Analytical solutions to some generalized matrix eigenvalue problems

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Abstract
We present analytical solutions to two classes of generalized matrix eigenvalue problems (GMEVPs) which naturally cover the matrix eigenvalue problems (MEVPs) with matrices such as the tridiagonal and pentadiagonal matrices. We refer to the matrices in the two classes as Toeplitz-regularized and corner-overlapped block-diagonal matrices. The first class generalizes the tridiagonal Toeplitz matrices to matrices with larger bandwidths where boundary entries are modified. For the second class, we decompose the problem into a lower two-block matrix problem where one of the blocks is a quadratic eigenvalue problem (QEVP). Analytical solutions are also obtained for these QEVPs. Moreover, we generalize the eigenvector-eigenvalue identity (rediscovered and coined recently for MEVPs) for GMEVPs and derive some trigonometric identities. Possible generalizations and applications to the design of better numerical methods for solving partial differential equations are discussed.

Keywords: matrix eigenvalue problem, eigenvalue, eigenvector, Toeplitz matrix, block-diagonal matrix, numerical spectral approximation

1. Introduction
It is well-known that the following tridiagonal Toeplitz matrix

\[ A = \begin{bmatrix}
  2 & -1 & & & \\
  -1 & 2 & -1 & & \\
  & \ddots & \ddots & \ddots & \\
  & & -1 & 2 & -1 \\
  & & & -1 & 2 \\
\end{bmatrix}_{n \times n} \]

has analytical eigenpairs \((\lambda_j, X_j)\) with \(X_j = (X_{j,1}, \ldots, X_{j,n})^T\) where

\[ \lambda_j = 2 - 2 \cos(j\pi h), \quad X_{j,k} = c \sin(j\pi kh), \quad h = \frac{1}{n+1}, \quad j, k = 1, 2, \ldots, n \]

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with \(0 \neq c \in \mathbb{R}\) (we assume that \(c\) is a nonzero constant throughout the paper); see, for example, [1, p. 514] or [2] for a general case of tridiagonal Toeplitz matrix. Following the constructive technique in [1, p. 515] for finding the analytical solutions to the matrix eigenvalue problem (MEVP) \(AX = \lambda X\), one can derive analytical solutions to the generalized matrix eigenvalue problem (GMEVP) \(AX = \lambda BX\) where \(B\) is an invertible tridiagonal Toeplitz matrix. For example, let

\[
B = \begin{bmatrix}
\frac{2}{3} & \frac{1}{6} & \frac{1}{6} & \cdots & \frac{1}{6} \\
\frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \cdots & \frac{1}{6} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\frac{1}{6} & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\
\end{bmatrix}_{n \times n},
\]

then, the GMEVP \(AX = \lambda BX\) has analytical eigenpairs \((\lambda_j, X_j)\) with \(X_j = (X_{j,1}, \ldots, X_{j,n})^T\) where (see [3, Sec. 4] for a scaled case; \(A\) is scaled by \(1/h\) while \(B\) is scaled by \(h\))

\[
\lambda_j = -6 + \frac{18}{2 + \cos(j \pi h)}, \quad X_{j,k} = c \sin(j \pi kh), \quad h = \frac{1}{n + 1}, \quad j, k = 1, 2, \ldots, n.
\]

These matrices or their scaled (by constants) versions arise from various applications. For example, the matrix \(A\) arises from the discrete discretizations of the 1D Laplace operator by the finite difference method (FDM, cf., [4, 5], scaled by \(1/h^2\)) or the spectral element method (SEM, cf., [6], scaled by \(1/h\)). The matrix \(B\) (and also \(A\)) arises from the discrete discretization of the 1D Laplace operator by the finite element method (FEM, cf., [7–9], scaled by \(h\)).

In general, it is difficult to find analytical solutions to the MEVPs. The work by Trench [10] gave analytical eigenpairs to the MEVPs for some symmetric and tridiagonal matrices. A new proof of these solutions was given by Da Fonseca [11]. Yueh [12] derived the eigenpairs of several tridiagonal matrices by the method of symbolic calculus. Based on Yueh’s work, Kouachi [13] extended the analytical results for a set of more general tridiagonal matrices. In [2], the author derived analytical solutions to some MEVPs for certain tridiagonal matrices with complex coefficients. The analytical eigenpairs are expressed in terms of trigonometric functions. We refer to the recent work [14] for a survey on the analytical eigenpairs of tridiagonal matrices. For pentadiagonal and heptadiagonal matrices, finding analytical eigenpairs becomes more difficult. The work [15] derived asymptotical results of eigenvalues for pentadiagonal symmetric Toeplitz matrices. Some spectral properties were found in [16] for some pentadiagonal symmetric matrices. The work [17] gave analytical eigenvalues (as the zeros of some complicated functions) for heptadiagonal symmetric Toeplitz matrices.

To the best of the author’s knowledge, there are no widely-known articles in literature which address the issue of finding analytical solutions to either the MEVPs with more general matrices (other than the ones mentioned above) or the GMEVPs. The articles [3, 18, 19] present analytical solutions (somehow implicitly) to GMEVPs for some tridiagonal and/or pentadiagonal matrices that are arising from the numerical spectral approximations (by isogeometric finite element methods) of the 1D Laplace operator. For heptadiagonal and more general matrices, no analytical solutions exist and the numerical
approximations in [18–20] can be considered as asymptotic results for certain structured matrices arising from the numerical discretizations of the differential operators.

In this paper, we present analytical solutions to GMEVPs for mainly two classes of matrices. The first class is the Toeplitz-regularized matrices while the second class is the corner-overlapped block-diagonal matrices. We give analytical solutions to the GMEVPs with these matrices. The main insights for the Toeplitz-regularized matrices are from the numerical spectral approximation techniques where a particular solution form such as, the well-known Bloch wave form, is sought. For the corner-overlapped block-diagonal matrices, we propose to decompose the original problem into a lower two-block matrix problem where one of the blocks is a quadratic eigenvalue problem (QEVP). We solve the QEVP by rewriting the problem and applying the analytical results from the tridiagonal Toeplitz matrices. Additionally, Denton, Parke, Tao, and Zhang in a recent work [21] rediscovered and coined the eigenvector-eigenvalue identity for certain MEVPs. We generalize this identity for the GMEVPs. Based on these identities, we derive some interesting trigonometric identities.

The rest of the article is organized as follows. Section 2 presents the main results, i.e., the analytical solutions to the two classes of matrices. Several examples are given and discussed. Section 3 generalizes the eigenvector-eigenvalue identity and derives some trigonometric identities. Section 4 considers the generalizations of the main results. The generalizations include the powers, tensor-products, and multiplications. Potential applications to the design of better numerical discretization methods for partial differential equations and other concluding remarks are presented in Section 5.

