Towards a theory of types III and IV non-Hermitian Weyl fermions

Zaur Z. Alsultanov
Institute of Physics of DFRS, Russian Academy of Sciences, Makhachkala, Russia

Edvin G. Idrisov
Department of Physics and Materials Science, University of Luxembourg, Luxembourg

(Dated: November 15, 2021)

We develop the non-Hermitian Hamiltonian formalism to describe Weyl fermions of type III and IV. The spectrum of Hamiltonian has an unusual type of anisotropy. Namely, the hermiticity of Hamiltonian strongly depends on the direction in momentum space: for some directions the spectrum is real, in contrast for other directions it becomes complex. This fact leads to non-trivial adiabatic evolution and fractional Chern number. Additionally, we demonstrate that the non-Hermitian Hamiltonian can be regarded as a one-particle problem in context of topological band theory.

The current theoretical progress in band theory allows to describe topologically protected quasiparticles in modern experimentally accessible materials \[1\]. Due to their unique properties, for instance chiral transport, the topological materials are expected to be promising for future electronics \[2\]. Among all, the topological materials with Dirac \[3,4\] and Weyl \[7,11\] points, as well as degenerate points \[12\] and lines \[13,15\] in Brillouin zone are of particular interest. The massless fermions in such materials are topologically protected, which results in quantum-electrodynamics effects known from high energy physics \[16\].

The idea of realization of Weyl points in condensed matter physics belongs back to C. Herring \[17\]. According to Herring’s description, the presence of Weyl points can be understood within two-band model. The corresponding Hamiltonian of in momentum representation can be presented in Pauli basis

\[ H(p) = \sigma_0 f_0(p) + \sum_{i=1}^{3} \sigma_i f_i(p), \]

where \( \sigma_0 \) is \(2 \times 2\) unit matrix, \( \sigma_i \) are Pauli matrices and \( p \) is a three-dimensional (3D) momentum vector. The Weyl point \( p_W \) arises due to crossing of bands, which satisfies to three algebraic equations \( f_i(p_W) = 0 \). Materials with these conditions are known as Dirac and Weyl semimetals \[18\]. Typically such kind of situation is easily realizable in 3D systems, and equations \( f_i(p_W) = 0 \) represent the two-dimensional (2D) surfaces in momentum space: three closed 2D surfaces can have many crossing points. In 2D system this condition corresponds to the crossing of three lines. This is not trivial and in order to have the Weyl point in 2D system, the certain types of symmetries (for example, \(C_6\) point symmetry) are required \[18\]. In the vicinity of Weyl point \( p_W \) one can expand the function \( f_i(p) \) in Taylor series, which results in low-energy linear spectrum, namely \( f_i(p) \propto |p| \), and corresponding excitations are known as massless (Weyl or Dirac) fermions \[18\]. The Eq. \(1\) describes conventional (non-tilted) type I Weyl fermions for zero and constant values of \( f_0(p) \).

Apart from Weyl semimetals of type I, it was proposed Weyl semimetals of type II \[19,22\]. Nowadays such kind of materials are accessible experimentally and WTe\(_2\) is one of the candidates for a realization of type II Weyl fermions \[23\]. In short the Hamiltonian for such systems can be modeled at the intersections of Fermi-pockets and the spectrum turns out to be tilted. The minimal Hamiltonian for this case has the following form

\[ H(p) = v_F \sigma \cdot p + \sigma_0 \omega \cdot p, \]

where \( v_F \) is Fermi velocity and \( \omega \) is a tilt vector. For \( v_F > \omega \) this Hamiltonian describes the Weyl semimetals of type I with tilted spectrum, and consequently for \( v_F < \omega \) the Hamiltonian corresponds to type II Weyl semimetals, where \( \omega = |\omega| \) is a modulus of vector. It is worth mentioning that Weyl semimetals of type II can be used for “modeling” of black and white holes, event horizon \[24,25\]. Indeed, assuming that the parameter of tilting, \( \omega(x) \), is a function of spatial coordinates, the Weyl fermions of type II can be described by action of massless spinor field \[24\]

\[ S_{II} = \int d^3x \bar{\psi} \left( i \gamma \cdot p \right) \psi \]

where \( \psi \) is a Dirac spinor and \( \gamma^\mu \) is the Dirac matrix in the Weyl representation \( \gamma^\mu = \left( \begin{array}{cc} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{array} \right) \) with \( \sigma^\mu = (\sigma_0, \sigma) \) and \( \bar{\sigma}^\mu = (\sigma_0, -\sigma) \) and \( \partial^\mu = (v_F^{-1} \partial_t, \partial) \). This is the possible basic (minimalistic) action for the description of massless spinor field in curved space with metrics \( ds^2 = (|\omega|^2 - 1) dt^2 - 2\omega \cdot dx dt + dx \cdot dx \), and allows to model phenomena in the vicinity of event horizon.

