The number of edges in $k$-quasi-planar graphs

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Abstract

A graph drawn in the plane is called $k$-quasi-planar if it does not contain $k$ pairwise crossing edges. It has been conjectured for a long time that for every fixed $k$, the maximum number of edges of a $k$-quasi-planar graph with $n$ vertices is $O(n)$. The best known upper bound is $n(\log n)^{O(\log k)}$. In the present note, we improve this bound to $(n \log n)^{2^{\alpha_k(n)}}$ in the special case where the graph is drawn in such a way that every pair of edges meet at most once. Here $\alpha(n)$ denotes the (extremely slowly growing) inverse of the Ackermann function. We also make further progress on the conjecture for $k$-quasi-planar graphs in which every edge is drawn as an $x$-monotone curve. Extending some ideas of Valtr, we prove that the maximum number of edges of such graphs is at most $2^{2^{\alpha_k(n)}} n \log n$.

1 Introduction

A topological graph is a graph drawn in the plane such that its vertices are represented by points and its edges are represented by non-self-intersecting arcs connecting the corresponding points. In notation and terminology, we make no distinction between the vertices and edges of a graph and the points and arcs representing them, respectively. No edge is allowed to pass through any point representing a vertex other than its endpoints. Any two edges can intersect only in a finite number of points. Tangencies between the edges are not allowed. That is, if two edges share an interior point, then they must properly cross at this point. A topological graph is simple if every pair of its edges intersect at most once: at a common endpoint or at a proper crossing. If the edges of a graph are drawn as straight-line segments, then the graph is called geometric.

Finding the maximum number of edges in a topological graph with a forbidden crossing pattern is a fundamental problem in extremal topological graph theory (see §2). It follows from Euler’s Polyhedral Formula that every topological graph on $n$ vertices and with no two crossing edges has at most $3n - 6$ edges. A graph is called $k$-quasi-planar if it can be drawn as a topological graph with no $k$ pairwise crossing edges. A graph is 2-quasi-planar if and only

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if it is planar. According to an old conjecture (see Problem 1 in Section 9.6 of [5]), for any fixed \( k \geq 2 \) there exists a constant \( c_k \) such that every \( k \)-quasi-planar graph on \( n \) vertices has at most \( c_k n \) edges. Agarwal, Aronov, Pach, Pollack, and Sharir [1] were the first to prove this conjecture for simple 3-quasi-planar graphs. Later, Pach, Radoiˇ ci´ c, and T´ oth [17] generalized the result to all 3-quasi-planar graphs. Ackerman [1] proved the conjecture for \( k = 4 \).

For larger values of \( k \), first Pach, Shahrokhi, and Szegedy [18] showed that every simple \( k \)-quasi-planar graph on \( n \) vertices has at most \( c_k n (\log n)^{2k-4} \) edges. For \( k \geq 3 \) and for all (not necessarily simple) \( k \)-quasi-planar graphs, Pach, Radoiˇ ci´ c, and T´ oth [17] established the upper bound \( c_k n (\log n)^{4k-12} \). Plugging into these proofs the above mentioned result of Ackerman [1], for \( k \geq 4 \), we obtain the slightly better bounds \( c_k n (\log n)^{2k-8} \) and \( c_k n (\log n)^{4k-16} \), respectively. For large values of \( k \), the exponent of the polylogarithmic factor in these bounds was improved by Fox and Pach [10], who showed that the maximum number of edges of a \( k \)-quasi-planar graph on \( n \) vertices is \( n (\log n)^{O(\log k)} \).

For the number of edges of geometric graphs, that is, graphs drawn by straight-line edges, Valtr [22] proved the upper bound \( O(n \log n) \). He also extended this result to simple topological graphs whose edges are drawn as \( x \)-monotone curves [23].

The aim of this paper is to improve the best known bound, \( n (\log n)^{O(\log k)} \), on the number of edges of a \( k \)-quasi-planar graph in two special cases: for simple topological graphs and for not necessarily simple topological graphs whose edges are drawn as \( x \)-monotone curves. In both cases, we improve the exponent of the polylogarithmic factor from \( O(\log k) \) to \( 1 + o(1) \).

