Ample vector bundles with sections vanishing on special varieties

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Abstract

Let $E$ be an ample vector bundle of rank $r$ on a complex projective manifold $X$ such that there exists a section $s \in \Gamma(E)$ whose zero locus $Z = \{s = 0\}$ is a smooth submanifold of the expected dimension $\dim X - r := n - r$. Assume that $Z$ is not minimal; we investigate the hypothesis under which the extremal contractions of $Z$ can be lifted to $X$. Finally we study in detail the cases in which $Z$ is a scroll, a quadric bundle or a del Pezzo fibration.

1 Introduction

A very classical and natural way of classifying complex projective manifolds $X$ consists in slicing $X$ with a number of general hyperplane sections obtaining in this way a complex manifold of smaller dimension which is likely classifiable. Then one should ascend the geometrical properties of this new manifold and obtain a complete description of $X$. To stress the classical flavor of this approach it is sometime called Apollonius method [Fuj90]. The hard part of the Apollonius method are the ascending properties; in this paper we will consider this problem in a slightly more general set up.

Let $E$ be an ample vector bundle of rank $r$ on $X$ such that there exists a section $s \in \Gamma(E)$ whose zero locus $Z = \{s = 0\}$ is a smooth submanifold of the expected dimension $\dim X - r := n - r$.

Assume that $Z$ is not minimal in the sense of Mori’s theory, that is $-K_Z$ is not nef; thus $Z$ has at least one extremal ray [Mor82, Cone Theorem] and an associated extremal contraction [KMM87, Kawamata-Shokurov base point free theorem].

Our question will then be, under which condition this contraction can be lifted to the ambient variety, determining its structure; this general situation is studied in section 3; suppose that $F_Z$ is an extremal face in $NE(Z)$ with supporting

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divisor $K_Z + \tau H_Z$; a lifting property is proved under the assumption that

$$H_Z \text{ is the restriction of an ample line bundle } H \text{ on } X. \quad (1.1)$$

Next we discuss some special situations in which the assumption (1.1) can be avoided. In the rest of the paper we consider some special cases, namely if $Z$ is a scroll, a quadric bundle or a del Pezzo fibration; the results are described in theorems (4.1), (5.1), (6.1) and corollaries.

These results generalize classical ones by L. Bădescu (see [Băd82a], [Băd81], [Băd82b]) and A.J. Sommese (see [Som76] and chapter 5 of [BS95], in particular Theorem 5.2.1 which should be compared with our results in section 3) and more recent ones by A. Lanteri and H. Maeda ([LM95], [LM96], [LM97]), who were the first ones to study the problem of special sections of ample vector bundles.

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## 2 Notations and generalities

We use the standard notation from algebraic geometry. In particular it is compatible with that of [KMM87] to which we refer constantly; we suggest to the reader also the survey [AW97]. We just explain some special definitions and propositions used frequently.

In this paper $X$ will always stand for a smooth complex projective variety of dimension $n$. Let $\text{Div}(X)$ the group of Cartier divisors on $X$; denote by $K_X$ the canonical divisor of $X$, an element of $\text{Div}(X)$ such that $O_X(K_X) = \Omega_X^n$.

Let $N_1(X) = \{\text{1-cycles}\} \otimes \mathbb{R}$, $N^1(X) = \{\text{divisors}\} \otimes \mathbb{R}$ and $<\text{NE}(X)> = \{\text{effective 1-cycles}\}$; the last is a closed cone in $N_1(X)$. Let also $\rho(X) = \dim_{\mathbb{R}} N^1(X)$.

Suppose that $K_X$ is not nef, that is there exists an effective curve $C$ such that $K_X \cdot C < 0$.

**Theorem 2.1** [KMM87] Let $X$ as above and $H$ a nef Cartier divisor such that $F := H^+ \cap <\text{NE}(X)> \setminus \{0\}$ is entirely contained in the set $\{Z \in N_1(X) : K_X \cdot Z < 0\}$, where $H^+ = \{Z : H \cdot Z = 0\}$. Then there exists a projective morphism $\varphi : X \to W$ from $X$ onto a normal variety $W$ which is characterized by the following properties:

i) For an irreducible curve $C$ in $X$, $\varphi(C)$ is a point if and only if $H \cdot C = 0$, if and only if $\text{cl}(C) \in F$.

ii) $\varphi$ has only connected fibers

iii) $H = \varphi^*(A)$ for some ample divisor $A$ on $W$. 

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**Definition 2.2** ([KMM87], definition 3-2-3). Using the notation of the above theorem, the map \( \varphi \) is called a Fano-Mori contraction (or an extremal contraction); the set \( F \) is an extremal face, while the Cartier divisor \( H \) is a supporting divisor for the map \( \varphi \) (or the face \( F \)). The contraction is of fiber type if \( \text{dim} W < \text{dim} X \), otherwise it is birational. If \( \text{dim} F = 1 \) the face \( F \) is called an extremal ray, while \( \varphi \) is called an elementary contraction.

**Remark 2.3** Note that a supporting divisor for a Fano-Mori contraction is of the form \( H = K_X + rL \) where \( r \) is a positive integer. In fact if \( H \) is a supporting divisor then \( H - K_X \) is an ample line bundle by Kleiman’s criterion.

**Remark 2.4** Let \( \pi : X \to V \) denote a contraction of an extremal face \( F \), supported by \( H = \pi^* A \). Let \( R \) be an extremal ray in \( F \) and \( \rho : X \to W \) the contraction of \( R \). Then \( \pi \) factors through \( \rho \) (this is because \( \pi^* A \cdot R = 0 \)).

**Remark 2.5** We have also [Mor82] that if \( X \) has an extremal ray \( R \) then there exists a rational curve \( C \) on \( X \) such that \( 0 < -K_X \cdot C \leq n + 1 \) and \( R = R[C] := \{ D \in \langle NE(X) \rangle : D \equiv \lambda C, \lambda \in \mathbb{R}^+ \} \). Such a curve is called an extremal curve. The last remark was generalized by P. Ionescu and J. Wiśniewski as in the follow.

**Definition 2.6** Let \( \varphi \) be a Fano-Mori contraction of \( X \) and let \( E = E(\varphi) \) be the exceptional locus of \( \varphi \) (if \( \varphi \) is of fiber type then \( E := X \)); let \( S \) be an irreducible component of a (non trivial) fiber \( F \). We define the positive integer \( l \) as
\[
l = \min \{-K_X \cdot C : C \text{ is a rational curve in } S \}.
\]
If \( \varphi \) is the contraction of a ray \( R \), then \( l \) is called the length of the ray.

**Proposition 2.7** [Wis91] In the set-up of the previous definition the following formula holds
\[
\text{dim} S + \text{dim} E \geq \text{dim} X + l - 1.
\]
In particular this implies that if \( \varphi \) is of fiber type then \( l \leq (\text{dim} Z - \text{dim} W + 1) \) and if \( \varphi \) is birational then \( l \leq (\text{dim} Z - \text{dim}(\varphi(E))) \).

If a manifold admits different extremal contractions, then the dimensions of different general fibers are bounded by the following
Theorem 2.8 ([Wis91, Theorem 2.2]) Let a manifold $X$ of dimension $n$ admit $k$ different contractions (of different extremal rays). If by $m_i, i = 1, 2, \ldots, k$ we denote dimensions of images of these contractions, then

$$
\sum_{i=1}^{k} (n - m_i) \leq n
$$

It is very useful to understand when a contraction is elementary; for this we will use in this paper the following result:

Proposition 2.9 ([ABW91]) Let $\pi : X \to W$ be a contraction of a face such that $\dim X > \dim W = m$. Suppose that for every rational curve $C$ in a general fiber of $\pi$ we have $-K_X \cdot C \geq (n + 1)/2$. Then $\pi$ is an elementary contraction except if

a) $-K_X \cdot C = (n + 2)/2$ for some rational curve $C$ on $X$, $W$ is a point, $X$ is a Fano manifold of pseudoindex $(n + 2)/2$ and $p(X) = 2$

b) $-K_X \cdot C = (n + 1)/2$ for some rational curve $C$, and $\dim W \leq 1$

Proof For the reader’s convenience we will give a sketch of the proof.

