Abstract

The concept of Generalized Inverse based Decoding (GID) is introduced, as an algebraic framework for the syndrome decoding problem (SDP) and low weight codeword problem (LWP). The framework has ground on two characterizations by generalized inverses (GIs), one for the null space of a matrix and the other for the solution space of a system of linear equations over a finite field. Generic GID solvers are proposed for SDP and LWP. It is shown that information set decoding (ISD) algorithms, such as Prange, Lee-Brickell, Leon, and Stern’s algorithms, are particular cases of GID solvers. All of them search GIs or elements of the null space under various specific strategies. However, as the paper shows the ISD variants do not search through the entire space, while our solvers do even when they use just one Gaussian elimination. Apart from these, our GID framework clearly shows how each ISD algorithm, except for Prange’s solution, can be used as an SDP or LWP solver. A tight reduction from our problems, viewed as optimization problems, to the MIN-SAT problem is also provided. Experimental results show a very good behavior of the GID solvers. The domain of easy weights can be reached by a very few iterations and even enlarged.

Keywords: Syndrome decoding, low weight codeword, information set decoding, generalized inverse.

Contents

1 Introduction 2
2 Preliminaries 5
3 The GI of a matrix over arbitrary fields 6
  3.1 Definitions and existence . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6
  3.2 Computing GIs . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7
4 The coset and subspace weight problems and the GI 8
  4.1 Generic GID solver for the coset weight problem . . . . . . . . . . . . . . . . . . . 8
    4.1.1 A general rank case . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 9
    4.1.2 Rank deficient matrices . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 11
    4.1.3 Full rank matrices . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 13
    4.1.4 Summarizing the results . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 14
5 Information set decoding and GID decoding

5.1 Information set decoding as a cryptographic attack

5.2 Prange’s approach

5.2.1 Prange’s algorithm

5.2.2 Prange’s approach as a GI-based approach

5.3 Lee-Brickell’s approach

5.3.1 Lee-Brickell’s algorithm

5.3.2 Lee-Brickell’s approach as a GI-based approach

5.4 Leon’s approach

5.4.1 Leon’s algorithm

5.4.2 Leon’s approach as a GI-based approach

5.5 Stern’s approach

5.5.1 Stern’s algorithm

5.5.2 Stern’s approach as a GI-based approach

5.6 Finiasz-Sendrier’s approach

5.6.1 Finiasz-Sendrier’s algorithm

5.6.2 Finiasz-Sendrier’s approach and a GI-based approach

5.7 Multiple decompositions

7 Experimental results

1. Introduction

McEliece’s cryptosystem. Rapid evolution of quantum computing, as well as Shor’s famous quantum algorithm for discrete logarithm and factorization, are urging public key cryptography to move away from number theoretic based solutions. Initiated in 2015 by the NIST, the post-quantum standardization process searches for quantum secure techniques for key exchange and digital signatures. Round 3 candidates for key exchange/key encapsulation mechanism are either code-based or lattice-based solutions. Amongst them, we discover one of the oldest public key encryption scheme, proposed in 1978 by McEliece. McEliece had the idea to generate a linear code that admits an efficient decoding algorithm (the private key) and to mask its structure (the masked code is the public key). Under the assumption that the public code is indistinguishable from a random code, breaking the scheme resumes to solving the syndrome decoding problem (SDP) for random codes, which is NP-complete. So, until one proves that there exist NP-complete problems that can be solved in polynomial time by quantum computers (such a result would be major breakthrough in complexity theory), the McEliece scheme is considered quantum secure.

As being the foundation of the McEliece scheme, a significant interest in finding algorithms for SDP emerged. Also, having a good estimation of the work factor required by algorithms for SDP is mandatory to assess the security level of the aforementioned scheme.
techniques, in term of time complexity. The decisional SDP takes as input the parity-check matrix $H \in M_{n-k,n}(F_q)$ of a linear code $C$ over a finite field with $q$ elements $F_q$, a syndrome vector $s \in F_q^n$ and an integer $t$, and asks if there is a solution to the equation $Hx = s$ satisfying $|x| \leq t$, where $|x|$ denotes the Hamming weight of the vector $x$. In SDP, syndrome vectors are always non-zero. When $s = 0$ the problem becomes LWP. The main idea behind the original ISD technique (Prange’s approach) is to pick a sufficiently large set of error-free coordinates such that the corresponding columns of $H$ form an invertible submatrix. This is equivalent to computing two matrices $P \in GL_{n-k}(F_q)$ and $Q \in S_n(F_q)$ such that $PHQ = (V \quad I_{n-k})$. This gives us $x = Q \left( \begin{array}{c} 0 \\ Ps \end{array} \right)$. The correct information set for SDP is found when $|Ps| \leq t$. In other words, this procedure will stop eventually, as there is a permutation which sends the support of the solution outside the information set. Relaxation of certain architectural constraints of Prange’s algorithm or the addition of optimizations, mainly based on the birthday paradox (also known as the meet-in-the-middle approach), has led to many improvements.

The difficulty of solving SDP highly depends on the range of values for the parameter $t$. In a cryptographic context, it is frequent to select hard instances, i.e., where $t$ is close to the Gilbert-Varshamov bound or sub-linear in the code-length. When $t$ is linear in $n$, optimizations to Prange’s algorithm have better complexity results even in the first term in the exponent. However, when $t$ is sub-linear in $n$, which is the case of all NIST code-based submissions, the advantage of all the improvements vanish asymptotically. To be more precise, for $t = o(n)$ and $n \rightarrow \infty$ the work factor of existing algorithms for SDP equals $2^{-t \log(1-k/n)(1+o(1))}$.

It worths mentioning that ISD is not the only technique for solving SDP. For example, statistical decoding has a quite different approach. However, it does not achieve performance comparable to even the simplest ISD techniques, e.g., Prange’s algorithm.

Both SDP and LWP can be seen as particular cases of some well-known generic problems, Coset Weight Problem (CWP) and Subspace Weight Problem (SWP). The difference between CWP and SDP, respectively between SWP and LWP, resides in the input matrix $A$ which for CWP and SWP is an arbitrary matrix from $M_{m,n}(F)$. Notice that when $m = n - k \leq n$ and rank($A$) = $n - k$, CWP becomes SDP. In order to stay as general as possible, in this paper we will present solutions for CWP/SWP and restrict to full rank matrices when discussions move towards coding theory.

Generalized inverse of a matrix. Since one has to solve a system of linear equations to find a solution for CWP, the idea of computing the inverse of the matrix $A$ comes natural in mind. However, as $A$ is not square, we can not apply this technique here. Nevertheless, the concept of matrix inverse exists in the case of non-square matrices. It is known as generalized inverse (GI).

Given a matrix $A \in M_{m,n}(F)$, a GI for $A$ is a matrix $X \in M_{n,m}(F)$ satisfying $AXA = A$. Several types of inverses are known, such as, reflexive, normalized, and pseudo-inverse (or Moore-Penrose inverse). In particular, the Moore-Penrose inverse is a helpful tool when minimum norm solutions are required over the field of real or complex numbers. However, when moving to finite fields, things change a lot, mainly due to the geometrical properties of the scalar product. Results regarding the GI in arbitrary and finite fields exist and are going to be used and extended here in the context of linear codes.

There were several attempts to use GI of a matrix in cryptography and coding theory. In 1998, Wu and Dawson have proposed a public-key cryptosystem based on GIs, but three years later it was cryptanalyzed. Dang and Nguyen have used pseudo-inverses, the strongest form of a GI, to design key exchange protocols and protocols for privacy-preserving auditing data in cloud.
The only reference of GIs with respect to SDP is by Finiasz [26] (\(H\) is the parity-check matrix, \(S\) is the syndrome, and the threshold is \(w\)):

“For instance, when \(w\) is larger than \(n/2\), solving SD becomes easy, as computing a pseudo-inverse \(H^{-1}\) of \(H\) and computing \(H^{-1} \cdot S\) will return a valid solution with large probability. However, for smaller values of \(w\), when a single solution exists, finding it becomes much harder.”

The pseudo-inverse of a matrix, when it exists, is unique. Its use in enumerating a space of possible values is doomed to failure. But if the inverse concept is relaxed, we can broaden the search spectrum, and things become affordable. But the question is: how affordable? Can we list the entire possible solution space? This is the question our paper wants to answer.

**Contributions.** We discuss below the contributions that our work makes.

**GI based solvers for CWP and SWP.** Our first contribution is to propose an algebraic formalism based on the GI of a matrix to address both CWP and SWP. This formalism allows us to have a unified vision of the two problems, and it provides the main tools for understanding all the algorithmic improvements for solving CWP and SWP.

We begin by a careful inspection of the solutions of a linear system of equations \(Ax = b\) with \(b \neq 0\), and prove that all its solutions can be obtained only by GIs. More precisely, we show that

\[
\{ x \in \mathbb{F}_q^n \mid Ax = b \} = \{ Xb \mid X \text{ is a GI of } A \}.
\]

This characterization allows us to attack CWP in a very direct way: sample GIs (by means of some strategy) until a solution with the desired Hamming weight is reached. For example, one could fix a transformation \((P, Q) \in \text{GL}_r(\mathbb{F}) \times S_n(\mathbb{F})\) with \(PAQ = (V I_m)\) for some \(V\), and then search solutions of the form \(Xb\), where

\[
X \in \left\{ Q \left[ \begin{array}{c} Z \\ I_r \end{array} \right] \right\} P \mid Z \in \mathcal{M}_{n-m,m}(\mathbb{F}) \}.
\]

This method covers the whole space of solutions of the system \(Ax = b\).

For SWP, we prove first that the null space of \(A\) can be characterized by

\[
\{ x \in \mathbb{F}_q^n \mid Ax = 0 \} = \{ (Y - X)b \mid Y \text{ is a GI of } A \},
\]

where \(X\) (\(b\), resp.) is an arbitrary but fixed GI of \(A\) (non-zero vector, resp.). Thus, this characterization allows us to design a generic algorithm for SWP as the one above: fix a GI \(X\) of \(A\) and a non-zero vector \(b\), and then sample GIs \(Y\) of \(A\) until a solution with the desired Hamming weight is reached. The sampling of GIs is with respect to some strategy. For instance, if we decompose \(A\) into \(PAQ = (V I_m)\), then the null space of \(A\) is

\[
\left\{ Q \left[ \begin{array}{c} Z \\ -VZ \end{array} \right] Pb \mid Z \in \mathcal{M}_{n-m,m}(\mathbb{F}) \} \right. \}
\]

So, the sampling can be on arbitrary matrices \(Z\).

We will use the terminology **GI based Decoding (GID)** to refer to any of the GI-based techniques presented above, and **GID solver** for any algorithm that falls under it.

**Information set decoding versus GID.** Our GID technique works as a common denominator for many existing information set decoding techniques, such as Prange, Lee-Brickell, Leon, Stern, Finiasz-Sendrier (and probably all). It explains the essence of all these methods in a very clear and unified way. For instance, we show in the paper that Prange’s algorithm computes
particular GIs until it finds the desired solution, without covering the entire space of solutions. More exactly, given \( H \in \mathcal{M}_{n-k,n}(F) \) a parity-check matrix of a linear code and a syndrome \( s \in F^{n-k} \), Prange’s algorithm generates solutions to the equation \( Hx = s \) of the form \( Xs \), where
\[
X \in \left\{ Q \begin{pmatrix} 0 \\ I_r \end{pmatrix} P \mid (P, Q) \in GL_r(F) \times S_n(F), (3V : PHQ = (V \ I_r)) \right\}.
\]

However, as we prove in the paper, \( P \) must be in \( GL_m(F) \) to cover the entire space of solutions. The same holds for the other ISD techniques discussed in paper, and probably for all techniques that share Prange’s idea.

In terms of GID, each ISD technique is just a strategy to search a partial subspace of the space of solutions or of the null space. This view allows us to easily convert each such ISD technique into one working for SDP or LWP.

**A tight reduction to MIN-SAT.** Both CWP and SWP can be viewed as decision problems associated to two optimization problems, namely the **minimum coset weight problem** (MIN-CWP) and the **minimum subset weight problem** (MIN-SWP). Our GI-based approach allows to tightly reduce these optimization problems (MIN-CWP and MIN-SWP) to the well-known MIN-SAT problem, when \( F = F_2 \). As the reduction is very tight, we expect many techniques working for MIN-SAT to apply to the two problems.

**Reaching easy weights by means of GID.** Our simulations have shown that for small length codes, GID solvers behave very similar to ISD decoders in terms of performance. We have also noticed through simulations that there is an interval of Hamming weights where the GID solvers manage to efficiently find solutions for CWP and SWP. For example when \( F = F_2 \) the interval is symmetric and centered in \( n/2 \) (see [18]). Our simulations suggest that it is rather easy in general to find solutions within this range. However, we know that hard instances exist even for this interval (see for example [66]). Our simulations show that for codes of length up to \( n = 3000 \), with just one \((P,Q)\) decomposition, we have reached solutions with Hamming weights in the range \([r^{q-1} - \sqrt{n}, r^{q-1} + n - r + \sqrt{n}]\) in only a few seconds on an ordinary laptop computer.

**Paper organization.** The article begins by setting the notation and basic definitions from coding theory (next section). Section 2 is dedicated to the GI. The two central problems, CWP and SWP, are treated in Section 3, where two generic GID solvers for them are presented. Moving forward to ISD, Section 5 starts with some historical considerations. After that, its focus is on the first ISD decoder, i.e., Prange’s algorithm (5.2). Till the end of Section 5, several variants of ISD are considered within the framework of GID. The GI allows us to make a closed reduction from CWP and SWP, viewed as optimization problems, to the well-known MIN-SAT problem (Section 6). Section 7 considers some practical issues, by providing experimental tests on a variety of code parameters.

2. Preliminaries

We fix the basic notation on linear algebra and coding theory that we will use in the paper (for details, the reader is referred to the standard textbooks such as [59, 30, 60, 35]).

Generic fields are denoted by \( F \). When we want to emphasize that a field is finite and has the order \( q \), we will write \( F_q \). \( F^n \) stands for the \( n \)-dimensional vector space over \( F \). The vectors of \( F^n \) will be denoted by lowercase letters, such as \( x \), and written in column form. The \( i \)th element of \( x \in F^n \) is denoted \( x(i) \), where \( 1 \leq i \leq n \), and the support of \( x \) is \( \text{Supp}(x) = \{ i \mid 1 \leq i \leq n, x(i) \neq 0 \} \). The cardinality of \( \text{Supp}(x) \) is the Hamming weight of \( x \). We shall simply denote
this as $|x|$. If $I$ is a non-empty subset of $\{1, \ldots, n\}$, $x_I$ stands generally for the restriction of $x$ to $I$, that is, the vector of size $|I|$ (the cardinality of $I$) that is obtained from $x$ by removing all entries on positions outside $I$. The operator “$\cdot$” is used both for the Hamming weight and the cardinality of a set. However, the distinction will always be clear from the context.

The set of $m \times n$ matrices with elements in $\mathbb{F}$ is denoted $\mathcal{M}_{m,n}(\mathbb{F})$. Matrices will be denoted by uppercase bold letters, such as $A$. $A(i,j)$ denotes the element of $A$ at the intersection of row $i$ and column $j$. $I_r$ stands for the identity matrix of size $r$, and $I_{m,n,r}$ is $I_r$ extended with zeroes to an $m \times n$ matrix, i.e., $I_{m,n,r} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. $\text{GL}_m(\mathbb{F})$ denotes the general linear group of order $m$ over $\mathbb{F}$ (i.e., the group of all invertible matrices $A \in \mathcal{M}_{m,m}(\mathbb{F})$). Its subgroup consisting of permutation matrices is denoted $S_n(\mathbb{F})$. $\pi_Q$ stands for the permutation induced by $Q \in S_n(\mathbb{F})$, and $\pi([i,j])$ is the image of the interval $[i,j]$ through the permutation $\pi$.

