Non-Analytic Extension of the Kinnersley-Chitre Group for Colliding Plane Gravitational Waves. I

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Abstract

A program is outlined concerning the set $\Sigma_{E}$ of all solutions of the hyperbolic Ernst equation on a 2-dimensional manifold whose underlying topological space $\{(r,s) : -1 \leq r < s \leq 1\}$ is the same as the domain of all Ernst potentials $E$ for colliding plane gravitational wave pairs (CPGWP's). The aim of the program is to construct and apply a non-trivial extension $\tilde{K}$ of the group of Kinnersley-Chitre transformations (i.e., the representation of the Geroch group which is due to W. Kinnersley and D. M. Chitre) such that $\tilde{K}$ is transitive on $\Sigma_{E}$. This is to be done by employing the formalism of a homogeneous Hilbert problem which generalizes one previously used to effect KC transformations [I. Hauser and F. J. Ernst, J. Math. Phys. 21, 1126 (1980); F. J. Ernst, A. García-Díaz and I. Hauser, J. Math. Phys. 29, 681 (1988)]. In this first paper of a series, the aforementioned program is completely carried out for the set $\Sigma_{\psi}$ of all real members $E = -\exp(2\psi)$ of $\Sigma_{E}$ and for an abelian subgroup of $\tilde{K}$ such that this subgroup is transitive on $\Sigma_{\psi}$. A simple Hilbert problem is used to...

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obtain a new form of the general solution for $\psi(r, s)$ and this general solution provides an independent derivation of the known result [U. Yurtsever, Phys. Rev. D 38, 1706 (1987)] for the asymptotic form of $\psi(r, s)$ as one approaches the axis \{(r, s) : -1 < r = s < 1\}. The necessary and sufficient condition that $\psi(r, s)$ have a continuous extension to the axis is $\sqrt{1 - \sigma g_3(\sigma)} = \sqrt{1 + \sigma g_2(\sigma)}$ ($-1 < \sigma < 1$) where $g_3(\sigma)$ and $g_2(\sigma)$ are those Abel transforms of the initial value function derivatives which were introduced by I. Hauser and F. J. Ernst, J. Math. Phys. 30, 872 (1989); 2322 (1989).
I. Introduction

This is the first of a series of papers on a non-trivial extension of the transformation group which was discovered by W. Kinnersley and D. M. Chitre (KC) and which is an effective realization of the symmetry group originally conceived by R. Geroch. The generalization we have in mind seems to be appropriate when the field equations are of hyperbolic type, as they are, for example, in the case of colliding plane gravitational waves.

A. Basic Concepts

The Ernst potentials which represent the regions of interaction of colliding plane gravitational wave pairs (CPGWP’s) are defined on certain two-dimensional differential manifolds. The underlying topological space of all of these manifolds is the $R^2$-subspace $\{(r, s) : -1 \leq r < s \leq 1\}$, and the charts which are usually employed are the familiar mappings

$$(r, s) \rightarrow (r(u), s(v))$$

into the null coordinate $(u, v)$-plane.

We shall be considering the set $\Sigma_E$ of all Ernst potentials which are defined on the same manifolds as those described above but which are not required to satisfy the colliding wave conditions. The set $\Sigma_E$ includes all Ernst potentials which represent the regions of interaction of CPGWP’s but it also includes Ernst potentials (such as those for the Kasner metrics) which do not represent any CPGWP’s.

It is the group of all KC transformations which induce permutations of $\Sigma_E$, i.e., which induce one-to-one mappings of $\Sigma_E$ onto $\Sigma_E$, that we shall be extending. We caution that each element of this KC group is not itself a permutation of $\Sigma_E$. It is a permutation of a certain set $\Sigma_F$ which is closely related to $\Sigma_E$. The KC group will accordingly be denoted by $K(\Sigma_F)$. The set $\Sigma_F$ is a special gauge of the $2 \times 2$ matrix spectral potentials which were originally introduced by KC. The definition of this special gauge is still being studied and will be covered in a later paper of the series. At present, it suffices to note that the domain of any member $F$ of $\Sigma_F$ will contain the set of all quadruples $(r, s, \tau, z_0)$ such that $r, s, z_0$ are real and $-1 \leq r < z_0 < s \leq 1$, while $\tau$ (the so-called spectral parameter) is any point in the extended complex plane $C$ minus the real axis interval $[r, s]$. Most
previous papers on KC transformations employ the parameter $t = (2\tau)^{-1}$ in place of $\tau$ and no previous paper employs the parameter $z_0$.

To enable us to present our key concepts and methods before having to cope with intricate mathematical problems stemming from the nonlinearity of the Ernst equation, we shall focus attention in this paper on the real members of $\Sigma_E$, i.e., on the Ernst potentials of vacuum Weyl metrics. Complex Ernst potentials will be covered in our next paper.

For the real members of $\Sigma_E$ which are the subject of the current paper, a factorization of $F$ due to KC yields

$$F(r, s, \tau, z_0) = e^{-\sigma_3 \psi(r, s)} F^{MS}(r, s, \tau) e^{-\sigma_3 \xi(r, s, \tau, z_0)}$$

where $\sigma_3$ is the usual Pauli spin matrix, $\psi$ is half of the natural logarithm of $-E$, the second factor $F^{MS}$ is the member of $\Sigma_F$ corresponding to the Minkowski space Ernst potential $E^{MS} = -1$ and $\xi$ is a complex-valued spectral potential. The above Eq. (1.1) will not be used in this paper but we shall define, thoroughly discuss and use $\xi$ in Secs. II, III and IV.

The spectral potential $F$ in the gauge $\Sigma_F$ is uniquely determined by the choice of $E \in \Sigma_E$ except perhaps for the gauge transformations $F \rightarrow FU$ where $U$ is any $2 \times 2$ unimodular matrix which is dependent only on $\tau$ and $z_0$, is a holomorphic function of $\tau$ throughout $C - \{z_0\}$ with no essential singularity, is real for real $\tau$ and is the unit matrix $I$ at $\tau = \infty$. Conversely, the Ernst potential $E \in \Sigma_E$ corresponding to a given $F \in \Sigma_F$ is uniquely determined from the second term of the following expansion in a neighborhood of $\tau = \infty$:

$$F(r, s, \tau, z_0) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} + (2\tau)^{-1}[H(r, s) + iB(z_0)] + O(\tau^{-2})$$

where $B(z_0)$ is a real symmetric matrix and $H(r, s)$ is a matrix generalization of the Ernst potential such that (with our conventions) $H(-1, 1) = -I$ and $E(r, s)$ is the lower diagonal element of $H(r, s)$. There is thus an obvious natural homomorphism of $K(\Sigma_F)$ onto a group $K(\Sigma_E)$ of permutations of the Ernst potential set $\Sigma_E$. It is worth recalling that the transformations $E(0) \rightarrow E$ resulting from almost every permutation in $K(\Sigma_E)$ are non-local and the direct calculation of $E$ in terms of $E(0)$ involves an unwieldy sequence (generally infinite) of iterated integrations over $r$ and $s$. In contrast, every transformation $F(0) \rightarrow F$ resulting from a permutation in $K(\Sigma_F)$ is local.
with respect to \( r, s, z \) and the calculation of \( F \) requires that one solve a linear integral equation of the Cauchy type in the \( \tau \)-plane.\(^{16}\) The kernel of this integral equation is a quadratic expression in \( F^{(0)} \).

Our extension of \( K(\Sigma_F) \) will also be a group of permutations of \( \Sigma_F \) and will be denoted by \( \tilde{K}(\Sigma_F) \). The corresponding extension of \( K(\Sigma_E) \) will be denoted by \( \tilde{K}(\Sigma_E) \). One way of defining \( \tilde{K}(\Sigma_F) \) involves two constructs which generalize the formalism\(^ {14} \) previously used by the authors for KC transformations and which will be motivated and detailed in later papers of the series. Brief descriptions of these two constructs follow in (i) and (ii) below:

(i) One construct will be a multiplicative group \( \tilde{K}_{(-1,1)} \) of \( 2 \times 2 \) unimodular real matrix functions \( v(\sigma,z_0) \) of real variables \( \sigma \) and \( z_0 \) such that \(-1 < \sigma < 1, \ -1 < z_0 < 1 \) and \( \sigma \neq z_0 \). These matrices will satisfy certain differentiability and/or Hölder conditions and, also, certain asymptotic conditions at \( \sigma = \pm 1, z_0 \) The specification of these conditions is still under study.

The subgroup of \( \tilde{K}_{(-1,1)} \) consisting of all of its elements which are independent of \( z_0 \) and are analytic functions of \( \sigma \) throughout \(-1 < \sigma < 1 \) will be denoted by \( K_{(-1,1)} \). Holomorphic continuations of the elements of \( K_{(-1,1)} \) into the \( \tau \)-plane were used by the authors in previous papers to represent KC transformations.\(^ {17} \) The factor group \( K_{(-1,1)}/\{I,-I\} \) is a faithful realization of \( K(\Sigma_F) \).

For the current paper, we shall require only that abelian subgroup of \( \tilde{K}_{(-1,1)} \) which consists of all of its elements of the form \( \exp[\sigma_3\lambda(\sigma,z_0)] \). The real-valued functions \( \lambda(\sigma,z_0) \) will be defined in Sec. IIG and further discussed in Secs. IIIB, IVA and IVC.

(ii) The second construct is a homogeneous Hilbert problem (HHP) or an equivalent linear integral equation whose kernel is simply constructed from any given \( v \in \tilde{K}_{(-1,1)} \) and \( F^{(0)} \in \Sigma_F \). We expect and shall endeavor to prove that (granting its existence) the solution \( F \) of the HHP is unique and is also a member of \( \Sigma_F \), whereupon it will be clear that the HHP defines a permutation \( \Sigma_F \to \Sigma_F \) for each given \( v \). We shall also endeavor to prove that the set of all of these permutations constitute a group which is isomorphic to \( \tilde{K}_{(-1,1)}/\{I,-I\} \) and which, of course, we denote by \( \tilde{K}(\Sigma_F) \).
For the special subsets of $\bar{K}_{\{-1,1\}}$ and $\Sigma_F$ which are to be considered in the current paper, the HHP is expressible as a simple (non-matrix) Hilbert problem whose general solution is known and has a finite closed form. The definition and solution of this simple Hilbert problem will be given in Sec. IIIE. A definitive version of the simple Hilbert problem will be given in Sec. VIE.

Our formalism is not expected to furnish any new efficacious methods of generating exact solutions of the Ernst equation. The advancement of exact solution productivity is not our aim.

We expect the enlarged group $\bar{K} (\Sigma_E)$ to include those permutations of $\Sigma_E$ which have already been successfully used to generate exact solutions but which are not in the original group $\bar{K}(\Sigma_E)$. For example, we hope to show that our extension includes the Kramer-Neugebauer involution$^{18,19,20,21}$ and those transformations of C. M. Cosgrove$^{22,23,24,25}$, D. Maison$^{26}$ and G. Neugebauer$^{27}$ which also involve a non-trivial permutation of the domain $\{ (r, s) : -1 \leq r < s \leq 1 \}$.

At an early stage we shall study in detail the precise relation between our extension of the KC group and the initial value problem for a CPGWP.$^{7,10,28,29}$

Our formalism also turns out, as we shall see in this paper, to be suited for studying the behavior near the axis (points at which the exterior product of the Killing vectors becomes zero) of the Ernst potential $\mathcal{E}$. This will be done in the present paper to provide an independent method of arriving at a result due to Yurtsever$^{30}$ for the collinear polarization case and we hope to extend our results to the noncollinear case in another paper of the series. We contemplate similar studies on the behavior of the Weyl conform tensor near the axis, on necessary and sufficient conditions for a horizon to exist on the axis and (when a horizon exists) continuation of the solution into the region in which the surface of transitivity of the Killing vectors is Lorentzian. We thus intend to reproduce, and possibly augment, Yurtsever’s results$^{31}$ on these topics.

So far, we have not provided a clear motivation for enlarging the KC group. Why is it necessary to do so?
B. Motivation

The KC group has been successfully used to generate numerous exact solutions of the Einstein field equations for stationary axisymmetric vacuum spacetimes and CPGWP’s. However, \( K(\Sigma_E) \) is “incomplete” in a sense which will be explained below in (i) and (ii):

(i) Consider those members of \( \Sigma_E \) which have continuous extensions to the domain
\[
\{(r, s) : -1 \leq r < s \leq 1 \text{ or } -1 < r = s < 1\}
\]
and let \( \Sigma_{ax}^E \) be the set of all of these continuous extensions. Let \( \Sigma_{AX}^E \) be the set of all \( \mathcal{E} \in \Sigma_{ax}^E \) such that the restriction of \( \mathcal{E} \) to \( \{(r, s) : -1 < r \leq s < 1\} \) is in the differentiability class \( C^2 \). This is the set for which no spacetime singularities occur on the axis \( \{(r, s) : -1 < r = s < 1\} \) and the curvature tensor is continuous throughout \( \{(r, s) : -1 < r \leq s < 1\} \). The Minkowski space Ernst potential \( \mathcal{E}^{MS} = -1 \) is in \( \Sigma_{AX}^E \) and so are the Ernst potentials for the Chandrasekhar-Xanthopoulos\(^{32} \) and Hoenselaers-Ernst\(^{33} \) CPGWP interaction regions (which are respectively isometric to different subregions of the Kerr metric ergosphere). On the other hand, the Ernst potentials for the Nutku-Hali\(^{34} \) spacetime and for any other CPGWP spacetime which has curvature singularities on the axis are in \( \Sigma_E - \Sigma_{AX}^E \).

Now let \( \mathcal{E}^{(0)} \in \Sigma_{AX}^E \) and let \( \mathcal{E}^{(0)} \rightarrow \mathcal{E} \) under the KC transformation corresponding to any given member \( v \) of the analytic matrix group \( K_{(-1,1)} \). Then methods which the authors have previously used for stationary axisymmetric vacuum spacetimes can be applied here to prove that \( \mathcal{E} \) is also in \( \Sigma_{AX}^E \) and that the following axis relation holds:

\[
\mathcal{E}(z) = i[v_1^1(z)]\mathcal{E}^{(0)}(z) + iv_2^1(z)]v_1^2(z)\mathcal{E}^{(0)}(z) + iv_2^2(z)]^{-1} \tag{1.3}
\]

where \( \mathcal{E}(z) := \mathcal{E}(z, z) \) and \(-1 < z < 1\).\(^{35,36} \)

It follows that no member of \( \Sigma_E - \Sigma_{AX}^E \) can be obtained from a member of \( \Sigma_{AX}^E \) (and vice versa) by applying an element of \( K(\Sigma_E) \).

(ii) Let \( \Sigma_{an}^E \) be the set of all \( \mathcal{E} \in \Sigma_E \) such that the initial data \( \mathcal{E}_3(r) = \mathcal{E}(r, 1) \) and \( \mathcal{E}_2(s) = \mathcal{E}(-1, s) \) are analytic functions throughout \(-1 < r < 1 \text{ and } -1 < s < 1\), respectively. The set \( \Sigma_{an}^E \) is clearly not
empty since the Ernst potentials for the Kasner metrics and for every published exact CPGWP solution with noncollinear polarizations lie in this set. The set \( \Sigma_E - \Sigma_{an}^E \) is also not empty even if we restrict ourselves to Ernst potentials which are \( C^\infty \) throughout their domains. The basic reason for this is that the Ernst equation for \( \Sigma_E \) is hyperbolic (unlike the Ernst equation for the stationary axisymmetric case). Now, a procedure like one previously used by the authors for stationary axisymmetric spacetimes can be used to prove that each element of \( K(\Sigma_E) \) maps \( \Sigma_{an}^E \) onto \( \Sigma_{an}^E \). Therefore, each element of \( K(\Sigma_E) \) maps \( \Sigma_E - \Sigma_{an}^E \) onto \( \Sigma_E - \Sigma_{an}^E \).

In summary, \( K(\Sigma_E) \) is not transitive. Therefore, \( K(\Sigma_F) \) is not transitive. This is our principal motivation for extending the KC group. We expect \( \bar{K}(\Sigma_F) \) to be transitive and we shall try to prove this in a future paper. In other words, we hope to prove that our extension can be used in principle to generate any \( E \in \Sigma_E \) from any known given \( E^{(0)} \in \Sigma_E \) (e.g., the Minkowski space Ernst potential \( E^{MS} = -1 \)). The proof will have at least two difficult parts:

1. A proof that the solution of the HHP, granted its existence, yields a member \( F \) of \( \Sigma_F \).

2. A proof that the solution of the HHP always exists and has the requisite differentiability properties.

Note that the conjecture that \( \bar{K}(\Sigma_F) \) is transitive resembles a conjecture due to Geroch\(^4\) for the stationary axisymmetric vacuum spacetimes. An explication of the Geroch conjecture was formulated and proven by the authors.\(^{36}\) An essential part of the proof was establishing the existence of a solution of the HHP used to effect KC transformations, i.e., a proof that the KC transformations actually exist. It is easy to forget that one must prove existence of the group.

