ON THE HEAT CONTENT OF A POLYGON

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Abstract. Let $D$ be a bounded, connected, open set in Euclidean space $R^2$ with polygonal boundary. Suppose $D$ has initial temperature 1 and the complement of $D$ has initial temperature 0. We obtain the asymptotic behaviour of the heat content of $D$ as time $t \downarrow 0$. We then apply this result to compute the heat content of a particular fractal polyhedron as $t \downarrow 0$.

1. Introduction

The conduction of heat or the diffusion of matter through a solid body is of importance in the physical and engineering sciences. The classic reference of Carslaw and Jaeger, [9], analyses many examples and applications. The mathematical tools used in [9] are centred around separation of variables and Laplace transforms and, in many cases, require properties of special functions. From a mathematical point of view, the heat equation, heat content and heat trace link the underlying geometry of the manifold and its boundary and boundary conditions to the spectral resolution of the Laplace operator. Over the last few decades, a considerable amount of progress has been made in understanding the asymptotic behaviour of the heat content for small time $t$, see [12].

It was discovered by Preunkert, [17], that even in the absence of boundary conditions the heat content of a ball $B$ in Euclidean space $R^m$ which is at initial temperature 1, while $R^m - B$ has initial temperature 0, has non-trivial asymptotic behaviour as $t \downarrow 0$. For small $t$, the initial condition on the complement of $B$ acts in a similar way to a Dirichlet 0 boundary condition. This was subsequently stated for bounded, open sets with $C^{1,1}$ boundary in [15], and proved in [16]. The discussion is simplified by the fact that the heat kernel on $R^m$ is known explicitly. The general situation for the heat content of a compact subdomain $\Omega$ in a compact Riemannian manifold $M$ was examined in [4]. The tools of pseudo-differential calculus used there rely heavily upon the smoothness assumptions on the boundary. Two-sided estimates for the heat content of non-compact sets in $R^m$ were obtained in [3]. These estimates are very different from the ones where Dirichlet 0 boundary conditions are imposed. See [1] and [2].

In this paper we denote the fundamental solution of the heat equation on $R^m$ by
\[
p(x, y; t) = (4\pi t)^{-m/2}e^{-|x-y|^2/(4t)},
\]
and for an open set $D \subset R^m$, we define
\[
(1) \quad u_D(x; t) = \int_D dy p(x, y; t).
\]
Then $u_D(x; t)$ satisfies the heat equation on $R^m$
\[
(2) \quad \Delta u_D = \frac{\partial u_D}{\partial t}, \quad x \in R^m, \ t > 0,
\]
(see Chapter 2 in [10]) and
\[
(3) \quad \lim_{t \downarrow 0} u_D(x; t) = 1_D(x), \quad x \in R^m - \partial D,
\]
where $\partial D$ is the boundary of $D$.

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We define the heat content of $D$ in $\mathbb{R}^m$ at time $t$ by

$$H_D(t) = \int_D dx u_D(x; t).$$

So by (1),

(4) \quad H_D(t) = \int_D dx \int_D dy p(x, y; t).

We denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}^m$ by $|A|$, its perimeter by $\mathcal{P}(A)$, and its $(m-1)$-dimensional Hausdorff measure by $H^{m-1}(A)$.

If $D$ is a bounded, open set in $\mathbb{R}^m$, $m \geq 2$, with $C^{1,1}$ boundary $\partial D$, then Theorem 2.4 of [10] implies that

(5) \quad H_D(t) = |D| - \mathcal{P}(D) \frac{t^{1/2}}{\sqrt{\pi}} + O(t^{1/2}), \quad t \downarrow 0.

In [5], we obtained explicit bounds for $H_D(t)$ for bounded, open sets $D$ in Euclidean space with $C^{1,1}$ boundary which are uniform in $t$ and in the geometric data of $D$. These bounds imply that

(6) \quad H_D(t) = |D| - \mathcal{P}(D) \frac{t^{1/2}}{\sqrt{\pi}} + O(t), \quad t \downarrow 0.

We observe that if $K$ is a closed set in $\mathbb{R}^m$ with $|K| = 0$ then by the definition of the perimeter (see [11]) and (4),

(7) \quad |D - K| = |D|, \quad \mathcal{P}(D - K) = \mathcal{P}(D), \quad H_{D - K}(t) = H_D(t).

The observations in (7) suggest that (5) holds for all open sets $D$ with finite perimeter and finite Lebesgue measure. The proof of such a statement is well beyond the scope of this paper.

In this paper, we focus on the heat content of a bounded, connected, open set $D \subset \mathbb{R}^2$ with polygonal boundary. We introduce some notation and then present the main result:

Theorem 1. Let $\gamma_1, \ldots, \gamma_n$ denote the interior angles of $\partial D$. Each such angle $\gamma_j$ is supported by two edges provided $\gamma_j < 2\pi$. By (7), we may exclude angles $2\pi$. We label the corresponding vertices by $V_1, \ldots, V_n$, and note that these $n$ vertices need not be pairwise disjoint. Let $W_j$ denote the infinite wedge of angle $\gamma_j$ with vertex $V_j$, such that $W_j \cap D \neq \emptyset$ and the boundary of the wedge contains the two edges which are adjacent to $V_j$ and have an angle $\gamma_j$. Let

(8) \quad \gamma = \{ \gamma_i : (\sin \gamma_i)^2 \leq (\sin \gamma_j)^2 \text{ for all } j \in \{1, 2, \ldots, n\} \}.

For $r > 0$, we also define the open sector

(9) \quad B_j(r) = \{ x \in W_j : d(x, V_j) < r \}

and

(10) \quad R = \frac{1}{2} \sup \left\{ r : B_\ell(r) \cap B_j(r) = \emptyset \text{ for all } \ell \neq j, \bigcup_{k=1}^n B_k(r) \subset D \right\}.

Theorem 1. Let $D \subset \mathbb{R}^2$ be a bounded, connected, open set with polygonal boundary $\partial D$ with $\gamma$ and $R$ as defined in (8) to (10). Then as $t \downarrow 0$,

(11) \quad H_D(t) = |D| - \mathcal{P}(D) \frac{t^{1/2}}{\sqrt{\pi}} + \sum_{j=1}^n g(\gamma_j) t

$$ + \sum_{j, \ell \in \{1, \ldots, n\} : j \neq \ell, V_j = V_\ell} k(\alpha_j, \gamma_j, \gamma_\ell) t + O(e^{-R^2(\sin \gamma)^2/(32t)}),$$

where $g : (0, 2\pi) \to \mathbb{R}$ is given by

(12) \quad g(\beta) = \begin{cases} \frac{1}{2} + \frac{1}{2} \cot \beta, & \beta \in (0, \pi) \cup (\pi, 2\pi); \\ 0, & \beta = \pi. \end{cases}
and $k : (0, \pi) \times (0, 2\pi) \times (0, 2\pi) \to \mathbb{R}$ is given by

\[ k(\alpha, \theta, \sigma) = \frac{1}{2\pi} \left( - (\sigma + \theta + \alpha - \pi) \cot(\sigma + \theta + \alpha) - (\alpha - \pi) \cot \alpha \right) + \frac{1}{2\pi} \left( (\sigma + \alpha - \pi) \cot(\sigma + \alpha) + (\theta + \alpha - \pi) \cot(\theta + \alpha) \right), \]

for $\sigma + \theta + \alpha \neq \pi$, $\alpha \neq \pi$, $\sigma + \alpha \neq \pi$, $\theta + \alpha \neq \pi$, where $\alpha$ denotes the smallest angle between $W_0$ and $W_\sigma$. In any of the remaining cases, such as $\alpha = \pi$, we define $k(\alpha, \theta, \sigma)$ by taking appropriate limits using l'Hôpital's rule.

The terms which involve the area and perimeter are as expected and agree with those in [6]. We see that the heat content has a non-trivial dependence on the interior angles with respect to $\pi$. That is

\[ g(\beta) = \frac{1}{2\pi} \left( (\pi - \beta) \cot(\pi - \beta) \right), \quad 0 < \beta < 2\pi. \]

By (12) and (14), we conclude that $g$ is non-negative. We remark that $k(\alpha, \theta, \sigma)$ is symmetric with respect to $\theta$ and $\alpha$ and that $k(\alpha, \theta, \sigma) = k(\pi - \theta - \sigma, \theta, \sigma)$. By Lemma 10 Section 3 it follows that $k$ is non-negative.

In addition, if $D$ is a regular $n$-gon in $\mathbb{R}^2$, then $\gamma_1 = \gamma_2 = \cdots = \gamma_n = \left( \frac{n-2}{n} \right) \pi$, and the angular contribution to the heat content is

\[ \frac{n}{\pi} + 2 \cot \left( \left( \frac{n-2}{n} \right) \pi \right) = O \left( \frac{1}{n} \right), \quad n \to \infty. \]

We observe that the coefficient of $\int_{\partial \Omega} L_{nn} t$ in the expansion of the heat content of a compact domain $\Omega$ with smooth boundary $\partial \Omega$ is also equal to $0$, see Theorem 1.6 in [4]. Here $L_{nn}$ is the trace of the second fundamental form when $\partial \Omega$ is oriented with a smooth inward-pointing unit normal vector field.

The expansion for the heat content of a polygon with Dirichlet 0 boundary conditions was obtained in [8]. There it was shown that if $v_D$ solves the heat equation $\Delta v = \frac{\partial v}{\partial n}$ on $D$, $t > 0$ with $\lim_{t \to 0} v(x; t) = 1$, $x \in D$ and satisfies a Dirichlet boundary condition $\lim_{x \to x_0} v(x; t) = 0$ for any $x_0 \in \partial D$ then

\[ \int_D dx \, v_D(x; t) = |D| - 2P(D)^{1/2} \frac{t^{1/2}}{\sqrt{\pi}} + \sum_{j=1}^n c(\gamma_j) t + O(e^{-R^2(\sin(\gamma_j/2))^2/(12t)}), \]

where

\[ c(\beta) = \int_0^\infty \frac{4 \sinh((\pi - \beta) x)}{(\sinh^2(\pi x))^{3/2}} dx. \]

We note that both (11) and (15) have angular contributions which are additive. However, in Theorem 1 there is an additional term in the case where vertices have multiplicity larger than 1. No such term is present in (15) since sectors based at the same vertex do not feel each other’s presence due to the Dirichlet 0 boundary condition.

The strategy to prove (15) is inspired by (14) and relies on some model computations. We use an analogous strategy to prove (11). For points $x \in D$ close to a vertex, say $x \in B_j(r)$ for some $j \in \{1, 2, \ldots, n\}$, $r > 0$, $u_D$ is approximated by $u_{W_j}$. For points $x \in D$ which have a distance at least $\delta$ to $\partial D$, for some $\delta > 0$, $u_D$ is approximated by 1. For the remaining points in $D$, $u_D$ is approximated by $u_H$, where $H$ is the half-plane which contains $D$ and whose boundary contains the edge of $\partial D$ nearest to $x$. As was the case in [8], the model computations involving the infinite wedge $W_j$ are the most difficult to carry out. However, in contrast to the case with Dirichlet 0 boundary conditions, we must also consider the contribution to the heat content from a vertex which belongs to the boundaries of more than one wedge. We deal with these computations in Lemma 9 and Lemma 10 in Section 3. In Section 4 we carry out the half-plane computations.

It is quite remarkable that, in contrast to the smooth case [5], the asymptotic expansion in half-powers of $t$ of $H_D(t)$ in Theorem 1 terminates after the term of order $t$, leaving an exponentially small remainder as $t \downarrow 0$. This agrees with the fact that there are no further locally computable invariants of $D$ and $\partial D$ available from which non-trivial quantities could be built. A similar phenomenon has been observed for the asymptotic expansion.
of the heat trace. See for example [7]. The precise form of the exponential remainder remains an open problem. The extension of the results in this paper to general polyhedra in $\mathbb{R}^3$ is another challenge beyond the scope of this paper.

