SUBSONIC FLOWS FOR THE FULL EULER EQUATIONS IN HALF PLANE

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ABSTRACT. We study the subsonic flows governed by full Euler equations in the half plane bounded below by a piecewise smooth curve asymptotically approaching $x_1$-axis. Nonconstant conditions in the far field are prescribed to ensure the real Euler flows. The Euler system is reduced to a single elliptic equation for the stream function. The existence, uniqueness and asymptotic behaviors of the solutions for the reduced equation are established by Schauder fixed point argument and some delicate estimates. The existence of subsonic flows for the original Euler system is proved based on the results for the reduced equation, and their asymptotic behaviors in the far field are also obtained.

1. Introduction

In this paper, we study subsonic polytropic flows governed by two-dimensional steady, full Euler equations:

\[
\begin{aligned}
\nabla \cdot \mathbf{m} &= 0, \\
\nabla \cdot \left( \frac{\mathbf{m} \otimes \mathbf{m}}{\rho} \right) + \nabla p &= 0, \\
\nabla \cdot (\mathbf{m}(E + p/\rho)) &= 0,
\end{aligned}
\]

where $\nabla$ is the gradient in $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathbf{m} = (m_1, m_2)$ the momentum, $\rho$ the density, $p$ the pressure, and

\[
E = \frac{|\mathbf{m}|^2}{2\rho^2} + \frac{p}{(\gamma - 1)\rho}
\]

the energy with adiabatic exponent $\gamma > 1$. The sonic speed of the flow is defined by

\[
c = \sqrt{\gamma p/\rho}.
\]

The flow is said to be subsonic if $|\mathbf{m}/\rho| < c$. 

2000 Mathematics Subject Classification. 35J15; 35M20; 76G25.

Key words and phrases. Subsonic flows; full Euler equations; polytropic gas; nonlinear elliptic equations.
To my best knowledge, no theoretical result was obtained for subsonic flows governed by full Euler system in unbounded domain. There are rich literatures of subsonic potential flows, which is a simplified model for Euler flows. Shiffman obtained the first existence result in [31], using variational method. In [1], Bers used complex analysis to show existence and uniqueness for the subsonic potential flows. Finn and Gilbarg [17, 18] solved the problem by PDE approach. Recently, Chen-Dafermos-Slemord-Wang [4] pushed the subsonic flows to the sonic limit, using the framework of compensated compactness. With in the same framework, Chen-Slemord-Wang [12] obtained transonic solutions by a vanishing viscosity method. Other results for subsonic or transonic flows of various models can be found in [2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 14, 25, 20, 29, 30, 32, 33, 34], and [15, 16, 19] provide related background and introduction.

The domain we study is the upper half plane with piecewise smooth boundary asymptotically flattened as $|x_1| \to \infty$. This setting can be viewed as a symmetric airfoil problem. For a airfoil symmetric about $x_1$-axis, we cut the exterior domain in half along the symmetry axis, and the upper part becomes our domain. More generally, we allow the boundary to be curved away from the profile. In this way, our setting also includes the model for the wind glancing the landscape.

Let $U = (m, p, \rho)$ be the solution for the subsonic flow. We prescribe an asymptotic limit $U_\infty$, close to a constant subsonic state $U_0$, for the flow in the far field. Unlike the setting for potential flows, the asymptotic behavior $U_\infty$ is not a constant state. Otherwise, the full Euler system can be reduced to a potential flow (cf. Proposition 3.1). To guarantee the convergence of the flows to $U_\infty$ in the far field, we need to obtain some decay property for $\psi - l$, which is the difference between the stream function and its limit behavior. For the whole plane, one knows that the fundamental solution for a Laplace equation has the form $\log |x|$. Therefore, in general, we can not expect the decay of a solution for an elliptic equation as $|x| \to \infty$ in the exterior domain of the whole plane. This is the main technical obstacle for us to obtaining the subsonic flows in the whole plane. However, when flows are restricted in the half plane, we can exclude the logarithmic growth of solutions by prescribing proper decay condition at the infinity.

We reduce the Euler system to a single elliptic equation (3.19) for a stream function $\psi$ by capturing some conservation properties of the system. More precisely, three properties are contained in (3.19): (1) existence of $\psi$ represents conservation of mass; (2) we use the fact $p/\rho^\gamma$ is constant along streamlines during the reduction, which implies entropy is conserved on each streamline. (3) to solve for $\rho$ in terms
of $\psi, \nabla \psi$, we use Bernoulli’s law, which relates to the conservation of energy. Usually, stagnation points occur in various situations and cause major difficulties (for instance, regular reflection for Euler equations in self-similar coordinates). Our reduction process enables us to bypass the difficulty and to obtain the existence of solutions. However, we do not have uniqueness result due to the existence of stagnation points and complex behaviors of streamlines. More details are explained in Remark 6.1.

Once the Euler system is reduced to the elliptic equation (3.19), the remaining work is to solve this nonlinear equation. In detail, we first truncate the original domain $\Omega$ by ball $B_R(O)$ centered at the origin with radius $R$. We solve (3.19) in bounded domain $\Omega_R = \Omega \cap B_R(O)$ with properly prescribed boundary condition. It is a standard method that we linearize (3.19), construct a map $T$ by solving the linearized equation, and prove the existence of a fixed point for $T$ by Schauder fixed point theorem. The fixed point $\psi_R$ is the solution for (3.19) in domain $\Omega_R$.

To take the limit of $\{\psi_R\}$ as $R \to \infty$ and obtain the solution in $\Omega$, the estimates should be independent of the radius $R$. It makes the estimates complicated that there is no sign for the coefficient $b_0$ in the linear equation (4.1). By choosing a proper barrier and using maximum principle with no restriction on the sign of $b_0$ (Lemma 4.2), we obtain uniform estimates, independent of $R$, for $\psi_R$. The barrier function we construct only works for the half plane, not the whole plane. Whether one can find a suitable barrier function for the whole plane is unclear at this moment.

The rest of the paper is organized as follows. In Section 2, we set up the subsonic flow problem, introduce the weighted norms, and state the main result. Section 3 explains the reduction of the full Euler system to a single elliptic equation for the stream function $\psi$. In Section 4, we prove a technical lemma for the linearized equation and obtain crucial estimates. In Section 5, we construct a iteration scheme to solve the nonlinear equation (3.19) in truncated domain $\Omega_R$. Schauder fixed point argument is used to prove the existence of the solution. In Section 6, we take the limit of subsequence of solutions $\psi_R$ in $\Omega_R$ and obtain the solution in the half plane $\Omega$. The relation between the original Euler system and the reduced system is explained.

2. Setup of the Subsonic Flow Problem

In order to describe the conditions and results of our subsonic problem, we need to use the following weighed Hölder norms: for any
\(x, x'\) in a two-dimensional domain \(\Omega\) and for a subset \(P\) of \(\partial \Omega\), define \(\delta_x := \min(\operatorname{dist}(x, P), 1)\), \(\delta_{x,x'} := \min(\delta_x, \delta_{x'}, 1)\), \(\Delta_x := \max(|x|, 1)\) and \(\Delta_{x,x'} := \max(|x|, |x'|, 1)\). Let \(\alpha \in (0, 1)\), \(\sigma, \beta \in \mathbb{R}\), and \(k\) be a nonnegative integer. Let \(k = (k_1, k_2)\) be a integer-valued vector, where \(k_1, k_2 \geq 0\), \(|k| = k_1 + k_2\) and \(D^k = \partial_{x_1}^{k_1} \partial_{x_2}^{k_2}\). We define

\[
[u]_{k,0;\alpha;\beta;\Omega}^{(\sigma;P)} = \sup_{x \in \Omega, |k| = k} (\delta_x^{\max(k+\sigma, 0)} \Delta_x^{\beta+k} |D^k u(x)|),
\]

\[
[u]_{k,\alpha;\beta;\Omega}^{(\sigma;P)} = \sup_{x, x' \in \Omega, x \neq x', \frac{|x|}{|k|} = k} (\delta_{x,x'}^{\max(k+\alpha+\sigma, 0)} \Delta_{x,x'}^{\beta+k+\alpha} \frac{|D^k u(x) - D^k u(x')|}{|x - x'|^\alpha}),
\]

\[
\|u\|_{k,\alpha;\beta;\Omega}^{(\sigma;P)} = \sum_{i=0}^{k} [u]_{i,0;\alpha;\beta;\Omega}^{(\sigma;P)} + [u]_{k,\alpha;\beta;\Omega}^{(\sigma;P)}.
\] (2.1)

For a vector-valued function \(u = (u_1, u_2, \ldots, u_n)\), we define

\[
\|u\|_{k,\alpha;\beta;\Omega}^{(\sigma;P)} = \sum_{i=1}^{n} \|u_i\|_{k,\alpha;\beta;\Omega}^{(\sigma;P)}.
\]

**Remark 2.1.** In the definition of the weighted norms, the lower index in the parenthesis represents the weight at the infinity and the upper index represents the weight to the set \(P\), which will be the set of some corner points on the boundary in the paper.

