Graded manifolds of type $\Delta$ and $n$-fold vector bundles\textsuperscript{1}

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Abstract

Vector bundles and double vector bundles, or 2-fold vector bundles, arise naturally for instance as base spaces for algebraic structures such as Lie algebroids, Courant algebroids and double Lie algebroids. It is known that all these structures possess a unified description using the language of supergeometry and graded manifolds of degree $\leq 2$. Indeed, a link has been established between the super and classical pictures by the geometrization process, leading to an equivalence of the category of graded manifolds of degree $\leq 2$ and the category of (double) vector bundles with additional structures.

In this paper we study the geometrization process in the case of $\mathbb{Z}^r$-graded manifolds of type $\Delta$, where $\Delta$ is a certain weight system and $r$ is the rank of $\Delta$. We establish an equivalence between a subcategory of the category of $n$-fold vector bundles and the category of graded manifolds of type $\Delta$.

1 Introduction

Graded manifolds of type $\Delta$. A graded manifold of type $\Delta$, a notion that we introduce here, is a natural generalization of the notion of a non-negatively $\mathbb{Z}$-graded manifold of degree $n$. We work in the category of smooth or complex-analytic graded manifolds and we use the language of sheaves and ringed spaces as in the theory of supermanifolds [L, Man].

Graded manifolds of degree $n$ were studied by various authors in for instance the context of the theories of Lie algebroids, Courant algebroids, double Lie algebroids and their higher generalizations [Vo1, R, BCMZ, LS, Vit, CM, JL, BGR]. We can define a non-negatively $\mathbb{Z}$-graded manifold of degree $n$ as a supermanifold which possesses an atlas with homogeneous coordinates with weights (or degrees) labeled by integers $0, 1, \ldots, n$, see [Vo2]. In this paper we study non-negatively $\mathbb{Z}^r$-graded manifolds, where $r \geq 1$.

In the case $r > 1$ the notion of a degree for graded manifolds is not sufficient to characterize the corresponding category. For example, consider the iterated tangent bundle $T(T(M))$ of a manifold $M$. The structure sheaf

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of $T(T(M))$ is naturally $\mathbb{Z}^2$-graded. Indeed, on $T(T(M))$ we can choose local charts with coordinates in the following form:

$$x_i, \ d_1(x_j), \ d_2(x_s), \ d_2(d_1(x_t)),$$

where $(x_i)$ are local coordinates on the manifold $M$ which we assume have weight $(0,0)$. Here $d_1$ and $d_2$ are the first and second de Rham differentials. We assume that the local coordinates $d_1(x_j)$ and $d_2(x_s)$ and $d_2(d_1(x_t))$ have weights $(1,0)$ and $(0,1)$ and $(1,1)$, respectively. We see that this is a graded manifold of degree $2 = 1 + 1$ with respect to the total degree. However in the $\mathbb{Z}^2$-graded case we can be more precise and consider graded manifolds of multi-degree $(n_1, n_2)$, in the $\mathbb{Z}^3$-graded case we should consider graded manifolds of multi-degree $(n_1, n_2, n_3)$ and so on. From this point of view $T(T(M))$ is a graded manifold of degree $(1,1)$.

Another observation here is that the numbers $(n_i)$ are also not sufficient to describe the whole picture. For example we can consider a category of graded manifolds of degree $(2,2)$ such that any object in this category possesses an atlas with local coordinates of degrees $(0,0), (2,0)$ and $(0,2)$. We see that in this case we can specify the definition of a graded manifold of degree $(2,2)$ and consider the category of graded manifolds of type $\Delta = \{(0,0), (2,0), (0,2)\}$. In addition we can think about $2$-tuples $(0,0), (2,0)$ and $(0,2)$ as vectors in $\mathbb{K}^2$, where for convenience we assume that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We introduce a monoid or a weight system $\Delta \subset \mathbb{K}^2$ that parametrizes degrees of local coordinates, see Definition 1 for details.

Summing up in this paper we study a more precise notion of a non-negatively $\mathbb{Z}^r$-graded manifold, i.e. the notion of a graded manifold of type $\Delta$. In addition we assume as in [Vo2] that local coordinates have parities that are related but not determined by the weights. This approach suggests a reduction of some questions about graded manifolds of type $\Delta$ to the study of the combinatorics of the monoid $\Delta$, as it is done for instance in this paper. A further example here is the following. We consider the root system $A_2$ of the Lie algebra $\mathfrak{sl}_3$. Then certain reflections in this root system correspond to the dualizations of double vector bundles [GrMa], see Section 3.5 for details. Another benefit of this approach is the possibility to give a more precise definition of an $n$-fold vector bundle. For instance this new definition distinguishes between the categories of double vector bundles and of double vector bundles with the trivial core. We introduce the notion of multiplicity free weight system $\Delta$ and study $n$-fold vector bundles of type $\Delta$, where $n$ is the rank of $\Delta$, see Definitions 4, 5 and 6.

**Geometrization process.** A geometrization process is a functor from the category of graded manifolds to the category of smooth (or holomorphic) manifolds. Such functors are well-known, for example the functor of points for
graded or supermanifolds and the linearisation functor [BGG]. Often it is interesting to ask which classical manifolds arise from graded manifolds. The goal of this paper is to answer this question for graded manifolds of type $\Delta$.

For motivation, consider the following table of correspondences.

| Geometric structures | Supergeometric structures |
|----------------------|---------------------------|
| base space / structure | base space / structure |
| vector bundles        | graded manifolds of degree $1$ |
|                      | $[Q, Q] = 0$ |
| Lie algebroids        | gr |
The second line of the table represents an equivalence between the categories of Courant algebroids and symplectic graded manifolds of degree 2 with a certain homological vector field $Q$. The result is due to P. Ševera [S] and D. Roytenberg [R], independently. In this case the geometrization process was used not directly. The category of graded manifolds of degree 2 (not necessary symplectic) was studied by D. Li-Bland in [LB]. His result corresponds to the third line of the table. This is an equivalence between the category of graded manifolds of degree 2 with a certain homological vector field $Q$ and the category of metric double vector bundles with the structure of VB-Courant algebroid (VB means “vector bundle”). In other words, to any graded manifold of degree 2, Li-Bland assigned a usual manifold, i.e. a metric double vector bundle, and he determined an additional structure that corresponds to the homological vector field $Q$. For more about applications of Courant algebroids and VB-algebroids, see [BCMZ, Cou, GrMe, Gu, KS, M2].

Other results in the direction of the third row of our table were obtained in [BCMZ, CM, JL]. In these papers the authors assigned to a graded manifold of degree 2 a usual manifold, or more precisely a double vector bundle with different types of additional structures. They also studied structures determined by a homological vector field $Q$.

A natural question is to investigate the last line of this table. Thus, in this paper we study more generally non-negatively $\mathbb{Z}_r$-graded manifolds of type $\Delta$. Due to the complexity we consider graded manifolds without any additional vector fields $Q$. This question is left for the future.

While this paper was in preparation, there appeared another result in this direction [BGR], in which the authors study graded bundles of degree $k$ that are the special case $r = 1$ of our graded manifolds of type $\Delta$. More precisely, in [BGR] the authors constructed a functor, which they called the full linearization functor, from the category of graded bundles of degree $k$ (graded manifolds of degree $k$ in our sense, i.e. the parities of coordinates do not necessary coincide with the parities of degrees) to the category of symmetric $k$-fold vector bundles with a family of morphisms that are parametrized by the symmetric group $S_k$. They showed that this functor is an equivalence of categories. Note that in the present paper we consider a different category of $k$-fold vector bundles, i.e. $k$-fold vector bundles of type $\Delta$ with a family of odd commuting vector fields.

**Main result.** Our results can be described as follows. We fix a weight system $\Delta \subset \mathbb{K}^r$ of rank $r$. Further we choose the parities of the basic weights, see Definition 1. Then we construct the corresponding multiplicity free weight system $\Delta' = \Delta'(\Delta)$ of rank $r'$, which is in general different from $r$. These two weight systems determine the category $\Delta\text{Man}$ of graded manifolds of
type $\Delta$ and the category $\Delta'^{\text{VB}}$ of $r'$-fold vector bundles of type $\Delta'$. Further we construct a functor $F : \Delta \text{Man} \to \Delta'^{\text{VB}}$, where the main idea is to use the $(r' - r)$-iterated tangent bundle $T \cdots T(N)$ of a graded manifold $N$. (The authors in [BGR] introduced independently a similar construction of the functor $F$, the linearisation functor, for the case $r = 1$.) We finally define the subcategory $\Delta'^{\text{VB Vect}}$ of the category $\Delta \text{Man}$, consisting of $r'$-fold vector bundles of type $\Delta'$ with $(r' - r)$ odd commuting homological vector fields. These vector fields arise from the iterated de Rham differentials on the structure sheaf of $T \cdots T(N)$. We prove that the image of $F$ coincides with $\Delta'^{\text{VB Vect}}$, and moreover that $F$ determines an equivalence of the categories $\Delta \text{Man}$ and $\Delta'^{\text{VB Vect}}$.

2 Graded manifolds of type $\Delta$

About $\mathbb{Z}$-graded manifolds of degree $n$ see for instance in [BGR, CM, GR, JL, LS, R, Vo1].

2.1 A weight system

Roughly speaking a weight system $\Delta$ is a monoid that parametrizes weights of local coordinates of a $\mathbb{Z}r$-graded manifold. Let us explain this notion in details.

We choose $r$ formal parameters $\alpha_1, \ldots, \alpha_r$, which we will call basic weights. It is convenient to think about $\alpha_i$ as about vectors in $\mathbb{R}^r$ or $\mathbb{C}^r$.

**Definition 1.** A weight system is a subset

$$\Delta \subset \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_r$$

satisfying the following properties:

1. $\Delta$ is finite;
2. $\{0\} \in \Delta$ and $\alpha_i \in \Delta$, where $i = 1, \ldots, r$;
3. if $\delta \in \Delta$ and $\delta = \sum a_i \alpha_i$, where $a_i \in \mathbb{Z}$, then $a_i \geq 0$.

The number $r$ is called the rank of $\Delta$.

We also will assign the parity $\tilde{\alpha}_i \in \{0, \bar{1}\}$ to any basic weight $\alpha_i$. If the parities of $\alpha_i$ are fixed for any $i$, the parities of all other elements from $\Delta$ are determined by the rule $\tilde{\delta_1} + \tilde{\delta_2} = \tilde{\delta_1} + \tilde{\delta_2}$. 

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In the next section we will introduce graded manifolds of type $\Delta$. This is graded manifolds with an atlas such that local coordinates are parameterized by elements from $\Delta$. The first condition of Definition 1 means that local coordinates of a graded manifold of type $\Delta$ may have only finite number of different weights. It is a natural agreement for a finite dimensional graded manifold. Further, the first part of the condition 2 means that, as in the theory of graded manifolds of degree $n$, we have an underlying manifold which structure sheaf is indicated by $0 \in \Delta$. The second part of the condition 2 is technical. The last condition shows that the structure sheaf of our graded manifold is non-negatively $\mathbb{Z}$-graded.

Examples of weight systems are:

$$\Delta_{D_2} := \{0, \alpha_1, \alpha_2, \alpha_1 + \alpha_2\}, \quad \Delta_{M_3} := \{0, \alpha_1, 2\alpha_1, 3\alpha_1\}. \quad (2)$$

The weight system $\Delta_{D_2}$ corresponds to a double vector bundle $D_2$ and the weight system $\Delta_{M_3}$ corresponds to a $\mathbb{Z}$-graded manifold $M_3$ of degree 3, see Section 3.

2.2 Definition of a graded manifold of type $\Delta$

Let us take a weight system $\Delta$ as in Definition 1 and let us assume that parities of the basic weights are fixed. Consider a finite dimensional vector space $V$ over $K$, where $K = \mathbb{R}$ or $\mathbb{C}$, with a decomposition into a direct sum of vector subspaces $V_\delta$, where $\delta \in \Delta$. In other words,

$$V = \bigoplus_{\delta \in \Delta} V_\delta.$$

We assume in addition that elements from $V_\delta \setminus \{0\}$ have weight $\delta$ and have the same parity as $\delta$. In other words, $V_\delta$ is a vector subspace in $V$ of parity $\overline{\delta}$ and of weight $\delta$. Further, we denote by $S^*(V)$ the super-symmetric power of $V$. Again we assume that the weight of a product is the sum of weights of factors. For example the weight of the product

$$v_{\delta_1} \cdot v_{\delta_2} \in V_{\delta_1} \cdot V_{\delta_2} \subset S^*(V),$$

where $v_{\delta_i} \in V_\delta_i$, is equal to $\delta_1 + \delta_2$. The same agreement holds for parities.

Consider the $\mathbb{Z}$-graded ringed space $\mathcal{U} = (U_0, \mathcal{O}_U)$, where $U_0 = V_0^*$, and the sheaf $\mathcal{O}_U$ is given by the following formula:

$$\mathcal{O}_U := F_{U_0} \otimes_K S^* \left( \bigoplus_{\delta \in \Delta \setminus \{0\}} V_\delta \right). \quad (3)$$
Here $\mathcal{F}_{U_0}$ is the sheaf of smooth (the case $\mathbb{K} = \mathbb{R}$) or holomorphic (the case $\mathbb{K} = \mathbb{C}$) functions on $U_0 = V_0^*$. The ringed space $\mathcal{U}$ is a non-negatively $\mathbb{Z}^r$-graded ringed space and $\Delta$ is the set of weights of its local coordinates. More precisely, let us choose a basis $(x_i)$ in $V_0$ and a basis $(\xi^\delta_j)$ in any $V_\delta$. Then we can consider the set $(x_i, \xi^\delta_j)_{\delta \in \Delta \setminus \{0\}}$ as the set of local coordinates on $\mathcal{U}$. We assign the weight 0 and the parity 0 to any $x_i$ and the weight $\delta$ and the parity $\bar{\delta}$ to any $\xi^\delta_j$. We see that the weight system $\Delta$ parametrizes the weights of local coordinates in $\mathcal{U}$. We will call the ringed space $\mathcal{U}$ a graded domain of type $\Delta$ and of dimension $\{\dim V_\delta\}_{\delta \in \Delta}$. Note that in this case the dimension is a set of numbers parametrized by the elements from $\Delta$.

**Definition 2.** • A graded manifold of type $\Delta$ and of dimension $\{\dim V_\delta\}_{\delta \in \Delta}$ is a $\mathbb{Z}^r$-graded ringed space $\mathcal{N} = (\mathcal{N}_0, \mathcal{O}_\mathcal{N})$, that is locally isomorphic to a graded domain of type $\Delta$ and of dimension $\{\dim V_\delta\}_{\delta \in \Delta}$.

• A morphism of graded manifolds of type $\Delta$ is a morphism of the corresponding $\mathbb{Z}^r$-graded ringed spaces.

We will denote the category of graded manifolds of type $\Delta$ by $\Delta\text{Man}$. Note that a graded manifold of type $\Delta$ is defined only if we determined parities of the basic weights in $\Delta$ and different parity agreements lead to different categories of graded manifolds.

We can describe a graded manifold of type $\Delta$ in terms of atlases and local coordinates. On a graded manifold $\mathcal{N}$ of type $\Delta$ there exists an atlas such that in any local chart we can chose local coordinates of weights $\delta \in \Delta$ and we require that transition functions between any two charts preserve all weights. Note that the structure sheaf of the underlying manifold $\mathcal{N}_0$ of $\mathcal{N}$ is equal to $(\mathcal{O}_\mathcal{N})_0$ and any homogeneous subsheaf $(\mathcal{O}_\mathcal{N})_\delta$ in $\mathcal{O}_\mathcal{N}$, where $\delta \in \Delta$, is a $(\mathcal{O}_\mathcal{N})_0$-locally free sheaf on $\mathcal{N}_0$.

### 3 Examples of graded manifolds of different types

#### 3.1 Example 1

*Graded manifolds of degree $n$.* An example of a graded manifold of type $\Delta$ is a $\mathbb{Z}$-graded manifold of degree $n$.

**Definition 3.** A graded manifold of degree $n$ is a graded manifold $\mathcal{M}_n$ of type $\Delta_{\mathcal{M}_n}$, where

$$\Delta_{\mathcal{M}_n} = \{0, \alpha_1, \ldots, n\alpha_1\} \subset \mathbb{Z}\alpha_1.$$  \hspace{1cm} (4)

The number $n$ is called the degree of $\mathcal{M}_n$. 

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3.2 Example 2

Double and r-fold vector bundles. Another example of graded manifolds of type $\Delta$ is a double and more general an r-fold vector bundle. For instance a double vector bundle is a graded manifold of type $\Delta_{D_2}$, see (2). A triple vector bundle $D_3$ has the weight system

$$\Delta_{D_3} := \{0, \alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$$

and so on. We can characterize the weight system $\Delta_{D_r}$ in the following way:

the weight system $\Delta_{D_r}$ has rank $r$ and it contains all linear combinations of $\alpha_i$ with coefficients 0 or 1, i.e. all linear combinations of $\alpha_i$ without multiplicities.

We will use the following definition of an r-fold vector bundle.

**Definition 4.** An r-fold vector bundle is a graded manifold of type $\Delta_{D_r}$.

**Remark.** This definition of an r-fold vector bundle is equivalent to a classical one. This result was obtained in [GR, Theorem 4.1], see also [Vo1].

In this paper we also will use a more general notion of an r-fold vector bundle: an r-fold vector bundle of type $\Delta$. For a motivation let us consider an example. Assume that a double vector bundle $D_2$ has trivial core (see [M1] for definitions). In our notations this means that this double vector bundle does not have local coordinates of weight $\alpha_1 + \alpha_2$ in a certain atlas. Hence we can assume that the weight system of any double vector bundle with trivial core has the form $\{0, \alpha_1, \alpha_2\}$. For our purpose it is convenient to distinguish these two categories: the category of double vector bundles and the category of double vector bundles with trivial core. So we will speak about the category of double vector bundles of type $\Delta_{D_2}$ and the category of double vector bundles of type $\{0, \alpha_1, \alpha_2\}$. Summing up, in this paper we will use the following definitions.

**Definition 5.** A weight system $\Delta$ is called multiplicity free if $\Delta$ contains only linear combinations of $\alpha_i$, where $i = 1, \ldots, r$, with coefficients 0 or 1. In other words it contains only linear combinations of $\alpha_i$ without multiplicities.

Clearly any multiplicity free weight system is contained in some $\Delta_{D_r}$ and $\Delta_{D_r}$ is the maximal multiplicity free system of rank $r$.

**Definition 6.**
- An r-fold vector bundle of type $\Delta$ is a graded manifold of type $\Delta$, where $\Delta$ is a multiplicity free weight system of rank $r$.
- A morphism of r-fold vector bundles of type $\Delta$ is a morphism of the corresponding graded manifolds.

