Collective excitation frequencies and vortices of a Bose-Einstein condensed state with gravity-like interatomic attraction

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We study the collective excitations of a neutral atomic Bose-Einstein condensate with gravity-like $1/r$ interatomic attraction induced by electromagnetic wave. Using the time-dependent variational approach, we derive an analytical spectrum for monopole and quadrupole mode frequencies of a gravity-like self-bound Bose condensed state at zero temperature. We also analyze the excitation frequencies of the Thomas-Fermi-gravity (TF-G) and gravity (G) regimes. Our result agrees excellently with that of Giovanazzi et al. [Europhysics Letters, 56, 1 (2001)], which is obtained within the sum-rule approach. We also consider the vortex state. We estimate the superfluid coherence length and the critical angular frequencies to create a vortex around the $z$-axis. We find that the TF-G regime can exhibit the superfluid properties more prominently than the G-regime. We find that the monopole mode frequency of the condensate decreases due to the presence of a vortex.

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I. INTRODUCTION

The discovery of Bose-Einstein condensation (BEC) in a dilute alkali atomic gas opens up new perspective in the field of many body physics \[^1\]\. Most of the properties of these dilute gas can be explained by considering only two-body short range interaction which is characterized by the s-wave scattering length $a$ \[^2\]\. Recently, a new kind of atomic BEC has been proposed by D. O’Dell et al. \[^4\]\. They have shown that the particular configuration of intense off-resonant laser beams gives rise to an effective gravity-like $1/r$ interatomic attraction between neutral atoms located well within the laser wavelength. This long range interaction potential is of the form, $U(r) = -u/r$, where $u = (11\pi/15)(I_0^2/\alpha^2\lambda^2)$. $I$ and $\lambda$ are the total laser intensity and wave length respectively. $\alpha$ is the atomic polarizability at the frequency $2\pi c/\lambda$. In this system, the gravity-like $1/r$ attraction balances the pressure due to the zero point kinetic energy and the short range interaction potential. For strong induced gravity-like potential, the BEC becomes stable even without the external trap potential \[^4\]\. There is a competition between the gravity-like potential either with the kinetic energy (G) or the two-body short range interatomic interaction potential characterized by the s-wave scattering length $a$ (TF-G), which gives two new regimes for new atomic BEC. These two regimes are obtained based on the Gaussian variational ansatz for the ground state wave function \[^4\]\. In the TF-G regime, collective excitation frequencies has been calculated numerically by solving the equations of collisionless hydrodynamics \[^7\]\. Moreover, in this regime, an analytic expression of the ground state density is obtained \[^7\]\. Within the sum-rule approach, collective excitation frequencies of a gravity-like self-bound Bose gas has been discussed in Ref. \[^7\]\. There has been no systematic and detail study on the collective excitation frequencies and vortices of a gravitationally self-bound Bose gas by using the time dependent variational approach. The purpose of this paper is to give an analytic expressions for collective excitation frequencies, superfluid coherence length and critical angular frequencies required to create a vortex of a rotating Bose condensed state and to compare qualitatively the results of the TF-G regime with the results obtained in the TF regime of an ordinary atomic BEC.

Here, by using the time-dependent variational method, we obtain the excitation spectrum of a gravity-like self-bound Bose gas. We also calculate the lower bound of the ground state energy, monopole and quadrupole mode frequencies of the TF-G and the G regimes. Our result agrees excellently with that of ref. \[^7\] which is obtained within the sum-rule approach. Next, we consider a rotating Bose condensate state with a single vortex along the $z$-axis. We estimate the superfluid coherence length and the critical angular frequencies required to create a vortex along the $z$-axis. We find that the TF-G regime of a gravitationally self-bound Bose condensed state should exhibit superfluid properties prominently than the G-regime. We find that the monopole mode frequency of the condensate decreases due to presence of the vortex.
II. TIME-DEPENDENT VARIATIONAL ANALYSIS

The equation of motion of the condensate wave function is described by the generalized Gross-Pitaevskii equation

\[ i\hbar \frac{\partial \psi(\vec{r}, t)}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + \frac{m\omega_0^2 r^2}{2} + V_H(r) \right] \psi(\vec{r}, t), \tag{1} \]

where \( V_H(r) \) is the self-consistent Hartree potential,

\[ V_H(\vec{r}) = \frac{4\pi a\hbar^2}{m} |\psi(\vec{r}, t)|^2 - u \int d^3r' \frac{|\psi'(\vec{r}, t)|^2}{|\vec{r} - \vec{r}'|} \tag{2} \]

The normalization condition for \( \psi \) is \( \int |\psi(\vec{r}, t)|^2 d^3r = N, N \) is the total number of particles in the condensed state. The original Gross-Pitaevskii equation can be obtained by putting \( u = 0 \).

