CONLEY INDEX OF INVARIANT SETS FOR STRONGLY DAMPED HYPERBOLIC EQUATIONS AT RESONANCE

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Abstract. We prove the existence of compact invariant sets for the strongly damped hyperbolic differential equation \( \ddot{u}(t) = -Au(t) - cA\dot{u}(t) + \lambda u(t) + F(t, u(t)) \) being at resonance at infinity, that is, \( A : X \supset D(A) \to X \) is a sectorial operator on a Banach space \( X \) and \( F : [0, +\infty) \times X^\alpha \to X \) is a continuous bounded map defined on the fractional space \( X^\alpha \), \( c > 0 \) is a damping factor and \( \lambda \) is an eigenvalue of \( A \). More precisely, we provide geometrical assumptions for the nonlinearity \( F \), that allow to obtain Conley index formulas stating that the Conley index for the associated semiflow, with respect to large ball, is equal to suspension of the sphere with dimension depending on what of the geometrical assumptions imposed on the nonlinearity is satisfied. It is also proved that the geometrical assumptions generalize well-known Landesman-Lazer conditions, and moreover, cover some other cases where the nonlinearity \( F \) exhibits a lower order resonance at infinity.

1. Introduction

We are concerned with the strongly damped hyperbolic equations of the form
\[
\ddot{u}(t) = -Au(t) - cA\dot{u}(t) + \lambda u(t) + F(u(t)), \quad t > 0
\] (1.1)
where \( c > 0 \) is a damping factor, \( \lambda \) is a real number, \( A : X \supset D(A) \to X \) is a sectorial operator on a Banach space \( X \) and \( F : X^\alpha \to X \) is a continuous map, where \( X^\alpha \) for \( \alpha \in (0, 1) \), is a fractional power space associated with \( A \). This equation is an abstract formulation of many partial differential equations including nonlinear heat equation
\[
u_{tt}(x,t) = \Delta u(x,t) + c\Delta u_t(x,t) + \lambda u(x,t) + f(x,u(x,t)), \quad t \geq 0, \quad x \in \Omega
\] (1.2)
where \( \Omega \) is an open subset of \( \mathbb{R}^n \) \((n \geq 1)\), \( \Delta \) is a Laplace operator with the Dirichlet boundary conditions and \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a continuous map. To see this, it is enough to take \( Au := -\Delta u \) and \( F(u) = f(\cdot, u(\cdot)) \).

The dynamics of strongly damped wave equation has been investigated by many authors in the last years (see e.g. [3], [5], [6], [7], [1], [2], [3], [4], [8], [9], [10], [11], [12]). All of these papers concerns the existence of global attractor: the maximal compact invariant set \( K \) with the property that any the limit set is arbitrary close to \( K \). The existence of global attractor is proved under dissipation assumption which roughly speaking says that there is a bounded set into which every orbit eventually enters and remains.

In this paper we intend to study the existence of compact invariant sets for the equation (1.1) being at resonance at infinity, that is, \( \text{Ker}(\lambda I - A) \neq \{0\} \) and \( F \) is a bounded map. The main difficulty lies in the fact that, in the presence of resonance, the dissipation condition is not satisfied and the problem of existence of compact invariant sets may not have solution for general nonlinearity \( F \). This fact has been explained in detail in Remark 4.1. To handle with this problem we impose geometrical assumptions \((G1)\) and \((G2)\) on the nonlinearity \( F \) (see page 112), that

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allow to obtain, the main result of this paper, Conley index formulas stating that
the Conley index the associated semiflow with respect to large ball is equal to the
suspension of the sphere with dimension depending on what of the two geometrical
assumptions is satisfied. The obtained abstract results are applied to derive criteria
on existence of compact invariant sets for strongly damped wave equation (1.2).

It turns out that if $F$ is a Niemytzki operator associated with the map $f : \Omega \times \mathbb{R} \to \mathbb{R}$,
then the introduced geometrical assumptions generalize well-known Landesman-
Lazer conditions (see [30], [17], [32], [33], [13] and [14]), and moreover, cover many
other cases where the nonlinearity $F$ exhibits even lower order resonance at infinity
(see [16], [12], [39]).

To explain our methods more precisely observe that the second order equation
(1.1) can be written as the first order equation
\[
\dot{w}(t) = -Aw(t) + F(w(t)), \quad t > 0,
\]
where $A : E \supset D(A) \to E$ is a linear operator on the space $E := X^\alpha \times X$
given by $D(A) := \{(x, y) \in E \mid x + cy \in D(A)\}$ and $A(x, y) := (-y, A(x + cy) - \lambda x)$ for $(x, y) \in D(A)$,
and $F : E \to E$ is the map defined by $F(x, y) := (0, F(x))$ for $(x, y) \in E$. Assume
that, for every initial data $(x, y) \in E$, the equation (1.3) admits a (mild) solution
$w(-; (x, y)) : [0, +\infty) \to E$ starting at $(x, y)$. We can define the semiflow $\Phi : [0, +\infty) \times E \to E$, by
\[
\Phi(t, (x, y)) := w(t, (x, y)) \quad \text{for} \quad t \in [0, +\infty), \ (x, y) \in E.
\]
In this paper we deal with an approach (see [35]) for finding compact invariant sets
for (1.1) that relies on seeking of Conley index of the semiflow $\Phi$ associated with
the equation (1.3). The Conley index, that is the main topological tool that we use
in this paper, was initially constructed for semiflows acting on finite dimensional
spaces, see [19], [35], [10]. However in the study of partial differential equations the
semiflow acts usually on the infinite dimensional functional spaces which is no longer
locally compact. In [35] and [39] the index theory were extended on arbitrary metric
space, which gave a rise to study the dynamics of partial differential equations.

The paper is organized as follows. In Section 2, we provides some spectral
properties of the hyperbolic operator $A$. We prove that the elements of spectrum
of $A$ with negative real part are actually eigenvalues of this operator. Subsequently
we describe spectral decomposition of the operator $A$ in the case when $\lambda$ is an
eigenvalue of the operator $A$. Here the main difficulties are caused by the fact
that the operator $A$ does not have compact resolvents, despite the fact that $A$ has
compact resolvents as we assumed. Crucial point for our considerations is to see the
relationship between spectral decomposition of operators $A$ and $A$ (see Theorem
2.6 (iii)).

Section 3 is devoted to the mild solutions for (1.1). First we provide the standard
facts concerning the existence and uniqueness for this equation. Furthermore, we
focus on continuity and compactness properties for the semiflow $\Phi$.

In Section 4 we provide geometrical assumptions for the nonlinearity $F$ and use
them to prove the Conley index formula that is the main result of this paper.

Finally, in section 6 we provide applications of the obtained abstract results partial
differential equations. First of all, in Theorems 5.3 and 5.5 we prove that if $F$
is a Niemytzki operator associated with a map $f$, then the well known Landesman-
Lazer (see [30]) and strong resonance conditions (see [16]) are actually particular
case of introduced geometrical assumption. Then, we provide the criteria on the
existence of $T$-periodic solution for the strongly damped wave equation in terms of
Landesman-Lazer and strong resonance type conditions.
2. Spectral properties of hyperbolic operator

Let \( A : X \supset D(A) \to X \) be a positive sectorial operator on a real Banach space \( X \) with norm \( \| \cdot \| \) such that following assumptions are satisfied:

(A1) the operator \( A \) has compact resolvents,

(A2) there is a Hilbert space \( H \) endowed with a scalar product \( \langle \cdot , \cdot \rangle_H \) and norm \( \| \cdot \|_H \) and a continuous injective map \( i : X \hookrightarrow H \),

(A3) there is linear self-adjoint operator \( \hat{A} : H \supset D(\hat{A}) \to H \) such that \( \text{Gr}(A) \subset \text{Gr}(\hat{A}) \), where the graph inclusion is understood in the sense of product map \( X \times X \xrightarrow{i \times i} H \times H \).

As we are working on a real space \( X \), by the spectrum \( \sigma(A) \) of the operator \( A \) we mean the sense of its complexification. To be more precise we put \( X_C := X \times X \) and, denoting \( x + iy := (x, y) \) for \( (x, y) \in X_C \), we define the operations of addition and multiplication by complex scalar

\[
(x_1 + iy_1) + (x_2 + iy_2) := (x_1 + x_2) + i(y_1 + y_2)
\]

\[
\lambda \cdot (x + iy) := (\lambda_1 x - \lambda_2 y) + i(\lambda_1 y + \lambda_2 x)
\]

for \( (x, y), (x_1, y_1), (x_2, y_2) \in X_C \) and \( \lambda = (\lambda_1 + \lambda_2 i) \in \mathbb{C} \). The function

\[
\|z\|_C := \sup_{\theta \in [0, 2\pi]} \|\sin \theta x + (\cos \theta) y\|
\]

for \( z = x + iy \in X_C \) defines a complete norm on \( X_C \). The complexification of \( A \) is a \( \mathbb{C} \)-linear operator \( A_C : X_C \supset D(A_C) \to X_C \) given by

\[
D(A_C) = D(A) \times D(A), \quad A_C(x + iy) := Ax + iAy \quad \text{for} \quad x + iy \in D(A_C).
\]

By the spectrum of the operator \( A \) we mean the spectrum of its complexification: \( \sigma(A) := \sigma(A_C) \). Similarly by the eigenvalue of \( A \) we mean eigenvalue of \( A_C \).

**Remark 2.1.** The spectrum \( \sigma(A) \) consists of the sequence (possibly finite) of real eigenvalues. Indeed, the operator \( A \) has compact resolvents which implies that

\[
\sigma(A) = \{ \lambda_k | \ i \geq 1 \} \subset \mathbb{C}
\]

and this set is finite or \( |\lambda_n| \to +\infty \) when \( n \to +\infty \). Furthermore, if \( \lambda \in \mathbb{C} \) is a complex eigenvalue of \( A \), then, by (A3), it is also a complex eigenvalue of the symmetric operator \( \hat{A} \) and hence \( \lambda \) is a real number. It follows that the spectrum of \( A \) can be exhibited as the increasing sequence of eigenvalues

\[
\lambda_1 < \lambda_2 < \ldots < \lambda_i < \lambda_{i+1} < \ldots
\]

which is finite or \( \lambda_i \to +\infty \) when \( i \to +\infty \). \( \square \)

Let \( X^n \) be a fractional space associated with \( A \) a let \( A : E \supset D(A) \to E \) be a linear operator on the space \( E := X^n \times X \) given by

\[
D(A) := \{(x, y) \in E = X^n \times X | x + cy \in D(A)\}, \quad A(x, y) := (-y, A(x + cy) - \lambda x) \quad \text{for} \quad (x, y) \in D(A), \tag{2.2}
\]

where \( c > 0 \), and \( \lambda \) is a real number. We assume that the space \( E \) is equipped with the norm

\[
\|(x, y)\|_E := \|x\|_n + \|y\| \quad \text{for} \quad (x, y) \in E.
\]

**Theorem 2.2.** The following assertions hold.

(i) The set \( \sigma(A) \setminus \{1/c\} \) consists of the eigenvalues of the operator \( A \).
(ii) If \( \lambda_k \leq \lambda < \lambda_{k+1}, k \geq 1 \) then \( \{ \mu \in \sigma_p(A) \mid \Re \mu \leq 0 \} = \{ \mu_i^- \mid 1 \leq i \leq k \} \), where
\[
\mu_i^\pm := \frac{\lambda c \pm \sqrt{(\lambda c)^2 - 4(\mu_i - \lambda)}}{2} \quad \text{for} \quad i \geq 1.
\] (2.3)

If \( \lambda < \lambda_1 \) then \( \{ \mu \in \sigma_p(A) \mid \Re \mu \leq 0 \} = \emptyset \).

**Proof.** Let \( A_C : (X^a \times X)_C \supset D(A_C) \to (X^a \times X)_C \) be a complexification of the operator \( A \). It can be easily checked that \( A_C \) is adjoint with the operator \( B : X_C \times X^a_C \supset D(B) \to X_C \times X^a_C \) given by
\[
D(B) := \{ (x,y) \in X^a_C \times X_C \mid x + cy \in D(A_C) \},
\]
\[
B(x,y) := (-y,A_C(x+cy)-\lambda x) \quad \text{for} \quad (x,y) \in D(B).
\]

Indeed, take \( C \)-linear isomorphism \( U : (Y \times X)_C \to X^a_C \times X_C \) given by
\[
U((x_1,y_1) + i(x_2,y_2)) := (x_1 + ix_2,y_1 + iy_2) \quad \text{for} \quad (x_1,y_1) + i(x_2,y_2) \in (Y \times X)_C.
\]

Then, it is not difficult to see that \( U \tilde{A}_C = BU \). Hence, without loss of generality one can take \( A_C := B \). In the proof we will use the following lemma.

**Lemma 2.3.** If \( \mu \in \mathbb{C} \setminus \{ 1/c \} \) then \( (x,y) \in \text{Ker}(\mu I - A_C) \) if and only if \( (x,y) = (w,-\mu w) \) for some \( w \in \text{Ker} \left( \frac{\lambda - \mu^2}{1 - c\mu} I - A_C \right) \).