Let $\Delta$ be a multiplicity free weight system of rank $r$. We will denote the category of r-fold vector bundles of type $\Delta$ by $\Delta\text{VB}$. 
3.3 Example 3

Vector bundles over a graded manifold of type $\Delta$. Let $\mathcal{N}$ be a graded manifold of type $\Delta$. In the category $\Delta\text{Man}$ we can define a vector bundle in the usual way: a vector bundle over $\mathcal{N}$ is a graded manifold $E$ of type $\Delta$ with a morphism $E \to \mathcal{N}$ that satisfies the usual condition of local triviality and that has $\mathcal{O}_\mathcal{N}$-linear transition functions between the trivial pieces. However sometimes it is more convenient to introduce an additional formal basic weight, say $\beta$, and to think about $E$ as about a graded manifold of type $\Delta_E$, where

$$\Delta_E := \Delta \cup \{\beta + \delta \mid \delta \in \Delta\}. \quad (5)$$

In more details, let us choose local sections $(e^\delta_j)$ of $E$ of weights $\delta \in \Delta$. (The weights of local sections are indicated by the superscript.) Now in any chart we replace the weight $\delta$ of $e^\delta_j$ by $\beta + \delta$. Clearly, this operation is well-defined and the $\mathcal{O}_\mathcal{N}$-linearity of transition functions means that the total weight of sections of $E$ is preserved. In the literature this operation is called the shift of parity or the shift of weight. In out case we speak about the shift by the weight $\beta$ in $E$.

Summing up, if $E$ is a vector bundle over a manifold $\mathcal{N}$ of type $\Delta$, we will assume that $E$ is of type (5) for some additional weight $\beta$. We see that the weight $\beta$ has no multiplicity in (5), i.e. $\beta$ is contained in weights from (5) with coefficients 0 or 1. Note that the converse statement is also true. Indeed, assume that $E$ is a graded manifold of type $\Delta$ and a certain basic weight $\alpha_i$ has no multiplicity in $\Delta$. Then $E$ is a vector bundle over a graded manifold $\mathcal{N}$ of type

$$\Delta' := \Delta \cap \bigoplus_{j \neq i} \mathbb{Z}\alpha_j.$$

The weight system $\Delta'$ satisfies conditions of Lemma 1, see below. Hence $\mathcal{N}$ is a well-defined graded manifold of type $\Delta'$.

3.4 Example 4

The tangent bundle of a graded manifold of type $\Delta$. For the tangent bundle $T\mathcal{N}$ of a graded manifold $\mathcal{N}$ of type $\Delta$ we will use a special weight agreement. Let us fix a basic weight $\alpha_i$. (Later we will use this procedure for different $i$. Here we show how it works for a certain fixed $i$.) Then we assume that the tangent bundle $T\mathcal{N}$ is a vector bundle shifted by $\beta - \alpha_i$, where $\beta$ is an additional formal weight. For instance, in this case the de Rham differential $d_{\text{dR}}$ must have the weight $\beta - \alpha_i$. Throughout this paper we also assume that the de Rham differential $d_{\text{dR}}$ is odd. Hence the weight $\beta$ has the opposite parity to $\alpha_i$.  

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To illustrate our agreement, let us consider a graded manifold $M_n$ of type $\Delta_{M_n}$, see (4). Here we have only one basic weight $\alpha_1$, hence the de Rham differential $d_{dR}$ has the weight $\beta - \alpha_1$, where $\beta$ is a formal additional basic weight, and the weight system $\Delta_{T M_n}$ of $T M_n$ is given by the following formula:

$$\Delta_{T M_n} = \{0, \alpha_1, \ldots, n\alpha_1, \beta - \alpha_1, \beta, \beta + \alpha_1, \ldots, \beta + (n - 1)\alpha_1\}.$$ 

We see that $T M_n$ is not a non-negatively graded manifold anymore. Indeed, the weight system $\Delta_{T M_n}$ contains weights with negative coefficients, for instance, $\beta - \alpha_1$. Such a weight agreement we will need to construct a functor from the category of graded manifolds to the category of $r$-fold vector bundles.

3.5 Example 5

*Root systems of rank 2 and the corresponding graded manifolds.*** There are the following types of root systems of rank 2:

$$A_1 \times A_1 \simeq D_2, \ A_2, \ B_2 \simeq C_2 \ and \ G_2.$$ 

Denote by $\Delta$ a system of positive roots in any of these root systems. Then we can consider the corresponding category of graded manifolds of type $\Delta \cup \{0\}$. Let us characterize these categories.
Case $A_2$. Let us choose a system of positive roots $\Delta = \{\alpha, \beta, \alpha + \beta\}$, see picture above. By our definition, see Example 2, a graded manifold of type $\Delta \cup \{0\}$ is a double vector bundle $D_2$:

$$
D_2 \rightarrow A \\
\downarrow \\
B \rightarrow M
$$

Here $D_2 \rightarrow A$, $D_2 \rightarrow B$, $A \rightarrow M$ and $B \rightarrow M$ are vector bundles, see [M1] for precise definition.

The root system $A_2 = \Delta \cup -\Delta \cup \{0\}$ has also a natural geometric interpretation. Let us choose a chart on $D_2$ with the following local coordinates:

$$x_i, \xi^\alpha_j, \xi^\beta_s, \xi^{\alpha+\beta}_t.$$

Here $x_i$ are local coordinates of weight 0 and $\xi^\delta_j$ are local coordinates of weight $\delta$, where $\delta \in \Delta$. We use here the standard agreement that $(x_i)$ are local coordinates on $M$, $(x_i, \xi^\alpha_j)$ are local coordinates on $A$, $(x_i, \xi^\beta_s)$ are local coordinates on $B$ and $(x_i, \xi^\alpha_j, \xi^\beta_s, \xi^{\alpha+\beta}_t)$ are local coordinates on $D_2$. Consider the cotangent space $T^*D_2$. It has the following local coordinates in the corresponding chart on $T^*D_2$:

$$\{x_i, \xi^\delta_j, \frac{\partial}{\partial x_i}, \frac{\partial}{\partial \xi^\delta_j}\}_{\delta \in \Delta}.$$

We may assume that the element $\frac{\partial}{\partial \xi^\delta_j}$ has the weight $-\delta$ and $\frac{\partial}{\partial x_i}$ has the weight 0. Hence, $T^*D_2$ is a graded manifold of type $A_2$.

Denote by $\hat{T}^*D_2$ the graded manifold of type $A_2$ with the structure sheaf $\mathcal{O}_{\hat{T}^*D_2}$ which is locally generated by the following elements:

$$\{x_i, \xi^\delta_j, \frac{\partial}{\partial \xi^\delta_j}\}_{\delta \in \Delta}$$

and with base $M$. (Clearly, $\mathcal{O}_{\hat{T}^*D_2}$ is a well-defined subsheaf in $\mathcal{O}_{T^*D_2}$.)

Further, the double vector bundle $D_2$ possesses two dualization operations in the direction $A$ and the direction $B$. We denote by $D_2^A$ the dual vector bundle in the direction $A$, i.e. $D_2^A \rightarrow A$ is the dual of the vector bundle $D_2 \rightarrow A$. Similarly we obtain $D_2^B$. In fact, $D_2^B \rightarrow B$ is the dual of the vector bundle $D_2 \rightarrow B$.

It is well-known (see [M1] and also [GrMa]) that $D_2^A$ and $D_2^B$ are again double vector bundles, hence graded manifolds. We can describe
these graded manifolds using the root system $A_2$. Indeed, the weight system of $D^*_2A$ is

$$\Delta^*_A \cup \{0\}, \quad \text{where} \quad \Delta^*_A = \{\alpha, -\alpha - \beta, -\beta\}. $$

Here we can take the weights $\alpha$ and $-\alpha - \beta$ as basic weights, then $-\beta = \alpha + (-\alpha - \beta)$. Similar picture we have for $D^*_2B$. The weight system of $D^*_2B$ is

$$\Delta^*_B \cup \{0\}, \quad \text{where} \quad \Delta^*_B = \{\beta, -\alpha - \beta, -\alpha\}. $$

We can take the weights $\beta$ and $-\alpha - \beta$ as basic weights. The structure sheaves of $D^*_2A$ and $D^*_2B$ are subsheaves in the structure sheaf of $\hat{T}^*D_2$.

Summing up, consider the picture for $A_2$ above, where we marked the roots $\alpha, \beta$ and $-\alpha - \beta$ by a cycle. We see that

*all double vector bundles that we can obtain from $D_2$ using dualizations up to isomorphism correspond to systems of positive roots in $A_2$ such that any of these systems contain exactly two marked roots.*

- **Case $B_2$.** Consider the following system of positive roots

$$\Delta = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\},$$

and the category of graded manifolds of type $\Delta \cup \{0\}$. We see that the weight $\beta$ has no multiplicity in $\Delta \cup \{0\}$, therefore any graded manifold $\mathcal{E}$ of this type is a graded vector bundle over a graded manifold $\mathcal{M}$ of type $\{0, \alpha\}$, see Example 3.

Consider the graded manifold $\hat{T}^*\mathcal{E}$ of type $B_2 = \Delta \cup -\Delta \cup \{0\}$ that is constructed as in case $A_2$. More precisely the structure sheaf of $\hat{T}^*\mathcal{E}$ is locally generated by

$$\left\{x_i, \xi_j, \frac{\partial}{\partial \xi_j} \right\}_{\delta \in \Delta}. $$

Further, we can take the dual vector bundle $\mathcal{E}^*$ of $\mathcal{E}$. It is a graded manifold of type $\Delta^* \cup \{0\}$, where

$$\Delta^* = \{\alpha, -2\alpha - \beta, -\beta, -\alpha - \beta\}. $$

The basic weights here are $\alpha$ and $-2\alpha - \beta$. Again we see that the weight system $\Delta^*$ can be obtained from $\Delta$ by a reflection in the root system $B_2$. Note that the structure sheaves of $\mathcal{E}$ and $\mathcal{E}^*$ are subsheaves in the structure sheaf of $\hat{T}^*\mathcal{E}$. 

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• **Case** $A_1 \times A_1$. Consider the system of positive roots $\Delta = \{\alpha, \beta\}$. This weight system is multiplicity free, hence it determines the category of certain double vector bundles. Such double vector bundles are called double vector bundles with trivial core, see [M1] for definitions. In other words it is just a sum of two vector bundles. Summing up, the category of graded manifolds of type $\Delta \cup \{0\}$ is the category of double vector bundles with trivial core. Again reflections $\alpha \mapsto -\alpha$ and $\beta \mapsto -\beta$ in the weight system $A_1 \times A_1$ correspond to dualizations of double vector bundles in different directions.

• **Case** $G_2$. The category of graded manifolds of type $\Delta \cup \{0\}$, where

\[
\Delta = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}.
\]

We do not know any geometric interpretation in this case.

4 Construction of graded manifolds of different types

4.1 Construction 1

Let us take a graded manifold $\mathcal{N}$ of type $\Delta$. We can associate to $\mathcal{N}$ a family of graded manifolds of different types. Let $\mathcal{N}_0$ be the underlying manifold of $\mathcal{N}$. In our notations, $(\mathcal{O}_\mathcal{N})_0$ is the structure sheaf of $\mathcal{N}_0$ and $(\mathcal{O}_\mathcal{N})_{\delta}$ are $(\mathcal{O}_\mathcal{N})_0$-locally free sheaves on $\mathcal{N}_0$, where $\delta \in \Delta$. Let us choose a subset $\Delta' \subset \Delta$ that satisfies the following property:

\[
\text{if } \delta \in \Delta' \text{ and } \delta = \sum_i \delta_i \text{ for some } \delta_i \in \Delta, \text{ then } \delta_i \in \Delta' \text{ for any } i. \quad (6)
\]

**Lemma 1.** Assume that a graded manifold $\mathcal{N}$ of type $\Delta$ is fixed. To any $\Delta'$ satisfying (6) we may assign the graded manifold $\mathcal{N}_{\Delta'}$ of type $\Delta'$.

**Proof.** Consider a local chart on $\mathcal{N}$ with the structure sheaf in the form (3). Clearly, we have the following inclusion of the sheaves

\[
\mathcal{F}_{\mathcal{U}\otimes \mathbb{K}} S^* \left( \bigoplus_{\delta \in \Delta \cup \{0\}} V_\delta \right) \hookrightarrow \mathcal{F}_{\mathcal{U}\otimes \mathbb{K}} S^* \left( \bigoplus_{\delta \in \Delta \cup \{0\}} V_\delta \right).
\]

By (6) the transition functions between any such charts preserve this inclusion. Gluing these charts together we get $\mathcal{N}_{\Delta'}$. $\square$
Lemma 2. Let us take two weight subsystem $\Delta'$ and $\Delta''$ in $\Delta$ satisfying (6). Then
\[ \Delta' \cap \Delta'' \quad \text{and} \quad \Delta' \cup \Delta'' \]
also satisfy (6) and determine graded manifolds of type $\Delta' \cap \Delta''$ and $\Delta' \cup \Delta''$.

Proof we leave for a reader. □

4.2 Construction 2

Another construction is the following. Let $\Delta$ be a weight system given by (1), $\alpha_i$ are basic weights and $\delta = \sum a_i \alpha_i \notin \Delta$, where $a_i \geq 0$, be a certain element from lattice (1). We set $\Delta' := \Delta \cup \{ \delta \}$.

Assume that a graded manifold $N_\Delta$ of type $\Delta$ is given. Our goal now is to “add” the weight $\delta$ and to construct a graded manifold $N_{\Delta'}$ of type $\Delta'$.

Let us take two $(\mathcal{O}_{N_{\Delta}})_0$-locally free sheaves $\mathcal{O}_\delta$ and $\mathcal{E}_\delta$ on the underlying space $(\mathcal{N}_\Delta)_0$ of $\mathcal{N}_\Delta$ such that the following sequence
\[ 0 \to (\mathcal{O}_{N_\Delta})_\delta \to \mathcal{O}_\delta \to \mathcal{E}_\delta \to 0 \] (7)
is exact. Here $(\mathcal{O}_{N_{\Delta}})_\delta$ is the subsheaf of weight $\delta$ in the structure sheaf $\mathcal{O}_{N_\Delta}$ of $\mathcal{N}_\Delta$. Assume that $\{U_i\}$ is an atlas on $(\mathcal{N}_\Delta)_0$ such that (7) is split over each $U_i$, i.e.
\[ \mathcal{O}_\delta|_{U_i} \simeq (\mathcal{O}_{N_\Delta})_\delta|_{U_i} \oplus \mathcal{E}_\delta|_{U_i}. \]
We assume in addition that the sheaf $\mathcal{E}_\delta|_{U_i}$ is free and
\[ U_i := (U_i, \mathcal{O}_{\mathcal{N}_\Delta}|_{U_i}) \]
is a local chart on $\mathcal{N}_\Delta$. Let us choose local coordinates $(x_i)$ in $U_i$, local homogeneous coordinates with non-trivial weights $(\xi^i_a)$ in $U_i$ and a basis of sections $(\eta^i_b)$ in $\mathcal{E}_\delta|_{U_i} \subset \mathcal{O}_\delta|_{U_i}$. To each $\eta^i_b$ we assign the weight $\delta$. We are ready to construct a local chart on $\mathcal{N}_{\Delta'}$. We set
\[ V_i := (U_i, \mathcal{F}_{U_i} \otimes \mathbb{R} S^* (\xi^i_a, \eta^i_b)), \]
and we define transition functions in any intersection $V_i \cap V_j$ in the following way: we take the transition functions of $\mathcal{N}_\Delta$ in the intersection of charts $U_i \cap U_j$ together with the transition functions $\eta^j_b = \eta^j_b(\xi^j_i, \eta^j_b)$ in $\mathcal{O}_\delta|_{U_i \cap U_j}$.
Clearly all total weights are preserved and we get a graded manifold $N_{\Delta'}$ of type $\Delta'$.

Further we will need the following observation. Let us take two graded manifolds $N$ and $N'$ of type $\Delta$ with the same underlying space, i.e. $N_0 = N'_0$. For simplicity of notations denote the structure sheaves of $N$ and $N'$ by $O$ and $O'$, respectively.

**Proposition 1.** Assume that the bundle isomorphisms are given $\varphi_\delta : O_\delta \to O'_\delta$ for any $\delta \in \Delta$. If for any $\delta \in \Delta$ the following diagram is commutative:

\[
\begin{array}{c}
\bigoplus_{\delta_1 + \delta_2 = \delta} O_{\delta_1} \cdot O_{\delta_2} \longrightarrow O_\delta \\
\varphi_{\delta_1} \cdot \varphi_{\delta_2} \\
\bigoplus_{\delta_1 + \delta_2 = \delta} O'_{\delta_1} \cdot O'_{\delta_2} \longrightarrow O'_\delta
\end{array}
\]

where the horizontal maps are natural inclusions and sums are taken over all $\delta_1, \delta_2 \in \Delta \setminus 0$, then $N$ and $N'$ are isomorphic as graded manifolds of type $\Delta$.

**Proof** is clear. $\square$

5 **A functor $F$ from $\Delta\text{Man}$ to $\Delta'\text{VB}$**

In this section we construct a functor $F : \Delta\text{Man} \to \Delta'\text{VB}$, where $\Delta'$ is a weight system that will be defined later. Recall that we denoted by $\Delta\text{Man}$ the category of graded manifolds of type $\Delta$ and by $\Delta'\text{VB}$ the category of $r'$-fold vector bundles of type $\Delta'$, where $\Delta'$ is a multiplicity free weight system and $r'$ is the rank of $\Delta'$. If $N$ is a graded manifold, we denote by $O_N$ its structure sheaf.

5.1 **Preliminaries**

Let $\Delta$ be a weight system of rank $r$, $\alpha_i$ be basic weights, see (1), and $N$ be a graded manifold of type $\Delta$. By definition of a weight system, any weight $\delta \in \Delta$ has the form

\[\delta = \sum_{i=1}^{r} a_i(\delta) \alpha_i,\]  

where $a_i(\delta)$ are non-negative integers. Denote by

\[n_i := \max_{\delta \in \Delta} \{a_i(\delta)\}, \quad i = 1, \ldots, r.\]
In other words \( n_i \) is the maximal multiplicity of the basic weight \( \alpha_i \) in the weight system \( \Delta \). Note that \( n_i > 0 \). To construct the functor \( F \), we will need the following set of additional formal weights:

\[
\{ \beta_{ji} \mid j = 2, \ldots, n_i, \ i = 1, \ldots, r \}. \tag{10}
\]

For our construction we will use sequentially the weights

\[
\beta_{21}, \ldots, \beta_{n_1}, \beta_{22}, \ldots, \beta_{n_2}, \ldots, \beta_{2r}, \ldots, \beta_{n_r}. \tag{11}
\]

Let us take the first weight from Sequence (11). For simplicity we assume that the first weight is \( \beta_{21} \). As above we denote by \( T_N \) the tangent space of \( N \) and we denote by \( d_{\beta_{21}} : \mathcal{O}_{TN} \to \mathcal{O}_{TN} \) the corresponding de Rham differential. We assume that the map \( d_{\beta_{21}} \) has the weight \( \beta_{21} - \alpha_1 \) and we indicate our assumption by the subscript \( \beta_{21} \) in \( d_{\beta_{21}} \). In other words, we assume that

\[
d_{\beta_{21}} \left( (\mathcal{O}_N)_{\delta} \right) \subset (\mathcal{O}_{TN})_{\delta + \beta_{21} - \alpha_1}.
\]

(Compare with Example 4, Section 3. We have seen there that such weight agreements is well-defined.) Here \( (\mathcal{O}_N)_{\delta} \) is the subsheaf in \( \mathcal{O}_N \) of weight \( \delta \). Note that \( \delta \) is not necessary from \( \Delta \) in this case. Using our assumption about the weight of \( d_{\beta_{21}} \), we see that the weight system \( \Delta_{TN} \) of \( TN \) is given by the following formula:

\[
\Delta_{TN} = \Delta \cup \{ \delta + \beta_{21} - \alpha_1 \mid \delta \in \Delta \}. \tag{12}
\]

Compare this with Formula (5). As in Example 4, we see that \( \Delta_{TN} \) is not a non-negatively graded manifold anymore. For instance \( \Delta_{TN} \) contains the weight \( \beta_{21} - \alpha_1 \) that has a negative coefficient.

