Research Article

A multidimensional Tauberian theorem for Laplace transforms of ultradistributions

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ABSTRACT

We obtain a multidimensional Tauberian theorem for Laplace transforms of Gelfand-Shilov ultradistributions. The result is derived from a Laplace transform characterization of bounded sets in spaces of ultradistributions with supports in a convex acute cone of $\mathbb{R}^n$, also established here.

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1. Introduction

In 1976, Vladimirov obtained an important multidimensional generalization of the Hardy-Littlewood-Karamata Tauberian theorem [1]. Multidimensional Tauberian theorems were then systematically investigated by him, Drozhzhinov, and Zav’yalov, and their approach resulted in a powerful Tauberian machinery for multidimensional Laplace transforms of Schwartz distributions. Such results have been very useful in probability theory [2] and mathematical physics [3–5]. Tauberian theorems for other integral transforms of generalized functions have been extensively studied by several authors as well, see e.g. [6–10]. We refer to the monographs [11–13] for accounts on the subject and its applications; see also the recent survey article [14].

The aim of this article is to extend the so-called general Tauberian theorem for the dilation group [13, Chapter 2] from distributions to ultradistributions. Our considerations apply to Laplace transforms of elements of $\mathcal{S}_t^\ast[\Gamma]$, the space of Gelfand-Shilov ultradistributions with supports in a closed convex acute cone $\Gamma$ of $\mathbb{R}^n$ where $\ast$ and $\dagger$ stand for the Beurling and Roumieu cases of weight sequences satisfying mild assumptions (see Section 2 for definitions and notation). In Section 3 we provide characterizations of bounded sets and convergent sequences in $\mathcal{S}_t^\ast[\Gamma]$ in terms of Laplace transform growth estimates; interestingly, our approach to the desired Laplace transform characterization...
is based on a useful convolution average description of bounded sets of $\mathcal{S}'_i(\mathbb{R}^n)$, originally established in [15] (cf. [16]) but improved here by relaxing hypotheses on the weight sequences. Those results are employed in Section 4 to derive a Tauberian theorem in which the quasiasymptotic behaviour of an ultradistribution is deduced from asymptotic properties of its Laplace transform. Finally, as a natural refinement of the main result of Section 3 when the weight sequences and the cone satisfy stronger regularity conditions, we prove in Section 5 that the Laplace transform is an isomorphism of locally convex spaces between $\mathcal{S}'_i[\Gamma]$ and a certain space of holomorphic functions on the tube domain $\mathbb{R}^n + i \text{int} \Gamma^*$, with $\Gamma^*$ the conjugate cone of $\Gamma$.

2. Preliminaries

We collect in this section several useful notions that play a role in the article.

2.1. Weight sequences

Given a weight sequence $\{M_p\}_{p \in \mathbb{N}}$ of positive real numbers, we associate to it the sequences $m_p = M_p/M_{p-1}$, $p \geq 1$, and $M_p^\star = M_p/p!$. Throughout this article we will often make use of some of the following conditions:

$$(M.1) \quad M_p^2 \leq M_{p-1}M_{p+1}, \quad p \geq 1;$$

$$(M.1)^* \quad (M_p^\star)^2 \leq M_{p-1}^\star M_{p+1}^\star, \quad p \geq 1;$$

$$(M.2) \quad M_{p+1} \leq A H^p M_p, \quad p \in \mathbb{N}, \text{ for certain constants } A, H \geq 1;$$

$$(M.2)' \quad M_{p+q} \leq A H^{p+q} M_p M_q, \quad p, q \in \mathbb{N}, \text{ for certain constants } A, H \geq 1;$$

$$(M.3)' \quad \sum_{p=1}^\infty 1/m_p < \infty;$$

$$(M.3) \quad \sum_{p=q}^\infty 1/m_p \leq c_0 q/m_q, \quad q \geq 1, \text{ for a certain constant } c_0. \quad (2.3)$$

The meaning of all these conditions is very well explained in [17]. Whenever we consider weight sequences, we assume they satisfy at least (M.1). For multi-indices $\alpha \in \mathbb{N}^n$, we will simply denote $M_{|\alpha|}$ by $M_\alpha$. As usual the relation $M_p < N_p$ between two such sequences means that for any $h > 0$ there is an $L = L_h > 0$ for which $M_p \leq L h^p N_p$, $p \in \mathbb{N}$. The associated function of the sequence $M_p$ is given by

$$M(t) := \sup_{p \in \mathbb{N}} \log \frac{t^p M_0}{M_p}, \quad t > 0, \quad (2.1)$$

and $M(0) = 0$. It increases faster than $\log t$ as $t \to \infty$ (cf. [17, p. 48]). The associated function of the sequence $M_p^\star$ will be denoted by $M^\star(t)$.

Throughout this text we shall often exploit the following bounds:

- If $M_p$ satisfies (M.2)', then for any $k > 0$

  $$M(t) - M(kt) \leq -\frac{\log(t/A) \log k}{\log H}, \quad t > 0. \quad (2.2)$$

- If $M_p$ satisfies (M.2) if and only if

  $$2M(t) \leq M(Ht) + \log(AM_0). \quad (2.3)$$
• If $M_p$ satisfies (M.1)*, we have, for some $A' > 0$,
\[
M^* \left( \frac{t}{4(m_1 + 1)M(t)} \right) \leq M(t) + A', \quad t \geq m_1 + 1.
\]  

(2.4)

Indeed the first and second statements are [17, Propositions 3.4 and 3.6], while the third one is shown in [18, Lemma 5.2.5, p. 96]. We shall also consider the following two sets
\[
\mathcal{R}^{(M_p)} := \{ (\ell_p)_{p \in \mathbb{N}^+} : \ell_p = \ell \text{ for some } \ell > 0 \},
\]
\[
\mathcal{R}^{\{M_p\}} := \{ (\ell_p)_{p \in \mathbb{N}^+} : \ell_p \not\in \infty \text{ and } \ell_p > 0, \forall p \in \mathbb{N} \},
\]
and use $\mathcal{R}^*$ as a common notation. Naturally, these two sets do not depend on $M_p$ at all, but it will be very convenient for us to make a notational distinction between the Beurling and Roumieu cases of the weight sequences. For any $(\ell \in \mathbb{R}^*)$ and use
\[
\mathcal{R}^{(M_p)} := \{ (\ell_p)_{p \in \mathbb{N}^+} : \ell_p = \ell \text{ for some } \ell > 0 \},
\]
\[
\mathcal{R}^{\{M_p\}} := \{ (\ell_p)_{p \in \mathbb{N}^+} : \ell_p \not\in \infty \text{ and } \ell_p > 0, \forall p \in \mathbb{N} \},
\]
and use $\mathcal{R}^*$ as a common notation. Naturally, these two sets do not depend on $M_p$ at all, but it will be very convenient for us to make a notational distinction between the Beurling and Roumieu case of a weight sequence when dealing with ultradistributions. For any $(\ell_p) \in \mathcal{R}^*$, we write $L_p = \prod_{j=1}^p \ell_j$ and denote the associated function of $M_pL_p$ as $M_{\ell_p}(t)$. The reader should keep in mind that whenever (M.1) holds one has the ensuing useful assertions [19, Lemma 4.5, p. 417] on the growth of a function $g : [0, \infty) \to [0, \infty)$
\[
\forall h > 0 : g(t) = O(e^{M(ht)}) \iff \exists (\ell_p) \in \mathcal{R}^{\{M_p\}} : g(t) = O(e^{M_{\ell_p}(t)})
\]  

(2.5)

and
\[
\forall (\ell_p) \in \mathcal{R}^{\{M_p\}} : g(t) = O(e^{-M_{\ell_p}(t)}) \iff \exists h > 0 : g(t) = O(e^{-M(ht)}).
\]  

(2.6)

It is also important to point out that if $M_p$ satisfies (M.2) or (M.2)', then for any given $(\ell_p) \in \mathcal{R}^*$ one can always find a $k_p \in \mathcal{R}^*$ such that $k_p \leq \ell_p, \forall p \in \mathbb{N}$, and $M_pK_p$ satisfies the same condition as $M_p$. For the $(M_p)$-case this is trivial, whereas the assertion for the $(\{M_p\})$-case directly follows from [20, Lemma 2.3].

