On the electromagnetic self-force of a point charge

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Abstract

We calculate the self-force of a point charge in rectilinear motion, using a local method and compare our results with those from the literature.

1. Introduction

The problem of the self-force acting on a moving point charged particle has been considered for a long time ([1], [2]). The most used formulas for the radiation reaction are those of Abraham-Lorentz-Dirac and Landau-Lifshitz, but there is still no consensus on which is more accurate [3]. There are difficulties regarding the pathological solutions of the Abraham-Lorentz-Dirac equation of motion (runaway solutions, preacceleration, postacceleration) which are still debated and different authors try to solve them either by modifying the equation of motion or by adding additional hypotheses regarding the initial conditions [3].

Most authors calculate the self-force using the energy-momentum conservation for the system consisting of the moving particle and its own field (global method). Still, a few authors calculate the self-force using a world-line calculation (or local method) ([5], [6]). This last method uses a Green function on the charge’s world-line that is the sum of a self-Green function and a radiation Green function ([6]). When introduced in the wave equation, this Green function gives a self-field and a radiation field, which can be written in terms of the advanced and retarded potentials.

The method presented in this paper for calculating the electromagnetic self-force is a local method. Still, we are able to manage using only the retarded field. We use delta-function formalism and all the derivatives in this paper are in the distributional sense. The only 'regularization' used in this paper is by performing the angular integration before the radial integration.

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This paper is organized as follows: in Section 2 we present our calculation of the self-force, in Section 3 we compare our result with other results from the literature, and the end section is devoted to conclusions.

2. Our method

For simplicity, we consider a point charge moving along the \(x\)-axis, but it should be possible to generalize the calculation for an arbitrary motion. The charge and current densities are given by:

\[
\rho(\vec{r}, t) = e\delta(\vec{r} - \vec{i}w(t)) = e \sum_{n=0}^{\infty} \frac{(-1)^n w^n(t)}{n!} \partial_x^n \delta(\vec{r}),
\]

\[
j_x(\vec{r}, t) = ew(t)\delta(\vec{r} - \vec{i}w(t)), \quad j_y(\vec{r}, t) = j_z(\vec{r}, t) = 0,
\]

where \(e\) is the electric charge, \(\vec{i}\) is the versor of the \(x\)-axis, \(\delta\) is the Dirac delta function, \(\vec{i}w(t)\) is the trajectory, \(\partial_x\) is the partial derivative with respect to \(x\), and the overdot denotes the derivative with respect to time. The time-dependence is arbitrary. The above Taylor expansion of the Dirac delta function suggests that we are considering small oscillations of the point charge about an equilibrium position. However, our final result for the self-force appears not to depend on the oscillation amplitude \(w(t)\), but only on its derivatives.

The scalar and vector potentials can be easily calculated, using integration by parts. One obtains:

\[
\phi(\vec{r}, t) = \int d\vec{r}' \frac{\rho(\vec{r}', t - \frac{|\vec{r} - \vec{r}'|}{c})}{|\vec{r} - \vec{r}'|} = e \sum_{n=0}^{\infty} \frac{(-1)^n w^n(t - \frac{r}{c})}{n!} \partial_x^n \phi(\vec{r}),
\]

\[
A_x(\vec{r}, t) = \frac{1}{c} \int d\vec{r}' \frac{j_x(\vec{r}', t - \frac{|\vec{r} - \vec{r}'|}{c})}{|\vec{r} - \vec{r}'|} = e \sum_{n=0}^{\infty} \frac{(-1)^n \partial_x^n \dot{w}(t - \frac{r}{c})}{n!} w^n(t - \frac{r}{c}) \partial_t^n.
\]