Further, let us take the next weight from Sequence (11). We assume that the next weight is \( \beta_{31} \). Denote by

\[
d_{\beta_{31}} : \mathcal{O}_{TTN} \to \mathcal{O}_{TTN}
\]

the de Rham differential on the tangent space \( TTN \) of \( TN \). We assume that \( d_{\beta_{31}} \) has the weight \( \beta_{31} - \alpha_1 \). We denote the tangent prolongation of \( d_{\beta_{21}} \) on \( \mathcal{O}_{TTN} \) also by \( d_{\beta_{21}} \). (The map \( d_{\beta_{21}} \) is a vector field on the graded manifold \( TN \). Hence the action of \( d_{\beta_{21}} \) is defined on all tensors on \( TN \) by the Lie derivative.) By definition of the tangent prolongation the vector fields \( d_{\beta_{21}} \) and \( d_{\beta_{31}} \) on \( \mathcal{O}_{TTN} \) commute. Summing up on \( TTN \) we have two odd homological commuting vector fields \( d_{\beta_{21}} \) and \( d_{\beta_{31}} \). In other words, we have

\[
d_{\beta_{21}} \circ d_{\beta_{21}} = 0, \quad d_{\beta_{31}} \circ d_{\beta_{31}} = 0, \quad [d_{\beta_{21}}, d_{\beta_{31}}] = 0.
\]
We continue this process. Altogether we iterate this procedure \( n \) times, where
\[
n := \sum_{i=1}^{r} n_i - r,
\]
using sequentially the de Rham differentials
\[
d_{\beta_{21}}, \ldots, d_{\beta_{n_1}}, \ d_{\beta_{22}}, \ldots, d_{\beta_{n_2}}, \ldots, \ d_{\beta_{2r}}, \ldots, d_{\beta_{n_r}}.
\]
We assume that the de Rham differential \( d_{\beta_{ji}} \) has the weight \( \beta_{ji} - \alpha_i \). The result of this procedure is the following iterated tangent bundle:
\[
\tilde{N} := T \cdots T(N)
\]
with \( n \) odd operators \( d_{\beta_{ji}} \) such that
\[
[d_{\beta_{ji}}, d_{\beta_{j'i'}},] = 0 \text{ for all } (ji) \text{ and } (j'i').
\]
Further let \( \mathcal{R} \) be a \( \mathbb{Z}^r \)-graded manifold that is not necessary non-negatively graded. In this paper we consider only the case, when \( \mathcal{R} \) is an iterated tangent bundle of a graded manifold of type \( \Delta \). Denote by \( \mathcal{J}_R^- \) the ideal in \( \mathcal{O}_\mathcal{R} \) that is generated by all elements with weights that have at least one negative coefficient. To simplify notations usually we will write \( \mathcal{J}^- \) instead of \( \mathcal{J}_R^- \).

### 5.2 Construction of \( F \)

We are ready to define the functor \( F \). Let \( \mathcal{N} \) be a graded manifold as above. Our goal now is to construct an \( r' \)-fold vector bundle \( D_{\mathcal{N}} \), where \( r' \) is defined below. Consider the sheaf \( \mathcal{O}_{\tilde{\mathcal{N}}}/\mathcal{J}^- \), where \( \tilde{\mathcal{N}} \) is defined in the previous section. Clearly this is the structure sheaf of a certain non-negatively graded manifold. We denote by \( \tilde{\Delta} \) the weight system of this graded manifold and by \( \Delta' \subset \tilde{\Delta} \) its maximal multiplicity free subsystem. That is \( \Delta' \) is the weight subsystem in \( \tilde{\Delta} \) that contains all weights from \( \tilde{\Delta} \) that have coefficients 0 or 1 before the basic weights \( \alpha_i \) and \( \beta_{ji} \). It is easy to see that \( \Delta' \) satisfies conditions of Lemma 1. We denote by \( D_{\mathcal{N}} \) the corresponding graded manifold of type \( \Delta' \), see the construction in Lemma 1.

Applying the rule (12), we see that \( \alpha_i, \beta_{ji} \in \tilde{\Delta} \) and we note that these weights are multiplicity free. Hence, \( \alpha_i, \beta_{ji} \in \Delta' \) and the rank of \( \Delta' \) is equal to \( r' := n + r \). Further, by definition the weight system \( \Delta' \) is multiplicity free, hence, the graded manifold \( D_{\mathcal{N}} \) is an \( r' \)-fold vector bundle, see Definition 6. We put
\[
F(\mathcal{N}) := D_{\mathcal{N}}.
\]
Note that $\Delta'$ depends only on $\Delta$, but not on a particular choice of $N$.

Further, let us take a morphism $\Phi : N \to N_1$ of two graded manifolds $N$ and $N_1$ of type $\Delta$. By definition, $\Phi$ preserves all weights. We have the corresponding map in the iterated tangent bundles

$$
( T \cdots T \Phi) : T \cdots T(N) \to T \cdots T(N_1),
$$

that preserves all weights. Therefore, the map

$$
(\Phi')^* : \mathcal{O}_{\tilde{N}_1}/\mathcal{J}_{\tilde{N}_1}^- \to \mathcal{O}_{\tilde{N}}/\mathcal{J}_{\tilde{N}}^-
$$

is well-defined. Here we use the following notations:

$$
\tilde{N} := T \cdots T(N), \quad \tilde{N}_1 := T \cdots T(N_1),
$$

and we denote by $\mathcal{J}_{\tilde{N}}^-$ and $\mathcal{J}_{\tilde{N}_1}^-$ the ideals in $\mathcal{O}_{\tilde{N}}$ and $\mathcal{O}_{\tilde{N}_1}$, respectively, that are defined in the previous section, i.e. these ideals are generated by all elements with weights that have at least one negative coefficient. Since $(\Phi')^*$ preserves all weights, we get

$$
(\Phi')^*(\mathcal{O}_{\mathcal{D}_{N_1}}) \subset \mathcal{O}_{\mathcal{D}_N}.
$$

Hence the map

$$
F(\Phi) : \mathcal{D}_N \to \mathcal{D}_{N_1}
$$

is defined. Clearly, the correspondence $\Phi \mapsto F(\Phi)$ sends a composition of morphisms to a composition of morphisms. Hence we obtain the following theorem.

**Theorem 1.** The correspondence $F$ is a functor from the category of graded manifolds of type $\Delta$ to the category of $r'$-fold vector bundles of type $\Delta'$.

### 5.3 Explicit description of $\Delta' = \Delta'(\Delta)$

Let us describe $\Delta' = \Delta'(\Delta)$ explicitly. We take $\delta \in \Delta$ with coefficients $a_i(\delta) \in \mathbb{Z}$ as in (8), and we put

$$
\beta_{I_i} := \sum_{s \in I_i} \beta_{si}, \quad \text{where } I_i \subset \{2, \ldots, n_i\}.
$$

**Proposition 2.** We have

$$
\Delta' = \bigcup_{\delta \in \Delta} \Delta'_\delta, \quad (17)
$$

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where $\Delta'_\delta$ is given by the following formula:

$$\Delta'_\delta := \left\{ \delta + \sum_{i=1}^{r} (\beta_i - |I_i| \alpha_i) \mid |I_i| = a_i(\delta) \text{ or } |I_i| = a_i(\delta) - 1 \right\}. \quad (18)$$

Here $I_i \subset \{2, \ldots, n_i\}$.

Proof. Consider the weight system $\Delta_{T\mathcal{N}}$ that is given by Formula (12). If we iterate this process we see that the weight system of $\tilde{\mathcal{N}}$ is given by

$$\Delta_{\tilde{\mathcal{N}}} = \bigcup_{\delta \in \Delta} \tilde{\Delta}_\delta, \quad \text{where} \quad \tilde{\Delta}_\delta := \left\{ \delta + \sum_{i=1}^{r} (\beta_i - |I_i| \alpha_i) \mid |I_i| = 0, 1, \cdots, n_i \right\}.\quad (18)$$

If we remove from $\tilde{\Delta}_\delta$ all weights with at least one negative coefficient and all weights with non-trivial multiplicities, we get (18). This finishes the proof. $\square$

6 Additional structures on $D_{\mathcal{N}} = \mathbb{F}(\mathcal{N})$

6.1 Odd commuting vector fields on $D_{\mathcal{N}}$

Recall that we denoted by $\mathcal{N}$ a graded manifold of type $\Delta$ and by $\tilde{\mathcal{N}}$ the $n$-times iterated tangent bundle of $\mathcal{N}$. It is a graded manifold of type $\Delta_{\tilde{\mathcal{N}}}$. Further, $n = r' - r$, where $r$ is the rank of $\Delta$ and $r'$ is the rank of $\Delta'$. On $\tilde{\mathcal{N}}$ there are $n$ odd commuting homological vector fields $d_{\beta_j}$ of weights $\beta_j - \alpha_i$, see Section 5.1. Our goal now is to show that these vector fields induce odd commuting homological vector fields on $D_{\mathcal{N}}$.

**Proposition 3.** The de Rham differentials (14) defined on $\tilde{\mathcal{N}}$ induce $n$ odd commuting homological vector fields on $D_{\mathcal{N}}$:

$$D_{\beta_1}, \ldots, D_{\beta_{n_1}}, D_{\beta_2}, \ldots, D_{\beta_{n_2}}, \ldots, D_{\beta_r}, \ldots, D_{\beta_{n_r}}. \quad (19)$$

Proof. By our weight agreement any vector field $d_{\beta_j}$ preserves the ideal $J^-$. Hence $d_{\beta_j}$ determines the vector field $D_{\beta_j}$ acting on the sheaf $O_{\tilde{\mathcal{N}}}/J^-$. Furthermore, by definition we have $O_{D_{\mathcal{N}}} \subset O_{\tilde{\mathcal{N}}}/J^-$. We need to show that $D_{\beta_j} ((O_{D_{\mathcal{N}}})_{\delta}) \subset O_{D_{\mathcal{N}}}$ for any $\delta \in \Delta'$. Consider the following inclusion:

$$D_{\beta_j} ((O_{D_{\mathcal{N}}})_{\delta}) = D_{\beta_j} ((O_{\tilde{\mathcal{N}}}/J^-)_{\delta}) \subset (O_{\tilde{\mathcal{N}}}/J^-)_{\delta + \beta_j - \alpha_i}.$$

Since $\delta$ is multiplicity free, the coefficient $a_i(\delta)$ before $\alpha_i$ is equal to 0 or 1, see (8) for notations. In case $a_i(\delta) = 0$, the weight $\delta + \beta_j - \alpha_i$ has a negative coefficient, hence

$$(O_{\tilde{\mathcal{N}}}/J^-)_{\delta + \beta_j - \alpha_i} = \{0\}.\quad (19)$$
In case \( a_i(\delta) = 1 \), the weight \( \delta + \beta_{ji} - \alpha_i \) has no negative coefficients. Since \( \beta_{ji} \) has no multiplicities in the weight system \( \Delta_N \), the sheaf \((O_N/J^-)_{\delta+\beta_{ji}-\alpha_i}\) is a product of subsheaves in \( O_N/J^- \) with multiplicity free weights. Hence it is a subsheaf in \( O_{D_N} \). The proof is complete. □

Some properties of the vector fields \( D_{\beta_{ji}} \) are described in the next propositions.

**Proposition 4.** The vector fields \( D_{\beta_{ji}} \) are \((O_{D_N})_0\)-linear.

**Proof.** Let us take \( f \in (O_N)_0 \) and a vector field \( d_{\beta_{ji}} \). Then the weight of \( d_{\beta_{ji}}(f) \) is equal to \( \beta_{ji} - \alpha_i \). It is a weight with a negative coefficient, therefore, \( D_{\beta_{ji}}(f) = 0 \). The result follows from the Leibniz rule. □

Let \( N \) and \( N_1 \) be two graded manifolds of type \( \Delta \). Denote by \( D_{\beta_{ji}} \) and \( D^1_{\beta_{ji}} \) the derivations on \( D_N \) and \( D_{N_1} \), respectively, as in Proposition 3.

**Proposition 5.** Let \( \psi : N \to N_1 \) be a morphism of graded manifolds of type \( \Delta \) and \( F(\psi) : D_N \to D_{N_1} \) be the corresponding morphism of \( r' \)-fold vector bundles. Then

\[
F(\psi)^* \circ D^1_{\beta_{ji}} = D_{\beta_{ji}} \circ F(\psi)^*.
\]

**Proof.** This follows from the definition of \( F(\psi) \) and the fact that all morphisms and the induced morphisms between tangent spaces commute with de Rham differentials. □

### 6.2 Description of \( D_N \) in local coordinates

Let us take \( \delta \in \Delta \) and \( \delta' = \sum_i a_i \alpha_i + \sum_{ji} b_{ji} \beta_{ji} \in \Delta'_\delta \), see (18) for the definition of \( \Delta'_\delta \). Then there exists the unique up to sign operator \( D_{\delta \to \delta'} \) that is equal to a composition of some \( D_{\beta_{ji}} \) or equal to the identity such that \( D_{\delta \to \delta'}(\delta) = \delta' \). The operator \( D_{\delta \to \delta'} \) is explicitly given by \( D_{\delta \to \delta'} = \pm D_{\beta_{ji_1}} \circ \cdots \circ D_{\beta_{jk_n}} \), where this composition is taken over all \( \beta_{ji} \)s such that \( b_{ji} \neq 0 \) in the expression for \( \delta' \). Let us choose a local chart \( U \) in \( N \) with local coordinates \( (\xi^\delta_i)_{\delta \in \Delta} \). Here the superscript \( \delta \) indicates the weight of coordinates. By our construction of \( D_N \) we obtain the following proposition.

**Proposition 6.** The ringed space \( (U_0, O_{D_N}|_{U_0}) \) is a local chart on \( D_N \) with the following local coordinates:

\[
\bigcup_{\delta \in \Delta} \{D_{\delta \to \delta'}(\xi^\delta_i) \mid \delta' \in \Delta'_\delta\}.
\]

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The coordinates (20) satisfy the following property. Let \( \delta, \delta' \) and \( D_{\delta \rightarrow \delta'} \) be as above. Then \( \eta^{\delta'}_i := D_{\delta \rightarrow \delta'}(\xi^{\delta'}_i) \) satisfies the equation \( D_{\beta_{j_{1}s}}(\eta^{\delta'}_i) = 0 \) for all \( D_{\beta_{j_{1}s}} \) such that \( b_{j_{1}s} \neq 0 \) in the expression for \( \delta' \). In other words, \( \eta^{\delta'}_i \in \text{Ker} D_{\beta_{j_{1}s}} \) for all such \( \beta_{j_{1}s} \). From this observation we get the following proposition.

**Proposition 7.** Let us take \( \delta' = \sum_i a_i \alpha_i + \sum ji b_{ji} \beta_{ji} \in \Delta' \). The sheaf \( (\mathcal{O}_{\mathcal{D}_N})^{\delta'} \) possesses the following decomposition:

\[
(\mathcal{O}_{\mathcal{D}_N})^{\delta'} = \bigoplus_{\delta'_1 + \delta'_2 = \delta'} (\mathcal{O}_{\mathcal{D}_N})^{\delta'_1}(\mathcal{O}_{\mathcal{D}_N})^{\delta'_2} + (\mathcal{O}_{\mathcal{D}_N})^{\delta'} \bigcap_{b_{j_{1}s} \neq 0} \text{Ker} D_{\beta_{j_{1}s}},
\]

where \( \delta'_i \neq 0 \).

### 6.3 Properties of the structure sheaf of \( \mathcal{D}_N \)

Recall that \( \mathcal{N} \) is a graded manifold of type \( \Delta \), \( \tilde{\mathcal{N}} \) is the \( n \)-times iterated tangent bundle of \( \mathcal{N} \), it is a graded manifold of type \( \Delta_{\mathcal{N}} \), and \( \mathcal{D}_N = F(\mathcal{N}) \) is a graded manifold of type \( \Delta' \), see Proposition 2 for the definition of \( \Delta' \). If \( \gamma \) is a certain weight in the weight lattice generated by \( \alpha_i, \beta_{ji} \), then by definition we put

\[
d_{\beta_{ji}}(\gamma) = D_{\beta_{ji}}(\gamma) := \gamma + \beta_{ji} - \alpha_i.
\]

Let us take a subset \( \Lambda = \{\gamma_1, \ldots, \gamma_s\} \) in the set (10). Denote by \( \bar{\Lambda} = (\gamma_1, \ldots, \gamma_s) \) the same set \( \Lambda \), but with a certain order, and by \( D^{\bar{\Lambda}} \) the following composition:

\[
D^{\bar{\Lambda}} : (\mathcal{O}_{\mathcal{N}}) \hookrightarrow (\mathcal{O}_{\tilde{\mathcal{N}}}) \twoheadrightarrow (\mathcal{O}_{\mathcal{N}})/J^-, \quad D^{\bar{\Lambda}} := d_{\gamma_1} \circ \cdots \circ d_{\gamma_s} \mod J^-.
\]

Note that the underlying spaces \( \mathcal{N}_0 \) and \( (\mathcal{D}_N)_0 \) of graded manifolds \( \mathcal{N} \) and \( \mathcal{D}_N \), respectively, coincide. Hence we can identify their structure sheaves \( (\mathcal{O}_{\mathcal{N}})_0 = (\mathcal{O}_{\mathcal{D}_N})_0 \). As in the proof of Proposition 4 we can see that the map of sheaves \( D^{\bar{\Lambda}} \) is \( (\mathcal{O}_{\mathcal{D}_N})_0 \)-linear. Hence, \( D^{\bar{\Lambda}} \) is a morphism of sheaves of \( (\mathcal{O}_{\mathcal{D}_N})_0 \)-modules.

Let us take a weight \( \delta = \sum_{i=1}^r a_i \alpha_i \), where \( a_i \geq 0 \), in the weight lattice generated by \( \alpha_i \). (Note that \( \delta \) is not necessary from \( \Delta \).

