Problems of interaction longitudinal shear waves with V-shape tunnels defect

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Abstract. The problem of determining the two-dimensional dynamic stress state near a tunnel defect of V-shaped cross-section is solved. The defect is located in an infinite elastic medium, where harmonic longitudinal shear waves are propagating. The initial problem is reduced to a system of two singular integral or integro-differential equations with fixed singularities. A numerical method for solving these systems with regard to the true asymptotics of the unknown functions is developed.

1. Introduction
At present, there exist numerous solutions of two-dimensional dynamical problems of the theory of elasticity for bodies with defects shaped as a segment of a straight line or an arc of a smooth curve. Examples of solving such problems can be found in [1–8]. However, the real defects can have corner points, be piecewise smooth, intersect one another, and bifurcate. Practically, the problems of determining the dynamical stress state in a neighborhood of such defects have not yet been solved. This is related to the difficulties arising when they are solved by using the method of boundary integral equations widely applied nowadays and the problems are reduced to singular integro-differential or hypersingular equations with fixed singularities. Most of the studies deal with static problems for bodies whose defects have corner points. First, we mention [9–12], where the exact solution was obtained by the Wiener–Hopf method, and an accurate value of the stress intensity factor was determined. These solutions and the results of [12] indicate that the presence of kernels with fixed singularities influences the singularity of the solutions in neighborhoods of the ends of integration intervals. The stress state near branched, broken, and edge cracks was also determined in [14–19], where the integral equations were numerically solved by the method of mechanical quadratures. This method is based on the application of the Gauss–Chebyshev quadrature formulas that give a root singularity of the solutions. In this case, the true asymptotics of the solution is not taken into account, or an additional condition leading to a singularity weaker than the root one is imposed on the solution. Another drawback of the numerical methods used in those works is the formal application of the Gauss–Chebyshev quadrature formulas to integrals with fixed singularities. As a result, the convergence is rather slow (in order to obtain results with an error less than 0.1%, several dozens of collocation points should be used). In the present paper, we solve the problem of determining the dynamical stress intensity factors for tunnel defects (cracks or thin rigid inclusions) which are V-shaped in a cross-section under the interaction with a harmonic longitudinal shear wave. The problem is reduced to a system of three singular integro-differential or integral equations solved by the
method of collocations. Within this method, the true singularity of the solution is considered, and special quadrature formulas are used to calculate the integrals with fixed singularities.

2. Statement of the problem and its reduction to a system of singular integral equations

We consider the unbounded isotropic elastic medium under the conditions of antiplane deformation with a through V-shaped defect in a cross-section in the plane $Oxy$ (figure 1).

It may be a crack or a thin rigid inclusion. The cross-section of the defect occupies two segments what are of length $2d_k$ and make angles $\alpha_k$, $k = 1, 2$, with the axis $Ox$. The defect interacts with a plane longitudinal shear wave. Its front makes an angle $\theta_0$ with the axis $Ox$ and causes the displacements

$$W_0^{(i)}(x, y) = A_0 \exp[i\kappa_2(x \cos \theta_0 + y \sin \theta_0)], \quad \kappa_2^2 = \frac{\rho\omega^2}{G},$$

in the medium along the axis $Oz$. In (1), the symbols $G$ and $\rho$ stand for the shear modulus and the medium density, and $\omega$ is the frequency of oscillations. The time dependence is given by the factor $\exp(-i\omega t)$ that is omitted here and below. Let the vector of displacements caused by the wave reflected from the defect have a single component $W(x, y)$ which is nonzero under an antiplane deformation. In the coordinate system $Oxy$, it satisfies the Helmholtz equation

$$\Delta W + \kappa_2^2 W = 0.$$  

To specify the boundary conditions on the defects, we associate the coordinate system with each of their segments. Let these systems $O_kx_ky_k$, $k = 1, 2$ be centered at the middles of segments, and let its centers have coordinates $O_k(d_k \cos \alpha_k, d_k \sin \alpha_k)$. Let

$$W_k(x_k, y_k) = W(a_k + x_k \cos \alpha_k - y_k \sin \alpha_k, b_k + x_k \sin \alpha_k + y_k \cos \alpha_k)$$

be the displacement in the coordinate systems related to the $k$th segment of the defect. In the case of a thin rigid inclusion, in view of its small thickness, we formulate the corresponding boundary conditions on its median surface. Assume that the condition of perfect contact is realized between the body and the inclusion

$$W_k^1(x_k, 0) = c - W_k^{(i)}(x_k, 0), \quad k = 1, 2,$$

where

$$W_k^{(i)}(x_k, y_k) = W_0^{(i)}((d_k + x_k) \cos \alpha_k - y_k \sin \alpha_k, (d_k + x_k) \sin \alpha_k + y_k \cos \alpha_k).$$
On the surface of the inclusion, the tangential stresses are discontinues and their jumps are denoted by

$$\tau_{zyk}(x_k, +0) - \tau_{zyk}(x_k, -0) = \chi_{1k}(x_k), \quad -d_k < x_k < d_k, \quad k = 1, 2. \quad (5)$$

In formula (4), $c$ is the amplitude of longitudinal oscillations of the inclusion in the equation of its motion as a solid body. In the case of harmonic oscillations, this equation becomes

$$-(m_1 + m_2)\omega^2 c = P + \sum_{k=1}^{2} \int_{-d_k}^{d_k} \chi_{1k}(\eta) \, d\eta, \quad m_k = 2\rho_0 d_k h, \quad (6)$$

where $h$ is the inclusion thickness and $\rho_0$ is the inclusion density.