As usual, $A^t$ (rank($A$)) stands for the transpose (inverse, rank) of $A \in \mathcal{M}_{m,n}(\mathbb{F})$. The range (null space) of $A$ is $R(A) = \{y \in \mathbb{F}^n \mid \exists x \in \mathbb{F}^m : x^tA = y\}$ (N($A$) = $\{x \in \mathbb{F}^n \mid Ax = 0\}$). We will use $\langle A \rangle$ to denote the vector space spanned by $A$.

A linear $[n,k]$ code over $\mathbb{F}_q$ is a vector subspace $C$ of $\mathbb{F}_q^n$ of dimension $k$. Any matrix $G \in \mathcal{M}_{k,n}(\mathbb{F}_q)$ whose rows form a basis for $C$ is a generator matrix for $C$. A parity-check matrix for $C$ is a generator matrix $H$ for the dual code $C^\perp$.

3. The GI of a matrix over arbitrary fields

The GI of a matrix has been much studied over the fields of real and complex numbers. Not all the results valid in this context remain valid when moving to an arbitrary field, especially to finite fields. Consequently, in this section, we will recall some results that are valid for matrices over arbitrary fields, and when needed, we will specialize them to finite fields. We will mainly follow [6, 52, 29], but we draw attention to the fact that some results will be presented in our own approach, which we consider appropriate for the case of finite fields.

3.1. Definitions and existence

**Definition 1.** Let $\mathbb{F}$ be an arbitrary field and $A \in \mathcal{M}_{m,n}(\mathbb{F})$ be a matrix. A GI of $A$ is any matrix $X \in \mathcal{M}_{n,m}(\mathbb{F})$ that fulfills

$$AXA = A.$$  

(1)

There are specialized cases of GI, but we do not mention them here because they are not used in our paper.

From the definition one can easily see that $A^{-1}$ is the only GI of $A$ when $A$ is non-singular. That is, in such a case, the GI exists and is unique. Before moving on to the analysis of the existence of the generalized inverse in the general case, let us analyze in more detail its definition and the connection with solving systems of linear equations. Let $\mathcal{GI}(A)$ stand for the set of GIs of $A$.

**Theorem 1 ([6]).** Let $A \in \mathcal{M}_{m,n}(\mathbb{F})$ and $X \in \mathcal{M}_{n,m}(\mathbb{F})$. Then,

$$X \in \mathcal{GI}(A) \iff (\forall b \in R(A))(Xb \text{ is a solution to } Ax = b).$$

**Theorem 2 ([6]).** Let $A \in \mathcal{M}_{m,n}(\mathbb{F})$, $b \in R(A)$, and $X \in \mathcal{GI}(A)$. Then, $x$ is a solution to $Ax = b$ if and only if $x = Xb + (I - XA)c$, for some $c \in \mathbb{F}^n$. 

6
It is also well known that $\mathcal{R}(I - XA) = \mathcal{N}(A)$ (see, for instance, [52, 58]). Therefore, by Theorem 2, $Xb$ is a solution to the system $Ax = b$ (when it is consistent) and any other solution can be obtained by adding arbitrary elements from the null space of $A$ to $Xb$.

GIs exist for all matrices over arbitrary fields [52, 29]. A first step in showing this is based on the following theorem.

**Theorem 3** ([3]). Let $A \in \mathcal{M}_{m,n}(\mathbb{F})$, $P \in \text{GL}_m(\mathbb{F})$, and $Q \in \text{GL}_n(\mathbb{F})$. Then, the function $f : \mathcal{GI}(A) \rightarrow \mathcal{GI}(PAQ)$ given by $f(X) = Q^{-1}XP^{-1}$, for any $X \in \mathcal{GI}(A)$, is a bijection.

**Proof.** It is straightforward to check that $f$ is well-defined and one-to-one. It remains to prove that any GI $Y$ of $PAQ$ is of the form $Q^{-1}XP^{-1}$, for some $X \in \mathcal{GI}(A)$.

If $Y \in \mathcal{GI}(PAQ)$, then $PAQYPAQ = PAQ$, which is equivalent to $AQYP = A$, since $P$ and $Q$ are non-singular matrices. However, this shows that $X = QYP \in \mathcal{GI}(A)$ and $Y = Q^{-1}XP^{-1}$. \hfill \Box

**Corollary 1.** Let $A, B \in \mathcal{M}_{m,n}(\mathbb{F})$. If $PAQ = B$ for some matrices $P \in \text{GL}_m(\mathbb{F})$ and $Q \in \text{GL}_n(\mathbb{F})$, then:

1. $\mathcal{GI}(A) = \{QXP \mid X \in \mathcal{GI}(B)\}$;
2. $|\mathcal{GI}(A)| = |\mathcal{GI}(B)|$.

**Proof.** Apply Theorem 3 to $B$ and $A = P^{-1}BQ^{-1}$. \hfill \Box

### 3.2. Computing GIs

To facilitate expression, a pair $(P, Q)$ of matrices as in Corollary 1 will often be called a *transformation* of $A$. The first part of this corollary shows that the set $\mathcal{GI}(A)$ does not depend on the transformation we apply to $A$. As a result, it suggests the following method for computing GIs of $A$:

- Transform the matrix $A$ through elementary operator matrices $P$ and $Q$ into a matrix $B = PAQ$ for which one can easily compute GIs;
- For each GI $X$ of $B$, $QXP$ is a GI of $A$. Besides, all GIs of $A$ are obtained in this way.

As an example, one may use the canonical form of $A$, $PAQ = I_{m,n,r}$ [30]. Thus, computing generalized inverses for $A$ is reduced to computing GIs for $I_{m,n,r}$. We present below some general constructions that also include this case. Even if the results are trivial to prove, we prefer to present them in the form of a proposition to highlight their usefulness further.

**Proposition 1.** Let $A \in \mathcal{M}_{m,n}(\mathbb{F})$ and $X \in \mathcal{M}_{n,m}(\mathbb{F})$ be matrices.

1. If $A$ and $X$ are divided into blocks of appropriate sizes, $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ and $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$, respectively, then $X$ is a GI of $A$ if and only if the following matrix equations are fulfilled:

$$\begin{align*}
(A_1X_1 + A_2X_3)A_1 + (A_1X_2 + A_2X_4)A_3 &= A_1 \\
(A_1X_1 + A_2X_3)A_2 + (A_1X_2 + A_2X_4)A_4 &= A_2 \\
(A_3X_1 + A_4X_3)A_1 + (A_3X_2 + A_4X_4)A_3 &= A_3 \\
(A_3X_1 + A_4X_3)A_2 + (A_3X_2 + A_4X_4)A_4 &= A_4
\end{align*}$$

(2)
2. If \( A \) and \( X \) are divided into blocks of appropriate sizes, \( A = (A_1 \ A_2) \) and \( X = (X_1 \ X_2) \), respectively, then \( X \) is a GI of \( A \) if and only if the following matrix equations are fulfilled:

\[
\begin{align*}
(A_1X_1 + A_2X_2)A_1 &= A_1 \\
(A_1X_1 + A_2X_2)A_2 &= A_2
\end{align*}
\]

(3)

3. If \( A \) and \( X \) are divided into blocks of appropriate sizes, \( A = (A_1 \ A_2) \) and \( X = (X_1 \ X_2) \), respectively, then \( X \) is a GI of \( A \) if and only if the following matrix equations are fulfilled:

\[
\begin{align*}
A_1(X_1A_1 + X_2A_2) &= A_1 \\
A_2(X_1A_1 + X_2A_2) &= A_2
\end{align*}
\]

(4)

**Proof.** Directly from (1).

**Example 1.** We present below some cases of application of Proposition 1:

1. If \( A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \), then the GIs of \( A \) have the form \( X = \begin{pmatrix} I_r & X_2 \\ X_3 & X_4 \end{pmatrix} \), where \( X_2, X_3, \) and \( X_4 \) are arbitrary matrices (of appropriate sizes) over \( F \). When \( F = F_q \) we have \( |GI(A)| = q^{m-n-r^2} \) (see [29]);

2. If \( A = \begin{pmatrix} I_r & A_2 \\ 0 & 0 \end{pmatrix} \), then the GIs of \( A \) have the form \( X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \), where \( X_1, X_2, X_3, \) and \( X_4 \) are arbitrary matrices (of appropriate sizes) over \( F \) that satisfy \( X_1 + A_2X_3 = I_r \).

3. If \( A = (A_1 \ I_r) \), then the GIs of \( A \) have the form \( X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \), where \( X_1 \) and \( X_2 \) are arbitrary matrices (of appropriate sizes) over \( F \) that satisfy \( A_1X_1 + X_2 = I_r \).

4. If \( A = (I_r \ 0) \), then the GIs of \( A \) have the form \( X = \begin{pmatrix} I_r \\ X_2 \end{pmatrix} \), where \( X_2 \) is a matrix (of appropriate size) over \( F \).

5. If \( A = \begin{pmatrix} I_r \\ 0 \end{pmatrix} \), then the GIs of \( A \) have the form \( X = (I_r \ X_2) \), where \( X_2 \) is a matrix (of appropriate size) over \( F \).

Even if the set of GIs of a matrix \( A \) is the same regardless of the transformation applied to the matrix, the equations that define them may be different. For instance, a matrix of rank \( r \) can be transformed into the form in Example 1(1) and in the form in Example 1(2). The equations that define the GI will be different, even if, in the end, we get the same set of GIs. But, as will be seen later, these equations will more or less facilitate working with GIs (enumeration, processing, convergence to solution). Due to this, given a transformation \((P, Q)\) for \( A \), we will denote by \( GI_{PQ}(A) \) the set of GIs of \( A \) obtained from the GIs of \( PAQ \). It is clear that \( GI_{PQ}(A) = GI(A) \).

4. **The coset and subspace weight problems and the GI**

This section will discuss possible applications of the generalized inverse in solving two closely related hard problems in coding theory. The first of them is the coset weight problem, and the second is the subspace weight problem, a subproblem of the first one.
4.1. Generic GID solver for the coset weight problem

The coset weight problem is as follows.

Coset Weight Problem (CWP)

Instance: $A \in \mathcal{M}_{m,n}(\mathbb{F})$, $b \in \mathbb{F}^m$, and positive integer $t$, where $\mathbb{F}$ is a finite field;

Question: Is there any solution $x_0 \in \mathbb{F}^n$ to $Ax = b$ such that $|x_0| \leq t$?

In coding theory, CWP occurs in the context of syndrome decoding, where $A$ is a full rank matrix of size $m \times n$ with $m < n$ and $b$ is a syndrome. For this reason, it is also called the syndrome decoding problem (more details about it are provided in Section 5.1). CWP is NP-complete when $\mathbb{F} = \mathbb{Z}_2$ [7]. However, both highlighting easy instances and constructing probabilistic polynomial-time algorithms to solve this problem can be of major importance when the problem is used to design secure cryptographic primitives.

In this section, we will analyze CWP through the generalized inverses of the matrix $A$. We will present the results, as much as possible, for the case of a general finite field $\mathbb{F}$. But, where necessary, we will restrict the analysis to $\mathbb{F} = \mathbb{Z}_2$.

The main strategy is the following. Given a CWP instance $(A, b, t)$, we will compute GIs $X$ of the matrix $A$ and check the solution’s weight. In fact, once a GI $X$ is computed, we have two approaches we can follow:

1. Consider only solutions $x = Xb$;
2. Consider solutions $x = Xb + (I - XA)c$, where $c \in \mathbb{F}^n$.

However, we will show that any solution to the system $Ax = b$ can be expressed in the form $Xb$, where $X \in \mathcal{GI}(A)$.

Theorem 4. Let $A \in \mathcal{M}_{m,n}(\mathbb{F}_q)$ with full row rank and $b \in \mathcal{R}(A)$ with $b \neq 0$. Then,

$$\{x \in \mathbb{F}^n_q \mid Ax = b\} = \{Xb \mid X \in \mathcal{GI}(A)\}.$$ 

Proof. The inclusion “$\supseteq$” follows simply from the fact that $Xb$ is a solution to $Ax = b$, for any $X \in \mathcal{GI}(A)$. Showing that the two sets have the same number of elements will end the proof.

It is straightforward to verify that $|\{x \in \mathbb{F}^n_q \mid Ax = b\}| = q^{n-m}$, since $A$ has full rank.

Let $PAQ = (V \ I_m)$. be a transformation of $A$. By Corollary[4] the sets $\mathcal{GI}(A)$ and $\mathcal{GI}(PAQ)$ are isomorphic and hence we can restrict to evaluate $|\{Xb \mid X \in \mathcal{GI}(PAQ)\}|$. Any GI of $PAQ$ has the form $X = \begin{pmatrix} X_1 \\ I_m - VX_1 \end{pmatrix}$, where $X_1 \in \mathcal{M}_{n-m,m}(\mathbb{F}_q)$ (Example[13]). As $b \neq 0$, $X_1b$ can take any value in $\mathbb{F}^{n-m}_q$. So, the number of solution $Xb = \begin{pmatrix} X_1b \\ (I_m - VX_1)b \end{pmatrix}$ is exactly $q^{n-m}$. This shows that the two sets have the same number of elements.

Hence, a generic GID solver for CWP samples $X \leftarrow \mathcal{GI}(A)$ until $|Xb| \leq t$. The main problem we get is how to do the sampling. Our approach is to apply transformations to $A$ to easily calculate GIs of the transformed matrix and to transfer the result to $A$.

Theorem[6] tells us that a single transformation of $A$ suffices to generate all its GIs. Our simulations and previous results on existing ISD algorithms show that it is more efficient to use different transformations for $A$ and run through several GIs for each transformation. According to this, we present below a generic GID solver for CWP.
Algorithm 1: GID solver for CWP

1: function CWGI\(\cdot\) \(\text{solve}(A, b, t)\)
2:     repeat
3:         Choose a transformation \((P, Q)\) of \(A\);
4:         \(X \leftarrow \text{GI}_{P, Q}(A)\) until \(|Xb| \leq t\) or no more sampling is allowed;
5:     until a solution \(Xb\) is found or no more transformation is allowed;
6:     return solution \(Xb\) or “fail”.

It should be understood that steps 3 and 4 are performed under various strategies, each leading to a variant of this generic algorithm. This will be clear in Section 5 when we discuss Prange, Lee-Brickell, Leon, Stern, and Finiasz-Sendrier’s algorithms.

In the following, we will analyze some transformations that can be applied to the matrix \(A\), focusing on the following three aspects:

- The general form of a GI \(X\) of \(A\);
- Particularities of the solution \(Xb\);
- Algorithmic issues.

4.1.1. A general rank case

Our first result here, regarding the form of the GI, follows directly from Corollary 1 and Proposition 1.

**Corollary 2.** Let \(A \in M_{m,n}(\mathbb{F})\), \(P \in \text{GL}_m(\mathbb{F})\), \(Q \in \text{GL}_n(\mathbb{F})\), and \(r > 0\) be an integer such that \(r \leq \text{rank}(A) \leq \min\{m, n\}\) and

\[
PAQ = \begin{pmatrix} I_r & A_2 \\ 0 & A_4 \end{pmatrix}.
\]  

Then, any GI \(X\) of \(A\) is of the form

\[
X = Q \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} P,
\]

where \(X_1\), \(X_2\), \(X_3\), and \(X_4\) are matrices of appropriate sizes that verify the matrix equations:

\[
\begin{cases}
X_1 + A_2X_3 = I_r \\
A_1X_3 = 0 \\
(X_2 + A_2X_4)A_4 = 0 \\
A_4X_4A_4 = A_4
\end{cases}
\]  

**Remark 1.** The transformation \([5]\) can always be obtained by Gaussian elimination. If no particular constraints are imposed on the matrices \(A_2\) and \(A_4\), \(Q\) can be obtained as a permutation matrix. It is also interesting to remark that \(X_4\) in \([6]\) must be a GI of \(A_4\) (see \([7]\)).

