The Instabilities of Bianchi Type IX Einstein Static Universes

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Abstract

We investigate the stability of the Einstein static universe as a non-LRS Bianchi type IX solution of the Einstein equations in the presence of both non-tilted and tilted fluids. We find that the static universe is unstable to homogeneous perturbations of Bianchi type IX to the future and the past.

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1 Introduction

The stability of the Einstein static universe is a problem with a history as old as relativistic cosmology. The static universe was the first cosmological solution of general relativity to be found, by Einstein [1], and the study of its stability, implicitly by Lemaître [2, 3], and then explicitly by Eddington [4], played an important role in establishing the concept of the expanding universe. These authors only showed that the Einstein static solution was unstable to spatially homogeneous and isotropic perturbations with a limited range of perfect fluid equations of state. Nevertheless, after 1930, it was widely believed that the Einstein static universe was always unstable and would evolve into a state of universal contraction or expansion if perturbed [5]. In 1967, Harrison [6] showed that all the perturbations which render the dust-filled Einstein static universe unstable are in fact oscillatory if the dust is replaced by black body radiation, and also showed that the Einstein static universe is neutrally stable against small inhomogeneous perturbations for a range of perfect fluids. Similar results were found for homogeneous and isotropic perturbations of all amplitudes by Gibbons [7, 8] who also provided a thermodynamic perspective and showed that if \( dp/d\rho > 1/5 \), where \( p \) is the pressure and \( \rho \) is the density of the contents of the
universe, then the static universe is stable. The physical reason for this surprising conclusion is that the Einstein static space is compact and when the sound speed exceeds $1/\sqrt{5}$ the Jeans length exceeds the size of the universe and the instability growth is quenched by the pressure. These studies are quite limited and were extended by Barrow, Ellis, Maartens and Tsagas [9] to consider the whole spectrum of inhomogeneous scalar, vector and tensor perturbations to the static universe, as well as the presence of a self-interacting scalar field. They found that the static universe is neutrally stable against all inhomogeneous adiabatic scalar perturbations if $dp/d\rho > 1/5$ and confirmed that the dust ($p = 0$) model is always unstable. Any initial vector perturbations remain frozen in and so there is neutral stability against vortical perturbations for all equations of state at linear order. Likewise, there is neutral stability with respect to transverse-traceless tensor perturbations for all equations of state. Two caveats are to be noted. Since the Einstein static universe has compact space sections and many Killing vectors, any perturbation analysis is susceptible to the problems of linearisation instability. In effect, Einstein static is a conical point in the space of all solutions to the Einstein equations and special higher-order constraints must be checked to ensure that the leading terms in any perturbative series expansion around it are the leading-order terms in series that converge to true solutions of the Einstein equations, see for example refs [10, 11, 9, 12]. Also, as an aside, we note that if ‘ghost’ fields with negative densities ($\rho < 0$) are permitted then Einstein’s static universe is a stable solution of the field equations and exact isotropic and homogeneous cosmological solutions can be found that exhibit stable oscillations of arbitrarily small amplitude around the static state [13] – in effect these are oscillating universes with non-zero minimum and maximum radii.

Einstein static universes are also particular solutions in a wide range of extensions of general relativity. Existence conditions are known, for example, for gravitation theories in which the lagrangian is an arbitrary function of the scalar, Ricci, and Riemann curvature invariants [14, 15]. There have been many studies of the Einstein static stability in other gravity theories [16, 17, 18, 19, 20, 21, 22, 23, 24], and from a thermodynamic point of view [25]. There are also possible cosmological initial states that rely upon stabilities of the static universe, for example that of the emergent universe scenario [20], which requires the Einstein static universe to be stable to the infinite past and expand like an Eddington-Lemaître universe to the future. Eddington favoured a cosmological scenario of this sort, with a infinite geometrical past but only a finite thermodynamic past when the universe was significantly out of equilibrium.

