\textbf{Abstract}

The purpose of this paper is to study cohomology and deformations of $\mathcal{O}$-operators on Lie triple systems. We define a cohomology of an $\mathcal{O}$-operator $T$ as the Lie-Yamaguti cohomology of a certain Lie triple system induced by $T$ with coefficients in a suitable representation. Then we consider infinitesimal and formal deformations of $\mathcal{O}$-operators from cohomological viewpoint. Moreover we provide relationships between $\mathcal{O}$-operators on Lie algebras and associated Lie triple systems.

**Key words:** Lie triple systems, $\mathcal{O}$-operator, representation, Lie-Yamaguti cohomology, deformation.

**Mathematics Subject Classification** (2020): 17A40, 17B56, 17B10, 17B38.

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1 Introduction

The concept of Lie triple system was introduced first by Jacobson [15]. The present formulation is due to Yamaguti [29]. Moreover, it appeared in Cartan’s work on Riemannian Geometry [8] and was strongly developed for Symmetric spaces and related spaces. Indeed, the tangent space of a symmetric space is a Lie triple system. It turns out that they have important applications in physics, in particular,
in elementary particle theory and the theory of quantum mechanics, as well as numerical analysis of differential equations. They have become an interesting subject in mathematics, their structure have been studied first by Lister in [20].

The concept of $O$-operators on Lie algebras appeared first, in a paper of B. A. Kupershmidt [17] as an operator analogue of classical $r$-matrices and Poisson structures. However, Rota-Baxter operators were introduced by G. Baxter [3] in his study of fluctuation theory in probability. Then developed by G.-C. Rota [24] in Combinatorics. They have important applications in the algebraic aspects of the renormalization in quantum field theory [7]. Rota-Baxter operators on Lie algebras (resp. associative algebras) are also related to the splitting of algebraic structures. In [4], the authors introduced the notion of a Rota-Baxter operator on a 3-Lie algebra. Recently, in order to construct solutions of the classical 3-Lie Yang-Baxter equation, the authors in [3] introduced a more general notion called relative Rota-Baxter operator on a 3-Lie algebra with respect to a representation.

The deformation theory using formal power series and suitable cohomologies was initiated by Gerstenhaber [11] for associative algebras. Nijenhuis and Richardson extended this study to Lie algebras [23]: Deformations of 3-Lie algebras and Lie triple systems were studied in [1] and [19, 7, 30] respectively. Recently deformations of certain operators, e.g. morphisms and Rota-Baxter operators ($O$-operators) were deeply studied, see [1, 8, 13, 21, 26]. One needs a cohomology to control deformations and extension problems of a given algebraic structure. Cohomologies of various algebraic structures, such as associative algebras, Lie algebras, Leibniz algebras, pre-Lie algebras and $n$-Lie algebras, are well known. Cohomology of Lie triple systems was introduced by K. Yamaguti in [29]. Recently, cohomology of Rota-Baxter operators ($O$-operators) on Lie algebras and associative algebras were developed in [28] and [7] respectively. In [28], the authors constructed a Lie 3-algebra whose Maurer-Cartan elements are $O$-operators on 3-Lie algebras, and studied cohomologies and deformations of $O$-operators on 3-Lie algebras, see also [13].

Inspired by these works, we tackle in this paper the cohomology theory of $O$-operators on Lie triple systems. We define a cohomology of an $O$-operator $T$ as the Lie-Yamaguti cohomology of a certain Lie triple system induced by $T$ with coefficients in a suitable representation. Moreover, we provide some connections between $O$-operators on Lie algebras and Lie triple systems. In Section 2, we summarize some basics and briefly recall representations and cohomology of Lie triple systems. In Section 3, we investigate $O$-operator and Nijenhuis operator on Lie triple systems. In Section 4, we define the cohomology of $O$-operator on Lie triple system using the underlying Lie triple systems of the $O$-operator. Section 5 deals with one-parameter deformations of $O$-operators using the cohomology theory established in Section 4. Finally, in Section 6, we describe some connections between cohomology of $O$-operators on Lie algebras and associated Lie triple systems.

In this paper all vector spaces are considered over a field $F$ of characteristic 0.

## 2 Preliminaries

In this section, we recall representations and cohomology theory of Lie triple systems.

**Definition 2.1.** A Lie triple systems (L.t.s) is a vector space $L$ endowed with a ternary bracket $[\cdot, \cdot, \cdot] : \wedge^2 L \otimes L \to L$ satisfying:

\[
[x, y, z] + [y, z, x] + [z, x, y] = 0, \tag{2.1}
\]

\[
[x, y, [z, t, e]] = [[x, y, z], t, e] + [z, [x, y, t], e] + [z, t, [x, y, e]], \tag{2.2}
\]

for any $x, y, z, t, e \in L$. 

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Example 2.1. Let \((L, [\cdot, \cdot])\) be a Lie algebra. We define \([\cdot, \cdot, \cdot] : (\wedge^2 L) \otimes L \to L\) by
\[
[x, y, z] = [[x, y], z], \quad \forall x, y, z \in L. \quad (2.3)
\]
Then \((L, [\cdot, \cdot, \cdot])\) becomes a L.t.s naturally.

A morphism \(f : (L, [\cdot, \cdot, \cdot]) \to (L', [\cdot, \cdot, \cdot'])\) of L.t.s is a linear map satisfying
\[
f([x, y, z]) = [f(x), f(y), f(z)]', \quad \forall x, y, z \in L.
\]
An isomorphism is a bijective morphism.

Denote by \(X = (x, y) \in L \otimes L\) and \(\mathcal{L}(X) : L \to L\) defined by \(\mathcal{L}(X)z_i = [x, y, z_i]\), then the identity (\ref{eq:2.2}) can be rewritten in the form
\[
\mathcal{L}(X)[z_1, z_2, z_3] = [\mathcal{L}(X)z_1, z_2, z_3] + [z_1, \mathcal{L}(X)z_2, z_3] + [z_1, z_2, \mathcal{L}(X)z_3],
\]
which means that \(\mathcal{L}(X)\) is a derivation with respect to the bracket \([\cdot, \cdot, \cdot]\).

In \cite{2}, K. Yamaguti introduced the notion of representation and cohomology theory of L.t.s. Later, the authors in \cite{3, 4} and \cite{5} studied the cohomology theory of L.t.s from a different point of view. K. Yamaguti's work can be described as follows.

Definition 2.2. A representation of a L.t.s \(L\) on a vector space \(V\) is a bilinear map \(\theta : \otimes^2 L \to \text{End}(V)\), such that the following conditions are satisfied:
\[
\begin{align*}
\theta(z, t)\theta(x, y) - \theta(y, t)\theta(x, z) - \theta(x, [y, z, t]) + D(y, z)\theta(x, t) &= 0, \\
\theta(z, t)D(x, y) - D(x, y)\theta(z, t) + \theta([x, y, z], t) + \theta(z, [x, y, t]) &= 0,
\end{align*}
\]
where \(D(x, y) = \theta(y, x) - \theta(x, y)\), for all \(x, y, z, t \in L\). We denote by \((L, [\cdot, \cdot, \cdot]; \theta)\), the L.t.sRep pair.

Remark 2.1. Let \(L\) be a L.t.s and \(\theta(x, y)\) be the linear map \(z \mapsto [z, x, y]\) of \(L\) into itself for all \(x, y \in L\). Using (\ref{eq:2.2}), then we can prove that \(L\) is an \(L\)-module. Moreover, by (\ref{eq:2.2}), \(D(x, y)\) becomes a linear map \(z \mapsto [x, y, z]\) (inner derivation). In this paper, we define \(\theta(x, y) := \mathcal{R}(x, y), D(x, y) := \mathcal{L}(x, y)\) where \(\mathcal{L}(x, y)(z) = [x, y, z], \mathcal{R}(x, y)(z) = [x, y, z]\).

As usual, we have a characterisation of a representation by a semi-direct product i.e. \((V, \theta)\) is a representation of a L.t.s \(L\) if and only if \(L \oplus V\) is a L.t.s under the following bracket
\[
[x + u, y + v, z + w]_{L \oplus V} = [x, y, z] + \theta(y, z)u - \theta(x, z)v + D(x, y)w,
\]
for any \(x, y, z \in L\) and \(u, v, w \in V\).

Let \((L, [\cdot, \cdot, \cdot]; \theta)\) be a L.t.sRep pair. For each \(n \geq 0\), we denote by \(C^{2n+1}(L, V)\), the vector space of \((2n + 1)\)-cochains of \(L\) with coefficients in \(V\): \(f \in C^{2n+1}(L, V)\) is a multilinear function of \(x^{2n + 1} L\) into \(V\) satisfying
\[
f(x_1, x_2, \ldots, x_{2n-2}, x, y) = 0,
\]
and
\[
f(x_1, x_2, \ldots, x_{2n-2}, x, y) + f(x_1, x_2, \ldots, x_{2n-2}, y, z) + f(x_1, x_2, \ldots, x_{2n-2}, z, x, y) = 0.
\]

The Yamaguti coboundary operator \(\delta^{2n-1} : C^{2n-1}(L, V) \to C^{2n+1}(L, V)\) is defined by
\[ \delta^{2n-1}f(x_1, x_2, \cdots, x_{2n+1}) = \theta(x_{2n}, x_{2n+1})f(x_1, x_2, \cdots, x_{2n-1}) - \theta(x_{2n-1}, x_{2n+1})f(x_1, x_2, \cdots, x_{2n-2}, x_{2n}) + \sum_{k=1}^{n} (-1)^{n+k}D(x_{2k-1}, x_{2k})f(x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, x_{2n+1}) \]
\[ + \sum_{k=1}^{n} \sum_{j=2k+1}^{2n+1} (-1)^{n+k+1}f(x_1, x_2, \cdots, \hat{x}_{2k-1}, \hat{x}_{2k}, \cdots, [x_{2k-1}, x_{2k}, x_j], \cdots, x_{2n+1}) \quad (2.7) \]

for all \( f \in C^{2n-1}(L, V) \), \( n \geq 1 \), where \( \hat{} \) denotes omission. The Yamaguti cochain forms a complex with this coboundary as follow

\[ C^1(L, V) \xrightarrow{\delta^1} C^3(L, V) \xrightarrow{\delta^3} C^5(L, V) \rightarrow \cdots, \]

and \( \delta^{2n+1} \circ \delta^{2n-1} = 0 \) for \( n = 1, 2, \cdots \) (see [29] for more details). Hence we get the Yamaguti cohomology group \( H^*(L, V) = Z^*(L, V)/B^*(L, V) \), where \( Z^*(L, V) \) is the space of cocycles and \( B^*(L, V) \) is the space of coboundaries.

