THE HESSIAN OF THE DISTANCE FROM A SURFACE IN THE
HEISENBERG GROUP

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Abstract. Given a smooth surface \( S \) in the Heisenberg group, we compute the Hessian of the function measuring the Carnot-Charathéodory distance from \( S \) in terms of the Mean Curvature of \( S \) and of an “imaginary curvature” which was introduced in [2] in order to find the geodesics which are metrically normal to \( S \). Explicit formulae are given when \( S \) is a plane or the metric sphere.

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1. Introduction

In this article we continue to study the properties of the function “signed distance from a surface \( S \)”, \( \delta_S \), in the Heisenberg group, started in [2]. In particular we are interested in the horizontal Hessian of \( \delta_S \) and in the related notions of curvature for \( S \). Let \( \mathbb{H} = \mathbb{H}^1 \) be \( \mathbb{R}^3 \) with the Heisenberg group structure, and let \( d \) be the associated Carnot-Charathéodory distance. See Section [2] for the definitions. If \( S \subset \mathbb{H} \) is closed, the distance from a point \( P \) to \( S \) is

\[
d_S(P) = \inf_{Q \in S} d(P, Q)
\]

Here, we consider the case where \( S = \partial \Omega \) is the \( C^3 \) boundary, in the Euclidean sense, of an open subset of \( \mathbb{H} \). It would be natural to consider \( C^2 \) surfaces in the Euclidean sense, since such surfaces satisfy a internal/external Heisenberg sphere condition at any non-characteristic point, see [2]. However, our elementary approach requires a degree more of regularity. This is a minor nuisance, since here we are interested in geometric properties of surfaces.

The signed distance from \( S \) is

\[
\delta_S(P) = \begin{cases} 
-d_S(P) & \text{if } P \in \Omega \\
 0 & \text{if } P \notin \Omega
\end{cases}
\]

Observe that, given \( S \), the signed distance \( \delta_S \) is defined modulo a sign, which corresponds to the choice of one of the two open sets having boundary \( S \), or, equivalently, to the choice of an orientation for \( S \).

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It was shown in [24] that, if $S$ is a closed subset of $\mathbb{H}$, then $d_S$ is $a.e.$ solution of the eikonal equation
\begin{equation}
|\nabla_H d_S| = 1,
\end{equation}
where $\nabla_H = (X,Y)$ is the horizontal gradient. More on this line of investigation is in [4], [6], [12], [31]. Recall that the left invariant vector fields
\[ X = \partial_x + 2y\partial_t, \quad Y = \partial_y - 2x\partial_t \]
are a orthonormal basis for the horizontal bundle $\mathcal{H}$ of $\mathbb{H}$. Concerning the properties of the distance function from a point and the distance function from a surface, we recall the paper by Agrachev and Gauthier, [1] and the contribution by Vershik and Gershkovich, which is surveyed in [33]. In [2], we proved several regularity results for the function $\delta_S$, in particular that, if $S$ is $C^2$ in the Euclidean sense, then $\nabla_H \delta_S$ is $C^1$ in the Euclidean sense in an open neighborhood of $S \setminus \text{Char}(S)$, where $\text{Char}(S)$ is the characteristic set of $S$.

In this article, we move a step forward in the understanding of the higher regularity of $\delta_S$. The \textbf{horizontal Hessian} of a smooth function $f$ defined on an open subset of $\mathbb{H}$ is the matrix
\[ Hess_H f = \begin{bmatrix} XXf & YXF \\ XYf & YYf \end{bmatrix}. \]
See, for instance, [10]. We denote by $I$ and $J$ the identity and the simplectic matrix, respectively,
\[ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \]

\textbf{Theorem 1.1.} Let $S = \partial \Omega$ be boundary of a $C^3$ open subset of $\mathbb{H}$ and let $P$ be a noncharacteristic point of $S$. Then,
\begin{equation}
Hess_H \delta_S (P) = v_P S \otimes v_P S \cdot (h_S(P) I + p_S(P) J),
\end{equation}
where $v_P S$ is the unit horizontal vector tangent to $S$ at $P$, $h_S(P)$ is the mean curvature of $S$ at $P$ and $p_S(P)$ is the curvature of the oriented metric normal to $S$ at $P$.

We shall also refer to $h_S(P)$ as the \textbf{real curvature} and to $p_S(P)$ as the \textbf{imaginary curvature} of $S$ at $P$, respectively. Some explanation about the terminology is in order. Complete definitions will be given in Section 2.

If $P$ is a noncharacteristic point of $S$, the intersection of the space tangent to $S$ at $P$, $T_PS$, with the horizontal fiber $\mathcal{H}_P$ is a one dimensional linear space $V_P S$, the \textbf{horizontal direction tangent to $S$ at $P$}. The \textbf{unit horizontal vector tangent to $S$ at $P$} is the unit vector spanning $V_P S$, which is defined modulo a sign. There is a rich literature on the calculus on surfaces in $\mathbb{H}$. Just to mention a few, we refer to [26], [13], [14], [15], [18], [28], [29]. In [14], in particular, surfaces are defined in terms of graphs of functions, with the aid of an implicit function theorem, an approach which is especially interesting in that it provides an intrinsic definition of $C^1$ surfaces in the Heisenberg sense. We remark that $v_P S$ is a first order differential object in both the Heisenberg and the Euclidean sense.

If the open set $\Omega \subset \mathbb{H}$ and $S = \partial \Omega$ have, near $P$, the smooth parametrizations $g(z,t) < 0$ and $g(z,t) = 0$, respectively, then the analytic expression for $p_S(P)$ is
\begin{equation}
p_S(P) = -\frac{[X,Y]g(P)}{|\nabla_{\mathbb{H}} g(P)|}.
\end{equation}
We say that the parametrization $g$ is \textit{compatible with the orientation} of $S$. Observe that $p_S(P)$ is a $C^1$ object in the Euclidean sense, since $[X,Y]g = -4\partial_t g$, while it is intermediate, in a sense, between $C^1$ and $C^2$ in the Heisenberg sense. It was shown in [2] how $p_S$ is a quantity peculiarly related to the Heisenberg geometry. Let $\mathcal{N}_P^+ S = \gamma$ be the \textbf{oriented metric normal to $S$ at $P$}, i.e., the geodesic arc $\gamma$ leaving $\Omega$ at $P$ such that, for small $\sigma > 0$, $\delta_S(\gamma(\sigma)) = \sigma$. The lifetime $\tau$ of the geodesic $\gamma$, the maximum amount of time over which $\gamma$ is length-minimizing,
is \( \tau = \frac{\pi}{|p_S(P)|} \). The sign of \( p_S(P) \) is positive or negative according to the fact that \( N^+_P S \) points “upward” (the \( t \)-coordinate of \( P^{-1} \cdot N^+_P S(\sigma) \) increases with \( \sigma \)) or “downward”. \( p_S(P) = 0 \) if \( N^+_P S \) is a Euclidean straight line. The number \( p_S(P) \) is a sort of curvature for \( N^+_P S \). Theorem 1.1 says, in a way, that \( p_S(P) \) is also a sort of curvature of the surface \( S \). Observe that the quantity \( p_S \) does not appear in the expression of \( \text{Hess}_{\mathbb{H}} \delta_S \), the symmetrized horizontal Hessian \( \text{Hess}_{\mathbb{H}} \delta_S \), the superscript \( tr \) denoting transposition of matrices.

The mean curvature \( h_S(P) \) of a surface \( S \) at \( P \) is an object much studied in the recent literature \([7, 13, 28, 29, 5]\). If \( g \) is a parametrization of \( S \) near the noncharacteristic point \( P \), which is compatible with the orientation of \( S \), the mean curvature is defined by

\[
h_S(P) = X \left( \frac{X g(P)}{\nabla_{\mathbb{H}} g(P)} \right) + Y \left( \frac{Y g(P)}{\nabla_{\mathbb{H}} g(P)} \right).
\]

Observe that, if \( S = \{(z, t) : g(z, t) = 0\} \), and \( \Omega = \{(z, t) : g(z, t) < 0\} \), then

\[
h_S(P) = \text{div}_{\mathbb{H}} N_P S,
\]

where \( N_P S = \frac{\nabla_{\mathbb{H}} g(P)}{|\nabla_{\mathbb{H}} g(P)|} \) is the intrinsic outer unit normal to \( S \) at \( P \). Observe that, by Theorem 1.1, it is tautological that

\[
\text{Trace}(\text{Hess}_{\mathbb{H}} \delta_S(P)) = h_S(P).
\]

In Section 4 with some applications in perspective, we show that there are other equivalent definitions for \( h_S(P) \). On the one hand, \( h_S(P) \) is the eigenvalue of a suitable horizontal Weingarten map on \( S \) at \( P \), in analogy with the Euclidean case (Theorem 4.2). On the other hand, if \( \Gamma \) is the unique horizontal curve on \( S \) through \( P \) and \( \pi(\Gamma) \) is its vertical projection onto the plane \( t = 0 \) in \( \mathbb{H} \), then \( h_S(P) \) is the Euclidean curvature of \( \pi(\Gamma) \) at \( \pi(P) \) (Theorem 4.1). Some of these facts have been independently proved in a similar form in \([3, 18, 7, 8]\). In \([18]\) it is also proved that \( h_S \) is the trace of a second order matrix which is different from the horizontal Hessian of \( \delta_S \). For a more general treatment of the parallel transport see, e.g., \([19]\) for the Riemannian case and \([21]\) for the subriemannian one.

As an application of Theorem 1.1 and Theorem 4.1 in Theorem 5.2, we give the explicit expression for the horizontal Hessian of the function measuring the Carnot-Charathéodory distance from a point.

There are two reasons why we restricted our analysis to the Heisenberg group. The first is out of necessity: the proofs in this paper are based on the results in \([2]\), which are stated and proved in this setting. The second is that the Heisenberg group provides the simplest nontrivial example of Carnot-Charathéodory group and it is important in itself, for instance in modelling the visual cortex \([9, 8, 30]\). It is interesting that these new applications require a finer study of surfaces in the Heisenberg group and of their curvatures.

Here is a short overview of the article. In Section 2 we provide some background and preliminary results. In Section 3 we prove Theorem 1.1. Section 4 is devoted to a discussion of the mean curvature, while in Section 5 we discuss several examples.

**Note on the proofs.** We will always assume implicitly that our surface \( S \) is locally parametrizable as \( 0 = g(x, y, t) = t - f(x, y) \). In particular, without loss of generality, we assume that \( \Omega \) lies below the graph of \( f \). The cases when \( S \) does not admit a parametrization of this kind, i.e., in the “near-Euclidean case” when the plane tangent to \( S \) is vertical at some point, can be dealt with in a way similar to that employed in \([2]\).

