Production, Supply, and Traffic Systems: A Unified Description

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Summary. The transport of products between different suppliers or production units can be described similarly to driven many-particle and traffic systems. We introduce equations for the flow of goods in supply networks and the adaptation of production speeds. Moreover, we present two examples: The case of linear (sequential) supply chains and the case of re-entrant production. In particular, we discuss the stability conditions, dynamic solutions, and resonance phenomena causing the frequently observed “bullwhip effect”, which is an analogue of stop-and-go traffic. Finally, we show how to treat discrete units and cycle times, which can be applied to the description of vehicle queues and travel times in freeway networks.

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Supply chain management is a major subject in economics, as it significantly determines the efficiency of production processes [1]. Many related studies focus on subjects like optimum buffer sizes and stock levels, but a stable dynamics and optimal network structure are also important subjects [2, 3]. Therefore, this scientific field is connected with the statistical physics of networks [4].

In this paper, we will investigate the dynamic properties and linear stability of supply chains by a generalization of “fluid-dynamic” production models, which have been inspired by traffic models [5–10]. Fluid-dynamic models take into account non-linear interactions and are suitable for on-line control, as they are numerically much more efficient than event-driven (Monte-Carlo) simulations. Moreover, unlike queueing theory, they are not mainly restricted to the treatment of stationary situations, but can reflect variations in the consumption rate and effects of machine breakdowns or changes in the production schedule.
The organization of this contribution is as follows: Section 1 focuses on the description of supply and production systems, in particular linear supply chains and re-entrant production. Section 2 treats freeway traffic as a queueing network and applies the formula for the cycle time to the determination of travel times.

1 Modeling production systems as supply networks

Our production model assumes \( u \) production units or suppliers \( j \) which deliver \( d_{ij} \) products of kind \( i \in \{1, \ldots, p\} \) per production cycle to other suppliers and consume \( c_{kj} \) goods of kind \( k \) per production cycle. The coefficients \( c_{kj} \) and \( d_{ij} \) are determined by the respective production process, and the number of production cycles per unit time (e.g. per day) is given by the production speed \( Q_j(t) \). That is, supplier \( j \) requires an average time interval of \( 1/Q_j(t) \) to produce or deliver \( d_{ij} \) units of good \( i \). The temporal change in the number \( N_i(t) \) of goods of kind \( i \) available in the system is given by the difference between the inflow

\[
Q_{i\text{in}}(t) = \sum_{j=0}^{u} d_{ij} Q_j(t) \tag{1}
\]

and the outflow

\[
Q_{i\text{out}}(t) = \sum_{j=1}^{u+1} c_{ij} Q_j(t). \tag{2}
\]

In other words, it is determined by the overall production rates \( d_{ij} Q_j(t) \) of all suppliers \( j \) minus their overall consumption rates \( c_{ij} Q_j(t) \):

\[
\frac{dN_i}{dt} = Q_{i\text{in}}(t) - Q_{i\text{out}}(t) = \sum_{j=1}^{u} (d_{ij} - c_{ij})Q_j(t) - Y_i(t). \tag{3}
\]

Herein, the quantity

\[
Y_i(t) = \frac{c_{i,u+1}Q_{u+1}(t)}{\text{consumption and losses}} - \frac{d_{i0}Q_0(t)}{\text{inflow of resources}} \tag{4}
\]

comprises the consumption rate of goods \( i \), losses, and waste (the “export” of material), minus the inflows into the considered system (the “imports”). In the following, we will assume that the quantities are measured in a way that \( 0 \leq c_{ij}, d_{ij} \leq 1 \) (for \( 1 \leq i \leq p, 1 \leq j \leq u \)) and the “normalization conditions”

\[
d_{i0} = 1 - \sum_{j=1}^{u} d_{ij} \geq 0, \quad c_{i,u+1} = 1 - \sum_{j=1}^{u} c_{ij} \geq 0 \tag{5}
\]

are fulfilled. Equations (3) can then be interpreted as conservation equations for the flows of goods.
1.1 Adaptation of production speeds

The production speeds \(Q_j(t)\) may be changed in time in order to adapt to varying consumption rates \(Y_i(t)\). We will assume that it takes a typical time period \(T\) to adapt the actual production speeds \(Q_j(t)\) to the desired production ones, \(W_j\):

\[
\frac{dQ_j}{dt} = \frac{1}{T} \left[ W_j(\{N_i(t)\}, \{dN_i/dt\}, \{Q_k(t)\}) - Q_j(t) \right].
\]

(6)

Here, the curly brackets indicate that the so-called management or control function \(W_j(\ldots)\) may depend on all inventories \(N_i(t)\) with \(i \in \{1, \ldots, p\}\), their derivatives \(dN_i/dt\), and/or all production speeds \(Q_k(t)\) with \(k \in \{1, \ldots, u\}\).

The resulting dynamics of the supply network can be investigated by means of a linear stability analysis, which has been carried out for linear supply chains in Ref. [2] and for supply networks with \(d_{ij} = \delta_{ij}\) in Ref. [3]. Some of the main results will be discussed in the following.

