

I. INTRODUCTION

The discrepancy in the value of the Hubble parameter as obtained from Cosmic Microwave Background (CMB) and Baryon Acoustic Oscillation (BAO) data (67.4 ± 0.5 km sec\(^{-1}\) Mpc\(^{-1}\) ) \([1, 2]\) and local distance ladder measurements (\(H_0 = 74.03 ± 1.42\) km s\(^{-1}\) Mpc\(^{-1}\) \([3]\)) has reached a level close to 6\(\sigma\) \([4–8]\) and is becoming a problem of the standard ΛCDM model. A similar issue, with lower significance level, appears when measuring the growth rate of cosmological perturbations using peculiar velocities (Redshift Space Distortions) \([9–14]\) and Weak Lensing \([15–19]\) cosmological data. Such measurements find a weaker growth rate of perturbations than anticipated in the context of the standard ΛCDM model \([9, 13, 14, 17, 18, 20]\). This weaker growth is expressed in the context of ΛCDM parameters as a lower best fit value of the matter density parameter \(Ω_{\text{m}} \approx 0.28 ± 0.03\) \([18, 21]\) than the one anticipated in the context of geometric probes including the CMB spectrum peak locations \([1, 2]\) and the BAO data \([22–24]\) in the context of flat ΛCDM model (\(Ω_{\text{m}} = 0.315 ± 0.007\)).

A wide range of models have been used to explain these tensions and properly extend ΛCDM using specific new degrees of freedom (for a quantitative measure of tensions see Refs. \([25–27]\)). For the Hubble tension these models include mechanisms that modify the scale of the sound horizon at last scattering using early dark energy \([28–32]\) or other types of early species \([33–35]\), interacting dark energy with matter \([36–41]\), screened fifth forces on the cosmic distance ladder \([42, 43]\), modified gravity \([44–47]\) and new properties of late dark energy including new types of dark energy equation of state parameter \([48–50]\). For the growth tension modified gravity \([13, 14, 51–54]\), running vacuum models \([55, 56]\) and modification of dark energy properties \([56–64]\) have also been considered.

In both types of tension it has become clear that new properties of dark energy may constitute the required missing degree of freedom. In particular, it has been shown that a mildly phantom dark energy with equation of state parameter evolving slightly below \(w = -1\) has the potential to resolve the Hubble tension by amplifying late time acceleration which leads to an increased best value of the Hubble parameter \(H_0\) in the context of the CMB data, thus bringing it close to the value obtained by local distance ladder measurements \([48, 50, 65–69]\). Most previous analyses along the above lines utilize evolving equation of state parameters that in many cases have sophisticated functional forms. Even though such functional forms of \(w(z)\), usually involve at most one new parameter, these approaches have two drawbacks: complexity of the \(w(z)\) considered forms and worse fit than ΛCDM to the Planck CMB TT power spectrum and other cosmological data (\(\Delta \chi^2 > 0\)). Thus, these models are usually not favoured \([70]\) compared to ΛCDM in the context of information criteria that penalize models with additional parameters if they do not improve the quality of fit to data. It would therefore be desirable to construct models/parametrizations with no new parameters that can
potentially resolve both the Hubble and growth tensions by modifying the dark energy properties.

In particular, the following questions need to be addressed:

- What are the properties of the new phantom degree of freedom required in order to increase the best fit value of \( H_0 \) in the context of CMB data to the level required for consistency with local measurements and resolution of the \( H_0 \) tension?
- What are the corresponding best fit values of cosmological parameters that emerge in the context this type of phantom dark energy and to what extend do they lead to improvement of the resolution of the growth tension?
- What is the quality of fit of these extended models to the CMB Planck and other cosmological data and how does it compare with the corresponding quality of fit of ΛCDM?

The goal of the present analysis is to address these questions using an approximate analytical method utilizing the degeneracies of cosmological parameters with respect to the form of the CMB power spectrum. In addition, we utilize more accurate numerical estimates of best fit cosmological parameters using Boltzmann and Markov Chain Monte Carlo (MCMC) codes. In the context of the analytical approximation we exploit the degeneracies of the CMB power spectrum among different cosmological parameter combinations and explore the consequences of variations of the dark energy equation of state parameter \( w(z) \) on other cosmological parameters and in particular on the Hubble parameter \( H_0 \) and the matter density parameter \( \Omega_{0m} \).