The \(x\)-component of the electric field is given by: \(E_x(\vec{r}, t) = E_x^1(\vec{r}, t) + E_x^2(\vec{r}, t)\), where:

\[
E_x^1(\vec{r}, t) = -\partial_x \phi(\vec{r}, t) = e \sum_{n=0}^{\infty} \frac{(-1)^{n+1} w^{n+1}(t - \frac{r}{c})}{n!} \partial_x^n \phi(\vec{r}),
\]

\[
E_x^2(\vec{r}, t) = -\frac{1}{c} \frac{\partial A_x}{\partial t} = -\frac{e}{c^2} \sum_{n=0}^{\infty} \frac{(-1)^n \partial_x^n \dot{w}(t - \frac{r}{c}) w^n(t - \frac{r}{c})}{n!} \partial_t^n.
\]
The $y$- and $z$- components of the electric field, and the magnetic field, do not give any contribution to the self-force for this type of motion. Note that the above derivatives are taken in the distributional sense. We have, for example:

$$
\frac{\partial^2 w(t - \frac{r}{c})}{r} = \frac{x^2}{c^2 r^2} \hat{w}(t - \frac{r}{c}) + \frac{1}{c^2} \left( \frac{3x^2}{r^2} - 1 \right) \hat{w}(t - \frac{r}{c}) + \frac{1}{c^2} \left( \frac{3x^2}{r^2} - 1 \right) \hat{w}(t - \frac{r}{c}) - \frac{4\pi}{3} w(t) \delta(\vec{r}),
$$

etc. We are not interested here in explicitly writing the delta-function contributions to the fields, but keep in mind their existence and interpret them as being the "field on the source" \cite{7}.

We calculate the self-force using the formula:

$$
F_x(t) = \int d\vec{r} \rho(\vec{r}, t) E_x(\vec{r}, t) = \frac{1}{(2\pi)^3} \int d\vec{k} \rho(\vec{k}, t) E_x(-\vec{k}, t),
$$

(7)

where $\rho(\vec{k}, t) = \int d\vec{r} e^{ik \cdot \vec{r}} \rho(\vec{r}, t)$, $E_x(-\vec{k}, t) = \int d\vec{r} e^{-ik \cdot \vec{r}} E_x(\vec{r}, t)$. The ambiguities of integrating the delta function and its derivatives on the semi-axis are circumvented by performing our integrals in the $\vec{r}$ and $\vec{k}$ spaces using symmetric spherical coordinates:

$$
\int d\vec{r} = \int_{-\infty}^{\infty} d\vec{r} \int_0^{\pi/2} d\theta \sin \theta \int_0^{2\pi} d\phi.
$$

We integrate in the Fourier space, using the formulas \cite{8}:

$$
\int_{-\infty}^{\infty} dk k^n \sin(kr) = 2\pi i^{n+1} \delta^{(n)}(r),
$$

for $n$ odd and:

$$
\int_{-\infty}^{\infty} dr \delta^{(n)}(r) w\left(t - \frac{r}{c}\right) = \frac{1}{c^n} w^{(n)}(t),
$$

where the upper index between parentheses is the order of the derivative with respect to the argument. We obtain:

$$
\frac{F_x(t)}{2e^2} = \sum_{n, p = 0}^{\infty} \frac{(-1)^n w^n(t)}{c^n + p + 3 n!(p + 1)! (n + p + 1)(n + p + 3)} \frac{d^{n+p+3} w^{p+1}(t)}{dt^{n+p+3}}. \quad (8)
$$

Although the force from the above formula seems to depend on $w(t)$, when we explicitly write the first few terms, we note that it depends only on the time derivatives of $w(t)$. Unfortunately, we have not been able to prove this dependence in the general case. The first few terms of (8) are:

$$
F_x(t) = \frac{2e^2}{3c^3} \hat{w}(t) + \frac{2e^2}{c^5} \hat{w}^2(t) + \frac{4e^2}{3c^5} \hat{w}^2 \hat{w} + O\left(\frac{1}{c^7}\right). \quad (9)
$$
3. Comparison with other results

