Existence of non-negative solutions for semilinear elliptic systems via variational methods

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Abstract. In this paper we consider a semilinear elliptic system with nonlinearities, indefinite weight functions and critical growth terms in bounded domains. The existence result of nontrivial nonnegative solutions is obtained by variational methods.

Keywords: variational methods, Robin boundary condition, first eigenvalue.

1 Introduction

In this paper we consider the existence results of the following two coupled semilinear equation

\begin{equation}
-\Delta u = \lambda au - (pav^2u|u|^{p-2} + 2av|v|^p), \quad x \in \Omega, \\
-\Delta v = \lambda av - (pau^2v|v|^{p-2} + 2au|u|^p), \quad x \in \Omega, \\
(1 - \alpha)\frac{\partial u}{\partial n} + \alpha u = (1 - \alpha)\frac{\partial v}{\partial n} + \alpha v = 0, \quad x \in \partial\Omega,
\end{equation}

where \(\alpha\) and \(\lambda\) are real parameters, \(p < 2^* - 2\), for \(2^* = \frac{2N}{N-2}\), \(\Omega\) is an open bounded domain in \(\mathbb{R}^N\), \(N \geq 3\) with a smooth boundary \(\partial\Omega\), and \(a : \Omega \to \mathbb{R}\) is a sign changing weight function.

This work is motivated by the results in the literature for the single equation case, namely the equation of the form

\begin{equation}
-\Delta u = \lambda g(x)(1 + |u|^p)u, \quad x \in \Omega, \\
(1 - \alpha)\frac{\partial u}{\partial n} + \alpha u = 0, \quad x \in \partial\Omega,
\end{equation}

with the sign changing weight function \(g\). See [1–4] and references therein for the case where \(\alpha \neq 0\), and [5] for the case where \(\alpha = 1\). Recently in [6] some existence results

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were established for the case when a single equation is replace by a system of equations. Also we refer to [7] where \( \Delta \) is replaced by \( \Delta_p \). In this work we extend this studies to classes of Robin boundary conditions. We prove our existence results via variational methods.

The system (1) is posed in the framework of the Sobolev space \( H = H^1(\Omega) \times H^1(\Omega) \) with the norm

\[
\| (u, v) \|_H = \left( \int_\Omega (|\nabla u|^2 + u^2) + \int_\Omega (|\nabla v|^2 + v^2) \right)^{\frac{1}{2}}.
\]

Moreover a pair of functions \((u, v) \in H\) is said to be a weak solution of the system (1) if

\[
\begin{align*}
\int_\Omega \nabla u \nabla \phi_1 + \int_\Omega \nabla v \nabla \phi_2 &- \lambda \int_\Omega a(u \phi_1 + v \phi_2) + \alpha \int_{\partial \Omega} (u \phi_1 + v \phi_2) \\
+ p \int_\Omega (av_2^p |u|^{p-2}u \phi_1 + au_2^p |v|^{p-2}v \phi_2) + 2 \int_\Omega (au|v|^p \phi_1 + av|u|^p \phi_2) &= 0
\end{align*}
\]

for all \((\phi_1, \phi_2) \in H\). Thus the corresponding energy functional to the system (1) is defined by

\[
J_{\lambda}(u, v) = \frac{1}{2} \left( \int_\Omega (|\nabla u|^2 - \lambda au^2) + \int_{\partial \Omega} u^2 \right) + \frac{1}{2} \left( \int_\Omega (|\nabla v|^2 - \lambda av^2) + \int_{\partial \Omega} v^2 \right) + \int_\Omega a(|u|^p + |v|^p)
\]

\[
= \frac{1}{2} (L(u) + L(v)) + G_1(u, v) + G_2(u, v),
\]

where

\[
L(t) = \int_\Omega (|\nabla t|^2 - \lambda at^2) + \alpha \int_{\partial \Omega} t^2
\]

for \( t = u \) or \( v \), and

\[
G_1(u, v) = \int_\Omega av^2 |u|^p
\]

and

\[
G_2(u, v) = \int_\Omega au^2 |v|^p.
\]

It is well known that the weak solutions of the system (1) are the critical points of the Euler functional \( J_\lambda \).

Let \( I \) be the Euler functional associated with an elliptic problem on a Banach space \( X \). If \( I \) is bounded below and has a minimizer on \( X \), thus this minimizer is a critical point.
of $I$, and so is a solution of the corresponding elliptic problem. However, the Euler functional $J_\lambda$ is not bounded below on the whole space $H$, but is bounded on an appropriate subset, and a minimizer on this set (if it exists) gives a solution to the system (1).

Then we introduce the following notation:

For any functional $f : H \to \mathbb{R}$ we denote by $f'(u,v)(h_1, h_2)$ the Gâteaux derivative of $f$ at $(u,v)$ in the direction of $(h_1, h_2) \in H$, and

$$f^{(1)}(u,v)h_1 = f'(u + \varepsilon h_1, v)|_{\varepsilon = 0},$$
$$f^{(2)}(u,v)h_2 = f'(u, v + \delta h_2)|_{\delta = 0}.$$

In fact we have

$$f'(u,v)(h_1, h_2) = f^{(1)}(u,v)h_1 + f^{(2)}(u,v)h_2.$$

## 2 Notations and preliminaries

First we consider the eigenvalue problem

$$\mu(\alpha, \lambda) = \inf \left\{ \int_\Omega (|\nabla u|^2 - \lambda au^2) + \frac{\alpha}{1 - \alpha} \int_\Omega u^2 \; ; \; u \in H^1(\Omega), \int_\Omega u^2 = 1 \right\}.$$  

Note that $\mu(\alpha, 0) > 0$ on $\alpha \in [\alpha_0, 1]$ for some small negative $\alpha_0$, and $\lambda \mapsto \mu(\alpha, \lambda)$ has exactly two zeroes $\lambda^-_\alpha$ and $\lambda^+_\alpha$ and those are principal eigenvalues of the following linear problem

$$-\Delta u = \lambda au, \quad x \in \Omega,$$
$$\frac{\partial u}{\partial n} + \alpha u = 0, \quad x \in \partial \Omega$$  

(4)  

(for more details, see [1]). Define

$$\|(u, v)\|_\Lambda = (L(u) + L(v))^{\frac{1}{2}}.$$  

We prove that for $\lambda \in (\lambda^-_\alpha, \lambda^+_\alpha)$, $\| . \|_\Lambda$ defines a norm in $H$ which is equivalent to the usual norm for $H$. Our proof is motivated by that of [3].

