (x) = \inf_{c \in \mathbb{R}} (cx - f_*(c)) = \inf_{c \in \mathbb{R}} (f (x) + \varepsilon(x;c)).
\] (3.16)
Let
\[
\varepsilon^* (x) := \inf \{ \varepsilon \geq 0 \mid \exists c \in \mathbb{R} \quad \text{s.t.} \quad f (y) + c (x - y) \leq f (x) + \varepsilon \quad \forall y \in \mathbb{R} \}
\] (3.17)
so that $\{ x \in \mathbb{R} \mid \varepsilon^* (x) > 0 \} = \{ x \in \mathbb{R} \mid \partial f (x) = \emptyset \}$. Hence we may conclude from (3.16) that when $\partial f (x) = \emptyset$, $f_{**} (x) = f (x) + \varepsilon^* (x) > f (x)$.

\[\square\]
Let \( W : [0, \infty) \to \mathbb{R} \) be defined as
\[
W(y) := \begin{cases} 
\left( \frac{G}{\psi} \right) \circ (F)^{-1}(y) & \text{for } y > 0 \\
\limsup_{x \downarrow a} \frac{G^+(x)}{\psi(x)} & \text{for } y = 0
\end{cases}
\]  
(3.18)
where \( F \) was defined in (2.2) and \( G^+ = G \lor 0 \). Similarly, let \( \tilde{W} \) be defined via
\[
\tilde{W}(y) := \begin{cases} 
\limsup_{x \uparrow b} \frac{G^+(x)}{\psi(x)} & \text{for } y = 0 \\
\left( \frac{G}{\psi} \right) \circ (\tilde{F})^{-1}(y) & \text{for } y < 0
\end{cases} .
\]  
(3.19)

The next result is the main result in this section and it shows that the value function \( V \) defined in (2.5) is such that \( V/\psi \) coincides with \( W_* \circ F \) where \( W_* \) is the concave biconjugate of the function \( W \). The value function \( V \) is also such that \( V/\varphi \) coincides with \( \tilde{W}_* \circ \tilde{F} \) where \( \tilde{W}_* \) is the concave biconjugate of the function \( \tilde{W} \). These transformations are used as the function \( y \mapsto W(y) \) (resp. \( y \mapsto \tilde{W}(y) \)) is concave if and only if \( x \mapsto (G/\psi)(x) \) is \( F \)-concave (resp. \( \tilde{F} \)-concave). The case that both boundaries are absorbing has been handled by showing that \( W_* (F(x)) \varphi(x) \) solves (2.6) in [19] Theorem 3.2.

**Theorem 3.** Assume that \( G : I \to \mathbb{R} \) is an upper-semicontinuous function satisfying the assumption (2.4) and such that \( W(0) = \tilde{W}(0) = 0 \). Consider the stopping problem (2.5), then
\[
V(x) = W_*(F(x)) \psi(x) = \tilde{W}_*(\tilde{F}(x)) \varphi(x) .
\]

The stopping time which attains the supremum in (2.5) is \( \tau^* = T_{a^*} \land T_{b^*} \) where
\[
a^* = \sup \{ a \leq x \mid W(F(a)) = W_*(F(a)) \} ,
\]  
(3.20)
\[
b^* = \inf \{ x \leq b \mid W(F(b)) = W_*(F(b)) \} ,
\]
so \( V(x) = E_{x}[G(X_{T_{a^*} \land T_{b^*}}^*) e^{-r(T_{a^*} \land T_{b^*})}] . \) Furthermore \( V \) solves (2.6).

**Proof.** The definition of \( W(0) \) and \( \tilde{W}(0) \) ensure that
\[
W_*(F(x)) \psi(x) \geq 0 ; \quad \tilde{W}_*(\tilde{F}(x)) \varphi(x) \geq 0
\]
for all \( x \in (a, b) \). Let \( (b_n)_{n \geq 1} \) be a sequence of real numbers such \( (a, b_n] \subset (a, b) \) for all \( n \geq 1 \) and \( (a, b_n) \to (a, b) \) as \( n \to +\infty \). Fix \( x \in (a, b) \) and for some sufficiently large \( N \), let \( x \in (a, b_n) \) for all \( n \geq N \). Denote by \( \delta : I \to \{ 0, +\infty \} \) the ‘characteristic function’ of a set \( A \) which is defined as
\[
\delta(x \mid x \in A) := \begin{cases} +\infty & \text{for } x \in A \\
0 & \text{for } x \notin A
\end{cases} ,
\]  
(3.21)
and introduce a function \( W_n \) defined as \( W_n(x) = W(x) - \delta(x \mid x \in I \setminus (a, b_n)] \) where \( W \) was defined in (3.18). Let \( X^n_t := X_{t \land T_{b_n}} \) and consider the following family of optimal stopping problems
\[
V^n(x) = \sup_{\tau} E_{x}[G(X^n_\tau) e^{-rt}] \]  
(3.22)
where \( G \) satisfies \( \limsup_{x \downarrow a} G^+(x) / \psi(x) = 0 \). An optimal stopping time for each of (3.22) exists (see [20] Chapter 1 Theorem 2.7) and is the exit time from an open set containing \( x \) so we may
write the optimal stopping time for [3.22] as \( \tau_n^* = T_{a_n} \wedge T_{b_n} \) for some \( a \leq a_n \leq x \leq b_n \leq b \).

Furthermore, each \( \tau_n^* \leq T_{b_n} \), \( \tau_n^* \leq \tau_{n+1}^* \) for all \( n \geq 1 \) and \( \lim_{n \to +\infty} \tau_n^* = \tau^* \) -a.s. for all \( x \in \mathbb{R} \). For all \( n \geq 1 \), \( X_{\tau_n}^n = X_{\tau_n} \) and the process \( X \) is left-continuous over stopping times so \( \lim_{n \to +\infty} X_{\tau_n}^n = X_{\tau^*} \) -a.s. for all \( x \in \mathbb{R} \). The assumption (2.4) implies that \( (G(X_t)e^{-rt})_{t \geq 0} \) is uniformly integrable so it follows that

\[
\lim_{n \to \infty} V^n(x) = \lim_{n \to \infty} E_x[G(X_{\tau_n}^n)e^{-r\tau_n}] = V(x).
\]

The process \( (e^{-r(\tau \wedge T_{b_n})}\varphi(X_{\tau}^n))_{t \geq 0} \) is a \( (P_x,F_{\tau \wedge T_{b_n}}) \)-martingale for each \( b_n > x \) so applying the optional sampling theorem yields \( E_x[\varphi(X_{\tau}^n)e^{-\tau\tau}] = \varphi(x) \) for all \( x \in (a,b_n] \) and all \( \tau \leq T_{b_n} \).

Hence for an arbitrary \( c \in \mathbb{R} \)

\[
V^n(x) = \sup_{\tau} E_x \left[ G(X_{\tau}^n) e^{-r\tau} \right] = c \varphi(x) + \sup_{\tau} E_x \left[ (c \varphi(X_{\tau}^n) - G(X_{\tau}^n)) e^{-r\tau} \right].
\]

Thus,

\[
V^n(x) = \inf_{c \in \mathbb{R}} \left( c \varphi(x) - \inf_{\tau} E_x \left[ (c \varphi(X_{\tau}^n) - G(X_{\tau}^n)) e^{-r\tau} \right] \right).
\]

The next step is to expand the right hand side of this expression in such a way that it converges to the concave biconjugate of \( W \) as \( n \to +\infty \). Using (2.3), the inner infimum in (3.23) can be written as

\[
\inf_{a \leq y \leq z \leq b_n} E_x \left[ (c \varphi(X_{\tau}^n_{T_{b_n}/T_{z}}) - G(X_{\tau}^n_{T_{b_n}/T_{z}})) e^{-r(T_{b_n}/T_{z})} \right]
\]

\[
= \inf_{a \leq y \leq z \leq b_n} \left( \frac{c \varphi(y) - G(y)}{\psi(y)} F(z) - F(x) + \frac{c \varphi(z) - G(z)}{\psi(z)} F(x) - F(y) \right) \psi(x).
\]

We claim that the right hand side of this expression converges to

\[
\inf_{y \in (a,b)} \left( \frac{c \varphi(y) - G(y)}{\psi(y)} \right) \psi(x) = \inf_{y \geq 0} (cy - W(y)) \psi(x) =: W_*(c) \psi(x)
\]

as \( n \to \infty \). To this end, take

\[
z^+ = \sup \{ z \in I \mid W_*(c) \psi(z) = c \varphi(z) - G(z) \}
\]

\[
y^- = \inf \{ y \in I \mid W_*(c) \psi(y) = c \varphi(y) - G(y) \}
\]

with the convention that \( \sup \emptyset = a \) and \( \inf \emptyset = b \). In general for all \( a \leq y \leq x \leq z \leq b_n \)

\[
W_*(c) \leq \frac{c \varphi(y) - G(y)}{\psi(y)} \frac{F(z) - F(x)}{F(z) - F(y)} + \frac{c \varphi(z) - G(z)}{\psi(z)} \frac{F(x) - F(y)}{F(z) - F(y)}.
\]

(3.24)

When \( y^- \leq x \leq z^+ \), (3.24) holds with equality for \( y = y^- \) and \( z = z^+ \); when \( y^- \geq x \) equality holds when \( y = a \) and \( z = y^- \) whereas if \( z^+ \leq x \) the inequality is strict as

\[
\frac{c \varphi(y) - G(y)}{\psi(y)} \frac{F(z) - F(x)}{F(z) - F(y)} + \frac{c \varphi(y^-) - G(y^-)}{\psi(y^-)} \frac{F(x) - F(y)}{F(y^-) - F(y)} > \frac{c \varphi(y^-) - G(y^-)}{\psi(y^-)}
\]

for all \( y \in [x,b_n] \). We may switch to the other ratio of the fundamental solutions using (2.3) on the left-hand-side of this equality and use \( \varphi(y) \to +\infty \) as \( y \uparrow b \) to conclude that

\[
\lim_{n \to \infty} \inf_{y \in [x,b_n]} \left( \frac{c \varphi(y) - G(y)}{\psi(y)} \frac{\bar{F}(z) - \bar{F}(x)}{\bar{F}(z) - \bar{F}(y)} + \frac{c \varphi(z) - G(z)}{\psi(z)} \frac{\bar{F}(x) - \bar{F}(y)}{\bar{F}(z) - \bar{F}(y)} \right) = W_*(c).
\]
Hence letting \( n \to \infty \) on both sides of (3.23) yields

\[
V(x) = \inf_{c \in \mathbb{R}} \left( \frac{\varphi(x)}{\psi(x)} + W_*(c) \right) \psi(x).
\]

The argument using the other ratio of fundamental solutions follows analogously. The statement about the optimal stopping time follows from the form of the value function provided as the stopping region \( D \) and continuation region \( C \) for (2.5) are

\[
C = \{ x \in I \mid V(x) > G(x) \} = \{ x \in I \mid W_{**}(F(x)) \psi(x) > G(x) \},
\]

\[
D = \{ x \in I \mid V(x) = G(x) \} = \{ x \in I \mid W_{**}(F(x)) \psi(x) = G(x) \}.
\]

Finally, \( W_{**} = -\text{cl}\left(\text{conv}(-W)\right) \) is the smallest concave function dominating \( W \), so \( V(x) = W_{**}(F(x)) \psi(x) \) is the smallest \( r \)-superharmonic function dominating the gains function \( G(x) \) and hence \( V(x) \) solves (2.6).

**Remark 4.** The function \( y \mapsto W(y) \) (resp. \( y \mapsto \widetilde{W}(y) \)) is defined at the natural boundary of \( (\text{resp. } b) \) in such a way as to ensure that \( V(a) = V(b) = 0 \). In [6] and [12] this condition is imposed by defining \( F(X_\tau(\omega)) = 0 \) on \( \{ \tau = +\infty \} \) for all Borel measurable functions \( F \).

**Remark 5.** If \( W(0) = \widetilde{W}(0) = 0 \) then a finite optimal stopping time for (2.5) exists whenever \( W \) is not strictly convex on \((0, +\infty) \) (resp. \( \widetilde{W} \) is not strictly convex on \((-\infty, 0) \)). When the gains function \( G \) satisfies (2.4) then we may assume that \( W(0) = \widetilde{W}(0) = 0 \) without loss of generality as when this is not true it can be achieved by subtracting the finite harmonic function \( h(x) = E_x \left[ \lim_{t \to \infty} G(X_t) e^{-rt} \right] \) from the function \( V(x) \) defined in (2.5). When (2.4) fails and \( G(x) \) grows faster than \( \psi \) as \( x \to a \) we will have \( W(0) = +\infty \) and no finite optimal stopping time exists as \( V(x) \geq \psi(x) \lim_{y \to a} G(y)/\psi(y) = +\infty \). When (2.4) fails and \( G(x) = O(\psi(x)) \) as \( x \to a \) then \( W(0) \) is finite but the harmonic function \( h \) need not be well-defined. The function \( W_{**}(F(x))\psi(x) \) remains the smallest \( r \)-superharmonic function above \( G \), and a sequence of stopping times \((\tau^*_n)_{n \geq 1} \) can be identified such that \( \lim_{n \to +\infty} \tau^*_n = \tau^* \) where \( \tau^* = \inf \{ t \geq 0 \mid W_{**}(F(X_t))\psi(X_t) = G(X_t) \} \). However, if the convention mentioned in Remark 4 is adopted it may occur that \( W_{**}(F(x))\psi(x) > E_x [G(X_{\tau^*})e^{-rt^*}] \).

For fixed \( x \in \mathbb{R} \) let

\[
l^r(c,p) = \sup \left\{ y \leq x \mid \left( \frac{\varphi}{\psi} \right)(y) - \frac{p}{\psi(x)} = c(F(y) - F(x)) \right\},
\]

\[
r^r(c,p) = \inf \left\{ y \geq x \mid \left( \frac{\varphi}{\psi} \right)(y) - \frac{p}{\psi(x)} = c(F(y) - F(x)) \right\},
\]

with the usual convention that \( \sup \emptyset = -\infty \) and \( \inf \emptyset = +\infty \). For a given \( x \in I \) consider the sets

\[
A_1(x) = \{ p \in [G(x), +\infty) \mid \exists c \in \mathbb{R} \text{ s.t. } (l^r(c,p), r^r(c,p)) \subseteq I \},
\]

\[
A_2(x) = \{ p \in [G(x), +\infty) \mid \exists c \in \mathbb{R} \text{ s.t. } r^r(c,p) = -l^r(c,p) = +\infty \}.
\]

In Lemma 2 it was shown that \( x \mapsto W_{**}(x) \) can be constructed by minimising over the ‘\( F \)-tangents’ of the form \( y \mapsto p/\psi(x) + c(F(y) - F(x)) \) which strictly dominate \( W \) on \( \text{dom}(W) \), i.e. \( W_{**}(x) = \inf A_2(x) \). The next corollary provides a short proof of the ‘dual interpretation’
provided in [19] which claims that $x \mapsto W^{**}(x)$ can be also constructed by maximising over the $F$-tangents which intercept $W$ on both sides of the point $x$ in dom$(W)$, i.e. $W^{**}(x) = \sup A_1(x)$. The latter formulation facilitates the construction of a maximising sequence of stopping times.