In this Letter, we make a further step to develop the theory for Weyl fermions of types III and IV. We show that these fermions can be described using the Hamiltonian formalism in the framework of non-Hermitian quantum theory. Moreover, the non-Hermitian Hamiltonian arises as a one-particle problem in the context of band
theory of Weyl semimetals. In order to study the properties of this Hamiltonian and associated Hilbert space, the acceptable theory of non-Hermitian systems is presented below. The spectrum of Weyl Hamiltonian under consideration turns out to be anisotropic in momentum space, namely for some directions the spectrum is real and for other directions it is complex quantity. The necessary and sufficient conditions for the real spectrum of the general Hamiltonian under consideration are formulated. The main feature of system under consideration is that “left” and “right” eigenvalues of Hamiltonian coincide with each other, which results in non-trivial abelian evolution and fractional Chern number.

Weyl fermions of type III and IV. In recent seminal papers, Ref. [26, 27], authors have suggested Weyl fermions of type III and IV. Briefly, using the tetrad formalism, there was provided the general action of massless spinor field of form

$$S = \int d^4x \sqrt{-g} [\bar{\psi} e^\nu_\mu \partial_\mu \psi],$$

where $g$ is the determinant of the metric tensor and $e^\nu_\mu$ is the tetrad tensor. One can see from Eq. (4), that Weyl fermions of type I and II are particular cases of the general tetrad action theory. Indeed, if one keeps only diagonal elements of tetrad tensor $e^\nu_\mu = v_F \delta^\nu_\mu$, then we get the action for Weyl fermions of type I. If one set $e^\nu_\mu = v_F \delta^\nu_\mu + \delta^\nu_\omega \delta^\omega_\mu$, then we get the action for type II Weyl fermions presented in Eq. (5). However, in more general case the Eq. (4) can contain other non-zero components of tetrad tensor, namely $e^\nu_\mu = v_F \delta^\nu_\mu + \delta^\nu_\omega \delta^\omega_\mu + \delta^\nu_i \delta^i_\mu$, where $\delta_i$ are the components of additional tilt parameter vector, $\vec{\theta}$. The presence of last term restores the symmetry with respect to rearrangement of indices $\mu$ and $\nu$. Note that this form corresponds to a spatially isotropic tetrad when there is no $\delta^\nu_i \delta^i_\mu$ term. According to Ref. [28], the case $v_F > \omega$ and $v_F < \theta$ corresponds to Weyl fermions of type III, and $v_F < \omega$ and $v_F < \theta$ is associated with type IV, where $\theta = |\vec{\theta}|$. Due to the last term, $\delta^\nu_i \delta^i_\mu$, the Lagrangian contains a new term with zero component of momentum $\sigma^i \delta^i_\mu p_\mu$, and $p_\mu = (p_0, \vec{p})$ with $p_0 = \varepsilon / v_F$. Particularly, the possible origin of this term and consequently Weyl points of type III and IV were associated with many-particle effects and thus can be worked out from self-energy [29]. It is worth mentioning, that in framework of Hermitian quantum mechanics the Hamilton formalism does not allow to take into account this term.

Nevertheless, we insist that the Hamiltonian formalism can be constructed, but in the framework of non-Hermitian quantum mechanics. Moreover, the non-Hermitian Hamiltonian formalism allows the alternative explanation, which does not necessarily include many-particle effects as the possible mechanism of explanation. In order to introduce the Hamiltonian, we first construct the wave equation

$$G^{-1}(p_0, p_\mu) |\Psi\rangle = 0,$$

where for the spatially isotropic case, which contains all important physics, the Greens function associated with action, Eq. (4) is given by

$$G^{-1}(p_0, p_\mu) = v_F \delta^0_0 p_0 - v_F \sigma^i p_i - \omega^0 p_0 - \sigma^i \partial_i p_0.$$

Using the above expression, the Eq. (5) can be easily rewritten in conventional form, where the Hamiltonian at left hand side is the function of only three dimensional momentum. After simple algebra one arrives to the following two wave equations

$$\mathcal{H} |\Psi^R\rangle = \varepsilon_R |\Psi^R\rangle, \quad \mathcal{H}^\dagger |\Psi^L\rangle = \varepsilon_L |\Psi^L\rangle,$$

where the non-Hermitian Hamiltonian takes form

$$\mathcal{H} = (i \Gamma + K)/(1 - \beta^2), \quad \Gamma = \sigma \cdot [p \times \theta],$$

$$K = [v_F \sigma + \sigma_0 (\omega + \theta)] \cdot p + (\sigma_0 (\omega + \theta) \cdot p) / v_F,$$

and $\beta = \partial / v_F$. This is the Hamiltonian of type III and IV non-Hermitian Weyl fermions and is our main result. This Hamiltonian can be also obtained directly from the action, Eq. (4), if one writes the wave equation as the Euler-Lagrange equation for the Dirac field. In Eq. (7) we have defined the $R$ (right) and $L$ (left) wave functions. It is worth mentioning, that use of so called dual basis $(R, L)$ is a convenient step for non-Hermitian systems, which allows to restore the common structure of Hilbert space [28, 30].