However, in Section 6, we use Theorem 1 to compute the heat content of a fractal polyhedron which is constructed as follows. Let $Q_0 \subset \mathbb{R}^3$ be an open cube of side-length 1. Let $0 < s < 1$. Attach a regular open cube $Q_{1,i}$ of side-length $s$, to the centre $c_{1,i}, i = 1, \ldots, 6$, of each face of $\partial Q_0$, and such that all the faces are pairwise-parallel. Now proceed by induction. For $j = 2, 3, \ldots$, attach $N(j) = 6 \cdot 5^{j-1}$ open cubes $Q_{j,1}, \ldots, Q_{j,N(j)},$ of side-length $s^j$ to the centres of the boundary faces of the cubes $Q_{j-1,1}, \ldots, Q_{j-1,N(j-1)}$, again with pairwise-parallel faces. We define the fractal polyhedron $D_s$ as

$$D_s = \text{interior} \left\{ Q_0 \cup \bigcup_{j \geq 1} \bigcup_{1 \leq i \leq N(j)} Q_{j,i} \right\}$$

for $0 < s < \sqrt{2} - 1$ (see Figure 1). We note that for this range of $s$, no cubes in the construction of $D_s$ overlap. For the two-dimensional construction, see Theorem 4 in [6]. In that paper, the critical value $s = \sqrt{2} - 1$, where the squares just touch, is allowed. This is due to the fact that the Dirichlet 0 boundary conditions guarantee the independence of the heat flow in these touching squares. In this paper, we do not impose Dirichlet 0 boundary conditions on $\partial D_s$, and if the cubes touch, then this could give rise to an extra term. For this reason, we only allow $0 < s < \sqrt{2} - 1$.

![Figure 1. The first two generations of $D_s$ with $s = \frac{1}{2}$](image)

We have that

$$|D_s| = \frac{1 + s^3}{1 - 5s^3},$$

and that the two-dimensional Hausdorff measure of the boundary is given by

$$\mathcal{H}^2(\partial D_s) = 6 \left( \frac{1 - s^2}{1 - 5s^2} \right).$$

Both quantities are finite for $0 < s < \sqrt{2} - 1$. Moreover, the total length of the edges of $\partial D_s$ is finite if and only if $0 < s < \frac{1}{5}$ and equals $12 \left( \frac{1 + s}{1 - 5s} \right)$. In addition to Theorem 1, in order to compute the heat content of $D_s$, we require the heat content of a sector of angle $\pi$ in an infinite wedge of angle $\frac{2\pi}{5}$ where this sector and wedge share one common edge and vertex. This will be computed in Section 5. In Section 6, we prove the following theorems which give the asymptotic expansion for the heat content of $D_s$ as $t \downarrow 0$. 
Theorem 2. Let \( d = \frac{3}{2} + \frac{1}{2} \log \frac{5}{s} \). Fix \( 0 < s < \sqrt{2} - 1 \), \( s \neq \frac{1}{2} \). There exists a periodic, continuous function \( p_s : \mathbb{R} \to \mathbb{R} \) with period \( \log(s^{-2}) \) such that

\[
H_D(t) = \frac{1 + s^3}{1 - 5s^4} - 6 \left( \frac{1 - s^2}{1 - 5s^2} \right)^{1/2} + \frac{12}{\pi} \left( \frac{1 + s}{1 - 5s} \right) t + 6p_s(t) - O(t^{3/2} \log(t^{-1}))^{3/2}, \quad t \downarrow 0.
\]

It is easy to see that if we write \( d = \frac{m - d_s}{2} \), \( m = 3 \), in Theorem 2 then \( d_s = \frac{\log 5}{\log(s^{-2})} \) is the interior Minkowski dimension of the vertices of \( \partial D_2 \), for \( 0 < s < \frac{1}{2} \), whereas for \( \frac{1}{2} < s < \sqrt{2} - 1 \), it is the interior Minkowski dimension of the edges of \( \partial D_2 \). Below we state the corresponding result for the critical case \( s = \frac{1}{2} \).

Theorem 3. For \( s = \frac{1}{2} \), there exists a periodic, continuous function \( p_{1/2} : \mathbb{R} \to \mathbb{R} \) with period \( \log 25 \) such that,

\[
H_{D,1/2}(t) = \frac{21}{20} - \frac{36}{5} \frac{t^{1/2}}{\sqrt{\pi}} + \frac{132}{5\pi} t - \frac{36}{5\pi \log 5} t \log t + 6p_{1/2}(t) - O(t^{3/2} \log(t^{-1}))^{3/2}, \quad t \downarrow 0.
\]

In Section 2 below we introduce some further notation and state and prove several lemmas.

2. Additional notation and lemmas.

Let \( D \) be as given in Theorem 1. In the proofs of Lemma 3 to Lemma 8 we use a variant of Kac’s principle of not feeling the boundary to reduce the computation of the heat content of \( D \) to a collection of model computations. For \( r > 0 \), \( \delta > 0 \), \( x \in D \), \( A \subset \mathbb{R}^2 \) we define

\[
d(x, A) = \inf \{|x - z| : z \in A\},
\]
and

\[
C(\delta, r) = \left\{ x \in D : d(x, \partial D) < \delta, x \notin \bigcup_{k=1}^n B_k(r) \right\},
\]
and

\[
D(\delta, r) = \left\{ x \in D : x \notin \bigcup_{k=1}^n B_k(r), x \notin C(\delta, r) \right\}.
\]

Choose \( R \) as in [9] and let \( \delta = \frac{d}{2} |\sin \gamma| \).

Lemma 4.

\[
\int_{D(\delta, R)} dx \int_D dy \frac{p(x, y; t)}{|D(\delta, R)|} = |D(\delta, R)| + O(e^{-R^2 (\sin \gamma)^2/(32)}) \downarrow 0.
\]

Proof. By [3] Proposition 9(1)], we have that

\[
1 - 2e^{-\delta^2/(8k)} \leq \int_D dy p(x, y; t) \leq 1,
\]
where \( \delta(x) = \min \{|x - y| : y \in \mathbb{R}^2 - D\} \). Since \( x \in D(\delta, R) \), \( \delta(x) \geq \delta \). Hence

\[
|D(\delta, R)| - 2|D(\delta, R)| e^{-\delta^2/(8k)} \leq \int_{D(\delta, R)} dx \int_D dy p(x, y; t) \leq |D(\delta, R)|
\]
as required. \( \square \)

We partition the region \( D - D(\frac{d}{2} |\sin \gamma|, R) \) into \( n \) sectors, \( B_i(R) \), of radius \( R \), \( n \) rectangles, \( S_n \), of height \( \frac{d}{2} |\sin \gamma| \) and \( 2n \) cusps of height \( \frac{d}{2} |\sin \gamma| \). The contributions to \( H_D(t) \) from these regions will be computed in Sections 3.4.1 and 4.2 respectively. Each sector has two neighbouring cusps. Each cusp is adjacent to a rectangle and a sector. The corresponding \( m \)-dimensional result to Lemma 7 is the following.
Lemma 5. Let $\bar{D}, F, G$ be non-empty, open subsets of $\mathbb{R}^m, m \geq 2$ such that $\bar{D} \cap F \neq \emptyset$ and $G \subset \bar{D} \cap F$. Let $E$ be a bounded, measurable subset of $G$. Then

$$\int_E dx \int_D dy p(x, y; t) = \int_E dx \int_F dy p(x, y; t) + O(e^{-t^2/(8\epsilon)}), t \downarrow 0,$$

where $\epsilon = \inf \{ |x - y| : x \in E, y \in (\bar{D} \cup F) \cap \partial G \}$.

Proof. We write

$$\int_E dx \int_D dy p(x, y; t) = \int_E dx \int_{D \cap F} dy p(x, y; t) + \int_E dx \int_{\bar{D} \setminus F} dy p(x, y; t)$$

(19)

$$= \int_E dx \int_F dy p(x, y; t) - \int_E dx \int_{D \cap F} dy p(x, y; t) + \int_E dx \int_{\bar{D} \setminus F} dy p(x, y; t).$$

By (19), we have that

$$\int_E dx \int_D dy p(x, y; t) \geq \int_E dx \int_F dy p(x, y; t) - \int_E dx \int_{\bar{D} \setminus F} dy p(x, y; t)$$

$$\geq \int_E dx \int_F dy p(x, y; t) - 2^{m/2} |E| e^{-t^2/(8\epsilon)},$$

and

$$\int_E dx \int_D dy p(x, y; t) \leq \int_E dx \int_F dy p(x, y; t) + \int_E dx \int_{\bar{D} \setminus F} dy p(x, y; t)$$

$$\leq \int_E dx \int_F dy p(x, y; t) + 2^{m/2} |E| e^{-t^2/(8\epsilon)}.$$

This completes the proof. □

Lemma 6. Let $i \in \{1, \cdots, n\}$ and $k \in \mathbb{N}$ such that $i + k \leq n$. Suppose $\gamma_i, \gamma_{i+1}, \cdots, \gamma_{i+k}$ are interior angles of $\partial D$ which are supported by edges which meet at the same vertex $V_i = V_{i+1} = \cdots = V_{i+k}$. Then

$$\int_{j_{i+k} B_j(R)} dx \int_D dy p(x, y; t)$$

(20)

$$= \sum_{j=1}^{i+k} B_j(R) \int_{W_j} dy p(x, y; t) + \sum_{j \neq i,j \neq i+1} \int_{W_j} dx \int_{W_t} dy p(x, y; t) + O(e^{-R^2(\sin \gamma)^2/(32\epsilon)}), t \downarrow 0.$$

Proof. By Lemma 5 with $\bar{D} = D, F = \bigcup_{j=1}^{i+k} W_j, E = \bigcup_{j=1}^{i+k} B_j(R)$ and $G = \{ z \in D : d(z, \bigcup_{j=1}^{i+k} B_j(R)) < \frac{\pi}{2} |\sin \gamma| \}$, we have that

$$\int_{j_{i+k} B_j(R)} dx \int_D dy p(x, y; t)$$

(21)

$$= \int_{j_{i+k} B_j(R)} dx \int_{j_{i+k} W_j} dy p(x, y; t) + O(e^{-R^2(\sin \gamma)^2/(32\epsilon)}), t \downarrow 0.$$

We also have that

$$\int_{j_{i+k} B_j(R)} dx \int_{j_{i+k} W_j} dy p(x, y; t)$$

(22)

$$= \sum_{j=i}^{i+k} B_j(R) \int_{W_j} dy p(x, y; t) + \sum_{j \neq i,j \neq i+1} \int_{W_j} dx \int_{W_t} dy p(x, y; t)$$

$$+ O(e^{-R^2/(8\epsilon)}), t \downarrow 0.$$

Combining (21) and (22) gives (20). □
Lemma 7. Let $H \subset \mathbb{R}^2$ denote the half-plane such that $\emptyset \neq \partial S_\gamma \cap \partial D \subset \partial H$ and $S_\gamma \subset H$. Then
\[
\int_{S_\gamma} dx \int_D dy \, p(x,y; t) = \int_{S_\gamma} dx \int_H dy \, p(x,y; t) + O(e^{-R^2(t+\gamma)^2/(32t)}), \quad t \downarrow 0.
\]

Proof. Using Lemma 5 with $\tilde{D} = D$, $F = H$, $E = S_\gamma$ and $G = \{ z \in D : d(z, S_\gamma) < \frac{R}{2} \sin \gamma \}$, the result follows. \hfill \square

Lemma 8. Let $C_\gamma$ denote the cusp which is adjacent to $S_\gamma$ and $B_\gamma(R)$. Let $H$ be the half-plane as in Lemma 7. Then
\[
\int_{C_\gamma} dx \int_D dy \, p(x,y; t) = \int_{C_\gamma} dx \int_H dy \, p(x,y; t) + O(e^{-R^2(t+\gamma)^2/(32t)}), \quad t \downarrow 0.
\]

Proof. Using Lemma 5 with $\tilde{D} = D$, $F = H$, $E = C_\gamma$ and $G = \{ z \in D : d(z, C_\gamma) < \frac{R}{2} \sin \gamma \}$, the result follows. \hfill \square

3. THE CONTRIBUTION TO THE HEAT CONTENT FROM POINTS CLOSE TO A VERTEX.

In this section we approximate $u_D$ by $u_{W_j}$. We then compute the contribution to the heat content of $D$ from a sector, $B_\gamma(R)$ with corresponding angle $\gamma_j = \beta$. We also compute the contribution to the heat content of $D$ from two disjoint wedges whose boundaries intersect in a vertex of $\partial D$.