One can write down the Lagrangian density corresponding to this system as follows,

\[ L = \frac{i\hbar}{2} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) + \frac{\hbar^2}{2m} |\nabla \psi|^2 + \frac{m\omega_0^2 r^2}{2} |\psi|^2 + \frac{2\pi a\hbar^2}{m} |\psi|^4 - \frac{u}{2} |\psi|^2 \int d^3r' \frac{|\psi'(\vec{r}, t)|^2}{|\vec{r} - \vec{r}'|}, \tag{3} \]

where * denotes the complex conjugation. The non-linear Schroedinger equation can be obtained from a minimization of the action, \( I = \int L d^3r dt \). The BEC of charged Bosons \( \Phi \) confined in an ion trap can be described by the above mentioned Lagrangian if we set \( u = e^2 \), where \( e \) is the electronic charge.

To calculate the excitations spectrum of an atomic BEC with gravity-like interaction, we will use the time-dependent variational method. This technique has been first used to calculate the low-lying excitations spectrum of a harmonically trapped atomic BEC in Ref. \( \Phi \). The result obtained from the variational method matches with Stringari's \( \Phi \) result within the sum-rule approach.

In Ref. \( \Phi \), it is shown that the oscillation frequencies obtained from the exact ground state and a Gaussian ansatz are in good agreement. In this system, a Gaussian ansatz is also a good variational wave function. In order to obtain the evolution of the condensate we assume the following variational wave function,

\[ \psi(\rho, z, t) = C(t) e^{-\frac{1}{2}(\alpha(t)\rho^2 + \beta(t)z^2)}, \tag{4} \]

where \( C(t) \) is the normalization constant which can be determined from the normalization condition. \( \rho \) and \( z \) are the variables in units of \( \Lambda \), where \( \Lambda = \sqrt{\hbar/m\omega_g} = \hbar^2/muN \) is the length scale in this system (similar to the harmonic oscillator length) when the harmonic trap is absent and \( \omega_g = m\omega N^2/\hbar^3 \) is the gravitational frequency. \( \beta \) is the two dimensional vector. \( \alpha(t) = 1/\alpha_1^2 + i\alpha_2 \) and \( \beta(t) = 1/\beta_1^2 + i\beta_2 \) are the dimensionless time dependent parameter. \( \alpha_1 \) and \( \beta_1 \) are the condensate widths in \( x \) \(- y \) plane and along the \( z \) direction respectively. The Gaussian variational wave function is an exact ground state when the two two-body interatomic interaction is absent.

Substituting (4) into (3) and integrating the Lagrangian density over the space co-ordinates. We get the following Lagrangian,

\[ L = \frac{SNu}{2a} \left[ (\alpha_1^2 \dot{\alpha}_2 + \frac{1}{2} \beta_1^2 \dot{\beta}_2) - \left( \frac{1}{\alpha_1^2} + \alpha_1^2 \alpha_2^2 \right) - \frac{1}{2} \left( \frac{1}{\beta_1} + \beta_1^2 \beta_2^2 \right) - \sqrt{2} \frac{S}{\pi} \left( \frac{F[\frac{1}{2}, 1; \frac{1}{2}, (1 - \frac{\alpha_1^2}{\beta_1^2})]}{\alpha_1^2 \beta_1} \right) \right], \tag{5} \]

where \( S = \frac{Na}{A} = \frac{Na_0^2 N^2}{6} \) is a dimensionless scattering parameter similar to the scattering parameter \( P = \frac{Na}{a_0} \) for an ordinary atomic BEC. Here, \( a_0 \) is the harmonic oscillator length. This \( S \) can be positive or negative depending on the sign of the scattering length \( a \). The scattering parameter \( S \) can also be written as \( S = (528\pi^2/105)I(Na/\lambda)^2 \), where \( I = I_0 \) and \( I_0 = (48\pi/7)(c^2 \gamma_0/m a_0^2) \) is the threshold laser intensity to create a self-bound condensate \( \Phi \).

