New oscillation criteria for discrete fractional order forced nonlinear equations

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Abstract. In this present work, new oscillation theorems for discrete forced nonlinear equations with fractional order of the form

$$\Delta \left[ \gamma(\ell) \phi(u(\ell)) \Delta^\mu u(\ell) \right] + q(\ell) F[G(\ell)] = \eta(\ell), \ell \geq \ell_0 > 0$$

is discussed. In the above equation $\mu(0 < \mu \leq 1)$ is the fractional order, $G(\ell) = \sum_{j=\ell_0}^{\ell-1+\mu} (\ell - j - 1)^{(-\mu)} u(j)$ and $\Delta^\mu$ is defined as the difference operator of the Riemann Liouville (RL) derivative. Based on the properties of RL derivative, inequalities and generalized Riccati type techniques, some new sufficient conditions that are essential for the oscillation of solutions of forced fractional order discrete nonlinear equations are established.

Keywords: Oscillation theory, fractional order difference equations, Riccati technique, forcing term.

1. Introduction

Fractional calculus is one of the fields in applied mathematics which involves integrals and derivatives of arbitrary orders. Moreover, students of engineering, economics, applied mathematics and science encounter with the differential calculus operators $\frac{d}{dx}, \frac{d^2}{dx^2}$, etc., but only few of them came to consider whether it is necessary for the derivative to be an integer order. The fundamental concept of fractional calculus is widely believed from a question raised by Marquis to Leibniz in the year 1695. A mathematical discipline dealing with integrals and derivatives of a fractional order is called fractional calculus.

Nowadays, various types of methods applied to estimate the fractional integral and derivative. Fractional calculus is the one of the most novel types of tools applied to engineering, science and technology, which having a broad area of applications. In the past few decades, many authors have been concerned with stability of solutions, existence and uniqueness of solutions of fractional differential equations. Recently, some papers examined oscillatory theorems for forced nonlinear fractional order differential equations, [1, 2] and there are books by Miller and Ross, Podlubny and Kilbas et al. [3, 4, 5] that summarize and organize on the subject of fractional differential equations theories and their applications.

Ever since Kuttner mentioned the fractional order differences for the first time in 1956, the theory of difference equations of fractional order has been evolving. Ross and Miller initiate to
study the Discrete Fractional Calculus and the authors of [6, 7] have developed the theoretical results for forward difference discrete fractional calculus. The first definitions of fractional order sums and difference were developed in the works of [8, 9]. The Theory of Discrete Fractional Calculus is elaborated by Atici and Eloe et al who obtained the important properties and results.

Many authors have focused to determining the solutions on the problems for various oscillation equations like ordinary differential equations, difference equation, partial differential equations, dynamic equations on timescales and functional differential equations [10, 11]. Recently, several authors investigated the qualitative properties of discrete fractional order equations which has been paid much attractive. The study of oscillation of solutions for fractional difference equations drew the interest of many researchers [12, 13, 14, 15] and see the references therein. One of the most useful technique for the study of solutions of oscillation of differential/difference equations are Riccati type transformations. Recently, paper [16] discussed the oscillatory properties of solutions to forced nonlinear fractional order differential equations of the form

\[ D^\alpha_t [r(t)\psi(x(t))] + q(t)f(x(t)) = e(t) \]

where \( D^\alpha_t \) is the Liouvillie fractional derivative and \( \alpha \) is the fractional order with respect to the variable \( t, t \geq t_0 > 0 \) and \( 0 < \alpha < 1 \). Only few papers has been published on the oscillatory behavior of forced nonlinear fractional order difference equations. The present work is motivated by [16] and we examine the new oscillation criteria for forced fractional order nonlinear difference equations of the form

\[ \Delta [\gamma(\ell)\phi(u(\ell))] + q(\ell)F[G(\ell)] = \eta(\ell), \ell \geq \ell_0 > 0 \]  

(1)

where \( G(\ell) = \sum_{j=\ell_0}^{\ell-1+\mu} (\ell - j - 1)(-\mu) u(j) \) and \( \Delta^\mu \) defined as the difference operator of the R-L derivative of order \( 0 < \mu \leq 1 \). In this work the following conditions to be hold throughout this paper:

(\( H_1 \)) \( q, \eta \in C([\ell_0, \infty), R) \) such that \( q(\ell) \geq 0 \);  

(\( H_2 \)) \( \gamma(\ell) \in C^1([\ell_0, \infty), R_+) \) such that \( \gamma(\ell) \leq \lambda \) for some \( \lambda > 0 \);  

(\( H_3 \)) \( 0 < \phi(u(\ell)) \leq m \) for some positive constant \( m \) and for all \( u \neq 0 \);  

(\( H_4 \)) \( F[G(\ell)] \geq 0 \) such that \( \frac{F[G(\ell)]}{G(\ell)} \geq \delta \), for some positive constant \( \delta \) and \( G(\ell) \neq 0, \ell \geq \ell_0 \).

