Abstract

We introduce the class of bounded variation (BV) functions in a general framework of strictly local Dirichlet spaces with doubling measure. Under the 2-Poincaré inequality and a weak Bakry–Émery curvature type condition, this BV class is identified with the heat semigroup based Besov class $B^{1,1/2}(X)$ that was introduced in our previous paper. Assuming furthermore a quasi Bakry–Émery curvature type condition, we identify the Sobolev class $W^{1,p}(X)$ with $B^{p,1/2}(X)$ for $p > 1$. Consequences of those identifications in terms of isoperimetric and Sobolev inequalities with sharp exponents are given.

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1 Introduction

In a metric measure space $X$ that is highly path-connected, the theory of Sobolev classes based on upper gradients provides an approach to calculus using a derivative structure that is strongly local. A weak upper gradient is an analog of $|\nabla f|$ when $f$ is a measurable function on the metric space; $|\nabla f|$ satisfies a variant of the fundamental theorem of calculus along most rectifiable curves in $X$, and it has the property that if $f$ is constant on a Borel set $E \subset X$, then $|\nabla f| = 0$ almost everywhere in $E$, see [43]. A corresponding approach to the theory of functions of bounded variation (BV) is also possible. Initially this was done in [77] under the assumption that the measure on $X$ is doubling and $X$ is sufficiently well-connected by paths to support a 1-Poincaré inequality controlling $f$ by $|\nabla f|$, but it was subsequently recognized that weaker assumptions still ensure the existence of a rich theory [9]. In such a setting, a fruitful exploration of the geometry of $X$ using BV functions and sets of finite perimeter (sets whose characteristic functions are BV functions) is possible.

However, there are metric measure spaces for which the preceding theory is degenerate [9, Corollary 7.5]; a typical situation occurs on a fractal like the Sierpinski gasket, where a paucity of rectifiable curves (e.g. in the sense of 1-modulus) causes the BV space coming from this approach to coincide with $L^1$. On the other hand, the theory of Dirichlet forms is well-developed in many such spaces, see for example [25,53,59,60,62,65,66,79,81,85–87], which is far from an exhaustive list of the literature. The theory of regular Dirichlet forms assumes as the fundamental object a topological space equipped with a $\sigma$-finite Borel measure, and a closed non-negative definite quadratic form $E$ with dense domain in the associated $L^2$-class on the space.

This paper is one of a sequence, including [2,3], in which we define certain Besov spaces from a Dirichlet form and a measure, and use them as tools to explore and elucidate notions of BV. The paper [2] introduces some general theory about these spaces, while [3] deals with some situations where the existing theory may degenerate as described above. In this paper we examine the application of this approach in a metric upper gradient setting connected to that in [6,9,77]. Specifically, we assume that the Dirichlet form provides an intrinsic metric with
respect to which \(\mu\) is a doubling measure, and where there is a 2-Poincaré inequality involving the Dirichlet energy measures (Definition 2.8). Under these conditions upper gradients exist and accordingly BV may be defined by the relaxation approach in [77]. However, rather than assuming a 1-Poincaré inequality as in [77] we introduce a geometrically-motivated assumption that we call a weak Bakry–Émery condition (Definition 2.13) and use it in establishing fundamental properties of BV. Some examples where one has 2-Poincaré and weak Bakry–Émery but where the validity of a 1-Poincaré inequality is unknown are in [15]. Our approach can also be compared to the measure-valued upper gradient structure introduced in [9] and adapted to Dirichlet spaces in [6]; we are indebted to the referee for pointing out that whether our approach lies in the scope of this theory depends on whether 2-Poincaré and our weak Bakry–Émery condition implies the \(\tau\)-regularity condition of [6, Definition 12.4]. The connection between the classical Bakry–Émery condition and Cheeger energy are discussed in some detail in [6, Section 10]; we do not know whether \(\tau\)-regularity is true in our setting. Moreover, we believe that the techniques used here may be of independent interest as tools in analyzing the BV space.

In a somewhat related direction to the present work, Sobolev type spaces constructed using Dirichlet forms have been shown in [66] to coincide with those constructed using upper gradients if the metric space supports a 2-Poincaré inequality. Moreover, from [25] it follows that in a doubling metric measure space supporting a \(p\)-Poincaré inequality for some \(1 \leq p \leq 2\), there is a Dirichlet form that is compatible with the upper gradient Sobolev class structure, see for example [65].

In summary, the goal of this paper is to develop a theory of a BV class in the specific setting of a locally compact, separable topological space \(X\), equipped with a Radon measure \(\mu\), a strictly local Dirichlet form \(\mathcal{E}\) on \(L^2(\mu)\) and its associated intrinsic metric \(d_{\mathcal{E}}\), such that \(\mu\) is \(d_{\mathcal{E}}\)-doubling and there is a 2-Poincaré inequality. The background and assumptions are established in Sect. 2. In Sect. 3 we propose a notion of BV functions and sets of finite perimeter and prove several fundamental properties following the approach of [77]: these include the Radon measure property of the BV energy seminorm and a co-area formula connecting sets of finite perimeter to BV energy (see Theorem 3.9). Section 4 is the heart of the paper. We begin by comparing the heat semigroup-based Besov class \(B^{\alpha/2}_{p,\infty}(X)\) introduced in [2] to a more classical Besov class \(B^{\alpha}_{p,\infty}(X)\) that was defined in [38] from the intrinsic metric \(d_{\mathcal{E}}\) rather than the heat semigroup \(P_t\). Under the standing assumptions (\(\mu\) is doubling and 2-Poincaré inequality) we show that \(B^{\alpha/2}_{p,\infty}(X)\) coincides with \(B^{\alpha}_{p,\infty}(X)\). This result is related to the correspondence of metric and heat semigroup Besov classes established in [80], but differs in that the latter makes the stronger assumption that \(\mu\) is Ahlfors regular. We then compare the class \(BV(X)\) to the heat semi-group Besov class \(B^{1,1/2}(X)\) and show that these coincide under the additional hypothesis that \(\mathcal{E}\) supports a weak Bakry–Émery curvature condition, see Theorem 4.4. We also explore a connection between the co-dimension 1 Hausdorff measure of the regular boundary of a set \(E\) of finite perimeter (meaning \(1_E \in BV(X)\)) to its perimeter measure \(\|D1_E\| = P(E, X)\), see Proposition 4.7. In the last part of Sect. 4 we show that if \(X\) supports a quasi Bakry–Émery curvature condition and \(p > 1\), then the heat semigroup-based Besov class \(B^{p,1/2}(X)\) coincides with the Sobolev space \(W^{1,p}(X)\), see Theorems 4.10, 4.12 and 4.18. Section 5 concludes with a discussion of Sobolev type embedding theorems for Besov and BV classes in the context of strictly local Dirichlet spaces satisfying the weak Bakry–Émery estimate. These parallel the classical Sobolev embedding theorems associated with the classical Sobolev and BV classes as in [1,75]. The tools of heat semigroup based Besov spaces that will be used were developed in [2]; nevertheless, the present paper can largely be read independently from the latter.
2 Preliminaries

2.1 Strictly local Dirichlet spaces, doubling measures, and standing assumptions

Throughout the paper, X will be a separable, locally compact topological space equipped with a Radon measure μ supported on X. Let \((\mathcal{E}, \mathcal{F} = \text{dom}(\mathcal{E}))\) be a Dirichlet form on X, meaning it is a densely defined, closed, symmetric and Markovian form on \(L^2(X)\). The book [35] is a classical reference on the theory of Dirichlet forms. We also refer to the foundational papers by Sturm [88–90].

We denote by \(C_c(X)\) the vector space of all continuous functions with compact support in X and \(C_0(X)\) its closure with respect to the supremum norm. A core for \((X, \mu, \mathcal{E}, \mathcal{F})\) is a subset \(\mathcal{C}\) of \(C_c(X) \cap \mathcal{F}\) which is dense in \(C_c(X)\) in the supremum norm and dense in \(\mathcal{F}\) in the \(\mathcal{E}_1\)-norm

\[
\|f\|_{\mathcal{E}_1} = \left(\|f\|_{L^2(X)}^2 + \mathcal{E}(f, f)\right)^{1/2}.
\]

The Dirichlet form \(\mathcal{E}\) is called regular if it admits a core. It is strongly local if for any two functions \(u, v \in \mathcal{F}\) with compact supports such that \(u\) is constant in a neighborhood of the support of \(v\), we have \(\mathcal{E}(u, v) = 0\) (see [35, p. 6]). We will assume that \((\mathcal{E}, \mathcal{F})\) is a strongly local regular Dirichlet form on \(L^2(X)\).

Since \(\mathcal{E}\) is regular, for every \(u, v \in \mathcal{F} \cap L^\infty(X)\) we can define the energy measure \(\Gamma(u, v)\) through the formula

\[
\int_X \phi \, d\Gamma(u, v) = \frac{1}{2}[\mathcal{E}(\phi u, v) + \mathcal{E}(\phi v, u) - \mathcal{E}(\phi, uv)], \quad \phi \in \mathcal{F} \cap C_c(X).
\]

Then \(\Gamma(u, v)\) can be extended to all \(u, v \in \mathcal{F}\) by truncation (see [26, Theorem 4.3.11]).

According to Beurling and Deny [21], one has then for \(u, v \in \mathcal{F}\)

\[
\mathcal{E}(u, v) = \int_X d\Gamma(u, v)
\]

and \(\Gamma(u, v)\) is a signed Radon measure.

Observe that the energy measures \(\Gamma(u, v)\) inherit a strong locality property from \(\mathcal{E}\), namely that \(1_U d\Gamma(u, v) = 0\) for any open subset \(U \subset X\) and \(u, v \in \mathcal{F}\) such that \(u\) is constant on \(U\). One can then extend \(\Gamma\) to \(\mathcal{F}_{\text{loc}}(X)\) defined as

\[
\mathcal{F}_{\text{loc}}(X) = \{u \in L^2_{\text{loc}}(X) : \forall \text{ compact } K \subset X, \exists v \in \mathcal{F} \text{ such that } u = v|_K \text{ a.e.}\}.
\]

We will still denote this extension by \(\Gamma\) and collect below some of its properties for later use. For proofs, we refer for instance to [35, Section 3.2] and also [89, Section 4].

- **Strong locality.** For all \(u, v \in \mathcal{F}_{\text{loc}}(X)\) and all open subset \(U \subset X\) on which \(u\) is a constant

  \[1_U d\Gamma(u, v) = 0.\]

- **Leibniz and chain rules.** For all \(u \in \mathcal{F}_{\text{loc}}(X), v \in \mathcal{F}_{\text{loc}}(X) \cap L^\infty_{\text{loc}}(X), w \in \mathcal{F}_{\text{loc}}(X)\) and \(\eta \in C^1(\mathbb{R})\), we have \(\eta(u) \in \mathcal{F}_{\text{loc}}\) and

  \[
d\Gamma(uv, w) = ud\Gamma(v, w) + vd\Gamma(u, w),
  \]

  \[
d\Gamma(\eta(u), v) = \eta'(u)d\Gamma(u, v).
\]
With respect to $E$ one can define the following intrinsic metric $d_E$ on $X$ by

$$d_E(x, y) = \sup\{u(x) - u(y) : u \in \mathcal{F} \cap C_0(X) \text{ and } d\Gamma(u, u) \leq d\mu\},$$

(3)

where the condition $d\Gamma(u, u) \leq d\mu$ means that $\Gamma(u, u)$ is absolutely continuous with respect to $\mu$ with Radon–Nikodym derivative bounded by 1. The term “intrinsic metric” is potentially misleading because in general there is no reason why $d_E$ is a metric on $X$ (it could be infinite for a given pair of points $x$, $y$ or zero for some distinct pair of points). However, the setting we work in here will, by definition, rule out this possibility. Namely, we will assume the Dirichlet space to be strictly local as defined e.g. in [73], which is based on the classical papers [22–24,88–90].

**Definition 2.1** A strongly local regular Dirichlet space is called strictly local if $d_E$ is a metric on $X$ and the topology induced by $d_E$ coincides with the topology on $X$.

**Example 2.2** In the context of a complete metric measure space $(X, d, \mu)$ supporting a 2-Poincaré inequality and where $\mu$ is doubling, one can construct a Dirichlet form $E$ with domain $N^{1,2}(X)$ by using a choice of a Cheeger differential structure as in [25]. This Dirichlet form is then strictly local and the intrinsic distance $d_E$ is bi-Lipschitz equivalent to the original metric $d$. We refer to [74] and the references therein for further details. This framework encompasses for instance the one of Riemannian manifolds with non-negative Ricci curvature and the one of doubling sub-Riemannian spaces supporting a 2-Poincaré inequality.

