λ-Toeplitz operators with analytic symbols

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Abstract

Let λ be a complex number in the closed unit disc \( \overline{D} \), and \( \mathcal{H} \) be a separable Hilbert space with the orthonormal basis, say, \( \mathcal{E} = \{ e_n : n = 0, 1, 2, \cdots \} \). A bounded operator \( T \) on \( \mathcal{H} \) is called a \( \lambda \)-Toeplitz operator if \( \langle T e_{m+1}, e_{n+1} \rangle = \lambda \langle T e_m, e_n \rangle \) (where \( \langle \cdot, \cdot \rangle \) is the inner product on \( \mathcal{H} \)). The subject arises naturally from a special case of the operator equation

\[ S^* AS = \lambda A + B, \]

where \( S \) is a shift on \( \mathcal{H} \), which plays an essential role in finding bounded matrix \( (a_{ij}) \) on \( l^2(\mathbb{Z}) \) that solves the system of equations

\[
\begin{align*}
    a_{2i,2j} &= p_{ij} + aa_{ij} \\
    a_{2i,2j-1} &= q_{ij} + ba_{ij} \\
    a_{2i-1,2j} &= v_{ij} + ca_{ij} \\
    a_{2i-1,2j-1} &= w_{ij} + da_{ij}
\end{align*}
\]

for all \( i, j \in \mathbb{Z} \), where \( (p_{ij}), (q_{ij}), (v_{ij}), (w_{ij}) \) are bounded matrices on \( l^2(\mathbb{Z}) \) and \( a, b, c, d \in \mathbb{C} \). In this paper, we study the essential spectra for \( \lambda \)-Toeplitz operators when \( |\lambda| = 1 \), and we will use the results to determine the spectra of certain weighted composition operators on Hardy spaces.

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1 Introduction

Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis, say, $\mathcal{E} = \{e_n : n = 0, 1, 2, \cdots \}$. Given $\lambda \in \mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$, a bounded operator $T$ is called a $\lambda$-Toeplitz operator if $\langle Te_{m+1}, e_{n+1} \rangle = \lambda \langle Te_m, e_n \rangle$ (where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{H}$). In terms of the basis $\mathcal{E}$, it is easy to see that the matrix representation of $T$ is given by

$$
\begin{pmatrix}
  a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} & \cdots \\
  a_1 & \lambda a_{0} & \lambda a_{-1} & \lambda a_{-2} & \lambda a_{-3} & \cdots \\
  a_2 & \lambda a_{1} & \lambda^2 a_{0} & \lambda^2 a_{-1} & \lambda^2 a_{-2} & \cdots \\
  a_3 & \lambda a_{2} & \lambda^2 a_{1} & \lambda^3 a_{0} & \lambda^3 a_{-1} & \cdots \\
  a_4 & \lambda a_{3} & \lambda^2 a_{2} & \lambda^3 a_{1} & \lambda^4 a_{0} & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

for some double sequence $\{a_n : n \in \mathbb{Z}\}$, and the boundedness of $T$ clearly implies that $\sum |a_n|^2 < \infty$. Therefore, it is natural to introduce the notation

$$
T = T_{\lambda, \varphi},
$$

where $\varphi \sim \sum_{n=\infty} a_n e^{in\theta}$ belongs to $L^2 = L^2(\mathbb{T})$, the Hilbert space of square integrable functions on the unit circle $\mathbb{T}$, with inner product

$$
\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f \overline{g} d\theta,
$$

and consider $T_{\lambda, \varphi}$ as an operator acting on $H^2$, i.e., the Hardy space defined by

$$
\left\{ f \in L^2 : \int_0^{2\pi} f(e^{i\theta})e^{in\theta} d\theta = 0, \ n < 0 \right\}
$$

