WELL-POSEDNESS OF COMPRESSIBLE MAGNETO-MICROPOLAR FLUID EQUATIONS

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ABSTRACT. We are concerned with compressible magneto-micropolar fluid equations (1.1)-(1.2). The global existence and large time behaviour of solutions near a constant state to the magneto-micropolar-Navier-Stokes-Poisson (MMNSP) system is investigated in $\mathbb{R}^3$. By a refined energy method, the global existence is established under the assumption that the $H^1$ norm of the initial data is small, but the higher order derivatives can be large. If the initial data belongs to homogeneous Sobolev spaces or homogeneous Besov spaces, we prove the optimal time decay rates of the solution and its higher order spatial derivatives. Meanwhile, we also obtain the usual $L^p$ type of the decay rates without requiring that the $L^p$ norm of initial data is small.

Keywords Magneto-micropolar fluid; Navier-Stokes-Poisson system; global existence; time decay rate; homogeneous Sobolev space; homogeneous Besov space.

1. Introduction

The dynamic of charged particles of one carrier type (e.g., electrons) in the effect of magnetic field can be described by the magneto-micropolar-Navier-Stokes-Poisson (MMNSP) system:

$$
\begin{cases}
\partial_t \rho + \text{div} (\rho u) &= 0, \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) - (\mu + \zeta) \Delta u - (\mu + \lambda - \zeta) \nabla \text{div} u + \nabla p &= 2\zeta \nabla \times w + (\nabla \times b) \times b + \rho \nabla \Phi, \\
\partial_t (\rho w) + \text{div} (\rho u \otimes w) - \mu' \Delta w - (\mu' + \lambda') \nabla \text{div} w + 4\zeta w &= 2\zeta \nabla \times u, \\
\partial_t b - \nabla \times (u \times b) &= -\sigma \nabla \times (\nabla \times b), \\
\Delta \Phi &= \rho - \bar{\rho}, \\
\text{div} b &= 0, \quad t > 0, x \in \mathbb{R}^3,
\end{cases}
$$

with initial data

$$
(\rho, u, b, w)|_{t=0} = (\rho_0, u_0(x), b_0(x), w_0(x)) \longrightarrow (\bar{\rho}, 0, 0, 0) \quad \text{as } |x| \longrightarrow \infty,
$$

where the unknowns $\rho = \rho(t, x) \geq 0$, $u = (u_1, u_2, u_3)(t, x)$, $w = (w_1, w_2, w_3)(t, x)$, $p = p(\rho)$ and $b = (b_1, b_2, b_3)(t, x)$ stand for the fluid density, velocity, micro-rational velocity, pressure and magnetic field, respectively. $\rho_0, u_0(x), b_0(x), w_0(x)$ are given and $b_0$ satisfies the compatibility condition, i.e. $\text{div} b_0 = 0$. The pressure $p = p(\rho)$ is a smooth function with $p'(\rho) > 0$. The parameters $\mu, \lambda, \zeta, \mu', \lambda'$ and $\sigma$ are constants denoting the viscosity coefficients of the flows satisfying

$$
\mu, \zeta, \mu', \sigma > 0, 2\mu + 3\lambda - 4\zeta \geq 0, 2\mu' + 3\lambda' \geq 0.
$$

In the motion of the fluid, due to the greater inertia the ions merely provide a constant charged background $\bar{\rho} > 0$. In particular, if $w = 0$, then (1.1)-(1.2) reduces to compressible magnetohydrodynamic equations (MHD), which has been studied extensively [11, 18, 19, 21, 48, 50, 54] and etc. If $w = b = 0$, then (1.1)-(1.2) reduces to compressible Navier-Stokes equations (NS), many work have been done on the existence, $L^p$-decay estimates with $p \geq 2$, stability and etc. for either non-isentropic or isentropic case, see e.g. [7, 8, 22, 23, 26, 34, 35] and the references therein.

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The main purpose of this paper is to investigate the influence of the electric field on the time decay rates of the solution to compressible magneto-micropolar-Navier-Stokes (MMNS) system. We first review some previous works related to MMNSP system. When \( \rho \) be a constant (e.g. \( \bar{\rho} \)), then (1.1)-(1.2) reduces to the incompressible MMNS system:

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u - \frac{2\chi}{p}\nabla \times w - \frac{1}{4}(\nabla \times b) \times b - \frac{\mu \lambda''}{p} \Delta u - \frac{\mu + \lambda' - \sigma}{p} \nabla \div u + \nabla P &= 0, \\
\partial_t w + u \cdot \nabla w - \frac{\mu}{p} \Delta w - \frac{\mu + \lambda' - \sigma}{p} \nabla \div w + \frac{2\chi}{p} w - \frac{2\chi}{p} \nabla \times u &= 0, \\
\partial_t b - \nabla \times (u \times b) &= -\sigma \nabla \times (\nabla \times b), \\
\div u &= \div b = 0, \quad t > 0, \quad x \in \mathbb{R}^3.
\end{aligned}
\] (1.3)

Such model was first proposed by Galdi-Rionero [13]. The system (1.3) enable us to study some physical phenomena that can not be treated by the classical NS equations for the viscous incompressible fluids, e.g., the motion of liquid crystal, animal blood and dilute aqueous polymer solutions, etc. Due to this important physical background, rich phenomenon, mathematical complex and challenge, there is a lot of literature devoted to the mathematical theory of (1.3). The existence and uniqueness of the strong solution was established by Rojas-Medar [42] in bounded domain \( \Omega_T := \Omega \times [0, T] \) with \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\) and \( 0 < T < \infty \). Further, Rojas-Medar and Boldrini [43] proved the existence of weak solution to (1.3) by the Galerkin method. In addition, the authors also showed that such weak solution is unique in two dimension. Soon after, Rojas-Medar et al. [41] derived global in time existence of strong solution to (1.3) by the Galerkin method. In 2010, S. Gala [12] established some improved regularity criteria of weak solutions to (1.3) in Morrey-Campanato spaces. Recently, Tan-Wu-Zhou [47] obtained the global in time existence of large time behaviour of the solutions to (1.3) in \( \mathbb{R}^3 \). Moreover, the authors also derived a weak solution in \( \mathbb{R}^2 \) with large initial data. The further literature on the the incompressible MMNS system is indeed huge and thus out of the scope of this paper, see [9,25,32,33,58,57] and the references therein.

Let us recall important mathematical characters on the incompressible MMNS system, the micropolar Navier-Stokes (MNS) system \((b = 0)\):

\[
\begin{aligned}
\partial_t u - (\chi_1 + \chi_2) \Delta u + u \cdot \nabla u + \nabla P - 2\chi \nabla \times w &= 0, \\
\partial_t w - \chi_3 \Delta w + u \cdot \nabla w + 4\chi w - \chi_4 \nabla \div w - 2\chi \nabla \times u &= 0, \\
\div u &= 0.
\end{aligned}
\] (1.4)

The constants \( \chi_i (i = 1, 2, 3, 4) \) are the viscosity coefficients. MNS system was first developed by Eringen [10] in 1966. In any bounded domain \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\), Galdi et al. [13] and Łukaszewicz [30] derived the existence of weak solution. Furthermore, in [30] (see also Yamaguchi [55]) the author also proved the existence and uniqueness of strong solution to (1.4). For the well-posedness of MNS system with full viscosity and partial viscosity in \( \mathbb{R}^2 \), one may refer to [3,50], respectively. In 2012, Miao et al. [2] proved the global well-posedness for the 3D MNS system in the critical Besov spaces by making a suitable transformation of the solutions and using the Fourier localization method. For more details, one can refer to [11,50,24,29,31,49] and the references therein.

For the compressible case, recently, Guo et al. [52] studied the global existence and optimal time decay rates of solution to MMNS system in \( \mathbb{R}^3 \) by combining the \( L^p - L^q \) estimates for the linearized equations and the Fourier splitting method. Later, Wu-Wang [53] gave a pointwise estimates for the 3D MNS system, which exhibited generalized Huygen’s principle for the fluid density and fluid momentum as the compressible NS equation. Besides, we would like to refer to [4,5,6,27,28,36,37,38,39] and the references therein.

Without loss of generality, in this paper, we set the constants \( \bar{\rho} = 1, \mu = \zeta = \frac{1}{2}, \lambda = \lambda' = \mu' = \sigma = 1 \), and note that

\[
\begin{aligned}
(\nabla \times b) \times b &= b \cdot \nabla b - \frac{1}{2} \nabla(|b|^2), \\
\nabla \times \nabla \times b &= \nabla \div b - \Delta b, \\
\nabla \times (u \times b) &= (b \cdot \nabla)u - (u \cdot \nabla)b + u \div b - b \div u.
\end{aligned}
\]
Now, we define \( \varrho := \rho - \bar{\rho} \), from above, then the system \((1.1)-(1.2)\) can be rewritten as
\[
\begin{align*}
\partial_t \varrho + \nabla u &= M_1, \\
\partial_t u + \varrho \nabla - \Delta u - \nabla \div u - \nabla \times w - \nabla \Phi &= M_2, \\
\partial_t w + 2w - \Delta w - 2\nabla \div w - \nabla \times u &= M_3, \\
\partial_t \Phi - \Delta b &= M_4, \\
\Delta \Phi &= \varrho, \\
\div b &= 0, t > 0, x \in \mathbb{R}^3,
\end{align*}
\]
with initial data
\[
(\varrho, u, w, b)(x, 0) = (\varrho_0, u_0, w_0, b_0) \rightarrow (0, 0, 0, 0) \quad \text{as } |x| \rightarrow \infty,
\]
where the nonlinear terms \( M_i \) \((i = 1, 2, 3, 4)\) are defined as
\[
M_1 = -\div (\varrho u), \\
M_2 = -u \cdot \nabla u - f(\varrho)[\Delta u + \nabla \div u + \nabla \times w] - h(\varrho) \nabla \varrho + g(\varrho) [b \cdot \nabla b - \frac{1}{2} \nabla (|b|^2)] + h(\varrho) \nabla \varrho, \\
M_3 = -u \cdot \nabla w - f(\varrho)[\Delta w + 2\nabla \div w - 2w + \nabla \times u], \\
M_4 = b \cdot \nabla u - u \cdot \nabla b - b \div u.
\]
Here,
\[
\gamma = \frac{p'(1)}{1}, \quad f(\varrho) = \frac{\varrho}{\varrho + 1}, \quad h(\varrho) = \frac{p'(\varrho + 1)}{\varrho + 1} - \frac{p'(1)}{1}, \quad g(\varrho) = \frac{1}{\varrho + 1},
\]
For simplicity, in the following, we set \( p'(1) = 1 \), that is \( \gamma = 1 \).

**Notation.** Throughout this paper, \( c \) denotes a general constant may vary in different estimate. If the dependence need to be explicitly stressed, some notations like \( c_k, c_N \) will be used. We use \( c_0 \) denotes the constants depending on the initial data \( k, N \) and \( s \). We may use \( A \sim B \), if there is \( c \) and \( c' \) such that \( cB \leq A \leq c'B \). We will use \( a \leq b \) or \( a \leq cb \). For simplicity, we denote
\[
\|\nabla^l (f, g, h)\|_{L^p} := \|\nabla^l f\|_{L^p} + \|\nabla^l g\|_{L^p} + \|\nabla^l h\|_{L^p},
\]
and \( \int f \, dx = \int_{\mathbb{R}^3} f \, dx \). In addition, \( \nabla^l \) with an integer \( l \geq 0 \) stands for the usual spatial derivations of order \( l \). When \( l < 0 \) or \( l \) is not a positive integer, \( \nabla^l \) stands for \( \Lambda^l \) defined by
\[
\Lambda^l f := \mathcal{F}^{-1}(|\xi|^l \mathcal{F} f),
\]
where \( \mathcal{F} \) is the usual Fourier transform operator and \( \mathcal{F}^{-1} \) is its inverse. We use \( H^s(\mathbb{R}^3) \), \( s \in \mathbb{R} \) to denote the homogeneous Sobolev spaces on \( \mathbb{R}^3 \) with norm \( \| \cdot \|_{H^s} \) defined by \( \| f \|_{H^s} = \| \Lambda^s f \|_{L^2} \), and we use \( H^s(\mathbb{R}^3) \) to denote the usual Sobolev spaces with norm \( \| \cdot \|_{H^s} \), and \( L^p(\mathbb{R}^3) \) \((1 \leq p \leq \infty)\) to denote the usual \( L^p \) spaces with norm \( \| \cdot \|_{L^p} \). Finally, we introduce the homogeneous Besov space, let \( \varphi \in C_0^\infty(\mathbb{R}^3) \) be a cut-off function such that \( \varphi(\xi) = 1 \) with \( |\xi| \leq 1 \), and \( \varphi(\xi) = 0 \) with \( |\xi| \geq 2 \). Let \( \psi(\xi) = \varphi(2\xi) - \varphi(2\xi) \) and \( \psi_j(\xi) = \psi(2^{-j}\xi) \) for \( j \in \mathbb{Z} \). Then, by the construction \( \sum_{j \in \mathbb{Z}} \psi_j(\xi) = 1 \) if \( \xi \neq 0 \), we set \( \Lambda_j f = \mathcal{F}^{-1} \psi_j f \), then for \( s \in \mathbb{R} \), we define the homogeneous Besov spaces \( \dot{B}_p^s(\mathbb{R}^3) \) with norm \( \| \cdot \|_{\dot{B}_p^s} \) by
\[
\| f \|_{\dot{B}_p^s(\mathbb{R}^3)} = \left\{ \sum_{\infty < j_{\lambda} < \infty} \left( \frac{\sum_{\infty < j_{\lambda} < \infty} \| \Lambda_j f \|^q_{L^p(\mathbb{R}^N)} \| \Lambda_j f \|^q_{L^p(\mathbb{R}^N)} \right)^\frac{q}{2} \right\}^{\frac{1}{q}}, \quad 1 \leq p \leq \infty, 1 < q < \infty,
\]
and the corresponding dissipation rate by
\[
\mathcal{D}_N(t) := \sum_{j=0}^{N} \| \nabla^j (\varrho, u, w, b) \|_{L^2}^2 + \sum_{j=0}^{N} \| \nabla^j (\varrho, u, w, b) \|_{L^2}^2.
\]
Theorem 1.1. Assume that initial data \((\nabla \Phi(0), q_0, u_0, w_0, b_0) \in H^N\) for an integer \(N \geq 3\) and

\[
\int q_0 dx = 0.
\]  
(1.10)

Then there exists a constant \(\delta_0 > 0\) such that if \(\delta_3(0) \leq \delta_0\), then the problem (1.5)-(1.6) admits a unique solution \((\nabla \Phi, q, u, w, b) \in C([0, \infty]; H^2(\mathbb{R}^3))\) satisfying that for all \(t > 0\)

\[
\sup_{0 \leq t \leq \infty} \delta_3(t) + \int_0^\infty \mathcal{D}_3(\tau) d\tau \leq c\delta_3(0).
\]  
(1.11)

Furthermore, if \(\delta_N(0) < \infty\) for any \(N \geq 4\), then (1.5)-(1.6) admits a unique solution \((\nabla \Phi, q, u, w, b) \in C([0, \infty]; H^N(\mathbb{R}^3))\), and for all \(t > 0\), there holds

\[
\sup_{0 \leq t \leq \infty} \delta_N(t) + \int_0^\infty \mathcal{D}_N(\tau) d\tau \leq c\delta_N(0).
\]  
(1.12)

Theorem [13] will be proved in Section 5 by using the strategy Guo [16]. The key point is that, by constructing some interactive energy functionals for \(k \geq 0\)

\[
\frac{d}{dt} \sum_{l=k}^{k+1} \int \nabla^l q \cdot \nabla^{l+1} q dx + \frac{1}{4} \sum_{l=k}^{k+1} (\|\nabla^l q\|_{L^2}^2 + \|\nabla^{l+1} q\|_{L^2}^2 + \|\nabla^l \nabla \Phi\|_{L^2}^2 + \|\nabla^{l+1} \nabla \Phi\|_{L^2}^2) \leq c\|q\|_{H^1} \sum_{l=k}^{k+1} (\|\nabla^{l+1} q\|_{L^2}^2 + \|\nabla^{l+2} w\|_{L^2}^2) + \sum_{l=k}^{k+1} (\|\nabla^{l+1} u\|_{L^2}^2 + 2\|\nabla^{l} w\|_{L^2}^2 + 4\|\nabla^{l+2} u\|_{L^2}^2),
\]

then one can derive the dissipative of \(q\) and \(\nabla \Phi\). This, together with Lemma [5.1] implies that for \(N \geq 3\)

\[
\frac{d}{dt} \delta_N(t) + \mathcal{D}_N(t) \leq \sqrt{\delta_3(t) \mathcal{D}_N(t)}.
\]  
(1.13)

In virtue of the smallness of \(\delta_3(0)\) and the argument of Theorem 5.1, the Theorem 1.1 follows from (1.13).

In addition, if the initial data belongs to Negative Sobolev or Besov spaces, we can derive some further decay rates of the solution and its higher order spatial derivatives to system (1.5)-(1.6). Based on the regularity interpolation method developed in Strain-Guo [46], Guo-Wang [17] and Sohinger-Strain [44], we can develop a general energy method, that is, by using a family of scaled energy estimates with minimum derivative counts and interpolation among them, we deduce that

\[
\frac{d}{dt} \delta_k^{k+2} + \mathcal{D}_k^{k+2} \leq \delta_3(t) \mathcal{D}^{k+2}_k + \delta_3(t) \sum_{l=k}^{k+1} \|\nabla^{l+1} (u, \nabla u, w)\|_{L^2}^2,
\]

where \(\delta_k^{k+2}\) and \(\mathcal{D}_k^{k+2}\) are defined by (5.7) and (5.8). Appealing to the properties of homogeneous Sobolev space \(H^{-s}\) or Besov space \(B_{2, \infty}^{-s}\), we can bound \(\|\nabla^k (\nabla \Phi, q, u, w, b)\|_{L^2}^2\) in term of \(\delta_k^{k+2}\). Hence, we are able to have

Theorem 1.2. Let all assumptions of Theorem 1.1 are in force. Let \((\nabla \Phi, q, u, w, b)(t)\) is the solution to (1.5)-(1.6) constructed in Theorem 1.1 with \(N \geq 3\). Suppose that \((\nabla \Phi(0), q_0, u_0, w_0, b_0) \in H^{-s}\) for some \(s \in [0, \frac{3}{2})\) or \((\nabla \Phi(0), q_0, u_0, w_0, b_0) \in B_{2, \infty}^{-s}\) for some \(s \in (0, \frac{3}{2})\), then we have

\[
\|(\nabla \Phi, q, u, w, b)(t)\|_{H^{-s}} \leq c_0,
\]  
(1.14)

or

\[
\|(\nabla \Phi, q, u, w, b)(t)\|_{B_{2, \infty}^{-s}} \leq c_0,
\]  
(1.15)

Moreover, for \(k \geq 0\), if \(N \geq k + 2\), there holds

\[
\|
abla^k (\nabla \Phi, q, u, w, b)(t)\|_{L^2} \leq c_0 (1 + t)^{-\frac{k+2}{2}},
\]  
(1.16)

and in particular, for \(0 \leq k \leq N - 3\)

\[
\|
abla^k q(t)\|_{L^2} \leq c_0 (1 + t)^{-\frac{k+3}{2}}.
\]  
(1.17)
We shall note that, in the usual $L^p - L^2$ approach of studying the optimal decay rates of the solution, it is difficult to show that the $L^p$-norm of the solution can be preserved along time evolution. An important feature is that the $H^{-s}$ or $B^{s}_{2,\infty}$ norm of the solution is preserved along time evolution. From Hardy-Littlewood-Sobolev inequality (Lemma 2.4), for $p \geq 2$, we infer that $L^p \subset H^{-s}$ with $s = 3(\frac{1}{p} - \frac{1}{2})$.