**Example 2.3** In the context of fractals, strictly local Dirichlet forms appear in [4,23,24,34,47,54,55,58,61,68,76,91,92] and play an important role in the analysis of first-order derivatives in these settings. Whether every local Dirichlet form admits a change of measure under which it becomes strictly local is an open question, though some natural conditions for this are discussed in [45,48], where it is also proved that $\Gamma$ is the norm of a well defined gradient that may be extended to measurable 1-forms, see [46]. Without giving details of this analysis, we mention that existence of a suitable collection of finite (Dirichlet) energy coordinate functions, which depend only on the Dirichlet form $E$, is essentially equivalent to the existence of a measure which is compatible with an intrinsic distance. In particular, [45] proves existence of a measure which is compatible with an intrinsic distance for any local resistance form in the sense of Kigami [58,60,62,63]. Thus, any fractal space with a local resistance form has an intrinsic metric and is a strictly local Dirichlet form for an appropriate choice of the measure.

Now suppose in addition to strict locality we know that open balls have compact closures and that $(X, d_E)$ is complete. In this setting we may apply [89, Lemma 1, Lemma 1′] to obtain that the distance function $\varphi_x : y \mapsto d_E(x, y)$ on $X$ is in $\mathcal{F}_{\text{loc}}(X) \cap C$ and $d\Gamma(\varphi_x, \varphi_x) \leq d\mu$. Then cut-off functions on intrinsic balls $B(x, r)$ of the form

$$\varphi_{x,r} : y \mapsto (r - d_E(x, y))_+$$

are also in $\mathcal{F}_{\text{loc}}(X) \cap C$ and $d\Gamma(\varphi_{x,r}, \varphi_{x,r}) \leq d\mu$ (for all $r > 0$ and $x \in X$). The following lemma will be useful. Its proof will be omitted because it is of a standard type related to that of the McShane extension theorem, see [42], using approximations of the form $f_j(x) := \inf\{f(q_i) + K \varphi_i(x) : i \in I \text{ with } i \leq j\}$ with $\{q_i : i \in \mathbb{N}\}$ a countable dense subset of $X$.

**Lemma 2.4** Let $f : X \to \mathbb{R}$ be locally Lipschitz continuous with respect to $d_E$. Then $f \in \mathcal{F}_{\text{loc}}(X)$ with $\Gamma(f, f) \ll \mu$. If $f$ is locally $K$-Lipschitz, then $\Gamma(f, f) \leq K^2 \mu$. 

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At many places in the paper we will need to approximate using locally Lipschitz functions and use locally Lipschitz cutoffs, so it is important that these functions are dense in $L^1(X)$. This is a consequence of density of simple functions and the fact that $\mu$ is Radon, for example by using the observation that if $K$ is compact and $U \supset K$ is open with $\mu(U \setminus K) < \varepsilon$ then $\|1_K - (1 - d(x, K)/d(U^c, K))\|_{L^1} < \varepsilon$ is a Lipschitz approximation of $1_K$.

Now we come to the final assumption which will be made throughout the paper, namely that $\mu$ is volume doubling.

**Definition 2.5** We say that the metric measure space $(X, d_\mathcal{E}, \mu)$ satisfies the volume doubling property if there exists a constant $C > 0$ such that for every $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \leq C \mu(B(x, r)).$$

It follows from the doubling property of $\mu$ (see [42]) that there is a constant $0 < Q < \infty$ and $C \geq 1$ such that whenever $0 < r \leq R$ and $x \in X$, we have

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq C \left(\frac{R}{r}\right)^Q. \quad (4)$$

Another well-known consequence of the doubling property is the availability of a maximally separated $\varepsilon$-covering as defined below.

**Definition 2.6** Let $U \subset X$ be a non-empty subset and let $\varepsilon > 0$. A maximally separated $\varepsilon$-covering is a family of balls $\{B_i^\varepsilon = B(x_i^\varepsilon, \varepsilon)\}_i$ such that

- The collection $\{B_i^{\varepsilon/2}\}_i$ is a maximal pairwise disjoint family of balls with radius $\varepsilon/2$;
- The collection $\{B_i^\varepsilon\}_i$ covers $U$, that is, $U = \bigcup_i B_i^\varepsilon$;
- For any $C > 1$ there exists $K = K(C) \in \mathbb{N}$ such that each point $x \in X$ is contained in at most $K$ balls from the family $\{B_i^{C\varepsilon}\}_i$.

We now summarize the assumptions that will be in force throughout the paper.

** Assumption 2.7 **

- The Dirichlet space $(X, \mu, \mathcal{E}, \mathcal{F})$ is strictly local, so $\mathcal{E}$ is strongly local and regular and $d_\mathcal{E}$ is a metric on $X$ that induces the topology on $X$;
- The metric space $(X, d_\mathcal{E})$ is complete;
- $\mu$ is volume doubling;

We will use the following consequences of these assumptions without further comment: closed and bounded subsets of $(X, d_\mathcal{E})$ are compact, and locally Lipschitz functions are dense in $L^1$. A commonly used notation throughout the paper is that if $\Gamma(f, f)$ is absolutely continuous with respect to $\mu$, as is the case for locally Lipschitz functions, then $|\nabla f|$ denotes the square root of its Radon–Nikodym derivative, so $d\Gamma(f, f) = |\nabla f|^2 d\mu$.

It should be noted that with the exception of some parts of Sect. 3, we will typically also assume existence of a 2-Poincaré inequality, which is discussed next.

### 2.2 The 2-Poincaré inequality

Let $(X, \mu, \mathcal{E}, \mathcal{F})$ be a strictly local regular Dirichlet space as in Sect. 2.1.

**Definition 2.8** We say that $(X, \mu, \mathcal{E}, \mathcal{F})$ supports the 2-Poincaré inequality if there are constants $C > 0$ and $\lambda \geq 1$ such that whenever $B$ is a ball in $X$ and $u \in \mathcal{F}$, we have

$$\frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \leq C \operatorname{rad}(B) \left(\frac{1}{\mu(\lambda B)} \int_{\lambda B} d\Gamma(u, u)\right)^{1/2},$$

where $\operatorname{rad}(B)$ is the radius of $B$ and $\lambda B$ denotes a concentric ball of radius $\lambda \operatorname{rad}(B)$.  

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Remark 2.9 The 2-Poincaré inequality does not need \( \mathcal{E} \) to be strictly local, but it does need it to be regular, in order for the measure \( \Gamma(u, u) \) representing the Dirichlet energy of \( u \in \mathcal{F} \) to exist, see [35] for more details. However we will always be considering strictly local forms.

Example 2.10 Example of strictly local Dirichlet spaces \((X, \mu, \mathcal{E}, \mathcal{F})\) that satisfy the volume doubling property and support the 2-Poincaré inequality include:

- Complete Riemannian manifolds with non-negative Ricci curvature or more generally RCD\((0, N)\) spaces in the sense of Ambrosio et al. [10];
- Carnot groups and other complete sub-Riemannian manifolds satisfying a generalized curvature dimension inequality (see [14,19]);
- Doubling metric measure spaces that support a 2-Poincaré inequality with respect to the upper gradient structure of Heinonen and Koskela (see [43,67,68]);
- Metric graphs with bounded geometry (see [40]).

When the 2-Poincaré inequality is satisfied, a standard argument due to Semmes tells us that locally Lipschitz continuous functions form a dense subclass of \( \mathcal{F} \), where \( \mathcal{F} \) is equipped with the norm \((1)\), see for example [43, Theorem 8.2.1]. Moreover, by [66], we know that if the 2-Poincaré inequality is satisfied and \( \mu \) is doubling, then the Newton–Sobolev class \((based on upper gradients, see also [66])\) is the same as the class \( \mathcal{F} \), with comparable energy seminorms.

The next lemma is used to define a length of the gradient in the current setting and shows that the Dirichlet form admits a carré du champ operator. In particular, the quantity \(|\nabla u|\) is an upper gradient of \( u \) in the sense of [66], and it follows that \( u \in \mathcal{F} \) satisfies some a-priori stronger Poincaré inequalities in which the integrability exponent for \(|u - u_B|\) in Definition 2.8 is higher, see [41].

Lemma 2.11 Suppose that \((X, \mu, \mathcal{E}, \mathcal{F})\) satisfies the doubling property and supports the 2-Poincaré inequality. Then for all \( u \in \mathcal{F} \), we have \( d\Gamma(u, u) \ll \mu \). The Radon–Nikodym derivative \( \frac{d\Gamma(u, u)}{d\mu} \) is denoted by \(|\nabla u|^2\).

Proof It is known [44, Lemma 2.1] that if \( u \in \mathcal{F} \) is the \( \mathcal{E}_1 \)-limit (i.e. limit in the norm \((1)\)) of functions \( v_j \in \mathcal{F} \) satisfying \( d\Gamma(v_j, v_j) \ll \mu \) then also \( d\Gamma(u, u) \ll \mu \). By the assumed regularity of the Dirichlet form, we may therefore assume \( u \in \mathcal{F} \cap C_c(X) \) and proceed to show that \( u \) is such an \( \mathcal{E}_1 \)-limit.

Fix \( \varepsilon > 0 \). Let \( \{B^\varepsilon_i = B(x^\varepsilon_i, \varepsilon) \}_i \) be a maximally separated \( \varepsilon \)-covering of \( X \) as in Definition 2.6, so that the family \( \{B^\varepsilon(C) \}_i \) has the bounded overlap property for any \( C > 1 \). Let \( \varphi^\varepsilon_i \) be a \((C/\varepsilon)\)-Lipschitz partition of unity subordinated to this cover: that is, \( 0 \leq \varphi^\varepsilon_i \leq 1 \) on \( X \), \( \sum_i \varphi^\varepsilon_i = 1 \) on \( X \), and \( \varphi^\varepsilon_i = 0 \) in \( X \setminus B^2\varepsilon_i \). We then set

\[
u^\varepsilon_i := \sum_i u_{B^\varepsilon_i} \varphi^\varepsilon_i,
\]

where \( u_{B^\varepsilon_i} = \int_{B^\varepsilon_i} u \, d\mu \).

Since \( u \in C_c(X) \) it is elementary that \( u^\varepsilon \to u \) in \( L^2(X, \mu) \). Indeed, using the bounded overlap of the balls \( B^2\varepsilon_i \) we see that, as \( \varepsilon \to 0 \),

\[
|u - u^\varepsilon|^2 \leq \sum_i \int_{B^2\varepsilon_i} |u(x) - u_{B^\varepsilon_i}|^2 \, d\mu \leq C \mu(spt(u)) \sup_i \sup_{x \in B^\varepsilon_i} |u(x) - u_{B^\varepsilon_i}|^2 \to 0.
\]

Now each \( \varphi^\varepsilon_i \) is Lipschitz, so we know that \( u^\varepsilon \) is locally Lipschitz and hence is in \( \mathcal{F}_{loc}(X) \). Indeed, for \( x, y \in B^\varepsilon_i \) we have from the 2-Poincaré inequality, the fact that \( B^2\varepsilon_i \cap B^2\varepsilon_j \neq \emptyset \) implies \( B^2\varepsilon_i \subset B^6\varepsilon_j \) and the volume doubling property, that

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Using the doubling measure property again, and the bounded overlap of \( \{B_i^\varepsilon\}_{i \in \mathbb{N}} \), we have

\[
|u_\varepsilon(x) - u_\varepsilon(y)| \leq \sum_{i : B_i^\varepsilon \cap B_{i+1}^\varepsilon \neq \emptyset} |u_{B_i^\varepsilon} - u_{B_{i+1}^\varepsilon}||\phi_i^\varepsilon(x) - \phi_i^\varepsilon(y)|
\]

\[
\leq C \frac{d(x, y)}{\varepsilon} \sum_{i : B_i^\varepsilon \cap B_{i+1}^\varepsilon \neq \emptyset} (|u_{B_i^\varepsilon} - u_{B_{i+1}^\varepsilon}| + |u_{B_{i+1}^\varepsilon} - u_{B_i^\varepsilon}|)
\]

\[
\leq C \frac{d(x, y)}{\varepsilon} \int_{B_{i+1}^\varepsilon} |u(y) - u_{B_{i+1}^\varepsilon}| d\mu(y)
\]

\[
\leq C d(x, y) \left(\frac{1}{\mu(B_i^{6\varepsilon})} \int_{B_i^{6\varepsilon}} d\Gamma(u, u)\right)^{1/2}.
\]

