PERIOD MAPPINGS FOR NONCOMMUTATIVE ALGEBRAS

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1. Introduction

Differential graded (dg) algebras naturally appear in the center of noncommutative geometry. One of main approaches to noncommutative geometry is to regard a pretriangulated dg category (or some kind of stable $(\infty, 1)$-categories) as a space and think that two spaces coincide if two categories are equivalent. This approach unifies many branches of mathematics such as algebraic geometry, symplectic geometry, representation theory and mathematical physics. In many interesting cases, such a category of interest admits a single compact generator, and the dg category is quasi-equivalent to the dg category of dg modules over a (not necessarily commutative) dg algebra. Thus, dg algebras can be viewed as incarnations of noncommutative spaces. For example, by [2] the dg category of quasi-coherent complexes on a quasi-compact and separated scheme admits a single compact generator.

The purpose of this paper is to construct a period mapping for deformations of a dg algebra. We generalize Griffiths’ period mapping for deformations of algebraic varieties to noncommutative setting. Before proceeding to describe formulations and results of this paper, we would like to briefly review the Griffiths’ construction of the period mapping. Let $X_0$ be a complex smooth projective variety (more generally, a complex compact Kähler manifold). Let $f : X \to S$ be a deformation of $X_0$ such that $X_0$ is the fiber over a point $0 \in S$, and $S$ is a base complex manifold. For each fiber $X_s$ over $s \in S$, consider the Hodge filtration $F^rH^i(X_s, \mathbb{C}) \subset H^i(X_s, \mathbb{C})$ on the singular cohomology. The derived pushforward $R^if_*\mathbb{C}_X$ of the constant sheaf $\mathbb{C}_X$ is a local system. We assume that $S$ is small so that $S$ is contractible. There is the natural identification $H^i(X_s, \mathbb{C}) \simeq H^i(X_0, \mathbb{C})$, and one has the pair

$$(F^rH^i(X_s, \mathbb{C}), H^i(X_s, \mathbb{C}) \simeq H^i(X_0, \mathbb{C})).$$

Therefore, to any point $s$ one can associate the subspace $F^rH^i(X_s, \mathbb{C})$ in $V := H^i(X_0, \mathbb{C})$ via $H^i(X_s, \mathbb{C}) \simeq H^i(X_0, \mathbb{C})$. It gives rise to the classifying map called the (infinitesimal) period mapping

$$P : S \to \text{Grass}(V, n)$$

where $n = \dim F^rH^i(X_0, \mathbb{C})$, and Grass$(V, n)$ is the Grassmannian which parametrizes the $n$-dimensional subspaces of $V$. 


In order to generalize $P$, it is necessary to consider a noncommutative analogue of Hodge filtrations. Remember that by Hochschild-Kostant-Rosenberg theorem the periodic cyclic homology $HP_*(X)$ of a smooth scheme $X$ over a field of characteristic zero can be described in terms of the algebraic de Rham cohomology of $X$. Namely, $HP_i(X) = \prod_{n \leq i} H^i_{dR}(X)$ for odd $i$, and $HP_i(X) = \prod_{n \geq i} H^i_{dR}(X)$ for even $i$. The image of the natural map $NH_*(X) \to HP_*(X)$ from the negative cyclic homology $NH_*(X)$ determines the subspace which can be expressed as a product of Hodge filtrations (see for instance [12], Remark 5.5). Note that cyclic homology theories can be defined for more general objects such as dg algebras, dg categories, etc. Let $A$ be a dg algebra over a field of characteristic zero. Let $CC_\bullet(A)$ and $CC_\bullet^{per}(A)$ denote its negative cyclic complex and its periodic cyclic complex respectively. It is for this reason that it is natural to think of the natural map of complexes $CC_\bullet(A) \to CC_\bullet^{per}(A)$ as a noncommutative analogue of Hodge complex. Hence, intuitively speaking, a period mapping should carry a deformation $\tilde{A}$ of $A$ to the deformation $CC_\bullet(\tilde{A}) \to CC_\bullet^{per}(\tilde{A})$ of $CC_\bullet(A) \to CC_\bullet^{per}(A)$ endowed with a trivialization of $CC_\bullet^{per}(A)$.

We employ the deformation theory by means of dg Lie algebras. The basic idea of so-called derived deformation theory is that any reasonable deformation problem in characteristic zero is controled by a dg Lie algebra. To a dg Lie algebra $L$ over a field $k$ of characteristic zero, one can associate a functor $\text{Spf}_L : \text{Art}_k \to \text{Sets}$, from the category of artin local $k$-algebras with residue field $k$ to the category of sets, which assigns to any artin local algebra $R$ with the maximal ideal $m_R$ to the set of equivalence classes of Maurer-Cartan elements of $L \otimes_R m_R$, see Section 3.3 (see [10], [34] for a homotopical foundation of this approach). Deformation theory via dg Lie algebras fits in with derived moduli theory in derived algebraic geometry (see e.g., the survey [39]). In fact, the Chevalley-Eilenberg cochain complex of a dg Lie algebra plays the role of the ring of functions on a formal neighborhood of a point of a derived moduli space. Another remarkable advantage is that it allows one to study deformation problems using homological methods of dg Lie algebras. That is to say, once one knows that a deformation space. Another remarkable advantage is that it allows one to study deformation problems using homological methods of dg Lie algebras. That is to say, once one knows that a deformation problem $\text{Art}_k \to \text{Sets}$ which assigns $R$ to the isomorphism classes of deformations to $R$ is of the form $\text{Spf}_L$, one can use homological algebra concerning $L$. It has been fruitful. For example, the famous constructions can be found in Goldman-Millson’s local study on certain moduli spaces [8] and Kontsevich’s deformation quantization of a Poisson manifold [21].

One of main dg Lie algebras in this paper is the dg Lie algebra $C^\bullet(A)[1]$, that is the shifted Hochschild cochain complex of a dg algebra $A$ over the field $k$. It controls curved $A_\infty$-deformations of $A$. Namely, the functor $\text{Spf}_{C^\bullet(A)[1]} : \text{Art}_k \to \text{Sets}$ associated to $C^\bullet(A)[1]$ can be identified with the functor which assigns to $R$ the set of isomorphism classes of curved $A_\infty$-deformations of $A$ to $R$, that is, curved $A_\infty$-algebras $A \otimes_R R$ over $R$ whose reduction $A \otimes_k R/m_R$ are identified with $A$ (see Section 3.3 for details). Let $k[t]$ be the dg algebra (with zero differential) generated by $t$ of cohomological degree 2. The complex $CC^\bullet(A)$ naturally comes equipped with the action of $k[t]$, and $CC_\bullet^{per}(A)$ comes equipped with the action of $k[t, t^{-1}]$. We then consider a pair $(W, \phi : CC_\bullet^{per}(A) \otimes_k R \simeq W \otimes_R R[t, t^{-1}])$ such that $W$ is a deformation of dg $k[t]$-module $CC_\bullet(A)$ to $R$, and $\phi$ is an isomorphism of dg $R[t, t^{-1}]$-modules. There is a dg Lie algebra $F$ such that $\text{Spf}_F(R)$ can be naturally identified with the set of isomorphism classes of such pairs. We can think of $\text{Spf}_F$ as a formal Sato Grassmannian generalized to the complex level (see Section 4.12).

From a naive point of view, our main construction of a period mapping may be described as follows (see Section 4. Theorem 4.17 for details):

**Theorem 1.1.** We construct an $L_\infty$-morphism $P : C^\bullet(A)[1] \to F$ such that the induced morphism of functors $\text{Spf}_P : \text{Spf}_{C^\bullet(A)[1]} \to \text{Spf}_F$ has the following modular interpretation: If one identifies $\alpha \in \text{Spf}_{C^\bullet(A)[1]}(R)$ with a curved $A_\infty$-deformation $\tilde{A}_\alpha$, then $\text{Spf}_P$ sends $\tilde{A}_\alpha$ to the
pair \((CC^-(\tilde{A}_\alpha), CC^\text{per}(A) \otimes k R) \simeq CC^-(\tilde{A}_\alpha) \otimes_{R[t]} R[t,t^{-1}]\) in \(\text{Spf}_F(R)\). (See Section 4.6 for \(L_{\infty}\)-morphisms)

Note that the pair
\[(CC^-(\tilde{A}_\alpha), CC^\text{per}(A) \otimes k R) \simeq CC^-(\tilde{A}_\alpha) \otimes_{R[t]} R[t,t^{-1}]\]
associated to \(\tilde{A}_\alpha\) in Theorem 1.1 is a counterpart of \((F^rH^i(X_s,\mathbb{C}), H^i(X_s,\mathbb{C}) \simeq H^i(X_0,\mathbb{C})\) appeared in the construction of the period mapping for deformations of \(X_0\). For this reason, we shall refer to \(\text{Spf}_F : \text{Spf}_{C^\bullet(A)[1]} \rightarrow \text{Spf}_F\) as the period mapping for \(A\) (one may think of the \(L_{\infty}\)-morphism \(\mathcal{P}\) as a Lie algebra theoretic realization of the period mapping). The construction of the \(L_{\infty}\)-morphism \(\mathcal{P} : C^\bullet(A)[1] \rightarrow \mathbb{F}\) (and the period mapping) is carried out in several steps. One of the key inputs is a modular interpretation of the Lie algebra action of the shifted Hochschild cochain complex \(C^\bullet(A)[1]\) on Hochschild chain complex \(C^\bullet(A)\). Note that if a dg algebra \(A\) is deformed, then it induces a deformation of Hochschild chain complex \(C^\bullet(A)\). Since deformations of the complex \(C^\bullet(A)\) is controled by the endomorphism dg Lie algebra \(\text{End}(C^\bullet(A))\), there should be a corresponding morphism \(C^\bullet(A)[1] \rightarrow \text{End}(C^\bullet(A))\) of dg Lie algebras. We find that it is given by the action \(L : C^\bullet(A)[1] \rightarrow \text{End}(C^\bullet(A))\) of \(C^\bullet(A)[1]\) on \(C^\bullet(A)\) which was studied by Tamarkin-Tsygan [38] generalizing the calculus structure on the pair \((HH^\bullet(A), HH^\bullet(A))\) given by Daletski-Gelfand–Tsygan, to the level of complexes (see Section 3). (It is crucial for our construction to work with complexes.) Also, it allows us to make use of homological algebra of homotopy calculus operad (see Section 4). Another input is a result on a nullity (Proposition 1.1). It says that the morphism \(L((t)) : C^\bullet(A)[1] \rightarrow \text{End}_{k[t,t^{-1}]}(CC^\text{per}(A))\) of dg Lie algebras induced by \(L\) (see Section 3.2) is null-homotopic. This fact plays a role analogous to Ehresmann fibration theorem.

Although we have discussed about general dg algebras, we would like to focus on the important case of interest where the dg algebra \(A\) is smooth and proper (see Section 5.1). In that case, the degeneration of an analogue of Hodge-to-de Rham spectral sequence on cyclic homology theories was conjectured by Kontsevich-Soibelman and was proved by Kaledin (see Section 5.2). As in the commutative world, the degeneration is pivotal. We obtain (see Section 5.4):

**Theorem 1.2.** Suppose that \(A\) is smooth and proper. Then the dg Lie algebra \(\mathbb{F}\) is equivalent to the dg Lie algebra \(\text{End}_{k[t,t^{-1}]}(HH^\bullet(A)((t))) / \text{End}_{k[t]}(HH^\bullet(A)\{t\})[-1]\) equipped with zero differential and zero bracket, and the \(L_{\infty}\)-morphism in Theorem 1.1 is
\[\mathcal{P} : C^\bullet(A)[1] \rightarrow \text{End}_{k[t,t^{-1}]}(HH^\bullet(A)((t))) / \text{End}_{k[t]}(HH^\bullet(A)\{t\})[-1].\]
Here \(HH^\bullet(A)\) is the graded vector space of Hochschild homology, and \(HH^\bullet(A)\{t\}\) is the graded vector space \(HH^\bullet(A) \otimes_k k[t]\) (the cohomological degree of \(t\) is \(2\)). The graded vector space \(HH^\bullet((t))\) is \(HH^\bullet(A) \otimes_k k[t,t^{-1}]\).

By using the properties on our period mapping and its codomain (so-called the period domain), one can deduce the property of the domain of the period mapping. Indeed, applying the period mapping as an \(L_{\infty}\)-morphism, we prove (see Section 6):

**Theorem 1.3.** Suppose that the dg algebra \(A\) is smooth, proper and Calabi-Yau. Then the dg Lie algebra \(C^\bullet(A)[1]\) is quasi-abelian. Namely, it is quasi-isomorphic to an abelian dg Lie algebra. In particular, a curved \(A_{\infty}\)-deformation of \(A\) is unobstructed.

Theorem 1.3 is a Bogomolov-Tian-Todorov theorem for \(C^\bullet(A)[1]\). It is also applicable to a dg category which is quasi-equivalent to the dg category of dg modules over \(A\). This unobstructedness was formulated in Katzarkov-Kontsevich-Pantev [21, 4.4.1] with the outline of an approach which uses an action of the framed little disks operad in the Calabi-Yau case. The proof of Theorem 1.3 in this paper is different from the approach in loc. cit., for example, we
do not employ framed little disks operad. Nevertheless, it would be interesting to compare them.

We obtain the following infinitesimal Torelli theorem as a direct but interesting consequence from our moduli-theoretic construction and Calabi-Yau property:

**Theorem 1.4.** Suppose that $A$ is smooth, proper and Calabi-Yau. The period mapping $Spf_P : Spf_{C^*(A)[1]} \to Spf_F$, which carries a curved $A_\infty$-deformation $\tilde{A}$ to the associated pair

$$(CC^*_-(\tilde{A}), CC^*_\text{gr}(A) \otimes_k R \simeq CC^*_-(\tilde{A}) \otimes_{R[t]} R[t, t^{-1}]),$$

is a monomorphism. Here as in Theorem 1.1 we implicitly identifies an element in $Spf_{C^*(A)[1]}(R)$ with a curved $A_\infty$-deformation.

It might be worth considering the case when the dg algebra $A$ comes from a (quasi-compact and separated) smooth scheme $X$ over the field $k$ (cf. Remark 3.3). The tangent space $Spf_{C^*(A)[1]}(k[\epsilon]/(\epsilon^2))$ of $Spf_{C^*(A)[1]}$ is isomorphic to the second Hochschild cohomology $HH^2(A)$ as vector spaces. According to Hochschild-Kostant-Rosenberg theorem, there is an isomorphism

$$Spf_{C^*(A)[1]}(k[\epsilon]/(\epsilon^2)) \simeq HH^2(A) \simeq H^0(X, \wedge^2 T_X) \oplus H^1(X, T_X) \oplus H^2(X, \mathcal{O}_X)$$

where $T_X$ is the tangent bundle, and $\mathcal{O}_X$ is the structure sheaf. Informally speaking, the space $H^1(X, T_X)$ parametrizes deformations of the scheme $X$ (in the commutative world), and the subspace $H^0(X, T_X^{\otimes 2}) \oplus H^2(X, \mathcal{O}_X)$ is involved with (twisted) deformation quantizations of $X$, which we think of as the noncommutative part. Thus, in that case, Theorem 1.4 means that the associated pair is deformed faithfully along deformations of the algebra not only to commutative directions but also to noncommutative directions.

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2. $A_\infty$-algebra and Hochschild complex

Our main interest lies in differential graded algebras and their deformations. But, if we consider (explicit) deformations of differential graded algebras, curved $A_\infty$-algebras naturally appear as deformed objects. Thus, in this Section we review the basic definitions and facts about curved $A_\infty$-algebras and Hochschild complexes which we need later. We also recall cyclic complexes of $A_\infty$-algebras.

**Convention.** When we consider $\mathbb{Z}$-graded modules, differential graded modules, i.e., complexes, etc., we will use the cohomological grading which are denoted by upper indices. But, the homological grading is familiar to homology theories such as Hochschild homology, cyclic homology. Thus, when we treat Hochschild, cyclic homology theories etc., we use the homological grading which are denoted by lower indices.

2.1. Let $k$ be a unital commutative ring. In this paper we usually treat the case when $k$ is either a field of characteristic zero or an artin local ring over a field of characteristic zero. Let $A$ be a $\mathbb{Z}$-graded $k$-module. Unless stated otherwise, we assume that each $k$-module $A^n$ is a free $k$-module. If $a \in A$ is a homogeneous element, we shall denote by $|a|$ the (cohomological) degree of $a$ (any element $a$ appeared in $|a|$ will be implicitly assumed to be homogeneous). We put $TA := \oplus_{n \geq 0} A^\otimes n$ where $A^\otimes n$ denotes the $n$-fold tensor product. We shall use the standard symmetric monoidal structure of the category of $\mathbb{Z}$-graded $k$-modules. In particular, the symmetric structure induces $a \otimes b \mapsto (-1)^{|a||b|} b \otimes a$. We often write $\otimes$ for the tensor product.
\( \otimes_k \) over the base \( k \). The graded \( k \)-module \( TA \) has a (graded) coalgebra structure given by a comultiplication \( \Delta : TA \to TA \otimes TA \) where

\[
\Delta(a_1, \ldots, a_n) = \sum_{i=0}^{n} (a_1, \ldots, a_i) \otimes (a_{i+1}, \ldots, a_n),
\]

and a counit \( TA \to A^{\otimes 0} = k \) determined by the projection. Here we write \( (a_1, \ldots, a_n) \) for \( a_1 \otimes a_2 \otimes \ldots \otimes a_n \).

A \( k \)-linear map \( f : TA \to TA \) is said to be a coderivation if two maps \( (1 \otimes f + f \otimes 1) \circ \Delta, \Delta \circ f : TA \Rightarrow TA \otimes TA \) coincide. Let us denote by \( \text{Coder}(TA) \) the graded \( k \)-module of coderivations from \( TA \) to \( TA \), where the grading is defined as the grading on maps of graded modules. The composition with the projection \( TA \to A \) induces an isomorphism

\[
\text{Coder}(TA) \to \text{Hom}_k(TA, A)
\]

where the right-hand side is the hom space of \( k \)-linear maps, and the inverse is given by \( \Sigma_i \otimes \Sigma_j \otimes f_i \otimes \Sigma^{n-i-j} \) for \( \{f_i\} \geq 0 \in \prod_{i \geq 0} \text{Hom}(A^{\otimes i}, A) \cong \text{Hom}(TA, A) \) (cf. [7 Proposition 1.2]).

2.2. For a graded \( k \)-module \( A \), the suspension \( sA = A[1] \) is the same \( k \)-module endowed with the shift grading \( (A[1])^i = A^{i+1} \). Put \( BA = T(A[1]) \). A curved \( A_{\infty} \)-structure on a graded \( k \)-module \( A \) is a coderivation \( b : BA \to BA \) of degree 1 such that \( b^2 = b \circ b = 0 \). Let \( b_i \) denote the composite \( (A[1])^{\otimes i} \to BA \xrightarrow{b} BA \to A[1] \) with the projection. A curved \( A_{\infty} \)-structure \( b \) on \( A \) is called an \( A_{\infty} \)-structure if \( b_0 = 0 \). A curved \( A_{\infty} \)-structure \( b \) on \( A \) is called a differential graded (dg for short) algebra structure if \( b_i = 0 \) and \( b_i = 0 \) for \( i > 2 \). A curved \( A_{\infty} \)-algebra (resp. \( A_{\infty} \)-algebra, dg algebra) is a pair \( (A, b) \) with \( b \) a curved \( A_{\infty} \)-structure (resp. an \( A_{\infty} \)-structure, dg algebra) and we often abuse notation by writing \( A \) for \( (A, b) \) if no confusion is likely to arise.

Let \( s : A \to A[1] \) be the “identity map” of degree –1 which identifies \( A^i \) with \( (A[1])^i \). The map \( s \) induces \( A^{\otimes n} \cong (A[1])^{\otimes n} \). Therefore we have a \( k \)-linear map \( m_n : A^{\otimes n} \to A \) such that the following diagram

\[
\begin{array}{ccc}
A^{\otimes n} & \xrightarrow{m_n} & A \\
\downarrow{s^{\otimes n}} & & \downarrow{s^{-1}} \\
(A[1])^{\otimes n} & \xrightarrow{b_n} & A[1].
\end{array}
\]

commutes. We adopt the sign convention [7] which especially implies

\[
(f \otimes g)(a \otimes a') = (-1)^{|a||g|} f(a) \otimes g(a')
\]

where \( a, a' \in A \) and \( f \) and \( g \) are \( k \)-linear maps of degree \( |f| \) and \( |g| \) respectively. To be precise, we write \( s^{\otimes n} \) for \( (s \otimes 1^{\otimes (n-1)}) \otimes (1 \otimes 1^{\otimes (n-2)}) \ldots (1 \otimes 1^{\otimes 1}) \) (in the literature the author adopts \( s^{\otimes n} = (1^{\otimes n-1} \otimes s)(1^{\otimes n-2} \otimes s^2 \otimes 1) \ldots (s \otimes 1^{\otimes n-1}) \) Put \( [a_1] \ldots [a_n] = sa_1 \otimes \ldots \otimes sa_n \). Applying this sign rule to \( s \) we have \( b_n[a_1] \ldots [a_n] = (-1)^{\sum_i (n-i)|a_i|} m_n(a_1, \ldots, a_n) \). A curved \( A_{\infty} \)-algebra \( (A, b) \) is said to be unital if there is an element \( 1_A \) of degree zero (called a unit) such that (i) \( m_1(1_A) = 0 \), (ii) \( m_2[a]1_A = m_2[1_A]a = a \) for any \( a \in A \), and (iii) for \( n \geq 3 \), \( m_n(a_1, \ldots, a_n) = 0 \) for one of \( a_i \) equals \( 1_A \).

2.3. An \( A_{\infty} \)-morphism \( f : (A, b) \to (A', b') \) between curved \( A_{\infty} \)-algebras is a differential graded (=dg) coalgebra map \( f : (BA, b) \to (BA', b') \), for which we often write an \( A_{\infty} \)-morphism \( f : A \to A' \). In the literature authors often impose the condition \( f(k) = k \subset BA' \). But we emphasize that when we treat curved \( A_{\infty} \)-algebras and deformation theory it is natural to
drop this condition (see also [31]). Consider the natural projection $BA' \to A'[1]$. Then the composition with the projection $BA' \to A'[1]$ gives rise to an isomorphism

$$\text{Hom}_{\text{coalg}}(BA, BA') \to \text{Hom}_k(BA, A'[1])$$

where the left-hand side indicates the hom set of graded coalgebra maps and right-hand side indicates hom set of $k$-linear maps. The inverse is given by $BA \xrightarrow{\Delta_n} (A'[1])^\otimes n$ for $f : BA \to A'[1]$ where $\Delta_n$ is the $(n-1)$-fold iteration of comultiplication for each $n \geq 0$. Therefore, a dg coalgebra map $f : BA \to BA'$ is determined by the family of graded $k$-linear maps $\{f_n : (A[1])^\otimes n \to BA \to BA' \to A'[1]\}_{n \geq 0}$ that satisfies a certain relation between $b_i$'s and $f_j$'s corresponding to the compatibility with respect to differentials. We shall refer to $f_n$ as the $n$-th component of $f$. If $f_1$ induces an isomorphism $A[1] \to A'[1]$ of graded $k$-modules, we say that $f$ is an $A_\infty$-isomorphism. Suppose that $(A, b)$ is an $A_\infty$-algebra. Then since the condition $b_0 = 0$ implies $b_1 = m_2 = 0$, we can define the cohomology $H^*(A, b_1)$ of $(A, b_1)$ which we regard as a graded $k$-module. An $A_\infty$-morphism $f : (A, b) \to (A', b')$ of two $A_\infty$-algebras is said to be quasi-isomorphism if $f$ induces an isomorphism $H^*(A, b_1) \to H^*(A', b'_1)$ of graded $k$-modules. For two unital curved $A_\infty$-algebras $A$ and $A'$, and an $A_\infty$-morphism $f : A \to A'$, we say that $f$ is unital if $f(1_A) = 1_{A'}$ and $f_n[a_1] \cdots [a_n] = 0$ if one of $a_i$ equals $1_A$.

2.4. We define a Hochschild cochain complex of an $A_\infty$-algebra. Let $(A, b)$ be a unital $A_\infty$-algebra. The graded $k$-module $\text{Coder}(BA)$ has a differential given by $\partial^{\text{Hoch}}(f) := [b, f] = b \circ f - (-1)^{|f|} f \circ b$ such that $\partial^{\text{Hoch}} \circ \partial^{\text{Hoch}} = 0$. We shall refer to

$$C^\bullet(A)[1] := \text{Coder}(BA) \simeq \text{Hom}_k(BA, A[1]) \simeq \prod_{n \geq 0} \text{Hom}_k((A[1])^\otimes n, A[1])$$

as the shifted Hochschild cochain complex of $A$. Its cohomology $H^*(C^\bullet(A)[1])$ is called the shifted Hochschild cohomology of $A$. We easily see that the bracket $[-, -]_G$ determined by the graded commutator and the differential $\partial^{\text{Hoch}}$ exhibit $C^\bullet(A)[1]$ as a dg Lie algebra. For the definition of dg Lie algebra, see e.g. [15, I. 3.5]. The bracket $[-, -]_G$ is call the Gerstenhaber bracket, and by an abuse of notation we usually write $[-, -]$ for $[-, -]_G$. Let us regard $f \in C^\bullet(A)[1]$ as a family of $k$-linear maps $\{f_n\}_{n \geq 0} \in \prod_{n \geq 0} \text{Hom}_k((A[1])^\otimes n, A[1])$ where $f_n : A[1])^\otimes n \to A[1]$. We say that $f$ is normalized if $f_n[a_1] \cdots [a_n] = 0$ if one of $a_i$ equals $1_A$ for $n \geq 1$. The normalized elements constitutes a dg Lie subalgebra $C^\bullet(A)[1]$ of $C^\bullet(A)[1]$ (we can easily check this by the definition of $\partial$ and the bracket). Moreover, by [29, Theorem 4.4] there is a deformation retract $C^\bullet(A)[1] \xrightarrow{\sim} C^\bullet(A)[1] \to C^\bullet(A)[1]$, which induces quasi-isomorphisms. If $A'$ is a dg algebra that is quasi-isomorphic to the $A_\infty$-algebra $A$, the complex $C^\bullet(A)[1]$ is quasi-isomorphic (via zig-zag) to $C^\bullet(A')[1]$ which is the usual (shifted) Hochschild cochain complex, that is quasi-isomorphic to the derived Hom complex $\text{RHom}_{A', \text{bimodule}}(A', A')[1]$ (see [29, Section 4]).

