Generating Mapping Class Groups by Involutions

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Abstract

Let $\Sigma_{g,b}$ denote a closed oriented surface genus $g$ with $b$ punctures and let $\text{Mod}_{g,b}$ denote its mapping class group. In [10] Luo proved that if the genus is at least 3, the group $\text{Mod}_{g,b}$ is generated by involutions. He also asked if there exists a universal upper bound, independent of genus and the number of punctures, for the number of torsion elements/involutions needed to generate $\text{Mod}_{g,b}$. Brendle and Farb in [1] gave a partial answer in the case of closed surfaces and surfaces with one puncture, by describing a generating set consisting of 7 involutions. Our main result generalizes the above result to the case of multiple punctures. We also show that the mapping class group can be generated by smaller number of involutions. More precisely, we prove that the mapping class group can be generated by 4 involutions if the genus $g$ is large enough. There is not a lot room to improve this bound because to generate this group we need at least 3 involutions. In the case of small genus (but at least 3) to generate the whole mapping class group we need a few more involutions.

1 Introduction

Let $\Sigma_{g,b}$ denote a closed, oriented surface of genus $g$ with $b$ punctures. By $\text{Mod}_{g,b}$ we will denote its mapping class group — the group orientation-preserving homeomorphisms preserving the set of punctures (i.e., we allow homeomorphisms of the surface which permute the punctures) modulo homotopy. Let us also denote by $\text{Mod}^0_{g,b}$ the subgroup of $\text{Mod}_{g,b}$, which fixes the punctures pointwise. It is clear that we have the exact sequence:

$$1 \to \text{Mod}^0_{g,b} \to \text{Mod}_{g,b} \to \text{Sym}_b,$$

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where the last projection is given by the restriction of a homeomorphism to its action on the puncture points.

The study of the mapping class group of a closed surface begun in the 1930-es. The first generating set of the mapping class group \( \text{Mod}_{g,0} \) for \( g \geq 3 \) was constructed by Dehn (see [2]). This set consists of \( 2g(g - 1) \) Dehn twists. Thirty years latter, Lickorish (see [8]) constructed a generating set for \( \text{Mod}_{g,0} \) consisting of \( 3g - 1 \) twists for any \( g \geq 1 \). This result was improved by Humphries (see [5]), who showed that a certain subset of Lickorish’s set consisting of \( 2g + 1 \) twists suffices to generate \( \text{Mod}_{g,0} \), and that this is in fact the minimal number of twist generators. This result was generalized in [6] by Johnson who proved that the same set of Dehn twists also generates \( \text{Mod}_{g,1} \). In the case of multiple punctures the mapping class group can be generated by \( 2g + b \) twists for \( b \geq 1 \) (see [3]).

It is possible to obtain smaller generating sets of \( \text{Mod}_{g,b} \) by using elements other than twists. N. Lu (see [9]) constructed a generated set of \( \text{Mod}_{g,0} \) consisting of 3 elements two of which are torsion. This result was improved by Wajnryb who found the smallest possible generating set of \( \text{Mod}_{g,0} \) consisting of 2 elements one of which is torsion see [14]. It is also known that in the case of closed surface the mapping class group can be generated by 3 torsion elements (see [11]), two of which are involutions. Recently Korkmaz (see [7]) showed that the mapping class group can be generated by 2 torsion elements (also in the case of a closed surface).

In [11], Maclachlan proved that the moduli space \( \mathcal{M}_g \) is simply connected as a topological space by showing that \( \text{Mod}_{g,0} \) is generated by torsion elements. Several years later Patterson generalized these results to \( \text{Mod}_{g,b} \) for \( g \geq 3, b \geq 1 \) see [13]. The question of generating mapping class groups by involutions was considered by McCarthy and Papadopoulos in [12]. Among other results, they proved that for \( g \geq 3 \), \( \text{Mod}_{g,0} \) is generated by infinitely many conjugates of a single involution.

Several year ago Luo, see [10], described the first finite set of involutions which generate \( \text{Mod}_{g,b} \) for \( g \geq 3 \). The size of his generating set depends linearly on both \( g \) and \( b \). Luo also proved that \( \text{Mod}_{g,b} \) is generated by torsion elements in all cases except \( g = 2 \) and \( b = 5k + 4 \), but this group is not generated by involutions if \( g \leq 2 \). In that paper Luo poses the question of whether there is a universal upper bound, independent of \( g \) and \( b \), for the number of torsion elements/involutions needed to generate \( \text{Mod}_{g,b} \).

