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Abstract. It is well-known that the Neumann initial-boundary value problem for the minimal-chemotaxis-logistic system

\[
\begin{align*}
    \frac{\partial u}{\partial t} &= \Delta u - \chi \nabla \cdot (u \nabla v) + au - bu^2, & x \in \Omega, t > 0, \\
    \tau \frac{\partial v}{\partial t} &= \Delta v - v + u, & x \in \Omega, t > 0,
\end{align*}
\]

in a bounded smooth domain $\Omega \subset \mathbb{R}^2$ doesn’t have any blow-ups for any $a \in \mathbb{R}, \tau \geq 0, \chi > 0$ and $b > 0$. Here, we obtain the same conclusion by replacing the logistic source $au - bu^2$ with a kinetic term $f(u)$ fulfilling $f(0) \geq 0$,

\[
\liminf_{s \to \infty} \left\{ -f(s) \cdot \frac{\ln s}{s^2} \right\} = \mu \in (0, \infty]
\]

as well as

\[
(\chi - \mu)^+ M < \frac{1}{2c^+ C_{GN}},
\]

where $c^+ = \max\{c, 0\}$, $C_{GN}$ is the Gagliardo-Nirenberg constant and

\[
M = \|u_0\|_{L^1(\Omega)} + |\Omega| \inf_{\eta > 0} \sup_{s > 0} \left\{ f(s) + \eta s : s > 0 \right\} \eta.
\]

In this setup, it is shown that this problem doesn’t have any blow-up by ensuring all solutions are global-in-time and uniformly bounded. Clearly, $f$ covers super-, logistic, sub-logistic sources like $f(s) = as - bs^\theta$ with $b > 0$ and $\theta \geq 2$, $f(s) = as - \frac{bs^2}{\ln(s+e)}$ with $b > 0$ and $\gamma \in (0, 1)$, and $f(s) = as - \frac{bs^2}{\ln(s+1)}$ with $b > 0$ etc. This indicates that logistic damping is not the weakest damping to guarantee boundedness for the 2D Keller-Segel minimal chemotaxis model.

1. Introduction

Chemotaxis is the directed movement of mobile species in response to chemical signal in their environment. To model this important process, in 1970s, Keller-Segel [15, 16] proposed a classical coupled parabolic partial differential system to describe the mobile cells (with density $u$) move towards the concentration gradient of a chemical substance $v$ produced by the cells themselves. Nowadays, this system

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is widely known as the minimal Keller-Segel chemotaxis model:

$$\begin{cases}
u_t = \Delta v - v + u, & x \in \Omega, t > 0, \\
\tau v_t = \Delta v - v + u, & x \in \Omega, t > 0, \\
\frac{\partial u}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\
u(x, 0) = u_0(x) \geq 0, \tau v(x, 0) = \tau v_0(x) \geq 0, & x \in \Omega, \\
\end{cases}$$  \hspace{1cm} (1.1)

where the habitat \( \Omega \subset \mathbb{R}^n(n \geq 1) \) is a bounded domain with the smooth boundary \( \partial \Omega, \tau \geq 0, \chi > 0 \) and \( \frac{\partial}{\partial \nu} \) stands for the outward normal derivative on \( \partial \Omega \).

Since this pioneering work, the minimal model (1.1) and its various variants have been received significant attention to understand chemotaxis mechanism in various contexts and, the chemotactic induced cross-diffusion has been shown to lead to finite/infinite-time blow-up under certain circumstances. The grand phenomenological picture is surveyed as follows: No finite/infinite time blow-up occurs in 1-D \( [11] [24] [32] \), critical mass blow-ups occur in 2-D: when the initial mass lies below the threshold solutions exist globally and converge to a single equilibrium in large time, \( [7, 24, 39] \), and, the chemotactic induced cross-diffusion has been shown to lead to blow-ups occur or global-in-time bounded solutions exist. See the review articles \( [11] [35] [3] \) for more progresses on (1.1) and its variants.

To see that the chemotactic induced cross-diffusion \(-\chi \nabla \cdot (u \nabla v)\) in model (1.1) has an aggregation effect, in the parabolic-elliptic case, i.e., \( \tau = 0 \), let us expand out the \( u \)-equation and then substitute the expression of \( \Delta v = v - u \) obtained from the \( v \)-equation, we see that the \( u \)-equation becomes

$$u_t = \Delta u - \chi \nabla u \nabla v + \chi u^2 - \chi uv. \hspace{1cm} (1.2)$$

This tells us that the chemotaxis term \(-\chi \nabla \cdot (u \nabla v)\) does play a chemotactic aggregation role and it behaves roughly like the quadratic growth \( \chi u^2 \), rigorous if \( v \) is \( L^\infty \)-bounded. Therefore, adding a logistic source of the form \( au - bu^2 \) with \( a \in \mathbb{R} \) and \( b > 0 \) to the \( u \)-equation in (1.1) is very much expected to eliminate such finite/infinite time blow-up phenomenon. The presence of logistic source indeed has been demonstrated to have an effect of preventing blow-ups for (1.1). More specifically, for the minimal-chemotaxis-logistic model (with the same boundary and initial data as (1.1) suspended for less writing and saving space)

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + au - bu^2, & x \in \Omega, t > 0, \\
\tau v_t = \Delta v - v + u, & x \in \Omega, t > 0, \end{cases}$$  \hspace{1cm} (1.3)

in the case of \( \tau = 0 \), if \( b > \chi \), then a replacement of the \( u \)-equation in (1.3) with Eq. (1.2) roughly shows that every solution to (1.3) with \( \tau = 0 \) is global-in-time and is uniformly bounded. In fact, subtle and rigorous studies, cf. \( [24] [25] [7] [10] \), have shown, for \( n = 1, 2 \), no matter \( \tau = 0 \) or \( \tau > 0 \), even arbitrarily small \( b > 0 \) will be enough to rule out any blow-up by ensuring all solutions to (1.3) are global-in-time and uniformly bounded for all reasonably initial data. This is even true in respective of global existence of classical solutions for (1.3) with \( \tau = 0 \) and the chemo-sensitivity \(-\chi \nabla \cdot (u \nabla v)\) replaced by a singular chemo-sensitivity \(-\chi \nabla \cdot (\frac{1}{2} \nabla v)\) and uniform boundedness is further ensured in \( [4] \) for sufficiently large \( a > 0 \). The same result has been recently extended to the fully parabolic case \( [43] \). These results
summarize that there is no difference between parabolic-elliptic case ($\tau = 0$) and parabolic-parabolic case ($\tau > 0$) in 2-D in respective of global existence and boundedness for (1.1). In 2-D setting, comparing the results for (1.1) with that of (1.3), we find that blow-up is fully precluded as long as a logistic source presents, and, in this case, there is no critical mass blow-up phenomenon. Based on these observations, we wonder

