Parity Decision Tree Complexity and 4-Party Communication Complexity of XOR-functions Are Polynomially Equivalent

Penghui Yao*
CWI, Amsterdam
phyao1985@gmail.com

June 30, 2015

Abstract

In this note, we study the relation between the parity decision tree complexity of a boolean function \( f \), denoted by \( D_\oplus(f) \), and the \( k \)-party number-in-hand multiparty communication complexity of the XOR functions \( F(x_1, \ldots, x_k) \overset{\text{def}}{=} f(x_1 \oplus \cdots \oplus x_k) \), denoted by \( \text{CC}^{(k)}(F) \). It is known that \( \text{CC}^{(k)}(F) \leq k \cdot D_\oplus(f) \) because the players can simulate the parity decision tree that computes \( f \). In this note, we show that

\[
D_\oplus(f) \leq O\left(\text{CC}^{(4)}(F)^5\right).
\]

Our main tool is a recent result from additive combinatorics due to Sanders \[San12\]. As \( \text{CC}^{(k)}(F) \) is non-decreasing as \( k \) grows, the parity decision tree complexity of \( f \) and the communication complexity of the corresponding \( k \)-argument XOR functions are polynomially equivalent whenever \( k \geq 4 \).

Remark: After the first version of this paper was finished, we discovered that Hatami and Lovett had already discovered the same result a few years ago, without writing it up.

*Supported by the European Commission FET-Proactive project Quantum Algorithms (QALGO) 600700.
1 Introduction

Communication complexity and the Log-Rank conjecture for XOR functions: Communication complexity quantifies the minimum amount of communication needed for computation when inputs are distributed among different parties [Yao79, KN96]. In the model of two-party communication, Alice and Bob hold inputs $x$ and $y$, respectively, and they are supposed to compute the value of a function $F(x, y)$ using as little communication as possible. One of the central problems in communication complexity is the Log-Rank conjecture. The conjecture proposed by Lovász and Saks in [LS88] asserts that the communication complexity of $F$ and log rank ($M_F$) are polynomially equivalent for any 2-argument total boolean function $F$, where $M_F = [F(x, y)]_{x,y}$ is the communication matrix of $F$. Readers may refer to [TWXZ13] for more discussion on the conjecture. The conjecture is notoriously hard to attack. It was shown over 30 years [MS82] that log rank ($M_F$) is a lower bound on the deterministic communication complexity of $F$. The state of the art is

$$CC^{(2)}(F) \leq O\left(\sqrt{\text{rank}(M_F) \log \text{rank}(M_F)}\right),$$

where $CC^{(2)}(F)$ stands for the two-party deterministic communication complexity of $F$. It is from a recent breakthrough due to Lovett [Lov14a]. The largest gap between $CC^{(2)}(F)$ and log rank ($M_F$) is $CC^{(2)}(F) \geq \Omega\left(\log \text{rank}(M_F)^{\log_3 6}\right)$ due to Kushilevitz in [NW95].

In [ZS10], Zhang and Shi initiated the study the Log-Rank conjecture for a special class of functions called XOR functions.

Definition 1.1. We say a $k$-argument function $F : \{0,1\}^k \rightarrow \{0,1\}$ is an XOR-function if there exists a function $f : \{0,1\}^n \rightarrow \{0,1\}$ such that $F(x_1, \ldots, x_k) = f(x_1 \oplus \ldots \oplus x_k)$ for any $x_1, \ldots, x_k \in \{0,1\}^n$, where $\oplus$ is bitwise xor.

XOR functions include many important examples, such as Equality and Hamming distance. The communication complexity of XOR functions has been studied extensively in the last decade [Zha09, LZ10, MO10, LLZ11, TWXZ13, Zha14]. A nice feature of XOR functions is that the rank of the communication matrix $M_F$ is exactly the Fourier sparsity of $f$.

Fact 1.2. [BC99] For XOR function $F(x, y) \overset{\text{def}}{=} f(x \oplus y)$, it holds that $\text{rank}(M_F) = \|\hat{f}\|_0$, where $\|\hat{f}\|_0$ is the Fourier sparsity of $f$ (see Section 2 for the definition) and $M_F$ is the communication matrix of $F$.

