SYMPLECTIC STRUCTURES ON FREE NILPOTENT LIE ALGEBRAS

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ABSTRACT. In this work we study the problem of existence of symplectic structures on free nilpotent Lie algebras. Necessary and sufficient conditions are given for even dimensional ones. The one dimensional central extension for odd dimensional free nilpotent Lie algebras is also considered.

1. INTRODUCTION

A symplectic structure on a $2n$-dimensional differentiable manifold $M$ is a closed 2-form $\omega$ such that $\omega^n$ is non singular. In the particular case of a nilmanifold $M = \Gamma \backslash G$, that is, a compact quotient of a nilpotent simply connected Lie group by a cocompact discrete subgroup $\Gamma$, the natural map from $H^i(g)$, $g$ the Lie algebra of $G$, to the de Rham cohomology group $H^i(M, \mathbb{R})$ is an isomorphism, $0 \leq i \leq 2n$ by Nomizu’s work ([11]). Thus, any symplectic form on $M$ is cohomologous to a left invariant form on $G$ and hence it is represented by a closed 2-form on the Lie algebra $g$.

Therefore, the study of existence of symplectic structures on the nilmanifold $\Gamma \backslash G$ reduces to the existence of a closed 2-form on the Lie algebra $g$ such that $\omega^n \neq 0$.

One possible approach to this existence problem is to perform a study case by case in those dimensions where nilpotent Lie algebras are classified. For instance, Morosov described all nilpotent Lie algebras up to dimension six in [9] and his tables were used by Goze and Bouyakoub in [2] to give the list of all symplectic Lie algebras of dimension $\leq 6$. Another alternative is to treat the problem separately on subfamilies of nilpotent Lie algebras. For example, in [3] the authors work with Heisenberg type nilpotent Lie algebras. Moreover, the classification of symplectic filiform Lie algebras, which are Lie algebras $g$ of nilpotency index $k = \dim g - 1$, is given in [8]. Among nilpotent Lie algebras associated to graphs, a complete description of the symplectic ones can be made in terms of the corresponding graph ([12]).

Following the second approach, we restrict ourselves to the family of free nilpotent Lie algebras. The goal of this work is to give an explicit description of those Lie algebras in

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the family that admit symplectic structures. More precisely, we prove the following result.

**Main Theorem.** Let $n_{m,k}$ be the free $k$-step nilpotent Lie algebra on $m$ generators.

1. If $\dim n_{m,k}$ is even then $n_{m,k}$ admits symplectic structures if and only if $(m, k) = (3, 2)$.

2. If $\dim n_{m,k}$ is odd, the one dimensional central extension $\mathbb{R} \oplus n_{m,k}$ admits symplectic structures if and only if $(m, k) = (2, 2)$.

In [1] Benson and Gordon give a necessary condition for the existence of symplectic structures on nilpotent Lie algebras. We develop that condition for free nilpotent Lie algebras using the Hall basis, throughout which we obtain specific restriction in that family enabling the proof of the Theorem above.

2. **Free nilpotent Lie algebras**

Let $\mathfrak{g}$ denote a real Lie algebra. The sequence of ideals of $\mathfrak{g}$, $\{C^r(\mathfrak{g})\}$, which for non-negative integers $r$ is given by

$$C^0(\mathfrak{g}) = \mathfrak{g}, \quad C^r(\mathfrak{g}) = [\mathfrak{g}, C^{r-1}(\mathfrak{g})]$$

is called the central descending series of $\mathfrak{g}$.

A Lie algebra $\mathfrak{g}$ is called $k$-step nilpotent if $C^k(\mathfrak{g}) = \{0\}$ but $C^{k-1}(\mathfrak{g}) \neq \{0\}$ and in this case $C^{k-1}(\mathfrak{g}) \subseteq \mathfrak{z}(\mathfrak{g})$, where $\mathfrak{z}(\mathfrak{g})$ denotes the center of the Lie algebra.

A particular family of nilpotent Lie algebras is constituted by the free nilpotent ones.

