O(N)-invariant Hierarchical Renormalization Group Fixed Points by Algebraic Numerical Computation and $\epsilon$-Expansion

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Abstract

Generalizing methods developed by Pinn, Pordt and Wieczerkowski for the hierarchical model with one component ($N = 1$) and dimensions $d$ between 2 and 4 we compute $O(N)$-symmetric fixed points of the hierarchical renormalization group equation for some $N$ and $d$ with $0 < d < 4$ and $-2 \leq N \leq 20$. The spectra of the linearized RG equation at the fixed points are calculated and the critical exponents $\nu$ are extracted from the spectrum and compared to Borel-Padé-resummed $\epsilon$-expansion.

Keywords: renormalization group, $O(N)$-invariance, hierarchical model, critical exponents
1 Introduction

Euclidean quantum field theories and systems of statistical mechanics are commonly defined by a generating functional of correlation functions

\[ Z(J) = \int d\mu_v(\Phi) e^{-V(\Phi)} e^{(J,\Phi)} \]  

(1)

where \( d\mu_v \) is a suitable Gaussian measure with covariance \( v \) on the space of fields under consideration. The Gaussian measure describes the free theory and \( e^{-V} \) describes the interaction and will be called the “Boltzmann factor”.

Let us consider a model with \( N \) local degrees of freedom. Famous examples of such models in statistical physics are the Ising model with \( N = 1 \) and the Heisenberg model with \( N = 3 \). These models correspond to \( \phi^4 \) theories for fields with \( N \) real components in the euclidean QFT setup. Here we are using a local variable \( \varphi \in \mathbb{R}^N \) to describe such situations. We analytically continue our calculations to noninteger and negative values of \( N \). The case \( N = 0 \) is an example of physical significance of this continuation, as it is used for the description of polymers.

\( N \)-component models have recently been reconsidered in the literature. In this paper we are doing so using the high accuracy scheme of the algebraic renormalization group. Instead of fine tuning critical parameters it is possible to calculate directly the fixed points by using Newton’s method. The critical exponents can then be calculated from the spectrum of the linearized RG equation at the fixed point. We give a list of scenarios for various values of \( N \) and \( d \).

We will consider hierarchical models in the sense of \([7, 4]\) as an approximation to “full” field theoretic models models by replacing the covariance \( v = (-\Delta + m^2)^{-1} \) by a translationally non-invariant hierarchical approximation of the massless covariance. They can also be regarded as a system of statistical physics with a peculiar structure. The hierarchical covariance simulates the long distance behaviour of the “full” covariance and preserves the locality of the Boltzmannians under renormalization group transformations (RGT). A Boltzmann factor is called local, if it factorizes

\[ e^{-V(\varphi)} = \prod_{x \in \Lambda} e^{-V(\varphi(x))}. \]

The hierarchical RG equation is a RGT of Boltzmannians \( Z = e^{-V} : \mathbb{R}^N \to \mathbb{R} \).
For $N$-component fields the hierarchical RG transformation is given by the non-linear integral transformation

$$RZ(\varphi) = \int d\mu_\gamma(1-\beta^2)(\psi)Z^{L^d}(\psi + \beta \varphi) ,$$

(2)

with the Gaussian measure $d\mu_\gamma(\varphi) = \frac{1}{\sqrt{2\pi \gamma}} e^{-\frac{1}{\gamma \phi^2}} d\phi_1 \ldots d\phi_N$, $\gamma > 0$, $\beta = L^{1-d}$. One may consider this equation as our starting point. A derivation of equation (2) for $N = 1$ can be found in [7]. The derivation for general $N$ can be done in an exactly analogous way.

In this paper we present the calculation of the $O(N)$-symmetric fixed points and their spectrum. Fixed points of the RGT determine the long distance behaviour of the system and are a tool to investigate the continuum limit and the thermodynamic limit of such theories. Recently also the stability of $O(N)$-symmetric fixed points of this equation under disturbances with cubic symmetry was investigated both for full models and in the hierarchical approximation [6].

The critical exponents are functions of $L$ in the hierarchical approximation but they do not vary very rapidly with $L$ unless $L$ is close to 1. The limit $L \to 1$ is perhaps singular. KOCH and WITTWER have shown the existence of a fixed point of this equation with the values $N = 1, d = 3$ and $L^d = 2$ [6]. For $L^d = 2$ the model is equivalent to Dyson’s version of the hierarchical model. Also a product of functions can be defined by

$$(f \cdot g)(\psi) := \int d\mu_\gamma(1-\beta^2)(\psi)f(\psi + \beta \varphi)g(\psi + \beta \varphi) .$$

(3)

The hierarchical RG equation is then given by

$$RZ = Z \cdot Z$$

(4)

and the linearisation $L_R$ of $R$ at the “point” $f$ is given by

$$L_R(f)Z = 2f \cdot Z .$$

(5)

Note that the product defined in this way is commutative but not associative [6, 7].

It is easy to see that there are two trivial fixed points of the transformation $R$ (apart from $Z_0 = 0$, which does not define a theory), namely $Z_{UV} = 1$ and a Gaussian fixed point

$$Z_{HT}(\varphi) = (2\beta^2)^{\frac{2\beta^2 - 1}{\gamma (1-\beta^2)}} e^{\frac{2}{\gamma (1-\beta^2)} \phi^2} .$$
$Z_{UV}$ belongs to the massless free theory and $Z_{HT}$ to the massive free theory. One important feature of the RG equation is

$$\mathcal{R}_{\beta,\gamma}(Z_{HT}Z) = Z_{HT}(\mathcal{R}_{\beta',\gamma'}Z)$$

with $\beta' = L^{-1-\frac{d}{2}} = 2^{-\frac{d+2}{d+4}}$ and $\gamma' = L^{-2}\gamma = 2^{-\frac{2}{d}}\gamma$. The RG equation with $\beta', \gamma'$ is called the high temperature picture, because the HT fixed point is splitted from the Boltzmannian. In the HT picture $Z = 1$ belongs to the HT fixed point and the inverse of $Z_{HT}$ belongs to the UV fixed point.

The linearisation at $Z_{UV}$ is given by

$$(L_R(Z_{UV})f)(\psi) = \int d\mu_{\gamma(1-\beta^2)}(\psi)1f(\psi + \beta\phi).$$

(6)

Normal ordering is in a sense inverse to Gaussian integration. Using this one is able to compute the spectrum of $L_R(Z_{UV})$ analytically. This will be done in a first section. After computing the spectrum we go on to discuss “algebraic” RG equations and their approximate solution using a computer. Another way to compute fixed points is by $\epsilon$-expansion. The $\epsilon$-expansion is presumably not convergent. We use Borel-Padé summation of the first terms of the series upon the hypothesis that the series is at least Borel summable. We conclude with a comparison of our results with those found in the literature.

2 The spectrum at $Z_{UV}$

Since the massive theory is not critical the massless fixed point is more interesting. The eigenfunctions at this fixed point are – as in the case $N = 1$ – Hermite (normal ordered) polynomials. Let $H_m$ be the $m$th Hermite polynomial given by

$$H_m(\varphi) = (-1)^m e^{\frac{\varphi^2}{2}} \frac{d^m}{d\varphi^m} e^{-\frac{\varphi^2}{2}} = \sum_{j=0}^{[\frac{m}{2}]} \frac{(-1)^j m!}{(m - 2j)!2^j j!} \varphi^{m - 2j} = \varphi^m;$$

(7)

and

$$H_m^{(\gamma)}(\varphi) := \gamma^m/2 H_m\left(\frac{\varphi}{\sqrt{\gamma}}\right) = \sum_{j=0}^{[\frac{m}{2}]} \frac{(-\gamma)^j m!}{(m - 2j)!2^j j!} \varphi^{m - 2j}.$$

(8)

Note that $H_m^{(0)}(\varphi) = \varphi^m$. 

