Local Scale Invariant Kaluza-Klein Reduction

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We perform the 4-dimensional Kaluza-Klein (KK) reduction of the 5-dimensional locally scale invariant Weyl-Dirac gravity. While compactification unavoidably introduces an explicit length scale into the theory, it does in such a way that the KK radius can be integrated out from the low energy regime, leaving the KK vacuum to still enjoy local scale invariance at the classical level. Imitating a $U(1) \times U(1)$ gauge theory, the emerging 4D theory is characterized by a kinetic Maxwell-Weyl mixing whose diagonalization procedure is carried out in detail. In particular, we identify the unique linear combination which defines the 4D Weyl vector, and fully classify the 4D scalar sector. The later consists of (using Weyl language) a co-scalar and two in-scalars. The analysis is performed for a general KK $m$-ansatz, parametrized by the power $m$ of the scalar field which factorizes the 4D metric. The no-ghost requirement, for example, is met provided $-\frac{1}{2} \leq m \leq 0$. An $m$-dependent dictionary is then established between the original 5D Brans-Dicke parameter $\omega_5$ and the resulting 4D $\omega_4$. The critical $\omega_5 = -\frac{1}{2}$ is consistently mapped into critical $\omega_4 = -\frac{3}{2}$. The KK reduced Maxwell-Weyl kinetic mixing cannot be scaled away as it is mediated by a 4D in-scalar (residing within the 5D Weyl vector). The mixing is explicitly demonstrated within the Einstein frame for the special physically motivated choice of $m = -\frac{1}{4}$. For instance, a super critical Brans-Dicke parameter induces a tiny positive contribution to the original (if introduced via the 5-dimensional scalar potential) cosmological constant. Finally, some no-scale quantum cosmological aspects are studied at the universal mini-superspace level.

INTRODUCTION

The idea of local scale invariance theories, which is over 100 years old, has been studied in many physical contexts [1,4]. Lately it has, once again, began to be the subject of intensive debate [5-10]. Only a few years after the introduction of general relativity, Weyl attempted the unification of electromagnetism and gravity by the introduction of general relativity, Weyl also discovered the so-called Weyl tensor which, under the transformation (1), has the transformation rule

$$g_{\mu\nu} \rightarrow e^{2\Omega(x)} g_{\mu\nu}; \quad K_{\mu}(x) \rightarrow K_{\mu}(x) - \partial_{\mu} \Omega(x),$$

(1)

with gravitation and electromagnetism thus unified by sharing a common $\Omega(x)$. In the course of developing his theory Weyl also discovered the so-called Weyl tensor which, under the transformation (1), has the transformation rule

$$C_{\mu\nu\sigma}^{\lambda} \rightarrow C_{\mu\nu\sigma}^{\lambda}$$

(2)

where all derivatives of $\Omega(x)$ drop out identically. The great appeal of this conformal symmetry is that its imposition actually leads to a unique choice of gravitational action, namely the Weyl action

$$S_W = -\zeta g \int C^{\lambda\mu\nu\sigma} C_{\lambda\mu\nu\sigma} \sqrt{-g} dx$$

(3)

where $\zeta_g$ has to be dimensionless. Thus conformal gravity possess a dimensionless coupling constant and forbids a presence of any fundamental cosmological term, which will arise from the symmetry breaking of the theory. The field equations of the theory yield a Schwarzschild-like metric with the addition of a linear term, which is suggested to be a solution to the galactic rotation curve problem [11-14]. Moreover it was suggested that by adding this term to a 'general' Standard Model the theory becomes "renormalizable" [15,16].

Once scalar-tensor theories enter the game $S_W$ is no longer unique. The simplest type of scalar-tensor theory is the Brans-Dicke theory [17] described by the action

$$S_{BD} = \int \left( \phi^2 R - 4\omega g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \right) \sqrt{-g} dx.$$  

(4)

In addition to fully enjoying a global scale invariance ($\Omega$ in this case is constant) i.e.

$$g_{\mu\nu} \rightarrow e^{2\Omega} g_{\mu\nu}; \quad \phi \rightarrow e^{-\Omega} \phi$$

(5)

this action may also enjoy a local scale symmetry under the transformation (1). This is only possible for the critical $\omega_c = -\frac{3}{2}$ case, where all $\Omega(x)$ derivatives drop out identically. Unfortunately this poses an issue as this critical case yields a kinetic "ghost" term which is nonphysical. Many of the recent local scale invariant theories involve the combining of a Brans-Dicke type Lagrangian with some standard model Lagrangian [18-22]. Scale-invariant theories are attracting increased interest due the strong physical motivation [23,24] found in low energy particle physics. One example being the classical action of the standard model if the Higgs mass term is dropped. This invited the idea that the mass term may emerge from the vacuum expectation value of an additional scalar field [26].

All the above make it apparent that local scale symmetry should be a fundamental symmetry in nature. Therefore
we will show how it is possible to build a local scale symmetric theory with an arbitrary Brans-Dicke $\omega$ by using the Weyl-Dirac action for an arbitrary dimension. Starting with a 5-dimensional Weyl-Dirac action we make a Kaluza-Klein type reduction \cite{27} to 4-dimensions, and the resulting action is a local scale invariant theory which includes two scale-less scalar fields and one field with the proper length units. At the mini-superspace level the no-scale $C^2$ conformal cosmology is empty, this encourages the use of the Weyl-Dirac cosmology and two scalar gravity-anti-gravity cosmology \cite{28,31}.

The latter local symmetry is translated into an additional constraint (on top of the Hamiltonian constraint). This allows the associated no-scale wave function of the Universe to solely depend on the scale-less fields. Near the Big Bang the wave function behavior is then governed by a scale-less scalar field \cite{32} which is a remanent of the 5D local scale symmetry.

