F-theory Duals of M-theory on $G_2$ Manifolds from Mirror Symmetry

Adil Belhaj

High Energy Physics Laboratory, Physics Department, Faculty of sciences
Avenue Ibn Battouta, PO Box 1014, Rabat, Morocco

March 24, 2022

Abstract

Using mirror pairs $(M_3, W_3)$ in type II superstring compactifications on Calabi-Yau threefolds, we study, geometrically, F-theory duals of M-theory on seven manifolds with $G_2$ holonomy. We first develop a way for getting Landau Ginzburg (LG) Calabi-Yau threefolds $W_3$, embedded in four complex dimensional toric varieties, mirror to sigma model on toric Calabi-Yau threefolds $M_3$. This method gives directly the right dimension without introducing non dynamical variables. Then, using toric geometry tools, we discuss the duality between M-theory on $S^1 \times \frac{M_3}{\mathbb{Z}_2}$ with $G_2$ holonomy and F-theory on elliptically fibered Calabi-Yau fourfolds with $SU(4)$ holonomy, containing $W_3$ mirror manifolds. Illustrating examples are presented.
1 Introduction

Over the few past years, there has been an increasing interest in studying string dualities. This interest is due to the fact that this subject allows to explore several connection between different models in string theory. Interesting examples are mirror symmetry between pairs of Calabi-Yau manifolds in type II superstrings [1, 2, 3, 4] and strong/weak coupling duality among this type and heterotic superstrings on $K3 \times T^2$ [5]. The most important consequence of the study of string duality is that all superstring models are equivalent in the sense that they correspond to different limits in the moduli space of the same theory, called M-theory [6, 7, 8, 9]. The latter, which is considered nowadays as the best candidate for the unification of the weak and strong coupling sectors of superstring models, is described, at low energies, by an eleven dimensional supergravity theory.

More recently, a special interest has been given to the study of the compactification of the M-theory on seven real manifolds $X_7$ with non trivial holonomy providing a potential point of contact with low energy semi realistic physics in our world [9, 10]. In particular, one can obtain four dimensional theory with $N = 1$ supersymmetry by compactifying M-theory on $R^{1,3} \times X_7$ where $X_7$ a seven manifold with $G_2$ exceptional holonomy [10-23]. This result can be understood from the fact that the $G_2$ group is the maximal subgroup of $SO(7)$ for which a eight dimensional spinor of it can be decomposed as a fundamental of $G_2$ and one singlet. In this regards, the $N = 1$ four dimensional resulting physics depends on geometric properties of $X_7$. For instance, if $X_7$ is smooth, the low energy theory contains, in addition to $N = 1$ supergravity, only abelian gauge group and neutral chiral multiplets. In particular, one has $b_2(X_7)$ abelian vector multiplets and $b_3(X_7)$ massless neutral chiral multiplets, $b_i$ denote the Betti numbers of $X_7$. The non abelian gauge symmetries with chiral fermions can be obtained by considering limits where $X_7$ develops singularities [17, 18, 21]. The $N = 1$ theory in four dimensions can be also obtained using alternative ways. A way is to consider the $E_8 \times E_8$ heterotic superstring compactifications. In this way, the compact manifold is a Calabi-Yau threefold with SU(3) holonomy with an appropriate choice of vector bundle over it braking the $E_8 \times E_8$ gauge symmetry [24].

Another way, which is dual to the heterotic compactification, is to compactify F-theory on elliptically fibered Calabi-Yau fourfolds with SU(4) holonomy group [25,26,27,28]. At this level, one might naturally ask the following questions. Is there a duality between M-theory on seven $G_2$ manifolds and F-theory on Calabi-Yau fourfolds and what will be the geometries behind this duality?
In this paper, we address these questions using toric geometry and mirror pairs in type II superstrings propagating on Calabi-Yau threefolds. In particular, we discuss the duality between M-theory on $G_2$ manifolds and F-theory on elliptically fibred Calabi-Yau fourfolds with $SU(4)$ holonomy. In this study, we consider M-theory on spaces of the form $\frac{S^1 \times M_3}{\mathbb{Z}_2}$ with $G_2$ holonomy where $M_3$ is a local Calabi-Yau threefolds described physically by $N = 2$ sigma model in two dimensions. The F-theory duals of such models can be obtained by the compactification on $\frac{T^2 \times W_3}{\mathbb{Z}_2}$, where $(M_3, W_3)$ are mirror pairs in type II superstring compactifications. More precisely, using toric geometry tools, we first develop a way for getting LG Calabi-Yau threefolds $W_3$ mirror to sigma model on toric Calabi-Yau $M_3$. This method is based on solving the mirror constraint eqs for LG theories in terms of the toric data of sigma model on $M_3$. Then we give a toric description of the above mentioned duality. In particular, we propose a special $\mathbb{Z}_2$ symmetry acting on the toric geometry angular variables producing seven manifolds with K3 fibrations in $G_2$ manifolds compactifications.

The organization of this paper is as follows. In section 2, using mirror symmetry in type II superstrings, we propose a possible duality between M-theory on $G_2$ manifolds and F-theory on elliptically fibered Calabi-Yau fourfolds involving mirror pairs of Calabi-Yau threefolds $(M_3, W_3)$. In section 3 we develop a method for getting LG Calabi-Yau threefolds $W_3$ mirror to sigma model on toric Calabi-Yau $M_3$. This method is based on solving the mirror constraint eqs for LG Calabi-Yau theories in terms of the toric data of sigma model on $M_3$, giving directly the right dimension of the mirror geometry without introducing non dynamical variables. In this way, the mirror LG Calabi-Yau threefolds $W_3$ can be described as hypersurfaces in four dimensional weighted projective spaces $\mathbb{WP}^4$, depending on the toric data of $M_3$. In section 4 we give a toric description of the above mentioned duality. In particular, we propose a special $\mathbb{Z}_2$ symmetry acting on the toric geometry coordinates leading to $G_2$ manifolds with K3 fibrations. Then we discuss others examples where $\mathbb{Z}_2$ acts trivially on mirror pairs $(M_3, W_3)$. In the last section, we give our conclusion.

2 On F-theory duals of M-theory on $G_2$-manifolds in four dimensions

In this section we want to study the duality between M-theory on $G_2$-manifolds and F-theory on elliptically fibered Calabi-Yau fourfolds. To start, recall that the duality between M-theory and F-theory was studied using different ways. For instance, this can be achieved using Mayr work based on the local mirror symmetry and special limits in the elliptic compactification
of F-theory on Calabi-Yau manifolds [29,30]. Alternative approach can be done using the Hořava–Witten compatification on spaces of the form $\frac{\mathbb{R}^1}{\mathbb{Z}_2} \times Y$, where $Y$ is a Calabi-Yau threefolds, giving rise to $N = 1$ supersymmetry in four dimensions [31]. The latter involves a weak coupling limit given by the heterotic superstring compactified on Calabi-Yau threefolds which may have F-theory dual on Calabi-Yau fourfolds. However, in this work we want to introduce manifolds with $G_2$ holonomy in the game. In particular we would like to discuss a new duality between M-theory on $G_2$-manifolds and F-theory on elliptically fibred Calabi-Yau fourfolds, with SU(4) holonomy group in four dimensions with $N = 1$ supersymmetry. Before going ahead, let us start by the first possible equivalence between M-theory and F-theory. This can appear in nine dimensions by help of T-duality in type II superstrings. Roughly speaking, this duality can be rewritten, using the above mentioned theories, as follows