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The $N$-component covariance is diagonal in color space. For a multiindex $\mu = (m_1, \ldots, m_N)$ define $H^{(\gamma)}_\mu$ by the $N$-fold tensor product of the $H_{m_i}$

$$H^{(\gamma)}_\mu(\varphi_1, \ldots, \varphi_N) = H^{(\gamma)}_{m_1}(\varphi_1) \ldots H^{(\gamma)}_{m_N}(\varphi_N) = \varphi_1^{m_1} \ldots \varphi_N^{m_N}. \gamma.$$ 

The normal ordered $O(N)$-invariant products $(\varphi^2)^n$ are then given by

$$:(\varphi^2)^n:\gamma := h^{(\gamma)}_n(\varphi) := \sum_{\nu_1 + \ldots + \nu_N = n} \begin{pmatrix} n \\ \nu_1 \ldots \nu_N \end{pmatrix} H^{(\gamma)}_{2\nu_1 \ldots 2\nu_N}(\varphi) = \sum_{\nu_1 + \ldots + \nu_N = n} \begin{pmatrix} n \\ \nu_1 \ldots \nu_N \end{pmatrix} \varphi_1^{2\nu_1} \ldots \varphi_N^{2\nu_N}. \gamma$$

In the following we will use not use the notation $:(\varphi^2)^n:\gamma$. Since normal ordering is the inverse of Gaussian integration in the following sense

$$\int d\mu(1 - \beta^2)(\psi) H^{(\gamma)}_\mu(\psi + \beta \varphi) = H^{(\gamma)(1 - \beta^2)}_\mu(\beta \varphi)$$

and

$$H^{(\beta^2)}(\beta \psi) = \beta |\mu| H^{(\gamma)}_\mu(\psi)$$

($|\mu| = m_1 + \ldots + m_N$) one finds that the $h^{(\gamma)}_n$ are a complete set of eigenfunctions of $L_R(Z_{UV})$ in the space $L_2(\mathbb{R}^N, d\mu_{\gamma})^{O(N)}$ of $O(N)$ symmetric functions. The corresponding eigenvalues are $\lambda_n = 2\beta^{2n} = 2^{1 + n + \frac{2 - d}{2}}$. The critical exponent $\nu$ is calculated by

$$\nu = \frac{\ln(L)}{\ln(\lambda_1)} = \frac{\ln(2)}{d \ln(\lambda_1)} = \frac{1}{2}.$$

We note that the spectrum in a space of $O(N)$ symmetric functions is non-degenerate and the same as for the case $N = 1$.

### 3 Algebraic Computation

We now go on to the task of computing non-trivial fixed points and their spectrum. For calculations using a computer it is convenient to use the so called algebraic RG equation. In this technical section we discuss the derivation of this equation.
One expands the functions under considerations into a series of Hermite polynomials or a power series as

\[ Z(\varphi) = \sum_{n=0}^{\infty} z_n h_n(\varphi) \]

or

\[ Z(\varphi) = \sum_{n=0}^{\infty} z_n (\varphi^2)^n. \]

These two possibilities to expand the functions minimize the necessary work. Consider a general expansion

\[ Z(\varphi) = \sum_{n=0}^{\infty} z_n e_n(\varphi), \]

where the \( e_n \) are a suitable system of functions. Then

\[ (\mathcal{R}Z)(\varphi) = \sum_{n,m=0}^{\infty} z_n z_m \int d\mu_{(1-\beta^2)}(\psi) e_n(\psi + \beta \varphi) e_m(\psi + \beta \varphi) \]

\[ = \sum_{n,m,k=0}^{\infty} z_n z_m C_k^{nm} \int d\mu_{(1-\beta^2)}(\psi) e_k(\psi + \beta \varphi). \]

For \( e_n(\varphi) = (\varphi^2)^n \) the expansion of \( e_n e_m \) is trivial and on the other hand if \( e_n \) is a Hermite polynomial the integral is trivial, since Gaussian integration is inverse to normal ordering.

### 3.1 Expansion in powers of \( \varphi^2 \)

Let us consider first the expansion in powers of \( \varphi^2 \). One has

\[ (\mathcal{R}Z)(\varphi) = \sum_{n,m} z_n z_m \int d\mu_{(1-\beta^2)}(\psi) ((\psi + \beta \varphi)^2)^{n+m} \]

\[ = \sum_{n,m} z_n z_m h_{n+m}^{(-\gamma(1-\beta^2))}(\beta \varphi). \]

The next step is to express (10) as a sum of powers of \( \varphi^2 \). To do the necessary calculations in a convenient way we introduce the generating function \( G^{(\gamma)} \) of the normal ordered products.

\[ G^{(\gamma)}(a, \varphi) := \sum_{n=0}^{\infty} \frac{a^n}{n!} h_n^{(\gamma)}(\varphi) \]
By reordering of the series for $N = 1$ and induction to general $N$ one finds

$$G^{(\gamma)}(a, \varphi) = \frac{1}{(1 + 2\gamma a)^N} \exp\left(\frac{a}{1 + 2\gamma a} \varphi^2\right).$$

Since $h^{(0)}_n(\psi) = (\psi^2)^n$ one expects $G^{(0)}(a, \varphi) = e^{a\varphi^2}$ which is easily verified.

As a first step we want to calculate the scalar products $\langle h^{(\gamma)}_n, h^{(\gamma)}_m \rangle_\gamma$ in $L_2(\mathbb{R}^N, d\mu_\gamma)$. By expanding

$$\langle G(a, \cdot), G(b, \cdot) \rangle_\gamma = \int d\mu_\gamma(\varphi)G(a, \varphi)G(b, \varphi) = (1 - 4\gamma^2 ab)^{-N/2},$$

in powers of $a$ and $b$ and comparing coefficients, we find

$$\langle h^{(\gamma)}_n, h^{(\gamma)}_m \rangle_\gamma = n! \prod_{i=0}^{n-1} (N + 2i)(2\gamma^2)^n \delta_{n,m}. \quad (11)$$

Equation (11) can be generalized to compute the coefficients of powers of $\varphi^2$ in equation (10). One calculates

$$\langle G^{(\gamma')}(a, \cdot), G^{(\gamma)}(b, \cdot) \rangle_\gamma = (1 - 2a(\gamma - \gamma') - 4\gamma^2 ab)^{-N/2}.$$ 

By expansion into Binomial series two times and comparing coefficients we find

$$\langle h^{(\gamma')}_k, h^{(\gamma)}_n \rangle_\gamma = \begin{cases} \prod_{i=0}^{k-1} (N + 2i)(2\gamma^2)^n (\gamma - \gamma')^{k-n} : & n \leq k \\ n! \prod_{i=0}^{n-k} (N + 2i)(2\gamma^2)^n (\gamma - \gamma')^{k-n} : & n > k \end{cases}. \quad (11)$$

If we want to get the coefficients $c_n^{(k)}$ in

$$h^{(\gamma')}_k = \sum_{n=0}^{\infty} c_n^{(k)} h^{(\gamma)}_n \quad (12)$$

as in equation (11) we can use the last result together with the orthogonality of the $h^{(\gamma)}_n$ (11) to find

$$c_n^{(k)} = \begin{cases} \prod_{i=0}^{k-1} (N + 2i)(2\gamma^2)^n (\gamma - \gamma')^{k-n} : & n \leq k \\ n! \prod_{i=0}^{n-k} (N + 2i)(2\gamma^2)^n (\gamma - \gamma')^{k-n} : & n > k \end{cases}. \quad (11)$$

The derivation is restricted to $\gamma, \gamma' > 0$, but by applying a Gaussian integration with covariance $\tilde{\gamma}$, the covariances can be changed to $\gamma - \tilde{\gamma}$ and $\gamma' - \tilde{\gamma}$. 