**Definition 2.3.** [28] Let \((V, \theta)\) be a representation of a L.t.s \( L \).‌

1. A linear map \( f \in C^1(L, V) \) is a 1-cocycle (closed) if

\[ D(x_1, x_2)f(x_3) - \theta(x_1, x_3)f(x_2) + \theta(x_2, x_3)f(x_1) - f([x_1, x_2, x_3]) = 0, \quad (2.8) \]

and a map \( f \in C^3(L, V) \) is a 3-coboundary if there exists a map \( g \in C^1(L, V) \) such that \( f = \delta^1 g \).

2. A map \( f \in C^3(L, V) \) is called a 3-cocycle if \( \forall x_1, x_2, x_3, y_1, y_2, y_3 \in L \),

\[ f(x_1, x_2) = 0, \quad (2.9) \]
\[ f(x_1, x_2, x_3) + f(x_2, x_3, x_1) + f(x_3, x_1, x_2) = 0, \quad (2.10) \]
\[ f(x_1, x_2, [y_1, y_2, y_3]) + D(x_1, x_2)f(y_1, y_2, y_3) = f([x_1, x_2, y_1], [y_2, y_3]) + f([x_1, x_2, y_2], [y_1, y_3]) + f([x_1, x_2, y_3], [y_1, y_2]) + \theta(y_2, y_3)f(x_1, x_2, y_1) - \theta(y_1, y_3)f(x_1, x_2, y_2) + D(y_1, y_2)f(x_1, x_2, y_3). \quad (2.11) \]

### 3 O-operators on Lie triple systems

In this section, we recall the notion of pre-Lie triple system structure introduced in [29] and define Rota-Baxter operators, \( O \)-operators on L.t.s, then we study their morphisms. Moreover, we give some characterisations of \( O \)-operators in terms of Nijenhuis operators and graphs.

**Definition 3.1.** Let \( L \) be a L.t.s. A linear map \( R : L \rightarrow L \) is said to be a Rota-Baxter operator of weight 0 if it satisfies

\[ [R(x), R(y), R(z)] = R([R(x), R(y), z] + [R(x), y, R(z)] + [x, R(y), R(z)]), \quad \forall x, y, z \in L. \]

The notion of \( O \)-operator (also called relative Rota-Baxter operator or Kupershmidt operator) is a generalization of Rota-Baxter operators in the presence of arbitrary representation.
**Definition 3.2.** Let \((L, [\cdot, \cdot; \theta])\) be a \(\text{L.t.sRep}\) pair. A linear map \(T : V \to L\) is called an \(O\)-operator of \(L\) with respect to \(\theta\) if it satisfies

\[
[Tu, Tv, Tw] = T\left(\theta(Tv, Tw)u + \theta(Tu, Tw)v - \theta(Tu, Tv)w\right), \quad \forall u, v, w \in V. \tag{3.1}
\]

If \(V = L\), then \(T\) is a Rota-Baxter operator of weight 0 on \(L\) with respect to the adjoint representation. Thus \(O\)-operators are generalization of Rota-Baxter operators.

Now we give the following characterization of an \(O\)-operator in terms of graphs.

**Proposition 3.1.** A linear map \(T : V \to L\) is an \(O\)-operator if and only if the graph \(Gr(T) = \{(Tu, u) \mid u \in V\}\) is a subalgebra of the semi-direct product \(L \oplus V\).

**Proof.** Let \((Tu, u), (Tv, v)\) and \((Tw, w) \in Gr(T)\). Then we have

\[
[Tu + u, Tv + v, Tw + w]_{L \oplus V} = [Tu, Tv, Tw] + \theta(Tv, Tw)u - \theta(Tu, Tw)v + D(Tu, Tv)w.
\]

Assume that \(Gr(T)\) is a subalgebra of the semi-direct product \(L \oplus V\), then we have

\[
[Tu, Tv, Tw] = T\left(\theta(Tv, Tw)u - \theta(Tu, Tw)v + D(Tu, Tv)w\right).
\]

On the other hand, if \(T\) is an \(O\)-operator, we obtain

\[
[Tu + u, Tv + v, Tw + w]_{L \oplus V} = T\left(\theta(Tv, Tw)u - \theta(Tu, Tw)v + D(Tu, Tv)w\right) + \theta(Tv, Tw)u - \theta(Tu, Tw)v + D(Tu, Tv)w \in Gr(T).
\]

Hence \(Gr(T)\) is a subalgebra of the semi-direct product \(L \oplus V\). \qed

In the following proposition, we show that an \(O\)-operator can be lifted up to a Rota-Baxter operator.

**Proposition 3.2.** Let \((L, [\cdot, \cdot; \theta])\) be a \(\text{L.t.sRep}\) pair and \(T : V \to L\) be a linear map. Define \(\hat{T} \in \text{End}(L \oplus V)\) by \(\hat{T}(x, u) = (T(u), 0)\). Then \(T\) is an \(O\)-operator if and only \(\hat{T}\) is a Rota-Baxter operator on \(L \oplus V\).

**Proof.** For any \(x, y, z \in L\) and \(u, v, w \in V\), we have

\[
[\hat{T}(x, u), \hat{T}(y, v), \hat{T}(z, w)]_{L \oplus V}
\]

\[
- \hat{T}\left([\hat{T}(x, u), \hat{T}(y, v), (z, w)]_{L \oplus V} + [\hat{T}(x, u), (y, v), \hat{T}(z, w)]_{L \oplus V} + [(x, u), \hat{T}(y, v), \hat{T}(z, w)]_{L \oplus V}\right)
\]

\[
= [(Tu, 0), (Tv, 0), (Tw, 0)]_{L \oplus V}
\]

\[
- \hat{T}\left([Tu, 0, (Tv, 0), (Tw, 0)]_{L \oplus V} + [(Tu, 0), (Tv, 0), (Tw, 0)]_{L \oplus V} + [(x, u), (Tv, 0), (Tw, 0)]_{L \oplus V}\right)
\]

\[
= (Tu, Tv, Tw, 0) - T(\theta(Tv, Tw)u - \theta(Tu, Tw)v + D(Tu, Tv)w, 0)
\]

\[
= (Tu, Tv, Tw) - T(\theta(Tv, Tw)u - \theta(Tu, Tw)v + D(Tu, Tv)w, 0).
\]

Then \(\hat{T}\) is a Rota-Baxter operator on \(L \oplus V\) if and only if \(T\) is an \(O\)-operator. \qed

**Example 3.3.** Let \(L\) be a 2-dimensional \(\text{L.t.s}\) with a basis \(\{e_1, e_2\}\) and a bracket defined by

\[
[e_1, e_2, e_2] = e_1.
\]

The operator \(T = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}\) is a Rota-Baxter operator on \(L\).
Example 3.4. Let $L$ be a 4-dimensional L.t.s with a basis \{e_1, e_2, e_3, e_4\} and a bracket defined by
\[[e_1, e_2, e_1] = e_4.\]

Then
\[T = \begin{pmatrix}
0 & a & 0 & 0 \\
0 & 0 & 0 & 0 \\
b & c & d & e \\
f & g & h & k
\end{pmatrix},\]
defines a Rota-Baxter operator on $L$.

Another characterization of an $O$-operator can be given in terms of Nijenhuis operators on L.t.s. First of all, we give the following definition of Nijenhuis operator on a L.t.s $L$.

Definition 3.3. A linear map $N : L \rightarrow L$ is called Nijenhuis operator on a L.t.s $L$ if $N$ satisfies the following condition
\[[N x, N y, N z] = N([N x, N y, z] + [x, N y, N z] + [N x, y, N z] - N[x, y, z]).\] (3.2)

Remark 3.1. The above definition of Nijenhuis operator on a L.t.s is similar to the definition of Nijenhuis operator on 3-Lie algebras introduced in\[18\] in the study of 2-order trivial deformations. On the other hand, in\[30\], the author has introduced another notion of a Nijenhuis operator on a L.t.s in the study of 1-order trivial deformations. In that definition, there is a quite strong condition $[N x_1, N x_2, N x_3] = 0$, whereas the above definition for $n = 3$ is the Eq. (3.2). So obviously, the above definition is different from the definition given in\[30\] and we think that is the right definition.

Lemma 3.5. If $N$ is a Nijenhuis operator on $L$. Then $(L, [\cdot, \cdot, \cdot]_N)$ is a L.t.s, where
\[[x, y, z]_N = [N x, N y, z] + [x, N y, N z] + [N x, y, N z] - N[x, y, z],\] (3.3)
and $N$ is a homomorphism from $(L, [\cdot, \cdot, \cdot])$ to $(L, [\cdot, \cdot, \cdot])$.

Proof. It follows from straightforward computations.

The following result is also straightforward, so we omit details.

Proposition 3.6. Let $(L, [\cdot, \cdot, \cdot]; \theta)$ be a L.t.sRep pair. A linear operator $T : V \rightarrow L$ is an $O$-operator if and only if
\[T = \begin{pmatrix}
0 & T \\
0 & 0
\end{pmatrix} : L \oplus V \rightarrow L \oplus V\]
is a Nijenhuis operator on the the semi-direct product L.t.s $L \oplus V$.

Next, we recall the notion of pre-Lie triple systems introduced in\[22\] which is the structure induced from $O$-operators.
Definition 3.4. Let $L$ be a vector space with a 3-linear map $\{\cdot,\cdot,\cdot\} : \otimes^3 L \to L$. The pair $(L,\{\cdot,\cdot,\cdot\})$ is called a pre-Lie triple system if the following identities hold
\[
\{x_5,x_1,[x_2,x_3,x_4]_C\} = \{\{x_5,x_1,x_2\},x_3,x_4\} - \{\{x_5,x_1,x_3\},x_2,x_4\} + \{x_5,x_2,x_3\}_C,
\]
\[
\{x_1,x_2,[x_3,x_4]_C\} = \{\{x_1,x_2,x_3\}^*,x_4\} + \{x_5,[x_1,x_2,x_3]_C,x_4\} + \{x_5,x_3,[x_1,x_2,x_4]_C\},
\]
where $\{\cdot,\cdot,\cdot\}^*$ and $[\cdot,\cdot,\cdot]_C$ are defined by
\[
\{x,y,z\}^* = \{y,z,x\} - \{z,x,y\},
\]
\[
[x,y,z]_C = \{x,y,z\} - \{z,x,y\} + \{y,z,x\} - \{y,x,z\}.
\]
for any $x,y,z \in L$.

It follows from (3.4) and (3.5) that the new operation $\{\cdot,\cdot,\cdot\}_C : \otimes^3 L \to L$ defined by Eq. (3.7) turns out to be a L.t.s.

