Random Embeddings of Graphs:
The Expected Number of Faces in Most Graphs is Logarithmic

Jesse Campion Loth\textsuperscript{1}, Kevin Halasz\textsuperscript{1}, Tomáš Masářík\textsuperscript{1,2}, Bojan Mohar\textsuperscript{1,11}, and Robert Šámal\textsuperscript{3,3}

\textsuperscript{1}Department of Mathematics, Simon Fraser University, Burnaby, BC, V5A 1S6, Canada, \{jcampion, khalasz, mohar\}@sfu.ca
\textsuperscript{2}Institute of Informatics, Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Warszawa, 02-097, Poland, masarik@mimuw.edu.pl
\textsuperscript{3}Computer Science Institute, Faculty of Mathematics and Physics, Charles University, Praha, 118 00, Czech Republic, samal@iuuk.mff.cuni.cz

Abstract

A random 2-cell embedding of a connected graph $G$ in some orientable surface is obtained by choosing a random local rotation around each vertex. Under this setup, the number of faces or the genus of the corresponding 2-cell embedding becomes a random variable. Random embeddings of two particular graph classes – those of a bouquet of $n$ loops and those of $n$ parallel edges connecting two vertices – have been extensively studied and are well-understood. However, little is known about more general graphs despite their important connections with central problems in mainstream mathematics and in theoretical physics (see [Lando & Zvonkin, Graphs on surfaces and their applications, Springer 2004]). There are also tight connections with problems in computing (random generation, approximation algorithms). The results of this paper, in particular, explain why Monte Carlo methods (see, e.g., [Gross & Tucker Local maxima in graded graphs of imbeddings, Ann. NY Acad. Sci 1979] and [Gross & Rieper, Local extrema in genus stratified graphs, JGT 1991]) cannot work for approximating the minimum genus of graphs.

In his breakthrough work ([Stahl, Permutation-partition pairs, JCTB 1991] and a series of other papers), Stahl developed the foundation of “random topological graph theory”. Most of his results have been unsurpassed until today. In our work, we analyze the expected number of faces of random embeddings (equivalently, the average genus) of a graph $G$. It was very recently shown [Campion Loth & Mohar, Expected number of faces in a random embedding of any graph is at most linear, arXiv 2022] that for any graph $G$, the expected number of faces is at most linear. We show that the actual expected number of faces is usually much smaller. In particular, we prove the following results:

\footnotesize
\begin{itemize}
    \item T.M. was supported by a postdoctoral fellowship at the Simon Fraser University through NSERC grants R611450 and R611368.
    \item B.M. was supported in part by the NSERC Discovery Grant R611450 (Canada) and by the Research Project J1-8130 of ARRS (Slovenia).
    \item R.S. was partially supported by grant 19-21082S of the Czech Science Foundation. This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 810115). This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 823748.
\end{itemize}
\[ \frac{1}{2} \ln n - 2 < \mathbb{E}[F(K_n)] \leq 3.65 \ln n, \text{ for } n \text{ sufficiently large.} \] This greatly improves Stahl’s \( n + \ln n \) upper bound for this case.

(2) For random models \( B(n, \Delta) \) containing only graphs, whose maximum degree is at most \( \Delta \), we show that the expected number of faces is \( \Theta(\ln n) \).

1 Introduction

1.1 Random embeddings of graphs in surfaces

Every 2-cell embedding of a graph \( G \) in an (orientable) surface can be described combinatorially up to homeomorphic equivalence by using a rotation system. This is a set of cyclic permutations \( \{R_v \mid v \in V(G)\} \), where \( R_v \) describes the clockwise cyclic order of edges incident with \( v \) in an embedding of \( G \) in an oriented surface. We refer to [33] for further details. In this way, a connected graph \( G \), whose vertices have degrees \( d(v) \) \( (v \in V(G)) \), admits precisely \( \prod_{v \in V(G)}(d(v)-1)! \) nonequivalent 2-cell embeddings.

Existing work on random embeddings of graphs in surfaces is mostly concentrated on the notion of the random genus of a graph. By considering the uniform probability distribution on the set \( \text{Emb}(G) \) of all (equivalence classes of) 2-cell embeddings of a graph in (orientable) closed surfaces, we can speak of a random embedding and ask what is the expected value of its genus. The initial hope of using Monte Carlo methods on the configuration space of all 2-cell embeddings to compute the minimum genus of graphs [18, 20] quickly vanished as empirical simulations showed that, in many interesting cases, the average genus is very close to the maximum possible genus in \( \text{Emb}(G) \). The work in [18] also showed that there can be arbitrarily deep local minima for the genus that are not globally minimum. One of the main outcomes of our work in this paper is a formal verification that the Monte Carlo approach cannot work for approximating the minimum genus of graphs. Still, random embeddings appear of sufficient interest not only in topological graph theory but also within several areas of pure mathematics and theoretical physics. They are ubiquitous as a fundamental concept in combinatorics (products of permutations, Hopf algebra, chord diagrams, random generation of objects), algebra (representations of the symmetric group), algebraic number theory (algebraic curves, Galois theory, Grothendieck’s “dessins d’enfants”, moduli spaces of curves and surfaces), knot theory (Vassiliev knot invariants), theoretical physics (quantum field theory, string theory, Feynmann diagrams, Korteweg and de Vries equation, matrix integrals), etc. We refer to [29] for additional in-depth information.

Unlike most previous works, we will not discuss the (average) genus but instead the (average) number of faces in random embeddings. Although the two variables are related linearly through Euler’s formula, it turns out that the study of the number of faces yields a more appreciative view of certain phenomena that occur in this area.

1.2 State-of-the-art

Two special cases of random embeddings are well understood. The first one is a bouquet of \( n \) loops (also called a monopole), which is the graph with a single vertex and \( n \) loops incident with the vertex. This family was first considered in a celebrated paper by Harer and Zagier [21] using representation theory. Several combinatorial proofs appeared later [7, 19, 23, 25, 42, 43]. By duality, the maps of the monopole with \( n \) loops correspond to unicellular maps [7] with \( n \) edges. The second well-studied case is the \( n \)-dipole, a two-vertex graph with \( n \) edges joining the two vertices; see
A more recent case gives an extension to the “multipoles” $[5]$ using a result of Stanley $[40]$. Random embeddings in all these cases are in bijective correspondence with products of permutations in two conjugacy classes. A notable generalization of these cases appears in a paper by Chmutov and Pittel $[11]$. Another well-studied case includes “linear” graph families, obtained from a fixed small graph $H$ by joining $n$ copies of $H$ in a path-like way, see $[17, 38]$ and references therein.

Here we discuss random graphs, including dense cases. One special case, which is of particular importance, is that of complete graphs. Looking at the small values of $n$, $K_3$ has only one embedding, which has two faces. It is easy to see that $K_4$ has two embeddings of genus 0 (with four faces) and all other embeddings have genus 1 and two faces. A brute force calculation using a computer gives the numbers for $K_5$ and $K_6$. They are collected in Table 1. The genus distribution of $K_7$ has been computed only recently $[3, 36]$ and there is no data for larger number of vertices. The computed numbers for $K_n$ show that for $n \leq 7$ most embeddings have a small number of faces. The results of this paper show that, similarly to the small cases, most embeddings of any $K_n$ will have large genus and the average number of faces is not only subquadratic but it is actually proportional to $\ln n$. This is a somewhat surprising outcome, because the complete graph $K_n$ has many embeddings with $\Theta(n^2)$ faces. In fact, it was proved by Grannell and Knor $[16]$ (see also $[14]$ and $[15]$) that there is a constant $c > 0$ such that the number of embeddings with precisely $\frac{1}{2}n(n - 1)$ faces is at least $n^{cn^2}$. All these embeddings are triangular (all faces are triangles) and thus of minimum possible genus. When we compare this result with the fact that

$$|\text{Emb}(K_n)| = ((n - 2)!)^n = n^{\Theta(n^2)},$$

we see that there is huge abundance of embeddings of $K_n$ with many more than logarithmically many faces.

Stahl $[37]$ introduced the notion of permutation-partition pairs with which he was able to describe partially fixed rotation systems. Through the linearity of expectation these became a powerful tool

---

1. This value was computed explicitly in $[36]$ Table 3.1.
2. We use $\ln n$ to denote the natural logarithm.
to analyze what happens in average. In particular, he was able to prove that the expected number of faces in embeddings of complete graphs is much lower than quadratic.

**Theorem 1.1** (Stahl [39, Corollary 2.3]). The expected number of faces in a random embedding of the complete graph $K_n$ is at most $n + \ln n$.

Computer simulations show that even the bound given in Theorem 1.1 is too high. In fact Mauk and Stahl conjectured the following.

**Conjecture 1.2** (Mauk and Stahl [32, page 289]). The expected number of faces in a random embedding of the complete graph $K_n$ is at most $2\ln n + O(1)$.

For general graphs, a slightly weaker bound than that of Theorem 1.1 was derived by Stahl using the same approach as in [39]; it had appeared in [38] a couple of years earlier.

**Theorem 1.3** (Stahl [38, Theorem 1]). The expected number of faces in a random embedding of any $n$-vertex graph is at most $n \ln n$.

The $n \ln n$ bound of Stahl was improved only recently. Campion Loth, Halasz, Masařík, Mohar, and Šámal [5] conjectured that the bound should be linear, which was then proved in [6].

**Theorem 1.4** (Campion Loth and Mohar [6, Theorem 3]). The expected number of faces in a random embedding of any graph is at most $\frac{\pi}{2} n$.

The bound of Theorem 1.4 is essentially best possible as there are $n$-vertex graphs whose expected number of faces is $\frac{4}{7} n$, see [6].

1.3 Our results

The first main contribution of our paper is the proof of Conjecture 1.2 with a slightly worse multiplicative factor.

**Theorem 1.5.** Let $n \geq 1$ be an integer and let $F(n)$ be the random variable whose value is the number of faces in a random embedding of the complete graph $K_n$. The expected value of $F(n)$ is at most $23\ln n$. For $n$ sufficiently large ($n \geq e^{e^{16}}$) the multiplicative constant is even better, namely:

$$\mathbb{E}[F(n)] \leq 3.65 \ln n.$$

The general bound of $23\ln n$ can be replaced by $5\ln n + 5$ with some additional work. We complement our upper bound with a lower bound showing that our result is tight up to the constant factor.

**Theorem 1.6.** Let $n \geq 1$ be an integer.

$$\mathbb{E}[F(n)] \geq \frac{1}{2} \ln(n) - 2.$$

In order to prove Theorem 1.5 we split the proof into ranges based on the value of $n$ and use a different approach for each range. In fact, we provide two theoretical upper bounds using a close examination of slightly different random processes. The first one is easier to prove, but it gives an asymptotically inferior bound. However it is useful for small values of $n$. In the bound we use the harmonic numbers $H_k := \sum_{j=1}^{k} \frac{1}{k}$, whose value is approximately equal to $\ln n$. 

4
Theorem 1.7. Let \( n \geq 10 \) be an integer. Then

\[ \mathbb{E}[F(n)] < H_{n-3}H_{n-2}. \]

Note that proof of Theorem 1.7 holds for \( n \geq 4 \), but with a very slightly worse bound (see Equation (1)), which we have not stated above. However, we used Equation (1) to estimate small values using computer.

The next theorem is our core result that proves Theorem 1.5 for \( n \geq 4158 \).

Theorem 1.8. For \( n \geq e^{16} \), \( \mathbb{E}[F(n)] \leq 3.65 \ln(n) \). For \( n \geq e^{30} \), \( \mathbb{E}[F(n)] \leq 5 \ln(n) \). For \( 4158 \leq n < e^{30} \), \( \mathbb{E}[F(n)] \leq 23 \ln(n) \).

For small values of \( n \leq 4157 \) we used a computer-assisted proof which is based on Theorem 1.7 and our general estimates given in the proof of Theorem 1.8 combined with pre-computed bounds for smaller values of \( n \) and Markov inequality. We will give more details on our computation in Section 5. We summarize the results of computer-calculated upper bounds in the following proposition. Note that having a small additive constant for small values of \( n \) helps us to keep smaller additive constants for middle values of \( n \) as our proof is inductive.

Proposition 1.9. For \( 1 \leq n \leq 4157 \), \( \mathbb{E}[F(n)] \leq 5 \ln(n) + 5 \).

In summary, the proofs of the above results for complete graphs are relatively long. A “log-square” improvement of Stahl’s linear bound is not that hard, but the \( O(\ln n) \) bound appears challenging and shows all difficulties that arise for more general dense graph classes.

In the second part of the paper, we turn to more general random graph families. First we state a general result for random embedding of random maps with fixed degree sequence. Some results of this flavor have been obtained earlier in the setup of “random chord diagrams”, see [10, 31].

Theorem 1.10. Let \((d_1, d_2, \ldots, d_n)\) be an admissible degree sequence for an \( n \)-vertex multigraph (possibly with loops) where \( 2 \leq d_i \) for all \( i \). The average number of faces in a random embedding of a random graph with degree sequence \((d_1, d_2, \ldots, d_n)\) is \( \Theta(\ln n) \).

In the case of random simple graphs with constant vertex degrees, we have a logarithmic bound as well.

Theorem 1.11. Let \( d \geq 3 \) be a constant, and let \((d_1, d_2, \ldots, d_n)\) be an admissible degree sequence for an \( n \)-vertex graph with \( 2 \leq d_i \leq d \) for all \( i \). The average number of faces in a random embedding of a random graph with degree sequence \((d_1, d_2, \ldots, d_n)\) is \( \Theta(\ln n) \).

In the light of the above theorems and our Monte Carlo experiments, we conjecture that a logarithmic upper bound should be achievable for any usual model of random graphs. However, extending our proof of Theorem 1.5 to arbitrary random graphs seems to require further ideas.

Conjecture 1.12. Let \( p = p(n) \) be the probability of edges in \( G(n, p) \). The expected number of faces in a random embedding of a random graph \( G \in G(n, p) \) is

\[ (1 + o(1))\ln(pn^2). \]
Structure of the paper. Before we dive into proofs we will present our common strategy and formalization used in Theorems 1.7 and 1.8 in Section 2. First, we present the easier proof of Theorem 1.7 in Section 3. Our main result (Theorem 1.8) on complete graphs can be found in Section 4. In Section 5, we describe how the estimates presented in Section 4 were used to compute the bounds for small values of \( n \) using computer evaluation. We conclude the complete graph sections with a short proof of our lower bound (Theorem 1.6) in Section 6. The paper closes with a proof of Theorem 1.11 in Section 7.

1.4 Preliminaries

Combinatorial maps. To describe 2-cell embeddings of graphs we need a formal definition of a map. Let \( G = (V, E) \) be a graph on \( n \) vertices, \( V = \{v_1, \ldots, v_n\} \). Each edge \( uv \in E \) is divided into two darts (half-edges), one incident with \( u \) and the other one incident with \( v \). The set of all darts is denoted by \( D \), and it can be partitioned into sets \( D_i \), which consist of all darts incident with vertex \( v_i \) for \( i \in [n] \). We can describe any 2-cell embedding on an orientable surface by describing the clockwise order of darts in \( D_i \) around each vertex. For each \( i \in [n] \), we let \( R_i \) be the unicyclic permutation of darts in \( D_i \)—in the clockwise order in a given embedding. So, \( R_i(d) \) is the dart following \( d \) in the clockwise order given by \( R_i \), and conversely \( R_i^{-1}(d) \) is the dart preceding \( d \) in this order. We let \( R \) be the permutation of \( D \), which is the union of all \( R_i \)'s. Next, we let \( L \) be the permutation of \( D \) consisting of 2-cycles—pairs of darts that correspond to the edges in \( E \). We call the triple \( M = (D, R, L) \) a combinatorial map (as introduced in [26, 35]). From an abstract point of view, \( D \) is any finite non-empty set, \( R \) is a permutation of \( D \), \( L \) is a fixed-point-free involution of \( D \), and the group \( \langle R, L \rangle \leq \text{Sym}(D) \) is transitive on \( D \) if we want the graph to be connected. Each combinatorial map corresponds to a unique (up to homeomorphisms) 2-cell embedding (see [35] or [33] for details) of the graph \( G \). Note that the vertices of \( G \) can be viewed as the orbits of \( R \), and the edges of \( G \) correspond to the cycles of the involution \( L \). We refer to \( G \) as the underlying graph of the map \( M \).

Faces. It is useful to observe that facial walks (or faces) of the implied embedding of a map \( M = (D, R, L) \) are given by the orbits of \( R \circ L \). One of our main interests in this paper will be the number of faces. It will be to our advantage to allow maps whose underlying graph is not connected. To allow for that case, we define the number of faces for maps that are not connected as the number of faces in each connected component minus the number of connected components plus one. This corresponds to the fact that one can always pick an arbitrary face \( f_1 \) in one connected component \( H_1 \) and an arbitrary face \( f_2 \) in another connected component \( H_2 \) and insert the whole embedding of \( H_2 \) inside \( f_1 \) using \( f_2 \) as a boundary. Note that each isolated vertex always contributes zero towards the number of faces. This notion enables us to define the genus \( g \) of the map \( M \) with underlying graph \( G = (V, E) \) via Euler’s formula, \( g = \frac{1}{2}(|E| - |V| - f + k + 1) \), where \( f \) is the number of faces and \( k \) is the number of connected components of \( G \).