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We see that equation (12) is correct for all values of $\gamma$ and $\gamma'$. Inserting the coefficients $c_n^{(k)}$ into (10) we get

$$(RZ)(\varphi) = \sum_{k=0}^{\infty} \sum_{n,m=0}^{\infty} z_n z_m c_k^{(n+m)} \beta^{2k} (\varphi^2)^k.$$ 

By replacing $z_n$ by $z_n (\gamma (1 - \beta^2))^{-1}$ that is using the expansion

$$Z(\varphi) = \sum_{n=0}^{\infty} \frac{z_n}{(\gamma (1 - \beta^2))^n} (\varphi^2)^n$$

we can eliminate the Term $(\gamma (1 - \beta^2))^{n+m-k}$ from the final algebraic RG equation.

$$(RZ)(\varphi) = \sum_{k=0}^{\infty} \sum_{n,m}^{\infty} z_n z_m \mathcal{S}_{k}^{nm}(N) \beta^{2k} (\gamma (1 - \beta^2))^{-k} ((\varphi^2)^k).$$

$$(RZ)_{k:=(z_k')}$$

$$\mathcal{S}_{k}^{nm}(N) := \left\{ \prod_{i=k}^{n+m-1} (N + 2i) \left( \begin{array}{c} n+m \\ k \end{array} \right) : \begin{array}{ll} k \leq n+m \\ k > n+m \end{array} \right\}$$

The algebraic RG equation is a mapping of a subset of $\mathbb{R}^\infty$ to $\mathbb{R}^\infty$ instead of an integral equation. It is given by

$$z = (z_0, z_1, z_2, \ldots) \mapsto R(z)$$

with

$$z_k \mapsto R(z)_k = \beta^{2k} \sum_{n,m}^{\infty} \mathcal{S}_{k}^{nm}(N) z_n z_m.$$ (13)

The dimension $d$ is only a numerical parameter in the integral equation. But equation (13) contains $N$ also only in numerical form. We can generalize to noninteger values of $N$ and investigate other values of $N$ which are of interest, e.g. $N = 0$. This is also possible for the integral equation by integrating out the $O(N)$-invariant part. The resulting equation is an integral equation with only one integration containing a Bessel function and $N$ as a parameter.
3.2 Solving the equation numerically

To compute approximate solutions using a computer one simply truncates the equation. We have to solve a finite system of nonlinear equations of the type (13). To avoid changing signs of the coefficients in the expansion of the nontrivial fixed point we do the calculations in the HT picture. Numerical experience suggested as in [7] to use a different normalization, namely

\[ Z(\varphi) = \sum_{n=0}^{\infty} \frac{z_n}{2^{2n} \sqrt{(2n)!}} (\gamma'(1 - \beta'^2))^{-n} (\varphi^2)^n. \]

The system of equations which was solved numerically was the one consisting of \( l_{\text{max}} + 1 \) equations (\( k \in \{0, \ldots, l_{\text{max}}\} \)) given by

\[ z_k \mapsto z_k - \beta'^{2k} \sum_{n,m=0}^{l_{\text{max}}} S_{nm}^m(N) z_n z_m \]

and

\[ S_{nm}^m(N) := \begin{cases} \prod_{i=0}^{n+m-1} (N+2i) \binom{n+m}{k} 4^{k-n-m} \sqrt{2n!(2m)!} : k \leq n+m \\ 0 : k > n+m \end{cases} \]

This was done by the procedure \texttt{C05PBF} of the NAG library, which is a combination of the newton method and the gradient method to find zeros of a set of nonlinear equations. The eigenvalues of the linearized RG equation have then been computed by the procedure \texttt{F02AFF}. We first reproduced the results of [7] to check the correctness of the implementation.

3.2.1 Effects of the truncation

For \( N = 1, d = 3 \) an approximation of this type converges for \( l_{\text{max}} \to \infty \) [6]. We investigated the effects of the truncation for general \( N \) and there is evidence that our approximation converges also. We found it necessary to increase \( l_{\text{max}} \) with \( N \), see table 1. For further discussion of the effects of the truncation see appendix 4.

4 Numerical results

We present here the observations made during the numerical investigation of the model. There is a bifurcation scenario of the hierarchical RG [2, 4, 9], which can be understood by means of the implicit function theorem. At
$d = 4$ a double-well fixed point bifurcates from the UV fixed point, which exists for $d < 4$. Generally at $d_k = \frac{2k}{k-1}, k \geq 2$ a $k$-well fixed points bifurcates from the UV fixed point, which exists for $d < d_k$. This can be understood more physically by observing that at $d_k$ the eigenvalue $\lambda_k$ of the linearized RG equation at $Z_{UV}$ becomes relevant ($\lambda_k > 1$). This behaviour has indeed been reproduced by our numerical results for $k = 2$ and for some cases of $k = 3$.

For plotting the fixed points one has to choose the parameter $\gamma'$. We have used the same convention as [1]:

$$Z(\varphi) = \sum_{l=0}^{l_{\text{max}}} \frac{z_l}{2^l \sqrt{(2l)!}} (\gamma'(1 - \beta^2))^{-l} \varphi^{2l}$$

with $\gamma' = \frac{1}{2}$. The plots show the potentials ($V = -\ln(Z)$) of the fixed points in the UV picture normed by

$$Z(0) = 1.$$ 

$O(N)$-invariance allows us to use only one variable for integer $N$. We define fixed points for noninteger $N$ by the formula above.

Close to the bifurcation points our truncation scheme becomes bad due to the flatness of the non-trivial fixed point. We present here the conjectured behaviour of the system, abstracted from the numerical results. The fixed point which bifurcates at $d = 4$ from the UV fixed point has shown the following behaviour.

- For $N = -2$ the potential is a single well and the critical exponent is 0.5. (Fig. 1, 3) The potential could be fitted to an approximatively quartic function $\varphi^a$ ($a \approx 3.982(2)$) for $\varphi = 0.01 \ldots 0.5$ at $d = 3$. First order $\epsilon$-expansion results in $v(\varphi) = c_1 + c_2 \varphi^4$.

| $l_{\text{max}}$ | $N = 3$ | $N = 10$ | $N = 20$ |
|------------------|---------|---------|---------|
| 10               | 0.77308443210437 | -       | -       |
| 20               | 0.76113986112499 | 0.92746045611772 | -       |
| 30               | 0.76113984902214 | 0.91861145661936 | 0.98467223300539 |
| 40               | 0.76113984902214 | 0.91861145661936 | 0.96069946588199 |
| 50               | -       | 0.91861145661842 | 0.9606806050581 |
| 60               | -       | -       | 0.96068060512759 |
| 70               | -       | -       | 0.96068060512757 |

Table 1: Effects of the truncation: $\nu$ as a function of $N$ and $l_{\text{max}}$. 
For $N > -2$ the potential is a double-well. The minimum becomes deeper with growing $N$ and shifts to bigger arguments. The critical exponent $\nu > \frac{1}{2}$ grows with $N$. $Z_N^{\frac{1}{2}}(\sqrt{N} \cdot \cdot \cdot )$ converges on a function $\zeta_*$ as $N \to \infty$. (Fig. 3, 6)

- $\nu \to \frac{1}{d-2}$ as $N \to \infty$ and $\nu \to \frac{1}{2}$ as $d \nearrow 4$ (Fig. 3).

- $N \geq 1$ (Fig. 3, 8)
  - The fixed point exists for $2 < d < 4$.
  - $\nu$ diverges as $d \searrow 2$, because the first fixed eigenvalue at the fixed point converges on 1: $\lambda_1 \to 1$ in $\nu = \frac{\log 2}{\log(\lambda_1)d}$.

- $0 < N < 1$ (Fig. 3, 8)
  - The fixed point exists for $d' < d < 4$ with $0 < d' < 2$.
  - $d'$ grows (non linearly) with $N$ from 0 to 2.
  - $\nu$ diverges as $d \searrow d'$, because of $\lambda_1 \to 1$ in $\nu = \frac{\log 2}{\log(\lambda_1)d'}$.