The structure of this paper is the following: In section II we review the well known degeneracies of the CMB TT power spectrum and identify the five cosmological parameter combinations that to a great extend uniquely determine the form of the spectrum. By demanding that these five combinations remain fixed to their Planck/ΛCDM values, we identify the expected change of the best fit cosmological parameters using Boltzmann and MCMC codes with Planck CMB data. We also compare the numerically obtained best fit cosmological parameter values with the corresponding values obtained in the context of the analytical approximation of section II for the same form of \( w(z) \). Finally, in section IV we summarize and discuss possible extensions of this analysis.

### II. CMB SPECTRUM DEGENERACIES AND THE \( H_0(w) \) DEPENDENCE

It is well known \([71, 72]\) that the form of the CMB temperature power spectrum is almost uniquely determined, if the following parameter combinations are fixed

- The matter density parameter combination \( \omega_m \equiv \Omega_{0m} h^2 \) where \( H_0 = 100 \ h \ km \ sec^{-1} \ Mpc^{-1} \).
- The baryon density parameter combination \( \omega_b \equiv \Omega_{0b} h^2 \) where \( \Omega_{0b} \) is the present day baryon density parameter.
- The radiation density parameter combination \( \omega_r \equiv \Omega_{0r} h^2 \) where \( \Omega_{0r} \) is the present day radiation density parameter.
- The primordial fluctuation spectrum.
- The curvature parameter \( \omega_k \equiv \Omega_{0k} h^2 \).
- The flat universe co-moving angular diameter distance to the recombination surface

\[
d_A(\omega_m, \omega_r, \omega_b, h, w(z)) = \left( \frac{dz}{H(z)} \right) \int_0^{z_r} \left( \frac{dz}{H(z)} \right) (2.1)
\]

where \( z_r \approx 1100 \) is the redshift of recombination provided to better accuracy as \([73]\)

\[
z_r = 1048(1 + 0.00124\omega_b^{-0.738})(1 + g_1\omega_m^2) \quad (2.2)
\]

\[
g_1 = 0.0783\omega_b^{-0.238} / (1 + 39.5\omega_b^{0.763})
\]

\[
g_2 = 0.560 / (1 + 21.1\omega_b^{1.81}).
\]

and \( H(z) \) is the Hubble parameter at redshift \( z \). The Hubble parameter takes the form

\[
H(z, \omega_m, \omega_r, \omega_b, h, w(z)) = H_0 \sqrt{\Omega_{0m}(1 + z)^3 + \Omega_{0r}(1 + z)^4 + \Omega_{0de}e^{3\int_0^z dz'(1+w(z'))/(1+z')}} \quad (2.3)
\]

where \( w(z) \) is the dark energy equation of state parameter at redshift \( z \) and \( \Omega_{0de} = 1 - \Omega_{0m} - \Omega_{0r} \) is the present day value of the dark energy density parameter. The product \( \sqrt{\omega_m} \cdot d_A \) is independent of \( H_0 \) and constitutes the well known shift parameter

\[
\frac{\sqrt{\omega_m} \cdot d_A}{H_0} \quad (2.4)
\]
FIG. 1: The predicted value of $h$ as a function of the fixed $w$ for the one parameter dark energy ($w$CDM) model. The orange line corresponds to the theoretically predicted best fit values of $h$ for different values of $w$ in the case of the $w$CDM model, whereas the dashed blue line corresponds to the linear fitting that has been made. The red points display the actual best fit values, including the errorbars, of $h$ for specific values of $w$ obtained by fitting these models to the CMB TT anisotropy via the MGCosmoMC (see Table II).