For a given intensity \( I = 1.5 \), the realistic, stable and self-trapped system (for sodium atoms) contains 40 to 400 atoms \( \Phi \) and the corresponding range of \( S \) is 1 to 100. This range can be alter by changing the scattering length \( a \) \( \Phi \). \( F[\frac{1}{2}, 1; \frac{1}{2}, (1 - \frac{\alpha_1^2}{\beta_1^2})] \) is the Hypergeometric function. The last term in the Eq. (5) is the mean-field energy of the gravity-like potential. We are interested to find out the excitation spectrum of a self-bound Bose gas as well as in the TF-G and G regimes. We have set \( V_{ext} = 0 \) because the system is stable even in the absence of an external trap potential.

The energy functional in terms of the variational parameter \( \alpha \) in an isotropic system is
\[ E = \frac{NuS}{2a} \left[ \frac{3}{2a^2} + \sqrt{\frac{2}{\pi}} \left( \frac{S}{\alpha^3} - \frac{1}{\alpha} \right) \right], \]

By minimizing the energy functional with respect to \( \alpha \), one can get the equilibrium point \( w \) which is given by,

\[ w = \frac{3}{2} \sqrt{\frac{\pi}{2}} \left[ 1 + \sqrt{1 + \frac{8S}{\delta \omega^2}} \right]. \]

The sound velocity is \( c_s^2 = \mu/m, \) where \( \mu = uS/2a [3/2w^2 + 2\sqrt{2/\pi}(S/w^3 - 1/w)] \) is the chemical potential. The sound velocity \( c_s \) vs. the dimensional scattering parameter \( S \) are shown in Fig.1.

Using the Euler-Lagrange equation, the time evolution of the widths are,

\[ \alpha_1 = \frac{1}{\alpha_1^3} + \sqrt{\frac{2}{\pi}} \left( \frac{S}{\alpha_1^4} \right) + \frac{F_{\alpha_1}[\alpha_1, \beta_1]}{2}, \]

\[ \beta_1 = \frac{1}{\beta_1^3} + \sqrt{\frac{2}{\pi}} \left( \frac{S}{\alpha_1^4} \right) + \frac{F_{\beta_1}[\alpha_1, \beta_1]}{2}. \]

\( F_{\alpha_1}[\alpha_1, \beta_1] \) is the derivative of \( F[1/3, 1; \frac{4}{3}, (1 - \frac{a^2}{\chi^2})]/\beta_1 \) with respect to \( \alpha_1 \). Similarly, \( F_{\beta_1}[\alpha_1, \beta_1] \) is the derivative of \( F[1/3, 1; \frac{4}{3}, (1 - \frac{a^2}{\chi^2})]/\beta_1 \) with respect to \( \beta_1 \). The exact form of \( F_{\alpha_1}[\alpha_1, \beta_1] \) and \( F_{\beta_1}[\alpha_1, \beta_1] \) are given in the Appendix A.

We are interested in the low-energy excitations of a gravity-like self-bound Bose condensate. The low-energy excitations of the condensate corresponds to the small oscillation \( S \) of the state around the equilibrium widths \( \alpha_{10} \) and \( \beta_{10} \). Therefore, we expand around the time dependent variational parameters around the equilibrium widths in the following way, \( \alpha_1 = \alpha_{10} + \delta \alpha_1 \) and \( \beta_1 = \beta_{10} + \delta \beta_1 \).

The time evolution of the widths around the equilibrium points are

\[ \delta \alpha_1 = -\left( \frac{3}{\alpha_{10}^3} + 3\sqrt{\frac{2}{\pi}} \frac{S}{\alpha_{10}^4 \beta_{10}} \right) \delta \alpha_1 - \sqrt{\frac{2}{\pi}} \frac{S}{\alpha_{10}^4 \beta_{10}^2} \delta \beta_1 + \frac{1}{2} \sqrt{\frac{2}{\pi}} F_{\alpha_1}[\alpha_{10}, \beta_{10}, \delta \alpha_1, \delta \beta_1], \]

\[ \delta \beta_1 = -2 \sqrt{\frac{2}{\pi}} \frac{S}{\alpha_{10}^4 \beta_{10}^2} \delta \alpha_1 - \left( \frac{3}{\beta_{10}^3} + 2 \sqrt{\frac{2}{\pi}} \frac{S}{\alpha_{10}^2 \beta_{10}^3} \right) \delta \beta_1 + \sqrt{\frac{2}{\pi}} F_{\beta_1}[\alpha_{10}, \beta_{10}, \delta \alpha_1, \delta \beta_1]. \]