**Proof.** Assume that \( \mu \in \mathbb{C} \setminus \{ 1/c \} \) and \( (x,y) = A_C(x,y) \) for some \( (x,y) \neq 0 \). It implies that \( x + cy \in D(A_C) \) and
\[
\mu x = -y, \quad \mu y = A_C(x+cy) - \lambda x.
\] (2.4)

Hence \( x \neq 0 \), \((1 - c\mu)x \in D(A_C)\) and \((\lambda - \mu^2)x = A_C((1 - c\mu)x)\), which gives
\[
(\lambda - \mu^2)/(1 - c\mu)x = A_Cx \quad \text{and} \quad (\lambda - \mu^2)/(1 - c\mu) = \lambda_i \quad \text{for some} \ i \geq 1.
\]

Then \( x \in \text{Ker}(\lambda_i I - A_C) \) and \((x,y) = (x,-\mu x)\). On the other hand, if \((x,y) = (w,-\mu w)\) for some \( w \in \text{Ker} \left( \frac{\lambda - \mu^2}{1 - c\mu} I - A_C \right)\), then \( x + cy \in D(A_C) \) and
\[
\mu y - A_C(x+cy) + \lambda x = (\lambda - \mu^2)w - A_C((1 - c\mu)w) = (\lambda - \mu^2)w - (1 - c\mu)A_Cw = 0.
\]

Consequently \( \mu(x,y) = A_C(x,y) \), which gives desired conclusion. \( \square \)

We return to the proof of Theorem 2.2. For the point (i), take \( \mu \in \sigma(A) \setminus \{ 1/c \} \) such that \( \text{Ker}(\mu I - A_C) = \{ 0 \} \). We show that for every \((f,g) \in X^a_C \times X_C\) there is \((x,y) \in D(A_C)\) such that \( \mu(x,y) - A_C(x,y) = (f,g) \). Assume for the moment that this is true. Then there is inverse operator \((\mu I - A_C)^{-1}\). Since \( A_C \) is closed, the inverse \((\mu I - A_C)^{-1}\) is bounded on \( X^a \times X \). Therefore \( \mu \in \sigma(A) \), which contradicts the assumption and proves that \( \text{Ker}(\mu I - A_C) \neq \{ 0 \} \). Take \((f,g) \in X^a_C \times X_C\) and consider the following equations
\[
\mu x = -y + f, \quad \mu y = A_C(x+cy) - \lambda x + g.
\] (2.5)

Multiplying the former equation by \( c\lambda - \mu \) and later by \( 1 - c\mu \) we have
\[
(c\lambda - \mu)\mu x = -(c\lambda - \mu)y + (c\lambda - \mu)f
\]
\[
(1 - c\mu)\mu y = (1 - c\mu)A_C(x+cy) - (1 - c\mu)\lambda x + (1 - c\mu)g,
\]
which after adding gives
\[
(\lambda - \mu^2)(x+cy) = (1 - c\mu)A_C(x+cy) + (1 - c\mu)h,
\]
where \( h = (c\lambda - \mu)/(1 - c\mu)f + g \), and hence
\[
\frac{\lambda - \mu^2}{1 - c\mu}(x+cy) = A_C(x+cy) + h.
\]
Note that \((\lambda - \mu^2)/(1 - c\mu)\) is an element from the resolvent set of the operator \(A_C\). Otherwise, there is \(w \neq 0\) such that \(w \in \text{Ker} ((\lambda - \mu^2)/(1 - c\mu)I - A_C)\). Since \(\mu \neq 1/c\), from point (i) it follows that
\[
(w, -\mu w) \in \text{Ker}(\mu I - A_C),
\]
which contradicts the assumption because \((w, -\mu w) \neq 0\). Therefore
\[
x + cy = b := \left(\frac{\lambda - \mu^2}{1 - \mu c}I - A_C\right)^{-1}h,
\]
which allows us to define
\[
x := \frac{1}{1 - \mu c}(b - cf) \quad \text{and} \quad y := \frac{1}{1 - \mu c}(f - \mu b).
\]
Since \(b \in D(A_C) \subset X^o\) and \(f \in X^o\), we have \((b - cf) \in X^o\) and hence \(x \in X^o\). Furthermore \(x + cy = b \in D(A_C)\), which implies that \((x, y) \in D(A_C)\). Therefore, it is enough to check that the equations (2.5) are satisfied. To this end observe that
\[
\mu x = \frac{\mu}{1 - \mu c}(b - cf) = -\frac{1}{1 - \mu c}(f - \mu b) + f = -y + f.
\]
Furthermore, we have the sequence of equivalent equalities
\[
\mu y = A_C(x + cy) - \lambda x + g
\]
\[
\frac{\mu}{1 - \mu c}(f - \mu b) = A_C b - \frac{\lambda}{1 - \mu c}(b - cf) + g
\]
\[
\frac{\lambda - \mu^2}{1 - \mu c}b - A_C b = \frac{c\lambda - \mu}{1 - \mu c}f + g
\]
\[
h = \frac{c\lambda - \mu}{1 - \mu c}f + g.
\]
The last equality is true and the proof of the point (i) is completed.

To verify (ii) let \(\mu \in \sigma_p(A)\) be such that \(\text{Re} \mu \leq 0\). Then, by Lemma 2.3 and Remark 2.4, we have \((\lambda - \mu^2)/(1 - c\mu) = \lambda_i\) for some \(i \geq 1\). Hence the equation \(\mu^2 - c\lambda_i \mu + \lambda_i - \lambda = 0\) is satisfied and computing its roots we infer that either \(\mu = \mu^+_i\) or \(\mu = \mu^-_i\). Since \(\lambda_k \leq \lambda < \lambda_{k+1}\) and \(\text{Re} \mu \leq 0\), it follows that \(\mu = \mu^-_i\) for some \(1 \leq i \leq k\) and hence \(\mu \in \sigma_p(A) \mid \text{Re} \mu \leq 0 \subset \{\mu_i \mid 0 \leq i \leq k\}\). In order to prove the opposite inclusion take \(\mu = \mu^+_i\) for \(1 \leq i \leq k\). Then \(\mu \leq 0\) and the equation \((\lambda - (\mu^-_i)^2)/(1 - c\mu^-_i) = \lambda_i\) is satisfied. Therefore Lemma 2.3 shows that \(\mu^-_i \in \sigma_p(A)\), which completes the proof of (ii).

**Theorem 2.4.** ([29, Theorem 2.3]) Let \(\lambda = \lambda_k\) for some \(k \geq 1\) and let \(X_0 := \text{Ker} (\lambda I - A)\). Then there are closed subspaces \(X_+\), \(X_-\) of \(X\) such that \(X = X_+ \oplus X_- \oplus X_0\) and the following assertions hold.

(i) We have inclusions \(X_- \subset D(A), \ A(X_-) \subset X_-, \ A(X_+ \cap \text{D}(A)) \subset X_+\). Furthermore \(X_-\) is a finite dimensional space such that \(X_- = \{0\}\) if \(k = 1\) and \(\text{Ker} (\lambda I - A) \oplus \ldots \oplus \text{Ker} (\lambda_{k-1} I - A)\) if \(k \geq 2\).

(ii) If \(A_+ : X_+ \supset D(A_+) \to X_+\) and \(A_- : X_- \supset D(A_-) \to X_-\) are parts of the operator \(A\) in \(X_+\) and \(X_-\), respectively, then \(\sigma(A_+) = \{\lambda_i \mid i \geq k + 1\}\) and \(\sigma(A_-) = \{\lambda_i \mid 1 \leq i \leq k - 1\}\) for \(k \geq 1\).

(iii) The spaces \(X_0, X_-\) are \(X_+\) mutually orthogonal, that is, \(\langle (u_l), i(u_m) \rangle_H = 0\) for \(u_l \in X_l\) and \(u_m \in X_m\) where \(l, m \in \{0, -, +\}, l \neq m\).
Remark 2.5. Let $P, Q_\pm : X \to X$ be projections given for any $x \in X$ by
\[ Px = x_0 \quad \text{and} \quad Q_\pm x = x_\pm \] (2.6)
where $x = x_+ + x_0 + x_-$ for $x_i \in X_i$, $i \in \{ 0, - , + \}$. Write $Q := Q_- + Q_+$. Since the inclusion $X^\alpha \subset X$ is continuous, one can decompose $X^\alpha$ on a direct sum of closed spaces $X^\alpha = X_0 \oplus X_- \oplus X^\alpha_+$, where $X^\alpha_+ := X^\alpha \cap X_+$, $X^\alpha_- := X^\alpha \cap X_-$. Therefore the projections $P$ and $Q_\pm$ can be also considered as continuous maps $P, Q_\pm : X^\alpha \to X^\alpha$ given for any $x \in X^\alpha$ by (2.6).
\[ \square \]

We proceed to the spectral decomposition of the operator $A$.

Theorem 2.6. Let $\lambda = \lambda_k$ for some $k \geq 1$ and let $E_0 := \text{Ker} (\lambda I - A) \times \text{Ker} (\lambda I - A)$.
Then there are closed subspaces $E_i, E_- \in E$ such that $E := E_- \oplus E_0 \oplus E_+$ and the following assertions hold.

(i) We have $E_- \subset D(A)$, $A(E_-) \subset E_-$ and $A(E_+ \cap D(A)) \subset E_+$. Furthermore $\dim E_- = 0$ if $k = 1$ and $\dim E_- = \sum_{i=1}^{k-1} \dim \text{Ker} (\lambda I - A)$ if $k \geq 2$.

(ii) If $A_+ : E_+ \supset D(A_+) \to E_+$ and $A_- : E_- \supset D(A_-) \to E_-$ are parts of $A$ in $E_- \subset E_+$, respectively, then $\sigma(A_+) = \{ z \in \mathbb{C} \mid \text{Re } z > 0 \}$ and $\sigma(A_-) = \{ \mu_i^- \mid 1 \leq i \leq k - 1 \}$.

(iii) If $P, Q_-, Q_+ : E \to E$ are projections on the spaces $E_0$, $E_-$ and $E_+$, respectively and $Q := Q_- + Q_-$, then
\[ P(x,y) = (Px, Py) \quad \text{and} \quad Q(x,y) = (Qx, Qy) \quad \text{for} \quad (x,y) \in E. \] (2.7)

In the proof we use the following lemmata

Lemma 2.7. Let $\mu_i^\pm$ for $i \geq 1$ be numbers given by (2.3). If $1 \leq i \leq k$ then the spaces $K_i^\pm := \{ (w, -\mu_i^\pm w) \mid w \in \text{Ker} (\lambda I - A) \}$, are such that $K_i^\pm \subset \text{Ker} (\mu_i^\pm I - A)$ and $\text{Ker} (\lambda I - A) \times \text{Ker} (\lambda I - A) = K_1^+ \oplus K_i^-$.

Proof. If $1 \leq i \leq k$, then $\mu_i^\pm \neq \mu_i^-$ which implies that $K_i^+ \cap K_i^- = \{ 0 \}$. Since $\dim \text{Ker} (\lambda I - A) < +\infty$ and $\dim E(\lambda_i) = 2 \dim \text{Ker} (\lambda I - A) = \dim K_1^+ + \dim K_1^-$, we deduce that $\text{Ker} (\lambda I - A) \times \text{Ker} (\lambda I - A) = K_1^+ \oplus K_1^-$. If $(x,y) \in K_i^+$ then $(x,y) = (w, -\mu_i^\pm w)$ for some $w \in \text{Ker} (\lambda I - A)$. It follows that $x + cy \in D(A)$ and
\[ \mu_i^\pm y - A(x + cy) + \lambda x = (\lambda - (\mu_i^\pm)^2)w - A((1 - c\mu_i^\pm)w) = (\lambda - (\mu_i^\pm)^2)w - (1 - c\mu_i^\pm)Aw = (\lambda - (\mu_i^\pm)^2)w - (1 - c\mu_i^\pm)\lambda_i w = -(\mu_i^\pm)^2 - \lambda_i c\mu_i^\pm + \lambda_i = 0, \]
where the last equality follows from the fact that $\mu_i^\pm$ are the roots of the equation $\mu^2 - \lambda_i \mu + \lambda_i = 0$. Hence $\mu_i^\pm(x,y) = (-y, A(x + cy) - \lambda x) = A(x,y)$ which implies that $K_i^+ \subset \text{Ker} (\mu_i^\pm I - A)$ and the proof is completed.
\[ \square \]

We will also need the following lemmata

Lemma 2.8. Let $B : V \to V$ be a linear operator on a real finite dimensional space $V$ such that $V = V_1 \oplus V_2 \oplus \ldots \oplus V_l$ ($l \geq 1$) and $Bx = \nu_i x$ for $x \in V_i$, where $\nu_i \in \mathbb{R}$ ($1 \leq i \leq l$). Then $\sigma(B) = \{ \nu_i \mid 1 \leq i \leq l \}$.

Proof. It is enough to prove that $\sigma(B) \subset \{ \nu_i \mid 1 \leq i \leq l \}$. The opposite inclusion is obvious. Let $\nu \in \mathbb{C}$ be such that $\nu z = B_z \nu z$ for some $z := x + iy \in V \subset V_z$, $z \neq 0$. Then $V_z = V_1 \times V_2 \times V_3 \times \ldots \times V_l$ and $B_z = \nu_i z$ for $z \in V_i \times V_i$ ($1 \leq i \leq l$). Hence $z = z_1 + z_2 + \ldots + z_l$ where $z_i \in V_i \times V_i$ ($1 \leq i \leq l$) and therefore $\nu z = B_z \nu z = \nu_1 z_1 + \nu_2 z_2 + \ldots + \nu_l z_l$. Since $z \neq 0$, there is $1 \leq i \leq l$ such that $z_i \neq 0$.
Corollary 2.9. $C$ generates an equicontinuous semigroup $\{S(t)\}_{t \geq 0}$ of bounded operators on $E$. Furthermore, the operator $A$ is sectorial (see e.g. [31], [21]) and hence $-A$ generates an equicontinuous $C_0$ semigroup $\{S_A(t)\}_{t \geq 0}$ of bounded operators on $E$. The following corollary is a simple consequence of Theorem 2.6 and [25] Theorem 1.5.3.

**Corollary 2.9.** Let $\lambda = \lambda_k$ for some $k \geq 1$ and let $E = E_- \oplus E_0 \oplus E_+$ be a direct sum decomposition obtained in Theorem 2.6. Then

$$S_A(t)E_i \subset E_i \quad \text{for} \quad t \geq 0, \quad i \in \{0, -, +\}.$$
3. Solutions and Compactness Properties of Hyperbolic Equations

Assume that \( A : X \supset D(A) \to X \) is a positively defined sectorial operator with compact resolvents on a separable Banach space \( X \). Consider the second order differential equation

\[
\ddot{u}(t) = -Au(t) - cA\dot{u}(t) + \lambda u(t) + F(s, u(t)), \quad t \geq 0 \tag{3.1}
\]

where \( \lambda \) is a real number, \( c > 0 \), \( s \in [0, 1] \) is a parameter and \( F : [0, 1] \times X^\alpha \to X \) is a continuous map satisfying the following assumptions

\( (F1) \) for every \( x \in X^\alpha \) there is an open neighborhood \( V \subset X^\alpha \) of \( x \) and constant \( L > 0 \) such that for any \( s \in [0, 1], x_1, x_2 \in V \)

\[
\|F(s, x_1) - F(s, x_2)\| \leq L\|x_1 - x_2\|_\alpha;
\]

\( (F2) \) there is a constant \( c > 0 \) such that

\[
\|F(s, x)\| \leq c(1 + \|x\|_\alpha) \quad \text{for } s \in [0, 1], \ x \in X^\alpha.
\]

\( (F3) \) \( F \) is completely continuous, that is, for any bounded set \( V \subset X^\alpha \) the set \( F(S \times V) \) is relatively compact in \( X \).

The second order equation (3.1) may be written in the following form

\[
w(t) = -Aw(t) + F(s, w(t)), \quad t > 0 \tag{3.2}
\]

where \( A : E \supset D(A) \to E \) is given by (2.2) and \( F : [0, 1] \times E \to E \) is defined by

\[
F(s, (x, y)) := (0, F(s, x)) \quad \text{for } s \in S, \ (x, y) \in E = X^\alpha \times X.
\]

Definition 3.1. Let \( J \subset \mathbb{R} \) be an interval. We say that a continuous map \( w : J \to E \) is a mild solution of equation (3.2), if

\[
w(t) = S_A(t - t')w(t') + \int_{t'}^t S_A(t - \tau)F(s, w(\tau)) \, d\tau
\]

for every \( t, t' \in J, t' < t \).

Remark 3.2. (a) Since \( X \) is separable, it is known that \( X^\alpha \) is also separable and hence \( E \) is separable as well.