An $O$-operator has an underlying pre-Lie triple system structure.

Proposition 3.7. Let $(L,\{\cdot,\cdot,\cdot\};\theta)$ be a L.t.sRep pair. Suppose that the linear map $T : V \to L$ is an $O$-operator associated to $(V,\theta)$. Then there exists a pre-Lie triple system structure on $V$ given by
\[
\{u,v,w\} = \theta(Tv,Tw)u, \ \forall u,v,w \in V.
\]

Next, we study morphism between $O$-operators.

Definition 3.5. Let $T$ be an $O$-operator on a L.t.sRep pair $(L,\{\cdot,\cdot,\cdot\};\theta)$. Suppose $(L',\{\cdot,\cdot,\cdot\};\theta')$ is another L.t.sRep pair. Let $T' : V' \to L'$ be an $O$-operator. A morphism of $O$-operators from $T$ to $T'$ consists of a pair $(\phi,\psi)$ of a L.t.s morphism $\phi : L \to L'$ and a linear map $\psi : V \to V'$ satisfying
\[
\phi \circ T = T' \circ \psi,
\]
\[
\psi\theta(x,y) = \theta(\phi(x),\phi(y))\psi, \ \forall x,y \in L.
\]
It is called an isomorphism if $\phi$ and $\psi$ are both bijective.

The proof of the following result is straightforward then we omit the details.

Proposition 3.8. A pair of linear maps $(\phi : L \to L',\psi : V \to V')$ is a morphism of $O$-operators from $T$ to $T'$ if and only if
\[
Gr((\phi,\psi)) = \{(x,u),(\phi(x),\psi(u)) \mid x \in L, u \in V\} \subset (L \oplus V) \oplus (L' \oplus V'),
\]
is a subalgebra, where $L \oplus V$ and $L' \oplus V'$ are equipped with semi-direct product structures of L.t.s.

In the rest of the paper, we will be most interested in morphism between $O$-operators on the same L.t.s with respect to the same module.

Proposition 3.9. Let $T$ and $T'$ be two $O$-operators on the L.t.sRep pair $(L,\{\cdot,\cdot,\cdot\};\theta)$. If $(\phi,\psi)$ is a morphism (resp. an isomorphism) from $T$ to $T'$, then $\psi : V \to V$ is a morphism (resp. an isomorphism) between induced pre-Lie triple system structures.

Proof. For all $u,v,w \in V$, by Eqs (3.5) and (3.11), we have
\[
\psi\{u,v,w\}_T = \psi(\theta(T(v),T(w))u) = \theta(\phi(T(v)),\phi(T(w)))\psi(u)
\]
\[
= \theta(T'(\psi(v)),T'(\psi(w)))\psi(u) = \{\psi(u),\psi(v),\psi(w)\}_T'.
\]
Hence the result follows.
4 Cohomology of $\mathcal{O}$-operators on Lie triple systems

In this section, we define a cohomology of an $\mathcal{O}$-operator $T$ on a L.t.s $L$ as the Lie-Yamaguti cohomology of a certain L.t.s with coefficients in a suitable representation on $L$. This cohomology will be used further, to study deformations of $T$.

It was proved in [2] that there is a pre-Lie triple system structure on $V$ as the underlying structure of an $\mathcal{O}$-operator on a L.t.s. Consequently, there is a subadjacent L.t.s structure on $V$ given as follows.

**Lemma 4.1.** Let $T : V \to L$ be an $\mathcal{O}$-operator on a L.t.sRep pair $(L, [\cdot, \cdot, \cdot]; \theta)$. Then $(V, [\cdot, \cdot, \cdot]_T)$ is a L.t.s, where

$$[u, v, w]_T := D(Tu, Tv)v + \theta(Tv, Tw)u - \theta(Tu, Tw)v.$$  

Moreover, $T$ is a homomorphism of L.t.s i.e. $T([u, v, w]_T) = [Tu, Tv, Tw]$.

Furthermore, there is a representation of the above L.t.s $(V, [\cdot, \cdot, \cdot]_T)$ on $L$:

**Proposition 4.2.** Let $T$ be an $\mathcal{O}$-operator on a L.t.sRep pair $(L, [\cdot, \cdot, \cdot]; \theta)$. Define $\theta_T : \otimes^2 V \to gl(L)$ by

$$\theta_T(u, v)x = [x, Tu, Tv] + T(\theta(x, Tv)u - D(x, Tu)v), \forall x \in L, u, v \in V.$$  

Then $(L, \theta_T)$ is a representation of the L.t.s $(V, [\cdot, \cdot, \cdot]_T)$ on $L$.

**Proof.** One can show it directly by a tedious computation. Here we take a different approach using Nijenhuis operators on L.t.s. Let $T$ be an $\mathcal{O}$-operator on a L.t.sRep pair $(L, [\cdot, \cdot, \cdot]; \theta)$. We define $\overline{T} : L \oplus V \to L \oplus V$ by

$$\overline{T}(x + u) = T(u), \forall x \in L, u \in V.$$  

Then $\overline{T}$ is a Nijenhuis operator on the semidirect product L.t.s $L \ltimes_\theta V$ and $\overline{T} \circ \overline{T} = 0$. Then by Eq. (3), there is a L.t.s structure $[\cdot, \cdot, \cdot]_{\overline{T}}$ on the vector space $L \oplus V$ given by

\[
\begin{align*}
[x + u, y + v, z + w]_{\overline{T}} &= \overline{T}(x + u), \overline{T}(y + v), z + w]_{L \oplus V} + [\overline{T}(x + u), y + v, \overline{T}(z + w)]_{L \oplus V} + [x + u, \overline{T}(y + v), \overline{T}(z + w)]_{L \oplus V} \\
&= [Tu, Tv, z + w]_{L \oplus V} + [Tu, y + v, Tw]_{L \oplus V} + [x + u, Tv, Tw]_{L \oplus V} \\
&- \overline{T}(\overline{T}(x + u), y + v, z + w)]_{L \oplus V} + [x + u, \overline{T}(y + v), z + w)]_{L \oplus V} \\
&= [Tu, Tv, z] + D(Tu, Tv)w + [Tu, y, Tv] - \theta(Tu, Tw)v + [x, Tu, Tv] + \theta(Tv, Tw)u \\
&- \overline{T}(\overline{T}(x, y, z)v + D(Tu, y)w + [x, Tv, z] + \theta(Tv, z)u + D(x, Tv)w) \\
&+ [x, y, Tv] + \theta(y, Tw)u - \theta(x, Tu)v \\
&= [Tu, Tv, z] - \overline{T}(x, y, z)v + D(Tu, y)w + \theta(Tv, z)u + D(x, Tv)w - \theta(Tu, Tw)v, \\
&\text{or by Eq. (2), we have} \\
&[Tu, Tv, z] = -[Tv, z, Tu] - [z, Tu, Tv] = [z, Tv, Tu] - [z, Tu, Tv].
\end{align*}
\]

Then we obtain that
\[
D_T(u, v)z = \theta_T(v, u)z - \theta_T(u, v)z
\]
Since a semidirect product of a L.t.s is equivalent to a representation of a L.t.s, we deduce that 

\[ \theta(z, T_v u - D(z, T_v) u) \]

Finally, we have

\[
[x + u, y + v, z + v]_T = [u, v, w]_T + \theta_T(v, w)x - \theta_T(u, w) y + D_T(u, v) z. \tag{4.3}
\]

Let \( (V, [\cdot, \cdot]_T; \theta_T) \) be a L.t.sRep pair. For each \( n \geq 0 \), we denote by \( C^{2n+1}(V, L) \), the vector space of \( (2n + 1) \)-cochains of \( V \) with coefficients in \( L \). An element \( f \in C^{2n+1}(V, L) \) is a multilinear map on \( \times^{2n+1} V \) into \( L \) satisfying

\[ f(v_1, v_2, \cdots, v_{2n-2}, v, u) = 0, \]

and

\[ f(v_1, v_2, \cdots, v_{2n-2}, v, u, w) + f(v_1, v_2, \cdots, v_{2n-2}, u, v, w) + f(v_1, v_2, \cdots, v_{2n-2}, w, v, u) = 0. \]

Let \( \delta_T^{2n-1} : C^{2n-1}(V, L) \to C^{2n+1}(V, L) \) be the corresponding coboundary operator on the L.t.s \( (V, [\cdot, \cdot]_T) \) with coefficients in the representation \( (L, \theta_T) \). More precisely, \( \delta_T : C^{2n-1}(V, L) \to C^{2n+1}(V, L) \) is given by

\[
\delta_T f(v_1, v_2, \cdots, v_{2n+1}) = \theta_T(v_{2n+1}, v_1) f(v_1, v_2, \cdots, v_{2n-1}) - \theta_T(v_{2n-1}, v_{2n+1}) f(v_1, v_2, \cdots, v_{2n-2}, v_{2n}) + \sum_{k=1}^{n} (-1)^{n+k} D_T(v_{2k-1}, v_2) f(v_1, v_2, \cdots, \hat{v}_{2k-1}, \hat{v}_{2k}, \cdots, v_{2n+1}) + \sum_{k=1}^{n} \sum_{j=2k+1}^{2n+1} (-1)^{n+k+1} f(v_1, v_2, \cdots, \hat{v}_{2k-1}, \hat{v}_{2k}, \cdots, \hat{v}_{2k+1}, v_{2k+1}, v_j, v_{2n+1}), \tag{4.4}
\]

for all \( f \in C^{2n-1}(V, L) \), \( n \geq 1 \), where \( \hat{\cdot} \) denotes omission. With this coboundary operator the Yamaguti cochain forms a complex

\[ C^1(V, L) \xrightarrow{\delta_T^1} C^3(V, L) \xrightarrow{\delta_T^3} C^5(V, L) \to \cdots, \]

and \( \delta_T^{2n+1} \circ \delta_T^{2n-1} = 0 \) for \( n = 1, 2, \cdots \).

**Definition 4.1.** Let \( (V, [\cdot, \cdot]_T; \theta_T) \) be a L.t.sRep pair.

1. A linear map \( f \in C^1(V, L) \) is called 1-cocycle if

\[ D_T(v_1, v_2) f(v_3) - \theta_T(v_1, v_3) f(v_2) + \theta_T(v_2, v_3) f(v_1) - f([v_1, v_2, v_3]_T) \]
Proof.  