Now one can easily obtain the spectrum of non-Hermitian and conjugated Hamiltonians from Eq. (7). The spectra are coincides with each other, namely

$$\varepsilon = \varepsilon_R = \varepsilon_L = \mathcal{A}(p) \pm \sqrt{\mathcal{B}(p)}, \quad \mathcal{A}(p) = (\theta + \omega) \cdot p,$$

$$\mathcal{B}(p) = [(\theta + \omega) \cdot p]^2 + (1 - \beta^2) \left[ v_F^2 p^2 - (\omega \cdot p)^2 \right],$$

where $p = |p|$. Note, that the Hamiltonian of form $\mathcal{H} = a \sigma_0 + (b + ic) \sigma$ has the spectrum $\varepsilon = a + \sqrt{b^2 - c^2 + 2ib \cdot c}$, which contains exceptional point due to term $b \cdot c$. In our case $b$ is proportional (collinear) to $p$ and $c$ is proportional to vector product $[p \times \vec{\theta}]$, thus $b \cdot c = 0$ and the expression under the root always is real. For simplicity, we set $\omega = 0$ for further discussion. This does not affect on outcomes, since the non-hermiticity is described by term $i \Gamma$. Then Hamiltonian from Eq. (8) can be rewritten in the form

$$\mathcal{H} = [\sigma \cdot (v_F e_p + i \theta \sin \chi e_\chi) p + \theta \cdot p] / (1 - \beta^2),$$

where the unit vector $e_p$ is directed along the momentum $p$ and the unit vector $e_\chi$ is directed along the direction of cross product $[p \times \theta]$. For values $\theta < v_F$
(β < 1) the parameter θ renormalizes the velocity ∂ε/∂p due to coefficient 1/(1 − β²) and results in slope (tilt).

For θ > v_F (β > 1), the spectrum in Eq. (9) is real in segment |sin χ| < β⁻¹. In contrast, for |sin χ| > β⁻¹ the spectrum consists of complex-conjugated branches. Therefore, we get the anisotropy with respect to spectrum’s (Hamiltonian’s) hermiticity (see Fig. 1). The eigenstates |Ψ^R⟩ and |Ψ^L⟩ are orthogonal in the segment |sin χ| > β⁻¹, where χ is the angle between vectors θ and p, i.e., ⟨Ψ^L | Ψ^R⟩ = 0. Indeed, from Eqs. (7), one has ⟨Ψ^L | H | Ψ^R⟩ = ε_R (⟨Ψ^L | Ψ^R⟩ = ε_L^* (⟨Ψ^L | Ψ^R⟩). Since ε_R ≠ ε_L^*, then ⟨Ψ^L | Ψ^R⟩ = 0. In the opposite segment |sin χ| < β⁻¹, where spectrum is real, generally speaking, these states are not orthogonal. Additionally, in Weyl point these states are not orthogonal as well, since ε_R = ε_L = 0. Further, it is easy to show, that in case of real spectrum ⟨Ψ^L ± | Ψ^R⟩ = 0 and ⟨Ψ^L ± | Ψ^R⟩ ≠ 0, where “±” corresponds to electron hole states in Eq. (9). At the same time, in the segment, where spectrum is complex one has ⟨Ψ^L ± | Ψ^R⟩ = 0 and ⟨Ψ^L ± | Ψ^R⟩ ≠ 0. Such a situation leads to non-trivial adiabatic evolution (see Sec. A and B in Ref. [31]). The existence of segment with real spectrum, despite the non-Hermitian nature of Hamiltonian, is surprising. Nevertheless, it can be proven the following statement: If the non-Hermitian Hamiltonian, \( \mathcal{H} \), can be brought to hermitian \( \mathcal{H}_h \), one, using the similarity transformations with non-degenerate matrix, then the eigenvalues of \( \mathcal{H} \) are real. Indeed, let apply the similarity transformation, \( \mathcal{H} S^{-1} \) with the linear operator \( S \) in the form of non-degenerate matrix, \( (S^{-1})^T = (S^1)^{-1} \), to Hamiltonian \( \mathcal{H} \). Next, we assume that the following equality holds

\[
\mathcal{H} S^{-1} = \mathcal{H}_h, \tag{11}
\]

Then \( \mathcal{H} S^{-1} |Ψ⟩ = \mathcal{H}_h |Ψ⟩ = \mathcal{E} |Ψ⟩ \), where \( \mathcal{E} \) is real since the Hamiltonian \( \mathcal{H}_h \) is hermitian. On the other hand due to linearity of operator \( S \), this equation can be written as \( \mathcal{H} (\overline{Ψ}) = \mathcal{E} (\overline{Ψ}) \), where \( \overline{Ψ} = S^{-1} |Ψ⟩ \). Thus, the eigenvalues of non-Hermitian Hamiltonian, which satisfies the condition Eq. (11), are real. The Eq. (11) can be rewritten in new form

\[
\mathcal{H}^1 = (S^1 S) \mathcal{H} (S S^1)^{-1} = η \mathcal{H} η^{-1}, \tag{12}
\]