(b) Consider the direct sum decomposition \( E := E_+ \oplus E_0 \oplus E_- \) obtained in Theorem 2.6 and let \( P : E \to E, Q_+ : E \to E \) and \( Q_- : E \to E \) be projections on spaces \( E_0, E_+ \) and \( E_- \), respectively. Since the components are closed subspaces, the projections are continuous. Furthermore, Corollary 2.6 gives

\[
S_A(t)Pz = PS_A(t)z \quad \text{and} \quad S_A(t)Q_\pm z = Q_\pm S_A(t)z \quad \text{for } t \geq 0, \ z \in E. \tag{3.3}
\]

Note that \( A(E_0) \subset E_0 \) and hence, if an operator \( A_0 : E_0 \supset D(A_0) \to E_0 \) is a part of \( A \) in the space \( E_0 \), then \( A_0(x, y) := (-y, c\lambda y) \) for \( (x, y) \in E_0 \) and

\[
S_A(t)z = S_{A_0}(t)z \quad \text{for } t \geq 0, \ z \in E_0. \tag{3.4}
\]

(c) Since \( F \) satisfies assumption (F1), straightforward calculations shows that \( F \) is also locally lipschitz. Indeed, if we take \( s \in S \) and \( (x, y) \in E \), then there is a neighborhood \( U \) of \( x \) in \( X^\alpha \) such that

\[
\|F(s, x_1) - F(s, x_2)\| \leq L\|x_1 - x_2\|_\alpha \quad \text{for } x_1, x_2 \in U,
\]

where \( L > 0 \) is a constant. Writing \( W := U \times X \) for the neighborhood of \( (x, y) \), we infer that, for any \( (x_1, y_1), (x_2, y_2) \in W \)

\[
\|F(s, (x_1, y_1)) - F(s, (x_2, y_2))\|_E = \|F(s, x_1) - F(s, x_2)\| \leq L\|x_1 - x_2\|_\alpha \leq L\|(x_1, y_1) - (x_2, y_2)\|_E
\]
and we see that $F$ satisfies assumption $(F1)$. Since $F$ satisfies assumption $(F2)$,
$$
\|F(s,(x,y))\|_E = \|F(s,x)\| \leq c(1 + \|x\|_\alpha) \leq c(1 + \|x,y\|_E),
$$
which shows that $F$ satisfies this assumption as well.

$(d)$ Observe that assumption $(F3)$ implies that the map $F$ is \textit{completely continuous}, that is, for any bounded $\Omega \subset E$ the set $F(S \times \Omega)$ is relatively compact in $E$. Indeed, since $\Omega$ is bounded there is a radius $R > 0$ such that $\Omega \subset \Omega_1 \times \Omega_2$, where $\Omega_1 := \{x \in X^\alpha \mid \|x\|_\alpha \leq R\}$ and $\Omega_2 := \{x \in X \mid \|x\| \leq R\}$. Furthermore
$$
F(S \times \Omega) = \{(0,F(s,x)) \mid s \in S, x \in \Omega_2\} = \{0\} \times F(S \times \Omega_1).
$$
By assumption $(F3)$, the set $\{0\} \times F(S \times \Omega_1)$ is relatively compact in $E$, which proves that $F$ is completely continuous.

The following theorem is quite standard and its proof is a consequence of\cite{25} Theorem 3.3.3, Corollary 3.3.5 and Remark 3.2 (c).

**Theorem 3.3.** For every $s \in [0,1]$ and $(x,y) \in E$, equation (3.2) admits a unique mild solution $w(t,s,(x,y)) : [0, +\infty) \to E$ starting at $(x,y)$.

Our aim is to prove the following theorem concerning the continuity of mild solution with respect to parameter and initial data.

**Theorem 3.4.** If sequences $(x_n,y_n)$ in $E$ and $(s_n)$ in $S$ are such that $(x_n,y_n) \to (x_0,y_0)$ in $E$ and $s_n \to s_0$ as $n \to +\infty$, then
$$
w(t,s_n,(x_n,y_n)) \to w(t,s_0,(x_0,y_0)) \quad \text{as} \quad n \to +\infty,
$$
for $t \geq 0$, and this convergence in uniform for $t$ from bounded subsets of $[0, +\infty)$.

In the proof we will use the following theorem.

**Theorem 3.5.** (see \cite{20} Proposition 2.7) Assume that $(x_n,y_n)$ in $E$ and $(s_n)$ in $S$ are sequences such that $(x_n,y_n) \to (x_0,y_0)$ and $s_n \to s_0$ as $n \to +\infty$. If for any $t \in [0, +\infty)$ the set $\{w(t; s_n, x_n) \mid n \geq 1\}$ is relatively compact in $E$, then the set
$$
\{w(\cdot; s_n, x_n, y_n) \mid n \geq 1\}
$$
is relatively compact in $C([0,T],E)$ where $T > 0$ is arbitrary.

We will also use the following lemma which is a simple consequence of Remark 3.2 (c) and the Gronwall inequality.

**Lemma 3.6.** Let $(s_n)$ be a sequence in $[0,1]$ and let $(x_n,y_n)$ be a bounded sequence in $E$. Then the set $\{w(t; s_n, x_n, y_n) \mid t \in [0,t_0], \ n \geq 1\}$ is bounded in $E$.

**Proof of Theorem 3.4.** Step 1. Write $w_n := w(\cdot; s_n, (x_n,y_n))$ for $n \geq 1$ and let $\beta$ be a Hausdorff measure of noncompactness on the space $E$. We show that the sequence $(w_n)$ is relatively compact in $C([0,T],E)$.

From the integral formula, we see that
$$
w_n(t) = S_\Delta(t)(x_n,y_n) + \int_0^t S_\Delta(t-\tau)F(s_n, w_n(\tau)) \, d\tau \quad \text{for} \quad t \geq 0.
$$
Hence, for any $t \geq 0$, we have
$$
\beta(\{w_n(t) \mid n \geq 1\}) \leq \beta(S_\Delta(t)\{(x_n,y_n) \mid n \geq 1\}) + D_t,
$$
where
$$
D_t := \beta \left( \left\{ \int_0^t S_\Delta(t-\tau)F(s_n, w_n(\tau)) \, d\tau \mid n \geq 1\right\} \right).
$$
From Lemma 3.6 it follows that the set
$$
\{w_n(\tau) \mid \tau \in [0,t], \ n \geq 1\}
$$
is bounded in $E$. Hence, (see e.g. [22], [26]), we find that the map \( \varphi : [0, t] \to \mathbb{R} \), given by \( \varphi(\tau) := \beta([S_A(t - \tau)F(s_n, w_n(\tau)) \mid n \geq 1]) \) for \( \tau \in [0, t] \), is integrable and
\[
\beta \left( \left\{ \int_0^t S_A(t - \tau)F(s_n, w_n(\tau)) \, d\tau \mid n \geq 1 \right\} \right) \leq \int_0^t \varphi(\tau) \, d\tau \quad \text{for} \quad t \geq 0. \tag{3.8}
\]
By Remark 3.2 (d), the map $F$ is completely continuous. Hence in view of the fact that the set in (3.7) is bounded we infer that the set \( \{F(s_n, w_n(\tau)) \mid n \geq 1\} \) is relatively compact in $E$ which implies that \( \varphi(\tau) = 0 \) for \( \tau \in [0, t] \). Hence, by 3.6 and 3.8, we obtain
\[
\beta(\{w_n(t) \mid n \geq 1\}) \leq \beta(\{x_n(y_n) \mid n \geq 1\}) = 0 \quad \text{for} \quad t \geq 0,
\]
where the last inequality follows from that fact that \( (x_n, y_n)^{\geq 1} \) is convergent. Hence, Theorem 3.7 says that, for any $T > 0$, the sequence \( (w_n) \) is relatively compact in $C([0, T], E)$.

**Step 2.** Take an arbitrary subsequence \( (w_{nk})_{k \geq 1} \) of \( (w_n)_{n \geq 1} \). By Step 1, we deduce that it contains a convergent subsequence \( (w_{nk}) \). Let \( v : [0, T] \to E \) be a continuous map such that \( u_{nk}(t) \to v(t) \) in $E$, uniformly for $t \in [0, T]$, as $l \to +\infty$. Hence, passing in
\[
w_{nk}(t) = S_A(t)(x_{nk}, y_{nk}) + \int_0^t S_A(t - \tau)F(s_{nk}, w_{nk}(\tau)) \, d\tau,
\]
to limit with $l \to +\infty$, for any $t \in [0, T]$, we have
\[
v(t) = S_A(t)(x_0, y_0) + \int_0^t S_A(t - \tau)F(s_0, v(\tau)) \, d\tau.
\]
In view of uniqueness of mild solutions we infer that \( v(t) = w(t; s_0, (x_0, y_0)) \) for \( t \in [0, t_0] \). Consequently \( w_{nk}(t) \to w(t; s_0, (x_0, y_0)) \) in $E$, uniformly for \( t \in [0, T] \), as $l \to +\infty$. Since the sequence \( (w_{nk})_{k \geq 1} \) is arbitrary, we deduce that \( w_n(t) \to w(t; s_0, (x_0, y_0)) \) in $E$, uniformly for \( t \in [0, T] \), as \( n \to +\infty \), which completes the proof.

**Theorem 3.7.** Let $N \subset E$ be a bounded subset and let sequences \( (s_n) \) in $[0, 1]$, \( (t_n) \) in $[0, +\infty)$ and \( (z_n) \) in $E$ are such that \( t_n \to +\infty \) as \( n \to +\infty \) and \( w([0, t_n] \times \{s_n\} \times \{z_n\}) \subset N \) for \( n \geq 1 \). Then the set \( \{w(t; s_n, z_n) \mid n \geq 1\} \) is relatively compact.

**Proof.** For any $n \geq 1$ write \( w_n := w(\cdot; s_n, z_n) \) and \( R := \sup_{z \in N} \|z\|_E \). By Corollary 2.9 there are constants $\delta, M > 0$ such that
\[
\|S_A(t)z\|_E \leq Me^{-\delta t}\|z\|_E \quad \text{for} \quad z \in E_+, \ t \geq 0. \tag{3.9}
\]
If $\varepsilon > 0$ is arbitrary, then there is $t_0 > 0$ such that $R\|Q_\varepsilon\|_{L(E)}e^{-\delta t_0} \leq \varepsilon$. Furthermore, one can take $n_0 \geq 1$ such that $t_n \geq t_0$ for $n \geq n_0$. Put
\[
D_t := \left\{ \int_0^t S_A(t - \tau)F(s_n, \tau, w_n(t_n - t_0 + \tau)) \, d\tau \mid n \geq 1 \right\}.
\]
In view of the fact that \( w_n(t_n) = \Phi^{s_n}(t_0, w_n(t_n - t_0)) \) for \( n \geq n_0 \), it follows that
\[
\beta(\{w_n(t_n)\}_{n \geq 1}) = \beta(\{w_n(t_n)\}_{n \geq n_0}) \leq \beta(\{w(t_0; s_n, w(t_n - t_0; s_n, z_n)) \mid n \geq n_0\}) \leq \beta(S_A(t_0) \{w(t_0 - t_0; s_n, z_n) \mid n \geq n_0\}) + \beta(D_{t_0}) \tag{3.10}
\]
\[
\leq \beta(S_A(t_0)N) + \beta(D_{t_0}).
\]
By Remark 3.2 (e) the map $F$ satisfies assumptions (F1) and (F2). Since the set
\[
\{w_n(t_n - t_0 + \tau) \mid \tau \in [0, t], n \geq 1\} \tag{3.11}
\]
is contained in $N$, it is bounded and hence
\[
\{S_A(t_0 - \tau)F(s_n, w_n(t_n - t_0 + \tau)) \mid \tau \in [0, t_0], \ n \geq 1\},
\]
is also bounded. Similarly as before the function $\varphi : [0, t] \to \mathbb{R}$ given by
\[
\varphi(\tau) := \beta(\{S_A(t_0 - \tau)F(s_n, w_n(t_n - t_0 + \tau)) \mid n \geq 1\}) \quad \text{for} \quad \tau \in [0, t]
\]
is of class $L^1([0, t])$ and
\[
\beta \left( \int_0^t S_A(t_0 - \tau)F(s_n, w_n(t_n - t_0 + \tau)) \, d\tau \mid n \geq 1 \right) \leq \int_0^t \varphi(\tau) \, d\tau.
\]
Therefore, in view of boundedness of (3.11) and the fact that $F$ is completely continuous (see Remark 3.2 (d)) we find that $\{F(s_n, w_n(t_n - t_0 + \tau)) \mid n \geq 1\}$ is relatively compact for any $\tau \in [0, t]$. Therefore
\[
\beta(D_\tau) = \varphi(\tau) = 0 \quad \text{for} \quad \tau \in [0, t_0].
\]
Put $P_- := P + Q_-$. By the standard properties of the measure $\beta$, we have
\[
\beta(S_A(t)N) \leq \beta(S_A(t)P_-) + \beta(S_A(t)Q_-) = \beta(S_A(t)N)
\]
for $t \geq 0$, where the last inequality follows from the fact that the set $S_A(t)P_-N$ is relatively compact as a bounded subset of finite dimensional space $E_0 \oplus E_-$. Let $\beta_{E_+}$ be a Hausdorff measure of noncompactness on the space $E_+$. By (3.9)
\[
\beta(S_A(t)N) \leq \beta(S_A(t)Q_+) \leq \beta_{E_+}(S_A(t)Q_+) N
\]
\[
\leq e^{-\delta t} \beta_{E_+}(Q_+) N \leq \|Q_+\|_{L(E)} e^{-\delta t}.
\]
for $t \geq 0$. Combining this with (3.10) and (3.12), we find that
\[
\beta(\{w_n(t) \mid n \geq 1\}) \leq R\|Q_+\|_{L(E)} e^{-\delta t} \leq \varepsilon,
\]
which completes the proof.

4. CONLEY INDEX FORMULA FOR INVARIANT SETS

We are interested in the study of the existence of bounded orbits for the differential equations of the form
\[
\ddot{u}(t) = -Au(t) - cAu(t) + \lambda u(t) + F(u(t)), \quad t \geq 0 \quad (4.1)
\]
where $c > 0$, $\lambda$ is an eigenvalue of the operator $A : X \supset D(A) \to X$ defined on a separable Banach space $X$ and $F : X^\alpha \to X$ is a continuous map. Assume that
(A1), (A2), (A3), (F1), (F3) are satisfied and the following condition holds
(F4) there is $m > 0$ such that $\|F(x)\| \leq m$ for $x \in X^\alpha$.

The second order equation (4.1) can be written in the following form
\[
\dot{w}(t) = -Aw(t) + F(w(t)), \quad t > 0 \quad (4.2)
\]
where $A : E \supset D(A) \to E$ is an operator given by (2.2) and $F : E \to E$ is defined by
\[
F(x, y) := (0, F(x)) \quad \text{for} \quad (x, y) \in E = X^\alpha \times X.
\]
From Remark 3.2 (c) and Theorem 3.3 it follows that, for any $(x, y) \in E$, there is a mild solution $w(\cdot; (x, y)) : [0, +\infty) \to E$ of the equation (4.1) starting at $(x, y)$. Let $\Phi : [0, +\infty) \times E \to E$ be a semiflow associated with this equation given by
\[
\Phi(t, (x, y)) := w(t; (x, y)) \quad \text{for} \quad t \in [0, +\infty), \ (x, y) \in E.
\]
Then, Theorems 3.4 and 3.7 imply that the semiflow $\Phi$ is a continuous map and any bounded subset of $E$ is admissible with respect to $\Phi$. 
Remark 4.1. If the equation (4.1) is at resonance at infinity, then the problem of existence of bounded orbits may not have solutions for general nonlinearity $F$.

Indeed, it is enough to take $F(x) = y_0$ for $x \in X^\alpha$, where $y_0 \in \text{Ker}(\lambda I - A) \setminus \{0\}$.