**Lemma 1.** With the above notation, if the system \(Ax = b\) is consistent, then

\[
Xb = Q \begin{pmatrix} X_1\bar{b}' + X_2\bar{b}'' \\ X_3\bar{b}' + X_4\bar{b}'' \end{pmatrix},
\]

where \(\bar{b}' = (Pb)_{[1,r]}\) and \(\bar{b}'' = (Pb)_{[r+1,m]}\).
Proof. Directly from (6).

Equation (8) gives us some flexibility in choosing the matrix $X$’s blocks to minimize the solution $Xb$’s weight. The following procedure could be used:

- Sample $X_4$ from $GI(A_4)$;
- Compute $X_2$ by $X_2 = -A_2X_4$;
- Generate a matrix $X_3$ such that $A_4X_3 = 0$;
- Compute $X_1$ by $X_1 = I_r - A_2X_3$.

The procedure is repeated as long as $|Xb|$ is greater than some given threshold $t$. The weight of $Xb$ depends on $P$, $Q$, and the matrix $X$’s blocks. If $Q$ is a permutation, then it can be neglected in choosing the matrix $X$’s blocks because it does not change the weight.

Remark 2. The above analysis is kept, with minor modifications, also for the case where $I_r$ occupies another position in (5). For example, if

$$PAQ = \begin{pmatrix} A_1 & I_r \\ A_3 & 0 \end{pmatrix}$$

then the blocks of the GI $X$ must verify the matrix equations:

$$
\begin{cases}
A_1X_1 + X_3 = I_r \\
A_3X_1 = 0 \\
(A_1X_2 + X_4)A_3 = 0 \\
A_3X_2A_3 = A_3
\end{cases}
$$

In this case, $X_2$ is a GI of $A_3$, and Lemma 1 holds true.

4.1.2. Rank deficient matrices

This is a sub-case of the previous case, which deals with rank deficient matrices. Our first result follows directly from Corollary 1 and Proposition 1.

Corollary 3. Let $A \in M_{m,n}(\mathbb{F})$, $P \in GL_m(\mathbb{F})$, and $Q \in GL_n(\mathbb{F})$ such that $\text{rank}(A) = r < \min\{m,n\}$ and

$$PAQ = \begin{pmatrix} I_r & A_2 \\ 0 & 0 \end{pmatrix}.$$ 

Then, any GI $X$ of $A$ is of the form

$$X = Q\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}P,$$

where $X_1$, $X_2$, $X_3$, and $X_4$ are matrices of appropriate sizes that verify the matrix equation

$$X_1 + A_2X_3 = I_r.$$

Remark 3. The transformation (11) can always be obtained by Gaussian elimination. If no requirement is imposed on the matrix $A_2$, $Q$ can be obtained as a permutation matrix.
Lemma 2. With the above notation, if the system $Ax = b$ is consistent, then
\[
Xb = Q \begin{pmatrix} X_1 b' \\ X_3 b' \end{pmatrix},
\]
(14)
where $b' = (Pb)_{[1,r]}$.

Proof. According to Lemma 1, we only have to prove that $b'' = 0$, where $b'' = (Pb)_{[r+1,m]}$.

Let $x_0$ be an arbitrary but fixed solution to $Ax = b$. That is, $Ax_0 = b$. Then,
\[
Pb = PAx_0
= PP^{-1} \begin{pmatrix} I_r & A_2 \\ 0 & 0 \end{pmatrix} Q^{-1}x_0
= \begin{pmatrix} \tilde{x}'_0 + A_2\tilde{x}'_0 \\ 0 \end{pmatrix},
\]
where $\tilde{x}'_0 = (Q^{-1}x_0)_{[1,r]}$ and $\tilde{x}'_0 = (Q^{-1}x_0)_{[r+1,n]}$. Therefore, $b'' = 0$. \qed

When $Q$ is a permutation, Lemma 2 allows us to obtain solutions to $Ax = b$ with any desired weight distribution on $\pi_Q([r+1,n])$, as the next theorem shows.

Theorem 5. Let $A \in M_{m,n}(\mathbb{F})$ with $\text{rank}(A) = r < \min\{m,n\}$, $b \in \mathbb{F}^m$, $P \in \text{GL}_m(\mathbb{F})$, and $Q \in S_n(\mathbb{F})$ such that $(Pb)_{[1,r]} \neq 0$ and
\[
P AQ = \begin{pmatrix} I_r & A_2 \\ 0 & 0 \end{pmatrix}
\]
(15)
Then, for any set $I \subseteq \pi_Q([r+1,n])$, a solution $x$ to $Ax = b$ with the property $\text{Supp}(x)_{\pi_Q([r+1,n])} = I$, can efficiently be computed.

Proof. Let $A$, $b$, $P$, and $Q$ as in the theorem’s hypothesis. Let $X$ be a GI of $A$. Then, Lemma 2 leads to
\[
Xb = Q \begin{pmatrix} X_1(Pb)_{[1,r]} \\ X_3(Pb)_{[1,r]} \end{pmatrix},
\]
where $X_1$ and $X_3$ verify the matrix equation $X_1 + A_2X_3 = I_r$.

Now, let $I$ be a subset of $\pi_Q([r+1,n])$, $i \in I$, and $1 \leq j \leq r$ be such that $(Pb)(j) \neq 0$. Then, there exists $r + 1 \leq k \leq n$ such that $\pi_Q(k) = i$. Moreover,
\[
(Xb)(i) = \sum_{j=1}^r X_3(k-r,j)(Pb)(j).
\]
We choose now the block $X_3$ by
\[
X_3(k-r,\ell) = \begin{cases} 1, & \text{if } \pi_Q(k) = i \in I \text{ and } \ell = j \\ 0, & \text{otherwise}, \end{cases}
\]
for all $r + 1 \leq k \leq n$ and $1 \leq \ell \leq r$. Then, $X_1 = I_r - A_2X_3$.

It is straightforward to check that $x = Xb$ verifies $\text{Supp}(x)_{\pi_Q([r+1,n])} = I$. Moreover, $x$ can be computed in polynomial time in the size of $A$. \qed
Remark 4. Under the conditions of Theorem 5, assuming in addition $A_2 = 0$, the solution $Xb$ will be of form

$$Xb = Q \begin{pmatrix} (Pb)[1, r] \\ X_3 (Pb)[1, r] \end{pmatrix}$$

because $X_1 = I_r$. Therefore, for any $\mathcal{I}$ chosen as in Theorem 5 we can compute a solution $x$ such that $\text{Supp}(x) = \mathcal{I} \cup \pi_Q(\text{Supp}((Pb)[1, r]))$.

Remark 5. When $PAQ$ takes the form

$$PAQ = \begin{pmatrix} A_1 & I_r \\ 0 & 0 \end{pmatrix}.$$  

the blocks of the GI must verify the matrix equation

$$A_1 X_1 + X_3 = I_r.$$  

The solution has the same form as in equation (14). This time however, in Theorem 5 we will consider $\mathcal{I} \subseteq \pi_Q([1, n - r])$ and in the proof we will assign $X_1$ instead of $X_3$.

A conclusion similar to that in Remark 4 can be drawn when $A_1 = 0$.

4.1.3. Full rank matrices

This is another particular case of the one in Section 4.1.1. As in previous cases, our first result follows directly from Corollary 1 and Proposition 1.

Corollary 4. Let $A \in M_{m,n}(\mathbb{F})$, $P \in \text{GL}_m(\mathbb{F})$, and $Q \in \text{GL}_n(\mathbb{F})$ such that $\text{rank}(A) = r = m < n$ and

$$PAQ = \begin{pmatrix} I_r & A_2 \end{pmatrix}.$$  

Then, any GI $X$ of $A$ is of the form

$$X = Q \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} P,$$

where $X_1$ and $X_2$ are matrices of appropriate sizes that verify the equation

$$X_1 + A_2 X_2 = I_r.$$  

Remark 6. 1. The transformation (19) can always be obtained by Gaussian elimination. If no requirement is imposed on the matrix $A_2$, $Q$ can be obtained as a permutation matrix.

2. It is straightforward to see that Lemma 2 and Theorem 5 remain valid in this case as well.

Please also remark that $\bar{\mathcal{V}}$ from Lemma 2 is now exactly $\bar{\mathcal{V}}$ because $r = m$.

Remark 7. If $A_2 = 0$ in Corollary 4 then the only constraint the GI must satisfy is $X_1 = I_r$. The solution $Xb$ is then

$$Xb = Q \begin{pmatrix} \bar{\mathcal{V}} \\ X_2 \end{pmatrix}$$

where for the choice of $X_3$ we have full flexibility. The conclusion in Theorem 5 can then be strengthen to “$\text{Supp}(x) = \mathcal{I} \cup \pi_Q(\text{Supp}((Pb)[1, r]))$”, as we did in Remark 4.
Remark 8. If we take
\[ PAQ = (A_1 \ I_r) \] (23)
then the blocks of the GI \( X \) should satisfy
\[ A_1X_1 + X_2 = I_r. \] (24)

The discussion on the solution \( Xb \) is then as in Remark 4 with \( Pb \) instead of \( (Pb)_{[1,r]} \).

A conclusion similar to that in Remark 4 can be drawn when \( A_1 = 0 \).

Remark 9. A similar discussion to that above stands for the case when \( \text{rank}(A) = r = n < m \).

If we assume that
\[ PAQ = \begin{pmatrix} I_r \\ A_2 \end{pmatrix} \] (25)
then any GI \( X \) of \( A \) is of the form
\[ X = Q(X_1 \ X_2)P, \] (26)
where \( X_1 \) and \( X_2 \) verify the equation
\[ X_1 + X_2A_2 = I_r. \] (27)

The solution \( Xb \) has the form \( Xb = Q(X_1(Pb)_{[1,r]} + X_2(Pb)_{[r+1,m]}) \) (please also see Lemma 4).

If we take
\[ PAQ = \begin{pmatrix} A_1 \\ I_r \end{pmatrix} \] (28)
then the blocks of the GI \( X \) should satisfy
\[ A_1X_1 + X_2 = I_r. \] (29)

4.1.4. Summarizing the results

The table in Figure 1 summarizes the form of solutions for the case of rank deficient and full rank matrices. The fourth column of the table also includes information about the matrices' sizes to help the reader get a pictorial view of them.

Theorem 7 (and the cases deduced from it) is vital in enumerating the solutions defined by the GI. We want to emphasize this here. Suppose that the GI is as in Figure 1 row 7. Let \( b \) be a non-zero vector. The matrix \( X_1 \) can be chosen so that \( X_1b \) has any distribution of its non-zero components. Therefore, to obtain a specific distribution or weight of \( X_2 \), we will have to look for a vector \( z_1 \) so that \( b - A_1z_1 \) is what we want. We can then easily compute \( X_1 \) to satisfy the relation \( X_1b = z_1 \).

4.2. The subspace weight problem and a generic GID solver for it

An important sub-problem of CWP is the one obtained by considering \( b = 0 \).

**Subspace Weight Problem (SWP)**

**Instance:** \( A \in M_{m,n}(\mathbb{F}) \) and positive integer \( t \), where \( \mathbb{F} \) is a finite field;

**Question:** Is there any solution \( x_0 \in \mathbb{F}^n \) to \( Ax = 0 \) such that \( |x_0| \leq t \)?
In coding theory, SWP is usually related to the low weight codeword problem that asks to find a codeword of small weight, of the code whose parity-check matrix is $A$ (more details about it are provided later in Section 5.1).

Even if SWP is a hard sub-problem of CWP, which is NP-complete, it is not immediately apparent that it is also NP-complete. It was conjectured in [7] that it is NP-complete, but the proof was later provided in [66].

Using GIs to solve SWP only gives us the solution $X0 = 0$. Characterizing the kernel of the matrix $A$ by $\mathcal{N}(A) = \mathcal{R}(I - XA)$, where $X$ is a GI of $A$, could be a solution. It requires sampling vectors $c$ and computing values $(I - XA)c$.

We will present below a method that we believe offers a good potential to approach the problem.

**Theorem 6.** Let $A \in \mathcal{M}_{m,n}(\mathbb{F}_q)$ with full row rank, $b \in \mathcal{R}(A)$ with $b \neq 0$, and $X \in \mathcal{GI}(A)$. Then,

$$\mathcal{N}(A) = \{(Y - X)b \mid Y \in \mathcal{GI}(A)\}. \quad (30)$$

| Transformation $PAQ$ | GI $X$ | Solution $Xb$ |
|---------------------|--------|---------------|
| 1. $(I_r A_2)$     | $Q\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}P$  
$X_1 + A_2X_3 = I_r$ | $Q\begin{pmatrix} X_1b' \\ r \\ X_3b' \\ n-r \end{pmatrix}$ |
| 2. $(I_r 0)$       | $Q\begin{pmatrix} I_r & X_2 \end{pmatrix}P$  
$X_1 + A_2X_2 = I_r$ | $Q\begin{pmatrix} r \\ X_2b' \\ n-r \end{pmatrix}$ |
| 3. $(I_r A_2)$     | $Q\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}P$  
$X_1 + A_2X_2 = I_r$ | $Q\begin{pmatrix} X_1b' \\ r \\ X_2b' \\ n-r \end{pmatrix}$ |
| 4. $(I_r 0)$       | $Q\begin{pmatrix} I_r & X_2 \end{pmatrix}P$  
$X_1 + A_2X_2 = I_r$ | $Q\begin{pmatrix} b' \\ X_2b' \\ n-r \end{pmatrix}$ |
| 5. $(A_1 I_r)$     | $Q\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}P$  
$A_1X_1 + X_3 = I_r$ | $Q\begin{pmatrix} X_1b' \\ n-r \\ X_3b' \\ r \end{pmatrix}$ |
| 6. $(0 I_r)$       | $Q\begin{pmatrix} X_1 & X_3 \\ I_r & X_4 \end{pmatrix}P$  
$A_1X_1 + X_3 = I_r$ | $Q\begin{pmatrix} X_1b' \\ n-r \\ r \end{pmatrix}$ |
| 7. $(A_1 I_r)$     | $Q\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}P$  
$A_1X_1 + X_2 = I_r$ | $Q\begin{pmatrix} X_1b' \\ n-r \\ X_2b' \\ r \end{pmatrix}$ |
| 8. $(0 I_r)$       | $Q\begin{pmatrix} X_1 & I_r \end{pmatrix}P$  
$A_1X_1 + X_2 = I_r$ | $Q\begin{pmatrix} X_1b' \\ r \\ X_2b' \\ n-r \end{pmatrix}$ |

Figure 1: $A \in \mathcal{M}_{m,n}(\mathbb{F}_q)$, $b \in \mathbb{F}^m$, $b = Pb$, $b' = (Pb)[1,r]$
Proof. It is straightforward to see that \( \{(Y - X)b \mid Y \in \mathcal{GI}(A)\} \subseteq \mathcal{N}(A) \).

Conversely, if \( v \in \mathcal{N}(A) \), then \( x = Xb + v \) is a solution to \( Ax = b \) (Theorem 2). On the other side, there exists a GI \( Y \) of \( A \) such that \( x = Yb \) (Theorem 4). Combining the two forms of \( x \) we get \( v = (Y - X)b \).