2.5. Following [7, Section 3] we recall the Hochschild chain complex of a unital curved $A_\infty$-algebra that generalizes the the Hochschild chain complex of a unital dg-algebra (see loc. cit. for details). Let $(A, b)$ be a unital curved $A_\infty$-algebra and $k \to A$ a unital $A_\infty$-morphism which sends $1 \in k$ to $1_A$. Put $\bar{A} = \text{Coker}(k \to A)$ and let $b : BA \to B\bar{A}$ be the curved $A_\infty$-structure on $\bar{A}$ associated to $(A, b)$. Consider the graded $k$-module

$$C_\bullet(A) := A \otimes B\bar{A}.$$  

We will define a differential on $C_\bullet(A)$. Note first that graded module $B\bar{A} \otimes A \otimes B\bar{A}$ has a natural $B\bar{A}$-bi-comodule structure given by $1_{B\bar{A} \otimes A} \otimes \Delta_{B\bar{A}}$ and $\Delta_{B\bar{A}} \otimes 1_{A \otimes B\bar{A}}$. We will define a differential

$$d : B\bar{A} \otimes A \otimes B\bar{A} \to B\bar{A} \otimes A \otimes B\bar{A}$$
which commutes with the coderivation of $B\bar{A}$, that is, the equality $(\tilde{d} \otimes 1 + 1 \otimes d) \circ (\Delta_{B\bar{A}} \otimes 1_{A \otimes B\bar{A}}) = (\Delta_{B\bar{A}} \otimes 1_{A \otimes B\bar{A}}) \circ d$ holds, and a similar condition holds for the right coaction (we think of it as a $\bar{A}$-bimodule structure on $A$). As in the case of coderivation on $BA$, this data is uniquely determined by the family of $k$-linear maps

$$ \{ d_{m,n} : (\bar{A}[1])^{\otimes m} \otimes A \otimes (\bar{A}[1])^{\otimes n} \rightarrow A \}_{m,n \geq 0}. $$

We define $d$ by giving the family $\{ d_{m,n} \}_{m,n \geq 0}$ such that the diagram

$$ \begin{array}{ccc}
(\bar{A}[1])^{\otimes m} \otimes A \otimes (\bar{A}[1])^{\otimes n} & \xrightarrow{d_{m,n}} & A \\
1^{\otimes m} \otimes 1^{\otimes n} & \downarrow s & \\
(\bar{A}[1])^{\otimes m} \otimes A[1] \otimes (\bar{A}[1])^{\otimes n} & \xrightarrow{b_{m+1+n}} & A[1].
\end{array} $$

commutes. The map $d_{m,n}$ sends $[a_1] \cdots [a_m] \otimes a \otimes [a_1'] \cdots [a_n']$ to

$$ (-1)^{\sum_{i=1}^{m}(|a_i|-1)} s^{1} b_{m+1+n}[a_1] \cdots [a_m] a [a_1'] \cdots [a_n']. $$

Let $\Delta^R$ and $\Delta^L$ denote the canonical right and left coactions of $BA \otimes \bar{A} \otimes BA$. Let $S_{2341}$ be the symmetric permutation

$$ BA \otimes (BA \otimes A \otimes BA) \rightarrow (BA \otimes A \otimes BA) \otimes BA, $$

and put $\Phi(A) := \text{Ker}(\Delta^R - S_{2341} \circ \Delta^L) \subset BA \otimes A \otimes BA$. Since $\Delta^R$ and $S_{2341} \circ \Delta^L$ commute with differentials on the domain and the target, $\Phi(A)$ inherits a differential from $BA \otimes A \otimes BA$. Moreover, $A \otimes BA \xrightarrow{1 \otimes \Delta_{BA}} A \otimes BA \otimes BA \xrightarrow{S_{112}} BA \otimes A \otimes BA$ is injective, and its image is $\Phi(A)$. Thus $C_\bullet(A) \cong \Phi(A)$ inherits a differential $\partial_{Hoch}$, and we define a Hochschild chain complex of $A$ to be a differential graded (dg) $k$-module $C_\bullet(A)$ endowed with $\partial_{Hoch}$. The Hochschild chain complex has a differential $B$ of degree $-1$ called Connes’ operator given by

$$ B(a \otimes [a_1] \cdots [a_n]) = \sum_{i=0}^{n} (-1)^{(\epsilon_i+1)\frac{1}{2}(|a_i|-1)} 1_A \otimes [a_i] \cdots [a_n] a [a_1] \cdots [a_{i-1}], $$

that satisfies $B \partial_{Hoch} + \partial_{Hoch}B = 0$ and $B^2 = 0$, where $\epsilon_i = \sum_{r=1}^{i} (|a_r|-1)$. The last two identities mean that $(C_\bullet(A), \partial_{Hoch}, B)$ is a mixed complex in the sense of [20]. Put another way, if we let $\Lambda$ be the dg algebra $k[\epsilon]/(\epsilon^2)$ with zero differential such that the (cohomological) degree of the generator $\epsilon$ is $-1$, then $(C_\bullet(A), \partial_{Hoch})$ is a dg-$\Lambda$-module where the action of $\epsilon$ is induced by $B$.

2.6. To the mixed complex $(C_\bullet(A), \partial_{Hoch}, B)$ we associate its negative cyclic chain complex and its periodic cyclic chain complex. Let

$$ CC^{-}_{\bullet}(A) := (C_\bullet(A)[[t]], \partial_{Hoch} + tB), \quad CC^{per}_{\bullet}(A) := (C_\bullet(A)((t)), \partial_{Hoch} + tB), $$

where $t$ is a formal variable of (cohomological) degree two. We consider the graded module $C_\bullet(A)[[t]]$ to be $\prod_{i \geq 0} C_\bullet(A) \cdot t^i$. If $C_{l}(A)$ denotes the part of (homological) degree $l$ of $C_\bullet(A)$, then the part of (cohomological) degree $r$ of $C_\bullet(A)[[t]]$ is $\prod_{l \geq 0, l-2r-l \geq 0} C_{r}(A) \cdot t^l$. The graded module $C_\bullet(A)((t))$ is regarded as $\bigcup_{l \in \mathbb{Z}} \prod_{l \geq 0} C_{r}(A) \cdot t^l$. The identities $\partial_{Hoch}^2 = B \partial_{Hoch} + \partial_{Hoch}B = B^2 = 0$ implies $(\partial_{Hoch} + tB)^2 = 0$. We call $CC^{-}_{\bullet}(A)$ (resp. $CC^{per}_{\bullet}(A)$) the negative cyclic complex of $A$ (resp. the periodic cyclic complex of $A$). The cohomology $HN_n(A) := H^{-n}(CC^{-}_{\bullet}(A))$ and $HP_n(A) := H^{-n}(CC^{per}_{\bullet}(A))$ is called the negative cyclic homology and periodic cyclic homology respectively. The periodic cyclic homology is periodic in the sense that $HP_n(A) = HP_{n+2i}(A)$ for any integer $i$. The module $HN_n(A)$ can be identified with the Ext-group $\text{Ext}_A(k, C_\bullet(A))$. To be precise, we let $k$ be the mixed complex of $k$ placed in degree zero equipped with the trivial action of $\epsilon \in \Lambda$. The category of dg $\Lambda$-module admits the projective
model structure where weak equivalences are quasi-isomorphisms, and fibrations are degreewise surjective maps. We choose a cofibrant resolution $K$ of $k$:

$$\cdots \xrightarrow{s} k \epsilon \xrightarrow{0} k \xrightarrow{s} k \epsilon \xrightarrow{0} k.$$ 

Then the graded module of Hom complex $\text{Hom}_\Lambda(K, C_\bullet(A))$ can naturally be identified with $C_\bullet(A)[t]$ (if we denote by $12k$ the unit $1 \in k$ placed in (cohomological) degree $-2k$, $f : K \to C_\bullet(A)$ corresponds to $\sum_{i=0}^\infty f(12i)t^i$ in $C_\bullet(A)[t]$). Unwinding the definition we see that the differential of the Hom complex corresponds to the differential $\partial_{\text{Hoch}} + tB$. Similarly, the Hom complex $\text{Hom}_\Lambda(K, k)$ is identified with $k[t]$. Here we regard $k[t]$ as the dg algebra with zero differential such that the (cohomological) degree of $t$ is 2. This dg algebra is the Koszul dual of $\Lambda$. Let $\iota : k \to K$ be a canonical section of the resolution $K \to k$. Then the natural $k[t]$-module structure on $CC_\bullet^-(A)[t]$ corresponds to

$$\text{Hom}_\Lambda(K, k) \otimes \text{Hom}_\Lambda(K, C_\bullet(A)) \to \text{Hom}_\Lambda(K, C_\bullet(A)), \quad \phi \otimes f \mapsto f \circ \iota \circ \phi.$$ 

### 3. Lie Algebra Actions and Modular Interpretation

#### 3.1. Let $k$ be a base field of characteristic zero. Let $A$ be a unital $A_\infty$-algebra or a unital dg algebra. In what follows we write $C_\bullet(A)[1]$ for the dg Lie algebra $\overrightarrow{C_\bullet(A)}[1]$ of the normalized Hochschild cochain complex, and $C_\bullet(A) := (C_\bullet(A)[1])[1]$. The dg Lie algebra $C_\bullet(A)[1]$ acts on the Hochschild chain complex $C_\Lambda(A)$. Our reference for this action is Tamarkin-Tsygan [38]. It is a part of data of a homotopy calculus algebra on $(C^\bullet(A), C_\bullet(A))$ (see Section 4.3 for algebra over calculi). We describe the action of the dg Lie algebra $C_\bullet(A)[1]$ on $C_\bullet(A)$ in detail.

Let $P \in C_\bullet(A)[1]$ be a homogeneous element, and suppose $P$ lies in $\text{Hom}((A[1])^\otimes, A[1]) \subset \oplus_{n \geq 0} \text{Hom}((A[1])^\otimes, A[1])$. Let $|P|$ be the degree of $P$ in the cochain complex $C_\bullet(A)$ (thus the degree of $P$ in $C^\bullet(A)[1]$ is $|P| - 1$). Let $\mu_i = \sum_{r=0}^i (\mu_r - 1) = (|a_0| - 1 + \epsilon_i)$. A linear map $L_P : C_\bullet(A) \to C_\bullet(A)$ of degree $|P| - 1$, which we regard as an element of the endomorphism complex $\text{End}_k(C_\bullet(A))$ of degree $|P| - 1$, is defined by the formula

$$L_P(a_0 \otimes [a_1| \ldots |a_n]) = \sum_{0 \leq i \leq j \leq j \leq n} (-1)^{(j+1)\mu_j}a_0 \otimes [a_1| \ldots |a_j|P[a_{j+1}| \ldots |a_{j+l}]| \ldots |a_n] + \sum_{\text{P includes a}_0} (-1)^{(\mu_n - \mu_i)}s^{-1}P([a_{i+1}| \ldots |a_n|a_0| \ldots |a_j]) \otimes [a_{j+1}| \ldots |a_l].$$

It gives rise to a graded $k$-linear map

$$L : C^\bullet(A)[1] \to \text{End}_k(C_\bullet(A))$$

which carries $P$ to $L_P$.

**Proposition 3.1.** (cf. [38] 3.3.2.) Let $b : BA \to BA$ denote the $A_\infty$-structure of $A$. Let $P, Q$ be elements in $C_\bullet(A)[1]$. The followings hold:

1. $L_{[P, Q]} = [L_P, L_Q]_{\text{End}} := L_P \circ L_Q - (-1)^{|L_P|L_Q}|L_Q \circ L_P,$
2. $\partial^{\text{End}} L_P - L_\partial_{\text{Hoch}} P = 0$ where $\partial^{\text{End}} L_P = \partial_{\text{Hoch}} \circ L_P - (-1)^{|L_P|L_P} \circ \partial_{\text{Hoch}}$,
3. $[B, L_P]_{\text{End}} = 0$.

**Remark 3.2.** By (1) and (2) of this Proposition $L : C^\bullet(A)[1] \to \text{End}_k(C_\bullet(A))$ is a map of dg Lie algebras, where $\text{End}_k(C_\bullet(A))$ is endowed with the bracket given by $[f, g] = f \circ g - (-1)^{|f||g|}g \circ f$. The condition (3) means that $L$ factors through the dg Lie algebra $\text{End}_\Lambda(C_\bullet(A))$ of the endomorphism dg Lie algebra of the $\Lambda$-module $C_\bullet(A)$.

**Proof.** (1) and (3) are nothing but [38] 3.3.2, (but unfortunately its proof is omitted). Thus, for the reader’s convenience we here give the proof since we will use these formulas.
We will prove (1). We first calculate $L_P \circ L_Q$. We may and will assume that $P$ and $Q$ belong to $\text{Hom}((A[1])^{\otimes n}, A[1])$ and $\text{Hom}((A[1])^{\otimes n}, A[1])$ respectively. For ease of notation we write $(a_0, a_1, \ldots, a_n)$ for $a_0 \otimes [a_1 \ldots, a_n]$ in $A \otimes (A[1])^{\otimes n}$.

\[ L_P \circ L_Q = \sum_{i,j} A(P, Q, i, j) + \sum_{i,j} B(P, Q, i, j) + \sum_{i,j} C(P, Q, i, j) + \sum_{i,j} D(P, Q, i, j) \]
\[ + \sum_{i,j} E(P, Q, i, j) + \sum_{i,j} F(P, Q, i, j) + \sum_{i,j} G(P, Q, i, j) + \sum_{i,j} H(P, Q, i, j), \]

where

\[ A(P, Q, i, j) = (-1)^{|Q|-1} \mu_i + |P|-1 \mu_j(a_0, \ldots, P[a_{j+1}] \ldots, Q[a_{i+1}] \ldots), \]
\[ B(P, Q, i, j) = (-1)^{|Q|-1} \mu_i + |P|-1 \mu_j(a_0, \ldots, P[a_{j+1}] \ldots|Q[a_{i+1}] \ldots), \]
\[ C(P, Q, i, j) = (-1)^{|Q|-1} \mu_i + |P|-1 \mu_j + |Q|-1(a_0, \ldots, Q[a_{i+1}] \ldots, P[a_{j+1}] \ldots), \]
\[ D(P, Q, i, j) = (-1)^{|Q|-1} \mu_i + \mu_j + |Q|-1\mu_n - \mu_j(s^{-1}P[a_{j+1}] \ldots|a_0 \ldots|Q[a_{i+1}] \ldots), \]
\[ E(P, Q, i, j) = (-1)^{|Q|-1} \mu_i + \mu_j + |Q|-1\mu_n - \mu_j(s^{-1}P[a_{j+1}] \ldots|a_0 \ldots, Q[a_{i+1}] \ldots), \]
\[ F(P, Q, i, j) = (-1)^{|Q|-1} \mu_i + \mu_j + |Q|-1\mu_n - \mu_j(s^{-1}P[a_{j+1}] \ldots|Q[a_{i+1}] \ldots|a_0 \ldots), \]
\[ G(P, Q, i, j) = (-1)^{|Q|-1} \mu_i + \mu_j + |Q|-1\mu_n - \mu_j(s^{-1}P[a_{j+1}] \ldots|Q[a_{i+1}] \ldots, P[a_{j+1}] \ldots), \]
\[ H(P, Q, i, j) = (-1)^{|Q|-1} \mu_i + \mu_j + |Q|-1\mu_n - \mu_j(s^{-1}P[a_{j+1}] \ldots|Q[a_{i+1}] \ldots, a_0 \ldots). \]

Here $i$ and $j$ run over an adequate range which depend on the type of terms in each summation. Interchanging $(P, i)$ and $(Q, j)$ we put

\[ A(Q, P, j, i) = (-1)^{|Q|-1} \mu_i + |P|-1 \mu_j(a_0, \ldots, Q[a_{i+1}] \ldots, P[a_{j+1}] \ldots). \]

Other terms $B(Q, P, j, i), C(Q, P, j, i), \ldots$ are defined in a similar way. For example,

\[ C(Q, P, j, i) = (-1)^{|P|-1} \mu_j + |Q|-1\mu_i + |P|-1(a_0, \ldots, P[a_{j+1}] \ldots, Q[a_{i+1}] \ldots). \]

Then we have

\[ L_Q \circ L_P = \sum_{i,j} A(Q, P, j, i) + \sum_{i,j} B(Q, P, j, i) + \sum_{i,j} C(Q, P, j, i) + \sum_{i,j} D(Q, P, j, i) \]
\[ + \sum_{i,j} E(Q, P, j, i) + \sum_{i,j} F(Q, P, j, i) + \sum_{i,j} G(Q, P, j, i) + \sum_{i,j} H(Q, P, j, i), \]

Notice that

\[ A(P, Q, i, j) - (-1)^{|P|-1}|Q|-1 C(Q, P, j, i) = 0, \]
\[ C(P, Q, i, j) - (-1)^{|P|-1}|Q|-1 A(Q, P, j, i) = 0, \]
\[ E(P, Q, i, j) - (-1)^{|P|-1}|Q|-1 G(Q, P, j, i) = 0, \]
\[ G(P, Q, i, j) - (-1)^{|P|-1}|Q|-1 E(Q, P, j, i) = 0. \]

Therefore, the terms $A, C, E$ and $G$ do not appear in $L_P \circ L_Q - (-1)^{|L_P||L_Q|} L_P \circ L_Q$.

Next we will calculate $L_P[Q]$. Note first that $L_P[Q] = L_{P,Q} - (-1)^{|P|-1}|Q|-1 L_Q \circ P$. The composition $P \circ Q \in \text{Hom}((A[1])^{\otimes n+1}, A[1])$ is given by

\[ P \circ Q[a_1 \ldots a_n] = \sum_i (-1)^{|Q|-1}|a_i|^{-1} P[a_1 \ldots a_i+1 \ldots |a_i+v] \ldots |a_{n+v-1}. \]
Using this we see that $L_{P\circ Q}(a_0, a_1, \ldots, a_n)$ is equal to
\[
\sum_{i,j} (-1)^{(|P|-1+|Q|-1)\mu_j+(|Q|-1)(\mu_i-\mu_j)}(a_{i+1}, \ldots, a_0, \ldots, P[a_{j+1}, \ldots |Q[a_{i+1}]| \ldots])
\]
\[
+ \sum_{i,j} (-1)^{\mu_j(\mu_n-\mu_j)+(|Q|-1)(\mu_i-\mu_j)}(s^{-1}P[a_{j+1}, \ldots |Q[a_{i+1}]| \ldots])
\]
\[
+ \sum_{i,j} (-1)^{\mu_j(\mu_n-\mu_j)+(|Q|-1)(\mu_i-\mu_j)}(s^{-1}P[a_{j+1}, \ldots |Q[a_{i+1}]| \ldots |a_0])
\]
\[
+ \sum_{i,j} (-1)^{\mu_j(\mu_n-\mu_j)+(|Q|-1)(\mu_i-\mu_j)}(s^{-1}P[a_{j+1}, \ldots |Q[a_{i+1}]| \ldots |a_0]| \ldots)
\]
where $i$ and $j$ run over adequate range in each summation. Thus by an easy computation of signs, we see that it is equal to
\[
\sum_{i,j} B(P, Q, i, j) + \sum_{i,j} D(P, Q, i, j) + \sum_{i,j} F(P, Q, i, j) + \sum_{i,j} H(P, Q, i, j).
\]
Similarly, we see that $L_{Q\circ P}$ is equal to
\[
\sum_{i,j} B(Q, P, j, i) + \sum_{i,j} D(Q, P, j, i) + \sum_{i,j} F(Q, P, j, i) + \sum_{i,j} H(Q, P, j, i).
\]
Therefore we deduce that $L_{[P,Q]} = [L_P, L_Q]$.

Next we prove (2). By a calculation of the differential $\partial_{\text{Hoch}}$ below (Lemma 3.11), $L_b = \partial_{\text{Hoch}}$.

Combined with $\partial_{\text{Hoch}}^P = [b, P]$, we see that (2) follows from (1).

Finally, we prove (3). We compare $B \circ L_P$ with $L_P \circ B$.

\[
B \circ L_P(a_0, a_1, \ldots, a_n)
\]
\[
= \sum_{i,j} (-1)^{(|P|-1)^{\mu_i}+(\mu_j+|P|-1)(\mu_n-\mu_j)}(1_A, a_{j+1}, \ldots, a_0, \ldots, P[a_{i+1}])
\]
\[
+ \sum_{i,j} (-1)^{(|P|-1)^{\mu_i}+(\mu_j+|P|-1)}(1_A, a_{j+1}, \ldots, P[a_{i+1}]])
\]
\[
+ \sum_{i,j} (-1)^{\mu_j(\mu_n-\mu_j)+(\mu_i-\mu_j)}(1_A, a_{j+1}, \ldots, P[a_{i+1}]|a_0])
\]

On the other hand, Since $P$ is normalized we have

\[
L_P \circ B(a_0, a_1, \ldots, a_n)
\]
\[
= \sum_{i,j} (-1)^{\mu_j(\mu_n-\mu_j)+(|P|-1)(\mu_n-\mu_j)+(\mu_i-\mu_j)}(1_A, a_{j+1}, \ldots, a_0, \ldots, P[a_{i+1}]])
\]
\[
+ \sum_{i,j} (-1)^{\mu_j(\mu_n-\mu_j)+(|P|-1)(\mu_i-\mu_j)}(1_A, a_{j+1}, \ldots, P[a_{i+1}]])
\]

where $P[\ldots]$ is allowed to include $a_0$ in the second summation. Comparing $B \circ L_P$ with $L_P \circ B$ we see that $B \circ L_P - (-1)^{|P|-1}L_P \circ B = 0$. 

3.2. We continue to assume that $(A, b)$ be a unital $A_{\infty}$-algebra (or a unital dg algebra). Consider the pair $(C^\bullet(A)[1], C_\bullet(A))$ of the shifted (normalized) Hochschild cochain complex and the Hochschild chain complex of $A$. According to Proposition 3.1 the dg Lie algebra $C^\bullet(A)[1]$ acts on the mixed complex $C_\bullet(A)$. That is, there is an (explicit) morphism $L : C^\bullet(A)[1] \to \text{End}_A(C_\bullet(A))$ of dg Lie algebras. Let us consider the morphism of dg Lie algebras $\text{End}_A(C_\bullet(A)) \to \text{End}_K(\text{Hom}_A(K, C_\bullet(A)))$ which carries $f \in C_\bullet(A) \to C_\bullet(A)$ to $\{ \phi \mapsto$
Definition 3.3. For a complex \((E,d)\) (i.e. a dg \(k\)-module) and \(R\) in \(\text{Art}_k\), a deformation of the complex \(E\) to \(R\) is a dg \(R\)-module \((E \otimes_k R, \tilde{d})\) such that the reduction of the differential

\[
f \circ \phi \in \text{End}_k(\text{Hom}_A(K, C_\bullet(A))), \text{ where } K \text{ is the cofibrant resolution of } k \text{ (see Section 2.6). Remember that there exists the natural isomorphism } \text{Hom}_A(K, C_\bullet(A)) \simeq (C_\bullet(A)[[t]], \partial_{\text{Hoch}} + tB). \text{ Moreover, } \{\phi \mapsto f \circ \phi\} \text{ commutes with the action of } \text{Hom}_A(K, k) \simeq k[t]. \text{ Consequently, we have the composition of maps of dg Lie algebras }

\[
C^\bullet(A)[1] \xrightarrow{L} \text{End}_A(C_\bullet(A)) \to \text{End}_{k[t]}(C_\bullet(A)[[t]]) := \text{End}_{k[t]}((C_\bullet(A)[[t]], \partial_{\text{Hoch}} + tB)).
\]

We easily observe that the image of \(P \in C^\bullet(A)[1]\) in \(\text{End}_{k[t]}(C_\bullet(A)[[t]])\) is \(L_P[[t]]\). We shall denote by \(L[[t]]\) this composite. We write \(k[t^\pm]\) for \(k[t, t^{-1}]\). By tensoring with \(\otimes_{k[t]} k[t^\pm]\) we also have

\[
L(t) : C^\bullet(A)[1] \to \text{End}_{k[t^\pm]}(C_\bullet(A)((t))) := \text{End}_{k[t^\pm]}((C_\bullet(A)((t)), \partial_{\text{Hoch}} + tB)).
\]

3.3. To a dg Lie algebra one can associate a deformation functor. There are several formalisms of deformation theories we shall employ in this paper. Let \(E\) be a nilpotent dg Lie algebra. An element \(x\) of degree one in \(E\) equipped with a differential \(d\) and a bracket \([-,-]\) is said to be a Maurer-Cartan element if the Maurer-Cartan equation \(dx + \frac{1}{2}[x,x] = 0\) holds. We denote by \(\text{MC}(E)\) the set of Maurer-Cartan elements. The space \(E^0\) of degree zero is a (usual) Lie algebra. Let \(\exp(E^0)\) be the exponential group associated to the nilpotent Lie group \(E^0\) whose product is given by the Baker-Campbell-Hausdorff product on \(E^0\). The Lie algebra \(E^0\) acts on \(E^1\) by \(E^0 \to \text{End}_k(E^1)\), \(\alpha \mapsto [\alpha, -] - d\alpha\). It gives rise to an action of \(\exp(E^0)\) on the space \(E^1\) given by

\[
\mu \mapsto e^{\alpha} \cdot \mu := e^{\text{ad}(\alpha)} \mu - \int_0^1 (e^{\text{ad}(s\alpha)} d\alpha) ds.
\]

for \(\mu\) in \(E^1\). This action preserves Maurer-Cartan elements. Let \(x, y \in \text{MC}(E)\). We say that \(x\) is gauge equivalent to \(y\) if both elements coincide in \(\text{MC}(E)/\exp(E^0)\). The element \(x\) is gauge equivalent to \(y\) if there is \(\alpha \in E^0 = \exp(E^0)\) such that \(e^{\alpha} \cdot x = y\). There is another more theoretical definition: If \(\Omega_1 = k[u, du]\) denotes the dg algebra of 1-dimensional polynomial differential forms (see Section 4.6), the coequalizer of two degeneracies \(d_0, d_1 : \text{MC}(\Omega_1 \otimes E) \rightrightarrows \text{MC}(E)\) determined by \(u = du = 0\) and \(u = 1, du = 0\) is isomorphic to \(\text{MC}(E)/\exp(E^0)\). This definition can also be applicable to \(L_\infty\)-algebras. Let \(\phi : E \to E'\) be a map of nilpotent dg Lie algebras (or more generally, an \(L_\infty\)-morphism, cf. Section 4.6). Then \(\phi\) induces \(\text{MC}(E) \to \text{MC}(E')\) which commutes with gauge actions. In particular, it gives rise to \(\text{MC}(E)/\exp(E^0) \to \text{MC}(E')/\exp((E')^0)\).

