STATISTICS OF N-BODY SIMULATIONS
I. EQUAL MASSES BEFORE CORE COLLAPSE

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ABSTRACT
We study the dynamical evolution of idealised stellar systems by averaging results from many \( N \)-body simulations, each having modest numbers of stars. For isolated systems with stars of uniform mass, we discuss aspects of evolution up to the point of core collapse: relaxation and its \( N \)-dependence, the evolution of the density profile, the development of the velocity dispersion and anisotropy, and the rate of stellar escape.

**Key words:** celestial mechanics, stellar dynamics - globular clusters: general.

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1 INTRODUCTION: THE STATISTICAL STUDY OF N-BODY SYSTEMS

There have been dramatic improvements in the performance of astrophysical N-body simulations since the pioneering work of von Hoerner (1960). Several, e.g. regularisation of close encounters, have taken place in the domain of software, i.e. developments of the programs used. Many of these are documented in Aarseth (1985), and indeed due to him, but others are beginning to have an impact on the field of collisional stellar dynamics, including tree codes (Barnes & Hut 1986, McMillan & Aarseth 1993).

Of comparable importance are improvements in hardware, many of which have been exploited quickly into collisional stellar dynamics. Among these are special-purpose devices such as Grape-2 and its descendants (Ito et al. 1991), and general-purpose machines such as the Cray Series supercomputers. The latter have, for example, been used effectively by McMillan, Hut & Makino (1990) in their studies of systems with primordial binaries. Even so, dedicated workstations are still competitive (Aarseth & Heggie 1992).

A further group of hardware devices suitable for stellar dynamics are parallel processors, such as the Connection Machine (Hut & Makino 1989). Other machines which have been used for problems in stellar dynamics include the Distributed Array Processor, which was used in a study of gravothermal behaviour (Heggie 1989), and transputer arrays (Sweatman 1993a). Similar machines with faster (vector) processors are now available, and have been used in the project described later in this paper.

The improvement in computational power over the last three decades can be exploited in several different ways. One aim is to study larger systems. This is driven by the fact that real globular clusters have many more stars than the largest useful computations, by at least an order of magnitude. It is not at all clear that this is necessarily the best way of using increased computing power, as the discussion of the following section shows.

Another aim is simply to carry out calculations more quickly. For example, workstation calculations on systems with 2500 stars and a substantial proportion of primordial binaries may still require as much as 2000 cpu hours (Aarseth & Heggie 1992).

A third aim is to exploit the power to produce better results, which means better statistics. It is well known that little can be done to improve the accuracy of the positions and velocities of the stars: errors in these quantities grow exponentially on a time scale which is about $t_{cr}/8$ (Goodman, Heggie & Hut 1993), where $t_{cr}$ is the crossing time. Therefore, on the timescales for interesting dynamical evolution (which are of order $10^2 t_{cr}$ for $N = 1000$) the positions and velocities of the stars are quite wrong. Nevertheless it is an article of faith among practitioners of N-body modelling that the statistical results are meaningful (e.g. Aarseth & Lecar 1975). Therefore, when we speak of improving the results, we mean improving the quality of statistical results. Furthermore, since statistical results are the only useful results from these simulations, it is surprising that in many cases we are content with a single measurement of the statistical results of importance. Of course, it may be that measurements taken at different times on the same system will allow a certain amount of statistical sampling, but it is by no means clear how independent these are. It is much safer, as stressed, for example, by McMillan, Hut & Makino (1991), to compute several simulations with identically distributed initial parameters, differing only in the random numbers used to compute each realisation.
This approach has been exploited previously by Casertano (1985), who studied aspects such as the distribution of escape energies, though for systems with 8 stars, and by McMillan, Caseranto & Hut (1988), who concentrated on an intensive study of the relaxation timescale in \( N \)-body systems with \( N \) in the range \( 16 < N < 1024 \). Our emphasis is on the wider range of dynamical processes, not excluding relaxation, which are of importance in astrophysical applications, such as the evolution of the spatial distribution of stars, anisotropy, escape, binary activity, and so on.

This paper continues with a summary of the software and hardware issues which are relevant to surveys of the kind that we have attempted. We then discuss a number of results for isolated systems in which all stars have the same mass: the evolution of the space distribution of the stars and the distribution of velocities (including anisotropy), and the rate of escape. In this paper we present and discuss data up to the end of core collapse. The next paper in this series continues the discussion well into the post-collapse regime, and adds some detailed results on the evolution of binaries, and their effects on the system. Subsequent papers will deal with systems of stars in which the dynamics is more complicated in various ways: systems with a spectrum of masses, tidally truncated systems, and systems affected by mass-loss through stellar evolution. Further papers will also consider the effects of rotation, which are less easily modeled by Fokker-Planck techniques, and mass segregation.

One purpose of these studies is to test the results of the valuable paper by Chernoff & Weinberg (1990), who studied a similar range of problems (up to the point of core collapse) with the Fokker-Planck method. More generally we aim to compare our \( N \)-body results with those of a variety of simple models and scaling arguments. It might be argued \textit{ab initio} that such an attempt is either bound to fail or to lack meaning, since these simple models usually apply in the limit of large \( N \), and it is not clear whether the small models we study lie sufficiently far into the asymptotic regime. But in cases where the \( N \)-body results do agree with the simple models, we can have some confidence that the simple models are an adequate description of the phenomenon in question and its underlying mechanisms. It might also be said that we take the approach of a theorist attempting to devise a model for a set of observations: starting from familiar physical principles or models (in this case, those of stellar dynamics), our aim is to construct the simplest consistent theory which accounts for the observations. We allow ourselves to adjust ill-determined parameters to optimise the fit, and include no detailed features which are not required by the data.

## 2 A PARALLEL STUDY OF \( N \)-BODY STATISTICS

### 2.1 Hardware

Much of the computational work reported below has been carried out on two different computers, both installed at the Edinburgh Parallel Computing Centre. One is the Meiko transputer array, the other the “Grand Challenge Machine”, also called maxwell, which is an array of 64 i860 processors, installed in the autumn of 1990. Though parallel computers are often thought of as being very hard to program, it is quite easy to mount a large number of independent \( N \)-body problems, as the only messages passed between processors involve simple tasks of assigning input parameters and gathering output. The difficulties are not
much greater than those involved in mounting an $N$-body program on any new single-
processor hardware.

