QUASIPOSITIVITY AND BRAID INDEX OF PRETZEL KNOTS

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Abstract. This short note is about three-stranded pretzel knots that have an even number of crossings in one of the strands. We calculate the braid index of such knots and determine which of them are quasipositive. The main tools are the Morton-Franks-Williams inequalities, and Khovanov-Rozansky concordance homomorphisms.

1. Introduction

Our protagonists are the $P(p,q,-2r)$ pretzel knots, where $p,q,r$ are integers, $p,q$ are odd and not equal to $\pm 1$, and $r$ is not 0. See Figure 1 for an example. Recently, Boileau, Boyer and Gordon proved that $P(p,q,-2r)$ is a strongly quasipositive knot if and only if all of $p,q,r$ are positive [BBG19]. The first result of this note gives a similar condition for these pretzel knots to be quasipositive.

Theorem 1. Let $p,q$ be odd integers not equal to $\pm 1$, and $r$ a non-zero integer. Then the $P(p,q,-2r)$ pretzel knot is quasipositive if and only if $p+q \geq 0$ and $r > 0$.

As a second result, which we need to prove the first, we explicitly define a braid $\beta(p,q,-2r)$ on $|r|+2$ strands with closure $P(p,q,-2r)$, and show that it is minimal, i.e. that it has the minimal number of strands among all braids with that closure.

Theorem 2. Let $p,q$ be odd integers not equal to $\pm 1$, and $r$ a non-zero integer. Then the braid index of $P(p,q,-2r)$ is $|r| + 2$. It is realized by the braid $\beta(p,q,-2r)$.

The braid $\beta(p,q,-2r)$ is defined in Section 2. To show its minimality and thus prove Theorem 2, we partially compute the Homflypt polynomial of $P(p,q,-2r)$, and then rely on the Morton-Franks-Williams inequalities.

To prove the ‘if’ direction of Theorem 1, we simply observe that $\beta(p,q,-2r)$ is quasipositive if $p+q \geq 0$ and $r > 0$. To show the ‘only if’ direction, we require the following obstruction to quasipositivity.

Lemma 3. [FK17, Lemma 3.6] Let $K$ be a quasipositive knot with braid index $b$. Let $w$ be the writhe of any minimal braid of $K$. Let $\phi$ be any slice-torus invariant. Then

$$1 + w - b = 2\phi(K).$$

The proof of Lemma 3 uses Jones’ conjecture, shown in [DP13, LM14], stating that all minimal braids of a knot $K$ have the same writhe. Comparing Lemma 3 with the previously computed values of the Khovanov-Rozansky $sl_3$-slice torus invariant [Lew14] reveals that $P(p,q,-2r)$ is not quasipositive if $p+q < 0$ or $r < 0$. The proof of Theorem 1 is contained in Section 3.

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The canonical diagram $D(p, q, -2r)$ of the $P(p, q, -2r)$ pretzel consists of two disks connected by three bands with $p, q, -2r$ half-twists, respectively. Note that by convention the signs of $p, q, -2r$ encode the handedness of twists. The signs of the crossings in the three bands in $D(p, q, -2r)$ are $\text{sgn}(p), \text{sgn}(q), \text{sgn}(r)$, respectively. As further background, let us list some results in a similar vein as Theorem 1:

- $P(p, q, -2r)$ is strongly quasipositive $\iff P(p, q, -2r)$ is positive $\iff p, q, r$ are all positive [BBG19].
- $P(p, q, -2r)$ is alternating $\iff D(p, q, -2r)$ is alternating $\iff p, q, -r$ all have the same sign [LT88].
- $P(p, q, -2r)$ is quasi-alternating $\iff p, q, -r$ all have the same sign, or \{p + q, p - 2r, q - 2r\} contains a negative and a positive number [Gre10].
- $P(p, q, -2r)$ is fibred $\iff p, q$ are of opposite sign, or $|r| = 1$ [Gab86].
- Conjecturally, $P(p, q, -2r)$ is slice $\iff p + q = 0$ [Lec15].

