On universal norms and the first layers of \( \mathbb{Z}_p \)-extensions of a number field

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Abstract For an odd prime \( p \) and a number field \( F \) containing a primitive \( p \)-th root of unity, we describe the Kummer radical \( A_F \) of the first layers of all the \( \mathbb{Z}_p \)-extensions of \( F \) in terms of universal norms of \( p \)-units along the cyclotomic tower of \( F \). We also study “twisted” radicals related to \( A_F \).

Introduction

Let \( p \) be an odd prime number and \( F \) be a number field containing a primitive \( p \)-th root of unity \( \zeta_p \). We denote by \( A_F \) the Kummer radical of the first layers of all the \( \mathbb{Z}_p \)-extensions of \( F \). Precisely \( A_F \) is the subgroup of \( F^*/F^{*p} \) consisting of classes \( a \mod F^{*p} \) such that the Kummer extension \( F(\sqrt[p]{a}) \) is contained in a \( \mathbb{Z}_p \)-extension of \( F \). The determination of the group \( A_F \) is an old problem which dates back to the beginnings of Iwasawa theory and has since been tackled by many authors. Here we present what we hope to be a satisfactory Iwasawa theoretical solution. In order to give meaning to this assertion, it is appropriate to recall that most of the (numerous) results obtained so far bring the study of \( A_F \) back to that of its orthogonal complement for the Kummer pairing, namely \( \text{tor}_{\mathbb{Z}_p} X_F / p \), where \( X_F \) denotes the Galois group of the maximal abelian pro-\( p \)-extension of \( F \) which is unramified outside \( p \)-adic primes.

Without pretending to be exhaustive, let us cite the following articles: [4], which uses the idelic description of \( X_F \) to compute \( \text{tor}_{\mathbb{Z}_p} X \) for certain quadratic fields;

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[9], which constructs from Artin symbols a “logarithm” map on $\mathcal{X}_F$ whose kernel is precisely the $\mathbb{Z}_p$-torsion $\text{tor}_{\mathbb{Z}_p} \mathcal{X}_F$; [13, 30], which approach $\mathcal{A}_F$ by a “dévissage” of $\text{tor}_{\mathbb{Z}_p} \mathcal{X}_F$ in a local-global perspective . . . From a cohomological point of view, and under Leopoldt’s conjecture, $\text{tor}_{\mathbb{Z}_p} \mathcal{X}_F$ is a “twisted dual” (see for example [24] and the references therein) of the $p$-primary part of the tame kernel of $K_2 F$ and a question raised by Coates [6] was whether $\mathcal{A}_F$ coincides with the Tate kernel of $F$, i.e. the subgroup $T_F$ of $F^* / F^* F_{\text{p}}$ consisting of classes $a \mod F^* F_{\text{p}}$ such that the symbol $\{ \pi_p, a \} = 0$ in $K_2 F$. In this direction and still without claim of exhaustiveness, let us cite the following articles: [21], which establishes a “wrong duality” between the elements of order $p$ in $\mathcal{X}_F$ and the quotient mod $p$ of the tame kernel $R_2 F$; [11], which performs Iwasawa descent on the “twisted duals” of the free part of the Galois group $\mathcal{X}_{F, \infty}$, where $F_{\infty}$ is the cyclotomic $\mathbb{Z}_p$-extension of $F$.

All these approaches, by class field theory or by $K$-theory, produce effective methods allowing to compute $\mathcal{A}_F$ from arithmetical parameters attached to $F$, such as the class group or the group of units of the field. But these descriptions of $\mathcal{A}_F$ can not be considered as theoretically complete when they lead to other arithmetical objects such as $\text{tor}_{\mathbb{Z}_p} \mathcal{X}_F$ or $T_F$ which are not necessarily better known than $\mathcal{A}_F$ itself. Thus $\text{tor}_{\mathbb{Z}_p} \mathcal{X}_F$ is, by Iwasawa theory, linked to the $p$-adic $L$-functions and its “local-global dévissage” [13, 24, 30] leads to an Iwasawa module of descent closely related to the class group. In the same way, the “local-global dévissage” of the Tate kernel $T_F$ eventually leads to the wild kernel, which is in turn isomorphic to a twisted version of the preceding Iwasawa module of descent. In addition, the intervention of an Iwasawa descent suggests that a satisfactory theoretical description of the Kummer radical $\mathcal{A}_F$ should include in some way an asymptotical ingredient. This is confirmed by Greenberg’s answer to the question of Coates [11, page 1242] : though on the ground level $\mathcal{A}_F$ and $T_F$ are not the same in general, they coincide when going sufficiently far up the cyclotomic tower. In short, a satisfactory (in our sense) description of $\mathcal{A}_F$ should take into account the following two remarks:

- the parameters which intervene should be arithmetically or at least effectively accessible
- the answer should be allowed to incorporate asymptotical ingredients, i.e. coming from high enough $F_n$ (for an explicit or computable $n$).

In this paper, we introduce as parameters some accessible norm subgroups of the pro-$p$-completion $\hat{U}_F$ of the group of $p$-units of $F$, more precisely, the subgroup $\hat{U}_F$ of (global) universal norms in the cyclotomic $\mathbb{Z}_p$-extension $F_{\infty} / F$, as well as the subgroup $\hat{U}_F$ of those which are locally universal norms in the local cyclotomic $\mathbb{Z}_p$-extensions $F_{v, \infty} / F_v$ at all $p$-adic primes $v$ (hence at all finite primes): $\hat{U}_F \subset \bar{U}_F \subset \hat{U}_F$. It is known ([18, 23, 27]) that every element of $\hat{U}_F$ starts a $\mathbb{Z}_p$-extension, i.e. $\hat{U}_F F^* F / F^* F_{\text{p}} \subset \mathcal{A}_F$. Our goal is to compare $\mathcal{A}_F$ with various radicals derived from $\hat{U}_F$ and $\hat{U}_F$.

As a first step, we bound the Kummer radical $\mathcal{A}_F$ “from below” by the radicals $\hat{U}_F F^* F / F^* F_{\text{p}}$ and $\mathcal{A}_F \cap (\hat{U}_F / p)$ and describe the deviations in terms of some asymptotic capitulation kernels (Corollary 2.5 and Proposition 2.7). Then, we give an “upper bound” for $\mathcal{A}_F$ in terms of the fixed points $(\hat{U}_F / p)^{G_n}$ for $n \gg 0$ (but accessible) with $G_n = \text{Gal}(F_n / F)$, and describe the deviation between them (Theorem 3.1). Finally,
we introduce a radical $B_F$ defined by conditions of $\mathbb{Z}_p$-embeddability locally everywhere, which contains the previous radicals $A_F$, $T_F$ and $\hat{U}_F/p$, and we determine the three respective quotients in Iwasawa theoretical terms. Since the modules $\hat{U}_{Fn}$ (resp. the capitulation kernels) are immediately (resp. asymptotically and effectively) accessible, our goal has been reached.

Although our approach has a theoretical orientation, it lends itself to effective or algorithmic calculations, as will be shown in the examples of Sect. 4, where we shall study the interrelationship between the three radicals $A_F$, $T_F$ and $\hat{U}_F/p$ for $p = 3$ and biquadratic fields of the form $F = \mathbb{Q}(\mu_3, \sqrt{d})$.

We shall always use the following general notations. If $n$ is a non-negative integer and $A$ is an abelian group, we denote by $A[n]$ the kernel of multiplication by $n$, and by $A/n$ the cokernel. For a prime number $p$, we denote by $A\{p\}$ the $p$-primary part of $A$. Also $Div(A)$ will denote the maximal divisible subgroup of $A$ and $A/\text{Div}(A)$ is simply written $A/\text{Div}$. If $M$ is a module over a ring $R$, $\text{tor}_R(M)$ is the $R$-torsion sub-module of $M$, and $\text{fr}_R(M) := M/\text{tor}_R(M)$. We always use the notation $(-)^* = \text{Hom}(-, \mathbb{Q}_p/\mathbb{Z}_p)$ for the Pontrjagin dual. For any number field $K$, we denote by $G_S(K)$ the Galois group over $K$ of the maximal extension of $K$ unramified outside places above our fixed odd prime $p$ and by $X_K := G_S(K)_{\text{ab}} \otimes \mathbb{Z}_p$ the Galois group over $K$ of the maximal abelian pro-$p$-extension of $K$ unramified outside $p$.

Finally, the notation $(-)'$ for a group indicates that we have factored out the $p$-primary roots of unity.

1 Norm subgroups and a radical at infinite level

In this section, we recall (and prove if necessary) a number of results, which are fragmented and more or less well-known, concerning some norm subgroups of $(p)$-units and the Kummer radical of an Iwasawa module related to our problem. They will not all be needed in the sequel, but we will give as complete an account as possible, relying essentially on theorems of Kuz’min [22]. If pressed for time, the reader can skip this section, coming back to it if necessary.

1.1 Global and local universal norms

Let $U_F$ be the group of $(p)$-units in our number field $F$. Its pro-$p$-completion, denoted by $\hat{U}_F$, is $U_F \otimes \mathbb{Z}_p$ since $U_F$ is finitely generated. Along the cyclotomic tower $F_\infty = \bigcup F_n$, we simply write $\hat{U}_n$ instead of $\hat{U}_{F_n}$ and put $\hat{U}_\infty = \lim \leftarrow U_n$ for norm maps. The $\mathbb{Z}_p$-torsion of $\hat{U}_n$ is the group $\mu_p(\mathbb{Z}_p) := \mu_p \cap F_n$ of $p$-primary roots of unity contained in $F_n$. By adding an apostrophe we indicate the $\mathbb{Z}_p$-free part of our modules:

$$\hat{U}'_n := \hat{U}_n/\mu_p(\mathbb{Z}_p).$$

Let $\Gamma := \text{Gal}(F_\infty/F)$ and $\Lambda := \mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[T]]$ be the Iwasawa algebra where the isomorphism is obtained by mapping a fixed topological generator $\gamma$ of $\Gamma$ to $1 + T$. Then both $\hat{U}_\infty$ and $\hat{U}'_\infty := \lim \leftarrow U'_n$ are endowed with a natural action of
The following result, due to Kuz’min [22], gives their $\Lambda$-structure. It is classically known.

