The First Temperature Corrections to the Fermi Liquid Fixed Point in Two Dimensions

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(March 22, 2022)

We calculate using perturbative calculations and Ward identities the basic parameters of the Fermi Liquid: the scattering vertex, the Landau interaction function, the effective mass, specific heat, and physical susceptibilities for a model of two-dimensional (2D) fermions with a short ranged interaction at non-zero temperature. The leading temperature dependence of the spin components of the scattering vertex, the Landau function, and the spin susceptibility is found to be linear. We find that the standard $T = 0$ relationships for a Galilean-invariant Fermi Liquid are violated by finite-temperature terms. The coefficients in the temperature corrections to these relationships involve a subtle interplay between contributions from small and large ($\sim 2k_F$) momentum processes. A connection with previous studies of the 2D Fermi-Liquid parameters is discussed. We conclude that the linear leading temperature dependence of the parameters is a generic feature of the 2D Fermi Liquid.

PACS numbers: 05.30.Fk, 71.10Ay, 71.10.-w, 71.10.Pm

I. INTRODUCTION

For more than a decade two-dimensional (2D) fermion systems have been a subject of great interest. One important motivation has been the non-Fermi-Liquid behavior observed in high-$T_c$ superconductors above $T_c$. In this context the existence and stability of the Fermi Liquid (FL) in $d > 1$ has been extensively investigated within various modern approaches. It seems fair to say that no compelling theoretical evidence has been found for non-Fermi-Liquid behavior in systems with short ranged interactions and in the absence of coupling to gauge fields. The FL is stable, provided that standard conditions are satisfied.

Given this a natural question arises: suppose 2D fermions are in the FL phase in the sense of the Landau’s Fermi Liquid Theory (FLT). We understand “zero order” FL quantities as those defined in the limit of zero temperature and zero momentum-energy transfers. Then what is the low-energy behavior of such quantities? In this paper we study the leading temperature corrections to the zero-order behavior of the the components of the Landau function and the scattering vertex, the quasiparticle’s effective mass, and the uniform response functions (compressibility, spin susceptibility). In Renormalization Group (RG) language we are studying the corrections coming from the leading irrelevant terms.

Interest in the issue grown in recent years also because these leading corrections provide the “bare” temperature dependence of the parameters in theories describing quantum critical phenomena in metals. In particular, puzzling data in several materials led to explanations based on an unusual underlying temperature, momentum or frequency dependence of electronic susceptibilities. The leading temperature corrections to the parameters of a stable FL been studied for many years. Rather surprisingly, the issue remains a subject of controversy. For example, it was found that the leading temperature correction to the specific heat coefficient $\gamma = C/T$ was $T^2 \ln T$ in $d=3$ spatial dimensions and $T$ in $d=2$. Recently these results were also rederived by multidimensional bosonization (see, e.g., Ref. [4]). Whether the spin and charge susceptibilities display similarly anomalous (i.e., non-$T^2$) temperature dependence is the subject of a lengthy controversy in the literature: see, e.g., Ref. [20], discussion and references therein. For a most recent reassessment of such results see Ref. [21]. The prevailing conclusion was that of Carneiro and Pethick who found no leading $T^2 \ln T$ correction to the spin susceptibility of the 3D FL. Their analysis implies that terms $\propto T$ are absent in 2D.

The heuristic argument which runs commonly through the literature to account a posteriori for the absence of anomalous terms (in $T$ or in $q$) in response functions is that although these terms are known to occur in the individual self-energy and vertex diagrams, they cancel in final results for symmetry reasons mathematically expressed via Ward identities. As we now discuss, this argument is misleading. Although exact symmetries of a model (e.g., global gauge and/or rotational invariance) result in relationships between the self-energy and the vertex through the Ward identities (cf. Sec. [V] below), these are usually not enough to close the system of equations and demonstrate explicitly...
the cancellation of the self-energy and vertex corrections. Additional symmetries like, e.g., the chiral symmetry of the 1D Tomonaga-Luttinger model, are needed in order to do it.\[2\] In $d > 1$ analogous extra asymptotic symmetries first noted by Haldane\[4\] occur in the fermionic Wilsonian low-energy effective action under some model’s restrictions. In the recent series of papers by Metzner, Castellani and Di Castro (reviewed in Ref. \[5\]) the relationship between the extra asymptotic symmetry of the effective action and the RPA (FLT) results for the response functions was clearly established. They also clarified a close connection between this approach and multidimensional bosonization.\[6\] The additional Ward identities are properties of a model in which all processes involving momentum transfers greater than a cutoff\[10\]

$$\Lambda_{ea} \ll k_F$$

are discarded. These restrictions appear naturally in the Wilsonian low-energy effective action with the UV cutoff\[11\] obtained from an RG approach such as that of Shankar,\[12\] in which interacting fermions are treated by progressive elimination of modes towards the Fermi surface. Only asymptotically (i.e., in the limit $\Lambda_{ea}/k_F \to 0$, $T \to 0$) this forward-scattering action possesses extra symmetries $U(1)\infty$ (and $SU(2)\infty$ for rotationally invariant case).\[13\] (It is assumed that the initial microscopic action (Hamiltonian) has only ordinary $U(1)$ [and $SU(2)$ symmetry].) However, these extra identities do not constrain possible $T$- or $q$- terms in physical quantities, coming from irrelevant terms. Even in 1D, e.g., the $SU(2)_L \otimes SU(2)_R$-breaking marginally irrelevant term coming from backscattering results in $1/\ln T$-leading correction to the uniform spin susceptibility of the Luttinger liquid.\[14\] In higher dimensions there are always [even in the limit\[15\]] special configurations of momenta near the Fermi surface which give rise to irrelevant terms in the effective action which strongly interfere with the forward scattering channel\[16\] and invalidate asymptotic symmetries. In the framework of the low-energy effective action approach there is also another potential source of temperature corrections, i.e., the possibility of $T$-dependence of the Wilsonian action’s vertices (couplings) developing on the previous stages of modes elimination before one reaches the effective action scale.\[17\] Apparently, this question has not been carefully studied.

We note that in the most of existing literature studying the leading FL corrections it is assumed that the crucial coupling is between quasiparticles and long-wavelength collective modes, i.e., only small momentum scales are taken into account. However the possibility of “$2k_F$ singularities”, i.e., anomalous temperature terms coming from processes involving large ($\sim 2k_F$) momentum transfers, has been pointed out by Misawa for 3D FL already in early 70s.\[20\] Apparently due to the lack of experimental evidence of a $T^2 \ln T$ term in the susceptibility of a generic 3D FL and also because Misawa’s results rely on the analysis of selected diagrams [cf. previous paragraph], they were widely disregarded in favor of those of Carneiro and Pethick. In the context of semiconductor physics Stern was the first to note\[19\] that in a 2D electron gas the electron scattering rate was proportional to $T$ due to $2k_F$ effects. The consequences of the $2k_F$ effects for the leading $T$-dependence of 2D FL quantities have not been considered in the literature until recently.

The issue of the leading correction to FLT has recently been revived by several papers. Belitz, Kirkpatrick and Vojta\[21\] presented perturbative calculations, mode-coupling arguments and power counting estimates which showed that the leading $g$ dependence of the spin susceptibility (but not the charge susceptibility) was $|g|$ in 2D ($q^2 \ln q$ in 3D). They did not find the analogous $T$-correction explicitly, but concluded that one should generally expect a linear $T$-term in the 2D FL susceptibility ($T^2 \ln T$ in 3D). This dependence has important implications for the theory of the quantum critical metallic ferromagnet.\[22\]

Sénéchal and one of us\[10\] predicted the occurrence of the linear $T$-corrections to the FL vertices from one-loop RG calculations based on a 2D effective action. Those RG calculations of the irrelevant corrections rely more on the low-energy effective action’s phase space constraints in their way to sort out the effective interactions and the scales involved, rather than on doing it according to the strength of the latter, as a perturbation theory does. That makes desirable to easily obtain a perturbative signal of the RG predictions. Also, an analogous RG calculation of other FL quantities was not done.

Hirashima and Takahashi\[23\] performed numerical analyses of perturbative expressions which appeared to confirm the prediction of Belitz et al\[21\] for 2D susceptibility. However due to numerical difficulties in handling divergences in some terms they were unable even to determine the sign of the coefficient in the leading $g$-term. Also contrary to Belitz et al who focused on the long-wavelength contributions, the authors of Ref. \[20\] emphasized the crucial role of $2k_F$ contributions in their findings. Following this Misawa conjectured a phenomenological form for the free energy which results in the linear $T$-term in the 2D spin susceptibility and in the coefficient $\gamma$, and agrees well with the numerical calculations in lowest order.\[24\]

This previous work has left a number of important questions unresolved, including analytic treatment of anomalous terms (important for verifying cancellations between different contributions), the relationship to FLT and to Ward identities, and the connection to the extensive literature from the 1960-70s. To elucidate the issues in the most transparent way, we apply the perturbation theory for 2D contact-interacting spin-$\frac{1}{2}$ fermions, starting from a microscopic
action. Although the Landau FLT is not a perturbative theory, for sufficiently weak interactions (assuming the interaction being repulsive and we are above the Kohn-Luttinger temperature) one should be able to find the parameters of the FL in terms of the coupling series.

We present what is apparently the first analytic calculation of the leading $T$-dependence of the effective mass, Landau parameters and response functions of a 2D electron gas, to second order in the interaction strength, including all channels and all momentum processes (scales). We take into account the Ward identities explicitly. As we show in this paper, the processes involving large ($\sim 2k_F$) momentum transfers are crucial to the anomalous temperature dependence of FL quantities.

The rest of this paper is organized as follows. Section II A defines the model. Section II B defines the four-point vertex to be calculated and its relationship to the Fermi Liquid parameters. We also give there the basic equation for that vertex in the one-loop approximation. In Section III we present and discuss our results for the scattering amplitude and the Landau function. In Section IV we give the Ward identities which are used for the following calculations. In Section V we calculate the effective mass. Sections VI, VII present the results for the compressibility and the spin susceptibility, respectively. The results are recapitulated in the concluding Section VIII. Appendices A and B contain detailed presentation of the calculations of the one-loop contributions entering the equation for the vertex. One of the goals of presenting these technicalities was to give the idea of how more involved two- and three-loop calculations of Secs. V-VII were carried out.

Some of the results presented here were announced in a previous short communication.