Define

\[
C^{(\sigma;P)}_{k,\alpha;\beta;\Omega}(\Omega) = \{u : \|u\|_{k,\alpha;\beta;\Omega}^{(\sigma;P)} < \infty\}. \tag{2.2}
\]

For the weighted norms of functions in one-dimensional space \(\Gamma = (a, b)\), with either \(a = -\infty\) or \(b = \infty\), we define

\[
[f]_{k,0;\alpha;\beta;\Gamma} = \sup_{x \in \Gamma} (|x| + 1)^{k+\beta} |f^{(k)}(x)|
\]

\[
[f]_{k,\alpha;\beta;\Gamma} = \sup_{x, x' \in \Gamma, x \neq x'} (\max(|x|, |x'|) + 1)^{k+\alpha+\beta} \frac{|f^{(k)}(x) - f^{(k)}(x')|}{|x - x'|^\alpha}
\]

\[
\|f\|_{k,\alpha;\beta;\Gamma} = \sum_{i=0}^{k} [f]_{k,0;\alpha;\beta;\Gamma} + [f]_{k,\alpha;\beta;\Gamma}.
\] (2.3)

Our domain \(\Omega\) is the upper half plane bounded below by a piecewise smooth curve consisting of three parts:

\[
\partial \Omega = \Gamma_- \cup A \cup \Gamma_+.
\] (2.4)

The following is the description of the three parts for \(\partial \Omega\) (see figure 1). Let \(\Gamma_\pm = \{x_2 = f_\pm(x_1)\}\), where \(f_-\) is defined on \((-\infty, -1)\) and \(f_+\) is
defined on \((1, \infty)\). Both \(\Gamma_-\) and \(\Gamma_+\) approach \(x_1\)-axis as \(|x_1|\) tends to \(\infty\). More precisely, we let
\[
\|f_\pm\|_{2, \alpha; (\alpha+\beta); (-\infty, -1)} \leq 1,
\]
\[
\|f_\pm\|_{2, \alpha; (\alpha+\beta); (1, \infty)} \leq 1.
\]

Let \(A_- = (-1, f_-(-1))\) and \(A_+ = (1, f_+(1))\) be the end points of \(\Gamma_-\), \(\Gamma_+\), respectively. The arch \(A\) connecting \(\Gamma_\pm\) at \(A_\pm\) can be parameterized by
\[
f(s) = (f_1(s), f_2(s)), \quad s \in (-1, 1),
\]
where \(f(-1) = A_-\), \(f(1) = A_+\) and \(f_1, f_2\) are \(C^{2,\alpha}\) smooth functions.

We assume the angles \(\theta^0_\pm\) between \(\Gamma_\pm\) and \(A\) at points \(A_\pm\) satisfies:
\[
\delta < \theta^0_\pm < \pi - \delta,
\]
for a fixed constant \(\delta\).

**Remark 2.2.** The above condition guarantees that stream function \(\psi\) for the flow is \(C^{1,\alpha}\) up to the corner points, which means the flow \(U\) is \(C^\alpha\) up to the corners. If we allow corner angles \(\theta^0_\pm \geq \pi\), \(\psi\) will be \(C^\alpha\) up to the corners, and \(U\) will blow up at corners. We exclude the latter situation just to avoid unimportant details.

Without loss of generality, we may also assume that
\[
A \subset B_{D_0}(0),
\]
i.e., $A$ is contained in the ball of radius $D_0$ centered at 0, and

$$f_-(x_1) > -\frac{1}{2}, \quad f_+(x_1) > -\frac{1}{2}, \quad f_2(s) > -\frac{1}{2}.$$  \hspace{1cm} (2.10)

This means that domain $\Omega$ is above the line $x_2 = -\frac{1}{2}$.

We prescribe slip condition on the boundary:

$$\mathbf{m} \cdot \nu|_{\partial \Omega} = 0,$$  \hspace{1cm} (2.11)

where $\nu$ is the outer normal on boundary $\partial \Omega$.

Let $(m_*, 0, p_0, \rho_0)$ be a constant subsonic solution for (1.1), i.e., $m_* / \rho_0$ less than $\sqrt{p_0 / \rho_0}$. Fix constants $p_0, \rho_0$ and let $m_0 \leq m_*$ be a sufficiently small constant to be determined later. So $U_0 = (m_0, 0, p_0, \rho_0)$, as our background state, is also a subsonic solution for (1.1). We define a vector-valued function $U_\infty = (m_\infty, 0, p_0, \rho_\infty)$ of variable $x_2$ as the asymptotic state for our solution $U = (\mathbf{m}, p, \rho)$ at the far field. We assume that $U_\infty$ is a small perturbation of the background solution $U_0$:

$$\|U_\infty - U_0\|_{2, \alpha; (0, \infty)} \leq \varepsilon m_0,$$  \hspace{1cm} (2.12)

where $0 < \varepsilon < \frac{1}{2}$ is a small parameter to be determined later.

Set $P = \{A_-, A_+\}$ as the set for the weight.

Now we state our main theorem about the existence of the subsonic flow in the half plane $\Omega$:

**Theorem 2.1.** Suppose the boundary $\partial \Omega$ satisfies (2.5)–(2.10) and $U_\infty$ satisfies (2.12). We fix $0 < \beta < \alpha < 1$, depending on $\delta$ in (2.8). For sufficiently small $m_0$, depending on $m_*, p_0, \rho_0, \delta, \alpha, \beta$ and the profile $A$, and sufficiently small $\varepsilon$, depending on $m_*, p_0, \rho_0, \delta, \alpha, \beta, A$ and $m_0$, there exists a subsonic solution $U \in C^{(-\alpha; P)}_{1, \alpha; (\beta)}(\Omega)$ for (1.1) with boundary condition (2.11), such that

$$\|U - U_\infty\|^{(-\alpha; P)}_{1, \alpha; (\beta); \Omega} \leq C m_0,$$  \hspace{1cm} (2.13)

where $C$ is a constant only depending on $m_*, p_0, \rho_0, \delta, \alpha, \beta$ and the profile $A$, but independent of $m_0$ and $\varepsilon$.

**Remark 2.3.** Estimate (2.13) immediately gives the asymptotic behavior of the flow $U$. That is $U$ approaches $U_\infty$ in $C^0$ norm at the rate $|x|^{-\beta}$ as $|x| \to \infty$.

### 3. Reduction of the Euler System

In this section, we use the conservation properties of the Euler equations (1.1) to reduce the four-equation system to one elliptic equation.
By the conservation of mass (first equation of (1.1)), we can find a potential function $\psi$ for the vector field $(-m_2, m_1)$, i.e.,

$$\psi_{x_1} = -m_2, \quad \psi_{x_2} = m_1.$$  

From (1.1), we can derive

$$(m_1 \partial_{x_1} + m_2 \partial_{x_2})(\gamma \ln \rho - \ln p) = 0,$$  

which implies that the quantity $\rho^\gamma / p$ is constant along streamlines, provided that the solution is $C^1$ smooth. This constant only depends on the stream function $\psi$. Thus, we have

$$p = \frac{\gamma - 1}{\gamma} A(\psi) \rho^\gamma.$$  

We will determine function $A$ later by $U_\infty$.