\( F_{\alpha_1}[\alpha_{10}, \beta_{10}, \delta \alpha_1, \delta \beta_1] \) is the first order fluctuations around the equilibrium points of \( F_{\alpha_1}[\alpha_1, \beta_1] \). Similarly, \( F_{\beta_1}[\alpha_{10}, \beta_{10}, \delta \alpha_1, \delta \beta_1] \) is the first order fluctuations around the equilibrium points of \( F_{\beta_1}[\alpha_1, \beta_1] \). The exact form of \( F_{\alpha_1}[\alpha_{10}, \beta_{10}, \delta \alpha_1, \delta \beta_1] \) and \( F_{\beta_1}[\alpha_{10}, \beta_{10}, \delta \alpha_1, \delta \beta_1] \) are given in the Appendix A.

We are looking for the solutions of \( e^{i\omega t} \) type. First we solve for these two equation in terms of \( \alpha_{10}, \beta_{10} \) and later we set \( \alpha_{10} = \beta_{10} = w \). For isotropic system, the excitations spectrum are

\[ \frac{\omega^2}{\omega_g^2} = \frac{3}{w^3} + \sqrt{\frac{2}{\pi}} \left( \frac{4S}{w^5} - \frac{2}{3w^3} \right), \]

\[ \frac{\omega^2}{\omega_g^2} = \frac{3}{w^3} + \sqrt{\frac{2}{\pi}} \left( \frac{S}{w^5} - \frac{1}{15w^3} \right). \]

where \( \omega_g = m \sqrt{\frac{2}{\pi}} \sqrt{\frac{S}{w^3}} \) is the gravitational frequency. The \( \omega_+ \) and \( \omega_- \) vs. the dimensionless scattering parameter \( S \) are shown in Fig.2. When \( S \) is small, the gravity-like \( 1/r \) attractive interaction dominates over the repulsive pseudopotential. In this limit, the system becomes more compressible. In other words, the system becomes less resistant to density changes. So one would expect that monopole mode lies below quadrupole mode. From the Fig.2, we identify that the upper branch of the excitation spectrum is quadrupole mode ( \( \omega_- = \omega_M \) ) and lower branch is monopole mode ( \( \omega_+ = \omega_M \) ). The monopole and quadrupole modes spectrum in Fig.2 matches very well with the spectrum obtained within the sum rule approach. When \( S = -1.169, \omega_M \) starts decreasing as shown in Fig.2. At \( S_c = -1.179 \), the system collapses. The value of \( S_c \) can also be obtained from the Eq. 10.

For large value of \( S \), the monopole mode lies above the quadrupole mode because the repulsive pseudopotential start dominates over the
attractive long-range interaction. At $S = 17.5$, there is a crossing between these two modes which is shown in inset of Fig. 2. Interestingly, this crossing of these two modes is also obtained from the time-dependent variational method.

**TF-G regime:** For large s-wave scattering length, the kinetic and the trap potential energy can be neglected. The gravity-like potential is balanced by the s-wave interaction strength. The total ground state energy is $E_0 = -0.9648 N^2 (u/A_J)$, where $A_J = 2\sqrt{\hbar^2/mu}$ is the Jeans wavelength which is the shortest wavelength to keep stable condensed state. The ground state energy per particle varies as $N$. The sound velocity $c_s$ varies as $N^{1/2}$ whereas $c_s \sim N^{1/5}$ for an ordinary atomic BEC in the TF approximation [13]. In this regime, we neglect the contribution of the kinetic energy term in Eqs. (8) and (9) and we find that the monopole and quadrupole frequencies are $\omega_M = 0.31995 \omega g S^{-3/4}$ and $\omega_Q = 0.202355 \omega g S^{-3/4}$. In this regime, the monopole and quadrupole frequencies are obtained by solving the hydrodynamic equations numerically in [5]. The monopole and quadrupole frequencies obtained from the variational approach are similar to the exact numerical values. For an ordinary atomic BEC in the TF regime, the $\omega_M$ and $\omega_Q$ are independent of the scattering length $a$. But, in this system, it still depends on the scattering length $a$. Here, the ratio $\omega_M/\omega_Q$ is $\sqrt{5/2} = 1.58114$. Remarkably, this ratio is identical to the result of [5] which is obtained within the sum-rule approach. This is also true for trapped atomic BEC without gravity-like interaction in the large $N$ limit. It was first pointed out in [5].