The transputer array (ECS, for Edinburgh Computing Surface) contains 400 pro-
cessors, each of which has communications and at least 4Mbytes of local storage. As the
machine is configured for multi-user operation, generally only smaller “domains”, of up
to 131 transputers, are available to each user. Programs are developed and run under
a software environment known as “CS-Tools”. Individual processors are rated at about
1Mflop. In practice, the code NBODY1 (Aarseth 1985) runs at about one fifth times the
speed obtained on a Sun ELC.

The Grand Challenge Machine has a similar structure and environment, which made
the task of porting code to this machine very quick and painless. It differs mainly in the
speed of the processors, which are roughly 100 times faster than those of the ECS. Each
i860 is a vector processor, with communications and local storage, which is capable in
principle of a performance of $10^2$ Mflops, and the entire machine has a peak performance
of about 5Gflops. It is also very cost-effective, at around $1M. The main purpose of the
machine is computations in high-energy physics, and performance approaching the peak
has been obtained in this application. But such performance is only available with suitable
hand-crafted code, and much more modest performance is obtained in routine FORTRAN
programs; NBODY1 runs at a speed about 5 times faster than on the ECS. On the other
hand, while it was being installed and tested (in the winter of 1990/91), much computing
time was available, and was used for some of the calculations described below (see §2.4).

Later in the course of this project a DEC Alpha superscalar machine (called “fringe”)
was installed on field test at the University of Edinburgh. This is a single-processor ma-
chine, but very fast, and data from large numbers of $N$-body simulations were built up
sequentially.

2.2 Statistical considerations

As already mentioned in §1, there are at least two ways in which the power of parallel
computers can be exploited, either in larger simulations or in more numerous simulations.
In order to assess the value of these two approaches let us consider the i860 machine, with
64 processors, and let us suppose that it is feasible to study a 1000-body problem on a
single processor.

First we consider what can be achieved by using the entire machine for a single large
calculation. With 64 processors it would be feasible to study a problem in which the
computational effort was 64 times larger (provided that the system could be used at 100%
efficiency). With a simple direct summation algorithm, however, the computational effort
grows roughly as $N^3$ (Aarseth 1985, with an extra power of $N$ because we are concerned
with processes occurring on the relaxation time scale), and so it is feasible actually to com-
pute a 4000-body system. Assuming that statistical results yield estimates with standard
deviation proportional to $N^{1/2}$, this means that the signal-to-noise ratio for these results
is improved by a factor of 2, compared with what can be achieved in a comparable time
with a single processor.

Now suppose the same machine is used to run 64 simultaneous (but independent)
1000-body simulations (and it is very easy to do so at virtually 100% efficiency). Then
the signal-to-noise ratio is improved by a factor of 8 compared with results for a single simulation. Therefore it is clear that the statistical quality of the results is optimised by adopting the second strategy. The exact figures, and especially the assumed number of bodies for a feasible single-processor simulation, are irrelevant. It is easy to see, assuming 100% efficiency, that the second strategy is better provided that the computational effort varies as $N^\alpha$ for any $\alpha > 1$. For Ahmad-Cohen schemes the empirical value of $\alpha$ is of order 2.6 (Aarseth 1985, modified as before), while a tree-based code yields $\alpha \geq 2$.

Before we conclude that it is always better to run large numbers of modest simulations than one large one, we should consider other issues than simply statistical quality. There are some phenomena which may well behave very differently in small and large systems - gravothermal effects, which manifest themselves only for systems containing many thousands of particle (Goodman 1987) are one. Another phenomenon which would be rather difficult to study by the use of large numbers of small simulations is the phenomenon of core motions (Makino & Sugimoto 1987, Heggie 1989, Sweatman 1993b), because such motions could not be expected to occur coherently in different systems. The statistical analysis required would be rather different from the kind that is considered in this paper. Possibly the estimation of autocorrelation functions from a large number of simulations would be a fruitful approach, but it is not pursued here. Nevertheless, for many phenomena of interest, such as those studied in this paper, there is no known qualitative difference between large and small systems, and the most scientifically productive way of exploiting parallelism is to run large numbers of simulations in parallel.

2.3 Software

In this project some pilot sets of calculations were computed using a version of Aarseth’s code NBODY1 (Aarseth 1985), which uses individual time steps. This code is quite effective up to the point where binary activity becomes a problem, and was used in a preliminary study of core collapse. Subsequently, a code including regularisation of binary encounters was adopted, along the lines described briefly in Heggie (1973), but extended to include “freezing” of hard isolated binaries (Aarseth 1985). This allowed the computation of new series of isolated models, with 250, 500 and 1000 equal-mass stars in each, which included the end of core collapse and subsequent reexpansion. This code was still too inefficient to deal with very hard interactions between stars and binaries and binaries themselves during the advanced collapse and post-collapse phases. Some cases were stopped because of the occurrence of excessive energy errors, and this resulted in substantial deterioration of overall statistics. Therefore our definitive sets of calculations of isolated systems with 250, 500, 1000 and 2000 equal-mass stars were carried out with Aarseth’s code NBODY5. It was also adopted, with some modifications, for calculations including tidal effects, mass loss through stellar evolution, a mass spectrum, etc., which will be described in later papers in this series.

One general point to made about software in a parallel environment is reliability. On a scalar machine it is possible, if irksome, to give individual attention to a model when some unforeseen combination of circumstances, or a programming error, leads to a serious degradation in accuracy. This is not practical if many models are computed simultaneously. Then it is necessary to write into the program instructions for automatically restarting a computation from a previously stored configuration, but with new parameters which will
allow a more accurate integration (McMillan et al. 1991).

For the organisation of parallel computations some additional programming considerations are relevant. The domain of processors is conceptually divided into a single ‘master’ and a number of ‘slaves’, one for each simulation in the set. The computation is initialised when the master issues initial data (e.g. $N$, and a seed for a random number generator) to each slave. Each slave runs a version of an $N$-body program. When the slave is ready to output analyzed data (which in our calculations was done every time unit) it does so to the master, which adds each such line of output to a single results file. Each line is identified with the time of the output (in $N$-body time units) and the identity of the slave. The calculations running on different slaves proceed at different rates, and so the lines are not in any definite time-order. Nevertheless, after all slaves have finished their simulations, the results file can be analyzed, all lines of output which refer to the same time (but from different slaves and simulations) being combined statistically.

Each slave also occasionally produces a file of data which is sufficient to restart the run, if this is required because of a degradation in accuracy, or if the run has been interrupted for any reason, or has to be scheduled over several production periods (in a shared environment). On the other hand this was not regarded as part of the “scientific” output of the project, as these data sets were regularly overwritten.

### 2.4 Initial conditions and data

In this section we discuss the initial parameters of the runs, summarise the data which was output, and present some statistics on the sets of runs which have been completed and are relevant to the results of this paper.