### 2.2. Ultradistributions

We now introduce the spaces of test functions and ultradistributions that we need in this work. Let $M_p$ and $N_p$ be two weight sequences. We will always assume that the sequence $M_p$ satisfies the conditions (M.1), (M.2)', and (M.3)'. On the other hand, our assumptions on $N_p$ are (M.1)* and (M.2). Furthermore, whenever considering the Beurling case we assume in addition that $N_p$ fulfills
\[
(NA) \quad p! \prec N_p.
\]

Note that these assumptions ensure that $N_{\ell_p}(t) = o(t)$ [17, Lemmas 3.8 and 3.10, p. 52–53], $N_{\ell_p}^*(t) < \infty$ for all $t \geq 0$, and $N_{\ell_p}^*(t) \to \infty$ as $t \to \infty$ for any sequence $(\ell_p) \in \mathcal{R}^*$. If stronger assumptions on the weight sequences are needed, this will be explicitly stated in the corresponding statement.

Let us now define Gelfand-Shilov spaces with respect to the sequences $M_p$ and $N_p$. We use the common notation $\ast = (M_p), [M_p]$ and $\dagger = (N_p), [N_p]$ for the Beurling and Roumieu cases of the weight sequences. For any $(a_p), (b_p) \in \mathcal{R}^*$ we consider the Banach
space of all $\varphi \in C^\infty(\mathbb{R}^n)$ such that

$$
\|\varphi\|_{(a_p), (b_p)} = \sup_{\alpha, \beta \in \mathbb{N}^n} \sup_{t \in \mathbb{R}^n} \frac{|t^\beta \varphi^{(\alpha)}(t)|}{A_\alpha M_\alpha B_\beta N_\beta},
$$

and denote it by $S_{N_p, (b_p)}^{M_p, (a_p)}(\mathbb{R}^n)$. Then, we define the test function spaces

$$
S^*_f(\mathbb{R}^n) := \lim_{(a_p), (b_p) \to \infty} S_{N_p, (b_p)}^{M_p, (a_p)}(\mathbb{R}^n),
$$

and consider their duals, the ultradistributions spaces $S'^*_f(\mathbb{R}^n)$ [16,18]. As classically done, the Roumieu type space $S^*_f(\mathbb{R}^n)$ could have also be introduced via an inductive limit, and that definition coincides (algebraically and topologically) with the projective description given here (see e.g. [21]). One has that $S_{(N_p)}^{(M_p)}(\mathbb{R}^n)$ is an (FS)-space, while $S_{(N_p)}^{(M_p)}(\mathbb{R}^n)$ is (DFS).

The subspace of $S^*_f(\mathbb{R}^n)$ consisting of compactly supported elements is denoted as usual as $\mathcal{D}^*(\mathbb{R}^n)$ (it is non-trivial [17] because of (M.3)' and we write $\mathcal{D}_K^*$ for those elements of $\mathcal{D}_K^*$ whose support is contained in a given compact subset $K \subset \mathbb{R}^n$. Similarly, $\mathcal{E}^*(\mathbb{R}^n)$ stands for the space of all $*$-ultradifferentiable functions on $\mathbb{R}^n$. These spaces are topologized in the canonical way [17,22].

### 2.3. Laplace transform

Throughout the article $\Gamma \subseteq \mathbb{R}^n$ stands for a (non-empty) closed, convex, and acute cone (with vertex at the origin). Acute means that its conjugate cone,

$$
\Gamma^* := \{ y \in \mathbb{R}^n : y \cdot u \geq 0, \ \forall \ u \in \Gamma \},
$$

has non-empty interior and we set $C = \text{int} \ \Gamma^*$. Note that $\Gamma^{**} = \Gamma$. The distance of a point $y \in \mathbb{R}^n$ to the boundary of $C$ is denoted as

$$
\Delta_C(y) := d(y, \partial C).
$$

We will often make use of the estimate (cf. [12, p. 61])

$$
y \cdot u \geq \Delta_C(y)|u|, \quad \forall \ u \in \Gamma, \ y \in C.
$$

(2.8)

The tube domain $T_C$ with base $C$ is the set

$$
T_C := \mathbb{R}^n + iC \subseteq \mathbb{C}^n.
$$

For any $\varepsilon > 0$, we denote by $\Gamma^\varepsilon$ the open set $\Gamma + B(0, \varepsilon)$. We define

$$
S^*_{f, \varepsilon}[\Gamma] = \{ f \in S^*_f(\mathbb{R}^n) : \text{supp} f \subseteq \Gamma \};
$$

it is a closed subspace of $S^*_f(\mathbb{R}^n)$. 

\[ \text{(2.7)} \]

\[ \text{(2.8)} \]
Let \( \eta : \mathbb{R}^n \to \mathbb{R} \) be a function such that \( \eta(\xi) = 1 \) for \( \xi \in \Gamma \) and \( \eta(\xi) e^{iz \cdot \xi} \in S^*_\uparrow(\mathbb{R}^n) \) for any \( z \in T^C \). The Laplace transform of \( f \in S^*_\uparrow[\Gamma] \) is then the holomorphic function

\[
L \{ f; z \} := \langle f(\xi), \eta(\xi) e^{iz \cdot \xi} \rangle, \quad z \in T^C.
\]

One can always find such an \( \eta \) (see e.g. Lemma 3.3 below) and the definition of the Laplace transform does not depend on this function. We write \( z = x + iy \) for complex variables.

### 3. Laplace transform characterization of bounded sets of \( S^*_\uparrow[\Gamma] \)

In this section we shall characterize those subsets of \( S^*_\uparrow[\Gamma] \) that are bounded (with respect to the relative topology inherited from \( S^*_\uparrow(\mathbb{R}^n) \)) via bounds on the Laplace transforms of their elements. The following theorem is our main result in this section.