**G regime:** In this regime, gravity-like potential is balanced by the kinetic energy. The trap potential and s-wave interaction can be neglected. This is analog of Boson-star (non-relativistic) [14]. The total ground state energy is $E_0 = -(N/19) \omega g$. Using the uncertainty relation, we estimate the total ground state energy is $E_0 = -(N/16) \omega g$ which is very close to the energy obtained from the variational approach. So our variational ansatz for wave function is good in this regime also. The ground state energy per particle varies as $N^2$. The sound velocity $c_s$ varies as $N$. In this regime, neglecting the contribution of the s-wave interaction potential in Eqs. (8) and (9), we get the monopole and quadrupole modes are $\omega_M = 0.070755 \omega g$ and $\omega_Q = 0.118363 \omega g$. These frequencies are very close to the frequencies obtained within the sum-rule approach [6]. Here, $\omega_M/\omega_Q = \sqrt{5/14} = 0.597615$. This ratio is also identical (up to five decimal) with that of [5], which is obtained within the sum-rule approach.

**III. VORTICES OF A GRAVITATIONALLY SELF-BOUND BOSE CONDENSATE STATE**

We consider a gravitationally self-bound Bose condensate state with a vortex along the $z$-axis. The experimental realization of a vortex state would be a direct signature of macroscopic phase coherence of this new atomic BEC with an attractive long-range interaction. One can use the time-dependent variational approach to describe the vortex state. In the previous section, we have explicitly shown that the monopole and quadrupole mode frequencies obtained by using the Gaussian ansatz coincides with the numerical results. So it is natural choice to assume a variational wave function of a self-bound BEC state with a vortex along the $z$-axis is,

$$\psi_q(r^2, t) = C_q(t) \rho \phi q \phi e^{-\frac{q}{\rho}(\frac{1}{\rho} + i\beta(t))}, \quad (14)$$

where $q$ is the vortex quantum number and $C_q(t)$ is the normalization constant. Also, $\rho^2 = x^2 + y^2, r^2 = x^2 + y^2 + z^2$ and $\phi = tan^{-1}(y/x)$. For simplicity, we consider only $q = 1$ and $q = 2$. By following the same procedure of the previous section, one would obtain the effective Lagrangian which is given by

$$L = \frac{N u S}{2a} [(q + \frac{3}{2}) \alpha^2 \beta - (q + \frac{3}{2})(\frac{1}{\alpha^2} + \alpha^2 \beta^2)] - \sqrt{\frac{2}{\pi}} (g_0 S - \frac{c_q}{\alpha}), \quad (15)$$

where $g_0 = \frac{2q!}{\pi^2 (q+1)!}, c_1 = 23/30$ and $c_2 = 37/56$.

The energy functional of the vortex state in terms of the variational parameter $\alpha$ is

$$E_q = \frac{N u S}{2a} [(q + \frac{3}{2})(\frac{1}{\alpha^2}) + \sqrt{\frac{2}{\pi}} (g_0 S - \frac{c_q}{\alpha})] \quad (16)$$

By minimizing the energy with respect to the variational parameter $\alpha$, one could obtain the equilibrium width $w_q$ which is given by

$$w_q = \sqrt{\frac{2\pi (q + \frac{3}{2})}{2c_q}} + \sqrt{\frac{2\pi (q + \frac{3}{2})^2 + 12g_0 S c_q}{2c_q}}, \quad (17)$$
where $g_q$ and $c_q$ are given above. The system collapses when $S_c = -8.53694$ for $q = 1$ and $S_c = -25.8875$ for $q = 2$. This critical value $S_c$ is increasing with increasing of the number of vorticity. The expectation value of the square of the system radius is $I_q = \sqrt{< r^2 >} = \sqrt{N(q+3/2)w_q}$. The energy functional satisfies the stability condition, 
\[ \frac{\partial^2 E_g}{\partial q^2} |_{q=w_q} > 0. \]