In the case of a crack, its surface is assumed to be free from stresses,

$$\tau_{zyk}(x_k, 0) = -\tau_{zyk}(x_k, 0), \quad -d_k < x_k < d_k, \quad k = 1, 2. \quad (7)$$

In addition, the displacement on the crack surface has a jump denoted by

$$W^l_k(x_k, +0) - W^l_k(x_k, -0) = \chi_{2k}(x_k), \quad -d_k < x_k < d_k, \quad k = 1, 2. \quad (8)$$

Also, the continuity of displacements along the edge yields

$$W_k(d_k, +0) = W_{k+1}(-d_{k+1}, +0), \quad W_k(d_k, -0) = W_{k+1}(-d_{k+1}, -0), \quad k = 1, 2. \quad (9)$$

We start solving the posed problem by constructing a discontinuous solution of the Helmholtz equation in the coordinate system $O_kx_ky_k, \ k = 1, 2$, for each link of the defect [6]. In the case of an inclusion, this is a solution with jumps (5):

$$W^{l1}_k(x_l, y_l) = \int_{-d_l}^{d_l} \frac{\chi_{1l}(\eta)}{G} r_2(\eta - x_l, y_l) \, d\eta, \quad l = 1, 2. \quad (9)$$

In the case of a crack, we construct the discontinuous solution with jumps (8):

$$W^d_k(x_k, y_k) = \frac{\partial}{\partial y_k} \int_{-d_k}^{d_k} \chi_k(\eta) r_2(\eta - x_k, y_k) \, d\eta. \quad (10)$$

In formulas (9), (10), we denote

$$r_2(\eta - x_l, y_l) = i \frac{\kappa_0}{4} H^{(1)}_0(\kappa_0 \sqrt{(\eta - x_l)^2 + y_l^2}),$$

where $H^{(1)}_0(z)$ is the Hankel function. Then the displacements of the reflected wave field can be given in the form

$$W(x, y) = \sum_{k=1}^{3} W^d_k(x, y), \quad (11)$$

where

$$W^d_k(x, y) = W^d_k((x - a_k) \cos \alpha_k + (y - b_k) \sin \alpha_k, -(x - a_k) \sin \alpha_k + (y - b_k) \cos \alpha_k).$$

In order to finally determine the displacements of the diffraction field, we should find the unknown jumps (5) or (8). To this end, we use conditions (4) or (7).
First, we consider the case of inclusion. To determinate jumps (5) from the boundary conditions (4), we obtain a system of singular integral equations. For the singular component of their kernels contain a singularity in the form of the Cauchy kernel, instead of (4), we should use [7] the conditions

\[ \frac{\partial W_k(x_k, 0)}{\partial x_k} = -\frac{\partial}{\partial x_k} W_k^{(i)}(x_k, 0), \quad W_k(-d_k, 0) = c, \quad k = 1, 2. \] (12)

The result of substitution of (3), (11), and (9) into (12) is the following singular integral equation with an auxiliary condition:

\[ \int_{-1}^{1} \varphi_1(\tau) \left[ \frac{1}{\tau - \xi} - R_{11}(\tau - \xi) \right] d\tau - \int_{-1}^{1} \varphi_2(\tau) [q_{12}(\tau, \xi) - R_{12}(\tau, \xi)] d\tau = f_1(\xi), \]
\[ \int_{-1}^{1} \varphi_2(\tau) \left[ \frac{1}{\tau - \xi} - R_{22}(\tau - \xi) \right] d\tau - \int_{-1}^{1} \varphi_1(\tau) [q_{21}(\tau, \xi) - R_{21}(\tau, \xi)] d\tau = f_2(\xi), \]
\[ \frac{1}{2\pi} \sum_{l=1}^{2} \gamma_l \int_{-1}^{1} \varphi_l(\tau) [\ln |\tau + (-1)^l| + D_l(\tau)] d\tau = c_0, \quad c_0 = \frac{c}{d}. \] (13)