**Theorem 1.1.** Let \( G = (V, E) \) be a \( k \)-quasi-planar simple topological graph with \( n \) vertices. Then \( |E(G)| \leq (n \log n)^{2^{\alpha(n)^{c_k}}} \), where \( \alpha(n) \) denotes the inverse of the Ackermann function and \( c_k \) is a constant that depends only on \( k \).

Recall that the Ackermann (more precisely, the Ackermann-Péter) function \( A(n) \) is defined as follows. Let \( A_1(n) = 2n \), and \( A_k(n) = A_{k-1}(A_k(n-1)) \) for \( k = 2, 3, \ldots \). In particular, we have \( A_2(n) = 2^n \), and \( A_3(n) \) is an exponential tower of \( n \) two’s. Now let \( A(n) = A_n(n) \), and let \( \alpha(n) \) be defined as \( \alpha(n) = \min \{ k \geq 1 : A(k) \geq n \} \). This function grows much slower than the inverse of any primitive recursive function.

**Theorem 1.2.** Let \( G = (V, E) \) be a \( k \)-quasi-planar (not necessarily simple) topological graph with \( n \) vertices, whose edges are drawn as \( x \)-monotone curves. Then \( |E(G)| \leq 2^{c_k^6 n \log n} \), where \( c \) is an absolute constant.

In both proofs, we follow the approach of Valtr [23] and apply results on generalized Davenport-Schinzel sequences. An important new ingredient of the proof of Theorem 1.1 is a corollary of a separator theorem established in [9] and developed in [8]. Theorem 1.2 is not only more general than Valtr’s result, which applies only to simple topological graphs, but its proof gives a somewhat better upper bound: we use a result of Pettie [20], which improves the dependence on \( k \) from double exponential to single exponential.

## 2 Generalized Davenport-Schinzel Sequences

The sequence \( u = a_1, a_2, \ldots, a_m \) is called \( l \)-regular if any \( l \) consecutive terms are pairwise different. For integers \( l, t \geq 2 \), the sequence

\[ S = s_1, s_2, \ldots, s_{lt} \]
of length $lt$ is said to be of type $up(l, t)$ if the first $l$ terms are pairwise different and

$$s_i = s_{i+l} = s_{i+2l} = \cdots = s_{i+(t-1)l}$$

for every $i, 1 \leq i \leq l$. For example,

$$a, b, c, a, b, c, a, b, c,$$

is a type $up(3, 4)$ sequence or, in short, an $up(3, 4)$ sequence. We need the following theorem of Klazar [13] on generalized Davenport-Schinzel sequences.

**Theorem 2.1** (Klazar). For $l \geq 2$ and $t \geq 3$, the length of any $l$-regular sequence over an $n$-element alphabet that does not contain a subsequence of type $up(l, t)$ has length at most

$$n \cdot l^{2(lt-3)} \cdot (10l)^{10a(n)^l}.$$ 

For $l \geq 2$, the sequence

$$S = s_1, s_2, \ldots, s_{3l-2}$$

of length $3l - 2$ is said to be of type $up-down-up(l)$ if the first $l$ terms are pairwise different and

$$s_i = s_{2l-i} = s(2l-2)+i$$

for every $i, 1 \leq i \leq i$. For example,

$$a, b, c, d, c, b, a, b, c, d,$$

is an $up-down-up(4)$ sequence. Valtr and Klazar [14] showed that any $l$-regular sequence over an $n$-element alphabet, which contains no subsequence of type $up-down-up(l)$, has length at most $2^{cl} n$ for some constant $c$. This has been improved by Pettie [20], who proved the following.

**Lemma 2.2** (Pettie). For $l \geq 2$, the length of any $l$-regular sequence over an $n$-element alphabet, which contains no subsequence of type $up-down-up(l)$, has length at most $2^{O(l^2)} n$.

For more results on generalized Davenport-Schinzel sequences, see [15, 20, 19].

