Eavesdropping of two-way coherent-state quantum cryptography via Gaussian quantum cloning machines

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We consider one of the quantum key distribution protocols recently introduced in Ref. [Pirandola et al., Nature Physics 4, 726 (2008)]. This protocol consists in a two-way quantum communication between Alice and Bob, where Alice encodes secret information via a random phase-space displacement of a coherent state. In particular, we study its security against a specific class of individual attacks which are based on combinations of Gaussian quantum cloning machines.

INTRODUCTION

Recently [1, 2], we have shown how two-way quantum communication can profitably be exploited to enhance the security of continuous variable quantum key distribution [3, 4, 5, 6]. In particular, we have investigated the security of two-way protocols in the presence of collective Gaussian attacks which are modelled by combinations of entangling cloners [4]. Even though this situation is the most important one from the point of view of the practical implementation, the effect of other kind of Gaussian attacks (i.e., not referable to entangling cloners) must also be analyzed. In this paper, we study the security of the two-way coherent-state protocol of Ref. [1] against individual attacks where an eavesdropper (Eve) combines two different Gaussian quantum cloning machines (also called Gaussian cloning machines). In particular, we are able to show the robustness of the two-way protocol when the first cloner is fixed to be symmetric in the output clones. This symmetry condition enables us to derive the results quite easily but clearly restricts our security analysis to a preliminary stage. For this reason, the optimal performance of Gaussian cloners against two-way quantum cryptography is still unknown at the present stage.

ADDITIVE GAUSSIAN CHANNELS AND GAUSSIAN CLONERS

Consider a stochastic variable $X$ with values $x \in \mathbb{R}$ distributed according to a Gaussian probability

$$G_{\Sigma^2}(x) = \frac{1}{\sqrt{2\pi\Sigma^2}} \exp\left[-\frac{x^2}{2\Sigma^2}\right],$$

with variance $\Sigma^2$. This variable is taken as input of a classical channel that outputs another stochastic variable $Y$ with values $y \in \mathbb{R}$. In particular, the classical channel is called additive Gaussian channel if, for every input $x$, the conditional output $y|x$ is Gaussianly distributed around $x$ with some variance $\sigma^2$. As a consequence, the output variable $Y$ is a Gaussian variable with zero mean and variance $\Sigma^2 + \sigma^2$. According to Shannon’s theory [5], the classical correlations between the input and output variables lead to a mutual information

$$I(X, Y) = \frac{1}{2} \log(1 + \gamma),$$

where $\gamma \equiv \Sigma^2/\sigma^2$ is the signal to noise ratio (SNR). This formula gives the maximal number of bits per Gaussian value that can be sent through a Gaussian channel with a given SNR (on average and asymptotically).

In quantum information theory, an example of additive Gaussian channel is provided by the Gaussian quantum cloning machine (GQCM) [9]. Consider a continuous variable (CV) system, like a bosonic mode, which is described by a pair of conjugate quadratures $\hat{x}$ and $\hat{p}$, with $[\hat{x}, \hat{p}] = i$, acting on a Hilbert space $\mathcal{H}$. Then, consider a coherent state $|\varphi\rangle$ with amplitude $\varphi = (x + ip)/\sqrt{2}$. A $1 \rightarrow 2$ GQCM is a completely-positive trace-preserving linear map

$$M : |\varphi\rangle\langle\varphi| \rightarrow \rho_{12} \in \mathcal{D}(\mathcal{H} \otimes \mathcal{H}),$$

such that the single clone states, $\rho_1 = \text{tr}_2(\rho_{12})$ and $\rho_2 = \text{tr}_1(\rho_{12})$, are given by a Gaussian phase-space modulation of the input state $|\varphi\rangle\langle\varphi|$, i.e.,

$$\rho_k = \int d\mu \Omega_{\sigma_k^2}(\mu) \hat{D}(\mu)|\varphi\rangle\langle\varphi| \hat{D}^\dagger(\mu), \quad k = 1, 2,$$

where

$$\Omega_{\sigma_k^2}(\mu) \equiv \frac{1}{\sigma_k^2} \exp\left[-\frac{|\mu|^2}{2\sigma_k^2}\right],$$

and

$$\hat{D}(\mu) = \exp(\mu \hat{a}^\dagger - \mu^* \hat{a}).$$

In Eq. 5, the quantities $\sigma_k^2$ are the error variances induced by the cloning process on both the $x$ and $p$ quadratures of the $k$-th clone. Notice that here we consider a
GQCM which clones symmetrically in the quadratures (in general, one can have a Gaussian cloner which is asymmetric both in the clones and the quadratures, with four different noise variances \( \sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2 \). The previous variances do not depend on the input state (universal GQCM) and satisfy the relation
\[
\sigma_1^2 \sigma_2^2 \geq 1/4,
\]
imposed by the uncertainty principle. In particular, the previous GQCM is said to be optimal if \( \sigma_1^2 \sigma_2^2 = 1/4 \). In terms of Shannon’s theory, each of the two real variables, \( x \) and \( p \), is subject to an additive Gaussian channel with noise equal to \( \sigma_k^2 \) during the cloning process from the input state to the output \( k \)-th clone.

