HIGHER-GENUS WALL-CROSSING IN LANDAU–GINZBURG THEORY

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Abstract. For a Fermat quasi-homogeneous polynomial, we study the associated weighted Fan–Jarvis–Ruan–Witten theory with narrow insertions. We prove a wall-crossing formula in all genera via localization on a master space, which is constructed by introducing an additional tangent vector to the moduli problem. This is a Landau–Ginzburg theory analogue of the higher-genus quasi-map wall-crossing formula proved by Ciocan-Fontanine and Kim. It generalizes the genus-0 result by Ross–Ruan and the genus-1 result by Guo–Ross.

1. Introduction

1.1. Overview. The goal of this paper is to prove the all-genera wall-crossing formula for the weighted Fan–Jarvis–Ruan–Witten (FJRW) invariants of a Fermat polynomial with extended narrow insertions.

Let $W$ be a Fermat type degree-$r$ quasi-homogeneous polynomial of weights $(w_1, \cdots, w_s) \in \mathbb{Z}^s$:

$$W(X_1, \cdots, X_s) = X_1^{r/w_1} + \cdots + X_s^{r/w_s}.$$ 

We assume that each $r/w_\alpha \geq 2$ is an integer and $\gcd(r, w_1, \cdots, w_s) = 1$. The polynomial $W$ defines a smooth hypersurface in the weighted projective space $\mathbb{P}(w_1, \cdots, w_s)$. We define charges $q_\alpha = w_\alpha/r$ and set $q = \sum q_\alpha$. When $q = 1$, $X_W$ is a Calabi–Yau orbifold and the Landau–Ginzburg/Calabi–Yau (LG/CY) correspondence relates the Gromov–Witten theory of $X_W$ to the FJRW theory of $(W, \langle J \rangle)$. These are the two phases of the gauged linear sigma model (GLSM) mathematically defined in [16].

There is a family of gauged linear sigma models interpolating between these two theories [16, 32], parametrized by a nonzero rational number $\epsilon$. As $\epsilon$ varies, the change of the theories gives a wall-and-chamber structure on the parameter space.

When $\epsilon > 0$, it is the CY side. The theory associated to sufficiently large $\epsilon$ is the Gromov–Witten theory of $X_W$: the theory associated to sufficiently small $\epsilon > 0$ is the stable quotient invariants [11, 13, 24, 25, 31]. In the recent work [12], Ciocan-Fontanine and Kim established an explicit wall-crossing formula relating these two theories for complete intersections in projective spaces.

When $\epsilon < 0$, it is the LG side and we have weighted FJRW theories. We call the theory associated to sufficiently negative $\epsilon$ the $\infty$-FJRW invariant theory; we call the theory associated to $\epsilon$ sufficiently close to 0 the 0-FJRW invariant theory. Our work is the LG analogue of the work by Ciocan-Fontanine and Kim [12]. We will prove a similar wall-crossing formula relating the the $\infty$-FJRW theory and the 0-FJRW theory, for extended narrow insertions.

We expect that the LG side and the CY side are more directly related near $\epsilon = 0$. Using wall-crossing Ross–Ruan proved the LG/CY correspondence in genus...
0 [29]. In genus 1, Guo–Ross used the wall-crossing formula to compute the FJRW invariants of the quintic 3-fold explicitly [18] and verified the genus-1 LG/CY correspondence [19]. Our result generalizes their wall-crossing formulas to all genera. We hope this will be useful for establishing all-genus LG/CY correspondence. The higher-genus wall-crossing formula is also proved by Clader–Janda–Ruan [14] using different methods.

1.2. The statement of the results. Let \( \phi_1, \ldots, \phi_r \) be formal symbols and \( H_W \) be the \( \mathbb{Q} \)-vector space spanned by \( \phi_1, \ldots, \phi_r \). Following [29], we call \( H_W \) the extended narrow state space.

We fix non-negative integers \( g, m, n \). For integers \( a_j, b_j \in \{1, \ldots, r \}, c_i \geq 0 \) indexed by \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \), we study the 0-FJRW invariants

\[
\langle \psi^{c_1} \phi_{a_1}, \ldots, \psi^{c_m} \phi_{a_m} | \phi_{b_1}, \ldots, \phi_{b_n} \rangle^0_{g, m | n} \in \mathbb{Q},
\]

which are defined via intersection theory on the moduli space parametrizing triples \( (C, L, p) \), where \( C \) is a Hassett-stable twisted curve with \( m \) orbifold markings \( x_1, \ldots, x_m \) of weight 1 and \( n \) non-orbifold markings \( y_1, \ldots, y_n \) of weight \( \epsilon \), for sufficiently small \( \epsilon > 0 \). \( L \) is a representable line bundle with appropriate monodromies at each orbifold marking \( x_i \); and \( p \) is a non-vanishing section

\[
p \in H^0( C, L^{-r} \otimes \omega_C \left( \sum_{i=1}^{m} x_i + \sum_{j \neq r} (1 - b_j)y_j + \sum_{b_j = r} y_j \right)).
\]

Here the \( \psi \) are the cotangent-line classes at the markings on the coarse curves. Because the weights of \( x_i \) are 1, they are called heavy markings; because the weights of \( y_j \) are arbitrarily small, they are called light markings. The monodromy of \( L \) at the orbifold point \( x_i \) is determined by the state \( \phi_{a_i} \) in (1). The light markings \( y_j \) are non-orbifold points and the \( b_j \) in (2) play the role of the monodromy (Section 2.6). We will give a self-contained definition of these invariants in Section 2. When \( 2g - 2 + m < 0 \) or \( 2g - 2 + m = n = 0 \), the moduli spaces are empty and we define the invariants (1) to be zero.

For any formal power series

\[
u = u_0 + u_1 \psi + u_2 \psi^2 + \cdots \quad \text{and} \quad t = \sum_{j=1}^{r} t_j \phi_j,
\]

we use (1) and multi-linearity to define

\[
\langle u, \cdots, u | t, \cdots, t \rangle^0_{g, m | n} \in \mathbb{Q}[\{u_{ij}, t_i\}]^n.
\]

We form a generating function of 0-FJRW invariants

\[
\mathcal{F}^0_g(u, t) = \sum_{m, n \geq 0} \frac{1}{m! n!} (u^m | t^n)^0_{g, m | n},
\]

where \( u^m \) means \( u, \ldots, u \), repeated \( m \)-times.

We call \( \phi_\alpha \) a narrow state if \( aq_\alpha \notin \mathbb{Z} \) for all \( \alpha = 1, \ldots, s \). This means that for each \( \alpha \), the line bundle \( L^{u_\alpha} \) has nontrivial monodromy at the marking where \( \phi_\alpha \) is “inserted”. Otherwise we call \( \phi_\alpha \) a broad state. The invariant (1) vanishes unless all the \( \phi_{a_i} \) are narrow (Lemma 3). This is referred to as the Vanishing Axiom in [28].
The $\infty$-FJRW theory is a special case of the 0-FJRW theory. It is by definition the 0-FJRW theory with no light markings. Thus we have
\[
\langle u^n \rangle_{g,m}^\infty = \langle u^n \rangle_{g,m}^0 [0 \in \mathbb{Q}[u_{ij}]
\]
and
\[
\mathcal{F}_g^\infty(u) = \mathcal{F}_g^0(u, 0).
\]

We now explain the analogy to the CY side. A map to $\mathbb{P}^n$ is a line bundle and $n$ sections without common zeros. In stable quotient theory, we allow some common zeros of those sections. In the 0-FJRW theory, assuming that $b_j = 2$ for all $j$, we can view the light markings as “common” zeros, as follows. We look at the image $\tilde{p}$ of $p$ under the natural inclusion
\[
H^0(C, L^r \otimes \omega_C \langle \sum_{i=1}^m x_i - \sum_{j=1}^r y_j \rangle) \to H^0(C, L^r \otimes \omega_C \langle \sum_{i=1}^m x_i \rangle).
\]
Then $y_j$ become the zeros of $\tilde{p}$. That is analogous to the stable quotient theory. While in the $\infty$-FJRW theory, there are no light markings. Hence $\tilde{p} = p$ is required to be non-vanishing. Hence it is analogous to the Gromov–Witten theory.

We now state the numerical wall-crossing formula. We first define an explicit $H_W$-valued series $\mu(t, z)$. As usual, for $x \in \mathbb{Q}$, we denote by $\lfloor x \rfloor$ the largest integer $\leq x$ and $\langle x \rangle = x - \lfloor x \rfloor$ the fractional part of $x$. For any $B_n = \{b_1, \cdots, b_n\}$, where $b_j \in \{1, \cdots, r\}$ for each $j$, we define $k_{B_n}$ to be the integer such that
\[
1 \leq k_{B_n} \leq r \quad \text{and} \quad k_{B_n} - 1 \equiv \sum_{j=1}^n (b_j - 1) \mod r.
\]
For each $\alpha = 1, \cdots, s$, we define
\[
\ell_{\alpha, B_n} = \left\lfloor \sum_{j=1}^n \langle q_{\alpha}(b_j - 1) \rangle \right\rfloor \quad \text{and} \quad k_{\alpha, B_n} = q_{\alpha} + \langle q_{\alpha}(k_{B_n} - 1) \rangle.
\]
For each integer $n$, we write $\lfloor x \rfloor_n = x(x + 1) \cdots (x + n - 1)$. We define
\[
\mu_{B_n}(z) = \prod_{\alpha=1}^s [k_{\alpha, B_n}]_{\ell_{\alpha, B_n}} z^{1 - n + \Sigma_n \ell_{\alpha, B_n}}
\]
and
\[
\mu(t, z) = \sum_{n \geq 1} \sum_{B_n} \frac{t_{b_1} \cdots t_{b_n}}{n!} \mu_{B_n}(z) \phi_{k_{B_n}}.
\]
Let $\mu^+(t, z)$ be the truncation of $\mu(t, z)$ consisting of all non-negative powers of $z$. The big $L$-function defined in [29] is our $z\phi_1 + \mu^+(t, z)$ (cf. Remark 9).

In this paper we prove

**Theorem 1.** For $g \geq 1$, we have
\[
\mathcal{F}_g^\infty(u, t) = \mathcal{F}_g^\infty(u + \mu^+(t, -\psi)).
\]

For $g = 0$, the same equation holds modulo linear terms in the variables $\{u_{ij}\}$.

This numerical wall-crossing formula can be generalized in two ways. We can allow $\psi$-class insertions at light marking and we can compare the virtual fundamental classes in the Chow groups. They are both included in Theorem 18.

This theorem is proved independently by Clader, Janda and Ruan [14]. Indeed their theorem includes the hybrid model case, assuming the existence of at least
one heavy marking. Our proof and the proof given in [14] both use master spaces, and use localization to derive explicit wall-crossing formulas. Apart from this, the two proofs took different directions. The master space used in [14] is constructed by introducing a new line bundle paired with additional sections, and the proof is by induction on genus.

The master space used in this proof is constructed by introducing an additional tangent vector at one light marking; the fixed-point components correspond to the correction terms in the wall-crossing formula, which allows us to obtain the wall-crossing term directly.

We now consider the Calabi–Yau case $q = 1$ and restrict ourselves to $t = t\phi_2$. Namely we consider those $B_n$ where all $b_j = 2$ and we set $t_2 = t$. It follows that $\mu^+(t\phi_2, z)$ only has degree-0 and degree-1 terms in $z$. Define power series $I_0(t) = 1 + O(t)$ and $I_1(t) = O(t)$ via

$$\mu^+(t\phi_2, z) = (I_0(t) - 1)z\phi_1 + I_1(t)\phi_2.$$ 

We set $u = 0$ in Theorem 1 and apply the dilation equation for the $\infty$-FJRW invariants [15] to get

**Corollary 2.** When $q = 1$,

$$I_0(t)^{2g-2} \sum_{d \geq 0} \frac{t^d}{d!} \langle \phi_2, \cdots, \phi_2 \rangle^0_{g,d} = \sum_{n \geq 0} \frac{1}{n!} \left( \frac{I_1(t)}{I_0(t)} \right)^n \langle \phi_2, \cdots, \phi_2 \rangle^\infty_{g,n}, \ g > 1;$$

and

$$\sum_{d \geq 1} \frac{t^d}{d!} \langle \phi_2, \cdots, \phi_2 \rangle^0_{1,d} = -\log(I_0(t))\langle \psi\phi_1 \rangle^\infty_{1,1} + \sum_{n \geq 1} \frac{1}{n!} \left( \frac{I_1(t)}{I_0(t)} \right)^n \langle \phi_2, \cdots, \phi_2 \rangle^\infty_{1,n}. \tag{3}$$

Formula (3) recovers Theorem 1.1 in [18]. Our definition is slightly different from theirs. This is discussed in Section 2.7. We can compute the genus-0 invariants and match the result of [29], this is explained in Section 7.

### 1.3. Plan of the paper

In Section 2, we briefly recall the construction of weighted FJRW invariants. In Section 3, we focus on the underlying coarse curves and construct a master space that contains various moduli spaces of Hassett-stable curves with weighted markings. The main result is the properness of the master space. In Sections 4, we define the $r$-spin master space. We introduce a $\mathbb{C}^*$-action on the master space and study the fixed-point components. In Section 5, we construct an equivariant perfect obstruction theory with an equivariant cosection. The localization formula will give us a relation among different $0$-FJRW theories. In Section 6, we collect and package the relations from the localization formula and prove our main theorems, including Theorem 1. In Section 7, we study the genus-0 invariants and match the main theorem of [29].

### 1.4. Conventions

In this paper we will work over the field of complex numbers. All scheme are assumed to be locally noetherian over $\mathbb{C}$. The genus of a curve is always the arithmetic genus. When $C$ is an orbifold curve and $x$ is a marking of orbifold index $r$ (i.e. the automorphism group of $x$ is cyclic of order $r$), the line bundle $O_C(x)$ has degree $1/r$, as opposed to 1.
For \( x \in \mathbb{Q} \), we denote by \( \lfloor x \rfloor \) the greatest integer no larger that \( x \) and \( \langle x \rangle = x - \lfloor x \rfloor \) the fractional part of \( x \). For an integer \( n \geq 0 \), we abbreviate \( x(x+1) \cdots (x+n-1) \) to \( [x]^n \).

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2. The weighted FJRW invariants

In this section, we give a self-contained description of the weighted FJRW theory within the scope of this paper. This is a special case of the gauged linear sigma models defined in \cite{16}. The construction of the virtual fundamental class follows \cite{7}.

2.1. The moduli spaces of stable \( r \)-spin curves with weighted markings.

We fix non-negative integers \( g, m, n \) such that \( 2g - 2 + m \geq 0 \) and \( (2g - 2 + m, n) \neq (0, 0) \). We will define the moduli spaces of genus-\( g \) stable \( r \)-spin curves with \( m \) heavy markings and \( n \) light markings. They are indexed by the discrete data \( \gamma = (a_1, \ldots, a_m, b_1, \ldots, b_n), \ a_i, b_j \in \{1, \ldots, r\} \).

Let \( S \) be any scheme.

Definition 1. An \( S \)-family of pre-stable \( r \)-spin curves of genus \( g \) with \( \gamma \)-weighted markings is the datum

\[
\xi = (C, \pi, x_1, \cdots, x_m; y_1, \ldots, y_n, L, p),
\]

where

- (curve) \( \pi : C \to S \) is an \( S \)-family of prestable twisted curves of genus \( g \) with balanced nodes and orbifold markings \( x_1, \cdots, x_m \);
- (heavy markings) \( x_1, \ldots, x_m \) are disjoint from each other;
- (light markings) \( y_1, \ldots, y_n \) are not necessarily disjoint markings contained in \( C_{\text{sm}} \{ x_1, \ldots, x_m \} \);
- (line bundle) \( L \) is a representable line bundle on \( C \);
- (p-field) \( p \in H^0(C, P) \) is non-vanishing, where

\[
P = L^{-r} \otimes \omega_{C/S}(\sum_{i=1}^{m} x_i + \sum_{b_j \neq r} (1 - b_j) y_j + \sum_{b_j = r} y_j).
\]

We will abbreviate \( x_1, \cdots, x_m \) to \( x \) and abbreviate \( y_1, \cdots, y_n \) to \( y \). Thus \( \xi = (C, \pi, x; y, L, p) \).