In the case of zero or one punctures, this question was answered by Brendle and Farb in [1], where they constructed a generating set of \( \text{Mod}_{g,b} \) for \( g \geq 3 \) and \( b \leq 1 \) consisting of 7 involutions. A detailed consideration of the generating set of involutions constructed by Brendle and Farb shows
that one of the involutions is redundant and the mapping class group can be generated by 6 involutions (at least in the case $g > 3$, or $g = 3$ and no punctures).

Our main result generalizes the construction in the above paper to the case of several punctures. We improve the bound given by Brendle and Farb for the smallest number of involutions needed to generate the mapping class group $\text{Mod}_{g,b}$.

**Theorem 1.** For all $g \geq 3$, the mapping class group $\text{Mod}_{g,b}$ can be generated by:

a) 4 involutions if $g > 7$ or $g = 7$ and the number of punctures is even;
b) 5 involutions if $g > 5$ or $g = 5$ and the number of punctures is even;
c) 6 involutions if $g > 3$ or $g = 3$ and the number of punctures is even;
d) 9 involutions if $g = 3$ and the number of punctures is odd.

**Remark 2.** From the work of Luo [10], it follows that the above theorem can not be generalized to the case of small genus, because the mapping class group is not generated by involutions if $g \leq 2$.

**Remark 3.** If one allows orientation-reversing involutions, it is possible to extend Theorem 1 to the extended mapping class group which includes orientation-reversing mapping classes and show that it is generated by the same number of involutions as the mapping class group.

**Remark 4.** Our result for genus 3 is weaker than the result by Brendle and Farb, but the construction in their paper does not work for $g = 3$ and one puncture. In [1], they wrote:

For simplicity of exposition, we provide explicit arguments only for $b = 0$. In the case $b = 1$ the arguments are the same, although some involutions must be replaced with certain conjugates which move the puncture to a fixed point of the involution.

The problem with this argument is that for $g = 3$ two of the involutions in their generating set (the pair swap involutions $J_1$ and $J_2$) do not have any fixed points.

**Remark 5.** It is interesting to find the smallest possible set of involutions which generated $\text{Mod}_{g,b}$. It is clear that such a set should contain at least 3 involutions, since every group generated by two involutions is dihedral. One should expect that the bound of 9 involution for the genus 3 case can be improved. The author believes that the bound of 4 involutions in the high genus case could not be improved.

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Figure 1: The embeddings of the surface $\Sigma_{g,b}$ in the Euclidian space used to define the involutions $\rho_1$ and $\rho_2$. 
2 Generating the mapping class group by 2 involutions and 3 Dehn twists

Let us embed our surface \( \Sigma = \Sigma_{g,b} \) in the Euclidean space in two different ways as shown on Figure 1. (In these pictures we will assume that genus \( g = 2k \) is even and the number of punctures \( b = 2l + 1 \) is odd. In the case of odd genus we only have to swap the top parts of the pictures, and in the case of even number of punctures we have to swap the lower parts of the pictures.)

In Figure 1 we have also marked the puncture points as \( x_1, \ldots, x_b \) and we have the curves \( \alpha_i, \beta_i, \gamma_i \) and \( \delta \). Using these curves one can construct an explicit homeomorphism between the two surfaces in Euclidiian space. The curve \( \delta \) separates our surface into two components: the first one, denoted by \( \Sigma' \), is a surface of genus \( g \) with boundary component and no punctures. The second one (denoted by \( D \)) is a disk with \( b \) puncture points. All other curves are simple non-separating curves on the surface \( \Sigma' \).

Each embedding gives a natural involution of the surface — the half turn rotation around its axis of symmetry. Let us call these involutions \( \rho_1 \) and \( \rho_2 \). The product \( R = \rho_2 \rho_1 \) of these involutions acts almost as a rotation on the surface \( \Sigma_{g,b} \), more precisely we have

\[
R\alpha_i = \alpha_{i+1}, \quad \text{for } 1 \leq i < g \\
R\beta_i = \beta_{i+1}, \quad \text{for } 1 \leq i < g \\
R\gamma_i = \gamma_{i+1}, \quad \text{for } 1 \leq i < g - 1^2.
\]

(1)

On the set of punctures \( R \) acts also as a long cycle

\[
Rx_1 = x_b \quad \text{and} \quad Rx_i = x_{i-1} \text{ for } 1 < i \leq b.
\]