- $N = 0$. (Fig. 1, 3, 3)
  - The fixed point exists for $0 < d < 4$.
  - $\nu$ diverges as $d \searrow 0$, because of $\frac{1}{d} \to \infty$ in $\nu = \frac{\log 2}{\log(\lambda_1)d}$. The first eigenvalue converges $\lambda_1 \to 2$.

- $-2 < N < 0$. (Fig. 3, 7)
  - The fixed point exists for $0 < d < 4$.
  - $\lambda_1$ diverges as $d \searrow 0$. There is a $0 < d' < 2$, with $\lambda_1 > \lambda_0 = 2$ for $0 < d < d'$. $d'$ decreases (non linearly) with growing $N$ from 2 to 0.
  - The limit of $\nu$ as $d \searrow 0$ exists, since the divergencies of $\lambda_1$ and $\frac{1}{d}$ cancel.

- At $d = 2$ $\nu$ diverges as $N \nearrow 1$, because of $\lambda_1 \to 1$. (Fig. 1, 3, 7, 8).

With a view on the theorem of implicit functions $\lambda_1 \to 1$ indicates possible points of further bifurcations or the end of existence of solutions. We did not find other solutions below the corresponding values of $d$ and is was also not possible to follow the fixed point below these values of $d$. Based on these observations the conclusion is, that these are the points where the
existence of the nontrivial fixed points ends. A similar conclusion seems to apply for the case \( d = 2, N \not> 1 \). The fixed point which exists below \( N = 1 \) does not exist for values of \( N \geq 1 \). In this case other fixed points are known to exist at \( N = 1, d = 2 \)

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
N & 0.01 & 0.1 & 0.25 & 0.5 & 0.75 & 0.9 \\
\hline
\text{value} & 0.66(2) & 1.14(2) & 1.46(2) & 1.75(2) & 1.91(2) & 1.97(2) \\
\hline
\end{array}
\]

Table 2: Values \( d' \) for which \( \lambda_1(d') = 1 \).

Furthermore we investigated 3-well fixed points and found the following behaviour.

- There exists a 3-well fixed point for \( N = 1 \) and \( 2 < d < 3 \). (Fig. 8 and the figures in [7])
- There exists a 3-well fixed point for \( N > 1 \) and \( d'(N) < d < 3 \), \( 2 \leq d'(N) < 3 \). It is unknown whether \( d' > 2 \). (Fig. 8)
- There exists a 3-well fixed point for \( N < 1 \) and \( d' < d < 3 \), \( 0 < d' < 2 \). (Fig. 2, 7)
- The second eigenvalue \( \lambda_2 \) of the linearized RG equation at the 3-well converges on 1 as \( d \searrow d' \). (Fig. 7, 8)

5 \( \epsilon \)-Expansion

Following closely the lines of [8], one can generalize the \( \epsilon \)-expansion to the model with \( N \) components (\( \epsilon = 4 - d \)). We work in the UV picture because of the simple structure of the UV fixed point,

\[
Z_{UV} = 1 = \sum_{k=0}^{\infty} z_n h_n^{(\gamma)}.
\]

This results in \( z_n = \delta_{n,0} \). Furthermore we expand in terms of hermite polynomials since they are the eigenfunctions at \( Z_{UV} \), see section 2. Again the first step is to calculate the algebraic RG transformation for this case. If we know the expansion of a product of two hermite polynomials we only have to integrate a hermite polynomial, which is trivial. We have to calculate the coefficients \( C_{nm} \) in

\[
h_n^{(\gamma)} h_m^{(\gamma)} = \sum_{k=0}^{\infty} C_{nm}^{kn} h_k^{(\gamma)}.
\]
This is again done by the method of generating functions. Using the orthog-
onality equation (11) we get
\[ \langle h_n(\gamma) h_m(\gamma), h_k(\gamma) \rangle_\gamma = k! \prod_{i=0}^{k-1} (N + 2i)(2\gamma^2)^k C_{nm}^k. \]
This leaves us with the calculation of the scalar product. We calculate
\[ \langle G(a, \cdot) G(b, \cdot), G(c, \cdot) \rangle_\gamma = (1 - 4\gamma^2(ab + bc + ca) - 16\gamma^3 abc)^{-N/2} \]
and do an a bit lengthy expansion of the right hand side of the equation to compare coefficients of \(a, b, c\) to find
\[ \langle h_n(\gamma) h_m(\gamma), h_k(\gamma) \rangle_\gamma = n!m!k!(4\gamma)^{n+m+k} \sum_{n,m,k \leq q \atop 2q \leq n+m+k} \frac{\prod_{i=0}^{q-1} (N + 2i)2^{-3q}}{(q-n)!(q-m)!(q-k)!(n+m+k-2q)!}. \]
The final result is
\[ C_{nm}^k(N) = n!m!4^{n+m+k} \gamma^{n+m-k} \sum_{n,m,k \leq q \atop 2q \leq n+m+k} \frac{\prod_{i=0}^{q-1} (N + 2i)2^{-3q-k}}{(q-n)!(q-m)!(q-k)!(n+m+k-2q)!}. \]
It was verified using Maple that for \(N = 1\) these coefficients are the same as those found by [7]. The integration of \(h_k(\gamma)\) in the RG equation is now trivial and only amounts to a multiplication with \(\beta^{2k}\).

To make the algebraic RG equation again independent of \(\gamma\) we use the expansion
\[ Z = \sum_{n=0}^{\infty} z_n \gamma^{-n} h_n(\gamma). \]
Inserting this expansion into the RG equation results in
\[ \mathcal{R}Z = \sum_{k=0}^{\infty} \sum_{m,n=0}^{\infty} \beta^{2k} C_{nm}^k(N) z_n z_m \gamma^{-n-m+k} \gamma^{-k} h_k(\gamma). \]
We define the \(\gamma\)-independent structure coefficients \(C_{nm}^m\) by
\[ C_{nm}^k := C_{nm}^m \gamma^{-n-m+k}. \]
and get the following algebraic RG equation \( z = (z_0, z_1, \ldots) \mapsto \mathcal{R}z \) with
\[
  z_k \mapsto (\mathcal{R}z)_k = \beta^{2k} \sum_{n,m=0}^{\infty} C_{k}^{nm} z_n z_m .
\]

\( \mathbf{(15)} \)

5.1 \( \epsilon \)-expansion of the fixed points

The \( \epsilon \)-expansion is an expansion of the coefficients \( z_l \) of a fixed point into an asymptotic series of the form
\[
  z_l = \sum_{k=0}^{\infty} z_l^{(k)} \epsilon^k .
\]

We write the fixed point equation in the form
\[
  \beta^{-2l} z_l = \sum_{n,m} C_{km}^{lm} z_n z_m .
\]

Expanding the left and the right hand side of the equation in powers of \( \epsilon \) yields recursive equations for the \( z_l^{(k)} \). The calculation of these equations does not differ in an essential way from the one in [7], and we will leave it out here. One only has to replace the structure coefficients by the \( C_{km}^{nm} \) we just calculated.

To first order we find for the fixed point
\[
  Z(\varphi) = 1 - \frac{\ln 2}{16(N + 8)} \gamma^{-2} \epsilon h_2^{(\gamma)}(\varphi) .
\]

For \( h_2^{(\gamma)} \) we have
\[
  h_2^{(\gamma)}(\varphi) = \varphi^4 - 2\gamma(N + 2)\varphi^2 + \gamma^2 N(N + 2) .
\]

We note that for \( N = -2 \) and \( \epsilon > 0 \) the fixed point has the form of a quartic single-well and for \( N > -2 \) it is a double-well.

Figure 9 presents some fixed points calculated to order 1, 2 and 3 by \( \epsilon \)-expansion compared to numerically calculated fixed points.