II. MODEL AND FORMALISM

A. Model

We treat interacting fermions at finite temperature in the standard path integral Grassmannian formalism. The partition function is

$$Z = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \ e^{S_0 + S_{\text{int}}}$$

(2)

where the free part of the action is

$$S_0 = \int (1) \bar{\psi}_\alpha(1) \left[ i\omega_1 + \mu - \epsilon(k_1) \right] \psi_\alpha(1)$$

(3)

and

$$\int (i) \equiv \int \frac{dk_i}{(2\pi)^d} \sum_{\omega_i} \beta \equiv (k_i, \omega_i) \equiv K_i$$

(4a)

(4b)

where $\beta$ is the inverse temperature, $\mu$ the chemical potential, $\omega_i$ the fermion Matsubara frequencies and $\psi_\alpha(i)$ a $N$-component Grassmann field with a spin (flavor, if $N \neq 2$) index $\alpha$. Summation over repeated indices is implicit throughout this paper. We set $k_B = 1$ and $\hbar = 1$. The $SU(N)$-invariant quartic interaction is

$$S_{\text{int}} = -\frac{1}{4} \int_{(1, 2, 3, 4)} \bar{\psi}_\alpha(1) \bar{\psi}_\beta(2) \psi_\gamma(3) \psi_\epsilon(4) U^{\alpha\beta}_{\gamma\epsilon} (1, 2; 3, 4) \delta^{(d+1)}(1 + 2 - 3 - 4)$$

(5)

Here the conservation of energy and momentum is enforced by the symbolic delta function

$$\delta^{(d+1)}(1 + 2 - 3 - 4) \equiv \beta(2\pi)^d \delta(k_1 + k_2 - k_3 - k_4) \delta_{\omega_1 + \omega_2 - \omega_3 - \omega_4, 0}$$

(6)

The $SU(N)$ extension of the physical $SU(2)$ symmetry is useful for different applications and the case of the spin-$\frac{1}{2}$ electrons is recovered by setting $N = 2$. We can decompose the interaction by factorizing its symmetric and antisymmetric frequency-momentum- and spin- dependent parts in the way which explicitly manifests the antisymmetry of the function $U$ under exchange:

$$U^{\alpha\beta}_{\gamma\epsilon} = U^A I^{\alpha\beta}_{\gamma\epsilon} + U^S T^{\alpha\beta}_{\gamma\epsilon}$$

(7)
where the functions $U^S$ and $U^A$ have the symmetry properties

$$U^A(1, 2; 3, 4) = -U^A(2, 1; 3, 4) = -U^A(1, 2; 4, 3) \quad (8a)$$
$$U^S(1, 2; 3, 4) = U^S(2, 1; 3, 4) = U^S(1, 2; 4, 3) \quad (8b)$$

while two operators $\hat{I}$ and $\hat{T}$, which are respectively symmetric and antisymmetric in the spin space, are defined as follows:

$$I_{\alpha \beta}^{\gamma \epsilon} \equiv \delta_{\alpha \epsilon} \delta_{\beta \gamma} + \delta_{\alpha \gamma} \delta_{\beta \epsilon} \quad (9a)$$
$$T_{\alpha \beta}^{\gamma \epsilon} \equiv \delta_{\alpha \epsilon} \delta_{\beta \gamma} - \delta_{\alpha \gamma} \delta_{\beta \epsilon} \quad (9b)$$

If we assume an instantaneous point interaction between electrons, then $U^A = 0$ and $U^S = U_0$. The sign of the interaction is chosen such that $U_0 > 0$ corresponds to repulsion. Then

$$S^c_{\text{int}} = -\frac{1}{4} U_0 \int_{(1,2,3,4)} \bar{\psi}_\alpha(1) \psi_\beta(2) \bar{\psi}_\gamma(3) \psi_\epsilon(4) T_{\gamma \epsilon}^{\alpha \beta}(d+1)(1 + 2 - 3 - 4) \quad (10)$$

In this paper we consider mainly 2D electrons with the bare spectrum of a free gas $\epsilon(k) = k^2/2m$ and the circular Fermi surface, interacting via (weak) contact repulsion \cite{[1]}, but discuss the consequences of generic spectra and a non-circular Fermi surface.

**B. Four-point 1PI vertex and parameters of Fermi Liquid**

We define the two-particle Green’s function as:

$$G_2(1, 2; 3, 4) = -\langle \psi_\alpha(1) \psi_\beta(2) \bar{\psi}_\gamma(3) \bar{\psi}_\epsilon(4) \rangle \quad (11)$$

The 1PI vertex $\hat{\Gamma}$ is related to $G_2$ as

$$G_2(1, 2; 3, 4) = \langle \psi_\alpha(1) \bar{\psi}_\gamma(3) \rangle \langle \bar{\psi}_\beta(2) \psi_\epsilon(4) \rangle - \langle \psi_\alpha(1) \bar{\psi}_\gamma(3) \rangle \langle \psi_\beta(2) \bar{\psi}_\epsilon(4) \rangle$$
$$+ \int_{1', 2', 3', 4'} \langle \psi_\alpha(1) \bar{\psi}_\beta(2) \bar{\psi}_\gamma(3') \bar{\psi}_\epsilon(4') \Gamma_{\gamma \epsilon}^{\alpha \beta} \rangle \langle \psi_\gamma(3') \bar{\psi}_\epsilon(4') \rangle \delta^{(d+1)}(1' + 2' - 3' - 4') \quad (12)$$

where due to the symmetries of the model we can write the one-particle Green’s function as

$$-\langle \psi_\alpha(1) \bar{\psi}_\beta(2) \rangle = G(1) \delta_{\alpha \beta} \delta^{(d+1)}(1 - 2) \quad (13)$$

Perturbatively, the vertex is constructed by considering only connected one-particle-irreducible (1PI) diagrams with amputated external legs. In lowest order $\hat{\Gamma}^{(0)}(1, 2; 3, 4) = \hat{U}(1, 2; 3, 4)$. To shorten notations we use the hat meaning that it is an operator in the spin space, and it also comprises two components. Along with the representation \cite{[1]} with the components $(\Gamma^A$, $\Gamma^S)$ for the vertex, which explicitly shows its antisymmetry, we will also use another representation more convenient for some applications, namely separating the vertex into charge and spin components via

$$\hat{\Gamma} \mapsto \Gamma^\alpha_{\gamma \epsilon} = \Gamma^A I_{\gamma \epsilon}^\alpha + \Gamma^S T_{\gamma \epsilon}^\alpha \quad (14)$$

where $\lambda^a$ ($a = 1,...,N^2-1$) are Hermitian traceless generators of the $SU(N)$ group, coinciding with the Pauli matrices for the case $N = 2$, and normalized such that

$$\text{tr}(\lambda^a \lambda^b) = 2 \delta^{ab} \quad (15)$$

The components of the vertex in different representations are related as

$$\Gamma_{\text{ch}} = (N - 1) \Gamma^S - (N + 1) \Gamma^A \quad (16a)$$
$$\Gamma_{\text{sp}} = -\Gamma^S - \Gamma^A \quad (16b)$$
Taking into account momentum and energy conservation, we write the vertex as:

$$\hat{\Gamma}(1, 2; 1 + Q, 2 - Q) \equiv \hat{\Gamma}(1, 2; Q),$$

with the transfer vector

$$Q = 3 - 1 \equiv (q, i\Omega)$$

where $i\Omega$ is a bosonic Matsubara frequency. In what follows we will frequently use the property of the components $\Gamma_{ch}$ and $\Gamma_{sp}$ which is a consequence of the vertex’s antisymmetry

$$\Gamma_{\gamma\varepsilon}^{\alpha\beta}(1, 2; Q) = \Gamma_{\varepsilon\gamma}^{\beta\alpha}(2, 1; -Q)$$

For the calculation of physical quantities of the FL we need $\hat{\Gamma}(1, 2; Q)$ at $Q = 0$. As is well known[1] the limit $Q \to 0$ is not unique, since the vertex is non-analytic function at $Q = 0$. Two vertices

$$\hat{\Gamma}^\chi(1, 2) = \lim_{q \to 0} \left[ \hat{\Gamma}(1, 2; Q) \right]_{\Omega=0},$$

$$\hat{\Gamma}^{\Omega}(1, 2) = \lim_{\Omega \to 0} \left[ \hat{\Gamma}(1, 2; Q) \right]_{q=0}$$

can be defined unambiguously at $1 \neq 2$ and then continued to $1 \to 2$. (See, e.g., Refs. [9,10] on this subtlety). In the calculation of the FLT vertices (real electrons, $N = 2$) the external momenta are chosen to lie on the Fermi surface, and the external Fermionic frequencies are put equal to the minimal Matsubara frequency $i\pi T$. In the case of a rotationally invariant Fermi surface the vertex dependence on the external momenta $k_1$ and $k_2$ lying on the Fermi surface can be parameterized by the relative angle $\theta_{12}$ between those momenta and then $\hat{\Gamma}^\chi(\theta_{12}), \hat{\Gamma}^{\Omega}(\theta_{12})$ can be identified with the scattering vertex and the Landau function, respectively. Namely, for the components of the scattering vertex ($A$, $B$) we have (cf., e.g., Ref.[9])

$$-2\nu_R Z^2 \Gamma^{\alpha\beta(0)}(\theta_{12}) = A(\theta_{12}) \delta_{\alpha\gamma} \delta_{\beta\varepsilon} + B(\theta_{12}) \lambda^a_{\alpha\gamma} \lambda^a_{\beta\varepsilon}$$

where $\nu_R = S_d K_F^{-2} m^*/(2\pi)^d$ is the $d$-dimensional renormalized electron density of states per spin on the Fermi level, $S_d$ is the $d$-dimensional area of the unit sphere (for $d = 2$: $\nu_R = m^*/2\pi$), and $Z$ is the field renormalization constant. So

$$A(\theta_{12}) = \nu_R Z^2 \Gamma^{\chi}_{ch}(\theta_{12})$$

$$B(\theta_{12}) = \nu_R Z^2 \Gamma^{\chi}_{sp}(\theta_{12})$$

Two components ($F$, $G$) of the Landau interaction function are defined by analogous equations, with the substitution $q \to \Omega, A \to F, B \to G$.

The one-loop approximation $\hat{\Gamma}^{(1)}$ of the straightforward perturbation theory for $\hat{\Gamma}(1, 2; Q)$ in diagrammatic form is given in Fig. 1. For the model given by (3,10) we obtain after some spin summations

$$\hat{\Gamma}^{(1)}(1, 2; Q) = -\frac{1}{2} L_{-} U_0^2 \hat{T} + \left\{ U_0 - \left[ \frac{1}{2} L_+ + C(1 + 2) \right] U_0^2 \right\} \hat{T}$$

with

$$L_{\pm} = L(Q) \pm L(Q')$$

We denote $Q' \equiv 2 - 1 - Q$. The functions $L$ and $C$ coming from the contributions of bubbles (ZS, ZS') and (BCS), respectively, are defined and calculated in Appendices [4] and [5]. From the one-loop approximation (25) for the 1PI vertex $\hat{\Gamma}$ valid for generic momenta, energies and transfer, we can obtain now the Fermi Liquid parameters (vertices) using the definitions [21,22,23,24].

