A CHARACTERIZATION OF RATIONALITY IN FREE SEMICIRCULAR OPERATORS

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ABSTRACT. For free semicircular elements realized on the full Fock space, we prove an equivalence between rationality of operators obtained from them and finiteness of the rank of their commutators with right annihilation operators. This is an analogue of the result for the reduced C*-algebra of the free group by G. Duchamp and C. Reutenauer which was extended by P. A. Linnell to densely defined unbounded operators affiliated with the free group factor. While their result was motivated by quantized calculus in noncommutative geometry, we state our results in terms of free probability theory.

1. INTRODUCTION

In 1881, Kronecker [11] found an interesting connection between Hankel matrices and rational functions. In terms of functional analysis, his result shows the equivalence between finite rank Hankel operators and bounded rational functions on the unit circle (see also Section 2.3). This connection is non-trivial and relies on a beautiful combination of analytic estimates and purely algebraic properties.

The argument of Kronecker’s theorem also appears in quantized calculus for noncommutative geometry. A. Connes conjectured an analogue of this theorem for the reduced free group C*-algebra $C^*_\text{red}(F_d)$ in his book [8, Section 4.5]. He considered the commutator $[F, \cdot]$ where $F$ arises from a free action of the free group on a tree, and it can be viewed as an analogue of bounded Hankel operators. His conjecture was that rationality of $a$, for any elements in $C^*_\text{red}(F_d)$, is equivalent to finiteness of the rank of $[F, a]$. This conjecture was solved by G. Duchamp and C. Reutenauer [10] and extended by P. A. Linnell [12] to densely defined closed operators affiliated with the free group factor. Their proof is based on facts from noncommutative rational series which are also related to Hankel matrices indexed by the free monoid (see [2, Theorem 2.1.6]).

In this paper, we prove a similar phenomenon for a $d$-tuple of free semicircular elements, instead of the generators of the free group. In free probability theory, free semicircular elements are of central importance. They are limit objects not only for free central limit theorem but also for empirical eigenvalue distributions of independent Gaussian unitary ensembles which are typical random matrix models (we recommend [18], [15], and [14] for the textbooks of free probability theory). Via the GNS-representation, they are represented on the full Fock space $F(H)$ by the left annihilation and creation operators. While the operator $F$ has a role to determine rationality of elements in the reduced free group C*-algebra, we prove that the right annihilation operators $\{r_i^*\}_{i=1}^d$ as well as the right creation operators $\{r_i\}_{i=1}^d$ characterize rationality of operators generated by free semicircular elements $s = (s_1, \ldots, s_d)$. 

1
Our main theorem can be stated as follows.

**Theorem 1.1** (Theorem 3.1). Let $a$ be in a von Neumann algebra $L^\infty(s)$ generated by $s$. Then \( \{ [r_i^*, a] \}_{i=1}^d \) are finite rank operators on $\mathcal{F}(H)$ if and only if $a \in C_{\text{div}}(s)$. In addition, we have

\[
C_{\text{div}}(s) = C_{\text{rat}}(s) \subset \overline{\mathbb{C}(s)}
\]

where $\overline{\mathbb{C}(s)}$ is the norm closure of noncommutative polynomials $\mathbb{C}(s)$ in $L^\infty(s)$.

In this theorem, we consider two notions of rationality, division closure $C_{\text{div}}(s)$ and rational closure $C_{\text{rat}}(s)$ of $\mathbb{C}(s)$ in $L^\infty(s)$ (see Definition 2.4). Roughly speaking, both closures describe bounded operators constructed from $\mathbb{C}(s)$ by combinations of algebraic operations $+,-,\times,-1$.

Moreover, as a consequence of this theorem, we prove an analogue of Linnell’s work which extends the above theorem to the algebra $\widetilde{L^\infty(s)}$ of closed densely defined (unbounded) linear operators affiliated with $L^\infty(s)$.

**Theorem 1.2** (Theorem 4.5). Let $u \in \widetilde{L^\infty(s)}$ represented by $u = f^{-1}a = bg^{-1}$ for $a, b, f, g \in L^\infty(s)$. Then \( \{ fr_i^*b - ar_i^*g \}_{i=1}^d \) are finite rank operators if and only if $u$ belongs to the division closure $D(s)$ of $\mathbb{C}(s)$ in $\widetilde{L^\infty(s)}$. Moreover, we have \( D(s) \cap L^\infty(s) = C_{\text{div}}(s) \), and for any $u \in D(s)$, we can take $a, b, f, g$ so that they belong to $C_{\text{div}}(s)$.

In the last few years, rational quotients have been the object of studies in terms of free probability theory. In [13], T. Mai, R. Speicher, and S. Yin found equivalent conditions for a tuple of operators so that one can evaluate any noncommutative rational functions in them. It is interesting to note that this condition is related to Voiculescu’s (non-microstates) free entropy dimension and some quantity introduced by A. Connes and D. Shlyakhtenko [9] in the context of their $L^2$-homology for von Neumann algebras. These studies have also been applied to random matrix models obtained from noncommutative rational functions in [7]. In [1], one can see how to estimate atoms of operators obtained from noncommutative rational functions evaluated in a free tuple of normal operators with the prescribed atoms. Our results show additional evidence that tools in free probability ought to play a natural role when we study rationality in a noncommutative setting.

This paper consists of four sections including the introduction. We explain the basics of the full Fock space and noncommutative rational series in Section 2. Then we prove our main result in Section 3. In Section 4, we extend this result to affiliated operators like in Linnell’s work.

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2. Preliminaries

2.1. Full Fock space. In the beginning of this section, we introduce free semicircular elements which are represented on the full Fock space. Let $d$ be a positive integer and $H$ be a complex Hilbert space with the dimension $d$ and an inner product $\langle \cdot, \cdot \rangle_H$. Then we consider the full Fock space as an orthogonal sum of Hilbert spaces $H^\otimes n$, 

$$F(H) = \bigoplus_{n=0}^{\infty} H^\otimes n$$

where $H^\otimes 0 = \mathbb{C}\Omega$ with a unit vector $\Omega$. $\langle \cdot, \cdot \rangle_{F(H)}$ denotes the inner product on $F(H)$. Note that $\langle \cdot, \cdot \rangle_{F(H)}$ satisfies

$$\langle \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_m, \eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_n \rangle_{F(H)} = \delta^n_m \prod_{i=1}^{m} \langle \xi_i, \eta_i \rangle_H$$

for $m, n \in \mathbb{N}$ and $\xi_i, \eta_i \in H$ where $\delta^n_m$ is the Kronecker’s delta. We consider a set of letters $[d] = \{1, \ldots, d\}$ and $[d]^*$ be the set of words which is the free monoid generated by $[d]$ with the empty word $\Omega$ (i.e. the identity in $[d]^*$). We denote by $|v|$ the length of a word $v \in [d]^*$.

We associate words in $[d]^*$ with an orthonormal basis of $F(H)$. Let $\{e_v\}_{v \in [d]}$ be an orthonormal basis of $H$ and we define $e_v = e_{v_1} \otimes e_{v_2} \otimes \cdots \otimes e_{v_n}$ for $v = v_1 v_2 \cdots v_n \in [d]^*$ and $e_\Omega = \Omega$. Then $\{e_v\}_{v \in [d]^*}$ is an orthonormal basis of $F(H)$.

Let $B(F(H))$ denote the set of bounded operators on $F(H)$. Now we introduce our main objects.

**Definition 2.1.** For $f \in H$, we define the left creation operator $l(f) \in B(F(H))$ by

$$l(f) e_v = f \otimes e_v,$$

and we also call its adjoint operator $l(f)^* \in B(F(H))$ the left annihilation operator.

In addition, we define the right creation operator $r(f) \in B(F(H))$ by

$$r(f) e_v = e_v \otimes f,$$

and the right annihilation operator by $r(f)^*$.

Note that $l(f)^*$ satisfies $l(f)^* e_v = \langle e_{v_1}, f \rangle_H e_{v_2} \cdots e_{v_n}$ and $r(f)^* e_v = \langle e_{v_n}, f \rangle_H e_{v_{n-1}} \cdots e_{v_1}$ for $v = v_1 v_2 \cdots v_n \in [d]^*$. Throughout this paper, we put $l_i = l(e_i)$ and $r_i = r(e_i)$ and $s_i = l_i + l_i^*$ for each $i \in [d]$.

Let $\mathbb{C}\langle s \rangle$ denote the *-algebra of noncommutative polynomials in $s = (s_1, \ldots, s_d)$; note that elements of $\mathbb{C}\langle s \rangle$ can be written as

$$\sum_{v \in [d]^*} \sum_{|v| \leq N} \alpha_v s^v$$

where $N \in \mathbb{Z}_{\geq 0}$, $\alpha_v \in \mathbb{C}$ and $s^v = s_{v_1} \cdots s_{v_n}$ for any $v = v_1 v_2 \cdots v_n \in [d]^*$.

We define $L^\infty(\mathbb{C}\langle s \rangle)$ as the von Neumann subalgebra of $B(F(H))$ generated by $s = (s_1, \ldots, s_d)$. In other words, $L^\infty(\mathbb{C}\langle s \rangle)$ is the closure of $\mathbb{C}\langle s \rangle$ in strong operator topology.
We remark that $\Omega$ is cyclic and separating for $L^\infty(s)$, i.e. $L^\infty(s)\Omega = F(H)$, and if $X \in L^\infty(s)$ satisfies $X\Omega = 0$, then we have $X = 0$.