**N-DIM LOCAL SCALE SYMMETRIC GRAVITY**

Our starting point is an n-dimensions Brans-Dicke-like action,

$$S = \int (\phi^2 R - 4\omega g^{\mu\nu} \phi_{,\mu}\phi_{,\nu}) \sqrt{-g} d^n x . \quad (6)$$

Where $R$ is the N-dim Ricci scalar. As is the case in 4D, this n-dimensional action fully enjoys a global scale invariance. Additionally for the critical case, with the problematic "ghost" term, this action enjoys a local scale invariance. However it is possible to overcome the issue of the "ghost" term. By utilizing Weyl’s geometry, it is possible to achieve local scale invariance even for an arbitrary $\omega$. The basic idea is to convert all tensors into co-tensors (denoted by the "star" notation) by supplementing them with the Weyl vector field $K_{\mu}$, with the transformation law

$$K_{\mu}(x) \rightarrow K_{\mu}(x) - \partial_{\mu} \Omega(x), \quad (7)$$

and its divergence. A co-tensor has the transformation law

$$Y^* \rightarrow e^{p\Omega} Y^* , \quad (8)$$

where $p$ is called the power of the co-tensor. We will denote square brackets as the power of the co-tensors, i.e., $[Y^*] = p$. In the case of $[Y^*] = 0$ we say that the co-tensor is an in-tensor. In n-dimensions the powers of the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ are

$$[g_{\mu\nu}] = 2 \iff [g^{\mu\nu}] = -2 . \quad (9)$$

Therefore the power of $\sqrt{-g}$ is

$$[\sqrt{-g}] = n . \quad (10)$$

Turning to our action \cite{6}, both curvature and the covariant scalar field derivative must be upgraded to their star counterparts, making the action scale invariant. Following Dirac and Weyl we write,

$$R^*_{\mu\nu} = R_{\mu\nu} + \frac{n - 2}{2} (K_{\mu\nu}K_{\nu\mu} + 2K_{\mu}K_{\nu}) + g_{\mu\nu}(K^\mu - (n - 2)K^n K_\mu) \quad (11)$$

$$R^* = g^{\mu\nu} R^*_{\mu\nu} = R + 2(n - 1)K^{\mu}_{\mu} - (n - 1)(n - 2)K^n K_\mu . \quad (12)$$

$R^*_{\mu\nu}$ is the Ricci co-tensor which is in an in-tensor. $R^*$ is the Ricci co-scalar. This means that for our action to be an in-scalar there is a need for a scalar field to be coupled to the starred curvature. The powers of the Ricci co-tensor and the scalar field are

$$[R^*] = -2 \quad [\phi] = (2 - n)/2 . \quad (13)$$

We summarize the transformation laws for all the above,

$$R^* \rightarrow e^{-2\Omega} R^*, \sqrt{-g} \rightarrow e^{\Omega} \sqrt{-g}, \phi \rightarrow e^{\frac{2-2\omega}{2}} \Omega \phi \quad (14)$$

$$\Rightarrow \phi^2 R^* \sqrt{-g} \rightarrow \phi^2 R \sqrt{-g} \quad (15)$$

We now turn our attention to the scalar field kinetic term. The co-covariant derivative is defined as

$$\phi_{*\mu} = \phi_{,\mu} - \frac{n - 2}{2} K_{\mu} \phi \quad (16)$$

with the transformation law

$$\phi_{*\mu} \rightarrow e^{\frac{2-2\omega}{2}} \Omega \phi_{*\mu} \quad (17)$$

similar to the scalar field. This means that the kinetic term is of the power, $[g^{\mu\nu} \phi_{*\mu}\phi_{*\nu}] = -n$ (Remember: $[\sqrt{-g}] = n$). We can now write the local scale invariant n-dimensional Brans-Dicke like action

$$S = \int (\phi^2 R^* - 4\omega g^{\mu\nu} \phi_{*\mu}\phi_{*\nu}) \sqrt{-g} d^n x . \quad (18)$$

A mandatory ingredient is a kinetic term for the Weyl vector field. Although, it is not directly required on plain local scale symmetry grounds, in its absence $K_{\mu}$ will stay non-dynamical in nature. The transformation law \cite{1} dictates the exact Maxwell structure, with the corresponding anti-symmetric differential 2-form given by

$$K_{\mu\nu} = K_{\mu\nu} - K_{\nu\mu} . \quad (19)$$

It is a simple exercise to show that the Weyl vector field strength tensor power is $[K_{\mu\nu}] = 0$. However the kinetic term’s power is $[K^{\mu\nu} K_{\mu\nu}] = -4$ and as such must be coupled to the scalar field with the appropriate power of $p = (8 - 2n)/(2 - n)$. Finally we can follow the Dirac
Notice the interesting choice of $\omega$ Brans-Dicke action.

For $n=4$ we find that $\omega_4 = -\frac{3}{2}$. This is the known $\omega_{BD} = -\frac{3}{2}$ which provides the local scale invariant Brans-Dicke action.

**5-DIMENSIONAL WEYL-DIRAC THEORY**

In five dimensions the Weyl-Dirac prescription reads,

$$S = \int \sqrt{-g} d^5 x \left( \phi^2 \hat{R}^* - 4\omega g^{\mu\nu} \phi_{*\mu} \phi_{*\nu} \right) - \frac{1}{4} \phi \hat{\nabla}^2 \phi. \tag{20}$$

We explicitly write the starred part of the Lagrangian,

$$\phi^2 R^* - 4\omega g^{\mu\nu} \phi_{*\mu} \phi_{*\nu} = \phi^2 R - 4\omega g^{\mu\nu} \phi_{*\mu} \phi_{*\nu} + (2-n) \omega (K^\mu \phi^2)_{*\mu} + (n-1) + (n-2) \omega (2K^\mu - (n-2)K K^\mu) \phi^2 \tag{21}$$

Notice the interesting choice of $\omega = \frac{1-n}{n-2}$. This yields, up to a total derivative, a local scale invariant action without the need for Weyl’s vector $K_{\mu}$, i.e.,

$$\phi^2 R^* - 4\omega g^{\mu\nu} \phi_{*\mu} \phi_{*\nu} = \phi^2 R - 4\omega g^{\mu\nu} \phi_{*\mu} \phi_{*\nu}. \tag{22}$$

$G^{MN}$ is the 5D metric (upper case letters denoting 5D coordinates), and $R^*$ is the, 5D, co-covariant Ricci scalar

$$\hat{R}^* = \hat{R} + 8K^M_N K_M - 12 K K^M K_M. \tag{24}$$

$\hat{R}$ denotes the 5D Ricci scalar. Furthermore $\phi_{*M}$ is the the 5D co-covariant derivative defined by

$$\phi_{*M} = \phi_{;M} + \frac{3}{2} K_M \phi . \tag{25}$$

We note that the scalar potential term in the action must be of a specific form, $V(\phi) = -2\Lambda \phi^{30/3}$, to keep our action an in-action. However if one wishes to explicitly break the Weyl symmetry, changing the potential is the simplest way this can be done. On pedagogical grounds we will continue with a general $V(\phi)$.