Firstly, we define
\[
V_{\beta}(t; R) = \int_{B_\gamma(R)} dx \int_{W_j} dy \, p(x,y; t).
\]

Lemma 9. For $\beta \in (0, \pi) \cup (\pi, 2\pi)$,
\[
V_{\beta}(t; R) = \frac{\beta R^2}{2} - 2R t^{1/2} + \left( \frac{1}{\pi} + \left( 1 - \frac{\beta}{\pi} \right) \cot \beta \right) t
\]
\[- \frac{1}{4\pi} t^{1/2} \int_0^R t^{1/2} \, t^{1/2} \left( -2Rx + R^2 \arcsin \left( \frac{x}{R} \right) + x \sqrt{R^2 - x^2} \right) e^{-x^2/(4t)}
\]+ $O(t e^{-R^2(t+\gamma)^2/(8t)})$, $t \downarrow 0$.

Proof. Changing to polar coordinates with $x = (r_1 \cos \theta_1, r_1 \sin \theta_1)$ and $y = (r_2 \cos \theta_2, r_2 \sin \theta_2)$ in (23) gives that
\[
V_{\beta}(t; R)
\]
\[\begin{align*}
&= (4\pi t)^{-1} \int_0^{\beta} d\theta_1 \int_0^{\beta} d\theta_2 \int_0^R dr_1 \int_0^{\infty} dr_2 (r_1 r_2) e^{-(r_1^2 + r_2^2)/(4t) + 2r_1 r_2 A/(4t)}
\end{align*}
\]
where $A = \cos(\theta_1 - \theta_2)$. The change of variable $r_2 - r_1 A = \rho$ gives that
\[
V_{\beta}(t; R)
\]
\[\begin{align*}
&= (4\pi t)^{-1} \int_0^{\beta} d\theta_1 \int_0^{\beta} d\theta_2 \int_0^R dr_1 \int_0^{\infty} dr_2 (r_1 r_2) e^{-(r_2 - r_1 A)^2/(4t) - r_1^2(1-A^2)/(4t)}
\end{align*}
\]
\[\begin{align*}
&= (4\pi t)^{-1} \int_0^{\beta} d\theta_1 \int_0^{\beta} d\theta_2 \int_0^{\infty} r \, dr \, \int_{-A}^{A} \, dp \, (\rho + Ar) e^{-\rho^2/(4t) - r^2(1-A^2)/(4t)}
\end{align*}
\]
\[\begin{align*}
&= I_1 + I_2.
\end{align*}
\]
We have that
\[
I_1 = (4\pi t)^{-1} \int_0^{\beta} d\theta_1 \int_0^{\beta} d\theta_2 \int_0^{\infty} r \, dr \, \int_{-A}^{A} \, dp \, p e^{-\rho^2/(4t) - r^2(1-A^2)/(4t)}
\]
\[\begin{align*}
&= \frac{\beta^2}{\pi} t(1 - e^{-R^2/(4t)}),
\end{align*}
\]
and

\[ I_2 = (4\pi t)^{-1} \int_0^\beta d\theta_1 \int_0^\beta d\theta_2 \int_0^R dr A r^2 \int_0^\infty d\rho e^{-\rho^2/(4\pi t) - r^2/(4\pi t)} \]

\[ = (4\pi t)^{-1} \int_0^\beta d\theta_1 \int_0^\beta d\theta_2 \int_0^R dr A r^2 e^{-r^2/(4\pi t)} \int_0^\infty d\rho e^{-\rho^2/(4\pi t)} \]

\[ + (4\pi t)^{-1} \int_0^\beta d\theta_1 \int_0^\beta d\theta_2 \int_0^R dr A^2 r^2 e^{-r^2/(4\pi t)} \int_0^\infty d\rho e^{-\rho^2/(4\pi t)} \]

\[ = (4\pi t)^{-1} \int_0^\beta d\theta_1 \int_0^\beta d\theta_2 \int_0^R dr A^2 r^2 e^{-r^2/(4\pi t)} \int_0^\infty d\rho e^{-\rho^2/(4\pi t)} \]

\[ + (4\pi t)^{-1} \int_0^\beta d\theta_1 \int_0^\beta d\theta_2 \int_0^R dr A^2 r^2 e^{-r^2/(4\pi t)} \int_0^\infty d\rho e^{-\rho^2/(4\pi t)} \]

\[ = I_3 + I_4 + I_5. \]

Via the change of variables \( \theta_1 - \theta_2 = -\eta \) and integrating by parts with respect to \( \theta \), we obtain

\[ I_3 = (4\pi t)^{-1} \int_0^\beta d\theta_1 \int_0^\beta d\theta_2 \int_0^R dr A r^2 \int_0^\infty d\rho e^{-\rho^2/(4\pi t)} \]

\[ = (4\pi t)^{-1/2} \int_0^\beta d\theta_1 \int_0^\beta d\theta_2 \int_0^R dr r^2 \cos(\theta_1 - \theta_2) e^{-r^2/(4\pi t)} \int_0^\infty d\rho e^{-\rho^2/(4\pi t)} \]

\[ = (4\pi t)^{-1/2} \int_0^\beta d\theta_1 \int_0^\beta d\theta_2 \int_0^R dr r^2 \cos \eta e^{-r^2/(4\pi t)} \int_0^\infty d\rho e^{-\rho^2/(4\pi t)} \]

\[ = (4\pi t)^{-1/2} \int_0^\beta d\theta_1 \int_0^\beta d\theta_2 \int_0^R dr r^2 \cos \eta e^{-r^2/(4\pi t)} \int_0^\infty d\rho e^{-\rho^2/(4\pi t)} \]

\[ = \beta (4\pi t)^{-1/2} \int_0^\beta d\eta \int_0^R dr r^2 \cos \eta e^{-r^2/(4\pi t)} \]

\[ - (4\pi t)^{-1/2} \int_0^\beta d\theta \int_0^R dr r^2 \cos \theta e^{-r^2/(4\pi t)}. \]

Similarly,

\[ I_4 = \beta (4\pi t)^{-1/2} \int_0^\beta d\eta \int_0^R dr r^2 |\cos \eta| e^{-r^2/(4\pi t)} \]

\[ - (4\pi t)^{-1/2} \int_0^\beta d\theta \int_0^R dr r^2 |\cos \theta| e^{-r^2/(4\pi t)}. \]

We first compute \( I_3 + I_4 \) and then we deal with \( I_5 \). By (25) and (26), we have that

\[ I_3 + I_4 = 2(4\pi t)^{-1/2} \int_0^R dr r^2 \int_{\beta}^\infty d\eta (\beta - \eta) \cos \eta e^{-r^2/(4\pi t)} \]

where \( B = [0, \beta] \cup ([0, \pi/2] \cup [3\pi/2, 2\pi]) \). First suppose \( \beta \in [0, \pi/2] \), then by (27) we have that

\[ I_3 + I_4 \]

\[ = 2(4\pi t)^{-1/2} \int_0^R dr r^2 \int_{\beta}^\infty d\eta (\beta - \eta) \cos \eta e^{-r^2/(4\pi t)} \]

\[ = 2(4\pi t)^{-1/2} \int_0^R dr r^2 \int_0^{\sin \beta} d\psi (\beta - \arcsin \psi) e^{-r^2\psi^2/(4\pi t)} \]

\[ = 2\beta (4\pi t)^{-1/2} \int_0^R dr r^2 \int_0^{\sin \beta} d\psi e^{-r^2\psi^2/(4\pi t)} - 2(4\pi t)^{-1/2} \int_0^R dr r^2 \int_0^{\sin \beta} d\psi \psi e^{-r^2\psi^2/(4\pi t)} \]

\[ - 2(4\pi t)^{-1/2} \int_0^R dr r^2 \int_0^{\sin \beta} d\psi (\arcsin \psi - \psi) e^{-r^2\psi^2/(4\pi t)} \]
Similarly, for $\beta \in \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right]$, we have that

\[ I_3 + I_4 \]

\[ = 2(4\pi t)^{-1/2} \int_0^R dr r^2 \int_0^\infty d\eta (\beta - \eta) \cos \eta e^{-r^2(\sin \eta)^2/(4\ell)} \]

\[ = 2(4\pi t)^{-1/2} \int_0^R dr r^2 \int_0^1 d\psi (\beta - \arcsin \psi) e^{-r^2\psi^2/(4\ell)} \]

\[ = 2\beta(4\pi t)^{-1/2} \int_0^R dr r^2 \int_0^\infty d\psi e^{-r^2\psi^2/(4\ell)} - 2(4\pi t)^{-1/2} \int_0^R dr r^2 \int_0^1 d\psi e^{-r^2\psi^2/(4\ell)} \]

\[ = 2\beta(4\pi t)^{-1/2} \int_0^R dr r^2 \int_0^\infty d\psi e^{-r^2\psi^2/(4\ell)} - 2(4\pi t)^{-1/2} \int_0^R dr r^2 \int_0^1 d\psi e^{-r^2\psi^2/(4\ell)} \]

\[ + 4t(4\pi t)^{-1/2} \int_0^R dr (e^{-r^2/(4\ell)} - 1) \]

\[ = \frac{\beta R^2}{2} - 2\beta t \int_0^\infty \frac{d\psi}{\psi^3} - \frac{2R}{\sqrt{\pi}} (1/2 + 2t) \int_0^1 \frac{d\psi}{\psi^3} \left( \arcsin \psi - \psi \right) \]

\[ + 2(4\pi t)^{-1/2} \int_0^1 d\psi (\arcsin \psi - \psi) \int_0^\infty dr r^2 e^{-r^2\psi^2/(4\ell)} + O(t e^{-R^2/(8\ell)}) \]

(29)
In order to have (30) in the same format as (28), we replaced 1 by $|\sin \beta|$ in (29). For $\beta \in \left[\frac{\pi}{4}, 2\pi\right]$, we have that
\[
I_3 + I_4 = 2(4\pi t)^{-1/2} \int_0^R dr \int_0^\frac{\pi}{2} d\eta \quad d(\beta - \eta) \cos \eta e^{-r^2 \sin^2 \eta^2/(4t)}
\]
\[
+ 2(4\pi t)^{-1/2} \int_0^R \int_0^\frac{\pi}{2} \left. d(\beta - \eta) \cos \eta e^{-r^2 \sin^2 \eta^2/(4t)} \right|_{\eta = 0}
\]
\[
= J_1 + J_2.
\]