$$\text{M-theory on } S^1 \times S^1(R) = \text{F-theory on } T^2 \times S^1\left(\frac{1}{R}\right),$$

where $R$ and $\frac{1}{R}$ are the radius of type II one circle compactifications. In this case, the $Sl(2,\mathbb{Z})$ symmetry in type IIB superstring can also have a geometric realization in terms of the M-theory on elliptic curve $T^2$. Duality (2.1) can be pushed further for describing the same phenomenon involving spaces that are more complicated than a circle, such as Calabi-Yau spaces in which the T-duality will be replaced by the mirror transformation. Indeed, using mirror symmetry duality in the Calabi-Yau compatifications, eq(2.1) can be extended to

$$\text{M-theory on } S^1 \times M_3 = \text{F-theory on } T^2 \times W_3,$$

where $(M_3, W_3)$ are mirror pairs manifolds whose Hodge numbers $h^{1,1}$ and $h^{2,1}$ satisfy

$$h^{1,1}(M_3) = h^{2,1}(W_3)$$
$$h^{2,1}(M_3) = h^{1,1}(W_3).$$

In this way, the complex (Kahler) structure moduli space of $M_3$ is identical to the Kahler (complex) structure moduli space of $W_3$ and the above four dimensional models are equivalent to type II superstrings compactified on mirror pairs $(M_3, W_3)$. Thus, equation (2.2) describes models with eight supercharges in four dimensions, ie $N = 2$ 4D. In what follows, we want to relate this duality to M-theory on manifolds with $G_2$ holonomy. However, to do this one has to break the half of the supersymmetry and should look for the expected holonomy groups which needed in both sides. It turns out that there are some possibilities to realize the first requirement. One way is to use the result of the string compactifications on Calabi-Yau manifolds with Ramond-Ramond fluxes [32]. Another method of doing, we are interested in
here, is to consider the modding of the above duality by $\mathbb{Z}_2$ symmetry. Note, in passing, that this operation has been used in many cases in string theory compatifications to break the half of the supercharges, in particular, in the case of type IIA propagating on $T^4$, which known by an orbifold limit of $K3$ surfaces. The four dimensional M-theory/F-theory dual pairs with $N = 1$ supersymmetry can be obtained by $\mathbb{Z}_2$ modding of corresponding dual pairs with $N = 2$ given in (2.2). Using this procedure, the resulting space in M-theory compactification is now a quotient space of the following form

$$X_7 = \frac{S^1 \times M_3}{\mathbb{Z}_2},$$  \hspace{1cm} (2.3)

where $\mathbb{Z}_2$ acts on $S^1$ as a reflection and non trivially on the Calabi-Yau threefolds $M_3$. The holonomy group of this geometry is now larger than $SU(3)$ holonomy of $M_3$, which is the maximal subgroup of $G_2$ Lie group. The superstrings propagating on this type of manifolds preserve 1/8 supercharges in three dimensions. Using the decompactification mechanism, M-theory on this geometry has similar feature of seven manifolds with $G_2$ holonomy. This type of manifolds has been a subject to an intensive interest during the last few years dealing with different problems in superstring theory. In particular, these involve, the computation of instantons superpotentials [11], the description of IIA superstring orientifold compactifications giving four dimensional $N = 1$ models [12] and the study two dimensional superconformal field theories [22]. In all those works, the $\mathbb{Z}_2$ acts on the complex homogeneous variables, defining the Calabi-Yau threefolds, by complex conjugation. However, in this present work we will use a new transformation acting on the toric geometry realizations of Calabi-Yau spaces. In section four, we will show that this procedure leads to $K3$ fibrations in $G_2$ manifold compactifications. In this way, we will be able to find heterotic superstring dual models using M-theory/heterotic duality in seven dimensions [10,17,18].

On F-theory side, the $N = 1$ dual model may be obtained using the same procedure by taking the following quotient space

$$W_4 = \frac{T^2 \times W_3}{\mathbb{Z}_2}. \hspace{1cm} (2.4)$$

In this equation, $\mathbb{Z}_2$ acts nontrivially on the mirror geometry $W_3$ and as a reflection on $T^2$ as follows

$$\mathbb{Z}_2 : \hspace{0.2cm} dz \rightarrow -dz \hspace{1cm} (2.5)$$

where $z$ is the complex coordinate of the elliptic curve $T^2$. If this symmetry has some fixed points, they need to be deformed for obtaining a smooth manifold. This manifold is then elliptically fibered over $\frac{W_3}{\mathbb{Z}_2}$. Like in M-theory side, the holonomy group of this quotient space
is larger than $SU(3)$ holonomy of $W_3$. However, the four dimensional field theories with $N = 1$ supersymmetry obtained from F-theory compactification require that $W_4$ is an elliptically fibred Calabi-Yau fourfolds with $K3$ fibration with $SU(4)$ holonomy. From these physical and mathematical arguments, we can propose, up some details, the following new duality

$$
\text{M-theory on } X_7(G_2) = \text{F-theory on } W_4(SU(4)), \quad (2.6)
$$

where in both sides involve mirror pairs of Calabi-Yau threefolds $(M_3, W_3)$. In what follows, we want to give a comment concerning the relation connecting the Betti numbers of the above quotient manifolds. This can be done just by knowing the results of the numbers of vector multiplets and massless neutral chiral multiplets obtained from both side. Using the results of [33, 34], we expect the following formula

$$
\begin{align*}
    b_2(X_7) &= h^{1,1}(W_4) - h^{1,1}(B_3) - 1 + h^{2,1}(B_3) \\
    b_3(X_7) &= h^{1,1}(B_3) - 1 + h^{2,1}(W_4) - h^{2,1}(B_3) + h^{3,1}(W_4),
\end{align*}
\tag{2.7}
$$

where $B_3 = \frac{W_3}{\mathbb{Z}_2}$. These Betti numbers, in general, depend on the framework of the geometric construction of manifolds and how the $\mathbb{Z}_2$ symmetry acts on the Calabi-Yau threefold. In this present context, the no zero Betti numbers correspond to the $\mathbb{Z}_2$ invariant forms of the above quotient spaces. In next section, we will use the toric geometry language, its relation to sigma model and LG mirror geometries to give expected relations and give some illustrating examples. In this analysis, one has the following Hodge constraint equation

$$
h^{2,1}(M_3) = h^{1,1}(W_3) = 0 \quad (2.8)
$$

reducing (2.7) to

$$
\begin{align*}
    b_2(X_7) &= h^{1,1}(W_4) - 1 + h^{2,1}(B_3) \\
    b_3(X_7) &= h^{2,1}(W_4) - 1 - h^{2,1}(B_3) + h^{3,1}(W_4). \quad (2.9)
\end{align*}
$$

More precisely, our strategy will be as follows:

(i) Building of three dimensional Calabi-Yau threefolds and theirs mirror in terms of hypersurfaces in four dimensional toric varieties using the toric geometry technics, its relation to sigma model and LG theories. In this steep, we give a tricky method for getting the mirror toric Calabi-Yau threefolds $W_3$, involved in F-theory compactifications, from the toric data of $M_3$ manifolds.