Therefore, the Log-Rank conjecture for XOR functions is equivalent to the question whether there exists a protocol computing $F$ with communication $\log^{O(1)} \|\hat{f}\|_0$. However, the Log-Rank conjecture is still difficult for this special class of functions. One nice approach proposed in [Zha09] is to design a parity decision tree (PDT) to compute $f$. PDTs allow query the parity of any subset of input variables. For any $k$-argument XOR function $F$ given in Definition 1.1 we can construct a communication protocol by simulating the PDT for $f$, with communication $k$ times the PDT complexity of $f$. It is therefore sufficient to show that $D_{\oplus} (f) \leq \log^{O(1)} \|\hat{f}\|_0$. Using such an approach, the Log-Rank Conjecture has been established for several subclasses of XOR functions [Zha09, MO10, TWXZ13].

One question regarding this approach is whether $D_{\oplus} (f)$ and $CC^{(2)}(F)$ are polynomially equivalent. Is it possible to design a protocol for $F$ much more efficient than simulating the parity decision tree of $f$?
Theorem 1.4. All logarithms in this note are base 2. Given $x, y \in \{0,1\}^n$, we define the inner product $x \cdot y \overset{\text{def}}{=} \sum_{i=1}^{n} x_i y_i \mod 2$. For simplicity, we write $x + y$ for $x \oplus y$.

Conjecture 1.3. There is a constant $c$ such that $\text{CC}^{(2)}(F) = \mathcal{O}(\text{D}_\oplus(F)^c)$ for any boolean function $f : \{0,1\}^n \to \{0,1\}$ and $F(x, y) \overset{\text{def}}{=} f(x \oplus y)$.

If this holds, then the Log-Rank conjecture for XOR-functions is equivalent to a question in parity decision tree. Namely, $\text{D}_\oplus(f) \leq \text{poly log } \left( \|f\|_0 \right)$. In this note, we prove a weaker variant of the above conjecture. Given a total boolean function $f$, we may also consider the communication complexity of the $k$-argument XOR-function $F_k(x_1, \ldots, x_k) \overset{\text{def}}{=} f(x_1 \oplus \cdots \oplus x_k)$ in the model of number-in-hand multiparty communication, which is denoted by $\text{CC}^{(k)}(F_k)$. It is easy to see that $\text{CC}^{(2)}(F_2) \leq \text{CC}^{(3)}(F_3) \leq \ldots$ and $\text{CC}^{(k)}(F_k) \leq k \cdot \text{D}_\oplus(f)$. Our main result in this note is that $\text{CC}^{(k)}(F_k)$ and $\text{D}_\oplus(f)$ are polynomially equivalent whenever $k \geq 4$.

Theorem 1.4. For any boolean function $f : \{0,1\}^n \to \{0,1\}$, we define a 4-argument XOR function by $F(x_1, x_2, x_3, x_4) = f(x_1 \oplus x_2 \oplus x_3 \oplus x_4)$. It holds that

$$\text{D}_\oplus(f) \leq \mathcal{O}(\text{cc}^{(4)}(F)^5).$$

Our techniques

To show the main theorem, it suffices to construct an efficient PDT for $f$ if the communication complexity of $F$ is small. We adapt a protocol introduced by Tsang et al. [TWXZ13]. The main step is to exhibit a large monochromatic affine subspace for $f$ if the communication complexity of $F$ is small. To this end, we adapt the quasipolynomial Bogolyubov-Ruzsa lemma [San12], which says that $4A \overset{\text{def}}{=} A + A + A + A$ contains a large subspace if $A \subseteq \mathbb{F}_2^n$ is large.

Related work

A large body of work has been devoted to the Log-Rank conjecture for XOR functions since it was proposed in [Zha09]. After almost a decade of efforts, the conjecture has been established for several classes of XOR function, such as symmetric functions [Zha09], monotone functions and linear threshold functions [MO10], constant $\mathbb{F}_2$-degree functions [TWXZ13].

A different line of work close to ours is the simulation theorem in [RM99, Zha09, She10, LMWZ15, PW15]. They study the relation between the (regular) decision tree complexity of function $f$ and the communication complexity of $f \circ g^n$ where $g$ is a 2-argument function of small size. The simulation theorem asserts that the optimal protocol for $f \circ g^n$ is to simulate the decision tree that computes $f$ if $g$ is a hard function. Simulation theorems have been established in various cases, when $g$ is bitwise AND or OR [She10], Inner-Product [LMWZ15], Index Function [RM99, PW15]. Our work gives a new simulation theorem when $g$ is an XOR function.