This, together with Theorem 1.2, we have the $L^p - L^2$ type of the optimal decay results. However, the imbedding theorem cannot cover the case $p = 1$. To amend this, Sohinger-Strain [44] instead introduced the homogeneous Besov space $B^{s}_{2,\infty}$ due to the fact that the endpoint imbedding $L^1 \subset B^{s}_{2,\infty}$ holds (Lemma 2.5). At this stage, by Theorem 1.2 we have the following corollary of the usual $L^p - L^2$ type of the decay results:

**Corollary 1.1.** Under the assumptions of Theorem 1.2 except that we replace $H^{-s}$ or $B^{s}_{2,\infty}$ assumption by $(\nabla \Phi_0, \xi_0, u_0, b_0, w_0) \in L^p$ for some $p \in [1, 2]$. Then, for any integer $k \geq 0$, if $N \geq k + 2$, then there holds

$$\|\nabla^k (\nabla \Phi, \xi, u, b, w)(t)\|_{L^2} \leq C_0(1 + t)^{-\frac{k + p}{2}},$$

where $s_p$ is defined by

$$s_p := 3\left(\frac{1}{p} - \frac{1}{2}\right).$$

The following are several remarks for our main results:

**Remark 1.1.** By the Poisson equation, it holds that

$$\|\xi_0\|_{H^s} + \|\nabla \Phi_0\|_{H^s} \sim \|\xi_0\|_{H^s} + \|\Lambda_{-1} \xi_0\|_{H^s}, \quad s \geq 0$$

and

$$\|\xi_0\|_{H^s} + \|\nabla \Phi_0\|_{H^s} \sim \|\xi_0\|_{H^s} + \|\Lambda_{-1} \xi_0\|_{H^s}, \quad p \in (1, 2].$$

Compared to the study of MMNS system, the norms of $\Lambda_{-1} \xi_0$ is additionally required. However, such assumption can be achieved by the natural neutral condition (1.10) (cf. [15]).

**Remark 1.2.** In Theorem 1.1/1.2 if $k = 0, 1$, we can remove the smallness of $\|\nabla \Phi(0)\|_{H^s}$ by assuming only $\nabla \Phi(0) \in H^s$. Then, we can also derive the time decay rates of (1.16) for $k = 0, 1$. Indeed, similar to (3.6)-(3.7), we can re-estimate the right-most term in (3.4) as following. If $l = 0$, we have

$$\int \nabla \Phi \cdot (\xi u) d x \leq \|\nabla \Phi\|_{L^6} \|\xi\|_{L^2} \|u\|_{L^3} \leq \|u\|_{H^s} (\|\nabla \Phi, \xi\|_{L^2}^2).$$

If $l \geq 1$, using Lemma 2.7 we arrive at

$$\int \nabla^l \nabla \Phi \cdot (\xi u) d x \leq \|\nabla \Phi\|_{L^6} \|\nabla^l (\xi u)\|_{L^2} \leq \left(\sum_{s = 0}^{l + 1} \|\nabla^s \xi\|_{L^2} \|\nabla^{l-s} u\|_{L^2} \right) \left(\sum_{s = 0}^{l + 1} \|\nabla^s \xi\|_{L^2} \|\nabla^{l-s} u\|_{L^2} \right) \leq c_1 \|\xi, u\|_{H^s} \|\nabla^l (\nabla \Phi, \xi)\|_{L^2}^2 + \|\nabla^l \xi\|_{L^2}^2,$$

where

$$\tilde{\theta} = \frac{1 - s}{l + 1}, \quad \tilde{\alpha} = \frac{l + 1}{2(l - s)} - 1 \in (\text{pending}), \quad \tilde{\theta} = \frac{l - s}{l}, \quad \tilde{\alpha} = 1 - \frac{l}{2s} \in (0, \frac{1}{2}].$$
Here, we may choose \( \alpha \in [0, 3] \), this implies \( l = 1 \). Combining both inequalities and (3.4) yields

\[
- \int \nabla^i \nabla \Phi \cdot \nabla^l u \, dx \geq \frac{1}{2\eta} \int \nabla^j \nabla \Phi \cdot \nabla^l u \, dx - c_1 ||(\partial_t u)||_{H^l} ||\nabla^{l+1} (\nabla \Phi, u)||_{L^2}^2
\]

with \( l = 0, 1 \). From above, then by a standard argument we can also obtain (3.6) and hence derive the arguments of Theorem [7,12] for \( k = 0, 1 \).

**Remark 1.3.** Note that both \( \dot{H}^{-s} \) and \( \dot{B}_2^{-s} \) norms enhance the decay rates of the solution. The constraint \( s \leq \frac{1}{2} \) in Theorem [7,2] comes from applying Lemma [2,3,4] to estimate the nonlinear terms in Section 2. For \( s > \frac{3}{2} \), the nonlinear estimates would not work, this in turn restricts \( p \geq 1 \) in Corollary [7,1]. Furthermore, let \( L^2 \) decay rate of higher order spatial derivatives of the solution is obtained in Corollary [7,1] then by Lemma [2,7] we can deduce that general optimal \( L^q (2 \leq q \leq \infty) \) time decay estimates of the solution.

The rest of our paper is organized as follows. First of all, in Section 2, we present some useful lemma which will be heavily used in our proof. Next, in Section 3, we prove the local existence of solution (\( \nabla \Phi, q, u, w, b \)). For this aim, we first derive the uniform a priori estimate of solution in subsection 3.1. Furthermore, by constructing the Cauchy sequence, we establish the local existence and the uniqueness of solution. In Section 4, we derive the evolution of the negative Sobolev and Besov norms of solution. Finally, we prove the main theorem in Section 5.

2. Preliminary

Throughout this section, we collect some auxiliary results, some of which have been proven elsewhere. First, we will extensively use the Sobolev interpolation of the Gagliardo-Nirenberg inequality.

**Lemma 2.1.** Let \( 0 \leq m, \alpha \leq l \), then we have

\[
||\nabla^\alpha f||_{L^p} \leq ||\nabla^m f||_{L^\frac{p}{r}} ||\nabla^l f||_{L^q}^\theta,
\]

where \( 0 \leq \theta \leq 1 \) and \( \alpha \) satisfies

\[
\frac{\alpha}{3} - \frac{1}{p} = \left( \frac{m}{3} - \frac{1}{q} \right) (1 - \theta) + \left( \frac{l}{3} - \frac{1}{r} \right) \theta.
\]

Here, when \( p = \infty \), we require that \( 0 < \theta < 1 \).

**Proof.** See e.g. P.125 in [40] or Lemma A.1 in [17]. \( \Box \)

The following commutator estimate will be used in Section 3.

**Lemma 2.2.** Let \( k \geq 1 \) be an integer and define the commutator

\[
[\nabla^k, g]h = \nabla^k (gh) - g\nabla^k h.
\]

Then we have

\[
||[\nabla^k, g]h||_{L^p} \leq c_k (||\nabla g||_{L^{p_1}}, ||\nabla^{k-1} h||_{L^{p_2}} + ||\nabla^k g||_{L^{p_3}}, ||h||_{L^{p_4}}),
\]

where \( p, p_2, p_3 \in (1, +\infty) \) and

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.
\]

**Proof.** If \( p = p_2 = p_3 = 2 \), then (2.2) can be proved by Lemma 2.1. For the general cases, one can refer to [20, 51]. \( \Box \)

**Lemma 2.3.** Let \( f(\cdot) \) be a smooth function, \( w : \mathbb{R}^3 \rightarrow \mathbb{R}^n (n \geq 1) \) satisfies \( ||w||_{H^1(\mathbb{R}^3)} \leq \delta \ll 1 \). If \( f(w) \sim w \) or \( f(0) = 0 \), then for any integer \( k \geq 0 \), we have

\[
||\nabla^k f(w)||_{L^\infty} \leq c_k ||\nabla^k w||_{L^2}^{\frac{1}{2}} ||\nabla^{k+2} w||_{L^2}^{\frac{1}{2}},
\]

or

\[
||\nabla^k f(w)||_{L^\infty} \leq c_k ||\nabla^k w||_{L^\infty},
\]

and

\[
||\nabla^k f(w)||_{L^2} \leq c_k ||\nabla^k w||_{L^2}.
\]
Lemma 2.4. Let that is (2.4). See Lemma 4.6 in [17].

Proof. For the proof of (2.3), one can refer Lemma 3.1 in [17]. Here, we only prove (2.4)-(2.5). It is trivial for the case \( k = 0 \). If \( k \geq 1 \) by (2.1) and the Hölder’s inequality, we get

\[
\| \nabla^k f(w) \|_{L^2} \leq \| \text{a sum of products} f^{(q_1, \ldots, q_m)} \nabla^{q_1} w \nabla^{q_2} w \cdots \nabla^{q_m} w \|_{L^2} \\
\leq \| \nabla^{q_1} w \|_{L^{2q_1/q_2}} \| \nabla^{q_2} w \|_{L^{2q_2/q_3}} \cdots \| \nabla^{q_m} w \|_{L^{2q_m/q_{m+1}}} \\
\leq \| \nabla w \|_{L^{2/q}} \| \nabla^k w \|_{L^{2/k}} \cdots \| \nabla w \|_{L^{2/q}} \| \nabla^k w \|_{L^{2/k}} \\
\leq \| w \|_{H^{m-1}} \| \nabla^k w \|_{L^2},
\]

(2.6)

with \( 1 \leq m \leq k, q_i \geq 1 \) and \( \sum_{i=1}^m q_i = k \). This implies (2.5).

By the same way, we infer that

\[
\| \nabla^k f(w) \|_{L^\infty} \leq \| \text{a sum of products} f^{(q_1, \ldots, q_m)} \nabla^{q_1} w \nabla^{q_2} w \cdots \nabla^{q_m} w \|_{L^\infty} \\
\leq \| \nabla^{q_1} w \|_{L^{2/q}} \| \nabla^{q_2} w \|_{L^{2/q}} \cdots \| \nabla^{q_m} w \|_{L^{2/q}} \\
\leq \| \nabla^k w \|_{L^\infty} \| \nabla w \|_{L^{2/q}} \cdots \| \nabla^k w \|_{L^\infty} \\
\leq \| w \|_{H^{m-1}} \| \nabla^k w \|_{L^\infty},
\]

that is (2.4). \( \square \)

If \( s \in [0, \frac{3}{2}) \), the Hardy-Littlewood-Sobolev theorem implies the following \( L^p \) type inequality.

Lemma 2.4. Let \( 0 \leq s < \frac{3}{2}, 1 \leq p \leq 2 \) with \( \frac{1}{p} + \frac{s}{2} = \frac{1}{p'} \), then

\[
\| f \|_{H^{s-\frac{1}{p}}} \leq \| f \|_{L^p}.
\]

(2.7)

Proof. See Theorem 1, p.119 in [45] or [14] for instance. \( \square \)

In addition, for \( s \in (0, \frac{3}{2}) \), we will use the following result.

Lemma 2.5. Let \( 0 < s \leq \frac{3}{2}, 1 \leq p \leq 2 \) with \( \frac{1}{p} + \frac{s}{2} = \frac{1}{p'} \), then

\[
\| f \|_{B^{s-\frac{1}{p}}_{2,\infty}} \leq \| f \|_{L^p}.
\]

(2.8)

Proof. See Lemma 4.6 in [17]. \( \square \)

The following two special Sobolev interpolation will be used in the proof of Theorem 1.2.

Lemma 2.6. Let \( s \geq 0 \), and \( l \geq 0 \), then we have

\[
\| \nabla f \|_{L^2} \leq \| \nabla^{l+1} f \|_{L^2} \| f \|_{H^{s-\frac{l}{l+1}}},
\]

(2.9)

where \( \theta = \frac{1}{l+1+s} \).

Proof. It follows directly by the Parseval theorem and the Hölder’s inequality. \( \square \)

Lemma 2.7. Let \( s > 0 \), and \( l \geq 0 \), then we have

\[
\| \nabla f \|_{L^2} \leq \| \nabla^{l+1} f \|_{L^2} \| f \|_{B^{s-\frac{l}{l+1}}_{2,\infty}},
\]

(2.10)

where \( \theta = \frac{1}{l+1+s} \).

Proof. See Lemma 4.5 in [44]. \( \square \)

3. Energy estimate and the local existence

3.1 Uniform a priori estimate. Theorem 1.1 will be proved by combining the local existence of \((\nabla \Phi, q, u, w, b)\) to (1.5)-(1.6) and some uniform a priori estimates as well as the communication argument. In this section, we aim to derive the uniform a priori estimates.
Lemma 3.1. Let $T > 0$. Suppose that
\begin{equation}
\sup_{0 \leq t \leq T} \| (\nabla \Phi, \rho, u, w, b) \|_{H^1} \leq \delta, \tag{3.1}
\end{equation}
for $0 \leq \delta \leq 1$, let all assumptions of Theorem 1.1 are in force. Then for any integer $k \geq 0$ and $t \geq 0$, there holds
\begin{align*}
\frac{1}{2} \frac{d}{dt} \sum_{l=k}^{k+2} \| \nabla^l (\nabla \Phi, \rho, u, w, b) \|_{L^2}^2 + \frac{1}{2} \sum_{l=k}^{k+2} \| \nabla^l w \|_{L^2}^2 + \sum_{l=k}^{k+2} \left( \frac{2}{3} \| \nabla^{l+1} u \|_{L^2}^2 + 3 \| \nabla^{l+1} w \|_{L^2}^2 + \| \nabla^{l+1} b \|_{L^2}^2 \right) \\
\leq c_{\delta} \sum_{l=k}^{k+2} \left( \| \nabla^l (\rho, w) \|_{L^2}^2 + \| \nabla^{l+1} (\nabla \Phi, u, w, b) \|_{L^2}^2 \right). \tag{3.2}
\end{align*}

Proof. For any integer $k \geq 0$, by the $\nabla^l$ ($l = k, k+1, k+2$) energy estimate, for (1.5), (1.5) on $\rho$ and $u$, we have
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int \nabla^l [\nabla^l \rho]^2 + |\nabla^l u|^2 \, dx + 2 \int \| \nabla^{l+1} u \|_{L^2}^2 \, dx - \int \nabla^l \nabla \Phi \cdot \nabla^l u \, dx \\
= \int \left[ - \nabla^l (\text{div} (\rho u)) \cdot \nabla \Phi + \nabla^l (\nabla \times w) \cdot \nabla^l u - \nabla^l (u \cdot \nabla u) \cdot \nabla^l u \\
- \nabla^l (f(\rho) \Delta u) \cdot \nabla^l u - \nabla^l (f(\rho) \nabla \text{div} u) \cdot \nabla^l u - \nabla^l (f(\rho) \nabla \times w) \cdot \nabla^l u \\
- \nabla^l (g(\rho) \nabla \rho) \cdot \nabla^l u + \nabla^l (g(\rho) \nabla \rho b \cdot \nabla b) \cdot \nabla^l u - \frac{1}{2} \nabla^l (g(\rho) \nabla (|b|^2)) \cdot \nabla^l u \\
- \nabla^l (\text{div} u) \cdot \nabla \rho - \nabla^l (\nabla \rho \cdot \nabla^l u) \, dx \\
:= \mathbb{I}_l. \tag{3.3}
\end{align*}

For the last term on the left hand side of of (3.3), by integration by parts, (1.5), and the Poisson equation (1.5), we deduce that
\begin{align*}
- \int \nabla^l \nabla \Phi \cdot \nabla^l u \, dx &= \int \nabla^l \Phi \cdot \nabla^l \text{div} u \, dx \\
&= - \int \nabla^l \Phi \cdot \nabla^l \partial_t \rho + \nabla^l \Phi \cdot \nabla^l (\rho u) \, dx \\
&= - \int \nabla^l \Phi \cdot \nabla^l \partial_t \Delta \Phi - \nabla^l \nabla \Phi \cdot \nabla^l (\rho u) \, dx \\
&= \frac{1}{2} \frac{d}{dt} \int |\nabla^l \nabla \Phi|^2 \, dx + \int \nabla^l \nabla \Phi \cdot \nabla^l (\rho u) \, dx. \tag{3.4}
\end{align*}