Partial embeddings, temporary faces, and partial faces. Given a map \( M = (D, R, L) \) with underlying graph \( G \), a submap of \( M \) is a map \( M' = (D', R', L') \) that corresponds to a subgraph \( G' \) of \( G \), where \( D' \subseteq D \) are the darts corresponding to the edges in \( G' \), \( L' = L |_{D'} \), and \( R'_i \) (for each \( v_i \in V(G') \)) is a restriction of each \( R_i \) to \( D' \) (the cyclic permutation formed from \( R_i \) by skipping the darts that are not in \( D' \)). Note that a submap of \( M \) is determined by the set of edges forming \( D' \), and the edges that are not part of the submap are removed. But sometimes, we would like to
keep all darts, without having them paired via the involution $L$. Then we define a partial submap as $(D, R, L^*)$, where compared to a submap we keep the whole local rotation information $R$ and, instead, only prescribe $L^* \subseteq L$ by deleting some orbits of $L$. We refer to darts that are not in $\bigcup L^*$ as unpaired darts, while the darts in $L^*$ are referred to as being paired. We define temporary faces of a partial submap $(D, R, L^*)$ as the faces of the submap $(D^*, R^*, L^*)$ of $(D, R, L^*)$, where $D^*$ are the paired darts of $D'$. Now, we look how the temporary faces in a partial submap $(D, R, L^*)$ look like. There can be unpaired darts inside each face. We say that a temporary face $f$ is $k$-open if $f$ contains exactly $k$ unpaired darts. That means, for each $d_1, d_2 \in f$ such that $R^*(d_1) = d_2$ we count all (unpaired) darts that appear between $d_1$ and $d_2$ in $R$. We also say that these unpaired darts belong to the face $f$. We say that a temporary face $f$ is strongly $2$-open if $f$ is $2$-open and the two unpaired darts in $f$ are incident with different vertices. Let $f$ be a $k$-open face and let $d_1, d_2, \ldots, d_k$ be the unpaired darts that belongs to $f$ in their clockwise order of appearance on $f$. For each $i$ ($1 \leq i \leq k$), the map $M$ contains a facial walk $f_i$ which contains $d_1$ and a segment of the temporary face $f$ from the dart $d_i$ following $f$ in clockwise order and ending with $d_{i+1}$ (indices modulo $k$). We call the whole segment of $f_i$ from $d_i$ to $d_{i+1}$ the partial facial walk (partial face) with initial dart $d_i$ and ending dart $d_{i+1}$. (We also say that this partial face leads from $d_i$ to $d_{i+1}$). Note that each unpaired dart is the initial dart for precisely one partial face and is also the ending dart of precisely one partial face.

By a random embedding of graph $G$ we mean an embedding where we select a random cyclic permutation $R_i$ (local rotation) of all darts $D_i$ incident with the vertex $v_i$, and this is done independently for each vertex.

We use $m^\underline{k} := m(m - 1)(m - 2) \cdots (m - k + 1)$ to denote the falling factorial. We will use the following precise estimate for the harmonic numbers $H_n = 1 + \frac{1}{2} + cdots + \frac{1}{n}$.

**Theorem 1.13** (Fast convergence of $H_n$ [13]). Let $n \geq 1$.

$$H_n = \ln \left( n + \frac{1}{2} \right) + \gamma + \varepsilon_n,$$

where $\frac{1}{24(n+1)^2} \leq \varepsilon_n \leq \frac{1}{24n^2}$ and $\gamma \approx 0.57721$ is the Euler–Mascheroni constant.

The above lower-bound works also for $n = 0$ since $H_0 = 0 \geq \ln \left( \frac{1}{2} \right) + \gamma + \frac{1}{21}$.

## 2 Our proof strategy for complete graphs

**General approach.** The definition of a random embedding requires choosing uniformly at random a cyclic permutation $R_i$ for each $i \in [n]$. However, we will work with a more delicate random process (still with the same distribution on embeddings) that will allow us to analyze the procedure step-by-step. In our case, a step will be a choice of $R_i(d)$ for some dart $d$. We build a combinatorial map $(D, R, L)$ whose underlying graph is $K_n$. Note that $D$ and $L$ are given by the graph, and our task will be to build $R$ uniformly at random. Our main tool is a simplified and refined approach of Stahl [39], which gives the best previously-known upper bound of $n + \ln(n)$ stated in Theorem 1.1.

Stahl’s strategy can be summarized as follows. First, we order the vertices of a graph $G$ arbitrarily. We represent the ordering as $v_1, v_2, \ldots, v_n$, and we process vertices one by one, starting with $v_n$. For each $i \in [n]$, we process all darts in $D_i$, one-by-one. In this way, we construct a decision tree of all possible embeddings. In each node $t$ of the decision tree, one of yet unprocessed darts $d$ is processed,
which means we enumerate all available choices for $R_i(d)$. For each such choice, we obtain a son of the node $t$ in the decision tree. It is straightforward to verify that at most one such choice actually completes a face, i.e., both $d$ and $R_i(d)$ are part of a face in $R \circ L$ which was completed once $R_i(d)$ was determined. We formalize this fact as the following observation, which can be attributed to Stahl [39].

**Observation 2.1** ([39] (reformulated)). Let $d$ be a dart of $G$ such that $R_i(d)$ is undefined. Among all valid choices for $R_i(d)$ at most one completes a face containing $d$.

Observe that the above-described procedure generates a decision tree where the leaves are uniformly random embeddings of $K_n$.

**Our refined approach.** In our refined strategy, we do one more complication to the described process, and we conclude by rather complicated computation. Instead of building a decision tree, we will look at what the graph before determining $R_i$ looks like and how probable it is. In order to do that, it is useful to also build $L$ in a step-by-step fashion. Of course, the resulting graph needs to be and still will be $K_n$, but we gain a more uniform description of how $(D, R, L)$ looks like during all the steps of the process. In fact, we give two proofs, each with different usage. One gives an asymptotically worse bound, but it will be useful to give the best estimates for small values of $n$. The other is more involved and requires rather tedious computation. Here we present an intuition on both proofs and introduce a bit more terminology.

First, we consider vertices $v_n$ and $v_{n-1}$. For those we can fix $R_1$ and $R_2$ arbitrarily and also we determine first element of $L$ which will be the $v_nv_{n-1}$ edge. So far, we have not completed any face. Now, imagine we are going to process vertex $v_k$, we refer to it as the $k$-th step. For each $k \in [n-2]$, we define the following terminology. Let $V^k$ be vertices $v_n, \ldots, v_{k+1}$ and $D_{V^k}$ be set of their darts. Recall that dart $d$ is unpaired if $L(d)$ is undefined. Now, we make the following random choice. For each $i > k$ we choose uniformly at random an unpaired dart $d_i \in D_i$ and we define $L(d_i) := d$ for some unpaired dart $d \in D_k$. We call all such newly paired darts active for this step. Observe that $k-1$ darts remain unpaired at vertex $v_k$ in this step. For the proof of Theorem 1.7, we now consider a fixed, cleverly chosen rotation $R_k$, and the randomness only comes from generating $L$. For the (more technically complicated) proof of Theorem 1.8, we make one more random choice that determines $R_k$ in a similar way as in Stahl’s approach, i.e., using a step-by-step definition of the function $R_i$ subject to some crafted order of darts in $D_i$. We describe the process in detail before each proof.

### 3 Log-square bound—Proof of Theorem 1.7

We start by proving Theorem 1.7.

**Theorem 1.7.** Let $n \geq 10$ be an integer. Then

$$\mathbb{E}[F(n)] < H_{n-3}H_{n-2}.$$ 

**Random process A.**

1. Order the vertices of the graph $v_n, \ldots, v_1$ arbitrarily, we process the vertices in this order.
2. Start with vertices \( v_n \) and \( v_{n-1} \). They belong to one temporary face and no face has been closed so far.

3. Consider vertex \( v_k \) for \( k \in [n-2] \). Label the darts of \( D_k \) as \( \{d_1, \ldots, d_{n-1}\} \) arbitrarily. We define \( R_k \) as this order, that is \( R_k(d_i) = d_{i+1} \) (indices modulo \( n-1 \)). Let \( C_k := \{\ell, n-1, \ldots, k+1, u, u, \ldots, u\} \) where there are \( k-1 \) copies of the symbol \( u \) that represent that the dart choosing this option becomes unpaired. This is the set of choices of where the darts may lead at the end of this step.

(a) Process darts in \( D_k \) in order \( d_1, d_2, \ldots, d_{n-1} \). If \( k > 1 \), we may assume \( d_1 \) is unpaired.

(b) Consider the dart \( d_\ell \) which is next in the order. Random choice 1a: Pick a symbol from the set \( C_k \) uniformly at random, then remove this choice from \( C_k \).

\[ \bullet \] Case 1: The choice was some \( i \geq k+1 \). Random choice 1b: Then pick an unpaired dart \( d' \) uniformly at random from those at \( v_i \). Then add the transposition \((d', d_\ell)\) to the permutation \( L \).

\[ \bullet \] Case 2: The choice was some \( u \). Then leave dart \( d_\ell \) unpaired.

Continue to the next dart in the order.

For each value of \( k \leq n-2 \), let \( F_k \) (\( F_k = F_k(n) \)) be the number of faces created at step \( k \). By this, we mean the facial walks that contain \( v_k \) and no vertex \( v_j \) with \( j < k \)—thus, they will stay unchanged until the end of the process. We need an upper bound on \( \mathbb{E}[F_k] \). By linearity of expectation, we have that \( \mathbb{E}[F(n)] = \sum_{k=1}^{n-2} \mathbb{E}[F_k(n)] \).

Suppose we are processing the dart \( d_\ell \) at step \( k \). Then we have two cases:

Case 1: \( \ell = 1 \), or the previous dart (that is \( d_{\ell-1} \)) was chosen to be unpaired: Then, we cannot complete a face when processing \( d_\ell \).

Case 2: \( d_{\ell-1} \) is paired: Let \( D \) consists of darts incident with vertices in \( V^\uparrow \) together with \( d_1, \ldots, d_{\ell-1} \). Then, in partial map \((D, R, L)\), there is a partial face \( f \) starting with \( d_{\ell-1} \) that leads to an unpaired dart \( d_u \in D \). This partial face is the only face that may now be completed by processing \( d_\ell \). Namely, by pairing \( d_\ell \) with \( d_u \). The probability we choose \( d_\ell \) to lead to vertex \( v_i \), for \( i > k \), is at most \( \frac{1}{n-\ell} \) as we have already chosen \( \ell-1 \) vertices in Random choice 1a. The probability that we choose dart \( d_u \) (and not another unused dart at \( v_i \)) to connect with \( d_\ell \) is \( \frac{1}{k} \) as there are \( k \) unpaired darts incident vertex \( v_i \) to choose from in Random choice 1b. Therefore the probability that we complete the face is at most \( \frac{\frac{k}{k(n-\ell)}}{\frac{n-k}{n-2}} \).

Case 3: \( k = 1 \): When processing \( d_{n-1} \) we can close two faces. One containing \( d_1 \) and \( d_{n-1} \) and the other containing \( d_{n-2} \) and \( d_{n-1} \).

Note that we started the process with an unpaired dart (if \( k \neq 1 \)). Thus, when processing the last dart around \( v_k \), we cannot complete another face containing both \( d_1 \) and \( d_{n-1} \). Assume now \( k > 1 \). Each dart (except for \( d_1 \)) has probability \( \frac{n-k}{n-2} \) of being paired. Thus a dart \( d_\ell \) (\( \ell \geq 3 \)) has the same probability \( \frac{\frac{n-k}{n-2}}{\frac{n-k}{n-2}} \) of being paired. Therefore, the probability that we close a face by pairing up \( d_\ell \) is at most \( \frac{\frac{n-k}{n-2}}{\frac{n-k}{n-2}} \frac{1}{k(n-\ell)} \).

For \( k = 1 \), all edges are connected to \( V^\uparrow \), thus the probability of closing a face by \( d_\ell \) (for \( \ell \geq 2 \) now) is \( \frac{1}{n-\ell} \). Moreover, the last dart \( d_{n-1} \) can close two faces as described in Case 3.
Summing over all values of $\ell$ we get for $k \geq 2$ and $n \geq 4$

$$\mathbb{E}[F_k] \leq \sum_{\ell=3}^{n-1} \frac{n-k}{n-2} \cdot \frac{1}{k(n-\ell)} = \frac{n-k}{k(n-2)} \cdot H_{n-3}. $$

Also,

$$\mathbb{E}[F_1] \leq 1 + \sum_{\ell=2}^{n-1} \frac{1}{n-\ell} = 1 + H_{n-2}. $$

Summing over all steps $k$ assuming $n \geq 4$ for all steps apart from the last one we obtain:

$$\mathbb{E}[F] = \mathbb{E}[F_1] + \sum_{k=2}^{n-2} \mathbb{E}[F_k]$$

$$\leq 1 + H_{n-2} + \sum_{k=2}^{n-2} \frac{n-k}{k(n-2)} H_{n-3}$$

$$= 1 + H_{n-2} + \frac{n}{n-2} H_{n-3} (H_{n-2} - 1) - \frac{n-3}{n-2} H_{n-3}$$

$$< H_{n-3} H_{n-2}. \quad \text{(for } n \geq 10)$$

4 Logarithmic bound—Proof of Theorem 1.8

Theorem 1.8. For $n \geq e^{e^{16}}$, $\mathbb{E}[F(n)] \leq 3.65 \ln(n)$. For $n \geq e^{30}$, $\mathbb{E}[F(n)] \leq 5 \ln(n)$. For $4158 \leq n < e^{30}$, $\mathbb{E}[F(n)] \leq 23 \ln(n)$.

We first introduce more notation that will be needed in the proof. We look more carefully at the $k$-th step. Before active darts for the $k$-th step are determined, the walks in $R \circ L$ can be split into two categories building on notation defined in Section 1.4:

1. Completed faces: cycles of $R \circ L$. Those are closed walks that corresponds to 0-open faces which will not change any more, and

2. Candidate walks: those are partial faces that originates at an unpaired dart $d_s$ and lead to an unpaired dart $d_e$ (possibly $d_s = d_e$).

Now, we determine the active darts for the $k$-th step. Observe that if a candidate walk starts with a dart $d_s$ and ends with $d_e$, then it can complete a face in step $k$ only if both $d_s$ and $d_e$ become active. We call such walks active in step $k$. We further partition the active walks into

(1) the ones where $d_s = d_e$. Observe that such are necessarily 1-open faces and so we refer to them as 1-open active faces, and

(2) the other (i.e., $d_s \neq d_e$), which we refer to as potential faces.

An active dart $d \in D_k$ is called 1-open if $L(d)$ is the dart incident with some 1-open face. An active dart $d \in D_k$ is called potential if $L(d)$ is incident with some potential face. We would like to give more intuition on our naming here. We will strengthen Observation 2.1 so that under certain circumstances, only potential faces may complete a face. Therefore, we call unpaired darts in $D_k$
together with darts that do not take part in any active walk non-contributing. Let $PF_k$ be a random variable representing the number of potential faces and $O_k$ be a random variable representing the number of 1-open active faces in step $k$, after active darts were chosen.

We now describe our random procedure in detail.

**Random process B.**

1. Label the vertices arbitrarily as $v_n, \ldots, v_1$ and process them in that order.
2. Start with vertices $v_n$ and $v_{n-1}$. They belong to one temporary face.
3. Consider vertex $v_k$ for $k \in [n-2]$.
   
   (a) **Random choice 1:** For each vertex in $V^\uparrow$ we choose uniformly at random one out of $k$ unpaired darts to lead to $v_k$ and update $L$ appropriately.
   
   (b) Process darts incident with $v_k$ in a special order $\sigma_k$ given by the type of walk the dart describes. We define $\sigma_k$ as follows:
   
   i. First, process 1-open darts in arbitrary order,
   
   ii. next, potential darts follow in arbitrary order, and
   
   iii. last, non-contributing darts are processed, again in arbitrary order.

   (c) **Random choice 2:** For each $d \in D_k$ in order $\sigma_k$ we choose uniformly at random one dart $d'$ among all possible options (those that do not violate the property that $R_k$ will define a single cycle eventually) and we set $R_k(d) := d'$.

Now, we define a function $q$, which will form an upper bound for the contribution of vertex $v_k$ to the average number of faces. The function is defined as follows. (Note that $H_0 = 0$.)

**Definition 4.1.** If $1 \leq t < n$ and $0 \leq \xi < n - 1 - t$, then

$$q(\xi, t) := \sum_{i=1}^{t} \frac{1}{n - \xi - i - 1} = H_{n-\xi-2} - H_{n-\xi-t-2}. \quad (2)$$

If $\xi + t = n - 1$ then

$$q(\xi, t) := \sum_{i=1}^{t-1} \frac{1}{n - \xi - i - 1} + 1 = H_{n-\xi-2} + 1. \quad (3)$$

It is easy to observe the following fact about Definition 4.1.

**Observation 4.2.** Let $a \geq 1$, $1 \leq t + a < n$, and $0 \leq \xi - a < n - 1 - t - a$. Then

$$q(\xi, t) \leq q(\xi - a, t + a).$$

Now, we state the crucial lemma that is a starting point of the upper bound computation.

**Lemma 4.3.** Given $PF_k = t$ and $O_k = \xi$, the average number of faces completed at vertex $v_k$ is at most $q(\xi, t)$. 

11
Note that $O_k + PF_k$ is never larger than $n - 1$ and therefore the value $q(\xi, t)$ is well-defined. Observe that $O_k + PF_k = n - 1$ if and only if $k = 1$ as there are exactly $n - k$ edges between $v_k$ and $V^\uparrow$.