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2. Notation and preliminaries

In this section, we collect some basic definitions and known facts about the structure and the geometry of $\mathbb{H}$. There is a vast literature on sub-Riemannian geometry and Carnot groups. Just to quote a few titles, we refer the reader to \[3, 13, 16, 22, 25, 26, 27, 32\].

The Heisenberg group $\mathbb{H} = \mathbb{H}^1$ is the Euclidean space $\mathbb{R}^3$ endowed with the noncommutative product

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + 2(x'y - xy')).$$

Sometimes it is convenient to think of the elements of $\mathbb{H}$ as $(z, t) \in \mathbb{C} \times \mathbb{R}$. The Carnot-Charathéodory distance in $\mathbb{H}$ is the sub-Riemannian metric that makes pointwise orthonormal the left invariant vector fields $X$ and $Y$,

$$X = \partial_x + 2y\partial_t, \quad Y = \partial_y - 2x\partial_t.$$  

The vector fields $X$, $Y$ do not commute, $[X, Y] = -4\partial_t$. The distance between two points $P$ and $Q$ in $\mathbb{H}$ is denoted by $d(P, Q)$. The span of the vector fields $X$ and $Y$ is called horizontal distribution, and it is denoted by $\mathcal{H}$. The fiber of $\mathcal{H}$ at a point $P$ of $\mathbb{H}$ is $\mathcal{H}_P = \text{span}\{X_P, Y_P\}$. The inner product in $\mathcal{H}_P$ is denoted by $\langle \cdot, \cdot \rangle$.

An important element of $\mathbb{H}$’s structure is the dilation group at the origin $\{\delta_\lambda : \lambda \neq 0\}$,

$$\delta_\lambda(z, t) = (\lambda z, \lambda^2 t), \quad z = x + iy$$

By left translation, a dilation group is defined at each point $P$ of $\mathbb{H}$.

The Heisenberg group is also endowed with a rotation group, which is useful in simplifying some calculations. For $\theta \in \mathbb{R}$, let

$$R_\theta(z, t) = (e^{i\theta}, t)$$

be the rotation by $\theta$ around the $t$-axis. Composing with left translation, one could define rotations around any vertical line $(x, y) = (a, b)$. $R_\theta$ is an isometry of $\mathbb{H}$ and its differential acts on the fiber $\mathcal{H}_O$ as a rotation by $\theta$. Under the usual identification between the Riemannian tangent space of $\mathbb{H}$ at $O$, $T_O\mathbb{H}$, and the Lie algebra $\mathfrak{h}$ of $\mathbb{H}$, the differential of $R_\theta$ can be thought of as a rotation on span$\{X, Y\}$, the first stratum of $\mathfrak{h}$. With respect to the basis $\{X, Y\}$,

$$dR_\theta V = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} V$$

whenever $V \in \text{span}\{X, Y\}$. With some abuse of notation, we denote $dR_\theta$ by $R_\theta$.

The distance between two points $P$ and $Q$ in $\mathbb{H}$ is defined as follows. Consider an absolutely continuous curve $\gamma$ in $\mathbb{R}^3$, joining $P$ and $Q$, which is horizontal. That is, $\gamma(t) = a(t)X_{\gamma(t)} + b(t)Y_{\gamma(t)}$ lies in $\mathcal{H}_{\gamma(t)}$. Its Carnot-Charathéodory length is $l_{\mathbb{H}}(\gamma)$,

$$l_{\mathbb{H}}(\gamma) = \int (a(t)^2 + b(t)^2)^{1/2} dt$$

The Carnot-Charathéodory distance between $P$ and $Q$, $d(P, Q)$, is the infimum of the Carnot-Charathéodory lengths of such curves. The infimum is actually a minimum, the distance between $P$ and $Q$ is realized by the length of a geodesic. By translation invariance, all geodesics are left translations of geodesics passing through the origin. The unit-speed geodesics at the origin \[23, 22\] are

$$\gamma_{O, \phi, W}(\sigma) = \begin{cases} x(\sigma) = \sin(\phi(W)) \frac{1 - \cos(\phi(\sigma))}{\phi} + \cos(\phi(W)) \frac{\sin(\phi(\sigma))}{\phi} \\ y(\sigma) = \sin(\phi(W)) \frac{\sin(\phi(\sigma))}{\phi} - \cos(\phi(W)) \frac{1 - \cos(\phi(\sigma))}{\phi} \\ t(\sigma) = 2 \frac{\phi(\sigma) - \sin(\phi(\sigma))}{\phi} \end{cases}$$

Here, $W$ is a unitary vector in $\mathcal{H}_O$ and $\phi(W) \in [0, 2\pi)$ is unique with the property $\dot{\gamma}_{O, \phi, W}(0) = W$. $\phi \in \mathbb{R}$, and the geodesic is length minimizing over any interval of length $2\pi/|\phi|$. In the case
\( \phi = 0 \), the geodesic is a straight line in the plane \( \{ t = 0 \} \),
\[
x(\sigma) = \cos(\alpha(W))\sigma, \quad y(\sigma) = \sin(\alpha(W))\sigma,
\]
and we say that the geodesic is straight.

From these equations we deduce the parametric equations of the boundary of the ball \( B(0, r) \).\( (z, t) \in \partial B(0, r) \) if and only if there is \( \phi \in [-2\pi/r, 2\pi/r] \) so that
\[
\begin{aligned}
|z| &= 2\frac{\sin(\phi r/2)}{\phi} \\
|t| &= 2\frac{\phi - \sin(\phi r)}{\phi^2}.
\end{aligned}
\]

If \( P = (z, t) \) and \( z \neq 0 \), then there exists a unique length minimizing geodesic connecting \( P \) and \( O \). If \( P = (0, t) \), \( t \neq 0 \), (i.e., if \( P \) belongs to the center of \( \mathbb{H} \)) then there is a one parameter family of length minimizing geodesics joining \( P \) and \( O \), obtained by rotation of a single geodesic around the \( t \)-axis.

We call curvature of \( \gamma_{O,\phi,W} \) the parameter \( \kappa(\gamma_{O,\phi,W}) = |\phi| \). We will see later how \( \kappa(\gamma_{O,\phi,W}) \) is related to the curvature of a surface in \( \mathbb{H} \). The notion of curvature extends to all geodesics by left translations, \( \kappa(L_P\gamma) = \kappa(\gamma) \), where \( L_P \) is left translation by \( P \in \mathbb{H} \) in \( \mathbb{H} \). Given points \( P = (z, t) \) and \( P' = (z', t') \), they are joined by a unique length minimizing geodesic, unless \( z = z' \).

Let
\[
\gamma_{P,\phi,\alpha} = L_P\gamma_{O,\phi,\alpha}
\]
The parameter \( \phi \) is geometric in the following sense. \( 2\pi/|\phi| \) is the length of \( \gamma_{P,\phi,W} \) and \( \text{sgn}(\phi) \) is positive if and only if the \( t \)-coordinate increases with \( \sigma \). Recall that in \( \mathbb{H} \) the orientation of the \( t \)-axis is an intrinsic notion, unlike the Euclidean space. If \( \gamma_{P,\phi,W} \) and \( \gamma_{P',\phi,W'} \) have an arc in common, then \( \phi = \pm \phi' \), while no such easy relation exists for the parameter \( W \). To change the orientation of a geodesic, observe that
\[
\gamma_{P,\phi,W}(\sigma) = \gamma_{P,-\phi,-W}(-\sigma)
\]
Unlike the Euclidean case, a geodesic \( \gamma \) leaving \( O \) is not determined by its tangent vector at the origin, \( \gamma(0) = W = \cos(\alpha(W))X_O + \sin(\alpha(W))Y_O \). The extra parameter we need is \( \kappa(\gamma) \).

The parameter \( \kappa(\gamma) \), if \( \phi = \kappa(\gamma) \neq 0 \), is related to the dilation group as follows. \( \delta_{\lambda} (\gamma_{O,\phi,W}) \) is a reparametrization of the geodesic \( \gamma_{O,\phi,\lambda W} \). That is, all geodesics \( \gamma \) leaving \( 0 \) and having fixed initial velocity \( \dot{\gamma}(0) = v \) in \( \mathcal{H}_O \) are dilated of each other. The case of the straight geodesics is the limiting one, corresponding to \( \lambda \rightarrow 0 \). In a precise sense, then, the set of geodesics at \( O \) is parametrized by the unit circle in \( \mathcal{H}_O \) and by the dilation group, a feature of \( \mathbb{H} \) with no Euclidean counterpart. This simple fact will be a recurrent theme in the following sections.

Let \( S \) be a smooth surface in \( \mathbb{H} \). We need some geometric notions about \( S \), some of which were studied in [2].

**Definition 2.1.** Let \( E \) be a closed subset of \( \mathbb{H} \), \( P \in E \). The metric normal to \( E \) at \( P \) is the set \( \mathcal{N}_P S \) of the points \( Q \in \mathbb{H} \) such that \( d(Q, E) = d(Q, P) \).

**Definition 2.2.** Let \( S \) be a \( C^1 \) surface in the Euclidean sense in \( \mathbb{H} \), which is the boundary of an open set \( \Omega \) and of \( \mathbb{H} - \overline{\Omega} \) and let \( P \in S \). The oriented metric normal to \( S \) at \( P \), \( \mathcal{N}_P^+ S \), is the unique parametrization of \( \mathcal{N}_P S \) such that \( \delta_S(\mathcal{N}_P^+ S(\sigma), P) = \sigma \).

**Definition 2.3.** Let \( S \) be a smooth surface in \( \mathbb{H} \) and let \( P \in S \) be a non-characteristic point. The Euclidean tangent space to \( S \) at \( P \) is denoted by \( T_P S \). The direction tangent to \( S \) at \( P \) is the 1-dimensional space \( V_P S = T_P S \cap \mathcal{H}_P \). The plane tangent to \( S \) at \( P \), \( \Pi_P S \), is the Euclidean plane in \( \mathbb{H} \), tangent to \( S \) at \( P \) in the Euclidean sense. The direction normal to \( S \) at \( P \) is \( N_P S = \mathcal{H}_P \oplus V_P S \), where \( \oplus \) is taken with respect to the subRiemannian metric.
The exterior normal to $S$ at $P$ is  
$$N_P S = (N_P S)'(0),$$
the vector tangent to $N_P S$ at $P$.