(ii) We give a toric description for the duality given in (2.6).
3 Toric geometry for Calabi-Yau threefolds and theirs mirrors

3.1 Toric geometry of Calabi-Yau threefolds $M_3$

Complex Calabi-Yau varieties are Ricci flat spaces with a vanishing first Chern class $c_1 = 0$. They are an important ingredient for constructing quasi-realistic superstring models in lower dimensions. As we have seen, they play also a crucial role in the study of the duality between superstring models and other theories; in particular in our proposition given in (2.6). A large class of these manifolds are usually constructed as hypersurfaces in toric varieties and are nicely described using toric geometry techniques. Calabi-Yau threefolds are the most important geometries in string theory as well as their mirrors which can be used in the type II duality, the geometric engineering, F-theory-heterotic duality and this proposed work [35, 36, 37, 38]. For this reason, we will focus our attention on this special Calabi-Yau geometry. In particular, we develop a tricky way for getting the mirror Calabi-Yau manifolds using techniques of toric geometry. In this way, we give the mirror manifolds as hypersurfaces in four dimensional weighted projective spaces $\mathbb{WP}^4$ of weights $(w_1, w_2, w_3, w_4, w_5)$. As we will see, this construction is based on the solving of the mirror constraint equations involved in the toric geometry in terms of the toric data of the Calabi-Yau threefolds in M-theory compactifications. Before doing this, let us start by describing $M_3$ in toric geometry framework and its relation to linear sigma model. This will be useful for our later analysis on geometries involved in the duality between M-theory on $G_2$ manifolds and F-theory compactifications on elliptically fibred Calabi-Yau fourfolds. We first note that toric geometry is a good tool for describing the essential one needs about the $n$-dimensional Calabi-Yau manifolds and their mirrors involved in the previous study. Roughly speaking, toric manifolds are complex $n$-dimensional manifolds with $T^n$ fibration over $n$ real dimensional base spaces with boundary [39,40,41,42]. They exhibit toric actions $U(1)^n$ allowing to encode the geometric properties of the complex spaces in terms of simple combinatorial data of polytopes $\Delta_n$ of the $R^n$ space. In this correspondence, fixed points of the toric actions $U(1)^n$ are associated with the vertices of the polytope $\Delta_n$, the edges are fixed one dimensional lines of a subgroup $U(1)^{n-1}$ of the toric action $U(1)^n$ and so on. Geometrically, this means that the $T^n$ fibres can degenerate over the boundary of the base. Note that in the case where the base space is compact, the resulting toric manifold will also be compact. The beauty of the toric representation is that it permits to learn the essential about the geometric features of toric manifolds by simply knowing the toric data of the corresponding polytope $\Delta_n$, involving the toric vertices and the Mori vector weights.
A simple example of toric manifold is $\mathbb{C}^n$, which can be parameterized by $z_i = |z_i|e^{i\theta_i}$, $i = 1, \ldots, n$ and endowed with Kahler form given by

$$J = i d\bar{z}_i \wedge dz_i = d(|z_i|^2) \wedge d\theta_i.$$  

(3.1)

This manifold admits $U(1)^n$ toric actions

$$z_i \rightarrow z_ie^{i\theta_i}$$

(3.2)

with fixed locus at $z_i = 0$. The geometry of $\mathbb{C}^n$ can be represented by a $T^n$ fibration over a $n$-dimensional real space parameterized by $|z_i|^2$. The boundary of the base is given by the union of the hyperplanes $|z_i|^2 = 0$. We have given a very simple example of toric manifolds. However, toric geometry is also a very useful for the building (local) Calabi-Yau manifolds, providing a way for superstring theory to interesting physics in lower dimensions in particular four dimensions. An interesting examples are the asymptotically local Euclidean (ALE) space with $ADE$ singularities. These are local complex two dimensions toric variety with a $SU(2)$ holonomy. However, for latter use, we will restrict ourselves to local three complex Calabi-Yau manifolds $M_3$ and their mirrors noted $W_3$ with a $SU(3)$ holonomy. The toric Calabi-Yau $M_3$, involved in M-theory compactification, can be represented by the following algebraic equation

$$\sum_{i=1}^{r+3} Q^a_i |z_i|^2 = R^a,$$  

(3.3)

together with the local Calabi-Yau condition

$$\sum_{i=1}^{r+3} Q^a_i = 0, \quad a = 1, \ldots, r.$$  

(3.4)

In eqs (3.3-4), $Q^a_i$ are integers defining the weights of the toric actions of the complex manifold in which the $M_3$ is embedded. Actually these equations, up some details on $Q^a_i$, generalize the one of the weighted projective space with weights $Q_i$ corresponding to $r = 1$. Each parameter $R^a$ is a Kahler deformation of the Calabi-Yau manifolds. The above geometry can be encoded in a toric diagram $\Delta(M_3)$ having $r+3$ vertices $v_i$ generating a finite dimensional sublattice of the $\mathbb{Z}^5$ lattice and satisfying the following $r$ relations given by

$$\sum_{i=1}^{r+3} Q^a_i v_i = 0, \quad a = 1, \ldots, r,$$  

(3.5)

with the local Calabi-Yau condition (3.4).

Equations (3.3) and (3.5) can be related to the so called D-term potential of two dimensional
$N = 2$ sigma model, for putting the discussion on a physical framework [43]. Indeed associating the previous variables $z_i$, or $v_i$ vertices in toric geometry language, to $(\phi_i)$ matter fields and interpreting the $Q^a_i$ integers as the quantum charges the $(\phi_i)$’s under a $U(1)^r$ symmetry, then the toric Calabi-Yau $M_3$ is now the moduli space of $2D \, N = 2$ supersymmetric linear sigma model. The $Q^a_i$’s obey naturally the neutrality condition, being equivalent to $c_1(M_3) = 0$, which means that the theory flows in the infrared to a non trivial superconformal model [43, 44]. In this way, equation (3.3) can be identified with the D-flatness conditions namely

$$\sum_{i=1}^{r+3} Q^a_i |\phi_i|^2 = R^a.$$  \hspace{1cm} (3.6)

In these eqs, $R^a$ are FT terms which can be complexified by the theta angles as follows

$$t^a = R^a + \theta^a,$$  \hspace{1cm} (3.7)

where $\theta^a$ have similar role of the B field in the string theory compactification on Calabi-Yau manifolds. The number of the independent FI parameters, or equivalent the number of $U(1)$ factors, equal $h^{1,1}(M_3)$.