Remark 1. Following \cite{4} (cf. \cite{9}), a family of twisted curves \( \pi : C \to S \) with balanced nodes and orbifold markings \( x_1, \cdots, x_m \) is a proper flat morphism of relative dimension 1 from a Deligne-Mumford stack \( C \) to \( S \), with closed substacks \( x_i \subset C \), such that
(1) the fibres are connected 1-dimensional with at worst nodal singularities, the coarse moduli of $C$ is a nodal curve of genus $g$ over $S$.

(2) a local model of a node is $[U/\mu_r] \to T$ with $T = \text{Spec } A, U = \text{Spec } A[z, w]/(zw-t)$ for some $t \in A$, and the action is given by $(z, w) \mapsto (z^r, z^{-1}w)$.

(3) each $x_i$ is a closed substack of the relatively smooth locus $C^{\text{sm}} \subset C$ and $\pi|_{x_i}$ is an étale gerbe banded by $\mu_{r_i}$.

(4) $C^{\text{sm}} \setminus \{x_1, \ldots, x_m\}$ is an algebraic space.

The line bundle $L$ being representable means that near each orbifold point $x$ of $C$, the automorphism group of $x$ acts faithfully on the fibre $L|_x$ of $L$ at $x$. That $L$ has monodromy $\frac{a}{r}$ at the orbifold marking $x_i$ means the generator of $\mu_{r_i}$ acts on $L|_{x_i}$ as multiplication by $\exp\left(\frac{2\pi i}{r} \right)$. Representability of $L$ implies that $\frac{r'}{r} = \frac{r}{\gcd(a_i, r)}$ determined by $a_i$ and $r$. To test the choice of the generator, we agree that near $x_i$, $L$ is isomorphic to $L'(\frac{a'}{r}x_i)$, where $L'$ is the pullback of some line bundle on the coarse moduli space of $C$.

**Definition 2.** The datum $\xi$ is said to be stable if the $\mathbb{Q}$-line bundle $\omega_{C/S}(\sum x_i + \epsilon \sum y_j)$ is relatively ample for all $\epsilon \in \mathbb{Q}_{>0}$.

We denote the moduli space parametrizing such stable datums $\xi$ by $\overline{M}_{g, \gamma}$. This is a smooth Deligne-Mumford stack of dimension $3g - 3 + m + n$.

### 2.2. The obstruction theory and the virtual fundamental class.

Let $\overline{M}_{g, \gamma}^{1/r, \varphi}$ be the stack of $S$-families of genus-$g$ stable $r$-spin curves with $\gamma$-weighted markings and $\varphi$-fields:

$$(C, \pi, x_1, \ldots, x_m; y_1, \ldots, y_n, L, p, \varphi_1, \ldots, \varphi_s),$$

where $$(C, \pi, x; y, L, p) \in \overline{M}_{g, \gamma}^{1/r}(S)$$

and

$$\varphi_\alpha \in H^0(C, \mathcal{L}^{w_\alpha}(D_\alpha)), \quad D_\alpha = - \sum_{q_\alpha a_i \in \mathbb{Z}} x_i + \sum_{b_j \neq r} q_\alpha (b_j - 1) y_j - \sum_{b_j = r} y_j.$$

The $\varphi_\alpha$ are called $\varphi$-fields.

Let $\pi : \mathcal{C} \to \overline{M}_{g, \gamma}^{1/r, \varphi}$ be the universal curve, $\omega_{\pi}$ be the relative dualizing sheaf and $\mathcal{L}$ be the universal line bundle. We use $x_i, y_j$ and $D_\alpha$ to denote the divisors on $\mathcal{C}$ as in the definition of $\overline{M}_{g, \gamma}^{1/r, \varphi}$.

Let $\tau : \overline{M}_{g, \gamma}^{1/r, \varphi} \to \overline{M}_{g, \gamma}^{1/r}$ be the forgetful map and $\mathbb{L}_\tau$ be its relative cotangent complex. Then $\tau$ admits a relative perfect obstruction theory

$$(\mathbb{L}_\tau)^\vee \to \bigoplus_{\alpha=1}^s R\pi_*\mathcal{L}^{w_\alpha}(D_\alpha).$$

The obstruction sheaf $\bigoplus_{\alpha=1}^s R^1\pi_*\mathcal{L}^{w_\alpha}(D_\alpha)$ has a cosection

$$\sigma : \bigoplus_{\alpha=1}^s R^1\pi_*\mathcal{L}^{w_\alpha}(D_\alpha) \to R^1\pi_*\omega_\pi \cong \mathcal{O}_{\overline{M}_{g, \gamma}^{1/r, \varphi}}$$

defined by

$$(\varphi_1, \ldots, \varphi_s) \mapsto p \sum_{i=1}^s \varphi_\alpha \partial_\alpha W(\varphi_1, \ldots, \varphi_s).$$
We now explain why \( \sigma \) is well-defined. Note that for each \( \alpha \) and each \( x_i \) such that \( q_\alpha a_i \not\in \mathbb{Z} \), \( L^{w_\alpha} \) has nontrivial monodromy at \( x_i \). Hence the section \( \varphi_\alpha \) must vanish along \( x_i \) and thus is a section of

\[
\pi_* (L^{w_\alpha}(-\sum_{i=1}^m x_i + \sum_{b_j \neq r} |q_\alpha (b_j - 1)| y_j - \sum_{b_j = r} y_j)).
\]

Hence \( p_\alpha \partial_\alpha W(\varphi_1, \cdots, \varphi_s) \) is a section of \( R^1 \pi_* \omega_\pi(-\Delta) \) where

\[
\Delta = \sum_{i=1}^m \left( \frac{r}{w_\alpha} - 2 \right) x_i + \sum_{q_\alpha a_i \in \mathbb{Z}} x_i + \sum_{b_j \neq r} (b_j - 1 - \frac{1}{q_\alpha} |q_\alpha (b_j - 1)|) y_j + \sum_{b_j = r} (\frac{r}{w_\alpha} - 1) y_j.
\]

Since we have assumed that \( r/w_\alpha \geq 2 \), we see that \( \Delta \) is effective and hence \( R^1 \pi_* \omega_\pi(-\Delta) \) naturally maps to \( R^1 \pi_* \omega_\pi \).

The degeneracy loci \( D(\sigma) \) of \( \sigma \) is defined to be the locus where \( \sigma \) is not surjective. As in [7], Serre duality implies that \( D(\sigma) \) is equal to \( \overline{M}_{g,\gamma}^{1/r,\varphi} \), viewed as a closed subset of \( \overline{M}_{g,\gamma}^{1/r,\varphi} \) where all the \( \varphi \)-fields are identically zero. The cosection lifts to the absolute obstruction theory and defines a cosection localized virtual fundamental class \( [\overline{M}_{g,\gamma}^{1/r,\varphi}]_{\text{vir}} \in A_*(\overline{M}_{g,\gamma}^{1/r,\varphi}) \) [7, 16].

2.3. Ramond vanishing. We say that the heavy marking \( x_i \) is narrow if \( q_\alpha a_i \not\in \mathbb{Z} \) for all \( \alpha \). Otherwise we say \( x_i \) is broad. Broad heavy markings will naturally appear in our construction even if we have started with narrow heavy markings only.

However, note that as the \( \phi_\alpha \) are required to vanish along the broad heavy markings, they are indeed “fake” broad markings, and we have the following vanishing result from [28]:

**Lemma 3** ([28]). The class \( [\overline{M}_{g,\gamma}^{1/r,\varphi}]_{\text{vir}} \) vanishes unless all heavy markings are narrow.

**Proof.** If there are no light markings, this follows from Theorem 2.1 of [28]. The equivalence between the cosection construction and Polishchuk’s construction is established in [7]. We are not able not find a direct and explicit reference in the presence of light markings. Instead, if \( g = 0 \), the lemma follows from the proof of Lemma 1.8 in [29]; if \( g \geq 1 \), we appeal to the wall-crossing in Theorem 18 and the lemma follows from the case when there are no light markings. \( \square \)

**Remark 2.** The main theorems in this paper does not depend on Lemma 3. Indeed Lemma 3 only implies that many terms in the wall-crossing formulas turn out to be zero. Hence there is no circular reasoning in the proof above.

2.4. The extended narrow state space and weighted FJRW invariants. We have introduced the \( \mathbb{Q} \)-vector space \( H_W \) whose basis consists of the formal symbols \( \phi_1, \cdots, \phi_r \). Let \( \gamma \) be as before and \( \psi_{x_i}, \psi_{y_j} \) be the cotangent-line classes on the coarse curves. We define the 0-FJRW invariants with descendants to be

\[
\langle \psi^{c_1} \phi_{a_1}, \cdots, \psi^{c_m} \phi_{a_m} | \psi^{d_1} \phi_{b_1}, \cdots, \psi^{d_n} \phi_{b_n} \rangle_{g,\gamma}^0
\]

(4)

\[
:= \epsilon_\gamma \cdot \int_{[\overline{M}_{g,\gamma}^{1/r,\varphi}]_{\text{vir}}} \psi^{c_1}_{x_1} \cdots \psi^{c_m}_{x_m} \psi^{d_1}_{y_1} \cdots \psi^{d_n}_{y_n}
\]

where the constant

\[
\epsilon_\gamma = \frac{1}{p^{g-1}} (-1)^{(2q-s)(g-1)} - \sum \alpha (q_\alpha (a_\alpha - 1)) - \sum \sum \alpha (q_\alpha (b_\alpha - 1))
\]
is introduced to be consistent with the original definition of Fan–Jarvis–Ruan. The sign is just \((-1)^{g,\gamma}(R\pi_*(\mathcal{L}^\omega_\alpha(D_\alpha)))\). The \(\infty\)-FJRW invariants are defined to be
\[
\langle \psi^{a_1}\cdots \psi^{a_m} \rangle_{g,\gamma} := \langle \psi^{a_1}\cdots \psi^{a_m} | \mathcal{O} \rangle^0_{g,\gamma}.
\]
Both of them are referred to as weighted FJRW invariants.

2.5. The twisted theory. We can also consider the twisted theory. Let
\[
(C, \pi, x, y, L, p)
\]
be the universal family over \(\overline{M}_{g,\gamma}^{1/r}\). Let \(D_\alpha\) be the divisor on \(C\) defined by
\[
D_\alpha = - \sum_{a, \alpha, \in \mathbb{Z}} x_i + \sum_{b, \neq r} |g_\alpha(b_j - 1)| y_j - \sum_{b, j = r} y_j.
\]
Let \(\mathcal{S} = (\mathbb{C}^*)^s\) be the torus that acts trivially on \(\overline{M}_{g,\gamma}^{1/r}\) and acts on the fibres of the twisted case in Section 2.4, with \([\overline{M}_{g,\gamma}^{1/r}]^\text{vir}_x\) replaced by \([\overline{M}_{g,\gamma}^{1/r}]^\text{vir}_\mathcal{S}\). We will focus on the untwisted case. The results of this paper also work in the twisted case if we modify one definition (Remark 8).

2.6. Orbifold markings v.s. non-orbifold markings. We have treated heavy markings as orbifold markings and light markings as non-orbifold markings. Actually orbifold markings and non-orbifold markings are equivalent in the following sense. Suppose \(\rho : C \to |\mathcal{C}|\) is the partial coarse moduli forgetting the orbifold structure only at \(x_1, \ldots, x_m\) but not at the nodes. Let \(\bar{x}_i, \bar{y}_j\) be the images of \(x_i, y_j\). Then we have an natural isomorphism
\[
\rho_*(\mathcal{L}^\omega_\alpha(D_\alpha)) \simeq \rho_*(\mathcal{L})^\omega_\alpha \otimes \mathcal{O}_{|\mathcal{C}|} \left( \sum_{a_i \neq r} [g_\alpha(a_i - 1)] \bar{x}_i - \sum_{a_i = r} \sum_{b_j \neq r} |g_\alpha(b_j - 1)| \bar{y}_j - \sum_{b_j = r} \bar{y}_j \right).
\]
Moreover for
\[
P = L^{-r} \otimes \omega_{|\mathcal{C}|} \left( \sum_{i=1}^m x_i + \sum_{b_j \neq r} (1 - b_j) y_j + \sum_{b_j = r} y_j \right),
\]
we have
\[
\rho_*P = (\rho_*(L)^{-r} \otimes \omega_{|\mathcal{C}|}) \left( \sum_{a_i \neq r} (1 - a_i) \bar{x}_i + \sum_{a_i = r} \bar{x}_i + \sum_{b_j \neq r} (1 - b_j) \bar{y}_j + \sum_{b_j = r} \bar{y}_j \right).
\]
In this sense we are treating heavy markings and light markings on an equal footing. By Theorem 4.2.1 of \cite{2}, there is a unique way to add the stack structure at \(x_i\). Hence in Definition 1, if we require that the curve has no orbifold structure at each \(x_i\), and replace the definition of \(P\) by the right hand side of (6) with \(L\) in place
of $\rho, L$, we get the same moduli space. Moreover, since $\rho$, is an exact functor, if we define the $\varphi$-fields as sections of the right hand side of (5), Section 2.2 works verbatim with the new definition and defines the same virtual fundamental class under the natural identification of the moduli spaces.

We can also replace only a subset of orbifold heavy markings by non-orbifold heavy markings. This will be useful when we “transform” a light marking into a heavy marking in Section 5.

**Remark 3.** We insist that the heavy markings are orbifold markings because when we “split” a node, we get a pair of orbifold heavy markings; we insist that the light markings are non-orbifold markings because we do not want to consider two colliding orbifold markings.

2.7. Various treatments of the light markings. There are at least two different ways to treat the light markings. In [29] and in this paper, the light markings are ordered. While in [16, 18], the light markings are unordered, meaning that we only remember the divisor $y_1 + \cdots + y_n$. These two treatments are equivalent, differing by a factor of $n!$. The argument is simple and we omit it here.

3. The moduli of stable curves with mixed weighted markings

In [20], Hassett defined the moduli of stable curves with weighted markings. In this paper, we consider the following special case: moduli space $\overline{M}_{g,m|n}$ of stable curves $\pi : C \to S$ with $m$ weight-1 markings $x_1, \ldots, x_m$ and $n$ weight-$\epsilon$ markings $y_1, \ldots, y_n$, where $\epsilon > 0$ is sufficiently small. The stability conditions for $\overline{M}_{g,m|n}$ are

- (1) $\pi : C \to S$ is a family of nodal curves and all the markings are contained in the relative smooth locus of $C$;
- (2) each $x_i$ does not intersect any other markings;
- (3) the $\mathbb{Q}$-line bundle $\omega_{C/S}(\sum x_i + \epsilon \sum y_j)$ is relatively ample for all $\epsilon \in \mathbb{Q}_{>0}$.

We call $x_i$ the heavy markings and $y_j$ the light markings.

In this section, we will construct the “master” moduli space $\widetilde{M}_{g,m|n}$ of genus-$g$ stable curves with mixed $(m,n)$-weighted markings $x_1, \ldots, x_m, y_1, \ldots, y_n$. In this moduli, the $x_i$ behave like heavy markings; $y_1, \ldots, y_n$ behave like light markings; $y_1$ is a “mixed weighted” marking. The stack $\overline{M}_{g,m|n}$ contains both $\widetilde{M}_{g,m|n}$ and $\overline{M}_{g,m+1|n-1}$ as closed substacks. We do not consider $r$-spin structures in this section. Thus we only consider non-orbifold curves.

3.1. The construction of $\widetilde{M}_{g,m|n}$. We fix non-negative integers $g, m, n$ such that $2g - 1 + m \geq 0, n \geq 1$ and $(2g - 1 + m, n - 1) \neq (0,0)$. Let $S$ be any scheme.