(Q) adding a logistic source may be more than enough to prevent blow-up for (1.1) in 2-D, and thus we wonder whether or not adding a sub-logistic source is already sufficient to prevent blow-up for (1.1) in 2-D? In this paper, we obtain a positive answer to (Q) by showing that a sub-logistic source is already enough to prevent blow-up for (1.1) in 2-D. To state our precise results, we replace the logistic source in (1.3) with a growth source and, for any $\sigma > 0$, there exists $C_\sigma = C(\sigma, u_0, \tau v_0, |\Omega|, \chi, f) > 0$ such that

$$
\|u(\cdot, t)\|_{L^1(\Omega)} + \|v(\cdot, t)\|_{W^{1, r}(\Omega)} \leq C, \quad \forall t \in (0, \infty)
$$

(1.8)

with this setup, our precise findings on blow-up prevention by sub-logistic sources for (1.1) in 2-D read as follows:

**Theorem 1.1.** [Blow-up prevention by sub-logistic sources for (1.1) in 2-D] Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain, $\tau \geq 0$, $\chi > 0$, the initial data $(u_0, v_0)$ satisfy $u_0 \in C(\Omega)$ and $v_0 \in W^{1, r}(\Omega)$ for some $r > 2$ in the case of $\tau > 0$, and, finally, let the kinetic function $f$ belonging to $W^{1, \infty}_0(\Omega)$ satisfy $f(0) \geq 0$ as well as

$$
\lim_{s \to \infty} \left\{ -f(s) \cdot \frac{\ln s}{s^\mu} \right\} = \mu \in (0, \infty].
$$

(1.5)

Assume either one of the following cases holds:

(B1) Sub-logistic, logistic or super-logistic source, i.e., $\mu = \infty$;

(B2) proper sub-logistic source dominates or cancels chemotactic aggregation, i.e., $0 < \mu < \infty$ and $\mu \geq \chi$;

(B3) chemotactic aggregation dominates proper sub-logistic source and small initial mass, i.e., $\chi > \mu$, $0 < \mu < \infty$ and

$$(\chi - \mu) M < \frac{1}{2C_{GN}}.$$ 

where $C_{GN}$ is the Gagliardo-Nirenberg constant, $M$ is finite and is given by

$$
M = \|u_0\|_{L^1(\Omega)} + |\Omega| \inf_{\eta > 0} \sup_{\eta} \{f(s) + \eta s : s > 0\}.
$$

(1.6)

Then the Keller-Segel chemotaxis-growth model (1.1) has a unique global-in-time classical solution which is uniformly-in-time bounded in the following ways: there exists a constant $C = C(u_0, \tau v_0, |\Omega|, \chi, f) > 0$ such that

$$
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1, r}(\Omega)} \leq C, \quad \forall t \in (0, \infty)
$$

(1.7)

and, for any $\sigma > 0$, there exists $C_\sigma = C(\sigma, u_0, \tau v_0, |\Omega|, \chi, f) > 0$ such that

$$
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C, \quad \forall t \in (\sigma, \infty).
$$

(1.8)
Remark 1.2. [Notes on blow-up prevention by sub-logistic sources in 2-D] Here and below, \( \mu \) is understood as an extended real positive number. Hence, via the positive part function \( a^+ = \max\{a,0\} \), the cases (B1), (B2) and (B3) can be unified as

\[
(\chi - \mu)^+ M < \frac{1}{2C_G^4}. \tag{1.9}
\]

The case that \( \mu = \infty \) covers sub-logistic, logistic or super-logistic sources; typical examples are \( f(s) = as - bs^\gamma \) with \( a \in \mathbb{R}, b > 0, \gamma \in (0,1) \) for \( s > 0 \), or \( f(s) = as - \frac{bs^2}{\ln(s+c)} \) with \( b > 0 \) etc. (sub-logistic sources); \( f(s) = as - bs^\theta \) with \( a \in \mathbb{R}, b > 0 \) and \( \theta \geq 2 \) (logistic or super-logistic sources). In the last case, the boundedness of solutions to (1.4) in 2-D is well-known as mentioned before.

We note also that not all sub-logistic sources are included in Theorem 1.1; for instance, growth sources like \( f(s) = as - bs^\alpha \) with \( a \in \mathbb{R}, b > 0 \) and \( 1 < \alpha < 2 \) fail to fulfill (1.5) since then \( \mu = 0 \). We leave this challenging problem (cf. remarks below) as future research to explore whether or not the radical opposite side of boundedness, namely, blow-up, will occur for the 2-D chemotaxis model (1.4). However, the main message that Theorem 1.1 conveys to us is that logistic damping is not the weakest damping to guarantee boundedness for the minimal chemotaxis-logistic model (1.4) in 2-D.

In higher dimensions, the logistic effect on boundedness becomes increasingly complex. For \( n \geq 3 \), so far, it is only known that properly strong logistic damping can prevent blow-up for (1.1). More specifically, in the case of \( \tau = 0 \), all classical solutions to (1.3) are uniformly bounded in time if

\[
b \geq \frac{(n-2)}{n} \chi. \tag{1.10}
\]

See, for instance, [32, 13, 17, 31].

