UNBOUNDED REPRESENTATIONS OF $q$-DEFORMATION OF CUNTZ ALGEBRA

VASYL OSTROVSKYI, DANIIL PROSKURIN, AND LYUDMILA TUROWSKA

To the memory of Leonid Vaksman

Abstract. We study a deformation of the Cuntz-Toeplitz $C^*$-algebra determined by the relations $a^*a_i = 1 + qa_ia_i^*$, $a_i^*a_j = 0$. We define well-behaved unbounded $*$-representations of the $*$-algebra defined by relations above and classify all such irreducible representations up to unitary equivalence.

Introduction

Many of the structures that have been studied recently arise as deformations of classical objects, e.g. deformations of the canonical commutation relations (CCR) and the canonical anti-commutation relations (CAR), quantum groups, quantum homogeneous spaces, non-commutative probability etc (see [3, 5, 9, 15, 21, 23, 24]). From the physical point of view important classes of objects come from the Fock space formalism forming algebras generated by raising and lowering operators and their numerous generalizations. Examples of such generalizations include $q$-deformed quantum oscillator algebra ([3, 9, 15]), twisted CCR ([21]), generalised deformed oscillator ([8]) and more general quadratic algebras with Wick ordering ([14]). Here unbounded representations arise naturally since most of the physical observables can not be realized by bounded operators. During the past 30 years many works concerning (topological) algebras of unbounded operators and their physical applications appeared in the literature (see e.g. [1, 10, 12, 22] and references therein).

One of the most known $q$-deformations of CCR is $q$-CCR with one degree of freedom, see [3, 15], i.e. the $*$-algebra which is generated by elements $a, a^*$ satisfying the commutation relation

\[ a^*a = 1 + qaa^*, \]

where $q \in [0, 1)$.

For many degree of freedom there exist several versions of $q$-CCR algebras, see [4, 9, 16, 21]. In this paper we consider a subclass of $q_{ij}$-CCR algebras introduced in [4], namely $*$-algebras, denoted later by $O_n^q$, which are generated by $a_i, a_i^*$, $i = 1, \ldots, n$, subject to the relations

\[ a_i^*a_i = 1 + qa_ia_i^*, \quad a_i^*a_j = 0, \quad a_i^*a_j = 0, \quad i \neq j, \quad i, j = 1, \ldots, n, \quad 0 < q < 1. \]

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Note that for \( q = 0 \) we obtain the \( \ast \)-algebra, \( O^0_n \), generated by \( n \) isometries with orthogonal ranges, i.e.

\[
O^0_n = \mathbb{C} \langle s_i, s^*_i s_j = \delta_{ij}1, \ i, j = 1, \ldots, n \rangle.
\]

Its enveloping \( C^* \)-algebra is an extension of the Cuntz \( C^* \)-algebra \( O_n \) by compact operators whose representation theory was extensively studied (see [6, 7] and references therein).

Our aim is to describe irreducible unbounded representations of \( O^q_n, n > 1 \). Note that each \( O^q_n, n > 1 \), has also bounded representations, however, since the corresponding universal enveloping \( C^* \)-algebra \( O^q_n \) is not of type I algebra and the problem of unitary classification of its irreducible \( \ast \)-representations is complicated. Nevertheless it turns out to be possible to classify all its “well-behaved” unbounded irreducible \( \ast \)-representations up to unitary equivalence. Unbounded representations are known to be a very delicate thing, since unbounded operators are not defined on the whole space. Depending on chosen domains of representations they can behave differently (see, e.g. [18, 22]). In the theory of \( \ast \)-representations of finite-dimensional Lie algebras the class of well-behaved representations form the representations which can be integrated to unitary representations of the corresponding simply connected Lie group. Nelson’s fundamental theorem (see [2, 18]) gives a criterion for the integrability in terms of the Laplace operator of the Lie algebra, requiring its essential self-adjointness on a common invariant dense domain. Our definition of well-behaved representation is motivated by this issue.

The well-known Stone-von Neumann theorem says that up to unitary equivalence there exists a unique irreducible representation of CCR which is unbounded and given by raising and lowering operators; the representation is often called the Fock representation. However, for \( q \)-CCR and, as we will see, for \( O^q_n, q \in (0,1) \), the Fock representation is bounded, and a whole bunch of irreducible unbounded representations which do not have any classical analogs, arises (for \( q \)-CCR see for example [19]).

The paper is organized as follows. In Preliminaries we recall a classification of irreducible representations of \( O^q_1 \) (or \( q \)-CCR) and prove an isomorphism of the enveloping \( C^* \)-algebras of \( O^q_n, q \in (0,1) \), to that of \( O^0_n \). As a consequence we get that the description of bounded \( \ast \)-representations of \( O^q_n \) is equivalent to that of \( \ast \)-representations of the Cuntz-Toeplitz algebra \( O^0_n \).

In Section 2.1 we give a definition of well-behaved unbounded \( \ast \)-representation of \( O^q_n \) in spirit of [21] and also present an equivalent one in terms of bounded operators.

Finally, in Section 2.2 we obtain a classification of all irreducible well-behaved unbounded \( \ast \)-representations of \( O^q_n \) up to unitary equivalence.

1. Preliminaries

Recall some facts on representation theory and properties of the universal enveloping \( C^* \)-algebra of \( q \)-CCR or \( O^q_1, 0 \leq q < 1 \), see [20]. If \( q = 0 \) we get the well-known \( \ast \)-algebra generated by a single isometry. Obviously any representation of \( O^q_1 \) is bounded. Moreover, any irreducible representation of \( O^0_1 \), up to a unitary equivalence, is either one-dimensional \( a_1 = \exp \varphi, \varphi \in [0,2\pi] \), or the
infinite-dimensional, called also the *Fock* representation, given by the action
\[ a_1 e_n = e_{n+1}, \quad n \in \mathbb{N}, \]
on an orthonormal basis \( \{ e_n : n \in \mathbb{N} \} \) in \( l_2(\mathbb{N}) \).

When \( 0 < q < 1 \), unbounded representations will also arise. Defining “well-behaved” unbounded representations as in Section 2.1 one has the following

**Proposition 1.** Any irreducible representation of \( O_q^1 \) with \( 0 < q < 1 \) is unitarily equivalent to exactly one listed below.