If $w : \mathbb{R} \to E$ is a mild solution of (4.2), then

$$w(t) = S_A(t)w(0) + \int_0^t S_A(t - \tau)(0, y_0) d\tau \quad \text{for} \quad t \in \mathbb{R}.$$  

Consider the direct sum decomposition $E := E_+ \oplus E_0 \oplus E_-$ obtained in Theorem 2.6 along with projections $P$, $Q_+$, $Q_-$ on its components. Acting on the equation by the operator $P$ and using (3.3), (3.4) and (2.7) we infer that

$$Pw(t) = SA(t)Pw(0) + \int_0^t SA(t - \tau)P(0, y_0) d\tau \quad \text{for} \quad t \in \mathbb{R}.$$  

which together with (3.3) and (2.7) implies that

$$Pw(t) = S_{A_0}(t)Pw(0) + \int_0^t S_{A_0}(t - \tau)(0, y_0) d\tau \quad \text{for} \quad t \in \mathbb{R}.$$  

Since $A_0$ is bounded, we infer that the map $(u_0, v_0) : \mathbb{R} \to E_0$ given by

$$(u_0(t), v_0(t)) := Pw(t) \quad \text{for} \quad t \geq 0,$$  

is of class $C^1$ and

$$\begin{cases} \dot{u}_0(t) = v_0(t), & t \in \mathbb{R}, \\ \dot{v}_0(t) = -c\lambda v_0(t) + y_0, & t \in \mathbb{R}. \end{cases}$$  

From the above equation it follows that

$$\frac{d}{dt}(\langle u_0(t), c\lambda y_0 \rangle_H + \langle v_0(t), y_0 \rangle_H) = \langle v_0(t), c\lambda y_0 \rangle_H + \langle -c\lambda v_0(t) + y_0, y_0 \rangle_H = \|y_0\|^2_H$$  

for $t \in \mathbb{R}$ and hence

$$\langle u_0(t), c\lambda y_0 \rangle_H + \langle v_0(t), y_0 \rangle_H - \langle u_0(0), c\lambda y_0 \rangle_H + \langle v_0(0), y_0 \rangle_H = t\|y_0\|^2_H \quad (4.3)$$  

for $t \in \mathbb{R}$. Therefore, if $w$ is bounded, then the left side of (4.3) is also bounded, a contradiction with $\|y_0\|_H > 0$. □

Recalling that $X^\alpha_+ \subset X^\alpha_0$ and $X^\alpha_-$ are subspaces from Remark 2.5, we overcome these difficulties by introducing the following geometric conditions for $F$ which will guarantee the existence of $T$-periodic solutions for the equation (4.2):

$$\begin{align*}
(\text{G1}) & \quad \{ F(x + y), x \}_H > -(F(x + y), z)_H \\
& \quad \text{for any balls } B_1 \subset X^\alpha_+ \oplus X^\alpha_0 \text{ and } B_2 \subset X_0 \text{ with } \|x\|_H \geq R.
\end{align*}$$  

$$\begin{align*}
(\text{G2}) & \quad \{ F(x + y), x \}_H < -(F(x + y), z)_H \\
& \quad \text{for any balls } B_1 \subset X^\alpha_+ \oplus X^\alpha_0 \text{ and } B_2 \subset X_0 \text{ with } \|x\|_H \geq R.
\end{align*}$$  

Now we proceed to the main result of this section, namely the index formula for bounded orbits. It is a tool to determining the Conley index for the maximal invariant set contained in appropriately large ball in terms of geometrical conditions (G1) and (G2). This theorem will be used to prove the existence of bounded orbits for the equation (4.2).

Theorem 4.2. Let $\lambda = \lambda_k$ for $k \geq 1$, be an eigenvalue of the operator $A$ and let $d_l := \sum_{i=1}^{l-1} \dim \text{Ker}(\lambda I - A)$ for $l \geq 1$ with the exceptional case $d_0 := 0$. Then there is a closed isolated neighborhood $N \subset E$, admissible with respect to $\Phi$ such that, for $K := \text{Inv}(N, \Phi)$, we have the following assertions:

- ...
Lemma 4.4. and where the linear homeomorphism $U$ which implies that $z \in [0, 1]$, $x \in X^\alpha$. (4.4)

In the proof of above theorem we will use the following differential equations

$$\dot{w}(t) = -Aw(t) + G(s,w(t)), \quad t > 0$$

(4.5)

where $G : [0, 1] \times E \rightarrow E$ is defined by

$$G(s,(x,y)) := (0,G(s,x)) \quad \text{for} \quad s \in [0, 1], \ (x,y) \in E.$$

**Remark 4.3.** (a) It is not difficult to see that $G$ satisfies assumption $(F1)$ and $(F2)$. In view of Remark 3.2, this implies that $G$ also satisfies assumptions $(F1)$ and $(F2)$. The later means that there is $m_0 > 0$ such that

$$\|G(s,x)\|_E \leq m_0 \quad \text{for} \quad s \in [0, 1], \ (x,y) \in E.$$  

(4.6)

(b) Since $F$ is completely continuous, it is not difficult to see that $G$ is also completely continuous. Hence, from Remark 3.2 $(d)$ we deduce that $G$ satisfies assumption $(F3)$, that is, the set $G([0, 1] \times [0, +\infty) \times \Omega)$ is a relatively compact in $E$, for any bounded $\Omega \subset E$. □

By the above remark and Theorem 3.3 for any $s \in [0, 1]$, we can define the semiflow $\Psi^s : [0, +\infty) \times X^\alpha \rightarrow X^\alpha$ given by

$$\Psi^s(t,(x,y)) := w(t;s,(x,y)) \quad \text{for} \quad t \in [0, +\infty), \ (x,y) \in E,$$

where $w(\cdot;s,(x,y)) : [0, +\infty) \rightarrow E$ is a mild solution of (4.3) starting at $(x,y)$. Theorems 3.4 and 5.7 imply that the family of semiflows $\{\Psi^s\}_{s \in [0,1]}$ is continuous and any bounded subset of $E$ is admissible. Furthermore, the family is a homotopy between $\Psi^1 = \Phi$ and $\Psi^0$. Note that every solution $(u,v) : [0, +\infty) \rightarrow E$ of the semiflow $\Psi^0$ satisfies the following integral formula

$$(u(t),v(t)) = S_A(t)(u(0),v(0)) + \int_0^t S_A(t-\tau)(0,PF(Pu(\tau)))d\tau \quad \text{for} \quad t \geq 0.$$  

(4.7)

Then for any $t \in [0, +\infty) \cap (x,y) \in E$ we have

$$\Psi^0(t,(x,y)) = \varphi_1(t,Q(x,y)) + \varphi_2(t,P(x,y)), $$

which implies that $\Psi^0$ is topologically equivalent with the cartesian product of $\varphi_1$ and $\varphi_2$, that is, for any $t \geq 0$ and $(z_1, z_2) \in (E_- \oplus E_+) \times E_0$

$$\Psi^0(t,U(z_1,z_2)) = U(\varphi_1(t,z_1),\varphi_2(t,z_2)), $$

(4.8)

where the linear homeomorphism $U : (E_- \oplus E_+) \times E_0 \rightarrow E$ is given by $U(z_1,z_2) = z_1 + z_2$ for $(z_1, z_2) \in (E_- \oplus E_+) \times E_0$.

The following lemma provides some a priori estimates on solutions of (4.5).

**Lemma 4.4.** There is a constant $R > 0$ such that the following assertions hold.

(i) If $w = w_s : \mathbb{R} \rightarrow E$, where $s \in [0, 1]$, is a full solution of (4.5) such that the set $\{Q_s, w(t) \mid t \leq 0\}$ is bounded in $E$, then

$$\|Q_s w(t)\|_E \leq R \quad \text{for} \quad t \in \mathbb{R}. $$

(4.9)
proof that, for $t \geq t'$, which implies that

$$w(t) = S_A(t - t')w(t') + \int_{t'}^t S_A(t - \tau)G(s, w(\tau)) d\tau \quad \text{for} \quad t \geq t'. \quad (4.11)$$

To verify point (i) we act on the above equation by $Q_+$. Then, by (2.3), we infer that there are constants $c, M > 0$ such that

$$\|S_A(t - t')Q_+ w(t')\|_E \leq M e^{-c(t-t')} \|Q_+ w(t')\|_E$$

for $t, t' \in \mathbb{R}, t > t'$. Hence the boundedness of $\{Q_+ w(t) | t \leq 0\}$ implies that

$$\|S_A(t - t')Q_+ w(t')\|_E \to 0 \quad \text{as} \quad t' \to -\infty. \quad (4.12)$$

By (2.3) and (4.6), we obtain

$$\|Q_+ w(t)\|_E \leq \|S_A(t - t')Q_+ w(t')\|_E + \int_{t'}^t \|S_A(t - \tau)Q_+ G(s, w(\tau))\|_E d\tau$$

$$\leq \|S_A(t - t')Q_+ w(t')\|_E + M \int_{t'}^t e^{-c(t-\tau)} \|Q_+ G(s, w(\tau))\|_E d\tau$$

$$\leq \|S_A(t - t')Q_+ w(t')\|_E + m_0 M \|Q_+\|_{L(E)} \int_{t'}^t e^{-c(t-\tau)} d\tau$$

$$= \|S_A(t - t')Q_+ w(t')\|_E + m_0 M \|Q_+\|_{L(E)} (1 - e^{-c(t-t')}/c,$$

and consequently

$$\|Q_+ w(t)\|_E \leq \|S_A(t - t')Q_+ w(t')\|_E + m_0 M \|Q_+\|_{L(E)} (1 - e^{-c(t-t')}/c.$$

Letting with $t' \to -\infty$ and using (4.12), we deduce that (4.9) is satisfied with $R := m_0 M \|Q_+\|_{L(E)} / c$.

To prove (ii), we act on (4.11) by the operator $Q_-$. Then we use (2.3) and obtain

$$Q_- w(t) = S_A(t - t')Q_- w(t') + \int_{t'}^t S_A(t - \tau)Q_- G(s, w(\tau)) d\tau$$

which implies that, for $t, t' \in \mathbb{R}$ and $t \geq t'$, we have

$$S_A(t' - t)Q_- w(t) = Q_- w(t') + \int_{t'}^t S_A(t' - \tau)Q_- G(s, w(\tau)) d\tau \quad (4.13)$$

because the semigroup $\{S_A(t)\}_{t \geq 0}$ can be extended on the space $E_-$ to a $C_0$ group. Then, by (2.3)

$$\|S_A(t' - t)Q_- w(t)\|_E \leq M e^{c(t'-t')} \|Q_- w(t)\|_E \quad \text{for} \quad t \geq t'$$

and therefore the boundedness of $\{Q_- w(t) | t \geq 0\}$ implies that

$$\|S_A(t' - t)Q_- w(t)\|_E \to 0 \quad \text{as} \quad t \to +\infty. \quad (4.14)$$
Here, using (4.10) and (2.9), we derive that
\[
\|Q_w(t')\|_E \leq \|S_A(t' - t)Q_w(t)\|_E + \int_{t'}^1 \|S_A(t' - \tau)Q_w(s, \tau)\|_E d\tau
\]
\[
\leq \|S_A(t' - t)Q_w(t)\|_E + \int_{t'}^1 M^{c(t' - \tau)} \|Q_w(s, \tau)\|_E d\tau
\]
\[
\leq \|S_A(t' - t)Q_w(t)\|_E + \int_{t'}^1 m_0 M \|Q_w\|_{L(E)} e^{c(t' - \tau)} d\tau
\]
\[
= \|S_A(t' - t)Q_w(t)\|_E + \frac{m_0 M \|Q_w\|_{L(E)}}{c} \left(1 - e^{c(t' - t)}\right).
\]

Letting \( t \to +\infty \) and using (4.14), we have
\[
\|Q_w(t')\|_E \leq m_0 M \|Q_w\|_{L(E)}/c \quad \text{for} \quad t' \in \mathbb{R},
\]
which implies that (4.10) is satisfied with \( R := m_0 M \|Q_w\|_{L(E)}/c \) and the proof is completed. \(\square\)

**Proof of Theorem 4.2.** Let \( \{e_1, e_2, \ldots, e_n\} \), where \( n = \dim \ker (\lambda I - A) \), be a basis of the space \( \ker (\lambda I - A) \) consisting of orthogonal and unitary vectors in the norm \( \|\cdot\|_H \). Assume that the space \( E_0 = \ker (\lambda I - A) \times \ker (\lambda I - A) \) is equipped with the scalar product and norm, given by
\[
\langle (x_1, y_1), (x_2, y_2) \rangle_E = \langle x_1, x_2 \rangle_H + \langle y_1, y_2 \rangle_H \quad \text{for} \quad (x_1, y_1), (x_2, y_2) \in E_0,
\]
\[
\| (x, y) \|_E = \left( \| x \|_H^2 + \| y \|_H^2 \right)^{1/2} \quad \text{for} \quad (x, y) \in E_0.
\]

Let \( \{f_1, f_2, \ldots, f_{2n}\} \) be a basis on the space \( E_0 \) given by
\[
f_i := \begin{cases} (\langle c\lambda \rangle^2 + 1)^{-1/2} (c\lambda e_i, e_i) & \text{for} \quad 1 \leq i \leq n, \\ (0, e_{i-n}) & \text{for} \quad n + 1 \leq i \leq 2n \end{cases}
\]
and let \( W : E_0 \to \mathbb{R}^n \times \mathbb{R}^n \) be a linear map defined by
\[
W(x, y) := \langle (x, y), f_1 \rangle_{E_0}, \langle (x, y), f_2 \rangle_{E_0}, \ldots, \langle (x, y), f_{2n} \rangle_{E_0} \quad \text{for} \quad (x, y) \in E_0.
\]

Then \( W(x, y) = (w_1, w_2) \) where
\[
w_1 := a \langle c\lambda x_1 + y_1, c\lambda x_2 + y_2, \ldots, c\lambda x_n + y_n \rangle \quad \text{and}
\]
\[
w_2 := (y_1, y_2, \ldots, y_n),
\]
where \( a := (\langle c\lambda \rangle^2 + 1)^{-1/2} \) and \( x_i := \langle x, e_i \rangle_H, y_i := \langle y, e_i \rangle_H \) for \( i = 1, 2, \ldots, n \).