From (1.1), we can also derive the Bernoulli’s law:

$$\frac{|\mathbf{m}|}{2\rho^2} + \frac{\gamma p}{(\gamma - 1)\rho} = B$$  

along the streamlines, where $B$ is the Bernoulli constant depending on $\psi$. With equation (3.3) and (3.1), the Bernoulli’s law (3.4) can be written as

$$\frac{1}{2} |\nabla \psi|^2 + A(\psi) \rho^{\gamma + 1} = B(\psi) \rho^2.$$  

In the subsonic region, we have

$$|\nabla \psi|^2 < c^2 \rho^2 = (\gamma - 1) A(\psi) \rho^{\gamma + 1}.$$  

Inequality (3.6) and the Bernoulli’s law (3.5) implies

$$\rho^{\gamma - 1} > \frac{2B}{(\gamma + 1)A}.$$  

Let $\chi = \frac{1}{2} |\nabla \psi|^2$ and $h(\rho, \psi) = B(\psi) \rho^2 - A(\psi) \rho^{\gamma + 1}$. Therefore, in subsonic region,

$$\frac{\partial h}{\partial \rho} = \rho A \left( \frac{2B}{(\gamma + 1)A} - \rho^{\gamma - 1} \right) < 0.$$  

Hence, we can uniquely solve

$$\chi = h(\rho, \psi) \equiv B(\psi) \rho^2 - A(\psi) \rho^{\gamma + 1}$$  

for $\rho = \rho(\chi, \psi)$ by implicit function theorem.
From (3.8), we can easily calculate
\[
\rho_\chi = -\frac{1}{(\gamma + 1)\rho^\gamma - 2B\rho}, \quad \text{(3.9)}
\]
\[
\rho_\psi = \frac{B'\rho - A'\rho^\gamma}{(\gamma + 1)\rho^\gamma - 2B}. \quad \text{(3.10)}
\]
Therefore, we compute
\[
\rho_{x_1} = \rho_\chi (\psi_{x_1} \psi_{x_1} + \psi_{x_2} \psi_{x_1}) + \rho_\psi \psi_{x_1} = \frac{-\psi_{x_1} \psi_{x_1} - \psi_{x_2} \psi_{x_1} + \psi_{x_1} (B'\rho^2 - A'\rho^{\gamma + 1})}{(\gamma + 1)\rho^\gamma - 2B\rho}, \quad \text{(3.11)}
\]
\[
\rho_{x_2} = \rho_\chi (\psi_{x_1} \psi_{x_1} + \psi_{x_2} \psi_{x_2}) + \rho_\psi \psi_{x_2} = \frac{-\psi_{x_1} \psi_{x_2} - \psi_{x_2} \psi_{x_2} + \psi_{x_2} (B'\rho^2 - A'\rho^{\gamma + 1})}{(\gamma + 1)\rho^\gamma - 2B\rho}. \quad \text{(3.12)}
\]
Now we can reduce the Euler system into one equation. We replace \textbf{m} in the second equation of (1.1) with \((-\psi_{x_2}, \psi_{x_1})\) according to (3.1). Multiplying the second equation of (1.1) by \((\gamma + 1)\rho^\gamma - 2B\rho\), and using the expressions (3.11) and (3.12), we obtain the following equation:
\[
\psi_{x_1} (a_{ij}(\psi, \nabla \psi) \psi_{x_1} - F(\psi, \nabla \psi)) = 0, \quad \text{(3.13)}
\]
where
\[
a_{11}(\psi, \nabla \psi) = (\gamma - 1)A(\psi)\rho^{\gamma + 1} - \psi_{x_2}^2, \quad \text{(3.14)}
\]
\[
a_{12}(\psi, \nabla \psi) = a_{21}(\psi, \nabla \psi) = \psi_{x_1} \psi_{x_2}, \quad \text{(3.15)}
\]
\[
a_{22}(\psi, \nabla \psi) = (\gamma - 1)A(\psi)\rho^{\gamma + 1} - \psi_{x_1}^2, \quad \text{(3.16)}
\]
\[
F(\psi, \nabla \psi) = \frac{\gamma - 1}{\gamma} \rho^{\gamma + 3} (\gamma A B' - 2A'B + A A' \rho^{\gamma - 1}). \quad \text{(3.17)}
\]
Similarly, the third equation of (1.1) gives rise to
\[
\psi_{x_2} (a_{ij}(\psi, \nabla \psi) \psi_{x_2} - F(\psi, \nabla \psi)) = 0. \quad \text{(3.18)}
\]
For a system without stationary points, i.e., \(\nabla \psi\) is nowhere 0, the original Euler system (1.1) can be reduced to the following equation for subsonic flows:
\[
a_{ij}(\psi, \nabla \psi) \psi_{x_i x_j} = F(\psi, \nabla \psi). \quad \text{(3.19)}
\]
Equation (3.19) can be written in divergence form:
\[
\nabla \cdot \left( \frac{\nabla \psi}{\rho} \right) = B'\rho - \frac{1}{\gamma} A'\rho^\gamma. \quad \text{(3.20)}
\]
Now we use the limit function \(U_\infty\) to determine \(A, B\) and the limit function \(l(x_2)\) of the stream function \(\psi\) as \(|x_1| \to \infty\).
Define
\[ l(x_2) = \int_0^{x_2} m_\infty(s) \, ds. \] (3.21)
By (2.12), we know that
\[ \frac{1}{2} m_0 < m_\infty = l' < 2m_0, \]
which implies that \( l \) is invertible and
\[ \frac{1}{2} m_0 x_2 < l(x_2) < 2m_0 x_2. \] (3.22)

Let
\[ \bar{A}(x_2) = \frac{\gamma p_0}{(\gamma - 1)\rho_\infty(x_2)}, \]
\[ \bar{B}(x_2) = \frac{m_\infty^2(x_2)}{2\rho_\infty^2(x_2)} + \frac{\gamma p_0}{\rho_\infty(x_2)}. \]

Then define
\[ A(s) = \bar{A}(l^{-1}(s)), \quad B(s) = \bar{B}(l^{-1}(s)). \] (3.23)

To describe the properties of \( A \) and \( B \), we need to modify the weighted norm in (2.3) as follows:
\[
\begin{align*}
[f]_{k,0;0}^{\prime} & = \sup_{x \in \Gamma} (|x| + m_0)^{k+\beta} |f^{(k)}(x)| \\
[f]_{k,0;0}^{\prime} & = \sup_{x,x' \in \Gamma, x \neq x'} (\max(|x|,|x'|) + m_0)^{k+\alpha+\beta} \frac{|f^{(k)}(x) - f^{(k)}(x')|}{|x - x'|^\alpha} \\
\|f\|_{k,0;0}^{\prime} & = \sum_{i=0}^{k} [f]_{i,0;0}^{\prime} + [f]_{k,0;0}^{\prime}. \quad (3.24)
\end{align*}
\]

Basically, we replace 1 in the weight in (2.3) with \( m_0 \) for the scaling reason.

Set
\[ A_0 = \frac{\gamma p_0}{(\gamma - 1)\rho_0^\gamma}, \quad B_0 = \frac{m_0^2}{2\rho_0^2} + \frac{\gamma p_0}{(\gamma - 1)\rho_0}. \]

By (2.12), we conclude that
\[
\begin{align*}
\|A - A_0\|_{2,0;0}^{\prime} & \leq C_0 \varepsilon_m m_0, \quad (3.25) \\
\|B - B_0\|_{2,0;0}^{\prime} & \leq C_0 \varepsilon_m m_0, \quad (3.26)
\end{align*}
\]
where \( C_0 \) is a constant depending only on \( m_*, p_0, \rho_0 \).