For any $T \theta - T v_1, T v_2, f(v_3)$ 

\[ -T \left( -D(f(v_2), T v_1, v_3) + D(f(v_1), T v_2, v_3) + \theta(T v_2, f(v_3)) v_1 + \theta(f(v_2), T v_3) v_1 \right) \]

\[ -\theta(T v_1, f(v_3)) v_2 - \theta(f(v_1), T v_3) v_2 \] 

\[ -f \left( D(T v_1, T v_2, v_3) + \theta(T v_2, T v_3) v_1 - \theta(T v_1, T v_3) v_2 \right), \quad (4.5) \]

and a map $f \in C^3(V, L)$ is called a 3-coboundary if there exists a map $g \in C^1(V, L)$ such that $f = \delta g$.

2. A map $f \in C^3(V, L)$ is called a 3-cocycle if $\forall v_1, v_2, v_3, u_1, u_2, u_3 \in V,$

\[ f(v_1, v_1, v_2) = 0, \quad (4.6) \]

\[ f(v_1, v_2, v_3) + f(v_2, v_3, v_1) + f(v_3, v_1, v_2) = 0, \quad (4.7) \]

\[ f(v_1, v_2, [u_1, u_2, u_3]) + D_T(v_1, v_2) f(u_1, u_2, u_3) \]

\[ = f([v_1, v_2, u_1]_T, u_2, u_3) + f(u_1, [v_1, v_2, u_2]_T, u_3) + f(u_1, u_2, [v_1, v_2, u_3]_T) \]

\[ + \theta_T(u_2, u_3) f(v_1, v_2, u_1) - \theta_T(u_1, u_3) f(v_1, v_2, u_2) + D_T(u_1, u_2) f(v_1, v_2, u_3). \quad (4.8) \]

For any $X \in L \wedge L$, we define $\partial_T(X) : V \to L$ by

\[ \partial_T(X) v = TD(X)v - [X, TV], \quad \forall v \in V. \]

**Proposition 4.3.** Let $T$ be an $O$-operator on a L.t.sRep pair $(L, [\cdot, \cdot, \cdot]; \theta)$. Then $\partial_T(X)$ is a 1-cocycle on the L.t.s $(V, [\cdot, \cdot, \cdot]; 2)$ with coefficients in $(L, \theta_T)$.

**Proof.** For any $v_1, v_2, v_3 \in V$, we have

\[
(\delta^2 \partial T(X))(v_1, v_2, v_3) \\
= [Tv_1, Tv_2, TD(X)v_3 - [X, TV]] + [Tv_1, TD(X)v_2 - [X, TV], TV] \\
= [TD(X)v_1 - [X, TV], TV] + TD(X)(D(Tv_1, Tv_2)v_3 + \theta(Tv_2, Tv_3)v_1) \\
- \theta(Tv_1, Tv_3)v_2 + [X, T(D(Tv_1, Tv_2)v_3 + \theta(Tv_2, Tv_3)v_1 - \theta(Tv_1, Tv_3)v_2)] \\
- T \left( \theta(Tv_2, TD(X)v_3 - [X, TV]) + \theta(TD(X)v_2 - [X, TV], TV) v_1 \right) \\
- T \left( -\theta(Tv_1, TD(X)v_3 - [X, TV]) v_2 - \theta(TD(X)v_1 - [X, TV], TV) v_2 \right)
\]

\[
= [[X, TV], TV] + [TD(X)v_1, TV, TV] - [Tv_1, [X, TV], TV] - [Tv_1, TV, [X, TV]] \\
+ [TD(X)v_1, TV, TV] + [Tv_1, TD(X)v_2, TV] + [Tv_1, TV, TD(X)v_2] \\
- TD(X)(D(Tv_1, Tv_2)v_3 + \theta(Tv_2, Tv_3)v_1 - \theta(Tv_1, Tv_3)v_2) + [X, [Tv_1, Tv_2, TV]] \\
- T \left( \theta(Tv_2, TD(X)v_3 - [X, TV]) v_2 - [X, TV, TV] v_1 \right) \\
- T \left( -\theta(Tv_1, TD(X)v_3 - [X, TV]) v_2 - \theta(TD(X)v_1 - [X, TV], TV) v_2 \right)
\]

\[
= [TD(X)v_1, TV, TV] + [Tv_1, TD(X)v_2, TV] + [Tv_1, TV, TD(X)v_2] \\
- TD(X)(D(Tv_1, Tv_2)v_3 + \theta(Tv_2, Tv_3)v_1 - \theta(Tv_1, Tv_3)v_2)
\]

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Thus, we deduce that $\delta^I_T \circ \partial_T(\mathcal{X}) = 0$. 

$\blacksquare$
Define the set of \((2n-1)\)-cochains by
\[
C^{2n-1}_T(V, L) = \begin{cases} 
C^{2n-1}(V, L), & n \geq 1, \\
L \wedge L, & n = 0.
\end{cases}
\tag{4.9}
\]

Define \(d_T : C^{2n-1}_T(V, L) \to C^{2n+1}_T(V, L)\) by
\[
d_T = \begin{cases} 
\delta_T, & n \geq 1, \\
\partial_T, & n = 0.
\end{cases}
\tag{4.10}
\]

We now give the cohomology of \(\mathcal{O}\)-operators on L.t.s.

**Definition 4.2.** Let \(T\) be an \(\mathcal{O}\)-operator on a L.t.s \(\text{Rep}\) pair \((L, [\cdot, \cdot]; \theta)\). Denote the set of cocycles by \(Z^*(V, L)\), the set of coboundaries by \(B^*(V, L)\) and the cohomology group by
\[
H^*(V, L) = Z^*(V, L)/B^*(V, L).
\]
The cohomology group which will be taken to be the cohomology group for the \(\mathcal{O}\)-operator \(T\).

We need the following statement to prove the functoriality of our cohomology theory.

**Proposition 4.4.** Let \(T\) and \(T'\) be two \(\mathcal{O}\)-operators on a L.t.s \(\text{Rep}\) pair \((L, [\cdot, \cdot]; \theta)\) and \((\phi, \psi)\) be a morphism from \(T\) to \(T'\). Then
\(\psi\) is a L.t.s morphism from the descendent L.t.s \((V, [\cdot, \cdot]; T)\) of \(T\) to the descendent L.t.s \((V, [\cdot, \cdot]; T')\) of \(T'\).

(ii) The induced representation \((L, \theta_T)\) of the L.t.s \((V, [\cdot, \cdot]; T)\) and the induced representation \((L, \theta_{T'})\) satisfy the following relation
\[
\phi \circ \theta_T(u, v) = \theta_{T'}(\psi(u), \psi(v)) \circ \phi, \quad \forall u, v \in V.
\tag{4.11}
\]
That is, for all \(u, v \in V\), the following diagram commutes:

\[
\begin{array}{ccc}
L & \xrightarrow{\phi} & L' \\
\theta_T(u, v) \downarrow & & \downarrow \theta_{T'}(\psi(u), \psi(v)) \\
L & \xrightarrow{\phi} & L
\end{array}
\]

**Proof.** By (3.9), (3.10), we have
\[
\psi([u, v, w]_T) = \psi\left(D(T(u), T(v))w - \theta(T(u), T(w))v + \theta(T(v), T(w))u\right)
= D(\phi T(u), \phi T(v))\psi(w) - \theta(\phi T(u), \phi T(w))\psi(v) + \theta(\phi T(v), \phi T(w))\psi(u)
= D(T'\psi(u), T'\psi(v))\psi(w) - \theta(T'\psi(u), T'\psi(w))\psi(v) + \theta(T'\psi(v), T'\psi(w))\psi(u)
= \psi(u), \psi(v), \psi(w)]_{T'}.
\]

Now, according to (3.9), (3.10) and (4.2), for all \(u, v \in V\), \(x \in L\), we have
\[
\phi(\theta_T(u, v)x) = \phi([x, T(u), T(v)] + \phi(T(\theta(x, T(v))u)) - \phi(T(D(x, T(u))v))
= [\phi(x), \phi(T(u)), \phi(T(v))] + T'(\psi(\theta(x, T(v))u)) - T'(\psi(D(x, T(u))v))
= [\phi(x), T'(\psi(u)), T'(\psi(v))] + T'(\theta(\phi(x), T'(\psi(v))))\psi(u) - D(\phi(x), T'(\psi(u)))\psi(v)
= \theta_{T'}(\psi(u), \psi(v))\phi(x).
\]
\[\Box\]
Let $T$ and $T'$ be two $\mathcal{O}$-operators on $L$ with respect to the representation $(V, \theta)$ and $(\phi, \psi)$ a morphism from $T$ to $T'$ with $\psi$ invertible. Denote by $C^{2n-1}_{T}(V_{T}, L)$ the space of $(2n-1)$-cochains of L.t.s $(V, [-, -]_{T})$ with coefficients on the representation $(L, \theta_{T})$. Define

$$
\gamma : C^{2n-1}_{T}(V_{T}, L) \to C^{2n-1}_{T'}(V_{T'}, L), \quad n \geq 1,
$$
$$
\gamma : L \wedge L \to L \wedge L, \quad n = 0.
$$

by

$$
\gamma(f)(u_1, \cdots, u_{2n-1}) = \phi(f(\psi^{-1}(u_1), \cdots, \psi^{-1}(u_{2n-1}))), \quad \forall u_i \in V, \quad n \geq 1,
$$
$$
\gamma(\mathbf{x}) = \phi(\mathbf{x}) = (\phi(x_1), \phi(x_2)), \quad \forall \mathbf{x} = (x_1, x_2) \in L \wedge L, \quad n = 0.
$$

**Theorem 4.5.** With the above notations, $\gamma$ is a cochain map from the cochain complex $(C^{*}_{T}(V_{T}, L), d_{T})$ to the cochain complex $(C^{*}_{T'}(V_{T'}, L), d_{T'})$. Consequently, it induces a morphism $\pi_{T}$ from the cohomology group $H^{2n-1}_{T}(V_{T}, L)$ to $H^{2n-1}_{T'}(V_{T'}, L)$, for all $n \geq 0$.