1.2 Modelling one-dimensional supply chains

For simplicity, let us investigate a model of one-dimensional supply chains, here, which corresponds to \(d_{ij} = \delta_{ij}\) and \(c_{ij} = \delta_{i+1,j}\), where \(\delta_{ij} = 1\) for \(i = j\) and \(\delta_{ij} = 0\) otherwise. This implies \(Q_i^{in}(t) = Q_i(t)\) and \(Q_i^{out}(t) = Q_{i+1}(t) = Q_{i+1}^{in}(t)\). The assumed model consists of a series of \(u\) suppliers \(i\), which receive products of kind \(i - 1\) from the next “upstream” supplier \(i - 1\) and generate products of kind \(i\) for the next “downstream” supplier \(i + 1\) at a rate \(Q_i(t)\) [7, 9]. The final products are delivered at the rate \(Q_u(t)\) and removed from the system with the consumption rate \(Y_u(t) = Q_{u+1}(t)\). The consumption rate is typically subject to perturbations, which may cause a “bullwhip effect” [11–18], i.e. growing variations in the stock levels and deliveries of upstream suppliers. This is due to delays in the adaptation of their delivery rates.

The inventory of goods of kind \(i\) changes in time \(t\) according to

\[
\frac{dN_i}{dt} = Q_i^{in}(t) - Q_i^{out}(t) = Q_i(t) - Q_{i+1}(t),
\]

(7)

while the temporal change of the delivery rate will be assumed as

\[
\frac{dQ_i}{dt} = \frac{1}{T} \left[ W_i(N_i(t), dN_i/dt, Q_i(t), Q_{i+1}(t)) - Q_i(t) \right].
\]

(8)

The management strategy

\[
\frac{dQ_i}{dt} = \frac{1}{T} \left\{ \frac{N_i^0 - N_i(t)}{\tau} - \beta \frac{dN_i}{dt} + \epsilon [Q_i^0 - Q_i(t)] \right\}
\]

(9)

appears to be appropriate to keep the inventories \(N_i(t)\) stationary, to maintain a certain optimal inventory \(N_i^0\) (in order to cope with stochastic variations due
to machine breakdowns etc.), and to operate with the equilibrium production rates $Q^0_i = Y^0_u$, where $Y^0_u$ denotes the average consumption rate. The supplier-independent parameter values $\tau$, $\beta$, and $\epsilon$ can be justified with suitable scaling arguments [3].

### 1.3 Dynamic solution and resonance effects

In the vicinity of the stationary state characterized by $N_i(t) = N^0_i$ and $Q_i(t) = Q^0_i$, it is possible to calculate the dynamic solution of the one-dimensional supply chain model [3, 8]. For this, let $n_i(t) = N_i(t) - N^0_i$ be the deviation of the inventory from the stationary one, and $q_i(t) = Q_i(t) - Q^0_i$ the deviation of the delivery rate. The linearized model equations read

$$\frac{dn_i}{dt} = q_i(t) - q_{i+1}(t)$$

(10)

and

$$\frac{dq_i}{dt} = -\frac{1}{T} \left( \frac{n_i}{\tau} + \beta \frac{dn_i}{dt} + \epsilon q_i \right).$$

(11)

Deriving Eq. (11) with respect to $t$ and inserting Eq. (10) results in the following set of second-order differential equations:

$$\frac{d^2q_i}{dt^2} + \frac{(\beta + \epsilon)}{T} \frac{dq_i}{dt} + \frac{1}{T^2} q_i(t) = \frac{1}{T} \left[ \frac{q_{i+1}(t)}{\tau} + \beta \frac{dq_{i+1}}{dt} \right] = f_i(t).$$

(12)

This corresponds to the differential equation for the damped harmonic oscillator with damping constant $\gamma$, eigenfrequency $\omega$, and driving term $f_i(t)$. The two eigenvalues of this system of equations are

$$\lambda_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega^2} = \frac{1}{2T} \left[ (\beta + \epsilon) \mp \sqrt{(\beta + \epsilon)^2 - 4T/\tau} \right],$$

(13)

i.e. for $(\beta + \epsilon) > 0$ their real parts are always negative, corresponding to a stable behavior in time. Nevertheless, we will identify a convective instability below, i.e. the oscillation amplitude can grow from one supplier to the next one upstream.

The set of equations (12) can be solved successively, starting with $i = u$ and progressing to lower values of $i$. For example, assuming periodic oscillations of the form $f_u(t) = f^0_u \cos(\alpha t)$, after a transient time much longer than $1/\gamma$ we find

$$q_u(t) = f^0_u F \cos(\alpha t + \varphi)$$

(14)

with

$$\tan \varphi = \frac{2\gamma \alpha}{\alpha^2 - \omega^2} = \frac{\alpha(\beta + \epsilon)}{\alpha^2 T - 1/\tau}$$

(15)
and
\[ F = \frac{1}{\sqrt{(\alpha^2 - \omega^2)^2 + 4\gamma^2\alpha^2}} = \frac{T}{\sqrt{[\alpha^2T - 1/\tau]^2 + \alpha^2(\beta + \epsilon)^2}}, \tag{16} \]
where the dependence on the eigenfrequency \( \omega \) is important to understand the occurring resonance effect. Equations (12) and (14) imply
\[ f_{u-1}(t) = \frac{1}{T}\left[ \frac{q_u(t)}{\tau} + \beta \frac{dq_u}{dt} \right] = f_{u-1}^0 \cos(\alpha t + \varphi + \delta) \tag{17} \]
with
\[ \tan \delta = \alpha \beta \tau \quad \text{and} \quad f_{u-1}^0 = f_u^0 \frac{F}{\sqrt{[1/\tau]^2 + (\alpha \beta)^2}}. \tag{18} \]