The group tangent to $S$ at $P$ is the 2-dimensional vector space $G_P S = V_P S \oplus T$, where $T = \{ (0, 0, t) : t \in \mathbb{R} \}$ is the center of $\mathbb{H}$ and $V_P S$ is the one parameter subgroup of $\mathbb{H}$ generated by $V_P S$. One point we want to make in the present paper is that $G_P S$ does not seem to capture the complexity of the geodesics’ set, while $T_P S$ does, in a precise sense.

The following facts are easily established by direct calculation.

**Proposition 2.1.** Let $S$ be a smooth surface in $\mathbb{H}$, implicitly defined by $g(x, y, t) = 0$. Let $P \in S$ be non-characteristic. Then,

$$V_P S = \text{span}\{ Y g \cdot X - X g \cdot Y \}, \quad N_P S = \text{span}\{ X g \cdot X + Y g \cdot Y \} = \text{span}\{ \nabla_{\mathbb{H}} g \}$$

Suppose that $S$ is locally parametrized by $g(x, y, t) = 0$. Let $P$ be a point in $S$ and let $C$ be the characteristic point of $\Pi_P S$, the euclidean plane in $\mathbb{H}$ which is tangent to $S$ at $P$. Then,

$$d(P, C) = \frac{|\nabla_{\mathbb{H}} g(P)|}{2|\partial g(P)|}$$

The equation of $N_P^+ S$ can be written in terms of $g$’s partial derivatives. $N_P^+ S = P \cdot \eta$ (left translation by $P$), where $\eta = (u, v, s)$ and

$$\eta(\sigma) = \begin{cases} 
  x'(\sigma) = \frac{1}{4u g} \left\{ Y g(P) \left(1 - \cos \left(\frac{4\partial g(P)\sigma}{|\nabla_{\mathbb{H}} g(P)|}\right)\right) + X g(P) \sin \left(\frac{4\partial g(P)\sigma}{|\nabla_{\mathbb{H}} g(P)|}\right) \right\} \\
  y'(\sigma) = \frac{1}{4u g} \left\{ -X g(P) \left(1 - \cos \left(\frac{4\partial g(P)\sigma}{|\nabla_{\mathbb{H}} g(P)|}\right)\right) + Y g(P) \sin \left(\frac{4\partial g(P)\sigma}{|\nabla_{\mathbb{H}} g(P)|}\right) \right\} \\
  t'(\sigma) = \frac{1}{|\nabla_{\mathbb{H}} g(P)|^2} \left\{ \frac{4\partial g(P)\sigma}{|\nabla_{\mathbb{H}} g(P)|} - \sin \left(\frac{4\partial g(P)\sigma}{|\nabla_{\mathbb{H}} g(P)|}\right) \right\}
\end{cases}$$

See p.10 in [2]. When $g(z, t) = t - f(z)$, we have

$$\begin{cases} 
  x' = \frac{1}{4}(X f \sin \alpha + Y f(1 - \cos \alpha)) \\
  y' = \frac{1}{4}(Y f \sin \alpha - X f(1 - \cos \alpha)) \\
  t' = \frac{1}{4\sqrt{f}}(\alpha - \sin \alpha),
\end{cases}$$

where $\alpha = \frac{4\pi}{\sqrt{f}}$. We introduce an exponential map $F : (u, v, \sigma) \mapsto F(u, v, \sigma)$, defined from an open subset of $\mathbb{R}^2 \times \mathbb{R}$ with values in $\mathbb{H}$,

$$F(u, v, \sigma) = N_{(u,v,f(u,v))}^+ S(\sigma).$$

We are going to use the coordinates

$$(x, y, t) = F(u, v, \sigma) = (u, v, f(u, v)) \circ (x', y', t')$$

where $(x', y', t') = \gamma_{u,v}(\sigma)$ and $\gamma_{u,v}$ is the metric normal’s left translate by $P^{-1}$. We have, then

$$\begin{cases} 
  x_u = 1 + x'_u \\
  y_u = y'_u \\
  t_u = f_u(u, v) - 2y' + t'_u + 2(vx'_u - uy'_u),
\end{cases}$$

See [2], p.15, Lemma 6.5.

At p.16 of [2] we find the following formulae, which we summarize in a vast lemma.

**Lemma 2.1.** We have the following equalities.

$$4x'_u = (X f)_u \sin \alpha + (Y f)_u(1 - \cos \alpha) + \alpha_u X f \cos \alpha + \alpha_u Y f \sin \alpha$$

$$= (X f)_u \sin \alpha + (Y f)_u(1 - \cos \alpha)$$

$$- \cos \alpha \frac{X f}{2 |\nabla_{\mathbb{H}} f|^3} \left(2X f(X f)_u + 2Y f(Y f)_u\right) - \sin \alpha \frac{X f}{2 |\nabla_{\mathbb{H}} f|^3} \left(2X f(X f)_u + 2Y f(Y f)_u\right)$$.
and so \( (x'_u)^\tau = 0 \). Analogously
\[
4x'_u = (Xf)_u \sin \alpha + (Yf)_u (1 - \cos \alpha) \nonumber \\
- \left( \cos \alpha Xf \frac{4\tau}{|\nabla_H f|^3} + \sin \alpha Yf \frac{4\tau}{|\nabla_H f|^3} \right) (Xf(Xf)_u + Yf(Yf)_u),
\]
and \( (x'_u)^\tau = 0 \);
\[
4x'_\tau = Xf \cos \alpha \frac{4}{|\nabla_H f|} + Yf \sin \alpha \frac{4}{|\nabla_H f|},
\]
and \( (x'_u)^\tau = 0 \);
\[
4y'_u = (Yf)_u \sin \alpha - (Xf)_u (1 - \cos \alpha) \nonumber \\
- \frac{4\tau}{|\nabla_H f|^3} \left( \cos \alpha Yf - \sin \alpha Xf \right) (Xf(Xf)_u + Yf(Yf)_u),
\]
and \( (y'_u)^\tau = 0 \);
\[
4y'_\tau = \frac{4\tau}{|\nabla_H f|^3} \left( \cos \alpha Yf - \sin \alpha Xf \right),
\]
and \( (y'_u)^\tau = 0 \);
\[
8t'_u = 2(Xf(Xf)_u + Yf(Yf)_u) (\alpha - \sin \alpha) - \frac{4\tau}{|\nabla_H f|} (Xf(Xf)_u + Yf(Yf)_u) (1 - \cos \alpha),
\]
and \( (t'_u)^\tau = 0 \);
\[
8t'_v = 2(Xf(Xf)_v + Yf(Yf)_v) (\alpha - \sin \alpha) - \frac{4\tau}{|\nabla_H f|} (Xf(Xf)_v + Yf(Yf)_v) (1 - \cos \alpha),
\]
and \( (t'_v)^\tau = 0 \), and, eventually
\[
8t'_\tau = 4 |\nabla_H f| (1 - \cos \alpha),
\]
and
\[
t'_\tau = 0.
\]

3. The Hessian of the function measuring the distance from a smooth surface

Let \( g \) be a smooth function from \( \mathbb{H} \) in \( \mathbb{R} \). We define the horizontal Hessian \( \text{Hess}_{\mathbb{H}} g \) and the symmetrized horizontal Hessian \( \text{Hess}^*_{\mathbb{H}} g \) of \( g \). See, e.g., \[10], \[20] for a discussion of second order differential operators in the Heisenberg group.

\[
(12) \quad \text{Hess}_{\mathbb{H}} g = \begin{pmatrix} XXg & XYg \\ XYg & YYg \end{pmatrix}
\]
Let \((XY)^* g = \frac{1}{2} (XYg + YXg)\).

\[
(13) \quad \text{Hess}^*_{\mathbb{H}} g = \begin{pmatrix} XXg & (XY)^* g \\ (XY)^* g & YYg \end{pmatrix}
\]
Next, we prove Theorem \[11] For ease of calculation, we restate it in a less geometric form.
Theorem 3.1. Let $S = \partial \Omega$ be a non-characteristic smooth surface in $\mathbb{H}$, where $\Omega$ locally defined by the equation $t - f(x, y) < 0$. Let

$$Q = XXf(Yf)^2 - 2(XY)^* f Xf Yf + YYf(Xf)^2$$

Then, the horizontal Hessian of $\delta_S$ is

$$\text{Hess}_H \delta_S = \left( \begin{array}{cc} \frac{Yf}{|\nabla f|^3}Q + 4 \frac{XfYf}{|\nabla f|^3} & -Yf \left[ \frac{Xf}{|\nabla f|^3}Q - 4 \frac{Yf}{|\nabla f|^3} \right] \\ -Xf \left[ \frac{Yf}{|\nabla f|^3}Q + 4 \frac{Xf}{|\nabla f|^3} \right] & \frac{Yf}{|\nabla f|^3}Q + 4 \frac{Yf}{|\nabla f|^3} \end{array} \right)$$

(14)

$$= v_P S \otimes v_P S \cdot (h_S(P)I + p_S(P)J)$$

Hence, the symmetrized horizontal Hessian of the distance function $\delta_S$, restricted to the surface $S$, is

$$\text{Hess}_H \delta_S = \left( \begin{array}{cc} \frac{(Yf)^2}{|\nabla f|^3}Q - 4 \frac{XfYf}{|\nabla f|^3} & \frac{XfYf}{|\nabla f|^3}Q + 2 \frac{(Xf)^2 - (Yf)^2}{|\nabla f|^3} \\ \frac{XfYf}{|\nabla f|^3}Q + 2 \frac{(Xf)^2 - (Yf)^2}{|\nabla f|^3} & \frac{(Xf)^2}{|\nabla f|^3}Q + 4 \frac{Xf}{|\nabla f|^3} \end{array} \right)$$

(15)

and, on $S$,

$$\Delta_H \delta_S = \frac{Q}{|\nabla f|^3} \frac{XXf(Yf)^2 - 2(XY)^* f Xf Yf + YYf(Xf)^2}{|\nabla f|^3}.$$

(16)

In particular, if $C$ is a characteristic point in $S$, then $|\text{Hess}_H \delta_S(P)| = O(d(P,C)^{-1})$ as $P \to C$ in $S$.

Elementary linear algebra applied to (14), or a direct calculation with (15), shows that the characteristic polynomial of $\text{Hess}_H \delta_S$ is

$$P(\lambda) = \lambda^2 - h_S(P)\lambda - \frac{1}{4}p_S^2(P).$$

We could consider the roots $\lambda_{1,2}$ of $P$, i.e. the eigenvalues of the symmetrized horizontal Hessian $\text{Hess}_H \delta_S$, to be the “horizontal principal curvatures” of $S$ at $P$. It would be interesting to have a direct geometric interpretation for these two quantities.