In this paper, we establish some new sufficient conditions that are essential for the oscillation of all solutions of forced fractional order discrete nonlinear equation (1) by using inequalities and generalized Riccati transformation techniques. Therefore we hope that this present work will contribute to the study of oscillation for discrete fractional order forced nonlinear equations. Some preliminary definitions of fractional derivative and basic lemma are given in section 2. Further the oscillation theorems of all solutions of forced nonlinear discrete fractional order equations (1) established in section 3.

2. Preliminaries

In order to use in the main results of this work, we give some definitions of fractional derivatives and their basic results in this section.

**Definition 1.** [17, 18] If a solution \( u(\ell) \) of (1) has arbitrarily large zeros, then it is called oscillatory solution, otherwise the solution \( u(\ell) \) of (1) is called nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.
Definition 2. [19] The RL $\kappa$th-order fractional sum $\Delta^{-\kappa}$ is defined by
\[
\Delta^{-\kappa}f(\ell) = \frac{1}{\Gamma(\kappa)} \sum_{\ell=a}^{\ell-\kappa} (\ell-1-j)^{(\kappa-1)} f(j), \ell \in N_a.
\]
For any $\kappa \geq 0$, the falling factorial is defined by
\[
\ell(\kappa) = \frac{\Gamma(\ell+1)}{\Gamma(\ell+1-\kappa)}.
\]

Definition 3. [19] The RL $\kappa$th-order fractional difference $\Delta^{\kappa}$ is defined by
\[
\Delta^{\kappa}u(\ell) = \Delta^{\beta} \Delta^{-(\beta-\kappa)}u(\ell), \ell \in N_a
\]
and so
\[
\Delta^{\kappa}u(\ell) = \frac{\Delta^{\beta}}{\Gamma(\beta-\kappa)} \sum_{\ell=a}^{\ell-\beta+\kappa} (\ell-1-j)^{(\beta-\kappa-1)} u(j), \ell \in N_a.
\]
Hence, the law of exponent for fractional sum is
\[
\Delta^{-\kappa}[\Delta^{-\sigma}u(\ell)] = \Delta^{-(\kappa+\sigma)}u(\ell) = \Delta^{-(\kappa-\sigma)}[\Delta^{-\kappa}u(\ell)].
\]

Lemma 4. [20] Let $u(\ell)$ be solution of (1) and $G(\ell) = \sum_{j=\ell_0}^{\ell-1+\mu} (\ell-1-j)^{(-\mu)} u(j)$, then
\[
\Delta G(\ell) = \Gamma(1-\mu) \Delta^{\mu} u(\ell).
\]
Proof.
\[
G(\ell) = \sum_{j=\ell_0}^{\ell-1+\mu} (\ell-1-j)^{(-\mu)} u(j) = \sum_{j=\ell_0}^{\ell-(1-\mu)} (\ell-j-1)^{(1-\mu)-1} u(j) = \Gamma(1-\mu) \Delta^{-(1-\mu)} u(j).
\]
Thus
\[
\Delta G(\ell) = \Gamma(1-\mu) \Delta^{\mu} u(\ell)
\]

3. Main results

Based on the properties of RL derivative, inequalities and generalized Riccati type techniques, some new sufficient conditions for all solutions of forced discrete fractional order nonlinear equation (1) to oscillate are established in this section.

