HEISENBERG UNIQUENESS PAIRS FOR THE FOURIER TRANSFORM ON THE HEISENBERG GROUP

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Abstract. In this article, we prove that a non-harmonic cone is Heisenberg uniqueness pair corresponding to the unit sphere for the symplectic Fourier transform on $\mathbb{C}^n$. Further, we derive that a sphere whose radius is not contained in the zero set of the Laguerre polynomials is a determining set for the spectral projections of the finite measure supported on the unit sphere. Finally, we give a proof of Benedick-Amrein-Berthier type theorem for the Heisenberg group.

1. Introduction

Let $\Gamma$ be a finite disjoint union of smooth curves in $\mathbb{R}^2$. Let $X(\Gamma)$ be the space of all finite complex-valued Borel measure $\mu$ in $\mathbb{R}^2$ which is supported on $\Gamma$ and absolutely continuous with respect to the arc length measure on $\Gamma$. For $(\xi, \eta) \in \mathbb{R}^2$, the Fourier transform of $\mu$ is defined by

$$\hat{\mu}(\xi, \eta) = \int_{\Gamma} e^{-i\pi (x \cdot \xi + y \cdot \eta)} d\mu(x, y).$$

In the above context, the function $\hat{\mu}$ becomes a uniformly continuous bounded function on $\mathbb{R}^2$. Thus, we can analyze the pointwise vanishing nature of the function $\hat{\mu}$.

Definition 1.1. Let $\Lambda$ be a set in $\mathbb{R}^2$. The pair $(\Gamma, \Lambda)$ is called a Heisenberg uniqueness pair for $X(\Gamma)$ if any $\mu \in X(\Gamma)$ satisfies $|\hat{\mu}|_{\Lambda} = 0$, implies $\mu = 0$.

In general, Heisenberg uniqueness pair (HUP) is a question of asking about the determining properties of the finite Borel measures which are supported on some lower dimensional entities whose Fourier transform also vanishes on lower dimensional entities. In particular, if $\Gamma$ is compact, then $\hat{\mu}$ is real analytic, having exponential growth, and hence $\hat{\mu}$ can vanishes on a very delicate set. Thus, the HUP problem becomes little easier in this case. However, this problem becomes immensely difficult when the measure is supported on a non-compact entity. It appears that the HUP problem is a natural invariant of the uncertainty principle for the Fourier transform.

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In addition, the concept of determining the Heisenberg uniqueness pair for a class of finite measures has a significant similarity with the celebrated result due to M. Benedicks and Amrein-Berthier (see [2], [6]). For \( f \in L^1(\mathbb{R}^n) \), denote \( \mathcal{E} = \{ x \in \mathbb{R}^n : f(x) \neq 0 \} \) and \( \mathcal{F} = \{ \xi \in \mathbb{R}^n : \hat{f}(\xi) \neq 0 \} \). Then both the set \( \mathcal{E} \) and \( \mathcal{F} \) cannot have finite Lebesgue measure simultaneously, unless \( f = 0 \). Later, various analogues of this result have been investigated in different aspects including the Heisenberg group and Euclidean motion groups (see [15, 19, 22, 31]).

In article [15], an analogous result appeared for the partial compactly supported function on Heisenberg group in terms of finite rank of the Fourier transform of the function. Further, Vemuri [31] has relaxed the compact support condition on the function by finite Lebesgue measure.

In this article, we give a new and simple proof of Benedick-Amrein-Berthier type theorem for the Heisenberg group as compare to [31]. Our proof evolves on the basic principles of Fourier analysis and measure theory. We also observe that this technique can be extended to prove an analogous result for general step two nilpotent Lie groups. Since, the proof for the latter case is analogous, we shall focus to the Heisenberg group with some remark in the general setups.

However, we first discuss the concept of HUP, which was introduced by Hedenmalm and Montes-Rodríguez in 2011. In the article [12], Hedenmalm and Montes-Rodríguez have shown that the pair (hyperbola, some discrete set) is a Heisenberg uniqueness pair. As a dual problem, a weak* dense subspace of \( L^\infty(\mathbb{R}) \) has been constructed to solve the Klein-Gordon equation. Further, Hedenmalm and Montes-Rodríguez (see [12]) have given a complete characterization of the Heisenberg uniqueness pairs corresponding to any two parallel lines.

Lev [14] and Sjölin [23] have independently shown that circle and certain system of lines are HUP corresponding to the unit circle \( S^1 \). Further, Vieli [32] has generalized HUP corresponding to circle in the higher dimension and shown that a sphere whose radius does not lie in the zero set of the Bessel functions \( J_{(n+2k-2)/2} \), \( k \in \mathbb{Z}_+ \), the set of non-negative integers, is a HUP corresponding to the unit sphere \( S^{n-1} \). In [27], the author has shown that a cone is a Heisenberg uniqueness pair corresponding to the sphere as long as the cone does not completely recline on the level surface of any homogeneous harmonic polynomial on \( \mathbb{R}^n \).

Further, Sjölin [24] has investigated some of the Heisenberg uniqueness pairs corresponding to the parabola. It has been extended to the case of paraboloid by Vieli [33]. Subsequently, Babot [5] has given a characterization of the Heisenberg uniqueness pairs corresponding to a certain system of three parallel lines. Thereafter, the authors in [11] have given some necessary and sufficient conditions for the Heisenberg uniqueness pairs corresponding to a system of four parallel lines. In the latter case, we observe a phenomenon of three totally
disconnected interlacing sets those are zero sets of three trigonometric polynomials. However, the question of the unique necessary and sufficient condition for the finitely many parallel lines as compared to three lines result [5] is still unsolved. In the article [11], the authors have also investigated some of the Heisenberg uniqueness pairs corresponding to the spiral, hyperbola, circle and the exponential curves.

In a major development, Jaming and Kellay [13] have given a unifying proof for some of the Heisenberg uniqueness pairs corresponding to the hyperbola, polygon, ellipse and graph of the functions $\varphi(t) = |t|^\alpha$, whenever $\alpha > 0$. Thereafter, Gröchenig and Jaming [10] have worked out some of the Heisenberg uniqueness pairs corresponding to the quadratic surface.

In this article, we work for an analogous problem on the Heisenberg group in various aspect. Firstly, we consider the symplectic Fourier transform on $\mathbb{C}^n$. We prove that a non-harmonic complex cone is HUP for the symplectic Fourier transform corresponding to $S^{2n-1}$. The above result has a sharp contrast with the similar result for the Euclidean Fourier transform on $\mathbb{R}^{2n}$. Since a non-trivial complex cone in $\mathbb{C}^n$ ($n \geq 2$) can have topological dimension at most $2n-2$, it follows that a $(2n-2)$-dimensional entity is a HUP for the symplectic Fourier transform corresponding to $S^{2n-1}$. Although, for the Euclidean Fourier transform on $\mathbb{R}^{2n}$, the least topological dimension required (in general) for a set to be HUP corresponding to the unit sphere $S^{2n-1}$ is $2n-1$. We also observe that the conclusion of the above result for symplectic Fourier transform holds good for a real non-harmonic cone in $\mathbb{C}^n$.

Thereafter, we consider the case of modified Fourier transform on the Heisenberg group. We prove that a finite measure supported on the cylinder $S^{2n-1} \times \mathbb{R}$ can be determined by any non-harmonic cone as well as the boundary of a bounded domain in $\mathbb{C}^n$.

Further, we consider a bit more interesting case of determining a finite measure $\mu$ which is supported on $S^{2n-1}$ in terms of its spectral projections. We prove that if the spectral projections $\varphi^n_k \times \mu$ vanish on the sphere whose radius is not contained in the zero set of the Laguerre polynomials, then $\mu$ is trivial. We observed that the above measure can also be determined by a non-harmonic complex cone. Though, the case of the real non-harmonic cone is yet to settle.

