Weil groups and $F$-isocrystals

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Introduction

The subject of this paper is an explicit formula for the fundamental class $u_{L/K}$ of a finite Galois extension $L/K$ of local fields with group $G$. This formula appears implicitly in in [7, Ch. XIII §5 Ex. 2] (see the discussion just before theorem 1.6.1), where Serre sketches a proof that this class is indeed the fundamental class; his argument uses the theory of class formations (and thus depends on the basic results of local class field theory). In this paper we give a proof of this assertion independent of any results of class field theory; we use instead the Dieudonné-Manin structure theorem for $F$-isocrystals over an algebraically closed field. The formula yields a direct proof of a generalization of Dwork’s formula for the norm residue symbol (this is also in [7]) and this in turn yields a simple proof that the cup product with $u_{L/K}$ is an isomorphism. In other words the cohomological treatment of local class field theory can dispense with the Tate-Nakayama theorem.

As a by-product of our main construction we get a construction of the relative Weil group $W_{L/K}$ as an automorphism group, and a simple proof of the local Weil-Shafarevich theorem. When $K = \mathbb{Q}_p$ this construction of $W_{L/K}$ is due to Morava [6], who expressed it in terms of Lubin-Tate groups. This construction leads directly to a direct proof of Shafarevich’s theorem (i.e. the local Shafarevich-Weil theorem); this answers a question raised in [6].

Our sign conventions for homological algebra are those of Bourbaki [1]. We will follow Deligne [3] and Tate [9] in normalizing the reciprocity law so that in an unramified extension the arithmetic Frobenius corresponds to the inverse of a uniformizer; this makes the formulas come out slightly simpler.

1 The Fundamental Class

We first recall a few general results about $F$-isocrystals on a field. In the next section we use these to construct the fundamental class of of a finite Galois extension of local fields.

1.1 $F$-isocrystals. In this section $K$ will denote a complete nonarchimedean discretely valued field with algebraically closed residue field $k$ of characteristic $p > 0$. We do not assume that $K$ itself has characteristic zero. We fix a power
q of p and assume that K has a lifting \( \sigma \) of the qth power Frobenius of \( k \). Since K is complete, \( \sigma \) fixes some uniformizer \( \pi \) of \( K \), which we also fix.

As usual an F-isocrystal in \( K \) is a \( K \)-vector space \( V \) of finite dimension endowed with a \( \sigma \)-linear automorphism \( F \), called the Frobenius structure of \( V \) (or of \( (V, F) \)). If it is necessary to specify \( \sigma \) (and \( q \)) we will say \( (\sigma, F) \)-isocrystal.

The basic structure theorem of Dieudonné and Manin for F-isocrystals, originally proven when \( K \) is absolutely unramified extend without modification to the present case, even when \( K \) has positive characteristic. Let \( K_\sigma[F] \) be the Dieudonné ring, i.e. the noncommutative polynomial ring in \( F \) with coefficients in \( K \) and multiplication defined by \( F a = a^\sigma F \). Any F-isocrystal \( (V, F) \) is a direct sum of simple objects; these all have the form

\[
V(\lambda) = K_\sigma[F]/K_\sigma[F](F^d - \pi^n)
\]

where \( \lambda = r/d \in \mathbb{Q} \) in lowest terms. There is one such simple object for every \( \lambda \in \mathbb{Q} \); if \( \lambda = 0 \) we take \( a = 0, b = 1 \). The rational number \( \lambda \) is the slope of \( V(\lambda) \). More generally the slopes of an F-isocrystal \( M \) on \( K \) are the \( \lambda \) such that \( V(\lambda) \) occurs as a summand of \( M \).

We denote by \( D(\lambda) \) the endomorphism algebra of \( V(\lambda) \). A theorem of Diedonné asserts that it is a simple division algebra over \( K \) with invariant \( -\lambda \). In particular if \( \lambda = r/d \) in lowest terms, \( D(\lambda) \) has dimension \( d^2 \) over \( K \). In fact it is not hard to check that \( D(\lambda) \) is a cyclic algebra over \( K \), but Diedonné’s original proof works just as well.

To compute the slopes of a simple F-isocrystal we will use the following proposition, which is an easy consequence of Katz’s “basic slope estimate” [5]. Let \( \mathcal{O}_K \) be the integer ring of \( K \). As always, a lattice in a (finite-dimensional) \( K \)-vector space \( V \) is a finite \( \mathcal{O}_K \)-module \( V_0 \) such that \( K \otimes \mathcal{O}_K V_0 \simeq V \).

**1.1.1 Proposition** The slopes of an F-isocrystal \( (V, F) \) are contained in the interval \([\lambda, \mu]\) if and only if for some lattice \( V_0 \subset V \) there are constants \( C, D \) such that

\[
\pi^{[n\mu]+C}V_0 \subseteq F^n(V_0) \subseteq \pi^{[n\lambda]+D}V_0
\]

independently of \( n \).

One can also prove this by observing first that if [1.1.2] holds for one lattice it holds for any lattice; this reduces to the case where \( V \) is one of the simple F-isocrystals [1.1.1] where it can be proven by direct computation.

Note that for any \( n > 0 \) the slopes of \( (V, F) \) are determined by the slopes of the \((\sigma^n, F)\)-isocrystal \( (V, F^n) \). We will use this trick in the section 1.3.

**1.2 The algebra \( L_K \).** From now on \( L/K \) is a Galois extension of nonarchimedean local fields (of any characteristic) with group \( G \) and degree \( d \). Denote by \( k_L/k \) the residual extension and set \( q = |k| = p^f \). We fix a completion \( K^{\text{ur}} \) of a maximal unramified extension of \( K \) and \( \sigma \) as before; we can of course assume that \( \pi \in K \).
The algebra
\[ L_K = K^{nr} \otimes_K L \] (1.2.1)
is a finite direct sum of finite extensions of \( K^{nr} \), in fact of copies of the completion of a maximal unramified extension \( L^{nr} \) of \( L \). Then \( G \) (resp. \( \sigma \)) act on \( L_K \) via the action on the right (resp. left) factor, and we use the same symbol \( \sigma \) for this extension of \( \sigma \) to \( L_K \). The actions of \( G \) and commute. For clarity we will usually use “pre-exponential” notation for the actions of \( \sigma \) and elements \( g \in G \), i.e. \( \sigma x \) and \( gx \) instead of \( \sigma(x) \) or \( g(x) \).