It follows from Lemma 2.4 that \( \Gamma(u_\varepsilon, u_\varepsilon) \ll \mu \) and we recall that the Radon–Nikodym measure is denoted by \( |\nabla u_\varepsilon|^2 \). Moreover, we also have on \( B_i^\varepsilon \) that

\[
d\Gamma(u_\varepsilon, u_\varepsilon) \leq C \left(\frac{1}{\mu(B_i^{6\varepsilon})} \int_{B_i^{6\varepsilon}} d\Gamma(u, u)\right) d\mu.
\]

Using the doubling measure property again, and the bounded overlap of \( \{B_i^{6\varepsilon}\}_{i \in \mathbb{N}} \), this yields

\[
\int_X |\nabla u_\varepsilon|^2 d\mu = \mathcal{E}(u_\varepsilon, u_\varepsilon) \leq \sum_i \int_{B_i^\varepsilon} d\Gamma(u_\varepsilon, u_\varepsilon) \leq C \sum_i \frac{\mu(B_i^{6\varepsilon})}{\mu(B_i^{6\varepsilon})} \int_{B_i^{6\varepsilon}} d\Gamma(u, u) \leq C \mathcal{E}(u, u).
\]

(5)

Take a sequence \( \varepsilon_n \to 0^+ \). From (5) and the reflexivity of \( \mathcal{F} \), there exists a subsequence of \( \{u_{\varepsilon_n}\}_{n \in \mathbb{N}} \) that is weakly convergent in \( \mathcal{F} \). By Mazur’s lemma, a sequence of convex combinations of \( u_{\varepsilon_n} \) (still denoted by \( u_{\varepsilon_n} \)) converges in the \( \mathcal{E}_1 \)-norm (1). Let the \( \mathcal{E}_1 \)-limit of \( \{u_{\varepsilon_n}\} \) be denoted \( v \). Since the \( L^2 \)-norm is part of this norm and we know that \( u_{\varepsilon_n} \to u \) in \( L^2(X) \), we have \( u = v \). Thus we have \( u_{\varepsilon_n} \to u \) in \( \mathcal{E}_1 \)-norm and \( d\Gamma(u_{\varepsilon_n}, u_{\varepsilon_n}) \ll \mu \), which concludes the proof.

\[
\square
\]

**Definition 2.12** Let \( 1 \leq p < \infty \). We say that \((X, \mu, \mathcal{E}, \mathcal{F})\) supports a \( p \)-Poincaré inequality if there are constants \( C > 0 \) and \( \lambda \geq 1 \) such that whenever \( B \) is a ball in \( X \) and \( u \in \mathcal{F} \), we have

\[
\frac{1}{\mu(B)} \int_B |u - u_B| d\mu \leq C \text{rad}(B) \left(\frac{1}{\mu(\lambda B)} \int_{\lambda B} |\nabla u|^p d\mu\right)^{1/p}.
\]

Of course, the \( p \)-Poincaré inequality for any \( p \neq 2 \) does not make sense if \( \mathcal{E} \) does not satisfy the condition of strict locality. The requirement that \( \mathcal{E} \) supports a \( 1 \)-Poincaré inequality is a significantly stronger requirement than supporting a \( 2 \)-Poincaré inequality.

Much of the current theory on functions of bounded variation in the metric setting requires a \( 1 \)-Poincaré inequality, though the theory can be constructed without this [9]. In this paper we will not require that \((X, \mu, \mathcal{E}, \mathcal{F})\) supports a \( 1 \)-Poincaré inequality but only the weaker \( 2 \)-Poincaré inequality. However in some of our analysis we will need an additional requirement called the weak Bakry–Émery curvature condition.

### 2.3 Sobolev classes \( W^{1,p}(X) \)

The theory of Sobolev spaces was first advanced in order to prove solvability of certain PDEs, see for example [32, 75]. When \( X \) is a Riemannian manifold, a function \( f \in L^p(X) \) is said
to be in the Sobolev class \( W^{1,p}(X) \) if its distributional derivative is given by a vector-valued function \( \nabla f \in L^p(X : \mathbb{R}^n) \). Extensions of this idea to sub-Riemannian spaces have been considered in [36]. However, in more general metric spaces where the distributional theory of derivatives (which relies on integration by parts) is unavailable, an alternate notion of derivatives needs to be found. Indeed, we do not need an alternative to \( \nabla f \), as long as we have a substitute for \( |\nabla f| \).

For metric spaces \( X \), Lipschitz functions \( f : X \to \mathbb{R} \) have a natural such alternative, \( \text{Lip} f \), given by

\[
\text{Lip} f(x) := \lim_{r \to 0^+} \sup_{y \in B(x, r)} \frac{|f(y) - f(x)|}{r}.
\]

Other notions such as upper gradients and Hajłasz gradients play this substitute role well, see for example [43]. In the current paper we consider another possible notion of \( |\nabla f| \) which has a more natural affinity to the heat semigroup and the Dirichlet form, as in Lemma 2.11.

So, in this paper, our definition of \( W^{1,p}(X) \), \( p \geq 1 \) is the following:

\[
W^{1,p}(X) = \left\{ u \in L^p(X) \cap \mathcal{F}_{\text{loc}}(X) : \Gamma(u, u) \ll \mu, |\nabla u| \in L^p(X) \right\}. 
\]

The norm on \( W^{1,p}(X) \) is then given by

\[
\|u\|_{W^{1,p}(X)} = \|u\|_{L^p(X)} + \|\nabla u\|_{L^p(X)}.
\]

Note, in particular, that \( W^{1,2}(X) = \mathcal{F} \). In the context of Sobolev spaces, Besov function classes arise naturally in two ways. Given a Sobolev class \( W^{1,p}(\mathbb{R}^{n+1}) \) and a bi-Lipschitz embedding of \( \mathbb{R}^n \) into \( \mathbb{R}^{n+1} \), there is a natural trace of functions in \( W^{1,p}(\mathbb{R}^{n+1}) \) to the embedded surface, and this trace belongs to a Besov class, see for example [51,52]. Besov classes also arise via real interpolations of \( L^p(\mathbb{R}^n) \) and \( W^{1,p}(\mathbb{R}^n) \), see for example [20,93]. In the present paper we will relate Sobolev classes \( W^{1,p}(X) \) to two types of Besov classes defined in our previous paper [2], see Theorems 4.12, 4.18, and 4.10. One of these types of Besov classes is defined from the heat semigroup, while the other uses only the metric structure of \( X \).

We note that previous metric characterizations of Sobolev spaces in the presence of doubling and 2-Poincaré have been studied in [29].

### 2.4 Bakry–Émery curvature conditions

Let \( \{P_t\}_{t \in [0, \infty)} \) denote the self-adjoint semigroup of contractions on \( L^2(X, \mu) \) associated with the Dirichlet space \((X, \mu, \mathcal{E}, \mathcal{F})\) and \( L \) the infinitesimal generator of \( \{P_t\}_{t \in [0, \infty)} \). The semigroup \( \{P_t\}_{t \in [0, \infty)} \) is referred to as the heat semigroup on \((X, \mu, \mathcal{E}, \mathcal{F})\). For classical properties of \( \{P_t\}_{t \in [0, \infty)} \), we refer to [2, Section 2.2]. In particular, it is known that the doubling property together with the 2-Poincaré inequality imply that the semigroup \( \{P_t\}_{t \in [0, \infty)} \) is conservative, i.e. \( P_1 1 = 1 \).

The work of Sturm [88,90] (see Saloff-Coste [82] and Grigor’yan [39] for earlier results on Riemannian manifolds) tells us that the doubling property together with the 2-Poincaré inequality are equivalent to the property that the heat semigroup \( P_t \) admits a heat kernel function \( p_t(x, y) \) on \([0, \infty) \times X \times X\) for which there are constants \( c_1, c_2, C > 0 \) such that whenever \( t > 0 \) and \( x, y \in X \),

\[
\frac{1}{C} \frac{e^{-c_1 d(x,y)^2/t}}{\sqrt{\mu(B(x, \sqrt{t})) \mu(B(y, \sqrt{t}))}} \leq p_t(x, y) \leq C \frac{e^{-c_2 d(x,y)^2/t}}{\sqrt{\mu(B(x, \sqrt{t})) \mu(B(y, \sqrt{t}))}}.
\]
The above inequalities are called Gaussian bounds for the heat kernel. Due to the doubling property, one can equivalently rewrite the Gaussian bounds as:

\[
\frac{1}{C} e^{-c_1 d(x,y)^2/4t} \leq p_t(x, y) \leq C e^{-c_2 d(x,y)^2/4t},
\]

for some different constants \(c_1, c_2, C > 0\). The combination of the doubling property and the 2-Poincaré inequality also implies the following Hölder regularity of the heat kernel

\[
|p_t(x, y) - p_t(z, y)| \leq \left( \frac{d(x,z)}{\sqrt{t}} \right)^\alpha \frac{C}{\mu(B(y, \sqrt{t}))},
\]

for some \(C > 0, \alpha \in (0, 1)\), and all \(x, y, z \in X\) (see for instance [83]). In some parts of this paper, we need a stronger condition than Hölder regularity for the heat kernel, in which case we will use the following uniform Lipschitz continuity property.

**Definition 2.13** We say that the Dirichlet metric space \((X, \mathcal{E}, d_\mathcal{E}, \mu)\) satisfies a weak Bakry–Émery curvature condition if, whenever \(u \in \mathcal{F} \cap L^\infty(X)\) and \(t > 0\),

\[
\|\| \nabla P_t u \|\|^2_{L^\infty(X)} \leq \frac{C}{t} \|u\|^2_{L^\infty(X)}.
\]

We refer to (9) as a weak Bakry–Émery curvature condition because, in many settings, its validity is related to the existence of curvature lower bounds on the underlying space.

**Example 2.14** The weak Bakry–Émery curvature condition is satisfied in the following examples:

- Complete Riemannian manifolds with non-negative Ricci curvature and more generally, the \(RCD(0, +\infty)\) spaces (see [50]);
- Carnot groups (see [16]);
- Complete sub-Riemannian manifolds with generalized non-negative Ricci curvature (see [14, 19]);
- On non-compact metric graphs with finite number of edges, the weak Bakry–Émery curvature condition has been proved to hold for \(t \in (0, 1]\) (see [18, Theorem 5.4]), and is conjectured to be true for all \(t\). If the graph is moreover compact, the weak Bakry–Émery estimate holds for every \(t > 0\) [18, Theorem 5.4].

Several statements equivalent to the weak Bakry–Émery curvature condition are given in [27, Theorem 1.2]. There are some metric measure spaces equipped with a doubling measure supporting a 2-Poincaré inequality but without the above weak Bakry–Émery condition, see for example [65]. It should also be noted that, in the setting of complete sub-Riemannian manifolds with generalized non-negative Ricci curvature in the sense of [17], while the weak Bakry–Émery curvature condition is known to be satisfied (see [14, 19]), the 1-Poincaré inequality is so far not known to hold, though the 2-Poincaré inequality is known to be always satisfied, see [15].

We will also sometimes need a condition that is stronger than (9).

**Definition 2.15** We say that the Dirichlet metric space \((X, \mathcal{E}, d_\mathcal{E}, \mu)\) satisfies a quasi Bakry–Émery curvature condition if there exists a constant \(C > 0\) such that for every \(u \in \mathcal{F}\) and \(t \geq 0\) we have \(\mu\) a.e.

\[
|\nabla P_t u| \leq C P_t |\nabla u|.
\]
The quasi Bakry–Émery curvature condition implies the weak one, as is demonstrated in the proof of Theorem 3.3 in [18]. Examples where the quasi Bakry–Émery estimate is satisfied include: Riemannian manifolds with non negative Ricci curvature (in that case $C = 1$, see [95]), some metric graphs like the Walsh spider (see [18, Example 5.1] and also [18, Theorem 5.4]), the Heisenberg group and more generally H-type groups (see [12,30]).

The quasi Bakry–Émery curvature condition, while stronger than the weak Bakry–Émery curvature condition (9), does not explicitly consider any dimension. In fact, it is weaker than a standard condition called the Bakry–Émery condition $BE(0, \infty)$ in strongly local Dirichlet spaces, see [84, Definition 3.1] and also [6,31]. The $BE(0, \infty)$ condition is said to be satisfied if, in the weak sense of Definition 3.1 in [84], the Bochner-inequality

$$\frac{1}{2} [L\Gamma(f, f) - 2\Gamma(f, Lf)] \geq 0,$$

holds, where $L$ is the infinitesimal generator associated with the Dirichlet form. Under some regularity condition, by [84, Corollary 3.5] the latter implies the gradient bound $|\nabla P_t u| \leq P_t |\nabla u|$, i.e. (10) with $C = 1$. Thus, in the setting of this paper, the quasi Bakry–Émery curvature condition (10) is indeed weaker than $BE(0, \infty)$.

