SOME BEST APPROXIMATION FORMULAS AND INEQUALITIES FOR WALLIS RATIO

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Abstract. In the paper, the authors establish some best approximation formulas and inequalities for Wallis ratio. These formulas and inequalities improve an approximation formula and a double inequality for Wallis ratio recently presented in “S. Guo, J.-G. Xu, and F. Qi, Some exact constants for the approximation of the quantity in the Wallis’ formula, J. Inequal. Appl. 2013, 2013:67, 7 pages; Available online at http://dx.doi.org/10.1186/1029-242X-2013-67”.

1. Introduction

Wallis ratio is defined as
\[ W_n = \frac{(2n-1)!!}{(2n)!!} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}, \]
where \( \Gamma \) is the classical Euler gamma function which may be defined by
\[ \Gamma(z) = \int_0^\infty u^{z-1} e^{-u} \, du, \quad \Re(z) > 0. \quad (1.1) \]

The study and applications of \( W_n \) have a long history, a large amount of literature, and a lot of new results. For detailed information, please refer to the papers [1, 4, 18, 21], related texts in the survey articles [17, 19, 20] and references cited therein. Recently, Guo, Xu, and Qi proved in [5] that the double inequality
\[ \sqrt{\frac{\pi}{e}} \left( 1 - \frac{1}{2n} \right)^{\frac{\sqrt{n}-1}{n}} < W_n \leq \frac{4}{3} \left( 1 - \frac{1}{2n} \right)^{\frac{\sqrt{n}-1}{n}} \quad (1.2) \]
for \( n \geq 2 \) is valid and sharp in the sense that the constants \( \sqrt{\frac{\pi}{e}} \) and \( \frac{4}{3} \) in (1.2) are best possible. They also proposed in [5] the approximation formula
\[ W_n \sim \chi_n := \sqrt{\frac{\pi}{e}} \left( 1 - \frac{1}{2n} \right)^{\frac{\sqrt{n}-1}{n}}, \quad n \to \infty. \quad (1.3) \]

The sharpness of the double inequality (1.2) was proved in [5] basing on the variation of a function which decreases on \([2, \infty)\) from \(4/3\) to \(\sqrt{\frac{\pi}{e}}\). As a consequence, the right-hand side of (1.2) becomes weak for large values of \(n\). Moreover, if we are interested to estimating \(W_n\) when \(n\) approaches infinity, then the constant \(\sqrt{\frac{\pi}{e}}\) should be chosen and inequalities using \(\sqrt{\frac{\pi}{e}}\) are welcome.

The aim of this paper is to improve the double inequality (1.2) and the approximation formula (1.3).

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2. A LEMMA

For improving the double inequality (1.2) and the approximation formula (1.3), we need the following lemma.

**Lemma 2.1 ([12, Lemma 1.1]).** If the sequence \( \{ \omega_n : n \in \mathbb{N} \} \) converges to 0 and
\[
\lim_{n \to \infty} n^k (\omega_n - \omega_{n+1}) = \ell \in \mathbb{R} \quad (2.1)
\]
for \( k > 1 \), then
\[
\lim_{n \to \infty} n^{k-1} \omega_n = \frac{\ell}{k-1}. \quad (2.2)
\]

**Remark 2.1.** Lemma 2.1 was first established in [15] and has been effectively applied in many papers such as [2, 3, 6, 7, 8, 9, 10, 11, 13, 14, 16].

3. A BEST APPROXIMATION FORMULA

With the help of Lemma 2.1, we first provide a best approximation formula of Wallis ratio \( W_n \).

**Theorem 3.1.** The approximation formula
\[
W_n \sim \sqrt{\frac{e}{\pi}} \left(1 - \frac{1}{2n}\right)^n \frac{1}{\sqrt{n}}, \quad n \to \infty \quad (3.1)
\]
is the best approximation of the form
\[
W_n \sim \sqrt{\frac{e}{\pi}} \left(1 - \frac{1}{2n}\right)^n \frac{\sqrt{n + a}}{n}, \quad n \to \infty, \quad (3.2)
\]
where \( a \) is a real parameter.