**Proposition 1.1** $\tilde{U}_\infty^\prime$ is $\Lambda$-free of rank $r_1 + r_2$. When $F$ contains $p$-th roots of unity $\mu_p$, then we have a $\Lambda$-module isomorphism $\tilde{U}^\prime_\infty \simeq \mathbb{Z}_p(1) \oplus \tilde{U}^\prime_\infty$.

**Definition 1.2** The universal norm subgroup of $\tilde{U}_F$ (resp. of $\tilde{U}^\prime_F$) is the intersection $\cap_{n \geq 0} N_n(\tilde{U}_F)$, denoted by $\tilde{U}_F$ (resp. of $\cap_{n \geq 0} N_n(\tilde{U}^\prime_F)$, denoted by $\tilde{U}^\prime_F$). Here $N_n$ is the norm map in $F_n/F$.

The usual compactness argument shows that $\tilde{U}_F$ is the image of the natural map

$$ (\tilde{U}_\infty)_\Gamma \rightarrow \tilde{U}_F. $$

Furthermore, this co-descent morphism is injective [22, Theorem 7.3] so that $\tilde{U}_F \simeq (\tilde{U}_\infty)_\Gamma$ is of $\mathbb{Z}_p$-rank equal to $r_1 + r_2$. Similarly, $\tilde{U}^\prime_F$ is isomorphic to $(\tilde{U}^\prime_\infty)_\Gamma$ and is of $\mathbb{Z}_p$-rank $r_1 + r_2$.

We are now going to define a second norm subgroup of $\tilde{U}_F$. For a prime $v$, let $F_v$ be the completion of $F$ at $v$. Let $\tilde{F}_v^\bullet := \lim_{\leftarrow} F_v^\bullet/F_v^{\prime p}$ denote the pro-$p$-completion of $F_v^\bullet$, and $\tilde{F}_v^\bullet$ denote the group of universal norms in the cyclotomic $\mathbb{Z}_p$-extension of $F_v$ (the local analogue of Definition 1.2). Let $X_\infty := \text{Gal}(L_\infty/F_\infty)$, where $L_\infty$ is the maximal unramified abelian pro-$p$-extension of $F_\infty$ in which every prime over $p$ (hence every prime) splits. We have the following Sinnott exact sequence (see [7,16,18, etc]):

$$ \tilde{U}_F \xrightarrow{g_F} \bigoplus_{v|p} \tilde{F}_v^\bullet/F_v^\bullet \xrightarrow{\text{Artin}} (X_\infty)_\Gamma \rightarrow A_F \quad (\text{Sinnott}), $$

where $g_F$ is naturally deduced from the diagonal map taking its values in the module $\bigoplus_{v|p} \tilde{F}_v^\bullet/F_v^\bullet$, which is defined as follows: let $F_{\infty,v}$ be the cyclotomic $\mathbb{Z}_p$-extension of $F_v$; then, by local class field theory, each $\tilde{F}_v^\bullet/F_v^\bullet$ is isomorphic to the Galois group $\text{Gal}(F_{\infty,v}/F_v)$ which can be considered as a subgroup of $\text{Gal}(F_{\infty}/F)$; now $\bigoplus_{v|p} \tilde{F}_v^\bullet/F_v^\bullet$ is the kernel of the natural map

$$ \bigoplus_{v|p} \tilde{F}_v^\bullet/F_v^\bullet \simeq \bigoplus_{v|p} \text{Gal}(F_{\infty,v}/F_v) \rightarrow \text{Gal}(F_{\infty}/F) $$

obtained by taking the product of the components.

**Definition 1.3** The everywhere local universal norm subgroup $\hat{U}_F$ of $\tilde{U}_F$ consists of those elements which are locally universal norms in the cyclotomic $\mathbb{Z}_p$-extensions $F_{\infty,v}/F_v$ for all finite primes $v$ in $F$. Since for a non-$p$-adic prime $v$, the cyclotomic $\mathbb{Z}_p$-extension $F_v$ is unramified, the group of universal norms in $F_{\infty,v}/F_v$ is precisely the group of units of $F_v$. Therefore

$$ \hat{U}_F = \text{Ker } g_F. $$

We obviously have the following inclusions $\hat{U}_F \subset \tilde{U}_F \subset \hat{U}_F$, as well as $\hat{U}_F \subset \tilde{U}_F \subset \hat{U}_F$, where an apostrophe indicates factoring out the $\mathbb{Z}_p$-torsion subgroup $\mu_{p\infty}(F)$.
Denote by $\bar{N}_v : \bar{F}_v^* \to \bar{\mathbb{Q}}_p^*$ the norm map associated to the Galois group $\text{Gal}(F_v/\mathbb{Q}_p)$. The next lemma gives a characterization of the Gross kernel $\hat{U}_F$ (see [18, Section 1]).

**Lemma 1.4** For an element $x \in \tilde{U}_F$, we have:

$$x \in \hat{U}_F \iff \bar{N}_v(x) \in p\mathbb{Z}_p \quad \forall v | p.$$  

This lemma implies that $\hat{U}_F$ is “accessible” in the sense of the introduction. According to the above Sinnott exact sequence, the $\mathbb{Z}_p$-rank of $\hat{U}_F$ is equal to $r_1 + r_2 + \delta$, where $\delta := rk_{\mathbb{Z}_p}(X_\infty)$. The number field $F$ satisfies Gross’ (generalized) Conjecture at $p$ if $\delta = 0$, namely if $(X_\infty)_{\Gamma}$ (or equivalently $X_{\Gamma}$, because $X_\infty$ is $\Lambda$-torsion) is finite. Gross’ Conjecture is known to hold for abelian extensions of $\mathbb{Q}$ [10,16]. In the inclusion tower $\tilde{U}_F \subset \hat{U}_F \subset \bar{U}_F$, we have

$$\bar{U}_F/\hat{U}_F \simeq \text{im } g_F \simeq \mathbb{Z}_p^{s-1-\delta},$$

where $s$ is the number of $p$-adic primes of $F$. Concerning the deviation between $\hat{U}_F$ and $\tilde{U}_F$ we have the following local-global result:

**Proposition 1.5** ([22, Proposition 7.5]) There exists a canonical exact sequence

$$0 \to \tilde{U}_F \to \hat{U}_F \to X_{\Gamma} \to 0.$$

Kuz’min’s proof is class field theoretic. For a proof with a more Iwasawa-theoretic flavour, see [20, Theorem 3.3]. See also [17, Section 3.2] for a cohomological proof. Our interest in $\tilde{U}_F$, as we recalled in the introduction, lies in the fact that, in the Kummerian situation, every element of $\tilde{U}_F$ starts a $\mathbb{Z}_p$-extension. We give a quick proof for the convenience of the reader:

**Lemma 1.6** When $\mu_p \subset F$, the Kummer radical $A_F$ contains $\tilde{U}_F F^p/F^p$.

**Proof** Recall Kuz’mín’s result that the natural codescent map gives an isomorphism $(\tilde{U}_\infty)_{\Gamma} \simeq \tilde{U}_F \subset \hat{U}_F \simeq H^1 \left( G_S(F), \mathbb{Z}_p(1) \right)$. On the other hand, codescent on the $(-1)$-twist gives a homomorphism $\tilde{U}_\infty(-1)_{\Gamma} \to \text{Hom}(G_S(F), \mathbb{Z}_p)$ [see [8, Theorem 3.7], where $NB(R_F, \mathbb{Z}_p) \simeq \tilde{U}_\infty(-1)_{\Gamma}$] and $G(R_F, \mathbb{Z}_p) \simeq \text{Hom} \left( G_S(F), \mathbb{Z}_p \right)$. Hence a composite homomorphism

$$\tilde{U}_\infty(-1)_{\Gamma}/p \to \text{Hom}(G_S(F), \mathbb{Z}_p)/p \xrightarrow{\text{nat}} \text{Hom} \left( G_S(F), \mathbb{Z}/p \right).$$

Since $\tilde{U}_\infty(-1)_{\Gamma}/p = (\tilde{U}_\infty)_{\Gamma}/p(-1) \simeq \tilde{U}_F/p(-1)$, it follows from Proposition 2.3, op. cit. that the induced map

$$\left( \tilde{U}_F/p \right)(-1) \longrightarrow \text{Hom}(G_S(F), \mathbb{Z}/p) = \text{Hom} \left( G_S(F), \mu_p \right)(-1)$$

is just the $(-1)$-twist of $\tilde{U}_F/p \longrightarrow \tilde{U}_F/p$. As it factors through $\text{Hom} \left( G_S(F), \mathbb{Z}_p \right)/p(-1)$, the proof is complete. □
In the above lemma, $\tilde{U}_p F^{\bullet p} / F^{\bullet p}$ is the Kummer radical of the first layers of $\mathbb{Z}_p$-extensions of a particular type. One can show from [8, Theorem 2.4] that they are the $\mathbb{Z}_p$-extensions $K_\infty = \bigcup_{n\geq 0} K_n$ of $F$, such that the rings of $p$-integers of all the $K_n$’s have normal bases which are coherent for the trace.

1.2 A Kummer radical at infinite level

Let $\mathcal{X}_\infty$ be the Galois group over $F_\infty$ of the maximal abelian $p$-extension of $F_\infty$ which is unramified outside $p$-adic primes. It is known that the $\Lambda$-rank of $\mathcal{X}_\infty$ is equal to $r_2$ (this is the weak Leopoldt conjecture, which holds in the case of the cyclotomic $\mathbb{Z}_p$-extension [33, Section 13.5]). Put $\text{fr}_\Lambda(\mathcal{X}_\infty)$ for its torsion-free part. When $\mu_p \subset F$, the Kummer radical of $\text{fr}_\Lambda(\mathcal{X}_\infty)$ is clearly related to the problem we are interested in. The determination of this Kummer radical has been performed independently by Kuz’min [22] and Kolster [18] using idelic methods. Here, we need a slightly sharper version of their result that we are going to prove using a direct approach.

First fix the following notations inside the “universal kummer radical” $\mathcal{K} = F_\infty^{\bullet} \otimes \mathbb{Q}_p / \mathbb{Z}_p$: $\mathcal{L} \cong \text{Hom}(\text{fr}_\Lambda(\mathcal{X}_\infty), \mu_{p^\infty})$, $\mathfrak{H} = \varprojlim (\bar{U}_n \otimes \mathbb{Q}_p / \mathbb{Z}_p)$, $\mathfrak{N} = \varprojlim (\hat{U}_n \otimes \mathbb{Q}_p / \mathbb{Z}_p)$ and $\mathfrak{F} = \varprojlim (\tilde{U}_n \otimes \mathbb{Q}_p / \mathbb{Z}_p)$.