**Theorem 3.1:** Let \( B \subseteq S^*_\uparrow[\Gamma] \).

(i) If \( B \) is a bounded set, then, there is \((\ell_p) \in \mathcal{P}^* \) for which, given any \( \varepsilon > 0 \), there is \( L_\varepsilon = L \) such that for all \( f \in B \)

\[
|L \{ f; z \}| \leq L \exp \left( \varepsilon |\text{Im } z| + M_{\ell_p}(|z|) + N_{\ell_p}^*(\frac{1}{\Delta C(\text{Im } z)}) \right), \quad z \in T^C. \tag{3.1}
\]

(ii) Conversely, suppose there are \( \omega \in \mathbb{C}, \sigma_0 > 0, L = L_B > 0, \) and \((\ell_p) \in \mathcal{P}^* \) such that

\[
|L \{ f; x + i\sigma \omega \}| \leq L \exp \left( M_{\ell_p}(|x|) + N_{\ell_p}^*(\frac{1}{\sigma}) \right), \tag{3.2}
\]

for all \( f \in B, x \in \mathbb{R}^n, \) and \( \sigma \in (0, \sigma_0] \), then \( B \) is a bounded subset of \( S^*_\uparrow[\Gamma] \).

Before proving Theorem 3.1, let us discuss an important consequence. Namely, we shall derive from it a characterization of convergent sequences of \( S^*_\uparrow[\Gamma] \). Notice first that if a sequence \( f_k \to g \) in \( S^*_\uparrow(\mathbb{R}^n) \) and \( \text{supp } f_k \subseteq \Gamma \) for each \( k \), one easily shows that

\[
\lim_{k \to \infty} L \{ f_k; z \} = L \{ g; z \},
\]

and this limit holds uniformly for \( z \) in compact subsets of \( T^C \); furthermore, by Theorem 3.1, the Laplace transforms of the \( f_k \) satisfy bounds of the form (3.1) uniformly in \( k \). The converse also holds. In fact, the next result might be interpreted as a sort of Tauberian theorem.

**Corollary 3.2:** Let \( (f_k)_{k \in \mathbb{N}} \) be a sequence in \( S^*_\uparrow[\Gamma] \). Suppose that there is a non-empty open subset \( \Omega \subseteq \mathbb{C} \) such that for each \( y \in \Omega \) the limit

\[
\lim_{k \to \infty} L \{ f_k; iy \} \tag{3.3}
\]

exists. If there are \( \omega \in \mathbb{C}, \sigma_0 > 0, \) and \((\ell_p) \in \mathcal{P}^* \) such that

\[
\sup_{k \in \mathbb{N}, x \in \mathbb{R}^n, \sigma \in (0, \sigma_0]} \exp \left( -M_{\ell_p}(|x|) - D_{\ell_p}^*(\frac{1}{\sigma}) \right)|L \{ f_k; x + i\sigma \omega \}| < \infty \tag{3.4}
\]
then
\[ \lim_{k \to \infty} f_k = g \quad \text{in } \mathcal{S}_i^{\ast} [\Gamma], \]  
for some \( g \in \mathcal{S}_i^{\ast} [\Gamma] \). In particular, the limit (3.3) is given by \( \mathcal{L} \{ g; iy \} \).

**Proof:** Notice first that if two subsequences converge, respectively, to ultradistributions \( g \) and \( h \), the limits (3.3) tell us \( \mathcal{L} \{ g; iy \} = \mathcal{L} \{ h; iy \} \) for all \( y \in \Omega \). By uniqueness of holomorphic functions and the injectivity of the Laplace transform (which follows from that of the Fourier transform), we conclude \( g = h \). It therefore suffices to show that every arbitrary subsequence of the \( f_k \) possesses a convergent subsequence in \( \mathcal{S}_i^{\ast} [\Gamma] \), but this follows from the fact that \( \mathcal{S}_i^{\ast} [\Gamma] \) is Montel because, in view of Theorem 3.1, the estimate (3.4) is equivalent to \( \{ f_k : k \in \mathbb{N} \} \) being bounded in \( \mathcal{S}_i^{\ast} [\Gamma] \) (and hence relatively compact).  

We shall prove Theorem 3.1 using several lemmas. For (i), we need the following concept. A family \( \{ \eta_{\varepsilon} \}_{\varepsilon > 0} \) of non-negative smooth functions \( \eta_{\varepsilon} : \mathbb{R}^n \to [0, \infty) \) is called a \( \ast \)-\( \Gamma \)-mollifier if for every \( \varepsilon > 0 \) the ensuing conditions hold

(a) \( \eta_{\varepsilon} (\xi) = 1 \) for \( \xi \in \Gamma^\varepsilon \) while \( \eta_{\varepsilon} (\xi) = 0 \) for \( \xi \notin \Gamma^{2\varepsilon} \);

(b) for every \( (\ell_p) \in \mathcal{R}^* \) there is a constant \( H_{\ell_p, \varepsilon} > 0 \) such that
\[
\left| \eta_{\varepsilon}^{(\alpha)} (\xi) \right| \leq H_{\ell_p, \varepsilon} L_\alpha M_\alpha, \quad \forall \xi \in \mathbb{R}^n, \quad \forall \alpha \in \mathbb{N}^n.
\]

**Lemma 3.3:** There are \( \ast \)-\( \Gamma \)-mollifiers.

**Proof:** The existence of such functions is guaranteed by non-quasianalyticity. Take any non-negative \( \varphi \in \mathcal{D}^* (\mathbb{R}^n) \) such that \( \text{supp} \varphi \subset B(0, 1/2) \) and \( \int_{\mathbb{R}^n} \varphi (\xi) \, d\xi = 1 \). Set \( \varphi_{\varepsilon} (\xi) := \varepsilon^{-n} \varphi (\xi/\varepsilon) \) and let \( \chi_{\Gamma^{3\varepsilon}/2} \) be the characteristic function of \( \Gamma^{(3/2)\varepsilon} \). Taking \( \eta_{\varepsilon} = \varphi_{\varepsilon} \ast \chi_{\Gamma^{3\varepsilon}/2} \), one easily verifies that \( \{ \eta_{\varepsilon} \}_{\varepsilon > 0} \) is a \( \ast \)-\( \Gamma \)-mollifier.  

**Lemma 3.4:** Let \( (a_p), (b_p) \in \mathcal{R}^* \) and \( \{ \eta_{\varepsilon} \}_{\varepsilon > 0} \) be a \( \ast \)-\( \Gamma \)-mollifier. Then there is \( (\ell_p) \in \mathcal{R}^* \) such that, for any \( \varepsilon > 0 \), we have
\[
\| \eta_{\varepsilon} (\xi) e^{iz \xi} \|_{(a_p), (b_p)} \leq H_{\ell_p, \varepsilon} \exp \left( 4\varepsilon |\text{Im} z| + M_{\ell_p} (|z|) + N_{\ell_p}^a \left( \frac{1}{\Delta_C (\text{Im} z)} \right) \right), \quad z \in \mathbb{C}.
\]
In particular, we have \( \eta_{\varepsilon} (\xi) e^{iz \xi} \in \mathcal{S}_{M_p(a_p)}^{M_p(b_p)} (\mathbb{R}^n) \) for all \( z \in \mathbb{C} \).