Let \( c \) be a simple non-separating curve on an oriented surface \( \Sigma \). We will denote by \( T_c \) the Dehn twist around the curve \( c \). For any homeomorphism \( h \) of the surface \( \Sigma \) the twists around the curves \( c \) and \( h(c) \) are conjugate in the mapping class group \( \text{Mod}(\Sigma) \),

\[
T_{h(c)} = hT_c h^{-1}.
\]

1In the case of no punctures \( R \) is homeomorphic to a rotation if one uses the flower embedding of the surface \( \Sigma_{g,0} \) in the Euclidian space

2We also have that \( R\beta_g = \beta_1 \), but \( R\alpha_g \neq \alpha_1 \) unless there are no punctures.

3We will not distinguish between the homeomorphisms and their images in the mapping class group \( \text{Mod}_{g,b} \).

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The relations \((1)\) imply that the Dehn twists around the curves \(\alpha_i\)-es, \(\beta_i\)-es and \(\gamma_i\)-es are conjugate, i.e.,
\[
RT_\alpha R^{-1} = T_{\alpha_{i+1}}, \quad RT_\beta R^{-1} = T_{\beta_{i+1}} \quad \text{and} \quad RT_\gamma R^{-1} = T_{\gamma_{i+1}}.
\]
(2)

The Dehn twists around these \(3g-1\) curves generate the whole mapping class group of a closed surface of genus \(g\) by a result of Lickorish, and of the surface \(\Sigma'\) of genus \(g\) and one boundary component by a result of Johnson (see [8, 6]). This together with relations \((2)\), shows that the subgroup \(G\) generated by \(\rho_1, \rho_2\) and 3 Dehn twists \(T_\alpha, T_\beta\) and \(T_\gamma\) around one of the curve in each family, contains

\[
\text{Mod}(\Sigma') \subset \text{Mod}(\Sigma_{g,b}).
\]

Our next step is to show that this subgroup actually contains all homeomorphisms of the surface \(\Sigma_{g,b}\) which preserve the punctures pointwise. In [3] it is shown that the group \(\text{Mod}^0(\Sigma_{g,b})\) is generated by the Dehn twists around the curves \(\alpha_i\)-es, \(\beta_i\)-es, \(\gamma_i\)-es and \(\delta_j\)-es, for \(j = 1, \ldots, b-1\), where the curves \(\delta_i\)-es are shown on Figure 2.

**Lemma 6.** The ‘rotation’ \(R\) acts on the curves \(\delta_j\) as follows:

\[
R^{-1}\delta_i = \eta_{i+1}
\]

**Proof.** The Figure 3 shows the action of \(\rho_1\) and \(\rho_2\) on the curve \(\delta_i\). It is clear from the picture that \(\eta_{i+1} = \rho_1 \rho_2(\delta_i) = R^{-1}\delta_i\). \(\square\)

Now we are ready to show that \(G\) contains \(\text{Mod}^0(\Sigma_{g,b})\).

**Lemma 7.** The group \(G\) contains the Dehn twists around the curves \(\delta_j\), for \(j = 0, \ldots, b-1\).

**Proof.** We will prove the lemma using induction on \(j\). The base case, \(j = 0\), is clear because by construction \(G\) contains \(T_{\delta_0} = T_{\alpha_1} \in G\). Suppose that \(G\) contains the twist \(T_{\delta_j}\). Using Lemma 6 we can see that the twist \(T_{\eta_{j+1}}\) also lies in \(G\), since it is conjugate to \(T_{\delta_j}\)

\[
T_{\eta_{j+1}} = R^{-1}T_{\delta_j}R \in G.
\]

To complete the induction step, we only need to notice that there exists a homeomorphism \(h_{j+1} \in \text{Mod}(\Sigma') \subset G\) such that \(h_{j+1}\eta_{j+1} = \delta_{j+1}\), because the curves \(\delta_{j+1}\) and \(\eta_{j+1}\) are homotopic in the disk \(D\). This gives us that

\[
T_{\delta_{j+1}} = h_{j+1}T_{\eta_{j+1}}h_{j+1}^{-1} \in G,
\]
which completes the induction step. \(\square\)
Figure 2: The curves $\delta_i$-es and $\eta_i$. 
Corollary 8. The group $G$ contains the subgroup $\text{Mod}^0(\Sigma_{g,b})$.