These parameter combinations also express the approximate degeneracy of the CMB with respect to various specific cosmological parameters. For example if the first four parameter combinations are fixed [eqs. (2.5) - (2.8)], the fifth constraint [eq. (2.9)] provides the analytically predicted best fit value of the Hubble parameter $H_0$ (or $h$) given the dark energy equation of state parameter $w(w_0, w_1, ..., z)$ where $w_0, w_1, ...$ are the parameters entering the $w(z)$ parametrization\(^1\). Thus, it is straightforward to use eqs. (2.1), (2.3), (2.5) and (2.9) to construct the function $h(w_0, w_1, ...)$ that gives semi-analytically the predicted best fit value of $h$ given a specific form of $w(z)$. This function is derived by solving the following equation with respect to $h$

$$d_A(\omega_m, \omega_r, h, w_0, w_1, ...) = d_A(\omega_m, \omega_r, h, w(z))$$ (2.10)

In the context of a simple one parameter parametrization where $w(z)$ remains constant in time and redshift, ($w$CDM model), eq. (2.3) takes the simple form

\[ H(z, \omega_m, \omega_r, \omega_b, h, w(z)) = H_0 \sqrt{\Omega_{0m}(1+z)^3 + \Omega_{0r}(1+z)^4 + (1 - \Omega_{0m} - \Omega_{0k})(1+z)^3(1+w)} \] (2.11)

and using the above described approach, solving eq. (2.10) it is straightforward to derive the degeneracy function $h(w)$ shown in Fig. 1 (continuous orange line). In the range $w \in [-1.5, -1]$, $h(w)$ is approximated as a

\(^1\) In the present analysis we assume a flat universe and fix $\omega_k = 0$. 

\[ R = \sqrt{\omega_m} \int_0^{z_r} \frac{dz}{H(z)} \] (2.4)
The value of \( h \) is constrained in the parameter space shown in Fig. 3. In particular, for the case of power-law dark energy \( w \), \( h \) is a key parameter. Specifically, it is straightforward to derive the degeneracy function \( h(w_0, w_1) \), by solving eq. (2.10). This is shown in Fig. 3. The dashed lines correspond to the parameter values that satisfy \( h(w_0, w_1) = 0.674 \) [the \( \Lambda CDM \) value which as expected goes through the point \((w_0, w_1) = (-1, 0)\)] and \( h(w_0, w_1) = 0.74 \) (the local distance ladder measurements value). The constant \( h \) lines shown in Fig. 3 are approximately straight lines in the range of the \( w_0 - w_1 \) parameter space shown in Fig. 3. In particular, for the case of the value \( h = 0.74 \), which alleviates the \( H_0 \) tension, this result is consistent with previous studies [65, 66]. In particular, a related analysis has been performed in [65], where the author points out that fixing the dark energy equation of state \( w \approx -1.3 \) or the effective number of relativistic species \( N_{\text{eff}} \approx 3.95 \) may lead to the relaxation of the \( H_0 \) tension. The novel feature of our work is the use of analytical methods to identify the qualitative features required for any form of \( w(z) \) to relax the \( H_0 \) tension.

This method for deriving the predicted dark energy properties required to resolve the \( H_0 \) tension may be extended to more parametrizations of \( w(z) \). For example, in the case of the two parameter CPL parametrization [75, 76] expansion of \( w(z) \)

\[
w = w_0 + w_1 (1 - a) = w_0 + w_1 z / (1 + z)
\]

eq. (2.3) is written as

\[
H(z) = H_0 \sqrt{\Omega_{m0}(1 + z)^3 + \Omega_r(1 + z)^4 + (1 - \Omega_{m0} - \Omega_r)(1 + z)^{3(1+w_0+w_1)}e^{-3\frac{1+z}{1+w_1}}} \quad (2.13)
\]

Using now eqs. (2.1), (2.5), (2.9) and (2.12) in the context of the above described method, it is straightforward to derive the degeneracy function \( h(w_0, w_1) \), by solving eq. (2.10). This is shown in Fig. 3. The dashed lines correspond to the parameter values that satisfy \( h(w_0, w_1) = 0.674 \) [the \( \Lambda CDM \) value which as expected goes through the point \((w_0, w_1) = (-1, 0)\)] and \( h(w_0, w_1) = 0.74 \) (the local distance ladder measurements value). The constant \( h \) lines shown in Fig. 3 are approximately straight lines in the range of the \( w_0 - w_1 \) parameter space shown in Fig. 3. In particular, for the case of the value \( h = 0.74 \), which alleviates the \( H_0 \) tension, this result is consistent with previous studies [65, 66]. In particular, a related analysis has been performed in [65], where the author points out that fixing the dark energy equation of state \( w \approx -1.3 \) or the effective number of relativistic species \( N_{\text{eff}} \approx 3.95 \) may lead to the relaxation of the \( H_0 \) tension. The novel feature of our work is the use of analytical methods to identify the qualitative features required for any form of \( w(z) \) to relax the \( H_0 \) tension.