Let $f_m$ denote the free Lie algebra on $m$ generators, $m \geq 2$ (notice that a unique element spans an abelian Lie algebra). The quotient Lie algebra $n_{m,k} = f_m/C^{k+1}(f_m)$ is the free $k$-step nilpotent Lie algebra on $m$ generators $n_{m,k}$. The image of a generator set of $f_m$ by the quotient map induces what is called a generator set of $n_{m,k}$. To each ordered set of generators $\{e_1, \ldots, e_m\}$ there is associated a basis of $n_{m,k}$, called a Hall basis (see [6, 4]). Its construction is as follows.

Define the length $\ell$ of each generator as 1. Take the Lie brackets $[e_i, e_j]$ for $i > j$, which by definition satisfies $\ell([e_i, e_j]) = 2$. Now the elements $e_1, \ldots, e_m$, $[e_i, e_j]$, $i > j$ belong to the Hall basis. Define a total order in that set by extending the order of the set of generators and so that $E > F$ if $\ell(E) > \ell(F)$. They allow the construction of the elements of length 3 and so on.

Recursively each element of the Hall basis of $n_{m,k}$ is defined as follows. The generators $e_1, \ldots, e_m$ are elements of the basis of length 1. Assume we have defined basic elements of lengths $1, \ldots, r - 1 \leq k - 1$, with a total order satisfying $E > F$ if $\ell(E) > \ell(F)$. 

If \( \ell(E) = s \) and \( \ell(F) = t \) and \( r = s + t \leq k \), then \( [E, F] \) is a basic element of length \( r \) if both of the following conditions hold:

1. \( E \) and \( F \) are basis elements and \( E > F \), and
2. if \( \ell(E) > 1 \) and \( E = [G, H] \) is the unique decomposition with \( G, H \) basic elements, then \( F \geq H \).

Fixed a Hall basis, denote by \( \mathfrak{p}(m, s) \) the subspace spanned by the elements of the basis of length \( s \). Hence \( n_{m,k} \) is a graded Lie algebra since

\[
[p(m, s), p(m, t)] \subseteq p(m, s + t) \quad \text{and} \quad n_{m,k} = \bigoplus_{s=1}^{k} p(m, s).
\]

The central descending series of a free nilpotent Lie algebra verifies

\[
C^r(n_{m,k}) = \bigoplus_{s=r+1}^{k} p(m, s).
\]

This property follows from the fact that every bracket of \( r + 1 \) elements of \( n_{m,k} \), is a linear combination of brackets of \( r + 1 \) elements in the Hall basis (see proof of Theorem 3.1 in [6]). This implies \( C^r(n_{m,k}) \subseteq \bigoplus_{s=r+1}^{k} p(m, s) \); the other inclusion is obvious. In particular,

\[
p(m, k) = C^{k-1}(n_{m,k}) \subseteq \mathfrak{j}(n_{m,k}).
\]

Denote as \( d_m(s) \) the dimension of \( p(m, s) \). Inductively one gets \([14]\)

\[
s \cdot d_m(s) = m^s - \sum_{r|s, r < s} r \cdot d_m(r), \quad s \geq 1.
\]

Hence for a fixed \( m \), one has \( d_m(1) = m \) and \( d_m(2) = m(m - 1)/2 \).

**Example 2.1.** Let \( n_{m,2} \) be the 2-step free nilpotent Lie algebra on \( m \) generators and let \( e_1, \ldots, e_m \) be a set of generators. The dimension of this Lie algebra is \( d_m(1) + d_m(2) = m + m(m - 1)/2 \) by equations \([11]\) and \([2]\).

From the construction described above, a Hall basis of \( n_{m,2} \) is

\[
\mathcal{B} = \{e_i, [e_j, e_k] : i = 1, \ldots, m, 1 \leq k < j \leq m\}.
\]

The center of \( n_{m,2} \) contains the first term of the central descending series \( C^1(n_{m,2}) = p(m, 2) \), since the Lie algebra is 2-step nilpotent. Even more,

\[
\mathfrak{j}(n_{m,2}) = p(m, 2).
\]

In fact, from \([3]\) above, any \( x \in \mathfrak{j}(n_{m,2}) \) can be written as

\[
x = \sum_{i=1}^{m} x_i e_i + \sum_{m \geq i > j \geq 1} y_{ij} [e_i, e_j].
\]

The bracket \( [x, e_1] \) is zero and equals \( \sum_{i=1}^{m} x_i [e_i, e_1] \). This implies that \( x_i = 0 \) for \( i = 2, \ldots, m \). By taking the Lie bracket \( [x, e_2] \) it turns out that \( x_1 = 0 \) and therefore \( x = 0 \).
Example 2.2. For the free 3-step nilpotent Lie algebra on $m$ generators $\mathfrak{n}_{m,3}$ the Hall basis associated to a set of generators has the form

\( \mathcal{B} = \{ e_i, [e_j, e_k], [[e_r, e_s], e_t], i = 1, \ldots, m, 1 \leq k < j \leq m, 1 \leq s < r \leq m, t \geq s \} \).