**TWO-WAY COHERENT-STATE PROTOCOL**

The protocol is sketched in Fig. 1 and consists of two configurations, ON and OFF, that can be selected by Alice with probabilities \( 1 - c \) and \( c \) respectively.

Let Bob prepare a reference coherent state \( |\beta\rangle \langle \beta| \), with amplitude \( \beta \) randomly chosen in the complex plane (e.g., according to a Gaussian distribution with a large variance). Such a state is sent to Alice on the forward use of the quantum channel. In the ON configuration, Alice encodes a signal on this reference state via a phase-space displacement \( \hat{D}(\alpha) \) whose amplitude \( \alpha \equiv (x_A + ip_A)/\sqrt{2} \) is chosen in the \( \mathbb{C} \)-plane according to a random Gaussian distribution \( \Omega_{\Sigma^2}(\alpha) \) with large variance \( \Sigma^2 \). Notice that

(i) The signal amplitude \( \alpha \) symmetrically encodes two signal quadratures, \( x_A \) and \( p_A \), i.e., two independent and real random variables distributed according to Gaussian distributions \( G_{\Sigma^2}(x_A) \) and \( G_{\Sigma^2}(p_A) \).

(ii) The output state
\[
\hat{D}(\alpha)|\beta\rangle\langle \beta|\hat{D}^\dagger(\alpha) = |\alpha + \beta\rangle \langle \alpha + \beta|,
\]
encodes the signal amplitude \( \alpha \), masked by the reference amplitude \( \beta \) chosen by Bob.

The state is finally sent back to Bob, who tries to guess the two Alice’s numbers \( x_A \) and \( p_A \) by a joint measurement of conjugate observables \( \{\alpha, p_A\} \). This is accomplished by a heterodyne detection \( \{\alpha, p_A\} \) of the state, which will give an outcome \( \zeta \approx \alpha + \beta \). After the subtraction of the known value \( \beta \), Bob achieves an estimate \( \alpha' \) of Alice’s complex amplitude \( \alpha \), i.e., \( x_A' \approx x_A \) and \( p_A' \approx p_A \).

In the case of a noiseless channel between Alice and Bob, the only noise in all the process is introduced by the heterodyne detection. This measurement can be seen as a further Gaussian additive channel at Bob’s site, which gives a Gaussian noise equal to 1 for each quadrature. Thus, according to Shannon’s formula, we have
\[
I_{AB} = I(x_A, x_A') + I(p_A, p_A') = \log(1 + \gamma_{AB}),
\]
with \( \gamma_{AB} = \Sigma^2/1 \).

Let us now consider a noisy channel adding Gaussian noise with variances \( \sigma^2 \) (in the forward path) and \( \sigma'^2 \) (in the backward path) for each quadrature. Then, the total noise of the channel is \( \sigma^2 + \sigma'^2 \) and the total noise which Bob tests, after detection, is equal to \( \sigma^2_{ch} = \sigma^2_{ch} + 1 \), giving a SNR \( \gamma_{AB} = \Sigma^2/\sigma^2_{ch} \). In the OFF configuration, Alice and Bob estimate the noise in the channel by performing two heterodyne detections. After receiving the reference state, Alice simply heterodynes it with outcome \( \beta' \) and then reconstructs a coherent state \( |\beta\rangle \). This state is sent to Bob, who gets the outcome \( \zeta \approx \beta \) after detection. In this way, Alice and Bob collect the pairs \( \{\beta, \beta'\} \) and \( \{\beta, \zeta\} \) from which they can estimate the two noises \( \sigma^2 \) and \( \sigma'^2 \) of the channel via public communications. Notice that here we are using the ON configuration to encode the key and the OFF configuration to check the noise of the channel. This means that we are implicitly assuming that Eve’s attack is disjoint between the two paths of the quantum communication (i.e., Eve is using two distinct one-mode GQCMs). More generally, in order to exclude joint attacks between the two paths, the ON and OFF configurations must be used symmetrically for encoding and checking \( \{\beta, \beta'\} \).