2. Main results

For matrices $A$ and $B$, the MEVP is to find the eigenpairs $(\lambda, X)$ such that
\[ AX = \lambda X \]  
while the GMEVP is to find the eigenpairs $(\lambda, X)$ such that
\[ AX = \lambda BX. \]

Throughout the paper, let $n$ be a positive integer. For simplicity, we denote the dimension of a matrix as $n$ or $2n+1$ in different scenarios which we specify later in the context. Since analytical eigenpairs for GMEVPs with small dimensions are easy to find, we assume that $n$ is large enough for the generalization of matrices to make sense. For simplicity, we slightly abuse the notation such as $A$ for a matrix and $(\lambda, X)$ for an eigenpair. Once analytic eigenpairs for a GMEVP are found, eigenpairs for MEVP $AX = \lambda X$ follows naturally by setting $B$ as an identity matrix.

2.1. The first class: Toeplitz-regularized matrices

In this section, we present analytical solutions to certain Toeplitz-type matrices. The main insights are from the proof in finding analytical solutions to a tridiagonal Toeplitz matrix in [1, p. 514] and the numerical spectral approximations of the Laplace operators (cf., [2, 18, 19]). The main idea is to seek eigenvectors in a particular form such as the Bloch wave form $e^{ax+ib}$, where $i^2 = -1$ as in [21, 25] and sinusoidal form as in [18].
Our contribution comes twofold: (1) we generalize these results from the linear algebra point of view and (2) we fix the boundary rows of the matrices to what we refer to as Toeplitz-regularized matrices and give analytical eigenpairs.

We denote \( m \geq 1 \) as the bandwidth of a matrix. In particular, when \( m = 1 \), the matrix is a tridiagonal matrix. For simplicity, we introduce the notation function \( \Xi^{(\ell)} \) for some matrix \( \Xi \). For example, \( B^{(\alpha)} \) means a matrix \( B \) with entries involving \( \xi \) replaced by \( \alpha \). In particular, we define \( G^{(\ell)}(\ell) = (G^{(\ell)}_{j,k}) \) as a square matrix with dimension \( n \) such that

\[
G^{(\ell)}_{j,k} = \xi |k|, \quad |k| \leq m, \quad j = 1, \ldots, n. \tag{2.3}
\]

Generalizing the results shown in the introduction, we have the following theorem.

**Theorem 1** (Analytical eigenvalues and eigenvectors, set 1). Let \( H^{(\ell)} = (H^{(\ell)}_{j,k}) \) be a zero square matrix with dimension \( n \). For \( m \geq 2 \), we modify \( H^{(\ell)} \) with

\[
H^{(\ell)}_{j,k} = H^{(\ell)}_{n-j+1, n-k+1} = \xi_{j+k}, \quad k = 1, \ldots, m - j, \quad j = 1, \ldots, m - 1. \tag{2.4}
\]

For \( m \geq 1 \), we define

\[
A = G^{(\alpha)} - H^{(\alpha)}, \quad B = G^{(\beta)} - H^{(\beta)}. \tag{2.5}
\]

Assume that \( B \) is invertible. Then, the GMEVP \( (2.2) \) has eigenpairs \( (\lambda_j, X_j) \) with \( X_j = (X_{j,1}, \ldots, X_{j,n})^T \) where

\[
\lambda_j = \frac{\alpha_0 + 2 \sum_{l=1}^m \alpha_l \cos(lj\pi h)}{\beta_0 + 2 \sum_{l=1}^m \beta_l \cos(lj\pi h)}, \quad X_j = c \sin(j\pi kh), \quad h = \frac{1}{m+1}, \quad j, k = 1, 2, \ldots, n. \tag{2.6}
\]

**Proof.** Following [1, p. 515], one seeks eigenvectors of the form \( c \sin(j\pi kh) \). Using the trigonometric identity \( \sin(a \pm b) = \sin(a) \cos(b) \pm \cos(a) \sin(b) \), one can verify that each row of the GMEVP \( (2.2) \), \( \sum_{k=1}^m A_{ik} X_{j,k} = \lambda \sum_{k=1}^m B_{ik} X_{j,k}, i = 1, \ldots, n \), reduces to 

\[
\alpha_0 + 2 \sum_{l=1}^m \alpha_l \cos(lj\pi h) = \lambda (\beta_0 + 2 \sum_{l=1}^m \beta_l \cos(lj\pi h)), \tag{2.4}
\]

which is independent of the row number \( i \). Thus, the eigenvectors \((\lambda_j, X_j)\) given in \( (2.6) \) satisfies \( (2.2) \). The GMEVP has at most \( n \) eigenpairs and the \( n \) eigenvectors are linearly independent. This proves the theorem. \( \square \)

**Remark 1.** All the matrices defined above are symmetric and persymmetric. The matrix \( G \) is a Toeplitz matrix while the matrix \( H \) is a zero matrix with modifications near the first and last few rows. For \( m = 1 \), both \( A \) and \( B \) are tridiagonal Toeplitz matrices. For \( m \geq 2 \), both \( A \) and \( B \) are not Toeplitz matrices as there are modifications to few rows of the matrices. As \( n \) gets larger, these matrices have more rows that are of Toeplitz structure. The first and last few rows are regularized/modified such that the specific solution form that the internal eigenvector entries admit extends naturally to the boundary entries. In particular, the first and last \( m - 1 \) rows and columns are modified. For this reason, we refer to these matrices as Toeplitz-regularized matrices.

We present the following example. Let \( m = 2, \alpha_0 = 1, \alpha_1 = -1/3, \alpha_2 = -1/6, \beta_0 = \)
11/20, β₁ = 13/60, β₂ = 1/120. Then, we have the following matrices

\[
A = \begin{bmatrix}
\frac{7}{6} & -\frac{1}{3} & -\frac{1}{6} \\
-\frac{1}{3} & 1 & -\frac{1}{3} \\
-\frac{1}{6} & -\frac{1}{3} & 1
\end{bmatrix}
B = \begin{bmatrix}
\frac{13}{60} & \frac{13}{120} & \frac{1}{120} \\
\frac{13}{120} & \frac{13}{60} & \frac{1}{120} \\
\frac{1}{120} & \frac{13}{60} & \frac{1}{120}
\end{bmatrix}
\]

The eigenpairs of the GMEVP (2.2) with these matrices are

\[
\lambda_j = -20 + \frac{240(3 + 2 \cos(j \pi h))}{33 + 26 \cos(j \pi h) + \cos(2 j \pi h)}, \quad X_{j,k} = c \sin(j \pi kh), \quad j, k = 1, \ldots, n. \quad (2.8)
\]

If we scale the matrix \(A\) by \(1/h\) while the matrix \(B\) by \(h\), then the new system is a GMEVP resulting from the isogeometric finite element approximation of the 1D Laplace operator with modified boundary entries; see, for example, [18, eqns. 118–123] (the first two rows have entries \(A_{11} = 4/3, A_{12} = A_{21} = -1/6, B_{11} = 1/3, B_{12} = B_{21} = 5/24\)). A Taylor expansion of the eigenvalues in (2.8) leads to the optimal eigenvalue errors of the isogeometric finite element approximation to the Laplace operator on a unit interval.

Similarly, if we shift the phase by a half and seek solutions of the form \(\sin(j \pi (k - \frac{1}{2}) h)\), we have the following result.