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III. TEMPERATURE DEPENDENCE OF THE SCATTERING VERTEX AND THE LANDAU FUNCTION

Taking the appropriate zero transfer limit we obtain from Eqs. (25, 23, 24) the components of the scattering vertex

\[ A(\theta) = u - u^2 + 2\chi(q')u^2 - C(q_s)u^2, \]

\[ B(\theta) = -u - u^2 + C(q_s)u^2 \]

with \( q' = 2k_F \sin \theta \) and \( q_s = 2k_F \cos \theta \). We absorbed the factor \( \nu_0 \equiv m/2\pi \) into the definition of coupling via

\[ u = \nu_0 U_0. \]

Definitions and detailed calculations of the functions involved in the equations above are given in Appendices A and B. For the components of the Landau function we obtain in the same way

\[ F(\theta) = u + 2\chi(q')u^2 - C(q_s)u^2 \]

\[ G(\theta) = -u + C(q_s)u^2 \]

Note that while calculating the FL vertices (23, 24) at the one-loop level we can put \( \nu_R = \nu_0 \) (i.e., \( m^* = m \)), \( Z = 1 \), and since \( L(\Omega, 0) = 0 \), the ZS contribution to \( F, G \) is zero in the limit (21).

Using the results (A27, A30, B30, B31) of the Appendices we obtain for the Fourier components of the two vertices:

\[ A_0 = u - u^2 \left( \ln 2\lambda - 1 + \frac{\pi^2}{96} T_0^2 \right) \]

\[ F_0 = u - u^2 \left( \ln 2\lambda - 2 + \frac{\pi^2}{96} T_0^2 \right) \]

\[ B_0 = -u + u^2 \left( \ln 2\lambda - 1 - \ln \frac{2}{2} T_0 \right) \]

\[ G_0 = -u + u^2 \left( \ln 2\lambda - \ln \frac{2}{2} T_0 \right) \]

\[ A_1 = F_1 = u^2 \left( \frac{1}{2} - \frac{\pi^2}{96} T_0^2 \right) \]

\[ B_1 = G_1 = u^2 \left( -\frac{1}{2} + \ln \frac{2}{2} T_0 \right) \]

The terms omitted in the above results are \( \mathcal{O}(u^2T_0^2) \) at most. Within one-loop accuracy the standard RPA-type relationships between the FL parameters \( A_n = F_n/(1 + F_n) \), \( B_n = G_n/(1 + G_n) \) hold.

Note that taken separately, each contribution of the ZS'- or BCS bubble has a leading linear temperature-dependent term in the first two Fourier components of the vertices [cf. Eqs. (A27, A30) and Eqs. (B30, B33), resp.]. However, a cancellation of such terms coming from two graphs occurs in the “charge sector” (i.e., in \( A, F \) components), while the linear T-terms survive in the “spin sector” (\( B, G \) components).

To understand where the temperature dependence of the vertices comes from, note that \( \chi(q') \) as a function of \( q' \) has some “temperature smearing” near \( q' \sim 2k_F \), otherwise it is virtually indistinguishable from its zero-temperature limit \( (A13) \), provided the condition \( (A25) \) applies. So the temperature dependence of the ZS' (exchange) contribution to the Fourier components of the vertices comes from integrating of the function \( \chi \) around the “effective transfer” \( q' \sim 2k_F \) in the ZS' graph, i.e., when incoming momenta \( k_1 \sim -k_2 \), while \( |k_1| = |k_2| = k_F \). Notice that the transfer \( q \) in the proper sense \( (B13) \) is zero. This source of temperature dependence was not considered by Pethick and Carneiro, for example. In the same vein, one can see from Eq. (B15) that the temperature dependence of the BCS contribution...
comes from regions of small $q_s$, i.e., again when $k_1 \sim -k_2$. The coefficients in the temperature expansions of the Fourier components of $\chi$ and $C$ are such [cf. Eqs. (A27,A30,B31)] that the cancellation occurs only for the linear terms in the charge components of the vertices, while the higher order terms in the temperature survive. Since to the best of our knowledge there is no particular reason for this cancellation of the linear terms to happen, we expect that linear $T$-term of the charge component(s) would come out of a more realistic calculations of the FL parameters.

Let us elaborate on the last point a bit more. First notice that in the perturbative approximation (23) for the vertex in the model with a constant couplings $U_0$ (bare vertex), all the three one-loop terms have the same factor $U_0^2$ in front of the bubble contributions, $L$, $C$. Had these coupling factors been different for each of the three graph, the cancellation of the temperature corrections would not have happened. On the other hand, even the simplest technique according to standard approaches.

The linear dependence of the leading temperature corrections to the FL parameters seems not to be sensitive to the actual form of the one-particle spectrum. The same dependence (apart from model-sensitive prefactors) was obtained in the previous RG analysis of the effective action model for 2D spinless fermions with a linearized spectrum. In that study the integration of the “effective” transfer in the exchange ($ZS'$) graph contribution to the RG flow was reproducible such development of the momentum dependence in couplings in the regime of effective action (1). This would prevent the cancellations between different terms, e.g., between the one-loop $ZS'$ and BCS contributions, which we have obtained in the simple perturbational framework.

Another artifact of the naive perturbative calculation of the vertex is its ultraviolet divergence coming from the BCS bubble. The $\ln \lambda$-term makes our perturbation theory applicable only if the interaction is small enough that the arbitrary, generally speaking, ultraviolet cutoff $\lambda$ can be chosen to satisfy $1 \ll \lambda < \frac{1}{4^u}$, ($u < 1$). However, we should not worry too much about this issue, since more sophisticated calculations based, e.g., on summations of the diagrammatic series or an RG would result in an effective screening of the repulsive interaction in the BCS channel. See, e.g., Refs. [8,9,2]. (Although providing the possibility of the infrared instability (divergence) at exponentially low temperatures (much lower than, say, we are working at) through the Kohn-Luttinger mechanism.

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**IV. WARD IDENTITIES**

In order to derive the Ward identities, we followed closely the methodology of Ref. [9] adapted to the path-integral technique according to standard approaches. The partition function is given by the path integral (2). In coordinate-imaginary-time space the free part of the action is

$$ S_0 = \int dX \tilde{\psi}_\alpha(X) \left[ -\frac{\partial}{\partial \tau} + \mu + \frac{\Delta}{2m} \right] \psi_\alpha(X) \tag{38} $$

We consider pair interactions which preserve the total spin and number of particles

$$ S_{\text{int}} = \int \psi_\alpha(X_1) \psi_\alpha(X_1) U(x_1 - x_2) \delta(\tau_1 - \tau_2) \psi_\beta(X_2) \psi_\beta(X_2) \tag{39} $$

Here $\int_X \equiv \int_0^\beta d\tau \int d\mathbf{x}$. Notice that interaction (10) is a special case of (39). Then the following charge Ward identity

$$ \frac{G^{-1}(1) - G^{-1}(1 - Q)}{i\Omega - \frac{a}{2m}(2k_1 - q)} = 1 - \frac{1}{N} \int_{(K)} \frac{i\Omega - \frac{a}{2m}(2k + q)}{i\Omega - \frac{a}{2m}(2k_1 - q)} \Gamma_{\alpha\beta}(1,K; -Q) G(K) G(K + Q) \tag{40} $$

and the spin Ward identity

$$ \frac{G^{-1}(1) - G^{-1}(1 - Q)}{i\Omega - \frac{a}{2m}(2k_1 - q)} = 1 - \frac{1}{2(N^2 - 1)} \int_{(K)} \frac{i\Omega - \frac{a}{2m}(2k + q)}{i\Omega - \frac{a}{2m}(2k_1 - q)} \chi_{\gamma\epsilon} \Gamma_{\beta\gamma} \lambda_{\beta\alpha}(1,K; -Q) \chi_{\beta\alpha} G(K) G(K + Q) \tag{41} $$

can be derived. From (14,15) one can find

$$ \Gamma_{\alpha\beta} = -NT_{\text{ch}} \tag{42a} $$

$$ \chi_{\gamma\epsilon} \Gamma_{\beta\gamma} \lambda_{\beta\alpha} = -2(N^2 - 1) \Gamma_{\chi\epsilon} \tag{42b} $$
These equations are helpful in order to unclutter the r.h.s of the Ward identities. A useful consequence of the Ward identities (40,41) is the identity:
\[ \int_{(K)} [\Gamma_{ch}(1, K; Q) - \Gamma_{sp}(1, K; Q)] G(K)G(K - Q) \frac{i\Omega - \frac{q}{2m}(2k - q)}{i\Omega - \frac{q}{2m}(2k_1 + q)} = 0 \iff \] (43a)
\[ \int_{(K)} [\Gamma^S(1, K; Q) - \Gamma^A(1, K; Q)] G(K)G(K - Q) \frac{i\Omega - \frac{q}{2m}(2k - q)}{i\Omega - \frac{q}{2m}(2k_1 + q)} = 0 \] (43b)

The above relationships are non-trivial, since \( \Gamma_{ch}(1, K; Q) \neq \Gamma_{sp}(1, K; Q) \) \( \Gamma^S(1, K; Q) \neq \Gamma^A(1, K; Q) \). There is also another Ward identity \[ \frac{k_1}{i} \frac{\partial G^{-1}(1)}{\partial i\Omega} = k_1 + \int_{(2)} k_2 \Gamma_{ch}(1, 2; -i\Omega) G(2) G(2 + i\Omega) \bigg|_{i\Omega \to 0} \] (44)
which follows from the Galilean invariance.

V. EFFECTIVE MASS

A. FLT calculation

In this subsection we calculate the effective mass using the Ward identities. The self-energy is defined such that
\[ G^{-1}(1) = G_0^{-1}(1) - \Sigma(1) \] (45)

From now on we assume \( G_0(1) \) in the above equation to be not the exact bare Green’s function of the non-interacting system, but that with the shifted chemical potential. We gauge the shift to be such that it removes all tadpole insertions to the Green’s functions in diagrammatics.