$$S = \int \sqrt{-g} d^5 x \left( \phi^2 \hat{R} - 4\omega g^{MN} \phi_{*M} \phi_{*N} + V(\phi) \right) + (4 + 3n) (2K^M_N - 3K K^M_K_N) \phi^2 - \frac{1}{4} \phi \hat{\nabla}^2 G^{MN} K_{MN}. \tag{26}$$

Associated with this, non-critical, Lagrangian and corresponding to variations with respect to $G_{MN}, \phi$ and $K_{\mu}$, respectively, are the following field equations:

$$\phi^2 \left( R_{MN} - \frac{1}{2} G_{MN} \hat{R} \right) =$$

$$-\phi^2 \partial^2_{MN} + G_{MN} (\phi^2)_M^N - 4\omega_5 \phi_{*M} \phi_{*N} + 2\omega_5 G_{MN} \phi_{*M} \phi_{*N} +$$

$$-\frac{1}{2} (4 + 3\omega_5) G_{MN} (2K^M_N - 3K K_N) \phi^2 - 3(4 + 3\omega_5) \phi^2 K K_N$$

$$- (4 + 3\omega_5) (K_N \phi_{*M} - K M \phi_{*N}) + (4 + 3\omega_5) G_{MN} (K^M \phi^2)_N$$

$$- \frac{1}{2} \phi \hat{\nabla}^2 K K^N K_N + \frac{1}{8} \phi \hat{\nabla}^2 G_{MN} K_{MN} K_{MN} + \frac{1}{2} G_{MN} V(\phi), \tag{27a}$$

$$\omega_5 G^{MN} (\phi_{*MN} - 2\phi_{*M} \phi_{*N}) =$$

$$-\phi^2 \hat{R} + \frac{1}{2} V'(\phi) + \frac{1}{12} \phi \hat{\nabla}^2 G^{MN} K_{MN}$$

$$- (4 + 3\omega_5) (2K^M_N - 3K K_N) \phi^2 . \tag{27b}$$

$$\left( \phi \hat{\nabla}^2 K_{MN} \right)_{;N} = 2 (4 + \omega_5) G^{MN} (\phi^2_{;N} + 3\phi^2 K_N). \tag{27c}$$

As the underlying symmetry has not been broken, as long as $V(\phi)$ allows it, the equations of motion are also
scale invariant. This allows them to be written in their "starred" form via some straightforward calculations. The starred variants of the 5D equations of motion for $G_{MN}, \phi$ and $K_{\mu}$, respectively, are:

$$\phi^2 \left( R_{MN}^* - \frac{1}{2} G_{MN} \hat{R}^* \right) =$$
$$\phi^2_{MN} - G_{MN} (\phi^2)_{\ast A} + 4 \omega_5 \phi_{\ast M} \phi_{\ast N}$$
$$- 2 \omega_5 G_{MN} G_{PQ} \phi_{\ast P} \phi_{\ast Q} + \frac{1}{2} \phi^2 \kappa_{MN} \kappa_{MN}$$
$$- \frac{1}{8} \phi^2 G_{MN} \kappa_{PQ} \kappa_{PQ} + \frac{1}{2} G_{MN} V(\phi) ,$$

(28a)

$$2 \omega_5 G_{MP} \phi^2_{\ast M} =$$
$$- \phi^2 \hat{R}^* + 4 \omega_5 G_{PQ} \phi_{\ast P} \phi_{\ast Q}$$
$$+ \frac{1}{12} \phi^2 \kappa_{MN} \kappa_{MN} + \frac{1}{2} V'(\phi) ,$$

(28b)

$$\left( \phi^2 \kappa_{MP} \right)_{\ast P} = 2 (4 + 3 \omega_5) G_{MP} \phi^2_{\ast P} .$$

(28c)

It is noteworthy to mention that although the "starred" equations are simpler in form than their "un-starred" counterparts, there is a price to pay. The staring of the equation removes our physical intuition of the problem. As such we must always perform our analysis in the more complicated "un-starred" version of the equations. Instead, we will make a Kaluza-Klein type reduction of the 5D action, vary it with respect to the relevant fields, and obtain the equations of motion in 4D. It is possible to perform the Kaluza-Klein reduction on the 5D equations of motion. However this will make the understanding of the embedded 4D theory more complicated.

**TRANSFORMATIONS AND THE K-K ANSATZ**

Working in higher dimensions, although very interesting, does not necessarily give us any intuition about our 4D world. In order to find the embedded 4D theory we must reduce the action from the higher dimension to 4D. This is done by decomposing the 5D elements into their 4D counter-parts. The decomposition is actually done according to the coordinate transformation laws of the 5D elements

$$G'_{MN} = \frac{\partial x_A}{\partial x'_M} \frac{\partial x_B}{\partial x'_N} G_{AB} ,$$

(29a)

$$K'_M = \frac{\partial x_A}{\partial x'_M} K_A ,$$

(29b)

$$\phi' = \phi .$$

(29c)

We denote the 5D elements with upper-case letters and the 4D elements with lower-case letters. Next we single out the 5th dimension and re-write $G_{MN}$ coordinate transformation laws

$$G'_{\mu\nu} = \frac{\partial x_A}{\partial x'_\mu} \frac{\partial x_B}{\partial x'_\nu} G_{AB} = \frac{\partial x_A}{\partial x'_\mu} \frac{\partial x_B}{\partial x'_\nu} G_{ab} + \frac{\partial x_A}{\partial x'_\mu} \frac{\partial x_B}{\partial x'_\nu} G_{a5} + \frac{\partial x_A}{\partial x'_\mu} \frac{\partial x_B}{\partial x'_\nu} G_{5b} + \frac{\partial x_A}{\partial x'_\mu} \frac{\partial x_B}{\partial x'_\nu} G_{55} ,$$

(30)

$$G'_{\mu 5} = \frac{\partial x_A}{\partial x'_\mu} \frac{\partial x_B}{\partial x'_5} G_{AB} = \frac{\partial x_A}{\partial x'_\mu} \frac{\partial x_B}{\partial x'_5} G_{ab} + \frac{\partial x_A}{\partial x'_\mu} \frac{\partial x_B}{\partial x'_5} G_{a5} + \frac{\partial x_A}{\partial x'_\mu} \frac{\partial x_B}{\partial x'_5} G_{5b} + \frac{\partial x_A}{\partial x'_\mu} \frac{\partial x_B}{\partial x'_5} G_{55} ,$$

