Modelling of non-equilibrium flow in the branched pipeline systems

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Abstract. This article presents a mathematical model and a numerical method for solving the task of water hammer in the branched pipeline system. The task is considered in the one-dimensional non-stationary formulation taking into account the realities such as the change in the diameter of the pipeline and its branches. By comparison with the existing analytic solution it has been shown that the proposed method possesses good accuracy. With the help of the developed model and numerical method the task has been solved concerning the transmission of the compression waves complex in the branching pipeline system when several shut down valves operate. It should be noted that the offered model and method may be easily introduced to a number of other tasks, for example, to describe the flow of blood in the vessels.

1. Introduction

When the valve actuates in the pipeline, as well as when the mode of operation of the transfer pump stops or changes the slowing down or complete stop of the medium movement in the pipeline takes place. In its turn, the breaking of the medium before the closing of valves or in front of the pump leads to the increase in pressure, the formation of the area with higher pressure in the pipeline and the subsequent spread of the area with higher pressure along the pipeline route. This propagation of waves with increasing pressure in pipes is called water hammer [1].

The water hammer is a significant threat to the integrity of the pipelines, as the growth of pressure can lead to rupture of the pipe or the formation of dangerous defects [2].

Thus, the water hammer has served as one of the causes of the accident with rupture of the pipeline at the tank farm "Sheskharis" (Novorossiysk) and pollution of the potaquatorium [3].

The danger of water hammer in pipeline systems is aggravated by the fact that the one-dimensional flows are usually realised in the pipes. Since there is a one-dimensional geometry, the pressure waves, which are propagating through the pipeline, attenuate rather slowly, and this leads to the creation of areas of potential danger for tens of kilometres.

The research of the water hammer has been carried out for decades, starting up with the work of Zhukovsky N.E.[1] and ending with modern monographs and scientific theses [2, 4-7]. It is common to consider water hammer in single line pipeline system. However, the real pipeline systems often have a more complex topology, which includes branching and looping. Branching structure of the pipeline may significantly impact on the development of water hammer.
The setting of the problem about the water hammer in a branching system is extremely essential from the practical point of view. Indeed, the effects of water hammer are the most dangerous in cases of rapid response of the valves. Such rapid (3-5 seconds) actuation of valves and complete shutoff of the flow often occur during shipment at the sea terminals. It is important to quickly cut off the flow, since emissions can get into the aquatic environment, which is fraught with huge ecological damage. The shipment of the large volumes (and at high velocity) often is carried out through several pipes in order to reduce the total time of shipment. As a result, when loading the tankers the occurrence of water hammer is possible in the branched system.

The purpose of this paper is to describe the models and numerical methods designed for modelling the water hammer in a branched pipeline system, as well as to illustrate the opportunities of the formulated approach.

2. Mathematical model

In the isothermal approximation the motion of the fluid in the tube of variable cross section is described by the system of following equations for one-dimensional non-stationary flow:

- Continuity equation:

\[
\frac{\partial (A \cdot \rho)}{\partial t} + \frac{\partial (\rho \cdot A \cdot u)}{\partial x} = 0, \tag{1}
\]

- Equation of momentum conservation:

\[
\frac{\partial (A \cdot \rho \cdot u)}{\partial t} + \frac{\partial (A \cdot \rho \cdot u^2)}{\partial x} = -A \cdot \frac{\partial p}{\partial x} - A \cdot \frac{\lambda(Re)}{2 \cdot D} \cdot \rho \cdot u \cdot |u| - A \cdot g \cdot \rho \cdot \beta, \tag{2}
\]

- Relation between the pressure and density (equation of state):

\[
p - p_0 = c^2 (\rho - \rho_0) \tag{3}
\]

where \( \rho, p, u \) - density, pressure and velocity of the fluid, averaged according to the cross section; \( t \) - time; \( x \) - the distance from the start of the pipeline; \( \lambda(Re) \) - the coefficient of friction as a function of the Reynolds number \( Re = Du / \nu \); \( A \) - cross sectional area of the pipeline, \( D \) - the diameter of the pipeline, \( g \) - acceleration of gravity power; \( \beta \) - sine of the angle of inclination of the route, which is determined by the elevations points of the pipeline \( h(x) \), \( \nu \) - kinematic viscosity; \( \rho_0 \) - density of the fluid at pressure \( p_0 \) and transportation temperature (usually \( p_0 = 10325 \) Pa), \( c \) - the speed of sound in the fluid.