The effective mass \( m^* \) is given by the following equation
\[ \frac{m^*}{m} = 1 - \left. \left[ \frac{\partial \Sigma(1)}{\partial \Omega} + \frac{m}{k_F} \frac{\partial \Sigma(1)}{\partial k} \frac{k_1}{k_F} \right] \right|_{k_1 \in S_F} \] (46)

Since we are working with the finite-temperature Matsubara Green’s functions and \( \omega_1 \) is a Fermionic Matsubara frequency, we understand \( i\omega_1 \to 0 \) in the sense that the zero-frequency limit will be taken after the appropriate analytical continuation of the final result for \( m^*(1) \). In the second order of the perturbation theory only the “sunrise” self-energy diagram contributes to the mass renormalization. In this approximation
\[ \frac{m^*}{m} = 1 - \left. \left[ \frac{\partial \Sigma(1)}{\partial \Omega} + \frac{m}{k_F} \frac{\partial \Sigma(1)}{\partial k} \frac{k_1}{k_F} \right] \right|_{k_1 \in S_F} + \mathcal{O}(U^3) \] (47)

The charge Ward identity can be written as
\[ \frac{\partial \Sigma(1)}{\partial k} = \int_{(2)} \frac{k_2}{m} \Gamma_{ch}(1, 2; -q) G(2) G(2 + q) \bigg|_{q \to 0} \] (48)
while the Galilean-invariance Ward identity (44) reads
\[ \frac{\partial \Sigma(1)}{\partial \Omega} = - \int_{(2)} \frac{k_1 k_2}{k_F^2} \Gamma_{ch}(1, 2; -i\Omega) G(2) G(2 + i\Omega) \bigg|_{i\Omega \to 0} \] (49)

Combining the three above equations together, we get
\[ \frac{m^*}{m} = 1 + \int_{(2)} \frac{k_1 k_2}{k_F^2} \left. \Gamma_{ch}(1, 2; -i\Omega) G(2) G(2 + i\Omega) \bigg|_{i\Omega \to 0} - \Gamma_{ch}(1, 2; -q) G(2) G(2 + q) \bigg|_{q \to 0} \right|_{k_1 \in S_F} \] (50)

Since the above equation has the accuracy \( \mathcal{O}(U^2) \), we can use the one-loop approximation for the vertex, while the Green’s functions can be approximated by \( G_0 \).

8
For the contact interaction \( (N = 2) \) the vertex's one-loop approximation (23) gives

\[
\Gamma_{\text{ch}}(1, 2; -Q) = U_0 + L(Q)U_0^2 - 2L(1 - 2 - Q)U_0^2 - \mathcal{C}(1 + 2)U_0^2
\]  

(51)

It is easy to verify that only the third and the fourth terms in the above expression (coming from ZS' and BCS graphs, resp.) give non-zero contributions to \( m^* \). Plugging in the formulas for the ZS' and BCS graph contributions and carrying out some manipulations, we obtain (recall that \( i\omega_1 \sim 0, \ k_1 \in S_F \))

\[
\frac{m^*}{m} = 1 + \frac{1}{4}U_0^2 \int \frac{k_1 k_2}{k_F^2} \left( \frac{\tanh(\frac{1}{2}\beta k_1) - \tanh(\frac{1}{2}\beta k_1 + k_2 - k_1)}{-i\omega_1 - \xi_{k_2} + \xi_{k_3} - \xi_{k_3 - k_2 - k_1}} \right) \frac{\beta}{\cosh^2(\frac{1}{2}\beta k_2)}
\]

(52)

Note that without applying the Ward identities, the above equation can be derived directly from Eq. (47) where the self-energy is given by the sunrise diagram. However, it takes much more tedious and lengthy calculations to verify the cancellation of other terms appearing at intermediate steps of the analysis.

Before proceeding further with the calculations, let us first comment on result (52). In the the zero-temperature limit \( \beta/\cosh^2(\frac{1}{2}\beta k_2) \propto \delta(k_2) \) enforces \( k_2 \) to lie on the Fermi surface, while the integration over the angle \( k_1 k_2 \) is equivalent to the calculation of the first Fourier harmonic of the expression in the curly brackets. The latter we have calculated already, and it is the sum of the ZS' and BCS contributions to the FL vertex. (Recall that the first order term and the ZS graph do not contribute to the first Fourier component of the vertex.) Then Eq. (52) in the the zero-temperature limit readily recovers the standard result of the FLT:

\[
\frac{m^*(T = 0)}{m} = 1 + F_1(T = 0)
\]

(53)

A straightforward extension of Eq. (53) to finite temperatures like \( m^*(T)/m = 1 + F_1(T) \) is not valid, since according to Eq. (22) \( m^*(T) \) contains an extra contribution from the "off-shell" integration over \( k_2 \) (\( \xi_{k_2} \)) normal to the Fermi surface, albeit the factor \( \beta/\cosh^2(\frac{1}{2}\beta k_2) \) makes this contribution well-localized near the Fermi surface. In other words the vertex entering the r.h.s. of Eq. (52) is not exactly the FL vertex \( F(T) \) as it is defined in the FLT (cf. definitions in Sec. II B), since one of its momenta (namely, \( k_2 \)) is not confined to the Fermi surface.

The calculation of the temperature expansion of the effective mass \( m^* \) is quite involved. We did not go beyond the linear-temperature terms. Evaluation of the contributions of two graphs to \( m^* \) denoted as

\[
\frac{m^*}{m} = 1 + u^2(M_{ZS'} + M_{BCS})
\]

(54)

gives

\[
M_{ZS'} = \frac{1}{4}(1 + \ln 2)T_0 + \mathcal{O}(T_0^2)
\]

(55a)

\[
M_{BCS} = \frac{1}{2} - \frac{1}{4}(1 + \ln 2)T_0 + \mathcal{O}(e^{-1/T_0})
\]

(55b)

Thus we get

\[
\frac{m^*}{m} = 1 + \frac{1}{2}u^2 + \mathcal{O}(u^2T_0^2)
\]

(56)

(Recall that dimensionless coupling and temperature are introduced by (29) and (32), resp.) In a close analogy with the cancellation of the linear-temperature terms in the Fourier components of the FL vertices \( A \) and \( F \), here the cancellation occurs between additive linear-\( T \) corrections coming from both "on-shell" (i.e., linear \( T \)-term coming from the \( 2k_F \)-contribution to the vertex) and "off-shell" (i.e., small-momentum contribution) integrations in two diagrams. Moreover, the "on-shell" ("off-shell") \( T \)-term of the ZS’ graph cancels the "on-shell" ("off-shell") \( T \)-term of the BCS graph, correspondingly.

**B. Alternative Calculation**

In this subsection we present an alternative direct evaluation of the leading temperature correction. The advantage of this evaluation is that it applies also to non-Galilean invariant situations and helps to more clearly establish the relation to previous studies. It also indicates possible generalizations.
We begin from the explicit expression for the self-energy correction given by the “sunrise” diagram

\[ \Sigma(P) = U_0^2 \int_{Q,K} G(P + Q + K)G(-K)G(Q) \]  

(57)

This choice of variables is convenient because the \( T \)-linear contributions to the effective mass will be seen to arise either from regions where \( q + k \) is small or from regions where it has magnitude \( 2p_F \). The bare electron Green function \( G \) is given by (A2). Evaluation of the Matsubara sums leads to

\[ \Sigma(p, i\omega) = U_0^2 \int_{k,q} \frac{[n_o(\xi_k) - n_o(\xi_q)][b(\xi_q - \xi_k) + n_o(\xi_{p+k+q})]}{i\omega + \xi_q - \xi_k} \]  

(58)

and by using \([n_o(\xi_k) - n_o(\xi_q)][b(\xi_q - \xi_k)] = n_o(\xi_q)[1 - n_o(\xi_k)]\) \((n_o \text{ and } b \text{ are the Fermi-Dirac and Bose-Einstein distribution functions, resp.})\), and relabelling variables we obtain

\[ \Sigma(p, i\omega) = U_0^2 \int_{k,q} \frac{n_o(\xi_q)}{i\omega + \xi_q - \xi_k - \xi_{p+k+q}} - n_o(\xi_q)n_o(\xi_k)J \]  

(59)

with \( J = 2/(i\omega + \xi_q - \xi_k - \xi_{p+k+q}) + 1/(-i\omega + \xi_q + \xi_k - \xi_{p+k+q}) \).

Now the first term in the expression for \( \Sigma \) cannot give rise to a term of order \( T \) because the integral of \( \xi_k \) is not confined to the region near the Fermi surface. Indeed, differentiating this term on temperature leads to a numerator containing the factor \( \xi_q \). Because the \( \xi_k \) integral is not confined to the region near the Fermi surface, obtaining a nonzero value for the \( \xi_q \) integral would require an additional factor of \( \xi_q \) divided by a scale of the order of \( E_F \), meaning that the integral would be of order \( T^2 \) as in the usual Sommerfeld expansion.

We therefore focus on possible \( T \)-linear contributions from the second term. To obtain these we will make the assumption, to be justified a posteriori, that all energies \((\xi_q, \xi_k, \xi_p)\) are near zero. We parameterize the \( q, k \) integrals by \( \xi_{k,q} \) and angles. We choose \( \theta_1 \) as the angle between \( p \) and \( q + k \) and \( \theta_2 \) as the angle between \( q \) and \( k \). For a spherical Fermi surface we have

\[ \xi_{p+k+q} = A + B \cos(\theta_1) \]  

(60)

with, \( \text{up to first order in deviations from the Fermi surface} \)

\[ A = [4E_F + \xi_q + \xi_k] \cos^2(\theta_2/2) + \xi_p \]  

(61)

\[ B = [4E_F + 2\xi_p + \xi_q + \xi_k] \cos(\theta_2/2) \]  

(62)

We may now do the integral over \( \theta_1 \) obtaining

\[ \Sigma(p, i\omega) = -\frac{u^2}{2\pi} \int d\xi_q d\xi_k d\theta_2 n_o(\xi_k)n_o(\xi_q)K \]  

(63)

with

\[ K = \frac{2 \text{sgn}(C_1)}{\text{Re} [\sqrt{C_1^2 - B^2}]} + \frac{\text{sgn}(C_2)}{\text{Re} [\sqrt{C_2^2 - B^2}]} \]  

(64)

and

\[ C_1 = \omega + \xi_q - \xi_k - \xi_p - [4E_F + \xi_q + \xi_k] \cos^2(\theta_2/2) \]  

(65)

\[ C_2 = -\omega + \xi_q + \xi_k - \xi_p - [4E_F + \xi_q + \xi_k] \cos^2(\theta_2/2) \]  

(66)

The crucial point is now that \( \text{after analytic continuation in } \omega \) the region of phase space in which the square root is real is very small, and in particular corresponds to \( \theta_2 \approx 0 \) or \( \theta_2 \approx \pm \pi \), as well as to small values of the energies, thus justifying the assumption made above. Let us consider these two regions separately.