1. The Fock representation acting on \( l_2(\mathbb{N}) \):
\[ \pi_F(a_1) e_n = \sqrt{\frac{1-q^n}{1-q}} e_{n+1}, \quad n \in \mathbb{N}. \]

2. One-dimensional representations:
\[ \pi_\varphi(a_1) e = \sqrt{\frac{1}{1-q}} \exp(i\varphi) e, \quad \varphi \in [0, 2\pi). \]

3. Unbounded representations acting on \( l_2(\mathbb{Z}) \):
\[ \pi_x(a_1) e_n = \sqrt{\frac{1-q^n}{1-q} + qx} e_{n+1}, \quad n \in \mathbb{Z}, \quad x \in (1 + qx_0, x_0] \]
where \( x_0 > \frac{1}{1-q} \) is fixed.

One of the fundamental facts on representation theory of \( O_q^1 \) is the Wold decomposition theorem, stating that any isometric operator is an orthogonal direct sum of a multiple of the unilateral shift and a unitary operator (see [17]). Using the description of irreducible representations of \( O_q^1 \) one can get a generalization of the Wold decomposition theorem to the case of linear operator satisfying \( q \)-canonical commutation relation (below we will refer to this fact as the \( q \)-Wold decomposition theorem).

**Theorem 1.** Let \( A : \mathcal{H} \to \mathcal{H} \) be a bounded linear operator satisfying for some \( q \in (0, 1) \) the \( q \)-commutation relation
\[ A^* A = 1 + q A A^*. \]
Then \( \mathcal{H} \) can be decomposed into orthogonal sum of subspaces \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_u \) invariant with respect to the actions of \( A, A^* \) and such that the restriction \( A|_{\mathcal{H}_u} \) is a multiple of the weighted shift on \( l_2(\mathbb{Z}_+) \) defined on the standard basis by \( A e_n = \sqrt{\frac{1-q^n}{1-q}} e_{n+1} \), and \( A|_{\mathcal{H}_u} = \frac{1}{\sqrt{1-q}} U \), where \( U \) is a unitary operator on \( \mathcal{H}_u \).

If we do not assume an operator \( A \) satisfying (1) to be bounded we can still decompose \( \mathcal{H} \) into orthogonal direct sum of invariant subspaces \( \mathcal{H}_0 \) and \( \mathcal{H}_u \) such that the restriction of \( A \) to \( \mathcal{H}_0 \) is a multiple of the Fock representation. However in general situation \( \mathcal{H}_u \) is decomposed into orthogonal sum \( \mathcal{H}_u = \mathcal{H}_1 \oplus \mathcal{H}_2 \), where \( A|_{\mathcal{H}_1} = \frac{1}{\sqrt{1-q}} U \) with unitary \( U \), and the restriction \( A|_{\mathcal{H}_2} \) is unbounded and given by a direct integral of unbounded irreducible representations. In particular, in the polar decomposition \( A|_{\mathcal{H}_2} = SC \) the isometric part \( S \) is unitary.

Note that the Fock representation \( \pi_F \) of \( O_q^1 \) is faithful and the universal enveloping \( C^\ast \)-algebra of \( O_q^1 \) is isomorphic to the \( C^\ast \)-algebra generated by \( \pi_F(a_1) \).
Definition 1. Let $A$ be a $*$-algebra. Assume that the set, $\Rep A$, of all its bounded representations is not empty and
\[
\sup_{\pi \in \Rep A} \|\pi(a)\| < \infty
\]
for any $a \in A$. The universal enveloping $C^*$-algebra of a $*$-algebra $A$ is the completion of $A/\mathcal{R}$ with respect to the following norm
\[
\|a + \mathcal{R}\| = \sup_{\pi \in \Rep A} \|\pi(a)\|
\]
where
\[
\mathcal{R} = \{a \in A \mid \pi(a) = 0, \pi \in \Rep A\}.
\]

The existence of the enveloping $C^*$-algebra of $\mathcal{O}^q_n$ follows from the fact that $\|\pi(a_i)\|^2 \leq \frac{1}{1-q}$, $i = 1, \ldots, n$ for any bounded representation $\pi$ of $\mathcal{O}^q_n$ (see Theorem 4). In what follows we denote this $C^*$-algebra by $\mathcal{O}^q_n$. It is known that $\mathcal{O}^q_1 \simeq \mathcal{O}^0_1$ for any $q \in (-1, 1)$. The same is true for $\mathcal{O}^q_n$, $n \in \mathbb{N}$.

Theorem 2. $\mathcal{O}^q_n \simeq \mathcal{O}^0_n$ for any $q \in (-1, 1)$.

Proof. Suppose that $\mathcal{O}^q_n$ is realized by linear operators on a Hilbert space. Consider the polar decompositions $a_i = s_i c_i$, $i = 1, \ldots, d$, where $c_i^2 = a_i^* a_i$ and $s_i$ are partial isometries such that $\ker s_i = \ker c_i$. Since the spectrum $\sigma(c_i^2)$ of $c_i^2$ is $\left\{\frac{1-q^n}{1-q}, n \in \mathbb{N}\right\} \cup \left\{\frac{1}{1-q}\right\}$, each $c_i$ is invertible and therefore each $s_i$ is an isometry. Moreover, $s_i = a_i (a_i^*)^\perp = C^*(a_i, a_i^*)$, $i = 1, \ldots, n$. Further from $a_i^* a_j = 0$ one has
\[
c_i s_i^* s_j c_j = 0, \quad \text{hence} \quad s_i^* s_j = 0, \quad i \neq j.
\]
Since in any irreducible bounded representation of $\mathcal{O}^q_1$ with $-1 < q < 1$ one has
\[
a_1 = s_1 \left(\sum_{n=0}^\infty q^n s_1^* s_1^n\right),
\]
the same equality holds in $\mathcal{O}^q_n$ for $a_i$ and $s_i$, $i = 1, \ldots, n$.

Therefore $a_i \in C^*(s_i, s_i^*)$, $i = 1, \ldots, n$, $i = 1, \ldots, n$, giving the statement of the theorem.

Since the Cuntz-Toeplitz algebra $\mathcal{O}^0_n$ is a not of type I algebra (see [6]), this theorem shows that the classification problem of all irreducible representations of $\mathcal{O}^q_n$ and therefore all irreducible bounded representations of $\mathcal{O}^q_n$ is very complicated. □
This definition is similar to the definition of unbounded representations of twisted canonical commutation relations given by Pusz and Woronowicz (21). It is motivated by the Nelson criterion of the integrability for representations of Lie algebras (2 [18]).