where we have introduced the Hermitian operator \( η = S^1 S \). The Eq. (12) is the necessary and sufficient condition for the spectrum of Hamiltonian \( \mathcal{H} \) to be real. If \( S \) is unitary matrix, then \( S^1 S = 1 \) and Eq. (12) recovers the hermiticity of Hamiltonian from hermitian quantum mechanics. One can show that the Hamiltonian Eq. (10) satisfies the condition (12) for \( |sin χ| < β^{-1} \) segment (see Sec. C of Ref. [31]). For non-degenerate matrix \( (S^{-1})^T = (S^1)^{-1} \), the condition in Eq. (12) coincides with the condition of pseudo-hermiticity [32,33]. This is the necessary condition for spectrum to be real. The sufficient condition for spectrum to be real is the existence of operator \( η \) in the form of \( S^1 S \). The condition in Eq. (12) at \( S^1 S = η \) is known as \( η \)-hermiticity. We call the Hamiltonians \( \mathcal{H} \) and \( \mathcal{H}_h \) from Eq. (11) equivalent, since they have the same real spectrum even if one of them is non-Hermitian. The equivalence of two Hamiltonians in that sense is the necessary and sufficient condition to have real valued spectrum.

**Topological protection.** We continue to investigate the spectrum of Weyl fermions of types III and IV. It is worth mentioning, that Weyl points are topologically protected with respect to external perturbations. The spectrum remains gapless, for the case of unit matrix perturbation,
where $p_0$ satisfies the condition $[(\theta + \omega) \cdot p_0 + U_0]^2 + [1 - \beta^2] \left[ v_F^2 p_0^2 - (\omega \cdot p_0 + U_0)^2 \right] = 0$. Next, the perturbation in the form of Pauli matrix, $H \rightarrow H + \sigma U$ results in the following spectrum

$$\varepsilon = \left[ \mathcal{E}_1(p) \pm \mathcal{E}_2(p) \right] / (1 - \beta^2),$$

(13)

where $\mathcal{E}_1(p) = A(p) + \theta \cdot U/v_F$ and $\mathcal{E}_2(p) = \sqrt{\xi_1^2(p)} + (1 - \beta^2)\xi(p)$, with $\xi(p) = (v_F p + U)^2 - (\omega \cdot p)^2$ and $A(p)$ is given in Eq. (4). This spectrum has a shift with respect to initial one, Eq. (9), but does not contain a gap.

It is as well useful to investigate the stability of spectrum with the help of topological invariant-Chern number. For non-Hermitian Hamiltonian the Chern number can be written as

$$N = \frac{1}{4\pi} \int d\mathbf{p} \times \langle \Psi^L | \partial_p | \Psi^R \rangle \cdot d\mathbf{S},$$

(14)

where the integration is carried out over a closed surface enclosing the Weyl point. The Chern number defined in Eq. (14) must vanish, if the Hamiltonian has certain symmetries, namely $H(p) = H^T(p)$, $\sigma_\alpha H(p) \sigma_z = H^T(-p)$ or $H(p) = H^T(p)$. The Hamiltonian does not satisfies any of these conditions, which indicates of topological protection of Chern invariant for the Hamiltonian under consideration.

The integration domain in Eq. (14) can be divided into two regions, $S_C$ and $S_R$, which correspond to complex and real spectrum of Hamiltonian in Eq. (10) (see Fig. 2). The region $S_C$ does not contribute to integral in Eq. (14) (see Sec.D of Ref. 36), therefore $N = \frac{1}{4\pi} \int_{S_R} i\partial_p \times \langle \Psi^L | \partial_p | \Psi^R \rangle \cdot d\mathbf{p}$, which results in fractional Chern number. This means that Weyl fermions of types III and IV are characterized with fractional Chern number. Fractional topological indexes have been considered earlier in literature, and typically are associated with exceptional points in spectrum [36], in contrast to the case under consideration with no exceptional points.

Discussion. Let investigate the spectrum in phase transition point $v_F = \theta$. For simplicity, we assume that vectors $\theta, \omega$ are parallel to $p$. Then one has $(\theta + \omega) \cdot p = \pm (\theta + \omega) p$, where $\pm$ correspond to positive and negative directions of momentum $p$. Let be $v_F > \theta$. This corresponds to phase transition between type I and III. The electron states with positive momentum and hole states with negative momentum in phase transition point can be associated with infinite group velocity $\partial \varepsilon / \partial p$. Indeed, in this case from Eq. (9) one has

$$\lim_{\varepsilon \rightarrow \infty} \varepsilon = \infty.$$

Such situation is described by vertical line. The second branch of spectrum originates from electron states with negative momentum and hole states with positive moment. For these states from Eq. (9) we have

$$\lim_{\varepsilon \rightarrow v_F} \varepsilon = -\frac{v_F^2 p^2 - (\omega \cdot p)^2}{2(\theta + \omega) \cdot p}.$$  

(16)