1.4 “Bull-whip effect” and stop-and-go traffic

The oscillation amplitude increases from one supplier to the next upstream one, if
\[ \frac{f_{u-1}^0}{f_u^0} = \left\{ 1 + \alpha^2\left[ \epsilon + 2\beta - 2T/\tau \right] + \alpha^4T^2 \right\}^{-1/2} > 1. \tag{19} \]
One can see that this resonance effect can occur for \( 0 < \alpha^2 < 2/(T \tau) - \epsilon(\epsilon + 2\beta)/T^2 \). Therefore, variations in the consumption rate are magnified under the instability condition
\[ T > \epsilon \tau (\beta + \epsilon/2). \tag{20} \]
Supply chains show this “bullwhip effect” (which corresponds to the phenomenon of convective, i.e. upstream moving instability), if the adaptation time \( T \) is too large, if there is no adaptation to the equilibrium production speed \( Q_0^i \), corresponding to \( \epsilon = 0 \), or if the management reacts too strong to deviations of the actual stock level \( N_i \) from the desired one \( N^0_i \), corresponding to a small value of \( \tau \). The latter is very surprising, as it implies that the strategy
\[ \frac{dQ_i}{dt} = \frac{1}{T \tau} [N_i^0 - N_i(t)], \tag{21} \]
which tries to maintain a constant work in progress \( N_i(t) = N_i^0 \), would ultimately lead to an undesirable bullwhip effect. In contrast, the management strategy
\[ \frac{dQ_i}{dt} = \frac{1}{T} \left\{ -\beta \frac{dN_i}{dt} + \epsilon (Q_i^0 - Q_i(t)) \right\} \tag{22} \]
would avoid this problem, but it would not maintain a constant work in progress. The control strategy (9) with a sufficiently large value of \( \tau \) would fulfill both requirements.

The “bullwhip effect” has, for example, been reported for beer distribution \([22, 23]\), but similar dynamical effects are also known for other distribution or transportation chains with significant adaptation times \( T \). It has, for example, some analogy to stop-and-go traffic \([3, 10]\), where delayed adaptation also leads
to an unstable behavior. In the case $\beta = 0$ and $\epsilon = 1$, the stability condition (20) agrees exactly with the one of the optimal velocity model [24], which is a particular microscopic traffic model. This car-following model assumes an acceleration equation of the form

$$\frac{dv_i(t)}{dt} = \frac{V_{\text{opt}}(d_i(t)) - v_i(t)}{T}$$  \hspace{1cm} (23)$$

and the complementary equation

$$\frac{dd_i(t)}{dt} = -[v_i(t) - v_{i+1}(t)].$$  \hspace{1cm} (24)$$

In contrast to the above supply chain model, however, the index $i$ represents single vehicles, $v_i(t)$ is their actual velocity of motion, $V_{\text{opt}}$ the so-called optimal (safe) velocity, which depends on the distance $d_i(t)$ to the next vehicle ahead. $T$ denotes again an adaptation time. Comparing this equation with Eq. (9), the velocities $v_i$ would correspond to the delivery rates $Q_i$, the optimal velocity $V_{\text{opt}}$ to the desired delivery rate $W_i(N_i) = Q_i^0 + (N_i^0 - N_i)/\tau$, and the inverse vehicle distance $1/d_i$ would approximately correspond to the stock level $N_i$ (apart from a proportionality factor). This shows that the analogy between supply chain and traffic models concerns only their mathematical structure, but not their interpretation, although both relate to transport processes. Nevertheless, this mathematical relationship can give us hints, how methods, which have been successfully applied to the investigation of traffic models before, can be generalized for the study of supply networks. Compared to traffic dynamics, supply networks and production systems have some interesting new features: Instead of a continuous space, we have discrete production units $j$, and the management strategy (6) is generally different from the velocity adaptation (23) in traffic. With suitable strategies, in particular with large values of $\tau$ and $\beta$, the oscillations can be mitigated or even suppressed. Moreover, production systems are frequently supply networks with complex topologies rather than one-dimensional supply chains, i.e. they have additional features compared to (more or less) one-dimensional freeway traffic. They are more comparable to street networks of cities [25].