**Proof:** We only employ here the assumptions (M.1) and (M.3)' for the sequence \( M_p \). Set \( \ell_p' := \min \{ a_p, b_p \} \). Due to the support assumption on \( \eta_{\varepsilon} \), we may assume below that \( \xi \in \Gamma^{2\varepsilon} \). Then for any \( z \in \mathbb{C}, \alpha, \beta \in \mathbb{N}^n \), we have
\[
\left| \sum_{0 \leq \alpha' \leq \alpha} \binom{\alpha}{\alpha'} (2|z|)^{\alpha'} M_{\ell_p'} (|z|) \right| \leq H_{\ell_p, \varepsilon} e^{M_{\ell_p} (|z|)} |\xi|^{\beta} e^{-y \xi} \frac{\left( \eta_{\varepsilon} (\xi) e^{iz \xi} \right)}{L_\beta N_\beta},
\]
\[
\left| \sum_{0 \leq \alpha' \leq \alpha} \binom{\alpha}{\alpha'} \left( \frac{2|\alpha' - \alpha|}{L_{\alpha' - \alpha} M_{\alpha - \alpha}} \right) \right| \leq H_{\ell_p, \varepsilon} e^{M_{\ell_p} (|z|)} |\xi|^{\beta} e^{-y \xi} \frac{\left( \eta_{\varepsilon} (\xi) e^{iz \xi} \right)}{L_\beta N_\beta},
\]
\[
\leq H_{\ell_p, \varepsilon} e^{M_{\ell_p} (|z|)} |\xi|^{\beta} e^{-y \xi} \frac{\left( \eta_{\varepsilon} (\xi) e^{iz \xi} \right)}{L_\beta N_\beta},
\]
where we have set $\ell_p := \ell'_p / 2$. Now $\xi = u + \nu$ for certain $u \in \Gamma$ and $\nu \in B(0, 2\varepsilon)$, so that by the Cauchy-Schwarz inequality

$$
\left| \frac{\xi^{\beta}}{L_\beta N_\beta} e^{-\gamma \xi} \right| \leq \left( \frac{|u| + 2\varepsilon}{L_\beta N_\beta} \right)^{\beta} e^{-\gamma u} \frac{e^{-\gamma \nu}}{L_\beta N_\beta} \leq \left( \frac{|u| + 2\varepsilon}{L_\beta N_\beta} \right)^{\beta} e^{-\Delta C(y)|u|} \frac{e^{2\varepsilon |y|}}{L_\beta N_\beta}
$$

$$
\leq \left( \frac{1}{\Delta C(y)} \right)^{\beta} \left( \frac{|\beta|}{e} \right)^{\beta} e^{2\varepsilon \Delta C(y) + 2\varepsilon |y|} \leq \exp \left( N_{\ell_p}^{\ast} \left( \frac{1}{\Delta C(y)} + 4\varepsilon |y| \right) \right),
$$

where we have used (2.8) and the elementary inequality $m^m \leq e^m m!$. ■

In preparation for the proof of part (ii), we first need to extend [15, Proposition 3.1] (cf. [16, Lemma 2.7]) by relaxing assumptions on the weight sequences. This provides a useful convolution characterization of bounded sets in $S_{\ast}^{\prime}(\mathbb{R}^n)$. Our approach to this convolution characterization employs the short-time Fourier transform (STFT) in the context of ultradistributions [23,24] and is inspired by the method from [25]. Given an ultradistribution and a window $\psi$ (a test function), the STFT of $f$ with respect to $\psi$ is given by the smooth function

$$
V_\psi f(x, \xi) = \langle f(t), \overline{\psi}(t-x) e^{-2\pi i t; \xi} \rangle,
$$

$(x, \xi) \in \mathbb{R}^{2n}$.

**Lemma 3.5:** A subset $B \subset S_{\ast}^{\prime}(\mathbb{R}^n)$ is bounded if and only if there exists $(\ell_p) \in \mathcal{R}^*$ such that

$$
\sup_{f \in B; x \in \mathbb{R}^n} e^{-N_{\ell_p}(|x|)} \left| (f * \psi)(x) \right| < \infty, \quad \forall \psi \in \mathcal{D}^* (\mathbb{R}^n).
$$

**Proof:** We only make use here of the assumptions (M.1) and (M.2)' on $N_p$. The necessity is easily obtained via the norms (2.7). Hence suppose that (3.7) holds for some $(\ell_p) \in \mathcal{R}^*$. We may assume the sequence $L_p N_p$ satisfies (M.2)'. We consider the weighted Banach space $X = \{ g \in C(\mathbb{R}^n) : g(\xi) = O(\exp(\epsilon |\xi|)) \}$ and fix a compact set $K \subset \mathbb{R}^n$ with non-empty interior.

The assumption (3.7) implies that for each $f \in B$ the mapping $L_f : \varphi \mapsto * \varphi$ is continuous from $\mathcal{D}^\ast(\mathbb{R}^n)$ into $X$, so that in particular, in view of the Banach-Steinhaus theorem, $\tilde{B} = \{(L_f)|_{\mathcal{D}^\ast_K} : f \in B \}$ is an equicontinuous subset of $L_b(\mathcal{D}^\ast_K, X)$. This implies that there is $(h_p) \in \mathcal{R}^*$ such that $\tilde{B} \subset L_b(\mathcal{D}^{\mathcal{M}_p}_{K,(h_p)}, X)$ and it is equicontinuous there, where $\mathcal{D}^{\mathcal{M}_p}_{K,(h_p)} = \{ \psi \in \mathcal{D}_K : \sup_{x \in K, \alpha \in \mathbb{N}^n} |\psi(\alpha)(x)|/(H_\alpha M_\alpha) < \infty \}$. Fix $\psi \in \mathcal{D}^{(\mathcal{M}_p)}_K$ with $\| \psi \|_{L^2(\mathbb{R}^n)} = 1$. Since $\{ e^{-M_{\mathcal{M}_p}(4\pi |\xi|)} e^{2\pi i \xi; \overline{\psi}} : \xi \in \mathbb{R}^n \}$ is a bounded family in $\mathcal{D}^{(\mathcal{M}_p)}_K$, we conclude that, for some $C_B > 0$, independent of $f \in B$,

$$
|V_\psi f(x, \xi)| = \left| e^{-2\pi i \xi; x} \left( f * (e^{2\pi i \xi; \overline{\psi}}) \right)(x) \right| \leq C_B \exp \left( N_{\ell_p}(|x|) + M_{\mathcal{M}_p}(4\pi |\xi|) \right).
$$

On the other hand, let now $\varphi \in S_{\ast}^{\prime}(\mathbb{R}^n)$. For any $(\ell'_p) \in \mathcal{R}^*$ it follows from [21, Proposition 1] that there is some $C_{\varphi} > 0$ such that

$$
|V_{-\varphi} \varphi(x, -\xi)| \leq C_{\varphi} \exp \left( -N_{\ell'_p}(|x|) - M_{\mathcal{M}_p}(4\pi |\xi|) \right).
$$
Moreover, according to the desingularization formula\(^1\) for the STFT [23, Equation (2.6)],

\[
\langle f, \varphi \rangle = \int_{\mathbb{R}^n} V_{\varphi} f(x, \xi) V_{\overline{\varphi}} \varphi(x, -\xi) \, dx \, d\xi.
\]

Let \( h > 0 \) be such that \( \log h / \log H \geq n + 1 \) (with \( H \) the corresponding constant occurring in (M.2)’ for \( L_p N_p \) and \( H_p M_p \)) and set \( \ell'_p := h^{-1} \min(\ell_p, (4\pi)^{-1} h_p) \), then applying (2.2) one gets

\[
\sup_{f \in B} \|f, \varphi\| \leq C_B C_{\varphi} \int_{\mathbb{R}^n} e^{M_{h_p}(4\pi|\xi|) - M_{\ell'_p}(\xi)} \, d\xi \int_{\mathbb{R}^n} e^{N_{h_p}(|x|) - N_{\ell'_p}(|x|)} \, dx < \infty,
\]

which concludes the proof of the sufficiency. \( \square \)

We are now ready to present a proof of Theorem 3.1.

