Distributions, First Integrals and Legendrian Foliations

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Abstract
We study germs of holomorphic distributions with “separated variables”. In codimension one, a well known example of this kind of distribution is given by

\[ dz = (y_1 dx_1 - x_1 dy_1) + \cdots + (y_m dx_m - x_m dy_m), \]

which defines the canonical contact structure on \( \mathbb{C}P^{2m+1} \). Another example is the Darboux distribution

\[ dz = x_1 dy_1 + \cdots + x_m dy_m, \]

which gives the normal local form of any contact structure. Given a germ \( D \) of holomorphic distribution with separated variables in \( (\mathbb{C}^n, 0) \), we show that there exists, for some \( \kappa \in \mathbb{Z}_{\geq 0} \) related to the Taylor coefficients of \( D \), a holomorphic submersion

\[ H_D : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^\kappa, 0) \]

such that \( D \) is completely non-integrable on each level of \( H_D \). Furthermore, we show that there exists a holomorphic vector field \( Z \) tangent to \( D \), such that each level of \( H_D \) contains a leaf of \( Z \) that is somewhere dense in the level. In particular, the field of meromorphic first integrals of \( Z \) and that of \( D \) are the same. Between several other results, we show that the canonical contact structure on \( \mathbb{C}P^{2m+1} \) supports a Legendrian holomorphic foliation whose generic leaves are dense in \( \mathbb{C}P^{2m+1} \). So we obtain examples of injectively immersed Legendrian holomorphic open manifolds that are everywhere dense.

Keywords Legendrian foliation · Holomorphic distribution · First integral

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1 Introduction

Let $M, N \in \mathbb{N}$, $M \geq 2$ and consider coordinates $x = (x_1, \ldots, x_M)$ and $z = (z_1, \ldots, z_N)$ in $\mathbb{C}^M$ and $\mathbb{C}^N$, respectively. Let $D$ be the distribution of dimension $M$ defined by the system of equations

$$
\begin{aligned}
dz_1 &= \omega_1 \\
dz_2 &= \omega_2 \\
&\quad \vdots \\
dz_N &= \omega_N,
\end{aligned}
$$

where $\omega_1, \ldots, \omega_N \in \Omega^1(\mathbb{C}^M, 0)$, that is, these forms depend only on the variables $(x_1, \ldots, x_M)$. An important example of this situation happens for $M = 2m$, $N = 1$ and the distribution

$$
\begin{aligned}
\,dz = (y_1 dx_1 - x_1 dy_1) + \cdots + (y_m dx_m - x_m dy_m),
\end{aligned}
$$

where we consider coordinates $(x_1, y_1, \ldots, x_m, y_m, z) \in \mathbb{C}^{2m+1}$—this distribution defines the canonical contact structure on $\mathbb{C}P^{2m+1}$. Another example is given by the Darboux contact form

$$
\begin{aligned}
\,dz = x_1 dy_1 + \cdots + x_m dy_m,
\end{aligned}
$$

which gives the normal local form of any contact structure, see Alarcón et al. (2017), Darboux (1882) or Godbillon (1969). The purpose of this work is to study the integrability properties of the distribution $D$ and their relationship with the dynamics of the holomorphic vector fields that are tangent to it. To begin with, we point out a situation where we easily find a holomorphic first integral for $D$: suppose that there exist $a_1, \ldots, a_N \in \mathbb{C}$, not all zero, such that

$$
a_1 d\omega_1 + \cdots + a_N d\omega_N = 0.
$$

Then, by Poincaré’s Lemma, we find a holomorphic function $g$ in $(\mathbb{C}^M, 0)$ with $g(0) = 0$ and such that

$$
dg = a_1 \omega_1 + \cdots + a_N \omega_N.
$$

Thus, if we set

$$
T(z) = a_1 z_1 + \cdots + a_N z_N,
$$

we have

$$
d(T(z) - g(x)) = a_1 (dz_1 - \omega_1) + \cdots + a_N (dz_N - \omega_N).
$$
It follows from this equation that $D$ is a subdistribution of

$$d \left( T(z) - g(x) \right) = 0,$$

so the function $T(z) - g(x)$ is a holomorphic first integral of $D$. This kind of first integrals of $D$ will be called elementary. Roughly speaking, our first result asserts that the elementary first integrals generate all the space of meromorphic first integrals of $D$ in $(\mathbb{C}^{M+N}, 0)$. In order to give a precise statement of our results, we briefly introduce some notions. Firstly we define

$$\omega = (\omega_1, \ldots, \omega_N), \quad d\omega = (d\omega_1, \ldots, d\omega_N), \quad \text{and} \quad \int_{\gamma} \omega = \left( \int_{\gamma} \omega_1, \ldots, \int_{\gamma} \omega_N \right),$$

where $\gamma$ is any piecewise smooth path in the domain of definition of $\omega$. We can express $d\omega$ as a series at $0 \in \mathbb{C}^M$ in the form

$$d\omega = \sum c_{ij}^{ij} x^K dx_i dx_j, \quad (1.3)$$

where $dx_i dx_j$ stands for $dx_i \wedge dx_j$, $c_{ij}^{ij} \in \mathbb{C}^N$ and the summation extends over

$$K = (k_1, \ldots, k_M) \in (\mathbb{Z}_{\geq 0})^M, \quad 1 \leq i < j \leq M.$$ 

Let

$$W_D \subset \mathbb{C}^N$$

be the complex vector space spanned by the coefficients $c_{ij}^{ij}$ and put

$$\kappa = \text{codim}(W_D).$$

If $\kappa \geq 1$, take a linear (surjective) map

$$T = (T_1, \ldots, T_\kappa): \mathbb{C}^N \to \mathbb{C}^\kappa$$

such that

$$\ker(T) = W_D.$$ 

The arguments explained above allow us to find a holomorphic map

$$g = (g_1, \ldots, g_\kappa): (\mathbb{C}^M, 0) \to (\mathbb{C}^\kappa, 0)$$
such that the functions

\[ h_j = T_j(z) - g_j(x), \quad j = 1, \ldots, \kappa \]

are elementary first integrals of \( \mathcal{D} \). Then the map

\[ H_{\mathcal{D}} := (h_1, \ldots, h_\kappa) : (\mathbb{C}^M, 0) \times \mathbb{C}^N \to (\mathbb{C}^\kappa, 0) \]

is a holomorphic first integral of \( \mathcal{D} \). Observe that \( \partial_x H_{\mathcal{D}} = T \), so that \( H_{\mathcal{D}} \) is a submersion. If \( \kappa = 0 \), that is, if \( W_{\mathcal{D}} = \mathbb{C}^N \), we define \( H_{\mathcal{D}} \) as the constant map from \( \mathbb{C}^{M+N} \) to the trivial vector space \( \mathbb{C}^0 \). The holomorphic functions on \( (\mathbb{C}^0, 0) \) are—by convention—the constants. Our first result states that \( H_{\mathcal{D}} \) is a primitive first integral of \( \mathcal{D} \), in the sense that any other meromorphic first integral of \( \mathcal{D} \) is a composition of \( H_{\mathcal{D}} \) with some meromorphic function.

**Theorem 1** A germ \( F \) of meromorphic function in \( (\mathbb{C}^{M+N}, 0) \) is a meromorphic first integral of \( \mathcal{D} \) if and only if

\[ F = f \circ H_{\mathcal{D}}, \]

where \( f \) is a germ of meromorphic function in \( (\mathbb{C}^\kappa, 0) \). In particular, if \( W_{\mathcal{D}} = \mathbb{C}^N \), the distribution \( \mathcal{D} \) has only constant meromorphic first integrals in \( (\mathbb{C}^{M+N}, 0) \).

**Remark 2** As a direct consequence of Theorem 1 we have the following fact: if \( \mathcal{D} \) has a non-constant meromorphic first integral, then \( \mathcal{D} \) has a holomorphic first integral that is a submersion.

Let \( F \) be a meromorphic first integral of \( \mathcal{D} \). Essentially, the proof of Theorem 1 is equivalent to showing that \( F \) is constant along the levels of \( H_{\mathcal{D}} \). This fact is a direct consequence of Theorem 4 below, which guarantees the existence of holomorphic vector fields tangent to \( \mathcal{D} \) having leaves that are somewhere dense in the levels of \( H_{\mathcal{D}} \). For ease of exposition, it will be convenient to adopt the following convention.

**Definition 3** Let \( \mathcal{D} \) be a holomorphic distribution on a complex manifold \( \mathcal{V} \). Let \( \mathcal{F} \) be a singular holomorphic foliation. We say that \( \mathcal{F} \) is a Legendrian foliation for \( \mathcal{D} \) if its leaves are tangent to \( \mathcal{D} \). If \( \dim \mathcal{F} = 1 \) and \( \mathcal{F} \) is generated by a holomorphic vector field \( Z \), we also say that \( Z \) is Legendrian for \( \mathcal{D} \).

If \( \mathcal{D} \) is a contact structure on \( \mathcal{V} \), the leaves of a Legendrian foliation \( \mathcal{F} \) are injectively immersed holomorphic isotropic varieties for \( \mathcal{D} \). If in addition we have \( 2 \dim \mathcal{F} = \dim \mathcal{D} \), the leaves of \( \mathcal{F} \) are injectively immersed holomorphic Legendrian manifolds for \( \mathcal{D} \). For more details on Legendrian varieties see for instance Arnold (1983, 1989) and Buczyński (2008). From now on and for the sake of simplicity, we say holomorphic foliation to mean singular holomorphic foliation.

**Theorem 4** Let \( \mathcal{D} \) and \( H_{\mathcal{D}} \) be defined by Eqs. (1.1) and (1.4) respectively. Then \( \mathcal{D} \) supports a Legendrian holomorphic vector field \( Z \), defined on the domain of definition of \( \mathcal{D} \), such that the following property holds: given a neighborhood \( \Delta \) of the origin in

\[ \mathbb{C}^N \]

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\(\mathbb{C}^{M+N}\), there is a nonempty open set \(\Delta^* \subset \Delta\) such that each leaf \(L\) of \(Z|_{\Delta^*}\) is dense in the level of \(H_D|_{\Delta^*}\) containing \(L\). In particular, if \(W_D = \mathbb{C}^N\), the leaves of \(Z|_{\Delta^*}\) are dense in \(\Delta^*\). Moreover, if \(M = 2\), there exists a hypersurface \(S\) through \(0 \in \mathbb{C}^{2+N}\) invariant by \(Z\) such that, given \(\Delta\), the set \(\Delta^*\) can be chosen in the form \(\Delta^* = B\setminus S\) for some neighborhood \(B\) of the origin.

The proof of this theorem is constructive and, as immediate consequence of this construction—see Proposition 18, we have the following additional properties about the vector field \(Z\).

**Proposition 5**  Under the same assumptions and notation of Theorem 4, we have the following additional properties:

(1) There exists a polynomial vector field

\[
X(x) = A_1(x) \frac{\partial}{\partial x_1} + \cdots + A_M(x) \frac{\partial}{\partial x_M}
\]

on \(\mathbb{C}^M\), depending only on a finite jet of the series (1.3) of \(d\omega\), such that the vector field \(Z\) is given by

\[
Z(x, z) = \sum_{i=1}^M A_i(x) \frac{\partial}{\partial x_i} + \sum_{j=1}^N [\omega_j(x) \cdot X(x)] \frac{\partial}{\partial z_j}.
\]

In particular, \(Z\) depends only on the variables \((x_1, \ldots, x_M) \in \mathbb{C}^M\). Here the dot product stands for the evaluation of a 1-form on a vector field.

(2) There exists a hypersurface \(S\), which is a union of hyperplanes through the origin of \(\mathbb{C}^M\), that is invariant by \(X\) and such that the orbits of \(X\) through \(\mathbb{C}^M \setminus S\) are everywhere dense.

Observe that, if the forms \(\omega_i\) are defined on a neighborhood \(U\) of the origin in \(\mathbb{C}^M\), the distribution \(D\) is defined on the set

\[U \times \mathbb{C}^N.\]

Thus, by Proposition 5, the vector field \(Z\) is also defined on the set \(U \times \mathbb{C}^N\). Nevertheless, note that Theorem 4 establishes a local property of the vector field \(Z\) near the origin, a fact which can be useful, as we show with a little example. Let \(\eta\) be a holomorphic 1-form on a neighborhood of the origin in \(\mathbb{C}^3\) such that \(\eta \wedge d\eta(0) \neq 0\).

By Darboux Theorem, the distribution \(\eta = 0\) near the origin is equivalent to the one defined by

\[dz = xdy.\]

Since this distribution is of type (1.1), from Theorem 4 the following corollary follows.
Corollary 1 Let \( \eta \) be a holomorphic 1-form on a neighborhood of the origin in \( \mathbb{C}^3 \) such that \( \eta \wedge d\eta(0) \neq 0 \). Then the distribution \( \eta = 0 \) supports a Legendrian holomorphic vector field on a neighborhood of the origin in \( \mathbb{C}^3 \), having an invariant hypersurface \( S \) through the origin, such that the following property holds: given any neighborhood \( \Delta \) of the origin in \( \mathbb{C}^3 \), there exists a neighborhood \( \Delta^* \subset \Delta \) of the origin in \( \mathbb{C}^3 \) such that any leaf of \( Z \) through a point in \( \Delta^* \setminus S \) is dense in \( \Delta^* \).

Now, we consider distributions \( D \) that are globally defined. If such a distribution is completely non-integrable, we show that it supports Legendrian holomorphic one-dimensional foliations with dense generic leaves.

Theorem 6 Assume that the holomorphic 1-forms \( \omega_i \) in the system (1.1) extend as meromorphic 1-forms on \( \mathbb{C}^M \). Let \( (\omega)^\infty \) be the union of pole sets of the \( \omega_i \), let \( Z \) be as given by Theorem 4, and \( S \) as given by Proposition 5. Then we have the following properties:

1. \( Z \) extends as a meromorphic vector field on \( \mathbb{C}^{M+N} \). If the forms \( \omega_i \) are rational (resp. polynomial), the vector field \( Z \) is also rational (resp. polynomial).
2. If \( W_D = \mathbb{C}^N \), then each leaf of \( Z \) passing through a point \( (x, z) \in \mathbb{C}^{M+N} \) is dense in \( \mathbb{C}^{M+N} \).

As an easy consequence we have the following corollary.

Corollary 2 Consider coordinates \( (x_1, \ldots, x_M, z) \) in \( \mathbb{C}^{M+1} \). Let \( D \) be the distribution defined by the equation

\[ dz = \omega, \]

where \( \omega \) is a rational non-closed 1-form in the variables \( (x_1, \ldots, x_M) \). Then \( D \) supports a Legendrian polynomial vector field whose generic leaf is dense in \( \mathbb{C}^{M+1} \). In particular, if \( M = 2m \) and \( \xi \) is the contact distribution on \( \mathbb{C}P^{2m+1} \) defined by Eq. (1.2), then \( \xi \) supports a Legendrian one-dimensional holomorphic foliation whose generic leaf is dense in \( \mathbb{C}P^{2m+1} \).

In relation with the contact structure \( \xi \) defined by Eq. (1.2), the leaves of the foliation given by Corollary 2 are examples of injectively immersed isotropic open holomorphic curves that are everywhere dense, although Corollary 2 does not specify the analytic type of these curves. Recently, in Alarcón et al. (2019, Corollary 6.11) the authors construct examples of everywhere dense injectively immersed Legendrian open curves of arbitrary analytic type in odd dimensional complex projective spaces. A slightly more elaborated application of our results allow us to obtain examples of Legendrian \( m \)-dimensional holomorphic foliation for \( \xi \) whose generic leaves are dense in \( \mathbb{C}P^{2m+1} \)—so we obtain examples of injectively immersed Legendrian holomorphic open manifolds for \( \xi \) that are everywhere dense.

Theorem 7 Let \( \xi \) be the canonical contact distribution on \( \mathbb{C}P^{2m+1} \) given by Eq. (1.2). Then \( \xi \) supports a Legendrian \( m \)-dimensional holomorphic foliation whose generic leaf is dense in \( \mathbb{C}P^{2m+1} \).
It is worth pointing out the relationship between our results and some classical connectivity properties of distributions in the real case. The problem of local connectivity by curves tangent to a distribution was studied initially in the real case by Carathéodory (codimension one) and by Chow (any codimension), see for instance Chow (1939) and Gromov (1996). Precisely, we have the following theorem.

**Theorem 8** (Chow Connectivity Theorem, Chow 1939; Gromov 1996) Let $X_1, \ldots, X_m$ be $C^\infty$ vector fields on a connected manifold $V$, such that successive commutators of these fields span each tangent space $T_v V$, $v \in V$. Then every two points on $V$ can be joined by a piecewise smooth curve in $V$ where each piece is a segment of an integral curve of one of the fields $X_i$.

As a direct consequence we have the total connectivity property by curves tangent to a distribution $\mathcal{D}$, provided it is completely non-integrable—in the sense that vector fields tangent to $\mathcal{D}$ Lie-generate the full tangent space. We will see in Remark 20 that, for a distribution given by a system of the form (1.1), this is exactly the case when $H_\mathcal{D}$ is a constant function.

With the aid of Chow Connectivity Theorem we can successively connect an arbitrary sequence of points; in this way we can construct dense piecewise smooth curves that are tangent to a completely non-integrable distribution $\mathcal{D}$. As can be expected, there are smoothing methods which allow us to deform such piecewise smooth curves to obtain smooth curves also tangent to the distribution—see Gromov (1996). Since, in general, the $C^\infty$ smoothing methods have no parallel in the holomorphic class, the existence of dense holomorphic curves tangent to completely non-integrable distributions is a more complicated problem, which will not be a direct consequence of a holomorphic version of the Connectivity Theorem—for a holomorphic version of the Connectivity Theorem in $\mathbb{C}^3$ we refer to Zorich (2019). Even more, we can look for the existence of holomorphic vector fields tangent to a totally non-integrable distribution with dense generic leaves. In this direction, Theorem 4 and Corollaries 1 and 2 can be regarded as partial positive answers.

## 2 Lifting Vector Fields

Consider the canonical projection
\[ \pi_1 : \mathbb{C}^{M+N} \to \mathbb{C}^M. \]
Let $X$ be a holomorphic vector field on a neighborhood $U$ of the origin in $\mathbb{C}^M$. We assume that $U$ is contained in the domain of definition of the 1-forms $\omega_i$ defining the distribution $\mathcal{D}$. The lifting of $X$ to the distribution $\mathcal{D}$ is the vector field $X^D$ on $U \times \mathbb{C}^N$ that is tangent to $\mathcal{D}$ and satisfies the equation
\[ d\pi_1 \cdot X^D = X. \]
If
\[ X = A_1 \frac{\partial}{\partial x_1} + \cdots + A_M \frac{\partial}{\partial x_M}, \]

\[ \square \] Springer
the lifting $X^D$ is explicitly given by
\[
X^D = \sum_{i=1}^{M} A_i(\pi_1) \frac{\partial}{\partial x_i} + \sum_{j=1}^{N} \left[ \omega_j(\pi_1) \cdot X(\pi_1) \right] \frac{\partial}{\partial z_j}.
\]

If $\mathcal{F}$ is the holomorphic foliation defined by $X$, we say that the holomorphic foliation defined by $X^D$ is the lifting of $\mathcal{F}$ to $\mathcal{D}$ and denote it by $\mathcal{F}^D$—this foliation is a Legendrian one-dimensional holomorphic foliation for $\mathcal{D}$.

The holomorphic vector $Z$ given by Theorem 4 will be constructed as the lifting of some vector field $X$ as above. Thus, the main task of our work is the study of foliations that are obtained by the lifting process. This kind of foliations defines holonomy maps which are strongly related with the values obtained by the integration of the forms $\omega_i$ along loops in $\mathbb{C}^M$, as we explain in the rest of the section.

**Holonomy**

Let $\mathcal{F}$ be as above. Fix a neighborhood $\Delta = \Delta_1 \times \Delta_2$ of $0 \in \mathbb{C}^{M+N}$, where $\Delta_1$ and $\Delta_2$ are—respectively—Euclidean balls centered at the origins of $0 \in \mathbb{C}^M$ and $0 \in \mathbb{C}^N$, and assume that $\mathcal{F}$ and $\mathcal{D}$ are defined on $\Delta$—in this case $\mathcal{F}$ and $\mathcal{D}$ are actually defined on $\Delta_1 \times \mathbb{C}^N$, as we have pointed out in the previous section. Let $p \in \Delta_1 \setminus \text{Sing}(\mathcal{F})$ and consider the fiber $\Delta_p := \{p\} \times \Delta_2$. Take any $\xi \in \Delta_p$ and denote by $\mathcal{L}$ the leaf of $\mathcal{F}^D|_{\Delta}$ passing through $\xi$. We are interested in the set of "returns" of $\mathcal{L}$ to the fiber $\Delta_p$, that is, we want to study the set $\mathcal{L} \cap \Delta_p$. A way to obtaining points in this set is by lifting some loops in $\Delta_1$ based at $p$, as we describe below. Let $\gamma : [0, 1] \to \Delta_1$ be a piecewise path with $\gamma(0) = p$. Then there exists a unique piecewise path $\tilde{\gamma}$ starting at $\xi \in \Delta_p$, tangent to $\mathcal{D}$ and such that $\pi_1 \circ \tilde{\gamma} = \gamma$. In fact, if $\xi = (p, z)$ and
\[
\gamma(t) = (x_1(t), \ldots, x_M(t))
\]
for $t \in [0, 1]$, we have
\[
\tilde{\gamma}(t) = (x_1(t), \ldots, x_M(t), z_1(t), \ldots, z_N(t)),
\]
where the functions $z_1, \ldots, z_N$ are the solutions of the system
\[
\begin{align*}
z_1' &= \omega_1(\gamma) \cdot \gamma' \\
&\vdots \\
z_N' &= \omega_N(\gamma) \cdot \gamma'
\end{align*}
\]
with the initial condition
\[
(z_1(0), \ldots, z_N(0)) = z.
\]
Now, suppose that $\gamma$ is contained in the leaf of $\mathcal{F}$ through $p$. Then
\[ \gamma'(t) = \theta(t)X(\gamma(t)) \]
for some piecewise function $\theta$ and, from System (2.1) and the definition of $X^D$, we deduce that $\tilde{\gamma}$ is contained in the leaf of $\mathcal{F}^D$ through $\zeta$. In principle, the path $\tilde{\gamma}$ is contained in $\Delta_1 \times \mathbb{C}^N$, it is not necessarily contained in $\Delta$. If $\gamma$ were a loop based at $p$ and we had $\tilde{\gamma} \subset \Delta$, then the ending point of $\tilde{\gamma}$ would be a return of $\mathcal{L}$ to the fiber $\Delta_p$. Lemma 9 below gives us an elementary criteria to guarantee that $\tilde{\gamma}$ is contained in $\Delta$. Recall that the forms $\omega_i$ can be considered as an $N$-tuple of forms
\[ \omega = (\omega_1, \ldots, \omega_N). \tag{2.2} \]
Given any $\xi \in \Delta$, we can see $\omega(\xi)$ as a linear map $\mathbb{C}^M \to \mathbb{C}^N$, so we consider its canonical norm $|\omega(\xi)|$. Moreover, we denote by $\ell(\gamma)$ the Euclidean length of $\gamma$.

**Lemma 9** Suppose that there exists $\mathcal{R} > 0$ such that $|\omega| \leq \mathcal{R}$ on
\[ \Delta = \Delta_1 \times \Delta_2. \]

Let $\gamma \subset \Delta_1$ be a piecewise path starting at $p$. Consider a point $\zeta = (p, z)$ in $\Delta$ and let $r > 0$ be the radius of $\Delta_2$. Then, if
\[ |z| + \mathcal{R}\ell(\gamma) < r, \]
the lifting of $\gamma$ to $\mathcal{D}$ starting at $\zeta$ is contained in $\Delta$.

**Proof** The lifting $\tilde{\gamma}$ of $\gamma$ to $\mathcal{D}$ starting at $\zeta$ has the form
\[ \tilde{\gamma}(t) = (\gamma(t), \tilde{z}(t)), \]
where $\tilde{z}(t) = (z_1(t), \ldots, z_M(t))$ satisfies System (2.1) with $\tilde{z}(0) = z$. Then
\[ \tilde{z}' = \omega(\gamma(t)) \cdot \gamma'(t), \]
so that, for any $s \in [0, 1]$ we have
\[ \tilde{z}(s) - z = \int_0^s \omega(\gamma(t)) \cdot \gamma'(t)dt. \]
Therefore
\[ |\tilde{z}(s)| \leq |z| + \int_0^s |\omega(\gamma(t)) \cdot \gamma'(t)| dt \]
\[ \leq |z| + \mathcal{R}\ell(\gamma) < r, \]
so $\tilde{\gamma}(s)$ belongs to $\Delta$ for all $s \in [0, 1]$. □
By Lemma 9, we conclude that, in order to obtain returns of the leaf $\mathcal{L}$ to the fiber $\Delta_p$, it suffices to lift suitable small loops $\gamma$ that are contained in the leaf of $\mathcal{F}|_{\Delta_1}$ through $p$.