Then the weight \( D^{\bar{\Lambda}}(\delta) \) is defined by (21). Assume that \( D^{\bar{\Lambda}}(\delta) \) is multiplicity free and does not have negative coefficients. Then we have the following morphism:

\[
D^{\bar{\Lambda}} : (\mathcal{O}_{\mathcal{N}})^\delta \to (\mathcal{O}_{\mathcal{D}_N})^{D^{\bar{\Lambda}}(\delta)},
\]
Note that in (23) the weight $D^\Lambda(\delta)$ is not necessary from $\Delta'$. However, for any multiplicity free weight $\theta$ we have $(\mathcal{O}_{D_N})_\theta = (\mathcal{O}_{\overline{N}}/J^-)_\theta$.

We will need the following proposition.

**Proposition 8.** Assume that $\delta = \sum_{i=1}^{r} a_i \alpha_i$, where $a_i \geq 0$, and the weight $D^\Lambda(\delta)$ is multiplicity free and does not have negative coefficients. Then the morphism (23) is injective.

**Proof.** The idea of the proof is to use the following fact: the kernel of the de Rham differential for supermanifolds (as for usual manifolds) restricted to functions coincides with the vector space of constant functions. A detailed proof can be found in Appendix. □

Let us demonstrate the result of Proposition 8 on an example. Let as take a graded domain $\mathcal{U}$ with coordinates $\xi_1, \xi_2$ of weight $\alpha_1$ and of parity $\overline{1}$. Assume that $\overline{\Lambda} = (\gamma_1, \gamma_2)$. Since,

$$D^\Lambda(\xi_1 \cdot \xi_2) = d_{\gamma_1} \circ d_{\gamma_2}(\xi_1 \cdot \xi_2) \mod \mathcal{J}^- = d_{\gamma_1} (d_{\gamma_2}(\xi_1) \cdot \xi_2 - \xi_1 \cdot d_{\gamma_2}(\xi_2)), \mod \mathcal{J}^-,$$

the restriction $D^\Lambda \mid (\mathcal{O}_{\mathcal{U}})_{2\alpha_1}$ is injective.

The next proposition describes the image of $D^\Lambda$ in some particular cases.

**Proposition 9.** Assume that $\delta = \sum_{i=1}^{r} a_i \alpha_i$, where $a_i \geq 0$, $D^\Lambda(\delta) = \sum_{i=1}^{r} a_i \alpha_i + \sum_{j} b_{ji} \beta_{ji}$ is multiplicity free, does not have negative coefficients and $D^\Lambda(\delta)$ satisfies the following property: if $b_{st} \neq 0$, then $a_t \neq 0$. Then we have

$$D^\Lambda \left((\mathcal{O}_{\mathcal{N}})_\delta\right) = \left((\mathcal{O}_{D_N})_{D^\Lambda(\delta)}\right) \bigcap_{k=1}^{s} \text{Ker} D_{\gamma_k}$$

and the map

$$D^\Lambda : (\mathcal{O}_{\mathcal{N}})_\delta \rightarrow \left((\mathcal{O}_{D_N})_{D^\Lambda(\delta)}\right) \bigcap_{k=1}^{s} \text{Ker} D_{\gamma_k}$$

is an isomorphism.

**Proof.** The idea of the proof is to use the Poincaré Lemma for supermanifolds: any closed differential form is locally exact. Details can be found in Appendix. □

Let us consider an example. Again we take the graded domain $\mathcal{U}$ with coordinates $\xi_1, \xi_2$ of weight $\alpha_1$ and of parity $\overline{1}$ and we put $\overline{\Lambda} = (\gamma_1, \gamma_2)$. 

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Assume that $\delta = 2\alpha_1$, then $D^\Lambda(\delta) = \gamma_1 + \gamma_2$. We see that $D^\Lambda(\delta)$ does not satisfy the property: if $b_{st} \neq 0$, then $a_t \neq 0$. In this case Proposition 9 is wrong since $\text{Im}(D^\Lambda)$ does not contain for example $d_\gamma_1(\xi_1) \cdot d_\gamma_2(\xi_2) \in \cap_{k=1}^2 \ker D_\gamma_k$.

We will need the following corollary:

**Corollary.** Assume that $\delta = \sum_{i=1}^r a_i \alpha_i + \sum_{ji} b_{ji} \beta_{ji} \in \Delta'$ and $a_{i_0}, b_{j_0 i_0} \neq 0$ for some indexes $i_0$ and $(j_0 i_0)$. Let us take

$$f \in (\mathcal{O}_{D_N})_{\delta} \bigcap_{b_{st} \neq 0} \ker D_{\beta_{st}}.$$  

Then there exists $F \in (\mathcal{O}_{D_N}/\mathcal{J}^-)_{\delta - \beta_{j_0 i_0} + \alpha_i}$ such that

$$(d_{\beta_{j_0 i_0}} \mod \mathcal{J}^{-})(F) = f \text{ and } (d_{\beta_{st}} \mod \mathcal{J}^{-})(F) = 0$$

for any $(st) \neq (j_0 i_0)$ such that $b_{st} \neq 0$.□

Proof follows from the proof of Proposition 9, see Appendix.□

Further properties of the commuting vector fields $D_{\beta_{ji}}$ are described in the following proposition.

**Proposition 10.** Let us take $\delta \in \Delta'$. If also $D_{\beta_{ji}}(\delta) \in \Delta'$, then

$$D_{\beta_{ji}} : (\mathcal{O}_{D_N})_{\delta} \to (\mathcal{O}_{D_N})_{D_{\beta_{ji}}(\delta)} \quad (24)$$

is an isomorphism of $(\mathcal{O}_{D_N})_0$-locally free sheaves. In particular, all maps

$$D_{\beta_{ji}} : (\mathcal{O}_{D_N})_{\alpha_i} \to (\mathcal{O}_{D_N})_{\beta_{ji}},$$

are isomorphisms of $(\mathcal{O}_{D_N})_0$-locally free sheaves.

**Proof.** Recall that $D_{\beta_{ji}}$ is $(\mathcal{O}_{D_N})_0$-linear by Proposition 4. Consider a chart $U$ on $\mathcal{M}$. Clearly this chart determines a chart on $D_N$. We choose coordinates $(x_p)$, $(\xi_q)$ and $(\eta_t)$ such that $x_p$ are local coordinates of weight 0, $\xi_q$ are coordinates with weights in the form $\alpha_i + \ldots$, and $\eta_t$ are other local coordinates.

Any $f \in (\mathcal{O}_{D_N})_{\delta}$ has the following form $f = \sum_{kl} f_{kl} \xi_k \eta^l$, where $I$ as a multi-index and $f_{kl}$ are functions of weight 0. Since $D_{\beta_{ji}}(\delta) \in \Delta'$ is multiplicity free, we see that $\delta$ does not depend on $\beta_{ji}$. The map (24) in coordinates is given by the following formula:

$$D_{\beta_{ji}}(f) = D_{\beta_{ji}} \left( \sum_{kl} f_{kl} \xi_k \eta^l \right) = \sum_{kl} f_{kl} D_{\beta_{ji}}(\xi_k) \eta^l.$$  

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We see that $D_{\beta ji}(\xi_k)$ and $\eta_q$ form a subset of independent local coordinates in $D_N$, since $D_{\beta ji}(\xi_k)$ and $\eta_q$ have different weights. Note that any function in $O_{D_N}$ of weight $D_{\beta ji}(\delta)$ has the form $\sum_{kI} f_{kI} D_{\beta ji}(\xi_k) \eta^I$. Therefore, the inverse map $\sum_{kI} f_{kI} D_{\beta ji}(\xi_k) \eta^I \mapsto \sum_{kI} f_{kI} \xi_k \eta^I$ of the map (24) is well-defined. □

The vector fields satisfying (24) we will call non-degenerate.

6.4 Combinatorial properties of odd commuting vector fields $D_{\beta ji}$

Some properties of odd commuting vector fields $D_{\beta ji}$ can be described using the combinatorics of the weight system $\Delta'$. Let us take $\delta, \delta' \in \Delta'$ and two vector fields $D_{\beta ji}$ and $D_{\beta si}$ such that

$$D_{\beta ji}(\delta) = D_{\beta si}(\delta') \in \Delta'.$$

Explicitly this means that $\delta = \alpha_i + \beta_{si} + \theta$ and $\delta' = \alpha_i + \beta_{ji} + \theta$ for a certain weight $\theta$. Then

$$D_{\beta ji} : (O_{D_N})_{\delta} \to (O_{D_N})_{D_{\beta ji}(\delta)} \quad \text{and} \quad D_{\beta si} : (O_{D_N})_{\delta'} \to (O_{D_N})_{D_{\beta si}(\delta)}$$

are isomorphisms, see Proposition 10. Hence the following isomorphism of sheaves is defined

$$D_{\beta si}^{-1} \circ D_{\beta ji} : (O_{D_N})_{\delta} \to (O_{D_N})_{\delta'}.$$

Explicitly on weights we have

$$D_{\beta si}^{-1} \circ D_{\beta ji}(\delta) = D_{\beta si}^{-1} \circ D_{\beta ji}(\alpha_i + \beta_{si} + \theta) = \alpha_i + \beta_{ji} + \theta = \delta'.$$

Assume that $\delta = \sum_{i=1}^{r} a_i \alpha_i + \sum_{ji} b_{ji} \beta_{ji}$ and $\delta' = \sum_{i=1}^{r} a'_i \alpha_i + \sum_{ji} b'_{ji} \beta_{ji}$. For $\delta \in \Delta'$ we put

$$S_\delta := \left( (O_{D_N})_{\delta} \bigcap \ker D_{\beta pt} \right). \quad (25)$$

Similarly we define the sheaf $S_{\delta'}$

We will need the following proposition.

**Proposition 11.** Assume that $b_{ji} = 0$ and $a_i, b_{si} \neq 0$ for indexes $i, (ji)$ and $(si)$. Then we have

$$(D_{\beta si}^{-1} \circ D_{\beta ji})(S_\delta) = S_{\delta'}.$$

**Proof.** It is enough to show only the following inclusion

$$(D_{\beta si}^{-1} \circ D_{\beta ji})(S_\delta) \subset S_{\delta'}.$$
Let us take \( f \in S_\delta \). In Corollary of Proposition 9, we have seen that since \( D_{\beta_\alpha}(f) = 0 \), there exists \( F \in (O_N^N/\mathcal{J})_{\delta-\beta_\alpha+\alphai} \) such that \( f = d_{\beta_\alpha}(F) \mod \mathcal{J}^- \).

We have

\[
(D_{\beta_\alpha}^{-1} \circ D_{\beta_{ji}})(f) = (D_{\beta_\alpha}^{-1} \circ D_{\beta_{ji}} \circ (d_{\beta_\alpha} \mod \mathcal{J}^-))(F) = - (D_{\beta_\alpha}^{-1} \circ D_{\beta_{ji}} \circ (d_{\beta_\alpha} \mod \mathcal{J}^-))(F) = - (d_{\beta_{ji}} \circ \mathcal{J}^-)(F) \in (O_{\mathcal{N}}^N)_\delta \cap \text{Ker } D_{\beta_{ji}}.
\]

Further, again by Corollary of Proposition 9, we have \( d_{\beta_{pt}}(F) \mod \mathcal{J}^- = 0 \), where \( b_{pt} \neq 0 \). Hence

\[
D_{\beta_{pt}} \circ (D_{\beta_\alpha}^{-1} \circ D_{\beta_{ji}})(f) = - D_{\beta_{pt}} \circ (d_{\beta_{ji}} \mod \mathcal{J}^-)(F) = 0.
\]

The proof is complete. \( \square \)

Let us take \( \delta, \delta_1, \delta_2 \in \Delta' \) in the following form:

\[
\delta = \alpha_i + \beta_{ji} + \theta, \quad \delta_1 = \alpha_i + \beta_{j_{1i}} + \theta, \quad \delta_2 = \alpha_i + \beta_{j_{2i}} + \theta,
\]

where \( j \neq j_1, j \neq j_2 \) and \( j_1 \neq j_2 \). Note that since \( \Delta' \) is multiplicity free, \( \theta \) does not depend on \( \alpha_i, \beta_{ji}, \beta_{j_{1i}} \) and \( \beta_{j_{2i}} \).

**Proposition 12.** Let \( \delta, \delta_1, \delta_2 \) be as above. Then we have:

\[
(D_{\beta_{ji}}^{-1} \circ D_{\beta_{j_{2i}}})(|_{O_{\mathcal{N}}^N\cap \text{Ker } D_{\beta_{ji}}}) = -(D_{\beta_{j_{1i}}}^{-1} \circ D_{\beta_{j_{2i}}})\circ (D_{\beta_{ji}}^{-1} \circ D_{\beta_{j_{2i}}})(|_{O_{\mathcal{N}}^N\cap \text{Ker } D_{\beta_{ji}}}). \tag{26}
\]

We will call (12) the **cocycle condition** for our vector fields.

**Proof.** Let us take \( f \in (O_{\mathcal{N}}^N)_{\delta} \cap \text{Ker } D_{\beta_{ji}} \). Again by Corollary of Proposition 9, we can find \( F \) such that \( f = (d_{\beta_{ji}} \mod \mathcal{J}^-)(F) \). We have

\[
(D_{\beta_{ji}}^{-1} \circ D_{\beta_{j_{2i}}})(d_{\beta_{ji}} \mod \mathcal{J}^-)(F) = -(d_{\beta_{j_{2i}}} \mod \mathcal{J}^-)(F).
\]

On the other hand,

\[
(D_{\beta_{j_{1i}}}^{-1} \circ D_{\beta_{j_{2i}}} \circ (D_{\beta_{ji}}^{-1} \circ D_{\beta_{j_{1i}}}) \circ (d_{\beta_{ji}} \mod \mathcal{J}^-))(F) = -(D_{\beta_{j_{1i}}}^{-1} \circ D_{\beta_{j_{2i}}})(d_{\beta_{j_{1i}}} \mod \mathcal{J}^-)(F) = (d_{\beta_{j_{2i}}} \mod \mathcal{J}^-)(F).
\]

The proof is complete. \( \square \)

**Remark.** Let \( \delta, \delta_1, \delta_2 \) be as in Proposition 12. Let us show that (26) does not hold for any \( f \in (O_{\mathcal{N}}^N)_{\delta} \). In other words the assumption \( f \in (O_{\mathcal{N}}^N)_{\delta} \cap \text{Ker } D_{\beta_{ji}} \) is essential. Let us take two variables \( \xi_1, \xi_2 \) of weight \( \alpha_i \). Then \( f =
\[ \xi_1 \cdot D_{\beta_{ji}}(\xi_2) \text{ has the weight } \alpha_i + \beta_{ji}. \quad \text{Further, } D_{\beta_{ji}}(f) = D_{\beta_{ji}}(\xi_1) \cdot D_{\beta_{ji}}(\xi_2) \neq 0. \] 

Hence, \( f \notin \text{Ker } D_{\beta_{ji}}. \) Applying the left hand side of (26), we get:

\[ (D^{-1}_{\beta_{ji}} \circ D_{\beta_{j2i}})(\xi_1 \cdot D_{\beta_{ji}}(\xi_2)) = D^{-1}_{\beta_{ji}}(D_{\beta_{j2i}}(\xi_1) \cdot D_{\beta_{ji}}(\xi_2)) = \pm D_{\beta_{j2i}}(\xi_1) \cdot \xi_2. \]

Further,

\[ (D^{-1}_{\beta_{ji}} \circ D_{\beta_{j2i}}) \circ (D^{-1}_{\beta_{ji}} \circ D_{\beta_{j1i}})(\xi_1 \cdot D_{\beta_{ji}}(\xi_2)) = \pm (D^{-1}_{\beta_{j1i}} \circ D_{\beta_{j2i}})(D_{\beta_{ji}}(\xi_1) \cdot \xi_2) = \pm \xi_1 \cdot D_{\beta_{j2i}}(\xi_2). \]

We see that the results are different.

7 Equivalence of categories

7.1 The category of \( r' \)-fold vector bundles with \( n \) odd commuting non-degenerate vector fields

In this section we introduce the category \( \Delta'\text{VBVect} \). This is a category of \( r' \)-fold vector bundles of type \( \Delta' \) with \( n \) odd commuting non-degenerate vector fields. More precisely, let \( \Delta' \) be a weight system with the following set of basic weights:

\[ \{ \alpha_i, \beta_{ji} \mid i = 1, \ldots, r, \ j = 2, \ldots, n_i \}, \]

where \( n_i \geq 2 \) and \( i = 1, \ldots, r \) are some non-negative integers. (See (1) for the definition of basic weights.) We put \( n := \sum_{i=1}^{r} n_i - r \) and \( r' = n + r \). Note that \( r' \) is the rank of \( \Delta' \). Let \( D \) be an \( r' \)-fold vector bundle of type \( \Delta' \). Assume in addition that on \( D \) we have \( n \) odd vector fields \( D_{\beta_{ji}} \) of weights \( \beta_{ji} - \alpha_i \). Assume that these vector fields have the following properties:

1. The vector fields \( D_{\beta_{ji}} \) are \((O_D)_0\)-linear.

2. The vector fields \( D_{\beta_{ji}} \) super-commute:

\[ [D_{\beta_{ji}}, D_{\beta_{j'i'}}] = 0 \]

for all \((ji)\) and \((j'i')\). In particular, any \( D_{\beta_{ji}} \) satisfy the condition \( D_{\beta_{ji}}^2 = 0 \).

3. The operators \( D_{\beta_{ji}} \) are non-degenerate in the following sense. Let us take \( \delta \in \Delta' \). As above we put \( D_{\beta_{ji}}(\delta) := \delta + \beta_{ji} - \alpha_i \). We call an odd vector field \( D_{\beta_{ji}} \) of weight \( \beta_{ji} - \alpha_i \) non-degenerate, if it satisfies
conditions of Proposition 10 for any $\delta$. More precisely, if $D_{\beta_{ji}}(\delta) \in \Delta'$ for a certain $\delta \in \Delta'$, then the following map

$$D_{\beta_{ji}} : (\mathcal{O}_D)_{\delta} \to (\mathcal{O}_D)_{D_{\beta_{ji}}(\delta)}$$

is an isomorphism of sheaves of $(\mathcal{O}_D)_0$-modules.

4. Let us take $\delta = \sum_i a_i \alpha_i + \sum_j \beta_{ji} \in \Delta'$. We assume that the sheaf $(\mathcal{O}_D)_{\delta}$ possesses the following decomposition:

$$(\mathcal{O}_D)_{\delta} = (\mathcal{O}_D)_{\delta} \bigcap \ker D_{\beta_{st}} + \bigoplus_{\delta_1 + \delta_2 = \delta} (\mathcal{O}_D)_{\delta_1} (\mathcal{O}_D)_{\delta_2},$$

where $\delta_1, \delta_2 \neq 0$.