Theorem 5. Suppose that conditions $(H_1) - (H_4)$ hold and assume that for any $L \geq \ell_0$, there exists $\alpha_1, \beta_1, \alpha_2, \beta_2$ such that $L \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2$ satisfying $\eta(\ell) \begin{cases} \leq 0, & \ell \in [\alpha_1, \beta_1] \\ \geq 0, & \ell \in [\alpha_2, \beta_2] \end{cases}$.
If there exists a positive function $\rho \in C^\alpha ([\ell_0, \infty), R_+)$, such that
\[
\lim_{\ell \to \infty} \frac{\Gamma(1-\mu)}{m\lambda} \sum_{j=\ell_0}^{\ell-1} \frac{1}{\rho(j)} = \infty,
\]
and
\[
\lim_{\ell \to \infty} \sum_{j=\ell_0}^{\ell-1} \left[ \delta \rho(j) q(j) - \frac{\Delta^2 \rho(j) m \lambda}{4 \rho(j) \Gamma(1 - \mu)} \right] = \infty. \tag{3}
\]

Then every solution of equation (1) is oscillatory.

**Proof.** Assume, for the sake of contradiction that \( u(\ell) \) is a non oscillatory solution of (1), say \( u(\ell) > 0 \) on \([L_0, \infty)\) for some sufficiently large \( L_0 \geq \ell_0 \). Define the following Riccati transformation function
\[
\omega(\ell) = -\frac{\gamma(\ell) \phi(u(\ell)) \Delta^\mu u(\ell)}{G(\ell)}, \quad \text{for } \ell \geq L_0 \tag{4}
\]

\[
\Delta \omega(\ell) = -\left[ \rho(\ell + 1) \Delta \left( \frac{\gamma(\ell) \phi(u(\ell)) \Delta^\mu u(\ell)}{G(\ell)} \right) + \frac{\gamma(\ell) \phi(u(\ell)) \Delta^\mu u(\ell)}{G(\ell)} \Delta \rho(\ell) \right]
\]
i.e.,
\[
\Delta \omega(\ell) = -\rho(\ell + 1) \left[ \frac{G(\ell + 1) \Delta \left[ \gamma(\ell) \phi(u(\ell)) \Delta^\mu u(\ell) \right]}{G(\ell) G(\ell + 1)} - \frac{\gamma(\ell + 1) \phi(u(\ell + 1)) \Delta^\mu u(\ell + 1)}{G(\ell) G(\ell + 1)} \Delta G(\ell) \right] - \frac{\gamma(\ell) \phi(u(\ell)) \Delta^\mu u(\ell)}{G(\ell)} \Delta \rho(\ell).
\]

Applying equations (1), (4) and Lemma 4 leads to
\[
\Delta \omega(\ell) = \frac{\omega(\ell)}{\rho(\ell)} \Delta \rho(\ell) - \frac{\rho(\ell + 1) \gamma(\ell + 1) \phi(u(\ell + 1)) \Delta^\mu u(\ell + 1) \Gamma(1 - \mu) \Delta^\mu u(\ell)}{G(\ell) G(\ell + 1)}
\]
i.e.,
\[
\Delta \omega(\ell) > \frac{\omega(\ell)}{\rho(\ell)} \Delta \rho(\ell) - \frac{\rho(\ell + 1) \gamma(\ell + 1) \phi(u(\ell + 1)) \Delta^\mu u(\ell + 1) \Gamma(1 - \mu) \Delta^\mu u(\ell)}{G(\ell) G(\ell + 1)}
\]

or
\[
\Delta \omega(\ell) > \frac{\Delta \rho(\ell)}{\rho(\ell)} \omega(\ell) - \frac{\rho(\ell + 1) \gamma(\ell + 1) \phi(u(\ell + 1)) \Delta^\mu u(\ell)}{G(\ell)} \eta(\ell) + \frac{\Delta \rho(\ell)}{\rho(\ell + 1) \gamma(\ell + 1) \phi(u(\ell + 1)) \Delta^\mu u(\ell)} \frac{\Delta \rho(\ell)}{\rho(\ell + 1) \gamma(\ell + 1) \phi(u(\ell + 1)) \Delta^\mu u(\ell)} \frac{\Delta \rho(\ell)}{\rho(\ell + 1) \gamma(\ell + 1) \phi(u(\ell + 1)) \Delta^\mu u(\ell)} \Gamma(1 - \mu)
\]

Using \((H_2) - (H_4)\) with equation (4), we have
\[
\Delta \omega(\ell) \geq \frac{\Delta \rho(\ell)}{\rho(\ell)} \omega(\ell) - \frac{\rho(\ell + 1) \gamma(\ell + 1) \phi(u(\ell + 1)) \Delta^\mu u(\ell)}{G(\ell)} \eta(\ell) + \delta \rho(\ell) q(\ell) + \frac{\Gamma(1 - \mu)}{\rho(\ell) m \lambda} \omega^2(\ell) \tag{5}
\]