**Proof of Lemma 4.3.** Recall that first, we determine which unpaired darts of $V^\uparrow$ lead to $v_k$ in order $R_k$ of $v_k$. As mentioned above, in order to do that, we will be constructing the function $R_k : D_k \rightarrow D_k$ step-by-step. We start with $R_k$ being undefined. We define a *forefather* of a dart $d \in D_k$ which is the furthest possible predecessor of $d$ in partially constructed $R_k$. If no predecessor of $d$ exists, then $d$ is its own forefather.

We label the darts of $D_k$ in order $\sigma_k$ as $d_1, \ldots, d_{n-1}$. Now, suppose we are about to process $d_i$ where $i \neq n - 1$. We pick uniformly randomly the next dart in rotation $R_k$, i.e., we choose $R_k(d_i)$. We are allowed to use any dart which does not have a predecessor (this rules out $i - 1$ choices) as well as the forefather of $d_i$ is disallowed (as such a choice would close the cycle $R_k$ prematurely). Observe that as there are $n - 1$ darts around $v_k$, for the $i$-th dart we have $n - 1 - (i - 1) - 1$ valid choices. In case $i = n - 1$, we do not have any choice and $R_k(d_{n-1})$ must be equal to the forefather of $d_{n-1}$. Observe that this process produces a uniformly random embedding.

We continue by calculating the probability that a face is formed by fixing some $R_k(d_i)$ for $i < n - 1$. If $d_i$ is of the first category, choosing its successor does never complete a new face as, so far, we only determined $R_k$ for 1-open darts. If $d_i$ is of the second category, we argue we can complete at most one face by determining $R_k(d_i)$. We follow $R \circ L$, and it leaves only one choice for the successor, which completes the face. Therefore, for each $d_i$ of the second category the probability that we complete a face is at most $\frac{1}{n-i-1}$. Here, $i$ goes from $\xi + 1$ to $\xi + t$. It is easy to see that for any $d_i$ in the last category, there is no choice $R_k(d_i)$ which completes a face. Therefore, if $k > 1$, then the third category is not empty and $R_k(n - 1)$ never completes a face. We conclude that we arrive at equation (2). If $k = 1$ fixing $R_i(n - 1)$ might complete a face and this accounts for the additional $+1$ in equation (3).

We define one more random variable. Let $T_{n-k}$ represent the number of temporary faces in $G[V^\uparrow]$ in step $k$ (before vertex $v_k$ is added). Note that $E[T_n]$ is, in other words, an average number of faces of $K_n$. Hence, the following lemma is the first step in the proof of the main theorem. The rest of the proof will provide an involved analysis of the right-hand side of Inequality (4).

**Lemma 4.4.** Let $n \geq 3$ and $PF_k, O_k$ be random variables as defined earlier.

$$
E[F] = E[T_n] \leq \sum_{k=1}^{n-2} E[q(O_{k}, PF_k)] = \sum_{k=1}^{n-2} \sum_{i=1}^{n-k} \sum_{j=0}^{n-k-i} q(j, i) \cdot Pr[O_k = j \land PF_k = i]. \quad (4)
$$
Proof. The equalities in (4) are clear, so we will only argue about the inequality. We execute Random process B as defined above. For the first two vertices $v_n$ and $v_{n-1}$ in the order, all choices are isomorphic. We process each other vertex as described in part 3 of the process description. Hence, the contribution of a single vertex is upper-bounded by Lemma 4.3.

Let $1/2 < \nu < 1$ be a constant and $\nu := 1 - \nu$. We will fix this value later on for different ranges of $n$ in order to optimise our bound. We split the above triple sum (Equation (4) in Lemma 4.4) into several parts:

- $S_1$ will contain the terms where $k = 1$.
- $S_2$ will contain the terms where $j < \nu n$ and $i < \frac{n-k}{k}$.
- $S_3$ will contain the terms where $j < \nu n$ and $i \geq \frac{n-k}{k}$.
- $S_4$ will contain the terms where $j \geq \nu n$.

Recall that we use $\gamma$ to denote the Euler-Mascheroni constant, as defined in Theorem 1.13. We now define $S_1$, $S_2$, $S_3$, and $S_4$. We will also state the bounds which we derive for each portion of the sum in the forthcoming subsections:

$$S_1 := \sum_{i=1}^{n-1} \sum_{j=0}^{n-1-i} q(j, i) \cdot \Pr[O_1 = j \land PF_1 = i] \leq H_{n-2} + 1 \leq \ln(n) + \gamma + 1. \quad (5)$$

For the rest, we first take the terms for which $O_k < \nu n$. Let $b = b(n, k, i) := \min(n - k - i, \lceil \nu n \rceil - 1)$. When writing down the terms for $S_2$, we used the fact that these terms do not occur if $\frac{n-k}{k} \leq 1$. Thus we have the summation range for $k$ only between 2 and $n/2$.

$$S_2 := \sum_{k=2}^{n/2} \sum_{i=1}^{\lceil \frac{n-k}{k} \rceil - 1} \sum_{j=0}^{b} q(j, i) \cdot \Pr[O_k = j \land PF_k = i] \leq \frac{1}{\nu} \ln(n) + \ln \left( \frac{\nu n - 3/2}{\nu n - 1/2 - \frac{n}{2}} \right) + \frac{1}{\nu} (\ln(\nu/2) - \ln(5\nu/2 - 1)). \quad (6)$$

$$S_3 := \sum_{k=2}^{n-2} \sum_{i=\lceil \frac{n-k}{k} \rceil}^{n-k} \sum_{j=0}^{b} q(j, i) \cdot \Pr[O_k = j \land PF_k = i] \leq \frac{n + 2}{n} \frac{\pi}{\nu} - \frac{1}{\nu^2} \ln(2\nu n) \left( 1 + \frac{4}{\nu n - 2} \right) + \left( 1 + \frac{4}{n} \right) (\ln(n) - 2\ln(n) + 11.5) + \frac{1}{\nu n - 5/2}. \quad (7)$$

Finally we take the remaining case where $O_k \geq \nu n$. There $\mu \in [1, 3]$ and $\kappa = \kappa(n)$ be the aditive constant (depending on $n$) in the upper bound on $\mathbb{E}[F(n)]$ expressed as $5\ln n + \kappa$ given by the induction or Proposition 1.9.

$$S_4 := \sum_{k=2}^{n-2} \sum_{i=\lceil \nu n \rceil}^{n-k} \sum_{j=0}^{b} q(j, i) \cdot \Pr[O_k = j \land PF_k = i] \leq 2 \frac{5 + \frac{\kappa}{\ln(n)}}{\nu^2 n} \ln^{1+\mu}(n) \ln(\nu n) + \nu n \ln(n) e^{-\frac{n\nu^2}{2}} + \left( \nu - \frac{2\ln n}{\nu n} \right) \ln^{\mu - 2}(n) \ln(n). \quad (8)$$
Lemma 4.4 together with the above analysis reformulates Theorem 1.8 as the following inductive theorem. The base case of the induction is computed using the computer analysis formulated as Proposition 1.9. Note that it is sufficient to assume \( n \geq 243 \) for the next theorem as the smaller values follows from Theorem 1.7 via computer-evaluation which is described later in Section 5.

**Theorem 4.5.** Let \( n \geq 22 \) be an integer. For \( 3 \leq m < n \), suppose that \( \mathbb{E}[F(m)] \leq 5 \ln(m) + s_n \) (Note that for a fixed \( m \) we can trade multiplicative and additive constant, so in this way we can fix the multiplicative constant to be always 5). Then we have:

\[
\mathbb{E}[F(n)] \leq S_1 + S_2 + S_3 + S_4
\]

where \( S_1, S_2, S_3, S_4 \) are defined below in Equations (5), (7), (8), (10).

**Organization of the remainder of the section.** First, we carefully compute our estimates and therefore we prove our main result. It remains to prove the bounds (5)–(11) on \( S_1, S_2, S_3, \) and \( S_4 \). In order to do that, we show estimates on first and second moment of random variable \( PF_k \). We follow by Subsections 4.1, 4.2, 4.3, and 4.4 where the bounds (5)–(11) are proven.

**Proof of Theorem 1.8.** Using the formulation of Theorem 4.5 and the estimates in Subsections 4.1, 4.2, 4.3, and 4.4 we conclude by analysis on different values of \( n \). For \( 4157 < n < e^{30} \) we set \( \nu := \frac{n}{10} \) (\( \nu = \frac{n}{10} \)), and \( \mu := 1.25 \). Note that we will maintain \( s_{n < e^{30}} \leq 134 \) throughout the induction. For \( n \geq e^{30} \) we set \( \nu := \frac{1}{2} \) (\( \nu = \frac{1}{2} \)), and \( \mu := 3 \). Note that \( s_{n > e^{30}} \leq 0 \) but we need to use \( s_{n \leq e^{30}} \leq 134 \) for the first step of the induction in this value range, i.e., for \( n = e^{30} \). For \( n \geq e^{1000} \) we set \( \nu := \frac{1}{1000} \) (\( \nu = \frac{1}{1000} \)), and \( \mu := 3 \). Note that \( s_{n > e^{30}} \leq 0 \). The start of the computation follows by Theorem 4.5 together with estimates in Subsections 4.1, 4.2, 4.3, and 4.4. We do some manipulation before we split into cases based on the ranges of \( n \) described above.

\[
\mathbb{E}[F(n)] \leq S_1 + S_2 + S_3 + S_4 \\
\leq \ln(n) + 1 + \gamma \\
\quad + \frac{1}{\nu} \ln(n) + \ln \left( \frac{\nu n - 3/2}{\nu n - 1/2 - \frac{2}{\nu}} \right) + \frac{1}{\nu} \left( \ln(\nu/2) - \ln(5\nu/2 - 1) \right) \\
\quad + \frac{n + 2}{n} \frac{e^6 - 1}{\nu^2} \ln(2n) \left( 1 + \frac{4}{\nu n - 2} \right) + \left( 1 + \frac{4}{\nu n} \right) (\ln(n) - 2\ln(n) + 11.5) + \frac{1}{\nu n - 5/2} \\
\quad + \frac{2(5 + \frac{8}{\ln(n)}) \ln^2(\nu n) \ln(n)}{\nu^2 n} + \frac{\nu n \ln(n) e^{-\frac{\nu^2}{2}}}{\nu n} + \left( \nu - 2 \ln(\nu n) \right) \cdot \frac{5 + \frac{\nu}{\ln^2(n)} \cdot \ln(n)}{\ln(n)}.
\]

We first regroup the the summands.

\[
\mathbb{E}[F(n)] \leq \ln(n) + \ln(n) + \frac{\pi^2/6 - 1}{\nu^2} \ln(n) + \frac{1}{\nu} \ln(n) \\
\quad + 1 + \gamma + \ln \left( \frac{\nu n - 3/2}{\nu n - 1/2 - \frac{2}{\nu}} \right) + \frac{1}{\nu} \left( \ln(\nu/2) - \ln(5\nu/2 - 1) \right) + \frac{\pi^2/6 - 1}{\nu^2} \ln(2\nu) \\
\quad + \frac{1}{\nu n - 5/2} + \frac{\pi^2/6 - 1}{\nu^2} \left( \frac{2}{n} + \frac{4}{\nu n - 2} + \frac{8}{n(\nu n - 2)} \right) \ln(2\nu n)
\]
\[ + \frac{4}{n} (\ln(n) - 2\ln(n) + 11.5) + \nu \ln(n) e^{-\frac{\nu n^2}{2}} \]  
\[ + (11.5 - 2\ln(n)) + \frac{2\left(5 + \frac{k}{\ln(n)}\right)\ln^{1+\mu}(n)\ln(\nu n)}{\nu^2 n} \]  
\[ + \left(\nu - \frac{2\ln^\mu(n)}{\nu n}\right) \cdot \frac{5 + \frac{k}{\ln(n)}}{\ln^{\mu-2}(n)} \cdot \ln(\nu n) \cdot \ln(n). \] (17)

Observe that Equation (12) contributes a logarithmic factor, based on the setup of \( \nu \). Equation (13) contributes a constant which for large values of \( n \) will be hidden within the multiplicative factor. Equations (14), (15) and (16) are decreasing with increasing \( n \). Note that the contribution of (14) is small (<0.13) even for \( n \geq 4158 \). When \( \mu = 3 \), Equation (17) decreases as \( n \) increases. When \( \mu = 1.25 \), Equation (17) is increasing with \( n \). However in this case we will measure its contribution to the sum in terms of multiples of \( \ln(n) \). Although its contribution increases, its coefficient of \( \ln(n) \) is decreasing.

We assume that \( n \geq 4158 \) and use the other variables set up as above.

\[ \mathbb{E}[F(n)] < 6.01 \ln(n) + 15.92 + 0.13 + 12.71 + 48.81 + 6.66 \ln(n) \leq 12.67 \ln(n) + 77.57 < 23 \ln(n). \]

We now assume that \( n \geq e^{30} \).

\[ \mathbb{E}[F(n)] < 3.75 \ln(n) + 9.5 < 5 \ln(n). \]

Finally, assume that \( n \geq e^{16} \).

\[ \mathbb{E}[F(n)] < 3.6453 \ln(n) + 5.44 - 5.82 < 3.65 \ln(n). \]  

Before we dive into case analysis of the upper bound of Inequality (4), we show an important lemma that bounds the first two moments of the random variable \( PF_k \). This will be used in the final computations. Recall that \( T_{n-k} \) represents the expected number of temporary faces in the random embedding of \( V^\uparrow \).

**Lemma 4.6.** Let \( n \geq 3 \) and \( k \leq n - 2 \) be natural numbers. Then

\[ \mathbb{E}[PF_k] \leq \frac{n - k}{k} \]

and

\[ \mathbb{E}[PF_k^2] \leq \frac{(n-k)\left(n + 2 - \frac{3}{k}\right) + 2\mathbb{E}(T_{n-k})}{k^2}. \]

**Proof.** There are precisely \( k(n-k) \) candidate walks \( W_1, W_2, \ldots, W_{k(n-k)} \). We decompose \( PF_k \) into a sum of \( k(n-k) \) indicator random variables \( X_i \), where each \( X_i \) corresponds to the candidate walk \( W_i \) in \( V^\uparrow \):

\[ PF_k = \sum_{i=1}^{k(n-k)} X_i. \]

More precisely, \( X_i = 1 \) if \( W_i \) is a potential face, and 0, otherwise. To determine that, we choose, for each vertex \( v_k \in V^\uparrow \), one of its unpaired darts and pair it with one of the darts incident with \( v_k \). Each possible dart at \( v_k \) is selected uniformly at random (Random choice 1) with probability \( \frac{1}{k} \).

This corresponds to Step 3a in the description of Random process B.

Now, we use the linearity of expectation to bound \( \mathbb{E}[PF_k] \) and \( \mathbb{E}[PF_k^2] \). For that we need to determine the values of \( \mathbb{E}[X_i], \mathbb{E}[X_i^2] \) and \( \mathbb{E}[X_iX_j] \) where \( i \neq j \).
Claim 4.7. For each $i \in [k(n - k)]$, we have

$$\mathbb{E}[X_i^2] = \mathbb{E}[X_i] \leq \frac{1}{k^2}.$$ 

Proof of claim. We just observe that $X_i^2 = X_i$, and that $X_i = 1$ if and only if the first and the last darts of the candidate walk $W_i$ are different and both active. If they are different and incident with the same vertex in $V^\uparrow$, then they cannot be both active; otherwise, each of them is active with probability $\frac{1}{k}$. This implies the claim.

Claim 4.7 gives an immediate conclusion about $\mathbb{E}[PF_k]$.

Two distinct candidate walks are consecutive if one originates with the dart that the other leads to. In other words, last dart of one candidate walk is the first dart of the other candidate walk.

Claim 4.8. Let $W_i$ and $W_j$ be candidate walks, where $i \neq j$. Then

- $\Pr[X_i = X_j = 1] = 0$ if $W_i, W_j$ are the two candidate walks on a 2-open face which is not strongly 2-open face.
- $\Pr[X_i = X_j = 1] \leq \frac{1}{k^2}$ if $W_i, W_j$ are the two candidate walks on a strongly 2-open face.
- $\Pr[X_i = X_j = 1] \leq \frac{1}{k^2}$ if $W_i, W_j$ are consecutive candidate walks not on a strongly 2-open face.
- $\Pr[X_i = X_j = 1] \leq \frac{1}{k^2}$ otherwise.

Proof of claim. If darts of two candidate walks on a 2-open face which is not strongly 2-open cannot both be active, so $X_iX_j = 0$. Suppose that $W_i$ and $W_j$ are the two candidate walks on a strongly 2-open face $f$ and let $d_1, d_2$ be the corresponding downward darts. As $f$ is strongly 2-open $d_1$ and $d_2$ cannot be incident with the same vertex of $V^\uparrow$. Hence, each of them is active with probability $\frac{1}{k}$, so $\Pr[X_i = X_j = 1] = \frac{1}{k^2}$.

Suppose now that $W_i$ and $W_j$ are consecutive candidate walks not on a 2-open face. Then $X_i = X_j = 1$ if and only if both are active. Since each is active with probability $\frac{1}{k}$, we conclude that $\Pr[X_i = X_j = 1] \leq \frac{1}{k^2}$.