by identifying $\mathcal{H}$ with $H^2$ and $e_n$ identified with the function $e^{in\theta}$, $n \geq 0$. Note that $H^2$ can be considered as the subspace of “analytic functions” in $L^2$ since it consists of elements in $L^2$ such that the negative terms of their Fourier coefficients are zero. Also note that when $\lambda = 1$ and $\varphi \in L^\infty = L^\infty(\mathbb{T})$, the matrix of $T_{1, \varphi}$ is the matrix of the bounded Toeplitz operator $T_{\varphi}$ on $H^2$. For the readers who are not familiar with the operator theory on $H^2$, the Toeplitz operator $T_{\varphi}$ with symbol $\varphi \in L^\infty$ is the operator defined by $T_{\varphi} f = P(\varphi f)$,
$f \in H^2$, where $P$ is the projection from $L^2$ on to $H^2$. Here we refer the reader to [8] and [14], both of which are excellent sources of information on the theory of Hardy spaces and Toeplitz operators.

Our interests in $T_{\lambda,\varphi}$ originated from the consideration of the following action on $\mathcal{B}(\mathcal{H})$: Let $S$ be the unilateral shift, i.e., $Se_n = e_{n+1}$, $n = 0, 1, 2, \cdots$, and define the mapping on $\mathcal{B}(\mathcal{H})$:

$$\phi(A) = S^* AS, \ A \in \mathcal{B}(\mathcal{H}).$$

Then it is not difficult to check, from the definition, that $\phi(T_{\lambda,\varphi}) = \lambda T_{\lambda,\varphi}$. Hence, the $\lambda$-Toeplitz operators are precisely the “eigenvectors” for $\phi$ associated with $\lambda$, and notice that the Toeplitz operators are just the special cases associated with $\lambda = 1$. Moreover, since $\|\phi\| \leq 1$ (in fact, the spectrum of $\phi$ is the closed unit disc), we accordingly restrict our attention to the case $|\lambda| \leq 1$. The motivation behind the action $\phi$ which prompted our interests in this subject came from the fact that this type of actions induced by shifts on Hilbert spaces is playing an important role in the study of the bounded matrix $(a_{ij})$ on $l^2(\mathbb{Z})$ (with respect to the canonical basis) which solves the system of equations

$$\begin{cases}
a_{2i,2j} = p_{ij} + aa_{ij} \\
a_{2i,2j-1} = q_{ij} + ba_{ij} \\
a_{2i-1,2j} = v_{ij} + ca_{ij} \\
a_{2i-1,2j-1} = w_{ij} + da_{ij}
\end{cases} \quad (\ast)$$

for all $i, j \in \mathbb{Z}$, where $(p_{ij})$, $(q_{ij})$, $(v_{ij})$, $(w_{ij})$ are bounded matrices on $l^2(\mathbb{Z})$ and $a, b, c, d \in \mathbb{C}$ (In the analysis of this system, however, the shift involved has infinite multiplicity. See [19]). For the details we refer the readers to [11] and [12].

There is a big overlap between $\lambda$-Toeplitz operators and the so-called Toeplitz-composition operators, i.e., operators which can be expressed as the products of Toeplitz operators and composition operators (See, for examples, [15] and [16]). To be specific, let $|\lambda| \leq 1$ and consider the operator $U_\lambda$ defined by $U_\lambda e_n = \lambda^n e_n$, $n = 0, 1, 2, \cdots$. Then $U_\lambda$ can also be considered as the composition operator

$$f \sim \sum_{n=0}^{\infty} a_n e^{i\theta} \rightarrow f \circ \tau \sim \sum_{n=0}^{\infty} a_n \lambda^n e^{i\theta}$$

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on $H^2$ induced by the map $\tau(z) = \lambda z$, $z \in \mathbb{D}$ (or, with symbol $\tau$). Notice here we write the image of $f$ as $f \circ \tau$ because it is convenient to do so (Analytic maps from $\mathbb{D}$ into $\mathbb{D}$ always induce bounded composition operators on $H^2$). For the theory of composition operators, we refer the reader to [7]). When $|\lambda| = 1$, $U_\lambda$ is unitary and it is easy to see that

$$T_{\lambda, \varphi} = U_\lambda T_{\varphi_{\lambda^+}} ,$$

where $T_{\varphi_{\lambda^+}}$ is the Toeplitz operator with symbol

$$\varphi_{\lambda^+} \sim \sum_{n=-\infty}^{\infty} b_n e^{i n \theta}, \quad b_n = \lambda^n a_n \text{ if } n \geq 0 \text{ and } b_n = a_n \text{ if } n < 0.$$