Remark 9. The group $G$ is not the whole group $\text{Mod}^0(\Sigma_{g,b})$ (if $b > 3$), because its image in the symmetric group $\text{Sym}_b$ is generated by two involutions and therefore is a dihedral group. It is easy to see that this image is the group $D_{2b}$ which is a proper subgroup of $\text{Sym}_b$. However in the next section we will show how to generated the Dehn twists $T_\alpha$, $T_\beta$ and $T_\gamma$ by several involutions and we can be flexible in choosing their actions on the set of punctures. We can choose these involutions so that their images in the symmetric group together with the images of $\rho_1$ and $\rho_2$ generate the whole symmetric group on the punctures.

3 Generating Mapping class group by involutions

In the previous section we have shown that we can generate almost the whole mapping class group by 2 involutions and 3 Dehn twists. Now we only need to explain how to generate the Dehn twists by involutions. The basic idea is to use the lantern relation and generate the Dehn twist $T_\alpha$ by four involutions.

3.1 The Lantern Relation

We begin by recalling the lantern relation in the mapping class group. This relation was first discovered by Dehn in the 1930s and later rediscovered by
Figure 4: Lantern and a pair swap involution

Johnson. Here we will use this relation to show how a Dehn twist can be expressed using 4 involutions. This argument is based on a result by Luo and Harer (see [10, 4]), who proved that a Dehn twist lie in a subgroup generated by 6 involutions. Brendle and Farb in [1] improved this argument by introducing pair swap involutions and showed that 4 involutions suffice to generate a Dehn twist.

Let the $S_{0,4}$ be a surface of genus 0 with 4 boundary components. Denote by $a_1, a_2, a_3$ and $a_4$ the four boundary curves of the surface $S_{0,4}$ and let the interior curves $x_1, x_2$ and $x_3$, be as shown in Figure 3.1.

The following relation:

$$T_{x_1}T_{x_2}T_{x_3} = T_{a_1}T_{a_2}T_{a_3}T_{a_4}$$

among the Dehn twists around the curves $a_i$ and $x_i$ is known as the lantern relation. Notice that the curves $a_i$ do not intersect any other curve and that the Dehn twists $T_{a_i}$ commute with every twist in this relation. This allows us to rewrite the lantern relation as follows

$$T_{a_4} = (T_{x_1}T_{a_1}^{-1})(T_{x_2}T_{a_2}^{-1})(T_{x_3}T_{a_3}^{-1})$$

The product $T_{x_1}T_{a_1}^{-1}$ can be expressed as a product of two involutions in the following way: let $\rho$ be an involution which maps $a_1$ to $x_1$, then we have

$$T_{x_1} = \rho T_{a_1} \rho$$

and

$$T_{x_1}T_{a_1}^{-1} = \rho T_{a_1} \rho T_{a_1}^{-1} = \rho (T_{a_1} \rho T_{a_1}^{-1})$$

Figure 3.1 shows the existence of a pair swap involutions $J_{12}$ on $\Sigma_{0,4}$, such that $J_{12}$ preserves the curves $a_4$ and $a_3$, and swaps the pairs $(a_1, x_1)$.
with \((a_2, x_2)\). There also exists another pair swap involution \(J_{13}\) with similar properties. This gives that

\[
T_{x_2} T_{a_2}^{-1} = J_{12} T_{x_1} T_{a_1}^{-1} J_{12} \\
T_{x_3} T_{a_3}^{-1} = J_{13} T_{x_1} T_{a_1}^{-1} J_{13}
\]

These relations together with relation 4 and 5 show that the twist around the curve \(a_4\) lies in the group generated by the involutions \(\rho, \bar{\rho} = T_{a_1} \rho T_{a_1}^{-1}\), \(J_{12}\) and \(J_{13}\), more precisely we have

\[
T_{a_4} = (\rho \bar{\rho})(J_{12} \rho \bar{\rho} J_{12})(J_{13} \rho \bar{\rho} J_{13}).
\]

After this preparation, we can start the prof of Theorem 1. We will begin with part c).

### 3.2 Genus \(g \geq 3\) and \(b\) even if \(g = 3\)

The argument which follows is the same as the one used in [1]. The only major difference is that we use a different lantern.

Recall that in the whole paper we assume that the genus of the surface is at least 3. In the argument which follows we will need an additional assumption that the number of punctures is even, if the genus \(g\) is exactly 3. For simplicity we will assume that the genus \(g = 2k\) is even. In the case of odd genus \(g = 2k + 1\), we can use the same lantern but use the involution \(\rho_2\) instead of \(\rho_1\).