5.2 \( \epsilon \)-expansion of eigenvalues

Given the fixed point we can calculate the linearisation of the RG equation at this fixed point. The linearisation is given by the matrix \( \mathbf{A}(z) \)
\[
  \mathbf{A}(z)_{ln} = 2\beta^{2l} \sum_{m=0}^{\infty} z_m C_{l}^{mn} .
\]
The eigenvalue equation is then

\[ A(z)v = \lambda v. \]

We get the \( \epsilon \)-expansion of \( A(z) \) by inserting the expansions of \( \beta^{2l} \) (with coefficients \( b^{(k)}_{-l} \)) and \( z_m \)

\[ A(z)_m = 2^{1-\frac{d}{2}} \sum_{k=0}^{\infty} \sum_{\nu=0}^{k} \sum_{m} C^{mn}_l(N) z^{(\nu)}_m b^{(k-\nu)}_{-l} \epsilon^k. \]  

(16)

Expanding also the eigenvectors

\[ v_{\mu} = \sum_{k=0}^{\infty} v^{(k)}_{\mu} \epsilon^k \]

and the corresponding eigenvalues

\[ \lambda_{\mu} = \sum_{k=0}^{\infty} \lambda^{(k)}_{\mu} \epsilon^k \]

we can use a generalisation of nondegenerate perturbation theory. For details we refer to [3, 4]. Summarizing we get the following equations (\( l_\ast = 2 \))

\[ A(z)^{(k)}_{\mu n} = 2^{1-\frac{d}{2}} \sum_{\nu=0}^{k} \sum_{m} C^{mn}_l(N) z^{(\nu)}_m b^{(k-\nu)}_{-l} \]

(17)

\[ v^{(0)}_{\mu} = (\delta_{\mu,\mu}) \quad \text{and} \quad \lambda^{(0)}_{\mu} = 2^{1-\frac{d}{2}} \epsilon, \quad \mu \in \mathbb{N}_0 \]

(18)

\[ 0 = (v^{(0)}_{\mu}, v^{(k)}_{\mu}) = \sum_{n=0}^{\infty} \delta_{\mu,n} v^{(k)}_{\mu,n} = v^{(k)}_{\mu,\mu} \]

(19)

\[ \lambda^{(k)}_{\mu} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} v^{(0)}_{\mu,\mu} v^{(k)}_{\mu,n} A^{(k-n)}_{\mu m} v^{(n)}_{\mu,j} = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} A^{(k-n)}_{\mu j} v^{(n)}_{\mu,j}, \]

(20)

\[ v^{(k)}_{\mu,\rho} = \frac{1}{2(2^{-\frac{d}{2}} - 2^{-\frac{d}{2}})} \sum_{n=0}^{k-1} \left( \sum_{m=0}^{\infty} \left( A^{(k-n)}_{\mu m} v^{(n)}_{\mu,m} \right) - \lambda^{(k-n)}_{\mu} v^{(n)}_{\mu,\rho} \right) \]

(21)

With these equations it is possible to calculate the \( \epsilon \)-expansion of the eigenvalues and eigenvectors recursively order by order. To each order the sums are finite.
The critical exponent $\nu$ belongs to $\lambda_1$. The first order coefficient is

$$\lambda_1^{(1)} = \sqrt{2} \ln 2 \left( \frac{1}{4} - \frac{N + 2}{N + 8} \right).$$

Summing up to first order

$$\lambda_1 = \sqrt{2} \left( 1 + \frac{\ln 2}{4} \left( \frac{1}{2} - \frac{N + 2}{N + 8} \right) \epsilon \right),$$

and expanding the logarithm in $\nu^{-1} = \frac{(d_k - \epsilon) \ln \lambda_1}{\ln 2}$ we get

$$\nu^{-1} = 2 \left( 1 - \frac{1}{2} \frac{N + 2}{N + 8} \epsilon \right) + O(\epsilon^2),$$

so that

$$\nu = \frac{1}{2} + \frac{1}{4} \frac{N + 2}{N + 8} \epsilon + O(\epsilon^2).$$

The $\epsilon$-expansion was calculated up to 6th order using Maple V Release 3. The results of [7] for the case $N = 1$ were reproduced. In appendix C some analytical results of the $\epsilon$-expansion are given.

| $d$  | $k = 1$  | $k = 2$  | $k = 3$  | $k = 4$  | $k = 5$  | numerical | BP |
|------|----------|----------|----------|----------|----------|-----------|----|
| 3.9  | 0.5065   | 0.5066   | 0.5066   | 0.5066   | 0.5066   | 0.50659   | 0.50659 |
| 3.8  | 0.5135   | 0.5140   | 0.5137   | 0.5139   | 0.5137   | 0.51380   | 0.51380 |
| 3.7  | 0.5211   | 0.5222   | 0.5211   | 0.5221   | 0.5208   | 0.52157   | 0.52157 |
| 3.6  | 0.5293   | 0.5313   | 0.5286   | 0.5319   | 0.5261   | 0.52990   | 0.52988 |
| 3.5  | 0.5382   | 0.5412   | 0.5359   | 0.5442   | 0.5266   | 0.53877   | 0.53873 |
| 3.4  | 0.5477   | 0.5521   | 0.5430   | 0.5602   | 0.5176   | 0.54820   | 0.54811 |
| 3.3  | 0.5580   | 0.5641   | 0.5496   | 0.5821   | 0.4945   | 0.55821   | 0.55761 |
| 3.2  | 0.5691   | 0.5771   | 0.5555   | 0.6128   | 0.4557   | 0.56884   | 0.56841 |
| 3.1  | 0.5810   | 0.5913   | 0.5607   | 0.6570   | 0.4045   | 0.58012   | 0.57946 |
| 3.0  | 0.5940   | 0.6068   | 0.5651   | 0.7236   | 0.3488   | 0.59209   | 0.59040 |

Table 3: $\nu(N = 0)$ calculated by different methods explained in the text.

We calculated $\nu$ first in the “naive” way, that is by summing the $\epsilon$-expansion of $\lambda$ and calculating $\nu = \frac{\ln 2}{d \ln \lambda_1}$. This is called “naive”, because the $\epsilon$-expansion is presumably not convergent. Upon the hypothesis that it is at least Borel-summable, we calculated the Borel sum of the expansion

$$B(\epsilon) = \sum_{k=0}^{\text{max. order}} \frac{\lambda_1^{(k)}}{k!} \epsilon^k.$$
Then we calculated the diagonal Padé approximation $Q(\epsilon)$ of this sum and did a numerical Borel transform of $Q$

$$\lambda_1(\epsilon) = \int_0^\infty Q(t\epsilon) \exp(-t) dt .$$

Table 3 shows values of the critical exponent $\nu$ calculated by “naive” summation of the $\epsilon$-expansion up to order $k$, by numerical computation of the eigenvalues and by Borel-Padé summation of the $\epsilon$-expansion up to 6th order.

Table 3 shows values of $\nu$ calculated by Borel-Padé approximation of the $\epsilon$-expansion up to 6. order. For small $\epsilon$ there is a very good agreement between the values calculated by solving the truncated system and those calculated by Borel-Padé summation of the $\epsilon$-expansion. If we go to $d = 3$ only 2 to 3 digits agree. Unfortunately we do not have error estimates for the Borel-Padé summation.