The eigenvalues of $\text{Hess}_H \delta_S$ have opposite sign, unless $p_S(P) = 0$. On the other hand, if there is an open subset $A$ of $S$ s.t. $p_S(P) = 0$ whenever $P \in A$, then $A$ must be a vertical subset of $S$, i.e. $A$ is defined as $\{(x,y,t) : \varphi(x,y) = 0\}$. Following [19] (see also [20] and [11]), we say that a function $k$ is mildly convex if $\text{Hess}_H \delta_S k$ is positive definite. As a consequence of the above, we have the following remark.

Remark 3.1. Let $S$ be an orientable surface in $\mathbb{H}$, which is free of characteristic points, and let $\delta_S$ the signed distance from $S$ associated to an orientation of $S$. Then, $\delta_S$ is mildly convex or concave in a neighborhood of $S$ if and only if $S$ is a vertical set $\{(x,y,t) : \varphi(x,y) = 0\}$ and $\{(x,y) : \varphi(x,y) = 0\}$ is a curve in $\mathbb{R}^2$ which does not change the sign of its Euclidean curvature.

In particular, the Carnot-Charathéodory distance in $\mathbb{H}$ is not mildly convex, since the metric spheres are not vertical sets. This could be compared with the result in [10], where it is proved that the gauge distance in $\mathbb{H}$ is mildly convex.

In order to prove the theorem, we need some preliminary results. We denote by $\text{Hess} k$ the Euclidean Hessian of a function $k$ from $\mathbb{R}^3$ in $\mathbb{R}$. If $K = (k_1, \ldots, k_m)$ maps $\mathbb{R}^3$ in $\mathbb{R}^m$, we write

$$\text{Hess} K = (\text{Hess} k_1, \ldots, \text{Hess} k_m)$$

Consider the equation

$$(x, y, t) = F(u, v, \tau)$$
Let us denote by Hess $x$, Hess $y$, Hess $t$ the Euclidean Hessian matrices of $x(u, v, \tau), y(u, v, \tau), t(u, v, \tau)$; let $\xi = (u, v, \tau)$ and let $\xi_x, \xi_y, \xi_t$ be the columns of the Jacobian matrix of $F^{-1}$ computed when $\tau = 0$.

Let $a = \xi_x + 2y\xi_t$ and $b = \xi_y - 2x\xi_t$.

**Lemma 3.1.** The entries of the horizontal Hessian of $\delta_S$ satisfy the following relations.

(i) We have

$$|\nabla_{\mathbb{H}} f| \cdot XX\delta_S = - \left| F_u | F_v | \text{Hess}(a, a) \right|$$

Here, $|c_1|c_2|c_3|$ denotes the determinant of the $3 \times 3$ matrix having columns $c_j$, $j = 1, 2, 3$.

(ii) We have

$$|\nabla_{\mathbb{H}} f| \cdot YY\delta_S = - \left| F_u | F_v | \text{Hess}(b, b) \right|$$

(iii) $XY\tau$ and $YX\tau$ satisfy

\[
\begin{cases}
X\delta_S \cdot XX\delta_S + Y\delta_S \cdot XY\delta_S = 0 \\
X\delta_S \cdot YX\delta_S + Y\delta_S \cdot YY\delta_S = 0
\end{cases}
\]

**Proof.** The Inverse Function Theorem applied to the equation $(x, y, t) = F(u, v, \tau)$ gives a set of equations for the second derivatives of $\tau = \delta_S$, w.r.t. $x, y, t$. Let $p, q$ be any two variables, possibly equal, chosen among $x, y, t$. Then

\[
\begin{align*}
x_u p_q + x_v p_q + x_\tau t_p q &= - \text{Hess}_x(\xi_p, \xi_q) \\
y_u p_q + y_v p_q + y_\tau t_p q &= - \text{Hess}_y(\xi_p, \xi_q) \\
t_u p_q + t_v p_q + t_\tau t_p q &= - \text{Hess}_t(\xi_p, \xi_q)
\end{align*}
\]

We solve the linear systems with respect to the second partial derivatives of $\tau$,

\[
\tau_{pq} = \frac{\begin{vmatrix} x_u & x_v & -\text{Hess}_x(\xi_p, \xi_q) \\ y_u & y_v & -\text{Hess}_y(\xi_p, \xi_q) \\ t_u & t_v & -\text{Hess}_t(\xi_p, \xi_q) \end{vmatrix}}{\begin{vmatrix} x_u & x_v & x_\tau \\ y_u & y_v & y_\tau \\ t_u & t_v & t_\tau \end{vmatrix}}
\]

From Lemma 2.1

\[
\text{det} \begin{vmatrix} x_u & x_v & x_\tau \\ y_u & y_v & y_\tau \\ t_u & t_v & t_\tau \end{vmatrix} = |\nabla_{\mathbb{H}} f| \bigg|_{\tau=0}
\]

Since $XX = \partial_{xx} + 4y\partial_{xt} + 4y^2\partial_{tt}$, since the determinant is linear with respect to each column and Hess$(U, V)$ is bilinear in $U$ and $V$, then (22) implies that, when $\tau = 0$,

$$|\nabla_{\mathbb{H}} f| \cdot XX\delta_S = \left| F_u | F_v | F_\tau \right| \cdot XX\delta_S$$

$$= \left| F_u | F_v | - \left[ \text{Hess}(\xi_x, \xi_x) + 4y\text{Hess}(\xi_x, \xi_t) + 4y^2\text{Hess}(\xi_t, \xi_t) \right] \right|$$

$$= - \left| F_u | F_v | \text{Hess}(a, a) \right|$$

and this shows (i). (ii) can be proved in the same way, starting from the expression

$$YY = \partial_{yy} - 4x\partial_{yt} + 4x^2\partial_{tt}.$$ 

(iii) follows taking derivatives w.r.t. $X$ and $Y$ of the eikonal equation (2). \qed
Proof. of Theorem 3.1 By (i) and (ii) in Lemma 3.1 the expression of \( XX\delta_S \) and \( YY\delta_S \) can be computed if we know \( \text{Hess}F \) restricted to \( \tau = 0 \) and \( J(F^{-1})(x, y, t)|_{\tau=0} = (\xi_x|\xi_y|\xi_t) \). By (iii), this allows us to compute \( XY\delta_S \) and \( YX\delta_S \) as well.

We apply the classical Implicit Function Theorem to

\[
(x, y, t) = F(u, v, \tau).
\]

Recall from Lemma 2.1 that the Jacobian of \( F \),

\[
JF(u, v, 0) = J \begin{pmatrix} x & y & t \\ u & v & \tau \end{pmatrix}_{\tau=0} = \begin{pmatrix} 1 & 0 & \frac{Xf}{\nabla_H f} \\ 0 & 1 & \frac{Yf}{\nabla_H f} \end{pmatrix},
\]

whose determinant is

\[
\det JF(u, v, 0) = |\nabla_H f|.
\]

Its inverse matrix is

\[
J(F^{-1})(x, y, t)|_{\tau=0} = J \begin{pmatrix} u & v & \tau \\ x & y & t \end{pmatrix}_{\tau=0} = \begin{pmatrix} 1 + \frac{Xf f_u}{\nabla^2 f} & \frac{Xf f_v}{\nabla^2 f} & -\frac{Xf f_{\tau}}{\nabla^2 f} \\ \frac{Yf f_u}{\nabla^2 f} & 1 + \frac{Yf f_v}{\nabla^2 f} & -\frac{Yf f_{\tau}}{\nabla^2 f} \end{pmatrix} = (\xi_x|\xi_y|\xi_t)
\]

If \( g(x, y, t) = t - f(x, y) \), we write \( XXf = XXg, XYf = XYg, \) and so on. The following relations will be used below.

\[
(Xf)_u = XXf (Yf)_u = XYf (Xf)_v = YXf (Yf)_v = YYf
\]

Recalling the derivatives of \( x, y, t \) computed in the proof of Lemma 2.1 we can compute

\[
x_{uu} = \frac{1}{4}(Xf)_{uu} \sin \alpha + \frac{1}{4}(Yf)_{uu}(1 - \cos \alpha) + \left( \frac{1}{4}(Xf)_u \cos \alpha + \frac{1}{4}(Yf)_u \sin \alpha \right) \alpha_u + \frac{1}{4} \alpha_u \sin \alpha \frac{Xf}{2 \sqrt{\nabla^2 f}} \frac{4\tau}{3} (2Xf(Xf)_u + 2Yf(Yf)_u)
\]

\[
- \frac{1}{4} \cos \alpha \frac{Xf}{2 \sqrt{\nabla^2 f}} \frac{4\tau}{3} (2Xf(Xf)_u + 2Yf(Yf)_u)_u
\]

\[
- \frac{1}{4} \cos \alpha \frac{Xf}{2 \sqrt{\nabla^2 f}} \frac{4\tau}{3} (2Xf(Xf)_u + 2Yf(Yf)_u)_v
\]

\[
- \frac{1}{4} \sin \alpha \frac{Xf}{2 \sqrt{\nabla^2 f}} \frac{4\tau}{3} (2Xf(Xf)_u + 2Yf(Yf)_u)_\tau,
\]

hence

\[
(x_{uu})_{\tau=0} = 0.
\]

\[
x_{vu} = \frac{1}{4}(Xf)_{vu} \sin \alpha + \frac{1}{4}(Yf)_{uv}(1 - \cos \alpha) + \left( \frac{1}{4}(Xf)_u \cos \alpha + \frac{1}{4}(Yf)_u \sin \alpha \right) \alpha_v + \frac{1}{4} \alpha_v \sin \alpha \frac{Xf}{2 \sqrt{\nabla^2 f}} \frac{4\tau}{3} (2Xf(Xf)_u + 2Yf(Yf)_u)
\]

\[
- \cos \alpha \frac{Xf}{2 \sqrt{\nabla^2 f}} \frac{4\tau}{3} (2Xf(Xf)_u + 2Yf(Yf)_u)_v
\]

\[
- \alpha_v \frac{1}{4} \cos \alpha \frac{Xf}{2 \sqrt{\nabla^2 f}} \frac{4\tau}{3} (2Xf(Xf)_u + 2Yf(Yf)_u)_\tau
\]

\[
- \frac{1}{4} \sin \alpha \frac{Xf}{2 \sqrt{\nabla^2 f}} \frac{4\tau}{3} (2Xf(Xf)_u + 2Yf(Yf)_u)_\tau,
\]
\[(x_{vv})_{\tau=0} = 0.\]