Since $\| . \|_\Lambda$ corresponds to the bilinear form

$$\langle (u_1, v_1), (u_2, v_2) \rangle = \int_\Omega (\nabla u_1 \nabla u_2 - \lambda au_1 u_2) + \frac{\alpha}{1 - \alpha} \int_\Omega u_1 u_2$$

$$+ \int_\Omega (\nabla v_1 \nabla v_2 - \lambda av_1 v_2) + \frac{\alpha}{1 - \alpha} \int_\Omega v_1 v_2,$$

in order to prove that $\| . \|_\Lambda$ is a norm, it is sufficient to prove that $\langle (u, v), (u, v) \rangle \geq 0$ for all $(u, v) \in H - \{(0, 0)\}$. It follows from the variational characterization of $\mu(\alpha, \lambda)$ that

$$\langle (u, v), (u, v) \rangle = L(u) + L(v) \geq \mu(\alpha, \lambda) \left( \int_\Omega (u^2 + v^2) \right).$$  

(5)

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Hence if $\lambda \in (\lambda_n^-, \lambda_n^+)$, then $\mu(\alpha, \lambda) > 0$ and so $\|(u, v)\|_\lambda > 0$ whenever $u, v \neq 0$, thus $\|\cdot\|_\lambda$ is a norm.

Moreover for $(u, v) \in H$, there exists a constant $k > 0$ such that

$$\|(u, v)\|_\lambda \leq k\|(u, v)\|_H.$$  

In fact by using the embedding of $(H, \|\cdot\|_H)$ into $L^2(\partial \Omega)$, we have

$$\|(u, v)\|_\lambda^2 \leq \left| \int \left( |\nabla u|^2 - \lambda \alpha u^2 \right) \, dx \right| + \frac{\alpha}{1 - \alpha} \int \left| \nabla v \right|^2 \, dx + \frac{\alpha}{1 - \alpha} \int \left| \nabla u \right|^2 \, dx \leq \left( \alpha c_1 \right) \left( \|u\|_H^2 + \|v\|_H^2 \right),$$

where $c_1$ is the best Sobolev constant of the embedding of $H_1(\Omega)$ into $L^2(\partial \Omega)$,

$$c_2 = \max \left\{ 1 + \alpha \frac{c_1}{1 - \alpha}, \lambda \lambda_+ a^+ \right\}$$

and $a^+ = \sup_{x \in \Omega} |\alpha(x)| > 0$. Now suppose that there exists a sequence $(u_n, v_n) \subseteq H$ such that $\|(u_n, v_n)\|_\lambda \to 0$ and $\|(u_n, v_n)\|_H = 1$. Since $\{(u_n, v_n)\}$ is bounded in $H$, there exists $(u, v) \in H$ such that $(u_n, v_n) \to (u, v)$ in $H$. Applying compactly embedding of $H$ in $L^2(\Omega) \times L^2(\Omega)$ and $L^2(\partial \Omega) \times L^2(\partial \Omega)$, we have $(u_n, v_n) \to (u, v)$ in $L^2(\Omega) \times L^2(\Omega)$ and $L^2(\partial \Omega) \times L^2(\partial \Omega)$, respectively. Taking $\|(u_n, v_n)\|_\lambda \to 0$ into account, from (5), we have $(u_n, v_n) \to (0, 0)$ in $L^2(\Omega) \times L^2(\Omega)$, and so $u = v = 0$. This implies $(u_n, v_n) \to (0, 0)$ in $L^2(\partial \Omega) \times L^2(\partial \Omega)$, which concludes that

$$\lim_{n \to \infty} \left( \int \left( |\nabla u|^2 + |\nabla v|^2 \right) \, dx \right) = 0.$$  

This contradicts with the fact $\|(u_n, v_n)\|_H = 1$ for all $n$. Hence, $\|\cdot\|_\lambda$ and $\|\cdot\|_H$ are equivalent norms.

Now we consider the Nehari minimizing problem

$$\mathcal{N}(\lambda) = \inf \left\{ J_\lambda(u, v); \ (u, v) \in M_\lambda \right\},$$

where

$$M_\lambda = \left\{ (u, v) \in H - \{(0, 0)\}; \ J^{(1)}_\lambda(u, v), (u, v) = J^{(1)}_\lambda(u, v)u + J^{(2)}_\lambda(u, v)v = 0 \right\}.$$  

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It is clear that all critical points of $J_\lambda$ must lie on $M_\lambda$, which is well-known as the Nehari manifold (see [2,8]). We will see below that local minimizers of $J_\lambda$ on $M_\lambda$ contains every non-zero solution of the system (1). First we claim that for $\lambda \in (\lambda_0^-, \lambda_0^+)-\{0\}$ we have $M_\lambda \neq \emptyset$. In fact, since the function $a$ changes sign, we can choose non-zero function $(u_0, v_0) \in H$ such that $G_1(u_0, v_0) = \int a|u_0|^2|u_0|^p > 0$ and $G_2(u_0, v_0) = \int a|u_0|^2|v_0|^p > 0$.

Let 

$$t^p = \frac{-L(u_0) - L(v_0)}{(2 + p)(G_1(u_0, v_0) + G_2(u_0, v_0))},$$

then 

$$\langle J'_\lambda(tu_0, tv_0), (tu_0, tv_0) \rangle = t^2 [L(u_0) + L(v_0) + t^p(p + 2)(G_1(u_0, v_0) + G_2(u_0, v_0))] = 0,$$

and so $(u, v) = t(u_0, v_0) \in M_\lambda$.

Define $F_\lambda(u, v) = \langle J'_\lambda(u, v), (u, v) \rangle$. Then for $(u, v) \in M_\lambda$,

$$\langle F_\lambda(u, v), (u, v) \rangle = 2(L(u) + L(v)) + (p + 2)^2(G_1 + G_2)(u, v)$$

$$= p(p + 2)(G_1 + G_2)(u, v) = -p(L(u) + L(v)).$$

Set 

$$K^+_\lambda = \inf \{ L(u) + L(v); \ (u, v) \in H, \ (G_1 + G_2)(u, v) = 1 \}$$

and 

$$K^-_\lambda = \inf \{ L(u) + L(v); \ (u, v) \in H, \ (G_1 + G_2)(u, v) = -1 \}.$$ 

By a similar way we can define $K^0_+$ and $K^-_0$.