We define the vacuum state $\tau_\Omega$ on $L^\infty(s)$ by

$$\tau_\Omega(\cdot) = \langle \cdot, \Omega \rangle_{F(H)}.$$  

Then it is known that $\tau_\Omega$ is a tracial state for $L^\infty(s)$, i.e. $\tau_\Omega(XY) = \tau_\Omega(YX)$ for $X, Y \in L^\infty(s)$.

Moreover, $\{s_i\}_{i=1}^d$ have free semicircle distributions with respect to $\tau_\Omega$. For these properties of $\Omega$ and $\tau_\Omega$, we refer to [3, Section 5.1].

Let $L^2(s)$ be the Hilbert space obtained from $L^\infty(s)$ by completion with respect to the inner product $\langle x, y \rangle_2 = \tau_\Omega(y^*x)$, $x, y \in L^\infty(s)$. Since the map

$$L^\infty(s) \to F(H)$$

$$X \mapsto \hat{X} = X\Omega$$

is an isometry and $\Omega$ is cyclic (i.e. $F(H) = L^\infty(s)\Omega$), this map can be extend to a unitary operator from $L^2(s)$ to $F(H)$.

Indeed, via the map $X \mapsto \hat{X}$, we can represent an orthonormal basis $\{e_v\}_{v \in [d]^*}$ of $F(H)$ as elements in $L^\infty(s)$ by using the Chebyshev polynomials of the second kind. Recall that the Chebyshev polynomials $U_n(X) \in \mathbb{C}[X]$ of the second kind are defined by the following recursion:

$$U_{-1}(X) = 0, \quad U_0(X) = 1, \quad U_{n+1}(X) = XU_n(X) - U_{n-1}(X).$$

For $w = i_1^{k_1}i_2^{k_2}\cdots i_n^{k_n}$ (where $i_1 \neq i_2 \neq \cdots \neq i_n$), we define an element $U_w$ in $L^\infty(s)$ by

$$U_w = U_{k_1}(s_{i_1})U_{k_2}(s_{i_2})\cdots U_{k_n}(s_{i_n})$$

and also define $U_0 = 1$. Then $U_w$ is defined for any $w \in [d]^*$. We can see by induction on the word length that for any $v \in [d]^*$,

$$\hat{U}_v = e_v.$$  

This equality is also remarked in [3, Section 5.1]. Since our main result focuses on noncommutative polynomials over $s = (s_1, \ldots, s_d)$, we will use $\hat{U}_v$ rather than $e_v$.

In order to represent annihilation operators $\hat{l}_i^*, \hat{r}_i^*$ in terms of word translations, we introduce the following operations which are also introduced in [2, Chapter 1].

**Definition 2.2.** Let 0 be a new letter. For $v \in [d]^* \cup \{0\}$ and $w \in [d]^*$, we define

$$v_{\cdot\cdot v^{-1}} = \begin{cases} v' & \text{if } v = v'w, \ v' \in [d]^* \\ \text{0 otherwise} \end{cases}$$

and also define

$$w_{\cdot\cdot \text{v}} = \begin{cases} v' & \text{if } v = \text{vw}', \ v' \in [d]^* \\ \text{0 otherwise} \end{cases}$$

Put $U_0 = 0$. One should be careful that we use the same notation $U_0$ for $U_0 = 0$ and $U_0(X) = 1$ ($U_0(X)$ corresponds with $U_{\Omega}$ in our definition). Then by using above notations we have for each $i \in [d]$

$$l_i^*(\hat{U}_v) = \hat{l}_{i-1}^*(v) \quad \text{and} \quad r_i^*(\hat{U}_v) = \hat{r}_{i-1}^*(v), \ v \in [d]^*.$$
2.2. Noncommutative rational series. In the sequel, we explain about noncommutative rational series. First, we give two definitions of rationality in a setting of unital algebras (over \(\mathbb{C}\)) as follows (see [2] Definition 6 or [13] Definition 4.6 and 4.8).

**Definition 2.3.** Let \(A\) be a unital algebra and \(B \subset A\) be a unital subalgebra of \(A\). We define the division closure of \(B\) in \(A\) as the smallest unital subalgebra \(C\) of \(A\) such that \(C\) contains \(B\) and satisfies
\[
x \in C \text{ is invertible in } A \implies x^{-1} \in C.
\]
In addition, we define the rational closure of \(B\) in \(A\) as the smallest (unital) subalgebra \(D\) of \(A\) such that \(D\) contains \(B\) and satisfies for any \(n \in \mathbb{N}\),
\[
X \in M_n(D) \text{ is invertible in } M_n(A) \implies X^{-1} \in M_n(D).
\]

Obviously, the division closure of any subalgebra is always contained in the rational closure of the same subalgebra, however, the converse is not necessarily true (Exercise 7.1.3 in [6]).

We will use facts for noncommutative rational series specific to our setting. The proofs of these results can be found in [2, Chapter 1]. We consider the algebra \(\mathbb{C}[[d]]\) of noncommutative formal power series with formal (noncommutative) variables \(\{X_i\}_{i \in [d]}\) like as \(\sum_{v \in [d]} \alpha_v X^v\) where \(X^v = X_{v_1} X_{v_2} \cdots X_{v_n}\) for \(v = v_1 v_2 \cdots v_n \in [d]^*\). Let \(\mathbb{C}[[d]]\) denote the subalgebra of noncommutative polynomials.

**Definition 2.4.** Let \(Z = \sum_{v \in [d]} \alpha_v X^v \in \mathbb{C}[[d]]\). We say \(Z\) is recognizable if there exists \(m \in \mathbb{N}\) and a linear representation \((\lambda, \mu, \gamma)\) of dimension \(m\) which consists of a multiplicative map \(\mu : [d]^* \to M_m(\mathbb{C})\) (i.e. \(\mu(vw) = \mu(v)\mu(w)\) for any \(v, w \in [d]^*\)) and \(\lambda, \gamma \in \mathbb{C}^m\) such that for any \(v \in [d]^*\)
\[
\alpha_v = \lambda \mu(v) \gamma.
\]

Let us say \(Z\) is rational if \(Z\) belong to the division closure of \(\mathbb{C}[[d]]\) in \(\mathbb{C}[[d]]\). Then the following theorem, known as the fundamental theorem, is crucial in this paper. This result is a collection of several works by Fliess, Jacobi, Kleene, and Schützenberger.

**Theorem 2.5** (Corollary 1.5.4 and Theorem 1.7.1 in [2]). Let \(Z = \sum_{v \in [d]} \alpha_v X^v \in \mathbb{C}[[d]]\). Then the following are equivalent.

1. A \(\mathbb{C}\)-vector subspace of \(\mathbb{C}[[d]]\) generated by \(\sum_{v \in [d]} \alpha_v X^{vw^{-1}} (w \in [d]^*)\) is finitely generated.

2. A \(\mathbb{C}\)-vector subspace of \(\mathbb{C}[[d]]\) generated by \(\sum_{v \in [d]} \alpha_v X^{w^{-1}v} (w \in [d]^*)\) is finitely generated.

3. \(Z\) is recognizable.

4. \(Z\) is rational.

Moreover, if a noncommutative formal power series is recognizable and its linear representation \((\lambda, \mu, \gamma)\) has the minimal dimension, then \(\mu\) is determined by its coefficients. This can be stated as follows.
Theorem 2.6 (Corollary 2.2.3 in [2]). Suppose $Z = \sum_{v \in [d]} \alpha_v X^v \in C \langle \langle [d] \rangle \rangle$ is recognizable with a linear representation $(\lambda, \mu, \gamma)$ which has the minimal dimension $m$. Then there exist $\{u_k\}_{k=1}^K, \{w_l\}_{l=1}^L \subset [d]^*$ and $c_{ij}^k \in C$ such that for any $v \in [d]^*$ and $1 \leq i, j \leq m$

$$\mu(v)_{ij} = \sum_{kl} c_{ij}^k \alpha_{u_k} v_{w_l}.$$ 

In addition, we use an operation between noncommutative formal power series. For $Z_1 = \sum_{v \in [d]} \alpha_v X^v, Z_2 = \sum_{v \in [d]} \beta_v X^v \in C \langle \langle [d] \rangle \rangle$, we define the Hadamard product $Z_1 \odot Z_2$ by

$$Z_1 \odot Z_2 = \sum_{v \in [d]^*} \alpha_v \beta_v X^v.$$ 

One of the connections between the Hadamard product and rationality can be stated as follows.

Theorem 2.7 (Theorem 1.5.5 in [2]). If $Z_1, Z_2 \in C \langle \langle [d] \rangle \rangle$ are rational, then $Z_1 \odot Z_2$ is also rational.

2.3. Kronecker’s theorem. We need to recall Kronecker’s theorem which basically tells us the equivalence between bounded rational functions and finite rank Hankel operators.

Let $\{\alpha_n\}_{n=0}^\infty \subset C$. We call a bounded operator $H$ on $l^2(Z \geq 0)$ the Hankel operator with respect to $\{\alpha_n\}_{n=0}^\infty$ if $H$ satisfies

$$\langle He_m, e_n \rangle = \alpha_{m+n}$$

for any $m, n \in Z \geq 0$ where $\{e_m\}_{m=0}^\infty$ is the standard orthonormal basis of $l^2(Z \geq 0)$. The following theorem is known as Kronecker’s theorem for the studies of Hankel operators (see [11] and [16 Theorem 3.11]).

