QUANTUM PRINCIPAL BUNDLES AND YANG–MILLS–SCALAR–MATTER FIELDS

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Abstract. This paper aims to develop a non–commutative geometrical version of the theory of Yang–Mills and space–time scalar matter fields. To accomplish this purpose, we will dualize the geometrical formulation of this theory, in which principal $G$–bundles, principal connections, and linear representations play the most important role. In addition, we will present the non–commutative geometrical Lagrangian of the system as well as the non–commutative geometrical associated field equations. At the end of this work, we show an illustrative example.

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1. Introduction

The Standard Model is one of the most successful and important theoretical achievement in modern physics. From a philosophical/mathematical point of view, it is another example of the intrinsic relations and interplay between Physics and Differential Geometry, which in this case, is given by the geometrical framework of principal bundles, their connections and the associated structures.

Despite all of this, it presents some basic and fundamental problems that it cannot solve. For example, a consistent and coherent description of the space–time at the level of the Plank scale. The need to investigate further is evident. Non–Commutative Geometry, also known as Quantum Geometry, arises as a kind of algebraic and physical generalization of geometrical concepts. There are a variety of reasons to believe that this branch of mathematics could solve some of the Standard Model’s fundamental problems.

The purpose of this paper is to develop a non–commutative geometrical version of the theory of Yang–Mills–Scalar–Matter fields, following the line of research of M. Durdevich and also in agreement with qvbH, lrz, Z, L. To accomplish this, we are going to dualize the geometrical formulation of the classical theory, in which principal $G$–bundles, principal connections, and linear representations play the most important role. In addition, we shall present a non–commutative geometrical Lagrangian for the system as well as the non–commutative geometrical associated field equations. At the end of this work, we are going to show a concrete example. This paper continues of the theory formulated in and in this way, we shall use its notation and all the concepts developed.

The importance of this paper lies in its geometrical approach, but also in the generality of this theory, which could be applied on too many quantum principal bundle and it works for all quantum principal connection, there is no need to assume that the connection is strong

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either regular BDH, D2. Furthermore, this paper opens the door to get a geometrical formulation of the Standard Model in the framework of Non–Commutative Geometry and all that this entails: Spin Geometry, characterization of Yang–Mills connections by characteristic classes, Higgs mechanism, and Higgs bundles, Etc. as well as the possibility of researching in Standard Model’s extensions.

Our work is organized into five sections. In the second one, we are going to present the theory of the left/right quantum star Hodge operator concerning quantum Riemannian metrics, as well as the left/right quantum codifferential. Furthermore, by considering associated left/right quantum vector bundles, we will present the non–commutative geometrical version of the adjoint operators of exterior covariant derivatives. By using the theory of the second section, in the third one, we will develop the theory of Yang–Mills and space–time scalar matter fields, starting with pure Yang–Mills fields, later dealing with n–multiples of space–time scalar matter fields and concluding with space–time scalar matter fields coupled to Yang–Mills fields.

To keep a correct size of this paper, in the fourth section, we are going to present just one example by using trivial quantum principal bundles in the sense of D2 to show that the theory developed in this paper is non–trivial. Especifically, we will use a quantum principal U(1)–bundle over the space of 2 × 2 matrices with complex coefficients. The last section is about some concluding comments. In Appendix A, the reader can appreciate a little summary of complementary basic concepts. If the reader is not familiar with Sa2, we highly recommend for start the reading of this paper with Appendix A.

In Sa3 we present another example of our theory using the quantum Hopf fibration also known as the q–Dirac monopole bundle and in Sa4 we present another example using a trivial quantum principal bundle with the two–point space as the base space and the symmetric group of order 2 as the structure group.

In the whole work, we will use Sweedler’s notation and we will use the symbols ⟨−, −⟩L, ⟨−, −⟩R to denote hermitian structures, quantum Riemannian metrics and their extensions.

For the aim of this paper, to define the Lagrangian of Yang–Mills–Scalar–Matter fields in Differential Geometry, it is necessary a closed Riemannian manifold (M, g), a principal G–bundle over M, an ad–invariant inner product of the Lie algebra g of G, a unitary finite–dimensional representation α of G on Vα, and a smooth function V : ℝ → ℝ. By using these elements, we define

\[ \mathcal{L}_{YMSM}(\omega, T) := \mathcal{L}_{YM}(\omega) + \mathcal{L}_{SM}(\omega, T), \]

with

\[ \mathcal{L}_{YM}(\omega) = -\frac{1}{2} \langle R^\omega, R^\omega \rangle, \quad \mathcal{L}_{SM}(\omega, T) := \frac{1}{2} (\langle \nabla_\alpha T, \nabla_\alpha T \rangle - V(T)), \]

where \( R^\omega \) is the canonical \( gM \)–valued differential 2–form of M associated to the curvature of the principal connection \( \omega \) (by means of the Gauge Principle KMS, SW), \( T \in \Gamma(M, V^\alpha M) \) is a section of the associated vector bundle with respect to \( \alpha \), \( \nabla_\alpha^\omega \) is the induced linear connection of \( \omega \) in \( V^\alpha M \) and \( V(\Phi) := V \circ \langle T, T \rangle \). This Lagrangian is gauge–invariant and

\[ 1 \text{Now it should be clear the definition of the corresponding maps } \langle -, - \rangle. \]
the critical points of its associated action

\begin{equation}
\mathcal{A}_{\text{YMSM}}(\omega, T) = \int_M \mathcal{L}_{\text{YMSM}}(\omega, T) \, d\text{vol}_g
\end{equation}

are pairs \((\omega, T)\) that satisfy

\begin{equation}
\langle d\nabla^* \omega | R^\omega \rangle = \langle \nabla^\omega T | \alpha'(\lambda)T \rangle,
\end{equation}

for all \(g\)-valued 1–form \(\lambda\); and

\begin{equation}
(\nabla^\omega \omega^\alpha \nabla^\omega \omega^\alpha - V'(T)) T = 0,
\end{equation}

where \(\nabla^\omega \omega^\alpha\) is the formal adjoint operator of \(\nabla^\omega \omega^\alpha\) and \(d\nabla^* \omega\) is the formal adjoint operator of the exterior covariant derivative associated to \(\nabla^\omega \omega^\alpha\). These equations are called Yang–Mills–Scalar–Matter equations and they represent the dynamical of space–time scalar matter particles coupled to gauge boson particles in the Riemannian space \((M, g)\). In Subsection 3.3 we show the non–commutative geometrical version of Equations.

Other viewpoints on quantum bundles can be found in the literature, for example in [BM], [BK], [Pl]. All these formulations are intrinsically related by the theory of Hopf–Galois extensions [KT]. Moreover, there are other proposals to bring Yang–Mills theory in Non–Commutative Geometry, for example [CR], [Dj], [CCM] in which the authors directly used quantum vector bundles, and the concept of spectral triples.

We have decided to use quantum principal bundles to develop this work because we believe that a Yang–Mills–Matter theory in Non–Commutative Geometry should be approached from the respective concepts of principal bundles and representations, just like in the classical case. In addition, we have decided to use Durdevich’s formulation of quantum principal bundles because of its purely geometrical–algebraic framework, which will be evident along this work.

2. The Quantum Hodge Operator and Adjoint Operators of Quantum Linear Connections

In this section, we are going to assume that \((M, \cdot, 1, \ast)\) is a \(*\)–subalgebra equipped with a \(C^*\)–norm (in other words, its corresponding completion is a \(C^*\)–algebra).

\textbf{Definition 2.1.} Given a quantum space \((M, \cdot, 1, \ast)\) and a graded differential \(*\)–algebra \((\Omega^* (M), d, \ast)\) generated by its degree 0 elements \(\Omega^0 (M) = M\) (quantum differential forms on \(M\)), we shall say that

\begin{enumerate}
\item \(M\) is oriented if for some \(n \in \mathbb{N}\),
\[\Omega^k (M) = 0\]
for all \(k > n\) and
\[\Omega^n (M) = M \, d\text{vol},\]
where \(0 \neq d\text{vol} \in \Omega^n (M)\) satisfies
\[p \, d\text{vol} = 0 \iff p = 0.\]

The element \(d\text{vol}\) is called quantum \(n\)–volume form and if we choose one, we are going to say that \(M\) has an orientation.
\end{enumerate}
(2) A left quantum Riemannian metric (lqrm) on $M$ is a family of hermitian structures (antilinear in the second coordinate)

$$\{\langle -, - \rangle^k_L : \Omega^k(M) \times \Omega^k(M) \rightarrow M\}$$

where for $k = 0$

$$\langle -, - \rangle^0_L : M \times M \rightarrow M$$

$$(\hat{p}, p) \mapsto \hat{p}p^*$$

and such that

$$\langle \hat{\mu}p, \mu \rangle_L^k = \langle \hat{\mu}, \mu p^* \rangle_L^k$$

and

$$\langle \mu, \mu \rangle_L^k = 0 \iff \mu = 0$$

for all $\hat{\mu}, \mu \in \Omega^k(M)$, $p \in M$ and $k \geq 1$. If $M$ has an orientation $d\text{vol}$, and

$$\langle -, - \rangle^k_L : \Omega^n(M) \times \Omega^n(M) \rightarrow M$$

$$(\hat{p} \ d\text{vol}, p \ d\text{vol}) \mapsto \hat{p}p^*,$$

then we will say that $d\text{vol}$ is a left quantum Riemannian $n$–volume form (lqr $n$–form).

Now it should be clear the dual definition of right quantum Riemannian metric (rqrm) on $M$

$$\{\langle -, - \rangle^k_R : \Omega^k(M) \times \Omega^k(M) \rightarrow M\}$$

and the right quantum Riemannian $n$–volume form (rqr $n$–form)

(3) If $M$ has an orientation $d\text{vol}$ and $s$ is a state of $M$, we define a quantum integral (qi) on $M$ as

$$\int_M : \Omega^n(M) \rightarrow \mathbb{C}$$

$$p \ d\text{vol} \mapsto s(p).$$

We can interpret that a given qi satisfies the Stokes theorem by explicitly defining

$$\int_{\partial M} : \Omega^{n-1}(M) \rightarrow \mathbb{C}$$

$$\mu \mapsto \int_M d\mu.$$

If $\text{Im}(d) \subseteq \text{Ker} \left(\int_M\right)$ we are going to say that $(M, \cdot, 1, \ast)$ is a quantum space without boundary (with respect to the given qi).

Better yet, it is easy to see that

$$\text{dvol} p = \varepsilon(p) \ d\text{vol}$$

for all $p \in M$, where $\varepsilon$ is a multiplicative unital linear isomorphism and the composition $\varepsilon \circ \ast$ is an involution. Notice that if the qi is a closed graded trace, it is possible to establish a link with the cyclic cohomology $\mathbb{C}$. Furthermore, by postulating the orthogonality between quantum forms of different degrees, we can induce riemannian structures in the whole graded space $\Omega^*(M)$; so we will not use superscripts anymore.
Given a quantum space \((M, \cdot, \mathbb{1}, \ast)\) with a qi, the maps

\[
\langle - | - \rangle_L := \int_M \langle - , - \rangle_L \ d\text{vol}, \quad \langle - | - \rangle_R := \int_M \langle - , - \rangle_R \ d\text{vol}
\]

are inner products for all \(k = 0, 1, \ldots, n\), and they are called the left/right quantum Hodge inner products, respectively.

**Remark 2.2.** Given a lqrm \(\{\langle - , - \rangle_L\}\) on \(M\), we can define a rqrm on \(M\) by means of

\[
\langle \hat{\mu} , \mu \rangle_R := \langle \hat{\mu}^\ast , \mu^\ast \rangle_L
\]

and viceversa.

From this moment on, we shall work just with lqrms; however, every single result presented has a counterpart for rqrms.

In many cases, Non–Commutative Geometry is too general in the sense that we have a lot of freedom to choose the appropriate structures, which is in a clear opposition with the classical theory. So in order to develop a meaningful theory, in many concrete situations we have to impose additional restrictions in some way. The reader should not worry about this because the theory keeps being non–trivial: there are still a lot of illustrative and rich examples, as we shall appreciate in the last section of this work and in [Sa3], [Sa4].

**Remark 2.3.** From this point on, we shall assume that \(M\) has a fixed qr \(n\)–form \(d\text{vol}\), and a qi for which \(M\) does not have boundary. Furthermore, we shall assume that for a given \(\mu \in \Omega^{n-k}(M)\), the left \(M\)–module map

\[
F_\mu : \Omega^k(M) \longrightarrow M
\]

\[
\hat{\mu} \longmapsto f_\mu(\hat{\mu}),
\]

where \(\hat{\mu} \mu = F_\mu(\hat{\mu}) \ d\text{vol}\), satisfies

\[
F_\mu = \langle -, \ast_L^{-1} \mu \rangle_L
\]

for a unique element \(\ast_L^{-1} \mu \in \Omega^k(M)\). We will suppose that this identification induces an antilinear isomorphism.