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\]
FIG. 3: The degeneracy with respect to the CMB spectrum in the parameter space \((w_0 - w_1)\). The dashed lines correspond to \(h = 0.674\) (\(\Lambda\)CDM value) and to \(h = 0.74\) (the value of Ref. [3]).

FIG. 4: The evolution of \(w(z)\) for various values of \((w_0, w_1)\) along the degeneracy \(h = 0.74\) line of Fig. 3. All these parameter values lead to a best fit value \(h = 0.74\) in the context of the CMB power spectrum. However, they do not have the same quality of fit to other cosmological data which can be used to break this model degeneracy. The common \((z, w)\) point of intersection of all the \(w(z)\) plots is \((0.31, -1.22)\).

all degenerate forms of CPL \(w(z)\) that relax the \(H_0\) tension go through the same point at \(z = 0.31\) crossing the \(w = -1.22\) line. This type of degeneracy in particular redshifts for cosmological parameters has been discussed in Ref. [77]. Also degenerate \(w(z)\) curves with \(w_0 < 1.22\) are increasing functions of \(z\), while those with \(w_0 > 1.22\) are decreasing functions of \(z\). This appears to be a general feature of all \(w(z)\) parametrizations that can relax the \(H_0\) tension. For example the PEDE parametrization [50] and the late dark energy transition hypothesis [78] with \(w(z \simeq 0) > -1.22\) are decreasing functions of the redshift \(z\) as predicted by the above degeneracy analysis. The identification of these properties opens up the possibility of a very late type phase transition at \(z \simeq 0.01\) from a phantom phase to a \(\Lambda\)CDM phase with a sharply increasing rather than decreasing function of \(w(z)\).

Even though the approximate parameter degeneracy exploited in this section is useful for the derivation of the forms of \(w(z)\) that can alleviate the \(H_0\) tension, an important fact that needs to be considered is the quality of fit of the preferred degenerate forms of \(w(z)\) to other cosmological data like SNIa, BAO and growth of perturbations data (Redshift Space Distortion \(f\sigma_8(z)\) and weak lensing data) as well as to actual CMB power spectrum data which may not fully respect the above exploited approximate degeneracy (especially at low \(l\)). Such a fit to cosmological data beyond the CMB is expected to break the above degeneracy obtained from the CMB spectrum. Even if particular forms of \(w(z)\) can lead to apparent alleviation of the \(H_0\) tension such a solution would not be preferable if the quality of fit to the actual CMB spectrum and to other cosmological data is significantly degraded compared to \(\Lambda\)CDM \((w = -1)\). Thus, in the next section we address the following questions:

- What is the quality of fit of the forms of \(w(z)\) that are predicted to resolve the \(H_0\) tension, on cosmological data involving SNIa, BAO, growth Redshift Space Distortion data and the actual Planck CMB TT power spectrum data? Is this quality of fit \((\chi^2)\) similar to the corresponding quality for \(\Lambda\)CDM?
- Is the \(H_0\) tension actually alleviated when the full CMB spectrum data are used in the context of a model with fixed \(w(z)\) to its predicted form \((e.g. w = -1.22)\) in the context of a constant \(w)\)?
- Is the growth tension partially relaxed in the context of the above preferred \(w(z)\) found?

These questions will be addressed mainly in the context of a redshift independent \(w\) but it is straightforward to generalize the analysis for more general forms of \(w(z)\).

III. NUMERICAL ANALYSIS OF DARK ENERGY MODELS

In order to test the resolution of the \(H_0\) tension and test the quality of fit to the CMB and other cosmological data of the models discussed in the previous section, we use the MGCosmoMC numerical package [79–81] with the Planck dataset. In particular, we use the Planck TT and lowP dataset, \(i.e.\) the TT likelihood for high-l multipoles \((l > 30)\) as well as the Planck temperature and polarization data for low multipoles \((l < 30)\). The priors that have been used as input can be seen in Table I.