In the context of RCD spaces, the relation between the conditions RCD($0, \infty$) and $BE(0, \infty)$ is discussed in detail in [5,10,37].

### 3 BV class and co-area formula

In this section we use the Dirichlet form and the associated family $\Gamma(\cdot, \cdot)$ of measures to construct a BV class of functions on $X$. To do so, we only need $\mu$ to be a doubling measure on $X$ and the class of locally Lipschitz functions to be dense in $L^1(X)$. So in this section we do not need the 2-Poincaré inequality nor do we need the weak Bakry–Émery curvature condition. In the second part of the section we prove a co-area formula for BV functions; such a co-area formula is highly useful in understanding the structure of BV functions, and underscores the importance of studying sets of finite perimeter (sets whose characteristic functions are BV functions).

#### 3.1 BV class

In this subsection we will construct a BV class based on Dirichlet forms. The motivation for this definition comes from the work of Miranda [77]. In particular, in the context of a doubling metric measure space $(X, d, \mu)$ supporting a 1-Poincaré inequality, where the Dirichlet form is given in terms of a Cheeger differential structure (see Example 2.2), the construction of $BV(X)$ and $\|Df\|$ is exactly that in [77]. It is also proved in [77] that this construction yields the usual notion of variation when applied to Riemannian or sub-Riemannian spaces.

We set the core of the Dirichlet form, $\mathcal{C}(X)$, to be the class of all $f \in \mathcal{F}_{loc}(X) \cap C(X)$ such that $\Gamma(f, f) \ll \mu$ and recall that the Sobolev class $W^{1,1}(X)$ is the class of all $f \in \mathcal{F}_{loc}(X) \cap L^1(X)$ for which $\Gamma(f, f) \ll \mu$ and $|\nabla f| \in L^1(X)$ (see Definition (6)).

**Definition 3.1** We say that $u \in L^1(X)$ is in $BV(X)$ if there is a sequence of local Lipschitz functions $u_k \in L^1(X)$ such that $u_k \to u$ in $L^1(X)$ and

$$\liminf_{k \to \infty} \int_X |\nabla u_k| \, d\mu < \infty.$$
We note that if the Dirichlet form supports a 1-Poincaré inequality, then the Sobolev space $W^{1,1}(X)$ is a subspace of $BV(X)$. 

**Definition 3.2** For $u \in BV(X)$ and open sets $U \subset X$, we set 

$$\|Du\|(U) = \inf_{u_k \in C(U), u_k \to u \text{ in } L^1(U)} \liminf_{k \to \infty} \int_U |\nabla u_k| \, d\mu.$$ 

We will see in the next part of this section that $\|Du\|$ can be extended from the collection of open sets to the collection of all Borel sets as a Radon measure, see Definition 3.5. The following lemmas are standard: the first can be proved by applying the Leibniz rule to the approximations $\eta u_k + (1 - \eta)v_k$ with $u_k$ and $v_k$ the sequences of functions from $\mathcal{F}$ that approximate $u$ and $v$, and the non-trivial part of the second is a partitioning argument.

**Lemma 3.3** If $u, v \in BV(X)$ and $\eta$ is a Lipschitz continuous function on $X$ with $0 \leq \eta \leq 1$ on $X$, then $\eta u + (1 - \eta)v \in BV(X)$ with 

$$\|D(\eta u + (1 - \eta)v)\|(X) \leq \|Du\|(X) + \|Dv\|(X) + \int_X |u - v| |\nabla \eta| \, d\mu.$$ 

**Lemma 3.4** Let $U$ and $V$ be two open subsets of $X$. If $u \in BV(X)$, then 

1. $\|Du\|({\emptyset}) = 0,$ 
2. $\|Du\|(U) \leq \|Du\|(V)$ if $U \subset V,$ 
3. $\|Du\|(\bigcup_i U_i) = \sum_i \|Du\|(U_i)$ if $\{U_i\}_i$ is a pairwise disjoint subfamily of open subsets of $X.$

We use the above definition of $\|Du\|$ on open sets in a Carathéodory construction.

**Definition 3.5** For $A \subset X$, we set 

$$\|Du\|^*(A) := \inf\{\|Du\|(O) : O \text{ is an open subset of } X, A \subset O\}.$$ 

By Lemma 3.4 $\|Du\|^*(A) = \|Du\|(A)$ when $A$ is open; abusing notation we re-name $\|Du\|^*(A)$ as $\|Du\|(A)$ for general $A$.

The main result of this section is that $\|Du\|$, as constructed above, is a Radon measure on $X$. The proof may be directly adapted from that for [77, Theorem 3.4], and relies on a lemma of De Giorgi and Letta [28, Theorem 5.1], see also [7, Theorem 1.53].

**Theorem 3.6** If $f \in BV(X)$, then $\|Df\|$ is a Radon outer measure on $X$ and the restriction of $\|Df\|$ to the Borel sigma algebra is a Radon measure which is the weak limit of $\|Duk\|$ for some sequence $uk$ of locally Lipschitz functions in $L^1(X)$ such that $uk \to f$ in $L^1(X)$.

**Example 3.7** There is a large class of fractal examples [59,60,70,92] with resistance forms $\mathcal{E}$, a so-called Kusuoka measure $\mu$, and a base of open sets $O$ with finite boundaries, such that $1_O \in BV(X)$ and $\|D1_O\|$ is absolutely continuous with respect to the counting measure on $\partial O$. Among these examples, the most notable are the Sierpinski gasket in harmonic coordinates [34,54,55,61,68,76,91] and fractal quantum graphs [4]. On the Sierpinski gasket [91, Proposition 4.14] shows how to make computations at the dense set of junction points. One might expect that if $u \in BV(X)$ then, following [46,47], $Du$ could be defined as a vector valued Borel measure, however the details of this construction are outside of the scope of this article. The long term motivation for this type of analysis comes from stochastic PDEs, see [13,49,56,57,78] and the references therein.
3.2 Co-area formula

We give a co-area formula that connects the BV energy seminorm of a BV function with the perimeter measure of its super-level sets.

**Definition 3.8** A function $u$ is said to be in $ BV_{\text{loc}}(X) $ if for each bounded open set $ U \subset X $ there is a compactly supported Lipschitz function $ \eta_U $ on $ X $ such that $ \eta_U = 1 $ on $ U $ and $ \eta_U u \in BV(X) $. We say that a measurable set $ E \subset X $ is of finite perimeter if $ 1_E \in BV_{\text{loc}}(X) $ with $ \| D1_E \| (X) < \infty $. For any Borel set $ A \subset X $, we denote by $ P(E, A) := \| D1_E \| (A) $ the perimeter measure of $ E $.

The proof of the following theorem is a direct adaptation of the corresponding result for the BV theory found in [77, Proposition 4.2].

**Theorem 3.9** The co-area formula holds true, that is, for Borel sets $ A \subset X $ and $ u \in L^1_{\text{loc}}(X) $,

$$ \| Du \| (A) = \int_\mathbb{R} P(\{ u > s \}, A) \, ds. $$

4 BV, Sobolev and heat semigroup-based Besov classes

Throughout the section, let $ (X, \mu, \mathcal{E}, \mathcal{F}, d_E) $ be a strictly local regular Dirichlet space that satisfies the general assumptions of Sect. 2 and the 2-Poincaré inequality. We stress that the 1-Poincaré inequality is not assumed.

4.1 Heat semigroup-based Besov classes

We first turn our attention to the study of Besov classes. In [2] the following heat semigroup-based Besov classes were introduced.

**Definition 4.1** ([2]) Let $ p \geq 1 $ and $ \alpha \geq 0 $. For $ f \in L^p(X) $, we define the Besov seminorm:

$$ \| f \|_{p,\alpha} = \sup_{t > 0} t^{-\alpha} \left( \int_X \int_X p_t(x, y) | f(x) - f(y) |^p d\mu(x) d\mu(y) \right)^{1/p}, $$

and the Besov spaces

$$ B^{p,\alpha}(X) = \{ f \in L^p(X) : \| f \|_{p,\alpha} < +\infty \}. \quad (11) $$

The norm on $ B^{p,\alpha}(X) $ is defined as:

$$ \| f \|_{B^{p,\alpha}(X)} = \| f \|_{L^p(X)} + \| f \|_{p,\alpha}. $$

It is proved in Proposition 4.14 and Corollary 4.16 of [2] that $ B^{p,\alpha}(X) $ is a Banach space for $ p \geq 1 $ and that it is reflexive for $ p > 1 $. In this section, we compare the spaces $ B^{p,\alpha}(X) $ to more classical notions of Besov classes that have previously been considered in the metric setting.

We recall the following definition from [38]. For $ 0 \leq \alpha < \infty $, $ 1 \leq p < \infty $ and $ p < q \leq \infty $, let $ B^{\alpha}_{p,q}(X) $ be the collection of functions $ u \in L^p(X) $ for which, if $ q < \infty $,

$$ \| u \|_{B^{\alpha}_{p,q}(X)} := \left( \int_0^\infty \left( \int_X \int_{B(x,t)} \frac{|u(y) - u(x)|^p}{t^{\alpha p} \mu(B(x,t))} d\mu(y) d\mu(x) \right)^{q/p} \frac{dt}{t} \right)^{1/q} < \infty. \quad (12) $$
and in the case \( q = \infty \)

\[
\|u\|_{B^p_{p,\infty}(X)} := \sup_{t > 0} \left( \int_X \int_{B(x,t)} \frac{|u(y) - u(x)|^p}{t^{ap} \mu(B(x,t))} \, d\mu(y) \, d\mu(x) \right)^{1/p} < \infty.
\]  

(13)

**Proposition 4.2** For \( 1 \leq p < \infty \) and \( 0 < \alpha < \infty \) we have

\[
B^{p,\alpha/2}(X) = B^\alpha_{p,\infty}(X),
\]

with equivalent seminorms.

**Proof** Since \( \mu \) is doubling and supports a 2-Poincaré inequality, we have the Gaussian double bounds (8) for \( p_t(x,y) \). Hence if \( u \in B^{p,\alpha}(X) \), we then must have

\[
\|u\|_{p,\alpha/2}^p \geq C^{-1} \sup_{t > 0} \int_X \int_X \frac{|u(y) - u(x)|^p}{t^{ap/2}} \frac{e^{-c d(x,y)^2/t}}{\mu(B(x,\sqrt{t}))} \, d\mu(y) \, d\mu(x)
\]

\[
\geq C^{-1} \sup_{\sqrt{t} > 0} \int_X \int_{B(x,\sqrt{t})} \frac{|u(y) - u(x)|^p}{t^{ap/2}} \frac{e^{-c d(x,y)^2/t}}{\mu(B(x,\sqrt{t}))} \, d\mu(y) \, d\mu(x)
\]

\[
\geq C^{-1} \sup_{\sqrt{t} > 0} \int_X \int_{B(x,\sqrt{t})} |u(y) - u(x)|^p \frac{1}{t^{ap/2} \mu(B(x,\sqrt{t}))} \, d\mu(y) \, d\mu(x)
\]

\[
= C^{-1} \|u\|_{B^p_{p,\infty}(X)}^p,
\]

and from this it follows that \( B^{p,\alpha/2}(X) \) embeds boundedly into \( B^\alpha_{p,\infty}(X) \).