**Definition 2.4.** For a given quantum space \((M, \cdot, \mathbb{1}, \ast)\), we define the left quantum Hodge star operator as

\[
\ast_L : \Omega^k(M) \longrightarrow \Omega^{n-k}(M)
\]

\[
\mu \longmapsto \ast_L \mu.
\]

By construction, for \(k = 0, \ldots, n\)

\[
\hat{\mu} \mu = \langle \hat{\mu} , \ast_L^{-1} \mu \rangle_L \ d\text{vol},
\]

with \(\hat{\mu} \in \Omega^k(M)\) and \(\mu \in \Omega^{n-k}(M)\) and \(\ast_L^{-1}\) is uniquely determined by the above equation.

The next result straightforwardly follows.

**Theorem 2.5.**

(1) For all \(\hat{\mu}, \mu \in \Omega^k(M)\) the following equality holds

\[
\hat{\mu} (\ast_L \mu) = \langle \hat{\mu}, \mu \rangle_L \ d\text{vol}.
\]
For all \( p \in M \) and \( \mu \in \Omega^{*}(M) \) we get
\[
\ast_{L}^{-1}(p \mu) = (\ast_{L}^{-1} \mu) p^*, \quad \ast_{L}^{-1}(\mu p) = \varepsilon(p)^{*} (\ast_{L} \mu),
\]
\[
\ast_{L}(\varepsilon(p)^{*} \mu) = (\ast_{L} \mu) p, \quad \ast_{L}(\mu p) = p^{*} (\ast_{L} \mu).
\]

We have
\[
\ast_{L} \text{dvol} = \text{vol}, \quad \ast_{L} \text{vol} = \text{vol}.
\]

For \( \tilde{\mu} \in \Omega^{m}(M) \), \( \hat{\mu} \in \Omega^{l}(M) \), \( \mu \in \Omega^{k}(M) \) such that \( m + l + k = n \)
\[
\langle \hat{\mu}, \ast_{L}^{-1}(\tilde{\mu} \mu) \rangle_{L} = \langle \hat{\mu}, \ast_{L}^{-1}(\tilde{\mu} \mu) \rangle_{L}.
\]

The following formula holds
\[
\langle \hat{\mu} \mid \mu \rangle_{L} = \int_{M} \hat{\mu}(\ast_{L} \mu)
\]
for all \( \hat{\mu}, \mu \in \Omega^{*}(M) \).

Our next and final step here is to present the construction of the non–commutative counterparts of the codifferential and the Laplace–de Rham operators.

\[\text{Definition 2.6.}\]

Let \((M, \cdot, \text{vol}, *)\) be a quantum space. By considering the left quantum Hodge star operator \(\ast_{L}\), we define the left quantum codifferential as the linear operator
\[
d^*_{L} := (-1)^{k+1} \ast_{L}^{-1} \circ d \circ \ast_{L} : \Omega^{k+1}(M) \to \Omega^{k}(M)
\]
\[
\mu \mapsto d^*_{L} \mu.
\]

For \( k + 1 = 0 \) we take \( d^*_{L} = 0 \).

Let \( \hat{\mu} \in \Omega^{k}(M) \), \( \mu \in \Omega^{k+1}(M) \). Then \( \ast_{L} \mu \in \Omega^{n-k-1}(M) \) and \( \hat{\mu} \ast_{L} \mu \in \Omega^{n-1}(M) \); so in the virtue of Theorem \[\text{2.5 point 1}\] and since \( M \) is a quantum space without boundary
\[
0 = \int_{M} d(\hat{\mu}(\ast_{L} \mu)) = \int_{M} (d\hat{\mu}) \ast_{L} \mu + (-1)^{k} \int_{M} \hat{\mu}(d \ast_{L} \mu)
\]
\[
= \int_{M} (d\hat{\mu}) \ast_{L} \mu - (-1)^{k+1} \int_{M} \hat{\mu}(\ast_{L} \ast_{L}^{-1} d \ast_{L} \mu)
\]
\[
= \int_{M} \langle d\hat{\mu}, \mu \rangle_{L} \text{vol} - \int_{M} \hat{\mu}(\ast_{L} d^*_{L} \mu)
\]
\[
= \int_{M} \langle d\hat{\mu}, \mu \rangle_{L} \text{vol} - \int_{M} \langle \hat{\mu}, d^*_{L} \mu \rangle_{L} \text{vol}
\]
and thus
\[
\langle d\hat{\mu} \mid \mu \rangle_{L} = \langle \hat{\mu} \mid d^*_{L} \mu \rangle_{L}.
\]

In other words, we have just proved

\[\text{Theorem 2.7.}\]

The map \( d^*_{L} \) is the adjoint operator of \( d \) concerning the left quantum Hodge inner product \( \langle - \mid - \rangle_{L} \).

Moreover, the following formulas hold

\[\text{a.2.7}\]

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\[\text{Theorem 2.7.}\]

The map \( d^*_{L} \) is the adjoint operator of \( d \) concerning the left quantum Hodge inner product \( \langle - \mid - \rangle_{L} \).

Moreover, the following formulas hold

\[d^*_{L} \circ d^*_{L} = 0,\]
Definition 2.8. Given a quantum space $(M, \cdot, \mathbb{1}, \ast)$ and the left quantum Hodge star operator $\ast_L$, the left quantum Laplace–de Rham operator is defined as
\[
\Delta_L := d \circ d^* + d^* \circ d = (d + d^*)^2 : \Omega^\bullet(M) \to \Omega^\bullet(M).
\]

Finally, it is easy to see that

Proposition 2.9. The left quantum Laplace–de Rham operator is symmetric and non–negative, i.e., $\langle \Delta_L \mu, \mu \rangle_L = \langle \mu, \Delta_L \mu \rangle_L$ and $\langle \Delta_L \mu, \mu \rangle_L \geq 0$.

Now it is possible to define left quantum harmonic differential forms, left quantum de Rham cohomology, and left quantum Hodge theory; but it is not the main focus of this work.

2.1. Adjoint Operators of Quantum Linear Connections. Let $\alpha$ be a finite–dimensional $G$–representaiton and $\zeta = (GM, M, GM\Phi)$ be a quantum principal bundle (qpb) with a quantum principal connection (qpc) $\omega$ (see Appendix A.2). We define the hermitian structure for left quantum vector bundle–valued differential forms (left qvb–valued differential forms, see Appendix A.3)
\[
\langle -, - \rangle_L : \Omega^\bullet(M) \otimes_M \Gamma^L(M, V^\alpha M) \times \Omega^\bullet(M) \otimes_M \Gamma^L(M, V^\alpha M) \to \mathbb{M}
\]
in such a way that $\langle \mu_1 \otimes_MT_1, \mu_2 \otimes_MT_2 \rangle_L = \langle \mu_1(T_1, T_2)_L, \mu_2 \rangle_L$. By using the previous definition and the qi we can define the map
\[
\langle -| - \rangle_L : \Omega^\bullet(M) \otimes_M \Gamma^L(M, V^\alpha M) \times \Omega^\bullet(M) \otimes_M \Gamma^L(M, V^\alpha M) \to \mathbb{C}
\]
by
\[
\langle \mu_1 \otimes_MT_1 | \mu_2 \otimes_MT_2 \rangle_L = \int_M \langle \mu_1(T_1, T_2)_L, \mu_2 \rangle_L \ \text{dvol}.
\]

Proposition 2.10. The map $\langle -| - \rangle_L$ is an inner product for left qvb–valued forms.

Proof. The only part of the statement that it is not trivial is the positive-definiteness; so let us proceed to prove it. Notice that it is enough to prove the statement for $\alpha \in \mathcal{T}$ and $\tau = \gamma^{-1}\alpha(\psi) = \sum_k \mu_k T_k \in \text{Mor}(\alpha, \Phi)$ and
\[
\psi = \sum_k \mu_k \otimes_MT_k = \sum_{i=1}^{d_\alpha} \mu_i^* \otimes_M \tau_i^L, \text{ where } \mu_k = \sum_{i=1}^{n_\alpha} \tau(e_i) x_{ki}^\alpha \ast \text{ (see Equation } \ref{\text{Appendix A.10})}. \text{ Hence}
\]
\[ 0 = \langle \psi, \psi \rangle_L = \sum_{i,j=1}^{d_n} \langle \mu_i^T \otimes_M T_i^L, \mu_j^T \otimes_M T_j^L \rangle_L = \sum_{i,j=1}^{d_n} \langle \mu_i^T \langle T_i^L, T_j^L \rangle_L, \mu_j^T \rangle_L \]

\[ = \sum_{i,j,k=1}^{d_n} \langle \mu_i^T x_{ik}^\alpha x_{jk}^\alpha \otimes_M \mu_j^T \rangle_L \]

\[ = \sum_{i,j,k,l=1}^{d_n} \langle \tau (e_i) x_{ik}^\alpha \otimes_M \mu_j^T \rangle_L \]

\[ = \sum_{j,k,l=1}^{d_n} \langle \tau (e_k) \delta_{lk} \otimes_M \mu_j^T \rangle_L \]

Since \((M, \cdot, 1, \ast)\) is a \ast–subalgebra of a \(C^*\)–algebra

\[ 0 \leq \langle \mu_j^T, \mu_j^T \rangle_L \leq \sum_{j=1}^{d_n} \langle \mu_j^T, \mu_j^T \rangle_L = 0 \implies \langle \mu_j^T, \mu_j^T \rangle_L = 0 \implies \mu_j^T = 0 \]

and therefore \(\psi = 0\). \(\blacksquare\)

Also we have

**4.2.14 Definition 2.11.** Considering the exterior covariant derivative associated to the induced qlc \(\nabla_\omega\), \(d^{\nabla_\omega}\) (see Appendix A.3), and the left quantum Hodge star operator \(\star_L\), we define

\[ d^{\nabla_\omega \ast_L} : \Omega^{k+1}(M) \otimes_M \Gamma^L(M, V^\alpha M) \rightarrow \Omega^k(M) \otimes_M \Gamma^L(M, V^\alpha M), \]

by

\[ d^{\nabla_\omega \ast_L} := (-1)^{k+1}((\ast_L^{-1} \circ \star) \otimes_M \text{id}_{\Gamma^L(M, V^\alpha M)}) \circ d^{\nabla_\omega} \circ ((\star \circ \ast_L) \otimes_M \text{id}_{\Gamma^L(M, V^\alpha M)}). \]

For \(k + 1 = 0\) we take \(d^{\nabla_\omega \ast_L} = 0\) and for \(k + 1 = 1\) we are going to write \(d^{\nabla_\omega \ast_L} := \nabla_\omega^\ast L\).

The following statement shows that our definition is in a total agreement with the classical theory.

**4.2.15 Theorem 2.12.** The operator \(d^{\nabla_\omega \ast_L}\) is the adjoint operator of \(d^{\nabla_\omega}\) with respect to the inner product for left qvb–valued forms for any qpc \(\omega\).

**Proof.** This proof consists of a large calculation. Let us first assume that \(\omega\) is real (see Appendix A.2). Notice that taking \(\nabla_\omega(T_2) = \sum_i \mu_i^{D^\omega(T_2)} \otimes_M T_i^L \in \Omega^1(M) \otimes_M \Gamma^L(M, V^\alpha M)\), one obtains