**Proof.** For $n = 0$, let $\mathbf{x} \in L \wedge L$ and $(\phi, \psi)$ be a morphism from $T$ to $T'$. Then we have

$$
(\partial_{T'}(\gamma(\mathbf{x}))(v) = T'(D(\gamma(\mathbf{x})))v - [\gamma(\mathbf{x}), T'v]
$$
$$
= T'(D(\phi(\psi^{-1}(1)(u_1), \cdots, \psi^{-1}(u_{2n-1})))) - [\phi(\mathbf{x}), T' \psi \circ \psi^{-1}(v)]
$$
$$
= \phi(T'(D(\phi(\psi^{-1}(1)(u_1), \cdots, \psi^{-1}(u_{2n-1})))) - \phi([\mathbf{x}, T' \psi \circ \psi^{-1}(v)])
$$
$$
= \phi(\partial_{T}(\mathbf{x})(\psi^{-1}(v))) = \gamma(\partial_{T}(\mathbf{x})(v)).
$$

Now, for any $f \in C^{2n-1}_{T}(V_{T}, L)(n \geq 1)$, by Proposition 4.3, we have

$$
(\delta_{T'}(\gamma(f))(u_1, u_2, \cdots, u_{2n+1})
$$
$$\theta_{T'}(u_2 n, u_{2n+1}) \gamma(f)(u_1, u_2, \cdots, u_{2n-1}) - \theta_{T'}(u_2 n, u_{2n+1}) \gamma(f)(u_1, u_2, \cdots, u_{2n-2}, u_{2n})
$$
$$+ \sum_{k=1}^{n} (-1)^{n+k} D_{T'}(u_{2k-1}, u_{2k}) \gamma(f)(u_1, u_2, \cdots, \tilde{u}_{2k-1}, \tilde{u}_{2k}, \cdots, u_{2n+1})
$$
$$+ \sum_{k=1}^{2n+1} (-1)^{n+k+1} \gamma(f)(u_1, u_2, \cdots, \tilde{u}_{2k-1}, \tilde{u}_{2k}, \cdots, [u_{2k-1}, u_{2k}, u_{j}]_{T'}, \cdots, u_{2n+1})
$$
$$= \theta_{T'}(u_2 n, u_{2n+1}) \phi(f(\psi^{-1}(1)(u_1), \psi^{-1}(u_2), \cdots, \psi^{-1}(u_{2n-1})))
$$
$$- \theta_{T'}(u_2 n, u_{2n+1}) \phi(f(\psi^{-1}(1)(u_1), \psi^{-1}(u_2), \cdots, \psi^{-1}(u_{2n-2}), \psi^{-1}(u_{2n}))
$$
$$+ \sum_{k=1}^{n} (-1)^{n+k} D_{T'}(u_{2k-1}, u_{2k}) \phi(f(\psi^{-1}(1)(u_1), \psi^{-1}(u_2), \cdots, \psi^{-1}(u_{2k-1}), \psi^{-1}(u_{2k}), \cdots, \psi^{-1}(u_{2n+1}))
$$
$$+ \sum_{k=1}^{2n+1} (-1)^{n+k+1} \phi(f(\psi^{-1}(1)(u_1), \psi^{-1}(u_2), \cdots, \psi^{-1}(u_{2k-1}), \psi^{-1}(u_{2k}), \cdots, [\psi^{-1}(u_{2k-1}), \psi^{-1}(u_{2k}), \psi^{-1}(u_{j}]_{T'}, \cdots, \psi^{-1}(u_{2n+1}))
$$
$$= \theta_{T'}(\psi \circ \psi^{-1}(1)(u_2), \psi \circ \psi^{-1}(u_{2n+1})) \phi(f(\psi^{-1}(1)(u_1), \psi^{-1}(u_2), \cdots, \psi^{-1}(u_{2n-1}))
$$
$$- \theta_{T'}(\psi \circ \psi^{-1}(1)(u_2), \psi \circ \psi^{-1}(u_{2n+1})) \phi(f(\psi^{-1}(1)(u_1), \psi^{-1}(u_2), \cdots, \psi^{-1}(u_{2n-2}), \psi^{-1}(u_{2n}))
$$
$$+ \sum_{k=1}^{n} (-1)^{n+k} D_{T'}(\psi \circ \psi^{-1}(1)(u_{2k-1}) \psi \circ \psi^{-1}(u_{2k})) \phi(f(\psi^{-1}(1)(u_1), \psi^{-1}(u_2), \cdots, \psi^{-1}(u_{2k-1}), \psi^{-1}(u_{2k})),
$$

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\[
\cdots, \psi^{-1}(u_{2n+1}) + \sum_{k=1}^{n} \sum_{j=2k+1}^{2n+1} (-1)^{n+k+1} \phi f(\psi^{-1}(u_1), \psi^{-1}(u_2), \cdots, \psi^{-1}(u_{2k}), \psi^{-1}(u_{2k})),
\]

\[
\cdots, [\psi^{-1}(u_{2k-1}), \psi^{-1}(u_{2k}), \psi^{-1}(u_{2k})]_T, \cdots, \psi^{-1}(u_{2n+1}) = \phi \left( \theta_T(\psi^{-1}(u_{2n})), \psi^{-1}(u_{2n+1}) \right) f(\psi^{-1}(u_1), \psi^{-1}(u_2), \cdots, \psi^{-1}(u_{2n-1}), \psi^{-1}(u_{2n}))
\]

\[
- \theta_T(\psi^{-1}(u_{2n-1}), \psi^{-1}(u_{2n+1})) f(\psi^{-1}(u_1), \psi^{-1}(u_2), \cdots, \psi^{-1}(u_{2n-2}), \psi^{-1}(u_{2n}))
\]

\[
+ \sum_{k=1}^{n} (-1)^{n+k} D_T(\psi^{-1}(u_{2k-1}), \psi^{-1}(u_{2k})) f(\psi^{-1}(u_1), \psi^{-1}(u_2), \cdots, \psi^{-1}(u_{2k-1}), \psi^{-1}(u_{2k})),
\]

\[
\cdots, \psi^{-1}(u_{2n+1}) + \sum_{k=1}^{n} \sum_{j=2k+1}^{2n+1} (-1)^{n+k+1} f(\psi^{-1}(u_1), \psi^{-1}(u_2), \cdots, \psi^{-1}(u_{2k-1}), \psi^{-1}(u_{2k})),
\]

\[
\cdots, [\psi^{-1}(u_{2k-1}), \psi^{-1}(u_{2k}), \psi^{-1}(u_{2k})]_T, \cdots, \psi^{-1}(u_{2n+1}) = \phi \left( \delta_T(f)(u_1, \cdots, u_{2n+1}) \right) = \gamma(\delta_T(f))(u_1, \cdots, u_{2n+1}).
\]

Thus \( \gamma \) is a cochain map. Consequently, it induces a morphism \( \nabla \) from the cohomology group \( H_1^{2n-1}(V_T, L) \) to \( H_{2n-1}^{2n}(V_T', L) \), for all \( n \geq 0 \). \( \square \)

## 5 Deformation of an \( O \)-operator on a Lie triple system

In this section, we will apply the classical deformation theory due to Gerstenhaber (infinitesimal and formal deformations) to an \( O \)-operator on a L.t.s. We will introduce the notion of Nijenhuis element associated to an \( O \)-operator that arise from trivial deformations. We also consider the rigidity of an \( O \)-operator and provide a sufficient condition in terms of Nijenhuis elements.

### 5.1 Infinitesimal deformations

Let \( (L, [\cdot, \cdot, \cdot], \theta) \) be a L.t.sRep pair. Suppose \( T : V \rightarrow L \) is an \( O \)-operator associated to representation \( (V, \theta) \).

**Definition 5.1.** A parametrized sum \( T_t = T + tT_1 \), for some \( T_1 \in Hom(V, L) \), is called an infinitesimal deformation of \( T \) if \( T_1 \) is an \( O \)-operator for all values of \( t \). In this case, we say that \( T_1 \) generates an infinitesimal deformation of \( T \).

Suppose \( T_1 \) generates an infinitesimal deformation of \( T \). Then we have

\[
[T_1(u), T_1(v), T_1(w)] = T_1(D(T_1(u), T_1(v))v + \theta(T_1(v), T_1(w))u - \theta(T_1(u), T_1(w))v), \quad \forall u, v, w \in V.
\]

This is equivalent to the following conditions

\[
[Tu, Tv, T_1(w)] + [Tu, T_1(v), Tw] + [T_1(u), Tv, Tw]
\]

\[
= T(D(Tu, T_1(v))w + D(T_1(u), Tv)w + \theta(Tv, T_1(w))u + \theta(T_1(v), Tw)u
\]

\[
- \theta(Tu, T_1(w))v - \theta(T_1(u), Tw)v) + T_1(D(Tu, Tv)w + \theta(Tv, Tw)u - \theta(Tu, Tw)v),
\]

\[
(5.1)
\]

\[
[Tu, T_1(v), T_1(w)] + [T_1(u), T_1(v), Tw] + [T_1(u), T_1(w), Tv]
\]
\[ T \left( D(T_1(u), T_1(v))w + \theta(T_1(v), T_1(w))u - \theta(T_1(u), T_1(w))v \right) \]
\[ + T_1 \left( D(Tu, T_1(v))w + D(T_1(u), Tv)w + \theta(Tv, T_1(w))u \right. \]
\[ + \theta(T_1(v), Tw)u - \theta(Tu, T_1(w))v - \theta(T_1(u), Tw)v \right), \quad (5.2) \]

and

\[ [T_1(u), T_1(v), T_1(w)] = T_1 \left( D(T_1(u), T_1(v))w + \theta(T_1(v), T_1(w))u - \theta(T_1(u), T_1(w))v \right). \quad (5.3) \]

Note that, by Eq. (5.1), we have

\[ [Tu, Tv, T_1(w)] = -[Tv, T_1(w), Tu] - [T_1(w), Tu, Tv] = [T_1(w), Tv, Tu] + [Tu, T_1(w), Tv]. \]

Then the identity (5.1) implies that \( T_1 \) is a 1-cocycle (closed) in the cohomology of \( T \). Hence, \( T_1 \) defines a cohomology class in \( H^1(V, L) \). Eq. (5.3) means that \( T_1 \) is an \( O \)-operator on the L.t.sRep pair \((L, [\cdot, \cdot, \cdot]; \theta)\).

**Definition 5.2.** Let \( T_t = T + tT_1 \) and \( T_t' = T + tT_1' \) be two infinitesimal deformations of an \( O \)-operator \( T \). We say that \( T_t \) and \( T_t' \) are equivalent if there exists an element \( X \in L \wedge L \) such that the pair

\[ \left( \phi_t = Id_L + t\mathcal{L}(X) = Id_L + t[X, -] \right), \quad \psi_t = Id_V + tD(X)(-) \], \quad (5.4) \]

defines a morphism of \( O \)-operators from \( T_t \) to \( T_t' \).