Next two theorems provide a criteria for representations to be well-behaved in terms of bounded operators. It will be important for later classification of well-behaved (irreducible) representations of $O_n^q$. Before stating the theorems we recall the definition of analytic and bounded vectors for an unbounded operator, which will be used in the proofs.

If $A$ is an operator in a Hilbert space $H$, then $u \in H$ is said to be an analytic vector (a bounded vector) for $A$, if

$$\sum_{n=0}^{\infty} \frac{||A^n u||}{n!} s^n < \infty,$$

for some $s > 0 (||A^n u|| \leq C^n$, for some $C > 0$ respectively). These concepts are fundamental in the theory of integrable representations of Lie algebras (2 [13] [18]).

**Theorem 3.** Let $\{A_i, i = 1, \ldots, n\}$ be a family of linear operators on $H$ defining a well-behaved representation of $O_n^q$ and let $A_i = S_i |A_i|$ be the polar decomposition of $A_i$ and $D_i = S_i |A_i| S_i^*$, $i = 1, \ldots, n$. Then

(a) $f(D_i^n)S_i = S_i f((1 + qD_i^n)$ for any real bounded measurable function $f$;

(b) for any $i$, $j$ the operators $|A_i|, |A_j|$ commute in the sense of resolutions of identity;

(c) $S_i^* S_j = \delta_{ij} I$.

**Proof.** Let $C_j^2$ be the closure of $A_j^* A_j$ on $D$, $j = 1, \ldots, n$. Clearly, $C_j^2$ is symmetric.

In order to show that all $C_j^2$ are selfadjoint and mutually commute in the sense of resolution of identity we prove first that

$$\Delta^n C_j^2 y = C_j^2 \Delta^n y, \ y \in D(\Delta^{n+2}).$$

Here and subsequently, $D(a)$ denotes the domain of an operator $a$.

We have

$$C_i^2 C_j^2 = (1 + qA_i A_i^*) (1 + qA_j A_j^*) = (1 + qA_i A_i^* + qA_j A_j^*)$$

$$= (1 + qA_j A_j^*) (1 + qA_i A_i^*) = C_j^2 C_i^2$$

on $D$. Thus if $x, y \in D$ then

$$(\Delta x, C_j^2 y) = (C_j^2 x, \Delta y) \text{ and } (C_j^2 \Delta x, y) = (C_j^2 x, \Delta y).$$

As $D$ is a core for $\Delta$ the second equality holds also for any $y \in D(\Delta)$.

We shall show next that $D(C_j^2) \supset D(\Delta^2)$. In fact, by (4)

$$\Delta^2 = \sum_{i=1}^{n} C_i^2 (\sum_{i=1}^{n} C_i^2) = \sum_{i=1}^{n} C_i^4 + \sum_{i \neq j} (1 + qA_i A_i^* + qA_j A_j^*)$$

giving $\Delta^2 \geq C_j^4$ on $D$. Since $D$ is a core for $\Delta^2$ we have that for $y \in D(\Delta^2)$ there exists $\{y_n\} \in D$ such that $y_n \to y$, $\Delta^2 y_n \to \Delta^2 y$ and

$$||C_j^2 (y_n - y_m)|| = \langle C_j^2 (y_n - y_m), C_j^2 (y_n - y_m) \rangle$$

$$= \langle C_j^2 (y_n - y_m), y_n - y_m \rangle$$

$$\leq \langle \Delta^2 (y_n - y_m), y_n - y_m \rangle.$$
Thus the sequence \( \{C_j^2 y_n\} \) converges to some \( z \in \mathcal{H} \) and by the closedness of \( C_j^2 \), 
\( y \in \mathcal{D}(C_j^2) \) and \( z = C_j^2 y \). Moreover,

(5) \[ C_j^4 \leq \Delta^2 \text{ on } \mathcal{D}(\Delta^2). \]

Let now \( y \in \mathcal{D}(\Delta^3) \). Then \( \Delta y \in \mathcal{D}(\Delta^2) \subset \mathcal{D}(C_j^2) \), \( y \in \mathcal{D}(\Delta^2) \subset \mathcal{D}(C_j^2) \) and by (4) 
\[ (\Delta x, C_j^2 y) = (x, C_j^2 \Delta y), x \in \mathcal{D}. \]

By the closedness argument the same equality holds for any \( x \in \mathcal{D}(\Delta) \) giving 
\( C_j^2 \mathcal{D}(\Delta^3) \subset \mathcal{D}(\Delta) \) and \( \Delta C_j^2 = C_j^2 \Delta \) on \( \mathcal{D}(\Delta^3) \). We proceed now by induction and suppose that for any \( y \in \mathcal{D}(\Delta^n) \), \( n \geq 3 \), \( C_j^2 y \in \mathcal{D}(\Delta^{n-2}) \) and

(6) \[ \Delta^{n-2} C_j^2 y = C_j^2 \Delta^{n-2} y. \]

In particular, if \( y \in \mathcal{D}(\Delta^{n+1}) \) then \( C_j^2 y \in \mathcal{D}(\Delta^{n-2}) \) and (6) holds. Let \( z = \Delta^{-2} y \). Then \( z \in \mathcal{D}(\Delta^3) \) and \( C_j^2 z = \Delta^{-2} C_j^2 y \in \mathcal{D}(\Delta) \). Therefore \( C_j^2 y \in \mathcal{D}(\Delta^{n-1}) \) and

\[ \Delta^{n-1} C_j^2 y = \Delta \Delta^{-2} C_j^2 y = \Delta C_j^2 \Delta^{-2} y = C_j^2 \Delta^{n-1} y. \]

Then by induction we obtain (6) for any \( n \geq 1 \).

Let \( \mathcal{D}_\omega \) be the space of analytic vectors for \( \Delta \). As \( \mathcal{D}_\omega \subset \mathcal{D}(\Delta^k) \), for any \( k \in \mathbb{N} \), by (5) and (6)

\[ ||\Delta^n C_j^2 x||^2 = ||C_j^2 \Delta^n x||^2 = (C_j^2 \Delta^n x, C_j^2 \Delta^n x) = (\Delta^n C_j^4 \Delta^n x, x) \leq (\Delta^n \Delta \Delta^{-2} \Delta^n x, x) = ||\Delta^{n+1} x||^2. \]

This shows that \( C_j^2 \mathcal{D}_\omega \subset \mathcal{D}_\omega \) and, moreover, \( C_j^2 \) mutually commute on \( \mathcal{D}_\omega \). The last can be seen by computing the scalar product of \( C_j^2 \) \( C_j^2 \) \( C_j^2 \mathcal{D}_\omega \) \( x \in \mathcal{D} \), with \( y \in \mathcal{D}_\omega \).

