OMEGA-LIMIT SETS CLOSE TO SINGULAR-HYPERBOLIC ATTRACTION

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ABSTRACT. We study the omega-limit sets $\omega_X(x)$ in an isolating block $U$ of a singular-hyperbolic attractor for three-dimensional vector fields $X$. We prove that for every vector field $Y$ close to $X$ the set $\{x \in U : \omega_Y(x) \text{ contains a singularity} \}$ is residual in $U$. This is used to prove the persistence of singular-hyperbolic attractors with only one singularity as chain-transitive Lyapunov stable sets. These results generalize well known properties of the geometric Lorenz attractor [GW] and the example in [MPu].

1. Introduction

The omega-limit set of $x$ with respect to a vector field $X$ with generating flow $X_t$ is the accumulation point set $\omega_X(x)$ of the positive orbit of $x$, namely

$$\omega_X(x) = \left\{ y : y = \lim_{t_n \to \infty} X_{t_n}(x) \text{ for some sequence } t_n \to \infty \right\}.$$ 

The structure of the omega limit sets is well understood for vector fields on compact surfaces. In fact, the Poincaré-Bendixon Theorem asserts that the omega-limit set for vector fields with finite many singularities in $S^2$ is either a periodic orbit or a singularity or a graph (a finite union of singularities an separatrices forming a closed curve). The Schwartz Theorem implies that the omega-limit set of a $C^\infty$ vector field on a compact surface either contains a singularity or an open set or is a periodic orbit. Another result is the Peixoto Theorem asserting that an open dense subset of vector fields on any closed orientable surface are Morse-Smale, namely their nonwandering set is formed by a finite union of closed orbits all of whose invariant manifolds are in general position. A direct consequence this result is that, for an open-dense subset of vector fields on closed orientable surfaces, most omega-limit sets are contained in the attracting closed orbits. This provides a complete description of the omega limit sets on closed orientable surfaces.

The above results are known to be false in dimension $> 2$. Hence extra hypotheses to understand the omega-limit sets are needed in general. An important one is the hyperbolicity introduced by Smale in the sixties. Recall that a compact
invariant set is hyperbolic if it exhibits contracting and expanding direction which together with the flow’s direction form a continuous tangent bundle decomposition. This definition leads the concept of Axiom A vector field, namely the ones whose non-wandering set is both hyperbolic and the closure of its closed orbits. The Spectral Decomposition Theorem describes the non-wandering set for Axiom A vector fields, namely it decomposes into a finite disjoint union of hyperbolic basic sets. A direct consequence of the Spectral Theorem is that for every Axiom A vector field $X$ there is an open-dense subset of points whose omega-limit set are contained in the hyperbolic attractors of $X$. By attractor we mean a compact invariant set $\Lambda$ which is transitive (i.e. $\Lambda = \omega_X(x)$ for some $x \in \Lambda$) and satisfies $\Lambda = \cap_{t \geq 0} X_t(U)$ for some compact neighborhood $U$ of it called isolating block. On the other hand, the structure of the omega-limit sets in an isolating block $U$ of a hyperbolic attractor is well known: For every vector field $Y$ close to $X$ the set 

$$\{x \in U : \omega_Y(x) = \cap_{t \geq 0} Y_t(U)\}$$

is residual in $U$. In other words, the omega-limit sets in a residual subset of $U$ are uniformly distributed in the maximal invariant set of $Y$ in $U$. This result is a direct consequence of the structural stability of the hyperbolic attractors.

There are many examples of non-hyperbolic vector fields $X$ with a large set of trajectories going to the attractors of $X$. Actually, a conjecture by Palis [P] claims that this is true for a dense set of vector fields on any compact manifold (although he used a different definition of attractor). A strong evidence is the fact that there is a residual subset of $C^1$ vector fields $X$ on any compact manifold exhibiting a residual subset of points whose omega-limit sets are contained in the chain-transitive Lyapunov stable sets of $X$ ([MPa2]). We recall that a compact invariant set $\Lambda$ is chain-transitive if any pair of points on it can be joined by a pseudo-orbit with arbitrarily small jump. In addition, $\Lambda$ is Lyapunov stable if the positive orbit of a point close to $\Lambda$ remains close to $\Lambda$. The result [MPa2] is weaker than the Palis conjecture since every attractor is a chain-transitive Lyapunov stable set but not vice versa.

In this paper we study the omega-limit sets in an isolating block of an attractor for vector fields on compact three manifolds. Instead of hyperbolicity we shall assume that the attractor is singular-hyperbolic, namely it has singularities (all hyperbolic) and is partially hyperbolic with volume expanding central direction [MPP1]. These attractors were considered in [MPP1] for a characterization of $C^1$ robust transitive sets with singularities for vector fields on compact three manifolds (see also [MPP3]). The singular-hyperbolic attractors are not hyperbolic although they have some properties resembling the hyperbolic ones. In particular, they do not have the pseudo-orbit tracing property and are neither expansive nor structural stable.

The motivation for our investigation is the fact that if $U$ is an isolating block of the geometric Lorenz attractor with vector field $X$ then for every $Y$ close to $X$ the set $\{x \in U : \omega_Y(x) = \cap_{t \geq 0} Y_t(U)\}$ is residual in $U$ (this is precisely the
same property of the hyperbolic attractors reported before). It is then natural to believe that such a conclusion holds if \( U \) is an isolating block of a singular-hyperbolic attractor. The answer however is negative as the example [MPu, Appendix] shows. Despite we shall prove that if \( U \) is the isolating block of a singular-hyperbolic attractor of \( X \), then the following alternative property holds: For every vector field \( Y \) close to \( X \) the set

\[
\{ x \in U : \omega_Y(x) \text{ contains a singularity} \}
\]

is residual in \( U \). In other words, the positive orbits in a residual subset of \( U \) look to be "attracted" to the singularities of \( Y \) in \( U \). This fact can be observed with the computer in the classical polynomial Lorenz equation [L]. It contrasts with the fact that the union of the stable manifolds of the singularities of \( Y \) in \( U \) is not residual in any open set. We use this property to prove the persistence singular-hyperbolic attractors with only one singularity as chain-transitive Lyapunov stable sets.

Now we state our result in a precise way. Hereafter \( M \) denotes a compact Riemannian three manifold unless otherwise stated. If \( U \subset M \) we say that \( R \subset U \) residual if it realizes as a countable intersection of open-dense subsets of \( U \). It is well known that every residual subset of \( U \) is dense in \( U \). Let \( X \) be a \( C^r \) vector field in \( M \) and let \( X_t \) be the flow generated by \( X \), \( t \in \mathbb{R} \). A compact invariant set is singular if it contains a singularity.