Let \(\text{Art}_k\) be the category of artin local \(k\)-algebras with residue field \(k\). For \(R\) in \(\text{Art}_k\), we write \(m_R\) for the maximal ideal of \(R\). Let \(E\) be a dg Lie algebra. The bracket on \(E \otimes m_R\) is defined by \([e \otimes m, e' \otimes m'] = [e, e'] \otimes mm'\). Note that the dg Lie algebra \(E \otimes m_R\) is nilpotent. For a morphism \(R \to R'\) in \(\text{Art}_k\), the induced map \(E \otimes_k m_R \to E \otimes_k m_{R'}\) determines a map \(\text{MC}(E \otimes m_R)/\exp(E^0 \otimes m_R) \to \text{MC}(E \otimes m_{R'})/\exp(E^0 \otimes m_{R'})\). We then define a functor

\[
\text{Spf}_E : \text{Art}_k \to \text{Sets}
\]

which carries \(R\) to \(\text{MC}(E \otimes m_R)/\exp(E^0 \otimes m_R)\), where Sets is the category of sets. By this notation we think of \(\text{Spf}_E\) as an analogue of formal schemes in scheme theory.

If \(\tilde{\phi} : E \to E'\) is a quasi-isomorphism of dg Lie algebras, the induced morphism \(\text{Spf}_\phi : \text{Spf}_E \to \text{Spf}_{E'}\) is an equivalence (this is a theorem stated and proved by Deligne, Goldman and Millson, Hinich, Kontsevich, Fukaya, Getzler and others).

Let us consider two examples of deformation problems.

**Definition 3.3.** For a complex \((E, d)\) (i.e. a dg \(k\)-module) and \(R\) in \(\text{Art}_k\), a deformation of the complex \(E\) to \(R\) is a dg \(R\)-module \((E \otimes_k R, \tilde{d})\) such that the reduction of the differential
$\tilde{d}$ of $E \otimes_k R$ to the complex $E \otimes_k R/m_R$ exhibits $E \otimes_k k$ as the initial complex $(E,\tilde{d})$. Let $(E \otimes_k R,\tilde{d}_1)$ and $(E \otimes_k R,\tilde{d}_2)$ be two deformations of $E$ to $R$. An isomorphism of these deformations is an isomorphism $E \otimes_k R \to E \otimes_k R$ of dg $R$-modules which induces the identity $E \otimes_k R/m_R \to E \otimes_k R/m_R$. Note that by nilpotent Nakayama lemma, every homomorphism $E \otimes_k R \to E \otimes_k R$ of dg $R$-modules which induces the identity $E \otimes_k R/m_R \to E \otimes_k R/m_R$, is an isomorphism.

Let us consider the functor $D_E : \text{Art}_k \to \text{Sets}$ which carries $R$ to the set of isomorphism classes of deformations of $E$ to $R$ (the functoriality is defined in the natural way, that is, a morphism $R \to R'$ in $\text{Art}_k$ induces the base change $E \otimes_k R \otimes_R R'$).

**Proposition 3.4.** There is a natural equivalence of functors $\text{Spf}_{\text{End}_k(E)} \to D_E$. For each $R$ in $\text{Art}_k$ it carries a Maurer-Cartan element $x$ in $\text{End}_k(E) \otimes m_R$ to a deformation $(E \otimes_k R, d \otimes_k R + x)$, where the differential is given by $(d \otimes_k R + x)(e \otimes r) = d(e) \otimes r + x(e)r$.

In Introduction, the following deformations are referred to as curved $A_{\infty}$-deformations of $A$.

**Definition 3.5.** Let $(A,b)$ be an $A_{\infty}$-algebra with unit $1_A$ (or a dg algebra) and $R$ an artin local $k$-algebra in $\text{Art}_k$. A deformation of $(A,b)$ to $R$ is a graded $R$-module $A \otimes_k R$ endowed with a curved $A_{\infty}$-structure $\tilde{b} : BA \otimes_k R \to BA \otimes_k R$ (over $R$) such that (i) the reduction $BA \otimes_k R/m_R \to BA \otimes R/m_R$ is $b : BA \to BA$ via the canonical isomorphism $BA \otimes_k R/m_R \simeq BA$, and (ii) the element $1_A \otimes 1_R \in A \otimes_k R$ is a unit. Here $BA \otimes_k R$ is a graded coalgebra over $R$. Let $(A \otimes_k R,\tilde{b}_1)$ and $(A \otimes_k R,\tilde{b}_2)$ be two deformations of $(A,b)$ to $R$. An isomorphism $(A \otimes_k R,\tilde{b}_1) \to (A \otimes_k R,\tilde{b}_2)$ of deformations is an $A_{\infty}$-isomorphism $BA \otimes_k R \to BA \otimes_k R$ (of dg coalgebras) over $R$, such that the reduction $BA \otimes_k R/m_R \to BA \otimes_k R/m_R$ is the identity. (As in the case of complexes, a homomorphism of dg coalgebras over $R$ whose reduction is the identity, is an isomorphism.) Note that a deformed $A_{\infty}$-structure is allowed to be curved.

Let $\text{DA}_{\infty}^A$ denote the functor $\text{Art}_k \to \text{Sets}$ which carries $R$ to the set of isomorphism classes of deformations of $(A,b)$ to $R$.

**Proposition 3.6.** There is a natural equivalence $\text{Spf}_{C^*(A)[1]} \to \text{DA}_{\infty}^A$. It carries a Maurer-Cartan element $x \in MC(C^*(A)[1] \otimes_k m_R)$ to a curved $A_{\infty}$-structure $b \otimes_k R + x : BA \otimes_k R \to BA \otimes_k R$. (Here $C^*(A)[1]$ is the normalized Hochschild cochain complex, $x : BA \to BA \otimes_k m_R$, and $(b \otimes_k R + x)(a \otimes r) = b(a) \otimes r + x(a)r$.)

**Remark 3.7.** The dg Lie algebra $C^*(A)[1]$ is derived Morita invariant (cf. [22]). Let $\text{Perf}_A$ be the dg category (or some model of a stable $(\infty,1)$-category) of perfect dg (left) $A$-modules. A dg $A$-module $M$ is said to be perfect if at the level of the homotopy category, $M$ belongs to the smallest triangulated subcategory of the triangulated category of dg $A$-modules, which contains $A$ and is closed under direct summands. If there is a quasi-equivalence $\text{Perf}_A \simeq \text{Perf}_B$, there is an equivalence $C^*(A)[1] \simeq C^*(B)[1]$ (more precisely, an $L_{\infty}$-quasi-isomorphism). In particular, $\text{Spf}_{C^*(A)[1]} \simeq \text{Spf}_{C^*(B)[1]}$. Thus, it seems that it is natural to describe $\text{Spf}_{C^*(A)[1]}$ in terms of category theory. Namely, it is worthwhile to attempt to describe $\text{Spf}_{C^*(A)[1]}$ as the functor of deformations of the dg category (or stable $(\infty,1)$-category) $\text{Perf}_A$ (see Keller-Lowen’s work [23] for the progress of this problem). At the time of writing this paper, the author does not know a category-theoretic formulation that fits into nicely with nilpotent deformations to curved $A_{\infty}$-algebras. Therefore, we choose the approach using curved deformations of the algebra $A$.

**Remark 3.8.** To get the feeling of deformations of a $A_{\infty}$-algebra $A$ (or a dg algebra), let us consider the situation when $A$ comes from quasi-compact separated scheme $X$ over $k$. According to [7] the derived category $D(X)$ of (unbounded) quasi-coherent complexes on $X$ admits a compact generator, that is, a compact object $C$ such that for any $D \in D(X)$ the condition
There is a natural bijective map \( f \) which carries a coderivation \( \partial \) to \( \mathcal{H}(\mathcal{O}_X) \). Indeed, one can choose \( \mathcal{A} \) to be the endomorphism dg algebra of \( C \) (after a suitable resolution). (Here we regard \( D(X) \) as a dg category.) For simplicity, suppose that \( X \) is smooth and proper over \( k \). On one hand, Hochschild-Kostant-Rosenberg theorem implies that \( HH^0(\mathcal{A}) = \oplus_{i+j=n} H^j(X, T_X^k) \). On the other hand, by the definition and Proposition 3.6

\[
\text{DA}_{\mathcal{A}}(k[e]/(e^2)) \simeq \text{Spf}_{C^\bullet(\mathcal{A})[1]}(k[e]/(e^2)) \simeq HH^2(\mathcal{A}),
\]

where \( \text{Spf}_{C^\bullet(\mathcal{A})[1]}(k[e]/(e^2)) \) is the tangent space of \( \text{Spf}_{C^\bullet(\mathcal{A})[1]} \). Consequently, we have

\[
\text{DA}_{\mathcal{A}}(k[e]/(e^2)) \simeq \text{Spf}_{C^\bullet(\mathcal{A})[1]}(k[e]/(e^2)) \simeq H^0(X, \wedge^2 T_X) \oplus H^1(X, T_X) \oplus H^2(X, \mathcal{O}_X).
\]

While there are several interpretations, we review one modular interpretation of the space of the right hand side, though details remain to be elucidated. The subspace \( H^1(X, T_X) \) parametrizes (first-order) deformations of the scheme \( X \) that induce deformations of \( D(X) \) (see Remark 3.3.1). Morally, the subspace \( H^0(X, \wedge^2 T_X) \oplus H^2(X, \mathcal{O}_X) \) should be considered to be the part of noncommutative deformations. For example, a Poisson structure on \( X \) belongs to \( H^0(X, \wedge^2 T_X) \), and it gives rise to deformation quantization of \( X \). The space \( H^2(X, \mathcal{O}_X) \) is identified with the space of liftings of the zero \( H^2(\mathcal{C}_\mathbb{A}(X, \mathcal{G}_m)) \) to \( H^2_\mathbb{C}(X \times_k k[e]/(e^2), \mathcal{G}_m) \). By a main theorem of [40], such a lifting can thought of as a (first-order) deformation of the structure sheaf \( \mathcal{O}_X \) as a derived Azumaya algebra. In other words, this sort of deformations may be described as deformations of twisted sheaves (see [43]).

3.4. Proposition 3.5 and 3.6 are well-known to experts. In particular, deformations of (curved) deformations of algebraic structure is one of main subjects of Hochschild cohomology. We present the proof for reader’s convenience because we are unable to find the literature that fits in with our curved \( A_\infty \)-setting. Proposition 3.6 is a consequence of the following two Claims.

Claim 3.8.1. Let \( \text{Def}_{\text{Alg}}(A, R) \) be the set of deformations \( \tilde{A} = (A \otimes_k R, \tilde{b}) \) of \((A, b) \) to \( R \). There is a natural bijective map

\[
\text{MC}(C^\bullet(A)[1] \otimes_k m_R) \overset{\sim}{\longrightarrow} \text{Def}_{\text{Alg}}(A, R)
\]

which carries a coderivation \( f : BA \to BA \otimes_k m_R \) of degree one satisfying the Maurer-Cartan equation \( \partial f + \frac{1}{2}[f, f] = 0 \) to a coderivation \( \partial f + f \otimes R : BA \otimes_k R \to BA \otimes_k R \).

Proof. Let \( f : BA \to BA \otimes_k m_R \) be a coderivation of degree one, that is, an element of \( (C^\bullet(A)[1] \otimes_k m_R)^1 \). We abuse notation by writing \( f \) for \( f \otimes R : BA \otimes_k R \to BA \otimes_k R \) given by \( f(x \otimes r) = f(x) r \). Then \( (b \otimes_k R + f)^2 = (b \otimes_k R)^2 + b \circ f + f \circ b + f \circ f \). Thus \( b \otimes_k R + f \) is a square-zero coderivation if and only if \( f \) is a Maurer-Cartan element. The map \( f \mapsto b \otimes_k R + f \) is injective. In addition, note that \( 1_A \otimes_k 1_R \) is a unit in \( (A \otimes_k R, b \otimes_k R + f) \) exactly when \( f \) is normalized. It remains to be proved that the map is surjective. To this end, it suffices to observe only that if a square-zero coderivation \( \tilde{b} : BA \otimes_k R \to BA \otimes_k R \) over \( R \) whose reduction is \( b \), then \( \tilde{b} - b \otimes_k R \) \circ \epsilon \) belongs to \( C^\bullet(A)[1] \otimes_k m_R \) where \( \epsilon : BA \otimes_k k \to BA \otimes_k R \) is the canonical inclusion. \( \square \)

Claim 3.8.2. Let \( \text{Aut}(A, R) \) be the group of automorphisms \( BA \otimes_k R \to BA \otimes_k R \) of the graded coalgebra \( BA \otimes_k R \) (not equipped with any coderivation) whose reduction \( BA \overset{\sim}{\to} BA \) is the identity. There is a natural isomorphism

\[
\exp(C^\bullet(A)[1] \otimes_k m_R)^0 \overset{\sim}{\longrightarrow} \text{Aut}(A, R)
\]

which carries a coderivation \( d : BA \to BA \otimes_k m_R \) of degree zero to an automorphism \( e^d = \sum_{i=0}^\infty \frac{1}{i!} d^i : BA \otimes_k R \to BA \otimes_k R \).
Consider the action of Aut(A, R) on Defor_{Alg}(A, R) given by \( \tilde{b} \mapsto c \tilde{b} \circ c^{-1} \) for \( c \in \text{Aut}(A, R) \). Then the gauge action of \( \exp(C^\bullet(A)[1] \otimes_k m_R)^0 \) on \( \text{MC}(C^\bullet(A)[1] \otimes_k m_R) \) commutes with the action of Aut(A, R) on Defor_{Alg}(A, R).

Proof. Since \( (\Delta_{BA} \otimes_k m_R) \circ d = (d \otimes 1 + 1 \otimes d) \circ \Delta_{BA} : BA \to BA \otimes_k BA \otimes_k m_R \), \( e^d = \sum_{i=0}^{\infty} \frac{1}{i!} d^i \) and \( e^d \otimes 1 + 1 \otimes e^d = e^d \otimes e^d \) satisfy the commutativity

\[
\begin{array}{ccc}
BA \otimes_k R & \cong & (BA \otimes_k R) \otimes_R (BA \otimes_k R) \\
\downarrow e^d & & \downarrow (e^d \otimes 1 + 1 \otimes e^d) \\
BA \otimes_k R & \cong & (BA \otimes_k R) \otimes_R (BA \otimes_k R)
\end{array}
\]

which means that \( e^d \) is an automorphism of the graded coalgebra \( BA \otimes_k R \). By the construction, the reduction \( BA \otimes_k R/m_R \to BA \otimes_k R/m_R \) is the identity.

Next we prove our claim by induction on the length of \( R \). The case \( R = k \) is obvious. Let

\[
0 \to I \to R \to R' \to 0
\]

be an exact sequence where \( R \to R' \) is a surjective map of artin local \( k \)-algebras and \( I \) is the kernel such that \( I \cdot m_R = 0 \). We assume that \( (C^\bullet(A)[1] \otimes_k m_R)^0 \to \text{Aut}(A, R') \), which carries \( d \) to \( e^d \), is an isomorphism. We will prove that \( (C^\bullet(A)[1] \otimes_k m_R)^0 \to \text{Aut}(A, R) \) is an isomorphism. Let \( d \in (C^\bullet(A)[1] \otimes_k m_R)^0 \) and \( h \in (C^\bullet(A)[1] \otimes_k I)^0 \). Then using \( I \cdot m_R = 0 \), we see that \( e^{d+h} = 1 + (d + h) + \frac{1}{2}(d + h)^2 + \cdots = e^d + h \). Combined this equality with the assumption on induction we deduce that \( (C^\bullet(A)[1] \otimes_k m_R)^0 \to \text{Aut}(A, R) \) is injective. To prove that the map is surjective, let \( f \in \text{Aut}(A, R) \) and take \( d \in C^\bullet(A)[1] \otimes_k m_R \) such that \( f = d \mod I \in (C^\bullet(A)[1] \otimes_k m_R)^0 \simeq \text{Aut}(A, R') \). If \( h = f - e^d : C \to C \otimes_k I \) is coderivation, then \( e^{d+h} = e^d + h = f \) implies that the map is surjective. Note that \( (f \otimes f - e^d \otimes e^d) \circ \Delta_{BA} = (\Delta_{BA} \otimes_k I) \circ (f - e^d) : BA \to BA \otimes BA \otimes_k I \). Put \( \tilde{f} = f - 1 \) and \( \tilde{e}^d = e^d - 1 \). Then

\[
\begin{align*}
\tilde{f} \otimes \tilde{f} - \tilde{e}^d \otimes \tilde{e}^d &= ((1 + \tilde{f}) \circ (1 + \tilde{f}) - (1 + \tilde{e}^d) \circ (1 + \tilde{e}^d) \\
&= (f - e^d) \otimes 1 + 1 \otimes (f - e^d) + \tilde{f} \otimes \tilde{f} - \tilde{e}^d \otimes \tilde{e}^d
\end{align*}
\]

The term \( \tilde{f} \otimes \tilde{f} - \tilde{e}^d \otimes \tilde{e}^d = (\tilde{f} - \tilde{e}^d) \otimes \tilde{f} + \tilde{e}^d \otimes (\tilde{f} - \tilde{e}^d) \) is zero since \( \tilde{f} - \tilde{e}^d \in C^\bullet(A)[1] \otimes_k I \), \( \tilde{f}, \tilde{e}^d \in C^\bullet(A)[1] \otimes_k m_R \) and \( I \cdot m_R = 0 \). Therefore, \( f - e^d \) is a coderivation.

Finally, we prove that the gauge action of \( \exp(C^\bullet(A)[1] \otimes_k m_R)^0 \) on \( \text{MC}(C^\bullet(A)[1] \otimes_k m_R) \) commutes with the action of \( \text{Aut}(A, R) \) on Defor_{Alg}(A, R). Let \( d \in (C^\bullet(A)[1] \otimes_k m_R)^0 \) and \( f \in \text{MC}(C^\bullet(A)[1] \otimes_k m_R) \). The automorphism \( e^d : BA \otimes_k R \to BA \otimes_k R \) acts on \( (b \otimes_k R + f) \) by \( (b \otimes_k R + f) \mapsto e^d \circ (b \otimes_k R + f) \circ e^{-d} \). Using \( e^x \circ e^y = e^{ad(x)}(y) \) and \( (b \otimes_k R + f)^2 = 0 \) we have

\[
e^d \circ (b \otimes_k R + f) \circ e^{-d} = e^{[d, \cdot]}(b \otimes_k R + f)
\]

\[
= \sum_{i \geq 0} \frac{[d, -]^i}{i!} (b \otimes_k R + f)
\]

\[
= b \otimes_k R + f + \sum_{i \geq 0} \frac{[d, -]^i}{(i+1)!} ([d, b \otimes_k R] + [d, f])
\]

\[
= b \otimes_k R + f + \sum_{i \geq 0} \frac{[d, -]^i}{(i+1)!} ([d, f] - \partial^\text{Hoch} \otimes_k R(d))
\]

\[
= b \otimes_k R + e^d \cdot f
\]

where \( e^d \cdot (-) \) indicates the (gauge) action of \( d \in \exp(C^\bullet(A)[1] \otimes_k m_R) \) on \( \text{MC}(C^\bullet(A)[1] \otimes_k m_R) \), see Section 5.3. Hence the desired compatibility follows. \qed
Proposition 3.3 follows from the following Claims. The proofs of them are analogous to those of Claims 3.8.1 and 3.8.2 and are easier.

Claim 3.8.3. Let $E$ be a complex (i.e., a dg $k$-module with a differential $d$). Let $\text{Defor}_{\text{com}}(E, R)$ be the set of deformations $\tilde{E} = (E \otimes_k R, \tilde{d})$ of $E$ to $R$. There is a natural bijective map

$$\text{MC}(\text{End}_k(E) \otimes_k m_R) \xrightarrow{\sim} \text{Defor}_{\text{com}}(E, R)$$

which carries $f : E \to E \otimes_k m_R$ of degree one satisfying the Maurer-Cartan equation to a differential $d \otimes_k R + f \otimes R : E \otimes_k R \to E \otimes_k R$.

Proof. As in Claim 3.8.1, our claim follows from the observation that if $\iota : E \to E \otimes R$ is the natural inclusion, a $R$-linear map $\tilde{d} : E \otimes R \to E \otimes R$ of degree 1, whose reduction is $d$, defines a deformation of $E$ if and only if $(\tilde{d} - d \otimes R) \circ \iota : E \otimes E \otimes R \to E \otimes m_R$ is a Maurer-Cartan element in $\text{End}_k(E) \otimes_k m_R$.

Claim 3.8.4. Let $\text{Aut}_{\text{gr}}(E, R)$ be the group of automorphisms $E \otimes_k R \to E \otimes_k R$ of the graded $R$-module $E \otimes_k R$ whose reduction $E \xrightarrow{\sim} E$ is the identity. There is a natural isomorphism of groups

$$\exp(\text{End}_k(E) \otimes_k m_R) \xrightarrow{\sim} \text{Aut}_{\text{gr}}(E, R)$$

which carries $g : E \to E \otimes_k m_R$ of degree zero to an automorphism $e^g : E \otimes_k R \to E \otimes_k R$. Moreover, the gauge action of $\exp(\text{End}_k(E) \otimes_k m_R)$ on $\text{MC}(\text{End}_k(E) \otimes_k m_R)$ commutes with the action of $\text{Aut}_{\text{gr}}(E, R)$ on $\text{Defor}_{\text{com}}(E, R)$ that is defined in a similar way to that of $\text{Aut}(A, R)$ on $\text{Defor}_{\text{Alg}}(A, R)$.

Proof. Given $\phi : E \otimes R \to E \otimes R$ in $\text{Aut}_{\text{gr}}(E, R)$, we let $f := \phi - \text{id} : E \otimes R \to E \otimes m_R$. Consider the $R$-linear map $g := \log(1 + f) = \sum_{n=1}^{\infty} (-1)^{n-1} f^n / n : E \otimes R \to E \otimes m_R$ (it is a finite sum since $m_R$ is nilpotent). Then we have $e^g = \phi$. Conversely, if $h$ belongs to $\text{End}_k(E) \otimes_k m_R$, then the exponential $e^{h \otimes R} : E \otimes R \to E \otimes R$ of $h \otimes R : E \otimes R \to E \otimes m_R$ defines an element in $\text{Aut}_{\text{gr}}(E, R)$. This correspondence yields the bijection between $\text{Aut}_{\text{gr}}(E, R)$ and $\text{End}_k(E) \otimes_k m_R$. The second claim can be proved in a similar way to Claim 3.8.2.

3.5. Let $\tilde{A} := (A \otimes R, \tilde{b})$ be a deformation of $(A, b)$ to $R$. Then it gives rise to

- the Hochschild chain complex $C_*(\tilde{A})$,
- the negative cyclic complex $C_*(-\tilde{A})[[t]]$,
- the periodic cyclic complex $C_*(\tilde{A})(\langle t \rangle)$.

Note that $C_*(\tilde{A})$ is a dg $R$-module (which is $(A \otimes_k B \tilde{A}) \otimes_k R$ as a graded $R$-module). It follows from the definition of Hochschild chain complexes in Section 2.3 that the the complex $C_*(\tilde{A})$ over $R$ is a deformation of the complex $C_*(A)$. Likewise, $C_*(A)[[t]]$ is a dg $R[t]$-module, and the dg $R[t]$-module $C_*(\tilde{A})[[t]]$ is a deformation of the dg $k[t]$-module $C_*(A)[[t]]$ to $R$. To be precise, by a deformation $\tilde{M}$ of the dg $k[t]$-module $C_*(A)[[t]]$ (resp. the dg $k[t^\pm]$-module $C_*(A)(\langle t \rangle)$) to $R$ we mean a graded $R[t]$-module $M = (C_*(A) \otimes_k R)[t]] \simeq C_*(A)[[t]] \otimes_k R$ (resp. a graded $R[t^\pm]$-module $M = (C_*(A) \otimes_k R)(\langle t \rangle) \simeq C_*(A)(\langle t \rangle) \otimes_k R$) equipped with a differential $\tilde{d}$ whose reduction $C_*(A) \otimes_k R / m_R[[t]] \simeq C_*(A)[[t]] \to (C_*(A) \otimes_k R / m_R)[t]] \simeq C_*(A)[[t]]$ (resp. $(C_*(A) \otimes_k R / m_R)(\langle t \rangle) \to (C_*(A) \otimes_k R / m_R)(\langle t \rangle)$) is $\partial_{\text{Hoch}} + t B$. An isomorphism $((C_*(A) \otimes_k R)[[t]], \partial_1) \to ((C_*(A) \otimes_k R)[[t]], \partial_2)$ of deformations of the dg $k[t]$-module $C_*(A)[[t]]$ to $R$ is an isomorphism of dg $R[t]$-modules whose reduction is the identity. An isomorphism of deformations of the dg $k[t^\pm]$-module $C_*(A)(\langle t \rangle)$ is defined in a similar way.