Let $T$ be a non breaking family of rational curves of $X$, which is dominant and whose dimension at every point is greater or equal to $-K_X \cdot l - 2$, where $l$ is a curve in $T$. The existence of this family follows as in [Mor79] from the fact that through a general point of a general fiber there passes a rational curve $l$ such that

$$-K_X \cdot l = -K_F \cdot l \leq n - m + 1$$

and that, on the other hand, by our assumptions, $-K_X \cdot l > (n - m + 1)/2$.

Suppose that $\pi$ is not an elementary contraction. Therefore there is a contraction $\varphi = \text{cont}_R$ where $R = \mathbb{R}_+[l_2]$ is an extremal ray contracted by $\pi$ and not containing $l$. Let $F$ be a fiber of $\varphi$; then $\dim F \geq l(R) - 1$ (2.7); let $T_p$ be the locus of curves from the family $T$ which pass through a given point $p$; we have $\dim T_p \geq -K_X \cdot l - 1$. But, by the non-breaking lemma (see [Wis90]), we must have

$$\dim F + \dim T_p \leq \dim X$$

if $T_p \cap F \neq \emptyset$ and for $p \notin F$, that is

$$-K_X \cdot l - K_X \cdot l_2 \leq n + 2$$

If $-K_X \cdot C > (n + 2)/2$ for every rational curve in a general fiber we have a contradiction.

If $-K_X \cdot C = (n + 2)/2$ then we have equality everywhere; in particular we have that $\dim F = -K_X \cdot l - 1 = n - \dim T - 1$. Then, by [BSW90, lemma 1.4.5] we have that $NE(X) = NE(F) + \mathbb{R}_+[l]$. Since $F$ is a positive dimensional fiber of an elementary contraction, we conclude that $NE(F) = \mathbb{R}_+$ and thus that
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\( \rho(X) = 2 \). Thus \( \rho(Y) = 0 \), i.e. \( Y \) is a point; therefore \( X \) is a Fano manifold of pseudoindex \((n+2)/2\).

If \(-K_X.C = (n+1)/2\) then a general fiber of \( \pi \), \( G \), is a Fano manifold of pseudoindex \( \geq (n+1)/2 \). If \( m \geq 2 \) we have that the pseudoindex of \( G \) is greater or equal to \((\dim G)/2 + 1\); therefore \( \rho(G) = 1 \) (see [Wis90]). This implies in particular that, if \( \pi \) is not an elementary contraction, then \( \varphi = \text{cont}_R \), as above, is birational and therefore that \( \dim F \geq -K_X.l_2 \geq (2.7) \); since \( \rho(X) \geq 3 \) it follows as above that

\[-K_X.l - K_X.l_2 > n,\]

and thus we arrive at the contradiction with \(-K_X.C \leq (n+1)/2\) using again [BSW90, lemma 1.4.5].

**Definition 2.10** Let \( L \) be an ample line bundle on \( X \). The pair \((X, L)\) is called a scroll (respectively a quadric fibration, respectively a del Pezzo fibration) over a normal variety \( Y \) of dimension \( m < n \) if there exists a surjective morphism \( \varphi : X \to Y \) such that

\[ K_X + (n - m + 1)L \approx p^*\mathcal{L} \]

(respectively \( K_X + (n - m)L \approx p^*\mathcal{L} \), respectively \( K_X + (n - m - 1)L \approx p^*\mathcal{L} \)) for some ample line bundle \( \mathcal{L} \) on \( Y \). \( X \) is called a classical scroll or a \( \mathbb{P} \)-bundle (respectively quadric bundle) over a projective variety \( Y \) of dimension \( r \) if there exists a surjective morphism \( \phi : X \to Y \) such that every fiber is isomorphic to \( \mathbb{P}^{n-r} \) (respectively to a quadric in \( \mathbb{P}^{n-r+1} \)) and if there exists a vector bundle \( \mathcal{E} \) of rank \( n - r + 1 \) (respectively of rank \( n - r + 2 \)) on \( Y \) such that \( X \approx \mathbb{P}(\mathcal{E}) \) (respectively exists an embedding of \( X \) over \( Y \) as a divisor of \( \mathbb{P}(\mathcal{E}) \) of relative degree 2).

**Remark 2.11** A scroll is a Fano Mori contraction of fiber type such that the inequality in (2.7) is actually an equality, i.e. \( l = (\dim X - \dim Y + 1) \) and moreover if \( C \) is a rational curve such that \(-K_XC = l\) then it exists an ample line bundle \( L \) such that \( LC = 1 \), i.e. \( C \) is a line with the respect to \( L \). The contrary is almost true in the sense that if \( \varphi \) is a Fano Mori contraction with the above properties then it factors through a scroll, that is the face which is contracted by \( \varphi \) contains a sub-face whose contraction is a scroll.

Similarly a quadric (respectively a del Pezzo) fibration is a Fano Mori contraction of fiber type such that \( l = (\dim X - \dim Y) \) (resp. \( l = (\dim X - \dim Y - 1) \)) and moreover if \( C \) is a rational curve such that \(-K_XC = l\) then it exists an ample line bundle \( L \) such that \( LC = 1 \), i.e. \( C \) is a line with the respect to \( L \).

**Theorem 2.12** [BS95, Proposition 3.2.1] Let \( p : X \to Y \) be a surjective equidimensional morphism onto a normal variety \( Y \) and let \( L \) be an ample line bundle on \( X \) such that \((F, L_F) \approx (\mathbb{P}^r, \mathcal{O}(1)) \) for the general fiber \( F \) of \( p \). Then \( p : X \to Y \) gives to \((X, L)\) the structure of a \( \mathbb{P}^d \)-bundle.
Remark 2.13 Let $\varphi : X \to Y$ be a scroll and let $\Sigma \subset Y$ be the set of points $y$ such that $\dim(\varphi^{-1}(y)) > k := \dim X - \dim Y$ then $\text{codim} \Sigma \geq 3$, thus if $\dim Y \leq 2$ then the scroll is a $\mathbb{P}^k$-bundle ([Som86, Theorem 3.3]).

Theorem 2.14 [ABW93, Theorem B] Let $(X, L)$ be a quadric fibration $\varphi : X \to Y$, with $L$ an ample line bundle on $X$; assume that $\varphi$ is an elementary contraction and that $\varphi$ is equidimensional. Then $E := p_* L$ is a locally free sheaf of rank $\dim X - \dim Y + 2$ and $L$ embeds $X$ into $\mathbb{P}(E)$ as a divisor of relative degree 2, i.e. $X$ is a classical quadric bundle.

Remark 2.15 Let $\varphi : X \to Y$ be a scroll (respectively a quadric fibration, respectively a del Pezzo fibration) and let $\dim X = n$, $\dim Y = m$; it follows directly from (2.9) that if $n \geq 2m - 1$ (respectively if $n \geq 2m + 1$ and $(m, n) \neq (0, 2), (1, 3)$, respectively if $n \geq 2m + 3$ and $(m, n) \neq (0, 4), (1, 5)$) then $\varphi$ is an elementary contraction, i.e. the contraction of an extremal ray.

Lemma 2.16 Let $E$ be an ample vector bundle of rank $r$ on a complex variety $X$. For any rational curve $C \subset X$ we have

$$(\det E).C \geq r.$$  

Moreover, if $C$ is smooth and the equality holds, then $E = \mathcal{O}_X(1)^{\oplus r}$.

Lemma 2.17 Let $Y$ be a complex projective variety of dimension $n$, $E$ an ample vector bundle on $Y$, $s$ a global section of $E$; denote with $V(s)$ the zero set of $s$. Then

$$\dim V(s) \geq n - r.$$  

Proof. See [Ful84, Example 12.1.3].

Proposition 2.18 Let $X$, $E$ and $Z$ be as before. Let $Y$ be a subvariety of $X$ of dimension $\geq r$. Then $\dim Z \cap Y \geq \dim Y - r$.