5. Let $\delta, D_{\beta_{ji}}, D_{\beta_{ji}'},$ and $D_{\beta_{ji}'}$ be as in Proposition 12. The vector fields $D_{\beta_{ji}}, D_{\beta_{ji}'}$ and $D_{\beta_{ji}'}$ satisfy the following cocycle condition:

$$(D_{\beta_{ji}'})^{-1} \circ D_{\beta_{ji}'} \big| (\mathcal{O}_D)_{\delta} \bigcap \ker D_{\beta_{ji}} = -(D_{\beta_{ji}}^{-1} \circ D_{\beta_{ji}'} \circ (D_{\beta_{ji}'})^{-1} \circ D_{\beta_{ji}'})) \big| (\mathcal{O}_D)_{\delta} \bigcap \ker D_{\beta_{ji}}.$$

6. Let $\delta, \delta' \in \Delta', D_{\beta_{ji}0}$ and $D_{\beta_{ji}0}$ be as in Proposition 11. Our vector fields preserve the kernels in the following sense:

$$(D_{\beta_{ji}0}^{-1} \circ D_{\beta_{ji}0}) \big| ((\mathcal{O}_D)_{\delta} \bigcap \ker D_{\beta_{st}}) = (\mathcal{O}_D)_{\delta'} \bigcap \ker D_{\beta_{st}}.$$

In other words this means that the operator $D_{\beta_{ji}0}^{-1} \circ D_{\beta_{ji}0}$ preserves the decomposition from item 4.

The category of $r'$-fold vector bundles of type $\Delta'$ with $n$ odd vector fields of weight $\beta_{ji} - \alpha_i$ satisfying the properties 1 – 6 we denote by $\Delta'\text{VBVect}$. A morphism in this category is a morphism in the category of $r'$-fold vector bundles of type $\Delta'$ that commutes with all vector fields.

It follows from Propositions 7, 4, 5, 10, 11 and 12 that the image of the functor $F$ is contained in $\Delta'\text{VBVect}$. In the next sections we will prove that $F$ defines an equivalence of categories. To do this we will use the following definition.

**Definition 7.** Two categories $\mathcal{C}$ and $\mathcal{C}'$ are called equivalent if there is a functor $F : \mathcal{C} \to \mathcal{C}'$ such that:

- $F$ is full and faithful, this is $\text{Hom}_\mathcal{C}(c_1, c_2)$ is in bijection with $\text{Hom}_{\mathcal{C}'}(Fc_1, Fc_2)$.
- $F$ is essentially surjective, this is for any $a \in \mathcal{C}'$ there exists $b \in \mathcal{C}$ such that $a$ is isomorphic to $F(b)$.
7.2 Graded manifolds of degree 2 and double vector bundles with an odd homological vector field

In this section we establish a correspondence between graded manifolds of degree 2 and double vector bundles with an odd non-degenerate homological vector field. Recall that graded manifolds of type \( \{0, \alpha, 2\alpha\} \) are usually called in the literature \textit{graded manifolds of degree 2}. Below we give two constructions. First of all we assign a double vector bundle to a graded manifold of degree 2 and then we reconstruct a graded manifold corresponding to a double vector bundle with an odd non-degenerate vector field.

The result of this section is equivalent to the statement about the equivalence of categories of graded manifolds of degree 2 and of double vector bundles with an involution (or a metric) obtained in [CM] (in [JL]).

Construction 1. Consider a graded manifold \( \mathcal{M}_2 \) of degree 2 or, in other words, a graded manifold \( \mathcal{M}_2 \) of type \( \Delta = \{0, \alpha, 2\alpha\} \). In this case \( \mathbb{F}(\mathcal{M}_2) =: \mathbf{D}_{\mathcal{M}_2} \) is a double vector bundle with basic weights \( \alpha := \alpha_1 \) and \( \beta := \beta_{21} \). The weight system of \( \mathbf{D}_{\mathcal{M}_2} \) has the following form:

\[ \Delta' = \{0, \alpha, \beta, \alpha + \beta\}. \]

On \( \mathbf{D}_{\mathcal{M}_2} \) we have an odd linear homological vector field \( D_\beta := d_\beta \mod J^- \) such that \( D_\beta \) is \textit{non-degenerate} and has \textit{weight} \( \beta - \alpha \). In this case the non-degeneracy of \( D_\beta \) means that the following map

\[ D_\beta : (\mathcal{O}_{\mathcal{M}_2})_\alpha \to (\mathcal{O}_{\mathcal{M}_2})_{\beta} \]

is an isomorphism of sheaves of \( (\mathcal{O}_{\mathcal{M}_2})_0 \)-modules.

Construction 2. Let us show that any double vector bundle \( \mathbf{D} \) or a graded manifold of type \( \Delta' = \{0, \alpha, \beta, \alpha + \beta\} \) with an odd non-degenerate linear homological vector field \( \mathbf{D} \) of weight \( \beta - \alpha \) is isomorphic to a double vector bundle in the form \( \mathbf{D}_{\mathcal{M}_2} := \mathbb{F}(\mathcal{M}_2) \), where \( \mathcal{M}_2 \) is a certain graded manifold of type \( \Delta = \{0, \alpha, 2\alpha\} \). We also will show that this isomorphism commutes with operators \( D_\beta \) and \( \mathbf{D} \), which are defined on \( \mathbf{D}_{\mathcal{M}_2} \) and \( \mathbf{D} \), respectively.

Step 1, exact sequence. Consider the subsheaf \( (\mathcal{O}_{\mathbf{D}})_{\alpha+\beta} \) in \( \mathcal{O}_{\mathbf{D}} \), where \( \mathcal{O}_{\mathbf{D}} \) is the structure sheaf of \( \mathbf{D} \). We have the following exact sequence of sheaves of \( (\mathcal{O}_{\mathbf{D}})_0 \)-modules:

\[ 0 \to (\mathcal{O}_{\mathbf{D}})_\alpha (\mathcal{O}_{\mathbf{D}})_\beta \to (\mathcal{O}_{\mathbf{D}})_{\alpha + \beta} \to \mathcal{E} \to 0. \]

Here \( \mathcal{E} \) is a certain locally free sheaf of \( (\mathcal{O}_{\mathbf{D}})_0 \)-modules. A standard argument shows that the following sequence is also exact

\[ 0 \to \text{Ker} \mathbf{D} \cap ((\mathcal{O}_{\mathbf{D}})_\alpha (\mathcal{O}_{\mathbf{D}})_\beta) \to \text{Ker} \mathbf{D} \cap (\mathcal{O}_{\mathbf{D}})_{\alpha + \beta} \to \text{Ker} \mathbf{D}' \to 0, \]
where
\[ D' : \mathcal{E} \to D((\mathcal{O}_D)_{\alpha+\beta})/D((\mathcal{O}_D)_\alpha(\mathcal{O}_D)_\beta) \]
is the map induced by D. Since D is non-degenerate and \( D((\mathcal{O}_D)_\beta) = D^2((\mathcal{O}_D)_\alpha) = 0 \), the following map
\[ D : (\mathcal{O}_D)_\alpha(\mathcal{O}_D)_\beta \to (\mathcal{O}_D)_\beta(\mathcal{O}_D)_\beta \]
is surjective. Since \( D \) has weight \( \beta - \alpha \) and since \( 2\beta \notin \Delta' \), we have
\[ D((\mathcal{O}_D)_{\alpha+\beta}) \subset (\mathcal{O}_D)_{2\beta} \text{ and } (\mathcal{O}_D)_{2\beta} = (\mathcal{O}_D)_\beta(\mathcal{O}_D)_\beta. \]
Hence,
\[ D'((\mathcal{O}_D)_{\alpha+\beta}) = (\mathcal{O}_D)_\beta(\mathcal{O}_D)_\beta, \]
the map \( D' \) is trivial and \( \ker D' = \mathcal{E} \). Therefore we have the following exact sequence:
\[
0 \to \ker D \cap ((\mathcal{O}_D)_\alpha(\mathcal{O}_D)_\beta) \to \ker D \cap (\mathcal{O}_D)_{\alpha+\beta} \to \mathcal{E} \to 0. \tag{27}
\]

**Step 2, construction of \( \mathcal{M}_2 \).** The idea is to show that the following data:
\[
\mathcal{O}_0 = (\mathcal{O}_D)_0, \quad \mathcal{O}_\alpha = (\mathcal{O}_D)_\alpha, \quad \mathcal{O}_{2\alpha} = (\mathcal{O}_D)_{\alpha+\beta} \cap \ker D
\]
defines a graded manifold \( \mathcal{M}_2 \) of degree 2. To simplify notations we denoted here by \( \mathcal{O} \) the structure sheaf of \( \mathcal{M}_2 \). First of all consider the graded manifold \( \mathcal{M}_1 \) of type \( \{0, \alpha\} \) with the structure sheaf \( S_{\mathcal{O}_0}^*(\mathcal{O}_\alpha) \) and the sheaf \( \mathcal{O}_{T\mathcal{M}_1}/\mathcal{J}^- \) with the vector field \( d \bmod \mathcal{J}^- \) defined as above, i.e induced by the de Rham differential on \( \mathcal{O}_{T\mathcal{M}_1} \). We set \( \mathcal{O}_\beta := (d \bmod \mathcal{J}^-)(\mathcal{O}_\alpha) \). By Proposition 9 or by a direct computation, we have
\[
(d \bmod \mathcal{J}^-)(\mathcal{O}_\alpha \cdot \mathcal{O}_\beta) = (\mathcal{O}_\alpha \cdot \mathcal{O}_\beta) \cap \ker(d \bmod \mathcal{J}^-). \tag{28}
\]
Now we can define an isomorphism of sheaves of ringed spaces
\[
\Theta : \mathcal{O}_{T\mathcal{M}_1}/\mathcal{J}^- = S^*(\mathcal{O}_\alpha \oplus \mathcal{O}_\beta) \to S^*((\mathcal{O}_D)_\alpha \oplus (\mathcal{O}_D)_\beta)
\]
in the following way:
\[
\Theta|_{\mathcal{O}_\alpha} := \text{id}, \quad \Theta|_{\mathcal{O}_\beta} := D \circ (d \bmod \mathcal{J}^-)^{-1}.
\]
Clearly, \( \Theta \) preserves all weights and \( \Theta \circ (d \bmod \mathcal{J}^-) = D \circ \Theta \). Therefore,
\[
\Theta((\mathcal{O}_\alpha \cdot \mathcal{O}_\beta) \cap \ker(d \bmod \mathcal{J}^-)) = ((\mathcal{O}_D)_\alpha(\mathcal{O}_D)_\beta) \cap \ker D. \tag{29}
\]
Combining (27), (28) and (29), we get the following exact sequence:

$$0 \rightarrow O_\alpha \cdot O_\alpha \xrightarrow{\chi} (O_D)_{\alpha+\beta} \cap \text{Ker } D \rightarrow E \rightarrow 0.$$  \hfill (30)

Here $\chi = \Theta \circ (\text{d mod } J^-)$ is an injective map. Further, we put

$$O_{2\alpha} := (O_D)_{\alpha+\beta} \cap \text{Ker } D.$$

By Construction 2, Section 4.2, the exact sequence (30) determines a graded manifold of degree 2, which we denote by $\mathcal{M}_2$.

**Step 3, construction of an isomorphism $\mathbb{F}(\mathcal{M}_2) \simeq D$.** To define an isomorphism $\mathbb{F}(\mathcal{M}_2) \simeq D$ we use Proposition 1. For simplicity of notations we denote the structure sheaf of $\mathbb{F}(\mathcal{M}_2)$ by $O'$. By definition and by properties of the functor $\mathbb{F}$ we have

$$O'_\alpha = O_\alpha, \quad O'_\beta = O_\beta, \quad O'_{\alpha+\beta} = O'_\alpha \cdot O'_\beta + (\text{d mod } J^-)(O_{2\alpha}).$$

The last equality follows from Proposition 7. We will use this decomposition to define an isomorphism $\mathbb{F}(\mathcal{M}_2) \rightarrow D$.

Consider the following commutative diagram:

$$
\begin{array}{c}
\text{d mod } J^-)(O_{2\alpha}) \, \xleftarrow{\text{d mod } J^-} \, O_{2\alpha} \quad \xrightarrow{\chi} \, (O_D)_{\alpha+\beta} \cap \text{Ker } D \\
\text{d mod } J^-)(O_\alpha \cdot O_\alpha) \, \xleftarrow{\text{d mod } J^-} \, O_\alpha \cdot O_\alpha \quad \xrightarrow{\chi} \, ((O_D)_\alpha(O_D)_\beta) \cap \text{Ker } D
\end{array}
$$

Note that the right square is commutative by definition. Here all horizontal maps are isomorphisms and all vertical maps are inclusions. By Proposition 1, we need to define isomorphisms $\varphi_\delta$, where $\delta \in \Delta'$. We put

$$\varphi_\alpha|_{O_\alpha} = \Theta|_{O_\alpha}, \quad \varphi_\beta|_{O_\beta} = \Theta|_{O_\beta}, \quad \varphi_{\alpha+\beta}|_{O'_\alpha O'_\beta} = \varphi_\alpha \varphi_\beta|_{O'_\alpha O'_\beta}, \quad \varphi_{\alpha+\beta}|(\text{d mod } J^-)(O_{2\alpha}) = (\text{d mod } J^-)^{-1}.$$

In the last line we use the identification $O_{2\alpha} = (O_D)_{\alpha+\beta} \cap \text{Ker } D$. Since the diagram above is commutative, the conditions of Proposition 1 hold and the injective map is defined. Since the sequence (27) is exact, $(O_D)_{\alpha+\beta} = (O_D)_{\alpha+\beta} \cap \text{Ker } D + (O_D)_\alpha(O_D)_\beta$. Hence the map $\varphi$ defined by $(\varphi_\delta)$ is an isomorphism. Clearly $\varphi$ commutes with $D_\beta = (\text{d mod } J^-)$ and $D$.

**Remark.** We have seen that the decomposition $(O_D)_{\alpha+\beta} = (O_D)_{\alpha+\beta} \cap \text{Ker } D + (O_D)_\alpha(O_D)_\beta$ follows from exactness of (27) in the case of graded manifolds of degree 2. Hence in this case it is enough to assume that the vector field $D$ is linear and non-degenerate. (Note that the vector field $D$ is homological due to the weight agreement.)
7.3 Two additional functors

Let us fix a weight system ∆ satisfying Definition 1 and the weight system
∆' = ∆'(∆) as in Proposition 2. Recall that we denoted by r and r' the
ranks of ∆ and ∆', respectively. There is a projection \( G : \Delta' \to \Delta \) that is
defined in the following way. Let us take
\[
\delta' = \sum_{k \in K} \alpha_{ik} + \sum_{(s,t) \in S \times T} \beta_{j_{st}} \in \Delta'
\]
for certain sets \( K, S \) and \( T \). We set
\[
G(\delta) := \sum_{k \in K} \alpha_{ik} + \sum_{(s,t) \in S \times T} \alpha_{it}.
\]
In other words, we replace any \( \beta_{j_{st}} \) by \( \alpha_{it} \).

**Proposition 13.** Let us take \( \delta \in \Delta \). We have
\[
G^{-1}(\delta) = \Delta'_b,
\]
where \( \Delta'_b \) is given by (18)

*Proof* follows from definitions. □

Denote by \( \Delta'_{<\beta_{ji}} \) and \( \Delta'_{=\beta_{ji}} \) the weight subsystems in \( \Delta' \) generated by the sets
\[
A_{<\beta_{ji}} = \{ \alpha_s, \beta_{st} \mid s = 1, \ldots, r, \ t < i \ \text{or} \ t = i \ \text{and} \ s < j \};
A_{=\beta_{ji}} = \{ \alpha_s, \beta_{st} \mid s = 1, \ldots, r, \ t < i \ \text{or} \ t = i \ \text{and} \ s \leq j \},
\]
respectively. We put \( \Delta'_{<\beta_{ji}} := G(\Delta'_{<\beta_{ji}}) \) and \( \Delta'_{=\beta_{ji}} := G(\Delta'_{=\beta_{ji}}) \).

In Section 5.2 we constructed the functor \( F \) from the category of graded
manifolds of type \( \Delta \) to the category of \( r' \)-fold vector bundles of type \( \Delta' \). Now
we need to construct in a similar way two additional functors \( F_{<\beta_{ji}} \) and \( F_{=\beta_{ji}} \).
The functor \( F_{<\beta_{ji}} \) is a functor from the category of graded manifolds of type
\( \Delta_{<\beta_{ji}} \) to the category of \( r' \)-fold vector bundles of type \( \Delta'_{<\beta_{ji}} \) and the functor
\( F_{=\beta_{ji}} \) is a functor from the category of graded manifolds of type \( \Delta_{=\beta_{ji}} \) to
the category of \( r' \)-fold vector bundles of type \( \Delta'_{=\beta_{ji}} \), respectively. Note that
always we deal with \( r' \)-fold vector bundles. Recall that to construct the
functor \( F \) we used the additional formal weights \( (\beta_{ji}), \) where \( j = 2, \ldots, n_i \)
and \( i = i, \ldots, r, \) see (10). Similarly we define the functor \( F_{<\beta_{ji}} \), using the
additional weights \( \beta_{st} \in A_{<\beta_{ji}} \) and the functor \( F_{=\beta_{ji}} \), using the additional
weights \( \beta_{st} \in A_{=\beta_{ji}} \).
More precisely, let us describe for example the functor $F_{<\beta_{ji}}$ in more details. We set $n' := \lvert A_{<\beta_{ji}} \rvert$, i.e. $n'$ is the number of elements in $A_{<\beta_{ji}}$. Further, we take a graded manifold $N_{<\beta_{ji}}$ of type $\Delta_{<\beta_{ji}}$. Then we define $F_{<\beta_{ji}}(N_{<\beta_{ji}})$ in the following way. We take $n'$-iterated tangent bundle $N'_{<\beta_{ji}}$ of $N_{<\beta_{ji}}$ using sequentially additional weights from $A_{<\beta_{ji}}$. Further, we consider the graded manifold with the structure sheaf $\mathcal{O}_{N'_{<\beta_{ji}}} / J_{-}$. Assume that it has type $\tilde{\Delta}_{<\beta_{ji}}$. We choose the maximal multiplicity free subset $F_{<\beta_{ji}}(\Delta_{<\beta_{ji}})$ in $\tilde{\Delta}_{<\beta_{ji}}$ and denote by $F_{<\beta_{ji}}(N_{<\beta_{ji}})$ the corresponding to $\Delta'_{<\beta_{ji}}$ graded manifold. The definition of the functor $F =_{\beta_{ji}}$ is similar: we should replace the set $A_{<\beta_{ji}}$ by $A_{=\beta_{ji}}$.

Note that the constructed functors $F_{<\beta_{ji}}$ and $F =_{\beta_{ji}}$ are from the categories of graded manifolds of type $\Delta_{<\beta_{ji}}$ and of type $\Delta_{=\beta_{ji}}$ to the categories of graded manifolds of type $F_{<\beta_{ji}}(\Delta_{<\beta_{ji}})$ and of type $F =_{\beta_{ji}}(\Delta_{=\beta_{ji}})$, respectively. The weight system $F_{<\beta_{ji}}(\Delta_{<\beta_{ji}})$ is defined by formulas (17) and (18), where we assume that $\beta_{st} \in A_{<\beta_{ji}}$. In the same way we define $F =_{\beta_{ji}}(\Delta_{=\beta_{ji}})$. However in fact we have the following equalities.