Since \( \phi_t = Id_L + t[X, -] \) is a L.t.s morphism of \((L, [\cdot, \cdot, \cdot])\), then we have the following conditions

\[ \begin{align*}
[z_1, [X, z_2], [X, z_3]] + [[X, z_1], z_2, [X, z_3]] + [[X, z_1], [X, z_2], z_3] &= 0, \\
[[X, z_1], [X, z_2], [X, z_3]] &= 0, \quad \text{for } z_1, z_2, z_3 \in L.
\end{align*} \quad (5.5) \]

The condition \( \psi_t(\theta(z_1, z_2))u = \theta(\phi_t(z_1), \phi_t(z_2))\psi_t(u) \) gives that

\[ \begin{align*}
\theta(z_1, [X, z_2])D(X) + \theta([X, z_1], z_2)D(X) + \theta([X, z_1], [X, z_2]) &= 0, \\
\theta([X, z_1], [X, z_2])D(X) &= 0.
\end{align*} \quad (5.6) \]

Finally, the condition \( \phi_t \circ T_t = T_t' \circ \psi_t \) is equivalent to

\[ \begin{align*}
T_1(u) + [X, Tu] &= TD(X)u + T_1'(u), \\
[X, T_1(u)] &= T_1'D(X)(u).
\end{align*} \quad (5.7) \]

Note that the above identities hold for all \( X \in L \wedge L \), \( z_1, z_2, z_3 \in L \) and \( u \in V \).

From the first condition of (5.7), we have

\[ T_1(u) - T_1'(u) = TD(X)u - [X, Tu] = (dT(X))(u). \]

Therefore, we get the following result.

**Theorem 5.1.** Let \( T_t = T + tT_1 \) and \( T_t' = T + tT_1' \) be two equivalent infinitesimal deformations of an \( O \)-operator \( T \). Then \( T_t \) and \( T_t' \) defines the same cohomology class in \( H^1(T_1(V; L)). \)

**Definition 5.3.** Let \( T_t = T + tT_1 \) be an infinitesimal deformation of an \( O \)-operator \( T \). The deformation \( T_t \) is said to be trivial if \( (\phi_t, \psi_t) \) is a morphism from \( T_t \) to \( T \).
Now we will define Nijenhuis elements associated to an \( O \)operator on a L.t.s.

**Definition 5.4.** Let \((L, [\cdot, \cdot, \cdot]; \theta)\) be a L.t.s\(Rep\) pair and \(T\) be an \( O \)-operator. An element \( \mathcal{X} \in L \wedge L \) is called a Nijenhuis element associated to \( T \) if \( \mathcal{X} \) satisfies Eqs. (5.5), (5.6) and the equation

\[
[\mathcal{X},TD(\mathcal{X})(u) - [\mathcal{X},T(u)]] = 0. \tag{5.8}
\]

The set of Nijenhuis elements associated to an \( O \)-operator \( T \) is denoted by \( \text{Nij}(T) \).

Our motivation to introduce the above definition is that a trivial infinitesimal deformation gives rise to a Nijenhuis element. We give in the next subsection a sufficient condition to the rigidity of an \( O \)-operator in terms of Nijenhuis elements.

### 5.2 Formal deformations

In this subsection, we consider formal deformations of \( O \)-operators generalizing the classical deformation theory of Gerstenhaber [1]. Let \((L, [\cdot, \cdot, \cdot]; \theta)\) be a L.t.s\(Rep\) pair and \( T : V \rightarrow L \) be an \( O \)-operator.

Let \( \mathbb{K}[[t]] \) be the ring of power series in one variable \( t \). For any \( \mathbb{K} \)-linear space \( L \), we denote by \( L[[t]] \) the vector space of formal power series in \( t \) with coefficients in \( V \). If in addition, we have a structure of L.t.s \((L, [\cdot, \cdot, \cdot])\) over \( \mathbb{K} \), then there is a L.t.s structure over the ring \( \mathbb{K}[[t]] \) on \( L[[t]] \) given by

\[
\left[ \begin{array}{c}
\sum_{i=0}^{+\infty} x_i t^i \\
\sum_{j=0}^{+\infty} y_j t^j \\
\sum_{k=0}^{+\infty} z_k t^k
\end{array} \right] = \left[ \begin{array}{c}
\sum_{s=0}^{+\infty} \sum_{i+j+k=s} \sum_{k=0}^{+\infty} [x_i, y_j, z_k] t^s,
\forall x_i, y_j, z_k \in L.
\end{array} \right] \tag{5.9}
\]

For any representation \((V, \theta)\) of \((L, [\cdot, \cdot, \cdot])\), there is a natural representation of the L.t.s \( L[[t]] \) on the \( \mathbb{K}[[t]] \)-module \( V[[t]] \), which is given by

\[
\theta \left( \sum_{i=0}^{+\infty} x_i t^i \right) \left( \sum_{j=0}^{+\infty} y_j t^j \right) \left( \sum_{k=0}^{+\infty} V_k t^k \right) = \sum_{s=0}^{+\infty} \sum_{i+j+k=s} \theta(x_i, y_j) v_k t^s, \forall x_i, y_j \in L, v_k \in V. \tag{5.10}
\]

Consider a power series

\[
T_t = \sum_{i=0}^{+\infty} T_i t^i, \quad T_i \in Hom_{\mathbb{K}}(V; L), \tag{5.11}
\]

that is, \( T_t \in Hom_{\mathbb{K}}(V; L)[[t]] = Hom_{\mathbb{K}}(V; L[[t]]) \). Extend it to be a \( \mathbb{K}[[t]] \)-module map from \( V[[t]] \) to \( L[[t]] \) which is still denoted by \( T_t \).

**Definition 5.5.** If \( T_t = \sum_{i=0}^{+\infty} T_i t^i \) with \( T_0 = T \) satisfies

\[
[T_t(u), T_t(v), T_t(w)] = T_t \left( D(T_t(u), T_t(v)) w + \theta(T_t(v), T_t(w)) u - \theta(T_t(u), T_t(w)) v \right), \tag{5.12}
\]

we say that \( T_t \) is a formal deformation of the \( O \)-operator \( T \).

Recall that a formal deformation of a L.t.s \((L, [\cdot, \cdot, \cdot])\) is a formal power series \( \omega_t = \sum_{k=0}^{+\infty} \omega_k t^k \) where \( \omega_k \in Hom((\wedge^2 L) \otimes L; L) \) such that \( \omega_0(x, y, z) = [x, y, z] \) for any \( x, y, z \in L \) and \( \omega_t \) defines a L.t.s structure over the ring \( \mathbb{K}[[t]] \) on \( L[[t]] \).

Based on the relationship between \( O \)-operators and L.t.s structure, we have
Proposition 5.2. Let $T_t = \sum_{i=0}^{+\infty} T_i t^i$ be a formal deformation of an $\mathcal{O}$-operator $T$ on the $L.t.sRep$ pair $(L, [\cdot, \cdot, \cdot]; \theta)$. Then $[\cdot, \cdot, \cdot]_{T_t}$ defined by

$$[u, v, w]_{T_t} = \sum_{k=0}^{+\infty} \sum_{i+j+k=i} (D(T_i(u), T_j(v))w + \theta(T_i(v), T_j(w))u - \theta(T_i(u), T_j(w))v) t^k,$$

for all $u, v, w \in V$, is a formal deformation of the associated $L.t.s$ $(V, [\cdot, \cdot, \cdot]_T)$ given in Lemma 5.4.

By applying Eqs. (5.9)-(5.11) to expand Eq. (5.12) and collecting coefficients of $t^s$, we see that Eq. (5.12) is equivalent to the following system of equations

$$\sum_{i+j+k=s \geq 2} [T_i(u), T_j(v), T_k(w)] = \sum_{i+j+k=s \geq 0} T_i \left( D(T_j(u), T_k(v))w + \theta(T_j(v), T_k(w))u - \theta(T_j(u), T_k(w))v \right),$$

for all $s \geq 0$, $u, v, w \in V$.

Proposition 5.3. Let $T_t = \sum_{i=0}^{+\infty} T_i t^i$ be a formal deformation of an $\mathcal{O}$-operator on the $L.t.sRep$ pair $(L, [\cdot, \cdot, \cdot]; \theta)$. Then $T_1$ is a 1-cocycle in the cohomology of an $\mathcal{O}$-operator $T$, that is, $d_T(T_1) = 0$.

Proof. Note that (5.13) holds for $s = 0$ as $T_0 = T$ is an $\mathcal{O}$-operator. For $s = 1$, we get

$$[Tu, Tv, T_1(w)] + [Tu, T_1(v), Tw] + [T_1(u), Tv, Tw] = T \left( D(Tu, T_1(v))w + D(T_1(u), Tv)w + \theta(Tv, T_1(w))u + \theta(T_1(v), Tw)u \right)$$

$$- \theta(Tu, T_1(w))v - \theta(T_1(u), Tv)v \right) + T_1 \left( D(Tu, Tv)w + \theta(Tv, Tw)u - \theta(Tu, Tw)v \right),$$

for all $u, v, w \in V$. This implies that $(d_T(T_1))(u, v, w) = 0$. Hence the linear term $T_1$ is a 1-cocycle in the cohomology of $T$. \qed

Definition 5.6. Let $T$ be an $\mathcal{O}$-operator on the $L.t.sRep$ pair $(L, [\cdot, \cdot, \cdot]; \theta)$. The 1-cocycle given in Proposition 5.3 is called the infinitesimal of the formal deformation $T_t = \sum_{i=0}^{+\infty} T_i t^i$ of $T$.

In the sequel, we discuss equivalent formal deformations.

Definition 5.7. Let $T_t = \sum_{i=0}^{+\infty} T_i t^i$ and $T'_t = \sum_{i=0}^{+\infty} T'_i t^i$ be two formal deformations of an $\mathcal{O}$-operator $T = T_0 = T'_0$ on a $L.t.sRep$ pair $(L, [\cdot, \cdot, \cdot]; \theta)$. They are said to be equivalent if there exists an element $X \in L \wedge L$, $\phi_i \in gl(L)$ and $\psi_i \in gl(V)$, $i \geq 2$, such that the pair

$$\left( \phi_t = Id_L + t[X, -] + \sum_{i=2}^{+\infty} \phi_i t^i, \psi_t = Id_V + tD(X)(-) + \sum_{i=2}^{+\infty} \psi_i t^i \right),$$

is a morphism of $\mathcal{O}$-operators from $T_t$ to $T'_t$. In particular, a formal deformation $T_t$ of an $\mathcal{O}$-operator $T$ is said to be trivial if there exists an $X \in L \wedge L$, $\phi_i \in gl(L)$ and $\psi_i \in gl(V)$, $i \geq 2$, such that $(\phi_t, \psi_t)$ defined by Eq. (5.14) gives an equivalence between $T_t$ and $T$, with the latter regarded as a deformation of itself.
\textbf{Theorem 5.4.} If two formal deformations of an $O$-operator $T$ on a $L.t.sRep$ pair $(L,[\cdot,\cdot];\theta)$ are equivalent, then their infinitesimals are in the same cohomology class in $H^1_F(V;L)$.