## 3 On intersection graphs of curves

In this section, we prove a useful lemma on intersection graphs of curves. It shows that every collection $C$ of curves, no two of which intersect many times, contains a large subcollection $C'$ such that in the partition of $C'$ into its connected components $C_1, \ldots, C_t$ in the intersection graph of $C$, each component $C_i$ has a vertex connected to all other $|C_i| - 1$ vertices.

For a graph $G = (V, E)$, a subset $V_0$ of the vertex set is said to be a separator if there is a partition $V = V_0 \cup V_1 \cup V_2$ with $|V_1|, |V_2| \leq \frac{2}{3}|V|$ such that no edge connects a vertex in $V_1$ to a vertex in $V_2$. We need the following separator lemma for intersection graphs of curves, established in [9].

**Lemma 3.1** (Fox–Pach). There is an absolute constant $c_1$ such that every collection $C$ of curves with $x$ intersection points has a separator of size at most $c_1\sqrt{x}$. 

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Call a collection $C$ of curves in the plane decomposable if there is a partition $C = C_1 \cup \ldots \cup C_t$ such that each $C_i$ contains a curve which intersects all other curves in $C_i$, and for $i \neq j$, the curves in $C_i$ are disjoint from the curves in $C_j$. The following lemma is a strengthening of Proposition 6.3 in [8]. Its proof is essentially the same as that of the original statement. It is included here, for completeness.

**Lemma 3.2.** There is an absolute constant $c > 0$ such that every collection $C$ of $m \geq 2$ curves such that each pair of them intersect in at most $t$ points has a decomposable subcollection of size at least $\frac{cm}{t \log m}$.

**Proof of Lemma 3.2** We prove the following stronger statement. There is an absolute constant $c > 0$ such that every collection $C$ of $m \geq 2$ curves whose intersection graph has at least $x$ edges, and each pair of curves intersect in at most $t$ points, has a decomposable subcollection of size at least $\frac{cm}{t \log m} + \frac{x}{m}$. Let $c = \frac{1}{5m^2}$, where $c_1 \geq 1$ is the constant in Lemma 3.1. The proof is by induction on $m$, noting that all collections of curves with at most three elements are decomposable. Define $d = d(m, x, t) := \frac{cm}{t \log m} + \frac{x}{m}$.

Let $\Delta$ denote the maximum degree of the intersection graph of $C$. We have $\Delta < d - 1$. Otherwise, the subcollection consisting of a curve of maximum degree, together with the curves in $C$ that intersect it, is decomposable and its size is at least $d$, and we are done. Also, $\Delta \geq 2 \frac{x}{m}$, since $2 \frac{x}{m}$ is the average degree of the vertices in the intersection graph of $C$. Hence, if $\Delta \geq 2 \frac{cm}{t \log m}$, then the desired inequality holds. Thus, we may assume $\Delta < 2 \frac{cm}{t \log m}$.

Applying Lemma 3.1 to the intersection graph of $C$, we obtain that there is a separator $V_0 \subset C$ with $|V_0| \leq c_1 \sqrt{tx}$, where $c_1$ is the absolute constant in Lemma 3.1. That is, there is a partition $C = V_0 \cup V_1 \cup V_2$ with $|V_1|, |V_2| \leq 2|V|/3$ such that no curve in $V_1$ intersects any curve in $V_2$. For $i = 1, 2$, let $m_i = |V_i|$ and $x_i$ denote the number of pairs of curves in $V_i$ that intersect, so that

$$x_1 + x_2 \geq x - \Delta |V_0| \geq x - 2 \frac{cm}{t \log m} c_1 \frac{x}{\sqrt{tx}}. \quad (1)$$

As no curve in $V_1$ intersects any curve in $V_2$, the union of a decomposable subcollection of $V_1$ and a decomposable subcollection of $V_2$ is decomposable. Thus, by the induction hypothesis, $C$ contains decomposable subcollection of size at least

$$d(m_1, x_1, t) + d(m_2, x_2, t) = \frac{cm_1}{t \log m_1} + \frac{x_1}{m_1} + \frac{cm_2}{t \log m_2} + \frac{x_2}{m_2} \geq \frac{c(m_1 + m_2)}{t \log (2m/3)} + \frac{(x_1 + x_2)}{2m/3}.$$

We split the rest of the proof into two cases.