**Proof:** Suppose \( B \subseteq S^*_t(\Gamma) \) is bounded in \( S^*_t(\mathbb{R}^n) \). By equicontinuity, there are certain \((a_p), (b_p) \in \mathfrak{R}^*\) such that \( B \subseteq (S_{N_p}(b_p))'(\mathbb{R}^n) \) and it is bounded there. Then, (3.1) follows directly from Lemma 3.4 (in particular, one does not employ (M.2) for \( N_p \) in this implication).

We now show that (3.2) is sufficient to guarantee boundedness. We are going to do this employing Lemma 3.5. We may assume that \((\ell_p)\) is such that \( L_p M_p \) satisfies (M.2)’ and \( L_p N_p^* \) fulfills (M.2) (the constants occurring in these conditions are denoted by \( A \) and \( H \) below). We may also suppose that \(|\omega| = 1\). Fix \( \varphi \in D^*(\mathbb{R}^n) \). Find \( R > 0 \) such that \( \text{supp} \varphi \subseteq B(0, R) \). We keep \( f \in B \). Take a bounded function \( \gamma : \mathbb{R}^n \to (0, \sigma_0] \), which will be specified later. Inverting the Laplace transform of \( f \ast \varphi \),

\[
(f \ast \varphi)(t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n+i\gamma(t)\omega}} e^{-iz \cdot t} \mathcal{L} \{f; z\} \mathcal{L} \{\varphi; z\} \, dz.
\]

By [17, Lemma 3.3, p. 49] and (2.6), we have that for any \((h_p) \in \mathfrak{R}^*\)

\[
|\mathcal{L} \{\varphi; x + i\gamma(t)\omega\}| \leq L_{\varphi} \exp\left(-M_{h_p}(|x|) + R\gamma(t)\right), \quad x \in \mathbb{R}^n.
\]

Choose \( h > 0 \) such that \( \log h \geq (n + 1) \log H \). Taking \( h_p = \ell_p / h \), the condition (M.2)’ in the form of estimate (2.2) yields

\[
M_{\ell_p}(|x|) - M_{h_p}(|x|) = M_{\ell_p}(|x|) - M_{\ell_p}(h|x|) \leq -(n + 1) \log(|x|/A),
\]

whence we infer the exponential function of this expression is integrable on \( \mathbb{R}^n \). Let \( d = \Delta_C(\omega) \). Employing (3.2) we then obtain

\[
\left| \int_{\mathbb{R}^{n+i\gamma(t)\omega}} e^{-iz \cdot t} \mathcal{L} \{f; z\} \mathcal{L} \{\varphi; z\} \, dz \right|
\leq L_B L_{\varphi} \exp\left(\gamma(t)(\omega \cdot t) + N_{h_p}^{*}\left(\frac{1}{d\gamma(t)}\right) + R\gamma(t)\right) \int_{\mathbb{R}^n} e^{M_{\ell_p}(|x|) - M_{h_p}(|x|)} \, dx
\leq L \exp\left(N_{\ell_p}^{*}\left(\frac{1}{d\gamma(t)}\right) + |t|\gamma(t)\right),
\]
for some $L > 0$. Note that $L_p N_p$ satisfies $(M.1)^*$, so that (2.4) holds for it. Also, since $N_{\ell_p}(t) = o(t)$, there is a sufficiently large $r_0$ such that

$$\frac{4(n_1 \ell_1 + 1)N_{\ell_p}(|t|)}{d|t|} \leq \sigma_0 \quad \text{for } |t| > r_0.$$  

Set $r = \max\{r_0, n_1 \ell_1 + 1\}$, we then define

$$\gamma(t) = \begin{cases} \sigma_0, & |t| < r, \\ \frac{4(n_1 \ell_1 + 1)N_{\ell_p}(|t|)}{d|t|}, & |t| \geq r. \end{cases}$$

For $|t| < r$ obviously

$$\exp\left(\frac{1}{d\gamma(t)} + |t|\gamma(t)\right) \leq \exp\left(r\sigma_0 + N_{\ell_p}^*\left(\frac{1}{\sigma_0 d}\right)\right).$$

If $|t| \geq r$, the inequality (2.4) yields

$$\exp\left(\frac{1}{d\gamma(t)} + |t|\gamma(t)\right) \leq \exp\left(N_{\ell_p}^*\left(\frac{|t|}{4(n_1 \ell_1 + 1)N_{\ell_p}(|t|)}\right) + \frac{4(n_1 \ell_1 + 1)}{d}N_{\ell_p}(|t|)\right) \leq \exp\left(2^k N_{\ell_p}(|t|) + A'\right)$$

for some $A' > 0$ and $k = \lceil \log_2 (1 + 4(n_1 \ell_1 + 1)/d) \rceil$. By repeated application of (2.3) for $N_{\ell_p}$, one obtains

$$\exp\left(2^k N_{\ell_p}(|t|)\right) \leq \exp(N_{\ell_p}(H^k|t|) + A''),$$

for some $A'' > 0$. Let $a_p = \ell_p H^{-pk}$. Summing up, we have shown that

$$\sup_{f \in B, t \in \mathbb{R}^n} e^{-N_{\ell_p}(|t|)} |(f * \varphi)(t)| < \infty.$$  

Since $\varphi$ was arbitrary, Lemma 3.5 applies to conclude that $B$ is bounded. 

4. The Tauberian theorem

We shall now use our results from the previous section to generalize the Drozhzhinov-Vladimirov-Zav’yalov multidimensional Tauberian theorem for Laplace transforms [12,13] from distributions to ultradistributions. Our goal is to devise a Laplace transform criterion for the so-called quasiasymptotics.

The quasiasymptotic behaviour was originally introduced by Zav’yalov [26] for distributions, but the definition of this concept naturally extends to ultradistributions or other duals [11,27,28] as follows. Assume that $\mathcal{X}$ is a (barreled) locally convex space of test functions on $\mathbb{R}^n$ provided with a continuous action of dilations. A generalized function
$f \in \mathcal{X}'$ is said to have \textit{quasiasymptotic behaviour} (at infinity) with respect to a (measurable) function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ if there is $g \in \mathcal{X}'$ such that
\[ f(\lambda \xi) \sim \rho(\lambda) g(\xi), \quad \text{as } \lambda \to \infty \tag{4.1} \]
in $\mathcal{X}'$, that is, if for each test function $\varphi \in \mathcal{X}$
\[ \langle f(\lambda \xi), \varphi(\xi) \rangle = \lambda^{-n} \langle f(\lambda \xi), \varphi(\xi/\lambda) \rangle \sim \rho(\lambda) \langle g(\xi), \varphi(\xi) \rangle. \]

The generalized function $g$ must be homogenous of some degree $\alpha \in \mathbb{R}$ and, if $g \neq 0$, the function $\rho$ must be regularly varying \cite{29} of index $\alpha \in \mathbb{R}$, namely,
\[ \lim_{\lambda \to \infty} \frac{\rho(\lambda a)}{\rho(\lambda)} = a^\alpha, \quad \text{for each } a > 0. \]

Note that since we are only interested in its terminal behaviour, one may assume \cite{29} without any loss of generality that the regularly varying function $\rho$ is continuous on $[0, \infty)$. We are exclusively interested in the case $\mathcal{X}' = \mathcal{S}'^*_{\Gamma}(\mathbb{R}^n)$.