2. SOME PRELIMINARIES

In this section, we describe some essential preliminary about Fourier transform on the Heisenberg group, Weyl transform and special Hermite expansion of function on $\mathbb{C}^n$. Finally, we mention some auxiliary results related to the bigraded spherical harmonics and non-harmonic cones.
The Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ is a step two nilpotent Lie group having center $\mathbb{R}$ that equipped with the group law

$$(z,t) \cdot (w,s) = \left( z + w, t + s + \frac{1}{2} \text{Im}(z \cdot \bar{w}) \right).$$

By the Stone-von Neumann theorem, the infinite dimensional irreducible unitary representations of $\mathbb{H}^n$ can be parameterized by $\mathbb{R}^* = \mathbb{R} \setminus \{ 0 \}$. That is, each of $\lambda \in \mathbb{R}^*$ defines a Schrödinger representation $\pi_\lambda$ of $\mathbb{H}^n$ by

$$\pi_\lambda(z,t) \varphi(\xi) = e^{i\lambda t} e^{i\lambda (x \cdot \xi + \frac{1}{2} z \cdot \bar{y})} \varphi(\xi + y),$$

where $z = x + iy$ and $\varphi \in L^2(\mathbb{R}^n)$. Hence, the group Fourier transform of $f \in L^1(\mathbb{H}^n)$, defined by

$$\hat{f}(\lambda) = \int_{\mathbb{H}^n} f(z,t) \pi_\lambda(z,t) \, dz \, dt$$

is a bounded operator, while for $f \in L^2(\mathbb{H}^n)$ it is a Hilbert-Schmidt operator. An important technique in many problems on $\mathbb{H}^n$ is to take partial Fourier transform in the $t$-variable to reduce matters to $\mathbb{C}^n$. Let

$$f^\lambda(z) = \int_{\mathbb{R}} f(z,t) e^{i\lambda t} \, dt$$

be the inverse Fourier transform of $f$ in the $t$-variable. The group convolution of the functions $f, g \in L^1(\mathbb{H}^n)$ is defined by

$$(2.1) \quad f \ast g(z,t) = \int_{\mathbb{H}^n} f((z,t) (-w, -s)) g(w,s) \, dw \, ds.$$ 

A simple calculation shows that

$$(f \ast g)^\lambda(z) = \int_{-\infty}^\infty f \ast g(z,t) e^{i\lambda t} \, dt$$

$$= \int_{\mathbb{C}^n} f^\lambda(z - w) g^\lambda(w) e^{i\lambda \text{Im}(z,w)} \, dw$$

$$= f^\lambda \times g^\lambda(z).$$

Thus, the group convolution $f \ast g$ on the Heisenberg group can be studied using the $\lambda$-twisted convolution $f^\lambda \times_\lambda g^\lambda$ on $\mathbb{C}^n$. For $\lambda \neq 0$, by scaling argument, it is enough to study the twisted convolution for the case $\lambda = 1$.

Now, we recall the Weyl transform, which is an important constituent of the group Fourier transform on the Heisenberg group. Denote $\pi_\lambda(z) = \pi_\lambda(z,0)$. Then $\pi_\lambda(z,t) = e^{i\lambda t} \pi_\lambda(z)$. For suitable function $g$ on $\mathbb{C}^n$, Weyl transform of $g$ can be expressed as

$$W_\lambda(g) = \int_{\mathbb{C}^n} g(w) \pi_\lambda(w) \, dw.$$
This implies, $\hat{f}(\lambda) = W_\lambda(f^\lambda)$. It is easy to see that $W_\lambda(g)$ is a bounded operator for $g \in L^1(\mathbb{C}^n)$. If $g \in L^2(\mathbb{C}^n)$, then $W_\lambda(g)$ is a Hilbert-Schmidt operator satisfying the Plancherel formula

$$\|W_\lambda(g)\|_{HS} = |\lambda|^{\frac{n}{2}}\|g\|_2.$$ 

Next, we describe the special Hermite expansion for functions on $\mathbb{C}^n$. Let

$$T = \frac{\partial}{\partial t}, \quad X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}y_j \frac{\partial}{\partial t} \quad \text{and} \quad Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2}x_j \frac{\partial}{\partial t}.$$ 

Then $\{T, X_j, Y_j : j = 1, \ldots, n\}$ forms a basis for the Lie algebra $\mathfrak{h}^n$ consists of all left-invariant vector fields on $\mathbb{H}^n$ and the representation $\pi_\lambda$ induces a representation $\pi_\lambda^* \alpha \beta$ of $\mathfrak{h}^n$ on the space of $C^\infty$ vectors in $L^2(\mathbb{R}^n)$ via

$$\pi_\lambda^*(X)f = \frac{dl}{dt} \bigg|_{t=0} \pi_\lambda(\exp tX)f.$$ 

It is easy to see that $\pi_\lambda^*(X_j) = i\lambda x_j$ and $\pi_\lambda^*(Y_j) = \frac{\partial}{\partial x_j}$. Hence for the sub-Laplacian $\mathcal{L} = -\sum_{j=1}^n (X_j^2 + Y_j^2)$, it follows that $\pi_\lambda^*(\mathcal{L}) = -\Delta_x + \lambda^2|\lambda|^2 =: \mathcal{H}_\lambda$, the scaled Hermite operator. Let $\phi_\alpha^\lambda(z) = |\lambda|^{\frac{n}{2}} \phi_\alpha(\sqrt{|\lambda|}z)$; $\alpha \in \mathbb{Z}^n$, where $\phi_\alpha$ are the Hermite functions on $\mathbb{R}^n$. Then $\phi_\alpha^\lambda$ is an eigenfunction of $\mathcal{H}_\lambda$ with eigenvalue $(2|\alpha| + n)|\lambda|$. Hence the entry functions $E^\alpha_{\alpha \beta}$'s of the representation $\pi_\lambda$ are eigenfunctions of the sub-Laplacian $\mathcal{L}$ satisfying

$$\mathcal{L} E^\alpha_{\alpha \beta} = (2|\alpha| + n)|\lambda| E^\alpha_{\alpha \beta},$$

where $E^\alpha_{\alpha \beta}(z, t) = \langle \pi_\lambda(z, t) \phi_\alpha^\lambda, \phi_\beta^\lambda \rangle$. Since $E^\alpha_{\alpha \beta}(z, t) = e^{i\lambda t} \langle \pi_\lambda(z) \phi_\alpha^\lambda, \phi_\beta^\lambda \rangle$, the eigenfunctions $E^\alpha_{\alpha \beta}$'s are not in $L^2(\mathbb{H}^n)$. However, for a fix $t$, they are in $L^2(\mathbb{C}^n)$. Now, define an operator $L_\lambda$ by $\mathcal{L}(e^{i\lambda t} f(z)) = e^{i\lambda t} L_\lambda f(z)$. Then the special Hermite functions

$$\phi_{\alpha \beta}^\lambda(z) = (2\pi)^{-\frac{n}{2}} \langle \pi_\lambda(z) \phi_\alpha^\lambda, \phi_\beta^\lambda \rangle$$

are eigenfunctions of $L_\lambda$ with eigenvalue $2|\alpha| + n$. We summarize by noting that the special Hermite functions $\phi_{\alpha \beta}^\lambda$'s forms an orthonormal basis for $L^2(\mathbb{C}^n)$ (see [30], Theorem 2.3.1, p.54). Thus, for $g \in L^2(\mathbb{C}^n)$, we have the expansion

$$g = \sum_{\alpha, \beta} \langle g, \phi_{\alpha \beta}^\lambda \rangle \phi_{\alpha \beta}^\lambda.$$ 

In order to further simplify the above expansion, denote $\varphi_{\alpha \beta}^{n-1}(z) = \varphi_{\alpha \beta}^{n-1}(\sqrt{|\lambda|}z)$, the Laguerre function of degree $k$ and order $n - 1$. Then the special Hermite functions $\phi_{\alpha \beta}^\lambda$ satisfy the relation

$$\sum_{|\alpha| = k} \phi_{\alpha, \alpha}^\lambda(z) = (2\pi)^{-\frac{n}{2}} |\lambda|^{\frac{n}{2}} \varphi_{\alpha \beta}^{n-1}(z).$$ (2.2)
Thus, \( g \in L^2(\mathbb{C}^n) \) can be expressed as
\[
g(z) = (2\pi)^{-n}|\lambda|^n \sum_{k=0}^{\infty} g \times \varphi^{n-1}_{k,\lambda}(z),
\]
whenever \( \lambda \in \mathbb{R}^n \), (see [30], p. 98). In particular, for \( \lambda = 1 \), we have
\[
g(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} g \times \varphi_k^{n-1}(z),
\]
which is the special Hermite expansion for \( g \). Hence \( g \) is completely determined by its spectral projections \( g \times \varphi_k^{n-1} \). Therefore, it is an interesting question to determine a finite measure \( \mu \), supported on a thin set in \( \mathbb{C}^n \), in terms of its spectral projections.

The following weighted functional relations can be obtained by considering the Hecke-Bochner identity for the spectral projections of compactly supported functions. For more details, see [30], p. 98.

**Lemma 2.1.** [30] For \( z \in \mathbb{C}^n \), let \( P \in H_{p,q} \) and \( d\nu_r \equiv Pd\mu_r \), where \( \mu_r \) is the surface measure on the sphere \( S_r \). Then
\[
\varphi_k^{n-1} \times \nu_r(z) = (2\pi)^{-n} \frac{\Gamma(k - q + 1)}{\Gamma(k + n + p)} 2^{(p+q)} \varphi_{k-q}^{n+p+q-1}(r)P(z) \varphi_{k-q}^{n+p+q-1}(z),
\]
if \( k \geq q \) and 0 otherwise.