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2. The braid index of pretzel knots

Definition 4. Let $p, q$ be odd integers not equal to $\pm 1$, and $r$ a non-zero integer. Let us define the following braids on $|r| + 2$ strands, denoting the standard generators of the braid group by $a_1, \ldots, a_{|r| + 1}$:

- $\gamma_r = a_3 a_4 \cdots a_{|r|} a_{|r| + 1}$,
- $\overline{\gamma_r} = a_{|r| + 1} a_{|r|} \cdots a_4 a_3$,
- $\beta(p, q, -2r) = a_1^q a_2 a_1^{-1} a_2^q a_1^{-1} a_2^{-r} a_1^{-1} a_2^{r-1}$ if $r > 0$,
- $\beta(p, q, -2r) = a_1^q a_2^{-1} a_1^{-1} a_2^{-1} a_1^{-1} a_2^{-1}$ if $r < 0$.

It has been shown in [DM20] that the closure of $\beta(p, q, -2r)$ is $P(p, q, -2r)$ (see also Figure 2 for an example). Note that in the simplest cases $r = \pm 1$, $\gamma_r$ and $\overline{\gamma_r}$ are empty and thus $\beta(p, q, -2) = a_1^q a_2 a_1^{-1} a_2$ and $\beta(p, q, 2) = a_1^q a_2^{-1} a_1 a_2^{-1}$.

Let us now prove Theorem 2 by showing that $\beta(p, q, -2r)$ realizes the braid index; for this, we will use the Morton-Franks-Williams inequalities [Mor86, FW87]:

$$1 + w(\beta) - k(\beta) \leq e(K) \leq E(K) \leq -1 + w(\beta) + k(\beta), \quad (1)$$

where $\beta$ is a braid with writhe $w(\beta)$, on $k(\beta)$ strands, and closure a knot $K$. Moreover, $e$ and $E$ are respectively the minimum and maximum exponent of $v$ appearing in the Homflypt polynomial $Q_K(v, z) \in \mathbb{Z}[v^\pm 1, z^\pm 1]$, which is defined by the skein relation $v^{-1}Q_{L^+} - vQ_{L^-} = zQ_{L_0}$ and setting $Q_U = 1$ (where $L_+$ is a link admitting a diagram $D$, such that $L_-$ arises from $D$ by changing a positive crossing to a negative one, and $L_0$ by orientedly resolving that crossing). Now (1) implies

$$2 \text{br}(K) \geq E(K) - e(K) + 2,$$
The term with lowest exponent of $v$ is $F(p + q + 1)F(q + 1)F(p + 1)$. The term with highest exponent of $v$ is $x^{p+q}$, provided $x := F(p + q + 1) + F(p + 1)F(q + 1)F(p + 1)F(q + 1) \neq 0$.

The reader is invited to spot the isotopy between the two diagrams.

where $\text{br}(K)$ denotes the braid index of $K$. For our purposes, it is sufficient to use the specialization $Q'_K(v) := Q_K(v, 1) \in \mathbb{Z}[v^{\pm 1}]$ of the Homflypt polynomial. Defining $e'$ and $E'$ as minimum and maximum exponent of $v$ appearing in $Q'$, we clearly have $E' \leq E$ and $e' \geq e$, and so

$$2 \text{br}(K) \geq E'(K) - e'(K) + 2. \tag{2}$$

This is the lower bound for the braid index that we will use. Let us now compute $e'$ and $E'$ of $P(p, q, -2r)$. We start by computing $Q'$ of the torus link $T(2, n)$ for all $n \in \mathbb{Z}$, which is the closure of the two-stranded braid $a_1^p$. One finds

$$Q'_{T(2, 0)} = v^{-1} - v,$$

$$Q'_{T(2, 1)} = 1,$$

$$Q'_{T(2, n+2)} = vQ'_{T(2, n+1)} + v^2Q'_{T(2, n)} \text{ for all } n \geq 0.$$  \tag{3}

Denote by $F(n)$ the $n$-th Fibonacci number, i.e.

$$F(0) = 0, F(1) = 1 \text{ and } F(n+2) = F(n+1) + F(n) \text{ for all } n \geq 0.$$  \tag{3}