**Definition 3.** An $S$-family of genus-$g$ stable curves with mixed $(m,n)$-weighted markings consists of $(C, \pi, x_1, \ldots, x_m; y_1, \ldots, y_n, N, v_1, v_2)$, where

- (1) $\pi : C \to S$ is a flat proper family of connected genus-$g$ nodal curves;
- (2) $x_1, \ldots, x_m, y_1, \ldots, y_n$ are markings contained in the relative smooth locus $C^{\text{sm}} \subset C$;
- (3) $N$ is a line bundle on $S$;
- (4) $v_1 \in H^0(S, T_{y_1} \otimes_{O_S} N)$ and $v_2 \in H^0(S, N)$, where $T_{y_1} = \omega_{C/S}^{\vee}|_{y_1}$.

such that

- (1) each $x_i$ is disjoint from all other markings;
(2) $v_1$ and $v_2$ do not have any common zero on $S$;
(3) the $\mathbb{Q}$-line bundle $\omega_{C/S}(\sum_{i=1}^{m} x_i + y_1 + \epsilon \sum_{j=2}^{n} y_j)$ is relatively ample for all $\epsilon \in \mathbb{Q}_{>0}$;
(4) when $v_1 = 0$, $y_1$ does not intersect other light markings $y_j, j = 2, \ldots, n$;
(5) when $v_2 = 0$, the $\mathbb{Q}$-line bundle $\omega_{C/S}(\sum_{i=1}^{m} x_i + \epsilon \sum_{j=1}^{n} y_j)$ is relatively ample for all $\epsilon \in \mathbb{Q}_{>0}$;

Let

$$\xi' = (C', \pi', x', y', N', v_1', v_2')$$

and

$$\xi = (C, \pi, x, y, N, v_1, v_2)$$

be two families of genus-$g$ stable curves with mixed $(m, n)$-weighted markings over $S'$ and $S$, respectively. An arrow $\xi' \to \xi$ consists of a fibred diagram

$$\begin{array}{ccc}
 C' & \xrightarrow{f} & C \\
 \pi' & & \pi \\
 S' & \xrightarrow{g} & S
\end{array}$$

and an isomorphism $\eta : N' \to g^*N$ of line bundles such that $f$ pulls back the markings to the corresponding markings, $\eta(v_1') = g^*v_1$ and $\eta(v_2') = g^*v_2$ and $(df_j') \otimes \eta(v_j') = g^*v_j$.

This defines the category $\overline{M}_{g,m|n}$ of genus-$g$ stable curves with mixed $(m, n)$-weighted markings. The category $\overline{M}_{g,m|n}$ is fibred in groupoids over the category of schemes.

**Theorem 4.** The category $\overline{M}_{g,m|n}$ is a smooth Deligne-Mumford stack of finite type over $\mathbb{C}$, of dimension $3g - 3 + m + n + 1$.

**Proof.** Let $\mathfrak{M}$ be the Artin stack of genus-$g$ nodal curves $C$ with $m + n$ not necessarily distinct markings $x_1, \ldots, x_m, y_1, \ldots, y_n$ in the smooth locus of $C$, such that $C$ has at most $2g - 2 + m + n$ irreducible components. This is a finite type smooth Artin stack of dimension $3g - 3 + m + n$. Let $T_{y_1}$ be the line bundle on $\mathfrak{M}$ formed by the tangent spaces to the curves at $y_1$. Let $\mathbb{P}$ be the projective bundle $\mathbb{P}_{\mathfrak{M}}(T_{y_1} \oplus \mathcal{O}_{\mathfrak{M}})$ over $\mathfrak{M}$. Then $\overline{M}_{g,m|n}$ is represented by an open substack of $\mathbb{P}$. Hence it is represented by an Artin stack of finite type. It is easy to see that each closed point of $\overline{M}_{g,m|n}$ has finite automorphisms. Hence $\overline{M}_{g,m|n}$ is a Deligne-Mumford stack. □

**Remark 4.** If $v_2$ is nowhere vanishing, it gives an isomorphism $N \simeq \mathcal{O}_S$ sending $v_2$ to 1. Hence the $(N, v_1, v_2)$ part of $\xi$ is equivalent to $v_1/v_2 \in H^0(S, T_{y_1})$. When $v_2 = 0$, $v_1$ is non-vanishing and gives an isomorphism $N \simeq T_{y_1}$. Thus at every closed point $s \in S$, we can view $(N, v_1, v_2)$ as a point of $T_{y_1} \cup \{\infty\}$.

**Remark 5.** The universal $v_1$ and $v_2$ are sections of line bundles over $\overline{M}_{g,m|n}$. The vanishing locus of $v_1$ is isomorphic to $\overline{M}_{g,m+1|n-1}$, where $y_1$ is a heavy marking; the vanishing locus of $v_2$ is isomorphic to $\overline{M}_{g,m|n}$, where $y_1$ is a light marking.

### 3.2. The properness of $\overline{M}_{g,m|n}$

**Theorem 5.** The stack $\overline{M}_{g,m|n}$ is proper.

**Proof.** We prove the properness by the valuative criterion. Let $R$ be a Henselian DVR with residue field $\mathbb{C}$. Let $B = \text{Spec} \, R$, $b \in B$ be the closed point and $B^* = B \backslash \{b\}$ be the generic point. Suppose $\xi^* = (C^*, \pi^*, x^*, y^*, N^*, v_1^*, v_2^*)$ is in $\overline{M}_{g,m|n}(B^*)$. We want to show that possibly after a finite base-change, we can
extend $\xi^*$ to a family $\xi = (C, \pi, x^*; y, N, v_1, v_2)$ over $B$, and the extension is unique up to unique isomorphisms.

We introduce some notations. When $(C, \pi, x^*; y, N, v_1, v_2)$ is a family over $B$, we denote by $C_b$ the special fibre $\pi^{-1}(b)$, and by $x_i(b)$ the $i$-th heavy marking of $C_b$, and similar for the light markings. By a rational tail (resp. bridge) $E \subset C_b$ we mean that $E$ is a smooth rational subcurve of $C_b$ intersecting the remainder of $C_b$ at one (resp. two) node(s) of $C_b$.

First we reduce to the case where $\pi^* : C^* \to B^*$ is smooth. By the standard stable reduction argument, possibly after finite base-change, the normalization of $C^*$ is a disjoint union $\coprod_{i=0}^k C_i^*$ of smooth curves over $B^*$. We view the preimages of $x^*$ and $y^*$ as heavy and light markings on $\coprod_{i=0}^k C_i^*$. We also view the preimages of the nodes of $C^*$ as heavy markings. Assume that the preimage of $y_1^*$ is in $C_0^*$. For $i > 0$, $C_i^*$ together with the markings forms a family $\xi_i^*$ of Hassett’s stable curves in some $\overline{M}_{g, m|n_1}(B^*)$. For $i = 0$, the map $C_0^* \to C^*$ induces an isomorphism of relative tangent sheaves near $y_1^*$. Thus the pointed curve $C_0^*$ together with $(N^*, v_1^*, v_2^*)$ is an object $\xi_0^*$ in some $\overline{M}_{g_0, m_0|n_0}(B^*)$. For $i > 0$, since Hassett’s moduli spaces are proper, possibly after finite base-change, $\xi_i^*$ extends uniquely to a $B$-family $\xi_i \in \overline{M}_{g_i, m_i|n_i}(B)$. Hence if possibly after finite base-change $\xi_0^*$ also extends uniquely to a $B$-family $\xi_0 \in \overline{M}_{g_0, m_0|n_0}(B^*)$, by gluing the $\xi_i$ along each pair of heavy markings coming from the nodes of $C^*$, we get a unique extension of $\xi^*$ to $\xi \in \overline{M}_{g, m|n}(B)$. Hence without loss of generality, we assume that $\pi^*$ is smooth.

If $v_1^* = 0$ or $v_2^* = 0$, then $\xi^*$ is equivalent to a family of Hassett’s stable curves with weighted markings. The theorem follows from the properness of Hassett’s moduli spaces. Hence we assume $v_1^* \neq 0$ and $v_2^* \neq 0$ on $B^*$.

We first consider the case $(g, m) \neq (0, 1)$. In this case $(C^*, \pi^*, x^*; y^*)$ is family of Hassett-stable curves with heavy markings $x^*$ and light markings $y^*$. Possibly after base-change we extend it to a $B$-family of Hassett-stable curves $(C, \pi; x; y)$. We claim that $(N^*, v_1^*, v_2^*)$ has a unique extension $(N, v_1, v_2)$ to $B$ such that $(v_1, v_2)$ have no common zero. Indeed, by fixing a trivialization of $T_{y_1}$, we can identify $T_{y_1}$ with $O_B$. Then $N^*$ together with the sections $(v_1^*, v_2^*)$ is equivalent to a map from $B^*$ to $\mathbb{P}^1$. This map has a unique extension to $B$. The extension of the map is equivalent to the extension of $(N^*, v_1^*, v_2^*)$ to $(N, v_1, v_2)$.

We now modify the family $(C, \pi; x; y; N, v_1, v_2)$ by iteratively blowing up $C$ at some smooth points of the special fibre to make this family stable. The only situation that violates the stability condition is when

$$(7) \quad v_1(b) = 0 \quad \text{and} \quad y_1(b) = y_j(b) \quad \text{for some} \quad j \neq 1.$$

If this happens, let $q : C' \to C$ be the blowup of $C$ at $y_1(b)$, and extend $(x^*; y^*, N^*, v_1^*, v_2^*)$ to a new $B$-family $\xi' = (C', x'; y'; N', v_1', v_2')$. We claim that $v_2'(b) \neq 0$ and the vanishing order of $v_1'$ at $b$ is exactly one less than that of $v_1$. To see this, notice that the map $q$ induces $q^*\Omega_{C/B} \simeq \Omega_{C'/B}(-E)$, where $E$ is the exceptional divisor. Hence we have an isomorphism $(\Omega_{C/B})_y|_{y_1}(\cdot) \simeq (\Omega_{C'/B})_y|_{y_1'}$ of line bundles on $B$, which restricts to the identity on $B^*$. Thus the claim follows immediately. We replace $\xi$ by $\xi'$, and repeat this procedure finitely many times until (7) does not happen. This gives us a chain of exceptional divisors $E_1, \cdots, E_k$. Then we blow down the maximal subchains of $E_1, \cdots, E_k$ that does not contain any
\( y_j(b) \) for \( j = 1, \ldots, n \). This gives a stable family over \( B \) and proves the existence in the case \( (g, m) \neq (0, 1) \).

Now we consider the case \( (g, m) = (0, 1) \). It follows that \( n \geq 2 \). In this case we can find an \( B^* \)-isomorphism between \( C^* \) and \( \mathbb{P}^1 \times B^* \), identifying \( y_1^* \) with \( \{0\} \times B^* \), \( x_1^* \) with \( \{\infty\} \times B^* \) and \( v_1/v_2 \) with the standard tangent vector \( \partial/\partial z \), where \( z \) is the coordinate on \( \mathbb{P}^1 \). We first take \( C = \mathbb{P}^1 \times B \) and \( N = \mathcal{O}_B \). We set \( v_2 \equiv 1 \) and \( v_1 \equiv \partial/\partial z \). Then we take the limit of the markings \( y_2^*, \ldots, y_n^* \) to get a family \( (C, \pi, x_1; y, N, v_1, v_2) \) over \( B \). The only situation that violates the stability condition is when

\[
(8) \quad x_1(b) = y_j(b) \text{ for some } j \neq 1.
\]

If this happens, we blow up \( C \) at \( x_1(b) \) to get a new family. In the new family \( x_1(b) \) lies on the exceptional divisor. If \( x_1(b) \) still intersects some \( y_j(b) \), we iterative this procedure until (8) does not happen. This only takes finitely many steps, since in the generic fibre \( x_1 \) does not intersect any light markings. After these blowups, the special fibre is a chain of smooth rational curves \( E_1, \ldots, E_k \), where \( y_1(b) \in E_1, x_1(b) \in E_k \) and \( E_i \) intersects \( E_{i+1} \) at a node for \( i = 1, \ldots, k - 1 \). Then we contract all the \( E_i \) that does not contain any \( y_j(b) \) for \( j = 2, \ldots, n \). This gives a stable family over \( B \) and proves the existence in the case \( (g, m) = (0, 1) \).

The following picture shows the change of the special fibre in a typical \( (g, m) = (0, 1) \) case. The subcurves drawn vertically are contracted.

For uniqueness, let \( \xi = (C, \pi, \mathbf{x}; y, N, v_1, v_2) \) and \( \xi' = (C', \pi', \mathbf{x}'; y', N', v_1', v_2') \) in \( \widetilde{M}_{g, m|n}(B) \) be two extensions of \( \xi^* \), we need to show that these two extensions are isomorphic, possibly after finite base-change. The proof is standard. We construct a third (unstable) \( B \)-family \( \widetilde{\xi} = (\widetilde{C}, \widetilde{\pi}, \widetilde{\mathbf{x}}; \widetilde{y}, \widetilde{N}, \widetilde{v_1}, \widetilde{v_2}) \) dominating \( \xi \) and \( \xi' \), and inducing isomorphisms over \( B^* \). Here we are only assuming that \( \widetilde{C} \) is a family of nodal curves, all markings are in the relative smooth locus and \( \widetilde{v}_1, \widetilde{v}_2 \) have no common zeros. We may also assume that \( \widetilde{C} \) is a regular surface. One can check that the maps \( q: \widetilde{C} \to C \) and \( q': \widetilde{C} \to C' \) contract the same set of rational subcurves in the special fibre, thus it induces an isomorphism between \( \xi \) and \( \xi' \). We will skip the details here. \( \square \)

**Remark 6.** We can actually prove that \( \widetilde{M}_{g, m|n} \) has a projective coarse moduli space. Since we will not need this result, we only sketch the proof here.

We apply induction on \( n \). When \( n = 1 \), \( y_1 \) is is not allowed to collide with any other markings since all other markings are heavy. Thus \( y_1 \) is equivalent to a heavy
marking. Hence $\tilde{M}_{g,m|1}$ is isomorphic to the projective bundle $\mathbb{P}_m(T_{y_1} \oplus \mathcal{O}_m)$, where $\mathcal{M} = \mathcal{M}_{g,m+1}$. Since $\mathcal{M}_{g,m+1}$ has a projective coarse moduli, so does $\tilde{M}_{g,m|1}$.

Let $\tilde{C}_{g,m|n}$ be the universal curve over $\tilde{M}_{g,m|n}$. If $\tilde{M}_{g,m|n}$ has a projective coarse moduli, so does $\tilde{C}_{g,m|n}$. For each $n \geq 1$, we look at the morphism $\tau_n : \tilde{M}_{g,m|n+1} \rightarrow \tilde{C}_{g,m|n}$ defined by viewing the last light marking $y_{n+1}$ as the distinguished point on the curve and then stabilizing the curve. It suffices to prove that $\tau_n$ is projective. The fibres of $\tau_n$ are either points or isomorphic to $\mathbb{P}^1$. The locus $Z \subset \tilde{M}_{g,m|n+1}$ where $y_1$ coincides with $y_{n+1}$ is a divisor, which intersects every one dimensional fibre at exactly one point. Hence $\mathcal{O}_{\tilde{M}_{g,m|n+1}}(Z)$ is ample when restricted to the fibres. Note that we can generalize [30, Tag 0D2S] to the case of algebraic spaces since we have the formal function theorem for algebraic spaces [22]. This implies that $\mathcal{O}_{\tilde{M}_{g,m|n+1}}(Z)$ is relatively ample, thus $\tau_n$ is projective.

4. The moduli of stable $r$-spin curves with mixed weighted markings

In this section we define the moduli space $\tilde{M}_{g,r}^{1/r}$ of genus-$g$ stable $r$-spin curves with mixed weighted markings. Then we introduce a $\mathbb{C}^*$-action on $\tilde{M}_{g,r}^{1/r}$ and study the fixed-point components.

4.1. The moduli space $\tilde{M}_{g,r}^{1/r}$ and its properness. In this subsection we assume that $2g - 1 + m \geq 0$, $n \geq 1$ and $(2g - 1 + m, n - 1) \neq (0,0)$. As before let

$$\gamma = \left(\frac{a_1}{r}, \ldots, \frac{a_m}{r} | \frac{b_1}{r}, \ldots, \frac{b_n}{r}\right), \quad a_i, b_j \in \{1, \ldots, r\}.$$ 

Definition 4. An $S$-family of stable $r$-spin curves with mixed $\gamma$-weighted markings is the datum

$$\xi = (C, \pi, x_1, \ldots, x_m; y_1, \ldots, y_n, N, L, v_1, v_2, p),$$

where $(C, \pi, x; y, L, p)$ is an $S$-family of genus-$g$ pre-stable curves with $\gamma$-weighted markings (Definition 1), such that if $\rho : C \rightarrow |C|$ is the coarse moduli of $C$ and $|\pi| : |C| \rightarrow S$ the induced projection, then

$$([C], |\pi|, \rho(x); \rho(y), N, (dp_{y_1} \otimes id_N)(v_1), v_2) \in \tilde{M}_{g,m|n}(S).$$

It is obvious how to pullback $\xi$ along any $S' \rightarrow S$, as in Section 3.1. This defines the category $\tilde{M}_{g,r}^{1/r}$ of genus-$g$ stable $r$-spin curves with mixed $\gamma$-weighted markings.

