On perpetual American put valuation and first-passage in a regime-switching model with jumps

Z. Jiang † M. R. Pistorius ‡

In memory of Yumin Jiang

Abstract
In this paper we consider the problem of pricing a perpetual American put option in an exponential regime-switching Lévy model. For the case of the (dense) class of phase-type jumps and finitely many regimes we derive an explicit expression for the value function. The solution of the corresponding first passage problem under a state-dependent level rests on a path transformation and a new matrix Wiener-Hopf factorization result for this class of processes.

Keywords American put option · matrix Wiener-Hopf factorization · phase-type · regime-switching · first-passage problem

Mathematics Subject Classification (2000) 60K15 · 90A09

JEL classification G13

1 Introduction
Consider a riskless bond and a stock whose price processes \( \{ B_t, t \geq 0 \} \) and \( \{ S_t, t \geq 0 \} \) are given by

\[
B_t = \exp \left( \int_0^t r(Z_s) ds \right), \quad S_t = \exp(X_t), \quad X_0 = x,
\]

with \( r(\cdot) \geq 0 \) the instantaneous interest rate, \( Z \) a finite state Markov process and \( X = \{ X_t, t \geq 0 \} \) a regime-switching phase-type Lévy process (that will be specified below in Section 2). When \( X_t \) is a Brownian motion with drift and

\[ * \text{Research supported by the Nuffield Foundation, grant NAL/00761/G, and EPSRC grant EP/D039053/1.} \]

\[ † \text{King’s College London, Department of Mathematics, Strand, London WC2R 2LS, UK} \]

\[ ‡ \text{Present address: School of Finance, Nanjing University of Finance and Economics, Nanjing, 210046, China} \]

\[ † \text{Email: zunjiang@yahoo.com.cn} \]

\[ ‡ \text{Email: martijn.pistorius@kcl.ac.uk} \]
is constant, the model reduces to the classical Black-Scholes model (BS). It has been well documented in the literature that the BS model is not flexible enough to accurately replicate observed market call prices simultaneously across different strikes and maturities.

To address some of the deficiencies of the BS model it was proposed to replace the geometric Brownian motion by an exponential Lévy process, modelling sudden stock price movements by jumps. A substantial literature has been devoted to the study and application of Lévy models in derivative pricing; popular models include the infinite jump activity models, such as the NIG [6], CGMY [10], KoBoL [8] and hyperbolic processes [12], and the finite activity, jump-diffusion models – see also Cont and Tankov [11] for an overview. In the latter category, for instance, Kou [21] investigated the pricing of European and barrier options in the case of double-exponential jumps; Asmussen et al. [4] considered perpetual American and Russian options under phase-type jumps.

In a parallel line of research the BS model was extended by allowing its parameters $\mu$, $\sigma$ and $r$ to be modulated by a finite state Markov chain $Z$. The process $Z$ models (perceived) changes in economic factors and their influence on the stock price. See Guo [15, 16] for background on this regime-switching model and further references. In the context of option pricing, Guo [14, 15] and Guo and Zhang [17] obtained closed form solutions of European, perpetual American put and lookback options for a two-state regime switching Brownian motion; For the case of $N$ states, Jobert and Rogers [18] considered the perpetual American put and numerically solved the finite time American put problem.

In the present study we consider the model which combines both the important features of regime-switching and jumps, motivated by the observation that Lévy models have been successfully calibrated to options with single, short time maturities whereas regime-switching models fit well longer dated options. The model allows, at least in principle, for a flexible specification of the jump-distribution, since the phase-type distributions are dense in the class of all distributions on a half-line (see [4, Prop. 1]). Under this model, we obtain explicit, analytically tractable results for the value function of a perpetual American put and corresponding optimal exercise strategy under this model. Guo & Zhang [17] and Jobert & Rogers [18] have shown that the optimal stopping time takes the form of the first-passage problem of $X_t$ under a level $k(Z_t)$ that depends on the current regime $Z_t$. We will show that the optimal stopping time still takes this form in our model and subsequently solve the corresponding first-passage problem. The solution of the latter rests on a path transformation and new matrix Wiener-Hopf factorization results, which extend the classical factorization results of London et al. [23]. The results also extend Asmussen et al. [4] who solved the first-passage problem across a constant level in the case of two regimes using methods different from ours.

To value a finite maturity American put under the model the solution of the perpetual American put problem may in principle be used as building block in an approximation procedure that we will briefly outline now. In the setting of the BS model, Carr [9] investigated the approximation of a finite maturity American put price by randomizing its maturity and showed, by nu-
numerical experiments, fast convergence of this algorithm. The proposed maturity randomization resulted in an iterative evaluation of a series of related perpetual-type American options, a procedure which was extended to the setting of jump-diffusions by Levendorskii [22]. The idea is then to combine our solution of the first-passage problem with Carr’s ideas to develop a pricing algorithm of finite maturity American put options under a regime-switching Lévy model – we leave further exploration of this idea for future research.

The rest of the paper is organized as follows. Section 2 is devoted to the problem formulation and the solution of the perpetual American put problem in terms of a first-passage problem. The solution to the first-passage problem under a state-dependent level is developed in Sections 3–6. Finally, in Section 7 the case of two regimes is considered in detail. Proofs that are not given in the text are deferred to the Appendix.

2 Problem formulation

2.1 Model

Let the bond and risky asset price processes be given as in (1) such that \( E[S_1] < \infty \), where \( Z = \{Z_t; t \geq 0\} \) is a continuous time irreducible Markov process with finite state space \( E^0 = \{1, \ldots, N\} \) and intensity-matrix \( G \), and \( X = \{X_t; t \geq 0\} \) is a regime-switching jump-diffusion given by

\[
X_t = x + \int_0^t \mu(Z_s)ds + \int_0^t \sigma(Z_s)dW_s + \sum_{i \in E^0} \int_0^t 1_{(Z_s = i)}dJ_i(s).
\]

Here \( 1_B \) is the indicator of the set \( B, x \in \mathbb{R} \), \( W = \{W_t; t \geq 0\} \) is a Wiener process, \( J_i = \{J_i(t); t \geq 0\} \) are independent compound Poisson processes with jumps arriving at rate \( \lambda_i \), and \( \mu \) and \( \sigma \) are real-valued functions on \( E^0 \) with \( \sigma(\cdot) > 0 \). The stochastic processes \( X \) and \( Z \) are defined on some filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \), where \( \mathcal{F}_t = \{\mathcal{F}_t\}_{t \geq 0} \) denotes the completed filtration generated by \( (X, Z) \). The jump sizes of the compound Poisson processes \( J_i \) are assumed to be distributed according to double phase-type distributions, the definition of which we will specify below. We first briefly review the definition of a phase-type distribution. A distribution \( F \) on \((0, \infty)\) is said to be of phase-type, if it is the distribution of the absorption time of a finite state Markov chain with one state \( \partial \) absorbing and the remaining states transient. One writes \( F \sim PH(\alpha, T) \) if this Markov chain, restricted to the transient states, has generator matrix \( T \) and initial distribution given by the (column) vector \( \alpha \). From Markov chain theory it follows that the density of \( F \) is given by

\[
f(x) = \alpha' e^{Tx}, \quad x > 0,
\]

where \( ' \) denotes transpose and \( t = (-T)1 \), with \( 1 \) a column vector of ones, is the vector of exit rates from a transient state to \( \partial \). The class of phase-type distributions is dense (in the sense of convergence in distribution) in the class

3
of all probability distributions on \((0, \infty)\). Examples of phase-type distributions include hyper-exponential and Erlang distributions. See Neuts [24] and Asmussen [1, 2, 3] for further background on phase-type distributions and their applications.

An extension to distributions supported on \(\mathbb{R}\) reads as follows:

**Definition 1** A continuous distribution \(H\) on \(\mathbb{R}\) is said to be of double phase-type with parameters \(p, \alpha, T, \beta, U\), and one writes \(H \sim \text{DPH}(p, \alpha, T, \beta, U)\), if its density \(h\) is of the form

\[
h(x) = pf_{\alpha,T}(x)1_{(x>0)} + (1-p)f_{\beta,U}(-x)1_{(x<0)},
\]

where \(p \in (0,1)\) and \(f_{\alpha,T}, f_{\beta,U}\) are PH\((\alpha, T)\) and PH\((\beta, U)\) densities respectively.

For each \(i \in E^0\) the jump size distribution \(F_i \sim \text{DPH}(p_i, \alpha_i^+, T_i^+, \alpha_i^-, T_i^-)\) of \(J_i\) is of double phase-type. Note that the positive and negative jumps of \(J_i\) arrive at rates \(\lambda_i^+ := p_i \lambda_i\) and \(\lambda_i^- := (1-p_i) \lambda_i\) and are distributed according to PH\((\alpha_i^+, T_i^+)\) and PH\((\alpha_i^-, T_i^-)\) distributions, respectively.

The market with price processes \((B, S)\) as specified above is arbitrage-free as there exists an equivalent martingale measure \(P^*\). Furthermore, there exists a \(P^*\) that is structure-preserving (i.e. \(X\) is still of the form \((2)\) but with different parameters) – a proof of this result is given in the Appendix. From now on we will assume that \(X\) admits a representation \((2)\) under a martingale measure \(P^*\), and we will write \(P\) for \(P^*\).