Theorem 2.8. Let $\{\alpha_n\}_{n=0}^\infty \subset C$. Then a formal Laurent series (in $z^{-1}$) $a(z) = \sum_{n=0}^\infty \alpha_n z^{-n-1}$ is a rational function (i.e. $a(z) = \frac{P(z)}{Q(z)}$ for some polynomials $P(z), Q(z)$) such that all poles of $a(z)$ are contained in $\{z \in C \mid |z| < 1\}$ if and only if $\{\alpha_n\}_{n=0}^\infty$ determines a finite rank Hankel operator. In this case, the number of poles on $f$ is equal to the rank of the Hankel operator.

Here, we explain a related recursion and estimate in Theorem 2.8 in order to explain Corollary 2.9 which we will use in the proof of Corollary 2.7. Indeed, if $a(z) = \sum_{n=0}^\infty \alpha_n z^{-n-1}$ is rational and the denominator of $a(z)$ written as $Q(z) = \sum_{k=0}^m \lambda_k z^k \ (\lambda_m \neq 0)$, then we have the following recursion for $\{\alpha_n\}_{n=0}^\infty$

$$\sum_{k=0}^m \lambda_k \alpha_{n+k} = 0,$$

where $\{\alpha_n\}_{n=0}^{m-1}$ are determined by the numerator of $a(z)$. This recursion is characterized by the poles of $a(z)$, and if we additionally assume $\lim_{n \to \infty} \alpha_n = 0$, we can see that all poles of $a(z)$ are contained in $\{z \in C \mid |z| < 1\}$ (see the proof of [16 Theorem 3.11]). Moreover, this implies $|\alpha_n|$ is bounded above by $Me^n$ where $M > 0$ and $c = \max\{|p| \mid p \text{ is a pole of } a(z)\}$.

By replacing $a(z)$ by $za(z^{-1})$, we obtain the following estimate from the above observation, which is used in the proof of [10 Lemma 10].
Corollary 2.9. Let \( a(z) = \sum_{n=0}^{\infty} \alpha_n z^n \) be a formal power series with \( \sum_{n=0}^{\infty} |\alpha_n|^2 < \infty \). If \( a(z) \) is rational, then there exists \( M > 0 \) and \( 0 < c < 1 \) such that we have for any \( n \in \mathbb{N} \)

\[
|\alpha_n| \leq Mc^n.
\]

3. Main results

Let \( C_{\text{div}}(s) \) denote the division closure of \( \mathbb{C}(s) \) in \( L^\infty(s) \) and \( C_{\text{rat}}(s) \) denote the rational closure of \( \mathbb{C}(s) \) in \( L^\infty(s) \).

Let us state our main theorem again.

Theorem 3.1. Let \( a \in L^\infty(s) \). Then \( \{r_i^*, a\}_{i=1}^d \) are finite rank operators on \( \mathcal{F}(H) \) if and only if \( a \in C_{\text{div}}(s) \). In addition, we have

\[
C_{\text{div}}(s) = C_{\text{rat}}(s) \subset \overline{\mathbb{C}(s)}
\]

where \( \overline{\mathbb{C}(s)} \) is the norm closure of \( \mathbb{C}(s) \) in \( L^\infty(s) \).

We basically follow the proof by G. Duchamp and C. Reutenauer [10]. The following two lemmas have important roles in proving our main theorem.

Lemma 3.2. For any \( i, j \in [d] \) and \( k \in \mathbb{N} \), we have

\[
[r_i^*, U_k(s_j)] = \delta_j^k \sum_{l=1}^k U_{l-1}(s_j)P_HU_{k-l}(s_j)
\]

where \( P_H \) is the orthogonal projection onto \( \hat{U}_\Omega = \Omega \).

Proof. This lemma is easily deduced from the property of Chebyshev polynomials (see [14] Exercise 10 in Section 8.8) and a dual system (see [17] Semicircular Example 5.13). However, we give a proof of this lemma for the purpose of self-containment.

First, we show for any \( i, j \in [d] \)

\[
[r_i^*, s_j] = \delta_j^i P_H.
\]

For any \( v \in [d]^* \) we have

\[
[r_i^*, s_j]U_v = [r_i^*, t_j^* + l_j]U_v = r_i^* (l_j^* + l_j)U_v - (t_j^* + l_j)r_i^* U_v = \hat{U}_{(j-1)v_i+1} - \hat{U}_{(j-1)v_i+1} - \hat{U}_{(j+1)v_i} - \hat{U}_{(j-1)v_i}.
\]

Note that \( [r_i^*, s_j]U_v = 0 \) except for \( v = \Omega \) and in this case we have

\[
[r_i^*, s_j]U_\Omega = \hat{U}_{ji^{-1}} = \delta_j^i \hat{U}_\Omega.
\]

Then we can compute \( [r_i^*, U_k(s_j)] \) by induction since by the Leibniz rule we have

\[
[r_i^*, U_{k+1}(s_j)] = [r_i^*, s_jU_k(s_j)] - [r_i^*, U_k(s_j)]
\]

\[
= [r_i^*, s_j](k_{s_j}U_k(s_j)) + s_j[r_i^*, U_k(s_j)] - [r_i^*, U_{k-1}(s_j)]
\]

\[
= \delta_j^i P_HU_k(s_j) + s_j[r_i^*, U_k(s_j)] - [r_i^*, U_{k-1}(s_j)]
\]
and also have by the recursion formula of $U_k$

$$s_j \sum_{l=1}^{k} U_{l-1}(s_j)P_lU_{k-l}(s_j) = \sum_{l=1}^{k} U_l(s_j)P_lU_{k-l}(s_j) + \sum_{l=2}^{k} U_{l-2}(s_j)P_lU_{k-l}(s_j)$$

$$= \sum_{l=2}^{k} U_{l-1}(s_j)P_lU_{k+1-l}(s_j) + \sum_{l=1}^{k-1} U_{l-1}(s_j)P_lU_{k-1-l}(s_j).$$

By multiplying by $\delta^i_j$ and using the induction hypothesis,

$$s_j[r^*_i, U_k(s_j)] = \delta^i_j \sum_{l=2}^{k+1} U_{l-1}(s_j)P_lU_{k+1-l}(s_j) + [r^*_i, U_{k-1}(s_j)],$$

which gives the asserted formula for $[r^*_i, U_{k+1}(s_j)]$. \qed

**Lemma 3.3.** For $v, w \in [d]^*$ and $i \in [d]$, we have

$$[r^*_i, U_v]\hat{U}_w = \hat{U}_{v(iw^*)^{-1}}$$

where $w^*$ is the transpose of $w$, in other words $w^* = i_{k_1}^n i_{k_2}^{n-1} \cdots i_{k_1}^1$ when $w = i_{k_1}^1 i_{k_2}^1 \cdots i_{k_n}^n$.

**Proof.** By Lemma 3.2 and the fact that $[r^*_i, \cdot]$ is a derivation, we have for $v = i_{k_1}^1 i_{k_2}^1 \cdots i_{k_n}^n$,

$$[r^*_i, U_v]\hat{U}_w$$

$$= \left( \sum_{m=1}^{n} U_{k_1}(s_{i_{k_1}}) \cdots U_{k_{m-1}}(s_{i_{k_{m-1}}}) [r^*_i, U_{k_m}(s_{i_{k_m}})] U_{k_{m+1}}(s_{i_{k_{m+1}}}) \cdots U_{k_n}(s_{i_{k_n}}) \right) \hat{U}_w$$

$$= \sum_{m=1}^{n} \sum_{j=1}^{k_m} \delta^{i_j}_m U_{k_1}(s_{i_{k_1}}) \cdots U_{k_{m-1}}(s_{i_{k_{m-1}}}) U_{j-1}(s_{i_{k_m}}) P_l U_{k_{m-j}}(s_{i_{k_m}}) U_{k_{m+1}}(s_{i_{k_{m+1}}}) \cdots U_{k_n}(s_{i_{k_n}}) \hat{U}_w$$

$$= \sum_{m=1}^{n} \sum_{j=1}^{k_m} \delta^{i_j}_m U_{k_1}^1 \cdots U_{k_{m-j}}^1 P_l U_{k_{m-j}}^1 \cdots \hat{U}_w.$$

Since we have

$$P_l U_{k_{m-j}}^1 \cdots \hat{U}_w = \langle U_{i_{k_{m-j}}^1} U_{i_{k_{m-j}}^1} \cdots \hat{U}_w, \Omega \rangle_{\mathcal{F}(H)} \Omega$$

and $\{\hat{U}_w\}_{w \in [d]^*}$ is an orthonormal basis, we conclude

$$[r^*_i, U_v]\hat{U}_w = \sum_{m=1}^{n} \sum_{j=1}^{k_m} \delta^{i_j}_m \langle \hat{U}_w, \hat{U}_{k_{m-j}}^1 \cdots \hat{U}_w \rangle_{\mathcal{F}(H)} \hat{U}_{i_{k_{m-j}}^1} \cdots \hat{U}_{i_{k_{m-j}}^{m-1}} = \hat{U}_{v(iw^*)^{-1}}.$$

\qed

Next we associate elements $\sum_{v \in [d]^*} \alpha_v \hat{U}_v$ in $\mathcal{F}(H)$ with noncommutative formal power series $\sum_{v \in [d]^*} \alpha_v X^v$. Since $U_v U_w \neq U_{vw}$ in general, we cannot directly connect $U_v$ with $X^v$ while keeping a multiplicative structure. However we can connect them by using a matrix representation, which may help us to prove our main theorem.
Lemma 3.4. For each \( i \in [d] \), we put
\[
S_i = E_{ii} \otimes \begin{pmatrix} s_i & 1 \\ 0 & 0 \end{pmatrix} + \sum_{j \neq i} E_{ji} \otimes \begin{pmatrix} s_i & 1 \\ 0 & 0 \end{pmatrix} \in M_{d}(\mathbb{C}) \otimes M_{2}(L^\infty(s))
\]
where \( E_{ji} \in M_{d}(\mathbb{C}) \) is a matrix whose \((j, i)\) entry is 1 and other entries are 0. Then for \( v = i_1^{k_1} i_2^{k_2} \cdots i_n^{k_n} \in [d]^n (i_1 \neq i_2 \neq \cdots \neq i_n) \) we have
\[
U_v = (1 \ 0) \left( t e_1 \otimes I_2 \right) S_{i_1}^{k_1} S_{i_2}^{k_2} \cdots S_{i_n}^{k_n} (e \otimes I_2) \left( \begin{array}{c} 1 \\ 0 \end{array} \right)
\]
where \( I_2 \) is the identity matrix and \( \{e_i\}_{i \in [d]} \subset \mathbb{C}^d \) is the standard basis of \( \mathbb{C}^d \), and we put \( e = \sum_{i=1}^{d} e_i \).