**Definition 1.1 (Attractor).** An attracting set of \( X \) is a compact, invariant, non-empty, set of \( X \) equals to \( \cap_{t>0} X_t(U) \) for some compact neighborhood \( U \) of it. This neighborhood is called isolating block. An attractor is a transitive attracting set.

**Remark 1.2.** [Hu] calls attractor what we call attracting set. Several definitions of attractor are considered in [Mi].

Denote by \( m(L) \) and \( Det(L) \) the minimum norm and the Jacobian of a linear operator \( L \) respectively.

**Definition 1.3.** A compact invariant set \( \Lambda \) of \( X \) is partially hyperbolic if there is a continuous invariant tangent bundle decomposition \( T_\Lambda M = E^s \oplus E^c \) and positive constants \( K, \lambda \) such that

1. \( E^s \) is contracting: \( \| DX_t(x)/E^s_x \| \leq Ke^{-\lambda t} \), for every \( \forall t > 0 \) and \( x \in \Lambda \);
2. \( E^s \) dominates \( E^c \): \( \frac{\| DX_t(x)/E^c_x \|}{m(DX_t(x)/E^c_x)} \leq Ke^{-\lambda t} \), for every \( \forall t > 0 \) and \( \forall x \in \Lambda \).

We say that \( \Lambda \) has volume expanding central direction if

\[
| Det(DX_t(x)/E^c_x) | \geq K^{-1} e^{\lambda t},
\]

for every \( t > 0 \) and \( x \in \Lambda \).

A singularity \( \sigma \) of \( X \) is hyperbolic if its eigenvalues are not purely imaginary complex number.
Definition 1.4 (Singular-hyperbolic set). A compact invariant set of a vector field \( X \) is singular-hyperbolic if it has singularities (all hyperbolic) and is partially hyperbolic with volume expanding central direction \([MPP1]\). A singular-hyperbolic attractor is an attractor which is also a singular-hyperbolic set.

Singular-hyperbolic attractors cannot be hyperbolic and the most representative example is the geometric Lorenz \([GW]\). Our result is the following.

**Theorem A.** Let \( U \) be an isolating block of a singular-hyperbolic attractor of \( X \). If \( Y \) is a vector field \( C^r \) close to \( X \), then \( \{ x \in U : \omega_Y(x) \text{ is singular} \} \) is residual in \( U \).

This result is used to prove

**Theorem B.** Singular-hyperbolic attractors with only one singularity in \( M \) are persistent as chain-transitive Lyapunov stable sets.

The precise statement of Theorem B (including the definition of chain transitive set, Lyapunov stable set and persistence) will be given in Section 7. This paper is organized as follows. In Section 2 we give some preliminary lemmas. In particular, Lemma 2.1 introduces the continuation \( A_Y \) of an attracting set \( A \) for nearby vector fields \( Y \). In Definition 2.3 we define the region of weak attraction \( A_w(Z, C) \) of \( C \), where \( C \) is a compact invariant sets of a vector field, as the set of points \( z \) such that \( \omega_Z(z) \cap C \neq \emptyset \). Lemma 2.4 proves that if \( U \) is a neighborhood of \( C \) and \( A_w(Z, C) \cap U \) is dense in \( U \), then \( A_w(Z,C) \cap U \) is residual in \( U \). We finish this section with some elementary properties of the hyperbolic sets. We present two elementary properties of singular-hyperbolic attracting sets in Section 3.

In Section 4 we introduce the Property (P) for compact invariant sets \( C \) all of whose closed orbits are hyperbolic. It requires that the unstable manifold of every closed orbit in \( C \) intersect transversely the stable manifold of a singularity in \( C \). This property has been proved for all singular-hyperbolic attractors \( \Lambda \) in \([MP1]\). In Lemma 4.3 we prove that it is open, namely it holds for the continuation \( A_Y \) of \( \Lambda \). The proof is similar to the one in \([MP1]\).

In Section 5 we study the topological dimension \([HW]\) of the omega-limit sets in an isolating block \( U \) of a singular-hyperbolic attracting set with the Property (P). In particular, Theorem 5.2 proves that if \( x \in U \) then the omega-limit set of \( x \) either contains a singularity or has topological dimension one provided the stable manifolds of the singularities in \( U \) do not intersect a neighborhood of \( x \). The proof uses the methods in \([M1]\) with the Property (P) playing the role of the transitivity. We need this theorem to apply the Bowen’s theory of one-dimensional hyperbolic sets \([Bo]\).

In Section 6 we prove Theorem A. The proof is based on Theorem 6.1 where it is proved that if \( U \) is an isolating block of a singular-hyperbolic attracting set with the Property (P) of a vector field \( Y \), then \( A_w(Y, Sing(Y, U)) \cap U \) is dense in \( U \) (here \( Sing(Y, U) \) denotes the set of singularities of \( Y \) in \( U \)). The
proof follows applying the Bowen’s theory (that can be used by Theorem \textsuperscript{5.2} and the arguments in [MPa1, p. 371]. It will follow from Lemma \textsuperscript{2.4} applied to \(C = \text{Sing}(Y, U)\) that \(A_w(Y, \text{Sing}(Y, U)) \cap U\) is residual in \(U\). Theorem \textsuperscript{A} follows because \(\omega_Y(x)\) is singular \(\forall x \in A_w(Y, \text{Sing}(Y, U)) \cap U\). In Section 7 we prove Theorem \textsuperscript{B} (see Theorem \textsuperscript{7.5}).

2. Preliminary lemmas

We state some preliminary results. The first one claims a sort of stability of the attracting sets. It seems to be well known and we prove it here for completeness.

\textbf{Lemma 2.1} (Continuation of attracting sets). Let \(A\) be an attracting set containing a hyperbolic closed orbit of a \(C^r\) vector field \(X\). If \(U\) is an isolating block of \(A\), then for every vector field \(Y\) \(C^r\) close to \(X\) the continuation

\[ A_Y = \cap_{t \geq 0} Y_t(U) \]

of \(A\) in \(U\) is an attracting set with isolating block \(U\) of \(Y\).