### 2.2 Perpetual American put

In the market \((1)\) we consider a perpetual American put with strike \(K > 0\), a contract that gives its holder the right to exercise it at any moment \(t\) and receive the payment \(K - S_t\). From standard theory of pricing American style options in [7, 19] it follows that, if \(S_0 = s = e^x\) and \(Z_0 = i\), an arbitrage-free price for this contract is given by

\[
V^*(s, i) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_{x,i}[B^{-1}_\tau(K - e^{X_\tau})^+],
\]

where \(u^+ = \max\{u, 0\}\), \(\mathcal{T}_{0,\infty}\) denotes the set of \(\mathbb{F}\)-measurable finite stopping times, and \(\mathbb{E}_{x,i}[\cdot] = \mathbb{E}[\cdot | X_0 = x, Z_0 = i]\). An optimal stopping time in \((5)\) reads as

\[
T(k^*) = \inf\{t \geq 0 : X_t \leq k^*(Z_t)\},
\]

for some function (or vector) \(k^* : E^0 \to \mathbb{R}\). This can be seen to be true as follows. In view of the fact that \((S, Z)\) is a Markov process, the general theory of optimal stopping in Shiryaev [27] implies that an optimal stopping time \(T^*\) in \((5)\) is given by

\[
T^* = \inf\{t \geq 0 : V^*(S_t, Z_t) \leq (K - S_t)^+\}.
\]
That $\tau^*$ is of the form (6) is a consequence of the fact that $V^*(s, i)$ is a positive, convex and decreasing function that dominates $(K - s)^+$ with $V^*(0, i) = K$. The latter follows in turn by the definition of $V^*$ and by observing that $s \rightarrow (K - se^{X_T - X_0})^+$ is convex and decreasing, and that subsequently taking the expectation and the supremum over the stopping times $\tau \in T_{0, \infty}$ preserves these properties.

The next result presents the solution to the valuation problem of the perpetual American put in the market (1):

**Theorem 1** The value function in (5) is given by $V^*(s, i) = V_{k^*}(s, i)$ where $V_{k}(s, i) = K v_{0,k}(x, i) - v_{1,k}(x, i), \quad i \in E^0, s = e^x, (7)$

with $v_{b,k}(x, i) = E_{x,i} \left[ B_T^{-1} e^{hX_T(k)} \right].$

An optimal stopping time in (5) is given by (6) where $k^* = (k^*_1, \ldots, k^*_N)$ satisfies $\lim_{x \downarrow k^*_j} V'_{k^*}(e^x, j) = -e^{k^*_j} \quad j \in E^0.$ (8)

### 2.3 First passage

To solve the American put problem we will consider the first-passage problem of $X_t$ under the level $k(Z_t)$, with $k : E^0 \rightarrow \mathbb{R}$, which amounts to finding the function $v_{b,k}(x, i) = E_{x,i} \left[ e^{-R_T + kX_T} h_0(Z_T) \right], \quad (9)$

where $R_T = \int_0^T a(Z_s) ds, a, h_0 : E^0 \rightarrow \mathbb{R}^+$, and $T = T(k) = \inf \{ t \geq 0 : X_t \leq k(Z_t) \}.$ (10)

Following [3, 4], the first step in the solution of (9) is to reformulate this problem as a first-hitting time problem for a related continuous Markov additive process $A$, called the fluid embedding of $X$. Informally, a path of $A$ is constructed from a path of $X$ by replacing the jumps of $X$ by linear stretches – see Figure 1. An explicit construction is given in Section 3.

A classical approach [20, 23] to solving the resulting first-hitting time problem rests on a characterization of the laws of corresponding up- and down-crossing ladder processes – see Figure 2. London et al. [23] developed matrix Wiener-Hopf factorization results for fluctuating additive processes (see also Rogers [26] for elegant martingale proofs and Brownian perturbations). By extending the results of [23, 26] to our setting, we solve the matrix Wiener-Hopf factorization problem for the embedding $A$, in Section 4.

To deal with the different ways in which first-passage in (9) can occur (see Figure 3), the Wiener-Hopf factorization is employed in Section 5 to calculate the distribution of the process $A$ at the first moment of leaving a finite interval or a regime-switch, whichever occurs first. In Section 6 the solution to (9) is derived by combining the foregoing results.
Figure 1: Shown is a sample path of $X$ until the first time $T^-$ that $X$ enters $(-\infty, k)$. The process $A$ has no positive jumps and always hits a level at first-passage.

3 Fluid Embedding

Let $Y$ be an irreducible continuous time Markov chain with finite state space $E \cup \partial$, where $\partial$ is an absorbing cemetery state, and denote by $A = \{A_t, t \geq 0\}$ the stochastic process given by

$$A_t = A_0 + \int_0^t s(Y_s) dW_s + \int_0^t m(Y_s) ds,$$

where $s$ and $m$ are functions from $E \cup \partial$ to $\mathbb{R}$ with $s(\partial) = m(\partial) = 0$. The process $A$ is the fluid-embedding of $X$ if the generator of $Y$ restricted to $E$ is equal to $Q_0$ where, in block notation,

$$Q_a = \begin{pmatrix} T^+ & t^+ & O \\ B^+ & G - D_a & B^- \\ O & t^- & T^- \end{pmatrix}. \tag{12}$$

Here $D_a$ is an $N \times N$ diagonal matrix with $(D_a)_{ii} = \lambda_i + a_i$, $O$ are zero matrices of appropriate sizes and, again in block notation,

$$B^\pm = \begin{pmatrix} \lambda_1^{\pm} \alpha_1^{\pm} \\ \vdots \\ \lambda_N^{\pm} \alpha_N^{\pm} \end{pmatrix}, \quad T^\pm = \begin{pmatrix} T_1^\pm \\ \vdots \\ T_N^\pm \end{pmatrix}, \quad t^\pm = \begin{pmatrix} t_1^\pm \\ \vdots \\ t_N^\pm \end{pmatrix}. \tag{13}$$
Figure 2: Pictured is a stylized sample path of the process $A$. The dashed vertical lines denote the jump times of $Y$. The horizontal dotted lines indicate the jump times of the associated Markov process $\tilde{Y}^+$. 

From the form of $Q_a$ it follows that $E$ can be partitioned as $E = E^+ \cup E^0 \cup E^-$ where $E^0$ is the state-space inherited from $Z$ and $E^+$ and $E^-$ are the states in which the path of $A$ is linear with slope +1 or −1 which originate from the positive and negative jumps of $X$, respectively. Similarly, we will write $E_i = E_i^+ \cup \{i\} \cup E_i^-$ for the subset of $E$ corresponding to the $i$th regime of $X$ with corresponding embedded positive and negative jumps. The functions $m(\cdot)$ and $s(\cdot)$ are then specified as follows:

$$s(j) = \begin{cases} 
\sigma_j & \text{if } j \in E^0 \\
0 & \text{otherwise}
\end{cases} \quad m(j) = \begin{cases} 
1 & \text{if } j \in E^+ \\
\mu_j & \text{if } j \in E^0 \\
-1 & \text{if } j \in E^-
\end{cases}$$

3.1 Path transformation

In Figure it is illustrated how a path of $A$ can be transformed to obtain a path of $X$. More formally, denoting by

$$T_0(t) = \int_0^t 1_{\{Y_s \in E^0\}} ds \quad \text{and} \quad T_0^{-1}(u) = \inf\{t \geq 0 : T_0(t) > u\}.$$
the time before $t$ spent by $Y$ in $E^0$ and its right-continuous inverse, respectively, it is not hard to verify that

$$ (A \circ T_0^{-1}, Y \circ T_0^{-1}) \text{ is in law equal to } (X, Z). $$

This implies in particular that the triplets $(T_0(T), A_T, Y_T)$ and $(T, X_T, Z_T)$ have the same distribution, where

$$ \tilde{T} = \tilde{T}(k) = \inf \left\{ t \geq 0 : Y_t \in E^0 \text{ and } A_t \leq \tilde{k}(Y_t) \right\} $$

with $\tilde{k} : E \to \mathbb{R}$ given by $\tilde{k}(j) = k(i)$ for $j \in E_i$. It is not hard to verify that state-dependent discounting (or ‘killing’) at rate $a(i)$ when $Y_t = i \in E^0$ can be included by replacing $Q_0$ by the generator $Q_a$ for the vector $a = (a(i), i \in E^0)$. Thus, it holds that

$$ v_{b,k}(x, i) = \mathbb{E}_{x,i} \left[ e^{bA_{\tilde{T}} h(Y_{\tilde{T}})} 1_{(\tilde{T} < \zeta)} \right], \quad (13) $$

with $Y$ now evolving according to the generator $Q_a$ and $E_i, x[\cdot] = \mathbb{E}[\cdot | Y_0 = i, A_0 = x]$. Here,

$$ \zeta = \inf \{ t \geq 0 : Y_t \notin E \} \quad (14) $$

with $\inf \emptyset = \infty$, and $h : E \to \mathbb{R}$ is given by $h(j) = h_0(i)$ for $j \in E_i$.