\[ d^{\nabla_\omega \ast_L}(\mu_2 \otimes_M T_2) = d^\ast \mu_2 \otimes_M x_2 + (-1)^{k+1} \sum_i \mu_i^{-1} \ast_L^{-1}(\mu_i^{D^\omega(T_2)} \ast (x_L \mu_2)) \otimes_M T_i^L \]
for all \( \mu_2 \in \Omega^{k+1}(M), \ T_2 \in \Gamma^L(M, V^a M) \). Now for \( \mu_1 \in \Omega^k(M) \) and \( T_1 \in \Gamma^L(M, V^a M) \) we get 
\[
\langle d\mu_1 \otimes_M x_1, \mu_2 \otimes_M T_1 \rangle_L = \langle d\mu_1(T_1, T_2)_L, \mu_2 \rangle_L \\
= \langle d(\mu_1(T_1, T_2)_L), \mu_2 \rangle_L + (-1)^{k+1} \langle \mu_1 d(T_1, T_2)_L, \mu_2 \rangle_L \\
= \langle d(\mu_1(T_1, T_2)_L), \mu_2 \rangle_L + (-1)^{k+1} \langle \mu_1 \nabla^\omega_\alpha(T_1, T_2)_L, \mu_2 \rangle_L \\
+ (-1)^{k+1} \langle \mu_1(T_1, \nabla^\omega_\alpha(T_2))_L, \mu_2 \rangle_L,
\]
since in this case, \(-,-\)_L and \(\nabla^\omega_\alpha\) are compatible (watch out with our abuse of notation!). By definition of our hermitian structures
\[
\langle \mu_1(\nabla^\omega_\alpha(T_1), T_2)_L, \mu_2 \rangle_L = \langle \mu_1 \nabla^\omega_\alpha(T_1), \mu_2 \otimes_M T_2 \rangle_L
\]
and
\[
\langle \mu_1(T_1, \nabla^\omega_\alpha(T_2))_L, \mu_2 \rangle_L = \sum_i \langle \mu_1 \otimes_M T_1, \ast_L^{-1}(\mu_i D^\omega(T_2) \ast (\ast_L \mu_2)) \otimes_M T^L_i \rangle_L.
\]
In fact
\[
\langle \mu_1(T_1, \nabla^\omega_\alpha(T_2))_L, \mu_2 \rangle_L = \sum_i \langle \mu_1(T_1, T^L_i)_L, \mu_i D^\omega(T_2) \ast (\ast_L \mu_2) \rangle_L;
\]
while by Theorem 2.5 point 4
\[
\sum_i \langle \mu_1 \otimes_M T_1, \ast_L^{-1}(\mu_i D^\omega(T_2) \ast (\ast_L \mu_2)) \otimes_M T^L_i \rangle_L = \sum_i \langle \mu_1(T_1, T^L_i)_L, \ast_L^{-1}(\mu_i D^\omega(T_2) \ast (\ast_L \mu_2)) \rangle_L =
\]
\[
\sum_i \langle \mu_1(T_1, T^L_i)_L, \mu_i D^\omega(T_2) \ast (\ast_L \mu_2)_L \rangle = \sum_i \langle \mu_1(T_1, T^L_i)_L, \mu_i D^\omega(T_2) \ast (\ast_L \mu_2) \rangle_L;
\]
thus the last assertion holds. Now taking into account these equalities and Theorem 2.7 we find 
\[
\langle d\nabla^\omega(\mu_1 \otimes_M T_1) \mid \mu_2 \otimes_M T_2 \rangle_L = \langle d\mu_1 \otimes_M T_1 \mid \mu_2 \otimes_M T_2 \rangle_L \\
+ (-1)^k \langle \mu_1 \nabla^\omega_\alpha(T_1), \mu_2 \otimes_M T_2 \rangle_L \\
+ \int_M \langle d(\mu_1(T_1, T_2)_L), \mu_2 \rangle_L \, dvol \\
+ (-1)^{k+1} \int_M \langle \mu_1(\nabla^\omega_\alpha(T_1), T_2)_L, \mu_2 \rangle_L \, dvol \\
+ (-1)^{k+1} \int_M \langle \mu_1(T_1, \nabla^\omega_\alpha(T_2))_L, \mu_2 \rangle_L \, dvol \\
+ (-1)^k \int_M \langle \mu_1 \nabla^\omega_\alpha(T_1), \mu_2 \otimes_M T_1 \rangle_L \, dvol \\
= \int_M \langle \mu_1(T_1, T_2)_L, d^{\ast_L} \mu_2 \rangle_L \, dvol \\
+ (-1)^{k+1} \int_M \langle \mu_1(T_1, \nabla^\omega_\alpha(T_2))_L, \mu_2 \rangle_L \, dvol \\
= \langle \mu_1 \otimes_M x_1 \mid d\nabla^\omega_L(\mu_2 \otimes_M T_2) \rangle_L
\]
and the statement in this case follows from linearity.

Since every real quantum connection displacement $\lambda$ (see Equations [4.2.2] and [4.2.3]) can written as $\lambda = \omega - \omega'$, where $\omega$, $\omega'$ are real qpcs, we have that the operator $\gamma_\alpha \circ K^\lambda \circ \gamma_\alpha^{-1}$ is adjointable, where

\[
4.f.2.24 \quad K^\lambda(\tau) := (D^{\omega} - D^{\omega'})(\tau) = -(-1)^k \tau^{(0)}(\pi(\tau^{(1)}))
\]

with $H^2(\pi(v)) = \tau^{(0)}(v) \otimes \tau^{(1)}(v)$ and $\text{Im}(\tau) \in \text{Hor}^kGM$. This implies that $\gamma_\alpha \circ i K^\lambda \circ \gamma_\alpha^{-1}$ is also adjointable. By Equation [4.2.4] we get that $D^{\omega} = D^{\omega'} + i K^\lambda$ for every qpc $\omega$ and the theorem follows.

Of course, there is a natural generalization of the left quantum Laplace–de Rham operator for left qvb–valued forms by means of

\[
4.f.2.25 \quad \Box^\omega_L := d^\nabla^\omega \circ d^\nabla^\omega * L + d^\nabla^\omega * L \circ d^\nabla^\omega.
\]

This operator satisfies

\[
\langle \Box^\omega_L \hat{\psi} \mid \psi \rangle_L = \langle \hat{\psi} \mid \Box^\omega_L \psi \rangle_L \quad \text{and} \quad \langle \Box^\omega_L \psi \mid \psi \rangle_L \geq 0
\]

for all $\hat{\psi}, \psi \in \Omega^*(M) \otimes^M \Gamma^L(M, V^\alpha M)$.

\[
4.f.2.26 \quad \langle -, - \rangle_R : \Gamma^R(M, V^\alpha M) \otimes^M \Omega^*(M) \times \Gamma^R(M, V^\alpha M) \otimes^M \Omega^*(M) \to M
\]

is given by $\langle T_1 \otimes^M \mu_1, T_2 \otimes^M \mu_2 \rangle_R = \langle \mu_1, \langle T_1, T_2 \rangle_R \mu_2 \rangle_R$ and the inner product

\[
4.f.2.27 \quad \langle -| - \rangle_R : \Gamma^R(M, V^\alpha M) \otimes^M \Omega^*(M) \times \Gamma^R(M, V^\alpha M) \otimes^M \Omega^*(M) \to \mathbb{C}
\]

is defined by

\[
\langle T_1 \otimes^M \mu_1 \mid T_2 \otimes^M \mu_2 \rangle_R = \int_M \langle \mu_1, \langle T_1, T_2 \rangle_R \mu_2 \rangle_R \text{dvol}.
\]

In the context of Remark 2.13 the right quantum Hodge star operator and the right quantum codifferential are given by

\[
4.f.2.28 \quad \ast_R = \ast \circ \ast_L \circ \ast, \quad d^R = (-1)^{k+1} \ast_{\ast_R}^{-1} \circ d \circ \ast_R = \ast \circ d^* \circ \ast;
\]

while the adjoint operator of the exterior covariant derivative of $\nabla^\omega_L$ is

\[
4.f.2.29 \quad d^\nabla^\omega_{\ast R} := (-1)^{k+1} (\text{id}_{\Gamma^R(M, V^\alpha M) \otimes^M \ast_R^{-1} \ast_R}) \circ d^\nabla^\omega \circ (\text{id}_{\Gamma^R(M, V^\alpha M) \otimes^M \ast R} \circ \ast R).
\]

For $k + 1 = 1$ we are going to write $d^\nabla^\omega_{\ast R} := \nabla^\omega_{\ast R}$. For the right structure we will use these relations.

3. Yang–Mills–Scalar–Matter Fields in Noncommutative Geometry

By using the theory developed in the previous section, we can accomplish our aim: the non–commutative geometrical version of the classical theory of Yang–Mills–Scalar–Matter fields. Examples will be presented in the next section.
3.1. Non–Commutative Geometrical Yang–Mills Fields. The next definition closely follows the classical formulation.

6.1.1 Definition 3.1 (Non–commutative geometrical Yang–Mills model). In Non–Commutative Geometry a Yang–Mills model (ncg YM model) consists of

1. A quantum space \((M, \cdot, \mathbb{1}, *)\) such that it is a \(*\)–algebra completable into \(C^*\)–algebra.
2. A quantum \(G\)–bundle over \(M\), \(\zeta = (GM, M, GM)\), with a differential Hodge star operator exists for the space of base forms.
3. The operators \(d^{\mathfrak{g}} := \mathbb{Y}_{\text{ad}} \circ S^* \circ \mathbb{Y}_1^{-1}\) and \(d^{\mathfrak{g}2} := \mathbb{Y}_{\text{ad}} \circ \hat{S}^* \circ \mathbb{Y}_1^{-1}\) are assumed to be adjointable for any \(\omega\) with respect to the inner products of \(\mathfrak{g}\)–valued forms, where \(\hat{S}^* = * \circ S^* \circ *\) (see Equation \ref{eq:adj}.

The first two points are necessary to guarantee Theorem \ref{thm:3.1}. Comments about the last point will be presented in the final section.

6.1.2 Definition 3.2 (Noncommutative Yang–Mills Lagrangian and its action). Given a ncg YM model, we define the non–commutative geometrical Yang–Mills Lagrangian (ncg YM Lagrangian) as the association

\[
\mathcal{L}_{\text{YM}} : \mathfrak{qGc}(\zeta) \rightarrow M
\]

\[
\omega \mapsto -\frac{1}{4} \left( \langle R^\omega, R^\omega \rangle_L + \langle \hat{R}^\omega, \hat{R}^\omega \rangle_R \right),
\]

where \(\langle R^\omega, R^\omega \rangle_L := \langle \mathbb{Y}_{\text{ad}} \circ R^\omega, \mathbb{Y}_{\text{ad}} \circ R^\omega \rangle_L\) and \(\langle \hat{R}^\omega, \hat{R}^\omega \rangle_R := \langle \hat{\mathbb{Y}}_{\text{ad}} \circ \hat{R}^\omega, \hat{\mathbb{Y}}_{\text{ad}} \circ \hat{R}^\omega \rangle_R\) (see Appendix A.2). We define its associated action as

\[
\mathcal{S}_{\text{YM}} : \mathfrak{qGc}(\zeta) \rightarrow \mathbb{R}
\]

\[
\omega \mapsto \int_M \mathcal{L}_{\text{YM}}(\omega) \, d\text{vol} = -\frac{1}{4} \left( \langle R^\omega | R^\omega \rangle_L + \langle \hat{R}^\omega | \hat{R}^\omega \rangle_R \right)
\]

and we shall call it the non–commutative geometrical Yang–Mills action (ncg YM action).

Let us consider the quantum gauge group (qgg) \(\mathfrak{qGg}\). If \(\mathcal{F}_f\) is a graded differential \(*\)–algebra morphism, then \(R^{S\omega} = \mathcal{F}_f \circ R^\omega\), and since the maps \(\mathcal{A}_f\) and \(\hat{\mathcal{A}}_f\) are unitary, a direct calculation shows that \(\mathcal{L}_{\text{YM}}(\omega) = \mathcal{L}_{\text{YM}}(f^{S\omega})\) for all \(\omega \in \mathfrak{qGc}(\zeta)\). It is important to observe that in general such relation does not hold for an arbitrary \(f \in \mathfrak{qGg}\).

6.1.3 Definition 3.3. We define the quantum gauge group of the Yang–Mills model as the group

\[\mathfrak{qGg}_{\text{YM}} := \{ f \in \mathfrak{qGg} | \mathcal{L}_{\text{YM}}(\omega) = \mathcal{L}_{\text{YM}}(f^{S\omega}) \text{ for all } \omega \in \mathfrak{qGc}(\zeta) \}\]

It is worth mentioning that every qgt induced by a graded differential \(*\)–algebra morphism \(\mathfrak{F}_f\) is an element of \(\mathfrak{qGg}_{\text{YM}}\), so \(\mathfrak{qGg}_{\text{YM}}\) always has at least one element.

Our next step is getting field equations for \(\omega \in \mathfrak{qGc}(\zeta)\) by postulating a variational principle for the ncg YM action, in total agreement with the classical case.

6.1.4 Definition 3.4 (Yang–Mills quantum principal connections). A stationary point of \(\mathcal{S}_{\text{YM}}\) is an element \(\omega \in \mathfrak{qGc}(\zeta)\) such that for any \(\lambda \in \mathfrak{qGc}(\zeta)\) (see Equation \ref{eq:stationary}),

\[
\left. \frac{\partial}{\partial z} \right|_{z=0} \mathcal{S}_{\text{YM}}(\omega + z \lambda) = 0.
\]
Stationary points are also called Yang–Mills qpcs (YM qpcs or non–commutative geometrical Yang–Mills fields). In terms of a physical interpretation, they should be considered as gauge boson fields without sources and possessing the symmetry $q\mathcal{G}_\text{YM}$.

Now we will proceed to find YM qpcs.

**Theorem 3.5.** A qpc $\omega$ is a YM qpc if and only if

\[
(3.1) \quad \langle \tilde{\Upsilon}\circ \lambda \mid (d\tilde{\nabla}_\text{ad}^\ast \sigma_L - d\tilde{S}^\ast \lambda_R) R^\omega \rangle_L + \langle \tilde{\Upsilon}\circ \lambda \mid (d\tilde{\nabla}_\text{ad}^\ast \sigma_R - d\tilde{S}^\ast \lambda_R) \tilde{R}^\omega \rangle_R = 0
\]

for all $\lambda \in \text{qpc}(\zeta)$, where $(d\tilde{\nabla}_\text{ad}^\ast \sigma_L - d\tilde{S}^\ast \lambda_R) R^\omega := (d\tilde{\nabla}_\text{ad}^\ast \sigma_L - d\tilde{S}^\ast \lambda_L) \circ \Upsilon_{\text{ad}} \circ R^\omega$, $(d\tilde{\nabla}_\text{ad}^\ast \sigma_R - d\tilde{S}^\ast \lambda_R) \tilde{R}^\omega := (d\tilde{\nabla}_\text{ad}^\ast \sigma_R - d\tilde{S}^\ast \lambda_R) \circ \Upsilon_{\text{ad}} \circ \tilde{R}^\omega$ and $d\tilde{S}^\ast \lambda_L$, $d\tilde{S}^\ast \lambda_R$ are the adjoint operators of $d\tilde{S}^\ast$, $d\tilde{S}^\ast$ respectively.