**Corollary 6.** Let

$$\hat{\varepsilon}(x;c) := \sup \{ \varepsilon \geq 0 \mid a \leq F^x(c, G(x) + \varepsilon) \leq x \leq r^x(c, G(x) + \varepsilon) \leq b \}$$

and set $A_c(x) := [F^x(c, G(x) + \varepsilon(x;c)), F^x(c, G(x) + \varepsilon(x;c))]$, then the value function of the optimal stopping problem (2.5) can be represented as

$$V(x) = \sup_{c \in \mathbb{R}} \sup_{y \in A_c(x)} (c(F(x) - y) + W(y)) \psi(x).$$

Moreover, $V(x) = \sup A_1(x)$ for each $x \in I$ where $A_1(x)$ is the set defined in (3.25).

**Proof.** For the first statement, note that for each $c \in \mathbb{R}$, by definition

$$\frac{G(x) + \hat{\varepsilon}(x;c)}{\psi(x)} + c(y - F(x)) \geq W(y)$$

for $y \in A_c(x)$ and equality holds for $y = F^x(c, G(x) + \hat{\varepsilon}(x;c))$. Hence,

$$\sup_{y \in A_c(x)} (W(y) - cy) = W(F(x)) + \frac{\hat{\varepsilon}(x;c)}{\psi(x)} - cF(x)$$

and it follows that

$$\sup_{c \in \mathbb{R}} \sup_{y \in A_c(x)} (c(F(x) - y) + W(y)) \psi(x) = \sup_{c \in \mathbb{R}} (G(x) + \hat{\varepsilon}(x;c))$$

Let

$$\varepsilon(x;c) := \inf \left\{ \varepsilon \geq 0 \mid \left( \frac{G}{\psi} \right)(y) + c(F(x) - y) \leq \frac{G(x) + \varepsilon}{\psi(x)} \quad \forall y \in \mathbb{R} \right\}.$$
For each \( c \in \mathbb{R} \), \( \varepsilon (x; c) \geq \hat{\varepsilon} (x; c) \) as illustrated in Figure 1 but to avoid contradicting their definitions \( \inf_{c \in \mathbb{R}} \varepsilon (x; c) = \sup_{c \in \mathbb{R}} \hat{\varepsilon} (x; c) \) and hence the result follows from Lemma 2. Furthermore, it follows from the definition of the set \( A_1 (x) \) that \( \sup A_1 (x) = G (x) + \sup_{c \in \mathbb{R}} \hat{\varepsilon} (x; c) = W_* (F(x)) \psi (x) = V (x). \)

The next corollary shows that Theorem 3 is capable of handling the case when one (or more) of the boundaries is absorbing.

**Corollary 7.** Take \( x \in [\alpha, \beta] \subset [a, b] \) and consider the stopped diffusion \( X^\alpha_\beta := X_{\tau \wedge T_{\alpha, \beta}} \) where \( T_{\alpha, \beta} = T_\alpha \wedge T_\beta \) and the corresponding optimal stopping problem

\[
V_{\alpha, \beta} (x) = \sup_{\tau} E_x \left[ G (X_{\tau \wedge T_{\alpha, \beta}}^\alpha_\beta) e^{-r \tau} \right] \tag{3.26}
\]

for \( G : (a, b) \to \mathbb{R} \) satisfying the assumptions in Theorem 3. Then

\[
V_{\alpha, \beta} (x) = (W_{[\alpha, \beta]})_{**} (F(x)) \psi (x) = (W_{[\alpha, \beta]})_{**} (\hat{F}(x)) \varphi (x)
\]

where

\[
W_{[\alpha, \beta]} (y) := W(y) - \delta (y | F^{-1} (y) \in I \setminus [\alpha, \beta]),
\]

\[
\bar{W}_{[\alpha, \beta]} (y) := \bar{W}(y) - \delta (y | \hat{F}^{-1} (y) \in I \setminus [\alpha, \beta]).
\]

Moreover, the stopping time which attains the supremum in (2.5) is \( \tau = T_{a*} \wedge T_{b*} \) where

\[
a^* = \sup \{ y \in [\alpha, x] \mid W_{[\alpha, \beta]} (F(y)) = (W_{[\alpha, \beta]})_{**} (F(y)) \},
\]

\[
b^* = \inf \{ y \in [x, \beta] \mid W_{[\alpha, \beta]} (F(y)) = (W_{[\alpha, \beta]})_{**} (F(y)) \}.
\]

**Proof.** An optimal stopping time for a problem of (3.26) exists (see Chapter 1 Theorem 2.7) and is the exit time from an open set containing \( x \) so we may write \( \tau^* = T_{a*} \wedge T_{b*} \) for some \( a \leq a^* \leq x \leq b^* \leq \beta \). Suppose that \( \beta < b \) then as shown in the proof of Theorem 3 the value function of the optimal stopping problem (3.26) can be written as

\[
V_{\alpha, \beta} (x) = \inf_{c \in \mathbb{R}} (c \varphi (x) - R_c (x)). \tag{3.27}
\]

where

\[
R_c (x) := \inf_{a \leq y \leq x \leq z \leq b} \left( \frac{c \varphi (y) - \tilde{G} (y) F(z) - F(x)}{\psi (y)} + \frac{c \varphi (z) - \tilde{G} (z) F(x) - F(y)}{\psi (z)} \right) \psi (x)
\]

and

\[
\tilde{G} (y) := G (a) I_{(a, \alpha]} + G (y) I_{(\alpha, b)} + G (b) I_{(\beta, b]}.
\]

As \( \varphi \) is \( r \)-harmonic on \( [y, b] \) for all \( y > a \), it follows that

\[
\frac{c \varphi (y) - \tilde{G} (y)}{\psi (y)} \psi (x) = E_x [ (c \varphi (X_{T_y}) - \tilde{G} (X_{T_y})) e^{-r T_y} ]
\]

\[
= c \varphi (x) - E_x [ \tilde{G} (X_{T_y}) e^{-r T_y} ]
\]

\[
> c \varphi (x) - G (a) E_x [e^{-r T_y}] = \frac{c \varphi (\alpha) - G (\alpha)}{\psi (\alpha)} \psi (x)
\]
for all \( y \in (a, \alpha) \). Likewise for \( y \in (\beta, b) \),
\[
\frac{c\varphi(y) - \tilde{G}(y)}{\psi(y)} \psi(x) > c\varphi(x) - G(\beta) E_x \left[ e^{-rT_x} \right] = \frac{c\varphi(\alpha) - G(\beta)}{\psi(\beta)} \psi(x).
\]
Hence, an argument similar to that presented in Theorem 3 can be used to show that
\[
(1) \quad \text{In this case}
\]
and let
\[
I = \text{the interval}
\]
\[
\text{such as (2.5) exhibit the 'continuous fit' property that}
\]
\[
(2) \quad \text{Let}
\]
\[
\text{for all}
\]
The argument using the other ratio of fundamental solutions follows analogously.

Thus (3.27) reads
\[
R_c(x) = \inf_{y \in (a,b)} \left( \frac{c\varphi(y) - \tilde{G}(y)}{\psi(y)} \right) \psi(x) = \inf_{y \in [a,\beta]} \left( \frac{c\varphi(y) - G(y)}{\psi(y)} \right) \psi(x) =: W_*(c).
\]
The case \( \beta = b \) can be handled using an approximating sequence of domains as in Theorem 3. The argument using the other ratio of fundamental solutions follows analogously.

It is well established that optimal stopping problems based on one-dimensional diffusions such as (2.5) exhibit the ‘continuous fit’ property that \( V(b) = G(b) \) for all \( b \in \partial D \) and the ‘smooth fit’ property \( V'(b) = G'(b) \) for all \( b \in \partial D \) such that \( G'(b) \) and \( F'(b) \) exist (see [20, Theorem 9.5]). The next proposition provides an alternative proof of some properties of the value function using properties of the concave biconjugate \( x \mapsto G_{**}(x) \).

**Proposition 8.** Take \( I' \subset \mathbb{R} \) and let \( G : I' \to \mathbb{R} \) to be a upper-semicontinuous function. Take \( X \) to be a Brownian motion with \( X_0 = x \) under the measure \( P_x \) which is absorbed upon exit from the interval \( I' \) and \( r = 0 \) in the optimal stopping problem (2.5).

(i) The function \( x \mapsto V(x) \) is continuous on \( \text{int}(I') \) where \( \text{int}(I') \) denotes interior of the set \( I' \). In particular, \( x \mapsto V(x) \) is continuous on \( \partial D \cap \text{int}(I') \);

(ii) Suppose that \( y \in C \) then \( V'(y) \) exists and \( \partial V(y) = \{ V'(y) \} \);

(iii) Suppose that \( y \in \partial D \) and \( G \in C^1([y-\varepsilon,y+\varepsilon]) \) for some \( \varepsilon > 0 \) then the ‘smooth-fit’ condition \( V'(y) = G'(y) \) holds.

**Proof.** (1) In this case \( V = G_{**} = -\text{cl} (\text{conv} (-G)) \) is by definition the smallest concave function lying above the obstacle \( G \) hence, the ‘continuous-fit’ property follows from [22, Theorem 10.1].

(2) Let \( \mathcal{A} := \{ x \in I' \mid \partial V(x) > 1 \} \) then by [22, Theorem 25.5], \( \mathcal{A}^c \) is a dense subset of \( I' \) upon which \( V \) is differentiable. Take \( x \in \mathcal{A} \) then by definition there exists a non-empty closed set \( \mu(x) \subseteq \text{dom}(G^*) \) such that for each \( c \in \mu(x) \), the tangent hyperplane \( h(c,y) := V(x) - c(x-y) \) dominates \( V \) on \( I' \). Take \( c' \in \text{int} (\mu(x)) \) then \( h(c',x) = V(x) \) and \( h(c',y) > V(y) \) for all \( y \in I' \setminus \{x\} \). Due to the upper semi-continuity of \( G \) there exists \( x' \in I' \) such that \( h(c',x') = G(x') \) and hence \( G(x) = V(x) \).

(3) When \( G \in C^1([y-\varepsilon,y+\varepsilon]) \) the derivative \( G' \) exists at \( y \in \partial D \) and the unique tangent hyperplane is \( h(x) = G(y) + G'(y)(x-y) \). Consequently, the characterisation of the continuation and stopping regions in Theorem 3 is equivalent to
\[
C = \{ x \in I' \mid \partial G(x) = \emptyset \} \quad ; \quad D = \{ x \in I' \mid \partial G(x) \neq \emptyset \}
\]
which implies that \( \partial V(y) = \partial G(y) = \{ G'(y) \} \) and hence \( V'(y) = G'(y) \).
The third part of the previous proposition shows that when \( G \in C^1 (I') \) that \( \partial V (y) = \{ G' (y) \} \) for all \( y \in \mathcal{D} \) and hence it follows from the second part that the set \( \mathcal{A} = \emptyset \) or equivalently \( V \in C^1 (I') \). The previous proposition can be extended to optimal stopping problems such as (2.5) when the diffusion has natural boundaries as explained in the next proposition.

**Proposition 9.** Assume that \( X \) is a one dimensional diffusion with \( X_0 = x \) under the measure \( P_x \) and such that the functions \( \varphi, \psi \in C^1 (I) \). Take \( G : I \to \mathbb{R} \) to be an upper-semicontinuous function satisfying assumption (2.4) and consider the optimal stopping problem (2.5).

(i) The function \( x \mapsto V (x) \) is continuous on \( I \) and in particular, \( x \mapsto V (x) \) is continuous at any point in \( \partial \mathcal{D} \).

(ii) Denote the superdifferential of \( W_{**} \) at \( F(y) \) by \( \partial W_{**} (F(y)) \), that is

\[
\partial W_{**} (F(y)) := \{ c \in \mathbb{R} \mid W_{**} (F(y)) \geq W_{**} (z) + c (F(y) - z) \quad \forall z \geq 0 \} .
\]

If \( y \in \mathcal{C} \) then \( V' (y) \) exists and \( \partial W_{**} (F(x)) = \{ \frac{d \psi}{d \psi} (V/\psi (x)) \} \).

(iii) Suppose that \( y \in \partial \mathcal{D} \) and \( G \in C^1 ([y - \varepsilon, y + \varepsilon]) \) for some \( \varepsilon > 0 \) then the ‘smooth-fit’ condition \( V' (y) = G' (y) \) holds.

**Proof.** Define a change of measure using the harmonic function \( x \mapsto \psi (x) \) solving the generator equation (2.1) using

\[
\tilde{P}_x (A) := \frac{1}{\psi (x)} E_x [\psi (X_t) e^{-rt} 1_A] \quad A \in \mathcal{F}_t \cap \{ T_\tau > t \} , t \geq 0
\] (3.28)

where \( T_\tau \sim \exp (r) \) is the killing time of the diffusion \( X \) under the original family of measures.