**Theorem 2** (Analytical eigenvalues and eigenvectors, set 2). Let \(H(\xi) = (H_{j,k}^{(\xi)})\) be a zero square matrix with dimension \(n\). We modify \(H(\xi)\) with

\[
H_{j,k}^{(\xi)} = H_{n-j+1,n-k+1}^{(\xi)} = \xi_{j+k-1}, \quad k = 1, \ldots, m - j + 1, \quad j = 1, \ldots, m. \quad (2.9)
\]

We define

\[
A = G^{(\alpha)} - H^{(\alpha)}, \quad B = G^{(\beta)} - H^{(\beta)}. \quad (2.10)
\]

Assume that \(B\) is invertible. Then, the GMEVP (2.2) has eigenpairs \((\lambda_j, X_j)\) with \(X_j = (X_{j,1}, \ldots, X_{j,n})^T\) where

\[
\lambda_j = \frac{\alpha_0 + 2 \sum_{l=1}^{\infty} \alpha_l \cos(l j \pi h)}{\beta_0 + 2 \sum_{l=1}^{\infty} \beta_l \cos(l j \pi h)}, \quad X_{j,k} = c \sin(j \pi (k - \frac{1}{2}) h), \quad h = \frac{1}{n}, \quad j, k = 1, 2, \ldots, n. \quad (2.11)
\]

The proof can be established similarly and we omit it for brevity. The entries of the matrices in [18, eqns. 118–123] satisfy the assumption of the above theorem, thus, without modifications to the boundary entries, the analytical solutions are given by (2.11). The eigenvectors of Theorems 1 and 2 correspond to the solutions of the Dirichlet eigenvalue problem on the unit interval. When \(m \geq 3\), the results in Theorems 1 and 2 provide an insight to remove the outliers for high-order isogeometric finite element spectral approximations. We refer to [2, 18] for the outlier behavior in the approximate spectrum and the design of technique which removes the outliers is subject to future work. Similarly, with the insights from the numerical methods for the Neumann eigenvalue problem on the unit interval, we have the following two sets of analytical eigenpairs.
Theorem 3 (Analytical eigenvalues and eigenvectors, set 3). Let \( H^{(\xi)} = (H_{j,k}^{(\xi)}) \) be a zero square matrix with dimension \( n \). We modify \( H^{(\xi)} \) with

\[
H_{1,1}^{(\xi)} = H_{n,n}^{(\xi)} = -\alpha_0/2, \\
H_{j+1,k+1}^{(\xi)} = H_{n-j,n-k}^{(\xi)} = \xi_{j,k}, \quad k = 1, \ldots, m-j, \quad j = 1, \ldots, m-1, \quad m \geq 2. \tag{2.12}
\]

We define

\[
A = G^{(\alpha)} + H^{(\alpha)}, \quad B = G^{(\beta)} + H^{(\beta)}. \tag{2.13}
\]

Assume that \( B \) is invertible. Let \( h = \frac{1}{n-1} \). Then, the GMEVP (2.2) has eigenpairs \((\lambda_j, X_j)\) with \( X_j = (X_{j,1}, \ldots, X_{j,n})^T \) where

\[
\lambda_{j+1} = \frac{\alpha_0 + 2 \sum_{l=1}^m \alpha_l \cos(\xi_{j,l} h)}{\beta_0 + 2 \sum_{l=1}^m \beta_l \cos(\xi_{j,l} h)}, \quad X_{j+1,k+1} = c \cos(\xi_{j,k} h), \quad j, k = 0, 1, \ldots, n-1. \tag{2.14}
\]

Theorem 4 (Analytical eigenvalues and eigenvectors, set 4). Let \( H^{(\xi)} = (H_{j,k}^{(\xi)}) \) be a zero square matrix with dimension \( n \). We modify \( H^{(\xi)} \) with

\[
H_{j+1,k+1}^{(\xi)} = H_{n-j,n-k}^{(\xi)} = \xi_{j,k}, \quad k = 1, \ldots, m-j, \quad j = 1, \ldots, m. \tag{2.15}
\]

We define

\[
A = G^{(\alpha)} + H^{(\alpha)}, \quad B = G^{(\beta)} + H^{(\beta)}. \tag{2.16}
\]

Assume that \( B \) is invertible. Let \( h = \frac{1}{n} \). Then, the GMEVP (2.2) has eigenpairs \((\lambda_j, X_j)\) with \( X_j = (X_{j,1}, \ldots, X_{j,n})^T \) where

\[
\lambda_{j+1} = \frac{\alpha_0 + 2 \sum_{l=1}^m \alpha_l \cos(\xi_{j,l} h)}{\beta_0 + 2 \sum_{l=1}^m \beta_l \cos(\xi_{j,l} h)}, \quad X_{j+1,k+1} = c \cos\left(\xi_{j,k} \left(\frac{1}{2} - \frac{1}{2}h\right)\right) \tag{2.17}
\]

with \( k = 1, 2, \ldots, n, \quad j = 0, 1, \ldots, n-1. \)

We present the following example. Let \( n = 4, m = 2, \alpha_0 = 7, \alpha_1 = 5, \alpha_2 = 2, \beta_0 = 5, \beta_1 = 3, \beta_2 = 1 \). Then, with the setting in Theorem 4 we have the following matrices

\[
A = \begin{bmatrix}
12 & 7 & 2 & 0 \\
7 & 7 & 5 & 2 \\
2 & 5 & 7 & 7 \\
0 & 2 & 7 & 12
\end{bmatrix}, \quad B = \begin{bmatrix}
8 & 4 & 1 & 0 \\
4 & 5 & 3 & 1 \\
1 & 3 & 5 & 4 \\
0 & 1 & 4 & 8
\end{bmatrix}. \tag{2.18}
\]

By direct calculations, the eigenpairs of (2.2) are

\[
\lambda_{1,2,3,4} = \frac{21}{13}, \quad \frac{5 + 4\sqrt{2}}{7}, \quad 1, \quad \frac{5 - 4\sqrt{2}}{7},
\]

\[
X_1 = (1, 1, 1, 1)^T, \quad X_2 = (1, \sqrt{2} - 1, 1 - \sqrt{2}, -1)^T, \quad X_3 = (1, -1, -1, 1)^T, \quad X_4 = (-1, 1 + \sqrt{2}, -1 - \sqrt{2}, 1)^T. \tag{2.19}
\]
2.2. The second class: Corner-overlapped block-diagonal matrices

The matrices can be complex-valued. For example, let \( t^2 = -1, n = 5, m = 2, \alpha_0 = 8 + 2i, \alpha_1 = 5 - i, \alpha_2 = 2i, \beta_0 = 6, \beta_1 = 3i, \beta_2 = 1 - i \). Then, with the setting in Theorem 3, we have the following matrices

\[
A = \begin{bmatrix}
4 + i & 5 - i & 2t & 0 & 0 \\
5 - i & 8 + 4t & 5 - i & 2t & 0 \\
2t & 5 - i & 8 + 2t & 5 - i & 2t \\
0 & 2t & 5 - i & 8 + 4t & 5 - i \\
0 & 0 & 2t & 5 - i & 4 + t \\
\end{bmatrix}
\]

By direct calculations, the eigenpairs of (2.2) are

\[
\lambda_{1,2,3,4,5} = \frac{\frac{t^2}{2}, 7 - 3t + (6 - 5i)\sqrt{2}}{9}, \frac{7}{5}, \frac{6}{\sqrt{8}}, \frac{5}{8}, \frac{7 - 3t - (6 - 5i)\sqrt{2}}{9},
\]

\[
X_1 = (1, 1, 1, 1, 1)^T, \\
X_2 = (1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1)^T, \\
X_3 = (1, 0, -1, 0, 1)^T, \\
X_4 = (1, -1, 1, -1, 1)^T, \\
X_5 = (-1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 1)^T,
\]

which verifies (2.14).

