KAZHDAN-LAUMON CATEGORY $\mathcal{O}$, BRAVERMAN-KAZHDAN SCHWARTZ SPACE, AND THE SEMI-INFINITE FLAG VARIETY

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Abstract. We define and study an analogue of Category $\mathcal{O}$ in the context of Kazhdan and Laumon’s gluing construction for perverse sheaves on the basic affine space. We explicitly describe the simple objects in this category, and we show its linearized Grothendieck group is isomorphic to a natural submodule of Lusztig’s periodic Hecke module. We then provide a categorification of these results by showing that the Kazhdan-Laumon Category $\mathcal{O}$ is equivalent to a full subcategory of a suitably-defined category of perverse sheaves on the semi-infinite flag variety.

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1. Introduction

In [KL88], with the aim of providing a new geometric construction of representations of finite Chevalley groups, D. Kazhdan and G. Laumon described a gluing construction for perverse sheaves on the basic affine space associated to a semisimple algebraic group $G$ split over a finite field or its algebraic closure, defining a “glued category” $\mathcal{A}$. This category was studied in detail by R. Bezrukavnikov and A. Polishchuk in [Pol01], and shown to be equivalent to the category of modules over global differential operators on the basic affine space in [BBP02], a connection which was further explored in [LS06].

In this paper, we study a subcategory $\mathcal{A}_B$ (and its mixed analogue $\mathcal{A}_P$) of $\mathcal{A}$ corresponding to $B$-equivariant perverse sheaves; this is a natural analogue of Category $\mathcal{O}$ in the context of Kazhdan and Laumon’s construction. We begin by studying this category in its own right and describing its simple objects explicitly. To state the following theorem, for any element $w$ in a Weyl group $W$ let $P(w)$ be
the standard parabolic subgroup of $W$ generated by all simple reflections $s$ with $\ell(ws) > \ell(w)$.

**Theorem 1.1.** The Kazhdan-Laumon Category $\mathcal{O}$ associated to the group $G$ with Weyl group $W$ has

\[ \sum_{w \in W} |P(w)\backslash W| \]

irreducible objects, described explicitly in Theorem 4.7. For any choice of $w \in W$ and coset $P(w)z$, the corresponding simple object can be written as a tuple $(G_y)_{y \in W}$ of irreducible objects in $\text{Perv}_B(G/U)$ such that

\[ G_y \cong \begin{cases} \text{IC}_w & y \in P(w)z \\ 0 & y \notin P(w)z \end{cases} . \]

The fact that objects of $A$ are tuples of perverse sheaves indexed by $W$ means that translation of the indices gives a natural action of $W$ on $A$ (and therefore also on $A_B$ and $A_P$) by functors $F_w$ for $w \in W$. Further, we show that $A_P$ admits a left action of the Hecke algebra $\mathcal{H}$ by convolution with mixed $B$-equivariant perverse sheaves on the flag variety $G/B$. These actions give $K_0(A_P) \otimes \mathbb{C}$ the structure of a $\mathcal{H}_q \otimes \mathbb{C}[W]$-module. We then describe this module explicitly in terms of a more familiar combinatorial object, namely G. Lusztig’s periodic Hecke module defined in [Lus97].

**Theorem 1.2.** There is an isomorphism $K_0(A_P) \otimes \mathbb{C} \cong M^0_{d,q}$ of $\mathcal{H}_q \otimes \mathbb{C}[W]$-modules, where $M^0_{d,q}$ is a submodule (introduced in Definition 3.8) of Lusztig’s periodic Hecke module from [Lus97].

In [BK99], A. Braverman and Kazhdan defined the Schwartz space $S$ of the basic affine space associated to an algebraic group over a non-Archimedean local field, equipped with a $\mathbb{C}[W]$-action by Fourier transform operators. They, too, found a combinatorial connection between $S$ and Lusztig’s module $M_{d,q}$, proving in loc. cit. that the subspace $S^{I \times T(O)}$ of Iwahori-invariants in $S$ is isomorphic to $M_{d,q}$ as a $\mathcal{H}_q \otimes \mathbb{C}[W]$-module. Combining their result with our Theorem 1.2 yields a description of $K_0(A_P) \otimes \mathbb{C}$ as a certain natural subspace $S^{I \times T(O)}$ of $S^{I \times T(O)}$.

**Theorem 1.3.** Composing the isomorphism from Theorem 1.2 with the morphism $S^{I \times T(O)} \to M_{d,q}$ from [BK99] gives an isomorphism $K_0(A_P) \otimes \mathbb{C} \cong S^{I \times T(O)}$ of $\mathcal{H}_q \otimes \mathbb{C}[W]$-modules.

Further, the isomorphism can be described directly in terms of Grothendieck’s sheaf-function dictionary: the diagram

\[ \begin{array}{c} K_0(A_P) \otimes \mathbb{C} \\
\downarrow \\
M_{d,q} \end{array} \xymatrix{ \ar[r] & S^{I \times T(O)} } \]

commutes, where the left and right maps are the morphisms from Theorem 1.2 and [BK99] respectively, and the top map is described in Section 6.3 in terms of lifts to the local field setting of functions obtained by taking the trace of a Frobenius endomorphism on objects of $A_P$. 

In [ABB+05], building off of work in [FM99], [FFKM99], and [BFGM02], the authors define a certain analogue of a category of perverse sheaves on the semi-infinite flag variety associated to \( G \). In 6.1.8 of [ABB+05], the authors discuss how the category of Iwahori-monodromic objects of their category can be viewed as a categorification of \( S^I \times T(O) \). Using their main result which relates such Iwahori-monodromic objects to graded modules over the small quantum group, they define a \( W \)-action via functors \( F_w \) which categorify the Fourier transforms appearing in [BK99]. With this in mind, we conclude this paper by upgrading Theorem 1.3 to an equivalence of categories.

**Theorem 1.4.** There exists a certain subcategory \( \tilde{\mathcal{P}} \) of Iwahori-monodromic perverse sheaves on the semi-infinite flag variety with \( K_0(\tilde{\mathcal{P}}) \otimes \mathbb{C} \cong S^I_0 \times T(O) \). There is an equivalence of categories between \( \mathcal{A}_P \) and \( \tilde{\mathcal{P}} \) which categorifies the isomorphism \( K_0(\mathcal{A}_P) \otimes \mathbb{C} \cong S^I_0 \times T(O) \).

This equivalence is compatible with convolution by perverse sheaves on \( G/B \), which categorifies the \( \mathcal{H}_q \)-action on each side, and intertwines the \( W \)-action by functors \( F_w \) on \( \mathcal{A}_P \) and \( \tilde{F}_w \) on \( \tilde{\mathcal{P}} \).

We hope that Theorem 1.4 will serve as a new perspective related to one of the goals stated in [ABB+05] to provide a geometric description of Braverman-Kazhdan’s Fourier transforms. In loc. cit., Fourier transform functors on perverse sheaves on the semi-infinite flag variety are only defined in the case of Iwahori-monodromic objects, as these functors come from the equivalence in loc. cit. to modules over the small quantum group. If suitable functors could be defined on the full category of perverse sheaves on the semi-infinite flag variety, we hope that Theorem 1.4 might generalize to a suitable equivalence of categories to the full Kazhdan-Laumon category \( A \), thereby elevating Kazhdan and Laumon’s construction from a tool intended for the study of finite Chevalley groups to an object with interesting connections to current objects of study related to a local geometric Langlands correspondence.

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2. Preliminaries

Let \( G \) be a semisimple algebraic group over \( \mathbb{Z} \). Letting \( \kappa = \mathbb{F}_q \) where \( q = p^n \) for \( p \) prime, we write \( G = G_\kappa \) for the base change to \( \mathbb{F}_q \). We assume that \( G \) is split, connected, and simply-connected. Let \( T \) be a Cartan subgroup, \( B \) a Borel subgroup containing \( T \), and \( U \) its unipotent radical. We use similar notation \( T, B, \) and \( U \) for their respective base changes to \( \kappa \). Let \( X = G/U \) be the basic affine space associated to \( G \), considered as a variety over \( \mathbb{F}_q \). Let \( W \) be the Weyl group,
and let $\hat{W}$ be the affine Weyl group associated to $G$, with $S \subset W$ and $S \subset \hat{W}$ their respective subsets of simple reflections. We denote by $w_0$ the longest element of $W$.

In Section 4, we will consider a local field $k$ with residue field $\kappa$. In this setting, we will denote by $G_k$ the base change of $G$ to $k$, and by $X_k$ the basic affine space over $k$. Let $\Pi$ be the set of simple roots, let $\Gamma$ (resp. $\Gamma^\vee$) be the coroot (resp. weight) lattice of $G$.

Let $\mathcal{H}$ be the usual Hecke algebra over $\mathbb{Z}[v,v^{-1}]$ generated by $\{T_w\}_{w \in W}$, normalized so that the quadratic relation satisfied by any $T_s$ for $s \in S$ takes the form

$$T_s^2 = (v^2 - 1)T_s + v^2. \tag{2.1}$$

For any $w \in W$, let $\hat{T}_w = v^{-l(w)}T_w$, so that for any $s \in S$,

$$\hat{T}_s - v)(\hat{T}_s + v^{-1}) = 0. \tag{2.2}$$

Following the conventions of Section 3.5 of [Wil03], recall the canonical basis $C_w$ of $\mathcal{H}$ given in terms of the Kazhdan-Lusztig polynomials $P_{x,w}$ by

$$C_w = \sum_{x \leq w} (-1)^{|(w) - (x)|} v^{|(x)| - |(w)|} P_{x_w, w} \hat{T}_x^{-1}. \tag{2.2}$$

We will sometimes consider $\mathcal{H}$ as a subalgebra of $\hat{\mathcal{H}}$, where $\hat{\mathcal{H}}$ is the affine Hecke algebra generated by $\{\hat{T}_w\}_{w \in W}$ with $\hat{T}_s$ satisfying the same relations as in (2.2) for any $s \in S$. Choosing once and for all a square root $q^{1/2}$ of $q$, we let $\mathcal{H}_q$ (resp. $\hat{\mathcal{H}}_q$) be the specialization of $\mathcal{H}$ (resp. $\hat{\mathcal{H}}$) at $v = q^{1/2}$.

Throughout the paper, when $\mathcal{C}$ is an abelian category we will denote by $K_0(\mathcal{C})$ the Grothendieck group of the derived category $D^b(\mathcal{C})$, and we will freely consider the action of left- or right-exact functors on $K_0(\mathcal{C})$ by identifying them with their right- or left-derived functors. When $\mathcal{C}$ is a category of perverse sheaves, we use the results of [Bei87] to freely identify its bounded derived category with the underlying constructible derived category.

3. Lusztig’s periodic Hecke module

In this section, we recall the periodic Hecke module considered in [Lus80] and [Lus97], following the notational conventions of the latter. Our main object of study is the related module $M_d$ defined in loc. cit. The module $M_d$ and its specialization $M_{d,q}$ will appear in every subsequent section, serving as the combinatorial data which underlies all three of the main objects of study in this paper (Kazhdan-Laumon Category $\mathcal{O}$, the Schwartz space of the basic affine space, and perverse sheaves on the semi-infinite flag variety).

3.1. Background and setup. To begin, we follow the setup of Section 4.2 of [BK99]. Let $\Xi$ denote the collection of all alcoves for the group $\hat{W}$ in the real Lie algebra $\mathfrak{k}$ of $T$. The set $\Xi$ admits two commuting actions of the group $\hat{W}$, one on the left and the other on the right. We will follow the conventions of [Lus97] so that the left action of any $s \in S$ takes an alcove $A$ to some neighboring alcove $sA$, with the right action defined so that each $s \in S$ reflects an alcove around the corresponding affine hyperplane $H_s$ in $\mathfrak{k}$, so that $A$ and $As$ are not, in general, neighboring. Let $\mathcal{C}^+ \subset \mathfrak{k}$ denote the dominant Weyl chamber. For any $\gamma \in \Gamma^\vee$, let $A^+\gamma$ denote the unique alcove $A \in \gamma + \mathcal{C}^+$ for which $\gamma \in \mathcal{A}$, and let $A^+ = A^+_0$, the “fundamental alcove.” We will also write $A_w = wA^+$ for any $w \in \hat{W}$. 
We also denote by $d : \Xi \times \Xi \to \mathbb{Z}$ (and $d_\alpha$ for $\alpha \in \Pi$) the “relative distance” functions defined in [Lus97], and we use also the definition of $\mathcal{L}(A)$ from 1.2 of loc. cit.; informally, for $A \in \Xi$, $\mathcal{L}(A)$ is the subset of $\hat{S}$ for which $A$ is “above” $sA$, where the direction is determined by that of $C^+$.

**Definition 3.1.** The module $M_\gamma$ is the $\mathbb{Z}[v, v^{-1}]$-span of $\Xi$ with an action of $\mathcal{H}$ defined for $s \in \hat{S}$ by

$$
\tilde{T}_s(A) = \begin{cases} 
  sA & \text{if } s \notin \mathcal{L}(A), \\
  sA + (v - v^{-1})A & \text{if } s \in \mathcal{L}(A),
\end{cases}
$$

and extended naturally into a left action. It also has an action of $\Gamma$ commuting with the $\mathcal{H}$-action defined by $\gamma \cdot A = A + \gamma$ for $\gamma \in \Gamma$.

**Definition 3.2.** Let $M_>$ be the “upward semi-infinite completion” of $M_\gamma$, consisting of all formal sums $\sum_{A \in \Xi} m_A A$ such that the set $\{A \in \Xi \mid m_A \neq 0\}$ is bounded below (in the sense of 4.13 of [Lus97]).