**Non-trivial Returns and Main Ideas**

Let $\gamma$ be a loop based at $p$, contained in the leaf of $\mathcal{F}|_{\Delta_1}$ through $p$ and such that the lifting of $\gamma$ to $\mathcal{D}$ is contained in $\Delta$—so it defines a return of $\mathcal{L}$ to $\Delta_p$. Naturally, we are interested in foliations $\mathcal{F}$ that are able to generate many returns of $\mathcal{L}$ to $\Delta_p$ that are different from $\zeta$ itself. Thus, as a first step, we want to estimate the difference—associated to the loop $\gamma$—between the starting point $\zeta = (p, z)$ and its return $\zeta^{\gamma} := (p, z^{\gamma})$ to $\Delta_p$ associated to the loop $\gamma$. From System (2.1) we easily obtain that

$$z^{\gamma} - z = \int_{\gamma} \omega.$$ 

Thus, our general objective is to study the integral of $\omega$ along loops $\gamma$ that are tangent to some holomorphic foliation $\mathcal{F}$. If $d\omega = 0$, we clearly obtain $\int_{\gamma} \omega = 0$ and we have no returns different from $\zeta$. This allows us to perceive the role played by the nonzero coefficients of $d\omega$ in its series expansion. On the other hand, if $\gamma$ were the boundary of a compact Riemann surface—with boundary—$S$, by Stokes Theorem we would obtain

$$\int_{\gamma} \omega = \int_{\gamma} \omega|_S = \int_S d(\omega|_S) = 0,$$

and this will happen even if the surface $S$ is singular. Thus, in order to obtain $\int_{\gamma} \omega \neq 0$ we have to take $\gamma$ and $\mathcal{F}$ such that:

1. $\gamma$ is not homotopically null in its leaf; otherwise $\gamma$ is the boundary of a complex disc in its leaf.
2. $\gamma$ is not the fundamental loop in a separatrix of $\mathcal{F}$; otherwise $\gamma$ is the boundary of a singular disc.

Given $\omega$ with $d\omega \neq 0$, we illustrate how we can obtain $\int_{\gamma} \omega \neq 0$ by looking for $\mathcal{F}$ in the class of linear foliations. For the sake of clarity, we suppose $M = 2$. Then we assume that $\mathcal{F}$ is defined by a linear vector field on $\mathbb{C}^2$. Since we need many loops tangent to $\mathcal{F}$ that are homotopically non trivial in its corresponding leaves, we take $\mathcal{F}$ being generated by the system

$$\begin{align*}
x'_1 &= m_1 x_1 \\
x'_2 &= -m_2 x_2,
\end{align*}$$

(2.3)
where $m_1$ and $m_2$ are coprime natural numbers. Take $(y_1, y_2) \in (\mathbb{C}^*)^2$ small and consider the loop
\[ \gamma : \partial \mathbb{D} \to \mathbb{C}^2 \]
\[ \xi \mapsto (y_1 \xi^{m_1}, y_2 \xi^{-m_2}), \]
which is contained in the leaf of $\mathcal{F}$ through $(y_1, y_2)$. We notice that $\gamma$ is the boundary of the parametrized singular disc
\[ D = \left\{ (y_1 w^{m_1}, y_2 \bar{w}^{m_2}) : w \in \mathbb{D} \right\}, \]
where $\mathbb{D} = \{ w \in \mathbb{C} : |w| \leq 1 \}$. Observe that $D$ is just a $\mathcal{C}^\infty$ singular disc.

The 2-form $d\omega$ can be expressed as
\[ d\omega = \sum c_{k_1k_2} y_1^{k_1} y_2^{k_2} dx_1 dx_2, \]
where $c_{k_1k_2} \in \mathbb{C}^N$. Then we have
\[
\int_\gamma \omega = \int_D d\omega = \sum c_{k_1k_2} \int_\mathbb{D} (y_1 w^{m_1})^{k_1} (y_2 \bar{w}^{m_2})^{k_2} d (y_1 w^{m_1}) d (y_2 \bar{w}^{m_2})
\]
\[ = \sum m_1 m_2 c_{k_1k_2} y_1^{k_1+1} y_2^{k_2+1} \int_\mathbb{D} w^{m_1(k_1+1)-1} \bar{w}^{m_2(k_2+1)-1} dw d\bar{w}. \quad (2.4) \]

We will use the following lemma, whose proof is a straightforward computation.

**Lemma 10** Let $p, q \in \mathbb{Z}_{\geq 0}$. Then
\[
\int_{\mathbb{D}} w^p \bar{w}^q dw d\bar{w} = \begin{cases} -\frac{2\pi \sqrt{-1}}{p+1} & \text{if } p = q, \\ 0 & \text{otherwise.} \end{cases}
\]

By this lemma we see that in the summation (2.4) only remain the terms corresponding to indexes in the set
\[ I = \left\{ (k_1, k_2) : m_1(k_1 + 1) = m_2(k_2 + 1) \right\} \]
and we obtain
\[
\int_\gamma \omega = \sum_{(k_1, k_2) \in I} \left( \frac{-2\pi m_2 \sqrt{-1}}{k_1 + 1} \right) c_{k_1k_2} y_1^{k_1+1} y_2^{k_2+1}. \quad (2.5) \]

From the equation above we see that $\int_\gamma \omega$ is an analytic function of $(y_1, y_2)$, hence $\int_\gamma \omega$ take a lot of values whenever the series in (2.5) is not null. To guarantee that this
series is not null we have to choose \((m_1, m_2)\) such that, for some \((k_1, k_2) \in I\), we have \(c_{k_1 k_2} \neq 0\). This choice is clearly possible if there exists some \(c_{k_1 k_2} \neq 0\): for this particular \((k_1, k_2)\) we can choose \((m_1, m_2)\) such that \(m_1 (k_1 + 1) = m_2 (k_2 + 1)\). In this case we can say that the linear singularity \((2.3)\) is adapted to the coefficient \(c_{k_1 k_2}\). The idea for the construction of the vector field \(Z\) in Theorem 4 is to find a foliation \(\mathcal{F}\) that “contains” many linear singularities adapted to each coefficient \(c_{k_1 k_2} \neq 0\). Actually, in this task we only need to consider a finite set coefficients \(c_{k_1 k_2}\) which spans the space \(W_D\). The term “contains” above is used to mean that such linear singularities are generated by \(\mathcal{F}\) after a blow-up of the origin. The construction of the foliation \(\mathcal{F}\) will be achieved in next section but we will finish this one with a result—Proposition 12—collecting the ideas sketched above in a more general setting.

**Returns Associated to Resonant Singularities**

Given \(x, y \in \mathbb{C}^M\), denote

\[
x \cdot y = x_1 y_1 + \cdots + x_M y_M
\]

and, for \(1 \leq i \leq M\), let \(e_i \in \mathbb{C}^M\) be the standard \(i^{th}\) canonical vector. Moreover, given \(L \in \mathbb{Z}^M\) and \(1 \leq i, j \leq M\), we denote

\[
L_{ij} = L + e_i + e_j.
\]

If \(y = (y_1, \ldots, y_M) \in (\mathbb{C}^*)^M\) and \(m = (m_1, \ldots, m_M) \in \mathbb{Z}^M\), the loop

\[
\zeta \in \partial \mathbb{D} \mapsto \left(y_1 \zeta^{m_1}, \ldots, y_M \zeta^{m_M}\right) \in \mathbb{C}^M
\]

will be called the \(m\)-loop based at \(y\); this loop is contained in a leaf of the holomorphic foliation defined by the linear system

\[
\begin{align*}
x'_1 &= m_1 x_1 \\
& \quad \vdots \\
x'_M &= m_M x_M.
\end{align*}
\]

Given \(\alpha \in \{1, \ldots, M\}\), we say that the vector \(m = (m_1, \ldots, m_M) \in \mathbb{Z}^M\) is \(\alpha\)-elementary if \(m_\alpha > 0\) and the other coordinates of \(m\) are negative. Let \(m \in \mathbb{Z}^M\) be \(\alpha\)-elementary and let \(\gamma\) be the \(m\)-loop based at \(y \in (\mathbb{C}^*)^M\). Observe that \(\gamma\) can be expressed as

\[
\zeta \in \partial \mathbb{D} \mapsto \left(y_1 \bar{z}^{1-m_1}, \ldots, y_{\alpha-1} \bar{z}^{1-m_{\alpha-1}}, y_\alpha z^{m_\alpha}, y_{\alpha+1} \bar{z}^{-m_{\alpha+1}}, \ldots, y_M \bar{z}^{-m_M}\right).
\]

Then \(\gamma\) is the boundary of the parametrized \((C^\infty)\) disc

\[
w \in \partial \mathbb{D} \mapsto \left(y_1 \bar{w}^{-m_1}, \ldots, y_{\alpha-1} \bar{w}^{-m_{\alpha-1}}, y_\alpha w^{m_\alpha}, y_{\alpha+1} \bar{w}^{-m_{\alpha+1}}, \ldots, y_M \bar{w}^{-m_M}\right),
\]

which will be denoted by \(D_\gamma\).
Lemma 11 Let \( m \in \mathbb{Z}^M \) be \( \alpha \)-elementary and let \( \gamma \) be the \( m \)-loop based at \( y \in (\mathbb{C}^*)^M \). Let \( K \in \left( \mathbb{Z}_{\geq 0} \right)^M \) and \( 1 \leq i < j \leq M \). Then the integral

\[
\int_{D_{\gamma}} x^K dx_i dx_j
\]
is nonzero only if \( \alpha \in \{i, j\} \) and \( m \cdot K_{ij} = 0 \). Moreover, in this case we have

\[
\int_{D_{\gamma}} x^K dx_i dx_j = \begin{cases} 
\frac{2\pi \sqrt{-1} \pm 1}{k_{\alpha} + 1} (m_j) y^{K_{ij}}, & \text{if } i = \alpha, \\
\frac{2\pi \sqrt{-1} \pm 1}{k_{\alpha} + 1} (-m_i) y^{K_{ij}}, & \text{if } j = \alpha.
\end{cases}
\] (2.8)

Proof If \( \alpha \notin \{i, j\} \), we see that

\[
D_{\gamma}^*(dx_i dx_j) = d \left( y_i \bar{w}^{-m_i} \right) d \left( y_j \bar{w}^{-m_j} \right) = 0.
\]

Then the integration of \( x^K dx_i dx_j \) over \( D_{\gamma} \) is nonzero only if \( i = \alpha \) or \( j = \alpha \). Suppose that \( i = \alpha \). Then

\[
\int_{D_{\gamma}} x^K dx_\alpha dx_j = \int_{\mathbb{D}} \left( y_1 \bar{w}^{-m_1} \right)^{k_1} \cdots \left( y_\alpha w^{m_\alpha} \right)^{k_\alpha} \cdots \left( y_M w^{-M} \right)^{k_M} d \left( y_\alpha w^{m_\alpha} \right) d \left( y_j \bar{w}^{-m_j} \right) = \int_{\mathbb{D}} y^K w^{m_\alpha k_\alpha} \bar{w}^{-m-K+m_\alpha k_\alpha} \left( m_\alpha y_\alpha w^{m_\alpha-1} d w \right) \left( -m_j y_j \bar{w}^{-m_j-1} d \bar{w} \right) = m_\alpha (-m_j) y^{K_{\alpha j}} \int_{\mathbb{D}} w^{m_\alpha k_\alpha + m_\alpha - 1} \bar{w}^{-m-K+m_\alpha k_\alpha - m_j - 1} d w d \bar{w},
\]

which, by Lemma 10, is nonzero only if

\[
m \cdot K_{\alpha j} = 0,
\]

and in this case (again by Lemma 10) we obtain

\[
\int_{D_{\gamma}} x^K dx_\alpha dx_j = \left( \frac{2\pi \sqrt{-1} m_j}{k_{\alpha} + 1} \right) y^{K_{\alpha j}}.
\]

If we have \( j = \alpha \), a similar computation gives us
\[
\int_D x^K d\alpha_i d\alpha = \left\{ \begin{array}{ll}
-\left(\frac{2\pi \sqrt{-1 \text{im}_i}}{k_{\alpha}+1}\right)^{y_{\alpha}}, & \text{if } m \cdot K_{\alpha} = 0 \\
0, & \text{otherwise}
\end{array} \right.
\]

which completes the proof. \(\square\)

Let \(L \in (\mathbb{Z}_{\geq 0})^M, \alpha \in \{1, \ldots, M\}\) and, given \(i, j \in \{1, \ldots, M\}\), denote
\[
L_{-ij} = L - e_i - e_j.
\]

Then define \(\mathcal{M}(L, \alpha)\) as the set composed of the monomial 2-forms
\[
\begin{xarray}{ll}
  x^{L_{-i\alpha}} d\alpha_i d\alpha & \text{for } 1 \leq i < \alpha, \text{ and } \\
  x^{L_{-\alpha j}} d\alpha \alpha d\alpha & \text{for } \alpha < j \leq M.
\end{xarray}
\]
\[(2.9)\]

We notice that \(\mathcal{M}(L, \alpha)\) has \((M - 1)\) monomials, although some of these monomials could have negative exponents. Let \(\omega\) be the \(N\)-tuple of forms as we have in (2.2) and, as in (1.3), set
\[
d\omega = \sum c_{ij}^K x^K d\alpha_i d\alpha_j.
\]

If \(\zeta\) denotes a monomial \(x^K d\alpha_i d\alpha_j\) with \(K \in \mathbb{Z}^M, 1 \leq i < j \leq M\), we define \(c(\zeta)\) as the coefficient of \(\zeta\) in \(d\omega\). Naturally, since \(\omega\) is holomorphic, if \(\zeta\) has some negative exponent we have \(c(\zeta) = 0\).

**Proposition 12.** Let \(m \in \mathbb{Z}^M\) be an \(\alpha\)-elementary vector for some \(\alpha \in \{1, \ldots, M\}\). Given \(y \in (\mathbb{C}^\ast)^M\), let \(\gamma_y\) be the \(m\)-loop based at \(y\). Then the function
\[
\rho(m): y \mapsto \int_{\gamma_y} \omega \in \mathbb{C}^N
\]
has a power series expansion
\[
\rho(m) = \sum a_L(m) y^L,
\]
where \(a_L(m) \in \mathbb{C}^N, L \in (\mathbb{Z}_{\geq 0})^M\), such that the following properties hold:

1. We have \(a_L(m) \neq 0\) only if \(m \cdot L = 0\).
2. Given \(L\) with \(m \cdot L = 0\), let \(\zeta_1, \ldots, \zeta_{M-1}\) be the monomials of \(\mathcal{M}(L, \alpha)\) in the order they appear in (2.9). Then there exist complex numbers
\[
\lambda_1(L, m), \ldots, \lambda_{M-1}(L, m),
\]
not depending on \(\omega\), such that
\[
a_L(m) = \lambda_1(L, m) c(\zeta_1) + \cdots + \lambda_{M-1}(L, m) c(\zeta_{M-1}).
\]
(3) Set \( L = (l_1, \ldots, l_M) \) and assume that \( l_\alpha \neq 0 \). Suppose that there exists \( \mathbb{C} \)-linearly independent \( \alpha \)-elementary vectors

\[
m_1, \ldots, m_{M-1} \in \mathbb{Z}^M
\]

such that

\[
m_1 \cdot L = \cdots = m_{M-1} \cdot L = 0.
\]

Then the matrix

\[
[\lambda_i(L, m_j)]_{1 \leq i, j \leq M-1}
\]

is invertible. In particular—from item (2)—the coefficients in \( d\omega \) of the monomials in \( \mathcal{M}(L, \alpha) \) are linear combinations of the \((M - 1)\) coefficients

\[
a_L(m_1), \ldots, a_L(m_{M-1})
\]

of the monomial \( y^L \) in the series

\[
\rho(m_1), \ldots, \rho(m_{M-1}).
\]

(4) In order to indicate the dependence of \( \rho(m) \) on \( \omega \), we will write

\[
\rho(m) = \rho(\omega, m) \text{ and } a_L(m) = a_L(\omega, m).
\]

From item (2) we see that \( a_L(\omega, m) \) depends only on the coefficients of \( d\omega \) corresponding to the monomials in \( \mathcal{M}(L, \alpha) \). Since these monomials have degree \((|L| - 2)\), we conclude the following: if \( d\omega_1 \) and \( d\omega_2 \) have series coinciding up to order \( k \), then we have

\[
a_L(\omega_1, m) = a_L(\omega_2, m), \text{ whenever } |L| \leq k + 2.
\]

**Remark 13** Let \( L \) be as in item (3). We will show that, if \( L \) has some nonzero coordinate besides the coordinate \( l_\alpha \), then those vectors \( m_1, \ldots, m_{M-1} \) always exist—this will be crucial in the proof of Theorem 4. Without loss of generality we can assume that \( \alpha = M \). Take vectors

\[
m_1', \ldots, m_{M-1}' \subset N^{M-1} \times \{0\} \subset \mathbb{C}^M
\]

which are \( \mathbb{C} \)-linearly independent. Multiplying these vectors by a suitable natural number if necessary, we can assume that the numbers

\[
m_1 = \frac{m_1' \cdot L}{l_M}, \ldots, m_{M-1} = \frac{m_{M-1}' \cdot L}{l_M}
\]
are non-negative integral numbers. Actually, since $L$ has a positive coordinate $l_\beta$ with $\beta \in \{1, \ldots, M-1\}$, the numbers $m_1, \ldots, m_{M-1}$ are naturals. Then

$$m_1 = -m_1' + m_1 e_M, \ldots, m_{M-1} = -m_{M-1}' + m_{M-1} e_M$$

are the desired vectors.

**Proof** By Stokes Theorem,

$$\int_{\gamma} \omega = \int_{\gamma'} d\omega = \sum c_{ij}^K \int_{\gamma} x^K dx_i dx_j.$$

By Lemma 11, each integral

$$\int_{\gamma} x^K dx_i dx_j$$  \hspace{1cm} (2.10)

is a monomial of $y = (y_1, \ldots, y_M)$. Moreover, given $L \in (\mathbb{Z}_{\geq 0})^M$, again by Lemma 11 we can see that the integral (2.10) is a monomial in $\mathbb{C}^* y^L$ only if the monomial $x^K dx_i dx_j$ belongs to the set

$$\mathcal{M} = \{x^K dx_i dx_j: \alpha \in \{i, j\}, \ m \cdot L = 0, \ K_{ij} = L\}. \hspace{1cm} (2.11)$$

Then we have

$$a_L(m)y^L = \sum \int_{\gamma} c_{ij}^K x^K dx_i dx_j, \hspace{1cm} (2.12)$$

where the summation extends over the monomials $x^K dx_i dx_j$ in the set $\mathcal{M}$. If $m \cdot L \neq 0$, the set $\mathcal{M}$ is empty, so item (1) follows. Suppose that $m \cdot L = 0$. Then we have $\mathcal{M} = \mathcal{M}(L, \alpha)$, so the summation (2.12) extends over the monomials $\varsigma_1, \ldots, \varsigma_{M-1}$. Thus—by Lemma 11—for each term of the summation (2.12) we have

$$\int_{\gamma} c_{ij}^K x^K dx_i dx_j = \begin{cases} \left( \frac{2\pi \sqrt{-1}}{l_\alpha} \right)^{\alpha} (-m_i c_{ij}^K y^L), & \text{if } j = \alpha \\ \left( \frac{2\pi \sqrt{-1}}{l_\alpha} \right)^{i} (m_j c_{ij}^K y^L), & \text{if } i = \alpha. \end{cases}$$

Then, if we set $\eta = \left( \frac{2\pi \sqrt{-1}}{l_\alpha} \right)$ and

$$(\lambda_1(L, m), \ldots, \lambda_{M-1}(L, m)) := \eta(-m_1, \ldots, -m_{\alpha-1}, m_{\alpha+1}, \ldots, m_M), \hspace{1cm} (2.13)$$
we obtain
\[ a_L(m) = \lambda_1(L, m) c(\zeta_1) + \cdots + \lambda_{M-1}(L, m) c(\zeta_{M-1}), \] (2.14)
so Item (2) is proved. For each \( i = 1, \ldots, M-1 \), put
\[ m_i = (m_i^1, \ldots, m_i^M). \]
Then, from (2.13) we have
\[ \left[ \lambda_i(L, m_j) \right]^T = \eta \begin{bmatrix} -m_1^1 & \ldots & -m_{\alpha-1}^1 & m_{\alpha+1}^1 & \ldots & m_M^1 \\ -m_1^2 & \ldots & -m_{\alpha-1}^2 & m_{\alpha+1}^2 & \ldots & m_M^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -m_1^{M-1} & \ldots & -m_{\alpha-1}^{M-1} & m_{\alpha+1}^{M-1} & \ldots & m_M^{M-1} \end{bmatrix}. \]
Thus, if we consider the projection
\[ \varpi(x_1, \ldots, x_M) = (x_1, \ldots, x_{\alpha-1}, x_{\alpha+1}, \ldots, x_M), \]
we have
\[ \det \left[ \lambda_i(L, m_j) \right] = \pm \eta^{M-1} \det \begin{bmatrix} \varpi(m_1) \\ \varpi(m_2) \\ \vdots \\ \varpi(m_{M-1}) \end{bmatrix}. \] (2.15)
Recall that the vectors \( m_1, \ldots, m_{M-1} \) belong to the hyperplane
\[ E = \left\{ (x_1, \ldots, x_M) \in \mathbb{C}^M : l_1x_1 + \cdots + l_Mx_M = 0 \right\}. \]
Since those vectors are linearly independent, in order to prove that the determinant in (2.15) is nonzero it is enough to show that the projection
\[ \varpi : \mathbb{C}^M \to \mathbb{C}^{M-1} \]
is injective on \( E \), which follows directly from the inequality \( l_\alpha \neq 0 \). Item (3) is proved.

3 Holomorphic Maps and Linear Dependence

In this section we show a number of lemmas which will be used in the proof of Theorem 4. These lemmas are mainly based on the vanishing properties of holomorphic functions and polynomials.
Lemma 14 Let \( d \in \mathbb{Z}_{\geq 0} \) and suppose that \( T \subset \mathbb{C} \) has at least \( d + 1 \) elements. Let

\[
f : \mathbb{C} \to \mathbb{C}^N
\]

be a polynomial map of degree \( \leq d \) such that \( f(T) \) is contained in some complex linear subspace \( W \subset \mathbb{C}^N \). Then the image of \( f \) is contained in \( W \).

**Proof** Let \( A : \mathbb{C}^N \to \mathbb{C}^n \) be a linear map such that \( W = \ker A \). Then \( A \circ f : \mathbb{C} \to \mathbb{C}^n \) is a polynomial map which vanishes on \( T \) and with coordinates of degree \( \leq d \). Whence each coordinate of \( A \circ f \) is null. \( \square \)

Lemma 15 Consider a germ of holomorphic function \( \rho : (\mathbb{C}^M, 0) \to (\mathbb{C}^N, 0) \) given by a series

\[
\rho(x) = \sum \theta_K x^K,
\]

where \( K = (k_1, \ldots, k_M), x = (x_1, \ldots, x_M), \theta_K \in \mathbb{C}^N \). Suppose that the image of \( \rho \) is contained in some complex linear subspace \( W \subset \mathbb{C}^N \). Then each coefficient \( \theta_K \) belongs to \( W \).

**Proof** Let \( A : \mathbb{C}^N \to \mathbb{C}^n \) be a linear map such that \( W = \ker A \). It follows from the hypothesis that \( A \circ \rho : (\mathbb{C}^M, 0) \to \mathbb{C}^n \) is null. On the other hand, given \( x \) in the domain of convergence of the series \( \rho \), it is easy to prove that

\[
A \left( \sum \theta_K x^K \right) = \sum A(\theta_K) x^K.
\]

Then the function \( \sum A(\theta_K) x^K \) vanishes, hence each \( A(\theta_K) \) is zero and the lemma follows. \( \square \)

Given \( \epsilon > 0 \), an \( \epsilon \)-basis of \( \mathbb{R}^k \) will be a basis whose elements have modulus smaller than \( \epsilon \).

Lemma 16 For each \( j = 1, \ldots, n \), let \( U_j \subset \mathbb{C}^M \) be an open connected set accumulate on the origin and let \( f_j : U_j \to \mathbb{C}^N \) be a holomorphic function such that \( f_j(z) \to 0 \) as \( z \to 0 \). Suppose that the set

\[
\Gamma = \{ f_j(z) : z \in U_j, j = 1, \ldots, n \}
\]

spans \( \mathbb{C}^N \) as a complex vector space. Then, for any \( \epsilon > 0 \), the set \( \Gamma \) contains an \( \epsilon \)-basis of \( \mathbb{C}^N \) as a real vector space.

**Proof** Let \( \epsilon > 0 \). For each \( j = 1, \ldots, n \), let \( U_j^* \subset U_j \) be a nonempty open set such that \( |f_j(x)| < \epsilon \) for all \( x \in U_j^* \). We only need to prove that

\[
\Gamma^* = \{ f_j(z) : z \in U_j^*, j = 1, \ldots, n \}
\]
spans $\mathbb{C}^N$ as a real vector space. Suppose that this is not true. Then $\Gamma^*$ is contained in some real hyperplane of $\mathbb{C}^N$. Thus we can find a nonzero complex linear map

$$A: \mathbb{C}^N \rightarrow \mathbb{C}$$

such that $\Gamma^* \subset \ker(\text{Re}(A))$, which means that $\text{Re}(A \circ f_j(x)) = 0$, $x \in U_j^*$; $j = 1, \ldots, n$. Then, since $A \circ f_j(x) \rightarrow 0$ as $x \rightarrow 0$, we have

$$(A \circ f_j)|_{U_j^*} \equiv 0, \quad j = 1, \ldots, n$$

and therefore, since the $U_j$ are connected, $A \circ f_j \equiv 0$, $j = 1, \ldots, n$. But this means that $\Gamma$ is contained in the codimension one complex hyperplane $\ker A \subset \mathbb{C}^N$, which is a contradiction. \qed

**Lemma 17** Let $\Gamma \subset \mathbb{R}^k$ be a set such that, for any $\epsilon > 0$, there exists an $\epsilon$-basis $u_1, \ldots, u_k$ of $\mathbb{R}^k$ with

$$\{\pm u_1, \ldots, \pm u_k\} \subset \Gamma.$$ 

Let $\Omega \subset \mathbb{R}^k$ be open, convex and bounded, and let $\Sigma \subset \mathbb{R}^k$ be such that $\Sigma \cap \Omega \neq \emptyset$. Suppose that there exists $\delta > 0$ such that:

$$z \in \Sigma, \quad \text{dist}(z, \Omega) < \delta \implies z + u \in \Sigma, \forall u \in \Gamma. \quad (3.1)$$

Then $\Omega$ is contained in the topological closure $\overline{\Sigma}$ of $\Sigma$.