Now by (29)
\[
(30)
J_1 = \frac{\beta R^2}{2} - \frac{2R}{\sqrt{\pi}} t^{1/2} + \left(\frac{\pi}{2} - \beta \right) t + 2(4\pi t)^{-1/2} \int_0^1 d\psi (\arcsin \psi - \psi) \int_R^\infty dr \int_0^\infty r^2 e^{-r^2 \psi^2/(4t)} + O(te^{-R^2/(8t)}), \quad t \downarrow 0.
\]

Via the change of variables $\eta' = 2\pi - \eta$, we also have that
\[
J_2 = 2(4\pi t)^{-1/2} \int_0^\frac{\pi}{2} d\eta (\beta - \eta) \cos \eta \int_0^R dr \int_0^\frac{\pi}{2} \left. d(\beta - \eta) \cos \eta e^{-r^2 \sin^2 \eta^2/(4t)} \right|_{\eta = 0}
\]
\[
= 2t \int_0^\frac{\pi}{2} d\eta (\beta + \eta - 2\pi) \cos \eta \int_0^R dr \int_0^\frac{\pi}{2} \left. d(\beta + \eta - 2\pi) \cos \eta e^{-r^2 \sin^2 \eta^2/(4t)} \right|_{\eta = 0}
\]
\[
= 2t \int_0^\frac{\pi}{2} d\eta (\beta + \eta - 2\pi) \cos \eta \int_0^R dr \int_0^\frac{\pi}{2} \left. d(\beta + \eta - 2\pi) \cos \eta e^{-r^2 \sin^2 \eta^2/(4t)} \right|_{\eta = 0}
\]
\[
(31)
= \left(\frac{3\pi}{2} - \beta - \cot \beta \right) t + O(te^{-R^2/(8t)}), \quad t \downarrow 0.
\]

Hence by (30) and (31), we see that for $\beta \in \left[\frac{3\pi}{4}, 2\pi\right]$ and $t \downarrow 0$
\[
(32)
I_3 + I_4 = \frac{\beta R^2}{2} - \frac{2R}{\sqrt{\pi}} t^{1/2} + (2\pi - 2\beta - \cot \beta)t
\]
\[
+ 2(4\pi t)^{-1/2} \int_0^1 d\psi (\arcsin \psi - \psi) \int_R^\infty dr \int_0^\infty r^2 e^{-r^2 \psi^2/(4t)} + O(te^{-R^2/(8t)}).
\]

It remains to compute $I_5$. Via the change of variables $\rho = r\rho'$, we see that
\[
I_5 = -(4\pi t)^{-1} \int_0^\beta \int_0^\beta d\theta_1 \int_0^\beta \int_0^\beta d\theta_2 \int_0^R dr A^2 r^2 e^{-r^2 (1-A^2)/(4t)} \int_1^\infty d\rho e^{-r^2 \rho^2/(4t)}
\]
\[
= -(4\pi t)^{-1} \int_0^\beta \int_0^\beta d\theta_1 \int_0^\beta \int_0^\beta d\theta_2 \int_0^R dr A^2 r^2 e^{-r^2 (1-A^2)/(4t)} \int_1^\infty d\rho e^{-r^2 A^2 \rho^2/(4t)}
\]
\[
= -(8\pi t)^{-1} \int_0^\beta \int_0^\beta d\theta_1 \int_0^\beta \int_0^\beta d\theta_2 \int_0^R dr A^2 r^2 \int_1^\infty d\rho e^{-r^2 (A^2 \rho^2 + 1 - A^2)/(4t)}
\]
\[
= -(8\pi t)^{-1} \int_0^\beta \int_0^\beta d\theta_1 \int_0^\beta \int_0^\beta d\theta_2 \int_0^R dr A^2 r^2 \int_1^\infty d\rho e^{-r^2 (A^2 \rho^2 + 1 - A^2)/(4t)}
\]
\[
+ (8\pi t)^{-1} \int_0^\beta \int_0^\beta d\theta_1 \int_0^\beta \int_0^\beta d\theta_2 \int_0^R dr A^2 r^2 \int_1^\infty d\rho e^{-r^2 (A^2 \rho^2 + 1 - A^2)/(4t)}
\]
\[
(33)
= \frac{2}{\pi} \int_0^\beta \int_0^\beta d\theta_1 \int_0^\beta \int_0^\beta d\theta_2 \int_1^\infty d\rho \int_1^\infty A^2 \left(\frac{A^2 \rho^2 + 1 - A^2}{2 \pi \rho^2 + 1 - A^2}ight)^{3/2} \arctan \left(\frac{\sqrt{1 - A^2}}{|A|}\right),
\]
\[
\int_0^\beta \int_0^\beta d\theta_1 \int_0^\beta \int_0^\beta d\theta_2 \int_1^\infty d\rho \int_1^\infty A^2 \left(\frac{A^2 \rho^2 + 1 - A^2}{2 \pi \rho^2 + 1 - A^2}ight)^{3/2} \arctan \left(\frac{\sqrt{1 - A^2}}{|A|}\right).
\]

Therefore by (33), (34) and the change of variables $\theta_2 - \theta_1 = \sigma$, we obtain that

$$-\frac{2t}{\pi} \int_0^\beta d\theta_1 \int_0^\beta d\theta_2 \int_0^\infty d\rho \left( \frac{A^2}{A^2 \rho^2 + 1 - A^2} \right)$$

$$= -\frac{2t}{\pi} \int_0^\beta d\theta \int_0^\theta d\sigma \left( -\left( \cot \sigma \right)^2 + \frac{\cos \sigma}{(\sin \sigma)^3} \arctan(\tan \sigma) \right).$$

Thus

$$I_5 = -\frac{2t}{\pi} \int_0^\beta d\theta \int_0^\theta d\sigma \left( -\left( \cot \sigma \right)^2 + \frac{\cos \sigma}{(\sin \sigma)^3} \arctan(\tan \sigma) \right) + O(t e^{-R^2/(4t^3)}), \ t \downarrow 0.$$

We note that

$$\arctan(\tan \sigma) = \sigma + U(\sigma),$$

where

$$U(\sigma) = \begin{cases} 0, & \text{if } \sigma \in (0, \frac{\pi}{2}); \\
-\pi, & \text{if } \sigma \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right); \\
-2\pi, & \text{if } \sigma \in \left(\frac{3\pi}{2}, 2\pi\right). \end{cases}$$

Hence it is necessary to compute

$$-\frac{2t}{\pi} \int_0^\beta d\theta \int_0^\theta d\sigma \left( -\left( \cot \sigma \right)^2 + \frac{\sigma \cos \sigma}{(\sin \sigma)^3} \right) = -\frac{2t}{\pi} \int_0^\beta d\theta \left( \frac{\cot \theta}{2} + \theta - \frac{t}{2(\sin \theta)^2} \right)$$

$$= \left( -\frac{\beta^2}{\pi} - \frac{\beta}{\pi} \cot \beta + \frac{1}{\pi} \right) t.$$

If $\beta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, then

$$-\frac{2t}{\pi} \int_0^\beta d\theta \int_0^\theta d\sigma \left( \frac{\cos \sigma}{(\sin \sigma)^3} \right) = 2t \int_0^\beta d\theta \int_0^{\frac{\pi}{2}} d\sigma \frac{\cos \sigma}{(\sin \sigma)^3} = \left( \beta + \cot \beta - \frac{\pi}{2} \right) t,$$

and if $\beta \in \left(\frac{3\pi}{2}, 2\pi\right)$, then

$$-\frac{2t}{\pi} \int_0^\beta d\theta \int_0^\theta d\sigma \left( \frac{\cos \sigma}{(\sin \sigma)^3} \right)$$

$$= 2t \int_\frac{\pi}{2}^{\frac{3\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} d\sigma \frac{\cos \sigma}{(\sin \sigma)^3} + 2t \int_0^{\frac{3\pi}{2}} d\theta \int_\frac{\pi}{2}^{\frac{3\pi}{2}} d\sigma \frac{\cos \sigma}{(\sin \sigma)^3} + 4t \int_0^{\frac{3\pi}{2}} d\theta \int_\frac{3\pi}{2}^{2\pi} d\sigma \frac{\cos \sigma}{(\sin \sigma)^3}$$

$$= (2\beta + 2 \cot \beta - 2\pi) t.$$

Hence

$$I_5 = \begin{cases} \left( -\frac{\pi}{2} \cot \beta + \frac{1}{2} \right) t, & \text{if } \beta \in (0, \frac{\pi}{2}); \\
\left( -\frac{\pi}{2} \cot \beta + \frac{1}{2} \beta \right) t, & \text{if } \beta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right); \\
\left( -\frac{\pi}{2} \cot \beta + \frac{1}{2} \beta \right) t, & \text{if } \beta \in \left(\frac{3\pi}{2}, 2\pi\right); \\
\left( -\frac{\pi}{2} \cot \beta + \frac{1}{2} \beta \right) t, & \text{if } \beta \in (\frac{3\pi}{2}, 2\pi). \end{cases}$$

Therefore, by combining (21), (35) and (28), (29), (32) respectively, we obtain that, as $t \downarrow 0$,

$$\mathcal{V}_\beta(t; R) = \frac{\beta R^2}{2} - \frac{2R t^{1/2}}{\sqrt{\pi}} \left( 1 + \frac{1 - \beta}{\pi} \cot \beta \right) t$$

$$+ 2(4\pi t)^{-1/2} \int_0^{\sin \beta} d\psi \left( \arcsin \psi - \psi \right) \int_R^\infty dr r^2 e^{-r^2/4} + O(t e^{-R^2/(\sin \beta)^2/8t^3}).$$
Via the change of variables $\rho = r\psi$ and integrating by parts with respect to $\psi$, we have

\[
2(4\pi t)^{-1/2} \int_0^{\left|\frac{\sin \beta}{\psi}\right|} d\psi (\arcsin \psi - \psi) \int_0^\infty dr \, r^2 \, e^{-r^2/4t/(4t)}
\]

\[
= 2(4\pi t)^{-1/2} \int_0^{\left|\frac{\sin \beta}{\psi}\right|} d\psi \left( \frac{\arcsin \psi - \psi}{\psi^3} \right) \int_0^\infty d\rho \, \rho^2 \, e^{-\rho^2/4t/(4t)}
\]

\[
= - (4\pi t)^{-1/2} R^3 \int_0^{\left|\frac{\sin \beta}{\psi}\right|} d\psi (\arcsin \psi - \psi) e^{-R^2\psi^2/4t/(4t)} + O(te^{-R^2(sin \beta)^2/(8t)})
\]

\[
(36)
\]

Again integrating by parts with respect to $\psi$, we see that

\[
(4\pi t)^{-1/2} \int_0^{\left|\frac{\sin \beta}{\psi}\right|} d\psi \left( \frac{1}{\sqrt{1 - \psi^2}} - \frac{1}{1 + \sqrt{1 - \psi^2}} \right) \int_0^\infty d\rho \, \rho^2 \, e^{-\rho^2/4t/(4t)}
\]

\[
= (4\pi t)^{-1/2} R^3 \int_0^{\left|\frac{\sin \beta}{\psi}\right|} d\psi (-\psi \sqrt{1 - \psi^2} + \psi) e^{-R^2\psi^2/4t/(4t)} + O(te^{-R^2(sin \beta)^2/(8t)})
\]

\[
(37)
\]

Hence by (36), (37) and the change of variables $\psi = \frac{\pi}{2}$, we have that

\[
2(4\pi t)^{-1/2} \int_0^{\left|\frac{\sin \beta}{\psi}\right|} d\psi (\arcsin \psi - \psi) \int_0^\infty dr \, r^2 \, e^{-r^2/4t/(4t)}
\]

\[
= - (4\pi t)^{-1/2} \int_0^{R|\sin \beta|} dx \left( R^2 \arcsin \left( \frac{x}{R} \right) - Rx \right) e^{-x^2/4t/(4t)}
\]

\[
+ (4\pi t)^{-1/2} \int_0^{R|\sin \beta|} dx \left( -x \sqrt{R^2 - x^2} + Rx \right) e^{-x^2/4t/(4t)} + O(te^{-R^2(sin \beta)^2/(8t)})
\]

\[
(38)
\]

This completes the proof of Lemma 9.