**Proof.** For a given $\lambda \in \text{qpc}(\zeta)$ we have

\[
\frac{\partial}{\partial z} \mid_{z=0} \langle R^{\omega + z \lambda} \mid R^{\omega + z \lambda} \rangle_L = \langle \Upsilon_{\text{ad}} \circ (d \circ \lambda - \langle \omega, \lambda \rangle - \langle \lambda, \omega \rangle) \mid R^\omega \rangle_L
\]

\[
= \langle \Upsilon_{\text{ad}} \circ (d \circ \lambda + [\lambda, \omega] - S^\omega \circ \lambda) \mid R^\omega \rangle_L
\]

\[
= \langle \Upsilon_{\text{ad}} \circ (D^\omega - S^\lambda) \circ \lambda \mid R^\omega \rangle_L
\]

\[
= \langle (d\tilde{\nabla}_\text{ad} - d\tilde{S}^\ast) \circ \Upsilon_{\text{ad}} \circ \lambda \mid R^\omega \rangle_L
\]

\[
= \langle \Upsilon_{\text{ad}} \circ \lambda \mid (d\tilde{\nabla}_\text{ad}^\ast \sigma_L - d\tilde{S}^\ast \lambda_L) \circ R^\omega \rangle_L.
\]

In the same way we get

\[
\frac{\partial}{\partial z} \mid_{z=0} \langle \tilde{R}^{\omega + z \lambda} \mid \tilde{R}^{\omega + z \lambda} \rangle_R = \langle \tilde{\Upsilon}_{\text{ad}} \circ \tilde{\lambda} \mid (d\tilde{\nabla}_\text{ad}^\ast \sigma_R - d\tilde{S}^\ast \lambda_R) \tilde{R}^\omega \rangle_R
\]

and the theorem follows. ■

We will refer to Equation 6.f1.1 as the non–commutative geometrical Yang–Mills field equation. It is worth mentioning that every flat qpc is a YM qpc since it satisfies trivially the equations. Of course, $q\mathcal{G}_\text{YM}$ acts on the space of YM qpcs.

**3.2. Non–Commutative Geometrical Multiples of Space–Time Scalar Matter Fields.** Like in the classical case, we shall introduce the necessary technical elements.

**Definition 3.6** (Non–commutative geometrical n–multiples for space–time scalar matter models). In Non–Commutative Geometry a n–multiples for a given space–time scalar matter model (ncg n–sm model) consists of

1. A quantum space $(M, \cdot, \mathbb{1}, *)$ closeable into a $C^*$–algebra.
2. A quantum $\mathcal{G}$–bundle over $M$, $\zeta = (GM, M_{GM}\Phi)$, with a differential calculus such that the left quantum Hodge star operator exists for the space of base forms.
3. The trivial $\mathcal{G}$–representation in $\mathbb{C}^n$.
4. A Fréchet differentiable $V : M \rightarrow M$ called the potential.

For the rest of this subsection, we shall consider $\alpha := \alpha_{\text{triv}}^{triv}$. It is worth mentioning that in this case the induced qpc $\nabla_\alpha$, $\tilde{\nabla}_\alpha$ do not depend on $\omega$ (where $\tilde{\alpha}$ is the complex conjugate representation of $\alpha$), they take the same values for every qpc; of course, this is because the representation is trivial.
**Definition 3.7** (Non–commutative geometrical $n$–space–time scalar matter Lagrangian and its action). Given a ncg $n$–sm model, we define its non–commutative geometrical Lagrangian as the association
\[ \mathcal{L}_{SM} : \Gamma^L(M, C^n M) \times \Gamma^R(M, \overline{C}^n M) \rightarrow M \]
given by
\[ \mathcal{L}_{SM}(T_1, T_2) = \frac{1}{4} \left( \langle \nabla^\omega_\alpha T_1, \nabla^\omega_\alpha T_1 \rangle_L - V_L(T_1) - \langle \nabla^\omega_\pi T_2, \nabla^\omega_\pi T_2 \rangle_R + V_R(T_2) \right) \]
where $V_L(T_1) := V \circ \langle T_1, T_1 \rangle_L$ and $V_R(T_2) := V \circ \langle T_2, T_2 \rangle_R$. We define its associated action as
\[ \mathcal{S}_{SM} : \Gamma^L(M, C^n M) \times \Gamma^R(M, \overline{C}^n M) \rightarrow \mathbb{C} \]
\[ (T_1, T_2) \mapsto \int_M \mathcal{L}_{SM}(T_1, T_2) \, d\text{vol}. \]

A direct calculation shows that
\[ \langle \nabla^\omega_\alpha T_1, \nabla^\omega_\alpha T_1 \rangle_L - V_L(T_1) = \sum_{i=1}^{n} \langle dp^T_i, dp^T_i \rangle_L - V(p^T_i(p^T_i)^*) \]
\[ \langle \nabla^\omega_\pi T_2, \nabla^\omega_\pi T_2 \rangle_R - V_R(T_2) = \sum_{i=1}^{n} \langle dp^T_2, dp^T_2 \rangle_L - V((p^T_2)^*p^T_2) \]

where $p^T_i = T_1(e_i), p^T_2 = T_2(\overline{\pi}) \in M$ and \{\$e_i$\}$_{i=1}^{n}$ is the canonical basis of $\mathbb{C}^n$. Since $\text{Im}(T) \subseteq M$ for all $T \in \text{Mor}(\alpha,_{GM} \Phi)$ and all $T \in \text{Mor}(\overline{\pi},_{GM} \Phi)$, taking any $f \in \mathfrak{g} \mathfrak{e} \mathfrak{g}$ we get $\tilde{f} \circ T = T$; so

**6.1.8 Proposition 3.8.** The Lagrangian $\mathcal{L}_{SM}$ is quantum gauge–invariant.

Like in the previous section, our next step is getting field equations postulating a variational principle for $\mathcal{S}_{SM}$, in agreement with the classical case.

**6.1.9 Definition 3.9** (Non–commutative geometrical $n$–multiples of scalar matter fields). A stationary point of $\mathcal{S}_{SM}$ is an element $(T_1, T_2) \in \Gamma^L(M, C^n M) \times \Gamma^R(M, \overline{C}^n M)$ such that for all $(U_1, U_2) \in \Gamma^L(M, C^n M) \times \Gamma^R(M, \overline{C}^n M)$

\[ \frac{\partial}{\partial z} \bigg|_{z=0} \mathcal{S}_{SM}(T_1 + z U_1, T_2 + z U_2) = 0. \]

In terms of a physical interpretation, stationary points should be considered as space–time scalar matter and antimatter fields.

As before, we will proceed to find stationary points.

**6.1.10 Theorem 3.10.** Assume that $(T_1, T_2) \in \Gamma^L(M, C^n M) \times \Gamma^R(M, \overline{C}^n M)$ satisfies
\[ \frac{\partial}{\partial z} \bigg|_{z=0} \int_M V_L(T_1 + z U_1) \, d\text{vol} = \int_M \frac{\partial}{\partial z} \bigg|_{z=0} V_L(T_1 + z U_1) \, d\text{vol}, \]
and
\[ \langle V_L'(T_1) U_1 \rangle_L = \langle U_1 | V_L'(T_1)^* T_1 \rangle_L \]
for all \((U_1, U_2) \in \Gamma^L(M, \mathbb{C}^n M) \times \Gamma^R(M, \overline{\mathbb{C}}^n M)\), where \(V'_L(T_1) := V' \circ \langle T_1, T_1 \rangle_L\) (and analogous assumptions for \(V'_R(T_2) := V' \circ \langle T_2, T_2 \rangle_R\)) with \(V'\) the derivative of \(V\). Then \((T_1, T_2)\) is a stationary point if and only if

\[
(3.2) \quad \nabla^\omega \cdot (\nabla^\omega T_1) - V'_L(T_1) * T_1 = 0 \quad \text{and} \quad \tilde{\nabla}^\omega \cdot (\tilde{\nabla}^\omega T_2) - T_2 V'_R(T_2) * U_2 = 0.
\]

**Proof.** For a given \((U_1, U_2) \in \Gamma^L(M, \mathbb{C}^n M) \times \Gamma^R(M, \overline{\mathbb{C}}^n M)\) we have

\[
\left. \frac{\partial}{\partial z} \right|_{z=0} \mathcal{S}_{SM}(T_1 + z U_1, T_2 + z U_2) = \frac{1}{4} \left( \langle U_1 | \nabla^\omega \cdot (\nabla^\omega T_1) - V'_L(T_1) * T_1 \rangle_L - \langle \tilde{\nabla}^\omega \cdot (\tilde{\nabla}^\omega T_2) - T_2 V'_R(T_2) * U_2 \rangle_R \right).
\]

According to Proposition 4.2.16 we get \(\left. \frac{\partial}{\partial z} \right|_{z=0} \mathcal{S}_{SM}(T_1 + z U_1, T_2 + z U_2) = 0\) for all \((U_1, U_2) \in \Gamma^L(M, \mathbb{C}^n M) \times \Gamma^R(M, \overline{\mathbb{C}}^n M)\) if and only if Equation 6.f1.2 holds. \(\blacksquare\)

Equation 6.f1.2 turns into

\[
(3.3) \quad \sum_{k=1}^n d^* d^T p_i^T - V'(p_i^T(p_i^T)^*)^* p_i^T = 0, \quad \sum_{k=1}^n d^* d(p_i^T)^* - V'((p_i^T)^* p_i^T)(p_i^T)^* = 0
\]

for all \(i = 1, \ldots, n\). Of course explicit solutions of the last equation depend completely on the form of \(V\) and the differential structure on the quantum base space; the quantum total space, the quantum group and their differential structures do not intervene explicitly.

### 3.3. Non–Commutative Geometrical Yang–Mills–Scalar–Matter Fields

**Definition 3.11** (Non–commutative geometrical Yang–Mills–Scalar–Matter model). In Non–Commutative Geometry a Yang–Mills–Scalar–Matter model (ncg YMSM model) will consist of

1. A quantum space \((M, \cdot, 1, \ast)\) such that it is a \(*\)–subalgebra of a \(C^\ast\)–algebra.
2. A quantum \(G\)–bundle over \(M\), \(\zeta = (GM, M, GM \Phi)\), with a differential calculus such that the left quantum Hodge star operator exists for the space of base forms.
3. The operators \(dS^\omega := Y_{ad} \circ S^\omega \circ Y_{ad}^{-1}\) and \(d\tilde{S}^\omega := \tilde{Y}_{ad} \circ \tilde{S}^\omega \circ \tilde{Y}_{ad}^{-1}\) are assumed to be adjointable for any \(\omega\) with respect to the inner products of qvb–valued forms, where \(\tilde{S}^\omega = \ast \circ S^\omega \circ \ast\).
4. A \(G\)–representation \(\alpha\) in a finite–dimensional \(\mathbb{C}\)–vector space \(V^\omega\).
5. A Fréchet differentiable map \(V : M \rightarrow M\) called the potential.

These conditions establish similar frameworks as the ones discuss in the previous subsections. Taking into account that the complex conjugate representation \(\overline{\alpha}\) of \(\alpha\) acts on \(\nabla\) we have

**Definition 3.12** (Non–commutative geometrical Yang–Mills–Scalar–Matter Lagrangian and its action). Given a ncg YMSM model, we define the non–commutative geometrical Yang–Mills–Scalar–Matter Lagrangian (ncg YMSM Lagrangian) as the association

\[
\mathcal{L}_{YMSM} : qpc(\zeta) \times \Gamma^L(M, V^\omega M) \times \Gamma^R(M, \overline{V}^\omega M) \rightarrow M
\]
given by
\[ \mathcal{L}_{YMSM}(\omega, T_1, T_2) = \mathcal{L}_Y(\omega) + \mathcal{L}_{GSM}(\omega, T_1, T_2), \]
where \( \mathcal{L}_Y \) is the ncg YM Lagrangian and \( \mathcal{L}_{GSM} \) is the non–commutative geometrical generalized space–time scalar matter Lagrangian (ncg GSM Lagrangian) which is given by (comparing with Definition 3.2)
\[ \mathcal{L}_{GSM}(\omega, T_1, T_2) = \frac{1}{4} \left( \langle \nabla^\omega T_1, \nabla^\omega T_1 \rangle_L - \langle \nabla^\omega T_2, \nabla^\omega T_2 \rangle_R + V_R(T_2) \right) \]
where \( V_L(T_1) := V \circ \langle T_1, T_1 \rangle_L \) and \( V_R(T_2) := V \circ \langle T_2, T_2 \rangle_R \). We define its associated action as
\[ \mathcal{S}_{YMSM} : qpc(\zeta) \times \Gamma^L(M, V^\alpha M) \times \Gamma^R(M, V^\alpha M) \to \mathbb{C} \]
\[ (\omega, T_1, T_2) \mapsto \int_M \mathcal{L}_{YMSM}(\omega, T_1, T_2) \text{dvol} \]
and we shall call it non–commutative geometrical Yang–Mills–Scalar–Matter action (ncg YMSM action).