**Proof** Take $z_0 \in \Sigma \cap \Omega$. Let $\epsilon > 0$ be arbitrary. It is sufficient to prove that there exists an $\epsilon$-basis $u_1, \ldots, u_k$ of $\mathbb{R}^k$ such that

$$\Omega \cap (z_0 + \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_k) \subset \Sigma.$$ 

By hypothesis we can find a basis $u_1, \ldots, u_k$ of $\mathbb{R}^k$ with $|u_j| < \min\{\epsilon, \delta/k\}$ for all $j = 1, \ldots, k$ and such that $\{\pm u_1, \ldots, \pm u_k\} \subset \Gamma$. Let $T$ be the tessellation of $\mathbb{R}^k$ generated by the basis $u_1, \ldots, u_k$ and the point $z_0$. Let $\mathcal{P}$ be the union of parallelopopes of $T$ that meet $\Omega$—then $\Omega \subset \mathcal{P}$ and, in particular, $z_0 \in \mathcal{P}$. Consider a point $z'$ in the set

$$\Omega \cap (z_0 + \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_k);$$

this point is clearly contained in $\mathcal{P}$. Since $\mathcal{P}$ is connected and $z_0 \in \mathcal{P}$, we can find points

$$w_0 = z_0, w_1, \ldots, w_n = z'$$

that are vertices of parallelopopes in $\mathcal{P}$ and such that

$$w_j - w_{j-1} \in \{\pm u_1, \ldots, \pm u_k\} \subset \Gamma, \forall j = 1, \ldots, n.$$
Observe that, since the parallelotopes in $T$ have Euclidean diameter smaller than
\[ |u_1| + \cdots + |u_k| < \delta, \]
any point $w$ belonging to $\mathcal{P}$—which therefore belongs to a parallelotope touching $\Omega$—satisfies
\[ \text{dist}(w, \Omega) < \delta. \]
In particular we have $\text{dist}(w_j, \Omega) < \delta$, $\forall j = 0, 1, \ldots, n$. Thus, since $(w_{j+1} - w_j) \in \Gamma$, if we had $w_j \in \Sigma$, by the hypothesis (3.1) we would obtain
\[ w_{j+1} = w_j + (w_{j+1} - w_j) \in \Sigma. \]
Therefore, since $w_0 = z_0 \in \Sigma$, we successively obtain $w_1, \ldots, w_n \in \Sigma$, so that $z' \in \Sigma$.

\section*{4 Proof of the Results}

As we have seen in previous sections, our construction of foliations that are tangent to the distribution $\mathcal{D}$ on $\mathbb{C}^{M+N}$ will be achieved by lifting to $\mathcal{D}$ some suitable holomorphic foliations in $\mathbb{C}^M$. The existence of these foliations in $\mathbb{C}^M$ is guaranteed by Proposition 18 below, whose proof is done in Sect. 6. If $k \in \mathbb{N}$, a diffeomorphism $h \in \text{Diff}(\mathbb{C}^M, 0)$ will be said $k$-tangent to the identity if the difference $(h - \text{id})$ has vanishing order greater than $k$ at $0 \in \mathbb{C}^M$.

**Proposition 18** Consider coordinates $(t_1, \ldots, t_M)$ in $\mathbb{C}^M$. Let $T \subset \mathbb{C}$ be a finite set, let $\lambda \colon T \to (\mathbb{C}^*)^M$ and let $k \in \mathbb{N}$ be given. Then, there exists a one dimensional singular holomorphic foliation $\mathcal{G}$ on $\mathbb{C}^M$ satisfying the following properties.

1. For a suitable choice of numbers $P \in \mathbb{N}$, $v_{ij} \in \mathbb{C}^*$, $b_j \in \mathbb{C}$, $i = 1, \ldots, M - 1$, $j = 1, \ldots, P$, such that the $b_j$ are pairwise distinct, the foliation $\mathcal{G}$ is defined by the non-autonomous system

\[
\begin{align*}
t'_1 &= \left( \sum_{j=1}^{P} \frac{v_{1j}}{t_{M} - b_j} \right) t_1 \\
t'_2 &= \left( \sum_{j=1}^{P} \frac{v_{2j}}{t_{M} - b_j} \right) t_2 \\& \hspace{1cm} \vdots \\
t'_{M-1} &= \left( \sum_{j=1}^{P} \frac{v_{(M-1)j}}{t_{M} - b_j} \right) t_{M-1} \\
t'_M &= 1.
\end{align*}
\]
Thus the singular set of $G$ on $\mathbb{C}^N$ is composed of the points

$$(0, \ldots, 0, b_j), \quad j = 1, \ldots, P.$$  

(2) The set $T$ is contained in $\{b_1, \ldots, b_P\}$ and, for each $\tau \in T$, there exists a diffeomorphism $h_\tau \in \text{Diff}(\mathbb{C}^M, 0)$ such that,

(a) $h_\tau$ is $k$-tangent to the identity; and
(b) if $\lambda(\tau) = (\lambda_1, \ldots, \lambda_M)$, the foliation $G$ at the singularity $p_\tau = (0, \ldots, 0, \tau) \in \mathbb{C}^M$ is generated by the pushforward of the linear system

$$s'_1 = \lambda_1 s_1$$
$$s'_2 = \lambda_2 s_2$$
$$\vdots$$
$$s'_M = \lambda_M s_M$$

by the diffeomorphism $f_\tau(s) := p_\tau + h_\tau(s)$.

(3) Since it is defined by a rational vector field, the foliation $G$ can be viewed as the strict transform of a singular holomorphic foliation $F$ by the punctual blow-up

$$\pi: \hat{\mathbb{C}}^M \to \mathbb{C}^M$$

at the origin. In other words, there exists a singular holomorphic foliation $F$ on $\mathbb{C}^M$—which is generated by a polynomial vector field—such that $F$ is the pushforward of $G$ by the map

$$(t_1, \ldots, t_M) \mapsto (x_1, \ldots, x_M) = (t_1, t_1 t_2, \ldots, t_1 t_M).$$

(4) Let $S$ be the hypersurface defined by the equation

$$x_1 \cdots x_{M-1} \prod_{j=1}^{P} (x_M - b_j x_1) = 0.$$  

Then

(a) $S$ is invariant by $F$;  
(b) the leaves of $F$ in $\mathbb{C}^M \setminus S$ are everywhere dense; and  
(c) given $\tau \in T$, the germ hypersurface $\pi^{-1}(S)$ at $p_\tau$ is the image by $f_\tau$ of the hypersurface $\{s_1 \cdots s_M = 0\}$.  

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Given \( \epsilon > 0 \) and a neighborhood \( \Delta \subset \mathbb{C}^M \) of the origin, there exists an open set \( \mathcal{U} \subset \Delta \setminus S \), such that the following properties hold.

(a) For each \( p \in \mathcal{U} \), there exists a set \( E \) dense in \( \mathcal{U} \), contained in the leaf \( L_p \) of \( \mathcal{F}|_{\Delta} \) through \( p \), such that any point \( q \in E \) can be connected to \( p \) by a curve in \( L_p \) whose Euclidean length is smaller than \( \epsilon \).

(b) The set \( \pi^{-1}(\mathcal{U} \cup S) \) is a neighborhood of each point in the exceptional divisor outside the tangent cone of

\[
\prod_{\tau \in T} (x_M - \tau x_1) = 0.
\]

(c) Let \( \tau \in T \) be such that there exists a real line through \( 0 \in \mathbb{C} \) separating \( \lambda_M(\tau) \) from the set

\[
\{\lambda_1(\tau), \lambda_1(\tau) + \lambda_2(\tau), \ldots, \lambda_1(\tau) + \lambda_{M-1}(\tau)\}.
\]

Then \( \pi^{-1}(\mathcal{U} \cup S) \) is a neighborhood of each point in the affine space \( \{ t_1 = 0, t_M = \tau \} \). Furthermore, if the property above takes place for all \( \tau \in T \), then \( \mathcal{U} \cup S \) is a neighborhood of the origin. If \( M = 2 \), the hypothesis above means that there exists a real line separating \( \lambda_2(\tau) \) from \( \lambda_1(\tau) \), which is equivalent to the condition \( \frac{\lambda_2(\tau)}{\lambda_1(\tau)} \in \mathbb{C}\setminus[0, +\infty) \).

The rest of this section is devoted to the proof of our main results.

**Proof of Theorem 4**

Let \( \pi : \widehat{\mathbb{C}^M} \to \mathbb{C}^M \) the blow-up at the origin and consider coordinates \( (t_1, \ldots, t_M) \) in \( \widehat{\mathbb{C}^M} \) such that

\[
\pi(t_1, \ldots, t_M) = (t_1, t_1t_2, \ldots, t_1t_M).
\]

Let \( \tilde{\omega} = \pi^*(\omega) := (\pi^*(\omega_1), \ldots, \pi^*(\omega_N)) \). Set

\[
d\tilde{\omega} = \sum \theta_{ij}^L t^L dt_i dt_j,
\]

where \( \theta_{ij}^L \in \mathbb{C}^N \), \( L = (l_1, \ldots, l_M) \in (\mathbb{Z}_{\geq 0})^M \) and the summation extends over \( |L| \geq 0, 1 \leq i < j \leq M \). Recall that

\[
d\omega = \sum c_{ij}^K x^K dx_i dx_j.
\]

In order to express the relation between the coefficients \( \theta_{ij}^L \) and the \( c_{ij}^K \), we analyze the pullback by \( \pi \) of a monomial 2-form \( x^K dx_i dx_j \) with \( 1 \leq i < j \leq M \). Suppose first that \( i > 1 \). Then

\[
\pi^*(x^K dx_i dx_j) = (t_1)^{k_1} (t_1t_2)^{k_2} \ldots (t_1t_M)^{k_M} d(t_1) d(t_1t_i) dt_j
= l_1^{k_1} \ldots l_M^{k_M} (t_1 t_i dt_j + t_1 t_i dt_i dt_j - t_1 dt_i dt_i dt_i).
\]
\[ t^1_{k_1^{1} + \cdots + k_M^{1}} \ldots t^N_{M} \left( \prod_{i=1}^{k_M} t^i_{M} + \prod_{i=1}^{k_{M-1}} t^i_{1} \right) dt_1 dt_j \]

\[ + \prod_{i=1}^{k_{M-2}} t^i_{2} \ldots t^i_{i} \left( \prod_{i=1}^{k_M} t^i_{M} + \prod_{i=1}^{k_{M-1}} t^i_{1} \right) dt_1 dt_j \]

\[ - \prod_{i=1}^{k_{M-2}} t^i_{2} \ldots t^i_{j} \left( \prod_{i=1}^{k_M} t^i_{M} + \prod_{i=1}^{k_{M-1}} t^i_{1} \right) dt_1 dt_i. \] (4.2)

On the other hand, if \( i = 1 \) we have

\[ \pi^* (x^K dx_1 dx_j) = (t^1_1 (t^1_1 t^1_2)^{k_2} \ldots (t^1_1 t^1_M)^{k_M} d(t^1_1) d(t^1_1 t^1_j)) \]

\[ = \prod_{i=1}^{k_{M-2}} t^i_{2} \ldots t^i_{i} \left( \prod_{i=1}^{k_M} t^i_{M} + \prod_{i=1}^{k_{M-1}} t^i_{1} \right) dt_1 dt_j. \] (4.3)

From Eqs. (4.2) and (4.3) we can deduce the following facts.

1. If \( i > 1 \), the monomial \( t^L dt_1 dt_j \) only appears in the pullback of the monomial \( x^K dx_1 dx_j \) with \( K \) such that

\[ L = \phi(K) : = (k_1 + \cdots + k_M + 2, k_2, \ldots, k_M) \]

and we have

\[ \theta^{ij}_{L} = c^{ij}_{\phi^{-1}(L)}. \]

2. If we define

\[ \varphi_1(K) : = (k_1 + \cdots + k_M + 1, k_2, \ldots, k_M) \]

\[ \varphi_j(K) : = (k_1 + \cdots + k_M + 1, k_2, \ldots, k_j + 1, \ldots, k_M), \quad j = 2, \ldots, M, \]

the monomial \( t^L dt_1 dt_j \) appears in the pullbacks of the monomials

\[ x^K dx_1 dx_j, \quad \varphi_i(K) = L, \quad i < j, \quad \text{and} \]

\[ x^K dx_j dx_i, \quad \varphi_i(K) = L, \quad i > j. \]

Thus, if we put \( c^{ij}_K = -c^{ij}_K \) for \( i > j \), we have

\[ \theta^{ij}_{L} = \sum_{i \neq j} c^{ij}_{\varphi^{-1}_i(K)}. \]

3. From items (1) and (2) above we can express the \( c^{ij}_K \) in terms of the \( \theta^{ij}_{L} \) and we obtain

\[ c^{ij}_K = \theta^{ij}_{\varphi_1(K)} - \sum_{i \neq 1, j} \theta^{ij}_{\phi \varphi^{-1}_i \varphi_1(K)}. \]

\[ c^{ij}_K = \theta^{ij}_{\phi(K)} (i \neq 1). \]
(4) Since all the relations founded above are linear, we conclude that the coefficients \( \theta_{ij}^L \) span the same space spanned by the \( c_{ij}^K \), that is, the space \( W_D \).

(5) From the Eqs. (4.2) and (4.3) we see that every monomial \( t_1^{l_1} \cdots t_M^{l_M} dt_i dt_j \) appearing after the blow-up with a nonzero coefficient is such that

\[
l_1 \geq l_2 + \cdots + l_M.
\]

Given \( \tau \in \mathbb{C} \), let \( \tilde{\omega}_\tau \) be the expression of \( \tilde{\omega} \) in the coordinates

\[
s : = (s_1, \ldots, s_M) = (t_1, t_2, \ldots, t_M - \tau).
\]

By a straightforward computation from Eq. (4.1) we can write

\[
d\tilde{\omega}_\tau = \sum \Theta_{ij}^L(\tau) s^L ds_i ds_j,
\]

where

\[
\Theta_{ij}^L(\tau) = \sum_{k \geq 0} \theta_{ij}^{(l_1, l_2, \ldots, l_M + k)} \left( \frac{l_M + k}{l_M} \right) \tau^k.
\]

From fact (5) above, it follows that the coefficient \( \theta_{ij}^{(l_1, l_2, \ldots, l_M + k)} \) can be nonzero only if \( k \leq l_1 \). Then \( \Theta_{ij}^L \) is a polynomial function of \( \tau \) such that

\[
\deg \Theta_{ij}^L \leq l_1 \leq |L|.
\]

Choose a finite set \( \mathcal{M} \) of monomials in \( \{ \theta_{ij}^L t^L dt_i dt_j \} \) whose coefficients span the space \( W_D \) and let \( d \) be the maximum degree of the monomials in \( \mathcal{M} \). It follows from the discussion above that for each \( \theta_{ij}^L t^L dt_i dt_j \) in \( \mathcal{M} \) we have

\[
\deg \Theta_{ij}^L \leq |L| \leq d.
\]

By Remark 13, we can take a finite set \( \Lambda \) in \( \mathbb{Z}^M \) with the following property:

for each monomial \( \theta_{ij}^L t^L dt_i dt_j \) in \( \mathcal{M} \), there exist \( \mathbb{C} \)-linearly independent \( i \)-elementary vectors \( m_1, \ldots, m_{M-1} \) in \( \Lambda \) such that

\[
m_1 \cdot \mathcal{L}_{ij} = \cdots = m_{M-1} \cdot \mathcal{L}_{ij} = 0,
\]

where, as defined in (2.6), we have

\[
\mathcal{L}_{ij} = \mathcal{L} + e_i + e_j.
\]

Take a finite set \( T \subset \mathbb{C} \) and a function \( \lambda : T \rightarrow \Lambda \) such that

for each \( m \in \Lambda \) the set \( \lambda^{-1}(m) \) has at least \( d + 1 \) points.
Let $G$ be the foliation given by Proposition 18 associated to $\lambda : T \to \mathbb{C}^M$ and $k = d + 1$. The foliation $F = \pi_\ast(G)$ is generated by a polynomial vector field $X \in \mathbb{C}^M$ and $k = d + 1$.

The foliation $F = \pi_\ast(G)$ is generated by a polynomial vector field $X$ with an isolated singularity at the origin. The vector field $Z$ mentioned in Theorem 4 will be taken being equal to the lift $X_{D}$ of $X$ to $D$, so the foliation defined by $Z$ is the lift $F_D$ of $F$ to $D$.

Without loss of generality we can suppose that $\Delta = \Delta_1 \times \Delta_2$, where $\Delta_1$ and $\Delta_2$ are balls centered at the origins of $\mathbb{C}^M$ and $\mathbb{C}^N$, respectively. We also can assume that the closure of $\Delta$ is contained in the domains of definition of $D$ and the first integral $H_D$—see (1.4). Let $\hat{r} > 0$ be such that $|\omega| \leq \hat{r}$ on $\Delta_1$. Let $r > 0$ be the radius of $\Delta_2$ and take $\epsilon > 0$ such that

$$6\hat{r}\epsilon < r. \quad (4.10)$$

Associated to the number $\epsilon$, we have an open set $\mathcal{U} \subset \Delta_1$ in $\mathbb{C}^M$ as given by (5) of Proposition 18. Set

$$\Delta' = \mathcal{U} \times \left\{ z \in \mathbb{C}^N : |z| < \frac{r}{2} \right\} \subset \Delta.$$

Given $p \in \Delta_1$, we set

$$\Delta_p = \{ p \} \times \Delta_2$$

and

$$\Delta'_p = \left\{ (p, z) : |z| < \frac{r}{2} \right\} \subset \Delta_p.$$

We clearly have $\Delta' = \bigcup_{p \in \mathcal{U}} \Delta'_p$. Let $L$ be a leaf of $\mathcal{F}_D|_\Delta$ passing through a point in $\Delta'$. Let $c \in \mathbb{C}^\kappa$ be the value taken by $H_D$ on $L$ and set

$$H_L = (H_D|_\Delta)^{-1}(c).$$

That is, $H_L$ is the level of $H_D|_\Delta$ containing $L$. We will show the following two assertions.

(I) If $L$ meets $\Delta'_p$ for some $p \in \mathcal{U}$, then $L \cap \Delta'_p$ is a dense subset of $H_L \cap \Delta'_p$.

(II) If $L$ contains a point $(p, z)$ with $p \in \mathcal{U}$ and $|z| < \frac{r}{3}$, then $L$ meets $\Delta'_q$ for every $q$ in a dense subset of $\mathcal{U}$.

**Claim** Assertions (I) and (II) imply Theorem 4.

**Proof** Consider

$$\Delta'' = \mathcal{U} \times \left\{ z \in \mathbb{C}^N : |z| < \frac{r}{3} \right\} \subset \Delta'$$

and define $\Delta^*$ as the saturation of $\Delta''$ by the foliation $\mathcal{F}_D|_\Delta$. Let $L$ be a leaf of $\mathcal{F}_D|_{\Delta^*}$. It follows from the definition of $\Delta^*$ that $L$ is a leaf of $\mathcal{F}_D|_\Delta$ passing through a point
in $\Delta''$. Then, by Assertion (II) we have that $\mathcal{L}$ meets $\Delta'_q$ for all $q$ in a set $E$ that is dense in $\mathcal{U}$. Therefore, by Assertion (I), the closure of $\mathcal{L}$ contains the set
\[
H_{\mathcal{L}} \cap \Delta'_q
\]
for all $q \in E$. Let us prove that $\overline{\mathcal{L}}$ contains the set
\[
H_{\mathcal{L}} \cap \Delta'_p
\]
for all $p \in \mathcal{U}$. Take any $p \in \mathcal{U}$ and consider a point
\[
(p, \bar{z}) \in H_{\mathcal{L}} \cap \Delta'_p.
\]
Since $H_D$ is expressed—see (1.4)—in the form $H_D = A(z) - g(x)$, the levels of $H_D$ are transverse to the fibers $x = cst$. In particular, the manifold $H_{\mathcal{L}}$ is transverse to the fibers $x = cst$. Thus, in a neighborhood of $(p, \bar{z})$ in $H_{\mathcal{L}}$, the sets $H_{\mathcal{L}} \cap \Delta'_q$, $q \in \mathcal{U}$ defines a locally trivial fibration, and therefore we can find a sequence of points $(q_n, \bar{z}_n) \in H_{\mathcal{L}}$, $n \in \mathbb{N}$ with $q_n \in E$, such that $(q_n, \bar{z}_n) \to (p, \bar{z})$. Since—as we have seen above—the closure of $\mathcal{L}$ contains the sets $H_{\mathcal{L}} \cap \Delta'_{q_n}$, we have that $\overline{\mathcal{L}}$ contains the point $(q_n, \bar{z}_n)$ for each $n \in \mathbb{N}$ and therefore $\overline{\mathcal{L}}$ contains the point $(p, \bar{z})$. Thus, we have that
\[
\overline{\mathcal{L}} \supset \bigcup_{p \in \mathcal{U}} H_{\mathcal{L}} \cap \Delta'_p = H_{\mathcal{L}} \cap \Delta' \supset H_{\mathcal{L}} \cap \Delta''.
\]
(4.11)

Now, since $\Delta^*$ is the saturation of $\Delta''$ we have that $\overline{\mathcal{L}}$ contains the set $H_{\mathcal{L}} \cap \Delta^*$. Finally, let us prove the last assertion of Theorem 4. Suppose that $M = 2$. Then the foliation $\mathcal{F}$ given by Proposition 18 is a foliation on $\Delta_1 \subset \mathbb{C}^2$. Given any $\tau \in T$, we have $\lambda(\tau) = (m_1, m_2) \in \Lambda$ and so, from the definition of $\Lambda$, we deduce that $m_1/m_2$ is a rational negative number. It follows from (5c) of Proposition 18 that $\mathcal{U} \cup S$ is a neighborhood of $0 \in \mathbb{C}^2$. Then
\[
B := (\mathcal{U} \cup S) \times \left\{ z \in \mathbb{C}^N : |z| < \frac{r}{3} \right\}
\]
is a neighborhood of $0 \in \mathbb{C}^{2+N}$. Thus, if we set $\mathcal{S} := S \times \Delta_2$, we can see that
\[
B \setminus \mathcal{S} = \mathcal{U} \times \left\{ z \in \mathbb{C}^N : |z| < \frac{r}{3} \right\} = \Delta'' \subset \Delta^*.
\]
Thus, Theorem 4 is reduced to the proof of assertions (I) and (II). \qed

\[\square\]}
Proof of Assertion (I)

Suppose that $L$ pass through a point $(p, \zeta)$ with $p \in \mathcal{U}$ and $|\zeta| < r$. Recall—see (1.4)—that

$$H_D(x, z) = A(z) - g(x).$$

Then we have

$$H_L \cap \Delta_p = \{(p, z): |z| < r, A(z) - g(p) = A(\zeta) - g(p)\}$$

$$= \{(p, z): |z| < r, A(z) = A(\zeta)\}$$

$$= \{(p, \zeta + w): |\zeta + w| < r, w \in \mathcal{W}_D\}.$$

Thus, since $L \cap \Delta_p \subset H_L \cap \Delta_p$, there exists a set $\Sigma \subset \mathcal{W}_D$ such that

$$L \cap \Delta_p = \{(p, \zeta + w): w \in \Sigma\}. \quad (4.12)$$

We clearly have $\Sigma \neq \emptyset$, because $0 \in \Sigma$. Let $\Omega \subset \mathcal{W}_D$ be defined by the equality

$$H_L \cap \Delta'_p = \{(p, \zeta + w): w \in \Omega\}. \quad (4.13)$$

It is sufficient to prove that $\Omega \subset \overline{\Sigma}$. This fact will be a consequence of Lemma 17, so we only have to verify the hypotheses of this lemma.