\textit{Proof.} Since \(A\) contains a hyperbolic closed orbit we have that \(A_Y \neq \emptyset\) for every \(Y\) close to \(X\) (use for instance the Hartman-Grobman Theorem \textsuperscript{dMP}). Since \(U\) is compact we have that \(A_Y\) also does. Then, to prove the lemma, we only need to prove that if \(Y\) is close to \(X\) then \(U\) is a compact neighborhood of \(A_Y\). For this we proceed as follows. Fix an open set \(D\) such that

\[ A \subset D \subset \text{clos}(D) \subset \text{int}(U) \]

and for all \(n \in \mathbb{N}\) we define

\[ U_n = \cap_{t \in [0,n]} X_t(U). \]

Clearly \(U_n\) is a compact set sequence which is nested \((U_{n+1} \subset U_n)\) and satisfies \(A = \cap_{n \in \mathbb{N}} U_n\). Because \(U_n\) is nested we can find \(n_0\) such that \(U_{n_0} \subset D\). In other words

\[ \cap_{t \in [0,n_0]} X_t(U) \subset D. \]

Taking complement one has

\[ M \setminus D \subset \cup_{t \in [0,n_0]} X_t(M \setminus U). \]

But \(X_t(M \setminus U)\) is open \((\forall t)\) since \(U\) is compact and \(X_t\) is a diffeomorphism. Hence \(\{X_t(M \setminus U) : t \in [0,n_0]\}\) is an open covering of \(M \setminus D\). Because \(D\) is open we have that \(M \setminus D\) is compact and so there are finitely many \(t_1, \ldots, t_k \in [0,n_0]\) such that

\[ M \setminus D \subset X_{t_1}(M \setminus U) \cup \cdots \cup X_{t_k}(M \setminus U). \]

By the continuous dependence of \(Y_t(U)\) on \(Y\) (with \(t\) fixed) one has

\[ M \setminus D \subset Y_{t_1}(M \setminus U) \cup \cdots \cup Y_{t_k}(M \setminus U) \]

\textbf{Remark.} This lemma generalizes a fact in [MPa1] where \(A\) is a hyperbolic attractor and \(X\) is a \(C^1\) vector field.
for all $Y$ close to $X$. By taking complement once more we obtain
\[ Y_{t_1}(U) \cap \cdots \cap Y_{t_k}(U) \subset D. \]
As $t_1, \cdots, t_k \geq 0$ one has $\cap_{t \in [0,n_0]} Y_t(U) \subset Y_{t_1}(U) \cap \cdots \cap Y_{t_k}(U)$ and then
\[ \cap_{t \in [0,n_0]} Y_t(U) \subset D \]
for every $Y$ close to $X$. On the other hand, it follows from the definition that $A_Y \subset \cap_{t \in [0,n_0]} Y_t(U)$ and so $A_Y \subset D$ for every $Y$ close to $X$. Because $\text{clos}(D) \subset \text{int}(U)$ we have that $A_Y \subset \text{int}(U)$. This proves that $U$ is a compact neighborhood of $A_Y$ and the lemma follows. □

Remark 2.2. The above proof shows that the compact set-valued map $Y \rightarrow A_Y$ is continuous in the following sense: For every open set $D$ containing $A$ one has $A_Y \subset D$ for every $Y$ close to $X$. Such a continuity is weaker than the continuity with respect to the Hausdorff metric. It follows from the above-mentioned continuity that if $A$ is a singular-hyperbolic attracting set of $X$ and $Y$ is close to $X$, then the continuation $A_Y$ in $U$ is a singular-hyperbolic attracting set of $Y$.

The following definition can be found in [BS, Chapter V].

Definition 2.3 (Region of attraction). Let $C$ be a compact invariant set of a vector field $Z$. We define the region of attraction and the region of weak attraction of $C$ by
\[ A(C) = \{ x \in M : \omega_X(p) \subset C \} \quad \text{and} \quad A_w(C) = \{ z : \omega_Y(z) \cap C \neq \emptyset \} \]
respectively. We shall write $A(Z, C)$ and $A_w(Z, C)$ to indicate dependence on $Z$.

The region of attraction is also called stable set. The inclusion below is obvious
\[ A(Z, C) \subset A_w(Z, C). \]

The elementary lemma below will be used in Section 6. Again we prove it for the sake of completeness.

Lemma 2.4. If $C$ a compact invariant set of a vector field $Z$ and $U$ is a compact neighborhood of $C$, then the following properties are equivalent:
1. $A_w(Z, C) \cap U$ is dense in $U$
2. $A_w(Z, C) \cap U$ is residual in $U$.

Proof. Clearly (2) implies (1). Now we assume (1) namely $A_w(Z, C) \cap U$ is dense in $U$. Defining
\[ W_n = \{ x \in U : Z_t(x) \in B_{1/n}(C) \text{ for some } t > n \} \quad \forall n \in \mathbb{N} \]
one has
\[ A_w(Z, C) \cap U = \cap_n W_n. \]
In particular $A_w(Z, C) \cap U \subset W_n$ for all $n$. Hence $W_n$ is dense in $U$ (for all $n$) since $A_w(Z, C) \cap U$ does. On the one hand, $W_n$ is open in $U$ [Tubular Flow-Box Theorem] because $B_{1/n}(T)$ is open. This proves that $W_n$ is open-dense in $U$ and the result follows. \hfill \Box

Next we state the classical definition of hyperbolic set.

**Definition 2.5 (Hyperbolic set).** A compact, invariant set $H$ of a $C^1$ vector field $X$ is hyperbolic if there are a continuous, tangent bundle, invariant, splitting $T_x M = E^s \oplus E^X \oplus E^u$ and positive constants $C, \lambda$ such that $\forall x \in H$ one has:

1. $E^s_x$ is the direction of $X(x)$ in $T_x M$.
2. $E^s$ is contracting: $\| DX_t(x)/E^s_x \| \leq C e^{-\lambda t}$, $\forall t \geq 0$.
3. $E^u$ is expanding: $\| DX_t(x)/E^u_x \| \geq C^{-1} e^{\lambda t}$, $\forall t \geq 0$.

A closed orbit of $X$ is hyperbolic if it is hyperbolic as a compact, invariant set of $X$. A hyperbolic set is saddle-type if $E^s \neq 0$ and $E^u \neq 0$.