There are two more objectives which are appropriate to mention at this point. One of these concerns the set \( \Sigma_{ax}^E \) which we defined above. We conjecture and hope to prove in a later paper that the element of \( \bar{K}(\Sigma_E) \) corresponding to any given \( v \in \bar{K}_{(-1,1)} \) maps \( \Sigma_{ax}^E \) onto \( \Sigma_{ax}^E \) if and only if \( v(\sigma, z_0) \) is independent of \( z_0 \). This is proven in Sec. IVC for those subsets of \( \Sigma_E \) and \( \bar{K}_{(-1,1)} \) which are treated in the current paper.
Another objective concerns the set $\Sigma^C_{\varepsilon}$ of all members of $\Sigma_{\varepsilon}$ for which the colliding wave conditions hold. More precisely, it concerns certain substantial subsets of $\Sigma^C_{\varepsilon}$ which are defined and discussed in Sec. VE of this paper. We have proven (unpublished as yet) that each of these subsets of $\Sigma^C_{\varepsilon}$ is mapped onto itself by the KC transformation corresponding to any given $v \in K_{(-1,1)}$ which is holomorphic on the closed interval $-1 \leq \sigma \leq 1$. In a later paper, we hope to extend this result to $\bar{K}(\Sigma_F)$ elements corresponding to members $v$ of $\bar{K}_{(-1,1)}$ which are not necessarily analytic but which do have certain “well behaved” extensions to the union of the intervals $-1 \leq \sigma < z_0$ and $z_0 < \sigma \leq 1$. For the special case which is treated in the current paper, this objective is fulfilled in Sec. VF.

C. The Principal Results for the Weyl Metrics

Before we launch into formal developments, we shall review the definition of $\mathcal{E}$ and then give a preview of the main results of this paper. Let the line element $(++++)$ be expressed as

$$ds^2 = dx^a dx^b g_{ab}(u,v) + 2dudvg_{34}(u,v),$$

where $a = 1,2$ and $b = 1,2$, where $x^1, x^2$ are the ignorable coordinates and where $x^3 = u$ and $x^4 = v$. Introduce a two dimensional duality operator $*$ on 1-forms which are linear combinations of $du$ and $dv$:

$$*du = du, *dv = -dv.$$  

Thus $** = 1$. The Ernst potential $\mathcal{E}$ is defined by

$$\mathcal{E} = f + i\chi, \ f := -g_{22},$$

where the twist potential $\chi$ is defined as the integral of

$$\rho d\chi = *f^2 d\omega,$$

where

$$\omega := g_{12}/g_{22}, \ \rho := \sqrt{\det g_{ab}}.$$  

From the above definition, $\mathcal{E}$ satisfies the Ernst equation

$$f \ d(*\rho d\mathcal{E}) = \rho d\mathcal{E}(*d\mathcal{E}).$$
Conversely, if a solution of Eq. (1.9) is given so that its real part $f$ is negative in value, then the metric components $g_{ab}(u, v)$ and $g_{34}(u, v)$ can be computed from $\mathcal{E}$ by employing Eqs. (1.6), (1.7) and (1.8) and other equations which will be given in Sec. VB. Unless we explicitly state otherwise, we shall fix the various constants of integration by adopting conventions so that $\mathcal{E}, g_{ab}$ and $g_{34}$ have the values $-1, \delta_{ab}$ and $-1$, respectively, at $u = v = 0$.

For the Weyl vacuum metrics, the metric components $g_{ab}$ in the line element (1.4) are expressible in the conventional forms

$$g_{12} = 0 \ , \ g_{22} = e^{2\psi} \ , \ g_{11} = \rho^2 e^{-2\psi} \ ,$$

(1.10)

whereupon

$$\mathcal{E} = -\exp (2\psi).$$

(1.11)

Note that the above usage of ‘$\psi$’ differs from that in some previous papers by the authors and others.\textsuperscript{10}

We recall that the vacuum field equations imply that\textsuperscript{37}

$$\rho(u, v) = \frac{1}{2} [s(v) - r(u)],$$

(1.12)

where

$$\dot{r}(u) > 0 \text{ for } 0 < u < u_0 \ , \ \dot{s}(v) < 0 \text{ for } 0 < v < v_0 ,$$

(1.13)

and

$$\dot{r}(0) = \dot{s}(0) = 0.$$

(1.14)

By appropriate scalings and choice of the arbitrary constant in $r(u)$ and $s(v)$, we impose the conventions:

$$r(0) = -1 \ , \ s(0) = 1 \ , \ \psi(0, 0) = 0 \ , \ r(u_0) = 1 \ , \ s(v_0) = -1 .$$

(1.15)

The domain of the chart in the region of interaction is

$$IV := \{(u, v) : 0 \leq u < u_0 \ , \ 0 \leq v < v_0 \ , \ -1 \leq r(u) < s(v) \leq 1\}.$$  

(1.16)

In a previous paper on the initial value problem for CPGWP’s, I. Hauser and F. J. Ernst\textsuperscript{7} introduced the notations

$$\psi_3(u) := \psi(u, 0) \ , \ \psi_2(v) := \psi(0, v)$$

(1.17)
for the initial values of \( \psi \), i.e., for the values in the plane wave regions III and II. An important role in our form of the solution of the initial value problem was played by the Abel transforms

\[
g_3(\sigma) := \int_{-1}^{\sigma} \frac{\dot{\psi}_3(r)}{\sqrt{\sigma - r}}, \quad g_2(\sigma) := \int_{\sigma}^{1} \frac{\dot{\psi}_2(s)}{\sqrt{s - \sigma}},
\]

where we are following a practice, which was discussed in detail in a previous paper\(^{28}\), of employing corresponding boldface letters for functions with the domain

\[
D_{IV} := \{(r,s) : -1 \leq r < s \leq 1\}
\]

such that, for example,

\[
\psi_3(r(u)) := \psi_3(u), \quad \psi_2(s(v)) := \psi_2(v), \quad \psi(r(u), s(v)) := \psi(u, v).
\]

Now let us introduce \((-1 < \sigma < 1)\)

\[
G_3(\sigma) := \sqrt{1 - \sigma}g_3(\sigma), \quad G_2(\sigma) := \sqrt{1 + \sigma}g_2(\sigma),
\]

and, for any given real number \(z_0\) such that \(-1 < z_0 < 1\), let

\[
\mathcal{G}(\sigma, z_0) := \begin{cases} 
G_3(\sigma) & \text{if } -1 < \sigma < z_0, \\
G_2(\sigma) & \text{if } z_0 < \sigma < 1,
\end{cases}
\]

and

\[
\lambda(\sigma, z_0) = -\frac{1}{\pi} \int_{-1}^{1} d\sigma' \frac{\mathcal{G}(\sigma', z_0)}{\sigma' - \sigma} \quad \text{(principal value)}
\]

for \(-1 < \sigma < 1\) and \(\sigma \neq z_0\). Then the general solution for \(\psi\) on the domain \(D_{IV}\) is expressible in the form

\[
\psi(r, s) = \frac{1}{\pi} \int_{r}^{s} d\sigma \frac{\lambda(\sigma, z_0)}{\sqrt{(\sigma - r)(s - \sigma)}} = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \lambda(z + \rho \sin \theta, z_0),
\]

where \(r < z_0 < s\), \(z = \frac{1}{2}(s + r)\), \(\rho = \frac{1}{2}(s - r)\). The above expression for \(\psi(r, s)\) is the principal result of this paper. Note that the integral is independent of \(z_0\).

As regards the connection with our new group realization, consider the additive group of all of the functions \(\lambda(\sigma, z_0)\) corresponding to all possible
initial data functions $\psi_3$ and $\psi_2$. Then the corresponding multiplicative group of all $2 \times 2$ matrix functions

$$v(\sigma, z_0) = \exp [\sigma_3 \lambda(\sigma, z_0)]$$

is a subgroup of the group realization. Here, $\sigma_3$ is the usual Pauli spin matrix.

Further study of the principal result (1.24) shows that the continuous extension to the axis points $(r, s) = (z, z), -1 < z < 1$, exists if and only if

$$G_3(\sigma) = G_2(\sigma) \quad \text{(necessary condition for a horizon to exist)},$$

whereupon $\lambda(\sigma, z_0)$ is independent of $z_0$ and

$$\lambda(\sigma) := \lambda(\sigma, z_0) = \psi(\sigma, \sigma).$$

Thus the connection between the axis values and the initial values of the potential is established. When the axis is not accessible, one obtains the result that

$$\psi(r, s) + \frac{1}{\pi} [G_3(z) - G_2(z)] \ln \rho$$

has a continuous extension to the axis points $-1 < z < 1$. This is consistent with Yurtsever’s result.

II. Spectral Potentials $\xi$ and Functions $\lambda$

A. Preliminaries

We shall be concerned in this paper only with the real Ernst potentials $E = -\exp(2\psi)$ or, equivalently, with the potentials $\psi$. Corresponding to each $\psi$, there is a gauge of potentials $\xi$ which depend on a complex (spectral) parameter $\tau$ as well as the coordinates $u$ and $v$, which have the property that $\xi(u, v, \tau)$ is holomorphic in a neighborhood of $\tau = \infty$ and which satisfy $\xi(u, v, \infty) = \psi(u, v)$. Spectral potentials which are closely related to $\xi$ were used in the treatment of the initial value problem for $\psi$ by I. Hauser and F. J. Ernst$^{7,10}$, and the spectral potentials $\xi$ will play a similar role in our exposition of the KC group extension for Weyl metrics.

The present section will be devoted to the concept of the $\xi$-potential and to some key properties of $\xi$. One important result will be a new form of the general solution for $\xi$ corresponding to any given initial data and a byproduct of this result will be the expression (1.24) for $\psi$. 

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Premise: In this section and in the next two, we shall assume only that the initial data functions \( r(u), \psi_3(u) \) and \( s(v), \psi_2(v) \) are in the differentiability class \( C^1 \) over the intervals \( 0 \leq u < u_0 \) and \( 0 \leq v < v_0 \), respectively, and that the inequality conditions (1.13) hold. The following known theorem then holds.

Theorem: There exists exactly one function \( \psi \) whose domain is IV such that the partial derivatives \( \psi_u(u,v), \psi_v(u,v) \) and \( \psi_{uv}(u,v) \) exist and are continuous throughout IV and such that

\[
\psi(u,0) = \psi_3(u), \quad \psi(0,v) = \psi_2(v),
\]

and \( d*(\rho d\psi) = 0. \)

B. The Sets \( D_{IV}, \bar{D}_{IV}, \sigma^\pm, (r,s)^\pm, \) and \( \text{dom} \mu \)

We shall follow the practice of a previous paper\(^7\) and regard the domain \( D_{IV} \) which was defined by Eq. (1.19) as a two-dimensional manifold for which the mapping of \( D_{IV} \) onto IV given by

\[
(r(u), s(v)) \to (u,v)
\]

is a chart. Therefore, concerning the function \( \psi \) which was defined by Eq. (1.20) and has \( D_{IV} \) as its domain, \( d\psi, d*d\psi \) and \( d^2\psi \) are well defined (in terms of the chart) even though the partial derivatives \( \psi_r(r,s) \) and \( \psi_s(r,s) \) do not generally exist at \( r = -1 \) and at \( s = 1 \), respectively. The following extensions of \( D_{IV} \) will be used later.

Definitions: Let

\[
D_{IV} := D_{IV} \cup \{(z,z) : -1 < z < 1\} \quad (2.1a)
\]

and

\[
\bar{D}_{IV} := D_{IV} \cup \{(-1,-1), (1,1)\}. \quad (2.1b)
\]

The Ernst equation for the Weyl metrics is equivalent to the linear equation

\[
d(\rho * d\psi) = 0, \quad (2.2)
\]
where \( \ast dr = dr \) and \( \ast ds = -ds \). A solution of Eq. (2.2) which is expressible as a function of \( r \) times a function of \( s \) is \( \mu(r, s, \tau)^{-1} \) where

\[
\mu(r, s, \tau) = \left( (\tau - r)(\tau - s) \right)^{\frac{1}{2}}
\]

and \( \tau \) is a separation parameter which will be assigned complex values. We shall not use the entire Riemann surface (for fixed \((r, s)\) in \( D_{IV} \)) of the analytic function of \( \tau \) on the right side of Eq. (2.3). The domain of \( \mu \) and various related concepts are defined below.

**Definitions:**

(i) We let \( C \) denote the extended complex plane, \([r, s]\) denote the closed real axis interval whose end points are \( r \) and \( s (r \leq s) \) and \([r, s)\) denote the open real axis interval whose end points are \( r \) and \( s (r < s) \).

(ii) Corresponding to each \( \sigma \) in \( ]-1, 1[ \), let \( \sigma^+ \) and \( \sigma^- \) denote the sets of all sequences

\[
\sigma + w_n \text{ and } \sigma - w_n \ (n = 1, 2, \ldots)
\]

respectively, such that \( \text{Im} w_n > 0 \) and \( w_n \to 0 \) as \( n \to \infty \). For any given \((r, s) \in D_{IV} \), let the sets \((r, s)^+\) and \((r, s)^-\) be

\[
(r, s)^\pm := \{ \sigma^\pm : r < \sigma < s \}.
\]

(iii) We shall employ that branch of the Riemann surface for (2.3) such that

\[
\text{dom} \mu := \{(r, s, t) : (r, s) \in D_{IV} \text{ and } \tau \in \left[ (C-[r, s]) \cup (r, s)^+ \cup (r, s)^- \right] \},
\]

such that, for fixed \((r, s)\), \( \mu(r, s, \tau) \) is a holomorphic function of \( \tau \) throughout \( C - [r, s] \) and satisfies

\[
\mu(r, s, \tau)/\tau = 1 \text{ when } \tau = \infty,
\]

such that, for fixed \((r, s)\) in \( D_{IV} \) and \( r < \sigma < s \),

\[
\mu(r, s, \sigma^\pm) := \lim_{{w \to 0}} \mu(r, s, \sigma \pm w) \\
= \pm i \sqrt{\left( \sigma - r \right) \left( s - \sigma \right)} \ (\text{Im} w > 0),
\]

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and such that 
\[ \mu(r, s, r) = \mu(r, s, s) = 0. \]

(iv) For fixed \((r, s)\) in \(D_{IV}\), there is an obvious one-to-one mapping \(\tau \rightarrow p(\tau)\) of the domain of \(\mu(r, s, \tau)\) onto the closure of one of the sheets of the Riemann surface for \([(|\tau - r| - |\tau - s|)|1/2]\) such that the points \(p(\sigma^\pm)\) constitute the two open line segments which form the bounding edges of the sheet and have common end points \(p(r)\) and \(p(s)\). We adopt that topology for the domain of \(\mu(r, s, \tau)\) such that the mapping \(\tau \rightarrow p(\tau)\) is a homeomorphism. The disjoint lines \((r, s)^+\) and \((r, s)^-\) will be called the upper and lower lips, respectively, of the real axis interval \([r, s]\) in the \(\tau\)-plane.

(v) For any point \(\tau\) in the union of \(C, (-1, 1)^+\) and \((-1, 1)^-\), let \(\mu(\tau)\) denote that function whose domain is
\[ \text{dom} \mu(\tau) := \{(r, s) : -1 \leq r \leq s \leq 1 \text{ and } \mu(r, s, \tau) \neq 0\} \]
and whose values are \(\mu(\tau)(r, s) := \mu(r, s, \tau)\). The function represented by the ratio \(\tau/\mu(\tau)\) is understood to have the value 1 when \(\tau = \infty\).

**Properties of \(\mu\):** It is clear that \(\mu\) is continuous throughout its domain and that
\[ \mu(z, z, \tau) = \tau - z \quad (2.7) \]
for all \(z\) in \([-1, 1]\). Also, \(\tau/\mu(\tau)\) is an analytic function of \((r, s)\) throughout its domain. When \(\tau = \sigma\) is real and \(-1 < \sigma < 1\), the domain of \(\mu(\sigma^+)\) is the same as the domain of \(\mu(\sigma^-)\), but the domain of \(\mu(\sigma)\) consists of two disjoint subregions of \(D_{IV}\). These domains are illustrated in Fig. 1.

**C. The Plemelj Theorem**\(^{38}\)

Domains like that of \(\mu\) will frequently be used for functions which are defined by integrals of Cauchy’s type. For example, let \(f(\sigma)\) be any real valued function defined for all \(\sigma\) on an open interval \(a < \sigma < b\) and summable on \([a, b]\). Suppose \(f\) also obeys a Hölder condition on every closed subinterval
\([c, d]\) such that \(a < c < d < b\); i.e., there exist positive real numbers \(M(c, d)\) and \(\nu(c, d)\) such that \(0 < \nu(c, d) \leq 1\) and

\[
|f(\sigma_2) - f(\sigma_1)| \leq M(c, d) |\sigma_2 - \sigma_1|^\nu(c,d)
\]

for all \(\sigma_1, \sigma_2\) in \([c, d]\). The number \(\nu(c, d)\) is called the index of the Hölder condition obeyed by \(f\) on \([c, d]\).