These earlier studies of the stability of Einstein static universes have examined the behaviour of small and large amplitude homogeneous and isotropic (conformal) perturbations, and small amplitude inhomogeneous perturbations. However, in our earlier study of the stability properties in general relativity, [9], we also pointed out that a key ingredient in understanding the stability of Einstein static is to understand its response to long-wavelength homogeneous gravitational waves of Bianchi type IX. We showed that there was instability with respect to the mode that controlled the volume expansion in the non-tilted
perfect fluid case. In this paper we will provide a full analysis of the stability of the Einstein universe with respect to the non-LRS Bianchi type IX degrees of freedom for tilted and non-tilted perfect fluids. This problem reduces to studying the stability properties of the special Einstein static universe solution of the Bianchi type IX (‘Mixmaster’) Einstein equations. This anisotropic metric describes the most general spatially homogeneous closed universe with Einstein static as a special case. We will show that the static universe is unstable to the past and to the future and we expect the solution to be unstable in other gravity theories as well if Mixmaster exact solutions exist. In particular, cosmological scenarios with a past-eternal ‘initial’ Einstein static are unstable in general relativity.

2 Non-tilted Bianchi Type IX Universes

We will proceed by showing that the Einstein static universe is a particular exact Bianchi type IX solution of the Einstein equations for spatially homogeneous cosmologies and then determine its stability. This will show any effects of the homogeneous gravitational degrees of freedom present in type IX universes. This metric possesses anisotropic spatial curvature and is the only member of the Bianchi classification of homogeneous spaces to contain the closed Friedmann universes as a special case. We will examine the situation with a non-tilted perfect fluid source first.

We follow Ellis and MacCallum [27] to write down the equations in a group invariant orthonormal frame. The geometry of Bianchi type IX space-time is characterised by the kinematic quantities of the homogeneous spatial hypersurface: bulk expansion rate $H$, trace-free shear tensor $\sigma_{\alpha\beta}$, an auxiliary three-vector $\Omega_{\alpha}$ that measures the rotation of the frame with respect to Fermi-propagated one and a symmetric three-tensor $n_{\alpha\beta}$ which determine the internal geometry of the spatial hypersurface. The spatial scalar curvature and trace-free part of Ricci tensor are given by

$$3R = -n_{\mu\nu}n^{\mu\nu} + \frac{1}{2}(n_\mu^\mu)^2$$
$$3S_{\alpha\beta} = 2n_{\alpha\mu}n_\beta^\mu - n_\mu^\mu n_{\alpha\beta} - \frac{1}{3} (2n_{\mu\nu}n^{\mu\nu} - (n_\mu^\mu)^2)\delta_{\alpha\beta}.$$

When the fluid is non-tilted, we can diagonalise $\sigma_{\alpha\beta}$ and $n_{\alpha\beta}$ simultaneously, which also implies $\Omega_\alpha = 0$. We assume the fluid satisfies

$$p = (\gamma - 1)\rho$$

with a constant $\gamma$. The cosmological constant $\Lambda$ can be regarded as another fluid with $\rho_\Lambda = -p_\Lambda = \Lambda$. Using the notations in [29], there are eight variables $\{H, \sigma_{\pm}, n_{1,2,3}, \rho, \rho_\Lambda\}$. For Bianchi type IX, $n_{1,2,3}$ are all non-zero and of a same signature, here taken to be positive.

Note that Einstein static is not a special case of the Kantowski-Sachs closed anisotropic universes, which have $S^1 \times S^2$ space sections.
To obtain dimensionless equations, let us define a new variable

\[ D \equiv \sqrt{H^2 + \frac{1}{4}(n_1 n_2 n_3)^2}, \]

which we will use to normalise the system of equations for Bianchi IX, following Hewitt, Ugga and Wainwright [29]. The advantage of using this variable is that the normalisation with respect to \( D \) is well-defined everywhere in the type IX state space since \( D \neq 0 \) as long as \( n_{1,2,3} \neq 0 \). Particularly, it is non-zero for Einstein static universe where \( H = 0 \).

We introduce dimensionless variables as follows:

\[ \hat{H} \equiv \frac{H}{D}, \quad \hat{\Sigma}_\pm \equiv \frac{\sigma_\pm}{D}, \quad \hat{N}_i \equiv \frac{n_i}{D}, \quad \hat{\Omega}_m \equiv \frac{\rho}{3D^2}, \quad \hat{\Omega}_\Lambda \equiv \frac{\Lambda}{3D^2}. \]  

(2)

We also introduce a new time variable \( \tau \) by defining

\[ \frac{dt}{d\tau} = \frac{1}{D} > 0 \]

and denote \( d/d\tau \) by \( ' \).