\[x_{vv} = \frac{1}{4} (Xf)_{vv} \sin \alpha + \frac{1}{4} (Yf)_{vv} (1 - \cos \alpha) + \frac{1}{4} (Xf)_{v} \cos \alpha + \frac{1}{4} (Yf)_{v} \sin \alpha - \left( \frac{1}{4} \cos \alpha Xf \frac{4 \tau}{|\nabla_H f|^{3}} + \frac{1}{4} \sin \alpha Yf \frac{4 \tau}{|\nabla_H f|^{3}} \right) (Xf(Xf)_{u} + Yf(Yf)_{u})),\]

\[(x_{vv})_{\tau=0} = 0.\]

\[x_{u\tau} = \frac{1}{4} (Xf \frac{4}{|\nabla_H f|})_{u} \cos \alpha - \frac{1}{4} Xf \frac{4}{|\nabla_H f|} \sin \alpha \alpha_{u} + \frac{1}{4} (Yf \frac{4}{|\nabla_H f|})_{u} \sin \alpha + \frac{1}{4} Yf \alpha_{u} \cos \alpha \frac{4}{|\nabla_H f|},\]

\[(x_{u\tau})_{\tau=0} = \frac{Yf}{|\nabla_H f|^{3}} ((Xf)_{u} Yf - Xf(Yf)_{u}) = \frac{Yf}{|\nabla_H f|^{3}} ((XXf)Yf - Xf(YYf)).\]

\[x_{\tau\tau} = (Xf \frac{1}{|\nabla_H f|})_{\tau} \cos \alpha - Xf \alpha_{\tau} \sin \alpha \frac{1}{|\nabla_H f|} + (Yf \frac{1}{|\nabla_H f|})_{\tau} \sin \alpha + Yf \alpha_{\tau} \cos \alpha \frac{1}{|\nabla_H f|},\]

and

\[(x_{\tau\tau})_{\tau=0} = \frac{4Yf}{|\nabla_H f|^{2}}.\]

\[x_{v\tau} = (Xf \frac{1}{|\nabla_H f|})_{v} \cos \alpha - Xf \alpha_{v} \sin \alpha \frac{1}{|\nabla_H f|} + (Yf \frac{1}{|\nabla_H f|})_{v} \sin \alpha + Yf \alpha_{v} \cos \alpha \frac{1}{|\nabla_H f|},\]

and

\[(x_{v\tau})_{\tau=0} = \frac{Yf}{|\nabla_H f|^{3}} ((Xf)_{v} Yf - Xf(Yf)_{v}) = \frac{Yf}{|\nabla_H f|^{3}} ((XXf)Yf - Xf(YYf)).\]

Moreover

\[t_{uu} = f_{uu} + \left( \frac{1}{4} (Xf(Xf)_{u} + Yf(Yf)_{u})_{u} (\alpha - \sin \alpha) + \frac{1}{4} (Xf(Xf)_{u} - Yf(Yf)_{u})_{u} (1 - \cos \alpha) \frac{\tau}{|\nabla_H f|^{3}} \right) + \frac{\tau}{2 |\nabla_H f|} (Xf(Xf)_{u})_{u} + Yf(Yf)_{u} (1 - \cos \alpha))_{u} + \left( \frac{\tau}{2 |\nabla_H f|} (Xf(Xf)_{u} + Yf(Yf)_{u})_{u} \right) \cdot (1 - \cos \alpha) + 2v x_{uu} - 2uy_{uu} - 2y_{u} - 2y_{u},\]
so that
\[(t_{uu})_{\tau=0} = (f_{uu} + 2vx'_u - 2uy'_u - 2y'_u)_{\tau=0} = f_{uu},\]
analogously
\[(t_{ev})_{\tau=0} = f_{ev},\]
and
\[(t_{vu})_{\tau=0} = f_{vu},\]
and
\[(t_{uv})_{\tau=0} = 2vx'u - 2uy'_u - 2y'_u,\]
and
\[(t_{vu})_{\tau=0} = 2vx'_u - 2uy'_v + 2x'_v,\]
and
\[(t_{uv})_{\tau=0} = 2vx'u - 2uy'_v.\]

Hence
\[Hess_{\tau=0} = \frac{Yf}{|\nabla_H f|^3} \begin{pmatrix} 0 & 0 & (XXf)Yf - Xf(YYf) \\ 0 & 0 & (XYf)Yf - Xf(Yf) \\ (XXf)Yf - Xf(YYf) & (XYf)Yf - Xf(Yf) & 4 |\nabla_H f| \end{pmatrix}.\]

Arguing in the same way we get
\[Hess_{y_{\tau=0}} = \frac{Xf}{|\nabla_H f|^3} \begin{pmatrix} 0 & 0 & (XYf)Xf - Yf(XXf) \\ 0 & 0 & (XYf)Xf - Yf(Yf) \\ (XYf)Xf - Yf(XXf) & (XYf)Xf - Yf(Yf) & -4 |\nabla_H f| \end{pmatrix}.\]

The horizontal Hessian matrix can be written now directly.

A direct calculation shows that
\[a = \left( \frac{(Yf)^2}{|\nabla_H f|^2}, \frac{Xf \cdot Yf}{|\nabla_H f|^2}, \frac{Xf}{|\nabla_H f|} \right)\]
(viewed as a column vector) and
\[b = \left( -\frac{Xf \cdot Yf}{|\nabla_H f|^2}, \frac{(Xf)^2}{|\nabla_H f|^2}, \frac{Yf}{|\nabla_H f|} \right)\]

We then compute
\[Hess_{x_{\tau=0}}(a, a) = 2 \frac{(Yf)^2 Xf}{|\nabla_H f|^6} Q + 4 \frac{(Xf)^2 Yf}{|\nabla_H f|^4} \]
\[Hess_{y_{\tau=0}}(a, a) = -2 \frac{(Xf)^2 Yf}{|\nabla_H f|^6} Q - 4 \frac{(Xf)^3}{|\nabla_H f|^4} \]
\[Hess_{t_{\tau=0}}(a, a) = -2 \frac{(Yf)^2}{|\nabla_H f|^4} Q - 4 \frac{Xf \cdot Yf}{|\nabla_H f|^2} + 8 \frac{(Xf)^2}{|\nabla_H f|^4} (vYf + uXf) + 4 \frac{Xf \cdot Yf}{|\nabla_H f|^6} (vYf + uXf)Q\]
Recall now that

$$-\left| \nabla H f \right| \cdot XX\delta_S = \begin{bmatrix} 1 & 0 & \text{Hess}_x|_{\tau=0}(a,a) \\ 0 & 1 & \text{Hess}_y|_{\tau=0}(a,a) \\ 2v - Xf & -2u - Yf & \text{Hess}_t|_{\tau=0}(a,a) \end{bmatrix}$$

$$= -\frac{(Yf)^2}{|\nabla H f|^4} Q - \frac{4Xf \cdot Yf}{|\nabla H f|^2}$$

Analogous calculations yield the value of $YY\delta_S$.

In order to compute the mixed derivatives, we use (20).

Finally, observe that, if $S$ is a $C^3$ surface, then

$$|\text{Hess}_\delta S|_{\tau=0} = O(|\nabla H f|^{-1}) = O(d(P,C)^{-1})$$

as $P$ tends to a characteristic point $C$ of $S$. Let us recall that

$$Q = XXf \ (Yf)^2 - 2(XY)^* f \ Xf \ Yf + YYf \ (Xf)^2$$

Then, the horizontal Hessian of $\delta_S$ is that given in (14),

$$-\text{Hess}_H \delta_S = \begin{bmatrix} -\frac{(Yf)^2}{|\nabla H f|^2} Q - 4\frac{Xf \ Yf}{|\nabla H f|^2} Yf & \frac{Xf}{|\nabla H f|^2} Q - 4\frac{Yf}{|\nabla H f|^2} \\ Xf \left[ \frac{Yf}{|\nabla H f|^2} Q + 4\frac{Xf}{|\nabla H f|^2} \right] & -\frac{(Xf)^2}{|\nabla H f|^2} Q - 4\frac{Yf}{|\nabla H f|^2} \end{bmatrix}$$

i.e.

$$-\text{Hess}_H \delta_S = \frac{Q}{|\nabla H f|^4} \left( \begin{bmatrix} \frac{(Yf)^2}{|\nabla H f|^2} & \frac{Xf \ Yf}{|\nabla H f|^2} \\ -\frac{Xf \ Yf}{|\nabla H f|^2} & \frac{(Xf)^2}{|\nabla H f|^2} \end{bmatrix} - \frac{4}{|\nabla H f|^2} \left( \begin{bmatrix} \frac{Xf}{|\nabla H f|^2} & \frac{Yf}{|\nabla H f|^2} \\ -\frac{Yf}{|\nabla H f|^2} & \frac{(Xf)^2}{|\nabla H f|^2} \end{bmatrix} \right) \right)$$

Now, let

$$M = \begin{bmatrix} \frac{(Yf)^2}{|\nabla H f|^2} & \frac{Xf \ Yf}{|\nabla H f|^2} \\ -\frac{Xf \ Yf}{|\nabla H f|^2} & \frac{(Xf)^2}{|\nabla H f|^2} \end{bmatrix}.$$  

Then, if we set $p = \frac{4}{|\nabla H f|}$ and

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

we have

$$\text{Hess}_H \delta_S = kM + pMJ = M(kI + pJ) = M \begin{bmatrix} k & p \\ -p & k \end{bmatrix}.$$  

Let us remark that keeping in mind (1) and that $g = t - f(x, y)$ we have that

$$d(P,C) = \frac{|\nabla H f|}{2},$$

so that

$$p = \frac{2}{d(P,C)},$$

that represents indeed a curvature. Moreover let us define

$$\cos \theta = \frac{Xf}{|\nabla H f|},$$

and

$$\sin \theta = \frac{Yf}{|\nabla H f|}.$$  

Then

$$M = \begin{bmatrix} \sin^2 \theta & -\sin \theta \cos \theta \\ -\sin \theta \cos \theta & \cos^2 \theta \end{bmatrix} = v_P S \otimes v_P S,$$

$\blacksquare$
The estimate for the Hessian of $\delta S$ on $S$ cannot be extended outside $S$. In Section 3, we see that, for instance, when $S$ is the plane $t = 0$, the behavior of $\text{Hess}_N \delta S$ is worst when we approach 0 near the line $x = y = 0$.

4. Mean Curvature

In this section, we make some considerations about the notion of mean curvature for a surface in $\mathbb{H}$ (see, e.g., [18]).

