We discuss classical dynamics of electron spin in two-dimensional semiconductors with a spin-split spectrum. We focus on a special case, when spin-orbit induced random magnetic field is directed along a fixed axis. This case is realized in III-V-based quantum wells grown in [110] direction and also in [100]-grown quantum wells with equal strength of Dresselhaus and Bychkov-Rashba spin-orbit couplings. We show that in such wells the long-time spin dynamics is determined by non-Markovian memory effects. Due to these effects the non-exponential tail $1/t^2$ appears in the spin polarization.

In spite of the large number of publications devoted to the study of non-Markovian transport phenomena, the role of memory effects in spin dynamics is not well understood. In this paper we discuss the slow down of the spin relaxation in 2D systems due to the non-Markovian memory. This effect is of particular interest for new rapidly growing branch of semiconductor physics, spintronics. The main goal of spintronics is the development of novel electronic devices that exploit the electron charge and spin on equal footing. For effective functioning of such devices, the lifetime of the non-equilibrium spin must be long compared to the device operation time. In III-V-based semiconductor nanostructures, this requirement is not easy to satisfy, since in such structures the spin polarization relaxes rapidly due to Dyakonov-Perel (DP) spin relaxation mechanism \cite{15}. This mechanism predicts the exponential relaxation of non-equilibrium spin with a certain characteristic time $\tau_S$. At small temperatures, this relaxation might slow down due to quantum interference effects \cite{16,17}. However, with increasing temperature, the interference effects are suppressed by inelastic scattering. Here we show that the classical non-Markovian effects which are not very sensitive to the temperature might lead to long-lived $1/t^2$ non-exponential tail in the spin polarization in analogy with velocity relaxation described by Eq. \ref{eq:1}.

The DP mechanism is based on the classical picture of the angular spin diffusion in random magnetic field induced by spin-orbit coupling. In 2D systems, the corresponding spin-relaxation time $\tau_S$ is inversely proportional to the momentum relaxation time $\tau$: $1/\tau_S \sim (\omega^2_p)/\tau$ \cite{18}. Here $\omega_p$ is the frequency of spin precession in a random magnetic field, $p$ is the electron momentum, and angular brackets denote averaging over momentum directions (for $p \approx p_F$). As a consequence, in high-mobility structures which are most promising for device applications, $\tau_S$ is especially short. However, in some special cases, the relaxation of one of the spin components can be rather slow even in a system with high mobility. In particular, a number of recent researches \cite{10,20,21,22,23,24} are
devoted to GaAs symmetric quantum wells (QW) grown in [110] direction. In such structures, \( \omega_p \) is perpendicular to the QW plane \(^2\) and depends on one component of the in-plane momentum (say \( x \)-component)

\[
\omega_p = \alpha p_z \hat{z}.
\]

Here \( \hat{z} \) is the unit vector normal to the well plane and \( \alpha \) characterizes the strength of the spin-orbit coupling. Also, the random magnetic field might be parallel to a fixed axis in an asymmetric \([100]\)-grown QW due to the interplay between the bulk \(^2\) and structural \(^2\) spin-orbit couplings \(^{27, 28, 29, 30, 31}\) (the structural coupling depends on the gate voltage \(^{32}\), so one can tune these two couplings to have equal strength). For such QW Eq. \((\text{2})\) is also valid, but in this case unit vector \( \hat{z} \) is parallel to the QW plane. In both cases, one component of the spin, \( s_z \), does not relax. Therefore, these structures are especially attractive for spintronics applications.

In this paper we discuss long-time dynamics of the spin polarization in such structures. We consider the relaxation of the vector \( \mathbf{s} = (s_x, s_y) \) which is perpendicular to the random magnetic field \((\mathbf{s} \perp \omega_p)\) and show that similar to the velocity autocorrelation function, the spin correlation function has long-lived \(1/t^2\) tail (both for the case of strong scatterers and for smooth potential). The analogy between velocity and spin relaxation is based on the following. As seen from Eq. \((\text{2})\), the spin rotation angle is proportional to the integral \( \varphi \sim \int p_z dt \) and is equal to zero for closed paths \(^{27}\). Thus when electron returns to the impurity its spin restores the original direction. This implies some kind of memory effects specific for the systems under discussion.