We can identify \( L_K \) with a direct sum of copies of \( L^{nr} \) in a canonical way by fixing a commutative diagram
\[ K^{nr} \quad \rightarrow \quad L^{nr} \]
\[ \downarrow \quad \quad \quad \downarrow \]
\[ K \quad \rightarrow \quad L \]
(1.2.2)
Then \( L \) and \( K^{nr} \) are identified with subfields of \( L^{nr} \), and the map
\[ L_K \rightarrow (L^{nr})^f \]
\[ x \otimes y \mapsto (\sigma^{-i}(x)y)_{0 \leq i < f} \] (1.2.3)
is an isomorphism. In fact if \( L_0/K \) is the maximal unramified extension of \( K \) in \( L \) then \([L_0 : K] = f\) and there is a similar decomposition \( K^{nr} \otimes_K L_0 \simeq (K^{nr})^f \), and the natural map \( K^{nr} \otimes_K L_0 \rightarrow L^{nr} \) is an isomorphism. With respect to \( 1.2.3 \) the \( K^{nr} \)-vector space structure of \( L_K \) is
\[ a(x_0, x_1, \ldots, x_{f-1}) = (ax_0, \sigma^{-1}(a)x_1, \ldots, \sigma^{-(f-1)}(a)x_{f-1}) \] (1.2.4)
and the natural embedding \( L \rightarrow L_K \) is
\[ a \mapsto (a, a, \ldots, a) \] (1.2.5)
Then \( \sigma|L = \sigma^f \) where \( \sigma|L \) is the lifting of Frobenius to \( L \), and \( \sigma : L_K \rightarrow L_K \) is
\[ (x_0, x_1, \ldots, x_{f-1}) \mapsto (\sigma(x_{f-1}), x_0, x_1, \ldots, x_{f-2}) \] (1.2.6)
with respect to \( 1.2.3 \).

When \( L/K \) is unramified we may identify \( K^{nr} \simeq L^{nr} \) canonically, and the diagram \( 1.2.2 \) identifies \( L \) with a subfield of \( K^{nr} \). The arithmetic Frobenius \( \sigma_{arith} = 1 \otimes (\sigma|L) \) is a generator of \( \text{Gal}(L/K) \) and for the decomposition \( 1.2.3 \) is given by
\[ \sigma_{arith}(a_0, a_1, \ldots, a_{f-1}) = (\sigma a_1, \sigma a_2, \ldots, \sigma a_{f-1}, \sigma a_0) \] (1.2.7)

The field norm \( N_{L/K} : L^\times \rightarrow K^\times \) extends to a homomorphism
\[ N_{L/K} : L_K^\times \rightarrow (K^{nr})^\times \] (1.2.8)
which is the norm for the ring extension $K^{nr} \to L$. If $x \in L_K^\times$ corresponds $(x_i)$ under the decomposition $\text{1.2.3}$ $N_{L/K}(a)$ is given by

$$N_{L/K}(x) = \prod_{0 \leq i < f} \sigma^i(N_{K^{nr}/K^{nr}}(x_i)). \quad (1.2.9)$$

We denote by $w_{L/K}: L^\times_K \to \mathbb{Q}$ the homomorphism

$$w_{L/K}(x) = [L : K]^{-1}v(N_{L/K}(x)) \quad (1.2.10)$$

where the numerical factor guarantees that $w_{L/K}$ extends the valuation of $K^\times \subset L_K^\times$. If $v_L$ is the unique valuation of $L^{nr}$ extending those of $L$ and $K^{nr}$,

$$w_{L/K}(x) = \sum_{0 \leq i < f} f^{-1}v_L(x_i). \quad (1.2.11)$$

### 1.3 Lubin-Tate F-isocrystals

By definition the fundamental class $u_{L/K} \in H^2(G, L^\times)$ is an element of invariant $1/d$; in other words it is the class of a central simple $K$-algebra of invariant $1/d$. From the discussion in the last section we can take this $K$-algebra the endomorphism algebra of any $F$-isocrystal $(V)$ on $K^{nr}$ of rank $d$ and invariant $-1/d$. Since we want to construct a $G$-valued 2-cocycle for $u_{L/K}$ we will need an explicit splitting of $\text{End}(V)$ over $L$, which amounts to giving and embedding $L \to \text{End}(V)$ over $K$.

If $\alpha \in L_K^\times$,

$$F_\alpha : L_K \to L_K \quad F_\alpha(x) = \alpha \cdot \sigma x \quad (1.3.1)$$

defines a Frobenius structure on $L_K$, and we denote by $L_K(\alpha)$ the F-isocrystal $(L_K, F_\alpha)$. We denote by $D(\alpha)$ the endomorphism ring of $L_K(\alpha)$. Since $\sigma \otimes 1$ is $L$-linear, right multiplication defines an embedding $L \to D(\alpha)$.

#### 1.3.1 Proposition

For any finite separable extension $L/K$ and $\alpha \in L_K^\times$, the F-isocrystal $L_K(\alpha)$ is isopentic of slope $w_{L/K}(\alpha)$.

**Proof.** As we observed at the end of section [1.1] it suffices to prove that the $d$th iterate of $L_K(\alpha)$ has slope $dw_{L/K}(\alpha)$ for any fixed $d$. We will take $d = [L : K]$. If $\alpha \in L_K^\times$ and $x \in L_K$ decompose as $\alpha = (\alpha_i)$ and $x = (x_i)$ according to [1.2.3] the Frobenius structure $F_\alpha$ is

$$F_\alpha(x_i) = (\alpha_0 \cdot \sigma x_{f-1}, \alpha_1 x_0, \ldots, \alpha_{f-1} x_{f-2})$$

by [1.2.6] iterating $f$ times yields

$$F_\alpha^f(x_i) = (\beta_1 \cdot \sigma^f x_i)$$

where $\sigma_L = \sigma^f$ as before, and

$$\beta_1 = \alpha_0 \alpha_1 \cdots \alpha_{f-1} \cdot \sigma_L(\alpha_{f-1} \cdots \alpha_1)$$
Since $\sigma_L$ does not change the valuation of an element of $L^{nr}$, all of the $\beta_i$ have the same valuation, namely

$$v_L(\beta_i) = \sum_{0 \leq j < f} v_L(\alpha_j).$$

Set $e = e_{L/K}$; then $d = ef$ and

$$F^d(x_i) = F^e(\beta_i, \sigma_L x_i) = (1 + \sigma_L + \cdots + \sigma_L^{f-1}) \beta_i \sigma_L x_i$$

and note that

$$v_L(1 + \sigma_L + \cdots + \sigma_L^{f-1}) = ev_L(\beta_i) = ef \sum_{0 \leq j < f} 1_f v_L(\alpha_j) = dw_{L/K}(\alpha).$$

Let $L_0 \subset L_K$ be the lattice corresponding to $(O_{L^{nr}})^f \subset (L^{nr})^f$. Since $dw_{L/K}(\alpha) \in \mathbb{Z}$, the last equality says that

$$F^d(L_0) = \pi^{dw_{L/K}(\alpha)} L_0$$

for any uniformizer $\pi$ of $K$. Then $F^{nd}(L_0) = \pi^{ndw_{L/K}} L_0$, and proposition 1.1.1 shows that $(L_K, F^d)$ has slope $dw_{L/K}$, as required.

### 1.3.1 The unramified case.

Let’s consider the case when $L/K$ unramified of degree $d = f$. Pick a $r \in \mathbb{Z}$ that is relatively prime to $d$ and set

$$\alpha = (\pi^r, 1, \ldots, 1) \quad (1.3.2)$$

where we identify $L_K \simeq (L^{nr})^f = (K^{nr})^d$ as before. By the proposition $L_K(\alpha)$ has slope $r/d$, and as it has rank $d$ it must be isomorphic to $V_K(r/d)$. In fact this is easy to write down an explicit isomorphism from explicit formula for $F_\alpha$ in the proof of proposition 1.3.1.