By a similar procedure as that one in the previous example one reaches $\mathfrak{z}(\mathfrak{n}_{m,3}) = \mathfrak{p}(m,3)$. Hence, by equation (2),

\[ \dim \mathfrak{z}(\mathfrak{n}_{m,3}) = d_m(3) = m(m^2 - 1)/3. \]

3. Free nilpotent Lie algebras and symplectic structures

In this section a necessary condition for a free nilpotent Lie algebra to admit a symplectic structure is proved. This condition relates the dimension of the center of $\mathfrak{n}_{m,k}$ with $m$, the amount of generators (see corollary 3.4). Afterwards, the proof of the Main Theorem is given.

A Lie algebra $\mathfrak{g}$ of even dimension $2n$ is called symplectic if it has a closed 2-form $\omega$ such that $\omega^n \neq 0$. Equivalently, $\omega$ as a skew symmetric bilinear form on $\mathfrak{g}$ is non degenerate.

Example 3.1. The one dimensional central extension of the free 2-step nilpotent Lie algebra on 2 generators, $\mathfrak{n}_{2,2}$, is symplectic.

The dimension of $\mathfrak{n}_{2,2}$ is three (recall example 2.1) and it is isomorphic to the Heisenberg Lie algebra. Its central extension $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{n}_{2,2}$ has a basis $\{ e_1, e_2, e_3, e_4 \}$ where the only non zero bracket is $[e_2, e_3] = e_4$. Let $\{ e^i \}_{i=1}^4$ be the dual basis. Then the Maurer-Cartan formula asserts that the differential $d : \mathfrak{g}^* \to \Lambda^2 \mathfrak{g}^*$ behaves in that basis in the following way

\[ d e^i = 0, \quad i = 1, 2, 3, \quad d e^4 = -e^2 \wedge e^3. \]

It is easy to verify that $\omega = e^1 \wedge e^2 + e^3 \wedge e^4$ is a symplectic structure on $\mathfrak{g}$.

Example 3.2. The 2-step free nilpotent Lie algebra on three generators, $\mathfrak{n}_{3,2}$ is symplectic. By example 2.1 the dimension of $\mathfrak{n}_{3,2}$ is six and it has a basis of the form $\{ e_1, \ldots, e_6 \}$ with non zero brackets

\[ [e_1, e_2] = e_4, \quad [e_1, e_3] = e_5, \quad [e_2, e_3] = e_6. \]

The differential of the Lie algebra in the dual basis $\{ e^1, e^2, \ldots, e^6 \}$ of one forms is

\[ \begin{cases} 
  d e^4 = -e^1 \wedge e^2 \\
  d e^5 = -e^1 \wedge e^3 \\
  d e^6 = -e^2 \wedge e^3 
\end{cases} \]

The 2-form $\omega = e^1 \wedge e^4 + e^2 \wedge e^6 + e^3 \wedge e^5$ is closed and $\omega^3 \neq 0$, hence it is a symplectic structure on $\mathfrak{n}_{2,3}$. 

The following necessary condition for nilpotent Lie algebras to admit symplectic structures was given by Benson and Gordon in [1]. Here we include the proof of Guan ([5]).