The considered model takes into account the following factors:

- non-stationary processes;
- variable cross-section of the pipeline;
- convective motion of the medium (the second temss are in the left parts (1) and (2);
- the appearance and circulation of the waves at a stop/start up of the pumps, closing valves (first term in the right part (2);
- the presence of friction on the pipe wall (the second term is in the right part (2);
- effect on the flow of gravity when the pipeline’s route goes along the complex terrain (the third term is in the right part (2).

The equations (1) - (3) are closed by initial and boundary conditions. As the boundary conditions the pressure is set on the inlet and outlet of the pipeline, this pressure corresponds to the pressure of the pumps or vessels at the ends of the pipeline. As an initial data the parameters are set for the stationary pumping, which can be obtained analytically from the solution of the system (1) - (3).

When the flow is shutoff by the valve the zero speed is set as a boundary condition.

The system (1) - (2) describes the motion in one linear sector. In the case of a branched pipeline system when the individual linear sectors are combined in a specific sequence, the system of equations (1) - (2) is recorded for each linear sector. The boundary conditions at the connection of two linear sectors are determined by the conditions of continuity of the flow and pressure at the cross-sections of the branches.
sectors are set in such a way that the flows of mass and impulses could be saved from one sector to another.

In this paper to determine the \( \lambda(Re) \) the dependence of Colebrook - White is applied [8], linking friction coefficient \( \lambda \) with Reynolds number \( Re \) and the characteristics of the pipeline (diameter \( D \) and roughness \( k \))

\[
\frac{1}{\sqrt{\lambda(Re)}} = -2\lg \left[ \frac{2.51}{Re \sqrt{\lambda(Re)}} + \frac{k}{3.71 \cdot D} \right]
\]  

(4)

here \( k \) – the size of the pipeline roughness.

Thus, the system of equations (1) - (4) allows a fairly full account of the real characteristics of the equipment functioning in the oil trunk pipeline: pipes, pumps, valves, etc.

3. Numerical Techniques

As a rule, when modelling the flows in pipelines the method of characteristics is applied [1], which is based on the assumption of constancy of the disturbance velocity of propagation in the pipeline. However, where this is no longer a constant speed, the use of the method of characteristics is problematic. In addition, under the common standard implementation the method of characteristics has extremely low accuracy - first-order accuracy. There are some works [8], in which the finite-difference methods are used to calculate the flows in pipelines. In these papers the derivatives in differential equations are replaced by difference analogues. The arithmetic expressions, obtained in such a way, can be more or less easily solved. These methods are more versatile compared with the method of characteristics. However, the difficulty with their practical implementation is that not all properties of the original differential equations are kept in the used arithmetic expression. This is why one needs to comply with complex procedures for providing even a rough match of numerical solutions with the exact ones.

The Godunov type method [9] has a higher versatility in comparison with the method of characteristics. It allows transferring the properties of exact solutions into numerical solutions by a natural way without major complications. Until recently, the major drawback of this method was the relatively long time needed for making calculations. But modern algorithmic and hardware/software solutions have made this method more comprehensible. As for now it is possible to confirm that the calculation time for tasks using the Godunov type method is not a limiting factor, as a rule.

The Godunov type method is offered in [10, 11] for solving the system of equations (1) - (4). The Godunov type method belongs to a class of integral difference methods, in which the main element is the calculation of fluxes (mass, impulse and energy) across the borders of certain elementary volume (difference cells). Summing up all flows along the borders for a certain period of time the values of the corresponding amounts after this interval of time are determined. Actually these fluxes are usually calculated on the basis of the solution of the Riemann problem for the decay of a discontinuity at the boundary of difference cells.

In practice Godunov type method is implemented as follows. The linear sector of the pipeline is divided into cells of size \( \Delta x \), in which the state at any moment of time \( n \Delta t \) (pressure \( p_i^n \), density \( \rho_i^n \), fluid velocity \( u_i^n \), velocity of sound \( c_i^n \), and velocity of perturbations transmission \( c_{w,i}^n \) ) was assumed to be constant in the entire volume of the cell.