From the assumption if \( x(\ell) > 0 \), then we can consider \( \alpha_1, \beta_1 \geq L_0 \) with \( \alpha_1 < \beta_1 \) such that \( \eta(\ell) \leq 0 \) on the interval \([\alpha_1, \beta_1]\). Similarly if \( x(\ell) < 0 \), then we can choose \( \alpha_2, \beta_2 \geq L_0 \) with \( \alpha_2 < \beta_2 \) such that \( \eta(\ell) \geq 0 \) on the interval \([\alpha_2, \beta_2]\). So \( \frac{\rho(\ell + 1) \gamma(\ell + 1) \phi(u(\ell + 1)) \Delta^\mu u(\ell)}{G(\ell)} \leq 0, \ell \in [\alpha_i, \beta_i], i = 1, 2 \). Now equation (5) reduce to
\[
\Delta \omega(\ell) \geq \frac{\Delta \rho(\ell)}{\rho(\ell)} \omega(\ell) + \delta \rho(\ell) q(\ell) + \frac{\Gamma(1 - \mu)}{\rho(\ell) m \lambda} \omega^2(\ell). \tag{6}
\]
Now summing up the above inequality from $\ell_1$ to $\ell - 1$ leads to

$$
\sum_{j=\ell_1}^{\ell-1} \Delta \omega(j) > \sum_{j=\ell_1}^{\ell-1} \left[ \frac{\Gamma(1-\mu)}{m\lambda \rho(j)} \omega^2(j) + \frac{\Delta \rho(j)}{\rho(j)} \omega(j) + \delta \rho(j) q(j) \right].
$$

(7)

Now equation (7) becomes

$$
\omega(\ell) - \omega(\ell_1) > \sum_{j=\ell_1}^{\ell-1} \delta \rho(j) q(j) + \left[ \left( \frac{\Gamma(1-\mu)}{m\lambda \rho(j)} \right)^{\frac{1}{2}} \omega(j) + \frac{\Delta \rho(j)}{2} \left( \frac{m\lambda}{\rho(j)\Gamma(1-\mu)} \right)^{\frac{1}{2}} \right]^2 - \sum_{j=\ell_1}^{\ell-1} \frac{\Delta^2 \rho(j)m\lambda}{4\rho(j)\Gamma(1-\mu)}
$$

$$
> \sum_{j=\ell_1}^{\ell-1} \left[ \left( \frac{\Gamma(1-\mu)}{m\lambda \rho(j)} \right)^{\frac{1}{2}} \omega(j) + \frac{\Delta \rho(j)}{2} \left( \frac{m\lambda}{\rho(j)\Gamma(1-\mu)} \right)^{\frac{1}{2}} \right]^2 + \sum_{j=\ell_1}^{\ell-1} \left[ \delta \rho(j) q(j) - \frac{\Delta^2 \rho(j)m\lambda}{4\rho(j)\Gamma(1-\mu)} \right]
$$

In view of equation (3), there exists a $\ell_2 > \ell_1$ such that

$$
\omega(\ell) > \sum_{j=\ell_1}^{\ell-1} \left[ \left( \frac{\Gamma(1-\mu)}{m\lambda \rho(j)} \right)^{\frac{1}{2}} \omega(j) + \frac{\Delta \rho(j)}{2} \left( \frac{m\lambda}{\rho(j)\Gamma(1-\mu)} \right)^{\frac{1}{2}} \right]^2
$$

Let us choose $Q_1(\ell) = \sum_{j=\ell_1}^{\ell-1} \left[ \left( \frac{\Gamma(1-\mu)}{m\lambda \rho(j)} \right)^{\frac{1}{2}} \omega(j) + \frac{\Delta \rho(j)}{2} \left( \frac{m\lambda}{\rho(j)\Gamma(1-\mu)} \right)^{\frac{1}{2}} \right]^2$, which implies that $\omega(\ell) > Q_1(\ell) > 0$. Now,

$$
\Delta Q_1(\ell) = \Delta \sum_{j=\ell_1}^{\ell-1} \left[ \left( \frac{\Gamma(1-\mu)}{m\lambda \rho(j)} \right)^{\frac{1}{2}} \omega(j) + \frac{\Delta \rho(j)}{2} \left( \frac{m\lambda}{\rho(j)\Gamma(1-\mu)} \right)^{\frac{1}{2}} \right]^2
$$