**Proposition 14.** We have

$$F_{<\beta_{ji}}(\Delta_{<\beta_{ji}}) = \Delta'_{<\beta_{ji}}, \quad F =_{\beta_{ji}}(\Delta_{=\beta_{ji}}) = \Delta'_{=\beta_{ji}}.$$ 

**Proof.** It is enough to prove only the second statement. Let us take

$$\delta' = \sum_k \alpha_{ik} + \sum_{st} \beta_{jst} \in F =_{\beta_{ji}}(\Delta_{=\beta_{ji}}).$$

Then $\alpha_{ik}, \beta_{jst} \in \Delta'_{=\beta_{ji}}$. Hence, $F =_{\beta_{ji}}(\Delta_{=\beta_{ji}}) \subset \Delta'_{=\beta_{ji}}$. On other hand assume that

$$\delta' = \sum_k \alpha_{ik} + \sum_{st} \beta_{jst} \in \Delta'_{=\beta_{ji}}.$$ 

Then $\delta' \in \Delta'_{=\beta_{ji}}$. Since $\delta'$ depends only on $\beta_{st} \in A_{=\beta_{ji}}$, we see that $\delta' \in F =_{\beta_{ji}}(\Delta =_{\beta_{ji}})$. 

**7.4 $F$ is an equivalence of categories**

In the previous section we constructed a functor $F$ from $\Delta \text{Man}$ to $\Delta \text{VBVect}$. Let us prove that $F$ is essentially surjective. This is the most difficult part of our paper.

Consider the set of additional weights $(\beta_{ji})$, see (10), with the lexicographical order: $\beta_{ji} < \beta_{j'i'}$ if $i < i'$ or $i = i'$ and $j < j'$. We will prove the essential surjectivity of $F$ by induction on this order.

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Let $D$ be an object in $\Delta^\prime \text{VBVect}$. Consider the weight systems $\Delta_{<\beta_{ji}}$, $\Delta_{<\beta_{ji}} := G(\Delta_{<\beta_{ji}})$, $\Delta_{=\beta_{ji}} := G(\Delta_{=\beta_{ji}})$ and the functors $F_{<\beta_{ji}}$, $F_{=\beta_{ji}}$ constructed in the previous section. Clearly the weight system $\Delta_{<\beta_{ji}}$ satisfies the conditions of Lemma 1. We denote by $D_{<\beta_{ji}}$ the graded manifold of type $\Delta_{<\beta_{ji}} \subset \Delta'$, see Construction 1, Section 4.1. Further, assume by induction that there exists a graded manifold $\mathcal{N}_{<\beta_{ji}}$ of type $\Delta_{<\beta_{ji}}$ such that

$$F_{<\beta_{ji}}(\mathcal{N}_{<\beta_{ji}}) \simeq D_{<\beta_{ji}},$$

and this isomorphism that we denote by $\varphi'$ commutes with all vector fields $D_{\beta_{ji}}$. Now our goal is to show that there exists a graded manifold $\mathcal{N}_{=\beta_{ji}}$ such that

$$F_{=\beta_{ji}}(\mathcal{N}_{=\beta_{ji}}) \simeq D_{=\beta_{ji}},$$

where $D_{=\beta_{ji}}$ is the graded manifold of type $\Delta'_{=\beta_{ji}} \subset \Delta'$, see Construction 1, Section 4.1.

Note that any graded manifold of type $\Delta_{<\beta_{ji}}$ is also a graded manifold of type $\Delta_{=\beta_{ji}}$. Hence we can apply functor $F_{=\beta_{ji}}$ to $\mathcal{N}_{<\beta_{ji}}$ and get the graded manifold $F_{=\beta_{ji}}(\mathcal{N}_{<\beta_{ji}})$ of type $\Delta_{=\beta_{ji}}$. However, $F_{=\beta_{ji}}(\mathcal{N}_{<\beta_{ji}})$ is also a graded manifold of type $F_{=\beta_{ji}}(\Delta_{<\beta_{ji}})$, where

$$F_{=\beta_{ji}}(\Delta_{<\beta_{ji}}) = \Delta'_{<\beta_{ji}} \cup \Delta'_{1} \cup \Delta'_{2} \subset \Delta'. \quad (31)$$

Here $\Delta_{1}'$ and $\Delta_{2}'$ are subsets in $\Delta'$ that are defined by the following formulas:

$$\Delta_{1}' = \{D_{\beta_{ji}}(\delta) \mid \delta \in \Delta_{<\beta_{ji}} : D_{\beta_{ji}}(\delta) \in \Delta_{1}'\};$$

$$\Delta_{2}' = \{\delta \in \Delta' \mid \exists \beta_{jui} < \beta_{ji} : D_{\beta_{jui}}(\delta) \in \Delta_{1}'\}. \quad (32)$$

Explicitly, $\delta_{1} \in \Delta_{1}'$ and $\delta_{2} \in \Delta_{2}'$ have the following form:

$$\delta_{1} = \beta_{ji} + \theta_{1} \quad \text{and} \quad \delta_{2} = \alpha_{i} + \beta_{ji} + \theta_{2}, \quad (33)$$

where $\theta_{i}$ are independent on $\beta_{ji}$ and $\alpha_{i}$. In addition we assume that for $\delta_{2}$ there exists $\beta_{jui} < \beta_{ji}$ such that $D_{\beta_{jui}}(\delta) \in \Delta_{1}'$. Again the weight system $F_{=\beta_{ji}}(\Delta_{<\beta_{ji}}) \subset \Delta'$ satisfies the conditions of Lemma 1. Denote by $D_{F_{=\beta_{ji}}(\Delta_{<\beta_{ji}})}$ the graded manifold of type $F_{=\beta_{ji}}(\Delta_{<\beta_{ji}})$, see Construction 1, Section 4.1.

For simplicity of notations we denote the structure sheaves of graded manifolds $F_{=\beta_{ji}}(\mathcal{N}_{<\beta_{ji}})$ and $D_{F_{=\beta_{ji}}(\Delta_{<\beta_{ji}})}$ by $\mathcal{O}$ and $\mathcal{O}'$, respectively. We also denote operators $D_{\beta_{ji}}$ in $\mathcal{O}$ and $\mathcal{O}'$ by the same letter.

**Proposition 15.** Let $\mathcal{N}_{<\beta_{ji}}$ and $D_{F_{=\beta_{ji}}(\Delta_{<\beta_{ji}})}$ be the graded manifolds of type $\Delta_{<\beta_{ji}}$ and $F_{=\beta_{ji}}(\Delta_{<\beta_{ji}})$ as above. Then there exists an isomorphism

$$\varphi : F_{=\beta_{ji}}(\mathcal{N}_{<\beta_{ji}}) \rightarrow D_{F_{=\beta_{ji}}(\Delta_{<\beta_{ji}})}$$
of graded manifolds of type $\mathbb{F}_{=\beta_j}(\Delta_{<\beta_j})$.

Proof. The idea of the proof is to extend the isomorphism $\varphi'$ using the vector fields $D_{\beta_{ji}}$. By Proposition 1 our goal is to construct compatible bundle maps $\varphi_{\delta}$ for all $\delta \in \mathbb{F}_{=\beta_j}(\Delta_{<\beta_j})$. We use the decomposition (31). By induction we have an isomorphism $\varphi' : F_{<\beta_j}(\Lambda_{<\beta_j}) \to D_{<\beta_j}$. For any $\delta \in \Delta_{<\beta_j}$ we put $\varphi_{\delta} := \varphi'_{\delta}$. Further, let us take $D_{\beta_{ji}}(\delta) \in \Delta'_{1}$. We put

$$\varphi_{D_{\beta_{ji}}(\delta)} := D_{\beta_{ji}} \circ \varphi_{\delta} \circ D^{-1}_{\beta_{ji}}. \quad (34)$$

Note that (34) is well-defined since $\delta \in \Delta'_{<\beta_j}$. If $\delta \in \Delta'_{2}$ we put

$$\varphi_{\delta} := D^{-1}_{\beta_{j_{0i}}} \circ \varphi_{D_{\beta_{j_{0i}}}(\delta)} \circ D_{\beta_{j_{0i}}}, \quad (35)$$

where $\beta_{j_{0i}}$ is as in the definition of $\Delta'_{2}$. Note that $\varphi_{D_{\beta_{j_{0i}}}(\delta)}$ is defined by (34), since $D_{\beta_{j_{0i}}}(\delta) \in \Delta'_{1}$. Combining (34) and (35), we get

$$\varphi_{\delta} := D^{-1}_{\beta_{j_{0i}}} \circ D_{\beta_{ji}} \circ \varphi_{\lambda} \circ D^{-1}_{\beta_{j_{1i}}} \circ D_{\beta_{j_{0i}}}. \quad \text{Explicitly, if } \delta = \alpha_{i} + \beta_{j_{0i}} + \theta_{2} \in \Delta'_{2}, \text{ see (33), then } \lambda = \alpha_{i} + \beta_{j_{0i}} + \theta_{2} \in \Delta'_{<\beta_j}. \quad \text{The formula (34) is well-defined, while (35) depends on the choice of } \beta_{j_{0i}}. \text{ Let us show that in fact this is not the case. Assume that}$$

$$\varphi_{\delta} = D^{-1}_{\beta_{j_{0i}}} \circ D_{\beta_{ji}} \circ \varphi_{\lambda} \circ D^{-1}_{\beta_{j_{1i}}} \circ D_{\beta_{j_{0i}}}, \quad \varphi_{\delta} = D^{-1}_{\beta_{j_{1i}}} \circ D_{\beta_{ji}} \circ \varphi_{\lambda'} \circ D^{-1}_{\beta_{j_{1i}}} \circ D_{\beta_{j_{1i}}}. \quad \text{We need to show that}$$

$$\varphi_{\lambda} = D^{-1}_{\beta_{j_{1i}}} \circ D_{\beta_{j_{0i}}} \circ D^{-1}_{\beta_{j_{1i}}} \circ D_{\beta_{ji}} \circ \varphi_{\lambda'} \circ D^{-1}_{\beta_{j_{1i}}} \circ D_{\beta_{j_{1i}}} \circ D^{-1}_{\beta_{j_{0i}}} \circ D_{\beta_{j_{1i}}}, \quad (36)$$

where $\lambda = \alpha_{i} + \beta_{j_{1i}} + \theta_{2}$ and $\lambda' = \alpha_{i} + \beta_{j_{1i}} + \theta_{2}$ such that $\lambda, \lambda' \in \Delta'_{<\beta_j}$. First of all consider $\varphi_{\lambda}|_{\text{Ker} D_{\beta_{j_{0i}}}}$. Then we can apply the cocycle condition for our vector fields:

$$(D^{-1}_{\beta_{j_{1i}}} \circ D_{\beta_{j_{0i}}} \circ D^{-1}_{\beta_{j_{1i}}} \circ D_{\beta_{j_{0i}}})|_{\text{Ker} D_{\beta_{j_{0i}}}} = - D^{-1}_{\beta_{j_{0i}}} \circ D_{\beta_{j_{1i}}}|_{\text{Ker} D_{\beta_{j_{0i}}}}.$$

Further, we use the relation:

$$(D^{-1}_{\beta_{j_{0i}}} \circ D_{\beta_{j_{0i}}})(\text{Ker} D_{\beta_{j_{0i}}}) = \text{Ker} D_{\beta_{j_{1i}}}. \quad \text{Therefore we can apply the cocycle condition again}$$

$$(D^{-1}_{\beta_{j_{1i}}} \circ D_{\beta_{j_{0i}}} \circ D^{-1}_{\beta_{j_{1i}}} \circ D_{\beta_{j_{0i}}})|_{\text{Ker} D_{\beta_{j_{1i}}}} = - D^{-1}_{\beta_{j_{1i}}} \circ D_{\beta_{j_{0i}}}|_{\text{Ker} D_{\beta_{j_{1i}}}}.$$
Now we can rewrite (36) in the following form
\[ \varphi_\lambda |_{\text{Ker} D_{\beta_{ji}}} = (D_{\beta_{ji}^{-1}} \circ D_{\beta_{ji}} \circ \varphi_{\lambda'} \circ D_{\beta_{ji}^{-1}} \circ D_{\beta_{ji}})|_{\text{Ker} D_{\beta_{ji}}} \]

or
\[ (D_{\beta_{ji}^{-1}} \circ D_{\beta_{ji}} \circ \varphi_{\lambda'})|_{\text{Ker} D_{\beta_{ji}}} = (\varphi_{\lambda'} \circ D_{\beta_{ji}^{-1}} \circ D_{\beta_{ji}})|_{\text{Ker} D_{\beta_{ji}}}. \]

This equation holds because \( \varphi_\lambda \) and \( \varphi_{\lambda'} \) commute with vector fields \( D_{\beta_{ji}} \) and \( D_{\beta_{ji}^{-1}} \), by induction.

To show (36) our next step is to use the decomposition
\[ \mathcal{O}_\lambda = \mathcal{O}_\lambda \cap \text{Ker} D_{\beta_{ji}} + \bigoplus_{\lambda_1 + \lambda_2 = \lambda} \mathcal{O}_{\lambda_1} \mathcal{O}_{\lambda_2}, \]
where \( \lambda_i \neq 0 \). (Note that the existence of such kind of decompositions follows from the definition of the category \( \Delta^\ast \text{VBVect} \)). Consider the following two cases:

1. Assume that \( \lambda_1 = \alpha_i + \gamma_1 \) and \( \lambda_2 = \beta_{ji} + \gamma_2 \) such that \( \lambda_1 + \lambda_2 = \lambda \), and \( f_i \) is a function of weight \( \lambda_i \), \( i = 1, 2 \). We have
\[ (D_{\beta_{ji}^{-1}} \circ D_{\beta_{ji}} \circ D_{\beta_{ji}^{-1}} \circ D_{\beta_{ji}})(f_1 f_2) = (D_{\beta_{ji}^{-1}} \circ D_{\beta_{ji}} \circ D_{\beta_{ji}^{-1}})(D_{\beta_{ji}}(f_1) \cdot f_2) = (-1)^{(f_1 + 1)} D_{\beta_{ji}^{-1}} \circ D_{\beta_{ji}}(D_{\beta_{ji}}(f_1) \cdot D_{\beta_{ji}^{-1}}(f_2)) = D_{\beta_{ji}^{-1}}(D_{\beta_{ji}}(f_1) \cdot D_{\beta_{ji}} \circ D_{\beta_{ji}^{-1}}(f_2)) = (f_1 \cdot D_{\beta_{ji}} \circ D_{\beta_{ji}^{-1}}(f_2)). \]

Further, since \( \varphi_{\lambda'} = \varphi_{\lambda'}' \), we have
\[ \varphi_{\lambda'}(f_1 \cdot D_{\beta_{ji}} \circ D_{\beta_{ji}^{-1}}(f_2)) = \varphi_{\lambda'}(f_1) \cdot \varphi_{\lambda'}(D_{\beta_{ji}} \circ D_{\beta_{ji}^{-1}}(f_2)), \]
where \( \lambda'_1 = \lambda_1 = \alpha_i + \gamma_1 \) and \( \lambda'_2 = \beta_{ji} + \gamma_2 \). Similarly we get
\[ (D_{\beta_{ji}^{-1}} \circ D_{\beta_{ji}} \circ D_{\beta_{ji}^{-1}} \circ D_{\beta_{ji}})(\varphi_{\lambda'}(f_1) \cdot \varphi_{\lambda'}(D_{\beta_{ji}} \circ D_{\beta_{ji}^{-1}}(f_2))) = \varphi_{\lambda'}(f_1) \cdot (D_{\beta_{ji}} \circ D_{\beta_{ji}^{-1}}) \circ \varphi_{\lambda'}(D_{\beta_{ji}} \circ D_{\beta_{ji}^{-1}}(f_2)) = \varphi_{\lambda'}(f_1) \cdot \varphi_{\lambda'}(f_2) = \varphi_{\lambda'}(f_1) \cdot \varphi_{\lambda'}(f_2). \]

2. Assume that \( \lambda_1 = \alpha_i + \beta_{ji} + \gamma_1 \) and \( \lambda_2 = \gamma_2 \), where \( \gamma_i \) are not depending on \( \alpha_i \) and \( \beta_{ji} \). Consider the restriction of (36) on \( \mathcal{O}_{\lambda_1} \cdot \mathcal{O}_{\lambda_2} \). We get
\[ D_{\beta_{ji}^{-1}} \circ D_{\beta_{ji}} \circ D_{\beta_{ji}^{-1}} \circ D_{\beta_{ji}} \circ \varphi_{\lambda'} \circ D_{\beta_{ji}^{-1}} \circ D_{\beta_{ji}} \circ D_{\beta_{ji}^{-1}} \circ D_{\beta_{ji}} |_{\mathcal{O}_{\lambda_1} \cdot \mathcal{O}_{\lambda_2}} = (D_{\beta_{ji}^{-1}} \circ D_{\beta_{ji}} \circ D_{\beta_{ji}^{-1}} \circ D_{\beta_{ji}} \circ \varphi_{\lambda'} \circ D_{\beta_{ji}^{-1}} \circ D_{\beta_{ji}} \circ D_{\beta_{ji}^{-1}} \circ D_{\beta_{ji}})|_{\mathcal{O}_{\lambda_1} \cdot \mathcal{O}_{\lambda_2}}. \]
where \( \lambda_1' = \alpha_i + \beta_{ji} + \gamma_1 \) and \( \lambda_2' = \lambda_2 \). Hence to prove (36) we need to prove

\[
\varphi \lambda_1 |_{\mathcal{O}_{\lambda_1}} = (D_{\beta_{ji}}^{-1} \circ D_{\beta_{ji}^{0}} \circ D_{\beta_{ji}} \circ \varphi \lambda_1' \circ D_{\beta_{ji}}^{-1} \circ D_{\beta_{ji}} \circ D_{\beta_{ji}^{0}} \circ D_{\beta_{ji}})|_{\mathcal{O}_{\lambda_1}}
\]

Since \( \lambda_2' \neq 0 \), the last equality follows by induction on length of \( \gamma_1 \), where the length of \( \gamma_1 \) is equal to the number of summands in \( \gamma_1 \). Note that in case \( \lambda_1' = \alpha_i + \beta_{ji}^{0} \) (in other words the length of \( \gamma_1 = 0 \), the basis of our induction) the result follows from the decomposition

\[
\mathcal{O}_{\alpha_i+\beta_{ji}^{0}} = (\mathcal{O}_{\alpha_i+\beta_{ji}^{0}}) \cap \text{Ker } D_{\beta_{ji}^{0}} + \mathcal{O}_{\alpha_i} \mathcal{O}_{\beta_{ji}^{0}}
\]

and the case 1. Hence we proved that \( \varphi_\delta \) is well-defined.