We may use the normalised equations given in [29] with a slight modification to accommodate two fluids. First of all, the evolution of \( D \) decouples as

\[ D' = -(1 + \tilde{q})\hat{H}D \]  

(3)

where \( \tilde{q} \) is algebraically determined by the normalised variables through Raychaudhuri equation

\[ \tilde{q} = 2(\hat{\Sigma}_+^2 + \hat{\Sigma}_-^2) + \frac{1}{2} (3\gamma - 2)\hat{\Omega}_M - \hat{\Omega}_\Lambda. \]  

(4)

The rest of the evolution equations read

\[ \hat{H}' = -(1 - \hat{H}^2)\tilde{q}, \quad \hat{\Sigma}_+ = -(2 - q)\hat{H}\hat{\Sigma}_\pm - \tilde{S}_\pm, \]  

(5)

\[ \hat{N}_1' = (\hat{H}\tilde{q} - 4\hat{\Sigma}_+) \hat{N}_1, \quad \hat{N}_2' = (\hat{H}\tilde{q} + 2\hat{\Sigma}_+ + 2\sqrt{3}\hat{\Sigma}_-) \hat{N}_2, \]  

(6)

\[ \hat{N}_3' = (\hat{H}\tilde{q} + 2\hat{\Sigma}_+ - 2\sqrt{3}\hat{\Sigma}_-) \hat{N}_3, \quad \hat{\Omega}_M' = (2\tilde{q} - 3\gamma + 2)\hat{H}\hat{\Omega}_M, \]  

(7)

\[ \hat{\Omega}_\Lambda' = 2(\tilde{q} + 1)\hat{H}\hat{\Omega}_\Lambda. \]  

(8)

The components of spatial Ricci tensor are given by

\[ \tilde{S}_+ = \frac{1}{6} \left[ (\hat{N}_2 - \hat{N}_3)^2 - \hat{N}_1(2\hat{N}_1 - \hat{N}_2 - \hat{N}_3) \right], \]  

\[ \tilde{S}_- = \frac{1}{2\sqrt{3}} \left[ (\hat{N}_2 - \hat{N}_3)(\hat{N}_1 - \hat{N}_2 - \hat{N}_3) \right]. \]
There are two constraint equations
\[ 1 = \tilde{\Sigma}_+^2 + \tilde{\Sigma}_-^2 + \tilde{V} + \tilde{\Omega}_M + \tilde{\Omega}_\Lambda, \] (12)
\[ 1 = \tilde{H}^2 + \frac{1}{4} \left( \tilde{N}_1 \tilde{N}_2 \tilde{N}_3 \right)^{\frac{2}{3}}, \] (13)
where
\[ \tilde{V} = \frac{1}{12} \left[ \tilde{N}_1^2 + \tilde{N}_2^2 + \tilde{N}_3^2 - 2\tilde{N}_1 \tilde{N}_2 - 2\tilde{N}_2 \tilde{N}_3 - 2\tilde{N}_3 \tilde{N}_1 + 3 \left( \tilde{N}_1 \tilde{N}_2 \tilde{N}_3 \right)^{\frac{2}{3}} \right]. \] (14)

3 The Stability of Einstein Static Universes

The only equilibrium point in the interior of type IX invariant set corresponds to the Einstein static solution, given by
\[ \tilde{H} = \tilde{\Sigma}_\pm = 0, \quad \tilde{N}_{1,2,3} = 2, \quad \tilde{\Omega}_M = \frac{2}{3\gamma}, \quad \tilde{\Omega}_\Lambda = 1 - \frac{2}{3\gamma}. \] (15)

We are interested in linear stability around it.

**Linearisation:**

We denote a small deviation of a quantity \( x \) from its value at the Einstein static solution, \( x_0 \), by \( \delta x = x - x_0 \). From (13), we have
\[ \delta \tilde{N}_1 + \delta \tilde{N}_2 + \delta \tilde{N}_3 = 0, \] (16)
and, with (12), this means that
\[ \delta \tilde{V} = 0, \quad \delta \tilde{\Omega}_M = -\delta \tilde{\Omega}_\Lambda. \] (17)

From the definitions of \( \tilde{q} \) and \( \tilde{S}_\pm \), we have
\[ \delta \tilde{q} = \frac{1}{2} (3\gamma - 2) \delta \tilde{\Omega}_M - \delta \tilde{\Omega}_\Lambda \] (18)
\[ \delta \tilde{S}_+ = -\frac{1}{3} (2\delta \tilde{N}_1 - \delta \tilde{N}_2 - \delta \tilde{N}_3) = \delta \tilde{N}_2 + \delta \tilde{N}_3 \] (19)
\[ \delta \tilde{S}_- = -\frac{1}{\sqrt{3}} (\delta \tilde{N}_3 - \delta \tilde{N}_2). \] (20)