The initial conditions for these models were drawn from a Plummer model, with equal masses, and no primordial binaries (except for any which occur by accident). The systems are isolated, with no external tidal effects. Units, which may be referred to below as “$N$-body units”, are standard (Heggie & Mathieu 1986), and lead to the following initial values: the crossing time is $2\sqrt{2}$, the virial radius is 1, and the initial rms 3-dimensional speed is $1/\sqrt{2}$. These units are achieved by first shifting to the barycentric frame and then rescaling the positions and velocities so that the initial kinetic and potential energies take the required values.

At each time unit, each model outputted a list of data. For these systems with equal masses it was not felt necessary to store data on individual particles, and so most of the items are statistical. To some extent the list evolved throughout the course of our project, as new aspects of interest emerged. The following list is comprehensive, but for the reason just stated, not all of this information is available for all sets of calculations.

1. the number of the run in the current series;
2. the time in $N$-body units;
3. the total mass of non-escapers (defined in §3.3);
4. the virial ratio (i.e. the ratio of kinetic to binding energy of all bound stars, excluding “internal” kinetic and potential energies of recognised binaries);
5. the central potential (i.e. the potential at the density centre);
6. the core radius $r_c$ (defined as in Casertano & Hut (1985)). In addition, the core radius was also computed from the definition $r_c^2 = 3v_c^2/4\pi G \rho_c$, where $v_c$ and $\rho_c$ are the central
three dimensional velocity dispersion and central density, which in turn were computed from information on stars within the first Lagrangian radius (cf. (13) and (31) below);

(7) the mass of the core, computed in two ways: (i) as in Casertano & Hut (1985), and (ii) by summing the masses of stars within a sphere of radius \( r_c \) (calculated by the second method outlined in (6) above) centered on the density centre, which in turn is defined below in §3.1.1);

(8) the ratio of the central density (defined as in Casertano & Hut (1985)) to the mean density within the half-mass radius;

(9) the number of stars in the core (computed in two ways, as in (7) above);

(10) the three-dimensional root mean square speed of stars in the core (except that a binary, defined here as a regularised bound pair, contributes only the speed of motion of its barycentre);

(11) the coordinates and velocity components of the density centre, the latter being defined by the change in coordinates over one time unit;

(12) the coordinates and velocities of the centre of mass;

(13) Lagrangian radii (radii of spheres containing fixed fractions of the mass of bound stars, relative to the density centre) for fractions 1, 2, 5, 10, 20, 30, 40, 50, 75 and 90%;

(14) the number of bound stars;

(15) the number of escaping single stars;

(16) the number of regularised binaries (the parameters which determine whether a pair is regularised taking standard values (Aarseth 1985); sometimes we loosely refer to such binaries as “hard”);

(17) the number of unregularised binaries with energy greater roughly than \( 0.1kT \), where \( kT \) is defined as \( 2/3 \) of the current mean kinetic energy of the single stars and of the barycentres of binary stars);

(18) the number of \emph{merged} pairs (defined as in Aarseth 1985 with standard parameters);

(19) the number of \emph{triple} or \emph{quadruple} configurations (defined as in (18) above);

(20) the total energy of escaping single stars;

(21) the total and maximum internal binding energy of all unregularised binaries with energies above \( 0.1kT \);

(22) the total and maximum internal binding energy of all regularised binaries;

(23) the total energy of the centres of mass of escaping regularised binaries;

(24) the total internal energy of escaping regularised binaries;

(25) the internal binding energy of \emph{triple} and \emph{quadruple} configurations (defined as in (19) above);

(26) the internal binding energy of \emph{mergers} (defined as in NBODY5);

(27) the total internal and external energy of the bound members of the system, defined as in Aarseth & Heggie (1992);

(28) the cumulative integration error in the total energy;

(29) the maximum and minimum internal energy and radius (relative to the density centre) of all regularised binaries;

(30) the names of stars forming the hardest binary;

(31) mean square radial and tangential velocities of stars between successive Lagrangian radii, and the same quantities corrected for escaping stars which have not yet been removed
from the $N$-body calculation.

The output from the various models in each series were collected in a single output file. For each series the typical size of an output file is typically 35Mbytes. The file for each series was later processed so as to produce statistics for all models in that series at each time. As each line is read, the time to which it refers is noted, and the data from that line are added to the statistical data for that time.

Table 1 summarises both the extent of the computational effort in these calculations and the numbers of particles and simulations. Smaller numbers of simulations with $N > 1000$ have also been included for comparative purposes. Further discussion and results, including some data from a 10000-body simulation, are considered in a paper by Giersz & Spurzem (1993). It is worth pointing out that the statistical value of these various sets of simulations are reasonably well balanced: for example, one set consisting of 40 simulations with 250 bodies is comparable with one simulation containing $10^4$ stars, as the total number of particles is the same.

Before discussing the results of these simulations we mention here some data about their accuracy, as measured by energy conservation. Throughout the core collapse period the mean of the absolute value of the total energy error is monotonically increasing and reaches at the time of collapse values between $10^{-5}$ to $10^{-3.8}$ for different $N$. As can be expected from simple theoretical arguments (e.g. the larger time to core collapse for larger $N$) the smaller value is for $N = 250$ and the larger for $N = 2000$. The energy error for individual cases never exceeded 0.002.

3 STATISTICAL RESULTS FROM $N$-BODY SIMULATIONS

3.1 Spatial evolution

3.1.1 Evolution of Lagrangian radii

We now discuss the results from these computations up to about the time of core collapse, and begin here with the distribution of mass. We assume spherical symmetry relative to the “density centre” (computed as in Aarseth & Heggie 1992). In what follows, “Lagrangian radii” are the radii of imaginary spheres containing a fixed fraction $f$ of the $N$ bound stars in the system, i.e. excluding escaping stars. Where this fraction corresponds to an integral number $fN$ of stars, the radius is that of the $fN$th star in order of distance from the density centre. Where $fN$ is fractional, the value is interpolated.

Combining data from a large number of runs may be done in several ways, and it is not clear at first how consistent the different results will be, or what is the most effective method. One could, for example, combine the data on individual stars from all cases at a given time, and then compute the Lagrangian radii for this enlarged ensemble of stars. Because of the limited data which we elected to output, however, this method could not be adopted, and we chose instead a suitable statistical estimator based on the set of Lagrangian radii obtained for each case individually. For example, Fig.1 shows data for the Lagrangian radius defined by the innermost 5% of the mass, for $N = 250$. These results indicate that the mean and median (over all cases computed) behave very similarly, and from now on we shall concentrate on the mean.