The proof of Theorem 6 uses Theorem 4. One can prove Theorem 6 first, based on a counting argument as in the proof of Theorem 4 and then derive Theorem 4.

The shape of \( A \)'s kernel elements in Theorem 6 depends on the shape of the GIs. Therefore, it depends on the transformation applied to the matrix \( A \) (please see our discussion on this topic at the end of Section 3.2). Corollary 5 below illustrates our discussion.

Corollary 5. Let \( A \in \mathcal{M}_{m,n}(\mathbb{F}_q) \) with full row rank, \( b \in \mathcal{R}(A) \) with \( b \neq 0 \), and \( PAQ = (V \ I_m) \) be a transformation of \( A \). Then,

\[
\mathcal{N}_{P,Q,b}(A) = \left\{ \left[ \begin{array}{c} Z \\ -VZ \end{array} \right] Pb \mid Z \in \mathcal{M}_{n-m,m}(\mathbb{F}_q) \right\}
\]  \hspace{1cm} (31)

Proof. Let \( X = Q \left[ \begin{array}{c} X_1 \\ I_m - VX_1 \end{array} \right] P \) be an arbitrary but fixed GI of \( A \). Then, according to Theorem 6

\[
\mathcal{N}_{P,Q,b}(A) = \{(Y - X)b \mid Y \in \mathcal{GI}(A)\}.
\]

For any \( Y \in \mathcal{GI}(A) \) there exits \( Y_1 \) such that \( Y = Q \left[ \begin{array}{c} Y_1 \\ I_m - VX_1 \end{array} \right] P \). Then,

\[
(Y - X)b = Q \left[ \begin{array}{c} Y_1 - X_1 \\ -V(Y_1 - X_1) \end{array} \right] Pb.
\]  \hspace{1cm} (32)

When \( Y_1 \) spans \( \mathcal{M}_{n-m,m}(\mathbb{F}_q) \), \( Z = Y_1 - X_1 \) spans the same space because \( X_1 \) is fixed. So, we get the result. \( \square \)

Theorem 6 allows us to design algorithms for SWP similar to those for CWP. Algorithm 2 below is a generic GID solver for SWP.

Algorithm 2 GID solver for SWP

1: function SWGLSOLVE\((A, t)\)
2: repeat
3: Choose a transformation \((P, Q)\) of \( A\);  
4: Choose \( b \in \mathcal{R}(A) \) with \( b \neq 0\);  
5: \( v \leftarrow \mathcal{N}_{P,Q,b}(A) \) until \(|v| \leq t\) or no more sampling is allowed;  
6: until a solution \( v \) is found or no more transformation is allowed;  
7: return solution \( v \) or “fail”.

It should be understood that steps 3 and 5 in Algorithm 2 are performed under various strategies, each leading to a variant of this generic algorithm. The vector \( b \) can be obtained by making the product between \( A \) and a randomly chosen vector \( x_0 \). \( b \) must be non-zero and preferably have as little weight as possible to reduce computational costs.

Algorithm 2 will be illustrated on Lee-Brickell, Leon, and Stern’s approaches in Section 5.
5. Information set decoding and GID decoding

In this section, we will show that many of the information set decoding (ISD) techniques used to solve the syndrome decoding problem in the theory of linear codes fit very well into the GID approach. In addition to the fact that the use of the GI provides a clearer overview of the problem, it also allows for improvements to the ISD techniques proposed so far.

5.1. Information set decoding as a cryptographic attack

“Information set decoding” refers to a class of techniques used in coding theory, especially in decoding problems, based on the concept of an information set. To better understand things, we will make a brief historical foray into developing the concept of information set decoding and its connection with cryptography.

The concept of information set was introduced in 1962 by Prange \[55\] in its attempt to propose a method of decoding cyclic codes. The basic idea was that, given a word \(w\) to be decoded, to build a set of code words that match \(w\) on a certain set of positions called information set. Formally, the concept of information set is as follows. A \textit{systematic code} is an \([n, k]\) code in which every code word can be separated into \(k\) information symbols and \((n-k)\) check symbols. The information symbols are identical with those of the source message before encoding. Thus, the process of encoding a systematic code involves the insertion of \((n-k)\) check symbols among the information symbols. The insertion positions must be the same for all the code words in the code. The set of positions of the information symbols is called an \textit{information set}. Every linear code can be arranged to be systematic.

Generally speaking, the techniques based on the concept of information set locate a potential information-set in a generator matrix, parity-check matrix, codeword, and perform various processing on it. Next, we will see the connection between these techniques and cryptanalysis.

Public-key cryptography was born in May 1975 (according to \[21\]) as the “child of two problems”, key distribution and digital signature. Diffie and Helmann’s 1976 paper \[22\] has hugely impacted cryptography and information security by proposing the key exchange method known today as the \textit{Diffie-Hellman key exchange} and suggesting the public-key cryptosystem and digital signature concepts. One immediately noted the importance and necessity of one-way functions to design public-key cryptographic primitives. It is not surprising that the 1976-1978 period was characterized by an effervescent proposal of public-key cryptographic primitives. Thus, in March 1977 (published May 1978), Berlekamp et al. showed that the problem of general decoding linear codes is NP-complete. In Jan-Feb 1978, McEliece proposes a public-key cryptosystem that bases its security on the problem of general decoding \[43\]. For a long time, this cryptosystem was seen only as an “alternative to public-key cryptography based on number theory”. Things changed significantly when it was understood that this cryptosystem could provide security against quantum attacks, which is not the case with the systems based on factorization or discrete logarithm. This fact has led to McEliece’s cryptosystem receiving much attention lately, being re-evaluated, re-analyzed, and looking for ways to make it more efficient.

\textbf{Cipher 1} (McEliece’s cryptosystem \[43\]). Let \(G \in M_{k,n}(\mathbb{F}_2)\) be a generator matrix for a \(t\)-error correcting \([n, k]\) linear code. Choose \(S \in \text{GL}_k(\mathbb{F}_2)\) and \(P \in \text{S}_n(\mathbb{F}_2)\) and compute \(\bar{G} = SGP\). \(\bar{G}\) is the public key. To encrypt a message \(m \in \mathbb{F}_2^n\) randomly generate an \(n\)-bit vector \(e\) with \(|e| \leq t\) and compute
\[
e' = m^t \bar{G} + e'.
\]
To decrypt \(c\), compute \(e'^t P^{-1}\) and then use a decoding algorithm for \(G\) to get rid of \(e'^t P^{-1}\) (remark that \(e'^t P^{-1}\) has Hamming weight at most \(t\)) and recover \(m^t S\). Finally, get \(m\) by means of \(S^{-1}\).
McEliece’s security analysis of the proposed cryptosystem links to information set decoding, even if not explicitly. Thus, he says in [43]: “A more promising attack is to select \( k \) of the \( n \) coordinates randomly in hope that none of the \( k \) are in error, and based on this assumption, to calculate \( u \).” The method was not much exploited by McEliece but is re-discussed in [57] and [1].

Let \( A_I \) denote the restriction of the matrix \( A \) to a given set \( I \) of column positions, that is, the matrix obtained from \( A \) by removing all columns whose index is not in \( I \). Similarly define \( v^I \) for vectors \( v \). Assume now that \( I \) has cardinality \( k \). Then, one can easily obtain

\[
c^I = m^I \tilde{G} + e^I.
\]

If \( I \) is a set of error free positions (that is, \( I \) is an information set), then \( e^I = 0^t \) and so, one can recover \( m \) by using \(( \tilde{G}^I )^{-1} \). As a result, the attack consists of repeating the following procedure:

- Randomly select \( k \) positions in the ciphertext and restrict it to them;
- Apply the inverse of the matrix \( \tilde{G}^I \).

If these \( k \) positions are error-free (that is, they form an information set), then the message \( m \) is obtained. But how can we know that the resulting message is \( m \)? We will not go into details on this question, but we refer the reader to [38] for an answer.

Let us consider now the parity-check matrix \( \tilde{H} \in M_{n-k,n}(\mathbb{F}_2) \) associated to \( \tilde{G} \). If we multiply \((33)\) to the right by \( \tilde{H}^t \), we obtain

\[
e^t \tilde{H}^t = s^t,
\]

where \( s^t = c^t \tilde{H}^t \) is the syndrome of \( c^t \) and \( e \) is the unknown vector.

The above attack now focuses on determining the error instead of determining the message. But this is a specific attack on Niederreiter’s cryptosystem [48].

**Cipher 2** (Niederreiter’s cryptosystem [48]). Let \( H \in M_{n-k,n}(\mathbb{F}_2) \) be a parity-check matrix for a \( t \)-error correcting \([n, k]\) linear code. Choose \( S \in GL_k(\mathbb{F}_2) \) and \( P \in S_n(\mathbb{F}_2) \) and compute \( H = SHP \). \( H \) is the public key. To encrypt \( e \in \mathbb{F}_2^n \) of weight \( t \), compute

\[
c = H e.
\]

To decrypt \( c \), compute \( S^{-1}c \) and then use a decoding algorithm to recover \( Pe \). Finally, get \( e \) by means of \( P^{-1} \).

So, what we have described above shows that if Niederreiter’s cryptosystem can be easily broken, then McEliece’s cryptosystem can be easily broken, and vice versa (for more details see [40]). As a result, solving matrix equation \((35)\) becomes critical to the security of both cryptosystems. Equation \((35)\) is an instance of the SDP in the theory of linear codes.

**Syndrome Decoding Problem (SDP)**

**Instance:** \( H \in M_{n-k,n}(\mathbb{F}_q) \) of full rank, syndrome \( s \in \mathbb{F}_q^{n-k} \), and positive integer \( t \);

**Question:** Is there a solution \( e \) of Hamming weight at most \( t \) to the equation \( He = s \)?

SDP is a particular case of CWP. Recall that, given an \([n, k]\) linear code \( C \), a generator matrix \( G \) and a parity-check matrix \( H \) of \( C \), we may write

\[
C = \{ m^t G \mid m \in \mathbb{F}_2^k \} = \{ c^t \in \mathbb{F}_2^n \mid Hc = 0 \}
\]

\[
C = \{ m^t G \mid m \in \mathbb{F}_2^k \} = \{ c^t \in \mathbb{F}_2^n \mid Hc = 0 \}
\]

18
We notice that the determination of a codeword of weight less than or equal to some positive integer $t$ is a sub-problem of SDP (when the syndrome is zero) or, more appropriate, it is a particular case of SWP. This is often met as the low weight codeword problem or the minimum weight codeword problem, even if “low weight” is not necessarily “minimum weight”.

**Low Weight Codeword Problem (LWP)**

**Instance:** Linear code $\mathcal{C}$ specified by $G$ or $H$ and positive integer $t$;

**Question:** Is there a nonzero codeword $c \in \mathcal{C}$ such that $|c| \leq t$?

LWP is NP-complete, while the computational version of it is NP-hard [66]. This problem, both in decisional and computational form, is important in coding theory and cryptography. For instance, easily solving it leads to an attack on McEliece and Niederreiter’s cryptosystems. Indeed, if $c$ is a McEliece ciphertext, then adding $c$ to the generator matrix $\bar{G}$ leads to a generator matrix $\tilde{G}$ for a $[k+1, n]$ linear code. In addition, there is a single codeword $d \in (G) \cap (\bar{G})$ with Hamming distance $d(c, d) = t$. As a result, $|c - d| = t$. Moreover, no other codeword has weight $t$. Once $c - d$ is computed, knowing $c$, we get $d$ which is undoubtedly the original message.

Over time, many algorithms for finding low-weight binary codewords have been proposed, based on the concept of information set (although this has not always been explicitly stated): [39, 62, 11, 27, 3, 41, 5, 42]. All these algorithms have been developed for the binary case. But there are also extensions to arbitrary finite fields, such as [13, 54, 44, 33, 47, 68, 34].

Throughout the next sections, we will denote by $(H, s, t)$ (or $(H, t)$) a generic instance of SDP (LWP), where $H \in M_{n-k,n}(\mathbb{F}_q)$ is a full rank matrix, $s \in \mathbb{F}_q^{n-k}$, and $t \leq n$ is a positive integer. We will also use the notation $r = n - k$, which is the rank of $H$.

### 5.2. Prange’s approach

We will describe in this section Prange’s approach [55] to SDP. We will then show that this is a particular case of our GID approach. We will show that Prange’s ISD algorithm iterates through a smaller solution space than the total solution space. At the end of the section, we suggest improvements to Prange’s method.

#### 5.2.1. Prange’s algorithm

The method proposed by Prange in the context of cyclic codes consists of the following. Having a word with possible errors, which we must decode, guess the error-free positions, and move them to the left, thus obtaining the error positions on the right. This method was also applied to McEliece’s cryptosystem (see Section 5.1) except that the cryptotext was restricted to error-free positions to decode it using the inverse of the public matrix (restricted to the same positions).

When using the parity-check matrix, Prange’s method is equivalent to determining the error in an equation $He = s$. In this case, packing the error-free positions to the left of $H$ means translating the error bits in $e$ to the base of the vector (leaving its top without error). This results immediately from the fact that the equation $He = s$ is equivalent to $(HQ)(Q^{-1}e) = s$, for any permutation matrix $Q$.

Specifically, Prange’s method in this context is as follows ($(H, s, t)$ is an instance of SDP):

1. Guess an information set $I \subseteq \{1, \ldots, n\}$ of size $k$ and assume that each $i \in I$ is an error-free position;
2. Guess a permutation matrix $Q \in S_n(F)$ and compute a non-singular matrix $P \in \text{GL}_r(F)$ so that all columns indexed by $I$ are packed to the left and $H$ is transformed into

$$PHQ = (V \quad I_r)$$  \hspace{1cm} (37)

for some matrix $V$, where $r = n - k$ (if this is not possible, a different permutation matrix is chosen). Remark that the information set $I$ in $H$ becomes now the set of the first $k$ positions in $PHQ$ ($V$ may not coincide with the sub-matrix of $H$ determined by $I$ due to the use of $P$);

3. The equivalences

$$Hx = s \iff P^{-1} (V \quad I_r) Q^{-1} x = s \iff (V \quad I_r) z = Ps$$

show that $x$ is a solution to $Hx = s$ if and only if $z = Q^{-1} x$ is a solution to

$$(V \quad I_r) z = Ps.$$  \hspace{1cm} (38)

However, (38) becomes

$$(V \quad I_r) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = Ps$$

and thus

$$Vz_1 + z_2 = Ps,$$  \hspace{1cm} (40)

where $z_1 \in F^k$, $z_2 \in F^r$, and $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$.

The assumption that $z_1$ is error-free because it corresponds to the information set after permutation, leads to $z_1 = 0$ and $z_2 = Ps$. Re-permuting $z$ we get $x = Q \begin{pmatrix} 0 \\ Ps \end{pmatrix}$, which is a solution to $Hx = s$. Moreover, remark that $|x| = |z| = |Ps|$ because $Q$ is a permutation.

Let us refer to a pair $(P, Q)$ of matrices as in (37) as a Prange transformation of $H$. A solution obtained employing a Prange transformation $(P, Q)$ will be referred to as a Prange $(P, Q)$-solution. Clearly, it is uniquely determined by $(P, Q)$. A Prange solution is a Prange $(P, Q)$-solution, for some Prange transformation $(P, Q)$.