\( \bullet \) \( \theta_2 \approx \pi \): These angles correspond to processes in which a small total momentum is transferred to the electron, i.e. to the processes which are usually considered to give rise to anomalous terms in \( C/T \), etc. In this regime we write \( \theta_2 = \pm \pi - y \); to account for \( \pm \pi \), \( y \) may have either sign. To leading order we have

\[ C_1 = \omega + \xi_q - \xi_k - \xi_p \]  

(67)

\[ C_2 = -\omega + \xi_q + \xi_k - \xi_p \]  

(68)

\[ B = 2E_F y \]  

(69)
Note that if all the \( \xi \) are small, then \( \theta_2 \) is confined to small angles, where the approximation is accurate. Performing the angle integral gives

\[
\Sigma^{\text{small}}(p, \omega) = -\frac{u^2}{4E_F} \int d\xi d\xi_n \delta(\xi_k) n_\omega(\xi_q) L^{\text{small}}
\]  \hspace{1cm} (70)

with

\[
L^{\text{small}} = 2 \text{sgn}(\omega + \xi_q - \xi_k - \xi_p) + \text{sgn}(-\omega + \xi_q + \xi_k - \xi_p)
\]  \hspace{1cm} (71)

The remainder of the evaluation is simple: to obtain the quasiparticle velocity renormalization we must take the sum of the \( \omega \) and \( \xi_p \) derivatives; thus the first term does not contribute at all while the second term gives \(-2\delta(\xi_q + \xi_k)\) for the \( \omega \) derivative and the same for the \( \xi_p \) derivative. Substituting this into the integrals and evaluating gives

\[
\frac{m^*}{m}|_{\text{small}} = \frac{u^2}{E_F} \int d\xi d\xi_n \delta(\xi_k) n_\omega(-\xi_k) = \frac{u^2 T}{E_F}
\]  \hspace{1cm} (72)

\[\bullet \theta_2 \approx 0: \text{These angles correspond to processes in which a momentum near } 2p_F \text{ is transferred to the electrons. In this regime we have } \text{sgn}(C_1) = \text{sgn}(C_2) = -1. \text{Further, as both } C_{1,2} \text{ and } B \text{ are of order } E_F, \text{ we write } C^2 - B^2 = (C + B) \ast (C - B) = -8E_F \ast (C + B). \text{ We have}
\]

\[
C_1 + B = \omega - \xi_k + \xi_q + \xi_p + \frac{1}{2}E_F \theta_2^2
\]  \hspace{1cm} (73)

\[
C_2 + B = -\omega + \xi_p + \xi_q + \xi_k + \frac{1}{2}E_F \theta_2^2
\]  \hspace{1cm} (74)

Again the \( \theta_2 \) integral may be done, yielding

\[
\Sigma^{2p_F}(p, \omega) = -\frac{u^2}{4E_F} \int d\xi d\xi_n \delta(\xi_k) n_\omega(\xi_q) L^{2p_F}
\]  \hspace{1cm} (75)

with

\[
L^{2p_F} = 2 \Theta(\omega - \xi_k + \xi_q + \xi_p) + \Theta(-\omega + \xi_p + \xi_q + \xi_k)
\]  \hspace{1cm} (76)

As above, one of the two terms give no contribution; the other has a factor of two but when differentiated gives only \( \delta \) not \( 2\delta \). The sign is opposite. Thus this term cancels the previous term, leading to no T-linear term in the effective mass enhancement.

An almost identical calculation of the order \( u^2 \) term in the free energy shows that there is no T-linear term in the specific heat coefficient.

As we see from these calculations, the coefficients of the linear T-contributions to the effective mass coming from small and nearly \( 2p_F \) processes are model-dependent. Any model modifications, e.g., momentum dependent interaction or non-circular Fermi surface, changes these coefficients, eliminating the cancellation and resulting in appearance of the linear T-term in the effective mass. The same conclusion applies to the specific heat coefficient. This agrees with the recent experimental data on liquid \(^3\)He films.[14]

**VI. DENSITY-DENSITY CORRELATION FUNCTION**

In this section we consider the density-density correlation function \( \Sigma(\mathbf{Q}) \), defined as follows:

\[
\delta^{(d+1)}(0) \Sigma(\mathbf{Q}) \equiv \langle \hat{\rho}(\mathbf{Q}) \hat{\rho}^\dagger(\mathbf{Q}) \rangle
\]  \hspace{1cm} (77)

where the density operator \( \hat{\rho}(\mathbf{Q}) \) is

\[
\rho(\mathbf{Q}) = 0 \int (1) \tilde{\psi}_\alpha(1) \psi_\alpha(1 + \mathbf{Q})
\]  \hspace{1cm} (78)

and \( \hat{\rho}(\mathbf{Q}) = \rho(\mathbf{Q}) - \langle \rho(\mathbf{Q}) \rangle \) stands for the density fluctuation. Using the definition for the 1PI four-point vertex ([12]), one obtains
\[ \kappa(Q) = -N \left[ \int_{(1)} G(1)G(1 - Q) + \int_{(1,2)} G(1)G(1 - Q)\Gamma_{ch}(1,2; -Q)G(2 + Q) \right] \]  

(79)

The correlation function has the properties

\[ \kappa(q = 0, \Omega \neq 0) = 0 \]  

(80a)

\[ \kappa(Q) = \kappa(-Q) \]  

(80b)

To derive (80a) it is enough to re-write the charge Ward identity as

\[ G(1 - Q)G(1) = \frac{G(1 - Q) - G(1)}{i\Omega - \frac{q(2k - q)}{2m}} - \int_{(2)} i\Omega - \frac{q(2k - q)}{2m} G(1)G(1 - Q)\Gamma_{ch}(1,2; -Q)G(2)G(2 + Q) \]  

(81)

Using the above expression for the first term in the brackets on the r.h.s. of Eq. (79), one can easily establish Eq. (80a). As to the second Eq. (80b), it can be proved with the use of Eqs. (19).

In order to find the compressibility of the system we need to calculate the density function in the non-trivial (static) zero transfer limit, which we denote \( \kappa_0 \). To do it in the second order over interaction by a naive perturbation theory, one needs to take into account the sunrise diagram for the self-energy in the Dyson equation (45) and to use such approximation for the total Green’s function entering the first term in the brackets on the r.h.s. of Eq. (79). For the second term in that equation one can use the “bare” Green’s functions [cf. comment after Eq. (45)] and the one-loop approximation for the vertex. In diagrammatic language this amounts to the calculation of a set of three-loop diagrams, which is very difficult in practice. The first problem is a proliferation of terms after doing each intermediate step of integration (summation) for a given three-loop diagram. The second is that some of these terms exhibit IR divergencies. It takes additional efforts to assure the final cancellation of those divergencies between different terms. As in the effective mass calculation, the use of the Ward identities drastically simplifies the problem. [ Cf. comment after Eq. (52)].

In applying the Ward identities for our calculations we will use the approaches developed in the classic derivations of FLT. We denote

\[ G^0_q(1) \equiv G(1)G(1 + q) \bigg|_{q \to 0} \]  

(82a)

\[ G^0_{\Omega}(1) \equiv G(1)G(1 + i\Omega) \bigg|_{i\Omega \to 0} \]  

(82b)

Let us also introduce two quantities \( \Delta_R \) and \( K \) as

\[ G^0_q(1) = G^0_{\Omega}(1) - \Delta_R(1) \]  

(83)

\[ \Gamma^0_{ch}(1,2) = \Gamma^0_{ch}(1,2) - K(1,2) \]  

(84)

Then from Eqs. (73, 54) we have

\[ -\frac{\kappa_0}{N} = \int_{(1)} G^2_q(1) + \int_{(1,2)} G^2_q(1)\Gamma^0_{ch}(1,2)G^2_q(2) - \kappa_2 \]  

(85)

where

\[ \kappa_2 = \int_{(1,2)} G^2_q(1)K(1,2)G^2_q(2) \]  

(86)

Eqs. (73, 80a) give

\[ \int_{(1)} G^2_q(1) + \int_{(1,2)} G^2_q(1)\Gamma^0_{ch}(1,2)G^2_q(2) = 0 \]  

(87)

Subtracting this from Eq. (83) and using (83), we obtain
\[- \frac{x^u}{N} = - \int_1 \Delta_R(1) \left[ 1 + 2 \int_2 \Gamma_{ch}(1, 2) G_{ch}^2(2) \right] + \int_1 \Delta_R(1) \Gamma_{ch}^1(1, 2) \Delta_R(2) - x_2 \] (88)

Note that from the charge Ward identity

\[ \frac{\partial \Sigma(1)}{\partial \Omega} = - \int_2 \Gamma_{ch}(1, 2) G_{ch}^2(2) \] (89)

So far we did not use any approximations, and Eq. (88) is exact. Now let us simplify it at the level \( O(U_0^2) \) for the model with interaction \( \langle \bar{d} d \rangle (N = 2) \). From the one-loop approximation for the vertex (51) we see that

\[ K(1, 2) = \frac{m}{2\pi} U_0^2 \] (90)

Also the Green’s functions in Eq. (88) can be approximated by their bare forms, resulting in

\[ x_2 = \frac{m}{2\pi} u^2 + O(u^3) \] (91)

After the standard analytical continuation to real frequencies, one can find for the parameter \( \Delta_R(1), 1 \equiv (k_1, \omega_1) \)

\[ \Delta_R(1) \approx \left( 1 + \frac{\partial \Sigma(1)}{\partial \omega} - \frac{m}{k_F} \frac{\partial \Sigma(1)}{\partial k_F} \right) \Delta(1) \] (92)

where

\[ \Delta(1) = \frac{1}{2} \xi_1 \delta(\omega_1 - \xi_{k_1}), \quad \xi_1 = \frac{\beta}{2} \cosh^2 \left( \frac{1}{2} \beta \xi_{k_1} \right) \] (93)

Eq. (92) is the extension to the finite-temperature of the well-known result \( \Delta_R(1, T = 0) = Z^2 m^*/k_F \delta(\omega_1 - k_1) \) (cf., e.g., Ref. [1]), combined with expansion of \( Z^2 m^* \) to order \( u^2 \). Up to the terms \( O(u^3) \) we have from Eqs. (88, 89, 92)

\[ - \frac{x^u}{2} = - \frac{m}{2\pi} u^2 + \int_1 \Delta(1) \left( \frac{\partial \Sigma(1)}{\partial \omega} + \frac{m}{k_F} \frac{\partial \Sigma(1)}{\partial k_F} \right) + \int_1 \Delta(1) \Gamma_{ch}^1(1, 2) \Delta(2) \] (94)

As we have shown before for the calculation of the effective mass, the first integral on the r.h.s. of Eq. (94) can be written via the Ward identities as the second term on the r.h.s. of Eq. (50) (With the only difference that in Eq. (94) \( (k_1, \omega_1) \) is arbitrary.) Since two limits of the vertex differ only by a constant term (90) which disappears after the integration over the angle \( k_1k_2 \), we can finally write

\[ x^u = \frac{m}{\pi} \left( 1 + u^2 + f_1 - f_0 \right) \] (95)

\[ f_1 = \frac{2\pi}{m} \int_1 \Delta(1) \Gamma_{ch}(1, 2) \Delta(2) \]

\[ f_0 = \frac{2\pi}{m} \int_1 \Delta(1) \Gamma_{ch}^1(1, 2) \Delta(2) \]