**3.2. An action of $W$.** For any $\alpha \in \Pi$, an operator $\theta_\alpha : M_> \to M_>$ is defined in [Lus97] as follows. First any $A \in \Xi$, $\alpha \in \Pi$, a sequence $A^n$, $n \geq 0$ is defined by the conditions that $A^0 = A_{\alpha}$, and $d_\alpha(A^n, A^{n+1}) > 0$, in addition to the condition that $A^n$ lie in the same “$\alpha$-strip” as $A$ (c.f. [Lus97] for a more precise definition). Then

$$
\theta_\alpha(A) = v^{-1}A^0 + \sum_{n=1}^{\infty} (-1)^n (v^{-n+1} - v^{-n-1}) A^n.
$$

We define for $w \in W$ the operator $\theta_w = \theta_{\alpha_1} \cdots \theta_{\alpha_k}$ for $w = s_{\alpha_1} \cdots s_{\alpha_k}$ a reduced expression, which yields a well-defined action of the Weyl group $W$ on $M_>$.

**3.3. The modules $M_d$ and $M^\theta_d$.** In [Lus97], Lusztig defines a duality operator $\tilde{b}$ on $M_>$ and exhibits a “canonical basis” for $M_>$ invariant under this operator.

**Proposition 3.3** (Theorem 12.2 in [Lus97]). For each $A \in \Xi$, there exists an associated element $A^\dagger \in M_>$ which satisfies $A^\dagger = A + v^{-1} \sum_B \mathbb{Z}[v^{-1}] B$ and $\tilde{b}(A^\dagger) = A^\theta$.

**Definition 3.4.** The module $M_d \subset M_>$ is the $\mathbb{Z}[v, v^{-1}]$ span of $\{A^\dagger\}_{A \in \Xi}$. It is invariant under the natural actions of $\mathcal{H}$ and $\Gamma$ obtained by extending the actions defined on $M_\gamma$. Let $M_{d,q} = M_d \otimes \mathbb{Z}[v, v^{-1}] \mathbb{C}$ be the specialization of $M_d$ at $v = q^\frac{1}{2}$.

We now introduce Lusztig’s $*$-action of $W$ on $\Xi$, which we will then use to describe the action of the $\theta_w$ on $M_d$.

**Definition 3.5** ([Lus80]). Let $F^\ast$ be the set of $\Gamma^\vee$-translates of the hyperplanes in the boundary of $C^+$. For any $\gamma \in \Gamma^\vee$, let $\Pi_\gamma$ be the unique connected component of $t_\Xi \setminus \cup_{H \in F^\ast} H$ containing $A^\dagger_\gamma$.

For any $s_\alpha \in S$ and $A \in \Xi$, we then define an alcove $s_\alpha \ast A$ as follows. First, choose the unique $\gamma \in \Gamma^\vee$ such that $A \in \Pi_\gamma$, and choose $w \in W$ so that $A = w A^\dagger_{s_\alpha(\gamma)}$. Then define $s_\alpha \ast A = w A^\dagger_{s_\alpha(\gamma)}$. We then define $w \ast A$ for any $w \in W$ by induction on $c(\theta_w)$.

**Proposition 3.6.** For any $s_\alpha \in S$,

$$
\theta_{s_\alpha}(A^\dagger) = (s_\alpha \ast A)^\dagger,
$$

(3.3)
Proof. Recall the elements $A^\flat \in M_d$ defined in [Lus97], noting that in our setup they agree with the elements $\hat{D}_A$ defined in 8.9 of [Lus80]. Additionally, recall the bilinear pairing $(-,-)$ on $M_d$ defined in Section 9 of [Lus97]. Corollary 8.9 of [Lus80] gives that for any $s_\alpha \in S$, $A \in \Xi$, $\theta_{s_\alpha}(A^\flat) = (s_\alpha \ast A^\flat)$. The elements $A^\flat$ are defined such that $(A^\flat, B^\flat) = \delta_{A,B}$. One can check directly that $(\theta_{s_\alpha}A, B) = (A, \theta_{s_\alpha}B)$ for any alcoves $A$ and $B$. So the continuity property in 3.3 of [Lus97] and the well-definedness of the $\theta_{s_\alpha}$ on $A^\flat$ guaranteed by 12.2 of loc. cit. means $(\theta_{s_\alpha}A^\flat, B^\flat) = (A^\flat, \theta_{s_\alpha}B^\flat)$ for any alcoves $A, B$. This means
\[
(A^\flat, \theta_{s_\alpha}(B^\flat)) = (\theta_{s_\alpha}(A^\flat), B^\flat)
\]
\[
= ((A * s_\alpha)^\flat, B^\flat)
\]
\[
= \delta_{s_\alpha * A,B}
\]
\[
= \delta_{A, s_\alpha * B}.
\]
Since $\theta_{s_\alpha}(B^\flat)$ satisfies the other defining properties in 12.2 of [Lus97], $\theta_{s_\alpha}(B^\flat) = (s_\alpha \ast B^\flat)$, yielding (3.3).

Definition 3.7. For any $z \in W$ and $\tilde{w} \in \tilde{W}$, let $\varepsilon_z(\tilde{w})$ be the element of $\tilde{W}$ such that $z \ast A_{\tilde{w}} = A_{\varepsilon_z(\tilde{w})}$.

Definition 3.8. Let $M^0_{d,q}$ be the $\mathbb{Z}[v,v^{-1}]$-submodule of $M_d$ generated by the finitely-many elements $\{A_w\}_{w \in W}$ and their images under the operators $\{\theta_w\}_{w \in W}$. Let $M^0_{d,q}$ be the specialization of $M^0_{d,q}$ at $v = q^{\frac{1}{2}}$.

Lemma 3.9. $M^0_{d,q}$ is a finite-dimensional $\mathcal{H}_q \otimes \mathbb{C}[W]$-submodule of $M_{d,q}$.

Proof. The fact that $M^0_{d,q}$ is a finite-dimensional $\mathbb{C}[W]$-submodule follows by definition. Since for any $w \in W \subset \tilde{W}$ and any $y \in W$ we have $\tilde{T}_y A_w \in \text{span}\{A_z\}_{z \in W}$ by the definition of the $\mathcal{H}_q$-action. Since this action commutes with the action of $\mathbb{C}[W]$, it follows that $M^0_{d,q}$ is closed under the action of $\mathcal{H}_q$. \hfill $\square$

3.4. Example in Type $A_1$. For $G = \text{SL}_2$, we can identify $\Xi$ with the set of intervals $(n, n+1) \subset \mathbb{R}$ for $n \in \mathbb{Z}$, which we denote by $A_n$. With this convention, $A_e = A_0$ and $A_{s_1} = A_{-1}$. It follows from the definitions in [Lus97] that
\[
s_1 \ast A_n = A_{-n}
\]
\[
A_n^\flat = A_n + \sum_{i=1}^{\infty} (-1)^i v^{-i} A_{n+i} \in M_d.
\]

A computation then shows that
\[
\tilde{T}_{s_1} A_n = \begin{cases} A_{n+1} & \text{if } n \text{ is odd,} \\ A_{n-1} + (v - v^{-1})A_n & \text{if } n \text{ is even,} \end{cases}
\]
\[
\tilde{T}_{s_1} A_n^\flat = \begin{cases} -v^{-1} A_n^\flat & \text{if } n \text{ is odd,} \\ v A_n^\flat + A_{n-1}^\flat + A_{n+1}^\flat & \text{if } n \text{ is even,} \end{cases}
\]
Further,
\[
\theta_{s_1}(A_n^\flat) = A_{-n}^\flat.
\]
4. Kazhdan-Laumon Category $\mathcal{O}$

In this section, we recall the Kazhdan-Laumon construction for gluing perverse sheaves on the basic affine space described in [KL88]. We define a subcategory $\mathcal{A}_B$ of the Kazhdan-Laumon category $\mathcal{A}$ analogous to Category $\mathcal{O}$ along with a mixed version $\mathcal{A}_\mathcal{P}$, and describe its simple objects explicitly.

4.1. Category $\mathcal{O}$, mixed sheaves, and the Hecke algebra. In this subsection, we recall the derived category $D^b_{B,m}(X,\mathbb{Q}_\ell)$ (for $\ell$ a prime number not equal to $p$) of $B$-equivariant mixed $\ell$-adic sheaves on $X$, with heart $\text{Perv}_{B,m}(X,\mathbb{Q}_\ell)$ under the perverse $t$-structure. We will follow the setup of Chapter 7 of [Ach21], adapted to $G/U$ instead of $G/B$ as is treated in loc. cit. Recalling our choice of a square root $q^{\frac{1}{2}}$ of $q$, we define the half-integer Tate twist $(\frac{1}{2})$ on $D^b_{B,m}(X,\mathbb{Q}_\ell)$. We then view $K_0(\text{Perv}_{B,m}(X,\mathbb{Q}_\ell))$ as a $\mathbb{Z}[v,v^{-1}]$-module where $v^{-1}$ acts by $(\frac{1}{2})$. When $\mathcal{C}$ is any category for which $K_0(\mathcal{C})$ is a $\mathbb{Z}[v,v^{-1}]$-module, we denote by $K_0(\mathcal{C}) \otimes \mathbb{C}$ the specialization $K_0(\mathcal{C}) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C}$ at $v = q^{\frac{1}{2}}$. We use $\mathbb{D}$ to denote the Verdier duality functor and $p^\mathbb{D}$ to denote the zeroth perverse cohomology functor.

Recall that $X$ is stratified by affine cells $\{X_w\}_{w \in W}$, and for each $w \in W$ we let $j_w : X_w \hookrightarrow X$ be the inclusion. We will use the following definitions of standard, costandard, and simple perverse sheaves in $\text{Perv}_{B,m}(X,\mathbb{Q}_\ell)$ with Tate twists (here our notation differs by a Tate twist from that of [Ach21]).

\begin{equation}
\Delta_w = j_w!(\mathbb{Q}_\ell[l(w)])(\ell(w)\frac{l(w)}{2})
\end{equation}

\begin{equation}
\nabla_w = j_w*(\mathbb{Q}_\ell[l(w)])(\ell(w)\frac{l(w)}{2})
\end{equation}

\begin{equation}
\text{IC}_w = j_{w!*}(\mathbb{Q}_\ell[l(w)])(\ell(w)\frac{l(w)}{2})
\end{equation}

The same notation will be used when referring to sheaves on $G/B$ depending on the context. As in 7.2 of [Ach21], we also have left and right convolution operators

\begin{equation}
*: D^b_{B,m}(G/B,\mathbb{Q}_\ell) \times D^b_{B,m}(X,\mathbb{Q}_\ell) \to D^b_{B,m}(X,\mathbb{Q}_\ell),
\end{equation}

\begin{equation}
*: D^b_{B,m}(X,\mathbb{Q}_\ell) \times D^b_{B,m}(G/B,\mathbb{Q}_\ell) \to D^b_{B,m}(X,\mathbb{Q}_\ell).
\end{equation}

Note that for the right convolution operator in (4.3), we first identify objects of $D^b_{B,m}(G/B,\mathbb{Q}_\ell)$ with their pullbacks to $D^b_{B,m}(G/U,\mathbb{Q}_\ell)$ and then take the usual convolution. Convolution gives a ring structure on $K_0(\text{Perv}_{B,m}(G/B,\mathbb{Q}_\ell))$, and it gives $K_0(\text{Perv}_{B,m}(X,\mathbb{Q}_\ell))$ the structure of a bimodule over the former. For $Y = G/B$ or $Y = X$, we make the following definition.

Definition 4.1. Let $\mathcal{P}(Y)$ be the full subcategory of $\text{Perv}_{B,m}(Y,\mathbb{Q}_\ell)$ generated by objects of the form $\text{IC}_w(\frac{m}{2})$ for $w \in W$, $m \in \mathbb{Z}$. When $Y = X$, we will write $\mathcal{P} = \mathcal{P}(X)$ for short.

Proposition 4.2 (c.f. Ex. 7.4.3 in [Ach21]). There is a unique isomorphism of rings

$$
\text{ch} : K_0(\mathcal{P}(G/B)) \cong \mathcal{H}
$$

such that $\text{ch}(\nabla_w) = \tilde{T}_w$ for all $w \in W$. Moreover, this map satisfies

\begin{equation}
\begin{array}{ll}
\text{ch}(\mathcal{F}(\frac{j}{2})) = v^{-1}\text{ch}(\mathcal{F}), & \text{ch}(\mathcal{D}(\mathcal{F})) = \overline{\text{ch}(\mathcal{F})}, \\
\text{ch}(\text{IC}_w) = \mathcal{C}_w, & \text{ch}(\nabla_w(\frac{j}{2})) = T_w.
\end{array}
\end{equation}
Similarly, $K_0(\mathcal{P}(X))$ is isomorphic to $\mathcal{H}$ as a left module over itself via the convolution action of $K_0(\mathcal{P}(\mathcal{G}/B))$, where this isomorphism again respects the equations in (4.4) for elements of $K_0(\mathcal{P}(X))$.