According to \cite{Sa1,sam}, if \( \mathfrak{f}_i \) is a graded differential \( * \)–algebra morphism, then \( R^{f^*} \omega = \mathfrak{f}_i \circ R^\omega \), and since the maps \( A_i \) and \( \hat{A}_i \) are unitary, a direct calculation shows that \( \mathcal{L}_{YMSM}(\omega, T_1, T_2) = \mathcal{L}(f^* \omega, A_i(T_1), \hat{A}_i(T_2)) \) for all \( \omega \in qpc(\zeta) \) and all \( T_1 \in \Gamma^L(M, V^\alpha M) \) and \( T_2 \in \Gamma^R(M, V^\alpha M) \). In agreement with the previous observations, in general it will be not true that any \( f \in q\mathfrak{g}\mathfrak{s}\mathfrak{g} \) is a Lagrangian symmetry.

**Definition 3.13.** We define the quantum gauge group of the Yang–Mills–Scalar–Matter model as the group \( q\mathfrak{g}\mathfrak{s}\mathfrak{g}_{YMSM} := \{ f \in q\mathfrak{g}\mathfrak{s}\mathfrak{g} \mid \mathcal{L}_{YMSM}(\omega, T_1, T_2) = \mathcal{L}(f^* \omega, A_i(T_1), \hat{A}_i(T_2)) \} \subseteq q\mathfrak{g}\mathfrak{s}\mathfrak{g} \).

It is worth mentioning that every qgt induced by a graded differential \( * \)–algebra morphism \( \mathfrak{f}_i \) is an element of \( q\mathfrak{g}\mathfrak{s}\mathfrak{g}_{YMSM} \), so \( q\mathfrak{g}\mathfrak{s}\mathfrak{g}_{YMSM} \) always has at least one element. Of course, the group \( q\mathfrak{g}\mathfrak{s}\mathfrak{g}_{YMSM} \) depends on the potential \( V \).

Like in previous subsections, the next step is getting the non–commutative geometrical field equations for \( (\omega, T_1, T_2) \in qpc(\zeta) \times \Gamma^L(M, V^\alpha M) \times \Gamma^R(M, V^\alpha M) \) by postulating a variational principle for \( \mathcal{S}_{YMSM} \). All of this in total agreement with the classical case.

**Definition 3.14** (Non–commutative geometrical Yang–Mills–Scalar–Matter field). A stationary point of \( \mathcal{S}_{YMSM} \) is a triplet \( (\omega, T_1, T_2) \in qpc(\zeta) \times \Gamma^L(M, V^\alpha M) \times \Gamma^R(M, V^\alpha M) \) such that for any \( (\lambda, U_1, U_2) \in qpc(\zeta) \times \Gamma^L(M, V^\alpha M) \times \Gamma^R(M, V^\alpha M) \)
\[ \frac{\partial}{\partial z} \mathcal{S}_{YMSM}(\omega + z \lambda, T_1, T_2) \bigg|_{z=0} = \frac{\partial}{\partial z} \mathcal{S}_{YMSM}(\omega, T_1 + z U_1, T_2 + z U_2) \bigg|_{z=0} = 0. \]

Stationary points are also called (non–commutative geometrical) Yang–Mills–Scalar–Matter fields (YMSM fields) and in terms of a physical interpretation, they can be interpreted as scalar matter and antimatter fields coupled to gauge boson fields with symmetry \( q\mathfrak{g}\mathfrak{s}\mathfrak{g}_{YMSM} \).

Now we are going to find the equations of motion.
Theorem 3.15. Assume that \((T_1, T_2) \in \Gamma^L(M, V^\alpha M) \times \Gamma^R(M, V^\alpha M)\) satisfies
\[
\frac{\partial}{\partial z} \bigg|_{z=0} \int_M V_L(T_1 + z U_1) \, dvol = \int_M \frac{\partial}{\partial z} \bigg|_{z=0} V_L(T_1 + z U_1) \, dvol,
\]
and
\[
\langle V'_L(T_1) U_1 | T_1 \rangle_L = \langle U_1 | V'_L(T_1)^* T_1 \rangle_L
\]
for all \((U_1, U_2) \in \Gamma^L(M, V^\alpha M) \times \Gamma^R(M, V^\alpha M)\), where \(V'_L(T_1) := V' \circ \langle T_1, T_1 \rangle_L\) (and analogous assumptions for \(V'_R(T_2) := V' \circ \langle T_2, T_2 \rangle_R\) with \(V'\) the derivative of \(V\). Then \((\omega, T_1, T_2) \in \text{qpc}(\zeta) \times \Gamma^L(M, V^\alpha M) \times \Gamma^R(M, V^\alpha M)\) is a YMSM field if and only if for all \(\lambda \in \text{qpc}(\zeta)\)
\[
(3.4)
\]
\[
\langle \gamma_\lambda \circ K^\lambda(T_1) | \nabla^\omega T_1 \rangle_L - \langle \tilde{\gamma}_\lambda \circ \tilde{K}^\lambda(T_2) | \nabla^\omega T_2 \rangle_R = 0.
\]
and
\[
(3.5)
\nabla^\omega (\nabla^\omega T_1 - V'_L(T_1)^* T_1) = 0, \quad \nabla^\omega (\nabla^\omega T_2) - T_2 V'_R(T_2)^* = 0.
\]

Proof. For a given \(\lambda \in \text{qpc}(\zeta)\) notice that
\[
\frac{\partial}{\partial z} \bigg|_{z=0} \mathcal{S}_\text{GSM}(\omega + z \lambda, T_1, T_2) = \frac{1}{4} \left( \langle \gamma_\lambda \circ K^\lambda(T_1) | \nabla^\omega T_1 \rangle_L - \langle \tilde{\gamma}_\lambda \circ \tilde{K}^\lambda(T_2) | \nabla^\omega T_2 \rangle_R \right),
\]
thus
\[
\frac{\partial}{\partial z} \bigg|_{z=0} \mathcal{S}_\text{YMSM}(\omega + z \lambda, T_1, T_2) = 0 \text{ if and only if Equation (3.4) holds.}
\]

We shall refer to Equations (3.4) and (3.5) as (the non-commutative geometrical) Yang–Mills–Scalar–Matter field equations (YMSM field equations). The reader is invited to compare these equations with their classical counterparts (see Equations (1.1)–(1.3)).

It is worth mentioning that in all cases, the variation of the action with respect to \(z^*\) produces the same field equations.

4. Example: Trivial Quantum Principal Bundles and Matrices

In the previous section, we had to impose some conditions to develop the theory. At the first instance, these conditions seem too restrictive, so it is necessary to present some examples to show that our theory is non-trivial and there are interesting spaces to study. It is worth mentioning that for the trivial corepresentation on \(\mathbb{C}^n\), the first part of Equation (3.1) equals to zero; thus the only way to satisfy Equation (3.4) is when \(\omega\) is a YM qpc. Moreover, Equation (3.5) reduces to Equation (3.3). In summary, for the trivial quantum representation on \(\mathbb{C}^n\) in any qpb, YMSM fields are triplets \((\omega, T_1, T_2)\) where \(\omega\) is a YM qpc and \((T_1, T_2)\) is a critical point of \(\mathcal{S}_\text{SM}\).

Now let us take a particular and illustrative trivial qpb (in the sense of \([\text{sald}1]\) and \([\text{sald}2]\)) by using \(U(1)\) as a structure group. In an abuse of notation, we will identify \(U(1)\) with the Laurent polynomial algebra. In \([\text{sald}3]\) the reader can check the results of this paper in the.
quantum Hopf fibration and in [Sa4] the reader can appreciated another example by using another trivial qpb.

For this example, the $*$-FODC on $U(1)$ will be given by the right ideal $\text{Ker}^2(\epsilon)$ and hence the universal differential envelope $*$-calculus $(\Gamma^\wedge, d, *)$ is the algebra of differential forms of $U(1)$. A Hamel basis of

$$\text{inv}\Gamma := \frac{\text{Ker}(\epsilon)}{\text{Ker}^2(\epsilon)}$$

is given by

$$\beta_{U(1)} = \{\varsigma := \pi(z)\},$$

where $\pi : U(1) \rightarrow \text{inv}\Gamma$ is the quantum germs map and it has the particularity that: $\varsigma \circ g = \epsilon(g) \varsigma$ for all $g \in U(1)$. Furthermore asking that $\beta_{U(1)}$ be an orthonormal set, the ad corpresentation, which in this case is given by

$$\text{ad}(\varsigma) = \varsigma \otimes \frac{B}{BD},$$

is unitary.

Now let us consider the $C^*$–algebra given by $2 \times 2$ matrices

$$(M := M_2(\mathbb{C}), \cdot, \text{Id}_2, \|\|_\text{op}, *),$$

where $\|\|_\text{op}$ is the standard operator norm and * is the complex transpose operation. A particular useful Hamel basis of $M$ is given by

$$\beta_M := \left\{\text{Id}_2, S_1 = \frac{1}{2}\sigma_1, S_2 = \frac{1}{2}\sigma_2, S_3 = \frac{1}{2}\sigma_3, \right\}$$

where $\{\sigma_1, \sigma_2, \sigma_3\}$ are the Pauli matrices and Id$_2$ is the identity matrix. Let us construct an appropriate differential calculus. By considering the $*$–Lie algebra $(\mathfrak{sl}(2, \mathbb{C}), [\cdot, \cdot])$ and the representation $\rho: \mathfrak{sl}(2, \mathbb{C}) \rightarrow \text{Der}(M)$, where $\text{Der}(M)$ is the space of derivations on $M$ [by DV], we can obtain the Chevalley–Eilenberg complex

$$\Omega^\bullet(M) := \Omega^\bullet_{\text{Der}}(M_2(\mathbb{C})), d, *)$$

and for comfort, its elements will be considered as $M$–valued alternating multilinear maps of $\mathfrak{sl}(2, \mathbb{C})$.

Due to $\{S_1, S_2, S_3\}$ is a Hamel basis of $\mathfrak{sl}(2, \mathbb{C})$, we can take its dual basis $\{h^1, h^2, h^3\}$ and get a left–right $M$–basis of $\Omega^\bullet(M)$ by means of

$$\beta_{\Omega^\bullet(M)} := \{h^{j_1 \ldots j_k} := h^{j_1} \wedge \ldots \wedge h^{j_k} \text{Id}_2 \mid 1 < j_1 < \ldots < j_k < 3\}.$$  

6.2.6 Proposition 4.1. The quantum space $(M, \cdot, \text{Id}_2, *)$ satisfies all the conditions mentioned in Remark 2.3 with respect to this graded differential $*$–algebra.

Proof. (1) The space $M$ is oriented since for all $k > 3$ we have $\Omega^k(M) = 0$ and

$$\text{dvol} := h^{1,2,3} = (h^1 \wedge h^2 \wedge h^3) \text{Id}_2$$

is a quantum $3$–volume form.
(2) A direct calculation shows that a lqrm can be defined on $M$ by means of

$$\langle -, - \rangle : M \times M \rightarrow M$$

$$( \hat{p} , p ) \mapsto \hat{p} p^* ;$$

$$\langle -, - \rangle : \Omega^1(M) \times \Omega^1(M) \rightarrow M$$

$$( \hat{\mu} , \mu ) \mapsto \sum_{k=1}^{3} \hat{p}_k \mu_k^* ;$$

if $\hat{\mu} = \sum_{k=1}^{3} h^k \hat{p}_k$, $\mu = \sum_{k=1}^{3} h^k p_k$;

$$\langle -, - \rangle : \Omega^2(M) \times \Omega^2(M) \rightarrow M$$

$$( \hat{\mu} , \mu ) \mapsto \sum_{1 \leq k < j \leq 3} \hat{p}_{kj} \mu_{kj}^* ;$$

if $\hat{\mu} = \sum_{1 \leq k < j \leq 3} h^{kj} \hat{p}_{kj}$, $\mu = \sum_{1 \leq k < j \leq 3} h^{kj} p_{kj}$ and finally

$$\langle -, - \rangle : \Omega^3(M) \times \Omega^3(M) \rightarrow M$$

$$( \hat{\mu} \ dvol , p \ dvol ) \mapsto \hat{p} p^* .$$

We have to remark that with this lqrm, $dvol$ is actually a lqr 3–form. In accordance with Remark 2.2.2, we get a rqrm with a rqr 3–form.

