Real-time propagator for high temperature dimensional reduction

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We discuss the extension of dimensional reduction in thermal field theory at high temperature to real-time correlation functions. It is shown that the perturbative corrections to the leading classical behavior of a scalar bosonic field theory are determined by an effective contour propagator. On the real-time-branch of the time-path contour the effective propagator is obtained by subtracting the classical propagator from the contour propagator of thermal field theory, whereas on the Euclidean branch it reduces to the non-static Matsubara propagator of standard dimensional reduction.

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I. INTRODUCTION

It has been argued that the dominant non-perturbative behavior of bosonic quantum fields at high temperature may be described by a classical effective theory [1]. The basic idea is to replace the full quantum theory by a scheme in which one solves classical equations of motion with initial conditions taken from a thermal ensemble.

In this letter we discuss the derivation of this classical approximation from the time-path formulation of thermal field theory. The thermal state is taken into account by extending the time evolution along a contour in the complex t-plane as shown in fig. 1. The contour consists of three parts: a forward time branch $C_1$ on the real axis, a backward branch $C_2$ and a third branch $C_3$ down the Euclidean path $t = t_{in} - i\tau$ from $\tau = 0$ to $\tau = \beta$, with $\beta = T^{-1}$ the inverse temperature [2]. The classical limit is obtained in the stationary phase approximation. On the real-time contour $C_{12} = C_1 \cup C_2$ this yields the classical equations of motion, whereas on the Euclidean branch the fields must be taken stationary and equal to the initial values for the real time evolution [3].

A systematic improvement involves the calculation of the effective action, both on the Euclidean, and the real-time contours. This is a generalization of the approach of dimensional reduction on the Euclidean contour, in which one constructs an effective 3D-theory for the static Matsubara mode. The bare parameters of the 3D-theory are matched to the corresponding physical parameters in the full thermal field theory [4,5], by a perturbative calculation of the static two- and four-point functions [6,7].

Below we present a scheme in which this procedure is generalized to the real-time contour. We show that the non-static Matsubara propagator generalizes to a contour propagator which is the difference of the contour propagator of thermal field theory and the classical propagator. The non-classical effective action on the real-time contour has loop contributions determined by this generalized non-static contour propagator.

II. CLASSICAL LIMIT

In the time-path formulation of thermal field theory [2], real-time correlation functions may be obtained from the generating functional

$$Z[j] = \int D\phi D\pi \exp iS[\phi, \pi] + j \cdot \phi .$$

The dot-notation is an abbreviation for $j \cdot \phi = i \int_C dt d^3x j(x) \phi(x)$. We consider the simple model of a scalar theory in Minkowski space $x = (t, \vec{x})$, with action

$$S[\phi, \pi] = \int_C dt \int d^3x \left[ \pi \phi - \frac{1}{2} \pi^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4 \right],$$

parameterized by a bare mass $m$ and a coupling constant $\lambda$, and we assume that the system is prepared in thermal equilibrium at some initial time $t_{in}$. The time-contour $C$ is depicted in fig 1. Occasionally we shall indicate fields on different parts of the contour as $\phi_r(t, \vec{x}) = \phi(t, \vec{x}), \; t \in C_r, \; r = 1, 2, 3$. In order that the KMS condition is satisfied, the fields are periodic with respect to the begin- and end-point of the contour: $\phi_1(t_{in}, \vec{x}) = \phi_3(t_{in} - i\beta, \vec{x})$. 

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In dimensional reduction one seeks to obtain from the generating functional (1) an effective theory for the static mode $\mathcal{P}\phi(x) = \Phi(\vec{x})$ and its conjugate momentum $\mathcal{P}\pi(x) = \Pi(\vec{x})$, where the projection operator is defined as

$$\mathcal{P}\phi(x) = iT \int_{C_3} dt \phi(t, \vec{x}) = T \int_0^\beta dt \phi(t_{in} - i\tau, \vec{x}) .$$

The effective theory is obtained by separating off the integration over the static fields and writing

$$Z[j] = \int D\Phi D\Pi \exp W[\Phi, \Pi; j] .$$

The effective action of the dimensionally reduced theory is formally given by the constrained generating functional $\omega(\vec{x})$ which contains the contributions of the non-static modes. The difference with the constraint effective potential defined to the Euclidean branch the fields may be decomposed into Matsubara modes $n$ modes on the contour and $\bar{n}$ modes arbitrary on the contour and $\bar{\pi}$ periodicity.