In system (13), we introduce the notation

\[ q_{kl}(\tau, \xi) = \frac{\gamma_l(\gamma_l \gamma_1 \cos \alpha_k + \gamma_k \xi_1 k)}{s_{kl}(\tau, \xi)}, \quad s_{kl}^2(\tau, \xi) = \gamma_1^2 \tau_1^2 + 2\gamma_1 \gamma_2 \tau_1 \xi_k \cos \alpha_k + \gamma_k^2 \xi_k^2, \quad k, l = 1, 2, \]
\[ \tau_1 = \tau - 1, \quad \tau_2 = \tau + 1, \quad \xi_1 = \xi - 1, \quad \xi_2 = \xi + 1, \]
\[ f_l(\xi) = -i\kappa_0 C_0 \cos \theta_{kl} \exp[i\kappa_0 \gamma_l(1 + \xi) \cos \theta_{kl}], \quad C_0 = \frac{A}{d}, \quad \kappa_0 = \kappa_2, \]
\[ \varphi_k(\tau) = G^{-1} \chi_k(d_k \tau), \quad \eta = d_k \tau, \quad x_k = d_k \xi, \quad \gamma_k = d^{-1} d_k, \quad d = \max(d_1, d_2), \quad k = 1, 2. \] (14)

In the case of a crack, the following system of singular integro-differential equations for jumps (8) and their derivatives is obtained after substitution of (3), (10) and (11) into (7):

\[ \frac{1}{2\pi} \int_{-1}^{1} \varphi_1(\tau) \left[ \frac{1}{\tau - \xi} + R_{11}(\tau - \xi) \right] d\tau + \frac{1}{2\pi} \int_{-1}^{1} \varphi_2(\tau) [q_{12}(\tau, \xi) + R_{12}(\tau, \xi)] d\tau \]
\[ + \frac{1}{2\pi} \int_{-1}^{1} \varphi_1(\tau) [-\gamma_2^2 \kappa_0^2 \ln |\tau - \xi| + U_{11}(\tau - \xi)] d\tau + \frac{1}{2\pi} \int_{-1}^{1} \varphi_2(\tau) U_{12}(\tau, \xi) d\tau = f_1(\xi), \]
\[ \frac{1}{2\pi} \int_{-1}^{1} \varphi_2(\tau) \left[ \frac{1}{\tau - \xi} + R_{22}(\tau - \xi) \right] d\tau + \frac{1}{2\pi} \int_{-1}^{1} \varphi_1(\tau) [q_{21}(\tau, \xi) + R_{21}(\tau, \xi)] d\tau \]
\[ + \frac{1}{2\pi} \int_{-1}^{1} \varphi_2(\tau) [-\gamma_1^2 \kappa_0^2 \ln |\tau - \xi| + U_{22}(\tau - \xi)] d\tau + \frac{1}{2\pi} \int_{-1}^{1} \varphi_2(\tau) U_{21}(\tau, \xi) d\tau = f_2(\xi). \] (15)

We used the following notation in system (15):

\[ q_{kl}(\tau, \xi) = \frac{(-1)^l \{ \gamma_l[1 + (-1)^l] \cos \alpha_k \xi_k + \gamma_k[1 + (-1)^l] \xi_1 \} \gamma_l}{s_{kl}(\tau, \xi)}, \quad \varphi_k(\tau) = d_k^{-1} \chi_k(d_k \tau), \]
\[ f_l(\xi) = -i\kappa_0 C_0 \sin \theta_{kl} \exp[i\kappa_0 \gamma_l(1 + \xi) \cos \theta_{kl}], \quad \theta_{kl} = \theta_k - \alpha_k, \quad k, l = 1, 2. \] (16)

As is seen in (14) and (16), the functions \( q_{kl}(\tau, \xi) \) have singularities at \( \tau = \pm 1, \xi = \pm 1 \). The functions \( R_{lk}, U_{lk}, R_{lk}(\tau, \xi), D_l(\tau), k, l = 1, 2, \) are bounded for \(-1 \leq \tau, \xi \leq 1\).
3. Solution of the problem in the case of an inclusion

The presence of the fixed singularity \( \tau = \pm 1 \), \( \xi = \pm 1 \) in the singular component of the system integral equations (13) affects the asymptotics of its solutions in neighborhoods of the points \( \tau = \pm 1 \). The singularity of the solutions in neighborhoods of these points can be determined by analyzing the asymptotic properties of singular integrals or by investigating the symbol of the singular kernel [13]. As a result, we established that the unknown function is to be sought in the form

\[
\varphi_1(\tau) = (1 + \tau)^{-1/2}(1 - \tau)^{-\delta} \psi_1(\tau), \quad \varphi_2(\tau) = (1 + \tau)^{-\delta}(1 - \tau)^{-1/2} \psi_2(\tau),
\]

where \( \psi_k(\tau), k = 1, 2 \), satisfy the Hölder condition, and the exponents are

\[
\delta = \begin{cases} 
\pi - 2\alpha & \text{for } 0 < \alpha \leq \frac{\pi}{2}, \\
\frac{2(\pi - \alpha)}{2\alpha - \pi} & \text{for } \frac{\pi}{2} < \alpha < \pi,
\end{cases} \quad \alpha = |\alpha_2 - \alpha_1|.
\]