Proof. Consider $E_Y$, the restriction of $E$ to $Y$ and $s_Y$, the restriction of $s$ to $\Gamma(Y, E_Y)$. Applying lemma (2.17) to $Y$ and $s_Y$ we get

$$\dim (Z \cap Y) = \dim V(s_Y) \geq \dim Y - r$$

3 Lifting of contractions

3.1 Let $E$ be an ample vector bundle of rank $r$ on $X$ such that there exists a section $s \in \Gamma(E)$ whose zero locus $Z = \{s = 0\}$ is a smooth submanifold of the expected dimension $\dim X - r := n - r$. Note that, with this assumptions, the restriction of $E$ to $Z$ is the normal bundle $N_XZ$ [Ful84, Example 6.3.4].
The idea of this section is to investigate the relation between $\text{NE}(X)$ and $\text{NE}(Z)$; one basic result in this direction is the following Lefschetz type theorem proved by Sommese in [Som78] and with slightly weaker assumptions in [LM95].

**Theorem 3.2** Let $X$, $E$ and $Z$ be as in 3.1 and let $i : Z \hookrightarrow X$ be the embedding. Then

1. $H^i(i) : H^i(X, Z) \rightarrow H^i(Z, Z)$ is an isomorphism for $i \leq \dim Z - 1$
2. $H^i(i)$ is injective and its cokernel is torsion free for $i = \dim Z$
3. $\text{Pic}(i) : \text{Pic}(X) \rightarrow \text{Pic}(Z)$ is an isomorphism for $\dim Z \geq 3$.
4. $\text{Pic}(i)$ is injective and its cokernel is torsion free for $\dim Z = 2$.
5. $\rho(X) = \rho(Z)$ for $\dim Z \geq 3$.

Note that, although the Picard groups are isomorphic, in general the ample cone of $X$ is properly contained in the ample cone of $Z$. However, in special cases, something can be said; the following proposition generalizes a result of J. A. Wiśniewski on divisors.

**Proposition 3.3** Let $X$ be a Fano manifold of dimension $n$, $E$ an ample vector bundle of rank $r$ on $X$ and $Z$ the zero locus of a section of $E$, smooth and of the expected dimension. If $\text{Pic}(X) \cong \text{Pic}(Z)$ and $X$ has no elementary extremal contractions with all fiber of dimension $\leq r$ then a line bundle on $X$ is ample if and only if its restriction to $Z$ is ample. The assumption is satisfied for instance if all the extremal rays of $X$ have length $l(R) \geq r + 2$ or $l(R) \geq r + 1$ if $R$ is not nef.

**Proof.** Observe that, since $X$ is Fano, a line bundle $L$ on $X$ is ample if and only if it has positive intersection with any extremal ray of $X$.

So take a line bundle $L_Z$ ample on $Z$; if we prove that every extremal ray of $X$ contains the class of a curve lying on $Z$ we can conclude that $L$ is ample on $X$.

By our assumption for every extremal ray of $X$ its associated contraction has a fiber $F$ of dimension $\geq r + 1$; thus

$$\dim F + \dim Z \geq n + 1$$

and therefore, by proposition [2.18] the intersection of $Z$ and $F$ contains a curve, which belongs to the ray $R$. Using [2.7] one shows immediately that the assumption on the length implies the lower bound on the fiber.

The same idea allows us to prove the following

**Theorem 3.4** (Lifting of contractions) Let $X$, $E$ and $Z$ be as in (3.1) and assume that $Z$ is not minimal. Let $F_Z$ be an extremal face of $Z$ and $D_Z = K_Z + \tau H_Z$ a good supporting divisor of $F_Z$. Assume that there exists an ample line bundle $H$ on $X$ which is the extension of $H_Z$. Then $D = K_X + \det E + \tau H$ is nef, but not ample; thus it defines an extremal face $F_X$ of $X$. Moreover, if $\tau \geq 2$ and $\dim Z \geq 3$, under the identification of $N_1(X)$ with $N_1(Z)$ we have $F_X = F_Z$ and the contraction of every ray spanning $F_Z$ lifts.
Proof. Suppose that $D$ is not nef. There exists a curve $C$ on $X$ such that $D.C < 0$; therefore there exists an extremal ray $R$ on $X$ on which $D$ is negative and s.t. $l(R) \geq r + \tau + 1$.
From the inequality (2.7) $\dim F \geq l(R) - 1$

where $F$ is a non trivial general fiber of the contraction of $R$ and this yields

$$\dim F + \dim Z \geq r + \tau + n - r = \tau + n \geq n + 1$$

so, recalling that we can assume $\tau \geq 1$ (2.3), in view of proposition (2.18) a curve of the ray $R$ lays on $Z$ and this is absurd, since $D|_Z$ is nef.

To prove the last claim, observe that every extremal ray $R$ in the face $F_X$ has length $l(R) \geq \tau + r$, so the general non trivial fiber of the contraction of $R$ has dimension $\dim F \geq \tau + r - 1 \geq r + 1$, so, in view of proposition (2.18) a curve of the ray $R$ lays on $Z$. $\square$

**Proposition 3.5** The hypothesis on the ampleness of $H$ is not necessary if $\dim Z \geq 2$ and $\text{Pic}(Z) \cong \mathbb{Z}$ or, more generally, if $\text{Pic}(i) : \text{Pic}(X) \to \text{Pic}(Z)$ is an isomorphism and $\text{NE}(Z) = \text{NE}(Z)_{K_Z \geq 0} + R$, i.e. if $Z$ has only one extremal ray. Example (4.10) shows that the hypothesis is necessary if $Z$ has at least two extremal rays.

**Proof.** There exists a line bundle $L$ which is ample on $X$; the restriction of this line bundle to $Z$, $L_Z$ is ample on $Z$, so, if $K_Z$ is not nef there exist a rational number $\sigma > 0$ such that $K_Z + \sigma L_Z$ is nef but not ample and it defines an extremal face $G_Z$ (KMM87, Kawamata rationality theorem). But we are supposing that on $Z$ there is only one extremal ray, thus $F_Z = G_Z$.

**Remark 3.6** Note that the proof of (3.5) actually shows there is always an extremal contraction on $Z$ which can be lifted to $X$.

**Lemma 3.7** If $\varphi : Z \to W$ is a $\mathbb{P}$-bundle contraction on a smooth minimal variety $W$ then $Z$ has only one extremal ray.

**Proof.** Suppose that $Z$ has another extremal ray, $R_1$; there exists a rational curve $C_0$ such that $-K_Z.C_0 > 0$ and $\varphi(C_0)$ is not a point. Let $C = \varphi(C_0)$, let $\nu : \mathbb{P}^1 \to C$ be the normalization of $C$ and consider the fiber product

$$\begin{array}{ccc}
Z \times_W \mathbb{P}^1 & \xrightarrow{\bar{\nu}} & Z \\
\varphi \downarrow & & \downarrow \varphi \\
\mathbb{P}^1 & \xrightarrow{\nu} & W
\end{array}$$

(3.8)
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\[ \varphi : Z_C := Z \times_W \mathbb{P}^1 \to \mathbb{P}^1 \] is a \( \mathbb{P} \)-bundle on \( \mathbb{P}^1 \) and so \( \rho(Z_C) = 2 \); the morphism \( \bar{\nu} \) induces a map of spaces of cycles \( N_1(Z_C) \to N_1(Z) \) which is an embedding. The Mori cone \( NE(Z_C) \) is contained in the intersection \( N_1(Z_C) \cap NE(Z) \) and so, since \( N_1(Z_C) \) is a plane in \( N_1(Z) \) and passes through two different extremal rays of \( Z \), \( NE(Z_C) \) is contained in the negative part of \( NE(Z) \).

By [KMM92, Corollary 2.8] \( -K_{Z_C/P^1} \) is not ample, so there exist an horizontal curve \( C_1 \) on \( Z_C \) such that \( -K_{Z_C/P^1}.C_1 \leq 0 \); noting that \( -K_{Z_C/P^1} = \bar{\nu}^* \phi^* K_W - \bar{\nu}^* K_Z \) we get
\[ \bar{\nu}^* \varphi^* K_W.C_1 = -K_{Z_C/P^1}.C_1 + \bar{\nu}^* K_Z.C_1 \leq K_Z.\varphi(C_1) < 0 \]
and therefore \( K_W.\varphi(\bar{\nu}(C_1)) < 0 \), which contradicts the minimality of \( W \).