In the parabolic-parabolic case (\( \tau > 0 \)), the issue becomes more delicate; for \( n \geq 4 \), sufficiently strong logistic damping can prevent blow-up for (1.1) (cf. [33, 42]), and, in the physically relevant case of \( n = 3 \), we have a compact formula:

\[
b > \begin{cases} 
\frac{2}{5} \chi, & \text{if } \tau = 1 \text{ and } \Omega \text{ is convex} \tag{33}, \\
\frac{3(1+2\tau)}{5(4\tau-2)} \chi, & \text{if either } \tau \neq 1 \text{ or } \Omega \text{ is not convex} \tag{41}.
\end{cases}
\]

Remark 1.3. From Theorem 1.1 and the text before it, we see that the 2-D boundedness doesn’t distinguish between parabolic-elliptic case and parabolic-parabolic case. However, by (1.10) and (1.11), we see that higher-dimensional boundedness distinguishes drastically between parabolic-elliptic case and parabolic-parabolic case (the convexity of \( \Omega \) and the equality of diffusion rates of \( u \) and \( v \) also matter a lot).

For other aspects, we mention, in 3-D bounded, smooth and convex domains, that logistic damping guarantees the existence of global weak solutions to (1.5) [15], and that sufficiently strong logistic damping can enhance ‘expected’ results such as global existence and stabilization toward constant equilibria, as well as more ‘unexpected’ behavior witnessing a certain strength of chemotactic destabilization for (1.3), cf. [2, 9, 6, 19, 29, 31, 36, 37, 38, 41].

Finally, we mention something that is related to Remark 1.2. One might think that blow-up may not be prevented by weak damping sources. However, since nice properties such as energy-like structure possessed by (1.1) are destroyed by the presence of growth source, blow-up has not been rigorously detected to occur in any
chemotaxis-growth system when $n = 2, 3$. Up to now, only when $n \geq 5$, radially symmetrical blow-up is known to be possible in a parabolic-elliptic simplified variant of (1.3) under a proper sub-quadratic damping source: for the parabolic-elliptic chemotaxis-growth system

$$u_t = \Delta u - \nabla \cdot (u \nabla v) + au - bu^\alpha, \quad v_t = \Delta v + u - m(t), \quad m(t) = \frac{1}{\Omega} \int_{\Omega} u,$$

radially symmetrical blow-up may occur for space-dimension $n \geq 5$ and exponents $1 < \alpha < \frac{3}{2} + \frac{1}{2n - 2}$ [34]. This doesn’t contradict Theorem 1.1 since $\mu = 0$ by (1.5).

Thus, the opposite side of Theorem 1.1, namely, the occurrence of blow-up is a challenging problem.

The rest of this paper is arranged as follows: in Section 2, we collect the standard Gagliardo-Nirenberg interpolation inequality and the local well-posedness of (1.4). In Section 3, we show the proof of Theorem 1.1; the key point consists in deriving a uniform-in-time estimate for $u \ln u$ under (1.9), afterward, we illustrate two commonly used approaches, cf. [23, 26, 40], to establish the $L^2$-boundedness $u$, and then a simple application of the widely known $L^{2+}$-boundedness criterion with $n = 2$ (cf. [1, 40]) obtained via Moser type iteration technique, we achieve the global existence and boundedness as in (1.7) and (1.8) of solutions to (1.4).

2. Preliminaries

For convenience, we state the well-known Gagliardo-Nirenberg inequality:

**Lemma 2.1.** (Gagliardo-Nirenberg interpolation inequality [4, 22]) Let $p \geq 1$ and $q \in (0, p)$. Then there exist a positive constant $C_{GN}$ depending on $p$ and $q$ such that

$$\|w\|_{L^p} \leq C_{GN} \left( \|\nabla w\|_{L^2}^{\delta} \|w\|_{L^q}^{(1-\delta)} + \|w\|_{L^s} \right), \quad \forall w \in H^1 \cap L^q,$$

where $s > 0$ is arbitrary and $\delta$ is given by

$$\frac{1}{p} = \delta \left( \frac{1}{2} - \frac{1}{n} \right) + \frac{1 - \delta}{q} \iff \delta = \frac{\frac{1}{q} - \frac{1}{p}}{\frac{1}{q} - \frac{1}{2} + \frac{1}{n}} \in (0, 1).$$

The local solvability and extendibility of the chemotaxis-growth system (1.4) is well-established by using a suitable fixed point argument and standard parabolic regularity theory; see, for example, [12, 30, 33].

**Lemma 2.2.** Let $\tau \geq 0, \chi \geq 0$ and let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with a smooth boundary. Suppose that the initial data $(u_0, v_0)$ satisfies $u_0 \in C(\overline{\Omega})$ and $v_0 \in W^{1,r}(\Omega)$ with some $r > n$ and that $f \in W^{1,\infty}_{loc}(\mathbb{R})$ with $f(0) \geq 0$. Then there is a unique, nonnegative, classical maximal solution $(u, v)$ of the IBVP (1.4) on some maximal interval $[0, T_m)$ with $0 < T_m \leq \infty$ such that

$$u \in C(\overline{\Omega} \times [0, T_m)) \cap C^{2,1}(\overline{\Omega} \times (0, T_m)),
\quad v \in C(\overline{\Omega} \times [0, T_m)) \cap C^{2,1}(\overline{\Omega} \times (0, T_m)) \cap L^\infty_{loc}([0, T_m); W^{1,r}(\Omega)).$$

In particular, if $T_m < \infty$, then

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,r}(\Omega)} \to \infty \quad \text{as } t \to T_m.$$
3. Blow-up prevention by sub-logistic sources

This section is devoted to the proof of Theorem 1.1. Unless otherwise specified, we shall assume all the conditions in Lemma 2.2 and Theorem 1.1 are fulfilled. As usual, we begin with the \( L_1 \)-norm of \( u \), and we have the following lemma:

**Lemma 3.1.** The \( L_1 \)-norm of \( u \) and \( L_2 \)-norm of \( v \) are uniformly bounded obeying

\[
\|u(t)\|_{L^1} \leq M, \quad \forall t \in (0, T_m),
\]

where \( M \) defined by \( 1.6 \), and, there exists \( C = C(u_0, \tau v_0, |\Omega|, f) > 0 \) such that

\[
\|v(t)\|_{L^2} \leq C, \quad \forall t \in (0, T_m).
\]