For degree reasons, $\tilde{M}_{g,r}^{1/r}$ is nonempty if and only if

$$2g - 2 + \sum_{i=1}^{m} (1 - a_i) + \sum_{j=1}^{n} (1 - b_j) \equiv 0 \mod r.$$ 

We assume that holds true from now on.

Theorem 6. The category $\tilde{M}_{g,r}^{1/r}$ is a smooth proper Deligne-Mumford stack.

Proof. The proof of properness is essentially the same as the proof of Theorem 1.5.1 in [3]. Let $\pi : \mathcal{C} \rightarrow \tilde{M}_{g,m|n}$ be the universal curve with universal markings $x; y$. Let $\mathcal{C}_r$ be the stack of $r$-th roots of the line bundle

$$\omega_{\mathcal{C}/\tilde{M}_{g,m|n}} \left( \sum_{i=1}^{m} x_i + \sum_{b_j \neq r} (1 - b_j)y_j + \sum_{b_j = r} y_j \right).$$
This is a proper Deligne-Mumford stack. Let $\beta$ be the class of a fibre of $\mathcal{C}_{rt} \to \widetilde{M}_{g,m,n}$. We consider $K_{g,m}^{\text{bal}}(\mathcal{C}_{rt}/\widetilde{M}_{g,m,n}, \beta)$, the stack of balanced twisted stable maps of genus $g$ and class $\beta$ into $\mathcal{C}_{rt}$ relative to $\widetilde{M}_{g,m,n}$. Then $\widetilde{M}_{g,\gamma}^{1/r}$ is isomorphic to the closed substack of $K_{g,m}^{\text{bal}}(\mathcal{C}_{rt}/\widetilde{M}_{g,m,n}, \beta)$ where the $i$-th marking is mapped to $x_i$, for all $i$. This is a proper Deligne-Mumford stack $[4]$.

The proof of smoothness is identical to the proof of Proposition 2.1.1 in $[3]$. $\Box$

4.2. The $\mathbb{C}^*$-action on $\widetilde{M}_{g,\gamma}^{1/r}$ and the fixed-point components. From now on we assume that $n \geq 1$ and $2g - 2 + m \geq 0$. We introduce a $\mathbb{C}^*$-action on $\widetilde{M}_{g,\gamma}^{1/r}$ via

$$\lambda \cdot (C, \pi, x; y, N, L, v_1, v_2, p) = (C, \pi, x; y, N, L, \lambda v_1, v_2, p), \quad \lambda \in \mathbb{C}^*.$$ 

This also defines an action on the universal family $(C, \pi, x; y, N, v_1, v_2, L, p)$: it defines a $\mathbb{C}^*$-action on $C$; $L$ is an equivariant line bundle on $\widetilde{M}_{g,m,n}$; all the markings, $p$ and $(v_1, v_2)$ are preserved by these actions.

The fixed-point components of $\widetilde{M}_{g,\gamma}^{1/r}$ are

1. $F_0^{1/r} = \{ \xi : v_1 = 0 \}$;
2. $F_\infty^{1/r} = \{ \xi : v_2 = 0 \}$;
3. For each $J \subset \{1, \cdots, n\}$ such that $\{1\} \not\subset J$, $F_J^{1/r}$ consists of

$$\xi = (C, \pi, x; y, N, L, v_1, v_2, p)$$

such that

- $C = C_J \cup E$ where $E$ is a smooth rational orbifold intersecting $C_J$ at a single node;
- $v_1 \neq 0$ and $v_2 \neq 0$;
- $y_1 \in E$ and $y_j = y_1$ for all $j \in J$;
- all other markings are on $C_J$.

It is easy to see that these are all the set-theoretic fixed-point components. Indeed, suppose $\xi \in \widetilde{M}_{g,m,n}(\mathbb{C})$ is $\mathbb{C}^*$-fixed and $v_1 \neq 0, v_2 \neq 0$. Let $E$ be the subcurve containing $y_1$. For each $1 \neq \lambda \in \mathbb{C}^*$, the isomorphism between $\xi$ and $\lambda \cdot \xi$ induces a nontrivial automorphism of $E$ fixing all the markings and nodes of $C$ contained $E$. It follows that $E$ is a smooth rational orbifold intersecting the remainder of $C$ at a single node, and all the markings on $E$ coincide with $y_1$. Hence $\xi$ is in some $F_J^{1/r}$.

A general $\xi$ in $F_J^{1/r}$ looks like (the $r$-spin structure is omitted):

```
E \cong \mathbb{P}^1
/  \bigg\arrow{1,1} \bigg\arrow{0,0} \bigg\arrow{-1,0} \bigg\arrow{0,1} \bigg\arrow{1,1} \bigg\arrow{0,0} \bigg\arrow{-1,0} \bigg\arrow{0,1}
\bullet \bigg\arrow{0,1} \bigg\arrow{1,1}
\bullet \bigg\arrow{-1,0} \bigg\arrow{0,0}
\bullet \bigg\arrow{-1,0} \bigg\arrow{0,0}
\bullet \bigg\arrow{-1,0} \bigg\arrow{0,0}
C_J
```

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We introduce some notations before describing the stack structure and the normal bundle of each fixed-point component. For any integer \( w \), let \( \mathbb{C}_w \) be the standard \( \mathbb{C} \)-action of weight \( w \):
\[
\lambda \cdot z = \lambda^w z, \quad \lambda \in \mathbb{C}^*, z \in \mathbb{C}.
\]

Let
\[
\gamma' = \left( \frac{a_1}{r}, \ldots, \frac{a_m}{r}, \frac{b_1}{r}, \ldots, \frac{b_n}{r} \right).
\]
Recall that \( T_{y_1} \) is the line bundle formed by the tangent spaces of the curves at \( y_1 \).

**Lemma 7.**

1. \( F_{0}^{1/r} \cong \overline{M}_{g,\gamma}^{1/r} \), where the heavy markings are \( x_1, \ldots, x_m, y_1 \) and the light markings are \( y_2, \ldots, y_n \). Its equivariant normal bundle is isomorphic to \( T_{y_1} \otimes \mathbb{C}_1 \).

2. \( F_{\infty}^{1/r} \cong \overline{M}_{g,\gamma}^{1/r} \), where the heavy markings are all the \( x_i \) and the light markings are all the \( y_j \). Its equivariant normal bundle is isomorphic to \( (T_{y_1} \otimes \mathbb{C}_1)^{1/r} \).

**Proof.** We prove (1); (2) is similar. Since \( v_1 \) vanishes identically on \( F_{0}^{1/r} \), \( y_1 \) does not intersect any other markings there. To prove \( F_{0}^{1/r} \cong \overline{M}_{g,\gamma}^{1/r} \), we only need to identify the non-orbifold marking \( y_1 \) of \( F_{0}^{1/r} \) with the last orbifold heavy marking of \( \overline{M}_{g,\gamma}^{1/r} \). This is worked out in Section 2.6. To compute the normal bundle, note that near \( F_{0}^{1/r} \), \( v_2 \) is never vanishing. Hence \( N \) is trivialized by \( v_2 \) and thus the datum \((N, v_1, v_2)\) is equivalent to \( v_1/v_2 \) (cf. Remark 4). Hence near \( F_{0}^{1/r} \), \( \overline{M}_{g,\gamma}^{1/r} \) is the total space of \( T_{y_1} \) over \( F_{0}^{1/r} \). The \( \mathbb{C}^* \)-action is identified with fibrewise multiplication. Hence the formula for the normal bundle follows. \( \square \)

We now come to \( F_{j}^{1/r} \). Let \( \xi = (C, \pi, x; y, N, L, v_1, v_2, p) \) be any closed point of \( F_{j}^{1/r} \). Recall that \( C = C_j \cup E \), where \( E \) is a smooth rational orbifold intersecting \( C_j \) at a node. The first observation is that the orbifold index of the node \( C_j \cap E \) and the monodromy of the line bundle \( L \) at the node are uniquely determined by the data \( \gamma \) and \( J \). Let \( k, a_{\infty} \in \{1, \ldots, r\} \) and \( \ell \in \mathbb{Z}_{\geq 0} \) be the integers such that
\[
10 \quad r\ell + k = 1 + \sum_{j \in J} (b_j - 1) \quad \text{and} \quad a_{\infty} \equiv -k \mod r,
\]

Let \( r' = r/\gcd(r, k) \).

**Lemma 8.** The orbifold index at the node \( C_j \cap E \) is \( r' \). The monodromy of \( L|_{C_j} \) at the node is \( \frac{k}{r} \). The monodromy of \( L|_{E} \) at the node is \( \frac{a_{\infty}}{r} \).

**Proof.** We have an isomorphism of line bundles
\[
(L|_{E})^{r} \cong \omega_{C}^{v} \left( \sum_{j \in J, b_j \neq r} (1 - b_j)y_j + \sum_{j \in J, b_j = r} y_j \right).
\]

Hence
\[
10 \quad r \deg(L|_{E}) \equiv \sum_{j \in J} (1 - b_j) - 1 \mod r.
\]

Since the node is the only orbifold point of \( E \), the fractional part of \( \deg(L|_{E}) \) must come from the orbifold structure at the node. The monodromy of \( L|_{E} \) at the node follows from the fact that \( L \) is representable; the monodromy of \( L|_{C_j} \) at the node follows from the fact that the node is balanced. \( \square \)
The stack structure of $F_{1/r}^J$ is more subtle due to the existence of the “ghost automorphisms”. We set
\[ \gamma_a = \left( \frac{a_1}{r}, \cdots, \frac{a_m}{r}, \frac{b_1}{r}(\frac{1}{r})_j \right) \text{ and } \gamma_E = \left( \frac{a_\infty}{r}, \frac{k}{r} \right). \]

Consider the moduli spaces $\overline{M}_{g, \gamma J}/r$ and $\overline{M}_{0, \gamma E}/r$, where the markings are labeled as $(x_1, \cdots, x_m, y_j)_j \in J$ and $(x_\infty, y_1)$ respectively.

We want to glue the two gerbes $x_j$ and $x_\infty$ to get a balanced node. For any scheme $S$, we define a category $\mathcal{I}(S)$. An object of $\mathcal{I}(S)$ is a 4-tuple $(\Sigma, q, L, s)$, where $q : \Sigma \to S$ is a gerbe banded by $\mu_r$; $L$ is a line bundle on $\Sigma$ with monodromy $\xi_s$; $s$ is a non-vanishing section of $L'$. An arrow $(\Sigma, q, L, s) \to (\Sigma', q', L', s')$ is a pair $(f, \eta)$, where $f : \Sigma \to \Sigma'$ is an $S$-isomorphism of banded gerbes, and $\eta : L \to f^*L'$ is an isomorphism of line bundles that takes $s$ to $f^*s'$. A priori, $\mathcal{I}(S)$ is only a 2-category. But representability of $L$ implies that the automorphism group of any arrow is trivial. Hence $\mathcal{I}(S)$ is equivalent to a category.

**Lemma 9.** The category $\mathcal{I}(S)$ is naturally equivalent to $B\mu_{r/r'}(S)$.

**Proof.** We have an isomorphism of stacks $B\mu_{r/r'} \simeq [C^*/C^*]$, where the action of $\lambda \in C^*$ is multiplication by $\lambda^{r/r'}$. Using this isomorphism, an object in $B\mu_{r/r'}(S)$ is a line bundle $M$ on $S$ and a non-vanishing section $t$ of $M^{r/r'}$. Given $(\Sigma, q, L, s)$, the line bundle $L'$ has trivial monodromy, hence descends to a line bundle $M$ on $S$. The section $s \in H^0(\Sigma, (L')^{r/r'})$ descends to a non-vanishing section $t$ of $M^{r/r'}$. Conversely, given $(M, t)$, let $q : \Sigma \to S$ be the stack of $r'$-th roots of $M$, and $L' \simeq q^*M$ be the universal $r'$-th root. The section $t$ pulls back to a non-vanishing section $s$ of $L'$. We can check that these two procedures are inverses to each other and define an equivalence of categories.

**Remark 7.** Actually $B\mu_{r/r'}$ is a connected component of the rigidified inertia stack $\mathcal{I}_{r} B\mu_{r}$, which classifies gerbes $\Sigma$ banded by $\mu_{r'}$ with representable $\Sigma \to B\mu_{r}$ [1].

Consider any $S$-families
\[ \xi_J = (C_J, \pi_J, x_1, \cdots, x_m, y_j)_j \in \overline{M}_{g, \gamma J}/r(S) \]
and
\[ \xi_E = (E, \pi_E, x_\infty; y_1, N, L_E, v_1, v_2, p_E) \in \overline{M}_{0, \gamma E}/r(S). \]
By definition, the marking $x_j$ is a gerbe banded by $\mu_{r'}$ over $S$. The restriction of the line bundle
\[ \omega_{C_{J/S}}(x_J + \sum_{i=1}^m x_i + \sum_{j \notin J, b_j \neq r} (1 - b_j)y_j + \sum_{j \notin J, b_j = r} y_j) \]
to $x_J$ is canonically trivial. Hence $p_J|_{x_J}$ is a non-vanishing section of $L_J|_{x_J}$. By Lemma 9, this defines an “evaluation” map
\[ ev_{x_J} : \overline{M}_{g, \gamma J}/r \to B\mu_{r/r'}. \]

Similarly, we have an evaluation map $\hat{ev}_{x_\infty}$ by “evaluating” at $x_\infty$ but reversing the banding of $x_\infty$
\[ \hat{ev}_{x_\infty} : \overline{M}_{0, \gamma E}/r \to B\mu_{r/r'}. \]
We now define a morphism

$$
i_j : \overline{M}_{g, \gamma}^{1/r} \times \widetilde{M}_{0, \gamma E}^{1/r} \to \widetilde{M}_{g, \gamma}^{1/r},$$

where the fibre product is formed via ev_{x_j} and ev_{x_\infty}. Consider any \(S\)-families \(\xi_j\) and \(\xi_E\) as above. We modify \(\xi_E\) to get a new family \(\xi'_E\) as follows. First let

$$L'_E = L_E(-c_{y_1}),$$

where

$$c = \begin{cases} \ell - \left| \{ j \in J : b_j = r \} \right| & \text{if } k \neq r \\ \ell - \left| \{ j \in J : b_j = r \} \right| + 1 & \text{if } k = r. \end{cases}$$

Then for each \(j \in J\), set \(y_j = y_1\). We have natural isomorphisms

$$L^{-r}_{E} \otimes \omega_{E/S}(x_\infty + (1-k)y_1) \simeq (L'_E)^{-r} \otimes \omega_{E/S}(x_\infty + \sum_{j \in J, b_j \neq r} (1-b_j)y_j + \sum_{j \in J, b_j = r} y_j),$$

if \(k \neq r\), or

$$L^{-r}_{E} \otimes \omega_{E/S}(x_\infty + y_1) \simeq (L'_E)^{-r} \otimes \omega_{E/S}(x_\infty + \sum_{j \in J, b_j \neq r} (1-b_j)y_j + \sum_{j \in J, b_j = r} y_j),$$

if \(k = r\).

Let \(p'_E\) be the image of \(p_E\) under either isomorphism, depending on whether \(k = r\). We get a new family

$$\xi'_E = (C, \bar{\pi}_E, x_\infty; (y_j)_{j \in J}, N, L'_E, v_1, v_2, p'_E).$$

By Lemma 9, an \(S\)-point of \(\overline{M}_{g, \gamma}^{1/r} \times \widetilde{M}_{0, \gamma E}^{1/r}\) consists of \((\xi_j, \xi_E)\) as above, and an \(S\)-isomorphism between

$$\theta_j = (x_j, \pi_j|_{x_j}, L_j|_{x_j}, p_j|_{x_j}) \quad \text{and} \quad \theta_E = (x_\infty, \bar{\pi}_E|_{x_\infty}, L_E|_{x_\infty}, p_E|_{x_\infty}).$$

Since to get \(L'_E\) we have only modified \(L_E\) near \(y_1\), \(\theta_E\) is naturally isomorphic to

$$\theta'_E = (x_\infty, \bar{\pi}_E|_{x_\infty}, L'_E|_{x_\infty}, p'_E|_{x_\infty}).$$

We use the isomorphism \(\theta_j \simeq \theta'_E\) to glue \(\xi_j\) and \(\xi'_E\) along \(x_j\) and \(x_\infty\), and get

$$\xi = (C, \bar{\pi}, x_1, \cdots, x_m; y_1, \cdots, y_n, N, L, v_1, v_2, p) \in \widetilde{M}_{g, \gamma}^{1/r}(S).$$

This defines the morphism \(i_j\). For more about gluing stacks and their morphisms, see [2]. See also Section 2.3 in [10] on gluing the spin structures.