In the remaining possibility, $W_i$ and $W_j$ are distinct candidate walks that are not consecutive. If they together involve fewer than 4 downward darts and are not in the cases treated above, then one of them (say $W_i$) involves just one downward dart, in which case $X_i = 0$ and the considered probability is 0. Otherwise, they involve four distinct downward darts, each of which is active with probability $\frac{1}{k}$. This implies that $\Pr[X_i = X_j = 1] \leq \frac{1}{k^2}$. 

Since the bounds in the claim are dependent on whether the walks are in strongly 2-open faces or not, we continue by estimating the number of strongly 2-open faces at a given step. Let $L_k$ denote the number of strongly 2-open faces at the start of step $k$. Let us first consider an upper bound of the expectation of $PF_k^2$ conditional on $L_k = \ell$.

Suppose that there are $\ell$ strongly 2-open faces. There are $k(n - k)$ candidate walks and there are at most $k(n - k)$ pairs of consecutive candidate walks since each pair has a unique downward dart in common. Therefore, there are at most $k(n - k) - 2\ell$ consecutive candidate walks that are not in strongly 2-open faces. Putting these facts together in combination with Claims 4.7 and 4.8 gives the following:
\[
2 \left( \sum_{i=1}^{k} \sum_{j=i+1}^{(n-k)} \Pr[X_i X_j = 1 \mid L_k = \ell] \right) \leq 2 \ell \frac{1}{k^2} + 2(k(n-k) - 2\ell) \frac{1}{k^3} + \left( k^2(n-k)^2 - k(n-k) - 2\ell - 2(k(n-k) - 2\ell) \right) \frac{1}{k^4}
\]

\[
= 2\ell \frac{k^2}{k^2} + \frac{2(n-k)}{k^2} - 4\ell \frac{1}{k^3} + \frac{(n-k)^2}{k^2} + \frac{2\ell}{k^4} - \frac{3(n-k)}{k^3}
\]

\[
\leq 2\ell \frac{k^2}{k^2} + \frac{2(n-k)}{k^2} \left( \frac{(n-k)^2}{k^2} - \frac{3(n-k)}{k^3} \right)
\]

To conclude the proof, we will use linearity of expectation. We will also use \( T_{n-k} \) as an upper bound on the number of strongly 2-open faces.

\[
\mathbb{E}[PF_k^2] = \sum_{i=1}^{k(n-k)} \sum_{j=1}^{k(n-k)} \mathbb{E}[X_i X_j] = \sum_{i=1}^{k(n-k)} \mathbb{E}[X_i^2] + 2 \sum_{i=1}^{k(n-k)} \sum_{j=i+1}^{k(n-k)} \Pr[X_i X_j = 1 \mid L_k = \ell] \Pr[L_k = \ell]
\]

\[
\leq k(n-k) \frac{1}{k^2} + \sum_{\ell} \left( \frac{2\ell}{k^2} + \frac{2(n-k)}{k^2} - \frac{(n-k)^2}{k^2} \right) \Pr[L_k = \ell]
\]

\[
= \frac{(n + 2 - \frac{3}{k})(n-k) + 2\mathbb{E}[L_k]}{k^2}
\]

\[
\leq (n + 2 - \frac{3}{k})(n-k) + 2\mathbb{E}[T_{n-k}]
\]

(18)

The proof of Theorem 1.8 follows by estimates on parts \( S_1 \), \( S_2 \), \( S_3 \), and \( S_4 \) which are given in the following subsections.

### 4.1 Estimate on \( S_1 \) (Equation (5))

We estimate the worst-case scenario for function \( q \) in the case when \( k = 1 \); see (Equation (5)) of Definition 4.1. Note that for the case \( k = 1 \), \( O_k + PF_k = n - 1 \).

\[
S_1 = \sum_{i=1}^{n-1} \sum_{j=0}^{n-1-i} q(j, i) \cdot \Pr[O_1 = j \wedge PF_1 = i] \leq q(0, n-1) \leq H_{n-2} + 1
\]

\[
< \ln(n) + 1 + \gamma.
\]

The last inequality follows from Theorem 1.13 (assuming \( n \geq 3 \)).

### 4.2 Estimate on \( S_2 \) (Equation (7))

First, we show a lemma we will be using in our estimates.
Lemma 4.9. Let \( n \geq 3 \), \( t \geq 1 \), and \( \xi \) be integers such that \( t + \xi \leq n - 2 \). Then
\[
q(\xi, t) = H_{n-\xi-2} - H_{n-\xi-t-2} \leq \ln \left( \frac{n - 3/2 - \xi}{n - 3/2 - \xi - t} \right) < \frac{t}{n - 3/2 - \xi - t}.
\] (20)

Proof. As \( t + \xi < n - 2 \) we use definition of function \( q \) in Equation (2). Note that the same together with \( t \geq 1 \) implies that \( H_{n-\xi-2} > H_0 \) and \( H_{n-\xi-t-2} \geq H_0 \). Hence, Theorem 1.13 yields the following estimate:
\[
H_{n-\xi-2} - H_{n-\xi-t-2} \leq \ln \left( \frac{n - 3/2 - \xi}{n - 3/2 - \xi - t} \right) = \ln \left( 1 + \frac{t}{n - 3/2 - \xi - t} \right)
\]
\[
< \frac{t}{n - 3/2 - \xi - t}.
\]
In the second inequality we used the fact that \( t \geq 1 \) and in the last one that \( \ln(1+x) \leq x \).

Now, we continue by showing that Inequality (7) holds. As \( q(j, i) \) is an increasing function in both \( i \) and \( j \), we can upper-bound it by the value for the largest \( i \) and largest \( j \). Therefore, we can factor it out of the sum and upper-bound the disjoint probabilities by 1. Recall that \( b = \min(n - k - i, \lfloor \nu n \rfloor - 1) \). The first inequality follows by Observation 4.2.

\[
S_2 = \sum_{k=2}^{\lfloor n/2 \rfloor} \sum_{i=1}^{\lfloor n-k \rfloor-1} \sum_{j=0}^{b} q(j, i) \cdot \Pr[O_k = j \land PF_k = i]
\]
\[
\leq \sum_{k=2}^{\lfloor n/2 \rfloor} q \left( \min \left( n - k - \lfloor \frac{n-k}{k} \rfloor, \lfloor \nu n \rfloor - 1, \lfloor \frac{n-k}{k} \rfloor \right) \right)
\]
\[
\leq \sum_{k=2}^{\lfloor n/2 \rfloor} q \left( \lfloor \nu n \rfloor - 1, \lfloor \frac{n-k}{k} \rfloor \right)
\]
\[
\leq \sum_{k=2}^{\lfloor n/2 \rfloor} q \left( \nu n, \frac{n-k}{k} \right)
\]
\[
\leq \sum_{k=2}^{\lfloor n/2 \rfloor} \ln \left( \frac{\nu n - 3/2}{\nu n - 1/2 - \frac{\nu}{k}} \right).
\]
Recall \( \nu \leq 5/11 \), \( k \geq 2 \), and \( n \geq 22 \). For the last inequality we used Lemma 4.9 as
\[
\nu n + \frac{n-k}{k} \leq \frac{5}{11} n + \frac{n-2}{2} \leq n - 2.
\]

Note that
\[
\frac{\nu n - 3/2}{\nu n - 1/2 - \frac{\nu}{k}} \leq \frac{\nu}{\nu - 1/k}
\]
when \( k \geq 3 \) as \( \nu > \frac{1}{2} \). Letting \( a := \ln \left( \frac{\nu n - 3/2}{\nu n - 1/2 - \frac{n}{2}} \right) \), we have:

\[
S_2 \leq \sum_{k=2}^{n/2} \ln \left( \frac{\nu n - 3/2}{\nu n - 1/2 - \frac{n}{2}} \right) \\
\leq \ln \left( \frac{\nu n - 3/2}{\nu n - 1/2 - \frac{n}{2}} \right) + \sum_{k=3}^{n/2} \ln \left( \frac{\nu}{\nu - 1/k} \right) \\
\leq a + \sum_{k=3}^{n/2} \frac{1}{\nu k - 1} = a + \sum_{k=3}^{n/2} \frac{1}{\nu k - 1} \\
\leq a + \int_{5/2}^{n/2+1/2} \frac{1}{\nu x - 1} dx \\
\leq a + \frac{1}{\nu} \int_{5\nu/2 - 1}^{\nu n/2} \frac{1}{z} dz \\
= a + \frac{1}{\nu} (\ln(\nu n/2) - \ln(5\nu/2 - 1)) \\
= \frac{1}{\nu} \ln(n) + \ln \left( \frac{\nu n - 3/2}{\nu n - 1/2 - \frac{n}{2}} \right) + \frac{1}{\nu} (\ln(\nu/2) - \ln(5\nu/2 - 1)).
\]

4.3 Estimate on \( S_3 \) (Equation (8))

In what follows, we will use an auxiliary function \( \tilde{q} \) with only one parameter \( 1 \leq t \leq n - k (\leq n - 2) \) which will be a worst-case upper-bound on the two-parameter function \( q \).

\[
\tilde{q}(t) := \begin{cases} 
\ln \left( \frac{\nu n - 3/2}{\nu n - 3/2 - t} \right), & \text{if } t \leq \nu n - 2, \\
\ln(2t + 1), & \text{if } t \geq \nu n - 2.
\end{cases}
\] (22)

Claim 4.10. Suppose that \( 2 \leq k \leq n - 2 \) and \( 1 \leq i \leq n - k \). Let \( b = b(n, k, i) = \min(n - k - i, \lceil \nu n \rceil - 1) \). Then \( q(b(n, k, i), i) \leq \tilde{q}(i) \).

Proof of claim. If \( n - k - i > \lceil \nu n \rceil - 1 \), then \( b = \lceil \nu n \rceil - 1 \) and \( i < n - k - \lceil \nu n \rceil + 1 \) (since \( k \geq 2 \)). Thus, \( i + b \leq \lceil \nu n \rceil - 2 + \lceil \nu n \rceil - 1 < n - 2 \) and Lemma 4.9 implies that:

\[
q(b, i) \leq \ln \left( \frac{n - 3/2 - \lceil \nu n \rceil + 1}{n - 3/2 - \lceil \nu n \rceil + 1 - i} \right) \leq \ln \left( \frac{\nu n - 3/2}{\nu n - 3/2 - i} \right) = \tilde{q}(i).
\]

Suppose now that \( b = n - k - i (\leq \lceil \nu n \rceil - 1) \). Again, \( i + b \leq n - k \leq n - 2 \) as \( k \geq 2 \). Therefore, Lemma 4.9 applies:

\[
q(b, i) \leq \ln \left( \frac{n - 3/2 - b}{n - 3/2 - b - i} \right) = \ln \left( \frac{i}{k - 3/2} \right) \leq \ln(2i + 1) = \tilde{q}(i).
\] \( \Box \)

Note that in the second case, we use quite a loose upper-bound because we want to keep function \( \tilde{q} \) continuous. It is easy to verify that for \( t = \nu n - 2 \) both expressions used in the definition of \( \tilde{q} \) coincide, so that \( \tilde{q}(\nu n - 2) = \ln(2\nu n - 3) \).
Using the function \( \tilde{q} \) we define new function for \( 1 \leq t \leq n - k \) (\( \leq n - 2 \)):

\[
f(t) := \frac{\tilde{q}(t)}{t^2}.
\]

First, we show that the function \( f \) is convex and few other properties.

**Lemma 4.11.** Let \( n \geq 3 \) and \( 2 \leq k \leq n - 2 \). The function \( f(t) \) is continuous. It is convex on the intervals \([1, \nu n - 2)\) and \([\nu n - 2, n - k]\). Moreover, if \( 1 \leq t < \frac{z_0}{z_0 + 1}(\nu n - 3/2) \), where \( z_0 \approx 2.51286 \) is the non-zero solution of the equation \( z^2 + 1 = e^{z/2} \), or if \( t > \nu n - 2 \), then \( f(t) \) is decreasing, while for \( \frac{z_0}{z_0 + 1}(\nu n - 3/2) < t < \nu n - 2 \) it is increasing.

**Proof.** For what follows, we observe that the function is continuous as it is continuous on two given intervals and the value at \( t = \nu n - 2 \) coincides in both expressions. It is also clear that \( f(t) \) is decreasing for \( t > \nu n - 2 \).

Suppose now that \( t < \nu n - 2 \). For simplicity, we make a linear substitution: \( x := \frac{t}{\nu n - 3/2} \). Now \( f \) is convex if and only the function \( g(x) = x^{-2} \ln(\frac{1}{1-x}) \) is convex for \( x \in \left( \frac{1}{\nu n - 3/2}, \frac{2}{\nu n - 3/2} \right) \). The result follows by examining Taylor series of \( \ln \left( \frac{1}{1-x} \right) = \sum_{j=1}^{\infty} \frac{x^j}{j} \). Now, the result follows since the (infinite) sum of convex functions is convex.

To see where \( f(t) \) is decreasing or increasing, we just need to see where its first derivative is negative. This is a routine task and is left to the reader.

For \( t \geq \nu n - 2 \), we examine the second derivative \( f''(t) \) and prove that it is positive. Again, this is a routine task and is left to the reader. \( \Box \)

Recall that \( b = \min(n - k - i, \lfloor \nu n \rfloor - 1) \). We start with Equation (8):

\[
S_3 = \sum_{k=2}^{n-2} \left( \sum_{i=\left\lceil \frac{n-k}{k} \right\rceil}^{n-k} b \sum_{j=0}^{b} q(j, i) \cdot \Pr[O_k = j \land PF_k = i] \right)
\]

\[
\leq \sum_{k=2}^{n-2} \left( \sum_{i=\left\lceil \frac{n-k}{k} \right\rceil}^{n-k} q(b, i) \cdot \Pr[O_k \leq b \land PF_k = i] \right)
\]

\[
\leq \sum_{k=2}^{n-2} \left( \sum_{i=\left\lceil \frac{n-k}{k} \right\rceil}^{n-k} q(b, i) \cdot \Pr[PF_k = i] \right)
\]

\[
\leq \sum_{k=2}^{n-2} \left( \sum_{i=\left\lceil \frac{n-k}{k} \right\rceil}^{n-k} \tilde{q}(i) \cdot \Pr[PF_k = i] \right).
\]

The last inequality uses the function \( \tilde{q} \) defined in (22) that upper-bounds \( q(b, i) \) by Claim 4.10. We transform it to an equivalent formulation:

\[
S_3 \leq \sum_{k=2}^{n-2} \sum_{i=\left\lceil \frac{n-k}{k} \right\rceil}^{n-k} f(i) \cdot i^2 \Pr[PF_k = i].
\]

By Lemma 4.11, the function \( f \) is convex on the interval \([1, \nu n - 2)\) and is decreasing on the interval \([\nu n - 2, n - k]\). This implies that

\[
M_k := \max \{ f(i) \mid \left\lfloor \frac{n-k}{k} \right\rfloor \leq i \leq n - k \} \leq \max \left\{ f \left( \left\lfloor \frac{n-k}{k} \right\rfloor \right), f(\nu n - 2) \right\}.
\]

(23)
**Lemma 4.12.** Let $2 \leq k < \left\lceil \frac{n}{2} \right\rceil$, $n \geq 22$, and $\frac{6}{11} \leq \nu < 1$.

If $k \leq \frac{\ln(2\nu n - 3)}{88\nu^2}$, then $$M_k \leq \ln(2\nu n) \cdot \nu^{-2}n^{-2}(1 + \frac{4}{\nu n - 4}).$$

If $k \geq \ln(2\nu n)\nu^{-1}(1 + \frac{4}{\nu n - 4})$, then $$M_k \leq \frac{k}{n(n - k)} \cdot \frac{1}{\nu - 1/k - 1/(2n)}.$$ 

For $\frac{\ln(2\nu n - 3)}{88\nu^2} < k < \min\left(\left\lceil \frac{n}{2} \right\rceil, \ln(2\nu n)\nu^{-1}(1 + \frac{4}{\nu n - 4})\right)$ we sum both estimates above to upper-bound $M_k$. For $k \geq \left\lceil \frac{n}{2} \right\rceil$, the maximum in (24) is attained at $f(1)$.

Consider first $k \geq \left\lceil \frac{n}{2} \right\rceil$ (assuming $n \geq 22$) then $$\frac{\ln(2\nu n - 3)}{(\nu n - 2)^2} \leq \frac{1}{\nu n - 3/2} \leq \ln \left(1 + \frac{1}{\nu n - 5/2}\right) = f(1).$$

Therefore, $k < \left\lceil \frac{n}{2} \right\rceil$. We first precompute some estimates on the function $f$.