We begin with the definition of the set $\Gamma$. Let $E$ be the set in the leaf $L_p$ of $\mathcal{F}|_{\Delta_1}$ through $p$ as given by (5a) of Proposition 18. Thus, given $q \in E$ there exists a curve $\beta = \beta_q$ in $L_p$, connecting $p$ with $q$ and such that $\ell(\beta) < \epsilon$. Fix $\tau$ in the set

$$T^* := \{\tau \in T: \lambda(\tau) \in \Lambda\}.$$

Set

$$p_\tau = (0, \ldots, 0, \tau) \in \mathbb{C}^M$$

and let $f_\tau = p_\tau + h_\tau$ be as given by (2) of Proposition 18. By (5a) of Proposition 18 the set $E$ is dense in $\mathcal{U}$. Moreover, from (5b) of Proposition 18 we deduce that $\pi^{-1}(\mathcal{U})$ has $p_\tau$ as a limit point. Then $\tilde{q} := \pi^{-1}(q)$ can be chosen close to $p_\tau$ so that $\tilde{q}$ belongs to the image $U_\tau$ of $f_\tau$. Then $\tilde{q} = f_\tau(\tilde{s})$, where $\tilde{s} \in \mathbb{C}^M$ is close to the origin. By (5) of Proposition 18 the set $\mathcal{U}$ is disjoint from the hypersurface $S$, so that $p \notin S$ and therefore $q \notin S$. Thus, it follows from (4) of Proposition 18 that $\tilde{s} \in (\mathbb{C}^*)^M$. Let

$$\lambda(\tau) = m = (m_1, \ldots, m_M) \in \Lambda$$
and let $\alpha = \alpha_q$ be the elementary $m$-loop based at $\tilde{s}$. As we have seen in Sect. 2, the
loop $\alpha$ is contained in a leaf of the system
\[
s'_1 = m_1 s_1 \\
\vdots \\
s'_M = m_M s_M.
\]
By (2) of Proposition 18, the system above coincides with the pullback foliation $f_\tau^*(G)$
in a neighborhood of the origin. Then $f_\tau \circ \alpha$ is a loop based at $\tilde{q}$ and contained in
the leaf of $\tilde{G}$ through $\tilde{q}$. Therefore, provided $\tilde{s}$ is small enough, $\pi \circ f_\tau \circ \alpha$ is a loop based
at $q$ and contained in the leaf of $\mathcal{F}|_{\Delta}$ through $q$, which coincides with the leaf $L_p$ of
$\mathcal{F}|_{\Delta}$ through $p$. Consider the loop
\[
y_q = \beta^{-1} \ast (\pi \circ f_\tau \circ \alpha) \ast \beta,
\]
where the symbol $\ast$ stands for the concatenation of paths. This loop is based at $p$ and
contained in $L_p$. We can assume that $\tilde{q}$ is close enough to $p_\tau$ such that the length of
$\pi \circ f_\tau \circ \alpha$ is less than $\epsilon$, so that the length of $y_q$ is less than $3\epsilon$. The inverse path $y_q^{-1}$
is also a loop based at $p$ of length smaller that $3\epsilon$. Consider the set
\[
\Gamma = \left\{ \int \gamma \omega : \gamma \in \{y_q, y_q^{-1}\}, q \in E, \tilde{q} \in U_\tau, \tau \in T^* \right\}.
\]
**Verification of the hypotheses of Lemma 17.** By (4.10) we can take
\[
\delta = \frac{r}{2} - 3\epsilon > 0.
\]
We start with the verification of the hypothesis (3.1) of Lemma 17. Consider $w \in \Sigma$
with $\text{dist}(w, \Omega) < \delta$ and $u \in \Gamma$—recall that $\Sigma$ and $\Omega$ are defined in (4.12) and (4.13). Then
\[
u = \int \gamma \omega,
\]
where $\gamma \in \{y_q, y_q^{-1}\}, q \in E, \tilde{q} \in U_\tau$ and $\tau \in T^*$. Take $w' \in \Omega$ such that $|w - w'| < \delta$.
Since $w' \in \Omega$ we have $(p, \tilde{z} + w') \in \Delta'_p$, hence
\[
|\tilde{z} + w'| < \frac{r}{2}
\]
and therefore
\[
|\tilde{z} + w| \leq |\tilde{z} + w'| + |w - w'| < \frac{r}{2} + \delta.
\]

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Then

$$|\tilde{z} + w| + \mathcal{R}\ell(\gamma) < \frac{r}{2} + \delta + \mathcal{R}(3\epsilon) = r,$$

so it follows from Lemma 9 that the lifting $\tilde{\gamma}$ of $\gamma$ to $\mathcal{D}$ starting at $(p, \tilde{z} + w)$ is well defined, it is contained in $\Delta$ and its ending point has the form

$$\left(p, \tilde{z} + w + \int_{\gamma} \omega\right) = (p, \tilde{z} + w + u).$$

Then the point $(p, \tilde{z} + w + u)$ belongs to the leaf of $\mathcal{F}_D|_{\Delta}$ through $(p, \tilde{z} + w)$. Observe that this leaf through $(p, \tilde{z} + w)$ coincides with the leaf $\mathcal{L}$ through $(p, \tilde{z})$, because $w \in \Sigma$. Then $(p, \tilde{z} + w + u) \in \mathcal{L}$ and therefore $w + u \in \Sigma$. Hypothesis (3.1) is verified.

It remains to prove the condition on $\Gamma$ required by Lemma 17. Since $0 \in \Sigma$, from Hypothesis (3.1) proved above we have that $0 + u \in \Sigma$ for all $u \in \Gamma$. Then $\Gamma \subset \Sigma$ and, in particular,

$$\Gamma \subset W_D. \quad (4.14)$$

Recall that, given $\tau \in T^*\Sigma$ with $\lambda(\tau) = m$ and $q \in E$ with $\tilde{q} f_\tau(\tilde{s}) \in U_\tau$,

$$\pm \int_{\tilde{q}} \omega \in \Gamma.$$

But

$$\int_{\tilde{q}} \omega = \int_{\beta^{-1} \circ \pi \circ f_\tau \circ \alpha \circ \beta} \omega = \int_{\pi \circ f_\tau \circ \alpha} \omega = \int_{f_\tau \circ \alpha} \tilde{\omega} = \int_{h_\tau \circ \alpha} \tilde{\omega}_\tau = \int_{\alpha} h_\tau^*(\tilde{\omega}_\tau) = \rho_\tau(\tilde{s}),$$

where

$$\rho_\tau : = \rho(h_\tau^*(\tilde{\omega}_\tau), m) \quad (4.15)$$

is as given by (4) of Proposition 12. Then

$$\tau \in T^*\Sigma, q \in E, \tilde{q} = f_\tau(\tilde{s}) \implies \pm \rho_\tau(\tilde{s}) \in \Gamma. \quad (4.16)$$

Since $p_\tau$ is a limit point of $\pi^{-1}(U)$,

$$U_\tau : = f_\tau^{-1}\left(\pi^{-1}(U) \cap U_\tau\right).$$
is a nonempty open subset of $\mathbb{C}^M$ accumulating at the origin. Recall that $q$ can take any value in the dense subset $E$ of $U$. Then $\tilde{q}$ can be arbitrarily chosen in a dense subset of $\pi^{-1}(U) \cap U_\tau$, so that, in view of (4.16), we find a dense subset $C_\tau \subset U_\tau$ such that

$$\pm \rho_\tau(\tilde{s}) \in \Gamma, \ \tilde{s} \in C_\tau.$$  \hfill (4.17)

Then, since $C_\tau$ is dense in $U_\tau$ and $\Gamma \subset \mathcal{W}_D$, we have

$$\pm \rho_\tau(\tilde{s}) \in \mathcal{W}_D, \ \tilde{s} \in U_\tau.$$  \hfill (4.18)

Thus, the set

$$\{ \rho_\tau(\tilde{s}): \tau \in T^*, \tilde{s} \in U_\tau \}$$  \hfill (4.19)

is contained in $\mathcal{W}_D$. Let $W \subset \mathcal{W}_D$ be the complex vector space spanned by the set in (4.19). Given $\epsilon > 0$, by Lemma 16 we can find a real $\epsilon$-basis of $W$ of the form

$$\rho_{\tau_1}(c_1), \ldots, \rho_{\tau_n}(c_n),$$

where $\tau_1, \ldots, \tau_n \in T^*$ and $c_1 \in U_{\tau_1}, \ldots, c_n \in U_{\tau_n}$. Since each $C_\tau$ is dense in $U_\tau$, we can find $c'_1 \in C_{\tau_1}, \ldots, c'_n \in C_{\tau_n}$ such that

$$\rho_{\tau_1}(c'_1), \ldots, \rho_{\tau_n}(c'_n)$$

is a real $\epsilon$-basis of $W$. Since each $\pm \rho_{\tau_j}(c'_j)$ belongs to $\Gamma$—see (4.17), we conclude that

$$\{ \pm \rho_{\tau_1}(c'_1), \ldots, \pm \rho_{\tau_n}(c'_n) \} \subset \Gamma.$$  

Therefore, in order to complete the verification of the condition on $\Gamma$ required by Lemma 17, it suffices to show that $\mathcal{W}_D = W$. With this objective in mind, since $\mathcal{W}_D \supset W$, it is enough to prove that the coefficients of the monomials in $\mathcal{M}$—they span $\mathcal{W}_D$—are all contained in $W$. Let $\theta_{ij}^{L} t^L dt_i dt_j$ be a monomial in $\mathcal{M}$. Since—see (4.5)—we have $\theta_{ij}^{L} = \Theta_{ij}^{L}(0)$, it suffices to show that $\Theta_{ij}^{L}(\tau)$ belongs to $W$ for all $\tau \in \mathcal{C}$. By condition (4.8), there exist $\mathcal{C}$-linearly independent $i$-elementary vectors $m_1, \ldots, m_{M-1}$ in $\Lambda$ such that

$$m_1 \cdot \xi_{ij} = \cdots = m_{M-1} \cdot \xi_{ij} = 0.$$  

Observe that the monomial $s^L ds_i ds_j$ belongs to the set $\mathcal{M}(\xi_{ij}, i)$ as defined in (2.9)—with $s$ instead of $x$. Then, by (3) of Proposition 12, the coefficient $\Theta_{ij}^{L}(\tau)$ of $s^L ds_i ds_j$ is a linear combination of the coefficients

$$a_{\xi_{ij}}(\hat{\omega}_\tau, m_1), \ldots, a_{\xi_{ij}}(\hat{\omega}_\tau, m_{M-1})$$
of $y^{L_{ij}}$ in the series

$$\rho(\tilde{\omega}_\tau, m_1), \ldots, \rho(\tilde{\omega}_\tau, m_{M-1}).$$

Then it suffices to show that the coefficients

$$a_{L_{ij}}(\tilde{\omega}_\tau, m_1), \ldots, a_{L_{ij}}(\tilde{\omega}_\tau, m_{M-1})$$

belong to $W$ for all $\tau \in \mathbb{C}$.

We continue the argumentation with the coefficient $a_{L_{ij}}(\tilde{\omega}_\tau, m_1)$—the proof is exactly the same for the other cases. By (2) of Proposition 12, we have that $a_{L_{ij}}(\tilde{\omega}_\tau, m_1)$ is a linear combination of the coefficients of the monomials in $M(L_{ij}, i)$. Observe that each monomial $\Theta^j_L(\tau)s^Lds_1ds_j$ with $s^Lds_1ds_j$ in $M(L_{ij}, i)$ is such that—see (4.7)

$$|L| = |L_{ij}| - 2 = |L| \leq d.$$

Then—see (4.6)—the coefficients of the monomials in $M(L_{ij}, i)$ have degree $\leq d$ as functions of $\tau$, and therefore $a_{L_{ij}}(\tilde{\omega}_\tau, m_1)$ has degree $\leq d$ as a polynomial function of $\tau$. Thus, by Lemma 14 and condition (4.9), it is enough to prove that $a_{L_{ij}}(\tilde{\omega}_\tau, m_1)$ belongs to $W$ for all $\tau$ in $\lambda^{-1}(m_1)$. Consider any $\tau \in T$ such that $\lambda(\tau) = m_1$.

By (2) of Proposition 18, the diffeomorphism $h_\tau$ is $(d+1)$-tangent to the identity, hence the expansion series of $d\tilde{\omega}_\tau$ and $dh^*_\tau(\tilde{\omega}_\tau) = h^*_\tau(d\tilde{\omega}_\tau)$ coincides up to order $d$. Then, by (4) of Proposition 12, we have that the series $\rho(\tilde{\omega}_\tau, m_1)$ and $\rho(h^*_\tau(\tilde{\omega}_\tau), m_1)$ coincides up to order $d+2$. Thus, since $|L_{ij}| = |L| + 2 \leq d + 2$, we conclude that the coefficient $a_{L_{ij}}(\tilde{\omega}_\tau, m_1)$ coincides with the coefficient $a_{L_{ij}}(h^*_\tau(\tilde{\omega}_\tau), m_1)$ of the series

$$\rho(h^*_\tau(\tilde{\omega}_\tau), m_1),$$

which is equal to the series $\rho_\tau$ defined in (4.15). Since—from the definition of $W$—we have that $\rho_\tau(s)$ belongs to $W$ for all $s$ in an open set that accumulates in the origin in $\mathbb{C}^N$, by Lemma 15 the coefficients of $\rho_\tau(s)$ belong to $W$, so that $a_{L_{ij}}(\tilde{\omega}_\tau, m_1)$ belongs to $W$. $\Box$

**Proof of Assertion (II)**

By hypothesis, $L$ contains a point $(p, z)$ with $p \in U$ and $|z| = r/3$. Let $E$ be the set in the leaf $L_p$ of $\mathcal{F}|_{\Delta_1}$ through $p$ given by (5) of Proposition 18. It is enough to prove that $L$ meets $\Delta'_q$ for all $q \in E$. Fix $q \in E$. By (5) of Proposition 18 there exists a path $\gamma$ in $L_p$ connecting $p$ with $q$ such that $\ell(\gamma) < \epsilon$. From (4.10) we obtain

$$|z| + \Re\ell(\gamma) < \frac{r}{3} + \Re\epsilon < \frac{r}{2}.$$
Therefore, it follows from Lemma 9 that the lifting \( \tilde{\gamma} \) of \( \gamma \) to \( D \) starting at \( (p, z) \) is contained in
\[
\left\{(x, z) \in \Delta : |z| < \frac{r}{2}\right\}.
\]
Then the ending point of \( \tilde{\gamma} \) belongs to the intersection of \( L \) with \( \Delta' \). The proof of Theorem 4 is complete. \( \square \)

**Remark 19** In the proof of Theorem 4, given \( \Delta = \Delta_1 \times \Delta_2 \), where \( \Delta_1 \) and \( \Delta_2 \) are balls centered at the origins of \( \mathbb{C}^M \) and \( \mathbb{C}^N \), such that \( \Delta' \) is contained in the domains of definition of \( D \) and \( H_D \), we have constructed the set \( \Delta^* \) containing a set \( \Delta'' \) of the form
\[
\Delta'' = U \times \left\{ z \in \mathbb{C}^N : |z| < \frac{r}{3}\right\},
\]
where we have the following facts.

1. The number \( r \) is the radius of the ball \( \Delta_2 \).
2. The set \( U \subset \Delta_1 \) is given by (5) of Proposition 18; it depends on the foliation \( F \), the ball \( \Delta_1 \) and the election—see (4.10)—of a positive number \( \epsilon > 0 \) such that \( 6\mathfrak{R}\epsilon < r \)—recall that \( \mathfrak{R} \) is a bound for \( |\omega| \) on \( \Delta_1 \).

Suppose that \( \Delta_1 \) is fixed. Since \( \omega \) depends only on \( (x_1, \ldots, x_M) \in \mathbb{C}^M \), we can take \( \Delta_2 \) having any radius \( r > 0 \). If we suppose \( r > 6\mathfrak{R} \), we can choose \( \epsilon = 1 \) and therefore the set \( U \) can be considered fixed and independent of \( r \). This fact will be used in the proof of Theorem 6.

**Proof of Theorem 1**

Let \( F \) be a germ of meromorphic first integral of \( D \) in \( (\mathbb{C}^{M+N}, 0) \). Let \( Z \) be as given by Theorem 4. Take a neighborhood \( \Delta \) of the origin in \( \mathbb{C}^{M+N} \) where \( F \) and \( Z \) are defined and let \( \Delta^* \subset \Delta \) be the corresponding set as given by Theorem 4. Let \( p \in \Delta \setminus \mathcal{S} \) outside the indeterminacy set of \( F \) and let \( F(p) = c \). Since \( F \) is a meromorphic first integral, \( F \) takes the value \( c \) along the leaf \( \mathcal{L} \) of \( Z|_\Delta \) through \( p \). Then, since Theorem 4 guarantees that \( (H_D|_{\Delta^*})^{-1}(c) \subset \mathcal{L} \), we have that \( F \) is constant on
\[
(H_D|_{\Delta^*})^{-1}(c).
\]
Since the point \( p \) can be arbitrarily chosen on an open set accumulating the origin of \( \mathbb{C}^{M+N} \), we conclude that \( F \)—whenever it is defined—is constant along the levels of \( H_D|_{\Delta^*} \). Since
\[
H_D : (\mathbb{C}^{M+N}, 0) \to (\mathbb{C}^\kappa, 0)
\]
is a submersion, there exists a neighborhood \( U \) of the origin in \( \mathbb{C}^\kappa \) and a holomorphic embedding
\[
g : U \to \Delta^*
\]
\( \square \) Springer
such that $H_D \circ g = \text{id}$. Consider the meromorphic function

$$f := F \circ g : U \to \mathbb{C}$$

We must show that $F = f \circ H_D$. Take $\xi \in \Delta^*$ such that $H_D(\xi) \in U$. Then

$$f \circ H_D(\xi) = f(\xi) = F(g \circ H_D(\xi)). \quad (4.20)$$

Since

$$H_D(g \circ H_D(\xi)) = H_D \circ g(H_D(\xi)) = H_D(\xi),$$

the points $g \circ H_D(\xi)$ and $\xi$—both belonging to $\Delta^*$—are contained in the same level of $H_D$, so $F$ takes the same value at these two points. Therefore

$$F(g \circ H_D(\xi)) = F(\xi)$$

and by (4.20) we obtain

$$F(\xi) = f \circ H_D(\xi).$$

The converse assertion of Theorem 1 is trivial, so the proof is complete. \qed

**Proof of Theorem 6**

Item (1) follows directly from (1) of Proposition 5, so we only prove Item (2). Suppose that $W_D = \mathbb{C}^N$. From the proof of Theorem 4, the holomorphic foliation defined by $Z$ is the lifting $\mathcal{F}^D$ of a holomorphic foliation $\mathcal{F}$, which is given by Proposition 18. Let $\mathcal{L}$ be a leaf of $\mathcal{F}^D$ through a point

$$(x, z) \in \mathbb{C}^{M+N}, \quad x \notin S \cup \omega_{\infty}$$

and let $U \subset \mathbb{C}^{M+N}$ be any open set. We must show that $\mathcal{L}$ meets $U$. Then, if we take

$$(x', z') \in U, \quad x' \notin S \cup \omega_{\infty},$$

it is enough to show that $\overline{\mathcal{L}}$ contains the leaf $\mathcal{L}'$ of $\mathcal{F}^D$ through the point $(x', z')$. Let $\Delta_1$ be a ball centered at the origin of $\mathbb{C}^M$ and disjoint of the pole set $(\omega)_{\infty}$. Let $\mathcal{U} \subset \Delta_1$ be the set as given by (5) of Proposition 18 for $\Delta = \Delta_1$ and $\epsilon = 1$. By Proposition 18, the leaf $L$ of $\mathcal{F}$ through the point $x$ is dense in $\mathbb{C}^M$. Then we can take a path $\gamma$ in $L$ connecting $x$ with a point $p \in \mathcal{U}$. Since $L$ meets $\Delta_1$ and $\Delta_1$ is disjoint of $(\omega)_{\infty}$, the leaf $L$ is not contained in $(\omega)_{\infty}$. Then, since $\dim L = 1$, the points of $(\omega)_{\infty}$ that belong to $L$ conform a discrete set in the intrinsic topology of $L$, hence we can assume
that the path $\gamma$ is disjoint of $(\omega)_{\infty}$. Thus, the lifting of $\gamma$ to $D$ starting at $(x, z)$ is well defined and its ending point belongs to the set

$$\Sigma_p := \{p \} \times \mathbb{C}^N.$$ 

We have proved that $\mathcal{L}$ contains some point $\zeta \in \Sigma_p$ for some $p \in \mathcal{U}$. Exactly the same arguments show that $\mathcal{L}'$ contains some point $\zeta' \in \Sigma_{p'}$ for some $p' \in \mathcal{U}$. As in the proof of Theorem 4, let $k$ be a bound for $|\omega|$ on $\Delta_1$. Take $r > 6k$ such that $\zeta$ and $\zeta'$ are contained in the set

$$\Delta'' = \mathcal{U} \times \left\{ z \in \mathbb{C}^N : |z| < \frac{r}{3} \right\}.$$ 

Set

$$\Delta = \Delta_1 \times \{ z \in \mathbb{C}^N : |z| < r \}$$

and let $\Delta^* \subset \Delta$ be as given by Theorem 4. By Remark 19 we have $\Delta'' \subset \Delta^*$. Then, since $W_D = \mathbb{C}^N$, the leaf of $\mathcal{F}_D|_{\Delta^*}$ passing through $\zeta$ is dense in $\Delta^*$. It follows that $\mathcal{L}$ is dense in $\Delta''$, so $\overline{L}$ contains $\zeta' \in \Delta''$ and therefore $\overline{L}$ contains $\mathcal{L}'$. \hfill \Box

**Proof of Corollary 2**

Since $\omega$ is non-integrable, then the series

$$d\omega = \sum c^{ij}_k dx_i dx_j$$

is not null. Therefore the linear subspace of $\mathbb{C}$ generated by the coefficients of $d\omega$ is the total space $\mathbb{C}$, that is, $W_D = \mathbb{C}$. Then the result follows directly from Theorem 6. \hfill \Box

**Proof of Theorem 7**

Let $Z$, $X$ and $S$ be as given by Theorem 4 and Proposition 5 for the distribution

$$dz = ydx - xdy$$

on $\mathbb{C}^3$. By Theorem 6, the leaves of $Z$ outside

$$\tilde{S} : = \{(x, y, z) : (x, y) \in S\}$$

are dense in $\mathbb{C}^3$. The vector field $Z$ is the lifting of a polynomial vector field

$$X = A(x, y) \frac{\partial}{\partial x} + B(x, y) \frac{\partial}{\partial y}$$

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Distributions, First Integrals and Legendrian Foliations

Recall that, in the coordinates $(x_1, y_1, \ldots, x_m, y_m, z) \in \mathbb{C}^{2m+1}$, the distribution $\xi$ is defined by the equation

$$dz = (y_1 dx_1 - x_1 dy_1) + \cdots + (y_m dx_m - x_m dy_m). \quad (4.22)$$

For each $j = 1, \ldots, m$ define the vector field $X_j$ on $\mathbb{C}^{2m}$ by the expression

$$X_j = A(x_j, y_j) \frac{\partial}{\partial x_j} + B(x_j, y_j) \frac{\partial}{\partial y_j}.$$

Let $Z_j$ be the lifting to $\xi$ of the vector field $X_j$, that is,

$$Z_j = A(x_j, y_j) \frac{\partial}{\partial x_j} + B(x_j, y_j) \frac{\partial}{\partial y_j} + \left[ y_j A(x_j, y_j) - x_j B(x_j, y_j) \right] \frac{\partial}{\partial z}.$$

It is easy to see that the vector fields $Z_j$ are pairwise commutative and generically linearly independent. Then the $Z_j$ define an $m$-dimensional singular holomorphic foliation $\mathcal{F}_m$ on $\mathbb{C}^{2m+1}$, which is clearly Legendrian for $\xi$. From (2) of Proposition 5, there is a polynomial $P \in \mathbb{C}[x, y]$ such that

$$\tilde{S} = \{(x, y, z): P(x, y) = 0\}.$$

Define the hypersurface $S_m \subset \mathbb{C}^{2m+1}$ by the equation

$$P(x_1, y_1) \cdots P(x_m, y_m) = 0.$$

Observe that $S_m$ is invariant by the vector fields $Z_j$, so that $S_m$ is invariant by $\mathcal{F}_m$. We will prove that the leaves of $\mathcal{F}_m$ outside $S_m$ are dense in $\mathbb{C}^{2m+1}$. We prove this property by induction on $m$. The property is clearly true for $m = 1$; in this case $\mathcal{F}_1$ is equivalent to the foliation defined by the vector field $Z$. Suppose that the property holds for some $m \geq 1$. Let $\mathcal{F}_m'$ be the foliation on the space

$$(x_1, y_1, \ldots, x_{m+1}, y_{m+1}, z)$$

generated by the vector fields

$$Z_j = A(x_j, y_j) \frac{\partial}{\partial x_j} + B(x_j, y_j) \frac{\partial}{\partial y_j} + \left[ y_j A(x_j, y_j) - x_j B(x_j, y_j) \right] \frac{\partial}{\partial z},$$

for $j = 1, \ldots, m$—we are considering the $Z_j$ as vector fields on $\mathbb{C}^{2m+3}$. Since these vector fields does not depend on the variables $(x_{m+1}, y_{m+1})$, they leave invariant each codimension 2 plane

$$\Sigma(a, b): x_{m+1} = a, \ y_{m+1} = b,$$
where \((a, b) \in \mathbb{C}^2\). Moreover, the foliation generated by the vector fields \(Z_1, \ldots, Z_m\) on each \(\Sigma(a, b)\) is equivalent to the foliation \(\mathcal{F}_m\) via the map

\[(x_1, y_1, \ldots, x_m, y_m, a, b, z) \mapsto (x_1, y_1, \ldots, x_m, y_m, z).\]

Let \(\mathcal{L}\) be a leaf of \(\mathcal{F}_{m+1}\) outside \(S_{m+1}\). Since \(\mathcal{F}_{m+1}\) is generated by the vector fields \(Z_1, \ldots, Z_m\)—which generate \(\mathcal{F}_m'\)—together with the vector field \(Z_{m+1}\), we have that the intersection of \(\mathcal{L}\) with a plane \(\Sigma(a, b)\)—if this intersection is not empty—is an invariant set of \(\mathcal{F}_m\). The property \(\mathcal{L} \cap S_{m+1} = \emptyset\) means that the points in \(\mathcal{L}\) do not satisfy the equation

\[P(x_1, y_1) \cdots P(x_m+1, y_{m+1}) = 0. \tag{4.23}\]

Then the points in \(\mathcal{L}\) do not satisfy the equation

\[P(x_1, y_1) \cdots P(x_m, y_m) = 0\]

and, in particular, the points in \(\mathcal{L} \cap \Sigma(a, b)\) do not belong to \(S_m\). Therefore, we can apply the inductive hypothesis to any leaf of \(\mathcal{F}_m\) contained in \(\mathcal{L} \cap \Sigma(a, b)\) to conclude that \(\mathcal{L} \cap \Sigma(a, b)\) is dense in \(\Sigma(a, b)\). Then, to guarantee the density of \(\mathcal{L}\) it suffices to show that \(\mathcal{L}\) meets \(\Sigma(a, b)\) for a dense set of points \((a, b)\) in \(\mathbb{C}^2\), that is, we must prove that the image of \(\mathcal{L}\) by the projection

\[(x_1, y_1, \ldots, x_m, y_m, x_{m+1}, y_{m+1}, z) \mapsto (x_{m+1}, y_{m+1})\]

is dense in \(\mathbb{C}^2\). Fix a point \(p \in \mathcal{L}\). This point belongs to some three-dimensional plane

\[\Sigma: x_1 = a_1, y_1 = b_1, \ldots, x_m = a_m, y_m = b_m,\]

for some \(a_1, b_1, \ldots, a_m, b_m \in \mathbb{C}\). The vector field \(Z_{m+1}\) leave the plane \(\Sigma\) invariant and the restriction \(Z_{m+1}|\Sigma\) is equivalent to the vector field \(Z\) on \(\mathbb{C}^3\) via the map

\[(a_1, b_1, \ldots, a_m, b_m, x, y, z) \mapsto (x, y, z).\]

Then the leaf \(\mathcal{L}\) of \(Z_{m+1}|\Sigma\) through \(p\) can be considered as a leaf of \(Z\). Since \(p\) does not satisfy Eq. (4.23), we have that \(p\) does not satisfy

\[P(x_{m+1}, y_{m+1}) = 0.\]

This means that, viewed as a point in \(\mathbb{C}^3\), the point \(p\) is not contained in the set \(\tilde{S}\). Therefore—by the density property of \(Z\)—the leaf \(\mathcal{L}\) is dense in \(\mathbb{C}^3\), so that image of \(\mathcal{L}\) by the projection

\[(x, y, z) \mapsto (x, y)\]
is dense in $\mathbb{C}^2$. Since $\mathcal{L}$ is contained in $\mathcal{L}$, we conclude that the image of $\mathcal{L}$ by the projection

$$(x_1, y_1, \ldots, x_m, y_m, x_{m+1}, y_{m+1}, z) \mapsto (x_{m+1}, y_{m+1})$$

is dense in $\mathbb{C}^2$. \qed

## 5 Final Remarks

Consider the distribution $\mathcal{D}$ defined by (1.1) and let

$$H_D : (\mathbb{C}^{N+M}, 0) \to (\mathbb{C}^k, 0)$$

be the associated holomorphic submersion defined in (1.4). Denote by $\mathcal{F}(H_D)$ the regular foliation defined by the levels of $H_D$. The following statement is a reinterpretation of Theorem 1.