Eqs. (88, 91) for the response function are manifestly free of spurious IR divergences and readily recover the standard FLT results at zero temperature. We recall [cf. Eq. (23)] that the components of the scattering vertex and the Landau function are just the appropriate limits of the vertex \( \Gamma_{ch} \) times the factor \( \nu_R Z^2 \) (\( \nu_R = m^*/2\pi \)). The latter reduces simply to \( m/2\pi \) in our approximation. Then at \( T = 0 \) [cf. Sec. 3 for the explanations of \( T \) \( \rightarrow \) limit, and note also that from Eq. (23) \( P_0^2 = u^2 + O(u^3) \)] we read off from Eq. (95)

\[ x^u(T = 0) = \frac{m}{\pi} (1 + F_0^2 + F_1 - F_0) \] (96)

which is nothing but the expansion to \( O(u^3) \) of 2D FLT result (cf., e.g., Ref. [3])

\[ x^{FLT}(T = 0) = \frac{m}{\pi} \frac{1 + F_1}{1 + F_0^2} \] (97)
Using Eq. (51) for the vertex entering the first integral term in Eq. (55), we obtain:

\[
f_1 = \frac{2\pi}{m} \int_0^{T_0^2} \frac{k_1 k_2 C_1 C_2}{k_1^2} \left\{ \frac{\tanh(\frac{i}{2} \beta \xi_{k_1}) - \tanh(\frac{i}{2} \beta \xi_{k_1+k_2-k_1})}{-\xi_{k_1} + \xi_{k_2} + \xi_{k_3} - \xi_{k_3+k_2-k_1}} - \frac{\tanh(\frac{i}{2} \beta \xi_{k_1})}{-\xi_{k_1} - \xi_{k_2} + \xi_{k_3} + \xi_{k_3-k_2-k_1}} \right\}
\]

where we indicated explicitly two contributions to the response function coming from the ZS' and BCS one-loop corrections of the vertex. For the second integral on the r.h.s. of Eq. (55) we have

\[
f_0 = \frac{m}{2\pi} U_0 + f_0^{ZS'} + f_0^{BCS}
\]

where the first term is the bare vertex contribution to \( \chi \), while the two other one-loop vertex corrections are given by the same formulas as in Eq. (55) with an obvious replacement \( k_1 k_2 / k_1^2 \mapsto 1 \).

We were able to analytically calculate the above terms in the leading order of their temperature dependence. The results are

\[
\begin{align*}
f_0^{ZS'} &= u^2(2 - \frac{1}{2} T_0) \\
f_0^{BCS} &= u^2(-\ln 2\lambda + \frac{1}{2} T_0) \\
f_1^{ZS'} &= \frac{1}{2} u^2 T_0 \\
f_1^{BCS} &= \frac{1}{2} u^2(1 - T_0)
\end{align*}
\]

up to the order \( O(u^2 T_0^0) \). Note that the BCS contribution to the term \( f_0 \) is ultraviolet divergent and we regularized it in the same way as we did it for that contribution to the vertex, i.e., by introducing an UV cutoff \( \lambda \gg 1 \). [Cf. Sec. III] Applying Eq. (53) for the term \( f_1 \) while passing from Eq. (54) to Eq. (57) may raise some questions from a cautious reader, since the ultraviolet regularization we used to calculate the BCS contribution would break the Galilean invariance, which is indispensable for the derivation of (49). However \( f_1 \) is UV-convergent, thus can be calculated in the limit of an infinite cutoff when the Galilean invariance is preserved.

Combining the above results together, we obtain

\[
\chi = \frac{m}{\pi}(1 - u - \frac{1}{2} u^2 + u^2 \ln 2\lambda) + O(u^2 T_0^0)
\]

Once again, the leading linear \( T \)-corrections which can be traced back to the ZS’- and BCS-loop contributions to the vertex cancel in each of the terms \( f_0 \) and \( f_1 \) separately. It is useful to keep in mind that albeit we expressed \( f_1 \) and \( f_0 \) in terms of the vertex only, these contributions entangle both “purely vertex” corrections and self-energy corrections to the response function. The latter are just expressed in terms of the vertex via the Ward identities.

Let us finally remind that the compressibility \( K = \chi / n^2 \), where \( n \) is electron density.

**VII. SPIN-SPIN CORRELATION FUNCTION**

To find the spin susceptibility in the \( SU(N) \) formalism, we consider the spin (flavor, if \( N \neq 2 \)) correlation function \( \chi^{ab}(Q) \), defined as follows:

\[
\delta^{(d+1)}(0) \chi^{ab}(Q) \equiv \langle S^a(Q) S^b(Q) \rangle
\]

where the flavor density operators

\[
\begin{align*}
S^a(Q) &= \frac{\sigma}{2} \int_{(1)} \lambda_{\alpha \beta}^a \tilde{\psi}_\alpha(1) \psi_\beta(1 + Q) \\
S^{\dagger a}(Q) &= \frac{\sigma}{2} \int_{(1)} \lambda_{\alpha \beta}^a \tilde{\psi}_\alpha(1) \psi_\beta(1 - Q)
\end{align*}
\]

and \( \sigma \) is the gyromagnetic ratio. Using Eqs. (42-44) one can find
\[
\chi^{ab}(Q) = -\frac{g^2}{2}\delta^{ab}\left[\int_{(1)} G(1)G(1-Q) + \int_{(1,2)} G(1)G(1-Q)\Gamma_{sp}(1,2; Q)G(2)G(2+Q)\right]
\]  

(104)

Note that in a paramagnetic state the response is the same along all the \((N^2 - 1)\) directions \(a\). Then we follow the same steps as in the previous section, with the only difference that instead of the charge Ward identity we use the spin identity \([11]\). The latter can be casted in the form of Eq. \((81)\) where \(\Gamma_{ch} \mapsto \Gamma_{sp}\). Function \(\chi^{aa}(Q)\) has the same properties \([80]\) as those of the density function. The uniform susceptibility is given by the static limit \(\chi^q = \chi^{aa}(Q \to 0, \Omega = 0)\) (no sum over \(a\)). The equation we obtain for \(2\chi^q/\delta^2\) is the same as the r.h.s of Eq. \((88)\), but \(\Gamma_{ch} \mapsto \Gamma_{sp}\). Then we use the consequence of the both spin and charge Ward identities \([cf. \text{ Eq. (43)}]\):

\[
\int_{(2)} \Gamma_{sp}^\Omega(1,2)G_2^\Omega(2) = \int_{(2)} \Gamma_{sp}^\Omega(1,2)G_2^\Omega(2)
\]  

(105)

and afterwards proceed as in the previous section. From the one-loop approximation \([25]\)

\[
\Gamma_{sp}(1,2; Q) = -U_0 + L(Q)U_0^2 + \mathcal{C}(1 + 2)U_0^2
\]  

(106)

Thus up to the terms \(O(u^3)\) we obtain

\[
\chi^q = \frac{m_0^2}{4\pi}\left\{1 + u^2 + f_1 - g_0\right\}
\]  

(107)

\[
g_0 = \frac{2\pi}{m} \int_{(1,2)} \Delta(1)\Gamma_{sp}^\Omega(1,2)\Delta(2)
\]  

(108)

Recalling once again what we said after Eq. \((54)\) and noting that from Eq. \((35)\), \(G_0^2 = u^2 + O(u^3)\), we easily see that in the zero-temperature limit the above equation gives

\[
\chi^q(T = 0) = \frac{m_0^2}{4\pi}(1 + G_0^2 + F_1 - G_0)
\]  

(109)

reproducing thus the second-order expansion of the 2D FLT result \([14]\)

\[
\chi^{FLT}(T = 0) = \frac{m_0^2}{4\pi}\frac{1 + F_1}{1 + G_0}
\]  

(110)

Luckily we do not have to do new calculations since

\[
g_0 = -u - f_0^{BCS}
\]  

(111)

So we find the spin susceptibility to have a linear leading temperature correction

\[
\chi^q = \frac{m_0^2}{4\pi}\left\{1 + u + \frac{3}{2}u^2 - u^2\ln2\lambda + \frac{T_0}{2}u^2\right\} + O(u^2T_0^2)
\]  

(112)

**VIII. SUMMARY AND DISCUSSION**

In this paper we have systematically examined the leading temperature corrections to FLT in two spatial dimensions. We find for the model of a 2D electron gas with a contact interaction that to order \(u^2\) the leading \(T\)-dependence of the FL parameters in the spin sector is linear in temperature, while for the parameters in the charge sector and for the effective mass a cancellation of the leading \(T\)-corrections occurs, and their expansions start from the terms quadratic in temperature.

The particularly interesting result we found is the leading linear temperature dependence of the spin susceptibility

\[
\frac{\chi^q(T)}{\chi^q(0)} \approx 1 + u^2\frac{T}{E_F}
\]  

(113)

According to the perturbative calculations of Belitz \textit{et al} \([2]\), the 2D FL susceptibility has a leading linear correction in \(|\mathbf{q}|\) at \(T = 0\) with a positive coefficient which is of the second order in interaction, i.e., their result has a structure of Eq. \((113)\). This also agrees with the phenomenology of Misawa \([1]\) and the numerical results \([4]\).
Our results reveal the crucial importance of $2k_F$ processes contributions into the low-energy parameters. We remind that recovering of the angular dependence of the FL vertices in the whole region where the angle varies, involves taking into account transfers varying from zero to $2k_F$. We have seen it from the direct bubbles evaluation, but the argument is non-perturbative. To appreciate this let us write the antisymmetry condition for the vertex as

$$\Gamma^{(\lambda)}(k_1, k_2; q) = \pm \Gamma^{(\lambda)}(k_1, k_2; k_2 - k_1 - q)$$

So, small- and large-transfer scales are intrinsically related due to the Pauli principle, and any calculation of the low-energy corrections should take this into account. Eventually the integration over whole range of the transfer values through a given loop results in the temperature corrections not only to the vertex, but to the response functions as well, when that loop is a part of a more complex diagram. The big transfer ($\sim 2k_F$) contributions essentially gave rise to the spin susceptibility result (113) in the three-loop approximation.

We find that the relationships known from the classical FLT derivations at $T = 0$ for the parameters of Galilean-invariant FL (e.g., the effective mass, response functions vz components of the Landau function) are violated by finite-temperature terms. The coefficients in the temperature corrections to these relationships subtly involve contributions from small and large ($\sim 2k_F$) momentum processes.

Concerning the argument for the cancellation of anomalous terms in the response functions due to Ward identities: We have calculated the vertices at the one-loop level $O(u^2)$. Through the Ward identities the self-energy corrections were taken into account with the same accuracy. There are no more terms of the order $O(u^2)$ to cancel the temperature dependence (113). Thus, the linear temperature dependence of susceptibility (or weaker temperature dependence of the compressibility) does not contradict the exact Ward identities known to us, moreover in our results for the response functions both vertex and self-energy corrections are included on the same footing by using the Ward identities.

For more realistic models of electrons in (quasi)-2D crystals, i.e., for various tight-binding spectra and fillings, the free-gas-like square-root $2k_F$ singularities (with $k_F$ depending on a chosen direction in $q$-space) are known to exist in the Lindhard functions. We think this is enough to result in linear $T$-terms in physical quantities analogous to what we found in this study. We argue that the cancellation of the $T$-terms in some FL parameters is special to second order perturbation theory and the model considered, while the leading linear temperature corrections are a generic feature of the 2D FL.