For all $n \geq 0$, define $V_n$ to fit into the following tautological short exact sequence

$$0 \to \tilde{U}_n \to \tilde{U}_n \to V_n \to 0.$$

We then have a commutative diagram

$$
\begin{array}{cccccc}
0 & \to & \tilde{U}_n & \to & \tilde{U}_n & \to & X_{\infty}^{\Gamma_n} & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \tilde{U}_n & \to & \tilde{U}_n & \to & V_n & \to & 0 \\
& & & & & & \downarrow & & \\
& & & & & & \oplus_{v|p} \tilde{F}_{n,v}^{\bullet} / \tilde{F}_{n,v}^{\bullet} \cong \mathbb{Z}_p^{s_n} & & \\
& & & & & & \downarrow & & \\
& & & & & & (X_\infty)^{\Gamma_n} & & \\
& & & & & & \downarrow & & \\
& & & & & & A_n & & \\
\end{array}
$$

where $A_n$ is the $p$-primary part of the $(p)$-class group of the layer $F_n$ and where $s_n$ is the number of $p$-adic primes in $F_n$. This immediately provides a short exact sequence

$$0 \to X_{\infty}^{\Gamma_n} \to V_n \to \mathbb{Z}_p^{s_n-1-\delta_n} \to 0$$

where $\delta_n := rk_{\mathbb{Z}_p}(X_\infty)^{\Gamma_n}$. In particular $\text{tor}_{\mathbb{Z}_p}(V_n) \cong \text{tor}_{\mathbb{Z}_p}(X_\infty^{\Gamma_n})$. 

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Note that \( \lim_{\to} \text{tor} \mathbb{Z}_p(X_{\infty}^{\Gamma_n} = X^\circ \) is the maximal finite submodule of \( X_{\infty} \) and the groups \( X_{\infty}^{\Gamma_n} \otimes \mathbb{Q}_p/\mathbb{Z}_p \) stabilize. Put \( \Delta_{\infty} := \lim(X_{\infty}^{\Gamma_n} \otimes \mathbb{Q}_p/\mathbb{Z}_p) \) for the “Gross asymptotical defect”.

**Proposition 1.7** ([18,22]) We have an exact sequence of \( \Gamma \)-modules

\[
0 \longrightarrow X^\circ \longrightarrow \hat{\mathfrak{N}} \longrightarrow \hat{\mathfrak{N}} \longrightarrow \Delta_{\infty} \longrightarrow 0
\]

In particular, \( \mathfrak{L} = \hat{\mathfrak{N}} \) precisely when all the layers \( F_n \) verify Gross’ conjecture.

*Proof* Tensoring with \( \mathbb{Q}_p/\mathbb{Z}_p \) the exact sequence \( 0 \to \tilde{U}_n \to \hat{U}_n \to X_{\infty}^{\Gamma_n} \to 0 \) of Proposition 1.5, we get:

\[
0 \longrightarrow \text{tor} \mathbb{Z}_p(X_{\infty}^{\Gamma_n} = \tilde{U}_n \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \hat{U}_n \otimes \mathbb{Q}_p/\mathbb{Z}_p \to X_{\infty}^{\Gamma_n} \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0.
\]

Hence, passing to the direct limit over \( n \) yields

\[
0 \to X^\circ \to \hat{\mathfrak{N}} \to \Delta_{\infty} \to 0.
\]

Now it remains to show that the cokernel of the first map \( X^\circ \to \hat{\mathfrak{N}} \) is precisely \( \mathfrak{L} \). Proceeding in the same way as before from the exact sequence

\[
0 \to \tilde{U}_n \to \hat{U}_n \to V_n \to 0
\]

we get:

\[
0 \to X^\circ \to \hat{\mathfrak{N}} \to \mathfrak{N} \to V := \lim(V_n \otimes \mathbb{Q}_p/\mathbb{Z}_p) \to 0
\]

which we split into two short exact sequences:

\[
0 \to X^\circ \to \hat{\mathfrak{N}} \to \mathfrak{L}' \to 0 \quad \text{and} \quad 0 \to \mathfrak{L}' \to \mathfrak{N} \to V \to 0.
\]

Write the Kummer dual of the first sequence:

\[
0 \to \text{Hom}(\mathfrak{L}', \mu_p) \to \text{Hom}(\hat{\mathfrak{N}}, \mu_p) \to \text{Hom}(X^\circ, \mu_p) \to 0.
\]

By definition, \( \tilde{U}_n = (\tilde{U}_{\infty})_{\Gamma_n} \). In view of the properties of \( \tilde{U}_{\infty} \) (Proposition 1.1), the module \( \hat{\mathfrak{N}} \) is a co-free module over \( \Lambda \) of co-rank \( r_2 \); \( \text{Hom}(\hat{\mathfrak{N}}, \mu_p) \cong \Lambda^{r_2} \). Therefore \( \text{Hom}(\mathfrak{L}', \mu_p) \) is \( \Lambda \)-torsion-free of the same rank \( r_2 \). Now, the Kummer dual of the second sequence:

\[
0 \to \text{Hom}(\mathfrak{N}, \mu_p) \to \text{Hom}(\mathfrak{N}, \mu_p) \to \text{Hom}(\mathfrak{L}', \mu_p) \to 0
\]
shows that $\text{Hom}(\mathcal{L}', \mu_{p\infty})$ is a quotient module of $\text{Hom}(\mathcal{N}, \mu_{p\infty})$, and it is known that $\text{fr}_\Lambda(\mathcal{X}_{\infty})$ is the maximal $\Lambda$-torsion free quotient of the Galois group $\text{Hom}(\mathcal{N}, \mu_{p\infty})$ ([15, Theorem 15]). Accordingly, we have a surjective map $\text{fr}_\Lambda(\mathcal{X}_{\infty}) \to \text{Hom}(\mathcal{L}', \mu_{p\infty})$, which must be an isomorphism since they both have $\Lambda$-rank $r_2$. Hence $\mathcal{L}' = \mathcal{L}$, as was to be shown.

Remark The structure theorem for finitely generated $\Lambda$-modules shows the existence of a finite module $H$ such that

$$0 \to \text{fr}_\Lambda(\mathcal{X}_{\infty}) \to \Lambda^{r_2} \to H \to 0.$$  

The above proof provides such an exact sequence in a canonical way as well as an isomorphism $H \simeq \text{Hom}(X^\circ, \mu_{p\infty})$. This Kummer duality between $H$ and $X^\circ$ was already implicit in [15]. It is also known (op. cit.) that $X^\circ$ is isomorphic to the (asymptotical) capitulation kernel $\text{Ker}(A_m \to \lim_{\to} A_n)$ for $m$ large [11, page 1240].

2 Lower bounds for the Kummer radical $A_F$

In this section we suppose that $F$ contains $\mu_p$. In order to get information about the radical $A_F$, we are first going to do “descent” from the module $\mathcal{L}$ in the same way as in [11] (but we will need somewhat sharper results). Recall the notations: $\tilde{U}_F = \tilde{U}/\mu_{p\infty}(F), \hat{U}_F = \hat{U}/\mu_{p\infty}(F)$ and let $\hat{A}'_F := A_F/(\mu_{p\infty}(F)/p)$, where $\mu_{p\infty}(F)/p \simeq \mu_{p\infty}(F)F^\bullet p/F^\bullet p$ is the Kummer radical of the first layer of the cyclotomic $\mathbb{Z}_p$-extension of $F$.

2.1 A lower bound in terms of universal norms

The starting point will be the exact sequence

$$0 \to X^\circ \to \tilde{\mathcal{N}} \to \mathcal{L} = \mathcal{L}' \to 0,$$

of Proposition 1.7, where we recall that $\tilde{\mathcal{N}} = \lim_{\to}(\tilde{U}_n \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ and $\mathcal{L} \cong \text{Hom}(\text{fr}_\Lambda(\mathcal{X}_{\infty}), \mu_{p\infty}) \subseteq F^\bullet_\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p$.

For all rational integers $i$, we consider the $i$-fold Tate-twists:

$$0 \to X^\circ(i) \to \tilde{\mathcal{N}}(i) \to \mathcal{L}(i) \to 0.$$  \hspace{1cm} (1)

**Theorem 2.1** For $i \in \mathbb{Z}$, we have an exact sequence:

$$0 \to X^\circ[i](p)(i) \to \tilde{U}'_F/p(i) \to \text{Div}(\mathcal{L}(i)^\Gamma)(p) \to X^\circ(i)^\Gamma/p \to 0$$

where $\text{Div}(-)$ denotes the maximal divisible subgroup of $(-)$.

Notice that, since $\mu_p \subset F$, the twist $i$ outside is purely cosmetic for the Galois action above $F$, but of course not for the action below $F$. 

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Proof The exact sequence (1) provides us with:

\[ 0 \to X^\circ(i)^\Gamma \to \hat{\mathcal{N}}(i)^\Gamma \to \mathcal{L}(i)^\Gamma \to X^\circ(i)^\Gamma \to \hat{\mathcal{N}}(i)^\Gamma \to \cdots \]

Let \( N_i \) be the cokernel of the first map on the left: \( 0 \to X^\circ(i)^\Gamma \to \hat{\mathcal{N}}(i)^\Gamma \to N_i \to 0 \). As noticed before (Proposition 1.1), the \( \Lambda \)-module \( \hat{\mathcal{N}} \) is cofree. Hence \( \hat{\mathcal{N}}(i)^\Gamma \) is trivial whereas \( \mathcal{N}(i)^\Gamma \) is divisible and therefore \( N_i = \text{Div}(\mathcal{L}(i)^\Gamma) \). Consequently, we have:

\[ 0 \to X^\circ(i)^\Gamma \to \hat{\mathcal{N}}(i)^\Gamma \to \text{Div}(\mathcal{L}(i)^\Gamma) \to 0. \]

\[ \square \]

Applying the snake lemma to multiplication-by-\( p \) gives

\[ 0 \to X^\circ(i)^\Gamma[p] = X^\circ\Gamma[p](i) \to \hat{\mathcal{N}}(i)^\Gamma[p] \]

\[ = \hat{\mathcal{N}}^\Gamma[p](i) \to \text{Div}(\mathcal{L}(i)^\Gamma)[p] \to X^\circ(i)^\Gamma/p \to 0. \]

To finish the proof of the Theorem, it remains to recognize \( \hat{\mathcal{N}}^\Gamma \):

Lemma 2.2 \( \hat{U}'_F \otimes \mathbb{Q}_p/\mathbb{Z}_p \simeq \hat{\mathcal{N}}^\Gamma \).