It remains to prove the compatibility condition of Proposition 1

\[
\varphi_\delta |_{\mathcal{O}_{\delta_1} \cdot \mathcal{O}_{\delta_2}} = (\varphi_{\delta_1} \cdot \varphi_{\delta_2}) |_{\mathcal{O}_{\delta_1} \cdot \mathcal{O}_{\delta_2}}
\]

for \( \delta = \delta_1 + \delta_2 \), where \( \delta_i \neq 0 \). Consider first the case \( \delta \in \Delta'_1 \). Without loss of generality we may assume that \( \delta_1 \) depends on \( \beta_{ji} \). For \( f_i \) of weight \( \delta_i \), we have

\[
\varphi_\delta (f_1 \cdot f_2) = (D_{\beta_{ji}} \circ \varphi'_{D_{\beta_{ji}}^{-1}(\delta)} \circ D_{\beta_{ji}}^{-1} (f_1) \cdot f_2) = (D_{\beta_{ji}} \circ \varphi'_{D_{\beta_{ji}}^{-1}(\delta)} \circ D_{\beta_{ji}}^{-1} (f_1) \cdot \varphi'_{\delta_2} (f_2)) = \varphi_{\delta_1} (f_1) \cdot \varphi_{\delta_2} (f_2).
\]

We used here the fact that \( \varphi' \) is an isomorphism by induction.

Further, assume that \( \delta = \delta_1 + \delta_2 \in \Delta'_2 \). Again without loss of generality we may assume that \( \delta_1 \) depends on \( \alpha_i \). Similarly we have for \( f_i \) of weight \( \delta_i \)

\[
\varphi_\delta (f_1 \cdot f_2) = (D_{\beta_{ji}^{0}} \circ \varphi_{D_{\beta_{ji}^{0}}^{-1}(\delta)} \circ D_{\beta_{ji}^{0}} (f_1) \cdot f_2) = (D_{\beta_{ji}^{0}} \circ \varphi_{D_{\beta_{ji}^{0}}^{-1}(\delta_1)} \circ D_{\beta_{ji}^{0}} (f_1) \cdot \varphi_{\delta_2} (f_2) = \varphi_{\delta_1} (f_1) \cdot \varphi_{\delta_2} (f_2).
\]

By Proposition 1, an isomorphism \( \varphi \) is defined by the collection \( (\varphi_\delta) \), where \( \delta \in F=\beta_{ji} \langle \Delta _{=\beta_{ji}} \rangle \). The proof is complete.\( \square \)

Note that in the proof of Proposition 15 we used the following decomposition

\[
\mathcal{O}_\delta = \mathcal{O}_\delta \bigcap_{\beta_{ji}^{0} \neq 0} \text{Ker } D_{\beta_{ji}} + \bigoplus_{\delta_1 + \delta_2 = \delta} \mathcal{O}_{\delta_1} \mathcal{O}_{\delta_2},
\]

where \( \delta_i \neq 0 \). The idea was the following. First we show a certain equality on kernels \( \mathcal{O}_\delta \bigcap_{\beta_{ji}^{0} \neq 0} \text{Ker } D_{\beta_{ji}} \) and then using induction on \( \bigoplus_{\delta_1 + \delta_2 = \delta} \mathcal{O}_{\delta_1} \mathcal{O}_{\delta_2} \).
Further we will use this idea several time. We will call this argument the **decomposition and induction argument**.

Our goal now is to prove that the constructed isomorphism $\varphi$ is a morphism in the category $\Delta'\text{VBVect}$.

**Proposition 16.** The isomorphism $\varphi$ from Proposition 15 commutes with operators $D_{\beta st}$.

**Proof.** We need to show that

$$D_{\beta st} \circ \varphi_\delta = \varphi_{D_{\beta st}(\delta)} \circ D_{\beta st}$$

for any operator $D_{\beta st}$ and for any $\delta$ from our weight lattice. Since $D_{\beta st}$ is a vector field, it is enough to show $(37)$ for $\delta \in \mathbb{F} = \beta_{ji}(\Delta_{<\beta_{ji}})$. We use the decomposition $(31)$. Consider the following cases.

1. **Case $\delta \in \Delta'_{<\beta_{ji}}$ and $(st) \neq (ji)$**. In this case $(37)$ holds by the assumption that $\varphi'$ commutes with all operators.

2. **Case $\delta \in \Delta'_{<\beta_{ji}}$ and $(st) = (ji)$**. If $\delta$ is independent on $\alpha_i$, the equality $(37)$ holds trivially. If $\delta = \alpha_i + \theta$, where $\theta$ is independent on $\beta_{ji}$, then $D_{\beta_{ji}}(\delta) \in \Delta'_{1}$ and $(37)$ holds by definition of $\varphi_{D_{\beta_{ji}}(\delta)}$.

3. **Case $\delta \in \Delta'_{1}$ and $(st) \neq (ji)$**. If $t = i$, the equality $(37)$ holds trivially, since $\delta$ does not depend on $\alpha_i$. Assume that $t \neq i$. Consider the decomposition of the weight $D_{\beta_{ji}}(\delta) = \gamma_1 + \gamma_2$, where $\gamma_1 = \beta_{ji} + \cdots \in \Delta'_{1}$ and $\gamma_2 = D_{\beta_{ji}}(\delta) - \gamma_1$. Note that $\gamma_2$ is independent on $\beta_{ji}$. Then the following bundle isomorphism is defined

$$D_{\beta_{ji}}^{-1}: O_{\gamma_1 + \gamma_2} \to O_{D_{\beta_{ji}}^{-1}(\gamma_1) + \gamma_2}, \quad D_{\beta_{ji}}^{-1}(f_1 \cdot f_2) := D_{\beta_{ji}}^{-1}(f_1) \cdot f_2,$$

where $f_i \in O_{\gamma_i}$. Therefore, we have

$$D_{\beta_{st}} \circ D_{\beta_{ji}}^{-1}|_{\mathcal{O}_h} = D_{\beta_{ji}}^{-1} \circ D_{\beta_{st}}|_{\mathcal{O}_h}.$$

Further,

$$\varphi_{D_{\beta_{st}}(\delta)} \circ D_{\beta_{st}} = (\varphi_{\gamma_1} \cdot \varphi_{\gamma_2}) \circ D_{\beta_{st}} = ((D_{\beta_{ji}} \circ \varphi_{D_{\beta_{ji}}^{-1}(\gamma_1)} \circ D_{\beta_{ji}}^{-1}) \circ \varphi_{\gamma_2}) \circ D_{\beta_{st}}.$$

On the other hand we have

$$D_{\beta_{st}} \circ \varphi_\delta = D_{\beta st} \circ (D_{\beta_{ji}} \circ \varphi_{D_{\beta_{ji}}^{-1}(\delta)} \circ D_{\beta_{ji}}^{-1}) \circ D_{\beta_{st}} = (D_{\beta_{ji}} \circ \varphi(D_{\beta_{st}} \circ D_{\beta_{ji}}^{-1}(\delta)) \circ D_{\beta_{ji}}^{-1}) \circ D_{\beta_{st}} =$$

$$(D_{\beta_{ji}} \circ \varphi_{D_{\beta_{ji}}^{-1}(\gamma_1)} \circ D_{\beta_{ji}}^{-1}) \circ D_{\beta_{st}}.$$

Hence, $(37)$ is proven for this case.

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4. Case $\delta \in \Delta'_2$ and $(st) \neq (ji)$. Assume that $t = i$ and $D_{\beta_{s_1}}(\delta)$ is a weight. In this case $D_{\beta_{s_1}}(\delta) \in \Delta'_1$. We have

$$D_{\beta_{s_1}} \circ \varphi_\delta = D_{\beta_{s_1}} \circ (D^{-1}_{\beta_{s_1}} \circ \varphi_{D_{\beta_{s_1}}(\delta)} \circ D_{\beta_{s_1}}) = \varphi_{D_{\beta_{s_1}}(\delta)} \circ D_{\beta_{s_1}}.$$

Hence, (37) holds.

Assume that $t = i$ and $D_{\beta_{s_1}}(\delta)$ is not a weight. Then since $D_{\beta_{s_1}}(\delta) \notin \mathbb{F}_{=\beta_{ji}}(\Delta_{<\beta_{ji}})$, it follows that $\delta = \beta_{s_i} + \ldots$, i.e. $\delta$ depends on $\beta_{s_i}$ non-trivially. Hence, (37) holds trivially. Assume that $t \neq i$. Then $D_{\beta_{s_1}}(\delta) = \gamma_1 + \gamma_2$, where $\gamma_1 \in \Delta'_1$. In this case (37) is equivalent to the following equality:

$$D_{\beta_{s_1}} \circ (D^{-1}_{\beta_{s_1}} \circ D_{\beta_{ji}} \circ \varphi_{D_{\beta_{ji}}(\delta)} \circ D^{-1}_{\beta_{s_1}} \circ D_{\beta_{s_1}}) = (D_{\beta_{s_1}} \circ \varphi_{D^{-1}_{\beta_{ji}}(\gamma_1)} \circ D^{-1}_{\beta_{ji}} \circ \varphi_{\gamma_2}) \circ D_{\beta_{s_1}},$$

where $j_0 \neq s, j$. Now we use the decomposition and induction argument. By Proposition 7, we have the following decomposition:

$$O_\delta = O_\delta \bigcap_{b_{s_1s_2}(\delta) \neq 0} \text{Ker } D_{b_{s_1s_2}} + \bigoplus_{\delta_1 + \delta_2 = \delta} O_{\delta_1} O_{\delta_2},$$

where $\delta_i \neq 0$ and $b_{s_1s_2}(\delta) \in \mathbb{C}$ is the multiplicity of $\beta_{s_1s_2}$ in $\delta$. Since $\delta$ depends on $\beta_{s_1}$ non-trivially and since the maps $D^{-1}_{\beta_{ji}} \circ D_{\beta_{s_1}}$ and $D^{-1}_{\beta_{s_1}} \circ D_{\beta_{ji}}$ preserve this decomposition, (38) follows from the previous cases and by induction.

Consider now the case $\delta \in \Delta'_2$ and $t \neq i$. If $\delta$ does not depends on $\alpha_t$, then (37) holds trivially. Assume that $\delta = \alpha_t + \ldots$ and $\delta$ does not depend on $\beta_{st}$. Then $D_{\beta_{st}}(\delta)$ is a weight. In this case we have:

$$D_{\beta_{st}} \circ (D^{-1}_{\beta_{st}} \circ \varphi_{D_{\beta_{st}}(\delta)} \circ D_{\beta_{st}}) = D^{-1}_{\beta_{st}} \circ D_{\beta_{st}} \circ \varphi_{D_{\beta_{st}}(\delta)} \circ D_{\beta_{st}} = (D^{-1}_{\beta_{st}} \circ \varphi_{D_{\beta_{st}}(\delta)} \circ D_{\beta_{st}}) \circ D_{\beta_{st}}.$$

Hence (37) holds. Further, assume that $\delta = \alpha_t + \beta_{st} + \ldots$. In this case (37) is equivalent to the following equality:

$$D_{\beta_{st}} \circ (D^{-1}_{\beta_{st}} \circ D_{\beta_{st}} \circ \varphi_{D^{-1}_{\beta_{st}} \circ D_{\beta_{st}}(\delta)} \circ D^{-1}_{\beta_{st}} \circ D_{\beta_{st}}) = (D^{-1}_{\beta_{st}} \circ D_{\beta_{st}} \circ \varphi_{D^{-1}_{\beta_{st}} \circ D_{\beta_{st}}(\delta)} \circ D^{-1}_{\beta_{st}} \circ D_{\beta_{st}}) \circ D_{\beta_{st}},$$

where $j_0 \neq j$. This holds by the decomposition and induction argument.

5. Case $\delta \in \Delta'_1 \cup \Delta'_2$ and $(st) = (ji)$. If $\delta \in \Delta'_1$, then (37) holds trivially. Further, assume that $\delta \in \Delta'_2$. In this case the result follows from the decomposition and induction argument. The proof is complete. □

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Recall that we denoted by $D_{\beta_{ji}}$ the graded manifold of type $\Delta'_{\beta_{ji}} \subset \Delta'$, see Construction 1, Section 4.1. Now we have the following situation. By induction we assumed that there exists an isomorphism $\varphi' : F_{<\beta_{ji}}(\mathcal{N}_{<\beta_{ji}}) \rightarrow D_{<\beta_{ji}}$ of graded manifolds of type $\Delta'_{<\beta_{ji}}$ that commutes with all operators. By Propositions 15 and 16 there exists an isomorphism $\varphi : F_{=\beta_{ji}}(\mathcal{N}_{<\beta_{ji}}) \rightarrow D_{=\beta_{ji}}(\Delta_{<\beta_{ji}})$ of graded manifolds of type $F_{=\beta_{ji}}(\Delta_{<\beta_{ji}})$ compatible with $\varphi'$ that also commutes with all operators. Our goal now is to prove the following proposition.

**Proposition 17.** There exists a graded manifold $\mathcal{N}_{=\beta_{ji}}$ of type $\Delta_{=\beta_{ji}}$ such that

$$F_{=\beta_{ji}}(\mathcal{N}_{=\beta_{ji}}) \simeq D_{=\beta_{ji}}$$

and this isomorphism say $\psi$ commutes with all operators.

**Proof.** First of all note that by Proposition 14, $\psi$ is an isomorphism of graded manifolds of type $\Delta'_{\beta_{ji}}$. Further, clearly we have

$$\Delta'_{=\beta_{ji}} = F_{=\beta_{ji}}(\Delta_{<\beta_{ji}}) \cup S,$$

where $S$ is the subset in $\Delta'$ that contains all weights in the form $\alpha_i + \sum_{q=2}^{j} \beta_{qi} + \gamma$. We prove this proposition by induction on the length of $\gamma$. (Recall that the length $|\theta|$ of a multiplicity free weight $\theta$ is the number of summands in $\theta$.)

If the set $S$ is empty by Propositions 15 and 16 we are done. Assume that $S \neq \emptyset$. Denote by $S_p$ the subset in $S$ such that $|\gamma| = p$. So we have $S = \cup_{p \geq 0} S_p$. Let us take $\delta' \in S_p$, where $p \geq 0$, satisfying the following property: if $\delta'$ depends on $\beta_{st}$, then $\delta'$ depends also on $\alpha_t$. (Note that by definition of $\Delta'$ we always can find such $\delta'$. ) In other words, if we put $\delta' = \sum_{s} a'_s \alpha_s + \sum_{pq} b'_{pq} \beta_{pq}$, then from $b'_{st} = 1$ it follows that $a'_t = 1$.

**Step 1, construction of graded manifold.** Consider the following exact sequence

$$0 \rightarrow \bigoplus_{\delta'_1 + \delta'_2 = \delta'} (\mathcal{O}_D)_{\delta'_1} (\mathcal{O}_D)_{\delta'_2} \rightarrow (\mathcal{O}_D)_{\delta'} \rightarrow \mathcal{E}_{\delta'} \rightarrow 0,$$

where $\delta'_i \neq 0$, $i = 1, 2$, and $\mathcal{E}_{\delta'}$ is a certain locally free sheaf. As in (25), for any $\delta' \in \Delta'$, we put

$$\mathcal{S}_{\delta'} := (\mathcal{O}_D)_{\delta'} \cap \operatorname{Ker} D_{\beta_{st}}.$$
By definition of the category $\Delta^\prime \text{VBVect}$, see Section 7.1, we have the following decomposition

$$
(\mathcal{O}_D)_{\delta'} = S_{\delta'} + \bigoplus_{\delta'_1 + \delta'_2 = \delta'} (\mathcal{O}_D)_{\delta'_1} (\mathcal{O}_D)_{\delta'_2},
$$

where $\delta_i \neq 0$. Hence the following sequence is also exact

$$
0 \to \bigoplus_{\delta'_1 + \delta'_2 = \delta'} (\mathcal{O}_D)_{\delta'_1} (\mathcal{O}_D)_{\delta'_2} \cap \text{Ker } D_{\beta st} \overset{i}{\to} S_{\delta'} \to E_{\delta'} \to 0.
$$

By induction we assume that there is a graded manifold $\mathcal{M}$ of type $\Delta_{\mathcal{M}} = G(\Delta^\prime_{\mathcal{M}})$, where $\Delta^\prime_{\mathcal{M}} = F_{\beta j} \cup \bigcup_{q < p} S_q$, and an isomorphism $\varphi_{\mathcal{M}} : F_{\beta j}(\mathcal{M}) \to D_{\Delta^\prime_{\mathcal{M}}}$ that is compatible with $\varphi$ and commutes with all operators. Here $D_{\Delta^\prime_{\mathcal{M}}}$ is again defined by Construction 1, Section 4.1.

We put $\delta := G(\delta')$. By Proposition 9, there is an isomorphism

$$
\varphi_{\mathcal{M}} \circ D^\lambda : (\mathcal{O}_\mathcal{M})_\delta \to \bigoplus_{\delta'_1 + \delta'_2 = \delta'} (\mathcal{O}_D)_{\delta'_1} (\mathcal{O}_D)_{\delta'_2} \cap \text{Ker } D_{\beta st}.
$$

Here we assume that $D^\lambda(\delta) = \delta'$ and that $\lambda$ has lexicographical order.

Note that $(\mathcal{O}_\mathcal{M})_\delta = \bigoplus_{\delta'_1 + \delta'_2 = \delta'} (\mathcal{O}_\mathcal{M})_{\delta'_1} (\mathcal{O}_\mathcal{M})_{\delta'_2}$, where $\delta_i \neq 0$, since $\delta \notin \Delta_{\mathcal{M}}$.

Hence the following sequence is exact

$$
0 \to \bigoplus_{\delta'_1 + \delta'_2 = \delta'} (\mathcal{O}_\mathcal{M})_{\delta'_1} (\mathcal{O}_\mathcal{M})_{\delta'_2} \overset{i \circ \varphi_{\mathcal{M}} \circ D^\lambda}{\to} S_{\delta'} \to E_{\delta'} \to 0.
$$

Now we use Construction 2, Section 4.2 to build a graded manifold $\tilde{\mathcal{M}}$ of type $\Delta_{\mathcal{M}} \cup G(S_p)$. In more details, we put

$$
(\mathcal{O}_{\tilde{\mathcal{M}}})_{\delta} := S_{\delta'}.
$$

If there is $\lambda' \in S_p$, where $\lambda' \neq \delta'$, satisfying the following property: if $\lambda'$ depends on $\beta_{st}$, then $\delta'$ depends also on $\alpha_t$, then we repeat this construction and define $(\mathcal{O}_{\tilde{\mathcal{M}}})_{\lambda}$, where $\lambda = G(\lambda')$. So we defined a graded manifold $\tilde{\mathcal{M}}$ of type $\Delta_{\mathcal{M}} \cup G(S_p)$.