We want to apply the argument in the previous section and find 4 involutions which generate the twist \(T_\alpha\). In order to do that we need to find an embedding of the lantern \(S_{0,4}\) into \(\Sigma_{g,b}\) and extend all involutions to the whole surface \(\Sigma_{g,b}\).

The Figure 5 shows one embedding of a lantern in our surface \(\Sigma_{g,b}\). Note that the compliment of this lantern is still a connected surface of genus \(g - 3\) with 4 boundary components and \(b\) punctures. The boundary components of this lantern are \(a_1 = \alpha_{k+1}\), \(a_2 = \alpha_{k-1}\), \(a_3 = \gamma_{k-1}\) and \(a_4 = \gamma_k\); and the middle curve \(x_1\) is \(\alpha_k\). Let denote \(S\) denotes this embedding of a lantern in \(\Sigma\).

Let us notice that the involution \(\rho_1\) (\(\rho_2\) in the case of odd \(g\)) maps \(a_1\) to \(x_1\). Let \(\rho_3\) denotes the conjugate of the involution \(\rho_1\) (or \(\rho_2\)) with \(T_{x_1}\). This gives us two of the four involutions needed to express \(T_{a_4} = T_{\gamma_k}\).

The compliment of the lantern is a surface \(S_1\) of genus \(g - 3\) with 4 boundary components and \(b\) punctures.
On a surface $S'_1$ (of genus $g-3$ with 4 boundary components) there exists an involution $\tilde{J}_{12}$ which preserves the two boundary components $a_3$ and $a_4$ and switches the other two. Moreover, we can arrange that this involution has $2(g-3)$ fixed points on the surface $S'_1$. We can find an invariant set (under the action of the involution $\tilde{J}_{12}$) of $b$ points/punctures $S'_1$, here we use the condition that $g > 3$ if $b$ is odd, which implies that there exist a finite $\tilde{J}_{12}$ invariant set with arbitrary number of points. This allows us to define an involution $\tilde{J}_{12}$ on the surface $S_1$, which preserves two of the boundary components and switches the other two. Finally, we can define an involution $I_{12}$ on the surface $\Sigma_{g,b}$ by gluing together the involutions $J_{12}$ and $\tilde{J}_{12}$ on the two parts of surface. In exactly the same way we can construct an involution $I_{13}$.

From the above construction it is clear that the twist $T_{\gamma_k}$ lies in the group $G_1$ generated by the involutions $\rho_1$, $\rho_2$, $\rho_3$, $I_{12}$ and $I_{13}$. Using conjugation with the element $R$ we can see that $G_1$ contains the twists around all curves $\gamma_i$-es.

By construction the involution $I_{13}$ takes the curve $\alpha_{k+1}$ to the curve
γ_{k-1}, i.e., we have
\[ \alpha_{k+1} I_{13} = I_{13} R \gamma_k R^{-1} I_{13} \in G_1. \]

So far we have used 5 involutions to generate a group \( G_1 \) which contains the twists around the curves \( \alpha_i \)-es and \( \gamma_i \)-es.

As shown on Figure 5, we can find a pair of pants (surface of genus 0, with 3 boundary components) on the surface \( \Sigma_{g,b} \), with boundary curves \( \alpha_k, \beta_{k-1} \) and some curve \( x \).

Using the same construction as above, we can find an involution \( I \) on the surface \( \Sigma_{g,b} \) which interchanges the curves \( \alpha_k \) and \( \beta_{k-1} \) and fixes the curve \( x \). Moreover, since the compliment of a pair of pants is a surface of genus \( g - 2 \geq 1 \), we can chose \( I \) so that it acts on the set of punctures as any involution in \( \text{Sym}_b \) with up to 3 fixed points.

**Lemma 10.** The group \( G_2 \) generated by the involutions \( \rho_1, \rho_2, \rho_3, I_{12}, I_{13} \) and \( I \) contains \( \text{Mod}^0(\Sigma_{g,b}) \).

**Proof.** We have already shown that \( G_1 \subset G_2 \) contains the Dehn twists around the curves \( \alpha_i \)-es and \( \gamma_i \)-es. Using the conjugation by the involution \( I \), we can see that \( G_2 \) also contains the twist around \( \beta_{i-1} \). This allows us to apply corollary and conclude that \( G_2 \) contains the subgroup \( \text{Mod}^0(\Sigma_{g,b}) \).