**Proof.** Integrating the \( u \)-equation in \( 1.4 \) and using the homogeneous Neumann boundary conditions, we obtain a Gronwall inequality, for any \( \eta > 0 \), that

\[
\frac{d}{dt} \int_{\Omega} u = \int_{\Omega} f(u) \leq -\eta \int_{\Omega} u + M_\eta |\Omega|,
\]

which simply yields

\[
\int_{\Omega} u \leq \int_{\Omega} u_0 + \frac{M_\eta}{\eta} |\Omega|.
\]

This, upon taking infimum over \( \eta > 0 \) and recalling the definition of \( M \) in \( 1.6 \), implies the \( L_1 \)-bound of \( u \) as stated in \( 3.1 \). Here, due to \( 1.5 \),

\[
M_\eta = \sup \left\{ f(s) + \eta s : s > 0 \right\} < \infty.
\]

Indeed, the definition of \( \mu \) in \( 1.5 \) gives rise to

\[
\exists \hat{s} \gg 1 \text{ s.t. } f(s) + \eta s \leq -\hat{\mu} s^2 \ln s, \quad \forall s \geq \hat{s},
\]

where \( \hat{\mu} = \frac{\mu}{2} \) if \( 0 < \mu < \infty \) and \( \hat{\mu} \) equals anything larger than \( \chi \) if \( \mu = \infty \).

This directly entails

\[
f(s) + \eta s \leq -\hat{\mu} s^2 \ln s + \eta s < 0, \quad \forall s \geq \hat{s},
\]

which in conjunction with the fact that \( f \) is bounded on any finite interval implies that \( M_\eta \) is finite.

Next, since \( \|u\|_{L^1} \) is bounded, the \( L_1 \)-boundedness of \( v \) follows from

\[
\tau \frac{d}{dt} \int_{\Omega} v + \int_{\Omega} v = \int_{\Omega} u \leq M.
\]

In the case of \( \tau = 0 \), the \( L_1 \)-boundedness of \( u \) and the elliptic estimate applied to the \( v \)-equation show immediately that \( \|v\|_{L^2} \) is bounded. In the case of \( \tau > 0 \), we (can alternatively use the Neumann heat semigroup type argument to get \( 3.3 \)) multiply the \( v \)-equation by \( v \) and integrate parts to deduce

\[
\tau \frac{d}{dt} \int_{\Omega} v^2 + \int_{\Omega} v^2 + \int_{\Omega} |\nabla v|^2 = \int_{\Omega} uv \leq \epsilon \int_{\Omega} v^3 + \frac{2}{3\sqrt{3\epsilon}} \int_{\Omega} u^{\frac{3}{2}}, \quad \forall \epsilon > 0,
\]

where we have applied the Young’s inequality with epsilon:

\[
ab 
\leq \frac{a^p}{(\epsilon p)^{\frac{q}{p}}} + \frac{b^q}{(\epsilon q)^{\frac{p}{q}}}, \quad p > 0, q > 0, \frac{1}{p} + \frac{1}{q} = 1, \quad \forall a, b \geq 0.
\]

By the standard Gagliardo-Nirenberg inequality in Lemma 2.1 with \( n = 2 \) and the \( L^1 \)-bound of \( v \), we conclude

\[
\|v\|_{L^3}^2 \leq C_{GN}(\|v\|_{L^2}^2 \|v\|_{L^1}^{\frac{4}{3}} + \|v\|_{L^1})^3 \leq C \|v\|_{L^2}^2 + C.
\]
On the other hand, it is simple to see from (3.11) that
\[ A_\epsilon := \sup \left\{ f(s) + s + \frac{2}{3\sqrt{3}\epsilon} s^\frac{2}{3} : s > 0 \right\} < \infty. \]

By taking sufficiently small \( \epsilon > 0 \), combing these two inequalities above with (3.8) and (3.10), we finally derive a differential inequality as follows:
\[ \frac{d}{dt} \int_\Omega \left( u + \frac{\tau}{2} v^2 \right) + \min\{1, \frac{2}{\tau} \} \int_\Omega \left( u + \frac{\tau}{2} v^2 \right) \leq C_s. \]

Solving this standard Gronwall inequality, we readily obtain the boundedness of \( \|u\|_{L^1} + \|v\|_{L^2} \), and so the \( L^2 \)-boundedness of \( v \) as in (3.2) follows.

Following the common way, cf. [23] [40], based on the \( L^1 \)-estimate of \( u \) as gained in Lemma 3.1, we proceed to show the uniform boundedness of \( \|u \ln u\|_{L^1} \).

**Lemma 3.2.** There exists \( C = C(u_0, \tau \nu_0, |\Omega|, \chi, f) > 0 \) such that
\[ \|(u \ln u(t))\|_{L^1} + \tau \chi \|\nabla v\|_{L^2}^2 \leq C, \quad \forall t \in (0, T_m). \] (3.7)

**Proof.** We use integration by parts to compute honestly from (1.4) that
\[ \frac{d}{dt} \int_\Omega u \ln u + 4 \int_\Omega |\nabla u|^2 = \chi \int_\Omega \nabla u \nabla v + \int_\Omega (\ln u + 1) f(u) \]
\[ = -\chi \int_\Omega u \Delta v + \int_\Omega (\ln u + 1) f(u). \] (3.8)

In the case of \( \tau = 0 \), we substitute the expression \( \Delta v = v - u \) from the \( v \)-equation in (1.4) into (3.8) and utilize the nonnegativity of \( u, v \) and \( \chi \) to obtain
\[ \frac{d}{dt} \int_\Omega u \ln u + 4 \int_\Omega |\nabla u|^2 \leq \int_\Omega [\chi u^2 + (\ln u + 1) f(u)]. \] (3.9)

In the case of \( \tau > 0 \), we multiply the \( v \)-equation by \( -\Delta v \) and then integrate by parts over \( \Omega \) to get
\[ \frac{\tau}{2} \frac{d}{dt} \int_\Omega |\nabla v|^2 + \int_\Omega |\nabla v|^2 + \int_\Omega |\Delta v|^2 = -\int_\Omega u \Delta v. \] (3.10)

We obtain through an obvious linear combination of identities (3.8) and (3.10) and a use of the Cauchy-Schawrz inequality that
\[ \frac{d}{dt} \int_\Omega \left( u \ln u + \frac{\tau \chi}{2} |\nabla v|^2 \right) + 4 \int_\Omega |\nabla u|^2 + \chi \int_\Omega |\nabla v|^2 + \chi \int_\Omega |\Delta v|^2 \]
\[ = -2\chi \int_\Omega u \Delta v + \int_\Omega (\ln u + 1) f(u) \]
\[ \leq \chi \int_\Omega |\Delta v|^2 + \int_\Omega \left[ \chi u^2 + (\ln u + 1) f(u) \right]. \] (3.11)

Now, the key is to digest the last integral on its right-hand side of (3.9) and (3.11).