Now, if we consider the function

\[
\psi_k(\tau) = \psi_{0k}(\tau) + 0.5(1 + \tau)\psi_k(1) + 0.5(1 - \tau)\psi_k(-1)
\]

then we obtain \( \psi_{0k}(\pm 1) = 0 \). This is why we can assume that

\[
\psi_{0k}(\tau) = (1 - \tau)^2g_k(\tau), \quad k = 1, 2,
\]

where \( g_k(\tau) \) are new unknown functions. Substituting (19) and (18) into (17), we obtain the following representation of the solutions of integrals equation (13):

\[
\varphi_k(\tau) = (1 - \tau)^{\sigma_k}(1 + \tau)^{\lambda_k}g_k(\tau) + 0.5(1 + \tau)^{\lambda_k}(1 - \tau)^{\sigma_k - 1}\psi_k(1) + 0.5(1 + \tau)^{\lambda_k - 1}(1 - \tau)^{\sigma_k}\psi_k(-1),
\]

Here \( \sigma_1 = \lambda_2 = 1 - \delta \), \( \sigma_2 = \lambda_1 = 1/2 \). Further, the approximate method of solution is based on approximation of the functions \( g_k(\tau) \) by the interpolation polynomials of degree \( (n - 1) \):

\[
g_k(\tau) \approx g_{k,n-1}(\tau) = \sum_{m=1}^{n} g_{mk}(\tau - \tau_{mk})\frac{Q_{mk}(\tau)}{Q_{mk}(\tau_{mk})}, \quad g_{nk} = g_k(\tau_{mk})
\]

where \( Q_{n1}(\tau) = P_n^{1-\delta,1/2}(\tau), \quad Q_{n2}(\tau) = P_n^{1/2,1-\delta}(\tau) \) are the Jacobi polynomials and \( \tau_{m1}, \tau_{m2} \) are roots of these polynomials. Then, for the singular integral with Cauchy kernel, the following quadrature formula holds [20]:

\[
\int_{-1}^{1} \frac{\varphi_k(\tau)}{\tau - \xi_{jk}} d\tau = \sum_{m=1}^{n} g_{mk} \frac{A_{mk}}{\tau_{mk} - \xi_{jk}} + \frac{\psi_k(-1)}{2} b^-_{jk} + \frac{\psi_k(1)}{2} b^+_{jk},
\]

where \( \xi_{j1}, \xi_{j2} \) \( (j = 1, \ldots, n + 1) \) are roots of the Jacobi functions of the second kind \( J_n^{1-\delta,1/2}(\xi), J_n^{1/2,1-\delta}(\xi) \), and \( A_{mk} \) are the coefficients of the corresponding Gauss–Jacobi quadrature formula. The use of representation (20) requires to calculate the following integrals with Cauchy kernel:

\[
b^+_{kj} = \int_{-1}^{1} \frac{(1 - \tau)^{\sigma_k}(1 + \tau)^{\lambda_k - 1}}{\tau - \xi_{kj}} d\tau, \quad b^-_{kj} = \int_{-1}^{1} \frac{(1 - \tau)^{\sigma_k - 1}(1 + \tau)^{\lambda_k}}{\tau - \xi_{kj}} d\tau.
\]

We obtain their values by the method described in [21] and based on their transformation to the Mellin convolution. Further, applying the theorem on convolution, we can represent these integrals as the sum of residuals at the poles of the integrands.
Then we calculate the integrals with a fixed singularity

\[ E^j_{kl} = \int_{-1}^{1} q_{kl}(\tau, \xi_{jk})(1 - \tau)^{\eta_1}(1 + \tau)^{\lambda_1} g_1(\tau) \, d\tau. \]  

(24)

Let \(0 < r_1 < 1\) be a positive number. In the case where \(1 \pm \xi_k > r_1\), the integral (24) is not singular, and it can be calculated by the Gauss–Jacobi quadrature formulas with the corresponding weight function \([22]\). If \(1 \pm \xi_k \rightarrow 0\), then we should replace \(\varphi_k(\tau)\) by expression (20) and use a method based on using the Mellin integral transformation \([21]\). We represent these integrals in terms of the following power series which converge for \(0 \leq 1 \pm \xi_k < r_1 < 1\):

\[
\int_{-1}^{1} q_{kl}(\tau, \xi_{jk}) \varphi_1(\tau) \, d\tau = \sum_{m=1}^{n} g_{m1} \frac{H^1_{m}}{s^+_kl(\tau_{ml}, \xi_{jk})} + \frac{\psi_1(1)}{2} s^+_jkl + \frac{\psi_1(-1)}{2} s^-jkl, \tag{25}
\]

\[
H^1_{m} = A_{ml}\{\gamma_l[\tau_{ml}+(1)^l] \cos \alpha_{kl} + \gamma_k[\xi_{jk}+(1)^k]\} - \frac{\gamma_l^2[\tau_{ml}+(1)^l]}{Q_{ml}(\tau_{ml})} h^2_{jl} - \frac{\gamma_l \gamma_k[\xi_{jk}+(1)^k]}{Q_{ml}(\tau_{ml})} h^3_{jl},
\]