The above identity thus puts $T_{\lambda, \varphi}$ into the category of Toeplitz-composition operators when $|\lambda| = 1$. When $|\lambda| < 1$, $T_{\lambda, \varphi}$ also falls into the same category (or the category of weighted composition operators, see, for example, [3], [4], [6], [9], [10], [17] and [18]) if $\varphi$ (or $\overline{\varphi}$) is in $H^2$. On the other hand, however, there is no reason for us to treat the $\lambda$-Toeplitz operators as a subclass of the Toeplitz-composition operators since in general, when $|\lambda| < 1$, we have

$$T_{\lambda, \varphi} = W_{\varphi_{\lambda^+}, \tau} + W_{\varphi_{-\lambda^+}, \tau}^*,$$

where $W_{\varphi_{\lambda^+}, \tau}$ is the weighted composition operator

$$W_{\varphi_{\lambda^+}, \tau} f := \varphi_{\lambda^+} \cdot (f \circ \tau)$$

and $W_{\varphi_{-\lambda^+}, \tau}^*$ is the adjoint of $W_{\varphi_{-\lambda^+}, \tau}$ (For the definition of the adjoint of an operator, see [5]):

$$W_{\varphi_{-\lambda^+}, \tau}^* f := (P(\varphi_- f)) \circ \tau \quad (W_{\varphi_{-\lambda^+}, \tau} f = \overline{\varphi}_- \cdot (f \circ \tau)), $$

with $\varphi_+ = P \varphi$, $\varphi_- = (I - P) \varphi$ and $\tau(z) = \lambda z$, $\overline{\tau}(z) = \lambda z$, $|z| < 1$ (Here $I$ is the identity map on $L^2$).

Perhaps the question an operator theorist is most likely to ask about this subject is that to what degree are $\lambda$-Toeplitz operators and Toeplitz operators related through the classical results of Toeplitz operators. While it may not be surprising to know that $\lambda$-Toeplitz operators and Toeplitz operators can be more or less connected through the classical Toeplitz operator theory if $|\lambda| = 1$ (e.g., Proposition 2.3 and Theorem 3.1), they are very different when $|\lambda| < 1$. Perhaps
1. For instance, one obvious difference between $\lambda$-Toeplitz operators and Toeplitz operators is that a nontrivial $\lambda$-Toeplitz operator may be compact, while a Toeplitz operator is compact if and only if it is the zero operator. In fact, one can easily show that $T_{\lambda, \varphi}$ is compact (or, even better, in the trace class) if and only if $|\lambda| < 1$ or $\varphi \equiv 0$, and with some more effort, that $T_{\lambda, \varphi}$ is of finite rank if and only if $\lambda = 0$ or $\varphi \equiv 0$ (See [13]). It is also worth mentioning here that if, in addition, $\varphi$ is analytic, then some of the statements about the compactness and the finite rank criteria for $\lambda$-Toeplitz operators follow directly from Gunatillake’s work in weighted composition operators on Hardy spaces (see, e.g., Theorem 1 in [9] and Theorem 2 in [10]).

In this paper, we shall concentrate on investigating the essential spectrum of $T_{\lambda, \varphi}$ when $|\lambda| = 1$, and we will apply the results to show that the spectrum of $T_{\lambda, \varphi}$ equals

$$\{ \mu \in \mathbb{C} : \mu^q \in \text{cl}(\hat{\phi}_{X_+}(\mathbb{D})) = \text{closure of } \hat{\phi}_{X_+}(\mathbb{D}) \}$$

if $\varphi$ (or, what is the same, $\varphi_{X_+}$) is analytic and $C^1$, and $\lambda = e^{2i\pi(p/q)}$ with the rational number $p/q$ in lowest terms, where