The improvement in the quality of the data obtained by combining results from many cases in this way is illustrated in Fig.2, which show the means for all Lagrangian radii
in one set of models, Fig.2a, and the results for a single, typical model, Fig.2b. The improvement in “signal-to-noise” is comparable to what one would expect from simple “root $n$” considerations. It is this improvement which will allow us to re-examine several quite old problems in collisional stellar dynamics which have lain relatively fallow for want of results of sufficient statistical quality.

3.1.2 $N$-dependence of dynamical evolution

The effect of the number of stars on the evolution of the Lagrangian radii is illustrated in Fig.3. The most obvious feature of this result is the increase of the time scale for core evolution with $N$. This is consistent with the assumption that the evolution is driven by two-body relaxation, and that the relaxation time increases nearly linearly with $N$ (cf. Spitzer 1987). More precisely, the theoretical result is that the relaxation time scale varies as $N/\ln(\gamma N)$, where $\gamma$ is a constant. In what follows we analyse this question quantitatively in a number of ways, though the order in which these are described differs from the order in which they were carried out.

One procedural point has to be mentioned first. Fig.4 shows, for $N = 250$, a comparison between the $N$-body results and two approximate (continuum) models, which are based on the theory of relaxation and described more precisely below. The point we wish to emphasise is the fact that the $N$-body results lie initially below those of the continuum models, which correctly give the value for a Plummer model. The reason for this discrepancy can be traced to several sources, some of which arise naturally from the way in which the initial conditions are scaled. For example, if 250 point masses are spatially distributed as in a Plummer model, the resulting potential will not be exactly that of the underlying Plummer model, and it is probable that the mean potential energy of a large number of simulations is biased away from its true value. Therefore, when the positions of the particles are scaled to the desired value of the total potential, there is a resulting bias in the spatial distribution. Another bias arises from use of the potential centre, which tends to be a point of overdensity in the system, and therefore Lagrangian radii tend to be smaller than they should be. The different values of the initial 2% Lagrangian radii for various $N$ are clearly visible on Fig.3.

Approximation correction for these biases has been carried out by adding a constant correction to each mean Lagrangian radius such as to bring the numerical data and the theoretical radius into optimal agreement at time $t = 0$, or for the average of the values at $t = 0$ and 1, or for the first 10% of the collapse phase. (In the latter two cases allowance was made for the evolution of the radii during this relatively short time interval.) It may be expected, from our discussion of the sources of these effects, that they are $N$-dependent, and indeed we find that the correction to the innermost Lagrangian radius generally decreases with increasing $N$, and is smaller (in absolute terms) at larger radii. In what follows we shall use these adjusted Lagrangian radii for a comparison of the various models, though we continue to use Fig.4 to illustrate the discussion.

Now we return to the question of how the time scale for evolution varies with $N$. Fig.5 illustrates, for two particular values of $N$, one way in which this can be done. It shows one curve for each Lagrangian radius, and was constructed as follows. At each value of $t$ in the 500-body models (which is the abscissa in Fig.5) the mean Lagrangian radius was computed, and then by interpolation we determined the corresponding time at which this
mean value of the radius was reached in the 2000-body models. (In fact there may be more than one such time, because statistical fluctuations result in evolution of the radius which is not quite monotonic. We chose the first such time, and checked that our results are insensitive to this choice.) According to the theory of relaxation the ratio of these times, which is the scale factor plotted in Fig.5, should be \( S_f = N_2 \ln(\gamma N_1)/N_1 \ln(\gamma N_2) \), where in this case \( N_1 = 500 \) and \( N_2 = 2000 \).

The results for all combinations of \( N_1 \) and \( N_2 \) show remarkable consistency. The values of \( S_f \) for the innermost Lagrangian radii (1 - 10%) and outermost Lagrangian radii (75 and 90%) are very close together. On the other hand the 20 - 40% Lagrangian radii lie below and the 50% radius lies above; this means that these intermediate radii evolve too fast in models with smaller \( N \). The explanation for this may be connected with the fact that binaries start to influence the evolution relatively earlier in core collapse in low-\( N \) systems than for larger \( N \) (cf. Giersz & Spurzem 1993 and also Paper 2 in this series). The effect of binaries would not be confined to intermediate radii, but that is where their influence might be most noticeable in this kind of analysis, because these radii evolve less quickly than those further in and further out.

Fig.5 shows clear evidence of the Coulomb logarithm in the relaxation time: without it we would predict \( S_f = 4 \) for the relevant values of \( N_1 \) and \( N_2 \). One would like to be able to use this data for an empirical determination of \( \gamma \), but this is very difficult because of the weak predicted dependence of \( S_f \) on \( \gamma \): fluctuations in \( S_f \) give unacceptably large fluctuations in \( \gamma \). To determine \( \gamma \), therefore, we adopted a somewhat more indirect method, which we now describe.

Two-body relaxation theory can be used to study the evolution of stellar systems in two ways. One is the gas model (Hachisu & Sugimoto 1978, Lynden-Bell & Eggleton 1980) and the other, which is usually thought to be more faithful to the picture underlying relaxation theory, is the Fokker-Planck model (cf. Spitzer 1987). Both models, however, share some significant simplifying assumptions, including spherical spatial symmetry, isotropy in the velocity distribution, and no loss of mass by escape, at least in the simplest formulations of these two models.

In order to use results from these models to evaluate the evolution of an \( N \)-body model, it is necessary to relate the time variables in the different models. Conversion between Fokker-Planck and \( N \)-body models depends only on the value of the coefficient \( \gamma \) in the expression for the relaxation time, i.e.

\[
  t_r = \frac{0.065(\sqrt{3}\sigma)^3}{\rho m G^2 \ln(\gamma N)}
\]

(cf. Hénon 1975, Spitzer 1987), where \( \sigma \) is the rms value of each component of velocity, \( \rho \) is the stellar mass density, and \( m \) is the individual stellar mass. Conversion between gas and \( N \)-body models requires in addition the specification of a dimensionless conductivity coefficient, denoted by \( C \) (Lynden-Bell & Eggleton 1980). A number of previous theoretical estimates are listed in Table 2.