For $\alpha \in [0, 1]$, we have that $\lambda \to K^\alpha_\lambda$ is a concave continuous curve on the interval $[\lambda_0^-, \lambda_0^+]$. By using a similar arguments we have these facts for $K^\alpha_\lambda$.

To state our main result, we now present some important properties of $K^\alpha_\lambda$ and $K^-_\lambda$.

**Lemma 1.** $K^+_0 > 0$ and $K^-_0 > 0$ for $\alpha \in (0, 1]$.

**Proof.** Suppose otherwise, that is there exists sequence $(u_n, v_n) \in H$ such that 

$$\lim_{n \to \infty} (L(u_n) + L(v_n)) = 0, \quad (G_1 + G_2)(u_n, v_n) = 1.$$ 

By the Sobolev embedding theorem and Hölder inequality, there exists $\varepsilon_0 < \frac{2}{\alpha - 2}$ such that 

$$1 = \left| \int \frac{a u_n^2 |u_n|^p + a u_n^2 |v_n|^p}{\varepsilon} \right| \leq a^+ \int \frac{v_n^2 |u_n|^p + u_n^2 |v_n|^p}{\varepsilon}$$

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Proof. Then \(\parallel\parallel\) is a concave continuous curve, we also have

\[
\phi \leq \lambda \rightarrow \infty \quad a_{\phi} < -\left(\phi^{u}, v^{v}\right) > K_{n} \leq \lambda \quad \lambda^{2} - 2(\alpha_{n} + 1)
\]

As \(n \rightarrow \infty\), which is a contradiction.

By the same argument we have \(K_{\lambda_{n}} > 0\).

Lemma 2. \(K_{\lambda_{n}}^{+} = K_{\lambda_{n}}^{-} = 0\).

Proof. Suppose that \(\phi^{+}\) is a positive eigenfunction of the linear problem (4) corresponding to the principal eigenvalue \(\lambda_{n}^{+}\). Then \(L(\phi^{+}) = 0\). On the other hand, if \(\lambda \neq 0\) be the principal eigenvalue of (4) with corresponding positive principal eigenfunction \(\phi\), then \(\lambda \int_{\Omega} a(\phi^{+}) > 0\) (see [3]). So in this case, since \(\lambda_{n}^{+} > 0\), we have \(\int_{\Omega} a(\phi^{+}) > 0\). We now let

\[
u = \frac{\phi^{+}}{2 \int_{\Omega} a(\phi^{+})^{p+2}}
\]

Then

\[(G_{1} + G_{2})(u, u) = 2 \int_{\Omega} a|u|^{p+2} = 1,
\]

and

\[L(u) + L(u) = \frac{2L(\phi^{+})}{2^{p+2}(\int_{\Omega} a(\phi^{+})^{p+2})} = 0,
\]

i.e., \(K_{\lambda_{n}^{+}} = 0\). By using a similar way we can prove that \(K_{\lambda_{n}^{-}} = 0\). Since \(\lambda \rightarrow \infty\), \(K_{\lambda_{n}^{+}}^{+}\) is a concave continuous curve, we also have \(0 < K_{\lambda_{n}^{+}}^{+} < K_{\lambda}^{-}\) for \(\lambda \in (0, \lambda_{n}^{+})\), and \(0 < K_{\lambda_{n}^{+}} < K_{\lambda}^{-}\) for all \(\lambda \in (\lambda_{n}^{-})\).

Lemma 3. For \(\lambda \in (\lambda_{n}^{-}, \lambda_{n}^{+})\), put

\[Y = \left\{u, v \in H: (G_{1} + G_{2})(u, v) = -\frac{1}{p + 2}\right\}.
\]

Then \(\|u, v\|_{\lambda}^{p} \in M_{\lambda}\) if \(u, v \in Y\), and \(\|u, v\|_{\lambda}^{p} \in p+2(p+2)\) if \(u, v \in M_{\lambda}\).

Proof. First suppose that \((u, v) \in Y\), then

\[
\mathcal{F}_{\lambda}(\|u, v\|_{\lambda}^{p}) = \langle J_{\lambda}(\|u, v\|_{\lambda}^{p} u, \|u, v\|_{\lambda}^{p} v), \|u, v\|_{\lambda}^{p} u, \|u, v\|_{\lambda}^{p} v \rangle = (\|u, v\|_{\lambda}^{p})^{p}(L(u) + L(v)) + \|u, v\|_{\lambda}^{p+2}(p+2)(G_{1} + G_{2})(u, v)
\]

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\[ \| (u, v) \|_{\lambda}^{\frac{2}{p}} (u, v) \in M_\lambda. \]

Now if \((u, v) \in M_\lambda\), we get
\[
L(u) + L(v) + (p + 2)(G_1 + G_2)(u, v) = \langle J'_\lambda(u, v)(u, v) \rangle = 0,
\]

thus
\[
\begin{align*}
(G_1 + G_2) \left( \| (u, v) \|_{\lambda}^{\frac{2}{p}} u, \| (u, v) \|_{\lambda}^{\frac{2}{p}} v \right) \\
= \| (u, v) \|_{\lambda}^{2} (G_1 + G_2)(u, v) = \| (u, v) \|_{\lambda}^{2} \frac{L(u) - L(v)}{p + 2} \\
= - \| (u, v) \|_{\lambda}^{2} \| (u, v) \|_{\lambda}^{2} = - \frac{1}{p + 2}.
\end{align*}
\]

i.e., \(\| (u, v) \|_{\lambda}^{\frac{2}{p}} (u, v) \in Y\).

For \(\lambda \in (\lambda^-_\alpha, \lambda^+_\alpha)\), we define \(Q_\lambda : Y \to \mathbb{R}\) by \(Q_\lambda(u, v) = \| (u, v) \|_{\lambda}^{\frac{2}{p}}\). Then for \((u, v) \in Y\) we have
\[
Q_\lambda(u, v) = \left( \frac{2(p + 2)}{p} J_\lambda \left( \| (u, v) \|_{\lambda}^{\frac{2}{p}} \right) \right)^{\frac{p}{p+2}}.
\]

Indeed,
\[
\begin{align*}
J_\lambda \left( \| (u, v) \|_{\lambda}^{\frac{2}{p}} (u, v) \right) \\
= \frac{1}{2} \left( \| (u, v) \|_{\lambda}^{\frac{2}{p}} (L(u) + L(v)) + \| (u, v) \|_{\lambda}^{2(p+2)} (G_1 + G_2)(u, v). \right)
\end{align*}
\]

Since \((G_1 + G_2)(u, v) = - \frac{1}{p+2}\) for \((u, v) \in Y\), we conclude
\[
J_\lambda \left( \| (u, v) \|_{\lambda}^{\frac{2}{p}} (u, v) \right) = \left( \frac{1}{2} - \frac{1}{p+2} \right) \| (u, v) \|_{\lambda}^{2(p+2)},
\]

which means that the claim is true.