$$
= \left[ \left( \frac{\Gamma(1-\mu)}{m\lambda \rho(\ell)} \right)^{\frac{1}{2}} \omega(\ell) + \frac{\Delta \rho(\ell)}{2} \left( \frac{m\lambda}{\rho(\ell)\Gamma(1-\mu)} \right)^{\frac{1}{2}} \right]^2
$$

$$
= \frac{\Gamma(1-\mu)}{m\lambda \rho(\ell)} \omega^2(\ell) + \frac{\omega(\ell)}{\rho(\ell)} \Delta \rho(\ell) + \frac{m\lambda \Delta^2 \rho(\ell)}{4\rho(\ell)\Gamma(1-\mu)}
$$

$$
\Delta Q_1(\ell) > \frac{\Gamma(1-\mu)}{m\lambda \rho(\ell)} \omega^2(\ell) > \frac{\Gamma(1-\mu)}{m\lambda \rho(\ell)} Q_1^2(\ell)
$$

$$
\frac{\Delta Q_1(\ell)}{Q_1^2(\ell)} > \frac{\Gamma(1-\mu)}{m\lambda \rho(\ell)}
$$

$$
\frac{\Gamma(1-\mu)}{m\lambda \rho(\ell)} < \frac{\Delta Q_1(\ell)}{Q_1^2(\ell)} < \frac{Q_1(\ell + 1)}{Q_1^2(\ell)} - \frac{Q_1(\ell)}{Q_1^2(\ell)}.
$$

Since $\Delta Q_1(\ell)$ is non decreasing, we have

$$
\frac{\Gamma(1-\mu)}{m\lambda \rho(\ell)} < \frac{Q_1(\ell + 1)}{Q_1^2(\ell + 1)} - \frac{1}{Q_1(\ell)} < \frac{1}{Q_1(\ell + 1)} - \frac{1}{Q_1(\ell)}
$$
Now summing up from $\ell_2$ to $\ell - 1$ yields

$$
\sum_{j=\ell_2}^{\ell-1} \frac{\Gamma(1-\mu)}{m \lambda \rho(j)} < \sum_{j=\ell_2}^{\ell-1} \left[ \frac{1}{Q_1(j+1)} - \frac{1}{Q_1(j)} \right] < \frac{1}{Q_1(\ell)} - \frac{1}{Q_1(\ell_2)}
$$
or

$$
\frac{\Gamma(1-\mu)}{m \lambda} \sum_{j=\ell_2}^{\ell-1} \frac{1}{\rho(j)} < \frac{1}{Q_1(\ell_2)}.
$$

Taking limit as $\ell \to \infty$, we obtain

$$
\lim_{\ell \to \infty} \frac{\Gamma(1-\mu)}{m \lambda} \sum_{j=\ell_2}^{\ell-1} \frac{1}{\rho(j)} < \frac{1}{Q_1(\ell_2)}
$$

which contradicts to (2). The proof of theorem is complete.

**Theorem 6.** Suppose that conditions $(H_1) - (H_4)$ hold and assume that for any $L \geq \ell_0$, there exists $\alpha_1, \beta_1, \alpha_2, \beta_2$ such that $L \leq \alpha_1 \leq \beta_1 \leq \alpha_2 < \beta_2$ satisfying $\eta(\ell) \left\{ \begin{array}{ll} \leq 0, & \ell \in [\alpha_1, \beta_1] \\ \geq 0, & \ell \in [\alpha_2, \beta_2] \end{array} \right.$.

If there exists a positive function $\rho(\ell)$ and a double positive sequence $H(\ell, j)$ such that,

$$
H(\ell, \ell) = 0 \quad \text{for} \quad \ell \geq \ell_0; \\
H(\ell, j) > 0 \quad \text{for} \quad \ell > j \geq \ell_0; \\
\Delta_2 H(\ell, j) = H(\ell, j+1) - H(\ell, j) \leq 0 \quad \text{for} \quad \ell > j \geq \ell_0.
$$

If

$$
\lim_{\ell \to \infty} \text{Sup} \frac{1}{H(\ell, \ell_0)} \sum_{j=\ell_0}^{\ell-1} \left[ \delta \rho(j) q(j) H(\ell, j) - \frac{h^2(\ell,j) \rho(j) \omega^2(j)}{4H(\ell,j) \Gamma(1-\mu)} \right] = \infty,
$$

where $h_+ (\ell, j) = \Delta_2 H(\ell, j) + \frac{\Delta \rho(j)}{\rho(j)} H(\ell, j)$, then $u(\ell)$ is oscillatory.