Knowing that \( G_2 \) contains \( \text{Mod}^0(\Sigma_{g,b}) \), we only need to make sure that all permutations of the punctures can be obtained as elements of \( G_2 \) to conclude that \( G_2 \) is the full mapping class group.

**Theorem 11 (6 involutions).** If \( g \geq 3 \), and \( b \) is even if \( g = 3 \), we can choose the involution \( I \) such that the group generated by \( \rho_1, \rho_2, \rho_3, I_{12}, I_{13} \) and \( I \) is the whole mapping class group \( \text{Mod}_{g,b} \).

**Proof.** To prove the theorem we only need to show that \( G_2 \) can be made to map surjectively onto the symmetric group \( \text{Sym}_b \), because \( G_2 \) contains \( \text{Mod}^0(\Sigma_{g,b}) \). This is equivalent to showing that the full symmetric group \( \text{Sym}_b \) can be generated by the involutions \( r_1 = (1, b)(2, b - 1)(3, b - 2) \ldots, r_2 = (1, b - 1)(2, b - 2) \ldots \) corresponding to \( \rho_1 \) and \( \rho_2 \) and another involution which have less then 4 fixed points.

**Lemma 12.** The symmetric group \( \text{Sym}_b \) is generated by \( r_1, r_2 \) and the involution \( r_3 = (2, b - 1)(3, b - 2) \ldots \).
Proof. The group generated by $r_i$ contains the long cycle $r_1r_2 = (12\ldots b)$ and the transposition $r_1r_3 = (1, b)$. These two elements generate the whole symmetric group, therefore the involutions $r_i$ generate $\text{Sym}_b$.

Thus we have shown that if $I$ acts on the punctures as $r_3$ then the group generated by $\rho_1, \rho_2, \rho_3, I_{12}, I_{13}$ and $I$ is $\text{Mod}_{g,b}$, which finishes the proof of part c) of Theorem 1.

### 3.3 Genus at least 5

Now we would like to refine the above argument and show that for large genus, we can avoid using the involution $I$, in order to generate the mapping class group. From now on we will assume that the genus $g$ of the surface is at least 5 (and that the number of punctures is even in the case $g = 5$).

We will start with the same lantern as above and define the involution $I_{13}$ in the same way. Now we know that the genus $g$ is at least 5 which allows us to make the action of $I_{13}$ on the set of punctures the same as the action of any involution with up to 3 fixed points. In particular we can make $I_{13}$ to act on the set of punctures as the involution $r_3$ used in the proof of Lemma 12.

We will slightly modify the construction involution $I_{12}$. The purpose of this modification is to make sure that $I_{12}$ sends some of the $\alpha_i$-es to some of the $\beta_i$-es, which will allow us to remove the involution $I$ for the generating set.

Let us glue to our lantern, two pairs of pants, to obtain a surface $S_2$ homeomorphic to a sphere with six boundary components. The first pair of pants is bounded by $a_1 = \alpha_{k+1}, \alpha_{k+2}$ and $\gamma_{k+1}$. The boundary components of the second pair of pants are $a_2 = \alpha_{k-1}, \beta_{k-2}$ and some curve $x$ (see Figure 6). From the figure is clear that the complement of $S_2$ in $\Sigma_{g,b}$ is a connected surface $S_3$ of genus $g - 5$ with 6 boundary components and $b$ punctures.

It is clear that we can extend the involution $J_{12}$ to a involution $\hat{J}_{12}$ of the surface $S_2$ which takes $\gamma_{k+1}$ to $\beta_{k-2}$ and $\alpha_{k+2}$ to $x$, and preserves the other two boundary components, $a_3 = \gamma_{k-1}$ and $a_4 = \gamma_k$. As before, we can construct an involution $\bar{J}_{12}$ on the $S_3$ which preserves two of the boundary components and swaps the other 4 components into two pairs. Gluing together $\hat{J}_{12}$ and $\bar{J}_{12}$ gives us the involution $I_{12}$ of the surface $\Sigma_{g,b}$.

**Theorem 13 (5 involutions).** If $g \geq 5$ and $b$ is even if $g = 5$, the group generated by $\rho_1, \rho_2, \rho_3, I_{12}$ and $I_{13}$ is the whole mapping class group $\text{Mod}_{g,b}$.