### 3.2 Solving the mirror constraint eqs for $W_3$

Toric geometry has been adopted to discussing mirror symmetry as well. The latter exchanges the Kahler structure parameters with the complex structure parameters. In general, given a toric realization of the manifold $M_3$, one can build its mirror manifold $W_3$. This will be also a toric variety which is obtained from $M_3$ by help of mirror symmetry. In this study, we will use the result of mirror symmetry in sigma model where the mirror Calabi-Yau $W_3$ will be a LG Calabi-Yau superpotentials, depending on the number of chiral multiples and gauge fields of dual sigma model on $M_3$. A tricky way to write down the equation of LG mirror Calabi-Yau superpotential is to use dual chiral fields $Y_i$ related to sigma model fields such that [45, 46, 47, 48]

$$\text{Re } Y_i = |\phi_i|^2, \quad i = 1, \ldots, k,$$  \hspace{1cm} (3.8)

and define the new variables $y_i$ as follows $y_i = e^{-Y_i}$. The $W_3$ LG mirror Calabi-Yau superpotential takes the form

$$\sum_{i=1}^{r+3} y_i = 0,$$  \hspace{1cm} (3.9)

where the fields $y_i$ must satisfy, up absorbing the complex Kahler parameters $t^a$, the following constraint equations

$$\prod_{i=1}^{r+3} y_i^{Q^a_i} = 1, \quad a = 1, \ldots, r.$$  \hspace{1cm} (3.10)
In toric geometry language, this means that the relation between the toric vertices of \( M_3 \) map to relations given by (3.9-10). To find an explicit algebraic equation for the local mirror geometry, one has to solve the constraint equation (3.10). It turns out that there many ways to solve these constraint equations. Here we present a tricky way, inspired from the Batyrev papers [49, 50] and [37], using the toric geometry representation of sigma model of \( M_3 \). This method can be proceed in some steps. First, we note that the \( y_i \)'s are not all independent variables, only 4 of them do. The latters can be thought as local coordinates of the weighted projective space \( \text{WP}^4(w_1, w_2, w_3, w_4, w_5) \) which parameterized by the following five homogeneous variables

\[
x_\ell = \lambda^{w_\ell} x_\ell, \quad \lambda \in \mathbb{C}^*, \ell = 1, \ldots, 5.
\] (3.11)

For instance, the four local variables can be obtained from the homogeneous ones using some coordinate patch. If \( x_5 = 1 \), the other \( x_\ell \) variables behave as four independent gauge invariants under \( \mathbb{C}^* \) action of \( \text{WP}^4(w_1, w_2, w_3, w_4, w_5) \). The second steep in our program is to find relations between the \( y_i \)'s and the \( x_i \) variables. A nice way of obtaining this is based on the using the toric data of the M-theory Calabi-Yau geometry for solving the mirror constraint eqs (3.10). In this method, the mirror geometry \( W_3 \) will be defined as a \( D \) degree homogeneous hypersurfaces in \( \text{WP}^4 \) with the following form

\[
p_D(x_1, x_2, x_3, x_4, x_5) = 0
\] (3.12)

satisfying

\[
p_D(\lambda^{w_\ell} x_\ell) = \lambda^D p_D(x_\ell).
\] (3.13)

To write down the explicit formula of this equation, one has to solve the mirror constraint equation (3.10) in terms of the \( WP^4 \) toric data. To that purpose, we consider a solution of the dual toric manifold \( M_3 \) of the form

\[
\sum_{i=1}^{r+3} Q_i^a n_i^\ell = 0, \quad a = 1, \ldots, r; \quad \ell = 1, \ldots, 5,
\] (3.14)

where \( n_i^\ell \) are integers specified later on. In patch coordinates \( x_5 = 1 \), one can parameterize the \( y_i \) gauge invariants in terms of the \( x_i \) as follows

\[
y_i = x_1^{(n_i^1-1)} x_2^{(n_i^2-1)} x_3^{(n_i^3-1)} x_4^{(n_i^4-1)} x_5^{(n_i^5-1)} = \prod_{\ell=1}^{5} x_\ell^{(n_i^\ell-1)},
\] (3.15)

where the deformation given by \( y_0 = 1 \) correspond \( (n_0^\ell-1) = 0 \). Using the toric geometry data of \( M_3 \), (3.10) is trivially satisfied by (3.15). Another thing we need in this analysis is that the
$y_i$ variables should be thought of as gauge invariants under the $\mathbf{WP}^4(w_1, w_2, w_3, w_4, w_5)$ projective action given by (3.11). Indeed under this transformation, the monomials $y_i$ transform as

$$y_i = \Pi_{\ell=1}^5 x_\ell^{(n_i^\ell - 1)} \rightarrow y_i' = y_i \Lambda_{w_i(n_i^\ell - 1)}$$

(3.16)

and their invariance are constrained by

$$\sum_{\ell=1}^5 w_\ell = D,$$  

(3.17)

$$\sum_{\ell=1}^5 w_\ell n_i^\ell = D.$$  

(3.18)

Equation (3.17) is a strong constraint which will be necessary for satisfying the Calabi-Yau condition in the mirror geometry; while equation (3.18) shows that the $n_i^\ell$ integers involved in (3.14-15) can be solved in terms of the partitions $d_i^\ell$ of the degree $D$ of the homogeneous polynomial $p_D(x_1, ..., x_5)$. Since $\sum_{\ell=1}^5 d_i^\ell = D$, one can see that $n_i^\ell = \frac{d_i^\ell}{w_i}$; and take, for $i = \ell$, the following property

$$n_i^\ell = \frac{D}{w_i}, \quad i = 1, \ldots, \ell.$$  

(3.19)

In this way the $v_i$ vertices can be chosen as follows

$$v_i^\ell = n_i^\ell - e_0^\ell = \frac{d_i^\ell}{w_i} - e_0^\ell,$$  

(3.20)

where $e_0^\ell = (1, 1, 1, 1, 1)$. This shifting will not influence to the toric realization (3.15) due to the Calabi-Yau condition (3.4). In this way the first 6 vertices and the corresponding monomials can be thought of as follows:

$$v_0 = (0, 0, 0, 0, 0) \rightarrow \Pi_{\ell=1}^5 (x_\ell)$$

$$v_1 = \left(\frac{D}{w_1} - 1, -1, -1, -1, -1\right) \rightarrow x_1^{\frac{D}{w_1}}$$

$$v_2 = \left(-1, \frac{D}{w_2} - 1, -1, -1, -1\right) \rightarrow x_2^{\frac{D}{w_2}}$$

$$v_3 = \left(-1, -1, \frac{D}{w_3} - 1, -1, -1\right) \rightarrow x_3^{\frac{D}{w_3}}$$

$$v_4 = \left(-1, -1, -1, \frac{D}{w_4} - 1, -1\right) \rightarrow x_4^{\frac{D}{w_4}}$$

$$v_5 = \left(-1, -1, -1, -1, \frac{D}{w_5} - 1\right) \rightarrow x_5^{\frac{D}{w_5}}.$$  

(3.21)
Using all the toric vertices in the M-theory geometry, the corresponding mirror polynomial should involve in the F-theory context takes the following form

\[
\sum_{\ell=1}^{5} x_{\ell}^5 + a_0 \prod_{\ell=1}^{5} (x_{\ell}) + \sum_{i=7}^{r+3} a_i \prod_{\ell=1}^{5} x_{\ell}^i = 0, \tag{3.22}
\]

where the \(a_i\)'s are complex moduli of the LG Calabi-Yau mirror superpotentials. For later use, we take \(a_i = 0\) and so the above geometry reduces to

\[
\sum_{\ell=1}^{5} x_{\ell}^5 + a_0 \prod_{\ell=1}^{5} (x_{\ell}) = 0. \tag{3.23}
\]

Actually, this geometry extends the quintic hypersurfaces in the ordinary \(\mathbb{P}^4\) projective space.