5.2.2. Prange’s approach as a GI-based approach

Prange’s approach is a particular case of our generic GID solver for CWP (Algorithm 1). To see that, let us consider the transformation (37). According to Corollary 4 and Remark 8 each GI of $H$ has the form

$$X = Q \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} P,$$  \hspace{1cm} (41)

where $X_1$ and $X_2$ are arbitrary matrices that verify $V X_1 + X_2 = I_r$ (please see the notation from (37)). If we take $X_1 = 0$ and $X_2 = I_r$, $x = Xs = Q \begin{pmatrix} 0 \\ Ps \end{pmatrix}$ is a solution to $Hx = s$. This is in fact the Prange $(P, Q)$-solution.

We can thus say that Prange’s algorithm iterates through a particular subset of GIs $X$ (the matrix $Q$ in the transformation $(P, Q)$ is a permutation, $X_1 = 0$, and $X_2 = I_r$) until it finds one with $|Xs| \leq t$. The fact that $Q$ is a permutation ensures that the Hamming weight of the solution is the Hamming weight of $Ps$. Therefore, Prange’s algorithm is a particular case of our GI-based approach, fact that is formally stated in the following.
Corollary 6. Let $H \in M_{n-k,n}(F)$ be a full rank matrix and $s \in F^{n-k}$. Prange’s algorithm generates solutions to the equation $Hx = s$ of the form $Xs$ with

$$X \in \left\{ Q \begin{pmatrix} 0 \\ I_r \end{pmatrix} P \mid (P, Q) \in GL_r(F) \times S_n(F), \ (\exists V : PHQ = (V \ I_r)) \right\}. \tag{42}$$

The question now is: does the space of the GIs computed by Prange’s algorithm cover the entire space $GI(H)$ or not? This question is justified because Corollary 4 shows us how to compute $GI(H)$ using only a transformation $(P, Q)$ arbitrarily chosen from $GL_r(F) \times GL_n(F)$. On the other side, Prange’s algorithm iterates on all transformations in $GL_r(F) \times GL_n(F)$ and, for each transformation $(P, Q)$, it computes the generalized inverse $Q \begin{pmatrix} 0 \\ I_r \end{pmatrix} P$.

To answer this question we need an intermediate result regarding the GIs of a matrix.

Theorem 7. Let $H \in M_{n-k,n}(F)$ be a full rank matrix. Then, the set $GI(H)$ consists of all matrices $X = Q \begin{pmatrix} 0 \\ I_r \end{pmatrix} P$, where $(P, Q) \in GL_{n-k}(F) \times GL_n(F)$ is a transformation of $H$ such that $PHQ = (V \ I_r)$, for some $V \in M_{n-k,k}(F)$.

Proof. Let $H \in M_{n-k,n}(F)$ be a full rank matrix and let $r$ denote $n-k$.

It is straightforward to show that all matrices $X$ as in the theorem are GIs of $H$.

Let $X$ now be a GI of $H$. There exists a transformation $(P, Q) \in GL_r(F) \times GL_n(F)$ such that $PHQ = (V \ I_r)$, for some matrix $V$. According to Remark 8 there are two matrices $X_1$ and $X_2$ such that $X = Q \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} P$ and $VX_1 + X_2 = I_r$. We have to show that $X$ can be written as $X = Q \begin{pmatrix} 0 \\ I_r \end{pmatrix} \bar{P}$, for some transformation $\bar{P}HQ = (V \ I_r)$.

We prove first that the rank of the matrix $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ is $r$. This simply follows from the relation

$$(V \ I_r) \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = I_r$$

and the well-known rank inequality $\text{rank}(AB) \leq \text{min}\{\text{rank}(A), \text{rank}(B)\}$.

Since, $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ is a full column rank matrix, we can apply a Gaussian column elimination, which is equivalent to computing two operator matrices $Z$ and $Y$ such that

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = Z \begin{pmatrix} 0 \\ I_r \end{pmatrix} Y.$$

Now it follows that

$$X = Q \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} P = QZ \begin{pmatrix} 0 \\ I_r \end{pmatrix} YP = Q \begin{pmatrix} 0 \\ I_r \end{pmatrix} \bar{P},$$

where $\bar{Q} = QZ$ and $\bar{P} = YP$. 

21
It remains to show that there exists a matrix $\bar{V}$ such that $PHQ = (\bar{V} \quad I_r)$. Decompose $Z$ into blocks $Z = (Z_1 \quad Z_2 \quad Z_3 \quad Z_4)$ of appropriate sizes and write:

$$PHQ = Y(\bar{V}I_r)Z$$

$$= Y(V \quad I_r) \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{pmatrix}$$

$$= (Y(VZ_1 + Z_3) \quad Y(VZ_2 + Z_4))$$

But,

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = Z \begin{pmatrix} 0 \\ I_r \end{pmatrix} Y = \begin{pmatrix} Z_2Y \\ Z_4Y \end{pmatrix}$$

and the matrix equation $VX_1 + X_2 = I_r$ leads then to $VZ_2 + Z_4 = Y^{-1}$. This proves that

$$PHQ = (\bar{V} \quad I_r),$$

where $\bar{V} = Y(VZ_1 + Z_3)$.

In what follows we will discuss several consequences of Theorem 7. First, it is good to face $\mathcal{GI}(H)$ both according to Remark 8 and to Theorem 7:

- **Computing $\mathcal{GI}(H)$ by fixing a transformation**: According to Remark 8, for a given transformation $(P, Q) \in GL_r(F) \times S_n(F)$ with $PHQ = (V \quad I_r)$ for some $V$, we have

$$\mathcal{GI}(H) = \left\{ Q \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} P \mid VX_1 + X_2 = I_r \right\};$$

- **Computing $\mathcal{GI}(H)$ by fixing a GI**: According to Theorem 7, we have

$$\mathcal{GI}(H) = \left\{ Q \begin{pmatrix} 0 \\ I_r \end{pmatrix} P \mid (P, Q) \in GL_r(F) \times GL_n(F), \ (\exists V : PHQ = (V \quad I_r)) \right\}.$$

**Remark 10.** In the proof of Theorem 7, it is not guaranteed that the matrix $Z$ is a permutation. As a result, $Q$ may not be a permutation. This shows us that Prange’s algorithm does not iterate through all the GIs of the matrix $H$, but only on those GIs $Q \begin{pmatrix} 0 \\ I_r \end{pmatrix} P$ for which $Q$ is a permutation.

**Remark 11.** Any Prange-like approach that generates solutions to the equation $Hx = s$ of the form $Xs$ with

$$X \in \left\{ Q \begin{pmatrix} X_1 \\ I_r - VX_1 \end{pmatrix} P \mid (P, Q) \in GL_r(F) \times S_n(F), \ (\exists V : PHQ = (V \quad I_r)) \right\}.$$

where $X_1$ is fixed, does not cover the set of all solutions to $Hx = s$. This can be obtained by a similar proof line to Theorem 5. As we shall see in the next subsections, this is the case of many ISD variants, such as, Lee-Brickell, Leon, Stern, etc.

**Remark 12.** According to the GI-based framework, Prange’s algorithm chooses one GI for each transformation it calculates. This can be improved in the spirit of Algorithm 2, as follows:
1. Choose a Prange transformation $PHQ = (V \ I_r)$ to take advantage of the fact that $Q$ is a permutation;

2. Iterate through some of the GIs

$$X = Q \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} P$$

with $V X_1 + X_2 = I_r$, until a solution $Xs$ with $|Xs| \leq t$ is found, or go to step 1 and choose another Prange transformation.

The second step above offers the advantage of computing several inverses using the same transformation. The idea is first to choose $X_1$ and compute $X_2$ by $X_2 = I_r - V X_1$. We note that $X_1s$ must be different from 0 because, otherwise, $X_2s = Ps$, which leads to a Prange solution.

5.3. Lee-Brickell’s approach

As mentioned in Section 5.1, a possible attack on McEliece’s cryptosystem can be mounted guessing the error-free positions. An improvement of this attack has been proposed in [38], where non-error-free positions are also allowed among the selected positions. This can equivalently be translated into Prange’s formalism, and ours too. Details are provided below. In addition, we will develop a Lee-Brickell-like method for LWP.

5.3.1. Lee-Brickell’s algorithm

In Prange’s ISD algorithm, error positions are only allowed outside the positions in the information set. With the notation of (40), this corresponds to the case $z_1 = 0$. Lee-Brickell’s approach [38] allows $z_1$ to contain errors, say at most $p$, where $p$ is a parameter to be chosen depending on $n$, $r$, and $t$. Once $z_1$ is chosen in this way, the equation (40) leads to $z_2$.

For $P$ and $Q$ given as in (37), the procedure above is iterated for each choice of $p$ positions from the first $k$ positions until the solution’s Hamming weight is less than or equal to $t$ (remark that the matrix $Q$ does not change the Hamming weight by pre-multiplication).

5.3.2. Lee-Brickell’s approach as a GI-based approach

Lee-Brickell’s approach is a particular case of the GI-based approach. We will discuss two cases here.

Case 1: Lee-Brickell’s approach for SDP. It is trivial to describe Lee-Brickell’s approach in terms of GIs for SDP. In fact, we only have to start from the generalized inverse calculated as in Prange’s approach and impose the new requirement. More precisely, given the GI (11) we do as follows:

- Given $p$ positions among the first $k$ positions, we choose the components of $X_1$ so that on those $p$ positions, $X_1 s$ has only 1 and 0 in rest. This always can be done, and even efficiently, when $s$ has at least one non-zero component (please see the proof of Theorem 5);

- The matrix $X_2$ can then be obtained from the equation $V X_1 + X_2 = I_r$.

So, Lee-Brickell’s approach is just a particular case of our generic GID solver for CWP (Algorithm 1).
Case 2: Lee-Brickell’s approach for LWP. As far as we know, Lee-Brickell’s approach has not been used to find solutions to LWP. However, one can quickly implement a Lee-Brickell-like strategy in Algorithm 2 to get a solution to this problem.

Algorithm 2 with transformations as in Corollary 5 outputs solutions of the form

\[ v = Q \left( \frac{Z}{-VZ} \right) Pb \]

(please see Algorithm 2 for notation).

As \( b \neq 0 \), we can easily choose \( Z \) to get any weight and distribution of the non-zero components of \( ZPb \). So, what we have to do is to choose a distribution that minimizes the Hamming weight of \(-V(ZPb)\). We may try any value of \( p \), starting with one up to some reasonable upper bound.

5.4. Leon’s approach

In 1988, Leon proposed a probabilistic algorithm for computing codewords of weight as small as possible from large error correction codes [39]. Without explicitly mentioning or referring to [35], Leon’s algorithm uses the concept of the information set. We will describe it in the following.

5.4.1. Leon’s algorithm

Let \( G \) be a generator matrix of an \([n, k]\) linear code for which we want to compute a low weight codeword. The basic idea of Leon’s algorithm is to repeat the following two steps until a solution is obtained or some halt criterion is fulfilled:

1. Apply a random permutation \( Q \) to its columns, followed by row transformations \( P \), so that the matrix \( G \) is brought in the form

\[ G' = PGQ = \begin{pmatrix} I_e & Z & C \\ 0 & 0 & D \end{pmatrix} \]

where the total length of the first two blocks is \( e + \ell \);

2. The following procedure is repeated on \( v \in \langle \begin{pmatrix} 0 & 0 & D \end{pmatrix} \rangle \) until the obtained vector has the desired weight, or all \( v \) were processed:

   • Add to \( v \) zero, one, or two rows from the first \( e \) rows but with the requirement that their sum on the positions \( \{e + 1, \ldots, e + \ell\} \) has weight 0.

   Step 2 can be implemented by a systematic enumeration of the space \( \langle \begin{pmatrix} 0 & 0 & D \end{pmatrix} \rangle \) or by random selection from it. In both cases, the procedure starts with a target weight \( t \), updated each time a lower weight vector is obtained (also saving this vector until the next weight update, if necessary). If the output is the vector \( v \), the codeword induced by it is \( c = Q^{-1}v \). We notice that the vector \( v \)’s weight on the first \( e \) positions will always be less than or equal to 2.

Ignoring \( Q \), which is a permutation and does not change the weight, we can say that Leon’s algorithm determines codewords (solutions of LWP) that have low weight, weight at most two on the first \( e \), and zero on the following \( \ell \) positions. Since this vector must be a solution of the equation \( Hx = 0 \), where \( H \) is the parity-check matrix of the code generated by \( G \), we deduce that Leon’s algorithm is looking for low weight solutions of LWP. These solutions have weight at most two on the first \( e \), and zero on the following \( \ell \) positions. Since the transformation from step 1 is a Prange transformation, we can view this algorithm as a variant of Lee-Brickell’s algorithm in which the solution is as above.
We can make a step further and generalize Leon’s algorithm to the SDP. To understand how this is done, let us consider \((H, s, t)\) an instance of this problem, \(PHQ = (V \ I_r)\) a Prange transformation of \(H\), and \(p\) and \(\ell\) two positive integers.

Using the notation in Section 5.2.1 we divide the matrices \(V, z_2,\) and \(\bar{s}\) into two blocks each, \(V = \begin{pmatrix} V_1 & V_2 \end{pmatrix}\), \(z_2 = \begin{pmatrix} z_1' \\ z_2' \end{pmatrix}\), and \(\bar{s} = \begin{pmatrix} \bar{s}' \\ \bar{s}'' \end{pmatrix}\), where the first block contains the first \(\ell\) rows in each and the second block the rest of the rows. Then, (40) can be written as

\[
\begin{pmatrix} V_1 z_1 \\ V_2 z_1 \end{pmatrix} + \begin{pmatrix} z_2' \\ z_2'' \end{pmatrix} = \begin{pmatrix} \bar{s}' \\ \bar{s}'' \end{pmatrix}
\] (44)

The requirements of Leon’s algorithm are now \(|z_1| \leq p\) and \(z_2' = 0\). Remark also that \(z_1\) must additionally satisfy \(V_1 z_1 = \bar{s}'\). Assuming that all these are fulfilled, \(z_2''\) is determined from \(V_2 z_1 + z_2'' = \bar{s}''\).

5.4.2. Leon’s approach as a GI-based approach

Leon’s approach is a particular case of the GI-based approach. We will discuss two cases here.

**Case 1: Leon’s approach for LWP.** Let \((H, t)\) be an instance of LWP. Algorithm 2 with transformations as in Corollary 5 outputs solutions of the form \(v = Q \begin{pmatrix} Z \\ -VZ \end{pmatrix} Pb\) (please see Algorithm 2 for notation). We further decompose \(VZ\) into two parts

\[
v = Q \begin{pmatrix} Z \\ -V_1 Z \\ -V_2 Z \end{pmatrix} Pb
\]

where \(V_1\) contains the first \(\ell\) rows and \(V_2\) the other rows of \(V\).

As \(b \neq 0\), we can easily choose \(Z\) to get any distribution of the non-zero components of \(ZPb\). As a result, Algorithm 2 iterates through Prange transformations of \(H\), and for each transformation, iterates through some values of \(Z\) until it finds one with the properties \(|ZPb| \leq p, |V_1 ZPb| = 0,\) and \(|v| \leq t\) (please also see the comments at the end of Section 4.1.3).

**Case 2: Leon’s approach for SDP.** Let \((H, s, t)\) be an instance of SDP. Algorithm 1 will iterate through Prange transformations \(PHQ = (V \ I_r)\), and for each transformation it will iterate through GIs of the form

\[
X = Q \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} P,
\] (45)

where \(X_1\) and \(X_2\) are arbitrary matrices that verify \(VX_1 + X_2 = I_r\).

