Gravitational waves in Fully Constrained Formulation in a dynamical spacetime with matter content

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Abstract. We analyze numerically the behaviour of the hyperbolic sector of the Fully Constrained Formulation (FCF) (Bonazzola et al. 2004). The numerical experiments allow us to be confident in the performances of the upgraded version of the CoCoNuT code (Dimmelmeier et al. 2005) by replacing the Conformally Flat Condition (CFC), an approximation of Einstein equations, by FCF. First gravitational waves in FCF in a dynamical spacetime with matter content will be shown.

1. Introduction
The 3+1 formalism (see, e.g., [1]) is commonly used in most numerical codes which solve Einstein equations in spacetimes containing astrophysical compact objects. The Fully Constrained Formalism (FCF) [2, 3, 4] is a constrained evolution formulation of full Einstein equations. FCF is a natural generalization of the Conformally Flat Condition (CFC) [5, 6], an approximation of Einstein equations. FCF includes a hyperbolic system governing the gravitational radiation. We present numerical simulations of this system and compare some gravitational waves extracted with other methods.

2. Formalism
Given an asymptotically flat spacetime $(\mathcal{M}, g_{\mu\nu})$ we consider a 3+1 splitting by spacelike hypersurfaces $\Sigma_t$. Latin (greek) indices go from 1 (0) to 3. $\gamma_{\mu\nu}$ denotes the 3-metric on $\Sigma_t$, $K_{\mu\nu}$ the extrinsic curvature, $N$ the lapse function and $\beta^i$ the shift vector. We denote $K = \gamma^{ij}K_{ij}$. A flat metric $f_{ij}$ is introduced. We define $\gamma := \det \gamma_{ij}$ and $f := \det f_{ij}$. We introduce the conformal decomposition $\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}$. We define $h^{ij} := \tilde{\gamma}^{ij} - f^{ij}$. Maximal slicing, $K = 0$, and the so-called generalized Dirac gauge, $D_k \tilde{\gamma}^{ki} = 0$, where $D_k$ is the Levi–Civita connection associated with $f_{ij}$, are chosen. Einstein equations become a coupled elliptic-hyperbolic system: the elliptic sector acts on $\psi$, $N$, and $\beta^i$, and the hyperbolic one on $h^{ij}$ [2]. If $h^{ij} = 0$ is imposed, CFC is recovered.

We introduce the decomposition $\tilde{A}^{ij} = \psi^{10} \left( K^{ij} - \frac{1}{3} K \gamma^{ij} \right)$ motivated by the local uniqueness properties of the elliptic equations shown in [4]. We rewrite the equation for $h^{ij}$ as a first order
evolution system for the tensors \((h^{ij}, \hat{A}^{ij}, w^{ij}_k)\), where \(w^{ij}_k := D_k \tilde{\gamma}^{ij}\).

3. Numerical simulations

We perform the numerical evolution of the tensors \((h^{ij}, \hat{A}^{ij}, w^{ij}_k)\) and the evolution of matter (only in the last test), with the CoCoNuT code [7, 8]. Some basic elements of the code are: 4th order scheme in space and time; axisymmetry and symmetry with respect to the equatorial plane; spherical orthonormal coordinates; a Sommerfeld condition at the outer boundary; and Kreiss-Oliger dissipative term.

3.1. Teukolsky waves

We evolve firstly axisymmetric Teukolsky waves [9], with an amplitude of \(10^{-5}\). They are linearized wave equation solutions in vacuum, satisfy the Dirac gauge and are traceless (linear approximation of unit determinant). The background is flat, i.e., \(N = \psi = 1\), and \(\beta^i = 0\).

We display in Fig. 1 the radial profile of the component \(h^{rr}\) at \(t = 6\), at the equator and at the pole, with different values of the number of radial and angular points. The analytical solution is recovered: velocity and amplitude of the wave, and its decay with the radius. We obtain an order of convergence close to the implemented 4th order for the tensor \(h^{ij}\).

3.2. Equilibrium configuration of rotating neutron stars

We consider an axisymmetric and uniformly rotating neutron star in equilibrium. The initial data have been obtained from LORENE [10]. The models have non-vanishing \(\beta^i\), \(N\), \(\psi\) and matter fields, which are all kept fixed during the evolution of the tensor \(h^{ij}\). It is possible to compute CFC and FCF initial models. During the evolution of a non equilibrium configuration we will consider the CFC approximation for the elliptic equations and the matter fields. The initial model is a neutron star with a 550 Hz rotation frequency, a \(1.6\ M_\odot\) baryon mass and a 12.86 km coordinate equatorial radius. Details about the radial logarithmic grid used, and the chosen outer boundary and extracting radius for the gravitational waves can be found in [11].