We replace \( \delta \tilde{N}_{1,2,3} \) by \( \delta \tilde{S}_\pm \) as the variables for the linearised equations. Note that the relation (16) was used to reduce the number of variables. Finally, the evolution of the perturbations are given by
\[ \delta \tilde{H}' = \frac{3\gamma}{2} \delta \tilde{\Omega}_\Lambda \] (21)
\[ \delta \tilde{\Omega}_\Lambda' = 2 \left( 1 - \frac{2}{3\gamma} \right) \delta \tilde{H} \] (22)
\[ \delta \tilde{\Sigma}_\pm' = -\delta \tilde{S}_\pm \] (23)
\[ \delta \tilde{S}_\pm' = 8 \delta \tilde{S}_\pm. \] (24)
The linear stability of the static solution is therefore decided by the eigenvalues:

\[ \lambda = \pm \sqrt{3\gamma - 2}, \quad \pm i2\sqrt{2}, \quad \pm i2\sqrt{2}. \]  

We see that there is always an instability in the direction of increasing and decreasing \( \tau \) time for \( \gamma > 2/3 \) because of the two real eigenvalues with opposite signs. Note that this mode is associated with the volume expansion (it is the one identified in ref [9]), namely the FLRW mode and would still be present in the isotropic limit where \( \tilde{N}_{1,2,3} = \tilde{\Sigma}_{\pm} = 0 \). The effects of the anisotropic curvature are not decisive at this order and produce the two pairs of purely imaginary eigenvalues. We note that two of the imaginary eigenvalues correspond to an axisymmetric mode \( (\delta \tilde{\Sigma}_+ - \delta \tilde{S}_+) \) mentioned by [30] who studied only the locally rotationally symmetric (LRS) case, although their analysis considered occurrence of chaos [31] following evolution away from the Einstein static universe.

### 4 The Extension to Tilted Fluids

We can add a tilt to the fluid velocity [33], so that the unit normal of the homogeneous hypersurfaces \( n^a \) and the fluid 4-velocity \( u^a \) are not coincident, but differ by the addition of a peculiar velocity vector \( v^a \), with associated scalar \( v \), so that

\[ u^a = \Gamma(n^a + v^a) \quad \Gamma = (1 - v^2)^{-\frac{1}{2}}. \]

Denote the energy density of the fluid seen by the observer \( u^a \) by \( \rho \) and it satisfies the equation of state (1). If we decompose its energy-momentum tensor with respect to \( n^a \) as

\[ T_{ab} = \mu n_a n_b + 2q(a n_b) + p(g_{ab} + n_a n_b) + \pi_{ab} \]

such that \( q_n n^a = \pi_n a^a = 0 \) and \( \pi_{ab} n^b = 0 \), then these quantities are given by

\[ \mu = \left( 1 + \gamma \Gamma^2 v^2 \right) \rho, \]  
\[ p = \left( \gamma - 1 + \frac{\gamma}{3} \Gamma^2 v^2 \right) \rho, \]  
\[ q_n = \gamma \Gamma^2 \rho v_n, \]  
\[ \pi_{ab} = \gamma \Gamma^2 \left[ v_a v_b - \frac{1}{3} v^2 (g_{ab} + n_a n_b) \right]. \]

In this case, the orthogonal equations in [29] are no longer usable and we start from the general equations for an arbitrary group invariant orthonormal frame given in [32]. Since the fluid velocity gives rise to an energy flux with respect to the observer \( u^a \), from the equations (1.93) in [32], we cannot diagonalise \( \sigma_{\alpha\beta} \) and \( n_{\alpha\beta} \) simultaneously. Although we could use the remaining gauge freedom to choose the spatial triad to diagonalise \( n_{\alpha\beta} \), this does not turn out to be the best gauge choice for analyzing the stability of isotopic equilibrium points, as we will see. Instead, we proceed without fixing the gauge, which means that we
need to modify the definition of the normalisation factor \( D \). Noting that for an arbitrary matrix \( A \),

\[
\frac{d}{dx} \det A = \det A \operatorname{tr} \left( A^{-1} \frac{dA}{dx} \right),
\]