We need the following basic facts about the bigraded spherical harmonics, (see [8, 9, 30] for details). Let \( K = U(n) \) be the unitary group and \( M = U(n - 1) \). Then, \( S^{2n-1} \cong K/M \) under the map \( kM \rightarrow k.e_n, k \in U(n) \) and \( e_n = (0, \ldots, 1) \in \mathbb{C}^n \). Let \( \hat{K}_M \) denote the set of all equivalence classes of irreducible unitary representations of \( K \) which have a nonzero \( M \)-fixed vector.

For a \( \delta \in \hat{K}_M \), which is realized on \( V_\delta \), let \( \{e_1, \ldots, e_{d(\delta)}\} \) be an orthonormal basis of \( V_\delta \) with \( e_1 \) as the \( M \)-fixed vector. Let \( t_{ij}^\delta(k) = \langle e_i, \delta(k)e_j \rangle, k \in K \). By the Peter-Weyl theorem, the set \( \{\sqrt{d(\delta)}t_{ij}^\delta : 1 \leq j \leq d(\delta), \delta \in \hat{K}_M\} \) form an orthonormal basis for \( L^2(K/M) \) (see [30], p.14 for details). Define
\[
Y_{ij}^\delta(\omega) = \sqrt{d(\delta)}t_{ij}^\delta(k), \quad \text{where} \quad \omega = k.e_n \in S^{2n-1}, k \in K.
\]
Then \( \{Y_{ij}^\delta : 1 \leq j \leq d(\delta), \delta \in \hat{K}_M\} \) becomes an orthonormal basis for \( L^2(S^{2n-1}) \).

For \( p, q \in \mathbb{Z}_+ \), let \( H_{p,q} = \{P \in P_{p,q} : \Delta P = 0\} \). Then \( H_{p,q} \) is \( K \)-invariant. Let \( \pi_{p,q} \) denote the corresponding representation of \( K \) on \( H_{p,q} \). Then \( \hat{K}_M \) can be identified with \( \{\pi_{p,q} : p, q \in \mathbb{Z}_+\} \). See [21], p.253, for details. Define the bi-graded spherical harmonic on \( S^{2n-1} \) by \( Y_{p,q}^{p,q}(\omega) = \sqrt{d(p, q)}t_{ij}^p(t_{ij}^q(k)) \). Then \( \{Y_{ij}^{p,q} : 1 \leq j \leq d(p, q), p, q \in \mathbb{Z}_+\} \) forms an orthonormal basis for \( L^2(S^{2n-1}) \).

Thus, a continuous function \( f \) on \( S^{2n-1} \) can be expressed as
\[
f(\omega) = \sum_{p, q \geq 0} \sum_{j=1}^{d(p, q)} Y_{ij}^{p,q}(\omega).
\]
For each $l \in \mathbb{Z}_+$, the space $H_l$ consists of spherical harmonic of degree $l$ is $SO(d)$ - invariant. When $d = 2n$, $H_l$ is $U(n)$ - invariant as well, and under this action of $U(n)$, the space $H_l$ breaks up into an orthogonal direct sum of $H_{p,q}$'s where $p + q = l$. (See [21], p. 255).

**Lemma 2.2.** [21]. Let $\omega \in S^{2n-1}$ and $Y_l \in H_l$. Then

$$Y_l(\omega) = \sum_{p+q=l} Y_{p,q}(\omega), \text{ where } Y_{p,q} \in H_{p,q}.$$  

**Definition 2.3.** A set $C \subset \mathbb{C}^n$ ($n \geq 2$) that satisfies the scaling condition $\lambda C \subseteq C$ for all $\lambda \in \mathbb{C}$, is called a complex cone.

We say a complex cone is *non-harmonic* if it is not contained in the zero set of any bi-graded homogeneous harmonic polynomial on $\mathbb{C}^n$ of the form

$$P(z) = \sum_{|\alpha|=p} \sum_{|\beta|=q} c_{\alpha\beta} z^\alpha \bar{z}^\beta,$$

where $p, q \in \mathbb{Z}_+$, the set of non-negative integers. Let $P_{p,q}$ denotes the space of all bi-graded homogeneous polynomials defined by (2.5).

An example of a non-harmonic complex cone was produced by the author (see [26]). The zero set of the polynomial $H(z) = az_1 \bar{z}_2 + |z|^2$, where $a \neq 0$ and $z \in \mathbb{C}^n$ is a complex cone which is not contained in the zero set of any bi-graded homogeneous harmonic polynomial.

In view of Lemma 2.2, it is easy to prove the following result that requires in the proof of our main results.

**Lemma 2.4.** Let $C$ be a complex cone and denote $\tilde{C} = \left\{ \frac{z}{|z|} : z \in C, \ z \neq 0 \right\}$.

Then $Y_l = 0$ on $\tilde{C}$ if and only if $Y_{p,q} = 0$ on $\Omega$, $\forall \ p, q \in \mathbb{Z}_+$ which lie on the diagonal $p + q = l$.

For each fixed $\xi \in S^{d-1}$, define a linear functional on $H_l$ by $\xi \mapsto Y_l(\xi)$. Then there exists a unique spherical harmonic, say $Z_l^{(l)} \in H_l$ such that

$$Y_l(\xi) = \int_{S^{d-1}} Z_l^{(l)}(\eta) Y_l(\eta) d\sigma(\eta).$$

The spherical harmonic $Z_l^{(l)}$ is a $K$ bi-invariant real-valued function, which is constant on the geodesics those are orthogonal to the line joining the origin and $\xi$. The spherical harmonic $Z_l^{(l)}$ is called the zonal harmonic of the space $H_l$ at the pole $\xi$. For more details, see [28], p. 143.

Let $f$ be a function in $L^1(S^{d-1})$. For each $l \in \mathbb{Z}_+$, we define the $l$th spherical harmonic projection of the function $f$ by

$$\Pi_l f(\xi) = \int_{S^{d-1}} Z_l^{(l)}(\eta) f(\eta) d\sigma(\eta).$$
The function $\Pi_l f$ is a spherical harmonic of degree $l$. If for a $\delta > (n - 2)/2$, we denote $A_l^m(\delta) = (m - l + \delta)(m + \delta)^{-1}$. Then the spherical harmonic expansion $\sum_{l=0}^{\infty} \Pi_l f$ is $\delta$-Cesaro summable to $f$. That is,

$$f = \lim_{m \to \infty} \sum_{l=0}^{m} A_l^m(\delta) \Pi_l f,$$

where limit in the right-hand side of (2.8) exists on $L^1 \left( S^{d-1} \right)$. For more details, see [25].

We would like to mention that the proof of our main result is being carried out by concentrating the problem to the unit sphere $S^{d-1}$ in terms of averages on its geodesic spheres. This is possible because the cone $C$ is closed under scaling.

For $\omega \in S^{d-1}$ and $t \in (-1, 1)$, the set $S^t_\omega = \{ \nu \in S^{d-1} : \omega \cdot \nu = t \}$ is a geodesic sphere on $S^{d-1}$ with a pole at $\omega$. Let $f$ be an integrable function on $S^{d-1}$. Then in view of Fubini’s Theorem, we can define the geodesic spherical means of the function $f$ by

$$\tilde{f}(\omega, t) = \int_{S^t_\omega} f d\nu_{d-2},$$

where $\nu_{d-2}$ is the normalized surface measure on the geodesic sphere $S^t_\omega$.

Since the zonal harmonic $Z_l^{(d)}(\xi)$ is $K$ bi-invariant, there exists a nice function $F$ on $(-1, 1)$ satisfying $Z_l^{(d)}(\eta) = F(\xi \cdot \eta)$. Hence the extension of the formula (2.6) for the functions $F \in L^1(-1, 1)$ is inevitable. This is known as the Funk-Hecke theorem. That is,

$$\int_{S^{d-1}} F(\xi \cdot \eta) Y_l(\eta) d\sigma(\eta) = C_l Y_l(\xi),$$

where the constant $C_l$ is given by

$$C_l = \alpha_l \int_{-1}^{1} F(t) G_l^{d-2}(t)(1 - t^2)^{d-3} dt$$

and $G_l^{\beta}$ stands for the Gegenbauer polynomial of degree $l$ and order $\beta$. As a consequence of the Funk-Hecke theorem, the geodesic means of the spherical harmonic $Y_l$ satisfy

$$\tilde{Y}_l(\omega, t) = D_l (1 - t^2)^{d-2} G_l^{d-2}(t) Y_l(\omega),$$

where the constant $D_l = |S^{d-2}|/G_l^{d-2}(1)$. Here $|S^{d-2}|$ denotes the surface area of the unit sphere in $\mathbb{R}^{d-1}$. For more details see [3], p. 459. To prove our main result, we need the following lemma which percolates the geodesic mean vanishing condition of $f \in L^1(S^{d-1})$ to each spherical harmonic component of $f$. For the class of continuous functions $C(S^{d-1})$, this lemma was proved in
In [27], the author has extended the result for functions in $L^1(S^{d-1})$ as a consequence of the Cesaro summation formula described by [28].