The Hamiltonian of the system is given by

\[
\hat{H} = \frac{\mathbf{p}^2}{2m} + \frac{\hbar}{2} \alpha p_z \sigma_z + U(\mathbf{r}),
\]

where \( U(\mathbf{r}) \) is a random potential, \( \sigma_z \) is the Pauli matrix and \( m \) is the electron effective mass.

In the Boltzmann approach, the classical dynamics of spin related to Hamiltonian \(^3\) is described by the kinetic equation \(^1\)

\[
\frac{\partial \mathbf{s}}{\partial t} + (\mathbf{v} \nabla) \mathbf{s} = \hat{J}_B \mathbf{s} + [\omega_p \times \mathbf{s}],
\]

where \( \mathbf{s}(\mathbf{r}, \mathbf{p}) \) is the spin density related to the averaged spin as \( \mathbf{S} = \int \mathbf{s}(\mathbf{r}, \mathbf{p}) \, d^2 \mathbf{r} \, d^2 \mathbf{p} / (2\pi \hbar)^2 \) and \( \hat{J}_B \) is the Boltzmann collision integral. Here we consider a case of degenerated electron gas \((T \ll E_F)\), assuming that the spin-polarized electrons have energies close to the Fermi energy \( E_F \). First we assume that electrons are scattered by strong scatterers randomly distributed in plane with average concentration \( n \). In this case

\[
\hat{J}_B \mathbf{s}(\theta) = n v_F \int \sigma(\theta - \theta') |\mathbf{s}(\theta') - \mathbf{s}(\theta)| d\theta',
\]

where \( \sigma(\theta) \) is differential cross-section of one scatterer (for electrons with energy \( E \approx E_F \)) and we used short-hand notation \( s(\theta) = s(\mathbf{r}, \mathbf{p}) \) (\( \theta \) is the angle of the vector \( \mathbf{p} \)). In Eq. \((\text{4})\) we neglected inelastic scattering. The role of such scattering will be briefly discussed below. To account for classical memory effects we will follow the method proposed in Refs. \(^{33}\) (calculation of the velocity correlation function by this method is presented in Ref. \(^{13}\)). The key idea is to replace \( n \to \sum \delta(\mathbf{r} - \mathbf{r}_i) = n + \nu(\mathbf{r}) \) in the collision integral, where \( \nu(\mathbf{r}) = \sum \delta(\mathbf{r} - \mathbf{r}_i) - n \). \( \langle \nu(\mathbf{r}) \rangle = 0 \) (averaging is taken over the position of the impurities). The collision integral becomes \( \hat{J}_B \to \hat{J}_B + \hat{J}_s \), where

\[
\hat{J}_s \mathbf{s}(\theta) = \nu(\mathbf{r}) v \int \sigma(\theta - \theta') [\mathbf{s}(\theta') - \mathbf{s}(\theta)] d\theta'.
\]

By the following transformation

\[
\mathbf{s} = \tilde{T}(x) \mathbf{s}', \quad \tilde{T} = \begin{bmatrix} \cos qx & -\sin qx \\ \sin qx & \cos qx \end{bmatrix}, \quad q = \alpha m
\]

we eliminate the spin rotation term \( [\omega_p \times \mathbf{s}] \) from Eq. \((\text{4})\)

\[
\frac{\partial \mathbf{s}'}{\partial t} + (\mathbf{v} \nabla) \mathbf{s}' = (\hat{J}_B + \hat{J}_s') \mathbf{s}',
\]