**Proof of Lemma 4.12**. We begin by bounding a value of the function $f$.

**Claim 4.13.** If $n \geq 8$, then $$f(\nu n - 2) \leq \ln(2\nu n) \cdot \nu^{-2}n^{-2}(1 + \frac{4}{\nu n - 4}).$$

**Proof of claim.**

$$f(\nu n - 2) = \frac{\ln(2\nu n - 3)}{(\nu n - 2)^2} \leq \frac{\ln(2\nu n)}{\nu^2n^2} \cdot \frac{\nu^2n^2}{(\nu n - 2)^2} = \frac{\ln(2\nu n)}{\nu^2n^2} \cdot \left(1 + \frac{4\nu n - 4}{\nu^2n^2 - 4\nu n + 4}\right)$$

$$= \frac{\ln(2\nu n)}{\nu^2n^2} \cdot \left(1 + \frac{4}{\nu n - 4}\right). \quad \diamond$$

**Claim 4.14.** Suppose that $2 \leq k < \left\lceil n/2 \right\rceil$ and $\nu \geq \frac{5}{11}$.

$$f\left(\frac{n-k}{k}\right) \leq \frac{k}{n(n - k)} \cdot \frac{1}{\nu - 1/k - 1/(2n)}.$$ 

**Proof of claim.**

$$f\left(\frac{n-k}{k}\right) = \frac{k^2}{(n-k)^2} \cdot \ln \left(1 + \frac{(n-k)/k}{\nu n - 1/2 - n/k}\right) \leq \frac{k^2}{(n-k)^2} \cdot \frac{(n-k)/k}{\nu n - 1/2 - n/k} = \frac{k}{n(n - k)} \cdot \frac{1}{\nu - 1/k - 1/(2n)} \cdot \quad \diamond$$

Now we will consider two situations:

**Case 1.** $2 \leq k < \frac{\ln(2\nu n - 3)}{88\nu^2}$. Then we will verify that $f\left(\frac{n-k}{k}\right) \leq f(\nu n - 2)$ and by the convexity of $f$ (Lemma 4.11) we conclude that $M_k$ is upper-bounded by Claim 4.13. We conclude by the following computation where, for the first inequality, we use Claim 4.13.
\[
f \left( \frac{n-k}{k} \right) \leq \frac{k}{n(n-k)} \cdot \frac{1}{\nu - 1/k - 1/(2n)} \leq \frac{2k}{n^2 (\nu - 1/k - 1/(2n))} \\
\leq \frac{88k}{n^2} \quad \text{as } k \geq 2 \text{ and } \nu \geq \frac{6}{11} \text{ and } n \geq 22. \\
\leq \frac{\ln(2\nu n - 3)}{\nu^2 n^2} \leq \frac{\ln(2\nu n - 3)}{(\nu n - 2)^2} = f(\nu n - 2).
\]

**Case 2.** \(k \geq \ln(2\nu n)\nu^{-1}(1 + \frac{4}{\nu n - 4})\). Then we will verify that \(f(\nu n - 2) \leq f \left( \frac{n-k}{k} \right) \) and by the convexity of \(f\) (Lemma 4.11) we conclude that \(M_k\) is upper-bounded by Claim 4.14. We conclude by the following computation where, for the first inequality, we use Claim 4.13.

\[
f(\nu n - 2) \leq \ln(2\nu n) \cdot \nu^{-2} n^{-2} (1 + \frac{4}{\nu n - 4}) \leq \frac{k^2}{(n-k)^2} \cdot \left( \frac{(n-k)/k}{(n-k)/k + \nu n - 1/2 - n/k} \right) \\
\leq \frac{k^2}{(n-k)^2} \cdot \ln \left( 1 + \frac{(n-k)/k}{\nu n - 1/2 - n/k} \right) = f \left( \frac{n-k}{k} \right).
\]

For the range between bounds, \(\frac{\ln(2\nu n - 3)}{88\nu^2} < k < \min(\ln(2\nu n)\nu^{-1}(1 + \frac{4}{\nu n - 4}), n/2)\), we will use the sum of the two bounds of Equation (24) (implied by Claims 4.13 and 4.14) as an upper bound on the maximum.

Below we will use the expectations \(E_k^2 := \mathbb{E}(PF_k^2) = \sum_{i=1}^{n-k} i^2 \Pr[PF_k = i]\) and their upper bound established in Lemma 4.6. Now we split the summation in [23], and use above inequalities from Lemma 4.12 to obtain the following:

\[
S_3 \leq \sum_{k=2}^{n-2} M_k \sum_{i=\left\lceil \frac{n-k}{k} \right\rceil}^{n-k} i^2 \Pr[PF_k = i] \\
\leq \sum_{k=2}^{n-2} M_k E_k^2 \\
\leq \ln(2\nu n) \cdot \nu^{-2} n^{-2} (1 + \frac{4}{\nu n - 4}) \sum_{k=2}^{\left\lceil \frac{\ln(2\nu n - 3)}{88\nu^2} \right\rceil} E_k^2 + \\
\sum_{k=\left\lceil \frac{\ln(2\nu n - 3)}{88\nu^2} \right\rceil}^{\left\lfloor \frac{(n-1)/2}{\nu n - 3} \right\rfloor} \frac{k}{n(n-k)} \cdot \frac{1}{\nu - 1/k - 1/(2n)} E_k^2 + f(1) \sum_{k=\lceil n/2 \rceil}^{n-2} E_k^2. \tag{26}
\]

It remains to estimate the following sums in the above estimate:

\[
A := \sum_{k=2}^{\left\lceil \frac{\ln(2\nu n - 3)}{88\nu^2} \right\rceil} E_k^2,
\]
Let us first recall from Lemma 4.6 that $k^2 E_k^2 \leq (n-k)(n+2-\frac{3}{k}) + 2\mathbb{E}(T_{n-k})$. Moreover, by using Theorem 1.7 (as $n \geq 10$), $2\mathbb{E}(T_{n-k}) \leq 2\mathbb{E}(F(n)) \leq H_{n-3}H_{n-2} \leq \frac{2}{k}(n-k) + k(n+2)$ so $k^2 E_k^2 \leq n(n+2)$. Using this, we get:

$$A = \sum_{k=2}^{\left\lfloor \ln(2\nu n - 3) \right\rfloor} \frac{k}{n(n-k)} \cdot \frac{1}{\nu - 1/k - 1/(2n)} E_k^2 \leq n(n+2) \sum_{k=2}^{\left\lfloor \ln(2\nu n - 3) \right\rfloor} \frac{1}{k^2} = n(n+2)(\frac{\pi^2}{6} - 1).$$

Similarly,

$$C = \sum_{k=\left\lceil n/2 \rightceil}^{n-2} \frac{E_k^2}{k^2} \leq n(n+2) \sum_{k=\left\lceil n/2 \rightceil}^{n-2} \frac{1}{k^2} \leq 2(n+2).$$

Next, we estimate $B$. Firstly note that:

$$\frac{1}{\nu - 1/k - 1/(2n)} \leq 1 - \frac{1}{1/k - 1/(2n)} = 1 + \frac{2+k/n}{2k-2-k/n} \leq 1 + \frac{3/2}{2k-5/2}$$

We may assume $k \geq 2$, which gives:

$$\frac{k}{n(n-k)} \cdot \frac{1}{\nu - 1/k - 1/(2n)} E_k^2 \leq \frac{k}{n(n-k)} \left( 1 + \frac{2}{k} \right) \frac{n(n+2)}{k^2} = \frac{(k+2)(n+2)}{(n-k)k^2}$$

Using this we may estimate $B$:

$$B \leq (n+2) \sum_{k=\left\lfloor \ln(2\nu n - 3) \right\rfloor}^{(n-1)/2} \frac{k+2}{k^2(n-k)}$$

$$= \frac{n+2}{n} \sum_{k=\left\lfloor \ln(2\nu n - 3) \right\rfloor}^{(n-1)/2} \left( 1 + \frac{1}{n} \right) \left( \frac{1}{k} + \frac{1}{n-k} \right) + \frac{1}{k^2}$$

$$\leq \frac{n+2}{n^2} \sum_{k=\left\lfloor \ln(2\nu n - 3) \right\rfloor}^{(n-1)/2} \left( 1 + \frac{1}{n} \right) \left( \frac{1}{k} + \frac{2}{n} \left( \frac{\pi^2}{6} - 1 \right) \right)$$

$$< \left( 1 + \frac{4}{n} \right) \left( H_n + 0.65 - 2 \ln(\ln(2\nu n - 3)/88\nu^2) \right)$$

$$< \left( 1 + \frac{4}{n} \right) \left( \ln(n) - 2 \ln \ln(n) + 11.5 \right)$$
Combining all the obtained estimates, we get:

\[ S_3 \leq \ln(2\nu n) \cdot \nu^{-2}n^{-2}(1 + \frac{4}{\nu n - 2}) \cdot A + B + f(1) \cdot C \]

\[ \leq \frac{n + 2}{n} \cdot \frac{\nu^6 - 1}{\nu^2} \cdot \ln(2\nu n) \left( 1 + \frac{4}{\nu n - 2} \right) + \left( 1 + \frac{4}{n} \right)(\ln(n) - 2\ln\ln(n) + 11.5) + \frac{1}{\nu n - 5/2}. \]

\[ S_3 \leq \ln(2\nu n)\nu^{-2}n^{-2}(1 + \frac{6}{n}) \cdot A + \nu^{-1}n^{-1}(1 + \frac{6}{n}) \cdot B + f(1) \cdot C \]

\[ \leq \frac{\pi^2/6 - 1}{\nu^2} \cdot (1 + \frac{9}{n}) \ln(2\nu n) + \frac{1}{\nu}(1 + \frac{9}{n})(\ln(n) - 2\ln\ln(n) + 11.17) + \frac{2}{\nu}(1 + \frac{7}{n}). \]

### 4.4 Estimate on \( S_4 \) (Equation (11))

The last estimate we need is for the value \( S_4 \), which counts what happens when \( O_k \) is large. Let us first recall that

\[ S_4 = \sum_{k=2}^{n-2} \sum_{i=1}^{n-k-i} \sum_{j=|\nu n|} q(j, i) \cdot \Pr[O_k = j \land PF_k = i] = \sum_{k=2}^{n-2} |\nu n|^{-k} \sum_{i=1}^{n-k-i} \sum_{j=|\nu n|} q(j, i) \cdot \Pr[O_k = j \land PF_k = i]. \]

To show the next lemma we will make use of Hoeffding’s Inequality.

**Theorem 4.15** (Hoeffding’s Inequality ([22], Theorem 1)). Let \( X_1, \ldots, X_d \) be independent random variables such that \( 0 \leq X_i \leq 1 \) for each \( i \) and let \( t > 0 \). Then the following holds:

\[ \Pr \left[ \sum_{i=1}^{n} X_i - \mathbb{E} \left[ \sum_{i=1}^{n} X_i \right] \geq nt \right] \leq e^{-2nt^2}. \]

**Lemma 4.16.** Let \( n \geq 4, \mu \in [1, 3] \), and let \( \kappa = \kappa_m \) be an additive constant given by the induction. For \( n > k \geq \frac{2}{\mu} \ln^\mu(n) \), we have:

\[ \Pr \left[ O_k > \frac{n}{\kappa} \right] \leq e^{-\frac{5}{2}n} + \frac{5 + \frac{\kappa}{\ln(n)}}{n \ln^{\mu-1}(n)}. \quad (27) \]

For \( 2 \leq k < \frac{2}{\mu} \ln^\mu(n) \)

\[ \Pr \left[ O_k > \frac{n}{\kappa} \right] \leq \frac{5 \ln(n) + \kappa}{\kappa n}. \]

**Proof.** Now suppose that \( k \geq \frac{2}{\mu} \ln^\mu(n) \). Let \( W_1, \ldots, W_{n-k} \) be indicator random variables where \( W_i \) describes whether vertex \( v_i \) is the first (and also the last) vertex in \( V^\uparrow \) forming a 1-open walk for vertex \( v_{k+i} \), for \( 1 \leq i \leq n - k \). It is readily seen that \( O_k = \sum_{i=1}^{n-k} W_i \). Let \( w_i \) be the number of downward darts incident to vertex \( v_{k+i} \) which form a 1-open face. Since each such dart must be in a different temporary face, we have \( \sum_{i=1}^{n-k} w_i \leq T_{n-k} \). By linearity of expectation

\[ \mathbb{E}[O_k] = \mathbb{E} \left[ \sum_{i=1}^{n-k} W_i \right] = \sum_{i=1}^{n-k} \mathbb{E}[W_i] = \sum_{i=1}^{n-k} w_i / k \leq \frac{T_{n-k}}{k}. \quad (28) \]

The proof follows as the sum of two conditional probabilities. First, we show that

\[ \Pr[O_k > \frac{n}{\kappa} | T_{n-k} \leq n \ln^\mu(n)] \leq \left( e^{\frac{5}{2}} \right)^{-n}. \]
By Inequality (28), \( \mathbb{E}[O_k] \leq \frac{T_{n-k}}{k} \leq \frac{n \ln^\mu(n)}{k} \). We apply Hoeffding’s Inequality (Theorem 4.15) on \( W_i \)'s:

\[
\Pr \left[ O_k > \nu n \mid T_{n-k} \leq n \ln^\mu(n) \right] \leq \Pr \left[ O_k - \mathbb{E}[O_k] \geq (n - k) \frac{\nu nk - n \ln^\mu(n)}{k(n - k)} \mid T_{n-k} \leq n \ln^\mu(n) \right]
\]

\[
\leq e^{-2(n-k) \frac{\nu^2(k-\ln^\mu(n))^2}{(n-k)^2k^2}}
\]

\[
\leq e^{-2n(\nu^2(k-\ln^\mu(n))^2)}.
\]

As \( k \geq \frac{2}{3} \ln^\mu(n) \),

\[
\leq e^{-\frac{2n(\frac{\nu^2}{k})^2}{k^2}} \leq e^{-\frac{\nu^2n}{3}} \leq \left( e^{\frac{\nu^2}{3}} \right)^{-n}.
\]

Second, we use Markov’s inequality with induction to conclude

\[
\Pr[T_{n-k} > n \ln^\mu(n)] \leq \frac{5 \ln(n) + \frac{8}{n \ln^\mu(n)}}{n \ln^\mu(n)} = \frac{5 + \frac{8}{\ln(n)}}{n \ln^\mu(n)}.
\]

The proof of the first part follows by trivial estimates as a sum of both cases. For \( k \leq \frac{2}{3} \ln^\mu(n) \) we use Markov’s inequality with induction to conclude

\[
\Pr[O_k > \nu n] \leq \frac{5 \ln(n) + \frac{8}{\nu n}}{\nu n}.
\]

We show one more useful lemma before concluding the proof.

**Lemma 4.17.** Let \( k \) be an integer satisfying \( \lfloor \nu n \rfloor > k \geq 2 \) and let \( q \) be the function defined in Definition 4.1. Then

\[
q(\lfloor \nu n \rfloor, \lfloor \nu n \rfloor - k) \leq \ln(\nu n) - \lfloor k \geq 3 \rfloor \ln(k - 1.5) + \lfloor k = 2 \rfloor,
\]

where the indicator function \( [\mathcal{P}(k)] \) has value 1 if the property \( \mathcal{P}(k) \) holds, and is 0 otherwise.

**Proof.** As \( k \geq 2 \) we have \( \lfloor \nu n \rfloor + \lfloor \nu n \rfloor - k < n - 1 \), hence, we use Equation (2):

\[
q(\lfloor \nu n \rfloor, \lfloor \nu n \rfloor - k) = H_{\lfloor \nu n \rfloor - 2} - H_{k-2}.
\]

If \( k = 2 \) then \( q(\lfloor \nu n \rfloor, \lfloor \nu n \rfloor - 2) \leq \ln(\nu n) + 1 \) by a trivial estimate. Otherwise, using Theorem 1.13 we conclude:

\[
H_{\lfloor \nu n \rfloor - 2} - H_{k-2} \leq \ln(\lfloor \nu n \rfloor - 1.5) + \frac{1}{24(\lfloor \nu n \rfloor - 2)^2} - \ln(k - 1.5) - \frac{1}{24(k - 1)^2}
\]

\[
\leq \ln(\lfloor \nu n \rfloor - 1.5) - \ln(k - 1.5)
\]

\[\square\]
With Lemma 4.16 and Inequality (5) in hands we estimate sum $S_4$ as follows; see Inequality (10). For different values of $n$, we can use a different values of $\mu \in [1, 3]$ as in Lemma 4.16. For the second equality we use the fact that for any $k \geq \lfloor \nu n \rfloor$ we must have $j \leq n - \lfloor \nu n \rfloor - 1 < \lceil \nu n \rceil$.

\[
S_4 = \sum_{k=2}^{n-1} \sum_{i=1}^{n-k} q(j, i) \cdot \Pr[O_k = j \land PF_k = i] \\
= \sum_{k=2}^{\lfloor \nu n \rfloor - 1} \sum_{i=1}^{n-k} \sum_{j=\lfloor \nu n \rfloor} q(j, i) \cdot \Pr[O_k = j \land PF_k = i] \\
\leq \sum_{k=2}^{\lfloor \nu n \rfloor - 1} q([\nu n], [\nu n] - k) \Pr[O_k \geq \nu n] \\
\leq \sum_{k=2}^{\lfloor \nu n \rfloor - 1} q(\nu n, \nu n - k) \Pr[O_k \geq \nu n] \\
\leq q(\nu n, \nu n - 2) \Pr[O_2 \geq \nu n] + \sum_{k=3}^{\lceil \frac{\ln \mu(n)}{\nu} \rceil - 1} q(\nu n, \nu n - k) \Pr[O_k \geq \nu n] \\
+ \sum_{k=\lceil \frac{\ln \mu(n)}{\nu} \rceil}^{\lfloor \nu n \rfloor - 1} q(\nu n, \nu n - k) \Pr[O_k \geq \nu n]
\]

Using Lemma 4.16 and Lemma 4.17 we conclude.

\[
S_4 \leq (\ln(\nu n) + 1) \frac{5 \ln(n) + \frac{k}{\nu n}}{\nu n} \\
+ \left( \frac{2}{\nu} \ln \mu(n) - 3 \right) \cdot \ln(\nu n) \cdot \frac{5 \ln(n) + \frac{k}{\nu n}}{\nu n} \\
+ \left( \lfloor \nu n \rfloor - 1 - \frac{2}{\nu} \ln \mu(n) + 1 \right) \cdot (\ln(\nu n) - \ln(k - 1.5)) \cdot \left( e^{-\frac{\nu^2}{2}} + \frac{5 + \frac{k}{\nu n}}{\ln \mu(n) - 1(n)} \right) \\
\leq \frac{2}{\nu} \ln \mu(n) - 1 \cdot \ln(\nu n) \cdot \frac{5 \ln(n) + \frac{k}{\nu n}}{\nu n} \\
+ \left( \nu n - \frac{2}{\nu} \ln \mu(n) \right) \cdot \ln(\nu n) \cdot \left( e^{-\frac{\nu^2}{2}} + \frac{5 + \frac{k}{\nu n}}{\ln \mu(n) - 1(n)} \right) \\
\leq 2 \left( \frac{5 + \frac{k}{\ln(n)}}{\nu^2 n} \right) \ln^{1+\mu(n)}(\nu n) + \nu n \ln(n) e^{-\frac{\nu^2}{2}} + \left( \nu - \frac{2 \ln \mu(n)}{\nu n} \right) \cdot \frac{5 + \frac{k}{\ln(n)}}{\ln(n) - 1(n)} \cdot \ln(\nu n).
\]

5 Computer-evaluated estimates for small values of $n$

Proposition 1.9. For $1 \leq n \leq 4157$, $E[F(n)] \leq 5 \ln(n) + 5$.