**Proof.** Define \( z_n(a) \) by
\[
W_n = \sqrt{\frac{e}{\pi}} \left(1 - \frac{1}{2n}\right)^n \frac{\sqrt{n + a}}{n} \exp z_n(a), \quad n \geq 1.
\]
It is not difficult to see that \( z_n(a) \to 0 \) as \( n \to \infty \). A direct computation gives
\[
z_n(a) - z_{n+1}(a) = -\frac{a}{2n^2} + \left(\frac{1}{2} + \frac{1}{2} a^2 + \frac{1}{12}\right) \frac{1}{n^3} + O\left(\frac{1}{n^4}\right)
\]
and
\[
\lim_{n \to \infty} \left\{ n^2 [z_n(a) - z_{n+1}(a)] \right\} = -\frac{a}{2}.
\]
Making use of Lemma 2.1, we immediately see that the sequence \( \{ z_n(a) : n \in \mathbb{N} \} \) converges fastest only when \( a = 0 \). The proof of Theorem 3.1 is complete. \( \square \)

**Remark 3.1.** The approximation formula (3.1) is an improvement of (1.3), since the approximation formula (1.3) is the special case \( a = -1 \) in (3.2).

4. AN ASYMPTOTIC SERIES ASSOCIATED TO (3.1)

In this section, by discovering an asymptotic series and a single-sided inequality for Wallis ratio, we further generalize the approximation formula (3.1) and improve the left-hand side of the double inequality (1.2).

**Theorem 4.1.** As \( n \to \infty \), we have
\[
W_n \sim \sqrt{\frac{e}{\pi}} \left(1 - \frac{1}{2n}\right)^n \frac{1}{\sqrt{n}} \exp \left(\frac{1}{24n^2} + \frac{1}{48n^3} + \frac{1}{160n^4} + \frac{1}{960n^5} + \cdots\right).
\]
Proof. Recall from [15] that, to an approximation formula \( f(n) \sim g(n) \), the following asymptotic series is associated

\[
f(n) \sim g(n) \exp \left( \sum_{k=1}^{\infty} \frac{a_k}{n^k} \right),
\]

where \( a_k \) for \( k \geq 2 \) is a solution of the following infinite triangular system

\[
a_1 - \binom{k-1}{1} a_2 + \cdots + (-1)^k \binom{k-1}{k-2} a_{k-1} = (-1)^k x_k
\]

(4.1)

and \( x_k \) are coefficients of the expansion

\[
\ln \frac{f(n)g(n+1)}{g(n)f(n+1)} = \sum_{k=2}^{\infty} x_k n^k.
\]

Replacing \( f(n) \) and \( g(n) \) by \( W_n \) and \( \sqrt{\frac{e}{n}} (1 - \frac{1}{2n})^n \), respectively, yields

\[
\ln \frac{f(n)g(n+1)}{g(n)f(n+1)} = \sum_{k=2}^{\infty} (-1)^k \left[ \frac{1 + (-1)^k}{(k+1)2^{k+1}} - \frac{1}{k+1} + \frac{1}{2k} \right] \frac{1}{n^k}.
\]

Hence, the system (4.1) becomes

\[
a_1 - \binom{k-1}{1} a_2 + \cdots + (-1)^k \binom{k-1}{k-2} a_{k-1} = \frac{1 + (-1)^k}{(k+1)2^{k+1}} - \frac{1}{k+1} + \frac{1}{2k}
\]

which has a solution

\[
a_1 = 0, \quad a_2 = \frac{1}{24}, \quad a_3 = \frac{1}{48}, \quad a_4 = \frac{1}{160}, \quad a_5 = \frac{1}{960}, \quad \ldots.
\]

The proof of Theorem 4.1 is complete. \( \square \)

**Theorem 4.2.** For every integer \( n \geq 1 \), we have

\[
W_n > \sqrt{\frac{e}{n}} \left( 1 - \frac{1}{2n} \right)^n \frac{1}{\sqrt{n}} \exp \left( \frac{1}{24n^2} + \frac{1}{48n^3} + \frac{1}{160n^4} + \frac{1}{960n^5} \right).
\]