In addition, by integration by parts, the H"{o}lder’s inequality and Lemma 2.1 it holds that
\begin{align*}
\int \nabla^l \nabla \Phi \cdot \nabla^l (\rho u) \, dx &= \int \nabla^l (\Delta \Phi u) \cdot \nabla^l \nabla \Phi \, dx \\
&= - \int \nabla^l (\nabla \Phi \cdot \nabla u) \cdot \nabla^l \nabla \Phi + \nabla^l (\nabla \Phi \cdot u) \cdot (\nabla \cdot \nabla^l \nabla \Phi) \, dx \\
&= - \int \sum_{j=0}^{l} C^l_j \left( (\nabla^j \nabla \Phi \cdot \nabla^{l-j} \nabla u) \cdot \nabla^l \nabla \Phi + (\nabla^j \nabla \Phi \cdot \nabla^{l-j} u) \cdot (\nabla \cdot \nabla^l \nabla \Phi) \right) \, dx \\
&\leq \int \sum_{j=0}^{l} \| \nabla^j \nabla \Phi \|_{L^2} \| \nabla^{l-j} u \|_{L^2} \| \nabla^l \nabla \Phi \|_{L^2} + \| \nabla^{l-j} u \|_{L^2} \| \nabla^l \nabla \Phi \|_{L^2} \\
&\leq \sum_{j=0}^{l} \| \nabla^j \nabla \Phi \|_{L^2} \| \nabla^{l-j} u \|_{L^2} \| \nabla^l \nabla \Phi \|_{L^2}. \tag{3.5}
\end{align*}
Making use of Lemma 2.1, we see that
\[
\sum_{s=0}^{l} \|\nabla^s \Phi\|_{L^2}^2 \|\nabla^{l+1-s} u\|_{L^2}^2 \leq \sum_{s=0}^{\lceil \frac{l}{2} \rceil} \|\nabla^s \Phi\|_{L^2}^\theta \|\nabla^{l+1} \Phi\|_{L^2}^{1-\theta} \|\nabla^{l+1} u\|_{L^2}^\theta + \sum_{s=\lceil \frac{l}{2} \rceil + 1}^{l} \|\nabla^s \Phi\|_{L^2}^\theta \|\nabla^{l+1} \Phi\|_{L^2}^{1-\theta} \|\nabla^{l+1} u\|_{L^2}^\theta,
\]
(3.6)
where $\bar{\alpha}, \tilde{\alpha}, \bar{\theta}, \tilde{\theta}$ satisfy
\[
\tilde{\theta} = 1 - \frac{s}{l + 1}, \quad \bar{\alpha} = \frac{k + 1}{2(k + 1 - s)} \in \left[\frac{1}{2}, 1\right], \quad \bar{\theta} = 1 - \frac{2s + 1}{2(l + 1)}, \quad \tilde{\alpha} = \frac{l + 1}{2s + 1} \in \left(\frac{1}{2}, 1\right).
\]
From (3.4)–(3.6), it follows that
\[
- \int \nabla^l \nabla \Phi \cdot \nabla^l \Phi, \quad \bar{\theta} \geq \frac{d}{2} \int \|\nabla^l \nabla \Phi \|^2 d\theta - c\|\nabla \Phi, u\|_{H^1} \|\nabla^{l+1} (\nabla \Phi, u)\|_{L^2}^2.
\]
(3.7)
Now, we concentrate our attention on estimating the terms $I_1 - I_{11}$. First, for the term $I_1$, we can see that
\[
I_1 = - \int \nabla^l (\nabla \Phi, u) \cdot \nabla^l \Phi d\theta - \int \nabla^l (u \cdot \nabla \Phi) \cdot \nabla^l \Phi d\theta
\]
:= $I_{1a} + I_{1b}$.
(3.8)
If $l = 0$, it is obvious that
\[
I_{1a} \leq \|\nabla \Phi\|_{H^1} \|\nabla, \nabla u\|_{L^2}^2
\]
(3.9)
If $l = 1$, we further obtain
\[
I_{1a} = - \int (\nabla \Phi, u) \cdot \nabla \Phi d\theta - \int (\nabla \Phi, u) \cdot \nabla \Phi d\theta
\leq \|\nabla \Phi\|_{L^2} \|\nabla \Phi\|_{L^2} \|\nabla u\|_{L^6} + \|\nabla \Phi\|_{L^2} \|\nabla \Phi\|_{L^2}
\leq c\|\nabla \Phi\|_{H^1} \|\nabla, \nabla^2 u\|_{L^2}^2.
\]
(3.10)
If $l \geq 2$, by the Leibniz formula and the Hölder’s inequality, we obtain
\[
I_{1a} = - \int \sum_{s=0}^{l} \nabla^s \Phi \nabla^{l-s} \Phi d\theta \cdot \nabla^l \Phi d\theta
\leq \sum_{s=0}^{\lceil \frac{l}{2} \rceil} \|\nabla^s \Phi\|_{L^2} \|\nabla^{l-s+1} \Phi\|_{L^2} \|\nabla^{l-s+1} \Phi\|_{L^2} + \sum_{s=\lceil \frac{l}{2} \rceil + 1}^{l} \|\nabla^s \Phi\|_{L^2} \|\nabla^{l-s+1} \Phi\|_{L^2} \|\nabla^{l-s+1} \Phi\|_{L^2}
\leq \|\nabla \Phi\|_{L^2} \|\nabla \Phi\|_{L^2} \|\nabla u\|_{L^6} + \|\nabla \Phi\|_{L^2} \|\nabla \Phi\|_{L^2}
\leq (I_{1aa} + I_{1ab}) \|\nabla^l \Phi\|_{L^2}.
\]
(3.11)
Making use of (2.1), it holds that
\[
I_{1aa} \leq \|\nabla^s \Phi\|_{L^2} \|\nabla^s (\nabla \Phi, u)\|_{L^2} \|\nabla^s (\nabla \Phi, u)\|_{L^2} \|\nabla^{l+1} \Phi\|_{L^2}
\leq c\|\nabla \Phi, u\|_{H^1} \|\nabla^l \Phi, \nabla^{l+1} \Phi\|_{L^2}.
\]
(3.12)
where
\[
\theta = \frac{l - s}{l} \in \left(\frac{1}{2}, 1\right), \quad \alpha = \frac{3l}{2(l - s)} \in \left(\frac{3}{2}, 3\right).
\]
Likewise, the term $I_{1ab}$ can be estimated as
\[
I_{1ab} \leq \|\nabla^l \Phi\|_{L^2} \|\nabla^l \Phi\|_{L^2} \|\nabla^l \Phi\|_{L^2} \|\nabla^{l+1} \Phi\|_{L^2}
\leq c\|\nabla \Phi, u\|_{H^1} \|\nabla^l \Phi, \nabla^{l+1} \Phi\|_{L^2},
\]
(3.13)
where
\[ \theta_1 = \frac{l - s}{l - 1}, \quad \alpha_1 = \frac{l - 1}{2(s - 1)} + 2 \in \left[ \frac{5}{2}, 3 \right]. \]
Inserting (3.12)-(3.13) into (3.11), and together with (3.9)-(3.10), we have
\[ I_{1a} \leq c \| (\mathcal{Q}, u) \|_{H^1} (\| \nabla^l \mathcal{Q} \|_{L^2}^2 + \| \nabla^{l+1} u \|_{L^2}^2). \] (3.14)
Employing Lemma 2.2 for the term \( I_{1b} \), we infer that
\[ I_{1b} = - \int (u \cdot \nabla \mathcal{Q} + [\nabla^l u] \cdot \nabla \mathcal{Q}) \cdot \nabla \mathcal{Q} dx \]
\[ \leq \| \nabla u \|_{L^\infty} \| \mathcal{Q} \|_{L^2}^2 + \| \nabla u \|_{L^6} \| \nabla \mathcal{Q} \|_{L^2} \| \nabla \mathcal{Q} \|_{L^2} \]
\[ \leq c \| (\mathcal{Q}, u) \|_{H^1} (\| \nabla^l \mathcal{Q} \|_{L^2}^2 + \| \nabla^{l+1} u \|_{L^2}^2). \] (3.15)
Plugging (3.14)-(3.15) into (3.8), yields that
\[ I_1 \leq c \| (\mathcal{Q}, u) \|_{H^1} (\| \nabla^l \mathcal{Q} \|_{L^2}^2 + \| \nabla^{l+1} u \|_{L^2}^2). \] (3.16)
Next, in virtue of the Hölder and Sobolev’s inequality, we are in a position to obtain
\[ I_3 = - \sum_{s=0}^{l} C_s^l \int (\nabla^s u \cdot \nabla \nabla^{l-s} u) \cdot \nabla u dx \]
\[ \leq \| \sum_{s=0}^{l} \nabla^s u \cdot \nabla^{l+1-s} u \|_{L^6} \| \nabla^{l+1} u \|_{L^2} \]
\[ \leq c \| (\mathcal{Q}, u) \|_{H^1} (\| \nabla^l \mathcal{Q} \|_{L^2}^2 + \| \nabla^{l+1} u \|_{L^2}^2). \] (3.17)
Making use of (3.1), we deduce that
\[ I_{3a} \leq \| \nabla^s u \|_{L^2} \| \nabla^{l+1-s} u \|_{L^2} \]
\[ \leq \| \nabla^{s_1} u \|_{L^2} \| \nabla^{l+1-s_1} u \|_{L^2} \]
\[ \leq c \| \nabla u \|_{H^1} \| \nabla^{l+1} u \|_{L^2}, \] (3.18)
where
\[ \theta_2 = \frac{l + 1 - s}{l + 1}, \quad \alpha_2 = \frac{l + 1}{2(l + 1 - s)} \in \left[ \frac{1}{2}, 1 \right]. \]
Similarly, we also have
\[ I_{3b} \leq \| \nabla^s u \|_{L^2} \| \nabla^{l+1-s} u \|_{L^2} \]
\[ \leq \| \nabla^s u \|_{L^2} \| \nabla^{l+1-s} u \|_{L^2} \]
\[ \leq c \| \nabla u \|_{H^1} \| \nabla^{l+1} u \|_{L^2}, \] (3.19)
where
\[ \theta_3 = \frac{l + 1 - s}{l + 1}, \quad \alpha_3 = \frac{l + 1}{2s} \in \left( \frac{1}{2}, 1 \right]. \]
As a consequence, from (3.17)-(3.19), it follows that
\[ I_3 \leq c \| u \|_{H^1} \| \nabla^{l+1} u \|_{L^2}^2. \] (3.20)
Now, we estimate the term \( I_4 \). If \( l = 0 \), appealing to the Hölder’s inequality, (2.1), (2.5) and Cauchy’s inequality, we obtain
\[ I_4 = \| f(\mathcal{Q}) \|_{L^2} \| \Delta u \|_{L^6} \|
\| \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^2}^2. \] (3.21)
If \( l = 1 \), by integration by parts, Lemma 2.3 and Lemma 2.1, we can see that
\[
I_4 = \int f(\varrho)|\Delta u|^2 \, dx
\]
\[
\leq ||f(\varrho)||_{L^\infty}||\Delta u||_{L^2}||\Delta u||_{L^2}
\]
\[
\leq ||u||_{H^1}(||\nabla q||_{L^2}^2 + ||\nabla u||_{L^2}^2).
\] (3.22)

Similarly, if \( l = 2 \), using Lemma 2.3 and Lemma 2.1, we obtain
\[
I_4 = \int (\nabla f(\varrho)\Delta u + f(\varrho)\nabla \Delta u) \cdot \text{div}\nabla^2 u \, dx
\]
\[
\leq ||\nabla f(\varrho)||_{L^\infty}||\Delta u||_{L^2}||\nabla^2 u||_{L^2} + ||f(\varrho)||_{L^3}||\nabla^3 u||_{L^2}
\]
\[
\leq ||(\varrho, u)||_{H^1}(||\nabla^2 q||_{L^2}^2 + ||\nabla^3 u||_{L^2}^2).
\] (3.23)

If \( l \geq 3 \), by integration by parts and the Hölder’s inequality, we deduce that
\[
I_4 = \int \nabla^{l-1}(f(\varrho)\Delta u) \cdot \text{div}\nabla^l u \, dx
\]
\[
= \sum_{s=0}^{l-1} C_{l-1}^s \int (\nabla^s f(\varrho)\nabla^{l-1-s} \Delta u) \cdot \text{div}\nabla^l u \, dx
\]
\[
\leq \sum_{s=0}^{l-1} ||\nabla^s f(\varrho)\nabla^{l-1-s} \Delta u||_{L^2}||\nabla^{l+1} u||_{L^2}.
\] (3.24)

If \( 0 \leq s \leq \frac{l-1}{2} \), by (2.24) and Lemma 2.1, we arrive at
\[
||\nabla^s f(\varrho)\nabla^{l-1-s} \Delta u||_{L^2} \leq ||\nabla^s f(\varrho)||_{L^\infty}||\nabla^{l+1-s} u||_{L^2}
\]
\[
\leq ||\nabla^s q||_{L^2}||\nabla^{l+s-1} u||_{L^2}
\]
\[
\leq ||\nabla^s q||_{L^2}^0 ||\nabla^s q||_{L^2}^{1-\theta_0} ||\Delta u||_{L^2}^{1-\theta_0} ||\nabla^{l+1} u||_{L^2}^{\theta_0}
\]
\[
\leq c_l||(\varrho, u)||_{H^1} (||\nabla^s q||_{L^2}^2 + ||\nabla^{l+1} u||_{L^2}^2),
\] (3.25)

where
\[
\theta_4 = \frac{l-1-s}{l-1} \in (\frac{1}{2}, 1], \quad \alpha_4 = \frac{3}{2} + \frac{s}{2(l-1-s)} \in (\frac{3}{2}, 2].
\]

By the same way, if \( \frac{l-1}{2} + 1 \leq s \leq l-1 \), in virtue of (2.1) and (2.5), we can see that
\[
||\nabla^s f(\varrho)\nabla^{l-1+s} \Delta u||_{L^2} \leq ||\nabla^s f(\varrho)||_{L^\infty}||\nabla^{l+1+s} u||_{L^2}
\]
\[
\leq ||\nabla^s q||_{L^2}||\nabla^{l+s-1} u||_{L^2}
\]
\[
\leq ||\nabla^s q||_{L^2}^0 ||\nabla^s q||_{L^2}^{1-\theta_0} ||\Delta u||_{L^2}^{1-\theta_0} ||\nabla^{l+1} u||_{L^2}^{\theta_0}
\]
\[
\leq c_l||(\varrho, u)||_{H^1} (||\nabla^s q||_{L^2}^2 + ||\nabla^{l+1} u||_{L^2}^2),
\] (3.26)

where
\[
\theta_5 = \frac{l-s}{l-2} \in (0, 1], \quad \alpha_5 = 3 - \frac{l-2}{2(s-2)} \in (2, 3).
\]

Inserting (3.25)-(3.26) into (3.24), by the Cauchy’s inequality, we obtain that for \( l \geq 3 \),
\[
I_4 \leq c_l||(\varrho, u)||_{H^1} (||\nabla^s q||_{L^2}^2 + ||\nabla^{l+1} u||_{L^2}^2).
\] (3.27)

Together (3.21)-(3.23) with (3.27), we finally obtain for \( l \geq 0 \)
\[
I_4 \leq c_l||(\varrho, u)||_{H^1} (||\nabla^s q||_{L^2}^2 + ||\nabla^{l+1} u||_{L^2}^2).
\] (3.28)

Similarly, for the term \( I_5 \), we deduce that for \( l \geq 0 \)
\[
I_5 \leq c_l||(\varrho, u)||_{H^1} (||\nabla^s q||_{L^2}^2 + ||\nabla^{l+1} u||_{L^2}^2).
\] (3.29)
Now, we estimate the term $I_6$, similar to (3.21)–(3.22), if $l = 0$, making use of (2.5) and Lemma 2.1, we can see that

$$I_6 = \int (f(q)\nabla \times w) \cdot ud\tau \leq \|q\|_{L^2} \|\nabla \times w\|_{L^2} \|u\|_{L^6} \leq \|q\|_{L^2} (\|\nabla w\|_{L^2} + \|w\|_{H^1}) \|\nabla u\|_{L^2} \leq \|(q, w)\|_{H^1} (\|q\|_{L^2}^2 + \|\nabla w, \nabla u\|_{L^2}^2). \tag{3.30}$$

If $l = 1$, by integration by parts, the Hölder’s inequality, Lemma 2.1 and Lemma 2.3, we infer that

$$I_6 = \int (f(q)\nabla \times w) \cdot \text{div} \nabla u \ud\tau \leq \|f(q)\|_{L^2} \|\nabla w\|_{L^2} \|\nabla^2 u\|_{L^2} \leq \|q\|_{H^1} \|\nabla^2 w, \nabla^2 u\|_{L^2}^2. \tag{3.31}$$

Next, for $l \geq 2$, similar to (3.24)–(3.26), there holds

$$I_6 = \sum_{s=0}^{l-1} \int (\nabla^s f(q)\nabla^{l-s-1} \nabla \times w) \cdot \text{div} \nabla u \ud\tau \leq \|\nabla^s f(q)\|_{L^2} \|\nabla^{l-s-1} \nabla \times w\|_{L^2} \|\nabla^{l+1} u\|_{L^2} \tag{3.32}$$

If $0 \leq s \leq \lfloor \frac{l-1}{2} \rfloor$, making use of the Hölder’s inequality, (2.1), we obtain

$$\|\nabla^s f(q)\nabla^{l-s-1} \nabla \times w\|_{L^2} \leq \|\nabla^s f(q)\|_{L^2} \|\nabla^{l-s-1} \nabla \times w\|_{L^6} \leq \|\nabla^s f(q)\|_{L^2} \|\nabla^{l-s-1} \nabla \times w\|_{L^6} \|\nabla^{l+1} w\|_{L^6}^{\theta_0} \leq \|(q, w)\|_{H^1} \|\nabla^s \theta\|_{L^6} \leq \|\nabla^{l+1} w\|_{L^2}, \tag{3.33}$$

where

$$\theta_0 = \frac{l-s}{l}, \quad \alpha_0 = \frac{l}{2(l-s)} \in [\frac{1}{2}, 1).$$

If $\lfloor \frac{l-1}{2} \rfloor + 1 \leq s \leq l-1$, applying Lemma 2.1 and Lemma 2.3, there holds

$$\|\nabla^s f(q)\nabla^{l-s-1} \nabla \times w\|_{L^2} \leq \|\nabla^s f(q)\|_{L^2} \|\nabla^{l-s-1} \nabla \times w\|_{L^6} \leq \|\nabla^s f(q)\|_{L^2} \|\nabla^{l-s-1} \nabla \times w\|_{L^6} \|\nabla^{l+1} w\|_{L^6}^{\theta_1} \leq c_l \|(q, w)\|_{H^1} \|\nabla^s \theta\|_{L^2}^2 + \|\nabla^{l+1} w\|_{L^2} \tag{3.34}$$

where

$$\theta_1 = \frac{l-1-s}{l-1}, \quad \alpha_1 = 2 - \frac{l-1}{2s} \in (1, \frac{3}{2}).$$