### 1.4 The fundamental extension.

We can now identify $D(\alpha)$ with a crossed product algebra associated to a certain an $L^\times$-valued 2-cocycle for $G$. In the case of the fundamental class we then identify this 2-cocycle as the class of a certain Yoneda 2-extension.

#### 1.4.1 Lemma

The sequence

$$1 \to L^\times \to L_K^\times \to L_K^{nr} \xrightarrow{w_{L/K}} d^{-1} \mathbb{Z} \to 0$$

is exact.

**Proof.** The only nonobvious point is that the image of $\sigma - 1$ is kernel of $w_{L/K}$, and it is clearly contained in the kernel. We first observe that the subgroup of $\mathbb{Z}^f$ consisting of $(n_i)$ such that $\sum \alpha_i = 0$ is spanned by elements with a 1 in position $i$, $-1$ in position $i + 1$ (addition modulo $f$) and zeros elsewhere. From
this it follows that any \((x_i) \in \ell_{K}\) in the kernel of \(w_{L/K}\) is congruent modulo the image of \(\sigma - 1\) to a \((x_i)\) such that \(v_L(x_i) = 0\) for all \(x\). Since \(\sigma^f - 1\) is a retraction of \((L^s)^\times\) onto its subgroup of elements of valuation zero, for each \(i\) there is a \(c_i\) such that \(b_i = \sigma^f(c_i)/c_i\). Then \(b = \sigma^f(c)/c\) in \(\ell_{K}\), and since

\[
\sigma^f - 1 = (\sigma - 1)(\sigma^{f-1} + \cdots + \sigma + 1)
\]

we conclude that \(b\) is in the image of \(\sigma - 1\).

Suppose now \(L/K\) is Galois with group \(G\). For any \(s \in G\), \(w_{L/K}(s^{-1} \alpha) = 0\) and the lemma shows that there is a \(\beta_s \in \ell_{K}\) such that

\[
\sigma^{-1} \beta_s = s^{-1} \alpha
\]

well defined up to a factor in \(L^\times\). From (1.4.1) we see that

\[
u_s : L_{K} \to L_{K} \quad u_s(x) = \beta_s \cdot ^sx
\]

commutes with \(F_{\alpha}\):

\[
F_{\alpha}(u_s(x)) = \alpha \cdot ^s(\beta_s \cdot ^sx)
= \alpha \cdot ^s \beta_s \cdot ^{s}s x = ^s \alpha \cdot ^s \beta_s \cdot ^{s}s x
= \beta_s \cdot ^s(^s(\alpha \cdot ^sx)) = u_s(F_{\alpha}(x)).
\]

and is thus an automorphism of \(L_{K}(\alpha)\).

A quick calculation shows that the \(u_s\) satisfy the relations

\[
u_s u_t = a_{s,t} u_{st}, \quad u_{s \ell} = ^s \ell u_s
\]

for all \(\ell \in L\) and \(s, t \in G\), where

\[
a_{s,t} = {}^s \beta_t \beta_{st}^{-1} \beta_s.
\]

1.4.1 Theorem The endomorphism algebra \(D_{L/K}(\alpha)\) is isomorphic to the crossed product algebra associated to the 2-cocycle (1.4.4) associated to \(\alpha\). The extension corresponding to \(D_{L/K}(\alpha)\) is isomorphic to

\[
1 \to L^\times \to W(\alpha) \xrightarrow{p} G_{L/K} \to 1
\]

where \(W(\alpha)\) the set of pairs \((\beta, s) \in \ell_{K}^\times \times G\) satisfying

\[
\sigma^{-1} \beta = s^{-1} \alpha
\]

with the composition law

\[
(\beta, s)(\gamma, t) = (\beta \cdot ^s \gamma, st)
\]

and \(i, p\) are the evident inclusion and projection.
Proof. The $1.4.3$ are the defining relations for the crossed product algebra $A$ defined by the cocycle $1.4.4$. The universal property of such algebras (e.g. [2, §16 a° 9 Prop. 12]) yields a nonzero $K$-homorphism $A \to D(\alpha)$ (since nonzero on $K$) which is necessarily an isomorphism since both sides have dimension $d^2$.

For the second part we need only remark that $i : K \to D_{L/K}(\alpha)$ is the canonical injection, the extension group corresponding to $D_{L/K}(\alpha)$ is the set of $u \in D_{L/K}(\alpha)$ satisfying $i \circ s = \text{ad}(u) \circ i$, and it is easily checked that these have the form $u = \beta s$ with $\beta \in L_K$ satisfying $1.4.5$.

1.4.1 Corollary For any finite Galois extension $L/K$ of degree $d$ and any $\alpha \in L_K^\times$ such that $w_{L/K}(\alpha) = 1/d$, the 2-cocycle $1.4.4$ represents the fundamental class of $L/K$.

When $w_{L/K}(\alpha) = 1/d$, the extension $W(\alpha)$ representing the fundamental class is by definition the Weil group of $L/K$ and denoted by $W_{L/K}$. Up to isomorphism of extensions it is independent of the choice of $\alpha$; in fact since $H^1(G, L^\times) = 0$ it is well-defined up to inner automorphisms by elements of $L^\times$.

We note finally that just as the Galois group $\text{Gal}(L/K)$ is (by definition!) an automorphism group, the same is true for $W(\alpha)$: by construction it consists of automorphisms $\sigma$ of $L_K(\alpha)$ satisfying $i \circ s = \text{ad}(u) \circ i$ for some $s \in G$. When $K = \mathbb{Q}_p$, this observation is due to Morava [6].

1.4.1.1 The unramified case again. Suppose $L/K$ is unramified, so that $f = d$ and we identify $K^{nr} \simeq L^{nr}$. The Galois group $G = \text{Gal}(L/K)$ is generated by the “arithmetic Frobenius” $\sigma_{\text{arith}} = 1 \otimes (\sigma|L)$ and we identify $G \simeq \mathbb{Z}/d\mathbb{Z}$ by means of the generator $\sigma_{\text{arith}}$; in the formulas to follow we also identify elements of $\mathbb{Z}/d\mathbb{Z}$ with integers in the range $[0, d)$. Choose $a \in K^\times$ and let $\alpha = (a^{-1}, 1, \ldots, 1)$. Then $1.2.7$ yields

$$\sigma_{\text{arith}}^{-1} a = (a, 1, \ldots, 1, a^{-1}, 1, \ldots, 1)$$

and a $\beta_i \in L_K^\times$ satisfying $\sigma^{-1} \beta_i = \sigma_{\text{arith}}^{-1} a$ is

$$\beta_i = (a^{-1}, a^{-1}, \ldots, a^{-1}, 1, \ldots, 1). \quad (1.4.7)$$

From this it follows easily that

$$\beta_i (\sigma_{\text{arith}}^i (\beta_j)) \beta_{i+j}^{-1} = \begin{cases} a^{-1} & i + j < d \\ 1 & i + j \geq d \end{cases}.$$ 

The coboundary of the constant 1-cochain with value $a$ is the constant 2-cocycle with value $a$. Adding this to the previous cocycle yields the well-known 2-cocycle

$$a_{i,j} = \begin{cases} 1 & i + j < d \\ a & i + j \geq d \end{cases}.$$
representing the fundamental class in the unramified case.