**Lemma 3.3.** ([1], [5]) Let $\omega$ be a symplectic structure on a nilpotent Lie algebra $\mathfrak{g}$, then

\[ \dim \mathfrak{j}(\mathfrak{g}) \leq \dim(n/C^1(\mathfrak{g})). \]

**Proof.** Let $\mathfrak{j}(\mathfrak{g})^\omega = \{ x \in \mathfrak{g} / \omega(x, z) = 0 , \forall z \in \mathfrak{j}(\mathfrak{g}) \}$ be the orthogonal space of the center with respect to the symplectic structure. Since $\omega$ is non degenerate it follows, $\dim \mathfrak{g} = \dim \mathfrak{j}(\mathfrak{g}) + \dim \mathfrak{j}(\mathfrak{g})^\omega$. Moreover, $\dim \mathfrak{g} = \dim C^1(\mathfrak{g}) + \dim(\mathfrak{g}/C^1(\mathfrak{g}))$. We claim that

\[ \dim C^1(\mathfrak{g}) \leq \dim \mathfrak{j}(\mathfrak{g})^\omega. \]

In fact, let $y = [y_1, y_2]$ in $C^1(\mathfrak{g})$. Since $\omega$ is closed, we have for any $z \in \mathfrak{j}(\mathfrak{g})$,

$\omega(z, y) = \omega(z, [y_1, y_2]) = \omega([z, y_2], y_1) + \omega([z, y_1], y_2) = 0.$

Hence $C^1(\mathfrak{g}) \subseteq \mathfrak{j}(\mathfrak{g})^\omega$ and equation (6) holds.

Since $\omega$ is non degenerate, we have

\[ \dim \mathfrak{j}(\mathfrak{g}) + \dim \mathfrak{j}(\mathfrak{g})^\omega = \dim C^1(\mathfrak{g}) + \dim(\mathfrak{g}/C^1(\mathfrak{g})). \]

and together with equation (6) we obtain the thesis. \qed

**Remark.** Condition (5) is not sufficient in general. In fact, any filiform Lie algebra $\mathfrak{g}$ satisfies equation (5) since $\dim \mathfrak{j}(\mathfrak{g}) = 1$ and $\dim(\mathfrak{g}/C^1(\mathfrak{g})) = 2$. Nevertheless, there are filiform Lie algebras admitting no symplectic structures (see for instance, [2]).

In [12], Poussele and Tirao showed that it is sufficient for the existence of symplectic structures in the family of nilpotent Lie algebras associated to graphs.

Clearly, $\dim(n_{m,k}/C^1(n_{m,k})) = m$ for $n_{m,k}$ the $k$-step free nilpotent Lie algebra on $m$ generators. Hence the equation (5) gives the following corollary.

**Corollary 3.4.** Let $n_{m,k}$ be the free $k$-step nilpotent Lie algebra on $m$ generators and consider the Lie algebra $\mathfrak{g} = \mathbb{R}^t \oplus n_{m,k}$ where $t = 0$ or $t = 1$ depending on whether $\dim n_{m,k}$ is even or odd. If $\mathfrak{g}$ admits a symplectic structure then

\[ \dim \mathfrak{j}(n_{m,k}) \leq m. \]

**Proof.** If $\dim n_{m,k}$ is even, then $\mathfrak{g} = n_{m,k}$ and equation (7) follows directly from Lemma 3.3. Let $\mathfrak{g}$ be the direct sum $\mathbb{R}^t \oplus n_{m,k}$ where $n_{m,k}$ has odd dimension. In this case, $\dim \mathfrak{j}(\mathfrak{g}) = \dim \mathfrak{j}(n_{m,k}) + 1$. Moreover, $\dim(\mathfrak{g}/C^1(\mathfrak{g})) = \dim(n_{m,k}/C^1(n_{m,k})) + 1$ since $C^1(\mathfrak{g}) = C^1(n_{m,k})$. Hence, condition (5) for $\mathfrak{g}$ is equivalent to (7). \qed
This condition on the dimension of the center is very restrictive for free nilpotent Lie algebras because, except for small \( m \) and \( k \), the dimension of \( \mathfrak{z}(n_{m,k}) \) is much bigger than the size of the generator set.

**Proof of the Main Theorem.** Let \( n_{m,k} \) denote the free \( k \)-step nilpotent Lie algebra on \( m \) generators. We treat separately the cases by the nilpotency index \( k \) of \( n_{m,k} \).

- \( k = 2 \). As shown in example 2.1, the dimension of the center \( n_{m,2} \), is \( m(m - 1)/2 \). Easy computations give that \( \dim \mathfrak{z}(n_{m,2}) > m \) except for \( m = 2 \) and \( m = 3 \). Hence, by (7) in Corollary 3.4, \( n_{m,2} \) or its one dimensional central extensions are not symplectic for all \( m \geq 4 \).