Plugging (3.33)–(3.34) into (3.32), we deduce that for $l \geq 2$,

$$I_6 \leq c_l \|(q, w)\|_{H^1} (\|\nabla^s \theta\|_{L^2}^2 + \|\nabla^{l+1} w, \nabla^{l+1} u\|_{L^2}^2). \tag{3.35}$$

Thus, combining (3.30)–(3.31) and (3.35), we conclude that for $l \geq 0$,

$$I_6 \leq c_l \|(q, w)\|_{H^1} (\|\nabla^s \theta\|_{L^2}^2 + \|\nabla^{l+1} w, \nabla^{l+1} u\|_{L^2}^2). \tag{3.36}$$

Noting that $h(q)$ is a smooth function, similar to (3.36), the term $I_7$ can be estimated as

$$I_7 \leq c_l \|(q, w)\|_{H^1} (\|\nabla^s \theta\|_{L^2}^2 + \|\nabla^{l+1} u\|_{L^2}^2). \tag{3.37}$$

In addition, observe that

$$g(q) = 1 - f(q), \quad \|\nabla^s (b \otimes b)\|_{L^2} \leq \|b\|_{H^1} \|\nabla^{l+1} b\|_{L^2}^2.$$
Thus, by integration by parts and the Hölder’s inequality, similar to the estimate of $I_6$, the term $I_8$ can be estimated as

$$I_8 = -\int (g(q)\nabla^{l-1} v \cdot (b \otimes b)) \cdot (\nabla \cdot \nabla u) dx + \sum_{i=1}^{l-1} C_i^f \int (\nabla^{l} f(q) \nabla^{l-1-i} v \cdot (b \otimes b)) \cdot (\nabla \cdot \nabla u) dx$$

$$\leq ||\nabla^{l} (b \otimes b)||_{L^2} ||\nabla^{l+1} u||_{L^2} + \sum_{i=1}^{l-1} ||\nabla^{l} f(q) \cdot \nabla^{l-1-i} v \cdot (b \otimes b)||_{L^2} ||\nabla^{l+1} u||_{L^2}$$

$$\leq c_i ||(q, b)||_{H^1} (||\nabla^{l} ||_{L^2} \nabla^{l+1} u||_{L^2}^2 + ||(\nabla^{l+1} b, \nabla^{l+1} u)||_{L^2}^2).$$

Likewise, for the term $I_9$, we can also obtain

$$I_9 \leq c_i ||(q, b)||_{H^1} (||\nabla^{l} ||_{L^2} \nabla^{l+1} u||_{L^2}^2 + ||(\nabla^{l+1} b, \nabla^{l+1} u)||_{L^2}^2).$$

Taking into account (3.7), (3.16), (3.20), (3.28), (3.29), (3.36)-(3.39) and note that $I_{10} + I_{11} = 0$, we finally obtain

$$\frac{1}{2} \frac{d}{dt} \int (||\nabla^{l} f||^2 + ||\nabla^{l} b||^2 + ||\nabla^{l} u||^2) dx + 2 \int ||\nabla^{l+1} u||^2 dx$$

$$\leq I_2 + c_i ||(q, b, q, u, w, b)||_{H^1} (||\nabla^{l} ||_{L^2} + ||\nabla^{l+1} (\nabla^{l} b, \nabla^{l+1} u)||_{L^2}^2).$$

Next, for any integer $k \geq 0$, by the $\nabla^l (l = k, k+1, k+2)$ energy estimate, for (1.5), (3.17), (3.18) on $w$ and $b$, there holds

$$\frac{1}{2} \frac{d}{dt} \int (||\nabla^{l} w||^2 + ||\nabla^{l} b||^2) dx + 2 ||\nabla^{l} w||^2 + 3 ||\nabla^{l+1} w||^2 \nabla^{l+1} b||^2_{L^2}$$

$$+ 2 \int ||\nabla^{l} f(q) \Delta w \cdot \nabla^{l} w dx - 2 \int \nabla^{l} (f(q) \Delta w) \cdot \nabla^{l} w dx$$

$$\leq I_3 + I_{11} + I_{12}.$$
where in the last inequality, we have used fact $|\nabla \times u|^2 \leq 2|\nabla u|^2$.

Combining (3.40) and (3.46)-(3.47), yields that

$$
\frac{1}{2} \frac{d}{dt} \sum_{i=k}^{k+2} ||\nabla(\nabla \Phi, \varrho, u, w, b)||_{L^2}^2 + \frac{1}{2} \sum_{i=k}^{k+2} ||\nabla w||_{L^2}^2 + \sum_{i=k}^{k+2} \left( \frac{1}{3} ||\nabla^{i+1} u||_{L^2}^2 + 3 ||\nabla^{i+1} w||_{L^2}^2 + ||\nabla^{i+1} b||_{L^2}^2 \right)
$$

$$
\leq c_l ||(\nabla \Phi, \varrho, u, w, b)||_{H^3}^2 \sum_{i=k}^{k+2} ||\nabla w||_{L^2}^2.
$$

This, together with (3.1), whence (3.2). Thus, we have completed the proof of Lemma 3.1 \qed

Note that in Lemma 3.1, we only derive the dissipation estimate of $u, w, b$. Now, we proceed to derive the dissipation estimate of $\nabla \Phi$ and $\varrho$ by constructing some interactive energy functions in the following lemma.

**Lemma 3.2.** Let all assumptions in Lemma 3.1 are in force, then for any $k \geq 0$, there holds

$$
\frac{1}{2} \frac{d}{dt} \sum_{i=k}^{k+1} \int \nabla \Phi \cdot \nabla l_{i+1} \varrho \, dx + \int \frac{1}{2} \sum_{i=k}^{k+1} (||\nabla l_{i+1} \varrho||_{L^2}^2 + ||\nabla^{i+1} \nabla \Phi||_{L^2}^2 + ||\nabla^{i+2} \nabla \Phi||_{L^2}^2)
$$

$$
\leq c_l \delta \sum_{i=k}^{k+1} (||\nabla^{i+1} \varrho||_{L^2}^2 + ||\nabla^{i+2} (w, b)||_{L^2}^2) + \sum_{i=k}^{k+1} (||\nabla^{i+1} u||_{L^2}^2 + 2 ||\nabla^{i+1} w||_{L^2}^2 + 4 ||\nabla^{i+2} u||_{L^2}^2). \tag{3.48}
$$

**Proof.** Applying $\nabla (l = k, k+1)$ to (1.5) and then taking the $L^2$ inner product with $\nabla^{i+1} \varrho$, we have

$$
\int \nabla \cdot \partial_t u \cdot \nabla^{i+1} \varrho \, dx + ||\nabla^{i+1} \varrho||_{L^2}^2
$$

$$
= \int \nabla \Delta u \cdot \nabla^{i+1} \varrho \, dx + \int \nabla \nabla \div u \cdot \nabla^{i+1} \varrho \, dx + \int \nabla \nabla \times w \cdot \nabla^{i+1} \varrho \, dx
$$

$$
- \int \nabla (u \cdot \nabla u) \cdot \nabla^{i+1} \varrho \, dx - \int \nabla (f(\varrho) \nabla u) \cdot \nabla^{i+1} \varrho \, dx - \int \nabla (f(\varrho) \div u) \cdot \nabla^{i+1} \varrho \, dx
$$

$$
- \int \nabla (f(\varrho) \nabla \times w) \cdot \nabla^{i+1} \varrho \, dx - \int \nabla (h(\varrho) \nabla \varrho) \cdot \nabla^{i+1} \varrho \, dx + \int \nabla (g(\varrho) b \cdot \nabla b) \cdot \nabla^{i+1} \varrho \, dx
$$

$$
- \frac{1}{2} \int \nabla (g(\varrho) |\varrho|^2) \cdot \nabla^{i+1} \varrho \, dx + \int \nabla \Phi \cdot \nabla \varrho \, dx
$$

$$
:= \sum_{j=1}^{11} J_j.
$$

On the other hand, it is clearly that

$$
\sum_{j=1}^{11} J_j = \frac{d}{dt} \int \nabla u \cdot \nabla^{i+1} \varrho \, dx - \int \nabla u \cdot \partial_t \nabla^{i+1} \varrho \, dx + ||\nabla^{i+1} \varrho||_{L^2}^2
$$

$$
= \frac{d}{dt} \int \nabla u \cdot \nabla^{i+1} \varrho \, dx + \int \nabla^{i+1} \div u \cdot \nabla u \, dx + \int \nabla^{i+1} \div (\varrho u) \cdot \nabla u \, dx + ||\nabla^{i+1} \varrho||_{L^2}^2
$$

$$
:= \frac{d}{dt} \int \nabla u \cdot \nabla^{i+1} \varrho \, dx - ||\nabla^{i+1} u||_{L^2}^2 + ||\nabla^{i+1} \varrho||_{L^2}^2 + J_{12}. \tag{3.49}
$$

Now, we concentrate our attention on estimating the terms $J_1 - J_{12}$. First, employing the Cauchy’s inequality, it holds that

$$
J_1 + J_2 \leq 4 ||\nabla^{i+2} u||_{L^2}^2 + \frac{1}{4} ||\nabla^{i+1} \varrho||_{L^2}^2,
$$

$$
J_3 \leq 2 ||\nabla^{i+1} w||_{L^2}^2 + \frac{1}{4} ||\nabla^{i+1} \varrho||_{L^2}^2,
$$

where in the last inequality, we have used the fact that $|\nabla \times w|^2 \leq 2|\nabla w|^2$. 

Moreover, taking into account (3.17)-(3.20), we are in a position to obtain
\[ J_4 = \int \text{div} \nabla (u \cdot \nabla u) \cdot \nabla q \, dx \]
\[ \leq ||\text{div} \nabla (u \cdot \nabla u)||_{L^4} ||\nabla q||_{L^6} \]
\[ \leq c||u||_{H^1}(||\nabla^{l+1} q||^2_{L^2} + ||\nabla^{l+1} u||^2_{L^2}). \]
Similar to (3.24), using the Hölder’s inequality, Lemma 2.1 and Lemma 2.3, the terms \( J_5, J_6 \) can be estimated as
\[ J_5 \leq \left\| \sum_{s=0}^{l} C_s^l \nabla^s f(q) \nabla^{l-s} \Delta u ||\nabla^{l+1} q||_{L^2} \right\|
\[ \leq c||q, u||_{H^1}(||\nabla^{l+1} q||^2_{L^2} + ||\nabla^{l+2} u||^2_{L^2}), \]
\[ J_6 \leq \left\| \sum_{s=0}^{l} C_s^l \nabla^s f(q) \nabla^{l-s} \nabla \text{div} u ||\nabla^{l+1} q||_{L^2} \right\|
\[ \leq c||q, u||_{H^1}(||\nabla^{l+1} q||^2_{L^2} + ||\nabla^{l+2} u||^2_{L^2}). \]
Furthermore, taking into account (3.36)-(3.37), applying the Hölder’s inequality, Lemma 2.1 and Lemma 2.3, we obtain
\[ J_7 \leq \left\| \sum_{s=0}^{l} \nabla^s f(q) \nabla^{l-s} \nabla \times \omega ||\nabla^{l+1} q||_{L^2} \right\|
\[ \leq c||q, w||_{H^1}(||\nabla^{l+1} q||^2_{L^2} + ||\nabla^{l+2} w||^2_{L^2}), \]
\[ J_8 \leq \left\| \sum_{s=0}^{l} \nabla^s h(q) \nabla^{l-s} \nabla \omega ||\nabla^{l+1} q||_{L^2} \right\|
\[ \leq c||q||_{H^1}(||\nabla^{l+1} q||^2_{L^2}). \]
Similar to the estimates of (3.28)-(3.29), we further obtain
\[ J_9 + J_{10} \leq \left( \sum_{s=0}^{l} \nabla^s g(q) \nabla^{l-s} \omega \cdot (b \otimes b) \right) ||\nabla^{l+1} q||_{L^2} \]
\[ \leq c||q, b||_{H^1}(||\nabla^{l+1} q||^2_{L^2} + ||\nabla^{l+2} b||^2_{L^2}). \]
Next, by integration by parts and (1.5), we infer that
\[ J_{11} = -\int \nabla^l \Delta \Phi \cdot \nabla q \, dx = -\int |\nabla^l q|^2 \, dx. \]
Furthermore, from (1.5), there holds
\[ ||\nabla^{l+1} \nabla \Phi||^2_{L^2} = ||\nabla^l \Delta \Phi||^2_{L^2} = ||\nabla^l q||^2_{L^2}, \quad ||\nabla^{l+2} \nabla \Phi||^2_{L^2} = ||\nabla^{l+1} q||^2_{L^2}. \quad (3.50) \]
Finally, by integration by parts and Lemma 2.1, we get
\[ -J_{12} = \int \nabla^{l+1} (q u) \cdot \nabla^{l+1} u \, dx \]
\[ = \int (\nabla^{l+1} q u) \cdot \nabla^{l+1} u \, dx + \int q \nabla^{l+1} u \cdot \nabla^{l+1} u \, dx + \int \sum_{s=1}^{l} C_s^l (\nabla^s q \nabla^{l+1-s} u) \cdot \nabla^{l+1} u \, dx \]
\[ \leq ||u||_{\infty}(||q||_{\infty})(||\nabla^{l+1} q||^2_{L^2} + ||\nabla^{l+1} u||^2_{L^2}) + \sum_{s=1}^{l} ||\nabla^s q \nabla^{l+1-s} u||_{L^2} ||\nabla^{l+1} u||_{L^2} \]
\[ \leq c||q, u||_{H^1}(||\nabla^{l+1} q||^2_{L^2} + ||\nabla^{l+1} u||^2_{L^2}). \]
Putting these estimations into (3.49), and summing up with $l = k, k + 1$, we finally obtain
\[
\frac{d}{dt} \sum_{l=k}^{k+1} \int \nabla^l u \cdot \nabla^{l+1} \varrho \, dx + \frac{1}{2} \sum_{l=k}^{k+1} \left( \|\nabla^l \varrho\|_{L^2}^2 + \|\nabla^{l+1} \varrho\|_{L^2}^2 + \|\nabla^{l+1} \nabla \Phi\|_{L^2}^2 + \|\nabla^{l+2} \nabla \Phi\|_{L^2}^2 \right) \leq c \|\varrho, u, w, b\|_{H^1} \sum_{l=k}^{k+1} \|\nabla^{l+1} \varrho\|_{L^2}^2 + (1 + c \|\varrho, u\|_{H^1}) \sum_{l=k}^{k+1} \|\nabla^{l+1} u\|_{L^2}^2 + c \|\varrho, w, b\|_{H^1} \sum_{l=k}^{k+1} \|\nabla^{l+2} (w, b)\|_{L^2}^2.
\]
This, together with (3.1) implies (3.48). Thus, we have completed the proof of Lemma 3.2. □

3.2. Local existence of solution. In this subsection, we devote to proving the local existence of solution $(\nabla \Phi, \varrho, u, w, b)$ in $H^1$-norm. Firstly, we construct the solution sequence $(\nabla \Phi^j, \varrho^j, u^j, w^j, b^j)_{j \geq 0}$ by solving iteratively the following Cauchy problem for $j \geq 0$:
\[
\begin{align*}
\partial_t \varrho^{j+1} + \text{div} \, u^{j+1} &= M_1^{j+1}, \\
\partial_t u^{j+1} + \nabla \varrho^{j+1} - \Delta u^{j+1} - \text{div} \, u^{j+1} - \nabla \times w^{j+1} - \nabla \Phi^{j+1} &= M_2^{j+1}, \\
\partial_t w^{j+1} + 2 w^{j+1} - \Delta w^{j+1} - 2 \text{div} \, w^{j+1} - \nabla \times u^{j+1} &= M_3^{j+1}, \\
\partial_t b^{j+1} - \Delta b^{j+1} &= M_4^{j+1}, \\
\Delta \Phi^{j+1} &= \varrho^{j+1}, \\
\text{div} \, b^{j+1} &= 0, \quad t > 0, \, x \in \mathbb{R}^3,
\end{align*}
\]
with initial data
\[
(\varrho^{j+1}, u^{j+1}, w^{j+1}, b^{j+1})(x, 0) = (\varrho_0, u_0, w_0, b_0) \rightarrow (0, 0, 0, 0) \quad \text{as} \quad |x| \rightarrow \infty, \tag{3.52}
\]
where the nonlinear terms $M_i^{j+1}$ $(i = 1, 2, 3, 4)$ are defined as
\[
\begin{align*}
M_1^{j+1} &= - (\varrho^{j} \text{div} \, u^{j+1} + u^{j} \cdot \nabla \varrho^{j+1}), \\
M_2^{j+1} &= - u^{j} \cdot \nabla u^{j+1} - f(\varrho^{j})[\Delta u^{j+1} + \nabla \text{div} \, u^{j+1} + \nabla \times w^{j+1}] \\
&\quad - h(\varrho^{j})[\nabla \varrho^{j+1} + g(\varrho^{j})(\nabla \times b^{j+1}) \times b^{j}], \\
M_3^{j+1} &= - u^{j} \cdot \nabla w^{j+1} - f(\varrho^{j})[\Delta w^{j+1} + 2 \text{div} \, w^{j+1} - 2 w^{j+1} + \nabla \times u^{j+1}], \\
M_4^{j+1} &= b^{j} \cdot \nabla u^{j+1} - u^{j} \cdot \nabla b^{j+1} - b^{j} \text{div} \, u^{j+1},
\end{align*}
\]
where $(\nabla \Phi^0, \varrho^0, u^0, w^0, b^0) \equiv (0, 0, 0, 0, 0)$ is set at initial step. In what follows, for simplicity, we may denote $(\nabla \Phi^j, \varrho^j, u^j, w^j, b^j)_{j \geq 0}$ and $(\nabla \Phi^0, \varrho_0, u_0, w_0, b_0)$ by $(\varphi^j)_{j \geq 0}$ and $\varphi_0$, respectively. Then, one has the following result.

Lemma 3.3. Let all assumptions in Theorem 3.1 hold for $N = 3$. Then, there holds
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\varphi^{j+1}(t)\|_{H^1}^2 + \left( \frac{\sqrt{2}}{\sqrt{3}} \|\nabla u^{j+1}\|_{H^1} + \sqrt{3} \|\nabla w^{j+1}\|_{H^1} \right) \|\varphi^{j+1}\|_{H^1}^2 + \frac{1}{2} \|w^{j+1}\|_{H^1}^2 \\
\leq c \|\varrho^{j}, u^{j}, b^{j}\|_{H^1}(\|\varphi^{j+1}\|_{H^1}^2 + \|\nabla \varrho^{j+1}\|_{H^1}^2 + \|\nabla u^{j+1}\|_{H^1}^2 + \|\nabla w^{j+1}\|_{H^1}^2) + c \|\varrho^{j}\|_{H^1}(\|\varphi^{j+1}\|_{H^1}^2 + \|\nabla \varphi^{j+1}\|_{H^1}^2), \tag{3.53}
\end{align*}
\]
where $c$ is a positive constant independent of $j$. 