For any \( z \in (a, x) \) the process the process \( (\psi (X_t) e^{-rt})_{t \geq 0} \) is a martingale and hence the measure (3.28) is an equivalent probability measure for random variables which are \( \mathcal{F}_{T_\tau} \)-measurable for some \( z > a \). Consider the auxiliary optimal stopping problem

\[
V_z (x) := \sup_{\tau \leq T_\tau} E_x [G(X_\tau) e^{-rt}] .
\]

This stopping problem can now be rewritten as problem without killing under the measure defined in (3.28), i.e.

\[
V_z (x) = \sup_{\tau \leq T_\tau} E_x \left[ G (X_\tau) e^{-rt} \right] = \psi (x) \sup_{\tau \leq T_\tau} \tilde{E}_x \left[ \left( \frac{G}{\psi} \right) (X_\tau) \right] .
\] (3.29)

where \( \tilde{E}_x \) denotes expectation under the measure defined in (3.28). The change of measure (3.28) is referred to as the ‘Doob h-transform’ and a property of this change of measure is that \( X \) remains a regular diffusion under \( \tilde{P}_x \) (see [4] pp. 33-34). Let \( S \) denote the scale function of \( X \) under \( P_x \) (see [4] pp. 14 for further details). The measure (3.28) is such that scale function of \( X \) under \( \tilde{P}_x, S \), solves \( \tilde{S}' (x) \psi^2 (x) = S' (x) \) (see [4] pp. 33-34). Furthermore, the Wronskian defined as

\[
W (\varphi, \psi) := \varphi' (x)\psi (x) + \psi' (x)\varphi (x) = \varphi^2 (x) S' (x) F' (x)
\]
is a constant (see [4] pp. 19) so the scale function of $X$ under $\tilde{P}_x$ is $\tilde{S} = F$ (modulo an affine transformation). Consequently, the process $(Y_t)_{t \geq 0} = (F(X_t))_{t \geq 0}$ is a continuous local martingale on $[0, T_x]$ with initial point $Y_0 := y = F(x)$ under $\tilde{P}_x$. We may rewrite (3.29) as

$$\left( \frac{V_z}{\psi} \right) (x) = \sup_{\tau \leq \tilde{T}_z} \tilde{E}_x \left[ (\frac{G}{\psi}) (X_\tau) \right] = \sup_{\tau \leq \tilde{T}_z} \tilde{E}_x \left[ (\frac{G}{\psi}) (X_{\tau \wedge T_x}) \right] = \sup_{\tau \leq \tilde{T}_z} \tilde{E}_x [W (Y_\tau)]$$

where $W (y) := (G/\psi) \circ F^{-1} (y)$. As $X$ is regular under $\tilde{P}_x$, $\lim_{t \to +\infty} [Y]_{t \wedge T_x} = [Y]_{T_x} < +\infty$ $\tilde{P}_x$-a.s. for all $x \in I$. Let

$$\sigma_s := \inf \{ t \geq 0 \mid [Y, Y]_t > s \}$$

then it follows from [21] Theorem 1.7 (pp. 181) that $B_s = Y_{\sigma_s}$ is a $(\mathcal{F}_{\sigma_s})$-Brownian motion on the time interval $[0, [Y]_{T_x}]$ and by expanding the probability space the process $B$ can be extended to be a Brownian motion on $\mathbb{R}$. Moreover, (3.30) can be rewritten as

$$\sup_{\tau \leq \tilde{T}_x} \tilde{E}_x [W (Y_\tau)] = \sup_{\sigma \leq \tilde{T}_{F(x)}} \tilde{E}_x [W (B_\sigma)] =: U_z (y),$$

where $\tilde{T}_y := \inf \{ t \geq 0 \mid B_t \leq y \}$. The value function of the optimal stopping problem $y \mapsto U_z (y)$ defined in (3.31) has the properties described in Proposition 8 on the interval $[z, \infty)$. Take a sequence of real numbers $(z_n)_{n \geq 1}$ such that $z_{n+1} < z_n$ for all $n \geq 1$ and $\lim_{n \to \infty} z_n = a$. For each $y \in \mathbb{R}^+$, the value functions $U_z (y)$ of the optimal stopping problems (3.31) form a monotone sequence so the limit $U (y) := \lim_{n \to +\infty} U_{z_n} (y)$ is well-defined but need not have a stochastic representation under $\tilde{P}_x$ as the measure change is not necessarily well-defined in the limit.

As a consequence of Corollary 7 for each $n \geq 0$ the function $U_{z_n} (y) = (W_{[z_n, \infty)})_{\operatorname{ss}} (y)$ for all $y \in [z_n, \infty)$ and can be extended to be a concave function on $\mathbb{R}^+$ by setting $U_{z_n} (y) = -\infty$ for $y \in [0, z_n)$. The function $U (y)$ is the point-wise limit of sequence of concave functions $(U_{z_n})_{n \geq 1}$ so is concave (see [22] Theorem 5.5) and inherits the properties described in Proposition 8.

The properties of the functions $\varphi, \psi$ imply that: (i) the function $V (x) = U (F(x)) \psi (x)$ inherits continuity from $U$; (ii) $V$ is differentiable at all points which $U$ is differentiable so the second assertion holds and (iii) the function $y \mapsto W (y)$ is differentiable at $y' \in \{ y \geq 0 \mid U (y) = W (y) \}$ if and only if $x \mapsto G (x)$ is differentiable at $F^{-1} (y')$.

This section concludes a basic example, the perpetual American put option.

**Example 10.** Take $\sigma \geq 0$ and define $X$ to be the solution to $dX_t = r X_t dt + \sigma X_t dW_t$ with $X_0 = x > 0$ where $W$ is a standard Brownian motion. Consider the perpetual American put option, the gains function of which is $G^+ (x) = (K - x)^+$ for a strike price $K > 0$. The risk neutral value of this option is

$$V (x) := \sup_{\tau} E_x \left[ (K - X_\tau)^+ e^{-r \tau} \right] = \sup_{\tau} E_x \left[ (K - X_\tau) e^{-r \tau} \right].$$

The equality follows since $\lim_{t \to -\infty} G^+ (X_t) e^{-rt} = 0$ and hence $V (x) \geq 0$. The infinitesimal generator associated with the geometric Brownian motion $X$ is

$$L_X u := rx \frac{du}{dx} + \frac{1}{2} \sigma^2 x^2 \frac{d^2 u}{dx^2}.$$
and the ODE $L_\chi u = ru$ has two fundamental solutions $\varphi(x) = x$ and $\psi(x) = x^{-2r/\sigma^2}$. Let $-\tilde{F}(x) = (\psi/\varphi)(x) = x^{-1/\alpha}$ where $\alpha := 1/(1 + 2r/\sigma^2) \in [0,1]$. The rescaled gains function associated with the problem (3.32) is

$$
\tilde{G}(y) := \begin{cases} 
\frac{\varphi(y)}{\psi} & \text{for } y > 0 \\
\limsup_{x \to +\infty} \frac{G(x)}{x} = 0 & \text{for } y = 0
\end{cases}
$$

Note that $\tilde{G}(y) = \tilde{W}(-y)$ where $\tilde{W}$ is as defined in (3.19). The function $y \mapsto \tilde{G}(y)$ has $\tilde{G}''(y) \leq 0$ for all $y \in \mathbb{R}_+$ and hence to find $y \mapsto \tilde{G}^{**}(y)$ we need only to find $y \geq 0$ such that

$$
\tilde{G}(y) - \tilde{G}'(y)y = 0.
$$

(3.33)

The unique solution to (3.33) is $y^* = 1/(K(1 - \alpha))^{1/\alpha}$ and

$$
\tilde{G}^{**}(y) = \begin{cases} 
\tilde{G}'(y^*) & \text{for } y \in [0,y^*] \\
\tilde{G}(y) & \text{for; } y > y^*
\end{cases}
$$

Applying Theorem 3

$$
V(x) = \tilde{G}^{**}(-\tilde{F}(x))\varphi(x) = \begin{cases} 
\frac{\sigma^2}{2r} \left( \frac{K}{1 + \sigma^2/2r} \right)^{1/\alpha} x^{-2r/\sigma^2} & \text{for } x \geq x^* \\
K - x & \text{for } x \in [0,x^*]
\end{cases}
$$

where $x^* = (y^*)^{\alpha} = K/(1 + \sigma^2/2r)$. Figure 2 illustrates the original and transformed payoff functions, the concave-biconjugate of the transformed payoff and the corresponding value function.

Figure 2: In the left figure, the transformed payoff function $\tilde{G}$ is draw along with its concave biconjugate. The original payoff $G(x) = (K - x)^+$ and the corresponding value function $V$ is in the figure on the right hand side.

Moreover, the continuation and stopping region for the problem (3.32) are

$$
D := \{ x \in \mathbb{R}_+ \mid V(x) = G(x) \} = [0,x^*] \quad ; \quad C := \{ x \in \mathbb{R}_+ \mid V(x) > G(x) \} = (x^*,+\infty).
$$

which coincides with the established solution (see for example [20] Section 25.1)
4 A modification of the Legendre transform

The purpose of this section is to extend the Legendre transform so that the observations about optimal stopping in the previous section can be applied to optimal stopping games. The function $G$ represents the payoff of the maximising agent while the function $H$ represents the payoff to the minimising agent. The two continuous gains functions $G, H$ are such that $G(x) \leq H(x)$ for all $x \in I$ and satisfy the assumptions (2.7) and (2.8) unless otherwise stated.

Inspired by the transformation used in Theorem 3 and Proposition 9 introduce a pair of rescaled gains functions $W^G : \mathbb{R}_+ \to \mathbb{R}$ and $W^H : \mathbb{R}_+ \to \mathbb{R}$ defined via

$$W^G(y) := \left\{ \begin{array}{ll}
\left( \frac{G}{\psi} \right) \circ F^{-1}(y) & y > 0 \\
\limsup_{x \downarrow a} \frac{G^+(x)}{\psi(x)} & y = 0
\end{array} \right. \quad (4.35)$$

and

$$W^H(y) := \left\{ \begin{array}{ll}
\left( \frac{H}{\psi} \right) \circ F^{-1}(y) & y > 0 \\
\limsup_{x \downarrow a} \frac{H^+(x)}{\psi(x)} & y = 0
\end{array} \right. \quad (4.36)$$

Under the assumptions (2.7) and (2.8) we have

$$\lim_{x \downarrow 0} (W^H(x) - W^G(x)) = \lim_{x \uparrow \infty} (W^H(x) - W^G(x)) = 0. \quad (4.37)$$

This assumption can be relaxed a little as discussed in Remark 13 below. These definitions could equally well be formulated with respect to the other ratio of the fundamental solutions but for clarity we shall focus only on these two expressions.

The aim of this section is to define and describe a version of the convex biconjugate of the function $W^H$ which is modified to ensure that it remains inside $\text{epi}(W^G)$. At the same time, we define a version of the concave biconjugate of the function $W^G$ which is modified to ensure that it remains inside $\text{cl}(\mathbb{R}^2 \setminus \text{epi}(W^H))$. This is achieved by defining a modification of the $\varepsilon$-sub/superdifferential typically used in convex analysis.

The main result in this section is Theorem 17 which is the purely analytical version of Theorem 4.1. Theorem 17 shows that the convex biconjugate respecting the lower barrier $W^H$ and the concave biconjugate respecting the upper barrier $W^G$ coincide. Although this ‘duality’ is hardly surprising considering the geometric method of construction presented here, this observation is used in Section 5 to provide a new (purely analytical) proof that the optimal stopping game (2.9)-(2.10) exhibits both a Stackelberg and a Nash equilibrium.

For a given function $f : I \to \mathbb{R}$, let

$$l^f_I(c, p) = \sup \{ z \leq x \mid f(z) - p = c(z - x) \},$$

$$r^f_I(c, p) = \inf \{ y \geq x \mid f(y) - p = c(y - x) \},$$

with the standard convention that $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$. For ease of notation, let $l^f_G(c, p) := l^f_{W^G}(c, p)$ and $l^f_H(c, p) := l^f_{W^H}(c, p)$. The point $l^f_G(c, W^H(x))$ (resp. $r^f_G(c, W^H(x))$) is the last (resp. first) time the line passing through $(x, W^H(x))$ with slope $c$, i.e. the function $y \mapsto W^H(x) + c(y - x)$, intercepts $W^G$ before (resp. after) $x$. 

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Define the subdifferential of $W_H$ in the presence of the lower boundary $W^G$ as
\[
\partial^G H(x) := \{c \in \mathbb{R} \mid W_H(y) - W_H(x) \geq c(y - x) \quad \forall y \in [l_G(c, W_H(x)), r_G(c, W_H(x))] \}. \tag{4.38}
\]
If the tangent $y \mapsto W_H(x) + c(y - x)$ minorises $y \mapsto W_H(y)$ prior to intercepting the lower boundary $y \mapsto W^G(y)$, then we refer to $c$ as a ‘subgradient of $W_H$ in the presence of $W^G$ at $x$', i.e. $c \in \partial_G H(x)$, as shown in Figure 3. Similarly, the superdifferential of $W^G$ in the presence of the upper boundary $W_H$ is
\[
\partial^H G(x) := \{c \in \mathbb{R} \mid W^G(y) - W^G(x) \leq c(y - x) \quad \forall y \in [l_H(c, W^G(x)), r_H(c, W^G(x))] \}. \tag{4.39}
\]
If the tangent $y \mapsto W^G(x) + c(y - x)$ majorises $y \mapsto W^G(y)$ prior to intercepting the upper boundary $y \mapsto W_H(y)$, then we refer to $c$ as a ‘subgradient of $G$ in the presence of $H$ at $x$', i.e. $c \in \partial^H G(x)$, as illustrated in Figure 3.

![Figure 3: The slope of the black line, $c$, is a supergradient of $W^G$ in the presence of $W_H$ at $x_1$ and a subgradient of $W_H$ in the presence of $W^G$ at $x_2$.](image)

The $\delta$-subdifferential of $W_H$ in the presence of the lower boundary $W^G$ is defined as
\[
\partial^G H(x) := \{c \in \mathbb{R} \mid W_H(y) - W_H(x) + \delta \geq c(y - x) \quad \forall y \in [l_G(c, W_H(x) - \delta), r_G(c, W_H(x) - \delta)] \}. \tag{4.40}
\]
When $c$ is a $\delta$-subgradient of $W_H$ in the presence of the lower boundary $W^G$, i.e. $c \in \partial^G H(x)$, it is possible to draw a line with slope $c$ through the point $(x, W_H(x) - \delta)$ which minorises $y \mapsto W_H(y)$ prior to intercepting the lower boundary $y \mapsto W^G(y)$. For example $0 \in \partial^G H(x_2)$ in Figure 4. Similarly, the $\epsilon$-superdifferential of $W^G$ in the presence of the upper boundary $W_H$ is defined as
\[
\partial^H G(x) := \{c \in \mathbb{R} \mid W^G(y) - W^G(x) - \epsilon \leq c(y - x) \quad \forall y \in [l_H(c, W^G(x) + \epsilon), r_H(c, W^G(x) + \epsilon)] \}. \tag{4.41}
\]
When \( c \) is an \( \varepsilon \)-supergradient of \( W_G \) in the presence of the upper boundary \( W_H \), i.e. \( c \in \partial^\varepsilon H(x) \), it is possible to draw a line with slope \( c \) through the point \( (x, W_G(x) + \varepsilon) \) which dominates \( y \mapsto W_G(y) \) prior to intercepting the upper boundary \( y \mapsto W_H(y) \). For example \( 0 \in \partial^\varepsilon G(x_1) \) in Figure 4.

Figure 4: \( c = 0 \) is a \( \varepsilon \)-supergradient of \( W_G \) in the presence of \( W_H \) at \( x_1 \) and a \( \delta \)-subgradient of \( W_H \) in the presence of \( W_G \) at \( x_2 \).

Let

\[
\varepsilon^*_c(x) := \inf \{ \varepsilon \in [0, W_H(x) - W_G(x)] \mid c \in \partial^\varepsilon H(x) \} \quad (4.42)
\]

and

\[
\delta^*_c(x) := \inf \{ \delta \in [0, W_H(x) - W_G(x)] \mid c \in \partial^\delta H(x) \} \quad (4.43)
\]

which are the smallest spike variations that can be made in \( W_G \), resp. \( W_H \) at \( x \) such that \( c \in \partial^\varepsilon G(x) \), resp. \( c \in \partial^\delta H(x) \). The quantities (4.42) and (4.43) are illustrated in Figure 5.