Now we focus on proving the converse embedding. From (4) and (8), we have

\[
\frac{1}{t^{ap/2}} \int_X \int_X |u(y) - u(x)|^p p_t(x,y) \, d\mu(y) \, d\mu(x)
\]

\[
\leq \frac{C}{t^{ap/2}} \int_X \sum_{i = -\infty}^\infty \int_{B(x,2^i \sqrt{t}) \setminus B(x,2^{i-1} \sqrt{t})} \frac{|u(y) - u(x)|^p}{\mu(B(x,\sqrt{t}))} e^{-c d(x,y)^2/t} \, d\mu(y) \, d\mu(x)
\]

\[
\leq \frac{C}{t^{ap/2}} \int_X \sum_{i = -\infty}^\infty \int_{B(x,2^i \sqrt{t}) \setminus B(x,2^{i-1} \sqrt{t})} \frac{|u(y) - u(x)|^p}{\mu(B(x,2^i \sqrt{t}))} e^{-c d(x,y)^2/t} \, d\mu(y) \, d\mu(x)
\]

\[
\leq \frac{C}{t^{ap/2}} \sum_{i = -\infty}^\infty e^{-c d(x,y)^2/t} \mu(B(x,2^i \sqrt{t})) \sum_{i=-\infty}^\infty \int_{B(x,2^i \sqrt{t})} \int_{B(x,2^i \sqrt{t})} \frac{|u(y) - u(x)|^p}{\mu(B(x,2^i \sqrt{t}))} \, d\mu(y) \, d\mu(x)
\]

\[
\leq C \|u\|_{B^p_{p,\infty}(X)}^p \sum_{i = -\infty}^\infty e^{-c d(x,y)^2/t} \mu(B(x,2^i \sqrt{t})) \leq C \|u\|_{B^p_{p,\infty}(X)}^p \sum_{i = -\infty}^\infty e^{-c d(x,y)^2/t} \mu(B(x,2^i \sqrt{t})) = C \|u\|_{B^p_{p,\infty}(X)}^p.
\]

Since

\[
\sum_{i = -\infty}^\infty e^{-c d(x,y)^2/t} \mu(B(x,2^i \sqrt{t})) \leq \sum_{i \in \mathbb{N}} e^{-c d(x,y)^2/t} \mu(B(x,2^i \sqrt{t})) + \sum_{i = 0}^{\infty} 2^{-iap} < \infty
\]

the desired bound follows. \( \square \)

### 4.2 Under the weak Bakry–Émery condition, \( B^{1,1/2}(X) = BV(X) \)

Recall from Definition 2.1 that \( u \in F_{\text{loc}}(X) \) if for each ball \( B \) in \( X \) there is a compactly supported Lipschitz function \( \varphi \) with \( \varphi = 1 \) on \( B \) such that \( u\varphi \in F \); in this case we can set \( |\nabla u| = |\nabla (u\varphi)| \) in \( B \), thanks to the strict locality property of \( E \).
Lemma 4.3 Suppose that the weak Bakry–Émery condition (9) holds. Then for \( u \in \mathcal{F} \cap W^{1,1}(X) \), we have
\[
\|P_t u - u\|_{L^1(X)} \leq C \sqrt{t} \int_X |\nabla u| \, d\mu.
\]
Hence, if \( u \in BV(X) \), then
\[
\|P_t u - u\|_{L^1(X)} \leq C \sqrt{t} \|Du\|(X).
\]

Proof To see the first part of the claim, we note that for each \( x \in X \) and \( s > 0 \), \( \frac{\partial}{\partial s} P_s u(x) \) exists, and so by the fundamental theorem of calculus, for \( 0 < \tau < t \) and \( x \in X \),
\[
P_t u(x) - P_\tau u(x) = \int_\tau^t \frac{\partial}{\partial s} P_s u(x) \, ds.
\]
Thus for each compactly supported function \( \varphi \in \mathcal{F} \cap L^\infty(X) \), by the facts that \( P_t u \) satisfies the heat equation and that \( P_s \) is a symmetric operator for each \( s > 0 \),
\[
\left| \int_X \varphi(x)[P_t u(x) - P_\tau u(x)] \, d\mu(x) \right| = \left| -\int_X \int_\tau^t \varphi(x) \frac{\partial}{\partial s} P_s u(x) \, ds \, d\mu(x) \right|
\]
\[
= \left| \int_\tau^t \int_X \frac{\partial}{\partial s} P_s u(x) \, ds \, d\mu(x) \right|
\]
\[
= \left| \int_\tau^t \int_X d\Gamma(\varphi, P_s u(x)) \, ds \right|
\]
\[
\leq \int_\tau^t \int_X |\nabla P_s \varphi| |\nabla u| \, d\mu \, ds
\]
\[
\leq \int_\tau^t \|\nabla P_s \varphi\|_{L^\infty(X)} \int_X |\nabla u| \, d\mu \, ds\quad
\]
An application of (9) gives
\[
\left| \int_X \varphi(x)[P_t u(x) - P_\tau u(x)] \, d\mu(x) \right| \leq C \|\varphi\|_{L^\infty(X)} \int_\tau^t \frac{1}{\sqrt{s}} ds \int_X |\nabla u| \, d\mu
\]
\[
\leq C \frac{1}{\sqrt{t}} \|\varphi\|_{L^\infty(X)} \int_X |\nabla u| \, d\mu.
\]
As the above holds for all compactly supported \( \varphi \in \mathcal{F} \cap L^\infty(X) \), we obtain
\[
\|P_t u - P_\tau u\|_{L^1(X)} \leq C \frac{1}{\sqrt{t}} \int_X |\nabla u| \, d\mu.
\]
Now by the fact that \( \{P_t\}_{t \in [0,\infty)} \) has an extension as a contraction semigroup to \( L^1(X) \) such that \( P_\tau u \to u \) as \( \tau \to 0^+ \) in \( L^1(X) \) (see [2, Section 2.2]), we have
\[
\|P_t u - u\|_{L^1(X)} \leq C \frac{1}{\sqrt{t}} \int_X |\nabla u| \, d\mu.
\]
Finally, if \( u \in BV(X) \), then we can find a sequence \( u_k \in \mathcal{F} \cap W^{1,1}(X) \) such that \( u_k \to u \) in \( L^1(X) \) and \( \lim_{k \to \infty} \int_X |\nabla u_k| \, d\mu = \|Du\|(X) \). By the contraction property of \( P_t \) on \( L^1(X) \), we have
\[
\|P_t u - u\|_{L^1(X)} \leq \|P_t (u - u_k)\|_{L^1(X)} + \|P_t u_k - u_k\|_{L^1(X)} + \|u_k - u\|_{L^1(X)}
\]
\[ \leq C\|u - u_k\|_{L^1(X)} + C\sqrt{t}\int_X |\nabla u_k| \, d\mu + \|u - u_k\|_{L^1(X)}. \]

Letting \( k \to \infty \) concludes the proof. \qed

Note from the results of [74, Theorem 4.1] that if the measure \( \mu \) is doubling and supports a 1-Poincaré inequality, then a measurable set \( E \subset X \) is in the BV class if

\[ \liminf_{t \to 0^+} \frac{1}{\sqrt{t}} \int_{E \setminus \gamma \setminus E} P_t 1_E \, d\mu < \infty. \]

Here \( E^\varepsilon = \bigcup_{x \in E} B(x, \varepsilon) \). Note that by the symmetry and conservativeness of the operator \( P_t \),

\[ \int_X |P_t 1_E - 1_E| \, d\mu = \int_E (1 - P_t 1_E) \, d\mu + \int_{X \setminus E} P_t 1_E \, d\mu \]

\[ = \int_X 1_E (1 - P_t 1_E) \, d\mu + \int_{X \setminus E} P_t 1_E \, d\mu \]

\[ = \int_X (P_t 1_E) 1_{X \setminus E} \, d\mu + \int_{X \setminus E} P_t 1_E \, d\mu = 2 \int_{X \setminus E} P_t 1_E \, d\mu. \]

Therefore,

\[ \int_{E \setminus \gamma \setminus E} P_t 1_E \, d\mu \leq \int_{X \setminus E} P_t 1_E \, d\mu = \frac{1}{2} \|P_t 1_E - 1_E\|_{L^1(X)}. \]

Thus if \( \mu \) is doubling and supports a 1-Poincaré inequality, and in addition

\[ \sup_{t > 0} \frac{1}{\sqrt{t}} \|P_t 1_E - 1_E\|_{L^1(X)} < \infty, \]

then \( E \) is of finite perimeter. In our framework, those results coming from [74] can not be used, since we do not assume the 1-Poincaré inequality. Instead we prove the following theorem, which is the main result of the section.

**Theorem 4.4** If the weak Bakry–Émery condition (9) holds, then \( B^{1,1/2}(X) = BV(X) \) with comparable seminorms. Moreover, there exist constants \( c, C > 0 \) such that for every \( u \in BV(X) \),

\[ c \limsup_{s \to 0} s^{-1/2} \int_X P_s (|u - u(y)|) (y) \, d\mu(y) \leq \|Du\|_{(X)} \]

\[ \leq C \liminf_{s \to 0} s^{-1/2} \int_X P_s (|u - u(y)|) (y) \, d\mu(y). \]

**Proof** First we assume that \( u \in BV(X) \). We can assume \( u \geq 0 \) a.e. We know that for almost every \( t \geq 0 \) the set \( E_t \) is of finite perimeter, where

\[ E_t = \{x \in X : u(x) > t\}, \]

and by the co-area formula for BV functions (see Theorem 3.9),

\[ \|Du\|_{(X)} = \int_0^{+\infty} \|D1_{E_t}\|_{(X)} \, dt. \]

For such \( t \), by Lemma 4.3 we know that

\[ \sup_{s > 0} \frac{1}{\sqrt{s}} \int_X |P_s 1_{E_t}(x) - 1_{E_t}(x)| \, d\mu(x) \leq C \|D1_{E_t}\|_{(X)} {\cdot} \]

\[ \square \] Springer
Now, setting $A = \{(x, y) \in X \times X : u(x) < u(y)\}$, we have for $s > 0$,

$$\int_X \int_X p_s(x, y) |u(x) - u(y)| \, d\mu(x) \, d\mu(y)$$

$$= 2 \int_{A} p_s(x, y) |u(x) - u(y)| \, d\mu(x) \, d\mu(y)$$

$$= 2 \int_{A} \int_{u(x)}^{u(y)} p_s(x, y) \, dt \, d\mu(x) \, d\mu(y)$$

$$= 2 \int_{X} \int_{X} \int_{0}^{+\infty} 1_{[u(x), u(y)]}(t) 1_{A}(x, y) \, p_s(x, y) \, dt \, d\mu(x) \, d\mu(y)$$

$$= 2 \int_{0}^{+\infty} \int_{X} \int_{X} 1_{E_t}(y) [1 - 1_{E_t}(x)] \, p_s(x, y) \, d\mu(x) \, d\mu(y) \, dt$$

$$= 2 \int_{0}^{+\infty} \int_{X \setminus E_t} P_s 1_{E_t}(x) \, d\mu(x) \, dt.$$

Observe that

$$\int_{X \setminus E_t} P_s 1_{E_t}(x) \, d\mu(x) = \int_{X \setminus E_t} |P_s 1_{E_t}(x) - 1_{E_t}(x)| \, d\mu(x) \leq \int_X |P_s 1_{E_t}(x) - 1_{E_t}(x)| \, d\mu(x).$$

Therefore we obtain

$$\int_{X} \int_{X} p_s(x, y) |u(x) - u(y)| \, d\mu(x) \, d\mu(y) \leq 2 \int_{0}^{+\infty} \|P_s 1_{E_t} - 1_{E_t}\|_{L^1(X)} \, dt.$$

An application of Lemma 4.3 now gives

$$\int_{X} \int_{X} p_s(x, y) |u(x) - u(y)| \, d\mu(x) \, d\mu(y) \leq C \sqrt{s} \int_{0}^{+\infty} \|D1_{E_t}\|(X) \, dt,$$

whence with the help of the co-area formula we obtain

$$\|u\|_{1, 1/2} \leq C \|Du\|(X),$$

that is, $u \in B^{1, 1/2}(X)$. Thus $BV(X) \subset B^{1, 1/2}(X)$ boundedly.

Now we show that $B^{1, 1/2}(X) \subset BV(X)$. This inclusion holds even when $\mathcal{E}$ does not support a Bakry--Émery curvature condition; only a 2-Poincaré inequality and the doubling condition on $\mu$ are needed.

Set $\Delta_\varepsilon = \{(x, y) \in X : d(x, y) < \varepsilon\}$ for some $\varepsilon > 0$. Suppose that $u \in B^{1, 1/2}(X)$. By (8), we have a Gaussian lower bound for the heat kernel:

$$p_t(x, y) \geq \frac{e^{-c \frac{d(x, y)^2}{t}}}{C\mu(B(x, \sqrt{t}))}.$$

Therefore for any $t > 0$ we get

$$\frac{1}{\sqrt{t}} \int_{X} \int_{X} p_t(x, y) |u(x) - u(y)| \, d\mu(x) \, d\mu(y)$$

$$\geq \frac{1}{\sqrt{t}} \int_{X} \int_{X} \frac{e^{-c \frac{d(x, y)^2}{t}}}{C\mu(B(x, \sqrt{t}))} |u(y) - u(x)| \, d\mu(y) \, d\mu(x).$$
Remark 4.5 As a byproduct of this proof, we also obtain that there exists a constant $H$ Poincaré inequality, that if $\|u\|_{1,1/2}$. It is known, see [64, Proposition 6.3], even without the assumption that $u \in BV(X)$. We point out here that although Theorem 3.1 in [74] assumes that $X$ supports a 1-Poincaré inequality, the second part of the proof does not need this assumption. In fact, the argument using discrete convolution there is valid also in our setting. It is this second part of the proof that we referred to above. We then obtain

$$\liminf_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int \int_{\Delta_{\varepsilon}} \frac{|u(y) - u(x)|}{\mu(B(x, \varepsilon))} d\mu(x) d\mu(y) < \infty. \quad (14)$$

Now an argument as in the second half of the proof of [74, Theorem 3.1] tells us that $u \in BV(X)$. It follows that $\|Du\|(X) \leq \liminf_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int \int_{\Delta_{\varepsilon}} \frac{|u(y) - u(x)|}{\mu(B(x, \varepsilon))} d\mu(x) d\mu(y) \leq \|u\|_{1,1/2}$.