### 4 \(Z_2\) symmetry in Toric geometry framework

#### 4.1 \(Z_2\) realization and K3 fibration

Here we would like to discuss the \(Z_2\) realization involved in the duality (2.6) using toric geometry tools. The latter, in the \(G_2\) holonomy sense, will act on \(M_3\), on the circle as the inversion, on the \(T^2\) and on \(W_3\) in the F-theory compactification. Due to richness of possibilities of the \(Z_2\) action in the Calabi-Yau manifold, we will focus our attention herebelow on giving a new \(Z_2\) transformation acting on the toric geometry variables. To this purpose, we first note that any local Calabi-Yau \(M_3\), described by the toric linear sigma model on (3.3) is, up some details, isomorphic to \(\mathbb{C}^r + 3/\mathbb{C}^\ast r\), or equivalently

\[
z_i \equiv \lambda Q_i^a z_i, \quad \sum_{i=1}^{r+3} Q_i^a = 0, \quad a = 1, \ldots, r. \tag{4.1}
\]

The latter has a \(T^3\) fibration obtained by dividing \(T^{r+3}\) by \(U(1)^r\) action generated by a simultaneous phase rotation of the coordinates

\[
z_i \rightarrow e^{iQ_i^a \vartheta_a} z_i, \quad a = 1, \ldots, r, \tag{4.2}
\]

where \(\vartheta_a\) are the generators of the \(U(1)\) factors. In the Calabi-Yau geometry, plus the reflection on the circle, we will consider a new \(Z_2\) symmetry acting on the toric geometry angular coordinates. The latter leads to \(G_2\) manifolds with K3 fibration in M-theory side. To see this feature, let us first consider the simple case corresponding to \(\mathbb{C}^3\), that is \(r = 0\); then we extend this feature to any 3-dimensional toric varieties. Indeed, \(\mathbb{C}^3\) has \(U(1)^3\) toric actions giving \(T^3\)
fibration in toric geometry realization of $C^3$. Besides these toric actions, considering now a $Z_2$ symmetry acting as follows

$$\theta \rightarrow -\theta_i \quad i = 1, 2, 3. \quad (4.3)$$

In this way, the toric actions become now

$$z_i \rightarrow z_i e^{i\theta_i},$$
$$\theta_i \rightarrow -\theta_i \quad i = 1, 2, 3, \quad (4.4)$$

where $z_i$ are the variables appearing in (3.3). This transformation is quite different to one given in the literature [11,12,22], because it acts on the angular variables of complex toric varieties. Note, in passing, that one can consider the following $Z_2$

$$\theta_i \rightarrow \theta_i + \pi \quad i = 1, 2, 3;$$

however this transformation is not interesting from physical argument. Indeed, first this action has no fixed points because the fixed loci are naturally identified with brane configurations [40]. Second, it does not leave the following constraint equation

$$\theta_1 + \theta_2 + \theta_3 = 0, \quad (4.5)$$

involved in the determination of special Lagrangian manifolds in $C^3$ [47].

Now we return to equation (4.4). The latter gives naturally $T^3/Z_2$ as a fiber space in the toric geometry realization of $C^3$, instead of before where we have just a $T^3$ fibration. This feature can be extended to any 3-dimensional toric complex manifolds, in particular local Calabi-Yau threefolds $M_3$. In this way, one has the following $Z_2$ symmetry acting, up the gauge transformation (4.2), on the Calabi-Yau $M_3$ variables as follows

$$z_i \rightarrow z_i e^{i\theta_i},$$
$$\theta_i \rightarrow -\theta_i \quad i = 1, \ldots, r + 3. \quad (4.6)$$

Geometrically, this transformation gives a local Calabi-Yau threefolds with $T^3/Z_2$ fibration. In this case, plus the toric geometry action fixed points we have now extra ones coming from the orbifold toric fibration. These fixed loci may be identified with brane configurations using the interplay between toric geometry and type II brane configurations [40]. For the moment we ignore the brane description and return to the geometric interpretation of the (4.6). Indeed, it is easy to see that, together with the action on the circle, these new toric actions lead to
G2 manifolds with $\frac{T^4}{\mathbb{Z}_2}$ fibration, being as the orbifold limit of the K3 surfaces, in the M-theory compactifications. The geometry given by (2.3) can be now viewed as a $G_2$ manifold with K3 fibration. In this way, we are able to find heterotic superstring dual models which could be used to support our proposed duality (2.6). Indeed, the moduli space of smooth compactifications can be obtained from the one of K3 followed by an extra compactification on a three dimensional space $Q_3$, down to four dimensions. This can be related directly to the heterotic superstring by fibering the M-theory/heterotic duality in seven dimensions on the same base $Q_3$. Locally, the moduli space of this compactification should have the following form

$$\mathcal{M}(K3) \times \mathcal{M}(Q_3)$$

where $\mathcal{M}(K3)$ is the moduli space of the M-theory on K3 in seven dimensions

$$\mathcal{M}(K3) = \mathbb{R}^+ \times \frac{SO(3,19)}{SO(3) \times SO(19)}$$

which is exactly the moduli space of heterotic strings on $T^3$. $\mathcal{M}(Q_3)$ describes the physical moduli coming after the extra compactification on $X_3$. This compactification describes the strong limit of heterotic superstrings on Calabi-Yau threefolds $Z$, being a $T^3$ fibration on $Q_3$. More recently, it was shown that M-theory on a $G_2$ manifolds with K3 fibration can give much more interesting physics that other superstrings derived models; in particular it leads to a theory similar to a four dimensional grand unified model [20]. Alternatively, the $\frac{T^4}{\mathbb{Z}_2}$ fiber space is locally isomorphic to $\frac{\mathbb{C}^2}{\mathbb{Z}_2}$ known by $A_1$ singularity, which can be determined algebraically in terms of the $\mathbb{Z}_2$ invariant coordinates on $\mathbb{C}^2$ as follows

$$z^2 = xy.$$  

M-theory on this local geometry singularity corresponds to two units of D6 branes. For general case where we have $A_n$ singularity, this geometry is equivalent to $n + 1$ D6 branes of type IIA superstring. In this way, the above compactification may have a four dimensional interpretation in terms of type IIA D6 branes.

Before going to F-theory, Now we want to discuss the Rahm cohomology of the M-theory quotient space. The Calabi-Yau condition $b_1(M_3) = 0$ and the $G_2$ holonomy condition require that $b_1(X_7) = 0$. Using eqs (4.6), the Kahler form now is odd under the above $\mathbb{Z}_2$ symmetry. Since there are no invariant 2 forms, we have the following constraint for the quotient space

$$b_2 = 0$$

$\textsuperscript{1}$The $G_2$ holonomy and the Calabi-Yau condition of K3 require that $b_1(Q_3) = 0$. 

13
However there are some invariant three forms; one type of them is given by

$$\phi = J \wedge dx$$

(4.10)

where $J$ is the Kahler form on $M_3$ and $x$ is a real coordinate parameterizing the circle. The number of these forms is given by $h^{1,1}(M_3)$ being the dimension of the complexified Kahler moduli space of $M_3$.