**Lemma 10.** The morphism \(i_j\) induces an isomorphism onto the substack \(F^{1/r}_j\).

**Proof.** An automorphism of \(\xi\) consists of an automorphism of \(\xi_j\) and automorphism of \(\xi'_E\) that respect the identification of \(\theta_j\) and \(\theta'_E\). Moreover, the automorphism groups of \(\xi'_E\) and \(\xi_E\) are naturally isomorphic. Hence the automorphisms of \(\xi\) are precisely the automorphisms of \((\xi_j, \xi_E, \theta_j \simeq \theta_E)\) in the fibre product of stacks. Hence the morphism \(i_j\) induces an isomorphisms of automorphism groups.

It is easy to see that \(i_j\) induces a bijection of closed points onto \(F^{1/r}_j\). Note that \(F^{1/r}_j\) is smooth since it is a fixed-point component in a smooth stack. Since \(i_j\) is representable and proper, it must be an isomorphism onto \(F^{1/r}_j\). \(\square\)
We now compute the equivariant normal bundle of \( F_{J}^{1/r} \). Recall that (9) defines the \( \mathbb{C}^* \)-action on \( \tilde{M}_{0,\gamma}^{1/r} \)

\[
\lambda \cdot (E, \pi_E, x_{\infty}; y_1, L, N, v_1, v_2, p) = (E, \pi_E, x_{\infty}; y_1, L, N, \lambda v_1, v_2, p), \ \lambda \in \mathbb{C}^*.
\]

This also defines an action on the universal curve \( \mathcal{E} \) over \( \tilde{M}_{0,\gamma}^{1/r} \). Moreover, the map \( \text{pr}_2^* \mathcal{E} \to \mathcal{C} \) induced by \( \iota_* \) is equivariant, where \( \mathcal{C} \) is the universal curve over \( \tilde{M}_{g,\gamma}^{1/r} \), and \( \text{pr}_2: \tilde{M}_{g,\gamma}^{1/r} \times \tilde{M}_{0,\gamma}^{1/r} \to \tilde{M}_{0,\gamma}^{1/r} \) is the second projection.

We now describe the \( \mathbb{C}^* \)-action on \( \mathcal{E} \). Note that \( \tilde{M}_{0,\gamma}^{1/r} \simeq B\mu_r \) and the \( \mathbb{C}^* \)-action on \( \tilde{M}_{0,\gamma}^{1/r} \) is trivial. The stability condition requires \( v_2 \) to be non-vanishing. Hence \((N, v_1, v_2)\) is equivalent to a tangent vector \( v_1/v_2 \) at \( y_1 \). Let \( z \) be the coordinate on the coarse moduli \( |\mathcal{E}| \) of \( \mathcal{E} \) so that \( y_1 = z = 0 \) and \( x_{\infty} = z = \infty \). Then the induced action on \( |\mathcal{E}| \) is given by

\[
\lambda \cdot z = \lambda^{-1} z, \ \lambda \in \mathbb{C}^*.
\]

Let \( T_{x,J} \) (resp. \( T_{x,\infty} \)) be line bundle on \( \tilde{M}_{g,\gamma}^{1/r} \) (resp. \( \tilde{M}_{0,\gamma}^{1/r} \)) formed by the tangent spaces of the coarse curves along the marking \( y \) (resp. \( x_{\infty} \)). Recall that for an integer \( w \), \( C_w \) is the standard \( \mathbb{C} \) with the \( \mathbb{C}^* \)-action of weight \( w \).

**Proposition 11.** The normal bundle of \( F_{J}^{1/r} \) in \( \tilde{M}_{g,\gamma}^{1/r} \) is isomorphic to

\[
N_{\text{node}} \oplus (O_{E_{J}^{1/r}} \otimes \mathbb{C}_{-1})^{\oplus(|J|-1)}.
\]

where \( N_{\text{node}} \) is a line bundle such that \( N_{\text{node}}^* \simeq T_{x,J} \boxtimes T_{x,\infty} \), and \(|J|\) is the cardinality of \( J \).

**Proof.** As in the proof of Proposition 2.1.1 in [3], the deformation and obstruction of stable \( r \)-spin curves with mixed \( \gamma \)-weighted markings are identical to those of the underlying twisted curves with mixed weighted markings. There is no obstruction in our case since \( \tilde{M}_{g,\gamma}^{1/r} \) is smooth. The deformation of \( \xi \in \tilde{M}_{g,\gamma}^{1/r} \) consists of two parts: the deformation of the underlying twisted curves and the deformation of the “tangent vector” \((v_1, v_2, N)\) (cf. Definition 4). For \( \xi \in F_{J}^{1/r} \) the underlying twisted curve has infinitesimal automorphisms that “cancel” with the deformation of \((v_1, v_2, N)\).

We first study the underlying twisted curves. Let \( M_{tw} \) be the Artin stack of twisted curves with balanced nodes and not necessarily distinct markings \( x_1, \ldots, x_m, y_1, \ldots, y_n \). Let \( \mathfrak{Z}_{tw} \subset M_{tw} \) be the closed substack where the curve has a node separating \( \{y_j\}_{j \in J} \) and \( \{x_1, \ldots, x_m\} \cup \{y_j\}_{j \notin J} \), and \( M_{tw}^{t} \subset M_{tw} \) be the closed substack where \( y_j \) is equal to \( y_1 \) for all \( j \in J \). Let \( \mathfrak{Z}_{tw}^{t} = \mathfrak{Z}_{tw} \times_{M_{tw}} M_{tw}^{t} \). Let \( M, \mathfrak{Z}, \mathfrak{Z}^{t} \) be the similarly defined moduli spaces of non-orbifold curves.

The local structure of \( M_{tw} \) was studied in [27]. The versal deformation of a twisted curve is in Remark 1.11 of [27]. It follows that the substacks \( M_{tw}^{t} \) and \( \mathfrak{Z}_{tw}^{t} \) intersect transversely along \( \mathfrak{Z}_{tw}^{t} \). Hence we have an isomorphism of normal bundles on \( \mathfrak{Z}_{tw}^{t} \)

\[
N_{\mathfrak{Z}_{tw}^{t}/M_{tw}^{t}} \simeq N_{\mathfrak{Z}_{tw}/M_{tw}} \oplus N_{M_{tw}^{t}/M_{tw}}.
\]

For simplicity, here (and after) we have suppressed the pullback notations.

We want to describe those normal bundles in terms of line bundles on the moduli of coarse curves. From the description of the versal deformation, we see that under
the forgetful morphism $\mathcal{M}^{tw} \to \mathcal{M}$, the smooth divisor $\mathfrak{z}$ pulls back to $\mathfrak{z}'^{tw}$. Let $\mathcal{N}_{\text{node}}$ be $\mathcal{N}_{\mathfrak{z}^{tw}/\mathcal{M}^{tw}}$, then we have an isomorphism on $\mathfrak{z}^{tw}$

\[(\mathcal{N}_{\text{node}})^{r'} \simeq \mathcal{N}_{\mathfrak{z}/\mathcal{M}}.\]

Similarly, we have

\[\mathcal{N}_{\mathfrak{z}^{tw}/\mathcal{M}^{tw}} \simeq T_{y_1}(J)-1,\]

where $T_{y_1}$ is the line bundle formed by the tangent spaces of the curves at $y_1$.

We claim that the normal bundle of $F^{1/r}_{1}$ is the pullback of $\mathcal{N}_{\mathfrak{z}^{tw}/\mathcal{M}^{tw}}$ via the forgetful morphism. We look at the $\mathbb{C}^*$-invariant open substack $U \subset \tilde{\mathcal{M}}^{1/r}_{g,m|n}$ of $\xi$ where $v_1 \neq 0$ and $v_2 \neq 0$. It contains $F^{1/r}_{1}$. Over $U$ the data $(N, v_1, v_2)$ is equivalent to $v_1/v_2$, a nonzero tangent vector at the light marking $y_1$. Since the $r$-spin structure has trivial deformation and obstruction, the forgetful morphisms $U \to \mathcal{M}^{tw}$ and $F^{1/r}_{1} \to \mathfrak{z}'^{tw}$ induce a distinguished triangle on $U$

\[(12) \quad T_{y_1} \to T_U \to T_{\mathcal{M}^{tw}} \xrightarrow{+1},\]

and a distinguished triangle on $F^{1/r}_{1}$

\[(13) \quad T_{y_1} \to T_{F^{1/r}_{1}} \to T_{\mathfrak{z}^{tw}} \xrightarrow{+1},\]

where $T$ means the tangent bundle and $T$ means the tangent complex. The distinguished triangle $(13)$ naturally maps to the restriction of $(12)$ to $F^{1/r}_{1}$. Taking the cones gives us an isomorphism $\mathcal{N}_{F^{1/r}_{1}/U} \simeq \mathcal{N}_{\mathfrak{z}^{tw}/\mathcal{M}^{tw}}$ on $F^{1/r}_{1}$.

To complete the proof, observe that on $F^{1/r}_{1}$ we have

\[\mathcal{N}_{\mathfrak{z}/\mathcal{M}} \simeq T_{x_{F^{1/r}_{1}}} \otimes T_{x_{\infty}} \quad \text{and} \quad T_{y_1} \simeq O_{F^{1/r}_{1}} \otimes \mathbb{C}_{-1}.\]

To see the latter, notice that $y_1$ comes from the factor $\tilde{\mathcal{M}}^{1/r}_{0,\gamma \xi} \simeq B\mu_{r}$, and $\mu_{r}$ acts trivially on the underlying curves. Hence $T_{y_1}$ is a constant line bundle on $F^{1/r}_{1}$. The $\mathbb{C}^*$-action is given by the tangent map of $(11)$. Hence is has weight $-1$. $\square$

5. THE PERFECT OBSTRUCTION THEORY AND LOCALIZATION

In this section we first define the virtual cycle on the master space via introducing $\varphi$-fields. That is parallel to Section 2.2. Then we apply virtual localization to get the basic wall-crossing formula.

5.1. The $\varphi$-fields and equivariant perfect obstruction theory. As before, we fix non-negative integers $g, m, n$ such that $2g - 2 + m \geq 0$ and $n \geq 1$. We also fix

\[\gamma = \left(\frac{a_1}{r}, \ldots, \frac{a_m}{r}, \frac{b_1}{r}, \ldots, \frac{b_n}{r}\right), \quad a_i, b_j \in \{1, \ldots, r\}.\]

**Definition 5.** An $S$-family of stable $r$-spin curves with mixed $\gamma$-weighted markings and $\varphi$-fields consists of

\[\left(C, \pi, x_1, \cdots, x_m, y_1, \cdots, y_n, N, L, v_1, v_2, p, \varphi_1, \cdots, \varphi_s\right),\]

where

\[\left(C, \pi, x; y, N, L, v_1, v_2, p\right) \in \tilde{\mathcal{M}}^{1/r}_{g,\gamma}(S),\]
and for $\alpha = 1, \cdots, s$,
\[
\varphi_\alpha \in H^0(C, L(D_\alpha)), \quad D_\alpha = -\sum_{q_\alpha a_\alpha \in \mathbb{Z}} x_i + \sum_{b_j \neq r} [q_\alpha (b_j - 1)] y_j - \sum_{b_j = r} y_j.
\]

As in [7], the category of such families is a Deligne-Mumford stack $\tilde{M}_{\gamma, g, \gamma}^{1/r}$ of finite type over $\mathbb{C}$, and we have a representable forgetful morphism
\[
\tau : \tilde{M}_{\gamma, g, \gamma}^{1/r} \to \tilde{M}_{g, \gamma}^{1/r}.
\]

We now define the perfect obstruction theory on $\tilde{M}_{\gamma, g, \gamma}^{1/r}$. We abbreviate $\tilde{M}_{1/r} = \tilde{M}_{g, \gamma}^{1/r}$ and $\tilde{M}_\varphi = \tilde{M}_{\gamma, g, \gamma}^{1/r}$.

Let $\pi : C \to \tilde{M}_\varphi$ be the universal curve, $\omega_\pi$ be the relative dualizing sheaf on $C$ and $L$ be the universal $r$-spin bundle. By abuse of notation, we will use $x_i, y_j$ and $D_\alpha$ to denote the divisors on $C$ as in Definition 5.

As in [7], $\tau$ has an equivariant relative perfect obstruction theory $E_\tau \to L_\tau$, where $L_\tau$ is relative cotangent complex of $\tau$ and $E_\tau = \bigoplus_{\alpha = 1}^s R\pi_* L^w_\varphi (D_\alpha)$.

It admits a cosection
\[
\sigma : h^1 (E_\tau^\vee) \to R^1 \pi_* \omega_\pi \simeq O_{\tilde{M}_\varphi}
\]
defined by
\[
\sigma(\dot{\varphi}_1, \cdots, \dot{\varphi}_s) = p \sum_{\alpha = 1}^s \dot{\varphi}_\alpha \partial_\alpha W(\varphi_1, \cdots, \varphi_s).
\]

Here $h^1$ means the cohomology sheaf in degree 1. The cosection is well-defined for the same reason as in Section 2.2.

Since $\tilde{M}_{1/r}$ is smooth, we get an absolute perfect obstruction theory on $\tilde{M}_\varphi$
\[
\mathbb{L}_{\tilde{M}_\varphi}^\vee \to E_\tilde{M}_\varphi^\vee
\]
which fits into the distinguished triangle
\[
\tau^* \mathcal{T}_{\tilde{M}_{1/r}} [-1] \to E_\tau^\vee \to E_{\tilde{M}_\varphi}^\vee \to 1.
\]

Taking cohomology sheaves, we have
\[
h^1 (E_\tau^\vee) \to h^1 (E_{\tilde{M}_\varphi}^\vee).
\]

**Lemma 12.** The cosection $\sigma$ factors through the absolute obstruction sheaf $h^1 (E_{\tilde{M}_\varphi}^\vee)$ and defines an equivariant cosection of the absolute obstruction theory.

**Proof.** The proof that $\sigma$ factors through the abstract obstruction sheaf is exactly the same as in [7]. Since the action is defined by scaling $v_1$ but the cosection $\sigma$ is independent of $v_1$, $\sigma$ is equivariant (cf. [8, Lemma 2.10]).

As in [7], Serre duality implies that the degeneracy locus $D(\sigma)$ is equal to $\tilde{M}_{g, \gamma}^{1/r}$, viewed as a closed subset of $\tilde{M}_\varphi$ where all the $\varphi$-fields are identically zero. By [21] we have a cosection localized equivariant virtual fundamental class
\[
[\tilde{M}_\varphi]_{\text{vir}}^{\text{loc}} \in A^*_{\delta(\gamma)} (\tilde{M}_{1/r}),
\]
where the virtual dimension
\[
\delta(\gamma) = (3 - s + 2q)(g - 1) + m + n + 1 - \sum_{a=1}^{s} \left( \sum_{i=1}^{m} q_{a}(a_{i} - 1) \right) - \sum_{j=1}^{n} q_{a}(b_{j} - 1))
\]
can be computed by the orbifold Riemann-Roch formula.

5.2. Virtual localization on $\tilde{M}_{g,\gamma}^{1/r,\varphi}$. The $\mathbb{C}^{*}$-action (9) on $\tilde{M}_{g,\gamma}^{1/r}$ lifts to $\tilde{M}_{g,\gamma}^{1/r,\varphi}$ by scaling the $v_{1}$ of $\tilde{M}_{g,\gamma}^{1/r,\varphi}$. The forgetful morphism $\tau : \tilde{M}_{g,\gamma}^{1/r,\varphi} \to \tilde{M}_{g,\gamma}^{1/r}$ is equivariant.

Lemma 13. The fixed-point components of $\tilde{M}_{g,\gamma}^{1/r,\varphi}$ are
\[
F_{\star}^{\varphi} := \tau^{-1}(F_{1/r}^{\varphi}), \quad \star = 0, \infty, J
\]
where $J$ runs over all subsets of $\{1, \ldots, n\}$ such that $\{1\} \not\subset J$.