For \( m = 2 \), the Lie algebra \( n_{2,2} \) has dimension three. In example 3.1 it was shown that its one dimensional central extension is symplectic.

The case \( m = 3 \), the Lie algebra \( n_{3,2} \) was treated in example 3.2 and it is also symplectic.

- \( k = 3 \). Example 2.2 asserts that \( \dim \mathfrak{z}(n_{m,3}) = m(m^2 - 1)/3 \). Then (7) does not hold for \( m > 2 \). Therefore, \( n_{m,3} \) and \( \mathbb{R} \oplus n_{m,3} \) are not symplectic if \( m > 2 \).

Despite the fact that \( \dim \mathfrak{z}(n_{2,3}) \) equals the amount of generators, the six dimensional Lie algebra \( g = \mathbb{R} \oplus n_{2,3} \) does not admit symplectic structures (see, for instance [2]).

To continue, we show that for any \( k \geq 4 \) and any \( m \geq 2 \) the dimension of \( \mathfrak{z}(n_{m,k}) \) is always grater than \( m \). This fact together with the previous corollary imply that neither \( n_{m,k} \) nor its one dimensional central extension is symplectic.

- \( k = 4 \). The subspace \( p(m,4) \) is contained in \( \mathfrak{z}(n_{m,4}) \), thus from (2):

\[
\dim \mathfrak{z}(n_{m,4}) \geq d_m(4) = \frac{1}{4}(m^4 - d_m(1) - 2d_m(2)) = \frac{m^2(m^2 - 1)}{4}.
\]

Notice that \( m^2(m^2 - 1)/4 > m \) whenever \( m \geq 2 \).

- \( k \geq 5 \). It is possible to give a lower bound of \( \dim \mathfrak{z}(n_{m,k}) \) by constructing different elements of length \( k \) in a Hall basis \( \mathcal{B} \) of \( n_{m,k} \).

Let \( \{e_1, \ldots, e_m\} \) a set of generators of \( n_{m,k} \) and consider the set

\[
\mathcal{U} = \{[[[i,j],k],e_m] : 1 \leq j < i \leq m, k \geq j\}.
\]

Any element in \( \mathcal{U} \) is basic and of length 4. Given \( x \in \mathcal{U} \), the bracket

\[
[x, e_m]^{(s)} := [[[x, e_m], e_m], \ldots, e_m] \quad s \geq 1
\]

is an element in the Hall basis if \( \ell([x, e_m]^{(s)}) \leq k \).

In fact if \( s = 1 \) then \( [x, e_m]^{(1)} = [x, e_m] \) and it holds:

(1) both \( x = [[[i,j],k],e_m] \in \mathcal{U} \) and \( e_m \) are elements of the Hall basis, and \( x > e_m \) because of their length;
(2) also \(x = [G, H]\) with \(G = [e_i, e_j, e_k]\) and \(H = e_m\) and we have \(e_m \geq H\).

So both conditions of the Hall basis definition are satisfied. Hence \([x, e_m]^{(1)} \in B\) and it also belongs to \(C^4(n_{m,k})\).

Inductively suppose \([x, e_m]^{(s-1)} \in B\), then clearly \([[[x, e_m], e_m] \cdots, e_m]^{(s-1)} > e_m\) and it is possible to write \([x, e_m]^{(s-1)} = [G, H]\) with \(H = e_m\). Thus \([x, e_m]^{(s)} \in B\). Notice that \([x, e_m]^{(s)} \in C^{s+3}(n_{m,k})\).

We construct the following set
\[
\tilde{U} := \{[x, e_m]^{(k-4)} : x \in U\} \subseteq C^{k-1}(n_{m,k}).
\]
It is contained in the center of \(n_{m,k}\) and it is a linearly independent set. Therefore

\[
\dim \tilde{U} \geq |U|.
\]

Clearly \(\tilde{U}\) and \(U\) have the same cardinal. Also, \(|U| = \sum_{j=1}^{m} (m-j+1)(m-j)\) since for every fixed \(j = 1, \ldots, m\), the amount of possibilities to choose \(k \geq j\) and \(i > j\) is \((m-j+1)\) and \((m-j)\) respectively.