Now let \(F\) by that function whose domain is

\([C - [a, b]] \cup (a, b)^+ \cup (a, b)^-\)

and which is defined by the integral

\[
F(\tau) := \frac{1}{\pi} \int_a^b d\sigma \frac{f(\sigma)}{\sigma - \tau},
\] (2.8)

where, for all \(a < \sigma < b\),

\[
F(\sigma^\pm) := \lim_{w \to 0} F(\sigma \pm w) \quad (\text{Im} w > 0).
\] (2.9)

A slight generalization of a theorem due to I. Plemelj asserts the following:

**The Plemelj Theorem:**

(i) The function \(F\) exists and so does

\[
\bar{F}(\sigma) := \frac{1}{\pi} \int_a^b d\sigma' \frac{f(\sigma')}{\sigma' - \sigma} \quad (a < \sigma < b).
\] (2.10)

The principal value is always to be understood in integrals like the above. The function \(F\) is holomorphic throughout \(C - [a, b]\), satisfies \(F(\infty) = 0\), and is continuous throughout the union of \(C - [a, b]\), \((a, b)^+\) and \((a, b)^-\).

(ii) The function \(F\) obeys a Hölder condition on any given closed subinterval \([c, d]\) of \([a, b]\). If \(\nu(c, d)\) is the index for \(f\), then the index for \(\bar{F}\) is also \(\nu(c, d)\) if \(\nu(c, d) < 1\) and is \(1 - \epsilon\) (for an arbitrarily small \(\epsilon > 0\)) if \(\nu(c, d) = 1\).

(iii) For all \(a < \sigma < b\),

\[
F(\sigma^\pm) = \bar{F}(\sigma) \pm if(\sigma).
\] (2.11)
In particular, we shall be employing Cauchy type integrals involving the functions \( G_3(\sigma) \) and \( G_2(\sigma) \) which are defined by Eqs. (1.18) and (1.21). The following pertinent theorems have been proven by I. Hauser and F. J. Ernst.\(^7,10\)

**Theorems:**

1. The functions \( g_3(\sigma)/\sqrt{c-\sigma} \) and \( g_2(\sigma)/\sqrt{\sigma-c} \) are summable on \([-1, c]\) and \([c, 1]\), respectively, where \( c \) is any real number such that \(-1 < c < 1\).

2. Also, \( g_3(\sigma) \) and \( g_2(\sigma) \) both obey Hölder conditions of index \( \frac{1}{2} \) on any given closed subinterval of the open interval \(-1 < \sigma < 1\).

The above two statements also hold if \( g_3 \) and \( g_2 \) are replaced by \( G_3 \) and \( G_2 \), respectively.

**D. The Special Gauge of Spectral Potentials \( \xi \)**

One readily proves that

\[
\rho \ast d[1/\mu(\tau)] = -d[(\tau - z)/\mu(\tau)].
\]  

(2.12)

Also, \( \omega \ast (\chi) = -(\ast \omega) \chi \) for any 1-forms \( \omega \) and \( \chi \) in \( D_{IV} \). Therefore, from Eq. (2.2)

\[
d \left[ \frac{(\tau - z - \rho \ast) d\psi}{\mu(\tau)} \right] = 0.
\]  

(2.13)

The domain (for given \( \tau \)) of the expression in brackets in the above equation is the intersection of the domain of \( \tau/\mu(\tau) \) with \( D_{IV} \).

For now, let us avoid the complications which arise when \( \tau \) is real and lies on \([-1, 1]\) (Fig. 1 illustrates some of these complications) by temporarily confining our discussion to the domain

\[
D_{IV} \times (C - [-1, 1]) = \{(r, s, \tau) : (r, s) \in D_{IV} \text{ and } \tau \in C - [-1, 1]\}.
\]  

(2.14)

Equation (2.13) clearly implies the existence of infinitely many functions \( \xi \) which have the above domain such that, if we let \( \xi(\tau) \) denote that function whose domain is \( D_{IV} \) and whose values are

\[
\xi(\tau)(r, s) := \xi(r, s, \tau),
\]
then $\xi(\tau)$ is a holomorphic function of $\tau$ throughout $C - [-1, 1]$, and $d\xi(\tau)$ exists and satisfies

$$d\xi(\tau) = \left[\frac{\tau - z - \rho^*}{\mu(\tau)}\right] d\psi$$

(2.15)

and

$$\xi(\infty) = \psi.$$  

(2.16)

There is exactly one solution $\xi$ of the above conditions and equations for each choice of

$$\xi_1(\tau) := \xi(-1, 1, \tau)$$

(2.17)

such that $\xi_1(\tau)$ is holomorphic throughout $C - [-1, 1]$ and satisfies $\xi_1(\infty) = 0$. Note that $\xi_1(\tau)$ are the values of $\xi(\tau)$ at the collision points of the colliding plane waves.

Now, what we require for the purpose of this paper is a very special choice of $\xi_1(\tau)$, viz., one which maximizes the domain of holomorphy of $\xi(r, s, \tau)$ in the $\tau$-plane. This selection principle and our previous experiences with specific examples led us to seek a choice of $\xi_1(\tau)$ such that, for fixed $(r, s)$, $\xi(r, s, \tau)$ has a holomorphic extension in the $\tau$-plane to the domain $C - [r, s]$. This is, of course, the same domain of holomorphy as that of $\mu(r, s, \tau)$ and is the maximal domain of holomorphy one can generally hope to achieve.

It turned out that our selection principle for $\xi_1(\tau)$ resulted in a more interesting structure than the one originally envisioned. We were inevitably led to a one-parameter family of choices of $\xi_1(\tau)$:

$$\xi_1(\tau, z_0) \text{ where } -1 < z_0 < 1$$

such that the corresponding $\xi$-potential $\xi(r, s, \tau, z_0)$ has a holomorphic extension to $C - [r, s]$ provided that one restricts $(r, s, z_0)$ to the range $-1 \leq r < z_0 < s \leq 1$.

For example, consider the Kasner metrics for which $\psi$ is given by

$$\psi^K(r, s) = \left(\frac{1 - n}{2}\right) \ln \rho$$

(2.18)

where $n$ is any real number and the Weyl canonical coordinates $z$ and $\rho$ are

$$z = \frac{1}{2}(s + r), \quad \rho = \frac{1}{2}(s - r).$$

(2.19)
We found the corresponding one-parameter family of spectral potentials

\[ \xi^K(r, s, \tau, z_0) = \left( \frac{1-n}{2} \right) \ln \left[ \frac{2(\tau - z_0)\rho}{\tau - z + \mu(r, s, \tau)} \right] \]  

(2.20)

where the parameter \( z_0 \) has the range

\[-1 < z_0 < 1\]

and where the cut is the union of \([r, z_0]\) and \([z_0, s]\), and the value at \( \tau = \infty \) is \( \psi^K(r, s) \). An alternative form is

\[ \xi^K(r, s, \tau, z_0) = \psi^K(r, s) + (n-1) \ln \left\{ \frac{1}{2} \left[ \left( \frac{\tau - r}{\tau - z_0} \right)^{1/2} + \left( \frac{\tau - s}{\tau - z_0} \right)^{1/2} \right] \right\} \]  

(2.21)

with branch cuts \([r, z_0]\) and \([z_0, s]\) for the respective square roots both of which have unit value at \( \tau = \infty \). For fixed \((r, s, z_0)\), it is clear that the above \( \xi^K(r, s, \tau, z_0) \) is a holomorphic function of \( \tau \) throughout \( C - [r, s] \) provided that \((r, s, z_0)\) is restricted to the range

\[-1 \leq r < z_0 < s \leq 1.\]

This restriction presents no problem since one can cover any given point \((r, s)\) in \( D_4 \) simply by selecting \( z_0 \) so that \( r < z_0 < s \).

There are other pertinent properties of the above one-parameter family of spectral potentials. Observe that \( \xi^K(r, s, \tau, z_0) \) is a continuous function of \( \tau \) over the domain

\( \{C - [r, s]\} \cup (r, z_0)^+ \cup (r, z_0)^- \cup (z_0, s)^+ \cup (z_0, s)^- \),

and that the singularity at \( \tau = z_0 \) is logarithmic.

We have found in fact that there exists a similar one-parameter gauge of spectral potentials for any given \( \psi \). This gauge will play a vital role in our extension of the KC group and will be called the special gauge of spectral potentials \( \xi \). The first step in formulating the definition of the special gauge will be to introduce some domains which are used in its definition.
E. The Sets $D(r, s, z_0)$, $D(\tau, z_0)$, $D$, and the Special Gauge of Spectral Potentials $\xi$

Definitions:

(i) For each choice of $(r, s, z_0)$ such that

$$-1 \leq r < z_0 < s \leq 1,$$

let

$$D(r, s, z_0) := \{ \tau : \tau \text{ is any point in the union of}$$

$$C - [r, s], (r, z_0)^+, (r, z_0)^-, (z_0, s)^+, (z_0, s)^-,$$

the singlet set $\{r\}$ if $r \neq -1,$

and the singlet set $\{s\}$ if $s \neq 1\}.$ \hspace{1cm} (2.22)

(ii) For each choice of $(\tau, z_0)$ such that $-1 < z_0 < 1$ and

$$\tau \in [(C - \{z_0\}) \cup (-1, z_0)^+ \cup (-1, z_0)^- \cup (z_0, 1)^+ \cup (z_0, 1)^-],$$

let $D(\tau, z_0)$ be defined as follows for various ranges of $\tau$:

$$D(\tau, z_0) := \{(r, s) : -1 \leq r < z_0 < s \leq 1\}$$

if $\tau \in (C - [-1, 1]),$ \hspace{1cm} (2.24a)

$$D(-1, z_0) := \{(r, s) : -1 < r < z_0 < s \leq 1\},$$

$$D(1, z_0) := \{(r, s) : -1 \leq r < z_0 < s < 1\},$$

$$D(\sigma^+, z_0) := \{(r, s) : -1 \leq r < z_0 < \sigma \leq s \leq 1\}$$

if $z_0 < \sigma < 1,$ \hspace{1cm} (2.24d)

$$D(\sigma, z_0) := \{(r, s) : -1 \leq r < z_0 < s \leq \sigma\}$$

if $z_0 < \sigma < 1,$ \hspace{1cm} (2.24e)

$$D(\sigma^+, z_0) := \{(r, s) : -1 \leq r \leq \sigma < z_0 < s \leq 1\}$$

if $-1 < \sigma < z_0,$ \hspace{1cm} (2.24f)

$$D(\sigma, z_0) := \{(r, s) : \sigma \leq r < z_0 < s \leq 1\}$$

if $-1 < \sigma < z_0.$ \hspace{1cm} (2.24g)

Figure 2 depicts $D(\tau, z_0)$ for the cases (2.24a), (2.24b) and (2.24c). Figure 3 depicts the cases (2.24d) and (2.24e). Figure 4 depicts
the cases (2.24f) and (2.24g). It is instructive to compare Figs. 3 and 4 with Fig. 1.

(iii) Let

\[
  D := \{(r, s, \tau, z_0) : -1 \leq r < z_0 < s \leq 1 \text{ and } \tau \in D(r, s, z_0)\}.
\]

Equivalently,

\[
  D := \{(r, s, \tau, z_0) : -1 < z_0 < 1, \tau \text{ is any point} \in \text{set (2.23)} \text{ and } (r, s) \in D(\tau, z_0)\}.
\]

(iv) If \( \xi \) denotes a function whose domain is \( D \), then corresponding to each choice of \((\tau, z_0)\) such that \(-1 < z_0 < 1\) and \(\tau\) is in the set (2.23), \(\xi(\tau, z_0)\) will denote that function which has domain \( D(\tau, z_0) \) and values

\[
  \xi(\tau, z_0)(r, s) := \xi(r, s, \tau, z_0).
\]

We next define the special gauge of spectral potentials \( \xi \). The definition is given without regard to the questions of existence and uniqueness. The existence and uniqueness of \( \xi \) for given \( \psi \) will be covered in Sec. III. In the meantime, existence will be granted.

**Definition:** For given \( \psi \), we shall henceforth let \( \xi \) denote a function whose domain is \( D \) and which satisfies the conditions (i) to (iv) given below.

(i) The function \( \xi(\tau, z_0) \) obeys the following integral equation for all \((r, s)\) and \((a, b)\) in \( D(\tau, z_0) \):

\[
  \xi(r, s, \tau, z_0) = \xi(a, b, \tau, z_0) + \int_{(a,b)}^{(r,s)} \left( \frac{\tau - z' - \rho' \ast}{\tau - z' + \rho' \ast} \right)^{\frac{1}{2}} d\psi(r', s')
\]

where note that

\[
  \left( \frac{\tau - z' - \rho' \ast}{\tau - z' + \rho' \ast} \right)^{\frac{1}{2}} = \frac{\tau - z' - \rho' \ast}{\mu(r', s', \tau)},
\]

since \((\tau - z' - \rho' \ast)(\tau - z' + \rho' \ast) = [\mu(r', s', \tau)]^2\), and where \((r', s')\) denotes any point on the integration path. The integration path
is any segmentally smooth path which lies entirely in \( D(\tau, z_0) \) and which has \((a, b)\) and \((r, s)\) as initial and final points, respectively. The integral in Eq. (2.28) is defined in the sense of Lebesgue.

(ii) For any given \((r, s, z_0)\) such that \(-1 < r < z_0 < s < 1\), \(\xi(r, s, \tau, z_0)\) is a continuous function of \(\tau\) throughout \( D(r, s, z_0) \), as defined by Eq. (2.22), and is holomorphic throughout \( C - [r, s] \).

(iii) The following relation holds for all \(-1 < z_0 < 1\):

\[
\xi_1(\infty, z_0) = 0
\]

where, for all \(\tau \in D(-1, 1, z_0)\),

\[
\xi_1(\tau, z_0) := \xi(-1, 1, \tau, z_0).
\]

(iv) Let \( C(\epsilon, c) \) denote any positively oriented circle in the complex plane with radius \(\epsilon\), with center \(c\), and with the points \(c \pm \epsilon\) deleted. Then, for all non-negative integers \(n\),

\[
\int_{C(\epsilon, c)} d\tau \left( \tau - c \right)^n \xi_1(\tau, z_0) \rightarrow 0 \quad \text{as} \quad \epsilon \rightarrow 0 \quad \text{when} \quad c = -1, 1, \text{and} \quad z_0.
\]

Note that the above integral exists since \(\xi_1(\tau, z_0)\) is continuous and bounded on \( C(\epsilon, c) \). This follows from the continuity [condition (ii)] of \(\xi_1(\tau, z_0)\) throughout \( D(-1, 1, z_0) \).

Notes on condition (i) in the above definition: When applying the integral equation (2.28), it is useful to know that \(*dr = dr\), \(*ds = -ds\), \(** = 1\), and that for both \(r < \sigma < z_0\) and \(z_0 < \sigma < s\),

\[
\left( \frac{\sigma^\pm - s}{\sigma^\pm - r} \right)^\frac{1}{2} = \pm i \frac{s - \sigma}{\sigma - r},
\]

while for both \(\sigma < r\) and \(\sigma > s\),

\[
\left( \frac{\sigma - s}{\sigma - r} \right)^\frac{1}{2} = \frac{s - \sigma}{\sigma - r},
\]

The existence of the integral on any segmentally smooth path in \(D(\tau, z_0)\) is manifest. Also, the independence of the integral on the choice of the
integration path which joins \((a, b)\) to \((r, s)\) requires only moderate effort to prove with the aid of Eq. (2.13). The only parts of the proof which may not be obvious are those for which the integration paths meet the dashed lines depicted in Figs. 3 and 4, and the reason is that \(\mu(r, s, \tau)\) vanishes on these dashed lines. This problem can be handled most easily by introducing appropriate new charts for effecting the integrations in neighborhoods of the dashed lines. For example, in a neighborhood of the dashed line in the domain \(D(\sigma^\pm, z_0)\) in Fig. 3, one can let

\[
\left(\frac{\sigma^\pm - z' - \rho' \ast}{\sigma^\pm - z' + \rho' \ast}\right)^{1/2} d\psi(r', s') = \\
\pm \left[ \frac{q}{\sqrt{\sigma - r(u')} du' \psi(u', v') - 2\sqrt{\sigma - r(u')} dq \psi_s(r(u'), \sigma + q^2) \right]
\]

where we used Eq. (2.33) and then introduced the new coordinates \(u'\) and \(q = \sqrt{s' - \sigma}\), with \(v'\) defined by \(s(v') = \sigma + q^2\). Observe that \(-1 \leq r(u') < z_0\) and \(q = 0\) on the dashed line which borders \(D(\sigma^\pm, z_0)\) in Fig. 3. We leave further details concerning the proof of path independence of the integral (2.28) for the interested reader. Also, we leave it to the reader to establish that Eq. (2.28) implies that \(\xi(\tau, z_0)\) is continuous throughout its domain \(D(\tau, z_0)\), and that \(d\xi(r, s, \tau, z_0)\) exists and satisfies Eq. (2.15) at all points \((r, s)\) at which \(\mu(r, s, \tau) \neq 0\).