Using the CFC approximation for the initial data for \(N\), \(\psi\) and \(\beta^i\), we find deviations in the components of the tensor \(h^{ij}\) during the evolution, which decay as \(1/r\) (all the components should decay as \(1/r^3\)). It can be observed in Fig. 2, where we display the radial profile of the component \(h^{rr}\) in logarithmic scale at the equator for different times. The left panel in Fig. 2 corresponds to our control simulation, in which we have kept the same accuracy as the one used (for the elliptic sector) in [12]. We have improved the accuracy of the elliptic sector, in order...
to reduce the offset, in the simulation showed in the right panel, and get the $1/r^3$ profile. The waves are generated initially when the CFC approximation is recovered in the code from the FCF initial data. Both simulations have 80 (16) radial (angular) grid points in the matter domain. The first reason for these deviations is, therefore, the accuracy in the numerical solutions of the elliptic equations.

We study now the influence of the accuracy of the evolution of the tensors $(h^{ij}, \hat{A}^{ij}, w^{ij}_k)$, keeping the CFC approximation for the elliptic equations and the same accuracy in their numerical solutions. In Fig. 3 we plot the evolution in time of the diagonal components of the tensor $h^{ij}$ at $r = 2.5 \times 10^7$ cm, for two different resolutions, in the CFC (FCF) case in the left (right) panel. Lower resolution corresponds to previous numerical grid, while the higher one has 160 (32) radial (angular) grid points in the already mentioned matter domain. In the CFC case, a deviation appears in $h^{rr}$ and $h^{\theta\theta}$, not corrected with resolution.

We compare simulations using CFC and FCF solutions of the elliptic equations. In the last case, a small deviation which decays as $1/r$ will appear, due to the numerical accuracy in the solutions of the elliptic equations, which is observed in $h^{rr}$. However, the largest errors occur in $h^{\theta\theta}$ and $h^{\varphi\varphi}$, and in the FCF case the absolute error is reduced around 1/4. We conclude that the numerical accuracy in the solution of the elliptic equations, firstly, and the terms including the $h^{ij}$ tensor in their sources, secondly, are the main reasons for the offset in the gravitational wave signal, that we will compute in the non stationary simulations.

Figure 2. Radial profile of $|h^{rr}|$ with lower (higher) resolution in left (right) panel. Grey lines correspond to the stationary solution (as reference). Dashed, dot-dashed and dotted lines correspond to the evolution of the initial stationary data at 0.1, 3.0 and 6.0 ms, respectively.

Figure 3. Evolution in time of the diagonal components of the $h^{ij}$ tensor, at coordinate radius $r = 2.5 \times 10^7$ cm, for two different resolutions, in the CFC approximation (FCF formulation) in the left (right) panel.
3.3. Perturbed equilibrium configuration of rotating neutron star

We evolve a perturbed neutron star. Hydrodynamic equations governing the matter evolution and elliptic equations are solved in the CFC approximation. In Fig. 4, we plot the real part of $\Psi_4$ Weyl scalar corrected with the numerical off-set of the stationary simulation, at $r = 2.5 \times 10^6$ cm, and the gravitational wave obtained within the quadrupole formula, for the already used resolutions. We observe an extremely well agreement. The amplitude of the wave is damped during the evolution due to numerical errors, but it is corrected with resolution.

4. Conclusions

CoCoNuT is a general-relativistic magneto-hydrodynamic numerical code which uses the CFC approximation. Next step is the replacement of the CFC approximation by the FCF. We show that the hyperbolic sector works properly through several tests and compare the extracted gravitational waveforms with the quadrupole formula.

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References

[1] Alcubierre M 2008 *Introduction to 3+1 numerical relativity* (Oxford: Oxford University Press)
[2] Bonazzola S, Gourgoulhon E, Grandclément P and Novak J 2004 *Phys. Rev. D* **70** 104007
[3] Cordero-Carrión I, Ibáñez J M, Gourgoulhon E, Jaramillo J L and Novak J 2008 *Phys. Rev. D* **77** 084007
[4] Cordero-Carrión I, Cerdá-Durán P, Dimmelmeier H, Jaramillo J L, Novak J and Gourgoulhon E 2009 *Phys. Rev. D* **79** 024017
[5] Isenberg J A 2008 *Int. J. Mod. Phys. D* **17** 265
[6] Wilson J R and Mathews G J 1989 *Relativistic hydrodynamics* (Frontiers in numerical relativity) (Cambridge: Cambridge University Press)
[7] http://www.mpa-garching.mpg.de/hydro/COCONUT
[8] Dimmelmeier H, Novak J, Font J A, Ibáñez J M and Müller E 2005 *Phys. Rev. D* **71** 064023
[9] Teukolsky S A 1982 *Phys. Rev. D* **26** 745
[10] http://www.lorene.obspm.fr/
[11] Cordero-Carrión I, Cerdá-Durán P and Ibáñez J M 2010 *J. Phys. Conf. Ser.* **228** 012055.
[12] Dimmelmeier H, Font J A and Mueller E 2002 *Astron. Astrophys.* **393** 523