(31)

$$G'_{5 5} = \frac{\partial x_A}{\partial x'_5} \frac{\partial x_B}{\partial x'_5} G_{AB} = \frac{\partial x_A}{\partial x'_5} \frac{\partial x_B}{\partial x'_5} G_{ab} + \frac{\partial x_A}{\partial x'_5} \frac{\partial x_B}{\partial x'_5} G_{a5} + \frac{\partial x_A}{\partial x'_5} \frac{\partial x_B}{\partial x'_5} G_{5b} + \frac{\partial x_A}{\partial x'_5} \frac{\partial x_B}{\partial x'_5} G_{55} .$$

(32)

Additionally we re-write $K_A$ coordinate transformation laws

$$K'_\mu = \frac{\partial x_A}{\partial x'_\mu} K_A = \frac{\partial x_A}{\partial x'_\mu} K_a + \frac{\partial x_5}{\partial x'_\mu} K_5 ,$$

(34)

$$K'_5 = \frac{\partial x_A}{\partial x'_5} K_A = \frac{\partial x_A}{\partial x'_5} K_a + \frac{\partial x_5}{\partial x'_5} K_5 .$$

(35)

Lastly, the scalar field does not go under a coordinate transformation, i.e., $\phi' = \phi$. Furthermore, using the Kaluza - Klein ansatz, for which the vacuum solely allows for $x_a(x'_\mu)$ and $x_5 = x'_5 + \Lambda(x'_\mu)$ we can write the $G_{MN}$ coordinate transformation as follows

$$G'_{\mu \nu} = \frac{\partial x_A}{\partial x'_\mu} \frac{\partial x_B}{\partial x'_\nu} G_{ab} + \frac{\partial x_A}{\partial x'_\mu} \frac{\partial x_B}{\partial x'_\nu} \frac{\partial \Lambda}{\partial x'_\mu} G_{a5} + \frac{\partial x_A}{\partial x'_\mu} \frac{\partial x_B}{\partial x'_\nu} \frac{\partial \Lambda}{\partial x'_\nu} G_{5b} + \frac{\partial x_A}{\partial x'_\mu} \frac{\partial x_B}{\partial x'_\nu} \frac{\partial \Lambda}{\partial x'_\nu} G_{55} ,$$

(36)

$$G'_{\mu 5} = \frac{\partial x_A}{\partial x'_\mu} G_{a5} + \frac{\partial x_A}{\partial x'_\mu} \frac{\partial \Lambda}{\partial x'_\nu} G_{55} ,$$

(37)
It is quite remarkable that the special combination\footnote{\textsuperscript{1}}
\[ G_{5\nu} = \frac{\partial x_b}{\partial x'_\nu} G_{5b} + \frac{\partial \Lambda}{\partial x'_\nu} G_{55} , \]
(38)

\[ G'_{55} = G_{55} . \]
(39)

These are accompanied by,
\[ K'_\mu = \frac{\partial x_a}{\partial x'_\mu} K_a + \frac{\partial \Lambda}{\partial x'_\mu} K_5 , \]
(40)

\[ K'_5 = K_5 . \]
(41)

Finally the 4D elements, embedded in the 5D elements, are recovered. We identify the coordinate transformation laws of:

- The 4D metric
  \[ g_{\mu\nu}' = \frac{\partial x_a}{\partial x'_\mu} \frac{\partial x_b}{\partial x'_\nu} g_{ab} . \]
(42)

- Two 4D vectors
  \[ \frac{G_{\mu 5}}{G_{55}} \rightarrow \frac{\partial x_a}{\partial x'_\mu} G_{a 5} + \frac{\partial \Lambda}{\partial x'_\mu} G_{5 5} , \quad \frac{K_5}{K_5} \rightarrow \frac{\partial x_a}{\partial x'_\mu} K_a + \frac{\partial \Lambda}{\partial x'_\mu} K_5 . \]
(43)

- A scalar field - \( g_{\mu\nu}' = G_{55} \).

It is quite remarkable that the special combination \( G_{5\nu} = \frac{\partial x_b}{\partial x'_\nu} G_{5b} - \frac{\partial \Lambda}{\partial x'_\nu} G_{55} \) is in-fact gauge invariant. Moreover this combination will later play a key part in our 4D theory.

We may more conveniently show off the 4D embedding in the 5D elements by writing
\[ G_{MN} = S^{m+1} \left( \frac{S^{-1} g_{\mu\nu} + A_\mu A_\nu}{A_\nu} \right) , \]
(44)

\[ K_M = s \left( \frac{V_\mu}{1} \right) . \]
(45)

With \( m \) being an arbitrary power of the scalar field. This, however, will not affect the underlying physics of the problem. Additionally it is possible to find a simple dictionary between different choices of \( m \).

We start by choosing \( S = \tilde{S}^p \) and \( g_{\mu\nu}' = \tilde{S}^p g_{\mu\nu} \) such that
\[ G_{MN} = \tilde{S}^p \left( \frac{\tilde{S}^q g_{\mu\nu} + \tilde{S}^p A_\mu A_\nu}{\tilde{S}^p A_\nu} \right) . \]

Wanting to preserve the original form we choose \( q = p-1 \)
\[ \tilde{S}^{p(m+1)} \left( \frac{\tilde{S}^{q-1} g_{\mu\nu} + \tilde{S} A_\mu A_\nu}{\tilde{S} A_\nu} \right) . \]
(46)

Thus a simple dictionary is found between the different \( m \),
\[ \tilde{m} = p(m+1) - 1 . \]
(48)

Note that the case of \( m \neq -1 \) is exceptional and cannot be transformed. In this case the action exclusively enjoys an extra local scale symmetry. This yields a \( K_\mu \) dependent local scale invariant action i.e the Brans-Dicke action with \( \omega_{BBP} = -3/2 \). We will continue our analysis under the assumption that \( m \neq -1 \).