(corresponding unitary transformation of Hamiltonian \(^3\) is presented in Refs. \(^{31, 32}\)). Following \(^{13}\), we solve equation \((\text{8})\) treating the term proportional to \( \nu(\mathbf{r}) \) as a small correction. In the second order of perturbation theory we obtain the following equation:

\[
\frac{\partial \mathbf{s}'}{\partial t} + (\mathbf{v} \nabla) \mathbf{s}' = \hat{J}_B \mathbf{s}' + \delta \hat{J} \mathbf{s}',
\]

where the kernel \( G(\mathbf{r}, \varphi, \varphi', t) \) of the operator \( \hat{G} \) obeys

\[
\frac{\partial G}{\partial t} + (\mathbf{v} \nabla) G = \hat{J}_B G + \hat{G}(\mathbf{r}) \delta(\varphi - \varphi') \delta(t).
\]

To calculate the average in the Eq. \((\text{10})\) we take into account that \( \langle \nu(\mathbf{r}) \nu(\mathbf{r}') \rangle = n \delta(\mathbf{r} - \mathbf{r}') \). As a result we get:

\[
\delta \hat{J} \mathbf{s}' = v n \int_0^\infty dt' d\theta' \delta(\theta - \theta', t') \left[ \mathbf{s}'(\theta', t-t') - \mathbf{s}'(\theta, t-t') \right],
\]

where

\[
\delta(\theta - \theta', t) = -\int d\varphi \sigma(\theta - \varphi - \varphi') G(0, \varphi, \varphi', t) d\varphi d\varphi'
\]

\[
\times \sigma_0(\varphi - \varphi, t) \sigma_0(\varphi' - \varphi', t) d\varphi d\varphi'
\]

Here \( \sigma_0 = \int d\varphi \sigma(\varphi) \) is the total cross-section and \( G(0, \varphi, \varphi', t) = G(\mathbf{r}, \varphi, \varphi', t)|_{r \to 0} \) is the probability for an electron starting in the direction \( \mathbf{n} = (\cos \varphi, \sin \varphi) \) to
return to the initial impurity after time $t$ along the direction $\mathbf{n}' = (\cos \varphi', \sin \varphi')$ (see Fig. 1). Four terms in the product $[\sigma(\theta - \varphi') - \sigma_0 \delta(\theta - \varphi)\delta(\varphi' - \theta')]$ correspond to four types of correlations shown in Fig. 1. Fig. 1b shows the process, where electron experiences two real scatterings on the same impurity. In Eq. (13) the corresponding contribution is presented by the term proportional to $\sigma_0 \delta(\theta - \varphi' - \theta')$. In the process shown in Fig. 1c, an electron passes twice the region with the size of the order of impurity size without scattering. Since the electron "keeps memory" about absence of impurity at a certain region of space, there exists a correlation which is accounted for by the term proportional to $\sigma_0^2 \delta(\theta' - \theta')$ in Eq. (13). The interpretation of the two other terms is based on the fact that in the Boltzmann picture, which neglects correlations, the following processes are allowed. An electron scatters on an impurity and later on passes through the region occupied by this impurity without a scattering (see Fig. 1a). Another process is shown in Fig. 1d. The contributions of the terms $\sigma_0 \sigma(\theta - \varphi') \delta(\varphi' - \theta')$ and $\sigma_0^2 \delta(\theta - \varphi') \sigma(\varphi' - \theta')$ in Eq. (13) correct the Boltzmann result by substrating probabilities of such unphysical events.