We fix \(w\) to the values of the points shown in Fig. 1 \((w = -1.0, -1.1, -1.2, -1.3)\) and construct the likelihood contours for the cosmological parameters of these four models. The resulting best fit values of \(h\) are shown in Table II (see also Fig. 1) and are in excellent agreement.
FIG. 5: The contour plots constructed with MGcosmoMC using the Planck TT and lowP likelihoods for ΛCDM and wCDM models. The gray contours correspond to the ΛCDM model. The green contours correspond to $w = -1.1$, the red ones to $w = -1.2$, while the blue to $w = -1.3$. For $w = -1.1$, the best fit value of $H_0$ is close to that of the Planck/ΛCDM measurement [1], while the $w = -1.2$ and $w = -1.3$ values shift $h$ closer to the local distance ladder measurements [3].

TABLE I: The MGcosmoMC priors that have been used in Figs. 5 and Fig. 7. We also set $A_{\text{lens}} = 1$ and $\Omega_k = 0$.

| Parameters | Priors |
|------------|--------|
| $\Omega_b h^2$ | [0.005, 0.1] |
| $\Omega_c h^2$ | [0.001, 0.99] |
| $100\theta_{MC}$ | [0.5, 10] |
| $\tau$ | [0.06, 0.8] |
| $\ln (10^{10} A_s)$ | [1.61, 3.91] |
| $n_s$ | [0.8, 1.2] |

with the expectations based on the parameter degeneracy analysis of the previous section (orange continuous line in Fig. 1). The corresponding likelihood contours are shown in Fig. 5.

Clearly, the likelihood contours for the Hubble parameter shift to higher best fit values as $w$ decreases in the phantom regime ($w < -1$). At the same time the best fit values of the matter density parameter $\Omega_{0m}$ decrease in accordance with the degenerate parameter combination $\Omega_{0m} h^2$.

This reduced value of the best fit $\Omega_{0m}$ would naively imply reduced growth of cosmological perturbations and thus resolution of the growth tension. However, the reduced best fit value of the matter density parameter $\Omega_{0m}$ matter density is not enough to soften the growth tension, since the best fit value of the parameter $\sigma_8$ (the present day rms matter fluctuations variance on scales of $8h^{-1}\text{Mpc}$) appears to increase more rapidly, as $w$ decreases in the phantom regime. Since this parameter is proportional to the initial amplitude of the matter perturbations power spectrum, its increase amplifies the growth of perturbations and tends to cancel the effect of the decrease of the best fit $\Omega_{0m}$ in the context of perturbations growth. This is demonstrated in Fig. 6 where we show the $\sigma_8$ likelihood contours obtained by fitting the models $w = -1$ (ΛCDM) and $w = -1.2$ to the growth $f\sigma_8$ data (we have used the conservative robust dataset of Table 2 of Ref. [82]). Superimposed we also show the corresponding likelihood contours obtained from the Planck CMB TT power spectrum obtained for each value of fixed $w$. Clearly, the tension between the RSD $f\sigma_8$ data and the Planck data increases in the context of the phantom model $w = -1.2$ compared to ΛCDM ($w = -1$).

In addition to the growth data we also fit the models $w = -1$ and $w = -1.2$ to a cosmological data com-
TABLE II: The analytically predicted CMB best fit values of $h$ and $\Omega_{0m}$ for fixed $w$, obtained by using the CMB parameter degeneracy arguments, as well as the ones obtained by the actual fit of the corresponding $w$ model to the Planck TT CMB anisotropy power spectrum. The quality of fit for each model compared to $\Lambda$CDM is also indicated by the value of $\Delta \chi^2$.

| $w$ | $\Omega_{0m}^{th}$ | $h_{th}$ | $\Omega_{0m}^{obs}$ | $h_{obs}$ | $\chi^2_{CMB}$ | $\Delta \chi^2_{CMB}$ |
|-----|-----------------|--------|-----------------|----------|----------------|----------------|
| $-1.0$ | 0.316 | 0.674 | 0.315 $\pm$ 0.013 | 0.673 $\pm$ 0.010 | 11265.516 | $-$ |
| $-1.1$ | 0.289 | 0.704 | 0.288 $\pm$ 0.013 | 0.704 $\pm$ 0.011 | 11265.530 | 0.014 |
| $-1.2$ | 0.265 | 0.735 | 0.263 $^{+0.012}_{-0.014}$ | 0.736 $\pm$ 0.013 | 11267.132 | 0.616 |
| $-1.3$ | 0.244 | 0.766 | 0.242 $^{+0.012}_{-0.014}$ | 0.768 $\pm$ 0.014 | 11266.520 | 0.004 |