Let us discuss the asymptotic behavior of \( U \) as \( |x_1| \to \infty \). We do not expect constant states at the infinity for general subsonic flows governed by full Euler equations. Actually, if the flow is uniform at
the infinity, we only get a potential flow. This fact is described by the following proposition.

**Proposition 3.1.** Suppose $U$ is a $C^1$ solution of (1.1) with no stagnation points ($\mathbf{m}$ is nowhere $0$). If $U_\infty$ is a constant state, the flow $U$ is potential, i.e., the velocity $\mathbf{u} = \mathbf{m}/\rho$ is irrotational.

**Proof.** If $U_\infty$ is constant, we immediately get $A, B$ are constants by the procedure of obtaining $A, B$. Hence, we have $A' = B' = 0$. Equation (3.20), which is equivalent to (1.1) under the assumption in the proposition, becomes

$$\nabla \cdot \left( \frac{\nabla \psi}{\rho} \right) = 0.$$

Since

$$\psi_{x_1} = -m_2 = -\rho u_2, \quad \psi_{x_2} = m_1 = \rho u_1,$nabla \cdot (u_1)_{x_2} - (u_2)_{x_1} = 0. \text{ Therefore, we have a potential flow. } \square $$

In general, the Euler system (1.1) and equation (3.19) are not equivalent, because the streamlines may not be nice enough for us to do the reduction of the system. However, the solution of (3.19) guarantees the existence of the solution for Euler equations (1.1). Therefore, we only need to solve (3.19) in order to prove Theorem 2.1.

By the definition of the stream function $\psi$, the slip condition (2.11) becomes the Dirichlet boundary condition for equation (3.19):

$$\psi|_{\partial \Omega} = 0. \quad (3.27)$$

We define $\Sigma$ as a set for the solutions of (3.19):

$$\Sigma = \{ u : \| u - l \|_{2,\alpha;2,\beta;\Omega} \leq C^* m_0 \}; \quad (3.28)$$

where constant $C^*$, depending on $m_*, p_0, \rho_0, \alpha, \beta, \delta, \mathcal{A}$, will be determined later in the estimates.

We state the following theorem, which implies Theorem 2.1.

**Theorem 3.1.** For sufficiently small $m_0$, depending on $m_*, p_0, \rho_0, \alpha, \beta, \delta, \mathcal{A}$, and sufficiently small $\varepsilon$, depending on $m_*, p_0, \rho_0, \alpha, \beta, \delta, \mathcal{A}$ and $m_0$, there exists a unique solution for equation (3.19) with boundary condition (3.27) in the set $\Sigma$ defined in (3.28).

We split the proof of Theorem 3.1 into the following steps:

1. Use bounded domain $\Omega_R := \Omega \cap B_R(O)$ to approach $\Omega$, where $B_R(O)$ is the ball with radius $R$ and centered at the origin. We linearize equation (3.19) and solve the linear equation in $\Omega_R$. 


With proper estimates for the linear equation, we solve the nonlinear equation (3.19) in bounded domain $\Omega_R$ using Schauder fixed point theorem.

Let $R \to \infty$, we prove the existence of solution for (3.19) in $\Omega_R$. Estimate the difference of any two solutions in $\Omega_R$ and then let $R \to \infty$ to obtain the uniqueness of the solution for (3.19).

Henceafter, we will use $C$ to denote generic constants, depending on the fixed data $m_*, p_0, \rho_0, \alpha, \beta, \delta$, and the profile $A$, but independent of $m_0, R$.

4. Estimates of a linear equation

In this section, we study a linear elliptic equation

$$a_{ij}(x)u_{x_ix_j} + b_i(x)u_{x_i} + b_0(x)u = 0,$$  \hspace{1em} (4.1)

in domain $\Omega_R$. The estimates of this equation will be used later for the linearized equation. For equation (4.1), we have the following assumptions for the coefficients:

$$a_{ij}(x)\xi_i\xi_j \geq \lambda(\xi_1^2 + \xi_2^2) \text{ for any } \xi_i \in \mathbb{R},$$  \hspace{1em} (4.2)

$$\|a_{ij} - \varepsilon \delta_{ij}\|_{C^\alpha(\Omega_R)} + \sum_{i=1}^3 \|b_i\|_{C^\alpha(\Omega_R)} \leq Cm_0,$$  \hspace{1em} (4.3)

$$(x_2 + 1)(|b_1| + |b_2|) + (x_2 + 1)^2|b_0| \leq Cm_0,$$  \hspace{1em} (4.4)

where $\lambda, \varepsilon$ are constants depending on $p_0, \rho_0$, and $\delta_{ij} = 1$ for $i = j$, otherwise $\delta_{ij} = 0$.

We let $R > D_0 + 1$ so that the boundary of $\Omega_R$ includes the whole profile $A$. Let $S_R = \{|x| = R\} \cap \partial \Omega_R$. Now we not only have the corner points $A_-, A_+$, but also have additional corners as the intersection of $S_R$ with $\partial \Omega$. These two points are denoted by

$$S_-^R = \Gamma_- \cap S_R, \quad S_+^R = \Gamma_+ \cap S_R.$$

Then we define the set of boundary points for the weight as:

$$\tilde{P} = \{A_-, A_+, S_-^R, S_+^R\}.$$

The boundary condition for (4.1) is:

$$u|_{\partial \Omega_R} = g,$$  \hspace{1em} (4.5)

where

$$\|g\|_{2, \alpha; (\alpha + \beta); \Omega_R} \leq Cm_0.$$  \hspace{1em} (4.6)

We then have the following lemma
Lemma 4.1. Suppose \( u \in C^{2,\alpha}(\Omega_R) \cap C(\overline{\Omega_R}) \) is a solution for (4.1) with boundary condition (4.5) and assumptions (4.2)–(4.6) hold. For sufficiently small \( m_0 \) independent of \( R \), we have the following estimate for \( u \):

\[
\|u\|_{2,\alpha(\beta); \Omega_R} \leq C^* m_0,
\]

where \( C^* \) is a constant independent of \( m_0, R \).

To prove Lemma 4.1, we need a maximum principle for the elliptic equation (4.1) without restriction on the sign for \( b_0 \). We take the following lemma from [24] (Theorem 2.11):

Lemma 4.2. Let elliptic operator \( L = a_{ij} \partial_{x_i} \partial_{x_j} + b_i \partial_{x_i} + b_0 \). For any bounded connected domain \( D \), assume \( a_{ij}, b_i \in C^0(\overline{D}) \) and \( a_{ij} \) satisfies ellipticity condition (4.2). Suppose there exists a function \( v \in C^2(D) \cap C^1(\overline{D}) \) such that \( v > 0 \) in \( D \) and \( Lv \leq 0 \) in \( D \). Suppose \( u \in C^2(D) \cap C(\overline{D}) \) satisfies \( Lu \geq 0 \) in \( D \). Then \( u/v \) achieves its nonnegative maximum on the boundary \( \partial D \).

With this maximum principle, we start to prove Lemma 4.1.

Proof. Notice that there are two different weights in the norm in (4.7): the weight with upper index \(-\alpha - 1\) is for the small scale near the corner points \( A_-, A_+, S_R^-, S_R^+ \) and the weight with lower index \( \beta \) is for the large scale away from the origin. We split the proof into three parts: Part 1 is for the estimate of maximum norm of \( u \) in the whole domain \( \Omega_R \); Part 2 is for the region near \( \mathcal{A} \); and Part 3 is for the region far away from the profile \( \mathcal{A} \). Let \( D = 2D_0 + 1 \), where \( D_0 \) is the radius to bound the profile \( \mathcal{A} \). Let \( \Omega_D = \Omega_R \cap \{ |x| \leq D \} \) be the region for Part 2, and \( \Omega_D^c = \Omega_R \cap \{ |x| > D - 1 \} \) for Part 3.