In Section 4.3, we deal with the remaining integral in (38).

**Lemma 10.** Let $W_1, W_2$ be two disjoint wedges in $\mathbb{R}^2$ with corresponding angles $\gamma_1, \gamma_2$ respectively such that $\partial W_1 \cap \partial W_2 = \{V_i\}$ for some $i \in \{1, \ldots, n\}$. Let $\alpha$ denote the angle between $W_1$ and $W_2$ such that $0 < \alpha \leq \pi$. Then

\[
\int_{\partial W_1} dx \int_{\partial W_2} dy p(x, y; t) = k(\alpha, \gamma_1, \gamma_2)t,
\]

where $k(\alpha, \gamma_1, \gamma_2)$ is as defined in (13).
Proof. By changing coordinates to polar coordinates, as in the proof of Lemma 9, we have that
\[
\int_{\gamma} dx \int_{\gamma} dy \; p(x, y; t) = (4\pi t)^{-1} \int_{\gamma} d\theta_1 \int_{\gamma} d\theta_2 \int_{0}^{\infty} dr_1 \int_{0}^{\infty} dr_2 \; e^{-(r_1^2 + r_2^2)/(4t)} e^{-(r_1^2 + r_2^2)/(4t)}
\]
\[
= \frac{4t}{\pi} \int_{\gamma} d\theta_1 \int_{\gamma} d\theta_2 \int_{0}^{\infty} dr_1 \int_{0}^{\infty} dr_2 \; e^{-(r_1^2 + r_2^2)/(4t)} e^{-(r_1^2 + r_2^2)/(4t)} \cos(\theta_1 - \theta_2)
\]
\[
= \frac{4t}{\pi} \int_{\gamma} d\theta_1 \int_{\gamma} d\theta_2 \int_{0}^{\infty} dr_1 \int_{0}^{\infty} dr_2 \; e^{-(r_1^2 + r_2^2)/(4t)} e^{-(r_1^2 + r_2^2)/(4t)} \cos(\theta_1 - \theta_2)
\]
\[
= \frac{2t}{\pi} \int_{\gamma} d\theta_1 \int_{\gamma} d\theta_2 \int_{0}^{\infty} dr_1 \int_{0}^{\infty} dr_2 \; e^{-(r_1^2 + r_2^2)/(4t)} e^{-(r_1^2 + r_2^2)/(4t)} \cos(\theta_1 - \theta_2)
\]
\[
\int_{\gamma} dx \int_{\gamma} dy \; p(x, y; t) = 2
\]
\[
\int_{\gamma} dx \int_{\gamma} dy \; p(x, y; t) = 4.
\]
Therefore (39) becomes
\[
\gamma_1 \gamma_2 t + \frac{2t}{\pi} \int_{\gamma_1} d\sigma \int_{\gamma_1} d\sigma \; \frac{\pi \cos \varphi}{4 |\sin \varphi|^3} \left( \arctan(|\cot \varphi|) + \frac{\arctan(|\cot \varphi|)}{2} \right) \frac{|\cos \varphi|}{|\sin \varphi|^3}
\]
\[
\gamma_1 \gamma_2 t + \frac{2t}{\pi} \int_{\gamma_1} d\sigma \int_{\gamma_1} d\sigma \; \frac{\pi \cos \varphi}{4 |\sin \varphi|^3} \left( \arctan(|\cot \varphi|) + \frac{\arctan(|\cot \varphi|)}{2} \right) \frac{|\cos \varphi|}{|\sin \varphi|^3}
\]
\[
\gamma_1 \gamma_2 t + \frac{2t}{\pi} \int_{\gamma_1} d\sigma \int_{\gamma_1} d\sigma \; \frac{\pi \cos \varphi}{4 |\sin \varphi|^3} \left( \arctan(|\cot \varphi|) + \frac{\arctan(|\cot \varphi|)}{2} \right) \frac{|\cos \varphi|}{|\sin \varphi|^3}
\]
(39)
\[
\gamma_1 \gamma_2 t + \frac{2t}{\pi} \int_{\gamma_1} d\sigma \int_{\gamma_1} d\sigma \; \frac{\pi \cos \varphi}{4 |\sin \varphi|^3} \left( \arctan(|\cot \varphi|) + \frac{\arctan(|\cot \varphi|)}{2} \right) \frac{|\cos \varphi|}{|\sin \varphi|^3}
\]
We see that
\[
S(\varphi) := \arctan(|\cot \varphi|) = \begin{cases} |\varphi - \frac{\pi}{2}|, & \text{if } 0 < \varphi \leq \pi; \\ |\varphi - \frac{3\pi}{2}|, & \text{if } \pi < \varphi \leq 2\pi. \end{cases}
\]
Hence
\[
\sin(2 \arctan(|\cot \varphi|)) |\cos \varphi| |\sin \varphi|^3 = \frac{1}{2} \left( \frac{\cos \varphi}{\sin \varphi} \right)^2.
\]
Therefore (39) becomes
\[
\gamma_1 \gamma_2 t + \frac{2t}{\pi} \int_{\gamma_1} d\sigma \int_{\gamma_1} d\sigma \; \frac{\pi \cos \varphi}{4 |\sin \varphi|^3} + \frac{1}{2} \left( \frac{\cos \varphi}{\sin \varphi} \right)^2 + \frac{S(\varphi)}{2 |\sin \varphi|^3}.
\]
By considering the function
\[
f(\varphi) = \frac{\pi \cos \varphi}{4 |\sin \varphi|^3} + \frac{1}{2} \left( \frac{\cos \varphi}{\sin \varphi} \right)^2 + \frac{S(\varphi)}{2 |\sin \varphi|^3}.
\]
for $\varphi$ in each of the four quadrants, we have that
\[
\frac{1}{2} \int_{\sigma}^{\sigma+\gamma_2} d\varphi \left( (\pi - \varphi) \frac{\cos \varphi}{(\sin \varphi)^2} + \frac{\cos \varphi}{\sin \varphi} \right)
\]
is a primitive for $f(\varphi)$ for all $\varphi \in (0, 2\pi)$. Integrating by parts with respect to $\varphi$, we obtain
\[
\frac{1}{2} \int_{\sigma}^{\sigma+\gamma_2} d\varphi \left( (\pi - \varphi) \frac{\cos \varphi}{(\sin \varphi)^2} + \frac{\cos \varphi}{\sin \varphi} \right) = \frac{1}{4} \left( (\gamma_2 + \sigma - \pi) - \cot(\gamma_2 + \sigma) - (\sigma - \pi) \frac{\cot \sigma - 2\gamma_2}{\sin \sigma^2} \right).
\]
(41)

Hence, by (40), (41) and integrating by parts with respect to $\sigma$, we have that
\[
\int W_1 \int_{W_2} dy p(x, y; t) = \frac{t}{2\pi} \left( (\gamma_2 + \gamma_1 + \alpha - \pi) \cot(\gamma_2 + \gamma_1 + \alpha) - (\alpha - \pi) \cot \alpha \right)
+ \frac{t}{2\pi} \left( (\gamma_2 + \alpha - \pi) \cot(\gamma_2 + \alpha) + (\gamma_1 + \alpha - \pi) \cot(\gamma_1 + \alpha) \right)
\]
as required. $\Box$

4. The contribution to the heat content from points close to an edge.

In this section, we consider points $x \in C(\frac{D}{2} \sin \gamma, R)$. We partition this region into $n$ rectangles $S_n$ and $2n$ cusps $C_i$, each of height $\frac{D}{2} \sin \gamma$. We approximate $u_D$ by $u_H$, where $H \subset \mathbb{R}^2$ is the half-plane such that $\emptyset \neq \partial S \cap \partial D \subset \partial H$ and $S \subset H$, as in Lemma 7.

4.1. The contribution to the heat content from a rectangle. We first compute the contribution to the heat content of $D$ from a rectangle, $S_n$, of height $\frac{D}{2} \sin \gamma$ and length $L$, where $L \in \mathbb{R}, L > 0$. We have that
\[
\int_H dy p(x, y; t) = \int_0^\infty dy_2 \int_0^\infty dy_1 (4\pi t)^{-1/2} e^{-(x_1 - y_1)^2/(4t)} \int_0^\infty dy_1 (4\pi t)^{-1/2} e^{-(y_2 - y_2)^2/(4t)}
= 1 - \int_{x_1}^\infty d\zeta (4\pi t)^{-1/2} e^{-\zeta^2/(4t)}.
\]

Let $x_1 \in (0, \frac{D}{2} \sin \gamma)$ and $x_2 \in (0, L)$. Then, by integrating by parts with respect to $x_1$, we obtain
\[
\begin{align*}
\int_{S_n} dx \int_H dy p(x; y; t) &= |S_n| - \int_{x_1}^\infty dx \int_{x_1}^\infty d\zeta (4\pi t)^{-1/2} e^{-\zeta^2/(4t)} \\
&= |S_n| - (4\pi t)^{-1/2} \int_0^L dx_2 \int_0^\infty d\zeta (4\pi t)^{-1/2} e^{-\zeta^2/(4t)} \\
&= |S_n| - (4\pi t)^{-1/2} \int_0^L dx_2 \int_0^\infty d\zeta (4\pi t)^{-1/2} e^{-\zeta^2/(4t)} + O(t^{1/2} e^{-R^2(\sin \gamma)^2/(32t)}) \\
&= |S_n| - \frac{L}{\sqrt{\pi}} t^{-1/2} + O(t^{1/2} e^{-R^2(\sin \gamma)^2/(32t)}), \ t \downarrow 0.
\end{align*}
\]
(42)

4.2. The contribution to the heat content from a cusp. We now compute the contribution to the heat content of $D$ from a cusp. Let $C_i$ denote the cusp which is
adjacent to $S_\gamma$ and $B_i(R)$. Then

$$\int_{C_i} dx \int_{H} dy \ p(x, y; t) = \int_0^{\frac{2\pi}{\sin \gamma}} dx \left( R - \sqrt{R^2 - x^2} \right) \int_0^\infty dy \ (4\pi t)^{-1/2} e^{-|x-y|^2/(4t)}$$

$$= \int_0^{\frac{2\pi}{\sin \gamma}} dx \left( R - \sqrt{R^2 - x^2} \right) \left( 1 - \int_x^\infty d\zeta (4\pi t)^{-1/2} e^{-\zeta^2/(4t)} \right)$$

$$= |C_i| - \int_0^{\frac{2\pi}{\sin \gamma}} dx \left( R - \sqrt{R^2 - x^2} \right) \int_x^\infty d\zeta (4\pi t)^{-1/2} e^{-\zeta^2/(4t)}.$$