Using a similar argument, if \((u, v) \in M_\lambda\), then
\[
J_\lambda(u, v) = \frac{p}{2(p+2)} Q_\lambda \left( \| (u, v) \|_{\lambda}^{\frac{2}{p}} (u, v) \right)^{\frac{p+2}{p}}.
\]

Indeed,
\[
\begin{align*}
\frac{p}{2(p+2)} Q_\lambda \left( \| (u, v) \|_{\lambda}^{\frac{2}{p}} (u, v) \right)^{\frac{p+2}{p}} \\
= \frac{p}{2(p+2)} \left( \| (u, v) \|_{\lambda}^{\frac{2}{p}} \| (u, v) \|_{\lambda}^{2} \right)^{\frac{p+2}{p}} = \frac{p}{2(p+2)} \left( \| (u, v) \|_{\lambda}^{\frac{2p}{p+2}} \right)^{\frac{p+2}{p}} \\
= \frac{p}{2(p+2)} \| (u, v) \|_{\lambda}^{2} = J_\lambda(u, v).
\end{align*}
\]

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The latest equality follows from that
\[ L(u) + L(v) + (p + 2)(G_1 + G_2)(u, v) = 0 \]
for \((u, v) \in M_\lambda\), i.e.,
\[ (G_1 + G_2)(u, v) = -\frac{1}{p + 2} (L(u) + L(v)) = -\frac{1}{p + 2} \| (u, v) \|_\Lambda^2, \]
and so
\[ J_\lambda(u, v) = \frac{1}{2} (L(u) + L(v)) + (G_1 + G_2)(u, v) \]
\[ = \frac{1}{2} \| (u, v) \|_\Lambda^2 - \frac{1}{p + 2} \| (u, v) \|_\Lambda^2 = \frac{p}{2(p + 2)} \| (u, v) \|_\Lambda^2. \]

In fact we proved the following result:

**Lemma 4.** Define \( q_\lambda = \inf_{(u, v) \in Y} Q_\lambda(u, v) \) and \( j_\lambda = \inf_{(u, v) \in M_\lambda} J_\lambda(u, v) \). Then, \( q_\lambda = \left( \frac{2(p + 2)}{p} j_\lambda \right)^{-\frac{1}{p+2}} \).

### 3 Main results

**Lemma 5.** \((u, v) = (0, 0)\) is not a limit point of \( M_\lambda \) if \( \lambda \in (\lambda^-_\alpha, \lambda^+_\alpha) \), \( \alpha \neq 0 \).

**Proof.** Let \( \{(u_n, v_n)\} \) in \( M_\lambda \), so that \( \| (u_n, v_n) \|_\Lambda \to 0 \) as \( n \to \infty \).

Now let \( (u'_n, v'_n) = \frac{(u_n, v_n)}{\| (u_n, v_n) \|_\Lambda} \), then \( \| (u'_n, v'_n) \|_\Lambda = 1 \), i.e., \( \{(u'_n, v'_n)\} \) is a bounded sequence in \( L^2(\Omega) \times L^2(\Omega) \) equipped with the norm
\[
\| (u, v) \|_{L^2(\Omega) \times L^2(\Omega)} = \left( \int_\Omega (|u|^2 + |v|^2) \right)^{\frac{1}{2}},
\]
therefore
\[
0 = \frac{J'_\lambda(u_n, v_n), (u_n, v_n)}{\| (u_n, v_n) \|_\Lambda^2}
= \frac{L(u_n) + L(v_n) + (p + 2)(G_1 + G_2)(u_n, v_n)}{\| (u_n, v_n) \|_\Lambda^2}
= 1 + (p + 2) \frac{(G_1 + G_2)(u_n, v_n)}{\| (u_n, v_n) \|_\Lambda^2}
= 1 + (p + 2) \frac{\| (u_n, v_n) \|_{L^\infty}^2 (G_1 + G_2)(u'_n, v'_n)}{\| (u_n, v_n) \|_\Lambda^2}.
\]
Moreover by the boundedness of \( \{(u'_n, v'_n)\} \) in \( H \), and applying the inequality mentioned in Lemma 1, we derive that the sequence \( \{(G_1 + G_2)(u'_n, v'_n)\} \) is bounded in \( H \), and so the right hand side of the last equality tends to 1. This contradiction proves the lemma. \( \Box \)
Theorem 1. Let $\alpha \in (0, 1]$ or $\int_\Omega a \neq 0$ if $\alpha = 0$. Then there exist two positive constants $\delta_1, \delta_2$ such that if $\{(u_n, v_n)\}$ is a minimizing sequence of $J_\lambda$ on $M_\lambda$, then

$$\liminf_{n \to \infty} \left| \int_\Omega a(u_n^2 + v_n^2) \right| > 0,$$

where $\lambda^-_\alpha < \lambda < \lambda^+_\alpha + \delta_1$ or $\lambda^+_\alpha - \delta_2 < \lambda < \lambda^+_\alpha$.

Proof. Let $\varphi^-$ and $\varphi^+$ be the corresponding eigenfunctions to the principal eigenvalues $\lambda^-_\alpha$ and $\lambda^+_\alpha$, respectively.