Part 1. In this part, we first construct a comparison function \( v \) and use the maximum principle (Lemma 4.2) in the whole domain \( \Omega_R \) to obtain the control of the maximum norm of \( u \).

Define the comparison function \( v \) by

\[
v(x) = r^{-\alpha-\beta}(x_2 + 1)^{\alpha},
\]

where \( r = \sqrt{x_1^2 + (x_2 + 1)^2} \).

We now verify the fact that \( Lv < 0 \).

First, it is easy to compute

\[
\Delta v = (\beta^2 - \alpha^2)r^{-\alpha-\beta-2}(x_2 + 1)^{\alpha} \\
-\alpha(1 - \alpha)r^{-\alpha-\beta}(x_2 + 1)^{\alpha-2} \\
\leq -\alpha(1 - \alpha)r^{-\alpha-\beta}(x_2 + 1)^{\alpha-2},
\]

(4.9)
noticing that $0 < \beta < \alpha < 1$. Also, one can verify that
\begin{align}
|Dv| &\leq Cr^{-\alpha - \beta}(x_2 + 1)^{\alpha - 1}, \\
|D^2v| &\leq Cr^{-\alpha - \beta}(x_2 + 1)^{\alpha - 2}.
\end{align}
(4.10)
(4.11)

We rewrite $Lv$ as
\[ Lv = (L - e\Delta)v + e\Delta v. \]

By assumptions (4.3) and (4.4), together with (4.10) and (4.11), we have
\[ |(L - e\Delta)v| \leq Cm_0r^{-\alpha - \beta}(x_2 + 1)^{\alpha - 2}. \]

The above estimate and (4.9) imply that $Lv < 0$ in $\Omega_R$, provided $m_0$ is small enough. Obviously, $v$ is positive. Hence, by the maximum principle (Lemma 4.2), we conclude that
\[ \frac{u}{v} \leq \max_{\partial\Omega_R} \frac{|g|}{v}. \]

By replacing $v$ with $-v$ and using Lemma 4.2 again, we have
\[ |u/v| \leq \max_{\partial\Omega_R} |g/v|. \]

This, with assumption (4.6), implies
\[ |u(x)| \leq Cm_0r^{-\beta}. \]
(4.12)

**Part 2.** For the region near the profile $A$, we need to take care of the corner points $A_-, A_+$. We use the weight up to $P$ and drop the lower index $\beta$ for the weight away from $A$. We treat the corner $A_-$ first, and $A_+$ can be dealt in the same way. For convenience, we move $A_-$ to the origin $O$. Assume the angle between $\Gamma_-$ and $x_1$-axis at $O$ (original $A_-$) is $\theta_-$, and the angle between $A$ and $x_1$-axis at $O$ is $\theta_0$. Therefore, the tangential directions of $\Gamma_-$ and $A$ at $O$ are
\[ \nu_- = (\cos \theta_-, \sin \theta_-), \quad \nu_0 = (\cos \theta_0, \sin \theta_0), \]
respectively. Let $\bar{u} = u - g(O) - c_1x_1 - c_2x_2$, where $c_1, c_2$ are linear combinations of $\frac{\partial g}{\partial \nu_-}(O)$ and $\frac{\partial g}{\partial \nu_0}(O)$ through solving the linear system
\[ \begin{cases}
(c_1, c_2) \cdot \nu_- = \frac{\partial g}{\partial \nu_-}(O) \\
(c_1, c_2) \cdot \nu_0 = \frac{\partial g}{\partial \nu_0}(O).
\end{cases} \]

Hence
\[ |c_1| + |c_2| \leq C|Dg(O)|, \]
and
\[ \bar{u}(O) = 0, \quad D\bar{u}(O) = (0, 0). \]
(4.13)
Choose $r_0 > 0$ small enough, such that $\Omega_D \cap B_{r_0}(O)$ is connected. Thus, for this fixed radius $r_0 < \min(D_0, 1)$,

$$
\bar{u}(x)|_{\partial \Omega_D \cap B_{r_0}(O)} \leq Cm_0|x|^{1+\alpha}.
$$

We know $\bar{u}$ satisfies the following equation

$$
\bar{L}\bar{u} \equiv a_{ij}\partial_i\partial_j \bar{u} + b_i\partial_i \bar{u} = F_0, \quad (4.14)
$$

where $F_0 = -b_ic_i - b_0u$. By estimate (4.12) and condition (4.4), we have

$$
|F_0| \leq Cm_0.
$$

Notice the elliptic operator in (4.14) does not contain $b_0\bar{u}$ term. So we can use standard maximum principle to control $\bar{u}$. The comparison function for $\bar{u}$ is defined in polar coordinates $(r, \theta)$ by

$$
v_1(r, \theta) = Cm_0r^{1+\alpha}\sin(\tau + (\theta - \theta_0)),
$$

for small positive $\tau$ depending on $\theta_- - \theta_0$ and $\alpha$. One can check that $\bar{L}v_1 < -Cm_0 < F_0 = \bar{L}\bar{u}$.

Also the boundary condition satisfies

$$
v_1|_{\partial(\Omega_D \cap B_{r_0}(O))} > Cm_0r^{1+\alpha} > \bar{u}|_{\partial(\Omega_D \cap B_{r_0}(O))}.
$$

By maximum principle, we conclude that

$$
|\bar{u}| \leq Cm_0r^{1+\alpha},
$$

for $|x| < r_0$.

Once we have the above estimate near the corner $A_-$, we use Schauder estimates with proper scaling to obtain the estimate near the corner $A_-(O)$:

$$
\|u\|_{2,\alpha;\Omega_D \cap B_{r_0}(A_-)} \leq Cm_0. \quad (4.15)
$$

The procedure is standard and related details can be found in chapter 6 of [22]. One can also refer to [5] (Lemma 4.2) for similar scaling argument. We sketch the proof as follows.

For any $x_0 \in \Omega_D \cap B_{r_0/2}$, let the angle between $\Gamma_-$ and the ray $A_-x_0$ be $\theta_{x_0}$, and the angle between $\Gamma_-$ and $A$ be $\theta^0$. Consider two cases: Case 1, $\theta_{x_0} > \pi/6$ and $\theta_- - \theta_{x_0} > \pi/6$; Case 2, otherwise. For Case 1, we know that $\bar{u}$ satisfies equation

$$
\bar{L}\bar{u} = -b_ic_i - b_0(g(O) + c_1x_1 + c_2x_2).
$$

Take the ball $B_{|x_0|/2}(x_0) \subset \Omega_D \cap B_{r_0/2}$ as the domain and by Schauder interior estimate (Theorem 6.2, [22]), we have

$$
\|\bar{u}\|_{2,\alpha;B_{|x_0|/2}(x_0)} \leq Cm_0|x_0|^{\alpha+1}. \quad (4.16)
$$
Here the upper index (0) is understood as the weight up to \( \partial B_{|x_0|}(x_0) \).

For Case 2, let \( x^* \) be a boundary point with the shortest distance from \( x_0 \), and \( d^* = |x_0| \sin(\frac{3}{4}\theta_0) \). Hence, \( B_{2d^*}(x^*) \) still contains \( x^* \).

We use Schauder boundary estimate (Lemma 6.4, [22]) in the domain \( \Omega_D \cap B_{d^*}(x^*) \) to obtain the estimate:

\[
\| \bar{u} \|_{2,\alpha; B_{d^*}(x^*)}^{(0)} \leq C m_0 |x_0|^{\alpha+1}.
\]  

(4.17)

Combining (4.16) and (4.17) gives the corner estimate (4.15). The other corner \( A_+ \) is treated in the same way. Together with standard Schauder estimates away from the corners, we conclude the estimate in \( \Omega_D \):

\[
\| u \|_{2,\alpha,\Omega_D}^{(-\alpha-1; P)} \leq C m_0.
\]  

(4.18)

Part 3. For the domain \( \Omega^c_D \), we also consider two kinds of estimates: one is near the corner points \( S_R^-, S_R^+ \), the other is away from the corners.