These two branches for the cases $v_F > \theta$, $v_F = \theta$ and $v_F < \theta$ are provided at Fig. 3 in case of type III (the qualitative picture for the case of type IV is similar). Let discuss these transitions in details. To do this, let consider the linear function $y = (a - b)x$. For $a > b$ this function describes the line with positive slope. At point $a = b$ the continuous transition from positive to negative slope occurs. At the same time the point $a = b$ corresponds to horizontal line, i.e. the transition occurs through $X$ axis. Is it possible to make a phase transition from positive slope to negative through $Y$ axis? This transition correspond to type I-III (II-IV) and it is completely different compared to previous one. The main difference is that during the phase transition there is the swapping of the electron and hole states. Such kind of transition can be described with function of form $y = x/(a - b)$. For $a > b$ this function describes the straight line with positive slope. For $a < b$ the slope is negative. The point $a = b$ correspond to transition from positive to negative slope. However, compared to previous example, $(a - b)x$, this is the singularity point. Thus, the transition from positive to negative slope is not continuous. In other words, in second example conditions $a > b$ and $a < b$ correspond to fundamentally different
phases, and the continuous transition between them is forbidden. Now, it become clear that types I and II are not topologically different since the continuous transition is allowed. Consequently, in this context, types I and III, I and IV are topologically different phases. Note that the discussion above is true only for one branch (in our case it is red line). For blue branch there is no inversion of electron and hole parts of spectrum. Indeed, the slope of blue branch of spectrum does not depend on the ratio between $v_F$ and $\vartheta$.

Finally, let us discuss the possible origin of non-Hermitian Weyl Hamiltonian of types III and IV. The authors of Ref. [26] suggest to associate the Weyl fermions with right hand side of Eq. (17). In the vicinity of Weyl point one has $f_i(p) \approx v_F p_i$ and $f_0(p) \approx \omega \cdot p$. If one requires the overlap matrix in Pauli basis has components $S^\mu_{\alpha\beta}(v_\nu) = [\mp \delta_\alpha \delta_\beta] / v_F$, then the effective Hamiltonian, $\tilde{H}(C_u, C_v)^T = \tilde{\mathcal{H}}(C_u, C_v)$, completely coincides with the Hamiltonian from Eq. (3) with the same prefactor $1/(1 - \beta^2)$. Thus the origin of this Hamiltonian can be related with the overlap between Bloch functions. For 1D chain such a Hamiltonian has been recently introduced by generalizing the Su-Schrieffer-Heeger model on non-hermitian case [41].

To summarize, we have provided the theory of type III and IV Weyl semimetals within non-Hermitian Hamiltonian formalism. The spectrum of this Weyl Hamiltonian exhibits an unusual type of anisotropy in momentum space, namely for some directions the spectrum is real, in contrast for other directions it is complex. The necessary and sufficient conditions for spectrum to be real is provided. Moreover, the anisotropy in spectrum results in anisotropy of the inner product of states, and thus leads to non-trivial adiabatic evolution and fractional Chern number. Additionally, we speculate the possible origin of non-Hermitian Hamiltonian.

We are grateful to G. E. Volovik, J. Nissinen and T. L. Schmidt for fruitful discussions. Z. A. acknowledges financial support from President Grant (MD 647.2020.2). E. I. acknowledges financial support from the National Research Fund Luxembourg under Grant CORE C19/MS/13579612/HYBMES.

\section*{References}

[1] Bansil, A. and Lin, Hsin and Das, Tanmoy, Rev. Mod. Phys. 88, 021004 (2016)
[2] Gilbert, Matthew J., Communications Physics 4, 70 (2021)
[3] Z. K. Liu and B. Zhou and Y. Zhang and Z. J. Wang and H. M. Weng and D. Prabhakaran and S.-K. Mo and Z. X. Shen and Z. Fang and X. Dai and Z. Hussain and Y. L. Chen, Science 343, 864867 (2014)
[4] Borsenko, Sergey and Gibson, Quinn and Evtushinsky, Danil and Zabolotnyy, Volodymyr and Büchner, Bernd and Cava, Robert J., Phys. Rev. Lett. 113, 027603 (2014)
[5] Yang, Bohm-Jung and Nagaosa, Naoto, Nature Commu-
Supplementary Material for “Towards a theory of types III and IV non-Hermitian Weyl fermions”

A: ANISOTROPY OF ORTHOGONALITY CONDITIONS

We write the spectrum from Eq. (9) in main text with additional subscripts as

$$\varepsilon_{R,\pm} = \varepsilon_{L,\pm} = \frac{A(p) \pm \sqrt{B(p)}}{1 - \beta^2},$$

(S-1)

where ”±" are associated with electron and hole states. The corresponding eigenvectors are denoted by $$| \Psi_{\pm}^{R,L} \rangle$$, which satisfy wave equations \( \mathcal{H} | \Psi_{\pm}^{R} \rangle = \varepsilon_{R,\pm} | \Psi_{\pm}^{R} \rangle \), and \( \mathcal{H}^\dagger | \Psi_{\pm}^{L} \rangle = \varepsilon_{L,\pm} | \Psi_{\pm}^{L} \rangle \). It is worth mentioning that \( \varepsilon_R = \varepsilon_L = \varepsilon \), where ”±" is omitted. In phases III and IV the spectrum is anysotropic (see Eq. (9) from main text) with respect to space orientation of momentum vector: for some directions the spectrum is real, in contrast, for other directions it is complex. The existence of domain, where \( \text{Im}(\varepsilon) \neq 0 \) is the main distinguishing feature of the new phases, III and IV. Apart from this, in domain of complex spectrum \( \varepsilon_{\pm}^* = \varepsilon_{\mp} \), which leads to important consequences. The main one is related with scalar product \( \langle \Psi_{\alpha}^L | \Psi_{\beta}^R \rangle \). Let consider the following 2 × 2 matrix

\[
\begin{pmatrix}
\langle \Psi_{\alpha}^L | \mathcal{H} | \Psi_{\beta}^R \rangle \\
\langle \Psi_{\alpha}^L | \mathcal{H}^\dagger | \Psi_{\beta}^R \rangle
\end{pmatrix}.
\]