Proof. Since Chebyshev polynomials \( U_n(X) \) satisfy for \( n \in \mathbb{N} \)
\[
\begin{pmatrix} U_n(X) \\ U_{n-1}(X) \end{pmatrix} = \begin{pmatrix} X & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} U_{n-1}(X) \\ U_{n-2}(X) \end{pmatrix},
\]
we can show that
\[
\begin{pmatrix} X & -1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} U_n(X) & -U_{n-1}(X) \\ U_{n-1}(X) & -U_{n-2}(X) \end{pmatrix}.
\]
In particular, we have for any \( i \in [d] \) and \( n \in \mathbb{Z} \geq 0 \)
\[
U_{i^n} = (1 \ 0) \left( \begin{array}{c} s_i \\ 1 \\ 0 \end{array} \right)^n \left( \begin{array}{c} 1 \\ 0 \end{array} \right).
\]
Then we have for \( v = i_1^{k_1} i_2^{k_2} \cdots i_n^{k_n} \)
\[
U_v = U_{i_1}^{k_1} U_{i_2}^{k_2} \cdots U_{i_n}^{k_n}
\]
\[
= \prod_{i=1}^{n} (1 \ 0) \begin{pmatrix} s_i & 1 \\ 0 & 0 \end{pmatrix}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
\[
= (1 \ 0) \left[ \prod_{i=1}^{n} P \begin{pmatrix} s_i & 1 \\ 0 & 0 \end{pmatrix}^k \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
where we put \( P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \). Note that \( P \begin{pmatrix} s_i & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} s_i & 1 \\ 0 & 0 \end{pmatrix} \). Since we have
\[
S_i^n = E_{ii} \otimes \begin{pmatrix} s_i & 1 \\ 0 & 0 \end{pmatrix}^n + \sum_{j \neq i} E_{ji} \otimes P \begin{pmatrix} s_i & 1 \\ 0 & 0 \end{pmatrix}^n,
\]
we obtain for \( i_1 \neq i_2 \neq \cdots \neq i_n \)
\[
S_{i_1}^{k_1} S_{i_2}^{k_2} \cdots S_{i_n}^{k_n} = \prod_{i=1}^{n} \left[ E_{ii} \otimes \begin{pmatrix} s_i & 1 \\ 0 & 0 \end{pmatrix}^{k_i} + \sum_{j \neq i} E_{ji} \otimes P \begin{pmatrix} s_i & 1 \\ 0 & 0 \end{pmatrix}^{k_i} \right].
\]
Thus we conclude (note that \((1 \ 0) P = (1 \ 0)\)),

\[
(1 \ 0) \left( ^t e_1 \otimes I_2 \right) S_{i_1}^{k_1} S_{i_2}^{k_2} \cdots S_{i_n}^{k_n} (e \otimes I_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1 \ 0) \left[ \prod_{l=1}^{n} P \begin{pmatrix} s_{i_l}^{-1} & -1 \\ 1 & 0 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} = U_v.
\]

Next step is to show convergence of \(\sum_{w \in [d]} \alpha_w S^w\) under certain assumptions. In order to estimate the operator norm of \(\sum_{w \in [d]} \alpha_w S^w\), we use the Haagerup type inequality for the full Fock space which was proved by M. Bożejko in terms of the \(q\)-deformed Fock space. One can get the inequality for the full Fock space as \(q = 0\). Here, we revisit a proof of this inequality in the case \(q = 0\) for reader’s convenience.

**Lemma 3.5 (Haagerup inequality).** Let \(m \in \mathbb{Z}_{\geq 0}\) and \(\{\alpha_v\}_{|v|=m}\) be a family of complex numbers. Then we have

\[
\left\| \sum_{|v|=m} \alpha_v U_v \right\| \leq (m + 1) \left\| \sum_{|v|=m} \alpha_v \hat{U}_v \right\|_{\mathcal{F}(H)},
\]

where \(\| \cdot \|\) is the operator norm on \(B(\mathcal{F}(H))\) and \(\| \cdot \|_{\mathcal{F}(H)}\) is the norm defined by \(\sqrt{\langle \xi, \xi \rangle_{\mathcal{F}(H)}}\) for \(\xi \in \mathcal{F}(H)\).

**Proof.** First, we show

\[
\max \left\{ \left\| \sum_{|v|=m} \alpha_v l_v \right\|, \left\| \sum_{|v|=m} \alpha_v \hat{l}_v \right\| \right\} \leq \left\| \sum_{|v|=m} \alpha_v U_v \right\|_{\mathcal{F}(H)},
\]

where \(l_v = l_{v_1} l_{v_2} \cdots l_{v_m}, \hat{l}_v = \hat{l}_{v_1} \hat{l}_{v_2} \cdots \hat{l}_{v_m}\) for \(v = v_1 v_2 \cdots v_m\). Since we have for \(\xi \in H^\otimes n\)

\[
\left\| \sum_{|v|=m} \alpha_v l_v \xi \right\|_{\mathcal{F}(H)}^2 = \left\| \sum_{|v|=m} \alpha_v (e_v \otimes \xi) \right\|_{\mathcal{F}(H)}^2 = \sum_{|v|=m} |\alpha_v|^2 \|\xi\|_{\mathcal{F}(H)}^2
\]

and \(\sum_{|v|=m} \alpha_v l_v(\xi)\) and \(\sum_{|v|=m} \alpha_v l_v(\eta)\) are orthogonal for \(\xi, \eta \in H^\otimes n\). We have \(\| \sum_{|v|=m} \alpha_v l_v \| \leq \| \sum_{|v|=m} \alpha_v \hat{U}_v \|_{\mathcal{F}(H)}\). Moreover, by taking
In order to prove this lemma, we use the following characterization of $\xi$ from the formula, we rewrite it as:

$$\sum_{|v|=m} \alpha_v l_v^* = \left( \sum_{|v|=m} \alpha_v l_v^* \right)^* = \left( \sum_{|v|=m} \overline{\alpha_v} l_v \right)^* = \sqrt{\sum_{|v|=m} |\alpha_v|^2} = \sum_{|v|=m} \alpha_v U_v$$

From this formula, we rewrite $\sum_{|v|=m} \alpha_v U_v$ by $\sum_{k=0}^{m} F(k)$ where $F(k)$ denotes

$$\sum_{|v|=m} \alpha_v l_v l_v^*$$

for $k = 0, \ldots, n$. We will show $\|F(k)\| \leq \| \sum_{|v|=m} \alpha_v U_v \|_{\mathcal{F}(H)}$ for any $k$. Since we have already proved this for $k = 0, n$, we fix $k = 1, \ldots, n-1$. In addition, since $F(k)(\xi)$ and $F(k)(\eta)$ are orthogonal when $\xi \in H^\otimes n$, $\eta \in H^\otimes n'$ where $n \neq n'$, it suffices to show that $\|F(k)(\xi)\|_{\mathcal{F}(H)} \leq \| \sum_{|v|=m} \alpha_v U_v \|_{\mathcal{F}(H)} \| \xi \|_2$ for $\xi \in H^\otimes n$ where $m \geq n - k$ (note that $F(k)(\xi) = 0$ when $n < m - k$). Then we have

$$\|F(k)(\xi)\|_{\mathcal{F}(H)}^2 = \left\langle \sum_{|u_1|=k} \alpha_{u_1} l_{u_1}^* \xi, \sum_{|v_1|=n-k} \alpha_{v_1} l_{v_1} \xi \right\rangle_{\mathcal{F}(H)}$$

$$= \sum_{|u_1|=k} \sum_{|v_1|=n-k} \alpha_{u_1} \alpha_{v_1} \left\langle l_{u_1}^* \xi, l_{v_1} \xi \right\rangle_{\mathcal{F}(H)}$$

$$= \sum_{|u_1|=k} \sum_{|v_1|=n-k} \alpha_{u_1} \alpha_{v_1} \left\langle l_{u_1}^* \xi, l_{v_1} \xi \right\rangle_{\mathcal{F}(H)}$$