Let $\mathcal{D}_C(A)[[t]]$ denote a functor $\text{Art}_k \to \text{Sets}$ which carries $R$ to the set of isomorphism classes of deformations of dg $k[t]$-module $C_*(A)[[t]]$ to $R$. Let $\mathcal{D}_C(A)(\langle t \rangle)$ denote a functor $\text{Art}_k \to \text{Sets}$ which carries $R$ to the set of isomorphism classes of deformations of dg $k[t^\pm]$-module $C_*(A)(\langle t \rangle)$.
to \(R\). The following is a version of Proposition 3.4, which follows from formal extensions of Claim 3.8.3 and Claim 3.8.4.

**Proposition 3.9.** There is a natural equivalence of functors \(\text{Spf}_{\text{End}_{k[t]}(C_\bullet(A)[[t]])} \to D_{C_\bullet(A)[[t]]}\) of \(\text{DAlg}_A\) to \(D_{C_\bullet(A)(t)}\), which carry a Maurer-Cartan element \(x\) in \(\text{End}_{k[t]}(C_\bullet(A)[[t]]) \otimes m_R\) to a deformation \(\big((C_\bullet(A) \otimes_k R)[[t]], \partial_{\text{Hoch}} + t B \big) \otimes_k R + x \otimes R\).

An isomorphism between two deformations \(\tilde{A}_1 = (A \otimes_k R, \tilde{b}_1) \sim \tilde{A}_2 = (A \otimes_k R, \tilde{b}_2)\) of \((A, b)\) induces an isomorphism of deformations \(C_\bullet(\tilde{A}_1) \sim C_\bullet(\tilde{A}_2)\) of the complex \(C_\bullet(A)\), and an isomorphism of deformations \(C_\bullet(\tilde{A}_1)[[t]] \sim C_\bullet(\tilde{A}_2)[[t]]\) of the dg \(k[t]\)-module \(C_\bullet(A)[[t]]\) to \(R\). Consequently, we obtain a natural transformation of functors

\[
\mathcal{P} : \text{DAlg}_A \to D_{C_\bullet(A)}
\]

which sends the isomorphism class of a deformation of \(\tilde{A}\) of \((A, b)\) to the isomorphism class of the deformation \(C_\bullet(\tilde{A})\) of \(C_\bullet(A)\) for each \(R\). Similarly, we define functors

\[
\mathcal{Q} : \text{DAlg}_A \to D_{C_\bullet(A)[[t]]} \quad \text{and} \quad \mathcal{R} : \text{DAlg}_A \to D_{C_\bullet(A)(t)}
\]

which carry a deformation of \(\tilde{A}\) of \((A, b)\) to the deformation \(C_\bullet(\tilde{A})[[t]]\) of \(C_\bullet(A)[[t]]\) and the deformation \(C_\bullet(\tilde{A})(t)\) of \(C_\bullet(A)(t)\) respectively for each \(R\).

**Proposition 3.10.** Through the equivalences \(\text{DAlg}_A \simeq \text{Spf}_{C_\bullet(A)[[1]]}, D_{C_\bullet(A)} \simeq \text{Spf}_{\text{End}_{k[t]}(C_\bullet(A)[[t]])}, D_{C_\bullet(A)(t)} \simeq \text{Spf}_{\text{End}_{k[t]}(C_\bullet(A)(t))}\), the functors \(\mathcal{P}, \mathcal{Q}\) and \(\mathcal{R}\) can be identified with the functors

\[
\begin{align*}
\text{Spf}_{L} : \text{Spf}_{C_\bullet(A)[[1]]} & \to \text{Spf}_{\text{End}_{k[t]}(C_\bullet(A))}, \\
\text{Spf}_{L[[t]]} : \text{Spf}_{C_\bullet(A)[[1]]} & \to \text{Spf}_{\text{End}_{k[t]}(C_\bullet(A)[[t]])}, \\
\text{Spf}_{L((t))} : \text{Spf}_{C_\bullet(A)[[1]]} & \to \text{Spf}_{\text{End}_{k[t]}(C_\bullet(A)(t))},
\end{align*}
\]

associated to \(L[[t]]\) and \(L((t))\) respectively.

We prove this Proposition. First, we find an explicit formula of the differential on Hochschild chain complex \(C_\bullet(A)\) (see Section 2.5). Let \((A, b)\) be a unital curved \(A_\infty\)-algebra. We denote by \(b_i : (A[1])^{(d)} \to A[1]\) the \(l\)-th component of \(b\).

**Lemma 3.11.** The differential on the Hochschild chain complex \(C_\bullet(A)\) of \(A\) is determined by the formula

\[
\partial_{\text{Hoch}}(a_0 \otimes [a_1| \ldots |a_n]) = \sum_{n+1 \leq t \leq n} (-1)^{\epsilon_i+|a_0|-1} \epsilon_i s^{-1} b_l[a_i+1] \ldots |a_n| a_0 |a_1| \ldots |a_{t-n+i-1}| \otimes |a_{t-n+i}| \ldots |a_i|
\]

Proof. We have an isomorphism \(A \otimes B \tilde{A} \simeq \Phi(A) \subset B \tilde{A} \otimes A \otimes B \tilde{A}\) given by \(S_{312} \circ (1 \otimes \Delta_{B \tilde{A}})\) (see Section 2.5). In explicit terms, it carries \(a \otimes [a_1| \ldots |a_n]\) to

\[
\sum_{i=0}^{n} (-1)^{\epsilon_i+|a_i|} \epsilon_n-\epsilon_i |a_i+1| \ldots |a_n| \otimes a \otimes [a_1| \ldots |a_i]
\]

where \(\epsilon_i = \sum_{l=1}^{i} (|a_l| - 1)\). The counit map \(\eta : B \tilde{A} \to k\) induces the inverse isomorphism \(\Phi(A) \subset B \tilde{A} \otimes A \otimes B \tilde{A} \eta \otimes 1 \subset A \otimes B \tilde{A}\). Let \(d\) be the differential on \(B \tilde{A} \otimes A \otimes B \tilde{A}\) given by
\{d_{m,n}\}_{m,n \geq 0}$ in Section 2.5.

d([a_1| \ldots |a_i] \otimes a \otimes [a'_1| \ldots |a'_j])
\begin{align*}
&= \sum_{s+l \leq i} (-1)^s [a_1| \ldots |a_s|b_1[a_{s+1}| \ldots |a_{s+l}]|a_i] \otimes a \otimes [a'_1| \ldots |a'_j]
\end{align*}
\begin{align*}
&+ \sum_{s+i \leq j} (-1)^s(-1)^{i-\epsilon_s}[a_1| \ldots |a_s] \otimes s^{-1}\cdot b_l[a_{s+i}| |a_i|a'_1| \ldots |a'_t|] \otimes [a'_{l+i}| \ldots |a'_j]
\end{align*}
\begin{align*}
&+ \sum_{t+l \leq j} (-1)^{i+|a|}(-1)^i[a_1| \ldots |a_i] \otimes a \otimes [a'_1| \ldots |a'_t|b_l[a_{t+i}| \ldots |a'_j]|a_{t+j}| \ldots |a_n].
\end{align*}

where $b_l$ is the $l$-th component of curved $A_\infty$-structure of $\tilde{A}$ (we abuse notation), and $\epsilon'_l = \sum_{i=1}^l (|a_i'| - 1)$. Taking account of the above two formulas we obtain the following formula of the differential on $C^\bullet(A) = A \otimes \tilde{B} A$.

\begin{align*}
\partial_{Hoch}(a_0 \otimes [a_1| \ldots |a_n]) &= (\eta \otimes 1 \otimes 1) \circ d \circ (S_{312} \circ 1 \otimes \Delta_{B,A})(a_0 \otimes [a_1| \ldots |a_n])
\end{align*}
\begin{align*}
&= \sum_{a \leq i \leq n} (-1)^{\epsilon_0+|a_0|}(\epsilon_n-\epsilon_i)\cdot s^{-1}\cdot b_l[a_{i+1}| \ldots |a_n|a_0|a_1| \ldots |a_{n+i-1}| \otimes [a_{i+n+i}| \ldots |a_i]
\end{align*}
\begin{align*}
&+ \sum_{j+l \leq n} (-1)^{1+|a_0|+\epsilon_0}a_0 \otimes [a_1| \ldots |b_l[a_{j+1}| \ldots |a_{j+l}|a_{j+l+1}| \ldots |a_n].
\end{align*}

\begin{proof}[Proof of Proposition 3.10]
In this proof, we will prove a more refined statement. Let $R$ be an artin local $k$-algebra in $\text{Art}_k$ and $m_R$ its maximal ideal. Let $MC_L : MC(C^\bullet(A)[1] \otimes_k m_R) \rightarrow MC(\text{End}_k(C^\bullet(A)) \otimes_k m_R)$ be the map induced by $L$. Claim 3.8.1 (resp. Claim 3.8.3) describes a canonical bijective correspondence between $MC(C^\bullet(A)[1] \otimes_k m_R)$ and the set of deformations of $(A, b)$ to $R$ (resp. between $MC(\text{End}_k(C^\bullet(A)) \otimes_k m_R)$ and the set of deformations of $C^\bullet(A)$ to $R$). Using these correspondences we can identify $MC_L$ with the map from the set of deformations of the $A_\infty$-algebra $(A, b)$ to that of the complex $C^\bullet(A)$, which carries $\tilde{A} = (A \otimes_k R, \tilde{b})$ to $C^\bullet(\tilde{A})$ as follows. Indeed, if we put $\tilde{b} = b \otimes_k R + f \otimes R$ with $f \in MC(C^\bullet(A)[1] \otimes_k m_R)$, then the differential on $C^\bullet(\tilde{A}) = (A \otimes_k R) \otimes_R (BA \otimes_k R)$ is given by the formula in Lemma 3.11 where each $b_l$ is replaced by $\tilde{b}_l = b_l \otimes_k R + f_l \otimes R$ where $f_l : (A[1])^{\otimes l} \rightarrow A[1] \otimes_k m_R$ is the $l$-th term of $f$. Observe that it coincides with $\partial_{Hoch} \otimes_k R + L_f \otimes R$ where $L_f : C^\bullet(A) \rightarrow C^\bullet(A) \otimes_k m_R$ is defined in Section 2.1.

Next through correspondences, the computation similar to Lemma 3.11 shows that $(C^\bullet(A)[1] \otimes_k m_R)^0 \rightarrow (\text{End}_k(C^\bullet(A)) \otimes_k m_R)^0$ induced by $L$ is identified with $\text{Aut}(A, R) \rightarrow \text{Aut}_{\text{gr}}(C^\bullet(A), R)$ which carries $BA \otimes_k R \rightarrow BA \otimes_k R$ (an automorphism of the graded coalgebras) to $C^\bullet(A) \otimes_k R \rightarrow C^\bullet(A) \otimes_k R$ (the induced automorphism of the graded $R$-module).

Finally,

$$
P : \text{DAlg}_A(R) \simeq \text{MC}(C^\bullet(A)[1] \otimes_k m_R)/\exp(C^\bullet(A)[1] \otimes_k m_R)^0 \rightarrow \text{MC}(\text{End}_k(C^\bullet(A)) \otimes_k m_R)/\exp(\text{End}_k(C^\bullet(A)) \otimes_k m_R)^0 \simeq D^*_C(A)(R)
$$

is naturally isomorphic to $\text{Spf}_L$. The cases of $Q$ and $R$ are similar.
\end{proof}

\section{Period mapping and homotopy calculus}

4.1. Let $A$ be a unital dg algebra over a field $k$ of characteristic zero. Henceforth dg algebras are assumed to be unital in this paper. In Section 3 we considered a morphism $P : \text{DAlg}_A \rightarrow D_C^*(A)$ of functors, and its cyclic versions $Q : \text{DAlg}_A \rightarrow D_C^*(A)[t]$, $R : \text{DAlg}_A \rightarrow D_C^*(A)(t)$. Proposition 3.10 allows us to consider them as the functors associated
to homomorphism $L, L[[t]], L((t))$ of dg Lie algebras. In particular, $\mathcal{R}$ can be identified with $Spf_{L((t))} : Spf_{C^\bullet(A)[1]} \to Spf_{End_k[t]^+(C^\bullet(A))}$. The purpose of this section is to construct a period mapping for deformations of a dg algebra which can be described in a moduli-theoretic fashion. We first prove the following:

**Proposition 4.1.** The morphism $L((t)) : C^\bullet(A)[1] \to \text{End}_{k[t]}(C^\bullet(A)((t)))$ of dg Lie algebras is null homotopic. See Section 4.6 for the mapping space between dg Lie algebras.

**Remark 4.2.** Proposition 4.1 especially means that any deformation of $A$ induces no non-trivial deformation of the periodic cyclic complex $C^\bullet(A)((t))$ at any level (including higher homotopy and derived structure). If we think of $C^\bullet(A)((t))$ as a “topological invariant” of $A$ (keep in mind that Hochschild-Kostant-Rosenberg theorem), this is an analogue of Ehresmann’s fibration theorem.

The proof of Proposition 4.1 is completed in the end of Section 4.10.

4.2. To prove Proposition 4.1 we need two more algebraic structures on $(C^\bullet(A), C^\bullet(A))$ (see [38]). We briefly review them. We adopt notation in Section 2.2. Recall that $C^\bullet(A) = C^\bullet(A)[1][-1] = \prod_{\geq 0} \text{Hom}_k((\bar{A}[1])^{\otimes n}, A)$ where $\bar{A} = A/k$. Let $P \in \text{Hom}_k((\bar{A}[1])^{\otimes p}, A) \simeq \text{Hom}_k((\bar{A}[1])^{\otimes p}, A[1])$ and $Q \in \text{Hom}_k((\bar{A}[1])^{\otimes q}, A) \simeq \text{Hom}_k((\bar{A}[1])^{\otimes q}, A[1])$ be two elements in $C^\bullet(A)$ (the isomorphism is given by $s : A \to A[1]$). The tensor product of maps, and compositions with the 2-nd component $b_2 : A[1] \otimes A[1] \to A[1]$ (of the $A_\infty$-structure) and $s^{-1} : A[1] \to A$ induce

$\text{Hom}_k((\bar{A}[1])^{\otimes p}, A[1]) \otimes \text{Hom}_k((\bar{A}[1])^{\otimes q}, A[1]) \to \text{Hom}_k((\bar{A}[1])^{\otimes p+q}, A[1]) \to \text{Hom}_k((\bar{A}[1])^{\otimes p+q}, A[1])$.

We define the cup product $P \cup Q$ to be the image of this composite. The cup product exhibits $C^\bullet(A)$ as a (non-commutative) dg algebra with unit 1 $\in \text{Hom}(k, A)$.

For $P$ in $\text{Hom}_k((\bar{A}[1])^{\otimes p}, A) \subset C^\bullet(A)$ we define a contraction map $I_P : C^\bullet(A) \to C^\bullet(A)$ of degree $|P|$ as follows.

\[
\text{Hom}_k((\bar{A}[1])^{\otimes p}, A) \otimes (A \otimes (\bar{A}[1])^{\otimes n}) \xrightarrow{s \otimes 1^{\otimes n+1}} \text{Hom}_k((\bar{A}[1])^{\otimes p}, A[1]) \otimes (A \otimes (\bar{A}[1])^{\otimes n}) \to A \otimes A[1] \otimes (\bar{A}[1])^{\otimes n-p} \xrightarrow{s^{-1} \otimes 1} A \otimes (\bar{A}[1])^{\otimes n-p}
\]

where the second arrow is given by the pairing of $\text{Hom}((\bar{A}[1])^{\otimes p}, A[1])$ and the first $p$ factors in $(\bar{A}[1])^{\otimes n}$ for $p \leq n$ (if otherwise it is defined to be zero). We denote the induced map by $I : C^\bullet(A) \to \text{End}_k(C^\bullet(A))$. $P \mapsto I_P$. Explicitly, $I_P(a_0 \otimes [a_1] \ldots [a_n])$ has the form $\pm a_0 P[a_1] \ldots [a_p] \otimes [a_{p+1}] \ldots [a_n]$ for $p \leq n$.

The five operations $\cup, [-, -], \wedge, I, B$ on $C^\bullet(A)[1]$, the dg Lie algebra map $L$, the contraction $I$, and Connes’ operator $B$ constitute a calculus structure on $(HH^\bullet(A), HH_\bullet(A))$ (cf. [38]). Here is the definition of calculus.

**Definition 4.3.**

1. A graded $k$-module $V$ is a Gerstenhaber algebra if it is endowed with a (graded) commutative and associative product $\wedge : V \otimes V \to V$ of degree zero and a Lie bracket $[-,-] : V \otimes V \to V$ of degree $-1$ which exhibits $V[1]$ as a graded Lie algebra. These satisfy the Leibniz rule

\[ [a, b \wedge c] = [a, b] \wedge c + (-1)^{|a|+1}b \wedge [a, c] \]

where $a, b, c$ are homogeneous elements of $V$. 

(2) A precalculus structure is a pair of a Gerstenhaber algebra \((V, \wedge, [-, -])\) and a graded \(k\)-module \(W\) together with a module structure
\[
i : V \otimes W \to W
\]
of the commutative algebra \((V, \wedge)\), and a module structure
\[
l : V[1] \otimes W \to W
\]
of (graded) Lie algebra \(V[1]\) such that
\[
i_a b - (-1)^{|a||b|-1} l_{b \delta a} = i_{[a,b]}, \quad \text{and} \quad l_{a \wedge b} = l_a i_b + (-1)^{|a|} i_a l_b
\]
where \(i_a : W \to W\) is determined by \(i(a \otimes (-))\) for \(a \in V\), and \(l_b : W \to W\) is determined by \(l(b \otimes (-))\) for \(b \in V\).

(3) A calculus is a precalculus \((V, W, \wedge, [-, -], i, l)\) endowed with a linear map \(\delta : W \to W\) of degree \(-1\) such that
\[
\delta^2 = 0 \quad \text{and} \quad \delta i_a - (-1)^{|a|} i_a \delta = l_a.
\]

We call the 7-tuple \((V, W, \wedge, [-, -], i, l, \delta)\) a calculus algebra (or we also refer to it as a structure of calculus on \((V, W)\)).

(4) A 5-tuple \((V, W, [-, -], l, \delta)\) is said to be a Lie\(^{\dagger}\)-algebra when \([-,-]\) determines a Lie bracket on \(V[1]\) as in (1), \(l : C[1] \otimes W \to W\) is an action of \(C[1]\) on \(W\), \(\delta : W \to W\) is a linear map of degree \(-1\) such that
\[
\delta^2 = 0 \quad \text{and} \quad \delta l_a - (-1)^{|a|-1} l_a \delta = 0.
\]

4.3. According to the work of Daletsky, Gelfand, Tamarkin and Tsygan (see \[38, 4, 5\]), \((\cup, [-, -]_G, I, L, B)\) determines a structure of calculus \((V = HH^*(A), W = HH_*(A))\), that is,
\[
(HH^*(A), HH_*(A), \cup, [-, -]_G, I, L, B)
\]
is an important example of a calculus algebra. The operations \((\cup, [-, -]_G, I, L, B)\) fail to determine a structure of calculus on \((C^\bullet(A), C_\bullet(A))\). For example, \(\cup\) and \([-,-]_G\) do not satisfy the Leibniz rule. To endow \((C^\bullet(A), C_\bullet(A))\) with algebraic structures coming from \((\cup, [-, -]_G, I, L, B)\) in a suitable way, one needs the machinery of operads.

Kontsevich and Soibelman constructed a 2-colored dg operad \(KS\) (consisting of two colors) and a natural action of \(KS\) on the pair \((C^*(A), C_\bullet(A))\) for an \(A_\infty\)-algebra \(A\), see \[25, 11.1, 11.2, 11.3\]. This action of the operad \(KS\) generalizes the solution of Deligne’s conjecture on an action of the little disks operad on \(C^\bullet(A)\). We also refer to Dolgushev-Tamarkin-Tsygan \[4\] Section 4 for the detailed study on \(KS\) in the case of (dg) algebras. The recent work of Horel \[13\] treats a generalization to the case of ring spectra, which is based on the factorization homology and Swiss-cheese operad conjecture. The operad \(KS\) is closely related to calculi. Let \textbf{calc} be the 2-colored graded operad defined by generators \(\wedge, [-, -], i, l, \delta\) and their relations in Definition \[3.3\] (we refer the reader to \[30\] for dg operads, \[12, 38\] Section 3.5, 3.6] for colored operads, modules over an algebra over an operad, and \[4\] for \textbf{calc}). The operad \textbf{calc} has the suboperad called Gerstenhaber operad, that is generated by \(\wedge\) and \([-,-]\) and relations in Definition \[1.3\] (1), see \[30, 13.3, 12\] for the Gerstenhaber operad. Let \textbf{Lie}\(^{\dagger}\) be the 2-colored graded suboperad of \textbf{calc} generated by \([-,-], l, \delta\) and their relations in Definition \[1.3\] (4). A Lie\(^{\dagger}\)-algebra is a Lie\(^{\dagger}\)-algebra. According to Proposition \[3.1\] \((C^*(A), C_\bullet(A), [-,-], L, B)\) is a Lie\(^{\dagger}\)-algebra in the category of complexes. Thanks to \[4\] Theorem 2, Theorem 3, Proposition 8, the homology operad \(H_*(KS)\) is quasi-isomorphic to \textbf{calc}. Moreover, \(KS\) is formal, namely, it is quasi-isomorphic to \(H_*(KS) \cong \textbf{calc}\). Let \textbf{Calc} denote a natural cofibrant replacement by the cobar-bar resolution \(\text{Calc} := \text{Cobar(Bar(calc))} \to \text{calc}\). There is a quasi-isomorphism \(\text{Calc} \to KS\). Thus, we obtain a \textbf{Calc}-algebra \((C^\bullet(A), C_\bullet(A))\). We summarize the result as
Proposition 4.4. There exists a Calc-algebra \((C^\bullet(A), C^\bullet_\bullet(A))\), that is, an action of Calc on \((C^\bullet(A), C^\bullet_\bullet(A))\) whose underlying \(H_*(\text{Calc}) \simeq \text{calc}\)-algebra is the calculus \((HH^*(A), HH_*(A))\). Moreover, the operad Calc has a suboperad \(\text{Lie}^\dagger\) endowed with a quasi-isomorphism \(\text{Lie}^\dagger \to \text{Lie}^\dagger\) which commutes with the counit quasi-isomorphism \(\text{Calc} \to \text{calc}\), such that the pullback of the \(\text{Calc}\)-algebra \((C^\bullet(A), C^\bullet_\bullet(A))\) to \(\text{Lie}^\dagger\) coincides with the pullback of the \(\text{Lie}^\dagger\)-algebra \((C^\bullet(A), C^\bullet_\bullet(A))\) to \(\text{Lie}^\dagger\).

Proof. The statement of the first half is \([4, \text{Theorem 2, 3, Proposition 8}]\). Namely, we can obtain an action of the operad Calc on the pair of complexes \((C^\bullet(A), C^\bullet_\bullet(A))\) from that of \(KS\) whose underlying action of \(H_*(KS) \simeq H_*(\text{Calc}) \simeq \text{calc}\) on \((HH^*(A), HH_*(A))\) can be identified with the above calculus \((HH^*(A), HH_*(A))\). According to \([1, \text{Theorem 4}]\) and its proof, the Calc-algebra \((C^\bullet(A), C^\bullet_\bullet(A))\) is quasi-isomorphic to another action of Calc on \((C^\bullet(A), C^\bullet_\bullet(A))\) whose restriction to \(\text{Lie}^\dagger\) comes from the \(\text{Lie}^\dagger\)-algebra given by operations \(L, [-, -], B\) on \((C^\bullet(A), C^\bullet_\bullet(A))\).

4.4. Let \((C, M, [-, -], l, \delta)\) be a \(\text{Lie}^\dagger\)-algebra in the symmetric monoidal category of complexes of \(k\)-modules (in particular, \(C, C', M\) and \(M'\) are complexes). In explicit terms, a \(\text{Lie}^\dagger\)-algebra amounts to data consisting of (i) a bracket \(C[1] \otimes C[1] \to C[1]\) of degree 0 which exhibits \(C[1]\) as a dg Lie algebra, (ii) \(\delta : M \to M\) of degree \(-1\) with \(\delta^2 = 0\) and \(d_M \delta + \delta d_M = 0\), (iii) an action of \(l : C[1] \otimes M \to M\) of the dg Lie algebra \(C[1]\) on the complex \(M\) which commutes with \(\delta\) (i.e., \(\delta l_c - (-1)^{|c|-1}l_c \delta = 0\) for \(c \in C\)). Here \(d_M\) is the differential of \(M\). The pair \((M, \delta)\) is a mixed complex. Thus it gives rise to a dg \(k[t^{\pm}]-\)module \((M((t)), d_M + t \delta)\). Roughly speaking, we may say that a \(\text{Lie}^\dagger\)-algebra \((C, M, [-, -], l, \delta)\) is a dg Lie algebra \(C[1]\) together with its action on a mixed complex \(M\). To a \(\text{Lie}^\dagger\)-algebra \((C, M, [-, -], l, \delta)\) we associate a dg Lie algebra map \(l((t)) : C[1] \to \text{End}_{k[t^{\pm}]}(M((t)))\) which carries \(c\) to \(l_c((t)) : M((t)) \to M((t))\), \(mt^n \mapsto l_c(m)t^n\).