**Proof.** By construction the group $G_3$ generated by these 5 involutions contains the Dehn twist around the curve $\gamma_k$ and using conjugation with powers
of $R$ we have that $G_3$ contains the twists around all $\gamma_i$-es. We have constructed the involution $I_{12}$ in such a way that it sends one of the curves $\gamma_i$-es into some $\beta_{k-2}$, also the involution $I_{13}$ sends some $\gamma_i$ to $\alpha_{k+1}$. This gives us that the Dehn twists $T_{\beta_{k-2}}$ and $T_{\alpha_{k+1}}$ lie in the group $G_3$. Using the Corollary we can conclude that $G_3$ contains the group $\text{Mod}^0(\Sigma_{g,b})$.

We also have that the group $G_3$ maps onto $\text{Sym}_b$ because its image contains the involutions $r_1$, $r_2$, and $r_3$, which proves that $G_3$ is the whole mapping class group.

3.4 High genus

Now we want to refine the above argument and show that for high genus we do not need the involution $I_{13}$ in order to generate the mapping class group. Assume that the genus of the surface is at least 7 and that the number of puncture points is even if $g = 7$.

We will construct an involution $I$, such that in the group generated by $I$ and $R$ there are elements $h_{12}$ and $h_{13}$ such that

$h_{12}a_1 = a_2, \ h_{12}x_1 = x_2$; and $h_{13}a_1 = a_3, \ h_{13}x_1 = x_3$. 

Figure 6: Extending a lantern with two pair of pants.
This would imply that the group generated by $\rho_i$ and $I$ contains the product of twists $T_{x_1}T_{a_1}^{-1} = \rho_1\rho_3$ and its conjugates

$$T_{x_2}T_{a_2}^{-1} = h_{12}T_{x_1}T_{a_1}^{-1}h_{12}^{-1} \quad \text{and} \quad T_{x_3}T_{a_3}^{-1} = h_{13}T_{x_1}T_{a_1}^{-1}h_{13}^{-1}$$

Therefore this group will contain the twist $T_{a_4} = T_{\gamma_k}$, and its conjugate $T_{a_{k+1}}$. If $I$ also send some $\gamma_i$ into $\beta_j$, it will imply that the group generated by $\rho_i$ and $I$ contains $\text{Mod}(\Sigma')$, which allows us to conclude that it is almost the whole mapping class group.

The lanterns $S$ and $R^2S$ have a common boundary component $a_1 = R^2a_2$ and their union is a surface $S_4$ homeomorphic to a sphere with 6 boundary components. There exists an involution $I$ of $S_4$ which takes $S$ to $R^2S$ and acts on the boundary components as follows

$$Ia_1 = R^2a_2 = a_1 \quad Ia_2 = R^2a_3 \quad Ia_3 = R^2a_1 \quad Ia_4 = R^2a_4$$

and therefore acts on the middle curves as

$$Ix_1 = R^2x_2 \quad Ix_2 = R^2x_3 \quad Ix_3 = R^2x_1.$$ 

Therefore, for any extension $\tilde{I}$ of $I$ to an involution of the surface $\Sigma$ we have:

$$T_{x_1}T_{a_1}^{-1} = \rho_1\rho_3 \quad T_{x_1}T_{a_1}^{-1} = R^{-2}\bar{I}\rho_1\rho_3\bar{R}^2 \quad T_{x_1}T_{a_1}^{-1} = \bar{I}R^2\rho_1\rho_3R^{-2}\bar{I}.$$

This shows that the group generated by $\rho_i$ and $\tilde{I}$ contains the twist $T_{a_4} = T_{\gamma_k}$ and by conjugation all twists around $\alpha_i$-es and $\gamma_i$-es. In order to conclude that this group is the whole mapping class group we need to make sure that $\tilde{I}$ sends some $\gamma_i$ to some $\beta_j$ and acts permutes the punctures in a nice way.

We will extend the surface $S_4$ by attaching two pairs of pants and obtain a surface $S_5$ homeomorphic to sphere with 8 boundary components, such that its compliment $S_6$ in $\Sigma_{g,b}$ is connected.

One of the pair of pants will be bounded by $\alpha_{k+3} = R^2a_1$, $\gamma_{k+3}$ and $\alpha_{k+4}$, and the second one will have boundaries $\gamma_{k-1} = a_3 = I(R^2a_1)$, $\beta_{k-2}$ and some curve $x$ (see Figure 7). The involution $I$ can be extended to $S_5$ with the property $I\beta_{k-2} = \gamma_{k+3}$.