Observe that
\[
|w_1| = a \| c\lambda x + y \|_H \quad \text{and} \quad |w_2| = \| y \|_H.
\]

**Step 1.** We define the isolating neighborhood for the family \( \{\Psi^s\}_{s \in [0, 1]} \). To this end, from Lemma 4.4 we obtain a constant \( R_1 > 0 \) with the property that, if \( w = (u, v) : \mathbb{R} \to \mathbf{E} \) is a bounded full solution of \( \Psi^s : \mathbf{E} \to \mathbf{E} \) for \( s \in [0, 1] \), then
\[
\|Qw(t)\|_E \leq R_1 \quad \text{for} \quad t \in \mathbb{R}.
\]

By assumption (F4) there is a constant \( m_1 > 0 \) such that
\[
\|PF(x)\|_H \leq m_1 \quad \text{for} \quad x \in X^n.
\]

Choose \( R_2 > 0 \) such that
\[
-c\lambda R_2^2 + m_1 R_2 < 0.
\]
and define the following sets

$$B_1 := \{ x \in X_0^\alpha \oplus X_0^\alpha \mid \| x \|_\alpha \leq R_1 + 1 \}$$

$$B_2 := \{ y \in \text{Ker} (M - A) \mid \| y \|_H \leq R_2 / (c\lambda) \}.$$  

Using geometrical conditions (G1), (G2) and orthogonality from Theorem 2.4 (iii), one can find $R_3 > 0$ such that

$$\langle PF(x + y), x \rangle_H > -\langle PF(x + y), z \rangle_H$$  (4.21)

for any $(y, z) \in B_1 \times B_2$, $x \in X_0$ with $\| x \|_H \geq R_3$, if condition (G1) is satisfied and

$$\langle PF(x + y), x \rangle_H < -\langle PF(x + y), z \rangle_H$$  (4.22)

for any $(y, z) \in B_1 \times B_2$, $x \in X_0$ with $\| x \|_H \geq R_3$, if the condition (G2) holds. Put

$$R_4 := ac\lambda R_3 + aR_2$$  (4.23)

and define the set $N \subset E$ as $N := N_1 \oplus N_2$, where

$$N_1 := \{ (x, y) \in E_+ \oplus E_+ \mid \| (x, y) \|_E \leq R_1 + 1 \},$$

$$N_2 := W^{-1}M \text{ dla } M := \{ (w_1, w_2) \in \mathbb{R}^{2n} \mid \| w_1 \| \leq R_4, \| w_2 \| \leq R_2 \}.$$  

**Step 2.** We show that $N$ is an isolating neighborhood for the family $\{ \Psi^s \}_{s \in [0, 1]}$. Suppose that $w : \mathbb{R} \to E$ is a full solution of the semiflow $\Psi^t : E \to E$ for some $s \in [0, 1]$ such that $w(\mathbb{R}) \subset N$ and $w(\mathbb{R}) \cap \partial N \neq \emptyset$. Without loss of generality we can assume that $w(0) \in \partial N$. Then either $Pw(0) \in \partial N_2$ and $Qw(0) \in N_1$ or $Pw(0) \in N_2$ and $Qw(0) \in \partial N_1$. The choice of $R_1$ and the boundedness of $w$ in the space $E$ gives $\| Qw(0) \|_E \leq R_1$ and consequently $Qw(0) \in \text{int} N_1$. It remains that $Pw(0) \in \partial N_2$ and $Qw(0) \in N_1$. From (4.17) we have $Qw(0) = (Qu(0), Qv(0))$, and hence (4.18) implies that $\| Qu(0) \|_\alpha + \| Qv(0) \| \leq R_1$, which in turn gives

$$\| Qu(0) \|_\alpha \leq R_1.$$  (4.24)

Let the map $(w_1, w_2) : \mathbb{R} \to \mathbb{R}^{2n}$ be given by the formula

$$(w_1(t), w_2(t)) := WP(w(t)) = W(Pu(t), Pv(t)) \text{ for } t \in \mathbb{R}.$$  

From (4.16) we obtain

$$w_1(t) := a(c\lambda u_1(t) + v_1(t), c\lambda u_2(t) + v_2(t), \ldots, c\lambda u_n(t) + v_n(t)), \quad w_2(t) := (v_1(t), v_2(t), \ldots, v_n(t)) \text{ for } t \in \mathbb{R},$$  (4.25)

where $a := ((c\lambda)^2 + 1)^{-1/2}$ and $u_i(t) := (Pu(t), e_i)_H$, $v_i(t) := (Pv(t), e_i)_H$ for $i = 1, 2, \ldots, n$. Acting by the operator $P$ on the equation

$$(u(t), v(t)) = S_A(t - t')(u(t'), v(t')) + \int_{t'}^t S_A(t - \tau)G(s, (u(\tau), v(\tau))) d\tau$$

where $t \geq t'$ and using (4.18), we infer that

$$P(u(t), v(t)) = S_A(t - t')P(u(t'), v(t')) + \int_{t'}^t S_A(t - \tau)PG(s, (u(\tau), v(\tau))) d\tau$$

for $t \geq t'$ and consequently, by (2.7) and (3.4), we have

$$(Pu(t), Pv(t)) = S_{A_0}(t - t')(Pu(t'), Pv(t')) + \int_{t'}^t S_{A_0}(t - \tau)(0, PF(sQu(\tau) + Pu(\tau))) d\tau$$

for $t \geq t'$, where $A_0 : E_0 \to E_0$ is the operator given by $A_0(x, y) := (-y, c\lambda y)$ for $(x, y) \in E_0$. Since $A_0$ is bounded and defined on finite dimensional space, the maps
If the former case holds, then, by \((4.25)\), we have
\[
P_t
\]
for
\[
\| H
\]
Hence we have
\[
c\lambda \hat{u}_i(t) + \dot{v}_i(t) = c\lambda \left\langle \frac{d}{dt} P u(t), e_i \right\rangle_H + \left\langle \frac{d}{dt} P v(t), e_i \right\rangle_H
\]
\[
= c\lambda v_i(t) - c\lambda v_i(t) + \langle PF(sQu(t) + Pu(t)), e_i \rangle_H
\]
\[
= \langle PF(sQu(t) + Pu(t)), e_i \rangle_H \quad \text{for} \quad t \in \mathbb{R}, \ 1 \leq i \leq n
\] (4.26)
and
\[
\dot{v}_i(t) = \left\langle \frac{d}{dt} P v(t), e_i \right\rangle_H = -c\lambda v_i(t) + \langle PF(sQu(t) + Pu(t)), e_i \rangle_H
\]
for \( t \in \mathbb{R}, \ 1 \leq i \leq n \). In view of the fact that \( P w(0) \in \partial N_2 \) we obtain
\[
(w_1(0), w_2(0)) = WP w(0) = W(Pu(0), Pv(0)) \in \partial M.
\]
Therefore either \( |w_1(0)| \leq R_4 \) and \( |w_2(0)| = R_2 \) or \( |w_1(0)| = R_4 \) and \( |w_2(0)| \leq R_2 \).
If the former case holds, then, by (4.25), we have
\[
\frac{d}{dt} \left( \frac{1}{2} |w_2(t)|^2 \right) = \dot{w}_2(t) \cdot w_2(t) = \sum_{i=1}^{n} \dot{v}_i(t) \cdot v_i(t)
\]
\[
= \sum_{i=1}^{n} -c\lambda v_i^2(t) + v_i(t) \langle PF(sQu(t) + Pu(t)), e_i \rangle_H
\]
\[
= -c\lambda |w_2(t)|^2 + \langle PF(sQu(t) + Pu(t)), P v(t) \rangle_H
\] (4.27)
for \( t \in \mathbb{R} \). Furthermore, from (4.17) and (4.19) it follows that
\[
\langle PF(sQu(t) + Pu(t)), P v(t) \rangle_H \leq m_1 \| P v(t) \|_H = m_1 |w_2(t)|,
\]
and hence, by (4.20) and (4.27), we deduce that
\[
\frac{d}{dt} \left( \frac{1}{2} |w_2(t)|^2 \right)_{t=0} = -c\lambda |w_2(0)|^2 + \langle PF(sQu(0) + Pu(0)), P v(0) \rangle_H
\]
\[
\leq -c\lambda |w_2(0)|^2 + m_1 |w_2(0)| = -c\lambda R_2^2 + m_1 R_2 < 0.
\]
Therefore, there is \( \delta > 0 \) such that \( |w_2(t)| > R_2 \) for \( t \in (-\delta, 0) \) which implies that \( P w(t) = (Pu(t), Pv(t)) \) \( \not\in N_2 \) for \( t \in (-\delta, 0) \), and contradicts the inclusion \( w(\mathbb{R}) \subset N \). Assume that \( |w_1(0)| = R_4 \) and \( |w_2(0)| \leq R_2 \). Then, by (4.25) and (4.26), we have
\[
\frac{d}{dt} \left( \frac{1}{2} |w_1(t)|^2 \right) = \dot{w}_1(t) \cdot w_1(t) = a^2 \sum_{i=1}^{n} (c\lambda \hat{u}_i(t) + \dot{v}_i(t))(c\lambda u_i(t) + v_i(t))
\]
\[
= a^2 \sum_{i=1}^{n} (c\lambda u_i(t) + v_i(t)) \langle PF(sQu(t) + Pu(t)), e_i \rangle_H
\]
\[
= a^2 \langle PF(sQu(t) + Pu(t)), c\lambda Pu(t) + P v(t) \rangle_H \quad \text{for} \quad t \in \mathbb{R}.
\]
From (4.17) it follows that \( \| P v(0) \|_H = |w_2(0)| \leq R_2 \) and, by (4.23), we have
\[
acR_3 + aR_2 = R_4 = |w_1(0)| = a|c\lambda Pu(0) + P v(0)|_H
\]
\[
\leq ac\lambda \| P u(0) \|_H + a \| P v(0) \|_H \leq ac\lambda \| P u(0) \|_H + aR_2,
\]
which implies that
\[
\| P u(0) \|_H \geq R_3 \quad \text{and} \quad \| P v(0) \|_H \leq R_2.
\] (4.28)
Therefore, if (G1) is satisfied, then by (1.24) and (1.28), the inequality (1.21) gives
\[
\frac{d}{dt} \left| w_1(t) \right|^2_{t=0} = a^2 \langle PF(sQu(0) + Pu(0)), c\lambda Pu(0) + Pv(0) \rangle_H \\
= a^2c\lambda \langle PF(sQu(0) + Pu(0)), Pu(0) \rangle_H \\
+ a^2 \langle PF(sQu(0) + Pu(0)), Pv(0) \rangle_H > 0.
\]
Consequently, there is \( \delta > 0 \) such that \( |w_1(t)| > R_4 \) for \( t \in (0, \delta) \). Hence \( Pw(t) = P(u(t), v(t)) \notin N_2 \) for \( t \in (0, \delta) \), which contradicts the inclusion \( w(\mathbb{R}) \subset N \). Similarly, if (G2) holds then
\[
\frac{d}{dt} \left| w_1(t) \right|^2_{t=0} < 0,
\]
that is, there is \( \delta > 0 \) such that \( |w_1(t)| > R_4 \) for \( t \in (-\delta, 0) \). Consequently \( Pw(t) = P(u(t), v(t)) \notin N_2 \) for \( t \in (-\delta, 0) \), contrary to the inclusion \( w(\mathbb{R}) \subset N \). Thus, we proved that \( N \) is an isolating neighborhood for the family \( \{\Psi^t\}_{t \in [0,1]} \).

**Step 3.** Now, we prove that, if condition (G1) is satisfied, then the set \( B := N_2 \) is an isolating block for \( \varphi_2 \) and the sets of strict egress points, strict ingress points and bounce off points are \( B^e = W^{-1}M^e \), \( B^c = W^{-1}M^c \) and \( B^b = W^{-1}M^b \), respectively, where
\[
M^e := \{ (w_1, w_2) \in \mathbb{R}^{2n} \mid |w_1| < R_4, |w_2| = R_2 \}, \\
M^c := \{ (w_1, w_2) \in \mathbb{R}^{2n} \mid |w_1| = R_4, |w_2| < R_2 \}, \\
M^b := \{ (w_1, w_2) \in \mathbb{R}^{2n} \mid |w_1| = R_4, |w_2| = R_2 \}.
\]
Furthermore, we show that, if condition (G2) is satisfied, then the set \( B := N_2 \) is an isolating block for the semiflow \( \varphi_2 \) and the boundary \( \partial B \) consists of strict ingress points.

Assume that condition (G1) is satisfied and let \((u, v) : [-\delta_2, \delta_1) \to E_0\), where \( \delta_1 > 0 \) and \( \delta_2 \geq 0 \), be a solution of the semiflow \( \varphi_2 \) such that \( (u(0), v(0)) \in W^{-1}M^t \). Let \((w_1, w_2) : \mathbb{R} \to \mathbb{R}^{2n}\) be a map given by
\[
(w_1(t), w_2(t)) := W(u(t), v(t)) \quad \text{for} \quad t \in [-\delta_2, \delta_1).
\]
From (4.16) it follows that
\[
w_1(t) := a(c\lambda u_1(t) + v_1(t), c\lambda u_2(t) + v_2(t), \ldots, c\lambda u_n(t) + v_n(t)), \\
w_2(t) := (v_1(t), v_2(t), \ldots, v_n(t)) \quad \text{for} \quad t \in [-\delta_2, \delta_1), \quad (4.29)
\]
where \( a := ((\lambda c)^2 + 1)^{-1/2} \) and \( u_i(t) := \langle u(t), e_i \rangle_H, \quad v_i(t) := \langle v(t), e_i \rangle_H \) for \( i = 1, 2, \ldots, n \). Since \((u, v)\) satisfies the equations
\[
\begin{cases}
\dot{u}(t) = v(t), \\
\dot{v}(t) = -c\lambda v(t) + PF(u(t)),
\end{cases} \quad t \in [-\delta_2, \delta_1), \quad (4.30)
\]
we infer that
\[
c\lambda \dot{u}_i(t) + \dot{v}_i(t) = c\lambda \langle \dot{u}(t), e_i \rangle_H + \langle \dot{v}(t), e_i \rangle_H \\
= c\lambda v_i(t) - c\lambda v_i(t) + \langle PF(u(t)), e_i \rangle_H \\
= \langle PF(u(t)), e_i \rangle_H \quad \text{for} \quad t \in [-\delta_2, \delta_1), \quad 1 \leq i \leq n \quad (4.31)
\]
and
\[
\dot{v}_i(t) = \langle \dot{v}(t), e_i \rangle_H = -c\lambda v_i(t) + \langle PF(u(t)), e_i \rangle_H \quad \text{for} \quad t \in [-\delta_2, \delta_1), \quad 1 \leq i \leq n.
\]
Then \((w_1(0), w_2(0)) = W(u(0), v(0)) \in M^t\) and
\[
\frac{d}{dt} \sum_{i=1}^n \|v_i(t)\|^2 = \sum_{i=1}^n \left(-c\lambda v_i(t)^2 + v_i(t)(PF(u(t)), e_i)_H\right)
\]
\[
= -c\lambda \|v(t)\|^2_H + \langle PF(u(t)), v(t)\rangle_H \quad \text{for } t \in [-\delta_2, \delta_1).
\]
Further, by (4.39), we have
\[
\langle PF(u(t)), v(t)\rangle_H \leq m_1\|v(t)\|^2_H = m_1|w_2(t)| \quad \text{for } t \in [-\delta_2, \delta_1),
\]
and hence, by (4.32) and (4.20), we deduce that
\[
\frac{d}{dt} \sum_{i=1}^n \|v_i(t)\|^2 + m_1\|w_2(t)\|^2 = -c\lambda R_2^2 + m_1 R_2 < 0.
\]
This implies that there are \(\varepsilon_1 \in (0, \delta_1)\) and \(\varepsilon_2 \in (0, \delta_2)\) such that \(|w_2(t)| > R_2\) for 
\(t \in [-\varepsilon_2, 0)\) (when \(\delta_2 > 0)\) and \(|w_2(t)| < R_2\) for 
\(t \in (0, \varepsilon_1]\). Taking \(\varepsilon_1 > 0\) smaller if necessary we have also \(w_1(t) \in B(0, R_4)\) for 
\(t \in (0, \varepsilon_1]\). Hence \((u(t), v(t)) \in int B\) for 
\(t \in (0, \varepsilon_1] and (u(t), v(t)) \notin B \) for \(t \in [-\varepsilon_2, 0)\), which proves that \(W^{-1}M^t\) is
contained in \(B^t\).