(S-2)

Applying the Hamiltonian inside matrix elements on right and left hand sides one obtains

\[
\begin{pmatrix}
\varepsilon_+ \langle \Psi_{\alpha}^L | \Psi_{\beta}^R \rangle \\
\varepsilon_+ \langle \Psi_{\alpha}^L | \Psi_{\beta}^R \rangle
\end{pmatrix} = \left( \begin{pmatrix}
\varepsilon_+^* \langle \Psi_{\beta}^L | \Psi_{\alpha}^R \rangle \\
\varepsilon_+^* \langle \Psi_{\beta}^L | \Psi_{\alpha}^R \rangle
\end{pmatrix} \begin{pmatrix}
\varepsilon_+ \langle \Psi_{\alpha}^L | \Psi_{\beta}^R \rangle \\
\varepsilon_+ \langle \Psi_{\alpha}^L | \Psi_{\beta}^R \rangle
\end{pmatrix}
\right).
\]

(S-3)

For real spectrum from the above equation it is obvious that \( \langle \Psi_{\beta}^L | \Psi_{\alpha}^R \rangle = 0 \) and \( \langle \Psi_{\beta}^L | \Psi_{\alpha}^R \rangle \neq 0 \). At the same time, for the domain, where the spectrum is complex one gets \( \langle \Psi_{\beta}^L | \Psi_{\alpha}^R \rangle = 0 \) and \( \langle \Psi_{\beta}^L | \Psi_{\alpha}^R \rangle \neq 0 \).

B: THE FORM OF EIGENSTATES

The Hamiltonian under consideration has the form

$$\mathcal{H} = A(p) \sigma_0 + [D(p) + iC(p)] \cdot \sigma,$$

(S-4)

where \( D \cdot C = 0 \). The spectrum of this Hamiltonian is given by

$$\varepsilon_{R,\pm} = \varepsilon_{L,\pm} = \varepsilon_{\pm} = A \pm \sqrt{D^2 - C^2},$$

(S-5)

where \( C = |C| \) and \( D = |D| \). Particularly, for the case under consideration in main text, the parameters are

\( A = (\omega \cdot p + \vartheta \cdot p)/(1 - \beta^2) \), \( D = (v_F p + \frac{1}{v_F} \vartheta (\omega \cdot p))/(1 - \beta^2) \) and \( C = |p \times \vartheta|/(1 - \beta^2) \). The eigenstates of Hamiltonian are given by

\[
|\Psi_{\pm}^{R} \rangle = \begin{pmatrix}
C_{\pm-\alpha} + \frac{\sqrt{D^2 - C^2}}{1}
C_{\pm-\alpha}(C_{\pm-\alpha} - D_{y}) + D_{y}
\end{pmatrix},
|\Psi_{\pm}^{L} \rangle = \begin{pmatrix}
C_{\pm-\alpha} + \frac{\sqrt{D^2 - C^2}}{1}
C_{\pm-\alpha}(C_{\pm-\alpha} + D_{y}) - D_{y}
\end{pmatrix},
\]

(S-6)

and

\[
\langle \Psi_{\pm}^{R} | = \begin{pmatrix}
C_{\pm-\alpha} + \frac{\sqrt{D^2 - C^2}}{1}
C_{\pm-\alpha}(C_{\pm-\alpha} - D_{y}) + D_{y}
\end{pmatrix}^*,
\langle \Psi_{\pm}^{L} | = \begin{pmatrix}
C_{\pm-\alpha} + \frac{\sqrt{D^2 - C^2}}{1}
C_{\pm-\alpha}(C_{\pm-\alpha} + D_{y}) - D_{y}
\end{pmatrix}^*.
\]

(S-7)