**Corollary 3** The field of meromorphic first integrals of $\mathcal{D}$ and that of $\mathcal{F}(H_D)$ are the same.

**Remark 20** If $H_D$ is constant, that is, if $\mathcal{D}$ has only constant first integrals, then the distribution is completely non-integrable. In fact, consider the vector fields $X_1, \ldots, X_M$, obtained by lifting to $\mathcal{D}$ the canonical basis on $\mathbb{C}^M$. Set $Y_{ij} := [X_i, X_j], i < j$. A simple computation shows that

$$Y_{ij} = \left(0, \sum_{K} c_{K}^{ij} x^K \right).$$

In particular, we have

1. $[Y_{ij}, Y_{kl}] = 0$ for all $i < j, k < l$;
2. $[X_l, Y_{ij}] = \left(0, \partial_l(\sum_{K} c_{K}^{ij} x^K)\right).$

For $i, j$ and $K_0 = (k_1, \ldots, k_M)$ fixed, we consider the vector field $X_{K_0}^{ij}$ obtained from $Y_{ij}$ by successive applications of the Lie bracket in the following form:

$$X_{K_0}^{ij} = \underbrace{[X_M, \ldots, [X_{M-1}, \ldots, [X_{M-k_{M-1}}, \ldots, [X_1, \ldots, [X_1, Y_{ij}] \ldots.}$$

It follows from item (2) that

$$X_{K_0}^{ij} = \left(0, \partial_{K_0}(\sum_{K} c_{K}^{ij} x^K)\right).$$
On the other hand we have
\[
\frac{1}{K_0} X^{ij}_{K_0} (0) = (0, c^{ij}_{K_0}).
\]
Therefore, since the $c^{ij}_{K}$ span all $\mathbb{C}^N$ we conclude our claim.

**Remark 21** Let $\mathcal{D}$ be a distribution on $\mathbb{P}^{M+N}$ and suppose that in some affine chart $(x_1, \ldots, x_M, z_1, \ldots, z_N)$, the distribution is defined by a system
\[
dz_i = \omega_i, \quad i = 1, \ldots, N,
\]
where each $\omega_i$ is a rational 1-form in the variables $(x_1, \ldots, x_M)$. Then, it follows from the construction of $H_D$ that its coordinates are Liouvillian functions, so the foliation $\mathcal{F}(H_D)$ is global on $\mathbb{P}^{M+N}$ and defined by Liouvillian first integrals—see Singer (1992).

The problem of describing the Kernel (algebra of regular first integrals) of a regular distribution has been studied by many authors, see for example Bonnet (2005), Nagata and Nowicki (1998), Nowicki (1994) and the references therein. The main result of Bonnet (2005) shows that for a distribution defined by a family of holomorphic vector fields $\{X_i\}$ there is a holomorphic vector field tangent to the distribution with the same algebra of regular first integrals. Clearly, the field of meromorphic first integral can be extremely large, however, as a consequence of Theorem 4 we can obtain a similar result when the distribution has separated variables.

**Corollary 4** Let $\mathcal{D}$ be the distribution on $(\mathbb{C}^{M+N}, 0)$ defined by (1.1). Then there exists a vector field tangent to $\mathcal{D}$ which has the same field of meromorphic first integrals as $\mathcal{D}$.

As can be expected, the separation of variables—that is, the existence of coordinates that reduces the distribution to the form (1.1)—is not always possible, as we see in the following example.

**Example 22** Let $\mathcal{D}$ be the distribution on $(\mathbb{C}^4, 0)$ generated by the vector fields
\[
X_1 = \partial_z, \quad X_2 = x \partial_x + y \partial_y + (1 + z) \partial_w,
\]
which is a regular rank 2 non-integrable distribution. It is easy to check that $\mathcal{D}$ is tangent to the codimension one foliation $\mathcal{F} : xdy - ydx = 0$, that is, the function $y/x$ is a rational first integral for $\mathcal{D}$. This means that the field of meromorphic first integrals near every point $p \in \mathbb{C}^4$ is not reduced to the constants. Thus, if the separation of variables can be achieved in $(\mathbb{C}^4, p)$, it follows from Remark 2 that $\mathcal{D}$ is tangent to a regular codimension one holomorphic foliation $\tilde{\mathcal{F}}$ in $(\mathbb{C}^4, p)$. If $\tilde{\mathcal{F}}$ were different from $\mathcal{F}$, the intersection of these foliations would contain $\mathcal{D}$ as a sub-distribution, whence $\mathcal{D}$ would be integrable. Then $\tilde{\mathcal{F}} = \mathcal{F}$ in $(\mathbb{C}^4, p)$ and therefore $\mathcal{F}$ is regular at $p$. We conclude that the separation of variables in $(\mathbb{C}^4, p)$ is impossible if $p \in \text{Sing}(\mathcal{F})$. 

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On the other hand, if \( p \not\in \text{Sing}(\mathcal{F}) \), the separation of variables in \((\mathbb{C}^4, p)\) is actually possible. For instance, near \( p = (1, 0, 0, 0) \) the distribution is given by the system

\[
dy = y \frac{dx}{x}, \quad dw = (1 + z) \frac{dx}{x}.
\]

Then, after the change \( x \leftarrow e^x \) the system is equivalent to

\[
dy = y dx, \quad dw = (1 + z) dx,
\]

which, by the new change \( y \leftarrow ye^x \), is taken to

\[
dy = 0, \quad dw = (1 + z) dx.
\]

**Question:** Let \( \mathcal{D} \) be a (singular) holomorphic distribution on a complex manifold \( M \). Is it always possible to separate variables at a generic point?

### Proof of Proposition 18

This section is devoted to the proof of Proposition 18. The proof is quite technical and will be organized with the aid of some lemmas that will be proved along the way: Lemmas 26, 29, 30, 31, 32, 33, 34 and 35. Before proceeding with the proof of this proposition, we need three previous lemmas.

**Lemma 23** Given \( m \in \mathbb{N} \), if we set \( n = 2m + 1 \) we have that, for \((\mu_{ij})\) chosen in a dense set of the space of \( m \times n \) complex matrices, the vectors

\[
\mu_j = \left( e^{2\pi \sqrt{-1} \mu_{1j}}, e^{2\pi \sqrt{-1} \mu_{2j}}, \ldots, e^{2\pi \sqrt{-1} \mu_{mj}} \right), \quad j = 1, \ldots, n
\]

generate a dense subgroup in the multiplicative group \((\mathbb{C}^*)^m\).

**Proof** If \( k_1, \ldots, k_n \in \mathbb{Z} \), we have

\[
\mu_1^{k_1} \mu_2^{k_2} \cdots \mu_n^{k_n} = \left( e^{2\pi \sqrt{-1} \sum_{j=1}^n \mu_{1j} k_j}, e^{2\pi \sqrt{-1} \sum_{j=1}^n \mu_{2j} k_j}, \ldots, e^{2\pi \sqrt{-1} \sum_{j=1}^n \mu_{mj} k_j} \right).
\]

Then \( \mu_1, \ldots, \mu_n \) generate a dense subgroup of \((\mathbb{C}^*)^m\) if the set

\[
\left\{ \left( \sum_{j=1}^n \mu_{1j} k_j, \ldots, \sum_{j=1}^n \mu_{mj} k_j \right) : k_1, \ldots, k_n \in \mathbb{Z} \right\} = \mathbb{Z} \mu_1 + \cdots + \mathbb{Z} \mu_n
\]

is dense in \( \mathbb{C}^m \). Take any \( \mu_1, \ldots, \mu_{2m} \) generating \( \mathbb{C}^m \) as a real linear space. It is well known that, if we set

\[
\mu_{m+1} := x_1 \mu_1 + \cdots + x_{2m} \mu_{2m}
\]
for real numbers $x_1, \ldots, x_{2m}$ such that $1, x_1, \ldots, x_{2m}$ are linearly independent over $\mathbb{Z}$, then

$$\mathbb{Z}\mu_1 + \cdots + \mathbb{Z}\mu_{2m+1}$$

is dense in $\mathbb{C}^m$. This proves the lemma. \hfill \qed

**Lemma 24** Let $m \in \mathbb{N}$, let $\Omega \subset \mathbb{C}$ be an open set, and let

$$f_1, \ldots, f_m : \Omega \rightarrow \mathbb{C}$$

be holomorphic functions. Suppose that the set of functions $\{f_1, \ldots, f_m\}$ have no linear relation. Then the function

$$(x_1, \ldots, x_m) \mapsto \det \begin{bmatrix} f_1(x_1) & f_1(x_2) & \cdots & f_1(x_m) \\ f_2(x_1) & f_2(x_2) & \cdots & f_2(x_m) \\ \vdots & \vdots & \ddots & \vdots \\ f_m(x_1) & f_m(x_2) & \cdots & f_m(x_m) \end{bmatrix}$$

does not vanish identically.

Set

$$f := (f_1, \ldots, f_m) : \Omega \rightarrow \mathbb{C}^m.$$ 

It is enough to show that there exist complex numbers $x_1, \ldots, x_m \in \Omega$ such that the vectors $f(x_1), \ldots, f(x_m)$ are linearly independent. If it is not the case, the set

$$W := \text{Span}\{f(x) : x \in \Omega\}$$

is a proper subspace of $\mathbb{C}^m$ and we will have a linear relation for $f_1, \ldots, f_m$. \hfill \qed

The following Lemma give us a particular set of function satisfying the hypotheses of Lemma 24.

**Lemma 25** If $k, m \in \mathbb{N}$ and $b_1, b_2, \ldots, b_m \in \mathbb{C}$ are pairwise distinct, the set of $km + 1$ functions

$$\left\{ 1, \frac{1}{(x - b_1)}, \frac{1}{(x - b_1)^2}, \ldots, \frac{1}{(x - b_1)^k}, \frac{1}{(x - b_2)}, \frac{1}{(x - b_2)^2}, \ldots, \frac{1}{(x - b_2)^k}, \ldots, \frac{1}{(x - b_m)}, \frac{1}{(x - b_m)^2}, \ldots, \frac{1}{(x - b_m)^k} \right\}$$

have no linear relation.
Proof Suppose that there exist complex numbers $c, c_{ij}, \ i = 1, \ldots, m, \ j = 1, \ldots, k$

such that the function

$$f(x) = c + \sum_{i,j} \frac{c_{ij}}{(x - b_j)^j}$$

vanishes. If $c_{ij}$ were nonzero for some $i, j$, the function $f$ would have a pole at $b_i$, which is not possible. Then the $c_{ij}$ are all zero and therefore $c$ is also zero. \hfill \Box

Let

$$T = \{b_1, b_2, \ldots, b_m\}, \quad (6.1)$$

where $m \in \mathbb{N}$ and the complex numbers $b_1, \ldots, b_m$ are pairwise distinct. By Lemma 23, for some $n \in \mathbb{N}$ we can find numbers

$${\mu}_{ij} \in \mathbb{C}, \ \ i = 1, \ldots, M - 1, \ j = 1, \ldots, n \quad (6.2)$$

with the following properties.

- For each $j = 1, \ldots, n$, we have

$$\text{Im}(\mu_{1j}), \text{Im}(\mu_{1j} + \mu_{2j}), \ldots, \text{Im}(\mu_{1j} + \mu_{(M-1)j}) > 0. \quad (6.3)$$

- The vectors

$$\mu_j = \left(e^{2\pi \sqrt{-1} \mu_{1j}}, e^{2\pi \sqrt{-1} \mu_{2j}}, \ldots, e^{2\pi \sqrt{-1} \mu_{(M-1)j}}\right), \quad j = 1, \ldots, n \quad (6.4)$$

generate a dense subgroup in the multiplicative group $(\mathbb{C}^*)^{M-1}$.

Consider the natural number $N = (k + 1)m + n + 1$ and take complex numbers $b_{m+1}, b_{m+2}, \ldots, b_N$, such that $b_1, b_2, \ldots, b_N$ are pairwise distinct. The choice of $b_{m+1}, b_{m+2}, \ldots, b_N$ satisfies some generic property that will be specified later. As mentioned in (1) of Proposition 18, for a suitable choice of numbers

$$\nu_{ij} \in \mathbb{C}, \quad i = 1, \ldots, (M - 1), \quad j = 1, \ldots, N,$$

the holomorphic foliation $\mathcal{G}$ will be defined by a rational vector field of the form

$$Y := \sum_{i=1}^{M-1} A_i(t_M) t_i \frac{\partial}{\partial t_i} + \frac{\partial}{\partial t_M}, \quad (6.5)$$
where

\[ A_i(t_M) = \sum_{j=1}^{N} \frac{v_{ij}}{t_M - b_j}, \quad i = 1, \ldots M - 1. \] (6.6)

This foliation has its singularities at the points

\[ p_j : = (0, \ldots, 0, b_j) \in \mathbb{C}^M, \quad j = 1, \ldots, N, \]

which are linearizable, as we explicitly show. Let \( l \in \{1, \ldots, N\} \) and consider coordinates

\[ (s_1, s_2, \ldots, s_M) = (t_1, t_2, \ldots, t_{M-1}, t_M - b_l) \] (6.7)

at the point \( p_l \). In these coordinates the vector field \( Y \) is explicitly given by

\[ Y_l : = \sum_{i=1}^{M-1} \left( \sum_{j=1}^{N} \frac{v_{ij}}{s_{M} - (b_j - b_l)} \right) s_i \frac{\partial}{\partial s_i} + \frac{\partial}{\partial s_{M}}. \]

Then, if we set

\[ g_{il}(s_M) : = \sum_{j=1, j \neq l}^{N} \frac{v_{ij}}{s_{M} - (b_j - b_l)}, \quad i = 1, \ldots, (M - 1), \]

we have

\[ Y_l = \sum_{i=1}^{M-1} \left( \frac{v_{il}}{s_{M}} + g_{il}(s_M) \right) s_i \frac{\partial}{\partial s_i} + \frac{\partial}{\partial s_{M}}. \]

Since the \( g_{il} \) are holomorphic near the origin, there exist functions \( h_{il} \) holomorphic on a neighborhood of the origin, such that \( h_{il}' = g_{il} \) and \( h_{il}(0) = 0 \) for \( i = 1, \ldots, M - 1 \). Define

\[ h_l(s_1, \ldots, s_M) = (s_1 e^{h_{1l}(s_M)}, \ldots, s_{M-1} e^{h_{(M-1)l}(s_M)}, s_M). \] (6.8)

A straightforward computation shows that

\[ h^*_l(Y_l) = \frac{1}{s_M} \left( \sum_{i=1}^{M-1} v_{il} s_i \frac{\partial}{\partial s_i} + s_M \frac{\partial}{\partial s_{M}} \right). \] (6.9)
That is, the foliation at $p_I$ is linearizable. Moreover, from the definition of $h_l$, we see that a sufficient condition for $h_I$ to be $k$-tangent to the identity is that the functions $g_{il}$ have orders at least $k - 1$ at the origin. The function $g_{il}$ can be expanded in the form

$$g_{il}(s_M) = - \sum_{j=1, j \neq l}^{N} \frac{v_{ij}}{b_j - b_l} \cdot \frac{1}{1 - \frac{s_M}{b_j - b_l}}$$

$$= - \sum_{j=1, j \neq l}^{N} \frac{v_{ij}}{b_j - b_l} \left( 1 + \frac{s_M}{b_j - b_l} + \left( \frac{s_M}{b_j - b_l} \right)^2 + \cdots \right),$$

whence the coefficients up to order $k - 1$ of $g_{il}$ are given by

$$c_{il0} = - \sum_{j=1, j \neq l}^{N} \frac{v_{ij}}{b_j - b_l}$$

$$c_{il1} = - \sum_{j=1, j \neq l}^{N} \frac{v_{ij}}{(b_j - b_l)^2}$$

$$\vdots$$

$$c_{il(k-1)} = - \sum_{j=1, j \neq l}^{N} \frac{v_{ij}}{(b_j - b_l)^k}.$$  \hspace{1cm} (6.10)

Therefore, a sufficient condition for $h_I$ to be $k$-tangent to the identity for $l = 1, \ldots, m$ is to have the equalities

$$c_{ila} = 0, \quad \text{for} \quad i = 1, \ldots, M - 1, \ l = 1, \ldots, m, \ \alpha = 0, \ldots, k - 1.$$  

For each $i = 1, \ldots, M - 1$, it will be convenient to define $F_i : \mathbb{C}^N \to \mathbb{C}^{km}$,

$$F_i = (c_{i10}, c_{i11}, \ldots, c_{i1(k-1)}, c_{i20}, \ldots, c_{i2(k-1)}, \ldots, c_{im0}, \ldots, c_{im(k-1)}),$$

as a function of the variable $v_i : = (v_{i1}, \ldots, v_{iN})$. The function $F_i$ is linear and—in view of (6.10)—it is represented by the $km \times N$ matrix
\[
\begin{pmatrix}
0 & \frac{1}{b_2-b_1} & \frac{1}{b_3-b_1} & \cdots & \frac{1}{b_m-b_1} & \cdots & \frac{1}{b_N-b_1} \\
0 & \frac{1}{(b_2-b_1)^2} & \frac{1}{(b_3-b_1)^2} & \cdots & \frac{1}{(b_m-b_1)^2} & \cdots & \frac{1}{(b_N-b_1)^2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \frac{1}{(b_2-b_1)^k} & \frac{1}{(b_3-b_1)^k} & \cdots & \frac{1}{(b_m-b_1)^k} & \cdots & \frac{1}{(b_N-b_1)^k} \\
\frac{1}{b_1-b_2} & 0 & \frac{1}{b_3-b_2} & \cdots & \frac{1}{b_m-b_2} & \cdots & \frac{1}{b_N-b_2} \\
\frac{1}{(b_1-b_2)^2} & 0 & \frac{1}{(b_3-b_2)^2} & \cdots & \frac{1}{(b_m-b_2)^2} & \cdots & \frac{1}{(b_N-b_2)^2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{1}{(b_1-b_2)^k} & 0 & \frac{1}{(b_3-b_2)^k} & \cdots & \frac{1}{(b_m-b_2)^k} & \cdots & \frac{1}{(b_N-b_2)^k} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{1}{(b_1-b_m)^2} & \frac{1}{(b_2-b_m)^2} & \frac{1}{(b_3-b_m)^2} & \cdots & \frac{1}{(b_m-b_m)^2} & \cdots & \frac{1}{(b_N-b_m)^2} \\
\frac{1}{(b_1-b_m)^k} & \frac{1}{(b_2-b_m)^k} & \frac{1}{(b_3-b_m)^k} & \cdots & \frac{1}{(b_m-b_m)^k} & \cdots & \frac{1}{(b_N-b_m)^k}
\end{pmatrix}
\]  \hspace{1cm} (6.11)

Since this matrix does not depend on \( i = 1, \ldots, M - 1 \), we will denote \( F_i = F \) for all \( i = 1, \ldots, M - 1 \). We consider any \( km \times km \) submatrix of the matrix in (6.11) disjoint of the first \( m \) columns: we take for example the submatrix

\[
J =
\begin{pmatrix}
\frac{1}{b_{m+1}-b_1} & \frac{1}{b_{m+1}-b_1} & \cdots & \frac{1}{b_{m+1}-b_1} \\
\frac{1}{(b_{m+1}-b_1)^2} & \frac{1}{(b_{m+1}-b_1)^2} & \cdots & \frac{1}{(b_{m+1}-b_1)^2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{(b_{m+1}-b_1)^k} & \frac{1}{(b_{m+1}-b_1)^k} & \cdots & \frac{1}{(b_{m+1}-b_1)^k} \\
\frac{1}{b_{m+1}-b_2} & \frac{1}{b_{m+1}-b_2} & \cdots & \frac{1}{b_{m+1}-b_2} \\
\frac{1}{(b_{m+1}-b_2)^2} & \frac{1}{(b_{m+1}-b_2)^2} & \cdots & \frac{1}{(b_{m+1}-b_2)^2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{(b_{m+1}-b_2)^k} & \frac{1}{(b_{m+1}-b_2)^k} & \cdots & \frac{1}{(b_{m+1}-b_2)^k} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{b_{m+1}-b_m} & \frac{1}{b_{m+1}-b_m} & \cdots & \frac{1}{b_{m+1}-b_m} \\
\frac{1}{(b_{m+1}-b_m)^2} & \frac{1}{(b_{m+1}-b_m)^2} & \cdots & \frac{1}{(b_{m+1}-b_m)^2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{(b_{m+1}-b_m)^k} & \frac{1}{(b_{m+1}-b_m)^k} & \cdots & \frac{1}{(b_{m+1}-b_m)^k} \\
\end{pmatrix}
\]  \hspace{1cm} (6.12)
Recall that \(b_1, \ldots, b_m\) are previously fixed pairwise distinct constants—see (6.1). As we have said above, the constants \(b_{m+1}, b_{m+2}, \ldots, b_N\) have to be chosen in a particular generic way; we explain now this choice. By Lemmas 24 and 25 we have that, as a function of \((b_{m+1}, b_{m+2}, \ldots, b_N)\), the determinant \(\det(J)\) does not vanish. In the same way, this property holds for any \(km \times km\) submatrix of (6.11) disjoint of the first \(m\) columns. Thus, we conclude that the set of

\[(b_{m+1}, b_{m+2}, \ldots, b_N) \in \mathbb{C}^{N-m}\]

such that all the already mentioned \(km \times km\) submatrices of (6.11) have nonzero determinant is a Zariski open set. Denote by \(\mathcal{U}\) this Zariski open set. On the other hand, we use the last \(km+1\) columns of (6.11) to construct the following \((km+1) \times (km+1)\) matrix:

\[
J_1 = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\frac{1}{b_N - km - b_1} & \frac{1}{b_N - km + 1 - b_1} & \cdots & \frac{1}{b_N - b_1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{b_N - km - b_2} & \frac{1}{b_N - km + 1 - b_2} & \cdots & \frac{1}{b_N - b_2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{b_N - km - b_m} & \frac{1}{b_N - km + 1 - b_m} & \cdots & \frac{1}{b_N - b_m} \\
\frac{(b_N - km - b_1)^k}{1} & \frac{(b_N - km + 1 - b_1)^k}{1} & \cdots & \frac{(b_N - b_1)^k}{1} \\
\frac{(b_N - km - b_2)^k}{1} & \frac{(b_N - km + 1 - b_2)^k}{1} & \cdots & \frac{(b_N - b_2)^k}{1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{(b_N - km - b_m)^k}{1} & \frac{(b_N - km + 1 - b_m)^k}{1} & \cdots & \frac{(b_N - b_m)^k}{1}
\end{bmatrix}.
\]

(6.13)

Again by Lemmas 24 and 25 we have that, as a function of \((b_{m+1}, b_{m+2}, \ldots, b_N)\), the determinant \(\det(J_1)\) does not vanish. Then the set of

\[(b_{m+1}, b_{m+2}, \ldots, b_N) \in \mathbb{C}^{N-m}\]

such that \(\det(J_1) \neq 0\) is a Zariski open set \(\mathcal{U}_1\). Finally, from now on the vector \((b_{m+1}, b_{m+2}, \ldots, b_N)\) will be supposed fixed in the Zariski open set \(\mathcal{U} \cap \mathcal{U}_1\). So that

- any \(km \times km\) submatrix of (6.11) disjoint of the first \(m\) columns has nonzero determinant, and
- the matrix \(J_1\) has nonzero determinant.