4 Matrix Wiener-Hopf factorization

The solution of the first-passage problem of the Markov process $(A, Y)$ across a constant level is closely linked to the up-crossing and down-crossing ladder processes $\tilde{Y}^+, \tilde{Y}^-$ of $(A, Y)$. These processes are defined as time changes of $Y$ that are constructed such that $Y$ is observed only when $A$ is at its maximum and at its minimum respectively, that is,

$$ \tilde{Y}^+_t = Y (\tau^+_t) \quad \text{and} \quad \tilde{Y}^-_t = Y (\tau^-_t), \quad (15) $$

where

$$ \tau^+_t = \inf \{ s \geq 0 : A_s > t \} \quad \text{and} \quad \tau^-_t = \inf \{ s \geq 0 : A_s < t \}. $$

It is easily verified that the ladder processes $\tilde{Y}^+$ and $\tilde{Y}^-$ are again Markov processes with state spaces $E^0 \cup E^+$ and $E^0 \cup E^-$, respectively. We will characterize the generators $Q^+_a$ and $Q^-_a$ of $\tilde{Y}^+$ and $\tilde{Y}^-$ along with the initial distributions $\eta^+$, defined by

$$ \eta^+(i, j) = \mathbb{P}_{0,i} \left[ \tilde{Y}^+_0 = j, \tau^+_0 < \zeta \right] \quad \text{for } i \in E^-, j \in E^+ \cup E^0 \quad (16) $$

$$ \eta^-(i, j) = \mathbb{P}_{0,i} \left[ \tilde{Y}^-_0 = j, \tau^-_0 < \zeta \right] \quad \text{for } i \in E^+, j \in E^- \cup E^0 \quad (17) $$

Denote by $\mathcal{Q}(n)$ the set of irreducible $n \times n$ generator matrices (matrices with non-negative off-diagonal elements and non-positive row sums) and write $\mathcal{P}(n, m)$
for the set of \( n \times m \) matrices whose rows are sub-probability vectors. Let \( V \) and \( \Sigma \) denote the \( |E| \times |E| \) diagonal matrices given by \( \text{diag}(m(i)) \) and \( \text{diag}(s(i)) \), respectively. The matrix \( Q_a \) is called recurrent if its rows sum up to zero; otherwise it is called transient. By considering the process \( A \) at the subsequent times it visits a certain state, \( r \in E \) say, and noting that this defines a random walk, we have that in the recurrent case either \( \sup_{t \geq 0} A_t = \infty \) or \( \lim_{t \rightarrow -\infty} A_t = -\infty \), \( P_0, i \)-a.s.

Write \( N, N^+ \) and \( N^- \) for the number of elements of \( E^0, E^+ \) and \( E^- \), respectively and let \( N_0^+ = N + N^+ \) and \( N_0^- = N + N^- \). We then define the Wiener-Hopf factorization of \((A, Y)\) as follows:

**Definition 2** Let \( G^+, C^+, G^- \) and \( C^- \) be elements of the sets \( Q(N_0^+) \), \( P(N^-, N_0^-) \), \( Q(N_0^-) \) and \( P(N^+, N_0^-) \), respectively. A quadruple \((C^+, G^+, C^-, G^-)\) is called a Wiener-Hopf factorization of \((A, Y)\) if

\[
\Xi (−G^+, W^+) = O \quad \text{and} \quad \Xi (G^-, W^-) = O, \tag{18}
\]

where, for matrices \( W \) with \( |E| \) rows,

\[
\Xi (S, W) = \frac{1}{2} \Sigma W S + V W S + Q_a W \tag{19}
\]

and \( W^+ \) and \( W^- \) are given in obvious block notation by

\[
W^+ = \begin{pmatrix} I^+ & O \\ O & I_0 \end{pmatrix} \quad \text{and} \quad W^- = \begin{pmatrix} C^- & 0 \\ 0 & C_0 \end{pmatrix}, \tag{20}
\]

where \( I_0, I^+ \) and \( I^- \) are identity matrices of sizes \( N \times N \), \( N^+ \times N^+ \) and \( N^- \times N^- \), respectively, and \( O \) denotes a zero matrix of the appropriate size.

In the following result the Wiener-Hopf factorization of \((A, Y)\) is identified:

**Theorem 2** (i) The quadruple \((\eta^+, Q^+, \eta^-, Q^-)\) is a Wiener-Hopf factorization of \((A, Y)\).

(ii) The Wiener-Hopf factorization \((\eta^+, Q^+, \eta^-, Q^-)\) is unique if \( Q \) is transient or if \( Q \) is recurrent and \( A \) oscillates (that is, \( \sup_t A_t = -\inf_t A_t = \infty \)).

(iii) If \( Q \) is recurrent and \( \lim_{t \rightarrow -\infty} A_t = -\infty \), there are precisely two Wiener-Hopf factorizations of \((A, Y)\) given by \((\eta^+, Q^+, \eta^-, Q^-)\) and

\[
(\eta^+(I - 1\mu) + \mu, Q^+(I - 1\mu), \eta^-, Q^-),
\]

where \( \mu \) is the left eigenvector of \( Q^+ \) corresponding to its largest eigenvalue, normalized such that \( \mu 1 = 1 \), where \( 1 \) denotes a column vector of ones.

**Proof:** For \( \ell \in \mathbb{R} \) let \( \Phi^+_{\ell} \) be given by the matrices

\[
\Phi^+_{\ell}(x) = W^+ \exp \left( Q^+(\ell - x) \right) \quad \text{and} \quad \Phi^-_{\ell}(x) = W^- \exp \left( Q^-(x - \ell) \right), \tag{21}
\]
where $W^+$ and $W^-$ are given by (20) with $C^+ = \eta^+$ and with $C^- = \eta^-$. The proof rests on the martingale property of

$$M_t^+ = f_+ \left( Y_{t \wedge \tau^+_\ell}, A_{t \wedge \tau^+_\ell} \right) \quad \text{and} \quad M_t^- = f_- \left( Y_{t \wedge \tau^-_\ell}, A_{t \wedge \tau^-_\ell} \right) \quad (22)$$

with

$$f_+(i, x) = e'_i \Phi^+_\ell(x) h_+ \quad \text{and} \quad f_-(i, x) = e'_i \Phi^-_\ell(x) h_-,$$

where $h_+$ and $h_-$ are $N_0^+ -$ and $N_0^- -$ column vectors, respectively. Here, and in the sequel, $e_i$ denotes a (column) vector of appropriate size with element $e_i(j) = 1$ if $j = i$ and zero otherwise. To verify that $M^+$ is a martingale, observe first that, since $Y^+$ is a Markov process with generator $Q^+ = Q_\ell^+$ and initial distribution $\eta^+$ defined by (16), Markov chain theory implies that for $x \leq \ell$

$$\mathbb{E}_{x, i} \left[ h \left( Y^+_\ell \right) 1_{(\tau^+_\ell < \zeta)} \right] = e_i' \Phi^+_\ell(x) h.$$

The martingale property of $M^+$ then follows from (23) – (24) as a consequence of the Markov property of $(A, Y)$. An application of Itô’s lemma shows that $f = (f_+(i, u), i \in E)$ satisfies for $u \leq \ell$

$$\frac{1}{2} e(i)^2 f''(i, u) + m(i) f'(i, u) + \sum_j q_{ij} (f(j, u) - f(i, u)) = 0, \quad (25)$$

where $f'$ and $f''$ denote the first and second derivatives of $f$ with respect to $u$. By substituting the expressions (21) – (24) into equation (25) we find, since $h$ was arbitrary, that $Q^+$ and $\eta^+$ satisfy the first set of equations of the system (18). The proof for $Q^-$ and $\eta^-$ is analogous and omitted. The proofs of Theorem 2 (ii), (iii) are deferred to the Appendix.

Example (Gerber-Shiu penalty function) Let $X_\ell$ in (2) with $x > 0$ be the surplus of an insurance company. Note that, for $\sigma = 0$, $N = 1$, and in the absence of positive jumps, (2) reduces to the classical Cramér-Lundberg model. Denote by $\rho = \inf \{ t \geq 0 : X_t < 0 \}$ the ruin time of $X$. Quantities of interest in this setting include the ruin probability $P_{x,i}(\rho < \infty)$ and the Gerber-Shiu [13] expected discounted penalty function which quantifies the severity of ruin by measuring the shortfall $X_\rho$ of $X$ at the ruin time $\rho$. Under the model (2), both these quantities can be expressed in terms of the functions $\Phi_\ell^\pm$ defined in (26).

For instance, the probability of ruin in regime $\ell \in E_0$ is given by

$$P_{x,i}(\rho < \infty, Z_\rho = \ell) = e'_i \Phi^-_0(x) f_{i(\ell)},$$

where $f_{i(\ell)}(j) = 1$ if $j \in E_\ell$ and zero else and $\Phi^-_0$ is given by (21) with $Q^- = Q_0^-$. More generally, for any non-negative function $\pi$ on $[0, \infty) \times E_0^0$ the Gerber-Shiu expected discounted penalty function reads as

$$\mathbb{E}_{x, i} \left[ e^{-R_\rho} \pi(X_\rho, Z_\rho) \right] = e'_i \Phi^-_0(x) \bar{g},$$

where $R_\rho = \inf \{ t \geq 0 : X_t = 0 \}$ is the ruin time and $\bar{g}$ is a vector defined in (21).
where $R_p = \int_0^p a(Z_s)ds$, $\Phi_0^+$ is given by (21) with $Q^- = Q_a^-$, and $\bar{g}$ is the $N^-$-vector with elements

$$\bar{g}(\ell) = \begin{cases} \pi(0, \ell) & \text{for } \ell \in E^0 \\ \int_0^{\infty} \pi(-s, m)e^{sT_m}t_mds & \text{for } \ell \in E_m^- . \end{cases}$$