1.5 Calculation of the cycle times

Apart from the productivity or throughput $Q_i$ of a production unit, production managers are highly interested in the cycle time $T_i$, i.e. the time interval between the beginning of the generation of a product and its completion. Let us assume that the queue length $L_i(t)$ of products waiting to be processed by production unit $i$ is given by the inventory $N_{i-1}(t)$ of product $i-1$. The change of the queue length $L_i(t)$ in time is then determined by the difference between the arrival rate $Q_i^{\text{arr}}(t)$, which corresponds to the inflow $Q_i^{\text{in}}(t) = Q_{i-1}(t)$ from production unit $i-1$, and the departure rate $Q_i^{\text{dep}}(t)$, which corresponds to the outflow $Q_i^{\text{out}}(t) = Q_i(t)$ to production unit $i$: 
\[
\frac{dL_i}{dt} = \frac{dN_{i-1}}{dt} = Q_{i-1}(t) - Q_i(t) = Q_i^{\text{arr}}(t) - Q_i^{\text{dep}}(t) .
\] (25)

On the other hand, the waiting products move forward \(Q_i^{\text{dep}}(t) = Q_i(t)\) steps per unit time, as \(Q_i(t)\) is the processing rate (production speed). For this reason, the overall time \(T_i(t)\) until having been processed is given by the implicit equation

\[
N_{i-1}(t) = L_i(t) = \int_t^{t+T_i(t)} dt' Q_i^{\text{dep}}(t') = \int_{-\infty}^{t+T_i(t)} dt' Q_i^{\text{dep}}(t') - \int_{-\infty}^t dt' Q_i^{\text{dep}}(t') ,
\] (26)

if the queue of length \(L_i(t)\) was joined at time \(t\). Accordingly, high inventories imply long cycle times, which favours just-in-time production with small stock levels. From Eqs. (25) and (26), one can finally derive a delay-differential equation for the waiting time under varying production conditions [10]:

\[
\frac{dT_i}{dt} = \frac{Q_i^{\text{arr}}(t)}{Q_i^{\text{dep}}(t + T_i(t))} - 1 = \frac{Q_{i-1}(t)}{Q_i(t + T_i(t))} - 1 .
\] (27)

This equation can be solved numerically as a function of the production rates \(Q_i(t)\), since the production initially starts with a cycle time of \(T_i(0) = 1/Q_i(0)\), corresponding to the average processing time \(1/Q_i(0)\) when the production unit \(i\) is started. In this way, it is possible to determine the process cycle times \(T_i\) and the overall production time as the sum of the cycle times of all single production steps, taking into account the respective time delays \(T_i\): When a specific product enters the queue before production unit \(i\) at time \(t = t_i\), it enters the queue before production unit \(i+1\) at time \(t_{i+1} = t_i + T_i(t_i)\).

The time of delivery to the customer is given by \(t_{u+1} = t_u + T_u(t_u)\).

1.6 Modeling of discrete units

The above “fluid-dynamic” model equations can not only be used to represent approximate mean values of large numbers of products. They can also be transferred to the treatment of discrete units (such as single units of a product), if their dynamics is sufficiently deterministic. For example, one could represent the time interval \(\Delta T\), during which a discrete unit occupies a certain production unit \(i\) by a step function. However, as step functions are not differentiable everywhere, we will replace them by smooth functions.

One possible specification uses a Fourier approximation of the step functions. To lowest order, one may take

\[
Q_i(t) = A \sum_k \left\{1 - \cos(\pi(t - t^k_i)/\Delta T)\right\}
\] (28)

where \(t^k_i\) denotes the starting time of occupation by object \(k\). The cosine function is set to zero for \(t < t^k_i\) and \(t > t^k_i + \Delta T\). The prefactor \(A\) is determined in a way that satisfies the normalization condition.
\[ \int_{t_i^k}^{t_i^k + \Delta T} dt' Q_i(t') = 1, \]  

which implies \( A = 1/\Delta T \). Under this condition, the integral

\[ N_i^{\text{tot}}(t) = \int_0^t dt' Q_i(t') \]

counts the number of units that have passed the cross section or production unit under consideration.

Another possible specification assumes

\[ Q_i(t) = B \sum_k (t - t_i^k)^2 (t - t_i^k - \Delta T)^2, \]

where we set the fourth order polynomial to zero for \( t < t_i^k \) and \( t > t_i^k + \Delta T \).

The normalization condition (29) implies \( B = 30/(\Delta T)^5 \).

Together with (28) or (31), the relationship (30) can also be applied to situations where several units occupy a production unit at the same time. As before, the management function can be chosen as a function of the inventories \( N_i(t) \), but in many cases, it would be reasonable to replace a reaction to temporal changes \( dN_i/dt \) in the inventories by exponentially smoothed values.

### 1.7 Re-entrant production

![Fig. 1. Illustration of the model of re-entrant production and its variables (see main text for details).](image)

Semiconductor production [19] relies on some extremely expensive (lithographic) production units, which are therefore used for many similar subsequent production steps \( i \), namely the production of the various layers of a chip. This is the reason for re-entrant production [20, 21] (see Fig. 1.7), posing particular control problems not only for the overall arrival rate \( Q_{\text{arr}}(t) \) in the re-entrant production area, but also concerning the fractions \( p_i(t) \) of products in the different production stages \( i \) fed into it. If \( N_i(t) \) denotes the
stock level of chips after the $i$th entry (production step), $Q_{i}^{\text{in}}(t)$ the respective inflow, and $Q_{i}^{\text{out}}(t)$ the outflow, the related balance equation is again