With the 4-vectors in mind we check how the original 5D local scale symmetry reflects on the 4D theory. Recalling our action (6) the 5D the transformation laws are,
\[ G_{MN} \rightarrow e^{2\Omega} G_{MN} , \quad K_M \rightarrow K_M + \Omega_{;M} , \quad \phi \rightarrow e^{-\frac{2\Omega}{m}} \phi . \]
(49)

Writing the metric explicitly,
\[ e^{2\Omega} G_{MN} = e^{2\Omega} S^{m+1} \left( \frac{S^{-1} g_{\mu\nu} + A_\mu A_\nu}{A_\nu} \right) , \]
(50)

we deduce the transformation laws of the 4D elements which constitute the 5D metric,
\[ S^{m+1} \rightarrow e^{2\Omega} S^{m+1} \Rightarrow S^m \rightarrow e^{2\Omega/m} S^m , \quad A_\mu \rightarrow A_\mu , \quad g_{\mu\nu} \rightarrow e^{2\Omega/m} g_{\mu\nu} , \]
(51)

where we have denoted \( \Omega \) as the 5D transformation parameter and \( \tilde{\Omega} \) as the the 4D transformation parameter. Additionally we emphasis that only \( \Omega(x_\mu) \) is allowed in order for the 4D theory to stay local scale invariant. This is owed to the fact that for the non-vacuum case all \( x_5 \) dependent fields need to be Fourier-expanded, including \( \Omega \). It is a simple exercise to show that once \( \Omega \) is Fourier-expanded alongside the rest of the fields the transformation laws are no longer defined by a single \( \Omega \). In this case the action does not remain local scale invariant and must be modified accordingly if one wishes for it to remain local scale invariant.

Since we know how \( S^m \) and \( g_{\mu\nu} \) transform both individually and as a product we can deduce that
\[ S^m g_{\mu\nu} \rightarrow e^{\frac{2\Omega}{m+1}} S^m g_{\mu\nu} \Rightarrow \Omega = \frac{m\Omega}{m+1} + \tilde{\Omega} . \]
(52)

Hence the relation between the 4D and 5D transformation parameters is
\[ \tilde{\Omega} = \frac{\Omega}{m+1} . \]
(53)

Next we write the 5D transformation of the Weyl gauge vector in detail,
\[ K_M \rightarrow K_M + \Omega_{;M} = s \left( \frac{V_\mu}{1} + \frac{\Omega_{;\mu}}{0} \right) . \]
(54)
Using the above relations we find the 4D transformation rules
\[
V_\mu \rightarrow V_\mu + \frac{(m+1)}{s} \tilde{\Omega}_\mu,
\]
\[
s \rightarrow s.
\]
Finally the scalar field 4D transformation is trivially given by \( \phi \rightarrow e^{-\frac{2}{s}(m+1)\tilde{\Omega}} \).

To summarize, after the reduction our 4D fields will have the following scale transformation laws
\[
g_{\mu\nu} \rightarrow e^{2\tilde{\Omega}} g_{\mu\nu}, \quad A_\mu \rightarrow A_\mu + S \rightarrow e^{2\tilde{\Omega}} S,
\]
\[
V_\mu \rightarrow V_\mu + \frac{(m+1)}{s} \tilde{\Omega}_\mu, \quad s \rightarrow s, \quad \phi \rightarrow e^{\frac{2}{s}(m+1)\tilde{\Omega}} \phi.
\]

However, the theory does not yet have a 4D Weyl gauge vector. Although \( V_\mu \) might appear as a good candidate it does not fill the requirements for the role. Unlike \( V_\mu \) the Weyl vector is gauge invariant. Consequently we recall the chosen ansatz for the metric,
\[
G_{MN} = S^{-m} \left( \begin{array}{c|c} g^{\mu\nu} & -A_\mu \\ \hline -A_\nu & A_\mu A_\nu + \frac{1}{S} \end{array} \right).
\]

and the Weyl gauge vector
\[
K_M = s \left( \frac{V_\mu}{1} \right),
\]
\[
K_M = S^{-m} \left( \frac{(m+1)k_\mu}{\frac{s}{5} - (m+1)k_\mu A_\mu} \right).
\]

We divide our action into four parts,
\[
S = \int (L_1 + L_2 + L_3 + L_{\text{kin}}) \, d^4x,
\]
where we have defined
\[
L_1 \equiv \phi^2 \hat{R} \sqrt{-G}, \quad L_2 \equiv -4\omega_5 G^{MN} \phi_M \phi_N \sqrt{-G},
\]
\[
L_3 \equiv (4 + 3\omega_5) (2K^M_{;\mu} - 3K^M K_M) \phi^2 \sqrt{-G},
\]
\[
L_{\text{kin}} \equiv -\frac{1}{4} \phi_\mu^2 K^{MN} K_{MN} \sqrt{-G}.
\]

Before we continue our analysis we wish to make a remark about the derivatives. In the 5D action most of the derivatives are not actually covariant derivatives but just partial derivatives, i.e \( \phi_\mu = \phi_M \). This is due to the fact the they are either acting on scalar fields or the anti-symmetry field strength tensor. We will be examining the Kaluza-Klein vacuum, thus all 5D partial derivatives are in fact identical to their 4D counter-parts. The only exception is the covariant derivative of the 5D Weyl gauge vector, \( K^M_{;\mu} \). During the reduction process this term is dealt with carefully, ensuring that after the reduction all derivatives written are their 4D variant.

| 4D | Maxwell | Weyl |
|----|---------|------|
| Gauge transformation | \( A_\mu \rightarrow A_\mu + \Lambda_\mu \) | \( k_\mu \rightarrow k_\mu + \tilde{\Omega}_\mu \) |
| Scale transformation | \( A_\mu \rightarrow A_\mu \) | \( k_\mu \rightarrow k_\mu + \tilde{\Omega}_\mu \) |

**KALUZA - KLEIN REDUCTION**

We start with the 5D Weyl-Dirac action in its unstarrred form \( [20] \). In this section we will omit the scalar potential term. The inclusion of a scalar potential term is a simple process which will be done when we discuss no-scale quantum cosmology at the mini-superspace level. Recall the chosen ansatz for the metric,
\[
G_{MN} = S^{m+1} \left( \begin{array}{c|c} S^{-1}g_{\mu\nu} + A_\mu A_\nu & A_\mu \\ \hline A_\nu & 1 \end{array} \right),
\]
\[
2k_\mu - 3(m+1)k_\mu + (1 + 3m)S^{-1}k_\mu S_\mu - \frac{3S^{-1}k_\mu}{m + 1}.
\]
Furthermore, $\omega_5$ has to be replaced with its 4D counter part. In order to do this we recall [21] in 4D,

$$
\phi^2 R^2 - 4 \omega_4 g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} = \\
\phi^2 R - 4 \omega_4 g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - 2\omega \left(K^\mu \phi^2\right)_{,\mu} + (3 + 2\omega_4) \left(2K^\mu - 2k^\mu K_\mu\right) \phi^2 . 
$$