$\square$

Remark 4.6 Another application of Proposition 4.2 is the following. It is in general not true that for every $u \in BV(X)$,

$$\sup_{\varepsilon > 0} \frac{1}{\varepsilon} \int \int_{\Delta_{\varepsilon}} \frac{|u(y) - u(x)|}{\mu(B(x, \varepsilon))} d\mu(x) d\mu(y) \leq \liminf_{\varepsilon \to 0^+} \frac{C}{\varepsilon} \int \int_{\Delta_{\varepsilon}} \frac{|u(y) - u(x)|}{\mu(B(x, \varepsilon))} d\mu(x) d\mu(y)$$

because both sides are comparable to $\|Du\|(X)$. Indeed, the fact that $\|Du\|(X)$ is dominated by the right hand side is directly from Theorem 4.4, which, together with Proposition 4.2 (the metric characterization of Besov spaces), implies that the left hand side can be bounded by $\|Du\|(X)$. This property of the metric measure space $(X, d_\varepsilon, \mu)$ can be viewed as an interesting consequence of the weak Bakry–Émery estimate.

4.3 Sets of finite perimeter

We introduce some notions from the paper of Ambrosio [8]. Given $A \subset X$ we set

$$\mathcal{H}(A) := \liminf_{\varepsilon \to 0^+} \left\{ \sum_i \frac{\mu(B_i)}{\text{rad}(B_i)} : A \subset \bigcup_i B_i, \text{ and } \forall i, \text{ rad}(B_i) < \varepsilon \right\}.$$ 

It is known, see [64, Proposition 6.3], even without the assumption that $X$ supports a 2-Poincaré inequality, that if $\mathcal{H}(\partial E) < \infty$, then $E$ is of finite perimeter.

Now let $E \subset X$ be a set of finite perimeter and define the measure-theoretic boundary by

$$\partial_{\mu}E = \left\{ x \in X : \limsup_{r \to 0^+} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > 0, \limsup_{r \to 0^+} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > 0 \right\}.$$
For \( \alpha \in (0, 1/2) \), define also

\[
\partial_\alpha E = \left\{ x \in X : \liminf_{r \to 0^+} \min \left\{ \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))}, \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} \right\} > \alpha \right\}.
\]

If \( X \) supports a 1-Poincaré inequality then, by the results of \([8, Theorems 5.3, 5.4]\), there is \( \gamma > 0 \) such that \( \mathcal{H}(\partial_\gamma E) = 0 \). Moreover, \( \mathcal{H}(\partial_\gamma E) < \infty \), and \( P(E, \cdot) \ll \mathcal{H}|_{\partial_\gamma E} \) with Radon-Nikodym derivative bounded below by some \( \delta > 0 \). Both \( \gamma \) and \( \delta \) depend solely on the doubling and the 1-Poincaré constants.

We are not assuming \( X \) supports a 1-Poincaré inequality, but only that \( \mu \) is doubling and \( X \) supports a 2-Poincaré inequality, in which case we have the following bound.

**Proposition 4.7** Suppose that \( E \subset X \) with \( \|1_E\|_{B^{1,1/2}(X)} < \infty \). Then for all \( 0 < \alpha < 1 \), we have \( \mathcal{H}(\partial_\alpha E) \leq \frac{C}{\alpha} P(E, X) \).

**Remark 4.8** According to \([71]\), if \( (X, d_E, \mu) \) is doubling and supports a 1-Poincaré inequality, then there is \( \alpha \), depending solely on the doubling and the Poincaré constants, such that finiteness of \( \mathcal{H}(\partial_\alpha E) \) implies that \( P(E, X) \) is finite and \( \mathcal{H}(\partial_\gamma E \setminus \partial_\alpha E) = 0 \).

**Proof** For \( r_0 > 0 \) and \( 0 < \alpha \leq 1/2 \) let

\[
\partial_\alpha^{r_0} E = \left\{ x \in X : \min \left\{ \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))}, \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} \right\} > \alpha \text{ for all } 0 < r \leq r_0 \right\}.
\]

\[
\Phi_{E, r_0} := \inf_{r \in \mathbb{Q} \cap (0, r_0]} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))}.
\]

Since \( \mu \) is Borel regular we know \( x \mapsto \mu(B(x, r) \cap E) \) is lower semicontinuous, so \( \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} \) and \( \Phi_{E, r_0} \) are Borel functions. It follows that the sets \( \partial_\alpha^{r_0} E \) are Borel, and we conclude by writing \( \partial_\alpha^{r_0} E = \bigcup_{\gamma \in [0,1]} \partial_\gamma E \) both that \( \partial_\alpha^{r_0} E \) is Borel and, by continuity of measure, that we need only prove \( \mathcal{H}(\partial_\gamma E) \leq \frac{C}{\alpha} P(E, X) \) for each \( r_0 \).

Using Theorem 4.4, if \( 1_E \in BV(X) \) then

\[
\sup_{r > 0} \frac{1}{\sqrt{t}} \int_X \int_X p_t(x, y) |1_E(x) - 1_E(y)| d\mu(x) d\mu(y) \leq C \|D1_E\|(X) = C P(E, X).
\]

Fix \( t < (r_0/3)^2 \leq 1/36 \) and let \( \{B_i\} \) be a maximally separated \( \sqrt{t} \)-covering of \( \partial_\gamma^{r_0} E \), so the balls \( 5B_i \) have bounded overlap (see Sect. 2.2). Observe that for \( x, y \in B_i \) the Gaussian lower bound for \( p_t(x, y) \) in (7) becomes \( p_t(x, y) \geq C \mu(B_i)^{-1} \). (In this calculation \( C \) denotes various constants that can change even within an expression, but depend only on the doubling and Poincaré constants of the space.) Thus

\[
\sqrt{t} P(E, X) \geq C \sum_i \int_{B_i \cap E} \int_{B_i \setminus E} p_t(x, y) d\mu(x) d\mu(y)
\]

\[
\geq C \sum_i \frac{1}{\mu(B_i)} \int_{B_i \cap E} \int_{B_i \setminus E} d\mu(x) d\mu(y)
\]

\[
\geq C \sum_i \frac{\mu(B_i \cap E)}{\mu(B_i)^2} \mu(B_i \setminus E) \mu(B_i) \geq C \alpha \sum_i \mu(B_i)\]
where the last inequality uses that at least one of $\mu(B_i \cap E)$ and $\mu(B_i \setminus E)$ is larger than $\mu(B_i)/2$ and the other is bounded below by $\alpha \mu(B_i)$ on $\partial_0^\alpha E$. However each $B_i$ has radius $\sqrt{t}$, so

$$P(E, X) \geq C \alpha \sum_i \frac{\mu(B_i)}{\text{rad}(B_i)}$$

and thus $P(E, X) \geq C \alpha \mathcal{H}(\partial_0^\alpha E)$, completing the proof. \qed

Proposition 4.7 gives us a way to control, from above, the $\mathcal{H}$-measure of $\partial_\alpha E$ for a set $E$ of finite perimeter. This should be contrasted with the following lower bound on the co-dimension 1 Minkowski measure of $\partial E$. For a set $A \subset X$, the co-dimension 1-Minkowski measure of $A$ is defined to be

$$M^{-1}(A) := \liminf_{\varepsilon \to 0^+} \frac{\mu(A_\varepsilon)}{\varepsilon},$$

where $A_\varepsilon = \bigcup_{x \in A} B(x, \varepsilon)$.

**Proposition 4.9** We have for a set $E$ of finite perimeter that

$$P(E, X) \leq M^{-1}(\partial E).$$

**Proof** We can assume $M^{-1}(\partial E) < \infty$. For each $\varepsilon > 0$, consider

$$u_\varepsilon(x) = \min\{1, \varepsilon^{-1} \text{dist}_{dE}(x, X \setminus A_\varepsilon)\},$$

where $\text{dist}_{dE}(x, A) = \inf\{|dE(x, y) : y \in A\}$ and $A = \partial E$. Lemma 2.4 together with the fact that $u_\varepsilon \to \chi_E$ as $\varepsilon \to 0^+$ give the desired result. \qed

### 4.4 Under the quasi Bakry–Émery condition, $B^{p,1/2}(X) = W^{1,p}(X)$ for $p > 1$

In this section we compare the Besov and Sobolev seminorms for $p > 1$. The case $p = 1$ was studied in detail in Sect. 4.2. Our main theorem in this section is the following:

**Theorem 4.10** Suppose that the quasi Bakry–Émery condition (10) holds. Then, for every $p > 1$, $B^{p,1/2}(X) = W^{1,p}(X)$ with comparable norms.

We will divide the proof of Theorem 4.10 in two parts. In the first part, Theorem 4.12, we prove that $B^{p,1/2}(X) \subset W^{1,p}(X)$. As we will see, this inclusion does not require the quasi Bakry–Émery condition (10). In the second part, Theorem 4.18 we will prove the inclusion $W^{1,p}(X) \subset B^{p,1/2}(X)$ and, to this end, will use the quasi Bakry–Émery condition. Before turning to the proof, we point out the following corollary regarding the Riesz transform.

**Corollary 4.11** Suppose that the quasi Bakry–Émery condition (10) holds. Let $p > 1$. Then for any $f \in B^{p,1/2}(X) \cap \mathcal{F}$,

$$\|f\|_{p,1/2} \simeq \|\sqrt{-L}f\|_{L^p(X)}.$$

Consequently, $B^{p,1/2}(X) = L^{1/2}_p$, where $L^{1/2}_p$ is the domain of the operator $\sqrt{-L}$ in $L^p(X)$ (see [2, Section 4.6] for the definition).
Proof In view of Theorem 4.10, we have that for any \( f \in B^{p,1/2}(X) \)
\[
\|f\|_{p,1/2} \simeq \|\nabla f\|_{L^p(X)}.
\]
On the other hand, it follows from [11, Theorem 1.4] that for any \( f \in \mathcal{L}_p^{1/2}, \)
\[
\|\sqrt{-L}f\|_{L^p(X)} \simeq \|\nabla f\|_{L^p(X)}.
\]
We conclude the proof by combining the above two facts. \( \square \)

4.4.1 \( B^{p,1/2}(X) \subset W^{1,p}(X) \)

**Theorem 4.12** Let \( p > 1. \) There exists a constant \( C > 0 \) such that for every \( u \in B^{p,1/2}(X), \)
\[
\|\nabla u\|_{L^p(X)} \leq C\|u\|_{p,1/2}.
\]

**Proof** Let \( u \in B^{p,1/2}(X). \) Then from Proposition 4.2, we see that for each \( \varepsilon > 0, \)
\[
\frac{1}{\varepsilon^{p}} \iint_{\Delta_{\varepsilon}} \frac{|u(x) - u(y)|^p}{\mu(B(x,\varepsilon))} \, d\mu(y) \, d\mu(x) \leq \|u\|_{p,1/2}^p \to < \infty.
\]

Fix \( \varepsilon > 0. \) As in the proof of Lemma 2.11, let \( \{B_j^\varepsilon = B(x_j^\varepsilon, \varepsilon)\}_i \) be a maximally separated \( \varepsilon \)-covering (Definition 2.6 and \( \{\varphi_i^\varepsilon\}_i \) be a \((C/\varepsilon)\)-Lipschitz partition of unity subordinated to this covering. We also set
\[
u_{\varepsilon} := \sum_i u_{B_j^\varepsilon} \varphi_i^\varepsilon.
\]