\[
h^1_{ml} = J^1_{ml}(\tau_{ml}) = A_{ml} Q^1_{ml}(\tau_{ml}), \quad h^2_{jl} = \frac{\Gamma\left(\frac{3}{2} + n\right)}{n!} \sum_{\beta = 0}^{\infty} z_{1p} y^p \cos((\pi - \alpha_{kl}) (\delta - p - 2)) - \cot(\pi \delta) \sum_{p=0}^{\infty} z_{2p} y^p \cos(\alpha_{kl} (p + 1)),
\]

\[
b_{kl}(y) = \frac{2^{2-\delta}}{n!} \left\{ \frac{y^{1-\delta}}{\sin(\pi \delta)} \sum_{p=0}^{\infty} z_{1p} y^p \cos((\pi - \alpha_{kl}) (\delta - p - 1)) + \cot(\pi \delta) \sum_{p=0}^{\infty} z_{2p} y^p \cos(\alpha_{kl} p) \right\};
\]

\[
z_{1p} = \frac{\Gamma(-0.5 - n + p) \Gamma(2 + n + p - \delta)(-1)^p}{p! \Gamma(2 + p - \delta)}, \quad z_{2p} = \frac{\Gamma(-1.5 - n + \delta + p) \Gamma(1 + n + p)}{p! \Gamma(\delta + p)}.
\]

The integrals

\[ s^+_jkl = \int_{-1}^{1} q_{kl}(\tau, \xi_{jk})(1 - \tau)^{\eta_1-1}(1 + \tau)^{\lambda_1} \, d\tau, \quad s^-jkl = \int_{-1}^{1} q_{kl}(\tau, \xi_{jk})(1 - \tau)^{\eta_1}(1 + \tau)^{\lambda_1-1} \, d\tau \]

are computed by a method similar to that used to calculate integrals (23).

To calculate the integrals with a logarithmic singularity, we use representation (20) and approximate the function \(g_k(\tau)\) by the interpolation polynomial (21), which we preliminarily transform according to the Darboux–Christoffel identity \([22]\). Thereafter we obtain the formula

\[
\int_{-1}^{1} \varphi_k(\tau) \ln |\tau + (-1)^l| \, d\tau = \sum_{m=1}^{n} A_{ml} g_{ml} \theta_{ml} + \frac{\psi_1(1)}{2} E^-_1 + \frac{\psi_1(-1)}{2} E^+_1,
\]  

(26)

where

\[
\theta_{ml} = \sum_{s=0}^{n-1} \frac{u_{s1} Q_{ml}(\tau_{ml})}{\sigma^2}, \quad u_{s1} = -a_s, \quad u_{s2} = (-1)^{s+1} a_s, \quad a_s = \frac{2^{5/2-\delta} \Gamma(1.5 + s) \Gamma(2 - \delta)}{\Gamma(3.5 - \delta + s)!},
\]

\[
E = \frac{2^{1/2-\delta} \Gamma(3+1)}{\Gamma(2.5-\delta)} \left[ \ln 2 + \Psi\left(\frac{3+1}{2} - \delta\right) - \Psi\left(\frac{5}{2} - \delta\right) \right].
\]

Here \(\Psi(x)\) is the logarithmic derivative for the \(\Gamma\) function.
The obtained formulas for the singular integrals (22), (25), (26) and Gauss–Jacobi quadrature formulas applied to the integrals with regular kernels allow us to replace (13) and (6) by a system of linear algebraic equations. As a result of solving this system, we obtain

\[ g_{m1} = g(\tau_{m1}), \quad g_{m2} = g(\tau_{m2}), \quad \psi_1(\pm 1), \quad \psi_2(\pm 1), \quad c_0. \]

Then the approximate solution of the system is determined by formulas (20) and (21). One of the important characteristics of the stress state near the inclusion is the stress intensity factor (SIF). It is determined from the known asymptotic representation [23] of stresses in neighborhoods of the end of the inclusion. We obtain the following approximate formulas for the dimensionless SIF:

\[
K_1 = G \sqrt{d_1} 2^{-s} \psi_1(1), \quad K_2 = G \sqrt{d_2} 2^{-s} \psi_2(1).
\] (27)

On the basis of the obtained formulas, the dependence of SIF on the dimensionless wave number \( \kappa_0 = \kappa d \) is investigated. For this purpose, the inclusion with links of equal length \( d \) symmetrical with respect to the axis \( Oy \) was considered (figure 2). The links of inclusion come from the origin of coordinates and make an angle \( 2\beta \) between them. Realizing the proposed method of solution, we first perform numerical investigation of its practical convergence. The calculation was carried out for the following data:

\[
\bar{\rho} = \rho_k / \rho = 1, \quad \varepsilon = h_k / d = 0.05, \quad \theta_0 = 90^\circ, \quad \beta = 30^\circ.
\]

Under these conditions, the equality \( K_1 = K_2 = K \) is executed. As results of computation by formula (27), we construct graphs of the dependence of the absolute value of the dimensionless SIF \( k = K / (G \sqrt{d}) \) on the dimensionless wave number, which is shown in figure 3.