4.2. Kazhdan-Laumon categories. In [KL88], the authors construct an abelian category $\mathcal{A}$ by “gluing” $|W|$-many copies of the category $\text{Perv}(X, \overline{\mathbb{Q}}_l)$ (which we henceforth denote simply by $\text{Perv}(X)$) via Fourier transforms $F_{s,!}$ indexed by simple reflections $s \in S$. We refer the reader to [Pol01] to a detailed explanation of these Fourier transforms. It is shown in loc. cit. that the functors $\{F_{s,!}\}_{s \in S}$ assemble to give an action of the generalized braid group $B_0$ on $\text{Perv}(X)$. Objects of $\mathcal{A}$ are then tuples indexed by $W$ of elements of $\text{Perv}(X)$ equipped with the additional structure explained in [Pol01]. For each $w \in W$, there is an exact functor $j_w^!: \mathcal{A} \to \text{Perv}(X)$ defined by $j_w^!((\mathcal{G}_w)_{w' \in W}) = \mathcal{G}_w$. Each of the functors $j_w^!$ has a left adjoint $j_w^*$ and a right adjoint $j_w^*$, each of which is a functor from $\text{Perv}(X) \to \mathcal{A}$. For a simple object $\mathcal{G}$ of $\text{Perv}(X)$ and a choice of $w \in W$ a simple object $j_w^!(\mathcal{G})$ is defined as loc. cit. as the image of the natural map $j_w^!(\mathcal{G}) \to j_w^*(\mathcal{G})$ obtained by adjunction. Finally, we also have the exact functors $\{F_w, F_{w!}\}_{w \in W}$, which act by $F_w((\mathcal{G}_w)_{w' \in W}) \mapsto (\mathcal{G}_w)_{w' \in W}$.

In Section 9 of [Pol01], this same gluing construction is applied to the category $\text{Perv}_m(X, \overline{\mathbb{Q}}_l)$ of mixed perverse sheaves on $X$ to construct a category $\mathcal{A}_m$. We will say an object $(\mathcal{G}_w)_{w \in W}$ of $\mathcal{A}_m$ is $B$-equivariant if each $\mathcal{G}_w$ is $B$-equivariant.

The following result follows from 6.3 of [Pol01].

**Proposition 4.3.** The functors $F_{w,!}$ are well-defined on $\text{Perv}_{B,m}(X, \overline{\mathbb{Q}}_l)$ and on $\mathcal{P}$. Indeed, for any $\mathcal{G} \in \text{Perv}_{B,m}(X, \overline{\mathbb{Q}}_l)$, and $s \in S$,

\[ F_{s,!}(\mathcal{G}) = pH^0(\mathcal{G} \ast \nabla_s(\frac{1}{2})) \]

**Definition 4.4.** Let $\mathcal{A}_{B,m}$ (resp. $\mathcal{A}_B$) be the Kazhdan-Laumon category obtained by applying the gluing procedure described in [KL88] and [Pol01] to $\text{Perv}_{B,m}(X, \overline{\mathbb{Q}}_l)$ (resp. $\text{Perv}_B(X, \overline{\mathbb{Q}}_l)$) and the functors $F_{w,!}$. Alternatively, Theorem 1.2.1 in [Pol01] ensures that $\mathcal{A}_B$ is the full subcategory of $B$-equivariant objects in $\mathcal{A}$.

Similarly, let $\mathcal{A}_P$ be the category obtained in this same way from $\mathcal{P}$.

We now recall that $K_0(\mathcal{P})$ admits an $\mathcal{H}$-action on the left by convolution.

**Lemma 4.5.** There is a left action of $\mathcal{H}$ on $K_0(\mathcal{P})$ given for $s \in S$ by

\[ \tilde{T}_s \cdot [\mathcal{G}] = [\nabla_s \ast \mathcal{G}] \]

Since this action commutes with convolution on the right by costandard objects and therefore with the functors $F_{w,!}$, it is straightforward to conclude that this left $\mathcal{H}$-action lifts to $K_0(\mathcal{A}_P)$.

**Proposition 4.6.** There is a left action of $\mathcal{H}$ on $K_0(\mathcal{A}_P)$ given for $s \in S$ by

\[ \tilde{T}_s \cdot [(\mathcal{A}_w)_{w \in W}] = [(\nabla_s \ast \mathcal{A}_w)_{w \in W}] \]

for any $s \in S$.

4.3. Simple objects in $\mathcal{A}_P$. We will now classify the simple objects in $\mathcal{A}_P$ up to Tate twist, which correspond to the simple objects in $\mathcal{A}_B$. We first state our main result. To do so, for any $w \in W$, let $P(w)$ denote the standard parabolic subgroup of $W$ generated by the simple reflections $s \in S$ for which $l(ws) > l(w)$. 

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The above text is a excerpt from a mathematical document discussing the construction and properties of certain categories and functors. The document delves into the theory of perverse sheaves and their actions, focusing on the Kazhdan-Laumon categories and their subcategories. The text includes technical definitions, propositions, and lemmas that are essential for understanding the structure and behavior of these objects within the categories under consideration.
Theorem 4.7. Up to Tate twist, any simple object in \( \mathcal{A}_P \) is of the form \( j_{z!*}(\text{IC}_w) \) for some \( w, z \in W \). Two such objects \( j_{z!*}(\text{IC}_{w_1}) \) and \( j_{z!*}(\text{IC}_{w_2}) \) are isomorphic if and only if \( w_1 = w_2 \) and \( z_2 \in P(w_1)z_1 \).

Accordingly, simple objects in \( \mathcal{A}_P \) up to Tate twist are in bijection with pairs \((w, \overline{w})\), where \( w \in W \), and \( \overline{w} \) is an element of \( P(w) \backslash W \).

To prove Theorem 4.7, we will need an intermediate result, Proposition 4.9. In the following, we freely use Lemma 1.3.1 in [Pol01], which guarantees that if \( \mathcal{G} \in \mathcal{A} \) is simple, then for all \( w \in W \), \( j_w^\ast(\mathcal{G}) \) is either zero or simple in \( \text{Perv}(X) \), and \( \mathcal{G} \cong j_w^\ast(j_w^\ast(\mathcal{G})) \) for any \( w \) for which it is nonzero. We begin with a lemma about Kazhdan-Lusztig polynomials \( P_{y,w} \) which we will need for our result. This is well-known; one concise proof is given in 4.3.2(ii) of [Wil03].

Lemma 4.8. If \( w \in W \) and \( s \) is a simple reflection such that \( l(ws) < l(w) \), then \( P_{ys,w} = P_{y,w} \) for all \( y < w \).

Proposition 4.9. If \( \mathcal{G} \in \mathcal{A}_P \) is simple and \( j_z^\ast \mathcal{G} = \text{IC}_w \) for some \( z \in W \) (in other words, \( \mathcal{G} \cong j_{z!*}(\text{IC}_w) \)), then for any \( y \in W \),

\[
\begin{cases}
\text{IC}_w & \text{if } y \in P(w) \\
0 & \text{if } y \notin P(w).
\end{cases}
\]

Proof. Suppose \( \mathcal{G} = j_{z!*}(\text{IC}_w) \). Pick some \( s \in P(w) \), and following [Pol01] let \( \phi_s \) denote the action of the convolution \(- \ast \nabla_s(\mathbf{1})\) on \( K_0(\text{Perv}_B(G/U)) \). By Lemma 1.3.1 in loc. cit., \( j_{z!*}^\ast \mathcal{G} \in \mathcal{P} \) is either simple or zero for every \( a \in W \). Writing \( [\mathcal{F}] = (c_a)_{a \in W} \) for \( c_a \in K_0(\mathcal{P}) \), Theorem 5.6.1 of [Pol01] ensures \( \phi_s c_z - c_{sz} \in p_s^\ast(K_0(\text{Perv}(G/Q_s))) \), where \( Q_s = [P_s, P_s] \) (for \( P_s \) the parabolic subgroup associated to \( s \) and \( p_s : G/U \to G/Q_s \) the projection). Further, an element of \( K_0(\mathcal{P}) \) is in \( p_s^\ast(K_0(\text{Perv}(G/Q_s))) \) if and only if it is in the subspace \( K_s \) of \( K_0(\mathcal{P}) \) spanned by Tate twists of \( \text{IC}_w \) for \( w' \) with \( l(w's) < l(w') \).

Using Lemma 4.8, it is easy to check that if \( l(ws) > l(w) \), then \( \phi_s[\text{IC}_w] - [\text{IC}_w] \in K_s \).

Since \( \phi_s c_z - c_{sz} \) and \( \phi_s[\text{IC}_w] - [\text{IC}_w] \) both lie in \( K_s \), it follows that \( c_{sz} - [\text{IC}_w] \in K_s \).

Since \( c_{sz} \) is the class in \( K_0(\mathcal{P}) \) of a simple object, we must have \( c_{sz} = [\text{IC}_w] \), and so \( j_{z!*} \mathcal{F} \cong \text{IC}_w \). By induction on \( l(y) \), this shows that \( j_{z!*}^\ast \mathcal{G} \cong \text{IC}_w \) whenever \( y \in P(w) \).

If instead \( y \notin P(w) \), we can write a reduced expression \( y = y_1 sy_2 \) where \( y_1 \in P(w) \), \( y_2 \in W \) and \( s \) is a simple reflection with \( l(ys) < l(v) \). We know that \( j_{y_1!*} j_{sy_2!} \mathcal{G} = j_{sy_2!*} \mathcal{G} \) by the previous case, so without loss of generality we can assume \( y_1 \) trivial. We now claim that

\[
F_{sy_2!}(\text{IC}_w) = pH^0(\text{IC}_w \ast \nabla_{sy_2}(\frac{sy_2}{2})) = 1.
\]

This will prove the result, since by the definition of \( j_{z!*} \), \( j_{sy_2!*}(j_{z!*}(\text{IC}_w)) \) is the image of \( j_{sy_2!}(j_{z!*}(\text{IC}_w)) \) under the natural map obtained by adjunction.

Indeed, \( \text{IC}_w \ast \nabla_{sy_2}(\mathbf{1}) \cong \text{IC}_w(1)[1] \) (see, e.g. 7.2.5 in [Akh21]), and \(- \ast \nabla_{sy_2} \) is right \( t \)-exact, meaning \( \text{IC}_w \ast \nabla_{sy_2} \cong \text{IC}_w(1)[1] \ast \nabla_{sy_2} \) can lie in perverse degrees at most \(-1 \). This means \( pH^0(\text{IC}_w \ast \nabla_{sy_2}(\frac{sy_2}{2})) = 0 \) as desired. \( \square \)

Proof of Theorem 4.7. The first statement follows from Lemma 1.3.2 of [Pol01] (since twists of \( \text{IC}_w \) are the simple objects of \( \mathcal{P} \)), so it remains only to determine
when simple objects of the form \( j_{w'}t_\alpha(I_{w'}) \) are isomorphic. Proposition 4.9 tells us that \( j_{w'}t_\alpha(I_{w'}) \) and \( j_{w''}t_\beta(I_{w''}) \) are isomorphic for any \( z \in P(w) \). Further, no other isomorphisms exist among the \( \{j_{w'}t_\alpha(I_{w'})\}_{w,w' \in W} \), since for any fixed \( w \), the same proposition tells us that the set of \( z \in W \) for which \( j^*_{z}(j_{w'}t_\alpha(I_{w'})) \) is nonzero depends only on the image of \( w' \) in \( P(w) \setminus W \).

**Example 4.10.** When \( G = \text{SL}_3 \), Figure 1 is an explicit list of the simple objects in \( A_B \). In this case \( w_0 = s_1s_2s_1 = s_2s_1s_2 \).

**Figure 1.** The 19 simple objects in \( A_B \) when \( G = \text{SL}_3 \).

| \( j_{cl \alpha}(IC_{cl}) \) | IC_{cl} | IC_{cl} | IC_{cl} | IC_{cl} | IC_{cl} | IC_{cl} |
|--------------------------|--------|--------|--------|--------|--------|--------|
| \( j_{cl \alpha}(IC_{s_1}) \) | IC_{s_1} | 0      | IC_{s_1} | 0      | 0      | 0      |
| \( j_{s_1 \alpha}(IC_{s_1}) \) | 0      | IC_{s_1} | 0      | 0      | IC_{s_1} | 0      |
| \( j_{s_1 \alpha}(IC_{s_2}) \) | IC_{s_2} | 0      | IC_{s_2} | 0      | 0      | 0      |
| \( j_{s_2 \alpha}(IC_{s_2}) \) | 0      | 0      | IC_{s_2} | IC_{s_2} | 0      | 0      |
| \( j_{s_1 \alpha}(IC_{s_2,s_1}) \) | IC_{s_2,s_1} | 0      | IC_{s_2,s_1} | 0      | 0      | 0      |
| \( j_{s_2 \alpha}(IC_{s_2,s_1}) \) | IC_{s_2,s_1} | 0      | 0      | IC_{s_2,s_1} | 0      | 0      |
| \( j_{s_1 \alpha}(IC_{s_1,s_2}) \) | IC_{s_1,s_2} | 0      | 0      | IC_{s_1,s_2} | 0      | 0      |
| \( j_{s_2 \alpha}(IC_{s_1,s_2}) \) | IC_{s_2,s_1} | 0      | IC_{s_2,s_1} | 0      | 0      | 0      |
| \( j_{s_1 \alpha}(IC_{s_1,s_2}) \) | IC_{s_1,s_2} | 0      | 0      | IC_{s_1,s_2} | 0      | 0      |
| \( j_{s_2 \alpha}(IC_{s_1,s_2}) \) | IC_{s_1,s_2} | 0      | 0      | IC_{s_1,s_2} | 0      | 0      |
| \( j_{w_0 \alpha}(IC_{w_0}) \) | IC_{w_0} | 0      | 0      | 0      | 0      | 0      |
| \( j_{s_1 \alpha}(IC_{w_0}) \) | IC_{w_0} | 0      | 0      | 0      | 0      | 0      |
| \( j_{s_2 \alpha}(IC_{w_0}) \) | IC_{w_0} | 0      | 0      | 0      | 0      | 0      |

4.4. Counting simple objects. We now use Theorem 4.7 to give an explicit formula for the number of simple objects in \( A_B \).