$$\frac{dN_{i}}{dt} = Q_{i}^{\text{in}}(t) - Q_{i}^{\text{out}}(t). \quad (32)$$

Assuming also a buffer before the first and a buffer after the last ($u$th) production step, this equation applies to $i \in \{0, 1, \ldots, u\}$, where $Q_{u}^{\text{out}}$ corresponds to the removal rate of products that have completed the re-entrant (lithographic) production steps. The outflow $Q_{i}^{\text{out}}(t)$ determines the arrival rate $Q_{i+1}^{\text{arr}}(t)$ for the $i$th re-entrant production step, while the departure rate $Q_{i}^{\text{dep}}(t)$ determines the inflow $Q_{i}^{\text{in}}(t)$ into the subsequent buffer. The overall stock level $N(t)$ of chips at various production stages in the re-entrant production area changes in time according to

$$\frac{dN}{dt} = \sum_{i=1}^{u} [Q_{i-1}^{\text{out}}(t) - Q_{i}^{\text{in}}(t)] = \sum_{i=1}^{u} [Q_{i}^{\text{arr}}(t) - Q_{i}^{\text{dep}}(t)] = Q_{i}^{\text{arr}}(t) - Q_{i}^{\text{dep}}(t). \quad (33)$$

The overall departure rate $Q_{i}^{\text{dep}}(t)$ from the re-entrant production area depends on the number $N(t)$ of products processed at the same time. Moreover, if $p_{i}(t)$ denotes the fraction of chips that have entered the re-entrant area for the $i$th production step at time $t$ and $T^{0}(t)$ is the related cycle time of these chips in the re-entrant area, their departure rate is given by

$$Q_{i}^{\text{dep}}(t) = p_{i}(t - T^{0}(t))Q_{i}^{\text{dep}}(N(t)) = Q_{i}^{\text{in}}(t). \quad (34)$$

As before, the delay-differential equation for the temporal change of the cycle time is

$$\frac{dT^{0}}{dt} = \frac{Q^{\text{arr}}(t)}{Q^{\text{dep}}(t + T^{0}(t))} - 1. \quad (35)$$

The overall arrival rate may be adapted according to

$$\frac{dQ_{i}^{\text{arr}}}{dt} = \frac{1}{T}W(N(t), dN/dt, Q^{\text{arr}}(t), Q^{\text{dep}}(t), N_{u}(t), dN_{u}/dt, Q_{u}^{\text{out}}(t) - Q^{\text{arr}}(t)),$$

where the management function $W(\ldots)$ depends not only on the variables $N(t)$ and $Q^{\text{dep}}(t)$ characterizing the re-entrant production area, but also on the (desired) removal rate $Q_{u}^{\text{out}}(t)$ and the stock level $N_{u}(t)$ of the final buffer. Finally, the specific arrival rates of chips for the $(i+1)$st re-entrant production step are

$$Q_{i+1}^{\text{arr}}(t) = p_{i+1}(t)Q_{i}^{\text{arr}}(t) = Q_{i}^{\text{out}}(t), \quad (37)$$

where their relative percentages

$$p_{i+1}(t) = \frac{W_{i}(N_{i}(t), dN_{i}/dt, Q_{i}^{\text{in}}(t), Q_{i}^{\text{out}}(t), Q_{u}^{\text{out}}(t))}{\sum_{j} W_{j}(N_{j}(t), dN_{j}/dt, Q_{j}^{\text{in}}(t), Q_{j}^{\text{out}}(t), Q_{u}^{\text{out}}(t))} \quad (38)$$
are controlled by another management function $W_i(\ldots)$, which depends on the stock levels $N_i(t)$ of the respectively preceding buffers and on the removal rate $Q_{\text{out}}^i(t)$. The management functions $W_i(\ldots)$ allow one to specify different priorities such as push or pull strategies. Specifications of the management function, which can avoid bullwhip (resonance) effects and long cycle times will be presented in a forthcoming paper. Generalizations to the simultaneous production of various products are possible, requiring the consideration of an individual number of re-entrant steps with potentially different production speeds and cycle times (“overtaking”) in the re-entrant area. The problem is similar to the treatment of heterogeneous multi-lane traffic [26]. In the following, we will present a simpler traffic model for uniform vehicles, which shows how to transfer the above approach from production processes to transportation processes in street networks.

## 2 Queueing model of vehicle traffic in freeway networks

In the following, we will propose a traffic model [25], which was inspired by the above model of supply networks. Here, the elements to be served are the vehicles. The formula for the queue length determines the density in a road section, and the cycle time corresponds to the travel time.

When we now specify road traffic as a queueing system, we will take into account essential traffic characteristics such as the flow-density relation or the properties of extended congestion patterns at bottlenecks. In fact, traffic congestion is usually triggered by spatial inhomogeneities of the road network [27], and queueing effects are normally not observed along sections of low capacity, but upstream of the beginning of a bottleneck. Therefore, we will subdivide roads into sections $i$ of homogeneous capacity and length $l_{i}^{\text{max}}$, which start at place $x_i$ and end with some kind of inhomogeneity (i.e. an increase or decrease of capacity) at place $x_{i+1} = x_i + l_{i}^{\text{max}}$. In other words, the end of a road section $i$ is, for example, determined by the location of an on- or off-ramp, a change in the number of lanes, or the beginning or end of a gradient.