Integrating by parts with respect to $x$, we obtain

$$- \int_0^{\frac{2\pi}{\sin \gamma}} dx \left( R - \sqrt{R^2 - x^2} \right) \int_x^\infty d\zeta (4\pi t)^{-1/2} e^{-\zeta^2/(4t)}$$

$$= -(4\pi t)^{-1/2} \int_0^{\frac{2\pi}{\sin \gamma}} dx \left( R x - \frac{R^2}{2} \arcsin \left( \frac{x}{R} \right) - \frac{x}{2} \sqrt{R^2 - x^2} \right) e^{-x^2/(4t)}$$

$$+ O(e^{-R^2(\sin \gamma)^2/(32t)}), \ t \downarrow 0.$$ 

Hence the contribution from each cusp is

$$\int_{C_i} dx \int_{H} dy \ p(x, y; t)$$

$$= |C_i| + (4\pi t)^{-1/2} \int_0^{\frac{2\pi}{\sin \gamma}} dx \left( -R x + \frac{R^2}{2} \arcsin \left( \frac{x}{R} \right) + \frac{x}{2} \sqrt{R^2 - x^2} \right) e^{-x^2/(4t)}$$

$$+ O(e^{-R^2(\sin \gamma)^2/(32t)}), \ t \downarrow 0.$$ 

4.3. The $O(t^{3/2})$ terms from the cusp and sector contributions. Finally, we deal with the remaining integrals from the sector and cusp contributions. Each sector has two neighbouring cusps so we are interested in the following integral

$$2(4\pi t)^{-1/2} \int_0^{\frac{2\pi}{\sin \gamma}} dx \left( -R x + \frac{R^2}{2} \arcsin \left( \frac{x}{R} \right) + \frac{x}{2} \sqrt{R^2 - x^2} \right) e^{-x^2/(4t)}$$

$$= (4\pi t)^{-1/2} \int_0^{\frac{2\pi}{\sin \gamma}} dx \left( -2R x + R^2 \arcsin \left( \frac{x}{R} \right) + x \sqrt{R^2 - x^2} \right) e^{-x^2/(4t)}.$$ 

By definition of $\gamma$, in (8), we can write the remaining integral from the sector contribution, as

$$- (4\pi t)^{-1/2} \int_0^{\frac{2\pi}{\sin \gamma}} dx \left( -2R x + R^2 \arcsin \left( \frac{x}{R} \right) + x \sqrt{R^2 - x^2} \right) e^{-x^2/(4t)}$$

$$= - (4\pi t)^{-1/2} \int_0^{\frac{2\pi}{\sin \gamma}} dx \left( -2R x + R^2 \arcsin \left( \frac{x}{R} \right) + x \sqrt{R^2 - x^2} \right) e^{-x^2/(4t)}$$

$$+ O(e^{-R^2(\sin \gamma)^2/(8t)}), \ t \downarrow 0.$$ 

Adding (43) and (44), we obtain

$$(4\pi t)^{-1/2} \int_0^{\frac{2\pi}{\sin \gamma}} dx \left( -2R x + R^2 \arcsin \left( \frac{x}{R} \right) + x \sqrt{R^2 - x^2} \right) e^{-x^2/(4t)}$$

$$- (4\pi t)^{-1/2} \int_0^{\frac{2\pi}{\sin \gamma}} dx \left( -2R x + R^2 \arcsin \left( \frac{x}{R} \right) + x \sqrt{R^2 - x^2} \right) e^{-x^2/(4t)}$$

$$= - (4\pi t)^{-1/2} \int_0^{\frac{2\pi}{\sin \gamma}} dx \left( -2R x + R^2 \arcsin \left( \frac{x}{R} \right) + x \sqrt{R^2 - x^2} \right) e^{-x^2/(4t)}$$

$$= O(e^{-R^2(\sin \gamma)^2/(12t)}), \ t \downarrow 0.$$ 

This completes the proof of Theorem 1.
5. The heat content of a π-sector in a 3π/2-wedge.

In this section, we compute the heat content of a π-sector in a 3π/2-wedge which share one common edge (and vertex). This is a crucial ingredient in the computation of the heat content of the fractal polyhedron $D_n$, which was constructed in Section 4 and will be used in Section 6.

**Lemma 11.** Let $B_ε(R) \subset W_{3π/2}$ such that $B_ε(R)$ and $W_{3π/2}$ share one common edge (and vertex). Then, for $t \downarrow 0$,

\[
\int_{B_ε(R)} dx \int_{W_{3π/2}} dy \, p(x, y; t) = \frac{πR^2}{2} - \frac{R}{\sqrt{π}} t^{1/2} + (4πt)^{-1/2} \int_0^1 d\psi (\arcsin \psi - \psi) \int_R^∞ dr \, r^2 e^{-r^2\psi^2/(4t)} + O(te^{-R^2/(8t)}).
\]

We remark that the coefficient of $t$ is equal to 0 in this case.

**Proof.** Similarly to the proof of Lemma 9, the left-hand side of (45) equals

\[
(4πt)^{-1} \int_0^π dθ_1 \int_0^{2π} dθ_2 \int_0^R dr_1 \int_0^∞ dr_2 (r_1 r_2) e^{-(r_1^2 + r_2^2)/(4t) + 2r_1 r_2 A/(4t)}
\]

\[
= (4πt)^{-1} \int_0^π dθ_1 \int_0^{2π} dθ_2 \int_0^R dr_1 \int_0^∞ dr_2 (r_1 r_2) e^{-(r_1^2 + r_2^2)/(4t) + 2r_1 r_2 A/(4t)}
\]

\[
- (4πt)^{-1} \int_0^π dθ_1 \int_0^{2π} dθ_2 \int_0^R dr_1 \int_0^∞ dr_2 (r_1 r_2) e^{-(r_1^2 + r_2^2)/(4t) + 2r_1 r_2 A/(4t)}
\]

\[
(46) \quad =: M_1 + M_2.
\]

Now

\[
M_1 = (4πt)^{-1} \int_0^π dθ_1 \int_0^{2π} dθ_2 \int_0^R dr_1 \int_0^∞ dr_2 (r_1 r_2) e^{-(r_1^2 + r_2^2)/(4t) + 2r_1 r_2 A/(4t)}
\]

\[
(47) \quad = \frac{πR^2}{2},
\]

and letting $r_2 - r_1 A = ρ$, we have that

\[
M_2 = -(4πt)^{-1} \int_0^π dθ_1 \int_0^{2π} dθ_2 \int_0^R dr_1 \int_0^∞ dr_2 (r_1 r_2) e^{-(r_1^2 + r_2^2)/(4t) + 2r_1 r_2 A/(4t)}
\]

\[
= -(4πt)^{-1} \int_0^π dθ_1 \int_0^{2π} dθ_2 \int_0^R dr_1 \int_0^∞ dr_2 (r_1 r_2) e^{-(r_1^2 + r_2^2)/(4t) + 2r_1 r_2 A/(4t)}
\]

\[
= -(4πt)^{-1} \int_0^π dθ_1 \int_0^{2π} dθ_2 \int_0^R dr_1 \int_0^∞ dr_2 (r_1 r_2) e^{-(r_1^2 + r_2^2)/(4t) + 2r_1 r_2 A/(4t)}
\]

\[
\frac{π}{2} t (1 - e^{-R^2/(4t)})
\]

\[
(48) \quad = -(4πt)^{-1} \int_0^π dθ_1 \int_0^{2π} dθ_2 \int_0^R dr_1 \int_0^∞ dr_2 (r_1 r_2) e^{-r^2/(4t) - r^2 (1 - A^2)/(4t)}.
\]

We also have that

\[
-(4πt)^{-1} \int_0^π dθ_1 \int_0^{2π} dθ_2 \int_0^R dr_1 \int_0^∞ dr_2 (r_1 r_2) e^{-r^2/(4t) - r^2 (1 - A^2)/(4t)}
\]

\[
= -(4πt)^{-1} \int_0^π dθ_1 \int_0^{2π} dθ_2 \int_0^R dr_1 \int_0^∞ dr_2 (r_1 r_2) e^{-r^2/(4t) - r^2 (1 - A^2)/(4t)}
\]

\[
- (4πt)^{-1} \int_0^π dθ_1 \int_0^{2π} dθ_2 \int_0^R dr_1 \int_0^∞ dr_2 (r_1 r_2) e^{-r^2/(4t) - r^2 (1 - A^2)/(4t)}
\]
\[
\begin{align*}
N_1 &= N_1 + N_2.
\end{align*}
\]

Now \( N_1 \) equals

\[
\begin{align*}
- \frac{(4\pi t)^{-1/2}}{2} & \int_0^\pi d\phi_1 \int_0^\pi d\phi_2 \int_0^R dr \ A_r^2 e^{-r^2(1-A^2)/(4t)} \\
&- \frac{(4\pi t)^{-1/2}}{2} \int_0^\pi d\phi_1 \int_0^\pi d\phi_2 \int_0^R dr \ A_r^2 e^{-r^2(1-A^2)/(4t)} \\
&+ \frac{(4\pi t)^{-1/2}}{2} \int_0^\pi d\phi_1 \int_0^\pi d\phi_2 \int_0^r dr \ A_r^2 e^{-r^2(1-A^2)/(4t)} \\
&- \frac{(4\pi t)^{-1/2}}{2} \int_0^\pi d\phi_1 \int_0^\pi d\phi_2 \int_0^r dr \ A_r^2 e^{-r^2(1-A^2)/(4t)} \\
&+ \frac{(4\pi t)^{-1/2}}{2} \int_0^\pi d\phi_1 \int_0^\pi d\phi_2 \int_0^r dr \ A_r^2 e^{-r^2(1-A^2)/(4t)}
\end{align*}
\]

(49)

by integrating by parts with respect to \( \theta \). In addition, as for the computation of \( I_5 \) (see \[33\]), we have that

\[
\begin{align*}
N_2 &= (4\pi t)^{-1} \int_0^\pi d\phi_1 \int_0^\pi d\phi_2 \int_0^\infty dr \ A_r^2 e^{-r^2(1-A^2)/(4t)} \\
&= \frac{2t}{\pi} \int_0^\pi d\phi_1 \int_0^\pi d\phi_2 \int_0^\infty dr \ \frac{A_r^2}{(A_r^2 + 1 - A^2)^2} + O(t e^{-R^2/(4t)}) \\
&= \frac{t}{\pi} \int_0^\pi d\phi_1 \int_0^\pi d\phi_2 \left( -\cot(\phi_1 - \phi_2)^2 + \frac{\cos(\phi_1 - \phi_2)}{\sin(\phi_1 - \phi_2)^3} \arctan(\tan(\phi_1 - \phi_2)) \right) + O(t e^{-R^2/(4t)}) \\
&= \frac{t}{\pi} \int_0^\pi d\phi_1 \int_0^\pi d\phi_2 \left( -\cot(\phi_1 + \phi_2)^2 + \frac{\cos(\phi_1 + \phi_2)}{\sin(\phi_1 + \phi_2)^3} \arctan(\tan(\phi_1 + \phi_2)) \right) + O(t e^{-R^2/(4t)}) \\
&= \frac{t}{\pi} \int_0^\pi d\phi_1 \int_0^\pi d\phi_2 \left( -\cot(\phi_1)^2 + \frac{\cos(\phi_1)}{\sin(\phi_1)^3} \arctan(\tan(\phi_1)) \right) + O(t e^{-R^2/(4t)}) \\
&= \frac{t}{\pi} \int_0^\pi d\phi_1 \int_0^\pi d\phi_2 \left( -\cot(\phi_1)^2 + \frac{\cos(\phi_1)}{\sin(\phi_1)^3} \arctan(\tan(\phi_1)) \right) + O(t e^{-R^2/(4t)}), \ t \downarrow 0.
\end{align*}
\]
Note that
\[
\arctan(\tan \eta) = \begin{cases} 
\eta & \text{if } \eta \in (0, \frac{\pi}{2}); \\
\eta - \pi & \text{if } \eta \in (\frac{\pi}{2}, \frac{3\pi}{2}).
\end{cases}
\]