After this work was put online, the author was informed that Hatami and Lovett discovered Theorem 1.4 (using the same idea) a couple of years ago without writing it up. Since our work is independent of theirs, we believe it is worth giving a complete proof to the main theorem.

2 Preliminaries

All logarithms in this note are base 2. Given $x, y \in \{0,1\}^n$, we define the inner product $x \cdot y \overset{\text{def}}{=} \sum_{i=1}^{n} x_i y_i \mod 2$. For simplicity, we write $x + y$ for $x \oplus y$.

Complexity measures. Given a boolean function $f : \{0,1\}^n \to \{0,1\}^n$, it can be viewed as a polynomial in $\mathbb{F}_2$, and $\text{deg}_2(f)$ is used to represent its $\mathbb{F}_2$-degree.
Definition 2.1. Given a function \( f : V \to \mathbb{F}_2 \), where \( V \) is an affine subspace of \( \mathbb{F}_2^n \), the parity certificate complexity of \( f \) on \( x \) is defined to be

\[
C_{\oplus} (f, x) \overset{\text{def}}{=} \min \{ \text{codim} (H) : H \subseteq V \text{ is an affine subspace where } f \text{ is constant and } x \in H \}
\]

where \( \text{codim} (H) \overset{\text{def}}{=} \dim V - \dim H \). The minimum parity certificate complexity for \( b \in \{0, 1\} \) is defined as

\[
C_{\oplus, \min}^b (f) \overset{\text{def}}{=} \min_{x \in f^{-1}(b)} C_{\oplus} (f, x),
\]

and \( C_{\oplus, \min} (f) \overset{\text{def}}{=} \min_x C_{\oplus} (f, x) \).

Definition 2.2. Given a boolean function \( f : \{0, 1\}^n \to \{0, 1\} \). We view it as a polynomial in \( \mathbb{F}_2 \). The linear rank of \( f \), denoted \( \text{rk} (f) \), is the minimum integer \( r \), such that \( f \) can be expressed as \( f = \sum_{i=1}^{r} l_i f_i + f_0 \), where \( \text{deg}_2 (l_i) = 1 \) for \( 1 \leq i \leq r \) and \( \text{deg}_2 (f_i) < \text{deg}_2 (f) \) for \( 0 \leq i \leq r \).

Definition 2.3. A parity decision tree (PDT) for a boolean function \( f : \{0, 1\}^n \to \{0, 1\} \) is a tree with internal nodes associated with a subset \( S \subseteq [n] \) and each leaf associated with an answer in \( \{0, 1\} \). To use a parity decision tree to compute \( f \), we start from the root and follow a path down to a leaf. At each internal node, we query the parity of the bits with the indices in the associated set and follow the branch according to the answer to the query. Output the associated answer when we reach the leaf. The deterministic parity decision tree complexity of \( f \), denoted by \( D_{\oplus} (f) \), is the minimum number of queries needed on a worst-case input by a PDT that computes \( f \) correctly.

Definition 2.4. In the model of number-in-hand multiparty communication, there are \( k \) players \( \{P_1, \ldots, P_k\} \) and a \( k \)-argument function \( F : (\{0, 1\}^n)^k \to \{0, 1\} \). Player \( P_i \) is given an \( n \)-bit input \( x_i \in \{0, 1\}^n \) for each \( i \in [k] \). The communication is in the blackboard model. Namely, every message sent by a player is written on a blackboard visible to all players. The communication complexity of \( f \) in this model, denoted by \( \text{CC}^{(k)} (F) \), is the least number of bits needed to be communicated to compute \( f \) correctly.

One way to design a protocol for the \( k \)-argument XOR-function \( F(x_1, \ldots, x_k) \overset{\text{def}}{=} f(x_1 + \cdots + x_k) \) to simulate a parity decision tree that computes \( f \).

Fact 2.5. Let \( f : \{0, 1\}^n \to \{0, 1\} \) be a boolean function and \( F \) be the \( k \)-argument XOR function defined as \( F(x_1, \ldots, x_k) \overset{\text{def}}{=} f(x_1 + \cdots + x_k) \). It holds that \( \text{CC}^{(k)} (F) \leq k \cdot D_{\oplus} (f) \).