Next we show that any \( x \in \mathcal{D}_\omega \) is also analytic for all \( C_j^2 \). In fact, as \( C_j^4 \leq \Delta^2 \) on \( \mathcal{D}_\omega \), by assuming by induction that \( C_j^{4n} \leq \Delta^{2n} \) on \( \mathcal{D}_\omega \) we obtain

\[ C_j^{4(n+1)} = C_j^2 C_j^{4n} C_j^2 \leq C_j^2 \Delta^{4n} C_j^2 \leq \Delta^{2n} C_j^4 \Delta^{2n} \leq \Delta^{2n+2} \Delta^2 = \Delta^{2(n+1)}. \]

This gives 
\[ ||(C_j^2)^n x||^2 = (x, C_j^{4n} x) \leq (x, \Delta^{2n} x) \leq ||\Delta^n x||^2, \]

i.e. \( \mathcal{D}_\omega \) is a subset of analytic vectors for all \( C_j^2 \). As \( \mathcal{D}_\omega \) is invariant with respect to all \( C_j^2 \) and \( C_j^2 \) mutually commute on \( \mathcal{D}_\omega \), we have that all \( C_j^2 \) are selfadjoint and mutually strongly commute, i.e. in the sense of resolutions of identity. In particular, we have proved that each \( C_j^2 \) is essentially selfadjoint on \( \mathcal{D} \), and \( C_j^2 = A_j^* A_j = |A_j|^2 \).

Next we prove that \( f(C_j^2) \mathcal{S}_i = S_i f(1+q C_j^2) \) for any bounded measurable function \( f \). Let \( R_i = C_j^2 \) for notation simplicity. As \( R_i A_i = A_i (1 + q R_i) \) and \( R_i A_i^* = A_i^* (R_i - 1)/q \) n \( \mathcal{D}_i \), using arguments similar to one given above one can show that for any non-negative integer \( n \) and \( x \in \mathcal{D}(R_i^{k-1}) \) we have that \( A_i^* x \in \mathcal{D}(R_i^k) \), \( A_i x \in \mathcal{D}(R_i^k) \) and

\[ R_i^k A_i x = A_i (1 + q R_i)^k x \text{ and } R_i^k A_i^* x = A_i^*((R_i - 1)/q)^k x. \]
Taking now $D_{i,\omega}$ the space of analytic vectors for $R_i$ and using that $R_i \leq (1+R_i)^2$ we obtain
\[
||R_i k A_i x||^2 = ||A_i(1 + qR_i k)x||^2 = ((1 + qR_i)^k A_i(1 + qR_i)^k x, x)
\]
\[
\leq (1 + qR_i)^k (1 + R_i)^2 (1 + qR_i)^k x, x
\]
\[
= ((1 + R_i)(1 + qR_i)^{2k}(1 + R_i)x, x)
\]
\[
\leq (1 + R_i)^{2(2k + 1)} x, x = ||(1 + R_i)^k x||^2
\]
giving that $A_i D_{i,\omega} \subset D_{i,\omega}$. Analogously, one proves that $A^*_i D_{i,\omega} \subset D_{i,\omega}$. Moreover, the relations $A^*_i A^*_i = 1 + qA_i A^*_i$ and $R_i A_i = A_i(1 + qR_i)$ hold on $D_{i,\omega}$ which can be shown analogously to commutation of $C_i^*$ on $D_\omega$ above. That $D_{i,\omega}$ is a core for $A_i$ and $A^*_i$ can be proved using the arguments in [21, Proposition 3.3]. The condition $f(C_i^2)S_i = S_i f(1 + qC_i^2)$ now follows from [19, Theorem 1, Theorem 2]. Furthermore,
\[
A^*_i A_i = 1 + qA_i A^*_i \iff C_i^2 = 1 + qS_i C_i^2 S^*_i \Rightarrow
\]
\[
S_i S^*_i C_i^2 = S_i S^*_i qS_i C_i^2 S^*_i = C_i^2 - 1
\]
on $D$ giving
\[
(1 - S_i S^*_i)C_i^2 = (1 - S_i S^*_i)
\]
As $D$ is a core for $C_i^2$ we obtain the equality on $(D(C_i^2))$. Similarly,
\[
(1 - S_i S^*_i)C_i^{2n} = (1 - S_i S^*_i)
\]
on $D(C_i^{2n})$ and in particular on $D_{i,\omega}$ for any $n \geq 1$. The arguments similar to one in [19, Theorem 1] give $(1 - S_i S^*_i) f(C_i^2) = f(1)(1 - S_i S^*_i)$ for any bounded Borel function $f$. This will also give that $S_i S^*_i$ commute with resolution of the identity of $C_i$ and from (7) we will get that $C_i \geq 1$ and since $\ker C_i = \ker S_i$, $S_i$ is an isometry.

To obtain $f(D_i^2)S_i = S_i f(1 + qD_i^2)$ we note that $A_i A^*_i D_i^2$ and $(A_i A^*_i) A_i = A_i(1 + qA_i A^*_i)$ on $D_{i,\omega}$. Moreover, clearly, any vector in $D_{i,\omega}$ is analytic for $A_i A^*_i$.

Using again [19, Theorem 1, Theorem 2] we obtain the desired equality.

What is left to prove is that $S^*_i S_j = 0$ if $i \neq j$. We have $A^*_i A_j = C_i S^*_i S_j C_j = 0$ on $D$ and $(S_i^* S_j C_j x, C_j y) = 0$ for any $x \in D$, $y \in D(C_i)$. As each $S_i$ is an isometry, the range of $C_i$ is dense implying $S^*_i S_j = 0$. □ □ □

Remark 1. Let $E_j(\cdot)$ be the resolution of identity of $D_j^2 = S_j |A_j|^2 S^*_j$, $j = 1, \ldots, n$. Then, by [19, (a)] is equivalent to
\[
E_j(\delta) S_j = S_j E_j(q^{-1}(\delta - 1)), j = 1, \ldots, n \text{ for any Borel } \delta \subset \mathbb{R}.\]

We will use the following notation when proving our next theorem.