We can assume that

$$\int_\Omega a|\varphi^+|^{p+2} = 1, \quad \int_\Omega a|\varphi^-|^{p+2} = -1,$$

$$\int_\Omega a(\varphi^+)^2 > 0 \quad \text{and} \quad \int_\Omega a(\varphi^-)^2 < 0.$$

Now let

$$\delta_2 = \lambda^+_\alpha = \frac{\int_\Omega |\nabla \varphi^+|^2 + \alpha \int_{\partial \Omega} (\varphi^+)^2 - K_0^+}{\int_\Omega a(\varphi^+)^2} = \frac{K_0^+}{\int_\Omega a(\varphi^+)^2}.$$

Then for $\lambda \in (\lambda^-_\alpha - \delta_2, \lambda^+_\alpha)$, if the sequence $\{(u_n, v_n)\}$ be a minimizing sequence of $J_\lambda$ on $M_\lambda$, we derive that

$$J_\lambda(u_n, v_n) = \frac{p}{2(p+2)} \left[ Q_\lambda(\| (u_n, v_n) \|_{\lambda}^{-\frac{2}{p+2}} (u_n, v_n)) \right]^{\frac{p+2}{p}}$$

$$= \frac{p}{2(p+2)} \left[ \| (u_n, v_n) \|_{\lambda}^{-\frac{2}{p+2}} (u_n, v_n) \right] \frac{(p+2)}{p}$$

$$= \frac{p}{2(p+2)} \left[ \| (u_n, v_n) \|_{\lambda}^{-\frac{4}{p+2}} (u_n, v_n) \|_{\lambda}^{\frac{2(p+2)}{p}} \right]$$

$$= \frac{p}{2(p+2)} \left[ \| (u_n, v_n) \|_{\lambda}^{\frac{p}{p+2}} (u_n, v_n) \right].$$

This implies $\{(u_n, v_n)\}$ is bounded in $H$ and so there exist a subsequence, which for convenience we again denote by $\{(u_n, v_n)\}$, and $\{(u, v)\} \subset H$ such that $(u_n, v_n) \to (u, v)$ in $H$. Since $H$ may be compactly embedded in $L^2(\Omega) \times L^2(\Omega)$, we have $(u_n, v_n) \to (u, v)$ in $L^2(\Omega) \times L^2(\Omega)$. For $\lambda > 0$, by using Lemma 4, if $\{(u_n, v_n)\}$ be a minimizing sequence of $J_\lambda$ on $M_\lambda$, then $\|(u_n, v_n)\|_{\lambda}^{\frac{p}{p+2}} \{(u_n, v_n)\}$ is a minimizing sequence of $Q_\lambda$ on $Y$, and so we get

$$\inf_Y Q_\lambda = \lim_{n \to \infty} Q_\lambda(\| (u_n, v_n) \|_{\lambda}^{-\frac{2}{p+2}} (u_n, v_n))$$

$$= \lim_{n \to \infty} \| (u_n, v_n) \|_{\lambda}^{-\frac{2}{p+2}} (L(u_n) + L(v_n))$$
$$= \lim_{n \to \infty} \left( L\left( \left\| (u_n, v_n) \right\|_\lambda^{\frac{2}{p+2}} u_n \right) + \left\| (u_n, v_n) \right\|_\lambda^{\frac{2}{p+2}} v_n \right)$$

for some $q > 0$. Also we have

$$< q < K + 0$$

and if

$$(G_1 + G_2) \left( \left\| (u_n, v_n) \right\|_\lambda^{\frac{2}{p+2}} u_n, \left\| (u_n, v_n) \right\|_\lambda^{\frac{2}{p+2}} v_n \right) = - \frac{1}{p+2}.$$ 

Now by Lemma 5, we obtain $\left\| (u_n, v_n) \right\|_\lambda \not\to 0$, i.e., $L(u_n) + L(v_n) \not\to 0$, and if $\int_\Omega a(u_n^2 + v_n^2) \to 0$, then we get

$$\left\| (u_n, v_n) \right\|_\lambda^{\frac{2}{p+2}} \left[ \int_\Omega (|\nabla u_n|^2 + |\nabla v_n|^2) + \int_{\partial \Omega} (u_n^2 + v_n^2) \right] \not\to 0.$$ 

So by taking $\lambda = 0$, we have

$$\lim_{n \to \infty} \left\| (u_n, v_n) \right\|_\lambda^{\frac{2}{p+2}} (L(u_n) + L(v_n)) = K_0^+ \leq q < K_0^+,$$

which is a contradiction. Therefore,

$$\lim_{n \to \infty} \left| \int_\Omega a(u_n^2 + v_n^2) \right| > 0.$$

Now let

$$\delta_1 = \frac{\int_\Omega |\nabla \varphi|^2}{\int_\Omega a \varphi^2} + \frac{\alpha}{1 - \alpha} \frac{\int_{\partial \Omega} (\varphi^-)^2 - K_0^-}{\int_\Omega a(\varphi^-)^2} - \lambda_0^- = - \frac{K_0^-}{\int_\Omega a \varphi^2}.$$ 

By a similar argument we get a same result for $\lambda \in (\lambda^-_0, \lambda_0^- + \delta_1)$. □

**Lemma 6.** The production of two Hilbert spaces, is a Hilbert space.

**Proof.** For Hilbert spaces $H_1$ and $H_2$, define

$$\langle (u_1, u_2), (v_1, v_2) \rangle_{H_1 \times H_2} = \langle u_1, v_1 \rangle_{H_1} + \langle u_2, v_2 \rangle_{H_2}.$$ 

It is easy to see that, the mentioned bilinear form defines an inner product on $H_1 \times H_2$. □

**Proposition 1.** Let $\Phi$ be a $C^1$-functional on a Hilbert space $X = X_1 \times X_2$, where $X_1$ and $X_2$ are Hilbert spaces, and let $\Gamma$ be a closed subset in $X$ such that for any $(u, v) \in \Gamma$ with $\Phi'(u, v) \neq 0$ and $\epsilon > 0$ small arbitrary, there exists Frechet differentiable function