**Corollary 4.11.** The number of simple objects in \( A_B \) is

\[
\sum_{w \in W} |P(w) \setminus W|.
\]

**Remark 4.12.** In Type \( A_n \), we can interpret the quantity in Corollary 4.11 as the number of pairs of permutations in the symmetric group \( S_{n+1} \) with no common rises in the sense of [AP78]. This is A000275 in the OEIS:

1, 3, 19, 211, 3651, 90921, 3081513, 136407699, \ldots
As a result, a generating function for the number of simple objects in $A_p$ in this case is provided by the coefficients of a Bessel function, i.e. the reciprocal of $J_0(z)$ as in loc. cit.

5. Kazhdan-Laumon categories and Lusztig’s module $M_d$

The Grothendieck group $K_0(A_P)$ carries a natural $W$-action along with an action of the Hecke algebra $H$. In this section, we use the description of simple objects in $A_P$ to show that there exists an isomorphism $\eta: K_0(A_P) \otimes \mathbb{C} \to M_0^d$ of $H \otimes \mathbb{C}[W]$-modules.

5.1. The modules $M^+_d$ and $\overline{M}^0_{d,q}$

**Definition 5.1.** Let $M^+_d$ be the $H \otimes \mathbb{C}[W]$-submodule of $M_{d,q}$ generated by $A^+_w$ for $\tilde{w} \in \tilde{W}$ with the property that for any $z \in W$, $\epsilon_z(\tilde{w})$ can be written as $w \cdot \lambda$ for some $w \in W, \lambda \geq 0$. It follows from [Lus97] that $M^0_{d,q} \subset M^+_d$.

Let $\overline{M}^0_{d,q}$ be the quotient of $M^+_d$ by the span of the elements $A^+_w$ such that there is no pair $z, w \in W$ for which $\epsilon_z(w) = \tilde{w}$. Since it is easy to see that this is a $H_q \otimes \mathbb{C}[W]$-submodule, we will continue to refer to the $H_q$-action and the $W$-action by $(\theta_q)_{y \in W}$ on the quotient module $\overline{M}^0_{d,q}$.

We use $\pi$ to refer to the quotient map from $M^+_d$ or $M^0_{d,q}$ to $\overline{M}^0_{d,q}$, and we write $\pi^d$ for $\pi(A^d)$ whenever $A^d \in M^+_d$.

5.2. The bijection. By the preceding section, recall that we can consider $K_0(A_P)$ as a $H \otimes \mathbb{Z}[v, v^{-1}][W]$ module, with $v$ acting by the half-Tate twist $(-\frac{1}{2})$, $H$ acting as in Proposition 4.6 and $W$ acting by the functors $F_w$. We will then consider $K_0(A_P) \otimes \mathbb{Z}[v, v^{-1}] \mathbb{C}$ where $v \mapsto q^{\frac{1}{2}}$.

**Theorem 5.2.** There is a well-defined isomorphism $\eta: K_0(A_P) \otimes \mathbb{C}[v, v^{-1}] \mathbb{C} \to M^0_{d,q}$ of $H_q \otimes \mathbb{C}[W]$-modules such that for any $w \in W$,

\[
\eta([j_{\tilde{c}t}(\Delta_w)]) = A_w.
\]

Further,

\[
(\pi \circ \eta)([j_{\tilde{c}t}(IC_w)]) = \pi^d
\]

in $\overline{M}^0_{d,q}$ for any $w \in W$.

We illustrate in Figure 2 the morphism $\pi \circ \eta: K_0(A_P) \otimes \mathbb{C} \to \overline{M}^0_{d,q}$ in the case $G = SL_3$. The next subsection will be devoted to the proof of this result.

5.3. Proof of Theorem 5.2

We begin by defining $\eta': K_0(A_P) \otimes \mathbb{C} \to \overline{M}^0_{d,q}$ for any $w, y \in W$ by

\[
\eta'( [j_{\tilde{c}t}(IC_w)]) \mapsto \theta_{y^{-1}} \left( \pi^d_w \right).
\]

To prove Theorem 5.2, we will first prove that $\eta'$ is a well-defined and bijective morphism of $\mathbb{C}[W]$-modules onto $\overline{M}^0_{d,q}$. Then we will show that $\pi$, too, is an isomorphism of $\mathbb{C}[W]$-modules from $M^0_{d,q}$ to $\overline{M}^0_{d,q}$. Then we will show that $\pi \circ \eta = \eta'$, which gives that $\eta$ is well-defined and bijective. From this, we will deduce the fact that $\eta$ is a morphism of $H_q$-modules.
Lemma 5.3. The map $\eta'$ is a well-defined isomorphism of $\mathbb{C}[W]$-modules.

Proof. This follows from the claim that the stabilizer of any $[j_{\ell,s}(\text{IC}_w)]$ under the action of the $\{F_w\}_{w \in W}$ is equal to the stabilizer of $A^w_0$ under the action of the $\{\theta_w\}_{w \in W}$. Note that Proposition 3.4 shows that the stabilizer of $A^w_0$ under the latter action corresponds to the stabilizer of $A_w$ under the $*$-action of $W$ on $\Xi$. One can check that this stabilizer is $P(w)$, which is the stabilizer of $[j_{\ell,s}(\text{IC}_w)]$ under the action of the $\{F_w\}_{w \in W}$ by Theorem 4.7.

Lemma 5.4. The morphism $\pi : M^0_{d,q} \to \overline{M}^0_{d,q}$ is an isomorphism of $\mathbb{C}[W]$-modules.

Proof. Note first that $\pi$ is clearly a surjective morphism of $\mathbb{C}[W]$-modules. To show it is an isomorphism, it is sufficient to check that $\dim M^0_{d,q} \leq \dim \overline{M}^0_{d,q}$. By Lemma
and Corollary 4.11.

\[ \dim M^0_{d,q} = \sum_{w \in W} |P(w)\backslash W|. \]

We now claim that \( M^0_{d,q} \) is spanned by elements of the form \( \theta_y^{-1}(A_w) \) where \( y \) is a the minimal-length representative of some coset \( P(w)y \) of \( P(w) \) in \( W \). Since there are \( \sum_{w \in W} |P(w)\backslash W| \) such elements, this will prove the inequality above.

We do this by induction on the Bruhat order: we claim that for any \( w \in W \), \( \text{span} \{ \theta_y^{-1}(A_w) \} \) \( y \in W, w \leq w \) is spanned by the subset of these elements with minimal-length \( y \) in the coset \( P(w)y \) as above. For \( w = w_0 \), this is trivial as \( P(w_0) = \emptyset \).

Suppose now that it is true for some \( w \). Then by further induction on the length of \( y \) it is enough to show that for any \( s \in P(w) \), \( \theta_s(A_w) \) is a linear combination of \( A_{ws}, A_w, \) and \( \theta_s(A_{ws}) \). Indeed, by the definition of \( \theta_s \) in 3.2 one can check that, since \( l(ws) > l(w) \),

\[ \theta_s(A_w) = A_w + v^{-1}A_{ws} - v^{-1}\theta_s(A_s), \]

completing the proof. \( \square \)

Now we define \( \eta = \pi^{-1} \circ \eta' \). By Lemmas 5.3 and 5.4 we know \( \eta \) is an isomorphism of \( \mathbb{C}[W] \)-modules. The following proposition will follow from Proposition 5.9 which is proved in the next subsection, and so we will delay its proof until then.

**Proposition 5.5.** For any \( w \in W \),

\[ \eta(\{j_z!(\Delta_w)\}) = A_w \]

Before we use Proposition 5.5 to show that \( \eta \) is a morphism of \( \mathcal{H}_q \)-modules to prove Lemma 5.7 below, we need the following straightforward consequence of Polishchuk’s work on “good representations” in \([Pol01]\).

**Lemma 5.6** ([Pol01], Theorem 11.5.1). The elements \( \{j_z!(\Delta_w)\}_{z, w \in W} \) generate \( K_0(A_P) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C} \) whenever \( v \) is specialized to a value which is not a root of unity (in particular, in our situation where \( v \mapsto q^{\pm} \)).

**Proof.** In Theorem 11.5.1 of \([Pol01]\), it is shown that \( K_0(A_P) \) must be generated by objects of the form \( j_z!(A) \) for \( A \in P \) so long as the action defined by the \( F_{w,!} \) functors factors through the Hecke algebra \( \mathcal{H}_q \). Although this is not true for the action of these functors on \( K_0(A) \), it holds for \( K_0(A_P) \) by Proposition 4.3. Since \( D^b(P) \) is generated by standard objects \( \{\Delta_w\}_{w \in W} \) and their Tate twists, this proves the lemma. \( \square \)

Note that it is not true that the elements in Lemma 5.6 generate \( K_0(A_P) \) alone before specializing at \( q^{\pm} \); this follows from the example provided in the appendix to \([Pol01]\). The question of whether the Grothendieck group \( K_0(A) \) of the full Kazhdan-Laumon category is generated by a similar collection is the subject of much of the work in \([Pol01]\); in future work, we hope to work beyond the case of Kazhdan-Laumon Category \( \mathcal{O} \) to address some of these conjectures in general.

**Lemma 5.7.** \( \eta \) is a morphism of \( \mathcal{H}_q \)-modules.
Comparing this with (3.1), we conclude

\[ \eta(\tilde{T}_s \cdot [j_{e!}(\Delta_w)]) = \eta([j_{e!}(\nabla_s \ast \Delta_w)]) \]

\[ = \begin{cases} 
\eta([j_{e!}(\Delta_{sw})]) & l(sw) < l(w), \\
\eta([j_{e!}(\Delta_{sw})]) + [j_{e!}(\Delta_w(-\frac{1}{2}))] - [j_{e!}(\Delta_w(\frac{1}{2}))] & l(sw) > l(w) 
\end{cases} \]

\[ = \begin{cases} 
A_{sw} & l(sw) < l(w), \\
A_{sw} + (q^{1/2} - q^{-1/2})A_w & l(sw) > l(w). 
\end{cases} \]

Comparing this with (3.1), we conclude

\[ (5.6) \quad \eta(\tilde{T}_s \cdot [j_{e!}(\Delta_w)]) = \tilde{T}_s \cdot \eta([j_{e!}(\Delta_w)]) \]

for any \( w \in W, s \in S. \)

By Lemma 5.3, the elements \([j_{e!}(\Delta_w)]\) for \( z, w \in W \) span the space \( K_0(A_P) \otimes \mathbb{C}. \) Since the \( W \)-action commutes with the action of \( H_q, \) the equation (5.6) implies that \( \eta \) is a \( H_q \)-module homomorphism on all of \( K_0(A_P) \otimes \mathbb{C}, \) as desired. \( \square \)

Combining Lemmas 5.3 and 5.7 (which, to recall, show together that \( \eta \) is a \( \mathbb{C}[W] \)-module isomorphism) with Proposition 5.5, Theorem 5.2 is proved. It remains to prove Proposition 5.7, this will be our main focus in the next subsection.

5.4. Restriction to fundamental alcoves. Let \( \Xi_\mathfrak{fn} \) denote the set of alcoves \( \{A_w\}_{w \in W} \) which we call the fundamental alcoves. In this subsection, we define a map \( J_e : M^0_{d,q} \to K_0(P) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C} \) (which can be interpreted as a sort of restriction to \( \Xi_\mathfrak{fn} \)) such that the diagram

\[ \begin{array}{ccc}
K_0(A_P) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C} & \xrightarrow{\eta} & M^0_{d,q} \\
\downarrow{\tilde{J}_e} & & \downarrow{J_e} \\
K_0(P) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C} & \xrightarrow{\rho} & \mathbb{C}
\end{array} \]

(5.7)

commutes. This gives an interpretation of the functor \( j^* \) on the \( M^0_{d,q} \) side of the bijection from Theorem 5.2 and will allow us to prove Proposition 5.7.

**Definition 5.8.** We define the map \( J_e : M^0_{d,q} \to K_0(P) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C} \) as the unique \( \mathbb{C} \)-linear map satisfying

\[ J_e(A_w) = \begin{cases} 
[\Delta_w] & \text{if } w \in W \\
0 & \text{if } w \in W \setminus W, 
\end{cases} \]

extending linearly to the domain \( M^0_{d,q}. \)

Let \( J^*_e : K_0(P) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C} \to M^0_{d,q} \) be the unique section of \( J_e \) whose image lies in the subspace of \( M^0_{d,q} \) spanned by \( \{A_w\}_{w \in W}. \) Let \( \xi = J^*_e \circ J_e \) be the projection onto the subspace of \( M^0_{d,q} \) spanned by \( \{A_w\}_{w \in W}. \) Finally, let \( \rho = \text{id} - \xi \) be the projection onto the subspace of \( M^0_{d,q} \) spanned by \( \{A_w\}_{w \in W \setminus W}. \)

**Proposition 5.9.** The diagram (5.7) commutes, i.e. \( J_e \circ \eta = j^*_e. \)

**Proof.** By the linearity of each of these maps, it is enough to check this statement on the elements \( \{[j_{e!}(\mathcal{I}C_w)]\}_{z,w \in W}, \) which span \( K_0(P) \) under \( \mathbb{Z}[v,v^{-1}] \) by Theorem...
By the same result, we know that
\[
j_*^w ([j_*^z(\text{IC}_w)]) = \begin{cases} \text{IC}_w & z \in P(w) \\ 0 & z \notin P(w), \end{cases}
\]
while \(J_\ast(\eta([j_*^z(\text{IC}_w)])) = J_\ast(\theta_{z^{-1}}(A^0_w))\). Since \(\pi \circ \eta = \eta'\) and it is easy to check that \(J_\ast\) is zero on any objects in the kernel of \(\pi\) by the definition of \(\overline{M}^0_{d,q}\), it is enough to show that
\[
J_\ast(\theta_{z^{-1}}(A^0_w)) = \begin{cases} \text{IC}_w & z \in P(w) \\ 0 & z \notin P(w). \end{cases}
\]

In the case where \(z \in P(w)\), we know by the proof of Lemma 5.3 that \(\theta_{z}(A^0_w) = A^0_w\), so this reduces to showing \(J_\ast(\theta_{z}(A^0_w)) = [\text{IC}_w]\). By the Kazhdan-Lusztig conjectures, formula (1.5.b) in [KL79] holds, and so our claim follows from the remark following 13.10 in [Lus80] combined with 11.19 of loc. cit. (for \(e = 0\)).