We call a cone $C$ \textit{solid} if it is non-empty and $\text{int} C' \neq \emptyset$.

After these preliminaries, we are ready to present our Tauberian theorem. It is inverse to the ensuing Abelian statement that readily follows from the definition: If an ultradistribution $f \in \mathcal{S}'^*_{\Gamma}[\Gamma]$ has quasiasymptotic behaviour (4.1) in $\mathcal{S}'^*_{\Gamma}(\mathbb{R}^n)$, then
\[ \lim_{r \to 0^+} \frac{r^n}{\rho(1/r)} \mathcal{L} \{ f ; rz \} = \mathcal{L} \{ g ; z \} \tag{4.2} \]
uniformly for $z$ in compact subsets of $T^C$. In the next theorem we write $\sharp = (A_p), \{ A_p \}$ for the Beurling and Roumieu cases of the specified weight sequence $A_p$.

**Theorem 4.1:** Assume that $M_p$ and $N_p$ both satisfy (M.1) and (M.2), while $M_p$ also satisfies (M.3). Set $A_p = M_p N_p$. Let $f \in \mathcal{S}'^*_{\Gamma}[\Gamma]$ and let $\rho$ be regularly varying of index $\alpha$. Suppose that there is a non-empty solid subcone $C' \subset C$ such that for each $y \in C'$ the limit
\[ \lim_{r \to 0^+} \frac{r^n}{\rho(1/r)} \mathcal{L} \{ f ; riy \} \tag{4.3} \]
exists. If there are $\omega \in C$ and $(\ell_p) \in \mathcal{R}^*$ such that
\[ \limsup_{r \to 0^+} \sup_{|x| + \sin^2 \theta = 1, \theta \in (0, \pi/2]} \frac{r^n e^{-A_p^* (1/\sin \theta)}}{\rho(1/r)} | \mathcal{L} \{ f ; r(x + i \sin \theta \omega) \} | < \infty, \tag{4.4} \]
then $f$ has quasiasymptotic behaviour with respect to $\rho$ in $\mathcal{S}'^*_{\Gamma}(\mathbb{R}^n)$.

**Proof:** In view of Corollary 3.2, it suffices to show that the Laplace transform of $f$ satisfies a bound of the form
\[ \frac{r^n}{\rho(1/r)} | \mathcal{L} \{ f ; r(x + i \sigma \omega) \} | \leq L \exp \left( M_{\ell_p^*}(|x|) + N_{\ell_p^*} \left( \frac{1}{\sigma} \right) \right) \tag{4.5} \]
for some $(\ell_p^*) \in \mathcal{R}^*, \sigma_0 > 0$ and all $x \in \mathbb{R}^n$ and $0 < \sigma < \sigma_0$. We may assume (M.2) holds for both $L_p M_p$ and $L_p N_p^*$ (with constants $A$ and $H$). We can also assume that $L_p \geq 1$ for all
thanlog
whenever \( |x|^2 + \sin^2 \theta = 1 \), where we always keep \( 0 < \theta < \pi/2 \). On the other hand, applying Theorem 3.1 to the singleton \( B = \{ f \} \) and possibly enlarging \( (\ell_p) \),
\[
|\mathcal{L} \{ f; r(x + i\sigma \omega) \} | \leq L_2 \exp \left( M_{\ell_p}(|x|) + A_{\ell_p}^* \left( \frac{1}{r \sigma} \right) \right),
\]
for any \( 0 < r < 1 \), \( x \in \mathbb{R}^n \) and \( \sigma < r_0 < 1 \). We may assume that \( \rho(\lambda) = 1 \) for \( \lambda < r_0 \).
Furthermore, Potter’s estimate [29, Theorem 1.5.4] yields
\[
\frac{\rho(\lambda t)}{\rho(\lambda)} \leq L_3 t^a \max\{t^{-1}, t\}, \quad t, \lambda > 0.
\]
We keep arbitrary \( r < 1 \), \( x \in \mathbb{R}^n \), \( 0 < \sigma < r_0 \), and write \( r' = \sqrt{|x|^2 + \sigma^2} \), \( x' = x/r' \), and \( \sin \theta = \sigma/r' \). If \( rr' < r_0 \), we obtain from (4.8), (4.6), and the fact that \( M_{\ell_p}(t) \) increases faster than \( \log t \),
\[
\frac{r^n}{\rho(1/r)} |\mathcal{L} \{ f; r(x + i\sigma \omega) \} | \leq L_4 \exp \left( M_{\ell_p}(h|x|) + A_{\ell_p}^* \left( \frac{h}{\sigma} \right) \right).
\]
We have found in all cases
\[
\frac{r^n}{\rho(1/r)} |\mathcal{L} \{ f; r(x + i\sigma \omega) \} | \leq L_4 \exp \left( M_{\ell_p}(h|x|) + A_{\ell_p}^* \left( \frac{h}{\sigma} \right) \right).
\]
for some \( L_4 \) and \( h = \max\{h', 2\} \). It remains to observe that \( A_{\ell_p}^* (h|x|/\sigma) \leq M_{\ell_p}(h|x|) + N_{\ell_p}^* (h/\sigma) \), so that (4.5) holds with \( \ell_p = \ell_p/(Hh) \), by (M.2).

5. Sharpening the bound (3.1)
If the sequence \( M_p \) and the cone \( \Gamma \) satisfy stronger conditions, it turns out that the bound (3.1) can be considerably improved. In fact, we shall show here how to remove the \( \varepsilon \) term from (3.1). Recall \( \Gamma \) is a solid cone if \( \text{int} \Gamma \neq \emptyset \).

We start with three lemmas, from which our improvement of Theorem 3.1 will follow.
Lemma 5.1: Let \( \{F_j\}_{j \in I} \) be a family of holomorphic functions on \( T^C \). Suppose that for some \( (\ell_p) \in \mathfrak{R}^* \) and each \( \varepsilon > 0 \) there is \( L = L_{\varepsilon} > 0 \) such that for all \( j \in I \)

\[
| (1 + |\text{Re}|)^{n+2} F_j(z) | \leq L \exp \left( \varepsilon |\text{Im} z| + N_{\ell_p}^* \left( \frac{1}{\Delta_C(\text{Im} z)} \right) \right), \quad z \in T^C. \tag{5.1}
\]

Then there are \( (h_p) \in \mathfrak{R}^* \) and \( f_j \in C^1(\mathbb{R}^n) \) with \( \text{supp} f_j \subseteq \Gamma, \forall j \in I \), such that \( \{e^{-N_{h_p}(1+\cdot)} f_j\}_{j \in I} \) is a bounded set in \( L^\infty(\mathbb{R}^n) \) and \( F_j(z) = \mathcal{L} \{f_j; z\}, j \in I \).