Let $\Lambda = \{0, \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \mid 1 \leq \alpha_i \leq n \ k \in \mathbb{N}\}$ be the set of all finite multi-indices. Introduce the transformations $\sigma$, $\sigma_k: \Lambda \rightarrow \Lambda$, $k = 1, \ldots, n$
\[
\sigma(\alpha_1, \ldots, \alpha_k) = (\alpha_2, \ldots, \alpha_k), \ \sigma(\alpha_1) = 0,
\]
\[
\sigma_k(\alpha_1, \ldots, \alpha_k) = (k, \alpha_1, \ldots, \alpha_k).
\]

Below having any family $u_1, \ldots, u_n$ of elements of some algebra and any nonempty multi-index $\alpha = (\alpha_1, \ldots, \alpha_k) \in \Lambda$ we will denote by $u_\alpha$ the product $u_{\alpha_1} u_{\alpha_2} \cdots u_{\alpha_k}$, it will be also convenient for us to put $u_0 = 1$

**Theorem 4.** Let $A_i = S_i |A_i|$, $i = 1, \ldots, n$, be the polar decompositions of closed operators $A_i$. If $|A_i|$, $D_j$, $S_i$, $i = 1, \ldots, n$ satisfy conditions (a)-(c) of Theorem □ then $\{A_i, i = 1, \ldots, n\}$ defines a well-behaved representation of $\mathfrak{O}_n^\mathbb{R}$. 
Proof. We construct the necessary domain.

The condition (S) implies that given a fixed $j$, the operators $A_j$, $A_j^*$ form a (well-behaved) representation of $q$-CCR relation with one degree of freedom. From the generalized Wold decomposition for representations of one-dimensional $q$-CCR we can write $\mathcal{H} = \mathcal{H}_0^{(j)} \oplus \mathcal{H}_s^{(j)} \oplus \mathcal{H}_u^{(j)}$, so that in $\mathcal{H}_0^{(j)}$ we have a multiple of the Fock representation, in $\mathcal{H}_a^{(j)}$ we have $D_j^2 = (1 - q)^{-1}$ and in $\mathcal{H}_u^{(j)}$ is unbounded component. Notice that in $\mathcal{H}_s^{(j)}$ and in $\mathcal{H}_u^{(j)}$ the operator $S_j$ is unitary. Let $\mathcal{H}_j$ be a span of the vectors $S_α x$, $x \in \mathcal{H}_u^{(j)}$, $α \in Λ$.

Lemma 1. 1. $\mathcal{H}_j$, $j = 1, \ldots, n$ are invariant subspaces.
2. $\mathcal{H}_j \perp \mathcal{H}_k$, $j \neq k$.
3. In the subspace $\mathcal{H}_0 = \mathcal{H} \ominus (\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n)$ the representation is bounded.

Proof. 1. It is sufficient to show that each $\mathcal{H}_j$ is invariant with respect to $S_k$, $S_k^*$, and $E_k(δ)$, $k = 1, \ldots, n$ for any measurable $δ$.

From $S_k S_α = S_{σ(α)}$ obviously follows the invariance with respect to $S_k$, $k = 1, \ldots, n$. For $α \neq 0$ the invariance with respect to $S_k^*$ follows as well since $S_k^* S_α = δ_{κα}, S_{σ(α)}$. For the vectors of the form $x \in \mathcal{H}_u^{(j)}$, since $S_j$ is unitary in $\mathcal{H}_u^{(j)}$, we have $x = S_j S_j^* x$ and therefore, $S_k^* x = δ_{jκ} S_j^* x \in \mathcal{H}_j$.

Take a measurable $δ$. Since $S_k S_k^*$ is the projection on the cokernel of $D_k^2$, then for $δ$ not containing $\{0\}$ we have $E_k(δ) S_j = E_k(δ) S_k S_k^* S_j = 0$, and $E_k(\{0\}) S_j = (1 - S_k S_k^*) S_j = S_j$, $j \neq k$. Therefore, for $0 ∈ δ$ we have $E_k(δ) S_j = S_j$, and for $0 \notin δ$ we have $E_k(δ) S_j = 0$, $j \neq k$. Thus we have that $E_k(δ) S_α = 0$, $α \neq k$. If $α_1 = k$, then we apply (S) and get $E_k(δ) S_α x = S_k E_k(q^{-1}(δ - 1)) S_{σ(α)} x ∈ \mathcal{H}_j$ by induction.

It remains to consider the case $α = 0$. Again, we can write $x = S_j S_j^* x$, and apply the arguments above.

2. Take arbitrary $x \in \mathcal{H}_u^{(j)}$, $y \in \mathcal{H}_u^{(k)}$, $j \neq k$.
   i) Since $x = S_j S_j^* x$, $y = S_k S_k^* y$, we have $(x, y) = (S_j^* x, S_j^* S_k S_k^* y) = 0$.
   ii) For any $α = (α_1, \ldots, α_t)$ we have $(S_α x, y) = (S_α^* y, (S_j^* x, S_j^* S_k S_k^* y)$. But since $y = S_k S_k^* y$ and $S_j^* S_k S_k^* = 0$, the latter scalar product is zero.
   iii) For any $α, β ∈ Λ$ we have $(S_α x, S_β y) = 0$ using quite similar arguments.

3. Is obvious, since unbounded component of any $A_j$ in its Wold decomposition generates $\mathcal{H}_j$. □

Let us continue the proof of the theorem. By Lemma 1 we decompose the representation space, $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \cdots \oplus \mathcal{H}_n$, in each component we construct the necessary domain $D_j$, $j = 0, \ldots, n$, and take $D = D_0 \oplus D_1 \oplus \cdots \oplus D_n$.

Since in $\mathcal{H}_0$ the operators are bounded, we can take $D_0 = \mathcal{H}_0$.

We fix some $j = 1, \ldots, n$ and construct the corresponding set $D_j \subset \mathcal{H}_j$.

Let $D_j$ be a span of vectors of the form $S_α E_j(δ) x$, where $x \in \mathcal{H}_u^{(j)}$, $α \in Λ$, $δ \subset R_+$ are bounded measurable sets.

1. $D_j$ is dense in $\mathcal{H}_j$. Indeed, choose $δ = [0, t]$, then for any $x \in \mathcal{H}_u^{(j)}$, $D_j$ contains $x_t = E_j(0, t] x$, which converges to $x$ strongly as $t \to \infty$. Applying the operators $S_α$, $α \in Λ$, we obtain total in $\mathcal{H}_j$ set.
2. $D_j \subset D(D_j^k)$, $k = 1, \ldots, n$, and consists of bounded vectors for these operators.
First we show that for any $z \in D_j$, the sequence $E_k([0,t])D_k^2 z$ converges in $\mathcal{H}_j$ as $t \to \infty$. But as noticed above $E_k([0,t])S_l = E_k(\{0\})S_l$, $k \neq l$, and hence $E_k([0,t])D_k^2 S_l = 0$, $k \neq l$. Therefore, $E_k([0,t])D_k^2 z = 0$ on any $z = S_\alpha E_j(\delta)x$, where $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\alpha_1 \neq k$.