On the other hand, if \(z \notin P(w)\), then \(\theta_{z^{-1}}(A^0_w) = (z^{-1} \ast A_w)^\sharp\). It is easy to see that if \(a \ast A_w \neq A_w\) for \(a, w \in W\), then \(a \ast A_w\) is not dominated by any alcove in \(\{A_w\}'\) occurring with nonzero coefficient in \((z \ast A_w)^\sharp\), and so \(J_\ast(\theta_{z^{-1}}(A^0_w)) = 0\) in this case, as desired.

**Corollary 5.10.** An element \(C \in M^0_d\) is zero if and only if \(J_\ast(\theta_{z}(C)) = 0\) for all \(z \in W\).

**Proof.** By the previous proposition and the fact that \(\eta\) is an isomorphism of \(C[W]\)-modules, this reduces to the fact that \([\mathcal{F}] = 0\) for \(\mathcal{F} \in \mathcal{A}_F\) if and only if \(j_*^z(\mathcal{F}) = 0\) for all \(z \in W\).

Now to prove Proposition 5.3 we will need the following; it is an easy computation from the definition of \(\theta_s\).

**Lemma 5.11.** For any \(w \in W\), \(s \in S\),
\[
J_\ast(\theta_s(A_w)) = [\Delta_w \ast \nabla_s(\frac{1}{\sharp})].
\]

We now wish to upgrade this result by induction and show that \(J_\ast(\theta_s(A_w)) = [\Delta_w \ast \nabla_s(\frac{1}{\sharp})]\) for any \(w, z \in W\). To do so, we will need to keep track of possible positions of alcoves after applying \(\theta_s\) to an alcove in \(\Xi_{\text{fin}}\). To make this precise, we introduce the following definition and subsequent lemmas.

**Definition 5.12.** For any \(v \in W\) and reduced word \(\overline{\tau}\) for \(v\), recall that we can define a subset of positive roots \(R(\overline{\tau})\). If \(\overline{\tau} = s_{i_1} \ldots s_{i_k}\), we set
\[
R(\overline{\tau}) = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \ldots, s_{i_{k-1}}(\alpha_{i_k})\}.
\]

Note that if \(\overline{\tau}\) is reduced and \(s\) is a simple reflection with \(l(sv) > l(v)\), then
\[
R(s\overline{\tau}) = sR(\overline{\tau}) \cup \{\alpha_s\}.
\]

We will sometimes use \(R(v)\) to denote \(R(\overline{\tau})\) for some choice of reduced word \(\overline{\tau}\) for \(v \in W\).

**Definition 5.13.** Given any subset \(R\) of positive roots, we define \(\Xi^+(R) \subset \Xi\) to be the set of alcoves \(A\) satisfying the following two conditions. First, we require that \(A \notin \Xi_{\text{fin}}\). Second, we require that \(A\) lies in the same Weyl chamber as some
element of $R$. By this definition, each each $\Xi^+(R)$ will be a union of Weyl chambers but with $\Xi_{\text{fin}}$ excised.

One can easily check for any two reduced expressions $w$ and $w'$ for $w \in W$ that $\Xi^+(R(w)) = \Xi^+(R(w'))$.

**Definition 5.14.** For any $w \in W$, let $\Xi^+_w = \Xi^+(R(w))$ for a choice of reduced expression $w$ for $w$.

The following is a straightforward computation which follows from the definition of $\theta_s$.

**Lemma 5.15.** If $v \in W$ and $s \in S$ with $l(sv) > l(v)$, then

$$\theta_s(\text{span}(\Xi^+_v \cup \Xi_{\text{fin}})) \subset \text{span}(\Xi^+_v \cup \Xi_{\text{fin}}),$$

$$\xi(\theta_s(\text{span}(\Xi^+_v))) = 0.$$ 

**Corollary 5.16.** For any $w, z \in W$,

$$\rho(\theta_z(A_w)) \in \text{span}(\Xi^+_z).$$ 

**Proof.** We go by induction on $l(z)$. For $l(z) = 0$, there is nothing to prove. Now suppose the result holds for $z$, and let $s \in S$ be such that $l(sz) > l(z)$. Then since $\rho + \xi = \text{id}$,

$$\rho(\theta_z(A_w)) = (\rho \circ \theta_s)[\rho(\theta_z(A_w)) + \xi(\theta_z(A_w))].$$

Clearly $\xi(\theta_z(A_w)) \in \text{span}(\Xi_{\text{fin}})$, and by our induction hypothesis, $\rho(\theta_z(A_w)) \in \text{span}(\Xi^+_z)$. So Lemma 5.15 guarantees that $\theta_s[\rho(\theta_z(A_w)) + \xi(\theta_z(A_w))]$ lies in the span of $\Xi^+_z \cup \Xi_{\text{fin}}$, and therefore its image under $\rho$ lies in $\text{span}(\Xi^+_z)$. \hfill $\square$

With these new definitions and lemmas in hand, we can now provide an inductive proof of Proposition 5.5.

**Proof of Proposition 5.5.** By the definition of $j_{c!}$,

$$j_{c!}^*(\mathcal{F}_z(j_{c!}^*(\Delta_w))) = \Delta_w \ast \nabla_{z^{-1}}(\frac{l(z)}{2}),$$

so by Corollary 5.10 to show that $\eta(j_{c!}^*(\Delta_w)) = A_w$ it is enough to show that

$$(5.8) \quad J_c(\theta_z(A_w)) = [\Delta_w \ast \nabla_{z^{-1}}(\frac{l(z)}{2})],$$

for all $w, z \in W$. We will fix $w$ and proceed by induction on $l(z)$. For $l(z) = 0$, this is immediate from the definition of $J_c$.

Suppose then that we know (5.8) holds for some $z$. Suppose $s \in S$ is such that $l(sz) > l(z)$. Then by this assumption along with Lemma 5.11 (extended by linearity, using linearity of the convolution $- \ast \nabla(\frac{1}{2})$) that

$$(J_c \circ \theta_s \circ \xi \circ \theta_z)(A_w) = J_c(\theta_s(J_c(\Delta_w \ast \nabla_{z^{-1}}(\frac{l(z)}{2}))))$$

$$= [\Delta_w \ast \nabla_{z^{-1}}(\frac{l(z)}{2}) \ast \nabla_s(\frac{1}{2})]$$

$$= [\Delta_w \ast \nabla_{sz^{-1}}(\frac{l(sz)}{2})].$$

To complete the proof, it remains to show that

$$J_c(\theta_{sz}(A_w)) = (J_c \circ \theta_s \circ \xi \circ \theta_z)(A_w).$$
By Corollary 5.16 we know $\rho(\theta_z(A_w)) \in \text{span}(\Xi^+_\alpha)$. By the first part of Lemma 5.15 we then have

$$J_\epsilon(\theta_xz(A_w)) - J_\epsilon(\theta_x(\xi(\theta_z(A_w)))) = (J_\epsilon \circ \theta_x)(\theta_z(A_w) - \xi(\theta_z(A_w)))$$

$$= (J_\epsilon \circ \theta_x)(\rho(\theta_z(A_w)))$$

$$\in J_\epsilon(\theta_x(\text{span}(\Xi^+_\alpha))),$$

which is zero by the second part of Lemma 5.15. \qed

6. The Schwartz space of the basic affine space

In this section, we recall the definition of the Schwartz space of the basic affine space $S$ as defined in [BK99]. We then focus on Iwahori-invariants $S^{T \times T(\mathcal{O})}$ of this space, and revisit a claim made in loc. cit. about its connection to $M_d$. This will then serve as a bridge to a more direct connection, which we will consider in the final subsection, between $S$ and the Kazhdan-Laumon category $A$.

6.1. Preliminaries from [BK99]. In this section, let $k$ be a non-Archimedean local field with residue field $\kappa = \mathbb{F}_q$, and $\mathcal{O}$ its ring of integers. Let $\pi$ be a uniformizer of $k$, and suppose the norm on $k$ is chosen so that $||\pi|| = q^{-1}$.

In this section only, we write $G_k$ for the base change of $G$ to $k$; we will consider the basic affine space $X(k) = (G/U)(k)$. Let $I \subset G(k)$ be an Iwahori subgroup. The setup of [BK99] begins with considering the space $S_c$ of locally constant compactly supported functions on $X(k)$; let $S \supset S_c$ be the Schwartz space of the basic affine space defined in loc. cit.

**Definition 6.1.** For any $n \in \mathbb{Z}$, let $\psi_n$ be a choice of additive character of conductor $n$ of the field $k$ (i.e. a character for which $\pi^n \mathcal{O} \subset \ker \psi_n$ but $\pi^{n-1} \mathcal{O} \not\subset \ker \psi_n$).

In [BK99], a $W$-action on $S$ is defined via Fourier transforms $\Phi_\alpha$ associated to any $s_\alpha \in S$ as follows. Let $P_\alpha(k)$ be the parabolic subgroup of $G(k)$ naturally corresponding to $s_\alpha$, and let $Q_\alpha(k) = [P_\alpha(k), P_\alpha(k)]$. Then $\pi_\alpha : X(k) \to G(k)/Q_\alpha(k)$ is a fibration with fibers isomorphic to $k^2 \setminus \{(0,0)\}$; the Fourier transform $\Phi_\alpha$ is defined fiberwise. Because the definition in [BK99] (c.f. the more explicit definition in [Kaz95]), which we will now slightly modify for our choice of the character $\psi_1$ depends on a choice of character and normalization, in the following definition we choose conventions explicitly for the present paper.

**Definition 6.2.** Define the Fourier transform operator $\Phi_\alpha : S \to S$ associated to a simple root $s_\alpha \in S$ by

$$\Phi_\alpha(f)(x) = q \int_{\pi^{-1}(\pi_\alpha(x))} \psi_1(\langle x, x' \rangle_{s_\alpha}) f(x') dx',$$

where we use the identification of the fiber $\pi^{-1}(\pi_\alpha(x))$ with $V_s(x) - \{0\}$, where $V_s(x)$ is a two-dimensional vector space with a volume form defined by the pairing $\langle \cdot, \cdot \rangle_{s_\alpha} : V \times V \to k$ (following the setup and notation of Section 1.1.2 of [Kaz95]).

We define $\Phi_w$ for any $w \in W$ by $\Phi_w = \Phi_{\alpha_1} \cdots \Phi_{\alpha_k}$ where $w = s_{\alpha_1} \cdots s_{\alpha_k}$ is a reduced expression.

**Remark 6.3.** The definition of the Fourier transform endomorphism $\Phi_\alpha$ in [BK99] depends on a choice of additive character of $k$. From now on, we will always use the character $\psi_1$ and the normalization (by $q$) chosen in Definition 6.2. This is the only choice of character for which Theorem 6.2 will hold as it is written. The
usual choice of character in the literature (as is used in Example 3.2 of [BK99] and throughout [Kaz95] and [Daw21], for example) is \( \psi \). However, this choice is not compatible with the normalizations chosen in the statement of Theorem 6.2, as we explain in Example 6.4. The reader should beware that our nonstandard choice of \( \psi \) is such that, in the \( SL_2 \) case,
\[
\Phi^\circ_{\alpha}(\chi_{O} \times \pi_{O}) = \chi_{O} \times \pi_{O},
\]
\[
\Phi^\circ_{\alpha}(\chi_{O} \times \pi_{\omega}) = q \chi_{O} \times \pi_{O} \neq \chi_{O} \times \pi_{O},
\]
which does not agree with the computation in Example 3.2 of [BK99] due to the discrepancy between the characters \( \psi_1 \) and \( \psi_0 \), as well as the constant \( q \) by which we multiply the integral in Definition 6.2.

The Iwahori subgroup \( I \) acts naturally on \( X(k) \) on the left, while \( T(O) \) acts on the right. We consider now the \( I \)-invariant subspace \( S^{I \times T(O)} \), as is studied in Section 4 of [BK99]. Recall, as explained in loc. cit., that the \( I \)-orbits on \( X(k) \) are indexed by \( \hat{W} \); for any \( w \in \hat{W} \) we can associate the \( I \)-orbit \( IwU(k) \subset X(k) \).

We also know that for all \( w \in \hat{W} \) and \( f \in S^{I \times T(O)} \), we can define the convolution \( \chi_{Iw} * f \), where \( \chi_{Iw} \) is the indicator function on the \( I \)-orbit \( IwI \subset G(k)/I \). This defines an action of the affine Hecke algebra \( H_q \), with \( T_w \) acting such that
\[
T_{w^{-1}}^{-1}(f) = (-q^{-1/2})^{l(w)} \chi_w * f.
\]

6.2. Connection to \( M_d \). In Lemma 4.3 and Corollary 4.6 of [BK99], it was claimed that \( M_{d,q} \equiv S(X)^{I \times T(O)} \) as \( H_q \otimes \mathbb{C}[W] \)-modules, where on the Schwartz space side the \( H_q \)-action is given by convolution and the \( W \)-action on the given by the \( \Phi_w \), while on the \( M_{d,q} \) side the \( W \)-action is given by the \( \theta_w \). We reiterate that this result is not true if the “usual” definition of the Fourier transforms using the character \( \psi_0 \) is used, as we explain in the following example.