Proof: We closely follow the proof of the lemma in [12, Section 10.5, p. 148]. We may assume that \( L_{\varepsilon} N_p \) satisfies (M.1) and (M.2). From (5.1) it follows in particular that

\[
(1 + |\cdot|) F_j(\cdot + iy) \in L^1(\mathbb{R}^n), \quad \forall y \in C, j \in I.
\]

From the Cauchy formula we obtain for each compact subset \( K \subseteq C \) and each \( j \in I \)

\[
\sup_{y \in K} \left| \frac{\partial}{\partial y_k} F_j(x + iy) \right| = O \left( \frac{1}{(1 + |x|)^{n+2}} \right), \quad k \in \{1, \ldots, n\}.
\]

Therefore,

\[
g_j(\xi, y) = (2\pi)^{-n} e^{\xi \cdot y} \mathcal{F} \{ F_j(\cdot + iy); \xi \} \in C^1(\mathbb{R}^n \times C), \quad j \in I,
\]

where \( \mathcal{F} \) stands for the Fourier transform. Furthermore, for each \( k \in \{1, \ldots, n\} \),

\[
\frac{\partial}{\partial y_k} g_j(\xi, y) = (2\pi)^{-n} e^{\xi \cdot y} \left[ \xi_k \mathcal{F} \{ F_j(\cdot + iy); \xi \} + i \mathcal{F} \left\{ \frac{\partial}{\partial x_k} F_j(\cdot + iy); \xi \right\} \right] = 0,
\]

so that the \( C^1 \) functions \( f_j(\xi) := g_j(\xi, y) \) do not depend on \( y \in C \). By (5.1), there is \( L' = L_{\varepsilon}' > 0 \) such that

\[
|f_j(\xi)| \leq L' \exp \left( \xi \cdot y + \varepsilon |y| + N_{\ell_p}^* \left( \frac{1}{\Delta_C(y)} \right) \right), \quad \xi \in \mathbb{R}^n, y \in C, j \in I. \tag{5.2}
\]

Take any \( \xi_0 \notin \Gamma \). As \( (\Gamma^*)^* = \Gamma \), there is some \( y_0 \in C \) such that \( \xi_0 \cdot y_0 = -1 \). Since \( \Delta_C(\lambda y_0) = \lambda \Delta_C(y_0) \) for \( \lambda > 0 \), we conclude from (5.2) for \( \varepsilon = (2|y_0|)^{-1} \) and \( y = \lambda y_0 \) that

\[
|f_j(\xi_0)| \leq L' \exp \left( -\frac{\lambda}{2} + N_{\ell_p}^* \left( \frac{1}{\lambda \Delta_C(y_0)} \right) \right), \quad \lambda > 0.
\]

By letting \( \lambda \to \infty \), it follows that this is only possible if \( f(\xi_0) = 0 \). We conclude that \( \text{supp} f_j \subseteq \Gamma \) for each \( j \in I \).

Now take an arbitrary \( y_0 \in C \) such that \( |y_0| = 1 \), then (5.2) gives us for \( \varepsilon = 1/2 \) and \( y = \lambda y_0, \lambda > 0 \),

\[
|f_j(\xi)| \exp \left( -(1 + |\xi|) \lambda - N_{\ell_p}^* \left( \frac{1}{\lambda \Delta_C(y_0)} \right) \right) \leq L' e^{-\lambda/2}.
\]

We now integrate this inequality with respect to \( \lambda \) on \( (0, \infty) \) in order to gain an estimate on the \( f_j \). The 1-dimensional case of [18, Lemma 5.2.6, p. 97] applied to the open cone \( (0, \infty) \),
yields the existence of constants $L''$, $c > 0$ such that
\[
\int_0^\infty \exp \left( -(1 + |\xi|)\lambda - N_{\ell_p}^* \left( \frac{1}{\lambda \Delta C(y_0)} \right) \right) \, d\lambda \geq L'' \exp \left( -N_{\ell_p}(c(1 + |\xi|)) \right).
\]
Hence, using [18, Lemma 2.1.3, p. 16], it follows for any $\xi \in \mathbb{R}^n$ and $j \in I$ that
\[
|f_j(\xi)| \leq \frac{2L'}{L''} \exp \left( N_{\ell_p}(c(1 + |\xi|)) \right) \leq \frac{2L'}{L''} \exp \left( N_{\ell_p}(2c(2c|\xi|)) \right).
\]
The proof is complete noticing that by the Fourier inversion theorem, yields the existence of constants $L''$, $c > 0$ such that
\[
\int_0^\infty \exp \left( -(1 + |\xi|)\lambda - N_{\ell_p}^* \left( \frac{1}{\lambda \Delta C(y_0)} \right) \right) \, d\lambda \geq L'' \exp \left( -N_{\ell_p}(c(1 + |\xi|)) \right).
\]

Recall an ultrapolynomial [17] of type $\ast$ is an entire function
\[
P(z) = \sum_{m=0}^{\infty} a_m z^m, \quad a_m \in \mathbb{C},
\]
where the coefficients satisfy $|a_m| \leq L/H_m M_m$ for some $(h_p) \in \mathcal{R}^*$ and $L > 0$. It is ensured by condition (M.2) that the multiplication of two ultrapolars is again an ultrapolynomial (cf. [17, Proposition 4.5, p. 58] and (2.5)).

**Lemma 5.2:** Let $\Gamma$ be a solid cone solid and let $(\ell_p) \in \mathcal{R}^*$. Suppose that $L_p M_p$ satisfies (M.1), (M.2), and (M.3). Then, there are an ultrapolynomial $P$ of type $\ast$ and constants $L, L' \geq 1$ such that
\[
e^{M_{\ell_p}(|z|)} \leq |P(z)| \leq L' e^{M_{\ell_p}(L|z|)}, \quad \forall z \in T^C.
\]

**Proof:** Set
\[
\tilde{P}(z) := \prod_{p=1}^{\infty} \left( 1 + \frac{z}{\ell_p m_p} \right), \quad z \in \mathbb{C},
\]
an ultrapolynomial of type $\ast$ satisfying a bound $P(z) = O(e^{M_{\ell_p}(L'|z|)})$ [17, Propositions 4.5 and 4.6, pp. 58–59]. Now for $\text{Re} z \geq 0$ as in [17, p. 89]
\[
|\tilde{P}(z)| \geq \sup_{p \in \mathbb{N}} \prod_{q=1}^{p} \left| \frac{z}{\ell_q m_q} \right| = \sup_{p \in \mathbb{N}} \frac{M_0 |z|^p}{L_p M_p} = e^{M_{\ell_p}(|z|)}.
\]