Proof At finite levels, we have \( \hat{U}'_n \hookrightarrow \hat{U}'_{n+1} \) (this is immediate from Lemma 1.4). Hence we also have \( \hat{U}'_n \hookrightarrow \hat{U}'_{n+1} \). By Definition 1.2, the norm maps \( \hat{U}'_n \to \hat{U}'_F \) are surjective. On the other hand, by Proposition 1.1, all the \( \hat{U}'_n \) are free \( \mathbb{Z}_p \)-modules. Hence \( \hat{U}'_F \) is a direct summand of \( \hat{U}'_n \) and the cokernel of the natural injection \( \hat{U}'_F \to \hat{U}'_n \) is torsion-free. Accordingly, the maps \( \hat{U}'_F \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \hat{U}'_n \otimes \mathbb{Q}_p/\mathbb{Z}_p \) are injective and so is the map: \( \hat{U}'_F \otimes \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \hat{\mathcal{N}}^\Gamma \).

Moreover, as previously explained, \( \hat{\mathcal{N}}^\Gamma \) is divisible and to show the equality all we need to do is compare the \( \mathbb{Z}_p \)-coranks of these two modules: the exact sequence

\[ 0 \to X^\circ \Gamma \to \hat{\mathcal{N}}^\Gamma \to \text{Div}(\mathcal{L}^\Gamma) \to 0 \]

shows that \( \text{corank}_{\mathbb{Z}_p}(\hat{\mathcal{N}}^\Gamma) = r_2 \). On the other hand:

\[
\text{rank}_{\mathbb{Z}_p}(\hat{U}'_F) = \text{rank}_{\mathbb{Z}_p}(\hat{U}'_F) - \text{rank}_{\mathbb{Z}_p}(X_\infty)^\Gamma \quad (\text{Kuz'min})
\]

\[ = \text{rank}_{\mathbb{Z}_p}(\hat{U}'_F) - (s-1) + \text{rank}_{\mathbb{Z}_p}(X_\infty)^\Gamma - \text{rank}_{\mathbb{Z}_p}(X_\infty)^\Gamma \quad (\text{Sinnott})
\]

\[ = r_2 + \text{rank}_{\mathbb{Z}_p}(X_\infty)^\Gamma - \text{rank}_{\mathbb{Z}_p}(X_\infty)^\Gamma, \]

where \( s \) is the number of \( p \)-adic primes in \( F \). Besides, since \( X_\infty \) is a \( \Lambda \)-torsion module, denoting by \( Y_\infty \) the “non-\( \mu \)-part” of \( X_\infty \), we have a “pseudo-exact” sequence

\[ 0 \to (X_\infty)^\Gamma \to Y_\infty \xrightarrow{y^{-1}} Y_\infty \to (X_\infty)^\Gamma \to 0 \]

of finitely generated \( \mathbb{Z}_p \)-modules which shows that \( (X_\infty)^\Gamma \) and \( (X_\infty)^\Gamma \) have the same \( \mathbb{Z}_p \)-rank. Hence finally the \( \mathbb{Z}_p \)-corank of \( \hat{U}'_F \otimes \mathbb{Q}_p/\mathbb{Z}_p \) is also \( r_2 \). \( \square \)
We now propose to give interpretations of \( \text{Div}(\mathcal{L}(i)^\Gamma)[p] \) in terms of radicals in the three special cases \( i = -1, 0, 1 \). We have already introduced \( \mathcal{A}_F' := \mathcal{A}_F/(\mu_{p^\infty}(F)/p) \) and \( (\hat{\mathcal{U}}_F/p) = (\hat{\mathcal{U}}_F/p)/(\mu_{p^\infty}(F)/p) \), where \( \mu_{p^\infty}(F)/p \cong \mu_{p^\infty}(F)^*/F^*p \) is the image of \( \mu_{p^\infty} \) in \( F^*/F^*p \). We also want to introduce a modified Tate kernel \( T'_F \).

Using elementary properties of symbols, one easily shows that

\[
\hat{\mathcal{U}}_F = \mathcal{K}^\Gamma, \quad \mathcal{K} = F_\infty^* \otimes \mathbb{Q}_p/\mathbb{Z}_p \cong H^1(F_\infty, \mu_{p^\infty}).
\]

\( \square \)

Recall that if Gross’ conjecture is valid for all the \( F_n \)’s, then \( \mathcal{L} = \hat{\mathcal{N}} \).

**Proposition 2.3** We have the following equalities:

(a) if \( i = -1 \) and we assume Leopoldt’s conjecture for \( F \), then \( \text{Div}(\mathcal{L}(-1)^\Gamma)[p](1) = \mathcal{A}_F' \).

(b) if \( i = 0 \) and we assume Gross’ conjecture for \( F \), then \( \text{Div}(\mathcal{L}^\Gamma)[p] = \hat{U}_F'/p \).

(c) if \( i = 1 \), then \( \text{Div}(\mathcal{L}(1)^\Gamma)[p](-1) = T'_F \).

For \( i = -1 \) this is Leopoldt’s conjecture and for \( i = 1 \) a consequence of results of Tate on \( K_2 \) (see [5, Theorem 2]). Greenberg’s conjecture does not concern the twist \( i = 0 \), since \( \mathcal{K}^\Gamma \) is of infinite co-type. Hence the assertion (b) requires a special treatment:

**Lemma 2.4** Assuming Gross’ conjecture for \( F \), we have an exact sequence

\[
0 \to \hat{U}_F^\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathcal{L}^\Gamma \to (X^\circ)^\Gamma \to 0.
\]

In particular, \( \hat{U}_F^\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p = \text{Div}(\mathcal{L}^\Gamma) \).

**Proof** We have two short exact sequences (Proposition 1.7 and its proof)

\[
0 \to X^\circ \to \hat{\mathcal{N}} \to \mathcal{L} \to 0 \quad \text{and} \quad 0 \to \mathcal{L} \to \hat{\mathcal{N}} \to \Delta_\infty \to 0.
\]

From the first, we have already derived:

\[
0 \to (X^\circ)^\Gamma \to \hat{\mathcal{N}}^\Gamma \to \mathcal{L}^\Gamma \to (X^\circ)^\Gamma \to 0.
\]

Since \( \mathcal{L} \) is the Kummer dual of \( \text{fr}_A(X_\infty^\infty) \), we have \( \mathcal{L}^\Gamma = 0 \) so that the second sequence implies:

\[
0 \to \mathcal{L}^\Gamma \to \hat{\mathcal{N}}^\Gamma \to \Delta_\infty^\Gamma \to 0.
\]

We compare this exact sequence to what we have at the level of \( F \) which is provided by Proposition 1.5 after tensoring with \( \mathbb{Q}_p/\mathbb{Z}_p \):

\[
0 \to \text{tor}_{\mathbb{Z}_p}(X_\infty^\Gamma) \to \hat{U}_F \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \hat{U}_F'/\mathbb{Q}_p/\mathbb{Z}_p \to X_\infty^\Gamma \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0.
\]
Namely, we break the above exact sequence into two exact sequences to obtain a commutative diagram:

\[
\begin{array}{ccccccc}
0 & \rightarrow & (X^\circ)^\Gamma & \rightarrow & \tilde{\Omega}^\Gamma & \rightarrow & \Omega^\Gamma & \rightarrow & (X^\circ)^\Gamma & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \text{tor}_{\mathbb{Z}_p}(X^\Gamma_\infty) & \rightarrow & \tilde{U}'_F \otimes \mathbb{Q}_p/\mathbb{Z}_p & \rightarrow & M & \rightarrow & 0
\end{array}
\]

leading to

\[
0 \rightarrow M \rightarrow \Omega^\Gamma \rightarrow (X^\circ)^\Gamma \rightarrow 0.
\]

To finish the proof, observe that under the Gross conjecture for \( F \), the tensor product \( X^\Gamma_\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p \) vanishes since \( X^\Gamma_\infty \) is finite. Hence in fact \( M = \hat{U}'_F \otimes \mathbb{Q}_p/\mathbb{Z}_p \). \( \square \)

One could also prove the statements (a) and (c) in the above proposition along the same lines. We leave this to the reader as an exercise. Since we are specifically dealing with the case (a) of the above Proposition, let us restate it after twisting once “à la Tate”.

**Corollary 2.5** If \( F \) verifies Leopoldt’s conjecture at \( p \), then we have a short exact sequence

\[
0 \rightarrow \hat{U}'_F F^{*p}/F^{*p} \rightarrow A_F \rightarrow (X^\circ(-1)^\Gamma/p)(1) \rightarrow 0.
\]

In particular, \( A_F = \hat{U}'_F F^{*p}/F^{*p} \) precisely when \( X^\circ = 0 \).

**Proof** The case \( i = -1 \) of Theorem 2.1 yields:

\[
0 \rightarrow \hat{U}'_F F^{*p}/F^{*p} \rightarrow A'_F \rightarrow (X^\circ(-1)^\Gamma/p)(1) = (X_\infty(-1)^\Gamma/p)(1) \rightarrow 0
\]

(The last equality comes from Leopoldt’s conjecture, which is known to be equivalent to the finiteness of \( X_\infty(-1)^\Gamma \)). Since \( \hat{U}'_F F^{*p}/F^{*p} \) and \( A'_F \) are obtained by taking the quotients of \( \hat{U}'_F F^{*p}/F^{*p} \) and \( A_F \) by the same submodule \( \mu_{p\infty}(F)/p \), the exact sequence of the corollary follows. \( \square \)

**Example 2.6** Consider a number field \( F \) which satisfies Leopoldt’s conjecture together with the following two properties:

(i) \( X^\circ = (0) \) and (ii) \( F \) contains only one \( p \)-adic prime

(for a cyclotomic field \( F := \mathbb{Q}(\mu_{p'}) \), the property (i) is implied by Vandiver’s conjecture and (ii) is of course automatically satisfied). Then \( A_F = \hat{U}_F F^{*p}/F^{*p} \) according to Proposition 1.5 and Corollary 2.5, and \( \overline{U}_F = \hat{U}_F \) according to (ii). It follows that \( A_F = \overline{U}_F / p \).