**EAVESDROPPING VIA GAUSSIAN CLONERS**

In the previous two-way quantum communication, the choices of the reference \( \beta \) and the signal \( \alpha \) are two independent processes. As a consequence, Eve has to extract information on both the reference \( \beta \) and the total displacement \( \alpha + \beta \) in order to access Alice’s encoding \( \alpha \) (this is true until the attack is disjoint). Let us consider two different attacks, one on the forward use of the channel and the other one in the backward use, by using two optimal GQCMs which we call \( M \) and \( M' \), respectively (see Fig. 1).
Since the reference $\beta$ and the signal $\alpha$ are chosen with large variances, such machines must be universal, and since the information is symmetrically encoded in the two quadratures, we consider equal cloning noises in $x$ and $p$. For these reasons, Eve’s GQCMs are exactly of the kind specified by Eq. [11] with $\sigma^2_1 \sigma^2_2 = 1/4$. After cloning, Eve must extract the information about $\alpha$ from her clones. She can directly heterodyne the clones. Alternatively, she can send the clones to a beam-splitter (BS), with suitable reflection and transmission coefficients $r$ and $t$, and then heterodynes the output ports.

In order to study the eavesdropping depicted in Fig. 1, it is not sufficient to consider the reduced states $\rho_k$ of the two single clones at the output of $M$, but we have to compute explicitly the whole bipartite state $\rho_{12}$ of modes $1$ and $2$. In fact, mode $1$ is sent to Alice (who displaces $\alpha$ specified by Eq. (4) with $F$ or these reasons, Eve’s GQCMs are exactly of the kind heterodynes the output ports.

One can prove that the bipartite state $\rho_{12}$ at the output of the optimal GQCM $M$ is a Gaussian state with correlation matrix (CM) equal to

$$ V = \frac{1}{2} \left( \begin{array}{cc} (1 + 2\sigma^2)I & I \\ I & \left(1 + 1/2\sigma^2\right)I \end{array} \right), \quad (10) $$

where $I$ is the $2 \times 2$ identity matrix. The CM of Eq. (10) has positive partial transpose for every $\sigma^2 \geq 0$, and, therefore, $\rho_{12}$ is always a separable state [12]. This means that Eve cannot exploit strategies based on the entanglement between her clones and the ones of Alice and Bob. In the particular case of symmetric cloning ($\sigma^2 = 1/2$), we can write the useful decomposition

$$ \rho_{12} = \int d^2 \mu \; \Omega_{1/2}(\mu) \times |\beta + \mu \rangle_1 \langle \beta + \mu | \otimes |\beta + \mu \rangle_2 \langle \beta + \mu | \quad (11) $$

Then, let us consider the case where the first cloner $M$ is optimal and symmetric ($\sigma^2_1 = \sigma^2_2 = 1/2$), while the second cloner $M'$ is optimal but asymmetric, with $\sigma^2_1 = \omega^2$ and $\sigma^2_2 = 1/4\omega^2$. In this case, at the output modes $+$ and $-$ of the BS, we have the bipartite state

$$ \rho_{+ -} = \int d^2 \mu \; \Omega_{1/2}(\mu) \; \chi(\mu) \quad (12) $$

where

$$ \chi(\mu) \equiv \int d^2 \lambda \; \Omega_{1/4\omega^2}(\lambda) \times |\theta_+ + \lambda r \rangle_+ \langle \theta_+ + \lambda r | \otimes |\theta_- + \lambda t \rangle_- \langle \theta_- + \lambda t | \quad (13) $$

and

$$ \theta_+ \equiv (\alpha + \beta)(t + r) + \alpha r, \quad \theta_- \equiv (\mu + \beta)(t - r) + \alpha t. \quad (14) $$

If we now take a balanced BS (i.e., $t = r = 1/\sqrt{2}$) we have $\theta_- \equiv \alpha/\sqrt{2}$ and, therefore, the output port $-$ does no longer contain the reference $\beta$. Here, the action of the BS is very similar to the sum$(\text{mod}2)$ performed over a binary key ($k$) and the corresponding encrypted message ($k \oplus m$), operation that reveals the message in the classical case ($k \oplus m \oplus k = m$). On the other hand, the other port $+$ still contains a mixing between $\alpha$ and $\beta$ and, therefore, does not provide further information about the signal. Tracing out this port, we have

$$ \rho_- = \int d^2 \lambda \; \Omega_{1/4\omega^2}(\lambda) \; |(\alpha + \lambda)/\sqrt{2}\rangle_- \langle(\alpha + \lambda)/\sqrt{2}| \quad (15) $$