4.5. Henceforce we will use some model category structures. Appropriate references are \([14, 32, \text{Appendix}]\). We emply the model category structure of the category of algebras over a dg operad \([12, 2.3.1, 2.6.1]\). The category of \(\text{Lie}^\dagger\)-algebras denoted by \(\text{Alg}_{\text{Lie}^\dagger}\) admits a combinatorial model category structure where a morphism \((C, M, [-, -], l, \delta) \to (C', M', [-, -]', l', \delta')\) is a weak equivalence (resp. a fibration) if both \(C \to C'\) and \(M \to M'\) are quasi-isomorphisms (resp. degreewise surjective maps) of complexes. Similarly, the category of dg Lie algebras (resp. dg \(k[t^{\pm}]-\)modules) admits a combinatorial model category structure where a morphism is a weak equivalence (resp. a fibration) if the underlying map of complexes is a quasi-isomorphism (resp. a degreewise surjective map). Note that every object is fibrant.

4.6. Let us recall the mapping space between two dg Lie algebras, cf. \([10]\). Let \(V\) and \(W\) be two dg Lie algebras. Let \(B_{\text{com}}V\) be the bar construction associated to \(V\), which is a unital cocommutative dg coalgebra over \(k\). Here coalgebras are assumed to be conilpotent. Explicitly, \(B_{\text{com}}V\) is the \(\oplus_{n \geq 0} \text{Sym}^n(V[1])\) as a graded cocommutative coalgebra. Here \(\text{Sym}^n\) indicates the \(n\)-fold symmetric product, and \(B_{\text{com}}(-)\) is different from \(B(-)\) in the previous sections. The differential is the sum of two differentials; the first comes from the differential of \(V\), the second is determined by \([-,-] : (V \wedge V)[2] \simeq V[1] \wedge V[1] \to V[1][1]\). It gives rise to a functor \(B_{\text{com}} : \text{dgLie} \to \text{dgcoAlg}\) from the category of dg Lie algebras to the category of unital cocommutative dg coalgebras. There is a left adjoint \(\text{Cob}_{\text{com}} : \text{dgcoAlg} \to \text{dgLie}\) of \(B_{\text{com}}\) given by the cobar construction (see e.g. \([10, 2.2.1]\)). For any dg Lie algebra \(V\), the counit map \(\text{Cob}_{\text{com}}B_{\text{com}}V \to V\) gives a (canonical) cofibrant replacement of \(V\). Let \(\Omega_n\) denote the commutative dg algebra of polynomial differential forms on the standard \(n\)-simplex. Namely, it is

\[
\Omega_n := k[u_0, \ldots u_n, du_0, \ldots du_n]/(\Sigma_{i=0}^n u_i - 1, \Sigma_{i=0}^n du_i)
\]
where \( k[u_0, \ldots, u_n, du_0, \ldots, du_n] \) is the free commutative graded algebra generated by \( u_0, \ldots, u_n \) and \( du_0, \ldots, du_n \) with \( |u_i| = 0, |du_i| = 1 \) for each \( i \), and the differential carries \( u_i \) to \( du_i \) (see e.g. [3]). If we consider the family \( \Omega_* = \{ \Omega_n \}_{n \geq 0} \) of commutative dg algebras, they form a simplicial commutative dg algebras in the natural way. Using the simplicial commutative dg algebra \( \Omega_* \) and the simplicial model category structure of \( \text{dgLie} \) [10, 2.4] we obtain a Kan complex \( \text{Hom}_{\text{dgLie}}(\text{Cob}_{\text{com}}B_{\text{com}}V, \Omega_* \otimes W) \), that is a model of the mapping space. In [10], the definition of simplicial model categories is slightly weaker than the standard one. But by [11, 1.4.2] this Kan complex is naturally homotopy equivalent to the Hom simplicial set in the simplicial category associated to the underlying model category via Dwyer-Kan hammock localization. We prefer to work with another presentation of this model. Let \( C \) be a dg coalgebra with a comultiplication \( \Delta : C \to C \otimes C \) and \( V \) a dg Lie algebra. Let \( \text{Hom}(C, V) \) be the Hom complex between the underlying complexes of \( C \) and \( V \). It is endowed with the following convolution Lie bracket:

\[
[f, g] : C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} V \otimes V \xrightarrow{[-,-]} V
\]

for \( f, g \in \text{Hom}(C, V) \). For the dg Lie algebra \( \text{Hom}(C, V) \) we define the set of Maurer-Cartan elements \( \text{MC}(C, V) := \text{MC}(\text{Hom}(C, V)) \). According to [10, 2.2.5] there is a natural isomorphism

\[
\text{Hom}_{\text{dgLie}}(\text{Cob}_{\text{com}}B_{\text{com}}V, \Omega_n \otimes W) \simeq \text{MC}(\text{com}_{\text{B}}V, \Omega_n \otimes W)
\]

where \( B_{\text{com}}V \) is the kernel of the counit \( B_{\text{com}}V \to k \). (The isomorphism is induced by the composition with the inclusion \( B_{\text{com}}V[-1] \hookrightarrow \text{Cob}_{\text{com}}B_{\text{com}}V \).) In particular,

\[
\text{Map}(V, W) := \text{Hom}_{\text{dgLie}}(\text{Cob}_{\text{com}}B_{\text{com}}V, \Omega_* \otimes W) \simeq \text{MC}(B_{\text{com}}V, \Omega_* \otimes W).
\]

In explicit terms, an element of the 0-th term \( \text{MC}(B_{\text{com}}V, W) \) of \( \text{MC}(B_{\text{com}}V, \Omega_* \otimes W) \) corresponds to a family of linear maps \( \text{Sym}^n(V[1]) \simeq (\Lambda^n V)[n] \to W[1] \) of degree zero \( (n \geq 1) \) that satisfies a certain relation coming from the Maure-Cartan equation. It is sometimes called an \( L_\infty \)-morphisms in the literature. Thus we refer to an element of \( \text{MC}(B_{\text{com}}V, W) \) as an \( L_\infty \)-morphisms. When \( V[1] = \text{Sym}^1(V[1]) \to W[1] \) is a quasi-isomorphism, we shall call it an \( L_\infty \)-quasi-isomorphism. Any equivalence class of a morphism from \( V \) to \( W \) of two dg Lie algebras can be represented by an \( L_\infty \)-morphisms. Since \( \text{Map}(V, W) \) is a Kan complex, for two \( L_\infty \)-morphisms \( f \) and \( g \) corresponding to two vertices of this Kan complex, the space of homotopies/morphisms from \( f \) to \( g \) makes sense (cf. [32, 1.2.2]).

Put \( \Omega_1 = k[u, du] \). We write \( W[u, du] \otimes W \) for \( k[u, du] \otimes W \) (do not confuse it with \( W \otimes k[u, du] \), that gives rise to the different sign rule). The degree of \( du \) is 1. An element of \( \text{MC}(B_{\text{com}}V, W[u, du]) \) represents a morphism (or a homotopy) between two \( L_\infty \)-morphisms. Face maps \( d_0, d_1 : \text{MC}(B_{\text{com}}V, W[u, du]) \Rightarrow \text{MC}(B_{\text{com}}V, W) \) are induced by the composition with maps \( p_0 \) and \( p_1 \) of dg Lie algebras:

\[
p_0 : W[u, du] \to W, \quad p_1 : W[u, du] \to W
\]

given by \( p_0(u) = p_0(du) = 0 \) and \( p_1(u) = 1, p_1(du) = 0 \). Thus, a homotopy/morphism from an \( L_\infty \)-morphisms \( f \) to an \( L_\infty \)-morphisms \( g \) can be represented by \( \phi \in \text{MC}(B_{\text{com}}V, W[u, du]) \) such that \( d_0(\phi) = f \) and \( d_1(\phi) = g \). We say that \( f \) is equivalent to \( g \) if a morphism exists between them.

4.7.

**Lemma 4.5.** Let \( (C, M, [-,-], l, \delta) \) and \( (C', M', [-,-]', l', \delta') \) be two \( \text{Lie}^\dagger \)-algebras in the category of complexes of \( k \)-modules. Let \( f : (C, M, [-,-], l, \delta) \to (C', M', [-,-]', l', \delta') \) be a weak equivalence of \( \text{Lie}^\dagger \)-algebras. If \( f \) is a trivial fibration (i.e., both underlying maps \( C \to C' \) and
$M \rightarrow M'$ are surjective quasi-isomorphisms), then there exist a dg Lie algebra $F$ and weak equivalences

$$\text{End}_{k[t^\pm]}(M((t))) \leftarrow F \rightarrow \text{End}_{k[t^\pm]}(M'(t)))$$

such that the left arrow is injective, and the right arrow is surjective.

Moreover, there exists a homotopy equivalence between the mapping space from $L((t)) : C[1] \rightarrow \text{End}_{k[t^\pm]}(M((t)))$ to the zero morphism (see Section [4.6]) and the mapping space from $L'(t)) : C'[1] \rightarrow \text{End}_{k[t^\pm]}(M'(t)))$ to the zero morphism via the zig-zag (see the proof for the construction).

Proof. We suppose that $f$ is a trivial fibration. It follows that the induced morphism $M((t)) \rightarrow M'(t))$ is a trivial fibration of dg $k[t^\pm]$-modules, i.e., a surjective quasi-isomorphism. Note also that every dg $k[t^\pm]$-module is cofibrant with respect to the projective model structure. Consider the pullback diagram of Hom complexes

\[
\begin{array}{ccc}
F & \longrightarrow & \text{End}_{k[t^\pm]}(M'(t))) \\
\downarrow & & \downarrow \\
\text{End}_{k[t^\pm]}(M((t))) & \longrightarrow & \text{Hom}_{k[t^\pm]}(M((t)), M'(t)))
\end{array}
\]

where the lower horizontal arrow and the right vertical arrow are induced by the composition with $M((t)) \rightarrow M'(t))$ respectively. Since the model category of dg modules admits a complicial model structure, the lower horizontal arrow is a trivial fibration, i.e., a surjective quasi-isomorphism. Thus, the upper horizontal arrow is also a trivial fibration. The right vertical arrow is also a weak equivalence, i.e., a (injective) quasi-isomorphism since $M((t)) \rightarrow M'(t))$ is so. By the 2-out-of-3 property, the left vertical arrow is a (injective) weak equivalence. Note that $F$ is a dg Lie subalgebra of $\text{End}_{k[t^\pm]}(M((t)))$ consisting of those linear maps $M((t)) \rightarrow M((t))$ such that the composite $M((t)) \rightarrow M((t)) \rightarrow M'(t))$ factors through the quotient $M((t)) \rightarrow M'(t))$. Moreover, the upper horizontal arrow is a morphism of dg Lie algebras. Thus, we see the first assertion. To prove the second claim, note first that $f$ is a trivial fibration between Lie$^\dagger$-algebras, so that $L((t)) : C[1] \rightarrow \text{End}_{k[t^\pm]}(M((t)))$ factors through $F \subset \text{End}_{k[t^\pm]}(M((t)))$. We then have the commutative diagram

\[
\begin{array}{ccc}
C[1] & \longrightarrow & F \\
\downarrow & & \downarrow \\
C'[1] & \longrightarrow & \text{End}_{k[t^\pm]}(M'(t)))
\end{array}
\]

where the left vertical arrow is the morphism of dg Lie algebras induced by $f$. Thus, we have a zig-zag of homotopy equivalences of mapping spaces of dg Lie algebras

$$\text{Map}(C[1], E) \leftarrow \text{Map}(C[1], F) \rightarrow \text{Map}(C[1], E') \leftarrow \text{Map}(C'[1], E')$$

where $E = \text{End}_{k[t^\pm]}(M((t)))$ and $E' = \text{End}_{k[t^\pm]}(M'(t)))$. Consequently, we obtain a homotopy equivalence between the mapping space from $L((t))$ to 0 and that of $L'(t))$ to 0.

Here is another $\infty$-categorical proof of the above Lemma.

**Lemma 4.6.** Let $(C, M, [-,-], l, \delta)$ and $(C', M', [-,-]', l', \delta')$ be two Lie$^\dagger$-algebras in the category of complexes of $k$-modules. Let $f : (C, M, [-,-], l, \delta) \rightarrow (C', M', [-,-]', l', \delta')$ be a weak equivalence of Lie$^\dagger$-algebras. Then there exists a natural homotopy equivalence between the mapping space from $L((t)) : C[1] \rightarrow \text{End}_{k[t^\pm]}(M((t)))$ to the zero morphism (see Section [4.6]) and the mapping space from $L'(t)) : C'[1] \rightarrow \text{End}_{k[t^\pm]}(M'(t)))$ to the zero morphism via the zig-zag (see the proof for the construction).
Proof. Let \( U : \text{dgLie} \simeq \text{dgAlg} \) be the adjoint pair where the right adjoint functor from the category of dg algebras \( \text{dgAlg} \to \text{dgLie} \) carries a dg algebra \( A \) to the dg Lie algebra \( A \) with the graded commutator. The left adjoint is given by the free functor of the universal envelopping algebras. If \( \text{dgAlg} \) and \( \text{dgLie} \) are endowed with the projective model structures respectively, the adjoint pair is a Quillen adjunction since the right adjoint preserves trivial fibrations and fibrations. The morphism of dg Lie algebras \( L((t)) : C[1] \to \text{End}_k[t^{\pm}](M((t))) \) and \( L'(((t)) : C'[1] \to \text{End}_k[t^{\pm}](M'(((t))) \) induce morphisms of dg algebras \( U(L((t))) : U(C[1]) \to \text{End}_k[t^{\pm}](M((t))) \) and \( U(L'(((t))) : U(C'[1]) \to \text{End}_k[t^{\pm}](M'(((t))) \) respectively. If necessary, we replace \( C[1] \) and \( C'[1] \) by the cofibrant models. By the Quillen adjunction, we may work with dg algebras instead of dg Lie algebras. Namely, it is enough to prove that there exists a homotopy equivalence between the mapping space from \( U(L((t))) : U(C[1]) \to \text{End}_k[t^{\pm}](M((t))) \) to \( U(0) : U(C[1]) \to \text{End}_k[t^{\pm}](M((t))) \) and the mapping space from \( U(L'(((t)))) : U(C[1]) \to U(C'[1]) \to \text{End}_k[t^{\pm}](M'(((t))) \) to \( U(0) : U(C[1]) \to \text{End}_k[t^{\pm}](M'(((t))) \). For simplify the notation, we put \( F = U(L((t))) \), \( F' = U(L'(((t)))) \) and \( A = U(C[1]) \). We introduce two categories. Let \( \text{Mod}_A \) be the category of dg \( k[t^{\pm}] \)-modules and let \( \text{Mod} \) be the category of dg \( \mathbb{k}[t^{\pm}]-\text{modules} \). We consider them to be combinatorial models which are endowed with projective model structures. (A morphism is a weak equivalence if the underlying map is a quasi-isomorphism, and a morphism is a fibration if the underlying map is a termwise surjective map. Every dg \( k[t^{\pm}]-\text{module} \) is cofibrant.). Taking the hammock localization, fibrant replacement of simplicial categories \([32, 1.1.4]\), and the simplicial nerve functor \([32, 1.1.5]\) we associate to the model categories \( \text{Mod}_A \) and \( \text{Mod} \)-categories \( \text{Map}_A^\infty \) and \( \text{Map}^\infty \) respectively. We also obtain the functor \( \text{Mod}_A^\infty \to \text{Mod}^\infty \) from the forgetful functor \( \text{Mod}_A \to \text{Mod} \). This functor can be decomposed as \( \text{Map}_A^\infty \overset{u}{\to} \text{Map}_A^\infty \overset{v}{\to} \text{Map}^\infty \) where \( u \) is a trivial cofibration, and \( v \) is a fibration with respect to Joyal model structure (see \([32, 2.2.5]\)).

Let \( \text{Mod}_A^\infty \times_{\text{Mod}^\infty} \{M((t))\} \) denote the homotopy fiber over \( M((t)) \in \text{Mod}^\infty \). We may take \( \text{Mod}_A^\infty \times_{\text{Mod}^\infty} \{M\} \) as one of its model. Let \( \text{Map}(A, \text{End}_k[t^{\pm}](M((t)))) \) denote the mapping space from \( A \) to \( \text{End}_k[t^{\pm}](M((t))) \) which we regard an \( \infty \)-category that is an \( \infty \)-groupoid. By the categorical characterization of the endomorphism algebras \([33, 6.1.2.41]\), there is a natural equivalence \( \text{Map}(A, \text{End}_k[t^{\pm}](M((t)))) \simeq \text{Mod}_A^\infty \times_{\text{Mod}^\infty} \{M((t))\} \) which carries a morphism \( \rho : k[t^{\pm}] \to \text{End}_k[t^{\pm}](M((t))) \) to the dg \( k[t^{\pm}]-\text{module} \) \( M(t) \) endowed with the \( k[t^{\pm}]-\text{module} \) structure given by \( \rho \) up to equivalence. We may consider that \( M((t)) \) belongs to \( \text{Mod}_A^\infty \times_{\text{Mod}^\infty} \{M((t))\} \). Similarly, we have \( \text{Map}(A, \text{End}_k[t^{\pm}](M'(((t)))) \simeq \text{Mod}_A^\infty \times_{\text{Mod}^\infty} \{M'(((t)))\} = \text{Mod}_A^\infty \times_{\text{Mod}^\infty} \{M'(((t)))\} \). The quasi-isomorphism \( M((t)) \to M'(((t)) \) induces the functorial equivalence

\[
\text{Mod}_A^\infty \times_{\text{Mod}^\infty} \{M((t))\} \to \text{Mod}_A^\infty \times_{\text{Mod}^\infty} \{M'(((t)))\}.
\]

To understand it, we let \( \text{Fun}([0,1], \text{Mod}) \) be the category of the functor category endowed with the projective model structure (see e.g. \([32, A. 2.8.2]\)). We here denote by \([0, 1]\) the category arising from the linearly ordered set \( 0 \to 1 \). Let \( \text{Fun}([0,1], \text{Mod})^\infty \) be the \( \infty \)-category associated to \( \text{Fun}([0,1], \text{Mod}) \) as above. There are two maps \( d_0, d_1 : \text{Fun}([0,1], \text{Mod})^\infty \to \text{Mod}^\infty \) given by the composition with \( \{0\} \to [0,1] \) and \( \{1\} \to [0,1] \) respectively, and the section \( \text{Mod}^\infty \to \text{Fun}([0,1], \text{Mod})^\infty \) induced by the constant functors. These three maps are (categorical) equivalences (cf. \([32, 4.2.4.4]\)). Then we consider

\[
H := \text{Mod}_A^\infty \times_{\text{Mod}^\infty} d_0 \times \text{Fun}(\Delta^1, \text{Mod})^\infty \times_{d_1, \text{Mod}^\infty} \{M((t))\}.
\]

Replacing \( M \) with \( M' \) we define \( H' \) in a similar way. By the canonical categorical equivalence \( \text{Mod}^\infty \to \text{Fun}([0,1], \text{Mod})^\infty \), the homotopy fiber \( \text{Fun}([0,1], \text{Mod})^\infty \overset{\Delta^1}{\to} \text{Mod}^\infty \) over \( \{M((t))\} \) is a contractible space. (There is a split injective \( \{M((t))\} \to \text{Fun}(\Delta^1, \text{Mod})^\infty \times_{d_1, \text{Mod}^\infty} \{M((t))\} \)
which makes \( \{M((t))\} \) the homotopy fiber. The composition with \( M((t)) \to M'(\langle t \rangle) \) induces
\[
\text{Fun}(\Delta^1, \text{Mod}) \times_{d_1, \text{Mod}} \{M((t))\} \to \text{Fun}(\Delta^1, \text{Mod}) \times_{d_1, \text{Mod}} \{M'(\langle t \rangle)\}
\]
which carries \( N \to M((t)) \) to \( N \to M((t)) \to M'(\langle t \rangle) \). It gives rise to a morphism \( H \to H' \). It is induced by the morphism from the following diagram \( D : I \to \text{Set}_{\Delta} \) of \( \infty \)-categories
\[
\begin{array}{ccc}
\text{Mod}_A^\infty & \xrightarrow{d_0} & \text{Mod}\mathbb{C} \\
\text{Fun}([0,1], \text{Mod})^\infty & \xrightarrow{d_1} & \text{Mod}^\infty \\
& \text{Mod}_A & \{M((t))\}
\end{array}
\]
to another diagram \( D' : I \to \text{Set}_{\Delta} \) in which \( M((t)) \) is replaced with \( M'(\langle t \rangle) \). Here \( I \) is the index category, and \( \text{Set}_{\Delta} \) is the category of simplicial sets. We use the injective model structure on the functor category \( \text{Fun}(I, \text{Set}_{\Delta}) \) (see e.g. \cite{32} A. 2.8.2, A. 2.8.7) and take functorial fibrant replacements \( D \to \overline{D} \) and \( D' \to \overline{D}' \) such that the diagram
\[
\begin{array}{ccc}
D & \to & D' \\
\downarrow & & \downarrow \\
\overline{D} & \to & \overline{D}'
\end{array}
\]
commutes. We let \( \overline{\mathcal{P}} \) and \( \overline{\mathcal{P}'} \) be the limits of diagram \( \mathcal{P} \) and \( \mathcal{P}' \) respectively. These limits are homotopy limits so that we take \( \overline{\mathcal{P}} \) and \( \overline{\mathcal{P}'} \) as homotopy fibers \( \text{Mod}_A^\infty \times_{\text{Mod}^\infty} \{M((t))\} \) and \( \text{Mod}_A^\infty \times_{\text{Mod}^\infty} \{M'(\langle t \rangle)\} \) respectively. The map \( \mathcal{P} \to \mathcal{P}' \) induces an equivalence \( \overline{\mathcal{P}} \to \overline{\mathcal{P}'} \). Let \( P = (M((t))_\rho, \text{id}_{M((t))}) : M((t)) \to M((t)) \) be the pair where \( M((t))_\rho \) endowed with the \( k[t^\pm] \otimes A \)-module structure determined by \( \rho \). It belongs to \( H \) (or \( \text{Mod}_A^\infty \times_{\text{Mod}^\infty} \{M((t))\} \)). Then \( H \to H' \) sends \( P \) to \( Q = (M((t))_\rho, M((t)) \to M'(\langle t \rangle)) \). The pair \( Q \) is equivalent to \( R = (M'(\langle t \rangle)_\rho, \text{id}_{M'(\langle t \rangle)}) : M'(\langle t \rangle) \to M'(\langle t \rangle)) \) in \( H' \), where \( M'(\langle t \rangle)_\rho \) is the \( k[t^\pm] \otimes A \)-module \( M'(\langle t \rangle) \) determined by \( F' \). The image of the last pair \( R \) in \( \overline{\mathcal{P}'} \) corresponds to \( F' : A \to \text{End}_{k[t^\pm]}(M'(\langle t \rangle)) \) up to equivalence (while the image of \( P \) in \( \overline{\mathcal{P}} \) corresponds to \( F \)). Consequently, we have
\[
\text{Map}(A, \text{End}_{k[t^\pm]}(M'(\langle t \rangle))) \simeq \overline{\mathcal{P}} \simeq \overline{\mathcal{P}'} \simeq \text{Map}(A, \text{End}_{k[t^\pm]}(M'(\langle t \rangle)))
\]
which carries \( F \) to \( F' \) up to equivalence. Hence our claim follows.

Let \( \text{Alg}_{\text{Calc}} \) (resp. \( \text{Alg}_{\text{Lie}^\dagger}, \text{Alg}_{\text{Calc}} \)) be the category of \( \text{Calc} \)-algebras (resp. \( \text{Lie}^\dagger \)-algebras, \( \text{calc} \)-algebras) in the category of chain complexes. As in the case of \( \text{Lie}^\dagger \)-algebras, by \cite{12} \( \text{Alg}_{\text{Calc}}, \text{Alg}_{\text{Lie}^\dagger}, \) and \( \text{Alg}_{\text{calc}} \) admit combinatorial model category structures where a morphism is a weak equivalence (resp. fibration) if it induces quasi-isomorphisms (resp. termwise surjective maps) of underlying complexes. The natural maps \( \text{Calc} \to \text{calc} \) and \( \text{Lie}^\dagger \to \text{calc} \) of operads induce the pullback functors
\[
\text{Alg}_{\text{calc}} \to \text{Alg}_{\text{Calc}} \quad \text{and} \quad \text{Alg}_{\text{calc}} \to \text{Alg}_{\text{Lie}^\dagger}
\]
which are right Quillen functors.

**Lemma 4.7.** Let \( A \) be a dg algebra. Consider an action of \( \text{Calc} \) on \( (C^\bullet(A), C_\bullet(A)) \) that satisfies the property in Proposition \( \ref{4.4} \). We denote by \( \mathfrak{A} \) this \( \text{Calc} \)-algebra. Then there exists a \( \text{calc} \)-algebra \( \mathfrak{B} := (C, M, [-, -], i, l, \delta) \) in the category of complexes such that its pullback along \( \text{Calc} \to \text{calc} \) is weak equivalence to \( \mathfrak{A} \), and its pullback \( i^* \mathfrak{B} \) along \( i : \text{Lie}^\dagger \to \text{calc} \) is weak equivalent to the \( \text{Lie}^\dagger \)-algebra \( (C^\bullet(A), C_\bullet(A), [-, -], L, B) \).