6 Comparison with other results and Conclusion

\[
\begin{array}{ccccccc}
N = -1 & N = 0 & N = 1 & N = 2 & N = 3 & N = 4 \\
0.541536 & 0.592086 & 0.649570 & 0.708225 & 0.761140 & 0.804364 \\
\hline
3 & - & - & 0.64957 & - & - & - \\
Pol & - & - & 0.6496 & 0.7082 & 0.7611 & 0.8043 \\
WH & - & 0.6066 & 0.6895 & 0.7678 & 0.8259 & 0.8648 \\
\hline
\hline
\text{p} & - & - & 0.6896 & 0.767 & 0.826 & 0.865 \\
\text{HT} & - & - & 0.6301(18) & 0.6734(28) & 0.7131(40) & 0.7361(68) \\
\text{pert.} & - & 0.5882(11) & 0.6304(13) & 0.6703(13) & 0.7073(30) & 0.7410(60) \\
\end{array}
\]

Table 4: Comparison of values of $\nu$ for $d = 3$.

Table 3 of [3] contains values of the critical exponent $\nu$ calculated from the Wegner-Houghton and the Polchinski RG equations. It is remarkable that our values coincide to three significant digits with those found by the Polchinski RG equation. See table 3. The results of the Wegner-Houghton equation differ from our results, but are compatible with those in [3]. Compared to results for the “full” model the hierarchical model overestimates the critical exponent. We refer to the results of [10] found by convergent high temperature expansion and of [3] calculated by perturbation theory. For $N \to \infty$ and $N \to -2$ the hierarchical model has the same exponents as the “full” model.
We have presented the calculation of $O(N)$ symmetric fixed points of the hierarchical RG transformation by $\epsilon$-expansion and by numerical methods based on the expansion of functions. The results of the $\epsilon$-expansion are compatible with those found by numerical computation when $\epsilon$ is not too large. Furthermore the results are compatible with results found by investigating the Polchinski equation.

Considering this, $O(N)$ symmetric solutions of the hierarchical approximation seem to be quite well understood. Mathematically there is still room for improvement since rigorous proofs of the existence of the fixed points calculated in this paper are to my knowledge still lacking for most values of $N$ and $d$ with the exception $N = 1, d = 3, L^d = 2$. Cassandro and Mitter have found other fixed points for $d < 2$ and $N \geq 1$ [2]. It should be possible to find these fixed points by algebraic computation. In this work no fixed points for these values of $N$ and $d$ have been calculated. It would be interesting to know if at least for $2 < d \leq 4$ the bifurcation scenario leads to all fixed points of the RG equation.

Other - more physical - routes from this point are systematic corrections to the hierarchical approximation and the study of fixed points with less symmetry, e.g. [8].

7 Acknowledgements

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A Effects of the truncation

Table 5 shows the first 10 coefficients of the fixed point $z_k^*$ for $N = 3, d = 3$. If we go to $l_{\text{max}} = 40$ the first 10 coefficients are the same as for $l_{\text{max}} = 30$. For $l_{\text{max}} = 40$ we found $z_{30}^* = 3.2209843597592 \cdot 10^{-20}$, $z_{30}^* = 1.0554235840436 \cdot 10^{-13}$ and $z_{20}^* = 5.0374367851155 \cdot 10^{-8}$. So we can use $l_{\text{max}} = 30$ for $N = 3$.

$N = 20$ was the greatest value of $N$ used during numerical calculations. See table 6 for a selection of coefficients of the fixed point. One can observe the increasing difficulty of finding the fixed point because the first 10 coefficients have the same order of magnitude and they are quite small. One has to go to $l_{\text{max}} = 60$ before the first 10 coefficients are constant under further increasing of $l_{\text{max}}$. Note further that the coefficients of the fixed point do not decrease as fast as for lower values of $N$. 

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| $r$ | $t_{\text{max}} = 10$ | $t_{\text{max}} = 20$ | $t_{\text{max}} = 30$ |
|-----|---------------------|---------------------|---------------------|
| 5  | 3.87160189377576 $10^{-4}$ | 3.9015294442769 $10^{-4}$ | 3.9015294869101 $10^{-4}$ |
| 6  | 2.7472799632497 $10^{-5}$ | 2.7488216314596 $10^{-5}$ | 2.7488216310166 $10^{-5}$ |
| 7  | 2.0464448832152 $10^{-6}$ | 2.047974444152 $10^{-6}$ | 2.0479744607890 $10^{-6}$ |
| 8  | 1.6922488066631 $10^{-7}$ | 1.693502475552 $10^{-7}$ | 1.6935024714472 $10^{-7}$ |
| 9  | 8.6657748574293 $10^{-8}$ | 8.925711448321 $10^{-8}$ | 8.9257114547286 $10^{-8}$ |
| 10 | 5.2731434885504 $10^{-9}$ | 5.27489679037 $10^{-9}$ | 5.2748967984464 $10^{-9}$ |
| 11 | 2.856592240763 $10^{-10}$ | 2.899930116763 $10^{-10}$ | 2.8999303106272 $10^{-10}$ |
| 12 | 1.477491557618 $10^{-11}$ | 1.507274120603 $10^{-11}$ | 1.5072741508427 $10^{-11}$ |
| 13 | 7.6520861450373 $10^{-12}$ | 7.686921974505 $10^{-12}$ | 7.686921862295 $10^{-12}$ |
| 14 | 3.1541868355194 $10^{-13}$ | 3.415786267627 $10^{-13}$ | 3.415786889480 $10^{-13}$ |
| 15 | 1.1396604499937 $10^{-14}$ | 1.507112742866 $10^{-14}$ | 1.507114902066 $10^{-14}$ |
|    | ...                  | ...                  | ...                  |
| 26 | -                    | 5.02756698651 $10^{-15}$ | 5.0374367835756 $10^{-15}$ |
|    | ...                  | ...                  | ...                  |
| 29 | -                    | 1.0553957852574 $10^{-17}$ | ...                  |

Table 5: Effects of the truncation for $N = 3$

| $r$ | $t_{\text{max}} = 40$ | $t_{\text{max}} = 50$ | $t_{\text{max}} = 60$ |
|-----|---------------------|---------------------|---------------------|
| 5  | 6.7135969167269 $10^{-4}$ | 6.713781100317 $10^{-4}$ | 6.713781326764 $10^{-4}$ |
| 6  | 5.672592416626 $10^{-5}$ | 5.672409186071 $10^{-5}$ | 5.6724091864667 $10^{-5}$ |
| 7  | 5.207412064560 $10^{-6}$ | 5.220951652976 $10^{-6}$ | 5.220951732299 $10^{-6}$ |
| 8  | 5.032432256809 $10^{-7}$ | 5.032539440537 $10^{-7}$ | 5.032539505618 $10^{-7}$ |
| 9  | 4.778154596898 $10^{-8}$ | 4.77935851117 $10^{-8}$ | 4.779358583032 $10^{-8}$ |
| 10 | 4.366461453968 $10^{-9}$ | 4.367808906417 $10^{-9}$ | 4.3678089559662 $10^{-9}$ |
| 11 | 4.014568939824 $10^{-10}$ | 4.016340398981 $10^{-10}$ | 4.016340478311 $10^{-10}$ |
| 12 | 3.581864226392 $10^{-11}$ | 3.581992679245 $10^{-11}$ | 3.581992698887 $10^{-11}$ |
| 13 | 3.072937145831 $10^{-12}$ | 3.073132889402 $10^{-12}$ | 3.073132901930 $10^{-12}$ |
| 14 | 2.540731653704 $10^{-13}$ | 2.540728246010 $10^{-13}$ | 2.540728251517 $10^{-13}$ |
| 15 | 2.072379082894 $10^{-14}$ | 2.072385126456 $10^{-14}$ | 2.072385126791 $10^{-14}$ |
|    | ...                  | ...                  | ...                  |
| 26 | -                    | 1.3348793621549 $10^{-15}$ | 1.3348793645545 $10^{-15}$ |
| 27 | 5.780032141111 $10^{-16}$ | 5.798999420249 $10^{-16}$ | 5.79899921995982 $10^{-16}$ |
| 28 | 3.91518449144 $10^{-17}$ | 3.929995707748 $10^{-17}$ | 3.9299957099882 $10^{-17}$ |
| 29 | 1.515856729964 $10^{-18}$ | 1.642687837244 $10^{-18}$ | 1.64268785956271 $10^{-18}$ |
| 30 | 7.383959815771 $10^{-19}$ | 7.2189285198076 $10^{-19}$ | 7.2189285200557 $10^{-19}$ |
|    | ...                  | ...                  | ...                  |
| 50 | -                    | 3.991120713494 $10^{-11}$ | 3.994960778607 $10^{-11}$ |
|    | ...                  | ...                  | ...                  |
| 60 | -                    | -                    | 6.021708064527 $10^{-18}$ |

Table 6: Effects of the truncation for $N = 20$
By increasing $l_{\text{max}}$ to 70 the first 10 coefficients are identical to those for $l_{\text{max}} = 60$. We found for $l_{\text{max}} = 70$: $z^*_{60} = 6.0220916546711 \cdot 10^{-18}$, $z^*_{50} = 3.3943995424453 \cdot 10^{-14}$ and $z^*_{40} = 7.2189530204572 \cdot 10^{-11}$.