The curves correspond to the number of interpolation nodes \( n = 5, 10, 15, 20 \) in approximation (21). We can see that, in the region of low frequencies \( \kappa_0 < 2 \), to attain a fairly high accuracy (the relative error does not exceed 1%), it suffices to use \( n = 5 \) interpolation nodes. In the whole frequency range \( 0 \leq \kappa_0 \leq 5 \), to obtain values of SIF with an error below 0.1%, it suffices to have \( n = 20 \) interpolation nodes in formula (21).

The graphs in figures 4 and 5 illustrate the influence of the angle \( \beta \) between the inclusion links on the dependence of SIF on the frequency \( \kappa_0 \). The curves correspond to the values:

1 — \( \beta = 5^\circ \), 2 — \( \beta = 45^\circ \), 3 — \( \beta = 60^\circ \), 4 — \( \beta = 87^\circ \). The graphs in figure 4 are constructed at the angle of wave propagation \( \theta_0 = 90^\circ \), and in figure 5, at the angle \( \theta_0 = 270^\circ \). We can see that a decrease in the angle between the inclusion links lead to a complication of frequency dependence of SIF. This complication is the appearance of local maxima and minima. The smallest values of SIF are observed for a rectilinear inclusion \( \beta \approx 90^\circ \).
4. Solution of the problem in the case of a crack

In case of a crack, we represent the derivatives of solutions of system (15) in the form

\[ \varphi'_l(\tau) = W_l(\tau)g_l(\tau), \quad l = 1, 2, \]

\[ W_1(\tau) = (1-\tau)^{-\sigma}(1+\tau)^{-1/2}, \quad W_2(\tau) = (1-\tau)^{-1/2}(1+\tau)^{-\sigma}, \quad (28) \]

where the functions \( g_l(\tau) \) (\( l = 1, 2 \)) satisfy the Hölder condition and the exponents are

\[ \sigma = \frac{\pi - \alpha}{2\pi - \alpha}, \quad \alpha = |\alpha_2 - \alpha_1|, \quad 0 \leq \alpha \leq \pi. \]

Further, as in the previous case we approximate the function \( g_l(\tau) \) by interpolation polynomials of degree \( (n-1) \) by formula (21), where \( Q_{1n}(\tau) = P_{n}^{\sigma - 1/2}(\tau) \), \( Q_{2n}(\tau) = P_{n}^{-1/2 - \sigma}(\tau) \) are the Jacobi polynomials and \( \tau_{m1}, \tau_{m2} \) are roots of these polynomials. Then, for the singular integral with Cauchy kernel, the following quadrature formula holds [20]:

\[ \int_{-1}^{1} \frac{\varphi'_l(\tau)}{\tau - \xi_{lj}} d\tau = \sum_{m=1}^{n} g_{lm} \frac{A_{lm}}{\tau_{lm} - \xi_{lj}}, \quad l = 1, 2, \quad j = 1, 2, \ldots, n. \quad (29) \]

where \( \xi_{j1}, \xi_{j2} \) (\( j = 1, 2, \ldots, n \)) are roots of the Jacobi functions of the second kind \( J_{n}^{\sigma - 1/2}(\tau) \), \( J_{n}^{-1/2 - \sigma}(\tau) \) and \( A_{lm} \) are the coefficients of the corresponding Gauss–Jacobi quadrature formula.

Further, we obtain similar formulas for the integrals with fixed singularities

\[ E_{kl}^j = \int_{-1}^{1} q_{kl}(\tau, \xi_{jk})\varphi'_l(\tau) d\tau. \quad (30) \]

If \( 1 \pm \xi > r_1 > 0 \), then it follows from relations (16) that the functions \( q_{kl}(\tau, \xi) \) are infinitely differentiable. Therefore, we can apply the Gauss–Jacobi quadrature formulas to integrals (30).

The principal difficulty is related to the calculation of these integrals as \( 1 \pm \xi \to 0 \). Then, as in the previous case, we should replace \( \varphi'_l(\tau) \) by expressions (28), (21) and again use the method based on the Mellin integral transformation [21]. The final formulas for calculating the integrals (30) become

\[ E_{kl}^j = \sum_{m=1}^{n} g_{ml} \frac{H_{ml}^{kl}}{s_{kl}(\tau_{ml}, \xi_{jk})}, \quad (31) \]

where the coefficients \( H_{ml}^{kl} \) are represented in terms of the following power series similar to (25) and converging for \( 0 \leq 1 \pm \xi < r_1 < 1 \).
Substituting expressions (28), (21) into formulas (32) and integrating, we obtain the representation
\[
\varphi_1(\tau) = \int_{-1}^{\tau} \varphi_1'(x) \, dx, \quad \varphi_2(\tau) = -\int_{\tau}^{1} \varphi_2'(x) \, dx.
\] (32)