It easily follows inductively that for all $n \geq 1$,

$$Q'_{T(2, n)} = F(n+1)v^{n-1} - F(n-1)v^{n+1}.$$  \tag{3}

Using $Q'_{T(2, -n)}(v) = (-1)^{n+1}Q'_{T(2, n)}(v^{-1})$, one may extend (3) to all $n$ by setting $F(-n) = (-1)^{n+1}F(n)$ for all positive $n$. Since $F(n) \neq 0$ for all $n \neq 0$, we find $e'(T(2, n)) = n - 1$ and $E'(T(2, n)) = n + 1$ for all $n \neq \pm 1$.

Let us proceed to the pretzels, starting with the case $r = 1$. Applying the skein relation to one of the two crossings in the band with 2 twists in $D(p, q, -2)$, and using that $Q'(K\#J) = Q'(K)Q'(J)$ for knots $K, J$ yields

$$Q'_{P(p, q, -2)} = vQ'_{T(2, p+q)} + v^2Q'_{T(2, p)}Q'_{T(2, q)}.$$  \tag{4}

In this polynomial, the term with highest exponent of $v$ is $F(p + 1)F(q + 1)v^{p+q+4}$. The term with lowest exponent of $v$ is $x^{p+q}$, provided

$$x := F(p + q + 1) + F(p + 1)F(q + 1) \neq 0.$$  \tag{4}

Let us show (4). We have $F(p+q+1) > 0$, $\text{sgn } F(p+1) = \text{sgn } p$, $\text{sgn } F(q+1) = \text{sgn } q$. So if $\text{sgn } p = \text{sgn } q$, then $F(p+1)F(q+1) > 0$, and (4) follows. If $\text{sgn } p \neq \text{sgn } q$, then $|p+1| > |p+q+1|$ or $|q+1| > |p+q+1|$, and thus $|F(p+1)| > |F(p+q+1)|$. 

Figure 2. On the left, the closure of $\beta(5, -3, -6) = a_1^3a_2a_1^{-3}a_2^{-1}a_2^{-1}a_2a_2a_3a_3$. On the right, the standard diagram $D(5, -3, -6)$ of the pretzel knot $P(5, -3, -6)$. The reader is invited to spot the isotopy between the two diagrams.
or $|F(q + 1)| > |F(p + q + 1)|$ (using monotonicity of the Fibonacci numbers), and so (4) also holds. It follows that $e'(P(p, q, -2)) = p + q$ and $E'(P(p, q, -2)) = p + q + 4$, so $\text{br}(P(p, q, -2)) \geq 3 = r + 2$ as desired.

Now for the case $r \geq 2$, applying the skein relation to one of the crossings in the band with $2r$ twists in $D(p, q, -2r)$ gives

$$Q'_{P(p, q, -2r)} = vQ'_{T(2,p+q)} + v^2Q'_{P(p,q,2-2r)}.$$  

It follows inductively that $Q'_{P(p,q,-2r)} = A + B$ for

$$A = (v + v^3 + \ldots + v^{2r-1}) \cdot Q'_{T(2,p+q)},$$
$$B = v^{2r} \cdot Q'_{T(2,p)} \cdot Q'_{T(2,q)}$$

and so

$$e'(A) = p + q,$$  
$$E'(A) = p + q + 2r$$
$$e'(B) = p + q + 2r - 2,$$  
$$E'(B) = p + q + 2r + 2.$$  

Thus $e'(P(p, q, -2r)) = e'(A)$ and $E'(P(p, q, -2r)) = E'(B)$, and it follows that $\text{br}(P(p, q, -2r)) \geq r + 2$.