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Let us assume \( t_{(u,v)}(\delta) = s_{(u,v)}(\delta - \frac{\Phi'(u,v)}{\|\Phi'(u,v)\|_X}) \) for \( 0 \leq \delta \leq \epsilon \), we have \( t_{(u,v)}(0) = 1 \), and
\[
 t_{(u,v)}(\delta) \left( u - \delta \frac{\Phi'(u,v)}{\|\Phi'(u,v)\|_{X_1}}, v - \delta \frac{\Phi'(u,v)}{\|\Phi'(u,v)\|_{X_2}} \right) \in \Gamma.
\]
If \( \Phi \) is bounded below on \( \Gamma \), then for any minimizing sequence \( \{(u_n, v_n)\} \) for \( \Phi \) in \( \Gamma \), there exists another minimizing sequence \( \{(u^*_n, v^*_n)\} \) for \( \Phi \) in \( \Gamma \) such that
\[
 \Phi(u^*_n, v^*_n) \leq \Phi(u_n, v_n), \quad \lim_{n \to \infty} \| (u^*_n, v^*_n) - (u_n, v_n) \| = 0
\]
and
\[
 \left\| \Phi'(u^*_n, v^*_n) \right\|_X \leq \frac{1}{n} \left( \sqrt{2} + \left( \left\| (u^*_n, v^*_n) \right\| \right) \left| t_{(u^*_n, v^*_n)}(0) \right| \right) + \left| t_{(u^*_n, v^*_n)}(0) \right| \left. \left\| \Phi'(u^*_n, v^*_n) \right\|_X \right| \left( (u^*_n, v^*_n) \right) \right|.
\]
Proof. Let \( \eta = \inf_{\gamma \in \Gamma} \Phi(\gamma) \). Using Ekeland variational principle, we get a minimizing sequence \( \{(u^*_n, v^*_n)\} \) in \( \Gamma \), which
\begin{enumerate}[(i)]  
  \item \( \Phi(u^*_n, v^*_n) \leq \Phi(u_n, v_n) < \eta + \frac{1}{n} \),  
  \item \( \lim_{t_n \to \infty} \| (u^*_n, v^*_n) - (u_n, v_n) \| = 0 \),  
  \item \( \Phi(x, y) \geq \Phi(u^*_n, v^*_n) - \frac{1}{n} \| (x, y) - (u^*_n, v^*_n) \| \) for all \( (x, y) \in \Gamma \).
\end{enumerate}
Let us assume \( \left\| \Phi'(u^*_n, v^*_n) \right\|_X > 0 \) for large \( n \). Apply the hypothesis on the set \( \Gamma \) with \( (u, v) = (u^*_n, v^*_n) \) to find the function
\[
 t_n(\delta) = t_{(u^*_n, v^*_n)}(\delta) = s_{(u^*_n, v^*_n)} \left( \delta - \frac{\Phi'(u^*_n, v^*_n)}{\left\| \Phi'(u^*_n, v^*_n) \right\|_X} \right).
\]
Then,
\[
 (x, y) = t_n(\delta) \left( u^*_n - \delta \frac{\Phi'(u^*_n, v^*_n)}{\left\| \Phi'(u^*_n, v^*_n) \right\|_{X_1}}, v^*_n - \delta \frac{\Phi'(u^*_n, v^*_n)}{\left\| \Phi'(u^*_n, v^*_n) \right\|_{X_2}} \right) \in \Gamma
\]
for all small enough \( \delta \geq 0 \). By the mean value theorem we have
\[
 \frac{1}{n} \left\| (x, y) - (u^*_n, v^*_n) \right\| \\
 \geq \Phi(u^*_n, v^*_n) - \Phi(x, y) = \left( \Phi'(x, y), (u^*_n, v^*_n) - (x, y) \right) + o(\delta) \\
 = \left( \Phi'(x, y), (u^*_n, v^*_n) \right) - \left( \Phi'(x, y), (x, y) \right) + o(\delta) \\
 = \Phi'(x, y, y), (u^*_n, v^*_n) - t_n(\delta) \Phi'(x, y, y), (u^*_n, v^*_n) \right) \\
 + \delta t_n(\delta) \left( \Phi'(x, y, y), \left( \frac{\Phi'(u^*_n, v^*_n)}{\left\| \Phi'(u^*_n, v^*_n) \right\|_{X_1}} \right) \left( \Phi(u^*_n, v^*_n) \right) \right) + o(\delta) \\
 = (1 - t_n(\delta)) \Phi'(x, y, y), (u^*_n, v^*_n) \\
 + \delta t_n(\delta) \left( \Phi'(x, y, y), \left( \frac{\Phi'(u^*_n, v^*_n)}{\left\| \Phi'(u^*_n, v^*_n) \right\|_{X_1}} \right) \left( \Phi(u^*_n, v^*_n) \right) \right) + o(\delta),
\]
where $\frac{o(\delta)}{\delta} \to 0$ as $\delta \to 0$. So

$$
\frac{1}{n\delta} \left( t_n(\delta)u_n^* - \delta t_n(\delta) \frac{\Phi^{(1)}(u_n^*, v_n^*)}{\|\Phi^{(1)}(u_n^*, v_n^*)\|_{X_1}} - u_n^*,
\right.
\left. t_n(\delta)v_n^* - \delta t_n(\delta) \frac{\Phi^{(2)}(u_n^*, v_n^*)}{\|\Phi^{(2)}(u_n^*, v_n^*)\|_{X_2}} - v_n^* \right)
\geq \frac{1 - t_n(\delta)}{\delta} \left( \Phi'(x_\delta, y_\delta), (u_n^*, v_n^*) \right)
+ t_n(\delta) \left( \Phi'(x_\delta, y_\delta), \left( \frac{\Phi^{(1)}(u_n^*, v_n^*)}{\|\Phi^{(1)}(u_n^*, v_n^*)\|_{X_1}}, \frac{\Phi^{(2)}(u_n^*, v_n^*)}{\|\Phi^{(2)}(u_n^*, v_n^*)\|_{X_2}} \right) \right) + o(\delta).
$$

Now we pass to the limit as $\delta \to 0$, and we derive

$$
\lim_{\delta \to 0} \frac{1}{n} \left( \left[ t_n(\delta) - 1 \right] u_n^*, \left[ t_n(\delta) - 1 \right] v_n^* \right)
+ \lim_{\delta \to 0} \frac{1}{n} \left( \left( \frac{\Phi^{(1)}(u_n^*, v_n^*)}{\|\Phi^{(1)}(u_n^*, v_n^*)\|_{X_1}}, \frac{\Phi^{(2)}(u_n^*, v_n^*)}{\|\Phi^{(2)}(u_n^*, v_n^*)\|_{X_2}} \right) \right)
= \frac{1}{n} |t_n'(0)| \left( \left( u_n^*, v_n^* \right) \right)
+ \frac{1}{n} \left[ \left( \left( \frac{\Phi^{(1)}(u_n^*, v_n^*)}{\|\Phi^{(1)}(u_n^*, v_n^*)\|_{X_1}}, \frac{\Phi^{(2)}(u_n^*, v_n^*)}{\|\Phi^{(2)}(u_n^*, v_n^*)\|_{X_2}} \right) \right)^2 \right]^{\frac{1}{2}}
\geq -t_n'(0) \left( \Phi'(u_n^*, v_n^*), (u_n^*, v_n^*) \right)
+ \frac{\Phi'(u_n^*, v_n^*), \left( \frac{\Phi^{(1)}(u_n^*, v_n^*)}{\|\Phi^{(1)}(u_n^*, v_n^*)\|_{X_1}}, \frac{\Phi^{(2)}(u_n^*, v_n^*)}{\|\Phi^{(2)}(u_n^*, v_n^*)\|_{X_2}} \right)}
+ \frac{\Phi'(u_n^*, v_n^*), \left( \frac{\Phi^{(1)}(u_n^*, v_n^*)}{\|\Phi^{(1)}(u_n^*, v_n^*)\|_{X_1}}, \frac{\Phi^{(2)}(u_n^*, v_n^*)}{\|\Phi^{(2)}(u_n^*, v_n^*)\|_{X_2}} \right)}
= -t_n'(0) \langle \Phi'(u_n^*, v_n^*), (u_n^*, v_n^*) \rangle
+ \frac{\Phi^{(1)}(u_n^*, v_n^*)}{\|\Phi^{(1)}(u_n^*, v_n^*)\|_{X_1}} + \frac{\Phi^{(2)}(u_n^*, v_n^*)}{\|\Phi^{(2)}(u_n^*, v_n^*)\|_{X_2}}.
$$