If the time variable of a set of \( N \)-body models is denoted by \( t_N \), the scaling between this variable and the time variable of a continuum model was carried out in a manner analogous to the determination of the scale factor \( S_f \) in Fig.5. For each value of \( t_N \) this
gives a single estimate for the scalings which bring the models into agreement. A typical result is shown in Fig.6, though since this illustrates scaling with the Fokker-Planck model the result has been expressed in terms of the equivalent value of $\gamma$. This case illustrates a fairly general finding, that the inner radii agree well within themselves, but that the half-mass and 75% radii are relatively discrepant. Also noticeable in the inner radii is a tendency for the value of $\gamma$ to decrease as $t_N$ increases; this is most noticeable in runs with the smallest $N$, and will be discussed further in §3.1.3.

We consider first the results for the radii well inside the half-mass radius. As can be inferred from Fig.6, individual estimates of $\gamma$ range widely, especially near the beginning of the evolution. Elsewhere, however, the range of values is smaller, and roughly independent of time and radius. Examination of results for other sets of calculations also show that it is roughly independent of $N$. The great bulk of the values lie in the range $0.07 \leq \gamma \leq 0.14$, and show greater consistency for larger values of $N$ than for $N = 250$. For the gas model the scale factor shows similar fluctuations, but their interpretation is more complicated, as the scaling depends on two parameters, viz. $\gamma$ and $C$.

These data can also be used to estimate global values (one for each value of $N$) for the scalings between the $N$-body models and the two continuum models. The results were then used to estimate the $N$-independent parameters $\gamma$ and $C$, with results shown in Table 2. For the gas models, the estimate of both parameters can be made from each pair of values of $N$, which was done in order to investigate the variation of these parameters. The values presented in Table 2 are, however, obtained from a least squares fit to the results for all $N$. The Fokker-Planck data yield one estimate of $\gamma$ for each $N$, but again the value in Table 2 is a mean. (Note, however, that the process is iterative, as the determination of the adjustment to the Lagrangian radii already requires a knowledge of the scaling between the $N$-body and Fokker-Planck models, i.e. a preliminary value of $\gamma$.)

The variations about these means are quite modest. For the Fokker-Planck determination of $\gamma$, results for the four values of $N$ range from about 0.108 to about 0.120. For the gas model the range of values of $C$ is also quite small, from about 0.099 to about 0.106, but the range for $\gamma$ is greater: 0.104 to 0.138. Note that these variations are correlated, larger values of $\gamma$ corresponding to (and partially compensating) smaller values of $C$. Despite these variations, the consistency is unexpected, especially when it is considered that $\gamma$ is a coefficient in the argument of a logarithm which is usually regarded as “slowly varying”. An indication of the overall success of the resulting parameters, at least for the inner radii, is provided by Fig.7, which for one value of $N$ shows the comparison between the gas, Fokker-Planck and $N$-body models.

It should be stressed again that our empirically determined values of $\gamma$ and $C$ are based mainly on data for the innermost Lagrangian radii.

Also shown in Table 2 are some earlier theoretical estimates of the parameters $\gamma$ and $C$. It is interesting to point out that one of Hénon’s values of $\gamma$ stemmed from a close reexamination of Chandrasekhar’s theory of relaxation, which he undertook in order to improve a previous comparison between the results of $N$-body and Fokker-Planck models (Aarseth, Hénon & Wielen 1974). It is only slightly greater than our value.

### 3.1.3 Differences between $N$-body and continuum results

We now discuss the discrepant results in Fig.6 (and related data), beginning with
the half-mass radius. It is well known that the half-mass radius is nearly constant in the collapse of a Plummer model, which can be understood in terms of energy conservation and the empirical fact that the half-mass radius, expressed as a fraction of the virial radius, is not sensitive to the model under consideration (Spitzer 1987). The implication of this near constancy, however, is that the scale factor from which $\gamma$ is computed is very sensitive to fluctuations in the N-body data. Nevertheless the difference between the behaviour of the half-mass radius and that of the inner radii is systematic, in the sense that the $r_h$ grows too quickly in the continuum models (Fig.7), a fact which was already noted in a previous small-scale statistical study of 100-body systems (Heggie 1991). As was mentioned in §3.1.2 similar behaviour is exhibited by the 20 and 75% Lagrangian radii, and this already becomes visible soon after the start of the evolution.

A possible reason for this is the development of anisotropy in the N-body models, which is discussed further in §3.2 below and in much detail by Giersz & Spurzem (1993). Their finding is that the evolution of $r_h$ is not even very well described by any reasonable anisotropic gaseous model. For the half mass radius the best agreement with the N-body data is provided by the anisotropic Fokker-Planck models of Stodólkiewicz (1982) who uses a Monte-Carlo technique. Since the only relevant published data from these models is diagrammatic, a quantitative measure of the agreement is not yet possible.

Another factor which is neglected in the continuum models is the change in mass ($M$) and energy ($E$) of the system caused by escape (cf. §3.3 below). We have corrected the values of $r_h$ in the gas models by using the actual values of mass and energy in the N-body models, assuming that $r_h \propto M^2/E$. The correction has a noticeable effect, but does not substantially improve the agreement between the two models.

It is easily seen in Fig.7 that the discrepancy is worse in the gas model than in the Fokker-Planck model, the former evolving more slowly beyond the 10% radius. This has nothing to do with the N-body models, of course, but is a deficiency in the gas model. In effect it means that the value of $C$ must increase slightly with radius, reinforcing the fact that the values in Table 2 refer to the innermost radii.

The other noticeable discrepant feature in Fig.6 is the slight decrease in the scaling factors with time. It is related to a systematic discrepancy between the evolution of the continuum and N-body radii which is visible particularly clearly in Fig.5. This trend is qualitatively consistent with two possible explanations, both of which are of dynamical interest. One is the assumption, discussed by Spitzer (1987), that, for the core, the argument of the Coulomb logarithm in eq. (1) should be proportional to $N_c$, the number of stars in the core, rather than $N$ (cf. Table 2). We discuss this possibility further below. The other possible explanation is the growing activity of binary stars, which, in such relatively small models, begins to influence the collapse of the core well before it brings the collapse to a close. (This process is not modeled at all in the Fokker-Planck and gas models used for this comparison.) We shall discuss this issue in the next paper in this series, which analyses the behaviour of binaries throughout the evolution, including the phase beyond core collapse. As a precaution, however, in the determination of $C$ and $\gamma$ we have discarded data at times when binaries have absorbed more than 1% of the total energy of the system.