We further decompose \(VX_1 + X_2 = I_r\) as

\[
\begin{pmatrix} V_1 X_1 \\ V_2 X_1 \end{pmatrix} + \begin{pmatrix} X_1' \\ X_2' \end{pmatrix} = \begin{pmatrix} I_r' \\ I_r'' \end{pmatrix}
\]

where \(V_1, X_1',\) and \(I_r'\) contain the first \(\ell\) rows of \(V, X_2,\) and \(I_r,\) respectively, while \(V_2, X_2',\) and \(I_r''\) contain the other rows.
Now, what we have to do is to compute $X_1$, $X'_2$, and $X''_2$ under the supplementary requirements $X'_2\bar{s} = \mathbf{0}$ and $|X_1\bar{s}| \leq p$. This can be implemented in various ways. For example, assuming $\bar{s} \neq \mathbf{0}$, we can choose $X_1$ so that $X_1\bar{s}$ is any $(n - r)$-dimensional vector (see, for instance, the proof of Theorem 5). So suffice it to say that this is a vector with at most $p$ positions 1. Then, for each such possibility, we check whether the sum of $V_1$’s columns corresponding to positions 1 in $X_1\bar{s}$ gives us $\bar{s}'$ (see the notation in Section 5.4.1). If so, we keep $X_1\bar{s}$ and compute $X''_2$. This is, for instance, the method in [4]. Tacking in both cases (Leon’s approach for SDP/LWP) $p = 2$ gives the original version proposed by Leon.

5.5. Stern’s approach

Independent and almost at the same time with Leon, Stern proposed an algorithm to determine low-weight codewords [63] that use the concept of the information set. We will describe it below and show that it fits very well in the GI-based approach.

5.5.1. Stern’s algorithm

Unlike Leon, the codewords in Stern’s algorithm are determined by the parity-check matrix $H \in \mathcal{M}_{n-k,n}(\mathbb{F}_2)$, so as solutions of the equation $Hx = \mathbf{0}$. To determine a zero linear combination of $H$’s columns, Stern proposed the following algorithm that uses two parameters $p$ and $\ell$ that are optimized later (please see Figure 2 for an illustration of the algorithm, and consider $\bar{s} = \mathbf{0}$):

1. Select $r = n - k$ columns from $H$ and, by row elementary transformations, form the columns of the unit matrix. For readability, we assume that these columns are packed to the right because the columns permutation does not change the solution’s weight (we point out that this requirement is not present in Stern’s algorithm). As a result, a Prange transformation is applied to $H$, $PHQ = (V I_r)$. Then, $x$ is a solution to $Hx = \mathbf{0}$ if and only if $z$ is a solution to $(V I_r)z = \mathbf{0}$, where $z = Q^{-1}x$;

2. The rest of the columns are randomly distributed in two sets of approximately equal size. Let $I_1$ and $I_2$ be the index sets of these column sets. With the notation from the previous step, we can say that $V$ is divided into two blocks of approximately the same size, $V = (V_1 V_2)$;

3. Randomly choose a set $L$ of $\ell$ row position;

4. Choose $J_1 \subseteq I_1$ and $J_2 \subseteq I_2$ of equal size $p$ such that the sum of the columns with index in $J_1$ is equal to the sum of the columns with index in $J_2$ on the positions in $L$. This will ensure that the sum of all these columns is 0 on all positions in $L$;

5. On certain positions outside the set $L$, the sum of the columns with the index in $J_1 \cup J_2$ can be 1. For each such position $j$, the column from the unit matrix will be added with 1 on position $j$. If we denote by $J_3$ the set of all these positions, the sum of the columns with the index in $J_1 \cup J_2 \cup J_3$ will be 0;

6. So the vector that has 1 only on the positions in $J_1 \cup J_2 \cup J_3$, and otherwise only zero will be a codeword with weight $2p + |J_3|$.

Stern’s algorithm is a solution to LWP. We can trivially extend it to be a solution to SD instance $Hx = s$. The equation we have to solve now is $(V I_r)z = \bar{s}$, where $\bar{s} = Ps$ (please see the diagram in Figure 2). This implies that on the positions in $L$ we must have satisfied the relation

$$ (V_1z'_1)_L + (V_2z'_2)_L = \bar{s}_L $$

(46)
Choosing $z_1'$ and $z_1''$ in this way, $z_2$ is obtained from
\[ V_1 z_1' + V_2 z_1'' + z_2 = s \]  
(47)

It should be noted that $z_2$ will have one on the positions in $J_3$. The solution $x$ is obtained by $x = Qz$.

The computation of $z_1'$ and $z_1''$, which satisfies equation (46), can be done this way. For each subset $J_1 \subseteq I_1$ of size $p$, the value $V_1 z_1' + s$ is stored in a list $L$, where $z_1'$ has 1 on all positions in $J_1$, and 0 in rest. Then, whenever it is necessary to find $z_1''$, a subset $J_2 \subseteq I_2$ of size $p$ is chosen. If the equation (46) is fulfilled, then $z_2$ is obtained from (47).

5.5.2. Stern’s approach as a GI-based approach

Stern’s approach is a particular case of the GI-based approach. We will discuss two cases here.

Case 1: Stern’s approach for LWP. Algorithm 2 with the transformations from Corollary 5 will output solutions for LWP of the form
\[ v = Q \left( \begin{array}{c} Z \\ -VZ \end{array} \right) P b \]  
(please see Algorithm 2 for notation). As $b \neq 0$, we can easily choose $Z$ to get any distribution on the non-zero components of $ZP b$. This corresponds to $z_1$ in Stern’s original approach. The second component, $-VZ$, corresponds to $z_2$ in Stern’s approach (please also see the comments at the end of Section 4.1.4).

So, Algorithm 2 in this case will mainly iterate through $Z$’s values and transformations ($P, Q$).

Case 2: Stern’s approach for SDP. Stern’s approach for SDP is a particular case of GI-based approach (Algorithm 1). To see that, consider the transformation (37). Each GI of $H$ has the form
\[ X = Q \left( \begin{array}{c} X_1 \\ X_2 \end{array} \right) P, \]  
(48)

where $X_1$ and $X_2$ are arbitrary matrices that verify $VX_1 + X_2 = I_r$ (please see the notation from (37)). We further decompose the matrices $V$ and $X_1$ into two parts each, $V = (V_1, V_2)$.
and \( X_1 = \begin{pmatrix} X'_1 \\ X''_1 \end{pmatrix} \). Then, we arrive at the equation

\[
V_1 X'_1 + V_2 X''_1 + X_2 = I_r
\]  

The choice of matrices \( X'_1 \) and \( X''_1 \) can be made on principles similar to those in the previous section.

5.6. Finiasz-Sendrier’s approach

In [27], Finiasz and Sendrier propose a “generalization” of Stern’s approach, making a partial Gaussian elimination on the matrix. We will discuss this approach below.

5.6.1. Finiasz-Sendrier’s algorithm

The transformation used on \( H \) in Finiasz-Sendrier’s approach is

\[
PHQ = \begin{pmatrix} V_1 & 0 \\ V_4 & I_{r-\ell} \end{pmatrix},
\]

where \( V_1 \in M_{k+k}(\mathbb{F}), \) \( V_2 \in M_{r-r}(\mathbb{F}) \), and \( \ell \geq 0 \).

The equation \( PHQz = Ps \) leads then to

\[
\begin{align*}
V_1 z_1 &= \bar{s}_1 \\
V_3 z_1 + z_2 &= \bar{s}_2
\end{align*}
\]

where \( z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \), \( z_1 \in \mathbb{F}^{k+\ell} \), \( z_2 \in \mathbb{F}^{r-\ell} \), \( \bar{s} = P s = \begin{pmatrix} \bar{s}_1 \\ \bar{s}_2 \end{pmatrix} \), \( \bar{s}_1 \in \mathbb{F}^{\ell} \), and \( \bar{s}_2 \in \mathbb{F}^{r-\ell} \).

The vector \( z_1 \) is divided into two parts, \( z'_1 \) and \( z''_1 \), that are computed as in Stern’s approach, while \( z_2 \) is obtained from the second equation in (51). At the end, \( x = Qz \).

5.6.2. Finiasz-Sendrier’s approach and a GI-based approach.

As with the other approaches, Finiasz-Sendrier’s approach reduces to a GI calculation. Any GI of the matrix in (50) has the form \( (X_1 X_2 X_3 X_4) \), where \( X_1 \in M_{k+k+\ell}(\mathbb{F}), \) \( X_2 \in M_{k+k,r-\ell}(\mathbb{F}), \)

\( X_3 \in M_{r-\ell,k}(\mathbb{F}), \) \( X_4 \in M_{r-\ell,r}(\mathbb{F}) \), and the following properties hold:

\[
\begin{align*}
V_1 X_1 V_1 &= V_1 \\
V_1 X_2 &= 0 \\
(V_3 X_1 + X_3) V_1 &= 0 \\
V_2 X_2 + X_4 &= I_{r-\ell}
\end{align*}
\]

The general solution to the system \( PHQz = Ps \) will then be

\[
z = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} X_1 \bar{s}_1 + X_2 \bar{s}_2 \\ X_3 \bar{s}_1 + X_4 \bar{s}_2 \end{pmatrix}
\]

One can easily that \( z_1 \) and \( z_2 \) derived as in (53) also satisfy (51). Indeed, first remark that, according to (52), \( X_1 \) is a GI of \( V_1 \) and so \( X_1 \bar{s}_1 \) is a solution to \( V_1 z_1 = \bar{s}_1 \). Then,

\[
V_1 z_1 = V_1 (X_1 \bar{s}_1 + X_2 \bar{s}_2) = V_1 X_1 \bar{s}_1 + V_1 X_2 \bar{s}_2 = \bar{s}_1
\]
and
\[ V_3z_1 + z_2 = V_2(X_1s_1 + X_2s_2) + X_3s_1 + X_4s_2 \]
\[ = (V_3X_1 + X_3)s_1 + (V_3X_2 + X_4)s_2 \]
\[ = (V_3X_1 + X_3)V_1X_1s_1 + I_{r-l}s_2 \]
\[ = 0X_1s_1 + s_2 \]
\[ = s_2. \]

Several strategies of selecting matrices in \([14]\) could be imagined. For example, setting \(X_2 = 0\) and \(X_3 = -V_3X_1\) implies \(X_4 = I_{r-l}\) and so, the solution has the form
\[ z = \begin{pmatrix} X_1s_1 \\ -V_3X_1s_1 + s_2 \end{pmatrix} \]  
(54)

In this case the following procedure could be applied:

- Sample \(X_1 \in GI(V_1)\) s.t. \(|X_1s_1| \leq t\);
- If \(|-V_3X_1s_1 + s_2| \leq t - w\) then output \(z\). Else, go to the previous step.

5.7. Multiple decompositions

Let \((H, s, t)\) be an instance of SDP and \((V, I_r)\) a Prange transformation of \(H\). If we look at the GI-based approach of the Leon and Finiasz-Sendrier’s algorithms, we find that \(V\) is vertically decomposed into two other blocks. The GI-based approach of Stern’s algorithm requires the horizontal decomposition of \(V\) to do a “meet-in-the-middle” (“birthday matching”) on sub-blocks. We can combine these two decompositions as follows.

Let
\[ PHQ = \begin{pmatrix} V_1 & I_{\ell} & 0 \\ V_2 & 0 & I_{r-l} \end{pmatrix}, \]  
(55)

where \(\ell\) is a positive integer. It is straightforward to see that any GI of \(H\) has the form
\[ X = Q \begin{pmatrix} X_1 \\ -V_2X_1 \\ I_{r-l} - V_2X_2 \end{pmatrix} P, \]  
(56)

where \(X_1 \in M_{k,\ell}(\mathbb{F})\) and \(X_2 \in M_{k,r-\ell}(\mathbb{F})\) are matrices of appropriate sizes. A solution will be then of the form
\[ Xs = Q \begin{pmatrix} X_1s' + X_2s'' \\ s' - V_1X_1s' + (-V_2X_1s'') \\ -V_2X_2s' + (s'' - V_2X_2s'') \end{pmatrix}, \]  
(57)

where \(s = Ps, s' = s[1,\ell],\) and \(s'' = s[\ell+1,r-\ell]\).

Assuming that \(s' \neq 0\) and \(s'' \neq 0\), \(X_1s'\) and \(X_2s''\) may independently define any linear combinations \(V_1\)'s columns. Therefore, the middle horizontal row in \(Xs\) is a generalization of the corresponding part in Leon and Stern’s algorithms. In addition, no higher computational costs are required than in these algorithms and leave a door open to other approaches.

We can generalize even more. For instance, the transformation
\[ PHQ = \begin{pmatrix} V_1 & I_{\ell_1} & 0 \\ V_2 & 0 & I_{\ell_2} \\ V_4 & 0 & I_{r-(\ell_1+\ell_2)} \end{pmatrix} \]  
(58)
leads to GLs of the form

\[
X = Q \begin{pmatrix}
X_1 & X_2 & X_3 \\
I_1 - V_1X_1 & -V_1X_2 & -V_1X_3 \\
-I_2 - V_2X_1 & I_2 - V_2X_2 & -V_2X_3 \\
-I_3 - V_3X_1 & -V_3X_2 & I_r - (I_1 + I_2 + I_3) - V_3X_3
\end{pmatrix} P
\] (59)

If we decompose \( \bar{s} \) into three corresponding parts, \( \bar{s}' \), \( \bar{s}'' \), and \( \bar{s}''' \), and assume that each of them is different than 0, then \( X_1\bar{s}' \), \( X_2\bar{s}'' \), and \( X_3\bar{s}''' \) may independently define any linear combination of \( V_1 \), \( V_2 \), and \( V_3 \)'s columns.

6. GI-based decoding and Boolean constraint satisfaction

This section will look at the CW and SWPs as optimization problems and link them to the MIN-SAT problem. To have a clearer picture of these problems, we recall first the concept of an optimization problem. For details, the reader is referred to [3, 15].

**Definition 2.** An optimization problem is a tuple \( A = (I, \text{Sol}, \mu, \text{Opt}) \), where:

1. \( I \) is a set of instances;
2. \( \text{Sol} \) is a function that maps each instance \( x \) to a set of feasible solutions of \( x \);
3. \( \mu \) is a measure function that associates to each pair \( (x, y) \in I \times \text{Sol}(x) \) a positive integer regarded as a measure of the instance \( x \)'s feasible solution \( y \);
4. \( \text{Opt} \in \{ \text{min}, \text{max} \} \) is the optimization criterion (minimization or maximization).

Given an optimization problem as above, we denote by \( \text{Sol}^*(x) \) the set of optimal solutions of \( x \) (that is, those feasible solutions from \( \text{Sol}(x) \) that fulfill the optimization criterion \( \text{Opt} \)). Moreover, \( \mu^*(x) \) stands for the value of the optimal solutions of \( x \), when they exist.

Each optimization problem \( A \) defines three related problems:

1. **Constructive problem** \( A_C \): Given an instance \( x \in I \), compute an optimal solution and its measure;
2. **Evaluation problem** \( A_E \): Given an instance \( x \in I \), compute \( \mu^*(x) \);
3. **Decision problem** \( A_D \): Given an instance \( x \in I \) and a positive integer \( t \), decide the predicate

\[
A_D(x, t) = \begin{cases} 
1, & (\text{Opt} = \text{min} \land \mu^*(x) \leq t) \lor \\
\text{ otherwise} & (\text{Opt} = \text{max} \land \mu^*(x) \geq t) \\
0, & \text{ otherwise}
\end{cases}
\] (60)

The underlying language of \( A \) is the set \( \{(x, t) \in I \times \mathbb{N} \mid A_D(x, t) = 1\} \).