Now we go on to the F-theory to give the corresponding geometries. Instead of being general, we will consider a concrete example describing the mirror quintic hypersurfaces obtained by taking

$$w_\ell = 1 \quad \forall \ell.$$

(4.11)

In this way, the general mirror geometry (3.23) reduces to

$$\sum_{\ell=1}^{5} x_\ell^5 + a_0 \prod_{\ell=1}^{5} (x_\ell) = 0.$$

(4.12)

This equation has any direct toric description of the $\mathbb{P}^4$ in which it is embedded. However a toric realization may be recovered if we consider the limit $a_0 \to \infty$ in the mirror description. In this limit, the defining equation of the mirror geometry becomes approximately, up scaling out the $a_0$, as

$$x_1 \ldots x_5 = 0.$$

(4.13)

This equation can be solved by taking one or more $x_i = 0$. In toric geometry language, this solution describes the union of the boundary faces of 4-simplex defining the polytope of the $\mathbb{P}^4$ projective space [40]. In this case, the F-theory mirror quintic is a $T^3$ fibration over 3-dimensional real space defined by the boundary faces of 4-simplex of $\mathbb{P}^4$. It consists of the intersection of 5 $\mathbb{P}^3$’s along 10 $\mathbb{P}^2$’s. This geometry has now a $U(1)^3$ toric action which can be deduced from the ones of $\mathbb{P}^4$ and can be thought of as follows

$$x_i \to e^{i\theta_i} x_i, \quad i = 1, 2, 3$$

$$x_i \to x_i, \quad i = 4, 5$$

(4.14)

In the F-theory geometry, $\mathbb{Z}_2$ symmetry will act on the mirror Calabi-Yau threefolds as follows

$$x_i \quad \rightarrow \quad x_i e^{i\theta_i}, \quad i = 1, 2, 3$$

$$\theta_i \quad \rightarrow \quad -\theta_i.$$

(4.15)

Here we repeat the same analysis of M-theory. In this case, the F-theory geometry can have also $T^4/\mathbb{Z}_2$ fibration over a four dimensional base space. However, a naive way to get an elliptically
K3 fibration, being the relevant geometry in the F-theory compatification for obtaining \( N = 1 \) in four dimensions, is to suppose that the \( \mathbb{Z}_2 \) symmetry acts trivially on the torus of \( W_4 \) Calabi-Yau fourfolds. In this way, the \( T^4 \) fibration reduces to \( T^2 \times \frac{T^2}{\mathbb{Z}_2} \) which is an elliptic model in the context of F-theory compactifications. This compactification should be interpreted in terms of type IIB on \( \frac{T^2}{\mathbb{Z}_2} \). By this limit, one can see that the orbifold (2.4) gives an elliptic K3 fibration in F-theory compactifications. In M-theory context, this geometry can be obtained using the following factorization in the Narain lattice

\[
\Gamma^{19,3} = \Gamma^{18,2} + \Gamma^{1,1}
\]

having a nice interpretation in terms of the action of \( \mathbb{Z}_2 \) symmetry on the moduli space of K3 surfaces [51].

### 4.2 More on the \( \mathbb{Z}_2 \) action

We would like to give comments regarding a particular realization of \( \mathbb{Z}_2 \) symmetry when it acts trivially on the Calabi-Yau threefolds. In this way, the geometries (2.3) and (2.4) reduce respectively to

\[
X_7 = \frac{S^1}{\mathbb{Z}_2} \times M_3
\]

\[
W_4 = \frac{T^2}{\mathbb{Z}_2} \times W_3.
\]

In M-theory, this compactification may be thought of as the Hořava–Witten compatification on spaces of the form \( \frac{S^1}{\mathbb{Z}_2} \times Y \), where \( Y = M_3 \) is a Calabi-Yau threefolds [52]. M-theory on this type of manifolds gives rise to \( N = 1 \) supersymmetry in four dimensions, having a weak coupling limit given by the heterotic superstring compactified on \( M_3 \). Using the toric description of \( M_3 \) where one has a \( T^3 \) fibration over three dimensional base space, the compactification (4.12) may be related to M-theory on \( G_2 \) manifolds with K3 fibration. This may be checked by the seven dimensional duality between M-theory on K3 and the heterotic superstring on \( T^3 \). In the end of this section we want to discuss the F-theory duals of this kind of the compactification. Instead of being general, we will consider concrete examples corresponding to \( M_3 \), described by \( N = 2 \) linear sigma model on the canonical line bundle over two complex dimensional toric varieties. In F-theory, the mirror map of these geometries are given by non compact Calabi-Yau threefolds LG superpotentials with an equation of the form

\[
W_3 : \quad f(x_1, x_2) = uv,
\]

15
where \( x_1, x_2 \) are \( C^* \) coordinates and \( u, v \) are \( C^2 \) coordinates. For illustrating applications, let us give two examples.

**i) \( \mathbb{P}^2 \) projective space**

The first example is the sigma model on the canonical line bundle over \( \mathbb{P}^2 \). In this way, the Calabi-Yau geometry in M-theory side is described by a \( U(1) \) linear sigma model with four matter fields \( \phi_i \) with the following vector charge

\[
Q_i = (1, 1, 1, -3),
\]

satisfying the Calabi-Yau condition (3.4). After solving the mirror constraint equations (3.10), the corresponding \( W_3 \) LG Calabi-Yau superpotential in F-theory compactification is given by the following equation

\[
W_3 : f(x_1, x_2) = 1 + x_1 + x_2 + \frac{e^{-t_1 x_1 x_2}}{x_1 x_2} = uv.
\]

**ii) Hirzebruch surfaces \( F_n \)**

As a second example, we consider a local model given by the canonical line bundle over the Hirzebruch surfaces \( F_n \). Recall by the way that the \( F_n \) surfaces are two dimensional toric manifolds, defined by a non-trivial fibration of a \( \mathbb{P}^1 \) fiber on a \( \mathbb{P}^1 \) base. The latter are realized as the vacuum manifold of the \( U(1) \times U(1) \) gauge theory with four chiral fields with charges

\[
Q_i^{(1)} = (1, 1, 0, -n)
\]

\[
Q_i^{(2)} = (0, 0, 1, 1).
\]

The canonical line bundle over these surfaces is a local Calabi-Yau threefolds described by an \( U(1) \times U(1) \) linear sigma model with five matter fields \( \phi_i \) with two vector charges

\[
Q_i^{(1)} = (1, 1, 0, -n, n - 2)
\]

\[
Q_i^{(2)} = (0, 0, 1, 1, -2).
\]

These satisfy naturally the Calabi-Yau condition (3.4). After solving (3.10), the defining equation for the LG mirror superpotential becomes

\[
W_3 \quad f_n(x_1, x_2) = uv,
\]

where \( f_n(x_1, x_2) = 1 + x_1 + \frac{e^{-t_1 x_1 x_2}}{x_1} + x_2 + \frac{e^{-t_2 x_4}}{x_4} \) and where \( t_i \) are complex parameters.

In F-theory side, eqs (4.21) and (4.24) describe elliptic fibration solutions of (4.19), where the elliptic curve fiber is given by

\[
f(x_1, x_2) = 0.
\]
By introducing an extra variable, this elliptic curve can have a homogeneous representation described by the cubic polynomial in $\mathbb{P}^2$, with the general form as follows

$$\sum_{i+j+k=3} a_{ijk} x^i y^j z^k = 0, \quad (4.26)$$

which can be related, up some limits in the complex structures, to the following Weierstrass form

$$y^2 z = x^3 + axz^2 + bz^3. \quad (4.27)$$

This form plays an important role in the construction of elliptic Calabi-Yau manifolds involved in F-theory compactifications [25, 26, 48].