**Remarks** (i) Lemma 2.4 was shown in [14] using a different approach.
(ii) Within the context of assertion (b), the exact sequence of Theorem 2.1 coincides with the one obtained by applying the snake lemma to multiplication-by-$p$ in Kuz’min’s exact sequence (Proposition 1.5).

(iii) For $i \neq 0$, the exact sequence of Theorem 2.1 comes also by applying the snake lemma to multiplication-by-$p$ in some descent exact sequences in Galois cohomology [20, Theorem 3.2bis].

(iv) For $i \neq 0$, the $\mathbb{F}_p$-modules $\text{Div}(\mathbb{L}(i)/\Gamma_1)[p]$ are the generalized Tate kernels studied in [1, 14, 19, 32] in connection with problems of capitulation. When $\mathbb{F} \supset \mu_{p^{e+1}}$, where $p^e$ is the exponent of $X^\circ$, then all the modules $\text{Div}(\mathbb{L}(i)/\Gamma_1)[p]$ are equal [1, 11] (see also Corollary 3.2 below). For a general comparison between these kernels when $i$ varies see [32, Theorem 2.7].

Under Leopoldt’s conjecture, the above Corollary 2.5 provides a good approximation of $\mathcal{A}_F$ by $\tilde{U}_F / p / F^* p$, whose dimension over $\mathbb{F}_p$ is $r_2 - h$, where $h := \text{dim } X^\circ \Gamma[p] = \text{dim } X^\circ (-1)^\Gamma[p] = \text{dim } (X^\circ(-1)^\Gamma / p)$. The difference $h$ is of an asymptotic nature and bounded in the cyclotomic tower. Interpreting $X^\circ$ as a capitulation kernel [11, 15], this parameter is theoretically and effectively accessible.

Nevertheless, the result is not entirely satisfactory as the $p$-units of $\tilde{U}_F / p$ are not immediately accessible. Kuz’min’s exact sequence (Proposition 1.5) suggests replacing $\tilde{U}_F$ by $\hat{U}_F$, which is easily accessible by Lemma 1.4.

2.2 A lower bound in terms of local universal norms

We want to “approximate” $\mathcal{A}_F$ by the intersection $\mathcal{A}_F \cap \tilde{U}_F / p$. In order to compute the deviation, let us come back to the exact sequence of Theorem 2.1, where the map $\sigma_i : \text{Div}(\mathbb{L}(i)/\Gamma_1)[p] \to X^\circ(i)/\Gamma_1 / p$ is given by the snake lemma and hence depends on the twist $i$. To compare the images of the $\sigma_i$’s, we must put them in a space which does not (at least for the action of Galois groups over $\mathbb{F}$) depend on $i$. From the exact sequence

$$0 \to X^\circ(i) / \Gamma_1 \to \mathfrak{N}(i) / \Gamma_1 \to \mathbb{L}(i)$$

(see Proposition 1.7), we derive

$$0 \to (X^\circ(i) / \Gamma_1)[p](i) \to \mathfrak{N}(i) / \Gamma_1[p](i) \to \mathbb{L}(i) / \Gamma_1[p](i).$$

Let $W$ be the cokernel of the map on the right, so that $W$ does not depend on $i$ (for the Galois action over $\mathbb{F}$) and we have a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (X^\circ(i) / \Gamma_1)[p](i) & \longrightarrow & \mathfrak{N}(i) / \Gamma_1[p](i) & \longrightarrow & \mathbb{L}(i) / \Gamma_1[p](i) & \longrightarrow & W & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \tau_i & & \\
0 & \longrightarrow & (X^\circ(i) / \Gamma_1)[p](i) & \longrightarrow & \mathfrak{N}(i) / \Gamma_1[p](i) & \longrightarrow & \text{Div}(\mathbb{L}(i)/\Gamma_1)[p] & \sigma_i & \longrightarrow & X^\circ(i) / \Gamma_1 / p & \longrightarrow & 0
\end{array}$$

where the right vertical map $\tau_i$ is defined tautologically and is injective. Finally, let $T_i := \tau_i(X^\circ(i) / \Gamma_1 / p)(-i)$, then with these notations we have

$\mathcal{A}_F$ Springer
Proposition 2.7 Suppose that Leopoldt’s and Gross’ conjecture are valid for F. Then we have the following exact sequence:

$$0 \rightarrow \mathcal{A}_F \cap \hat{U}_F / p \rightarrow \mathcal{A}_F \rightarrow T_{-1} / T_0 \cap T_{-1} \rightarrow 0.$$  

Proof Write $D_i := \text{Div } \mathcal{L}(i)^\Gamma[p]$ for short. Then $D_i(-i)$, for $i = 0, -1$, can be identified with the Kummer radicals in $F^*/F^{*p}$ of the Kummer extensions $F \left( \sqrt[p]{\hat{U}_F} \right)/F$ and $F \left( \sqrt[p]{\mathcal{A}_F} \right)/F$ respectively (obvious notations). According to Theorem 2.1, $\tau_i(X^\circ(i)^\Gamma/p)$ for $i = 0, -1$ can be identified with the Kummer radicals in $F \left( \sqrt[p]{\hat{U}_F} \right)$ of $F \left( \sqrt[p]{\mathcal{A}_F} \right)$ respectively (see the diagram). Hence the statement of the proposition by elementary Kummer theory.

Remark An analogous result holds when replacing the pair $(\mathcal{A}_F, \hat{U}_F/p)$ by any pair taken from $\left\{ \mathcal{A}_F, \hat{U}_F/p, T_F \right\}$ or by any pair $(D_i, D_j), i \neq j$. This should be compared with [32, Theorem 2.7] which states (in our notations) that $D_i/D_i \cap D_j \simeq p^t \Delta_{i,j}$, where $t$ is the maximal integer such that $i \equiv j \mod [F(\mu_{p^t}) : F]$ and $\Delta_{i,j} \subseteq H^2 \left( G_S(F), \mathbb{Z}_p(j) \right)$ is the image, by corestriction, of $H^2_w \left( G_S(F_{\infty}), \mathbb{Z}_p(j) \right)$.

We already noticed [Remark (v) following Corollary 2.5] that $\mathcal{A}_F$ and $\hat{U}_F/p$ coincide when $F$ contains enough $p$-primary roots of unity (see also Corollary 3.2 below). Proposition 2.7 shows that this is not the case in general, but their deviation, which is of an asymptotical nature, goes to zero when we go up the cyclotomic tower.

3 Upper bounds for the Kummer radical $\mathcal{A}_F$

We are going to give two “upper bounds” for $\mathcal{A}_F$. The first will be a “norm” radical which is accessible in the sense of the introduction. The second, via a local-global
3.1 Bounding from above by a norm radical

We keep the notations of the preceding sections.

**Theorem 3.1** Suppose that $F$ contains $\mu_p$ and the layers $F_n$’s verify Gross’ conjecture (i.e. $\Delta_\infty = 0$). Take $m$ large enough for $\Gamma_m$ to act trivially on $X^\circ$ and put $n = m + 1$. Then, for every $i \in \mathbb{Z}$, we have an exact sequence:

$$0 \to \text{Div}(\hat{\mathbb{N}}(i)^\Gamma)[p] \to (\hat{U}'_n/p)^{G_n}(i) \to X^\circ(i)^\Gamma[p] \to 0.$$  

In particular, if $F$ also satisfies Leopoldt’s conjecture, we have:

$$0 \to \mathcal{A}'_F \to (\hat{U}'_n/p)^{G_n} \to (X^\circ(-1))^{\Gamma}[p](1) \to 0,$$

where $G_n = \text{Gal}(F_n/F)$.

**Proof** Let us start from the exact sequence which appeared at the beginning of the proof of Theorem 2.1 (notice that by the hypothesis $\Delta_\infty = 0$, we have $\mathcal{L} = \hat{\mathbb{N}}$):

$$0 \to X^\circ(i)^\Gamma \to \hat{\mathbb{N}}(i)^\Gamma \to \hat{\mathbb{N}}(i)^\Gamma \to X^\circ(i)^\Gamma \to 0$$

from which we derive by the snake lemma applied to the $p$-th power map:

$$0 \to \text{Div}(\hat{\mathbb{N}}(i)^\Gamma)[p] \to \hat{\mathbb{N}}(i)^\Gamma[p] \to X^\circ(i)^\Gamma[p] \to 0.$$  

It remains to give an adequate expression of $\hat{\mathbb{N}}(i)^\Gamma[p]$. But $\hat{\mathbb{N}}(i)^\Gamma[p] = \hat{\mathbb{N}}^\Gamma[p](i)$ since $F$ contains $\mu_p$. Now, by Lemma 2.4, we have an exact sequence

$$0 \to \hat{U}'_n \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \hat{\mathbb{N}}^\Gamma_n \to (X^\circ)_\Gamma_n \to 0,$$

for all $n$. Hence also

$$0 \to \hat{U}'_F/p \to \hat{\mathbb{N}}^\Gamma_n[p] \to (X^\circ)_\Gamma_n[p] \to 0.$$  

Take the fixed points by $G_n := \text{Gal}(F_n/F)$ and compare with the level of $F$. By Lemma 1.4, the natural map $\hat{U}'_F/p \to (\hat{U}'_n/p)^{G_n}$ is injective, hence we have a commutative diagram:
Let $\omega_n := \gamma^n - 1$ where, recall that $\gamma$ is the fixed topological generator of $\Gamma$. The vertical map on the right (which corresponds to restriction) is multiplication by $\omega_n/\omega_0$. Choose $m$ such that $\Gamma_m$ acts trivially on $X^\circ$, and take $n = m + 1$, so that $\omega_n/\omega_m$ annihilates $(X^\circ)^\Gamma_m[p] = X^\circ[p]$. Then $\omega_n/\omega_m = \omega_n/\omega_m$ is the zero map on $(X^\circ)^\Gamma[p]$ and we also get an isomorphism $(\hat{\mathcal{U}}_n'/p)^G_n \simeq \hat{\mathcal{H}}^\Gamma[p].$

The above Theorem gives a satisfactory description of the radical $\mathcal{A}_F$ in the terms set out in the introduction. The module $(\hat{\mathcal{U}}_n'/p)^G_n$ is effectively accessible and the asymptotical deviation $(X^\circ(\lambda - 1))_\Gamma[p](1)$ is bounded in the cyclotomic tower.