$$\phi_{X_+} = \prod_{j=0}^{q-1} \varphi_{X_+} \circ \tau_j, \quad \tau(e^{i\theta}) = \lambda e^{i\theta}$$

and $\hat{\phi}_{X_+}$ is the analytic function, called the Gelfand transform of $\phi_{X_+}$, defined by (See Chap. 6, [8])

$$\hat{\phi}_{X_+}(z) := \frac{1}{2\pi} \int_0^{2\pi} \frac{\phi_{X_+}(e^{i\theta})}{1 - ze^{-i\theta}} d\theta, \quad |z| < 1.$$

This result generalizes, under this $C^1$ restriction for the symbols, a well-known result for the spectra of analytic Toeplitz operators due to Wintner (See Theorem 7.21, [8] and [21]). As a consequence, we obtain the spectrum for the weighted composition operator $W_{\varphi, \rho}$ with $\varphi$ being continuously differentiable on $\mathbb{T}$ and analytic, and $\rho$ being an elliptic analytic automorphism on $\mathbb{D}$ of finite order (Theorem 3.2).

2 Essential spectrum of $T_{\lambda, \varphi}, \ |\lambda| = 1$

We begin with a brief introduction about the Fredholm operators on Hilbert spaces. Let $A$ be a bounded operator on $\mathcal{H}$ and $A^*$ be its adjoint. We say
that $A$ is Fredholm (or $A \in \Phi$) if $\dim(\ker A), \dim(\ker A^*) < \infty$ and both $A$ and $A^*$ have closed ranges. For $A \in \Phi$, the index of $A$, denoted by $\text{ind}(A)$, is defined by the integer

$$\dim(\ker A) - \dim(\ker A^*).$$

The essential spectrum for a bounded operator $A$, on the other hand, is defined by

$$\sigma_e(A) = \{ \alpha \in \mathbb{C} : A - \alpha \notin \Phi \},$$

and it is a standard property that $\sigma_e(A) = \sigma_e(A + K)$, whenever $K$ is a compact operator (For the Fredholm theory of bounded operators, we refer the readers to [3]).

Now Let $|\lambda| = 1$. The following lemma gives the connection between $\sigma_e(T_{\lambda, \varphi})$ and $\sigma_e(T_{\varphi, \lambda^+})$ when $\lambda$ is of finite order:

**Lemma 2.1** For any bounded $\lambda$-Toeplitz operator $T_{\lambda, \varphi}$, we have

$$T_{\lambda, \varphi}^k = U_{\lambda}^k T_{\varphi, \lambda, \varphi} \circ \tau^{k-1} T_{\varphi, \lambda, \varphi} \circ \tau^{k-2} \cdots T_{\varphi, \lambda, \varphi} \quad (\tau(e^{i\theta}) = \overline{\lambda} e^{i\theta})$$

for $k = 1, 2 \cdots$, and as a consequence, we have

$$\sigma_e(T_{\lambda, \varphi})^q = \sigma_e(T_{\varphi, \lambda^+}) = \sigma_e(T_{\varphi, \lambda^+})$$

if $\lambda = e^{2\pi (p/q)}$ and $\varphi_{\lambda^+} \in C(\mathbb{T})$.

**Proof** For any $\psi \in L^\infty(\mathbb{T})$, we have

$$\psi U_{\lambda} f = U_{\lambda}((\psi \circ \tau)f) \quad (f \in H^2).$$

Since obviously $U_{\lambda}$ commutes with $P$, we have $T_{\psi} U_{\lambda} = U_{\lambda} T_{\psi, \varphi}$, and therefore the first identity follows by repeated application of this to

$$T_{\lambda, \varphi}^k = \underbrace{U_{\lambda} T_{\varphi, \lambda, \varphi} U_{\lambda} T_{\varphi, \lambda, \varphi} \cdots U_{\lambda} T_{\varphi, \lambda, \varphi}}_{k}. $$