**Case 1.** $x \geq t^{-1} \left(12 c_1 \frac{m}{\log m}\right)^2$. In this case, by (1), we have $x_1 + x_2 \geq \frac{5}{6}x$ and hence there is a decomposable subcollection of size at least

$$d(m_1, x_1, t) + d(m_2, x_2, t) \geq \frac{c(m_1 + m_2)}{t \log m} + \frac{5x}{4m} = d + \frac{x}{4m} - \frac{c(m - (m_1 + m_2))}{t \log m} \geq d + \frac{x}{4m} - \frac{c_1 \sqrt{tx}}{t \log m} > d,$$

completing the analysis.
Case 2. \( x < t^{-1}\left(12c_1c\frac{m}{\log m}\right)^2 \). There is a decomposable subcollection of size at least

\[
d(m_1, x_1, t) + d(m_2, x_2, t) \geq \frac{c(m_1 + m_2)}{t \log(2m/3)} \geq \frac{c}{t} \left(m - c_1\sqrt{tx}\right) \left(\frac{1}{\log m} + \frac{1}{2\log^2 m}\right)
\]

\[
\geq \frac{c}{t} \left(\frac{m}{\log m} + \frac{m}{2\log^2 m} - \frac{2c_1\sqrt{tx}}{\log m}\right) \geq \frac{c}{t} \left(\frac{m}{\log m} + \frac{m}{4\log^2 m}\right)
\]

\[
\geq \frac{c}{t} \left(\frac{m}{\log m} + \frac{m}{4\log^2 m}\right) \geq \frac{cm}{t \log m} + \frac{x}{m} = d,
\]

where we used \( c = \frac{1}{576c_1}\). \(\square\)

4 Simple Topological Graphs

In this section, we prove Theorem 1.1. The following statement will be crucial for our purposes.

Theorem 4.1. Let \( G = (V, E) \) be a \( k \)-quasi-planar simple topological graph with \( n \) vertices. Suppose that \( G \) has an edge that crosses every other edge. Then we have \( |E| \leq n \cdot 2^{\alpha(n)k} \), where \( \alpha(n) \) denotes the inverse Ackermann function and \( c_k \) is a constant that depends only on \( k \).

Proof of Theorem 4.1. Let \( k \geq 5 \) and \( c_k = 40 \cdot 2^{k^2 + 2k} \). To simplify the presentation, we do not make any attempt to optimize the value of \( c_k \). Label the vertices of \( G \) from 1 to \( n \), i.e., let \( V = \{1, 2, \ldots, n\} \). Let \( e = uv \) be the edge that crosses every other edge in \( G \). Note that \( d(u) = d(v) = 1 \).

Let \( E' \) denote the set of edges that cross \( e \). Suppose without loss of generality that no two of elements of \( E' \) cross \( e \) at the same point. Let \( e_1, e_2, \ldots, e_{|E'|} \) denote the edges in \( E' \) listed in the order of their intersection points with \( e \) from \( u \) to \( v \). We create two sequences of vertices \( S_1 = p_1, p_2, \ldots, p_{|E'|} \) and \( S_2 = q_1, q_2, \ldots, q_{|E'|} \subset V \), as follows. For each \( e_i \in E' \), as we move along edge \( e \) from \( u \) to \( v \) and arrive at the intersection point with \( e_i \), we turn left and move along edge \( e_i \) until we reach its endpoint \( u_i \). Then we set \( p_i = u_i \). Likewise, as we move along edge \( e \) from \( u \) to \( v \) and arrive at edge \( e_i \), we turn right and move along edge \( e_i \) until we reach its other endpoint \( w_i \). Then we set \( q_i = w_i \). Thus, \( S_1 \) and \( S_2 \) are sequences of length \( |E'| \) over the alphabet \( \{1, 2, \ldots, n\} \). See Figure 1 for a small example.