**Remark 3.9** We found the idea of the proof of (3.7) in [SW90a] and [KMM92].

For the rest of this section we will be in the hypothesis of theorem (3.4) and we will denote by \( \varphi : Z \to W \) the contraction of the face \( F_Z \) and by \( \phi : X \to Y \) the contraction of \( F_X \). Let also \( m = \dim W \).

By the adjunction formula \( -K_Z = -(K_X + \det E)_Z \), so \( -K_Z \) is \( \phi \)-ample.

On \( Z \) we have thus two contractions, \( \varphi \) and \( \phi_Z \). Now we are going to investigate the relation between them. Clearly we have a commutative diagram

\[ \begin{array}{ccc}
X & \xrightarrow{i} & Z \\
\text{\varphi} \downarrow & & \downarrow \phi_Z \\
Y & \xleftarrow{\pi} & W
\end{array} \]  \( (3.10) \)

**Lemma 3.11** \( \phi_Z(Z) \supseteq \phi(E(\phi)) \).

**Proof.** We reason as in the proof of (3.4): since \( \phi \) is the contraction of a ray of length \( l(R) \geq r + \tau \), a non trivial fiber of \( \phi \) has dimension \( \geq \tau + r - 1 \) and thus it has nonempty intersection with \( Z \).

**Proposition 3.12** If the contraction \( \varphi \) is of fiber type then also \( \phi \) is of fiber type.

**Proof.** If \( \varphi \) is of fiber type the commutativity of the diagram \( (3.10) \) implies that also \( \phi_Z \) is of fiber type, so \( Z \) is contained in the exceptional locus of \( \phi \), \( E(\phi) \), and by lemma \( (3.11) \) \( \phi_Z(Z) = \phi(E(\phi)) \).

Suppose that \( \phi \) is birational; in this case \( E(\phi) \subsetneq X \) and
\[ \dim \phi(E(\phi)) = \dim \phi_Z(Z) < \dim Z = n - r. \]
Y has dimension $n$, so it is possible to find a subvariety $Y' \subset Y$ of dimension $r$ which has empty intersection with $\phi(E(\phi))$; away from $E(\phi)$, $\phi$ is an isomorphism, so $X' = \phi^{-1}(Y') \subset X$ is a subvariety of $X$ of dimension $r$ which has empty intersection with $E(\phi)$ and therefore with $Z$, but this is absurd by proposition (2.18). □

**Proposition 3.13** If $\varphi$ is of fiber type or if $\tau \geq 2$, $\phi_Z$ has connected fibers. Moreover $\phi_Z$ factors as $\phi_Z = \sigma \circ \varphi$ where $\sigma : W \to \phi_Z(Z)$ is the normalization morphism. In particular, if $\varphi$ is of fiber type then $\phi_Z = \varphi$.

**Proof.** The fibers of $\phi_Z$ are of the form $Z \cap F$ with $F$ fiber of $\phi$. If $\varphi$ is of fiber type, then the same is for $\phi$ (see 3.12) whose fibers have thus dimension $\geq \dim X - \dim Y = n - m$; so $\dim Z \cap F \geq n - r - m \geq 1$. If $\tau \geq 2$, reasoning as in the proof of theorem (3.12) we again have $\dim Z \cap F \geq 1$. So theorem (3.1.1) applies to $F$ and $E_F$ and gives $H^0(Z \cap F, Z) \cong H^0(F, Z) \cong \mathbb{Z}$. Using the Universal Coefficient Theorem we get $H_0(Z \cap F) \cong \mathbb{Z}$.

Let $\sigma : W' \to \phi_Z(Z)$ be the normalization of $\phi_Z(Z)$; by the universal property $\phi_Z$ factors through $\tilde{\phi}_Z : Z \to W'$ and $\sigma$; note also that, since $\phi_Z$ has connected fibers, the same is true for $\phi_Z$.

Let $C \subset Z$ be any curve contracted by $\phi_Z$ (and hence by $\tilde{\phi}_Z$); thus $(K_X + \det E + \tau H_Z)_C = 0$ which is equivalent to $(K_Z + \tau H_Z)_C = 0$, i.e. $C$ is contracted by $\varphi$. By the commutativity of the diagram every curve contracted by $\varphi$ is contracted by $\phi_Z$ (and hence by $\tilde{\phi}_Z$). Therefore $\varphi$ and $\tilde{\phi}_Z$ are two Fano Mori contractions which contract the same extremal face, so they are the same morphism.

To prove the last claim recall that, by lemma (3.11) and proposition (3.12) if $\varphi$ is of fiber type then $\phi_Z(Z) = Y$ and hence $\phi_Z(Z)$ is normal.

**Proposition 3.14** If the contraction $\varphi$ is birational and $\tau \geq 2$, then also $\phi$ is birational.

**Proof.** Suppose $\phi$ is of fiber type; reasoning again as in the proof of (3.4) we can prove that $\dim Z \cap F \geq 1$ for the generic fiber of $\phi$, so $\phi_Z = \varphi$ is of fiber type, a contradiction in view of proposition (3.13).

**Remark 3.15** If $\varphi$ is birational and $\tau = 1$ then $\phi$ can be of fiber type (see case 3. of proposition (4.12)).

4 Scrolls and $\mathbb{P}^d$-bundles

**Theorem 4.1** Let $X$, $E$ and $Z$ be as in (3.4) with $\dim Z \geq 2$. We assume that $Z$ has a scroll contraction $\varphi : Z \to W$ with respect to an ample line bundle on $Z$, $H_Z$, which is the restriction of an ample line bundle $H$ on $X$. Then $X$ has a Fano-Mori contraction $\phi : X \to W$ which is of fiber type and with supporting divisor $D = K_X + \det E + (n - m - r + 1)H$. The general fiber of $\phi$ is isomorphic to $\mathbb{P}^{n-m}$ and $E$ restricted to it is $\oplus \mathcal{O}_Z(1)$. 

If φ is elementary or \( \dim X = n \geq 2m - 1 = 2 \dim W - 1 \) (this is always the case if \( \dim W \leq 3 \)) then φ is elementary and it is a scroll contraction (i.e., it is supported by the divisor \( K_X + (n - m + 1)H \)) and moreover in the second case even φ had to be elementary.

**Proof.** The morphism φ is a contraction supported by \( K_X + (n - m + 1)H \) so, applying theorem (3.4), we get a contraction \( \phi : X \to Y \), defined by an high multiple of \( D = K_X + \det E + (n - m - r + 1)H \); this contraction is of fiber type and \( Y = W \) by proposition (3.13). Let \( F \) be a general fiber of \( \phi \) then \( F \) is a smooth Fano manifold of dimension \( n - m \) such that \( -K_F = (\det E + (n - m - r + 1)H) \). Moreover, for any line in a general fiber \( (\det E - rH) \cdot l = 0 \).

Assume now that \( \dim X \geq 2 \dim W - 1 \); note that \( \dim X \geq \dim Z + 1 \geq 2 \dim W + 2 \), so the inequality holds for \( \dim W \leq 3 \).

By the proposition (2.9) the contraction \( \phi : X \to W \) is an elementary contraction and \( \det E = rH + \phi^*B \)

that is \( \phi \) is supported by \( K_X + (n - m + 1)H \); note that also \( \varphi : Z \to W \) had to be elementary, by the last claim of theorem (3.4) if \( \dim Z \geq 3 \) and by (2.3) if \( \dim Z = 2 \).

**Corollary 4.2** Assume now that \( Z = \mathbb{P}(\mathcal{F}) \) for some vector bundle \( \mathcal{F} \) on \( W \), and its tautological bundle is the restriction of an ample line bundle on \( X \); then \( X = \mathbb{P}(\mathcal{G}) \) for some vector bundle \( \mathcal{G} \) on \( W \) which admits \( \mathcal{F} \) as a quotient; in this case \( \mathcal{E} = \xi_{\mathcal{G}} \otimes \phi^*\mathcal{T} \) where \( \mathcal{T} \) fits into the exact sequence

\[
0 \to \mathcal{T} \to \mathcal{G} \to \mathcal{F} \to 0
\]

and \( \phi : X \to W \) is the \( \mathbb{P} \)-bundle contraction.

**Proof.** The theorem gives us a contraction \( \phi : X \to W \); we claim that \( \phi \) is equidimensional; in fact if it has any fiber of dimension \( n - m \) then, by proposition (2.13), even \( Z \to W \) should have a fiber of dimension \( (n - m - r) \).