The corner estimates are similar to those in Part 2. In brief, consider the corner \( S_R^- \) for instance. If \( x_0 \in \Omega^c_D \cap B_{r_0}(S_R^-) \), we have the following estimate

\[
\| u \|_{2,\alpha,\Omega_D \cap B_{r_0}(S_R^-)}^{(-1-\alpha; \{S_R^-, S_R^+\})} \leq C m_0 R^{-\beta}.
\]  

(4.19)

If \( x_0 \in \Omega^c_D \) and \( |x_0| < R/2 \), the ball \( B_{\frac{|x_0|}{2}}(x_0) \) has no intersection with the outer boundary \( S_R = \{|x| = R\} \) or the profile \( A \). Using conditions (4.2), (4.3), (4.6) and estimate (4.12), by Schauder interior estimates (see Theorem 6.2 in [22]), we have

\[
\| u \|_{2,\alpha; B_{|x_0|}(x_0)} \leq C m_0,
\]  

(4.20)

where no upper index in the norm means no weight up to \( \hat{P} \). For \( |x_0| \geq R/2 \), we use Schauder boundary estimates (Lemma 6.4 in [22]) with the boundary condition (4.6) and estimate (4.19) to obtain

\[
\| u \|_{2,\alpha; B_{|x_0|}(x_0) \cap \Omega^c_D}^{(-1-\alpha; \{S_R^-, S_R^+\})} \leq C m_0.
\]  

(4.21)

Estimates (4.18), (4.20) and (4.21) imply estimate (4.7) in the lemma. \( \square \)

By the continuity method, one can prove the existence of solutions for (4.1) with estimate (4.7). The uniqueness is simply the result of the maximum principle, Lemma 4.2 with the aid of the comparison function \( v \) constructed in Lemma 4.1. Since the procedure is standard, we omit the proof and only state the result as follows.
Lemma 4.3. Assume (4.2)–(4.6) holds. For sufficiently small $m_0$, equation (4.11) with boundary condition (4.5) admits a unique solution $u \in C^2(\Omega_R) \cap C(\overline{\Omega_R})$.

5. NONLINEAR EQUATION IN BOUNDED DOMAIN

In this section, we will solve equation (3.19) in the bounded domain $\Omega_R$ with boundary condition given below.

We want to prescribe the boundary data for $\psi$ such that (3.27) hold on $\partial \Omega \cap \Omega_R$ and $\psi - l$ vanishes away from $\partial \Omega$. We will define function $g$ such that

$$g(x) = -\eta(x_2) \left( (1 - \eta(x_1)) l(f_{\text{sign}(x_1)}(x_1)) + \eta(x_1) l(x_2) \right),$$  \hspace{1cm} (5.1)

where $\text{sign}(x_1) = -$ for $x_1 < 0$ and $\text{sign}(x_1) = +$ for $x_1 > 0$.

First it is easy to construct a smooth cutoff function $\eta(s)$ such that $\eta(s) = 1$ for $|s| \leq D_0$ and $\eta(s) = 0$ for $|s| \geq D_0 + 1$. We also assume that $\|\eta\|_{C^{2,\alpha}(\mathbb{R})} \leq 10$. Let

$$g(x) = -\eta(x_2) \left( (1 - \eta(x_1)) l(f_{\text{sign}(x_1)}(x_1)) + \eta(x_1) l(x_2) \right),$$  \hspace{1cm} (5.2)

where $\text{sign}(x_1) = -$ for $x_1 < 0$ and $\text{sign}(x_1) = +$ for $x_1 > 0$.

It is easy to check that $g|_{\partial \Omega} = -l|_{\partial \Omega}$, $g = 0$ for $x_2 > D_0 + 1$, and also $g$ satisfies condition (4.6). Now, let constant $C^*$ in (3.28) be the same as in estimate (4.7) in Lemma 4.1. The set for solutions of (3.19) in $\Omega_R$ is

$$\Sigma_R = \{ u : \|u - l\|_{2,\alpha;\partial \Omega_R} \leq C^* m_0 \}. \hspace{1cm} (5.3)$$

We state our lemma for the solution of (3.19) in $\Omega_R$:

**Lemma 5.1.** Equation (3.19) in $\Omega_R$ with boundary condition (5.1) admits a unique solution in the set $\Sigma_R$, provided $m_0, \varepsilon$ are sufficiently small.

**Proof.** To prove the lemma, we first linearize the nonlinear equation (3.19). By solving the linearized equation, we construct a map $T$ in the set $\Sigma_R$. The solution of the nonlinear equation (3.19) is a fixed point of $T$.

We start the proof with the linearization of (3.19). We know that the limit function $l$ satisfies (3.19) by its definition. That means the following equation holds:

$$a_{ij}(l, 0, l') l_{x_i x_j} = F(l, 0, l'),$$  \hspace{1cm} (5.4)

where $a_{ij}, F$ are defined in (3.14)–(3.16). Taking the difference of equations (3.19) and (5.4) leads to

$$a_{ij}(\psi, \nabla \psi)(\psi - l)_{x_i x_j} + l''(a_{22}(\psi, \nabla \psi) - a_{22}(l, 0, l')) = F(\psi, \nabla \psi) - F(l, 0, l').$$  \hspace{1cm} (5.5)
Denote \((l + s(\psi - l), \nabla(l + s(\psi - l)))\) by \(t_s^\psi\), and let

\[
\begin{align*}
a_{ij}^\psi &= a_{ij}(\psi, \nabla \psi), \\
b_0^\psi &= l'' \int_0^1 (a_{22})_\psi(t_s^\psi) ds - \int_0^1 F_\psi(t_s^\psi) ds, \\
(b_1^\psi, b_2^\psi) &= l'' \int_0^1 (a_{22})_{\nabla \psi}(t_s^\psi) ds - \int_0^1 F_{\nabla \psi}(t_s^\psi) ds.
\end{align*}
\]

Then we linearize equation (5.5) as follows:

\[
a_{ij}^\psi(\tilde{\psi} - l)_{x_i x_j} + b_0^\psi(\tilde{\psi} - l) + b_i^\psi(\tilde{\psi} - l) = 0.
\]

We solve the above equation by applying Lemma 4.1. In the following, we will check the conditions (4.2)–(4.4) for \(\psi \in \Sigma_R\). In fact, (4.2) will be satisfied if (4.3) holds with sufficiently small \(m_0\). We let \(e\) in (4.3) be \(a_{11}(m_0 x_2, 0, m_0) = a_{22}(m_0 x_2, 0, m_0) = \gamma p_0 m_0\). By the expressions for \(a_{ij}\), (3.14)–(3.16), and (3.10), (3.9), it is not hard to verify that

\[
\begin{align*}
\| (a_{ij})_\psi(t_s^\psi) \|_{C^0(\Omega_R)} &\leq \frac{\varepsilon}{m_0} CC^*, \\
\| (a_{ij})_{\nabla \psi}(t_s^\psi) \|_{C^0(\Omega_R)} &\leq \frac{\varepsilon}{m_0} CC^*,
\end{align*}
\]

for \(\psi \in \Sigma\) and \(m_0\) small.

In the proof of this lemma, the generic constants \(C\) are independent of \(C^*\).