**Lemma 2.5.** [27] Let $f \in L^1(S^{d-1})$. Then $\hat{f}(\omega, t) = 0$ for all $t \in (-1, 1)$ if and only if $\Pi_t f(\omega) = 0$ for all $l \in \mathbb{Z}_+$. 

Notice that as a corollary to Lemma 2.5 it can be deduced that if $\hat{f}(\omega, t) = 0$ for all $(\omega, t) \in \mathbb{C} \times (-1, 1)$, then $f = 0$ on $S^{d-1}$ if cone $C$ is not contained in the zero set of any homogeneous harmonic polynomial.

A set $C$ in $\mathbb{R}^d$ ($d \geq 2$) which satisfies $\lambda C \subseteq C$, for all $\lambda \in \mathbb{R}$ is called a real cone.

An example of such a cone had been produced by Armitage, (see [4]). Let $0 < a < 1$ and $C^k_a(x)$ denotes Gegenbauer polynomial of degree $k$ and order $\lambda$. Then $K_a = \{x \in \mathbb{R}^d : |x|^2 = a^2|x|^2\}$ is a non-harmonic cone if and only if $D^m C^k_a (a) \neq 0$, for all $0 \leq m \leq k - 2$, where $D^m$ denotes the $m$th derivative.

**3. Uniqueness Pair for the Symplectic Fourier Transform**

In this section, we prove that a non-harmonic complex cone is Heisenberg uniqueness pair corresponding to the unit sphere $S^{2n-1}$ for the symplectic Fourier transform on $\mathbb{C}^n$.

Let $X(S^{2n-1})$ be the space of all finite Borel measure $\mu$ in $\mathbb{C}^n$ which is supported on $S^{2n-1}$ and absolutely continuous with respect to the arc length of $S^{2n-1}$. Then by Radon-Nikodym theorem, there exists $f \in L^1(S^{2n-1})$ such that $d\mu = f d\sigma$.

Define the symplectic Fourier transform of a measure $\mu \in X(S^{2n-1})$ by

$$\hat{\mu}(z) = \int_{S^{2n-1}} e^{-\frac{i}{2} \text{Im} z \cdot \xi} f(\xi) d\sigma(\xi),$$

where $z = x + iy \in \mathbb{C}^n$ and $\zeta = \xi + i\eta \in \mathbb{C}^n$. Hence, the function $\hat{\mu}$ becomes a bounded uniformly continuous function on $\mathbb{C}^n$. Thus, we can analyze the pointwise vanishing nature of the function $\hat{\mu}$.

In other words, we can express

$$(3.1) \quad \hat{\mu}(x, y) = \int_{S^{2n-1}} e^{-\frac{i}{2} (-x \eta + y \xi)} f(\xi, \eta) d\sigma(\xi, \eta).$$

We prove the following result.

**Theorem 3.1.** Let $C$ be a non-harmonic complex cone in $\mathbb{C}^n$. If $\mu \in X(S^{2n-1})$ satisfies $\hat{\mu}(z) = 0$, for all $z \in C$, then $\mu = 0$.

**Proof.** Let $(x, y) = r \omega$, where $\omega = (\omega_1, \ldots, \omega_n, \omega'_1, \ldots, \omega'_n) \in S^{2n-1}$. Denote $\tilde{\omega} = (\omega'_1, \ldots, \omega'_n, -\omega_1, \ldots, -\omega_n)$. Then from (3.1), it implies that

$$(3.2) \quad \int_{S^{2n-1}} e^{-\frac{i}{2} r \tilde{\omega}(\xi, \eta)} f(\xi, \eta) d\sigma(\xi, \eta) = 0,$$
for all $r\omega \in C$. Since $C$ is closed under complex scaling, $r\omega \in C$ implies $r\bar{\omega} \in C$. By decomposing the integral in (3.2) over geodesic spheres at pole $\omega$, we obtain

$$\int_{-1}^{1} \left( \int_{S^\nu_{\omega}} e^{-\frac{1}{2}r\omega \cdot \nu} f(\nu) d\sigma_{2n-2}(\nu) \right) dt = 0,$$

where $S^\nu_{\omega} = \{ \nu \in S^{2n-1} : \omega \cdot \nu = t \}$. That is,

$$\int_{-1}^{1} e^{-\frac{1}{2}rt} \tilde{f}(\omega, t) dt = 0,$$

for all $r > 0$. Hence $\tilde{f}(\omega, t) = 0$, for all $t \in (-1, 1)$. Thus by Lemma 2.5, it follows that $\Pi_l(f)(\omega) = 0$ for all $l \in \mathbb{Z}_+$. Further by Lemma 2.4, we get $\Pi_{p,q} f(\omega) = 0$ for all $p, q \in \mathbb{Z}_+$. Thus, by the given condition that $w \notin Y^{-1}_{pq}(0)$ for any $p, q \in \mathbb{Z}_+$, it follows that $f = 0$. That is, $\mu = 0$. \hfill \Box

**Remark 3.2.** (a) Further, we observed that Theorem 3.1 holds for a non-harmonic real cone. Let $C$ be a non-harmonic real cone. Write $\hat{\omega} = \sigma_\omega \omega$, where $\sigma_\omega$ is the symplectic matrix that belongs to $U(n) \subset O(2n)$. Suppose $\mu \in X(S^{2n-1})$ satisfies $\hat{\mu}|_C = 0$. Then $\Pi_l(f(\sigma_\omega \omega)) = 0$ for all $l \in \mathbb{Z}_+$. Since $\sigma_\omega^{-1} \cdot \Pi_l f$ is a spherical harmonic, we infer that $C$ is a HUP corresponding to $S^{2n-1}$ for the symplectic Fourier transform.

(b) It is easily followed by the Euclidean result [32] that $(S^{2n-1}, J^{(n+k-1)}_0)$ is a HUP for the symplectic Fourier transform as long as $r \notin J^{(n+k-1)}_0$ for any $k \in \mathbb{Z}_+$.

4. **Uniqueness pairs for the modified Fourier transform on $H^n$**

In this section, we prove that a finite measure supported on the cylinder $S^{2n-1} \times \mathbb{R}$ can be determined by any non-harmonic cone as well as the boundary of a bounded domain in $\mathbb{C}^n$.

We know that the modified Fourier transform of $f \in L^1(H^n)$ is defined by

$$\hat{f}(\xi, \lambda) = \pi_{\lambda}(\xi) W_\lambda(f^\lambda) \pi_{\lambda}(-\xi),$$

where $W_\lambda(f^\lambda)$ is the Weyl transform of $f^\lambda$ and $(\xi, \lambda) \in \mathbb{C}^n \times \mathbb{R}^*$. This, in turn, can be expressed as

$$\hat{f}(\xi, \lambda) = \int_{\mathbb{C}^n} \pi_{\lambda}(\xi) \pi_{\lambda}(z) f^\lambda(z) \pi_{\lambda}(-\xi) dz,$$

$$= \int_{\mathbb{C}^n} e^{-i\lambda \text{Im}\xi \cdot \bar{z}} f^\lambda(z) \pi_{\lambda}(z) dz.$$

Consider the measure $\mu \in X(S_r \times \mathbb{R})$. Then there exists $f \in L^1(S_r \times \mathbb{R})$ such that $d\mu(\xi, t) = f(\xi, t) d\xi dt$. Define the modified Fourier transform of $\mu$ by

$$\tilde{\mu}(\xi, \lambda) = \int_{S_r} e^{-i\lambda \text{Im}\xi \cdot \bar{z}} f^\lambda(z) \pi_{\lambda}(z) dz.$$

Let $\Lambda = K \times \mathbb{R}^*$, where $K$ is a real/complex cone.
Proposition 4.1. Let \( \mu \in X(S_r \times \mathbb{R}) \). If \( \hat{\mu}(\xi, \lambda) \) is a finite rank operator for all \((\xi, \lambda) \in \Lambda \). Then \( \mu = 0 \) if and only if \( K \) is non-harmonic.

Proof. Since \( \hat{\mu}(\xi, \lambda) \) is a finite rank operator, there exists \( k \in \mathbb{N} \) such that

\[
\langle \hat{\mu}(\xi, \lambda) \varphi, \phi_\alpha \rangle = 0,
\]

whenever \(|\alpha| > k \) and \( \varphi \in L^2(\mathbb{R}^n) \). Set \( \varphi = \phi_\alpha \) and \( \psi = \phi_\alpha, \alpha \in \mathbb{Z}^n \). We know that

\[
\langle \pi_\lambda(z) \varphi, \psi \rangle = c_\alpha z^\alpha e^{-\frac{|z|^2}{4}}.
\]

For the above formula, we refer [29], p.19. Hence from (4.1), we have

\[
\int_{S_r} e^{-i\lambda \text{Im} z \xi} f^\lambda(z) c_\alpha z^\alpha e^{-\frac{|z|^2}{4}} dz = 0
\]

for all \((\xi, \lambda) \in \Lambda \). This reduces to the case of symplectic Fourier transform on \( \mathbb{C}^n \). Hence in view of Theorem 3.1 and Remark 3.2 (a), we infer that \( f^\lambda = 0 \) if and only if \( K \) is non-harmonic. Thus, \( f = 0 \).

\[\square\]

Theorem 4.2. Let \( \partial \Omega \) be the boundary of the bounded domain \( \Omega \) in \( \mathbb{C}^n \). Suppose \( \mu \in X(S_r \times \mathbb{R}) \) satisfies \( \hat{\mu}(\xi, \lambda) = 0 \) for all \((\xi, \lambda) \in \partial \Omega \times \mathbb{R}^\ast \). Then \( \mu = 0 \).