Proof. Similar to Lemma 3.1 for $0 \leq l \leq N$, $N = 3$, for $(3.51)_1-(3.51)_2$ on $q^{j+1}$ and $u^{j+1}$, we get

$$\frac{1}{2} \frac{d}{dt} \int |\nabla q^{j+1}|^2 + |\nabla u^{j+1}|^2dx + 2 \int |\nabla u^{j+1}|^2dx - \int \nabla \nabla \Phi^{j+1} \cdot \nabla u^{j+1}dx$$

$$= \int -\nabla (u^j \cdot \nabla q^{j+1}) \cdot \nabla q^{j+1} + \nabla (\nabla \Phi^{j+1}) \cdot \nabla \Phi^{j+1} \cdot \nabla u^{j+1}$$

$$- \int \nabla (u^j \cdot \nabla u^{j+1}) \cdot \nabla u^{j+1} + \nabla (f(q^j)\nabla u^{j+1}) \cdot \nabla u^{j+1}$$

$$- \nabla (f(q^j) \cdot \nabla \cdot w^{j+1}) \cdot \nabla u^{j+1} + \nabla [g(q^j)(\nabla \Phi^{j+1}) \cdot \nabla u^{j+1} + \nabla [g(q^j)(\nabla \Phi^{j+1}) \cdot \nabla u^{j+1}.$$ 

$$:= \sum_{i=1}^{9} H_i. \tag{3.54}$$

First of all, by integration by parts and $\ref{3.51}_5$, we infer that

$$\int \nabla \Phi^{j+1} \cdot \nabla (\nabla \Phi^{j+1})d_2x = \int \nabla \Phi^{j+1} \cdot \nabla (\nabla \Phi^{j+1})d_2x$$

$$= -\int \nabla \Phi^{j+1} \cdot \nabla \Phi^{j+1} \cdot \nabla (\nabla \Phi^{j+1})d_2x$$

$$= -\int \nabla \Phi^{j+1} \cdot \nabla \Phi^{j+1} \cdot \nabla (\nabla \Phi^{j+1})d_2x$$

$$= \frac{1}{2} \frac{d}{dt} \int |\nabla \Phi^{j+1}|^2dx - \int \nabla \Phi^{j+1} \cdot \nabla (\nabla \Phi^{j+1})d_2x. \tag{3.55}$$

Applying Lemma 2.1 and the Young’s inequality, it is easy to see that

$$\int \nabla \Phi^{j+1} \cdot \nabla (\nabla \Phi^{j+1})d_2x \leq ||\nabla \Phi^{j+1}||_{L^2} ||\nabla (\nabla \Phi^{j+1})||_{L^2}$$

$$\leq c|\nabla q^j||_{H^s}||\nabla \Phi^{j+1}||_{H^s}^2 + ||\nabla u^{j+1}||_{H^s}^2. \tag{3.56}$$

On the other hand, if $l = 0$, it holds that

$$\int \Phi^{j+1} \cdot (u^j \cdot \nabla \Phi^{j+1})d_2x = \int \Phi^{j+1} \cdot (u^j \cdot \nabla \Phi^{j+1})d_2x$$

$$\leq ||\Phi^{j+1}||_{L^2} ||u^j||_{L^\infty} ||\nabla \Phi^{j+1}||_{L^2} + ||\Phi^{j+1}||_{L^2} ||u^j||_{L^\infty} ||\nabla \Phi^{j+1}||_{L^2}$$

$$\leq c||u^j||_{H^s}^2 ||\nabla \Phi^{j+1}||_{L^2} + ||\Phi^{j+1}||_{L^2}^2. \tag{3.57}$$

If $l \geq 1$, then we obtain

$$\int \nabla \Phi^{j+1} \cdot \nabla (u^j \cdot \nabla \Phi^{j+1})d_2x = \int \nabla \Phi^{j+1} \cdot \nabla (u^j \cdot \nabla \Phi^{j+1})d_2x$$

$$\leq ||\nabla \Phi^{j+1}||_{L^2} ||u^j||_{L^\infty} ||\nabla \Phi^{j+1}||_{L^2}$$

$$\leq c||u^j||_{H^s}^2 ||\nabla \Phi^{j+1}||_{L^2} + ||\Phi^{j+1}||_{L^2}^2. \tag{3.58}$$

Inserting (3.56)-(3.58) into (3.55), we have

$$- \int \nabla \nabla \Phi^{j+1} \cdot \nabla u^{j+1}dx \geq \frac{1}{2} \frac{d}{dt} \int |\nabla \Phi^{j+1}|^2dx - c||\nabla q^j, u^j||_{H^s}^2 ||\Phi^{j+1}||_{H^s}$$

$$- c||\nabla q^j, u^j||_{H^s} ||\nabla q^j, u^j||_{H^s} \tag{3.59}.$$
Likewise, we can also obtain
\[
H_2 \leq c\|q'\|_{H^1}\|\nabla u^{j+1}\|_{H^1}^2, \tag{3.61}
\]
\[
H_4 \leq c\|u'\|_{H^1}\|\nabla u^{j+1}\|_{H^1}^2. \tag{3.62}
\]

Next, for the term $H_5$, by the Hölder’s inequality and Lemma 2.3 it holds that
\[
H_5 = \int \nabla^{l-1} (f(q^j)\Delta u^{j+1}) \cdot \text{div} \nabla u^{j+1}
\leq \sum_{s=0}^{l-1} \||\nabla^s f(q^j)\cdot \nabla^{l-1-s} \Delta u^{j+1}\|_{L^2} \|\nabla^{l-s} u^{j+1}\|_{L^2} \leq c\|q'\|_{H^2}\|\nabla u^{j+1}\|_{H^1}^2. \tag{3.63}
\]

Similarly, we further obtain
\[
H_6 \leq c\|q'\|_{H^1}\|\nabla u^{j+1}\|_{H^1}^2, \tag{3.64}
\]
\[
H_7 \leq c\|q'\|_{H^1}\|\nabla (w^{j+1}, u^{j+1})\|_{H^1}^2 \tag{3.65}
\]
\[
H_8 \leq c\|q'\|_{H^2}\|\nabla\Phi^{j+1}\|_{H^1}^2. \tag{3.66}
\]

Finally, for the term $H_9$, note that $g(q) = 1 - f(q)$, using the Hölder and Young’s inequality, we deduce that
\[
H_9 = \int g(q^j)\nabla^l((\nabla \times b^{j+1}) \times b^j) \cdot \nabla^l u^{j+1} \, dx - \sum_{s=1}^{l-1} C^s_1 \int [\nabla^s f(q^j)\nabla^{l-s}((\nabla \times b^{j+1}) \times b^j)] \cdot \nabla^l u^{j+1} \, dx
\leq \|\nabla((\nabla \times b^{j+1}) \times b^j)\|_{L^2} \|\nabla u^{j+1}\|_{L^2} + \sum_{s=1}^{l-1} \||\nabla^s f(q^j)\|_{L^2} \|\nabla^{l-s}((\nabla \times b^{j+1}) \times b^j)\|_{L^2} \|\nabla^l u^{j+1}\|_{L^2}
+ \sum_{s=1}^{l-1} \||\nabla^s f(q^j)\|_{L^2} \|\nabla^{l-s}((\nabla \times b^{j+1}) \times b^j)\|_{L^2} \|\nabla^l u^{j+1}\|_{L^2}
\leq c\|b'\|_{H^1}\|\nabla (u^{j+1}, b^{j+1})\|_{H^1}^2 + c\|q'\|_{H^2}\|\nabla\Phi^{j+1}\|_{H^1}^2. \tag{3.67}
\]

Plugging (3.59)-(3.67) into (3.54), we conclude that
\[
\frac{1}{2} \frac{d}{dt} \int \|\nabla u^{j+1}\|^2 + \|\nabla v^{j+1}\|^2 + \|\nabla \Phi^{j+1}\|^2 \, dx + 2 \int |\nabla^{j+1} u^{j+1}|^2 \, dx
\leq H_3 + c\|q'\|_{H^1}\|\nabla (u^{j+1}, b^{j+1})\|_{H^1}^2 + c\|q'\|_{H^2}\|\nabla\Phi^{j+1}\|_{H^1}^2. \tag{3.68}
\]

On the other hand, from (3.51)3-(3.51)4, we can see that for $1 \leq l \leq 3$
\[
\frac{1}{2} \frac{d}{dt} \int \|\nabla w^{j+1}\|^2 + |\nabla b^{j+1}|^2 \, dx + \|\nabla (\sqrt{3}w^{j+1}, b^{j+1})\|_{L^2}^2 + 2\|\nabla w^{j+1}\|_{L^2}^2
= \int \nabla l((\nabla \times u^{j+1}) \cdot \nabla w^{j+1} - \nabla l(u^{j+1} \cdot \nabla w^{j+1}) - \nabla l((f(q^j)\Delta w^{j+1}) \cdot \nabla w^{j+1})
- 2\nabla l(f(q^j)\text{div} w^{j+1}) \cdot \nabla w^{j+1} + 2\nabla l(f(q^j)w^{j+1}) \cdot \nabla w^{j+1} - \nabla l(f(q^j)\nabla \times u^{j+1}) \cdot \nabla w^{j+1}
+ \nabla l(b^{j+1} \cdot \nabla u^{j+1}) \cdot \nabla b^{j+1} - \nabla l(u^{j+1} \cdot \nabla b^{j+1}) \cdot \nabla b^{j+1} - \nabla l(b^{j+1} \cdot \text{div} u^{j+1}) \cdot \nabla b^{j+1} \, dx
\leq \sum_{i=10}^{19} H_i. \tag{3.69}
\]

Similar to (3.60)-(3.62), we arrive at
\[
H_{11} \leq c\|u'\|_{H^1}\|\nabla w^{j+1}\|_{H^1}^2, \tag{3.70}
\]
\[
H_{17} \leq c\|u'\|_{H^1}\|\nabla b^{j+1}\|_{H^1}^2, \tag{3.71}
\]
Furthermore, in virtue of (2.3) and the Hölder’s inequality, for the term $H_{16}$, we are in a position to obtain

$$H_{16} + H_{18} \leq c||b'||_{H^1}||\nabla(u^{i,j+1}, b^{i,j+1})||^2_{H^1}. \tag{3.72}$$

Next, taking into account (3.63), we are in a position to obtain

$$H_{12} + H_{13} \leq c||q'||_{H^2}||\nabla w^{j+1}||^2_{H^1}. \tag{3.73}$$

Furthermore, in virtue of (2.3) and the Hölder’s inequality, for the term $H_{14}$, we can find that

$$H_{14} = 2 \sum_{j=0}^{l} C_i^j \int \nabla^j f(q^j) \nabla^{l-j} w^{j+1} \cdot \nabla^j w^{j+1} dx + 2 \sum_{j=0}^{l} C_i^j \int \nabla^j f(q^j) \nabla^{l-j} w^{j+1} \cdot \nabla^j w^{j+1} dx$$

$$\leq \sum_{j=0}^{l} ||\nabla^j f(q^j)||_{L^\infty} ||\nabla^{l-j} w^{j+1}||_{L^2} ||\nabla^j w^{j+1}||_{L^2} + \sum_{j=0}^{l} ||\nabla^j f(q^j)||_{L^\infty} ||\nabla^{l-j} w^{j+1}||_{L^2} ||\nabla^j w^{j+1}||_{L^2}$$

$$\leq c||q'||_{H^2} (||w^{j+1}||^2_{H^3} + ||\nabla w^{j+1}||^2_{H^2}). \tag{3.74}$$

Similar to (3.65), the term $H_{15}$ can be estimated as

$$H_{15} \leq c||q'||_{H^2} ||\nabla(u^{i,j+1}, w^{j+1})||^2_{H^1}. \tag{3.75}$$

Inserting (3.70)-(3.75) into (3.69), we conclude that

$$\frac{1}{2} \frac{d}{dt} ||\nabla'(w^{j+1}, b^{j+1})||^2_{L^2} + ||\nabla'(\sqrt{q} w^{j+1}, b^{j+1})||^2_{L^2} + 2||\nabla w^{j+1}||^2_{L^2}$$

$$\leq c(||q'||_{H^2} ||\nabla(u^{i,j+1}, w^{j+1}, b^{j+1})||^2_{H^3} + c||q'||_{H^2} ||w^{j+1}||^2_{H^2} + H_{10}. \tag{3.76}$$

Now, combining (3.68) and (3.76), taking into account that $H_3 + H_{10} = 2H_{10}$ and (3.47), then we have (3.53). This completes the proof of Lemma 3.3. \square

Similar to Lemma 3.2, now, we are going to deduce the dissipative estimate of $\nabla \Phi^{j+1}, q^{j+1}$.

**Lemma 3.4.** Let all assumptions in Theorem 1.1 hold for $N = 3$. Then, for $l = 0, 1, 2$, there holds

$$\frac{d}{dt} \sum_{j=0}^{l} \int \nabla u^{j+1} \cdot \nabla q^{j+1} dx + \frac{1}{4} ||(q^{j+1}, \nabla q^{j+1}, \nabla^2 q^{j+1}, \nabla^3 \Phi^{j+1})||^2_{H^2}$$

$$\leq ||\nabla u^{j+1}||^2_{H^2} + 2||\nabla w^{j+1}||^2_{H^2} + 4||\nabla w^{j+1}||^2_{H^2} + c||q'||_{H^2} ||b'||_{H^2} ||\nabla(q^{j+1}, b^{j+1})||^2_{H^2}$$

$$+ c||q'||_{H^2} ||(\nabla q^{j+1}, u^{j+1}, w^{j+1}, b^{j+1})||^2_{H^2} + ||\nabla^2 u^{j+1}||^2_{H^2}. \tag{3.77}$$

**Proof.** Applying $\nabla^l$ ($0 \leq l \leq 2$) to (3.51) and then taking the $L^2$ inner product with $\nabla^{l+1} q^{j+1}$, we obtain

$$\int \nabla^l \partial_t u^{j+1} \cdot \nabla^{l+1} q^{j+1} dx + ||\nabla^{l+1} q^{j+1}||^2_{L^2} = \int \nabla^l \Delta u^{j+1} \cdot \nabla^{l+1} q^{j+1} dx$$

$$+ \int \nabla^l \nabla \text{div} u^{j+1} \cdot \nabla^{l+1} q^{j+1} dx + \int \nabla^l \nabla \times w^{j+1} \cdot \nabla^{l+1} q^{j+1} dx - \int \nabla^l (u^{j+1} \cdot \nabla u^{j+1}) \cdot \nabla^{l+1} q^{j+1} dx$$

$$- \int \nabla^l (f(q^j) \Delta u^{j+1}) \cdot \nabla^{l+1} q^{j+1} dx - \int \nabla^l (f(q^j) \nabla \text{div} u^{j+1}) \cdot \nabla^{l+1} q^{j+1} dx$$

$$- \int \nabla^l (f(q^j) \nabla w^{j+1}) \cdot \nabla^{l+1} q^{j+1} dx - \int \nabla^l (b(q^j) \nabla q^{j+1}) \cdot \nabla^{l+1} q^{j+1} dx$$

$$+ \int \nabla^l (q^j) (\nabla \times b^{j+1}) \cdot \nabla^{l+1} q^{j+1} dx + \int \nabla^l \Phi^{j+1} \cdot \nabla^{l+1} q^{j+1} dx$$

$$:= \sum_{i=1}^{10} W_i.
On the other hand, we can see that
\[
\sum_{i=1}^{10} W_i = \frac{d}{dt} \int \nabla^i u^{j+1} \cdot \nabla^{j+1} e^{j+1} \, dx - \int \nabla^i u^{j+1} \cdot \nabla^{j+1} \partial_t e^{j+1} \, dx + \|\nabla^{j+1} e^{j+1}\|_{L^2}^2
\]
\[
= \frac{d}{dt} \int \nabla^i u^{j+1} \cdot \nabla^{j+1} e^{j+1} \, dx + \int \nabla^i u^{j+1} \cdot \nabla^{j+1} \text{div} \, u^{j+1} \, dx
\]
\[
+ \int \nabla^i u^{j+1} \cdot \nabla^{j+1} (\partial_t e^{j+1} + u^j \cdot \nabla e^{j+1}) \, dx + \|\nabla^{j+1} e^{j+1}\|_{L^2}^2
\]
\[
:= \frac{d}{dt} \int \nabla^i u^{j+1} \cdot \nabla^{j+1} e^{j+1} \, dx - \|\nabla^{j+1} u^{j+1}\|_{L^2}^2 + \|\nabla^{j+1} e^{j+1}\|_{L^2}^2 + W_{11}. \quad (3.78)
\]

Now, we turn to estimate the terms \( W_1 - W_{11} \). First, by the Cauchy’s inequality, we can see that \( W_1 - W_3 \) can be estimated as
\[
W_1 + W_2 \leq \frac{1}{4} \|\nabla^{j+1} e^{j+1}\|_{L^2}^2 + 4\|\nabla^{j+2} u^{j+1}\|_{L^2}^2
\]
\[
\leq \frac{1}{4} \|\nabla e^{j+1}\|_{H^2}^2 + 4\|\nabla^2 u^{j+1}\|_{H^2}^2
\]
\[
W_3 \leq \frac{1}{4} \|\nabla^{j+1} e^{j+1}\|_{L^2}^2 + 2\|\nabla^{j+1} w^{j+1}\|_{L^2}^2
\]
\[
\leq \frac{1}{4} \|\nabla e^{j+1}\|_{H^2}^2 + 2\|\nabla^2 w^{j+1}\|_{H^2}^2.
\]

Furthermore, applying the Hölder’s inequality and Lemma \( 2.1 \) for the term \( W_4 \), we have
\[
W_4 \leq c\|u^j\|_{H^2} \|\nabla (q^{j+1}, u^{j+1})\|_{H^2}^2.
\]

Similar to \( 3.63 \), we obtain
\[
W_5 + W_6 \leq c\|q^j\|_{H^2} (\|\nabla q^{j+1}\|_{H^2}^2 + \|\nabla^2 u^{j+1}\|_{H^2}^2).
\]

In addition, taking into account \( 3.65 \), we deduce that
\[
W_7 + W_8 \leq c\|q^j\|_{H^2} \|\nabla (q^{j+1}, w^{j+1})\|_{H^2}^2.
\]

Next, similar to \( 3.67 \), it holds that
\[
W_9 = \int g(q^j) \nabla^i ((\nabla \times b^{j+1}) \times b^j) \cdot \nabla^{j+1} e^{j+1} \, dx - \sum_{s=1}^{I} C_s^j \int [\nabla^s f(q^j) \nabla^{j-s} ((\nabla \times b^{j+1}) \times b^j)] \cdot \nabla^{j+1} e^{j+1} \, dx
\]
\[
\leq c\|b^j\|_{H^2} \|\nabla (q^{j+1}, b^{j+1})\|_{H^2}^2 + c\|q^j\|_{H^2} \|b^j\|_{H^2} \|\nabla (q^{j+1}, b^{j+1})\|_{H^2}^2.
\]

Moreover, by \( 3.51 \), it is obvious that
\[
W_{10} = -\int |\nabla^i e^{j+1}|^2 \, dx.
\]

Finally, the term \( W_{11} \) can be estimated as
\[
W_{11} = -\int (\nabla \cdot \nabla u^{j+1}) \cdot \nabla^i (q^j \text{div} \, u^{j+1} + u^j \cdot \nabla e^{j+1}) \, dx
\]
\[
\leq c\|(q^j, u^j)\|_{H^2} \|\nabla (q^{j+1}, u^{j+1})\|_{H^2}^2.
\]

Putting these estimates into \( 3.78 \) and summing up with \( l = 0, 1, 2 \), we finally obtain \( 3.77 \). This completes the proof of Lemma \( 3.4 \) \( \square \)

Based on Lemma \( 3.3 \) and \( 3.4 \), we then immediately have the following result.