**B Numerical results: the exponent $\nu$**

| $d$ | $N = -1$ | $N = 0$ | $d$ | $N = -1$ | $N = 0$ |
|-----|----------|----------|-----|----------|----------|
| 0   |          |          | 2   | 0.572496 | 0.766551 |
| 0.1 |          |          | 2.1 | 0.570806 | 0.742861 |
| 0.2 |          |          | 2.2 | 0.568709 | 0.720999 |
| 0.3 |          |          | 2.3 | 0.566240 | 0.700752 |
| 0.4 | 0.528092 | 2.557642 | 2.4 | 0.563437 | 0.681937 |
| 0.5 | 0.534783 | 2.094339 | 2.5 | 0.560333 | 0.664404 |
| 0.6 | 0.541065 | 1.794838 | 2.6 | 0.556966 | 0.648025 |
| 0.7 | 0.546882 | 1.585792 | 2.7 | 0.553369 | 0.632691 |
| 0.8 | 0.552197 | 1.431311 | 2.8 | 0.549577 | 0.618308 |
| 0.9 | 0.556982 | 1.312021 | 2.9 | 0.546222 | 0.604796 |
| 1   | 0.561217 | 1.216665 | 3   | 0.541536 | 0.592086 |
| 1.1 | 0.564886 | 1.138306 | 3.1 | 0.537351 | 0.580118 |
| 1.2 | 0.567979 | 1.066139 | 3.2 | 0.533094 | 0.568842 |
| 1.3 | 0.570489 | 1.016113 | 3.3 | 0.528794 | 0.558214 |
| 1.4 | 0.572419 | 0.967166 | 3.4 | 0.524479 | 0.548199 |
| 1.5 | 0.573773 | 0.924112 | 3.5 | 0.520177 | 0.538766 |
| 1.6 | 0.574364 | 0.885845 | 3.6 | 0.515917 | 0.529896 |
| 1.7 | 0.574809 | 0.851529 | 3.7 | 0.511728 | 0.521574 |
| 1.8 | 0.574528 | 0.820526 | 3.8 | 0.507645 | 0.513801 |
| 1.9 | 0.573748 | 0.792332 | 3.9 |          | 0.506593 |

Table 7: Numerically calculated values of $\nu$

Table 7 and table 8 show some numerically calculated results of the critical exponent $\nu$ for different values of $N$ and $d$.

**C Some results of the $\epsilon$ expansion**

In this appendix we present some of the results of the $\epsilon$ expansion calculated using Maple.

**C.1 $\epsilon$ expansion of the coefficients $z_m$ up to 2. order**

$T = \sqrt{2}$ und $R = \ln 2$

$z_{(0)}^0 = 1$
Table 8: Numerically calculated values of $\nu$

| $d$ | $N = 1$  | $N = 2$  | $N = 3$  | $N = 5$  | $N = 10$ |
|-----|----------|----------|----------|----------|----------|
| 2.1 | 2.088862 | 9.193920 | -        | -        | -        |
| 2.2 | 1.362344 | 4.045225 | -        | -        | -        |
| 2.3 | 1.099150 | 2.288147 | 2.889098 | 3.126743 | -        |
| 2.4 | 0.957038 | 1.559028 | 2.030739 | 2.287148 | -        |
| 2.5 | 0.865337 | 1.214032 | 1.535818 | 1.782761 | 1.908031 |
| 2.6 | 0.799849 | 1.022244 | 1.236507 | 1.449465 | 1.574345 |
| 2.7 | 0.749927 | 0.901226 | 1.046530 | 1.217683 | 1.337054 |
| 2.8 | 0.710110 | 0.817609 | 0.918937 | 1.051532 | 1.160436 |
| 2.9 | 0.677294 | 0.755940 | 0.828437 | 0.929516 | 1.024914 |
| 3   | 0.649570 | 0.708225 | 0.761140 | 0.837755 | 0.918611 |
| 3.1 | 0.625700 | 0.669949 | 0.709101 | 0.767069 | 0.833820 |
| 3.2 | 0.604841 | 0.638382 | 0.667563 | 0.711338 | 0.765245 |
| 3.3 | 0.586400 | 0.611780 | 0.635455 | 0.666445 | 0.691111 |
| 3.4 | 0.569946 | 0.588980 | 0.605103 | 0.629584 | 0.662617 |
| 3.5 | 0.555163 | 0.561766 | 0.586926 | 0.598813 | 0.623711 |
| 3.6 | 0.541815 | 0.551801 | 0.560104 | 0.572765 | -        |
| 3.7 | 0.529725 | 0.536443 | -        | 0.546388 | -        |
| 3.8 | 0.518769 | 0.522810 | 0.526130 | 0.531208 | -        |
| 3.9 | 0.508867 | 0.510099 | 0.512199 | 0.514498 | -        |

\[
\begin{align*}
\bar{z}_0^{(1)} &= 0 \\
\bar{z}_0^{(2)} &= -NR^2(N+2) \\
\bar{z}_1^{(0)} &= 0 \\
\bar{z}_1^{(1)} &= 0 \\
\bar{z}_1^{(2)} &= \frac{R^2(N+2)}{8(T-2)(N+8)^2} \\
\bar{z}_2^{(0)} &= 0 \\
\bar{z}_2^{(1)} &= \frac{-R}{16(N+8)} \\
\bar{z}_2^{(2)} &= R \left( -8(N+8)^2 + 4(-N^2 + 44N + 200)R + 6(N+8)^2T + (-92N - 424 + 3N^2)TR \right) / \left( (N+8)^3(-384T + 512) \right) \\
\bar{z}_3^{(0)} &= 0 \\
\bar{z}_3^{(1)} &= 0 \\
\bar{z}_3^{(2)} &= \frac{R^2}{32(N+8)^2(T-1)} \\
\bar{z}_4^{(0)} &= 0
\end{align*}
\]
\( z_4^{(1)} = 0 \)
\( z_4^{(2)} = \frac{R^2}{512(N + 8)^2} \)

C.2 \( \epsilon \) expansion of the exponent \( \nu \)

\[
\begin{align*}
\nu^{(0)} &= \frac{1}{2} \\
\nu^{(1)} &= \frac{1}{4} \frac{N + 2}{N + 8} \\
\nu^{(2)} &= -\frac{(N + 2)}{16(N + 8)^3(4 - 3T)} \cdot \\
& \quad \left( -8(N + 8)(N + 2) + 12(7N + 20)R \\
& \quad + 6(N + 8)(N + 2)T - 7(7N + 20)RT \right) \\
\nu^{(3)} &= -\frac{(N + 2)}{32(N + 8)^5(4 - 3T)^2} \cdot \\
& \quad \left( -68(N + 8)^2(N + 2)^2 + 45(N + 8)(7N + 20)(5N + 16)R \\
& \quad + (681N^3 - 440N^2 - 2648N + 42016)R^2 \\
& \quad + 48(N + 8)^2(N + 2)^2T \\
& \quad - 32(N + 8)(7N + 20)(5N + 16)RT \\
& \quad - 6(79N^3 - 60N^2 - 432N + 4544)R^2T \right)
\end{align*}
\]