**Remark 11.** It follows from these definitions that for all \( \delta > \delta^*_c(x) \)

\[
W_H(y) > W_H(x) - \delta + c(y - x) \quad \forall y \in [l^\varepsilon_G(c, W_H(x) - \delta), r^\varepsilon_G(c, W_H(x) - \delta)].
\]

Whereas, for \( \delta < \delta^*_c(x) \) there exists \( x' \in [l^\varepsilon_G(c, W_H(x) - \delta), r^\varepsilon_G(c, W_H(x) - \delta)] \) such that \( W_H(x') > W_H(x) - \delta + c(x' - x) \). These two statements imply that either:

(a) There exists \( z \in [l^\varepsilon_G(c, W_H(x) - \delta^*_c(x)), r^\varepsilon_G(c, W_H(x) - \delta^*_c(x))] \) such that \( c \in \partial^\delta G(z) \) and/or

(b) \( \delta^*_c(x) = W_H(z) - W_G(z) \) for \( z = l^\varepsilon_G(c, W_H(x) - \delta^*_c(x)) \) and/or \( z = r^\varepsilon_G(c, W_H(x) - \delta^*_c(x)) \).

The top left panel of Figure 5 shows a point where only condition (a) holds, whereas the bottom left panel illustrates a situation where only condition (b) holds. Similarly, for all \( \varepsilon > \varepsilon^*_c(x) \)

\[
W_G(y) < W_G(x) + \varepsilon + c(y - x) \quad \forall y \in [l^\varepsilon_H(c, W_G(x) + \varepsilon), r^\varepsilon_H(c, W_G(x) + \varepsilon)].
\]

Whereas, for \( \varepsilon < \varepsilon^*_c(x) \) there exists \( x' \in [l^\varepsilon_H(c, W_G(x) - \varepsilon), r^\varepsilon_H(c, W_G(x) - \varepsilon)] \) such that \( W_H(x') < W_G(x) + \varepsilon + c(x' - x) \). These two statements imply that:

\[
\begin{align*}
\text{Figure 5:} & \quad \text{The top left panel of Figure 5 shows a point where only condition (a) holds, whereas the bottom left panel illustrates a situation where only condition (b) holds. Similarily, for all } \varepsilon > \varepsilon^*_c(x) \text{ these two statements imply that:}
\end{align*}
\]
(a) There exists \( z \in [l^r_H(c,W^G(x) + \varepsilon^*_c(x)), r^r_H(c,W^G(x) + \varepsilon^*_c(x))] \) such that \( c \in \partial H G(z) \) and/or
(b) \( \delta^*_c(z) = W_H(z) - W^G(z) \) for \( z = l^r_G(c,W_H(x) - \delta^*_c(x)) \) and/or \( z = r^r_G(c,W_H(x) - \delta^*_c(x)) \).

The top right panel of Figure 5 shows a point where only condition (a) holds, whereas the bottom right panel illustrates a situation where only condition (b) holds.

Figure 5: The two figures on the left illustrate \( \delta^*_c(x) \) and the two figures on the right illustrate \( \varepsilon^*_c(x) \).

Furthermore, let
\[
\varepsilon^*(x) := \inf_{c \in \mathbb{R}} \varepsilon^*_c(x) \quad ; \quad \delta^*(x) := \inf_{c \in \mathbb{R}} \delta^*_c(x)
\]
which are the smallest spike variation that can be made in \( W^G \) (resp. \( W_H \)) at \( x \) such that the set \( \partial H G(x) \) (resp. \( \partial G H(x) \)) is non-empty. The first result in this section shows that there is ‘no gap’ between the minimum spike variation in \( W_H \) downwards admitting a \( \delta \)-subgradient and the minimum spike variation in \( W^G \) upwards admitting a \( \varepsilon \)-supergradient.

Lemma 12. Suppose that \( W^G : \mathbb{R}_+ \to \mathbb{R} \) and \( W_H : \mathbb{R}_+ \to \mathbb{R} \) are as defined in (4.35) and (4.36) and that (4.37) holds, then: for all \( x \geq 0 \) and \( c \in \mathbb{R} \)
\[
W^G(x) + \varepsilon^*_c(x) \geq W_H(x) - \delta^*_c(x)
\]
and
\[
W^G(x) + \varepsilon^*(x) = W_H(x) - \delta^*(x).
\]

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Proof. Fix $x \geq 0$, $c \in \mathbb{R}$ and take $\delta > \delta^*_c(x)$. It follows from the definition of $\delta^*_c(x)$ that

$$W_H(y) > W_H(x) - \delta + c(y - x) \quad \forall y \in [l_H^c(c, W_H(x) - \delta), r_H^c(c, W_H(x) - \delta)]$$

and

$$[l_H^c(c, W_H(x) - \delta), r_H^c(c, W_H(x) - \delta)] \subset [l_H^c(c, W_H(x) - \delta), r_H^c(c, W_H(x) - \delta)]$$

where the inequality and inclusion are strict to avoid contradicting that $\delta > \delta^*_c(x)$. Due to the properties of the line $y \mapsto W_H(x) - \delta^*_c(x) + c(y - x)$ described in Remark 11 so is the only line passing through $(x, p)$. When (4.37) fails one (or more) of the inequalities in both of these statements can hold with equality which invalidates the following

$$\exists y \in [l_H^c(c, W_H(x) - \delta), r_H^c(c, W_H(x) - \delta)] \quad \text{s.t.} \quad W_H(x) - \delta + c(y - x) < W^G(y)$$

Let $\varepsilon' := W_H(x) - W^G(x) - \delta$ so that (4.48) is equivalent to

$$\exists y \in [l_H^c(c, W_H(x) - \delta), r_H^c(c, W_H(x) - \delta)] \quad \text{s.t.} \quad W^G(x) - \varepsilon' + c(y - x) < W^G(y).$$

It follows from the definition of $\varepsilon^*_c(x)$ that $\varepsilon'(x) < \varepsilon^*_c(x)$ or equivalently

$$W_H(x) - \delta < W^G(x) + \varepsilon^*_c(x) \quad \forall \delta > \delta^*_c(x).$$

Take any sequence of real numbers $(\delta_n)_{n \geq 1}$ such that $\delta_n \in (0, \delta^*_c(x))$ for all $n \geq 1$ and $\lim_{n \to \infty} \delta_n = \delta^*_c(x)$ then (4.49) shows that $W_H(x) - \delta_n < W^G(x) + \varepsilon^*_c(x)$ for all $n \geq 1$. Taking the limit as $n \to \infty$ of both sides of this inequality we obtain (4.50).

Suppose that for some $c \in \mathbb{R}$ that $W^G(x) + \varepsilon^*_c(x) = W_H(x) - \delta^*_c(x)$. When the assumption (4.37) holds the line $g(y) := W^G(x) + \varepsilon^*_c(x) + c(y - x)$ has both sets of properties described in Remark 11 so is the only line passing through $(x, p)$ for some $p \in (W^G(x), W_H(x))$ which both dominates $y \mapsto W^G(y)$ on $[l_H^c(c, p), r_H^c(c, p)]$ and minorises $y \mapsto W_H(y)$ on $[l_H^c(c, p), r_H^c(c, p)]$. When (4.37) fails, $y \mapsto g(y)$ may not be the only line with these properties and (4.46) may not hold, this case is examined further in Remark 13 below. Consequently, it follows from the continuity of $G$ and $H$ and the assumption (4.37) that for any $c' \neq c$ the line $y \mapsto h(y) = W^G(x) + \varepsilon^*_c(x) + c'(y - x)$ must either satisfy both

$$l_H^c(c', W_H(x) - \delta^*_c(x)) < l_H^c(c', W_H(x) - \delta^*_c(x)),$$

$$r_H^c(c', W_H(x) - \delta^*_c(x)) < r_H^c(c', W_H(x) - \delta^*_c(x)), \quad (4.50)$$

or

$$l_H^c(c', W_H(x) - \delta^*_c(x)) > l_H^c(c', W_H(x) - \delta^*_c(x)),$$

$$r_H^c(c', W_H(x) - \delta^*_c(x)) > r_H^c(c', W_H(x) - \delta^*_c(x)). \quad (4.51)$$

In the right panel of Figure 6 the line marked (b) has the properties (4.50) whereas in the left panel the line marked (b) has the properties (4.51). When (4.37) fails one (or more) of the inequalities in both of these statements can hold with equality which invalidates the following argument. Next the line $y \mapsto h(y)$ is shifted upwards so that it passes through $(x, W_H(x) - \delta)$ for some $\delta \in [0, \delta^*_c(x)]$. As is illustrated by the lines marked (c) in Figure 6, the properties (4.51) imply that

$$l_H^c(c', H(x) - \delta) \in [l_H^c(c', H(x) - \delta^*_c(x)), x]$$
and hence \( l_H^c(c', H(x) - \delta) > l_G^c(c', H(x) - \delta) \). Whereas if (4.50) holds then

\[
    r_H^c(c', H(x) - \delta) \in [x, r_H^c(c', H(x) - \delta_c(x))],
\]

which implies that \( r_H^c(c', H(x) - \delta) < r_G^c(c', H(x) - \delta) \). In both cases \( c' \notin \partial^\delta H(x) \) for arbitrary \( \delta < \delta_c^c(c) \). Moreover, as \( c' \) was arbitrary we have shown that \( \partial^\delta H(x) = \emptyset \) for all \( \delta < \delta_c^c(c) \). We may conclude that when for some \( c \in \mathbb{R} \) equality holds in (4.45) that \( \delta_c^c(x) = \delta^*(x) \). A symmetric argument can be used to show that when equality holds for some \( c \in \mathbb{R} \) equality holds in (4.45) that \( \varepsilon_c^*(x) = \varepsilon^*(x) \) from which we deduce (4.46) holds.

Figure 7: When \( W_H(0) > W^G(0) \) it may be the case that condition (4.46) in Lemma 12 fails.

Remark 13. In the previous lemma it is essential to assume that (4.37) holds. When this is not the case it may be the case that \( W^G(x) + \varepsilon_c^*(x) < W_H(x) + \delta^*(x) \) as is illustrated in Figure 7. Suppose that as in the illustration \( W^G(0) < W_H(0) \) and there is a \( x > 0 \) such that for some \( c' \in \mathbb{R}, c' \in \partial^H G(x) \) but \( t^c_G(c', W^G(x)) = t_H^c(c', W^G(x)) = -\infty \). For this \( x \), \( W^G(x) - \)
\( c'x < W_H(0) \). Take \( a \in (W_G(x) - c', W_H(0)) \) such that \( \exists b < c' \) with the properties: (i) \( a + bx \in (W_G(x), W_H(x)) \), (ii) \( r^*_G(b, e + bx) < r^*_H(b, e + bx) \) and (iii) \( b \in \partial H(r^*_G(b, a + bx)) \). Let \( \delta'(x) := W_H(x) - (a + bx) \) then \( b \in \partial G(0) H(x) \) and moreover \( \delta'(x) < \delta' \leq W_H(x) - (a + bx) \leq W_H(x) - W_G(x) \). Thus, by construction \( \partial H(x) \neq \emptyset \) so \( \varepsilon^*(x) = 0 \) which implies

\[
W_G(x) + \varepsilon^*(x) < W_H(x) - \delta^*(x),
\]

so in this case \( (4.46) \) fails.

**Remark 14.** The assumption that \( (4.37) \) holds in Lemma 12 can be relaxed if we re-define \( W_G(0) = W_H(0) \) to take some value \( w_0 \in \left[ \lim_{x \downarrow a} \frac{G^+(x)}{\psi(x)}, \lim_{x \downarrow a} \frac{H^+(x)}{\psi(x)} \right] \)

and consider the function \( y \mapsto W_G(y) \) (and/or \( y \mapsto W_H(y) \)) to be multivalued at \( y = 0 \) taking all values in the interval \( [\lim_{x \downarrow a} G^+(x)/\psi(x), w_0] \) (resp. \( [w_0, \lim_{x \uparrow b} H^+(x)/\psi(x)] \)). However, the choice of \( w_0 \) not only completely determines \( \delta^*(0) \) and \( \varepsilon^*(0) \) but the values of the functions \( x \mapsto \delta^*(x) \) and \( x \mapsto \varepsilon^*(x) \) on \( [0, z] \) for some \( z \geq 0 \).

![Figure 8: The sets \( \hat{A}_G^x(c) \) and \( \hat{A}_H^x(c) \) are defined in (4.52) and (4.53) are marked when \( c \) is the slope of the pair of parallel lines.](image)

For a given \( x \geq 0 \), define two sets of admissible neighborhoods using

\[
\hat{A}_G^x(c) := \bigcup_{p \in [G(x), H(x)]} \{ y \geq 0 \mid l_H^*(c, p) \leq l_G^*(c, p) \leq y \leq r_H^*(c, p) \leq r_G^*(c, p) \}, \quad (4.52)
\]

\[
\hat{A}_H^x(c) := \bigcup_{p \in [G(x), H(x)]} \{ y \geq 0 \mid l_H^*(c, p) \leq l_G^*(c, p) \leq y \leq r_H^*(c, p) \leq r_G^*(c, p) \}. \quad (4.53)
\]

These admissible neighborhoods are illustrated in Figure 8. The next result describes a pair of functions which coincide with \( y \mapsto W_G(y) + \varepsilon^*(y) \) and \( y \mapsto W_H(y) - \delta^*(y) \).
Proposition 15. Define a pair of functions

\[
(W^G)_{H}^{\star\star}(x) := \sup_{c \in \mathbb{R}} \sup_{y \in \hat{A}^G_c(c)} (z(x - y) + W^G(y)), \tag{4.54}
\]

\[
(W_H)^G_{\star\star}(x) := \inf_{z \in \mathbb{R}} \inf_{y \in \hat{A}^H_c(c)} (z(x - y) + W_H(y)), \tag{4.55}
\]

then \((W^G)_H(x) = W_H(x) - \delta^*(x)\) and \((W_H)^G_{\star\star}(x) = W^G(x) + \varepsilon^*(x)\).

Proof. Fix a \(x \geq 0\) and \(c \in \mathbb{R}\), it follows from the definition of \(\delta^*_G(x)\) and the continuity of \(G\) that the set (4.52) satisfies

\[
\hat{A}^G_c(c) = [l^G_c(c, W_H(x) - \delta^*_G(x)), r^G_c(c, W_H(x) - \delta^*_G(x))].
\]

By definition \(c \in \partial^G_H(x)\) for all \(\delta > \delta^*_G(x)\), which is equivalent to

\[
W_H(x) - \delta^*_G(x) + c(y - x) \geq W^G(x) \quad \forall y \in \hat{A}^G_c(c) \tag{4.56}
\]

and equality holds in (4.56) for \(y = l^G_c(c, W_H(x) - \delta^*_G(x))\) and \(y = r^G_c(c, W_H(x) - \delta^*_G(x))\). Hence,

\[
\sup_{y \in \hat{A}^G_c(c)} (W^G(y) - cy) = W^G(r^G_c(c, W_H(x) - \delta^*_G(x))) - c r^G_c(c, W_H(x) - \delta^*_G(x))
\]

and

\[
(W^G)_{H}^{\star\star}(x) := \sup_{c \in \mathbb{R}} \sup_{y \in \hat{A}^G_c(c)} (z(x - y) + W^G(y))
= \sup_{c \in \mathbb{R}} \left( c(x - r^G_c(c, W_H(x) - \delta^*_G(x))) + W^G(r^G_c(c, W_H(x) - \delta^*_G(x))) \right)
= \sup_{c \in \mathbb{R}} (W_H(x) - \delta^*_G(x)) = W_H(x) - \delta^*(x).
\]

Thus we have shown that \((W^G)_{H}^{\star\star}(x) = W_H(x) - \delta^*(x)\) for all \(x \geq 0\). A symmetric argument can be used to show that \((W_H)^G_{\star\star}(x) = W^G(x) + \varepsilon^*(x)\).