The Invariant Manifold Theory [HPS] says that through each point $x \in H$ pass smooth injectively immersed submanifolds $W^{ss}(x), W^{uu}(x)$ tangent to $E^s_x, E^u_x$ at $x$. The manifold $W^{ss}(x)$, the strong stable manifold at $x$, is characterized by $y \in W^{ss}(x)$ if and only if $d(X_t(y), X_t(x))$ goes to 0 exponentially as $t \to \infty$. Similarly $W^{uu}(x)$, the strong unstable manifold at $x$, is characterized by $y \in W^{uu}(x)$ if and only if $d(X_t(y), X_t(x))$ goes to 0 exponentially as $t \to -\infty$. These manifolds are invariant, i.e. $X_t(W^{ss}(x)) = W^{ss}(X_t(x))$ and $X_t(W^{uu}(x)) = W^{uu}(X_t(x))$, $\forall t$. For all $x, x' \in H$ we have that $W^{ss}(x)$ and $W^{ss}(x')$ either coincides or are disjoint. The maps $x \in H \to W^{ss}(x)$ and $x \in H \to W^{uu}(x)$ are continuous (in compact parts). For all $x \in H$ we define

$$W^s_X(x) = \cup_{t \in \mathbb{R}} W^{ss}(X_t(x)) \quad \text{and} \quad W^u_X(x) = \cup_{t \in \mathbb{R}} W^{uu}(X_t(x)).$$

Note that if $O \subset H$ is a closed orbit then

$$A(X, O) = W^s_X(O)$$

but $A_w(X, O) \neq W^s_X(O)$ in general. If $H$ is saddle-type and $dim(M) = 3$, then both $W^s_X(x), W^u_X(x)$ are one-dimensional submanifolds of $M$. In this case given $\epsilon > 0$ we denote by $W^{ss}_X(x, \epsilon)$ an interval of length $\epsilon$ in $W^{ss}_X(x)$ centered at $x$ (this interval is often called the local strong stable manifold of $x$).

**Definition 2.6.** Let $\{O_n : n \in \mathbb{N}\}$ be a sequence of hyperbolic periodic orbits of $X$. We say that the size of $W^s_X(O_n)$ is uniformly bounded away from zero if there is $\epsilon > 0$ such that the local strong stable manifold $W^s_X(x_n, \epsilon)$ is well defined for every $x_n \in O_n$ and every $n \in \mathbb{N}$.

**Remark 2.7.** Let $O_n$ be a sequence of hyperbolic periodic orbits of a vector field $X$. It follows from the Stable Manifold Theorem for hyperbolic sets [HPS] that the size of $W^s_X(O_n)$ is uniformly bounded away from zero if all the periodic orbits $O_n$ ($n \in \mathbb{N}$) are contained in the same hyperbolic set $H$ of $X$. 
3. Two Lemmas for Singular-Hyperbolic Attracting Sets

Hereafter we denote by $M$ a compact three manifold. Recall that $\text{clos}(\cdot)$ denotes the closure of $(\cdot)$. In addition, $B_\delta(x)$ denotes the (open) $\delta$-ball in $M$ centered at $x$. If $H \subset M$ we denote $B_\delta(H) = \cup_{x \in H} B_\delta(x)$. For every vector field $X$ on $M$ we denote by $\text{Sing}(X)$ the set of singularities of $X$ and if $B \subset M$ we define $\text{Sing}(X, B) = \text{Sing}(X) \cap B$.

Lemma 3.1. Let $\Lambda$ be a singular-hyperbolic attracting set of a $C^r$ vector field $Z$ on $M$. Let $U$ be an isolating block of $\Lambda$. If $x \in U$ and $\omega_Z(x)$ is non-singular, then every $k \in \omega_Z(x)$ is accumulated by a hyperbolic periodic orbit sequence $\{O_n : n \in \mathbb{N}\}$ such that the size of $W^s_Z(O_n)$ is uniformly bounded away from zero.

Proof. For every $\epsilon > 0$ we define

$$\Lambda_\epsilon = \cap_{t \in \mathbb{R}} Z_t(\Lambda \setminus B_\epsilon(\text{Sing}(Z, \Lambda))).$$

Clearly $\Lambda_\epsilon$ is either $\emptyset$ or a compact, invariant, non-singular set of $Z$. If $\Lambda_\epsilon \neq \emptyset$, then $\Lambda_\epsilon$ is hyperbolic [MPP2]. Observe that $\omega_X(x)$ is non-singular by assumption. Then, there are $\epsilon > 0$ and $T > 0$ such that

$$Z_t(x) \notin \text{clos}(B_\epsilon(\text{Sing}(Z, U))), \quad \forall t \geq T.$$

It follows that $\omega_Z(x) \subset \Lambda_\epsilon$ and so $\Lambda_\epsilon \neq \emptyset$ is a hyperbolic set. In addition, for every $\delta > 0$ there is $T_\delta > 0$ such that

$$Z_t(x) \in B_\delta(\Lambda_\epsilon),$$

for every $t > T_\delta$. Pick $k \in \omega_Z(x)$. The last property implies that for every $\delta > 0$ there is a periodic $\delta$-pseudo-orbit in $B_\delta(\Lambda_\epsilon)$ formed by paths in the positive $Z$-orbit of $x$. Applying the Shadowing Lemma for Flows [HK, Theorem 18.1.6 pp. 569] to the hyperbolic set $\Lambda_\epsilon$ we arrange a periodic orbit sequence $\{O_n : n \in \mathbb{N}\}$ accumulating $k$. Then, Remark 2.7 applies since $H = \Lambda_{\epsilon/2}$ is hyperbolic and contains $O_n$ (for all $n$). The lemma is proved.

The following is a minor modification of [M2, Theorem A].

Lemma 3.2. If $U$ is an isolating block of a singular-hyperbolic attractor of a $C^r$ vector field $X$ in $M$, then every attractor in $U$ of every vector field $C^r$ close to $X$ is singular.

Proof. Let $\Lambda$ be the singular-hyperbolic attractor of $X$ having $U$ as isolating block. By [M2, Theorem A] there is a neighborhood $D$ of $\Lambda$ such that every attractor of every vector field $Y$ $C^r$ close to $X$ is singular. By Remark 2.2 we have that $\cap_{t \geq 0} Y_t(U) \subset D$ for all $Y$ close to $X$. Now if $A \subset U$ is an attractor of $Y$, then $A \subset \cap_{t \geq 0} Y_t(U)$ by invariance. We conclude that $A \subset D$ and then $A$ is singular for all $Y$ close to $X$. This proves the lemma.
4. Property (P)

First we state the definition. As usual we write $S \pitchfork S' \neq \emptyset$ to indicate that there is a transverse intersection point between the submanifolds $S, S'$.

**Definition 4.1** (The Property (P)). Let $\Lambda$ be a compact invariant set of a vector field $X$. Suppose that all the closed orbits of $\Lambda$ are hyperbolic. We say that $\Lambda$ satisfies the Property (P) if for every point $p$ on a periodic orbit of $\Lambda$ there is $\sigma \in \text{Sing}(X, \Lambda)$ such that

$$W^u_Y(p) \pitchfork W^s_Y(\sigma) \neq \emptyset.$$  

The lemma below is a direct consequence of the classical Inclination-lemma [dMP] and the transverse intersection in Property (P).