A class in \( \text{Br}(L/K) \) gives rise, via the isomorphisms
\[
\text{Br}(L/K) \cong H^2(G, L^\times) \cong \text{Ext}_G^2(\mathbb{Z}, L^\times)
\]
(1.4.8)
to a Yoneda 2-extension of \( \mathbb{Z} \) by \( L^\times \). We can identify the extension corresponding to the fundamental class:

1.4.2 Theorem Suppose \( [L : K] = d \) and write \( w = w_{L/K} \). The class of \( u_{L/K} \in \text{Br}(L/K) \) is the class of the extension
\[
0 \to L^\times \to L^\times_K \xrightarrow{\sigma^{-1}} L^\times_K \xrightarrow{dw} \mathbb{Z} \to 0.
\]
(1.4.9)
in \( \text{Ext}_G^2(\mathbb{Z}, L^\times) \).

Proof. By lemma 1.4.1, 1.4.9 is exact and we just need to recall the usual recipe (e.g. \([1, \S 7\text{ no.}\ 3]\) for associating to it a class in \( \text{Ext}_G^2(\mathbb{Z}, L^\times) \cong H^2(G, L^\times) \). If \( P \to \mathbb{Z} \) is a resolution of \( \mathbb{Z} \) by projective \( \mathbb{Z}[G] \)-modules, there is a morphism of complexes
\[
\begin{array}{ccccccccccc}
P_3 & \xrightarrow{d_3} & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & \mathbb{Z} & \to 0 \\
\downarrow{f_3} & & \downarrow{f_2} & & \downarrow{f_1} & & \downarrow{f_0} & & \downarrow{} & \\
1 & \to & L^\times & \to & L^\times_K & \xrightarrow{\sigma^{-1}} & L^\times_K & \xrightarrow{dw} & \mathbb{Z} & \to 0
\end{array}
\]
(1.4.10)
unique up to homotopy. The map \( f_2 \) lies in the kernel of \( \text{Hom}(P_2, L^\times) \to \text{Hom}(P_3, L^\times) \), and its image in \( H^2(G, L^\times) \) is the class of 1.4.9. For \( P \), we take the standard bar complex:
\[
d_0(\[\]) = 1, \quad d_1([s]) = s[\ ] - [\ ], \quad d_2([s][t]) = s[t] - [st] + [s].
\]
If we set
\[
f_0(\[\]) = \alpha, \quad f_1([s]) = \beta_s, \quad f_2([s][t]) = a_{s,t}
\]
the commutativity of 1.4.10 shows that
\[
dw(\alpha) = 1, \quad \sigma^{-1}\beta_s = s^{-1}\alpha, \quad a_{s,t} = s\beta_t\beta_{st}^{-1}\beta_s.
\]
The first equality shows that \( D_{L/K}(\alpha) \) represents the fundamental class \( u_{L/K} \), and the others show that \( (a_{s,t}) \) represents \( D_{L/K}(\alpha) \) in \( H^2(G, L^\times) \). Since the class of 1.4.9 is \( (a_{s,t}) \) we are done.

1.5 The norm residue symbol. We denote by
\[
\eta_{L/K} : G^{ab} \to K^\times/N_{L/K}(L^\times)
\]
(1.5.1)
the map induced by the cup product
\[
\cup u_{L/K} : \hat{H}^2(G, \mathbb{Z}) \to \hat{H}^0(G, L^\times)
\]
and the standard identification of these Tate cohomology groups with those in 
1.5.1 We know that \( \eta_{L/K} \) is an isomorphism and we denote its inverse, the 
norm residue symbol by
\[
\theta_{L/K} = \eta_{L/K}^{-1} : K^\times / N_{L/K}(L^\times) \to G^{ab}.
\]
We will also write \( \theta_{L/K} \) for the composite
\[
K^\times \to K^\times / N_{L/K}L^\times \to G^{ab}.
\]
and we will also use the traditional notation for
\[
(a, L/K) = \theta_{L/K}(a)
\]
where \( a \) is an element of either \( K^\times \) or \( K^\times / N_{L/K}L^\times \).

That \( \eta_{L/K} \) is an isomorphism is of course a consequence of the Tate-Nakayama 
theorem, but in fact this follows from the exact sequence 1.4.9 and standard 
arguments (as in [7, Ch. XIII §5 Ex. 2]) once it is shown that \( L_K^\times \) is a cohomologically trivial \( G \)-module. In fact if \( I \subseteq G \) is the inertia subgroup, \( L_K^\times \) is the \( G \)-module induced from the \( I \)-module \((L^{nr})^\times\), and the latter is cohomologically trivial since the norm \((L^{nr})^\times \to (K^{nr})^\times\) is surjective. We should point out that 
the role of the extension 1.4.9 is obscured in [7] since in the case that Serre is 
considering (quasi-finite residue fields) the sequence \( L_K^\times \xrightarrow{1-F} L_K^\times \to \mathbb{Z} \) is 
not always exact. However the argument works since the cohomology group here is 
cohomologically trivial; c.f. [7, loc. cit.].

1.5.1 Theorem Suppose \( L/K \) is a Galois extension of local fields with group 
\( G \) and set \( d = [L : K] \). Let \( \alpha \) be an element of \( L_K^\times \) such that \( w_{L/K}(\alpha) = 1/d \), 
and let \( s \in G \). If \( \beta \in L_K^\times \) satisfies
\[
\sigma^{-1} \beta = s^{-1} \alpha
\]
then
\[
\eta_{L/K}(s) = N_{L/K}(\beta) \mod N_{L/K}L^\times.
\]
Proof. This follows from the formula [7] for the cup product and the definition 
1.4.4
\[
\bar{s} \cup u_{L/K} = \prod_{t \in G} a_{t,s} = \prod_{t \in G} t^{\beta_s \beta_{ts}^{-1}} = N_{L/K}(\beta_s)
\]
since we can take \( \beta = \beta_s \).

When \( L/K \) is totally ramified we may identify \( L_K = L^{nr} \) and take \( \alpha = \pi^{-1} \) 
for any uniformizer \( \pi \) of \( L^{nr} \); in this case 1.5.4 is equivalent to the formula 
proven by Dwork [4].
1.6 Successive extensions. The functorial behavior of the norm residue symbol follows from theorem 1.5.1 by direct computations. This really could have been left as an exercise, but these constructions will be needed in section 2.2.

We first need to extend the constructions in section 1.2. Fix a finite Galois extension $E/K$ with group $G_{E/K}$ and a subfield $L$ of $G$ containing $K$. The Galois group of $E/L$ will be denoted by $G_{E/L}$, and if $L/K$ is Galois its group is $G_{L/K}$.