Let \(\varepsilon < \frac{m_0}{(C^*)^2}\), we have

\[
\begin{align*}
\| a_{ij}^\psi - e\hat{\delta}_{ij} \|_{C^0(\Omega_R)} &\leq \frac{m_0}{(C^*)^2} \int_0^1 \| (a_{ij})_\psi(t_s^\psi) \|_{C^0(\Omega_R)} ds \| \psi - m_0 x_2 \|_{C^0(\Omega_R)} \\
&\quad + \int_0^1 \| (a_{ij})_{\nabla \psi}(t_s^\psi) \|_{C^0(\Omega_R)} ds \| \nabla(\psi - m_0 x_2) \|_{C^0(\Omega_R)} \\
&\leq \varepsilon C(C^*)^2 \\
&\leq C m_0.
\end{align*}
\]

In the same manner, we can obtain

\[
\| b_i \|_{C^0(\Omega_R)} \leq C m_0, \quad i = 0, 1, 2.
\]

The above estimates lead to condition (4.3). Now we verify condition (4.4). For \(b_0^\psi\), by its expression, we need to estimate \((a_{22})_\psi\) and \(F_\psi\). By the definition of \(F\), (3.17), and estimates (3.25), (3.26), we have

\[
|F_\psi(\psi, \nabla \psi)| \leq \frac{\varepsilon m_0 C}{(m_0 + |\psi|)^2}.
\]
Let \( u_s = l + s(\psi - l) \). For any \( \psi \in \Sigma \), we consider two cases: **Case 1**, \( |\psi - l| \leq \frac{1}{4}m_0x_2 \); **Case 2**, otherwise. For **Case 1**,
\[
u_s \geq l - \frac{1}{4}m_0x_2 \geq \frac{1}{4}m_0x_2,
\]
noticing (3.22). Therefore,
\[
|F_\psi(t_s^\psi)| = |F_\psi(u_s, \nabla u_s)| \leq \frac{\varepsilon m_0 C}{(m_0 + |u_s|)^2} \leq \frac{\varepsilon C m_0}{(1 + x_2)^2}.
\]
for \( \varepsilon < m_0^2 \). For **Case 2**, i.e., \( |\psi - l| > \frac{1}{4}m_0x_2 \), since \( \psi \in \Sigma_R \), we have
\[
\frac{1}{4}m_0x_2 < |\psi - l| \leq \frac{C^* m_0}{|x|^\beta}.
\]
This implies that \( x_2 < (4C^*)^{-\frac{1}{\beta + 1}} \equiv R_0 \). Hence, for \( \varepsilon < (m_0/(1 + R_0))^2 \), we have
\[
|F_\psi(t_s^\psi)| \leq \frac{\varepsilon C}{m_0} \leq \frac{\varepsilon C m_0(1 + R_0)^2}{(m_0)^2(1 + x_2)^2} < \frac{C m_0}{(1 + x_2)^2}.
\]
The above analysis about both **Case 1** and **Case 2** gives rise to
\[
\left| \int_0^1 F_\psi(t_s^\psi) ds \right| \leq \frac{C m_0}{(1 + x_2)^2}. \tag{5.10}
\]
Similarly, we have
\[
|(a_{22})_\psi(t_s^\psi)| \leq \frac{C m_0}{1 + x_2}.
\]
Together with the fact that \( |l''(x_2)| < \frac{\varepsilon C m_0}{1 + x_2^2} \), we conclude that
\[
|b_0^\psi| < \frac{C m_0}{(1 + x_2)^2}.
\]
Same arguments apply to the estimates for \( b_1^\psi, b_2^\psi \):
\[
|b_1^\psi| + |b_2^\psi| < \frac{C m_0}{1 + x_2}.
\]
Therefore, we have verified condition (4.4). Thus, we apply Lemma 4.1 to solve (5.9) for \( \tilde{\psi} \), where \( u = \tilde{\psi} - l \) in Lemma 4.1. By estimate (4.7), we know that \( \tilde{\psi} \in \Sigma_R \). Therefore, we can define a map \( T \) from \( \Sigma_R \) to itself by \( T\psi \equiv \tilde{\psi} \). It is obvious that a solution for the nonlinear equation (3.19) is a fixed point of \( T \). In order to prove the existence of a fixed point of \( T \), we apply Schauder fixed point theorem, which says: if \( \Sigma_R \) is a compact convex set of a Banach space \( \mathcal{B} \), and \( T : \Sigma_R \rightarrow \Sigma_R \) is continuous in \( \mathcal{B} \), \( T \) has a fixed point.
Hence, we verified the continuity of $T(5.9)$. This contradicts with the uniqueness of the solution for $(5.9)$. 

Let $\| \cdot \|_2$ be the weighed norm $\| \cdot \|_{2, \alpha'(\beta), \Omega_R}$ and $c_0$ be a fixed constant. Since $\{T\psi_n\}$ is compact in $B$, there exists a subsequence, still denoted by $\{T\psi_n\}$, convergent to $\bar{\psi} \in \Sigma$. On the other hand, $\psi_n \to \psi$ implies that $a_{ij}^{\psi_n} \to a_{ij}^{\psi}(i, j = 1, 2)$, $b_i^{\psi_n} \to b_i^{\psi}(i = 0, 1, 2)$ in $C^{\alpha'}$. Let $k \to \infty$ and we see that $\bar{\psi}$ is also a solution of $(5.9)$ with the same boundary condition $(4.5)$. Inequality $(5.11)$ implies that $\| \bar{\psi} - \tilde{\psi} \|_2 \geq c_0 > 0$, which means $\bar{\psi}, \tilde{\psi}$ are two distinct solutions for $(5.9)$. This contradicts with the uniqueness of the solution for $(5.9)$. Hence, we verified the continuity of $T$ in $\Sigma_R$.

By Schauder fixed point theorem, there exists a fixed point $\psi_R$ for $T$. So $\psi_R$ is a solution for the nonlinear equation $(3.19)$ in $\Omega_R$ with boundary condition $(5.1)$. The uniqueness of the solution for $(3.19)$ is proved by the maximum principle Lemma 4.2 as follows.

For any two solutions $\psi_1, \psi_2 \in \Sigma_R$ of $(3.19)$, $(5.1)$, we take the difference of the two equations and obtain

$$a_{ij}^{\psi_1}(\psi_1 - \psi_2)_{x_i x_j} + (\psi_2)_{x_i x_j}(a_{ij}^{\psi_1} - a_{ij}^{\psi_2}) = F(\psi_1, \nabla \psi_1) - F(\psi_2, \nabla \psi_2).$$

Similar to the notations in $(5.3)$ - $(5.8)$, we set

$$t_s = (\psi_2 + s(\psi_1 - \psi_2), \nabla(\psi_2 + s(\psi_1 - \psi_2)))$$

and let $u = \psi_1 - \psi_2$. Then we derive the following equation:

$$a_{ij}^{\psi_s}u_{x_i x_j} + b_i u_{x_i} + b_0 u = 0,$$  \hspace{1cm} (5.12)

where

$$b_0 = (\psi_2)_{x_i x_j} \int_0^1 (a_{ij})_{\psi}(t_s) ds - \int_0^1 F_{\psi}(t_s) ds, \hspace{1cm} (5.13)$$

and

$$(b_1, b_2) = (\psi_2)_{x_i x_j} \int_0^1 (a_{ij})_{\nabla \psi}(t_s) ds - \int_0^1 F_{\nabla \psi}(t_s) ds. \hspace{1cm} (5.14)$$

Notice that the factor $(\psi_2)_{x_i x_j}$ in $(5.13)$ and $(5.14)$ blows up at the corner points in $\bar{P} = \{A_-, A_+, S^R_-, S^R_+\}$. Lemma 4.2 requires continuity of coefficients $a_{ij}, b_i$ up to the boundary. Therefore, in order to apply
the lemma, we truncate small neighborhoods around corners from $\Omega_R$. Define

$$B_{\mathcal{P},r} = \bigcup_{I \in \mathcal{P}} B_r(I),$$

(5.15)

where $\mathcal{P}$ is a set of boundary points on $\Omega_R$. Let

$$\Omega_{\mathcal{P},r}^R = \Omega_R \setminus \overline{B_{\mathcal{P},r}}, \quad S_{\mathcal{P},r} = \partial B_{\mathcal{P},r} \cap \Omega_R.$$  

(5.16)

Now, we know $b_i \in C^\alpha(\overline{\Omega}_{\mathcal{P},r}^R)$, $i = 0, 1, 2$. Define $r_I = |x - I|$, for any corner point $I \in \tilde{\mathcal{P}}$. Then we have the following estimates for $b_i$:

$$(x_2 + 1)(|b_1| + |b_2|) + (x_2 + 1)^2|b_0| \leq Cm_0(1 + \sum_{I \in \tilde{\mathcal{P}}} r_I^{\alpha - 1}).$$