Then \( \nu_{\varepsilon} \) is locally Lipschitz and hence is in \( \mathcal{F}_{\text{loc}}(X). \) Indeed, for \( x, y \in B_j^\varepsilon \) we see that
\[
|\nu_{\varepsilon}(x) - \nu_{\varepsilon}(y)| \leq \sum_{i:2B_j^\varepsilon \cap 2B_i^\varepsilon \neq \emptyset} |u_{B_j^\varepsilon} - u_{B_i^\varepsilon}| |\varphi_i^\varepsilon(x) - \varphi_i^\varepsilon(y)|
\]
\[
\leq \frac{C \, d(x,y)}{\varepsilon} \sum_{i:2B_j^\varepsilon \cap 2B_i^\varepsilon \neq \emptyset} \left( \int_{2B_j^\varepsilon} \int_{B(x,2\varepsilon)} |u(y) - u(x)|^p \, d\mu(y) \, d\mu(x) \right)^{1/p}.
\]

Therefore, by Lemma 2.4, we see that
\[
|\nabla \nu_{\varepsilon}| \leq \frac{C}{\varepsilon} \sum_{i:2B_j^\varepsilon \cap 2B_i^\varepsilon \neq \emptyset} \left( \int_{2B_j^\varepsilon} \int_{B(x,2\varepsilon)} |u(y) - u(x)|^p \, d\mu(y) \, d\mu(x) \right)^{1/p}
\]
\[
\leq C \left( \int_{2B_j^\varepsilon} \int_{B(x,2\varepsilon)} \frac{|u(y) - u(x)|^p}{\varepsilon^p} \, d\mu(y) \, d\mu(x) \right)^{1/p},
\]
and so by the bounded overlap property of the collection \( 2B_j^\varepsilon, \)
\[
\int_X |\nabla \nu_{\varepsilon}|^p \, d\mu \leq \sum_j \int_{B_j^\varepsilon} |\nabla \nu_{\varepsilon}|^p \, d\mu
\]
\[
\leq C \sum_j \int_{B_j^\varepsilon} \int_{B(x,2\varepsilon)} \frac{|u(y) - u(x)|^p}{\varepsilon^p} \, d\mu(y) \, d\mu(x)
\]
\[
\leq C \int_X \int_{B(x,2\varepsilon)} \frac{|u(y) - u(x)|^p}{\varepsilon^p} \, d\mu(y) \, d\mu(x)
\]
\[ \leq C \frac{1}{\varepsilon^p} \int_{\Delta_{2\varepsilon}} \frac{|u(x) - u(y)|^p}{\mu(B(x, \varepsilon))} d\mu(y) d\mu(x) \leq C \| u \|_{p,1/2}^p. \]

Hence we have
\[ \sup_{\varepsilon > 0} \int_X |\nabla u_\varepsilon|^p d\mu \leq C \| u \|_{p,1/2}^p. \] (15)

In a similar manner, we can also show that
\[ \int_X |u_\varepsilon(x) - u(x)|^p d\mu(x) \leq C\varepsilon^p \int_{\Delta_{2\varepsilon}} \frac{|u(x) - u(y)|^p}{\varepsilon^p \mu(B(x, \varepsilon))} d\mu(y) d\mu(x) \leq C \varepsilon^p \| u \|_{p,1/2}^p, \]
that is, \( u_\varepsilon \to u \) in \( L^p(X) \) as \( \varepsilon \to 0^+ \).

Take a sequence \( \varepsilon_n \to 0^+ \). From (15) and the reflexivity of \( L^p(X) \), there exists a subsequence of \( \{\nabla u_{\varepsilon_n}\}_n \) that is weakly convergent in \( L^p(X) \). By Mazur’s lemma, a sequence of convex combinations of \( u_{\varepsilon_n} \) converges in the norm of \( W^{1,p}(X) \). Since it converges to \( u \) in \( L^p(X) \), we conclude that \( u \in W^{1,p}(X) \) and hence
\[ \| |\nabla u| \|_{L^p(X)} \leq C \| u \|_{p,1/2}. \]

\[ \square \]

### 4.4.2 \( W^{1,p}(X) \subset B^{p,1/2}(X) \)

We now turn to the proof of the upper bound for the Besov seminorm in terms of the Sobolev seminorm and assume that the quasi Bakry–Émery condition (10) holds.

A first important corollary of the quasi Bakry–Émery estimate is the following Hamilton’s type gradient estimate for the heat kernel. This type of estimate is well-known on Riemannian manifolds with non-negative Ricci curvature (see for instance [69]), but is new in our general framework.

**Theorem 4.13** There exists a constant \( C > 0 \) such that for every \( t > 0 \), \( x, y \in X \),
\[ |\nabla_x \ln p_t(x, y)|^2 \leq \frac{C}{t} \left( 1 + \frac{d(x, y)^2}{t} \right). \]

**Proof** The proof proceeds in two steps.

**Step 1:** We first collect a gradient bound for the heat kernel. Observe that (10) implies a weaker \( L^2 \) version as follows
\[ |\nabla P_t u|^2 \leq C P_t (|\nabla u|^2), \]

and hence the following pointwise heat kernel gradient bound (see [11, Lemma 3.3]) holds:
\[ |\nabla_x p_t(x, y)| \leq \frac{C}{\sqrt{t}} \frac{e^{-c d(x,y)^2/t}}{\sqrt{\mu(B(x, \sqrt{t})) \mu(B(y, \sqrt{t}))}}. \]

In particular, we note that \( |\nabla_x p_t(x, \cdot)| \in L^p(X) \) for every \( p \geq 1 \).

**Step 2:** In the second step, we prove a reverse log-Sobolev inequality for the heat kernel. Let \( \tau, \varepsilon > 0 \) and \( x \in X \) be fixed. We denote \( u = p_\tau(x, \cdot) + \varepsilon \). One has, from the chain rule for strictly local forms [35, Lemma 3.2.5],
\[ P_t(u \ln u) - P_t u \ln P_t u = \int_0^t \partial_s (P_s(P_{t-s}u \ln P_{t-s}u)) ds \]
\[
\begin{align*}
&= \int_0^t L P_s (P_{t-s} u \ln P_{t-s} u) - P_s (L (P_{t-s} u \ln P_{t-s} u)) ds
\end{align*}
\]

where the above computations may be justified by using the Gaussian heat kernel estimates for the heat kernel and the Gaussian upper bound for the gradient of the heat kernel obtained in Step 1. In particular, we point out that the commutation mates for the heat kernel and the Gaussian upper bound for the gradient of the heat kernel where the above computations may be justified by using the Gaussian heat kernel estimate, one concludes that

\[
E \leq L P_s (L (P_{t-s} u \ln P_{t-s} u)) - P_s (L (P_{t-s} u \ln P_{t-s} u)) ds
\]

\(\int_0^t (P_s (|\nabla P_{t-s} u|^2)) ds, \quad (16)\)

Our desired inequality follows by rescaling \(t\), adjusting the constant \(C\) and using the symmetry of \(p_t(x,y)\) in \(x\) and \(y\) in the whole above argument. \(\square\)

**Corollary 4.14** Let \(p > 1\). There exists a constant \(C > 0\) such that for every \(u \in L^p(X)\),

\[
|\nabla P_t u| \leq \frac{C}{t^{1/p}} |u|^p.
\]

**Proof** Let \(p > 1, q\) be the conjugate exponent and \(u \in L^p(X)\). One has from Hölder’s inequality

\[
\begin{align*}
|\nabla P_t u|(x) &\leq \int_X |\nabla_x p_t(x,y)||u(y)|d\mu(y) \\
&\leq \left( \int_X \frac{|\nabla_x p_t(x,y)|q}{p_t(x,y)q/p} d\mu(y) \right)^{1/q} (P_t |u|^p)^{1/p} \\
&\leq \left( \int_X |\nabla_x \ln p_t(x,y)|q p_t(x,y)d\mu(y) \right)^{1/q} (P_t |u|^p)^{1/p}.
\end{align*}
\]
The proof follows then from Theorem 4.13 and the Gaussian upper bound for the heat kernel.

Note that by integrating over \( X \) the previous proposition immediately yields:

**Lemma 4.15** Let \( p > 1 \). There exists a constant \( C > 0 \) such that for every \( u \in L^p(X) \)

\[
\| |\nabla P_t u| |^2_{L^p(X)} \leq \frac{C}{t} \| u \|^2_{L^p(X)}.
\]

From this estimate we obtain the following result.

**Lemma 4.16** Let \( p > 1 \). There exists a constant \( C > 0 \) such that for every \( u \in L^p(X) \cap \mathcal{F} \) with \( |\nabla u| \in L^p(X) \)

\[
\| P_t u - u \|_{L^p(X)} \leq C \sqrt{t} \| \nabla u \|_{L^p(X)}.
\]

**Proof** With the previous lemma in hand, the proof is similar to the one in Lemma 4.3, with \( \varphi \) in \( \mathcal{F} \cap L^q(X) \) and compactly supported, where \( p^{-1} + q^{-1} = 1 \). As compactly supported functions in \( \mathcal{F} \cap L^q(X) \) form a dense subclass of \( L^q(X) \) we recover the \( L^p \)-norm of \( P_t u - u \) by taking the supremum over all such \( \varphi \) with \( \int_X |\varphi|^q \, d\mu \leq 1 \).

**Lemma 4.17** Let \( p > 1 \), then for every \( u \in L^p(X) \cap \mathcal{F} \) with \( |\nabla u| \in L^p(X) \)

\[
\left( \int_X \int_X |P_t u(x) - u(y)|^p p_t(x, y) \, d\mu(x) \, d\mu(y) \right)^{1/p} \leq C \sqrt{t} \| \nabla u \|_{L^p(X)}.
\]

**Proof** Let \( u \in L^p(X) \) and \( t > 0 \) be fixed in the above argument. By an application of Fubini’s theorem we have

\[
\left( \int_X \int_X |P_t u(x) - u(y)|^p p_t(x, y) \, d\mu(x) \, d\mu(y) \right)^{1/p} = \left( \int_X P_t(|P_t u(x) - u|^p)(x) \, d\mu(x) \right)^{1/p}.
\]

The main idea now is to adapt the proof of [12, Theorem 6.2]. As above, let \( q \) be the conjugate of \( p \). Let \( x \in X \) be fixed. Let \( g \) be a function in \( L^\infty(X) \) such that \( P_t(|g|^q)(x) \leq 1 \).

We first note that from the chain rule:

\[
\partial_s \left[ P_s((P_{t-s}u)(P_{t-s}g))(x) \right] = LP_s((P_{t-s}u)(P_{t-s}g))(x) - Ps((LP_{t-s}u)(P_{t-s}g))(x) - Ps((P_{t-s}u)(LP_{t-s}g))(x) - Ps((P_{t-s}u)(P_{t-s}g))(x) - 2P_s(\Gamma(P_{t-s}u, P_{t-s}g)).
\]

Therefore we have

\[
P_t((u - P_t u(x))g)(x) = P_t(ug)(x) - P_t u(x) P_t g(x)
\]

\[
= \int_0^t \partial_s \left[ P_s((P_{t-s}u)(P_{t-s}g))(x) \right] \, ds
\]

\[
= 2 \int_0^t P_s \left( \Gamma(P_{t-s}u, P_{t-s}g) \right)(x) \, ds
\]

\[
\leq 2 \int_0^t P_s \left( |\nabla P_{t-s}u||\nabla P_{t-s}g| \right) \, ds
\]

\[
\leq 2 \int_0^t P_s \left( |\nabla P_{t-s}u|^p \right)^{1/p}(x) P_s \left( |\nabla P_{t-s}g|^q \right)^{1/q}(x) \, ds.
\]
Now from the strong Bakry–Émery estimate and Hölder’s inequality we have
\[ P_s \left( |\nabla P_{-s} u|^p \right)^{1/p} (x) \leq C P_s \left( |P_{-s} (|\nabla u|^p)|^{1/p} \right) (x) = C P_t (|\nabla u|^p)^{1/p} (x). \]

On the other hand, Corollary 4.14 gives
\[ P_s \left( |\nabla P_{-s} g|^q \right)^{1/q} (x) \leq \frac{C}{(t-s)^{q/2}} P_t (|g|^q)^{1/q} (x) \leq \frac{C}{(t-s)^{1/2}}. \]

One concludes
\[ P_t((u - P_t u(x))g)(x) \leq C \sqrt{t} P_t (|\nabla u|^p)^{1/p} (x). \]

Thus by \( L^p - L^q \) duality in \( (X, P_t(\cdot, y)\mu(dy)) \), one concludes
\[ P_t(|u - P_t u(x)|^p)^{1/p} \leq C \sqrt{t} P_t (|\nabla u|^p)^{1/p} (x) \]
and finishes the proof by integration over \( X \). 