First, the definition of \( \mu \) in (3.3) allows us to deduce that
\[ \forall \epsilon \in (0, \mu), \exists s_c > 1 \text{ s.t. } f(s) \leq -\epsilon + \frac{(\ln s + 1)^2}{\ln s}, \quad \forall s \geq s_c. \] (3.12)

Here and below, \( \mu \) is understood as any finite number larger than \( \chi \) in the case of \( \mu = \infty \). Then it follows trivially from (3.12) that
\[ \chi s^2 + (\ln s + 1) f(s) \leq \chi s^2 - (\mu - \epsilon) \frac{(\ln s + 1)^2}{\ln s} s^2 \leq (\chi - \mu) s^2 + \epsilon s^2, \quad \forall s \geq s_c. \]
Therefore, we have
\[
\int_{\Omega} \left[ \chi u^2 + (\ln u + 1)f(u) \right] = \int_{\{u \leq s_1\}} \left[ \chi u^2 + (\ln u + 1)f(u) \right] + \int_{\{u > s_1\}} \left[ \chi u^2 + (\ln u + 1)f(u) \right] \quad (3.13)
\]
\[
\leq \sup_{0 < s < s_1} \left[ \chi s^2 + (\ln s + 1)f(s) \right] |\Omega| + (\chi - \mu)^+ \int_{\Omega} u^2 + \epsilon \int_{\Omega} u^2.
\]
Now, we apply the well-known two dimensional Gagliardo-Nirenberg inequality in Lemma 2.1 with \( n = 2 \) and the \( L^1 \)-bound of \( u \) in (3.1) to estimate
\[
\int_{\Omega} u^2 = \|u^\frac{1}{2}\|^4_L \leq C_{GN}^4 \left( \|\nabla u^\frac{1}{2}\|_{L^2}^2 \|u^\frac{1}{2}\|^2_{L^2} + \|u^\frac{1}{2}\|_{L^2}^4 \right)^\frac{1}{2} 
\]
\[
\leq 8C_{GN}^4 \left( M \|\nabla u^\frac{1}{2}\|_{L^2}^2 + M^2 \right),
\]
where we applied the elementary inequality \( (a + b)^4 \leq 2^3(a^4 + b^4) \) for all \( a, b \geq 0 \).

Next, noticing that
\[
u \ln u \leq cu^2 + L_\epsilon, \quad L_\epsilon = \sup_{s \geq 0} \{ s \ln s - es^2 : s > 0 \} < \infty, \quad (3.15)
\]
we conclude from (3.14), (3.16) and (3.17) that
\[
\int_{\Omega} \left[ \chi u^2 + (\ln u + 1)f(u) + u \ln u \right] \leq 8M C_{GN}^4 \left( (\chi - \mu)^+ + 2\epsilon \right) \int_{\Omega} |\nabla u^\frac{1}{2}|^2 + N_\epsilon, \quad (3.16)
\]
where
\[
N_\epsilon = \sup_{0 < s < s_1} \left[ \chi s^2 + (\ln s + 1)f(s) \right] |\Omega| + L_\epsilon |\Omega| + 8M^2 C_{GN}^4 \left( (\chi - \mu)^+ + 2\epsilon \right) < \infty.
\]

Now, thanks to (1.9), we can fix, for instance, \( \epsilon \) according to
\[
\epsilon = \epsilon_0 = \frac{1}{2} \min \left\{ \mu, \frac{1}{2MC_{GN}^2} - (\chi - \mu)^+ \right\} > 0
\]
\[
\iff 8M C_{GN}^4 \left( (\chi - \mu)^+ + 2\epsilon_0 \right) \leq 4,
\]
and then, in the case of \( \tau = 0 \), combing (3.16) with (3.14), we readily infer a differential inequality as follows:
\[
\frac{d}{dt} \int_{\Omega} u \ln u + \int_{\Omega} u \ln u \leq N_{\epsilon_0},
\]
this easily entails that
\[
\int_{\Omega} u \ln u \leq \left( \int_{\Omega} u_0 \ln u_0 \right) + N_{\epsilon_0}.
\]
Hence, the fact \(-s \ln s \leq e^{-1}\) for all \( s > 0 \) further entails
\[
\int_{\Omega} |u \ln u| = \int_{\Omega} u \ln u - 2 \int_{\{u \leq 1\}} u \ln u \leq \int_{\Omega} u_0 \ln u_0 + (N_{\epsilon_0} + 2e^{-1}|\Omega|). \quad (3.18)
\]
In the case of \( \tau > 0 \), collecting (3.14), (3.16) and (3.17), we deduce another key differential inequality:
\[
\frac{d}{dt} \int_{\Omega} \left( u \ln u + \frac{\tau \chi}{2} |\nabla v|^2 \right) + \min \{ 1, \frac{2}{\tau} \} \int_{\Omega} \left( u \ln u + \frac{\tau \chi}{2} |\nabla v|^2 \right) \leq N_{\epsilon_0},
\]
which yields simply
\[
\int_{\Omega} \left( u \ln u + \frac{\tau}{2} |\nabla v|^2 \right) \leq \int_{\Omega} \left( u_0 \ln u_0 + \frac{\tau}{2} |\nabla v_0|^2 \right) + \frac{N_u}{\min\{1, \frac{\tau}{4}\}}.
\]

This together with (3.18) gives our desired estimate in (3.7). \hfill \square

For the standard logistic source \( f(s) = as - bs^2 \), based on the \( L^1 \)-boundedness of \( u \ln u \), there are two common methods, cf. \[23, 26, 40\], to obtain \( L^2\)-boundedness of \( u \). To make our argument self-contained and for completeness, we here sketch these two methods for sub-logistic sources satisfying (1.5).