**Lemma 26** For each \(i = 1, \ldots, M - 1\), there exists

\[v_i = (v_{i1}, \ldots, v_{iN}) \in \mathbb{C}^N\]
such that the following properties hold.

(1) $\nu_{ij} \neq 0$, for all $i = 1, \ldots, M - 1$, $j = 1, \ldots, N$.

(2) $F(v_1) = \cdots = F(v_{M-1}) = 0$.

(3) $(v_1, \ldots, v_{i(m+n)}) = \left( \frac{\lambda_i(b_1)}{\lambda_M(b_1)}, \ldots, \frac{\lambda_i(b_m)}{\lambda_M(b_m)}, \mu_{i1}, \ldots, \mu_{in} \right)$, where the $\mu_{ij}$ are as defined in (6.2).

(4) For each $j = (m + n + 1), \ldots, N$, there exists a real line through the origin in $\mathbb{C}$ separating $1 \in \mathbb{C}$ from the set

$$\{ v_{1j}, v_{1j} + v_{2j}, \ldots, v_{1j} + v_{(M-1)j} \}.$$ 

(5) The numbers

$$\tilde{v}_1 := -1 - \sum_{j=1}^{N} v_{1j},$$

$$\tilde{v}_2 := 1 - \sum_{j=1}^{N} v_{2j},$$

$$\vdots$$

$$\tilde{v}_{M-1} := 1 - \sum_{j=1}^{N} v_{(M-1)j},$$

are all nonzero and there exists a real line through the origin in $\mathbb{C}$ separating $1 \in \mathbb{C}$ from the set

$$\{ \tilde{v}_1, \tilde{v}_1 + \tilde{v}_2, \ldots, \tilde{v}_1 + \tilde{v}_{(M-1)} \}.$$ 

Remark 27 By (3) of Lemma 26 we have

$$v_{i(m+1)} = \mu_{i1}, \ v_{i(m+2)} = \mu_{i2}, \ldots, \ v_{i(m+n)} = \mu_{in}.$$ 

Then, it follows from (6.3) that, for each $j = m + 1, \ldots, m + n$, there exists a real line through the origin in $\mathbb{C}$ separating $1 \in \mathbb{C}$ from the set

$$\{ v_{1j}, v_{1j} + v_{2j}, \ldots, v_{1j} + v_{(M-1)j} \}.$$ 

Therefore, (4) of Lemma 26 will actually hold true whenever $j \notin \{1, \ldots, m\}$.

Proof Since $N = (m+n) + (km+1)$, it will be convenient to consider $F$ as a function of

$$(u, v) \in \mathbb{C}^{m+n} \times \mathbb{C}^{km+1}.$$
For each $i = 1, \ldots, M - 1$, we start by finding vectors satisfying the properties (2) and (3) of the lemma. Thus we are looking for solutions of the equation $F(u, v) = 0$. The property established above about the $km \times km$ submatrices of (6.11) implies that, in the equation $F(u, v) = 0$, any set of $km$ variables chosen between the $km + 1$ coordinates of $v$ can be expressed as linear functions of the $m + n + 1$ remaining variables; so these remaining variables can be arbitrarily chosen. Since the $m + n + 1$ remaining variables includes the $m + n$ coordinates of $u$, we can find solutions of $F(u, v) = 0$ with any prescribed value of $u$. Then there exist

$$(u_1, v_1), \ldots, (u_{M-1}, v_{m-1}) \in \mathbb{C}^{m+n} \times \mathbb{C}^{N-m-n}$$

such that

$$F(u_1, v_1) = \cdots = F(u_{M-1}, v_{m-1}) = 0,$$

and

$$u_i = \left( \frac{\lambda_i(b_1)}{\lambda_M(b_1)}, \ldots, \frac{\lambda_i(b_m)}{\lambda_M(b_m)}, \mu_{i1}, \ldots, \mu_{in} \right), \quad i = 1, \ldots, M - 1. \tag{6.15}$$

Let $G : \mathbb{C}^{km+1} \to \mathbb{C}^{km}$ be the linear function defined by

$$G(v) = F(0, v).$$

We write

$$v = (v_1, \ldots, v_{km+1}) \in \mathbb{C}^{km+1},$$

$$\text{Im} v = (\text{Im} v_1, \ldots, \text{Im} v_{km+1}) \in \mathbb{R}^{km+1}$$

and consider the linear function

$$S(v) = v_1 + v_2 + \cdots + v_{km+1}.$$

We will find $v \in \mathbb{C}^{km+1}$ such that

$$G(v) = 0, \quad \text{Im} S(v) \neq 0, \quad \text{Im}(v) \in (\mathbb{R}^*)^{km+1}. \tag{6.16}$$

Since the matrix of $G$ is composed of the last $km + 1$ columns of the matrix (6.11), their $km \times km$ submatrices have nonzero determinant. Then, in the equation $G(v) = 0$, any set of $km$ coordinates between the $km + 1$ coordinates of $v$ can be expressed as linear functions of the remaining coordinate; so this remaining coordinate of $v$ can be arbitrarily chosen. In particular, given any $j = 1, \ldots, (km + 1)$, we can find a solution of $G(v) = 0$ with $\text{Im} v_j \neq 0$; that is,

$$G^{-1}(0) \nsubseteq \{ \text{Im} v_j = 0 \}, \quad j = 1, \ldots, (km + 1). \tag{6.17}$$

On the other hand, from (6.13) we can see that the matrix of $G$ is given by the matrix $J_1$ with its first row deleted. Thus, the fact of $J_1$ being non-singular implies that
ker(G) ⊄ ker S. Therefore, since ker(G) is a complex linear space, and since the
maximal complex linear space in the real space \{Im S = 0\} is ker S, we deduce that
ker(G) can not be contained in \{Im S = 0\}, that is,

$$G^{-1}(0) \not\subset \{\text{Im } S = 0\}.$$  

It follows from this and (6.17)—since \(G^{-1}(0)\) is irreducible—that

$$G^{-1}(0) \not\subset \{\text{Im } S = 0\} \cup \bigcup_{j=1}^{km+1} \{\text{Im } v_j = 0\},$$

and therefore we can choose \(\nu \in \mathbb{C}^{km+1}\) satisfying (6.16). Since

$$\text{Im}(\nu_i + s\nu) = \text{Im } \nu_i + s \text{Im } \nu, \quad s \in \mathbb{R}$$  \hspace{1cm} (6.18)

and no component of \(\text{Im } \nu\) is zero, we can fix \(s > 0\) so big that

\(\star\) for each \(i = 1, \ldots, M - 1\) and each \(j = 1, \ldots, (km + 1)\), the \(j\)th component of
\(\text{Im}(\nu_i + s\nu) \in \mathbb{R}^{km+1}\) have the same sign as the \(j\)th component of \(\text{Im } \nu\).

Now, for each \(i = 1, \ldots, M - 1\), define

$$\nu_i = (u_i, v_i + s\nu).$$

Clearly, from (6.15) the property (3) of the lemma holds. Property (3) of the lemma
together with \(\star\) imply the property (1) of the lemma. From (6.14) and (6.16) we have

$$F(u_i, v_i + s\nu) = F(u_i, v_i) + s G(v) = 0,$$  \hspace{1cm} (6.19)

so the property (2) of Lemma 26 also holds. Now, let \(j \in \{m + n + 1, \ldots, N\}\). From
(\(\star\)) we deduce that the numbers \(\text{Im } v_{1j}, \text{Im } v_{2j}, \ldots, \text{Im } v_{(M-1)j}\) have the same sign.
This means that the numbers

$$v_{1j}, v_{2j}, \ldots, v_{(M-1)j}$$

are simultaneously contained in the upper half plane or in the lower half plane of \(\mathbb{C}\). Then the same holds for the numbers

$$v_{1j}, v_{1j} + v_{2j}, \ldots, v_{1j} + v_{(M-1)j},$$

which in turns implies the property (4) of the lemma. In the same way, the property
(5) of the lemma will be guaranteed if the numbers

$$\text{Im } \tilde{\nu}_1, \text{Im } \tilde{\nu}_2, \ldots, \text{Im } \tilde{\nu}_{M-1}$$
have the same sign. Given $i = 1, \ldots, M - 1$, we can express

$$\text{Im} \tilde{v}_i = - \text{Im} \sum_{j=1}^{N} v_{ij} = c_i - s \text{ Im } S(v),$$

for some constant $c_i \in \mathbb{R}$. Therefore, by taking $s > 0$ bigger if necessary, $\text{Im} \tilde{v}_i$ has the same sign as $- \text{ Im } S(v)$ for all $i = 1, \ldots, M - 1$, which finishes the proof. \hfill \Box

From now on we fix the vectors $v_1, \ldots, v_{M-1} \in \mathbb{C}^N$ according to Lemma 26. Then the foliation $\mathcal{G}$ is already defined as stated by (1) of Proposition 18. Let us start the proof of the remaining assertions of Proposition 18.

Let $\tau \in T$. Then $\tau = b_{l_0}$ for some $l_0$ in $\{1, \ldots, m\}$. By the choice of the $v_i$ we have

$$F(v_i) = 0, \quad i = 1, \ldots, M - 1,$$

that is

$$c_{i\alpha l} = 0, \text{ for } i = 1, \ldots, M - 1, \ l = 1, \ldots, m, \ \alpha = 0, \ldots, k - 1.$$ 

As we have seen above, these equalities guarantee that $h_l$ is $k$-tangent to the identity for $l = 1, \ldots, m$. Thus, if we define

$$h_\tau := h_{l_0},$$

the assertion (2a) of Proposition 18 holds. On the other hand, it follows from (6.9) and (3) of Lemma 26 that

$$h_\tau^* (Y_{l_0}) = \frac{1}{s_M} \left( \sum_{i=1}^{M-1} v_{i l_0} s_i \frac{\partial}{\partial s_i} + s_M \frac{\partial}{\partial s_M} \right) = \frac{1}{s_M} \left( \sum_{i=1}^{M-1} \frac{\lambda_i(\tau)}{\lambda_M(\tau)} s_i \frac{\partial}{\partial s_i} + s_M \frac{\partial}{\partial s_M} \right) = \frac{1}{\lambda_M(\tau)s_M} \left( \sum_{i=1}^{M} \frac{\lambda_i(\tau)}{\lambda_M(\tau)} s_i \frac{\partial}{\partial s_i} \right),$$

which means that the foliation defined by $Y_{l_0}$ at $0 \in \mathbb{C}^M$ is the pushforward of the foliation generated by the linear system

$$s_1' = \lambda_1(\tau) s_1$$

$$s_2' = \lambda_2(\tau) s_2$$

$$\vdots$$

$$s_M' = \lambda_M(\tau) s_M.$$
Thus, since $Y_{t_0}$ is nothing but the vector field $Y$ after the change of coordinates

$$(t_1, \ldots, t_M) = (s_1, \ldots, s_M) + (0, \ldots, 0, b_{t_0}),$$

the assertion (2b) of Proposition 18 follows.

The assertion (3) of Proposition 18 is easily verified, so it needs no additional details. Thus we proceed with the proof of the remaining assertions.

It is easy to see that the line

$$C : = \{ t \in \mathbb{C}^M : t_1 = 0, \ldots, t_{M-1} = 0 \}$$

is invariant by $\mathcal{G}$, so $C^* = \mathbb{C}\backslash \{p_1, \ldots, p_N\}$ is a leaf of $\mathcal{G}$. We are interested in the holonomy of this leaf, which—due to the special form of the vector field $Y$—has a global nature and can be easily computed, as we show next. Consider the hyperplane

$$\Sigma = \{ t \in \mathbb{C}^M : t_M = a \},$$

where

$$a \in \mathbb{C}\backslash \{b_1, \ldots, b_N\}.$$ 

In view of the natural identification $\mathbb{C} \cong \mathbb{C}$, a curve in $\mathbb{C}^*$ can be thought of as curve in $\mathbb{C}\backslash \{b_1, \ldots, b_N\}$. Thus, we consider a smooth curve

$$\gamma : [0, 1] \to \mathbb{C}\backslash \{b_1, \ldots, b_N\}, \quad \gamma(0) = a. \quad (6.20)$$

Then, given $z = (z_1, \ldots, z_{M-1}, a) \in \Sigma$, there exists a unique curve

$$\gamma^z : [0, 1] \to \mathbb{C}^M, \quad \gamma^z(0) = z$$

with last coordinate given by $\gamma$ and that is tangent to $\mathcal{G}$: in view of (6.5), this curve is explicitly given by

$$\gamma^z(s) = \left( z_1 \exp \int_{\gamma|[0,s]} A_1(u) du, \ldots, z_{M-1} \exp \int_{\gamma|[0,s]} A_{M-1}(u) du, \gamma(s) \right). \quad (6.21)$$

Thus, the holonomy associated to $\gamma$ is the map

$$\mathcal{H}_\gamma : z \mapsto \gamma^z(1), \quad (6.22)$$
which is a linear isomorphism between $\Sigma = \{t_M = a\}$ and $\{t_M = \gamma(1)\}$. This map $H_\gamma$ can be computed if $\gamma$ is a closed curve, because it depends only on the integrals

$$\int_\gamma A_i(u) du, \quad i = 1, \ldots, M - 1.$$ 

For each $j = 1, \ldots, N$, choose a smooth positively oriented closed simple curve $\gamma_j : [0, 1] \to \mathbb{C}\{b_1, \ldots, b_N\}, \quad \gamma(0) = \gamma(1) = a,$

(6.23)

whose interior domain intersects $\{b_1, \ldots, b_N\}$ exactly at $b_j$. Then

$$\int_{\gamma_j} A_i(u) du = \int_{\gamma_j} \sum_{l=1}^{N} \frac{v_{ij}}{w - b_j} dw = \int_{\gamma_j} \frac{v_{ij}}{w - b_j} = 2\pi \sqrt{-1} v_{ij},$$

so that

$$H_{\gamma_j}(z) = \left( z_1 e^{2\pi \sqrt{-1} v_{1j}}, \ldots, z_{M-1} e^{2\pi \sqrt{-1} v_{(M-1)j}}, a \right),$$

(6.24)

which can be identified with the linear map

$$z \mapsto \left( e^{2\pi \sqrt{-1} v_{1j} z_1}, \ldots, z_{M-1} e^{2\pi \sqrt{-1} v_{(M-1)j}} \right)$$

(6.25)

or even with its corresponding diagonal matrix. Then the holonomy group of $\mathbb{C}^*$ is generated by the diagonal matrices $H_{\gamma_1}, \ldots, H_{\gamma_N}$ in $\text{GL}(M - 1, \mathbb{C})$ and so, in particular, this group is abelian.

Before going ahead with the rest of the proof, we establish some notation and conventions.

**Definition 28** We know that the blow-up $\pi : \widehat{\mathbb{C}^M} \to \mathbb{C}^M$ is a biholomorphism between $\mathbb{C}^M \setminus \pi^{-1}(0)$ and $\mathbb{C}^M \setminus \{0\}$. Thus, if no confusion arise we can identify objects in $\mathbb{C}^M \setminus \pi^{-1}(0)$ with their corresponding images in $\mathbb{C}^M \setminus \{0\}$. Then, $w \in \mathbb{C}^M \setminus \pi^{-1}(0)$ and $W \subset \mathbb{C}^M \setminus \pi^{-1}(0)$ are identified with $\pi(w)$ and $\pi(W)$. The foliation $\mathcal{G}$ on $\mathbb{C}^M \setminus \pi^{-1}(0)$ is identified with the foliation $\mathcal{F}$ on $\mathbb{C}^M \setminus \{0\}$. Given a point $w \in \mathbb{C}^M \setminus \pi^{-1}(0)$, the leaf of $\mathcal{G}$ through $w$ is identified with the leaf of $\mathcal{F}$ through $\pi(w)$, so this leaf is simply referred to as the leaf through $w$ with no reference to the foliation. A curve $\alpha : [0, 1] \to \mathbb{C}^M \setminus \pi^{-1}(0)$ is identified with the curve $\pi \circ \alpha$ in $\mathbb{C}^M \setminus \{0\}$; furthermore, we define the length of the curve $\alpha$ as the Euclidean length of $\pi \circ \alpha$, and this number will be denoted by $\ell(\alpha)$. If the curve $\alpha$ is contained in a leaf, we will say that $\alpha$ is an integral curve. Finally, given $w \in \mathbb{C}^M \setminus \pi^{-1}(0)$, we denote by $\|w\|$ the Euclidean norm of $\pi(w)$, that is, $\|w\| := |\pi(w)|$.

**Lemma 29** If $\gamma$ is a curve as in (6.20), there exists $K_\gamma > 0$ such that the following properties hold.
(1) If $z \in \Sigma, z_1 \neq 0$ and $\|z\| < 1$, then
\[ \ell(\gamma^z) \leq K_\gamma \|z\|. \]

(2) The number $K_\gamma$ depends continuously on $\gamma$ if we consider the $C^1$-topology in the space of curves $\gamma$.

As a direct consequence of the first assertion above, there exists a constant $K > 0$ such that
\[ z \in \Sigma \setminus \pi^{-1}(0), \|z\| < 1, \ j = 1, \ldots, N \implies \ell(\gamma^z_j) \leq K \|z\|. \quad (6.26) \]

**Proof** Write $\gamma^z(s) = (\gamma_1(s), \ldots, \gamma_{M-1}(s), \gamma), s \in [0, 1]$. It follows from (6.21) that
\[ |\gamma_i(s)| \leq K_1|z_i|, \ i = 1, \ldots, M - 1, \quad (6.27) \]
where
\[ K_1 := \max \left\{ \left| \exp \int_{\gamma'[0, s]} A_i(u) \, du \right| : s \in [0, 1], i = 1, \ldots, M - 1 \right\}. \]

Also from (6.21) we have
\[ |\gamma_i'(s)| \leq \left| \gamma_i'(s) A_i(\gamma(s)) \gamma_i(s) \right|, \ i = 1, \ldots, M - 1, \]
which together with (6.27) leads to
\[ |\gamma_i'(s)| \leq K_2 K_1|z_i|, \ i = 1, \ldots, M - 1, \quad (6.28) \]
where
\[ K_2 := \max \left\{ \left| \gamma_i'(s) A_i(\gamma(s)) \right| : s \in [0, 1], i = 1, \ldots, M - 1 \right\}. \]

Observe that
\[ |(\pi \circ \gamma^z)'| \leq |\gamma_1'| + |(\gamma_1 \gamma_2)'| + \cdots + |(\gamma_1 \gamma_{M-1})'| + |(\gamma_1 \gamma)'|. \quad (6.29) \]

Moreover, from (6.27) and (6.28) we obtain
\[ |(\gamma_1 \gamma_i)'| \leq |\gamma_i'| |\gamma_i| + |\gamma_1| |\gamma_i'| \leq \left( (K_2 K_1|z_1|)(K_1|z_i|) + (K_1|z_1|)(K_2 K_1|z_i|) \right), \]
whence
\[ |(\gamma_1 \gamma_i)'| \leq 2K_2 K_1^2 |z_1||z_i|, \ i = 1, \ldots, M - 1. \quad (6.30) \]
On the other hand,

\[ |(\gamma_1 \gamma)'| \leq |\gamma_1'||\gamma| + |\gamma_1||\gamma'| \leq (K_2 K_1 |z_1|) |\gamma| + (K_1 |z_1|) |\gamma'| , \]

so we have

\[ |(\gamma_1 \gamma)'| \leq K_3 |z_1| , \quad (6.31) \]

where

\[ K_3 : = \max_{s \in [0,1]} \left( K_2 K_1 |\gamma(s)| + K_1 |\gamma'(s)| \right) . \quad (6.32) \]

By using (6.28), (6.30) and (6.31) in (6.29), we obtain

\[ |(\pi \circ \gamma^z)'| \leq K_2 K_1 |z_1| + 2 K_2 K_1^2 \sum_{i=2}^{M-1} |z_1||z_i| + K_3 |z_1| \]

\[ \leq K_2 K_1 ||z|| + 2 K_2 K_1^2 \sum_{i=2}^{M-1} ||z|| + K_3 ||z|| \]

\[ \leq \left( K_2 K_1 + (M - 2)2 K_2 K_1^2 + K_3 \right) ||z|| , \]

\[ \leq K_\gamma ||z|| , \quad (6.34) \]

where

\[ K_\gamma : = K_2 K_1 + (M - 2)2 K_2 K_1^2 + K_3 . \]

Therefore,

\[ \ell(\gamma^z) = \int_0^1 |(\pi \circ \gamma^z)'| ds \leq K_\gamma ||z|| . \]

Finally, the last assertion of the lemma follows from the fact of \( K_1, K_2 \) and \( K_3 \) depending continuously on \( \gamma \).

Between the curves \( \gamma_1, \ldots, \gamma_N \)—they are defined in (6.23)—we distinguish the loops

\[ \xi_1 : = \gamma_{m+1}, \quad \xi_2 : = \gamma_{m+2}, \ldots, \; \xi_n : = \gamma_{m+n} \]

and denote their holonomies by

\[ g_1 : = H_{\xi_1}, \quad g_2 : = H_{\xi_2}, \ldots, \; g_n : = H_{\xi_n} . \]
Lemma 30 \textit{Let }G\textit{ be the group generated by }g_1, \ldots, g_n\textit{—this is a subgroup of the holonomy group of the leaf }\mathbb{C}^*\textit{.}

(1) \textit{The orbit by }G\textit{ of any}

$$z \in (\mathbb{C}^*)^{M-1} \times \{a\} \subset \Sigma$$

\textit{is dense in }\Sigma\textit{.}

(2) \textit{There exists }\delta \in (0, 1)\textit{ such that}

$$\|g_j(z)\| < \delta \|z\|, \quad z \in \Sigma \setminus \pi^{-1}(0), \quad j = 1, \ldots, n.$$\

\textbf{Proof} From (6.24) and (3) of Lemma 26, for each \(j = 1, \ldots, n\) and \(z \in \Sigma \setminus \pi^{-1}(0)\) we obtain

$$g_j(z) = (\varsigma_1 z_1, \varsigma_2 z_2, \ldots, \varsigma_{M-1} z_{M-1}, a), \quad (6.35)$$

where

$$\varsigma_1 = e^{2\pi i \mu_1j}, \quad \varsigma_2 = e^{2\pi i \mu_2j}, \ldots, \varsigma_{M-1} = e^{2\pi i \mu(M-1)j}.$$\

So the first assertion of the lemma follows directly from (6.4).