The elliptic K3 fibration of (4.8) may be obtained by using only the orbifold of the torus in F-theory. A tricky way to see this is as follows. First, we consider the elliptic curve $T^2$ as a fiber circle $S^1_f$, of radius $R_f$, fibered on a $S^1_b$ base circle, of radius $R_b$. After that we take a $\mathbb{Z}_2$ symmetry acting only on the $S^1_b$ and leaving the $S^1_f$ fiber invariant. In this way, the orbifold $T^2/\mathbb{Z}_2$ can be seen as a $S^1_f$ bundle over a line segment, with two fixed points zero and $\pi R_b$. This space, up some details, has similar feature of the toric realization of $\mathbb{P}^1$. To see this, assuming that the radius of the $S^1_f$ fiber vary on the base space as follows

$$R_f \sim \sin \frac{x}{R_b} \quad (4.28)$$

where $x$ is the coordinate on the interval which runs from zero to $\pi R_b$. In this case, the $S^1_f$ can shrink at the two end-points of the line segment. With this assumption, the resulting geometry is identified with the toric geometry realization of the $\mathbb{P}^1$ [39,40]. By this limit in the $\mathbb{Z}_2$ orbifold, the F-theory geometry given by (4.18) can be viewed as a $\mathbb{P}^1$ fibration over on $W_3$. Since $W_3$ is elliptic model, this fibration, in presence of $\mathbb{P}^1$, can give an elliptic K3 fibration which is the relevant geometry for getting $N = 1$ models in four dimensions from F-theory compactifications.

5 Discussions and conclusion

In this paper, we have studied, geometrically, the $N = 1$ four dimensional duality between M-theory on $G_2$ manifolds and F-theory on elliptically fibred Calabi-Yau fourfolds with $SU(4)$ holonomy. In this study, we have considered M-theory on spaces of the form $S^1 \times M_3/\mathbb{Z}_2$, where $M_3$ is a Calabi-Yau threefolds described physically by $N = 2$ sigma model in two dimensions. In particular, using mirror symmetry, we have discussed the possible duality between M-theory
on such spaces and F-theory on $\mathbb{T}_2^2 \times \mathbb{Z}_2^3$, where $(M_3, W_3)$ are mirror pairs in type II superstring compatifications on Calabi-Yau threefolds. In this work, our results are summarized as follows:
(1) First, we have developed a way for getting LG Calabi-Yau threefolds $W_3$ mirror to sigma model on toric Calabi-Yau $M_3$. This method is based on solving the mirror constraint eqs for LG theories in terms of the toric data of sigma model on $M_3$. Actually, this way gives directly the right dimensions of the mirror geometry. In particular, we have shown that the mirror LG Calabi-Yau threefolds $W_3$ can be described as hypersurfaces in four dimensional weighted projective spaces $\mathbb{WP}^4$, depending on the toric data of $M_3$. In this way, the $\mathbb{WP}^4$ may be determined in terms of the toric geometry data of $M_3$.
(2) Using these results, we have shown, at least, there exist two classes of F-theory duals of M-theory on $S^1 \times M_3$. These classes depend on the possible realizations of the $\mathbb{Z}_2$ actions on $M_3$. In the case, where $\mathbb{Z}_2$ acts no trivially on $M_3$, we have given a toric description of the above mentioned duality. In particular, we have proposed a special $\mathbb{Z}_2$ symmetry acting on the toric geometry angular variables. This way gives seven manifolds with K3 fibrations of $G_2$ holonomy. For the case, where $\mathbb{Z}_2$ acts trivially on $M_3$, we have given some illustrating examples of the F-theory duals of M-theory. Finally, since every supersymmetric intersecting brane dynamics is expected to lift to M-theory on local $G_2$ manifolds [53], it would be interesting to explore this physics using intersecting D 6-7 branes in Calabi-Yau manifolds. Moreover, the $N = 1$ model studied in this work can be related to heterotic superstrings on Calabi-Yau threefolds with the elliptic fibration over del Pezzo surfaces $dP_k$, the $\mathbb{P}^2$’s blown up in $k$ points with $k \leq 9$. In a special limit these surfaces may be specified by the elliptic fibration $\frac{W_3}{\mathbb{Z}_2} \rightarrow dP_k$ with $\mathbb{P}^1$ fibers. It should be also interesting to explore physical realizations, via branes, of this fibration. Progress in this direction will be reported elsewhere.

Acknowledgments
I would like to thank many peoples. I would like to thank Instituto de Fisica Teorica, Universidad Autonoma de Madrid for kind hospitality during the preparation of this work. I am very grateful to C. Gómez for discussions, encouragement and scientific helps. I would like to thank A.M. Urenga for many valuable discussions and comments on this work during my stay at IFT-UAM. I am very grateful to E. H. Saidi for discussions, encouragement and scientific helps. I would like to thank the organizers of the Introductory School on String theory (2002), the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, for kind hospitality. I would like to thank K. S. Narain, M. Blau and T. Sarkar for discussions at ICTP. I am very grateful to V. Bouchard for the reference [14]. I would like to thank J. McKay and A.
Sebbar for discussions, encouragement and scientific helps. This work is partially supported by SARS, programme de soutien à la recherche scientifique; Université Mohammed V-Agdal, Rabat. I would like to thank my family for helps.
References

[1] D. R. Morrison, M. R. Plesser; Nucl.Phys. B440 (1995) 279-354, hep-th/9412236.

[2] B. R. Greene, D. R. Morrison, M. R. Plesser; Commun.Math.Phys. 173 (1995) 559-598, hep-th/9402119.

[3] S. Ferrara, J. Harvey, A. Strominger and C. Vafa, Second Quantized Mirror Symmetry; Phys. Lett. 361B (1995) 59–65, hep-th/9505162.

[4] C. Vafa, Lectures on Strings and Dualities, hep-th/9702201.

[5] S. Kachru, C. Vafa, Exact Results for N=2 Compactifications of Heterotic Strings; Nucl. Phys. B450 (1995) 69-89, hep-th/9505105.

[6] E. Witten, String theory dynamics in various dimensions; Nucl.Phys. B 443 (1995)85, hep-th/9503124.

[7] P.K. Townsend, The eleven-dimensional supermembrane revisited; Phys. Lett B 350 (1995)184, hep-th/9501068.

[8] C. Gómez, Lectures presented at the Workshop on Noncommutative Geometry, Superstrings and Particle Physics. Rabat -Morocco, (May 11-12 2001).

[9] P.K. Townsend and G. Papadopoulos, Compactification of D=11 supergravity on spaces of exceptional holonomy; Phys. Lett B 357 (1995)472, hep-th/9506150.

[10] B. Acharya, On realizing N = 1 super Yang-Mills in M-theory, hep-th/0011089.

[11] J. A. Harvey, G. Moore, Superpotentials and Membrane Instantons, hep-th/9907026.

[12] S. Kachru, J. McGreevy, M-theory on Manifolds of G_2 Holonomy and Type IIA Orientifolds, JHEP 0106 (2001) 027, hep-th/0103223

[13] M.F. Atiyah and E. Witten, M-theory dynamics on a manifold of G_2 Holonomy; hep-th/0107177.

[14] M. Cvetic, G. Shiu, A. M. Uranga, Chiral Type II Orientifold Constructions as M Theory on G_2 holonomy spaces, hep-th/0111179.

[15] A. M. Uranga, Localized instabilities at conifolds, hep-th/0204079.
[16] D. Joyce, Compact manifolds of Special Holonomy, Oxford University Press, 2000.

[17] E. Witten, Anomaly Cancellation On Manifolds Of G_2 Holonomy, hep-th/0108165.

[18] B. Acharya, E. Witten, Chiral Fermions from Manifolds of G_2 Holonomy, hep-th/0109152.

[19] G. Curio, Superpotentials for M-theory on a G_2 holonomy manifold and Triality symmetry, hep-th/0212211.