Remark 2. Theorems 1–4 give analytical eigenvalues and eigenvectors for four different sets of GMEVPs. The eigenvalues are of the same form while the eigenvectors are of different forms. For a fixed bandwidth \( m \), the internal entries of the matrices defined in Theorems 1–4 are the same while the modifications to the boundary rows are slightly different. This small discrepancy leads to different eigenvectors. It is well-known that for a complex-valued Hermitian matrix, the eigenvalues are real and eigenvectors are complex. The matrices in (2.20) are complex-valued. They are symmetric but not Hermitian. The eigenvalues are complex while the eigenvectors are real.

2.2. The second class: Corner-overlapped block-diagonal matrices

In this section, we consider the following type of matrix, that is, \( G^\Omega = (G^\Omega_{j,k}) \) with

\[
G^\Omega = \begin{bmatrix}
\xi_3 & \xi_1 \\
\xi_1 & \xi_0 & \xi_1 \\
\xi_1 & \xi_3 & \xi_1 \\
\xi_2 & \xi_1 & \xi_0 & \xi_1 & \xi_2 \\
\xi_1 & \xi_3 & \xi_1 & \xi_2 & \xi_2 \\
\xi_2 & \xi_1 & \xi_0 & \cdots & \xi_2 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\xi_2 & \xi_1 & \xi_0 & \cdots & \xi_2 \\
\xi_1 & \xi_3 & \xi_1 & \xi_2 & \xi_2 \\
\xi_2 & \xi_1 & \xi_0 & \cdots & \xi_2 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}.
\]
We assume that the dimension of the matrix is \((2n + 1) \times (2n + 1)\). Figure 1 shows the structure and sparsity of the matrix. It is a block-diagonal matrix where the corners of blocks are overlapped. Therefore, we refer to this type of matrices as the corner-overlapped block-diagonal matrices.

![Figure 1: The sparsity and structure of a corner-overlapped block-diagonal matrix.](image)

In this section, we derive their analytical eigenpairs. To illustrate our idea, we consider the following matrix

\[
A = \begin{bmatrix}
\alpha_3 & \alpha_1 & \alpha_1 & \alpha_2 \\
\alpha_1 & \alpha_0 & \alpha_1 & \alpha_2 \\
\alpha_1 & \alpha_3 & \alpha_1 & \alpha_1 \\
\alpha_2 & \alpha_1 & \alpha_0 & \alpha_1 \\
\alpha_1 & \alpha_1 & \alpha_3 & \alpha_1 \\
\end{bmatrix}
\]  

(2.23)

and its MEVP (2.1). A direct symbolic calculation leads to the analytical eigenvalues

\[
\lambda_1 = \alpha_3,
\lambda_{2,3} = \frac{1}{2} \left( \alpha_0 + \alpha_2 + \alpha_3 \pm \sqrt{12\alpha_1^2 + (\alpha_0 + \alpha_2 - \alpha_3)^2} \right),
\lambda_{4,5} = \frac{1}{2} \left( \alpha_0 - \alpha_2 + \alpha_3 \pm \sqrt{4\alpha_1^2 + (\alpha_0 - \alpha_2 - \alpha_3)^2} \right).
\]  

(2.24)

Alternatively, to find its eigenvalues, we note that the first and the last rows of (2.1) lead to

\[
\begin{aligned}
\alpha_3 X_1 + \alpha_1 X_2 &= \lambda X_1, \\
\alpha_1 X_4 + \alpha_3 X_5 &= \lambda X_5.
\end{aligned}
\]  

(2.25)
On one hand, we solve these equations to arrive at
\begin{align}
\alpha_1 X_2 &= (\lambda - \alpha_3)X_1, \\
\alpha_1 X_4 &= (\lambda - \alpha_3)X_5,
\end{align}
which is then substituted into the third equation in (2.21) to get
\begin{equation}
(\lambda - \alpha_3)(X_5 - X_1 + X_1) = 0. \tag{2.27}
\end{equation}
On the other hand, we substitute (2.26) and the third equation of (2.21) into the second and the fourth equations in (2.21) to get
\begin{align}
(2\alpha_1^2 + \alpha_0(\lambda - \alpha_3) - \lambda(\lambda - \alpha_3))X_2 + (\alpha_1^2 + \alpha_2(\lambda - \alpha_3))X_4 &= 0, \\
(\alpha_1^2 + \alpha_2(\lambda - \alpha_3))X_2 + (2\alpha_1^2 + \alpha_0(\lambda - \alpha_3) - \lambda(\lambda - \alpha_3))X_4 &= 0. \tag{2.28}
\end{align}
We now see that the MEVP (2.29) is decomposed into two subproblems; that is, one MEVP and one quadratic eigenvalue problem (QEVP) as follows
\begin{equation}
\tilde{A}X = \lambda X, \quad \text{where} \quad A = [\alpha_3]_{1 \times 1} \tag{2.29}
\end{equation}
and
\begin{equation}
\lambda^2 I X - \lambda BX - CX = 0, \tag{2.30}
\end{equation}
where \( I \) is the identity matrix (we assume that the dimension of \( I \) is adaptive to its occurrence, in this case, it is \( 2 \times 2 \)),
\begin{equation}
B = \begin{bmatrix} \alpha_0 + \alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_0 + \alpha_3 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2\alpha_1^2 - \alpha_0\alpha_3 & \alpha_1^2 - \alpha_2\alpha_3 \\ \alpha_1^2 - \alpha_2\alpha_3 & 2\alpha_1^2 - \alpha_0\alpha_3 \end{bmatrix}. \tag{2.31}
\end{equation}
Both matrices \( B \) and \( C \) are symmetric. The MEVP (2.29) has an analytical eigenvalue \( \lambda = \alpha_3 \) which is one of the eigenvalues in (2.21). The characteristic polynomial of the QEVP (2.30) is
\begin{equation}
\chi(\lambda) = \det(\lambda^2 I + \lambda B + C), \tag{2.32}
\end{equation}
which is a polynomial of order four. From fundamental theory of algebra, it has four roots. It is easy to verify that \( \lambda_j, j = 2, 3, 4, 5 \) given in (2.24) are the four roots of the equation \( \chi(x) = 0 \).
Now, we propose the following idea. Assuming that \( \lambda \neq 0 \), we rewrite the QEVP (2.30) as
\begin{equation}
\Xi X = \lambda X, \tag{2.33}
\end{equation}
where
\begin{equation}
\Xi = B + \frac{1}{\lambda} C = \begin{bmatrix} \alpha_0 + \alpha_3 + \frac{2\alpha_1^2 - \alpha_0\alpha_3}{\lambda} & \alpha_2 + \frac{\alpha_1^2 - \alpha_2\alpha_3}{\lambda} \\ \alpha_2 + \frac{\alpha_1^2 - \alpha_2\alpha_3}{\lambda} & \alpha_0 + \alpha_3 + \frac{2\alpha_1^2 - \alpha_0\alpha_3}{\lambda} \end{bmatrix}, \tag{2.34}
\end{equation}
which is a symmetric matrix. For a fixed \( \lambda \) the matrix \( \Xi \) can be viewed as a constant matrix. We apply Theorem 4 with bandwidth \( m = 1 \) and dimension \( n = 2 \). Thus, the eigenvalues of \( \Xi \) satisfy
\begin{equation}
\lambda_j = \left( \alpha_0 + \alpha_3 + \frac{2\alpha_1^2 - \alpha_0\alpha_3}{\lambda_j} \right) + 2\left( \alpha_2 + \frac{\alpha_1^2 - \alpha_2\alpha_3}{\lambda_j} \right) \cos(j\pi/3), \quad j = 1, 2, \tag{2.35}
\end{equation}
The corresponding eigenvectors are