Now suppose that $\lambda = e^{2\pi (p/q)}$ and $\varphi_{\lambda^+} \in C(\mathbb{T})$. Since $U_{\lambda} = I$ and $\varphi_{\lambda^+} \in C(\mathbb{T})$, we have

$$T_{\lambda, \varphi}^q = T_{\varphi, \lambda^+, \varphi} \circ \tau^{q-1} T_{\varphi, \lambda^+, \varphi} \circ \tau^{q-2} \cdots T_{\varphi, \lambda^+, \varphi} = T_{\varphi, \lambda^+} + K$$

for $q = 1, 2, \cdots$.
with some compact operator $K$ since Toeplitz operators with continuous symbols commute modulo compact operators (See Proposition 7.22 in [8]). Therefore by the spectral mapping theorem the proof is complete. Notice that Lemma 2.1 implies, in particular, 

$$\|T_{\lambda,\varphi}^k\|_e = \|\varphi_{\lambda+}^{(k)}\|_\infty,$$

where

$$\varphi_{\lambda+}^{(k)} = \prod_{j=0}^{k-1} \varphi_{\lambda+} \circ \tau^j.$$

Hence, in terms of ergodic theory, we obtain, through the spectral radius formula, that (See [20])

**Corollary 2.2** The essential spectral radius of $T_{\lambda,\varphi}$ equals

$$\sup_{\mu \in M_\tau(T)} \exp \left( h_\tau(\nu) + \int_{\tau} \log |\varphi_{\lambda+}| d\nu \right)$$

if $\varphi_{\lambda+} \in C(\mathbb{T})$, where $M_\tau(T)$ is the set of $\tau$-invariant Borel probability measures on $\mathbb{T}$ and $h_\tau(\mu)$ is the entropy of $\mu$ with respect to $\tau$.

One important fact pointed out by Lemma 2.1 is that if $\mu \in \sigma_e(T_{\lambda,\varphi})$ and $\varphi_{\lambda+}$ is continuous, then $\varphi_{\lambda+} - \mu^\tau$ is not invertible. We will show that the converse is true if, in addition, $\varphi_{\lambda+}$ is $C^1$ (Proposition 2.3). For this, we need some detailed information from the Fredholm theory for the Toeplitz-Composition algebra, i.e., the $C^*$-algebra generated by the Toeplitz algebra $\mathcal{A}$ and the composition operator $C_\rho$ induced by an analytic automorphism $\rho$ on $\mathbb{D}$, and $\mathcal{A} \rtimes_\rho \mathbb{Z}$ is the crossed product of the action of symbols on $\mathbb{T}$ (Theorem 2.1, [15]). Note that Jury's result generalizes the famous Coburn's exact sequence (See [1], [2])

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{A} C_\rho \rightarrow C(\mathbb{T}) \rtimes_\rho \mathbb{Z} \rightarrow 0$$

where $\mathcal{A} C_\rho = C^*(\mathcal{A}, C_\rho)$ is the Toeplitz-Composition algebra generated by $\mathcal{A}$ and the composition operator $C_\rho$ is induced by an analytic automorphism $\rho$ on $\mathbb{D}$, and $C(\mathbb{T}) \rtimes_\rho \mathbb{Z}$ is the crossed product of the action of symbols on $\mathbb{T}$ (Theorem 2.1, [15]).
Then he proceeded to show (Theorem 3.1, [15]) that if $\rho$ is an elliptic auto-
morphism of order $q$, then $T = \sum_{j=0}^{q-1} T_{j} C_{\rho}^{j}$, with $f_{j} \in C^{1}(\mathbb{T})$ for each $j$, is
Fredholm if and only if the $C(\mathbb{T})$-valued determinant
\[
h_{T} = \begin{vmatrix}
    f_{0} & f_{1} & \cdots & f_{q-1} \\
    f_{q-1} \circ \rho & f_{0} \circ \rho & \cdots & f_{q-2} \circ \rho \\
    \cdots & \cdots & \cdots & \cdots \\
    f_{1} \circ \rho^{q-1} & \cdots & \cdots & f_{0} \circ \rho^{q-1}
\end{vmatrix}
\]
is nonvanishing on $\mathbb{T}$, and, in this case, the Fredholm index $\text{ind}(T)$ equals
\[
\text{wn}(h_{T}) = -\frac{1}{2\pi i q} \int_{\mathbb{T}} \frac{dh_{T}}{h_{T}},
\]
where $\text{wn}(h_{T})$ is the \textit{winding number} for the curve $h_{T}(\mathbb{T})$.