**Definition 4.1.** Let $S = \partial \Omega$ be a smooth surface in $\mathbb{H}$, with $\Omega$ locally defined by the inequality $g(x, y, t) < 0$. The **mean curvature** of $S$ at $P \in S$ is

$$h_S(P) = X \left( \frac{Xg(P)}{|\nabla_H g(P)|} \right) + Y \left( \frac{Yg(P)}{|\nabla_H g(P)|} \right)$$

(30)

The curvature is well defined for non characteristic points only.

Here are some characterizations of mean curvature. The first is in terms of the distance function and it is an immediate consequence of Theorem 3.1.

**Corollary 4.1.** Let $S = \partial \Omega$ be the boundary of an open, $C^3$ set in $\mathbb{H}$. Then, for any non characteristic point $P$ in $S$,

$$\Delta_H \delta_S(P) = h_S(P)$$

R. Monti pointed out to us that the same result can be obtained by an explicit calculation, which does not rely on the knowledge of the distance’s Hessian. In [18], it is shown that the mean curvature is the trace of a certain matrix $M$, which is not the Hessian of the distance function.

The second is in terms of horizontal curves on $S$. First, we state an immediate consequence of the existence theorem for O.D.E.’s.

**Lemma 4.1.** Let $S$ be a smooth surface in $\mathbb{H}$ and let $P \in S$ be non-characteristic. Then, modulo restrictions and reparametrizations, there exists a unique horizontal curve $\Gamma$ through $P$ on $S$. Suppose that $S$ is defined by $g(x, y, t) = 0$. A parametrization of the curve is the solution of the O.D.E.’s system

$$\begin{align*}
\dot{\Gamma}(\tau) &= Yg(\Gamma(\tau)) \cdot X(\Gamma(\tau)) - Xg(\Gamma(\tau)) \cdot Y(\Gamma(\tau)) \\
\Gamma(0) &= P
\end{align*}$$

(32)

Before we state the theorem relating the horizontal curves $\Gamma$ and the mean curvature of $S$, we fix some notation. We give $\mathcal{H} = \text{span}\{X, Y\}$ the orientation for which the oriented angle from $X$ to $Y$ has amplitude $\pi/2$. Let $P \in S = \partial \Omega$. The **positive horizontal direction** $V^+_P S$ tangent to $S$ is $V^+_P S$, the unit vector in the direction $V_P S$ such that the angle from $N_P S$ to $V^+_P S$ has amplitude $-\pi/2$, i.e.

$$V^+_P S = dR_{-\pi/2} N_P S,$$

denoting the differential of $R_{-\pi/2}$. If the curve $\Gamma$ of Lemma 4.1 is oriented in such a way that $\Gamma$ has the same orientation of $V^+_P S$, we say that $\Gamma$ is a **positively oriented horizontal curve** on $S$. Let $p$ be the projection of $\mathbb{H}$ onto the $z$-plane, $p(z, t) = z$ and let $\gamma = p(\Gamma)$ be the projection of $\Gamma$. $\gamma$ inherits the orientation of $\Gamma$. The **Euclidean curvature** $k$ of the oriented curve $\gamma$ at $\gamma(t_0)$ is defined (locally in $t_0$) by

$$\frac{d}{dt}|_{t=t_0} (n(\gamma(t))) = k \cdot \gamma(t_0),$$

where

$$n(\gamma(t)) = \frac{dR_{\pi/2} \gamma(t)}{|dR_{\pi/2} \gamma(t)|}$$
is the unit normal to the oriented curve $\gamma$ at $\gamma(t)$, which is locally well defined even if $\gamma$ has self intersections.

In a picture, $\gamma$ has positive curvature if it borders a convex region on its right hand side.

**Theorem 4.1.** Let $S = \partial \Omega$ be a smooth surface and let $P \in S$ be non-characteristic. Let $\Gamma$ be the oriented horizontal curve on $S$ through $P$. Then, $h_S(P)$ is the curvature of the oriented curve $p(\Gamma)$ in $\mathbb{R}^2$ at $p(P)$.

**Proof.** Suppose first that $T_PS$ is a non-vertical plane, that is, that in a neighborhood of $P$, $S$ admits the parametrization $S = \{f(x, y) = t\}$. The horizontal curve $\Gamma$ on $S$ through $P$ satisfies equation (32), hence $\Gamma = (x, y, t)$ is a solution of
\[
\begin{align*}
\dot{x} &= Yf(\Gamma) \\
\dot{y} &= -Xf(\Gamma) \\
\dot{t} &= 2yYf(\Gamma) + 2xXf(\Gamma)
\end{align*}
\]
Since $Xf$ and $Yf$ are independent of $t$, we solve for $x$ and $y$, independently of $t$, and we find the differential equation for $\gamma = p(\Gamma)$,
\[
\begin{align*}
\dot{x} &= Yf(\Gamma) \\
\dot{y} &= -Xf(\Gamma)
\end{align*}
\]
Suppose that $Yf \neq 0$ at $P$ (otherwise, we can let $Xf \neq 0$ at $P$, since $P$ is non characteristic). Parametrizing $\gamma$ as $y = \gamma(x)$, we find
\[
y' = -\frac{Xf}{Yf}
\]
The curvature of $\gamma$ in the Euclidean plane is defined as
\[
k = -\frac{d}{dx} \left( \frac{y'}{\sqrt{1 + (y')^2}} \right) = -\frac{y''}{(1 + (y')^2)^{3/2}}
\]
\[
= \frac{(Yf)^3}{|\nabla_Hf|^3} \frac{d}{dx} \left( \frac{Xf}{Yf} \right)
\]
\[
= \frac{Yf}{|\nabla_Hf|^3} \left[ (\partial_x Xf + \partial_y Xfy') Yf - (\partial_x Yf + \partial_y Yfy') Xf \right]
\]
\[
= \frac{XXf (Yf)^2 - 2(XY)^* f Xf Yf + YYf (Xf)^2}{|\nabla_Hf|^3}
\]
\[
= \Delta_H \delta_S = h_S(P)
\]
by Corollary [43].

The case when $T_PS$ is a vertical plane can be dealt with similarly.

\[\square\]

In the Heisenberg group, vectors belonging to different sections of the tangent bundle can be compared by means of left translations. Under the identification $\mathbb{H} \equiv \mathbb{R}^3$, hence of the tangent space at $P \in \mathbb{H}$ with $\mathbb{R}^3$, the differential of the left translation by $Q = (a, b, c)$, $\mathcal{L}_Q : P \mapsto Q \circ P$, is the matrix
\[
d\mathcal{L}_Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2b & -2a & 1 \end{bmatrix}
\]
which is independent of the point where the differential is calculated. By definition of Carnot-Carathéodory distance, $d\mathcal{L}_Q$ maps $\mathcal{H}_P$ in $\mathcal{H}_{Q \circ P}$, isometrically.
Fix $P \in \mathbb{H}$ and let $\phi : I \to \mathbb{H}$ be a smooth, horizontal curve, such that $\phi(0) = P$. Let $W$ be a section of the horizontal bundle along $\phi$: $W(s) \in \mathcal{H}_{\phi(s)}$ for $s \in I$, and $W$ is a smooth map. We define the **derivative** of $W$ along $\phi$ at $P$ as

$$\frac{d_{\phi}^HW(s)}{ds}_{s=0} := \lim_{s \to 0} \frac{dL_{(\phi(0))\circ(-\phi(s))}W(s) - W(0)}{s}$$

We give the notion of derivative of a vector field with respect to a vector in the Heisenberg group.

**Definition 4.2.** Let $W$ be a section of the horizontal bundle $\mathcal{H}$. Let $V \in \mathcal{H}_P$ and $\phi : I \to \mathbb{H}$ be any horizontal curve such that $\phi(0) = P$ and $\phi'(0) = V$. We set

$$\nabla^H_VW(P) := \frac{d^H(W \circ \phi)(s)}{ds}_{s=0}.$$

**Lemma 4.2.** Let $W = aX + bY$ be a section of the horizontal bundle $\mathcal{H}$ and $\phi : I \to \mathbb{H}$ be any horizontal curve such that $\phi(0) = P$ and $\phi'(0) = V$. Then

$$\nabla^H_VW(P) = Va(P)X(P) + Vb(P)Y(P)$$

where $Va$ and $Vb$ are the derivatives of the functions $a$ and $b$ in the direction $V$, respectively.

**Proof.** Indeed

$$\frac{dL_{(\phi(0))\circ(-\phi(s))}W(s) - W(0)}{s}$$

$$= \frac{dL_{(\phi(0))\circ(-\phi(s))}(a(\phi(s)), b(\phi(s), 2\phi_2(s)a(\phi(s)) - 2\phi_1(s)b(\phi(s))) - W(\phi(0))}{s}$$

$$= \frac{(a(\phi(s)), b(\phi(s), 2\phi_2(0)a(\phi(s)) - 2\phi_1(0)b(\phi(s))) - W(\phi(0))}{s}$$

$$= \frac{a(\phi(s)) - a(\phi(0))}{s}X(\phi(0)) + \frac{b(\phi(s)) - b(\phi(0))}{s}Y(\phi(0))$$

and as a consequence

$$\lim_{s \to 0} \frac{dL_{(\phi(0))\circ(-\phi(s))}W(\phi(s)) - W(\phi(0))}{s} = Va(P)X(P) + Vb(P)Y(P)$$

□

Let $N_S(P)$ the unit vector normal to $S = \partial\Omega$ at $P$, in the Heisenberg sense, and pointing outside $\Omega$. If $g(x, y, t) = 0$ is a local parametrization of $S$, then

$$N_S(P) = \left(\frac{Xg(P)}{|\nabla^Hg(P)|}X_P + \frac{Yg(P)}{|\nabla^Hg(P)|}Y_P\right)$$

depending on the orientation of $S$.

**Theorem 4.2.** The linear operator $M_P : V \mapsto \nabla^H_VN_S(P)$ maps $V_P$ into $V_P$. $M_P$ acts on $V_P$ as multiplication times $h_S(P)$.

Modulo a sign, the map $M_P$ is the Weingarten map for surfaces in the Heisenberg group.