**Lemma 4.2.** Let $\Lambda$ a compact invariant set with the Property (P) of a vector field $Z$ in a manifold $M$ and $I$ be a submanifold of $M$. If there is a periodic orbit $O \subset \Lambda$ of $Z$ such that $I \pitchfork W^s_Z(O) \neq \emptyset$, then

$$I \cap \left( \bigcup_{\sigma \in \text{Sing}(Z, \Lambda)} W^s_Z(\sigma) \right) \neq \emptyset.$$  

The Property (P) was proved in [MPa1, Theorem 5.1] for all singular-hyperbolic attractors. Here we prove that such a property is open, namely it holds for the continuation in Lemma 2.1 of a singular-hyperbolic attractor.

**Lemma 4.3** (Openness of the Property (P)). Let $U$ be an isolating block of a singular-hyperbolic attractor of a $C^r$ vector field $X$ on $M$. Then, the continuation

$$\Lambda_Y = \cap_{t \geq 0} Y_t(U)$$

has the Property (P) for every vector field $Y$ $C^r$ close to $X$.

**Proof.** By Lemma 2.1 we have that $\Lambda_Y$ is an attracting set with isolating block $U$ since $\Lambda$ has a hyperbolic singularity. Now let $p$ be a point of a periodic orbit $\gamma \subset \Lambda_Y$ of $Y$. Then

$$\text{clos}(W^u_Y(p)) \subset \Lambda_Y$$

since $\Lambda_Y$ is attracting. We claim

$$\text{clos}(W^u_Y(p)) \cap \text{Sing}(Y, U) \neq \emptyset.$$  

Indeed suppose that it is not so, i.e. there is $Y$ $C^r$ close to $X$ such that $\text{clos}(W^u_Y(p)) \cap \text{Sing}(Y, U) = \emptyset$ for some $p$ in a periodic orbit of $Y$ in $U$. It follows from [MPP2] that $\text{clos}(W^u_Y(p))$ is a hyperbolic set. Since $W^u_Y(p)$ is a two-dimensional submanifold we can easily prove that $\text{clos}(W^u_Y(p))$ is an attracting set of $Y$. This attracting set necessarily contains a hyperbolic attractor $A$ of $Y$. Since $A \subset \text{clos}(W^u_Y(p)) \subset \Lambda_Y \subset U$ we conclude that $A \subset U$. By Lemma 3.2 we have that $A$ is singular as well. We conclude that $A$ is an attracting singularity of $Y$ in $U$. This contradicts the volume expanding condition at Definition 1.4 and
the claim follows. One completes the proof of the lemma using the claim as in [MPa1, Theorem 5.1]. □

5. Topological dimension and the Property (P)

We study the topological dimension of the omega-limit set in an isolating block of a singular-hyperbolic attracting set with the Property (P). First of all we recall the classical definition of topological dimension [HW].

Definition 5.1. The topological dimension of a space $E$ is either $-1$ (if $E = \emptyset$) or the last integer $k$ for which every point has arbitrarily small neighborhoods whose boundaries have dimension less than $k$. A space with topological dimension $k$ is said to be $k$-dimensional.

The result of this section is the following.

Theorem 5.2. Let $U$ be an isolating block of a singular-hyperbolic attracting set with the Property (P) of a $C^r$ vector field $Y$ on $M$. If $x \in U$ and there is $\delta > 0$ such that $B_\delta(x) \cap (\cup_{\sigma \in \text{Sing}(Y,U)}W^s_Y(\sigma)) = \emptyset$, then $\omega_Y(x)$ is either singular or a one-dimensional hyperbolic set.

Proof. Let $\Lambda_Y$ be the singular-hyperbolic attracting set of $Y$ having $U$ as isolating block. Obviously $\text{Sing}(Y,U) = \text{Sing}(Y,\Lambda_Y)$. Let $x, \delta$ be as in the statement. Define

$$H = \omega_Y(x).$$

We shall assume that $H$ is non-singular. Then $H$ is a hyperbolic set by [MPP2]. To prove that $H$ is one-dimensional we shall use the arguments in [M1]. However we have to take some care because $\Lambda$ is not transitive. The Property (P) will supply an alternative argument. Let us present the details.

First we note that by Lemma 3.1 every point $k \in H$ is accumulated by a periodic orbit sequence $O_n$ satisfying the conclusion of that lemma. Second, by the Invariant Manifold Theory [HPS], there is an invariant contracting foliation \{\mathcal{F}^s(w) : w \in \Lambda_Y\} which is tangent to the contracting direction of $Y$ in $\Lambda_Y$. A cross-section of $Y$ will be a 2-disk transverse to $Y$. When $w \in \Lambda_Y$ belongs to a 2-disk $D$ transverse to $Y$, we define $\mathcal{F}^s(w, D)$ as the connected component containing $w$ of the projection of $\mathcal{F}^s(w)$ onto $D$ along the flow of $Y$. The boundary and the interior of $D$ (as a submanifold of $M$) are denoted by $\partial D$ and $\text{int}(D)$ respectively. $D$ is a rectangle if it is diffeomorphic to the square $[0,1] \times [0,1]$. In this case $\partial D$ as a submanifold of $M$ is formed by four curves $D^h_v, D^v_h, D^l_v, D^r_v$ ($v$ for vertical, $h$ for horizontal, $l$ for left, $r$ for right, $t$ for top and $b$ for bottom). One defines vertical and horizontal curves in $D$ in the natural way.

Now we prove a sequence of lemmas corresponding to lemmas 1-4 in [M1] respectively.
Lemma 5.3. For every regular point \( z \in \Lambda_Y \) of \( Y \) there is a rectangle \( \Sigma \) such that the properties below hold:
1. \( z \in \text{int}(\Sigma) \);
2. If \( w \in \Lambda_Y \) then \( F^s(w, \Sigma) \) is a horizontal curve in \( \Sigma \);
3. If \( \Lambda_Y \cap \Sigma^h \neq \emptyset \) then \( \Sigma^h = F^s(w, \Sigma) \) for some \( w \in \Lambda_Y \cap \Sigma \);
4. If \( \Lambda_Y \cap \Sigma^b \) then \( \Sigma^b = F^s(w, \Sigma) \) for some \( w \in \Lambda_Y \cap \Sigma \).

Proof. The proof of this lemma is similar to [M1, Lemma 1]. Observe that the corresponding proof in [M1] does not use the transitivity hypothesis. \( \square \)

Definition 5.4. If \( w \in H \cap \Sigma \) we denote by \((H \cap \Sigma)_w\) the connected component of \( H \cap \Sigma \) containing \( w \).