We need two lemmas. The first one is due to Valtr [23].

Lemma 4.2 (Valtr). For \( l \geq 1 \), at least one of the sequences \( S_1, S_2 \) defined above contains an \( l \)-regular subsequence of length at least \( |E'|/(4l) \). \(\square\)

Since each edge in \( E' \) crosses \( e \) exactly once, the proof of Lemma 4.2 can be copied almost verbatim from the proof of Lemma 4 in [23] and is left to the reader as an exercise.

For the rest of this section, we set \( l = 2^{k^2 + k} \) and \( t = 2^k \).

Lemma 4.3. Neither of the sequences \( S_1 \) and \( S_2 \) has a subsequence of type \( up(l, t) \).

Proof. By symmetry, it suffices to show that \( S_1 \) does not contain a subsequence of type \( up(l, t) \). The argument is by contradiction. We will prove by induction on \( k \) that the existence of such a sequence would imply that \( G \) has \( k \) pairwise crossing edges. The base cases \( k = 1, 2 \) are trivial. Now assume the statement holds up to \( k - 1 \). Let
Since a point on the edge $e$ corresponds to $s_{t+1}$ for $j = 0, 1, 2, \ldots, t-1$. We will think of $s_{t+1}$ as a point on $a_{i,j}$ very close but not on edge $e$. For simplicity, we will let $s_{t+q} = s_q$ for all $q \in \mathbb{N}$ and $a_{i,j} = a_{i,j'}$ for all $j \in \mathbb{Z}$, where $j' \in \{0, 1, 2, \ldots, t-1\}$ is such that $j \equiv j' \pmod{t}$. Hence there are $l$ distinct vertices $v_1, \ldots, v_l$, each vertex of which has $l$ arcs emanating from it to the edge $e$.

Consider the arrangement formed by the $t$ arcs emanating from $v_1$ and the edge $e$. Since $G$ is simple, these arcs partition the plane into $t$ regions. By the pigeonhole principle, there is a subset $V' \subset \{v_1, \ldots, v_l\}$ of size

$$\frac{l-1}{t} = \frac{2k^2+k-1}{2^k}$$

such that all of the vertices of $V'$ lie in the same region. Let $j_0 \in \{0, 1, 2, \ldots, t-1\}$ be an integer such that $V'$ lies in the region bounded by $a_{1,j_0}, a_{1,j_0+1}$, and $e$. See Figure [2]. In the case $j_0 = t-1$, the set $V'$ lies in the unbounded region.

Let $v_i \in V'$ and $a_{i,j_0+j_1}$ be an arc emanating from $v_i$ for $j_1 \geq 1$. Notice that $a_{i,j_0+j_1}$ cannot cross both $a_{1,j_0}$ and $a_{1,j_0+1}$, since $G$ is a simple topological graph. Suppose that $a_{i,j_0+j_1}$ crosses $a_{1,j_0+1}$. Then all arcs emanating from $v_i$,

$$A = \{a_{i,j_0+1}, a_{i,j_0+2}, \ldots, a_{i,j_0+j_1-1}\}$$

must also cross $a_{1,j_0+1}$. Indeed, let $\gamma$ be the simple closed curve created by the arrangement

$$a_{i,j_0+j_1} \cup a_{1,j_0+1} \cup e.$$

Since $a_{i,j_0+j_1}, a_{1,j_0+1},$ and $e$ pairwise intersect at precisely one point, $\gamma$ is well defined. We define points $x = a_{i,j_0+j_1} \cap a_{1,j_0+1}$ and $y = a_{1,j_0+1} \cap e$, and orient $\gamma$ in the direction from $x$ to $y$ along $\gamma$. 

Figure 1: In this example, $S_1 = v_1, v_3, v_4, v_3, v_2$ and $S_2 = v_2, v_2, v_1, v_5, v_5$. 