**Notes on conditions (iv) in the definition of the special gauge:** The combined conditions (ii), (iii) and (iv) enable us to regard \(\xi_1(\tau, z_0)\) as if it is the usual complex representation of a two-dimensional electrostatic field whose sources all lie on the real axis interval \([-1, 1]\). The conditions (ii) inform us that there are continuous summable (Lebesgue integrable) source distributions on the open intervals \([-1, z_0[\) and \(]z_0, 1[\) but that there are no other sources on these open intervals. The conditions (iv), as given by (2.32), reassure us that there are no point multipole sources (of any order, whatsoever) at the endpoints of these open intervals. In other words, there are no sources which produce poles or isolated essential singularities of \(\xi_1(\tau, z_0)\) at the points \(\tau = -1, 1, z_0\). So, in effect, the only sources for \(\xi_1(\tau, z_0)\) are the continuous summable line distributions on \([-1, z_0[\) and \(]z_0, 1[\).
Nevertheless, the limits of $\xi_1(\tau, z_0)$ as $\tau \to -1, 1$ or $z_0$ do not generally exist. This is not surprising if we employ the electrostatic analogue. However, a few explanations are in order. Let us start with $\tau = \pm 1$.

When the initial data functions $\psi_3(r)$ and $\psi_2(s)$ have continuous first derivatives at $r = -1$ and $s = 1$, respectively, then the limits of $\xi_1(\tau, z_0)$ as $\tau \to -1$ and as $\tau \to 1$ turn out to exist. For example, these limits exist for the Kasner metric $\xi$-potentials given by Eqs. (2.20) and (2.21). However, they do not generally exist. For example, they do not exist for any CPGWP.\textsuperscript{39}

As regards the singularity at $\tau = z_0$, we shall see that it is always logarithmic when it exists, that it arises from a finite step discontinuity of the source distribution at $\tau = z_0$, and that it does not exist if and only if $\psi(r, s)$ has a limit as $(r, s) \to (z_0, z_0)$.

F. The Function $\Xi$ and a Theorem

We shall next give some simple results which revolve around the fact that the difference $\xi(r, s, \tau, z_0) - \xi_1(\tau, z_0)$ is independent of $z_0$.

Definition: For any

$$\tau \in (C - [-1, 1]) \cup (-1, 1)^+ \cup (-1, 1)^- ,$$  \hspace{1cm} (2.35)

let

$$D(\tau) := D_{IV} \text{ if } \tau \in C - [-1, 1] ,$$

$$D(\sigma^\pm) := \{(r, s) \in D_{IV} : r \leq \sigma \leq s\} \text{ if } -1 < \sigma < 1 .$$ \hspace{1cm} (2.36)

The domain $D(\sigma^\pm)$ is depicted in Fig. 5.

Definition: For given $\psi$, let $\Xi$ denote the function whose domain is

$$\text{dom} \Xi := \{(r, s, \tau) : \tau \text{ is in the set (2.35) and } (r, s) \in D(\tau)\}$$ \hspace{1cm} (2.37)

and whose values are

$$\Xi(r, s, \tau) := \int_{(r, s)}^{(r', s')} \left( \frac{\tau - z' - \rho^*}{\tau - z' + \rho^*} \right)^{\frac{1}{2}} d\psi(r', s')$$ \hspace{1cm} (2.38)
where \((r', s')\) is any point on the integration path which joins \((-1, 1)\) to \((r, s)\) and which lies in \(D(\tau)\). Statements similar to those made for \(\xi\) [after Eq. (2.28) and after Eq. (2.34)] are also applicable to \(\Xi\), though allowances must be made for the fact that the domains of \(\xi\) and \(\Xi\) differ.

**Theorem:**

(i) If \((r, s, \tau) \in \text{dom} \Xi\), then

\[
\xi(r, s, \tau, z_0) - \xi_1(\tau, z_0) = \Xi(r, s, \tau)
\]

for all \(z_0\) such that \(-1 < z_0 < s\), and such that \(z_0 \neq \sigma\) when \(-1 < \sigma < 1\) and \(\tau = \sigma \pm\). In particular, note the choice \(z_0 = z = (s + r)/2\).

(ii) If \((r, s) \in D_{IV}\), then

\[
\xi(r, s, \infty, z_0) = \psi(r, s)
\]

for all \(z_0\) such that \(r < z_0 < s\). Moreover,

\[
\xi(r, s, \infty, z) = \Xi(r, s, \infty) = \psi(r, s).
\]

**Proof:** The proof is simple and uses Eqs. (2.28), (2.30), (2.38) and our convention \(\psi(-1, 1) := 0\). The depiction of the various domains in Figs. 2 to 5 may be helpful.

**G. The Spectral Function \(\lambda\)**

**Theorem:** Suppose \(\tau = \sigma\) and \(-1 < \sigma < z_0\) or \(z_0 < \sigma < 1\). Then

\[
\lambda(\sigma, z_0) := \xi(\sigma, s, \sigma, z_0) \text{ if } -1 < \sigma < z_0
\]

(2.42a)

and

\[
\lambda(\sigma, z_0) := \xi(r, \sigma, \sigma, z_0) \text{ if } z_0 < \sigma < 1
\]

(2.42b)

are independent of \(s\) over the range \(z_0 < s \leq 1\) and of \(r\) over the range \(-1 \leq r < z_0\), respectively.
Proof: The above theorem follows easily from Eq. (2.28) and the relations
\( r = z - \rho \) and \( s = z + \rho \).

Note that the theorem contains a definition of the function \( \lambda \) which appeared in Eqs. (1.23) and (1.24). Those equations will be derived in Sec. III. As a point of interest for those who are acquainted with previous work on KC transformations, Eqs. (2.42a) and (2.42b) are generalizations of the axis relation for the special case of the vacuum Weyl metrics. As we have already stated in Sec. I and as we shall show in Sec. IV, \( \psi \) has a continuous extension to \( D_{IV} \) (see Eq. (2.1a)) if and only if \( \xi(r, s, \tau, z_0) \) is independent of \( z_0 \). In that case, \( \lambda(\sigma, z_0) \) is independent of \( z_0 \) and it will be shown that

\[
\lambda(\sigma) := \lambda(\sigma, z_0) = \psi(\sigma, \sigma) \tag{2.43}
\]

The above Eq. (2.43) is the axis relation for the special case of the vacuum Weyl metrics.

The definition of \( \lambda \) which was given above was suggested by Figs. 3 and 4, which depict the \((r, s)\)-plane for fixed \((\sigma, z_0)\). The set of points on which \( \xi(r, s, \sigma, z_0) \) and \( \xi(r, s, \sigma^\pm, z_0) \) take on the values \( \lambda(\sigma, z_0) \) are represented by the dashed lines in those figures.

An equivalent definition can be based on a picture of the \((\tau, z_0)\)-space for fixed \((r, s)\). Specifically, given that \((r, s) \in D_{IV} \),

\[
\lambda(r, z_0) := \xi(r, s, r, z_0) \tag{2.44a}
\]

and

\[
\lambda(s, z_0) := \xi(r, s, s, z_0) \tag{2.44b}
\]

for all \( z_0 \) such that \( r < z_0 < s \). Thus, \( \lambda(r, z_0) \) and \( \lambda(s, z_0) \) are the values of \( \xi(r, s, \tau, z_0) \) at the end point \( r \) of the pair of lips \((r, z_0)^\pm\) and at the end point \( s \) of the pair of lips \((z_0, s)^\pm\), respectively.

III. The Spectral Function \( \xi \) as a Solution of a Hilbert Problem

A. A Preliminary Theorem on Cauchy Integrals

Definition of \( \Gamma^+ \) and \( \Gamma^- \) for a given closed contour \( \Gamma \): Suppose \( \Gamma \) denotes a simple, segmentally smooth, positively oriented closed contour
in the complex plane $\mathbb{C}$. Then $\Gamma^+$ and $\Gamma^-$ will denote those disjoint open subsets of $\mathbb{C}$ which are bounded and unbounded, respectively, which have $\Gamma$ as their common boundary, and which satisfy $C = \Gamma \cup \Gamma^+ \cup \Gamma^-$. This is depicted by Fig. 6.

**Definitions of $[a,b[$ and $]a,b]$**: Suppose $a$, $b$ and $d$ are real numbers such that $a < d < b$. Then

$$[a,d[ := \{\sigma : a \leq \sigma < d\}$$

and

$$]d,b] := \{\sigma : d < \sigma \leq b\}.$$

**Theorem**: Consider the function $F(\tau)$ which is defined by the Cauchy integral of Eq. (2.8), where the summable function $f(\sigma)$ in the integrand has $]a,b[$ as its domain and obeys a Hölder condition on any closed subinterval of $]a,b[$. Let $\Gamma$ denote any simple, segmentally smooth, positively oriented closed contour in $\mathbb{C}$ such that $d \in \Gamma$, $[a,d[ \subset \Gamma^+$ and $]d,b] \subset \Gamma^-$. (3.1)

Then, for any non-negative integer $n$,

$$\int_{\Gamma-\{d\}} d\tau (\tau - a)^n F(\tau) = -2i \int^d_a d\sigma (\sigma - a)^n f(\sigma). \quad (3.2)$$

If, instead, $\Gamma$ is chosen so that $d \in \Gamma$, $]d,b] \subset \Gamma^+$ and $[a,d[ \subset \Gamma^-$. (3.3)

then Eq. (3.2) still holds provided one makes the replacements $i \to -i$ and $a \to b$. NOTE: The integral on the left side of Eq. (3.2) exists since $F(\tau)$ is continuous and bounded on $\Gamma - \{d\}$. This follows from the Plemelj theorem in Sec. IIC.

**Proof**: Let $a'$ and $b'$ be any real points such that $a < a' < d < b' < b$. Then

$$\int_{\Gamma-\{d\}} d\tau (\tau - a)^n F(\tau) = \frac{1}{\pi} \int_{\Gamma-\{d\}} d\tau \int^a_{a'} d\sigma \frac{(\tau - a)^n f(\sigma)}{(\sigma - \tau)}$$

$$+ \frac{1}{\pi} \int_{\Gamma-\{d\}} d\tau \int^b_{b'} d\sigma \frac{(\tau - a)^n f(\sigma)}{(\sigma - \tau)}$$

$$+ \int_{\Gamma-\{d\}} d\tau (\tau - a)^n F(a',b',\tau), \quad (3.4)$$
where
\[ F(a', b', \tau) := \frac{1}{\pi} \int_{a'}^{b'} d\sigma \frac{f(\sigma)}{\sigma - \tau}. \] (3.5)

The orders of integration in the first two iterated integrals on the right side of Eq. (3.4) can be interchanged. Upon interchanging the orders of integration and completing the integrations over \( \tau \), one obtains
\[ \int_{\Gamma - \{d\}} \frac{d\tau}{(\tau - a)^n} F(\tau) = -2i \int_a^{a'} d\sigma (\sigma - a)^n f(\sigma) \]
\[ + \frac{1}{\pi} \int_{\Gamma - \{d\}} d\tau (\tau - a)^n F(a', b', \tau). \] (3.6)

From the Plemelj theorem in Sec. IIIC, \( F(a', b', \tau) \) is a holomorphic function of \( \tau \) throughout \( C - [a', b'] \) and is a continuous function of \( \tau \) throughout the union of \( C - [a', b'], (a', b')^+ \) and \( (a', b')^- \) such that
\[ F(a', b', \sigma^+) - F(a', b', \sigma^-) = 2i f(\sigma) \] (3.7)
for all \( a' < \sigma < b' \). Another well known theorem of Plemelj tells us that the singularity of \( F(a', b', \tau) \) at \( \tau = a' \) is merely logarithmic. Therefore, the path of integration \( \Gamma - \{d\} \) in the second term on the right side of Eq. (3.6) can be deformed (without changing the value of the integral) until it becomes
\[ \frac{1}{\pi} \int_{d}^{a'} d\sigma (\sigma - a)^n [F(a', b', \sigma^+) - F(a', b', \sigma^-)]. \]

So, from Eqs. (3.6) and (3.7), we obtain our final conclusion (3.2). The proof for the alternative case when the conditions (3.3) hold is similar.

**Corollary:** Let \( c = a \) or \( b \), and let \( C(\epsilon, c) \) denote the positively oriented circle with radius \( \epsilon \), with center \( c \), and with the points \( c \pm \epsilon \) deleted. Then, for any non-negative integer \( n \), the function \( F(\tau) \) defined by the Cauchy integral in Eq. (2.8) satisfies:
\[ \int_{C(\epsilon, c)} d\tau (\tau - a)^n F(\tau) \to 0 \text{ as } \epsilon \to 0. \] (3.8)
B. A Hilbert Problem on \([-1, 1]\)

In Sec. III, as before, our only premises concerning the initial data functions will be those given in Sec. IIA. Also, we shall continue to assume that \(\xi\) exists until Sec. IIIC.

We now apply the integral equation (2.28) to the straight line paths

\[ (r, \sigma) \to (r, s) \text{ where } -1 \leq r < z_0 < \sigma \leq s \leq 1 \]

and

\[ (\sigma, s) \to (r, s) \text{ where } -1 \leq r \leq \sigma < z_0 < s \leq 1 \]

in the domains \(D(\sigma^\pm, z_0)\) which are depicted in Figs. 3 and 4, respectively. With the aid of Eqs. (2.33), (2.42a) and (2.42b), we obtain

\[ \xi(r, s, \sigma^\pm, z_0) = \lambda(\sigma, z_0) \mp i \int_{s'}^{s} ds' \sqrt{\frac{\sigma - r}{s' - \sigma}} \psi_{s'}(r, s') \quad (3.9a) \]

for \(z_0 < \sigma < s\), and

\[ \xi(r, s, \sigma^\pm, z_0) = \lambda(\sigma, z_0) \pm i \int_{r'}^{r} dr' \sqrt{\frac{s - \sigma}{\sigma - r'}} \psi_{r'}(r', s) \quad (3.9b) \]

for \(r < \sigma < z_0\). The above Eqs. (3.9a) and (3.9b) will be used both here and in Sec. IIIE.

Here, we apply Eqs. (3.9a) and (3.9b) when \((r, s) = (-1, 1)\). From the definitions (1.18) and (1.21) of \(G_j(\sigma)\) in terms of the initial data functions, from the definition (1.22) of \(G(\sigma, z_0)\), and from the definition (2.31) of \(\xi_1(\tau, z_0)\),

\[ \xi_1(\sigma^\pm, z_0) = \lambda(\sigma, z_0) \mp i G(\sigma, z_0) \quad (3.10) \]

or, equivalently,

\[ \xi_1(\sigma^+, z_0) - \xi_1(\sigma^-, z_0) = -2i G(\sigma, z_0) \quad (3.11) \]

and

\[ \xi_1(\sigma^+, z_0) + \xi_1(\sigma^-, z_0) = 2\lambda(\sigma, z_0). \quad (3.12) \]

The task of finding \(\xi_1(\tau, z_0)\) so that Eq. (3.11) is satisfied is an example of a simple kind of Hilbert problem.
A Hilbert Problem on $[-1, 1]$: Let the function $G(\sigma, z_0)$ be given in terms of the initial data functions. We seek a function $\xi_1$ with the domain
$$\{(\tau, z_0) : -1 < z_0 < 1 \text{ and } \tau \in D(-1, 1, z_0)\}$$
such that conditions (ii), (iii) and (iv) in Sec. IIE are satisfied by $\xi_1$ and such that Eq. (3.11) holds.

**Theorem:** There exists exactly one solution
$$\xi_1(\tau, z_0) = -\frac{1}{\pi} \int_{-1}^{1} d\sigma \frac{G(\sigma, z_0)}{\sigma - \tau} \quad (3.13)$$
of the preceding Hilbert problem.

**Proof:** Employ the two theorems in Sec. IIC and the corollary [Eq. (3.8)] in Sec. IIIA to prove that Eq. (3.13) is a solution of the Hilbert problem. As regards uniqueness, let
$$\Delta := \xi_1' - \xi_1$$
for any given solutions $\xi_1$ and $\xi_1'$ of the Hilbert problem. Conditions (ii) and (iii) in Sec. IIE, taken together with Eq. (3.11), imply that $\Delta(\tau, z_0)$ is a continuous function of $\tau$ throughout
$$(C - [-1, 1]) \cup (-1, z_0)^+ \cup (-1, z_0)^- \cup (z_0, 1)^+ \cup (z_0, 1)^-,$$
that $\Delta(\tau, z_0)$ is holomorphic throughout $C - [-1, 1]$, that $\Delta(\infty, z_0) = 0$, and that $\Delta(\sigma^+, z_0) = \Delta(\sigma^-, z_0)$ for all $-1 < \sigma < z_0$ and $z_0 < \sigma < 1$. A well known theorem of Riemann then implies that $\Delta(\tau, z_0)$ has a holomorphic continuation to the domain $C - \{-1, 1, z_0\}$. However, condition (iv) [Eq. (2.32)] in Sec. IIE tells us that there are no isolated singularities at $\tau = -1, 1, z_0$. So, from Liouville’s theorem and the equation $\Delta(\infty, z_0) = 0$, it follows that $\Delta := \xi_1' - \xi_1$ is identically zero.