**Lemma 3.3.** For \( \tau = 0 \), there exists \( C = C(u_0, |\Omega|, \chi, f) > 0 \), and, for any \( q \in (0, \infty) \), there exists \( C_q = C(q, u_0, |\Omega|, \chi, f) > 0 \) such that
\[
\|u(\cdot, t)\|_{L^2} \leq C, \quad \|\nabla v(\cdot, t)\|_{L^2} \leq C_q, \quad \forall t \in (0, T_m).
\]

**Proof.** By testing the \( u \)-equation by \( u \) and using the \( v \)-equation with \( \tau = 0 \) in the chemotaxis-growth model (1.4), upon integration by parts, we find that
\[
\frac{d}{dt} \int_{\Omega} u^2 + 2 \int_{\Omega} |\nabla u|^2 \leq \chi \int_{\Omega} u^3 + 2 \int_{\Omega} uf(u). \tag{3.20}
\]

Thanks to the uniform boundedness of \( \|u \ln u\|_{L^1} \) by (3.18) of Lemma 3.2 and the extended version of Gagliardo-Nirenberg inequality involving logarithmic functions from \[27\] Lemma A.5, we can easily deduce that
\[
\int_{\Omega} u^3 \leq \eta \int_{\Omega} |\nabla u|^2 + C_\eta, \quad \forall \eta > 0. \tag{3.21}
\]

On the other hand, it follows from (3.1) that \( \sup\{2sf(s) + s^2 : s > 0\} < \infty \). Hence, by taking \( \eta \in (0, \frac{\chi}{2}) \) in (3.21), we deduce from (3.20) an ODE for \( \int_{\Omega} u^2 \):
\[
\frac{d}{dt} \int_{\Omega} u^2 + \int_{\Omega} u^2 \leq C(u_0, |\Omega|, \chi, f),
\]
which quickly gives rise to the \( L^2 \)-boundedness of \( u \) as in (3.19).

Then the \( \dot{W}^{2,p} \)-elliptic estimate applied to \( -\Delta v + v = u \) implies the boundedness of \( \|v\|_{\dot{W}^{2,2}} \), and then the Sobolev embedding gives the boundedness of \( v \) as described in the second part of (3.19). \hfill \square

**Lemma 3.4.** For \( \tau > 0 \), there exists \( C = C(u_0, v_0, |\Omega|, \chi, \tau, f) > 0 \) such that
\[
\|u(\cdot, t)\|_{L^2} + \|\nabla v(\cdot, t)\|_{L^2} + \|\Delta v(\cdot, t)\|_{L^2} \leq C, \quad \forall t \in (0, T_m). \tag{3.22}
\]

**Proof.** We multiply the \( u \)-equation in (1.4) by \( u \) and integrate by parts to infer from Hölder’s inequality that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 + \int_{\Omega} |\nabla u|^2 = \chi \int_{\Omega} u \nabla u \nabla v + \int_{\Omega} uf(u)
\]
\[
= -\frac{\chi}{2} \int_{\Omega} u^2 \Delta v + \int_{\Omega} uf(u). \tag{3.23}
\]

Now, we have two choices to obtain the \( L^2 \)-estimate of \( u \) by coupling (3.23) with two energy identities associated with the \( v \)-equation. The first choice is to multiply the \( v \)-equation by \( \Delta^2 v = \Delta(\Delta v) \) and then integrate over \( \Omega \) by parts to obtain that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\Delta v|^2 + \int_{\Omega} |\nabla v|^2 + \int_{\Omega} |\nabla \Delta v|^2 = -\int_{\Omega} \nabla u \nabla \Delta v. \tag{3.24}
\]
The second choice is to take gradient of the $v$-equation and then multiply it by $\nabla v |\nabla v|^2$ and, finally integrate by parts to see that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla v|^4 + \int_{\Omega} |\nabla |\nabla v|^2|^2 + 2 \int_{\Omega} |\nabla v|^2 |D^2 v|^2 + 2 \int_{\Omega} |\nabla v|^4$$

$$= \int_{\partial \Omega} |\nabla v|^2 \frac{\partial}{\partial \nu} |\nabla v|^2 - \int_{\Omega} u \Delta v |\nabla v|^2 - \int_{\Omega} u \nabla v |\nabla |\nabla v|^2|^2. \quad (3.25)$$

**Method I:** We combine (3.23) with (3.24) to get, for any $\epsilon > 0$, that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} (u^2 + |\Delta v|^2) + \int_{\Omega} |\nabla u|^2 + \int_{\Omega} |\Delta v|^2 + \int_{\Omega} |\nabla |\nabla v|^2|^2$$

$$= \frac{1}{2} \int_{\Omega} u^2 \Delta v - \int_{\Omega} \nabla u \nabla \Delta v + \int_{\Omega} u f(u)$$

$$\leq \epsilon \int_{\Omega} |\Delta v|^3 + \frac{\chi^2}{3\sqrt{6}\epsilon} \int_{\Omega} u^3 + \frac{1}{2} \int_{\Omega} |\nabla u|^2$$

$$+ \frac{1}{2} \int_{\Omega} |\nabla \Delta v|^2 + \int_{\Omega} u f(u), \quad (3.26)$$

where we have applied the Young’s inequality with epsilon (3.6).