5 First exit from a finite interval

The two-sided exit problem of $A$ from the interval $[k, \ell]$ for $-\infty < k < \ell < +\infty$ is to find the distribution of the position of $(A_t, Y_t)$ at the first-exit time

$$\tau = \tau_{k, \ell} = \inf\{t \geq 0 : A_t \notin [k, \ell]\} .$$

By considering appropriate linear combinations of the martingales $M^+$ and $M^-$ defined in (22) we will now show that the two-sided exit problem can be solved explicitly in terms of $(\eta^+, Q^+, \eta^-, Q^-)$. To this end, introduce

$$Z^+ = \begin{pmatrix} O \\ \eta^+ \end{pmatrix} e^{Q^+(\ell-k)}, \quad Z^- = \begin{pmatrix} \eta^- \\ I_0 \end{pmatrix} e^{Q^-(\ell-k)},$$

and define the matrices

$$\Psi^+(x) = \begin{pmatrix} W^+e^{Q^+(\ell-k)} - W^-e^{Q^-(x-k)}Z^+ \\ (I - Z^-Z^+)^{-1} \end{pmatrix}, \quad (26)$$

$$\Psi^-(x) = \begin{pmatrix} W^-e^{Q^-(x-k)} - W^+e^{Q^+(\ell-x)}Z^- \\ (I - Z^+Z^-)^{-1} \end{pmatrix}, \quad (27)$$

$$\Psi^0(s, x) = \begin{pmatrix} e^{sxI} - e^{s\Psi^+(x)J^+} - e^{sk\Psi^-(x)J^-} & [-K(s)]^{-1} \end{pmatrix}, \quad (28)$$

where $I$ is an identity matrix, $W^\pm$ are given in (20) with $C^+ = \eta^+$ and $C^- = \eta^-$, $J^\pm$ is the transpose of (20) with $C^\pm$ replaced by zero matrices, and

$$K(s) = \frac{1}{2}(\Sigma^2 s^2 + Vs + Q_a). \quad (29)$$

We will write $\Psi^+_k, \Psi^-_k, \Psi^0_k$ if we wish to clarify their dependence on $k$ and $\ell$.

The complete solution of the two-sided exit problem reads as follows:

**Proposition 1** Let $h^+, h^-$ and $h^1$ be functions that map $E^0 \cup E^+, E^0 \cup E^-$ and $E$ to $\mathbb{R}$. If

$$s(i)^2 b^2 + m(i)b < -q_i$$

for all $i \in E$, \hspace{1cm} (30)

then it holds for $x \in [k, \ell]$ and $i \in E$ that

$$\mathbb{E}_{x,i} \left[ h^+ (Y_r) 1_{(A_r = \ell, \tau < \zeta)} \right] = e^i_\psi \Psi^+(x)h^+, \quad (31)$$

$$\mathbb{E}_{x,i} \left[ h^- (Y_r) 1_{(A_r = k, \tau < \zeta)} \right] = e^i_\psi \Psi^-(x)h^-, \quad (32)$$

$$\mathbb{E}_{x,i} \left[ e^{bA^-_r} h^1 (Y_{\tau^-}(Y_r) 1_{(\tau^- < \tau)} \right] = e^i_\psi \Psi^0(b, x)\Delta h^1 q_a, \quad (33)$$

where $q_a = (-Q_a)1$, $\zeta$ is defined by (14) and $\Delta h^1$ is the diagonal matrix with elements $h^1(j)$.
Proof: Define \( g_+ \) and \( g_- \) by the right-hand sides of (31) and (32) respectively. It is straightforward to verify from (26) – (27) that it holds that
\[
\begin{align*}
g_+ (i, x) &= \begin{cases} 
h^+(i) & \text{if } x = \ell, i \in E^+ \cup E^0 \\
0 & \text{if } x = k, i \in E^- \cup E^0 
\end{cases} \\
g_- (i, x) &= \begin{cases} 
0 & \text{if } x = \ell, i \in E^+ \cup E^0 \\
h^-(i) & \text{if } x = k, i \in E^- \cup E^0 
\end{cases}
\end{align*}
\]

In view of these boundary conditions and the fact that any linear combination of \( M^+ \) and \( M^- \), defined in (22), is a bounded martingale, Doob’s optional stopping theorem gives that
\[
g_+ (i, x) = \mathbb{E}_{x,i} [g_+ (Y_{\tau}, A_{\tau}) 1_{(\tau < \zeta)}] = \mathbb{E}_{x,i} [h^+(Y_{\tau}) 1_{(A_{\tau} = \ell, \tau < \zeta)}],
\]
where \( \tau = \tau_{k, \ell} \). Similarly, it follows that
\[
g_- (i, x) = \mathbb{E}_{x,i} [h^-(Y_{\tau}) 1_{(A_{\tau} = k, \tau < \zeta)}].
\]

To prove the third identity, consider the map \( h^* : E \rightarrow \mathbb{R} \) given by
\[
h^*(i) = \mathbb{E}_{0,i} \left[ e^{sA_\zeta} - h^+(Y_{\zeta}^-) \right].
\]

By conditioning on the first jump epoch \( \xi \) of \( Y \) it is straightforward to verify that
\[
h^*(i) = -\frac{h^+(i) q_{i \partial} + \sum_{j \neq i} q_{ij} h^*(j)}{s(i)^2 s^2 / 2 + m(i)s + q_{ii}},
\]
where \( q_{ij} [q_{i \partial}] \) denotes the intensity of a transition \( i \rightarrow j \) \([i \rightarrow \partial]\). After reordering and writing the above expression in matrix form, it follows that
\[
K(s) h^* = \Delta_{h^*} Q_a 1,
\]
so that, for \( s \) satisfying \( s(i)^2 s^2 + m(i)s < -q_{ii}, i \in E \),
\[
h^* = [-K(s)]^{-1} \Delta_{h^*} (-Q_a) 1. \tag{34}
\]

In view of the strong Markov property and (31) – (32), it follows that
\[
\begin{align*}
\mathbb{E}_{x,i} [e^{sA_\zeta} 1_{(\zeta < \tau)}] &= \mathbb{E}_{x,i} [e^{sA_\zeta^-}] - \mathbb{E}_{x,i} [e^{sA_\zeta^-} 1_{(\zeta > \tau)}] \\
&= e^{s h^* (i)} - \mathbb{E}_{x,i} [e^{sA_\zeta^-} 1_{(\tau < \zeta)} h^*(Y_{\tau})] \\
&= e^s [e^{s h^*} - e^{s \Psi^- (x) J^- h^*} - e^{s \Psi^+ (x) J^+ h^*}], \tag{35}
\end{align*}
\]
Inserting (34) into (35) finishes the proof of (33).
\( \square \)
Figure 3: First-passage under the state-dependent level \((k_2, k_1)\) takes place while \(Y \in E^0\) and can take place in two ways: \(A\) hits the level \(k_i\) while \(Y = i, i = 1, 2\) (illustrated in (a) and (b)) or by a jump of \(Y\) (illustrated in (a), (b) and (c)). The different line styles of the paths of \(A\) correspond to the different states of \(Y\).

6 First-passage under state-dependent levels

By combining the ingredients from the previous sections the first-passage function \(v_{k,b}(x,i)\) of \(A_t\) under the level \(k(Y_t)\)

\[
v_{b,k}(x,i) = E_{x,i} \left[ e^{bA_{\tilde{T}(k)}} h \left( Y_{\tilde{T}(k)} \right) 1(\tilde{T}(k) < \zeta) \right],
\]

with \(h : E \to \mathbb{R}_+\) and

\[
\tilde{T}(k) = \inf \left\{ t \geq 0 : Y_t \in E^0 \text{ and } A_t \leq \tilde{k}(Y_t) \right\}
\]
can be explicitly expressed in terms of the matrix Wiener-Hopf factorization found in Theorem 2. For simplicity we will assume that the levels are ordered as \(k_1 > k_2 > \ldots > k_N\) (the general case of possibly equal levels follows by a similar reasoning). As the first-passage over \(k\) can only occur when \(Y\) is in \(E^0\), it follows that \(Y\) can cross the boundary \(k\) before \(A\) exits the interval \([k_j, k_{j-1}]\) in two ways: either \(Y\) jumps into a state \(\{1, \ldots, j - 1\}\) or \(A\) hits the level \(k_j\) while \(Y\) is in state \(j\) – see Figure 3. We are thus led to considering the processes

\[
Y^{(j)} = Y\big|_{\tilde{E}_j}, \quad \text{where } \tilde{E}_j := E \setminus \cup_{i=1}^{j-1} E_i
\]

(with \(Y^{(1)} = Y\)). Clearly, the \(Y^{(j)}\) are themselves Markov processes with generators \(Q_{(j)}\) given by the corresponding restrictions of \(Q_a\); in block notation the resulting partitions read as

\[
Q_a = \begin{pmatrix} R^{(j)} & Q^{(j)} \\ q^{(j)} & Q_{(j)} \end{pmatrix}, \quad j = 2, \ldots, N,
\]
for some matrix $R^{(j)}$, where $q^{(j)}$ is the matrix of exit rates from the sub-space $E_j$. By the strong Markov property, Proposition 1 and Theorem 2, the value of $v_{b,k}(x, i)$ can be expressed in terms of the unknowns $v_{b,k}(k_j, i)$. For these unknowns a system of equations can be derived by invoking smoothness and continuity properties of $v_{b,k}$ above the barrier $k$.