The transitivity isomorphism $K^{nr} \otimes_K E \simeq (K^{nr} \otimes_K L) \otimes_L E$ may be written

$$E_K \simeq L_K \otimes_L E.$$ \hspace{1cm} (1.6.1)

This shows that $E_K$ has a canonical structure of a free $L_K$-module, and we denote by

$$N^K_{E/L}: E_K^\times \to L_K^\times$$ \hspace{1cm} (1.6.2)

the norm map. When $E/L/K$ is $L/K/K$ this is the norm map 1.2.8 introduced earlier. The norm $N^K_{E/L}$ is equivariant for the action of $\sigma$ on $E_K$ and $L_K$. If $E/K$ is Galois with group $G_{E/K}$, it is also equivariant for the action of $G_{E/K}$ in the sense that

$$E_K^\times \xrightarrow{N^K_{E/L}} L_K^\times \xrightarrow{s}$$ \hspace{1cm} (1.6.3)

commutes for any $s \in G_{E/K}$. Finally, for a 3-fold extension $F/E/L/K$ the norm is transitive:

$$N^K_{E/L} \circ N^K_{F/E} = N^K_{F/L}.$$ \hspace{1cm} (1.6.4)

Applying this in the case $E/L/K/K$ and invoking 1.2.10, we find that

$$w_{L/K}(N_{E/L}(\beta)) = [E:L]w_{E/K}(\beta)$$ \hspace{1cm} (1.6.5)

for $\beta \in E_K$.

The decomposition 1.2.3 extends in an obvious way to a decomposition of $E_K$ as an $E_L$-module. As before we fix a completion $L^{nr}$ of a maximal unramified extension of $L$ and a morphism $K^{nr} \to L^{nr}$ making 1.2.2 commutative. Tensoring the isomorphism of $L$ with $E$ yields a direct sum decomposition

$$E_K \simeq L_K \otimes_L E \simeq \bigoplus_{0 \leq i < f} (\sigma^i)^* L^{nr} \otimes_L E \simeq \bigoplus_{0 \leq i < f} (\sigma^i)^* E_L$$ \hspace{1cm} (1.6.6)

of $K^{nr} \otimes_K E$-bimodules. This direct sum decomposition is, like 1.2.3, canonically determined by a choice of diagram 1.2.2. The isomorphism 1.6.6 is equivariant for the action of $G_{E/L}$ on both sides.
For any 3-fold extension $F/E/L/K$ of local fields the norms $N^K_{F/E}$ and $N^L_{F/E}$ are related as follows: if

$$F_K \simeq \bigoplus_{0 \leq i < f} (\sigma^i)^* F_L$$

$$E_K \simeq \bigoplus_{0 \leq i < f} (\sigma^i)^* E_L$$

are the decompositions (1.6.6) for $F$ and $E$ and

$$\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{f-1}) \in F_K \simeq \bigoplus_{0 \leq i < f} (\sigma^i)^* F_L$$

then

$$N^K_{F/E}(\gamma) = (N^L_{F/E}(\gamma_0), N^L_{F/E}(\gamma_1), \ldots, N^L_{F/E}(\gamma_{f-1})$$  (1.6.7)

for the above decomposition of $E_K$. This can be seen by fixing a basis of $F$ as an $E$-vector space and using the fact that determinants relative to a direct sum of rings respect the direct sum decomposition.

Recall that for the decomposition 1.2.3 of $L_K$ the action of $\sigma$ is given by 1.2.6 where $\sigma_L$ is the Frobenius of $L^{nr}$ with respect to its own residue field. It follows that the action of $\sigma$ as $\sigma \otimes 1$ on $E_K = K^{nr} \otimes K E$ is then given by

$$(\beta_0, \beta_1, \ldots, \beta_{f-1}) \mapsto (\sigma_L(\beta_{f-1}), \beta_0, \beta_1, \ldots, \beta_{f-2})$$  (1.6.8)

for the decomposition 1.6.6, where now $\sigma_L$ denotes the extension of $\sigma_L$ acting on $L^{nr}$ to $\sigma_L \otimes 1$ acting on $E_L$.

We now define homomorphisms

$$\iota^E_{L/K}, \delta^E_{L/K} : E_L^\times \to E_K^\times$$

by

$$\iota^E_{L/K}(\alpha) = (\alpha, 1, \ldots, 1)$$

$$\delta^E_{L/K}(\beta) = (\beta, \beta, \ldots, \beta)$$  (1.6.9)

with respect to the decomposition 1.6.6. Since the decomposition 1.6.6 is equivariant for the action $G_{E/L}$, so are $\iota^E_{L/K}$ and $\delta^E_{L/K}$. Corresponding to the formula 1.6.3 we have, for $\alpha \in E_L^\times$,

$$w_{E/K}(\iota^E_{L/K}(\alpha)) = \frac{1}{[L : K]} w_{L/K}(\alpha).$$  (1.6.10)

For any 3-fold extension $F/E/L/K$ the maps $\iota$ and $\delta$ are compatible with the norms. In fact 1.6.7 implies that the diagrams

$$\begin{array}{ccc}
F^\times_L & \xrightarrow{\delta^E_{L/K}} & F^\times_K \\
\downarrow N^E_{F/E} & & \downarrow N^E_{F/E} \\
E^\times_L & \xrightarrow{\delta^E_{L/K}} & E^\times_K
\end{array}$$

$$\begin{array}{ccc}
F^\times_L & \xrightarrow{\iota^E_{L/K}} & F^\times_K \\
\downarrow N^E_{F/E} & & \downarrow N^E_{F/E} \\
E^\times_L & \xrightarrow{\iota^E_{L/K}} & E^\times_K
\end{array}$$  (1.6.11)
are commutative.

We need one more fact about these maps. The composite

\[ L_L^\times \xrightarrow{\delta_{L/K}} L_K^\times \xrightarrow{N_{L/K}^K} K_K^\times \]

is a homomorphism \((L^{nr})^\times \to (K^{nr})^\times\). It is not the usual norm map for \(L^{nr}/K^{nr}\), however:

**1.6.1 Lemma** For \(\ell \in L^\times \subset (L^{nr})^\times\),

\[ N_{L/K}^K(\delta_{L/K}^L(\ell)) = N_{L/K}(\ell) \tag{1.6.12} \]

where the norm on the right is the norm for \(L/K\).

**Proof.** Since \(N_{L/K}^K\) is the map 1.2.3, the equality 1.2.9 shows that

\[ N_{L/K}^K(\delta_{L/K}^L(\ell)) = \prod_{0 \leq i < f} \sigma_i(N_{L^{nr}/K^{nr}}(\ell)). \]

As before let \(L_0\) be the maximal unramified extension of \(K\) in \(L\), and identify it with a subfield of \(K^{nr}\). Since \(L^{nr} \simeq L \otimes_{L_0} K^{nr}\), \(N_{L^{nr}/K^{nr}}(\ell) = N_{L/L_0}(\ell)\) and the product on the right hand side is

\[ \prod_{0 \leq i < f} \sigma_i(N_{L/L_0}(\ell)) = N_{L_0/K}(N_{L/L_0}(\ell)) = N_{L/K}(\ell) \]

since \(\sigma|L_0\) generates the Galois group of \(L_0/K\).