(30)
it is easy to see from the equation (1.96) in [32] that

\[
\frac{d}{dt} (\det n_{\alpha\beta}) = \det n_{\alpha\beta} (n^{-1})_{\mu\nu} \frac{dn_{\mu\nu}}{dt} = -3H \det n_{\alpha\beta}
\]

and so if we redefine \( \hat{D} \) as

\[
\hat{D} \equiv \sqrt{H^2 + \frac{1}{4} (\det n_{\alpha\beta})^2},
\]

then the normalisation will work as before. Let us first define the normalised geometrical quantities by

\[
\hat{H} \equiv \frac{H}{\hat{D}}, \quad \hat{\Sigma}_{\alpha\beta} \equiv \frac{\sigma_{\alpha\beta}}{\hat{D}}, \quad \hat{N}_{\alpha\beta} \equiv \frac{n_{\alpha\beta}}{\hat{D}}.
\]

There now exists an angular velocity, \( \Omega_\alpha \), of the frame with respect to the Fermi-propagated frame. This is non-dynamical and is just a manifestation of the gauge freedom. It is normalised by

\[
\hat{R}_\alpha \equiv \frac{\Omega_\alpha}{\hat{D}}.
\]

The fluid density parameter seen by the observer \( n^a \) is defined by

\[
\hat{\Omega}_M \equiv \frac{\mu}{3 \hat{D}^2} = \left(1 + \gamma \Gamma^2 v^2\right) \frac{\rho}{3 D^2}.
\]

Since the tilt velocity, \( v_\alpha \), is already dimensionless, it will not be normalised. The cosmological constant represents another dynamical degree of freedom through the variable

\[
\hat{\Omega}_\Lambda \equiv \frac{\Lambda}{3 \hat{D}^2}.
\]

We can now derive a new system of equations from the equations (1.90)-(1.100) in [32].

First of all, the evolution of \( \hat{D} \) decouples as before;

\[
\hat{D}' = -(1 + \hat{q}) \hat{H}^2 \hat{D}
\]

(31)
where \( ' \) now denotes \( d/d\hat{\tau} = \hat{D}^{-1} d/dt \) and \( \hat{q} \) is determined by

\[
\hat{q} = 2 \hat{\Sigma}^{\mu\nu} \hat{\Sigma}_{\mu\nu} + \frac{3 \gamma - 2 - (\gamma - 2)v^2}{2(1 + (\gamma - 1)v^2)} \hat{\Omega}_M - \hat{\Omega}_\Lambda.
\]

(32)
The rest of the Einstein equations read

\[
\begin{align*}
\dot{\Sigma}_{\alpha\beta} & = -(2 - \tilde{q})\hat{H}\hat{\Sigma}_{\alpha\beta} + 2\epsilon^{\mu\nu}_{\alpha}(\hat{\Sigma}_{\beta\mu})\hat{R}_{\nu} - \dot{\hat{\Sigma}}_{\alpha\beta} + \dot{\hat{\Pi}}_{\alpha\beta}, \\
1 & = \hat{\Sigma}^{\mu\nu}\hat{\Sigma}_{\mu\nu} + \dot{\hat{V}} + \hat{\Omega}_M + \hat{\Omega}_\Lambda, \\
0 & = \frac{3\gamma\hat{\Omega}_M}{2(1 - (\gamma - 1)v^2)}v_\alpha + \epsilon_\alpha^\mu\dot{\hat{\Sigma}}^{\lambda\mu}_\beta\hat{N}_{\beta\nu},
\end{align*}
\]

(33)

where \(\epsilon_{\mu\nu\sigma}\) is the spatial Levi-Civita symbol and

\[
\begin{align*}
\dot{\hat{S}}_{\alpha\beta} & = 2\hat{N}_\alpha^\mu\hat{N}_{\beta\mu} - \frac{2}{3}\hat{N}_{\mu\nu}\hat{N}^{\mu\nu}\delta_{\alpha\beta} - \hat{N}^\lambda_\mu\left[\hat{N}_{\alpha\beta} - \frac{1}{3}\hat{N}_\mu^\mu\delta_{\alpha\beta}\right], \\
\dot{\hat{\Pi}}_{\alpha\beta} & = \frac{3\gamma\hat{\Omega}_M}{2(1 - (\gamma - 1)v^2)}\left[v_\alpha v_\beta - \frac{1}{3}v^2\delta_{\alpha\beta}\right], \\
\dot{\hat{V}} & = \frac{1}{4}\left(\det\hat{N}_{\alpha\beta}\right)^{\frac{3}{2}} + \frac{1}{6}\hat{N}_{\mu\nu}\hat{N}^{\mu\nu} - \frac{1}{12}\left(\hat{N}_\mu^\mu\right).
\end{align*}
\]