Proof. Since \( \hat{\mu} \) can be extended holomorphically to a function \( F(., \lambda) \) on \( \mathbb{C}^{2n} \) taking values in \( L^2(\mathbb{R}^n) \), it follows that \( F(., \lambda)|_{\mathbb{R}^{2n}} = \hat{\mu} \) is a real analytic function. Consider

\[
\hat{\mu}(\xi, \lambda) = \int_{S_r} e^{-i\lambda \text{Im} z \xi} f^\lambda(z) \pi_\lambda(z) dz.
\]

Then

\[
\frac{\partial}{\partial \xi_j} \hat{\mu}(\xi, \lambda) = -\frac{\lambda}{2} \int_{S_r} \bar{z}_j e^{-i\lambda \text{Im} z \xi} f^\lambda(z) \pi_\lambda(z) dz
\]

and

\[
\frac{\partial^2}{\partial \xi_j \partial \xi_j} \hat{\mu}(\xi, \lambda) = -\frac{\lambda^2}{4} \int_{S_r} \bar{z}_j z_j e^{-i\lambda \text{Im} z \xi} f^\lambda(z) \pi_\lambda(z) dz.
\]

It follows that

\[
\Delta_{\xi} \hat{\mu}(\xi, \lambda) + (r\lambda)^2 \hat{\mu}(\xi, \lambda) = 0.
\]

Now, for \( \varphi, \psi \in L^2(\mathbb{R}^n) \) we have

\[
\Delta_{\xi} \langle \hat{\mu}(\xi, \lambda) \varphi, \psi \rangle + (r\lambda)^2 \langle \hat{\mu}(\xi, \lambda) \varphi, \psi \rangle = 0.
\]

Let \( g(\xi, \lambda) = \langle \hat{\mu}(\xi, \lambda) \varphi, \psi \rangle \). Then \( g \) is a real analytic function, which satisfies

\[
\Delta_{\xi} g(\xi, \lambda) + (r\lambda)^2 g(\xi, \lambda) = 0.
\]

Hence \( g(., \lambda); \lambda \in \mathbb{R}^\ast \) are eigenfunctions of the Dirichlet problem. By the discreetness of eigenvalues of the Dirichlet problem in the bounded domain, it follows that \( g(., \lambda) = 0 \) for all most all \( \lambda \in \mathbb{R}^\ast \). Since \( g(., \lambda) \) is continuous in \( \lambda \), we infer that \( g(\xi, \lambda) = 0 \) for all \((\xi, \lambda) \in \mathbb{C}^n \times \mathbb{R}^\ast \). Thus, \( \mu = 0 \).

\[\square\]
5. Uniqueness pair for the spectral projections

In this section, we derive that a sphere whose radius is not contained in the zero set of any Laguerre polynomial determines the spectral projections of those finite measure on \( C^n \) which are supported on \( S^{2n-1} \). Further, we deduce that non-harmonic complex cone as well as \( NA \)-set, also determine the spectral projections of the above class of measures.

Let \( S_r = \{ z \in C^n : |z| = r \} \). For \( \mu \in X(S_r) \), we define the spectral projection of \( \mu \) by

\[
\varphi_k^{n-1} \times \mu(z) = \int_{S_r} \varphi_k^{n-1}(z - w)e^{i\text{Im}(z,w)}d\mu(w).
\]

We prove the following result.

**Theorem 5.1.** Let \( \mu \in X(S_{r_1}) \) be such that \( \varphi_k^{n-1} \times \mu(z) = 0 \) for all \( k \in \mathbb{Z}_+ \) and for all \( z \in S_{r_2} \). Then \( \mu = 0 \) provided \( r_i \notin (\varphi_{k-q}^{n+p+q-1})^{-1}(0) \), for \( i = 1, 2 \) and for all \( k, p, q \in \mathbb{Z}_+ \).

**Proof.** Since \( \mu \in X(S_{r_1}) \), there exists \( f \in L^1(S_{r_1}) \) such that \( d\mu = f d\sigma \). Thus,

\[
(5.1) \quad \varphi_k^{n-1} \times \mu(z) = \int_{S_{r_1}} \varphi_k^{n-1}(z - w)e^{i\text{Im}(z,w)}f(w)dw = 0,
\]

\( \forall k \in \mathbb{Z}_+ \) and for all \( z \in S_{r_2} \). As \( f \in L^1(S_{r_1}) \), \( f \) will satisfy

\[
f = \lim_{m \to \infty} \sum_{l=0}^{m} A_l^m(\delta)\Pi_l f,
\]

where \( A_l^m(\delta) = \left( \frac{m-l+\delta}{\delta} \right) \left( \frac{m+\delta}{\delta} \right)^{-1} \) and \( \delta > n - 1 \). Further, from the Lemma 2.2, it follows that

\[
f = \lim_{m \to \infty} \sum_{p+q=0}^{m} A_{p+q}^m(\delta)\Pi_{p,q} f.
\]

Now from condition \((5.1)\), it follows that

\[
\sum_{p+q=0}^{m} A_{p+q}^m(\delta)\varphi_k^{n-1} \times \Pi_{p,q} f(z) = \sum_{p+q=0}^{m} A_{p+q}^m(\delta)\varphi_k^{n-1} \times \Pi_{p,q} f(z) - \varphi_k^{n-1} \times f(z)
\]

\[
\leq M_k(z) \int_{S_{r_1}} \sum_{p+q=0}^{m} A_{p+q}^m(\delta)\Pi_{p,q} f(w) - f(w) \, dw,
\]

where \( |\varphi_k^{n-1}(z - w)| \leq M_k(z) \). Hence in view of \((2.8)\), we deduce that

\[
(5.2) \quad \lim_{m \to \infty} \sum_{p+q=0}^{m} A_{p+q}^m(\delta)\varphi_k^{n-1} \times \Pi_{p,q} f(z) = 0,
\]
whenever \( k \in \mathbb{Z}_+ \) and \( z \in S_{r_2} \). From Lemma 2.1, we get
\[
\int_{S^{2n-1}} \varphi_k^{n}(z - r_1 \eta) e^{z r_1 \text{Im}(z \cdot \eta)} \pi_{p,q}^c(\eta) d\eta = B_n^{k,\gamma}(r_1) \varphi_k^{\gamma-1}(r_1) \varphi_k^{\gamma-1}(z) P_{p,q}(z),
\]
where \( B_n^{k,\gamma} = (2\pi)^{-n} \Gamma(k - q + 1) \Gamma(k + n + p) \) and \( \gamma = n + p + q \). Let \( z = r_2 \xi \) and \( \xi \in S^{2n-1} \).

From (5.2), we have
\[
\lim_{m \to \infty} A_{p,q}(\delta) B_n^{k,\gamma}(r_1 r_2) \varphi_k^{\gamma-1}(r_1) \varphi_k^{\gamma-1}(r_2) \Pi_{p,q} f(\xi) = 0.
\]

Since the bi-graded spherical harmonic projections \( \Pi_{p,q} f \) are orthogonal among themselves and \( \lim_{m \to \infty} A_{p,q}(\delta) = 1 \) holds for every choice of \( p, q \in \mathbb{Z}_+ \), from (5.4) we infer that
\[
\varphi_k^{\gamma-1}(r_1) \varphi_k^{\gamma-1}(r_2) \| \Pi_{p,q} f \|_2 = 0.
\]

Hence, we conclude that \( \Pi_{p,q} f = 0 \) if \( r = (\varphi_k^{\gamma-1})^{-1}(0) \) for all \( k, \gamma \in \mathbb{Z}_+ \). Thus, \( f = 0 \).

**Remark 5.2.** A set, which is determining set for any real analytic function, is called \( NA \) - set. For instance, the spiral is an \( NA \) - set in the plane (see [18]). Since the spectral projection \( \varphi_k^{n-1} \times \mu \) can be extended holomorphically on \( \mathbb{C}^{2n} \), the function \( \varphi_k^{n-1} \times \mu \) must be real analytic on \( \mathbb{C}^n \).