More specifically, denote, for some constants $C_j(\ell), D_j(\ell)$, the vectors $h_j^-$ and $h_j^+ : \bar{E}_j \to \mathbb{R}$ by

$$h_j^+(\ell) = \begin{cases} C_{j-1}(\ell) & \text{if } \ell \in E^0 \setminus \{1, \ldots, j - 1\}, \\ D_{j-1}(\ell) & \text{else} \end{cases}, \quad (j = 2, \ldots, N), \quad (37)$$

$$h_j^-(\ell) = \begin{cases} e^{bk_j} h(j) & \text{if } \ell = j \\ C_j(\ell) & \text{if } \ell \in E^0 \setminus \{1, \ldots, j\}, \\ D_j(\ell) & \text{else} \end{cases}, \quad (j = 1, \ldots, N), \quad (38)$$

respectively, and set

$$h_j^\dagger = (h(\ell), \ell \in \bar{E}_j).$$

We shall write $\Psi_j^+, \Psi_j^-$ and $\Psi_j^\circ$ as shorthand for $\Psi_{(k_j, k_{j-1})}^+, \Psi_{(k_j, k_{j-1})}^-$ and $\Psi_{(k_j, k_{j-1})}^\circ$, respectively, and denote by $f(x^+)$ and $f(x^-)$ the right- and left-limits of the function $f$ at $x$. Then the following characterization of $v$ holds true:

**Theorem 3** Assume that $s(i)^2 b^2 / 2 + m(i)b < -q_{ii}$ for all $i \in E$. The function $v_{b,k}$ is given by

$$v_{b,k}(x, i) = \begin{cases} e_i^j \Phi^-(x) h_j^- & \text{if } x > k_j, \\ e_i^j \left[ \Psi_j^+(x) h_j^+ + \Psi_j^-(x) h_j^- + \Psi_j^\circ(b, x) \Delta h_j^\dagger q^{(j)} \right] & \text{if } j = 2, \ldots, N, \quad k_{j-1} < x \leq k_j. \end{cases} \quad (39)$$

where $\Phi^-, \Phi_{k_1}^-$ is given by (2.4) and $h_j^+, h_j^-$ are specified by (2.1) – (2.3) with $C_j(\ell)$ and $D_j(\ell)$ satisfying the following system of equations:

$$v_{b,k}'(k_j, \ell) = v_{b,k}'(k_{j-1}, \ell), \quad \ell \in E^0 \setminus \{1, \ldots, j\}, \quad (40)$$

$$v_{b,k}(k_j, \ell) = v_{b,k}(k_{j-1}, \ell), \quad \ell \in \bar{E}_j \setminus [E^0 \cup E^-], \quad (41)$$

$$(D_j(\ell), \ell \in E^-) = e^{bk_j} (bI - T_j^-)^{-1} t_j^- h(j), \quad (42)$$

where $j = 1, \ldots, N - 1$, and $'$ denotes differentiation with respect to $x$.

**7 Example: two regimes**

To illustrate the results derived in previous sections we consider next the model (2) in the case of two regimes. Suppose that $Z$ is a Markov chain with state
space $E^0 = \{1, 2\}$ and transition matrix

$$G = \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix}$$

and that $X$ evolves as a Brownian motion with drift $\mu_1 t + \sigma_1 W_t$ when $Z$ is in state 1 and as the jump-diffusion $\mu_2 t + \sigma_2 W_t - J_t$ when $Z$ is in state 2, with $J$ a compound Poisson process with intensity rate $\lambda$ and exponential jumps with mean $1/\alpha$. Then the embedding of $(X, Z)$ has state space $E = \{1, 2, 2^*\}$, say, with corresponding transition matrix

$$Q_r = \begin{pmatrix} -q_1 - r_1 & q_1 & 0 \\ q_2 & -q_2 - r_2 - \lambda & \lambda \\ 0 & \alpha & -\alpha \end{pmatrix}. \quad (43)$$

We will consider the stopping time $T(k_1^*, k_2^*)$ for the three different configurations of the optimal levels: $k_1^* < k_2^*$, $k_1^* = k_2^*$ and $k_1^* > k_2^*$. For $x > \max\{k_1^*, k_2^*\}$, the value function of the put is determined by the generator matrix $Q_r^-$ of the corresponding down-crossing ladder process, which we determine by invoking the matrix Wiener-Hopf factorization results from Section 4. Noting that $E^+ = \emptyset$ and $E^- = \{2^*\}$, it follows that $Q^- = Q_r^-$ satisfies

$$\frac{1}{2} \Sigma^2(Q^-)^2 + VQ^- + Q_r = O, \quad (44)$$

where

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Denoting by $\beta[\theta]$ a right-eigenvector of $Q_r^-$ corresponding to eigenvalue $\theta$ it follows by right-multiplying (44) with $\beta[\theta]$ that the matrix

$$K(\theta) = \frac{1}{2} \Sigma^2 \theta^2 + V \theta + Q_r$$

is singular and $K(\theta) \beta[\theta] = 0$. It is a matter of algebra to verify that $\theta$ satisfies

$$g(\theta) = 0$$

where

$$g(\theta) = F_1(\theta) ((\alpha + \theta) F_2(\theta) - \lambda \theta) - q_1 q_2 (\alpha + \theta), \quad (45)$$

with

$$F_j(\theta) = \frac{1}{2} \sigma_j^2 \theta^2 + \mu_j \theta - q_j - r_j, \quad j = 1, 2. \quad (46)$$

The following result lists the properties of the roots of $g(\theta) = 0$:

**Lemma 1** Suppose that $r_1, r_2 > 0$. Then $g(\theta)$ has five different real roots which satisfy the distribution characteristics $\theta_1 < \theta_2 < \theta_3 < 0 < \theta_4 < \theta_5$. As a consequence, $Q^- = Q_r^-$ has three distinct eigenvalues $\theta_1, \theta_2$ and $\theta_3$. 15
Since the eigenvectors $\beta[\theta_i]$ corresponding to the different eigenvalues $\theta_i$, $i = 1, 2, 3$, are linearly independent, the matrix $Q^-$ explicitly reads as

$$
Q^- = (\theta_1\beta[\theta_1] \quad \theta_2\beta[\theta_2] \quad \theta_3\beta[\theta_3]) (\beta[\theta_1] \quad \beta[\theta_2] \quad \beta[\theta_3])^{-1}.
$$

(47)

1. **Case** $k = k_1^* = k_2^*$. For $x > k$, the value function of the American put reads as

$$W(e^x, i) = e^i e^{Q^- (x-k)} H(k) \quad \text{with} \quad H(k) = \begin{pmatrix} K - e^k \\ K - e^k \\ K - e^k \frac{\alpha}{\alpha+1} \end{pmatrix},$$

where $Q^-$ is given in (47) and $k$ solves

$$-e^k = e^i Q^- H(k) \quad i = 1, 2.$$

2. **Case** $k_2^* > k_1^*$. To deal with the case that $k_1^* < x < k_2^*$ and $Z = 1$, we note that, if the process $Z$ is restricted to state 1, $X$ is equal to a Brownian motion with drift, $\mu_1 t + \sigma_1 W_t$, killed at rate $q_1 + r_1$. The generator matrices of this restriction of $Z$ and the corresponding ladder processes, denoted by $\bar{Q}, -\bar{Q}^+, \bar{Q}^-$, reduce in this case to scalars, given by

$$\bar{Q} = -q_1 - r_1$$

and the positive and negative root of the equation

$$\frac{1}{2} \sigma_1^2 x^2 + \mu_1 x - q_1 - r_1 = 0.$$

The associated two-sided exit probabilities from the interval $[k_1^*, k_2^*]$ are

$$\Psi_1^-(x) = \frac{e^{\bar{Q}^- (x-k)} - e^{\bar{Q}^+ (\ell-x)} e^{\bar{Q}^- (\ell-k)}}{1 - e^{\bar{Q}^+ (\ell-k)} e^{\bar{Q}^- (\ell-k)}},$$

$$\Psi_1^+(x) = \frac{e^{\bar{Q}^+ (\ell-x)} - e^{\bar{Q}^- (x-k)} e^{\bar{Q}^+ (\ell-k)}}{1 - e^{\bar{Q}^- (\ell-k)} e^{\bar{Q}^+ (\ell-k)}},$$

with $k = k_1^*$ and $\ell = k_2^*$. Putting everything together shows that the value function of the American put in this case is given by

$$W(e^x, i) = \begin{cases} 
 e^i e^{Q^- (x-k_2^*)} \bar{H}(k_2^*), & x \geq k_2^*, \ i = 1, 2, \\
 H^+(x) + \Psi_1^+(x) [C - H^+(k_2^*)] + \Psi_1^-(x) H^-(k_i^*), & k_1^* < x \leq k_2^*, \ i = 1,
\end{cases}$$

where

$$\bar{H}(k) = \begin{pmatrix} C \\ K - e^k \\ K - e^k \frac{\alpha}{\alpha+1} \end{pmatrix}, \quad H^-(k) = e^{kT_1} \frac{q_1}{q_1}, \quad H^+(x) = K - e^x \frac{q_1 + r_1}{q_1}.$$
using equation A.3. Here C is determined by
\[ W'(e^{k_2}, -1) = e'_1 Q^- \tilde{H}(k_2) \]
and the levels \( k_1^* \) and \( k_2^* \) satisfy the smooth fit equations
\[
\begin{align*}
\Psi_1^-(k_1) H^-(k_1) + \Psi_1^+(k_1) [C - H^+(k_2)] &= \frac{r_1}{q_1} e^{k_1}, \\
e'_2 Q^- \tilde{H}(k_2) &= -e^{k_2},
\end{align*}
\]
where prime in the first equation denotes differentiation with respect to \( x \).