The model can be derived from the “fluid-dynamic” continuity equation

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial Q_i(x,t)}{\partial x} = \text{source terms} \quad (39)$$

describing the conservation of the number of vehicles. Here, $\rho(x,t)$ denotes the vehicle density per lane at place $x$ and time $t$, and $Q_i(x,t)$ the traffic flow per lane. The source terms originate from ramp flows $Q_{i}^{\text{ramp}}(t)$, which enter the road at place $x_{i+1}$. Let us define the arrival rate at the upstream end of road section $i$ (the “inflow”) by $Q_{i}^{\text{arr}}(t) = I_i Q_i(x_i + dx, t)$, where $dx$ is a differential space interval and $I_i$ the number of lanes of road section $i$. Analogously, the departure rate from the downstream end of this section is defined by $Q_{i}^{\text{dep}}(t) = I_i Q_i(x_{i+1} - dx, t)$. (Note that this “outflow” from section
$i$ is to be distinguished from the outflow $Q_{\text{out}}$ per lane from congested traffic. The conservation of the number of vehicles implies that the departure rate plus the ramp flow determine the arrival rate in the next downstream section $i + 1$:

$$Q_{i+1}^{\text{arr}}(t) = Q_i^{\text{dep}}(t) + Q_i^{\text{ramp}}(t).$$

In order to guarantee non-negative flows, we will demand for the ramp flows that the consistency condition $-Q_i^{\text{dep}}(t) \leq Q_i^{\text{ramp}}(t) \leq Q_{i+1}^{\text{arr}}(t)$ is always met.

Integrating the continuity equation over $x$ with $x_i < x < x_{i+1}$ provides a conservation equation for the number $N_i(t) = \int_{x_i}^{x_{i+1}} dx I_i \rho(x, t)$ of vehicles in road section $i$. It changes according to

$$\frac{dN_i(t)}{dt} = Q_i^{\text{arr}}(t) - Q_i^{\text{dep}}(t) = Q_i^{\text{dep}}(t) + Q_i^{\text{ramp}}(t) - Q_i^{\text{dep}}(t).$$

In the terminology of queueing theory, equation (41) reflects the change in the number $N_i(t)$ of vehicles waiting to be served by the downstream end of the section, and the number $I_i$ of lanes corresponds to the number of “channels” serving in parallel. Moreover, $N_i(t)$ corresponds to the “queue length”. However, this does not necessarily mean that we have congested traffic as, in the terminology of traffic theory, “queue length” refers to something else, namely the spatial extension $l_i(t)$ of a congested road section. In the following, we will try to express the traffic dynamics and the travel times only through the flows at the cross sections $x_i$, taking into account the features of traffic flow in a simplified way.

For free flow, i.e. below some critical vehicle density $\rho_{\text{cr}}$ per lane, the relation between the traffic flow $Q_i$ per lane and the density $\rho$ per lane can be approximated by an increasing linear relationship, while above it, a falling linear relationship is consistent with congested flow-density data (in particular, if the average time gap $T$ is treated as a time-dependent, fluctuating variable) [28]. This implies $Q_i(x, t) \approx Q_i(\rho(x, t))$ with

$$Q_i(\rho) = \begin{cases} Q_i^{\text{free}}(\rho) = \rho V_i^0 & \text{if } \rho < \rho_{\text{cr}} \\ Q_i^{\text{cong}}(\rho) = \frac{1 - \rho}{\rho_{\text{jam}}} / T & \text{otherwise}. \end{cases}$$

Here, $V_i^0$ denotes the average free velocity, $T$ the average time gap, and $\rho_{\text{jam}}$ the density per lane inside of traffic jams. Moreover, we define the free and congested densities by

$$\rho_i^{\text{free}}(Q_i) = Q_i / V_i^0 \quad \text{and} \quad \rho_i^{\text{cong}}(Q_i) = (1 - T Q_i) / \rho_{\text{jam}}.$$  

The quantity $Q_{\text{out}} = (1 - \rho_{\text{cr}} / \rho_{\text{jam}}) / T$ corresponds to the outflow per lane from congested traffic [29]. Depending on the parameter specification, the model describes a continuous flow-density relation (for $\rho_{\text{cr}} V_i^0 = Q_{\text{out}}$) or a capacity drop at the critical density $\rho_{\text{cr}}$ and high-flow states immediately before (if $\rho_{\text{cr}} V_i^0 > Q_{\text{out}}$).

According to shock wave theory [30], density variations at place $x$ propagate with velocity $C(t) = [Q_i(x + dx, t) - Q_i(x - dx, t)] / [\rho(x + dx, t) - \rho(x - dx, t)]$.
Accordingly, the propagation velocity is $C = V_i^0$ in free traffic, and $C = -c = -1/(T\rho_{\text{max}})$ in congested traffic. Therefore, it takes the time period $T_i^{\text{free}} = l_i^{\text{max}}/V_i^0$ for a perturbation to travel through free traffic, while it takes the time period $T_i^{\text{cong}} = l_i^{\text{max}}/c$, when the entire road section $i$ is congested.