Let \( \Lambda \) be an \( NA \)-set for real analytic functions on \( \mathbb{C}^n \). If \( \mu \in X(S_r) \) satisfies \( \varphi_k^{n-1} \times \mu |_\Lambda = 0 \) for all \( k \in \mathbb{Z}_+ \), then \( \varphi_k^{n-1} \times \mu(z) = 0 \) for all \( z \in \mathbb{C}^n \). Now, let \( z = s \xi \), where \( s > 0 \) and \( \xi \in S^{2n-1} \). Then in view of (5.5), we get
\[
\varphi_k^{\gamma-1}(r) \varphi_k^{\gamma-1}(s) \| \Pi_{p,q} f \|_2 = 0
\]
for all \( s > 0 \). Hence, we infer that \( \Pi_{p,q} f = 0 \) if \( r \notin (\varphi_k^{\gamma-1})^{-1}(0) \) for all \( k, \gamma \in \mathbb{Z}_+ \). Thus, \( f = 0 \).

Next, we shall prove that spectral projections of a finite measure supported on a sphere are determined by a non-harmonic complex cone.

**Theorem 5.3.** Let \( \Lambda \) be a complex cone in \( \mathbb{C}^n \). Suppose \( \mu \in X(S_r) \) satisfies \( \varphi_k^{n-1} \times \mu |_\Lambda = 0 \) for all \( k \in \mathbb{Z}_+ \). Then \( \mu = 0 \) if and only if \( \Lambda \) is non-harmonic.

**Proof.** From (5.4), it follows that
\[
\lim_{m \to \infty} \sum_{p+q=0}^{m} A_{p,q}(\delta) B_n^{k,\gamma}(r s) \varphi_k^{\gamma-1}(r) \varphi_k^{\gamma-1}(s) \Pi_{p,q} f(\xi) = 0,
\]
for all \( s \xi \in \Lambda \) and for all \( k \in \mathbb{Z}_+ \). Since the complex cone \( \Lambda \) is closed under complex scaling, on replacing \( \xi \) by \( e^{i\theta} \xi \), we obtain
\[
\lim_{m \to \infty} \sum_{p+q=0}^{m} A_{p,q}(\delta) B_n^{k,\gamma}(r s) e^{i(p-q)\theta} \varphi_k^{\gamma-1}(r) \varphi_k^{\gamma-1}(s) \Pi_{p,q} f(\xi) = 0,
\]
for all \( \theta \in \mathbb{R} \). Therefore, we conclude that \( \mu = 0 \) if and only if \( \Lambda \) is non-harmonic.
for all $s\xi \in \Lambda$ and for all $k \in \mathbb{Z}_+$. Now, by induction on $k$, we show that all of the projections $\Pi_{p,q}$ are zero on $\Lambda$. For $k = 0$, the choice for $q = 0$. Since the set $\{e^{i\theta p} : p \in \mathbb{Z}_+\}$ is an orthonormal set, we infer that $\Pi_{p,0}(f)(\xi) = 0$. Similarly, for $k = 1$, the choices for $q = 0, 1$. The case $q = 0$ is already settled. Now, for $q = 1$, the set $\{e^{i(p-1)\theta} : p \in \mathbb{Z}_+\}$ is an orthonormal set. Hence $\Pi_{p,1}(f)(\xi) = 0$. This, in turn, implies that each set of horizontal projections are vanishing on $\Lambda$. Hence, we conclude that $\Pi_{p,q} f|_\Lambda = 0$. Thus, $f = 0$ if and only if $\Lambda$ is non-harmonic.

6. Benedick-Amrein-Berthier type theorem

In this section, we prove Benedick-Amrein-Berthier type theorem for the Heisenberg group using twisted translations on $\mathbb{C}^n$.

For $\lambda = 1$, we identify $W_1(g)$ with $W(g)$. Let $g \in L^2(\mathbb{C}^n)$ and $W(g)$ be a finite rank operator. Then there exists an orthonormal basis $\{e_1, e_2, \ldots\}$ of $L^2(\mathbb{R}^n)$ such that $\mathcal{R}(W(g)) = \text{span}\{e_1, \ldots, e_N\}$, where $\mathcal{R}$ stands for the range. Define an orthogonal projection on $L^2(\mathbb{R}^n)$ by $P_N\varphi = \sum_{j=1}^{N} c_j e_j$, where $\varphi \in L^2(\mathbb{R}^n)$ and $\varphi = \sum_{j=1}^{\infty} c_j e_j$. For a measurable set $A$ in $\mathbb{C}^n$, we define a pair of orthogonal projections $E_A$ and $F_N$ on $L^2(\mathbb{C}^n)$ by

$$E_A g = \chi_A g \quad \text{and} \quad W(F_N g) = P_N W(g),$$

where $\chi_A$ is the characteristic function. Then we can express the range of $E_A$ and $F_N$ by $\mathcal{R}(E_A) = \{g \in L^2(\mathbb{C}^n) : g = \chi_A g\}$ and

$$\mathcal{R}(F_N) = \{g \in L^2(\mathbb{C}^n) : \mathcal{R}(W(g)) \subset \text{span}\{e_1, \ldots, e_N\}\}.$$

First we prove that $E_A F_N$ is a Hilbert-Schmidt operator that satisfies

$$\|E_A F_N\|^2_{HS} = m(A)N.$$

Throughout this section, we shall assume that $A$ is a set of finite Lebesgue measure.

**Lemma 6.1.** $E_A F_N$ is an integral operator on $L^2(\mathbb{C}^n)$.

**Proof.** For $g \in L^2(\mathbb{C}^n)$, we have $W(F_N g) = P_N W(g)$. By inversion formula for the Weyl transform

$$(F_N g)(z) = \text{tr}(\pi(z)^* W(F_N g)) = \text{tr}(\pi(-z) P_N W(g))$$

$$= \text{tr}(P_N W(g) \pi(-z))$$

$$= \int_{\mathbb{C}^n} g(\omega) \text{tr}(P_N \pi(\omega) \pi(-z)) \, d\omega.$$
Hence it follows that

\[(E_A F_N g)(z) = \chi_A(z)(F_N g)(z) = \chi_A(z) \int_{\mathbb{C}^n} g(\omega) \text{tr}(P_N \pi(\omega) \pi(-z)) \, d\omega\]

\[= \int_{\mathbb{C}^n} g(\omega) K(z, \omega) \, d\omega,\]

where \(K(z, \omega) = \chi_A(z) \text{tr}(P_N \pi(\omega) \pi(-z)).\) We infer that \(E_A F_N\) is an integral operator with kernel \(K.\)

\[\square\]

**Lemma 6.2.** \(E_A F_N\) is a Hilbert-Schmidt operator with \(\|E_A F_N\|^2_{HS} = m(A)N.\)

**Proof.** From Lemma 6.1, we know that \(E_A F_N\) is an integral operator with kernel \(K(z, \omega).\) Therefore,

\[\|E_A F_N\|^2_{HS} = \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |K(z, \omega)|^2 \, d\omega \, dz\]

\[= \int_{\mathbb{C}^n} |\chi_A(z)|^2 \left( \int_{\mathbb{C}^n} |\text{tr}(P_N \pi(\omega) \pi(-z))|^2 \, d\omega \right) \, dz\]

\[= \int_{\mathbb{C}^n} \chi_A(z) \left( \int_{\mathbb{C}^n} \left| \sum_{j=1}^N \langle \pi(\omega) \pi(-z) e_j, e_j \rangle \right|^2 \, d\omega \right) \, dz\]

\[(6.1)\]

\[= \int_{\mathbb{C}^n} \chi_A(z) \left( \int_{\mathbb{C}^n} \left| \sum_{j=1}^N \langle \pi(-z) e_j, \pi(-\omega) e_j \rangle \right|^2 \, d\omega \right) \, dz.\]