Up to $n \leq 7$ the exact values are known, see also Table 1b in the introduction. They can be computed by exhaustive enumeration of all possible embeddings. Computing higher values might require additional insight to cut down the size of the search space. Therefore for the computation of
small values of \( n > 7 \), we used a different approach. We provide a simple program in Sage\(^3\) that is used only to numerically compute the exact upper bounds as derived in the preceding proofs, using previously computed values for smaller numbers of vertices.

For \( n \leq 242 \), we use bound provided by Theorem 1.7. In fact, for computer computation, we used a very slightly sharper bound which appears in the proof as Equation (1). In this range of parameters, \( \mathbb{E}[F(n)] < 5 \ln(n) + 5 \); see Figure 1.

For \( 154 \leq n \leq 4157 \), we used partial estimates from the proof in order to minimize the accumulation of overestimation in our analysis. As in the proof of Theorem 4.5 we express \( \mathbb{E}[F] \leq S_1 + S_2 + S_3 + S_4 \). Recall that some estimates use induction and, hence, in such cases, we used computer-calculated upper bounds. That is, we use bounds for \( n' < n \) that we already computed (denoted as \( \beta(n') \)). We now describe what we used in our program to upper bound those quantities. A similar applies to the value of \( 1/2 < \nu < 1 \), which is a split point between the cases. In principle, \( \nu \) can be different for each \( n \). However, to reduce running time, we only considered a couple of values around the value \( \nu \) that have performed the best for \( n - 1 \). For \( S_1 \), we used a simple estimate in Equation (19). For \( S_2 \), we used estimate given by Equation (21). For \( S_3 \), we used estimate given by Equation (25), where \( M_k \) was estimated by Equation (24) and \( E_k^2 = \mathbb{E}[P F_k^2] \) as estimated by Equation (18). For \( S_4 \), we used Equation (31), where \( Pr[O_k \geq \nu n] \) is estimated in Lemma 4.16. There, we do one more optimization. We find a minimum value of sum in Equation (27) by checking all admissible \( x \) in the following equation:

\[
h(x) := e^{-2(n \nu k - x)(n - k)k^2} + \frac{\beta(n - k)}{x}.
\]

---

\(^3\)Available in the sources of our arxiv submission, file Num_bounds.sage. We also provide computed data using this program for \( 7 \leq n \leq 4157 \) in file data.txt.
Observe that in Lemma 4.16 this equation originates in Equation (29) and Equation (30), where the $x$ is fixed to be $n \ln \mu(n)$. As a final upper bound of $\Pr[O_k \geq \nu n]$, we take the smaller from $h(x)$ and $\beta(n-k)\nu n$.

The upper bound given by this part of computation is $5 \ln(n) + 5$ for $243 \leq n \leq 558$ and for $559 \leq n \leq 4157$ even $5 \ln(n)$, see Figure 2 for details. All together, the analysis and computations described in this section prove Proposition 1.9.

6 Lower bound for complete graphs

In this section, we provide a counterpart to Theorem 1.5—a logarithmic lower-bound on the expected number of faces.

**Theorem 1.6.** Let $n \geq 1$ be an integer.

$$\mathbb{E}[F(n)] \geq \frac{1}{2} \ln(n) - 2.$$  

**Proof.** We partition the set of possible (oriented) faces according to their length and we only count those that are easy to count: Let $F_k'$ be the number of faces having $k$ vertices and $k$ edges on their boundary. There are $\frac{1}{k}n(n-1) \cdots (n-k+1)$ possibilities for such a face. Each of them becomes a face of a random embedding with probability $(n-2)^{-k}$. Together, we get (using Bernoulli’s inequality):

$$\mathbb{E}[F'_k] = \frac{1}{k} \prod_{i=0}^{k-1} \frac{n-i}{n-2} \geq \frac{1}{k} \prod_{i=0}^{k-1} \left(1 - \frac{i}{n}\right) \geq \frac{1}{k} \left(1 - \sum_{i=0}^{k-1} \frac{i}{n}\right) \geq \frac{1}{k} \left(1 - \frac{k^2}{2n}\right) = \frac{1}{k} - \frac{k-1}{2n}.$$  

Let $m := \lfloor \sqrt{2n} \rfloor$. Then $F \geq F_3' + F_4' + \cdots + F_m'$, and

$$\mathbb{E}[F] \geq \sum_{k=3}^{m} \mathbb{E}[F'_k] \geq \sum_{k=3}^{m} \left(\frac{1}{k} - \frac{k-1}{2n}\right) = H_m - \frac{3}{2} - \frac{1}{2n}(2 + 3 + \cdots + (m-1)) \geq H_m - 2$$

$$= H_{\lfloor \sqrt{2n}\rfloor} - 2 \geq \ln(\sqrt{2n}) + \left(\ln(\lfloor \sqrt{2n}\rfloor) - \ln(\sqrt{2n})\right) - 2 + \gamma$$

$$\geq \frac{1}{2} \ln(n) + \frac{1}{2} \ln(2) + \ln(1/2) - 2 + \gamma > \frac{1}{2} \ln(n) - 2$$

We have used estimate $H_m \geq \ln(m) + \gamma$ (implied by Theorem 1.13) and $\lfloor \sqrt{2n}\rfloor / \sqrt{2n} \geq 1/2$. \qed
7 The $\Theta(\ln(n))$ bounds for small values of $p$.

For small values of $p$, we first refer to a result of Chmutov and Pittel [11]. The authors consider a random surface obtained by gluing together polygonal disks. Taking the dual embeddings of this problem shows this is equivalent to studying random embeddings of random graphs with a fixed degree sequence, where we allow loops and multiple edges. In this case, a corollary to their main result gives that the expected number of faces is asymptotic to $\ln(n) + O(1)$. Their method of proof uses representation theory. In particular, they use representation theory of the symmetric group and recent character bounds of Larsen and Shalev [30]. We start by giving a combinatorial proof that the expected number of faces in this model is $\Theta(\ln(n))$. We then extend this result to random simple graphs with a fixed degree sequence. This second result is not equivalent to a conjugacy class product in the symmetric group. Therefore standard representation theoretic techniques do not apply, but we can still make use of our combinatorial reformulation.

We do not use a random process. Instead, we count two different things and combine them to give estimates on $E[F]$. Firstly, we count all the different possible faces which could appear in a random embedding on a fixed degree sequence. Then for each possible face, we estimate the number of embeddings that contain it. When we study random multigraphs, these numbers can be estimated directly. When we restrict to simple graphs, we will appeal to a result of Bollobás and McKay [4]. This will help us estimate the fraction of total faces and embeddings which are simple.

7.1 Random multigraphs

In this subsection, we proof the following result.

**Theorem 1.10.** Let $(d_1, d_2, \ldots, d_n)$ be an admissible degree sequence for an $n$-vertex multigraph (possibly with loops) where $2 \leq d_i$ for all $i$. The average number of faces in a random embedding of a random graph with degree sequence $(d_1, d_2, \ldots, d_n)$ is $\Theta(\ln(n))$.

We are interested in random graphs with a fixed degree sequence generated using the configuration model (see [41] for an in-depth description of this model). Fix an arbitrary sequence of integers $d = (\deg(1), \deg(2), \ldots, \deg(n))$ satisfying $2 \leq \deg(1) \leq \deg(2) \leq \cdots \leq \deg(n)$ and $\sum_{i=1}^{n} \deg(i) \equiv 0 \pmod{2}$. Whereas the second condition on $d$ is satisfied by the degree sequence of all graphs, the first condition eliminates vertices that do not affect the number of faces in a random embedding. We also fix the integer $m := \frac{1}{2} \sum_{i=1}^{n} d_i$ corresponding to the number of edges in a graph with degree sequence $d$.

Given a set $D$ of $2m$ darts and a partition $\lambda \vdash 2m$, we write $C_\lambda$ for the conjugacy class in $\text{Sym}(D)$ comprised of all permutations with cycle type $\lambda$. Notice that we can think of $d$ as a partition of $2m$, so that a random map with degree sequence $d$ is defined by a map $M = (D, R, L)$ satisfying $D = \{1, \ldots, 2m\}$, $R \in C_d$, and $L \in C_{2m}$.

Recall that the expected number of faces in a random map with degree sequence $d$ is just the expected number of cycles in a product $R \circ L$ of a pair of random permutations $R \in C_d$ and $L \in C_{2m}$. Because we are picking uniformly from each conjugacy class, the number of cycles in $R \circ L$ does not depend on the particular permutations $R$ and $L$. We may therefore fix $R \in C_d$ while letting $L$ range over all possibilities in $C_{2m}$. Let

$$\mathcal{M}_d := \{(D, R, L) \mid L \in C_{2m}, \{(D, R, L) \mid L \in C_{2m}, R = R_0\}$$

29
denote the set of all maps with degree sequence $d$ where, by the discussion above, we fixed an arbitrary rotation scheme $R_0 \in C_d$. We define the set of possible faces (of maps with rotation scheme $R \in C_d$) to be

$$\Phi_R = \{ f \mid f \text{ is a face of } R \circ L \text{ for some } L \in C_{2m} \}.$$  

When it is clear from the context, we omit the subscript $R$. In what follows, we use two different measures for the size of a face in $\Phi_R$.

**Definition 7.1** (Face length). Given a possible face, $f \in \Phi$, define:

- $l(f)$ is the length of the face in the usual sense (the length of the facial walk defining $f$).
- $u(f)$ is the unique length of the face and is defined as the number of different edges in the face.

For example, suppose a face has length $k$, visits $k - 2$ edges once and visits one edge twice by traveling on either side of this edge. Then this face has unique length $u(f) = k - 1$ as it visits $k - 1$ different edges. We will enumerate faces using their unique length.

The natural setting for our analysis is multigraphs (allowing both loops and multiple edges), as restricting the question to simple graphs means restricting $R$ and $L$ to subsets of their conjugacy classes. Moreover, by allowing parallel edges and loops we get a very simple formula for the number of maps containing a given element of $\Phi$. First, we look what happens when we fix a particular face $f \in \Phi$ with $u(f) = k$.

**Lemma 7.2.** Each face $f \in \Phi$ with $u(f) = k$ appears in $|C_{2m-k}|$ embeddings.

**Proof.** Recall that we have fixed a permutation $R \in C_d$ referring to the rotation schemes of the darts at the vertices. We are therefore counting the number of permutations $L \in C_{2m}$, such that $R \circ L$ contains the given face $f$. In order for $L$ to give face $f$, $k$ different edges of $f$ must all appear in $L$.

Now the key observation here is that the remaining darts can be joined in any way in order to make an embedding containing this face. This means we have free choice for an edge permutation on the remaining darts. Since there are $k$ unique edges in $f$, we, therefore, have free choice for the other $m - k$ edges. There are $|C_{2m-k}|$ possible edge schemes on this number of darts, giving the result. \qed

Notice that the number of embeddings in the above lemma only depends on the unique length of the face and not the structure of the face. This means we can use it to enumerate the total number of faces across all maps in $M_d$ in the following manner: For a permutation $\tau \in S_n$, define $c(\tau)$ as the number of cycles in this permutation. Then the expected number of faces in a random element of $M_d$ (denoted as $E(F_d)$) is given by a simple counting over all possible embeddings:

$$E(F_d) = \frac{1}{|C_{2m}|} \sum_{L \in C_{2m}} c(RL). \quad (33)$$

We define $h_k$ as the number of faces $f \in \Phi_R$ such that $u(f) = k$. Using this notion, we formulate the expected number of faces.
Lemma 7.3. Let \( \mathbf{d} \) be a degree sequence. Let \( E(\mathbf{d}) \) denote the expected number of faces of a random map \( M \in \mathcal{M}_d \). Then:

\[
E(\mathbf{d}) = \sum_{k=1}^{m} \frac{h_k}{(2m-1)(2m-3)(2m-5)\ldots(2m-2k+1)},
\]

where \( m \) denote the number of edges in any \( M \) with degree sequence \( \mathbf{d} \).

Proof. We start with Equation (33), which we can rearrange using \( \Phi_R \) summing over all possible faces \( (f \in \Phi_R) \) instead of \( L \in C_{2m}^2 \):

\[
\frac{1}{|C_{2m}|} \sum_{f \in \Phi_R} |\{L \in C_{2m}^2 \mid f \in R \circ L\}|.
\]

We can then rearrange the previous formula in terms of \( h_k \) using Lemma 7.2 for any possible unique length of \( k \):

\[
\frac{1}{|C_{2m}|} \sum_{k=1}^{m} h_k|C_{2m-k}|.
\]

(34)

Firstly, we calculate the fraction of the two sizes of conjugacy classes. It is straightforward to see that

\[
|C_2| = \frac{(2j)!}{j!2^j}.
\]

Hence we have:

\[
\frac{|C_{2m-k}|}{|C_{2m}|} = \frac{(2m-2k)!}{(2m-1)(2m-3)(2m-5)\ldots(2m-2k+1)}.
\]

Therefore, Equation (34) can be rewritten as follows:

\[
\sum_{k=1}^{m} \frac{h_k}{(2m-1)(2m-3)(2m-5)\ldots(2m-2k+1)}.
\]

We say that a face \( f \) together with one marked dart \( d \in f \) such that \( R^{-1}(d) \in f \) is a rooted face and \( d \) is called its root. Let \( g_k \) denote the number of rooted faces \( f \) of unique length \( k \) such that \( f \in \Phi_R \). We will calculate \( g_k \) then use the following simple relation between \( h_k \) and \( g_k \):

Observation 7.4. Let \( 1 \leq k \leq m \). \( \frac{1}{2k}g_k \leq h_k \leq \frac{1}{2}g_k \).

Proof. Let \( f \) be a face with \( u(f) = k \). Consider an edge \( (d_1, d_2) = e \in f \). If \( e \) appears only once on \( f \) then exactly one of \( R^{-1}(d_1) \) or \( R^{-1}(d_2) \) is in \( f \) and the other is not in \( f \). So only one of them can be the root. If \( e \) appears twice on \( f \) then both \( d_1 \) and \( d_2 \) can serve as the root. Hence,

\[
kh_k \leq g_k \leq 2kh_k.
\]

31
In the following lemmas we show quite tight upper- and lower-bounds on $g_k$ that will be close to $(2m - 1)(2m - 3)(2m - 5) \ldots (2m - 2k + 1)$. We will compute how many options there are to construct a rooted face with $k$ unique edges by fixing $L$ step-by-step. We will look at the darts of one face $f$ of unique length $k$ in the order given by $R \circ L$ starting with the root of $f$ denoted as $d_1$. More precisely, we say that darts $d_1, d_2, \ldots, d_{2k}$ form a rooted unique sequence for some rooted face $f$ with $u(f) = k$ if they are the sequence of darts in order of appearance on $f$ starting with root $d_1$ and $d_2 = L(d_1)$ excluding any repeats (obtained by traversing an edge the second time).

Recall the definition of a partial face from Section 1.4. A part of a rooted unique sequence $d_1, d_2, \ldots, d_k$ for $1 \leq i \leq k < u(f)$ can be viewed as a partial face starting with $R^{-1}(d_1)$ and leading to $d_{2i}$ which is an unpaired dart at the moment. Given a partially constructed $L$ (that is edges defined by $d_1, d_2, \ldots, d_{2i}$), we will define permutation $U$ as a clockwise permutation of the unpaired darts of $f$. We will extend this definition to allow also the paired dart as arguments of $U$. In that case, for a paired dart $d$, $U(d)$ is defined as $U(d')$, where $d'$ is the starting unpaired dart of the partial face we are constructing (that is the one defined with $d_1, d_2, \ldots, d_{2i}$). In other words, for a dart $d_i \in L$ where $i$ is odd, $U(d_i)$ is the first unpaired dart of the walk defined by $R \circ L$ (first applying $L$ on $d_i$) starting with dart $d_i$.

Lemma 7.5 (Upper-bound on $g_k$). We have $g_1 = 2m$. For $2 \leq k \leq m$,

$$g_k \leq 2(2m)(2m - 1)(2m - 3)(2m - 5) \ldots (2m - 2k + 3).$$

Proof. If $k = 1$ then there are $2m$ choices for $d_1$. Then, in order to close a face with $d_2$, we have only one choice.