Since \( \varphi : Z \to W \) is elementary \( \phi \) is a scroll with the respect to \( H \). The first part of the corollary is proven by (2.13). The second part is a well known fact about vector bundles (see [Ful84 B.5.6]).

**Example 4.3** Let \( \mathcal{F} \) be an ample vector bundle on a smooth curve \( C \) of genus \( g > 0 \). If \( \mathcal{F} \) is decomposable into a sum of \( r \) bundles \( \mathcal{F}_i \), by [Fuj80 Corollary 4.20] each \( \mathcal{F}_i \) fits into an exact sequence

\[
0 \to \mathcal{O}_C \to \mathcal{G}_i \to \mathcal{F}_i \to 0
\]

with \( \mathcal{G}_i \) ample; so we can construct an exact sequence

\[
0 \to \oplus^r \mathcal{O}_C \to \mathcal{G} = \bigoplus_{i=1}^r \mathcal{G}_i \to \mathcal{F} = \bigoplus_{i=1}^r \mathcal{F}_i \to 0.
\]

On \( X = \mathbb{P}(\mathcal{G}) \) the vector bundle \( \mathcal{E} = \xi_{\mathcal{G}} \otimes p^* (\oplus^r \mathcal{O}_C) = \oplus^r \xi_{\mathcal{G}} \) is ample and has a section vanishing on \( \mathbb{P}(\mathcal{F}) \).
Corollary 4.4 Let $X$, $E$ and $Z$ be as in (3.1) with $\dim Z \geq 2$. We assume that $Z$ is a $\mathbb{P}$-bundle over a smooth variety $W$ and also that $W$ is minimal. Then $X$ is a $\mathbb{P}$-bundle over $W$ and $E|_F = \oplus \mathcal{O}(1)$ for every fiber $F$ of $\phi : X \to W$.

**Proof.** The assumption on the tautological bundle is not necessary in this case as noted in (3.7).

**Remark 4.5** In case $r = 1$ the last corollary shows that [BS92, Conjecture 5.5.1] is true if $b \geq 3$, $X$ is smooth and $B$ is minimal.

Corollary 4.6 Let $X$, $E$ and $Z$ be as in (3.1) with $\dim Z \geq 2$. We assume that $Z$ has a scroll contraction $\phi : Z \to W$ with $\dim W \leq 1$ (or equivalently that $Z$ is a $\mathbb{P}$-bundle over a smooth variety $W$ of dimension $\leq 1$). Then $X$ is a $\mathbb{P}$-bundle over $W$ and $E|_F = \oplus \mathcal{O}(1)$ for every fiber $F$ of $\phi : X \to W$ except possibly for $W = \mathbb{P}^1$ and $Z = \mathbb{P}(\oplus(\mathcal{O}_{\mathbb{P}^1})) = \mathbb{P}^1 \times \mathbb{P}^{(n-r-1)}$ or $Z = \mathbb{P}(\oplus(n-r-1)\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$.

**Proof.** The corollary will follow if we prove that under the assumptions $Z$ has only one extremal ray. If $\dim W = 0$ then $Z = \mathbb{P}^{(n-r)}$ and thus $Z$ has only one extremal ray. If $\dim W = 1$ then $\rho(Z) = 2$, thus $Z$ has one extremal ray or it is Fano. But if $Z$ is a Fano manifold then $0 = h^1(O_Z) = g(W)$, thus $W = \mathbb{P}^1$. Therefore we can assume that $Z = \mathbb{P}(E)$ for a vector bundle $E$ on $\mathbb{P}^1$ with $\text{rank}(E) = s = n-r$ and $0 \leq c_1(E) \leq s-1$. But since $-K_Z = s\xi + (2 - c_1(E))H$, with $\xi$ the tautological bundle and $H$ the pull back of a point in $\mathbb{P}^1$, if $c_1(E) \geq 2$ then $\xi$ and thus $E$ would be ample. This is in contradiction with $c_1(E) \leq s-1$. Thus $0 \leq c_1(E) \leq 1$ which gives our claim.

**Remark 4.7** Even if $W$ is not minimal the $\mathbb{P}$-bundle contraction can be some-time lifted to $X$. For instance if $Z$ is not Fano and $\text{Pic}(W) = \mathbb{Z}$ the same proof of the above corollary gives the lifting. In particular if $W = \mathbb{P}^m$ and $\mathcal{F}$ is a vector bundle with $0 \leq c_1\mathcal{F} \leq (m-1)$ which is not spanned by global section then by [SW90], $Z$ is not Fano and the $\mathbb{P}$-bundle contraction $Z = \mathbb{P}(\mathcal{F}) \to W$ can be lifted.

**Remark 4.8** The above theorem and corollaries extend the results of Lanteri and Maeda; their papers inspired and motivated ours. Summarizing we have a precise description of $X$ when $Z$ is a scroll or a $\mathbb{P}$-bundle satisfying assumptions (1.1) in the introduction (Theorem (4.1) and corollary (4.2)). We have a good result also if we drop the assumption but $W$ is minimal. If $W$ is not minimal the situation is much more complicated; cases that occur if $\dim W \leq 1$ are described in the rest of this section; other simple cases are described in remark (4.7).
Proposition 4.9 Suppose that $Z = \mathbb{P}(\oplus (n-r)\mathcal{O}_F) = \mathbb{P}^1 \times \mathbb{P}^{n-r-1}$ is the zero set of a section of an ample vector bundle $\mathcal{E}$ of rank $r$ on a smooth $X$ of dimension $n$, then

1. $X$ is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ and $\mathcal{E}_F = \oplus^r \mathcal{O}_F(1)$ for any fiber.
2. $X$ is a $\mathbb{P}^{r+1}$-bundle over $\mathbb{P}^{n-r-1}$ and $\mathcal{E}_F = \oplus^r \mathcal{O}_F(1)$, for any fiber.
3. $(X, \mathcal{E})$ is $(\mathbb{P}^n, \oplus n^3 \mathcal{O}_F(1) \oplus \mathcal{O}_F(2))$ or $(\mathbb{P}^n, \oplus n^3 \mathcal{O}_Q(1))$ and $Z$ is a smooth quadric surface.

Proof. Let $Z = \mathbb{P}^1 \times \mathbb{P}^{(n-r-1)}$ and $p_1, p_2$ the projections on the two factors; $Pic(Z) \simeq Z < p_1^* \mathcal{O}_{\mathbb{P}^1}(1) > \oplus < p_2^* \mathcal{O}_{\mathbb{P}^{n-r-1}}(1) >$; $Z < L_1 > + Z < L_2 >$; $p_1$ and $p_2$ are two Fano-Mori contractions with supporting divisors $(aL_1, 0)$ and $(0, bL_2)$ respectively $(a, b > 0)$; on $Z$ there is a third Fano-Mori contraction, $p$, which is the contraction of $Z$ to a point.

Suppose that $Z$ is the zero locus of a section of an ample vector bundle $\mathcal{E}$ on $X$; by remark $(3.6)$ at least one of the extremal contractions of $Z$ lifts to $X$.

Suppose that $p_1$ lifts; as in the proofs of theorem $(4.1)$ and corollary $(4.2)$ we obtain that $X$ is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ and $\mathcal{E}_F = \oplus^r \mathcal{O}_F(1)$.

Suppose now that $p_2$ lifts; as above we get that $X$ is a $\mathbb{P}^{r+1}$-bundle over $\mathbb{P}^{n-r-1}$ and $\mathcal{E}_F = \oplus^r \mathcal{O}_F(1)$.

Finally, suppose that $p$ lifts. We start with the case in which $Pic(i) : Pic(X) \to Pic(Z)$ is an isomorphism and $\rho(X) = \rho(Z)$. We want to show that in this case both $p_1$ and $p_2$ would lift, but this is not possible, in view of theorem $(2.3)$.