(69)

Allowing us to identify the dictionary between $\omega_5$ and $\omega_4$

$$(4 + 3\omega_5)(m + 1) = (3 + 2\omega_4) \quad \Rightarrow \quad \omega_5 = \frac{3 + 2\omega_4}{3(m + 1)} - \frac{4}{3} .
$$

(70)

To verify self-consistency, we check the $\omega_4 = -3/2$ critical case. This yields $\omega_5 = -4/3$ just as it should be. Using this found dictionary we finally get

$$
\mathcal{L}_3 = S^{\frac{1+2m}{2}} \phi^2 \sqrt{-g} (3 + 2\omega_4) \left(2k^\mu - 3(m + 1)k_\mu\right) + \phi^2 S^{\frac{1+2m}{2}} \sqrt{-g} (3 + 2\omega_4) \left((1 + 3m)S^{-1}k^\mu S_{,\mu} - \frac{3S^{-1}s^2}{m + 1} - \frac{4(1 + 4m - 2\omega_4)}{3(3 + 2\omega_4)(m + 1)} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu}\right) .
$$

(71)

We have denoted the field strength tensor for the Weyl gauge vector as $k_{\mu\nu}$. In this form the Weyl-Maxwell mixing is hard to miss. The mixing term

$$
\frac{1}{4} \left(\phi S^{\frac{3(1+m)}{4}}\right)^{\frac{3}{2}} s (m + 1) k_{\mu\nu} F_{\mu\nu}
$$

is coupled only to in-scalars, who do not undergo transformations, making this term un-gaugable.

Finally we integrate over $x_5$ and write the final form of the reduced 4D action, up to a total derivative, in it’s full glory

$$
\mathcal{L}_{\text{kin}} \sqrt{-G} = \\
\frac{1}{4} \left(\phi S^{\frac{3(1+m)}{4}}\right)^{\frac{3}{2}} ((m + 1) k_{\mu\nu} + s F_{\mu\nu}) ((m + 1) k_{\mu\nu} + s F_{\mu\nu}) \\
+ \frac{1}{4} \left(\phi S^{\frac{3(1+m)}{4}}\right)^{\frac{3}{2}} \frac{3}{2} g_{\mu\sigma} s_{\sigma s_{,\nu}} .
$$

(73)

We make a few remarks concerning our reduced action,

- If one wishes that the action have no ghost terms $m$ cannot be completely arbitrary. The strongest $m$ constraint is due to the $S$ field kinetic term coefficient where it must be that $m(1 + 2m) <

0 \Rightarrow -\frac{1}{2} < m < 0 .

- Assuming $x_5$ - independence at the Kaluza-Klein vacuum level, we can trivially integrate out the fifth dimension.
THE $m = -\frac{4}{3}$ CASE

An interesting case to examine is the $m = -\frac{4}{3}$ one. First we note the elegant $\omega_4$ to $\omega_5$ dictionary,

$$\omega_4 = \omega_5 - \frac{1}{6}.$$  \hspace{1cm} (75)

Secondly we denote the Kaluza-Klein radius in a more convenient manner - $l \equiv l_{KK}$. For this case the action,

$$S = l \int \sqrt{-g} d^4x \left( \phi^2 R - \phi^2 S \frac{1}{4} F^\mu \nu F_\mu \nu 
- \frac{1}{6} \phi^2 S - 2 g^{\mu \nu} S_{\mu \nu} \phi^2 + \frac{1}{3} \phi S^\frac{2}{3} \left( \frac{2}{3} s_{\mu \nu} F_{\mu \nu} \right) \right),$$

features no coupling between $S$ and the curvature. It is possible, yet not very informative, to star this action. The starring will leave us with a simpler action.

It is possible, yet not very informative, to star this action. The starring will leave us with a simpler action, with the transformation rules

$$\sigma \rightarrow \sigma ; \quad s \rightarrow s ; \quad \phi \rightarrow e^{-\Omega} \phi.$$  \hspace{1cm} (77)

Unit wise - our action is not composed of "normal" scalar (with length unit $L^{-1}$). In-addition, following our ansatz, all vector fields are also not "normal" (for further details see table I). This stems from the reduction where the 5D length length scales are set to make sure that the action is length-less. For proper length units we must add a length scale, the Kaluza-Klein radius, into the ansatz

$$G_{MN} = S^{m+1} \left( \frac{S^{-1} g_{\mu \nu} + l^2 A_{\mu} A_{\nu}}{|l A_{\mu}|} \right),$$  \hspace{1cm} (79)

with the transformation rules

$$\sigma \rightarrow \sigma ; \quad s \rightarrow l^2 s ; \quad \phi \rightarrow e^{-\Omega} \phi.$$  \hspace{1cm} (80)

This leads to the reduced action,

$$S = l \int \left( \phi^2 R - 4 \omega_4 g^{\mu \nu} \phi_{\mu} \phi_{\nu} - \frac{1}{4} \sigma F^\mu \nu F_\mu \nu 
- \frac{1}{6} \phi^2 g^{\mu \nu} \sigma_{\mu \nu} \sigma_{\mu \nu} \right) \left( 3 + 2 \omega_4 \right) \sigma^2 \phi^4 + \frac{1}{2} \sigma^2 \left( 2 k_{\mu \nu} + s F_{\mu \nu} \right) \right),$$

with the transformation rules

$$s \rightarrow l^2 s, \quad \phi \rightarrow l^2 \phi, \quad S \rightarrow l^2 S.$$  \hspace{1cm} (81)
allowing us to find the proper powers which leave the action \( l \) - independent. Let us look at following terms,

\[
\frac{l^3}{4} \sigma F^\mu\nu F_{\mu\nu} \rightarrow \frac{1}{4} l^{3+2B} \sigma F^\mu\nu F_{\mu\nu} \tag{83}
\]

\[
l\phi^2 R^* \rightarrow l^{1+2B} \phi^2 R^* \tag{84}
\]

\[
\frac{9}{2} (3 + 2\omega_4) \frac{s^2}{\sigma^2} \phi^4 \rightarrow \frac{9}{2} (3 + 2\omega_4) l^{1+2A-C+2B} \frac{s^2}{\sigma} \phi^4 . \tag{85}
\]