Fourier analysis. For any real function \( f : \{0, 1\}^n \to \mathbb{R} \), the Fourier coefficients are defined as \( \hat{f} (s) \overset{\text{def}}{=} \frac{1}{2^n} \sum_x f(x) \chi_s (x) \) for \( s \in \{0, 1\}^n \), where \( \chi_s (x) \overset{\text{def}}{=} (-1)^{s \cdot x} \). The function \( f \) can be decomposed as \( f = \sum_s \hat{f} (s) \chi_s \). The \( \ell_p \) norm of \( \hat{f} \) for any \( p \geq 1 \) is defined as \( \| \hat{f} \|_p \overset{\text{def}}{=} \left( \sum_s |\hat{f} (s)|^p \right)^{1/p} \). The Fourier sparsity \( \| \hat{f} \|_0 \) is the number of nonzero Fourier coefficients of \( f \).

Let \( V \subseteq \mathbb{F}_2^n \) be an affine subspace and \( f : V \to \mathbb{F}_2 \) be a boolean function. A complexity measure of \( f \) \( m(f) \) is downward non-increasing if \( m(f') \leq m(f) \) for any subfunction \( f' \) obtained by restricting \( f \) to an affine subspace of \( V \). For instance, \( \text{deg}_2 (\cdot) \) is downward non-increasing.

Fact 2.6. \cite{TWXYZ13} If \( \text{rk} (\cdot) \leq m(\cdot) \) for some downward non-increasing complexity measure \( m \), then it holds that \( D_{\oplus} (f) \leq m(f) \cdot \log \| \hat{f} \|_0 \).
Additive combinatorics. Given two sets $A, B \subseteq \mathbb{F}_2^n$ and an element $x \in \mathbb{F}_2^n$, $A + B \overset{\text{def}}{=} \{a + b : a \in A, b \in B\}$ and $x + A \overset{\text{def}}{=} \{x + a : a \in A\}$. For any integer $t$, $tA \overset{\text{def}}{=} A + \ldots + A$ where the summation includes $A$ for $t$ times. Studying the structure of $tA$ for small constant $t$ is one of the central topics in additive combinatorics. Readers may refer to the excellent textbook [TV09]. The following is the famous quasi-polynomial Bogolyubov-Ruzsa lemma due to Sanders [San12]. It asserts that $4A$ contains a large subspace if $A \subseteq \mathbb{F}_2^n$ is large. Readers may refer to the nice exposition [Lov14a] by Lovett.

**Fact 2.7.** [TWXZ13] For all non-constant $f : \mathbb{F}_2^n \to \mathbb{F}_2$, it holds that $\text{rk}(f) \leq C_{\oplus, \min}(f)$.

**3 Main result**

**Lemma 3.1.** Let $1 \leq c \leq n$, $A_1, A_2, A_3, A_4 \subseteq \mathbb{F}_2^n$ be subsets of size at least $2^n - c$. Then there exists an affine subspace $V \subseteq A_1 + A_2 + A_3 + A_4$ of $\mathbb{F}_2^n$ such that

$$\text{codim}(V) = O\left(\log^4(\alpha^{-1})\right).$$

**Proof.** The lemma is trivial if $c \geq n^{1/4}$. We assume that $c < n^{1/4}$. As $|A_1 + A_2| \leq 2^n$, there exists an element $a \in \mathbb{F}_2^n$ such that $a = a_1 + a_2$ for at least $2^{n-2c}$ pairs $(a_1, a_2) \in A_1 \times A_2$. Then we have $|A_1 \cap (A_2 + a)| \geq 2^{n-2c}$. For the same reason, there exists an element $a' \in \mathbb{F}_2^n$ such that $|A_3 \cap (A_4 + a')| \geq 2^{n-2c}$. Note that $|(A_1 \cap (A_2 + a)) + (A_3 \cap (A_4 + a'))| \leq 2^n$. Thus there exists an element $a'' \in \mathbb{F}_2^n$ such that $a'' = a_3 + a_4$ for at least $2^{n-4c}$ pairs $(a_3, a_4) \in (A_1 \cap (A_2 + a)) \times (A_3 \cap (A_4 + a'))$. Set

$$A = A_1 \cap (A_2 + a) \cap ((A_3 \cap (A_4 + a')) + a'') = A_1 \cap (A_2 + a) \cap (A_3 + a'') \cap (A_4 + a'' + a').$$