We are finally in a position to prove the inclusion of the Sobolev space \( W^{1,p}(X) \) into the Besov class \( \mathbf{B}^{p,1/2} \), which in turn completes the proof of Theorem 4.10, which is the main result of this section.

**Theorem 4.18** Let \( p > 1 \). There exists a constant \( C > 0 \) such that for every \( u \in W^{1,p}(X) \),
\[ \|u\|_{p,1/2} \leq C \|\nabla u\|_{L^p(X)}. \]

**Proof** We first assume \( u \in L^p(X) \cap \mathcal{F} \) with \( |\nabla u| \in L^p(X) \). One has
\[
\left( \int_X \int_X |u(x) - u(y)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p} \\
\leq \left( \int_X \int_X |u(x) - P_t u(x)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p} + \left( \int_X \int_X |P_t u(x) - u(y)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p} \\
\leq \|P_t u - u\|_{L^p(X)} + \left( \int_X \int_X |P_t u(x) - u(y)|^p p_t(x, y) d\mu(x) d\mu(y) \right)^{1/p} \\
\leq 2C \sqrt{t} \|\nabla u\|_{L^p(X)},
\]
where in the last step we applied Lemma 4.16 to the first term and Lemma 4.17 to the second term. Thus
\[ \|u\|_{p,1/2} \leq C \|\nabla u\|_{L^p(X)}. \]

Now let \( u \in W^{1,p}(X) \) and choose an increasing sequence of functions \( \phi_n \in C^\infty((0, \infty)) \) such that \( \phi_n \equiv 1 \) on \( [0, n] \), \( \phi_n \equiv 0 \) outside \( [0, 2n] \), and \( |\phi'_n| \leq \frac{2}{n} \). Let \( x_0 \in X \). If \( h_n(x) = \phi_n(d(x_0, x)) \) then \( h_n u \in \mathcal{F} \), \( h_n \not\rightarrow 1 \) on \( X \) as \( n \to \infty \), and \( \|\nabla(h_n u)\|_{L^p(X)} \to \|\nabla u\|_{L^p(X)} \). Taking the limit in the inequality
\[ \|h_n u\|_{p,1/2} \leq C \|\nabla(h_n u)\|_{L^p(X)} \]
yields the result. \( \square \)
4.5 Continuity of $P_t$ in the Besov spaces and critical exponents

We first note the following straightforward continuity property of $P_t$ in the Besov spaces.

**Proposition 4.19** Suppose that the quasi Bakry–Émery condition (10) holds. Let $p > 1$. There exists a constant $C_p > 0$ such that for every $f \in L^p(X)$ and $t > 0$

$$
\|P_t f\|_{p,1/2} \leq \frac{C_p}{t^{1/2}} \|f\|_{L^p(X)}.
$$

**Proof** This is a consequence of Lemma 4.15 and Theorem 4.18. \hfill \Box

**Remark 4.20** The above result is true without the quasi Bakry–Émery condition for $1 < p \leq 2$ on very general Dirichlet spaces, see [2, Theorem 5.1].

For $p \geq 1$, as in [2], we define the $L^p$ Besov density critical exponent $\alpha^*_p(X)$ and triviality critical exponent $\alpha^#_p(X)$ as follows:

$$
\alpha^*_p(X) = \sup \{ \alpha > 0 : B^{p,\alpha}(X) \text{ is dense in } L^p(X) \},
$$

$$
\alpha^#_p(X) = \sup \{ \alpha > 0 : B^{p,\alpha}(X) \text{ contains non-constant functions} \}.
$$

**Theorem 4.21** Suppose that the weak Bakry–Émery condition (9) holds, then for $1 \leq p \leq 2$,

$$
\alpha^*_p(X) = \alpha^#_p(X) = \frac{1}{2}.
$$

Furthermore, if the quasi Bakry–Émery condition (10) holds, then for every $p > 2$,

$$
\alpha^*_p(X) = \alpha^#_p(X) = \frac{1}{2}.
$$

**Proof** Assume that the weak Bakry–Émery condition (9) holds and begin with the case $p = 1$. Let $f \in B^{1,\alpha}(X)$ with $\alpha > 1/2$. Since $B^{1,\alpha}(X) \subset B^{1,1/2}(X) = BV(X)$, we deduce that $f$ is a BV function. Now since $f \in B^{1,\alpha}(X)$, one has for every $t > 0$,

$$
\int_X \int_X p_t(x, y)|f(x) - f(y)|d\mu(x)d\mu(y) \leq t^\alpha \|f\|_{1,\alpha}.
$$

By using the gaussian heat kernel lower bound we obtain

$$
\liminf_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{\Delta_\varepsilon} |f(y) - f(x)|d\mu(x)d\mu(y) = 0,
$$

so $\|Df\|(X) = 0$, and from Remark 4.6 one gets that $f$ is constant. It follows that $\alpha^#_1(X) \leq 1/2$. On the other hand, from Corollary 4.8 in [2], $B^{1,1/2}(X)$ is dense in $L^1(X)$, so $\alpha^*_1(X) = \alpha^#_1(X) = \frac{1}{2}$. From Proposition 5.6 in [2], one has:

1. Both $p \mapsto \alpha^*_p(X)$ and $p \mapsto \alpha^#_p(X)$ are non-increasing;
2. For $1 \leq p \leq 2$ we have $\alpha^*_p(X) \geq \alpha^#_p(X) \geq \frac{1}{2}$.

Therefore, for $1 \leq p \leq 2$ we also have $\alpha^*_p(X) = \alpha^#_p(X) = \frac{1}{2}$.

Now let $p > 2$ and assume the quasi Bakry–Émery condition (10). In that case, according to Proposition 4.19, for every $f \in L^p(X)$ and $t > 0$ one has $P_t f \in B^{p,1/2}(X)$. Thus, $B^{p,1/2}(X)$ is dense in $L^p(X)$ by strong continuity of the semigroup $P_t$ in $L^p(X)$. Hence $\alpha^*_p(X) \geq 1/2$. Using again the fact that both $p \mapsto \alpha^*_p(X)$ and $p \mapsto \alpha^#_p(X)$ are non-increasing and moreover that $\alpha^*_2(X) = \alpha^#_2(X) = \frac{1}{2}$, one concludes that for every $p > 2$, $\alpha^*_p(X) = \alpha^#_p(X) = \frac{1}{2}$. \hfill \Box
5 Sobolev and isoperimetric inequalities

Combining the conclusions in this paper with the results in [2, Section 6], we immediately obtain the following results that generalize the Sobolev embedding theorems from the classical Euclidean setting (see for example [75]) and metric upper gradient setting (see for example [43] and [41]) to the setting of Dirichlet forms and BV functions.

The following proposition is a weak-type version of the standard Sobolev embedding theorem. It gives weak-$L^q$ control of the Besov function $f$, with $q$ the Sobolev conjugate of $p$, and can therefore be used to control the $L^q$-norm of $f$ in terms of the Besov norm of $f$ when $1 \leq s < pQ/(Q - p)$.

**Proposition 5.1** If the volume growth condition $\mu(B(x, r)) \geq C_1 r^Q$, $r \geq 0$, is satisfied for some $Q > 0$ then one has the following weak type Besov space embedding. Let $0 < \delta < Q$ and $1 \leq p < Q/\delta$. Then there exists a constant $C_{p, \delta} > 0$ such that for every $f \in B^{p, \delta/2}(X)$,

$$
\sup_{s \geq 0} s \mu \left( \{x \in X : |f(x)| \geq s\} \right)^{1/\delta} \leq C_{p, \delta} \sup_{r > 0} \frac{1}{r^{\delta + Q/p}} \left( \int \int_{\{x, y\in X \times X : d(x, y) < r\}} |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y) \right)^{1/p},
$$

where $q = \frac{pQ}{Q - p \delta}$. Furthermore, for every $0 < \delta < Q$, there exists a constant $C_{iso, \delta}$ such that for every measurable $E \subset X$, $\mu(E) < +\infty$,

$$
\mu(E)^{\frac{Q - \delta}{Q}} \leq C_{iso, \delta} \sup_{r > 0} \frac{1}{r^{\delta + Q}} (\mu \otimes \mu) \left( (x, y) \in E \times E^c : d(x, y) \leq r \right).
$$

**Proof** From the heat kernel upper bound (7), the volume growth condition $\mu(B(x, r)) \geq C_1 r^Q$, $r \geq 0$, implies the ultracontractive estimate

$$
p_t(x, y) \leq \frac{C}{t^{Q/2}}.
$$

We are therefore in the framework of Theorem 6.1 in [2], from which one obtains that there is a constant $C_{p, \delta} > 0$ such that for every $f \in B^{p, \delta/2}(X)$,

$$
\sup_{s \geq 0} s \mu \left( \{x \in X : |f(x)| \geq s\} \right)^{1/\delta} \leq C_{p, \delta} \|f\|_{p, \delta/2}
$$

where $q = \frac{pQ}{Q - p \delta}$. The conclusion follows from Theorem 4.2. \qed

**Example 5.2** Assume that $X = \mathbb{R}^d$ is equipped with the standard Dirichlet form and the Lebesgue measure $\lambda^d$. If $E$ is a Borel set whose boundary $\partial E \subset \mathbb{R}^d$ is closed and $m$-rectifiable, by [33, Theorem 3.2.39] we have

$$
\limsup_{r \to 0^+} \frac{1}{r^{d-m}} \lambda^d((\partial E)_r) = \frac{2\lambda^m(\partial E)\Gamma\left(\frac{1}{2}\right)^m}{m\Gamma\left(\frac{d}{2}\right)},
$$

where $(\partial E)_r$ denotes the $r$-neighborhood of $\partial E$. This implies $1_E \in B^{1, \frac{d-m}{m}}(\mathbb{R}^d)$ and proposition 5.1 (17), is satisfied with $Q = d$, $\delta = d - m$. For instance, if $E$ is the so-called Koch snowflake domain in $\mathbb{R}^2$ then $d = 2$ and $m = \frac{\log 4}{\log 3}$.  

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In Euclidean space there is a standard method for using the above weak-type Sobolev embedding to obtain the usual Sobolev embedding theorem, in which the weak-$L^q$ control of $f$ is replaced by the strong-$L^q$ control. However this approach uses locality properties which need not be valid for the Besov seminorm $\| \cdot \|_{p,\alpha}$. We direct the interested reader to [41] for more details on this topic.

The one circumstance we have investigated in which the Besov seminorm has a locality property arose in Theorem 4.4, see also Remark 4.5, for the space $B^{1,1/2}$ under the assumption of a weak Bakry–Émery estimate, in which case we had $B^{1,1/2} = BV(X)$. This locality property lets us obtain a standard Sobolev embedding in which the $L^q$ norm is controlled by the BV norm. We may view this as an extension of known results on Riemannian manifolds with non-negative Ricci curvature (see Theorem 8.4 in [72]) or on Carnot groups (see [94]) to our metric measure Dirichlet setting under the further hypothesis that there is a weak Bakry–Émery estimate.

**Theorem 5.3** Suppose that the weak Bakry–Émery estimate (9) is satisfied. If the volume growth condition $\mu(B(x, r)) \geq C_1 r^Q$, $r \geq 0$, is satisfied for some $Q > 0$, then there exists a constant $C_2 > 0$ such that for every $f \in BV(X)$,

$$
\|f\|_{L^q(X)} \leq C_2 \|Df\|(X)
$$

where $q = \frac{Q}{Q-1}$. In particular, if $E$ is a set with finite perimeter in $X$, then

$$
\mu(E) \frac{q-1}{q} \leq C_2 P(E, X).
$$

**Proof** Observe that as in the above proof, the heat kernel satisfies the ultracontractive estimate (18). From Theorem 4.4 we have

$$
\|f\|_{1,1/2} \leq C \liminf_{s \to 0} s^{-1/2} \int_X P_s(|f - f(y)|)(y) d\mu(y).
$$

This verifies a condition denoted by $(P_{1,1/2})$ in Definition 6.7 of [2]), putting us in the framework of [2, Theorem 6.9] with $p = 1$, $\alpha = 1/2$ and $\beta = Q/2$. Notice also that $\|f\|_{1,1/2} \leq C \|Df\|(X)$ from Theorem 4.4, so we have

$$
\|f\|_{L^q(X)} \leq C \|f\|_{1,1/2} \leq C_2 \|Df\|(X),
$$

where $q = \frac{Q}{Q-1}$. Taking $f = 1_E$ then yields

$$
\mu(E) \frac{q-1}{q} \leq C_2 P(E, X).
$$

\[\square\]

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