This leads to

\[
3 + C + 2B = 0 , \quad 1 + 2B = 0 , \quad 1 + 2A - C + 2B = 0 . \tag{86}
\]

Solving these we find \( A = -1 \), \( B = -\frac{1}{2} \), and \( C = -2 \). We can finally write the reduced, diagonalized, action

\[
S = \int \sqrt{-g} d^4x \left( \phi^2 R^* - 4\omega_4 g^\mu\nu \phi_{s\mu} \phi_{s\nu} - \frac{1}{4} \sigma F^\mu\nu F_{\mu\nu} \right.
\]

\[
- \frac{1}{6} \phi^2 g^\mu\nu \sigma_{s\mu} \sigma_{s\nu} - \frac{9}{2} (3 + 2\omega_4) \frac{s^2}{\sigma} \phi^4 + \frac{1}{2} \phi^2 g^\mu\nu \sigma_{s\mu} \sigma_{s\nu}
\]

\[
+ \frac{1}{4} \left( \frac{2}{3} k^\mu\nu + sF^\mu\nu \right) \left( \frac{2}{3} k_{\mu\nu} + sF_{\mu\nu} \right) \frac{\sigma^2}{\sigma} \right) . \tag{87}
\]

With the KK radius properly absorbed into the scalar fields the action is composed of two in-scalars (\( \sigma \) and \( s \)) and one co-scalar (\( \phi \)). Length-wise each in-scalar is unit-less as it brings no scale into the action, as such they are not "normal" scalar fields with the proper \( L^{-1} \) length unit (see table II). On the other hand the co-scalar has the proper scalar field length units, this invites the idea, that a mass term may emerge from the vacuum expectation value of this field.

**NO-SCALE KALUZA-KLEIN QUANTUM COSMOLOGY**

Our starting point is the \( m = -1/3 \) reduced action. If we wish to follow Hartle and Hawking our action must include a scalar potential term. It must also stem from the 5D scalar potential (Recall \( V(\phi) = -2\Lambda \phi^{10/3} \)), which is trivially reduced to \( V(\phi) = -\frac{2\Lambda}{(\sigma S)^{2/3}} \phi^4 \). Plugging this into our action

\[
S = \int \left( \phi^2 R^* - 4\omega_4 g^\mu\nu \phi_{s\mu} \phi_{s\nu} - \frac{1}{4} \sigma F^\mu\nu F_{\mu\nu} \right.
\]

\[
- \frac{1}{6} \phi^2 g^\mu\nu \sigma_{s\mu} \sigma_{s\nu} + \frac{1}{2} \phi^2 g^\mu\nu s_{s\mu}s_{s\nu} + \frac{1}{4} \phi^2 \left( \frac{2}{3} k^\mu\nu + sF^\mu\nu \right) \left( \frac{2}{3} k_{\mu\nu} + sF_{\mu\nu} \right)
\]

\[
- \left( \frac{9}{2} (3 + 2\omega_4) \frac{s^2}{\sigma} + 2\Lambda \frac{\sigma}{\sigma^{1/3}} \right) \phi^4 \right) \sqrt{-g} d^4x . \tag{88}
\]

At the mini-superspace level, cosmology can only tolerate the pure gauge configurations

\[
A_\mu = (A(t), 0, 0, 0) , \quad k_\mu = (v(t), 0, 0, 0) . \tag{89}
\]

As a result, \( F_{\mu\nu} = k_{\mu\nu} = 0 \) leaving the extraordinary Weyl-Maxwell mixing out of the game. At the moment we will abandon our \( \sigma \) notation, leaving us with three scalar fields \( s, S, \phi \). Following the standard procedure of integrating out over the maximally symmetric space

\[
\int \mathcal{L} \sqrt{-g} dtd^3x \rightarrow \int \mathcal{L}_{mini} dt , \tag{90}
\]

and up to a total derivative, the mini superspace Lagrangian is

\[
\mathcal{L}_{mini} = \frac{6a^2 \phi^2}{\sqrt{S} S} + \frac{12a^2 \phi^2 \dot{\phi}}{n n} - \frac{6n a^3 \phi^2}{6n S^2} \frac{S^{1/3}}{S} - \frac{3n S}{3n} - \frac{3n}{S} - \frac{3n}{S^{1/3}} - \frac{3n}{S^{2/3}} , \tag{91}
\]

while reviving the lapse function \( n(t) \) to keep track of the underlying diffeomorphism. We translate the mini-superspace Lagrangian to the Hamiltonian formalism. Using the Legendre transform \( H = p_\phi \dot{\phi} - L \) and the canonical momentum

\[
p_a = \frac{12a^2 \phi \dot{a}}{n} + \frac{12a^2 \phi \dot{\phi}}{n} , \tag{92a}
\]

\[
p_\phi = \frac{12a^2 \phi \dot{a}}{n} - \frac{2a^3 \phi \dot{S}}{3n S} - \frac{4a^3 (1 + 6\omega_4) \phi}{3n} \]

\[
- \frac{(3 + 2\omega_4) 4a^3 \phi^2 \dot{v}}{n} , \tag{92b}
\]

\[
p_S = -\frac{2a^3 \phi \dot{S}}{6n S^2} , \tag{92c}
\]