**Corollary:** The solution for $\lambda(\sigma, z_0)$ is given by Eq. (1.23). Moreover, $\lambda(\sigma, z_0)$ obeys a Hölder condition of index $\frac{1}{2}$ on every closed subinterval of the open intervals $-1 < \sigma < z_0$ and $z_0 < \sigma < 1$. Finally,
$$\xi(r, s, \sigma^+, z_0) + \xi(r, s, \sigma^-, z_0) = 2\lambda(\sigma, z_0) \quad (3.14)$$
for all $(r, s)$ in $D(\sigma^\pm, z_0)$.

**Proof:** Use the two theorems in Sec. IIC as well as Eqs. (3.12) and (3.13). Equation (3.14) is implied directly by Eqs. (3.9a) and (3.9b).
C. The Existence and Uniqueness of $\xi$

**Theorem:** For given $\psi$, a spectral potential $\xi$ in the special gauge exists.

Proof: In this proof, $\xi_1(\tau, z_0) := \xi(-1, 1, \tau, z_0)$ will be defined by Eq. (3.13) for all $\tau$ in $D(-1, 1, z_0)$. [See Eq. (2.22).]

The first step is to introduce the function $\xi^{(1)}$ whose domain is

$$\text{dom}\xi^{(1)} := \{ (r, s, \tau, z_0) : (\tau, z_0) \in \text{dom}\xi_1$$

and $(r, s) \in D(\tau, z_0)$

$$= \{ (r, s, \tau, z_0) : -1 \leq r < z_0 < s \leq 1$$

and $\tau \in D(-1, 1, z_0) \} \quad (3.15)$$

and whose values are

$$\xi^{(1)}(r, s, \tau, z_0) := \xi_1(\tau, z_0) + \Xi(r, s, \tau), \quad (3.16)$$

where $D(\tau, z_0)$ is defined by Eq. (2.24a) and depicted by Fig. 2 when $\tau \in C - [-1, 1]$, where $D(\sigma^\pm, z_0)$ is defined by Eq. (2.24d) and depicted by Fig. 3 when $z_0 < \sigma < 1$, where $D(\sigma^\pm, z_0)$ is defined by Eq. (2.24f) and depicted by Fig. 4 when $-1 < \sigma < z_0$, and where $\Xi$ is defined in Sec. IIF. By using alternative paths of integration in the defining equation (2.38) for $\Xi$, we obtain

$$\Xi(r, s, \tau) = \int_{-1}^{r} dr' \left( \frac{\tau - 1}{\tau - r'} \right)^{\frac{1}{2}} \psi_3(r') + \int_{1}^{s} ds' \left( \frac{\tau - r}{\tau - s'} \right)^{\frac{1}{2}} \psi_{s'}(r', s') \quad (3.17)$$

and

$$\Xi(r, s, \tau) = \int_{1}^{s} ds' \left( \frac{\tau + 1}{\tau - s'} \right)^{\frac{1}{2}} \psi_2(s') + \int_{-1}^{r} dr' \left( \frac{\tau - s}{\tau - r'} \right)^{\frac{1}{2}} \psi_{r'}(r', s), \quad (3.18)$$

where recall that $\psi_3(r) := \psi(r, 1)$ and $\psi_2(s) := \psi(-1, s)$. From the above Eq. (3.16), we obtain with the aid of Eqs. (1.18), (1.21), (1.22), (2.33), (2.34) and (3.11):

$$\xi^{(1)}(r, s, \sigma^+, z_0) - \xi^{(1)}(r, s, \sigma^-, z_0) = 0 \text{ for } -1 < \sigma \leq r \quad (3.19)$$

by employing Eq. (3.17), and

$$\xi^{(1)}(r, s, \sigma^+, z_0) - \xi^{(1)}(r, s, \sigma^-, z_0) = 0 \text{ for } s \leq \sigma < 1 \quad (3.20)$$
by employing Eq. (3.18).

The second phase of the proof uses Eqs. (3.19) and (3.20) to define a new function \( \xi^{(2)} \) whose domain is [compare with Eq. (3.15)]

\[
\text{dom} \xi^{(2)} := \{(r, s, \tau, z_0) : -1 \leq r < z_0 < s \leq 1 \; \text{and} \; \tau \in [D(r, s, z_0) - \{-1, 1\}] \}
\]  
(3.21)

and whose values are

\[
\xi^{(2)}(r, s, \tau, z_0) := \xi^{(1)}(r, s, \tau, z_0)
\]

when \( \tau \in C - [-1, 1] \),

\[
\text{or when} \; \tau = \sigma^\pm \; \text{and} \; r < \sigma < s,
\]  
(3.22)

and, using Eqs. (3.19) and (3.20),

\[
\xi^{(2)}(r, s, \sigma, z_0) := \xi^{(1)}(r, s, \sigma^\pm, z_0)
\]

when \(-1 < \sigma \leq r \) or \( s \leq \sigma < 1 \).

(3.23)

From its definition by Eqs. (3.15) and (3.16), \( \xi^{(1)}(r, s, \tau, z_0) \) is (for fixed \( r, s, z_0 \)) holomorphic throughout \( C - [-1, 1] \) and continuous throughout

\[
(C - [-1, 1]) \cup (-1, z_0)^+ \cup (-1, z_0)^- \cup (z_0, 1)^+ \cup (z_0, 1)^-.
\]

Therefore, from Eq. (3.21) to Eq. (3.23) and a well known theorem of Riemann, \( \xi^{(2)}(r, s, \tau, z_0) \) is holomorphic throughout

\[
C - ([r, s] \cup \{-1, 1\})
\]  
(3.24)

and is a continuous function of \( \tau \) throughout

\[
C - ([r, s] \cup \{-1, 1\}) \cup (r, z_0)^+ \cup (r, z_0)^- \cup (z_0, 1)^+ \cup (z_0, 1)^-.
\]

Note that the above set includes the point \( \tau = r \) when \( r > -1 \) and the point \( \tau = s \) when \( s < 1 \).

The final phase of the proof considers the two isolated points

\[
\tau = -1 \; \text{when} \; r > -1, \; \tau = 1 \; \text{when} \; s < 1
\]

which are excluded from the domain of holomorphy (3.24). Upon applying the corollary [Eq. (3.8)] in Sec. IIIA to the expression for \( \xi_1 \) in Eq. (3.13)
and upon using Eqs. (3.16), (3.17), (3.18), (3.22) and (3.23), we see that $\xi^{(2)}(r, s, \tau, z_0)$ has no pole and no isolated essential singularity at either of these points.

Now let $\xi$ denote that function whose domain is [compare with $\text{dom}\xi^{(2)}$ in Eq. (3.21)] given by Eq. (2.25) and whose values are

$$\xi(r, s, \tau, z_0) := \xi^{(2)}(r, s, \tau, z_0) \quad \text{when} \quad (r, s, \tau, z_0) \in \text{dom}\xi^{(2)} \quad (3.25)$$

and

$$\xi(r, s, -1, z_0) := \lim_{\tau \to -1} \xi^{(2)}(r, s, \tau, z_0) \quad \text{when} \quad r > -1, \quad (3.26a)$$

$$\xi(r, s, 1, z_0) := \lim_{\tau \to 1} \xi^{(2)}(r, s, \tau, z_0) \quad \text{when} \quad s < 1. \quad (3.26b)$$

The above limits exist, of course. We leave it to the enterprising reader to verify that the function $\xi$ which has just been defined satisfies all of the requirements specified in the definition of the special gauge in Sec. IIE.

**Theorem:** For given $\psi$, the spectral potential $\xi$ in the special gauge is unique.

**Proof:** We proved that $\xi_1$ is unique in Sec. IIIB. The uniqueness of $\xi$ then follows from Eq. (2.39) in the theorem of Sec. IIF.

**D. The Spectral Potential $\phi$**

**Definition:** Let

$$\phi := \xi/\mu. \quad (3.27)$$

The domain of $\phi$ is the domain of $\xi$ minus those points $(r, s, \tau, z_0)$ at which $\mu(r, s, \tau) = 0$, i.e., at which $\tau = r$ or $\tau = s$. From Eqs. (2.12) and (2.15), one derives:

$$d*(\rho d\phi) = 0. \quad (3.28)$$

So $\phi$ satisfies the same hyperbolic equation (2.2) as $\psi$. 
E. A Hilbert Problem on \([r, s]\)

With the aid of Eqs. (2.6) and (3.27), one sees that Eqs. (3.9a) and (3.9b) imply that

\[
\phi(r, s, \sigma^+, z_0) - \phi(r, s, \sigma^-, z_0) = \frac{-2i\lambda(\sigma, z_0)}{\sqrt{(\sigma - r)(s - \sigma)}}. \tag{3.29}
\]

This equation is the basis for another simple Hilbert problem. Both in the statement of the Hilbert problem and in the derivation of its solution, we shall grant the relation

\[
\lambda(\sigma, z_0) = \Lambda(\sigma, z_0) + \frac{1}{\pi}\left[G_3(z_0) - G_2(z_0)\right] \ln |\sigma - z_0| \tag{3.30}
\]

where \(\Lambda(\sigma, z_0)\) is defined throughout the open interval \(-1 < \sigma < 1\) and obeys a Hölder condition of index \(\frac{1}{2}\) on every closed subinterval of \([-1, 1]\). Equation (3.30) will be proven in Sec. IV. From Eq. (3.30), it follows that the right side of Eq. (3.29) obeys a Hölder condition of index \(\frac{1}{2}\) on every closed subinterval of the open intervals \(r < \sigma < z_0\) and \(z_0 < \sigma < s\). It also follows that the right side of Eq. (3.29) is summable on \([r, s]\) if \(-1 < r < z_0 < s < 1\).

A Hilbert Problem on \([r, s]\): At present, assume that \(-1 < r < z_0 < s < 1\). Let \(\lambda(\sigma, z_0)\) be given in terms of the initial data functions by Eqs. (1.18) to (1.23). Then, for fixed \((r, s, z_0)\), we seek \(\phi(r, s, \tau, z_0)\) such that it is a holomorphic function of \(\tau\) throughout \(C - [r, s]\), it is a continuous function of \(\tau\) throughout \(D(r, s, z_0) - \{r, s\}\), it satisfies Eq. (3.29) and has the value 0 at \(\tau = \infty\), and it satisfies the condition that

\[
\mu(r, s, \tau)|\tau - z_0|^\epsilon \phi(r, s, \tau, z_0)
\]

(for arbitrary \(\epsilon > 0\) remain bounded as \(\tau \to r\), as \(\tau \to s\), and as \(\tau \to z_0\).

The solution of the above Hilbert problem is well known, is unique and is given by

\[
\phi(r, s, \tau, z_0) = -\frac{1}{\pi} \int_r^s d\sigma \frac{\lambda(\sigma, z_0)}{(\sigma - \tau)\sqrt{(\sigma - r)(s - \sigma)}}. \tag{3.31}
\]
Equation (3.29) can be extended to include $r = -1$ and $s = 1$ by using the fact that $\phi(r,s,\tau,z_0)$ is continuous at $r = -1$ when $\tau \neq -1$ and at $s = 1$ when $\tau \neq 1$:

$$\phi(-1,s,\tau,z_0) = \lim_{r\to-1} \phi(r,s,\tau,z_0) \text{ when } \tau \neq -1,$$

$$\phi(r,1,\tau,z_0) = \lim_{s\to1} \phi(r,s,\tau,z_0) \text{ when } \tau \neq 1. \quad (3.32)$$

From Eqs. (3.27) and (3.31)

$$\xi(r,s,\tau,z_0) = -\frac{\mu(r,s,\tau)}{\pi} \int_r^s d\sigma \frac{\lambda(\sigma,z_0)}{(\sigma-\tau)(\sigma-r)(s-\sigma)} \quad (3.33)$$

whereupon Eq. (1.24) follows from Eqs. (2.5) and (2.41). The alternative expression for $\psi$ in Eq. (1.24) is obtained by the substitution $\sigma = z + \rho \sin \theta$. The same substitution yields an alternative expression for the integral in Eq. (3.33). By setting $(r,s) = (-1,1)$ in Eq. (3.33), we obtain

$$\xi_1(\tau,z_0) = -\frac{\mu(-1,1,\tau)}{\pi} \int_{-1}^1 d\sigma \frac{\lambda(\sigma,z_0)}{(\sigma-\tau)(\sqrt{1-\sigma^2})} \quad (3.34)$$

From the Plemelj theorem in Sec. IIC and from Eqs. (3.11) and (2.6), Eq. (3.34) yields

$$G(\sigma,z_0) = \frac{\sqrt{1-\sigma^2}}{\pi} \int_{-1}^1 d\sigma' \frac{\lambda(\sigma',z_0)}{(\sigma'-\sigma)(\sqrt{1-\sigma'^2})}. \quad (3.35)$$

The above Eq. (3.35) is the inverse relation of Eq. (1.23).

IV. The Singularities of $\xi$ and $\psi$ at $\rho = 0$

A. The Singularities of $\xi_1(\tau,z_0)$ at $\tau = z_0$ and of $\lambda(\sigma,z_0)$ at $\sigma = z_0$

Observe that $G(\sigma,z_0)$ as defined by Eq. (1.22) generally has a step discontinuity at $\sigma = z_0$. Let

$$\tilde{G}(\sigma,z_0) := \begin{cases} G_3(\sigma) - G_3(z_0) & \text{if } \sigma \leq z_0 \\ G_2(\sigma) - G_2(z_0) & \text{if } z_0 \leq \sigma. \end{cases} \quad (4.1)$$
Then $\tilde{G}(\sigma, z_0)$ is continuous at $\sigma = z_0$ and, from the second theorem in Sec. IIC and the definition of a Hölder condition, $\tilde{G}(\sigma, z_0)$ obeys a Hölder condition of index $\frac{1}{2}$ on any given closed subinterval of $-1 < \sigma < 1$. Upon substituting from Eq. (4.1) into Eqs. (1.23) and (3.13), we obtain

$$\lambda(\sigma, z_0) = \tilde{\lambda}(\sigma, z_0) - \frac{1}{\pi} \tilde{G}_3(z_0) \ln \left( \frac{1}{\sigma + 1} \right) - \frac{1}{\pi} \tilde{G}_2(z_0) \ln \left( \frac{1 - \sigma}{\sigma - z_0} \right)$$

(4.2)

and

$$\xi_1(\tau, z_0) = \tilde{\xi}_1(\tau, z_0) - \frac{1}{\pi} \tilde{G}_3(z_0) \ln \left( \frac{\tau - z_0}{\tau + 1} \right) - \frac{1}{\pi} \tilde{G}_2(z_0) \ln \left( \frac{\tau - 1}{\tau - z_0} \right),$$

(4.3)

where

$$\tilde{\lambda}(\sigma, z_0) := -\frac{1}{\pi} \int_{-1}^{1} d\sigma' \frac{\tilde{G}(\sigma, z_0)}{\sigma' - \sigma}$$

(4.4)

and

$$\tilde{\xi}_1(\tau, z_0) := -\frac{1}{\pi} \int_{-1}^{1} d\sigma \frac{\tilde{G}(\sigma, z_0)}{\sigma - \tau}.$$  

(4.5)

The principal values of the logarithms are understood in Eq. (4.2). The cuts for the logarithms in Eq. (4.3) are $[-1, z_0]$ and $[z_0, 1]$, respectively, and their values at $\tau = \infty$ are zero. From the Plemelj theorem in Sec. IIC, we obtain the following statements.

**Properties of $\tilde{\lambda}(\sigma, z_0)$ and $\tilde{\xi}_1(\tau, z_0)$ for fixed $z_0$:**

1. The function $\tilde{\lambda}(\sigma, z_0)$ obeys a Hölder condition of index $\frac{1}{2}$ on any given closed subinterval of $-1 < \sigma < 1$.

2. The function $\tilde{\xi}_1(\tau, z_0)$ is holomorphic throughout $\mathbb{C} - [-1, 1]$, is continuous throughout

$$(C - [-1, 1]) \cup (-1, 1)^+ \cup (-1, 1)^-$$

and satisfies the Plemelj relation

$$\tilde{\xi}_1(\sigma^\pm, z_0) = \tilde{\lambda}(\sigma, z_0) \mp i\tilde{G}(\sigma, z_0).$$  

(4.6)
For computational purposes, it is convenient to recast the expression (4.2) into the following form:

$$\lambda(\sigma, z_0) = \Lambda(\sigma, z_0) - 1 \pi [G_3(z_0) - G_2(z_0)] \ln |\sigma - z_0| \quad (4.7)$$

where

$$\Lambda(\sigma, z_0) := \tilde{\lambda}(\sigma, z_0) + 1 \pi G_3(z_0) \ln(1 + \sigma) - 1 \pi G_2(z_0) \ln(1 - \sigma). \quad (4.8)$$

Since the logarithms of $1 \pm \sigma$ are differentiable in the open interval $-1 < \sigma < 1$, $\Lambda(\sigma, z_0)$ obeys a Hölder condition of index $\frac{1}{2}$ on any given closed subinterval of $-1 < \sigma < 1$. We shall grant this fact below.