The term linear in the fields $\bar{\phi}$ and $\bar{\pi}$ is then determined by the classical background field $\bar{\phi}(\vec{x})$, $\bar{\pi}(t_{in}, \vec{x}) = \Pi(\vec{x})$ required by periodicity.

It is convenient to separate off all terms in the shifted action $S[\phi, \phi, \pi]$ that are of second-and-higher order in the fields $\phi(x)$ and $\pi(x)$. This defines a new action

$$\tilde{S}[\phi, \pi] = S[\phi, \phi, \pi] + i\delta_\phi S[\phi, \pi] \cdot \phi + i\delta_\pi S[\phi, \pi] \cdot \pi - S[\phi, \pi] .$$

The term linear in $\pi(x)$ can be made to vanish by imposing

$$\delta_\pi S[\phi, \pi] = \tilde{\pi} = 0 , \quad t \in C_{12} .$$

On the Euclidean branch the contribution of this term vanishes on account of the fact that the background fields are time independent there and the constraints on the integration in (2). For the same reason the term linear in $\phi$ on the right-hand side of (3) gives no contribution on $C_3$. As a result of these manipulations we find for the constraint action defined in (3)

$$W[\Phi, \Pi; j] = iS[\phi, \pi] + \tilde{W}[J] + j \cdot \phi .$$

The second term is entirely determined by the shifted action (3)

$$\tilde{W}[J] = \log \int D\phi D\pi \delta (\mathcal{P}\phi) \delta (\mathcal{P}\pi) \exp i\tilde{S}[\phi, \pi] + J \cdot \phi .$$

The source has the value $J = j + \delta_\phi S[\phi]$ on the real part $C_{12}$ of the contour, where $S[\phi] = S[\phi, \dot{\phi}]$ is the ordinary action of a $\lambda\phi^4$-theory.

If we identify the background fields with the solution to the stationary phase approximation for (1), the source $J$ is equal to zero, and we are left with the first and last term on the right-hand side of (3). This constitutes the classical approximation of thermal field theory (3) (4) (5). The time evolution of the field $\phi(x)$ is then determined by the classical equations of motion $j + \delta_\phi S[\phi] = 0$ in Minkowski space. The remaining functional integral in (3) averages over the initial conditions with thermal weight $\exp iS[\Phi, \Pi] = \exp -\beta H[\Phi, \Pi]$.}

III. QUANTUM CORRECTIONS

The interaction of the classical background field $\tilde{\phi}(x)$ with the quantum and thermal fluctuations is contained in the generating functional (1). These quantum corrections will be determined perturbatively. We insert an integral
representation $\delta(P\pi) = \int D\chi \exp \chi \cdot \pi$, for the momentum delta-function in (3). The auxiliary field $\chi(x)$ is zero everywhere except on $C_3$ where it has a spatial dependence: $\chi(t_{in} - i\tau, \vec{x}) = \chi(\vec{x})$. Interchanging the order of integration, we first integrate out the momentum variable $\pi(x)$ and subsequently the auxiliary field $\chi(\vec{x})$. The relevant Gaussian integral is

$$\int D\chi \int D\pi \exp i \int_C dt d^3x \left[ -\frac{1}{2} \pi^2 + (\chi + \dot{\phi}) \pi \right] = \exp \left( \frac{1}{2} \phi \cdot \phi - \frac{1}{2} P\phi \cdot P\phi \right),$$

(10)

where the second factor comes from the constraint on the momentum integration. Recalling definition (3) for the projector, we deduce

$$P\phi(x) = iT \int_{C_3} dt \dot{\phi}(x) = iT[\phi_1(t_{in}, \vec{x}) - \phi_2(t_{in}, \vec{x})].$$

(11)

The boundary condition imposes periodicity on the fields over the whole contour $C$, but not on the Euclidean segment $C_3$ separately. Hence, the integral (11) gives a finite contribution which is equal to the difference of the field values at the begin and end points of the real-time contour $C_{12}$; see fig 1.

Substituting the result (10) into (9), we obtain

$$e^{\tilde{W}[J]} = \int D\phi \delta(P\phi) \exp \left( i\tilde{S}[\phi] - \frac{1}{2} P\phi \cdot P\phi + J \cdot \phi \right).$$

(12)

In this expression the action $\tilde{S}[\phi] = S_0[\phi] + \tilde{S}_I[\phi]$ is the ordinary action for a scalar $\lambda\phi^4$-theory translated by a background field $\phi(x)$, with quadratic part $S_0[\phi]$ and interaction part

$$\tilde{S}_I[\phi] = \frac{1}{4} \phi \cdot \phi + \frac{1}{3!} \phi \cdot \phi \cdot \phi + \frac{1}{4!} \phi \cdot \phi \cdot \phi \cdot \phi.$$  

(13)

The striped lines denote the background field $\bar{\phi}$ and the solid lines denote the quantum field $\phi$. The interactions are treated in the usual perturbative manner.

$$\tilde{W}[J] = e^{i\tilde{S}[-i\delta J]} e^{\tilde{W}_0[J]} \bigg|_{con}.  