Now, we shall simplify the integrand of the above integral (6.1). So, for \(z = x + iy\) and \(\omega = u + iv\) in \(\mathbb{C}^n,\)

\[\sum_{j=1}^N \langle \pi(z)e_j, \pi(\omega)e_j \rangle^2 = \left| \sum_{j=1}^N \int_{\mathbb{R}^n} \pi(z)e_j(\xi) \cdot \pi(\omega)e_j(\xi) \, d\xi \right|^2\]

\[= \left| \sum_{j=1}^N e^{\frac{i}{2} (x - y - u - v)} \int_{\mathbb{R}^n} e^{-i\xi \cdot (u - x)} \cdot e^{yv}(\xi) \, d\xi \right|^2\]

\[= \left| e^{\frac{i}{2} (x - y - u - v)} \sum_{j=1}^N \widehat{e^y}_{j \pi}(u - x) \right|^2\]

\[(6.2)\]

\[= \left| \sum_{j=1}^N \widehat{e^y}_{j \pi}(u - x) \right|^2,\]

where \(e^y_j(\xi) = e_j(y + \xi) \overline{e_j(v + \xi)}.\) For \(m, n \in \mathbb{N},\) we shall show that

\[\int_{\mathbb{R}^{2n}} \widehat{e^y}_{m}(u - x) \cdot \overline{\widehat{e^y}_{m}(u - x)} \, du dv = \delta_{mn}.\]

\[(6.3)\]
By the Parseval identity
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \overline{e_{m}^{\nu}(u - x)} \cdot e_{n}^{\nu}(u - x) dv \, du = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e_{m}^{\nu}(u - x) \cdot \overline{e_{n}^{\nu}(u - x)} dv \, du
\]
\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e_{m}(y + u - x) e_{m}(v + u - x) \cdot e_{n}(y + u - x) e_{n}(v + u - x) dv \, du
\]
\[
= \int_{\mathbb{R}^n} e_{m}(y + u - x) e_{n}(y + u - x) \left( \int_{\mathbb{R}^n} e_{m}(v + u - x) e_{n}(v + u - x) dv \right) du.
\]
Hence, from (6.16.3), we infer that
\[
\|E_{A} F_{N}\|_{HS}^{2} = \int_{\mathbb{R}^n} \chi_{A}(-z) \left( \int_{\mathbb{R}^n} \left| \sum_{j=1}^{N} \langle \pi(z) e_{j}, \pi(\omega) e_{j} \rangle \right|^{2} \, dv \right) \, dz
\]
\[
= \int_{\mathbb{R}^n} \chi_{A}(-z) \left( \int_{\mathbb{R}^n} \left| \sum_{j=1}^{N} e_{j}^{\nu}(u - x) \right|^{2} \, dv \right) \, dz = m(A) N.
\]
\]

We need the following result from \cite{2}, which describes an interesting property of a measurable set. Denote \(\omega A = \{ z \in \mathbb{C}^n : z - \omega \in A \}\).

**Lemma 6.3.** \cite{2} Let \(C\) be a measurable set in \(\mathbb{C}^n\) with \(0 < m(C) < \infty\). If \(C_0\) be a measurable subset of \(C\) with \(m(C_0) > 0\), then for \(\epsilon > 0\) there exists \(\omega \in \mathbb{C}^n\) such that
\[
m(C) < m(C \cup \omega C_0) < m(C) + \epsilon.
\]

For orthogonal projections \(E\) and \(F\) on a Hilbert space \(\mathcal{H}\), let \(E \cap F\) denote the orthogonal projection on \(\mathcal{H}\) onto \(\mathcal{R}(E) \cap \mathcal{R}(F)\). Then we have the relation
\[
\|E \cap F\|_{HS}^{2} = \dim \mathcal{R}(E \cap F) \leq \|EF\|_{HS}^{2}.
\]
We abbreviate \(A' = \mathbb{C}^n \setminus A\) and \(F_{N}^{\perp} = I - F_{N}\).

**Proposition 6.4.** Let \(A\) be a measurable subset of \(\mathbb{C}^n\) having finite Lebesgue measure. Then \(E_{A} \cap F_{N} = 0\).

**Proof.** We shall prove the result by contradiction. Suppose there exists a non-zero function \(g_0 \in \mathcal{R}(E_{A} \cap F_{N})\). Then for \(\varphi \in L^{2}(\mathbb{R}^n)\),
\[
W(g_{0}) \varphi = c_{1} e_{1} + \cdots + c_{N} e_{N},
\]
for some \(c_{i} \in \mathbb{C}\) depending on \(\varphi\). Let \(A_{0} = \{ x \in A : g_{0}(x) \neq 0 \}\). Clearly, \(0 < m(A_0) < \infty\). Choose \(s \in \mathbb{N}\) that satisfies \(s > 2m(A_0) N\). We shall define an increasing sequence of measurable sets \(\{ A_{l} : l = 1, \ldots, s \}\) containing \(A_{0}\). By applying Lemma 6.3 for \(\epsilon = \frac{1}{2N}\), \(C_{0} = A_{0}\) and \(C = A_{l-1}\), there exists \(\omega_{l} \in \mathbb{C}^n\) such that
\[
m(A_{l-1}) < m(A_{l-1} \cup \omega_{l} A_{0}) < m(A_{l-1}) + \frac{1}{2N}.
\]
Write $A_l = A_{l-1} \cup \omega_l A_0$. In view of (6.4) and Lemma 6.2, we obtain
\[
\dim \mathcal{R}(E_{A_l} \cap F_N) \leq m(A) N \leq \left( m(A_0) + \frac{s}{2N} \right) N < s.
\]
Now, we construct $s+1$ linearly independent functions in $\mathcal{R}(E_{A_l} \cap F_N)$ with the help of twisted translations on $\mathbb{C}^n$. Let $g_l(z) = e^{\frac{i}{2} l m(z - \omega_l)} g_0(z - \omega_l)$. We show that $g_l \in \mathcal{R}(F_N)$ for each $l = 1, \ldots, s$. Since $\pi(z) \pi(\omega) = e^{\frac{i}{2} l m(z \cdot \omega)} \pi(z + \omega)$, for $\varphi \in L^2(\mathbb{R}^n)$ and $j > N$, we have
\[
\langle W(g_l) \varphi, e_j \rangle = \int_{\mathbb{C}^n} g_l(z) \langle \pi(z) \varphi, e_j \rangle \, dz
= \int_{\mathbb{C}^n} e^{\frac{i}{2} l m(z \cdot \omega_l)} g_0(z - \omega_l) \langle \pi(z) \varphi, e_j \rangle \, dz
= \int_{\mathbb{C}^n} e^{\frac{i}{2} l m(z \cdot \omega_l)} g_0(z) \langle \pi(z + \omega_l) \varphi, e_j \rangle \, dz
= \int_{\mathbb{C}^n} g_0(z) \langle \pi(z) \pi(\omega_l) \varphi, e_j \rangle \, dz
= \int_{\mathbb{C}^n} g_0(z) \langle \pi(z) \psi, e_j \rangle \, dz
= \langle W(g_0) \psi, e_j \rangle = 0.
\]
Hence $\mathcal{R}(W(g_l)) \subset \text{span}\{e_1, \ldots, e_N\}$. Since $A_m = A_0 \cup \omega_1 A_0 \cup \cdots \cup \omega_m A_0$ and $g_l(z) = 0$ on $(\omega_l A_0)'$, we have $E_{A_m} g_l = g_l$ for $l = 1, \ldots, m$. Furthermore, $E_{A_m} \setminus A_{m-1} g_l = 0$ for $l = 1, \ldots, m-1$ and $E_{A_m} \setminus A_{m-1} g_m \neq 0$. Therefore, it shows that $g_m$ is not a linear combination of $g_0, \ldots, g_{m-1}$. Hence, $g_0, \ldots, g_s$ are linearly independent functions in $\mathcal{R}(E_{A_s} \cap F_N)$, which is a contradiction.

**Remark 6.5.** If $0 < m(A) < \infty$, then $\dim \mathcal{R}(E_A) = \infty$. Now, in view of Proposition 6.4 and the fact that $E_A = (E_A \cap F_N) + (E_A \cap F_N^c) = (E_A \cap F_N^c)$, it follows that $\dim \mathcal{R}(E_A \cap F_N^c) = \infty$. Since $m(A') = \infty$, there exists a measurable set $B \subset A'$ satisfying $0 < m(B) < \infty$. Hence $\mathcal{R}(E_A \cap F_N^c) \supset \mathcal{R}(E_B \cap F_N^c)$. This implies $\dim \mathcal{R}(E_A \cap F_N^c) = \infty$. Similarly, $\dim \mathcal{R}(E_A \cap F_N^c) = \infty$.

The following theorem is the main result of this section which is analogous to Benedick-Amrein-Berthier theorem for the Heisenberg group.

**Theorem 6.6.** Let $A \subset \mathbb{C}^n$ be a set of finite Lebesgue measure. Suppose $f \in L^1(\mathbb{H}^n)$ and \{$(z, t) \in \mathbb{H}^n : f(z, t) \neq 0$\} $\subset A \times \mathbb{R}$. If $\hat{f}(\lambda)$ is a finite rank operator for each $\lambda \in \mathbb{R}^*$, then $f = 0$.

In order to prove Theorem 6.6, it is sufficient to prove the following result for the Weyl transform which is the most non-commutative constituent of the group Fourier transform on the Heisenberg group.

**Proposition 6.7.** Let $g \in L^1(\mathbb{C}^n)$ and \{ $z \in \mathbb{C}^n : g(z) \neq 0$ \} $\subset A$, where $m(A)$ is finite. Let $\lambda \in \mathbb{R}^*$ and $W_\lambda(g)$ has finite rank. Then $g = 0$. 
Since $W_{\lambda}(g)$ is a finite rank operator, by the Plancheral theorem for the Weyl transform, $g \in L^{2}(\mathbb{C}^{n})$. Hence, it is enough to prove Proposition 6.7 for $g \in L^{2}(\mathbb{C}^{n})$ and $\lambda = 1$. Proposition 6.7 follows from Proposition 6.4.