$S = s_1, s_2, ..., s_{lt}$ be our up($l,t$) sequence of length $lt$ such that the first $l$ terms are pairwise distinct and for $i = 1, 2, \ldots, l$ we have

$$s_i = s_{i+l} = s_{i+2l} = s_{i+3l} = \cdots = s_{i+(t-1)t}.$$
Figure 2: Vertices of $V'$ lie in the region enclosed by $a_{1,j_0}, a_{1,j_0+1}, e$.

In view of the fact that $a_{i,j_0+j_1}$ intersects $a_{1,j_0+1}$, the vertex $v_i$ must lie to the right of $\gamma$. Moreover, since the arc from $x$ to $y$ along $a_{1,j_0+1}$ is a subset of $\gamma$, the points corresponding to the subsequence

$$S' = \{s_q \in S \mid 2 + (j_0 + 1)l \leq q \leq (i - 1) + (j_0 + j_1)l\}$$

must lie to the left of $\gamma$. Hence, $\gamma$ separates vertex $v_i$ and the points of $S'$. Therefore, using again that $G$ is simple, each arc from $A$ must cross $a_{1,j_0+1}$ (these arcs cannot cross $a_{i,j_0+j_1}$). See Figure 4.

By the same argument, if the arc $a_{i,j_0-j_1}$ crosses $a_{1,j_0}$ for $j_1 \geq 1$, then the arcs emanating from $v_i$,

$$a_{i,j_0-1}, a_{i,j_0-2}, \ldots, a_{i,j_0-j_1+1}$$

must also cross $a_{1,j_0}$. Since $a_{i,j_0+t/2} = a_{i,j_0-t/2}$, we have the following observation.

**Observation 4.4.** For half of the vertices $v_i \in V'$, the arcs emanating from $v_i$ satisfy

1. $a_{i,j_0+1}, a_{i,j_0+2}, \ldots, a_{i,j_0+t/2}$ all cross $a_{1,j_0+1}$, or
2. $a_{i,j_0-1}, a_{i,j_0-2}, \ldots, a_{i,j_0-t/2}$ all cross $a_{1,j_0}$.

Since $t/2 = 2^{k-1}$ and

$$\frac{|V'|}{2} \geq \frac{l - 1}{2l} = \frac{2k^2 + k - 1}{2 \cdot 2^k} \geq 2^{(k-1)^2 + (k-1)},$$

by Observation 4.4 we obtain a $up(2^{(k-1)^2 + (k-1)}, 2^{k-1})$ sequence such that the corresponding arcs all cross either $a_{1,j_0}$ or $a_{1,j_0+1}$. By the induction hypothesis, it follows that there exist $k$ pairwise crossing edges.

Now we are ready to complete the proof of Lemma 4.1. By Lemma 4.2 we know that, say, $S_1$ contains an $l$-regular subsequence of length $|E'|/(4l)$. By Theorem 2.1 and Lemma 4.3 this subsequence has length at most

$$n \cdot l2^{(l-3)} \cdot (10l)^{10\alpha(n)l}.$$

Therefore, we have

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(a) The case when $j_0 + j_1 \mod t \leq t - 1$.

(b) $\gamma$ defined from Figure 3(a).

(c) The case when $j_0 + j_1 \mod t < j_0$. Recall $a_{i,j_0+j_1} = a_{i,j_0+j_1 \mod 2k}$.

d) $\gamma$ defined from Figure 3(c).

Figure 3: Defining $\gamma$ and its orientation.

$$\frac{|E'|}{4 \cdot l} \leq n \cdot 2^{(t-3)} \cdot (10l)^{10\alpha(n)t},$$

which implies

$$|E'| \leq 4 \cdot n \cdot 2^{(t-3)} \cdot (10l)^{10\alpha(n)t}.$$ 

Since $c'_k = 40 \cdot lt = 40 \cdot 2^{k^2+2k}$, $\alpha(n) \geq 2$ and $k \geq 5$, we have

$$|E| = |E'| + 1 \leq n \cdot 2^{c'_k(n)},$$

which completes the proof of Lemma 4.1.