Proof. Since $\tau$ is equivariant, it suffices to show that the $\mathbb{C}^{*}$-action on $F_{\star}^{\varphi}$ is trivial. For $J = 0, \infty$, it follows immediately from the definition of $F_{1/r}^{\varphi}$. For $\star = J$, the underlying curve $C$ can be written as $C_{J} \cup E$ where $E$ is the smooth rational subcurve containing $y_{1}$. The $\mathbb{C}^{*}$ acts trivially on $C_{J}$. For degree reasons, the $\varphi$-fields vanish on $E$. Hence the $\mathbb{C}^{*}$-action on $\tau^{-1}(F_{J}^{1/r})$ is trivial. □

By [17, 21], the restriction of $\mathbb{E}_{\tilde{M}_{g,\gamma}^{1/r,\varphi}}^{\gamma}$ to $F_{\star}^{\varphi}$ decomposes as the direct sum of its fixed part and moving part. The fixed part gives a perfect obstruction theory of $F_{\star}^{\varphi}$. The cosection $\sigma$ restricts to a cosection on $F_{\star}^{\varphi}$. This gives a cosection localized virtual fundamental class
\[
[F_{\star}^{\varphi}]_{\text{vir}}^{\text{loc}} \in A_{\star}(F_{1/r}^{\varphi}).
\]
The moving part is the virtual normal bundle $N^{\varphi}_{\star}$. Let $\iota_{\star} : F_{\star}^{1/r} \to \tilde{M}_{g,\gamma}^{1/r}$ be the inclusion. We have the virtual localization formula
\[
(15) \quad [\tilde{M}_{g,\gamma}^{1/r,\varphi}]_{\text{vir}}^{\text{loc}} = \sum_{\star} (\iota_{\star})_{\star} \left( \left( [F_{\star}^{\varphi}]_{\text{vir}}^{\text{loc}} \right) \right) \quad \text{in} \quad A_{\star}(\tilde{M}_{g,\gamma}^{1/r}) \otimes \mathbb{Q}[z, z^{-1}],
\]
and where $z \in A^{1}(BC^{*})$ is the first Chern class of $\mathbb{C}_{1}$, the standard $\mathbb{C}$ with weight-1 action.

We have a stabilization map
\[
(16) \quad \text{st} : \tilde{M}_{g,\gamma}^{1/r} \to \mathbb{M}_{g,\gamma}^{1/r}
\]
defined by forgetting the $(N, v_{1}, v_{2})$ and then contracting the unstable rational subcurves. To construct this map, one can use the alternative description of $r$-spin curves as balanced maps to Deligne-Mumford stacks in the proof of Theorem 6 (or in [3]). Then one can use Corollary 9.1.3 of [4].

The stabilization map st is $C^{*}$-equivariant, when $C^{*}$ acts on $\mathbb{M}_{g,\gamma}^{1/r}$ trivially. For any equivariant Chow cohomology class $\alpha \in A_{\star}^{\text{vir}}(\tilde{M}_{g,\gamma}^{1/r})$. We can cap both sides of (15) with $\alpha$ and then pushforward by the map st. This gives us an equation in
\[
A_{\star}^{\text{vir}}(\mathbb{M}_{g,\gamma}^{1/r}) \otimes \mathbb{Q}[z, z^{-1}] = A_{\star}(\mathbb{M}_{g,\gamma}^{1/r}) \otimes \mathbb{Q}[z, z^{-1}].
\]
Since $\text{st}_{\star}(\alpha \cap [\tilde{M}_{g,\gamma}^{1/r,\varphi}]_{\text{vir}}^{\text{loc}})$ lies in $A_{\star}^{\text{vir}}(\mathbb{M}_{g,\gamma}^{1/r}) = A_{\star}(\mathbb{M}_{g,\gamma}^{1/r}) \otimes \mathbb{Q}[z]$, we have
Proposition 14. For any $\alpha \in A_*^\circ(\overline{M}_{g,\gamma}^{1/r})$, the coefficients of negative degree powers of $z$ in
\begin{equation}
(17) \quad \text{st}_* \left( \sum (i_*)_* \left( (i_*)^* \alpha \cap [F^\varphi_0^*|^\vir_{\text{loc}}] \right) \right) \in A_*((\overline{M}_{g,\gamma}^{1/r}) \otimes \mathbb{Q}[z, z^{-1}])
\end{equation}
are zero.

5.3. The contribution from each fixed-point component. In this subsection we determine the contribution of each fixed-point component $F^\varphi_0$ to (17). We will use the distinguished triangle (14) to compute the fixed and moving parts of the restriction of $\widetilde{E}^\varphi_0$ to each $F^\varphi_0$.

We first consider $* = 0, \infty$. Let
\[ \gamma' = \left( \frac{a_1}{r}, \cdots, \frac{a_m}{r}, \frac{b_1}{r}, \cdots, \frac{b_n}{r} \right). \]
Parallel to Lemma 7, we have isomorphisms
\[ F^\varphi_0 \simeq \overline{M}^{1/r, \varphi}_{g,\gamma'} \quad \text{and} \quad F^\varphi_\infty \simeq \overline{M}^{1/r, \varphi}_{g,\gamma}. \]
The first isomorphism adds orbifold structure of index $r' = r / \gcd(r, k)$ along the $(m + 1)$-th marking $y_1$. By abuse of notation we will also use $st$ to denote the stabilization map $st : \overline{M}^{1/r}_{g,\gamma'} \to \overline{M}^{1/r}_{g,\gamma}$. This map is the restriction of the previous stabilization map (16) to $F^\varphi_0 \simeq \overline{M}^{1/r}_{g,\gamma'}$.

Lemma 15. Under the isomorphisms above, we have
\[ \frac{[F^\varphi_0^*|^\vir_{\text{loc}}]}{c_{\text{top}}^\varphi(\overline{N}^\varphi_0 )^{F^\varphi_0^*}]} = \frac{[\overline{M}^{1/r, \varphi}_{g,\gamma'}|^\vir_{\text{loc}}]}{z - \psi_{y_1}} \quad \text{and} \quad \frac{[F^\varphi_\infty^*|^\vir_{\text{loc}}]}{c_{\text{top}}^\varphi(\overline{N}^\varphi_\infty )^{F^\varphi_\infty^*}]} = \frac{[\overline{M}^{1/r, \varphi}_{g,\gamma}]|^\vir_{\text{loc}}}{-z + \psi_{y_1}} \]
where $\psi_{y_1}$ means the cotangent-line class at $y_1$ on the coarse curves.

Proof. For $* = 0, \infty$, we restrict the distinguished triangle (14) to $F^\varphi_0$. The complex $E^\varphi_i|_{F^\varphi_0^*}$ has only the fixed part, since the action on the $r$-spin curves is trivial over $F^\varphi_0$. This is identified with the relative obstruction theory of $\overline{M}^{1/r, \varphi}_{g,\gamma'}$ or $\overline{M}^{1/r, \varphi}_{g,\gamma}$, compatible with the cosection in Section 2.2. The fixed and moving part of $\tau^*T_{\overline{M}^{1/r}_{g,\gamma'}}|_{F^\varphi_0^*}$ is the pullback of the tangent and normal bundle of $F^\varphi_0^*$, respectively. The normal bundle of $F^\varphi_0^*$ in $\overline{M}^{1/r}_{g,\gamma}$ is computed in Lemma 7. Hence the lemma follows. \hfill $\Box$

We now come to $* = J$ for $\{1\} \not\subseteq J \subset \{1, \cdots, n\}$. Recall that
\[ F^\varphi_J^{1/r} \simeq \overline{M}^{1/r}_{g,\gamma_J, B_{\mu_{J'}}} \times \overline{M}^{1/r}_{0,\gamma_E}, \]
where $r' = r / \gcd(r, k)$. Since the $\varphi$-fields vanish on the rational tails for degree reasons, we have the following fibred diagram
\[ \begin{array}{ccc}
F^\varphi_J^{1/r} & \xrightarrow{\text{pr}_J^J} & \overline{M}^{1/r, \varphi}_{g,\gamma_J} \\
\downarrow & & \downarrow \\
F^\varphi_J^{1/r} & \xrightarrow{\text{pr}_J^1} & \overline{M}^{1/r}_{g,\gamma_J}
\end{array} \]
Define
\[ \ell_{\alpha,j} = \left\lfloor \sum_{j \in J} (q_\alpha(b_j - 1)) \right\rfloor \quad \text{and} \quad k_{\alpha,j} = q_\alpha + \left\lfloor \sum_{j \in J} q_\alpha(k - 1) \right\rfloor, \]
where \( \langle x \rangle = x - \lfloor x \rfloor \) denotes the fractional part of \( x \). Define
\[ \mu J(z) = \prod_{\alpha=1}^s [k_{\alpha,j}]_{\ell_{\alpha,j}} z^{1-|J|+\sum_\alpha \ell_{\alpha,j}}, \]
where \( [x]_n = x(x+1) \cdots (x+n-1) \). Let \( \mu J^+(z) \) be the truncation of \( \mu J(z) \) consisting of all non-negative powers of \( z \).

**Lemma 16.** We have
\[ \frac{[F_J^-]^{\vir}_{\text{loc}}}{C_{\text{top}}(N^J_{\ell}/F_I)} = \text{pr}_1^* \left( (-1)^{\sum_\alpha \ell_\alpha} \frac{\mu J^+(-z)}{z - \psi_{x,j}} \cap [\overline{M}_{g,\gamma_J}^{1/r,\varphi}]_{\text{vir}} \right), \]
where \( \psi_{x,j} \) is the cotangent-line class at the new heavy marking \( x,j \) on the coarse curves.

**Proof.** The distinguished triangle (14) restricts to a distinguished triangle on \( F_J^- \). We will compute the fixed and moving part of the first and third terms. The fixed parts will give us \( \text{pr}_1^*(\overline{M}_{g,\gamma_J}^{1/r,\varphi})_{\text{vir}} = [F_J^-]^{\vir}_{\text{loc}} \). To show this, by Proposition 7.5 of [6], it suffices to show that the fixed part of \( E^\vee_{F_J^-} \) is the pull back of the perfect obstruction theory on \( \overline{M}_{g,\gamma_J}^{1/r,\varphi} \) defined in Section 2.2, compatible with the cosections.

We introduce some notations. Let \( \mathcal{C} = \mathcal{C}_J \cup \mathcal{E} \to F_J^- \) be the pullback of the universal curve, where \( \mathcal{C}_J \) is the pullback of the universal curve over \( \overline{M}_{g,\gamma_J}^{1/r,\varphi} \) and \( \mathcal{E} \) is the pullback of the universal curve over \( \overline{M}_{0,\gamma_J}^{1/r,\varphi} \). Let \( \mathcal{L} \) be the restriction of the universal \( r \)-spin line bundle to \( \mathcal{C} \). Let \( x_i, y_j, D_\alpha \) denote the divisors on \( \mathcal{C} \) as in Definition 5. Recall that
\[ D_\alpha = -\sum_{q_\alpha a_i \in \mathbb{Z}} x_i + \sum_{b_j \neq r} (q_\alpha(b_j - 1)) y_j - \sum_{b_j = r} y_j. \]

We first study the relative perfect obstruction theory
\[ E^\vee_{F_J^-} \simeq R\pi_* (\oplus_{\alpha=1}^s \mathcal{L}^{w_\alpha}(D_\alpha)). \]
For each \( \alpha \), consider the short exact sequence
\[ 0 \to \mathcal{L}^{w_\alpha}(D_\alpha)_{|C_J}(-x,J) \to \mathcal{L}^{w_\alpha}(D_\alpha) \to \mathcal{L}^{w_\alpha}(D_\alpha)_{|E} \to 0 \]
where the first and third terms are understood as their pushforward to \( \mathcal{C} \). This induces the distinguished triangle
\[ R\pi_* (\mathcal{L}^{w_\alpha}(D_\alpha)_{|C_J}(-x,J)) \to R\pi_* (\mathcal{L}^{w_\alpha}(D_\alpha)) \to R\pi_* (\mathcal{L}^{w_\alpha}(D_\alpha)_{|E}) \xrightarrow{+1}. \]

We will compute \( R\pi_* (\mathcal{L}^{w_\alpha}(D_\alpha)_{|E}) \) and show that it is in the moving part. Since \( \mathcal{L}^{w_\alpha}(D_\alpha)_{|E} \) has negative fibre-wise degree, we have
\[ R\pi_* (\mathcal{L}^{w_\alpha}(D_\alpha)_{|E}) = R^1 \pi_* (\mathcal{L}^{w_\alpha}(D_\alpha)_{|E})[-1]. \]

---

1 Proposition 7.5 of [6] also works in our cosection localization setting. This is because \( \text{pr}_1 \) is flat and the intrinsic normal cone pulls back to the intrinsic normal cone.
The datum $E$, $L|_E$ and so on are pulled back from the universal curve over $\tilde{M}^{1/r,\varphi}_{g,\gamma,J} \cong B\mu_r$. Moreover we only care about Chow groups with rational coefficients, hence we can compute on a fixed smooth rational orbifold curve $E$, with $r$-spin bundle $L$.\footnote{The $\mathbb{C}^*$-action on $L$ is only well-defined up to $\mu_r$. Nevertheless we can compose the action with the $r$-th power map $\mathbb{C}^* \to \mathbb{C}^*$ so that it’s well-defined.} Fixing $E$ and $L$ amounts to pulling back everything along the degree-$r$ cover $\text{Spec } \mathbb{C} \to B\mu_r$. Let $x_\infty, y_1$ still denote the markings on $E$. Thus $E$ only has an orbifold point of index $r'$ at $x_\infty$. We choose the coordinate on the coarse moduli of $E$ such that $y_1$ is at 0 and $x_\infty$ is at $\infty$. Thus the action on $E$ is given by (11). We define a divisor on $E$

$$D_{\alpha,E} = \sum_{j \in I, b_j \neq r} [q_\alpha(b_j - 1)] y_1 - \sum_{j \in I, b_j = r} y_1.$$ 

Up to the degree-$r$ cover introduced above, the line bundle $L|_E$ is the pullback of $L$, and the divisor $D_{\alpha,E}$ is the pullback of $D_{\alpha,E}$. We have isomorphisms of equivariant line bundles

$$L' \cong \omega_E(x_\infty + \sum_{j \in I, b_j \neq r} (1-b_j)y_1 + \sum_{j \in I, b_j = r} y_1), \quad \omega_E \cong \mathcal{O}_E(-x_\infty - y_1).$$

Hence we have

$$(L^w_\alpha(D_{\alpha,E}))^{-1} \otimes \omega_E \cong \mathcal{O}_E(-r x_\infty + r(k_{\alpha,j} + \ell_{\alpha,j} - 1)y_1) \cong \mathcal{O}_E((r'k_{\alpha,j} - 1)x_\infty + (\ell_{\alpha,j} - 1)y_1).$$

Note that $\mathcal{O}_E(r_k_{\alpha,j}y_1 - r'r'k_{\alpha,j}x_\infty)$ is the trivial bundle with the $\mathbb{C}^*$-action of weight $r_k_{\alpha,j}$. The equivariant Picard group of $E$ has no torsion. Hence we have

$$(L^w_\alpha(D_{\alpha,E}))^{-1} \otimes \omega_E \cong \mathcal{O}_E((r'k_{\alpha,j} - 1)x_\infty + (\ell_{\alpha,j} - 1)y_1).$$

Since $0 \leq r'k_{\alpha,j} - 1 < r'$ and $x_\infty$ is an orbifold point of index $r'$, we can compute

$$H^1(E, \mathcal{L}^{w_\alpha}(D_E)) \cong \left( \left. H^0(E, (L^w_\alpha(D_E))^{-1} \otimes \omega_E) \right| \right) \cong \mathbb{C}_{k_{\alpha,j}}.$$ 

Hence it only has moving part and the equivariant top Chern class is

$$(18) \quad c^\alpha_{\text{top}} \left( R\pi_* (\mathcal{L}^{w_\alpha}(D_{\alpha,E}))^{\bullet} \right) = \left( z^{\ell_{\alpha,j}} [k_{\alpha,j}]_{\ell_{\alpha,j}} \right)^{-1}. $$

The term $R\pi_* (\mathcal{L}^{w_\alpha}(D_{\alpha,E})|_{C_J}(-x_J))$ is in the fixed part, since the $\mathbb{C}^*$-action on $C_J$ is trivial. We claim that it is the relative obstruction theory of $\tilde{M}^{1/r,\varphi}_{g,\gamma,J}$ in Section 2.2. Indeed, if $\mathcal{L}^{w_\alpha}$ has trivial monodromy at $x_J$ (i.e. $kq_\alpha \in \mathbb{Z}$), then by definition, the $\varphi$-fields of $\tilde{M}^{1/r,\varphi}_{g,\gamma,J}$ are sections of $\mathcal{L}^{w_\alpha}(D_{\alpha})|_{C_J}(-x_J)$; otherwise $\mathcal{L}^{w_\alpha}$ has nontrivial monodromy at $x_J$, and thus we have

$$R\pi_* (\mathcal{L}^{w_\alpha}(D_{\alpha})|_{C_J}(-x_J)) \cong R\pi_* (\mathcal{L}^{w_\alpha}(D_{\alpha})|_{C_J}).$$

It is easy to see that the isomorphism is compatible with the cosection. Then we consider the $\tau^r T\tilde{M}$ term of the distinguished triangle (14). By Lemma 13, the fixed part is $\tau^r T^{F_j/r}_{g,\gamma,J}$. Moreover, since $pr_1$ is étale, we have $T^{F_j/r}_{g,\gamma,J} \cong pr_1^* T^{F_j/r}_{g,\gamma,J}$. This finishes the proof that $pr_1^* \left( \left[ \tilde{M}^{1/r,\varphi}_{g,\gamma,J} \right]^{\bullet}_{\text{loc}} \right) = \left[ F_j^{\bullet} \right]^{\bullet}_{\text{loc}}$.\footnote{The $\mathbb{C}^*$-action on $L$ is only well-defined up to $\mu_r$. Nevertheless we can compose the action with the $r$-th power map $\mathbb{C}^* \to \mathbb{C}^*$ so that it’s well-defined.}
Similarly, the moving part of \( \tau^* T_M |_{F^j} \) is the pullback of the normal bundle of \( F^j_{1/r} \) described in Proposition 11. Its equivariant top Chern class is
\[
(19) \quad (z)^{|j|-1} (z - w_{x,j}).
\]
Putting (18) and (19) together, we get the desired formula for the virtual normal bundle.