$X$ is a Fano variety and $-K_X = det \mathcal{E} + H$; this implies that the two extremal rays $R_1, R_2$ of $X$ have length $\geq r + 1$; if $R_i \ (i = 1, 2)$ is not nef, then the fibers of the associated contraction have dimension $\geq r + 1$, and so, by proposition $(2.13)$, there exists a curve in the ray belonging to $Z$. If $R_i \ (i = 1, 2)$ is nef, then the fibers of the associated contraction $P_i$ have dimension $\geq r$ by $(2.7)$; if a fiber has dimension $\geq r + 1$ then its intersection with $Z$ contains at least a curve and we are done, so we can suppose that $P_i$ is equidimensional and all the fibers have dimension $r$. Recalling that, for the general fiber

$$K_F = (K_X)_F = -(det \mathcal{E})_F - H_F$$

we have that $(F, H_F) \simeq (\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$, so, by theorem $(2.12)$, $X$ is a $\mathbb{P}^r$-bundle on a normal variety $W$. This fact and the previous expression of $K_F$ yield $(F, \mathcal{E}_F) \simeq (\mathbb{P}^r, \oplus r \mathcal{O}_{\mathbb{P}^r}(1))$ for every fiber; this implies that $Z \cap F$ is a point or contains a curve; in the second case we are done, while the first is impossible, since $Z$ would be isomorphic to $W$, which has different Picard number.

So suppose that $Pic(i) : Pic(X) \to Pic(Z)$ is not an isomorphism or $\rho(X) \neq \rho(Z)$; by $(3.4)$ this is possible only for dim $Z = 2$ and in this case we have
By the discussion above, it is clear that in this example, the contraction cannot be lifted; as in the proof of (4.1) and corollary (4.2) we obtain that 2 cannot be the restriction on an ample line bundle on $X$ which cannot be lifted; as in the proof of theorem (3.4) and in the case of ample divisors this is not possible by a result of Sommese [BS92, Theorem 5.2.1].

**Example 4.10** The effectiveness of case 3. is clear; to see the effectiveness of case 1. consider the sequence
\[ 0 \longrightarrow \oplus^{n} \mathcal{O}_{p_{1}} \longrightarrow \oplus^{n}(\mathcal{O}_{p_{1}}(a) \oplus \mathcal{O}_{p_{1}}(s-a)) \longrightarrow \oplus^{n} \mathcal{O}_{p_{1}}(s) \longrightarrow 0 \]
which is exact in view of [Bad81, Remark 1, p.170] and choose $a, s$ in such a way that $0 < a - s < a$; the construction in [Ful84, B.5.6] applies and gives $\mathbb{P}^{1} \times \mathbb{P}^{n-1}$ as a section of the ample vector bundle $\mathcal{E} = \oplus^{n} \mathcal{O}_{2}$ on $X = \mathbb{P}(\mathcal{G})$.

By the discussion above, it is clear that in this example, the contraction $p_{2}$ cannot be lifted; $p_{2}$ is supported by $K_{Z} + H_{Z} = bL_{2} \ (b > 0)$; recalling that $K_{Z} = -2L_{1} - (n - r)L_{2}$ we have that $H_{Z} = 2L_{1} + (n - r + b)L_{2}$ is an ample line bundle on $Z$ which cannot be the restriction on an ample line bundle on $X$.

**Remark 4.11** The effectiveness of case 2. looks uncertain; we note that for $r = 1$, i.e. in the case of ample divisors this is not possible by a result of Sommese [BS92, Theorem 5.2.1]..

**Proposition 4.12** Suppose that $Z = \mathbb{P}(\oplus^{(n-r-1)} \mathcal{O}_{p_{1}} \oplus \mathcal{O}_{p_{1}}(1))$ is the zero set of a section of an ample vector bundle $\mathcal{E}$ of rank $r$ on a smooth $X$ of dimension $n$, then

1. $X$ is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^{1}$ and $\mathcal{E}_{F} = \oplus^{n-1} \mathcal{O}_{F}(1)$ for any fiber.
2. $X$ is a scroll over $\mathbb{P}^{n-r}$ and $\mathcal{E}_{F} = \oplus^{r} \mathcal{O}_{F}(1)$ for the general fiber.
3. $X$ is both as in 1 and in 2 and it is $\mathbb{P}^{1} \times \mathbb{P}^{n-1}$, $r = 1$ and $\mathcal{E} = \mathcal{O}(1,1)$.

**Proof.** Let $Z = \mathbb{P}(\oplus^{(n-r-1)} \mathcal{O}_{p_{1}} \oplus \mathcal{O}_{p_{1}}(1))$ = blow-up of $\mathbb{P}^{(n-r)}$ along a linear subspace of codimension 2. The Mori cone of $Z$ is two dimensional, and it is spanned by two extremal rays, $R_{1}$ and $R_{2}$; let $p_{i} \ i = 1, 2$ be the extremal contraction associated to $R_{i}$; $p_{1}$ is the $\mathbb{P}^{n-r-1}$-bundle map on $\mathbb{P}^{1}$, while $p_{2}$ is the blow down to $\mathbb{P}^{n-r}$; moreover, let $p$ be the contraction of $Z$ to a point, associated to the extremal face generated by $R_{1}$ and $R_{2}$. By (3.6), there is an extremal contraction that lifts to $X$.

Suppose that $p_{1}$ lifts; as in the proof of (1.1) and corollary (1.2) we obtain that $X$ is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^{1}$ and $\mathcal{E}_{F} = \oplus^{r} \mathcal{O}_{p_{n-1}}(1)$.

Suppose now that $p_{2}$ lifts to an extremal contraction $P_{2}$ on $X$. 

$\text{Pic}(X) \simeq \mathbb{Z}$. By the proof of theorem (3.4), $K_{X} + \text{det} \mathcal{E} + 2H = \mathcal{O}_{X}$ for some ample line bundle $H$ such that $2H_{Z} = -K_{Z}$. Note that there are curves $C$ on $Z$ such that $H.C = 1$, so $H$ is the ample generator of $\text{Pic}(X)$; write $\text{det} \mathcal{E} = sH$; since the index of a Fano manifold is at most $n + 1$ we must have $s = r, r + 1$. So $(X, \mathcal{E})$ is either $(\mathbb{P}^{n}, \oplus^{n-3} \mathcal{O}_{p}(1) \oplus \mathcal{O}_{p}(2))$ or $(\mathbb{Q}^{n}, \oplus^{n-2} \mathcal{O}_{p}(1))$ and $Z$ is a smooth quadric surface. \(\square\)
Lemma 4.13 The contraction $P_2$ is of fiber type.

Proof. Assume by contradiction that $P_2$ is birational.

Claim. $P_2$ is divisorial and it is the blow up of a smooth $X'$ along a subvariety of codimension $r + 2$.

The contraction $P_2$ is supported by a divisor of the form $K_X + \text{det} \mathcal{E} + H$, and this implies $\dim F \geq l(R) \geq r + 1$, so, reasoning as in the proof of (3.13), we can prove that $P_{2Z}$ has connected fibers. The fact that $P_{2Z}$ factors through $p_2$ and the normalization of $P_{2Z}(Z)$ yields that all the nontrivial fibers of $P_{2Z}$ have dimension 1; this implies that all the nontrivial fibers of $P_2$ have dimension at most $r + 1$ by proposition (2.18); so we can conclude that the contraction $P_2$ is equidimensional and all the nontrivial fibers have dimension $r + 1$; by (2.7) we have

$$\dim E(P_2) \geq l(R) \geq r + 2 \geq n - 1$$

so $P_2$ is divisorial and $\dim E(P_2)(E(P_2)) = (n - 1) - (r + 1) = n - r - 2$. In view of (3.2) the contraction $P_2$ is elementary, and so we can apply \[AW93\] Theorem 4.1 and Corollary 4.11, proving the claim.

The Picard group of $Z$ is generated by the tautological line bundle $\xi$ and $f$, a fiber of the projection on $\mathbb{P}^1$, but also by $E(P_2)$ and $p_2^* O(1)$; we have $p_2^* O(1) = \xi$ and $E(p_2) = \xi - f$. Using the isomorphism $\text{Pic}(i) : \text{Pic}(X) \to \text{Pic}(Z)$ and the fact that $p_2$ and $P_2$ are elementary contractions, we get that the restriction homomorphism $\text{Pic}(j) : \text{Pic}(X') \to \text{Pic}(\mathbb{P}^{n-2})$ is an isomorphism. So there exists an ample generator of $\text{Pic}(X')$, $H'$ whose restriction to $\mathbb{P}^{n-2}$ is $O_p(1)$.