In particular, there exist a cofibrant \( \text{Lie}^\dagger \)-algebra \( \mathfrak{I} \) and a diagram of morphisms of \( \text{Lie}^\dagger \)-algebras
\[
i^* \mathfrak{B} \xleftarrow{i} \mathfrak{I} \xrightarrow{c} : (C^\bullet(A), C_\bullet(A), [-, -], L, B)
\]
where both arrows are trivial fibrations.
Proof. We first note that by [12, 2.4.5] \( \text{Alg}_{\text{calc}} \to \text{Alg}_{\text{Calc}} \) induces an equivalence of homotopy categories. It implies the first claim. Suppose that \( \mathcal{B} \) is a \( \text{calc} \)-algebra whose pullback to \( \text{Alg}_{\text{Calc}} \) is weak equivalent to \( \mathfrak{A} \). It follows that its pullback to \( \text{Alg}_{\text{Lie}^+} \) is weak equivalent to the pullback of \( \mathfrak{C} \) to \( \text{Alg}_{\text{Lie}^+} \). Notice that \( \text{Lie}^+ \to \text{Lie}^1 \) is a weak equivalence and thus the pullback functor \( \text{Alg}_{\text{Lie}^+} \to \text{Alg}_{\text{Lie}^+} \) induces an equivalence of homotopy categories. It follows that \( \iota^* \mathcal{B} \) is weak equivalent to \( \mathfrak{C} \).

The final statement follows from the model category structure of \( \text{Alg}_{\text{Lie}^+} \). Indeed, take a cofibrant replacement \( r : \mathcal{Y} \to \iota^* \mathcal{B} \) that is a trivial fibration, and choose a weak equivalence \( \mathcal{Y} \to \mathfrak{C} \). The weak equivalence \( \mathcal{Y} \to \mathfrak{C} \) is decomposed into \( \mathcal{Y} \xrightarrow{\alpha} \mathcal{Z} \xrightarrow{\beta} \mathfrak{C} \) where \( \alpha \) is a trivial cofibration and \( \beta \) is a trivial fibration. Choose \( s : \mathcal{Z} \to \iota^* \mathcal{B} \) of \( \alpha \) such that \( r = s \circ \alpha \). Then we have the desired diagram \( \iota^* \mathcal{B} \xleftarrow{s} \mathcal{Z} \xrightarrow{\beta} \mathfrak{C} \).

4.8.

Lemma 4.8. We adopt notation in Lemma 4.7. Suppose that \( \mathcal{B} = (C, M, \cdot, [-, -], i, l, \delta) \) is a \( \text{calc} \)-algebra. Then

\[
\begin{align*}
[ia, ib, ic] &= 0, \\
[ia, ib, lc] &= 0
\end{align*}
\]

for any \( a, b, c \in C \).

Proof.

\[
[ia, ib, ic] = ia[ib, ic] - (-1)^{|a||b|}ib[ia, ic] + ia[ib, ic] = ia[ib, ic] - (-1)^{|a||b|+|c|}ib[ia, ic] = ia[ib, ic] - (-1)^{|a||b|+|c|}ib[ia, ic]
\]

where we use the module structure \( i : C \otimes M \to M \) in the second equation. Therefore it will suffice to prove that \( a \cdot [b, c] - (-1)^{|a||b|+|c|} [b, c] \cdot a = 0 \). It is a direct consequence of the Koszul sign rule and the graded commutativity. Finally, \( [ia, ib, lc] = 0 \) follows from (1) and \( [ib, lc] = ia[ib, lc] \).

We associate a morphism \( l((t)) : C[1] \to \text{End}_{k[t^\pm]}(M((t))) \) of dg Lie algebras to \( \mathcal{B} = (C, M, \cdot, [-, -], i, l, \delta) \) (see Section 4.3). Let \( i((t)) : C \to \text{End}_{k[t^\pm]}(M((t))) \) be a morphism of dg algebras which carries \( c \) to \( ic : M((t)) \to M((t)) \) defined by \( ic(mt^n) = ic(m)t^n \).

Proposition 4.9. We have

\[
e^{-\frac{1}{t}i((t))} \bullet 0 = \sum_{n=0}^{\infty} \frac{[-\frac{1}{t}i((t)) - 1]^n}{(n+1)!} (-1)^n [-\frac{1}{t}i((t)), 0] + d_H(\frac{1}{t}i((t))) = l((t))
\]

in the Hom complex \( H := \text{Hom}(\overline{B}\text{com}(C[1]), \text{End}_{k[t^\pm]}(M((t)))) \). Here \( d_H \) denotes the differential of the Hom complex.

Proof. For ease of notation, we put \( I := i((t)) \) and \( L := l((t)) = L \) in this proof. Also, if \( c \in C \), we write \( I_c \) and \( L_c \) for \( ic((t)) \) and \( lc((t)) \) of degree \(-1\) and \( 0\) respectively. We first calculate \([I, 0] + d_H(\frac{1}{t}I) = d_H(\frac{1}{t}I)\). Let \( \delta \) denote the differential of \( M \). Let \( d_E(-) = [\partial + t\delta, -] \)
be the differential of $\text{End}_{k[t^\pm]}(M((t)))$. Let $d_B$ be the differential of $B_{\text{com}}(C[1])$. Then we have

$$d_H\left(\frac{1}{t}I\right) = d_E\left(\frac{1}{t}I\right) - (-1)^0|d_B|\frac{1}{t}I \circ d_B$$

$$= \frac{1}{t}[\partial, I] + [t\delta, \frac{1}{t}I] - \frac{1}{t}I \circ d_B$$

$$= L + \frac{1}{t}((\partial, I) - I \circ d_B).$$

Note that we here used $L = [\delta, I]$ and $\frac{1}{t}I$ has degree 0 in $H$. Notice also that $I$ kills all higher part $\oplus_{n \geq 1} \text{Sym}^n(C[2])$, and remember that $d_B$ is the sum $d_1 + d_2$: $d_1$ comes from that of $C[2]$, $d_2$ is generated by $Q_2: \text{Sym}^2(C[2]) \xrightarrow{(s-1)^{\otimes 2}} C[1] \wedge C[1] \xrightarrow{[-,-]} C[1] \cong C[2]$. Here $s: C[1] \to C[1][1]$ is the obvious “identity” map. (See [28, 37] for the detailed account on sign issue.) We deduce that $I \circ d_B = I_{dC(-)} - I_{[-,-]}$. Note that $i: C \to \text{End}(M)$ is a dg map. It follows that $[\partial, I] = I_{dC(-)}$. Hence

$$d_H\left(\frac{1}{t}I\right) = L + \frac{1}{t}((\partial, I) - I_{dC(-)} + I_{[-,-]}) = L + \frac{1}{t}I_{[-,-]}.$$

Therefore, $e^{-\frac{1}{t}I} \bullet 0$ equals to

$$L + \frac{1}{t}I_{[-,-]} - \frac{1}{2t}I_L^H + \text{(terms of the forms $\frac{1}{m}[I[\cdots[I, I_{[-,-]}]\cdots]]$ and $\frac{1}{4m}[I[\cdots[I, L]\cdots]]$)}$$

Here the bracket appearing in $I_{[-,-]}$ is that of $C[1]$, and the bracket $[-,-]^H$ is the convolution Lie bracket of $H$. According to Lemma 4.8 the terms of $n$-ary operations ($n \geq 3$) vanish.

Using the definition of convolution Lie bracket we have the formula

$$[I, L]_H(a \otimes b) = [I_a, L_b]_E + (-1)^{|a||b| + |a|+|b|}[I_b, L_a]_E$$

for $a \otimes b \in C[1] \wedge C[1]$ where $[-,-]_E$ is the bracket of $\text{End}_{k[t^\pm]}(M((t)))$, and $|a|, |b|$ are degrees in $C$. Combined with $[I_a, L_b]_E = I_{i(a,b)}$ coming from the structure of calculus we conclude that $2I_{[-,-]} = [I, L]_H$. Hence $e^{-\frac{1}{t}I} \bullet 0 = 0$. \hfill $\Box$

The dg Lie algebra $H = \overline{\text{Hom}(B_{\text{com}}(C[1]), \text{End}_{k[t^\pm]}(M((t))))}$ is pro-nilpotent. Indeed, put $F^nH = \{f \in H \mid \text{restriction } \oplus_{n \geq 1} \text{Sym}^i(C[2]) \to \text{End}_{k[t^\pm]}(M((t))) \text{ is zero}\}$. Each $F^nH$ is a dg ideal and $[F^nH, F^mH] \subset F^{n+m}H$. We have a natural isomorphism $H \cong \lim H/F^nH$ such that each quotient $H/F^nH$ is nilpotent. Thus the exponential action of $H^0$ on $MC(H)$ makes sense. More generally, we may consider the action $\exp(H^0[u, du])$ on $MC(H[u, du])$. The element $-\frac{1}{t}i((t))$ in $H^0[u, du]$ acts on 0 in $MC(H[u, du])$, and we obtain $e^{-\frac{1}{t}i((t))} \bullet 0$ in $MC(H[u, du])$.

Let us consider the map

$$\text{MC}(\text{Hom}(B_{\text{com}}(C[1]), E)|u, du\rangle) \to \text{MC}(\text{Hom}(B_{\text{com}}(C[1]), E)|u, du\rangle),$$

induced by the canonical map $k[u, du] \otimes \text{Hom}(B_{\text{com}}(C[1]), E) \to \text{Hom}(B_{\text{com}}(C[1]), k[u, du] \otimes E)$ of dg Lie algebras. Here $E = \text{End}_{k[t^\pm]}(M((t)))$. Let $\Phi$ be the image of $e^{-\frac{1}{t}i((t))} \bullet 0$ in $MC(\text{Hom}(B_{\text{com}}(C[1]), k[u, du] \otimes E))$. Consider two face maps

$$d_0, d_1 : \text{MC}(\text{Hom}(B_{\text{com}}(C[1]), E)|u, du\rangle) \to \text{MC}(\text{Hom}(B_{\text{com}}(C[1]), E)),$$

where $d_0$ (resp. $d_1$) is determined by $u = du = 0$ (resp. $u = 1$ and $du = 0$). By Proposition 4.9 we see that $d_0(\Phi) = 0$ and $d_1(\Phi) = i((t))$. Consequently, we have

**Proposition 4.10.** The element $\Phi$ (constructed from $-\frac{1}{t}i((t))$) gives a morphism/homotopy from 0 to $i((t))$. 
4.9. We continue to consider the \textbf{calc}-algebra \( \mathfrak{B} = (C, M, \cdot, [-,-], i, l, \delta) \) (see Lemma 4.7). Let us consider the natural morphism of dg Lie algebras

\[ j : \text{End}_{k[t]}(M[[t]]) \to \text{End}_{k[t^\pm]}(M((t))) \]

induced by the base change \( \otimes_{k[t]}k[t^\pm] \). Consider a homotopy fiber of this morphism. It is well-defined in the \( \infty \)-category or the model category of dg Lie algebras (or \( L_\infty \)-algebras). Let \( E_\geq := \text{End}_{k[t]}(M[[t]]) \) and \( E := \text{End}_{k[t^\pm]}(M((t))) \). We use the product of dg Lie algebras

\[ F_j := \{(e, f(u, du)) | f(0,0) = 0, f(1,0) = j(e) \} \subset E_\geq \times E[u, du] \]

together with the first projection to \( E_\geq \) as a model of the homotopy fiber (the differential is given by \( d(e, e') = (de, de') \)). The underlying complex of \( F_j \) is quasi-isomorphic to the standard mapping cocone \( E_\geq \oplus E[-1] \) endowed with the differential \( d(e, e') = (de, j(e) - de') \) over \( E \).

(4.10) \( \text{Lemma 4.10.} \) There is a homotopy pullback exactly when the underlying complex is a homotopy pullback at the level of complexes.) An explicit quasi-isomorphism \( \iota : E_\geq \oplus E[-1] \to F_j \) is given by the formula \( (e, e') \mapsto (e, u \otimes e + du \otimes e') \). It has a homotopy inverse \( \pi : F_j \to E_\geq \oplus E[-1] \) defined by \( (e, f(u) \otimes e'_1 + g(u)du \otimes e'_2) \mapsto (e, e'_1 \int_0 u g(u)du) \) (cf. \cite{6} Section 3).

We take the morphism/homotopy \( \Phi \), constructed in Proposition 4.10 from \( 0 : C[1] \to \text{End}_{k[t^\pm]}(M((t))) \) to \( l(t) : C[1] \to \text{End}_{k[t^\pm]}(M((t))) \). Actually, \( \Phi \) is an \( L_\infty \)-morphism \( \overline{B}_{\text{com}}(C[1]) \to E[u, du] \). It gives rise to an \( L_\infty \)-morphism \( l[[t]] \times \Phi : B_{\text{com}}(C[1]) \to F_j \subset E_\geq \times E[u, du] \) such that the diagram

\[
\begin{array}{ccc}
C[1] & \xrightarrow{l[[t]] \times \Phi} & E_\geq \\
\downarrow & & \downarrow \pi_1 \\
C[1] & \xrightarrow{l[[t]]} & E_\geq
\end{array}
\]

commutes where arrows are implicitly considered to be \( L_\infty \)-morphisms.

Remark 4.11. Observe that the morphism \( l[[t]] \times \Phi : C[1] \to F_j \) at the level of complexes without bracket can be represented by \( \frac{1}{r} i((t)) : C[1] \to (\text{End}_{k[t^\pm]}(M((t))))/ \text{End}_{k[t]}(M[[t]]))[-1] \).

In fact, if we think of \( C[1] \to E[u, du] \) as a map of complexes by forgetting the non-linear terms, and consider \( \text{Hom}(C[1], E[u, du]) \) to be the dg Lie algebra with the trivial bracket, the straightforward computation of the gauge action of \( \frac{1}{r} i((t)) \) on \( E \) shows that the underlying map \( C[1] \to E[u, du] \) is given by \( u \otimes l(t) + du \otimes \frac{1}{r} i((t)) \). The composite \( C[1] \xrightarrow{l[[t]] \times \Phi} F_j \xrightarrow{\pi} E_\geq \oplus E[-1] \) is \( l[[t]] \times \frac{1}{r} i((t)) \) (\( \pi \) is the quasi-isomorphism). Since \( E_\geq \to E \) is injective, there is a natural quasi-isomorphism \( E_\geq \oplus E[-1] \to (E/E_\geq)[-1] \) given by the second projection.

Namely, \( (E/E_\geq)[-1] \) is another model of the mapping cocone. It follows that \( l[[t]] \times \Phi \) at the level of complexes gives rise to \( C[1] \to (E/E_\geq)[-1] \) which carries \( P \) to \( \frac{1}{r} i P ((t)) \).

4.10. We now apply the construction in Section 4.9 to the \textbf{Lie}^\dagger- algebra

\[ \mathcal{C} = (C^\bullet(A), C^\bullet(A), [-,-]_G, L, B) \]

considered in Lemma 4.7. Consider the morphisms of dg Lie algebras

\[ L[[t]] : C^\bullet(A)[1] \to \text{End}_{k[t]}(C^\bullet(A)[[t]]), \quad \text{and} \quad L((t)) : C^\bullet(A)[1] \to \text{End}_{k[t^\pm]}(C^\bullet(A)((t))) \]

(see Section 3.2).

Lemma 4.12. There is a homotopy \( \Psi \) defined in \( \text{MC}(C^\bullet(A)[1], \text{End}_{k[t^\pm]}(C^\bullet(A)((t))))[u, du] \), from \( 0 \) to \( L((t)) \) that comes from \( \Phi \).

Proof. We apply Lemma 4.7 and Lemma 4.7 (or one can simply use Lemma 4.6). Namely, by Lemma 4.7 there is a diagram \( i^* \mathfrak{B} \leftarrow \mathfrak{I} \to \mathcal{C} \) of trivial fibrations of \textbf{Lie}^\dagger-algebras. Then
by Lemma 4.10 using a zig-zag of homotopy equivalences (see the proof of Lemma 4.10) we can transfer $\Phi$ in Proposition 4.10 to a 1-simplex $\Psi$ of the Kan complex

$$\text{MC}(C^\bullet(A)[1], \Omega_\ast \otimes \text{End}_{k[t^\pm]}(C_\bullet(A)((t)))).$$

$\square$

**Proof of Proposition 4.1.** Proposition 4.1 follows from Lemma 4.12 $\square$

4.11. For simplicity, we put $E_{\geq \cdot} = \text{End}_{k[t]}(C_\bullet(A)([t]))$, $E = \text{End}_{k[t^\pm]}(C_\bullet(A)(([t])))$ and $E[u, du] = k[u, du] \otimes E$. Let $\iota: E_{\geq \cdot} \to E$ be the natural injective morphism of dg Lie algebras induced by the base change $\otimes_{k[t]} k[t^\pm]$. Let $F := F_\ast \subset E_{\geq \cdot} \times E[u, du]$ be a (model of) homotopy fiber of $\iota$ defined in a similar way as $F_j$ in Section 4.9.

$$F = \{ (p, q(u, du)) | d_0(q(u, du)) = 0, d_1(q(u, du)) = \iota(p), \text{ i.e., } q(0, 0) = 0, \ q(1, 0) = \iota(p), \}. $$

Note that $F$ depends on $C_\bullet(A)([t])$, and we here abuse notation by omitting the subscript that indicates $C_\bullet(A)([t])$. As in Section 4.9, we define an $L_\infty$-morphism

$$L[[t]] \times \Psi : B_{com}(C^\bullet(A)(1)) \to F \subset E_{\geq \cdot} \times E[u, du]$$

where $\Psi$ is the $L_\infty$-morphism in Lemma 4.12. For ease of notation we write

$$P := L[[t]] \times \Psi : C^\bullet(A)(1) \longrightarrow F$$

for this $L_\infty$-morphism. In summary, we have the following commutative diagram:

$$\begin{array}{ccc}
C^\bullet(A)(1) & \xrightarrow{L[[t]]} & F \\
\downarrow{\Psi} & & \downarrow{pr_1} \\
E[u, du] & \xrightarrow{d_4} & E.
\end{array}$$

The $L_\infty$-morphism $P : C^\bullet(A)(1) \longrightarrow F$ plays the main role in this paper. In the next Section, it turns out that this $L_\infty$-morphism “amounts to” a period mapping for infinitesimal (curved) deformations of $A$ via a moduli-theoretic interpretation. Note that there is another map $d_0 : E[u, du] \to E$ which is a part of data of the path object $E \hookrightarrow E[u, du]$ and the composite $C^\bullet(A)(1) \to E[u, du] \xrightarrow{d_0} \ E \times \ E$, and the zero morphism.

4.12. The dg Lie algebra $F$ is a homotopy fiber of $\iota : E_{\geq \cdot} \to E$, and $E_{\geq \cdot}$ and $E$ “represent” the deformations of negative and periodic cyclic complexes respectively (cf. Section 3.3). The purpose of Section 4.12 is to explain a modular interpretation of $\text{Spf}_F$ associated to the homotopy fiber $F$. It turns out that $\text{Spf}_F$ is a generalization of formal Sato Grassmannian to the level of complexes.

**Definition 4.13.** Let $R$ be an artin local $k$-algebra with residue field $R/mR \simeq k$, that is, $R$ belongs to $\text{Art}_k$. Let $Z$ be a dg $k[t]$-module. Let $\tilde{Z}$ be a deformation of $Z$ to $R$, that is, a dg $R[t]$-module $\tilde{Z}$ such that the underlying graded $R[t]$-module of $\tilde{Z}$ is $Z \otimes_{k[t]} R[t]$, and its reduction $\tilde{Z} \otimes R[t] \otimes_{k[t]} R[t]$ is the dg $k[t]$-module $Z$ (namely, the differential on $\tilde{Z}$ induces that of $Z$ via the canonical identification $\tilde{Z} \otimes_{k[t]} R[t] \otimes_{k[t]} R[t] \simeq R$ of graded complexes, see Section 3.5). A periodically trivialized deformation of $\tilde{Z}$ to $R$ is a pair

$$ (\tilde{Z}, \phi : Z \otimes_{k[t]} R[t^\pm] \overset{\sim}{\to} \tilde{Z} \otimes_{R[t]} R[t^\pm]) $$
where $\tilde{Z}$ is a deformation of $Z$, and $Z \otimes_{k[t]} R[\pm]^t$ denotes the trivial deformation of the dg $k[\pm]$-module $Z \otimes_{k[t]} k[\pm]$, and $\phi$ is an isomorphism of deformations of dg $k[\pm]$-modules to $R$.

Suppose that we are given two periodically trivialized deformations $(Z_1, \phi_1)$ and $(Z_2, \phi_2)$. An isomorphism $(Z_1, \phi_1) \to (Z_2, \phi_2)$ is an isomorphism $h : Z_1 \to Z_2$ of deformations such that there exists some $a$ in $(\text{End}_{k[\pm]}(Z \otimes_{k[t]} k[\pm])) \otimes m_R)^{-1}$ such that the diagram

$$
\begin{array}{ccc}
Z \otimes_{k[t]} R[\pm]^t & \xrightarrow{e^a} & Z \otimes_{k[t]} R[\pm]^t \\
\downarrow \phi_1 & & \downarrow \phi_2 \\
Z_1 \otimes R[t] R[\pm]^t & \xrightarrow{0} & Z_2 \otimes R[t] R[\pm]^t
\end{array}
$$

commutes where $d$ is the differential of $\text{End}_{k[\pm]}(Z \otimes_{k[t]} k[\pm]) \otimes m_R$.

**Remark 4.14.** There is a more conceptual description of the commutativity of the square in Definition 4.13 (however, we will not need it in this paper). Let $V$ be a deformation of the dg $k[\pm]$-module $Z \otimes_{k[t]} k[\pm]$ to $R \in \text{Art}_k$. We denote by $Z \otimes_{k[t]} R[\pm]^t$ the trivial deformation as above. Let $\psi_0 : Z \otimes_{k[t]} R[\pm]^t \to V$ and $\psi_1 : Z \otimes_{k[t]} R[\pm]^t \to V$ be isomorphisms of deformations. A homotopy from $\psi_0$ to $\psi_1$ is defined to be a $k[u, du] \otimes R[\pm]$-linear isomorphism $h : k[u, du] \otimes (Z \otimes_{k[t]} R[\pm]) \to k[u, du] \otimes R[\pm] V$ such that (i) its reduction $k[u, du] \otimes Z \otimes_{k[t]} k[\pm] \to k[u, du] \otimes Z \otimes_{k[t]} k[\pm]$ is the identity, (ii) the reduction by $u = du = 0$ is $\psi_0$, and (iii) the reduction by $u = 1, du = 0$, is $\psi_1$. Put $\psi_0 = e^f$ and $\psi_1 = e^g$ where $f, g : Z \otimes_{k[t]} k[\pm] \to Z \otimes_{k[t]} k[\pm]$ is cofibrant with respect to the projective model structure. Similarly, we have a Kan complex $\text{Hom}_{\text{End}_{k[\pm]}(Z \otimes_{k[t]} k[\pm]) \otimes m_R)(l^0(u) + du \otimes l^{-1}(u) = l(u, du) = (-k[u, du] \otimes f) \cdot n(u, du)$ (the multiplication “$\cdot$” is given by the Baker-Campbell-Hausdorff product). Then $e^{l^0(u)} \cdot 0 = 0$ corresponds to $dl(u, du) = 0$ where $d$ is the differential. It is equivalent to simultaneous equations

$$
dl^0(u) = 0 \quad \text{and} \quad dl^0(u) = dl^{-1}(u)
$$

where $l^0(u)'$ denotes the formal derivative of the polynomial. Note also that $l(0, 0) = (-f) \cdot f = 0$. It follows that there is an element $a$ of degree $-1$ in $\text{End}_{k[\pm]}(Z \otimes_{k[t]} k[\pm]) \otimes m_R$ such that $l(1, 0) = da$. Thus, $e^{-f} \circ e^g = e^a$. To see the “if” direction, suppose that we have $\psi_1 = \psi_0 \circ e^{da}$. If we put $n(u, du) = e^f \circ e^{u \cdot da + da \sigma a}$, then $n(u, du)$ gives a homotopy. Consequently, we see that the commutativity of the square in Definition 4.13 amounts to the homotopy between $h \otimes R[t] R[\pm] \circ \phi_1$ and $\phi_2$.

Finally, let us consider homotopies from the viewpoint of the space of morphisms of deformations. Let $\text{Hom}_{\text{End}_{k[\pm]}(Z \otimes_{k[t]} k[\pm]) \otimes m_R)}(\Omega_n \otimes Z \otimes_{k[t]} k[\pm]), \Omega_n \otimes Z \otimes_{k[t]} k[\pm])$ be the hom set of morphisms of dg $\Omega_n \otimes k[\pm]$-modules. These sets form a simplicial set $\text{Hom}_{\text{End}_{k[\pm]}(Z \otimes_{k[t]} k[\pm]) \otimes m_R)}(\Omega_n \otimes Z \otimes_{k[t]} k[\pm]), \Omega_n \otimes Z \otimes_{k[t]} k[\pm])$, which is a Kan complex of the mapping space because $Z \otimes_{k[t]} k[\pm]$ is cofibrant with respect to the projective model structure. Similarly, we have a Kan complex $\text{Hom}_{\text{End}_{k[\pm]}(Z \otimes_{k[t]} k[\pm]) \otimes m_R)}(\Omega_n \otimes Z \otimes_{k[t]} k[\pm], \Omega_n \otimes Z \otimes_{k[t]} k[\pm]), \Omega_n \otimes Z \otimes_{k[t]} k[\pm])$. It is a Kan fibration since $V \to Z \otimes_{k[t]} k[\pm]$ is a surjective morphism of dg $R[\pm]$-modules (i.e., a fibration), and the (projective) model category of dg $R[\pm]$-modules is simplicial in the sense
of \([11, 1.4.2]\). The fiber \(F\) of this Kan fibration over the constant simplicial set determined by the identity map of \(Z \otimes_{k[t]} k[t^\pm]\) is a homotopy fiber which is a Kan complex. This Kan complex should be regarded as the \(\infty\)-groupoid of morphisms of deformations from \(Z \otimes_{k[t]} R[t^\pm]^\text{tr}\) to \(V\). An edge in this homotopy fiber \(F\) is a homotopy \(h : k[u, du] \otimes_k (Z \otimes_{k[t]} R[t^\pm]^\text{tr}) \to k[u, du] \otimes_k V\) defined above.