**Remark 8.** In the case of twisted theories, the same lemma holds with \( \mu_j(z) \) replaced by
\[
\mu_j^G(z) = z^{1-|j|} \prod_{\alpha=1}^s \prod_{k=1}^{|j|} (kz - w_{\alpha,j}).
\]
The rest of the paper works for twisted theories with this new definition.

### 5.4. The basic wall-crossing formula.

To describe the contribution of \( F^j_{1/r} \) in terms of weighted FJRW invariants, we define a map
\[
(20) \quad \beta_j : \overline{M}^1_{g,\gamma,j} \rightarrow \overline{M}^1_{g,\gamma}.
\]
The map replaces the last heavy marking \( x_j \) by the set of light markings \( \{y_j\}_{j \in J} \). More precisely, consider any family \( \xi' = (C', \pi', x'_1, \ldots, x'_m, x_j; \{y'_j\}_{j \notin J, L', p'}) \) in \( \overline{M}^1_{g,\gamma,j}(S) \), let \( \rho : C' \rightarrow C \) be the partial coarse moduli only forgetting the orbifold structure at \( x_j \) and \( \pi : C \rightarrow S \) be the induced projection. Let \( x_i = \rho(x'_i) \) for \( i = 1, \ldots, m \), \( y_j = \rho(y'_j) \) for \( j \notin J \) and \( y_j = \rho(x_j) \) for \( j \in J \). Recall that \( r\ell + k = 1 + \sum_{j \notin J} (b_j - 1) \), where \( \ell \geq 0, 1 \leq k \leq r \) are integers. Let \( L = \rho_*(L')(cy_1) \), where
\[
(21) \quad c = \begin{cases} 
\ell - |\{ j \in J : b_j = r \}| & \text{if } k \neq r \\
\ell - |\{ j \in J : b_j = r \}| + 1 & \text{if } k = r.
\end{cases}
\]
Then we have a natural isomorphism
\[
\begin{align*}
&f_* \left( (L')^{-r} \otimes \omega_C \left( x_j + \sum_{i=1}^m x'_i + \sum_{j \notin J, b_j \neq r} (1 - b_j)y'_j + \sum_{b_j = r} y'_j \right) \right) \\
&\cong L^{-r} \otimes \omega_C \left( \sum_{i=1}^m x_i + \sum_{b_j \neq r} (1 - b_j)y_j + \sum_{b_j = r} y_j \right).
\end{align*}
\]
Thus we get a possibly unstable family \( \xi = (C, \pi, x_1, \ldots, x_m; y_1, \ldots, y_n, L, f_*(p')) \). The family \( \beta_j(\xi') \) is obtained by stabilizing \( \xi \).

From the concrete description of the maps, we have a commutative diagram
\[
\begin{array}{ccc}
F^1_{1/r} & \overset{\text{pr}_1}{\longrightarrow} & \overline{M}^1_{g,\gamma,j} \\
\downarrow & & \downarrow \text{st} \\
\overline{M}^1_{g,\gamma,j} & \underset{\beta_j}{\longrightarrow} & \overline{M}^1_{g,\gamma}
\end{array}
\]
Now recall the constant \( \epsilon_\gamma \) defined in (4). Let \( d_1, \ldots, d_n \) be any non-negative integers. We use the short-hand notation \( \Sigma d_j \) for \( \sum_{j \in J} d_j \). We can use the projection formula to put Proposition 14, Lemma 15 and Lemma 16 together:
Corollary 17.

\[
\prod_{j=1}^{n} \psi_{y_j}^{d_j} \cap \epsilon_\gamma \left[ \overline{M}_{g,\gamma}^{1/r,\varphi} \right]_{\text{vir}}^\text{loc} - \text{st}_* \left( \prod_{j=1}^{n} \psi_{y_j}^{d_j} \cap \epsilon_\gamma \left[ \overline{M}_{g,\gamma}^{1/r,\varphi} \right]_{\text{vir}}^\text{loc} \right) = \sum_{J} \beta_J \left( \mu_J^+(-\psi_{x_J}) \psi_{x_J}^{\Sigma_J} \prod_{j \notin J} \psi_{y_j}^{d_j} \cap \epsilon_\gamma \left[ \overline{M}_{g,\gamma_J}^{1/r,\varphi} \right]_{\text{vir}}^\text{loc} \right),
\]

where the summation is over all \( J \subset \{1, \cdots, n\} \) such that \( \{1\} \not\subset J \); the \( \psi_{y_j} \) are the cotangent-line classes on the coarse curves.

Proof. The equivariant \( \psi \)-classes on the master space \( \overline{M}_{g,\gamma}^{1/r} \) restrict to the non-equivariant \( \psi \)-classes on the fixed-point components, except for the following cases: for each \( j \in J \), the class \( \psi_{y_j} \) on \( \overline{M}_{g,\gamma}^{1/r} \) restricts to \( (-z) \) on \( F_j^{1/r} \). The corollary then follows from taking \( \alpha = \prod_{j=1}^{n} \psi_{y_j}^{d_j} \), then taking the coefficient of \( z^{-1} \)-term in (17). Note that the map \( \text{pr}_1 \) is finite flat of degree \( 1/r' \). This cancels with the factor \( r' \) in Lemma 16. For the constants we have \( \epsilon_\gamma = \epsilon_{\gamma'} \) and \( (-1)^{\Sigma_\alpha} \epsilon_{\gamma_J} = \epsilon_\gamma \). \( \square \)

We now rewrite the lemma for later use. We make the convention that when \( J = \{1\} \), we have \( \gamma_J = \gamma' \), \( \beta_J = \text{st}, \mu_J(z) = 1 \), and \( x_J = y_1 \), then Corollary 17 can be rewritten as

\[
\prod_{j=1}^{n} \psi_{y_j}^{d_j} \cap \epsilon_\gamma \left[ \overline{M}_{g,\gamma_J}^{1/r,\varphi} \right]_{\text{vir}}^\text{loc} = \sum_{J} \beta_J \left( \mu_J^+(-\psi_{x_J}) \psi_{x_J}^{\Sigma_J} \prod_{j \notin J} \psi_{y_j}^{d_j} \cap \epsilon_\gamma \left[ \overline{M}_{g,\gamma_J}^{1/r,\varphi} \right]_{\text{vir}}^\text{loc} \right),
\]

where the summation is over all \( 1 \in J \subset \{1, \cdots, n\} \).

6. The wall-crossing formulas

6.1. The Chow version of wall-crossing formulas. In this subsection, we consider

\[
\gamma^- = \left( \frac{a_1}{r}, \cdots, \frac{a_m}{r}, \frac{b_1}{r}, \cdots, \frac{b_h+1}{r}, \cdots, \frac{b_n}{r} \right)
\]

and

\[
\gamma^+ = \left( \frac{a_1}{r}, \cdots, \frac{a_m}{r}, \frac{b_1}{r}, \cdots, \frac{b_h}{r}, \cdots, \frac{b_n}{r} \right).
\]

Assume that \( h \neq 0 \), \( 2g-2+m \geq 0 \) and \( a_i, b_j \in \{1, \cdots, r\} \) as before. In both situations we denote the markings by

\[
(x_1, \cdots, x_m, y_1, \cdots, y_h, y_{h+1}, \cdots, y_n).
\]

As before we have a stabilization map

\[
\text{st} : \overline{M}_{g,\gamma^+}^{1/r} \rightarrow \overline{M}_{g,\gamma^-}^{1/r}.
\]

Moreover we can generalize \( \beta_J \) as follows. For any collection \( J \subset 2 \{1, \cdots, n\} \) of disjoint subsets of \( \{1, \cdots, n\} \), we denote \( \bigcup_{J \in \mathcal{J}} J \) by \( \cup \mathcal{J} \), and define

\[
\gamma_{\mathcal{J}} = \left( \frac{a_1}{r}, \cdots, \frac{a_m}{r}, \frac{k_J}{r} \right)_{J \in \mathcal{J}} \left( \frac{b_J}{r} \right)_{J \in \mathcal{J}'},
\]

where \( \mathcal{J}' \) is the complement of \( \mathcal{J} \) in \( \{1, \cdots, n\} \).
where \( k_j \) is the integer such that

\[
1 \leq k_j \leq r \quad \text{and} \quad k_j - 1 \equiv \sum_{j \in J} (b_j - 1) \mod r.
\]

We denote the \( J \)-th new heavy marking by \( x_J \) and form a map

\[
\beta_J : \overline{M}^{1/r}_{g, \gamma, J} \rightarrow \overline{M}^{1/r}_{g, \gamma, J}.
\]

The definition of the map \( \beta_J \) is similar to that of \( \beta_J \) in (20). For each \( J \in \mathcal{J} \), we replace the heavy marking \( x_J \) by the light markings \( \{y_j\}_{j \in J} \). Then we modify the line bundle as in (20), successively for each \( J \in \mathcal{J} \). Finally we stabilize the family by contracting all unstable rational subcurves.

For various \( \mathcal{J} \), the maps \( \beta_J \) are compatible with each other in the following sense. Suppose \( \mathcal{J} \) is the disjoint union of \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \). If we view \( \mathcal{J}_1 \) as a collection of pairwise disjoint subsets of \( \{1, \cdots, n\} \setminus \mathcal{J}_2 \), then we can compose \( \beta_{\mathcal{J}_1} \) and \( \beta_{\mathcal{J}_2} \).

Hence \( \beta_{\mathcal{J}} \) is the composition of a sequence of \( \beta_J \).

We now have the wall-crossing formula relating \( \gamma^+ \) and \( \gamma^- \). For any integers \( d_1, \cdots, d_n \geq 0 \), recall that \( \Sigma d_J = \sum_{j \in J} d_j \).

**Theorem 18 (Main Theorem).**

\[
\prod_{j=1}^n \psi_{d_j}^{j, \gamma} \cap \epsilon_{\gamma^-} \left[ \overline{M}^{1/r}_{g, \gamma, J} \right]_{\text{loc}}^{\text{vir}} = \sum_{\mathcal{J}} \beta_J \left( \left( \prod_{j \in \mathcal{J}} \psi_{d_j}^{j, \gamma} \right) \left( \prod_{J \in \mathcal{J}} \mu_J^+ (-\psi_{x_J}) \psi_{x_J}^{\Sigma d_J} \right) \cap \epsilon_{\gamma_J} \left[ \overline{M}^{1/r}_{g, \gamma, J} \right]_{\text{loc}}^{\text{vir}}, \right)
\]

where \( \mathcal{J} \) runs over all subsets of the power set \( 2^{\{1, \cdots, n\}} \) such that

1. the sets \( J \in \mathcal{J} \) are disjoint from each other,
2. \( \{1, \cdots, h\} \subset \cup \mathcal{J} \),
3. for each \( J \in \mathcal{J}, J \cap \{1, \cdots, h\} \neq \emptyset \).

**Proof.** We prove this by induction on \( h \). We will suppress the pushforward notations and everything will be pushed forward to \( \overline{M}^{1/r}_{g, \gamma} \).

The case \( h = 1 \) is Corollary 17, reformulated as (22). We now assume that \( h > 1 \) and the theorem is true for the wall-crossing between \( \gamma^- \) and

\[
\gamma_0 = \left( \frac{a_1}{r}, \cdots, \frac{a_m}{r}, \frac{b_1}{r}, \cdots, \frac{b_{h-1}}{r}, \frac{b_h}{r}, \frac{b_{h+1}}{r}, \cdots, \frac{b_n}{r} \right).
\]

This gives

(23)

\[
\prod_{j=1}^n \psi_{d_j}^{j, \gamma} \cap \epsilon_{\gamma^-} \left[ \overline{M}^{1/r}_{g, \gamma, J} \right]_{\text{loc}}^{\text{vir}} = \sum_{J_1} \prod_{j \in \cup \mathcal{J}_1} \psi_{d_j}^{j, \gamma} \prod_{J \in \mathcal{J}_1} \left( \mu_J^+ (-\psi_{x_J}) \psi_{x_J}^{\Sigma d_J} \right) \cap \epsilon_{\gamma_{J_1}} \left[ \overline{M}^{1/r}_{g, \gamma_{J_1}} \right]_{\text{loc}}^{\text{vir}},
\]

where \( \mathcal{J}_1 \subset 2^{\{1, \cdots, n\}} \) runs over all collections of pairwise disjoint subsets such that

\[
\{1, \cdots, h-1\} \subset \cup \mathcal{J}_1 \quad \text{and} \quad J \cap \{1, \cdots, h-1\} \neq \emptyset, \forall J \in \mathcal{J}_1.
\]
For each $\mathcal{J}_i$ such that $h \not\in \cup \mathcal{J}_i$, $\frac{1}{r}M_{g,\gamma,\mathcal{J}_i}$ has $y_h$ as a light marking. We can apply (22) to replace it with a heavy marking. This gives

$$\prod_{j \not\in \cup \mathcal{J}_i} \psi_{y_j}^d \prod_{j \in \mathcal{J}_i} (\mu_j^+(-\psi_{x_j})\psi_{x_j}^{\Sigma d_j}) \cap \epsilon_{\gamma,\mathcal{J}_i} \left[ \frac{1}{r}M_{g,\gamma,\mathcal{J}_i} \right]_{\text{vir}}$$

$$= \sum_{J_2} \left( \prod_{j \not\in \cup \mathcal{J}_i} \psi_{y_j}^d \right) \left( \mu_{J_2}^+(-\psi_{x_{J_2}})\psi_{x_{J_2}}^{\Sigma d_{J_2}} \right) \prod_{j \in \mathcal{J}_i} (\mu_j^+(-\psi_{x_j})\psi_{x_j}^{\Sigma d_j}) \cap \epsilon_{\gamma,\mathcal{J}_i} \left[ \frac{1}{r}M_{g,\gamma,\mathcal{J}_2} \right]_{\text{vir}},$$

where $J_2 = \mathcal{J}_1 \cup \{J_2\}$ and $J_2$ runs over all subsets of $\{1, \ldots, n\} \setminus (\cup \mathcal{J}_i)$ such that $h \in J_2$. Since $\mathcal{J}_i$ and $J_2$ exactly run over all $\mathcal{J}$ in the statement of the theorem, the proof is complete.