**Step 2, construction of an isomorphism.** Our goal now is to construct an isomorphism $\tilde{\psi} : F_{\beta j}(\tilde{\mathcal{M}}) \to \tilde{D}$, where $\tilde{D}$ is a graded manifold of type $\Delta^\prime_{\mathcal{M}} \cup S_p$ that is defined by Construction 1, Section 4.1. We put $\tilde{\psi}_\theta = (\varphi_{\mathcal{M}})_\theta$
for any \( \theta \in \Delta'_{\mathcal{M}} \). To define \( \tilde{\psi}_{\delta'} \), where \( \delta' \in S_p \) satisfy the following property: if \( \delta' \) depends on \( \beta_{st} \), then \( \delta' \) depends also on \( \alpha_t \), we use the decomposition:

\[
(O_{x_{\beta_{ji}}(\mathcal{M})})_{\delta'} = D^\Lambda ((O_{x_{\alpha_{ji}}})_{\delta}) + \bigoplus_{\delta'_1 + \delta'_2 = \delta'} (O_{x_{\beta_{ji}}(\mathcal{M})})_{\delta'_1} \cdot (O_{x_{\beta_{ji}}(\mathcal{M})})_{\delta'_2},
\]

where \( \delta'_i \neq 0 \). Note that \( \tilde{\psi}_{\delta'} \) is already defined on the second summand. Further we put

\[
\tilde{\psi}_{\delta'} |_{D^\Lambda ((O_{x_{\alpha_{ji}}})_{\delta})} : D^\Lambda ((O_{x_{\alpha_{ji}}})_{\delta}) \to S_{\delta'},
\]

(39)

Let us show that \( \tilde{\psi}_{\delta'} \) is well-defined. Assume that

\[
f \in D^\Lambda ((O_{x_{\alpha_{ji}}})_{\delta}) \cap \bigoplus_{\delta'_1 + \delta'_2 = \delta'} (O_{x_{\beta_{ji}}(\mathcal{M})})_{\delta'_1} \cdot (O_{x_{\beta_{ji}}(\mathcal{M})})_{\delta'_2}.
\]

Then we have

\[
\tilde{\psi}_{\delta'}(f) = (D^\Lambda)^{-1}(f) = \iota \circ \varphi_{\mathcal{M}} \circ D^\Lambda((D^\Lambda)^{-1}(f)) = \iota \circ \varphi_{\mathcal{M}}(f).
\]

Now our goal is to define \( \tilde{\psi}_{\delta'} \) for other \( \theta' \in S_p \). Let us take any \( \theta' \in S_p \). Then there exists operators \( D_{\gamma_1}, \ldots, D_{\gamma_t} \) and a weight \( \delta' \) as above such that \( \theta' = D_{\gamma_1} \circ \cdots \circ D_{\gamma_t}(\delta') \). In this case we put

\[
\tilde{\psi}_{\delta'} := (D_{\gamma_1} \circ \cdots \circ D_{\gamma_t}) \circ \tilde{\psi}_{\delta'} \circ (D_{\gamma_1} \circ \cdots \circ D_{\gamma_t})^{-1}.
\]

Note that the composition \( D_{\gamma_1} \circ \cdots \circ D_{\gamma_t} \) is unique up to sign. Hence \( \tilde{\psi}_{\delta'} \) is well-defined. It can be easily shown that \( (\tilde{\psi}_{\delta'}) \) satisfies the conditions of Proposition 1. Therefore the following morphism of graded manifolds

\( \tilde{\psi} = (\tilde{\psi}_{\delta'}) \), where \( \theta' \in \Delta'_{\mathcal{M}} \cup S_p \), is defined.

**Step 3, isomorphism commutes with the operators.** It remains to show that

\[
D_{\beta_{st}} \circ \tilde{\psi}_{\delta'} = \tilde{\psi}_{D_{\beta_{st}}(\theta')} \circ D_{\beta_{st}} \quad (40)
\]

for \( (s,t) \leq (j,i) \). If \( \theta' \in \Delta'_{\mathcal{M}} \), then (40) holds by induction. Assume that \( \theta' \in S_p \) and consider the following cases:

1. Assume that \( D_{\beta_{st}}(\theta') \in \Delta' \). Then (40) follows by definition.

2. Assume that \( D_{\beta_{st}}(\theta') \notin \Delta' \). We also may assume that \( \theta' \) depends on \( \alpha_t \), since otherwise (40) holds trivially. In this case \( \theta' \) depends on \( \beta_{st} \) since otherwise \( D_{\beta_{st}}(\theta') \in \Delta' \).
2.1. Assume that \( \theta' \) satisfy the following property: if \( \theta' \) depends on \( \beta_{si} \), then \( \theta' \) depends also on \( \alpha_i \). By (39), the equality (40) holds by the decomposition and induction argument.

2.2. Assume that \( \theta' = (D_{\gamma_1} \circ \cdots \circ D_{\gamma_p})(\theta'_1) \), where \( \theta'_1 \) is from (2.1). Then we have

\[
D_{\beta_{st}} \circ \tilde{\psi}_{\theta'} = D_{\beta_{st}} \circ (D_{\gamma_1} \circ \cdots \circ D_{\gamma_p}) \circ \tilde{\psi}_{\theta'} \circ (D_{\gamma_1} \circ \cdots \circ D_{\gamma_p})^{-1} =
(D_{\gamma_1} \circ \cdots \circ D_{\gamma_p}) \circ \tilde{\psi}_{\beta_{st}(\theta')} \circ (D_{\gamma_1} \circ \cdots \circ D_{\gamma_p})^{-1} \circ D_{\beta_{st}} = \tilde{\psi}_{D_{\beta_{st}}(\theta')} \circ D_{\beta_{st}}.
\]

The proof is complete. \( \square \)

**Proposition 18.** Let \( D \) be an \( r' \)-fold vector bundle of type \( \Delta' \) with a family of \( (r' - r) \) odd commuting non-degenerate operators \( D_{\beta_{ji}} \) of weights \( \beta_{ji} - \alpha_i \), where \( i = 1, \ldots, r \) and \( j = 1, \ldots, n_i \), satisfying properties 1–6, Section 7.1. Then there exists a graded manifold \( \mathcal{N} \) of type \( \Delta = G(\Delta') \) such that \( F(\mathcal{N}) \simeq D \).

**Proof.** The proof follows by induction from Propositions 15, 16 and 17. \( \square \)

It is remaining to show that \( F \) is full and faithful, see Definition 7.

**Proposition 19.** The functor \( F \) is full and faithful.

**Proof.** Let us take two objects in the category \( \Delta' VBVect \), i.e. two \( r' \)-fold vector bundles \( D_1 \) and \( D_2 \) of type \( \Delta' \) and a morphism \( \Psi : D_1 \to D_2 \) that commutes with all vector fields \( D_{\beta_{ji}} \). We have seen in Proposition 18 that there exist graded manifolds \( \mathcal{N}_i \) of type \( \Delta \) such that \( D_i \simeq F(\mathcal{N}_i) \), where \( i = 1, 2 \).

Further, let us take two charts \( U_1 \) and \( U_2 \) on \( D_1 \) and \( D_2 \), respectively, such that we can consider the restriction \( \Psi : U_1 \to U_2 \). Denote by \( \mathcal{V}_1 \) the corresponding to \( U_1 \) chart on \( \mathcal{N}_1 \). Assume that \( \delta \) and \( \delta' \) are as in Proposition 9. By Proposition 6, we can chose coordinates \( \zeta^\delta \) and \( \zeta^{\delta'} \) in \( \mathcal{V}_2 \) and \( \mathcal{V}_2 \), respectively, and there exists unique up to even permutation the operator \( D^\delta \) such that \( D^\delta(\zeta^{\delta'}) = \zeta^\delta \). Consider \( f = \Psi^*(\zeta^{\delta'}) \). By Proposition 9 there exists unique function \( F \in \mathcal{O}_{\mathcal{V}_1} \) such that \( D^\delta(F) = f \). Now we can define the morphism \( \Phi|_{\mathcal{V}_1} : \mathcal{V}_1 \to \mathcal{V}_2 \) by \( \Phi(\zeta^\delta) := F \). (Note that for any \( \delta \in \Delta \) there exists \( \delta' \in \Delta' \) as in Proposition 9.) Since function \( F \) if unique, the morphisms \( \Phi|_{\mathcal{V}_i} \) coincide in all intersections \( \mathcal{V}_s \cap \mathcal{V}_t \) and define the global morphism \( \Phi \). Clearly, \( F(\Phi) = \Psi \). The proof is complete. \( \square \)

Now we can formulate our main result.

**Theorem 2.** The categories \( \Delta Man \) and \( \Delta' VBVect \) are equivalent.

**Proof.** The proof follows from Propositions 18 and 19. \( \square \)
8 Appendix

In this section we will prove Propositions 8 and 9.

Proof of Proposition 8. Let us fix an operator $d_{\gamma^p}$, where $p \in \{1, \ldots, s\}$, from Sequence (22). Clearly, we can rewrite $D_{\Lambda}$ in the following form

$$D_{\Lambda} = (d_{\gamma^1 \mod J^-}) \circ \cdots \circ (d_{\gamma^p \mod J^-}) \circ \cdots \circ (d_{\gamma^s \mod J^-}).$$

We put

$$D_{\Lambda} = D_1 \circ (d_{\gamma^p \mod J^-}) \circ D_2,$$

where the notations $D_1$ and $D_2$ have obvious meaning. Assume that $D_2(f) \neq 0$ in $O_{\tilde{N}}/J^-$, where $f \in (O_N)_{\delta}$. Our goal is to show that

$$(d_{\gamma^p \mod J^-})(D_2(f)) \neq 0.$$ 

Assume that $\gamma^p = \beta^j_i$. We work locally in a chart $\mathcal{U}$ on the non-negatively graded manifold $\mathcal{M} := (N^0, O_{\tilde{N}}/J^-)$. Note that we can divide all local homogeneous coordinates in $\mathcal{U}$ into three groups: coordinates with weight 0; coordinates with weights of the form $c\alpha_i + \cdots$, where $c > 0$; and all other coordinates. Hence we can write $D_2(f)$ in the following form:

$$D_2(f) = \sum_{IJ} f_{IJ} \xi^{I} \eta^{J}.$$ 

Here $I$ and $J$ are multi-indexes, $f_{IJ} \in (O_N)_0$, $\xi^{I}$ are monomials in local homogeneous coordinates from the second group, and $\eta^{J}$ are monomials in local homogeneous coordinates from the third group. Since $D_{\Lambda}(\delta)$ does not have negative coefficients, we observe that the weight of $D_2(f)$ also does not have negative coefficients. Hence the weight $D_2(f)$ is a weight of the form $c\alpha_i + \cdots$, where $c > 0$. It follows that $D_2(f)$ depends on coordinates from the second group non-trivially. Let us apply $(d_{\beta^j_i \mod J^-})$ to the function $D_2(f)$. Since $d_{\beta^j_i}(f_{IJ}) \in J^-$ and $d_{\beta^j_i}(\eta^J) \in J^-$, we get

$$(d_{\beta^j_i \mod J^-}) \left( \sum_{IJ} f_{IJ} \xi^{I} \eta^{J} \right) = \sum_{IJ} f_{IJ} d_{\beta^j_i}(\xi^{I}) \eta^{J} \mod J^-.$$ 

Assume that

$$\sum_{IJ} f_{IJ} d_{\beta^j_i}(\xi^{I}) \eta^{J} \mod J^- = 0. \quad (41)$$ 

Since $D_{\Lambda}(\delta)$ is multiplicity free and the weight of $d_{\beta^j_i}(\xi^{I})$ contains the summand $\beta^j_i$, we see that in the expression for weights of $\eta^J$ we do not have the
summand $\beta_{ji}$. Therefore, the equation (41) is equivalent to the vanishing of all coefficients before $\eta^J$:

$$\sum f_{IJ} d_{\beta_J} (\xi^I) = 0 \quad \text{for any } J.$$ 

Note that here we do not need to assume that the equality holds mod $J^{-}$. Further, $f_{IJ}$ is a function of weight 0. In other words it is a usual smooth (or holomorphic) function that is defined in $U_0$. Let us take a point $x \in U_0$ and evaluate the function $f_{IJ}$ at $x$. We get

$$\sum f_{IJ}(x) d_{\beta_J}(\xi^I) = d_{\beta_J}(\sum_{I} f_{IJ}(x) \xi^I) = 0 \quad \text{for any } x \text{ and } J.$$ 

Since $d_{\beta_J}$ is the de Rham differential and we can consider $f_{IJ}(x)\xi^I$ as an exterior form of degree 0 (or just a function) for this operator, we conclude from this equation that $f_{IJ}(x)\xi^I$ is a constant function. Therefore $f_{IJ}\xi^I$ does not depend on $\xi^I$ for any $x$. This contradicts to the fact that the weight of $\mathcal{D}_2(f)$ depends on $\alpha_i$ non-trivially and $(d_{\gamma_p} \mod J^{-}) (\mathcal{D}_2(f)) \neq 0$ is proven. Since this holds for any $p$, the result follows. □

**Proof of Proposition 9.** Since $d_{\beta_J} \circ d_{\beta_J} = 0$, see (15), and therefore $d_{\beta_J} \circ d_{\beta_J} \mod J^{-} = 0$, we have

$$\mathcal{D}^\Lambda ((\mathcal{O}_N)_\delta) \subset \left( (\mathcal{O}_{\mathcal{D}_N})_{\mathcal{D}^\Lambda(\delta)} \right) \bigcap_{k=1}^{s} \ker \mathcal{D}_{\gamma_k}. \quad (42)$$

Our goal is to prove that the inclusion (42) is in fact the equality. As in the proof of Proposition 8, let us write the sequence (22) in the following form:

$$\mathcal{D}_1 \circ (d_{\gamma_p} \mod J^{-}) \circ \mathcal{D}_2,$$

where $d_{\gamma_p} = d_{\beta_J}$, and let us take any

$$f' \in \left( (\mathcal{O}_{\mathcal{D}_N})_{\mathcal{D}^\Lambda(\delta)} \right) \bigcap_{k=1}^{s} \ker \mathcal{D}_{\gamma_k}.$$ 

Assume by induction that we found an element

$$f \in (\mathcal{O}_{\mathcal{N}}/J^{-})_{(d_{\gamma_p} \circ \mathcal{D}_2)(\delta)}$$

such that $\mathcal{D}_1(f) = f'$ and $(d_{\gamma_q} \mod J^{-})(f) = 0$ for $q = p, p+1, \ldots, s$. We have to show that there exists $F \in (\mathcal{O}_{\mathcal{N}}/J^{-})_{\mathcal{D}_2(\delta)}$ such that

$$(d_{\beta_J} \mod J^{-})(F) = f \quad \text{and} \quad (d_{\gamma_q} \mod J^{-})(F) = 0.$$
for any $q = p + 1, p + 2, \ldots, s$. 

Consider a chart $\mathcal{U}$ on $\mathcal{M}$ as in the proof of Proposition 8 with coordinates $(x_i), (\xi_j)$ and $(\eta_l)$ from the groups 1, 2 and 3, respectively. We can write $f$ locally in the following form:

$$f = \sum_{I,J,u} f_{IJu} \xi^I d_{\beta_{ji}}(\xi_u) \eta^J.$$  

Here $I$ and $J$ are multi-indexes, $f_{IJu}$ are functions of weight 0; $\xi^I$ and $\eta^J$ are monomials in local homogeneous coordinates from the second and third groups, respectively; and $d_{\beta_{ji}}(\xi_u)$ are local coordinates from the third group which weights contain $\beta_{ji}$. By our assumption, we have

$$(d_{\beta_{ji}} \mod J^-)(f) = \sum_{I,J,u} f_{IJu} d_{\beta_{ji}}(\xi^I) d_{\beta_{ji}}(\xi_u) \eta^J = 0 \mod J^-.$$

Note that to obtain the first equality in (43), we use the following facts

$$d_{\beta_{ji}}(f_{IJu}) \in J^-, \quad d_{\beta_{ji}}(\eta^J) \in J^- \quad \text{and} \quad d_{\beta_{ji}} \circ d_{\beta_{ji}} = 0.$$ 

Since weights of monomials $\eta^J$ do not contain $\beta_{ji}$, the equality (43) is equivalent to

$$d_{\beta_{ji}} \left( \sum_{I,u} f_{IJu}(x_0) \xi^I d_{\beta_{ji}}(\xi_u) \right) = 0 \quad \text{for any} \ J \ \text{and any} \ x_0 \in \mathcal{U}_0.$$

We see that $\sum_{I,u} f_{IJu}(x_0) \xi^I d_{\beta_{ji}}(\xi_u)$ is a closed 1-form in the superdomain with coordinates $(\xi_j)$, with respect to the de Rham differential $d_{\beta_{ji}}$. By the Poincaré Lemma for supermanifolds, for any $x_0$ and $J$ there exists

$$F_J(x_0) = \sum_K F_{KJ}(x_0) \xi^K \quad \text{such that} \quad d_{\beta_{ji}}(F_J(x_0)) = \sum_{I,u} f_{IJu}(x_0) \xi^I d_{\beta_{ji}}(\xi_u).$$

Here $K$ is a multi-index. In particular we have

$$\frac{\partial}{\partial \xi_u} (F_J(x_0)) = \sum_I f_{IJu}(x_0) \xi^I.$$

Note that $F_J(x_0)$ is defined up to a constant. However, if we assume that

$$\text{weight}(F_J(x_0)) = \text{weight} \left( \sum_I f_{IJu}(x_0) \xi^I \right) - (\beta_{ji} - \alpha_i),$$

then $F_J(x_0)$ is unique.
Now we need to show that $F_{J}(x_0)$ is a smooth (or holomorphic) function in $x_0$. Consider the following equality

$$F_{KJ} = \frac{\partial}{\partial \xi_K}(F_{J}),$$

where $\frac{\partial}{\partial \xi_K}$ is the corresponding to $\xi^K$ differential operator. Using (44), we see that the function $F_{KJ}$ is a certain iterated derivative of $\sum f_{IJu} \xi^I$, and hence it is smooth (or holomorphic).

Summing up, we constructed the functions $F_{J} = \sum_{K} F_{KJ} \xi^{K}$ such that

$$d_{\beta_{ji}}(F_{J}) = \sum_{Iu} f_{IJu} \xi^I d_{\beta_{ji}}(\xi_u) \mod J^{-}.$$

We put

$$F := \sum_{KJ} F_{KJ} \xi^{K} \eta^{J}.$$ 

Clearly the weight of $F$ is $D_{2}(\delta)$ and we have $d_{\beta_{ji}}(F) \mod J^{-} = f$.

It is remaining to show that $d_{\gamma_{q}}(F) = 0 \mod J^{-}$ for $q = p + 1, \ldots, s$. Consider the function $H := (d_{\gamma_{q}} \mod J^{-})(F)$, where $q = p + 1, \ldots, s$. The weight of $H$ does not contain $\beta_{ji}$ since $D_{1}^{\Lambda}(\delta)$ is multiplicity free, and it has the form $c_{0} \alpha_{i} + \ldots$, where $c > 0$, since our assumption that the weight $D_{1}^{\Lambda}(\delta)$ depends on $\alpha_{i}$ non-trivially for any $i = 1, \ldots, r$. Therefore, $H$ is a non-constant function for the de Rham differential $d_{\beta_{ji}}$. Further,

$$(d_{\beta_{ji}} \mod J^{-})(H) = (d_{\beta_{ji}} \mod J^{-}) \circ (d_{\gamma_{q}} \mod J^{-})(F) = -(d_{\gamma_{q}} \mod J^{-}) \circ (d_{\beta_{ji}} \mod J^{-})(F) = -(d_{\gamma_{q}} \mod J^{-})(f) = 0.$$

Hence $H = 0$.

If we iterate our construction, we get a function

$$F' \in (\mathcal{O}_{\mathcal{N}}^\delta)_{/J^{-}} = (\mathcal{O}_{\mathcal{N}})_{\delta}$$

such that $D_{1}^{\Lambda}(F') = f'$. The proof is complete. □

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