We apply Definition 4.13 to the dg \(k[t]\)-module \(C_* (A)[[t]]\) of negative cyclic complex. We shall denote by \((C_* (A) \otimes R)((t))\)^{tr} the trivial deformation \(((C_* (A) \otimes R)((t)), (\partial_{\text{Hoch}} + tB) \otimes R)\) of the dg \(k[t^\pm]\)-module \(C_* (A)((t))\) (see Section 3.5). Let

\[
(Q = ((C_* (A) \otimes R)[[t]]), \tilde{\partial}), \phi : (C_* (A) \otimes R)((t))^{\text{tr}} \xrightarrow{\sim} Q \otimes_{R[[t]]} R[t^\pm])
\]

be a pair where \(Q\) is a deformation of the dg \(k[t]\)-module \(C_* (A)[[t]]\) to \(R\), and \(\phi\) is an isomorphism of deformations of the dg \(k[t^\pm]\)-module \(C_* (A)((t))\). Let \(\text{Gr}(R)\) be the set of isomorphism classes of periodically trivialized deformations of \(C_* (A)[[t]]\) to \(R\). The assignment \(R \mapsto \text{Gr}(R)\) gives rise to a functor

\[
\text{Gr} : \text{Art}_k \to \text{Sets}.
\]

Again, \(\text{Gr}\) depends on \(C_* (A)[[t]]\) though we omit the subscript indicating \(C_* (A)[[t]]\).

The following is the moduli-theoretic presentation of \(\text{Spf}_F\) in explicit terms:

**Proposition 4.15.** There is a natural equivalence \(\text{Spf}_F \to \text{Gr}\) of functors. This equivalence carries an element \(c \in \text{Spf}_F(R)\) represented by a Maurer-Cartan element \((\alpha, \beta)\) in \(\text{MC} (\mathcal{F} \otimes m_R) \subset \text{MC} (E_\geq \otimes m_R) \times \text{MC} (E[u, du] \otimes m_R)\) to the class of a periodically trivialized deformation of the form

\[
(Q_\alpha, (C_* (A) \otimes R)((t))^{\text{tr}} \xrightarrow{\sim} Q_\alpha \otimes_{R[[t]]} R[t^\pm]).
\]

Here \(Q_\alpha\) denotes the element in \(D_{C_* (A)[[t]]} (R)\) which corresponds to \(\alpha\) (see Proposition 4.14).

**Proof of Proposition 4.15.** By \([6]\), there exist an \(L_\infty\)-structure on the mapping cone \(C := E_\geq \oplus E[-1]\) of \(\iota : E_\geq \to E\) and an \(L_\infty\)-quasi-isomorphism \(C \to F\). (In this paper, an \(L_\infty\)-structure on a graded vector space \(V\) is defined to be a unital graded cocommutative coalgebra \(B_{\text{com}} V\) endowed with a square-zero coderivation \(b : B_{\text{com}} V \to B_{\text{com}} V\), see Section 4.6 for the notation. The coderivation \(b : B_{\text{com}} V \to B_{\text{com}} V\) amounts to \(\{b_n : \text{Sym}^n (V[1]) \to V[1]\}_{n \geq 1}\) satisfying certain identities.) Moreover, according to \([6\, \text{Theorem 2}]\) the set of Maurer-Cartan elements is given by

\[
\text{MC} (C \otimes m_R) = \{(\alpha, a) \in \text{MC} (E_\geq \otimes m_R) \times \text{MC} (E^0 \otimes m_R) \mid e^{-a} \cdot \iota = 0\}.
\]

A Maurer-Cartan element \((\alpha, a)\) is gauge equivalent to another element \((\beta, b)\) if and only if there exist \(m \in E^0 \otimes m_R\) and \(q \in E^{-1} \otimes m_R\) such that \(e^m \cdot \alpha = \beta\) and \(b = dq \cdot a \cdot (-\iota(m))\) where in the last formula we use Baker-Campbell-Hausdorff product. Notice that there is a natural equivalence between \(\text{Spf}_C\) and \(\text{Gr}\). In fact, there is a natural bijective map \(\text{Spf}_C (R) \to \text{Gr}(R)\) which carries the class represented by \((\alpha, a)\) to the class represented by

\[
(Q_\alpha, e^{-a} : (C_* (A) \otimes R)((t))^{\text{tr}} \xrightarrow{\sim} Q_\alpha \otimes_{R[[t]]} R[t^\pm]).
\]

By the invariance of \(\text{Spf}\) with respect to \(L_\infty\)-quasi-isomorphisms, the \(L_\infty\)-quasi-isomorphism \(C \to F\) induces an equivalence \(\text{Spf}_C \to \text{Spf}_F\) of functors. In addition, the composition \(\text{Spf}_C \to \text{Spf}_F \to \text{Spf}_{E_\geq}\) sends the class represented by \((\alpha, a)\) to the class of \(\alpha\) (cf. \([6\, \text{Remark 5.3}]\)). Choose an \(L_\infty\)-quasi-isomorphism \(F \to C\) which is a homotopy inverse of \(C \to F\). Then \(F \to C\) induces the inverse \(\text{Spf}_C \to \text{Spf}_F\) of \(\text{Spf}_C \to \text{Spf}_F\). We have the equivalence \(\text{Spf}_F \to \text{Spf}_C \simeq \text{Gr}\), as desired. \(\square\)

We should think of the functor \(\text{Spf}_F \simeq \text{Gr}\) as a formal neighborhood of a point on a generalized Sato Grassmannian. To understand it, we begin by reviewing the Sato Grassmannian. Let \(V\) be a finite dimensional \(k\)-vector space. Let \(R\) be a commutative \(k\)-algebra. Consider a pair
(W, f : (V ⊗ R)((u)) ≃ W ⊗_{R[[u]]} R((u))) such that W is a finitely generated projective \( R[[u]] \)-module, and \( f \) is an isomorphism of \( (u) \)-modules. Here \( R[[u]] \) is the ring of formal power series, and \( R((u)) \) is the ring of formal Laurent series. An isomorphism \((W, f) \to (W', f')\) of such pairs is defined to be an isomorphism \( W \to W' \) of \( R[[u]] \)-modules that commutes with \( f \) and \( f' \). Let \( SGr(R) \) be the set of isomorphism classes of pairs. If \( CAlg_k \) denotes the category of commutative \( k \)-algebras, the Sato Grassmannian as a functor is the functor \( CAlg_k \to \text{Sets} \) which assigns to each \( R \) the set \( SGr(R) \). Moreover, it is well-known that this functor \( SGr \) is represented by the colimit of the sequence of closed immersions of schemes \( \lim_{n \in \mathbb{N}} X_n \). Fix a pair \((W, f : V((u)) \simeq W \otimes_{k[[u]]} k((u))) \) in \( SGr(k) \). Let \( R \) be in \( Art_k \). A deformation of the pair \((W, f)\) is a pair \((\tilde{W}, \phi : (V \otimes R)((u)) \simeq \tilde{W} \otimes_{R[[u]]} R((u)))\) such that \( \tilde{W} \) is a deformation of \( W \), that is, a finitely generated projective \( R[[u]] \)-module endowed with \( W \otimes_{R[[u]]} k[[u]] \simeq W \), and \( \phi : (V \otimes R)((u)) \simeq \tilde{W} \otimes_{R[[u]]} R((u)) \) is an isomorphism of \( R((u)) \)-modules whose reduction to \( k((u)) \) is \( f \). An isomorphism of deformations is defined in the obvious way. Then consider the functor \( SGr_{(W,f)} : Art_k \to \text{Sets} \) which carries \( R \) to the set of isomorphism classes of deformations of \((W, f)\). Informally, this functor \( Art_k \to \text{Sets} \) can be viewed as a formal neighborhood (i.e., a formal scheme) at a point \((W, f)\) on \( SGr \) (the proper interpretation is left to the interested reader). Now we generalize it to the level of complexes. We replace \( k[[u]] \) by the dg algebra \( k[t] \) where the cohomological degree of \( t \) is 2. Note that \( k[t] \) is \( t \)-adically complete in the sense that \( k[t] \) is a homotopy limit of the sequence \( \cdots \to k[t]/(t^{n+1}) \to \cdots \to k[t]/(t^2) \to k[t]/(t) \). Consider the \( k[t] \)-module \( C_*(A)[[t]] \), and the natural isomorphism \( C_*(A)((t)) \overset{\sim}{\to} C_*(A)[[t]] \otimes_{k[t]} k[[t]] \) of \( k[[t]] \)-modules instead of \((W, f)\). It is natural to think of \( Gr \simeq \text{Spf} \) as a “complicial generalization” of \( SGr_{(W,f)} \), and regard the dg Lie algebra \( \mathcal{F} \) as the Lie-theoretic presentation (\( \mathcal{F} \) also has the “derived structure”). Thus, we call \( Gr \simeq \text{Spf} \) the formal complicial Sato Grassmannian or the complicial Sato Grassmannian simply.

**Remark 4.16.** It is natural to ask for a global complicial Sato Grassmannian. It might be realized as a geometric object in derived algebraic geometry developed by Toën-Vezzosi and Lurie, whose \( R \)-valued points informally parametrize the space of pairs \((W, (C_*(A) \otimes_R R)((t)) \simeq W \otimes_{R[t]} R[[t]] \) consisting of a compact \( R[t] \)-module \( W \) and an equivalence of \( R[[t]] \)-modules.

### 4.13. Moduli-theoretic construction of \( CAlg_A \to Gr \) of functors.

The construction is based on the moduli-theoretic interpretation of dg Lie algebras and their morphisms. Let \( \mathcal{A} \) be a deformation of the dg algebra \( A \) to \( R \). It gives rise to the negative cyclic complex \( C_*(\mathcal{A})[[t]] \) that is a deformation of the dg \( k[[t]] \)-module \( C_*(A)[[t]] \). Namely, we associate to \( \mathcal{A} \) a deformation \( C_*(\mathcal{A})[[t]] \) of the dg \( k[[t]] \)-module \( C_*(A)[[t]] \) to \( R \). Let \( \alpha \) be the Maurer-Cartan element in \( MC(C_*(A)[1] \otimes m_R) \) that corresponds to \( \mathcal{A} \) (see Claim 3.8.1). By Proposition 3.10, the morphism of dg Lie algebras \( L[[t]] : C_*(A)[1] \to \text{End}_{k[[t]]}(C_*(A)[[t]]) \) sends \( \alpha \) to the class represented by \( L[[t]](\alpha) \) in \( \text{Spf(End}_{k[[t]]}(C_*(A)[[t]])(R) \) that corresponds to the isomorphism class of the deformation \( C_*(A)[[t]] \) in \( D_{C_*(A)[[t]]}(R) \) (see also Proposition 3.9). Since \( L[[t]] \) has the lift \( \mathcal{P} : C_*(A)[1] \to \mathcal{F} \), the element \( \text{Spf}(\mathcal{P}(\alpha)) \) lying in \( Gr(R) \) through the isomorphism \( \text{Spf}(R) \simeq Gr(R) \) in Proposition 4.15 is represented by a periodically trivialized deformation of the form

\[
((C_*(\mathcal{A})[[t]], (C_*(A) \otimes_R R)((t))^{tr} \sim C_*(\mathcal{A})((t))).
\]

We refer to it as (the isomorphism class of) the periodically trivialized deformation associated to \( \mathcal{A} \). We obtain a morphism

\[
P(R) : DAlg_A(R) \to Gr(R), \quad \mathcal{A} \mapsto (C_*(\mathcal{A})[[t]], (C_*(A) \otimes_R R)((t))^{tr} \sim C_*(\mathcal{A})((t))).
\]

We refer to this morphism of functors as the period mapping for \( A \).
Unwinding our construction based on results about modular interpretations of morphisms of dg Lie algebras we can conclude:

**Theorem 4.17.** Through the identifications $\text{DAlg}_A \simeq \text{Spf}_{C^\bullet(A)[1]}$ and $\text{Gr} \simeq \text{Spf}_F$, the morphism $P : \text{DAlg}_A \to \text{Gr}$ can be identified with $\text{Spf}_P : \text{Spf}_{C^\bullet(A)[1]} \to \text{Spf}_F$.

We shall call both morphisms $P : \text{DAlg}_A \to \text{Gr}$ and $\text{Spf}_P : \text{Spf}_{C^\bullet(A)[1]} \to \text{Spf}_F$ the period mapping for $A$.

**Remark 4.18.** In Introduction, for simplicity, we do not distinguish $P : \text{DAlg}_A \to \text{Gr}$ from $\text{Spf}_P : \text{Spf}_{C^\bullet(A)[1]} \to \text{Spf}_F$.

5. **Hodge-to-de Rham spectral sequence and complicial Sato Grassmannians**

In the last section, we constructed a period mapping from the dg Lie algebra representing deformations of an algebra to the dg Lie algebra of the complicial Sato Grassmannian. In good cases, it turns out that the complicial Sato Grassmanian admits a simple and nice structure. Our main interest lies in the situation where a non-commutative analogue of Hodge-to-de Rham spectral sequence for cyclic homology theories degenerates. Hence we start with a spectral sequence for cyclic homology theories.

5.1. Let $A$ be a dg algebra over a field $k$ of characteristic zero. We write $C_\bullet := C_\bullet(A)$, $C_\bullet[t] := (C_\bullet(A)[t], \partial_{\text{Hoch}} + tB)$ and $C_\bullet((t)) := (C_\bullet(A)((t)), \partial_{\text{Hoch}} + tB)$ for the Hochschild chain complex, the negative cyclic complex and the periodic cyclic complex respectively (see Section 2.4). (Note that the degree of $t$ is two.) We let $C_\bullet[t^{-1}] := (C_\bullet(A)[t^{-1}], \partial_{\text{Hoch}} + tB)$ be the cyclic complex of $A$. There is an exact sequence of complexes

$$0 \to C_\bullet[t] \to C_\bullet((t)) \to C_\bullet[t^{-1}] \cdot t^{-1} \to 0,$$

whose associated long exact sequence

$$\cdots \to HN_n(A) \to HP_n(A) \to HC_{n-2}(A) \to \cdots$$

relates the negative and periodic cyclic homology with the cyclic homology $HC_\bullet(A) = H_*(C_\bullet[t^{-1}])$.

We define a decreasing filtration of $C_\bullet((t))$ by the formula

$$F^lC_\bullet((t)) \cap C_\bullet((t)) = \prod_{r \geq l} C_{n+2r}(A) \cdot t^r \subset C_\bullet((t)) \cap C_{n+2r}(A) \cdot t^r$$

where $C_l(A)$ is the homologically $l$-th term of $C_\bullet(A)$. This filtration forms a family of subcomplexes of $C_\bullet((t))$. Note that $F^lC_\bullet((t))/F^{l+1}C_\bullet((t))$ is isomorphic to $C_\bullet \cdot t^l$. This filtration gives rise to a spectral sequence which we denote by

$$HH_\bullet(A)((t)) \Rightarrow HP_\bullet(A).$$

Each term in the $E_1$-stage is of the form $HH_j(A) \cdot t^i$. The restriction $F^lC_\bullet[[t]] = C_\bullet[[t]] \cap F^lC_\bullet((t))$ also yields a spectral sequence $HH_*A[[t]] \Rightarrow HN_\bullet(A)$. Since $C_\bullet[t^{-1}] \simeq C_\bullet((t))/t \cdot C_\bullet[[t]]$, the filtration $F^lC_\bullet((t))/t \cdot C_\bullet[[t]]$ gives rise to a spectral sequence $HH_\bullet(A)[t^{-1}] \Rightarrow HC_\bullet(A) = H_*(C_\bullet[t^{-1}])$. We call these spectral sequences the Hodge-to-de Rham spectral sequences. We refer the reader to [18] for the relation to the classical Hodge-to-de Rham spectral sequence (see also Remark 5.5). We now recall some properties of dg algebras.

**Definition 5.1.** Let $A$ be a dg algebra over $k$

1. $A$ is proper if the underlying complex has finite dimensional cohomology, and $H^n(A) = 0$ for $|n| >> 0$.
2. $A$ is smooth if $A$ is a compact object in the triangulated category of dg $A \otimes A^{op}$-modules.
Remark 5.2. Put another way, the condition (2) in Definition 5.1 is equivalent to the condition that \( A \) belongs to the smallest triangulated subcategory which includes free \( A \otimes A^{op} \)-modules of finite rank and is closed under direct summands.

Remark 5.3. If \( A \) is smooth and proper, it is easy to see that \( HH_n(A) \) is finite dimensional for \( n \in \mathbb{Z} \), and \( HH_n(A) = 0 \) for \( |n| >> 0 \). (Keep in mind that \( HH_n(A) \) is \( \text{Tor}_n^{A \otimes A^{op}}(A, A) \).

5.2. In the rest of this section, we assume that the dg algebra \( A \) is smooth and proper. In the smooth proper case, a conjecture of Kontsevich and Soibelman predicts that the Hodge-to-de Rham spectral sequences degenerate at \( E_1 \)-stage [25]. On the basis of his ingenious generalization of the Deligne-Illusie approach by positive characteristic methods to a noncommutative setting, Kaledin proved this degeneration conjecture in [18] under some technical condition, and in [19] without any assumption:

Theorem 5.4 ([17, 18, 19]). (Suppose that the dg algebra \( A \) is smooth and proper.) Then the Hodge-to-de Rham spectral sequences

\[
HH_*((t)) \Rightarrow HP_*(A), \quad HH_*[t^{-1}] \Rightarrow HC_*(A)
\]
degenerate at \( E_1 \)-stage.

The filtration on \( HP_n(A) \) is defined by \( F^i HP_n(A) = \text{Image}(H_n(F^i C_•((t)))) \rightarrow H_n(C_•((t))) \). Filtrations \( F^i HN_n(A) \) and \( F^i HC_n(A) \) are defined in a similar way. By the degeneration, identifying \( \text{Gr}^i_F HP_n(A) \) with \( HH_{n+i}(A) \cdot t^i \) we obtain a non-canonical isomorphism of vector spaces

\[
\bigoplus_i HH_{n+2i}(A) \cdot t^i \cong HP_n(A).
\]

We here forget the degree of \( t^i \), and hope that no confusion is likely to arise. Likewise, we have a non-canonical isomorphism \( \bigoplus_{i \leq 0} HH_{n+2i}(A) \cdot t^i \cong HC_n(A) \). By the long exact sequence

\[
\cdots \rightarrow HN_n(A) \rightarrow HP_n(A) \rightarrow HC_{n-2}(A) \rightarrow \cdots
\]
and reason of dimension, we also have \( \bigoplus_{i \geq 0} HH_{n+2i}(A) \cdot t^i \cong HN_n(A) \) where we identify \( HH_{n+2}(A) \cdot t^i \) with \( \text{Gr}^i_F HH_n(A) \) (i.e., for dimension reasons, the spectral sequence for \( HN_*(A) \) degenerates). The filtration \( \{ F^i HP_n(A) \} \) may be considered as a noncommutative analogue of Hodge filtration on the de Rham cohomology of a smooth projective variety (Remark 5.5). The 0-th part \( F^0 HP_n(A) \) can be identified with the image of \( HN_*(A) \hookrightarrow HP_*(A) \).

Remark 5.5. To illustrate the analogy with the classical situation, let us recall the comparison results. For simplicity, \( X \) is a smooth projective variety over the complex number field, though the results hold more generally. By Hochschild-Kostant-Rosenberg theorem, there is an isomorphism

\[
HN_n(X) \cong \bigoplus_{i \in \mathbb{Z}} \mathbb{H}^{2i-n}(X, \Omega^n_X), \quad \text{and} \quad HP_n(X) \cong \bigoplus_{i \in \mathbb{Z}} H^{2i-n}_{dR}(X) = \bigoplus_{i \in \mathbb{Z}} \mathbb{H}^{2i-n}(X, \Omega^n_X).
\]

where \( \Omega^n_X \) is the algebraic de Rham complex, \( \mathbb{H} \) indicates the hypercohomology, and \( NH_n(X) \) and \( HP_n(X) \) are the negative cyclic homology and the periodic cyclic homology of \( X \) respectively. The classical Hodge theory implies that \( \bigoplus_{i \in \mathbb{Z}} \mathbb{H}^{2i-n}(X, \Omega^n_X) \) is equal to \( \bigoplus_{i \in \mathbb{Z}} F^i H^{2i-n}_{dR}(X) \) where \( F^i \) is the Hodge filtration.

5.3. Let us recall the smoothness of functors.

Definition 5.6. Let \( X, Y : \text{Art}_k \rightarrow \text{Sets} \) be functors. Let \( F : X \rightarrow Y \) be a morphism. We say that \( F \) is formally smooth if for any surjective morphism \( R' \rightarrow R \) in \( \text{Art}_k \), the following natural map is surjective:

\[
X(R') \rightarrow X(R) \times_{Y(R)} Y(R')
\]
induced by the morphisms \( X(R') \rightarrow X(R) \) and \( X(R') \rightarrow Y(R') \).
Let $*$ be the functor $\text{Art}_k \to \text{Sets}$ such that $*(R)$ is the set consisting of a single element for any $R$. A functor $X$ is said to be formally smooth if the natural morphism $X \to *$ is formally smooth.

**Theorem 5.7.** The functor $\text{Gr}$ is formally smooth.

Let $HH_*(A)((t))$ be the dg $k[t^\pm]$-module with zero differential whose term of cohomological degree $-n$ is the vector space $\oplus_{i \leq 0} HH_{n+2i}(A) \cdot t^i$ (we abuse notation by forgetting the degree of $t^i$). Let $HH_*(A)[[t]]$ be the dg $k[t]$-module with zero differential whose term of cohomological degree $-n$ is $\oplus_{i \geq 0} HH_{n+2i}(A) \cdot t^i$. In other words, $HH_*(A)((t)) = HH_*(A) \otimes_k k[t^\pm]$ and $HH_*(A)[[t]] = HH_*(A) \otimes_k k[t]$ since $HH_n(A) = 0$ for $|n| > 0$.

We need some technical Lemmata.

**Lemma 5.8.** There is an injective quasi-isomorphism of dg $k[t^\pm]$-modules

$$HH_*(A)((t)) \to C_*(((t))).$$

Similarly, there is an injective quasi-isomorphism of dg $k[t]$-modules

$$HH_*(A)[[t]] \to C_*[t].$$

Moreover, one can choose a quasi-isomorphism in such a way that $HH_*(A)((t)) \to C_*(((t)))$ is obtained from $HH_*(A)[[t]] \to C_*[t]$ by tensoring with $k[t^\pm]$.

**Proof.** We construct a quasi-isomorphism $HH_*(A)((t)) \to C_*(((t)))$. Suppose that $HH_m(A) = 0$ for $m > N$, and $HH_N(A) \neq 0$ (if $HH_0(A) = 0$, the assertion is obvious). We will construct an injective linear map $\oplus_{r \in \mathbb{Z}} HH_{2r}(A) \to \prod_{r} C_{2r}(A) \cdot t^r = C_*(((t)))$. Let $h$ be the integer such that $2h \leq N$. Choose a section

$$q_h : HH_{2h}(A) \cdot t^h \simeq H_0(F^h C_*(((t)))) \to \prod_{r \geq h} C_{2r}(A) \cdot t^r = F^h C_*(((t))).$$

where (the existence of) the first isomorphism follows from the degeneration and the vanishing of the higher term of Hochschild homology. Next consider the long exact sequence

$$\cdots \to HH_{2h+1}(A) \cdot t^{h+1} \simeq H_0(F^h C_*(((t)))) \rightarrow H_0(F^{h+1} C_*(((t)))) \rightarrow H_0(F^{h+1}/F^h) \simeq HH_{2h+2}(A) \cdot t^{h+1} \to \cdots$$

arising from $0 \to F^h C_*(((t))) \to F^{h+1} C_*(((t))) \to F^{h+1}/F^h \to 0$ (in fact, it is a short exact sequence by degeneration). Choose sections $HH_{2h+2}(A) \cdot t^{h+1} \rightarrow H_0(F^h C_*(((t))))$ and $H_0(F^{h+1} C_*(((t)))) \to F^{h+1} C_*(((t)))_0$ that extends $H_0(F^h C_*(((t)))) \to F^h C_*(((t)))_0$. Let $q_{h-1} : HH_{2h-2}(A) \cdot t^{h-1} \to F^{h-1} C_*(((t)))$ be the composite. Then we have

$$q_h \oplus q_{h-1} : HH_{2h}(A) \cdot t^h \oplus HH_{2h-2}(A) \cdot t^{h-1} \to F^{h-1} C_*(((t))).$$

By the construction it is injective. We repeat this procedure to obtain an injective linear map

$$Q_0 := \bigoplus_{h \geq 0} q_h : H_0(HH_{2h}(A) \cdot t^h) \to C_*(((t))).$$

Moreover, the image of $Q_0$ is contained in the kernel of the differential of $C_*(((t)))$, and the image of each $q_h$ is contained in $F^h C_*(((t)))$. For any $z \in \mathbb{Z}$, we define

$$Q_{2z} = Q_0 \cdot t^{-z} : H_0(HH_{2i}(A) \cdot t^i) \to C_*(((t)))_{2z} = t^{-z} \cdot C_*(((t))).$$

Consider odd degrees. As in the case of degree 0, we construct an injective linear map

$$P_1 = \bigoplus_{N \geq 2i+1} p_i : H_0(HH_{2i+1}(A) \cdot t^i) \to C_*(((t)))_1$$

such that the image of $P_1$ is contained in the kernel of the differential of $C_*(((t)))$, and the image of each $p_i : H_0(HH_{2i+1}(A) \cdot t^i) \to C_*(((t)))_1$ is contained in $F^i C_*(((t)))$. For an arbitrary odd degree $2z+1$ we put

$$P_{2z+1} = P_1 \cdot t^{-z} : \bigoplus_{N \geq 2i+1} HH_{2i+1}(A) \cdot t^{i-z} \to C_*(((t)))_{2z+1} = t^{-z} \cdot C_*(((t)))_1.$$
Since $\oplus_i HH_{n+2i}(A) \cdot i \cong HP_n(A)$, these linear maps $\{Q_{2z}, P_{2z+1}\}_{z \in \mathbb{Z}}$ define an injective quasi-isomorphism $HH_*(A)([t]) \to C_*(t))$. Note also that it is a dg $k[t^\pm]$-module map. The case of $C_*(t)$ is similar. We define $Q_{2z} : \oplus_{0 \leq i \leq z} HH_{2i}(A) \cdot t^{i-z} \to C_*(t)|_{2z}$ and $P_{2z+1} : \oplus_{i \leq z} HH_{2i+1}(A) \cdot t^{i-z} \to C_*(t)|_{2z+1}$ to be the restrictions of $Q_{2z}$ and $P_{2z+1}$. We then obtain an injective quasi-isomorphism $HH_*(A)|[t] \to C_*(t)$ which has the desired compatibility.