We have $|A| \geq 2^{n-4c} > 0$ since $c < n^{1/4}$. Thus there exists a subspace $V \subseteq 4A$ of codimension $\text{codim}(V) \leq O\left(c^3\right)$ by Theorem 2.8. Note that $4A \subseteq A_1 + A_2 + A_3 + A_4 + a + a'$. The affine subspace $V + a + a'$ serves the purpose. \qed

We define a downward non-increasing measure which is an upper bound on $\text{rk}(\cdot)$.

**Definition 3.2.** Given a function $f : V \to \mathbb{F}_2$, where $V$ is an affine subspace of $\mathbb{F}_2^n$ and $t \overset{\text{def}}{=} \dim(V)$, let $L : \mathbb{F}_2^t \to \mathbb{F}_2^t$ be an affine map satisfying that $L(\mathbb{F}_2^t) = V$. Set $F : (\mathbb{F}_2^n)^4 \to \mathbb{F}_2$ by $F(x_1, x_2, x_3, x_4) \overset{\text{def}}{=} f(L(x_1 + x_2 + x_3 + x_4))$. The complexity of $f$ is defined to be $M(f) \overset{\text{def}}{=} \text{CC}^4(F)$.

Note that the affine map is invertible. The complexity $M(f)$ does not depend on the choice of the affine map.

**Lemma 3.3.** $M(\cdot)$ is downward non-increasing.
Thus the players simulate the protocol that computes $O(V)$. The main theorem follows.

**Lemma 3.4.** For any $f : V \to \mathbb{F}_2$, where $V$ is an affine subspace of $\mathbb{F}_2^n$, it holds that $C_{\oplus, \min}(f) = O(M(f)^4)$.

**Proof.** We assume w.l.o.g. that $V = \mathbb{F}_2^n$. Let $F(x_1, x_2, x_3, x_4) \stackrel{\text{def}}{=} f(x_1 + x_2 + x_3 + x_4)$. Let $c \stackrel{\text{def}}{=} CC^{(4)}(F)$. The optimal protocol partitions the domain into at most $2^c$ monochromatic hyperrectangles. Thus there exists a monochromatic hyperrectangle $A_1 \times A_2 \times A_3 \times A_4$ satisfying $|A_1 | \times |A_2 | \times |A_3 | \times |A_4 | \geq 2^{n-c}$. Hence $|A_i | \geq 2^n - c$ for $1 \leq i \leq 4$. Using Lemma 3.1, there exists an affine subspace $V \subseteq A_1 + A_2 + A_3 + A_4$ satisfying $\text{codim}(V) = O(c^4)$. It implies that $C_{\oplus, \min}(f) \leq O(c^4)$. The result follows.

Combining Fact 2.7, Lemma 3.3 and Lemma 3.4, we have

$$D_{\oplus}(f) \leq O(M(f)^4 \cdot \log \|f\|_0).$$

By Definition 3.2 $M(f) \leq CC^{(4)}(F)$. Note that $\log \|f\|_0 \leq CC^{(4)}(F)$. The main theorem follows.

**Open problems**

Here we list two open problems towards proving the Log-Rank Conjecture for XOR functions.

1. The most interesting work along this line is to show that the PDT complexity of $f$ and the communication complexity of the corresponding 2-argument XOR-function $F_2$ are polynomially equivalent.

2. Can we extend Theorem 1.4 to the randomized communication complexity?

**Acknowledgement**

I would like to thank Ronald de Wolf for helpful discussion and improving the presentation. I also thank Shengyu Zhang for his comments and Shachar Lovett for informing us about his unpublished proof with Hatami.

**References**

[BC99] Anna Bernasconi and Bruno Codenotti. Spectral analysis of boolean functions as a graph eigenvalue problem. *Computers, IEEE Transactions on*, 48(3):345–351, Mar 1999.
[BSRZW14] Eli Ben-Sasson, Noga Ron-Zewi, and Julia Wolf. Sampling-based proofs of almost-periodicity results and algorithmic applications. In *Proceedings of the 41st international conference on Automata, languages and programming*, ICALP’14, Berlin, Heidelberg, 2014. Springer-Verlag.