Suppose that \((u, v) : [-\delta_2, \delta_1) \to \mathbb{E}_0\), where \(\delta_1 > 0\) and \(\delta_2 \geq 0\), is a solution for the
semiflow \(\varphi_2\) such that \((u(0), v(0)) \in W^{-1}M^e\). Similarly as before, we define
a map \((w_1, w_2) : [-\delta_2, \delta_1) \to \mathbb{R}^{2n}\) by the formula \((w_1(t), w_2(t)) = W(u(t), v(t))\) for 
\(t \in [-\delta_2, \delta_1)\). Then \((w_1(0), w_2(0)) = W(u(0), v(0)) \in M^e\) which together with
(4.29) and (4.31) gives
\[
\frac{d}{dt} \sum_{i=1}^n \|v_i(t)\|^2 = \sum_{i=1}^n (c\lambda u_i(t) + v_i(t))(c\lambda u_i(t) + v_i(t))
\]
\[
= a^2 \sum_{i=1}^n (c\lambda u_i(t) + v_i(t))(PF(u(t)), e_i)
\]
\[
= a^2 \langle PF(u(t)), c\lambda u(t) + v(t)\rangle \quad \text{for } t \in [-\delta_2, \delta_1).
\]
From (4.17) it follows that \(\|v(0)\|^2_H = \|w_2(0)\|^2 \leq R_2\) and
\[
ac\lambda R_4 + aR_2 = R_4 = |w_1(0)| = a|c\lambda u(0) + v(0)|_H
\]
\[
\leq ac\lambda \|u(0)\|^2_H + a\|v(0)\|^2_H \leq ac\lambda \|u(0)\|^2_H + aR_2,
\]
which implies that
\[
\|u(0)\|^2_H \geq R_3 \quad \text{and} \quad \|v(0)\|^2_H \leq R_2.
\]
Therefore, by (4.34), the inequality (4.21) yields
\[
\frac{d}{dt} \sum_{i=1}^n \|v_i(t)\|^2 |_{t=0} = a^2 \langle PF(u(0)), c\lambda u(0) + v(0)\rangle
\]
\[
= a^2 c\lambda \langle PF(u(0)), u(0)\rangle + a^2 \langle PF(u(0)), v(0)\rangle.
\]
From this there are \(\varepsilon_1 \in (0, \delta_1)\) and \(\varepsilon_2 \in (0, \delta_2)\) such that \(|w_1(t)| > R_4\) for \(t \in (0, \varepsilon_1]\)
and \(|w_1(t)| < R_4\) for \(t \in (0, -\varepsilon_2)\) (when \(\delta_2 > 0)\). Taking again \(\varepsilon_2 > 0\) smaller if
necessary, we have also \(w_2(t) \in B(0, R_2)\) for \(t \in [-\varepsilon_2, 0)\). Therefore \((u(t), v(t)) \in int B\) for 
\(t \in [-\varepsilon_2, 0) and (u(t), v(t)) \notin B \) for \(t \in (0, \varepsilon_1]\) which implies that the set
\(W^{-1}M^e\) is contained in \(B^e\).

Let \((u, v) : [-\delta_2, \delta_1) \to \mathbb{E}_0\), where \(\delta_1 > 0\) and \(\delta_2 \geq 0\), be a solution of the
semiflow \(\varphi_2\) such that \((u(0), v(0)) \in W^{-1}M^h\). Define the map \((w_1, w_2) : [-\delta_2, \delta_1) \to \mathbb{R}^{2n}\) by
\((w_1(t), w_2(t)) = W(u(t), v(t)) \) for \(t \in [-\delta_2, \delta_1)\). Then \((w_1(0), w_2(0)) =
Combining (4.40) we find that implies that and the point (ii) contains invariance of the Conley index we have

\[ h(\Phi, K) = h(\Psi^1, K_1) = h(\Psi^0, K_0). \]

(4.36)

Let \( K'_1 := \text{Inv}(\varphi_1, N_1) \) and \( K'_2 := \text{Inv}(\varphi_2, N_2) \). In view of (2.8), (2.9) and Theorem 11.1 from we follow that \( K'_1 = \{0\}, K'_1 \in S(E_{-} \oplus E_{+}) \) and furthermore

\[ h(\varphi_1, K'_1) = \Sigma^{d_{k-1}}, \]

(4.37)

where the last equality is a consequence of Theorem 2.4 (i). Further, by Step 3, we infer that \( K'_2 \in S(E_0) \) is an isolated invariant set. Hence the multiplication property of the homotopy index yields \( K'_1 \times K'_2 \in S(\varphi_1 \times \varphi_2, (E_{-} \oplus E_{+}) \times E_0) \) and

\[ h(\varphi_1 \times \varphi_2, K'_1 \times K'_2) = h(\varphi_1, K'_1) \wedge h(\varphi_2, K'_2). \]

(4.38)

Write \( K' := \text{Inv}(\varphi_1 \times \varphi_2, N_1 \times N_2) \). Then \( K' = K'_1 \times K'_2 \) and (4.8) implies that \( U(K') = \text{Inv}(\Psi^0, N) = K_0 \). Therefore, by the topological invariance of Conley index we find that

\[ h(\Psi^0, K_0) = h(\varphi_1 \times \varphi_2, K'_1 \times K'_2). \]

(4.39)

Combining (4.36), (4.39), (4.38) and (4.37) yields

\[ h(\Phi, K) = h(\varphi_1, K'_1) \wedge h(\varphi_2, K'_2) = \Sigma^{d_{k-1}} \wedge h(\varphi_2, K'_2). \]

(4.40)

If (G1) hold, then the pair \((B, B^-)\) is homeomorphic with \((M, M^-)\), where

\[ M := \{ (w_1, w_2) \in \mathbb{R}^{2n} \mid |w_1| \leq R_4, |w_2| \leq R_2 \}, \]

\[ M^- := \{ (w_1, w_2) \in \mathbb{R}^{2n} \mid |w_1| = R_4, |w_2| \leq R_2 \} \]

and therefore

\[ h(\varphi_2, K'_2) = \Sigma^n, \]

(4.41)

which together with (4.40) gives

\[ h(\Phi, K) = \Sigma^{d_{k-1}} \wedge \Sigma^n = \Sigma^{d_k}, \]

(4.42)

and the proof of (i) is completed. If condition (G2) is satisfied, then \( B := N_2 \) is an isolating block for the semiflow \( \varphi_2 \) with that boundary \( \partial B \) consisting of strict ingress points. In this case the pair \((B, B^-)\) is homeomorphic with \((M, \emptyset)\), which implies that

\[ h(\varphi_2, K'_2) = \Sigma^0. \]

(4.43)

Combining this with (4.40) we find that

\[ h(\Phi, K) = \Sigma^{d_{k-1}} \wedge \Sigma^0 = \Sigma^{d_{k-1}}, \]

(4.44)

and the point (ii) follows. \( \square \)
5. Applications

We assume that $\Omega \subset \mathbb{R}^n$ where $n \geq 1$, is an open bounded set with the boundary $\partial \Omega$ of class $C^\infty$. Let $A$ be a second order differential operator with a Dirichlet boundary conditions:

$$A\hat{u}(x) = -\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i\hat{u}(x)) \quad \text{for} \quad \hat{u} \in C^2(\overline{\Omega}),$$

such that $a_{ij} = a_{ji} \in C^2(\overline{\Omega})$ for $1 \leq i, j \leq n$ and there is $c_0 > 0$ such that

$$\sum_{1 \leq i,j \leq n} a_{ij}(x)|\xi|^2 \geq c_0 |\xi|^2 \quad \text{for} \quad x \in \Omega, \; \xi \in \mathbb{R}^n.$$

Furthermore, assume that $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a continuous map satisfying:

(E1) there is $L > 0$ such that for $x \in \Omega$, $s_1, s_2 \in \mathbb{R}$ and $y_1, y_2 \in \mathbb{R}^n$ we have

$$|f(x, s_1, y_1) - f(x, s_2, y_2)| \leq L(|s_1 - s_2| + |y_1 - y_2|).$$

(E2) there is $m > 0$ such that

$$|f(x, s, y)| \leq m \quad \text{for} \quad x \in \Omega, \; s \in \mathbb{R}, \; y \in \mathbb{R}^n.$$

Write $X := L^p(\Omega)$ where $p \geq 1$. With the operator $A$ we can associate the operator $A_p : X \supset D(A_p) \to X$, where

$$D(A_p) := W^{2,p}_0(\Omega) := \text{cl}_{W^{2,p}(\Omega)} \{ \phi \in C^2(\overline{\Omega}) \mid \phi|_{\partial \Omega} = 0 \},$$

$$A_p \hat{u} := A\hat{u} \quad \text{for} \quad \hat{u} \in D(A_p).$$

It is known (see e.g. [18, 41]) that $A_p$ is positively defined sectorial operator on $X$. Let $X^\alpha := D(A^\alpha_p)$ for ($\alpha \in (0, 1)$) be a fractional space with the norm

$$\|\hat{u}\|_\alpha := \|A^\alpha_p \hat{u}\| \quad \text{for} \quad \hat{u} \in X^\alpha.$$

From now on we assume that

(E3) $p \geq 2n$ and $\alpha \in (3/4, 1)$.

**Remark 5.1.** (a) Observe that $A_p$ satisfies assumptions (A1), (A2) and (A3). Since $A_p$ has compact resolvent (see e.g. [13, 41]), the assumption (A1) holds. Take $H := L^2(\Omega)$ equipped with the standard inner product and norm

$$\langle \hat{u}, \hat{v} \rangle_{L^2} := \int_\Omega \hat{u}(x)\overline{\hat{v}(x)} \; dx, \quad \|\hat{u}\|_{L^2} = \left( \int_\Omega |\hat{u}(x)|^2 \; dx \right)^{1/2} \quad \text{for} \quad \hat{u}, \hat{v} \in H$$

and put $\hat{A} := A_2$. Then we see that the boundedness of $\Omega$ and the fact that $p \geq 2$ imply that there is a continuous embedding $i : L^p(\Omega) \hookrightarrow L^2(\Omega)$ and the assumption (A2) is satisfied. Furthermore we have $D(A_p) \subset D(\hat{A})$ and $\hat{A}\hat{u} = A_p\hat{u}$ and $\hat{u} \in D(A_p)$. This shows that $A_p \subset \hat{A}$ in the sense of the inclusion $i \times i$. Since the operator $\hat{A}$ is self-adjoint (see e.g. [18]) we see that the assumption (A3) is also satisfied.

(b) Remark [2.1] shows that the spectrum $\sigma(A_p)$ of the operator $A_p$ consists of sequence of positive eigenvalues

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_i < \lambda_{i+1} < \ldots \quad \text{for} \quad i \geq 1,$$

and furthermore $(\lambda_i)$ is finite or $\lambda_i \to +\infty$ when $i \to +\infty$.

(c) Note that the following inclusion is continuous

$$X^\alpha \subset C^1(\overline{\Omega}).$$

(5.2)
Indeed, according to assumption (E3) we have \( \alpha \in (3/4, 1) \) and \( p \geq 2n \), and hence \( 2n - \frac{\alpha}{p} > 1 \). Therefore, the assertion is a consequence of [25, Theorem 1.6.1].

(d) If \( 1 \geq \alpha > \beta \geq 0 \) then the inclusion \( X^\alpha \subset X^\beta \) is continuous and compact as [25, Theorem 1.4.8] says.

By Remark 5.1 (c) we can define a map \( F : X^\alpha \to X \) given, for any \( \bar{u} \in X^\alpha \), as
\[
F(\bar{u})(x) := f(x, \bar{u}(x), \nabla \bar{u}(x)) \quad \text{for } x \in \Omega. \tag{5.3}
\]
We call \( F \) the Niemytzki operator associated with \( f \). Inclusion (5.2) along with simple calculations can be used to obtain the following lemma.

**Lemma 5.2.** The map \( F \) is well defined, continuous and satisfies assumption (F1), (F2) and (F4).

5.1. **Resonant properties of Niemytzki operator.** In this section, our aim is to examine what assumptions should satisfy the mapping \( f \) so that the associated Niemytzki operator \( F \) meets the introduced earlier geometrical conditions. We start with the following theorem which says that well known Landesman-Lazer conditions introduced in [30] are actually particular case of conditions (G1) and (G2).

**Theorem 5.3.** Let \( f_+ : \Omega \to \mathbb{R} \) be continuous functions such that
\[
f_+(x) = \lim_{s \to +\infty} f(x, s, y) \quad \text{and} \quad f_-(x) = \lim_{s \to -\infty} f(x, s, y)
\]
for \( x \in \Omega \), uniformly for \( y \in \mathbb{R}^n \). Let \( B_1 \subset X^\alpha \oplus X^\alpha \) and \( B_2 \subset X_0 \) be bounded subsets in norms \( \lVert \cdot \rVert \) and \( \lVert \cdot \rVert_{L^2} \), respectively.