With this definition we shall prove the following lemma.

Lemma 5.5. If \( w \in H \cap \Sigma \) and \((H \cap \Sigma)_w \neq \{w\}\), then \((H \cap \Sigma)_w\) contains a non-trivial curve in the union \( F^s(w, \Sigma) \cup \partial \Sigma \).

Proof. We follow the same steps of the proof of Lemma 2 in [M1]. First we observe that \((H \cap \Sigma)_w \cap (\text{int}(\Sigma) \setminus F^s(x, \Sigma)) \neq \emptyset \). Hence we can fix \( w' \in (H \cap \Sigma)_w \cap (\text{int}(\Sigma) \setminus F^s(x, \Sigma)) \). Clearly \( F^s(w', \Sigma) \) is a horizontal curve which together with \( F^s(w, \Sigma) \) form the horizontal boundary curves of a rectangle \( R \in \Sigma \). One has that \( H \cap \text{int}(B) \neq \emptyset \) for, otherwise, \( w \) and \( w' \) would be in different connected components of \( H \cap \Sigma \) a contradiction. Hence we can choose \( h \in H \cap \text{int}(B) \). Since \( H = \omega_Y(y) \) we have that there is \( y' \) in the positive \( Y \)-orbit of \( y \) arbitrarily close to \( h \). In particular, \( y' \in \text{int}(B) \). By the continuity of the foliation \( F^s \) we have that \( F^s(y', \Sigma) \) is a horizontal curve separating \( \Sigma \) in two connected components containing \( w \) and \( w' \) respectively. Since \( w, w' \) belong to the same connected component of \( H \cap \Sigma \) we conclude that there is \( k \in F^s(y', \Sigma) \cap H \neq \emptyset \).

On one hand, by Lemma 5.4, \( k \in H \) is accumulated by a hyperbolic periodic orbit sequence \( O_n \) such that the size of \( W^s_Y(O_n) \) is uniform bounded away from zero. On the other hand \( y' \) belongs to the positive orbit of \( y \) and \( y \in B_\delta(x) \). By the uniform size of \( W^s_Y(O_n) \) one has \( B_\delta(x) \cap W^s_Y(O_n) \neq \emptyset \) for some \( n \in \mathbb{N} \). Since \( B_\delta(x) \) is open we conclude that
\[
B_\delta(x) \cap W^s_Y(O_n) \neq \emptyset
\]
Then,
\[
B_\delta(x) \cap \left( \bigcup_{\sigma \in \text{Sing}(Y,U)} W^s_Y(\sigma) \right) \neq \emptyset
\]
by Lemma 4.2 since \( \Lambda_Y \) has the Property (P). This is a contradiction which proves the lemma. \( \square \)

Lemma 5.6. For every \( w \in H \) there is a rectangle \( \Sigma_w \) containing \( w \) in its interior such that \( H \cap \Sigma_w \) is 0-dimensional.

Proof. This lemma corresponds to Lemma 3 in [M1] with similar proof. Let \( \Sigma_w = \Sigma \) where \( \Sigma \) is given by Lemma 5.5. Let \( J \subset F^s(w, \Sigma) \cap \partial \Sigma \) be the curve
in the conclusion of this lemma. We can assume that \( J \) is contained in either \( F^s(w, \Sigma) \) or \( \partial \Sigma \). If \( J \subset F^s(w, \Sigma) \) we can prove as in the proof of [M3, Lemma 3] that \( y \in H \) and so \( y \) is accumulated by periodic orbits whose unstable and stable manifolds have uniform size. We arrive a contradiction by Lemma 4.3 as in the last part of the proof of Lemma 5.5. Hence we can assume that \( J \subset \partial \Sigma \). We can further assume that \( J \subset \Sigma_{w} \) (say) for otherwise we get a contradiction as in the previous case. Now if \( J \subset \Sigma_{w} \) then we can obtain a contradiction as before again using the Property (P) and Lemma 4.2. This proves the result. □

The following lemma corresponds to [M1, Lemma 4].

**Lemma 5.7.** \( H \) can be covered by a finite collection of closed one-dimensional subsets.

*Proof.* If \( w \in H \) we consider the cross-section \( \Sigma_{w} \) in Lemma 5.7. By saturating forward and backward \( \Sigma_{w} \) by the flow of \( Y \) we obtain a compact neighborhood of \( w \) which is one-dimensional (see [HW, Theorem III 4 p. 33]). Hence there is a neighborhood covering of \( H \) by compact one-dimensional sets. Such a covering has a finite subcovering since \( H \) is compact. Such a subcovering proves the result. □

Theorem 5.2 now follows from Lemma 5.7 and [HW, Theorem III 2 p. 30]. □

6. **Proof of Theorem A**

The proof is based on the following result.

**Theorem 6.1.** Let \( U \) be an isolating block of a singular-hyperbolic attracting set with the Property (P) of a vector field \( Y \) on \( M \). Then \( A_w(Y, \text{Sing}(Y, U)) \cap U \) is residual in \( U \).

*Proof.* By Lemma 2.4 it suffices to prove that \( A_w(Y, \text{Sing}(Y, U)) \cap U \) is dense in \( U \). Let \( \Lambda_Y \) be the singular-hyperbolic attracting set of \( Y \) having \( U \) as isolating block. Obviously \( \text{Sing}(Y, U) = \text{Sing}(Y, \Lambda_Y) \). To simplify the notation we write \( R_Y = A_w(Y, \text{Sing}(Y, U)) \cap U \). Suppose by contradiction that \( R_Y \) is not dense in \( U \). Then, there is \( x \in U \) and \( \delta > 0 \) such that \( B_{\delta}(x) \cap R_Y = \emptyset \). In particular, \( \omega_Y(x) \cap \text{Sing}(Y, U) = \emptyset \) and so \( \omega_Y(x) \) is non-singular. Recalling the inclusion Eq.(1) at Section 2 one has

\[
U \cap \left( \bigcup_{\sigma \in \text{Sing}(Y, U)} W^s_Y(\sigma) \right) \subset R_Y.
\]

Thus

\[
B_{\delta}(x) \cap \left( \bigcup_{\sigma \in \text{Sing}(Y, U)} W^s_Y(\sigma) \right) = \emptyset.
\]

(2)