Now, remember that congestion in section $i$ starts to form upstream of a bottleneck, i.e. at place $x_{i+1}$. Let $l_i(t)$ denote the length of the congested area and $x(t) = x_{i+1} - l_i(t) = x_i + l_i^{\text{max}} - l_i(t)$ the location of its upstream front. Then, we have free traffic between $x_i$ and $x_i + l_i^{\text{max}} - l_i(t)$, i.e. $Q_i(x - dx, t) = Q_i^{\text{arr}}(t - (x - x_i)/V_i^0)$ (considering $dx \to 0$), and congested traffic downstream of $x(t)$, i.e. $Q_i(x + dx, t) = Q_i^{\text{dep}}(t - (x_{i+1} - x)/c)$. With $dx/dt = -dl_i/dt = C(t)$ and Eq. (43) we find

$$dl_i/dt = \frac{Q_i^{\text{dep}}(t - l_i(t)/c)/I_i - Q_i^{\text{arr}}(t - [l_i^{\text{max}} - l_i(t)]/V_i^0)/I_i - \rho_i^{\text{cong}}(Q_i^{\text{dep}}(t - l_i(t)/c)/I_i - \rho_i^{\text{free}}(Q_i^{\text{arr}}(t - [l_i^{\text{max}} - l_i(t)]/V_i^0)/I_i).}$$

(44)

The capacity of a congested road section $i$ is approximated as the outflow $Q_{\text{out}} = (1 - \rho_i/\rho_{\text{jam}})/T$ from congested traffic per lane times the number $I_i$ of lanes, minus the maximum bottleneck strength at the end of this section. This may be given by an on-ramp flow $Q_i^{\text{ramp}}(t) > 0$ or analogously by $(I_i - I_{i+1})Q_{\text{out}}$ in case of a reduction $I_{i+1} - I_i < 0$ in the number of lanes, or in general by some time-dependent value $\Delta Q_i(t)$ in case of another bottleneck such as a gradient:

$$Q_i^{\text{cap}}(t) = I_iQ_{\text{out}} - \max[Q_i^{\text{ramp}}(t), (I_i - I_{i+1})Q_{\text{out}}, \Delta Q_i(t), 0].$$

(45)

Analogously, the maximum capacity $Q_i^{\text{max}}(t)$ of the road section $i$ under free flow conditions is given by the maximum flow $I_i\rho_{\text{cap}}V_i^0$ minus the reduction by bottleneck effects:

$$Q_i^{\text{max}}(t) = I_i\rho_{\text{cap}}V_i^0 - \max[Q_i^{\text{ramp}}(t), (I_i - I_{i+1})\rho_{\text{cap}}V_i^0, \Delta Q_i(t), 0].$$

(46)

Moreover, we have to specify the departure rate $Q_i^{\text{dep}}(t)$ as a function of the respective traffic situation. Focussing on the cross section at location $x_{i+1}$ and considering the directions of information flow (i.e. the propagation direction of density variations), we can distinguish three different cases:

1. If we have free traffic in the upstream section $i$ and free or partially congested traffic in the downstream section $i + 1$, density variations propagate downstream and the departure rate $Q_i^{\text{dep}}(t)$ at time $t$ is given as the arrival rate $Q_i^{\text{arr}}(t - T_i^{\text{free}}) = Q_{i-1}^{\text{dep}}(t - T_i^{\text{free}}) + Q_{i-1}^{\text{ramp}}(t - T_i^{\text{free}})$, since the vehicles entering section $i$ at time $t - T_i^{\text{free}}$ leave the section after an average travel time $T_i$ of $T_i^{\text{free}}$.

2. In the case of partially or completely congested traffic upstream and free or partially congested traffic downstream, the departure rate $Q_i^{\text{dep}}(t)$ is given by the capacity $Q_i^{\text{cap}}(t)$ of the congested road section $i$. 

3. In the case of congested traffic on the entire downstream road section $i + 1$, the departure rate $Q_{i+1}^{\text{dep}}(t)$ is given by the departure rate $Q_{i+1}^{\text{dep}}(t - T_{i+1}^{\text{cong}})$ from the downstream section at time $t - T_{i+1}^{\text{cong}}$ minus the ramp flow $Q_i^{\text{ramp}}(t)$ entering at location $x_{i+1}$.

Summarizing this, we have

$$Q_{i+1}^{\text{dep}}(t) = \begin{cases} Q_i^{\text{arr}}(t - T_i^{\text{free}}) & \text{if } l_{i+1}(t) < l_{i+1}^{\text{max}} \text{ and } l_i(t) = 0 \\ Q_i^{\text{dep}}(t) & \text{if } l_{i+1}(t) < l_{i+1}^{\text{max}} \text{ and } l_i(t) > 0 \\ Q_{i+1}^{\text{dep}}(t - T_{i+1}^{\text{cong}}) - Q_i^{\text{ramp}}(t) & \text{if } l_{i+1}(t) = l_{i+1}^{\text{max}}. \end{cases}$$

(47)