We now apply Jury’s result to get

**Proposition 2.3** Let $\lambda = e^{2\pi i (p/q)}$ with the rational number $p/q$ in lowest
terms. Suppose, in addition, that $\phi_{\lambda}^{-}$ is continuously differentiable. Then
the essential spectrum of $T_{\lambda,\varphi}$ is
\[
\{ \mu \in \mathbb{C} : \phi_{\lambda}^{-} - \mu^{q} \text{ is not invertible} \},
\]
i.e., $\mu \in \sigma_{e}(T_{\lambda,\varphi})$ if and only if $\mu^{q} \in \phi_{\lambda}^{-}(\mathbb{T})$, where
\[
\phi_{\lambda}^{-} = \prod_{j=0}^{q-1} \varphi_{\lambda}^{-} \circ \tau_{j}, \quad \tau(e^{i\theta}) = \lambda e^{i\theta}.
\]
Moreover, we have $\text{ind}(T_{\lambda,\varphi} - \mu) = -q^{-1} \text{wn}(\phi_{\lambda}^{-} - \mu^{q})$.

**Proof** Since the unitary $U_{\tau}$ is in fact $C_{\tau}$, and $U_{\tau} T_{\lambda,\varphi} = T_{\varphi_{\lambda}^{-},\tau}$, we see that
$T_{\lambda,\varphi} - \mu$ is Fredholm if and only if $T_{\varphi_{\lambda}^{-},\tau} - \mu C_{\tau}$ is Fredholm. Therefore,
by applying Jury’s criterion to $f_{0} = \varphi_{\lambda}^{-}, f_{j} = 0$ for $j = 1, 2, \cdots, q-2,$
$f_{q-1} = -\mu$, $\rho = \tau$ and $T = T_{\varphi_{\lambda}^{-},\tau} - \mu C_{\tau}$, one has

\[
h_{T} = \begin{vmatrix}
    \varphi_{\lambda}^{-} & 0 & \cdots & \cdots & \cdots & -\mu \\
    -\mu & \varphi_{\lambda}^{-} \circ \tau & 0 & \cdots & 0 \\
    0 & -\mu & \cdots & \cdots & 0 \\
    0 & 0 & \cdots & \cdots & 0 \\
    0 & 0 & \cdots & \cdots & -\mu & \varphi_{\lambda}^{-} \circ \tau^{q-1} \\
    \varphi_{\lambda}^{-} \circ (\varphi_{\lambda}^{-} \circ \tau) \cdots (\varphi_{\lambda}^{-} \circ \tau^{q-1}) + (-1)^{q+1}(-\mu)^{q}
\end{vmatrix}
\]

\[
= \phi_{\lambda}^{-} - \mu^{q},
\]

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and from this and the Jury’s index formula, the index formula for $T_{\lambda,\varphi} - \mu$ is derived immediately.

3 Applications to some weighted composition operators

Let $f \in H^2$ and recall the Gelfand transform of $f$:

$$\hat{f}(z) := \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - ze^{-i\theta}} d\theta, \ |z| < 1.$$ 

In 1929, A. Wintner showed in [21] that if $\varphi \in L^\infty$ is analytic and $\hat{\varphi}$ is bounded on $D$ (or, simply, $\varphi \in H^\infty$), then the spectrum of the Toeplitz operator $T_\varphi$ is $\text{cl}(\hat{\varphi}\lambda(D))$. Notice that $T_\varphi = T_{1,\varphi}$ and the order of 1 is one.