**Theorem 3.1.** Let all assumptions in Theorem \( 2.7 \) hold for \( N = 3 \). There are constants \( \varepsilon_1 > 0 \), \( T_1 > 0 \), \( M_0 > 0 \) such that if \( \|\varphi_0\|_{H^2} \leq \varepsilon_1 \), then for each \( j \geq 0 \), \( \varphi^j \in C([0, T_1]; H^3) \) is well defined and
\[
\sup_{0 \leq t \leq T_1} \|\varphi^j(t)\|_{H^2} \leq M_0, \quad j \geq 0. \quad (3.79)
\]
Furthermore, \((\mathcal{A}^j)_{j\geq 0}\) is a Cauchy sequence in the Banach space \(C([0,T_1]; H^3)\), the corresponding limit function denoted by \(\mathcal{A}(t)\) belongs to \(C([0,T_1]; H^3)\) with
\[
\sup_{0\leq t \leq T_1} \|\mathcal{A}(t)\|_{H^3} \leq M_0, \tag{3.80}
\]
and \(\mathcal{A} = (\nabla \Phi, \Theta, u, w, b)\) is a solution over \([0, T_1]\) to the Cauchy problem \((3.7) - (4.7)\). Finally, the Cauchy problem \((3.3) - (3.6)\) admits at most one solution in \(C([0,T_1]; H^3)\) satisfying \((3.80)\).

**Proof.** First of all, we shall prove \((3.79)\) by induction. The trivial case is \(j = 0\) since \(\mathcal{A}^0 = 0\) by the assumption at initial step. Suppose that it is true for \(j \geq 0\) with \(M_0 > 0\) small enough to be determined later. Now, we propose to prove it for \(j + 1\), we need some energy estimates on \(\mathcal{A}^j\).

By the induction assumptions and Lemma 3.3-3.4, we have
\[
1 \frac{d}{dt} \|\mathcal{A}^j(t)\|^2_{H^3} + \|\nabla\mathcal{A}^j(t)\|^2_{H^3} + \|w^j(t)\|^2_{H^3} 
\leq c M_0 \|\nabla^2 \Phi^j, \Theta^j, u^j, w^j, b^j\|^2_{H^3} + c M_0 \|\nabla(u^j, b^j)\|^2_{H^3} + c M_0 \|w^j\|^2_{H^3}
\]
and
\[
\frac{d}{dt} \sum_{l=0}^{2} \int \nabla u^j \cdot \nabla \mathcal{A}^j dx + \frac{1}{4} \|\nabla \mathcal{A}^j\|^2_{H^3} + \|\nabla u^j\|^2_{H^3} + \|\nabla w^j\|^2_{H^3} + c M_0 \|\nabla(u^j, b^j)\|^2_{H^3}
\]
Thus, by linear combination of \((3.81)\) and \((3.82)\), we deduce that there exists an instant energy functional \(\|\mathcal{A}^j(t)\|^2_{H^3}\) such that
\[
\frac{d}{dt} \|\mathcal{A}^j(t)\|^2_{H^3} + \|\nabla \mathcal{A}^j(t)\|^2_{H^3} + \|w^j(t)\|^2_{H^3} + \|\nabla \Phi^j, \Theta^j, u^j, w^j, b^j\|^2_{H^3} + \|\nabla(u^j, b^j)\|^2_{H^3}
\]
for some \(\lambda \in (0, 1)\). Further, by the Gronwall’s inequality and the property of \(\mathcal{A}^j(t)\), it holds that
\[
\|\mathcal{A}^j(t)\|^2_{H^3} + \lambda \int_0^t \|\nabla \mathcal{A}^j(s)\|^2_{H^3} + \|w^j(s)\|^2_{H^3} ds 
\leq c M_0 \left[ \varepsilon_1^2 + c M_0 \int_0^t \|\nabla \mathcal{A}^j(s)\|^2_{H^3} + \|\nabla \Phi^j, \Theta^j, u^j, w^j, b^j\|^2_{H^3} ds \right]
\]
for any \(0 \leq t \leq T_1\). From above, we can take suitable small \(\varepsilon_1 > 0\), \(T_1 > 0\) and \(M_1 > 0\) such that
\[
\|\mathcal{A}^j(t)\|^2_{H^3} + \lambda \int_0^t \|\nabla \mathcal{A}^j(s)\|^2_{H^3} + \|w^j(s)\|^2_{H^3} ds \leq M_0^2
\]
for any \(0 \leq t \leq T_1\) and \(\lambda' \in (0, \lambda]\) be a constant. This implies that \((3.79)\) holds for \(j + 1\) if so for \(j\). Hence \((3.79)\) is proved for all \(j \geq 0\).

Next, from \((3.83)\) and the equivalence of \(\mathcal{A}^j(t)\) and \(\mathcal{A}^j(s)\), we can see that
\[
\|\mathcal{A}^j(t)\|^2_{H^3} - \|\mathcal{A}^j(s)\|^2_{H^3} = \int_s^t d\tau \|\mathcal{A}^j(\tau)\|^2_{H^3} d\tau 
\leq c \int_s^t \|\mathcal{A}^j(\tau)\|^2_{H^3} (\|\nabla \mathcal{A}^j(\tau)\|^2_{H^3} + \|\nabla \Phi^j, \Theta^j, u^j, w^j, b^j(\tau)\|^2_{H^3}) d\tau
\]
and
\[
\|\mathcal{A}^j(t)\|^2_{H^3} - \|\mathcal{A}^j(s)\|^2_{H^3} \leq c M_0 \int_s^t \|\nabla \mathcal{A}^j(\tau)\|^2_{H^3} + \|\nabla \Phi^j, \Theta^j, u^j, w^j, b^j(\tau)\|^2_{H^3} d\tau,
\]
for any $0 \leq s \leq t \leq T_1$. Here, the time integral in the last inequality is finite due to (3.84), and hence $\|\omega^{j+1}(t)\|_{H^3}^2$ is continuous in $t$ for each $j \geq 0$.

In order to consider the convergence of the sequence $(\omega^j)_{j \geq 0}$, by taking the difference of (3.51) for each $j$ and $j-1$, we arrive at

\[
\begin{align*}
\partial_t(\omega^{j+1} - \omega^j) + \text{div} (u^{j+1} - u^j) &= M_{1,j}^{j+1} - M_{1,j}^j, \\
\partial_t(u^{j+1} - u^j) + \nabla(\omega^{j+1} - \omega^j) - \Delta(u^{j+1} - u^j) - \nabla \text{div} (u^{j+1} - u^j) \\
&= -\nabla \times (w^{j+1} - w^j) + 2(w^{j+1} - w^j) - \Delta(w^{j+1} - w^j) - 2\nabla \text{div} (w^{j+1} - w^j) \\
- \nabla \times (u^{j+1} - u^j) &= M_{2,j}^{j+1} - M_{2,j}^j, \\
\partial_t(w^{j+1} - w^j) + 2(w^{j+1} - w^j) - \Delta(w^{j+1} - w^j) - 2\nabla \text{div} (w^{j+1} - w^j) \\
- \nabla \times (u^{j+1} - u^j) &= M_{3,j}^{j+1} - M_{3,j}^j, \\
\partial_t(b^{j+1} - b^j) - \Delta(b^{j+1} - b^j) &= M_{4,j}^{j+1} - M_{4,j}^j, \\
\Delta(\Phi^{j+1} - \Phi^j) &= \omega^{j+1} - \omega^j, \\
\text{div} (b^{j+1} - b^j) &= 0, \quad t > 0, \quad x \in \mathbb{R}^3,
\end{align*}
\]

where the nonlinear terms $M_{i,j}^{j+1} - M_{i,j}^j$ $(i = 1, 2, 3, 4)$ are defined as

\[
\begin{align*}
M_{1,j}^{j+1} - M_{1,j}^j &= -[\omega^j \text{div} (u^{j+1} - u^j) + (\omega^j \omega^{j+1} - \omega^j) \text{div} u^j + u^j \cdot \nabla(\omega^{j+1} - \omega^j) + (u^j - u^{j-1}) \cdot \nabla \omega^j], \\
M_{2,j}^{j+1} - M_{2,j}^j &= -[u^j \cdot \nabla(u^{j+1} - u^j) + (u^j - u^{j-1}) \cdot \nabla u^j] - f(\omega^j)[\Delta(u^{j+1} - u^j) + \text{div} (u^{j+1} - u^j) \\
&+ \nabla \times (w^{j+1} - w^j)] - [f(\omega^j) - f(\omega^{j+1})][\Delta u^j + \text{div} u^j + \nabla \times w^j - h(\omega^j)\nabla(\omega^{j+1} - \omega^j)] \\
&- (h(\omega^j) - h(\omega^{j+1}))\nabla \omega^j + g(\omega^j)\nabla(\nabla(b^{j+1} - b^j) \times b^j + \nabla \times b^j \times (b^j - b^{j-1})) \\
&+ (g(\omega^j) - g(\omega^{j+1})) \nabla \times b^j \times b^{j-1}, \\
M_{3,j}^{j+1} - M_{3,j}^j &= -u^j \cdot \nabla(w^{j+1} - w^j) - (u^j - u^{j-1}) \cdot \nabla w^j \\
&- f(\omega^j)[\Delta(w^{j+1} - w^j) + \text{div} (w^{j+1} - w^j) - 2(w^{j+1} - w^j) + \nabla \times (u^{j+1} - u^j)] \\
&- (f(\omega^j) - f(\omega^{j+1}))[\Delta w^j + \text{div} w^j - 2w^j + \nabla \times u^j], \\
M_{4,j}^{j+1} - M_{4,j}^j &= b^j \cdot \nabla(u^{j+1} - u^j) + (b^j - b^{j-1}) \cdot \nabla u^j - u^j \cdot \nabla(b^{j+1} - b^j) \\
&- (u^j - u^{j-1}) \cdot \nabla b^j - b^j \text{div}(u^{j+1} - u^j) - (b^j - b^{j-1}) \text{div} u^j,
\end{align*}
\]

By using the same energy estimates as before, we infer that

\[
d\frac{d}{dt}\|\omega^{j+1} - \omega^j\|_{H^3}^2 + \lambda''\|\nabla(\nabla(\Phi^{j+1} - \Phi^j), u^{j+1} - u^j, w^{j+1} - w^j, b^{j+1} - b^j)\|_{H^3}^2
\]

\[
+ \lambda''\|\omega^{j+1} - \omega^j, w^{j+1} - w^j\|_{H^3}^2 
\leq c\|\omega^{j+1} - \omega^{j-1}\|_{H^3}^2 \|\nabla(\Phi^{j+1} - \Phi^j, u^j, w^j, b^j)\|_{H^3} + c\|\omega^j, u^j\|_{H^3} \|\nabla(\Phi^{j+1} - \Phi^j)\|_{H^3}
\]

\[
+ c\|\omega^{j+1}\|_{H^3} \|\nabla(\Phi^{j+1} - \Phi^j, u^{j+1} - u^j, w^{j+1} - w^j, b^{j+1} - b^j)\|_{H^3}^2
\]

with $\lambda'' \in (0, 1)$ be a constant. Taking into account (3.84), the further time integration gives

\[
\left\|
\begin{array}{c}
\omega^{j+1}(t) - \omega^j(t) \\
\omega^{j+1} - \omega^j, u^{j+1} - u^j, w^{j+1} - w^j, b^{j+1} - b^j \\
\omega^{j+1} - \omega^j, w^{j+1} - w^j, \Phi^{j+1} - \Phi^j, u^{j+1} - u^j, w^{j+1} - w^j, b^{j+1} - b^j
\end{array}
\right\|_{H^3}(t)
\]

\[
+ \lambda'' \int_0^t \|\nabla(\Phi^{j+1} - \Phi^j, u^{j+1} - u^j, w^{j+1} - w^j, b^{j+1} - b^j, s)\|_{H^3} ds 
\leq e^{\lambda M_0} \sup_{0 \leq s \leq T_1} \|\omega^j(s) - \omega^{j-1}(s)\|_{H^3}^2
\]

\[
+ cM_0 \int_0^t \|\nabla(\Phi^{j+1} - \Phi^j, u^{j+1} - u^j, w^{j+1} - w^j, b^{j+1} - b^j, s)\|_{H^3}^2 ds
\],

for any $0 \leq t \leq T_1$. By the smallness of $M_0$ and $T_1$, then there exists a constant $\beta \in (0, 1)$ such that

\[
\sup_{0 \leq t \leq T_1} \|\omega^{j+1}(t) - \omega^j(t)\|_{H^3}^2 \leq \beta \sup_{0 \leq t \leq T_1} \|\omega^j(t) - \omega^{j-1}(t)\|_{H^3}^2
\]

(3.86)
Lemma 4.1. Let all assumptions in Lemma 3.1 be in force. Then, for \( s \geq 0 \) is a Cauchy sequence in the Banach space \( C([0, T_1]; H^3) \). Thus, the limit function
\[
\mathcal{A} = \mathcal{A}^0 + \lim_{n \to \infty} \sum_{j=0}^{n} (\mathcal{A}^{j+1} - \mathcal{A}^j)
\]
indeed exists in \( C([0, T_1]; H^3) \) and satisfies
\[
\sup_{0 \leq t \leq T_1} \|\mathcal{A}(t)\|_{H^1} \leq \sup_{0 \leq t \leq T_1} \|\mathcal{A}^j(t)\|_{H^1} \leq M_0,
\]
that is \( (3.80) \).

Finally, suppose that \( \mathcal{A}(t) \) and \( \mathcal{A}(t) \) are two solutions in \( C([0, T_1]; H^3) \) satisfying \( (3.80) \). By applying the same process as in \( (3.80) \), we deduce that
\[
\sup_{0 \leq t \leq T_1} \|\mathcal{A}(t) - \mathcal{A}(t)\|_{H^1}^2 \leq \beta_1 \sup_{0 \leq t \leq T_1} \|\mathcal{A}(t) - \mathcal{A}(t)\|_{H^1}^2,
\]
for \( \beta_1 \in (0, 1) \) be a constant, which implies \( \mathcal{A}(t) = \mathcal{A}(t) \) holds. This proves the uniqueness and thus completes the proof of Theorem 3.1.

Remark 3.1. With a few modifications to Lemma 3.2, 3.4, we can deduce the local existence of \( \mathcal{A}(t) \) in \( H_N^I \) norm \( (N \geq 4) \) without the assumption that \( \|\mathcal{A}_0\|_{H_N^I} \) small enough. In fact, by re-estimating Lemma 3.3, 3.4 carefully, we can see that
\[
\frac{1}{2} \frac{d}{dt} \sum_{i=4}^{N} \|\nabla \mathcal{A}^{j+1}(t)\|_{L^2}^2 + \sum_{i=4}^{N} \|\nabla \mathcal{A}^{j+1}(t)\|_{L^2}^2 \leq \frac{2}{3} \frac{d}{dt} \sum_{i=4}^{N} \|\nabla \mathcal{A}^{j+1}(t)\|_{L^2}^2 + \frac{1}{2} \sum_{i=4}^{N} \|\nabla \mathcal{A}^{j+1}(t)\|_{L^2}^2,
\]
and
\[
\frac{d}{dt} \sum_{i=4}^{N} \|\nabla \mathcal{A}^{j+1}(t)\|_{L^2}^2 \leq \frac{1}{4} \|\nabla \mathcal{A}^{j+1}(t)\|_{L^2}^2 + \frac{1}{4} \|\nabla \mathcal{A}^{j+1}(t)\|_{L^2}^2 + \frac{1}{2} \|\nabla \mathcal{A}^{j+1}(t)\|_{L^2}^2 + \frac{1}{2} \|\nabla \mathcal{A}^{j+1}(t)\|_{L^2}^2,
\]
where \( \beta_i \in (0, 1) \), \( i = 1, \cdots, 7 \). For any \( M > 0 \) be fixed. Suppose that \( \|\mathcal{A}_0\|_{H_N^I} \leq M \), then applying the smallness of \( M_0 \) in Theorem 3.1, by the same process as the proof of Theorem 3.1, one can verify that \( \|\mathcal{A}(t)\|_{H_N^I} \leq M \), for any \( t \in [0, T_1] \) and \( T_1 \) was determined in Theorem 3.1. Furthermore, we can also conclude that the limit function of \( \mathcal{A}(t) \) is indeed the solution over \( [0, T_1] \) to (3.8)-(1.6).