For $\alpha_1 = k$ we have $E_k([0,t])D_k^2 z = S_k(1 + qD_k^2)E_k([0, q^{-1}(t - 1)])S_{\sigma^m(\alpha)}E_j(\delta)x$ and for $\alpha$ with $\alpha_1 = \ldots = \alpha_m = k$

$$E_k([0,t])D_k^2 S_k^m = E_j(\delta)x$$

If $\sigma^m(\alpha) \neq \emptyset$ and $\alpha_{m+1} \neq k$ the expression in (9) is equal to

$$1 - \frac{q^m - \delta_1 - I + q^m D_k^2 E_k([0, (1 - \delta_1 q^{-1})])S_{\sigma^m(\alpha)}E_j(\delta)x.$$ 

Here we use the equalities $D_k^2 E_k(\{0\}) = 0$ and $E_k(\{0\})S_j = S_j$. Similarly, in the case $\sigma^m(\alpha) = \emptyset$ for $k \neq j$ using $x = S_j S^*_j x$ we have

$$E_k([0,t])D_k^2 S_j^m E_j(\delta)x = 1 - \frac{q^m}{1 - q}S_j^m E_j(\delta)x.$$ 

For $k = j$, $\sigma^m(\alpha) = \emptyset$, and large $t$ we have $E_j(\delta)E_j([0, (1 - \delta_1 q^{-1})]) = E_j(\delta)$ since $\delta_1$ is bounded, therefore in this case (9) does not depend on $t$ as above and obviously converges in $\mathcal{H}_j$ as $t \to \infty$.

Finally, for a bounded $\delta$ and $x \in \mathcal{H}_u^{(j)}$ we have the following expressions:

$$D_k^2 S_\alpha E_j(\delta)x = 0, \quad \alpha_1 \neq k;$$
$$D_k^2 S_\alpha E_j(\delta)x = \frac{1 - q^m}{1 - q}S_\alpha E_j(\delta)x, \quad \alpha = (k, \ldots, k, \alpha_{m+1}, \ldots, \alpha_l), \quad \alpha_{m+1} \neq k;$$
$$D_k^2 S_{\alpha^m} E_j(\delta)x = \frac{1 - q^m}{1 - q}S_{\alpha^m} E_j(\delta)x, \quad k \neq j;$$

$$D_j^2 S_j^m E_j(\delta)x = S_j^m E_j(\delta)\left(1 - \frac{q^m}{1 - q}I + q^m D_j^2 E_j(\delta)x,\right)$$

where in the last formula we used

$$E_j(\delta)\left(1 - \frac{q^m}{1 - q}I + q^m D_j^2 E_j(\delta) = \left(1 - \frac{q^m}{1 - q}I + q^m D_j^2 \right) E_j(\delta).$$

From (10) we conclude that $D_k$, $k \neq j$, is bounded in $\mathcal{H}_j$ with $\|D_k^2\|_{\mathcal{H}_j} = (1 - q)^{-1}$.

For $k = j$ we have $\|D_j^2 S_\alpha E_j(\delta)x\| \leq (1 - q)^{-1}\|E_j(\delta)x\|$, which means that $D_j$ consists of bounded vectors of $D_j^2$.

3. $D_j$ is invariant with respect to $S_k$, $S^*_k$, $k = 1, \ldots, n$. The invariance with respect to $S_k$ is obvious. For $z = S_\alpha E_j(\delta)x$ with $x \in \mathcal{H}_u^{(j)}$ we have $S_k^* z = \delta_{k\alpha_1} S_{\sigma^m(\alpha)} E_j(\delta)x$ if $\alpha \neq 0$. For $z = E_j(\delta)x$ we have $z = S_j S^*_j x$ and

$$S_kz = \delta_{kj} S^*_j x = \delta_{kj} E_j(q^{-1}(\delta - 1)) S^*_j x \in D_j.$$ 

4. Define $A_k$, $k = 1, \ldots, n$ as a closure of $D_k S_k$ from $D$. Then $D \subset D(A_k), D(A^*_k)$, $k = 1, \ldots, n$ and (2) holds on $D$. This follows directly.
5. One can easily see that $D$ consists of bounded vectors for the operators $S_j^* D_k^2 S_j$ and using their commutation, for $\Delta$ as well. Therefore $\Delta$ is essentially selfadjoint on $D$. \hfill \Box \Box

Remark 2. In fact it follows from unitarity of $S_j$ on $\mathcal{H}^{(j)}_\Lambda$ that $\mathcal{H}_j$ coincides with the closure of the span of the family $\{S_\alpha x, \, x \in \mathcal{H}^{(j)}_\Lambda, \, \alpha \in \Lambda_j\}$, where $\Lambda_j = \{\emptyset, (\alpha_1, \ldots, \alpha_k), \, 1 \leq \alpha_i \leq n, \, \alpha_i \neq \alpha_j \neq j, \, k \in \mathbb{N}\}.$

Remark 3. It follows from the considerations above that $S_\alpha \mathcal{H}^{(j)}_\Lambda$ are invariant with respect to $D^2_k$ and if $\alpha \neq \emptyset$, $\alpha \in \Lambda_j$, then restriction of $D^2_k$ to $S_\alpha \mathcal{H}^{(j)}_\Lambda$ is bounded. In fact, for nonempty $\alpha \in \Lambda_j$ and any $k = 1, \ldots, n$ one has

\begin{equation}
D^2_k x = \frac{1 - q^{m_k(\alpha)}}{1 - q} x, \quad x \in S_\alpha \mathcal{H}^{(j)}_\Lambda,
\end{equation}

where the function $m_k(\alpha) = \mathbb{Z}_+$, $\alpha \neq \emptyset$, is determined by the condition

\begin{equation}
\sigma_k^{m_k(\alpha)}\sigma_k^{m_k(\alpha)}(\alpha) = \alpha, \quad \sigma_k^{m_k(\alpha)+1}\sigma_k^{m_k(\alpha)+1}(\alpha) \neq \alpha.
\end{equation}

Recall also that $D^2_k x = 0$, for $x \in \mathcal{H}^{(j)}_\Lambda$, $k \neq j$, and with $m_k(\emptyset) = 0$ the formula (11) becomes true for $x \in \mathcal{H}^{(j)}_\Lambda$ and $k \neq j$ also.