We now generalize this result to $\lambda$-Toeplitz operators, with an additional assumption that $\varphi$ is smooth:

**Theorem 3.1** Let $\lambda = e^{2\pi i (p/q)}$ with the rational $p/q$ in lowest terms. Then $\sigma(T_{\lambda,\varphi})$, the spectrum of $T_{\lambda,\varphi}$, equals

$$\{ \mu \in \mathbb{C} : \mu^q \in \text{cl}(\hat{\varphi}_\lambda(D)) \}$$

if $\varphi$ is analytic and continuously differentiable.

**Proof** We will prove the theorem by showing that $T_{\lambda,\varphi} - \mu$ is invertible if and only if $T_{\varphi,\lambda} - \mu^q$ is. The conclusion then follows directly from Wintner’s theorem for analytic Toeplitz operators.

It is well-known in the Fredholm theory of Toeplitz operators that if $T_\psi$ is Fredholm, then $T_\psi$ is invertible if and only if $\text{ind}(T_\psi) = 0$, and $\text{ind}(T_\psi) = -\text{wn}(\psi)$ if $\psi$ is continuous (See Corollary 7.25, [K]). On the other hand, by elementary operator theory, we know that if $A$ is not invertible, then only one of the following three possibilities may occur: (1) $A \notin \Phi$ (2) $A \in \Phi, \ \text{ind}(A) \neq 0$ (3) $A \in \Phi, \ \text{ind}(A) = 0$, but not invertible. Hence the spectrum of $T_{\varphi,\lambda}$ equals

$$\left\{ \alpha : T_{\varphi,\lambda} - \alpha \notin \Phi \right\} \cup \left\{ \alpha : T_{\varphi,\lambda} - \alpha \in \Phi, \ \text{ind}(T_{\varphi,\lambda} - \alpha) \neq 0 \right\}$$

$$= \sigma_e(T_{\varphi,\lambda}) \cup \left\{ \alpha : T_{\varphi,\lambda} - \alpha \in \Phi, \ \text{wn}((\varphi,_{\lambda} - \alpha) \neq 0 \right\}$$
since $\phi_{\lambda+}$ is continuous (Note that $T_{\psi} - \alpha = T_{\psi - \alpha}$). So, by the Fredholm criterion and the index formula in Proposition 2.3, the only thing we need to show is that if $T_{\lambda, \varphi} - \mu \in \Phi$ and $\text{ind}(T_{\lambda, \varphi} - \mu) = 0$, then $T_{\lambda, \varphi} - \mu$ is invertible.

Since $\varphi_{\lambda+} = \varphi \circ \tau$ if $\varphi$ is analytic, $\varphi \in H^\infty$ if and only if $\varphi_{\lambda+} \in H^\infty$. Therefore,

$$T^q_{\lambda, \varphi} = T_{\varphi_{\lambda+} \circ \tau^{k-1}} T_{\varphi_{\lambda+} \circ \tau^{k-2}} \cdots T_{\varphi_{\lambda+}} = T_{\varphi_{\lambda+}} \mod \mathcal{K}$$

in Lemma 2.1 is actually an equality. Thus we have

$$\sigma(T_{\lambda, \varphi})^q = \sigma(T_{\varphi_{\lambda+}}).$$

Now by Proposition 2.3 if $T_{\lambda, \varphi} - \mu \in \Phi$ and $\text{ind}(T_{\lambda, \varphi} - \mu) = 0$, then $\varphi_{\lambda+} - \mu^q$ is invertible and $\text{wn}(\varphi_{\lambda+} - \mu^q) = 0$. This means that, $T_{\varphi_{\lambda+}} - \mu^q$ must be invertible, which implies immediately that $\mu \notin \sigma(T_{\lambda, \varphi})$.