**Proof.** Assume, for the sake of contradiction, that $u(\ell)$ is non oscillatory solution of (1). We can assume that $u(\ell) \neq 0$ on $[L_0, \infty)$ for sufficiently large $L_0 \geq \ell_0$. Proceeding as Theorem 5. and apply the assumptions $(H_1) - (H_4)$, we arrive at equation (6). Now by multiplying the inequality (6) by $H(\ell, \ell)$ and summing up from $\ell_1$ to $\ell - 1$, we obtain

$$
\sum_{j=\ell_1}^{\ell-1} \Delta \omega(j) H(\ell, j) > \sum_{j=\ell_1}^{\ell-1} \left[ \frac{\Gamma(1-\mu)}{\rho(j) m \lambda} \omega^2(j) + \frac{\Delta \rho(j)}{\rho(j)} \omega(j) + \delta \rho(j) q(j) \right] H(\ell, j)
$$

$$
- \sum_{j=\ell_1}^{\ell-1} \delta \rho(j) q(j) H(\ell, j) > - \sum_{j=\ell_1}^{\ell-1} \Delta \omega(j) H(\ell, j) + \sum_{j=\ell_1}^{\ell-1} \left[ \frac{\Gamma(1-\mu)}{\rho(j) m \lambda} \omega^2(j) + \frac{\Delta \rho(j)}{\rho(j)} \omega(j) \right] H(\ell, j)
$$
i.e.,

$$
- \sum_{j=\ell_1}^{\ell-1} \mu \rho(j) q(j) H(\ell, j) > \sum_{j=\ell_1}^{\ell-1} \Delta \omega(j) H(\ell, j) + \sum_{j=\ell_1}^{\ell-1} \left[ \frac{\Gamma(1-\mu)}{\rho(j) m \lambda} \omega^2(j) H(\ell, j) + \frac{\Delta \rho(j)}{\rho(j)} \omega(j) H(\ell, j) \right].
$$
Application of summation by parts formula yields

\[- \sum_{j=\ell_1}^{\ell-1} \Delta \omega(j) H(\ell, j) = H(\ell, \ell_1) \omega(\ell_1) + \sum_{j=\ell_1}^{\ell-1} \omega(j + 1) \Delta_2 H(\ell, j). \tag{10}\]

Using equation (10) in equation (9) yields

\[- \sum_{j=\ell_1}^{\ell-1} \delta \rho(j) q(j) H(\ell, j) > H(\ell, \ell_1) \omega(\ell_1) + \sum_{j=\ell_1}^{\ell-1} \omega(j + 1) \Delta_2 H(\ell, j) + \sum_{j=\ell_1}^{\ell-1} \left[ \Gamma(1 - \mu) \right] \frac{\omega^2(j) H(\ell, j) + \Delta \rho(j)}{\rho(j)} \omega(j) H(\ell, j) \right].

Since, $\Delta \omega(\ell) < 0$, which is equivalent to $\omega(\ell) > \omega(\ell + 1)$, we obtain

\[- \sum_{j=\ell_1}^{\ell-1} \delta \rho(j) q(j) H(\ell, j) > H(\ell, \ell_1) \omega(\ell_1) + \sum_{j=\ell_1}^{\ell-1} \omega(j + 1) \Delta_2 H(\ell, j) + \sum_{j=\ell_1}^{\ell-1} \left[ \Gamma(1 - \mu) \right] \frac{\omega^2(j + 1) H(\ell, j) + \Delta \rho(j)}{\rho(j)} \omega(j + 1) H(\ell, j) \right].