3. Case \( k_1^* > k_2^* \) For \( k_2^* < x < k_1^* \) and \( Z = 2 \), we are led to consider the
Markov process \( Y^{(2)} \) with state space \( \{2, 2^*\} \) and generator matrix
\[
Q^{(2)} = \begin{pmatrix}
-q_2 - r_2 - \lambda & \lambda \\
\alpha & -\alpha
\end{pmatrix}.
\]
In this case it can be checked from (18), (19), and (20) with \( Q_a \) replaced by \( Q^{(2)} \), that
\[
-\tilde{Q}^+(2) \quad \text{is a scalar given by the positive root of}
\frac{\sigma^2}{2} x^2 + \mu_2 x + \frac{\lambda}{x + \alpha} = q_2 + r_2 + \lambda
\]
and that
\[
\eta^+ = \alpha / [-Q^+ + \alpha].
\]
We can calculate \( \tilde{Q}^- \) in a similar way as we calculated \( Q^- \) above. Writing \( \tilde{Q}^+ = Q^{2+} \) and \( \tilde{Q}^- = Q^{2-} \) the corresponding matrices of two-sided exit probabilities from \([k_2^*, k_1^*]\) read as
\[
\begin{align*}
\Psi_2^+(x) &= \frac{1}{c} \left[ \left( \begin{array}{c}
1 \\
\eta^+
\end{array} \right) e^T(x-x) - e^T(x-k) \left( \begin{array}{c}
1 \\
\eta^+
\end{array} \right) e^T(x-k) \right], \\
\Psi_2^-(x) &= \left[ e^T(x-k) - e^T(x-k) M e^T(x-k) \right] \left[ I - e^T(x-k) M e^T(x-k) \right]^{-1},
\end{align*}
\]
with \( k = k_2^* \) and \( \ell = k_1^* \), where
\[
c = 1 - (1 \ 0) e^T(x-k) \left( \begin{array}{c}
1 \\
\eta^+
\end{array} \right) e^T(x-k), \quad M = \begin{pmatrix} 1 & 0 \\ \eta^+ & 0 \end{pmatrix}.
\]
The value function is
\[
W(e^x, i) = \begin{cases} 
 e^T e^T(x-k) \tilde{H}(k_1), & x \geq k_1^*, \ i = 1, 2, \\
\tilde{H}^+(x) + f \left[ \Psi_2^+(x) [C - \tilde{H}^+(k_1)] + \Psi_2^-(x) \tilde{H}^-(k_2) \right], & k_2^* < x \leq k_1^*, \ i = 2,
\end{cases}
\]
where \( f \) is the row vector \( f = (1\ 0) \) and

\[
\tilde{H}(k) = \begin{pmatrix} K - e^k \\ C \\ D \end{pmatrix}, \quad \tilde{H}^+(x) = K - e^x \frac{q_2 + r_2}{q_2}, \quad \tilde{H}^-(k) = e^k \frac{r_2}{q_2} \left( \frac{1}{\alpha + 1} \right),
\]

where we used again equation (A.3). Here \( C \) and \( D \) are determined by the two linear equations

\[
W(e^{k_1} -_s, 2^*) = W(e^{k_1} +_s, 2^*), \\
W'(e^{k_1} -_s) = W'(e^{k_1} +_s),
\]

and the levels \( k_1^* \) and \( k_2^* \) satisfy the two equations

\[
f \left[ \Psi_2^-(k_2) \tilde{H}^-(k_2) + \Psi_2^+(k_2) \{C - \tilde{H}^+(k_1)\} \right] = e^{k_2} \frac{r_2}{q_2}, \\
e_1^Q \tilde{H}^-(k_1) = -e^{k_1}.
\]

Acknowledgements

We would like to thank Florin Avram for inspiring conversations. We also thank an anonymous referee and an associate editor for their numerous useful suggestions and comments that led to considerable improvements of the presentation of the paper.

Appendix

A Proofs

A.1 Cramér Martingale Measure

In this section and the next we present a construction of an equivalent martingale measure for the process \((X, Z)\) and show how the parameters change under this change of measure. For a background of Markov additive processes we refer to Asmussen (3).

An important role in the construction of the change of measure is played by the process \( X_a = \{X_a(t); t \geq 0\} \) defined by

\[
X_a(t) = \int_0^t a(Z_s) dX_s
\]

for some function \( a \) to be specified below. It is straightforward to verify that the process \( X_a \) is still of the form (2), but with changed parameters; its characteristic matrix is \( K_a[s] = G + \Delta_a[s] \), where \( \Delta_a[s] \) is the diagonal matrix with elements \( \kappa_i(a_i s) \). Write \( h \) and \( \lambda \) for the Perron-Frobenius right-eigenvector and eigenvalue of \( K_a[1] \), respectively, and define the candidate change of measure \( L = \{L_t; t \geq 0\} \) by

\[
L(t) = e^{X_a(t) - \lambda t} h(Z(t))/h(Z(0)), \quad (A.1)
\]
with $h(i) = h_i$ the $i$th coordinate of $h$. If $E[S_1] < \infty$, then $\kappa_i(1) < \infty$ for $i \in E^0$, and a solution $a_i, i \in E^0$, exists of the equations

$$\kappa_i(a_i + 1) = r(i) + \kappa_i(a_i),$$

(A.2)

with $\kappa_i(a_i) < \infty$, where $\kappa_i(s) = \log E[e^{sX_i}]$ is the Laplace exponent of $X_i = \mu_i t + \sigma_i W_t + J_i(t)$. It is shown in the following result that the measure $\mathbb{P}^*$ with Radon-Nikodym derivative

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} \bigg|_{F_t} = L(t)$$

is indeed an equivalent martingale measure:

**Proposition 2** Suppose that $E[S_1]$ is finite.

(i) The process $L = \{L_t, \mathcal{F}_t; t \geq 0\}$ is a positive mean one martingale and $\mathbb{P}^*$ is a probability measure;

(ii) Under $\mathbb{P}^*$, $\exp \left( X - \int_0^t r(Z_s)ds \right) = B_t^{-1}S_t$ is a martingale.

In view of Proposition 2 the market (1) with price processes as specified in (2) is arbitrage-free if $E[S_1] < \infty$. It is shown below in Proposition 3 that, under $\mathbb{P}^*$, $X$ is still of the form (2) but with changed parameters. The process $B_t^{-1}S_t$ is martingale if the following restriction holds for the parameters of $X$ (see also [4, Sec. 2]):

$$\sigma^2(i) + \mu(i) + \lambda_i(\tilde{F}_i(1) - 1) = r(i), \quad i \in E^0,$$

(A.3)

where

$$\tilde{F}_i(s) = p_i \alpha_i(-sI - T_i)^{-1} t_i + q_i \beta_i(sI - U_i)^{-1} u_i$$

denotes the moment-generating function of $f_i$, the pdf of the jump-sizes of $X$ in state $i$.

**Proof:** of Proposition 2 (i) Let $(X, Z)$ be of the form (2), with corresponding characteristic matrix $K$ and suppose that $g$ is a $E^0$-row vector. Asmussen and Kella [5] have shown that

$$e^{bX_t - ct} g(Z_t) - e^{bX_0} g(Z_0) - \int_0^t e^{bX_u - cu} g(Z_u) du (K[b] - cI),$$

(A.4)

is a row vector of martingales for $c \geq 0$ and $b$ such that the diagonal elements of $K[b]$ are finite. Choosing in (A.4) the process $X$ to be equal to $X_a$, $b = 1$, $c = \lambda$ and $g = h$, it follows that $e^{X_a(t) - \lambda t} h(Z_t) - h(Z_0)$ is a zero mean martingale. As $h$ is positive, the process $L$ in (A.1) is thus a positive mean one martingale. The proof of (ii) can be found in the next section. \qed
A.2 Change of measure

Proposition 3 Under $\mathbb{P}^*$, the process $X$ is still of the form \( \mathbb{E} \) with $\sigma^*(i) = \sigma(i)$, 
\[
\mu^*(i) = \mu(i) + \alpha_i \sigma_i^2 - \int_0^1 y(1 - e^{\alpha_i y}) \lambda_i^{(+)} F_{i}^{(+)}(dy) 
\]
and with $J_i^*$ compound Poisson processes with changed jump rates 
\[
\lambda_i^{(+)*} = \lambda^{(+)} \hat{F}^{(+)}[\gamma] \quad \text{and} \quad \lambda_i^{(-)*} = \lambda^{(-)} \hat{F}^{(-)}[\gamma] 
\]
and distributions of the positive and negative jumps of phase-type with representations 
\[
(\alpha_i^{(+)*}, T_i^{(+)*}) = (\alpha_i^{(+,a_i)}, T_i^{(+,a_i)}) \quad \text{and} \quad (\alpha_i^{(-)*}, T_i^{(-)*}) = (\alpha_i^{(-,a_i)}, T_i^{(-,a_i)}), 
\]
respectively, where the parameters are transformed according to 
\[
(\alpha^{(+,\gamma)}, T^{(+,\gamma)}) = (\alpha^{(+)} \Delta_+/\hat{F}^{(+)}[\gamma], \Delta_+^{-1} T^{(+)} \Delta_+ + \gamma I), \quad (A.5) \\
(\alpha^{(-,\gamma)}, T^{(-,\gamma)}) = (\alpha^{(-)} \Delta_-/\hat{F}^{(-)}[\gamma], \Delta_-^{-1} T^{(-)} \Delta_- - \gamma I), \quad (A.6) 
\]
where $\Delta_+$ and $\Delta_-$ are the diagonal matrices with respectively $k_+$ and $k_-$ on the diagonal such that $k_+ = (-\gamma I - T^{(+)})^{-1} t^{(+)}$, $k_- = (\gamma I - T^{(-)})^{-1} t^{(-)}$ and $I$ is an identity matrix of appropriate size.