At $t \gg \tau$ the return probability reads

$$G(0, \varphi - \varphi', t) = \frac{1}{8\pi^2 D\tau} \left(1 - \frac{l^2}{2D\tau} \mathbf{nn}'\right),$$

where $l^{-1} = nv\sigma_\tau$ and $\sigma_\tau = \int d\varphi \sigma(\varphi)(1 - \cos \varphi)$. Integrating Eq. (14) over angles we get $1/4\pi D\tau$ which is the probability to return to the initial point with arbitrary angle (diffusive return). The second term in Eq. (14) is a small angle-dependent correction which is responsible for the effect under discussion. Indeed, one can see that the first term in Eq. (14) gives zero contribution to Eq. (13). Calculating the contribution of the second term we get

$$\delta \sigma^{(\theta, t)} = \frac{\sigma_\tau^2}{4\pi^2 v} \cos \theta.$$

It worth noting that $\delta \sigma(\theta, t)dt$ has a dimension of length and can be interpreted as a correction to the scattering cross-section due to diffusive returns taking the time lying in the interval $[t, t + dt]$.

The diffusion equation can be obtained by standard means from kinetic equation (9) with the use of Eq. (16). As a result we find the diffusion-like equation for the isotropic part of the spin density $s_\tau = \langle s(x, \mathbf{r}, t)\rangle_\theta$ (averaging is taken over momentum directions):

$$\frac{\partial s_\tau}{\partial t} = D\Delta \left( s_\tau - \frac{n\sigma_\tau^2}{4\pi} \int_0^\infty s_\tau(t - t')dt'\right),$$

where $\Delta = \hat{T(x)\Delta\hat{T}(x)^{-1}} = (\partial/\partial x + q\hat{\epsilon})^2 + \partial^2/\partial y^2$ and $\hat{\epsilon}$ is the antisymmetric tensor: $\hat{\epsilon} = [e_x \times s]$. Eq. (16) describes the spin dynamics in the diffusion approximation. It simplifies in the homogeneous case:

$$\frac{\partial s_\tau}{\partial t} = -\frac{s_\tau}{\tau_\tau} + \frac{n\sigma_\tau^2\tau}{4\pi^2} \int_0^\infty s_\tau(t - t')dt'/t^2.$$

Here $1/\tau_\tau = Dq^2 = (\alpha_{PF}/2)\tau/2$ is the Dyakonov-Perel' spin relaxation rate. The initial condition for (17) is $s_\tau(0) = s_1$ (we also assume that $s_\tau(t) = 0$ for $t < 0$). Neglecting the second term in the rhs of Eq. (17) we get the exponential relaxation $s_\tau(t) = \exp(-t/\tau_\tau)s_1$. This solution is valid until $\exp(-t/\tau_\tau) \sim n\sigma_\tau^2\tau_\tau S/t^2$. For larger times, $\int_0^t dt's_\tau(t - t')/t^2 \approx s_1\tau_\tau S/t^2$ and one can neglect the term $\partial s_\tau/\partial t$ in Eq. (17). As a result we find that the polarization has a long-lived tail

$$s_\tau(t) \approx \frac{n\sigma_\tau^2\tau_\tau S}{4\pi^2}\frac{s_1}{t},$$

which is positive in contrast to Eq. (11). Eq. (13) was derived for the case of strong scatterers with low concentration ($n\sigma^2 \ll 1$). The opposite limiting case (weak scatterers, $n\sigma^2 \gg 1$) corresponds to the smooth random potential with the correlation function $\langle U(x)U(x') \rangle = \int dq \exp(iq(x - x'))/2\pi$. In this case, the collision integral can be written as a sum of the Boltzmann collision integral $\hat{J}_B = (1/\tau)(\partial^2/\partial \varphi^2)$ and

$$\hat{J} = \hat{J} - \hat{J}_B = \frac{1}{\tau} \left(\int_0^\infty dt[n \times f(r)]/[n \times f(r - vt)] - 1\right) \frac{\partial}{\partial \varphi} \hat{J}_B,$$

where $1/\tau = (1/2m^2v^2) \int dq \kappa_q dq $ and $f = -\nabla U/r/mv$. One can check that $\langle \hat{J} \rangle = 0$. Substituting Eq. (13) into Eq. (10) and accounting for two types of correlations we get