Thus, we have

$$
\|\Phi'(u_n^*, v_n^*)\|_X
= \|\Phi^{(1)}(u_n^*, v_n^*) + \Phi^{(2)}(u_n^*, v_n^*)\|_X \leq \|\Phi^{(1)}(u_n^*, v_n^*)\|_{X_1} + \|\Phi^{(2)}(u_n^*, v_n^*)\|_{X_2}
\leq \frac{1}{n} \left( \sqrt{2} + |t_n'(0)| \|\Phi(u_n^*, v_n^*)\| \right) + t_n'(0) \langle \Phi'(u_n^*, v_n^*), (u_n^*, v_n^*) \rangle.
$$

This completes the proof. \(\square\)

Let $\alpha \in (0, 1]$ or that $\int_{\Omega} a \neq 0$ for $\alpha = 0$. Then for $\lambda \in (\lambda_n^-, \lambda_n^+)$ we have the following results:

**Lemma 7.** $J_\lambda$ is bounded below on $M_\lambda$.  

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Proof. Firstly we note that for \((u, v) \in M_\lambda,\)
\[(G_1 + G_2)(u, v) = -\frac{1}{p + 2}(L(u) + L(v)).\] (6)

Now for \(\lambda \in (\lambda^-_\alpha, \lambda^+_\alpha),\) let \(w = \frac{u}{\langle f_\Omega, u^2 \rangle^{\frac{1}{2}}} \). Then
\[
\mu(\alpha, \lambda) \leq L(w) = \frac{L(u)}{\int_{\Omega} u^2} = \frac{\int_{\Omega} (\langle \nabla u \rangle^2 - \lambda au^2) + \frac{\alpha}{4-p} \int_{\partial \Omega} u^2}{\int_{\Omega} u^2},
\]
and so \(L(u) > 0\) for \(\lambda \in (\lambda^-_\alpha, \lambda^+_\alpha).\) A similar argument shows that \(L(v) > 0\) for \(\lambda \in (\lambda^-_\alpha, \lambda^+_\alpha),\) and (6) implies \((G_1 + G_2)(u, v) < 0\) for \((u, v) \in M_\lambda.\) Hence, \(J_\lambda(u, v) = -\frac{1}{2}(G_1 + G_2)(u, v) > 0,\) and \(J_\lambda\) is bounded below on \(M_\lambda.\)

\section*{Theorem 2}
There exists a minimizing sequence \(\{u^*_n, v^*_n\}\) of \(J_\lambda\) on \(M_\lambda\) such that
\[
\lim_{n \to \infty} \|J'_\lambda(u^*_n, v^*_n)\|_\lambda = 0.
\]

Proof. For \((u, v) \in M_\lambda,\) let \(\Psi : \mathbb{R} \times H \to \mathbb{R}\) such that
\[
\Psi(s, (w_1, w_2)) = \mathcal{F}_\lambda(sw - w_1, sw - w_2).
\]

Since \((u, v) \in M_\lambda,\) we have \(\Psi(1, (0, 0)) = 0.\) Also,
\[
\frac{\partial}{\partial s} \Psi(1, (0, 0)) = 2(L(u) + L(v)) + (p + 2)^2(G_1 + G_2)(u, v)
\]
\[
= 2(-p + 2)(G_1 + G_2)(u, v) + (p + 2)^2(G_1 + G_2)(u, v)
\]
\[
= p(p + 2)(G_1 + G_2)(u, v).
\]

Now we want to apply the Implicit function theorem at \((1, (0, 0))\) to get for any \(\delta > 0\) small enough, a differentiable function \(s_{(u, v)} : B_\delta \to \mathbb{R}\) such that
\[
s_{(u, v)}(0, 0) = 1,\quad s_{(u, v)}(w_1, w_2)((u, v) - (w_1, w_2)) \in M_\lambda,
\]
\[
\left( s_{(u, v)}(0, 0), (w_1, w_2) \right) = \left( \frac{\mathcal{F}'_\lambda(u, v)(w_1, w_2)}{\mathcal{F}'_\lambda(u, v)(u, v)} \right)
\]
for all \((w_1, w_2) \in B_\delta,\) where \(B_\delta\) is defined before.

Now for every \((u, v) \in H,\) let
\[
(x, y)_{(u, v)} = \frac{J'_\lambda(u, v)}{\|J'_\lambda(u, v)\|_\lambda},
\]
\[
t_{(u, v)}(\rho) = s_{(u, v)}(\rho(x, y)_{(u, v)}) \quad \text{for } 0 \leq \rho \leq \delta.
\]