Since $\{e_v\}_{v \in [d]}$ is an orthonormal basis of $\mathcal{F}(H)$, the last term is equal to

$$\sum_{|u_1|=k} \sum_{|v_1|=n-k} \alpha_{u_1} \alpha_{v_1} \left\langle l_{u_1}^* \xi, l_{v_1} \xi \right\rangle_{\mathcal{F}(H)} = \sum_{|u|=k} \sum_{|v|=n-k} \alpha_{uv} l_u \xi, \sum_{|v|=n-k} \alpha_{uv} l_v \xi \right\rangle_{\mathcal{F}(H)}$$

$$= \sum_{|u|=k} \sum_{|v|=n-k} \alpha_{uv} \xi^2 \left\|_{\mathcal{F}(H)}^2 \right.$$
Since we have \( \|\sum_{|v|=n-k} a_{uv} l_v^*\| \leq \sqrt{\sum_{|v|=n-k} |a_{uv}|^2} \), we obtain

\[
\|F^{(k)}(\xi)\|^2 \leq \sum_{|v|=k} \sum_{|v|=n-k} |a_{uv}|^2 \|\xi\|^2 \mathcal{F}(H) = \left( \sum_{|v|=m} \alpha_v \mathcal{U}_v \right)^2 \|\xi\|^2 \mathcal{F}(H).
\]

Thus we conclude

\[
\left\| \sum_{|v|=m} \alpha_v U_v \right\| = \left\| \sum_{k=0}^m F^{(k)} \right\| \\
\leq \sum_{k=0}^m \|F^{(k)}\| \\
\leq (m + 1) \left\| \sum_{|v|=m} \alpha_v \hat{U}_v \right\| \mathcal{F}(H).
\]

\[
\square
\]

**Lemma 3.6.** Let us take \( S_1, \ldots, S_d \in \mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_d(L^\infty(\mathbb{R})) \) as in Lemma 3.4. Then we have for any \( m \in \mathbb{Z}_{\geq 0} \)

\[
\left\| \sum_{|v|=m} \alpha_v S^v \right\| \leq 4d^2 (m + 1) \left\| \sum_{|v|=m} \alpha_v \hat{U}_v \right\| \mathcal{F}(H).
\]

**Proof.** When \( m = 0, 1 \), one can easily derive the above inequality from Lemma 3.4 and Lemma 3.5. Thus we may suppose from now on that \( m \geq 2 \). Recall from the proof of Lemma 3.4 that we have for \( v = i_{k_1} i_{k_2} \ldots i_{k_n} \)

\[
S^v = E_{i_{k_1} i_{k_2}} \otimes \left[ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \prod_{l=2}^{n} P \left( \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right)^{k_l} \right] + \sum_{j \neq i_1} E_{j_{k_n}} \otimes \left[ \prod_{l=1}^{n} P \left( \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right)^{k_l} \right].
\]

Since we have

\[
\left( \begin{array}{rr}
 s_{i_1} & -1 \\
 0 & 1
\end{array} \right)^{k_i} \prod_{l=2}^{n} \left( \begin{array}{rr}
 s_{i_l} & -1 \\
 0 & 1
\end{array} \right)^{k_l} = \left( \begin{array}{cccc}
 U_{i_{k_1}} & 0 \\
 0 & 1
\end{array} \right) \left( \prod_{l=2}^{n} U_{i_{k_l}} \right) \left( \begin{array}{cccc}
 U_{i_{k_n}} & -U_{i_{k_n-1}} \\
 0 & 0
\end{array} \right),
\]

\[
S^v \text{ can be written as the following form,}
\]

\[
S^v = E_{i_{k_1} i_{k_2}} \otimes \left( \begin{pmatrix} U_v & -U_{v_{i_{k_1}}^{-1}} \\
 U_{i_{k_1}^{-1}} & -U_{i_{k_1}^{-1} v_{i_{k_1}}^{-1}} \end{pmatrix} \right) + \sum_{j \neq i_1} E_{j_{k_n}} \otimes \left( \begin{pmatrix} U_v & -U_{v_{i_{k_n}}^{-1}} \\
 0 & 0 \end{pmatrix} \right).
\]
Thus we have
\[ \sum_{|v|=m} \alpha_v S^v = \]
\[ \sum_{i,j \in [d]} \sum_{|v|=m-2} \alpha_{ivj} E_{ij} \otimes \left( U_{ivj} - U_{ivj} \right) + \sum_{k \neq i} E_{kij} \otimes \left( U_{ivj} - U_{ivj} \right) \]
\[ = \sum_{i,j \in [d]} E_{ij} \otimes \left( \sum_{k \in [d]} \sum_{|v|=m-2} \alpha_{kvj} U_{kvj} - \sum_{|v|=m-2} \alpha_{ivj} U_{ivj} \right) \]
\[ = \sum_{i,j \in [d]} E_{ij} \otimes \left( \sum_{|v|=m-1} \alpha_{ivj} U_{ivj} - \sum_{|v|=m-2} \alpha_{ivj} U_{ivj} \right), \]

Note that all entries of \( \sum_{|v|=m} \alpha_v S^v \) are sums of \( U_v \) \( (|v| = m, m-1, m-2) \) whose coefficients are subsequences of \( \{\alpha_v\}_{|v|=m} \). Therefore by Lemma 3.5, operator norms of all entries of \( \sum_{|v|=m} \alpha_v S^v \) are bounded by \( (m+1) \| \sum_{|v|=m} \alpha_v U_v \| \|U\| \) and we obtain a desired estimate by the triangle inequality.

By the same argument in the Lemma 10 of [10], we have the following corollary.

**Corollary 3.7.** Let \( \{\alpha_v\}_{v \in [d]} \) be a family of complex numbers such that \( \sum_{v \in [d]} |\alpha_v|^2 < \infty \) and \( \sum_{v \in [d]} \alpha_v X^v \) is rational as a noncommutative formal power series. We put \( a_m = \sum_{|v|=m} \alpha_v S^v \in M_d(\mathbb{C}) \otimes M_2(L^\infty(\mathfrak{a})) \). Then \( \sum_{m=0}^{\infty} a_m \) converges in the operator norm.

**Proof.** Note that \( \sum_{v \in [d]} \alpha_v X^v \) is also rational (i.e. recognizable) by taking a complex conjugate of each entry of a linear representation of the recognizable series \( \sum_{v \in [d]} \alpha_v X^v \). Since the Hadamard product of two rational series is also rational by Lemma 2.7, \( \sum_{v \in [d]} |\alpha_v|^2 X^v \) is also rational as a noncommutative formal power series. By evaluating \( X_1, X_2, \ldots, X_d \) in one variable \( z \) (i.e. \( X_1 = X_2 = \cdots = X_d = z \)), we can use the argument of Kronecker (see Corollary 2.9) for the formal power series
\[ \sum_{m=0}^{\infty} \left( \sum_{|v|=m} |\alpha_v|^2 \right) z^m. \]

Thus there exists \( M > 0 \) and \( 0 < c < 1 \) such that
\[ \sum_{|v|=m} |\alpha_v|^2 \leq Me^m. \]

By using Lemma 3.5, we can estimate the operator norm of \( a_m \) as
\[ \|a_m\| \leq 4d^2(m+1) \left( \sum_{|v|=m} |\alpha_v|^2 \right) \leq M'(m+1)c^m \]
for some constant \( M' > 0 \) and \( 0 < c' < 1 \). Thus \( \sum_{m=0}^{\infty} a_m \) converges in operator norm.

We also use the following technical lemma.

**Lemma 3.8** ([Lemma 11 in [10]). Let \( n \in \mathbb{N} \) and \( A \) be a Banach algebra. If \( x \in M_n(A) \) satisfies \( \lim_{m \to \infty} \|x^m\| = 0 \), then we have
(1) \( \sum_{n=0}^{\infty} x^n \) converges in the operator norm to \((1 - x)^{-1} \in M_n(\mathcal{A})\).
(2) All entries of \((1 - x)^{-1}\) belong to the division closure of the subalgebra generated by all entries of \(x\) in \(\mathcal{A}\).

**Proposition 3.9.** Let \(a \in L^\infty(s)\). If \(\{[r^*_i, a]\}_{i=1}^d\) are finite rank operators on \(\mathcal{F}(H)\), then \(a \in C_{\text{div}}(s)\).

**Proof.** Let \(\hat{a} = \sum_{v \in [d]^*} \alpha_v \hat{U}_v\) be the expansion of \(\hat{a}\) and \(M\) be a \(\mathbb{C}\)-submodule of \(\mathbb{C}\langle\langle[d]\rangle\rangle\) generated by \(\sum_{v \in [d]^*} \alpha_v X^{vw}^{-1} \ (w \in [d]^*)\). Thanks to Lemma 3.8, we have for each \(i \in [d]\)

\[
[r^*_i, a] \hat{U}_w = \sum_{v \in [d]^*} \alpha_v \hat{U}_{(iw)^{-1}}.
\]

Note that the linear map from \(\mathcal{F}(H)\) to \(\mathbb{C}\langle\langle[d]\rangle\rangle\) which maps \(\hat{U}_v\) to \(X^v\) for each \(v \in [d]^*\) is injective. Therefore \(M\) is finitely generated if \(\{[r^*_i, a]\}_{i=1}^d\) are finite rank operators. Thus the noncommutative formal power series \(\sum_{v \in [d]^*} \alpha_v X^v\) is a recognizable series by Theorem 2.5. In other words, there exists a linear representation which consists of a multiplicative morphism \(\mu : [d]^* \to M_m(\mathbb{C})\) and vectors \(\lambda, \gamma \in \mathbb{C}^m\) such that \(\alpha_v = \lambda \mu(v) \gamma\). Moreover by choosing a linear representation such that its dimension is minimal, we may assume from Theorem 2.6 that there exists \(\{u_k\}_{k=1}^K, \{w_l\}_{l=1}^L \subset [d]^*\) such that

\[
\mu(v)_{ij} = \sum_{kl} c_{ij}^{kl} \alpha_{u_k v w_l}
\]

for any \(v \in [d]^*\) and \(1 \leq i, j \leq m\). We put \(V(\mathbf{X}) = \sum_{i \in [d]} \mu(i) X_i\). Note that since \(\mu\) is multiplicative, \(V(\mathbf{X})\) satisfies

\[
V(\mathbf{X})^m = \sum_{|v|=m} \mu(v) X^v
\]

and, on the level of formal power series, we have

\[
\sum_{v \in [d]^*} \alpha_v X^v = t^\lambda \left[ \sum_{m=0}^{\infty} V(\mathbf{X})^m \right]^{-1} \gamma = t^\lambda [1 - V(\mathbf{X})]^{-1} \gamma.
\]