Example 6.4. Suppose \( G = SL_2 \), so \( X(k) \cong \mathbb{A}_k^2 \setminus \{(0,0)\} \). For each \( w \in \tilde{A}_1 = \langle s_1, s_0 \rangle \), there is an Iwahori-orbit \( IwU(k) \subset X(k) \). These sets are either of the form \( \pi^nO \times \pi^nO \setminus \pi^{n+1}O \) or \( \pi^nO \times \pi^{n+1}O \setminus \pi^{n+1}O \times \pi^{n+1}O \) for some \( n \in \mathbb{Z} \).

Lemma 4.3 in [BK99] asserts that the map \( \gamma : S(X)^{I \times T(O)} \to M_{d,q} \) takes some multiple of the indicator function on each Iwahori orbit \( IwU(k) \) to some alcove \( A \in \Xi \). Accordingly, set \( A^* = \gamma(-q^{-1/2} \chi_{IwU(k)}) \). (Note that \( IwU(k) = O \times O \setminus \pi O \).)

Following the computations in Section 3.4, one can compute that we must have \( \gamma(-q^{-1/2} \chi_{O} \times \pi_{O}) = (A^*)^s \). Since \( \chi_{O} \times \pi_{O} \) is \( G(O) \)-invariant, we know that \( T_{s_1}(\chi_{O} \times \pi_{O}) = -q^{-1/2} \chi_{O} \times \pi_{O} \). So if \( \gamma \) is to be an isomorphism of \( H_q \)-modules, we must have that \( T_{s_1}((A^*)^s) = -q^{-1/2} (A^*)^s \), meaning \( A^* = A_n \) for some odd \( n \), by the computations in Section 3.4.

This is impossible, however, since if the character \( \psi_0 \) is used, then \( \Phi_{\alpha}(\chi_{O} \times \pi_{O}) = \chi_{O} \times \pi_{O} \), and so we expect \( \theta_{s_1}(A_n^s) = A_n^s \). But this only holds for \( n = 0 \), which is not odd, contradicting the existence of such a \( \gamma \). This example is intended to justify Definition 6.2 and specifically our nonstandard choice of additive character, showing that such a choice is necessary if Corollary 4.6 in [BK99] is to hold.

We can fix this by our use of the character \( \psi_1 \) in Definition 6.2, thereby yielding the following result. For ease of notation, we denote by \( \delta_w \) the function \((-q^{1/2}d(A_n,A_n)) \chi_{IwU(k)} \in S^{I \times T(O)} \).
Theorem 6.5 (BK99). The map \( \Psi : M_c \to S^1 \times T(\mathcal{O}) \) defined on alcoves \( A_w \) by
\[
\Psi(A_w) = \delta_w,
\]
extends to an isomorphism between \( M_{d,q} \) and \( S^1 \times T(\mathcal{O}) \) of \( \mathcal{H}_q \otimes \mathbb{C}[\Gamma \times W] \)-modules.

Example 6.6. We now explain this bijection explicitly in the case of \( \mathbf{SL}_2 \); its proof in general follows from this computation. In this case, \( W = \langle s_1 \rangle \); we also denote by \( s_0 \) the simple reflection associated to the affine root. Let \( A_n \) be the alcove \((n, n+1)\). Note that for any \( n \), \( A_{2n} = A_{(s_1 s_0)^n} \), while \( A_{2n-1} = A_{s_1 (s_1 s_0)^n} \).

The action of \( \mathcal{H}_q \) on \( M_{d,q} \) is given by the formulas of Section 6.3 specialized at \( v = q^{1/2} \). Comparing these with the action of \( T_{s_1} \) on \( S^1 \times T(\mathcal{O}) \) whose inverse is given by convolution with \( -q^{-1/2} \chi_{s_1 t} \), one can check that the actions agree; the same goes for the action of \( \Gamma \).

It remains to check that the \( W \)-actions agree, which is only true when using the character \( \psi_1 \) in the definition of \( \Phi_{s_1} \). By applying \( \Psi \) to the formula (3.3),
\[
\Psi(A^\pm_n) = \begin{cases} (-q^{1/2})^n \chi \times (\pi(n/2) \times (\pi(n/2+1) \mathcal{O}) & \text{if } n \text{ is even,} \\ (-q^{1/2})^n \chi \times (\pi(n+1/2) \times (\pi(n+1/2) \mathcal{O}) & \text{if } n \text{ is odd.} \end{cases}
\]
Since for \((x, y) \in k \times k\),
\[
\Phi_{s_1}(\chi \times \mathcal{O} \times \pi \mathcal{O}) = q \int_{a \in \pi \mathcal{O}} \int_{b \in \pi \mathcal{O}} \psi_1(xb - ya) = q \psi_1(-ya) \int_{b \in \pi \mathcal{O}} \psi_1(xb) = q \cdot q^{-2k-1} \chi \times \pi \mathcal{O}(y) = q^{-2k} \chi \times \pi \mathcal{O}(x, y),
\]
which implies that
\[
\Phi_{s_1} \Psi(A^\pm_n) = \Psi(A^\pm_n) = \Psi(\theta_{s_1}(A^\pm_n))
\]
for \( n \) even, while a similar proof works for \( n \) odd.

6.3. Schwartz space and Kazhdan-Laumon categories. We now translate Theorem 6.2 into a statement about the Schwartz space of the basic affine space, since we can relate \( M_{d,q} \) to \( S^{1} \times T(\mathcal{O}) \) by Theorem 6.5.

Theorem 6.7. There is an injection \( \Theta : K_0(\mathcal{A}_P) \otimes \mathbb{Z}[v, n^{-1}] \mathcal{O} \to S^{1} \times T(\mathcal{O}) \) of \( \mathcal{H}_q \otimes \mathbb{C}[W] \)-modules given by
\[
(6.3) \quad \Theta([j_{c!}(\Delta_w)]) = \delta_w
\]
for all \( w \in W \).

Proof. This follows directly from Theorems 6.5 and 6.2. The last part follows in particular by comparing equation (5.1) with Theorem 6.5 and the definition of \( \delta_w \).

Now we place Theorem 6.2 in this context, rephrasing the result in terms of a more direct connection between the Kazhdan-Laumon category \( \mathcal{A} \) containing \( P \), and the Schwartz space \( S \supseteq S^{1} \times T(\mathcal{O}) \).

Grothendieck’s sheaf-function correspondence tells us that for any \( F \in \mathcal{P} \), we can produce a function \( \text{tr}(F) : (G/U)(\mathbb{F}_q) \to \mathbb{Q}_\ell \). Applying this to the standard
sheaves $\Delta_w \in \mathcal{P}$, it follows that $\text{tr}(\Delta_w) = (-q^{-1/2})^{l(w)} \chi_w$, where $\chi_w$ is the indicator function on the $\mathbb{F}_q$-points of the Bruhat cell $BwU \subset G/U$. Returning to the setting where $k$ is a local field, $\mathcal{O}$ its ring of integers, $\pi$ the uniformizer, and $\kappa$ the quotient, one can construct an “Eisenstein map” as follows.

Let $X(\mathcal{O})_{\text{fin}} \subset X(\mathcal{O})$ denote the union of the Bruhat cells $IwU$ across all $w$ in the finite Weyl group $W$. The subset $X(\mathcal{O})_{\text{fin}}$ has the property that the projection $\mathcal{O} \twoheadrightarrow \kappa$ induces a well-defined and surjective map $X(\mathcal{O})_{\text{fin}} \rightarrow X(\kappa)$.

We then have the diagram

$$
\begin{array}{ccc}
X(\mathcal{O})_{\text{fin}} & \xrightarrow{i} & X(\kappa) \\
\downarrow p & & \downarrow \\
X(k) & \xrightarrow{\iota} & X(\kappa)
\end{array}
$$

which we obtain from the natural maps $k \leftarrow \mathcal{O} \twoheadrightarrow \kappa$. One can then consider $\iota \circ p^*$ as a map from the space $C(X(\kappa))$ of functions on $X(\kappa)$ to $S_c \subset S_c(X(\kappa))$.

**Definition 6.8.** Let $S_0$ be the subspace of $S$ generated under the Fourier transforms $\Phi_w$ by the image of $\iota ! \circ p^*$.

Then (6.3) in Theorem 6.7 tells us the following.

**Proposition 6.9.** The map $\Theta$ is an isomorphism onto $S^I \times T(\mathcal{O})$. For $F \in \mathcal{P}$,

$$
\Theta([j_!(F)]) = (\iota \circ p^*)(\text{tr}(F)) \in S^I \times T(\mathcal{O}),
$$

This concludes the proof of Theorem 1.3 from the introduction, giving a direct interpretation of the composition $K_0(A_P) \otimes \mathbb{C} \rightarrow M_{d,q} \rightarrow S^I \times T(\mathcal{O})$ (each of which is originally described combinatorially) in terms of Grothendieck’s sheaf-function dictionary via the map $\iota \circ p^*$.

7. Perverse sheaves on the semi-infinite flag manifold

7.1. Preliminaries. In this section, we follow the technical setup of [ABB+05]. In loc. cit., the authors define a category $\text{Perv}(\mathcal{F}^{\mathcal{T}})^{\text{fr}}$, intended to be an analogue of a category of Iwahori-monodromic perverse sheaves on the semi-infinite flag variety by using the Drinfeld space $B\text{un}_N$. (In this section, we use the notation $N^-$ for the maximal unipotent subgroup opposite $U$ to be consistent with loc. cit.) This construction builds off of earlier work in [FM99], [FFKM99] and [BFGM02].

It is shown in [ABB+05] that the category $\text{Perv}(\mathcal{F}^{\mathcal{T}})^{\text{fr}}$ has simple objects $IC_{\mathcal{W}}$ indexed by $\mathcal{W} \in \mathcal{W}$. For each such $\mathcal{W}$, the standard and costandard objects $\nabla_{\mathcal{W}}$ and $\Delta_{\mathcal{W}}$ are also defined using maps $i_{\mathcal{W}}$ corresponding to inclusion of the appropriate stratum indexed by $\mathcal{W}$ (we use bold symbols for standard and costandard objects in $\text{Perv}(\mathcal{F}^{\mathcal{T}})^{\text{fr}}$ to distinguish them from the usual standard and costandard objects $\Delta_w$ and $\nabla_w$ in $\text{Perv}_B(G/B)$; note also that their labelling as standard and costandard sheaves respectively, as opposed to the reverse, is nonstandard but follows the conventions of [ABB+05]). One of the main results in loc. cit. is an equivalence between the category of Artinian objects in $\text{Perv}(\mathcal{F}^{\mathcal{T}})^{\text{fr}}$ and a certain category of graded modules over the small quantum group. This result categorifies a known connection between the Grothendieck group of the latter category and Lusztig’s periodic Hecke module which appears in [AJS94], c.f. Theorem 17.8. The latter category admits a $W$-action given in Lemma 1.1.16 of loc. cit. As a result, there
is a natural $W$-action on simple objects in $\text{Perv}(\mathcal{F}_\ell \mathfrak{X})$ given as follows. We note that equation (55) in Section 6.1.8 of loc. cit. describes this same result in terms of “restricted irreducible” objects, but there is a typo in the indices in the equation referenced, so we reproduce a corrected version here employing instead the theory of “restricted irreducible” objects, but there is a typo in the indices in the equation (55) in Section 6.1.8 of loc. cit. describes this same result in terms of the periodic Hecke module follows the conventions of [Soe97], which differs from those of [Lus80] which we use in the present paper, hence the presence of the sign change on $\lambda$ in the results in this section. We also make reference to the $\mathcal{H}$ action on $K_0(\text{Perv}(\mathcal{F}_\ell \mathfrak{X}))$ discussed in [ABB+05] and [FFKM99]. Translating the result to our conventions, we arrive the following.

**Theorem 7.2 ([FFKM99]).** There is an isomorphism of $\mathcal{H}_q \otimes \mathbb{C}[W]$-modules

$$K_0(\mathcal{P}^{\mathfrak{X}}) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C} \to M_{d,q}$$

such that for any $\tilde{w} = w \cdot \lambda$ (for $w \in W$, $\lambda \in \Gamma$),

$$[\mathcal{I}_{\tilde{w}}] \mapsto A^\mathfrak{X}_{w^{\cdot}(-\lambda)}$$

$$[\nabla_{\tilde{w}}] \mapsto A^\mathfrak{X}_{w^{\cdot}(-\lambda)}.$$

**Definition 7.3.** Let $W_{\geq} \subset \tilde{W}$ be the set of $\tilde{w} \in \tilde{W}$ such that for any $y \in W$, $\epsilon_y(\tilde{w})$ can be written as $w \cdot \tilde{v}$ for some $w \in W$, and some coweight $\tilde{v}$ for which $\tilde{v} \leq 0$.

Let $W' \subset W_{\geq} \subset \tilde{W}$ be the set of all $\tilde{w} \in \tilde{W}$ such that there exists $w, z \in W$ with $\mathcal{I}_{\tilde{w}} = F_z(\mathcal{I}_{i(w)})$. In other words, it is the orbit of $W$ in $\tilde{W}$ under the action defined by $i(\tilde{w}) \mapsto \epsilon_w(i(\tilde{w}))$ for $w \in W$.

Finally, let $T = W_{\geq} - W'$ be the subset of $W_{\geq}$ consisting of all elements which do not lie in $W'$.