The superfluid coherence length, $\xi$, is a distance over which the condensate wave function can heal. In the case of a vortex, it corresponds to the distance over which the wave function increases from zero, on the vortex axis, to the bulk density. It can be calculated by equating the kinetic energy to the interaction energies. The kinetic energy term can not be neglected even for large $S$, since it determines the structure of the vortex core. The system exhibits superfluid properties if the coherence length is small compared to the size of the condensed state, otherwise it will be less prominent to observe the superfluid properties. Equating the kinetic energy to the interaction energies,

\[ \frac{\hbar^2}{2m\xi^2} = \frac{4\pi a_N \hbar^2}{mR^3} - \frac{uN}{\xi}, \]

where $R = aN\sqrt{F}/S$ is the radius of the condensed state at $S$ and,

\[ F = \frac{3}{2w^2} + \sqrt{\frac{S}{\pi}} \left( \frac{S}{w^3} - \frac{1}{w} \right). \]

$w$ is given in the Eq. (18). After rescaling the above equation, we get a quadratic equation of $\xi$ whose solution is,

\[ \frac{\xi}{R} = \frac{F + \sqrt{F^2 + 8SF}}{8\pi S}. \]

The superfluid coherence length $\xi/R$ vs. $S$ is shown in Fig.3 for a wide range of $S$. The vortex state play an important role in characterizing the superfluid properties of Bose system. The critical angular frequency required to produce a vortex state is

\[ \Omega_q = \frac{(E_q - E_0)}{N\hbar q}, \]

where $E_q$ is the energy of a vortex states with vortex quantum number $q$ and $E_0$ is the energy with no vortex. The critical angular frequency $\Omega_1$ vs. the dimensionless scattering parameter $S$ is shown in Fig.4. The critical angular frequency decreases with increasing $S$ (or $N$). For attractive interaction, $\Omega_q$ increases with increasing of $S$. This is also true for an ordinary BEC in the TF regime.

The monopole mode frequency for the vortex state with a vortex number $q$ is

\[ \frac{\omega^2}{\omega^2_g} = \frac{3}{w_q^2} + \sqrt{\frac{2}{\pi}} \left( \frac{1}{q} + \frac{3}{2} \right) \left( \frac{6S_{g0}}{w_q} - \frac{c_g}{w_q^2} \right), \]

where $w_q$ is given in the Eq. (17).

**TF-G regime:** For large s-wave scattering length, kinetic energy can be neglected. The superfluid coherence length can be obtain from the Eq. (21). When $S$ is large, one would gets $F = 0.6142S^{-1/2}$ from Eq. (13). Then, $\xi/R = 0.1765S^{-5/8}$. When is $S$ is very large, coherence length is very small compared to the size of the system. The TF-G regime should exhibit the superfluid properties. The mean size of the condensate with $q = 1$ and $q = 2$ are $I_1 = 2.21163\sqrt{S}$ and $I_2 = 2.4412\sqrt{S}$ respectively. The size of a condensate state with vortices increases with the number of vorticity. The critical angular frequencies for $q = 1$ and $q = 2$ are $\Omega_1 = \omega_0 0.0077S^{-1/2}$ and $\Omega_2 = \omega_0 0.0094S^{-1/2}$ respectively. The monopole mode frequencies for one and two vortices are $\omega_1 = 0.2999\omega_0 S^{-3/4}$ and $\omega_2 = 0.266\omega_0 S^{-3/4}$, respectively. These two monopole mode frequencies are less than the monopole mode frequency of the vortex free condensate. So, in the TF-G limit, monopole mode frequency of the condensate decreases due to the presence of the vortex. The monopole mode frequency for an ordinary atomic BEC in the TF regime is independent of the vortex.

**G regime:** In this regime, we neglect the contribution from the s-wave interaction energy. The superfluid coherence length $\xi$ can be obtained by equating the kinetic energy to the gravity-like interaction energy, $\hbar^2/2m\xi^2 = uN/\xi$, which gives $\xi = 0.8862R_g$ which is almost equal to the radius ($R_g$) of the condensate state in this regime. As we know that if the coherence length is comparable to the radius of the condensate state, it is difficult to exhibit the superfluid properties. In this regime, $I_1 = 12.92$ and $I_2 = 24.84$. Here, $I_2 >> I_1$. The size of a condensate state (with vortices) expanding abruptly with increase of number of the vorticity. For example, $I_1/I_0 \sim 2.8$ and $I_2/I_0 \sim 5.39$, where
$I_0 = 4.604$ is the mean size of the vortex free condensate. The critical angular frequencies for $q = 1$ and $q = 2$ are $\Omega_1 = 0.0343\omega_g$ and $\Omega_2 = 0.0215\omega_g$, respectively. Here, $\Omega_2 < \Omega_1$. In this regime, the condensate state with vortex of $q = 2$ is unbounded because $|\mu_2|/\hbar < 2\Omega_2$, where $\mu_2 = -0.0297uS/a$ is the chemical potential in the rotating frame. So the vortex of $q \geq 2$ cannot be created in this regime unless there is an additional repulsive potential. Note that although there is an indication of instability of vortex with $q = 2$, this may be just an artifact of the variational approach. The monopole mode frequency for $q = 1$ is $\omega_1 = 0.0149\omega_g$. The $\omega_1$ is also less than the monopole mode frequency $\omega_M$ in the vortex free condensate.