FIG. 6: The $1\sigma - 4\sigma$ contours in the parametric space $\Omega_{0m} - \sigma_8$. The blue contours correspond to the best fit growth compilation of Ref. [82], while the red to the $1\sigma - 4\sigma$ confidence contours for $w = -1$ (left panel) and $w = -1.2$ (right panel) obtained from the Planck data.

bination including the Pantheon SnIa [83], BAO data [23, 84, 85], CMB data [1], as well as the prior of the Hubble constant published by Riess et al. [3] and obtain for $\Lambda$CDM $\chi^2 = 12319.2$, while for $w = -1.2$ we obtain $\chi^2 = 12332.7$. We thus find $\Delta \chi^2 = 13.5$. This difference of $\Delta \chi^2 = 13.5$ for the phantom model, indicates a significantly reduced quality of fit compared to $\Lambda$CDM in agreement with previous studies [86]. The corresponding likelihood contours are shown in Fig. 7. It is therefore clear that the particular fixed $w$ models considered here lead to an apparent resolution of the Hubble tension since they increase the best fit value of $H_0$ in the context of the CMB data but the resolution is not viable since the growth tension gets worse while the quality of fit of these models to the SnIa and BAO data is not as good as for $\Lambda$CDM. This result is consistent with previous studies [66]. It is, however, worth mentioning that for the combination of the CMB Planck data and the Riess Hubble constant prior the quality of the fit improves drastically for $w = -1.2$, with $\Delta \chi^2 = -10.7$ in respect to $w = -1$. The exploitation of the CMB spectrum degeneracy of more complicated forms of $w(z)$ however may lead to better fits to growth, SnIa and BAO cosmological data.

IV. CONCLUSION - OUTLOOK

We have used analytical degeneracy relations among cosmological parameters and numerical fits to cosmological data to identify the qualitative and quantitative features of dark energy models that have the potential to relax the $H_0$ tension of the $\Lambda$CDM model. We have found that mildly phantom models with mean equation of state parameter $w \simeq -1.2$ have the potential to alleviate this tension. The models may be constructed in such a way that there are no extra parameters compared to $\Lambda$CDM by using fixed parametrizations of $w(z)$. In practice however these models involve more fine tuning compared to $\Lambda$CDM and are clearly less natural than the standard model. In addition the quality of fit of the simplest of such models to cosmological data beyond the CMB is not as good as the corresponding quality of fit of $\Lambda$CDM. However, it is straightforward to construct physical models involving either phantom scalar field with negative...
kinetic terms or modified gravity models that naturally produce the required phantom behavior of dark energy. Despite the usual stability issues of such models it is possible to construct ghost free versions [87].

Interesting extensions of the present analysis include the following:

- A comparative analysis of phantom models identified using the degeneracy analytical method proposed here involving also redshift dependence of $w(z)$. Such an analysis would rank these models according to their quality of fit on cosmological data.

- The construction of physical models that can reproduce the forms of $w(z)$ required to relax the $H_0$ and possibly the growth tension as well, while providing a better fit to the cosmological data than the fit of $\Lambda$CDM. The construction of stable theories with phantom behavior is possible in the context of modified gravity theories. In many such theories however including $f(R)$ and scalar-tensor theories, it is not possible to combine stability, with the weaker gravity and phantom behaviour [88, 89] required for the resolution of the $H_0$ and growth tensions.

The analytical approach for the $H_0$-$w(z)$ degeneracy pointed out in the present analysis offers a new method to systematically search and design $w(z)$ forms that can combine the proper features required to consistently relax the tension while keeping a good fit to other cosmological data. Our goal here was only to introduce the method and apply it to the simplest cases while also pointing out the difficulties in resolving the tension. In a subsequent full application and extension of the method we plan to exploit its full potential in identifying possible forms of $w(z)$ that can actually resolve the tension while keeping good fit to other cosmological data.

**Numerical Analysis Files:** The numerical files for the reproduction of the figures can be found in [90].

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