The previous result holds without the need to assume (4.37) holds. At first glance, the notation used in the previous result may appear to be the wrong way around but when (4.37) holds Lemma 12 implies that \((W^G)_{H}^{\star\star}(x) = W^G(x) + \varepsilon^*(x)\) and \((W_H)^G_{\star\star}(x) = W_H(x) - \delta^*(x)\).

Remark 16. For a given \(x \geq 0\) consider the sets

\[
A_1(x) := \left\{ p \in [W^G(x), W_H(x)] \mid \exists c \in \mathbb{R} \text{ s.t. } [l^G_c(c, p), r^G_c(c, p)] \subseteq \hat{A}^G_c(c) \right\},
\]

\[
A_2(x) := \left\{ p \in [W^G(x), W_H(x)] \mid \exists c \in \mathbb{R} \text{ s.t. } [l^H_c(c, p), r^H_c(c, p)] \subseteq \hat{A}^H_c(c) \right\}.
\]

The ‘dual interpretation’ provided in (19) illustrates that \((W^G)_{H}^{\star\star}(x)\) can be constructed by maximising over functions of the form \(y \mapsto p + c(y - x)\) which hit \(W^G\) before \(W_H\), i.e. \((W^G)_{H}^{\star\star}(x) = \sup A_1(x)\) as shown in Proposition 12 or by minimising over the functions of the form \(y \mapsto p + c(y - x)\) which hit \(W_H\) before \(W^G\) i.e. \((W^G)_{H}^{\star\star}(x) = \inf A_2(x)\).
Theorem 17. Suppose that (4.37) holds, then the functions those used in [19] and used in the next result which is an analytical complement to [19] Theorem 4.1. Moreover, let

$$L^x(c,p) = \sup \{z \leq x \mid p + c(z - x) \notin \text{cl(epi}(W^G) \setminus \text{epi}(W_H))\},$$
$$R^x(c,p) = \inf \{y \geq x \mid p + c(y - x) \notin \text{cl(epi}(W^G) \setminus \text{epi}(W_H))\}.$$ 

For a given $x \geq 0$, let $p(x) = W_H(x) - W^G(x)$ and define two sets of admissible neighborhoods which are larger than (4.52) and (4.53) using

$$\mathcal{A}_G^x(c) := \bigcup_{\delta \in [0,p(x)]} \bigcup_{c \in \partial_G^H(x)} \{y \in \mathbb{R}_+ \mid L^x(c, W_H(x) - \delta) \leq y \leq R^x(c, W_H(x) - \delta)\},$$
$$\mathcal{A}_H^x(c) := \bigcup_{\varepsilon \in [0,p(x)]} \bigcup_{c \in \partial_G^H(x)} \{y \in \mathbb{R}_+ \mid L^x(c, W^G(x) + \varepsilon) \leq y \leq R^x(c, W^G(x) + \varepsilon)\}.$$ 

These admissible neighborhoods are illustrated in Figure 9. These neighborhoods coincide with those used in [19] and used in the next result which is an analytical complement to [19] Theorem 4.1.

**Theorem 17.** Suppose that (4.37) holds, then the functions $x \mapsto (W^G)^{**}(x)$ and $x \mapsto (W_H)^{**}(x)$ defined in (4.54) and (4.55) satisfy

$$(W^G)^{**}(x) = \inf_{z \in \mathbb{R}} \sup_{y \in \mathcal{A}_G^x(z)} (z(x - y) + W^G(y)),$$
$$(W_H)^{**}(x) = \sup_{z \in \mathbb{R}} \inf_{y \in \mathcal{A}_H^x(z)} (z(x - y) + W_H(y)).$$

Moreover, $(W^G)^{**}(x) = (W_H)^{**}(x)$ for all $x \geq 0$ and

$$\{x \in \mathbb{R}_+ \mid (W^G)^{**}(x) = W^G(x)\} = \{x \in \mathbb{R}_+ \mid \partial^G G(x) \neq \emptyset\},$$
$$\{x \in \mathbb{R}_+ \mid (W_H)^{**}(x) = W_H(x)\} = \{x \in \mathbb{R}_+ \mid \partial^H H(x) \neq \emptyset\}.$$
Proof. Fix \( x \geq 0 \) and \( c \in \mathbb{R} \) and define
\[
(W_G^*)_H^*(c) := \inf_{y \in \mathcal{A}_H^c(c)} (cy - W^G(y)).
\]
By definition, the line \( f(y) := W^G(x) + \varepsilon^*(x) + c(y - x) \) dominates \( y \mapsto W^G(y) \) on \( \hat{\mathcal{A}}_H^*(c) = [l_H^*(c, W^G(x)), r_H^*(c, W^G(x))] \subseteq \mathcal{A}_H^*(c) \). If there exists \( z \in \mathcal{A}_H^*(c) \) such that \( f(z) = W^G(z) \) then either \( c \in \partial^H G(z) \) or \( z \) is on the boundary of \( \mathcal{A}_H^*(c) \), i.e. \( z = L_H^*(c, W^G(x)) \) or \( z = R_H^*(c, W^G(x)) \). Thus we may conclude that \((W_G^*)_H^*(c) = cx' - W^G(x')\) for some \( x' \) such that \( c(y - x') + W^G(x') = W^G(x) + \varepsilon^*_z(x)\).

Hence
\[
\inf_{z \in \mathbb{R}} \sup_{y \in \mathcal{A}_G^c(z)} (z(x - y) + W^G(y)) = \inf_{z \in \mathbb{R}} (W^G(x) + \varepsilon^*_z(x))
\]
so it follows from the definition of \( \varepsilon^*(x) \) and Proposition[15] that
\[
\inf_{z \in \mathbb{R}} \sup_{y \in \mathcal{A}_G^c(z)} (z(x - y) + W^G(y)) = W^G(x) + \varepsilon^*(x) = (W_G^*)_H^*(x).
\]
A symmetric argument can be used to show that
\[
\inf_{z \in \mathbb{R}} \sup_{y \in \mathcal{A}_G^c(z)} (z(x - y) + W^G(y)) = W_H(x) - \delta^*(x) = (W_H)_G^*(x).
\]
It follows from Lemma[12] that \((W_G^*)_H^*(x) = (W_H)_G^*(x)\) and the final statement follows from the definition of \( \varepsilon^*(x) \) and \( \delta^*(x) \).

The function \( y \mapsto (W_G^*)_H^*(y) \) can be viewed as the ‘concave biconjugate of \( W^G \) in the presence of \( y \mapsto W_H(y) \)’ (or constrained to remain within \( \text{cl}(\mathbb{R}^2 \setminus \text{epi}(W_H)) \)) for the following three reasons:

(i) Theorem[17] illustrates that the function \( y \mapsto (W_G^*)_H^*(y) \) can be expressed as
\[
(W_G^*)_H^*(x) = \inf_{z \in \mathbb{R}} \left( zx - \inf_{y \in \mathbb{R}} (zy - W^G(y) + \delta(y | \mathcal{A}_G^c(z))) \right),
\]
where \( \delta(y | \mathcal{A}_G^c(z)) \) is the ‘characteristic function’ of the set \( \mathcal{A}_G^c(z) \), i.e. \( \delta(y | \mathcal{A}_G^c(z)) = 0 \) if \( y \in \mathcal{A}_G^c(z) \) and \( \delta(y | \mathcal{A}_G^c(z)) = \infty \) otherwise. The difference with the standard concave-biconjugate is that this characteristic function depends on the choice of \( c \) and \( x \). However, this dependence seems essential to ensure that \( \text{epi}(W_H) \subseteq \text{epi}((W_G^*)_H^*) \).

(ii) For any measurable function \( f : \mathbb{R}^+ \rightarrow \mathbb{R} \), the superdifferential of \( f \) is as defined in (3.14) and the \( \varepsilon \)-superdifferential is defined as
\[
\partial_\varepsilon f(x) = \{ c \in \mathbb{R} | f(y) - f(x) - \varepsilon \geq c(y - x) \ \forall y \in \text{dom}(f) \}.
\]
Hence \( f_{\ast \ast}(x) - f(x) = \inf\{ \varepsilon \geq 0 | \partial_\varepsilon f(x) \neq \emptyset \} \). This definition of an \( \varepsilon \)-superdifferential coincides with the definition (4.41) when \( W_H \) is taken to be the multivalued function
\[
W_H(y) = \begin{cases} [W^G(0), +\infty] & y = 0 \\ [W^G(y), +\infty] & y \in (0, +\infty) \\ \infty & y = +\infty \end{cases}.
\]
Moreover, when \((4.37)\) holds, it follows from Theorem \[\text{17}\] that \((W^G)^{**}_{\tilde{H}}(x) - W^G(x) = \varepsilon^*(x)\) (where \(\varepsilon^*(x)\) is as defined in \((4.44)\)) so \(y \mapsto (W^G)^{**}_{\tilde{H}}(y)\) and \(y \mapsto f^{**}(y)\) can both be characterised in terms of a smallest spike variation.

(iii) In Corollary \[\text{6}\] it was shown that the concave biconjugate of \(y \mapsto (G/\psi) \circ F^{-1}(y)\) is such that \(((G/\psi) \circ F^{-1})^{**}(x) = \sup A_1(x)\) where the set \(A_1(x)\) is defined in \((3.25)\). This characterisation of the concave biconjugate uses the same method of construction as is used in Proposition \[\text{15}\] to define \(y \mapsto (W^G)^{**}_{\tilde{H}}(y)\).

The function \(y \mapsto (W^G)^{**}_{\tilde{H}}(y)\) will be referred to as the ‘convex biconjugate of \(W^G\) in the presence of \(y \mapsto W^H(y)\) (or constrained to remain within \(\text{epi}(W^G)\)) for directly similar reasons.

Furthermore, Theorem \[\text{17}\] can be reformulated in terms of the \(F\)-concavity (resp. \(F\)-convexity) of the functions \(G\) and \(H\) determining \(W^G\) and \(W^H\). To see this observe that it follows from the definitions of \(y \mapsto W^G(y)\) and \(y \mapsto W^H(y)\) in \((4.35)\) and \((4.36)\) that

\[
(G)^{**}_{\tilde{H}}(x) := (W^G)^{**}_{\tilde{H}}(F(x))\psi(x) = \inf_{z \in \mathbb{R}} \sup_{F(y) \in A^G_{\tilde{H}}(z)} \left( z(F(x) - F(y)) + \frac{G}{\psi}(y) \right) \psi(x),
\]

\[
(H)^{**}_{\tilde{H}}(x) := (W^H)^{**}_{\tilde{H}}(F(x))\psi(x) = \inf_{z \in \mathbb{R}} \sup_{F(y) \in A^G_{\tilde{H}}(z)} \left( z(F(x) - F(y)) + \frac{H}{\psi}(y) \right) \psi(x),
\]

which are the modified \(F\)-concave/\(F\)-convex-biconjugates used in \[\text{19}\].

5 Nash equilibrium in optimal stopping games

In this section the duality between the modified convex and concave biconjugates proved in Theorem \[\text{17}\] is used to show that the optimal stopping game \((2.9) - (2.10)\) exhibits a Nash equilibrium. The first step in this direction is to show the optimal stopping game \((2.9) - (2.10)\) exhibits a Stackelberg equilibrium. This is achieved by showing that \((G)^{**}_{\tilde{H}} \geq \bar{V}\) and \((H)^{**}_{\tilde{H}} \leq \bar{V}\) as Theorem \[\text{17}\] then implies that \(V = \bar{V} = (G)^{**}_{\tilde{H}}\).

**Theorem 18.** Suppose that \((2.7)\) holds and consider the optimal stopping game \((2.9) - (2.10)\). The optimal stopping game has a Stackelberg equilibrium, i.e. the games has a value \(V\) which can be represented as

\[V(x) = (G)^{**}_{\tilde{H}}(x) = (W^G)^{**}_{\tilde{H}}(F(x))\varphi(x).\]

Moreover, when \((2.8)\) hold, it follows that \(V = (G)^{**}_{\tilde{H}} = (H)^{**}_{\tilde{H}}\).

**Proof.** Let

\[
\tilde{H}(x) = \begin{cases} H(x) & x \in (a, b) \\ [G(x), H(x)] & x \in \{a, b\}. \end{cases}
\]