In order to investigate the effect of a variable Coulomb logarithm we have repeated the computation of gas models but have chosen for the Coulomb logarithm the expression
\ln(\gamma N_c) \text{ if } N(r) < N_c \text{ and } \ln(\gamma N(r)) \text{ otherwise, where } N_c \text{ is the number of stars in the core (Spitzer 1987, p.149) and } N(r) \text{ is the number of stars within radius } r. \text{ The value of } \gamma \text{ chosen was } 0.1 \text{ (cf. Table 2), and the computation carried out for } N = 250, \text{ for which the problems with fitting the Lagrangian radii by results of gas computations seemed most difficult. The rationale for the choice of Coulomb logarithm starts from the fact that the argument of the logarithm should be the ratio of the largest effective impact parameter to the } 90^\circ \text{ deflection distance. The latter, which depends on the velocity dispersion, varies rather slowly throughout the system. The former may be taken as being comparable with the radius of that part of the system where the density is not much smaller than the local value. Within the core this distance will be of order } r_c, \text{ the core radius, and outside the core it is of order the radius } r \text{ itself. In a nearly isothermal halo (outside the core) } r \propto N(r) \text{ approximately.}

The result of this calculation showed an overall slowing of the evolution, since the modification of the Coulomb logarithm from } \ln(\gamma N) \text{ lengthens the relaxation time. But the effect decreases as one moves to larger radii. Compared with the } N\text{-body models with } N = 250, \text{ it was found that the gas model with variable Coulomb logarithm gave results for the scale factor (between time variables for the gas and } N\text{-body models) which were a little more consistent for different radii (but still excluding the half-mass radius). On the other hand the time-dependence of the scale factor was virtually unimproved, which suggests that this has some other explanation, such as binary evolution.}

In conclusion, there is some evidence favoring use of a variable Coulomb logarithm. In addition, the comparison between } N\text{-body and Fokker-Planck data gives some indication that } \gamma \text{ may increase slightly with increasing radius, which is qualitatively consistent with our prescription for a variable Coulomb logarithm. Unfortunately this result cannot be conclusive because of the large fluctuations in the values of } \gamma \text{ (cf. Fig.6).}

### 3.2 The velocity distribution

Fig.8 shows some typical results for the velocity dispersions (radial and tangential, or the sum of these). The general increase with time in the inner parts of the cluster (Fig.8a) is to be expected from core collapse, and the results from smaller } N\text{ models show a very similar trend with greater noise. The illustrated comparison with the results of gas calculations is satisfactory until late in the core collapse phase, and there is even a suggestion that the } N\text{-body models exhibit the initial cooling, which is also found in Fokker-Planck models (cf. Cohn 1980, Fig.7). There is no discernible anisotropy in the innermost shell, and slight evidence (seen in a graph of the quantity } \langle v_t^2 \rangle/\langle v_r^2 \rangle, \text{ which is not, however, reproduced here) between the Lagrangian radii for 20 and 50\% of the mass. Anisotropy clearly grows, however, in the outer shells (Fig.8b). There is a slight decline in the radial velocity dispersion, but the tangential dispersion decreases much more rapidly. The overall downward trend of the mean velocity dispersion agrees well with the predictions of an isotropic gas model and is associated with the general expansion of this zone, but the radial dispersion is maintained by the ejection of stars from the inner parts of each system. The growth of anisotropy (measured by } A = 2 - \langle v_t^2 \rangle/\langle v_r^2 \rangle \text{) with time is quite linear, and it reaches values between 0.9 and 1.1 (the larger value corresponding to the smallest value of } N \text{) at the end of core collapse. Detailed comparisons with anisotropic gaseous models are presented in Giersz & Spurzem (1993).}
3.3 Escape

3.3.1 Review of previous work

The rate at which stars escape from an isolated clusters of equal stars is a problem with a long history (Table 3). Early theoretical estimates (Ambartsumian 1938, Chandrasekhar 1943) were based on the theory of relaxation, and led to predictions of a fractional escape rate per relaxation time which is independent of \( N \). This translates to an escape rate per crossing time which is proportional to \( \log N \). Relaxation being understood as a diffusive process, Hénon (1961) (who also lists some other theoretical estimates of the escape rate) then pointed out the difficulty of diffusing stars across the energy of escape, when their periods should become extremely long. He therefore gave an estimate of escape from two-body interactions, though the result depends on the assumed distribution of velocities; it is

\[
\frac{dN}{dt} = \frac{-256 \sqrt{2\pi} G^2 m^2}{3} \int_0^\infty r^2 dr \int \int \frac{(\varepsilon + \varepsilon' - \phi)^{3/2}}{\varepsilon'^2} f(\varepsilon) f(\varepsilon') d\varepsilon d\varepsilon',
\]  

where \( m \) is the individual stellar mass, \( r \) is the distance from the centre of the cluster, \( f(\varepsilon) \) is the phase-space density expressed as a function of the specific energy \( \varepsilon \), \( \phi(r) \) is the potential, and the integration is taken over the range \( \varepsilon < 0 \), \( \varepsilon' < 0 \), \( \varepsilon + \varepsilon' > \phi \). Hénon’s point of view was further modified by Spitzer & Shapiro (1972), who pointed out that the distribution itself evolves on a relaxation time scale, and then a star which has diffused to energies a little below the escape limit can escape in a single two-body encounter in the core. Some previous numerical estimates for the escape rate are also summarised in Table 3.

3.3.2 Results from the present \( N \)-body simulations

Escaping stars in our \( N \)-body models are identified according to the following definition. Their energy (computed in the rest frame of the centre of mass of the entire system) must be positive, and their distance from the density centre must exceed a boundary radius which is chosen as 10 or 20 times the half-mass radius. The results presented in this paper are for boundary radius equal to 20 times \( r_{h} \), except where stated in §3.3.3.

Two typical sets of results are shown (along with a theoretical comparison, which we discuss below) in Fig.9. Note that these are means over many runs. We have checked that the mean agrees quite well with the median number of escapers, but there is a wide range about the mean. For \( N = 250 \), for example (not shown), the mean number of escapers at the end of core collapse is approximately 4.0, but in individual models may range from 0 to 9. Note the clear increase in the rate of escape with time; such an increase was already noted in the Monte Carlo models of Spitzer & Shull (1972), who found that it increased by a factor of about four during the course of core collapse. We found that it increased by a factor of about 6 for \( N = 250 \). For higher \( N \) the increase is even larger (Fig.10).