Hence
\[
N_2 = \frac{t}{\pi} \int_0^\pi d\theta \int_0^{\frac{\pi}{2} + \theta} d\eta \left(-\left(\cot \eta\right)^2 + \frac{\cos \eta}{(\sin \eta)^2} \arctan(\tan \eta)\right) + O(te^{-R^2/(4\epsilon)})
\]
\[
= \frac{t}{\pi} \int_0^\pi d\theta \int_0^{\frac{\pi}{2} + \theta} d\eta \left(-\left(\cot \eta\right)^2 + \frac{\eta \cos \eta}{(\sin \eta)^3}\right)
\]
\[
+ \frac{t}{\pi} \int_0^\pi d\theta \int_0^{\frac{\pi}{2} + \theta} d\eta \left(-\left(\cot \eta\right)^2 + \frac{(\eta - \pi) \cos \eta}{(\sin \eta)^3}\right)
\]
\[
+ \frac{t}{\pi} \int_0^\pi d\theta \int_0^{\frac{\pi}{2} + \theta} d\eta \left(-\left(\cot \eta\right)^2 + \frac{(\eta - \pi) \cos \eta}{(\sin \eta)^3}\right) + O(te^{-R^2/(4\epsilon)})
\]
\[
= \frac{t}{2\pi} + \frac{t}{2\pi} + \frac{t}{\pi} \left(\frac{\pi^2}{4} - 1\right) + O(te^{-R^2/(4\epsilon)})
\]
(51)

Thus, by (48), (49), (50) and (51), we have that, as \(t \downarrow 0\),
(52)
\[
M_2 = -\frac{R}{\sqrt{\pi}} t^{1/2} + (4\pi t)^{-1/2} \int_0^1 d\psi (\arcsin \psi - \psi) \int_R^\infty dr r^2 e^{-r^2 \psi^2/(4\epsilon)} + O(te^{-R^2/(8\epsilon)}).
\]

Therefore, by (46), (47) and (52), as \(t \downarrow 0\),
\[
(4\pi t)^{-1} \int_0^\pi d\theta \int_0^{\frac{3\pi}{2}} d\theta_2 \int_0^R d\rho_1 \int_{R}^\infty dr_2 (r_1 r_2) e^{-(r_1 \cos \theta_1 - r_2 \cos \theta_2)^2/(4\epsilon)} - (r_1 \sin \theta_1 - r_2 \sin \theta_2)^2/(4\epsilon)
\]
\[
= \frac{\pi R^2}{2} - \frac{R}{\sqrt{\pi}} t^{1/2} + (4\pi t)^{-1/2} \int_0^1 d\psi (\arcsin \psi - \psi) \int_R^\infty dr r^2 e^{-r^2 \psi^2/(4\epsilon)} + O(te^{-R^2/(8\epsilon)}).
\]

\(\square\)

We note that
\[
\mathcal{V}_{\frac{3\pi}{2}}(t; R) = \int_{B_{\frac{3\pi}{2}}} dx \int_{W_{\frac{3\pi}{2}}} dy p(x, y; t)
\]
\[
= \int_{B_{\frac{3\pi}{2}}} dx \int_{W_{\frac{3\pi}{2}}} dy p(x, y; t) + \int_{B_{\frac{3\pi}{2}}} dx \int_{W_{\frac{3\pi}{2}}} dy p(x, y; t).
\]
By Theorem 1 and Lemma 11 this implies that, as \(t \downarrow 0\),
\[
\int_{B_{\frac{3\pi}{2}}} dx \int_{W_{\frac{3\pi}{2}}} dy p(x, y; t)
\]
\[
= \frac{\pi R^2}{4} - \frac{R}{\sqrt{\pi}} t^{1/2} + \left(\frac{3\pi}{2}\right) t
\]
\[
+ (4\pi t)^{-1/2} \int_0^1 d\psi (\arcsin \psi - \psi) \int_R^\infty dr r^2 e^{-r^2 \psi^2/(4\epsilon)} + O(te^{-R^2/(8\epsilon)})
\]
(53)
\[
= \frac{\pi R^2}{4} - \frac{R}{\sqrt{\pi}} t^{1/2} + \frac{t}{\pi}
\]
\[
+ (4\pi t)^{-1/2} \int_0^1 d\psi (\arcsin \psi - \psi) \int_R^\infty dr r^2 e^{-r^2 \psi^2/(4\epsilon)} + O(te^{-R^2/(8\epsilon)}).
\]

The result of Lemma 11 and formula (53) will be used in Section 6 below.
6. The heat content of the fractal polyhedron $D_s$.

In this section, we use Theorem 1 to compute the heat content of the fractal polyhedron $D_s$ which was constructed in Section 1. To do this, we adopt the scheme of (6) to the three-dimensional setting below. The key step in (6) was to obtain a renewal equation by making a suitable Ansatz for the heat content. The corresponding Ansatz has been made here in (60) and (61) for $0 < s < \sqrt{2} - 1$, $s \neq \frac{1}{2}$, and in (63) and (65) for $s = \frac{1}{2}$.

In order to derive the required renewal equation, we need to compute the contribution to the heat content $H_{D_s}(t)$ from $Q_0$ and $Q_{1,1}$. We do this below in Lemma 12 and Lemma 13, respectively. In what follows, for $A \subset \mathbb{R}^3$, $d(x, A)$ is the 3-dimensional analogue of (58).

We make the following approximations for $u_{D_s}(x; t)$.

Let $0 < \delta \leq \min \left\{ \frac{s^2}{\pi}, \frac{s}{(1-s)} \right\}$ and let $x \in D_s$. If $d(x, \partial D_s) \geq \delta$, then we have that

$$|u_{D_s}(x; t) - 1| \leq 2^{3/2} e^{-\delta^2/(8t)},$$

by the principle of not feeling the boundary, Proposition 9(i)]. We define

$$\hat{F} = \{ x \in D_s : d(x, \partial D_s) < \delta, d(x, e) > \delta \text{ for all edges } e \in \partial D_s \}.$$ If $x \in \hat{F}$, then we have that

$$|u_{D_s}(x; t) - u_H(x; t)| \leq 2^{3/2} e^{-\delta^2/(8t)},$$

where $u_H(x; t) = (4\pi t)^{-1/2} \int_{-\delta(x, \partial D_s)}^{\delta(x, \partial D_s)} \, d\zeta \, e^{-\zeta^2/(4t)}$, i.e. $H$ is a half-space whose boundary contains the face of $\partial D_s$ nearest to $x$. Let

$$\tilde{E} = \{ x \in D_s : d(x, e) < \delta \text{ for some edge } e \in \partial D_s, d(x, v) > \delta \text{ for all vertices } v \in \partial D_s \}.$$ If $x \in \tilde{E}$, then we have that

$$|u_{D_s}(x; t) - u_W(x; t)| \leq 2^{3/2} e^{-\delta^2/(8t)},$$

where $W$ is the infinite wedge $W^\infty$ for entrant edges and $W$ is the infinite wedge $W^\infty$ for en- trant edges. (See the proof of Lemma 12 for further details). The estimates (54) follow by similar arguments to those given in the proof of Lemma 5 with $\hat{D} = D_s$, $F = H, W, E = \hat{F}, \tilde{E}$ respectively and $G = \{ x \in D_s : d(x, E) < \delta \}$. It remains to approximate $u_{D_s}(x; t)$ for $x$ near a vertex of $\partial Q_0 \cap \partial D_s$, $\partial Q_{1,1} \cap \partial D_s$ respectively. We only require the contribution to the heat content $H_{D_s}(t)$ from these vertices to derive the required renewal equation. The relevant approximation to make here is via a one-sided infinite cone $C_v$ with vertex $v \in \partial D_s$ such that $\partial C_v \supseteq \{ x \in \partial D_s : d(x, v) < \delta \}$. Definition aside, no viable expressions are known for $u_{C_v}(x; t)$ in this 3-dimensional setting. For our purposes, it is sufficient to approximate the neighbourhood of each vertex $v \in (\partial Q_0 \cup \partial Q_{1,1}) \cap \partial D_s$ by a cube $S_v$. Each cube $S_v$ centred at $v$ has side-length $2\delta$ and is chosen such that the faces of $\partial S_v$ are pairwise parallel to those of $\partial Q_0$. We are interested in the contribution to the heat content $H_{D_s}(t)$ from the region $S_v \cap D_s$. There are two cases to consider. Either the vertex $v$ is entrant and $S_v \cap D_s$ is $\frac{1}{2}$ of $S_v$, or the vertex $v$ is re-entrant and $S_v \cap D_s$ is $\frac{2}{3}$ of $S_v$. If $v$ is entrant, then the coefficient of $t^{3/2}$ in the expansion for $H_{D_s}(t)$ is equal to $\frac{12}{\pi} \delta$ by separation of variables. Unfortunately, we were unable to compute the coefficient of $t^{3/2}$ for a re-entrant vertex. However, the contribution to the heat content $H_{D_s}(t)$ from each region $S_v \cap D_s$ is $O(t^{3/2} (\log(t^{-1}))^{3/2})$. We note that this choice of $\delta$ gives $O(e^{-\delta^2/(8t)}) = O(t^t)$.

**Lemma 12.** Let $0 < s < \sqrt{2} - 1$. Then

$$\int_{Q_0} \, dx \, u_{D_s}(x; t) = 1 - 6(1 - s^2)\frac{t^{1/2}}{\sqrt{\pi}} + \frac{12}{\pi} t + O(t^{3/2} (\log(t^{-1}))^{3/2}), \, t \downarrow 0.$$
Proof. Partition $Q_0$ into the following sets.

(i) $\partial Q_0 \cap \partial D$, has 32 vertices; $v_k$, $k = 1, \ldots, 32$. At each vertex $v_k$, consider a cube $S_k$ of side-length $2\delta$ centred at $v_k$. Let $\mathcal{S} = S_k \cap Q_0$, $k = 1, \ldots, 32$ and $\mathcal{S} = \cup_{k=1}^{32} S_k$.

(ii) $\partial Q_0 \cap \partial D_s$ has 36 edges; $e_j$, $j = 1, \ldots, 36$. Let $\tilde{E}_j = \{x \in Q_0 : d(x, e_j) < \delta, x \notin \tilde{S}\}$.

(iii) $\mathcal{F} = \{x \in Q_0 : d(x, \partial Q_0 \cap \partial D_s) < \delta, x \notin (\mathcal{S} \cup \bigcup_{j=1}^{36} \tilde{E}_j)\}$.

(iv) The interior of $Q_0$ minus (i), (ii) and (iii); an open polygon $P_0$ with distance at least $\delta$ to $\partial D_s$.

(v) The remainder, which has measure zero.

The contribution to the heat content from (iv) is $|P_0| + O(e^{-\delta^2/(8t)}) = |P_0| + O(t^{3/2})$, $t \downarrow 0$.

To compute the contribution from (ii), there are two types of edges to consider. Each edge $e_j$ is the intersection of two faces of $\partial D_s$. For fixed $j$, let $\Pi_j$ denote the plane which is orthogonal to these faces and intersects $e_j$ in exactly one point. Apply Lemma 9 with $\tilde{D} = D_s \cap \Pi_j$, $F = \tilde{W}^\perp$, $\tilde{W}^\perp$ respectively, $E = \tilde{E}_j \cap \Pi_j$ and $G = \{x \in D_s \cap \Pi_j : d(x, \tilde{E}_j \cap \Pi_j) < \delta\}$ so that either:

(I) the contribution from $\tilde{E}_j \cap \Pi_j$ can be approximated by that from a sector $B_s(\tilde{\pi})$ in a wedge $\tilde{W}^\perp$, or

(II) the contribution from $\tilde{E}_j \cap \Pi_j$ can be approximated by that from a sector $B_s(\pi)$ in a wedge $W^\perp$.