It then follows from Theorem 5.2 that \( H = \omega_Y(x) \) is a one-dimensional hyperbolic set. This allows to apply the Bowen’s Theory of one-dimensional hyperbolic sets. More precisely there is a family of (disjoint) cross-sections \( S = \{S_1, \cdots, S_r\} \) of small diameter such that \( H \) is the flow-saturated of \( H \cap \text{int}(S') \), where \( S' = \cup S_i \).
exists because

large. Now, by Lemma 3.1, I

point of sequence S

R

This contradiction proves that there is x in the positive orbit of x contained in the interior of Jn. We can fix S = Si ∈ S in order to assume that Jn ⊂ S for every n. Let w ∈ S be a limit point of x. Then w ∈ H ∩ int(S′). Because I is tangent to E′ the interval sequence Jn converges to an interval J ⊂ WY,w (w) in the C1 topology (WY,w (w) exists because w ∈ H and H is hyperbolic). J is not trivial since the length of Jn is ≥ δ′. It follows from this lower bound that Jn intersects WY,w (w) for some n large. Now, by Lemma 3.1 w is accumulated by periodic orbits O n satisfying the conclusion of this lemma. The continuous dependence in compact parts of the stable manifolds implies Jn ∩ WY,w (O n) ≠ ∅. Since Jn is in the positive orbit of I and I ⊂ Bδ(x) we obtain

Bδ(x) ∩ WY,w (O n) ≠ ∅.

Then,

Bδ(x) ∩ (∩σ∈Sing(Y;U) WY,σ (σ)) ≠ ∅

by Lemma 4.2 since AY has the Property (P). This is a contradiction by Eq. (2). This contradiction proves that RY is dense in U for all Y C′ close to X.

Proof of Theorem A: Let U be an isolating block of a singular-hyperbolic attractor of a C′ vector field X on M. By Lemma 2.1 we have that AY = ∩l>0 Yl(U) is a singular-hyperbolic attracting set with isolating block U for all vector field Y C′ close to X. In addition, AY has the Property (P) by Lemma 4.3. It follows from Theorem 6.1 that Aw(Y, Sing(Y, U)) ∩ U is residual in U. The result follows because ωY(x) is singular ∀x ∈ Aw(Y, Sing(Y, U)) ∩ U (recall Definition 2.3).

Remark 6.2. Let Y be a vector field in a manifold M. In [BS, Chapter V] it was defined a weak attractor of Y as a closed set C ⊂ M such that Aw(Y, C) is a neighborhood of C. Similarly one can define a generic weak attractor of Y as a closed set C ⊂ M such that A(Y, C) ∩ U is residual in U for some neighborhood U of C (compare with the definition of generic attractor [Mi, Appendix 1 p.186]). A direct consequence of Theorem 6.1 is that the set of singularities of a singular-hyperbolic attractor of Y is a generic weak attractor of Y.
7. Persistence of singular-hyperbolic attractors

In this section we prove Theorem B as an application of Theorem A. The idea is to address the question below which is a weaker local version of the Palis’s conjecture [P].

**Question 7.1.** Let \( \Lambda \) an attractor of a \( C^r \) vector field \( X \) on \( M \) and \( U \) be an isolating block of \( \Lambda \). Does every vector field \( C^r \) close to \( X \) exhibit an attractor in \( U \)?

This question has positive answer for hyperbolic attractors, the geometric Lorenz attractors and the example in [MPu]. In general we give a partial positive answer for all singular-hyperbolic attractors with only one singularity in terms of chain-transitive Lyapunov stable sets.

**Definition 7.2.** A compact invariant set \( \Lambda \) of a vector field \( X \) is **Lyapunov stable** if for every open set \( U \supset \Lambda \) there is an open set \( \Lambda \subset V \subset U \) such that \( \bigcup_{t>0} X_t(V) \subset U \).

Recall that \( B_\delta(x) \) denotes the (open) ball centered at \( x \) with radius \( \delta > 0 \).

**Definition 7.3.** Given \( \delta > 0 \) we define a \( \delta \)-chain of \( X \) as a pair of finite sequences \( q_1, ..., q_{n+1} \in M \) and \( t_1, ..., t_n \geq 1 \) such that
\[
X_{t_i}(B_\delta(q_i)) \cap B_\delta(q_{i+1}) \neq \emptyset, \quad \forall i = 1, \cdots, n.
\]
The \( \delta \)-chain joins \( p, q \) if \( q_1 = q \) and \( q_{n+1} = p \). A compact invariant set \( \Lambda \) of \( X \) is **chain-transitive** if every pair of points \( p, q \in \Lambda \) can be joined by a \( \delta \)-chain, \( \forall \delta > 0 \).

Every attractor is a chain-transitive Lyapunov stable set but not vice versa. The following generalizes the concept of robust transitive attractor (see for instance [MPa4]).

**Definition 7.4.** Let \( \Lambda \) be a chain-transitive Lyapunov stable set of a \( C^r \) vector field \( X \), \( r \geq 1 \). We say that \( \Lambda \) is **\( C^r \) persistent** if for every neighborhood \( U \) of \( \Lambda \) and every vector field \( Y \) \( C^r \) close to \( X \) there is a chain-transitive Lyapunov stable set \( \Lambda_Y \) of \( Y \) in \( U \) such that \( A(Y, \Lambda_Y) \cap U \) is residual in \( U \).

Compare this definition with the one in [Hu] where it is required the continuity of \( Y \to \Lambda_Y \) (with respect to the Hausdorff metric) instead of the residual condition of the stable set. Another related definition is that of \( C^r \) weakly robust attracting sets in [CMP]. The result of this section is the following one. It is precisely the Theorem B stated in the Introduction.

**Theorem 7.5.** Singular-hyperbolic attractors with only one singularity for \( C^r \) vector fields on \( M \) are \( C^r \) persistent.

**Proof.** Let \( \Lambda \) be a singular-hyperbolic attractor of a \( C^r \) vector field \( X \) on \( M \). Suppose that \( \Lambda \) contains a unique singularity \( \sigma \). Let \( U \) be a neighborhood of \( \Lambda \).
We can suppose that $U$ is an isolating block. Let $\sigma(Y)$ the continuation of $\sigma$ for every vector field $Y$ close to $X$. Note that $\sigma(X) = \sigma$. Clearly $\text{Sing}(Y,U) = \{\sigma(Y)\}$ for every $Y$ close to $X$.

For every vector field $Y \in C^\alpha$ close to $X$ one defines

$$\Lambda(Y) = \{q \in \Lambda_Y : \forall \delta > 0 \exists \delta\text{-chain joining } \sigma(Y) \text{ and } q\}.$$  

Recall that $\Lambda_Y$ is the continuation of $\Lambda$ in $U$ for $Y$ close to $X$ as in Lemma 2.1.

We note that $\Lambda(Y) \neq \Lambda_Y$ in general [MPu].