Substituting expressions (28), (21) into formulas (32) and integrating, we obtain the representation
\[
\varphi_1(\tau) = \left[1 - (-1)^l \tau\right]^{1/2} \sum_{m=1}^{n} A_{lm} \psi_{lm}(\tau), \quad l = 1, 2,
\]
\[
S_{lm}(\tau) = \sum_{m=1}^{n} A_{lm} \psi_{lm}(\tau),
\]
\[
\psi_{lm}(\tau) = F\left(\frac{1}{2}, \frac{3}{2}, \frac{1}{2} \left[1 + (-1)^l \tau\right]^{1/2} - \left[1 + (-1)^l \tau\right]^{1/2} \sum_{j=1}^{n} Q_{lj}(\tau_{lm})Q_{lj}^{(1)}(\tau),
\]
which \(Q_{lj}(\tau) = P_j^{-\sigma,1/2}(\tau), \quad Q_{lj}(\tau) = P_j^{1/2,1-\sigma}(\tau)\) are the Jacobi polynomials and \(F\) is a hypergeometric function. Representation (33) is the basis for the following quadrature formulas for integrals with unknown functions:
\[
\int_{-1}^{1} \varphi_k(\tau) \left[ U_{kk}^{(1)}(\tau - \zeta_{kj}) \right] d\tau = \sum_{m=1}^{n} A_{km} \psi_{km} \left[ U_{jm}^{(1)} \right],
\]
\[
\left[ U_{jm}^{(1)} \right] = \sum_{p=1}^{n} B_{kp} S_{km}(\zeta_{kp}) \left[ U_{ik}^{(1)}(\zeta_{kp}, \zeta_{kj}) \right],
\]
where \(\zeta_{kp}\) are roots of the Jacobi polynomials \(P_n^{0,1/2}(z_{1p}) = 0, \quad P_n^{1/2,0}(z_{2p}) = 0\) and \(B_{1p} = A_{p}^{0,1/2}, \quad B_{2p} = A_{p}^{1/2,0}\) are the coefficients of the corresponding Gauss-Jacobi quadrature formula.

Let us consider the integrals with logarithmic difference kernel from system (16). As a result of the integration by parts and the use of representations (28) and (21) for the derivatives of the unknown functions, we obtain the quadrature formulas
\[
\int_{-1}^{1} \varphi_k(\tau) \ln |\tau - \zeta_{lj}| \, d\tau = \sum_{m=1}^{n} A_{lm} H_{jm}^{(l)}, \quad l = 1, 2,
\]
\[
H_{jm}^{(l)} = (-1)^l[1 + (-1)^l \zeta_{lj}][\ln |1 + (-1)^l \zeta_{lj}| - 1] - (\tau_{lm} - \zeta_{lj})(\ln |\tau_{lm} - \zeta_{lj}| - 1).
\]
Let us use quadrature formulas (29), (31), (34), (35), the Gauss-Jacobi quadrature formulas, and the roots of Jacobi polynomials as points of collocation. Then system (15) can be reduced to a system of linear algebraic equations for \(g_{mn} = g_{km}(\tau_{mn})\). By solving this system, we obtain the following formulas for the approximate values of SIF:
\[
K_i = -G \sqrt{\frac{d}{d}} 2^{-(1+\sigma)}Q_{lm}((-1)^l) \sum_{m=1}^{n} p_{lm}(\tau_{lm})[1 - (-1)^l \tau_{lm}].
\]
The numerical realization of the proposed method is carried out for the crack shown in figure 3. The calculation was carried out for the same data as in the case of inclusion. The results of studying the practical convergence of the proposed method are given in figure 6 as graphs of the absolute value of SIF. Here \(k = K_i/(G\sqrt{\overline{d}})\) as a function of the dimensionless wave number \(\kappa_0 = \kappa_2d\). The curves correspond to the numbers of interpolation nodes.
Figure 6. Numerical investigation of the practical convergence.

Figure 7. Dependence of SIF on the frequency $(\theta_0 = 90^\circ)$.

$n = 5, 10, 15, 20, 25$ in approximation (21). These results show that, in the region of low frequencies $\kappa_0 < 1$, to attain a fairly high accuracy (the relative error does not exceed 1%), it suffices to use $n = 5$ interpolation nodes. In the whole considered frequency range $0 \leq \kappa_0 \leq 6$, to obtain values of SIF with an error below 0.1%, it suffices to have $n = 20$ interpolation nodes in formula (21). We also numerically studied the influence of the angle $\beta$ between the crack links on the dependence of SIF on the frequency. The results of calculations are given in figure 7. The curves correspond to values: 1 — $\beta = 15^\circ$, 2 — $30^\circ$, 3 — $45^\circ$, 4 — $60^\circ$, 5 — $87^\circ$ and are constructed at the angle of wave propagation $\theta_0 = 90^\circ$. We can see that, at low frequencies $0 \leq \kappa_0 \leq 2$, as the angle increases, the value of SIF also increases. The largest values of SIF are observed as the crack is transformed in a segment of the straight line $\beta \approx 90^\circ$.

In the further increase in the frequency $2 \leq \kappa_0 \leq 6$, the dependence becomes more complicated and has many maxima and minima. This complication is related to the multiple reflections of waves from the sides of the angle made by the cracks links.