One essential point is to be specified in this situation: the point, which is not always clearly perceived. Generally speaking, the flow in the pipeline, even considered in the one-dimensional formulation, is substantially determined by a combination of multi-dimensional effects. Such factors as the operation of the valves, change of the diameter pipelines, branching, and etc. make the dimensiality obvious. At that gaps may even occur in the one-dimensional flow, which are the consequence of the real multi-dimensional effects. However, even simple propagation of the compression wave in the pipeline is accompanied by certain two-dimensional effects. As far back N.E.
Zhukovsky [1] showed that due to the change of the diameter of the pipe because of the pressure change in the pipe, the velocity of disturbance propagation in the system "liquid-thin-walled tube" \( c_w \) changes, and does not coincide with the velocity of disturbance propagation in an infinite volume of fluid (the velocity of sound \( c \)). According to Zhukovsky formula [1, 5] \( c_w \) the velocity of disturbance propagation along the pipeline is determined by the following equation:

\[
\frac{1}{2} \sqrt{\frac{\rho}{K + \frac{\rho D}{E \delta}}} \]

(5)

here \( \delta \) - pipe wall thickness, \( E \) - Young's module of the pipe material, \( K \) - the elastic module of the liquid medium, transported through the pipeline.

Therefore, it is the speed of the waves, which is calculated by the formula (5), is applied in the numerical method used when solving the Riemann problem. It is clear that the use of (5) corresponds to the solution of the Riemann problem in the acoustic approximation, which is quite acceptable for the flows of the liquid medium within the pressure of several tens of atmospheres.

As a matter of fact the numerical method for solving problems consists of two stages, built on the principle of splitting in physical processes. That is, the equations, obtained from the system (1) - (2), are solved consequently, and only some physical processes are described. Primarily the flow is modelled without taking into account the friction and the gravity, and at the stage II the action of friction and gravity is modelled.

**Stage I.** The systems of equations (1) - (3) are solved, in which there are no components that are responsible for the action of viscous forces and the forces of gravity. At this stage the flows of mass and impulses are directed in each \( i \)-cell, and after that there is a new state in the cell. The flows are calculated with the help of solving the problem of the decay of discontinuity. As the liquid, in specified circumstances, can be considered as little compressible, the solution of the problem of decay of a discontinuity at the boundary of two cells is allowed in a sound approximation. Writing down the equations for waves on the right and left borders, we’ll get the following system for finding the parameters of cells on the border of the cells:

\[
\begin{align*}
&p_r - p_{r+1} = \rho^n_{r+1} \cdot c_{w+1} \cdot (u_r - u_{r+1}) \\
&p_r - p_n = -\rho^n \cdot c_w \cdot (u_r - u_n)
\end{align*}
\]

(6)

\[
\begin{align*}
&p_l - p_{l+1} = \rho^n_{l+1} \cdot c_{w+1} \cdot (u_l - u_{l+1}) \\
&p_{l-1} - p^n_l = -\rho^n_{l-1} \cdot c_{w-1} \cdot (u_{l-1} - u^n_l)
\end{align*}
\]

(7)

where the index "r" means belonging to the right parameters and the index "l" - to the left border of the \( i \)-cell.

The systems of equations (6) and (7) uniquely determine the flow parameters on the left and right boundaries of the \( i \)-cell of the \( n \)-time step:

\[
\begin{align*}
&u_r = \frac{p_r - p_{r+1} + \rho^n_{r+1} \cdot c_{w+1} \cdot u_{r+1} + \rho^n \cdot c_w \cdot u_r}{\rho^n_{r+1} \cdot c_{w+1} + \rho^n \cdot c_w} \\
&p_r = \frac{p_{r+1} + \rho^n_{r+1} \cdot c_{w+1} \cdot u_{r+1} - p_r - \rho^n_{r+1} \cdot c_{w+1} \cdot u_{r+1}}{\rho^n_{r+1} \cdot c_{w+1} + \rho^n \cdot c_w}
\end{align*}
\]