**1.7 Functorial properties.** We can now deduce the behavior of the norm residue symbol with respect to field automorphisms and successive extensions, starting with the latter. As before we suppose that \(L/K\) is Galois, and denote by

\[ \pi_{E/L}^K : G_{E/K} \to G_{L/K} \]

the canonical homomorphism. To calculate \(\eta_{E/K}\) we choose \(\alpha \in E_L^\times\) such that \(w_{E/K}(\alpha) = 1/[E : K]\); if \(s \in G_{E/K}\) we pick \(\beta \in E_K^\times\) such that \(\sigma^{-1}\beta_s = s^{-1}\alpha\); then \(\eta_{E/K}(s) = N_{L/K}(\beta_s)\) (note that \(N_{L/K} = N_{E/K}^L\) here). On the other hand 1.6.5 says that

\[ w_{L/K}(N_{E/L}^K(\alpha)) = \frac{[E : L]}{[E : K]} = \frac{1}{[L : K]} \]

and we may use \(N_{E/L}(\alpha)\) to calculate \(\eta_{L/K}\). By equivariance

\[ \sigma^{-1}N_{E/L}(\beta) = s^{-1}N_{E/L}(\alpha) \tag{1.7.1} \]

so from 1.6.3 and 1.5.4 we get

\[ \eta_{E/K}(s) = \eta_{L/K}(\pi_{E/L}^K(s)) \pmod{N_{L/K}L^\times} \tag{1.7.2} \]
which is equivalent to the formula
\[ \pi^K_{E/L}(a, E/K) = (a, L/K). \] (1.7.3)

We next allow \( L \) to be any intermediate extension and take \( s \in G_{E/L} \). To compute \( \eta_{E/L}(s) \) we choose \( \alpha \in E \times L \) such that \( \wp_{E/L}(\alpha) = [E : L]^{-1} \) and \( \beta \in E^\times_L \) such that \( \sigma^{-1}\beta = s^{-1}\alpha \); then \( \eta_{E/L}(s) \) is the class of \( N_{E/L}(\beta) \) in \( L^\times/N_{E/L}E^\times \) (recall here that \( N_{E/L} \) here is \( N_{L^\times/E} \)). On the other hand by 1.6.10 we have \( \wp_{E/K}(\iota^E_{L/K}(\alpha)) = 1/[E : K] \), so we can use \( \iota^E_{L/K}(\alpha) \) to compute \( \eta_{E/K}(s) \). The formula 1.2.6 for the action of \( \sigma \) on \( L^\times K^\times \) shows that
\[ \sigma^{-1}\delta^K_{E/L}(\beta) = (\sigma^{-1}\beta, 1, \ldots, 1) = (s^{-1}\alpha, 1, \ldots, 1) = s^{-1}\iota^E_{L/K}(\alpha) \]
and thus
\[ \eta_{E/K}(s) = N^K_{E/K}(\delta^K_{L/K}(\beta)) \mod N_{E/K}E^\times. \]
To evaluate the right hand side we consider the diagram

\[
\begin{array}{c}
E^\times_L \\
\downarrow N_{L/K}^L \\
L^\times_L \\
\downarrow N_{L/K}^L \\
K^\times_L
\end{array}
\quad \begin{array}{c}
E^\times_K \\
\downarrow N_{E/K}^L \\
L^\times_K \\
\downarrow N_{E/K}^L \\
K^\times_K
\end{array}
\]

in which the square is the case \( E/L/L/K \) of the commutative square on the left of 1.6.11 and the right hand triangle commutes by the transitivity of norms. From this we see that
\[ N^K_{E/K}(\delta^K_{E/L}(\beta)) = N^K_{L/K}(\delta^L_{E/K}(N^L_{E/L}(\beta))) = N^K_{L/K}(\delta^L_{E/K}(\eta_{E/L}(s))). \]
Since \( N_{E/L}(\beta_L) \in L^\times \), lemma 1.6.1 shows that
\[ N^K_{L^\times/K}(\delta^L_{E/K}(\eta_{E/L}(s))) = N_{L/K}(\eta_{E/L}(s)) \]
and combining the previous equalities yields
\[ \eta_{E/K}(s) = N_{L/K}(\eta_{E/L}(s)) \mod N_{E/K}E^\times \] (1.7.4)
which is equivalent to
\[ (a, E/L) = (N_{L/K}(a), E/K) \] (1.7.5)
for \( a \in L^\times \).

Again with \( s, t \in G_{E/K} \) and \( \sigma^{-1}\beta_s = t^{-1}\alpha \), the equality
\[ \sigma^{-1}(t\beta_s) = t(\sigma^{-1}\beta_s) = ts^{-1}\alpha = tsts^{-1}(\alpha) \]
implies that
\[ \eta_{E/t(L)}(tsts^{-1}) = t\eta_{E/L}(s) \] (1.7.6)
modulo appropriate subgroups. This says that
\[ s(a, E/L)s^{-1} = (s(a), E/s(L)) \]  
(1.7.7)

for \( s \in G_{E/K} \) and \( a \in L \).

The last compatibility asserts that the transfer \( \text{Ver} : G_{E/K}^b \to G_{L/K}^b \) corresponds via the norm residue symbol to the map \( K^\times/N_{E/K}E^\times \to L^\times/N_{E/L}E^\times \) induced by the inclusion. From our point of view this is best understood in terms of the ideas of the next section (proposition 2.2.2).

## 2 Weil Groups

Usually the absolute Weil group is defined directly in terms of the absolute Galois group; the formalism of the relative Weil groups is an immediate consequence. In this section we show how this formalism follows from our construction of the relative Weil groups. This leads to a proof of Shafarevich’s theorem.

### 2.1 The formalism of Weil groups

We fix a Galois extension \( E/K \) with group \( G_{E/K} \) and let \( L \) be an extension of \( K \) in \( E \) (not necessarily Galois over \( K \)). As before \( G_{E/L} \) is the Galois group of \( E/L \), and similarly for \( G_{L/K} \) when \( L/K \) is Galois. Our aim in this section is to define morphisms \( \iota : W_{E/L} \to W_{E/K} \) and, when \( L/K \) is Galois, \( \pi_{E/L} : W_{E/K} \to W_{L/K} \). Since \( W_{L/K} \) by definition is only defined up to inner automorphisms it will be necessary to “rigidify” it by a particular choice of “realization” \( W(\alpha) \), which as before is the set of pairs \((\beta, s) \in L^\times \times G_{L/K} \) satisfying 1.4.5 with the composition law 1.4.6. To indicate the fields involved we write \( W_{L/K}(\alpha) \) for \( W(\alpha) \), so a homomorphism such as \( \pi_{E/L} : W_{E/K} \to W_{L/K} \) should be understood as a homomorphism \( W_{E/K}(\alpha) \to W_{L/K}(\alpha) \) for particular choices (usually implicit) of \( \alpha, \alpha' \) appropriate to \( L/K \) and \( E/K \).