Conclusions

In the presence of angular points in cracks and thin inclusions the method of integral equations reduce boundary value problems to singular integral or integro-differential equations with fixed singularities. For the numerical solution of such equations one must take into account the real property of unknown functions and apply special quadrature formulas for singular integrals. It ensures rapid convergence of numerical method of solution. On the example of V-shaped defects it can be seen the shape and type of defect substantially influence on stress state in their region. In particular, the angle formed by the sides of the defect substantially influence on the SIF dependence on the frequency. So, in the low frequency region. In the case of crack, the largest values of the SIF are observed for strait lines cracks. If the defect is inclusion for this angle values of the SIF will be the smallest. The presence of maximus resonance type in the region of high frequencies caused by the multiple reflection of the waves from the defects sides is shown.

References

[1] Kit H S, Kunets Ya I, and Yemets V F 1999 Elastodynamic scattering from a thin-walled inclusion of low rigidity Int. J. Engng Sci. 37 331–45

[2] Olsson P 1986 Elastodynamic scattering by fluid-filled nonplanar cracks J. Nondestruct. Eval. 5 (3) 161–8

[3] Olsson P, Datta S, and Boström A 1990 Elastodynamic scattering from inclusion surrounded by thin interface layers J. Appl. Mech. 57 (3) 672–6
[4] Kunets’ Ya I, Matus Ya I, and Porokhov’s’kyi V V 2000 Scattering of pulses of elastic SH-waves on a thin-walled elastic curvilinear inclusion Mat. Met. Fiz.-Mekh. Polya 43 (4) 150–4
[5] Takahuda Kuzuo, Tukizava Yasushi, Koizumi Tukashi, and Shibuya Toshikazu 1984 Dynamic interactions between cracks. Diffraction of SH waves being incident of Griffith cracks in an infinite body Trans. Jpn. Soc. Mech. Eng. Ser. A 50 (452) 799–804
[6] Popov V G 1992 Investigation of fields of displacements and stresses under diffraction of elastic shear waves on a thin rigid delaminated inclusion Izv. Ross. Akad. Nauk. Mekh. Tverd. Tela No. 3 139–46
[7] Popov V G 1993 Comparison of displacement fields and stress fields under diffraction of elastic shear waves on different defects: crack and thin rigid inclusion Dinam. Sistemy 12 35–41
[8] Gross D and Zhang Ch 1988 Diffraction of SH waves by a system of cracks: Solution by integral equation method Int. J. Solids Struct. 24 (1) 41–9
[9] Afyan B A 1984 On the integral equations with fixed singularities in the theory of branched cracks Dokl. Akad. Nauk Arm. SSR 79 (4) 177–81
[10] Vitek V 1977 Plane strain stress intensity factors for branched cracks Int. J. Fract. 13 (4) 481–501
[11] Atkinson C 1973 Some ribbon-like inclusion problems Int. J. Engng Sci. 11 (2) 243–66
[12] Stullybrass M P 1970 A crack perpendicular to an elastic half-plane Int. J. Engng Sci. 8 (5) 351–62
[13] Duduchava R V 1979 Integral Equations in Convolution with Discontinuous Presymbols, Singular Integral Equations with Fixed Singularities, and Their Applications to Some Problems of Mechanics (Tbilisi: Metsniereba)
[14] Osiv P N and Savruk M P 1983 Determination of stresses in an infinite plate with broken or branching crack J. Appl. Mech. Techn. Phys. 24 (2) 266–71
[15] Savruk M P 1981 Two-dimensional problems of elasticity for bodies with cracks (Kiev: Naukova Dumka) p 323 [in Russian]
[16] Isida M and Noguchi H 1992 Stress intensity factors at tips of branched cracks under various loadings Int. J. Fract. 54 (4) 293–316
[17] Li Z and Achenbach J 1991 Interaction of rayleigh wave with a disband in a material interphase normal to a free surface Ultrasonics 29 (1) 45–52
[18] Gupta G D and Erdogan F E 1974 The problem of edge cracks in an infinity strip Appl. Mech. 41 (4) 1001–6
[19] Erdogan F, Gupta G, and Cook T 1973 Numerical solution of singular integral equations In Methods of Analysis and Solutions of Crack Problems. Mechanics of Fracture ed G C Sih vol 1 (Dordrecht: Springer) 368–425
[20] Andreev A V 2005 Direct numerical method for solving singular integral equations of the first kind with generalized kernels Mech. Solids 40 (1) 104–19
[21] Popov G Ya 1982 Concentration of Elastic Stresses near Stamps, Cuts, Thin Inclusions and Reinforcements (Moscow: Nauka) p 342 [in Russian]
[22] Krylov V I 1967 The Approximate Calculation of Integrals (Moscow: Nauka) p 500 [in Russian]
[23] Sulym H 2007 Fundamentals of the Mathematical Theory of Thermoelastic Equilibrium of Deformable Solid Bodies with Thin Inclusions (Lviv: Shevchenko Scientific Society) p 715 [in Ukrainian]