(3) By defining the linear map

$$\int_{M} : \Omega^3(M) \rightarrow \mathbb{C}$$

$$p \ dvol \mapsto \frac{1}{2} \ tr(p) ,$$

where $tr$ denotes the trace operator, it should be clear that it is a qi. Furthermore, the elements of $\text{Im}(d|_{\Omega^2(M)})$ are trace–zero, so $(M, \cdot, \text{Id}_2, \star)$ is a quantum space without boundary (with respect to this qi).

(4) A direct calculation shows

$$\star_L p = p^* \ dvol$$

for all $p \in M$;

$$\star_L (p \ dvol) = p^*$$

for all $p \ dvol \in \Omega^3(M)$;

$$\star_L \mu = h^{1,2} p_3^* - h^{1,3} p_2^* + h^{2,3} p_1^* .$$

for all $\mu = \sum_{l=1}^{3} h^l p_l \in \Omega^1(M)$ and finally

$$\star_L \mu = h^1 p_{23}^* - h^2 p_{13}^* + h^3 p_{12}^* .$$
for all $\mu = \sum_{1 \leq i < j \leq 3} h^{ij}p_{ij} \in \Omega^2(M)$. To define $\star_R$ it is enough to consider the Equation (6.2.28).

It is worth mentioning that $\varepsilon = \text{id}_M$ and $\star_L \circ \star_L = (-1)^{k(n-k)}\text{id}_{\Omega^k(M)}$ (Equation (6.2.1)).

A direct calculation shows

\[d^{\star_L} \mu = -\sum_{k=1}^{3} i[S_k, p_k]\]

for $\mu = \sum_{k=1}^{3} h^k p_k \in \Omega^1(M)$;

\[d^{\star_R} \mu = \sum_{k=1}^{3} h^k p_k\]

with $p_1 = i[S_2, p_{12}] + i[S_3, p_{13}] + p_{23}$, $p_2 = -i[S_1, p_{12}] + i[S_3, p_{23}] - p_{13}$, $p_3 = -i[S_1, p_{13}] - i[S_2, p_{23}] + p_{12}$, for $\mu = \sum_{1 \leq k < j \leq 3} h^{kj} p_{kj} \in \Omega^2(M)$ and

\[d^{\star_R} \mu = \sum_{1 \leq k < j \leq 3} h^{kj} p_{kj}\]

with $p_{12} = -i[S_3, p]$, $p_{13} = i[S_2, p]$, $p_{23} = -i[S_1, p]$, if $\mu = p \text{dvol} \in \Omega^3(M)$. To define $d^{\star_R}$ we can apply the Equation (4.2.12).

Consider now the trivial quantum principal $U(1)$–bundle $\zeta^{\text{triv}}$ with the trivial differential calculus formed by all these spaces (\[micho2\]). Qpcs are characterized by the \textit{non–commutative gauge potentials}, linear maps $A^\omega: \text{inv} \Gamma \longrightarrow \Omega^1(M)$ such that

\[\omega = \omega^{\text{triv}} + (A^\omega \otimes \text{id}_{U(1)}) \circ \text{ad},\]

where $\omega^{\text{triv}}(\theta) = 1 \otimes \theta$ (for all $\theta \in \text{inv} \Gamma$) is the trivial qpc, i.e., every element of $\text{qpc}(\zeta^{\text{triv}})$ is of the form $(A^\omega \otimes \text{id}_{U(1)}) \circ \text{ad}$ for some $A^\omega$. In particular, it is easy to see that $\omega$ is regular if and only if $A^\omega(\varsigma)$ is a linear combination of $\{h^j_1 \text{Id}_2\}_{j=1}^3$.

The only possible embedded differential (\[stheve\]) is

\[\delta : \text{inv} \Gamma \longrightarrow \text{inv} \Gamma \otimes \text{inv} \Gamma\]

given by $\delta = 0$; which implies that $d^{\delta \varsigma} = d^{\delta \zeta} = 0$ and consequently, its adjoint operators are zero as well.

In this way, the \textit{non–commutative field strength} $F^\omega(\zeta^{\text{triv}})$ is given by $F^\omega(\zeta) = dA^\omega(\varsigma)$. 
4.1. Non–commutative geometrical Yang–Mills Fields. We claim that every YM qpc is flat. Indeed, a direct calculation shows that

\[
\frac{\partial}{\partial z} J_{\text{YM}}(\omega + z \lambda') = -\frac{1}{4} (\langle A'(\varsigma) \mid d^\omega F^\omega(\varsigma) \rangle_L + \langle A'(\varsigma)^* \mid d^\omega F^\omega(\varsigma)^* \rangle_R)
\]

\[
= -\frac{1}{4} (\langle dA'(\varsigma) \mid F^\omega(\varsigma) \rangle_L + \langle dA'(\varsigma)^* \mid F^\omega(\varsigma)^* \rangle_R)
\]

\[
= -\frac{1}{2} \langle dA'(\varsigma) \mid dA^\omega(\varsigma) \rangle_L
\]

where \( \lambda'(\varsigma) = A'(\varsigma) \otimes 1 \). Since \( \langle - \mid - \rangle_L \) is an inner product we conclude that any YM qpc has to satisfy \( dA^\omega(\varsigma) = F^\omega(\varsigma) = 0 \). This result is similar to the one obtained in Differential Geometry for a trivial U(1)–bundle with a Riemannian metric on the base space.

A direct calculation shows that

(4.6) \[ qG\mathcal{G}_{\text{YM}} = \{ f \in qG\mathcal{G} \mid f \circ \omega^{\text{triv}} \text{ is flat} \}. \]

In addition, in accordance with \( \text{Sald} \) U(1) \( \subset qG\mathcal{G}_{\text{YM}} \) and the non–commutative gauge potential of a YM qpc is always given by

\[ A^\omega(\varsigma) = dp \]

for some \( p \in M \) because the first cohomology group of the Chevalley–Eilenberg complex is trivial.

4.2. Non–commutative geometrical \( n \)–multiples of Space–Time Scalar Matter Fields. By Proposition \( \text{Sald} \), the differential algebra of the Equation \( ? \) can be used, and for all \( p = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} \in M \) we have

\[ d^\omega dp = \begin{pmatrix} p_1 - p_4 & 2p_2 \\ 2p_3 & -p_1 + p_4 \end{pmatrix} \]

so taking \( V = \text{const} \) the pair \( (T_1, T_2) \) with \( p^{T_1} = \lambda_1 \text{Id}_2, p^{T_2} = \lambda_2 \text{Id}_2, \lambda_1, \lambda_2 \in \mathbb{C} \) is a stationary point. As another example, if \( V \) is such that \( V'(\text{Id}_2) = \frac{1}{2} \text{Id}_2 \), then the pair \( (T_1, T_2) \) with \( p^{T_1} = p^{T_2} = S_1 \) is a stationary point.

4.3. Non–commutative geometrical Yang–Mills–Scalar–Matter Fields. It is well–known that a complete set of mutually inequivalent irreducible unitary U(1)–representation \( \tau \) is in bijection with \( \mathbb{Z} \). The trivial representation on \( \mathbb{C} \) is given by \( n = 0 \), so let us consider \( n \neq 0 \). In all these cases, the left–right \( M \) basis given by Proposition \( \text{Sald} \) has just one element defined by

\[ T^n : \mathbb{C} \rightarrow M \otimes U(1) \]

\[ w \mapsto w \text{Id}_2 \otimes z^n \]

and hence, every \( T \in \text{MOR}(n, GM\Phi) \) is of the form \( T = p^T T^n = T^m p^T \) where \( p^T = T(1)(\text{Id}_2 \otimes z^n) \).
In general, for a qpc $\omega$ with $\omega(\varsigma) = A^\omega(\varsigma) \otimes 1 + \text{Id}_2 \otimes \varsigma$ and with $A^\omega(\varsigma) = \sum_{i=1}^{3} h^i p_i$ we get that Equation 6.f2.4 reduces to

$$-\frac{1}{n} (p_1^* dp_1 - p_2 dp_2^*) + p_1 p_1 A^\omega(\varsigma) - p_2 p_2 A^\omega(\varsigma) - 2 d^* d A^\omega(\varsigma) = 0$$

for $T_1 = \frac{1}{n} p_1 T^n$, $T_2 = -\frac{1}{n} T^{-n} p_2$; while Equation 6.f1.5 becomes

$$\nabla_n^\omega \times L \left( \nabla_n \omega (T_1) \right) = \left[ \frac{1}{n} d^* d p_1 + \star_L^{-1} (d (\star_L A^\omega(\varsigma) p_1^*)) \right] + \star_L^{-1} (A^\omega(\varsigma) \star_L dp_1) + n \star_L^{-1} (A^\omega(\varsigma) \star_L A^\omega(\varsigma) p_1^*)] T^n$$

$$\nabla^\omega R \left( \nabla^\omega \right) = T^{-n} \left[ -\frac{1}{n} d^* d p_2 - \star_R^{-1} (d (\star_R A^\omega(\varsigma))^*)) \right]$$

$$\star_R^{-1} (\star_R dp_2) A^\omega(\varsigma) + n \star_R^{-1} (p_2^* \star_R A^\omega(\varsigma)^*) A^\omega(\varsigma))$$

Now it is possible to look for YMSM fields. For example, for $n = 1$ the triplet $(\omega_{\text{triv}}, T_1, T_2)$, where $T_1(1) = (S_1 + S_2 + S_3) \otimes z$, $T_2(1) = (S_1 + S_2 + S_3) \otimes z^*$, is a YMSM field for a potential $V$ such that $V(\frac{3}{4} \text{Id}_2) = 2 \text{Id}_2$, for example $V(p) := 2p$ for all $p \in M$.

Also for $n = 1$, the triplet $(\omega, \sqrt{3} T^1, T^{-1})$, where $\omega(\varsigma) = \sum_{j=1}^{3} S_j h^j \otimes 1 + 1 \otimes \varsigma$, is again a YMSM field for a potential $V$ such that $V'(3 \text{Id}_2) = V'(\text{Id}_2) = -\frac{3}{4} \text{Id}_2$, for example $V(p) := -\frac{3}{4} p$ for all $p \in M$.

It is important to mention that in this case $\omega$ is not a YM qpc or a regular qpc and actually, $\sum_{j=1}^{3} S_j h^j$ is an eigenvector of $d^* \circ d$. Of course, there are more YMSM fields; however, they all in general depend on the form of $V$.

At least we can ensure that

$$\{ f \in qS_6 | f(z^n) = e^{it} \text{Id}_2, f(z^{*n}) = e^{is} \text{Id}_2, f(\Omega^1(M)) = 0 \text{ with } t, s \in \mathbb{R} \}$$

is a relative large subgroup of $qS_6_{\text{YMSM}}$ for any $V$.

We just used $M = M_2(\mathbb{C})$ just to develop a concrete example. However, it is possible to use $M_n(\mathbb{C})$ and the corresponding Chevalley–Eilenberg complex and having different results; although, YM qpcs are always flat.

5. Concluding Comments

Durdevich’s theory of qpbs is really general in the sense that one has the freedom to choose so many structures (giving us a much richer theory), and the theory presented in this paper follows the same line. Despite our classically motivated notation, it is important to notice
the incredible dual similarity with Differential Geometry since [D1, D2]. Furthermore, it presents the quantum version of the major result for principal $G$–bundles in [SW]. Clearly, due to the generality of the theory, it has a number of essential differences when we compare this work to its classical counterpart. Moreover, there are differences with the formulations presented in other papers, although they maintain a similar research philosophy [HM, LRZ, Z, L]. One of the most important differences with these other approaches is the absence of the fundamental operator $S^{ω}$ and a lack of the systematical use of the left/right associated qvbs.

The operator $S^{ω}$ is completely quantum in the sense that it does not have a classical counterpart: in Differential Geometry, every principal connection is regular and hence $S^{ω} = 0$. It is worth mentioning that in our theory we just assume the existence of $dS^{ω} L$, $d^{\hat{S}}^{ω} R$, not a specific form of them. In Differential Geometry, the element $d^{ω} L$ fulfills $d^{ω} L = 0$. This equation is known as the continuity equation. In Non–Commutative Geometry this equation turns into

\[(d^{ω} L - d^{S^{ω}} L)^2 R^{ω} = (d^{\hat{S}}^{ω} R - d^{\hat{S}}^{ω} R)^2 \hat{R}^{ω} = 0.\]

In our example the above equation holds; however, this is simply because of $S^{ω} = 0$ (since the only possible embedded differential is $δ = 0$). In [Sa1], we have $(d^{ω} L - d^{S^{ω}} L)^2 = 0$, $(d^{\hat{S}}^{ω} R - d^{\hat{S}}^{ω} R)^2 = 0$; nevertheless, the previous equalities do not hold in a trivial qpb with matrices as the space of base forms and with $S_2$ as the structure group. In terms of a physical interpretation, the continuity equation tells us that a quantity is conserved. In this sense, the non–commutative geometrical continuity equation could be used to identify physical fields (together with the fact that only real connections have physical sense) in more realistic examples. We consider this quite motivating to keep our research alive and going on.