**B. The Singularities of the Potentials at $\rho = 0$**

We shall now employ the results obtained in Sec. IIF. From Eqs. (2.39) and (3.33) and from the fact that we may set $z_0 = z$ in Eq. (2.39),

$$\xi(r, s, \tau, z_0) - \xi_1(\tau, z_0) = \Xi(r, s, \tau)$$

$$= -\xi_1(\tau, z) - \frac{\mu(r, s, \tau)}{\pi} \int_r^s d\sigma \frac{\lambda(\sigma, z)}{(\sigma - \tau)(\sigma - r)(s - \sigma)}. \quad (4.9)$$

We substitute from Eq. (4.8) into the above Eq. (4.9). The term involving $\ln |\sigma - z|$ may be integrated explicitly. One obtains different forms of the integral for the two cases $\rho < |\tau - z|$ and $\rho > |\tau - z|$. Here, we are interested only in the case $\rho < |\tau - z|:

$$\xi(r, s, \tau, z_0) - \xi_1(\tau, z_0) = \Xi(r, s, \tau) = \xi_1(\tau, z)$$

$$+ \frac{1}{\pi} [G_3(z) - G_2(z)] \left\{ \frac{\mu(r, s, \tau)}{\tau - z} \ln \left[ \frac{\tau - z + \mu(r, s, \tau)}{\tau - z} \right] - \ln \rho \right\}$$

$$- \frac{\mu(r, s, \tau)}{\pi} \int_r^s d\sigma \frac{\Lambda(\sigma, z)}{(\sigma - \tau)(\sigma - r)(s - \sigma)}. \quad (4.10)$$

for $\rho < |\tau - z|$. Upon employing the identity

$$(\sigma - r)(s - \sigma) = \rho^2 - (\sigma - z)^2$$
and introducing a new integration variable, we obtain the following alternative expression for the integral in Eq. (4.10):

\[ \int_{r}^{s} \frac{d\sigma}{(\sigma - \tau)\sqrt{(\sigma - r)(s - \sigma)}} \Lambda(\sigma, z)(\sigma - \tau)\sqrt{(\sigma - r)(s - \sigma)} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \frac{\Lambda(z + \rho \sin \theta, z)}{z + \rho \sin \theta - \tau}. \quad (4.11) \]

From Eqs. (2.5), (2.30) and (2.41), we further obtain

\[ \psi(r, s) = -\frac{1}{\pi} [G_3(z) - G_2(z)] \ln\left(\frac{\rho}{2}\right) + \frac{1}{\pi} \int_{r}^{s} d\sigma \frac{\Lambda(\sigma, z)}{\sqrt{(\sigma - r)(s - \sigma)}} \]

\[ = -\frac{1}{\pi} [G_3(z) - G_2(z)] \ln\left(\frac{\rho}{2}\right) + \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \Lambda(z + \rho \sin \theta, z). \quad (4.12) \]

Thus the statement made in Sec. I that the expression (1.28) has a continuous extension to

\[ \mathcal{D}_{IV} := \mathcal{D}_{IV} \cup \{(z, z) : -1 < z < 1\} \quad (4.13) \]

is proven. From Eq. (4.10), it is also apparent that

\[ \xi(r, s, \tau, z_0) + \frac{1}{\pi} [G_3(z) - G_2(z)] \ln\left(\frac{\rho}{2}\right) \]

has a continuous extension to \( \mathcal{D}_{IV} \) and one obtains

\[ \{\psi(r, s) + \frac{1}{\pi} [G_3(z) - G_2(z)] \ln\left(\frac{\rho}{2}\right)\}_{\rho = 0} \]

\[ = \{\xi(r, s, \tau, z_0) - \xi_1(\tau, z_0) + \xi_1(\tau, z) \]

\[ + \frac{1}{\pi} [G_3(z) - G_2(z)] \ln\left(\frac{\rho}{2}\right)\}_{\rho = 0} \]

\[ = \Lambda(z, z). \quad (4.14) \]

As we have already indicated in Sec. IF, the above results are in agreement with those already obtained by Yurtsever; i.e., our results supply the same asymptotic expression for \( \psi \) as \( \rho \to 0 \). For those who wish to make a comparison, here are some correspondences between his notations and ours. His notations are on the left sides of the equations:

\[ U = -\ln \rho, \quad V = -2\psi + \ln \rho, \]

\[ \alpha = \rho, \quad \beta = z, \quad r_{\text{his}} = -r, \quad s_{\text{his}} = s. \quad (4.15) \]
Of course, our expressions for $\xi$ are completely new (since Yurtsever made no use of the spectral potential concept) and our integral expressions in Eqs. (4.9) to (4.12) are new.

A Comparison of Eq. (4.12) with the Kasner $\psi$-potential of Eq. (2.18) shows why Yurtsever was led to regard $\psi$ as having a Kasner-like dependence (for fixed $z$) near $\rho = 0$.

C. The Potentials when the Axis ($\rho = 0$) is Accessible

Theorem:

(i) A necessary and sufficient condition for $\psi$ to have a continuous extension to $D_{IV}$ [Eq. (4.13)] is that

$$G_3(\sigma) = G_2(\sigma) = G(\sigma) \text{ for all } -1 < \sigma < 1 \quad (4.16)$$

which is equivalent to the statement that $\xi_1(\tau, z_0)$ is independent of $z_0$ or, alternatively, to the statement that $\lambda(\sigma, z_0)$ is independent of $z_0$.

(ii) When the condition (4.16) holds, $G(\sigma)$ is integrable over $[-1, 1]$ and obeys a Hölder condition of index $\frac{1}{2}$ on any given closed subinterval of $]-1, 1[$. Also

$$\xi_1(\tau) = -\frac{1}{\pi} \int_{-1}^{1} d\sigma \frac{G(\sigma)}{\sigma - \tau} \quad (4.17)$$

and

$$\psi(\sigma, \sigma) = \lambda(\sigma) = -\frac{1}{\pi} \int_{-1}^{1} d\sigma' \frac{G(\sigma')}{\sigma' - \sigma} \quad (4.18)$$

Furthermore,

$$\xi(r, s, \tau) = -\frac{\mu(r, s, \tau)}{\pi} \int_{r}^{s} d\sigma \frac{\lambda(\sigma)}{(\sigma - \tau)\sqrt{(\sigma - r)(s - \sigma)}} \quad (4.19)$$

and

$$\psi(r, s) = \frac{1}{\pi} \int_{r}^{s} d\sigma \frac{\lambda(\sigma)}{\sqrt{(\sigma - r)(s - \sigma)}} = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \lambda(z + \rho \sin \theta). \quad (4.20)$$
Proof: Use Eqs. (1.22), (1.23), (3.13), (4.7), (4.10), (4.11) and (4.12). Also, use the second theorem in Sec. IIC.

The question of what happens when we take partial derivatives with respect to \( r \) and \( s \) of the above integrals will have to be saved for a future paper (in which we shall adopt stronger premises than those given in Sec. IIA).

V. A Broad Class of CPGWP’s

A. Premises and Colliding Wave Conditions

For an arbitrary real member of \( \Sigma_E \), we do not know if \( \lambda(\sigma, z_0) \) has asymptotic forms as \( \sigma \to -1 \) and as \( \sigma \to 1 \). The purpose of this section is to introduce a two-parameter family of subsets \( \Sigma(n_3, n_2) \subset \Sigma_E \) for which the asymptotic forms exist, are easy to deduce and are simple. Moreover, \( \Sigma(n_3, n_2) \subset \Sigma_E^{CW} \) if and only if each of the real parameters \( n_j \) is either equal to 1 or is in the range \( 2 \leq n_j < \infty \). For all other cases, \( \Sigma(n_3, n_2) \) and \( \Sigma_E^{CW} \) are disjoint.

At present, there is no need to specialize to real Ernst potentials and it will serve the objectives of later papers as well as this one to consider any member \( \mathcal{E} \) of \( \Sigma_E \) until we reach Sec. VF. The subsets \( \Sigma(n_3, n_2) \subset \Sigma_E \) will be defined in Sec. VE. Until then, we shall introduce some preliminary concepts. We start by replacing the premises given in Sec. IIA by the following ones which imply those given in Sec. IIA and do not, therefore, effect the validity of any results obtained in previous sections.

**Premises:** Until we reach Sec. VF, we shall assume that \( r(u), \mathcal{E}_3(u) \) and \( s(v), \mathcal{E}_2(v) \) are \( C^2 \) throughout \( 0 \leq u < u_0 \) and \( 0 \leq v < v_0 \), respectively, with the possible exception of a finite number of finite step discontinuities in \( \dot{\mathcal{E}}_3(u) \) and \( \dot{\mathcal{E}}_2(v) \). Also, we continue to assume that the conditions (1.13) hold, i.e., \( \dot{r}(u) > 0 \) and \( \dot{s}(v) < 0 \) throughout \( 0 < u < u_0 \) and \( 0 < v < v_0 \), respectively.

The initial data functions \( r(u), \mathcal{E}_3(u) \) and \( s(v), \mathcal{E}_2(v) \) for which the corresponding Ernst potentials \( \mathcal{E} \) (with domain \( D_{IV} \)) are members of \( \Sigma_E^{CW} \) are those which satisfy the following additional conditions. \( ^{10} \)

**Colliding Wave Conditions:**

1. Equations (1.14) \( \dot{r}(0) = \dot{s}(0) = 0 \) hold.
2. The limits of the ratios

\[ \frac{[\ddot{r}(u) - 2(1 - r(u))|\dot{E}_3(u)|^2]{/\dot{r}(u)} }{\dot{E}_3(u)} \]

and

\[ \frac{[\ddot{s}(v) + 2(1 + s(v))|\dot{E}_2(v)|^2]{/\dot{s}(v)} }{\dot{E}_2(v)} \]

as \( u \to 0 \) and \( v \to 0 \) (from above), respectively, exist.

The origin of the above conditions (2) will become a little clearer below.

B. The Functions \( \gamma, \gamma_3 \) and \( \gamma_2 \)

So far, we have not discussed the metric component \( g_{34} \) in the line element (1.1). Let

\[ \gamma := \frac{1}{2} \ln(fg_{34}) , \ f := \text{Re} E = -g_{22} . \]  

(5.1)

In our current work, the vacuum field equations for the line element (1.1) are all expressed in terms of the functions \( \rho, E \) and \( \gamma \). In some previous papers by the authors and others, the functions

\[ E := (\rho + ig_{12})/g_{22} , \ \Gamma := \frac{1}{2} \ln(-\rho g_{34}) \]  

(5.2)

were used in place of \( E \) and \( \gamma \). (Recall that \( E \), like \( E \), is a solution of the Ernst equation.) However, we have found that the employment of \( E \) and \( \gamma \) is more suitable for applying and extending the KC group and for analyzing the gravitational field in a neighborhood of the axis \( (\rho = 0) \).

The vacuum field equations expressed in terms of the triad of functions \( \rho, E, \gamma \) are obtained from those expressed in terms of \( \rho, E, \Gamma \) simply by substituting \( E \) for \( E \) and \( \gamma \) for \( \Gamma \). When these substitutions are applied to a result\(^{10}\) previously derived by the authors for the function \( \Gamma \), one obtains the following solution for \( \gamma \) in terms of \( E \) and the initial data functions \( r, E_3, s, E_2 \):

\[ \gamma(u, v) = \gamma_3(u) + \gamma_2(v) - \text{Re} \int_0^u da \int_0^v db \left\{ \left[ \frac{E_a(a, b)}{2f(a, b)} \right]^* \cdot \left[ \frac{E_b(a, b)}{2f(a, b)} \right] \right\} \]  

(5.3)

where \( \gamma_3 \) and \( \gamma_2 \) are determined up to arbitrary constant by the equations

\[ \dot{\gamma}_3 = [\ddot{r} - 2(1 - r)|\dot{E}_3/2f_3|^2]/2\dot{r} \]  

(5.4a)
and

\[ \dot{\gamma}_2 = [\dot{s} + 2(1 + s)|\dot{E}_2/2f_2|^2]/2\dot{s} \quad (5.4b) \]

which generally hold over the domains \(0 < u < u_0\) and \(0 < v < v_0\), respectively. If the right sides of the above equations are summable over the intervals \([0, u]\) and \([0, v]\), respectively, then \(\gamma_3(u)\) and \(\gamma_2(v)\) are defined as the integrals of the right sides of the equations over these intervals. Thereupon, \(\gamma_3(u)\) and \(\gamma_2(v)\) are continuous throughout \(0 \leq u < u_0\) and \(0 \leq v < v_0\), respectively, and

\[ \gamma(0, 0) = 0, \quad \gamma_3(u) = \gamma(u, 0), \quad \gamma_2(v) = \gamma(0, v). \quad (5.5) \]

With our premises, note that the pair of colliding wave conditions (2) in Sec. VA is equivalent to the statement that \(\gamma_3(u)\) and \(\gamma_2(v)\) are \(C^1\) throughout \(0 \leq u < u_0\) and \(0 \leq v < v_0\), respectively.

C. The Variables \(p, q\) and the Functions \(\mathcal{E}_j, \beta_j\) and \(\psi_j\)

The symbols ‘\(p\)’ and ‘\(q\)’, like ‘\(r\)’ and ‘\(s\)’, will have two different but related uses. However, the context will always indicate which use is intended.

Definitions: In some equations, \(p\) and \(q\) will denote independent variables which both have the range \([0, 1]\). The variables \(r\) and \(s\) will often be expressed as functions of \(p\) and \(q\), respectively, as follows:

\[ r(p) := -1 + 2p^2, \quad s(q) = 1 - 2q^2. \quad (5.6) \]

In other equations, \(p\) and \(q\) will denote functions whose domains are \(0 \leq u < u_0\) and \(0 \leq v < v_0\), respectively, and whose values are given by

\[ p(u) := \sqrt{\frac{1 + r(u)}{2}}, \quad q(v) := \sqrt{\frac{1 - s(v)}{2}}. \quad (5.7) \]

Definitions: Let \(\mathcal{E}_3\) and \(\mathcal{E}_2\) denote those functions whose domains are \(0 \leq p < 1\) and \(0 \leq q < 1\), respectively, such that

\[ \mathcal{E}_3(p) := \mathcal{E}_3(r(p)), \quad \mathcal{E}_2(q) := \mathcal{E}_2(s(q)). \quad (5.8) \]
Also, let $\beta_3$ and $\beta_2$ denote those functions whose domains are the same as those of $\dot{\varepsilon}_3$ and $\dot{\varepsilon}_2$, respectively, such that

$$\beta_j := \frac{\dot{\varepsilon}_j}{2f_j} \text{ where } f_j := \text{Re}\varepsilon_j.$$  

(5.9)

Note that, for real $\varepsilon = -\exp(2\psi),$

$$\beta_j = \dot{\psi}_j, \, \psi_3(p) := \psi_3(r(p)) \, , \, \psi_2(q) := \psi_2(s(q)).$$  

(5.10)

D. The Functions $K_j, K_j, u_j$ and $u_j$

The functions which we shall define below in Sec. VD will be used to prove a theorem in the next Sec. VE.

**Definition:** Null coordinate transformations $u \rightarrow u'$ and $v \rightarrow v'$ will be called standard if they are $C^2$ and satisfy $du'/du > 0$ and $dv'/dv > 0$ throughout $0 \leq u < u_0$ and $0 \leq v < v_0$, respectively.

The truth of the premises in Sec. VA is invariant under standard null coordinate transformations and so is the truth or falsity, as the case may be, of each of the colliding wave conditions in Sec. VA. Also, the values of the functions $K_j$ which are defined below are invariant under standard null coordinate transformations.

**Definitions:** Let $K_3$ and $K_2$ denote those functions whose domains are $0 < u < u_0$ and $0 < v < v_0$, and whose values are given by

$$K_3(u) := \frac{2[1 + r(u)]\exp[2\gamma_3(u)]}{\dot{r}(u)}, \quad K_2(v) := \frac{2[1 - s(v)]\exp[2\gamma_2(v)]}{-\dot{s}(v)}.$$  

(5.11)

Let $K_3$ and $K_2$ denote those functions whose domains are $0 < p < 1$ and $0 < q < 1$, respectively, such that

$$K_3(p(u)) := K_3(u) \, , \, K_2(q(v)) := K_2(v).$$  

(5.12)

**Theorem:** If $\gamma_3(u)$ and $\gamma_2(v)$ are continuous throughout $0 \leq u < u_0$ and $0 \leq v < v_0$, respectively, then

$$K_j(0) := K_j(0) := \lim_{a \rightarrow 0} K_j(a) = 0$$  

(5.13)

exists and (as indicated above) equals zero. Thereupon, $K_j(x)$ is continuous throughout $0 \leq x < 1$. 