2.2. Irreducible unbounded representations. In this section we will give a classification of irreducible unbounded representations of $\mathcal{O}^\alpha_n$. We will keep notation from the previous section and consider only well-behaved representations of $\mathcal{O}^\alpha_n$.

Let $\{A_1, \ldots, A_n\}$ be a family of closed operators on $\mathcal{H}$ defining a representation $\pi$ of $\mathcal{O}^\alpha_n$. Let $A_i = A_i^* A_i$ be the polar decomposition of $A_i$ and $D_i = S_i C_i S_i^*$. Denote by $E_i(\cdot)$ the resolution of identity of $D_i$ and let $\mathcal{B}(\mathbb{R})$ be the $\sigma$-algebra of all Borel subsets of $\mathbb{R}$.

Definition 3. The representation $\pi$ will be called irreducible if the only operator $C \in B(\mathcal{H})$ which commutes with all $S_i, \, S_i^*$ and $E_i(\delta), \, \delta \in \mathcal{B}(\mathbb{R}), \, i = 1, \ldots, n$, is a multiple of unity, or equivalently, the space $\mathcal{H}$ can not be decomposed into a direct sum of two non-trivial subspaces which are invariant with respect to $S_i, \, S_i^*$, and $E_i(\delta), \, \delta \in \mathcal{B}(\mathbb{R}), \, i = 1, \ldots, n$.

It follows from Lemma 4 that for irreducible unbounded $\pi$ the representation space $\mathcal{H}$ coincides with the closed span $\mathcal{H}_j$ of $\{S_\alpha \mathcal{H}^{(j)}_\Lambda, \, \alpha \in \Lambda_j\}$ for some $j \in \{1, \ldots, n\}$.

Lemma 2. Let $\pi$ be an irreducible unbounded representation of $\mathcal{O}^\alpha_n$ on $\mathcal{H}_j$ for some fixed $j$. Then the restriction $A_j|_{\mathcal{H}^{(j)}_\Lambda}$ determines an irreducible representation of $\mathcal{O}^\alpha_1$ on $\mathcal{H}^{(j)}_\Lambda$.

Proof. Assume that $\mathcal{H}^{(j)}_\Lambda = H^1_\Lambda \oplus H^2_\Lambda$ where $H^i_\Lambda$, $i = 1, 2$, are invariant with respect to $S_j, \, S_j^*$, and $E_j(\delta)$ for any Borel $\delta \in \mathbb{R}$. Let $\mathcal{H}_i$, $i = 1, 2$, be the closed linear span of $\{S_\alpha H^i_\Lambda, \, \alpha \in \Lambda_j\}$. Then following arguments in the proof of Theorem 2 we obtain that $\mathcal{H} = H_1 \oplus H_2$ where each of $H_i$ is invariant with respect to $S_k, \, S_k^*$, $E_k(\delta)$ for any $k = 1, \ldots, n$ and $\delta \in \mathbb{R} \in \mathcal{B}(\mathbb{R})$ contradicting the irreducibility of $\pi$. \hfill \Box
Theorem 5. Let $j \in \{1, \ldots, n\}$ and $x \in (1 + qx_0, x_0]$, where $x_0 > \frac{1}{1-q}$ is fixed. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e^s_{\alpha}, s \in \mathbb{Z}, \alpha \in \Lambda_j\}$ and let $A_k$, $k = 1, \ldots, n$ be linear operators on $\mathcal{H}$ given by

$$A_k e^s_{\alpha} = \sqrt{\frac{1 - q^{m_k(\sigma_k(\alpha))}}{1 - q}} e^s_{\sigma_k(\alpha)}, \quad k \neq j$$

$$A_j e^s_{\alpha} = \sqrt{\frac{1 - q^{m_j(\sigma_j(\alpha))}}{1 - q}} e^s_{\sigma_j(\alpha)}, \quad \alpha \neq \emptyset$$

$$A_j e^s_\emptyset = \sqrt{\frac{1 - q^s}{1 - q}} + q^s x \ e^{s+1}_\emptyset$$

(13)

where $m_k(\alpha)$ are defined by (12). Then $\{A_1, \ldots, A_k\}$ defines an irreducible representation $\pi(x,j)$ of $\mathcal{O}_n$. Moreover any unbounded irreducible representation is unitarily equivalent to exactly one representation $\pi(x,j)$.

Proof. Let $\mathcal{U}$ be the closure of span of $\{e^s_\emptyset, s \in \mathbb{Z}\}$. Then $\mathcal{U}$ is invariant with respect to $A_j, A_j^*$ and the restrictions of these operators to $\mathcal{U}$ determine a well-behaved irreducible representation of $\mathcal{O}_n$. Consider the polar decompositions $A_k = S_k C_k$, and put $D_k = S_k C_k S_k^*, k = 1, \ldots, n$. Then

$$D_k^2 e^s_\emptyset = A_k A_k^* e^s_\alpha = \frac{1 - q^{m_k(\alpha)}}{1 - q} e^s_\alpha, \quad \alpha \in \Lambda_j, \quad \alpha \neq \emptyset$$

$$D_k^2 e^s_\emptyset = A_k A_k^* e^s_\emptyset = 0, \quad k \neq j, \quad s \in \mathbb{Z}$$

and $S_k e^s_{\alpha} = e^s_{\sigma_j(\alpha)}$ if either $k \neq j$ or $\alpha \neq \emptyset$, and $S_j e^s_\emptyset = e^{s+1}_\emptyset$; $S_k^* e^s_\emptyset = \delta_{k,1} e^s_{\sigma_j(\alpha)}$ if $\alpha \neq \emptyset$, $S_j^* e^s_\emptyset = e^{s-1}_\emptyset$ and $S_k^* e^s_\emptyset = 0, k \neq j$. In particular, $e^s_\emptyset = S_\alpha e^s_{\alpha}$, for any non-empty $\alpha \in \Lambda_j$ and $s \in \mathbb{Z}$. It is a routine to verify that the conditions of Theorem 3 are satisfied and formulas (13) determine a well-behaved representation $\pi(x,j)$ of $\mathcal{O}_n$ with $\mathcal{U} = \mathcal{H}_u$ and $\pi(x,j)(a_j) = A_j$.