The zeroth and first order of the \( \epsilon \) expansion are the same for the hierarchical and the full model. For the full model we have

\[
\nu_f^{(2)} = \frac{N + 2}{8(N + 8)^3}(N^2 + 23N + 60)
\]

and

\[
\nu_f^{(3)} = \frac{N + 2}{32(N + 8)^5}(2N^4 + 89N^3 + 1412N^2 + 5904N + \\
8640 - 192(5N + 22)(N + 8)t)
\]

with \( t \approx 0.60103 \) [11]. Note that the prefactors \( \frac{N + 2}{(N + 8)^3} \) and \( \frac{N + 2}{(N + 8)^5} \) are the same in both expansions and that the limit \( N \to \infty \) is the same for both models and equals the coefficients of the expansion of \( \frac{1}{a - 2} = \frac{1}{2} \frac{1}{1 - \epsilon/2} \).

\( \nu^{(1)} \to \frac{1}{4}, \nu^{(2)} \to \frac{1}{8} \) and \( \nu^{(3)} \to \frac{1}{16} \)
### Table 9: \( \nu \): Diagonal Borel-Padé summation of the \( \epsilon \)-expansion up to 6. order.

| \( \lambda \) | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| -2 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.49996 | 0.49994 |
| -1 | 0.50764 | 0.51172 | 0.51586 | 0.51995 | 0.52371 | 0.52744 | 0.53299 | 0.53266 | - |
| 0 | 0.51380 | 0.52157 | 0.52988 | 0.53873 | 0.54811 | 0.55761 | 0.56841 | 0.57946 | 0.59040 |
| 1 | 0.51877 | 0.52973 | 0.54183 | 0.55525 | 0.56972 | 0.58469 | 0.60113 | 0.62597 | - |
| 2 | 0.52281 | 0.53644 | 0.55181 | 0.56920 | 0.58817 | 0.61332 | 0.63773 | 0.6734 | - |
| 3 | 0.52613 | 0.54199 | 0.56010 | 0.58093 | 0.60520 | 0.63391 | - | - | - |
| 4 | 0.52889 | 0.54660 | 0.56699 | 0.59064 | 0.61836 | 0.65126 | 0.69084 | 0.73922 | 0.79939 |
| 5 | 0.53121 | 0.55046 | 0.57274 | 0.59872 | 0.62928 | 0.66562 | 0.70931 | 0.76246 | 0.82801 |
| 6 | 0.53318 | 0.55372 | 0.57757 | 0.60546 | 0.63834 | 0.67747 | 0.72449 | 0.78161 | 0.85185 |
| 7 | 0.53487 | 0.55650 | 0.58167 | 0.61113 | 0.64589 | 0.68727 | 0.73696 | 0.79724 | 0.87121 |
| 8 | 0.53633 | 0.55890 | 0.58517 | 0.61593 | 0.65224 | 0.69543 | 0.74724 | 0.81000 | 0.88690 |
| 9 | 0.53761 | 0.56098 | 0.58818 | 0.62004 | 0.65761 | 0.70227 | 0.75578 | 0.82050 | 0.89964 |
| 10 | 0.53873 | 0.56279 | 0.59079 | 0.62337 | 0.66220 | 0.70806 | 0.76293 | 0.82919 | 0.91008 |
| 11 | 0.53973 | 0.56439 | 0.59308 | 0.62664 | 0.66615 | 0.71300 | 0.76897 | 0.83466 | 0.91908 |
| 12 | 0.54061 | 0.56581 | 0.59510 | 0.62933 | 0.66958 | 0.71725 | 0.77412 | 0.84258 | 0.92585 |
| 13 | 0.54141 | 0.56707 | 0.59688 | 0.63169 | 0.67258 | 0.72093 | 0.77664 | 0.84778 | 0.93187 |
| 14 | 0.54212 | 0.56821 | 0.59847 | 0.63378 | 0.67522 | 0.72415 | 0.78237 | 0.85225 | 0.93697 |
| 15 | 0.54277 | 0.56923 | 0.59990 | 0.63655 | 0.67755 | 0.72698 | 0.78752 | 0.85611 | 0.94133 |
| 16 | 0.54336 | 0.57015 | 0.60119 | 0.63732 | 0.67963 | 0.72949 | 0.78806 | 0.85948 | 0.94510 |
| 17 | 0.54390 | 0.57099 | 0.60235 | 0.63883 | 0.68150 | 0.73172 | 0.79126 | 0.86243 | 0.94938 |
| 18 | 0.54440 | 0.57176 | 0.60341 | 0.64019 | 0.68318 | 0.73372 | 0.79357 | 0.86504 | 0.95125 |
| 19 | 0.54485 | 0.57246 | 0.60438 | 0.64143 | 0.68409 | 0.73551 | 0.79564 | 0.86737 | 0.95379 |
| 20 | 0.54527 | 0.57311 | 0.60526 | 0.64256 | 0.68607 | 0.73714 | 0.79751 | 0.86945 | 0.95605 |

\( d = 2 \) 0.55556 | 0.58823 | 0.62500 | 0.66667 | 0.71429 | 0.76923 | 0.81333 | 0.90909 | 1.00000

D Figures
Figure 1: Potentials of the fixed points ($N = 0, d = 0.3$ to $3.9$, steps of $\frac{1}{10}$). Deeper potentials belong to lower values of $d$.

Figure 2: Potentials of the 3-well fixed points ($N = 0, d = 2.0$ to $2.9$). The local extremum grows with decreasing $d$. 
Figure 3: The critical exponent $\nu$ for $0 < d < 4$ and (from the lower left to the upper right) $N = -1, -0.5, 0, 1, 1.24, 1.5, 2, 3, 4, 5, 10, 15$. The plotted line $\frac{1}{d-2}$ is the asymptotic value of $\nu$ for $N \to \infty$.

Figure 4: The critical exponent $\nu$ for $d = 2$ and $N = -2$ to 1.
Figure 5: Potentials of the fixed points \((d = 2, N = -2 \text{ to } 0.9, \text{ steps of } \frac{1}{10})\). Deeper potentials belong to bigger values of \(N\).

Figure 6: Potentials of the fixed points \((d = 3, N = 1 \text{ to } 20)\) after rescaling. The potential deviating most from the “limit potential” belongs to \(N = 1\).
Figure 7: Eigenvalues for $N \leq 0$. The unbroken lines belong to the eigenvalues at the UV fixed point, continued to $d < 2$. The (· · ·) lines belong to the eigenvalues at the 2-well fixed point. The picture for $N = -1.5$ shows additionally the eigenvalues at the HT fixed point (· · ·) and the picture for $N = 0$ the eigenvalues of the 3-well fixed point (· · ·), which bifurcates at $d = 3$ from the UV fixed point. At every fixed point we also have a “volume eigenvalue” $\lambda_0 = 2$. 
Figure 8: Eigenvalues for $N > 0$. The unbroken lines belong to the eigenvalues at the UV fixed point. The ($\cdots$) lines belong to the eigenvalues at the 2-well fixed point. The picture for $N = 10$ shows additionally the eigenvalues at the $N \to \infty$ fixed point ($\dashdashed$) and the picture for $N = 1$ and $N = 5$ the eigenvalues of the 3-well fixed point ($\cdots$), which bifurcates at $d = 3$ from the UV fixed point. At every fixed point we also have a “volume eigenvalue” $\lambda_0 = 2$. 
Figure 9: $\epsilon$-expansion to 1. (---), 2. (-- --) and 3. order (···) compared to numerically calculated fixed points.

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