(36)

The evolution for the matter variables is obtained by substituting (26)-(29) into the contracted Bianchi identities:

\[
\begin{align*}
\dot{\hat{\Omega}}_M & = 2(1 + \hat{q})\hat{H}\hat{\Omega}_M - \frac{\gamma}{2(1 - (\gamma - 1)v^2)}\left[(3 + v^2)\hat{H} + \hat{\Sigma}_{\alpha\beta}v^\alpha v^\beta\right]\hat{\Omega}_M, \\
v_\alpha' & = \frac{1}{2}\left(\hat{\Sigma}_{\alpha\beta}v^\beta - \epsilon_{\alpha\mu\nu}\hat{R}^{\mu\nu} + \epsilon_{\alpha\mu\nu}\hat{N}_\beta^{\mu\beta}v^\nu\right), \\
\dot{\hat{\Omega}}_\Lambda & = 2(1 + \hat{q})\hat{H}\hat{\Omega}_\Lambda
\end{align*}
\]

(40)

Finally, by the definition of the normalisation, we have

\[
\begin{align*}
\hat{H}' & = -(1 - \hat{H}^2)\hat{q}, \\
1 & = \hat{H}' + \frac{1}{4}\left(\det\hat{N}_{\alpha\beta}\right)^{\frac{3}{2}}.
\end{align*}
\]

(43)

The normalised fluid density, \(\hat{\Omega}_M\), is usually eliminated by the constraint (34) and then there are 16 dynamical variables (taking into account the traceless condition for \(\hat{\Sigma}_{\alpha\beta}\)). We still have 3 degrees of gauge freedom and 4 constraints ((35) and (44)), so the dynamical system is 9-dimensional, which is consistent with the counting of the ref [34] because now we have one more degree of freedom coming from addition of the cosmological constant to the 8 in general single-fluid tilted models.

To find equilibrium points in the system, we need consider the gauge. Unless we exhaust this freedom, an equilibrium point can be time dependent. A typical
choice is to diagonalize $\hat{N}_{\alpha\beta}$. It leads to the following defining equations for the non-dynamical $\hat{R}_\alpha$:

$$
\begin{align*}
\hat{R}_1 &= \frac{\hat{\Sigma}_{23}(\hat{N}_{22} + \hat{N}_{33})}{\hat{N}_{22} - \hat{N}_{33}} \\
\hat{R}_2 &= \frac{\hat{\Sigma}_{13}(\hat{N}_{33} + \hat{N}_{11})}{\hat{N}_{33} - \hat{N}_{11}} \\
\hat{R}_3 &= \frac{\hat{\Sigma}_{12}(\hat{N}_{11} + \hat{N}_{22})}{\hat{N}_{11} - \hat{N}_{22}}.
\end{align*}
$$

If we substitute those expressions into the evolution equations, they become singular when the denominators vanish, which includes the isotropic case. Thus this gauge is not suited for the stability analysis of Einstein static universe. The situation is the same for a gauge where $\hat{\Sigma}_{\alpha\beta}$ is diagonal.

Fortunately, in the Einstein static universe all the dynamical variables have to be time independent since all the gauge dependent variables are zero. While there is ambiguity in the values of $\hat{R}_\alpha$ which are arbitrary (potentially time-dependent) since they merely represent the rotation of the spatial frame aside from it, the equilibrium point is gauge invariantly characterised by

$$
\dot{H} = 0, \quad \dot{\hat{\Sigma}}_{\alpha\beta} = 0, \quad \dot{\hat{N}}_{\alpha\beta} = 2\delta_{\alpha\beta}, \quad \hat{\Omega}_M = \frac{2}{3}\gamma = 1 - \hat{\Omega}_\Lambda, \quad v_\alpha = 0.
$$