\[ \lambda_j = \frac{\alpha_3}{\beta_3}, \quad \lambda_{j+1} = \frac{-\hat{b} \pm \sqrt{\hat{b}^2 - 4\hat{c}\hat{d}}}{2\hat{d}}, \quad j = 1, 2, \ldots, n. \]  

(2.37)

where

\[ \hat{a} = \beta_0\beta_3 - 2\beta_1^2 + 2(\beta_2\beta_3 - \beta_2^2) \cos(j\pi h), \]
\[ \hat{b} = 4\alpha_1\beta_1 - \beta_0\alpha_3 - \alpha_0\beta_3 - 2(\beta_2\alpha_3 - 2\alpha_1\beta_1 + \alpha_2\beta_3) \cos(j\pi h), \]
\[ \hat{c} = \alpha_0\alpha_3 - 2\alpha_1^2 + 2(\alpha_2\alpha_3 - \alpha_1^2) \cos(j\pi h). \]

(2.38)

The corresponding eigenvectors are \( X_j = (X_{j,1}, \ldots, X_{j,2n+1})^T \) with

\[ X_{2n+1,1} = e(-1)^j, \quad X_{2n+1,2} = 0, \quad j = 1, \ldots, n, \]
\[ X_{2n+1,2k+1} = \frac{\alpha_1 - \lambda_j\beta_1}{\lambda_j\beta_3 - \alpha_3}(X_{2k+1} + X_{j,2k+2}), \]
\[ X_{2n+1,2k} = e\sin\left(\frac{f}{n+1}\right), \quad h = \frac{1}{n+1}, \quad j = 1, \ldots, 2n, \quad k = 0, 1, \ldots, n, \]  

(2.39)

where \([ \cdot ]\) is the ceiling function.

Proof. For simplicity, we denote \((\lambda, X = (X_1, \ldots, X_{2n+1}))^T\) as a generic eigenpair of the GMEVP \((2.2)\). We assume that \( X_0 = X_{2n+2} = 0 \). One one hand, the \((2j+1)\)-th row of \((2.2)\) leads to

\[ \alpha_1X_{2j} + \alpha_3X_{2j+1} + \alpha_1X_{2j+2} = \lambda(\beta_1X_{2j} + \beta_3X_{2j+1} + \beta_1X_{2j+2}), \quad j = 0, 1, \ldots, n, \]  

(2.40)

which is simplified to

\[ (\alpha_1 - \lambda\beta_1)(X_{2j} + X_{2j+2}) = (\lambda\beta_3 - \alpha_3)X_{2j+1}, \quad j = 0, 1, \ldots, n. \]  

(2.41)

Using \((2.1)\) recursively and \( X_0 = X_{2n+2} = 0 \), we calculate

\[ 0 = (\alpha_1 - \lambda\beta_1)X_{2n+2} \]
\[ = (\lambda\beta_3 - \alpha_3)X_{2n+1} - (\alpha_1 - \lambda\beta_1)X_{2n} \]
\[ = \cdots \]
\[ = (\lambda\beta_3 - \alpha_3)\left(\sum_{j=0}^{n}\left(-1\right)^{n-j}X_{2j+1}\right). \]

(2.42)
One the other hand, the \((2j + 2)\)-th row of (2.2) leads to
\[
\alpha_2 X_{2j} + \alpha_1 X_{2j+1} + \alpha_0 X_{2j+2} + \alpha_1 X_{2j+3} + \alpha_2 X_{2j+4} = \lambda (\beta_2 X_{2j} + \beta_1 X_{2j+1} + \beta_0 X_{2j+2} + \beta_1 X_{2j+3} + \beta_2 X_{2j+4}),
\]
where \(j = 0, 1, \ldots, n - 1\). Using (2.41) and \(X_0 = X_{2n+2} = 0\), this equation simplifies to
\[
\tilde{\alpha}_1 X_{2j} + \tilde{\alpha}_0 X_{2j+2} + \tilde{\alpha}_1 X_{2j+4} = \lambda (\tilde{\beta}_1 X_{2j} + \tilde{\beta}_0 X_{2j+2} + \tilde{\beta}_1 X_{2j+4}), \quad j = 0, 1, \ldots, n - 1,
\]
where
\[
\tilde{\alpha}_0 = \alpha_0 + \frac{2 \alpha_1 (1 - \lambda \beta_1)}{\lambda \beta_3 - \alpha_3}, \quad \tilde{\beta}_0 = \beta_0 + \frac{2 \beta_1 (1 - \lambda \beta_1)}{\lambda \beta_3 - \alpha_3},
\]
\[
\tilde{\alpha}_1 = \alpha_2 + \frac{\alpha_1 (1 - \lambda \beta_1)}{\lambda \beta_3 - \alpha_3}, \quad \tilde{\beta}_1 = \beta_2 + \frac{\beta_1 (1 - \lambda \beta_1)}{\lambda \beta_3 - \alpha_3}.
\]

The GMEVP (2.2) with the matrices defined in this theorem is then decomposed to a block form
\[
\begin{pmatrix}
A_{cc} & 0 \\
A_{oc} & A_{oo}
\end{pmatrix}
\begin{pmatrix}
X_c \\
X_o
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix},
\]
where \(X_c = (X_2, \ldots, X_{2n})^T\), \(X_o = (X_1, \ldots, X_{2n+1})^T\), the first block \(A_{cc}\) are formed from (2.44), and the blocks \(A_{cc}\) and \(A_{oo}\) are formed from (2.41). The characteristic polynomial of (2.46) is
\[
\chi(\lambda) = \det(A_{cc}) \det(A_{oo}).
\]
The roots of \(\chi(\lambda) = 0\) give the eigenvalues. Firstly, using (2.42), \(\det(A_{oo}) = 0\) leads to the eigenpair which we denote it as \((\lambda_{2n+1}, X_{2n+1})\) with the eigenvector \(X_{2n+1} = (X_{2n+1,1}, \ldots, X_{2n+1,2n+1})^T\) and
\[
\lambda_{2n+1} = \frac{\alpha_3}{\beta_3}, \quad X_{2n+1,2j+1} = c(-1)^j, \quad X_{2n+1,2j} = 0, \quad j = 1, \ldots, n.
\]