Applying the Gagliardo-Nirenberg interpolation inequality with $n = 2$, Sobolev interpolation inequality and the boundedness of $\|v\|_{H^1}$ implied by (3.2) and (3.7), we derive (see details, for instance, in [23, 40]) that

$$\int_{\Omega} |\Delta v|^3 \leq C \int_{\Omega} |\nabla \Delta v|^2 + C. \quad (3.27)$$

Now, since $\sup \{s f(s) : s > 0\} < \infty$ clearly implied by (1.4), based on (3.27), (3.21) and (3.26), one can easily deduce a Gronwall inequality of the form:

$$\frac{d}{dt} \int_{\Omega} (u^2 + |\Delta v|^2) + \int_{\Omega} (u^2 + |\Delta v|^2) \leq C(u_0, v_0, |\Omega|, \chi, \tau, f),$$

yielding directly the uniform boundedness of $\|u\|_{L^2} + \|\Delta v\|_{L^2}$ as stated in (3.22).

**Method II:** We combine (3.25) with (3.26) to derive that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} (u^2 + |\nabla v|^4) + \int_{\Omega} |\nabla u|^2 + \int_{\Omega} |\nabla |\nabla v|^2|^2$$

$$+ 2 \int_{\Omega} |\nabla v|^2 |D^2 v|^2 + 2 \int_{\Omega} |\nabla v|^4$$

$$= \chi \int_{\Omega} u \nabla u \nabla v - \int_{\Omega} u \Delta v |\nabla v|^2 - \int_{\Omega} u \nabla v |\nabla |\nabla v|^2|^2$$

$$+ \int_{\partial \Omega} |\nabla v|^2 \frac{\partial}{\partial \nu} |\nabla v|^2 + \int_{\Omega} u f(u). \quad (3.28)$$
Next, we shall estimate the integrals on the right in terms of the dissipation terms on its left using the very common ideas:

\[
\chi \int_{\Omega} u \nabla u \nabla v - \int_{\Omega} u \Delta v |\nabla v|^2 - \int_{\Omega} u \nabla v \nabla |\nabla v|^2 \\
\leq \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{\chi}{2} \int_{\Omega} u^2 |\nabla v|^2 + \int_{\Omega} |\Delta v|^2 |\nabla v|^2 + \frac{1}{4} \int_{\Omega} u^2 |\nabla v|^2 \\
+ \frac{1}{2} \int_{\Omega} |\nabla |\nabla v|^2|^2 + \frac{1}{2} \int_{\Omega} u^2 |\nabla v|^2 \\
\leq \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{2} \int_{\Omega} |\nabla |\nabla v|^2|^2 + 2 \int_{\Omega} |D^2 v|^2 |\nabla v|^2 + (1 + \chi^2) \int_{\Omega} u^2 |\nabla v|^2 \\
\text{and the Young’s inequality with epsilon (3.6) shows} \\
\int_{\Omega} u^2 |\nabla v|^2 \leq \epsilon \int_{\Omega} |\nabla v|^6 + \frac{2}{3\sqrt{3\epsilon}} \int_{\Omega} u^3, \quad \forall \epsilon > 0.
\] (3.30)

In view of the boundedness of $||\nabla v||_{L^2}$ by (3.7), the 2-D GN inequality entails

\[
\int_{\Omega} |\nabla v|^6 = ||\nabla v|^2||_{L^3}^3 \leq \left(C_{GN}(||\nabla |\nabla v|^2||_{L^2}^{\frac{3}{2}} ||\nabla v|^2||_{L^1}^{\frac{3}{2}} + ||\nabla v^2||_{L^1})\right)^3 \\
\leq C||\nabla v|^2||_{L^2}^3 + C.
\] (3.31)

As for the boundary integral in (3.28), there are a couple of known ways to bound it in terms of the boundedness of $||\nabla v||_{L^2}$, cf. [14, 28, 41]; the final outcome is

\[
\int_{\partial \Omega} |\nabla v|^2 \frac{\partial}{\partial \nu} |\nabla v|^2 \leq \epsilon \int_{\Omega} |\nabla |\nabla v|^2|^2 + C_{\epsilon} \left( \int_{\Omega} |\nabla v|^2 \right)^2 \\
\leq \epsilon \int_{\Omega} |\nabla |\nabla v|^2|^2 + C_{\epsilon}, \quad \forall \epsilon > 0.
\] (3.32)

Inserting the estimates (3.29), (3.30), (3.31), (3.32) and (3.28) into (3.28) and then choosing sufficiently small $\epsilon > 0$, as before, we obtain an ODE as follow:

\[
\frac{d}{dt} \int_{\Omega} \left( u^2 + |\nabla v|^4 \right) + \int_{\Omega} \left( u^2 + |\nabla v|^4 \right) \leq C(u_0, v_0, |\Omega|, \chi, \tau, f),
\]

which directly establishes the uniform boundedness of $||u||_{L^2} + ||\nabla v||_{L^4}$. \qed

**Proof of Theorem (1.1)**: In light of the uniform $L^2$-boundedness of $u$ provided by Lemmas 3.3 and 3.3, the quite known $L^{2+}$-boundedness criterion with $n = 2$ in [11, 12] obtained via Moser type iteration technique shows $T_m = \infty$ and the uniform boundedness as stated in (1.1). Notice that $(u(\cdot, \sigma), v(\cdot, \sigma)) \in (C^2(\Omega))^2$ for any $\sigma > 0$. Whence, performing a small time shift and treat $t = \sigma$ as the new "initial time" and replacing $(u_0, v_0)$ by $(u(\cdot, \sigma), v(\cdot, \sigma))$, applying the same $L^{2+}$-boundedness criterion with $n = 2$, we conclude the uniform boundedness as stated in (1.3). \qed

**References**

[1] N. Bellomo, A. Bellouquid, Y. Tao, M. Winkler Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues, Math. Models Methods Appl. Sci. 25 (2015), 1663–1763.

[2] X. Cao, Global bounded solutions of the higher-dimensional Keller-Segel system under smallness conditions in optimal spaces, Discrete Contin. Dyn. Syst. 35 (2015), 1891–1904.

[3] E. Feireisl, P. Laurençot and H. Petzeltova, On convergence to equilibria for the Keller-Segel chemotaxis model, J. Differential Equations, 236 (2007), 551–569.