$\text{Pic}(X)$ is thus generated by $P_2^* H$ and $E(P_2)$. Write $K_{X'} = kH'$ for some $k \in \mathbb{Z}$.

Let $l$ be a line in $f$; we obtain

$$-(n - r) = K_{Z}.l = (K_X + \text{det} \mathcal{E})_{Z}.l = (P_2^* K_{X'} + (r + 1)E(P_2) + \text{det} \mathcal{E})_{Z}.l = (kP_2^* H' + (r + 1)E(P_2) + \text{det} \mathcal{E})_{Z}.l = ((k + r + 1)\xi - (r + 1)f + \text{det} \mathcal{E})_{Z}.l = k + r + 1 + \text{det} \mathcal{E}_{Z}.l \geq k + 2r + 1$$

and so

$$k \leq -(n - r - 1)$$

which is absurd, since the index of a Fano variety is not greater than $n + 1$, and the lemma is proven.

On $X$ we thus have a fiber type elementary contraction, supported by an high multiple of $K_X + \text{det} \mathcal{E} + H$ on a variety of dimension $n - r$ by (3.11). For the general fiber of $P_2$ we have

$$K_F + \text{det} \mathcal{E}_F + H_F = O_F$$
and so \((F, E_F) = (P^n, \oplus^r \mathcal{O}_p(1))\); therefore \(Z \cap F\) is a point for the generic fiber, and thus \(P_{2Z}\) is generically one-to-one and therefore coincides with \(p_2\) (see (3.13)). So the conclusion is that, if \(p_2\) lifts, \(X\) is a scroll over \(\mathbb{P}^{n-r}\).

As a final case suppose that \(p\) lifts; if \(\text{Pic}(i) : \text{Pic}(X) \to \text{Pic}(Z)\) is an isomorphism and \(\rho(X) = \rho(Z)\); as in the proof of proposition (4.9) we can prove that also \(p_1\) \(p_2\) lift, so \(X\) is a Fano variety which has a \(\mathbb{P}^{n-1}\)-bundle contraction on \(\mathbb{P}^1\) and a scroll contraction on \(\mathbb{P}^{n-r}\). The only possibility is that \(r = 1\) and \(X = \mathbb{P}_1(\oplus^r \mathcal{O}) = \mathbb{P}^1 \times \mathbb{P}^{n-1}\).

If \(\text{Pic}(i)\) is not an isomorphism or \(\rho(X) \neq \rho(Z)\), by theorem (3.2) \(\text{Pic}(X) \simeq \mathbb{Z}\) and \(\dim Z = 2\), so \(Z = \mathbb{F}_1\) and \(X\) is a Fano variety; this case is ruled out in [LM97, Section 2]. \(\Box\)

Example 4.14 Cases 1. and 3. are effective; examples for the first case can be constructed as in example (4.11) starting with the exact sequence

\[
0 \to \oplus^n \mathcal{O}_{p_1} \to \oplus^{n-1}(\mathcal{O}_{p_1}(a) \oplus \mathcal{O}_{p_1}(s-a)) \oplus \mathcal{O}_{p_1}(a) \oplus \mathcal{O}_{p_1}(s+1-a) \to \ldots
\]

\[
\ldots \to \oplus^{n-1} \mathcal{O}_{p_1}(s) \oplus \mathcal{O}_{p_1}(s+1) \to 0
\]

while in case 3. easy computations show that a smooth \(Z\) in the linear system \(|\mathcal{O}(1, 1)|\) must be a Fano variety with a \(\mathbb{P}\)-bundle contraction on \(\mathbb{P}^1\) and a birational contraction on \(\mathbb{P}^{n-1}\).

Remark 4.15 The example of case 3. shows that the assumption \(\tau \geq 2\) in (3.14) is necessary.

Remark 4.16 As in the case \(Z = \mathbb{P}^1 \times \mathbb{P}^{n-r-1}\) the effectiveness of case 2. looks uncertain; we note that for \(r = 1\), i.e. in the case of ample divisors this is not possible except for the trivial case \(X = \mathbb{P}^{n-1} \times \mathbb{P}^1, \mathcal{E} = \mathcal{O}_{X=\mathbb{P}^{n-1} \times \mathbb{P}^1}(1, 1)\). This is a very well known fact that descends by a result of Sommese [BS92, Theorem 5.2.1] and classical results of Bădescu [Bå81, Bå82].

5 Quadric fibrations and quadric bundles

Theorem 5.1 Let \(X, \mathcal{E}\) and \(Z\) be as in (3.14) with \(\dim Z \geq 3\). We assume that \(Z\) has a quadric fibration contraction \(\varphi : Z \to W\) with respect to an ample line bundle on \(Z\), \(H_Z\), which is the restriction of an ample line bundle \(H\) on \(X\). Then \(X\) has a Fano-Mori contraction \(\phi : X \to W\) which is of fiber type and with supporting divisor \(D = K_X + \det \mathcal{E} + (n-m-r)H\) with \(n = \dim X\) and \(m = \dim W\). For the general fiber of \(\phi\), \(F\) we have either \((F, \mathcal{E}_F) \simeq (\mathbb{P}^{n-m}, \oplus^{r-1} \mathcal{O}_p(1) \oplus \mathcal{O}_p(2))\) or \((F_1, \mathcal{E}_F) \simeq (\mathbb{Q}^{n-m}, \oplus^r \mathcal{O}_q(1))\).

If \(\varphi\) is elementary then also \(\phi\) is elementary and it is either a scroll contraction or a quadric fibration contraction (i.e. it is supported by the divisor \(K_X + (n-m+1)H\) or by the divisor \(K_X + (n-m)H\)). The last result holds replacing the
Assumption on $\varphi$ with the strongest assumption $\dim X \geq 2m + 1 = 2\dim W + 1$ (this is always the case if $\dim W \geq 1$).

**Proof.** The morphism $\varphi$ is a contraction supported by $K_Z + (n - r - m)H_Z$, so, applying theorem (3.4), we get a contraction $\phi : X \to Y$, defined by an high multiple of $D = K_X + detE + (n - m - r)H$; this contraction is of fiber type and $Y = W$ by proposition (3.13). Let $F$ be a general fiber of $\phi$; then $F$ is a smooth Fano manifold of dimension $n - m$ such that $-K_F = (detE + (n - m - r)H)_{|F}$. Thus, by [PSW92, Theorem 0.1] applied to the ample vector bundle $E_1 = (\mathcal{E} \oplus n - m - r H)_{|F}$ either $F = \mathbb{P}^{n - m}$ and $\mathcal{E}$ restricted to it is $\oplus r^{-1} \mathcal{O}_F(1) \oplus \mathcal{O}_F(2)$ or $F = \mathbb{P}^{n - m}$ and $\mathcal{E}$ restricted to it is $\oplus r \mathcal{O}_F(1)$. Moreover, for any line in a general fiber

$$(detE - (r + \epsilon)H). l = 0 \quad (5.2)$$

with $\epsilon = 1, 0$. By theorem (3.2) $\rho(X/W) = \rho(Z/W)$ and so, if $\varphi$ is elementary, also $\phi$ is so and, by (2.2) $detE = (r + \epsilon)H + \phi^*B$, that is $\phi$ is supported by $K_X + (n - m + \epsilon)H$. Assume now that $\dim X \geq 2 \dim W + 1$; by proposition (2.9) the contraction $\phi : X \to W$ is an elementary contraction and so, again by (3.2) also $\varphi : Z \to W$ had to be elementary. $\square$

**Corollary 5.3** Assume now that there exists a vector bundle $\mathcal{F}$ on $W$ and an embedding of $Z$ into $\mathbb{P}(\mathcal{F})$ as a divisor of relative degree 2; assume moreover that $K_Z + (n - r - m)(\xi\mathcal{F})_Z$ is a good supporting divisor of a quadric bundle elementary contraction and $(\xi\mathcal{F})_Z$ is the restriction of an ample line bundle on $X$. Then either there exists an ample vector bundle $\mathcal{G}$ of rank $n - m + 1$ such that $X = \mathbb{P}(\mathcal{G})$ or there exists a vector bundle $\mathcal{G}$ of rank $n - m + 2$ and an embedding of $X$ into $\mathbb{P}(\mathcal{G})$ as a divisor of relative degree two.