Now we are in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $G = (V, E)$ be a $k$-quasi-planar simple topological graph on $n$ vertices. By Lemma 3.2 there is a subset $E' \subset E$ such that $|E'| \geq c|M|/\log |E|$, where $c$ is an absolute constant and $E'$ is decomposable. Hence, there is a partition...
\[ E' = E_1 \cup E_2 \cup \cdots \cup E_t \]
such that each \( E_i \) has an edge \( e_i \) that intersects every other edge in \( E_i \), and for \( i \neq j \), the edges in \( E_i \) are disjoint from the edges in \( E_j \). Let \( V_i \) denote the set of vertices that are the endpoints of the edges in \( E_i \), and let \( n_i = |V_i| \). By Lemma 4.1, we have

\[
|E_i| \leq n_i 2^{\alpha' k}(n_i) + 2n_i,
\]

where the \( 2n_i \) term accounts for the edges that share a vertex with \( e_i \). Hence,

\[
\frac{c|E|}{\log |E|} \leq \sum_{i=1}^{t} n_i 2^{\alpha' k}(n_i) + 2n_i \leq n 2^{\alpha' k}(n) + 2n,
\]

Therefore, we obtain

\[
|E| \leq (n \log n)2^{\alpha' k}(n),
\]

for a sufficiently large constant \( c_k \).

## 5 \( x \)-Monotone Topological Graphs

The aim of this section is to prove Theorem 1.2.

**Proof of Theorem 1.2** For \( k \geq 2 \), let \( g_k(n) \) be the maximum number of edges in a \( k \)-quasi-planar topological graph whose edges are drawn as \( x \)-monotone curves. We will prove by induction on \( n \) that

\[
g_k(n) \leq 2^{ck^6} n \log n
\]

where \( c \) is a sufficiently large absolute constant.

The base case is trivial. For the inductive step, let \( G = (V, E) \) be a \( k \)-quasi-planar topological graph whose edges are drawn as \( x \)-monotone curves, and let the vertices be labeled \( 1, 2, \ldots, n \). Let \( L \) be a vertical line that partitions the vertices into two parts, \( V_1 \) and \( V_2 \), such that \( |V_1| = \lfloor n/2 \rfloor \) vertices lie to the left of \( L \), and \( |V_2| = \lceil n/2 \rceil \) vertices lie to the right of \( L \). Furthermore, let \( E_1 \) denote the set of edges induced by \( V_1 \), let \( E_2 \) denote the set of edges induced by \( V_2 \), and let \( E' \) be the set of edges that intersect \( L \). Clearly, we have

\[
|E_1| \leq g_k(\lfloor n/2 \rfloor) \quad \text{and} \quad |E_2| \leq g_k(\lceil n/2 \rceil).
\]

It suffices that show that

\[
|E'| \leq 2^{ck^6/2}n,
\]

since this would imply

\[
g_k(n) \leq g_k(\lfloor n/2 \rfloor) + g_k(\lceil n/2 \rceil) + 2^{ck^6/2}n \leq 2^{ck^6} n \log n.
\]

In the rest of the proof, we only consider the edges belonging to \( E' \). For each vertex \( v_i \in V_1 \), consider the graph \( G_i \) whose vertices are the edges with \( v_i \) as a left endpoint, and two vertices
in $G_i$ are adjacent if the corresponding edges cross at some point to the left of $L$. Since $G_i$ is an incomparability graph (see [7], [11]) and does not contain a clique of size $k$, $G_i$ contains an independent set of size $|E(G_i)|/(k-1)$. We keep all edges that correspond to the elements of this independent set, and discard all other edges incident to $v_i$. After repeating this process for all vertices in $V_1$, we are left with at least $|E'|(k-1)$ edges.

Now we continue this process on the other side. For each vertex $v_j \in V_2$, consider the graph $G_j$ whose vertices are the edges with $v_j$ as a right endpoint, and two vertices in $G_j$ are adjacent if the corresponding edges cross at some point to the right of $L$. Since $G_j$ is an incomparability graph and does not contain a clique of size $k$, $G_j$ contains an independent set of size $|E(G_j)|/(k-1)$. We keep all edges that corresponds to this independent set, and discard all other edges incident to $v_j$. After repeating this process for all vertices in $V_2$, we are left with at least $|E'|(k-1)^2$ edges.