**Definition 3.** An optimization problem \( A = (I, \text{Sol}, \mu, \text{Opt}) \) belongs to the class NPO if:

1. Its set of instances is recognizable in polynomial time;
2. There exists a polynomial \( p \) such that for each instance \( x \), its feasible solutions have size at most \( p(|x|) \). Moreover, for each \( y \) of size at most \( p(|x|) \), it is decidable in polynomial time whether \( y \) is a feasible solution of \( x \);
3. The measure function is computable in polynomial time.

Now is the time to introduce the MIN-CWP and MIN-SWP.

| MIN COSET WEIGHT PROBLEM (MIN-CWP) |
|-------------------------------------|
| **Instance:** Matrix $A \in \mathcal{M}_{m,n}(F)$ with full row rank and vector $b \in \mathbb{F}^m$, where $F$ is a finite field; |
| **Sol:** Solution $x_0 \in \mathbb{F}^n$ to $Ax = b$; |
| **Measure:** Hamming weight of $x_0$; |
| **Opt:** $\min$. |

If we choose $b = 0$ in MIN-CWP we get the MIN-SWP. We distinguish between these problems because the methods for solving them are different (see Section 4.1 for the CWP and Section 4.2 for SWP).

Remark that the matrix $A$ in MIN-CWP and MIN-SWP has full row rank. That is particularly important for the rest of this section from several points of view: we may use Theorems 4 and 6 to characterize the set of feasible solutions of an instance, and we have a suitable form of the GIs. On the other hand, this also corresponds to SDP and LWP. Of course, one may extend the framework if necessary.

**Proposition 2.** The following properties hold:

1. MIN-CWP and MIN-SWP are NPO problems.
2. CWP (SWP) is the decision problem of MIN-CWP (MIN-SWP).
3. The constructive, evaluation, and the decision problems of MIN-CWP (MIN-SWP) are all polynomial-time Turing-equivalent.

**Proof.** We will prove the theorem in the case of the MIN-CWP.

(1) It is quite clear that each MIN-CWP instance $(A, b)$ is recognizable in polynomial time. Given an instance $(A, b)$, its set of feasible solutions consists of all solutions to the equation $Ax = b$. According to Theorem 4, this set is 

$$\text{Sol}(A, b) = \{ Xb \mid X \in \mathcal{G}(A) \}.$$ 

Clearly, each feasible solution is linear in the size of the instance. Moreover, given a vector $c \in \mathbb{F}^n$, we can decide in polynomial time if it is a solution to $Ax = b$ (that is, whether it is a feasible solution or not of the instance $(A, b)$). Finally, the measure of a feasible solution can be computed in linear time with respect to the size of the solution.

(2) follows directly from definitions.

(3) We know that CWP is an NP-complete problem [7]. According to Theorem 1.5 in [8], all the problems associated to an NPO problem are polynomial-time Turing-equivalent if the decision problem is NP-complete. So, we get the result. □

We will establish further a helpful connection between the MIN-CWP and MIN-SWPs over $\mathbb{F}_2$, denoted MIN-CWP($\mathbb{F}_2$) and MIN-SWP($\mathbb{F}_2$), and the constraints satisfaction framework. Let us first recall the classical constraint satisfaction framework [10].

Let $D$ be a finite set of cardinality at least two that we consider fixed during this section. A constraint over a finite set $\mathcal{V}$ of variables is a pair $\sigma = (f, (x_1, \ldots, x_\ell))$ consisting of a function $f : D^\ell \to \{0, 1\}$, for some $\ell \geq 1$, and a list $(x_1, \ldots, x_\ell)$ of variables from $\mathcal{V}$ that take values in $D$. 31
The function $f$ is called the constraint function, and the list of variables, the constraint scope of $\sigma$. The variables in the constraint scope may not be pairwise distinct.

An instance of the constraint satisfaction problem over a set $V$ of variables is a finite set $\Sigma$ of constraints over $V$. Distinct constraints in $\Sigma$ may have distinct constraint functions or scopes.

Let $\gamma : V \rightarrow D$ be an assignment of the variables in $V$. A constraint $\sigma = (f, (x_1, \ldots, x_\ell))$ over $V$ is satisfied by $\gamma$ if $f(\gamma(x_1), \ldots, \gamma(x_\ell)) = 1$.

Now, we are able to formulate the following problem.

**MIN Constraint Satisfaction Problem (MIN-CSP)**

- **Instance:** Finite set $\Sigma$ of constraints over a finite set $V$ of variables;
- **Goal:** Assignment $\gamma : V \rightarrow D$;
- **Measure:** Number of constraints in $\Sigma$ satisfied by $\gamma$;
- **Opt:** $\text{min}$.

When $D = \{0, 1\}$ and each constraint function is a Boolean function, MIN-CSP is usually referred to as the MIN-SAT problem. When the constraints are affine, that is, expressions of the form

$$x_1 \oplus \cdots \oplus x_n = b,$$

where $n \geq 1$, $x_1, \ldots, x_n \in V$, and $b \in \{0, 1\}$, the problem is usually denoted MIN-SAT(affine).

We will show that MIN-CWP($\mathbb{F}_2$) and MIN-SWP($\mathbb{F}_2$) are reducible in a very strict way to the MIN-SAT problem. Recall first the concept of reducibility between optimization problems.

**Definition 4.** Let $A = (I_A, \text{Sol}_A, \mu_A, \text{Opt}_A)$ and $B = (I_B, \text{Sol}_B, \mu_B, \text{Opt}_B)$ be two NPO problems. A reduction from $A$ to $B$ is a pair $(f, g)$ of polynomial-time computable function such that:

1. $f(x) \in I_B$, for any $x \in I_A$;
2. $g(x, y) \in \text{Sol}_A(x)$, for any $x \in I_A$ and $y \in \text{Sol}_B(f(x))$.

**Definition 5.** Let $A, B$ be two NPO problems and $(f, g)$ be a reduction from $A$ to $B$. Then $(f, g)$ is called an $S$-reduction from $A$ to $B$ if the following two properties hold:

1. $\mu_B^*(f(x)) = \mu_A^*(x)$, for any $x \in I_A$;
2. $\mu_A(x, g(x, y)) = \mu_B(f(x), y)$, for any $x \in I_A$ and $y \in \text{Sol}_B(f(x))$.

**Theorem 8.** MIN-CWP($\mathbb{F}_2$) and MIN-SWP($\mathbb{F}_2$) are $S$-reducible to MIN-SAT(affine).

**Proof.** We will discuss first the case of the MIN-CWP($\mathbb{F}_2$) problem.

Let $I = (A, b)$ be an instance of the MIN-CWP($\mathbb{F}_2$) problem, where $A \in \mathcal{M}_{m,n} (\mathbb{F}_2)$, $b \in \mathbb{F}_2^n$, $b \neq 0$, and $\text{rank}(A) = m < n$. We associate to $I$ the following instance of MIN-SAT(affine).

First, by Gaussian elimination, transform $A$ into $PAQ = (A_1 \ I_m)$, for some matrices $P \in \text{GL}_m(\mathbb{F})$, $Q \in \text{S}_n(\mathbb{F})$, and $A_1$. Consider then a set $V = \{z_1, \ldots, z_{n-m}\}$ of $n-m$ Boolean variables and define the set $\Sigma$ of affine constraints

$$\begin{align*}
\Sigma : \\
&z_1 = 1 \\
&\cdots \\
&z_{n-m} = 1 \\
&Pb - A_1 z = 1,
\end{align*}$$

(61)
where \( z = \begin{pmatrix} z_1 \\ \vdots \\ z_{n-m} \end{pmatrix} \) and \( 1 \) is a vector of 1’s. Each row in \( Pb - A_1z = 1 \) is a constraint, and so \( \Sigma \) contains exactly \( n \) constraints. Thus, \( (V, \Sigma) \) is an instance of the MIN-SAT(affine) problem, which can be constructed in polynomial-time from \( I \).

So, we have defined a polynomial-time computable function \( f \) that, on an instance \( I \) of MIN-CWP(\( \mathbb{F}_2 \)), associates an instance \( f(I) = (V, \Sigma) \) of MIN-SAT(affine).

Let \( \gamma \in \text{Sol}_B(V, \Sigma) \) be a feasible solution for \( (V, \Sigma) \). Extend \( \gamma \) homomorphically and component-wise to affine expressions and systems of affine expressions. Let \( \bar{\gamma} \) be the extension. Define now \( x_\gamma \in \mathbb{F}_n^m \) as follows:

- \( (x_\gamma)_{[1,n-m]} = \bar{\gamma}(z) \in \mathbb{F}_2^{n-m} \);
- \( (x_\gamma)_{[n-m+1,n]} = \bar{\gamma}(Pb - A_1z) \in \mathbb{F}_2^m \).

We will prove that \( Qx_\gamma \) is a feasible solution for \( I \). First, recall that each solution to \( Ax = b \) is of the form \( Xb \), where \( X \) is a GI of \( A \) (Theorem 4). Then, \( X \) may be chosen of the form \( X = Q \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} P \), where \( X_1 \) and \( X_2 \) satisfy the property \( A_1X_1 + X_2 = I_m \).

As \( b \neq 0 \), we may choose \( X_1 \) so that \( X_1Pb = \bar{\gamma}(z) = (x_\gamma)_{[1,n-m]} \) (see the discussion at the end of Section 4.1.4). Then,

\[
X_2Pb = Pb - A_1X_1Pb = Pb - A_1\bar{\gamma}(z) = \bar{\gamma}(Pb - A_1z) = (x_\gamma)_{[n-m+1,n]}
\]

is a solution to \( Ax = b \).

So, we have defined a polynomial-time computable function \( g \) that, on an instance \( I \) and a feasible solution \( \gamma \) for \( f(I) \), returns a feasible solution \( g(I, \gamma) = Qx_\gamma \) for \( I \).

To complete the proof, we have to show that \( (f, g) \) is an S-reduction. Directly from the construction, we obtain

\[
\mu_{CW}(I, Qx_\gamma) = \mu_{SAT}(V, \Sigma, \gamma)
\]

and \( \mu_{SAT}^*(V, \Sigma) \geq \mu_{CW}^*(I) \), where \( \mu_{CW} \) and \( \mu_{SAT} \) are the measure functions of MIN-CWP(\( \mathbb{F}_2 \)) and MIN-SAT(\( \mathbb{F}_2 \)), respectively.

Let \( I = (A, b) \) be an instance of MIN-CWP(\( \mathbb{F}_2 \)). All feasible solutions for \( I \) are of the form \( Xb \), where \( X \) is a generalized inverse of \( A \) as that above. Going in the opposite direction on the reasoning already done, we notice that for any feasible solution \( Xb \) for \( I \), there is a feasible solution \( \gamma \) for \( f(I) \) so that \( g(I, \gamma) = Xb \). So, \( g \) maps \( \text{Sol}_{SAT}(f(I)) \) onto \( \text{Sol}_{CW}(I) \). Moreover, it is straightforward to see that \( g \) is also one-to-one. Therefore, the property \( \mu_{SAT}^*(V, \Sigma) = \mu_{CW}^*(I) \) must hold.

The proof in the case of the MIN-SWP(\( \mathbb{F}_2 \)) problem is similar to the previous one, but this time we will use Corollary 5. So, define the set of constraints by

\[
\Sigma : \begin{cases} 
    z_1 & = 1 \\
    \vdots & \\
    z_{n-m} & = 1 \\
    -A_1z & = 1
\end{cases} \quad (62)
\]

and \( x_\gamma \) by
\[
(x)_{[1, n - m]} = \bar{\gamma}(z) \in F_{2}^{n - m};
\]
\[
(x)_{[n - m + 1, n]} = \bar{\gamma}(-A_{1}z) \in F_{2}^{m}.
\]

The proof can now be accomplished using Corollary 5.

The proof of the theorem is constructive, providing the tightest reduction between problems. We can then propose the following algorithms to solve our problems. Both of them use a MIN-SAT solver denoted MIN-SAT\_solve.

**Algorithm 3** SAT solver for MIN-CWP\((F_{2})\)

1. **function** MIN-CWP\_solve\((A, b)\)
2. Choose a transformation \(PAQ = (A_{1} I_{m})\);
3. Choose a vector \(z\) of \(n - m\) variables;
4. Compute de MIN-SAT(affine) instance \(\Sigma = \begin{cases} z & = 1 \\ Pb - A_{1}z & = 1 \end{cases}\);
5. \(\gamma := \text{MIN-SAT\_solve}(z, \Sigma)\);
6. Compute \((x)_{[1, n - m]} := \bar{\gamma}(z)\) and \((x)_{[n - m + 1, n]} := \bar{\gamma}(Pb - A_{1}z)\);
7. **return** \(Qx\).

**Algorithm 4** SAT solver for MIN-SWP\((F_{2})\)

1. **function** MIN-SWP\_solve\((A)\)
2. Choose a transformation \(PAQ = (A_{1} I_{m})\);
3. Choose a vector \(z\) of \(n - m\) variables;
4. Compute de MIN-SAT(affine) instance \(\Sigma = \begin{cases} z & = 1 \\ -A_{1}z & = 1 \end{cases}\);
5. \(\gamma := \text{MIN-SAT\_solve}(z, \Sigma)\);
6. Compute \((x)_{[1, n - m]} := \bar{\gamma}(z)\) and \((x)_{[n - m + 1, n]} := \bar{\gamma}(-A_{1}z)\);
7. **return** \(Qx\).

The solver for MIN-SWP can be thought of as a particular case of the solver for MIN-CWP, namely the case where \(b = 0\). That is not true for SWP and CWP solvers (see Algorithms 1 and 2).

As we said, the reduction of MIN-CWP and MIN-SWP to MIN-SAT is very close and probably optimal. Only one Gaussian elimination is needed, after which a constraint is associated for each component of the solution vector. Moreover, the first \(n - m\) constraints are costless, of the form \(z = 1\). The other constraints are obtained by simply multiplying a matrix with a vector of Boolean variables. The solution returned by the result provided by the MIN-SAT solver is just as simple. The first \(n - m\) components of the solution vector are exactly the assignment of the Boolean variables provided by the MIN-SAT solver. The following components are obtained by multiplying a matrix by the assigned variable vector.

Currently, considerable effort is being made to streamline MIN-SAT solvers. They are widely used in optimization problems and have significant advantages over MAX-SAT solvers [2]. These advantages are not the result of encodings but the result of the techniques used in MIN-SAT, some of which are not valid for MAX-SAT.
7. Experimental results

We have implemented several GID algorithms in order to verify, test, and confront our theoretical results. The implementation was done in Magma V2.25-3 running on a regular laptop equipped with Intel(R) Xeon(R) E-2176M processor with CPU @ 2.70GHz.

Experiment 1. This is an experiment for SDP. The instances are \((H, s, t)\), where \(H \in \mathcal{M}_{n-k,n}(F_q)\) and \(s \in \mathbb{F}_q^{n-k}\). We have chosen \(n = 500\), \(k = 250\), and \(q \in \{2, 3, 5, 7\}\). The steps of the experiment are:

1. We have generated exactly one Prange transformation \(PHQ = (V \ I_r)\);
2. We have performed ten iterations. They are represented, in a cumulative way, on the abscissa of the graphs in Figure 3;
3. For each iteration, \(k\) GIs were generated \(X = Q \left( X_1 X_1^T \right) P\) so that \(|X_1 \bar{s}|\) takes all the values in \(\{1, \ldots, k\}\);

Step 3 when implemented in practice does the following. It selects a random subset \(E\) of \(\text{Supp}(\bar{s})\) of cardinality \(i \leq k\). It computes the second part of the solution \(x_2 = \bar{s} - \sum_{i \in E} V(i) r_i\) where \(r_i\) is a random non-zero element from \(\mathbb{F}_q\). It stores the weight of the solution \(i + |x_2|\).