(5.17)

We construct a comparison function $\tilde{v}$ as follows: Let

$$v_I = r_I^{-\frac{3}{2}\beta}(x_2 + 1)^{\frac{\beta}{2}}.$$  

(5.18)

Define $\tilde{v} = \sum_{I \in \tilde{\mathcal{P}}} v_I$. By (4.9)–(4.11), we can verify that

$$a_{ij}(v_I)_{x_ix_j} \leq -c_0(r_I^{-\frac{3}{2}\beta - 2}(x_2 + 1)^{\frac{\beta}{2}} + r_I^{-\frac{3}{2}\beta}(x_2 + 1)^{\frac{\beta}{2} - 2}),$$

(5.19)

$$\sum_{i=1,2} b_i(v_I)_{x_i} + b_0v_I \leq Cm_0(1 + \sum_{I \in \tilde{\mathcal{P}}} r_I^{\alpha - 1})r_I^{-\frac{3}{2}\beta}(x_2 + 1)^{\frac{\beta}{2} - 2},$$

(5.20)

where $c_0 > 0$ is a constant only depending on $\beta$. Inequalities (5.19) and (5.20) directly imply that

$$a_{ij}(v_I)_{x_ix_j} + b_i v_{x_i} + b_0 \tilde{v} < 0.$$  

By Lemma 4.2, we have

$$\sup_{\Omega_{\mathcal{P},r}^R} \frac{|u|}{\tilde{v}} = \max_{\partial \Omega_{\mathcal{P},r}^R} \frac{|u|}{\tilde{v}}.$$  

(5.21)

Since $u|_{\partial \Omega_R} = 0$ and $u \in C^{1,\alpha}(\Omega_R)$, we know that

$$\max_{\partial \Omega_{\mathcal{P},r}^R} \frac{|u|}{\tilde{v}} = \max_{S_{\mathcal{P},r}} \frac{|u|}{\tilde{v}} \leq Cr^\alpha R^{\frac{3}{2}\beta}.$$  

Hence, we have

$$\sup_{\Omega_{\mathcal{P},r}^R} |u| \leq CR^{\frac{3}{2}\beta}r^\alpha \sup_{\Omega_{\mathcal{P},r}^R} \tilde{v} \leq CR^{\frac{3}{2}\beta}r^\alpha(1 + r^{-\frac{3}{2}\beta}).$$

Therefore, by letting $r \to 0$, the above inequality implies $\sup_{\Omega_R^R} |u| \leq 0$. This shows the uniqueness of the solution for (3.19) in $\Omega_R$. Hence, the proof of this lemma is complete.

\[\square\]
6. Subsonic Flow in Half Plane

After we solve (3.19) in $\Omega_R$, we let $R$ tend to infinity to prove Theorem 3.1 as follows.

**Proof of Theorem 3.1.** By Lemma 5.1, for a given radius $R$, we can find a unique solution $\psi_R \in \Sigma_R$. By a diagonal process, one can choose proper sequence $R_n \to \infty$ as $n \to \infty$, such that $\psi_{R_n}$ converges to some function $\psi$ in $\| \cdot \|_{2,\alpha;\Omega}^{(-\alpha-1;P)}$ norm, for any fixed $Q > D_0 + 1$. Since $\psi_{R_n} \in \Sigma_{R_n}$, we have

$$\|\psi_{R_n} - l\|_{2,\alpha;\Omega}^{(-\alpha-1;P)} \leq C^*m_0, \quad (6.1)$$

for any $R_n > 2(D_0 + 1)$. Let $n \to \infty$ in (6.1), we obtain estimate

$$\|\psi - l\|_{2,\alpha;\Omega}^{(-\alpha-1;P)} \leq C^*m_0, \quad (6.2)$$

which implies $\psi \in \Sigma$. This completes the existence of solutions for (3.19).

To prove the uniqueness of the solution, we will use the asymptotic behavior of solutions described by set $\Sigma$. We still use the truncated domain $\Omega_R$ and follow the same strategy as in the uniqueness part of Lemma 5.1. Now, the situation here is slightly different from that in Lemma 5.1: (1) we have no singularity for $(\psi_2)_{x_i x_j}$ at corners $S_{R^-}, S_{R^+}$; (2) $u = \psi_1 - \psi_2$ does not vanish on the boundary portion $S_R = \{ |x| = R \} \cap \Omega$.

Similarly as in Lemma 5.1, we have the same equation (5.12) for $u$. We define the comparison function $\bar{v}$ by $\bar{v} = v_{A_-} + v_{A_+}$, where $v_{A_-}, v_{A_+}$ are defined in (5.18). The domain we consider here is $\Omega^{P,r}_R$ defined by (5.15). By the same computation as in Lemma 5.1, we have

$$a_{ij}^{\psi_k} v_{x_i x_j} + b_i v_{x_i} + b_0 \bar{v} < 0.$$ We know that $|u| \leq CR^{-\beta}$ on $S_R$ by the definition of $\Sigma$. Also, we see

$$|u|_{S_{P,r}} \leq Cr^{\alpha}, \quad \bar{v}|_{S_R} \geq R^{-\frac{3}{2}\beta}, \quad \bar{v}|_{S_{P,r}} \geq r^{-\frac{3}{4}\beta}.$$ Therefore, we conclude that

$$\sup_{\Omega^{P,r}_R} \frac{|u|}{\bar{v}} \leq C \max\left(r^{\alpha + \frac{3}{2}\beta}, R^{-\frac{1}{4}\beta}\right). \quad (6.3)$$

Let $r = R^{-\frac{1}{2}}$ and we obtain

$$\sup_{\Omega^{P,r}_R} |u| \leq CR^{-\frac{1}{2}\beta} \sup_{\Omega^{P,r}_R} \bar{v} \leq CR^{-\frac{1}{4}\beta}(1 + R^\beta) \leq CR^{-\frac{\beta}{2}}.$$
Letting $R \to \infty$ gives rise to $\sup_\Omega |u| \leq 0$, which implies the uniqueness of the solution for (3.19) in $\Sigma$. This finishes the proof of Theorem 3.1.

Once we proved Theorem 3.1 define $U = (m, p, \rho)$ by

$$m = (\psi_{x_2}, -\psi_{x_1}), \quad \rho = \rho(\psi, \nabla \psi), \quad p = \gamma^{-1} A(\psi) \rho^\gamma,$$

where $\rho$ is uniquely solved from Bernoulli’s law (3.5). Hence, by (3.13), (3.18) and (3.5), we can recover the original Euler equations (1.1). It is easy to check that $U$ satisfies (2.13) with the aid of estimate (6.2).

**Remark 6.1.** The uniqueness of the solution for Euler system (1.1) can not be obtained from Theorem 3.1 due to two obstacles. One is the existence of stagnation points, which disqualifies equivalence between equation (3.19) and the two momentum equations (3.13), (3.18). The corners $A_-, A_+$ on $\partial \Omega$ are necessarily stagnation points, because $\nabla \psi$ is continuous up to the corners. Whether or what kind of stagnation points may appear inside domain $\Omega$ is not clear. The other problem is about the complexity of streamline topology. During the reduction in section 3 we assume streamlines have simple topology, which means that streamlines in $\Omega$ extend from $-\infty$ to $\infty$ in $x_1$, so that information about $A, B$ can be carried along streamlines and reach the whole domain $\Omega$. However, the geometry of profile $A$ may be complicated and cause nontrivial topology of streamlines, such as closed orbits or intersection of streamlines at stagnation points. The above reasons prevent us from obtaining the uniqueness for the Euler flows out of Theorem 3.1.

**Acknowledgments.** I would like to thank Gui-Qiang Chen, Mikhail Feldman and Marshall Slemrod for their constructive suggestions and stimulating discussions. The paper was supported in part by the National Science Foundation under Grants DMS-0354729.

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