**Remark.** It is not difficult to see the connection between the set $\hat{\varphi}_{\lambda+}(D)$ and $\varphi_{\lambda+}$ since for $f \sim \sum_0^\infty a_n e^{i\theta} \in H^2$, the Taylor series of $\hat{f}$ at the origin is $\sum_0^\infty a_n z^n$, $|z| < 1$. Furthermore, by a theorem of Fatou, $\hat{f}(re^{i\theta}) \to f(e^{i\theta})$ a.e. $\theta$ and in $L^2$ (See [8] and [14]). Hence we may consider $f$ as the “boundary value” of $\hat{f}$. Let us use this understanding in the following example:

Let $r(A)$ and $r_e(A)$ denote the spectral radius and essential spectral radius of $A$, respectively. Let $\lambda = e^{i2\pi/3}$, and consider $\varphi(e^{i\theta}) = e^{i\theta} - \sqrt{2}$. Then $\varphi_{\lambda+}(e^{i\theta}) = e^{i4\pi/3} e^{i\theta} - \sqrt{2}$ and $\varphi_{\lambda+}(e^{i\theta}) = e^{i3\theta} - 2$. Therefore

$$r_e(T_{\lambda, \varphi}) = r(T_{\lambda, \varphi}) = \sqrt{3} < 1 + \sqrt{2} = ||\varphi_{\lambda+}||_\infty = ||T_{\lambda, \varphi}||.$$

This means that the identity

$$r(T_{\varphi}) = r_e(T_{\varphi}) = ||T_{\varphi}||, \ \varphi \in C(T)$$

for Toeplitz operators is no longer valid for $\lambda$-Toeplitz operators.

The same example also shows that a well-known result about the connectedness of the spectrum of Toeplitz operators (due to Windom) no longer holds for $\lambda$-Toeplitz operators since $\hat{\varphi}_{\lambda+}(z) = z^3 - 2$ and so by Theorem 3.1

$$\sigma(T_{\lambda, \varphi}) = \{\mu \in \mathbb{C} : \mu^3 \in \text{cl}(\hat{\varphi}_{\lambda+}(D))\} = \{\mu \in \mathbb{C} : |\mu^3 + 2| \leq 1\}.$$

Clearly, this set is not connected.

We finish with an application of Theorem 3.1 to certain weighted composition operators:
**Theorem 3.2** Let $\varphi \in H^\infty \cap C^1$ and $\rho$ be an elliptic analytic automorphism of $\mathbb{D}$ of order $q$. Then the spectrum of the weighted composition $W_{\varphi, \rho}$ is

$$\{ \mu \in \mathbb{C} : \mu^q \in cl(\hat{\psi}(\mathbb{D})) \},$$

where

$$\psi = \prod_{k=0}^{q-1} (\varphi \circ \zeta^{-1}) \circ \tau_k,$$

$\zeta$ is an analytic automorphism of $\mathbb{D}$ such that $\zeta^{-1}(\rho(\zeta(z))) = \lambda z$ for some $\lambda \in \mathbb{T}$ with order $q$, and $\tau(z) = \overline{\lambda}z$.

**Proof** Since $\rho$ is an elliptic analytic automorphism of $\mathbb{D}$ of order $q$, we can find analytic automorphism $\zeta$ of $\mathbb{D}$ such that $\zeta^{-1}(\rho(\zeta(z))) = \lambda z$, $|z| < 1$ for some $\lambda \in \mathbb{T}$ with order $q$. Now since the composition operator $C_\zeta$ is invertible and $C_{\zeta^{-1}} = C_{\zeta^{-1}}$, we have

$$C_{\zeta^{-1}}W_{\varphi, \rho}C_\zeta = C_{\zeta^{-1}}W_{\varphi, \rho}C_\zeta = W_{\varphi \circ \zeta^{-1}, \tau}, \quad \tau(z) = \lambda z,$$

which means that $W_{\varphi, \rho}$ is similar to $W_{\varphi \circ \zeta^{-1}, \tau}$.

Now since $\varphi \in H^\infty \cap C^1$ implies $\varphi \circ \zeta^{-1} \in H^\infty \cap C^1$, $W_{\varphi \circ \zeta^{-1}, \tau}$ is the same with $T_{\lambda, \varphi \circ \zeta^{-1}}$, and hence the result follows from Theorem 3.1 since we also have $f_{\lambda, \tau} = f \circ \tau$ if $f$ is analytic. \hfill \blacksquare

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