(8)
\[ u_i = \frac{p_{i-1} - p_i^n + \rho_i^n \cdot c_{w_i}^n \cdot u_i^n + \rho_{i-1} \cdot c_{w_{i-1}} \cdot u_{i-1}}{\rho_i^n \cdot c_{w_i}^n + \rho_{i-1} \cdot c_{w_{i-1}}}, \]
\[ p_i = p_i^n + \rho_i^n \cdot c_{w_i}^n \cdot \left( \frac{p_{i-1}^n - p_i^n + \rho_{i-1} \cdot c_{w_{i-1}} \cdot (u_i^n - u_{i-1})}{\rho_i^n \cdot c_{w_i}^n + \rho_{i-1} \cdot c_{w_{i-1}}} \right) \]

Using the obtained according to (8) - (9) the pressure values \( p_i \) and \( p_r \) to (3) are values of densities \( \rho_i \) and \( \rho_r \) are calculated at the boundaries of the cell. Knowing \( p_i, \rho_i, u_i \) and \( p_r, \rho_r, u_r \), one can obtain the total flows of mass \( \Phi_{mi} \) and impulse \( \Phi_{pi} \) in each cell:

\[ \Phi_{mi} = u_i \cdot \rho_i - u_r \cdot \rho_r, \quad \Phi_{pi} = u_i^2 \cdot \rho_i - p_i + u_r^2 \cdot \rho_r - p_r. \]

The final values of the parameters in \( i \)-cell at \( n+1 \) time step are calculated with the following formula:

\[ \rho_i^{n+1} = \frac{m_i^{n+1}}{\Delta x} = \frac{m_i^n + \Phi_{mi} \cdot \Delta t}{\Delta x}, \quad u_i^{n+1} = \frac{R_i^{n+1}}{m_i^{n+1}} = \frac{R_i^n + \Phi_{pi} \cdot \Delta t}{m_i^n}, \]

where \( m_i^n \) and \( m_i^{n+1} \) - masses, \( R_i^n \) and \( R_i^{n+1} \) - impulses in \( i \)-cell at \( n+1 \) and \( n \) time steps. The pressure in \( i \)-cell at \( n+1 \) time step is determined according to the correlation (3).

**Stage II.** The forces of friction and gravity are taken into account. The velocity in the \( i \)-cell is recalculated according to the formula

\[ u_i^{n+1} = u_i^{n+1} - \lambda \left( Re_i^{n+1} \right) \cdot \frac{|u_i|^n \cdot |u_i^{n+1}|}{2D} \cdot \Delta t - g \beta \cdot \Delta t \]

It should be noted that this article considers the isothermal flow of poorly compressible fluid, but the approach of the Godunov type method can be generalized to the cases of multiphase flows with heat exchange, as well as of more complicated equations of the medium state. The described option of the numerical calculation method is characterized by the first order of accuracy; however, the accuracy can be improved easily. Finally, the offered method is easily extended to non-regular grid.

### 4. Test calculations and comparison with exact solutions

The proposed method has been implemented as a computer program for a personal computer. When using any numerical method, a natural question arises - what is the accuracy of the obtained decisions? One of the ways to assess the quality of the solution of any given numerical method is the comparison with the solutions of the problems that have analytical solution. To check the accuracy of the obtained numerical solutions and the applicability of this approach several problems have been solved that have the analytical solution:

- about stationary motion in a horizontal pipe with friction against the walls;
- about stationary motion in the pipe without wall friction but with altitude difference on the track;
- about filling the tube with liquid when one end is closed.

The numerical solutions received for these test problems coincide with a high degree of accuracy with the analytical solutions.

For branching systems there are few problems that have analytical solutions. One of such problem is described in [6]. In [6] the exact solution is presented for the flow near the branching when the wave of water hammer comes out from the pipe in the diverge of two pipes; in one of these pipes the medium moves, and in the other – does not. The friction against the pipe wall is not taken into account in this decision. The scheme of this problem according to [6] is shown in figure 1.
Figure 1. The scheme of water hammer in the branching system where in one of the pipes (bottom bend) the medium does not move initially.