For a first comparison between our own results and those of theory we have adopted a synthesis of the general theoretical result of Hénon with the ideas of Spitzer & Shapiro. What we have done is to apply Hénon’s result to the time-dependent distribution function given by the (isotropic) Fokker-Planck model. Results of such a comparison are shown in Fig.9, which shows that the agreement is rather poor, except initially. Up to the end of
core collapse the mean number of escapers in the $N$-body models exceeds the theoretical prediction by a factor of about 1.8 and 2.8 for $N = 500$ and 1000, respectively. The results for other values of $N$ are qualitatively similar, and the agreement at early times is clearer in the larger models than smaller one (as we can see in Fig.9b). For all models (but particularly for this with low $N$) the comparison is complicated by the fact that stars are only deemed to escape after they have reached a sufficiently large radius. This depresses the escape rate at very early times (cf. also Fig.10), and makes the number of escapers depend on the definition of the boundary radius.

The disagreement between $N$-body and Fokker-Planck results is not surprising, because eq.(2) is based on the assumption that the distribution of velocities is isotropic. This is true of all our theoretical comparisons, but it is especially critical in this case, because escape tends to take place for stars which are already loosely bound and yet have sufficiently small angular momenta to pass through the core. In addition to anisotropy, which we discuss further below, further possible explanations of this discrepancy are explored in Paper II.

Fig.11 shows the situation for the energy carried off by escapers, i.e. their asymptotic kinetic energy at infinity. What is plotted here is, as usual, the mean value over all the simulations at this value of $N$. Use of the median would be particularly unsuitable for escaper statistics, as a substantial fraction of cases may have no escapers for a significant part of the core collapse phase.

In Fig.11 the theoretical result is based on a general expression given by Hénon (1969) for isotropic distribution functions, and evaluated by him for a Plummer model with unequal masses. As before we applied it to the evolving distribution function produced by an isotropic Fokker-Planck code. Again there is agreement at early times, but a faster rise throughout much of core collapse. Even more dramatic is the abrupt rise towards the close of core collapse, and it is natural to interpret this in terms of escapers produced in three-body encounters. Again discussion of this process is relegated to Paper II in this series.

### 3.3.3 The effect of anisotropy

In order to understand the influence of anisotropy on the rate of escape it would be desirable to calculate the evolution of the distribution function using an anisotropic model, and then repeat our computation of the escape rate using a suitable anisotropic generalisation of Hénon’s formula, eq. (2). In fact anisotropic gas models have been developed by several authors (Larson 1970, Bettwieser 1983, Louis and Spurzem 1991), and a recent version of such a code is compared with the results of our and other $N$-body data in Giersz & Spurzem (1993). Unfortunately, however, the gas model does not directly yield the distribution function in phase space, and so these results cannot be applied to the discussion of the escape rate.

Since the effect of anisotropy on escape appears to be quite unknown, we have formulated a model problem which should provide a semi-quantitative guide. What we have done is to use a Monte Carlo method to compute the escape rate in a sequence of anisotropic models. The sequence we chose was devised by Dejonghe (1987), and has the benefit that all the models in the sequence have the same spatial density distribution as in Plummer’s model. This allows us to compare our results with Hénon’s well known results for this case.
(Hénon 1961, 1969). More importantly, the models have the same density distribution as our initial conditions; thus the study of this sequence of models allows us to determine the effects of anisotropy, while our study of the Fokker-Planck model allows us to study the effect of dynamical evolution. Dejonghe’s models have one scale-free free parameter, \( q \), which we took in the range from 0 (Plummer’s model) to 1.5. It is related to the anisotropy \( A \) defined in §3.2 by

\[
q = A(1 + 1/r^2).
\] (3)

In order to compute the escape rate the following procedure was used. First, the radius \( r \) is selected from a probability density function proportional to \( 4\pi r^2 n(r)^2 \), where \( n \) is the number-density. Next, the velocities of two stars are selected from the local distribution of velocities. Then the impact parameter \( p \) is chosen with probability density proportional to \( p \) up to a maximum value \( p_{\text{max}} \), and an angle which determines the relative orientation of the two stars at the moment when they would be at their minimum separation during an encounter, if their paths were undeflected by it. Then it is possible to determine whether either star has a speed, after the encounter, above the local escape speed. Then the escape rate is the average value of \( v \), the relative speed of the two stars before the encounter, suitably normalised. The resulting value is subject to statistical uncertainty, and depends on the choice of \( p_{\text{max}} \). Experimentation with different values, and different numbers of trials, allowed us to determine satisfactory choices.

The results of our calculations are shown in Figs.12 and 13. Units are standard, i.e. those in which \( M = 1, G = 1 \) and \( E = -1/4 \). For \( q = 0 \) the results may be compared with those given by Hénon, i.e. \( \dot{N} = 0.00942 \) and \( \dot{E} = 0.000741 \). Evidently the presence of anisotropy can lead to a large increase in the rate of escape and the flux of energy carried off by escaping stars.

In applying these results to a comparison with our \( N \)-body data we have made two rather arbitrary assumptions. One is that the effects of evolution (discussed in the previous section) and anisotropy can be combined multiplicatively. The second is that we can estimate the effect of anisotropy by using the analytical (Plummer-like) model which has a value of \( q \) given by eq.(3), in which \( A \) is taken to be the average anisotropy in the outer half of the cluster and \( r \) is the 75% Lagrangian radius. When this is done it is found that the predicted number and energy of escapers agree with those from the \( N \)-body simulations quite well as can be seen in Figs.9 and 11. The agreement is particularly good for \( N \geq 1000 \). For smaller \( N \) the predicted number and energy of escapers are larger then those obtained from the \( N \)-body data, and for \( N = 250 \) the discrepancy is about 40\% of the \( N \)-body value. The largest discrepancy that we have noticed is the energy flux at late times in core collapse; as already mentioned it rises rather suddenly as the end of core collapse is approached. It is probable that the increased contribution from three-body escape events (i.e. those associated with binaries; cf. Paper II) is beginning to make a significant contribution. Another explanation which we cannot rule out is the inappropriateness of one or other of the assumptions mentioned earlier in this paragraph, though the abruptness of the rise suggests otherwise.

As we mentioned before the \( N \)-body data depend on the definition of the boundary radius. If the boundary radius is reduced to \( 10r_h \) the predicted values agree with those from \( N \)-body models to better than 25\% throughout most of core collapse. But now, for
all values of $N$ that we have studied, they are smaller than the $N$-body values. Therefore we can expect that in the case of instantaneous removal of escapers from $N$-body models (which is in effect what happens in the theoretical model) the discrepancies between computed and predicted values will be larger than for the model with boundary radius $10r_h$. But at any rate, our conclusion is that anisotropy has a large enough effect on the escape rate that it may be quite enough to reconcile theoretical predictions with our numerical results.