**Definition 7.4.** Let $\mathcal{P}^{\mathfrak{X}}_{\geq}$ (resp. $T$) be the Serre subcategory of $\mathcal{P}^{\mathfrak{X}}$ generated by half-integer twists of $\{\mathcal{I}_{\tilde{w}}\}_{\tilde{w} \in W_{\geq}}$ (resp. $\{\mathcal{I}_{\tilde{w}}\}_{\tilde{w} \in T}$) under extensions. Let $\mathcal{Q}$ be the Serre quotient $\mathcal{P}^{\mathfrak{X}}_{\geq}/T$.

Let $\mathcal{P}^{\mathfrak{X}}, T^{\mathfrak{X}}$, and $\mathcal{Q}^{\mathfrak{X}}$ be the non-mixed analogues, e.g. $\mathcal{P}^{\mathfrak{X}}$ is the Serre subcategory of $\text{Perv}(\mathcal{F}_\ell \mathfrak{X})$ generated by $\{\mathcal{I}_{\tilde{w}}\}_{\tilde{w} \in W'}$. 

**Proposition 7.1 (Section 6.18 in [ABB+05]).** The category $\text{Perv}(\mathcal{F}_\ell \mathfrak{X})$ admits a $W$-action by endofunctors $\{F_w\}_{w \in W}$. These functors send simple objects to simple objects, and in particular

$$F_w(\mathcal{I}_{i(\tilde{w})}) = \mathcal{I}_{w(\tilde{w})}$$

for any $s \in S$, $\tilde{w} \in \tilde{W}$ (c.f. Definition 3.1), where $i : \tilde{W} \to \tilde{W}$ denotes the involution sending $w \cdot \lambda \mapsto w \cdot (-\lambda)$ for $w \in W$, $\lambda \in \Gamma$. 

In Section 6.13 of [FFKM99], the authors explain a direct connection between perverse sheaves on the semi-infinite flag variety and Lusztig’s periodic Hecke module $M_\Gamma$. This is made precise for the category $\text{Perv}(\mathcal{F}_\ell \mathfrak{X})$ by [ABB+05], c.f. Theorem 4.3.6. In 6.1.8 of loc. cit., the authors remark that there exists a category $\mathcal{P}^{\mathfrak{X}}$ consisting of “mixed $D$-modules of Hodge-Tate type in $\text{Perv}(\mathcal{F}_\ell \mathfrak{X})$” equipped with a half-Tate twist endofunctor ($\frac{1}{2}$) corresponding to $q^{-1/2}$ generated by the simple objects $\mathcal{I}_{\tilde{w}}$ for all $\tilde{w} \in \tilde{W}$ and $m \in \mathbb{Z}$. As before, the corresponding Grothendieck group is then a $\mathbb{Z}[v,v^{-1}]$-module where $v^{-1}$ acts by $(\frac{1}{2})$.

Note that in the original result of [FFKM99] loc. cit., the setup and notation of the periodic Hecke module follows the conventions of [Soe97], which differs from those of [Lus80] which we use in the present paper, hence the presence of the sign change on $\lambda$ in the results in this section. We also make reference to the $\mathcal{H}$ action on $K_0(\text{Perv}(\mathcal{F}_\ell \mathfrak{X}))$ discussed in [ABB+05] and [FFKM99].
Our aim in the subsequent sections will be to categorify Theorem 7.2 by showing that $Q^\circ$ and $A_B$ (resp. $Q$ and $A_P$) are equivalent as categories, and then showing that the Serre quotient map $\pi : P_\leq \to P_\leq / T$ admits a fully faithful right adjoint. By composing this adjoint functor with the equivalence of categories claimed above, we will then get an equivalence of categories between $A_B$ (resp. $A_P$) and a full subcategory $\mathcal{P}$ of $\mathcal{P}^+_{\leq}$ as claimed in Theorem 1.4.

7.2. Setup. We will continue to refer to the setup and notation of $[\text{ABB}^+05]$. In loc. cit., the authors define the category $\text{Perv}(\mathcal{F}_{T^\circ})^0_{\leq}$ as the subcategory of sheaves in $\text{Perv}(\mathcal{F}_{T^\circ} \text{Bun}_{N^-})$ satisfying certain conditions. For any coweight $\check{\nu}$, they define $\mathcal{F}_{T^\circ} \text{Bun}_{N^-} \subset \mathcal{F}_{T^\circ} \text{Bun}_{N^-}$, denoting by $\check{\tau}_{\check{\nu}}$ the inclusion map. We will use this map in the case where $\check{\nu} = 0$. Similarly, they define $\mathcal{F}_{T^\circ} \text{Bun}_{N^-} \subset \mathcal{F}_{T^\circ} \text{Bun}_{N^-}$.

For any $\check{w} \in W^\circ$, it is easy to see that the strata indexed by $\check{\nu}$ on which $\text{IC}_{\check{w}}$ is supported must be such that $\check{\nu} \leq 0$. (This is the statement that $W^\circ \subset W_\leq$, which follows purely from the combinatorics of $M_d$ and the definition of the operators $\theta_w$; note that the orderings $\leq$ and $\geq$ are switched when translating from $\text{Perv}(\mathcal{F}_{T^\circ})^0_{\leq}$ to $\mathcal{M}$, as we explain in Section 7.1.)

Lemma 7.5 ([\text{ABB}^+05, Section 4.1.3]). The closure of $\check{\tau}_{\check{w}}^0 \text{Bun}_{N^-}$ in $\mathcal{F}_{T^\circ} \text{Bun}_{N^-}$ is $\check{\tau}_{\check{w}}^0 \text{Bun}_{N^-}$.

Corollary 7.6. The functor $\check{\tau}_{\check{w}}^0 : \check{P}_\leq \to \text{Perv}(\mathcal{F}_{T^\circ} \text{Bun}_{N^-})$ is exact.

Proof. By the preceding discussion, any element of $\check{P}_\leq$ is supported in $\check{\tau}_{\check{w}}^0 \text{Bun}_{N^-}$. It is shown in $[\text{ABB}^+05]$ that $\check{\tau}_{\check{w}}^0 \text{Bun}_{N^-}$ is an open subset of $\check{\tau}_{\check{w}}^0 \text{Bun}_{N^-}$, from which the result follows. \qed

Now the following result, also shown in $[\text{ABB}^+05]$, will be crucial for our purposes.

Proposition 7.7 ([\text{ABB}^+05, Section 4]). There exists an equivalence of categories $\text{Perv}_B(G/U) \to \text{Perv}(\check{\tau}_{\check{w}}^0 \text{Bun}_{N^-})$, where $\text{Perv}(\check{\tau}_{\check{w}}^0 \text{Bun}_{N^-}) \subset \text{Perv}(\mathcal{F}_{T^\circ} \text{Bun}_{N^-})$ is the subcategory satisfying the same conditions as in the definition of $\text{Perv}(\mathcal{F}_{T^\circ})^0_{\leq}$ in $\text{Perv}(\mathcal{F}_{T^\circ} \text{Bun}_{N^-})$. This equivalence sends the standard sheaves $\Delta_w$ to the sheaves $\nabla_w$ described in 4.4.3 of $[\text{ABB}^+05]$.

From now on, by means of the equivalence in this proposition, we will implicitly identify $\text{Perv}_B(G/U)$ with $\text{Perv}(\check{\tau}_{\check{w}}^0 \text{Bun}_{N^-})$, viewing $\text{Perv}_B(G/U)$ as the target (resp. source) of the functor $\check{\tau}_{\check{w}}^0$ (resp. $\check{\tau}_{\check{w}}^{-}\check{\tau}_{\check{w}}^0$).

Proposition 7.8. The functor $\check{\tau}_{\check{w}}^0$ is zero when restricted to $T$. Further, the category $\mathcal{T}$ is closed under the action of the $\{F_w\}_{w \in W}$. As a result, the functor $\check{\tau}_{\check{w}}^0$ and the functors $\{F_w\}_{w \in W}$ are each well-defined on the quotient category $Q$.

Proof. By the definition of $\mathcal{T}$, it is generated by a collection of irreducible objects $\text{IC}_{\check{w}}$, each of which having the property that $\check{w} = w \cdot \check{\nu}$ for $w \in W$ and $\check{\nu}$ some coweight with $\check{\nu} < 0$. This means $\check{\tau}_{\check{w}}^0 \text{IC}_{\check{w}} = 0$ for each such irreducible, and therefore $\check{\tau}_{\check{w}}^0(\mathcal{T}) = 0$.

Next, note that by the definition of $W'$, irreducible objects $\text{IC}_{\check{w}}$ for $\check{w} \in W_{\leq} - W'$ are sent to other such irreducible objects under the functors $\{F_w\}_{w \in W'}$: i.e. these functors send irreducible objects in $\mathcal{T}$ to other irreducible objects in $\mathcal{T}$. Since these functors are exact, this shows that $\mathcal{T}$ is closed under each of them. \qed
7.3. An equivalence of categories. Here, we recall the following result from \cite{Pol01} (specialized to the case of $A_B$); see also \cite{BBP02} for a more general statement.

**Theorem 7.9 (Pol01).** Suppose $K$ is an abelian category equipped with exact functors $j_w^*: K \to \text{Perv}_B(G/U)$, each of which has a left adjoint $j_w!$. Suppose that these functors satisfy $j_w^*j_w! = F_{w^\prime - 1}$. Assume also that if an object $A \in K$ satisfies $j_w^*A = 0$ for all $w \in W$, then $A = 0$. Then there is a natural equivalence $K \cong A_B$.

By the previous section, we know that $\overline{T}_0$ can be considered as a functor from $Q$ to $P$. Further, it is exact by Lemma 7.6. Accordingly, we make the following definition.

**Definition 7.10.** Let $j_w^*: Q \to P$ be the functor $\overline{T}_0$. Further, for any $w \in W$, let

\[
j_w^* := i_0 \circ F_w, \\
j_w! := F_{w^{-1}} \circ i_0
\]

**Proposition 7.11.** Each functor $j_w^*$ has a left adjoint given by

\[
(7.1) \quad \pi \circ F_{w^{-1}} \circ \overline{T}_{0!}
\]

where $\pi : \overline{P} \to Q$ is the Serre quotient functor.

**Proof.** First note that for any $F \in P$ and any $Y \in Q$, we have a functorial isomorphism

\[
(7.2) \quad \text{Hom}_P(F, (i_0^* \circ F_w)(Y)) \cong \text{Hom}_{P_{\leq}}((F_{w^{-1}} \circ i_0^*)(F), Y)
\]

by adjointness of $(\overline{T}_0, i_0)$. It remains to show that

\[
(7.3) \quad \text{Hom}_{P_{\leq}}((F_{w^{-1}} \circ i_0^*)(F), Y) \cong \text{Hom}_Q((F_{w^{-1}} \circ i_0^*)(F), Y).
\]

But note that $(F_{w^{-1}} \circ i_0^*)(F)$ admits no nontrivial quotient which lies in $T$, by the adjointness of $(F_{w^{-1}} \circ i_0^*, F_w \circ i_0^*)$ and by the fact that $\overline{T}_0$ is trivial on $T$. By the definition of morphisms in Serre quotient categories, this means any morphism $\overline{T} \in \text{Hom}_Q((F_{w^{-1}} \circ i_0^*)(F), Y)$ lifts to a genuine morphism $f : (F_{w^{-1}} \circ i_0^*)(F) \to Y/Y'$, where $Y'$ is some subobject of $Y$ lying in $T$. But again by the adjointness of $(F_{w^{-1}} \circ i_0^*, F_w \circ i_0^*)$ and by the fact that $\overline{T}_0$ is trivial on $T$, any such morphism lifts to a morphism from $(F_{w^{-1}} \circ i_0^*)(F)$ to $Y$, and this gives the bijection in (7.3).  

We know that the functors $j_w^*$ are exact (since the $F_w$ are exact), that $\pi \circ j_w!$ are their left adjoints, and that

\[
(7.4) \quad j_w^*j_w! = i_0^* \circ F_{w^{-1}} \circ \overline{T}_{0!}
\]

So to show that the functors $(j_w^*)_w \in W$ as defined here satisfy the conditions in Theorem 7.9 it remains to show Lemmas 7.12 and 7.13 below.

**Lemma 7.12.** When considered as endofunctors of $P$, $i_0^* \circ F_w \circ \overline{T}_{0!}$ and $F_{w^{-1}}$ are naturally isomorphic.

**Proof.** Recall that for any $F \in P$, we can write $F_{w^{-1}}(F) = F \ast \nabla_w((\overline{T}_w))$. By 4.4.1 of \cite{ABB03}, convolution on the left with objects of $\text{Perv}_B(G/U)$ commutes with the functors $\overline{T}_{0!}$ and $i_0^!$. By the definition of the $F_w$ in loc. cit., it is also clear that they too commute with convolution on the left (on the level of $K_0$, this is simply
the fact that the $W$ and $\mathcal{H}$-actions on the periodic Hecke module $M_d$ commute with one another). This means

\begin{align}
\phi_0 \circ F_w \circ \tau_0(\mathcal{F}) &= \phi_0 \circ F_w \circ \tau_0(\mathcal{F} \ast \Delta_c) \\
&= \mathcal{F} \ast (\phi_0 \circ F_w \circ \tau_0(\Delta_c)),
\end{align}

(7.6)

so it remains only to show that $\tau_0 \circ F_w \circ \tau_0(\Delta_c) \cong \nabla_w(\frac{t(w)}{2})$. Note that since the action of $F_w$ on the Grothendieck group agrees with Lusztig’s $\theta_w$ operator on $M_d$,

$$[\Delta_w(\frac{t(w)}{2}) \ast (\tau_0 \circ F_w \circ \tau_0(\Delta_c))] = [\tau_0 \circ F_w \circ \tau_0(\Delta_w(\frac{t(w)}{2}))] = [\Delta_c].$$

Since $\Delta_c$ is irreducible and the left-hand side is perverse (since $\tau_0(\Delta_w) = \nabla_w$ and $F_w$ is $t$-exact), this means $\Delta_w(\frac{t(w)}{2}) \ast (\tau_0 \circ F_w \circ \tau_0(\Delta_c)) \cong \Delta_c$, which gives a natural isomorphism $\tau_0 \circ F_w \circ \tau_0(\Delta_c) \cong \nabla_w(\frac{t(w)}{2})$, as desired. \hfill $\square$

**Lemma 7.13.** If $A \in Q$ satisfies $j_w^* A = 0$ for all $w \in W$, then $A = 0$.

**Proof.** Note that $K_0(Q) = \overline{M}^!_{d,q}$. Our discussion in Section 5.2 shows that for any nonzero $B \in \overline{M}^!_{d,q}$, there exists at least one $w \in W$ for which $j_c(\theta_w(B)) \neq 0$ which is equivalent to the desired statement under this identification. \hfill $\square$

As a result, all of the conditions for Theorem 7.9 are satisfied, yielding the following as a corollary.