We order the remaining edges $e_1, e_2, \ldots, e_m$ in the order in which they intersect $L$ from bottom to top. (We assume without loss of generality that any two intersection points are distinct.) Define two sequences, $S_1 = p_1, p_2, \ldots, p_m$ and $S_2 = q_1, q_2, \ldots, q_m$, such that $p_i$ denotes the left endpoint of edge $e_i$ and $q_i$ denotes the right endpoint of $e_i$. We need the following lemma.

**Lemma 5.1.** Neither of the sequences $S_1$ and $S_2$ has subsequence of type up-down-up($k^3 + 2$).

**Proof.** By symmetry, it suffices to show that $S_1$ does not have a subsequence of type up-down-up($k^3 + 2$). Suppose for contradiction that $S_1$ does contain such a subsequence. Then there is a sequence

$$S = s_1, s_2, \ldots, s_{k^3+2}$$

such that the integers $s_1, \ldots, s_{k^3+2}$ are pairwise distinct and

$$s_i = s_{2(k^3+2)-i} = s_{2(k^3+2)}-2+i$$

for $i = 1, 2, \ldots, k^3 + 2$.

For each $i \in \{1, 2, \ldots, k^3 + 2\}$, let $v_i \in V_1$ denote the label (vertex) of $s_i$ and let $x_i$ denote the $x$-coordinate of the vertex $v_i$. Moreover, let $a_i$ be the arc emanating from vertex $v_i$ to the point on $L$ that corresponds to $s_{2(k^3+2)-i}$. Let $A = \{a_2, a_3, \ldots, a_{k^3+1}\}$. Note that the arcs in $A$ are enumerated downwards with respect to their intersection points with $L$, and they correspond to the elements of the “middle” section of the up-down-up sequence. We define two partial orders on $A$ as follows.

$$a_i \prec_1 a_j \text{ if } i < j, \quad x_i < x_j \quad \text{and the arcs } a_i, a_j \text{ do not intersect},$$

$$a_i \prec_2 a_j \text{ if } i < j, \quad x_i > x_j \quad \text{and the arcs } a_i, a_j \text{ do not intersect}.$$

Clearly, $\prec_1$ and $\prec_2$ are partial orders. If two arcs are not comparable by either $\prec_1$ or $\prec_2$, then they must cross. Since $G$ does not contain $k$ pairwise crossing edges, by Dilworth’s theorem, there exist $k$ arcs $\{a_1, a_2, \ldots, a_k\}$ such that they are pairwise comparable by either $\prec_1$ or $\prec_2$. Now the proof falls into two cases.

**Case 1.** Suppose that $a_1 \prec_1 a_2 \prec_1 \cdots \prec_1 a_k$. Then the arcs emanating from $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ to the points corresponding to $s_{2(k^3+2)-2+i_1}, s_{2(k^3+2)-2+i_2}, \ldots, s_{2(k^3+2)-2+i_k}$ are pairwise crossing. See Figure 4

**Case 2.** Suppose that $a_1 \prec_2 a_2 \prec_2 \cdots \prec_2 a_k$. Then the arcs emanating from $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ to the points corresponding to $s_{i_1}, s_{i_2}, \ldots, s_{i_k}$ are pairwise crossing. See Figure 5.
We are now ready to complete the proof of Theorem 1.2. By Lemma 4.2, we know that, $S_1$, say, contains a $(k^3 + 2)$-regular subsequence of length

$$|E'| \leq 4(k^3 + 2)(k - 1)^2.$$ 

By Lemmas 2.2 and 5.1 this subsequence has length at most $2c'k^6 n$, where $c'$ is an absolute constant. Hence, we have

$$\frac{|E'|}{4(k^3 + 2)(k - 1)^2} \leq 2c'k^6 n,$$

which implies that

$$|E'| \leq 4k^5 2c'k^6 n \leq 2ck^6 / 2 n$$

for a sufficiently large absolute constant $c$. \qed

Figure 4: Case 1.
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