It should be noted that the vortex has two different length scales, condensate radius and core radius, whereas the trial wave function (14) has only one variational length scale ($\alpha$). In this variational approach, the various numerical values computed for the energies of the two regimes and for the collapse are just indicative but more accurate values can be obtained by other rigorous methods.

IV. SUMMARY AND CONCLUSIONS

In this paper, we have derived an analytic expressions of the monopole and quadrupole excitation frequencies of a self-bound Bose gas induced by the electromagnetic wave for a wide range of the dimensionless scattering parameter $S$. Later, we consider the two new regimes, namely, TF-G and G regimes. In these regimes, we have calculated the lower bound of the ground state energy, sound velocity, monopole and quadrupole mode frequencies. Our results are in excellent agreement with the result obtained by using the sum-rule approach. Interestingly, the ratio $\omega_M/\omega_Q$ is identical (in both the regimes) with that of which is obtained within the sum-rule approach. In the TF regime of an ordinary atomic BEC, the monopole and quadrupole mode frequencies are independent of the scattering length $a$. On the other hand, in the TF-G regime, the monopole and quadrupole mode frequencies depend on the scattering length $a$. The local sound velocity $c_s$ varies as $N^{1/2}$ in the TF-G regime, whereas $c_s \sim N^{1/5}$ for an ordinary atomic BEC in the TF regime. For harmonic trapped Bose system, the excitation frequencies are determined by the oscillator frequency of the trap potential. But, in this system, the monopole and quadrupole mode frequencies are fixed by the gravitational frequency $\omega_g$.

In section III, based on the time-dependent variational method and simple ansatz for the wave function (equ. 14), we have studied a rotating gravity-like self-bound BEC states with vortices along the $z$-axis. We derived an analytic expressions for the coherence length and the critical angular frequency to create a vortex in the condensed state. We found that the coherence length in the TF-G regime is very very small compared to the radius of the system. On the other hand, the coherence length in the G-regime is comparable to the radius of the system. We could say that the TF-G regime should exhibit the superfluid properties more prominently than the G-regime. In the TF regime of an ordinary atomic BEC, the monopole mode frequency of the condensate does not change due to the presence of the vortex. But, the monopole mode frequency in the TF-G regime as well as in the G regime of this new BEC decreases due to the presence of the vortex.

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APPENDIX A:

Here, we shall give the exact form of $F[\alpha_1, \beta_1] = F[\frac{1}{2}, 1; \frac{3}{2}; (1 - \frac{\alpha^2}{\beta^2})]/\beta_1$, its derivative with respect to $\alpha_1$, $\beta_1$ and the first order deviation from the equilibrium functions $F_{\alpha_1}[\alpha_{10}, \beta_{10}]$ and $F_{\beta_1}[\alpha_{10}, \beta_{10}]$.

$$F[\alpha_1, \beta_1] = \frac{1}{\beta_1}[1 + \frac{1}{3}(1 - \frac{\alpha^2}{\beta^2}) + \frac{1}{5}(1 - \frac{2\alpha^2}{\beta^2} + \frac{3\alpha^4}{2\beta^4}) + \frac{1}{7}(1 - \frac{3\alpha^2}{\beta^2} + 3\frac{\alpha^4}{\beta^4} - \frac{\alpha^6}{\beta^6}) + \ldots].$$ (A1)