Figure 3: Hamming weights outside the range of efficiently computed solutions \((n = 500, k = 250)\).
The experiment took less than one second and the results can be seen in Figure 3. It is noteworthy that only one Prange transformation with a very small number of iterations produces solutions whose Hamming weight cover a rich range

\[ \left[ r \frac{q-1}{q}, r \frac{q-1}{q} + n - r \right] \]

(the missing part of the line next to the \( i \) value on the abscissa).

Our experiment confirms the easy domain \([\omega_{\text{easy}}, \omega'_{\text{easy}}]\) established in [18]. We would like to stress that we have only used one Prange transformation.

In order to see how the interval of easy weights behaves when multiple decompositions are used, we have repeated Experiment 1 for 20 different decompositions. We have stored an updated list of solution weights that are not retrieved and plot it in Figure 4 (in green). The same experiment was performed using a single decomposition and plotted (in black) in the same figure. We notice the following facts.

- For a fixed number of iterations, the algorithm using multiple decompositions tend to produce the same interval of weights as the algorithm using a single decomposition.

- For \( n = 500, k = 250 \) the value where we move from single solution to multiple solutions corresponds to \( t = 57 \) for \( \mathbb{F}_2 \) and \( t = 123 \) for \( \mathbb{F}_3 \). Our algorithm finds solutions with weight within the interval \([102, 397]\) for \( \mathbb{F}_2 \) and \([146, 437]\) for \( \mathbb{F}_3 \). This corresponds to a larger interval than what in given in [18], i.e., \([r \frac{q-1}{q} - \sqrt{n}, r \frac{q-1}{q} + n - r + \sqrt{n}]\), only using a small number of iterations. We have verified our statement for other code length values. For example for \( n = 1000, k = 500 \) over \( \mathbb{F}_3 \) the value of \( t = 242 \) and our simulations produced the interval \([300, 862]\) while \([r \frac{q-1}{q} - \sqrt{n}, r \frac{q-1}{q} + n - r + \sqrt{n}] = [301, 801]\). We also wanted to illustrate how the easy weights interval behaves for other code dimensions, fact that we plot in Figure 5 for codes of length \( n = 2000 \) and dimension increasing from \( n - k = 500 \) to \( n - k = 1000 \). We stress out that even for larger values of \( n \) the algorithm is fast, in less than 30 seconds we obtained the results for all the code dimensions.

Experiment 2. There are many variations of LWP [66]. One of them requires finding a codeword whose weight should be in some given interval \([w_1, w_2]\) (the code is specified by a parity check
matrix). This problem is NP-complete \cite{50}. However, it has many easy instances as our results show. Moreover, an instance’s hardness could depend on the positioning of the interval \( (w_1, w_2) \). Our simulations have shown that, as in the case of SDP, the interval of weights that are reached using a small number of iterations tends to \( \left[ r \frac{a-1}{q} - \sqrt{n}, r \frac{a-1}{q} + n - r + \sqrt{n} \right] \). Our simulations should be interpreted as follows:

- Regarding the intractability of SDP and LWCP, as simulations point our, we have good reasons to believe that in average both these problems are difficult when the parameter \( t \) is in a vicinity of the Gilbert-Varshamov bound.

- Regarding the intractability of the half-length weight codeword problem \cite{20}, our simulations show that this might not be a difficult problem in average. Indeed, we were not able to find codes for which the GID could not find a solution of weight \( n/2 \) to the system \( Hx = s \).

**Acknowledgments**

This work was supported by a grant of the Ministry of Research, Innovation and Digitization, CNCS/CCCDI – UEFISCDI, project number PN-III-P1-1.1-PD-2019-0285, within PNCDI III.

**References**

[1] Adams, C. M. and Meijer, H. (1987). Security-related comments regarding mceliece’s public-key cryptosystem. In *A Conference on the Theory and Applications of Cryptographic Techniques on Advances in Cryptology, CRYPTO ’87*, page 224–228, Berlin, Heidelberg. Springer-Verlag.

[2] Argelich, J., Li, C.-M., Manyà, F., and Zhu, Z. (2013). Minsat versus maxsat for optimization problems. In Schulte, C., editor, *Principles and Practice of Constraint Programming*, pages 133–142, Berlin, Heidelberg. Springer Berlin Heidelberg.

[3] Ausiello, G., Crescenzi, P., Gambosi, G., Kann, V., Marchetti-Spaccamela, A., and Protasi, M. (2003). *Complexity and Approximation: Combinatorial Optimization Problems and Their Approximability Properties*. Springer-Verlag, Berlin, Heidelberg, 2nd corrected printing edition.
[4] Baldi, M., Barenghi, A., Chiaraluce, F., Pelosi, G., and Santini, P. (2019). A finite regime analysis of information set decoding algorithms. *Algorithms*, 12(10).

[5] Becker, A., Joux, A., May, A., and Meurer, A. (2012). Decoding random binary linear codes in 2n/20: How 1 + 1 = 0 improves information set decoding. In *Proceedings of the 31st Annual International Conference on Theory and Applications of Cryptographic Techniques*, EUROCRYPT’12, page 520–536, Berlin, Heidelberg. Springer-Verlag.

[6] Ben-Israel, A. and Greville, T. (2006). *Generalized Inverses: Theory and Applications*. CMS Books in Mathematics. Springer New York.

[7] Berlekamp, E., McEliece, R., and van Tilborg, H. (1978). On the inherent intractability of certain coding problems. *IEEE Trans. Inf. Theor.*, 24(3):384–386.

[8] Bernstein, D. J., Lange, T., and Peters, C. (2011). Smaller decoding exponents: Ball-collision decoding. In Rogaway, P., editor, *Advances in Cryptology – CRYPTO 2011*, pages 743–760, Berlin, Heidelberg. Springer Berlin Heidelberg.

[9] Bernstein, E. and Vazirani, U. (1997). Quantum complexity theory. *SIAM J. Comput.*, 26(5):1411–1473.

[10] Both, L. and May, A. (2018). Decoding linear codes with high error rate and its impact for LPN security. In Lange, T. and Steinwandt, R., editors, *Post-Quantum Cryptography - 9th International Conference, PQCrypto 2018, Fort Lauderdale, FL, USA, April 9-11, 2018, Proceedings*, volume 10786 of *Lecture Notes in Computer Science*, pages 25–46. Springer.

[11] Canteaut, A. and Chabaud, F. (1998). A new algorithm for finding minimum-weight words in a linear code: application to McEliece’s cryptosystem and to narrow-sense BCH codes of length 511. *IEEE Transactions on Information Theory*, 44(1):367–378.

[12] Canto Torres, R. and Sendrier, N. (2016). Analysis of information set decoding for a sub-linear error weight. In *Proceedings of the 7th International Workshop on Post-Quantum Cryptography - Volume 9606*, PQCrypto 2016, page 144–161, Berlin, Heidelberg. Springer-Verlag.

[13] Coffey, J. and Goodman, R. (1990). The complexity of information set decoding. *IEEE Transactions on Information Theory*, 36(5):1031–1037.

[14] Coffey, J. T., Goodman, R. M., and Farrell, P. G. (1991). New approaches to reduced-complexity decoding. *Discrete Appl. Math.*, 33(1–3):43–60.

[15] Creignou, N., Khanna, S., and Sudan, M. (2001). *Complexity Classifications of Boolean Constraint Satisfaction Problems*. Society for Industrial and Applied Mathematics, USA.

[16] Crescenzi, P. (1997). A short guide to approximation preserving reductions. In *Proceedings of Computational Complexity, Twelfth Annual IEEE Conference*, pages 262–273.

[17] Dang, V. H. and Nguyen, T. D. (2017). Construction of pseudoinverse matrix over finite field and its applications. *Wirel. Pers. Commun.*, 94(3):455–466.

[18] Debris-Alazard, T., Sendrier, N., and Tillich, J.-P. (2019). Wave: A new family of trapdoor one-way preimage sampleable functions based on codes. In *ASIACRYPT*.

[19] Debris-Alazard, T. and Tillich, J.-P. (2017). Statistical decoding. In *2017 IEEE International Symposium on Information Theory (ISIT)*, pages 1798–1802.
[20] Diaconis, P. and Graham, R. L. (1985). The radon transform on $\mathbb{Z}_k^2$. Pacific J. Math., 118:176–185.

[21] Diffie, W. (1988). The first ten years of public-key cryptography. Proceedings of the IEEE, 76(5):560–577.

[22] Diffie, W. and Hellman, M. (1976). New directions in cryptography. IEEE Transactions on Information Theory, 22(6):644–654.

[23] Dumer, I. (1989). Two decoding algorithms for linear codes. Probl. Inform. Transm., 25(1):17–23.

[24] Dumer, I. (1991). On minimum distance decoding of linear codes. In Proc. 5th Joint Soviet-Swedish Int. Workshop Inform. Theory, pages 50–52, Moscow.

[25] Esser, A. and Bellini, E. (2021). Syndrome decoding estimator. IACR Cryptol. ePrint Arch., 2021:1243.

[26] Finiasz, M. (2005). Syndrome decoding in the non-standard cases. Available on-line at the address https://finiasz.net/research/2006/finiasz-clc06.pdf.

[27] Finiasz, M. and Sendrier, N. (2009). Security bounds for the design of code-based cryptosystems. In Matsui, M., editor, Advances in Cryptology – ASIACRYPT 2009, pages 88–105, Berlin, Heidelberg. Springer Berlin Heidelberg.

[28] Fossorier, M. P. C., Kobara, K., and Imai, H. (2007). Modeling bit flipping decoding based on nonorthogonal check sums with application to iterative decoding attack of McEliece cryptosystem. IEEE Trans. Inf. Theor., 53(1):402–411.

[29] Fulton, J. D. (1978). Generalized inverses of matrices over a finite field. Discrete Mathematics, 21(1):23 – 29.

[30] Gentle, J. E. (2017). Matrix Algebra. Theory, Computations and Applications in Statistics. Springer Texts in Statistics. Springer International Publishing.

[31] Gilbert, E. N. (1952). A comparison of signalling alphabets. The Bell System Technical Journal, 31(3):504–522.

[32] Hamdaoui, Y. and Sendrier, N. (2013). A non asymptotic analysis of information set decoding. Cryptology ePrint Archive, Report 2013/162. Available on-line at https://ia.cr/2013/162.

[33] Hirose, S. (2016). May-ozerov algorithm for nearest-neighbor problem over $\mathbb{Z}_q$ and its application to information set decoding. In Bica, I. and Reyhanitabar, R., editors, Innovative Security Solutions for Information Technology and Communications, pages 115–126, Cham. Springer International Publishing.

[34] Horlemann-Trautmann, A.-L. and Weger, V. (2021). Information set decoding in the lee metric with applications to cryptography. Advances in Mathematics of Communications, 15(4):677–699.

[35] Huffman, W., Kim, J.-L., and Solé, P. (2021). Concise Encyclopedia of Coding Theory. Chapman and Hall/CRC, first edition.
[36] Jabri, A. A. (2001). A statistical decoding algorithm for general linear block codes. In Honary, B., editor, *Cryptography and Coding*, pages 1–8, Berlin, Heidelberg. Springer Berlin Heidelberg.

[37] Larsen, M. V., Guo, X., Breun, C. R., Neergaard-Nielsen, J. S., and Andersen, U. L. (2021). Deterministic multi-mode gates on a scalable photonic quantum computing platform. *Nature Physics*, 17(9):1018–1023.

[38] Lee, P. J. and Brickell, E. F. (1988). An observation on the security of McEliece’s public-key cryptosystem. In *Lecture Notes in Computer Science on Advances in Cryptology - EUROCRYPT’88*, page 275–280, Berlin, Heidelberg. Springer-Verlag.

[39] Leon, J. S. (1988). A probabilistic algorithm for computing minimum weights of large error-correcting codes. *IEEE Trans. Inf. Theor.*, 34(5):1354–1359.

[40] Li, Y. X., Deng, R. H., and Wang, X. M. (1994). On the equivalence of mceliece’s and niederreiter’s public-key cryptosystems. *IEEE Transactions on Information Theory*, 40(1):271–273.

[41] Moore, E. H. (1920). On the reciprocal of the general algebraic matrix (abstract). *Bulletin of American Mathematical Society*, 26:394–395.

[42] Niederreiter, H. (1986). Knapsack-type cryptosystems and algebraic coding theory. *Problems of Control and Information Theory*, 15:159–166.

[43] Ntafos, S. and Hakimi, S. (1981). On the complexity of some coding problems (corresp.). *IEEE Transactions on Information Theory*, 27(6):794–796.
[51] Overbeck, R. (2006). Statistical decoding revisited. In Batten, L. M. and Safavi-Naini, R., editors, Information Security and Privacy, pages 283–294, Berlin, Heidelberg. Springer Berlin Heidelberg.

[52] Pearl, M. H. (1968). Generalized inverses of matrices with entries taken from an arbitrary field. Linear Algebra and its Applications, 1(4):571 – 587.

[53] Penrose, R. (1955). A generalized inverse for matrices. Mathematical Proceedings of the Cambridge Philosophical Society, 51(3):406–413.

[54] Peters, C. (2010). Information-set decoding for linear codes over fq. In Proceedings of the Third International Conference on Post-Quantum Cryptography, PQCrypto’10, page 81–94, Berlin, Heidelberg. Springer-Verlag.

[55] Prange, E. (1962). The use of information sets in decoding cyclic codes. IRE Transactions on Information Theory, 8(5):5–9.

[56] Preskill, J. (2018). Quantum computing in the nisq era and beyond. Quantum, 2:79.

[57] Rao, T. R. N. and Nam, K.-H. (1987). Private-key algebraic-coded cryptosystems. In Odlyzko, A. M., editor, Advances in Cryptology — CRYPTO’ 86, pages 35–48, Berlin, Heidelberg. Springer Berlin Heidelberg.

[58] Rohde, C. A. (2003). Linear algebra and matrices. Lecture notes.

[59] Roman, S. (2007). Advanced Linear Algebra. Graduate Texts in Mathematics. Springer New York.

[60] Roth, R. (2006). Introduction to Coding Theory. Cambridge University Press.

[61] Shor, P. (1994). Algorithms for quantum computation: Discrete logarithms and factoring. In Goldwasser, S., editor, FOCS, pages 124–134.

[62] Simon, D. (1994). On the power of quantum computation. In Proceedings 35th Annual Symposium on Foundations of Computer Science, pages 116–123.

[63] Stern, J. (1988). A method for finding codewords of small weight. In Proceedings of the 3rd International Colloquium on Coding Theory and Applications, page 106–113, Berlin, Heidelberg. Springer-Verlag.

[64] Sun, H.-M. (2001). Cryptanalysis of a public-key cryptosystem based on generalized inverses of matrices. IEEE Communications Letters, 5(2):61–63.

[65] Takeda, S. and Furusawa, A. (2019). Toward large-scale fault-tolerant universal photonic quantum computing. APL Photonics, 4(6):060902.

[66] Vardy, A. (1997). The intractability of computing the minimum distance of a code. IEEE Transactions on Information Theory, 43(6):1757–1766.

[67] Varshamov, R. R. (1957). Estimate of the number of signals in error correcting codes. Dokl. Acad. Nauk SSSR, 117:739–741.

[68] Weger, V., Battaglioni, M., Santini, P., ChiaraKuce, F., Baldi, M., and Persichetti, E. (2020). Information set decoding of lee-metric codes over finite rings. ArXiv, abs/2001.08425.

[69] Wu, C.-k. and Dawson, E. (1998). Generalised inverses in public key cryptosystem design. In Proc. IEE-Comput. Digit. Tech, volume 145, pages 321–326, Berlin, Heidelberg.