We hope our results may be experimentally tested in real 2D FL systems. For example, a very naive fit of the temperature dependence of the spin susceptibility in Sr$_2$RuO$_4$ system when it is in the 2D metallic regime (above 3D crossover temperature) shows that the data are compatible with the form (113). We expect our results stimulate a more detailed examination of the leading temperature dependences of response functions in 2D systems.

ACKNOWLEDGMENTS

Stimulating conversations with S. Lukyanov are gratefully acknowledged. We thank V. Oudovenko for his help with the numerical tests of our results. This work is supported by NSF DMR0081075.

APPENDIX A: ZERO SOUND (ZS) AND PEIERLS (ZS') GRAPHS

1. Finite-temperature Lindhard function

We define the Lindhard function as

$$L(Q) = \frac{1}{\beta} \sum_{\omega_m} \int \frac{dK}{(2\pi)^2} G_0(K, \omega_m) G_0(K + Q, \omega_m + \Omega_n)$$

(A1)

It corresponds to the contribution of the particle-hole loop in graph ZS or ZS’, depending on the actual value of $Q$ we put. For the purpose of notational economy in the Appendices we will reserve capital letters for momentum vectors, while small letters correspond to their dimensionless counterparts defined below. The free electron Green’s function entering the loop is given by

$$G_0^{-1}(K, \omega_n) = i\omega_n + \mu - \epsilon(K)$$

(A2)

with its spectrum.
\[ \epsilon(K) = \frac{K^2}{2m}, \quad \mu = \frac{K^2}{2m}, \quad \xi_K = \epsilon(K) - \mu \]  

(A3)

A straightforward summation over Matsubara frequencies results in

\[ L(Q) = \int \frac{dK}{(2\pi)^2} \frac{n_\circ(\xi_K) - n_\circ(\xi_{K+Q})}{i\Omega_n + \xi_K - \xi_{K+Q}} \]  

(A4)

We denote by \( n_\circ \) the Fermi-Dirac distribution function

\[ n_\circ(x) \equiv \frac{1}{e^{\beta x} + 1} = \frac{1}{2} - \frac{1}{2} \tanh \left( \frac{\beta x}{2} \right) \]  

(A5)

We choose 2D polar axis along the transfer vector such that \( KQ = KQ \cos \vartheta \). Then the Lindhard function can be transformed to the expression

\[ L(Q, 0) = -\frac{m}{\pi^2} \int_0^\infty KdK n_\circ(\xi_K) \int_0^{2\pi} d\vartheta \frac{1}{Q^2 - 4K^2 \cos^2 \vartheta} \]  

(A6)

The angular integration above is understood in the sense of the principal value. We define the dimensionless Lindhard function \( \chi \) at zero frequency transfer as

\[ L(Q, 0) \equiv -\nu_0 \chi(Q), \quad \nu_0 = \frac{m}{2\pi} \]  

(A7)

where \( \nu_0 \) is the free 2D electron density of states per spin. Let us also introduce the new (dimensionless) variables

\[ \beta_0 = \frac{\beta K_F^2}{4m}, \quad q = \frac{Q}{2K_F}, \quad k = \frac{K}{K_F} \]  

(A8)

In order to proceed we use the following result for the principal value of the integral \( I_a \) at \( a > 0 \):

\[ I_a = \int_0^{2\pi} \frac{d\vartheta}{a^2 - \cos^2 \vartheta} = \frac{2\pi}{a\sqrt{a^2 - 1}} \Theta(a^2 > 1) \]  

(A9)

where \( \Theta \) is the event (Heaviside) function. After the angular integration we obtain

\[ \chi(q) = \frac{1}{2q} \int_0^q \frac{kdk}{\sqrt{q^2 - k^2}} \left\{ 1 - \tanh \beta_0 (k^2 - 1) \right\} \]  

(A10)

The above formula can be easily casted into the final form

\[ \chi(q) = \frac{1}{2} + \frac{1}{2} \tanh \beta_0 - \frac{\beta_0}{2} \int_{-1}^{q^2 - 1} \frac{dz}{\cosh^2 \beta_0 z} \sqrt{1 - \frac{1 + z}{q^2}} \]  

(A11)

To make zero-temperature check of our result the following formula is useful

\[ \beta_0 \to \infty : \quad \frac{\beta_0}{2 \cosh^2 \beta_0 z} \to \delta(z) \]  

(A12)

Then from Eq. (A11) we easily recover the result known for \( T = 0 \), which was first found by Stern:

\[ q^2 < 1 : \quad \chi(q) = 1 \]  

(A13a)

\[ q^2 > 1 : \quad \chi(q) = 1 - \sqrt{1 - \frac{1}{q^2}} \]  

(A13b)

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2. Fourier components

Let us consider now the Lindhard function at the “effective” transfer

\[ Q' = K_F^2 - K_F^1 - Q \]  

(A14)

which occurs in the ZS’ loop when the “real” transfer in the vertex is \( Q \). Parameterizing vectors \( K_F^2 \) and \( K_F^1 \) by their angles \((\theta_1, \theta_2)\) relatively to the direction of vector \( Q \), we get

\[ Q'^2 = Q^2 + 4K_F^2 \sin^2 \frac{\theta_2 - \theta_1}{2} + 4QK_F \sin \frac{\theta_2 - \theta_1}{2} \sin \frac{\theta_2 + \theta_1}{2} \]  

(A15)

In terms of new angles

\[ \theta = \frac{\theta_2 - \theta_1}{2}, \quad \psi = \frac{\theta_2 + \theta_1}{2} \]  

(A16)

and of the dimensionless variables (A8) one reads

\[ q'^2 = q^2 + \sin^2 \theta + 2q \sin \theta \sin \psi \]  

(A17)

Let us consider now \( \chi(q') \) at zero vertex transfer \( q = 0 \), i.e., \( q'^2 = q_{12}^2 = \sin^2 \theta \). For the Fourier transform

\[ \chi_n = \frac{1}{\pi} \int_0^\pi d\theta \cos(2n\theta)\chi(\theta) \]  

(A18)

of \( \chi(\theta) \) given by Eq. (A11), we have

\[ \chi_n = \left( \frac{1}{2} + \frac{1}{2} \tanh \beta_0 \right) \delta_n - \delta \chi_n \]  

(A19)

where

\[ \delta \chi_n = \frac{\beta_0}{2\pi} \int_0^\pi d\theta \cos(2n\theta) \int_{-1}^{\sin^2 \theta - 1} \frac{dz}{\cosh^2 \beta_0 z} \sqrt{1 + \frac{1}{\sin^2 \theta}} \]  

(A20)

Then

\[ \delta \chi_0 = \frac{\beta_0}{\pi} \int_0^1 \frac{dx}{x \sqrt{1 - x^2}} \int_{-1}^{x^2 - 1} \frac{dz}{\cosh^2 \beta_0 z} \sqrt{x^2 - (1 + z)} = \frac{\beta_0}{\pi} \int_{-1}^0 \frac{dz}{\cosh^2 \beta_0 z} \int_{1 + z}^{1} \frac{dx}{x \sqrt{1 - x^2}} \sqrt{x^2 - (1 + z)} \]  

(A21)

Using the result

\[ \int_a^1 \frac{dx}{x \sqrt{1 - x^2}} \sqrt{x^2 - a^2} = \frac{\pi}{2}(1 - a) \]  

(A22)

we get

\[ \delta \chi_0 = \frac{1}{2} \tanh \beta_0 - \frac{\beta_0}{2} \int_{-1}^0 \frac{dz}{\cosh^2 \beta_0 z} \sqrt{1 + z} \]  

(A23)

Thus

\[ \chi_0 = \frac{1}{2} + \frac{\beta_0}{2} \int_0^1 \frac{dz}{\cosh^2 \beta_0 z} \sqrt{1 - z} = \frac{1}{2} + \frac{1}{2} \int_0^{\beta_0} \frac{dz}{\cosh^2 \beta_0 z} \sqrt{1 - T_0 z} \]  

(A24)

The dimensionless temperature \( T_0 \) is

\[ T_0 = \frac{1}{\beta_0} \equiv \frac{4mT}{K_F^2} \ll 1 \]  

(A25)
Taylor-expanding the square root under the integral, we can then extend the upper limit to the infinity with exponential accuracy. Thus,

\[
\chi_0 \approx \frac{1}{2} + \frac{1}{2} \int_0^\infty \frac{dz}{\cosh^2 z} \left( 1 - \frac{1}{2} T_0 z - \frac{1}{8} T_0^2 z^2 - \frac{1}{16} T_0^3 z^3 + \mathcal{O}(T_0^4) \right)
\] (A26)

After simple integration we arrive to the sought result:

\[
\chi_0 = 1 - \frac{\ln 2}{4} T_0 - \frac{\pi^2}{192} T_0^2 - \frac{9\zeta(3)}{256} T_0^3 + \mathcal{O}(T_0^4)
\] (A27)

For the next Fourier component using \(\cos 2\theta = 1 - 2\sin^2 \theta\) we obtain

\[
\chi_1 = -\delta \chi_0 + \frac{2\beta_0}{\pi} \int_{-1}^0 \frac{dz}{\cosh^2 \beta_0 z} \int_{\sqrt{1-z^2}}^{\sqrt{1+z^2}} \sqrt{1-x^2} - (1+z) = -\delta \chi_0 + \frac{\beta_0}{2} \int_0^1 \frac{zd\zeta}{\cosh^2 \beta_0 z}
\] (A28)

Thus,

\[
\chi_1 = -\delta \chi_0 + \ln 2 \frac{1}{4} T_0 + \mathcal{O}(e^{-1/T_0})
\] (A29)

and combining the last formula with the previous results, we have

\[
\chi_1 = \ln 2 \frac{1}{4} T_0 - \frac{\pi^2}{192} T_0^2 - \frac{9\zeta(3)}{256} T_0^3 + \mathcal{O}(T_0^4)
\] (A30)

APPENDIX B: BCS GRAPH

1. BCS loop contribution at finite temperature

We define the contribution of the particle-particle loop in the BCS graph as

\[
\mathbb{C}(Q_s) \equiv \mathbb{C}(Q_s, \Omega_s) = \frac{1}{\beta} \sum_{\omega_m} \int \frac{dK}{(2\pi)^2} G_0(K, \omega_m) G_0(-K + Q_s, -\omega_m + \Omega_s)
\] (B1)

where

\[
Q_s \equiv K_1 + K_2, \quad \Omega_s \equiv \omega_1 + \omega_2
\] (B2)

After summation of the Matzubara frequencies we get

\[
\mathbb{C}(Q_s) = -\int \frac{dK}{(2\pi)^2} \frac{n_o(\xi_K) - n_o(-\xi_K + Q_s)}{-i\Omega_s + \xi_K + \xi_K - Q_s} = \frac{1}{2} \int \frac{dK}{(2\pi)^2} \frac{\tanh \left( \frac{1}{2} \beta \xi_K \right) + \tanh \left( \frac{1}{2} \beta \xi_K + Q_s \right)}{-i\Omega_s + \xi_K + \xi_K + Q_s}
\] (B3)