In fact everything we need is contained in the formulas 1.7.1 and 1.7.4. The first of them shows that \( N_{E/L} : \hat{E}_K^\times \to \hat{E}_E^\times \) extends to a homomorphism
\[ \pi_{E/L} : W_{E/K}(\alpha) \to W_{L/K}(N_{E/L}(\alpha)) \]
(2.1.1)

making commutative the diagram
\[
\begin{array}{ccc}
1 & \to & E^\times & \to & W_{E/K} & \to & G_{E/K} & \to & 1 \\
& & \downarrow N_{E/L} & & \downarrow \iota_{E/L} & & \downarrow \pi_{E/L} & \\
1 & \to & L^\times & \to & W_{L/K} & \to & G_{L/K} & \to & 1
\end{array}
\]  
(2.1.2)

The transitivity of the norm shows that the maps \( \pi \) just constructed are transitive:
\[ \pi_{E/L}^K \circ \pi_{F/E}^K = \pi_{E/F}^K \]  
(2.1.3)
for a 3-fold extension $F/E/L/K$; here of course the groups $W_{F/K}$, $W_{E/K}$ and $W_{L/K}$ are realized as $W_{F/K}(\alpha)$, $W_{E/K}(N_{F/L}(\alpha))$ and $W_{L/K}(N_{F/L}(\alpha))$ for appropriate $\alpha$; from now on we will not be explicit about this.

In the special case when $E/L/K$ is $L/K/K$, we have $W_{K/K} = K^\times$, and the definition of $\pi_{L/K}^K$ shows that the diagram

$$
\begin{array}{c}
W_{L/K} \\ \pi_{L/K}^K \downarrow \\
K^\times \\
\end{array} \quad \begin{array}{c}
G_{L/K} \\ \eta_{L/K} \downarrow \\
K/N_{L/K}L^\times \\
\end{array}
$$

(2.1.4)

commutes, where the horizontal maps are the natural projections. In fact if $\alpha \in L_K^\times$ is used to define $W_{L/K}$, $(\beta, s) \in W_{L/K}$ implies that $\sigma^{-1}\beta = s^{-1}\alpha$, so that

$$(\pi_{L/K}(\beta, s) \mod N_{L/K}L^\times) = (N_{L/K}(\beta) \mod N_{L/K}L^\times) = \eta_{L/K}(s).$$

That (2.1.4) is commutative could be expressed by saying that $\pi_{L/K} : W_{L/K} \to K^\times$ is a lifting of the inverse reciprocity map. It could also be rephrased as saying that the diagram

$$
\begin{array}{c}
W_{L/K}^{ab} \\ \pi_{L/K}^{ab} \downarrow \\
K^\times \\
\end{array} \quad \begin{array}{c}
G_{L/K}^{ab} \\ \theta_{L/K} \downarrow \\
\end{array}
$$

(2.1.5)

is commutative.

Now [1,7.3] shows that $\delta_{E/L} : L_K^\times \to E_K^\times$ extends to a homomorphism

$$
i_{E/L}^K : W_{L/K}(\alpha) \to W_{E/K}(i_{E/L}(\alpha))$$

$$i_{E/L}^K(\beta, s) = (\delta_{E/K}(\beta), s)$$

(2.1.6)

making commutative a diagram

$$
\begin{array}{c}
1 \\ \downarrow \\
L^\times \\
\end{array} \quad \begin{array}{c}
W_{L/K} \\ \pi_{E/L}^K \downarrow \\
G_{L/K} \\
\downarrow \\
1
\end{array} \quad \begin{array}{c}
1 \\ \downarrow \\
E^\times \\
\end{array} \quad \begin{array}{c}
W_{E/K} \\ \pi_{E/L}^K \downarrow \\
G_{E/K} \\
1
\end{array}
$$

(2.1.7)

where the extreme vertical arrows are the canonical inclusions. For a 3-fold extension $F/E/L/K$ we have

$$\delta_{F/E}^K \circ \delta_{E/L}^K = \delta_{F/L}^K$$

(2.1.8)
which shows that the $i$ are transitive:

$$i^K_{F/E} \circ i^K_{E/L} = i^K_{F/L}. \quad (2.1.9)$$

Finally the right hand diagram of 1.6.11 implies that for any 3-fold extension $F/E/L/K$ the diagram

$$
\begin{array}{ccc}
W_{F/L} & \xrightarrow{\pi^K_{F/E}} & W_{E/L} \\
\downarrow{i^K_{E/K}} & & \downarrow{i^K_{L/K}} \\
W_{F/K} & \xrightarrow{\pi^K_{F/E}} & W_{E/K}
\end{array}
$$

is commutative. This can be expressed by saying that if we use the maps $i_{E/L}$ to identify $W_{E/L}$ with a subgroup of $W_{E/K}$, then these identifications are compatible with the canonical projection maps $\pi$ (assuming as always that compatible choices of $\alpha$ are made in each place).

2.2 The Shafarevich-Weil theorem. It is evident from the construction that $i^K_{E/L} : W_{E/L} \to W_{E/K}$ is injective. It is also true that $\pi^K_{E/L} : W_{E/K} \to W_{E/L}$ is surjective, although this is not completely obvious from our construction.

2.2.1 Proposition The connecting homomorphism $\partial : G_{E/L} \to L^\times/N_{E/L}^\times$ arising from the diagram 2.1.2 is the inverse norm residue homomorphism $\eta_{E/L}$.

Proof. For $s \in G_{E/L}$, $\partial(s)$ is computed as follows: lift $s$ to an element $(\beta, s) \in W_{E/K}$; then $\pi_{E/L}(\beta, s) \in W_{L/K}$ lies in $L^\times \subset W_{L/K}$ and $\partial(s)$ is the image of $\pi_{E/L}(\beta, s)$ in $L^\times/N_{E/L}E^\times$. The commutative diagram 2.1.10 applied to $E/L/L/K$ is

$$
\begin{array}{ccc}
W_{E/L} & \xrightarrow{\pi_{E/L}} & W_{L/L} \\
\downarrow{i_{L/K}} & & \downarrow{i_{L/K}} \\
W_{E/K} & \xrightarrow{\pi_{E/L}} & W_{L/K}
\end{array}
$$

which shows that $\partial(s)$ can also be computed by lifting $s$ to $(\beta, s) \in W_{E/L}$, and then $\partial(s)$ is the class of $\pi_{E/L}(\beta, s) \in W_{L/L} = L^\times$ in $L^\times/N_{E/L}E^\times$. Now $\pi_{E/L}(\beta, s) = N_{E/L}(\beta)$ where $N_{E/L}$ is the norm $E_L^\times \to (L^nr)^\times$, and if $\alpha \in E_L^\times$ is used to define $W_{E/L}$, $\beta$ satisfies $s^{-1}\beta = s^{-1}\alpha$, so $\partial(s) = \eta_{E/L}(s)$ by Dwork’s formula.

2.2.1 Corollary For any successive extension $E/L/K$ of local fields with $E/K$ and $L/K$ Galois, the homomorphism $\pi^K_{E/L} : W_{E/K} \to W_{L/K}$ is surjective.