On the other hand, from (6.3) we find \(\tilde{\delta} > 0\) such that

$$\text{Im}(\mu_1j), \text{Im}(\mu_1j + \mu_2j), \ldots, \text{Im}(\mu_1j + \mu(M-1)j) > \tilde{\delta}.$$\

Then

$$e^{2\pi i \text{Im}(\mu_1j)}, e^{2\pi i \text{Im}(\mu_1j + \mu_2j)}, \ldots, e^{2\pi i \text{Im}(\mu_1j + \mu(M-1)j)} < e^{2\pi i \tilde{\delta}},$$

whence

$$|\varsigma_1|, |\varsigma_1 \varsigma_2|, \ldots, |\varsigma_1 \varsigma_{M-1}| < \delta := e^{2\pi i \tilde{\delta}}.$$\

Therefore, from (6.35) we have

$$\|g_j(z)\| = \sqrt{|\varsigma_1 z_1|^2 + |\varsigma_1 z_1 \varsigma_2 z_2|^2 + \cdots + |\varsigma_1 z_1 \varsigma_{M-1} z_{M-1} |^2 + |\varsigma_1 z_1 a|^2}$$

$$< \sqrt{\delta^2 |z_1|^2 + \delta^2 |z_1 z_2|^2 + \cdots + \delta^2 |z_1 z_{M-1}|^2 + \delta^2 |a|^2},$$

that is,

$$\|g_j(z)\| < \delta \|z\|. \quad \square$$
Proof of (4) of Proposition 18 It follows directly from the expression of the foliation $G$ given in (1) of Proposition 18 that the hyperplanes

$$\{ t_2 = 0 \}, \ldots, \{ t_{M-1} = 0 \}, \{ t_M = b_1 \}, \ldots, \{ t_M = b_N \}$$

are invariant by $G$. Therefore the hyperplanes

$$\{ x_2 = 0 \}, \ldots, \{ x_{M-1} = 0 \}, \{ x_M = b_1 x_1 \}, \ldots, \{ x_M = b_N x_1 \}$$

are invariant by $F$. The invariance of the hyperplane $\{ x_1 = 0 \}$ by $F$ is equivalent to the invariance by $G$ of the hyperplane $\{ t_M = \infty \}$ in the space $\pi^{-1}(\mathbb{C}^M)$. The local expression of the foliation $G$ near $\{ t_M = \infty \}$ is given in the proof of Lemma 35 at the end of the section. It will be clear from that expression—together with (5) of Lemma 26—that $\{ t_M = \infty \}$ is actually invariant by $G$, so we leave this verification to the reader—which will prove (4a) of Proposition 18. Given $\tau \in T$, the hypersurface $\pi^{-1}(S)$ near $p_\tau$ is given by the equation

$$t_1 \cdots t_{M-1}(t_M - \tau) = 0.$$

On the other hand, it follows from the construction of $h_\tau$—see (6.8)—that the map $f_\tau$ respectively maps the sets

$$\{ s_1 = 0 \}, \ldots, \{ s_{M-1} = 0 \}, \{ s_M = 0 \}$$

into the sets

$$\{ t_1 = 0 \}, \ldots, \{ t_{M-1} = 0 \}, \{ t_M - \tau = 0 \}.$$

This means that, in the local coordinates $(s_1, \ldots, s_M)$, the hypersurface $\pi^{-1}(S)$ is given by $\{ s_1 \cdots s_M = 0 \}$, which proves (4c) of Proposition 18. Recall that, given any $a' \in \mathbb{C}$ different from $b_1, \ldots, b_N$ and given any curve in $\mathbb{C}\setminus\{ b_1, \ldots, b_N \}$ connecting $a'$ with $a$, we obtain an associated holonomy map

$$h: \{ t_M = a' \} \to \{ t_M = a \},$$

which is a linear isomorphism. Thus, it follows from (1) of Lemma 30 that the leaf through a point $w \in \{ t_M = a' \}$ is everywhere dense provided

$$h(w) \in \Sigma^* = \{ t_M = a \}\setminus\{ t_1 \cdots t_{M-1} = 0 \}.$$

Thus, since the hyperplanes $\{ t_1 = 0 \}, \ldots, \{ t_{M-1} = 0 \}$ are invariant by $G$, the leaf through $w$ is dense if $w$ is not contained in any of these hyperplanes. Therefore we conclude that any leaf outside the hyperplanes
\{t_1 = 0\}, \ldots, \{t_{M-1} = 0\}, \{t_M = b_1\}, \ldots, \{t_M = b_N\}

is everywhere dense, so (4b) of Proposition 18 is proved. \(\square\)

Let \(\epsilon > 0\) and let \(\Delta \subset \mathbb{C}^M\) be a neighborhood of the origin—as given in (5) of Proposition 18. Choose \(\epsilon \in (0, \epsilon)\) such that

\[x \in \mathbb{C}^M, \ |x| < 3\epsilon \implies x \in \Delta. \quad (6.36)\]

**Lemma 31** There exists \(\rho \in (0, \epsilon)\) with the following property: if \(z \in \Sigma \setminus \pi^{-1}(0)\), \(\|z\| < \rho\), \(g \in G\) and \(\|g(z)\| < \rho\), then there exists an integral curve connecting \(z\) with \(g(z)\) of length smaller than \(\epsilon/3\).

**Proof**

Set

\[\rho = \min \left\{ \frac{(1 - \delta)\epsilon}{6Kn}, 1 \right\}\]

where \(K\) is as given in Lemma 29, \(\delta\) as in Lemma 30, and let \(z \in \Sigma \setminus \pi^{-1}(0), \|z\| < \rho\). Suppose that \(g\) is of the form \(g = g_i^l\) for some \(l \in \mathbb{N}, i \in \{1, \ldots, n\}\). Without loss of generality we can assume \(i = 1\). Since \(g_1 = \mathcal{H}_{\xi_1}\), the curve \(\xi_1^z\) connects \(z\) with \(g_1(z)\), the curve \(\xi_1^{g_1(z)}\) connects \(g_1(z)\) with \(g_1^2(z)\), the curve \(\xi_1^{g_1^2(z)}\) connects \(g_1^2(z)\) with \(g_1^3(z)\), and so on. Moreover, by (6.26) and (2) of Lemma 30, we have

\[\ell \left( \xi_1^{g_1^j(z)} \right) \leq K \|g_1^j(z)\| \leq K \delta^j \|z\|, \quad j \geq 0. \quad (6.37)\]

Then the integral curve

\[\alpha_1 := \xi_1^z \ast \xi_1^{g_1(z)} \ast \cdots \ast \xi_1^{g_1^{(l-1)}(z)},\]

connects \(z\) with \(g_1^l(z)\) and, from (6.37),

\[\ell(\alpha_1) = \ell(\xi_1^z) + \ell \left( \xi_1^{g_1(z)} \right) + \cdots + \ell \left( \xi_1^{g_1^{(l-1)}(z)} \right) \leq K \|z\| + K \delta \|z\| + \cdots + K \delta^{l-1} \|z\|,\]

\[\ell(\alpha_1) \leq \frac{K \|z\|}{1 - \delta}.\]

Suppose now that

\[g = g_n^{ol_n} \circ g_{n-1}^{ol_{n-1}} \circ \cdots \circ g_1^{ol_1},\]

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where \( l_1, \ldots, l_n \in \mathbb{N} \). By the construction above, we find an integral curve \( \alpha_1 \) connecting \( z \) with \( g_1^{ol_1}(z) \), such that

\[
\ell(\alpha_1) \leq \frac{K \| z \|}{1 - \delta} < \frac{K \rho}{1 - \delta},
\]

whence

\[
\ell(\alpha_1) < \frac{\varepsilon}{6n}.
\]

Again by the construction above—with \( g_1^{ol_1}(z) \) instead of \( z \)—there is an integral curve \( \alpha_2 \) connecting \( g_1^{ol_1}(z) \) with \( g_2^{ol_2}\left(g_1^{ol_1}(z)\right) \), such that

\[
\ell(\alpha_2) < \frac{K \| g_1^{ol_1}(z) \|}{1 - \delta} \leq \frac{K \| z \|}{1 - \delta} < \frac{K \rho}{1 - \delta},
\]

whence

\[
\ell(\alpha_2) < \frac{\varepsilon}{6n}.
\]

Then \( \alpha_1 \ast \alpha_2 \) is an integral curve that connects \( z \) with \( g_2^{ol_2} g_1^{ol_1} \) and satisfies

\[
\ell(\alpha_1 \ast \alpha_2) = \ell(\alpha_1) + \ell(\alpha_2) < 2 \left( \frac{\varepsilon}{6n} \right).
\]

By iterating this argument, we find curves \( \alpha_1, \ldots, \alpha_n \), such that

\[
\overline{\alpha} := \alpha_1 \ast \cdots \ast \alpha_n
\]

is an integral curve, connects \( z \) with \( g_n^{ol_n} \circ \cdots \circ g_1^{ol_1}(z) = g(z) \) and satisfies

\[
\ell(\overline{\alpha}) = \ell(\alpha_1) + \cdots + \ell(\alpha_n) < n \left( \frac{\varepsilon}{6n} \right),
\]

\[
\ell(\overline{\alpha}) < \frac{\varepsilon}{6}.
\]

If \( g \) is of the type above, that is, if

\[
g = g_n^{ol_n} \circ g_{n-1}^{ol_{n-1}} \circ \cdots \circ g_1^{ol_1},
\]

where \( l_1, \ldots, l_n \in \mathbb{N} \), we will say that \( g \) is contractive. Thus, we have proved that, if \( g \) is contractive and \( \| z \| < \rho \), then there exists an integral curve \( \overline{\alpha} \) that connects \( z \) with \( g(z) \), such that \( \ell(\overline{\alpha}) < \frac{\varepsilon}{6} \). Now, consider any \( g \in G \) such that \( \| g(z) \| < \rho \). Since \( G \) is abelian, we can find \( g_1, g_2 \in G \) contractive such that \( g = g_2^{-1} g_1 \). Since
\( g_1 \) is contractive and \( ||z|| < \rho \), there exists an integral curve \( \bar{\alpha}_1 \) connecting \( z \) with \( g_1(z) \), such that \( \ell(\bar{\alpha}_1) < \frac{\epsilon}{6} \). If we set \( z' = g(z) \), since \( g_2 \) is contractive and \( ||z'|| < \rho \), again we find an integral curve \( \bar{\alpha}_2 \) connecting \( z' = g(z) \) with \( g_2(z') = g_1(z) \), such that \( \ell(\bar{\alpha}_2) < \frac{\epsilon}{6} \). Finally, the curve

\[
\alpha := \bar{\alpha}_1 \ast \bar{\alpha}_2^{-1}
\]
is an integral curve connecting \( z \) with \( g(z) \), such that

\[
\ell(\alpha) = \ell(\bar{\alpha}_1) + \ell(\bar{\alpha}_2) < \frac{\epsilon}{3},
\]
so Lemma 31 is proved. \( \square \)

**Proof of (5a) of Proposition 18** Set

\[
\Sigma^* := \{ z \in \Sigma : ||z|| < \rho, z_1 \neq 0, \ldots, z_{M-1} \neq 0 \},
\]
where \( \rho \) is as given by Lemma 31. Let \( \mathcal{U} \) be the set of points that can be connected to a point in \( \Sigma^* \) by an integral curve of length smaller than \( \epsilon/3 \). Since \( \Sigma^* \) is transverse to the foliation, it is easy to see that \( \mathcal{U} \) is open. Moreover, since \( \Sigma^* \) is disjoint of the exceptional divisor \( \pi^{-1}(0) \), so is \( \mathcal{U} \). Then the set \( \pi(\mathcal{U}) \), which we identify with \( \mathcal{U} \), will be the set mentioned in (5) of Proposition 18. Fix \( p \in \mathcal{U} \). Then there exists an integral curve of length smaller than \( \epsilon/3 \), connecting \( p \) with a point \( z_0 \in \Sigma^* \). Set

\[
\mathcal{E} := \{ z \in \Sigma^* : z = g(z_0), g \in G \}.
\]

It follows from (1) of Lemma 30 that \( \mathcal{E} \) is dense in \( \Sigma_\rho = \{ z \in \Sigma : ||z|| < \rho \} \). By Lemma 31, each \( z \in \mathcal{E} \) is connected to \( z_0 \) by an integral curve of length smaller than \( \epsilon/3 \). Then, since \( z_0 \) in turn is connected to \( p \) by an integral curve of length smaller than \( \epsilon/3 \), each \( z \in \mathcal{E} \) is connected to \( p \) by an integral curve of length smaller than \( 2\epsilon/3 \). Then, if \( E \) is defined as the set of points that can be connected to a point in \( \mathcal{E} \) by an integral curve of length smaller than \( \epsilon/3 \), we conclude each point \( q \in E \) is connected to \( p \) by an integral curve of length smaller than \( \epsilon \leq \epsilon \). It is evident that \( E \) is contained in the leaf through \( p \): in fact, \( q \in E \) is connected to \( p \) by the integral curve \( \beta_q \). Moreover, it follows directly from the definitions of \( E \) and \( \mathcal{U} \) that \( E \subset \mathcal{U} \). Let us show that \( E \) is dense in \( \mathcal{U} \). Let \( U \subset \mathcal{U} \) be an open set. Fix a point \( w \in U \). Then there exists an integral curve \( \alpha \) connecting \( w \) with a point \( z \in \Sigma^* \), such that \( \ell(\alpha) < \epsilon/3 \). Since \( \Sigma^* \) is transverse to the foliation, if \( \Omega \subset U \) is a small enough neighborhood of \( w \), there exists a submersion \( h : \Omega \rightarrow \Sigma^* \) with \( h(w) = z \), such that each point \( w' \in \Omega \) can be connected to \( h(w') \in \Sigma^* \) by an integral curve \( \alpha' \) close enough to \( \alpha \) so that \( \ell(\alpha') < \epsilon/3 \). Thus, since \( \mathcal{E} \) is dense in \( \Sigma^* \), for a suitable choice of \( w' \in \Omega \) we have \( h(w') \in \mathcal{E} \). Then \( w' \in \Omega \cap E \), so \( E \) is dense in \( \mathcal{U} \). At this point, (5a) of Proposition 18 is almost proved: it only remains to verify the following two facts.

(1) \( \mathcal{U} \subset \Delta \). Rigorously speaking this means that \( \pi(\mathcal{U}) \subset \Delta \).
(2) For each \( q \in E \), the curve \( \beta_q \) above is contained in \( \Delta \). Rigorously speaking: \( \pi(\beta_q) \) is contained in \( \Delta \).

Let \( w \) be any point in \( U \). By the definition of \( U \) there exists an integral curve \( \alpha \) connecting \( w \) with a point \( z \in \Sigma^* \), such that \( \ell(\alpha) < \varepsilon/3 \). Thus, since \( \ell(\alpha) \) is actually the Euclidean length of \( \pi(\alpha) \), we have

\[
|\pi(w) - \pi(z)| < \frac{\varepsilon}{3}.
\]

Then

\[
|\pi(w)| < |\pi(z)| + \frac{\varepsilon}{3} < \rho + \frac{\varepsilon}{3} < \varepsilon + \frac{\varepsilon}{3},
\]

so

\[
|\pi(w)| < \frac{4\varepsilon}{3}.
\]

(6.38)

Thus, it follows from (6.36) that \( \pi(w) \in \Delta \), therefore the first fact above is proved.

Now, let \( \zeta \) be any point in \( \beta_q \). Then, since \( \ell(\beta_q) < \varepsilon \),

\[
|\pi(\zeta) - \pi(p)| < \varepsilon.
\]

(6.39)

Therefore, since from inequality (6.38) we obtain \( |\pi(p)| < \frac{4\varepsilon}{3} \), we find that

\[
|\pi(\zeta)| < \frac{7\varepsilon}{3}.
\]

(6.40)

Thus, again by (6.36) we see that \( \pi(\zeta) \in \Delta \). \( \square \)

We start with the proof of the two last assertions of Proposition 18.

**Lemma 32** Let \( D \subset \mathbb{C}\{b_1, \ldots, b_N\} \) be a closed disc. Then there exists \( \eta_D > 0 \) such that each point in

\[
\Omega_D := \left\{ t \in \mathbb{C}^M : t_1 \cdots t_{M-1} \neq 0, t_M \in D, \|t\| < \eta_D \right\}
\]

is connected to a point in \( \Sigma^* \) by an integral curve of length smaller that \( \varepsilon/6 \). In particular, \( \Omega_D \subset U \).

**Proof** For each \( c \in D \), we can choose a smooth curve

\[
\gamma_c : [0, 1] \to \mathbb{C}\{b_1, \ldots, b_N\}, \quad \gamma_c(0) = a, \quad \gamma_c(1) = c,
\]

depending smoothly on the parameter \( c \in D \). From Lemma 29 we have that there exists \( K_{\gamma_c} > 0 \) depending continuously on \( \gamma_c \), such that

\[
\ell(\gamma_c^z) < K_{\gamma_c}\|z\|, \quad z \in \Sigma^*.
\]
Then, since $D$ is compact, we can find a positive number $\varrho < \rho$ such that
\[ \ell(y_c^z) < \frac{\varepsilon}{6}, \quad z \in \Sigma^*, \ |z| < \varrho, \ c \in D. \tag{6.41} \]

For each $c \in D$, let
\[ h_c : \{t_M = c\} \to \{t_M = a\} \]
be the holonomy associated to $\gamma_c^{-1}$. Thus, given $w \in \{t_M = c\}$, we have $h_c(w) \in \{t_M = a\}$ and the integral curve $y_c^{\gamma(w)}$ connects $h(w)$ with $w$. Observe that, when $w_1, \ldots, w_{M-1} \neq 0$ and the hypersurface $\{t_1 \cdots t_{M-1} = 0\}$ is invariant, we have $h_c(w) \in \Sigma^*$. As we have seen before—see (6.21) and (6.22)—the holonomy $h_c$ is a linear map of the form
\[ h_c(t_1, \ldots, t_{M-1}, c) = (\vartheta_1(c)t_1, \ldots, \vartheta_{M-1}(c)t_{M-1}, a), \tag{6.42} \]
where
\[ \vartheta_i(c) = \exp \int_{\gamma_c^{-1}} A_i(u)du, \quad i = 1, \ldots, M - 1. \]

The closure of the image of the hyperplane $\{t_M = c\}$ by the blow-up map $\pi$ defines the linear subspace
\[ V_c : = \{(x_1, \ldots, x_M) \in \mathbb{C}^M : x_M = cx_1\}. \]

In the same way, $\{t_M = a\}$ defines the linear subspace
\[ V_a : = \{(x_1, \ldots, x_M) \in \mathbb{C}^M : x_M = ax_1\}. \]

It is easy to see from (6.42) that $\pi \circ h_c \circ \pi^{-1}$ extends as a linear isomorphism
\[ h_c : V_c \to V_a, \]
which depends continuously on $c \in D$. Then, since $D$ is compact, we can take $\eta_D > 0$ such that
\[ x \in V_c, \ |x| < \eta_D, \ c \in D \implies |h_c(x)| < \varrho. \]

Thus, applying this fact to $x = \pi(w)$ for any $w \in \Omega_D$ we obtain
\[ \|h_c(w)\| < \varrho, \quad w \in \Omega_D. \tag{6.43} \]

Moreover, since $w \in \Omega_D$ is outside the hypersurface $\{t_1 \cdots t_{M-1} = 0\}$, which is invariant by the foliation, we deduce that $h_c(w) \in \Sigma^*$. Then it follows from (6.41)
that the integral curve $\gamma_c^{h(w)}$, which connects $w$ with $h(w) \in \Sigma^*$, has length smaller than $\varepsilon/6$. $\square$

For each $j = 1, \ldots, N$, choose $r_j > 0$ such that the disc

$$D_j := \{ u \in \mathbb{C} : |u - b_j| < r_j \}$$

intersects $\{ b_1, \ldots, b_N \}$ exactly at $b_j$. Moreover, let $R > 0$ be such that the disc

$$D_R := \{ u \in \mathbb{C} : |u| < R \}$$

contains $\overline{D_1} \cup \cdots \cup \overline{D_N}$.

**Lemma 33** Set

$$\mathcal{K} := \overline{D_R} - D_1 \cup \cdots \cup D_N.$$

There exists $\eta_{\mathcal{K}} > 0$ such that each point in the set

$$\Omega_{\mathcal{K}} := \left\{ t \in \mathbb{C}^M : t_1 \cdots t_{M-1} \neq 0, t_M \in \mathcal{K}, \|t\| < \eta_{\mathcal{K}} \right\}$$

is connected to a point in $\Sigma^*$ by an integral curve of length smaller than $\varepsilon/6$. In particular, $\Omega_{\mathcal{K}} \subset \mathcal{U}$.

**Proof** Since $\mathcal{K}$ is compact and contained in $\mathbb{C} \setminus \{ b_1, \ldots, b_N \}$, we can cover $\mathcal{K}$ by finitely many compact discs $D_1, \ldots, D_n$, $n \in \mathbb{N}$, each of them contained in $\mathbb{C} \setminus \{ b_1, \ldots, b_N \}$. From Lemma 32 each point in the union of $\Omega_{D_1}, \ldots, \Omega_{D_n}$ is connected to a point in $\Sigma^*$ by an integral curve of length smaller than $\varepsilon/6$. Thus, since the choice $\eta_{\mathcal{K}} := \min\{ \eta_{D_1}, \ldots, \eta_{D_n} \}$ implies

$$\Omega_{\mathcal{K}} \subset \Omega_{D_1} \cup \cdots \cup \Omega_{D_n},$$

the lemma follows. $\square$

**Lemma 34** Let $l \in \{ 1, \ldots, N \}$. Suppose there exists a real line through the origin in $\mathbb{C}$ separating $1 \in \mathbb{C}$ from the set

$$\{ \nu_1l, \nu_1l + \nu_2l, \ldots, \nu_1l + \nu_{(M-1)l} \}.$$  

Then there exists $\eta_l > 0$ such that $\mathcal{U}$ contains the set

$$\Omega_l := \left\{ t \in \mathbb{C}^M : t_1 \cdots t_{M-1} \neq 0, t_M \neq b_l, t_M \in D_l, \|t\| < \eta_l \right\}.$$

In view of (4) of Lemma 26 and Remark 27, the conclusion of this lemma takes place if $b_l \notin T$.  

\begin{center} Springer \end{center}
Proof Recall that the vector field $Y$ defining $G$ is given by

$$Y = \sum_{i=1}^{M-1} A_i(t_M)t_i \frac{\partial}{\partial t_i} + \frac{\partial}{\partial t_M},$$

where

$$A_i(t_M) = \sum_{j=1}^{N} \frac{v_{ij}}{t_M - b_j}, \quad i = 1, \ldots, M - 1.$$  

Write

$$A_i(u) = \frac{v_{ii}}{u - b_i} + g_i(u), \quad (6.45)$$

where

$$g_i(u) = \sum_{j=1, j \neq i}^{N} \frac{v_{ij}}{u - b_j}.$$  

Since $g_i$ is holomorphic on a neighborhood of $\overline{D}_l$ we can find $K_1 > 1$ such that, for each $i = 1, \ldots, M - 1$, we have

$$|g_i(u)| \leq K_1, \quad u \in \overline{D}_l, \quad \text{and}$$

$$|A_i(u)| \leq \psi(u) : = K_1 + \frac{K_1}{|u - b_i|}, \quad u \in \overline{D}_l - \{b_l\}. \quad (6.47)$$

It follows from (4) of Lemma 26 that there exists $\theta \in \mathbb{C}^*$ such that

$$\text{Re}(\theta) > 0, \quad \text{and}$$

$$\text{Re}(\theta v_{1l}), \text{Re}(\theta [v_{1l} + v_{2l}]), \ldots, \text{Re}(\theta [v_{1l} + v_{(M-1)l}]) < 0. \quad (6.48)$$

Set $\Gamma = \left\{1, v_{1l}, (v_{1l} + v_{2l}), \ldots, (v_{1l} + v_{(M-1)l})\right\}$ and

$$K_2 := \max_{\zeta \in \Gamma} \frac{2K_1(r_l + 1)|\theta|e^{\frac{2K_1r_l|\theta|}{\text{Re}(\theta)}}}{|\text{Re}(\xi \theta)|}, \quad (6.49)$$

and take $\eta_l > 0$ such that

$$\eta_l(M + |b_l| + r_l)e^{\frac{2K_1r_l|\theta|}{\text{Re}(\theta)}} < \eta_3, \quad \text{and}$$

$$\eta_l(M + |b_l| + r_l)K_2 < \frac{\varepsilon}{6}. \quad (6.51)$$
Now, fix a point $\sigma = (\sigma_1, \ldots, \sigma_M) \in \Omega_I$. Thus

$$\|\sigma\| < \eta, \quad 0 < |\sigma_M - b| < r_l, \quad \sigma_1 \cdots \sigma_{M-1} \neq 0.$$  

(6.52)

Set

$$\theta_\sigma := \frac{\log \left( \frac{r_l}{|\sigma_M - b|} \right) \theta}{\text{Re}(\theta)}$$  

(6.53)

and consider the curve

$$\gamma(s) = b_l + (\sigma_M - b_l)e^{\theta_\sigma s}, \quad s \in [0, 1].$$  

(6.54)

It is easy to check that

$$\gamma(0) = \sigma_M, \quad \gamma(1) \in \partial \mathcal{D}_l.$$  

(6.55)

As we have seen—see (6.21)—the curve

$$\gamma^\sigma(s) := (\gamma_1(s), \ldots, \gamma_{M-1}(s), \gamma(s)), \quad s \in [0, 1],$$  

(6.56)

where

$$\gamma_i(s) = \sigma_i \exp \int_{\gamma|[0,s]} A_i(u)du, \quad i = 1, \ldots, M - 1,$$  

(6.57)

is an integral curve connecting $\sigma$ with

$$\gamma^\sigma(1) = (\gamma_1(1), \ldots, \gamma_{M-1}(1), \gamma(1)).$$  

(6.58)

Let us prove that $\gamma^\sigma(1) \in \Omega_\mathcal{K}$. Firstly, since $\partial \mathcal{D}_l \subset \mathcal{K}$, it follows from (6.55) that $\gamma(1) \in \mathcal{K}$. Secondly, since the hyperplanes

$$\{t_1 = 0\}, \ldots, \{t_{M-1} = 0\}$$

are invariant by $\mathcal{G}$ and none of these hyperplanes contains $\sigma$, we conclude

$$\gamma^\sigma(1) \notin \{t_1 \cdots t_{M-1} = 0\}.$$  

Thus it is enough to prove that $\|\gamma^\sigma(1)\| < \eta_\mathcal{K}$.

\[ \text{ Springer} \]
From (6.45) we have

\[
\int_{\gamma[0,s]} A_i(u) = \int_{\gamma[0,s]} \frac{v_i\ell}{u-b_l} du + \int_{\gamma[0,s]} g_i(u) du
\]

\[
= \int_{0}^{s} \frac{v_i\ell}{\gamma(s)-b_l} \gamma'(s) ds + \int_{0}^{s} g_i(\gamma(s)) \gamma'(s) ds,
\]

(6.59)

so that

\[
\int_{\gamma[0,s]} A_i(u) = (v_i\ell\theta_\sigma)s + \int_{0}^{s} g_i(\gamma(s)) \gamma'(s) ds.
\]

(6.60)

Let us show that the function

\[
B_i(s) := \int_{0}^{s} g_i(\gamma(s)) \gamma'(s) ds
\]

has a bound independent of \(\sigma\). From (6.46),

\[
|B_i(s)| \leq K_1 \int_{0}^{s} |\gamma'(s)| ds,
\]

so it suffices to show that the length \(\ell(\gamma) = \frac{1}{0} \int |\gamma'(s)| ds\), has a bound independent of \(\sigma\). We have

\[
\int_{0}^{1} |\gamma'(s)| ds = \int_{0}^{1} |\theta_\sigma(\sigma_M-b_l)e^{\theta_\sigma s}| ds = \int_{0}^{1} |\theta_\sigma(\sigma_M-b_l)| e^{\text{Re}(\theta_\sigma)s} ds,
\]