Now, suppose that $k \geq 2$. There are $2m$ darts in total, and therefore, $2m$ choices for $d_1$. Then there are $2m - 1$ choices for $d_2$. As $R$ is fixed, $d_3$ is determined.

Now suppose that the sequence of darts is currently $d_1, d_2, \ldots, d_{2i}$, i.e., first $2i$ darts of a rooted unique sequence are fixed, for $1 \leq i \leq k - 2$. We follow the facial walk from $d_{2i}$ to $R(d_{2i})$ possibly along any other edges until we reach a dart not equal to $d_1, d_2, \ldots, d_{2i}$. This dart is denoted as $d_{2i+1}$ in the rooted unique sequence. In other words $U(d_i) = d_{2i+2}$ when $L$ consist only of edges defined by $d_1, d_2, \ldots, d_{2i}$. We then have at most $2m - (2i + 1)$ choices for $d_{2i+2}$, as it cannot be any of the previous darts in the facial walk.

After our facial walk passes through $k - 1$ distinct edges, we split into two cases. As before, $d_{2k-1}$ is the first dart on the facial walk not equal to $d_1, d_2, \ldots, d_{2k-2}$. We illustrate these choices in Figure 3.

Case 1: $d_{2k-1} \neq U^{-1}(d_1)$. In this case, there is at most one choice of $d_{2k}$ which closes this facial walk into a face of unique length $k$. This choice is setting $d_{2k}$ as $U^{-1}(d_1)$. Hence, there are $(2m)(2m - 1)(2m - 3)(2m - 5) \ldots (2m - 2k + 3)$ facial walks of length $k - 1$, and then at most one choice to close it.

Observe that in Case 1, the last unique edge $d_{2k-1}, d_{2k}$ always appears on $f$ only once. However, this does not need to be always the case; see Figure 5c for such an example. There, in particular, it is not true that we have at most one choice when choosing the last edge.

Case 2: $d_{2k-1} = U^{-1}(d_1)$. Let $L'$ be the set of edges defined by our choices of $d_1, d_2, \ldots, d_{2k-1}$. A dart $d \notin \{d_1, d_2, \ldots, d_{2k-1}\}$ is called 1-open if $d \neq d_{2k-1}$ is the only unpaired dart on a 1-open temporary face in $R \circ L'$. Observe that by choosing $d_{2k}$ we can close the face if and only if $d_{2k}$ is 1-open dart. We therefore need an upper bound on the number of 1-open darts. Recall that for all $1 \leq i \leq k - 1$, we made the first $i$ choices of edges in the walk uniformly at random. Let $O_i$ be the random variable on this space representing the number of $1$-open darts after $i$ choices are made.
(a) There are 16 total darts, so there are 16 choices for \(d_1\). There are then 15 choices for \(d_2\). Once \(d_2\) is chosen, \(d_3\) must be the next dart in the rotation scheme at that vertex. There are 13 choices for \(d_4\).

(b) There are 11 choices for \(d_6\), and once this is chosen \(d_7\) is determined as shown in the diagram. Now suppose that \(k = 4\). Then there is precisely one choice of \(d_8\) which completes this walk into a face, and that choice is \(R^{-1}(d_1)\).

(c) We give an example of a sequence \(d_1, d_2, \ldots, d_{13}\) for \(k = 7\). In this case, any choice of \(d_{14}\) will close this walk into a face. This is because \(d, d'\) and \(d''\) are exactly the 1-open darts in this partial map.

Figure 3: An illustration of the argument in Lemma 7.5.
Proof of claim. We proceed by induction on \( i \). Initially, we have only one dart in the face and \( O_1 \) is always equal to 0.

Assume that \( E[O_{i-1}] \leq 1 \). When picking dart \( d_{2i} \) in the walk, we have \( 2m - 2i + 1 \) choices. We claim that at most one of them adds a new 1-open face. Namely choosing \( d_{2i} = U^2(d_{2i-1}) \) if it exists. There, \( U(d_{2i-1}) \) is the unpaired 1-open dart. Any other choice will not create 1-open face using dart \( d_{2i-1} \) as, in particular, there will be at least two unpaired darts \( U(d_{2i-1}) \) and \( U^2(d_{2i-1}) \). Also, choosing \( d_{2i} \) as any dart which belongs to a 1-open face, will remove that 1-open face. Therefore there will be \( O_{i-1} \) choices which remove a 1-open face. Putting these facts together gives:

\[
E[O_i] \leq E[O_{i-1}] + \frac{1}{2m - 2i + 1} \left( 1 - \sum_j j Pr[O_{i-1} = j] \right) = E[O_{i-1}] \left( 1 - \frac{1}{2m - 2i + 1} \right) + \frac{1}{2m - 2i + 1} \leq 1.
\]

Putting [Case 1] and [Case 2] we will estimate how many faces we close by a choice of \( d_{2k} \):

\[
1 \cdot Pr[Case 1] + E[O_{k-1} | Case 2] \cdot Pr[Case 2] \leq Pr[Case 1] + E[O_{k-1}] \leq 2.
\] (35)

We concluded the computation above by Claim 7.6. This completes the proof.

\[ \square \]

Note that we were over-counting since the proof above counts walks which visit the last dart early and walks where there is no choice for \( d_{2k} \) that leads to \( d_1 \) using only \( k \) unique edges. We are also over-estimating in Equation (35).

**Lemma 7.7** (Lower-bound on \( g_k \)). \( g_i = 2m \). For \( 2 \leq k \leq m \),

\[
g_k \geq (2m)(2m - 4)(2m - 6)(2m - 8) \ldots (2m - 2k).
\]

We refer to Figures 4 and 5 for an illustration of arguments in Lemma 7.7.

**Proof.** There are \( 2m \) darts in total, and therefore, \( 2m \) choices for \( d_1 \). Let \( d' := R^{-1}(d_1) \) and \( d'' := U^{-1}(d') \) unless \( R^{-1}(d') = d_1 \). In that case, \( d'' := \emptyset \). The intuition is to keep \( d' \) and \( d'' \) reserved, so \( d' \) is available to be picked as \( d_{2k} \) in the rooted unique sequence (which will also be the last edge of the rooted face we are constructing). Moreover, not using \( d'' \) as \( d_i \) for \( i \) even will not force us to use \( d' \) as \( d_j \) for some \( j \) odd. However, we cannot prevent \( d'' \) to be chosen as \( d_i \) for \( i \) odd. In this case, we will redefine \( d'' \) so that \( d' \) is still available to be picked as \( d_{2k} \). To be able to follow the strategy above, we also need to make choices that avoids creation of 1-open darts. Because, using 1-open dart later might force us to use \( d'' \) as \( d_i \) for odd \( i \) and so using \( d' \). So in addition to the above we will forbid \( d'' := U^2(d_{2i-1}) \) which is the only choice that can create 1-open face when paired with \( d_{2i-1} \). Consult Figure 4a for the illustration.

The choices \( d_1, d', d'' \) are not allowed for \( d_2 \). We also disallow the choice of \( R^2(d_1) = U^2(d_1) \), as this will add a 1-open face incident with \( R(d_1) \). We therefore have \( 2m - 4 \) choices for \( d_2 \). As before, \( R \) being fixed means \( d_3 \) is determined. Now suppose that the sequence of darts is currently \( d_1, d_2, \ldots, d_{2i} \), e.i., first \( 2i \) darts of a rooted unique sequence are fixed, for \( 1 \leq i \leq k - 2 \). We also suppose that \( d' \) and \( d'' \) are not among \( d_1, d_2, \ldots, d_{2i} \) and no 1-open darts were created. We follow
Theorem 7.8. with specified constants.

Following computation:

We set and continue. Let \( d(j) \in d \) be the degree of vertex \( j \) incident to dart \( d_{2i+2} \). In case \( \deg(j) = 2 \), we set \( d'' = \emptyset \). In case \( \deg(j) \neq 2 \), we set \( d'' := U^{-1}(d') \). Consult Figures 4b, 4c, and 5a for the illustration.

After our facial walk passes through \( k - 1 \) distinct edges, as before, \( d_{2k-1} \) is the first dart on the facial walk not equal to \( d_1, d_2, \ldots, d_{2k-2} \). Since the choice of \( d'' \) will prevent us from choosing \( d_i \) for \( 1 \leq i \leq 2k - 1 \), we can choose the last edge of the constructed face as \( d_{2k-1}, d' \). Such a choice is always valid and closes the face after exactly \( k \) unique edges were determined.

In the estimate above, besides the obvious loss of not counting \( d'' \), we do not count the option that the last edge appears of the face twice.

We are now ready to put these lemmas together to get the final theorem which is Theorem 7.10 with specified constants.

**Theorem 7.8.** Let \( m \) denote the number of edges in the graph. Then:

\[
\frac{1}{2} (H_m - 1) \leq \frac{E(F_d)}{m} \leq 4H_m + 4.
\]

**Proof.** In both estimates we use Lemma 7.3 as a base for computation of \( E(F_d) \) and Observation 7.4.

For the lower-bound we start of with estimate on \( g_k \) given by Lemma 7.7 and conclude by the following computation:

\[
\frac{1}{2} (H_m - 1) \leq \sum_{k=1}^{m} \frac{1}{2k} \frac{m-k}{m} \leq \sum_{k=1}^{m} \frac{1}{2k} \frac{2m(2m-2k)}{(2m-1)(2m-3)}
\]

\[
\leq \sum_{k=1}^{m} \frac{1}{2k} \frac{(2m)(2m-4)(2m-6)(2m-8) \ldots (2m-2k)}{(2m-1)(2m-3)(2m-5) \ldots (2m-2k+1)}
\]

\[
\leq \sum_{k=1}^{m} \frac{1}{2k} \frac{g_k}{(2m-1)(2m-3)(2m-5) \ldots (2m-2k+1)}
\]

\[
\leq \sum_{k=1}^{m} \frac{h_k}{(2m-1)(2m-3)(2m-5) \ldots (2m-2k+1)} = E(F_d).
\]

For the upper-bound we use the estimate on \( g_k \) given by Lemma 7.5 and we conclude that:

\[
E(F_d) = \sum_{k=1}^{m} \frac{h_k}{(2m-1)(2m-3)(2m-5) \ldots (2m-2k+1)}
\]

\[
\leq \sum_{k=1}^{m} \frac{1}{k} \frac{g_k}{(2m-1)(2m-3)(2m-5) \ldots (2m-2k+1)}
\]

\[
\leq \sum_{k=1}^{m} \frac{1}{k} \frac{2(2m)(2m-1)(2m-3)(2m-5) \ldots (2m-2k+3)}{(2m-1)(2m-3)(2m-5) \ldots (2m-2k+1)} = 2 \sum_{k=1}^{m} \frac{1}{k} \frac{2m}{2m-2k+1}
\]

\[
< 4 + 2 \sum_{k=1}^{m-1} \frac{m}{k(m-k)} = 4 + 2 \sum_{k=1}^{m} \left( \frac{1}{k} + \frac{1}{m-k} \right) \leq 4 + 4H_m.
\]

\( \square \)
(a) From the start of the process, $d'$ and $d''$ are set as shown in the top picture. We disregard the choices to pair $d_i$ with $d'$, $d''$, and $d^o = U^2(d_5)$ (this option would create 1-open dart $U(d_5)$) as described.

(b) Darts $d'$ and $d''$ stays the same until we choose $d_6$ to be $R^{-1}(d'')$. Then $d''$ became $d_7$ and we redefine $d''$ as $\emptyset$.

(c) If $d''$ is $\emptyset$ once we choose $d_8$, we let $d'' = U^{-1}(d_8)$. This happens unless degree of vertex where $d_8$ was chosen has degree two.

Figure 4: An illustration of the argument in Lemma 7.7, (Part I.)
(a) In the case a vertex incident with $d_8$ has degree two then $d''$ stays $\emptyset$. Later on, when $d_{10}$ is chosen, then $d''$ is set as $U^{-1}(d')$. Moreover, observe that $d''$ is a forbidden choice due to another reason: the same dart is also $d^{\prime} = U^2(d_{11})$.

Figure 5: An illustration of the argument in Lemma 7.7 (Part II.).

Theorem 7.8 is a direct analogue of Theorem 1.11 for multigraphs with loops.

Corollary 7.9. Let $G$ be a random multi graph with degree sequence $d$. Then the probability that the number of faces in a random embedding of $G$ is greater than $c(\log(n) + 1)$ is less than $\frac{4}{n}$.

Proof. Observe that picking a random multi graph with degree sequence $d$ then randomly embedding it gives a uniform at random chosen element from $M_d$. Therefore, the result follows from Theorem 7.8 and Markov’s inequality.

7.2 Random simple graphs

In this section we prove the following theorem.

Theorem 1.11. Let $d \geq 3$ be a constant, and let $(d_1, d_2, \ldots, d_n)$ be an admissible degree sequence for an $n$-vertex graph with $2 \leq d_i \leq d$ for all $i$. The average number of faces in a random embedding of a random graph with degree sequence $(d_1, d_2, \ldots, d_n)$ is $\Theta(\ln n)$.

Since Conjecture 1.12 concerns the random graph model $G(n, p)$, we are mostly interested in simple graphs. For larger degree sequences, the majority of random embeddings generated in the model of Chmutov and Pittel will not be simple. Therefore we will be focusing on degree sequences with bounded parts while we allow $n$ to grow to infinity. Given a degree sequence $d = \deg(1), \deg(2), \ldots, \deg(n)$ let $m_d = \frac{1}{2} \sum_i \deg(i)$ and $\lambda_d := \frac{1}{2m_d} \sum_{i=1}^n \binom{\deg(i)}{2}$. We omit the subscript when $d$ is clear from the context. Janson [27] showed that a random multigraph with degree sequence $d$ is asymptotically almost surely not simple unless $\lambda_d = O(1)$. This means, for example, that the probability of a $d$-regular multigraph on $n$ vertices being simple is bounded away from 0 only if $d$ is constant (while $n$ grows arbitrarily).

Restricting our attention to the case where vertex degrees are bounded by an absolute constant, Janson’s result tells us that simple graphs make up a nontrivial fraction of all multigraphs with a given degree sequence. In fact, this special case of Janson’s result was obtained over 30 years earlier by Bender and Canfield [2]. Let us fix some notation to be used throughout this section.

If $\deg(i) \leq d$ for all $i$, we refer to $d$ as a $d$-bounded degree sequence.

As before we may fix a rotation scheme $R \in C_d$, then let $M^s_d$ denote the collection of simple maps with the fixed rotation $R$ (and therefore degree sequence $d$). Let $\Phi^s(R)(k)$ denote the collection
of possible faces of unique length $k$ in $\mathcal{M}_d$. Moreover, let $G(n, d)$ and $G^s(n, d)$ denote, respectively, the collection of multigraphs and the collection of simple graphs on $n$ vertices with degree sequence $d$. Bender and Canfield [2] showed that a random multigraph with degree sequence $d$ is simple with probability $(1 + o(1))e^{-\lambda_d - \lambda_d^2}$. In particular, this tells us that

$$|G^s(n, d)| = (1 + o(1))e^{-\lambda_d - \lambda_d^2}|G(n, d)|.\quad (36)$$

We continue by using a theorem of Bollobás and McKay to determine the number of maps containing a given $f \in \Phi^n_k(k)$. Index the vertices in our model by $\{v_1, v_2, \ldots, v_n\}$ so that vertex $v_i$ has degree $\deg(i)$. We say that $v_iv_j \in E(f)$ if a dart incident to $v_i$ is paired with a dart incident to $v_j$ in the face $f$. For each $f \in \Phi^n$ we define

$$\mu_f := \frac{1}{2m} \sum_{v_i v_j \in E(f)} \deg(i) \deg(j).$$

The following is a special case of Theorem 1 from [4] which we will reformulate as an analog of Lemma 7.2 for simple graphs; see Corollary 7.11 below.

**Theorem 7.10** (Bollobás and McKay [4]). Suppose $d$ is a $d$-bounded degree sequence of length $n$ such that $m = m_d > n$. Let $f$ be a face on degree sequence $f_1, \ldots, f_n$ (i.e., degrees of vertices within the face $f$), and recall the definition of $\mu_f$. Let $\deg(i)' = \deg(i) - f_i$ for $i = 1, \ldots, n$ and let $d' = \deg(1)', \ldots, \deg(n)'$. Then if we pick a map uniformly at random from those in $\mathcal{M}_d$ which contain $f$, the probability that this map is simple is:

$$(1 + o(1))e^{-\lambda_{d'} - \lambda_{d'}^2 - \mu_f}$$

We want to obtain a bound for the number of maps containing a face with unique length $k$, so we give the following simple corollary.

**Corollary 7.11.** Let $f \in \Phi^n_k(k)$, then the number of simple maps with a $d$-bounded degree sequence $d$ containing $f$ is at most $|C_{2m-k}|$ and at least $(1 + o(1))e^{-(\frac{d}{2})^2 - \frac{d^2}{2}}|C_{2m-k}|$.

**Proof.** Let $f$ be a face on degree sequence $f_1, \ldots, f_n$, let $\deg(i)' = \deg(i) - f_i$ for $i = 1, \ldots, n$ and let $d' = \deg(1)', \ldots, \deg(n)'$. The number of (not necessarily simple) maps on degree sequence $d'$ is $|C_{2m-k}|$ for any $d'$. The number of simple maps on this degree sequence is therefore bounded by this also, proving the upper bound.