We can use Lemma 9 to deduce that the contribution from the edges of type (I) is

\[
12(1 - 2\delta) \left( |\tilde{E}_j \cap \Pi_j| - 2\delta \frac{t^{1/2}}{\sqrt{\pi}} + \frac{1}{\pi} \right) + O(t^{3/2})
\]

\[
= 12 \left( |\tilde{E}_j| - 2\delta (1 - 2\delta) \frac{t^{1/2}}{\sqrt{\pi}} + \frac{1}{\pi} (1 - 2\delta)^t \right) + O(t^{3/2}), \ t \downarrow 0.
\]

In addition, Lemma 11 gives that the contribution from the edges of type (II) is

\[
24(s - 2\delta) \left( |E_j \cap \Pi_j| - \delta \frac{t^{1/2}}{\sqrt{\pi}} \right) + O(t^{3/2})
\]

\[
= 24 \left( |E_j| - \delta (s - 2\delta) \frac{t^{1/2}}{\sqrt{\pi}} \right) + O(t^{3/2}), \ t \downarrow 0.
\]

Each cross-section of $\tilde{F}$ is a union of rectangles and cuspidal regions. Thus by Section 4 we have that the contribution from $\tilde{F}$ equals

\[
|\tilde{F}| - H^2(\partial \tilde{F}) \frac{t^{1/2}}{\sqrt{\pi}} + O(t^{3/2}), \ t \downarrow 0.
\]

For edges of type (I), the sector $\tilde{E}_j \cap \Pi_j$ has two cuspidal neighbours. For edges of type (II), the sector $E_j \cap \Pi_j$ has one cuspidal neighbour but by Lemma 11 the term of order $t^{3/2}$ is half that of Lemma 5. Even though the sector and cusp terms of order $t^{3/2}$ cancel out, the remainder is dominated by the contribution from (i), which is $O(t^{3/2}(\log(t^{-1}))^{3/2})$. This completes the proof of Lemma 12. □

Following the strategy of [1], in order to compute $\int_{D_s - Q_0} dx u_{D_s}(x; t)$, we introduce a model solution which approximates $u_{D_s}$ in one of the six components of $D_s - Q_0$. Consider the half-space $H = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 < 0\}$ and attach one of the six components of $D_s - Q_0$ to $H$. The resulting set is

\[
H_s = \text{interior} \left\{ H \cup \bigcup_{j \geq 1} \bigcup_{1 \leq i \leq \frac{1}{N}(j)} Q_{j,i} \right\}.
\]

Let $u_{H_s}$ be the unique solution of (2) and (3) with $D = H_s$. Define

\[
E(t) = \int_{H_s - H} dx u_{H_s}(x; t).
\]
Applying Lemma 5 with \( \tilde{D} = D_s, F = H_s, E = H_s - H \) and \( G = \{ x \in H_s : d(x, E) < \epsilon \} \), where \( \epsilon = \frac{(1+t^2/2^2)(1-2s-s^2)}{2(1-s)} \), we have

\[
\int_{D_s-Q_0} dx u_{D_s}(x; t) = 6 \int_{H_s-H} dx u_{D_s}(x; t) = 6 \int_{H_s-H} dx u_{H_s}(x; t) + O(e^{-\epsilon^2/(8t)}) = 6E(t) + O(e^{-\epsilon^2/(8t)}), t \downarrow 0.
\]

(57)

In contrast to [6], where the temperature of the boundary is fixed for all \( t > 0 \), we must account for the fact that \( H_s - H \) and \( H_s - D_s \) feel each other’s presence. The choice of \( \epsilon \) above is a lower bound for the distance between \( H_s - H \) and \( H_s - D_s \).

Similarly to Lemma 12 we have;

**Lemma 13.** Let \( 0 < s < \sqrt{2} - 1 \). Then

\[
\int_{Q_{1,1}} dx u_{H_s}(x; t) = s^3 - 5s^2(1-s)^2 \frac{t^{1/2}}{\sqrt{\pi}} + \frac{12s}{\pi} t + \tilde{h}(t), t \downarrow 0,
\]

where \( |\tilde{h}(t)| \leq \tilde{C} t^{3/2}(\log(t^{-1}))^{3/2} \) for \( t \leq s^2 \) and some constant \( \tilde{C} > 0 \).

**Proof.** The proof of Lemma 13 is analogous to that of Lemma 12. We note that in this case there is an additional type of edge to consider. Namely, the edges where the contribution from \( \tilde{E} \cap Q_{1,1} \cap \Pi_1 \) can be approximated by that from a sector \( B_2(\delta) \) in a wedge \( W_{s,t} \).

There are 4 such edges so the contribution to the heat content is

\[
4(s-2\delta) \left( |\tilde{E} \cap Q_{1,1} \cap \Pi_1| - \delta \frac{t^{1/2}}{\sqrt{\pi}} + \frac{1}{\pi} t \right) + O(t^{3/2})
\]

\[
= 4 \left( |\tilde{E} \cap Q_{1,1}| - \delta(s-2\delta) \frac{t^{1/2}}{\sqrt{\pi}} + \frac{(s-2\delta)}{\pi} t \right) + O(t^{3/2}), t \downarrow 0,
\]

by (56). This completes the proof of Lemma 13 by our choice of \( \delta, (56) \). \( \square \)

Below we state and prove the corresponding 3-dimensional result to [6 Proposition 4] for completeness.

**Lemma 14.** Fix \( 0 < s < \sqrt{2} - 1 \). Then

\[
E(t) = 5s^3E \left( \frac{t}{s^2} \right) + s^3 - 5s^2(1-s^2) \frac{t^{1/2}}{\sqrt{\pi}} + \frac{12s}{\pi} t + h(t),
\]

where \( |h(t)| \leq \tilde{C} t^{3/2}(\log(t^{-1}))^{3/2} \) for \( t \leq s^2 \) and some constant \( \tilde{C} > 0 \).

**Proof.** Re-write \( E(t) \) as

\[
E(t) = \int_{Q_{1,1}} dx u_{H_s}(x; t) + \int_{H_s-(H_s \cup Q_{1,1})} dx u_{H_s}(x; t).
\]

Then \( H_s \cup (H \cup Q_{1,1}) \) consists of 5 copies of \( H_s - H \) scaled by a factor \( s \), say \( A_1, \ldots, A_5 \). Each of these copies has a face \( f_i \) connecting it to \( Q_{1,1} \). Let \( H_{A_1} \) be the half-space such that \( \partial H_{A_1} \cap f_i \) and \( H_{A_1} \cap Q_{1,1} \). Put \( F_i = \cup_{A \neq A_i} \cup H_{A_1} \cup f_i \). Then \( F_i \) is a copy of \( H_s \) scaled by a factor \( s \). Thus, by scaling, we have

\[
\int_{A_i} dx u_{F_i}(x; t) = s^3 \int_{H_s-H} dx u_{H_s}(x; t/s^2) = s^3 E(t/s^2).
\]

Define \( G_i = \{ x \in H_s : d(x, A_i) < \tilde{r} \} \), where \( \tilde{r} = \frac{4(1-2s-s^2)}{2(1-s)} \). Applying Lemma 5 with \( \tilde{D} = H_s, F = F_i, E = A_i, \) and \( G = G_i \), we have

\[
\int_{A_i} dx u_{H_s}(x; t) = \int_{A_i} dx u_{F_i}(x; t) + O(e^{-\tilde{r}^2/(8t)}), t \downarrow 0.
\]
Lemma 15. Let \( q(61) \)

Similarly to \([6, \text{Proposition 5}]\), define

From (60), we also have \( q(61) \)

Substitute (58) into (60) to obtain

Hence

Combining this with Lemma [13] gives the result. \( \square \)

We must now consider the different regimes for \( s \).

Lemma 16. Let \( d = \frac{3}{2} + \frac{1}{2} \log \frac{t}{\log s} \) and fix \( 0 < s < \sqrt{2} - 1, s \neq \frac{1}{3} \). Then there exists a periodic, continuous function \( p_s : \mathbb{R} \rightarrow \mathbb{R} \) with period \( \log(s^{-3}) \) such that as \( t \downarrow 0 \),

\[
E(t) = \frac{s^3}{1 - 5s^3} - \frac{5s^2(1 - s^2)}{1 - 5s^2} \frac{t^{1/2}}{\sqrt{\pi}} + \frac{12s}{\pi(1 - 5s)} t + p_s(\log t)t^d + O(t^{3/2}(\log(t^{-1}))^{3/2}).
\]

Proof. Define

Substitute (58) into (60) to obtain

From (60), we also have

which implies

or equivalently,

Similarly to \([6]\) Proposition 5], define \( p_s(\log t) \) by

Then

hence

Since there is a constant \( \hat{C} > 0 \) such that \( |h(t)| \leq \hat{C}t^{3/2}(\log(t^{-1}))^{3/2} \) for \( t \leq s^2 \), there is a constant \( \tilde{C} > 0 \) such that

\[
\sum_{j=1}^{\infty} h(t s^2)(t s^2)^{-d} \leq \tilde{C}t^{(3/2)-d} \left( (\log(t^{-1}))^{3/2} \sum_{j=1}^{\infty} (s^{3-2d})^j + (-\log s)^{3/2} \sum_{j=1}^{\infty} (2j)^{3/2}(s^{3-2d})^j \right)
\]

(62)

Combining (60) with (61) and (62) gives (59). \( \square \)

Lemma 16. Let \( s = \frac{1}{3} \). Then there exists a periodic, continuous function \( p_{\frac{1}{3}} : \mathbb{R} \rightarrow \mathbb{R} \) with period \( \log 25 \) such that as \( t \downarrow 0 \),

\[
E(t) = \frac{1}{120} - \frac{6}{25} \frac{t^{1/2}}{\sqrt{\pi}} - \frac{6}{5\pi} t \log t + \frac{12}{5\pi} t + p_{\frac{1}{3}}(\log t)t + O(t^{3/2}(\log(t^{-1}))^{3/2}).
\]
Proof. For $s = \frac{1}{5}$, $d = 1$. Define

$$q_\pi(t) = E(t) - \frac{1}{120} \left( \frac{t^{1/2}}{25 \sqrt{\pi}} \right) + \frac{6}{5 \pi \log 5} t \log t - \frac{12}{5 \pi}. \tag{63}$$

Substitute (58) with $s = \frac{1}{5}$ into (63) to obtain

$$q_\pi(t) = \frac{1}{25} E(25t) - \frac{1}{3000} \left( \frac{t^{1/2}}{25 \sqrt{\pi}} \right) + \frac{6}{5 \pi \log 5} t \log t + h(t).$$

By considering (63) with $t$ replaced by $25t$, we obtain that

$$q_\pi(25t) = q_\pi(t) - h(t),$$

or equivalently,

$$q_\pi(t) = q_\pi(25t) + h(t)t^{-1}. \tag{64}$$

Define $p_\pi(\log t)$ by

$$p_\pi(\log t) = p_\pi(\log 25t) - \sum_{j=1}^{\infty} h(t(25)^{-j})(t(25)^{-j})^{-1}. \tag{65}$$

Then, as in the proof of Lemma 15, we have $p_\pi(\log t) = p_\pi(\log 25t)$ via (64). Since (62) holds for $s = \frac{1}{5}$, we obtain the same remainder estimate as in Lemma 15. □

Combining Lemma 21, 57 and Lemma 15, Lemma 16 we obtain (16), (17) respectively.

Using the fact that $t \mapsto H_{D_t}(t)$ is continuous, it can be shown that $t \mapsto p_\pi(\log t)$ is continuous for $0 < s < \sqrt{2} - 1$, see [6] Section 5.1.

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