On the other hand, in order to talk about the left/right structures we have to start with Equations 2.35, 3.35. These equations allow us to define associated left/right qvbs as finitely generated projective left/right $M$–modules. To define the Lagrangians, we used both structures; in addition, we have to emphasize that in the Lagrangians of Subsections 3.2, 3.3, we used a representation $α$ and its complex conjugate representation $\overline{α}$, making them a little different that their classical counterpart: now it looks like if in the quantum case left particles and right antiparticles cannot be separated; they appear naturally interconnected.

The importance of this change becomes more explicit when we play with the quantum Hopf fibration [Sa2]. For example, if we do not consider the right structure, Equation 3.4 becomes

\[\langle γ_n \circ K^\lambda(T_1) | \nabla_n^{ω} T_1 \rangle_L = 0,\]

which does not have solutions for an arbitrary $n$. Furthermore, the fact that $\nabla_n^{ω} L \nabla_n^{ω}$ and $\nabla_n^{\hat{S}} R \nabla_n^{\hat{S}}$ are mutually different (they do not share all their eigenvalues and eigenspaces) is another strong motivating reason to consider the left/right structure: it appears that ignoring one of the structures leaves to losing relevant information about the quantum spaces [Sa3].

It is worth emphasizing that the theory presented here is almost entirely algebraic: the only assumption about continuity or norms is in the potential $V$, and when we ask that the
quantum space $M$ be $C^*$-clouvable; and as the reader should have already noticed, we have used this hypothesis just to guarantee that
\[ \sum_i p_i p_i^* = 0 \iff p_i = 0. \]
This is a clear difference with other non–commutative geometrical Yang–Mills theories; for example, the reader can check [CCM] in which $C^*$-algebras and spectral triples play fundamental roles. In this sense, our theory is purely geometric–algebraic. Using the spectral triplets can be a way to relate this theory with Connes’ formulations as well as adding a kind of non–commutative geometrical spin geometry to our theory. Other lines of research can be studied from this paper in order to complete the whole non–commutative geometrical description of the Standard Model and the mathematics that it involves.

The presented formalism can be easily generalized in order to add quantum Pseudo–Riemannian closed orientable spaces by weakening Definition A.2.1 point 2. In fact, one can define a left quantum Pseudo–Riemannian metric (lqprm) on a quantum space $(M, \cdot, 1, \ast)$ as a family of $M$–valued symmetric sesquilinear maps
\[ \{\langle -, - \rangle^k : \Omega^k(M) \times \Omega^k(M) \to M \} \]
such that for $k = 0$
\[ \langle - , - \rangle^0 : M \times M \to M \]
\[ (\hat{p}, p) \mapsto \hat{p} p^* \]
and such that for $k \geq 1$
\[ \langle \hat{p}, \mu \rangle^k = \langle \hat{\mu}, \mu p^* \rangle^k \quad \text{and} \quad \langle \hat{\mu}, \mu \rangle^k = 0 \quad \forall \hat{\mu} \in \Omega^k(M) \iff \mu = 0. \]
It should be clear how to define the left quantum Pseudo–Riemannian $n$–volume form (lqpr $n$–form) and the right structure. Of course, we would also have to impose that with this lqprm, the symmetric sesquilinear map given in Equation A.2.1 is non–degenerate, as well as the existence of Hodge operators.

**APPENDIX A. NOTATION AND BASIC CONCEPTS**

In this appendix we are going to show a little summary about matrix compact quantum groups, the universal differential envelope $*$–calculus, quantum principal bundles, and associated quantum vector bundles. The reader always can consult the original work [W1], [W2], [D1], [D2], [D3], [So1], [D5], [Sa1], [Sa2].

**A.1. Compact Matrix Quantum Groups.** The concept of compact matrix quantum group (cmqg) was developed by S. L. Woronowicz in [W1], [W2]. A cmqg will be denoted by $G$; while its dense $*$–Hopf (sub)algebra will be denoted by $G^\infty := (G, \cdot, 1, \phi, \epsilon, \kappa, *)$, where $\phi$ is the comultiplication, $\epsilon$ is the counity, and $\kappa$ is the coinverse.

A (smooth right) $G$–representation on a $\mathbb{C}$–vector space $V$ is a linear map
\[ \alpha : V \to V \otimes G \]
such that
\[
\begin{array}{ccc}
V & \xrightarrow{\alpha} & V \otimes G \\
\downarrow{id_V} & \circlearrowleft & \downarrow{id_V \otimes \epsilon} \\
V & \cong & V \otimes \mathbb{C}
\end{array}
\]
and
\[
\begin{array}{ccc}
V & \xrightarrow{\alpha} & V \otimes G \\
\downarrow{\alpha} & \circlearrowleft & \downarrow{id_V \otimes \phi} \\
V \otimes G & \xrightarrow{\alpha \otimes \text{id}_G} & V \otimes \mathcal{G} \otimes \mathcal{G}.
\end{array}
\]

We say that the representation is finite-dimensional if \(\dim_{\mathbb{C}}(V) < |\mathbb{N}|\). \(\alpha\) usually receives the name of (right) coaction or (right) corepresentation of \(\mathcal{G}\) on \(V\).

Given two \(\mathcal{G}\)–representations \(\alpha, \beta\) acting on \(V, W\), respectively, a representation morphism is a linear map
\[
T : V \rightarrow W
\]
such that the following diagram holds
\[
\begin{array}{ccc}
V & \xrightarrow{\alpha} & V \otimes G \\
\downarrow{T} & \circlearrowleft & \downarrow{T \otimes \text{id}_G} \\
W & \xrightarrow{\beta} & W \otimes \mathcal{G}.
\end{array}
\]

If \(\alpha, \beta\) are two representations, we define the set of all representation morphisms between them as
\[
\text{Mor}(\alpha, \beta)
\]
and the set of all finite–dimensional \(\mathcal{G}\)–representations will be denoted by
\[
\text{Obj}(\text{Rep}_\mathcal{G}).
\]

It is important to mention that Woronowicz proved in W1 the non–commutative version of Weyl’s representation theory. Another important result is the next one

**Theorem A.1.** Let \(\mathcal{T}\) be a complete set of mutually non–equivalent irreducible unitary (necessarily finite–dimensional) \(\mathcal{G}\)–representations with \(\alpha^{\text{triv}}_\mathbb{C} \in \mathcal{T}\) (the trivial corepresentation on \(\mathbb{C}\)). For any \(\alpha \in \mathcal{T}\) that acts on \((V^\alpha, \langle-|-\rangle)\),
\[
\alpha(e_i) = \sum_{j=1}^{n_\alpha} e_j \otimes g^\alpha_{ij},
\]
where \(\{e_i\}^{n_\alpha}_{i=1}\) is an orthonormal basis of \(V^\alpha\) and \(\{g^\alpha_{ij}\}^{n_\alpha}_{i,j=1} \subseteq G\). Then \(\{g^\alpha_{ij}\}_{a,i,j}\) is a Hamel basis of \(G\), where the index \(\alpha\) runs on \(\mathcal{T}\) and \(i, j\) run from 1 to \(n_\alpha\).

Taking a bicovariant first order differential \(*\)–calculus (\(*\)–FODC \(\text{stheve}\) \([\text{So}1]\) \((\Gamma, d)\) on \(G\), the universal differential envelope \(*\)–calculus \((\Gamma^\wedge, d, *)\) is given by
\[
\Gamma^\wedge := \otimes_G^* \Gamma / \mathcal{Q}, \quad \otimes_G^* \Gamma := \bigoplus_k (\otimes_G^k \Gamma) \quad \text{with} \quad \otimes_G^k \Gamma := \underbrace{\otimes_G \cdots \otimes_G}_{k \text{ times}} \Gamma.
\]
where $Q$ is the bilateral ideal of $\otimes_G^\Gamma$ generated by $\sum g_i \otimes_G dh_i$ such that $\sum g_i dh_i = 0$ with $g_i, h_i \in G$. This space is interpreted as quantum differential forms on $G$.

Let us consider $inv^\Gamma = \{ \theta \in \Gamma \mid \Phi^\Gamma(\theta) = 1 \otimes \theta \}$, where $\Phi^\Gamma$ is the extension of the canonical left representation of $G$ in $\Gamma$. This space is a graded $C^\ast$-vector space and it is well-known that $inv := inv^\Gamma \cong ker(\epsilon) / R$, where $R \subseteq ker(\epsilon)$ is the canonical right $G$-ideal of $G$ associated to $(\Gamma, d)$. The canonical right corepresentation of $G$ on $\Gamma$ leaves $inv$ invariant and denoting it by $a.f$ we have

$$ad : inv \Gamma \longrightarrow inv \Gamma \otimes G$$

(A.1)

we have

$$ad \circ \pi = (\pi \otimes id_G) \circ Ad,$$

where $Ad$ is the (right) adjoint action of $G$ and $\pi : G \longrightarrow inv \Gamma$ is the quantum germs map which is defined by $\pi(g) = \kappa(g^{(1)})dg^{(2)}$. There is a right $G$-module structure in $inv \Gamma$ given by $\theta \circ g = \kappa(g^{(1)}\theta g^{(2)}) = \pi(hg - \epsilon(h)g)$ if $\theta = \pi(h)$.

A.2. Quantum Principal Bundles. Let $(M, \cdot, 1, \ast)$ be a quantum space and let $\mathcal{G}$ be a cmqg. A quantum principal $\mathcal{G}$-bundle over $M$ (qpb) is a quantum structure formally represented by the triplet $\zeta = (GM, M, GM \Phi)$, where $(GM, \cdot, 1, \ast)$ is a quantum space called the quantum total space with $(M, \cdot, 1, \ast)$ as quantum subspace, which receives the name of quantum base space, and

$$GM \Phi : GM \longrightarrow GM \otimes G$$

is a $\ast$-algebra morphism that satisfies

1. $GM \Phi$ is a $\mathcal{G}$-representation.
2. $GM \Phi(x) = x \otimes 1$ if and only if $x \in M$.
3. The linear map $\beta : GM \otimes GM \longrightarrow GM \otimes \mathcal{G}$ given by

$$\beta(x \otimes y) := x \cdot GM \Phi(y) = (x \otimes 1) \cdot GM \Phi(y)$$

is surjective.

Given a qpb $\zeta$ over $M$, a differential calculus on it is

1. A graded differential $\ast$-algebra $(\Omega^\ast(GM), d, \ast)$ generated by $\Omega^0(GM) = GM$ (quantum differential forms on $GM$).
2. A bicovariant $\ast$-FODC (first order differential $\ast$-calculus) over $G (\Gamma, d)$.
3. The map $GM \Phi$ is extendible to a graded differential $\ast$-algebra morphism

$$\Omega \Psi : \Omega^\ast(GM) \longrightarrow \Omega^\ast(GM) \otimes \Gamma^\wedge,$$

where $(\Gamma^\wedge, d, \ast)$ is the universal differential envelope $\ast$-calculus (quantum differential forms on $G$).

The space of horizontal forms is defined as

$$\text{Hor}^\ast GM := \{ \varphi \in \Omega^\ast(GM) \mid \Omega \Psi(\varphi) \in \Omega^\ast(GM) \otimes G \},$$
it is a graded \(\ast\)-subalgebra of \(\Omega^\ast(GM)\) and the map \(\Phi := \Omega\Phi|_{\text{Hor}^\ast GM}\) is a \(G\)-representation on \(\text{Hor}^\ast GM\). Also one can define the space of base forms (quantum differential forms on \(M\)) as

\[
\Omega^\ast(M) := \{\mu \in \Omega^\ast(GM) \mid \Omega\Phi(\mu) = \mu \otimes 1\}.
\]

In this way, a quantum principal connection (qpc) is a linear map

\[
\omega : \text{inv} \Gamma \rightarrow \Omega^1(GM)
\]

that satisfies \(\Omega\Phi(\omega(\theta)) = (\omega \otimes \text{id}_G)\text{ad}(\theta) + 1 \otimes \theta\).

For every qpb, there always exist qpcs \(\Phi\). In analogy with the classical case, the set

\[\text{qpc}(\zeta) := \{\omega : \text{inv} \Gamma \rightarrow \Omega^1(GM) \mid \omega \text{ is a qpc on } \zeta\}\]

is an affine space modeled by the \(C\)-vector space of all quantum connection displacements

\[\text{qpc}(\zeta) := \{\lambda : \text{inv} \Gamma \rightarrow \Omega^1(GM) \mid \lambda \text{ is a linear map such that } \Phi \circ \lambda = (\lambda \otimes \text{id}_G) \circ \text{ad}\}\].

Let us consider the involution

\[
\wedge = \text{qpc}(\zeta) \rightarrow \text{qpc}(\zeta)
\]

\[
\omega \mapsto \hat{\omega} := * \circ \omega \circ *
\]

We define the dual qpc of \(\omega\) as \(\hat{\omega}\). A qpc \(\omega\) is real if \(\hat{\omega} = \omega\) and we say that it is imaginary if \(\hat{\omega} = -\omega\).