For the lower bound, by Theorem 7.10 the probability of a map in $\mathcal{M}_d$ being simple is $(1 + o(1))e^{-\lambda_{d'} - \lambda_{d'}^2 - \mu_f}$. Since $d$ is $d$-bounded, we have $\lambda_{d'} \leq \frac{1}{2m} \sum_{i=1}^n \binom{d_i}{2} \leq \binom{d}{2}$. Similarly,

$$\mu_f \leq \frac{1}{2m} \sum_{v_i v_j \in E(f)} d_i^2 = \frac{d^2}{2}. \quad \Box$$

Recall that in the previous section we defined $f_k$ as the number of faces of unique length $k$, and $g_k$ as the number of rooted faces of unique length $k$. We define a simple face as a face which has no loops or multiple edges in it. Then define $h_k^s$ as the number of simple faces of unique length $k$, and $g_k^s$ as the number of rooted simple faces of unique length $k$.  

38
Here there are 18 choices for \(d_1\). There are only 14 choices for \(d_2\). This is because all of the other choices at the starting vertex are disallowed, since our graph is simple and cannot have loops.

In this example there are only 3 choices for \(d_{10}\). The other two darts at the same vertex as \(d_9\) aren’t allowed as the graph cannot have loops. The darts at the top left and bottom right vertices are also disallowed, as we cannot have multiple edges.

Figure 6: An illustration of the proof of Lemma 7.12
Lemma 7.12. Let $2 \leq k \leq m - d^2$. Then:

$$h_k^s \geq \frac{1}{4k} 2m(2m - d^2)(2m - d - 2) \ldots (2m - d^2 - 2k + 4)$$

We follow a very similar proof as in Lemma 7.7, see Figure 6 for an example of the process. The difference is that at each step when picking $d_{2i}$ we also disallow any choices which add a parallel edge or loop.

Proof of Lemma 7.12. We will count rooted simple faces. Then, using Observation 7.4 we will obtain $h_k^s \geq g_k^s/2k$.

There are $2m$ choices for $d_1$. Since we are not allowing any loops in the face, we cannot choose any other darts at the vertex incident with $d_1$. This means that we have at least $2m - d > 2m - d^2$ total choices for $d_2$. As before, $R$ being fixed means $d_3$ is determined. Let $d' := R^{-1}(d_1)$ and $d'' := R^{-1}(d')$.

Now suppose that the sequence of darts is currently $d_1, d_2, \ldots, d_{2i}$. We follow the facial walk from $d_{2i}$ to $R(d_{2i})$ possibly along any other edges until we reach a dart not equal to $d_1, d_2, \ldots, d_{2i}$. This dart is denoted as $d_{2i+1}$ in the rooted unique sequence. There are several different choices of $d_{2i+2}$ which we disallow:

- Any of the choices $d_1, d_2, \ldots, d_{2i+1}$.
- The choices $d', d''$.
- Any choice which adds a loop or multiple edge to the face.
- Any choice which adds a 1-open face.

We upper bound the total number of disallowed choices. Suppose $d_{2i+1}$ is at vertex $v$, then there are at most $d - 1$ other darts present at $v$. Pairing into any unpaired dart at $v$ will create a loop. Pairing into any dart at a vertex $u$ for which there is already an edge between from $v$ to $u$ will add a multiple edge. There are therefore at most $(d - 1)^2$ choices which add a loop or multiple edge. There is at most one choice which adds a 1-open face, by the same reasoning as in the proof of Lemma 7.7. In total, we have at most $(2i + 1) + (d - 1)^2 + 2 + 1 \leq d^2 + 2i$ disallowed choices for $d_{2i+1}$ as $d \geq 2$. Therefore we have at least $2m - d^2 - 2i$ choices for $d_{2i+2}$. Redefine $d'$ (if needed) as in the proof of Lemma 7.7 and continue to the next choice.

We need a little extra analysis for the final step, as we must ensure that when completing the face we do not add a loop or multiple edge. After our facial walk passes through $k - 2$ distinct edges, we have the sequence $d_1, d_2, \ldots, d_{2k-3}$ and must make a choice for $d_{2k-2}$. At this step, we disallow all the $d^2 + 2(k - 1)$ choice from the previous case. We use $v$ to denote the vertex incident with the starting dart $d_1$. When choosing $d_{2k+2}$ we disallow any darts incident with $v$, and any darts incident with vertices $u$ where there is an edge between $u$ and $v$. Therefore we have at least $2m - 2d^2 - 2(k - 2) > \frac{1}{4}(2m - d^2 - 2(k - 2))$ choices at this step. This number is always strictly greater than zero, since we set $k \leq m - d^2$.

Now at the final step, we choose $d_{2k} = d'$. Our disallowed choices mean that this choice is always possible, as we have not yet used $d'$ in the sequence. Also, we chose $d_{2k-2}$ so that the edge $(d_{2k-1}, d_{2k})$ will not add a loop or multiple edge. □
Proof of Theorem 7.11. Select a uniformly random \( M \in \mathcal{M}_d \). For each \( f \in \Phi_k \), let \( X_f \) denote the indicator random variable for the event “\( f \) appears in \( M \).” Using Corollary 7.11 and Equation 36 we get:

\[
\mathbb{E}[F_d] = \sum_f \mathbb{E}[X_f] \\
\leq \sum_{k=1}^m h_k^s \frac{|C_{2m-k}|}{|G(n, d)|} \\
= \frac{1}{(1 + o(1))e^{-\lambda_d - \frac{\lambda_d^2}{2}}} \sum_{k=1}^m h_k^s \frac{|C_{2m-k}|}{|G(n, d)|}
\]

Using the trivial bound \( h_k^s \leq h_k \) and Theorem 7.8 we obtain the upper bound:

\[
\mathbb{E}[F_d] \leq \frac{1}{(1 + o(1))e^{-\lambda_d - \frac{\lambda_d^2}{2}}} \sum_{k=1}^m h_k \frac{|C_{2m-k}|}{|C_{2m}|} \\
= \frac{1}{(1 + o(1))e^{-\lambda_d - \frac{\lambda_d^2}{2}}} \mathbb{E}[F_d] = O(\log(n)).
\]

For the lower bound, recall from Lemma 7.3 that

\[
\frac{|C_{2m-k}|}{G(n, d)} = \frac{1}{(2m-1)(2m-3)(2m-5)\ldots(2m-2k+1)}
\]

. Combining this with Lemma 7.12 we obtain the following for \( k \leq (m-d^2)/2 \):

\[
\frac{4kh_k^s |C_{2m-k}|}{|C_{2m}|} \geq \frac{2m(2m-d^2)(2m-d^2-2)\ldots(2m-d^2-2k+4)}{(2m-1)(2m-3)\ldots(2m-2k+1)} \\
\geq \frac{1}{2m-3} \frac{1}{2m-5} \ldots \frac{1}{2m-d^2+2} \frac{1}{2m-d^2+4} \ldots \frac{1}{2m-2k+3} \ldots \frac{1}{2m-2k+1} \\
\geq \left(1 - \frac{2k-2}{2m-3}\right) \left(1 - \frac{2k-2}{2m-5}\right) \ldots \left(1 - \frac{2k-2}{2m-2k+1}\right) \\
\geq \left(1 - \frac{2k-2}{2m-d^2+2}\right)^{d^2/2} \\
\geq \left(1 - \frac{m-d^2-2}{2m-d^2+4}\right)^{d^2/2} \\
\geq \frac{1}{2}
\]

Putting this together with Corollary 7.11 gives the required result:

\[
\mathbb{E}[F_d] \geq \frac{(1 + o(1))e^{-\lambda_d} - \frac{\lambda_d^2}{2} - \frac{d^2}{2}}{(1 + o(1))e^{-\lambda_d - \frac{\lambda_d^2}{2}}} \sum_{k=1}^m h_k^s |C_{2m-k}| \\
\geq \frac{(1 + o(1))e^{-\lambda_d} - \frac{\lambda_d^2}{2} - \frac{d^2}{2}}{4(1 + o(1))e^{-\lambda_d - \frac{\lambda_d^2}{2}}} \left(\frac{1}{2}\right)^{d^2/2} H_{(m-d^2)/2} = \Omega(\log(n))
\]

\( \square \)
References

[1] George E. Andrews, David M. Jackson, and Terry I. Visentin. A hypergeometric analysis of the genus series for a class of 2-cell embeddings in orientable surfaces. *SIAM J. Math. Anal.*, 25(2):243–255, March 1994. doi:10.1137/s0036141092229549

[2] Edward A. Bender and E. Rodney Canfield. The asymptotic number of labeled graphs with given degree sequences. *J. Combin. Theory Ser. A*, 24(3):296–307, 1978. doi:10.1016/0097-3165(78)90059-6.

[3] Stephan Beyer, Markus Chimani, Ivo Hedtke, and Michal Kotrbčík. A practical method for the minimum genus of a graph: models and experiments. In *Experimental algorithms*, volume 9685 of *Lecture Notes in Comput. Sci.*, pages 75–88. Springer, [Cham], 2016. doi:10.1007/978-3-319-38851-9_6

[4] Béla Bollobás and Brendan D. McKay. The number of matchings in random regular graphs and bipartite graphs. *J. Combin. Theory Ser. B*, 41(1):80–91, 1986. doi:10.1016/0095-8956(86)90029-8.

[5] Jesse Campion Loth, Kevin Halasz, Tomáš Masařík, Bojan Mohar, and Robert Šámal. Random 2-cell embeddings of multistars. *Proc. Amer. Math. Soc.*, 150(9):3699–3713, 2022. doi:10.1090/proc/15899

[6] Jesse Campion Loth and Bojan Mohar. Expected number of faces in a random embedding of any graph is at most linear. *arXiv preprint*, 2022. doi:10.48550/ARXIV.2202.07746

[7] Guillaume Chapuy. A new combinatorial identity for unicellular maps, via a direct bijective approach. *Adv. in Appl. Math.*, 47(4):874–893, 2011. doi:10.1016/j.aam.2011.04.004

[8] Ricky X. F. Chen. Combinatorially refine a Zagier-Stanley result on products of permutations. *Discrete Math.*, 343(8):111912, 5, 2020. doi:10.1016/j.disc.2020.111912

[9] Ricky X. F. Chen and Christian M. Reidys. Plane permutations and applications to a result of Zagier-Stanley and distances of permutations. *SIAM J. Discrete Math.*, 30(3):1660–1684, 2016. doi:10.1137/15M1023646

[10] Sergei Chmutov and Boris Pittel. The genus of a random chord diagram is asymptotically normal. *J. Combin. Theory Ser. A*, 120(1):102–110, 2013. doi:10.1016/j.jcta.2012.07.004

[11] Sergei Chmutov and Boris Pittel. On a surface formed by randomly gluing together polygonal discs. *Adv. in Appl. Math.*, 73:23–42, 2016. doi:10.1016/j.aam.2015.09.016

[12] Robert Cori, Michel Marcus, and Gilles Schaeffer. Odd permutations are nicer than even ones. *European J. Combin.*, 33(7):1467–1478, 2012. doi:10.1016/j.ejc.2012.03.012

[13] Duane W. DeTemple. A quicker convergence to Euler’s constant. *Amer. Math. Monthly*, 100(5):468–470, 1993. doi:10.1080/00029890.1993.11990433

[14] M. J. Grannell and T. S. Griggs. A lower bound for the number of triangular embeddings of some complete graphs and complete regular tripartite graphs. *J. Combin. Theory Ser. B*, 98(4):637–650, 2008. doi:10.1016/j.jctb.2007.10.002
[15] M. J. Grannell and T. S. Griggs. Embedding and designs. In *Topics in topological graph theory*, volume 128 of *Encyclopedia Math. Appl.*, pages 268–288. Cambridge Univ. Press, Cambridge, 2009. doi:10.1017/cbo9781139087223.016

[16] M. J. Grannell and M. Knor. A lower bound for the number of orientable triangular embeddings of some complete graphs. *J. Combin. Theory Ser. B*, 100(2):216–225, 2010. doi:10.1016/j.jctb.2009.08.001

[17] Jonathan L. Gross, Imran F. Khan, Toufik Mansour, and Thomas W. Tucker. Calculating genus polynomials via string operations and matrices. *Ars Math. Contemp.*, 15(2):267–295, 2018. doi:10.26493/1855-3974.939.77d

[18] Jonathan L. Gross and Robert G. Rieper. Local extrema in genus-stratified graphs. *J. Graph Theory*, 15(2):159–171, 1991. doi:10.1002/jgt.3190150205.

[19] Jonathan L. Gross, David P. Robbins, and Thomas W. Tucker. Genus distributions for bouquets of circles. *J. Combin. Theory Ser. B*, 47(3):292–306, 1989. doi:10.1016/0095-8956(89)90030-0.

[20] Jonathan L. Gross and Thomas W. Tucker. Local maxima in graded graphs of embeddings. In *Second International Conference on Combinatorial Mathematics (New York, 1978)*, volume 319 of *Ann. New York Acad. Sci.*, pages 254–257. New York Acad. Sci., New York, 1979.

[21] John Harer and Don Zagier. The Euler characteristic of the moduli space of curves. *Invent. Math.*, 85(3):457–485, 1986. doi:10.1007/BF01390325.

[22] Wassily Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, 58(301):13–30, March 1963. doi:10.1080/01621459.1963.10500830.

[23] David M. Jackson. Counting cycles in permutations by group characters, with an application to a topological problem. *Trans. Amer. Math. Soc.*, 299(2):785–801, 1987. doi:10.2307/2000524.

[24] David M. Jackson. Algebraic and analytic approaches for the genus series for 2-cell embeddings on orientable and nonorientable surfaces. In *Formal Power Series and Algebraic Combinatorics*, pages 115–132, 1994.

[25] David M. Jackson. On an integral representation for the genus series for 2-cell embeddings. *Trans. Amer. Math. Soc.*, 344(2):755–772, February 1994. doi:10.1090/s0002-9947-1994-1236224-5.

[26] André Jacques. Constellations et graphes topologiques. *Combinatorial Theory and Applications*, 2:657–673, 1970.

[27] Svante Janson. The probability that a random multigraph is simple. *Combin. Probab. Comput.*, 18(1-2):205–225, 2009. doi:10.1017/S0963548308009644.

[28] Jin Ho Kwak and Jaeun Lee. Genus polynomials of dipoles. *Kyungpook Math. J.*, 33(1):115–125, 1993. URL: https://www.koreascience.or.kr/article/JAK0199325748114657.page
[29] Sergei K. Lando and Alexander K. Zvonkin. *Graphs on surfaces and their applications*, volume 141 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004. With an appendix by Don B. Zagier, Low-Dimensional Topology, II. doi:10.1007/978-3-540-38361-1

[30] Michael Larsen and Aner Shalev. Characters of symmetric groups: sharp bounds and applications. *Inventiones mathematicae*, 174(3):645–687, 2008. doi:10.1007/s00222-008-0145-7

[31] Nathan Linial and Tahl Nowik. The expected genus of a random chord diagram. *Discrete Comput. Geom.*, 45(1):161–180, 2011. doi:10.1007/s00454-010-9276-x

[32] Clay Mauk and Saul Stahl. Cubic graphs whose average number of regions is small. *Discrete Math.*, 159(1-3):285–290, 1996. doi:10.1016/0012-365X(95)00089-F

[33] Bojan Mohar and Carsten Thomassen. *Graphs on surfaces*. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 2001.

[34] Robert G. Rieper. *The enumeration of graph imbeddings*. PhD thesis, Western Michigan University, Kalamazoo, MI, 1987.

[35] Gerhard Ringel. *Map color theorem*, volume 209. Springer Science & Business Media, 2012. doi:10.1007/978-3-642-65759-7

[36] Peter Schmidt. Algoritmicke vlastnosti vnorení grafov do plôch. Bachelor’s thesis, Univerzita Komenského v Bratislave, Fakulta matematiky, fyziky a informatiky, 2012. URL: https://opac.crzp.sk/?fn=detailBiblioForm&sid=19793BA85A47B7FB6C934D42DA1

[37] Saul Stahl. Permutation-partition pairs: A combinatorial generalization of graph embeddings. *Trans. Amer. Math. Soc.*, 259(1):129–129, January 1980. doi:10.1090/S0002-9947-1980-0561828-2

[38] Saul Stahl. An upper bound for the average number of regions. *J. Combin. Theory Ser. B*, 52(2):219–221, 1991. doi:10.1016/0095-8956(91)90063-P

[39] Saul Stahl. On the average genus of the random graph. *J. Graph Theory*, 20(1):1–18, 1995. doi:10.1002/jgt.3190200102

[40] Richard P. Stanley. Two enumerative results on cycles of permutations. *European J. Combin.*, 32(6):937–943, 2011. doi:10.1016/j.ejc.2011.01.011

[41] Nicholas C. Wormald. Models of random regular graphs. In *Surveys in combinatorics, 1999 (Canterbury)*, volume 267 of *London Math. Soc. Lecture Note Ser.*, pages 239–298. Cambridge Univ. Press, Cambridge, 1999. doi:10.1017/CBO9780511721335.010

[42] Don Zagier. On the distribution of the number of cycles of elements in symmetric groups. *Nieuw Arch. Wisk. (4)*, 13(3):489–495, 1995.

[43] Don Zagier. Applications of the representation theory of finite groups. In *Graphs on surfaces and their applications*, pages 399–427. Springer, 2004. URL: https://people.mpim-bonn.mpg.de/zagier/files/tex/ApplicRepTheoryFiniteGroups/fulltext.pdf