The derivative of $F[\alpha_1, \beta_1]$ with respect $\alpha_1$ is given by

$$F_{\alpha_1}[\alpha_1, \beta_1] = \frac{1}{\beta_1}(-\frac{2\alpha_1}{3\beta_1} + \frac{1}{5}(\frac{3\alpha_1^3}{\beta_1^3} + \frac{4\alpha_1^3}{\beta_1^3}) + \frac{1}{7}(\frac{6\alpha_1^3}{\beta_1^3} + 12\frac{\alpha_1^3}{\beta_1^3} - \frac{6\alpha_1^5}{\beta_1^5}) + \ldots).$$ (A2)

Expanding $F_{\alpha_1}[\alpha_1, \beta_1]$ around the equilibrium width $\alpha_{10}$ and $\beta_{10}$. The first order deviation from the equilibrium function $F_{\alpha_1}[\alpha_{10}, \beta_{10}]$ is,

$$F_{\alpha_1}[\alpha_{10}, \beta_{10}, \delta\alpha_1, \delta\beta_1] = \frac{1}{\beta_{10}}[-\frac{2}{3}\frac{\alpha_1}{\beta_{10}^2} + \frac{1}{5}(\frac{6\alpha_{10}}{\beta_{10}^2} + 12\frac{\alpha_{10}^2}{\beta_{10}^4}) + \frac{1}{7}(\frac{6\alpha_{10}}{\beta_{10}^2} + 36\frac{\alpha_{10}^2}{\beta_{10}^4} - 30\frac{\alpha_{10}^4}{\beta_{10}^6}) + \ldots] \delta\alpha_1$$ (A3)
\[ F_{\beta_1} [\alpha_1, \beta_1] = -\frac{1}{\beta_1^2} \left[ \frac{2}{3} \alpha_1 \beta_1 + \frac{1}{5} \left( -\frac{4 \alpha_1}{\beta_1} + \frac{4 \alpha_3}{\beta_1} \right) + \frac{1}{7} \left( -\frac{6 \alpha_1^3}{\beta_1^3} + 12 \frac{\alpha_3^3}{\beta_1^3} - 6 \frac{\alpha_5}{\beta_1^5} \right) + \ldots \right] \delta \beta_1 \\
+ \frac{1}{\beta_1^3} \left[ \frac{4 \alpha_1}{3 \beta_1} + \frac{1}{5} \left( \frac{8 \alpha_1}{\beta_1} - 16 \frac{\alpha_3}{\beta_1} \right) + \frac{1}{7} \left( 12 \frac{\alpha_1^3}{\beta_1^3} - 48 \frac{\alpha_3^3}{\beta_1^3} + 36 \frac{\alpha_5}{\beta_1^5} \right) + \ldots \right] \delta \beta_1. \]

The derivative of \( F_{\alpha_1, \beta_1} \) with respect to \( \beta_1 \) is given by

\[ F_{\beta_1} [\alpha_1, \beta_1] = -\frac{1}{\beta_1^2} \left[ 1 + \frac{1}{3} (1 - \frac{\alpha_1^2}{\beta_1^2}) + \frac{1}{5} (1 - 2 \frac{\alpha_1^2}{\beta_1^2} + \frac{\alpha_4}{\beta_1^4}) + \frac{1}{7} (1 - 3 \frac{\alpha_1^2}{\beta_1^2} + 3 \frac{\alpha_4}{\beta_1^4} - \frac{\alpha_6}{\beta_1^6}) + \ldots \right] \delta \beta_1 \\
+ \frac{1}{\beta_1^3} \left[ \frac{2}{3} \alpha_1^2 \beta_1 + \frac{1}{5} \left( \frac{4 \alpha_1^2}{\beta_1^2} - \frac{4 \alpha_4}{\beta_1^4} \right) + \frac{1}{7} \left( 6 \frac{\alpha_1^2}{\beta_1^2} - 12 \frac{\alpha_4}{\beta_1^4} + 6 \frac{\alpha_6}{\beta_1^6} \right) + \ldots \right]. \]
FIG. 1. The sound velocity $c_s$ as a function of the dimensionless scattering parameter $S$.

FIG. 2. Monopole and quadrupole mode frequencies vs. the scattering parameter $S$, for positive as well as negative values of $\alpha$. The solid and dashed lines correspond to the quadrupole and monopole mode frequencies, respectively. The crossing between two modes is shown in the inset of this figure.
FIG. 3. The superfluid coherence length $\xi$ as a function of the dimensionless scattering parameter $S$.

FIG. 4. The critical angular frequency $\Omega_1$ as a function of the dimensionless scattering parameter $S$. 