For the case we are interested in (i.e., zero sum frequency \(\Omega_s\) and vectors \(K_1, K_2\) lying on the Fermi surface, cf. also notations [A14, A16]) we have

\[
Q_s = |K_F^{(2)} + K_F^{(1)}| = 2K_F \cos \theta, \quad \Omega_s = 0
\] (B4)

To regularize the momentum integral in (B3) we will introduce an ultraviolet cutoff \(\Lambda\) which we assume to be the largest momentum scale of the problem. Under this condition we shift the momentum integration in the second term of the sum on the r.h.s. of Eq. (B3), obtaining the expression

\[
\mathbb{C}(Q_s) = \frac{1}{2} \int \frac{dK}{(2\pi)^2} \left( \frac{\tanh \left( \frac{1}{2} \beta \xi_K \right)}{-i\Omega_s + \xi_K + \xi_K + Q_s} + \frac{\tanh \left( \frac{1}{2} \beta \xi_K \right)}{i\Omega_s + \xi_K + \xi_K + Q_s} \right)
\] (B5)

Two-dimensional integration in (B5) reads as:
\[ \int d\mathbf{K} = \int_0^\Lambda K \, dK \int_0^{2\pi} d\theta \]  
(B6)

Using the previously defined dimensionless parameters \([\text{cf. notations (A8)}]\) along with new variables
\[ q_s \equiv \frac{Q_s}{2K_F}, \quad \lambda \equiv \frac{\Lambda}{K_F} \gg 1 \]  
(B7)

and the dimensionless function \( C \) \([\text{cf. notations (A7)}]\)
\[ C(Q_s, 0) \equiv \nu_0 C(q_s) \]  
(B8)

one gets
\[ C(q_s) = \frac{1}{4\pi} \int_0^\Lambda 2k dk \tanh \beta_0 (k^2 - 1) \int_0^{2\pi} d\theta \frac{k^2 - 1 + 2q_s^2}{(k^2 - 1 + 2q_s^2)^2 - 4k^2q_s^2 \cos^2 \theta} \]  
(B9)

or
\[ C(q_s) = \frac{1}{4\pi} \int_{-1}^{\lambda^2 - 1} dz \tanh \beta_0 z \frac{z + 2q_s^2}{4(z + 1)q_s^2} I_a \]  
(B10)

We denote \( I_a \) according to Eq.(A9) with
\[ a^2 \equiv \frac{(z + 2q_s^2)^2}{4(z + 1)q_s^2} \geq 0 \]  
(B11)

Note that
\[ a^2 > 1 \iff z^2 > 4q_s^2(1 - q_s^2) \]  
(B12)

After some simple manipulations we get
\[ C(q_s) = \frac{1}{2} \int_{-1}^{\lambda^2 - 1} dz \frac{\tanh \beta_0 z}{\sqrt{z^2 - 4q_s^2(1 - q_s^2)}} \text{sign}(z + 2q_s^2) \Theta(z^2 > 4q_s^2(1 - q_s^2)) \]  
(B13)

The square root is chosen to be positive on the real axis of integration \( z \). Recalling that \( q_s^2 = \cos^2 \theta \) and, consequently,
\[ 4q_s^2(1 - q_s^2) = \sin^2 2\theta , \]  
(B14)

we see that the sign- and event functions play their role on the patch \( z \in [-1, 1] \) only. Due to the symmetry properties of function \( C(q_s = \cos \theta) \) it suffices to consider the angle range \( 0 \leq \theta \leq \pi/2 \). After some analysis one can obtain
\[ C(q_s) = \frac{1}{2} \int_{-1}^{\lambda^2 - 1} dz \frac{\tanh \beta_0 z}{\sqrt{z^2 - \sin^2 2\theta}} + \Theta\left(\frac{\pi}{4} < \theta \leq \frac{\pi}{2}\right) \int_{\sin 2\theta}^{\lambda^2 - 1} dz \frac{\tanh \beta_0 z}{\sqrt{z^2 - \sin^2 2\theta}} \]  
(B15)

The function \( C(q_s) \) is continuous. It can be easily calculated at zero temperature, giving the known result (See. e.g., Ref. [39]):
\[ C(q_s) = \frac{1}{2} \ln \frac{\lambda^2}{q_s^2}, \quad 0 < q_s < 1 , \]  
(B16)

as well as for zero total incoming momentum \( q_s = 0 \) at low \( (T_0 \ll 1) \) temperature, giving with exponential precision
\[ C(0) = \ln \left( \frac{4e^\gamma}{\pi} \frac{\lambda}{T_0} \right) \]  
(B17)

where \( \gamma = 0.577.. \) is Euler’s constant. For the latter calculation cf., e.g., section 33.3 of Ref. [8]. In the above formulas and in what follows we track \( \lambda \) up to terms \( \mathcal{O}(1) \).
2. Fourier components

From Eq. (B15) it is easy to calculate Fourier components of function $C(q_s)$ given by

$$C_n = \frac{2}{\pi} \int_0^{\pi} d\theta \cos 2n\theta \ C(\theta) \quad (B18)$$

For the zeroth component after some manipulations we have

$$C_0 = \frac{1}{\pi} \int_0^1 dz \ \tanh \beta_0 z \ \int_0^{\arcsin z} \frac{d\psi}{\sqrt{z^2 - \sin^2 \psi}} + \frac{1}{\pi} \int_0^{\frac{\pi}{2}} d\psi \int_0^{\lambda^2 - 1} dz \ \frac{\tanh \beta_0 z}{\sqrt{z^2 - \sin^2 \psi}} \equiv C_0^{(1)} + C_0^{(2)} \quad (B19)$$

The first term can be written as

$$C_0^{(1)} = \frac{1}{\pi} \int_0^1 dz \ \tanh \beta_0 z \ \cdot K(z) \quad (B20)$$

where $K(z)$ is the complete elliptic integral of the first kind, which has the following Taylor expansion:

$$K(z) = \frac{\pi}{2} \left( 1 + \frac{1}{4} z^2 + \frac{9}{64} z^4 + O(z^6) \right) \quad (B21)$$

To proceed further we notice that

$$\int K(z) = \frac{\pi}{2} z \cdot 3F_2 \left[ \left\{ \frac{1}{2}, \frac{1}{2} \right\}, \left\{ \frac{1}{2}, \frac{3}{2} \right\}; z^2 \right] \equiv \mathcal{F}(z) \quad (B22)$$

where the Taylor expansion of the hypergeometric function can be readily read from the term by term integration of the expansion (B21), i.e.,

$$\mathcal{F}(z) = \frac{\pi}{2} \left( z + \frac{1}{12} z^3 + \frac{9}{320} z^5 + O(z^7) \right) \quad (B23)$$

Integrating by parts the r.h.s. of Eq. (B20) we obtain

$$C_0^{(1)} = \frac{1}{\pi} \left( \mathcal{F}(1) \cdot \tanh \beta_0 z - \int_0^1 \frac{\beta_0 dz}{\cosh^2 \beta_0 z} \mathcal{F}(z) \right) \quad (B24)$$

Using the same (exponentially accurate at low temperature) approximations which lead us to Eq. (A26), we end up

$$C_0^{(1)} \approx \frac{1}{\pi} \mathcal{F}(1) - \frac{1}{2} \int_0^\infty \frac{dz}{\cosh^2 z} \left( T_0 z + \frac{T_0^3}{12} z^3 + \frac{9 T_0^5}{320} z^5 + O(T_0^7) \right) \quad (B25)$$

In order to calculate $\mathcal{F}(1)$ it is convenient to come back to the integral representation of the elliptic integral, and then to evaluate it via changing the order of integration.

$$\mathcal{F}(1) = \int_0^1 dz \int_0^z \frac{dx}{\sqrt{1 - x^2}} \sqrt{z^2 - x^2} = \int_0^1 \frac{dx}{\sqrt{1 - x^2}} \int_x^1 \frac{dz}{\sqrt{z^2 - x^2}} = 2 \int_0^{\frac{\pi}{2}} d\theta \ln \cot \theta = 2 \mathcal{G} \quad (B26)$$

Here $\mathcal{G} = 0.91569...$ is Catalan’s constant. Finally

$$C_0^{(1)} = \frac{2}{\pi} \mathcal{G} - \ln \frac{2}{T_0} - \frac{3\zeta(3)}{64} T_0^3 - O(T_0^5) \quad (B27)$$

For the second contribution $C_0^{(2)}$ one integration by parts gives

$$C_0^{(2)} = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} d\psi \left( \ln \frac{2 \lambda^2}{1 + \cos \psi} - \int_0^{\lambda^2 - 1} \frac{\beta_0 dz \ln(z + \sqrt{z^2 - \sin^2 \psi})}{\cosh^2 \beta_0 z} \right) \quad (B28)$$

It is easy to see that the second term on the r.h.s of Eq. (B28) is exponentially small, while the first one can be readily evaluated resulting in
\[ C_0^{(2)} = \ln 2\lambda - \frac{2}{\pi} G \]  

Combining our results together, we have

\[ C_0 = \ln 2\lambda - \frac{\ln 2}{2} T_0 - \frac{3\zeta(3)}{64} T_0^3 - O(T_0^5) \]  

Calculation of the first Fourier component \( C_1 \) is very simple, since the first term on the r.h.s. of Eq. (B31) gives zero after the angular integration, while the second one can be evaluated exactly, resulting in

\[ C_1 = -\frac{1}{2\beta_0} \ln \cosh \beta_0 = -\frac{1}{2} + \ln \frac{2}{2} T_0 + O(e^{-1/T_0}) \]  

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31 It should pointed out that we are not saying that in the case when the running vertices are not constants, but functions of the momenta, cancellations between different loop contributions could not take place. For example, as one can easily check from Fig.1, when $k_1 = k_2$ momentum-antisymmetric parts of the direct (ZS) and the exchange (ZS') graphs cancel each other for any running coupling function which has the antisymmetric properties (8,9), thus preserving at the one-loop
level the property $\Gamma^\dagger(\mathbf{k}_i, \mathbf{k}_i; \mathbf{q}) = 0$ following from the Pauli principle. (For more on this issue see Ref. [10].) What we are saying is that even at special configurations of the vertex momenta, (i.e., $\mathbf{k}_1 = -\mathbf{k}_2$, $\mathbf{q} \to 0$) whence the loops temperature dependence comes mostly, we cannot see analogous “algebraic” cancellations between ZS’ abd BCS contributions for generic coupling function satisfying (7,8).

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FIG. 1. Diagrammatic equation for the four-point vertex at one-loop level. The one-loop graphs are called ZS, ZS’, and BCS in the order they appear on the r.h.s. of this equation.