The flow in the pipe with the velocity $u_0$ of the medium with the density $\rho_0$ is considered. The diameter of this pipe is $d_1$ (at Fig. 1 this pipe is represented horizontally). The pipe has a divergence with the diameter $d_2$ (Fig. 1 shows this pipe downward). The medium in the divergence is calm, its density is $\rho_0$. At some point in time right behind the divergence the valve operates, which immediately cuts off the flow. As a result, the flow on the valve stops; and two shock waves propagate along the main pipe and the divergence.

According to [6] two waves are formed propagating in the two pipes. Pressure $p^*$ in the main pipe and the branching pipeline, which occurs under such setting of the problem, is described by the following equation for the pressure increase:

$$
\Delta p = p^* - p_0 = \rho_0 u_0 c \frac{d_1^2}{d_1^2 + d_2^2}
$$

The results of solving this problem, obtained numerically using the described above method, are shown in Table 1. Table 1 shows the calculated $\Delta p_{\text{calc}}$, and defined by the formula (13) $\Delta p_{\text{exact}}$ of the pressure increase. The calculations were performed for the medium with a density $\rho_0 = 840$ kg/m$^3$ and the velocity of propagation of perturbations $c = 1300$ m/s.

| №  | $u_0$, m/s | $d_1$, m | $d_2$, m | $\Delta p_{\text{calc}}$, kPa | $\Delta p_{\text{exact}}$, kPa | Error, % |
|----|------------|----------|----------|-------------------------------|-------------------------------|----------|
| 1  | 0.245      | 1        | 0.7071   | 181.21                        | 178.36                        | 1.59     |
| 2  | 0.5        | 1        | 1        | 271.91                        | 273.00                        | 0.40     |
| 3  | 1          | 1        | 1.4142   | 359.7                         | 364.00                        | 1.19     |

As it is seen from the Table 1 the error of calculation makes approximately 1%, and this may be referred to as pretty low inaccuracy.

5. Calculations of water hammer: one valve actuation

Let us consider the following model configuration of the branching pipeline (see figure 2). From the beginning of the pipeline (point A) to the branching point (point O) the pipe with the length of 20 km has a diameter of 1 m, at the point O the pipeline is diverged into two identical pipes with the length of 20 km and the diameter of $1/(2)^{0.5}$ m each, i.e., the cross-sectional area of the two divergences equals to the cross-sectional area of the main pipe. All pipes have the same roughness - 0.3 mm, and are laid on a flat terrain. At the beginning of the pipe has the pressure of 10 atmospheres. The pipe-end branches have the pressure of 3 atm., and the slide valves are installed, which completely stop the flow within 10 seconds.
The following scenario of the valves shutdown is considered. Initially the valve number 2 operates, and then, after a while, the valve №1 operates. When making calculations the time count down starts from the beginning of the valve №2 closing.

It was assumed when making calculation that the speed of disturbance propagation coincides with the speed of sound and is 1300 m/s.

The figure 3 shows the pressure profiles at different time points in the case where only the valve №2 operates (the response time of the valve №1 is referred to the infinity). At this figure the main pipe corresponds to the section from the beginning (point A) to the valve №1, the divergence in this figure corresponds to the section from the point O to the valve №2.

Figure 2. The scheme of branching pipeline system.

Figure 3. Pressure profiles in the pipeline at time points of 10, 20 seconds (a) and 200, 400, 600, 1400 seconds (b) after operation of the valve №2 at the end of the divergence.
From the figure 3 a) it is clearly seen as the wave formed in the divergence (see. Fig. 3 a), 10) goes into the main pipe (see Fig. 3 a), 20). The Figure 3 b) shows perfectly well how after the circulation of the waves in the system the pressure gradually increases in all parts of the pipeline. Most significant increase of the pressure comes on the section from the point O to the valve №2, where the flow stops completely. In the other divergence (from the point O to the valve №1) and the main pipe (from the beginning to the point O) the growth of the pressure is less significant by the relative value.

6. Conclusion
The paper proposes to apply the Godunov type method for the calculation of water hammer in branched pipeline systems. The possibility to disseminate this method among the branched pipeline systems has been successfully demonstrated.
The example that has an analytic solution of the problem of water hammer in the diverging of the pipeline system shows high accuracy of the numerical method used.
The example of water hammer in branched pipeline system was presented. This example demonstrates the good quality of solution using the proposed numerical method.

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