[4] A. Friedman, Partial differential equations. Holt, Rinehart and Winston, New York-Montreal, Que.-London, 1969.
[5] K. Fujie, M. Winkler and T. Yokota, Blow-up prevention by logistic sources in a parabolic-elliptic Keller-Segel system with singular sensitivity, Nonlinear Anal. 109 (2014), 56–71.

[6] X. He and S. Zheng, Convergence rate estimates of solutions in a higher dimensional chemotaxis system with logistic source, J. Math. Anal. Appl. 436 (2016), 970–982.

[7] T. Hillen and A. Potapov, The one-dimensional chemotaxis model: global existence and asymptotic profile, Math. Methods Appl. Sci. 27 (2004), 1783–1801.

[8] T. Hillen and K. Painter, A user’s guide to PDE models for chemotaxis, J. Math. Biol., 58 (2009), 183–217.

[9] T. Hillen and K. Painter, Spatio-temporal chaos in a chemotaxis model, Phys. D 240 (2011), 363–375.

[10] D. Horstmann and G. Wang, Blow-up in a chemotaxis model without symmetry assumptions, European J. Appl. Math. 12 (2001), 159–177.

[11] D. Horstmann, From 1970 until now: the Keller-Segal model in chemotaxis and its consequence I, Jahresber DMV, 105 (2003), 103–165.

[12] D. Horstmann and M. Winkler, Boundedness vs. blow-up in a chemotaxis system, J. Differential Equations 215 (2005), 52–107.

[13] B. Hu and Y. Tao, Boundedness in a parabolic-elliptic chemotaxis-growth system under a critical parameter condition, Appl. Math. Lett. 64 (2017), 1–7.

[14] S. Ishida, K. Seki and T. Yokota, Boundedness in quasilinear Keller-Segel systems of parabolic-parabolic type on non-convex bounded domains, J. Differential Equations 256 (2014), no. 8, 2993–3010.

[15] E. Keller and L. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theoret Biol., 26 (1970), 399–415.

[16] E. Keller and L. Segel, Model for chemotaxis, J. Theor. Biol., 30 (1971), 225–234.

[17] K. Kang and A. Stevens, Blowup and global solutions in a chemotaxis-growth system, Nonlinear Anal. 135 (2016), 57–72.

[18] J. Lankeit, Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source, J. Differential Equations 258 (2015), 1158–1191.

[19] J. Lankeit, Chemotaxis can prevent thresholds on population density, Discrete Contin. Dyn. Syst. Ser. B 20 (2015), 1499–1527.

[20] T. Nagai, Blowup of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains, J. Inequal. Appl. 6 (2001), 37–55.

[21] T. Nagai, T. Senba and K. Yoshida, Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis, Funkcial. Ekvac. 40 (1997), 411–433.

[22] L. Nirenberg, An extended interpolation inequality, Ann. Scuola Norm. Sup. Pisa. (3) 20 (1966), 733-737.

[23] K. Osaki,T. Tsujikawa, A. Yagi, M. Mimura, Exponential attractor for a chemotaxis-growth system of equations, Nonlinear Anal. 51, 119-144 (2002).

[24] K. Osaki and A. Yagi, Finite dimensional attractor for one-dimensional Keller-Segel equations, Funkcial. Ekvac. 44 (2001), 441–469.

[25] T. Senba and T. Suzuki, Parabolic system of chemotaxis: blowup in a finite and the infinite time, Methods Appl. Anal. 8 (2001), 349–367.

[26] Y. Tao and M. Winkler, Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity, J. Differential Equations 252 (2012), 692–715.

[27] Y. Tao and M. Winkler, Energy-type estimates and global solvability in a two-dimensional chemotaxis-haptotaxis model with remodeling of non-diffusible attractant, J. Differential Equations 257 (2014), 784–815.

[28] Y. Tao and M. Winkler, Boundedness and decay enforced by quadratic degradation in a three-dimensional chemotaxis-fluid system, Z. Angew. Math. Phys. 66 (2015), 2555–2573.

[29] Y. Tao and M. Winkler, Persistence of mass in a chemotaxis system with logistic source, J. Differential Equations 259 (2015), 6142–6161.

[30] J. Tello and M. Winkler, A chemotaxis system with logistic source, Comm. Partial Differential Equations 32 (2007), 849–877.

[31] Z. Wang and T. Xiang, A class of chemotaxis systems with growth source and nonlinear secretion, arXiv:1510.07204, 2015.

[32] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model, J. Differential Equations, 248 (2010), 2889–2905.

[33] M. Winkler, Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source, Comm. Partial Differential Equations 35 (2010), 1516–1537.
[34] M. Winkler, Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction, J. Math. Anal. Appl. 384 (2011), 261–272.
[35] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, J. Math. Pures Appl. 100 (2013), 748–767.
[36] M. Winkler, Global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with strong logistic dampening, J. Differential Equations 257 (2014), 1056–1077.
[37] M. Winkler, How far can chemotactic cross-diffusion enforce exceeding carrying capacities? J. Nonlinear Sci. 24 (2014), 809–855.
[38] M. Winkler, Emergence of large population densities despite logistic growth restrictions in fully parabolic chemotaxis systems, Discrete Contin. Dyn. Syst. Ser. B 22 (2017), 2777–279.
[39] T. Xiang, On effects of sampling radius for the nonlocal Patlak-Keller-Segel chemotaxis model, Discrete Contin. Dyn. Syst. 34 (2014), 4911-4946.
[40] T. Xiang, Boundedness and global existence in the higher-dimensional parabolic-parabolic chemotaxis system with/without growth source, J. Differential Equations 258 (2015), 4275–4323.
[41] T. Xiang, How strong a logistic damping can prevent blow-up for the minimal Keller-Segel chemotaxis system? J. Math. Anal. Appl. 459 (2018), 1172–1200.
[42] C. Yang, X. Cao, Z. Jiang and S. Zheng, Boundedness in a quasilinear fully parabolic Keller-Segel system of higher dimension with logistic source, J. Math. Anal. Appl. 430 (2015), 585–591.
[43] X. Zhao and S. Zheng, Global boundedness to a chemotaxis system with singular sensitivity and logistic source, Z. Angew. Math. Phys. 68 (2017), no. 1, Art. 2, 13 pp.

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