Heterodyning such a state, Eve can estimate the value of $\alpha$ up to a Gaussian noise with variance

$$ \sigma^2_E = 2 + (4\omega^2)^{-1}, \quad (16) $$

for each quadrature. For Bob, instead, we have a total noise

$$ \sigma^2_B = 1 + \sigma^2_{ch} \quad (17) $$

equal to the sum of the heterodyne noise (1) and the total channel noise

$$ \sigma^2_{ch} = 1/2 + \omega^2. \quad (18) $$

According to Shannon, Bob ($B$) and Eve ($E$) will share with Alice ($A$) a mutual information equal to $I_{AX} = \log(1 + \gamma_{AX})$ with $\gamma_{AX} \equiv \Sigma^2 / \sigma^2_X$ for $X = B, E$. Since

$$ I_{AB} \geq I_{AE} \iff \gamma_{AB} \geq \gamma_{AE} \iff \sigma^2_{ch} \leq \sigma^2_{E} \quad (19) $$

we can easily compute a security threshold for this kind of attack, which is equal to

$$ \sigma^2_{ch} = (3 + \sqrt{5})/4 \simeq 1.3. \quad (20) $$

Such a threshold must be compared with the security threshold (6) which characterizes one-way coherent-state protocols [4, 5] against individual GQCM attacks.

**CONCLUSION**

In this paper we have considered one of the two-way protocols introduced in [1]. Then, we have explicitly studied its security in the presence of particular kind of individual attacks which are based on combinations of one-mode Gaussian cloners. Our analysis indicates that the superadditive behavior of the security threshold should also hold against this kind of Gaussian attacks. However, our analysis is far to be complete since we have considered only particular combinations of cloners and we have also excluded the possibility of a two-mode cloner (acting coherently on both the paths of the quantum communication). Furthermore, the analysis covers the case of direct reconciliation only. Despite these restrictions, the present work represents the first step in the security analysis of two-way protocols against more exotic kind of Gaussian interactions.
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[1] S. Pirandola, S. Mancini, S. Lloyd, and S. L. Braunstein, “Continuous variable quantum cryptography using two-way quantum communication,” Nature Physics 4, 726 (2008).
[2] S. Pirandola, S. Mancini, S. Lloyd, and S. L. Braunstein, “Security of two-way quantum cryptography against asymmetric attacks,” Proc. SPIE, Vol. 7092, 709215 (2008). See also arXiv:0807.1937.
[3] T. C. Ralph, “Continuous variable quantum cryptography,” Phys. Rev. A 61, 010303(R) (2000); T. C. Ralph, “Security of continuous-variable quantum cryptography,” Phys. Rev. A 62, 062306 (2000); M. D. Reid, “Quantum cryptography with a predetermined key using continuous-variable Einstein-Podolsky-Rosen correlations,” Phys. Rev. A 62, 062308 (2000).
[4] D. Gottesman, and J. Preskill, “Secure quantum key distribution using squeezed states,” Phys. Rev. A 63, 022309 (2001); S. Iblisdir, G. Van Assche, and N. J. Cerf, “Security of quantum key distribution with coherent states and homodyne detection,” Phys. Rev. Lett. 93, 170502 (2004).
[5] F. Grosshans, G. Van Assche, J. Wenger, R. Brouni, N. J. Cerf, and P. Grangier, “Quantum key distribution using Gaussian-modulated coherent states,” Nature 421, 238 (2003); F. Grosshans, and Ph. Grangier, “Continuous variable quantum cryptography using coherent states,” Phys. Rev. Lett. 88, 057902 (2002).
[6] C. Weedbrook et al., “Quantum cryptography without switching,” Phys. Rev. Lett. 93, 170504 (2004); A. M. Lance et al., “No-switching quantum key distribution using broadband modulated coherent light,” Phys. Rev. Lett. 95, 180503 (2005).
[7] More properly, this channel is called additive Gaussian noise channel. For the general theory of these channels see, e.g., T. M. Cover and J. A. Thomas, “Elements of Information Theory” (Wiley, 2006).
[8] C. E. Shannon, “A Mathematical Theory of Communication,” Bell Syst. Tech. J. 27, 623 (1948).
[9] N. J. Cerf, A. Ipe, and X. Rottenberg, “Cloning of continuous quantum variables,” Phys. Rev. Lett. 85, 1754 (2000).
[10] E. Arthurs and J. L. Kelly, “On the simultaneous measurement of a pair of conjugate observables,” Bell Syst. Tech. J. 44, 725 (1965).
[11] H.P. Yuen and J.H. Shapiro, “Optical communication with two-photon coherent states - part III: Quantum measurements realizable with photoemissive detectors,” IEEE Trans. Inf. Theory IT-26, 78-92 (1980).
[12] R. Simon, “Peres-Horodecki separability criterion for continuous variable systems,” Phys. Rev. Lett. 84, 2726 (2000).
[13] I. Csiszár and J. Körner, “Broadcast channels with confidential messages,” IEEE Trans. Inf. Theory IT-24, 339 (1978).