For completeness, let us now reprore along the same lines a slightly improved version of a result of Greenberg, generalized by Vauclair:

**Corollary 3.2** ([11, page 1242], [31, Theorem 2.2]) Let $p^a$ be the exponent of $X^\circ$. Assume that $\mu_{p^a}(F) = \mu_{p^a}$ with $a \geq e$, and that the conditions of Theorem 3.1 hold. Then $\mathcal{A}_F = \hat{\mathcal{U}}_F/p$ if $a > e$. If $a = e$ and $\Gamma$ acts trivially on $X^\circ$, then $\mathcal{A}_F \neq \hat{\mathcal{U}}_F/p$

**Proof** According to the proof of Theorem 3.1, we have two exact sequences

$$0 \rightarrow \mathcal{A}_F^\prime \rightarrow \mathcal{L}(-1)^\Gamma[p](1) = \mathcal{L}^\Gamma[p] \rightarrow (X^\circ(-1))_\Gamma[p](1) \rightarrow 0$$

\[0 \rightarrow (\hat{\mathcal{U}}_F'/p) \rightarrow \hat{\mathcal{H}}^\Gamma[p] \rightarrow (X^\circ)^\Gamma[p] \rightarrow 0\]

which we need to link by vertical maps becoming equalities. These exact sequences were obtained by Kummer duality from

$$0 \rightarrow H(1)^\Gamma/p \xrightarrow{\sigma^1} ((fr_\Lambda \mathcal{X}_\infty(1))(1))_\Gamma/p \rightarrow fr_{\mathbb{Z}_p}((fr_\Lambda \mathcal{X}_\infty(1)))_\Gamma/p \rightarrow 0$$

$$0 \rightarrow H^\Gamma/p \xrightarrow{\sigma} (fr_\Lambda \mathcal{X}_\infty)_\Gamma/p \rightarrow fr_{\mathbb{Z}_p}((fr_\Lambda \mathcal{X}_\infty)_\Gamma/p \rightarrow 0$$

where the maps $\sigma$ and $\sigma_1$ originate from the snake lemma. More precisely the map $\sigma : H^\Gamma/p \rightarrow (fr_\Lambda \mathcal{X}_\infty)_\Gamma/p = (fr_\Lambda \mathcal{X}_\infty)/(\omega, p)$ (where $\omega = \gamma - 1$) is defined from the exact sequence

$$0 \rightarrow fr_\Lambda \mathcal{X}_\infty \rightarrow \Lambda^\Gamma \rightarrow H \rightarrow 0$$

in the following way: for $h \in H^\Gamma$, let $\lambda$ be any lift of $h$ in $\Lambda^\Gamma$. Then $\omega(\lambda) \in fr_\Lambda \mathcal{X}_\infty$ and $\sigma$ sends $h \mod p$ to $\omega(\lambda) \mod (\omega, p)$. Likewise, since $H^\Gamma = H(1)^\Gamma$ by hypothesis, we start with the same $h$ that we lift to the same $\lambda \in \Lambda^\Gamma$. Put $\omega^{(1)} := \kappa(\gamma)\gamma - 1$, where $\kappa$ is the cyclotomic character. Then $\omega^{(1)}(\lambda) \in fr_\Lambda \mathcal{X}_\infty$ and $\sigma_1$ sends $h \mod p$
to \( \omega(1)(\lambda) \mod (\omega(1), p) \). Now, \((\omega(1), p) = (\omega, p)\) and \((\omega(1) - \omega)(\lambda) \in p^{e+1} \Lambda \Gamma \) by hypothesis. Therefore \((\omega(1) - \omega)(\lambda) \in p \mathfrak{fr}_A \mathfrak{x}_\infty \) since \( p^e \) annihilates \( \mathcal{H} \). We then have the following commutative square

\[
\begin{array}{ccc}
H(1)^\Gamma / p & \xrightarrow{\sigma_1} & (\mathfrak{fr}_A \mathfrak{x}_\infty(1))^\Gamma / p \\
\| & = & \|
\end{array}
\]

which implies the equality \( \mathfrak{fr}_{Z_p} ((\mathfrak{fr}_A \mathfrak{x}_\infty(1))^\Gamma / p) = \mathfrak{fr}_{Z_p} ((\mathfrak{fr}_A \mathfrak{x}_\infty)^\Gamma / p) \), i.e. the equality \( A_F' = \hat{U}_F / p \). Hence \( A_F = \hat{U}_F / p \).

Suppose now that \( a = e \) and \( \Gamma \) acts trivially on \( X^\circ \). Then \( H(1)^\Gamma = H^1 = H \). Keeping the same notations, choose \( h \in H \) of maximal order \( p^e \). Then \((\omega(1), p) = (\omega, p)\) and \((\omega(1) - \omega)(\lambda) \in p^{e+1} \Lambda \Gamma \) and as before \((\omega(1) - \omega)(\lambda) \in \mathfrak{fr}_A \mathfrak{x}_\infty \). Putting \( \kappa(\gamma) = 1 + up^e \), with \( u \in \mathbb{Z}_p^* \), we have \( x := (\omega(1) - \omega)(\lambda) = up^e \gamma(\lambda) \in \mathfrak{fr}_A \mathfrak{x}_\infty \).

If \( x \in p \mathfrak{fr}_A \mathfrak{x}_\infty \), we would get \( p^{e-1} \gamma(\lambda) \in \mathfrak{fr}_A \mathfrak{x}_\infty \), contrary to the choice of \( h \). We have thus shown that \( \sigma_1(h) \neq \sigma(h) \) \( \Box \)

**Remark** Of course, an analogous result holds when replacing the pair \((A_F, \hat{U}_F / p)\) by any pair taken from \( \{A_F, \hat{U}_F / p, T_F\} \) or by any pair \((D_i, D_j)\) as in the remark following Proposition 2.7, satisfying Greenberg’s conjecture alluded to in the proof of Proposition 2.3. For a general exponent \( p^k \), see [31, Theorem 2.2].

### 3.2 Bounding from above by the Bertrandias–Payan radical

In this subsection, we introduce a certain Kummer radical \( B_F \) (see [24]) coming from global-local considerations concerning embeddability in cyclic \( p \)-extensions of arbitrary degrees. The radical \( B_F \) contains all the previous ones \( A_F, T_F \) and \( \hat{U}_F / p \). The determination of the respective quotients sheds additional light on the interrelationship between the three radicals themselves.

**Definition 3.3** A (necessarily cyclic) \( p \)-extension \( K/F \) is called infinitely embeddable (resp. \( \mathbb{Z}_p \)-embeddable) if it can be embedded in cyclic \( p \)-extensions of arbitrarily large degrees (resp. in a \( \mathbb{Z}_p \)-extension) of \( F \).

The compositum of all the infinitely embeddable \( p \)-extensions of \( F \) is called the field of Bertrandias–Payan \( F^{BP} \) (in reference to [3]). An infinitely embeddable extension is necessarily unramified outside \( p \)-adic primes: \( F^{BP} \) is contained in the maximal abelian pro-\( p \)-extension of \( F \) unramified outside the primes above \( p \). Hence the Galois group \( BP_F := \text{Gal}(F^{BP}/F) \) is a quotient of \( \mathfrak{x}_F \). Moreover \( F^{BP} \) obviously contains the compositum of all the \( \mathbb{Z}_p \)-extensions of \( F \) so that

\[
\mathfrak{fr}_{Z_p}(BP_F) = \mathfrak{fr}_{Z_p}(\mathfrak{x}_F).
\]

The following criterion is a consequence of class field theory (see [3]):

**Proposition 3.4** Assume that \( F \) contains a \( p \)-th root of unity \( \xi_p \) and let \( K := F(\sqrt[p]{a}) \), \( a \in F^* \) be a cyclic extension of degree \( p \). The following conditions are equivalent:

\( \Box \) Springer
(i) $F(\sqrt[p]{a})/F$ is infinitely embeddable;
(ii) $F_v(\sqrt[p]{a})/F_v$ is $\mathbb{Z}_p$-embeddable for all finite primes $v$;
(iii) $a \in F^*_v\hat{F}^*_v$ for all finite primes $v$ (note that $\hat{F}^*_v = U_v$ for $v \nmid p$);
(iv) $(a, \zeta_p)_v = 1$ for all finite primes $v$. Here $(\cdot, \cdot)_v$ stands for the maximal degree Hilbert symbol attached to the local field $F_v$;
(v) the symbol $(a, \zeta_p)$ belongs to the wild kernel $WK_2(F)$, i.e. the intersection in $K_2(F)$ of the kernels of all the Hilbert symbols.

**Definition 3.5** The radical of Bertrandias–Payan $\mathcal{B}_F$ is, by definition, the subgroup of $F^*/F^{*p}$ consisting of classes $a$ mod $F^{*p}$ such that $a$ verifies the preceding equivalent conditions.

**Remark** Clearly $\mathcal{B}_F$ contains $\mathcal{A}_F$, $\hat{U}_F/p$ and the Tate kernel $T_F$. Condition (i) means that $\mathcal{B}_F = \text{Hom}(B_P^F, \mu_p)$ and the quotient $\mathcal{B}_F/A_F$ measures the obstruction between “global $\mathbb{Z}_p$-embeddability” and “everywhere local $\mathbb{Z}_p$-embeddability”. Condition (iii) [resp. (iv), resp. (v)] says that $\mathcal{B}_F$ coincides with the radical denoted by $D_F^{(1)}$ (resp. $B/F^{*p}$, resp. $E/F^{*p}$) in [18] (resp. [11], resp. [21, section 3]).

**Theorem 3.6** Suppose that $F$ contains $\mu_p$. Then

(i) $\mathcal{B}_F/T_F \simeq X_{\infty}(1)\Gamma[p](-1)$
(ii) $\mathcal{B}_F/A_F \simeq X_{\infty}(-1)\Gamma[p](1)$
(iii) $\mathcal{B}_F/(\hat{U}_F/p) \simeq X_{\infty}(0)\Gamma[p](0)$.