$$\delta \hat{J}_s^{(\varphi, \tau)} = \frac{2\pi^2 d^2}{\tau} \int_0^\infty dt'[G''(0, t')\partial^2 s(\varphi, t - t')/\partial^2 \varphi],$$

and

$$G''(\varphi, t')\partial^2 s(\varphi + \pi, t - t')/\partial^2 \varphi.$$
where $G''(\phi, t) = \partial^2 G(0, \phi, t)/\partial \phi^2$. The equations analogous to the case of strong scattering centers yield

$$s_0(t) = \frac{d^2}{d^2} \frac{\tau \rho_s}{\tau^2} s_i. \tag{21}$$

In Eqs. (20) and (21), $d = \sqrt{2 \int_0^\infty \kappa q^3 dq / \int_0^\infty \kappa q^2 dq}$.

Above we assumed that $s_i(t)$ is homogenous. For slowly varying $s_i(r, t)$, the derived equations relate $s_0(r, t)$ with $s_i(r)$ provided that the spatial scale of inhomogeneity $L$ is large compared to $\sqrt{\tau \rho_s} \sim 1/\kappa$ [37]. One can show that in the opposite case $l \ll L \ll 1/\kappa$, these equations are also valid relating $\int d\mathbf{r} \ s_0(\mathbf{r}, t)$ with $\int d\mathbf{r} \ s_i(\mathbf{r})$.

Let us briefly discuss the role of electron-electron interaction. Such interaction manifests itself both in inelastic scattering and in the additional, with respect to the diffusion process, decay of density fluctuation due to Maxwell relaxation. Such a relaxation partially suppresses the long-lived tail in velocity correlation function leading to the faster decay [38]:

$$1/\tau \rightarrow \tau_B^2 / \tau^3,$$

where $\tau_B$ is the 2D screening length, which coincides with the Bohr radius. In contrast to this, the Maxwell relaxation has no effect on the spin dynamics because in the classical approximation the spin fluctuations are not coupled to the charge fluctuations, and, as a consequence, do not lead to creation of long-range electrical field responsible for Maxwell relaxation.

As for electron-electron collisions, their characteristic time $\tau_{ee}$ is inversely proportional to $T^2$. For relatively small temperatures, $\tau_{ee} \gg \tau$ and electron-electron collisions do not have any effect on the spin relaxation in the classical approximation. In the opposite limiting case, $\tau_{ee} \ll \tau$, electron-electron collisions might suppress spin relaxation [39]. The detailed discussion of this case is out of the scope of this paper. We believe that $1/\tau^2$ dependence of long-time polarization is also valid for this case, while the coefficient in this dependence might change.

Finally, we compare non-Markovian tail in the spin polarization (Eqs. (19), (21)) with the long-lived tail induced by weak localization [17]: $s_0(t) = s_i(\tau_s / \pi k_F l t) \exp(-t/\tau_{\varphi})$, where $\tau_{\varphi}$ is the phase-breaking time. At $T = 0$, when $\tau_{\varphi} = \infty$, long-time spin dynamics is determined by weak localization. However, for $T \neq 0$ classical memory effects dominate at $t \gg \tau_{\varphi}$.

In conclusion, we developed a theory of long-time spin dynamics for 2D system, where spin-orbit-induced magnetic field is parallel to a fixed axis. We showed that independently on the type of disorder the non-equilibrium spin polarization in such a system decays as $1/t^2$ (for $t \rightarrow \infty$) due to purely classical memory effects.

This work has been supported by RFBR, a grant of the RAS, a grant of the Russian Scientific School, and a grant of the foundation "Dynasty"-ICFPM.

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[36] In a purely classical case $\sigma_0 = \infty$ (with an exception for Lorentz gas model) due to divergent contribution of small-angle scattering. However, our final result only depends on $\sigma_0$ (see Eq. (15)) thus justifying the approach based on classical consideration.
[37] Actually, collision integral also contains terms with spatial derivatives. However, they give negligible contribution to velocity and spin correlation functions.
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