Theorem \[\text{17}\] and Remark \[\text{14}\] imply it is sufficient to show that

\[
(H)^{**}_{\tilde{H}} \leq V \leq (G)^{**}_{\tilde{H}}
\]

where \(\bar{V}\) and \(\bar{V}\) are defined in \((2.9)\) and \((2.10)\) respectively.
For the first inequality in \([5.63]\) let \(\hat{\mathcal{A}}_x = \bigcup_{z \in \mathbb{Z}} \mathcal{A}^C_{\hat{\mathcal{G}}}(z)\) and \(\hat{\tau} = \inf\{t \geq 0 \mid X_t \notin \hat{\mathcal{A}}_x\}\). Let \(a' = \inf \hat{\mathcal{A}}_x\) and \(b' = \sup \hat{\mathcal{A}}_x\) and suppose that \(b' < b\) then
\[
\forall \sigma \leq \hat{\tau} \quad \mathbb{E}_x \left[ \hat{H}(X_\sigma) e^{-\tau \sigma} \right] =: V^- (x),
\]
where \(\hat{H} (\cdot) := H(\cdot) \mathbb{1}_{(a',b')} + G(\cdot) \mathbb{1}_{[x=a'[\cup x=b']}\). The stopping time which attains this infimum is of the form \(\hat{\tau} = T_{a'} \wedge T_{b'}\) for some \(a' \leq a^* \leq x \leq b' \leq b\). As we have assumed that \(\sup \hat{\mathcal{A}}_x < b\) the process \((e^{-\tau (t\wedge \hat{\tau})} \varphi(X_{t\wedge \hat{\tau}}))_{t \geq 0}\) is a \((P_x, \mathcal{F}_{t\wedge \hat{\tau}})\)-martingale. The optional sampling theorem implies that \(E_y [\varphi(X_\sigma) e^{-\tau \sigma}] = \varphi(y)\) for all \(y \in \hat{\mathcal{A}}_x\) and all \(\sigma \leq \hat{\tau}\). Hence for arbitrary \(c \in \mathbb{R}\),
\[
V^- (x) = \inf_{\sigma \leq \hat{\tau}} \mathbb{E}_x \left[ \hat{H}(X_\sigma) e^{-\tau \sigma} \right] = c \varphi(x) + \inf_{\sigma \leq \hat{\tau}} \mathbb{E}_x \left[ (\hat{H}(X_\sigma) - c \varphi(X_\sigma)) e^{-\tau \sigma} \right].
\] 
Since \((5.64)\) holds for all \(c \in \mathbb{R}\), it follows that
\[
V^- (x) = \sup_{c \in \mathbb{R}} \left( c \varphi(x) + \inf_{\sigma \leq \hat{\tau}} \mathbb{E}_x \left[ (\hat{H}(X_\sigma) - c \varphi(X_\sigma)) e^{-\tau \sigma} \right] \right).
\] 
The optimal stopping problem on the right hand side of \((5.65)\) can be expressed as
\[
R_c (x) := \inf_{\sigma \leq \hat{\tau}} \mathbb{E}_x \left[ (\hat{H}(X_\sigma) - c \varphi(X_\sigma)) e^{-\tau \sigma} \right]
\]
\[
= \inf_{a' \leq y \leq x \leq b'} \left( \frac{\hat{H}(y) - c \varphi(y)}{\psi(y)} \frac{F(z) - F(x)}{\psi(z)} + \frac{\hat{H}(z) - c \varphi(z)}{\psi(z)} \frac{F(x) - F(y)}{\psi(z)} \right) \psi(x)
\]
\[
\geq \inf_{y \in \hat{\mathcal{A}}_x} \left( \frac{\hat{H}(y) - c \varphi(y)}{\psi(y)} \right) \psi(x).
\]
Hence, from \((5.65)\) we obtain
\[
V^- (x) \geq \sup_{c \in \mathbb{R}} \left( c \varphi(x) + \inf_{y \in \hat{\mathcal{A}}_x} \left( \frac{\hat{H}(y) - c \varphi(y)}{\psi(y)} \right) \right) \psi(x).
\]
Let
\[
c(x) = \inf \{c \in \mathbb{R} \mid \inf \hat{\mathcal{A}}_x = \inf \hat{\mathcal{A}}^C_{\hat{\mathcal{G}}}(c)\}, \quad \tau(x) = \sup \{c \in \mathbb{R} \mid \sup \hat{\mathcal{A}}_x = \sup \hat{\mathcal{A}}^C_{\hat{\mathcal{G}}}(c)\}.
\]
Define \(\delta^c(x)\) and \(\delta^\tau(x)\) as in \((4.43)\) and \((4.44)\) with \(H\) replaced by \(\hat{H}\) where necessary. Due to the definition of \(\hat{\mathcal{A}}_x\) it follows that for all \(c \in \mathbb{R}\)
\[
cF(x) + \inf_{y \in \hat{\mathcal{A}}_x} \left( \frac{\hat{H}(y) - c \varphi(y)}{\psi(y)} \right) \leq \frac{\hat{H}(x)}{\psi(x)} - \delta^c(F(x)).
\]
and equality holds for all \(c \in [c(x), \tau(x)]\). In particular, it follows from the definition of \(\hat{\mathcal{A}}_x\) that \(\delta^c(F(x)) = \delta^\tau(F(x))\) for some \(c \in [c(x), \tau(x)]\) so we may conclude that
\[
V^- (x) \geq \frac{\hat{H}(x)}{\psi(x)} - \delta^\tau(F(x)) \psi(x) = (W^G_{\hat{H}})_{\delta^\tau(F(x))} \psi(x) = (\hat{H})^G_{\delta^\tau}(x).
\]
which is the first inequality in \((5.63)\). The case that \(b' = b\) can be handled by using an approximating sequence of domains as in Theorem 3.
For the final inequality in (5.63), let \( A_x = \bigcup_{x \in \mathbb{R}} \tilde{A}^H_x(z) \) and let \( \hat{\sigma} = \inf \{ t \geq 0 \mid X_t \notin A_x \} \).

Let \( a' = \inf A_x \) and \( b' = \sup A_x \) and suppose that \( b' < b \). There is a natural asymmetry in the value function defined in (2.11) as \( R_x(\tau, \tau) = E_x[G(X_\tau)e^{-r\tau}] \). Thus

\[
\tilde{V}(x) = \inf_{\sigma} \sup_{\tau} E_x \left[ \left( G(X_\tau) \mathbb{1}_{[\tau \leq \sigma]} + \tilde{H}(X_\sigma) \mathbb{1}_{[\tau > \sigma]} \right) e^{-r(\tau \wedge \sigma)} \right]
\leq \sup_{\tau \leq \hat{\sigma}} E_x \left[ G(X_\tau) e^{-r\tau} \right] =: V^+(x).
\]

Proceeding as above it is possible to show

\[
V^+(x) \leq \inf_{c \in \mathbb{R}} \left( c \varphi(x) + \sup_{y \in A_x} \left( \frac{G(y) - c \varphi(y)}{\psi(y)} \right) \right) \psi(x),
\]

when \( b < b' \). Let

\[
\underline{c}(x) = \inf \{ c \in \mathbb{R} \mid \sup A_x = \sup \tilde{A}^H_x(c) \}, \quad \bar{c}(x) = \sup \{ c \in \mathbb{R} \mid \inf A_x = \inf \tilde{A}^H_x(c) \}.
\]

Define \( \epsilon^*_c(x) \) and \( \epsilon^*(x) \) as in (4.42) and (4.44) with \( H \) replaced by \( \tilde{H} \) where necessary. Due to the definition of \( A_x \) it follows that for all \( c \in \mathbb{R} \)

\[
cF(x) + \sup_{y \in A_x} \left( \frac{G(y) - c \varphi(y)}{\psi(y)} \right) \geq \frac{G(x)}{\psi(x)} + \epsilon^*_c(F(x)).
\]

and equality holds for for all \( c \in [\underline{c}(x), \bar{c}(x)] \). In particular, it follows from the definition of \( A_x \) that \( \epsilon^*_c(F(x)) = \epsilon^*(F(x)) \) for some \( c \in [\underline{c}(x), \bar{c}(x)] \) so we may conclude that

\[
V^+(x) \leq G(x) - \epsilon^*(F(x)) \psi(x) = (W^G)^*_H(F(x)) \psi(x) = (G)^{\tilde{H}}_H(x).
\]

which is the final inequality in (5.63). The case that \( b' = b \) can be handled by using an approximating sequence of domains as in Theorem 3.

The second statement follows immediately since when (2.8) holds we have \( H(x) = \tilde{H}(x) \) for all \( x \in [a, b] \). \( \square \)

**Remark 19.** In the previous Theorem the assumption (2.7) has only been used to rule out the degenerate case that \( \limsup_{x \to a} (G/\psi)(y) = +\infty \) (resp. \( \limsup_{x \to b} (G/\psi)(y) = +\infty \)) as in this case the optimal stopping time for the maximising agent is not finite and \( \overline{V}(x) = V(x) = +\infty \) for all \( x \in I \). In this case the functions \( (G)^*_H \) and \( (H)^*_G \) are no longer related to the value function of the optimal stopping game.

In Theorem 18 the functions \( H \) and \( G \) are defined on the entire domain of the one-dimensional diffusion \( X \). Theorem 18 can handle the case that \( X \) is absorbed upon exit from a set \([c, d] \in I \) by setting \( G(x) = H(x) \) for all \( x \in I \setminus [c, d] \). The value function in Theorem 18 can be thought of as an elastic cord which is tied to \( \limsup_{y \to 0} W^G(y) \) and pulled towards infinity between the two obstacles \( W^G \) and \( W_H \). On a compact domain it is the shortest path between these two obstacles as shown in 19.
The next two lemmata are used to prove that the game with upper- and lower-value (2.9)-(2.10) has a Nash equilibrium. The first step is to show that \( (G_H)^{**} \in \text{Sup}[G,H] \cap \text{Sub}(G,H) \) where

\[
\text{Sup}[G,H] = \{ F : I \rightarrow [G,H] | F \text{ is finely continuous and } r\text{-superharmonic on } \{ F > G \} \text{ and } \{ F > V \} \},
\]

\[
\text{Sub}(G,H) = \{ F : I \rightarrow [G,H] | F \text{ is finely continuous and } r\text{-subharmonic on } \{ F < H \} \text{ and } \{ F < V \} \}.
\]

are the admissible sets in the dual problems defined in (2.12). The second step is to show that \( (G_H)^{**} = \tilde{V} \) and \( (H_H)^{**} = \tilde{V} \).

**Lemma 20.** The concave biconjugate of \( G \) in the presence of the upper barrier \( H \) defined in (4.59) is in the admissible sets for the dual problems, i.e.

\[
(G_H)^{**} \in \text{Sup}[G,H] \cap \text{Sub}(G,H).
\] (5.66)

Moreover, the convex biconjugate of \( H \) in the presence of the lower barrier \( G \) defined in (4.60) is in the admissible sets for the dual problems, i.e.

\[
(H_H)^{**} \in \text{Sup}[G,H] \cap \text{Sub}(G,H).
\]

**Proof.** First we show that \( (G_H)^{**} \in \text{Sup}[G,H] \). Take \( x \in I \) such that \( (G_H)^{**}(x) < H(x) \) and let

\[
z_-(x) := \sup \{ y \leq x | (G_H)^{**}(y) = H(y) \} ; \quad z_+(x) := \inf \{ y \geq x | (G_H)^{**}(y) = H(y) \}.
\] (5.67)

Let \( \tilde{G}(y) := G(y)(z_-(x),z_+(x)) + H(x)(y=z_-(x)) \cup (y=z_+(x)) \) and define a function \( \tilde{W} \) using

\[
\tilde{W}(y;x) := \left( \frac{\tilde{G}}{\psi} \right) \circ F^{-1}(y).
\]

For \( z \in [F(z_-(x)), F(z_+(x))] =: J^x \) and \( c \in \mathbb{R} \) let

\[
\hat{\varepsilon}_c(z) := \inf \left\{ \varepsilon \geq 0 \left| \tilde{W}(y;x) \leq W^G(z) + \varepsilon + c(y-z) \quad \forall y \in J^x \right. \right\}.
\]

Define \( \hat{\varepsilon}(z) := \inf_{c \in \mathbb{R}} \hat{\varepsilon}_c(z) \), this infimum is attained at some \( \hat{c} \in \mathbb{R} \) because \( G \) is continuous and proper on \( I \). We claim that \( \hat{\varepsilon}(z) = \varepsilon^*(z) \) where \( \varepsilon^*(z) \) is defined in (4.42). To prove this first suppose that

\[
\hat{\varepsilon}(z) = \hat{\varepsilon}_c(z) < \inf_{c \in \mathbb{R}} \varepsilon_c^*(z),
\] (5.68)

then by definition

\[
h(y;z) := W^G(z) + \hat{\varepsilon}(z) + \hat{\varepsilon}(y-z) \geq \tilde{W}(y;x) \quad \forall y \in J^x
\]

and in particular \( h(F(z_-(x));F(x))\psi(z_-(x)) \geq H(z_-(x)) \) and \( h(F(z_+(x));F(x))\psi(z_+(x)) \geq H(z_+(x)) \). Thus the continuity of \( y \mapsto (H/\psi)(y) \) implies that for all \( z \in J^x \)

\[
[l_H(\hat{\varepsilon}, W^G(z) + \hat{\varepsilon}(z)), r_H^z(\hat{\varepsilon}, W^G(x) + \hat{\varepsilon}(z))] \subseteq [z_-(x), z_+(x)]
\]

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and moreover \( \hat{c} \in \partial^H G(\hat{z}(z)) \) which contradicts (5.68) so \( \hat{z}(z) \geq \varepsilon^*(z) \).

Suppose that \( \hat{z}(z) > \varepsilon^*(z) \) and take \( c' \in \partial^H G(\hat{z}(z)) \) so that by assumption

\[
    h(y) := W^G(z) + \varepsilon^*(z) + c'(y - z) < W^G(y)
\]

for some \( y \in J^x \setminus [l_H^r(\hat{c}, W^G(z) + \varepsilon^*(z)), r_H^r(\hat{c}, W^G(x) + \varepsilon^*(z))] \). Let \( \hat{c} = \inf\{ c \in \mathbb{R} \mid c \in \partial^G H(F(z_+(x))) \} \) then either: (i)

\[
    W_H(r_H^r(c', W^G(z) + \varepsilon^*(z))) \geq W_H(F(z_+(x))) + \hat{c}(r_H^r(c', W^G(z) + \varepsilon^*(z)) - F(z_+(x)))
\]

which implies that

\[
    h(y) > W_H(F(z_+(x))) + \hat{c}(y + \varepsilon^*(z)) - F(z_+(x)) \geq W^G(y)
\]

for all \( y \in [r_H^r(c', W^G(z) + \varepsilon^*(z)), F(z_+(z))] \); or (ii) there exists \( \hat{y} \in [r_H^r(c', W^G(z) + \varepsilon^*(z)), F(z_+(z))] \) and \( b \in \mathbb{R} \) such that

\[
    r_H^r(c', W^G(z) + \varepsilon^*(z)) = l_H^r(b, W^G(\hat{y}))
\]

and hence \( h(y) > W^G(\hat{y}) + b(y - \hat{y}) \geq W^G(y) \) for all \( y \in [r_H^r(c', W^G(z) + \varepsilon^*(z)), F(z_+(z))] \). These two cases are illustrated in Figure 10. We may conclude that \( h(y) \geq W^G(y) \) for all \( y \in [r_H^r(c', W^G(z) + \varepsilon^*(z)), F(z_+(z))] \). A similar argument can be used to show that \( h(y) \geq W^G(y) \) for all \( y \in [F(z_-(z)), l_H^r(c', W^G(z) + \varepsilon^*(z))] \). We may conclude that \( \hat{z}(z) = \varepsilon^*(z) \) for all \( z \in J^x \).

![Figure 10](image_url)

Figure 10: In the figure on the left the line \( h \) intercepts \( W_H \) at a point above the line \( \hat{y} \) whereas in the the figure on the right this is not the case. However, in the figure on the right \( h \) intercepts \( W_H \) at the same point as the tangent to \( W^G \) at \( \hat{y} \) intercepts \( W_H \).

Using Lemma 2 for all \( z \in J^x \) the concave biconjugate of \( \tilde{W} \) may be written as

\[
    \tilde{W}_{**}(z; x) = \inf_{c \in \mathbb{R}} \left( cz - \tilde{W}_*(c; x) \right) = \inf_{c \in \mathbb{R}} (W^G(z) + \hat{c}(z)) = W^G(z) + \varepsilon^*(z) = (W^G)_H^*(z).
\]

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By definition $z \mapsto \tilde{W}_{**}(z;x)$ is concave on $J^F$ which implies that $x \mapsto (G)_{H}^{**}(x)$ is $F$-concave on $[z_-(x), z_+(x)]$. Consequently, $(G)_{H}^{**}$ is $F$-superharmonic on $[z_-(x), z_+(x)]$ and since $x$ was arbitrarily chosen $(G)_{H}^{**}$ is $F$-superharmonic on each connected section of $\{x \in I \mid (G)_{H}^{**}(x) < H(x)\}$. The concave biconjugate $y \mapsto \tilde{W}_{**}(y;x)$ is continuous on $J^F$ (see [22] Theorem 10.1) and
\[
\tilde{W}_{**}(F(z_-(x);x)\psi(z_-(x)) = H(z_-(x)) \quad \tilde{W}_{**}(F(z_+(x);x)\psi(z_+(x)) = H(z_+(x))
\]
so the continuity of $x \mapsto H(x)$ implies that $x \mapsto (G)_{H}^{**}(x)$ is continuous. We may conclude that $(G)_{H}^{**} \in \text{Sup}(G,H)$ since for one dimensional diffusions continuity in the fine topology is equivalent to continuity in the Euclidean topology.