\[
p_s = -\frac{a^3 \phi \dot{\phi}^{2/3}}{n S^{2/3}} . \tag{92d}
\]
The resulting Hamiltonian,
\[
H = \frac{v}{a} \left[ 24 \phi^2 \frac{S^2 p_b^2}{2a^2 \phi^2} + \frac{S^2 p_s^2}{2a^2 \phi^2} - \frac{(ap_a - \phi p_b + 2S p_s)^2}{8(3 + 2\omega a^2 \phi^2)} + 6a^2 \phi^2 - \left( \frac{9(3 + 2\omega_4)s^2}{2S^2 \phi^2} + \frac{2\Lambda}{S^2 \phi^2} \right) a^4 \phi^4 \right],
\]
is linear in \(v, n\). This gives rise to two first class constraints. We are mainly interested in the a quantum non-scale cosmology. Thus we will proceed directly to the pair of Schrödinger equations, skipping the classical equations of motion. We immediately recognize the coefficient of \(v\) in Eq.(93) as the \(n\)-independent scale symmetry constraint, leading to
\[
\left( a \frac{\partial}{\partial a} - \phi \frac{\partial}{\partial \phi} + 2S \frac{\partial}{\partial S} \right) \psi(a, \phi, S, s) = 0. \tag{94}
\]
This constraint forces the wave function to depend solely on Dirac's in-scalars,
\[
a^\alpha \phi^\beta S^\gamma \quad \text{with} \quad \alpha - \beta + 2\gamma = 0. \tag{95}
\]
Without losing generality, the simplest choice would be
\[
\psi(a, \phi, S, s) = \psi(a\phi, \log(S\phi^2), s), \tag{96}
\]
for which we can now use the short hand in-scalar notations
\[
b = a\phi, \quad z = \log S\phi^2 = \log \sigma. \tag{97}
\]
Note how \(\sigma\) re-enters the theory, as was anticipated, being a fundamental building block of the theory. The associated Hamiltonian constraint, identified as the coefficient of \(n\) in Eq.(93), eventually becomes the zero energy Schrödinger equation
\[
-\frac{\hbar^2}{24} \frac{\partial^2 \psi}{\partial b^2} + \frac{3\hbar^2}{2b^2} \frac{\partial^2 \psi}{\partial z^2} + \frac{\hbar^2}{2b^2} \frac{\partial^2 \psi}{\partial s^2} + V(b, z, s) \psi = 0, \tag{98}
\]
with the accompanying potential being
\[
V(b, z, s) = 6\kappa b^2 - \left( \frac{9}{2} (3 + 2\omega_4)e^{-z}s^2 + 2e^{-\frac{3}{2}z} \Lambda \right) b^4. \tag{99}
\]
In some respects, the \(b^4\) coefficient can be thought to be taking the part of \(A_{eff}(s, \Lambda)\). It is quite remarkable how even in the case of \(\Lambda = 0\) the \(b^4\) coefficient, provided \(3 + 2\omega_4 > 0\), can still be positive. The door is now widely open for a variety of spacial cases. On simplicity grounds, we choose to analyze the case of a constant Kaluza-Klein in-radius. In the standard Kaluza-Klein theory, with the line element \(ds_5^2 = S^{-\frac{4}{3}} ds_2^2 + S^{-\frac{2}{3}} b^2 (dt + A_\mu dx^\mu)^2\), the invariant 5D radius is given by \(S^{-\frac{2}{3}}\ell\). However in the original works of Kaluza and Klein the radius was chosen to be constant, \(S = 1\), in order for the theory to resemble the Einstein-Maxwell action. Unfortunately, such a choice does not make any sense once local scale symmetry is applied. Instead, the closest choice one can make is by freezing a tenable in-scalar (which has nothing to do with gauge fixing), with the most obvious choice being
\[
\sigma = 1. \tag{100}
\]
This will allow us to focus on the special role played by the Weyl 4D in-scalar \(s\). The corresponding wave function, \(\psi(b, s)\), obeys the Hartle-Hawking equation
\[
-\frac{\hbar^2}{24} \frac{\partial^2 \psi}{\partial b^2} + \frac{3\hbar^2}{2b^2} \frac{\partial^2 \psi}{\partial s^2} + V(b, s) \psi = 0, \tag{101}
\]
\[
V(b, s) = 6\kappa b^2 - \left( \frac{9}{2} (3 + 2\omega_4)s^2 + 2\Lambda \right) b^4. \tag{102}
\]
We begin our analysis by looking at the critical case \(\omega_4 = -\frac{3}{2}\). This is the simplest choice we can make as it allows for the separation of variables \(\psi(b, s) = f(b)g(s)\). \(f(b)\) serves as a modified Hartle-Hawking wave function subject to the effective potential
\[
V_{eff}(b) = \frac{\eta \hbar^2}{2b^2} + 6\kappa b^2 - 2\Lambda b^4. \tag{103}
\]
The constant \(\eta\) governs the equation
\[
g''(s) = \eta g(s). \tag{104}
\]
The sign of \(\eta\) dictates the behavior of our theory. Especially interesting is the behavior near the Big Bang origin \(b \to 0\). There are three possible cases:

1. \(\eta = 0\) - yields a fully recovered Hartle-Hawking model, accompanied by \(g(s) = const\). In addition the no-boundary proposal is recovered for \(f(b) \sim b\).

2. \(\eta > 0\) - yields an unbounded \(g(s) = e^{\pm \sqrt{\eta} s}\), which might imply a non-physical case.

3. \(\eta < 0\) - yields a well behaved \(g(s) = e^{\pm i\sqrt{\eta} s}\). Furthermore, if \(\eta \hbar^2 \Lambda + 16\kappa^3 > 0\) the cosmic evolution undergoes an embryonic era, as is evident in this case by the shape of the effective potential. The no-boundary proposal is not recovered, however both solutions \(f_{1,2}(b) \sim b^{1/2}\) (with \(0 < Re(\delta_{1,2}) < 1\)) vanish asymptotically at the origin. This is, in-fact, the deWitt initial condition emerging from the theory automatically.

For further details, see Fig.[1].
FIG. 1: The 3D (right) and contour (left) plots of the critical, namely $2\omega_4 + 3 = 0$, no-scale cosmological wave function $\psi(b,s)$, which is subjected to the automatic deWitt initial condition $\psi(0,s) = 0$. This case is plotted for $\eta < 0$, which strongly reminds us (up to $s$-periodicity) of the Hartle-Hawking solution.

In comparison, the non-critical case is characterized by an effective cosmological constant

$$\Lambda_{\text{eff}}(s) = \Lambda + \frac{9}{4}(3 + 2\omega_4)s^2.$$  \hfill (105)

The additional term is positive for a super-critical Brans-Dicke parameter $\omega_4 > -\frac{3}{2}$ (including, in particular, the ghost-free case $\omega_4 \geq 0$). Sadly the separation of variables method does not work any more. However, the structure of the Schrodinger equation makes the value of $\omega_4$ irrelevant to the behavior of the wave function near the Big Bang. The larger $\omega_4$, the more concentrated is the wave function around $s^2 \ll 1$. For further details, see Fig.(2).

FIG. 2: The 3D (right) and contour (left) plots of the super critical, namely $2\omega_4 + 3 > 0$, no-scale cosmological wave function $\psi(b,s)$, which is subjected to the automatic deWitt initial condition $\psi(0,s) = 0$. In addition to surviving the $\Lambda \to 0$ limit, this case favors the small values of $s$.

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