**Proof.** If $W = N_S(P)$, we have

$$\nabla^H_VN_S(P) \in V_P$$

In fact, differentiating $|N_S(P) \circ \phi(s)|^2 = 1$ with respect to $s$, $N_S(P) = \nu_1(P)X_P + \nu_2(P)Y_P$, we obtain

$$0 = \nu_1(\phi(s)) \frac{d}{ds}\nu_1(\phi(s)) + \nu_2(\phi(s)) \frac{d}{ds}\nu_2(\phi(s))$$
For $s = 0$, this relation becomes
\[ 0 = \langle N_S(P), \nabla_V^H N_S(P) \rangle_H \]
Since $N_S(P) = \frac{Xg(P)}{|\nabla g(P)|} X_P + \frac{Yg(P)}{|\nabla g(P)|} Y_P$ we get, for $V = \alpha_1 X(P) + \alpha_2 Y(P) \in V_P S$,
\begin{equation}
\nabla_V^H N_S(P) = \begin{bmatrix} X(\frac{Xg(P)}{|\nabla g(P)|}) & Y(\frac{Xg(P)}{|\nabla g(P)|}) \\ X(\frac{Yg(P)}{|\nabla g(P)|}) & Y(\frac{Yg(P)}{|\nabla g(P)|}) \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}
\end{equation}

The linear map $V \mapsto \nabla_V^H N_S(P)$ is defined on the one-dimensional linear space $V_P S$, hence it acts as multiplication by a constant $k$. Testing (33) on $(\alpha_1, \alpha_2) = (-Yg(P), Xg(P)) = \dot{\Gamma}(t)$, if $\Gamma(t) = P$, i.e. on a basis of $V_P S$, we see that
\[ k = \frac{Q}{|\nabla_H g|} \]
is the mean curvature of $S$ at $P$. \hfill \Box

Observe that multiplication by the matrix in (33) acts on $V_P S$ as multiplication by the matrix’ trace.

5. Examples

In this section, we see two basic examples of surfaces in $\mathbb{H}$. First, we consider the case of the plane $\{ t = 0 \}$. Then, we consider the case of the ball with respect to the Heisenberg distance. In particular, we explicitly compute the Hessian of the function ”distance from a point”.

Calculations are easier if we exploit the symmetries of $\mathbb{H}$.

**Lemma 5.1.** Let $g$ be a smooth function on $\mathbb{H}$, which is homogeneous of degree $m$ with respect to the dilation group at the origin. Let $W_j$ be left invariant vector fields on $\mathbb{H}$, $j = 1, \ldots, n$. Then, $W_1 \ldots W_n g$ is homogeneous of degree $m - n$,
\begin{equation}
W_1 \ldots W_n g(\delta \lambda P) = \lambda^{m-n} g(P)
\end{equation}

In particular, if $S$ is a smooth surface in $\mathbb{H}$ and $\sigma$ denotes the distance function from $S$, then $\sigma$, $\nabla_H \sigma$ and $\text{Hess}_H \sigma$ are homogeneous of degree 1, 0 and $-1$, respectively.

Recall that $R_\theta$ denotes both the rotation by $\theta$ around the $t$-axis in $\mathbb{H}$ and the rotation by $\theta$ in $\text{span}\{X,Y\}$. We use the same symbol for the matrix
\[ R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \]

**Lemma 5.2.** Let $f$ be a smooth function on $\mathbb{H}$, which is invariant under the rotation group at the origin,
\[ f(r_\theta P) = f(P) \]
for $\theta$ in $\mathbb{R}$. Then, $\nabla_H$ and $\text{Hess}_H$ commute with rotations. Namely,
\begin{equation}
\nabla_H f(R_\theta P) = R_\theta \nabla_H f(P)
\end{equation}
and
\begin{equation}
\text{Hess}_H f(R_\theta P) = R_\theta \text{Hess}_H f(P)
\end{equation}

In (36), the multiplication on the right hand side is a multiplication of $2 \times 2$ matrices.
5.1. **Nonvertical planes.** Let $\Pi$ be the plane \( \{ t = 0 \} \). Observe that all nonvertical planes are isometric to $\Pi$. In \cite{2} we computed the metric normal to $\Pi$ at $P = (x, y, 0) = (z, 0)$,

\[
\gamma_P(\sigma) = (u(\sigma), v(\sigma), s(\sigma)) = \begin{cases} \frac{x}{2} \left( 1 + \cos \left( \frac{2\sigma}{11} \right) \right) + \frac{y}{2} \sin \left( \frac{2\sigma}{11} \right) \\ \frac{y}{2} \left( 1 + \cos \left( \frac{2\sigma}{11} \right) \right) - \frac{x}{2} \sin \left( \frac{2\sigma}{11} \right) \end{cases}
\]

The geodesic $\gamma_P$ realizes the distance from $\Pi$ when $|\sigma| \leq \frac{\pi}{2}$. By Lemma 5.1, $\delta_\Pi$ is homogeneous of degree 1, $\nabla_\Pi \delta_\Pi$ is homogeneous of degree 0 and $\text{Hess}_\Pi \delta_\Pi$ is homogeneous of degree $-1$. Since $\delta_\Pi$ is invariant under rotations around the $t$-axis, $\text{Hess}_\Pi \delta_\Pi$ commutes with the same group of rotations.

Let $\beta = \frac{\sigma}{|\Pi|} \in (-\pi/2, \pi/2)$, $R = |z| \geq 0$ and $\alpha \in [0, 2\pi)$. Here, we use on $\mathbb{H}$ the coordinates \((u, v, s) = (R \cos \beta \cos \alpha, R \cos \beta \sin \alpha, R^2(\beta + 1/2 \sin(2\beta)))\).

Hence

\[
J = \begin{pmatrix} u & v & s \\ R & \alpha & \beta \end{pmatrix} = \begin{bmatrix} \cos \alpha \cos \beta, & -R \cos \beta \sin \alpha, & -R \sin \beta \cos \alpha \\ \cos \beta \sin \alpha, & R \cos \beta \cos \alpha, & -R \sin \beta \sin \alpha \\ 2R(\beta + \frac{1}{2} \sin(2\beta)), & 0, & 2R^2 \cos^2 \beta \end{bmatrix}
\]

and

\[
\det J = 2R^3 \cos \beta (\cos \beta + \sin \beta).
\]

As a consequence

\[
\det J = \begin{pmatrix} R & \alpha & \beta \\ u & v & s \end{pmatrix} = \begin{bmatrix} \cos^2 \beta \cos \alpha & \cos \beta \sin \alpha & \sin \beta \\ \cos \beta \cos \alpha & R \cos \beta & 2R(\cos \beta + \sin \beta) \\ -\cos \beta \sin \alpha & \cos \beta \sin \alpha & 2R^2(\cos \beta + \sin \beta) \end{bmatrix}
\]

A long, but elementary calculation shows now that, in the new coordinates,

\[
X = \frac{\cos \beta \cdot \cos(\beta - \alpha)}{\cos \beta + \beta \sin \beta} \partial_R - \frac{\sin \alpha}{R \cos \beta} \partial_\alpha - \frac{\beta \cos \alpha + \cos \beta \cdot \sin(\beta - \alpha)}{R(\cos \beta + \beta \sin \beta)} \partial_\beta
\]

and

\[
Y = -\frac{\cos \beta \cdot \sin(\beta - \alpha)}{\cos \beta + \beta \sin \beta} \partial_R + \frac{\cos \alpha}{R \cos \beta} \partial_\alpha - \frac{\beta \sin \alpha + \cos \beta \cdot \cos(\beta - \alpha)}{R(\cos \beta + \beta \sin \beta)} \partial_\beta.
\]

Hence,

**Lemma 5.3.** Let $\Pi$ be the plane $\{ t = 0 \}$. Then

\[
X \delta_\Pi = -\sin(\beta - \alpha)
\]

and

\[
Y \delta_\Pi = -\cos(\beta - \alpha).
\]

**Proof.** Recalling (37) and (38) we get $X \delta_\Pi = -\sin(\beta - \alpha)$ and $Y \delta_\Pi = -\cos(\beta - \alpha)$. \hfill \Box

**Remark 5.1.** After the computation of $X \delta_\Pi$, we can deduce the value of $Y \delta_\Pi$ keeping in mind (35).

**Theorem 5.1.** Let $\Pi$ be the plane $\{ t = 0 \}$. Then

\[
\text{Hess}_\Pi \delta_\Pi = \frac{1}{R \cos \beta (\cos \beta + \beta \sin \beta)} \begin{bmatrix} \cos(\alpha - \beta)A, & \cos(\alpha - \beta)B \\ \sin(\alpha - \beta)A, & \sin(\alpha - \beta)B \end{bmatrix},
\]

where

\[
A = -(\sin \alpha \cos \beta + \beta \cos(\alpha + \beta) + \cos^2 \beta \sin(\alpha - \beta))
\]
and

\[ B = (\cos \alpha \cos \beta + \beta \sin (\alpha + \beta) + \cos^2 \beta \cos (\alpha - \beta)). \]

Moreover for \( P \in \Pi, P \neq O \), then

\[-(\Delta \delta_{\Pi})_{\sigma=0} = 0,\]

i.e. the mean curvature is 0.

Proof. Recalling Lemma 5.3 we get by straightforward computation:

\[
XX\delta_{\Pi} = -\frac{\sin \alpha}{R \cos \beta} \cos (\alpha - \beta) + \frac{\beta \cos \alpha - \cos \beta \sin (\alpha - \beta)}{R (\cos \beta + \beta \sin \beta)} \cos (\alpha - \beta) = \frac{\cos (\alpha - \beta)}{R \cos \beta} A
\]

\[
YX\delta_{\Pi} = \frac{\cos \alpha}{R \cos \beta} \cos (\alpha - \beta) + \frac{\sin \alpha + \cos \beta \cos (\alpha - \beta)}{R (\cos \beta + \beta \sin \beta)} \cos (\alpha - \beta) = \frac{\cos (\alpha - \beta)}{R \cos \beta} B
\]

\[
XY\delta_{\Pi} = -\frac{\sin \alpha}{R \cos \beta} \sin (\alpha - \beta) + \frac{\beta \cos \alpha - \cos \beta \sin (\alpha - \beta)}{R (\cos \beta + \beta \sin \beta)} \sin (\alpha - \beta) = \frac{\sin (\alpha - \beta)}{R \cos \beta} A
\]

\[
YY\delta_{\Pi} = \frac{\cos \alpha}{R \cos \beta} \sin (\alpha - \beta) + \frac{\beta \sin \alpha + \cos \beta \cos (\beta - \alpha)}{R (\cos \beta + \beta \sin \beta)} \sin (\alpha - \beta) = \frac{\sin (\alpha - \beta)}{R \cos \beta} B.
\]

In particular, we have

\[
\text{Trace}(\text{Hess}_H \delta_{\Pi})_{\sigma=0} = \text{Trace} \left( \frac{1}{R} \left[ \begin{array}{cc} -2 \cos \alpha \sin \alpha, & 2 \cos^2 \alpha \\ -2 \sin^2 \alpha, & 2 \cos \alpha \sin \alpha \end{array} \right] \right) = 0.
\]