(6.61)

\[
\int_{0}^{1} |\gamma'(s)| ds = \frac{|\theta_\sigma||\sigma_M-b_l|}{\text{Re}\theta_\sigma} \cdot (e^{\text{Re}(\theta_\sigma)} - 1)
\]

(6.62)

Thus, since from (6.53)

\[
|\theta_\sigma| = \frac{|\theta|}{|\text{Re}\theta|} \log \frac{r_l}{|\sigma_M-b_l|} \quad \text{and} \quad \text{Re}\theta_\sigma = \log \frac{r_l}{|\sigma_M-b_l|}
\]
we have
\[
\int_0^1 |\gamma'(s)| \, ds = \frac{|\theta|}{|\text{Re} \theta|} |\sigma_M - b_l| \left( \frac{r_l}{|\sigma_M - b_l|} - 1 \right)
\] (6.63)
so that
\[
\int_0^1 |\gamma'(s)| \, ds < \frac{r_l|\theta|}{|\text{Re} \theta|}.
\] (6.64)
Therefore
\[
|B_i(s)| \leq \frac{K_1r_l|\theta|}{|\text{Re} \theta|}, \quad i = 1, \ldots, M - 1
\] (6.65)
and from (6.60) we obtain
\[
\text{Re} \int_{\gamma'[0,s]} A_i(u) \leq \text{Re}(\nu_1l\theta\sigma) s + \frac{K_1r_l|\theta|}{|\text{Re} \theta|}.
\] (6.66)
Then, since
\[
|\gamma_i(s)| = \sigma_i \exp \int_{\gamma'[0,s]} A_i(u) \, du = |\sigma_i| \exp \text{Re} \int_{\gamma'[0,s]} A_i(u) \, du,
\] (6.67)
we obtain
\[
|\gamma_i(s)| \leq |\sigma_i| e^{\frac{K_1r_l|\theta|}{|\text{Re} \theta|}} e^{\text{Re}(\nu_1l\theta\sigma) s}.
\] (6.68)
On the other hand, from (6.48), (6.53) and the inequality $|\sigma_M - b_l| < r_l$ we obtain that
\[
\text{Re} \left( \nu_1l\theta\sigma \right), \text{Re} \left( \nu_1l + \nu_2l\theta\sigma \right), \ldots, \text{Re} \left( \nu_1l + \nu(M-1)l\theta\sigma \right) < 0.
\] (6.69)
Then, from this and (6.68) we have
\[
|\gamma_1| \leq |\sigma_1| e^{\frac{K_1r_l|\theta|}{|\text{Re} \theta|}} e^{\text{Re}(\nu_1l\theta\sigma) s} \leq |\sigma_1| e^{\frac{K_1r_l|\theta|}{|\text{Re} \theta|}}, \\
|\gamma_1\gamma_2| \leq |\sigma_1\sigma_2| e^{\frac{2K_1r_l|\theta|}{|\text{Re} \theta|}} e^{\text{Re}(\nu_1l + \nu_2l\theta\sigma) s} \leq |\sigma_1\sigma_2| e^{\frac{2K_1r_l|\theta|}{|\text{Re} \theta|}}, \\
\vdots \\
|\gamma_1\gamma_{M-1}| \leq |\sigma_1\sigma_{M-1}| e^{\frac{K_1r_l|\theta|}{|\text{Re} \theta|}} e^{\text{Re}(\nu_1l + \nu(M-1)l\theta\sigma) s} \leq |\sigma_1\sigma_{M-1}| e^{\frac{K_1r_l|\theta|}{|\text{Re} \theta|}}.
\] (6.70)
Therefore
\[ \| \gamma^\sigma (1) \| \leq | \gamma_1 (1) | + | \gamma_1 (1) \gamma_2 (1) | + \cdots + | \gamma_1 (1) \gamma_{M-1} (1) | + | \gamma_1 (1) \gamma (1) | \]
\[ \leq | \sigma_1 | e^{ K_1 | \Re \theta |} + | \sigma_1 \sigma_2 | e^{ 2 K_1 | \Re \theta |} + \cdots + | \sigma_1 \sigma_{M-1} | e^{ K_1 | \Re \theta |} + | \sigma_1 | e^{ - K_1 | \Re \theta |} (| b_I | + r_I) \]
\[ \leq \| \sigma \| \left( e^{ K_1 | \Re \theta |} + (M - 2) e^{ - K_1 | \Re \theta |} + e^{ K_1 | \Re \theta |} (| b_I | + r_I) \right) \]
\[ \leq \| \sigma \| (M + | b_I | + r_I) e^{ K_1 | \Re \theta |}, \]
so that, by (6.52) and (6.50),
\[ \| \gamma^\sigma (1) \| < \eta_K, \]
which shows that \( \gamma^\sigma (1) \in \Omega_\infty \). Since Lemma 33 guarantees that \( \gamma^\sigma (1) \) is connected to a point in \( \Sigma^* \) by an integral curve of length smaller than \( \varepsilon / 6 \), and \( \gamma^\sigma \) connects \( \sigma \) with \( \gamma^\sigma (1) \), Lemma 34 will be proved if we show that \( \ell (\gamma^\sigma) < \varepsilon / 6 \). To see this we write
\[ \ell (\gamma^\sigma) = \int_0^1 | (\pi \circ \gamma^\sigma)' (s) | ds \leq \int_0^1 | \gamma_1' | ds + \int_0^1 | (\gamma_1 \gamma_2)' | ds + \cdots \]
\[ \cdots + \int_0^1 | (\gamma_1 \gamma_{M-1})' | ds + \int_0^1 | (\gamma_1 \gamma)' | ds. \]  
(6.71)

Let us find bounds for each of the integrals above. From (6.57) and (6.47), we have
\[ \int_0^1 | \gamma_1' | ds = \int_0^1 | A_1 (\gamma) \gamma' \gamma_1 | ds \leq \int_0^1 | \psi (\gamma) \gamma' \gamma_1 | ds. \]  
(6.72)

Observe that, since \( \Re \theta_\sigma = \log \frac{r_I}{| \sigma_M - b_I |} > 0 \),
\[ | \psi (\gamma) \gamma' | = \left( K_1 + \frac{K_1}{| \gamma - b_I |} \right) | \gamma' | \]
\[ = \left( K_1 + \frac{K_1}{| \sigma_M - b_I | e^{ \Re \theta_\sigma s} \right) | \sigma_M - b_I | e^{ \Re \theta_\sigma s} \]
\[ \leq \left( K_1 | \sigma_M - b_I | e^{ \Re \theta_\sigma s} + K_1 \right) | \sigma_M - b_I | e^{ \Re \theta_\sigma s} \]
\[ \leq \left( K_1 | \sigma_M - b_I | e^{ \log \frac{r_I}{| \sigma_M - b_I |} } + K_1 \right) | \sigma_M - b_I | e^{ \Re \theta_\sigma s}. \]
\[ |\psi(\gamma)\gamma'| \leq K_1(r_l + 1)|\theta_{\sigma}|. \quad (6.73) \]

Moreover, from (6.69) we have \( \text{Re}(v_{1l}\theta_{\sigma}) < 0 \) and it follows from (6.53) that

\[
\frac{|\theta_{\sigma}|}{|\text{Re} v_{1l}\theta_{\sigma}|} = \frac{|\theta|}{|\text{Re} v_{1l}\theta|}. 
\]

Using these facts together with (6.73) and (6.68) in (6.72), we obtain

\[
\int_0^1 |\gamma_1'| ds \leq \int_0^1 |\psi(\gamma)\gamma'| \gamma_1| ds \leq K_1(r_l + 1)|\theta_{\sigma}| \int_0^1 |\sigma_1| e^{\frac{K_{1|\gamma'|}}{|\text{Re} v_{1l}\theta|}} e^{\text{Re}(v_{1l}\theta_{\sigma})s} ds \\
= \frac{K_1(r_l + 1)|\theta_{\sigma}| |\sigma_1| e^{\frac{K_{1|\gamma'|}}{|\text{Re} v_{1l}\theta|}}}{|\text{Re}(v_{1l}\theta_{\sigma})|} \left( 1 - e^{\text{Re}(v_{1l}\theta_{\sigma})} \right) \\
\leq K_1(r_l + 1)|\theta_{\sigma}| |\sigma_1| e^{\frac{K_{1|\gamma'|}}{|\text{Re} v_{1l}\theta|}},
\]

so that,

\[
\int_0^1 |\gamma_1'| ds \leq K_2|\sigma|, \quad (6.74) 
\]

were \( K_2 \) is as defined in (6.49).

Now, for each \( i = 2, \ldots, M - 1 \), from (6.70) we have

\[
|\gamma_{1i}| \leq |\sigma| |e^{2 \frac{K_{1|\gamma'|}}{|\text{Re} v_{1l}\theta|}} e^{\text{Re}(v_{1l} + v_{2l})\theta_{\sigma}}|. \quad (6.75) 
\]

This together with (6.57) and (6.73) leads to

\[
\int_0^1 |(\gamma_1\gamma_i)'| ds = \int_0^1 |\gamma_1\gamma_i + \gamma_i'\gamma_1| ds = \int_0^1 |A_1(\gamma)\gamma'\gamma_1\gamma_i + A_i(\gamma)\gamma'\gamma_i\gamma_1| ds \\
\leq \int_0^1 2|\psi(\gamma)\gamma'\gamma_1\gamma_i| ds \\
\leq \int_0^1 2K_1(r_l + 1)|\theta_{\sigma}| |\sigma| |e^{2 \frac{K_{1|\gamma'|}}{|\text{Re} v_{1l}\theta|}} e^{\text{Re}(v_{1l} + v_{2l})\theta_{\sigma}}| ds \\
\leq 2K_1(r_l + 1)|\theta_{\sigma}| |\sigma| |e^{2 \frac{K_{1|\gamma'|}}{|\text{Re} v_{1l}\theta|}} e^{\text{Re}(v_{1l} + v_{2l})\theta_{\sigma}} - \frac{1}{\text{Re}(v_{1l} + v_{2l})\theta_{\sigma}}|. \quad (6.76) 
\]
Again by (6.69) and (6.53), we see that \( \Re (\nu_1 + \nu_2) \theta_\sigma < 0 \) and
\[
\frac{|\theta_\sigma|}{|\Re (\nu_1 + \nu_2) \theta_\sigma|} = \frac{|\theta|}{|\Re (\nu_1 + \nu_2) \theta|}.
\]
Thus, from (6.76) we obtain
\[
\int_0^1 |(\gamma_1)'| ds \leq \frac{2K_1 (r_l + 1)|\theta_\sigma| \|\sigma\| e^{2K_1|\nu_1| \Re |\theta_\sigma|}}{|\Re (\nu_1 + \nu_2) \theta_\sigma|} \left(1 - e^{\Re (\nu_1 + \nu_2) \theta_\sigma} s\right)
\]
(6.77)
\[
\leq \frac{2K_1 (r_l + 1)|\theta| \|\sigma\| e^{2K_1|\nu_1| \Re |\theta|}}{|\Re (\nu_1 + \nu_2) \theta|},
\]
(6.78)
so that
\[
\int_0^1 |(\gamma_1)'| ds \leq K_2 \|\sigma\|, \quad i = 2, \ldots, M - 1.
\]
(6.79)

On the other hand,
\[
\int |(\gamma_1)'| ds = \int |\gamma_1'| ds + \int |\gamma_1\gamma'| ds \leq \int |\gamma_1'| ds + \int |\gamma_1\gamma'| ds.
\]

Then, from (6.70), (6.74) and (6.64) we have
\[
\int |(\gamma_1\gamma')| ds \leq \int |\gamma_1'| |(b_l| + r_l) ds + \int e^{K_1|\nu_1| \Re |\theta|} \|\sigma\| |\gamma'| ds
\]
\[
\leq K_2 \|\sigma\| (|b_l| + r_l) + e^{K_1|\nu_1| \Re |\theta|} \|\sigma\| \int |\gamma'| ds
\]
\[
\leq K_2 \|\sigma\| (|b_l| + r_l) + e^{K_1|\nu_1| \Re |\theta|} \frac{r_l|\theta|}{|\Re |\theta|} \|\sigma\|.
\]

Thus, since the inequality \( K_1 > 1 \) implies
\[
e^{K_1|\nu_1| \Re |\theta|} \frac{r_l|\theta|}{|\Re |\theta|} \leq K_2,
\]
we obtain
\[
\int |(\gamma_1\gamma')| ds \leq K_2 (|b_l| + r_l + 1) \|\sigma\|.
\]
(6.80)

Using (6.80), (6.79) and (6.74) in (6.71), we obtain
\[ \ell(\gamma^\sigma) \leq \int |\gamma_1'|ds + \int |(\gamma_1\gamma_2')|ds + \cdots + \int |(\gamma_1\gamma_{M-1}')|ds + \int |(\gamma_1\gamma')|ds \]
\[ \leq K_2\|\sigma\| + \cdots + K_2\|\sigma\| + K_2(|b_i| + r_i + 1)\|\sigma\| \]
and it follows from (6.52) and (6.51) that
\[ \ell(\gamma^\sigma) < \varepsilon. \]

Lemma 34 is proved. \( \Box \)

Now, we want to establish a result analogous to Lemma 34, but near \( t_M = \infty \). To do so, we consider coordinates \((s_1, \ldots, s_M)\) in \( \pi^{-1}(\mathbb{C}^M) \) such that
\[ t = \phi(s) := \left( s_1s_M, \frac{s_2}{s_M}, \ldots, \frac{s_{M-1}}{s_M}, \frac{1}{s_M} \right). \]

Note that
\[ \pi(\phi(s)) = (s_1s_M, s_1s_2, \ldots, s_1s_{M-1}, s_1). \]

Thus, if we define
\[ \tilde{\pi} : (s_1, \ldots, s_m) \mapsto (\tilde{x}_1, \ldots, \tilde{x}_M) = (s_1, s_1s_2, \ldots, s_1s_M), \]
we can see that
\[ |\tilde{\pi}(s)| = |\pi(\phi(s))|. \]

Therefore it will be consistent to define
\[ \|s\| := |(s_1, s_1s_2, \ldots, s_1s_M)|. \]

Let \( R \) be as before—see (6.44)—and consider the disc
\[ \widehat{D} := \{ s_M \in \mathbb{C} : |s_M| \leq R \}. \]

**Lemma 35** There exists \( \tilde{\eta} > 0 \) such that \( \mathcal{U} \) contains the set
\[ \tilde{\Omega} := \left\{ s \in \mathbb{C}^M : s_1 \cdots s_{M-1}s_M \neq 0, \ s_M \in \widehat{D}, \ \|s\| < \tilde{\eta} \right\}. \]

**Remark 36** Note that in the coordinates \((t_1, \ldots, t_M)\) the set \( \tilde{\Omega} \) is given by
\[ \tilde{\Omega} = \left\{ t \in \mathbb{C}^M : t_1 \cdots t_{M-1} \neq 0, \ |t_M| > R, \ \|t\| < \tilde{\eta} \right\}. \]
Proof We can check that $\pi \circ \phi \circ \tilde{\pi}^{-1}$ is given by

$$(\tilde{x}_1, \ldots, \tilde{x}_M) \mapsto (x_1, \ldots, x_M) = (\tilde{x}_M, \tilde{x}_2, \ldots, \tilde{x}_{M-1}, \tilde{x}_1),$$

which is an Euclidean isometry. This means that, given a curve $\alpha$ in the coordinates $(s_1, \ldots, s_M)$, the Euclidean length of $\tilde{\pi}(\alpha)$ coincides with the Euclidean length of $\pi(\phi(\alpha))$, that is, we can define $\ell(\alpha)$ as the length of $\tilde{\pi}(\alpha)$ and we will have $\ell(\alpha) = \ell(\phi(\alpha))$. Thus, we are able to work in the coordinates $(s_1, \ldots, s_M)$ following the same steps of the proof of Lemma 34. Let us do that. In the coordinates $(s_1, \ldots, s_M)$, the foliation $\mathcal{G}$ is generated by the rational vector field

$$\tilde{Y} : = -\frac{1}{s_M^2} \phi^*(Y),$$

which—by a straightforward computation—is expressed in the form

$$\tilde{Y} = \sum_{i=1}^{M-1} \tilde{A}_i(s_M) s_i \frac{\partial}{\partial s_i} + \frac{\partial}{\partial s_M},$$

where

$$\tilde{A}_i(s_M) = \left( \tilde{v}_i + \tilde{g}_i(s_M) \right), \quad i = 1, \ldots, M - 1,$$

$$\tilde{v}_1 = -1 - \sum_{j=1}^{N} v_{1j},$$

$$\tilde{v}_2 = 1 - \sum_{j=1}^{N} v_{2j},$$

$$\vdots$$

$$\tilde{v}_{M-1} = 1 - \sum_{j=1}^{N} v_{(M-1)j},$$

and $\tilde{g}_1, \ldots, \tilde{g}_{M-1}$ are holomorphic on a neighborhood $\tilde{D}$. Now, we choose $\tilde{K}_1 > 1$ such that, for each $i = 1, \ldots, M - 1$, we have

$$|\tilde{g}_i(u)| \leq \tilde{K}_1, \quad u \in \tilde{D}, \text{ and}$$

$$|\tilde{A}_i(u)| \leq \tilde{\psi}(u) : = \tilde{K}_1 + \frac{\tilde{K}_1}{|u|}, \quad u \in \tilde{D}.$$
From (5) of Lemma 26, we can find \( \tilde{\theta} \in \mathbb{C} \) such that

\[
\text{Re}(\tilde{\theta}) > 0, \quad \text{and} \quad \text{Re}(\tilde{\theta} \tilde{v}_1), \text{Re}(\tilde{\theta} [\tilde{v}_1 + \tilde{v}_2]), \ldots, \text{Re}(\tilde{\theta} [\tilde{v}_1 + \tilde{v}_{M-1}]) < 0.
\]

Set

\[
\tilde{\Gamma} := \{1, \tilde{v}_1, (\tilde{v}_1 + \tilde{v}_2), \ldots, (\tilde{v}_1 + \tilde{v}_{(M-1)})\}
\]

and

\[
\tilde{K}_2 := \max_{\zeta \in \tilde{\Gamma}} \frac{2\tilde{K}_1(R + 1)|\tilde{\theta}|}{|\text{Re}(\zeta \tilde{\theta})|},
\]

and take \( \tilde{\eta} > 0 \) such that

\[
\tilde{\eta}(M + R)e^{-\frac{2\tilde{K}_1 R |\tilde{\theta}|}{\text{Re} \tilde{\theta}}} < \eta \zeta, \quad \text{and} \quad \tilde{\eta}(M + R)\tilde{K}_2 < \frac{\epsilon}{6}.
\]

From this point forward, the proof follows exactly as in Lemma 34, so we leave the details to the reader. \( \Box \)

**Proof of (5b) of Proposition 18** Let

\[\zeta := [\xi_1 : \cdots : \xi_M]\]

be a point in the exceptional divisor \( \pi^{-1}(0) \) outside the tangent cone of

\[\prod_{\tau \in T} (x_M - \tau x_1) = 0.\]

Suppose first that \( \xi_1 \neq 0 \). Then the point \( \zeta \) is given in the coordinates \((t_1, \ldots, t_M)\) as

\[\zeta = \left(0, \frac{\xi_2}{\xi_1}, \ldots, \frac{\xi_M}{\xi_1}\right).\]

If \( \frac{\xi_M}{\xi_1} \notin \{b_1, \ldots, b_N\} \), we take a closed disc

\[D \subset \mathbb{C} \setminus \{b_1, \ldots, b_N\}\]

centered at \( \frac{\xi_M}{\xi_1} \) and so, by Lemma 32, the corresponding set \( \Omega_D \) is contained in \( \mathcal{U} \). Recall that we are identifying the sets \( \mathcal{U} \) and \( \pi^{-1}(\mathcal{U}) \). Since in the statement of Proposition 18 the notation \( \mathcal{U} \) refers to the set before the blow-up, rigorously speaking Lemma 32 shows that \( \Omega_D \subset \pi^{-1}(\mathcal{U}) \). Therefore, the set \( \Omega_D \cup \pi^{-1}(S) \), which is a neighborhood
of \( \zeta \), is contained in \( \pi^{-1}(U \cup S) \). If \( \frac{\tau_l}{\tau_M} = \frac{b_l}{b_M} \) for some \( l = 1, \ldots, N \) with \( b_l \not\in T \), by Lemma 34 we have \( \Omega_l \subset \pi^{-1}(U) \). Therefore the set \( \Omega_l \cup \pi^{-1}(S) \), which is a neighborhood of \( \zeta \), is contained in \( \pi^{-1}(U \cup S) \).

Suppose now that \( \tau_1 = 0 \). In this case \( \tau_M \neq 0 \), otherwise \( \zeta \) belongs to the tangent cone above. Then, by (6.81) and (6.82) the point \( \zeta \) is given in the coordinates \( (s_1, \ldots, s_M) \) as

\[
\zeta = \left( 0, \frac{x_2}{x_{\tau_1}}, \ldots, \frac{x_{\tau_{M-1}}}{x_{\tau_M}}, \frac{x_1}{x_{\tau_M}} \right).
\]

Thus, by Lemma 35 we have \( \tilde{\Omega} \subset \pi^{-1}(U) \). Therefore the set \( \tilde{\Omega} \cup \pi^{-1}(S) \), which is contained in \( \pi^{-1}(U \cup S) \), is a neighborhood of any point in \( \{ t_1 = 0, t_M = \tau \} \).

**Proof of (5c) of Proposition 18** Since \( \tau \in T \) we have \( \tau = \frac{b_l}{b_M} \) for some \( l = 1, \ldots, m \). Thus, it follows from Lemma 34 that \( \Omega_l \subset \pi^{-1}(U) \). Therefore the set \( \Omega_l \cup \pi^{-1}(S) \), which is contained in \( \pi^{-1}(U \cup S) \), is a neighborhood of any point in \( \{ t_1 = 0, t_M = \tau \} \). If the property about \( \tau \) takes place for all \( \tau \in T \), then the conclusion of Lemma 34 holds for each \( l = 1, \ldots, N \). This together with Lemmas 33 and 35 leads to the inclusion

\[
\Omega_1 \cup \cdots \cup \Omega_N \cup \Omega_{\tilde{\Omega}} \cup \tilde{\Omega} \subset \pi^{-1}(U).
\]

But if we take

\[
\eta : = \min(\eta_1, \ldots, \eta_N, \eta_{\tilde{\Omega}}, \tilde{\eta}),
\]

it follows from the definitions of the sets \( \Omega_1, \ldots, \Omega_N, \Omega_{\tilde{\Omega}}, \tilde{\Omega} \) and Remark 36 that

\[
\Omega_1 \cup \cdots \cup \Omega_N \cup \Omega_{\tilde{\Omega}} \cup \tilde{\Omega}
\]

contains the set

\[
\Omega : = \left\{ t \in \mathbb{C}^M : t_1, \ldots, t_{M-1} \neq 0, t_M \neq b_l, l = 1, \ldots, N, \| t \| < \eta \right\},
\]

whence \( \Omega \subset \pi^{-1}(U) \). Therefore the set \( \Omega \cup \pi^{-1}(S) \), which is a neighborhood of the exceptional divisor, is contained in \( \pi^{-1}(U \cup S) \). The proof of Proposition 18 is finished. \( \square \)

**References**

Alarcón, A., Forstneric, F., López, F.J.: Holomorphic Legendrian curves. Compos. Math. 153, 1945–1986 (2017)

Alarcón, A., Forstneric, F., Larusson, F.: Holomorphic Legendrian curves in \( \mathbb{CP}(3) \) and superminimal surfaces in \( S^4 \). \texttt{arXiv:1910.12996} (2019)

Arnold, V.I.: Geometrical Methods in the Theory of Ordinary Differential Equations. Grundlehren der mathematischen Wissenschaften, vol. 250. Springer, New York (1983)
Arnold, V.I.: Mathematical Methods of Classical Mechanics. Graduate Texts in Mathematics, vol. 60. Springer, New York (1989)
Bonnet, P.: Families of k-derivations on k-algebras. J. Pure Appl. Algebra 199(1–3), 11–26 (2005)
Buczynski, J.: Algebraic Legendrian varieties. arXiv:0805.3848v2 (2008)
Chow, W.L.: Uber Systeme von linearen partiellen Differentialgleichungen erster Ordnung. Math. Ann. 117, 98–105 (1939)
Darboux, G.: Sur le problème de Pfaff. Bull. Sci. Math. 6, 14–36, 49–68 (1882)
Godbillon, C.: Géométrie différentielle et mécanique analytique, p. 183. Hermann, Paris (1969)
Gromov, M.: Carnot-Carathéodory Spaces Seen from Within, in Sub-Riemannian Geometry. Progr. Math., vol. 144. Birkhauser, Basel, pp. 79–323 (1996)
Nagata, M., Nowicki, A.: Rings of constants for k-derivations in k[x1, . . . , xn]. J. Math. Kyoto Univ. 28(1), 111–118 (1998)
Nowicki, A.: Rings and fields of constants for derivations in characteristic zero. J. Pure Appl. Algebra 96(1), 47–55 (1994)
Singer, M.F.: Liouvillian first integrals of differential equations. Trans. Am. Math. Soc. 333, 673–688 (1992)
Zorich, V.A.: Holomorphic distributions and connectivity by integral curves of distributions. arXiv:1907.05610 (2019)

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