We evaluate \(\mathbf{X} = (X_1, \ldots, X_d)\) in \(\mathbf{S} = (S_1, \ldots, S_d)\), where the \(S_i\)'s are defined like in Lemma 3.4. Then we can see that \(\sum_{m=0}^{\infty} V(\mathbf{S})^m = M_m(\mathbb{C}) \otimes M_d(\mathbb{C}) \otimes M_2(L^\infty(s))\) converges in the operator norm from Corollary 3.7 since all entries of \(\mu(v)\) are given by finite linear spans of \(\alpha_{uvw}\) for some words \(u, w\). Thus we can conclude

\[
\sum_{v \in [d]^*} \alpha_v \hat{U}_v = (1 \ 0) \left( \epsilon_1 \otimes I_2 \right) \sum_{v \in [d]^*} \alpha_v S^v (e \otimes I_2) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \Omega
\]

\[
= (1 \ 0) \left( \epsilon_1 \otimes I_2 \right) t^\lambda [1 - V(\mathbf{S})]^{-1} \gamma (e \otimes I_2) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \Omega.
\]

Note that \(\lim_{m \to \infty} \|V(\mathbf{S})^m\| = 0\), and we can apply Lemma 3.8 to \(V(\mathbf{S})\). Since \(\Omega\) is a separating vector for \(L^\infty(s)\), we conclude \(a \in C_{\text{div}}(s)\). \(\square\)

**Proof of Theorem 3.7.** Let \(\mathcal{A}\) be a subset of \(L^\infty(s)\) such that \(\{[r^*_i, a]\}_{i=1}^d\) are finite rank operators on \(\mathcal{F}(H)\) for any element \(a \in \mathcal{A}\). We will show \(\mathcal{A}\) is a subalgebra of
which contains $C(s)$ and satisfies for any $n \in \mathbb{N}$,

$$X \in M_n(A) \text{ is invertible in } M_n(L^\infty(s)) \implies X^{-1} \in M_n(A).$$

Note that $s_i \in A$ for any $i \in [d]$ since $[r_i^*, s_j] = \delta_j^i P_{11}$ is a finite rank operator for any $i, j \in [d]$. If $a, b \in A$, then the following operators

\[ [r_i^*, a + b] = [r_i^*, a] + [r_i^*, b] \]
\[ [r_i^*, ab] = [r_i^*, a]b + a[r_i^*, b] \]

are finite rank operators for each $i \in [d]$. Thus $A$ is a subalgebra of $L^\infty(s)$ which contains $C(s)$. Let $n \in \mathbb{N}$ be given and assume $X \in M_n(A)$ is invertible in $M_n(L^\infty(s))$. Then we have for any $i \in [d]$ and $1 \leq j, k \leq n$

\[ [r_i^*, t_{e_j}X^{-1}e_k] = t_{e_j}[I_n \otimes r_i^*, X^{-1}]e_k \]
\[ = -t_{e_j}X^{-1}[I_n \otimes r_i^*, X]X^{-1}e_k \]

where $I_n \otimes r_i^* \in M_n(\mathbb{C}) \otimes B(\mathcal{F}(H)) \cong M_n(B(\mathcal{F}(H)))$ is the operator such that all diagonal entries are $r_i^*$ and other entries are zero. Since $X \in M_n(A)$, all entries of $[I_n \otimes r_i^*, X]$ are finite rank operators and therefore $X^{-1} \in M_n(A)$. Since $C_{\text{rat}}(s)$ is the smallest subalgebra satisfying above properties, we obtain $C_{\text{rat}}(s) \subseteq A$.

Moreover, we have $A \subseteq C_{\text{div}}(s) \subseteq \overline{C(s)}$ by Proposition 3.9 and thus $C_{\text{rat}}(s) = C_{\text{div}}(s) = A \subseteq \overline{C(s)}$. 

\begin{remark}
Let us see what happens when we take $r_i$ and consider $[r_i, a]$ instead of $[r_i^*, a]$ for $a \in L^\infty(s)$. Indeed, for any $i \in [d]$, $[r_i, a]$ is a finite rank operator if and only if $[r_i^*, a]$ is also a finite rank operator since $[r_i + r_i^*, a] = 0$ and therefore $[r_i, a] = -[r_i^*, a]$ for any $a \in L^\infty(s)$. This is deduced from the commutativity of the left multiplication with the right multiplication. One can also see this directly via the following equalities

\[ [r_i + r_i^*, s_j] = [r_i, s_j] + [r_i^*, s_j] \]
\[ = -(r_i^*, s_j)^* + \delta_j^i P_{11} \]
\[ = -\delta_j^i P_{11} + \delta_j^i P_{11} = 0. \]

for any $i, j \in [d]$ where we use $[a, b]^* = b^*a^* - a^*b^* = -[a^*, b^*]$ and $[r_i^*, s_j] = \delta_j^i P_{11}$. Then we have $[r_i + r_i^*, a] = 0$ for any $a \in C(s)$ and thus for any $a \in L^\infty(s)$.

\end{remark}

\begin{remark}
We remark that a tuple of operators $(r_1^*, r_2^*, \ldots, r_d^*)$ is known as a dual system which is introduced by D. Voiculescu in [17].

In our setting, $(D_1, \ldots, D_d) \in B(\mathcal{F}(H))$ is called a dual system for $s$ if we have for any $i, j \in [d]$

\[ [D_i, s_j] = \delta_j^i P_{11}. \]

From the proof of Lemma 3.2, $(r_1^*, r_2^*, \ldots, r_d^*)$ is obviously a dual system and we have

\[ [D_i, a] = [r_i^*, a] \]

for each $i \in [d]$ and $a \in L^\infty(s)$. Thus Theorem 3.1 holds if we change $\{r_i^*\}_{i=1}^d$ by any dual system for $s$.

One can also see that $(D_1, \ldots, D_d) \in B(\mathcal{F}(H))$ is a dual system for $s$ if and only if $r_i^* - D_i$ belongs to $L^\infty(s)'$ for each $i \in [d]$ where $L^\infty(s)'$ is the commutant of $L^\infty(s)$.

We have not proved Theorem 3.1 for a general tuple of operators with a dual system yet and we leave it for future works.
4. Rationality criterion for affiliated operators

In this section, we extend our main result in the previous section to affiliated operators, which follows results of Linnell [12]. Let us denote by $\widehat{\mathcal{L}}(s)$ the $\star$-algebra of closed densely defined (unbounded) linear operators affiliated with $\mathcal{L}(s)$. Note that any element $u \in \widehat{\mathcal{L}}(s)$ can be written as $u = f^{-1}a = bg^{-1}$ by using some $a, b, f, g \in \mathcal{L}(s)$ where $f, g$ are nonzero divisors (i.e. $fx, gx \neq 0$ for any $x \in \mathcal{L}(s) \setminus \{0\}$) and thus invertible in $\widehat{\mathcal{L}}(s)$. For example, we can take $f = (1 + uu^*)^{-1}$, $a = (1 + uu^*)^{-1}u$, $b = u(1 + u^*u)^{-1}$, $g = (1 + u^*u)^{-1}$. We focus on bounded operators $\{fr_i^*b - ar_i^*g\}_{i=1}^d$ instead of commutators $\{[r_i^*, u]\}_{i=1}^d$. Note that we have formally $fr_i^*b - ar_i^*g = f[r_i^*, u]g$ since $u = f^{-1}a = bg^{-1}$.

The following lemma tells us that we can find a common denominator of two affiliated operators.

Lemma 4.1. Let $u_1, u_2 \in \widehat{\mathcal{L}}(s)$. Then there exist $a_1, a_2, b_1, b_2 \in \mathcal{L}(s)$ and $f, g \in \mathcal{L}(s)$ such that $u_1 = f^{-1}a_1 = bg^{-1}$ and $u_2 = f^{-1}a_2 = bg^{-1}$.

Proof. Let $u_k = f_k^{-1}a_k = bg_k^{-1}$ for $k = 1, 2$ where $a_k, b_k, f_k, g_k \in \widehat{\mathcal{L}}(s)$. Then we can write $f_1f_2^{-1} = x^{-1}y$ for some $x, y \in \mathcal{L}(s)$. Note that $f_1^{-1} = (yf_2)^{-1}x$ and $f_2^{-1} = (xf_1)^{-1}y = yf_2$. We put $f = xf_1 = yf_2$. Then we have $u_1 = f^{-1}xa_1$ and $u_2 = f^{-1}ya_2$. Similarly by representing $g_1^{-1}g_2 = xy^{-1}$ for some $x, y \in \mathcal{L}(s)$, we have $u_1 = b_1xy^{-1}$ and $u_2 = b_2yg^{-1}$ where $g = g_1x = g_2y$. $\square$

We also use the following lemmas for bounded operators and affiliated operators (see [12]).

Lemma 4.2 (Lemma 2.1 in [12]). Let $\theta : H \to K$ and $\phi : K \to H$ be bounded linear maps between Hilbert spaces.