As a consequence the mean curvature of \( \Pi - O \) is 0. This fact could also be proved as follows. The punctured plane \( \Pi - O \) is the union of straight half line leaving \( O \), hence its Heisenberg curvature at a point \( P \) is the Euclidean curvature of a straight line. by Theorem 4.3. □

Remark 5.2. Notice that \( \text{Hess}_H \delta_{\Pi} \) is singular as \( R \to 0 \) and as \( \beta \to \pm \pi/2 \). For instance, if \( \alpha = 0 \), then \( YY\delta_{\Pi} \) diverges as \( \beta \to \pm \pi/2 \) and \( R \to 0 \). Indeed

\[
(Hess_H \delta_{\Pi})_{\alpha=0} = \frac{1}{R \cos \beta (\cos \beta + \beta \sin \beta)} \left[ \begin{array}{cc} -(\beta - \cos \beta \sin \beta) \cos^2 \beta, & (\cos \beta + \beta \sin \beta + \cos^3 \beta) \cos \beta \\ (\beta - \cos \beta \sin \beta) \sin \beta \cos \beta, & -(\cos \beta + \beta \sin \beta + \cos^3 \beta) \sin \beta \end{array} \right]
\]

and

\[
(YY \delta_{\Pi})_{\alpha=0} = -\frac{\tan \beta \cos \beta + \beta \sin \beta + \cos^3 \beta}{R \cos \beta + \beta \sin \beta} = -\frac{\tan \beta}{R} \left( 1 + \frac{\cos^3 \beta}{\cos \beta + \beta \sin \beta} \right).
\]

The singularity as \( R \to 0 \) is expected, by Theorem 3.1. The singularity as \( \beta \to \pm \pi/2 \) shows that the estimate at the end of Theorem 3.1 cannot be extended outside the surface.

5.2. The Carnot-Charathéodory sphere and the Hessian of the distance function.

Here we consider \( \Sigma = \partial B(0,1) \), the sphere of unitary radius with respect to the Carnot-Charathéodory metric. The poles of \( \Sigma \) are \( NP = (0,0,1/\pi) \) and \( SP = (0,0,1/\pi) \). The calculation of the horizontal Hessian of \( \delta_\Sigma \) will be accomplished in several steps. First, we find the equations of the horizontal curves lying on \( \Sigma \). Via Theorem 4.3, this allows us to compute the mean curvature of \( \Sigma \) at every point. By Theorem 5.1 (14) and (16), and Corollary 4.1 the second order terms of the formula relating \( \text{Hess}_H \) to the equation of \( \Sigma \) are all contained in the expression for the mean curvature of \( \Sigma \). Hence, in order to compute \( \text{Hess}_H \delta_\Sigma \) we are left with the calculation of some first order derivatives.

Finally, we note that the calculation of \( \text{Hess}_H \delta_\Sigma \) immediately leads to an expression for the horizontal Hessian of the Carnot-Charathéodory distance in \( H \), \( \text{Hess}_H d(O, \cdot) \).
We can parametrize $\Sigma$ in several useful ways. Let $(z, t) \in \Sigma$. Then, there are $\phi \in [0, 2\pi)$ and $u \in [-2\pi, 2\pi]$ such that

$$
\begin{aligned}
 z(u) & = e^{i\phi} 2^{\sin(u/2)} \\
 t(u) & = 2^{u - \sin(u)}
\end{aligned}
$$

(39)

The parametric representation in (39) can be easily deduced by the geodesic equations (6).

Below we consider the upper half of $\Sigma$, corresponding to $u \geq 0$, so that $B(O, 1)$ is below the surface.

We see, now, how $\Sigma$ is ruled by horizontal curves. Incidentally, this provides a differently useful parametrization of $\Sigma$.

**Proposition 5.1.** The surface $\Sigma - \{NP, SP\}$ is the disjoint union of horizontal curves $\gamma_\theta(u) = (z(u), t(u)) \ (|u| < 2\pi)$,

$$
\begin{aligned}
 z(u) & = e^{i(\psi(u/2) + \theta)} 2^{\sin(u/2)} \\
 t(u) & = 2^{u - \sin(u)}
\end{aligned}
$$

(40)

where $\psi$ satisfies the conditions $\psi(0) = 0$ and

$$
\psi'(p) = \cot^2 p - \cot p
$$

(41)

These are the only horizontal curves on $\Sigma$.

In (41), observe that $\psi'$ is well defined for $p = 0$ as well.

**Proof.** First, note that, by (39), any curve parametrized as in (40) lies on $\Sigma$. Let now $\gamma$ be the curve in (40) corresponding to $\theta = 0$ (by rotation invariance, $R_\theta \gamma = \gamma$ is horizontal if and only if $\gamma$ is). The curve $\gamma = (x, y, t)$ is horizontal if and only if

$$
\dot{t}(u) = 2\dot{x}(u)y(u) - 2x(u)\dot{y}(u).
$$

(42)

Here,

$$
\begin{aligned}
 x(u) & = \cos(\psi(u/2)) 2^{\sin(u/2)} \\
 y(u) & = \sin(\psi(u/2)) 2^{\sin(u/2)}
\end{aligned}
$$

Let $p = u/2$ We make (42) into an explicit O.D.E. A calculation shows that

$$
2\dot{x}(2p)y(2p) - 2x(2p)\dot{y}(2p) = -\psi'(p) 2^{\sin^2(p)}
$$

and that

$$
\dot{t}(2p) = \cos(p) 2^{\sin(p) - p \cos(p)}
$$

Equation (41) follows immediately. \[\square\]

It can be proved that $\psi(p) \to \infty$ as $p \to \pi$, i.e. that the projection of $p(\gamma_\theta)$ onto the plane $\{t = 0\}$ spirals infinitely many times around the origin. It can also be shown that the Heisenberg length of $\gamma_\theta$, i.e. the Euclidean length of $p(\gamma_\theta)$, is infinite.

Next, we compute the mean curvature of $\Sigma$.

**Proposition 5.2.** The curvature of $\Sigma$ at $P = (z, t)$, where

$$
\begin{aligned}
 |z| & = 2^{\sin(u/2)} \\
 t & = 2^{u - \sin(u)}
\end{aligned}
$$

is the number

$$
h_\Sigma(P) = \frac{u \cos u - \sin u}{\frac{u}{2} \cos \left(\frac{u}{2}\right) - \sin \left(\frac{u}{2}\right)} 4 \sin \left(\frac{u}{2}\right)
$$

(43)
Proof. By Theorem 4.1, we only need to apply the formula for the computation of the mean curvature of a curve in parametric form (see, e.g., [17]) to the curves described in Proposition 5.1.

By Proposition 5.2, in order to compute the Hessian of \( \delta \Sigma \), we are left with the calculation of a number of first order derivatives of \( f \).

Lemma 5.4. Let \( t = f(x,y) \) be the equation of the upper half of \( \Sigma \), let \((z,t)\) be as in (39) and \( u = 2v \). Then,

\[
\begin{align*}
Xf(z,t) &= 2 \cos \phi \frac{\cos(\theta)}{v} + 2 \sin \phi \frac{\sin(v)}{v} = \frac{2}{v} \cos(\phi - v) \\
Yf(z,t) &= 2 \sin \phi \frac{\cos(\theta)}{v} - 2 \cos \phi \frac{\sin(u)}{v} = \frac{2}{v} \sin(\phi - v)
\end{align*}
\]

Proof. Let \( t = f(x,y) \). By the chain rule,

\[
J \left( \begin{array}{c} t \\ x \\ y \\ \phi \end{array} \right) = J \left( \begin{array}{c} t \\ u \\ \phi \end{array} \right) \cdot J \left( \begin{array}{c} x \\ u \\ \phi \\ \phi \end{array} \right)^{-1}
\]

By (39),

\[
J \left( \begin{array}{c} t \\ u \\ \phi \end{array} \right) = \left( \cos(u/2) (u/2)^3 \sin(u/2) - u/2 \cos(u/2) \right), 0 = \left( -\frac{\cos(u/2)}{u/2} h(u/2), 0 \right)
\]

where

\[
h(s) = \frac{s \cos(s) - \sin(s)}{s^2}
\]

and

\[
J \left( \begin{array}{c} x \\ u \\ \phi \\ \phi \end{array} \right) = \left[ \begin{array}{cc} \frac{1}{2} h(u/2) \cos \phi & -\frac{\sin(u/2)}{u/2} \sin \phi \\ \frac{1}{2} h(u/2) \sin \phi & \frac{\sin(u/2)}{u/2} \cos \phi \end{array} \right]
\]

so that

\[
J \left( \begin{array}{c} x \\ u \\ \phi \\ \phi \end{array} \right)^{-1} = \frac{1}{\frac{1}{2} h(u/2) \frac{\sin(u/2)}{u/2}} \left[ \begin{array}{cc} \frac{\sin(u/2)}{u/2} \cos \phi & \frac{\sin(u/2)}{u/2} \sin \phi \\ -\frac{1}{2} h(u/2) \sin \phi & \frac{1}{2} h(u/2) \cos \phi \end{array} \right]
\]

Using (45), we obtain

\[
J \left( \begin{array}{c} t \\ x \\ y \\ \phi \end{array} \right) = \left( \begin{array}{c} 2 \cos \phi \\ -\frac{2 \sin \phi}{u} \end{array} \right)
\]

and the conclusion of the lemma follows immediately.

We have now all the ingredients for the calculation of \( \text{Hess}_{\Sigma} \delta \Sigma \).

Theorem 5.2. Let \( P = (z,t) \) be the point of the upper half of \( \Sigma \),

\[
\begin{align*}
z(u) &= 2 \sin(u/2) \\
t(u) &= 2 \frac{u - \sin(u)}{u^2}
\end{align*}
\]

Then,

\[
\text{Hess}_{\Sigma} \delta \Sigma(P) = v_P \Sigma \otimes v_P \Sigma \cdot (h_\Sigma(P) I + p_\Sigma(P) J),
\]

where

\[
v_P \Sigma = \sin(\phi - v) \cdot X_P - \cos(\phi - v) \cdot Y_P,
\]

\[
h_\Sigma(P) = \frac{u \cos u - \sin u}{\frac{u}{2} \cos \left( \frac{u}{2} \right) - \sin \left( \frac{u}{2} \right) 4 \sin \left( \frac{u}{2} \right)},
\]

and

\[
p_\Sigma(P) = 2u.
\]

Proof. The theorem follows immediately from Theorem 4.1, Proposition 5.2, Lemma 5.4, (4) and the definition of \( v_P \Sigma \).
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