**Proof** Idelic proofs of properties (i) and (iii) can be found in [18, pages 18 and 14]. Here we prove only property (ii). The exact sequence

$$0 \to \text{tor}_{\mathbb{Z}_p}(B_P^F) \to B_P^F \to \text{fr}_{\mathbb{Z}_p}(B_P^F) = \text{fr}_{\mathbb{Z}_p}(X_F) \to 0,$$

yields, by Kummer duality, an exact sequence

$$0 \to A_F \to B_F \to \text{Hom}(\text{tor}_{\mathbb{Z}_p} B_P^F, \mu_p) \to 0.$$

It remains to determine $\text{Hom}(\text{tor}_{\mathbb{Z}_p} B_P^F, \mu_p) \simeq (\text{tor}_{\mathbb{Z}_p} B_P^F/p)^*(1) \simeq (\text{tor}_{\mathbb{Z}_p} B_P^F)^*[p](1)$. We will follow the proof of Theorem 4.2 of [24] (but without appealing to Leopoldt’s conjecture). The equivalence between the first two parts of Proposition 3.4 shows that

$$(B_P^F)^* = \{ \chi \in X_F^*/\chi_v \in \text{Div}(X_v^*), \forall v \mid p \}.$$

Here $X_v$ is the Galois group over $F_v$ of the maximal abelian pro-$p$-extension of $F_v$ and $\chi_v$ is the character obtained by restricting $\chi$ to $X_v$. Then

$$(\text{tor}_{\mathbb{Z}_p} B_P^F)^* \simeq (B_P^F)^*/\text{Div}(X_F^*)$$

$$\simeq \text{Ker}(X_F^*/\text{Div} \to \oplus_{v \mid p} X_v^*/\text{Div})$$

$$\simeq \text{Ker}(H^1(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p)/\text{Div} \to \oplus_{v \mid p} H^1(F_v, \mathbb{Q}_p/\mathbb{Z}_p)/\text{Div}).$$
By a result of Tate in $p$-adic cohomology [29, Section 2], the boundary map induces an isomorphism $H^1(G_S(F), \mathbb{Q}/\mathbb{Z})/\text{Div} \cong \operatorname{tor}_{\mathbb{Z}_p} H^2(G_S(F), \mathbb{Z}_p)$ (and similarly for the local cohomology groups), hence

$$(\operatorname{tor}_{\mathbb{Z}_p} B P_{\mathcal{F}})^*[p] \cong \text{Ker} \left( H^2(G_S(F), \mathbb{Z}_p) \to \bigoplus_{v \mid p} H^2(F_v, \mathbb{Z}_p) \right)[p].$$

By Poitou-Tate’s duality, $\text{Ker} \left( H^2(G_S(F), \mathbb{Z}_p) \to \bigoplus_{v \mid p} H^2(F_v, \mathbb{Z}_p) \right)$ is dual to $\text{Ker} \left( H_2(G_S(F), \mathbb{Z}_p) \to \bigoplus_{v \mid p} H_2(F_v, \mathbb{Z}_p) \right)[p]$. Thus (ii) is proved. □

Here again, one could also prove the statements (i) and (iii) along the same lines and we leave this to the reader as an exercise. Also, compare (ii) with the main result of [27].

4 A case study

Although our approach is essentially geared towards theory, it lends itself to effective or algorithmic calculations, as will be shown in the examples below.

4.1 The case where $X^\circ$ is of order $p$

We continue to assume the two standard conjectures: Leopoldt’s for our base field $F$ and Gross’ for all the layers $F_n$. Assuming that $X^\circ$ is of order $p$, the exact sequence (2) in the proof of Theorem 3.1 yields:

$$0 \to X^\circ(i) \to \mathfrak{N}(i)^\Gamma[p] = \mathfrak{N}[p](i) \to \text{Div}(\mathfrak{N}(i)^\Gamma)[p] \to X^\circ(i) \to 0.$$ 

We are interested in $\text{Div}(\mathfrak{N}(i)^\Gamma)[p]$ for which the above exact sequence only provides a hyperplane. But in the exact sequence (2), the surjectivity of $\mathfrak{N}(i)^\Gamma \to \text{Div}(\mathfrak{N}(i)^\Gamma)$ shows that each $\beta_i \in \text{Div}(\mathfrak{N}(i)^\Gamma)[p]$ has a pre-image $\tilde{\beta}_i \in \mathfrak{N}(i)^\Gamma[p^2]$. In fact, we have a commutative diagram

$$0 \to X^\circ(i) \to \mathfrak{N}(i)^\Gamma[p] \to \text{Div}(\mathfrak{N}(i)^\Gamma)[p] \to X^\circ(i) \to 0$$

and we are going to devise an algorithm determining $D_i := f_i^{-1}\left( \text{Div}(\mathfrak{N}(i)^\Gamma)[p] \right)$.

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To this end, we notice first that
\[
\hat{\mathfrak{N}}(i)^\Gamma[p^2] = (\hat{\mathfrak{N}}(i)[p^2]^{\Gamma_1})^{G_1} = (\hat{\mathfrak{N}}^{\Gamma_1}[p^2](i))^{G_1} \simeq (\tilde{U}_1/p^2)(i)^{G_1} \quad \text{(Lemma 2.2)}
\]
where \( \Gamma_1 := \text{Gal}(F_\infty/F_1) \) and \( G_1 := \text{Gal}(F_1/F) \). Since \( \tilde{U}_1/p^2 \simeq (\tilde{U}_1)^{\Gamma_1}/p^2 \) is free as a module over \( \mathbb{Z}/p^2[G_1] \) (Proposition 1.1 and the injectivity of \( (U_\infty)^r \to \tilde{U}_F \) noticed in Sect. 1.1), the fixed points \( (\tilde{U}_1/p^2)(i)^{G_1} \) can be identified with \( v(i) \tilde{U}_1/p^2 \), where \( v \) is the norm (trace in additive notation) relative to \( G_1 \) and \( v(i) \) is its \( i \)-twist. The above diagram (3) then shows that \( \mathcal{D}_i \) consists of all the \( v(i)(\tilde{x}), \tilde{x} \in \tilde{U}_1/p^2 \) such that \( p \left( g_i(v(i)(\tilde{x})) \right) = 0 \), or, according to the third line of (3), such that \( p \left( v(i)(\tilde{x}) \right) \in X^0(i) \). Thus, we are left to express \( X^0(i) \) in this context. We suppose that \( \tilde{U}_F \) is known (by Lemma 1.4) hence also \( \tilde{U}_F' \) by Proposition 1.5. In fact, since they are both free \( \mathbb{Z}_p \)-modules with \( [\tilde{U}_F' : \tilde{U}_F] = p \), there exists a \( \mathbb{Z}_p \)-basis \( (e_1, e_2, \ldots) \) of \( \tilde{U}_F' \) such that \( (pe_1, e_2, \ldots) \) is a \( \mathbb{Z}_p \)-basis of \( \tilde{U}_F \). The first line of (3) shows immediately that \( X^0(i) \) is the cyclic group generated by the class (in \( \tilde{U}_F'/p \)) of \( pe_1 \), which can be written as \( pe_1 \otimes 1/p \in \hat{\mathfrak{N}}(i)^\Gamma[p] \).

Now, writing \( \tilde{x} \in \tilde{U}_1/p^2 \) as \( x \otimes 1/p^2 \in \hat{\mathfrak{N}}(i)^\Gamma[p^2], \) with \( x \in \tilde{U}_1' \), we conclude our algorithm:

**Proposition 4.1** The pre-image \( \mathcal{D}_i = f_i^{-1}(\text{Div}(\hat{\mathfrak{N}}(i)^\Gamma[p])) \) consists of the elements \( v(i)(x) \otimes 1/p^2 \in \hat{\mathfrak{N}}(i)^\Gamma[p^2], x \in \tilde{U}_1' \) such that \( v(i)(x) \otimes 1/p \) is collinear to \( pe_1 \otimes 1/p \).

**Remarks**

(i) The collinearity factor above is in \( \mathbb{Z}/p \). It is not zero precisely when \( v(i)(x) \otimes 1/p^2 \) is of order \( p^2 \).

(ii) For practical purposes, it is important to stress that the maps \( f_i \) and \( g_i \) in diagram (3) are induced by the natural map \( \hat{\mathfrak{N}} \to \hat{\mathfrak{N}} \).

(iii) Our algorithm can be extended to the general case (i.e. when \( X^0 \) is not necessarily of order \( p \)), but the calculations of course become heavier, because \( F_1 \) must be replaced by some higher level \( F_n \), and the preliminary determination of \( X^0 \) (considered for example as a capitulation kernel; see the remark at the end of Sect. 2) is not a trivial matter.

(iv) As in the already existing effective ([9, 13]) or algorithmic ([30]) calculations of \( \mathcal{A}_F \), we need information on the \( p \)-units (via \( \tilde{U}_F' \)) and the \( p \)-class groups (via \( X^0 \)). But note that the methods based on class field theory (op. cit.) require, in addition to this, an appeal to explicit reciprocity laws (via Hilbert symbols).

### 4.2 Biquadratic fields for \( p = 3 \)

The biquadratic fields \( F := \mathbb{Q}(\mu_3, \sqrt{d}) \) for \( p = 3 \) with \( d \in \mathbb{Z} \) squarefree and \( 3 \nmid d \) were first studied by Kramer and Candiotti [21] who determined \( \mathcal{A}_F \) for \( |d| < 200 \) and showed that \( \mathcal{A}_F = \mathcal{T}_F \), except for the five critical values \( d = -107, 67, 103, 106 \) and 139 which they did not treat. Immediately after, Greenberg [11] showed that \( \mathcal{A}_F \neq \mathcal{T}_F \) for \( d = 67 \). The corresponding \( \mathcal{A}_F \) for these critical cases were computed by Hémand [13] using an idelic calculation of the \( \mathbb{Z}_p \)-torsion of the Bertrandias–Payan module \( BP_F \) (see Sect. 3.2). A similar approach allowed Thomas [30] to devise an
algorithm computing $A_F$ for a wide range of $d$. We could redo these calculations using the algorithm of Sect. 4.1, but it seems more interesting to use our approach to determine the Tate kernel $T_F$ because, curiously enough, we know of no such systematic computation in the literature. For $p = 3$, there are only three relevant kernels $U_F/3$, $A_F$ and $T_F$ corresponding to the twists $0, -1$ and $1$ respectively. For the biquadratic fields $F$, we aim to derive the Tate kernel from the knowledge of the two other kernels.