(4.2)

Proof. It suffices to prove

\[
\alpha_n = n \ln \left( 1 - \frac{1}{2n} \right) - \frac{1}{2} \ln n - \ln \frac{(2n-1)!!}{(2n)!!} + \ln \sqrt{\frac{e}{n}} + h(n) < 0,
\]

where

\[
h(x) = \frac{1}{24x^2} + \frac{1}{48x^3} + \frac{1}{160x^4} + \frac{1}{960x^5}.
\]

Because \( \alpha_n \) converges to 0, it is sufficient to show that the sequence \( \{\alpha_n : n \in \mathbb{N}\} \) is strictly increasing. It is not difficult to obtain \( \alpha_{n+1} - \alpha_n = s(n) \), where

\[
s(x) = (x+1) \ln \left( 1 - \frac{1}{2x+2} \right) - x \ln \left( 1 - \frac{1}{2x} \right)
\]

\[
- \frac{1}{2} \ln \left( 1 + \frac{x}{1} \right) - \ln \frac{2x+1}{2x+2} + h(x) - h(x),
\]

\[
s''(x) = \frac{C(x-1)}{32x^7(x+1)^2(2x+1)^2(2x-1)^2}
\]

and

\[
C(x) = 4913 + 33387x + 98177x^2 + 164799x^3 + 174543x^4
\]

\[
+ 121173x^5 + 55197x^6 + 15920x^7 + 2640x^8 + 192x^9.
\]
Accordingly, the function \( s(x) \) is strictly convex on \([1, \infty)\). Combining this with the fact that \( \lim_{x \to \infty} s(x) = 0 \) reveals that the function \( s(x) \) on \([1, \infty)\), and so the sequence \( \{s(n) : n \in \mathbb{N}\} \), is positive. The proof of Theorem 4.2 is complete. \( \Box \)

5. A NEW APPROXIMATION FORMULA AND A DOUBLE INEQUALITY

Finally we will find a new approximation formula and a double inequality for Wallis ratio \( W_n \).

**Theorem 5.1.** As \( n \to \infty \), we have

\[
W_n \sim \mu_n := \sqrt{\frac{e}{\pi}} \left[ 1 - \frac{1}{2(n + 1/3)} \right]^{n+1/3} \frac{1}{\sqrt{n}}.
\]

**Proof.** Motivated by (3.1), we now ask for the best approximation of the form

\[
W_n \sim \sqrt{\frac{e}{\pi}} \left[ 1 - \frac{1}{2(n + b)} \right]^{n+c} \frac{1}{\sqrt{n}}, \quad n \to \infty,
\]

where \( b \) and \( c \) are real parameters. For this, let

\[
W_n = \sqrt{\frac{e}{\pi}} \left[ 1 - \frac{1}{2(n + b)} \right]^{n+c} \frac{1}{\sqrt{n}} \exp \beta_n(b, c).
\]

Then an easy calculation leads to

\[
\beta_n(b, c) - \beta_{n+1}(b, c) = \frac{1}{2}(c - b) \frac{1}{n^2} + \left( b^2 - bc - \frac{1}{4}c + \frac{1}{12} \right) \frac{1}{n^3}
\]
\[
+ \left( \frac{1}{4}c - \frac{1}{8}b + \frac{3}{4}bc - \frac{3}{8}b^2 - \frac{3}{2}b^3 + \frac{3}{2}b^2c - \frac{1}{16} \right) \frac{1}{n^4} + O\left( \frac{1}{n^5} \right).
\]

This implies that

\[
\lim_{n \to \infty} \left\{ n^2[\beta_n(b, c) - \beta_{n+1}(b, c)] \right\} = \frac{c - b}{2}
\]

and

\[
\lim_{n \to \infty} \left\{ n^2[\beta_n(b, b) - \beta_{n+1}(b, b)] \right\} = \frac{3b - 1}{12}.
\]

By Lemma 2.1, it follows that the sequence \( \{\beta_n(b, c) : n \in \mathbb{N}\} \) converges fastest only when \( b = c = \frac{1}{3} \). The proof of Theorem 5.1 is complete. \( \Box \)

**Remark 5.1.** We note that the approximation formula (5.1) is the most accurate possible among a class of approximation formulas mentioned above. The numerical computation in Table 1 shows the superiority of (5.1) over (1.3).