Let $\tilde{H}(x) = H(x)I_{(a,b)} + G(x)I_{[x=a] \cup [x=b]}$ then by reversing the roles of $G$ and $H$ and replacing $H$ with $\tilde{H}$ in the arguments above it can be shown that $(\tilde{H})_{**} \in \text{Sub}(G, \tilde{H}) \subset \text{Sub}(G, H)$ . The first part of the lemma then follows from Theorem [17] as $(\tilde{H})_{**}^{G} = (G)_{H}^{**} = (G)_{H}^{**}$. The second part can be derived using a symmetric argument.

As $(G)_{H}^{**} \in \text{Sup}(G,H)$ from the definition of the dual problems $\hat{V} \leq (G)_{H}^{**}$ and $(H)_{**}^{G} \in \text{Sub}(G,H)$ implies $(H)_{**}^{G} \leq \hat{V}$. The next lemma shows that the reverse inequalities hold.

**Lemma 21.** The concave biconjugate of $G$ in the presence of the upper barrier $H$ defined in [4.67] coincides with the value of one of the dual problems defined in [2.12] and the convex biconjugate of $H$ in the presence of the lower barrier $G$ defined in [4.62] coincides with the other dual problem, i.e.
\[
(G)_{H}^{**} = \hat{V} \quad ; \quad (H)_{**}^{G} = \hat{V}.
\]

**Proof.** For the first statement it is sufficient to show that $\hat{V} \geq (G)_{H}^{**}$. Fix $x' \in I$ such that $(G)_{H}^{**}(x') < H(x')$ and suppose that $\hat{V}(x') < (G)_{H}^{**}(x') = (W^{G})_{H}^{**}(F(x'))\psi(x')$. Define another set
\[
\tilde{\text{Sup}}(G,H) := \{ F : \mathbb{R}_+ \rightarrow [G,H] \mid F \text{ is continuous and concave on } \{ F > W^{G} \} \cup \{ F > (W^{G})_{H}^{**} \} \}
\]
Let $x = F(x')$ then by assumption there exists $f \in \tilde{\text{Sup}}(G,H)$ such that $f(x) = W^{G}(x) + \varepsilon$ for some $\varepsilon < \varepsilon^*(x)$ where $\varepsilon^*$ is as defined in [4.42]. Let
\[
l_f(x) := \sup\{ y \leq x \mid f(y) = W^{G}(y) \} \quad ; \quad r_f(x) := \inf\{ y \geq x \mid f(y) = W^{G}(y) \}
\]
so that the function $f$ is concave on $[l_f(x), r_f(x)]$. A tangent to $f$ at $x$ can be expressed as
\[
u(y;x) := f(x) + c'(y-x) \quad \text{for} \quad c' \in \left[ \frac{d^+}{dy}f(x), \frac{d^-}{dy}f(x) \right].
\]
Consider
\[
l_\eta := l_\eta^{G}(c', f(x) + \eta) \lor l_\eta^{H}(c', f(x) + \eta) \quad , \quad r_\eta := r_\eta^{G}(c', f(x) + \eta) \land r_\eta^{H}(c', f(x) + \eta).
\]
As $\partial^H f(x) \subseteq \partial^H G(x) = \emptyset$ it follows that either $l_\eta = l_\eta^{G}(c', f(x))$ and/or $r_\eta = r_\eta^{G}(c', f(x))$. Consequently, for sufficiently small $\eta > 0$, either $l_\eta = l_\eta^{G}(c', f(x) + \eta)$ and/or $r_\eta = r_\eta^{G}(c', f(x) + \eta)$. Thus $f(y) = f(x) + \eta + c'(y-x)$ for some $y \in [l_f(x), r_f(x)]$ which contradicts that $f$ is concave
on \([l_f(x), r_f(x)]\). We may conclude that it is not possible to construct a function \(f \in \text{Sup}(G, H)\) passing through \(p \in [W^G(x), W^G(x) + \varepsilon^*(x)]\) or equivalently, it is not possible to construct a function \(\tilde{f} \in \text{Sup}(G, H)\) passing through \(\tilde{p} \in [G(x'), G(x') + \varepsilon^*(F(x'))\) for arbitrary \(x' \in I\). Hence \((G)_H^* \geq \tilde{V}\) and a symmetric argument can be used to show that assuming \((H)_G^* < \hat{V}\) leads to a similar contradiction.

The next theorem is the main result in this section and shows when (2.7) and (2.8) hold, \(\hat{V} = \tilde{V}\) is equivalent to the existence of a Nash equilibrium in the optimal stopping game defined in (2.9)-(2.10). As such it is a purely analytical version of one direction of [19] Theorem 2.1. Take

\[
D^* := \{x \in I \mid \partial^H G(F(x)) \neq \emptyset\}; \quad D^- := \{x \in I \mid \partial^G H(F(x)) \neq \emptyset\} \quad (5.69)
\]
as the candidate stopping regions.

**Theorem 22.** Let

\[
\tau^* = \inf\{t \geq 0 \mid X_t \in D^*\} \quad \sigma^* = \inf\{t \geq 0 \mid X_t \in D^-\} \quad (5.70)
\]

and suppose that assumptions (2.7) and (2.8) hold. The optimal stopping game (2.9)-(2.10) has a Nash equilibrium and \((\tau^*, \sigma^*)\) is a saddle point, i.e. for any \(\tau, \sigma\)

\[
R_x(\tau, \sigma^*) \leq R_x(\tau^*, \sigma^*) \leq R_x(\tau^*, \sigma) \quad (5.71)
\]

for all \(x \in I\).

**Proof.** When (2.7) and (2.8) hold, it was shown in Theorem 18 that the optimal stopping game has a value and \(V = (G)_H^* = (H)_G^*\). The candidate stopping regions (5.69) suggest the following candidates for the optimal stopping times

\[
\tau^* = \inf\{t \geq 0 \mid (G)_H^*(X_t) = G(X_t)\} \quad \sigma^* = \inf\{t \geq 0 \mid (G)_H^*(X_t) = H(X_t)\}.
\]

It follows from Lemmata 20 and 21 that \(V = \hat{V} = \tilde{V}\) is \(r\)-harmonic on \(I \setminus (D^+ \cup D^-)\) and \((G)_H^*(x) = R_x(\tau^*, \sigma^*)\). Let

\[
x_+ = \inf\{z \geq x \mid \partial^H G(F(z)) \neq \emptyset\} \quad x_- = \inf\{y \leq x \mid \partial^H G(F(y)) \neq \emptyset\} \quad (5.72)
\]
and

\[
y_+ = \inf\{z \geq x \mid \partial^G H(F(z)) \neq \emptyset\} \quad y_- = \inf\{y \leq x \mid \partial^G H(F(y)) \neq \emptyset\} \quad (5.73)
\]
so that \(\tau^* = T_{x+} \wedge T_{x-}\) and \(\sigma^* = T_{y+} \wedge T_{y-}\). Take any other \(\sigma = T_{x+} \wedge T_{x-}\) and consider the set \(A_x = [x_+, x_-] \cap [z_+, z_-]\), the following three cases are illustrated in Figure 11. First suppose that \(A_x = [x_+, x_-]\) then

\[
h(y) = \left(\frac{G}{\psi}\right)(x_-) + \left(\frac{G}{\psi}\right)(x_+) = \frac{(x_+ - x_-)}{F(x_+) - F(x_-)}(y - F(x_-)) \geq \frac{(G)_H^*(y)}{\psi(y)}
\]

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for all $y \in A_x$, moreover, $h$ is the largest $F$-convex function with $h(x_+) = (G/\psi)(x_+)$ and $h(x_-) = (G/\psi)(x_-)$ so $R_x(\tau^*, \sigma) = h(F(x))\psi(x) \geq R_x(\tau^*, \sigma^*)$. Secondly suppose that $A_x = [z_-, z_+]$ and let

$$
g(y) = \left( \frac{\mathcal{H}}{\psi} \right) (z_-) + \frac{\left( \frac{\mathcal{H}}{\psi} \right) (z_+)}{F(z_+)} - \frac{\left( \frac{\mathcal{H}}{\psi} \right) (z_-)}{F(z_-)} (y - F(z_-))
$$

for $y \in A_x$. Define $\tilde{H}(y) := H(y)I_{(x_-, x_+)} + G(y)I_{[y=x_-] \cup [y=x_+]}$ for $y \in [x_+, x_-]$, then the convex biconjugate of $\tilde{W}_H(y) := (\tilde{H}/\psi) \circ F^{-1}(y)$ satisfies $g(y) \geq \tilde{W}_H^{**}(y)$ for all $y \in A_x$. Moreover, Lemma 21 implies that $\tilde{W}_H^{**}(y) = (W^G)_H^{**}(y)$ so $R_x(\tau^*, \sigma) = g(F(x))\psi(x) \geq R_x(\tau^*, \sigma^*)$. Thirdly, let $A_x = [x_+, z_-]$ then

$$
f(y) = \left( \frac{\mathcal{G}}{\psi} \right) (x_-) + \frac{\left( \frac{\mathcal{G}}{\psi} \right) (x_+)}{F(x_+)} - \frac{\left( \frac{\mathcal{G}}{\psi} \right) (x_-)}{F(x_-)} (y - F(x_-))
$$

and as in the previous case $f(y) \geq \tilde{W}_H^{**}(y)$ and $R_x(\tau^*, \sigma) = f(F(x))\psi(x) \geq (G)_H^{**}(x) = R_x(\tau^*, \sigma^*)$. The final case that $A_x = [z_+, x_-]$ follows by a symmetric argument. Thus we have shown that the second inequality in (5.71) holds. The first inequality in (5.71) can be shown using a similar argument and thus the stopping times (5.70) are a saddle point.

\[\square\]

Figure 11: In the figure on the left the line $h$ connects $(W^G(F(x_-)), F(x_-))$ and $(W^G(F(x_+)), F(x_+))$ so dominates the black line $\tilde{W}_H^{**}$. In the right hand figure the line $g$ connects $(W_H(F(z_-)), F(z_-))$ to $(W_H(F(z_+)), F(z_+))$ and the line $f$ connects $(W^G(F(x_-)), F(x_-))$ to $(W_H(F(z_+)), F(z_+))$ so both dominate $\tilde{W}_H^{**}$.

The next corollary relaxes the assumptions (2.7) and (2.8) and is a purely analytical version of Theorems 4.4 and 4.5.

**Corollary 23.** Define $F(X_+(\omega)) = 0$ on the event $\{\tau = +\infty\}$ for any Borel measurable function $F$ and let

$$
l_a := \lim_{x \downarrow a} \frac{G^+(x)}{\psi(x)} \quad , \quad l_b := \lim_{x \uparrow b} \frac{G^+(x)}{\varphi(x)}.
$$

(5.74)
Then:

(i) When \( l_a = l_b = 0 \) the optimal stopping game \((2.9)-(2.10)\) has a Nash equilibrium and \((\tau^*, \sigma^*)\) is a saddle point.

(ii) When \( l_a > 0, l_b = 0 \) and there exists \( l > a \) such that \((G^H)^*_{\tilde{H}}(x) > G(x)\) for all \( x \in (a, l) \) then the optimal stopping game \((2.9)-(2.10)\) does not have a Nash equilibrium. Similarly, if \( l_b > 0, l_a = 0 \) and there exists \( k < b \) such that \((G^H)^*_{\tilde{H}}(x) > G(x)\) for all \( x \in (k, b) \) then the optimal stopping game \((2.9)-(2.10)\) does not have a Nash equilibrium.

(iii) When \( l_a > 0, l_b = 0 \) and there exists \( l > a \) such that \((G^H)^*_{\tilde{H}}(x) = G(x)\) and/or \((G^H)^*_{\tilde{H}}(x) = H(x)\) for all \( x \in (a, l) \) then the optimal stopping game \((2.9)-(2.10)\) has a Nash equilibrium and \((\tau^*, \sigma^*)\) is a saddle point. When \( l_b > 0, l_a = 0 \) and there exists \( k < b \) such that \((G^H)^*_{\tilde{H}}(x) = G(x)\) and/or \((G^H)^*_{\tilde{H}}(x) = H(x)\) for all \( x \in (k, b) \) then the optimal stopping game \((2.9)-(2.10)\) has a Nash equilibrium and \((\tau^*, \sigma^*)\) is a saddle point.

Proof. Replacing \( H(x) \) with \( \tilde{H}(x) = H(x)I_{(a,b)} + G^+(x)I_{|x=a|\cup|x=b|} \) and \( G(x) \) with \( \tilde{G}(x) = G(x)I_{(a,b)} + G^+(x)I_{|x=a|\cup|x=b|} \) in Theorem 18 shows that \( V = (G^H)^*_{\tilde{H}} \). The function \( V \) is extended onto the closed interval \([a, b]\) using the convention at the boundaries. When \( l_a = l_b = 0 \) the function \( V \) is continuous on \([a, b]\), so the first case follows in exactly the same way as in Theorem 22.