Proof. Since $\partial = \eta_{E/L}$ is surjective, this follows from the snake lemma.
2.2.1 Lemma For any Galois extension of local fields the transfer \( \text{Ver} : W_{L/K}^{ab} \to L^\times \) is the composite

\[
W_{L/K}^{ab} \xrightarrow{\pi_{L/K}^{ab}} K^\times \to L^\times
\]

where we identify \( W_{K/K} \simeq K^\times \), and the second map is the inclusion.

Proof. Recall the definition of the transfer \( G^{ab} \to H^{ab} \) for a subgroup \( H \subseteq G \) of finite index: choose a section \( \theta : H \setminus G \to G \) of the projection \( G \to H \setminus G \), and for \( s, t \in G \) define \( x_{s,t} \in H \) by \( \theta(Ht)s = x_{t,s} \theta(Hts) \); then

\[
\text{Ver}(s) = \prod_{t \in H \setminus G} x_{t,s}
\]

where the product is in \( H^{ab} \). In our case \( G = W_{L/K} \) and we identify \( H = L^\times \) with its abelianization. For \( t \in G_{L/K} = G/H \) we take \( \theta(t) = (\beta_t, t) \) for some appropriate \( \beta_t \in L_{K}^\times \); then

\[
\theta(t)(\beta, s) = (\beta_t, t)(\beta, s) = (\beta_t \cdot t, ts) = (\beta_t \beta_t^{-1}, 1) \theta(ts)
\]

where the last equality is justified since \( \beta_t \beta_t^{-1} \in L^\times \). For this \( \theta \) then we have \( x_{t,s} = (\beta, 1) \), and then

\[
\text{Ver}(\beta, s) = \prod_{t \in G_{L/K}} (\beta_t \beta_t^{-1}, 1) = (N_{L/K}(\beta), 1)
\]

which proves the assertion. \( \square \)

We can now prove the last compatibility for the norm residue symbol.

2.2.2 Proposition For any 2-fold extension \( E/L/K \) of local fields with \( E/K \) Galois,

\[
K^\times \xrightarrow{\theta_{E/K}} L^\times \quad \quad (2.2.1)
\]

\[
G^{ab}_{E/K} \xrightarrow{\text{Ver}} G^{ab}_{E/L}
\]

is commutative, where the upper horizontal map is the inclusion.

Proof. In view of the diagram 2.1.5 applied to \( E/K \) and \( E/L \), and since for any Galois extension \( L/K \) the projection \( \pi_{E/K} : W_{E/K} \to K^\times \) is surjective, it suffices to show that the diagrams

\[
W_{E/K}^{ab} \xrightarrow{\text{Ver}} W_{E/L}^{ab} \quad \quad (2.2.2)
\]

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are commutative. The first just expresses the functoriality of the transfer. The second can be embedded in the diagram

\[
\begin{array}{c}
W_{ab}^{E/K} \xrightarrow{\text{Ver}} W_{ab}^{E/L} \\
\pi_{ab}^{E/K} \downarrow \quad \downarrow \pi_{ab}^{E/L} \\
K \times \xrightarrow{i} L \times \xrightarrow{j} E^x
\end{array}
\]

in which \(i\) and \(j\) are the inclusions. By lemma 2.2.1 \(i \circ \pi_{E/K}^{ab}\) and \(j \circ \pi_{E/L}^{ab}\) are the transfer homomorphisms for the subgroups \(E^x \subset W_{E/K}\) and \(E^x \subset W_{E/L}\) respectively. Since the transfer is transitive for chains of subgroups, the outside square in this diagram is commutative, and since \(L^x \to E^x\) is injective, the inside square is as well.

In terms of the norm residue symbol, the commutativity of 2.2.1 says that

\[
\text{Ver}(a, E/K) = (a, E/L)
\]

for \(a \in K^\times\). Proposition 2.2.2 is less elementary than the previous compatibilities since the homomorphism \(G_{E/K}^{ab} \to G_{E/L}^{ab}\) is not induced by a homomorphism \(G_{E/K} \to G_{E/L}\).

We can now prove Shafarevich’s theorem [9]. Suppose that \(E/L/K\) are extensions of local fields with \(E/K\) and \(L/K\) Galois. We define a map

\[
\text{Sh} : G_{E/K} \to W_{L/K}/NE_{L/E}^x
\]

as follows: for \(s \in G_{E/K}\) choose \(x \in W_{E/K}\) mapping to \(s\) under the natural projection \(W_{E/K} \to G_{E/K}\). Then \(\pi_{E/L}^{K}(x) \in W_{L/K}\) is well-defined up to a factor in \(NE_{L/E}^x \subset L^x \subset W_{L/K}\), and we define \(\text{Sh}(s)\) to be the class of \(\pi_{E/L}^{K}(x)\) in \(W_{L/K}/NE_{L/E}^x\).

### 2.2.1 Theorem (Shafarevich)

Suppose that \(E/L/K\) are extensions of local fields with \(E/K\) and \(L/K\) Galois. The diagram

\[
\begin{array}{c}
1 \qquad G_{E/L} \quad G_{E/K} \quad G_{L/K} \quad 1 \\
\downarrow \eta_{E/L} \quad \downarrow \quad \downarrow \text{Sh} \\
1 \qquad L^x/NE_{L/E}^x \quad W_{L/K}/NE_{L/E}^x \quad G_{L/K} \quad 1
\end{array}
\]

is commutative. In particular if \(E/L\) is abelian, \(\text{Sh}\) is an isomorphism.

**Proof.** The commutatvity of the right hand square is clear from the construction. As to the one on the left, suppose \(W_{E/L}\) has been defined by some
\( \alpha \in \mathbb{E}_L^\times \) and that \( W_{E/K} \) is defined by \( \iota_{E/K}^E(\alpha) \), as in our construction of \( i : W_{E/L} \to W_{E/K} \). Then if \( s \in G_{E/L} \), any lifting of it to \( W_{E/K} \) lies in the image of \( \iota_{E/K}^E \) and we may suppose it has the form \( (\delta_{L/K}^E(\beta), s) \), with \( \beta \in L_K^\times \) satisfying \( \sigma^{-1} \beta = s^{-1} \alpha \). Now the diagram 2.1.10 in the case when \( F/E/L/K \) is \( \mathbb{E}/\mathbb{L}/\mathbb{L}/K \) is

\[
\begin{array}{ccc}
W_{E/L} & \xrightarrow{\pi_{E/L}^K} & W_{L/L} \\
\iota_{E/K}^E & \downarrow & \iota_{L/K}^L \\
W_{E/K} & \xrightarrow{\pi_{E/L}^K} & W_{L/K}
\end{array}
\]

which shows that \( \text{Sh}(s) \) can be identified with the image of

\[
\pi_{E/L}^K(\beta, s) = (N_{E/L}(\beta), 1) \in W_{L/K}.
\]

Since \( N_{E/L}(\beta) = \eta_{E/L}(s) \) the left hand square is commutative. Finally if \( E/L \) is abelian \( \eta_{E/L} \) is an isomorphism, hence so is \( \text{Sh} \).

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