4. Energy evolution of negative Sobolev or Besov norms

In this section, we shall show the evolution of the negative Sobolev and Besov norms of the solution
\( (\nabla \Phi, \varrho, u, w, b) \) to (1.5)-(1.6). In order to estimate the nonlinear terms, we shall restrict ourselves to that
\( s \in (0, \frac{1}{2}) \). Firstly, for the homogeneous Sobolev space, we have the following result.

Lemma 4.1. Let all assumptions in Lemma 3.7 be in force. Then, for \( s \in (0, \frac{1}{2}) \), we have
\[
\frac{1}{2} \frac{d}{dt} \|\nabla \mathcal{A} \|_{H^{s+1}}^2 + \frac{2}{3} \|\nabla u \|_{H^{-s}}^2 + \frac{2}{3} \|\nabla u \|_{H^{-s}}^2 + 3 \|\nabla u \|_{H^{s+1}} + 3 \|\nabla u \|_{H^{s+1}} + \frac{1}{2} \|\nabla u \|_{H^{s}} + \frac{1}{2} \|\nabla u \|_{H^{s}} + \frac{1}{2} \|\nabla b \|_{H^{-s}}^2 \leq (\|\nabla (\varrho, u, w, b) \|_{H^{s+1}}^2 + \|\nabla (\varrho, u, w, b) \|_{H^{s+1}}^2) \|\nabla \mathcal{A} \|_{H^{s+1}},
\]
(4.1)
Similarly, we further obtain
\begin{align*}
\frac{1}{2} \frac{d}{dt} \||\dot{\nabla} \Phi, q, u, w, b\||^2_{H^{-\frac{s}{2}}} + \frac{2}{3} \||\nabla u||^2_{H^{-\frac{s}{2}}} + 3 \||\nabla w||^2_{H^{-\frac{s}{2}}} + \frac{1}{2} \||w||^2_{H^{-\frac{s}{2}}} + \||\nabla b||^2_{H^{-\frac{s}{2}}} \\
\leq \|(q, u, w, b)||^{s+\frac{1}{2}}_{L^2} \||\dot{\nabla} \Phi, q, u, w, b||^{\frac{s+1}{2}}_{L^2} \|\|\dot{\nabla} \Phi, q, u, w, b\|| \|\|\dot{\nabla} \Phi, q, u, w, b\||_{H^{-\frac{s}{2}}} .
\end{align*}
(4.2)

Proof. By the $\Lambda^{-s}$ ($s > 0$) energy estimate of (4.1)-(4.4), we obtain
\begin{align*}
\frac{1}{2} \frac{d}{dt} \||(q, u, w, b)||^2_{H^{-s}} + 2 \||\nabla u||^2_{H^{-s}} + 3 \||\nabla w||^2_{H^{-s}} + 2 \||w||^2_{H^{-s}} + \||\nabla b||^2_{H^{-s}} \\
= \int \Lambda^{-s} [-\text{div } u - \text{div } (qu)] \cdot \Lambda^{-s} \dot{q} \, dx + \int \Lambda^{-s} \left\{ \nabla \times w - \nabla q - u \cdot \nabla - f(q) \right\} \dot{\nabla} \Phi \cdot \Lambda^{-s} \dot{q} \, dx \\
+ \int \Lambda^{-s} \left\{ \nabla \times u - u \cdot \nabla w - f(q) \right\} \Lambda^{-s} \dot{w} \, dx \\
+ \int \Lambda^{-s} \left\{ \nabla \times (u \cdot \nabla b) - b \cdot \text{div } u \right\} \Lambda^{-s} b \, dx \\
\leq \|\text{div } (qu)||_{H^{-s}} \|\dot{q}\|_{H^{-s}} + \| - u \cdot \nabla - f(q) \| \|\dot{\nabla} \Phi\|_{H^{-s}} \|u\|_{H^{-s}} \\
+ \|g(q)\| \|b \cdot \text{div } u\| \|\dot{\nabla} \Phi\|_{H^{-s}} \|u\|_{H^{-s}} + \int \Lambda^{-s} \nabla \Phi \cdot \Lambda^{-s} \dot{u} \, dx \\
+ \| - u \cdot \nabla w - f(q) \| \|\dot{\nabla} \Phi\|_{H^{-s}} \|w\|_{H^{-s}} + \|\nabla \times u - 2w\| \|\dot{\nabla} \Phi\|_{H^{-s}} \|w\|_{H^{-s}} \\
+ \|\nabla \times (u \cdot \nabla b) - b \cdot \text{div } u\| \|\dot{\nabla} \Phi\|_{H^{-s}} \|u\|_{H^{-s}} + \int \Lambda^{-s} \nabla \times u \cdot \Lambda^{-s} \dot{u} \, dx \\
+ \int \Lambda^{-s} \nabla \times u \cdot \Lambda^{-s} \dot{u} \, dx ,
\end{align*}
(4.3)
where in the last inequality, we have taken into account that
\[\int \Lambda^{-s} (-\text{div } u) \cdot \Lambda^{-s} \dot{q} \, dx + \int \Lambda^{-s} (-\nabla q) \cdot \Lambda^{-s} \dot{u} \, dx = 0.\]

Now, we concentrate our attention on estimating the terms on the right hand side of (4.3), and we distinguish the argument by the value of $s$. First, if $s \in (0, \frac{1}{2})$, then $\frac{1}{2} + \frac{s}{2} < 1$ and $\frac{1}{2} \geq 6$.

In virtue of (2.7), Sobolev inequality, the Hölder and Young’s inequality, we obtain
\begin{align*}
\|\text{div } (qu)\|_{H^{-s}} &\leq \|\text{div } u\|_{H^{-s}} + \|\nabla q \cdot u\|_{H^{-s}} \\
&\leq \|\text{div } u\| \frac{1}{L^{-s}} \|\nabla q \cdot u\| \frac{1}{L^{-s}} \\
&\leq \|q\|_{L^s} \|\nabla u\|_{L^2} + \|u\|_{L^s} \|\nabla q\|_{L^2} \\
&\leq \|\nabla q\|^{1+s}_{L^3} \|\nabla^2 q\|^{1-s}_{L^3} \|\nabla u\|_{L^2} + \|\nabla u\|^{1+s}_{L^3} \|\nabla^2 u\|^{1-s}_{L^3} \|\nabla q\|_{L^2} \\
&\leq \|\nabla q\|_{H^{-s}}^2 + \|\nabla u\|_{H^s}^2 .
\end{align*}
(4.4)

Similarly, we further obtain
\begin{align*}
\|u \cdot \nabla u\|_{H^{-s}} &\leq \|\nabla u\|_{H^s}^2 ,
\|u \cdot \nabla w\|_{H^{-s}} &\leq \|\nabla w\|_{H^s}^2 + \|\nabla u\|_{H^s}^2 ,
\|b \cdot \nabla u\|_{H^{-s}} &\leq \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{H^s}^2 ,
\|u \cdot \nabla b\|_{H^{-s}} &\leq \|\nabla b\|_{L^2}^2 + \|\nabla u\|_{H^s}^2 ,
\|b \cdot \text{div } u\|_{H^{-s}} &\leq \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{H^s}^2 .
\end{align*}
(4.5-4.9)

In addition, using Lemma 2.3, there holds
\[\|f(q) \Delta u\|_{H^{-s}} + \|f(q) \nabla \text{div } u\|_{H^{-s}} \leq \|f(q)\|_{L^s} \|\Delta u\|_{L^2} \leq \|\nabla q\|_{H^s} + \|\nabla u\|_{H^s}^2 .
\] (4.10)
By the same way, we infer that
\[
\|f(q)\nabla \times w\|_{H^2} \leq \|\nabla q\|_{H^2}^2 + \|\nabla w\|_{H^2}^2, \tag{4.11}
\]
\[
\|f(q)\nabla \times u\|_{H^2} \leq \|\nabla u\|_{H^2}^2 + \|\nabla q\|_{H^2}^2, \tag{4.12}
\]
\[
\|f(q)w\|_{H^2} \leq \|w\|_{L^2}^2 + \|\nabla q\|_{H^2}^2, \tag{4.13}
\]
\[
\|f(q)\Delta w\|_{H^2} + \|f(q)\nabla \nabla w\|_{H^2} \leq \|\nabla q\|_{H^2}^2 + \|\nabla w\|_{H^2}^2, \tag{4.14}
\]
\[
\|b(q)\nabla q\|_{H^2} \leq \|\nabla q\|_{H^2}^2, \tag{4.15}
\]

Next, note that \(g(q) \in (0, 1)\), then we can see that
\[
\|g(q)(b \cdot \nabla b)\|_{H^2} \leq \|g(q)b\|_{L^2} \|\nabla b\|_{L^2} \leq \|b\|_{L^2} \|\nabla b\|_{L^2} \leq \|\nabla b\|_{H^2}^2. \tag{4.16}
\]

Similarly, there holds
\[
\|g(q) \cdot \nabla (|b|^2)\|_{H^2} \leq \|\nabla b\|_{H^2}^2. \tag{4.17}
\]

Moreover, taking into account (3.47), by the Young’s inequality, we have
\[
\int \Lambda^{-s} \nabla \times w \cdot \nabla^{-s} u \, dx + \int \Lambda^{-s} \nabla \times u \cdot \nabla^{-s} w \, dx
= 2 \int \Lambda^{-s} \nabla \times u \cdot \nabla^{-s} w \, dx \leq \frac{4}{3} \|\Lambda^{-s} \nabla u\|_{L^2}^2 + \frac{3}{2} \|\Lambda^{-s} w\|_{L^2}^2. \tag{4.18}
\]

Finally, for the Poisson term, by integration by parts, it holds that
\[
- \int \Lambda^{-s} \nabla \Phi \cdot \nabla^{-s} u \, dx = \int \Lambda^{-s} \Phi \nabla^{-s} \nabla u \, dx
= \int \Lambda^{-s} \Phi \nabla^{-s} (-\partial_3 \Phi - \div (q\Phi)) \, dx
= \int -\Lambda^{-s} \Phi \nabla^{-s} \partial_3 \Phi + \Lambda^{-s} \nabla \Phi \cdot \nabla^{-s} (q\Phi) \, dx
= \frac{1}{2} \frac{d}{dt} \int \|\Lambda^{-s} \nabla \Phi\|_{L^2}^2 + \int \Lambda^{-s} \nabla \Phi \cdot \nabla^{-s} (q\Phi) \, dx. \tag{4.19}
\]
If \(s \in (0, \frac{1}{2})\), using Lemma 2.1 and Lemma 2.4, we infer that
\[
\|\Lambda^{-s} (q\Phi)\|_{L^2} \leq \|\Phi\|_{L^2} \|\nabla q\|_{L^2} \|\Phi\|_{L^2} \leq \|\Phi\|_{L^2}^2 + \|\nabla u\|_{H^2}^2. \tag{4.20}
\]
Thus, from (4.3) and (4.20), it follows (4.1) for \(s \in (0, \frac{1}{2})\).
On the other hand, if \(s \in (\frac{1}{2}, \frac{3}{2})\), we have \(\frac{1}{2} + \frac{s}{3} < 1, 2 < \frac{1}{2} < 6\). Then, we obtain
\[
\|\div (q\Phi)\|_{H^2} \leq \|\nabla q\|_{H^2} + \|\div u\|_{H^2} \leq \|q\|_{L^2} \|\nabla u\|_{L^2} + \|\div u\|_{H^2} \leq \|q\|_{L^2} \|\nabla u\|_{L^2} + \|q\|_{L^2} \|\nabla u\|_{L^2} + \|\div u\|_{H^2} \|\nabla u\|_{L^2}. \tag{4.21}
\]
Likewise, we can also obtain
\[
\|u \cdot \nabla u\|_{H^2} \leq \|u\|_{L^2} \|\nabla u\|_{L^2} + \|\div u\|_{H^2} \|\nabla u\|_{L^2} \leq \|u\|_{L^2} \|\nabla u\|_{L^2} + \|\div u\|_{H^2} \|\nabla u\|_{L^2}. \tag{4.22}
\]
\[
\|u \cdot \nabla u\|_{H^2} \leq \|u\|_{L^2} \|\nabla u\|_{L^2} + \|\div u\|_{H^2} \|\nabla u\|_{L^2} \leq \|u\|_{L^2} \|\nabla u\|_{L^2} + \|\div u\|_{H^2} \|\nabla u\|_{L^2}. \tag{4.23}
\]
\[
\|(b \cdot \nabla u)\|_{H^2} \leq \|b\|_{L^2} \|\nabla u\|_{L^2} + \|\div u\|_{H^2} \|\nabla u\|_{L^2} \leq \|b\|_{L^2} \|\nabla u\|_{L^2} + \|\div u\|_{H^2} \|\nabla u\|_{L^2}. \tag{4.24}
\]
\[
\|(u \cdot \nabla b)\|_{H^2} \leq \|u\|_{L^2} \|\nabla b\|_{L^2} + \|\div u\|_{H^2} \|\nabla b\|_{L^2} \leq \|u\|_{L^2} \|\nabla b\|_{L^2} + \|\div u\|_{H^2} \|\nabla b\|_{L^2}. \tag{4.25}
\]
\[
\|(b \Div u)\|_{H^2} \leq \|b\|_{L^2} \|\nabla u\|_{L^2} + \|\Div u\|_{H^2} \|\nabla u\|_{L^2} \leq \|b\|_{L^2} \|\nabla u\|_{L^2} + \|\Div u\|_{H^2} \|\nabla u\|_{L^2}. \tag{4.26}
\]
Taking into account Lemma 2.3, we further obtain

\[
\|f(q)\Delta u\|_{H^{s}} + \|f(q)\nabla \text{div } u\|_{H^{s}} \lesssim \|\varepsilon\|_{L^{2}T}^{\frac{1}{2}} \|\nabla \varepsilon\|_{L^{2}T}^{\frac{1}{2}} \|\nabla^{2} u\|_{L^{2}},
\]

(4.27)

\[
\|f(q)\Delta t\|_{H^{s}} + \|f(q)\nabla \text{div } u\|_{H^{s}} \lesssim \|\varepsilon\|_{L^{2}T}^{\frac{1}{2}} \|\nabla \varepsilon\|_{L^{2}T}^{\frac{1}{2}} \|\nabla^{2} w\|_{L^{2}},
\]

(4.28)

\[
\|f(q)\nabla \times w\|_{H^{s}} + \|f(q)\nabla \times u\|_{H^{s}} \lesssim \|\varepsilon\|_{L^{2}T}^{\frac{1}{2}} \|\nabla \varepsilon\|_{L^{2}T}^{\frac{1}{2}} \|\nabla w\|_{L^{2}} + \|\nabla u\|_{L^{2}},
\]

(4.29)

\[
\|f(q)w\|_{H^{s}} + \|h(q)\nabla \varepsilon\|_{H^{s}} \lesssim \|\varepsilon\|_{L^{2}T}^{\frac{1}{2}} \|\nabla \varepsilon\|_{L^{2}T}^{\frac{1}{2}} \|\nabla w\|_{L^{2}} + \|\nabla \varepsilon\|_{L^{2}},
\]

(4.30)

In addition, similar to (4.16)-(4.17), there holds

\[
\|g(q)(b \cdot \nabla b)\|_{H^{s}} + \|g(q) \cdot \nabla |b|^2\|_{H^{s}} \lesssim \|b\|_{L^{2}T}^{\frac{1}{2}} \|\nabla b\|_{L^{2}}^{\frac{3}{2}} \|\nabla b\|_{L^{2}}.
\]

(4.31)

While for the right-most term in (4.19), we can see that

\[
\|\Lambda^{-s}(\partial u)\|_{L^{2}} \lesssim \|\varepsilon\|_{L^{2}T}^{\frac{1}{2}} \|\nabla \varepsilon\|_{L^{2}T}^{\frac{1}{2}} \|\partial u\|_{L^{2}}^{\frac{3}{2}}.
\]

(4.32)

Combining (4.3), (4.18)-(4.19) and (4.21)-(4.32), we have (4.2) for \(s \in \left(\frac{1}{2}, \frac{3}{2}\right)\). This completes the proof of Lemma 4.1.