\textit{Proof.} Let $(\phi_t,\psi_t)$ be the two maps defined by Eq. (5.14) which gives an equivalence between two deformations $T_t = \sum_{i=0}^{+\infty} T_i t^i$ and $T'_t = \sum_{i=0}^{+\infty} T'_i t^i$ of an $O$-operator $T$. By $(\phi_t \circ T_t)(u) = (T'_t \circ \psi_t)(u)$, we have

$$T_t(u) = T'_t(u) + TD(\mathfrak{x})u - [\mathfrak{x},Tu]_\theta$$

$$= T'_t(u) + (dT)(u), \ \forall u \in V,$$

which implies that $T_t$ and $T'_t$ are in the same cohomology class. \hfill \Box

\textbf{Definition 5.8.} An $O$-operator $T$ is said to be rigid if any formal deformation of $T_t$ is trivial.

Now, we give a cohomological characterization of rigidity of an $O$-operator involving the set of Nijenhuis elements.

\textbf{Theorem 5.5.} Let $T$ be an $O$-operator. If $Z^1_F(V,L) = d_T(Nij(T))$, then $T$ is rigid.

\textit{Proof.} Let $T_t = \sum_{i=0}^{\infty} t^i T_i$ be any formal deformation of $T$. We have seen in Proposition 5.3 that $T_1$ is a 1-cocycle in the cohomology of $T$, i.e. $T_1 \in Z^1_F(V,L)$. Thus, from the hypothesis, we get a Nijenhuis element $\mathfrak{x} \in Nij(T)$ such that $T_1 = d_T(\mathfrak{x})$. Then, setting

$$\phi_t = Id_L + t[\mathfrak{x}, -], \quad \psi_t = Id_V + tD(\mathfrak{x})(-)$$

and define $T'_t = \phi_t \circ T_t \circ \psi_t^{-1}$, one obtains $T'_t$ is a deformation equivalent to $T_t$. Moreover, we have

$$T'_t(u) = (Id_L + t[\mathfrak{x}, -])(T_t(Id_V - tD(\mathfrak{x}) + t^2 D^2(\mathfrak{x}) + \cdots + (-1)^i t^i D^i(\mathfrak{x}) + \cdots)(u))$$

$$= T(u) + t(T_1(u) - TD(\mathfrak{x})(u) + [\mathfrak{x}, T(u)]) + t^2 T'_2(u) + \cdots$$

$$= T(u) + t^2 T'_2(u) + \cdots \quad \text{(as } T_1(u) = d_T(\mathfrak{x})(u).)$$

Hence the coefficient of $t$ in the expression of $T'_t$ is trivial. By applying the same process repeatedly, we get that $T_t$ is equivalent to $T$. Therefore, $T$ is rigid. \hfill \Box

6 \ From cohomology groups of $O$-operators on Lie algebras to those on Lie triple systems

Motivated by the construction of $L.t.s$ from Lie algebras. We give some connections between $O$-operators on Lie algebras and $L.t.s$.

A representation of a Lie algebra $(L,[\cdot,\cdot])$ on a vector space $V$ is a linear map $\rho : L \rightarrow End(V)$ such that

$$\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x), \ \forall x, y \in L.$$

Similarly, we call the pair of Lie algebra $(L,[\cdot,\cdot])$ and the representation $\rho$ a LieRep pair, denoted by $(L,[\cdot,\cdot];\rho)$. Let $(L,[\cdot,\cdot],\rho)$ be a LieRep pair. Define a bilinear bracket $[\cdot,\cdot]_\rho$ on $L \oplus V$ by

$$[x + u, y + v]_\rho = [x,y] + \rho(x)v - \rho(y)u, \ \forall x,y \in L, \ u,v \in V.$$
Then \((L \oplus V, [\cdot, \cdot]_\rho)\) is a semi-direct product Lie algebra, denoted by \(L \ltimes_\rho V\).

Let \((L, [\cdot, \cdot]; \rho)\) be a LieRep pair. The space of \(p\)-cochains is \(C^p_{\text{Lie}}(L, V) = \text{Hom}(\wedge^p L, V)\) for \(p \geq 0\). The coboundary operator \(\partial_\rho : C^p_{\text{Lie}}(L, V) \to C^{p+1}_{\text{Lie}}(L, V)\) is defined by

\[
\partial_\rho(f)(x_1, \ldots, x_{p+1}) = \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} f([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{p+1})
\]

\[
+ \sum_{i=1}^{p+1} (-1)^{i+1} \rho(x_i)f(x_1, \ldots, \hat{x}_i, \ldots, x_{p+1}),
\]

for \(f \in C^p_{\text{Lie}}(L, V)(p \geq 0)\). We denote the corresponding \(p\)-th cohomology group by \(H^p_{\text{Lie}}(L, V)\).

**Definition 6.1.** A linear map \(T : V \to L\) is called an \(O\)-operator on a LieRep pair \((L, [\cdot, \cdot]; \rho)\) if \(T\) satisfies

\[
[Tu, Tv] = T\left(\rho(Tu)v - \rho(Tv)u\right), \quad \forall u, v \in V.
\]

(6.1)

Let \(T\) be an \(O\)-operator on a LieRep pair \((L, [\cdot, \cdot]; \rho)\). There is a Lie algebra structure on \((V, [\cdot, \cdot]_T)\), where the bracket \([\cdot, \cdot]_T : V \times V \to V\) is given by

\[
[u, v]_T = \rho(Tu)v - \rho(Tv)u \quad \forall u, v \in V.
\]

(6.2)

Now, we recall some results from [3]. There is a representation of \((V, [\cdot, \cdot]_T)\), \(\rho_T : V \to gl(L)\), defined by

\[
\rho_T(u)x = [Tu, x] + T\rho(x)u, \quad \forall u \in V, x \in L.
\]

Let \(T\) be an \(O\)-operator on a LieRep pair \((L, [\cdot, \cdot]; \rho)\). Consider the set of \(p\)-cochains \(C^p_T(V, L) = \text{Hom}(\wedge^p V, L)\). Let \(\tilde{d}_T : C^p_T(V, L) \to C^{p+1}_T(V, L)(p \geq 0)\) be the corresponding coboundary operator of the Lie algebra \((V, [\cdot, \cdot]_T)\) with coefficients in the representation \((L, \rho_T)\), defined for all \(f \in C^p_T(V, L)\) and \(u_1, \ldots, u_{p+1} \in V\) by

\[
\tilde{d}_T(f)(u_1, \ldots, u_{p+1}) = \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} f(\rho(Tu_i)u_j - \rho(Tu_j)u_i, u_1, \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, u_{p+1})
\]

\[
+ \sum_{i=1}^{p+1} (-1)^{i+1} \left(\sum_{i=1}^{p+1} (-1)^{i+1} \left(Tu_i, f(u_1, \ldots, \hat{u}_i, \ldots, u_{p+1})\right) - T\rho(f(u_1, \ldots, \hat{u}_i, \ldots, u_{p+1}))(u_i)\right).
\]

Then the cochain complex \((C^p_T(V, L), \cdot\)\) is called a cochain complex of the \(O\)-operator \(T\) on a LieRep pair \((L, [\cdot, \cdot]; \rho)\). We denote the corresponding \(p\)-th cohomology group by \(H^p_T(V, L)\).

As we know, there is a method of constructing L.t.s from Lie algebras given in Example 2.

**Theorem 6.1.** Let \((L, [\cdot, \cdot]; \rho)\) be a LieRep pair. Define \(\theta_\rho : \otimes^2 L \to gl(V)\) by

\[
\theta_\rho(x, y) = \rho(y)\rho(x).
\]

(6.3)

Then \((L, [\cdot, \cdot] = [\cdot, \cdot] \circ ([\cdot, \cdot] \otimes Id_L) : \theta_\rho)\) is a L.t.sRep pair.
Proof. Let \((L, [\cdot, \cdot]; \rho)\) be a LieRep pair. We know that there is a L.t.s structure on \(L\) given by the bracket 
\([\cdot, \cdot, \cdot] = [[\cdot, \cdot], \cdot]\). Now, for all \(x, y, z, t, e \in L\) and \(u, v, w \in V\), we have
\[
[x + u, y + v, z + w]_{L \oplus V} = [[x + u, y + v]_{L \oplus V}, z + w]_{L \oplus V}
\]
\[
= [[x, y] + \rho(x)(v) - \rho(y)(u), z + w]_{L \oplus V}
\]
\[
= [[x, y], z] + \rho([x, y])(w) - \rho(z)\rho(x)(v) + \rho(z)\rho(y)(u)
\]
\[
= [x, y, z] + \left(\rho(x)\rho(y) - \rho(y)\rho(x)\right)(w) - \rho(z)\rho(x)(v) + \rho(z)\rho(y)(u)
\]
\[
= [x, y, z] + \left(\theta_{\rho}(y, x) - \theta_{\rho}(x, y)\right)(w) - \theta_{\rho}(x, z)(v) + \theta_{\rho}(y, z)(u)
\]
\[
= [x, y, z] + D_{\rho}(x, y)(w) - \theta_{\rho}(x, z)(v) + \theta_{\rho}(y, z)(u).
\]
Then by Eq. (L.7), \((L, [\cdot, \cdot, \cdot] = [[\cdot, \cdot], \cdot] \circ ([\cdot, \cdot] \otimes Id_L); \theta_{\rho})\) is a L.t.sRep pair.

Theorem 6.2. Every 1-cocycle for the cohomology of LieRep pair \((L, [\cdot, \cdot]; \rho)\) is a 1-cocycle for the cohomology of the L.t.sRep pair \((L, [\cdot, \cdot, \cdot] = [[\cdot, \cdot], \cdot] \circ ([\cdot, \cdot] \otimes Id_L); \theta_{\rho})\).

Proof. Let \(\varphi\) be a 1-cocycle of the cohomology of the LieRep pair \((L, [\cdot, \cdot], \rho)\), then
\[
\forall x, y \in L, \theta_{\rho}(\varphi)(x, y) = \rho(x)\varphi(y) - \rho(y)\varphi(x) - \varphi([x, y]) = 0.
\]
Now, for any \(x, y, z \in L\), we have
\[
d^{1}(\varphi)(x, y, z) = D_{\rho}(x, y)\varphi(z) - \theta_{\rho}(x, z)\varphi(y) + \theta_{\rho}(y, z)\varphi(x) - \varphi([x, y, z])
\]
\[
= \rho([x, y])\varphi(z) - \rho(z)\rho(x)\varphi(y) + \rho(z)\rho(y)\varphi(x) - \varphi([x, y, z])
\]
\[
= \rho([x, y])\varphi(z) - \rho(z)\rho(x)\varphi(y) + \rho(z)\rho(y)\varphi(x) - \rho([x, y])\varphi(z)
\]
\[
+ \rho(z)\rho(x)\varphi(y) - \rho(z)\rho(y)\varphi(x)
\]
\[
= 0,
\]
which means that \(\varphi\) is a 1-cocycle for the cohomology of the L.t.sRep pair \((L, [\cdot, \cdot, \cdot] = [[\cdot, \cdot], \cdot] \circ ([\cdot, \cdot] \otimes Id_L); \theta_{\rho})\).