Since we assumed $\text{int} \Gamma \neq \emptyset$, there is a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ such that $e_j \in \text{int} \Gamma$ for $1 \leq j \leq n$. Find also $\lambda > 0$ such that $\lambda \min j |e_j \cdot z| \geq |z|$ for all $z \in C$. Now define
\[
P(z) := \prod_{j=1}^{n} \tilde{P}(-\lambda n^{1/2} i e_j \cdot z),
\]
which is an ultrapolynomial of type $\ast$ as well and the upper bound in (5.3) holds because of (2.3) applied to $M_{\ell_p}$. Since for any $z \in T^C$ we have $\text{Re}(-nie_j \cdot z) > 0$, $1 \leq j \leq n$, one then obtains for any $z$ in the tube domain
\[
|P_{\ell_p}(z)| \geq \exp \left( \sum_{j=1}^{n} M_{\ell_p} \left( n^{1/2} \lambda |e_j \cdot z| \right) \right) \geq \exp \left( M_{\ell_p}(|z|) \right).
\]

\[\blacksquare\]
Lemma 5.3: Let $(\ell_p) \in \mathcal{R}^*$. It holds that for any $y \in C$

$$\sup_{\xi \in \Gamma} \exp \left( N_{\ell_p}(\ell_p(\xi) - y \cdot \xi) \right) \leq \exp \left( N_{\ell_p}^*(\frac{1}{\Delta_C(y)}) \right).$$  \tag{5.4}$$

Proof: We only make use of $(M.1)^*$. Using the estimate (2.8), we obtain for any $y \in C$

$$\sup_{\xi \in \Gamma} e^{N_{\ell_p}(\ell_p(\xi) - y \cdot \xi)} \leq \sup_{t \geq 0} e^{N_{\ell_p}(t) - \Delta_C(y) t},$$

so that (5.4) follows from [30, Lemma 5.6]

$$\sup_{t > 0} \left| N_{\ell_p}(t) - st \right| \leq N_{\ell_p}^*(\frac{1}{s}), \quad s > 0.$$  \hfill \blacksquare

Theorem 5.4: Suppose that the cone $\Gamma$ is solid, $M_p$ and $N_p$ both satisfy $(M.1)$ and $(M.2)$, and $M_p$ also satisfies $(M.3)$. Then, a set $B \subset S^*_1(\Gamma)$ is bounded if and only if there are $L > 0$ and $(\ell_p) \in \mathcal{R}^*$ such that for all $f \in B$

$$|L \{f; z\}| \leq L \exp \left( M_{\ell_p}(|z|) + N_{\ell_p}^* \left( \frac{1}{\Delta_C(\text{Im} z)} \right) \right), \quad z \in T^C. \tag{5.5}$$

Proof: We only need to show that if $B = \{f_j\}_{j \in I}$ is bounded then (5.5) holds. By Theorem 3.1, there is $(\ell_p) \in \mathcal{R}^*$ such that for any $\varepsilon > 0$ there is $L = L_\varepsilon > 0$ such that for all $j \in I$

$$|L \{f_j; z\}| \leq L \exp \left( \varepsilon |y| + M_{\ell_p}(|z|) + N_{\ell_p}^* \left( \frac{1}{\Delta_C(y)} \right) \right), \quad \forall z \in T^C.$$  

We may assume $L_p M_p$ satisfies $(M.1)$, $(M.2)$ and $(M.3)$. Let $P$ be the ultrapolynomial constructed as in Lemma 5.2. Fix $k \geq H^{n+2}$, where $H$ is the constant occurring in $(M.2)'$ for $L_p M_p$. We consider the ultrapolynomial $Q(z) = P(kz)$, so that it satisfies the bounds $e^{M_{\ell_p}(kz)} \leq |Q(z)| \leq L' e^{M_{\ell_p}(vz)}$ for all $z \in T^C$ and some $v > 0$. Set now $F_j(z) = L \{f_j; z\}$, which are holomorphic functions on $T^C$. In view of (2.2), the family $\{F_j/Q\}_{j \in I}$ satisfies the conditions of Lemma 5.1, so that there are $g_j \in C^1(\mathbb{R}^n)$ with $\text{supp} g_j \subseteq \Gamma$ for which there is some $(\ell_p') \in \mathcal{R}^*$ such that $\{\exp(-N_{\ell_p'}(|\cdot|) g_j)\}_{j \in I}$ is a bounded subset of $L^\infty(\mathbb{R}^n)$ and $F_j(z) = Q(z)L \{g_j; z\}$ for each $j \in I$. Now, taking into account (2.2) (we may assume $H$ is the same constant for both $L_p M_p$ and $L'_p N_p$) and Lemma 5.3, there are some $L'', L''' > 0$ such that for all $j \in I$

$$|F_j(z)| \leq L'' e^{M_{\ell_p}(vz)} \int_\Gamma e^{N_{\ell_p}(\ell_p(\xi) - y \cdot \xi)} d\xi$$

$$\leq L'' e^{M_{\ell_p}(vz)} \sup_{\xi \in \Gamma} e^{N_{\ell_p}(k\xi) - y \cdot \xi} \int_\Gamma \frac{d\xi}{(1 + |\xi|)^{n+2}}$$

$$\leq AL' \left( \int_\Gamma \frac{d\xi}{(1 + |\xi|)^{n+2}} \right) \exp \left( M_{\ell_p}(v|z|) + N_{\ell_p}^* \left( \frac{k}{\Delta_C(y)} \right) \right).$$

Hence, we obtain a bound of type (5.5) for the sequence $k_p = \min\{\ell_p/v, \ell_p'/k\}. \quad \blacksquare$
Theorem 5.4 can be used to draw further topological information. In fact, it leads to an isomorphism between $S_{r}^{\ast} [\Gamma]$ and analogs of the Vladimirov algebra [12, Chapter 12] $H (T^{C})$ of holomorphic functions on $T^{C}$. Given $\ell > 0$, we define the Banach space $\mathcal{O}_{\ell} (T^{C})$ of all holomorphic functions $F$ on the tube domain $T^{C}$ that satisfy the bounds

$$
\| F \|_{\ell} = \sup_{z \in T^{C}} | F(z) | e^{-M(\ell |z|) - N^{*}(\ell / \Delta_{C}(\text{Im } z))} < \infty.
$$

We then introduce the $(DFS)$ and $(FS)$ spaces

$$
\mathcal{O}_{(N_{p})}^{(M_{p})} (T^{C}) = \lim_{\ell \to \infty} \mathcal{O}_{\ell} (T^{C}) \text{ and } \mathcal{O}_{[N_{p}]}^{[M_{p}]} (T^{C}) = \lim_{\ell \to 0} \mathcal{O}_{\ell} (T^{C}).
$$

The arguments we have given above actually show that the Laplace transform maps $S_{r}^{\ast} [\Gamma]$ bijectively into $\mathcal{O}^{\ast} (T^{C})$ and that this mapping and its inverse map bounded sets into bounded sets (cf. the property (2.5) in the Roumieu case). Since the spaces under consideration are all bornological, we might summarize the results from this section as follows,

**Theorem 5.5:** Let $\Gamma$ be a solid convex acute cone and suppose that $M_{p}$ and $N_{p}$ both satisfy $(M.1)$ and $(M.2)$, while $M_{p}$ also satisfies $(M.3)$. Then, the Laplace transform

$$
\mathcal{L} : S_{r}^{\ast} [\Gamma] \to \mathcal{O}^{\ast} (T^{C})
$$

is an isomorphism of locally convex spaces.

**Note**

1. This is stated in [23] under the assumptions $(M.1)$ and $(M.2)$, but one can relax $(M.2)$ to $(M.2)'$ using the continuity result [21, Proposition 1] and adapting the arguments given in [25, Section 3] or [23].

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