Let us consider $h_+(\ell, j) = \Delta_2 H(\ell, j) + \frac{\Delta \rho(j) H(\ell, j)}{\rho(j)}$

\[- \sum_{j=\ell_1}^{\ell-1} \delta \rho(j) q(j) H(\ell, j) > H(\ell, \ell_1) \omega(\ell_1) + \sum_{j=\ell_1}^{\ell-1} \left[ \Gamma(1 - \mu) \right] \frac{\omega^2(j + 1) H(\ell, j) + h_+(\ell, j) \omega(j + 1)}{\rho(j)} \omega(j) H(\ell, j) \right].

or

\[- \sum_{j=\ell_1}^{\ell-1} \delta \rho(j) q(j) H(\ell, j) > H(\ell, \ell_1) \omega(\ell_1) - \sum_{j=\ell_1}^{\ell-1} \frac{h^2_+(\ell, j)}{4} \frac{\rho(j) m \lambda}{H(\ell, j) \Gamma(1 - \mu)} + \sum_{j=\ell_1}^{\ell-1} \left[ \left( \frac{H(\ell, j) \Gamma(1 - \mu)}{\rho(j) m \lambda} \right)^{1/2} \omega(j + 1) + \frac{h_+(\ell, j)}{2} \left( \frac{\rho(j) m \lambda}{H(\ell, j) \Gamma(1 - \mu)} \right)^{1/2} \right] \omega(j + 1) \right]

i.e.,

\[- \sum_{j=\ell_1}^{\ell-1} \delta \rho(j) q(j) H(\ell, j) > H(\ell, \ell_1) \omega(\ell_1) - \sum_{j=\ell_1}^{\ell-1} \frac{h^2_+(\ell, j)}{4} \frac{\rho(j) m \lambda}{H(\ell, j) \Gamma(1 - \mu)}. \tag{11}\]

Since, $\Delta_2 H(\ell, j) \leq 0$ for $\ell > j \geq \ell_0$ and $0 < H(\ell, \ell_1) \leq H(\ell, \ell_0)$ for $\ell > j \geq \ell_0$. Now the inequality (11) becomes,

\[- \sum_{j=\ell_1}^{\ell-1} \delta \rho(j) q(j) H(\ell, j) < - H(\ell, \ell_1) \omega(\ell_1) + \sum_{j=\ell_1}^{\ell-1} \frac{h^2_+(\ell, j)}{4} \frac{\rho(j) m \lambda}{H(\ell, j) \Gamma(1 - \mu)} \]

or

\[- \sum_{j=\ell_1}^{\ell-1} \left[ \delta \rho(j) q(j) H(\ell, j) - \frac{h^2_+(\ell, j)}{4} \frac{\rho(j) m \lambda}{H(\ell, j) \Gamma(1 - \mu)} \right] < H(\ell, \ell_0) \omega(\ell_1). \tag{12}\]
Since, $0 < H(\ell, j) \leq H(\ell, \ell_0)$ for $\ell > j \geq \ell_0$ and $0 < \frac{H(\ell, j)}{H(\ell, \ell_0)} \leq 1$ for $\ell > j \geq \ell_0$. Hence it follows from that

$$\frac{1}{H(\ell, \ell_0)} \sum_{j=\ell_0}^{\ell-1} \left[ \delta \rho(j) q(j) H(\ell, j) - \frac{h^2_+(\ell, j)}{4} \frac{j \rho(j) m \lambda}{H(\ell, j) \Gamma(1-\mu)} \right] \leq \frac{1}{H(\ell, \ell_0)} \sum_{j=\ell_0}^{\ell-1} \left[ \delta \rho(j) q(j) H(\ell, j) - \frac{h^2_+(\ell, j)}{4} \frac{j \rho(j) m \lambda}{H(\ell, j) \Gamma(1-\mu)} \right] + \omega(\ell_1) \text{ using (12)}$$

or

$$\frac{1}{H(\ell, \ell_0)} \sum_{j=\ell_0}^{\ell-1} \left[ \delta \rho(j) q(j) H(\ell, j) - \frac{h^2_+(\ell, j)}{4} \frac{j \rho(j) m \lambda}{H(\ell, j) \Gamma(1-\mu)} \right] \leq \omega(\ell_1) + \sum_{j=\ell_0}^{\ell-1} \delta \rho(j) q(j).$$

Now letting $\ell \to \infty$, we get

$$\lim_{\ell \to \infty} \sup_{\ell_0} \frac{1}{H(\ell, \ell_0)} \sum_{j=\ell_0}^{\ell-1} \left[ \delta \rho(j) q(j) H(\ell, j) - \frac{h^2_+(\ell, j)}{4} \frac{j \rho(j) m \lambda}{H(\ell, j) \Gamma(1-\mu)} \right] < \omega(\ell_1) + \sum_{j=\ell_0}^{\ell-1} \mu \rho(j) q(j)$$

which contradicts to (8). Hence the proof of theorem is complete.

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