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Proof: Equations (5.4a) and (5.4b) are expressible in the forms

\[ e^{2\gamma_3} \frac{d}{du}(\dot{r}e^{-2\gamma_3}) = 2(1 - r)|\dot{\hat{E}}_3/2f_3|^2 \]  \hspace{1cm} (5.14a)

and

\[ e^{2\gamma_2} \frac{d}{dv}(-\dot{s}e^{-2\gamma_2}) = 2(1 + r)|\dot{\hat{E}}_2/2f_2|^2 . \]  \hspace{1cm} (5.14b)

Therefore, \( \dot{r}\exp(-2\gamma_3) \) and \( -\dot{s}\exp(-2\gamma_2) \) are non-decreasing functions. The rest of the proof uses the mean value theorem of the differential calculus and is left to the reader.

It is easy to show that Eqs. (5.14a) and (5.14b) are expressible in the form

\[ 2 - [x/K_j(x)]\dot{K}_j(x) = (1 - x^2)|\beta_j(x)|^2 \]  \hspace{1cm} (5.15)

where we have used Eqs. (5.9), (5.11) and (5.12) and where \( x = p \) when \( j = 3 \) and \( x = q \) when \( j = 2 \). Equation (5.15) will be used in Sec. VE.

From our premises in Sec. VA and from Eqs. (5.4a) and (5.4b), \( \gamma_3(u) \) and \( \gamma_2(v) \) are always \( C^1 \) throughout \( 0 < u < u_0 \) and \( 0 < v < v_0 \), respectively.

**Definitions:** Suppose \( \gamma_3(u) \) and \( \gamma_2(v) \) are \( C^1 \) throughout \( 0 \leq u < u_0 \) and \( 0 \leq v < v_0 \), respectively. (This is true if and only if the colliding wave conditions (2) in Sec. VA hold.) Then let \( u_3 \) and \( u_2 \) denote those functions whose domains are \( 0 \leq u < u_0 \) and \( 0 \leq v < v_0 \), respectively, and whose values are given by

\[ u_3(u) := \int_0^u da \exp[2\gamma_3(a)] , \ u_2(v) := \int_0^v db \exp[2\gamma_2(b)] . \]  \hspace{1cm} (5.16)

Let \( u_3 \) and \( u_2 \) denote those functions whose domains are \( 0 \leq p < 1 \) and \( 0 \leq q < 1 \), respectively, such that

\[ u_3(p(u)) := u_3(u) , \ u_2(q(v)) := u_2(v) . \]  \hspace{1cm} (5.17)

Equivalently, as one can see from Eqs. (5.7), (5.11), (5.12) and (5.16),

\[ u_j(x) := \int_0^x dy y^{-1}K_j(y) . \]  \hspace{1cm} (5.18)

Clearly, \( u_3 \) and \( u_2 \) exist and are \( C^2 \) throughout \( 0 \leq u < u_0 \) and \( 0 \leq v < v_0 \), respectively. The null coordinate transformations \( u \to u_3 \) and \( v \to v_3 \) are standard. The corresponding transformations for \( \gamma_3 \) and \( \gamma_2 \) are \( \gamma_3 \to 0 \) and \( \gamma_2 \to 0 \).
In the remainder of Sec. V, we shall no longer take for granted that null coordinates $u, v$ exist such that the premises in Sec. VA hold and we shall not grant any colliding wave conditions. The only initial data functions which are assumed to be given are $E_3(r)$ and $E_2(s)$ or, equivalently, $E_3(p)$ and $E_2(q)$ which are defined by Eqs. (5.6) and (5.8).

**Premise:** The functions $E_j(x)$ are $C^1$ throughout $0 \leq x < 1$.

Granted the above premise, it has been proven as a special case of a broad theorem that a function $E$ with domain

$$\text{dom}E := \{(p, q) : 0 < \rho(p, q) \leq 1\}$$

(5.19)

exists such that

$$\rho(p, q) = 1 - p^2 - q^2, \ E(p, 0) = E_3(p), \ E(0, q) = E_2(q),$$

(5.20)

such that the partial derivatives $E_p, E_q$ and $E_{pq}$ exist and are continuous throughout $\text{dom}E$ and such that $E$ is a solution of the Ernst equation throughout $\text{dom}E$.\(^{29}\)

**Definition:** Let real numbers $n_3$ and $n_2$ be defined by the equation

$$|\beta_j(0)|^2 = 2 - (1/n_j)$$

(5.21)

where we recall the definition of $\beta_j$ by Eq. (5.9). Thus, $-\infty < n_j < 0$ or $\frac{1}{2} \leq n_j \leq \infty$.

At this point, the authors considered a variety of additional premises which led to broad classes of CPGWP’s that appeared to deserve further study. The simplest of these additional premises and the only ones which can be treated here in a reasonable space is contained in the following theorem.

**Theorem:** Suppose $E_3(p)$ and $E_2(q)$ are $C^2$ throughout $0 \leq p < 1$ and $0 \leq q < 1$, respectively, with the possible exception of a finite number of finite step discontinuities in the second derivatives $\ddot{E}_3(p)$ and $\ddot{E}_2(q)$.

Then the following statements (i) and (ii) are equivalent to each other.
(i) There exist null coordinates \( u \) and \( v \) such that, if we let
\[
\mathcal{E}_3(u) := \mathcal{E}_3(p(u)) , \quad \mathcal{E}_2(v) := \mathcal{E}_2(q(v)) , \tag{5.22}
\]
then \( r(u) \), \( \mathcal{E}_3(u) \) and \( s(v) \), \( \mathcal{E}_2(v) \) satisfy all of the premises and colliding wave conditions given in Sec. VA.

(ii) Each of the numbers \( n_3 \) and \( n_2 \) satisfies the condition
\[
n_j = 1 \text{ or } 2 \leq n_j < \infty . \tag{5.23}
\]

**Proof:** We first prove that statement (i) implies statement (ii). The proof will be given in three parts \((\alpha), (\beta)\) and \((\gamma)\).

\((\alpha)\) From Eq. (5.9), \( \beta_j(x) \) is \( C^1 \) throughout \( 0 \leq x < 1 \) with the possible exception of a finite number of finite step discontinuities in \( \beta_j(x) \). Therefore, from the theorem which contains Eq. (5.13) and from Eqs. (5.15) and (5.21),
\[
K_j(x) = c_j x^{(1/n_j)} e^{S_j(x)} \tag{5.24}
\]
where \( c_j \) is a positive constant,
\[
S_j(x) := \int_0^x dy \ y^{-1} \left[ 2 - (1/n_j) - (1 - y^2) |\beta_j(y)|^2 \right] \tag{5.25}
\]
is \( C^1 \) throughout \( 0 \leq x < 1 \), and
\[
\frac{1}{2} \leq n_j < \infty . \tag{5.26}
\]

\((\beta)\) Considering the statements immediately after Eq. (5.18), we employ \( u_3 \) and \( u_2 \) as our null coordinates and obtain the following results from Eqs. (5.7), (5.17), (5.18), (5.24) and (5.25):
\[
\begin{align*}
\dot{r}(u_3) &\quad - \dot{s}(u_2) \quad = 4c_j^{-1} x^{2-(1/n_j)} e^{-S_j(x)} \tag{5.27} \\
\end{align*}
\]
and
\[
\begin{align*}
\dot{r}(u_3) &\quad - \dot{s}(u_2) \quad = 4(1 - x^2)|\beta_j(x)|^2 \left[ c_j^{-1} x^{1-(1/n_j)} e^{-S_j(x)} \right]^2 \tag{5.28}
\end{align*}
\]
with the understanding that $x = p(u_3)$ when $j = 3$ and $x = q(u_2)$ when $j = 2$. Also,

$$\dot{E}_j(u_j) = E_j(x) \left[ e^{-S_j(x)} \right]$$

and

$$\ddot{E}_j(u_j) = \left[ c_j^{-1} e^{-S_j(x)} \right]^2 \times \left\{ \dot{E}_j(x)[x^{1-(1/n_j)}]^2 + E_j(x)(1 - x^2)|\beta_j(x)|^2 - 1|x^{1-(2/n_j)}| \right\}.$$

(γ) The colliding wave conditions $\dot{r}(0) = \dot{s}(0) = 0$ and Eqs. (5.9), (5.21), (5.26) and (5.27) imply

$$\frac{1}{2} < n_j < \infty, \quad \beta_j(0) = -\frac{1}{2}\dot{E}_j(0) \neq 0$$

where we have used the convention $E_j(0) = -1$. So, Eqs. (5.28) and the premise of statement (i) that $r(u)$ and $s(v)$ are $C^2$ throughout $0 \leq u < u_0$ and $0 \leq v < v_0$, respectively, imply that

$$1 \leq n_j < \infty.$$

Note that the above condition on $n_j$ is consistent with Eq. (5.29a) and the premise of statement (i) that $E_3(u)$ and $E_2(v)$ are $C^2$ throughout $0 \leq u < u_0$ and $0 \leq v < v_0$, respectively, with the possible exception of a finite number of finite step discontinuities in the second derivatives. Equation (5.29b), taken together with Eq. (5.21) and the inequality $\dot{E}_j(0) \neq 0$ of (5.30), further imply that $n_j$ must lie in the range (5.23). We have thus proved that statement (i) implies statement (ii).

(δ) We next prove that statement (ii) implies (i). Let new null coordinates $u$ and $v$ be defined by the equations

$$p(u) = u^{n_3}, \quad q(v) = v^{n_2} \text{ where } 0 \leq u < 1 \text{ and } 0 \leq v < 1.$$  

Then one uses Eqs. (5.7) and (5.21), together with the premise that $E_j(x)$ is $C^2$ throughout $0 \leq x < 1$ (with the possible exception of a finite number of finite step discontinuities in its second derivative), to prove that the conditions (5.23) imply that $r(u)$, $E_3(u)$ and $s(v)$, $E_2(v)$ satisfy all of the premises and colliding wave conditions given in Sec. VA. We leave details to the reader.
Definition: For any given numbers $n_3$ and $n_2$ such that $-\infty < n_j < 0$ or $\frac{1}{2} \leq n_j \leq \infty$ ($j = 3, 2$), let $\Sigma(n_3, n_2)$ denote the set of all members of $\Sigma_{\mathcal{E}}$ for which $\mathcal{E}_3(p)$ and $\mathcal{E}_2(q)$ are $C^2$ with the possible exception of a finite number of finite step discontinuities in $\mathcal{E}_3(p)$ and $\mathcal{E}_2(q)$ throughout $0 \leq p < 1$ and $0 \leq 1 < 1$, respectively, and for which $|\beta_j(0)|^2 = |\dot{\beta}_j(0)/2|^2 = 2 - (1/n_j)$.

Corollary: Suppose $\mathcal{E} \in \Sigma(n_3, n_2)$. Then the preceding theorem is equivalent to the statement that $\mathcal{E} \in \Sigma_{\mathcal{E}}^{\mathcal{C}}$ if and only if $n_3$ and $n_2$ both lie in the range $\{1\} \cup [2, \infty[. Furthermore, if $\mathcal{E} \in \Sigma_{\mathcal{E}}^{\mathcal{C}}$ and null coordinates $u$ and $v$ are defined by Eqs. (5.31), then the transformations $u_3 \rightarrow u$ and $u_2 \rightarrow v$ are standard.

Corollary: Suppose $\mathcal{E} \in \Sigma(n_3, n_2)$ and $\mathcal{E} \in \Sigma_{\mathcal{E}}^{\mathcal{C}}$. Then the plane wave front at $u = 0$ is impulsive if and only if both $n_3 = 1$ and $\dot{\mathcal{E}}_3(0) \neq 0$, or $n_3 = 2$. Furthermore, a shock front occurs at $u = 0$ if and only if

$$\text{both } n_3 = 1 \text{ and } \dot{\mathcal{E}}_3(0) \neq 0, \text{ or } n_3 = 2. \quad (5.32)$$

Likewise, a shock front occurs at $v = 0$ if and only if both $n_2 = 1$ and $\dot{\mathcal{E}}_2(0) \neq 0$, or $n_2 = 2$.

Proof: From Eqs. (5.31),

$$\dot{\mathcal{E}}_3(u) = \dot{\mathcal{E}}_3(p(u))n_3u^{n_3-1} \quad (5.33)$$

and

$$\ddot{\mathcal{E}}_3(u) = \ddot{\mathcal{E}}_3(p(u))[n_3u^{n_3-1}]^2 + \dot{\mathcal{E}}_3(p(u))n_3(n_3 - 1)u^{n_3-2}, \quad (5.34)$$

where we note from (5.30) that $\dot{\mathcal{E}}_3(0) \neq 0$. The rest of the proof follows from the well known facts that $\dot{\mathcal{E}}(0) \neq 0$ is necessary and sufficient for an impulsive wave front at $u = 0$, whereas $\ddot{\mathcal{E}}_3(0) \neq 0$ is necessary and sufficient for a shock front at $u = 0$. The proof for the wave front at $v = 0$ is similar.

F. The Spectral Functions $G_j(\sigma)$ and $\lambda(\sigma, z_0)$ for the Real Members of $\Sigma(n_3, n_2)$

We now specialize to real $\mathcal{E} = -\exp(2\psi)$, whereupon Eqs. (5.10) hold. It is convenient to let

$$p_\sigma := \sqrt{\frac{1+\sigma}{2}}, \quad q_\sigma := \sqrt{\frac{1-\sigma}{2}}. \quad (5.35)$$
Note that \( p^2 + q^2 = 1 \) and that the Abel transforms (1.18) are expressible as
\[
G_3(\sigma) = q\sigma \int_{\pi/2}^{\pi} d\theta \beta_3(p\sigma \sin \theta)
\] (5.36a)
and
\[
G_2(\sigma) = -p\sigma \int_{\pi/2}^{\pi} d\theta \beta_2(q\sigma \sin \theta)
\] (5.36b)
where \( \dot{\psi}_j(x) = \beta_j(x) \) and where we have used Eqs. (1.21). The inverses of the above transforms are expressible as
\[
\psi_3(p) = \frac{2\sqrt{2p}}{\pi} \int_{0}^{\pi/2} d\theta \sin \theta g_3(-1 + 2p^2 \sin^2 \theta)
\] (5.37a)
and
\[
\psi_2(q) = -\frac{2\sqrt{2q}}{\pi} \int_{0}^{\pi/2} d\theta \sin \theta g_2(1 - 2q^2 \sin^2 \theta).
\] (5.37b)
Recall the definition of \( n_j \) by Eq. (5.21). If one grants that \( \psi_j(x) \) is \( C^1 \) throughout \( 0 \leq x < 1 \), then Eqs. (5.36a) and (5.36b) imply that \( G_3(\sigma) \) is continuous throughout \(-1 \leq \sigma < 1\), that \( G_2(\sigma) \) is continuous throughout \(-1 < \sigma \leq 1\) and that
\[
G_3(-1) = \frac{\pi}{2} \beta_3(0), \quad G_2(1) = -\frac{\pi}{2} \beta_2(0)
\] (5.38)
where
\[
|\beta_j(0)| = \sqrt{2 - (1/n_j)}.
\] (5.39)
If \( \psi_j(x) \) is \( C^2 \) throughout \( 0 \leq x < 1 \), then Eqs. (5.35), (5.36a) and (5.36b) imply that \( G_3(-1 + 2p^2) \) is a \( C^1 \) function of \( p\sigma \) over the domain \( 0 \leq p\sigma < 1 \) and that \( G_2(1 - 2q^2) \) is a \( C^1 \) function of \( q\sigma \) over the domain \( 0 \leq q\sigma < 1 \). The following result is clear.

**Theorem:** If \( \psi_3(p) \) is \( C^2 \) throughout \( 0 \leq p < 1 \), then \( G_3(\sigma) \) obeys a Hölder condition of index \( \frac{1}{2} \) on any given closed subinterval of \(-1 \leq \sigma < 1\). If \( \psi_2(q) \) is \( C^2 \) throughout \( 0 \leq q < 1 \), then \( G_2(\sigma) \) obeys a Hölder condition of index \( \frac{1}{2} \) on any given closed subinterval of \(-1 < \sigma \leq 1\).

Note that Eqs. (5.37a) and (5.37b) imply that a sufficient (but not necessary) condition for \( \psi_j(x) \) to be \( C^2 \) throughout \( 0 \leq x < 1 \) is that \( g_3(-1 + 2p^2) \)
and $g_2(1 - 2q_2^2)$ be $C^2$ functions of $p_\sigma$ and $q_\sigma$ throughout $0 \leq p_\sigma < 1$ and $0 \leq q_\sigma < 1$, respectively.