A numerical solution of the above defined section-based queueing-theoretical traffic model is carried out as follows: First, calculate the new arrival and departure rates by means of Eqs. (40) and (47), taking into account the boundary conditions for the flows at the open ends of the road network. Second, determine the queue lengths $l_i(t)$ in all road sections $i$: (i) If traffic in the road section flows freely ($l_i(t) = 0$) and the maximum capacity $Q_i^{\text{max}}(t)$ is not reached, i.e. $Q_i^{\text{arr}}(t - T_i^{\text{free}}) < Q_i^{\text{max}}(t)$, we have $dl_i(t)/dt = 0$ and the traffic flow in the road section remains free. (ii) If the road section is completely congested ($l_i(t) = l_{i+1}^{\text{max}}$) and the arrival rate $Q_i^{\text{arr}}(t)$ is not below the departure rate at time $t - T_i^{\text{cong}}$, i.e. $Q_{i+1}^{\text{dep}}(t) - Q_i^{\text{ramp}}(t) < Q_i^{\text{arr}}(t)$, the road section $i$ stays fully congested and $dl_i(t)/dt = 0$. (iii) In other cases, we have partially congested traffic in road section $i$ and the length $l_i(t)$ of the congested area changes according to Eq. (44). Next, one continues with the first step for the new time $t + dt$, and so on. It is obvious, that this numerical solution is significantly more simple and robust than the numerical solution of the Lighthill-Whitham model, as shock waves (i.e. the interfaces between free and congested traffic) are treated analytically and the propagation velocities of perturbations within the free and congested regions are constant.

The travel time $T_i(t)$ of a vehicle that enters road section $i$ at time $t$ can be calculated analogously to Eq. (27) [25], i.e. via the delay-differential equation

$$\frac{dT_i(t)}{dt} = \frac{Q_i^{\text{arr}}(t)}{Q_i^{\text{dep}}(t + T_i(t))} - 1 = Q_{i-1}^{\text{dep}}(t) + Q_{i-1}^{\text{ramp}}(t) + \frac{Q_i^{\text{dep}}(t)}{Q_i^{\text{dep}}(t + T_i(t))} - 1.\quad(48)$$

According to this, the travel time $T_i(t)$ increases with time, when the arrival rate $Q_i^{\text{arr}}(t)$ at the time of entry exceeds the departure rate $Q_i^{\text{dep}}(t)$ at the leaving time $t + T_i(t)$, while it decreases when it is lower. It is remarkable that this formula does not explicitly depend on the velocities on the road section, but only on the arrival and departure rates. The calculation of the travel time based on the velocity is considerably more complicated: Let $v(t)$ be the velocity and $x(t) = x_i + \int_{t_0}^{t} v(t') dt'$ the location of a vehicle at time $t$, when it enters section $i$ at time $t_0$. Its travel time $T_i(t_0)$ on section $i$ is given by the implicit equation $x(t + T_i(t_0)) = x_{i+1}$, which says that this vehicle reaches place $x_{i+1} = x_i + l_i^{\text{max}}$ at time $t_0 + T_i(t_0)$. The vehicle speed $v(t)$ is also difficult to determine, as it depends on the (free or congested) traffic state at
its respective location $x(t)$: It is $v(t) = V_0^i$ in free flow, i.e. for $V_0^i(t-t_0) < l_i^{\max} - l_i(t)$. In congested flow, i.e. for $t_0 + [l_i^{\max} - l_i(t)]/V_0^i \leq t \leq t_0 + T_i(t_0)$, it is determined via $v(t) = Q_i(x(t), t)/\rho(x(t), t) = [1/\rho(x(t), t) - 1/\rho_{\max}] / T$ with $\rho(x(t), t) = \rho(x_{i+1}, t-[x_{i+1} - x(t)]/c) = \{1-TQ_i^{\text{dep}}(t-[x_{i+1} - x(t)]/c)/I_i\}/\rho_{\text{jam}}$.

3 Summary and conclusions

In this contribution, traffic and production systems have been treated in a uniform way as dynamic queueing networks, since it does not matter whether one treats the transport of goods from one production unit to the next one or of vehicles from one cross section of the road network to the next. Consequently, the stability conditions for supply chains looked similar to the ones of some specific traffic models. Moreover, we have presented a delay-differential equation for the determination of cycle (production) times, which can be also applied to calculate the travel times of vehicles. Interestingly enough, this formula did not require to calculate the vehicle speed on freeway sections, but only the in- and outflows at certain cross sections of the street network, namely, where the road capacity changed.

The derived instability conditions allow one to choose appropriate management strategies which can avoid the well-known bullwhip effect. This effect describes an amplification of variations in the delivery rate and inventory from one supplier to the next one upstream. Such a convective instability can occur despite of a stable behavior in time because of the possibility of resonance effects. Apart from this, we have sketched the treatment of re-entrant production processes and of discrete units.

The advantage of “fluid-dynamic” models of traffic, supply, and production networks is their great numerical efficiency and their consideration of non-linear interaction effects. Therefore, in contrast to most classical queueing theoretical approaches and to event-driven (Monte Carlo) simulations, they are suitable for on-line control and the treatment of variations in the consumption rate or in the production program.

Present research focusses on the effect of the topology of supply networks on their dynamics [3]. The dynamics of production processes can even be chaotic [22, 31–33]. Other studies concentrate on the subject of optimal control (including chaos control) [33–39], which is particularly challenging for re-entrant production [20, 21]. Finally, the presented traffic model is now being applied to the simulation of city networks with adaptive traffic light control and to dynamic assignment problems.

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