The well-known Lorentz-Dirac radiation-reaction force is given by the spatial part of the Abraham-Lorentz four-vector \[9, 10, 11, 12\]:

\[\Gamma^\mu = \frac{2e^2}{3c^3} \left( \gamma^2 \ddot{v} + \frac{3\gamma^4}{c^2} \ddot{\vec{v}} \cdot \dot{\vec{v}} + \frac{\gamma^4}{c^2} \dot{\vec{v}} \cdot \dddot{\vec{v}} + \frac{3\gamma^6}{c^4} (\vec{v} \cdot \dddot{v})^2 \right), \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \tag{10}\]

where \(\vec{v}\) is the three-velocity and the overdots denote derivatives with respect to the laboratory time \(t\) (\(dt = \gamma d\tau\)). For rectilinear motion along the \(x\)-axis, in our notation, the above equation reads:

\[F_{x}^{\text{A-L}} = \frac{2e^2}{3c^3} \left( \gamma^2 \ddot{w} + \frac{3\gamma^4}{c^2} \dot{w} \ddot{w} + \frac{\gamma^4}{c^2} \dot{w} \dddot{w} + \frac{3\gamma^6}{c^4} (\dot{w}^2 \dot{w})^2 \right). \tag{11}\]

Keeping only the terms up to \(1/c^5\), this gives:

\[F_{x}^{\text{A-L}}(t) = \frac{2e^2}{3c^3} \ddot{w}(t) + \frac{2e^2}{c^5} \dot{w}(t) \dddot{w}(t) + \frac{4e^2}{3c^3} \dot{w}^2(t) \dot{w}(t) + O \left( \frac{1}{c^7} \right), \tag{12}\]

which coincides with Eq. \((9)\).

It is worth comparing our method of calculation with the elegant method of Gordeyev \([13, 14]\), which uses the Lagrange expansion for the Lienard-Wiechert potentials. This is an exact expansion, useful for retarded functions. As the Taylor expansion used by us is also exact, our electric field should coincide with Gordeyev’s field. In the following we shall verify this assertion.

We start from our Eq. \((5)\) and use the Taylor expansion:

\[w^n \left( t - \frac{\vec{r}}{c} \right) = \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\gamma^k}{c^k} \dot{w}^{(k)}(t) \right)^n = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} e^k \frac{\gamma^k}{c^k} \partial^k \partial t^n \dot{w}^{(k)}(t). \]

Our Eq. \((5)\) becomes:

\[E_1^x(\vec{r}, t) = e \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!} \frac{\partial^k}{\partial t^k} \frac{\partial}{\partial x^n} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \dot{w}^{(n)}(t) \partial x^n r^{k-1}. \tag{13}\]

Now, using that

\[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} w^n(t) \partial x^n r^{k-1} = |\vec{r} - \vec{w}(t)|^{k-1}, \tag{14}\]
we obtain
\[
E_1^1(\vec{r}, t) = -e \nabla \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \sqrt{c}} \frac{\partial^k}{\partial t^k} |\vec{r} - \vec{r}_w(t)|^{k-1},
\]
which coincide with the electric field obtained from the scalar potential given in Eq. (7) of Ref. [13]. One can also verify the coincidence of our result Eq. (6) with the electric field obtained from the vector potential given in Eq. (7) of Ref. [13].

We underline here the important fact that in Gordeyev’s calculation, as in other calculations of the point-charge self-force, there appear divergent terms, which are eliminated from the final result by the so-called ”mass renormalization”. In our calculation, such divergent terms and those that some authors ([2], [5]) get rid of by employing the half-difference of the advanced and retarded fields (as done first by Dirac [2]) do not appear.

4. Conclusions

We obtained the electromagnetic self-force of a point particle in rectilinear motion, in the form of an infinite series. It should be possible to generalize this result for an arbitrary motion. We used a simple method, which employs the delta-function formalism, Taylor expansions and integrations by parts, working with the retarded fields only. No divergencies have emerged in our calculations. The first terms of the force, up to the $1/c^5$ order, coincide with those of the Abraham-Lorentz four vector. Unfortunately, we cannot sum the series and verify whether our force completely coincides with the Lorentz-Dirac force.

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