**Table 1.** Numerical computation

| \( n \)  | \( W_n - \chi_n \)   | \( W_n - \mu_n \) |
|---------|---------------------|---------------------|
| 50      | 8.0124 × 10^{-4}    | 4.4198 × 10^{-9}    |
| 100     | 2.8269 × 10^{-4}    | 3.9124 × 10^{-10}   |
| 250     | 7.1425 × 10^{-5}    | 1.5850 × 10^{-14}   |
| 1000    | 8.9225 × 10^{-6}    | 1.2388 × 10^{-13}   |

**Theorem 5.2.** For every integer \( n \geq 1 \), we have

\[
\sqrt{\frac{e}{\pi}} \left[ 1 - \frac{1}{2(n + 1/3)} \right]^{n+1/3} \frac{1}{\sqrt{n}} < W_n
\]
\[
< \sqrt{\frac{e}{\pi}} \left[ 1 - \frac{1}{2(n + 1/3)} \right]^{n+1/3} \frac{1}{\sqrt{n}} \exp\left( \frac{1}{144n^3} \right). \quad (5.2)
\]
Proof. It is sufficient to prove
\[ b_n = \left( n + \frac{1}{3} \right) \ln \left( 1 - \frac{1}{2(n + 1/3)} \right) - \frac{1}{2} \ln n - \ln \left( \frac{2n - 1}{(2n)!} \right) + \ln \sqrt{\frac{e}{\pi}} < 0 \]
and
\[ c_n = b_n + \frac{1}{144n^3} > 0. \]
Because \( b_n \) and \( c_n \) converge to 0, it suffices to show that \( b_n \) is strictly increasing and \( c_n \) is strictly decreasing. For this, we discuss the differences \( b_{n+1} - b_n = p(n) \) and \( c_{n+1} - c_n = q(n) \), where
\[
p(x) = \left( x + \frac{4}{3} \right) \ln \left( 1 - \frac{1}{2(x + 4/3)} \right) - \left( x + \frac{1}{3} \right) \ln \left( 1 - \frac{1}{2(x + 1/3)} \right) - \frac{1}{2} \ln \left( 1 + \frac{1}{x} \right) - \ln \frac{2x + 1}{2x + 2} \]
and
\[
q(x) = p(x) + \frac{1}{144(x + 1)^3} - \frac{1}{144x^3}. \]
Since
\[
p''(x) = \frac{A(x - 1)}{2x^2(3x + 1)(3x + 4)(x + 1)^2(2x + 1)^2(6x - 1)^2(6x + 5)^2} > 0 \]
and
\[
q''(x) = -\frac{B(x - 1)}{12x^5(3x + 1)(3x + 4)(2x + 1)^2(6x - 1)^2(x + 1)^5(6x + 5)^2} < 0, \]
where
\[
A(x) = 351068 + 1516131x + 2684091x^2 + 2495340x^3 + 1285956x^4 + 348624x^5 + 38880x^6 \]
and
\[
B(x) = 6780036 + 50421819x + 166596550x^2 + 322415601x^3 + 405307306x^4 + 346439295x^5 + 20449525x^6 + 82629900x^7 + 45508200x^8 + 305208x^9 + 7776x^{10} + 7776x^{11}, \]
it follows that \( p(x) \) is strictly convex and \( q(x) \) is strictly concave on \([1, \infty)\). As a result, considering the fact that \( \lim_{x \to \infty} p(x) = \lim_{x \to \infty} q(x) = 0 \), we derive that \( p(x) > 0 \) and \( q(x) < 0 \) on \([1, \infty)\). Consequently, the sequences \( \{ p(n) : n \in \mathbb{N} \} \) and \( \{ q(n) : n \in \mathbb{N} \} \) are positive. The proof of Theorem 5.2 is complete. □

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