The compliment $S_6$ is a surface of genus $g - 7 \geq 2$ with 8 boundary components and $b$ punctures. Using the arguments from the previous sections it can be seen that there exists an involution $\bar{I}$ on $S_6$ which swaps the boundary components in 4 pairs and acts on the punctures as the involution $r_3^4$. Let $J$ be the involution obtained by gluing together $I$ and $\bar{I}$.

\footnote{Here we use that we can find an involution $\bar{I}$ of $S_6$ with $s = 2(g - 7) + 2$ fixed points. The number $s \geq 2$ is greater than the number of fixed points of $r_3$, which is either 2 or 3.}
Figure 7: Extending the lanterns $S$ and $R^2S$ with two pairs of pants.

**Theorem 14 (4 involutions).** If $g \geq 7$ for $b$ even and $g > 7$ for $b$ odd, the group generated by $\rho_1$, $\rho_2$, $\rho_3$ and $J$ is the whole mapping class group $\text{Mod} g,b$.

**Proof.** Let $G_5$ be the group generated by these 4 involutions. As explained above, this group contains the twist $T_{\alpha_k} = T_{\gamma_k}$, and therefore the twists around all curves $\alpha_i$, $\beta_i$ and $\gamma_i$ because all these curves lie in the orbit of $\gamma_k$ under the group generated by $I$ and $R$. This allows us to apply Corollary 12 and conclude that $G_5$ contains $\text{Mod}^0(\Sigma_{g,b})$. By construction this group also maps onto $\text{Sym} b$, because its image contains the involution $r_1$, $r_2$ and $r_3$, therefore the group $G_5$ is the whole mapping class group.

**Remark 15.** The symmetric group $\text{Sym} b$ can be generated by the involutions $r_1$, $r_2$ and an other involution with 1 fixed point if $b = 4k + 3$, which shows that Theorem 14 can be extended to the case $g = 7$ and $b = 4k + 3$. If $b = 4k + 1$ this is not possible because every involution with 1 fixed point is even. However, we believe that if one slightly modify the action of $\rho_2$ on the disk $D$ and make $\rho_2$ acts on punctures with 3 fixed points, it is possible to obtain that $\text{Mod} 7,b$ is generated by 4 involutions for all values of $b$. 

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3.5 Genus 3 and odd number of punctures

Finally we will sketch the case of genus 3 with odd number of punctures. In this case it is not possible to extend the pair swap involutions $J_{12}$ and $J_{23}$ to involutions of the surface $\Sigma_{3,b}$. Therefore, in order to generate the twist $T_\alpha$ we need 6 involutions (one of which is $\rho_1$). We also need 2 more involutions which move some curve $\alpha$ to some curve $\beta$ and $\gamma$, respectively. These, together with the involution $\rho_2$, give us a generating set consisting of 9 involutions.

**Theorem 16 (9 involutions).** If $g = 3$ and $b$ is odd, the mapping class group $\text{Mod}_{g,b}$ can be generated by 9 involutions.

This finishes the proof of the last part of Theorem 1.

**Remark 17.** One can try to find a small generating set of the group $\text{Mod}^0_{g,b}$ consisting of involutions. It is not clear that this group is generated by involutions – if the number of punctures $b$ is more than $2g + 2$ then there are no involutions in the group $\text{Mod}^0_{g,b}$. It is easy to see that if $g \geq 3$ and $b \leq 2(g - 2)$ then the group $\text{Mod}^0_{g,b}$ is generated by finitely many involutions. It is interesting to see for which values of $g$ and $b$ is this group generated by finitely many involutions and what is the smallest number of involutions needed to generate this group.

The constructions used in the proof of Theorem 1 can be generalized to show that if $b \leq 2(g - 2)$, the group $\text{Mod}^0_{g,b}$ can be generated by $Cg$ involutions for some constant $C$, and that there is a function $f(\lambda)$ for $\lambda < 2$ such that if $b \leq \lambda g$ then $\text{Mod}^0_{g,b}$ the group $\text{Mod}^0_{g,b}$ can be generated by $f(\lambda)$ involutions. It is interesting whether there exists a constant $C$ such that the mapping class group $\text{Mod}^0_{g,b}$ can be generated by $C$ involutions for all values of $g$ and $b$ such that this group is generated by involutions.

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