Straightforward computations give \(|\tilde{U}| = 1/3 m^3 + m^2 + 2/3 m\) which together with \((8)\) proves that for any \(m\) and \(k \geq 5\)

\[
\dim \tilde{U} \geq 1/3 m^3 + m^2 + 2/3 m.
\]

The right hand side is greater than \(m\) for all \(m \geq 2\).

\[\square\]

Remark. Our Main Theorem here extends the results in \([3, \text{Example 4.9}]\). In fact, they prove the non existence of symplectic structures on 2-step free nilpotent Lie algebras with different techniques. Those do not apply for every degree of nilpotency.

Denote by \(N_{m,k}\) the simply connected Lie group corresponding to \(n_{m,k}\), the free \(k\)-step nilpotent Lie algebra on \(m\) generators. It is well known that the structure constants of \(n_{m,k}\) relative to a Hall basis are rational (see for instance \([13]\)). A result due to Malcev \((17)\) asserts that the Lie group \(N_{m,k}\) admits a cocompact discrete subgroup \(\Gamma\). Recall the correspondence between symplectic structures on the nilmanifold \(M = \Gamma \setminus N_{m,k}\) and symplectic structures on \(n_{m,k}\). Therefore, the Main Theorem can be stated in terms of the nilmanifolds \(N_{m,k}\), namely:

**Theorem** Let \(N_{m,k}\) be the simply connected Lie group with Lie algebra \(n_{m,k}\) and \(\Gamma\) a cocompact subgroup.

1. If \(\dim N_{m,k}\) is even then the nilmanifold \(M = \Gamma \setminus N_{m,k}\) admits symplectic structures if and only if \((m, k) = (3, 2)\).
If \( \dim N_{m,k} \) is odd, the nilmanifold \( S^1 \times \Gamma \backslash N_{m,k} \) admits symplectic structures if and only if \((m, k) = (2, 2)\).

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References

[1] C. Benson, C. Gordon, Kähler and symplectic structures on nilmanifolds, Topology, 27 (4) (1988), 513-518.

[2] A. Bouyakoub, M. Goze, Sur les algèbres de Lie munies d’une forme symplectique, Rend. Sem. Fac. Sci. Univ. Cagliari, 57 (1987), 85-97.

[3] I. Dotti, P. Tirao, Symplectic structures on Heisenberg-type nilmanifolds, Manuscripta Mathematica, 102 (2000), 387-341.

[4] M. Grayson, R. Grossman, Models for free nilpotent Lie algebras, J. Algebra 35 (1990), 177–191. See draft in [http://users.lac.uic.edu/~grossman/trees.htm](http://users.lac.uic.edu/~grossman/trees.htm)

[5] Z.-D. Guan, Toward a classification of compact nilmanifolds with symplectic structures, International Mathematics Research Notices 49 (2010), 4377-4384.

[6] M. Hall, A basis for free Lie rings and higher commutators in free groups, Proc. Amer. Math. Soc. 1 (1950), 575-581.

[7] A.I. Malcev, On a class of homogeneous spaces, reprinted in Amer. Math. Soc. Trans. Ser., 9 (1) (1962), 276-307.

[8] D. Millionschikov, Graded filiform Lie algebras and symplectic nilmanifolds, Advances in the Mathematical Sciences (AMS), 5 (2004), 259-279.

[9] V. Morosov, Classification of nilpotent Lie algebras of order 6, Izv. Vyssh. Uchebn. Zaved. Math., 4 (1958), 161-171.

[10] O. Myasnichenko, Nilpotent (3,6) sub-Riemannian problem, J. Dyn. Control Syst. 8 (4) (2002), 573–597.

[11] K. Nomizu, On the cohomology of compact homogeneous spaces of nilpotent Lie groups, Annals of Mathematics 59 (1954), 531-538.

[12] H. Poussele, P. Tirao, Compact symplectic nilmanifolds associated with graphs, Journal of Pure and Applied Algebra, 213 (2009), 1788-1794.

[13] C. Reutenauer, Free Lie algebras, London Mathematical Society Monographs 7, The Clarendon Press Oxford University Press (1993).

[14] J. P. Serre, Lie algebras and Lie groups, Lecture Notes in Math. 1500, Springer Verlag (1992).