Theorem 6.3. Let \(\varphi \in Z_{Lie}^{2}(L, V)\). Then \(\omega(x, y, z) = \varphi([x, y], z) - \rho(z)\varphi(x, y)\) is a 3-cocycle of the L.t.sRep pair \((L, [\cdot, \cdot, \cdot] = [[\cdot, \cdot], \cdot] \circ ([\cdot, \cdot] \otimes Id_L); \theta_{\rho})\).

Proof. Let \(\varphi \in Z_{Lie}^{2}(L, V)\). According to (L.8) in Definition 2.3, we have
\[
\omega(x, y, z) = \varphi([x, y], z) - \rho(z)\varphi(x, y)
\]
\[
= - \left(\varphi([y, x], z) - \rho(z)\varphi(y, z)\right) = -\omega(y, x, z).
\]
and
\[
\omega(x, y, z) + \omega(y, z, x) + \omega(z, x, y)
\]
\[
= \varphi([x, y], z) - \rho(z)\varphi(x, y) + \varphi([y, z], x) - \rho(x)\varphi(y, z) + \varphi([z, x], y) - \rho(y)\varphi(z, x)
\]
\[
= \theta_{\rho}(\varphi)(x, y, z) = 0.
\]
Now, for any \(x, y, z, t, e \in L\), we have
\[
d^{3}(\omega)(x, y, z, t, e)
\]
According to Lemma 6.4, which means that \( \omega \) defined by:

\[
\omega(x, y, [z, t, e]) = \omega(x, y, z, t, e) - \omega(z, [x, y, t], e)
\]

\[
- \omega(z, t, [x, y, e]) - \theta_p(t, e)\omega(x, y, z) + \theta_p(z, e)\omega(x, y, t) - D_\rho(z, t)\omega(x, y, e)
\]

\[
= \varphi(x, y, [z, t, e]) - \rho([z, t, e])\varphi(x, y) + \rho([x, y])\varphi([z, t, e]) - \rho([x, y])\rho(e)\varphi(z, t)
\]

\[
- \varphi([[x, y], z, t], e) + \rho(e)\varphi([[x, y], z, t]) - \varphi([x, y, z, t], e) + \rho(e)\varphi(z, [x, y, t])
\]

\[
- \varphi([z, t], [[x, y], e]) + \rho([[x, y], e])\varphi(z, t) - \rho(e)\rho(t)\varphi([x, y], z) + \rho(e)\rho(z)\varphi([x, y], t)
\]

\[
+ \rho(e)\rho([t, z])\varphi(x, y) - \rho([z, t])\varphi(x, y) + \rho([z, t])\rho(e)\varphi(x, y)
\]

\[
= \varphi(x, y, [z, t, e]) - \varphi([z, t], [x, y, e]) + \varphi([[x, y], [t, z], e]) + \rho([x, y])\varphi([z, t], e)
\]

\[
- \rho([z, t])\varphi(x, y, e) + \rho(e)\varphi([x, y, e]) + \rho(e)\varphi([z, z, e]) - \rho(e)\rho(t)\varphi([x, y], z)
\]

\[
+ \rho(e)\rho(z)\varphi([x, y, t], e) - \rho(e)\rho([x, y])\varphi(z, t)
\]

\[
= \varphi(x, y, [z, t, e]) - \varphi([z, t], [[x, y], e]) + \varphi([[x, y], [t, z], e]) + \rho([x, y])\varphi([z, t], e)
\]

\[
- \rho([z, t])\varphi(x, y, e) + \rho(e)\varphi([x, y, [z, t], e])
\]

Since \( \varphi \in Z^2_{Lie}(L, V) \), we get \( \delta^3(\omega)(x, y, z, t, e) = 0 \).

Lemma 6.4. Let \( \alpha \in C^1_{Lie}(L, V) \). Then

\[
\delta^1(\alpha)(x, y, z) = \partial_\rho(\alpha)([x, y], z) - \rho(z)\partial_\rho(\alpha)(x, y).
\]

Proof. For any \( x, y, z \in L \), we have

\[
D_\rho(x, y)\alpha(z) - \theta_\rho(x, z)\alpha(y) + \theta_\rho(y, z)\alpha(x) - \alpha([x, y, z])
\]

\[
= \rho([x, y])\alpha(z) - \rho(z)\rho(x)\alpha(y) + \rho(z)\rho(y)\alpha(x) - \alpha([x, y, z])
\]

\[
= \rho([x, y])\alpha(z) - \rho(z)\alpha([x, y]) - \alpha([x, y, z])
\]

\[
- \rho(z)^2(\rho(x)\alpha(y) - \rho(y)\alpha(x) - \alpha([x, y]))
\]

\[
= \partial_\rho(\alpha)([x, y], z) - \rho(z)\partial_\rho(\alpha)(x, y).
\]

Proposition 6.5. Let \( \varphi_1, \varphi_2 \in Z^2_{Lie}(L, V) \). If \( \varphi_1, \varphi_2 \) are in the same cohomology class then \( \omega_1, \omega_2 \) defined by:

\[
\omega_i(x, y, z) = \varphi_i([x, y], z) - \rho(z)\varphi_i(x, y), \quad i = 1, 2
\]

are in the same cohomology class of the associated L.t.s.

Proof. Let \( \varphi_1, \varphi_2 \in Z^2_{Lie}(L, V) \) be two cocycles in the same cohomology class, that is

\[
\varphi_2 - \varphi_1 = \partial_\rho(\alpha), \quad \alpha \in C^1_{Lie}(L, V).
\]

and

\[
\omega_i(x, y, z) = \varphi_i([x, y], z) - \rho(z)\varphi_i(x, y), \quad i = 1, 2.
\]

According to Lemma 6.4, we have

\[
\omega_2(x, y, z) - \omega_1(x, y, z) = (\varphi_2 - \varphi_1)([x, y], z) - \rho(z)(\varphi_2 - \varphi_1)(x, y)
\]

\[
= \partial_\rho(\alpha)([x, y], z) - \rho(z)\partial_\rho(\alpha)(x, y)
\]

\[
= \delta^1(\alpha)(x, y, z), \quad \alpha \in C^1_{Lie}(L, V),
\]

which means that \( \omega_1 \) and \( \omega_2 \) are in the same cohomology class.
**Proposition 6.6.** Let $T : V \to L$ be an $O$-operator on a LieRep pair $(L, [\cdot, \cdot]; \rho)$. Then $T$ is also an $O$-operator on the $L.t.sRep$ pair $(L, [\cdot, \cdot, \cdot] = [\cdot, \cdot] \circ ([\cdot, \cdot] \otimes \text{Id}_L); \theta_{\rho})$.

**Proof.** For any $u, v, w \in V$, we have

$$[Tu, Tv, Tw] = [(Tu, Tv), Tw] = [T(\rho(Tu)v - \rho(Tv)u), Tw]$$

$$= [T\rho(Tu)v, Tw] - [T\rho(Tv)u, Tw]$$

$$= T\left(\rho(\rho(Tu)v)w - \rho(Tw)\rho(Tu)v\right) - T\left(\rho(\rho(Tv)u)w - \rho(Tw)\rho(Tv)u\right)$$

$$= T\left(\rho([Tu, Tv])w - \rho(Tw)\rho(Tv)u + \rho(Tw)\rho(Tv)u\right)$$

$$= T\left(D_{\rho}(Tu, Tv)w + \theta_{\rho}(Tv, Tw)u - \theta_{\rho}(Tu, Tw)v\right).$$

Hence $T$ is an $O$-operator on the $L.t.sRep$ pair $(L, [\cdot, \cdot, \cdot] = [\cdot, \cdot] \circ ([\cdot, \cdot] \otimes \text{Id}_L); \theta_{\rho})$.

Now there are two methods of constructing a $L.t.s$ structure on $V$ from a LieRep pair $(L, [\cdot, \cdot]; \rho)$. On the one hand, we induce a $L.t.s$ structure $(V, [\cdot, \cdot, \cdot]_T)$ of the $O$-operator $T$ on the corresponding $L.t.sRep$ pair $(L, [\cdot, \cdot, \cdot] = [\cdot, \cdot] \circ ([\cdot, \cdot] \otimes \text{Id}_L); \theta_{\rho})$, where

$$[u, v, w]_T = D_{\rho}(Tu, Tv)w + \theta_{\rho}(Tv, Tw)u - \theta_{\rho}(Tu, Tw)v. \tag{6.6}$$

On the other hand, we firstly induce a Lie algebra $(V, [\cdot, \cdot; T]$ of $T$ on the LieRep pair $(L, [\cdot, \cdot]; \rho)$, and then we give a $L.t.s$ structure induced from the Lie algebra $(V, [\cdot, \cdot; T]$ by the bracket $[\cdot, \cdot, \cdot]_T = [\cdot, \cdot]_T \circ ([\cdot, \cdot] \otimes \text{Id}_V)$. These two methods give us the same L.t.s structure on $V$.

The following results hold for the cohomology groups of an $O$-operator and here we omit the proofs.

**Corollary 6.7.** Every 1-cocycle for the cohomology of an $O$-operator $T$ on a LieRep $(L, [\cdot, \cdot]; \rho)$ is a 1-cocycle for the cohomology of an $O$-operator on the $L.t.sRep$ pair $(L, [\cdot, \cdot, \cdot] = [\cdot, \cdot] \circ ([\cdot, \cdot] \otimes \text{Id}_L); \theta_{\rho})$.

**Corollary 6.8.** Let $\varphi \in Z^2_{\rho}(V, L)$ be a 2-cocycle on the cohomology of an $O$-operator $T$ on a LieRep $(L, [\cdot, \cdot]; \rho)$. Then $\omega(u, v, w) = \varphi([u, v], T, w) - \rho_T(w)\varphi(u, v)$ is a 3-cocycle of the cohomology of an $O$-operator $T$ on the $L.t.sRep$ pair $(L, [\cdot, \cdot, \cdot] = [\cdot, \cdot] \circ ([\cdot, \cdot] \otimes \text{Id}_L); \theta_{\rho})$.

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