Let us consider the five critical cases $F := \mathbb{Q}(\mu_3, \sqrt{d})$, $d = -107, 67, 103, 106$ and $139$ with $p = 3$. Using the PARI package, we can see that $X^\circ(F^+)$ is of order 3, hence the algorithm of Sect. 4.1 applies. It is easy to check that $F(\sqrt{5})$ is the first layer of the anticyclotomic $\mathbb{Z}_3$-extension of $F$ and 3 is a global universal norm. The three distinct kernels $U'_F/3$, $A'_F$ and $T'_F$ have dimension 2 and contain $3 := 3 \mod F^\bullet$. Let us treat in detail a specific example, say $d = 67$. The fundamental unit of the quadratic subfield $F^+ = \mathbb{Q}(\sqrt{67})$ is $\epsilon = 48842 + 5967\sqrt{67}$ and Lemma 1.4 shows that $\tilde{\epsilon} \in U'_F/3$ (but $\epsilon \notin U'_F$). Then $\tilde{U}'_F/3 = \langle 3, \epsilon \rangle$ whereas $A'_F = \langle 3, \tilde{\epsilon}^2\eta \rangle$ ([13, page 371]) with $\eta = 8 + \sqrt{67}$ being a generator of a 3-adic prime in $F^+$. These radicals are hyperplanes of the $\mathbb{Z}/3$-vector space $\tilde{U}'_F/3 = \langle 3, \tilde{\epsilon}, \tilde{\eta} \rangle$, but also of the $\mathbb{Z}/3$-vector space $(\tilde{U}'_1/3)^{G_1}$ according to Theorem 3.1. Their explicit form shows that a basis of $(\tilde{U}'_1/3)^{G_1}$ will be $\langle 3, \tilde{\epsilon}, \tilde{\epsilon}^2\eta \rangle$, or better $\langle 3, \tilde{\epsilon}, \tilde{\eta} \rangle$. In particular, $(\tilde{U}'_1/3)^{G_1}$ coincides with the image of the natural injection $\tilde{U}'_F/3 \hookrightarrow \tilde{U}'_1/3$. Obviously the third kernel $T'_F$ will be of the form $\langle 3, \eta \rangle$ or $\langle 3, \tilde{\eta} \epsilon \rangle$. But the choice between these two forms will unexpectedly be non trivial.

Since the rest of the calculation will essentially be elementary linear algebra, it will be more convenient to use additive notation for the three kernels $D_i := \text{Div}(\tilde{\mathfrak{N}}(i)^F)[3]$. They could be put on the same level and computed using the algorithm of Sect. 4.1, but here we want to deduce the last one from the two others. Let us first fix some general notations: $D_i = \langle 3 \otimes \frac{1}{3}, \beta_i \rangle$, $\beta_0 = \epsilon \otimes \frac{1}{3}$, $\beta_{-1} = (\eta - \epsilon) \otimes \frac{1}{3}$. According to diagram (3) of Sect. 4.1, $D_i = f_i^{-1}(D_i)$ is of type $(3^2, 3)$ and $f_i(D_i[3]) = f_i(\tilde{\mathfrak{N}}(i)^F[3])$ is the cyclic group generated by $3 \otimes \frac{1}{3}$. We look for pre-images (necessarily of order $3^2$) $\tilde{\beta}_i \in D_i$ of $\beta_i$ such that $D_i = \langle 3 \otimes \frac{1}{3}, \tilde{\beta}_i \rangle$.

Recall that Proposition 4.1 gives us the general form of the $\tilde{\beta}_i$’s, but here we directly know (by our chosen approach) $\beta_0$ and $\beta_{-1}$. Let us fix $\epsilon' \in \tilde{U}'_F$ such that $\epsilon' \otimes \frac{1}{3} = \epsilon \otimes \frac{1}{3}$ in $\tilde{\mathfrak{N}}$. We have just seen that $f_0(\tilde{\mathfrak{N}}(0)^F[3]) = \langle 3 \otimes \frac{1}{3}, \epsilon' \otimes \frac{1}{3} \notin \tilde{\mathfrak{N}}^0[3]$ or equivalently $\epsilon' \notin \tilde{U}'_F$. Because $3\epsilon' \in \tilde{U}'_F$, a pre-image $\tilde{\beta}_0$ of $\beta_0$ will be $3\epsilon' \otimes \frac{1}{3}$. Similarly, consider $\eta \otimes \frac{1}{3} \in (\tilde{U}'_1/3)^{G_1}$ and fix $\eta' \in \tilde{U}'_1$ such that $\eta' \otimes \frac{1}{3} = \eta \otimes \frac{1}{3}$. We have seen that $(\tilde{U}'_1/3)^{G_1} = \nu(\tilde{U}'_1/3)$ and therefore $(\tilde{U}'_1/3)^{G_1} \simeq \tilde{U}'_F/3$ since the (arithmetic) norm of $\tilde{U}'_1/3$ is $\tilde{U}'_F/3$. The same argument as for $\epsilon'$ allows us to show that $\nu(\eta' - \epsilon') \otimes \frac{1}{3}$ to $(\eta - \epsilon) \otimes \frac{1}{3}$, but be aware that $3(\eta' - \epsilon') \otimes \frac{1}{3}$ is not a priori in $\tilde{\mathfrak{N}}(-1)^F[3^2]$. Actually, Proposition 4.1 shows that there exists an element $\nu^{(-1)}(x) \otimes \frac{1}{3} \in \mathcal{D}_{-1}$ such that $g_{-1}(\nu^{(-1)}(x) \otimes \frac{1}{3}) = (\eta - \epsilon) \otimes \frac{1}{3}$, hence $\nu^{(-1)}(x) \otimes \frac{1}{3}$ and $3(\eta' - \epsilon') \otimes \frac{1}{3}$ differ by an element of order 3 in $\tilde{\mathfrak{N}}[3^2]$. Therefore, we can
Proposition 4.1 then reads: $3(\eta - \epsilon') \otimes \frac{1}{3} + \delta_{-1}$ with $\delta_{-1} \in \hat{\mathcal{N}}[3]$. The additional condition in Proposition 4.1 then reads: $3(\eta - \epsilon') \otimes \frac{1}{3} = 0$. The sign $-1$ on the right hand side would mean that $3\eta' \otimes \frac{1}{3} = 0$ in $\breve{U}'/3$, i.e. $\eta' \in \breve{U}'$ since $\breve{U}'$ is torsion free: a contradiction. Finally, $\tilde{\beta}_{-1} = 3(\eta - \epsilon') \otimes \frac{1}{3} + \delta_{-1}$ and $3\tilde{\beta}_{-1} = 3\epsilon' \otimes \frac{1}{3}$.

We must now give a general expression for the elements of order $3^2$ in $D_1$. If $\tilde{\beta}_1$ is such an element, any other is obtained by adding an element of order $3$ to $\tilde{\beta}_1$. So we start by constructing a particular $\tilde{\beta}_1$. Fix $x_0, x_{-1} \in \breve{U}'$ such that $\nu(x_0) \otimes \frac{1}{3} = \tilde{\beta}_0$ and $\nu(-1)(x_{-1}) \otimes \frac{1}{3} = \tilde{\beta}_{-1}$ (see Proposition 4.1), and put $x_1 = x_0 + \lambda x_{-1}$, $\lambda = \pm 1$.

An elementary calculation on Tate twists shows that $\nu^{(1)}(x)$ differs from $\nu^{(i)}(x)$ by an element of $(\breve{U}')^3$, hence $\tilde{\beta}_1 := \nu^{(1)}(x_1) \otimes \frac{1}{3} = \tilde{\beta}_0 + \lambda \tilde{\beta}_{-1} + \delta_1$, with $\delta_1 \in \hat{\mathcal{N}}[3]$. The additional condition in Proposition 4.1 reads: $3\tilde{\beta}_1 = 3\tilde{\beta}_0 + \lambda 3\tilde{\beta}_{-1} = \pm (3\epsilon' \otimes \frac{1}{3})$.

Since $3\tilde{\beta}_0 = 3\epsilon' \otimes \frac{1}{3} = 3\tilde{\beta}_{-1}$ (see the previous calculations), we get $1 + \lambda = \pm 1$. The only possibility is $\lambda = 1$. Hence $\tilde{\beta}_1 = \tilde{\beta}_0 + \tilde{\beta}_{-1} + \delta_1$ and $f_1(\tilde{\beta}_1) = \eta \otimes \frac{1}{3} + f_1(\delta_1)$, so that any element of $D_1$ will be of the form $\pm \eta \otimes \frac{1}{3} + \delta'_1$, where $\delta'_1$ is in the image of the natural map $\hat{\mathcal{N}}[3] \to \hat{\mathcal{N}}[3]$. We can now choose between the two possibilities $\eta \otimes \frac{1}{3}$ or $(\eta + \epsilon) \otimes \frac{1}{3}$. In the second case, we would have $\epsilon \otimes \frac{1}{3}$ or $(\epsilon - \eta) \otimes \frac{1}{3} = \delta'_1$, whence $\tilde{\beta}_0$ or $\tilde{\beta}_{-1}$ would be of order $3$ (because the kernel $X^0$ is of order $3$): a contradiction.

In conclusion, the Tate kernel in our example is $\mathcal{T}_F = \langle \tilde{3}, \tilde{\eta} \rangle$: the symbol $\{\zeta_3, \eta\}$ is trivial in $K_2(F)$ whereas $\{\zeta_3, \epsilon \eta\} = \{\zeta_3, \epsilon\}$ is a non-trivial element of the wild kernel $WK_2(F)$.

**Remark** The above result could of course be reached by describing the whole tame kernel $K_{20F}$ by generators and relations. Algorithms for such a calculation exist in the literature [2]. However, Karim Belabas kindly pointed out to us that in the case of the above example, the generators are U-units, where $S$ contains all the primes of norm less than 5096520 so by brute force calculation it would take years to find the relations between them.

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