To prove the theorem we shall prove that $\Lambda(Y)$ satisfies the following properties ($\forall Y \in C^\alpha$ close to $X$):

1. $\Lambda(Y)$ is Lyapunov stable.
2. $\Lambda(Y)$ is chain-transitive.
3. $A(Y,\Lambda(Y)) \cap U$ is residual in $U$.

One can easily prove (1). To prove (2) we pick $p,q \in \Lambda(Y)$ for $Y$ close to $X$ and fix $\delta > 0$. By Theorem A there is $x \in B_\delta(p)$ such that $\omega_Y(x)$ contains $\sigma(Y)$. Hence there is $t > 1$ such that $X_t(x) \in B_\delta(\sigma)$. On the other hand, since $q \in \Lambda(Y)$, there is a $\delta$-chain $\{(t_1, \cdots, t_n), \{q_1, \cdots, q_{n+1}\}\}$ joining $\sigma$ to $q$. Then (2) follows since the $\delta$-chain $\{(t, t_1, \cdots, t_n), \{x, q_1, \cdots, q_{n+1}\}\}$ joints $p$ and $q$. To finish we prove (3). It follows from well known properties of Lyapunov stable sets [BS] that $\Lambda(Y) = \cap_n O_n$ where $O_n$ is a nested sequence of positively invariant open sets of $Y$. Obviously we can assume that $O_n \subset U$ for all $n$. Clearly the stable set of $O_n$ is open in $U$. Let us prove that such a stable set is dense in $U$. Let $O$ be an open subset of $U$. By Theorem 5.2 there is $x \in O$ such that $\omega_Y(x)$ contains $\sigma(Y)$. Hence there is $t > 0$ such that $X_t(x) \in O_n$. The last implies that $x$ belongs to the stable set of $O_n$. This proves that the stable set of $O_n$ is dense for all $n$. But the stable set of $\Lambda(Y)$ is the intersection of $W_s^Y(O_n)$ which is open-dense in $U$. We conclude that the stable set of $\Lambda(Y)$ is residual and the proof follows.

Theorem 7.5 gives only a partial answer for Question 7.1 (in the one singularity case) since chain-transitive Lyapunov stable set are not attractors in general. However a positive answer for the question will follow (in the one singularity case) once we give positive answer for the questions below.

**Question 7.6.** Is a singular-hyperbolic, Lyapunov stable set an attracting set?

**Question 7.7.** Is a singular-hyperbolic, chain-transitive, attracting set a transitive set?

As it is well known these questions have positive answer replacing singular-hyperbolic by hyperbolic in their corresponding statements. Besides it, a positive answer for Question 7.6 holds provided the two branches of the unstable manifold of every singularity of the set are dense on the set [MPa3].
References

[Bo] Bowen, R., Symbolic dynamics for hyperbolic flows, Amer. J. Math. 95 (1973), 429-460.
[BS] Bhatia, N., Szego, G., Stability theory of dynamical systems, Die Grundlehren der
mathematischen Wissenschaften, Band 161 Springer-Verlag, New York-Berlin (1970).
[CMP] Carballo, C., Morales, C., Pacifico, M., Maximal transitive sets with singularities for
generic $C^1$ vector fields, Bol. Soc. Brasil. Mat. (N.S.) 31 (2000), no. 3, 287-303.
[dMP] de Melo, W., Palis, J., Geometric theory of dynamical systems. An introduction., Translated from the Portuguese by A. K. Manning. Springer-Verlag, New York-Berlin, 1982
[GW] Guckenheimer, J., Williams, R., Structural stability of Lorenz attractors, Publ Math
HES 50 (1979), 59-72.
[HPS] Hirsch, M., Pugh, C., Shub, M., Invariant manifolds, Lec. Not. in Math. 583 (1977), Springer-Verlag.
[HW] Hurewicz, W., Wallman, H., Dimension Theory, Princeton Mathematical Series 4, Princeton University Press, Princeton, N. J., (1941).
[Hu] Hurley, M., Attractors: persistence, and density of their basins, Trans. Amer. Math. Soc. 269 (1982), no. 1, 247-271.
[HK] Hasselblatt, B., Katok, A., Introduction to the modern theory of dynamical systems.
With a supplementary chapter by Katok and Leonardo Mendoza, Encyclopedia of Math-
ematics and its Applications, 54. Cambridge University Press, Cambridge (1995).
[L] Lorenz, E., Deterministic nonperiodic flow, Journal of Atmospheric Sciences 20 (1963),
130-141.
[M] Milnor, J., On the concept of attractor, Comm. Math. Phys. 99 (1985), no. 2, 177-195.
[M1] Morales, C., Singular-hyperbolic sets and topological dimension, Dyn. Syst. 18 (2003),
no. 2, 181-189.
[M2] Morales, C., The explosion of singular-hyperbolic attractors, Preprint (2003). On web
http://front.math.ucdavis.edu/math.DS/0303253
[MPa1] Morales, C., Pacifico, M., Mixing attractors for 3-flows, Nonlinearity, 14 (2001), 359-378.
[MPa2] Morales, C., Pacifico, M., Lyapunov stability of $\omega$-limit sets, Discrete Contin. Dyn. Syst. 8 (2002), no. 3, 671-674.
[MPa3] Morales, C., Pacifico, M., A dichotomy for three-dimensional vector fields, to appear
in Ergodic Theory Dynam. Systems.
[MPa4] Morales, C., Pacifico, M., Sufficient conditions for robustness of attractors, Preprint IMPA Serie A 2003/211. On web
http://www.preprint.impa.br/cgi-bin/MMMbrowse.cgi.
[MPP1] Morales, C., Pacifico, M., Pujals, E., On $C^1$ robust singular transitive sets for three-
dimensional flows, C. R. Acad. Sci. Paris, 326 (1998), Série I, 81-86.
[MPP2] Morales, C., Pacifico, M., Pujals, E., Singular Hyperbolic Systems, Proc. Amer. Math. Soc. 127 (1999), 3393-3401.
[MPP3] Morales, C. A., Pacifico, M. J., Pujals, E. R., Robust transitive singular sets for 3-flows
are partially hyperbolic attractors or repellers, to appear in Annals of Math..
[MPu] Morales, C., Pujals, E., Singular strange attractors on the boundary of Morse-Smale systems, Ann. Ec. Norm. Sup., 30, (1997), 693-717.
[P] Palis, J., A global view of dynamics and a conjecture on the denseness of finitude of attractors, Geometrie complexe et systemes dynamiques (Orsay, 1995). Asterisque 261 (2000), xiii-xiv, 335-347.
[PT] Palis, J., Takens, F., Hyperbolicity and sensitive chaotic dynamics at homoclinic bifur-
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