where "*" means the complex conjugation. Using the form of eigenstates one can show that \( \langle \Psi_{\pm}^{L} | \Psi_{\pm}^{R} \rangle = 0 \) for \( D > C \) and \( \langle \Psi_{\pm}^{L} | \Psi_{\pm}^{R} \rangle = 0 \) for \( D < C \). Further, the direct calculations give \( \langle \Psi_{\pm}^{L} | \Psi_{\pm}^{R} \rangle = C^2 - D^2 - |D^2 - C^2| + 2 i (C_{\pm} - D_{\pm}) \sqrt{D^2 - C^2} \). Therefore, for \( D < C \) we get \( \langle \Psi_{\pm}^{L} | \Psi_{\pm}^{R} \rangle = 0 \). On the other hand, \( \langle \Psi_{\pm}^{L} | \Psi_{\pm}^{R} \rangle = C^2 - D^2 + |D^2 - C^2| + 2 (C_{\pm} - D_{\pm}) \text{Im} \sqrt{D^2 - C^2} \). Thus, for \( D > C \) we have \( \langle \Psi_{\pm}^{L} | \Psi_{\pm}^{R} \rangle = 0 \).
From the main text, Eq. (12), it is clear, that $[\eta, \mathcal{H}] = \eta \mathcal{H} - \mathcal{H} \eta \neq 0$, for non-Hermitian $\mathcal{H}$. Indeed, otherwise $[\eta, \mathcal{H}] = 0$ and one arrives to contradiction, namely the conjugated Hamiltonian $\mathcal{H}^\dagger = \eta \mathcal{H} \eta^{-1} = \mathcal{H}$ is Hermitian. As an example, let consider the simple non-Hermitian matrix $M = \begin{pmatrix} 1 & 1 + \lambda \\ 1 - \lambda & -1 \end{pmatrix}$, where $\lambda$ is the arbitrary number. Next, let find the matrix $\eta$, which satisfies the condition $M^\dagger = \eta M \eta^{-1}$. One can show that the $\eta$ has the following form $\eta = \begin{pmatrix} \frac{1 + \lambda}{1 - \lambda} & q \\ \frac{1 - \lambda}{1 + \lambda} & -q \end{pmatrix}$, where $p, q$ are arbitrary numbers as well. Further, in order to have the real eigenvalues for matrix $M$, there must be possibility to present $\eta = S^\dagger S$. From these condition, one consequently obtains that $q, r \in \mathbb{R}$ and $\eta^\dagger = \eta$. Further, it is obvious, that the condition det($SS^\dagger$) > 0 must be satisfied. This condition set limits on the choice of elements of matrix $\eta$. In our case, this condition brings us to inequality for $p, q$ and $\lambda$, namely $\frac{q^2 - r^2}{\lambda^2 - 1} - q^2 > 0$. This inequality does not work for $\lambda^2 > 2$. Thus, the eigenvalues of matrix $M$ are real for $\lambda^2 < 2$.

Let now investigate our Hamiltonian in the same manner above. For simplicity, we consider the case of $\theta = (0, 0, \vartheta)$. In this case, the Hamiltonian in Eq. (10) from main text is written as

$$\mathcal{H} = \frac{v_F}{1 - \beta^2} \left[ \begin{pmatrix} p_z \\ (1 + \beta)(p_x + ip_y) \\ -p_z \end{pmatrix} (1 - \beta) \begin{pmatrix} p_z - ip_y \\ p_z \end{pmatrix} + \beta \sigma_0 \right],$$

(S-8)

where $\beta = \theta/v_F$ and $\sigma_0$ is the identity matrix. The matrix $\eta$ has the form $\eta = \begin{pmatrix} a & b \\ \bar{b}^* & d \end{pmatrix}$, where $a = \frac{r - \beta p_x}{1 + \beta(p_x - ip_y)}$ and $d = \frac{r + \beta p_x}{1 + \beta p_x - ip_y}$, $r, b \in \mathbb{C}$. From condition $\eta = S^\dagger S$ one gets that $q, r > 0 \in \mathbb{R}$ and the condition $\det(SS^\dagger) > 0$ results in inequality for other parameters, $ad - |b|^2 > 0 \in \mathbb{R}$. Using these conditions at $\beta < 1$, one can show that matrix $\eta = S^\dagger S$ exists always. For example, $\eta = \begin{pmatrix} \frac{q}{\beta} & 0 \\ 0 & \frac{1}{\beta + \beta} \end{pmatrix}$, where $q > 0 \in \mathbb{R}$. For $\beta > 1$ the matrix $\eta = S^\dagger S$ exists only for $\frac{p_x^2}{p_x^2 + p_y^2} < \beta^2 - 1$, which turns out to be $|\sin \chi| < \beta^{-1}$ in spherical coordinates, and this condition coincides with the one we have provided in Eq. (9) from main text. In this case the matrix has the form $\eta = \begin{pmatrix} \frac{p_z}{\beta - 1} & -p_z + ip_y \\ -p_z - ip_y & \frac{p_z}{\beta + 1} \end{pmatrix}$.

In other words, for $|\sin \chi| < \beta^{-1}$ the Hamiltonian above, Eq. (S-8) can be brought to hermitian Hamiltonian using similarity transformations. Namely $S^{-1} \mathcal{H} S$ is the Hermitian operator, despite the fact that $\mathcal{H}$ is non-Hermitian. Additionally, it is worth pointing out, that redefinition of scalar product in Hilbert space (unitary theorem) in the following form $\langle \Psi | \bar{\Psi} \rangle_S = \langle \Psi | S^\dagger S | \Psi \rangle$ does depend on time, if Hamiltonian satisfies the condition Eq. (12) from main text. Thus, this means that the probability density does not depend on time, and the unitary condition is suited.