(1) If $\ker \phi = \{0\}$ and $\phi \theta$ has finite rank, then $\theta$ also has finite rank.

(2) If $\text{Im} \theta$ is dense in $K$ and $\phi \theta$ has a finite rank, then $\phi$ also has a finite rank.

The following lemma is proved in Lemma 2.2 in [12] in terms of the free group, and the proof can be also applied to our setting.

Lemma 4.3 (Lemma 2.2 in [12]). Let $\theta \in \mathcal{L}(s)$. If $\theta$ is a nonzero divisor, then $\ker \theta = \{0\}$ and $\text{Im} \theta$ is dense in $\mathcal{F}(H)$.

We define $R(s)$ and $R'(s)$ as subsets of $\widehat{\mathcal{L}}(s)$. We say $u \in R(s)$ if $\{fr_i^*b - ar_i^*g\}_{i=1}^d$ are finite rank operators for any expression $u = f^{-1}a = bg^{-1}$ where $a, b, f, g \in \mathcal{L}(s)$. We say $u \in R'(s)$ if we can write $u = f^{-1}a = bg^{-1}$ for some $a, b, f, g \in \mathcal{L}(s)$ such that $\{fr_i^*b - ar_i^*g\}_{i=1}^d$ are finite rank operators. Note that we have $R(s) \subset R'(s)$ by definition.

To define rationality, we consider the division closure $\mathcal{D}(s)$ of $\mathbb{C}\langle s \rangle$ in $\widehat{\mathcal{L}}(s)$. From the results in [13], $\mathcal{D}(s)$ forms the free skew field of fractions of $\mathbb{C}\langle s \rangle$. Note that the rational closure of $\mathbb{C}\langle s \rangle$ in $\mathcal{L}(s)$ coincides with $\mathcal{D}(s)$ since $\mathcal{D}(s)$ is a skew field (see [13] Proposition 4.9)].

Remark 4.4. Let $i \in [d]$ and $u = f^{-1}a = bg^{-1} \in \widehat{\mathcal{L}}(s)$ where $a, b, f, g \in \mathcal{L}(s)$ and assume $fr_i^*a - br_i^*g$ is a finite rank operator. Then thanks to Remark 3.10 we
Therefore, if \( f \) we see that is closed under taking inverse. By Lemma 4.1, we can write other hand, if \( u \) are finite rank operators. Thanks to Theorem 3.1, we have 4.2 and 4.3; hence, we see that \( u \) divisors, \( f \) and thus we obtain in the same way as before that
\[
(f r_i^* b - a r_i^* g) y' = (f r_i^* b - a r_i^* g) x'.
\]
By combining them, we have
\[
y(f r_i^* b - a r_i^* g) y' = x(f r_i^* b - a r_i^* g) y' = x(f r_i^* b - a r_i^* g) x'.
\]
Since \( f r_i^* b - a r_i^* g \) is a finite rank operator for any \( i \in [d] \) and \( y, y' \) are non-zero divisors, \( f r_i^* b - a r_i^* g \) is also a finite rank operator for any \( i \in [d] \) by Lemmas 4.2 and 4.3; hence, we see that \( u \in R(s) \). This shows \( R'(s) \subset R(s) \) and thus we conclude \( R(s) = R'(s) \). 

**Remark 4.7.** If \( u \in R(s) \cap L^\infty(s) \), since we can write \( u = u 1^{-1} = 1^{-1} u \), \( \{x r_i^*, u \}^{d}_{i=1} \) are finite rank operators. Thanks to Theorem 3.1 we have \( u \in C_{div}(s) \). On the other hand, if \( u \in C_{div}(s) \), then \( \{x r_i^*, u \}^{d}_{i=1} \) are finite rank operators, and thus \( u \in R'(s) \) by the same theorem. By using Lemma 4.6 we have
\[
R(s) \cap L^\infty(s) = R'(s) \cap L^\infty(s) = C_{div}(s).
\]

We will prove four lemmas in order to deduce that \( R(s) \) is a *-subalgebra which is closed under taking inverse.

**Lemma 4.8.** If \( u_1, u_2 \in R(s) \), then \( u_1 + u_2 \in R(s) \).

**Proof.** By Lemma 4.1 we can write \( u_k = f^{-1} a_k = b_k g^{-1} \) for \( k = 1, 2 \). Then \( u_1 + u_2 = f^{-1}(a_1 + a_2) = (b_1 + b_2) g^{-1} \). Since \( u_1, u_2 \in R(s) \) and for any \( i \in [d] \)
\[
fr_i^* (b_1 + b_2) - (a_1 + a_2) r_i^* g = (fr_i^* b_1 - a_1 r_i^* g) + (fr_i^* b_2 - a_2 r_i^* g),
\]
we see that \( fr_i^* (b_1 + b_2) - (a_1 + a_2) r_i^* g \) is a finite rank operator for any \( i \in [d] \). Therefore \( u_1 + u_2 \in R(s) \) by Lemma 4.6.

**Lemma 4.9.** If \( u_1, u_2 \in R(s) \), then \( u_1 u_2 \in R(s) \).
Since $u_1 = f_k^{-1}a_k = b_kg_k^{-1}$ where $a_k, b_k, f_k, g_k \in L^\infty(s)$ for $k = 1, 2$. Let $a_1f_2^{-1} = x^{-1}y$ and $g_1^{-1}b_2 = pq^{-1}$ where $p, q, x, y \in L^\infty(s)$. Then $u_1u_2 = f_1^{-1}a_1f_2^{-1}a_2 = (xf_1)^{-1}ya_2$ and $u_1u_2 = b_1g_1^{-1}b_2g_2^{-1} = b_1p(g_2q)^{-1}$. Since $xa_1 = yf_2$ and $g_1p = b_2g_2$, we have
\[x_1r_1^*b_1p - y_2r_2^*g_2 = x(f_1r_1^*b_1 - a_1r_1^*g_1)p + y(f_2r_2^*b_2 - a_2r_2^*g_2)q.\]

Since $u_1, u_2 \in R(s)$, this operator is a finite rank operator for any $i \in [d]$, and thus $u_1u_2 \in R(s)$ by Lemma 4.9.

**Lemma 4.10.** If $u \in R(s)$ is invertible, then $u^{-1} \in R(s)$.

**Proof.** If $u \in R(s)$, then we can write $u = f^{-1}a = b^{-1}$ and $fr_i^*b - ar_i^*g$ has a finite rank for any $i \in [d]$. In addition if $u$ is invertible, we have $u^{-1} = a^{-1}f = gb^{-1}$. Since $ar_i^*g - fr_i^*b = -(fr_i^*b - ar_i^*g)$ for each $i \in [d]$, $u \in R(s)$ by Lemma 4.6.

**Lemma 4.11.** If $u \in R(s)$, then $u^* \in R(s)$.

**Proof.** If $u \in R(s)$, then we can write $u = f^{-1}a = b^{-1}$ and $fr_i^*b - ar_i^*g$ is a finite rank operator for any $i \in [d]$. Since $u^* = g^{-1}b^* = a^*f^{-1}$, we need to check that $g^*r_i^*a^* - b^*r_i^*f^*$ is a finite rank operator.

Since $T^*$ is a finite rank operator if $T$ is a finite rank operator on a Hilbert space and $fr_i^*b - ar_i^*g$ is a finite rank operator by Remark 4.3, $g^*r_i^*a^* - b^*r_i^*f^* = -(fr_i^*b - ar_i^*g)^*$ is also a finite rank operator for any $i \in [d]$. Thus we conclude $u^* \in R(s)$ by Lemma 4.6.

**Proof of Theorem 4.3.** By Lemmas 4.8, 4.9, 4.10, we see that $R(s)$ is a subalgebra of $\hat{L}^\infty(s)$ which contains $C(s)$ and is closed under taking inverse. Thus $D(s) \subset R(s)$.

Now, let $u \in R(s)$. Since $R(s)$ is also closed under the involution by Lemma 4.11, $a = (1 + uu^*)^{-1}u$ and $f = (1 + uu^*)^{-1}$ belong to $R(s) \cap L^\infty(s) = C_{div}(s)$ (see Remark 4.7) and therefore $u = f^{-1}a$ belongs to the division closure of $C_{div}(s)$ in $\hat{L}^\infty(s)$. Since $D(s)$ is the division closed subalgebra of $L^\infty(s)$ which contains $C_{div}(s)$, it also contains the division closure of $C_{div}(s)$ in $\hat{L}^\infty(s)$ (both coincide actually). Thus $u \in D(s)$.

In Theorem 4.3, we show an equivalent condition to $u \in D(s)$ by using bounded operators $\{fr_i^*b - ar_i^*g\}_{i=1}^d$ instead of commutators $\{r_i^*, u\}_{i=1}^d$. As we remark in the beginning of Section 3, both operators $fr_i^*b - ar_i^*g$ and $[r_i^*, u]$ are formally connected by $fr_i^*b - ar_i^*g = f[r_i^*, u]g$.

In the following proposition, we give another characterization of $u \in D(s)$ by using commutators $\{[r_i^*, u]\}_{i=1}^d$, which is an analogue of Proposition 1.2 in [12].

**Proposition 4.12.** Let $u \in \hat{L}^\infty(s)$. Then $u \in D(s)$ if and only if there exists a linear subspace $M$ of finite codimension in $F(H)$ such that $M \cap \bigcap_{i \in [d]} \text{dom}(ur_i^*) = M \cap \text{dom}(u)$ and $r_i^*u = ur_i^*$ on $M \cap \text{dom}(u)$ for each $i \in [d]$, where $\text{dom}(u)$ denotes the domain of $u$.