Each representation $\pi(x,j)$ is irreducible. In fact, let $C \in B(\mathcal{H})$ be a selfadjoint operator commuting with $S_k, E_k(\delta), k = 1, \ldots, n, \delta \in \mathcal{B}(\mathbb{R})$. One can easily see that $\mathcal{H}_u(\delta) = E_j(\Delta_x)$, where $\Delta_x = \{\frac{xq^n}{1-q} + q^n x, n \in \mathbb{Z}\}$, giving that $\mathcal{H}_u(\delta)$ is invariant with respect to $C$. Put $C_\emptyset = C|\mathcal{H}_u(\emptyset)$; since the representation of $\mathcal{O}_n$ defined by the restriction of $\pi(x,j)(a_j)$ to $\mathcal{H}_u(\emptyset)$ is irreducible, one has $C_\emptyset = \lambda_\emptyset 1, \lambda_\emptyset \in \mathbb{C}$. Further, using the commutation of $C$ with all $S_k$ and induction on length of $\alpha \in \Lambda_j$, we get that $S_\alpha \mathcal{H}_u(\emptyset)$ is invariant with respect to $C$ for any $\alpha \in \Lambda_j$. Denoting by $C_\emptyset$ the corresponding restriction we obtain again by induction $C_\alpha = \lambda_\alpha 1$ for any $\alpha \in \Lambda_j$.

As $\mathcal{H} = \oplus_{\alpha \in \Lambda_j} s_\alpha \mathcal{H}_u(\emptyset)$ we conclude that $C = \lambda_\emptyset 1_{\mathcal{H}}$ and $\pi(x,j)$ is irreducible.

Next we show that $\pi(x,j)$ are non-equivalent representations. Since $\pi(x,j)(a_k)$, are bounded if $j \neq k$, we have that $\pi(x,j)$ is not unitarily equivalent to $\pi(y,k)$ when $j \neq k$.

Let $x, x' \in (1 + qx_0, x_0], j \in \{1, \ldots, n\}$. Suppose that $\pi(x,j) \simeq \pi(x',j)$. Denote the corresponding representation spaces by $\mathcal{H}$ and $\mathcal{F}$ and let $V: \mathcal{H} \to \mathcal{F}$ be a unitary operator giving the equivalence of the representations. Then $V$ gives the equivalence of representations of $\mathcal{O}_n$ defined by the actions of $\pi(x,j)(a_j)$ and $\pi(x',j)(a_j)$ on $\mathcal{H}$ and $\mathcal{F}$ respectively. Consider the decompositions $\mathcal{H} = \mathcal{H}_u(\emptyset) \oplus (\mathcal{H}_u(\emptyset))^\perp$ and $\mathcal{F} = \mathcal{F}_u(\emptyset) \oplus (\mathcal{F}_u(\emptyset))^\perp$. Recall that summands in these decompositions are invariant with
respect to \( \pi(x,j)(a_j), \pi(x,j)(a^*_j) (\pi(x',j)(a_j), \pi(x',j)(a^*_j) \) respectively), the restriction, \( \pi_1(a_j), \) of \( \pi(x,j)(a_j) \) (and \( \pi'_1(a_j) \) of \( \pi(x',j)(a_j) \)) to \( (F^{(j)}_u)^\perp \) \(( (F^{(j)}_u)^\perp \) respectively) is bounded and

\[
\pi_2(a_j) = \pi(x,j)(a_j)|_{F^{(j)}_u} = \pi_x(a), \quad \pi'_2(a_j) = \pi(x',j)(a_j)|_{F^{(j)}_u} = \pi_{x'}(a)
\]

Since \( \pi_1 \) and \( \pi_2 \) are disjoint and the same are \( \pi'_1 \) and \( \pi'_2 \) one has that \( V\pi(x,j)(a_j)V^* = \pi(x',j)(a_j) \) implies \( V = V_2 \oplus V_1 \) and \( V_2 \pi_2(a_j)V_2^* = \pi'_2(a_j) \), hence by Proposition 4 we get \( x = x' \).

Finally, by Lemma 2 any irreducible well-behaved representation \( \pi \) of \( O^q_n \) acting on Hilbert space \( H \) corresponds to some fixed \( j = 1, \ldots, n \) such that the restriction of \( \pi(a_j) \) to \( H^{(j)}_u \) determines an irreducible well-behaved representation of \( O^q_j \). Then decomposing \( H = H^{(j)}_u \oplus (H^{(j)}_u)^\perp \) and taking a unitary operator \( V \) of the form \( V = V_1 \oplus 1 \), where \( V_1 \) is unitary acting on \( H^{(j)}_u \) such that \( V^*_1 \pi(a_j)|_{H^{(j)}_u}V_1 = \pi_x(a) \) for some \( x \in (1 + qx_0, x_0) \), we obtain by Remark 3 that \( V \) gives the unitary equivalence of \( \pi \) to \( \pi(x,j) \).

Remark 4. Using the same arguments we can describe irreducible representations of \( O^q_n \) such that some of \( S_j \) is not a pure isometry (i.e. its Wold decomposition consists of the unitary part). In this case we have the following two possibilities: either the corresponding representation is unbounded as described in Theorem 5 or it is unitarily equivalent to one determined by the following formula

\[
\begin{align*}
A_k e_\alpha &= \sqrt{1 - q^{m_k(\sigma_k(\alpha))}} e_{\sigma_k(\alpha)}, \quad k \neq j \\
A_j e_\alpha &= \sqrt{1 - q^{m_j(\sigma_j(\alpha))}} e_{\sigma_j(\alpha)}, \quad \alpha \neq 0 \\
A_j e_\emptyset &= \exp(2\pi i \phi_j) \sqrt{\frac{1}{1 - q}} e_\emptyset, \quad \phi_j \in [0, 1)
\end{align*}
\]

on \( H \) with orthonormal basis \( \{ e_\emptyset, e_\alpha, \alpha \in A_j \} \). Representations corresponding to different \( j = 1, \ldots, n \) or \( \phi_j \in [0, 1) \) are non-equivalent.

In particular, for \( q = 0 \) we get a classification of all irreducible representations of \( O_n \) such that one of the generators is not a pure isometry.

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E-mail address: vo@imath.kiev.ua

Institute of Mathematics, National Academy of Sciences of Ukraine

E-mail address: prosk@univ.kiev.ua

Kyiv National Taras Shevchenko University, Faculty of Cybernetics

E-mail address: turowska@math.chalmers.se

Chalmers University of Technology, Department of Mathematics