[20] T. Friedmann, E. Witten, Unification Scale, Proton Decay, And Manifolds Of G_2 Holonomy, hep-th/0211269.

[21] A. Belhaj, Manifolds of G_2 Holonomy from N = 4 Sigma Model; Phys. A35 (2002) 8903-8912, hep-th/0201155.

[22] R. Blumenhagen, V. Braun, Superconformal Field Theories for Compact G_2 Manifolds; JHEP 0112 (2001) 006, hep-th/0110232.
R. Roiban, C. Romelsberger, J. Walcher, Discrete Torsion in Singular G_2-Manifolds and Real LG, hep-th/0203272.

[23] S. Gukov, S. T. Yau, E. Zaslow, Duality and Fibrations on G_2 Manifolds, hep-th/0203217.

[24] E. Witten, New Issues in Manifolds of SU(3) Holonomy; Nucl. Phys. B 268 (1986) 79–112.
A. Sen, Heterotic string theory and Calabi Yau manifolds in the Green Schwarz Formalism; Nucl. Phys. B355 (1987) 423.

[25] C. Vafa; Nucl. Phys. B 469 (1996)403.

[26] C. Vafa and D. Morrison; Nucl. Phys. B 476 (1996)437.

[27] P. Berglund and P. Mayr, Heterotic String/F-theory Duality from Mirror Symmetry hep-th/9811217. P. Berglund, P. Mayr, Stability of vector bundles from F-theory; JHEP 9912 (1999) 009, hep-th/9904114.

[28] W. Lerche, On the Heterotic/F-Theory Duality in Eight Dimensions, contribution to the proceedings of Cargese 1999, hep-th/9910207.
B. Andreas, G.Curio, Horizontal and Vertical Five-Branes in Heterotic/F-Theory Duality; JHEP 0001 (2000) 013, hep-th/9912025.
[29] P. Mayr, Non-Perturbative N=1 String Vacua, lectures delivered at the spring workshop on superstrings and related matters, ICTP Trieste (March 1999).

[30] P. Mayr, *N=1 Heterotic string Vacua from Mirror Symmetry*, hep-th /9904115.

[31] M. Marquart, D. Waldram, *F-theory duals of M-theory on \( \frac{\mathbb{Z}_2}{\mathbb{Z}_2} \times T^4 / \mathbb{Z}_N \)*, hep-th/0204228.

[32] T.R. Taylor, C. Vafa, *RR Flux on Calabi-Yau and Partial Supersymmetry Breaking*; Phys. Lett. **B474** (2000) 130-137, hep-th/9912152.

S. Gukov, M. Haack, IIA String Theory on Calabi-Yau Fourfolds with Background Fluxes, hep-th/0203267.

[33] B. Andreas, G. Curio, D. Lust, *N=1 Dual String Pairs and their Massless Spectra*; Nucl.Phys. **B507** (1997) 175-196, hep-th/9705174.

[34] G. Curio, D. Lust, *A Class of N=1 Dual String Pairs and its Modular Superpotential*; Int.J.Mod.Phys. A12 (1997) 5847-5866, hep-th/9703007.

[35] S. Katz, P. Mayr and C. Vafa, *Mirror symmetry and exact solution of 4d N=2 gauge theories I*; Adv. Theor. Math. Phys **1**(1998)53.

[36] P. Mayr, *Geometric construction of N=2 of Gauge Theories*, Fortsch Phys. 47 (1999)39-63. P. Mayr, N=2 of Gauge theories, lectures presented at Spring school on superstring theories and related matters, ICTP, Trieste, Italy, (1999).

[37] A. Belhaj, A. E. Fallah and E. H. Saidi, *On the non-simply mirror geometries in type II strings*; CQG 17 (2000)515-532.

[38] A. Belhaj and E.H. Saidi, *Toric Geometry, Enhanced non Simply Laced Gauge Symmetries in Superstrings and F-theory Compactifications*; hep-th/0012131.

[39] W. Fulton, *Introduction to Toric varieties*; Annals of Math. Studies, No .131, Princeton University Press, 1993.

[40] N.C. Leung and C. Vafa; Adv .Theo. Math. Phys 2(1998) 91, hep-th/9711013.

[41] D. Cox, *the homogeneous coordinate Ring of a toric variety*, J.Alg.geom. **4** (1995)17.

[42] M. Kreuzer, H. Skarke, *Reflexive polyhedra, weights and toric Calabi-Yau fibrations*, math.AG/0001106.
A.C. Avram, M. Kreuzer, M. Mandelberg, H. Skarke, *The web of Calabi-Yau hypersurfaces in toric varieties*; Nucl.Phys. B505 (1997) 625-640, hep-th/9703003.

M. Kreuzer, H. Skarke, *Calabi-Yau 4-folds and toric fibrations*; J.Geom.Phys. 26 (1998) 272-290, hep-th/9701175.

F. Anselmo, J. Ellis, D.V. Nanopoulos and G. Volkov, *Universal Calabi-Yau Algebra: Towards an Unification of Complex Geometry*; hep-th/0207188.

[43] E. Witten, Nucl Phys **B403** (1993)159-22, hep-th/9301042.

[44] A. Belhaj and E. H. Saidi, *Hyper-Kahler Singularities in Superstrings Compactification and 2d N=4 Conformal Field Theory*; C.Q.G. 18 (2001) 57-82, hep-th/0002205. A. Belhaj, E.H. Saidi, *On HyperKahler Singularities*; Mod. Phys. Lett. A, Vol. 15, No. 29 (2000) pp. 1767-1779, hep-th/0007143

[45] K. Hori, C.Vafa, *Mirror Symmetry*, hep-th/0002222.

[46] K. Hori, A.Iqbal, C.Vafa, *D-Branes And Mirror Symmetry*, hep-th/0005247.

[47] M. Aganagic and C. Vafa, *Mirror Symmetry, D-branes and Counting Holomorphic Discs*, hep-th/0012041.

M. Aganagic, C. Vafa, *Mirror Symmetry and a G2 Flop*, hep-th/0105225.

M. Aganagic, A. Klemm, C. Vafa, *Disk Instantons, Mirror Symmetry and the Duality Web*, hep-th/0105045.

[48] A. Belhaj, *Mirror symmetry and Landau- Ginzburg Calabi-Yau superpotentials in F-theory compactifications*; J.Phys. A35 (2002) 965-984, hep-th/0112005.

[49] V. V. Batyrev, E. N. Materov, *Toric Residues and Mirror Symmetry*, math.AG/0203216

[50] V. V. Batyrev, *Toric Degenerations of Fano Varieties and Constructing Mirror Manifolds*, alg-geom/9712034.

V. V. Batyrev, *Dual Polyhedra and mirror symmetry for Calabi-Yau Hypersurfaces in Toric Varities*, J.Algebraic.Geom 3 (1994)493, Duke. Math.J **75** (1994)293.

[51] F.A. Cachazo, C. Vafa, *Type I’ and Real Algebraic Geometry*, hep-th/0001029.

[52] P. Horava, E. Witten, *Eleven-Dimensional Supergravity on a Manifold with Boundary*; Nucl.Phys. **B475** (1996) 94-114, hep-th/9603142.
[53] R. Blumenhagen, V. Braun, B. Kors, D. Lust, *Orientifolds of K3 and Calabi-Yau Manifolds with Intersecting D-branes*, hep-th/0206038.