4 DISCUSSION AND CONCLUSIONS

We have computed large numbers ($\sim 200$) of isolated $N$-body simulations with $N$ in the range from 250 to 2000. All stars have equal masses, and initially the systems are in dynamic equilibrium. The purpose of these studies is to improve the quantitative results of such computations by statistically combining data from many simulations.

We find that the evolution of the spatial distribution of stars, up to the end of core collapse, is quantitatively quite consistent with the theory of relaxation. Indeed we have been able to estimate a rather reliable value for the Coulomb logarithm. Small deviations from the predictions of isotropic models based on the theory of relaxation are present at small radii (where they are probably associated with incipient binary activity) and at the 50 and 75% Lagrangian radii (where anisotropy of the velocity dispersion may be responsible).

Inside the radius containing half the mass of the cluster, the evolution of the velocity dispersion is also consistent with simplified models in which the distribution of velocities is assumed to be isotropic, but there is strong growth of anisotropy in the outer half of the mass. A quantitative comparison with the anisotropy predicted by a simplified theory of relaxation is presented elsewhere (Giersz & Spurzem 1993).

The rate of escape of stars is roughly consistent with a hybrid model based on existing theory for escape in evolving, isotropic systems, with a somewhat ad hoc but important modification due to anisotropy. The escape is assumed to take place by two-body interactions from stars whose distribution function slowly changes in accordance with the usual theory of relaxation.
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Table 1

OUTLINE OF THE COMPUTATIONS

| $N$  | number of cases | cpu (hrs) | machine     |
|-----|-----------------|-----------|-------------|
| 250 | 56              | 50        | super       |
| 500 | 56              | 200       | super       |
| 1000| 40              | 190       | maxwell     |
| 1000| 50              | 1200      | fringe      |
| 2000| 16              | 1350      | super       |
| 2500| 2               | 13000     | copernicus (Sun ELC) |

Note: For parallel machines (super and maxwell) the total number of processor-hours is the product of columns 2 and 3.
| Parameter | Source | Notes |
|-----------|--------|-------|
| C         | γ      |       |
| — 0.25    | Ambartsumian (1938) | |
| — 0.35    | Chandrasekhar (1942) | |
| — 0.15    | Hénon (1975) | Depends slightly on velocity distribution |
| — 0.2     | McMillan (pers. comm.) | |
| — 0.4     | Spitzer (1987) | |
| — $2N_c/N$| Spitzer (1987) | For the core; $N_c =$ number of stars in core |
| 0.104     | — Heggie & Stevenson (1988), Heggie (1985), Goodman (1987) | |
| 0.104     | 0.11 | This project (gas model) |
| — 0.115   | This project (F-P model) | |
| Source               | $-t_{cr}dN/dt$ | Notes                                      |
|---------------------|----------------|--------------------------------------------|
| Ambartsumian (1938) | 0.16 $\ln \Lambda$ | Relaxation; $\ln \Lambda = \text{Coulomb logarithm}$ |
| Chandrasekhar (1943)| 0.33 $\ln \Lambda$ | Relaxation and dynamical friction          |
| Hénon (1961)        | 0.027          | Plummer model; 2-body encounters           |
| Spitzer & Shull (1972) | 0.06 $\ln \Lambda$ | Mean value in core collapse              |
| Wielen (1974)       | $\simeq 0.1$   | $50 \leq N \leq 250$                     |
| Aarseth & Lecar (1975) | 0.2-0.3      | Plummer model ($N = 250$)                 |
| Stodólkiewicz (1982)| 0.3           | Mean value in core collapse               |
| Heggie (1991)       | 0.09           | $N = 100$                                 |
FIGURE CAPTIONS

**Fig.1** Statistics of the 5% Lagrangian radius for 56 cases with $N = 250$. The curves correspond to the minimum, median, mean and maximum at each time.

**Fig.2** Mean Lagrangian radii for (a) 56 cases with $N = 250$, compared with (b) a single typical case.

**Fig.3** Mean Lagrangian radii for all series, for a mass fraction of 2%. For a Plummer model the value is 0.166.

**Fig.4** Comparison between $N$-body, gas (IGM) and Fokker-Planck (F-P) models for the 1% Lagrangian radius and $N = 250$. The time scales of the continuum models have been scaled empirically to provide a reasonable global fit; the values used were $C = 0.104$, $\gamma = 0.11$ (cf. Table 2).

**Fig.5** The scale factor in time which brings $N = 500$ and $N = 2000$ models into agreement. All ten Lagrangian radii are shown. The discrepant ones are 20% − 40% radii (below the others) and half-mass radius (above the others). The thick line shows scale factor for $\gamma = 0.11$. The abscissa is the time variable in the 500-body calculations.

**Fig.6** The value of $\gamma$ which brings Fokker-Planck and $N$-body results into agreement at each time, for $N = 500$. The discrepant Lagrangian radii (below the others) are the 50% and 75% Lagrangian radii.

**Fig.7** Comparison between the gas (IGM), Fokker-Planck (F-P) and $N$-body models with 7 Lagrangian radii for $N = 1000$, to indicate the overall success of our fitted values of the relaxation parameters $\gamma$ and $C$ (cf. Table 2).

**Fig.8** Velocity dispersions for $N$-body models with $N = 1000$. (a) Total velocity dispersion between Lagrangian radii for 1% and 2% of the mass, compared with a gas model. (b) Radial and transverse dispersions between Lagrangian radii for 75% and 90%.

**Fig.9** Number of escapers as a function of time for different sets of models. F-P-I - isotropic Fokker-Planck model, P-A-C - anisotropic Plummer model with evolution (see text for discussion). (a) for $N = 500$, (b) for $N = 1000$.

**Fig.10** The escape rate as a function of time for $N = 1000$. The squares represent data given by Spitzer & Shull (1972) for a Monte-Carlo model. Their time unit has been scaled to $N$-body time for $N = 1000$. The triangle represents data obtained by Stodolkiwicz (1982) for a Monte-Carlo model. The dot represents the result given by Hénon (1961) for a Plummer model. The labels $10r_h$ and $20r_h$ correspond to the boundary radii chosen in different sets of $N$-body simulations (see text for discussion).

**Fig.11** Energy of escapers as in Fig.9.

**Fig.12** Escape rate for an analytic sequence of anisotropic models (the anisotropy being determined by $q$). Error bars are estimated 1-$\sigma$ confidence limits. The dashed line and the solid curve are linear and quadratic fits of the form $a + bq$ and $a + bq + cq^2$ respectively. The linear fit is adequate for all except the lowest value of $q$.

**Fig.13** Rate at which energy is carried off by escapers in a sequence of analytical models. For other details see the caption to Fig.12.