This eigenvalue \(\lambda_{2n+1}\) is simple due to the nonzero block matrix \(A_{cc}\). Now, the first block part of (2.46) that leads to \(A_{cc}X_c = 0\) can be written as a GMEVP with tridiagonal matrices defined using parameters in (2.44). Applying Theorem 1 with \(m = 1\), we obtain that the eigenpairs \((\lambda_j, X_j)\) with \(X_j = (X_{j,1}, \ldots, X_{j,2n})^T\) and
\[
\lambda_j = \frac{\tilde{\alpha}_0 + 2 \tilde{\alpha}_1 \cos(j \pi h)}{\tilde{\beta}_0 + 2 \tilde{\beta}_1 \cos(j \pi h)}, \quad X_{j,2k} = c \sin(j \pi kh), \quad h = \frac{1}{n + 1}, \quad j, k = 1, 2, \ldots, n.
\]

Using (2.45) and rearranging the index of the eigenpairs accordingly (due to the quadratic feature in the eigenvalue, one eigenpair \((\lambda_j, X_j)\) becomes two eigenpairs \((\lambda_{2j-1}, X_{2j-1})\) and \((\lambda_{2j}, X_{2j})\)), we have
\[
\lambda_{2j-1,2j} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad j = 1, 2, \ldots, n,
\]
where
\[
\tilde{a} = \beta_0 \beta_3 - 2 \beta_1^2 + 2 (\beta_2 \beta_3 - \beta_1^2) \cos(j \pi h),
\]
\[
\tilde{b} = 4 \alpha_1 \beta_1 - \beta_0 \alpha_3 - \alpha_0 \beta_3 - 2 (\beta_2 \alpha_3 - 2 \alpha_1 \beta_1 + \alpha_2 \beta_3) \cos(j \pi h),
\]
\[
\tilde{c} = \alpha_0 \alpha_3 - 2 \alpha_1^2 + 2 (\alpha_2 \alpha_3 - \alpha_1^2) \cos(j \pi h).
\]
Associated with the eigenvalue $\lambda_{2j-1}$ and $\lambda_{2j}$, the eigenvectors are then denoted as $X_{2j-1}$ and $X_{2j}$, respectively. From (2.39), we have

$$X_{2j-1,2k} = X_{2j,2k} = c\sin(j\pi kh), \quad j, k = 1, 2, \ldots, n.$$  

The odd entries of the eigenvectors are given by (2.41) as

$$X_{2j,2k+1} = \frac{\alpha_1 - \lambda_{2j-1}\beta_1}{\lambda_{2j-1}\beta_3 - \alpha_3}(X_{2k} + X_{2k+2}),$$

$$X_{2j,2k+1} = \frac{\alpha_1 - \lambda_{2j}\beta_1}{\lambda_{2j}\beta_3 - \alpha_3}(X_{2k} + X_{2k+2}), \quad k = 0, 1, \ldots, n, \quad j = 1, 2, \ldots, n.$$  

This completes the proof.

**Remark 3** (Analytical solutions to quadratic eigenvalue problems). The nonlinear eigenvalue problem $A_{xx}X_x = 0$ defined in the proof of Theorem 7 is a QEVP. In particular, the problem is written as follows

$$\lambda^2 BX + \lambda CX + DX = 0,$$

where $B = G^{(\gamma)}$ with $\gamma_0 = \beta_0\beta_3 - 2\beta_1^2, \gamma_1 = \beta_2\beta_3 - \beta_1^2$, $C = G^{(\gamma)}$ with $\gamma_0 = 4\alpha_1\beta_1 - \beta_0\alpha_3 - \alpha_0\beta_3, \gamma_1 = 2\alpha_1\beta_1 - 2\alpha_2\beta_3 - \alpha_2\beta_3$, and $D = G^{(\gamma)}$ with $\gamma_0 = \alpha_0\alpha_3 - 2\alpha_1^2, \gamma_1 = \alpha_2\alpha_3 - \alpha_1^2$. Herein, $G^{(\gamma)}$ is the notation function in (2.3) with $m = 1$. The analytical eigenpairs are derived based on the Theorem 7 with $m = 1$. Generalization of this result is possible and it is the subject of future work.

### 3. Trigonometric identities

In this section, we derive some trigonometric identities based on the eigenvector-eigenvalue identity that was rediscovered and coined recently in [21]. The eigenvector-eigenvalue identity for the MEVP (2.41) is (see [21] Theorem 1)

$$|X_{j,k}|^2 \prod_{l=1, l \neq j}^n (\lambda_j - \lambda_l) = \prod_{l=1}^{n-1} (\lambda_j - \mu_l^{(k)}), \quad j, k = 1, \ldots, n,$$

where $A$ is a Hermitian matrix with dimension $n$, $(\lambda_j, X_j), j = 1, \ldots, n$, are eigenpairs of $AX = \lambda X$ with normalized eigenvectors $X_j = (X_{j,1}, \ldots, X_{j,n})^T$, and $\mu_l^{(k)}$ is an eigenvalue of $A^{(k)} Y = \mu^{(k)} Y$ with $A^{(k)}$ being the minor of $A$ formed by removing the $k^{th}$ row and column. We generalize this identity for the GMEVPs as follows.

**Theorem 6** (Eigenvector-eigenvalue identity for the GMEVP). Let $A$ and $B$ be Hermitian matrices with dimension $n \times n$. Assume that $B$ is invertible. Let $(\lambda_j, X_j), j = 1, \ldots, n$, be the eigenpairs of the GMEVP (2.3) with normalized eigenvectors $X_j = (X_{j,1}, \ldots, X_{j,n})^T$. Then, there holds

$$|X_{j,k}|^2 \prod_{l=1, l \neq j}^n (\lambda_j - \lambda_l) = \prod_{l=1, l \neq j}^{n-1} \eta_l^{(k)} \prod_{l=1}^{n-1} (\lambda_j - \mu_l^{(k)}), \quad j, k = 1, \ldots, n,$$

where $\mu_l^{(k)}$ is an eigenvalue of $A^{(k)} Y = \mu^{(k)} B^{(k)} Y$ with $A^{(k)}$ and $B^{(k)}$ being minors of $A$ and $B$ formed by removing the $k^{th}$ row and column, respectively, $\eta_l$ is an eigenvalue of $BY = \eta Y$, and $\eta_l^{(k)}$ is an eigenvalue of $B^{(k)} Y = \eta^{(k)} Y$. 