Proposition 4 Under $\mathbb{P}^*$, $Z$ has intensity matrix $G^*$ with elements 
\[
g^{ij}_{\alpha} = g_{ij} h(j)/h(i), \quad i \neq j, \quad \text{and} \quad g^{ii}_{\alpha} = - \sum_{j \neq i} g_{ij} h(j)/h(i), 
\]
where $g_{ij}$ is the $ij$th element of $G$;

Proof: of Propositions 3 and 4. We first show how to find the characteristic matrix of $X$ under $\mathbb{P}^*$. Denote by $X_{\alpha}$ the process $X_{\alpha} + bX$, where $b$ is such that the elements of the characteristic matrix of $X_{\alpha}$ are finite, and let $f$ be a function that maps $E^0$ to $\mathbb{R}$. Applying Itô’s lemma to $e^{X_{\alpha}(t)} h(Z_t)f(Z_t)$ shows that 
\[
e^{X_{\alpha}(t) - \lambda t} h(Z_t)f(Z_t) - e^{X_{\alpha}(0)} h(Z_0)f(Z_0) - \int_0^t \lambda e^{X_{\alpha}(u)} - \lambda u h(Z_u)f(Z_u)du \\
\times \left\{ \sum_{j \in E^0} (1_{(Z_{\alpha} = j)} h(i)f(i) \kappa_j(a_i + b) + \sum_{j \neq i} g_{ij}(h(j)f(j) - h(i)f(i)) \right\}, 
\]
is a $\mathbb{P}$-martingale, where we wrote $\lambda = \lambda_{\alpha}$ and $h = h_{\alpha}$. Since 
\[
\lambda_{\alpha}(h_{\alpha})_i = (K_{\alpha}[1] h_{\alpha})_i = \sum_{j \neq i} g_{ij}(h_{\alpha}(j) - h_{\alpha}(i)) + \kappa_i(a_i) h_{\alpha}(i), 
\]
where $K_{\alpha}$ are the positive matrices with respectively
it follows from taking expectations and rearranging terms that, in vector notation,

$$
\mathbb{E}_{0,i}[L_t e^{bX(t)}1_{Z_i}] = 1_i + \int_0^t \mathbb{E}_{0,i}[L_u e^{bX(u)}1_{Z_i}] (G^* + \Delta^*[b])du,
$$

(A.7)

where $G^*$ is as in the statement of the proposition and $\Delta^*$ is the diagonal matrix with elements $\Delta^*_i = \kappa_i(a_i + b) - \kappa_i(a_i)$. Writing $F^*_0[b]$ for the matrix with elements $\mathbb{E}_{0,i}[e^{bX_i}1_{Z_i=s}]$ and differentiating (A.7) with respect to $t$, we arrive at the matrix differential equation

$$
F^*_t[b] = F^*_0[b](G^* + \Delta^*), \quad F^*_0[b] = I,
$$

where $t$ denotes the time derivative. Solving this system shows that the characteristic matrix of $X$ under $P^*$ is given by $K^*[b] = G^* + \Delta^*$. See [1] for a proof of (A.5) and (A.6); the rest of the statements of (i) and (ii) directly follow from Proposition 3 in [24].

Proof: of Proposition 3 (ii) From Proposition 2 it follows that the $a_i$'s have been chosen in such that, under $P^*$, $X^i$ have cumulant-generating functions satisfying $\kappa^*_i(1) = \kappa_i(a_i + 1) - \kappa_i(a_i) = \eta_i$, so that the characteristic matrix $K^*$ of $(X - \int_0^\infty r(Z_s)ds, Z)$ under $P^*$ satisfies $K^*[1] = G^*$ and $1$ is an eigenvector of $K^*[1]$ with eigenvalue $0$. Setting $g \equiv 1$ in (A.4) and taking $A = 0$, it thus follows that the process $e^{X_1 - \int_0^\infty r(Z_s)ds}$ is a martingale under $P^*$.

A.3 Wiener-Hopf factorization

Proof: of Theorem 2 (ii) Now we turn to the proof of the uniqueness of the Wiener-Hopf factorization. To this end, let $(Z^+, G^+, Z^-, G^-)$ be a Wiener-Hopf factorization and define the function $\tilde{f}$ as in (21), but replacing $\eta^+$ and $Q^+$ by $Z^+$ and $G^+$ respectively. Since $(Z^+, G^+)$ satisfies the first equation of (18), it follows from an application of Itô’s lemma, that $\tilde{f}(Y_t, A_t)$ is a local martingale that is bounded on $\{t \leq \tau^+_t\}$, so that Doob’s Optional Stopping Theorem implies that

$$
\tilde{f}(j, x) = \mathbb{E}_{x,j}[\tilde{f}(Y_{t\wedge \tau^+_t}, A_{t\wedge \tau^+_t})] \nonumber
\quad = \mathbb{E}_{x,j}[\tilde{f}(Y^+, A_{\tau^+_t})1_{(\tau^+_t < \infty)}] + \lim_{t \to \infty} \mathbb{E}_{x,j}[\tilde{f}(Y_t, A_t)1_{(\tau^+_t = \infty)}].
$$

(A.8)

By the definition of $\tilde{f}$ and the absence of positive jumps of $A$, the first expectation in (A.8) is equal to $f(j, x)$. Note that the second term in (A.8) is zero if $Q$ is transient or $Q$ is recurrent and sup$_t A_t = +\infty$. Indeed, in the latter case, $\tau^+_t$ is finite a.s. whereas in the former case $P_{x,t}(Y_t \in E)$ converges to zero. Thus $f = \tilde{f}$ for all $h$ and we deduce that $G^+ = Q^+$ and $Z^+ = \eta^+$. Similarly, one can show that $G^- = Q^-$ and $Z^- = \eta^-$ and the uniqueness is proved.

Proof: of Theorem 2 (iii) Assume that $Q$ is recurrent but $A_t \to -\infty$. As $Q^+$ inherits the irreducibility property of $Q$, it follows from the Perron-Frobenius theorem that the matrix $Q^+$ has a probability vector $\mu$ as left-eigenvector with
its largest eigenvalue. Since the quadruple \((\eta^+, Q^+, \eta^-, Q^-)\) satisfies (15), it is straightforward to check that this remains the case if we replace \((\eta^+, Q^+)\) by \((\eta^+(I - 1\mu) + \mu, Q^+(I - 1\mu))\). We are left to show that these are the only two factorizations of \((A, Y)\). As in the proof of Theorem 2(ii), it follows that any factorization quadruple of \((A, Y)\) must contain \(\eta^-\) and \(Q^-\). Letting \((\eta^+, G^+)\) and \(\tilde{f}(j, x)\) be as in the proof of Theorem 2, we distinguish between the cases that \(G\) is recurrent or transient. In the latter case \(\tilde{f}(j, x)\) tends to zero if \(x \to -\infty\) and we deduce from (A.8) that \(f = \tilde{f}\) and thus \(G = Q^+\) and \(Z^+ = \eta^+\). In the former case, we note that, as \(G^+\) inherits the irreducibility property of \(Q\), it has a unique invariant distribution \(v\) given by the left-eigenvector of \(G\) with eigenvalue 0. Thus \(\tilde{f}(j, x)\) converges to \(e_{j}^1 \nu h = \nu h\) as \(x \to -\infty\). The right-hand side of (A.8) is thus equal to

\[
\tilde{f}(j, x) = f(j, x) + \mathbb{P}_{x,j}(\tau_k^+ = \infty) \nu h
\]

\[
= f(j, x) + (1 - e_j^1 W^+ \exp(Q^+(k - x)) I) \nu h. \tag{A.9}
\]

By differentiation of (A.9) with respect to \(x\), we deduce that \(G = Q^+(I - 1\nu)\). In particular, it follows that \(v\) is a left-eigenvector of \(Q^+\). Since the Perron-Frobenius eigenvector is the unique eigenvector with the largest eigenvalue, it follows that \(\mu = \nu\) and then also that \(Z^+ = \eta^+(I - 1\mu) + \mu\), which completes the proof. \(\Box\)

### A.4 First-passage under state-dependent levels

**Proof: of Theorem 3** For brevity of notation we will drop the subscript and write \(v\) for \(v_{b,k}\). Appealing to the strong Markov property, it follows that, for \(x > k_1\),

\[
v(x, i) = \mathbb{E}_{x,i}[v(k_1, Y_{\tau^-})1_{(\tau^- < \zeta)}] = \mathbb{E}_{x,i}[h^-_1(Y_{\tau^-})1_{(\tau^- < \zeta)}],
\]

where \(\tau^- = \tau_{k_1}^+\), and for \(k_j < x < k_{j-1}\),

\[
v(x, i) = \mathbb{E}_{x,i}[v(k_{j-1}, Y_{\tau^-})1_{(\tau^- < \zeta, A_+ = k_{j-1})}] + \mathbb{E}_{x,i}[v(k_j, Y_{\tau^-})1_{(\tau^- < \zeta, A_+ = k_j)}] + \mathbb{E}_{x,i}[v(A^-_{\zeta}, Y_{\tau^-})1_{(\tau^- < \zeta)}].
\]

Invoking results from Proposition 4 yields that (36) is valid for some vectors \(h^-_j, h^+_j\) and \(h^0_j\). To finish the proof we have to show that the stated form of these vectors is correct. We start with noting that, by the structure of the process \((A, Y)\),

\[
v(k_j, j) = e^{bk_j} \quad \text{and} \quad v(k_j, \ell) = e^{bk_j} e^{e_{j}^\ell(sI - T^-_{\ell})^{-1} t^-_{\ell}} \quad \text{for} \quad \ell \in E^-_{j}.
\]