7. Benedick-Amrein-Berthier type theorem for step two nilpotent Lie groups

In this section, we give a summary of analogous results for general step two nilpotent Lie groups.

Let $G$ be connected, simply connected Lie group with real step two nilpotent Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ has the orthogonal decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$. Let $\{X_{1},\ldots,X_{m}\}$ and $\{T_{1},\ldots,T_{k}\}$ be orthonormal basis of $\mathfrak{b}$ and $\mathfrak{z}$ respectively. Since $\mathfrak{g}$ is nilpotent, the exponential map $\exp : \mathfrak{g} \rightarrow G$ is surjective. Hence $G$ can be identified with $\mathfrak{b} \oplus \mathfrak{z}$. Thus, we can write $X + T \in \mathfrak{b} \oplus \mathfrak{z}$ for $\exp(X + T)$ and denote it by $(X,T) \in \mathbb{R}^{m} \times \mathbb{R}^{k}$. Since $[\mathfrak{b},\mathfrak{b}] \subset \mathfrak{z}$ and $[\mathfrak{b},[\mathfrak{b},\mathfrak{b}]] = 0$, by the Baker-Campbell-Hausdorff formula group law can be expressed as

$$(X,T)(X',T') = (X + X', T + T' + \frac{1}{2}[X,X'])$$

for all $X,X' \in \mathfrak{b}$ and $T,T' \in \mathfrak{z}$. Let $\mathfrak{z}^{*}$ be the real dual of $\mathfrak{z}$. For each $\lambda \in \mathfrak{z}^{*}$, define the bilinear form $B_{\lambda}$ on $\mathfrak{b}$ by

$$B_{\lambda}(X,Y) = \lambda([X,Y]), \text{ for all } X,Y \in \mathfrak{b}.$$ 

Let $m_{\lambda}$ be the orthogonal complement of $r_{\lambda} = \{X : B_{\lambda}(X,Y) = 0, \forall Y \in \mathfrak{b}\}$ in $\mathfrak{b}$. Then $\Lambda = \{\lambda \in \mathfrak{z}^{*} : \dim m_{\lambda} \text{ is maximum}\}$ is a Zariski open subset of $\mathfrak{z}^{*}$. Now, general step two nilpotent Lie groups can be studied in two different cases. For more details, please refer to [7, 16, 17].

Step two nilpotent Lie groups with MW-condition: In this case, $r_{\lambda} = \{0\}$ for each $\lambda \in \Lambda$ and the irreducible unitary representations are parameterized by $\Lambda$. This is called M´etivier group.

Step two nilpotent Lie groups without MW-condition: In this case $r_{\lambda} \neq \{0\}$ for each $\lambda \in \Lambda$ and $B_{\lambda}|_{m_{\lambda}}$ is non-degenerate, hence $\dim m_{\lambda}$ even. Let $\{X_{1}(\lambda),\ldots,X_{n}(\lambda),Y_{1}(\lambda),\ldots,Y_{n}(\lambda),Z_{1}(\lambda),\ldots,Z_{r}(\lambda)\}$ be an orthonormal basis of $\mathfrak{b}$ and $d_{j}(\lambda) > 0$ be satisfying

1. $r_{\lambda} = \text{span}\{Z_{1}(\lambda),\ldots,Z_{r}(\lambda)\}$,
2. $\lambda([X_{i}(\lambda),Y_{j}(\lambda)]) = \delta_{ij}d_{i}(\lambda)$, for $1 \leq i,j \leq n$ and
   $\lambda([X_{i}(\lambda),X_{j}(\lambda)]) = 0, \lambda([Y_{i}(\lambda),Y_{j}(\lambda)]) = 0$, for $1 \leq i,j \leq n$.

Since the main result for M´etivier group will be similar to the case “without MW-condition,” we will not discuss it.

Let $\xi_{\lambda} = \text{span}\{X_{1}(\lambda),\ldots,X_{n}(\lambda)\}$ and $\eta_{\lambda} = \text{span}\{Y_{1}(\lambda),\ldots,Y_{n}(\lambda)\}$. Then we have the decomposition $\mathfrak{g} = \xi_{\lambda} \oplus \eta_{\lambda} \oplus r_{\lambda} \oplus \mathfrak{z}$. For $\lambda \in \Lambda, \mu \in r_{\lambda}^{\perp}$ the
irreducible unitary representation $\pi_{\lambda,\mu}$ of $G$ realized on $L^2(\eta_\lambda)$.

\[
(\pi_{\lambda,\mu}(x, y, z, t)\varphi)(\xi) = e^{i\sum_{j=1}^n \lambda_j t_j + i\sum_{j=1}^r \mu_j z_j + i\sum_{j=1}^n d_j(\lambda)(x_j \xi_j + z_j y_j)}\varphi(\xi + y),
\]
where $\varphi \in L^2(\eta_\lambda)$. Then the Fourier transform of $f \in L^1(G)$ can be expressed as

\[
\hat{f}(\lambda, \mu) = \int_{\Lambda} \int_{\eta_\lambda} \int_{\xi_\lambda} f(x, y, z, t)\pi_{\lambda,\mu}(x, y, z, t)dxdydzdt.
\]

Let $Pf(\lambda) = \prod_{j=1}^n d_j(\lambda)$. Consider inverse Fourier transform of $f$ in $t$ and $(z, t)$ variables as follows.

\[
f^\lambda(x, y, z) = \int_{\Lambda} e^{i\sum_{j=1}^n \lambda_j t_j} f(x, y, z, t)dt,
\]

\[
f^{\lambda,\mu}(x, y) = \int_{\Lambda} \int_{\eta_\lambda} \int_{\xi_\lambda} e^{i\sum_{j=1}^n \lambda_j t_j + i\sum_{j=1}^r \mu_j z_j} f(x, y, z, t)dtdz.
\]

If $f \in L^1 \cap L^2(G)$, then $\hat{f}(\lambda, \mu)$ is a Hilbert Schmidt operator on $L^2(\eta_\lambda)$ and satisfies (see [20])

\[
(7.1) \quad Pf(\lambda) \parallel \hat{f}(\lambda, \mu) \parallel^2_{HS} = (2\pi)^n \int_{\eta_\lambda} \int_{\xi_\lambda} |f^{\lambda,\mu}(x, y)|^2dxdy.
\]

For $f \in L^2(G)$ we have the Plancherel formula (see [16])

\[
(7.2) \quad \int_{\Lambda} \int_{\eta_\lambda} Pf(\lambda) \parallel \hat{f}(\lambda, \mu) \parallel^2_{HS} d\mu d\lambda = (2\pi)^\gamma \int_{\Lambda} \int_{\eta_\lambda} \int_{\xi_\lambda} |f(x, y, z, t)|^2dxdydzdt,
\]

where $\gamma = n + r + k$.

The following theorem is analogous to Benedick-Amrein-Berthier theorem for step two nilpotent Lie groups.

**Theorem 7.1.** Let $f \in L^2(G)$ with $\{(x, y, z, t) \in G : f(x, y, z, t) \neq 0\} \subset A \times \mathbb{R}^r \times \mathbb{R}^k$, where $m(A)$ is finite. If $\hat{f}(\lambda, \mu)$ is a finite rank operator for all $\lambda \in \Lambda$ and for all $\mu \in r^*_\Lambda$, then $f = 0$.

For each $\lambda \in \Lambda$, $\mu \in r^*_\Lambda$ and $g \in L^1 \cap L^2(\xi_\lambda \oplus \eta_\lambda)$ define Weyl transform of $g$ by

\[
W_{\lambda,\mu}(g) = \int_{\xi_\lambda} \int_{\eta_\lambda} g(x, y)\pi_{\lambda,\mu}(x, y)dxdy,
\]

where $\pi_{\lambda,\mu}(x, y) = \pi_{\lambda,\mu}(x, y, 0, 0)$. Then inversion formula of $W_{\lambda,\mu}$ will be of the form

\[
(7.3) \quad g(x, y) = (2\pi)^{-n} Pf(\lambda)\text{tr}(\pi_{\lambda,\mu}(x, y)W_{\lambda,\mu}(g)).
\]

**Proposition 7.2.** Let $\lambda \in \Lambda$, $\mu \in r^*_\Lambda$ and $g \in L^1(\xi_\lambda \oplus \eta_\lambda)$ with $\{(x, y) : g(x, y) \neq 0\} \subset A$, where $m(A)$ is finite. If $W_{\lambda,\mu}(g)$ has finite rank, then $g = 0$. 

Proof of Proposition 7.2 is similar to Proposition 6.7 and hence we omit it here. Finally, from Proposition 7.2, we get $f_{\lambda, \mu} = 0$ for all $\lambda \in \Lambda, \mu \in r^*_\lambda$. Thus, the proof of Theorem 7.1 will be followed from (7.1) and (7.2).

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