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Proof. We follow the proof of (3.1) for $AX = \lambda X$ while using perturbative analysis in [21 Sect. 2.4]. Firstly, since $B$ is invertible, $\det(B) \neq 0$ and hence $\eta_l \neq 0, l = 1, \cdots, n$. Let $Q(\lambda)$ be the characteristic polynomial of the GMEVP (2.2). Then,

$$Q(\lambda) = \det(\lambda B - A) = \det(B) \det(\lambda I - B^{-1} A) = \det(B) \prod_{l=1}^{n}(\lambda - \lambda_l).$$

(3.3)

The derivative $Q'(\lambda_j)$ of $Q(\lambda)$ at $\lambda = \lambda_j$ is

$$Q'(\lambda_j) = \det(B) \prod_{l \neq j}^{n}(\lambda_j - \lambda_l).$$

(3.4)

Similarly, let $P^{(k)}(\mu^{(k)})$ be the characteristic polynomial of the GMEVP $A^{(k)}Y = \mu^{(k)} B^{(k)} Y$. Then,

$$P^{(k)}(\mu^{(k)}) = \det(B^{(k)}) \prod_{l=1}^{n-1}(\mu^{(k)} - \mu_l^{(k)}).$$

(3.5)

Now, with the limiting argument, we assume that $A$ has simple eigenvalues. Let $\epsilon$ be a small parameter and we define the perturbed matrix

$$A^{\epsilon,k} = A + \epsilon \epsilon_k \epsilon_k^T, \quad k = 1, \cdots, n,$$

(3.6)

where $\{\epsilon_k\}_{k=1}^{n}$ is the standard basis. The perturbed GMEVP is defined as

$$A^{\epsilon,k} X^{\epsilon} = \lambda^{\epsilon,k} B X^{\epsilon}.$$  

(3.7)

Using (3.5) and cofactor expansion, the characteristic polynomial of this perturbed GMEVP can be expanded as

$$Q'(\lambda) = \det(\lambda B - A^{\epsilon,k}) = Q(\lambda) - \epsilon P^{(k)}(\lambda) + O(\epsilon^2).$$

(3.8)

With $X_j$ being a normalized eigenvector, one has

$$X_j^T X_j = 1, \quad X_j^T B X_j = \eta_j, \quad j = 1, \cdots, n.$$ 

(3.9)

Using this normalization, from perturbation theory, the eigenvalue $\lambda_j^\epsilon$ of (3.7) can be expanded as

$$\lambda^{\epsilon,k} = \lambda_j + \frac{\epsilon}{\eta_j} |X_j,k|^2 + O(\epsilon^2).$$

(3.10)

Applying the Taylor expansion and $Q(\lambda_j) = 0$, we rewrite

$$0 = Q'(\lambda^{\epsilon,k}) = Q(\lambda^{\epsilon,k}) - \epsilon P^{(k)}(\lambda^{\epsilon,k}) + O(\epsilon^2)$$

$$= Q(\lambda_j) + \frac{\epsilon}{\eta_j} |X_j,k|^2 Q'(\lambda_j) - \epsilon P^{(k)}(\lambda_j) + O(\epsilon^2)$$

$$= \frac{\epsilon}{\eta_j} |X_j,k|^2 Q'(\lambda_j) - \epsilon P^{(k)}(\lambda_j) + O(\epsilon^2),$$

(3.11)

which the linear term in $\epsilon$ leads to

$$|X_j,k|^2 Q'(\lambda_j) = \eta_j P^{(k)}(\lambda_j).$$

(3.12)

Applying (3.4) and (3.5) with $\det(B) = \prod_{l=1}^{n} \eta_l$ and $\det(B^{(k)}) = \prod_{l=1}^{n-1} \eta_l^{(k)}$ to (3.12) completes the proof.


Remark 4. The identity \((3.2)\) can be rewritten in terms of the characteristic polynomials as \((3.12)\). A similar identity in terms of determinants, eigenvalues, and rescaled eigenvectors was presented in \([22, \text{ eqn. 18}]\) for real-valued matrices. The identity \((3.2)\) is in terms of only eigenvalues and eigenvectors.

Based on these two identities \((3.1)\) and \((3.2)\), one can easily derive the following trigonometric identities. For MEVP, using \((3.1)\) and applying Theorem I with \(m = 1\) and \(B\) being an identity matrix, we have

\[
\frac{2}{n + 1} \sin^2 \frac{k\pi}{n + 1} = \frac{\prod_{j=1}^{n} \left( \cos \frac{k\pi}{n+1} - \cos \frac{j\pi}{n+1} \right)}{\prod_{j=1,j\neq k}^{n} \left( \cos \frac{k\pi}{n+1} - \cos \frac{j\pi}{n+1} \right)}, \quad n \geq 2, \quad k = 1, \ldots, n. \tag{3.13}
\]

We note that this identity is independent of the matrix entries \(a_{00}\) and \(a_{11}\). The left hand side can be written in terms of a cosine function as \(\frac{1}{n+1} (1 - \cos \frac{2\pi}{n+1})\) to have an identity in terms of only cosine functions. For example, let \(n = 2, k = 1\), then the identity boils down to

\[
\frac{1}{2} = \frac{2}{3} \sin^2 \frac{\pi}{3} = \frac{\frac{\cos(\pi/3) - \cos(\pi/2)}{\cos(\pi/3) - \cos(2\pi/3)}}{1/2 + 1/2} = \frac{1}{2}.
\]

Similarly, we have for \(n \geq 2, \quad k = 1, \ldots, n, \quad l = 2, \ldots, n - 1\)

\[
\frac{2}{n + 1} \sin^2 \frac{kl\pi}{n + 1} = \frac{\prod_{j=1}^{t-1} \left( \cos \frac{k\pi}{n+1} - \cos \frac{j\pi}{n+1} \right) \prod_{j=t}^{n-1} \left( \cos \frac{k\pi}{n+1} - \cos \frac{(j-l+1)\pi}{n+1} \right)}{\prod_{j=1,j\neq k}^{n} \left( \cos \frac{k\pi}{n+1} - \cos \frac{j\pi}{n+1} \right)}, \tag{3.15}
\]

If we introduce the notation that \(\prod_{j=1}^{0} \left( \cdot \right) = 1\), then \((3.13)\) can be written as \((3.15)\) with \(l = 1\) or \(l = n\).

For the GMEVP, using \((3.2)\) and applying Theorem I with \(m = 1\), we have for \(n \geq 2, \quad l, k = 1, \ldots, n, \quad l \neq k\)

\[
\frac{2}{n + 1} \sin^2 \frac{kl\pi}{n + 1} = \frac{\prod_{j=1}^{t-1} \left( \beta_0 + 2\beta_1 \cos \frac{j\pi}{n+1} \right) \prod_{j=t}^{n-1} \left( \beta_0 + 2\beta_1 \cos \frac{(j-l+1)\pi}{n+1} \right)}{\prod_{j=1,j\neq k}^{n} \left( \beta_0 + 2\beta_1 \cos \frac{j\pi}{n+1} \right)}, \tag{3.16}
\]

where

\[
\Pi_1 = \prod_{j=1}^{l-1} \left( \frac{\alpha_0 + 2\alpha_1 \cos \frac{k\pi}{n+1}}{\beta_0 + 2\beta_1 \cos \frac{k\pi}{n+1}} - \frac{\alpha_0 + 2\alpha_1 \cos \frac{j\pi}{n+1}}{\beta_0 + 2\beta_1 \cos \frac{j\pi}{n+1}} \right),
\]

\[
\Pi_2 = \prod_{j=l}^{n-1} \left( \frac{\alpha_0 + 2\alpha_1 \cos \frac{k\pi}{n+1}}{\beta_0 + 2\beta_1 \cos \frac{k\pi}{n+1}} - \frac{\alpha_0 + 2\alpha_1 \cos \frac{(j-l+1)\pi}{n+1}}{\beta_0 + 2\beta_1 \cos \frac{(j-l+1)\pi}{n+1}} \right), \tag{3.17}
\]

\[
\Pi_3 = \prod_{j=1,j\neq k}^{n} \left( \frac{\alpha_0 + 2\alpha_1 \cos \frac{k\pi}{n+1}}{\beta_0 + 2\beta_1 \cos \frac{k\pi}{n+1}} - \frac{\alpha_0 + 2\alpha_1 \cos \frac{j\pi}{n+1}}{\beta_0 + 2\beta_1 \cos \frac{j\pi}{n+1}} \right).
\]

It is obvious that \((3.16)\) reduces to \((3.15)\) when \(B\) is an identity matrix (or multiplied by a nonzero constant).