Of course, the operation \(\wedge\) can be defined in \(\text{qpc}(\zeta)\). In such a way, every real quantum connection displacement \(\lambda\) can be written as

\[
\lambda = \omega - \omega'
\]

where \(\omega, \omega'\) are real elements; while for any qpc \(\omega\)

\[
\omega = \omega' + i \lambda'.
\]

with \(\omega', \lambda'\) real elements.

A qpc is called regular if for all \(\varphi \in \text{Hor}^k GM\) and \(\theta \in \text{inv} \Gamma\) we have

\[
\omega(\theta) \varphi = (-1)^k \varphi^{(0)}(0) \omega(\theta \circ \varphi^{(1)}),
\]

where \(\Phi(\varphi) = \varphi^{(0)} \otimes \varphi^{(1)}\); and it is called multiplicative if

\[
\omega(\pi(g^{(1)})) \omega(\pi(g^{(2)})) = 0
\]

for all \(g \in R\) with \(\phi(g) = g^{(1)} \otimes g^{(2)}\).

For any \(\ast\)-algebra \((\mathcal{X}, m, \text{Id}, \ast)\) and linear maps \(T_1, T_2 : \text{inv} \Gamma \rightarrow \mathcal{X}\), let us define

\[
\langle T_1, T_2 \rangle := m \circ (T_1 \otimes T_2) \circ \delta : \text{inv} \Gamma \rightarrow \mathcal{X}
\]

\[
[T_1, T_2] := m \circ (T_1 \otimes T_2) \circ c^T : \text{inv} \Gamma \rightarrow \mathcal{X}
\]

where \(\delta\) is an embedded differential and \(c^T\) is the transposed commutator. In this way, the curvature of a qpc is defined as

\[
\text{R}^\omega := d \circ \omega - \langle \omega, \omega \rangle : \text{inv} \Gamma \rightarrow \Omega^2(GM)
\]

If \(\text{R}^\omega = 0\), it is common to say that \(\omega\) is flat. Finally the covariant derivative of a qpc \(\omega\) is the first–order linear map

\[
D^\omega : \text{Hor}^\ast GM \rightarrow \text{Hor}^\ast GM
\]
such that for every \( \varphi \in \text{Hor}^{k}GM \)
\[
D^{\omega}(\varphi) = d\varphi - (-1)^{k}\varphi^{(0)}\omega(\pi(\varphi^{(1)}));
\]
while the dual covariant derivative of \( \omega \) is the first–order linear map
\[
\hat{D}^{\omega} := \ast \circ D^{\omega} \circ \ast.
\]

Let \( T \) a complete set of mutually non–equivalent irreducible \( \mathcal{G} \)–representations with \( \alpha^{\text{triv}}_{\mathbb{C}} \in T \). In order to develop the theory of associated qvbs, we have to assume that for a given \( \zeta = (GM, M, GM\Phi) \) and each \( \alpha \in T \) there exists
\[
\{T^{L}_{k}\}_{k=1}^{d_{\alpha}} \subseteq \text{Mor}(\alpha, GM\Phi)
\]
for some \( d_{\alpha} \in \mathbb{N} \) such that
\[
\text{(A.5)} \quad \sum_{k=1}^{d_{\alpha}} x^{\alpha}_{k1} x^{\alpha}_{kj} = \delta_{ij} \mathbb{1},
\]
with \( x^{\alpha}_{ki} := T^{L}_{k}(e_{i}) \), where \( \{e_{i}\}_{i=1}^{n_{\alpha}} \) is an orthonormal basis. Also we are going to assume that the following relation holds
\[
\text{(A.6)} \quad W^{\alpha} T X^{\alpha \ast} = \text{Id}^{\alpha}_{n_{\alpha}}, \quad \text{where} \quad W^{\alpha} = (w^{\alpha}_{ij}) = Z^{\alpha} X^{\alpha} C^{\alpha \ast \ast}
\]
for each \( \alpha \). Here \( X^{\alpha} = (x^{\alpha}_{ij}) \in M_{d_{\alpha} \times n_{\alpha}}(GM) \), \( X^{\alpha \ast} = (x^{\alpha \ast}_{ij}) \), while \( \text{Id}^{\alpha}_{n_{\alpha}} \) is the identity element of \( M_{n_{\alpha}}(GM) \) and \( Z^{\alpha} = (z^{\alpha}_{ij}) \in M_{n_{\alpha}}(\mathbb{C}) \) is a strictly positive element. Finally \( C^{\alpha} \in M_{n_{\alpha}}(\mathbb{C}) \) is the matrix written in terms of the basis \( \{e_{i}\}_{i=1}^{n_{\alpha}} \) of the canonical representation isomorphism between \( \alpha \) and \( \alpha^{\text{cc}} := (\text{id}_{V^{\alpha}} \otimes \kappa^{2})\alpha \), and \( W^{\alpha \ast \ast} \) is the transpose matrix of \( W^{\alpha} \).

It is worth mentioning that in terms of the theory of Hopf–Galois extensions \([KT]\), the first condition guarantees that \( GM \) is principal \([BDH]\). Furthermore, the second condition implies the existence of a right \( M \)–linear right \( \mathcal{G} \)–colinear splitting of the multiplication \( GM \otimes M \longrightarrow GM \). However, we have decided to use Equations \([BD2]\) because in this way, it is possible to do explicit calculations as the reader have already noticed.

Finally, for a qpc \( \omega \) and every \( \tau \in \text{Mor}(\text{ad}, \mathbb{H}\Phi) \) such that \( \text{Im}(\tau) \in \text{Hor}^{k}GM \), let us define
\[
\text{(A.7)} \quad S^{\omega}(\tau) := \langle \omega, \tau \rangle - (-1)^{k}\langle \tau, \omega \rangle - (-1)^{k}[\tau, \omega] \in \text{Mor}(\text{ad}, \mathbb{H}\Phi)
\]
There is a non–commutative geometrical version of the Bianchi identity:
\[
\text{(A.8)} \quad (D^{\omega} - S^{\omega}) R^{\omega} = \langle \omega, \langle \omega, \omega \rangle \rangle - \langle \langle \omega, \omega \rangle, \omega \rangle.
\]
When \( \omega \) is regular, \( S^{\omega} = 0 \) and if \( \omega \) is multiplicative \( \langle \omega, \langle \omega, \omega \rangle \rangle - \langle \langle \omega, \omega \rangle, \omega \rangle = 0 \) \([BD2]\), so if \( \omega \) is regular and multiplicative (for example, for classical principal connections) we have \( D^{\omega} R^{\omega} = 0 \).

A.3. **Associated Quantum Vector Bundles.** Let us start by taking a quantum \( \mathcal{G} \)–bundle \( \zeta = (GM, M, GM\Phi) \) and a \( \mathcal{G} \)–representation \( \alpha \in T \) acting on \( V^{\alpha} \). The \( \mathbb{C} \)–vector space \( \text{Mor}(\alpha, GM\Phi) \) has a natural \( M \)–bimodule structure given by multiplication with elements of \( M \) and by Equation A.4.1 it is a finitely generated projective left \( M \)–module; while under the assumption of Equation A.4.2 it is a finitely generated projective right \( M \)–module. We define the associated left quantum vector bundle (associated left qvb) to \( \zeta \) with respect to \( \alpha \) as the finitely generated projective left \( M \)–module
\[
\zeta^{L}_{\alpha} := (T^{L}(M, V^{\alpha}M) := \text{Mor}(\alpha, GM\Phi), +, \cdot).
\]
Let $\omega$ be a qpc. Then the map

$$\Upsilon_{\alpha}^{-1} : \Omega^\bullet(M) \otimes_M \Gamma^L(M, V^\alpha M) \to \text{Mor}(\alpha, H\Phi)$$

such that

$$\Upsilon_{\alpha}^{-1}(\mu \otimes_M T) = \mu T$$

is a graded $M$–bimodule isomorphism, where $\text{Mor}(\alpha, H\Phi)$ has the $M$–bimodule structure similar to the one of $\text{Mor}(\alpha, G\Phi)$; and its inverse is given by

$$\Upsilon_{\alpha}(\tau) = \sum_{k=1}^{d_\alpha} \mu_k^T \otimes_M T_k$$

and

$$\mu_k^T = \sum_{i=1}^{n_\alpha} \tau(e_i) x_{ki}^\alpha \in \Omega^1(M).$$

Elements of this tensor product can be interpreted as left qvb–valued differential forms. Thus the linear map

$$\nabla^\omega : \Gamma^L(M, V^\alpha M) \to \Omega^1(M) \otimes_M \Gamma^L(M, V^\alpha M)$$

$$T \mapsto \Upsilon_{\alpha} \circ D^\omega \circ T,$$

is called the induced quantum linear connection (induced qlc) in $c^L_{\alpha}$. Now we define the associated right quantum vector bundle (associated right qvb) to $\zeta$ with respect to $\alpha$ as as the finitely generated projective right $M$–module

$$\zeta_{\alpha}^R := (\Gamma^R(M, V^\alpha M) := \text{Mor}(\alpha, G\Phi), +, \cdot)$$

The map

$$\widetilde{\Upsilon}_{\alpha}^{-1} : \Gamma(M, V^\alpha M) \otimes_M \Omega^\bullet(M) \to \text{Mor}(\alpha, H\Phi)$$

such that

$$\widetilde{\Upsilon}_{\alpha}^{-1}(T \otimes_M \mu) = T \mu$$

is a graded $M$–bimodule isomorphism as well, with the inverse given by

$$\widetilde{\Upsilon}_{\alpha}(\tau) = \sum_{k=1}^{d_\alpha} T_k \otimes_M \tilde{\mu}_k^T$$

with

$$\tilde{\mu}_k^T = \sum_{i,j=1}^{d_\alpha, n_\alpha} y_{ij}^\alpha w_{ij}^\alpha \tau(e_j) \in \Omega(M).$$

where $Y^\alpha = (y_{ij}^\alpha) \in M_{d_\alpha}(\mathbb{C})$ is the inverse matrix of $Z^\alpha$ and $T_k^R = \sum_{i=1}^{d_\alpha} z_{ki}T_i^L$. Elements of this tensor product can be interpreted as right qvb–valued differential forms. The linear map

$$\widehat{\nabla}^\omega : \Gamma^R(M, V^\alpha M) \to \Gamma^R(M, V^\alpha M) \otimes_M \Omega^1(M)$$

$$T \mapsto \widehat{\Upsilon}_{\alpha} \circ \ast \circ D^\omega \circ \ast \circ T,$$

is called the induced quantum linear connection (induced qlc) in $c^R_{\alpha}$. All these constructions can be extended in a very natural way by using the direct sum operator for every $\alpha \in \text{Obj}(\text{Rep}_G)$. The following formulas for exterior covariant derivatives hold

$$d^{\nabla^\omega} = \Upsilon_{\alpha} \circ D^\omega \circ \Upsilon_{\alpha}^{-1}, \quad d^{\widehat{\nabla}^\omega} = \widehat{\Upsilon}_{\alpha} \circ \ast \circ D^\omega \circ \ast \circ \widehat{\Upsilon}_{\alpha}^{-1}$$

The canonical hermitian structure on $\zeta^L_{\alpha}$ is the map given by
\[ \langle -, - \rangle_L : \Gamma^L(M, V^\alpha M) \times \Gamma^L(M, V^\alpha M) \rightarrow M \]

\[ (T_1, T_2) \mapsto \sum_{k=1}^{n_\alpha} T_1(e_k) T_2(e_k)^*; \]

while the canonical hermitian structure on \( \zeta^\alpha \) is the map given by

\[ \langle -, - \rangle_R : \Gamma^R(M, V^\alpha M) \times \Gamma^R(M, V^\alpha M) \rightarrow M \]

\[ (T_1, T_2) \mapsto \sum_{k=1}^{n_\alpha} T_1(e_k)^* T_2(e_k), \]

where \( \{ e_i \}_{i=1}^{n_\alpha} \) is any orthonormal basis of \( V^\alpha \). It is worth mentioning that \( \langle -, - \rangle_L, \langle -, - \rangle_R \) are non–singular and \( \nabla^\alpha, \tilde{\nabla}^\alpha \) are hermitian if \( \omega \) is real.

In accordance with [Br2], \( GM \Box^G V^{\alpha*} \cong \Gamma(M, V^\alpha M) \) (for the natural left action on the dual space of \( V^\alpha, V^{\alpha*} \)), which is the commonly accepted construction of the associated qvb. Nevertheless, we have decided to use \( \Gamma(M, V^\alpha M) \) because in this way, the definitions of \( \nabla^\alpha, \tilde{\nabla}^\alpha \) are completely analogous to its classical counterparts; not to mention that it is easier to work with, since it will allow us to do explicit calculations. In addition, by using intertwining maps, the definition of the canonical hermitian structure looks more natural.

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