Let us consider a perturbation around this background without fixing the gauge. In this way, we expect to have 12 eigenvalues taking into account the four constraints mentioned above. From now on, the quantities without $\delta$ prefixes are understood to take the background values. The constraint (35) becomes

$$
2\delta v_\alpha = -\epsilon^{\mu\nu}_{\alpha} \delta \hat{\Sigma}^{\beta}_{\mu} \hat{N}_{\beta\nu} \equiv 0.
$$

Thus, the velocity perturbation to linear order automatically vanishes, and so $\delta\Pi_{\alpha\beta} = 0$. Secondly, the constraint (44) implies

$$
0 = \delta (\det \hat{N}_{\alpha\beta}) = \frac{\partial \det \hat{N}_{\alpha\beta}}{\partial \hat{N}_{\mu\nu}} \delta \hat{N}_{\mu\nu} = \det \hat{N}_{\alpha\beta} \text{tr} \left( (\hat{N}^{-1})^\rho_{\alpha} \frac{\partial \hat{N}_{\rho\beta}}{\partial \hat{N}_{\mu\nu}} \right) \delta \hat{N}_{\mu\nu}
$$

where we used (30) again. Noting that

$$
(\hat{N}^{-1})^\alpha_{\alpha\beta} = \frac{1}{2} \delta_{\alpha\beta}
$$

and

$$
\frac{\partial \hat{N}_{\rho\sigma}}{\partial \hat{N}^\mu_{\mu\nu}} = \delta^\mu_{\rho} \delta^\nu_{\sigma},
$$

9
we can see that
\[ \delta \hat{N}_\mu^\nu = 0, \]
namely, \( \delta \hat{N}_{\alpha\beta} \) is trace-free. This ensures that \( \delta \hat{V} = 0 \). It’s also easy to see that \( \delta \hat{S}_{\alpha\beta} = 2\delta \hat{N}_{\alpha\beta} \). Therefore, the linearised evolution equations are quite simple:

\[
\begin{align*}
\delta \hat{\Sigma}_{\alpha\beta}' &= -2\delta \hat{N}_{\alpha\beta} + 2e_{(\alpha}^{\mu\nu} \delta \hat{\Sigma}_{\beta)\mu} \hat{R}_{\nu}, \\
\delta \hat{N}_{\alpha\beta}' &= 4\delta \hat{\Sigma}_{\alpha\beta} + 2e_{(\alpha}^{\mu\nu} \delta \hat{N}_{\beta)\mu} \hat{R}_{\nu}, \\
\delta \hat{H}' &= \frac{3\gamma}{2} \delta \hat{\Omega}_\Lambda, \\
\delta \hat{\Omega}_\Lambda' &= 2 \left( 1 - \frac{2}{3\gamma} \right) \delta \hat{H}.
\end{align*}
\]

The equations decouple into two. The scalar part is exactly the same as the non-tilted case and gives one positive and one negative eigenvalue and so the Einstein static universe is also unstable under the same conditions as in the non-tilted case. Therefore there is instability of the static solution for increasing and decreasing times. The tensorial part can be brought into a standard form

\[
\begin{align*}
\delta \hat{\Sigma}_{\alpha\beta}' &= -2\delta \hat{N}_{\alpha\beta} \\
\delta \hat{N}_{\alpha\beta}' &= 4\delta \hat{\Sigma}_{\alpha\beta}
\end{align*}
\]

by an appropriate rotation of the spatial frame, leaving five pairs of coupled equations, and therefore five sets of eigenvalues \( \pm i 2\sqrt{2} \). Although not all of them are physical as we haven’t fix the gauge, this is unimportant because all the eigenvalues are purely imaginary. This means the nature of the tensorial perturbations is also the same as in the non-tilted case.

5 Conclusions

In this study of the stability of the Einstein static universe against Bianchi type IX modes we have extended the earlier studies of ref [9], that considered only the destabilizing effects of a single homogeneous mode in the presence of comoving fluids, and those of ref [30], who considered only the evolution of LRS type IX models with non-tilted fluid. We investigated the general type IX evolution in the vicinity of the Einstein static model in the presence of a fluid with non-tilted and tilted perfect fluids with \( \rho + 3p > 0 \). We also found that the imaginary eigenvalues corresponding to the perturbative effects of anisotropic curvature (Mixmaster modes) and fluid tilt generalise the oscillatory behaviour of the finite wavelength vector and tensor perturbations found in early studies of small amplitude perturbations.
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