Furthermore, we claim that \(v(\cdot, i)\) is continuous. Indeed, from the Markov property it follows that for \(\ell \in E^n, m \in E^0\)

\[
v(z, \ell) = \int_0^\infty v(z + y, m) e^{e_{\ell}^y T^-_{m}} t^-_{m} dy, \tag{A.10}
\]
so that, in particular, it holds that \( v(\cdot, \ell) \) is continuously differentiable on \((k_m, \infty)\). Similarly, it follows that \( v(\cdot, \ell) \in C^1(k_m, \infty) \) for \( \ell \in E^+_m \). The continuity of \( v(\cdot, \ell) \) for \( \ell \in E^0 \) follows directly from its definition. As a consequence it follows that the equation (11) holds true. Let \( \ell > j, \ell \in E^0 \) and consider \( v(x, \ell) \) for \( x \in [k_j - \epsilon, k_j + \epsilon] \). By a Feynman-Kac argument it follows that, on 
\([k_j - \epsilon, k_j + \epsilon]\) for \( \epsilon > 0 \) small enough (such that \( k_j - \epsilon > k_l \)), \( v(\cdot, \ell) \) is equal to the unique \( C^2 \) solution of the ODE
\[
\frac{\sigma^2(\ell)}{2} f'' + \mu(\ell)f' - c(\ell)f = g, \quad f(k_j \pm \epsilon) = v(k_j \pm \epsilon, \ell),
\]
for some continuous function \( g \) and some constant \( c(\ell) \). In particular, \( v(\cdot, \ell) \) is continuously differentiable at \( k_j \) and it follows that (11) holds true. \( \square \)

A.5 American put

Proof: of Theorem 1 The proof of this result follows a standard approach for solving perpetual American option pricing problems. As argued above the optimal stopping time must be of the form \((\ell)\). Therefore, the value function is given by \( V_{k^*} \) for some vector \( k^* \in (-\infty, \log K)^N \). The vector \( k^* \) can subsequently be found by optimisation. At this point we note that the condition (A.3) implies that for the embedding \( s(i)^2/2 + m(i) < -q_{ii} \) is satisfied for all \( i \in E \), so that we can apply Theorem 3. Since, for fixed \((x, i)\), \( k \mapsto V_k(x, i) \) is continuously differentiable it follows that \( k^* \) satisfies
\[
\frac{\partial V_k}{\partial k_j}(e^{x}, i) \bigg|_{k=k^*} = 0 \quad \text{for all } (x, i), j = 1, \ldots, N. \tag{A.11}
\]
Consider next the finite difference \( [V_k(e^{k_j+h}, j) - V_k(e^{k_j}, j)]/h \) and note that it is equal to the sum
\[
\frac{V_k(e^{k_j+h}, j) - V_k(e^{k_j}, j)}{h} + \frac{V_{k+h}(e^{k_j+h}, j) - V_k(e^{k_j}, j)}{h}. \tag{A.12}
\]
Letting \( h \downarrow 0 \), it follows from (A.11), that the first term converges to zero, while the second term converges to \(-e^{k_j}\). Thus we see that the smooth fit equations (8) hold true. By a martingale argument it also follows that \( V_{k^*} = V^* \) for any solution \( k^* \in (-\infty, \log K)^N \) of (8). \( \square \)

Proof: of Lemma 1 Suppose first that
\[
\alpha \neq [\mu_1 + \sqrt{\mu_1^2 + 2(r_1 + q_1)\sigma_1^2}] / \sigma_1^2.
\]
From the definitions of \( g(\theta), F_1(\theta), \) and \( F_2(\theta) \), we have that
\[
g(+\infty) = +\infty, \quad g(-\infty) = -\infty, \quad g(0) = \alpha[(q_1 + r_1)(q_2 + r_2) - q_1q_2] > 0.
\]
Note that \( F_1(\theta) \) has two different real roots \( \theta_{0,1} > 0 > \theta_{0,2} \) with
\[
\theta_{0,2} = -[\mu_1 + \sqrt{\mu_1^2 + 2(r + q_1)\sigma_1^2}] / \sigma_1^2,
\]
23
we then have \( \theta_{0,2} \neq -\alpha \). Also,
\[
g(\theta_{0,1}) = -q_1 q_2 (\alpha + \theta_{0,1}) < 0,
\]
because \( q_1, q_2, \alpha, \theta_{0,1} > 0 \). Therefore, we further have that
\[
g(\theta_{0,2}) = \begin{cases} 
- q_1 q_2 (\alpha + \theta_{0,2}) < 0, & \text{if } \theta_{0,2} > -\alpha, \\
- q_1 q_2 (\alpha + \theta_{0,2}) > 0, & \text{if } \theta_{0,2} < -\alpha.
\end{cases}
\]
\[
\lim_{\theta \to -\alpha} g(\theta) = \begin{cases} 
\lambda \alpha F_1(-\alpha; r) > 0, & \text{if } \theta_{0,2} > -\alpha, \\
\lambda \alpha F_1(-\alpha; r) < 0, & \text{if } \theta_{0,2} < -\alpha.
\end{cases}
\]

In view of the intermediate value theorem the proof of the first assertion is complete. Since \( Q^- \) is a generator matrix, it is negative semi-definite and the final assertion follows.

If
\[
\alpha = [\mu_1 + \sqrt{\mu_1^2 + 2(r_1 + q_1)\sigma_1^2}/\sigma_1^2],
\]
\( \theta = -\alpha \) is a root. By a similar reasoning applied to \( h(\theta) = g(\theta)/(\theta + \alpha) \) and \( h(-\alpha) < 0 \) it can be shown that \( h \) has four distinct roots (two positive and two negative ones). \( \square \)

References

[1] Asmussen, S.: Exponential families generated by phase-type distributions and other Markov lifetimes. Scand. J. Statist. 16, 319–334, 1989.

[2] Asmussen, S.: Phase–type representations in random walk and queueing problems. Ann. Probab. 20, 772–789, 1992.

[3] Asmussen, S.: Ruin Probabilities. World Scientific, 2000.

[4] Asmussen, S., Avram, F. and Pistorius, M.R.: Russian and American put options under phase-type Lévy models. Stoch. Proc. Appl. 109, 79–111, 2004.

[5] Asmussen, S. and Kella, O.: A multi–dimensional martingale for Markov additive processes and its applications, Adv. Appl. Probab. 32, 376–393, 2000.

[6] Barndorff-Nielsen, O. E. Processes of normal inverse Gaussian type. Finance Stoch. 2, 41–68, 1998.

[7] Bensoussan, A. On the theory of option pricing, Acta applicandae mathematicae 2, 139–158, 1984.

[8] Boyarchenko, S.I., and Levendorskii, S.Z.: Perpetual American options under Lévy processes. SIAM Journal on Control and Optimization 40, 1663–1696, 2002.
[9] Carr, P.: Randomization and the American Put. Rev. Fin. Studies 11, 597–626, 1998.

[10] Carr, P., Geman, H., Madan, D.P. and Yor, M.: The fine structure of asset returns, Journal of Business 75, 305–332, 2002.

[11] Cont, R. and Tankov, P. Financial modelling with jump-diffusions. Chapman/CRC Press, 2004.

[12] Eberlein, E. and Keller, U: Hyperbolic Distributions in Finance. Bernoulli 1, 281–299, 1995.

[13] Gerber, H.U., Shiu, E.S.W. On the time value of ruin. North American Actuarial Journal 2, 48–78, 1998.

[14] Guo, X. An explicit solution to an optimal stopping problem with regime switching. J. Appl. Prob. 38, 464–481, 2001.

[15] Guo, X. Inside information and stock fluctuations. PhD dissertation, Department of Mathematics, Rutgers University, Newark, NJ, 1999.

[16] Guo, X. Information and option pricing, Quantitative Finance, 1, 38–44, 2001.

[17] Guo, X. and Q. Zhang: Closed-form solutions for perpetual American put options with regime switching. SIAM J. Appl. Math. 64, 2034–2049, 2004.

[18] Jobert, A. and Rogers, L.C.G.: Option pricing with Markov-modulated dynamics. Siam J. Control Optim. 44, 2063–2078, 2006.

[19] Karatzas, I. On the pricing of American options. Applied mathematics and optimization 17, 37-60, 1988.

[20] Kemperman, J. H. B. The passage problem for a stationary Markov chain. Statistical Research Monographs, Vol. I. The University of Chicago Press, Chicago, 1961.

[21] Kou, S. G.: A jump-diffusion for option pricing. Management Science 48, pp. 1086–1101, 2002.

[22] Levendorskii, S.Z.: Early exercise boundary and option pricing in Lévy driven models; Quantitative Finance 4, 525–547, 2004.

[23] London, R. R.; McKean, H. P.; Rogers, L. C. G.; Williams, David A martingale approach to some Wiener-Hopf problems. I, II. Seminar on Probability, XVI, pp. 41–67, 68–90, Lecture Notes in Math., 920, Springer, Berlin-New York, 1982.

[24] Neuts, M. F.: Matrix-Geometric Solutions in Stochastic Models, John Hopkins, 1981.
[25] Palmowski, Z. and Rolski, T.: A technique of exponential change of measure for Markov processes, Bernoulli 8, 767–785, 2002.

[26] Rogers, L.C.G.: Fluid models in Queueing theory, Ann. Appl. Probab. 4, 390-413, 1994.

[27] Shiryaev, A. N. Optimal stopping rules. Applications of Mathematics, Vol. 8. Springer-Verlag, New York-Heidelberg, 1978.