**Proof.** If $Z$ is a quadric bundle then $\phi$ is equidimensional; in fact if it has any fiber of dimension $> n - m$ then, by proposition (2.18), even $Z \to W$ should have a fiber of dimension $> (n - m - r)$. Since $\varphi : Z \to W$ is elementary $\phi$ is a scroll or a quadric fibration with the respect to $H$. We conclude by (2.12) and (2.14).

**Remark 5.4** Theorem (6.1) extends [LM09], Theorem C] and rules out the doubtful case (3).

### 6 Del Pezzo fibrations

**Theorem 6.1** Let $X, \mathcal{E}$ and $Z$ be as in (3.4) with $\dim Z \geq 3$. We assume that $Z$ has a del Pezzo fibration contraction $\varphi : Z \to W$ with respect to an ample line bundle on $Z$, $H_Z$, which is the restriction of an ample line bundle $H$ on $X$. Then $X$ has a Fano-Mori contraction $\phi : X \to W$ which is of fiber type and with supporting divisor $D = K_X + det\mathcal{E} + (n - m - r - 1)H$, with $n = \dim X$ and $m = \dim W$. If $n - m \geq 5$, for the general fiber of $\phi$, $F$, we have either $(F, \mathcal{E}_F) \simeq (\mathbb{P}^{n - m}, \oplus r^{-1} \mathcal{O}_F(1) \oplus \mathcal{O}_F(3))$, $(F, \mathcal{E}_F) \simeq (\mathbb{P}^{n - m}, \oplus r^{-2} \mathcal{O}_F(1) \oplus \mathcal{O}_F(2))$,
or \((F, \mathcal{E}_F) \simeq (\mathbb{Q}^{n-m}, \oplus^r \mathcal{O}(1) \oplus \mathcal{O}(2))\) or \(F\) is a del Pezzo manifold with \(b_2 = 1\) and \(\mathcal{E}_F \simeq \oplus^r \mathcal{O}(1)\), where \(\mathcal{O}(1)\) is the ample generator of \(\text{Pic}(F)\).

If \(\varphi\) is elementary then \(\phi\) is elementary and it is either a scroll contraction or a quadric fibration contraction or a del Pezzo fibration contraction (i.e. it is supported by the divisor \(K_X + (n - m + \epsilon)H\) with \(\epsilon = 1, 0\) or \(-1\)). The last result holds replacing the assumption on \(\varphi\) with the strongest assumption \(\dim X = n \geq 2m + 3 = 2\dim W + 3\).

**Proof** The morphism \(\varphi\) is a contraction supported by \(K_Z + (n-r-m-1)H\), so, applying theorem (3.4), we get a contraction \(\phi : X \rightarrow Y\), defined by an high multiple of \(D = K_X + \det \mathcal{E} + (n-m-r-1)H\); this contraction is of fiber type and \(Y = W\) by proposition (3.13). Let \(F\) be a general fiber of \(\phi\); then \(F\) is a smooth Fano manifold of dimension \(n - m\) such that \(-K_F = (\det \mathcal{E} + (n-m-r)H)_F\). Thus, applying [PSW92, Main theorem] to the ample vector bundle \(\mathcal{E}_1 = (\mathcal{E} \oplus n-m-r-1)H\) we get the description of \(F\) and \(\mathcal{E}_F\).

Moreover, for any line in a general fiber

\[
(\det \mathcal{E} - (r+\epsilon)H).l = 0
\]  

(6.2)

with \(\epsilon = 1, 0\) or \(-1\).

By theorem (3.3) \(\rho(X/W) = \rho(Z/W)\) and so, if \(\varphi\) is elementary, also \(\phi\) is so, by (3.2) \(\det \mathcal{E} = (r+\epsilon)H + \phi^* B\), that is \(\phi\) is supported either by \(K_X + (n-m+\epsilon)H\). Assume now that \(\dim X \geq 2 \dim W + 3\); by proposition (2.9) the contraction \(\phi : X \rightarrow W\) is an elementary contraction and so, again by (3.2) also \(\varphi : Z \rightarrow W\) had to be elementary. □

**Remark 6.3** If \(n - m = 4\) the rank of \(\mathcal{E}\) can be 1 or 2 and, according to [PSW92, proposition 7.4] the possibilities for the general fiber are those listed in the theorem plus

1. \((F, \mathcal{E}_F) \simeq (\mathbb{P}^2 \times \mathbb{P}^2, \oplus^2 \mathcal{O}(1,1))\).
2. \((F, \mathcal{E}_F) \simeq (\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(1,1))\).
3. \((F, \mathcal{E}_F) \simeq (\mathbb{Q}^4, S(2))\).
4. \(F\) is a Fano 4-fold with \(b_2 = 1\) and index 1.

**Remark 6.4** If \(n - m = 3\) the rank of \(\mathcal{E}\) must be 1 and, according to [PSW92, Theorem 0.4] the possibilities for the general fiber are those listed in the theorem plus

1. \((F, \mathcal{E}_F) \simeq (\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{O}(2,1))\).
2. \((F, \mathcal{E}_F) \simeq (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1,1))\).
3. \(F\) is a del Pezzo manifold with \(b_2 \geq 2\) and \(\mathcal{E} = \mathcal{O}_F(1)\).
Corollary 6.5 Let $X$, $E$ and $Z$ be as in (3.1) and let $Z$ be a del Pezzo manifold with $b_2 = 1$. Then one of the following occurs:

1. $X \simeq \mathbb{P}^n$ and $E$ is either $\bigoplus^2 \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)$ or $\mathcal{O}_{\mathbb{P}^n}(3) \oplus \bigoplus^{r-1} \mathcal{O}_{\mathbb{P}^n}(1)$.

2. $X \simeq \mathbb{Q}^n$ and $E$ is $\mathcal{O}_{\mathbb{Q}^n}(2) \oplus \bigoplus^{r-1} \mathcal{O}_{\mathbb{Q}^n}(1)$.

3. $X$ is a del Pezzo manifold with $b_2 = 1$ and $E \simeq \bigoplus^r \mathcal{O}_X(1)$ where $\mathcal{O}_X(1)$ is the ample generator of $\text{Pic}(X)$.

**Proof.** The hypothesis on $H$ is not necessary in this case as noted in (3.5) and the cases in (6.3) and (6.4) are ruled out, because of the isomorphism $\text{Pic}(Z) \simeq \text{Pic}(X) \simeq Z$.

7 Some final considerations

Using the same arguments we can consider the case in which $Z$ has an extremal contraction on $W$ whose general fiber $F$ is a Fano variety of index $\leq \dim F - 2$. However, in these cases it is very difficult to provide a good description of the vector bundle $E$ and to construct non trivial examples. These difficulties already show up in the case in which $W$ is a point and $\text{Pic}(Z) \simeq Z$; we recall that a line on $Z$ in this case is a rational curve which is a line with respect to a the generator of $\text{Pic}(Z)$. The existence of a line on $Z$ is proved if the index of $Z$ is $\geq (n-2)$, by recent results of M. Mella. A line exists also if $2 \text{index } Z > \dim Z + 1$.

The following proposition summarize the simplest cases.

**Proposition 7.1** Let $X$, $E$ and $Z$ be as in (3.1); we assume that $Z$ is a Fano variety of dimension $\geq 2$ with $\text{Pic}(Z) \simeq Z$ and that $Z$ has a line. Then $X$ is a Fano variety with $\text{Pic}(X) \simeq Z$ and $\text{coindex}(Z) \geq \text{coindex}(X)$.

**Proof.** Let $H$ be a generator of $\text{Pic}(X) = \text{Pic}(Z)$ and let $s$ and $\tau$ be positive integers such that $\text{det } E = sH$ and $-K_Z = \tau H_Z$. Therefore, by adjunction formula and (3.13), we have that

$$K_X + \text{det } E + \tau H = K_X + (s + \tau)H = \mathcal{O}_X,$$

thus $X$ is a Fano manifold. If $C$ is a line of $Z$ then $\text{det } E \cdot C = sH \cdot C = s$; thus, by (2.16), $s \geq r$. In particular this gives $n + 1 \geq \text{index}(X) = s + \tau \geq r + \tau$.

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