(14)

So we can focus our attention on the free part of the generating functional (12). Using again an auxiliary field, we write

$$e^{\tilde{W}_0[J]} = e^{-\frac{1}{2} P\phi \cdot P\phi} \int D\phi e^{iS_0[\phi] + (J + \chi) \cdot \phi}.  

(15)

We have brought out the term quadratic in

$$P\phi(x) = T \left[ \frac{\delta}{\delta J_1(t_{in}, \vec{x})} - \frac{\delta}{\delta J_2(t_{in}, \vec{x})} \right].$$

(16)

The Gaussian integral over the field $\phi$ can be performed and leads to

$$e^{\tilde{W}_0[J]} = Z_0[0] e^{-\frac{1}{2} P\phi \cdot P\phi} \int D\chi \exp \frac{1}{2} (J + \chi) \cdot D\phi \cdot (J + \chi),$$

(17)

with $D_C(x - x')$ the well-known free contour propagator of thermal field theory (2). In the next section we will show by explicitly evaluating (17) that the free generating functional is a standard bilinear functional of $J$ as for free fields

$$e^{\tilde{W}_0[J]} = N e^{\frac{1}{2} J \cdot \Delta_C \cdot J},$$

(18)

but with a new contour propagator

$$\Delta_C(x - x') = D_C(x - x') - S_C(x - x'),$$

(19)

where $S_C(x - x')$ turns out to be the propagator of the classical theory on the real-time contour and the static Matsubara propagator on $C_3$.  

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IV. CONTOUR PROPAGATOR

In this section we calculate the free action $\tilde{W}_0[J]$ given in (17). Writing out the Gaussian exponent, we encounter the 3D propagator $D_{3D} = 1/\omega_k^2$ (the static Matsubara propagator), and the time-integrated propagator

$$\tilde{f}_C(t, \vec{k}) = i T \int_{C_3} dt' \tilde{D}_C(t - t', \vec{k}) = \begin{cases} \frac{T}{\omega_k} \cos \omega_k(t_{in} - t) & t \in C_{12} \\ \frac{T}{\omega_k} & t \in C_3 \end{cases}, \quad (20)$$

with $\omega_k^2 = \vec{k}^2 + m^2$. Here we used the explicit expression for the contour propagator in (9), formula (3.74). By the usual procedure of shifting the integration variable, we obtain the result

$$e^{\tilde{W}_0[J]} = Z_0[0] e^{-\frac{i}{2} \hat{P}_\phi \cdot \hat{P} \phi e^{\frac{i}{2} J \cdot D_C \cdot J}}, \quad (21)$$

with a subtracted contour propagator $D^I_C(x, x') = D_C(x - x') - S^I_C(x, x'),$

$$S^I_C(x, x') = -\beta \int \frac{d^3 k}{(2\pi)^3} e^{i \vec{k} \cdot (\vec{x} - \vec{x}')} \omega_k^2 \tilde{f}_C(t, \vec{k}) \tilde{f}_C(t', \vec{k}). \quad (22)$$

Note that this expression is not time-translation invariant.

We will now show that the exponential factor quadratic in $\hat{P} \phi(\vec{x})$ in eq.(21) leads to a further subtraction of the contour propagator. Using (10) we first calculate

$$\frac{1}{2} \hat{P} \phi \cdot (J \cdot D_C \cdot J) = -T \int d^4 x' [D^I_{1s}(t_{in}, t') - D^I_{2s}(t_{in}, t')] J_s(x'), \quad (23)$$

with $D_{ss}(t, t') = D_C(t, t')$ with $t \in C_r, t' \in C_s$. We suppressed the spatial dependence of the propagators. Their difference is easily calculated. On the Euclidean contour it vanishes. Using the Fourier transform we find for $t' \in C_{12}$

$$\tilde{g}_C(t', \vec{k}) = \tilde{D}^I_{1s}(t_{in} - t') - \tilde{D}^I_{2s}(t_{in} - t') = \frac{i}{\omega_k} \sin \omega_k(t_{in} - t'), \quad s = 1, 2. \quad (24)$$

The subtraction (22) gives no contribution. Next we notice the property

$$\frac{1}{2} \hat{P} \phi \hat{P} \phi \cdot (J \cdot D_C \cdot J) = 0. \quad (25)$$