6.2. The invariants and potentials. In this subsection we prove the wall-crossing formula for the generating functions of weighted FJRW invariants. Let $2g - 2 + m \geq 0$, $(2g - 2 + m, n) \neq (0, 0)$ as before and consider

$$\gamma^- = \left( \frac{a_1}{r}, \ldots, \frac{a_m}{r}, \frac{b_1}{r}, \ldots, \frac{b_n}{r}, \emptyset \right) \quad \text{and} \quad \gamma^+ = \left( \frac{a_1}{r}, \ldots, \frac{a_m}{r}, \frac{b_1}{r}, \ldots, \frac{b_n}{r}, \emptyset \right).$$

Theorem 18 implies

**Corollary 19.**

$$\langle \psi^{c_1} \phi_{a_1}, \ldots, \psi^{c_m} \phi_{a_m}, \psi^{d_1} \phi_{b_1}, \ldots, \psi^{d_n} \phi_{b_n} \rangle^0_{g,m|n} =$$

$$\sum_{h=1}^\infty \frac{1}{h!} \sum_{J_1, \ldots, J_h} \left( \psi^{c_1} \phi_{a_1}, \ldots, \psi^{c_m} \phi_{a_m}, \mu_h^+(-\psi)\psi^{\Sigma d_{J_1}} \phi_{k_{J_1}}, \ldots, \mu_h^+(-\psi)\psi^{\Sigma d_{J_h}} \phi_{k_{J_h}} \right)^\infty_{g,m+h},$$

where the sequence $(J_1, \ldots, J_h)$ runs over all length-$h$ partitions of $\{1, \ldots, n\}$.

**Proof.** The $\psi$-classes at the have markings pullback along the maps $\beta_j$. The corollary is proved by capping both sides of the equation in Theorem 18 by $\psi^{c_1} \cdots \psi^{c_m}$ and then taking the degrees of the classes on both sides. In Theorem 18, the $\mathcal{J}$ is a set; while here $(J_1, \ldots, J_h)$ is a sequence. This accounts for the factor $1/h!$. $\square$

We can package these formulas into generating functions. We only consider primary insertions at the light markings. Recall that we have defined

$$F^0_g(u, t) = \sum_{m,n \geq 0} \frac{1}{m!n!} (u^m | t^n) \langle 0^0 \rangle_{g,m|n} \quad \text{and} \quad F^\infty_g(u) = F^0_g(u, 0),$$

where

$$u = u_0 + u_1 \psi + u_2 \psi^2 + \cdots \quad \text{and} \quad t = \sum_j t_j \phi_j, \quad \text{where} \quad u_i = \sum_{j=1}^r u_{ij} \phi_j.$$

The unstable terms are defined to be zero.

We have also defined

$$\mu(t, z) = \sum_{n=1}^\infty \sum_{B_n} \frac{t_{b_1} \cdots t_{b_n}}{n!} \mu_{B_n}^+(z) \phi_{k_{B_n}}.$$ 

Some explanation of the notations is in order. When we have fixed a sequence of positive integers $B = (b_1, \ldots, b_n), b_j \in \{1, \ldots, r\}$ and an index set $J \subset \{1, \ldots, n\}$, we define $k_J$ and $\ell_J$ to be the integers such that $\ell_J \geq 0, 1 \leq k_J \leq r$ and $r \ell_J +
\[ k_J = 1 + \sum_{j \in J} (b_j - 1). \] When we simply write \( k_B \) and \( \ell_B \), we mean \( k_J, \ell_J \) with \( J = \{1, \ldots, n\} \). The same rule applies to \( \mu_J(z) \) and \( \mu_B(z) \).

Now we are ready to prove the numerical wall-crossing formula (Theorem 1)

\[ \mathcal{F}_g^0(u, t) = \mathcal{F}_g^\infty(u + \mu^+(t, -\psi)). \]

When \( g = 0 \), this is only true modulo linear terms in the \( \{u_{ij}\} \).

**Proof of Theorem 1.** We fix some \( n \geq 0 \) and \( b_1, \ldots, b_n \) in \( \{1, \ldots, r\} \). We compare the coefficients of \( t_{b_1} \cdots t_{b_n} \) on both sides of (24). For each \( m \geq 0 \), we looked at the \( \{u_{ij}\} \)-degree-\( m \) part. Assume \( m \geq 2 \) when \( g = 0 \) and \( m \geq 1 \) when \( g = 1 \). The left hand side of (24) gives

\[ \frac{1}{|\text{Aut}(b_1, \ldots, b_n)|} \left\langle \frac{1}{u^{m}} | \phi_{b_1}, \ldots, \phi_{b_n} \right\rangle_{g, m | n}, \]

where \( \text{Aut}(b_1, \ldots, b_n) \) is the group of permutations of \( \{1, \ldots, n\} \) that fix \( (b_1, \ldots, b_n) \).

By Corollary 19, this equals to

\[ \frac{1}{|\text{Aut}(b_1, \ldots, b_n)|} \sum_{h=1}^{\infty} \frac{1}{h!} \sum_{J_1, \ldots, J_h} \left\langle \frac{u^{m}}{J_1!}, \ldots, \frac{u^{m}}{J_h!} \right\rangle_{g, J_1, \ldots, J_h} \]

\[ = \frac{1}{|\text{Aut}(b_1, \ldots, b_n)|} \sum_{h=1}^{\infty} \frac{1}{h!} \sum_{J'_1, \ldots, J'_h} \left\langle \frac{u^{m}}{|J_1'|!}, \ldots, \frac{u^{m}}{|J_h'|!} \right\rangle_{g, J'_1, \ldots, J'_h}, \]

where \( (J_1, \ldots, J_h) \) runs over all partitions of \( \{1, \ldots, n\} \) of length \( h \). The \( (J'_1, \ldots, J'_h) \) also runs over all partitions of \( \{1, \ldots, n\} \) of length \( h \) but each \( J'_i \) is viewed as a sequence.

The right hand side of (24) gives

\[ \sum_{h=1}^{\infty} \frac{1}{h!} \sum_{B_1, \ldots, B_h} \left\langle \frac{u^{m}}{|B_1|!}, \ldots, \frac{u^{m}}{|B_h|!} \right\rangle_{g, B_1, \ldots, B_h}, \]

where the \( (B_1, \ldots, B_h) \) runs over all sequences of numbers in \( \{1, \ldots, r\} \) such that the juxtaposition of \( B_1, \ldots, B_h \) is equal to \( (b_1, \ldots, b_n) \) up to permutation.

The proof is now complete by noticing that given \( B_1, \ldots, B_h \), there are exactly \( |\text{Aut}(b_1, \ldots, b_n)| \) many choices of \( J'_1, \ldots, J'_h \) such that the sequence \( B_i \) is equal to the sequence \( (b_j)_{j \in J'_j} \) for all \( i = 1, \ldots, h \).

**Remark 9.** Since the invariants with broad heavy insertions vanish (Lemma 3), the generating function \( \mathcal{F}_g^0(u, t) \) is independent of \( u_{ij} \) unless

\[ j \alpha \notin \mathbb{Z}, \text{ for all } \alpha = 1, \ldots, s. \]

In \( \mathcal{F}_g^\infty(u + \mu^+(t, -\psi)) \), the invariants involving broad \( \phi_{kB_n} \) will be zero. Hence we can redefine

\[ k_{\alpha, B_n} := \left\langle \frac{1}{q_\alpha k_{B_n}} \right\rangle, \]

so that in \( \mu(t, z) \), the coefficients of narrow \( \phi_{kB_n} \) are unchanged and the coefficients of broad \( \phi_{kB_n} \) become zero. Thus the wall-crossing formula remains unchanged.
7. THE GENUS-0 WALL-CROSSING FORMULAS

7.1. The moduli space and virtual localization. In this section we consider the case \( g = 0 \) with one heavy marking and \( n \) light markings. We assume that \( n \geq 2 \) and consider
\[
\gamma = \left( \frac{a}{r}, \frac{b_1}{r}, \ldots, \frac{b_n}{r} \right).
\]
In this case, the moduli space \( \overline{M}_{0,\gamma}^{1/r} \) does not exist. However the master space \( \widetilde{M}_{0,\gamma}^{1/r} \) exists if and only if
\[
-2 + (1 - a) + \sum_{b_j \neq r} (1 - b_j) + \sum_{b_j = r} 1 \equiv 0 \mod r.
\]

Theorem 6 applies in this case. The perfect obstruction theory and localization also work. Compared to the list of fixed-point components in Section 4.2, we have a slightly different list here:

1. \( F_0^{1/r} = \{ (\xi : v_1 = 0) \} \) remains the same;
2. There is no analogue of \( F_{\infty}^{1/r} \). Indeed \( v_2 \) never vanishes.
3. For each \( \{1\} \notin J \subset \{1, \ldots, n\} \), we have
   - if \( J \neq \{1, \ldots, n\} \), \( F_j^{1/r} \) remains the same;
   - if \( J = \{1, \ldots, n\} \), \( F_j^{1/r} \) consists of \( (C, \pi, x_1; y, N, L, v_1, v_2, p) \), where \( \pi : C \to S \) is smooth; \( y_j = y_1 \) for all \( j = 1, \ldots, n \); \( v_1, v_2 \) are both non-vanishing.

The description of \( F_j^{1/r} \) (Lemma 7) and \( F_{\infty}^{1/r}, J \neq \{1, \ldots, n\} \) (Lemma 10) still holds. For \( J = \{1, \ldots, n\} \), it is easy to see that \( F_j^{1/r} \simeq B\mu_r \).

The \( \varphi \)-fields identically vanish in the genus-0 case for degree reasons. The proof is the same as that of Lemma 1.5 of [29]. Hence we have \( \overline{M}_{0,\gamma}^{1/r, e} = \widetilde{M}_{0,\gamma}^{1/r} \). We still have the localization formula (15). Let \( \mathcal{N}_r^\varphi \) be the virtual normal bundle of \( F_r^\varphi \). For \( \star = 0 \) or \( \star = J \neq \{1, \ldots, n\} \), the \( \mathcal{N}_r^\varphi \) are the same as those in Lemma 15 and Lemma 16.

Lemma 20. For \( J = \{1, \ldots, n\} \), we have
\[
\frac{1}{c_{\text{top}}(\mathcal{N}_r^\varphi)} = (-1)^{\sum \ell_\alpha, j} \mu_B(-z).
\]

Proof. The computation of the normal bundle is almost the same as that in Lemma 16. The relative obstruction theory is the same. The moving part of \( \tau^*T_{\overline{M}_{0,\gamma}} \) is \( T_{y_1}^{\oplus(n-1)} \). Hence
\[
\frac{1}{c_{\text{top}}(\mathcal{N}_r^\varphi)} = (-1)^{\sum \ell_\alpha, j} (-z)^{1-n+\sum \ell_\alpha, j} \prod_{\alpha=1}^{s} \left[ k_{\alpha, j} \right]_{\ell_{\alpha, j}} = (-1)^{\sum \ell_\alpha, j} \mu_B(-z).
\]

7.2. The wall-crossing formulas. We do not have the stabilization maps used in Proposition 14, due to the absence of \( \overline{M}_{0,\gamma}^{1/r} \). However, we can pushforward everything to a point instead. For any non-negative integer \( c \) this gives
\[
\sum_{j} \left\langle \psi^c \phi_\alpha, \mu_j^\varphi(-\psi)|\phi_\beta_j\right\rangle_{0,2(|J| - 1)}^{0} = \left\{ \mu_B(z) \right\}_{z^{-c-1}}.
\]
where the summation is over all \( 1 \in J \subset \{1, \ldots, n\} \), and \( \ast \) means the coefficient of \( z^{-c-1} \) in \( \ast \).

For \( a \in \{1, \ldots, r\} \), let \( \phi^a = \phi_{a'} \) where \( a + a' \equiv 0 \mod r \). Let \( \mu_B^-(z) \) and \( \mu^{-}(t, z) \) be the truncation of \( \mu_B(z) \) and \( \mu(t, z) \) consisting of all the negative powers of \( z \).

**Corollary 21.**

\[
\sum_{n \geq 2} \sum_{a=1}^{r} \frac{\phi^a}{n!} \left( \frac{\phi_a}{z - \psi}, (\mu^+(t, -\psi))^n \right)_{0,1+n} = \mu^-(t, z).
\]

*Proof.* We multiply both sides of (27) by \( z^{-c-1} \) and sum over all \( c \geq 0 \). This gives

(28)

\[
\sum_{J} \left( \frac{\phi_a}{z - \psi}, \mu^+(t, -\psi) \right)_{0,2(n-|J|)} = \mu_B^-(z),
\]

where the summation is over all \( 1 \in J \subset \{1, \ldots, n\} \).

For this proof only, we redefine the unstable terms

\[
\langle \cdots \rangle_{0,2}^\infty := 0 \quad \text{and} \quad \langle \frac{\phi_a}{z - \psi}, \phi_{b_1}, \ldots, \phi_{b_n} \rangle_{0,1[n]}^0 := \mu_B^-(z).
\]

Thus (28) becomes

\[
\sum_{J} \left( \frac{\phi_a}{z - \psi}, \mu^+(t, -\psi) \right)_{0,2(n-|J|)} = \langle \frac{\phi_a}{z - \psi}, \phi_{b_1}, \ldots, \phi_{b_n} \rangle_{0,1[n]}^0,
\]

where the summation is over all \( 1 \in J \subset \{1, \ldots, n\} \). This is parallel to (22) and the combinatorics is exactly the same as before. Hence we conclude that

\[
\sum_{n \geq 2} \frac{1}{n!} \left( \frac{\phi_a}{z - \psi}, (\mu^+(t, -\psi))^n \right)_{0,1+n} = \sum_{n \geq 2} \sum_{B_n} \frac{t_{b_1} \cdots t_{b_n}}{n!} \mu_B^-(z),
\]

where \( B_n = (b_1, \ldots, b_n) \) runs over all \( n \)-tuples such that \( b_j \in \{1, \ldots, r\} \) and \( k_{B_n} + a = 0 \mod r \). The latter condition follows from (26).

Hence we have \( \phi^a = \phi_{k_{B_n}} \) for each \( B_n \) and thus

\[
\sum_{n \geq 2} \sum_{a=1}^{r} \frac{\phi^a}{n!} \left( \frac{\phi_a}{z - \psi}, (\mu^+(t, -\psi))^n \right)_{0,1+n} = \sum_{n \geq 2} \sum_{B_n} \frac{t_{b_1} \cdots t_{b_n}}{n!} \mu_B^-(z) \phi_{k_{B_n}},
\]

where \( B_n = (b_1, \ldots, b_n) \) runs over all \( n \)-tuples such that \( b_j \in \{1, \ldots, r\} \). Notice that \( \mu_B^-(z) = 1 \) and thus \( \mu_B^-(z) = 0 \). Hence we can extend the summation on the right hand side over all \( n \geq 1 \) and the Corollary follows. \( \square \)

### 7.3. The big \( J \)-function and Givental’s formalism.

Now we define the big \( J \)-function

(29)

\[
J^0(u, t, z) = u(-z) + z \phi_1 + \mu(t, z) + \sum_{m \geq 1} \sum_{n \geq 0} \frac{1}{m! n!} \phi^a \left( \frac{\phi_a}{z - \psi}, u^m | t^n \right)_{0,1+m|n},
\]

where the unstable term \( (m, n) = (1, 0) \) is defined to be zero. We have

**Corollary 22.**

\[
J^0(u, t, z) = u(-z) + \mu^+(t, z) + z \phi_1 + \sum_{n \geq 0} \frac{r}{n!} \phi^a \left( \frac{\phi_a}{z - \psi}, (u + \mu^+(t, -\psi))^n \right)_{0,1+n},
\]

where the unstable terms \( n = 0, 1 \) are defined to be zero.
Proof. We first write \( \mu(t, z) = \mu^+(t, z) + \mu^-(t, z) \) in (29). Putting Corollary 21 and Theorem 1 together, we get

\[
\mu^-(t, z) + \sum_{m \geq 1, n \geq 0} \sum_r \frac{\phi^a}{m! n!} \left( \frac{\phi^a}{z - \psi}, \psi^m \right) \left( u^n \right)_{0,1+m+n}^0
\]

\[
= \sum_{m \geq 0, n \geq 0} \sum_r \frac{\phi^a}{m! n!} \left( \frac{\phi^a}{z - \psi}, \psi^m \right) \left( u^n \right)_{0,1+m+n}^\infty
\]

This completes the proof. \( \square \)

We can restrict our formula to the narrow state space since \( \phi^a \) is narrow if and only if \( \phi^a \) is narrow. This means that we set \( u_{ij} = t_j = 0 \) if \( j \phi^a \in \mathbb{Z} \) for some \( \alpha = 1, \ldots, s \). The big \( \mathbb{I} \)-function defined in [29] is our \( z \phi^a + \mu^+(t, z) \) (cf. Remark 9). If we set \( \hat{u}(z) = u(z) + \mu(t, -z) \), then \( \mathbb{I}(u, t, -z) \) satisfies Definition 1.10 in [29] and thus lies on the Lagrangian cone, matching [29, Theorem 1].

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