(i) Assume that
\[
(\text{LL1}) \quad \int_{\{u \geq 0\}} f_+(x) \bar{u}(x) \, dx + \int_{\{u < 0\}} f_-(x) \bar{u}(x) \, dx > 0 \tag{5.5}
\]
for \( \bar{u} \in \text{Ker} (\mathcal{I} - A_p) \setminus \{0\} \). Then there is \( R > 0 \) such that for any \( (\bar{w}, \bar{v}, \bar{u}) \in B_1 \times B_2 \times X_0 \), with \( \lVert \bar{u} \rVert_{L^2} \geq R \), we have the following inequality:
\[
\langle F(\bar{w} + \bar{u}), \bar{u} \rangle_{L^2} > -\langle F(\bar{w} + \bar{u}), \bar{v} \rangle_{L^2}.
\]

(ii) Assume that
\[
(\text{LL2}) \quad \int_{\{u \geq 0\}} f_+(x) \bar{u}(x) \, dx + \int_{\{u < 0\}} f_-(x) \bar{u}(x) \, dx < 0 \tag{5.5}
\]
for \( \bar{u} \in \text{Ker} (\mathcal{I} - A_p) \setminus \{0\} \). Then there is \( R > 0 \) such that for any \( (\bar{w}, \bar{v}, \bar{u}) \in B_1 \times B_2 \times X_0 \), with \( \lVert \bar{u} \rVert_{L^2} \geq R \), we have the following inequality:
\[
\langle F(\bar{w} + \bar{u}), \bar{u} \rangle_{L^2} < -\langle F(\bar{w} + \bar{u}), \bar{v} \rangle_{L^2}.
\]

**Proof.** Since the proofs of points (i) and (ii) are analogous, we focus only on the first one. Suppose, contrary to the point (i), that there are sequences \( (\bar{w}_n) \) in \( B_1 \), \( (\bar{v}_n) \) in \( B_2 \) and \( (\bar{u}_n) \) in \( X_0 \) such that \( \lVert \bar{u}_n \rVert_{L^2} \to \infty \) when \( n \to \infty \) and
\[
\langle F(\bar{w}_n + \bar{u}_n), \bar{u}_n \rangle_{L^2} \leq -\langle F(\bar{w}_n + \bar{u}_n), \bar{v}_n \rangle_{L^2} \quad \text{for } n \geq 1. \tag{5.4}
\]
For \( n \geq 1 \), we define \( \bar{z}_n := \bar{u}_n/\lVert \bar{u}_n \rVert_{L^2} \). Since \( X_0 \) is finite dimensional space, with out loss of generality we can assume that there is \( \bar{z}_0 \in X_0 \) such that \( \bar{z}_n \to \bar{z}_0 \) in \( L^2(\Omega) \) and furthermore \( \bar{z}_n(x) \to \bar{z}_0(x) \) for a.a. \( x \in \Omega \) as \( n \to \infty \). In view of the fact that \( A_p \) has compact resolvents, Remark 5.1 (d) says that \( X^\alpha \) is compactly embedded in \( X \). Therefore, the boundedness of \( (\bar{w}_n) \) in \( X^\alpha \), implies that this sequence is relatively compact in \( X \). Hence, passing if necessary to a subsequence, we can also assume that \( \bar{w}_n \to \bar{w}_0 \) in \( X \) where \( \bar{w}_0 \in X = L^p(\Omega) \) and furthermore \( \bar{w}_n(x) \to \bar{w}_0(x) \) for a.a. \( x \in \Omega \) as \( n \to \infty \). From (5.3), we have
\[
\langle F(\bar{w}_n + \bar{u}_n), \bar{z}_n - \bar{z}_0 \rangle_{L^2} + \langle F(\bar{w}_n + \bar{u}_n), \bar{z}_0 \rangle_{L^2} \leq -\langle F(\bar{w}_n + \bar{u}_n), \bar{v}_n \rangle_{L^2}/\lVert \bar{u}_n \rVert_{L^2} \tag{5.5}
\]
for $n \geq 1$. Furthermore, by Lemma 5.2 (ii), the map $F$ is bounded and hence the convergence $\bar{z}_n \to \bar{z}_0$ in $L^2(\Omega)$, implies that
\[
\langle F(\bar{w}_n + \bar{u}_n), \bar{z}_n - \bar{z}_0 \rangle_{L^2} \leq \|F(\bar{w}_n + \bar{u}_n)\|_{L^2} \|\bar{z}_n - \bar{z}_0\|_{L^2} \to 0 \quad (5.6)
\]
as $n \to +\infty$. Since $F$ is a bounded map and the sequence $(\bar{v}_n)$ is bounded in $L^2(\Omega)$,
\[
\langle F(\bar{w}_n + \bar{u}_n), \bar{v}_n \rangle_{L^2} / \|\bar{u}_n\|_{L^2} \to 0 \quad \text{as} \quad n \to +\infty. \quad (5.7)
\]
If we define $\Omega_+ := \{ x \in \Omega \mid \bar{z}_0(x) > 0 \}$ and $\Omega_- := \{ x \in \Omega \mid \bar{z}_0(x) < 0 \}$, then
\[
\langle F(\bar{w}_n + \bar{u}_n), \bar{z}_0 \rangle_{L^2} = \int_{\Omega} f(x, \bar{w}_n(x) + \bar{u}_n(x), \nabla \bar{w}_n(x) + \nabla \bar{u}_n(x)) \bar{z}_0(x) \, dx
\]
\[
= \int_{\Omega_+} f(x, \bar{v}_n(x) + \bar{u}_n(x), \nabla \bar{v}_n(x) + \nabla \bar{u}_n(x)) \bar{z}_0(x) \, dx
\]
\[
+ \int_{\Omega_-} f(x, \bar{v}_n(x) + \bar{u}_n(x), \nabla \bar{v}_n(x) + \nabla \bar{u}_n(x)) \bar{z}_0(x) \, dx \quad (5.8)
\]
for $n \geq 1$. Observe that the equation
\[
\bar{w}_n(x) + \bar{u}_n(x) = \bar{u}_n(x) + \|\bar{u}_n\|_{L^2} \bar{z}_n(x) \quad \text{for a.a.} \quad x \in \Omega_+ \text{ and } n \geq 1
\]
leads to the convergence
\[
\bar{w}_n(x) + \bar{u}_n(x) \to +\infty \quad \text{for a.a.} \quad x \in \Omega_+ \text{ as } n \to \infty,
\]
which together with assumption (E2) and dominated convergence theorem gives
\[
\int_{\Omega_+} f(x, \bar{v}_n(x) + \bar{u}_n(x), \nabla \bar{v}_n(x) + \nabla \bar{u}_n(x)) \bar{z}_0(x) \, dx \to \int_{\Omega_+} f_+(x) \bar{z}_0(x) \, dx \quad (5.9)
\]
when $n \to +\infty$. Proceeding in the similar way, we infer that
\[
\int_{\Omega_-} f(x, \bar{v}_n(x) + \bar{u}_n(x), \nabla \bar{v}_n(x) + \nabla \bar{u}_n(x)) \bar{z}_0(x) \, dx \to \int_{\Omega_-} f_-(x) \bar{z}_0(x) \, dx \quad (5.10)
\]
when $n \to +\infty$. Hence, combining (5.9), (5.10) and (6.3) yields
\[
\langle F(\bar{w}_n + \bar{u}_n), \bar{z}_0 \rangle_{L^2} \to \int_{\Omega_+} f_+(x) \bar{z}_0(x) \, dx + \int_{\Omega_-} f_-(x) \bar{z}_0(x) \, dx \quad \text{as} \quad n \to \infty. \quad (5.11)
\]
Therefore, letting $n \to \infty$ in (5.11) and using (5.6), (5.7) we infer that
\[
\int_{\Omega_+} f_+(x) \bar{z}_0(x) \, dx + \int_{\Omega_-} f_-(x) \bar{z}_0(x) \, dx \leq 0, \quad (5.12)
\]
which contradicts condition (LL1), because $\|\bar{z}_0\|_{L^2} = 1$. Thus the proof of point (i) is completed. \hfill \Box

**Example 5.4.** Suppose that $A_p u = -u_{xx}$ is an operator defined on $W^{2,p}(0,1) \cap W_0^{1,2}(0,1)$ and $f : \mathbb{R} \to \mathbb{R}$ is a map given by $f(s) := \arctan(s)$ for $s \in \mathbb{R}$. Then the spectrum of $A_p$ consists of eigenvalues $(\lambda_i)_{i \geq 1}$ where $\lambda_i = (i\pi)^2$. Furthermore Ker $(\lambda I - A_p) = \{ r \sin(\pi r) \mid r \in \mathbb{R} \}$ and $f_+(x) = \pm \pi/2$ for $x \in (0,1)$. Hence it is not difficult to verify that condition (LL1) is satisfied. Furthermore, if $f(s) := -\arctan(s)$ for $s \in \mathbb{R}$, then condition (LL2) holds. \hfill \Box

The following lemma proves that conditions (G1) and (G2) are also implicated by the strong resonance conditions, studied for example in [16, 42, 39].

**Theorem 5.5.** Assume that there is a continuous function $f_\infty : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^n \ (n \geq 3)$, such that
\[
f_\infty(x) = \lim_{|s| \to +\infty} f(x, s, y) \cdot s
\]
for \( x \in \Omega \), uniformly for \( y \in \mathbb{R}^n \). Let \( B_1 \subset X^\circ_1 \oplus X^\circ_2 \) and \( B_2 \subset X_0 \) be bounded sets in norms \( \| \cdot \|_{\alpha} \) and \( \| \cdot \|_{L^2} \), respectively.

(i) If the following condition is satisfied

\[
\text{(SR1)} \quad \begin{cases}
\text{there is } h \in L^1(\Omega) \text{ such that } \\
\int_{\Omega} f(\infty)(x) \, dx > 0,
\end{cases}
\]

then there is \( R > 0 \) such that for \((\bar{w}, \bar{v}, \bar{u}) \in B_1 \times B_2 \times X_0\), with \( \| \bar{u} \|_{L^2} \geq R \), we have the inequality \( \langle F(\bar{w} + \bar{u}), \bar{u} \rangle_{L^2} > -\langle F(\bar{w} + \bar{u}), \bar{v} \rangle_{L^2} \).

(ii) If the following condition is satisfied

\[
\text{(SR2)} \quad \begin{cases}
\text{there is a function } h \in L^1(\Omega) \text{ such that } \\
\int_{\Omega} f(\infty)(x) \, dx < 0,
\end{cases}
\]

then there is \( R > 0 \) such that for \((\bar{w}, \bar{v}, \bar{u}) \in B_1 \times B_2 \times X_0\), with \( \| \bar{u} \|_{L^2} \geq R \), we have the inequality \( \langle F(\bar{w} + \bar{u}), \bar{u} \rangle_{L^2} < -\langle F(\bar{w} + \bar{u}), \bar{v} \rangle_{L^2} \).

Remark 5.6. Observe that under the assumptions of the previous theorem we have

\[
f_\pm(x) := \lim_{s \to \pm \infty} f(x, s, y) = 0
\]

for \( x \in \Omega \), uniformly for \( y \in \mathbb{R}^n \). It follows that the Landesman-Lazer conditions (LL1) and (LL2) used in Theorem 5.3 are not valid. \( \square \)

Proof of Theorem 5.5. It suffices to prove the first point, as the proof of the second one goes analogously. We argue by contradiction and assume that there are sequences \((\bar{w}_n)\) in \( B_1 \), \((\bar{v}_n)\) in \( B_2 \) and \((\bar{u}_n)\) in \( X_0 \) such that \( \| \bar{u}_n \|_{L^2} \to +\infty \) and

\[
\langle F(\bar{w}_n + \bar{u}_n), \bar{u}_n \rangle_{L^2} \leq -\langle F(\bar{w}_n + \bar{u}_n), \bar{v}_n \rangle_{L^2} \quad \text{for } n \geq 1.
\]

(5.12)

Since \( B_1 \subset X^\circ_1 \) is a bounded set and the inclusion \( X^\circ_2 \subset X \) is compact, passing if necessary to subsequence, we can assume that there is \( \bar{w}_0 \in X \) such that \( \bar{w}_n \to \bar{w}_0 \) in \( X \) and \( \bar{w}_n(x) \to \bar{w}_0(x) \) for a.a. \( x \in \Omega \) as \( n \to +\infty \). For any \( n \geq 1 \), define \( \bar{z}_n := \bar{u}_n/\| \bar{u}_n \|_{L^2} \). Since \( X_0 \) is a finite dimensional space we can also assume that there is \( \bar{z}_0 \in X_0 \) such that \( \bar{z}_n \to \bar{z}_0 \) and \( \bar{z}_n(x) \to \bar{z}_0(x) \) for a.a. \( x \in \Omega \) as \( n \to +\infty \).

Put \( \tilde{c}_n := \bar{w}_n + \bar{u}_n \) for \( n \geq 1 \) and take \( x \in \Omega_+ := \{ x \in \Omega \mid \bar{z}_0(x) > 0 \} \). Then

\[
\tilde{c}_n(x) = \bar{w}_n(x) + \bar{u}_n(x) = \bar{w}_n(x) + \| \bar{u}_n \|_{L^2} \bar{z}_n(x) \to +\infty,
\]

when \( n \to +\infty \). If we take \( x \in \Omega_- := \{ x \in \Omega \mid \bar{z}_0(x) < 0 \} \) we infer that

\[
\tilde{c}_n(x) = \bar{w}_n(x) + \bar{u}_n(x) = \bar{w}_n(x) + \| \bar{u}_n \|_{L^2} \bar{z}_n(x) \to -\infty
\]

when \( n \to +\infty \). Using (5.12) we derive that

\[
\langle F(\bar{w}_n + \bar{u}_n), \bar{w}_n + \bar{u}_n \rangle_{L^2} \leq \langle F(\bar{w}_n + \bar{u}_n), \bar{w}_n - \bar{v}_n \rangle_{L^2}
\]

(5.15)

for any \( n \geq 1 \). Note that for the both conditions (SR1) and (SR2) we have

\[
\int_{\Omega_+} f(x, \tilde{c}_n(x), \nabla \tilde{c}_n(x)) \tilde{c}_n(x) \, dx \geq -\| h \|_{L^1} \quad \text{and}
\]

\[
\int_{\Omega_-} f(x, \tilde{c}_n(x), \nabla \tilde{c}_n(x)) \tilde{c}_n(x) \, dx \geq -\| h \|_{L^1} \quad \text{for } n \geq 1.
\]

(5.16)
Since $z_0 \neq 0$, from \cite[Theorem 1.1]{23} and \cite[Proposition 3]{23} it follows that the Lebesgue measure of the set $\Omega_0 := \{x \in \Omega \mid z_0(x) = 0\}$ is equal to zero. Therefore, applying the inequalities (5.10), we infer that
\[
\liminf_{n \to +\infty} (\bar{\Omega} := \sup \{\|\bar{w}_n - \bar{v}_n\|_L^2 \mid n \geq 1\}).
\]

From the boundedness of $B_1$ and $B_2$, it follows that there is a constant $r < +\infty$ such that $r := \sup \{|\bar{w}_n - \bar{v}_n| \mid n \geq 1\}$. Then, for any $n \geq 1$,
\[
(F(\bar{w}_n + \bar{u}_n), \bar{w}_n - \bar{v}_n)_{L^2} \leq \|F(\bar{w}_n + \bar{u}_n)\|_{L^2} \|\bar{w}_n - \bar{v}_n\|_{L^2}.
\]

Note that, from the assumptions of lemma, we have
\[
\lim_{|s| \to +\infty} f(x, s, y) = 0
\]
for $x \in \Omega$, uniformly for $y \in \mathbb{R}^d$. Furthermore, combining (5.13), (5.14) and (5.19), yields
\[
f(x, \bar{c}_n(x), \nabla \bar{c}_n(x)) \to 0 \quad \text{for a.a.} \quad x \in \Omega_+ \cup \Omega_-
\]
Since $\Omega_0$ is of Lebesgue measure zero, the boundedness of $f$ (assumption (E2)) and dominated convergence theorem imply that
\[
\|F(\bar{w}_n + \bar{u}_n)\|_{L^2}^2 = \int_{\Omega_+} |f(x, \bar{c}_n(x), \nabla \bar{c}_n(x))|^2 \, dx + \int_{\Omega_-} |f(x, \bar{c}_n(x), \nabla \bar{c}_n(x))|^2 \, dx \to 0,
\]
when $n \to +\infty$. Hence the inequality (5.18) implies
\[
(F(\bar{w}_n + \bar{c}_n), \bar{w}_n - \bar{v}_n)_{L^2} \to 0 \quad \text{as} \quad n \to +\infty,
\]
which along with (5.15) and (5.17), leads to
\[
0 \geq \liminf_{n \to +\infty} (F(\bar{w}_n + \bar{u}_n), \bar{w}_n - \bar{v}_n)_{L^2} \geq \int_{\Omega} f(x) \, dx.
\]
This contradicts condition (SR1) and hence the proof of point (i) is completed. \(\Box\)
Example 5.7. If \( f : \mathbb{R} \to \mathbb{R} \) is a map given by \( f(s) := s/(1 + s^2) \) for \( s \in \mathbb{R} \) then \( f(s) \cdot s \to 1 \) as \( |s| \to +\infty \). Hence condition (SR1) is satisfied. Furthermore, if \( f \) is given by \( f(s) := -s/(1 + s^2) \) for \( s \in \mathbb{R} \), then \( f(s) \cdot s \to -1 \) as \( |s| \to +\infty \) and consequently condition (SR2) holds.