The two results (24) and (25) together imply that the first exponent $-\frac{1}{2} \hat{P} \phi \cdot \hat{P} \phi$ in (21) may be replaced by $-\frac{1}{2} J \cdot S^I_C \cdot J$, with the propagator

$$S^I_C(x, x') = T \int \frac{d^3 k}{(2\pi)^3} e^{i \vec{k} \cdot (\vec{x} - \vec{x}')} \tilde{g}_C(t, \vec{k}) \tilde{g}_C(t', \vec{k}). \quad (26)$$

The integrand is given by (24) for $t \in C_{12}$, and is zero for $t \in C_3$.

This all leads to the following conclusion: the subtraction term $S_C(x - x')$ in (13) is given by the sum of (22) and (26). Writing this out on the various parts of the contour, we find

$$\tilde{S}_C(t - t', \vec{k}) = \begin{cases} \frac{T}{\omega_k} \cos \omega_k(t - t') & t, t' \in C_{12} \\ \frac{T}{\omega_k} \cos \omega_k(t_{in} - t) & t \in C_3, t' \in C_{12} \\ \frac{T}{\omega_k} \cos \omega_k(t_{in} - t_{in}) & t \in C_{12}, t' \in C_3 \\ \frac{T}{\omega_k} & t, t' \in C_3 \end{cases} \quad (27)$$

We recognize the first term in this list as the familiar classical free two-point function (10). The last term is the static Matsubara propagator. The second and third term connect the vertical Euclidean time branch $C_3$ with the real-time branch $C_{12}$. In thermal field theory it is shown that in the limit $t_{in} \to -\infty$, the two real-time contours decouple from the Euclidean branch, provided that the infinitesimal damping coefficient $\epsilon$ in the spectral density is kept finite till the end of the calculations (11,13).
V. CONCLUSIONS AND OUTLOOK

We have argued that for the case of a scalar bosonic field, the scheme of dimensional reduction may be extended to real-time correlation functions. Like in imaginary-time dimensional reduction, the IR behavior of the effective theory is improved because on internal lines classical modes are excluded. This allows a perturbative calculation of the quantum corrections to the effective action

\[ \Gamma[\bar{\phi}] = iS[\bar{\phi}, \bar{\pi}] + \Gamma_{\bar{\phi}}[0]. \] (28)

The Feynman rules for calculating the loop contributions in the background field \( \bar{\phi}(x) \) are the usual ones of thermal field theory, save for the fact that solid lines connecting the vertices in eq\((13)\), represent the contour propagator \((19)\). If the external background field is conveniently chosen to be equal to the expectation value of the quantum field, the background field satisfies the equation of motion

\[ \frac{\partial \Gamma[\bar{\phi}]}{\partial \bar{\phi}(x)} = j(x), \] (29)

on the real-time contour. The equation of motion \((29)\) has to be solved with the initial conditions \( \bar{\phi}(t_{in}, \vec{x}) = \Phi(\vec{x}), \\dot{\bar{\phi}}(t_{in}, \vec{x}) = \Pi(\vec{x}) \). This may be compared with the effective action as obtained by Greiner and Muller \([13]\) by integrating out hard modes, \( k^2 > k_c^2 \sim \lambda T^2 \). As a consequence their vertex functions are cut-off dependent.

As an example let us consider the case of linear relaxation for a given external source \( j(x) \) which is the same on both branches. This leads to the equation of motion \([14]\)

\[ -\partial^2 \bar{\phi}(x) + m^2 \bar{\phi}(x) + \int d^4 x' \Sigma(x-x') \bar{\phi}(x') = j(x), \] (30)

where \( \Sigma(x-x') \) is the real-time retarded self-energy on \( C_1 \) and the advanced self-energy on \( C_2 \), both calculated with the contour propagator \((19)\). Compared to the explicit expression for the retarded self energy to one-loop-order in the full quantum theory \([14]\), the zero-mode contribution is seen to drop out. This "matches" the mass of the quantum theory to the mass of the effective classical theory \( m^2 \rightarrow m^2 + \lambda T^2/24 - (c.t.)_{3D} \). The last term is the counterterm for the 3D-theory that cancels the linear divergence in the one-loop two-point function. A calculation of the coupling constant for the effective theory gives the standard dimensional reduction result \([6]\). So we find an effective theory which, in the local approximation and to one-loop-order, reduces to the classical theory with the matching relations as proposed in \([4, 5, 9]\).

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Fig. 1. The time contour $C$. 