As one can observe from Eq. (12) from main text, in general the realness of spectrum is defined not only by the properties of operator by itself, but as well it influences on Hilbert space. It worth mentioning, that the $\mathcal{PT}$ symmetry for non-Hermitian quantum systems introduced by Bender [1], says that $\mathcal{PT}$ operator and Hamiltonian have the same eigenvectors. In this case Hamiltonian has the real spectrum, despite the fact that it is non-Hermitian. This requirements is the particular case of Eq. (12) from main text. Namely, to show this, let change a bit the matrix we have considered above, $M = \begin{pmatrix} 1 & \lambda - 1 \\ \lambda + 1 & -1 \end{pmatrix}$. This matrix is not hermitian and it does not have $\mathcal{PT}$ symmetry. However, the eigenvalues of this matrix are real for any real $\lambda$. This is related with the fact that matrix $\begin{pmatrix} 1 & \lambda - 1 \\ \lambda + 1 & -1 \end{pmatrix}$ is equivalent to hermitian matrix $\begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$. Namely, there always exists the similarity transformation $R$, therefore $R \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} R^{-1} = \begin{pmatrix} 1 & \lambda - 1 \\ \lambda + 1 & -1 \end{pmatrix}$.

D: FRACTIONAL CHERN NUMBER

The Chern number is given by

$$N_\alpha = \frac{1}{4\pi} \int \Omega_\alpha \cdot dS,$$

(S-9)
where \( \Omega_\alpha \) is Berry curvature of band \( \alpha \), and the integration is carried out with respect to closed surface, enclosing the Weyl point. We rewrite the Eq. (S-9) as

\[
N_\alpha = \frac{1}{4\pi} \left( \int_{S_C} \Omega_\alpha \cdot dS + \int_{S_R} \Omega_\alpha \cdot dS \right),
\]

(S-10)

where \( S_C \) and \( S_R \) are domains of surface corresponding to complex and real spectra (see Fig. 4 from main text). The Berry curvature for non-Hermitian Hamiltonian is given by

\[
\Omega_\alpha = i \partial_p \times \langle \psi^L_\alpha | \partial_p | \psi^R_\alpha \rangle.
\]

(S-11)

One can show, that \( \int_{S_C} \Omega_\alpha \cdot dS = 0 \). To do this, we consider the time-evolution of eigenstates at \( D = C \). In this case the spectrum has only one band, \( \varepsilon_R = \varepsilon_L = A(p) \) and the eigenstates are given by

\[
|\psi^R_\pm\rangle = \left( \frac{C_x + i D_x}{1} \right), \quad |\psi^L_\pm\rangle = \left( \frac{C_x + i D_x}{1} \right).
\]

(S-12)

From the above equations one gets \( \langle \psi^L | |\psi^R\rangle = \langle \psi^R | |\psi^L\rangle = 0 \), and for adiabatic time evolution we have \( i \partial_t |\psi^R(t)\rangle = \mathcal{H}(t)|\psi^R(t)\rangle = \varepsilon(t)|\psi^R(t)\rangle \). Therefore

\[
i \langle \psi^L | \partial_t |\psi^R\rangle = \varepsilon(t) \langle \psi^L | |\psi^R\rangle = 0, \quad \text{or} \quad \langle \psi^L | \partial_p |\psi^R\rangle \cdot \partial_t p = 0.
\]

(S-13)

Consequently, the line integral \( \oint \langle \psi^L | \partial_p |\psi^R\rangle \cdot dp \), corresponding to equation \( D = C \) vanishes. Indeed

\[
\oint \langle \psi^L | \partial_p |\psi^R\rangle \cdot dp = \int_0^T \langle \psi^L | \partial_p |\psi^R\rangle \cdot dt = 0,
\]

(S-14)

where the cyclic evolution around closed path is denoted by times 0 and \( T \). The line (contour) for \( D = C \) is presented at Fig. 4 in main text. According to Stokes’s theorem

\[
\oint_{l_1 + l_2} \langle \psi^L_\alpha | \partial_p |\psi^R_\alpha\rangle \cdot dp = \int_{S_C} (\partial_p \times \langle \psi^L_\alpha | \partial_p |\psi^R_\alpha\rangle) \cdot d^2p = 0,
\]

(S-15)

since the contour \( l_1 + l_2 \) is the boundary of domain \( S_C \) (see Fig. 4 in main text). It is worth mentioning, that the Stokes’s theorem can be applied only to domain \( S_C \), because the orthogonality condition \( \langle \psi^L_\alpha | \psi^R_\alpha\rangle = 0 \) is the same as for contour \( l_1 + l_2 \). Thus, the domain of momentum space, \( S_C \), corresponding to complex spectrum does not contribute to integral, Eq. (S-15). Consequently, it follows that the Chern number is given by

\[
N_\alpha = \frac{1}{4\pi} \int_{S_R} \Omega_\alpha \cdot d^2p.
\]

(S-16)

Thereby, only part of the sphere in momentum space contributes to integral, which results in fractional Chern number. Thus, the Weyl fermions of types III and IV are characterized by fractional Chern number.

[1] C. M. Bender, *PT Symmetry In Quantum and Classical Physics*, (World Scientific, Singapore, 2019)