We have shown for all positive $r$ that the braid index of $P(p, q, -2r)$ equals $|r| + 2$. The braid index of the knots $P(p, q, -2r)$ and $P(-p, -q, 2r)$ agrees, since they are mirror images of one another. Thus we have completed the proof of Theorem 2. □

### 3. The quasipositivity of pretzel knots

As sketched in the introduction, the proof of Theorem 1 relies on Lemma 3, which we restate below for the reader’s convenience. Here, a braid is quasipositive if it equals a product of conjugates of the standard generators $a_i$ of the braid group. A knot is quasipositive if it is the closure of some quasipositive braid. A slice-torus invariant $\phi$ is a homomorphism from the smooth knot concordance group to $\mathbb{R}$, such that $\phi(K) \leq g_4(K)$ holds for all knots $K$ (where $g_4$ denotes the smooth slice genus), and $\phi(T(p, q)) = g_4(T(p, q))$ holds for all positive torus knots $T(p, q)$ [Liv04] (see also [Lew14, FLL22a]).

**Lemma 3.** [FK17, Lemma 3.6] Let $K$ be a quasipositive knot with braid index $b$. Let $w$ be the writhe of any minimal braid of $K$. Let $\phi$ be any slice-torus invariant. Then

$$1 + w - b = 2\phi(K).$$

While the proof of Lemma 3 uses Jones’ conjecture, it is worth noting that if the Morton-Franks-Williams inequality (2) is an equality for a knot $K$ (as it is for the pretzel knots we are considering), then the statement of Jones’ conjecture for $K$ can be easily deduced directly from (1).

Let us now prove Theorem 1. We consider four exhaustive and mutually exclusive cases. Note that the writhe $w(\beta(p, q, -2r))$ equals $p + q + r + \text{sgn } r$. We will use the statement of Theorem 2 that $\beta(p, q, -2r)$ is a minimal braid representative for $P(p, q, -2r)$.

- **Case $p + q \geq 0$ and $r > 0$.** Then, $\beta(p, q, -2r)$ is clearly quasipositive. This settles the ‘if’ direction of Theorem 1. The remaining three cases cover the ‘only if’ direction.
The class of squeezed knots includes quasipositive, quasinegative, and alternating invariants [Lew14]. Since all slice-torus invariants take the same value on a fixed cases. In the first case that \( \text{sgn}(\phi) \neq 0 \), the braid is strongly quasipositive if and only if \( p,q,r > 0 \). On the other hand, one observes that for \( p,q,r > 0 \), the braid \( \beta(p,q,−2r) = a_r^0a_1^2a_2^0b_{2,r+1}1 \) is strongly quasipositive.

In summary we have:

- \( P(p,q,−2r) \) braid positive \( \iff \beta(p,q,−2r) \) positive braid \( \iff p,q > 0, r = 1 \).
- \( P(p,q,−2r) \) str. quasipos. \( \iff \beta(p,q,−2r) \) str. quasipos. \( \iff p,q,r > 0 \).
- \( P(p,q,−2r) \) quasipositive \( \iff \beta(p,q,−2r) \) quasipositive \( \iff p + q \geq 0, r > 0 \).

A knot \( K \) is called squeezed if it appears as a slice of a genus-minimizing oriented connected smooth cobordism between a positive and a negative torus knot [FLL22b]. The class of squeezed knots includes quasipositive, quasinegative, and alternating knots. This guarantees the squeezedness of \( P(p,q,−2r) \) unless \( \text{sgn}(p + q) = −\text{sgn} r \) and \( \text{sgn} p = −\text{sgn} q \). For pretzels satisfying those equations, let us distinguish two cases. In the first case that \( \text{sgn}(p − 2r) = −\text{sgn}(q − 2r) \), e.g. as for \( P(5,−3,2) \), the Khovanov-Rozansky \( s_3 \)-slice torus invariant \( s_3 \) is not equal to the Rasmussen invariant [Lew14]. Since all slice-torus invariants take the same value on a fixed

4. Further properties of Pretzel knots

Let us have a quick look at further notions of positivity. A knot is called braid positive if it is the closure of a positive braid, i.e. a braid that can be written as product of positive powers of the standard generators. Such knots are fibered [Sta78] and positive. Recalling from the introduction which product of positive powers of the standard generators. Such knots are fibered [Sta78]

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squeezed knot, the knots in that case are not squeezed. I have not been able to
determine the squeezedness of the pretzel knots of the second case, namely those
with $\text{sgn}(p - 2r) = \text{sgn}(q - 2r)$, such as $P(5, -3, 4)$.

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