**Proof.** We use the well-known fact that for any subspace $M$ of finite codimension in a linear space $H$ and for any linear map $T$ on $H$, the preimage $T^{-1}(M)$ is also a subspace of finite codimension in $H$ (since $T$ induces an injective linear map from the quotient subspace $H/T^{-1}(M)$ to $H/M$ which is finite-dimensional).

In addition, an intersection $M_1 \cap M_2$ of two subspaces $M_1, M_2$ of finite codimension in $H$ is also a subspace of finite codimension (since $(M_1 + M_2)/M_2$ is
isomorphic to $M_1/(M_1 \cap M_2)$ and the two quotient spaces $H/M_1$, $(M_1 + M_2)/M_2$ are finite-dimensional).

Now we suppose $M$ is a subspace of finite codimension such that $r_i^* u = ur_i^*$ on $M \cap \text{dom}(u) = M \cap \bigcap_{i \in [d]} \text{dom}(ur_i^*)$ for any $i \in [d]$. We can write $u$ as $u = f^{-1} a = bg^{-1}$ where $a, b, g \in L^\infty(s)$. Note that $N = g^{-1}(M)$ is a subspace of finite codimension in $\mathcal{F}(H)$ such that $gN \subset M \cap \text{dom}(u) = M \cap \bigcap_{i \in [d]} \text{dom}(ur_i^*)$. Thus we have for any $\xi \in N$

$$(fr_i^* b - ar_i^* g)\xi = f(r_i^* ug\xi - ur_i^* g\xi) = 0.$$  

Since $N$ has a finite codimension in $\mathcal{F}(H)$, $fr_i^* b - ar_i^* g$ is a finite rank operator for each $i \in [d]$ and thus $u \in D(s)$ by Theorem 4.5.

On the other hand, if $u \in D(s)$, then by Theorem 4.5 there exists $a, f \in C_{\text{div}}(s)$ such that $u = f^{-1} a$. Note that $f^{-1} a$ forms a closed operator even though we see it as a composition of unbounded operators (we do not have to take closure). Thus we can write $\text{dom}(u) = \{\xi \in \mathcal{F}(H); a\xi \in f\mathcal{F}(H)\}$. Since $a, f \in C_{\text{div}}(s)$, $[r_i^*, f], [r_i, f], [r_i^*, a]$ are finite rank operators for each $i \in [d]$. Then kernels of these operators have finite codimensions. We put for each $i \in [d]$

$$N_i = \ker[r_i^*, f] \cap \ker[r_i, f]$$

and define $N$ as

$$N = \bigcap_{i \in [d]} N_i.$$ 

Note that $N$ is a subspace of finite codimension in $\mathcal{F}(H)$ and there exists a subspace $M_1$ of finite codimension in $\mathcal{F}(H)$ such that $M_1 \cap f\mathcal{F}(H) \subset fN$.

For example, we can take $M_1$ as a direct sum of $fN$ and a complementary subspace of $f\mathcal{F}(H) \subset \mathcal{F}(H)$. Since $r_i fN = fr_i N$ for each $i \in [d]$, there exists a subspace $M_2$ of finite codimension in $\mathcal{F}(H)$ for each $i \in [d]$ such that

$$M_2 \cap r_i f\mathcal{F}(H) \subset \mathcal{F}(H).$$

We can take $M_1$ either as a direct sum of $r_i fN$ and complementary subspace of $f\mathcal{F}(H) \subset \mathcal{F}(H)$ as above, or as $(r_i^*)^{-1}(M_1)$. Then we put $M$ by

$$M = a^{-1}[(\mathcal{C}\Omega)^{\perp}] \cap a^{-1}(M_1) \cap \bigcap_{i \in [d]} \ker[r_i^*, a] \cap \bigcap_{i \in [d]} (r_i r_i^* a)^{-1}(M_2).$$

Then $M$ is obviously a subspace of finite codimension in $\mathcal{F}(H)$.

Let us show $M \cap \bigcap_{i \in [d]} \text{dom}(ur_i^*) = M \cap \text{dom}(u)$. If $\xi \in M \cap \text{dom}(u)$, then $a\xi = f\eta$ for $\eta \in N$ since $a\xi \in M_1 \cap f\mathcal{F}(H) \subset fN$. Since $\xi \in \ker[r_i^*, a]$ and $\eta \in N$, we obtain $ar_i^* \xi = fr_i^* \eta$ which implies $\xi \in \text{dom}(ur_i^*)$ for any $i \in [d]$. On the other hand, if $\xi \in M \cap \bigcap_{i \in [d]} \text{dom}(ur_i^*)$, then $r_i^* a\xi = ar_i^* \xi \in f\mathcal{F}(H)$ for each $i \in [d]$. By multiplying by $r_i$, we have $r_i r_i^* a\xi \in M_2 \cap r_i f\mathcal{F}(H) \subset f\mathcal{F}(H)$ for each $i \in [d]$. Then there exists $\eta_i \in \mathcal{F}(H)$ for each $i \in [d]$ such that $r_i r_i^* a\xi = f\eta_i$. Since $a\xi \in (\mathcal{C}\Omega)^{\perp}$, we have

$$a\xi = \sum_{i \in [d]} r_i r_i^* a\xi = f \sum_{i \in [d]} \eta_i$$

which implies $\xi \in \text{dom}(u)$. Therefore we have $M \cap \bigcap_{i \in [d]} \text{dom}(ur_i^*) = M \cap \text{dom}(u)$. 

...
In order to see $ur_i^* = r_i^*u$ on $M \cap \text{dom}(u)$, we take $\xi \in M \cap \text{dom}(u)$. Then, as shown above, there exists $\eta \in N$ such that $ar_i^*\xi = fr_i^*\eta$; hence $ar_i^*u\xi = r_i^*u\eta$ for each $i \in [d]$. Moreover since $ar_i^*\xi = fr_i^*\eta$, as shown above, we conclude $ur_i^*\xi = r_i^*\eta = r_i^*u\xi$.

\[\square\]

Remark 4.13. The proof of Proposition 4.12 also works when we take $\{r_i\}_{i \in [d]}$ instead of $\{r_i^*\}_{i \in [d]}$. In this case, we can say that there exists a subspace of finite codimension in $\mathcal{F}(H)$ such that $M \cap \text{dom}(u) = M \cap \text{dom}(ur_i)$ and $ur_i = r_iu$ on $M \cap \text{dom}(u)$ for each $i \in [d]$.

References

[1] O. Arizmendi, G. Cébron, R. Speicher, and Sheng Yin. Universality of free random variables: atoms for non-commutative rational functions arXiv:2107.11507, 2021.
[2] J. Berstel and C. Reutenauer. Noncommutative Rational Series with Applications (Encyclopedia of Mathematics and its Applications 137). Cambridge, UK: Cambridge University Press, 2011.
[3] P. Biane and R. Speicher. Stochastic calculus with respect to free Brownian motion and analysis on Wigner space. Probability Theory and Related Fields, 112:373–409, 1998.
[4] M. Bożejko. Ultracontractivity and Strong Sobolev Inequality For $q$-Ornstein–Uhlenbeck Semigroup ($-1 < q < 1$). Infinite Dimensional Analysis, Quantum Probability and Related Topics, 2(2):203-220, 1999.
[5] M. Bożejko, B. Kümmerer and R. Speicher. $q$-Gaussian processes: Non-commutative and classical aspects. Comm. Math. Phys. 185 (1997) 129–154.
[6] P. M. Cohn. Free ideal rings and localization in general rings, volume 3. Cambridge university press, 2006.
[7] B. Collins, T. Mai, A. Miyagawa, F. Parraud and S. Yin. Convergence for noncommutative rational functions evaluated in random matrices. arXiv:2103.05962, 2021.
[8] A. Connes Noncommutative geometry San Diego, CA, Academic Press Inc., 1994.
[9] A. Connes and D. Shlyakhtenko. $L^2$-homology for von Neumann algebras Journal für die reine und angewandte Mathematik, 586:125-186, 2005.
[10] G. Duchamp and C. Reutenauer. Un critère de rationalité provenant de la géométrie non commutative. Invent. math., 128, 613–622 (1997).
[11] L. Kronecker. Zur Theorie der Elimination einer Variablen aus zwei algebraischen Gleichungen. Monatsber. Königl. Preussischen Acad Wies, Berlin, 535-600, 1881.
[12] P. A. Linnell. A rationality criterion for unbounded operators. J. Funct. Anal., 171(1):115–121, 2000.
[13] T. Mai, R. Speicher and S. Yin. The free field: realization via unbounded operators and Atiyah property. arXiv:1905.08187, 2019.
[14] J. A. Mingo and R. Speicher. Free Probability and Random Matrices (Fields Institute Monographs 35). Springer, 2017.
[15] A. Nica and R. Speicher. Lectures on the Combinatorics of Free probability (London Mathematical Society Lecture Note series 335). Cambridge, UK: Cambridge University Press, 2006.
[16] J. R. Partington. An Introduction to Hankel Operators (London Mathematical Society Student Texts 13). Cambridge, UK: Cambridge University Press, 1988.
[17] D. Voiculescu. The analogues of entropy and of Fisher’s information measure in free probability theory V. Noncommutative Hilbert Transforms Inventiones mathematicae, 132(1):189-227, 1998.
[18] D. Voiculescu, K. Dykema, and A. Nica Free Random Variables (CRM Monograph series) Providence, RI, Amer. Math. Soc., Vol. 1, 1992.

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