Let us next consider $\lambda(\sigma, z_0)$ as given by Eqs. (1.22) and (1.23). We have, from Eqs. (5.38):

$$\lambda(\sigma, z_0) = -\frac{1}{2} \beta_3(0) \ln \left( \frac{z_0 - \sigma}{1 + \sigma} \right) - \frac{1}{\pi} \int_{-1}^{z_0} d\sigma' \frac{G_3(\sigma') - G_3(-1)}{\sigma' - \sigma}$$

$$- \frac{1}{\pi} \int_{z_0}^{1} d\sigma' \frac{G_2(\sigma')}{\sigma' - \sigma}$$  \quad (5.40a)

for $-1 < \sigma < z_0$ and

$$\lambda(\sigma, z_0) = -\frac{1}{2} \beta_2(0) \ln \left( \frac{\sigma - z_0}{1 - \sigma} \right) - \frac{1}{\pi} \int_{z_0}^{1} d\sigma' \frac{G_2(\sigma') - G_2(1)}{\sigma' - \sigma}$$

$$- \frac{1}{\pi} \int_{-1}^{z_0} d\sigma' \frac{G_3(\sigma')}{\sigma' - \sigma}$$  \quad (5.40b)

for $z_0 < \sigma < 1$. Suppose $\psi_3(p)$ is $C^2$ throughout $0 \leq p < 1$. Then, from the preceding theorem and from a theorem of Plemelj\textsuperscript{38}, the first integral on the right side of Eq. (5.40a) defines a function of $\sigma$ which obeys a H"older condition of index $\frac{1}{2}$ on any given closed subinterval of $-1 \leq \sigma < z_0$. As regards the second integral on the right side of Eq. (5.40a), it is an analytic function of $\sigma$ throughout $-1 \leq \sigma < z_0$. Similar remarks are clearly applicable to the integrals in Eq. (5.40b) when $\psi_2(q)$ is $C^2$ throughout $0 \leq q < 1$.

The reader can readily pursue further specializations such as granting that $\psi_j(x)$ is $C^\infty$ or even analytic throughout $0 \leq x < 1$. In any case, except when $\beta_3(0) = 0$ and $\beta_2(0) = 0$, the function $\lambda(\sigma, z_0)$ has those logarithmic singularities at $\sigma = -1$ and $\sigma = 1$ which are displayed by Eqs. (5.40a) and (5.40b). For $\Sigma(n_3, n_2) \subset \Sigma_{C^k}$, $n_3, n_2$ are accessible,

$$|\beta_j(0)| = 2 \text{ or } \sqrt{3}/2 < |\beta_j(0)| < \sqrt{2}.$$  \quad (50)

So there is no escape from the endpoint logarithmic singularities for the class of CPGWP’s discussed in this section. In particular, when the axis ($\rho = 0$ and $-1 < z < 1$) is accessible, $\psi(z, z) = \lambda(z)$ [see Sec. IVC] has logarithmic singularities at $z = \mp 1$.
VI. On the Next Paper

A. Preliminary

In the next paper of this series, we shall extend our formalism to cover the entire Ernst potential set $\Sigma_\xi$. The concepts and methods which were roughly described in Sec. I will be detailed, and we shall attempt to give the reader an appreciation of how the formalism can be used by means of a few examples. At the same time, we shall avoid obscuring the key ideas by an excess of formal mathematical topics. In particular, some difficult existence proofs will be reserved for later papers.

Let us become more specific about the structure of the next paper. The main new concepts in that paper will be generalizations of five constructs which occur in the current paper and which will be reviewed below.

B. The First Construct

In Eqs. (2.13) to (2.17), we introduced the concept of the spectral potential $\xi$ in an arbitrary gauge (not the special gauge which we chose later). This concept will be generalized in our next paper by employing a factorization of the $F$-potential of W. Kinnersley and D. M. Chitre.\textsuperscript{1,2,3} For the case of a real member of $\Sigma_\xi$, this factorization is given by Eq. (1.1) and the last factor in Eq. (1.1) is $\exp(-\sigma_3 \xi)$. For an arbitrary member of $\Sigma_\xi$, the last factor of a product similar to the one in Eq. (1.1) will be a generalized version of $\exp(-\sigma_3 \xi)$ which we shall define and discuss in our next paper. The analysis of this generalized version of $\exp(-\sigma_3 \xi)$ will lead in a natural way to the second construct.

C. The Second Construct

The starting points of the major deliberations in the current paper were Eqs. (3.9a) and (3.9b) which can be expressed in the unified form

$$\xi(r, s, \sigma^\pm, z_0) = \lambda(\sigma, z_0) \mp i\mathcal{G}(r, s, \sigma, z_0)$$

(6.1)

where

$$\mathcal{G}(r, s, \sigma, z_0) = \begin{cases} 
\mathcal{G}^{(3)}(r, s, \sigma) & \text{when } r < \sigma < z_0, \\
\mathcal{G}^{(2)}(r, s, \sigma) & \text{when } z_0 < \sigma < s.
\end{cases}$$

(6.2)
and where $G^{(3)}$ and $G^{(2)}$ are defined by the integrals on the right sides of Eqs. (3.9b) and (3.9a), respectively.\textsuperscript{40} Note that $G(-1,1,\sigma,z_0)$ is the function $G(\sigma,z_0)$ which was defined by Eqs. (1.18) and (1.21). We can obtain a better feeling for the functions $G^{(j)}$ by setting $r' = (1-x^2)\sigma + x^2 r$ and $s' = (1-x^2)\sigma + x^2 s$ in Eqs. (3.9b) and (3.9a), respectively, where $x$ is a new integration variable such that $0 \leq x \leq 1$. Then

$$G^{(j)}(r,s,\sigma) = -\sqrt{(\sigma - r)(s - \sigma)}\bar{\phi}^{(j)}(r,s,\sigma)$$

(6.3)

where, for $-1 < r \leq \sigma < s \leq 1$ and $-1 = r < \sigma < s \leq 1$,

$$\bar{\phi}^{(3)}(r,s,\sigma) = -2\int_0^1 dx \psi_r'((1-x^2)\sigma + x^2 r, s)$$

(6.4a)

and, for $-1 \leq 4 < \sigma \leq s < 1$ and $-1 \leq r < \sigma < s = 1$,

$$\bar{\phi}^{(2)}(r,s,\sigma) = -2\int_0^1 dx \psi_s'(r, (1-x^2)\sigma + x^2 s).$$

(6.4b)

It can be shown from Eq. (2.13) that, for $r < \sigma < s$, the difference

$$\Delta(\sigma) := G^{(3)}(r,s,\sigma) - G^{(2)}(r,s,\sigma)$$

(6.5)

is independent of $(r,s)$. Note that $G^{(3)}(-1,1,\sigma)$ and $G^{(2)}(-1,1,\sigma)$ are the functions $G_3(\sigma)$ and $G_2(\sigma)$ which were defined by Eqs. (1.21). We recall from Sec. IVC that the vanishing of $\Delta(\sigma)$ is necessary and sufficient for $\psi(r,s)$ to have a continuous extension to the axis points $-1 < r = s < 1$.

When $-1 < r < s < 1$, one can see that the function of $\sigma$ given by $G(r,s,\sigma,z_0)$ has a continuous extension to the union of $[r,z_0[ \text{ with } ]z_0,s]$ and satisfies

$$G(\sigma,s,\sigma,z_0) = G(r,\sigma,\sigma,z_0) = 0.$$ 

(6.6)

The second major new construct of the next paper will be a $2 \times 2$ matrix generalization of $G$ with the same domain. This generalization of $G$ will be used to formulate a third construct.

D. The Third Construct

In Sec. IIIB of the current paper, we introduced a simple Hilbert problem on the real axis interval $[-1,1]$ of the complex plane. This is a special case
of a simple Hilbert problem on the real axis interval \([r, s]\) where \(-1 \leq r < z_0 < s \leq 1\). It is convenient at this stage to restrict the Hilbert problem to the range \(-1 < r < z_0 < s < 1\). The values of the Hilbert problem solutions corresponding to \(r = -1\) or to \(s = 1\) can always be obtained by computing the limits of the solutions as \(r \to -1\) or as \(s \to 1\), respectively.

**First Hilbert problem on \([r, s]\) when \(-1 < r < z_0 < s < 1\):** Suppose that \(\psi\) is given. Then problem is to find \(\xi(r, s, \tau, z_0)\) and \(\lambda(\sigma, z_0)\) such that the following two sets of conditions are satisfied for any fixed \((r, s, z_0)\) for which \(-1 < r < z_0 < s < 1\):

1. The function of \(\tau\) given by \(\xi(r, s, \tau, z_0)\) is continuous throughout \(D(r, s, z_0)\) (the domain defined by Eq. (2.22)), is holomorphic throughout \(C - [r, s]\), has the value \(\psi(r, s)\) at \(\tau = \infty\) and, for any given \(\epsilon > 0\),

\[
|\tau - z_0|^{\epsilon} \xi(r, s, \tau, z_0) \to 0
\]

as \(\tau \to z_0\) throughout any sequence of values in \(D(r, s, z_0)\).

2. Equation (6.1) holds, where \(G\) is computed from \(\psi\) by using Eqs. (3.9a), (3.9b) and (6.2).

The solution of the above Hilbert problem is unique and, according to the Plemelj theorem, exists. Thus, the Hilbert problem defines \(\xi\) (in our gauge) and \(\lambda\) for any given \(\psi\). The solution for \(\xi\) is

\[
\xi(r, s, \tau, z_0) = \psi(r, s) - \frac{1}{\pi} \int_r^s d\sigma' \frac{G(r, s, \sigma', z_0)}{\sigma' - \tau}
\]

(6.7)

and \(\lambda(\sigma, z_0)\) is given by the principal value of the above expression with \(\tau\) replaced by \(\sigma\). Alternatively, as can be seen from Eqs. (6.1) and (6.6), \(\lambda(\sigma, z_0)\) is given by Eqs. (2.42a) and (2.42b).

The third major new construct of the next paper will be a \(2 \times 2\) matrix generalization of the above Hilbert problem. This will be an HHP on \([r, s]\) and its solution will define \(2 \times 2\) matrix generalizations of \(\exp(-\sigma_3 \xi)\) and \(\exp(\sigma_3 \lambda)\) corresponding to any given \(E \in \Sigma_E\). The \(2 \times 2\) matrix generalizations of \(\exp(-\sigma_3 \xi)\) will be used, in turn, to construct (by simple algebraic means) the special gauge \(\Sigma_F\) of spectral potentials which were briefly described in Sec. IA.
E. The Fourth and Fifth Constructs

The fourth construct in the next paper will be the group $K_{(-1,1)}$ of $2 \times 2$ matrix functions $v(\sigma, z_0)$ which were briefly described in Sec. IA. This group will be defined algebraically in terms of our generalization of the set of functions $\exp[\sigma_3 \lambda(\sigma, z_0)]$ (though we caution that $K_{(-1,1)}$ is not identical with the set and that the relation between the matrices $v$ and the generalization of the functions $\exp(\sigma_3 \lambda)$ is of a subtle character). The group $K_{(-1,1)}$ will be used to formulate the fifth construct which we now introduce.

In Sec. IIIE of the current paper, we defined a second Hilbert problem on $[r, s]$. In this Hilbert problem, it is $\lambda(\sigma, z_0)$ which is assumed to be given and the idea is to find $\phi = \xi/\mu$. An alternative version of the Hilbert problem is given below. This version is slightly more complex than the one in Sec. IIIE but it is closer to the generalization in our next paper.

**Second Hilbert problem on** $[r, s]$ **when** $-1 < r < z_0 < s < 1$: Let $\lambda(\sigma, z_0)$ be given. The problem is to find $\phi(r, s, \tau, z_0)$ and $\bar{\phi}(r, s, \sigma, z_0)$ such that the following three sets of conditions are satisfied for any fixed $(r, s, z_0)$ for which $-1 < r < z_0 < s < 1$:

1. The domain of $\phi(r, s, \tau, z_0)$ is $D(r, s, z_0) - \{r, s\}$ but

$$\mu(r, s, \tau)(\tau - z_0) \phi(r, s, \tau, z_0)$$

has a continuous extension to the union of $\{z_0\}$ and $D(r, s, z_0)$. Furthermore, $\phi(r, s, \tau, z_0)$ is holomorphic throughout $C - [r, s]$ and is zero at $\tau = \infty$.

2. The function of $\sigma$ given by $\bar{\phi}(r, s, \sigma, z_0)$ is continuous throughout $r \leq \sigma < z_0$ and $z_0 < \sigma \leq s$ and has (at most) a finite step discontinuity at $\sigma = z_0$.

3. The equation

$$\phi(r, s, \sigma^\pm, z_0) = \bar{\phi}(r, s, \sigma, z_0) \mp \frac{i\lambda(\sigma, z_0)}{\sqrt{(\sigma - r)(s - \sigma)}}, \quad (6.8)$$

holds for all $\sigma$ in the ranges $r < \sigma < z_0$ and $z_0 < \sigma < s$.\[54\]
The solution for $\phi$ is given by Eq. (3.31) and the corresponding expression for $\bar{\phi}$ is the principal value of (3.31).

The fifth major new construct of the next paper will be a $2 \times 2$ matrix generalization of the above Hilbert problem and will, in fact, be the HHP which was briefly described in Sec. IA. The next paper will also include some simple examples of this HHP corresponding to various choices of $v \in \tilde{K}_{(-1,1)}$. We shall derive the solutions of these examples in the next paper.

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7. I. Hauser and F. J. Ernst, J. Math. Phys. 30, 2322 (1989). See Sec. IIA. (In this reference, $\gamma$ denoted the object which we now denote by $\psi - \frac{1}{2} \ln \rho$. This use of ‘$\gamma$’ was completely dropped in all of our subsequent papers. See Ref. 10 for another notational change.)
8. I. Hauser and F. J. Ernst, J. Math. Phys. 31, 871 (1990). See Sec. I.
9. The Ernst potential $E$ is not to be confused with the Ernst potential $\mathbf{E}$ which the authors used in some previous papers (e.g., Ref. 10) and which will be reviewed in Sec. VB of this paper.
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11. See Sec. VB of Ref. 7 for an explanation of why the Ernst potentials which represent no CPGWP’s are of interest for the study of CPGWP’s.
12. This is a large subgroup of the KC group but it is not the total KC group. The total KC group includes elements which transform members of $\Sigma_\mathcal{E}$ into Ernst potentials whose domains are proper subspaces of $\{(r, s) : -1 \leq r < s \leq 1\}$ (and which are not, therefore, in $\Sigma_\mathcal{E}$). We shall have more to say about such elements in a future paper.
13. The $F$-potentials were originally introduced by KC (Ref. 2) as generating functions for their hierarchy of potentials $H^{(n)}$. They restricted their spectral parameter $t = (2\tau)^{-1}$ to real values and a few of their conventions (e.g., the labeling of the matrix elements) differ from ours. Though KC focused on the stationary axisymmetric electrovac spacetimes, their formalism is readily extended to cover the case where the commuting Killing vectors which characterize the spacetime are both spacelike.

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35. I. Hauser and F. J. Ernst, J. Math. Phys. 22, 1051 (1981). The axis relation is proven in Sec. 5 of Ref. 35, where the matrix $u(t)$ is related to $v(\tau)$ as described above in Ref. 17. (Part of the Geroch conjecture proof in Ref. 35 is invalid since Eqs. (61) and (62) in Ref. 35 are not equivalent as claimed. This defect is remedied in Ref. 36.)

36. I. Hauser and F. J. Ernst, “A new proof of an old conjecture,” in Gravitation and Geometry, a Volume in Honor of Ivor Robinson, edited by W. Rindler and A. Trautman (Bibliopolis, Naples, 1987) pp. 165-214. The methods of Refs. 35 and 36 will be extended to $\Sigma^\xi$ (and proofs of the above assertions concerning $\Sigma^A$ and $\Sigma^m$ can then be provided) in later papers of the current series.

37. See, e.g., Sec. II of Ref. 10.

38. A good reference for this topic is N. I. Muskhelishvili, Singular Integral Equations (Noordhoff, Groningen, Holland, 1953), Ch. 2. The author’s $\xi^\pm$ is our $\xi(\sigma^\pm)$.

39. See, e.g., Sec. IIA of Ref. 7.

40. The integrals in Eqs. (3.9a) and (3.9b) are Abel transforms such as those employed and analysed in Refs. 7 and 10.
Figure 1: The domains of $\mu(\sigma^\pm)$ and $\mu(\sigma)$ when $-1 < \sigma < 1$. The domain of $\mu(\sigma)$ is $\bar{D}_\sigma^{(3)} \cup \bar{D}_\sigma^{(2)}$. The dashed lines separating the domains consist of the points on which $\mu(r, s, \sigma) = 0$. 
Figure 2: The rectangular domain $D(\tau, z_0)$ when $\tau \in C \setminus \{-1, 1\}$. Delete the points $(-1, s)$ when $\tau = -1$. Delete the points $(r, 1)$ when $\tau = 1$. 

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Figure 3: The rectangular domains $D(\sigma^\pm, z_0)$ and $D(\sigma, z_0)$ when $z_0 < \sigma < 1$. Note that $D(\sigma^\pm, z_0)$ and $D(\sigma, z_0)$ both contain their common boundary (the dashed line).
Figure 4: The rectangular domains $D(\sigma^\pm, z_0)$ and $D(\sigma, z_0)$ when $-1 < \sigma < z_0$. Note that $D(\sigma^\pm, z_0)$ and $D(\sigma, z_0)$ both contain their common boundary (the dashed line).
Figure 5: The domain $D(\sigma^\pm)$ where $-1 < \sigma < 1$. 
Figure 6: The open sets $\Gamma^+$ and $\Gamma^-$ which have a given contour $\Gamma$ as their common boundary such that $\Gamma$, $\Gamma^+$ and $\Gamma^-$ are disjoint and have $C'$ as their union.