When replace the homogeneous Sobolev space by the homogeneous Besov space. Now, we proceed to derive the evolution of the negative norms of the solution \((\nabla \Phi, q, u, w, b)\) to (1.5)-(1.6). Precisely, we have

**Lemma 4.2.** Let all assumptions in Lemma 3.7 hold. Then, for \(s \in (0, \frac{1}{2}]\), we have

\[
\frac{1}{2} \frac{d}{dt} \|\nabla \Phi, q, u, w, b\|_{B^{s}_{2,2}}^{2} + 2\|\nabla u\|_{B^{s}_{2,2}}^{2} + 3\|\nabla w\|_{B^{s}_{2,2}}^{2} + \frac{1}{2}\|\nabla b\|_{B^{s}_{2,2}}^{2} \lesssim (\|\nabla (q, u, w, b)\|_{H^{s}}^{2} + \|\nabla (q, w)\|_{H^{s}}^{2})(\|\nabla \Phi, q, u, w, b\|_{B^{s}_{2,2}}),
\]

(4.33)

and for \(s \in \left(\frac{1}{2}, \frac{3}{2}\right)\), we have

\[
\frac{1}{2} \frac{d}{dt} \|\nabla \Phi, q, u, w, b\|_{B^{s}_{2,2}}^{2} + 2\|\nabla u\|_{B^{s}_{2,2}}^{2} + 3\|\nabla w\|_{B^{s}_{2,2}}^{2} + \frac{1}{2}\|\nabla b\|_{B^{s}_{2,2}}^{2} \lesssim (\|q, u, w, b\|_{L^{2}}^{\frac{1}{2}} \|\nabla (q, u, w, b)\|_{L^{2}}^{\frac{1}{2}})(\|\nabla \Phi, q, u, w, b\|_{B^{s}_{2,2}}).
\]

(4.34)

**Proof.** Applying \(\hat{\Delta}_{j}\) energy estimate of (1.5)-(1.6) with multiplication of \(2^{-2sj}\) and then taking the supremum over \(j \in \mathbb{Z}\), we infer that

\[
\frac{1}{2} \frac{d}{dt} \|q, u, w, b\|_{B^{s}_{2,2}}^{2} + 2\|\nabla u\|_{B^{s}_{2,2}}^{2} + 3\|\nabla w\|_{B^{s}_{2,2}}^{2} + \frac{1}{2}\|\nabla b\|_{B^{s}_{2,2}}^{2} \lesssim \sup_{j \in \mathbb{Z}} 2^{-2sj} \int \hat{\Delta}_{j}[-\text{div } u - \text{div } (\partial u) \cdot \hat{\Delta}_{j} \partial u] dx
\]

\[ + \sup_{j \in \mathbb{Z}} 2^{-2sj} \int \hat{\Delta}_{j}[\nabla \times w - \nabla q - u \cdot \nabla u - f(q)[\Delta u + \nabla \text{div } u + \nabla \times w]
\]

\[ - h(q)\nabla \Phi + g(q) \cdot b \cdot \nabla b - \frac{1}{2} \nabla |b|^2[\nabla \Phi] \cdot \hat{\Delta}_{j} \partial u dx
\]

\[ + \sup_{j \in \mathbb{Z}} 2^{-2sj} \int \hat{\Delta}_{j}[\nabla \times u - u \cdot \nabla w - f(q)[\Delta w + 2\nabla \text{div } w + \nabla \times u - w)] \cdot \hat{\Delta}_{j} \partial u dx
\]

\[ + \sup_{j \in \mathbb{Z}} 2^{-2sj} \int \hat{\Delta}_{j}[(b \cdot \nabla)u - (u \cdot \nabla)b - b(\text{div } u)] \cdot \hat{\Delta}_{j} \partial b dx
\]
exists an instant energy functional \( \tilde{\mathcal{E}}_3 \), \( \text{(3.48)} \) of Lemma \( 3.2 \) and summing up, we deduce that for any \( t \in [0, T] \)

\[
\frac{1}{2} \frac{d}{dt} \sum_{l=0}^{3} \| \nabla^l (\nabla \phi, q, u, w, b) \|_{L_2}^2 + \frac{2}{3} \sum_{l=0}^{3} \| \nabla \phi \|_{L_2}^2 \leq c_3 \delta \sum_{l=0}^{3} \left( \| \nabla^{l+1} (\nabla \phi, u, w, b) \|_{L_2}^2 + \| \nabla^{l+1} (\nabla \phi, u, w, b) \|_{L_2}^2 \right). 
\]

(5.1)

In addition, taking \( k = 0, 1 \) in \( (3.48) \) of Lemma \( 3.2 \) and summing up, we obtain

\[
\frac{d}{dt} \sum_{l=0}^{2} \int \nabla^l u \cdot \nabla^{l+1} q \, dx + \frac{2}{3} \sum_{l=0}^{2} (\| \nabla^l \phi \|_{L_2}^2 + \| \nabla^{l+1} \nabla \phi \|_{L_2}^2 + \| \nabla^{l+2} \nabla \phi \|_{L_2}^2) \leq c_2 \delta \sum_{l=0}^{2} (\| \nabla^{l+1} \nabla \phi \|_{L_2}^2 + \| \nabla^{l+2} (w, b) \|_{L_2}^2) + \sum_{l=0}^{2} (\| \nabla^{l+1} u \|_{L_2}^2 + 2\| \nabla^{l+1} w \|_{L_2}^2 + 4\| \nabla^{l+2} u \|_{L_2}^2). 
\]

(5.2)

Taking into account the smallness of \( \delta \), by linear combination of (5.1) and (5.2), we deduce that there exists an instant energy functional \( \tilde{\mathcal{E}}_3 \) equivalent to \( \tilde{\mathcal{E}}_3 \) such that

\[
\tilde{\mathcal{E}}_3(t) + \int_{0}^{t} \mathcal{E}_3(s) \, ds \leq \tilde{\mathcal{E}}_3(0), \quad \forall t \in [0, T]. 
\]

(5.3)

In what follows, we denote \( \tilde{\mathcal{E}}_3(t) \) by \( \mathcal{E}_3(t) \) due to the equivalence of \( \tilde{\mathcal{E}}_3(t) \) and \( \mathcal{E}_3(t) \). Now, choose a positive constant \( \varepsilon_0 := \min \{ \delta, \varepsilon_1 \} \), where \( \delta \) and \( \varepsilon_1 \) are given in Lemma \( 3.1 \) and Theorem \( 3.1 \) respectively. Further, choose initial data \( (\nabla \phi(0), q_0, u_0, w_0, b_0) \) and small constant \( \delta_0 \) such that

\[
\sqrt{\mathcal{E}_3(0)} \leq \sqrt{\delta_0} := \frac{\varepsilon_0}{2 \sqrt{1 + \varepsilon_3}}. 
\]

Define the lifespan to Cauchy problem \( (1.5)-(1.6) \) by

\[
T := \sup \left\{ t : \sup_{0 \leq s \leq t} \sqrt{\mathcal{E}_3(s)} \leq \varepsilon_0 \right\}. 
\]

Since

\[
\sqrt{\mathcal{E}_3(0)} \leq \frac{\varepsilon_0}{2 \sqrt{1 + \varepsilon_3}} \leq \frac{\varepsilon_0}{2} < \varepsilon_0 \leq \varepsilon_1,
\]

5. Proof of main theorem

In this section, based on the assumption that \( H^2 \) norm of initial data is small, we shall combine all energy estimates that we have derived in the previous two sections to prove the global existence of \( (\nabla \phi, q, u, w, b) \) to \( (1.5)-(1.6) \).

Proof of Theorem \( (1.7) \). For simplicity, we divide the proof into several steps.

Step 1. Global small \( \mathcal{E}_3 \)-solution.

We first close the energy estimates at the \( H^2 \)-level by assuming a priori that \( \sqrt{\mathcal{E}_3(t)} \leq \delta \) is sufficiently small. Thus, from Lemma \( 3.1 \) taking \( k = 0, 1 \) in \( (3.2) \) and summing up, we deduce that for any \( t \in [0, T] \)

\[
\frac{1}{2} \frac{d}{dt} \sum_{l=0}^{3} \| \nabla^l (\nabla \phi, q, u, w, b) \|_{L_2}^2 + \frac{3}{2} \sum_{l=0}^{3} \| \nabla \phi \|_{L_2}^2 \leq c_3 \delta \sum_{l=0}^{3} \left( \| \nabla^{l+1} (\nabla \phi, u, w, b) \|_{L_2}^2 + \| \nabla^{l+1} (\nabla \phi, u, w, b) \|_{L_2}^2 \right). 
\]

(4.35)

According to the Lemma \( 2.5 \) and \( (4.35) \), the remaining proof of Lemma \( 4.1 \) is exactly same with the proof of Lemma \( 4.2 \) except that we allow \( s = \frac{1}{2} \) and replace Lemma \( 2.4 \) with Lemma \( 2.5 \) \( H^2 \)-norm by \( \mathcal{B}_{s, \infty} \) norm. \( \square \)
then $T > 0$ holds true from the local existence result Theorem 3.1 and the continuation argument. If $T$ is finite, as a consequence, from the definition of $T$, it follows that
\[ \sup_{0 \leq s \leq T} \sqrt{\varepsilon_3(s)} = \varepsilon_0, \]
which is a contradiction to the fact from uniform a priori that
\[ \sqrt{\varepsilon_3(s)} \leq \sqrt{\varepsilon_3(0)} \leq \frac{\varepsilon_0 \sqrt{\varepsilon_3}}{2 \sqrt{1 + \varepsilon_3}} \leq \frac{\varepsilon_0}{2}. \]

Therefore, $T = \infty$. This implies that the local solution $(\nabla \Phi, \varrho, u, w, b)$ obtained in Theorem 3.1 can extent to infinite time. Thus, the Cauchy problem (1.5)-(1.6) admits a unique solution $(\nabla \Phi, \varrho, u, w, b) \in C([0, \infty]; H^1)$. Finally, (1.11) follows from (5.3). This proves the existence of unique global $\varepsilon_3$-solution.

**Step 2.** Global $\varepsilon_N$ solution.

From Remark 3.1 and the global existence of $\varepsilon_3$ solution, we can deduce the global existence of $\varepsilon_N$ solution. Thus, for $N \geq 4$, $t \in [0, \infty)$, applying Lemma 3.1 and taking $k = 0, \cdots, N - 2$, we infer that
\[
\begin{align*}
&\frac{1}{2} \frac{d}{dt} \sum_{l=0}^{N} \|\nabla^{l} (\nabla \Phi, \varrho, u, w, b)\|^2_{L^2} + \frac{1}{2} \sum_{l=0}^{N} \|\varrho w\|^2_{L^2} + \sum_{l=0}^{N} \left( \frac{2}{3} \|\nabla^{l+1} u\|^2_{L^2} + 3 \|\nabla^{l+1} w\|^2_{L^2} + \|\nabla^{l+1} b\|^2_{L^2} \right) \\
&\leq c_{N, \varepsilon_0} \sum_{l=0}^{N} \left( \|\nabla^{l} (\varrho, w)\|^2_{L^2} + \|\nabla^{l+1} (\nabla \Phi, u, w, b)\|^2_{L^2} \right). \quad (5.4)
\end{align*}
\]

Furthermore, by Lemma 3.2 and taking $k = 0, \cdots, N - 2$, we have
\[
\begin{align*}
&\frac{d}{dt} \sum_{l=0}^{N-1} \int \nabla^{l} u \cdot \nabla^{l+1} \varrho dx + \frac{1}{4} \sum_{l=0}^{N-1} \left( 2 \|\nabla^{l+1} \varrho\|^2_{L^2} + \|\nabla^{l+1} \nabla \Phi\|^2_{L^2} + 4 \|\nabla^{l+2} \nabla \Phi\|^2_{L^2} \right) \\
&\leq c_{N-1, \varepsilon_0} \sum_{l=0}^{N-1} \left( \|\nabla^{l+1} \varrho\|^2_{L^2} + \|\nabla^{l+2} (w, b)\|^2_{L^2} \right) + \sum_{l=0}^{N-1} \left( \|\nabla^{l+1} u\|^2_{L^2} + 4 \|\nabla^{l+1} w\|^2_{L^2} + \|\nabla^{l+2} u\|^2_{L^2} \right). \quad (5.5)
\end{align*}
\]

By linear combination of (5.4) and (5.5), we infer that there exists an instant energy functional $\varepsilon_N$ is equivalent to $\varepsilon_N$, such that
\[
\frac{d}{dt} \varepsilon_N + \lambda \varepsilon_N(t) \leq 0, \quad (5.6)
\]
for some $\lambda \in (0, 1)$. This implies (1.12). Thus, we have completed the proof of Theorem 1.1.

According to the conclusion of Theorem 1.1, Lemma 4.1-4.2 now, we proceed to prove the various time decay rates of the unique global solution to (1.5)-(1.6).

**Proof of Theorem 1.2** In what follows, for the convenience of presentations, we define a family of energy functionals and the corresponding dissipation rates as
\[
\begin{align*}
\varepsilon_{k+2}^N := & \sum_{l=k}^{k+2} \|\nabla^{l} (\nabla \Phi, \varrho, u, w, b)\|^2_{L^2}, \\
\varepsilon_{k+2}^{N+1} := & \sum_{l=k}^{k+2} \|\nabla^{l} (\varrho, w)\|^2_{L^2} + \sum_{l=k}^{k+2} \|\nabla^{l+1} (\nabla \Phi, u, w, b)\|^2_{L^2}. 
\end{align*} \tag{5.7}
\]

and
\[
\begin{align*}
\varepsilon_{k+2}^{N+2} := & \sum_{l=k}^{k+2} \|\nabla^{l} (\varrho, w)\|^2_{L^2} + \sum_{l=k}^{k+2} \|\nabla^{l+1} (\nabla \Phi, u, w, b)\|^2_{L^2}. 
\end{align*} \tag{5.8}
\]

Taking into account Lemma 3.1-3.2 and Theorem 3.1 we have that for $k = 0, 1, \cdots, N - 2$,
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \sum_{l=k}^{k+2} \|\nabla^{l} (\nabla \Phi, \varrho, u, w, b)\|^2_{L^2} + \sum_{l=k}^{k+2} \|\varrho w\|^2_{L^2} + \sum_{l=k}^{k+2} \left( \frac{2}{3} \|\nabla^{l+1} u\|^2_{L^2} + 3 \|\nabla^{l+1} w\|^2_{L^2} + \|\nabla^{l+1} b\|^2_{L^2} \right) \\
&\leq c_4 \varepsilon_0 \sum_{l=k}^{k+2} \left( \|\nabla^{l} (\varrho, w)\|^2_{L^2} + \|\nabla^{l+1} (\nabla \Phi, u, w, b)\|^2_{L^2} \right). \quad (5.9)
\end{align*}
\]
and

\[
\frac{d}{dt} \sum_{l=0}^{k+1} \int \nabla^l u \cdot \nabla^{l+1} \mathcal{Q} d\Omega + \frac{1}{4} \sum_{l=0}^{k+1} \left( \| \nabla^l \mathcal{Q} \|^2_{L^2} + \| \nabla^{l+1} \mathcal{Q} \|^2_{L^2} + \| \nabla^l \nabla \mathcal{Q} \|^2_{L^2} + \| \nabla^{l+1} \nabla \mathcal{Q} \|^2_{L^2} \right)
\leq c_\Omega \epsilon_0 \sum_{l=0}^{N-1} \left( \| \nabla^{l+1} \mathcal{Q} \|^2_{L^2} + \| \nabla^{l+2} (w, b) \|^2_{L^2} \right) + \sum_{l=0}^{k+1} \left( \| \nabla^{l+1} u \|^2_{L^2} + 2\| \nabla^{l+1} w \|^2_{L^2} + 4\| \nabla^{l+2} u \|^2_{L^2} \right).
\] (5.10)

By linear combination of (5.9) and (5.10), since \( \epsilon_0 \) is small, we deduce that there exists an instant energy functional \( \mathcal{E}^{k+2} \) is equivalent to \( \mathcal{E}^{k+2} \) such that

\[
\frac{d}{dt} \mathcal{E}^{k+2} + \mathcal{G}^{k+2} \leq 0.
\] (5.11)

Noting that \( \mathcal{G}^{k+2} \) is weaker than \( \mathcal{E}^{k+2} \), which prevents the exponent decay of the solution. We need to bound the missing terms in the energy, that is, \( \| \nabla^k (\nabla \Phi, u, b) \|^2_{L^2} \) in terms of \( \mathcal{E}^{k+2} \). From which, then we can derive the time decay rate from (5.11). For this aim, we need the Sobolev interpolation between the negative and positive Sobolev norms. From now on, we assume for the moment that we have proved (1.14) and (1.15). Using Lemma 4.1, we propose to prove (1.14).

First, it is trivial for the case \( s \in [0, \frac{1}{2}] \). Next, let \( s \in (\frac{1}{2}, 1) \), note that, the arguments for the case \( s \in (0, \frac{1}{2}] \) can not be applied to this case. However, observing that we have \( (\nabla \Phi(0), u_0, w_0, b_0) \in H^{-s} \) due to the fact \( H^{-s} \cap L^2 \subset H^{-q} \) for any \( q \in [0, s] \). At this stage, from (1.16), it holds that for \( k \geq 0 \) and \( N \geq k + 2 \)

\[
\| \nabla^k (\nabla \Phi, u, w, b) \|^2_{L^2} \leq c_0 (1 + t)^{-\frac{k + \frac{1}{2}}{s}}.
\] (5.16)
Thus, integrating (4.2) in time for $s \in (\frac{1}{2}, 1)$ and applying (5.16), yields that
\[
\|(\nabla \Phi, \varrho, u, w, b)\|^2_{H^{-s}} \leq (\|(\nabla \Phi(0), \varrho_0, u_0, w_0, b_0)\|^2_{H^{-s}} + \int_0^t \sqrt{\mathcal{D}_A(\tau)} \|(\nabla \Phi, \varrho, u, w)\|_{L^2}^\cdot \|\varrho\|_{L^2}^\cdot \|u\|_{L^2}^\tau \, d\tau \\
\times \|(\nabla (\varrho, u, w, b))\|_{L^2} \|\nabla \Phi, \varrho, u, w, b)\|_{H^{-s}} \, d\tau \\
\leq c(1 + \sup_{0 \leq \tau \leq T} \|(\nabla \Phi, \varrho, u, w)\|_{H^{-s}} \int_0^t (1 + \tau)^{-\frac{3}{2}}(1 + \tau) \, d\tau) \\
\leq c_0(1 + \sup_{0 \leq \tau \leq T} \|(\nabla \Phi, \varrho, u, w, b)\|_{H^{-s}}),
\]
(5.17)
where in the last inequality, we have used the fact $s \in (\frac{1}{2}, 1)$, so that the time integral is finite. By the Cauchy’s inequality, this implies (1.14) for $s \in (\frac{1}{2}, 1)$, from which, we also verify (1.16) for $s \in (\frac{1}{2}, 1)$. Finally, let $s \in [1, \frac{3}{2})$, we choose $\varsigma_0$ such that $s - \frac{1}{2} < \varsigma_0 < 1$. Then $\|(\nabla \Phi(0), \varrho_0, u_0, w_0, b_0)\|_{H^{-\varsigma_0}}$, and from (1.16), the following estimate holds
\[
\|(\nabla \Phi, \varrho, u, w, b)\|_{H^{-s}} \leq c_0(1 + t)^{\frac{\varsigma_0}{2}n} 
\]
(5.18)
for $k \geq 0$ and $N \geq k + 2$. Therefore, similar to (5.17), using (5.18) and (4.2) for $s \in (1, \frac{3}{2})$, we conclude that
\[
\|(\nabla \Phi, \varrho, u, w, b)\|_{H^{-s}}^2 \leq c(1 + \sup_{0 \leq \tau \leq T} \|(\nabla \Phi, \varrho, u, w, b)\|_{H^{-s}} \int_0^t (1 + \tau)^{\varsigma_0 + \frac{3}{2} - s} \, d\tau) \\
\leq c_0(1 + \sup_{0 \leq \tau \leq T} \|(\nabla \Phi, \varrho, u, w, b)\|_{H^{-s}}).
\]
(5.19)
Here, we have taken into account $s - \varsigma_0 < \frac{1}{2}$, so that the time integral in (5.19) is finite. This implies (5.16) for $s \in (1, \frac{3}{2})$ and thus we have proved (1.16) for $s \in (1, \frac{3}{2})$. The rest of the proof is exactly same with above, we only need to replace Lemma [2.6] and Lemma [4.1] by Lemma [2.7] and Lemma [4.2] respectively. Then, we can deduce (1.15) for $s \in (0, \frac{3}{2})$, and thus verify (1.17) for $s \in (0, \frac{3}{2})$. Here, we just skip it. Thus, we have completed the proof of Theorem 1.2.

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