(2) In the case that \( l_a > 0 \) extending the function \( V = (G^H)^*_{\tilde{H}} \) onto \([a, b]\) using the convention at the boundaries results in \( V \) being lower-semicontinuous on \([a, b]\). In this case there exists a maximising sequence of stopping times but their limiting value need not be attained. To see this, suppose that \( x_\pm, y_\pm \) are defined as in \((5.72)-(5.73)\) and that \( x_- \vee y_- = a \). Take \( \tau_n = T_{a+1/n} \wedge T_{x_+ \wedge y_+} \) then by assumption \( \lim_{n \to \infty} \tau_n = \tau^* \) and the approach used in the proof of Theorem 22 shows that \( R_x(\tau_n, \sigma^*) \leq (G^H)^*_{\tilde{H}}(x) \) for all \( n \geq 1 \) and \( x \in (a, x_+ \wedge y_+) \) and by construction \( \lim_{n \to \infty} R_x(\tau_n, \sigma^*) = (G^H)^*_{\tilde{H}}(x) \). However, using the convention at the boundary,

\[
R_x(\tau^*, \sigma^*) = E_x \left[ (G(X_{T_{x_+}})I_{x_+ \leq y_+} + H(X_{T_{y_+}})I_{y_+ < x_+})e^{-\tau_{x_+ \wedge y_+} \wedge \tau_{y_+}}I_{T_{x_+ \wedge y_+} \leq \tau_{y_+}} \right] = \left( \frac{G(x_+)}{\psi(x_+)} \frac{F(x) - F(a)}{\psi(x)} I_{x_+ \leq y_+} + \frac{H(y_+)}{\psi(y_+)} \frac{F(x) - F(a)}{\psi(y)} I_{y_+ < x_+} \right) \psi(x)
\]

for all \( x \in (a, x_+ \wedge y_+) \). Moreover,

\[
R_x(\tau_n, \sigma^*) = G(a + 1/n) \frac{\psi(x)}{\psi(a + 1/n)} \frac{F(x_+ \wedge y_+) - F(x)}{F(x_+ \wedge y_+) - F(a + 1/n)} + R_x(\tau^*, \sigma^*).
\]

Since \( F(a+) = 0 \) it follows that

\[
\lim_{n \to \infty} R_x(\tau_n, \sigma^*) = l_a \left( 1 - \frac{F(x)}{F(x_+ \wedge y_+)} \right) \psi(x) + R_x(\tau^*, \sigma^*) > R_x(\tau^*, \sigma^*)
\]

as the first term is strictly positive as the function \( F \) is strictly increasing. Thus a saddle point does not exist as the limit of the maximising sequence is not attained.
exist then

\begin{align*}
\text{Proposition 25.} & \quad \text{Assume that (2.7) and (2.8) hold and suppose} \\
& \quad \partial F \text{ in (4.38) and (4.39) now follows from Lemma 21 and Theorem 22. Let} \\
& \quad 14 \text{ the optimal stopping game (2.9)-(2.10) has a saddle point in all the above cases.}
\end{align*}

\begin{align*}
& \text{Remark 24. The previous result is heavily dependent on the assumption that } F(X_\tau(\omega)) = 0 \text{ on} \\
& \quad \text{the event } \{ \tau = +\infty \} \text{ as under this assumption } (G)^*_H \text{ can have a discontinuity at the boundaries of } I. \text{ If instead we extend } G \text{ and } H \text{ onto the closure of } I \text{ using the approach discussed in Remark} \\
& \quad 14 \text{ the optimal stopping game (2.9)-(2.10) has a saddle point in all the above cases.}
\end{align*}

A generalisation of the smooth-fit property in terms of the super- and subdifferentials defined in (4.38) and (4.39) now follows from Lemma 21 and Theorem 22. Let \( y \mapsto W^V(y) = (V/\psi) \circ F^{-1}(y) \) where \( V \) is the value of the optimal stopping game (2.9)-(2.10) and define \( \partial^H V(x) \) (resp. \( \partial^G V(x) \)) by replacing \( W^G \) with \( W^V \) in (4.38) (resp. replacing \( W_H \) with \( W^V \) in (4.39).)

\begin{align*}
& \text{Proposition 25. Assume that (2.7) and (2.8) hold and suppose } x \in D^+ \text{ and that } d^\pm (G/\psi)/dF(x) \text{ exist then} \\
& \quad \partial^H V(F(x)) \subseteq \left[ \frac{d^+}{dy} W^G(F(x)), \frac{d^-}{dy} W^G(F(x)) \right] = \left[ \frac{d^+}{dF} \left( \frac{G}{\psi} \right) (x), \frac{d^-}{dF} \left( \frac{G}{\psi} \right) (x) \right] \quad (5.75)
\end{align*}

which holds with equality when \( x \in D^+ \setminus \partial D^+ \). Smooth fit is said to hold at \( x \in \partial D^+ \) when (5.75) holds with equality. Similarly, suppose \( x \in \partial D^- \) and that \( d^\pm H/dF(x) \) exist then

\begin{align*}
& \quad \partial G V(x) \subseteq \left[ \frac{d^-}{dy} W_H(F(x)), \frac{d^+}{dy} W_H(F(x)) \right] = \left[ \frac{d^-}{dF} \left( \frac{H}{\psi} \right) (x), \frac{d^+}{dF} \left( \frac{H}{\psi} \right) (x) \right] \quad (5.76)
\end{align*}

which holds with equality when \( x \in D^- \setminus \partial D^- \). Smooth fit is said to hold at \( x \in \partial D^- \) when (5.76) holds with equality.

\textbf{Proof.} It follows from Theorem 18 and Proposition 15 that the value of the game is \( V(x) = G(x) + \varepsilon^*(F(x))\psi(x) \) where \( \varepsilon^* \) is defined in (4.42) so \( \partial^H V(x) = \partial^H \varepsilon(x) G(x) = \partial^H \varepsilon^*(x) G(x) \) for all \( x \in F(I) \). Moreover, the stopping regions (5.69) can be written as

\[ D^+ := \{ x \in I \mid \varepsilon^*(F(x)) = 0 \} \quad \text{,} \quad D^- := \{ x \in I \mid \delta^*(F(x)) = 0 \}. \]

and hence \( \partial^H V(x) \subseteq \partial^H G(x) \) for all \( x \in D^+ \). Suppose that \( x \in D^+ \) then \( y \mapsto W^V(y) \) is concave on \([F(z_-(x)), F(z_+(x))]\) where \( z_-(x), z_+(x) \) are defined in (5.67) which implies that \( W^V \) is almost everywhere differentiable on \([F(z_-(x)), F(z_+(x))]\) (see 22 Theorem 25.5) and the left- and right-derivatives of \( W^V \) exist for all \( y \in [F(z_-(x)), F(z_+(x))] \). Thus,

\[ \partial^H V(F(x)) = \left[ \frac{d^+}{dy} W^V(F(x)), \frac{d^-}{dy} W^V(F(x)) \right] \quad \forall x \in D^+. \]

For all \( x \in D^+ \setminus \partial D^+ \)

\[ \partial^H V(F(x)) = \left[ \frac{d^+}{dy} W^G(F(x)), \frac{d^-}{dy} W^G(F(x)) \right] \]
as for sufficiently small $\varepsilon > 0$, $V(x + \delta) = G(x + \delta)$ for $\delta$ such that $|\delta| \leq \varepsilon$. Suppose that $x \in \partial D^+$ then either, (i) $\exists \varepsilon > 0$ such that $V(x + \delta) > G(x + \delta)$ for all $\delta \in (0, \varepsilon)$ and/or (ii) $\exists \varepsilon > 0$ such that $V(x - \delta) > G(x - \delta)$ for all $\delta \in (0, \varepsilon)$. In the first case

$$
\frac{\left(\frac{G}{\psi}\right)(x + \delta) - \left(\frac{G}{\psi}\right)(x)}{F(x + \delta) - F(x)} < \frac{\left(\frac{V}{\psi}\right)(x + \delta) - \left(\frac{V}{\psi}\right)(x)}{F(x + \delta) - F(x)},
$$

whereas, in the second case

$$
\frac{\left(\frac{G}{\psi}\right)(x - \delta) - \left(\frac{G}{\psi}\right)(x)}{F(x - \delta) - F(x)} > \frac{\left(\frac{V}{\psi}\right)(x + \delta) - \left(\frac{V}{\psi}\right)(x)}{F(x + \delta) - F(x)}.
$$

Letting $\delta \downarrow 0$ in the first case and $\delta \uparrow 0$ in the second case we obtain

$$
\frac{d^+}{dF} \left(\frac{V}{\psi}\right)(x) \geq \frac{d^+}{dF} \left(\frac{G}{\psi}\right)(x), \quad \frac{d^-}{dF} \left(\frac{V}{\psi}\right)(x) \leq \frac{d^-}{dF} \left(\frac{G}{\psi}\right)(x).
$$

and hence

$$
\partial^H V(F(x)) \subseteq \left[\frac{d^+}{dy} W^G(F(x)), \frac{d^-}{dy} W^G(F(x))\right] \quad \forall x \in D^+.
$$

When $F, G, H \in C^1(I)$, the previous corollary implies that the stopping regions (5.69) are equivalent to

$$
D^+ := \left\{ x \in I \mid \frac{d}{dF} \left(\frac{G}{\psi}\right)(x) \in \partial^H G(F(x)) \right\}, \quad D^- := \left\{ x \in I \mid \frac{d}{dF} \left(\frac{H}{\psi}\right)(x) \in \partial^G H(F(x)) \right\}.
$$

We conclude by examining the Israeli-\(\delta\) put-option introduced in [17].

**Example 26 (Israeli-\(\delta\) Put).** Suppose that $X$ is a geometric Brownian motion. That is $X$ solves the SDE

$$
dX_t = rX_t \, dt + \sigma X_t \, dW_t
$$

where $r > 0$ is the discount rate and $\sigma > 0$ is a volatility parameter. In this case

$$
\text{L}_X u(x) = r^2 \frac{du}{dx}(x) + \frac{1}{2} \sigma^2 \frac{d^2u}{dx^2}(x)
$$

and the two fundamental solutions to $\text{L}_X u = ru$ are $\varphi(x) = x$ and $\psi(x) = x^{-2r/\sigma^2}$. Moreover,

$$
-\tilde{F}(x) = \left(\frac{\psi}{\varphi}\right)(x) = x^{-(1+2r/\sigma^2)}, \quad F(x) = \left(\frac{\varphi}{\psi}\right)(x) = x^{1+2r/\sigma^2}.
$$

and $(-\tilde{F})^{-1}(y) = y^{-\alpha}, \; F^{-1}(y) = y^\alpha$ where $\alpha = 1/(1 + 2r/\sigma^2) \in [0, 1]$. Let

$$
G(x) = (K - x)^+, \quad H(x) = (K - x)^+ + \delta.
$$
so that the maximising agent has bought a perpetual American put option with strike $K > 0$ from
the minimising agent but the minimising player retains the right to cancel the option by paying
a fixed penalty of $\delta > 0$. The gains functions are rescaled by taking
\[
\overline{W}^G(y) = \left( \frac{G}{\varphi} \right) \circ (-\bar{F})^{-1}(y) = (Ky^\alpha - 1)^+,
\]
\[
\overline{W}_H(y) = \left( \frac{H}{\varphi} \right) \circ (-\bar{F})^{-1}(y) = (Ky^\alpha - 1)^+ + \delta y^\alpha.
\]
The functions $G, H$ are illustrated in the top panel of Figure 12 and the functions $\overline{W}^G$ and $\overline{W}_H$
are illustrated in the bottom panel of Figure 12. The functions $\overline{W}^G$ and $\overline{W}_H$ are concave on the
interval $[K^{-1/\alpha}, +\infty)$ and $\overline{W}_H$ is concave on the interval $[0, K^{-1/\alpha}]$.

The problem is degenerate in the sense that the inf-player will choose $\sigma^* = +\infty$ and the
game has the same value as the perpetual put option examined in Example 10 if the concave
biconjugate of $\overline{W}^G$ minorises $\overline{W}_H$, i.e. $\overline{W}^G_{**}(y) \leq \overline{W}_H(y)$ for all $y \in \mathbb{R}_+$ which is equivalent to
\[
\delta \geq \overline{W}^G_{**}(K^{-1/\alpha}) = \frac{\sigma^2}{2r} \frac{K}{(1 + \frac{\sigma^2}{2r})^{1+2r/\sigma^2}} =: \delta^*.
\]
To avoid this case assume that $\delta \in (0, \delta^*)$. On $[0, K^{-1/\alpha})$ the function $\overline{W}_H$ is concave so
\[
(\overline{W}^G)^*_{**}(y) \leq \left( \frac{\overline{W}_H(K^{-1/\alpha})}{K^{-1/\alpha}} \right) y = \frac{\delta}{K} K^{1/\alpha} y \quad \forall y \in [0, K^{-1/\alpha}].
\]
In fact this inequality holds with equality as for all $c < \overline{W}_H(K^{-1/\alpha}) / K^{-1/\alpha}$
\[
cy \vee \overline{W}^G(y) < \overline{W}_H(y) \quad \forall y \in (0, \infty).
\]
On $[K^{-1/\alpha}, \infty)$ we can use an approach similar to that used in Example 10. There exists $y^*$
such that
\[
\overline{W}_H(K^{-1/\alpha}) = \overline{W}^G(y^*) = \frac{d}{dy} \overline{W}^G(y^*)(y - K^{-1/\alpha}).
\]
This $y^*$ solves the polynomial
\[
\frac{\delta}{K} + 1 = K(y)^\alpha(1 - \alpha) - \alpha(y)^\alpha K^{(\alpha-1)/\alpha}.
\]
Let $x^*$ be defined as $y^* = (x^*)^{-1/\alpha}$, then the constant $x^*$ satisfies
\[
\left(1 + 2r/\sigma^2\right) \left(1 - \frac{\delta}{K}\right) \left(\frac{x}{K}\right) = 2r/\sigma^2 + \left(\frac{x}{K}\right)^{1+2r/\sigma^2}
\]
which is the same condition as in [17]. Moreover, $(\overline{W}^G)^*_{**}(y) = \overline{W}^G(y)$ for $y \geq y^*$ and on
$[K^{-1/\alpha}, y^*)$ we have
\[
(\overline{W}^G)^*_{**}(y) = \frac{\delta}{K} + K(y^*)^{(\alpha-1/\alpha)} \frac{y - K^{-1/\alpha}}{y^* - K^{-1/\alpha}}
\]
\[
= \frac{\delta}{K} \frac{(x^*)^{-1/\alpha} - y}{(x^*)^{-1/\alpha} - K^{-1/\alpha}} + \frac{1}{x^*(K - x^*)} \frac{y - K^{-1/\alpha}}{(x^*)^{-1/\alpha} - K^{-1/\alpha}}.
\]
Figure 12: The top figure illustrates the payoff functions of the game. The black line is the value function of the game. The lower panel illustrates the transformed payoffs $\tilde{W}_H$ and $\tilde{W}_G$ the black line is the function $(\tilde{W}_G)^{**}_H$.

The function $y \mapsto (\tilde{W}_G)^{**}_H(y)$ is the black line in the lower panel of Figure 12. According to Theorem 18 the value of this game option satisfies $V(x) = (\tilde{W}_G)^{**}_H(-\tilde{F}(x))\varphi(x)$ so

$$V(x) = \begin{cases} 
\delta \left( \frac{x}{K} \right) \frac{(x^*)^{1/\alpha} - x^{1/\alpha}}{(x^*)^{1/\alpha} - K^{1/\alpha}} + (K - x^*) \left( \frac{x}{x^*} \right) \frac{x^{1/\alpha} - K^{1/\alpha}}{(x^*)^{1/\alpha} - K^{1/\alpha}} & \text{for } x < x^* \\
\delta \left( \frac{x}{K} \right)^{\alpha/(1-\alpha)} & \text{for } x \in [x^*, K] \\
\delta \left( \frac{x}{K} \right)^{\alpha/(1-\alpha)} & \text{for } x > K
\end{cases}$$

which can be rearranged into the form derived in [17]. In this example assumptions (2.7) and (2.8) hold so Theorem 22 tells us that a saddle point of the game option is

$$\tau^* = \inf \{ t \geq 0 \mid X_t \leq x^* \} \quad , \quad \sigma^* = \inf \{ t \geq 0 \mid X_t = K \}.$$

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