5.2. Criteria on existence of bounded orbits. We consider the second order differential equation of the form

\[
\ddot{u}(t) = -A_p u(t) - A_p \dot{u}(t) + \lambda u(t) + F(u(t)), \quad t > 0, \quad x \in \Omega
\]

where \( c > 0, \lambda \) is a real number and \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a continuous map satisfying assumptions (E1), (E2) and (E3). Then the second order equation (5.21) may be written in the abstract form

\[
\ddot{u}(t) = -A_p u(t) - A_p \dot{u}(t) + \lambda u(t) + F(u(t)), \quad t > 0.
\]

Let \( A_p : \mathcal{E} \supset D(A_p) \to \mathcal{E} \) be a linear operator on a Banach space \( \mathcal{E} := X^\alpha \times X \) given by

\[
D(A_p) := \{(\bar{u}, \bar{v}) \in X^\alpha \times X \mid \bar{u} + c\bar{v} \in D(A_p)\}
\]

and let \( F : \mathcal{E} \to \mathcal{E} \) be a map, given by \( F(\bar{u}, \bar{v}) := (0, F(\bar{u})) \) for \((\bar{u}, \bar{v}) \in \mathcal{E}\). The equation (5.22) can be written as the first order equation

\[
\dot{w}(t) = -A_p w(t) + F(w(t)), \quad t > 0.
\]

Let \( \Phi : [0, +\infty) \times \mathcal{E} \to \mathcal{E} \) be a map associated with (5.23). We start with the following criterion with Landesman-Lazer conditions.

Theorem 5.8. Let \( f_+, f_- : \Omega \to \mathbb{R} \) be continuous functions such that

\[
f_+(x) = \lim_{s \to +\infty} f(x, s) \quad \text{and} \quad f_-(x) = \lim_{s \to -\infty} f(x, s)
\]

for \( x \in \Omega \). If \( \lambda = \lambda_k \) for some \( k \geq 1 \), then there is a compact full solution \( w : \mathbb{R} \to \mathcal{E} \) of (5.21) provided either (LL1) or (LL2) holds.

Proof. By Remark 5.1 (a) and Lemma 5.2 we deduce that assumptions (A1), (A2), (A3), (F1), (F3) and (F4) are satisfied. From Theorem 5.3 it follows that condition (LL1) implies (G1) and that condition (LL2) implies (G2). Hence application Theorem 5.2 and the existence property of Conley index completes the proof.

Now we proceed to the following criterion with strong resonance conditions.

Theorem 5.9. Let \( \Omega \subset \mathbb{R}^n \) where \( n \geq 3 \), be an open bounded set and assume that there is a continuous function \( f_\infty : \Omega \to \mathbb{R} \) such that

\[
f_\infty(x) = \lim_{|s| \to +\infty} f(x, s) \cdot s
\]

for \( x \in \Omega \). If \( \lambda = \lambda_k \) for some \( k \geq 1 \), then there is a compact full solution \( w : \mathbb{R} \to \mathcal{E} \) of (5.21) provided either (SR1) or (SR2) holds.

Proof. By Remark 5.1 (a) and Lemma 5.2 we deduce that assumptions (A1), (A2), (A3), (F1), (F3) and (F4) are satisfied. From Theorem 5.5 it follows that condition (SR1) implies (G1) and that condition (SR2) implies (G2). Hence the proof is completed by application of Theorem 5.2 and the existence property of Conley index.
6. Appendix

We say that continuous map $\Phi : [0, +\infty) \times E \rightarrow E$ is a semiflow on $E$ provided

(a) $\Phi(0, (x, y)) = (x, y)$ for $(x, y) \in E$,
(b) $\Phi(t + t', (x, y)) = \Phi(t, \Phi(t', (x, y)))$ for $t, t' \geq 0$, $(x, y) \in E$.

A map $\sigma : [-\delta_1, \delta_2) \rightarrow E$, where $\delta_2 > 0$ and $\delta_1 \geq 0$, is said to be a solution of the semiflow $\Phi$, if

$$\Phi(t, \sigma(s)) = \sigma(t + s) \quad \text{for} \quad t \geq 0, \; s \in [-\delta_1, \delta_2), \; t + s \in [-\delta_1, \delta_2).$$

If $\sigma$ is defined on $\mathbb{R}$, then $\sigma$ is called a full solution for $\Phi$. Let $K \subset E$ be a subset. We say that $K$ is invariant with respect to $\Phi$, if for every $(x, y) \in K$ there is full solution $\sigma$ of $\Phi$ such that $\sigma(0) = (x, y)$ and $\sigma(\mathbb{R}) \subset K$. If $N \subset E$ then we define maximal invariant set as

$$\text{Inv}(N, \Phi) := \left\{(x, y) \in N \mid \text{there is a solution } \sigma : \mathbb{R} \rightarrow E \text{ of } \Phi \text{ such that } \sigma(0) = (x, y) \text{ and } \sigma(\mathbb{R}) \subset N \right\}.$$

A closed invariant set $K \subset E$ is called isolated, if there is a closed set $N \subset E$ such that $K = \text{Inv}(N) \subset \text{int} N$. In this case $N$ is called isolating neighborhood for $K$.

Let $\Phi^s : [0, +\infty) \times E \rightarrow E$ for $s \in [0, 1]$, be a family of semiflows. We say that $N \subset E$ is admissible with respect to $\{\Phi^s\}_{s \in [0, 1]}$, if for every sequences $s_n \in [0, 1]$, $z_n$ in $E$ and $(t_n)$ in $[0, +\infty)$ such that $t_n \rightarrow +\infty$ when $n \rightarrow \infty$, the inclusion

$$\Phi_{s_n}(\{[0, t_n] \times \{z_n\}) \subset N \quad \text{for} \quad n \geq 1,$$

implies that the set $\{\Phi^s_{s_n}(t_n, z_n) \mid n \geq 1\}$ is relatively compact in $E$. Furthermore the family $\{\Phi^s\}_{s \in [0, 1]}$ is continuous provided $\Phi^s_{s_n}(t_n, z_n) \rightarrow \Phi^s(t_0, z_0)$ as long as $s_n \rightarrow s_0$, $t_n \rightarrow t_0$ and $z_n \rightarrow z_0$ as $n \rightarrow +\infty$.

A subset $N \subset E$ is admissible with respect to the single semiflow $\Phi$, if it is admissible with respect to family consisting from constant semiflow $\Phi$. From now on we write $S(E) = S(E, \Phi)$ for a class of invariant sets admitting an admissible isolated neighborhood with respect to $\Phi$. A special case of isolated neighborhood is an isolating block. To define it assume that $B \subset E$ is a closed set and let $(x, y) \in \partial B$.

We say that $(x, y)$ is a strict egress point (resp. strict ingress point, resp. bounce off point), if for any solution $\sigma : [-\delta_1, \delta_2) \rightarrow E$, where $\delta_1 \geq 0$ and $\delta_2 > 0$, of the semiflow $\Phi$ such that $\sigma(0) = (x, y)$ the following holds:

(a) there is $\varepsilon_2 \in (0, \delta_2]$ such that $\sigma(t) \notin B$ (resp. $\sigma(t) \in \text{int} B$, resp. $\sigma(t) \notin B$) for $t \in (0, \varepsilon_2]$;
(b) if $\delta_1 > 0$ then there is $\varepsilon_1 \in (0, \delta_1)$ such that $\sigma(t) \in B$ (resp. $\sigma(t) \notin B$, resp. $\sigma(t) \notin B$) for $t \in [-\varepsilon_1, 0)$.

We write $B^e$, $B^i$ and $B^b$ for the sets of strict egress points, strict ingress points and bounce off points, respectively. Furthermore, put $B^- := B^e \cup B^b$.

Definition 6.1. A close set $B \subset E$ is an isolating block, if $\partial B = B^e \cup B^i \cup B^c$ and the set $B^c$ is closed.

For $K \in S(E)$ we define the Conley index (the homotopy index) of $K$ as the homotopy type $h(\Phi, K)$ of pointed space given by

$$h(\Phi, K) := \begin{cases} [B/B^-, [B^-]] & \text{if } B^- \neq \emptyset; \\ [B \cup \{c\}, c] & \text{if } B^- = \emptyset \end{cases}$$

where $B/B^-$ is the quotient space and $B \cup \{c\}$ is a disjoint sum of $B$ and the one point space $\{c\}$. It is known that the homotopy index is independent from the choice of isolating block of $K$ has the following properties:

(H1) (Existence) If $K \in S(E)$ and $h(\Phi, K) \neq \emptyset$, then $K \neq \emptyset$. 


(H2) (Addition) If $K_1, K_2 \in \mathcal{S}(E)$ and $K_1 \cap K_2 = \emptyset$, then $K_1 \cup K_2 \in \mathcal{S}(E)$ and
\[ h(\varphi, K_1 \cup K_2) = h(\Phi, K_1) \vee h(\Phi, K_2). \]

(H3) (Multiplication) Let $\Phi^i : [0, +\infty) \times E \to E$, for $i=1,2$ be semiflows. If $K_1, K_2 \in \mathcal{S}(E)$ then $K_1 \times K_2 \in \mathcal{S}(\Phi^1 \times \Phi^2, E \times E)$ and
\[ h(\Phi^1 \times \Phi^2, K_1 \times K_2) = h(\Phi^1, K_1) \wedge h(\Phi^2, K_2). \]

(H4) (Homotopy invariance) Let $\{ \Phi^s : [0, +\infty) \times E \to E \}_{s \in [0,1]}$ be a continuous family of semiflows and let the set $N \subset X$ be admissible with respect to this family. If for any $s \in [0,1]$ the set $N$ is an isolating neighborhood of $K_s := \text{Inv}(\Phi^s, N)$, then $K_s \in \mathcal{S}(\Phi^s, E)$ for $s \in [0,1]$ and
\[ h(\Phi^0, \text{Inv}(\Phi^0, N)) = h(\Phi^1, \text{Inv}(\Phi^1, N)). \]

References

[1] Dell'Oro, Filippo Global attractors for strongly damped wave equations with subcritical-critical nonlinearities, Commun. Pure Appl. Anal. 12 (2013), no. 2, 10151027.
[2] Araruna, Fgner Dias; Bezerra, Flank David Morais Rate of attraction for a semilinear wave equation with variable coefficients and critical nonlinearities, Pacific J. Math. 266 (2013), no. 2, 257282.
[3] Sun, Chunyou; Yang, Lu; Duan, Jinqiao Asymptotic behavior for a semilinear second order evolution equation, Trans. Amer. Math. Soc. 363 (2011), no. 11, 60856109.
[4] Yang, Meihua; Sun, Chunyou; Yan, Lu Rate of attraction for a semilinear wave equation with critical nonlinearities, Pacific J. Math. 266 (2013), no. 2, 257282.
[5] Belleri, Veronica; Pata, Vittorino Attractors for a semilinear strongly damped wave equation, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 36 (2003), no. 2, 257282.
[20] Aleksander Cwiszewski, Topological degree methods for perturbations of operators generating compact $C_0$ semigroups, J. Differential Equations 220 (2006), no. 2, 434–477.

[21] Aleksander Cwiszewski oraz Krzysztof P. Rybakowski, Singular dynamics of strongly damped beam equation, J. Differential Equations 247 (2009), no. 12, 3202–3233.

[22] Klaus Deimling, Multivalued differential equations, de Gruyter Series in Nonlinear Analysis and Applications, vol. 1, Walter de Gruyter & Co., Berlin, 1992.

[23] Djairo G. de Figueiredo oraz Jean-Pierre Gossez, Strict monotonicity of eigenvalues and unique continuation, Comm. Partial Differential Equations 17 (1992), no. 1-2, 339–346.

[24] Nicola Garofalo oraz Fang-Hua Lin, Unique continuation for elliptic operators: a geometric-variational approach, Comm. Pure Appl. Math. 40 (1987), no. 3, 347–366.

[25] Daniel Henry, Geometric theory of semilinear parabolic equations, Lecture Notes in Mathematics, vol. 840, Springer-Verlag, Berlin, 1981.

[26] Mikhail Kamenskii, Valeri Obukhovskii, oraz Pietro Zecca, Condensing multivalued maps and semilinear differential inclusions in Banach spaces, De Gruyter Series in Nonlinear Analysis and Applications, vol. 7, Walter de Gruyter & Co., Berlin, 2001.

[27] Wioletta Karpinska, A note on bounded solutions of second order differential equations at resonance, Topol. Methods Nonlinear Anal. 14 (1999), no. 2, 371–384.

[28] Piotr Kokocki, Periodic solutions for nonlinear evolution equations at resonance, Journal of Mathematical Analysis and Applications, vol. 392, no. 1 (2012), 55–74.

[29] Piotr Kokocki, Averaging principle and periodic solutions for nonlinear evolution equations at resonance, Nonlinear Analysis: Theory, Methods and Applications vol. 85, (2013), 253-278.

[30] E. M. Landesman oraz A. C. Lazer, Nonlinear perturbations of linear elliptic boundary value problems at resonance, J. Math. Mech. 19 (1969/1970), 609–623.

[31] Paul Massatt, Limiting behavior for strongly damped nonlinear wave equations, J. Differential Equations 48 (1983), no. 3, 334–349.

[32] Jean Mawhin oraz James R. Ward, Jr., Bounded solutions of some second order nonlinear differential equations, J. London Math. Soc. (2) 58 (1998), no. 3, 733–747.

[33] Sigeru Mizohata, The theory of partial differential equations, Cambridge University Press, New York, 1973, Translated from the Japanese by Katsumi Miyahara.

[34] Rafael Ortega oraz Antonio Tineo, Resonance and non-resonance in a problem of boundedness, Proc. Amer. Math. Soc. 124 (1996), no. 7, 2089–2096.

[35] Krzysztof P. Rybakowski, On the homotopy index for infinite-dimensional semiflows, Trans. Amer. Math. Soc. 269 (1982), no. 2, 351–382.

[36] Krzysztof P. Rybakowski, The homotopy index and partial differential equations, Universitext, Springer-Verlag, Berlin, 1987.

[37] Krzysztof P. Rybakowski, Trajectories joining critical points of nonlinear parabolic and hyperbolic partial differential equations, J. Differential Equations 51 (1984), no. 2, 182–212.

[38] Dietmar Salamon, Connected simple systems and the Conley index of isolated invariant sets, Trans. Amer. Math. Soc. 291 (1985), no. 1, 1–41.

[39] Martin Schechter, Nonlinear elliptic boundary value problems at resonance, Nonlinear Anal. 14 (1990), no. 10, 889–903.

[40] Joel Smoller, Shock waves and reaction-diffusion equations, Grundlehren der Mathematischen Wissenschaften, vol. 258, Springer-Verlag, New York, 1983.

[41] H. Triebel, Interpolation theory, function spaces, differential operators, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[42] José Valdo A. Gonçalves, On bounded nonlinear perturbations of an elliptic equation at resonance, Nonlinear Anal. 5 (1981), no. 1, 57–60.

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