Remark 5. Other similar trigonometric identities can be established. Moreover, Theorems I and II give various analytical eigenpairs. An application of the eigenvector-eigenvalue identity \((3.1)\) along with these analytical results set up a system of equations governing the eigenvalues of the minors of the original matrices. Thus, the eigenvalues of these minors can be found by solving this system of equations.
4. Generalizations

This section concerns the possible generalizations of finding analytical eigenpairs to MEVPs and GMEVPs with more general matrices. We consider the following cases.

4.1. Constant scaling

Let \( c_1 \neq 0, c_2 \neq 0 \) be constants, then the GMEVP \( c_1 AX = \lambda c_2 BX \) has eigenvalues \( \lambda = \tilde{\lambda} c_2/c_1 \) where \( \tilde{\lambda} \) is an eigenvalue of the GMEVP \( AX = \lambda BX \) for certain matrices considered in Section 2. The eigenvectors remain the same. This constant scaling has applications for various numerical spectral approximations of the differential operators. For example, for FEM, the scaling is usually in the form \( (1/h)AX = \lambda h BX \) where \( h \) is the size of a uniform mesh (cf., \([5, 9]\)). For FDM, the scaling is \( c_1 = 1/h^2, c_2 = 1 \) on a uniform grid.

4.2. Powers and products

We have the following observations based on \([5, \text{Section 5}]\). For MEVP of the form (2.1), the powers \( A^k, k \in \mathbb{Z} \) has eigenvalues \( \lambda_j^k \), where \( \lambda_j, j = 1, \cdots, n \) are eigenvalues of (2.1). The eigenvectors remain the same. For the MEVPs \( AX = \lambda X \) and \( BX = \mu X \), if \( A \) commutes with \( B \), that is, \( AB = BA \), then the two MEVPs have the same eigenvectors and the eigenvalues of \( AB \) (or \( BA \)) are \( \lambda \mu \). Additionally, in this case, \( A + B \) has eigenvalues \( \lambda + \mu \).

For GMEVP of the form (2.2), similar results are obtained for the matrix \( B^{-1}A \). If the matrices entries defined in Section 2 are such that \( AB = BA \) and \( B \) is invertible, it is easy to see that

\[
B^{-1}A = AB^{-1}.
\]

With this in mind, the GMEVP

\[
A^k X = \mu B^k X
\]

has eigenvalues \( \mu = \lambda^k \) where \( \lambda \) is an eigenvalue of \( AX = \lambda BX \). Similar results can be obtained for products and additions. We remark that the matrices defined in (2.7) are commutative. More generally, the stiffness and mass matrices (resulting from the isogeometric finite element method) that are regularized according to Theorems 1–4 are commutative.

4.3. Tensor-product matrices

Now, with the insights from the numerical spectral approximation of multi-dimensional Laplace operator; see, for example, \([14, 21]\). We have the following results.

**Theorem 7** (Eigenvalues and eigenvectors for tensor-product matrices). Let \((\lambda_j, X_j), j = 1, \cdots, n, \) be the eigenpairs of the GMEVP (2.2) and \((\mu_k, Y_k), k = 1, \cdots, m, \) be the eigenpairs of the GMEVP \( CY = \mu DY \). Then, the GMEVP

\[
(A \otimes D + B \otimes C)Z = \eta(B \otimes D)Z
\]

has eigenpairs \((\eta_{(j,k)}, Z_{(j,k)})\) with

\[
\eta_{(j,k)} = \lambda_j + \mu_k, \quad Z_{(j,k)} = X_j \otimes Y_k, \quad j = 1, \cdots, n, \quad k = 1, \cdots, m.
\]
Proof. Let \((\lambda, X)\) be a generic eigenpair of \(AX = \lambda BX\) and \((\mu, Y)\) be a generic eigenpair of \(CY = \mu DY\). Let \(Z = X \otimes Y\), we calculate that

\[
(A \otimes D + B \otimes C)Z = (A \otimes D + B \otimes C)(X \otimes Y)
= (AX \otimes DY + BX \otimes CY)
= \lambda BX \otimes DY + BX \otimes \mu DY
= (\lambda + \mu)(BX \otimes DY)
= (\lambda + \mu)(B \otimes D)(X \otimes Y)
= (\lambda + \mu)(B \otimes D)Z,
\]

which completes the proof.

\(\square\)

**Remark 6.** Once the two sets of the eigenpairs are found, either numerically or analytically, the eigenpairs for the GMEVP in the form (4.3) can be derived. A FEM (FDM, SEM, or isogeometric FEM) discretization of the two-dimensional Laplace operator on unit square domain with a uniform tensor-product mesh leads to the GMEVP in the form of (4.3). For three- or higher-dimensional problems, this result can be generalized.

5. Concluding remarks

We first remark that the ideas for finding analytical solutions to the GMEVPs with Toeplitz-regularized and corner-overlapped block-diagonal matrices can be generalized to other problems where a particular solution form is sought. For example, for the GMEVP arising from the Laplace operator on the unit interval with mixing Dirichlet and Neumann boundary conditions, one may seek eigenvectors of the form \(\sin \left(\frac{1}{2} - 1/j + jk\pi h\right)\). In this case, the resulting matrices are symmetric but not persymmetric. Other applications include matrix presentations of differential operators such as the Schrödinger operator in quantum mechanics [23] and the 2n-order operators [24]. Moreover, the results for QEVP can be generalized to other nonlinear eigenvalue problems (NEVPs). By using the results in this work, finding the eigenvalues of certain NEVPs reduces to finding solutions to nonlinear equations that are in terms of only one variable. This is an interesting direction for future work.

The boundary modifications for the Toeplitz-regularized matrices give new insights for designing better numerical methods. For example, the high-order isogeometric finite element method (cf., [3]) produce outliers in the high-frequency region of the spectrum. A method which modifies the boundary terms that lead to the Toeplitz-regularized matrices will be outliers free. This modification improves the overall performance. Similarly, by modifying the terms associated with the boundaries, one can design isogeometric collocation methods to eliminate the outliers in their spectra. For FDM, the structure of the Toeplitz-regularized matrices give insights to the design better approximations near the domain boundaries. Lastly, we remark that the corner-overlapped block-diagonal matrices have applications in the FEMs and the discontinuous Galerkin methods.

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