Then we have the following results:
(i) \(t_{(u, v)}(0) = 1,\)
(ii) \(t'_{(u, v)}(0) = \langle s'_{(u, v)}(0, 0), (x, y)_{(u, v)} \rangle,\)
from Proposition 1, there exists a minimizing sequence \(\{u_n^*, v_n^*\}\) for \(J_\lambda\) such that
\[
J_\lambda(u_n^*, v_n^*) \leq J_\lambda(u_n, v_n) < \inf_{M_\lambda} J_\lambda + \frac{1}{n},
\]
and
\[
\|J'_\lambda(u_n^*, v_n^*)\|_{\lambda} \leq \frac{1}{n} (\sqrt{2} + \|(u_n^*, v_n^*)\|_{\lambda} |t'_{(u_n^*, v_n^*)}(0)|) + |t'_{(u_n^*, v_n^*)}(0)F_\lambda(u_n^*, v_n^*)|.
\]
Since
\[
J_\lambda(u_n^*, v_n^*) = \frac{p}{2(p + 2)} \|(u_n^*, v_n^*)\|_{\lambda}^2 < \inf_{M_\lambda} J_\lambda + \frac{1}{n},
\]
so that \(\{u_n^*, v_n^*\}\) is a bounded sequence in \(H\), i.e., there exists \(c_1 > 0\) such that \(\|(u_n^*, v_n^*)\|_{\lambda} < c_1\) for all \(n\). Then,
\[
\|J'_\lambda(u_n^*, v_n^*)\|_{\lambda} \leq \frac{1}{n} (\sqrt{2} + |t'_{(u_n^*, v_n^*)}(0)|c_1).
\]
Moreover,
\[
|t'_{(u_n^*, v_n^*)}(0)| = \left| \langle s'_{(u_n^*, v_n^*)}(0), (x, y)(u_n^*, v_n^*) \rangle \right| = \frac{\|\langle F'_{\lambda}(u_n^*, v_n^*), (x, y)(u_n^*, v_n^*) \rangle \|_{\lambda}}{\|\langle F'_{\lambda}(u_n^*, v_n^*), (x, y)(u_n^*, v_n^*) \rangle \|_{\lambda}} \frac{\|F'_{\lambda}(u_n^*, v_n^*), (x, y)(u_n^*, v_n^*) \|_{\lambda}}{p \|(u_n^*, v_n^*)\|_{\lambda}^2}
\]
and
\[
\liminf_{n \to \infty} \|(u_n^*, v_n^*)\|_{\lambda} > 0,
\]
since \((0, 0)\) is not a limit point of \(M_\lambda\).

So, if we show that \(|t'_{(u_n^*, v_n^*)}(0)|\) is uniformly bounded on \(n\), we are done.

By using the Hölder inequality, Sobolev embedding theorem, boundedness of the sequence \(\{u_n^*, v_n^*\}\) and \(\|(x, y)(u_n^*, v_n^*)\| = 1\), we have
\[
\|\langle F'_{\lambda}(u_n^*, v_n^*), (x, y)(u_n^*, v_n^*) \rangle \|_{\lambda} \leq c_2 \|(u_n^*, v_n^*)\|_{\lambda} + c_3.
\]
This proves the theorem.

We can now prove the main result of the paper:

**Theorem 3.** Let \(\alpha \in (0, 1]\) or that \(\int_{\Omega} a \neq 0\) for \(\alpha = 0\). For any \(\lambda \in (\lambda^+_a, \lambda_a^- + \delta_1) \cup (\lambda_a^- - \delta_2, \lambda_a^+)\), \(\lambda \neq 0\), the system (1) has a nontrivial nonnegative solution.
Proof. Let \( c = \inf J_\lambda(M_\lambda) \) and \( \{(u_n, v_n)\} \) be a sequence in \( M_\lambda \) such that

\[
\lim_{n \to \infty} J_\lambda(u_n, v_n) = c.
\]

By using Theorem 2, we have \( \lim_{n \to \infty} \| J'_\lambda(u_n, v_n) \|_\lambda = 0 \). Then \( \{(u_n, v_n)\} \) is bounded and we can find a weak limit point of the sequence in \( H \), i.e., \( u_n \to u \) and \( v_n \to v \) both in \( H^1(\Omega) \), for some \( u, v \in H^1(\Omega) \), and so \( u_n \to u \) and \( v_n \to v \) both in \( L^q(\Omega) \) for \( q < \frac{2N}{N-2} \). In particular for any \( (h_1, h_2) \in H \),

\[
\langle J'_\lambda(u_n, v_n), (h_1, h_2) \rangle = \int_\Omega \nabla u_n \nabla h_1 + \int_\Omega \nabla v_n \nabla h_2 - \lambda \int_\Omega a(u_n h_1 + v_n h_2) + \frac{\alpha}{1 - \alpha} \int_{\partial \Omega} (u_n h_1 + v_n h_2) + p \int_\Omega a(v_n^2 |u_n|^{p-2} u_n h_1 + u_n^2 |v_n|^{p-2} v_n h_2) + 2 \int_\Omega a(u_n |v_n|^{p-1} h_1 + v_n |u_n|^{p-1} h_2),
\]

which converges to

\[
\langle J'_\lambda(u, v), (h_1, h_2) \rangle = \int_\Omega \nabla u \nabla h_1 + \int_\Omega \nabla v \nabla h_2 - \lambda \int_\Omega a(u h_1 + v h_2) + \frac{\alpha}{1 - \alpha} \int_{\partial \Omega} (u h_1 + v h_2) + p \int_\Omega a(v^2 |u|^{p-2} u h_1 + u^2 |v|^{p-2} v h_2) + 2 \int_\Omega a(u |v|^{p-1} h_1 + v |u|^{p-1} h_2)
\]

as \( n \to \infty \). So, we derive

\[
|\langle J'_\lambda(u, v), (h_1, h_2) \rangle| = \lim_{n \to \infty} \langle J'_\lambda(u_n, v_n), (h_1, h_2) \rangle \leq \lim_{n \to \infty} \| J'_\lambda(u_n, v_n) \|_\lambda \| (h_1, h_2) \| = 0,
\]

that means \( \langle J'_\lambda(u, v), (h_1, h_2) \rangle = 0 \) for all \( (h_1, h_2) \in Y \). Therefore, \( (u, v) \) is a weak solution for the system (1).

In particular, \( \langle J'_\lambda(u, v), (u, v) \rangle = 0 \). Since \( \lim \inf \int_{\Omega} (|u_n|^2 + |v_n|^2) > 0 \), we have \( (u, v) \neq (0, 0) \). Hence, \( (u, v) \in M_\lambda \).

On the other hand, \( J_\lambda \) is weakly lower semicontinuous, and so we have

\[
c \leq J_\lambda(u, v) \leq \lim_{n \to \infty} J_\lambda(u_n, v_n) = c,
\]

which follows that \( J_\lambda(u, v) = c \) and that \( \|(u_n, v_n)\|_\lambda \to \|(u, v)\|_\lambda \) which implies that \( u_n \to u \) and \( v_n \to v \) both in \( H^1(\Omega) \).

Since \( J'_\lambda \) is continuous at \( (u, v) \), we get \( J'_\lambda(u, v) = 0 \). One can check that \( J_\lambda(u, v) = J_\lambda(|u|, |v|) \), so \( (u, v) \) is a nontrivial nonnegative solution for the system (1).
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