**Proposition 7.14.** There is an equivalence of categories between $\mathcal{A}_B$ and $Q^\circ$ (resp. $\mathcal{A}_P$ and $Q$). This categorifies the morphism $\eta' : K_0(\mathcal{A}_P) \otimes C \to \overline{M}^!_{d,q}$.

Now we use the following result to upgrade this result and obtain a fully faithful functor from $\mathcal{A}_B$ to $D^\mathbb{C}$.

**Proposition 7.15.** The Serre quotient functor $\pi : \mathcal{P}_\leq \to Q$ admits a right adjoint $\sigma$, which is then necessarily fully faithful. Further, $\sigma$ is also left adjoint to $\pi$, and is therefore exact.

**Proof.** Using Proposition 7.14 we can let $\phi : Q \to \mathcal{A}_B$ be the equivalence of categories described therein. To complete the proof of this proposition, it is enough to show that $\phi \circ \pi$ admits a right adjoint $\sigma$ which is also its left adjoint.

First, we claim that for any $w \in W$, $j_w^* \phi \pi F_w^{-1} = \tau_0$ as functors from $\mathcal{P}_\leq^\circ \to P^\circ$.

Indeed, by the definition of $\phi$ we have $j_w^* \phi = \tau_0 \circ F_w$, and so the result follows from Proposition 7.8 which says that $F_w$ descends to a map on $Q$ and therefore commutes with $\pi$.

By the dual result to Lemma 0793 in [Sta23], it is enough to show that there exists a subset $\mathcal{J} \subseteq \text{Ob}(\mathcal{A}_B)$ such that

1. Every object of $\mathcal{A}_B$ is a subobject of an object in $\mathcal{J}$,
2. For every $J \in \mathcal{J}$ there exists a choice of object $\sigma(J)$ in $\mathcal{P}_\leq$ such that $\text{Hom}_{\mathcal{P}_\leq}(X, \sigma(J)) \cong \text{Hom}_{\mathcal{A}_B}(\phi(X), J)$ functorially for every $X$ in $\mathcal{P}_\leq$.

Let $\mathcal{J} \subseteq \text{Ob}(\mathcal{A}_B)$ be the collection of objects of the form $j_w^* (\mathcal{F})$ for $w \in W$, $\mathcal{F} \in P^\circ$.

First note that (1) holds because for any $A \in \mathcal{A}_B$, the adjunctions $(j_w^*, j_w^*)$ give a morphism

\begin{equation}
A \to \bigoplus_{w \in W} j_w^* j_w^* A
\end{equation}
which is easily seen to be an inclusion.

Now to meet condition (2), for any object of the form $j_w^!(\mathcal{F})$, we let $\sigma(j_w^!(\mathcal{F})) = F_{w^{-1}}(\tilde{\mathcal{F}}_0(\mathcal{F}))$. Then for any $X \in \tilde{\mathcal{P}}_\leq$,

$$\text{Hom}_{\mathcal{P}_\leq}(X, \sigma(j_w^!(\mathcal{F}))) = \text{Hom}_{\mathcal{P}_\leq}(X, F_{w^{-1}}(\tilde{\mathcal{F}}_0(\mathcal{F})))$$

$$= \text{Hom}_{\mathcal{P}_\leq}(F_{w}(X), \tilde{\mathcal{F}}_0(\mathcal{F}))$$

$$= \text{Hom}_{\mathcal{P}}(\tilde{\mathcal{F}}_0(F_w(X)), \mathcal{F})$$

$$= \text{Hom}_{\mathcal{P}}(j_w^*\phi\pi(X), \mathcal{F})$$

$$= \text{Hom}_{\mathcal{A}_B}((\phi \circ \pi)(X), j_w^!(\mathcal{F}))$$

with the step from the third to the fourth line using the fact that $j_w^*\phi\pi F_{w^{-1}} = \tilde{\mathcal{F}}_0$ as functors from $\mathcal{P}_\leq$ to $\mathcal{P}$, with each of the isomorphisms in the above being functorial.

This means the assignment $\sigma$ above extends to a functor $\sigma : \mathcal{Q} \to \tilde{\mathcal{P}}_\leq$ which is right-adjoint to the Serre quotient functor $\pi$. Since the quotient functor $\pi$ is a localization functor and $\sigma$ is its right adjoint, we then have that $\sigma$ is necessarily fully faithful.

Finally, we show that $\sigma$ is also left-adjoint to $\phi\pi$, and is therefore exact. By the same result as used above, we can repeat the above argument with quotients rather than subobjects, using the fact that for any $A \in \mathcal{A}_B$, there is a surjective morphism

$$\bigoplus_{w \in W} j_w^! A \to A.$$ 

Now to meet the dual version of condition (2) above and show that $\sigma$ is left-adjoint to $\pi$, it is enough to check that for any $\mathcal{F} \in \mathcal{P}$,

$$\text{Hom}_{\mathcal{P}_\leq}(\sigma(j_w^!(\mathcal{F})), Y) \cong \text{Hom}_{\mathcal{A}_B}(j_w^!(\mathcal{F}), \phi\pi Y)$$

for every $Y \in \tilde{\mathcal{P}}_\leq$. By dualizing the argument as above, this amounts to showing that $\sigma(j_w!\mathcal{F}) = \tilde{\mathcal{F}}!(\mathcal{F})$ for any such $\mathcal{F}$. By 4.4.1 of [ABB+05], the compatibility of $\tilde{\mathcal{F}}_0$ and $\sigma$ (by means of means of $\tilde{\mathcal{F}}_0^*$) with convolution with $\mathcal{F}$ means it is sufficient to show that $\sigma(j_w!(\Delta_c)) = \Delta_c$. Note that $j_w!(\Delta_c) = j_{w^*}(\nabla_w)$ in $\mathcal{A}_B$, and we know $\sigma(j_{w^*}(\nabla_w)) = F_{w^!}(\nabla_w)$. By Proposition 3.2.10 of [ABB+05], $F_{w^!}(\nabla_w) \cong \Delta_c$, as desired. \hfill \Box

**Definition 7.16.** Let $\tilde{\mathcal{P}}$ be the image of the functor $\tilde{\sigma}$, which is an exact full subcategory of $\mathcal{P}^\perp$ by Proposition 7.15.

As a corollary to Proposition 7.15 we get the following elaboration on Theorem 1.14. Note that the fact that the equivalence of categories induced by the fully faithful functor $\sigma$ described in Proposition 7.15 categorifies the isomorphism $\eta$ follows from the fact that $\sigma(j_w!(\Delta_w)) = \nabla_w$.

**Theorem 7.17.** There exists an equivalence of categories between $\mathcal{A}_B$ and $\tilde{\mathcal{P}}^\perp$ (resp. $\mathcal{A}_P$ and $\tilde{\mathcal{P}}$). This categorifies the isomorphism $\eta : K_0(\mathcal{A}_P) \otimes \mathbb{C} \to M^0$, and respects convolution on the left by the standard and costandard objects $\Delta_w$ and $\nabla_w$ of $\text{Perv}_{\mathcal{A}_B}(G/B)$ for all $w \in W$. The $W$-action on $\mathcal{A}_P$ by functors $F_w$ is mapped to the $W$-action on $\tilde{\mathcal{P}}$ by the functors $F_w$. 
References

[ABB+05] S. Arkhipov, A. Braverman, R. Bezrukavnikov, D. Gaitsgory, and I. Mirković. Modules over the small quantum group and semi-infinite flag manifold. Transform. Groups, 10(3-4):279–362, 2005.

[Ach21] Pramod N. Achar. Perverse sheaves and applications to representation theory, volume 258 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2021.

[AJS94] H. H. Andersen, J. C. Jantzen, and W. Soergel. Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p. Astérisque, (220):321, 1994.

[AP78] Morton Abramson and David Promislow. Enumeration of arrays by column rises. J. Combinatorial Theory Ser. A, 24(2):247–250, 1978.

[BBP02] Roman Bezrukavnikov, Alexander Braverman, and Leonid Positselski. Gluing of abelian categories and differential operators on the basic affine space. J. Inst. Math. Jussieu, 1(4):543–557, 2002.

[Bei87] A. A. Beilinson. On the derived category of perverse sheaves. In K-theory, arithmetic and geometry (Moscow, 1984–1986), volume 1289 of Lecture Notes in Math., pages 27–41. Springer, Berlin, 1987.

[BFGM02] A. Braverman, M. Finkelberg, D. Gaitsgory, and I. Mirković. Intersection cohomology of Drinfeld’s compactifications. Selecta Math. (N.S.), 8(3):381–418, 2002.

[BK99] Alexander Braverman and David Kazhdan. On the Schwartz space of the basic affine space. Selecta Math. (N.S.), 5(1):1–28, 1999.

[Daw21] Stefan Dawydiak. On Lusztig’s asymptotic Hecke algebra for SL_2. Proc. Amer. Math. Soc., 149(1):71–88, 2021.

[FFKM99] Boris Feigin, Michael Finkelberg, Alexander Kuznetsov, and Ivan Mirković. Semi-infinite flags. II. Local and global intersection cohomology of quasimaps’ spaces. In Differential topology, infinite-dimensional Lie algebras, and applications, volume 194 of Amer. Math. Soc. Transl. Ser. 2, pages 113–148. Amer. Math. Soc., Providence, RI, 1999.

[FM99] Michael Finkelberg and Ivan Mirković. Semi-infinite flags. I. Case of global curve P^1. In Differential topology, infinite-dimensional Lie algebras, and applications, volume 194 of Amer. Math. Soc. Transl. Ser. 2, pages 81–112. Amer. Math. Soc., Providence, RI, 1999.

[Kaz95] David Kazhdan. “Forms” of the principal series for GL_n. In Functional analysis on the eve of the 21st century, Vol. 1 (New Brunswick, NJ, 1993), volume 131 of Progr. Math., pages 153–171. Birkhäuser Boston, Boston, MA, 1995.

[KL79] David Kazhdan and George Lusztig. Representations of Coxeter groups and Hecke algebras. Invent. Math., 53(2):165–184, 1979.

[KL88] D. Kazhdan and G. Laumon. Gluing of perverse sheaves and discrete series representation. J. Geom. Phys., 5(1):63–120, 1988.

[LS06] Thierry Levasseur and J. T. Stafford. Differential operators and cohomology groups on the basic affine space. In Studies in Lie theory, volume 243 of Progr. Math., pages 377–403. Birkhäuser Boston, Boston, MA, 2006.

[Lus80] George Lusztig. Hecke algebras and Jantzen’s generic decomposition patterns. Adv. in Math., 37(2):121–164, 1980.

[Lus97] G. Lusztig. Periodic W-graphs. Represent. Theory, 1:207–279, 1997.

[Pol01] Alexander Polischchuk. Gluing of perverse sheaves on the basic affine space. Selecta Math. (N.S.), 7(1):83–147, 2001. With an appendix by R. Bezrukavnikov and the author.

[Soe97] Wolfgang Soergel. Kazhdan-Lusztig polynomials and a combinatoric for tilting modules. Represent. Theory, 1:83–114, 1997.

[Sta23] The Stacks project authors. The stacks project, 2023.

[Wil03] Geordie Williamson. Mind your P and Q-symbols: Why the Kazhdan-Lusztig basis of the Hecke algebra of Type A is cellular, 2003.
Expanding on Figure 2 but suppressing the labels, we now provide illustrations in rank 2 for $\overline{M}_d^0$ and $K_0(A_B)$. More precisely, in the figures below, the shaded alcoves are the alcoves $A \in \Xi$ for which $A^\circ \in \overline{M}_d^0$. By Theorem 5.2, the shaded alcoves index a basis for $K_0(A_B)$. The colors indicate the orbits of the $A^\circ$ in $\overline{M}_d^0$ under the operators $\{\theta_w\}_{w \in W}$, which correspond to the orbits of simple objects in $A_B$ under the operators $\{F_w\}_{w \in W}$. As in Figure 2, the alcove $A_e$ is always pictured in red, and the fundamental alcoves $\{A_w\}_{w \in W}$ are outlined with a bold stroke. (To draw these pictures, we used Proposition 3.6 which provides a description, in terms of alcoves and reflections along root hyperplanes, of the orbits of $\{A^\circ_w\}_{w \in W}$ under the operators $\{\theta_w\}_{w \in W}$.)

By Theorem 5.2, the number of shaded alcoves in each example is the number of simple objects in $A_B$. We hope that these rank 2 illustrations demonstrate the underlying alcove geometry behind the formula in Corollary 4.11.

Figure 3. Type $A_2$
Figure 4. Type $B_2$

Figure 5. Type $G_2$