[KN96] Eyal Kushilevitz and Noam Nisan. *Communication Complexity*. Cambridge University Press, 1996.

[LLZ11] Ming Lam Leung, Yang Li, and Shengyu Zhang. Tight bounds on communication complexity of symmetric XOR functions in one-way and SMP models. In *Proceedings of the 8th Annual Conference on Theory and Applications of Models of Computation*, TAMC’11, pages 403–408, Berlin, Heidelberg, 2011. Springer-Verlag.

[LMWZ15] Shachar Lovett, Raghu Meka, Thomas Watson, and David Zuckerman. Rectangles are nonnegative juntas. In *Proceedings of the 47th Annual ACM Symposium on Theory of Computing*, STOC ’15, to appear, New York, NY, USA, 2015. ACM.

[Lov14a] Shachar Lovett. Communication is bounded by root of rank. In *Proceedings of the 46th Annual ACM Symposium on Theory of Computing*, STOC ’14, pages 842–846, New York, NY, USA, 2014. ACM.

[Lov14b] Shachar Lovett. An exposition of Sanders’ quasi-polynomial Freiman–Ruzsa theorem. 6(6):1–14, 2014.

[LS88] L. Lovász and M. Saks. Lattices, Möbius functions and communications complexity. In *Foundations of Computer Science, 1988., 29th Annual Symposium on*, pages 81–90, Oct 1988.

[LZ10] Troy Lee and Shengyu Zhang. Composition theorems in communication complexity. In *Proceedings of the 37th International Colloquium Conference on Automata, Languages and Programming*, ICALP’10, pages 475–489, Berlin, Heidelberg, 2010. Springer-Verlag.

[MO10] Ashley Montanaro and Tobias Osborne. On the communication complexity of XOR functions. *CoRR*, abs/0909.3392v2, 2010.

[MS82] Kurt Mehlhorn and Erik M. Schmidt. Las Vegas is better than determinism in VLSI and distributed computing (extended abstract). In *Proceedings of the Fourteenth Annual ACM Symposium on Theory of Computing*, STOC ’82, pages 330–337, New York, NY, USA, 1982. ACM.

[NW95] Noam Nisan and Avi Wigderson. On rank vs. communication complexity. *Combinatorica*, 15(4):557–565, 1995.

[PW15] Toniann Pitassi and Thomas Watson. Deterministic communication vs. partition number. *Electronic Colloquium on Computational Complexity (ECCC)*, 15:050, 2015.

[RM99] Ran Raz and Pierre McKenzie. Separation of the monotone NC hierarchy. *Combinatorica*, 19(3):403–435, 1999.
[San12] Tom Sanders. On the Bogolyubov–Ruzsa lemma. *Analysis and PDE*, 5(3):627–655, 2012.

[She10] Alexander A. Sherstov. On quantum-classical equivalence for composed communication problems. *Quantum Info. Comput.*, 10(5):435–455, May 2010.

[TV09] Terence Tao and Van Vu. *Additive Combinatorics*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2009.

[TWXZ13] Hing Yin Tsang, Chung Hoi Wong, Ning Xie, and Shengyu Zhang. Fourier sparsity, spectral norm, and the log-rank conjecture. In *Proceedings of the 2013 IEEE 54th Annual Symposium on Foundations of Computer Science*, FOCS ’13, pages 658–667, Washington, DC, USA, 2013. IEEE Computer Society.

[Yao79] Andrew C. Yao. Some complexity questions related to distributive computing (preliminary report). In *Proceedings of the eleventh annual ACM Symposium on Theory of Computing*, STOC ’79, pages 209–213, New York, NY, USA, 1979. ACM.

[Zha09] Shengyu Zhang. On the tightness of the Buhrman-Cleve-Wigderson simulation. In Yingfei Dong, Ding-Zhu Du, and Oscar Ibarra, editors, *Algorithms and Computation*, volume 5878 of *Lecture Notes in Computer Science*, pages 434–440. Springer Berlin Heidelberg, 2009.

[Zha14] Shengyu Zhang. Efficient quantum protocols for XOR functions. In *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1878–1885, 2014.

[ZS10] Zhiqiang Zhang and Yaoyun Shi. On the parity complexity measures of boolean functions. *Theor. Comput. Sci.*, 411(26-28):2612–2618, June 2010.