Lemma 5.9. There exist a dg Lie algebra $E$ and quasi-isomorphisms of dg Lie algebras

$$\text{End}_{k[t^\pm]}(C_*(t)) \leftarrow E \to \text{End}_{k[t^\pm]}(HH_*(A)(t)),$$

If we replace $(t)$ by $[[t]]$, then the same assertion holds. Namely, there exist a dg Lie algebra $E_\geq$ and quasi-isomorphisms of dg Lie algebras

$$\text{End}_{k[[t]]}(C_*(t)) \leftarrow E_\geq \to \text{End}_{k[[t]]}(HH_*(A)([[t]])).$$

Moreover, there is a morphism of dg Lie algebras $E_\geq \to E$ such that the diagram

$$\text{End}_{k[[t]]}(C_*(t)) \quad \text{E} \quad \text{End}_{k[[t]]}(HH_*(A)([[t]]))$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\text{End}_{k[t^\pm]}(C_*(t)) \quad E \quad \text{End}_{k[t^\pm]}(HH_*(A)(t))$$

commutes where the right and left vertical arrows are induced by tensoring with $k[t^\pm]$.

Proof. Let us define $E$ to be the pullback in the diagram of complexes

$$E \quad \text{End}_{k[t^\pm]}(HH_*(A)(t))$$

$$\downarrow \quad \downarrow$$

$$\text{End}_{k[t^\pm]}(C_*(t)) \quad \text{Hom}_{k[t^\pm]}(HH_*(A)(t), C_*(t))$$

where the lower horizontal map and the right vertical map are induced by the composition with the quasi-isomorphism $i : HH_*(A)(t)) \cong C_*(t))$ in Lemma 5.9. Note that $E$ is a dg Lie subalgebra of $\text{End}_{k[t^\pm]}(C_*(t))$ which consists of linear maps preserving $HH_*(A)(t))$. The upper horizontal arrow and the left vertical arrow are morphisms of dg Lie algebras. Since $HH_*(A)(t))$ is cofibrant, the right vertical arrow is a quasi-isomorphism. (We here employ the complicial model category with $h$-model structure on the category of dg $k[t^\pm]$-modules (see [1] Theorem 3.5)), where a morphism is a $h$-weak equivalence if the map of dg $k[t^\pm]$-modules is a homotopy equivalence. Every object is $h$-cofibrant and $h$-fibrant.) Observe that the injective map $i : HH_*(A)(t)) \cong C_*(t))$ is a trivial cofibration. It follows that the lower horizontal arrow is a trivial fibration, and thus the left vertical arrow and the upper horizontal arrow are quasi-isomorphisms. To see that $i$ is a trivial cofibration, by [1] Proposition 3.7 it is enough to show that $HH_*(A)(t)) \cong C_*(t))$ is isomorphic to $HH_*(A)(t)) \to HH_*(A)(t)) \oplus C_*(t))$ which is contractible. For this purpose, take decompositions $C_0 = dC_1 \oplus H_0 \oplus N_0$ and $C_1 = dC_2 \oplus H_1 \oplus N_1$ where $C_i$ (resp. $H_i$) denotes the homologically $i$-th degree of $C_*(t))$ (resp. $HH_*(A)(t))$, and $d$ is the differential. We here choose subspaces $N_0$ and $N_1$. By $C_{2z} = t^{z} \cdot C_0$ and $C_{2z+1} = t^{z} \cdot C_1$ for $z \in \mathbb{Z}$, we put $C_{2z} = t^{z} \cdot (dC_1 \oplus H_0 \oplus N_0)$ and $C_{2z+1} = t^{z} \cdot (dC_2 \oplus H_1 \oplus N_1)$. Then $t^{z} \cdot H_0 = H_{2z} \to t^{z} \cdot (H_0 \oplus (dC_1 \oplus N_0))$ and $t^{z} \cdot H_1 = H_{2z+1} \to t^{z} \cdot (H_1 \oplus (dC_2 \oplus N_1))$ determines an injective homotopy equivalence with a splitting $C_*(t)) \to HH_*(A)(t))$. The case of $C_*(t)$ is a variant of the above proof ($E_\geq$ is the dg Lie subalgebra of $\text{End}_{k[t^\pm]}(C_*(t))$ which consists of linear maps preserving $HH_*(A)([[t]])$. \qed
Theorem 5.7 is a consequence of the following Proposition:

**Proposition 5.10.** The dg Lie algebra $\mathbb{F}$ is quasi-isomorphic to an abelian dg Lie algebra. Here by abelian we mean the vanishing of the bracket.

**Proof.** By Lemma 5.9 we may replace $\varepsilon : \text{End}_k(t)(C_\bullet([t])) \to \text{End}_k(t^{\pm})(C_\bullet((t)))$ by the natural injective morphism $\varepsilon' : \text{End}_k(t)(HH_\ast(A)[[t]]) \to \text{End}_k(t^{\pm})(HH_\ast(A)((t)))$. Then according to [16 Proposition 3.4] a homotopy fiber of $\varepsilon'$ is quasi-isomorphic (as dg Lie algebras or $L_\infty$-algebras) to an abelian dg Lie algebra. \hfill $\square$

**Proof of Theorem 5.7.** By Proposition 5.10 we may replace $\mathbb{F}$ by an abelian dg Lie algebra. Thus we will assume that $\mathbb{F}$ is abelian. Then for $R \in \text{Art}_k$, $\text{MC}(\mathbb{F} \otimes m_R) = Z^1(\mathbb{F}) \otimes m_R$. Here $Z^1(\cdot)$ is the space of closed elements of degree one. Therefore, for any surjective homomorphism $R' \to R$ in $\text{Art}_k$, we see that $\text{MC}(\mathbb{F} \otimes m_{R'}) \to \text{MC}(\mathbb{F} \otimes m_R)$ is surjective. \hfill $\square$

5.4. We conclude this Section by some observations which reveals a simple structure of $\mathbb{F}$ and $\text{Gr}$. According to Lemma 5.9, a homotopy fiber of the natural injective morphism

$$\text{End}_k(t)(HH_\ast(A)[[t]]) \to \text{End}_k(t^{\pm})(HH_\ast(A)((t)))$$

is equivalent (quasi-isomorphic) to the homotopy fiber $\mathbb{F}$ of $\text{End}_k(t)(C_\bullet([t])) \to \text{End}_k(t^{\pm})(C_\bullet((t)))$. The underlying complex of a homotopy fiber of $\text{End}_k(t)(HH_\ast(A)[[t]]) \to \text{End}_k(t^{\pm})(HH_\ast(A)((t)))$ is quasi-isomorphic to the mapping cone

$$\text{End}_k(t)(HH_\ast(A)[[t]]) \oplus \text{End}_k(t^{\pm})(HH_\ast(A)((t)))[-1]$$

with differential $(a, b) \mapsto (0, a)$. There is a natural projection to the graded vector space with the zero differential:

$$\text{End}_k(t^{\pm})(HH_\ast(A)((t)))/\text{End}_k(t)(HH_\ast(A)[[t]])[-1],$$

which is a quasi-isomorphism. In addition, by Proposition 5.10 the homotopy fiber is equivalent to an abelian dg Lie algebra. Consequently, we can choose the homotopy fiber to be the graded vector space $G := \text{End}_k(t^{\pm})(HH_\ast(A)((t)))/\text{End}_k(t)(HH_\ast(A)[[t]])[-1]$ endowed with the zero differential and the zero bracket. We take an $L_\infty$-morphism

$$C^\ast(A)[1] \to G$$

which is equivalent to the period map $P : C^\ast(A)[1] \to \mathbb{F}$ (via an $L_\infty$-quasi-isomorphism $\mathbb{F} \to G$). By Remark 4.11 the induced map

$$HH^\ast(A)[1] \to \text{End}_k(t^{\pm})(HH_\ast(A)((t)))/\text{End}_k(t)(HH_\ast(A)[[t]])[-1]$$

carries $P$ to $H_\ast(\frac{1}{t}I_P((t)))$ modulo $\text{End}_k(t)(HH_\ast(A)[[t]])$. Therefore, we can summarize this observation:

**Proposition 5.11.** There is an $L_\infty$-morphism

$$C^\ast(A)[1] \to G = \text{End}_k(t^{\pm})(HH_\ast(A)((t)))/\text{End}_k(t)(HH_\ast(A)[[t]])[-1]$$

which is equivalent to the period map $P : C^\ast(A)[1] \to \mathbb{F}$ constructed in Section 4.11. The induced morphism

$$HH^\ast(A)[1] \to G = \text{End}_k(t^{\pm})(HH_\ast(A)((t)))/\text{End}_k(t)(HH_\ast(A)[[t]])[-1]$$

carries $P$ in $HH^\ast(A)[1]$ to $H^\ast(\frac{1}{t}I_P((t)))$. 
Remark 5.12. The graded vector space $\operatorname{End}_{k[t^\pm]}(HH_*(A)((t)))/\operatorname{End}_{k[t]}(HH_*(A)[[t]])$ is isomorphic to

$$\bigoplus_{i \in \mathbb{Z}, j \in \mathbb{Z}, r < 0} \operatorname{Hom}_{k}(HH_{i}(A), HH_{j}(A) \cdot t^{r}).$$

where $\operatorname{Hom}_{k}(-, -)$ indicates the space of $k$-linear maps, and the (cohomological) degree of elements in $\operatorname{Hom}_{k}(HH_{i}(A), HH_{j}(A) \cdot t^{r})$ is $2r - j + i$. Notice that an element of the image of $HH^{*}(A)[1] \to \operatorname{End}_{k[t^\pm]}(HH_*(A)((t)))/\operatorname{End}_{k[t]}(HH_*(A)[[t]])[-1]$ does not preserve the filtration on $HH_*(A)((t)) \simeq HP_*(A)$. However, $H^{*}(\frac{1}{2} \mathcal{I}_{P}(t))$ carries $F^{r}HP_{*}(A)$ to $F^{r-1}HP_{*}(A)$. It can be thought of as the Griffiths’ transversality in the noncommutative situation.

6. UNOBSTRUCTEDNESS OF DEFORMATIONS OF ALGEBRAS

We apply our period mapping to study unobstructedness of deformations, and quasi-abelian properties of Hochschild cochain complexes. We prove a noncommutative generalization of the Bogomolov-Tian-Todorov theorem (cf. Corollary 6.2). The argument is based on the idea of a purely algebraic proof of Bogomolov-Tian-Todorov theorem by Iacono and Manetti [16]. We continue to assume that $A$ is smooth and proper (so that by the theorem of Kaledin, Hodge-to-de Rham degenerate).

Theorem 6.1. Suppose that the following condition: the linear map

$$HH^{*}(A) \to \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{k}(HH_{i}(A), HH_{i-s}(A))$$

given by $P \mapsto I_{P}$ is injective for any integer $s$. We here abuse notation by writing $I_{P}$ for $H^{*}(I_{P})$. Then $C^{*}(A)[1]$ is quasi-abelian, namely, it is quasi-isomorphic to an abelian dg Lie algebra. In particular, the functor $\operatorname{Spf}C^{*}(A)[1] \simeq \operatorname{DA}l_{A}$ is formally smooth.

Proof. Note first that the linear map

$$H^{*}(\mathcal{P}) : HH^{*}(A)[1] \to G = \operatorname{End}_{k[t^\pm]}(HH_*(A)((t)))/\operatorname{End}_{k[t]}(HH_*(A)[[t]])[-1]$$

induced by the period mapping (see Proposition 5.11) can be naturally identified with the linear map $HH^{*}(A) \to \bigoplus_{i \in \mathbb{Z}, j \in \mathbb{Z}, r < 0} \operatorname{Hom}_{k}(HH_{i}(A), HH_{j}(A) \cdot t^{r})$ which sends $P$ to $\frac{1}{2} I_{P}$ (see Remark 5.12). Thus, our condition amounts to the injectivity of the first linear graded map $H^{*}(\mathcal{P})$. Clearly, this graded map admits a left inverse (graded) map $G \to HH^{*}(A)[1]$. If we equip $HH^{*}(A)[1]$ with the zero bracket, then this inverse is a morphism of dg Lie algebras. It gives rise to

$$C^{*}(A)[1] \to \operatorname{End}_{k[t^\pm]}(HH_*(A)((t)))/\operatorname{End}_{k[t]}(HH_*(A)[[t]])[-1] \to HH^{*}(A)[1]$$

where the left morphism is the period mapping in Proposition 5.11 which is an $L_{\infty}$-morphism. The composite is an $L_{\infty}$-quasi-isomorphism. Thus $C^{*}(A)[1]$ is quasi-isomorphic to $HH^{*}(A)[1]$ with the zero bracket. The final assertion follows from the same argument with the proof of Theorem 5.7. □

Let us recall Calabi-Yau condition on $A$. We say that a smooth dg algebra $A$ is Calabi-Yau of dimension $d$ if there is an isomorphism $f : A \to \operatorname{RHom}_{A^{c}}(A, A \otimes A)[-d] = A^{c}[-d]$ in the triangulated category of dg $A$-bimodules (i.e. left dg $A^{c} := A \otimes A^{op}$-modules), such that $f = f^{t}[d]$ (see [19]). Here $A$ has the $A$-bimodule structure given by the left and right multiplications, $A \otimes A$ is endowed with the outer $A$-bimodule action of $A \otimes A$, i.e. $a \cdot (a_{1} \otimes a_{2}) \cdot a' := a_{1}a_{2}a'$, and $\operatorname{RHom}_{A}(A, A \otimes A)$ is a derived Hom complex. We use the projective model structure of left/right dg $A \otimes A^{op}$-modules. The $A$-bimodule structure on $\operatorname{RHom}_{A^{c}}(A, A \otimes A)$ is given by the inner bi-module action, i.e. $a \cdot (a_{1} \otimes a_{2}) \cdot a' := a_{1}a'a_{2}$. 

Corollary 6.2. Let $A$ be a Calabi-Yau algebra of dimension of $d$. Then $C^{*}(A)[1]$ is quasi-abelian.
Proof. It is enough to prove that the Calabi-Yau condition implies the hypothesis in Theorem 6.1. Assume that $A$ is Calabi-Yau of dimension $d$. Then by Van den Bergh duality [41, Theorem 1] there is an element $\pi \in HH_d(A)$ such that

$$ HH^s(A) \to HH_{d-s}(A), $$

which carries $P$ to $H_s(IP(\pi))$, is an isomorphism for $s \in \mathbb{Z}$. Indeed, this isomorphism is given by

$$ HH^s(A) = H^s(R\text{Hom}_{A^e}(A, A)) $$

$$ \simeq H^s(R\text{Hom}_{A^e}(A, A \otimes A) \otimes_{A^e}^L A) $$

$$ \simeq H^s(A[-d] \otimes_{A^e}^L A) $$

$$ \simeq H^{s-d}(A \otimes_{A^e}^L A) $$

$$ = HH_{d-s}(A), $$

where the third identification is induced by a fixed morphism $f : A \to A^d$, and $\otimes^L$ is the derived tensor product. Let $\pi$ be the image of $1_A \in HH^0(A)$ under $HH^0(A) \simeq HH_d(A)$. (It is natural to think that $\pi$ is the fundamental class of $A$.) As observed in [31, Theorem 3.43 (i)] and its proof, the above isomorphism carries a class $P$ in $HH^s(A)$ to $H_s(IP(\pi))$ in $HH_{d-s}(A)$. Thus, the Calabi-Yau condition implies that the hypothesis in Theorem 6.1.

Example 6.3. Let $X$ be a smooth projective Calabi-Yau variety over $k$. Here Calabi-Yau means that the canonical bundle is trivial. There is a dg algebra $A$ such that the dg category (or $\infty$-category) of dg $A$-modules is equivalent to that of (unbounded) quasi-coherent complexes on $X$ (see Remark 6.5). Then $A$ is an example of a Calabi-Yau algebra. In this case, it seems that Corollary 6.2 may be proved by using Kontsevich formality type result for the Hochschild cochain complex $C^\bullet(X)[1]$ of $X$ and an analytic method by $\partial\bar{\partial}$-lemma (over the complex number). However, our proof is purely algebraic. Moreover, it is also a "non-commutative proof" of the noncommutative problem in the sense that the dg Lie algebra $C^\bullet(A)[1]$ depends only on the derived Morita equivalence class of $A$, and the proof does not rely on the commutative world.

Example 6.4. There are interesting constructions of Calabi-Yau algebras due to the work of Kuznetsov (see [26] and references therein). We consider the famous example that comes from a smooth cubic fourfold $X \subset \mathbb{P}^5$. Let Perf($X$) be the triangulated category (or the enhancement by a dg category) of perfect complexes on $X$, that is, the derived category $D^b(X)$. Let

$$ A_X = \{ F \in \text{Perf}(X) \mid R\text{Hom}_{\text{Perf}(X)}(O_X(i), F) \simeq 0 \text{ for } i = 0, 1, 2 \} $$

be the orthogonal subcategory to the set of line bundles $O_X$, $O_X(1)$, $O_X(2)$. Then $A_X$ is a smooth and proper $2$-dimensional Calabi-Yau category, and thus $A_X$ is equivalent to the triangulated category (or the dg category) of perfect dg modules over a (smooth and proper) $2$-dimensional Calabi-Yau algebra $A$ (see Remark 6.7 for perfect modules). The Hochschild homology coincides with that of K3 surfaces, i.e., $\dim HH_0(A) = 22$, $\dim HH_2(A) = 1$ and the other terms are zero. Moreover, there is a smooth cubic fourfold $X$ which does not admit a K3 surface $S$ such that $A_X \simeq \text{Perf}(S)$. These categories/algebras are called "noncommutative K3 surfaces". The fascinating other examples can be found in [26, 27]. We refer the reader also to [33] for the various constructions and facts on smooth and proper algebras.

7. Infinitesimal Torelli theorem

The purpose of this Section is to prove the following infinitesimal Torelli theorem:
Theorem 7.1. Let $P : \text{DAlg}_A \to \text{Gr}$ be the period mapping constructed in Section 4.12. Suppose that $A$ is Calabi-Yau of dimension $d$. Then $P$ is a monomorphism. Namely, for each $R \in \text{Art}_k$ the induced map

$$P(R) : \text{DAlg}_A(R) \to \text{Gr}(R)$$

sending the isomorphism class of a deformation $\tilde{A}$ of $A$ to $R$ to the isomorphism class of the associated periodically trivialized deformation

$$(C^*(\tilde{A})[[t]], (C^*(A) \otimes R)((t)^\alpha) \simeq C^*(\tilde{A})[[t]] \otimes_{R[[t]]} R[[t]]^\alpha)$$

is injective.

Proof. We first consider the case when $R = k[e]/(e^2)$. According to Theorem 4.17 we can interpret $P(R) : \text{DAlg}_A(R) \to \text{Gr}(R)$ as $\text{Spf}_P(R) : \text{Spf}_{C^*(A)[1]}(R) \to \text{Spf}_P(R)$. In addition, we may and will replace $P$ by

$$C^*(A)[1] \to \text{End}_{k[e]}(HH_*(A)((t)))/\text{End}_{k[e]}(HH_*(A)[[t]])[-1]$$

in Proposition 5.1. We denote by $P$ this $L_{\text{nc}}$-morphism. Since the bracket on $C^*(A)[1] \otimes m_R$ is zero, there is the natural isomorphism $C^*_P(A)[1] \simeq HH^2(A)$. By Proposition 5.11 $\text{Spf}_P$ sends an element $P$ in $HH^2(A)$ to $I_{\text{I}}(t)$ in $\text{End}_{k[e]}(HH_*(A)((t)))/\text{End}_{k[e]}(HH_*(A)[[t]])$. As in the proof of Theorem 6.1 and Corollary 6.2 the Calabi-Yau condition implies that

$$\text{Spf}_P(R) = HH^2(A) \to \text{End}_{k[e]}(HH_*(A)((t)))/\text{End}_{k[e]}(HH_*(A)[[t]])$$

is injective. Hence $P(R)$ is injective when $R = k[e]/(e^2)$. Next we prove the general case by induction on the length of the maximal ideal $m_R$ of $R$. To this end, let

$$0 \to (e) / (e^2) \to R \to R' \to 0$$

be an exact sequence where $(e) / (e^2)$ is a nonunital square-zero 1-dimensional $k$-algebras which is the kernel of the surjective homomorphism $R \to R'$ of Artin local $k$-algebras. Suppose that $\text{Spf}_P(R')$ is injective. It is enough to prove that $P(R) \simeq \text{Spf}_P(R)$ is injective. To simplify the notation, put $L := C^*(A)[1]$ and $E := \text{End}_{k[e]}(HH_*(A)((t)))/\text{End}_{k[e]}(HH_*(A)[[t]])[-1]$. Let $\alpha, \beta$ be two elements in $\text{Spf}_L(R)$ such that $\text{Spf}_P(R)(\alpha) = \text{Spf}_P(R)(\beta)$ in $\text{Spf}_E(R)$. Note that $(\text{Spf}_P(R)(\alpha), \text{Spf}_P(R)(\beta))$ belongs to $\text{Spf}_E(R) \times_{\text{Spf}_E(R')} \text{Spf}_E(R)$. It follows from the injectivity of $\text{Spf}_P(R')$ that the images of $\alpha$ and $\beta$ in $\text{Spf}_L(R')$ coincide. Namely, $(\alpha, \beta)$ lies in $\text{Spf}_L(R) \times_{\text{Spf}_E(R')} \text{Spf}_L(R)$. Note that $\text{Spf}_L$ satisfies the Schlessinger’s condition “$(H_1)$” (see [17] 2.21, [15] I.3.31). In particular, the natural map $\text{Spf}_L(R \times_{R'} R) \to \text{Spf}_L(R) \times_{\text{Spf}_E(R')} \text{Spf}_L(R)$ is surjective. Here $R \times_{R'} R$ is not the tensor product but the fiber product of aritin local $k$-algebras. We write $(r, \tilde{r} + a \epsilon)$ for an element in $R \times_k k[e]/(e^2)$ where $\tilde{r}$ is the image of $r$ in $k \simeq R/m_R$. There is an isomorphism of aritin local $k$-algebras $R \times_k k[e]/(e^2) \simeq R \times_{R'} R$ which carries $(r, \tilde{r} + a \epsilon)$ to $(r, r + a \epsilon)$. We identify $\text{Spf}_L(R \times_{R'} R)$ with $\text{Spf}_L(R \times_k k[e]/(e^2)) \simeq \text{Spf}_L(R) \times \text{Spf}_L(k[e]/(e^2))$. We choose an element $(\alpha, q)$ in $\text{Spf}_L(R) \times \text{Spf}_E(k[e]/(e^2))$ which is a lift of $(\alpha, \beta)$. It will suffice to prove that $q$ is zero. Since $\text{Spf}_E(k[e]/(e^2))$ is injective, we are reduced to showing that $\text{Spf}_E(k[e]/(e^2))(q)$ is zero in $\text{Spf}_E(k[e]/(e^2))$. For this, notice that $E$ has the zero bracket and the zero differential, so that the natural map

$$\text{Spf}_E(R) \times \text{Spf}_E(k[e]/(e^2)) \simeq \text{Spf}_E(R \times_{R'} R) \to \text{Spf}_E(R) \times \text{Spf}_E(R') \text{Spf}_E(R)$$

is an isomorphism. The composition $\text{Spf}_E(R) \times \text{Spf}_E(k[e]/(e^2)) \to \text{Spf}_E(R)$ with the first projection (resp. the second projection) $\text{Spf}_E(R) \times \text{Spf}_E(R') \text{Spf}_E(R) \to \text{Spf}_E(R)$ is induced by the first projection (resp. $R \times_k k[e]/(e^2) \to R$ defined by $(r, \tilde{r} + a \epsilon) \mapsto r + a \epsilon$). Thus, $(\text{Spf}_P(R)(\alpha), \text{Spf}_P(R)(\beta))$ corresponds to $(\text{Spf}_P(R)(\alpha), 0)$. Taking account of the functoriality of $\text{Spf}_P$, we conclude that $\text{Spf}_P(k[e]/(e^2))(q)$ is zero. □
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