On the number of hyperbolic manifolds of complexity $n$

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Abstract

We consider hyperbolic manifolds with boundary, which admit an ideal triangulation with $n$ ideal triangles and one edge. We improve the bound of the number of these manifolds, proving it to be at least subexponential.

Introduction

Let $A_n$ be the set of connected graphs with $n$ vertices without loops and multiple edges, so that the degree of each vertex is 4. Bollobas estimated the number $|A_n|$ using deep technique of random graphs:

$$|A_n| \geq e^{-\frac{15}{4}(4n)!}$$

In [2] Frigerio, Martelli and Petronio considered a class $M_n$ of 3-dimensional oriented hyperbolic manifolds with boundaries as manifolds whose special spines (in Matveev’s sense [3]) have exactly one 2–dimensional cell and $n$ vertices. They proved that the complexity of manifolds $M_n$ equals $n$, that the manifolds could be supplied with hyperbolic metrics with geodesic boundaries, and that there is at least $O(\frac{n^2}{n})$ manifolds in $M_n$. The aim of the current work is to estimate the number of manifolds in $M_n$ more precisely. It is enough to estimate the number of orientable special spines with $n$ vertices and one two–dimensional cell, because such spines are thickened to non-homeomorphic manifolds.

Definition. A special spine is a finite connected two–dimensional cell complex, such that each vertex is incident to 4 edges (with multiplicities) and each edge is incident to three two–dimensional cells (with multiplicities). The regular neighborhood of the inner point of an edge is homeomorphic to a “book with 3 pages”, the regular neighborhood of a vertex is homeomorphic to a cone on the edges of tetrahedron. A special spine is orientable, if it could be immersed into an oriented manifold.

The main result of the present paper is the following

For $n \geq 8$ we have $|M_n| \geq |A_{n-3}|$.

This means that we improve the lower bound of the number $|M_n|$, proving that it is at least subexponential. It is also clear that $|M_n| \leq 18^n g(n)$, where $g(n)$ is the number of connected graphs with $n$ vertices of degree 4 (maybe with loops and multiple edges).

Main part.

For $G \in A_n$ let $P(G)$ be a class of oriented special spines with singularity graph $G$ and with minimal number of cells (among all oriented special spines with singularity graph $G$).

For a spine $S \in P(G)$ let us choose two cells $(e_i, e_j)$ and count the number $v(e_i, e_j)$ of vertices, such that all incident to them edges belong to chosen cells. Let $t(S)$ be the maximum of $v(e_i, e_j)$ for all pairs of cells $(e_i, e_j)$.

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We shall consider a neighbourhood of $G$ in a spine $S$ with $r$ two–dimensional cells as a graph $G$ with $r$ glued cylinders, were one circle of cylinder is mapped into $G$ and the other is called the boundary line.

**Lemma 1.** Let $S \in P(G)$ Then every edge of $G$ belong to at most 2 two–dimensional cells.

**Proof.** Let us suppose that an edge $e \in G$ belongs to 3 two–dimensional cells. We could cut the $e$ at the middlepoint and cut the boundary lines, which pass near $e$. If we fix an orientation of $e$, then we obtain a cyclic order of boundary lines near $e$. We rotate the parts of boundary lines clockwise on the one part of $e$ in such a way that a boundary line will glue with the next boundary line according to cyclic order. We glue rotated parts of boundary lines with unrotated ones. So we get a new oriented spine with the same singularity graph, and the number of cells decrease. It is a contradiction to $S \in P(G)$.

**Lemma 2.** Let $S \in P(G)$ and an edge $e \in G$ belongs to 2 two–dimensional cells. Then if we choose an orientation on the boundary lines of each two–dimensional cell, then two boundary lines along $e$, which belong to one two–dimensional cell, will have parallel orientation.

**Proof.** Let us suppose the contrary. Analogously to the proof of lemma 1 we cut the edge $e$ in the middlepoint and cut the boundary lines. We rotate the parts of boundary lines clockwise on the one part of $e$ in such a way that a boundary line will glue with the next boundary line according to cyclic order. We glue rotated parts of boundary lines with unrotated ones. So we get a new oriented spine with the same singularity graph, and the number of cells decrease. It is a contradiction to $S \in P(G)$.

**Lemma 3.** If $S \in P(G)$, then for every vertex $v \in G$ all incident to $v$ edges belong to at most 2 two–dimensional cells.

**Proof.** Let us suppose the contrary. Let $OA, OB, OC$ and $OD$ be 4 edges incident to a vertex $O \in G$. Let $OA$ belongs to 2 two–dimensional cells $e_1$ and $e_2$, and boundary line of one of them passes through $AOC$ and boundary line of the other pass through $AOB$ and $AOD$ correspondingly.

If $OB$ belongs to the cell $e_2$ only, then $OC$ belongs to $e_1$ and $e_2$ and to at least one more two–dimensional cell by assumption. It is a contradiction to lemma 1.

If $OB$ belongs to the cells $e_1$ and $e_2$, then $OD$ belongs to $e_1$ and $e_2$ and to at least one more two–dimensional cell by assumption. It is a contradiction to lemma 1.

If $OB$ belongs to $e_2$ and $e_3$, then we have the following cases:

1. $e_3$ contains $COB$
2. $e_3$ contains $COB$ and $DOB$
3. $e_3$ contains $DOB$.

In the case (1) $COD$ cannot belong to $e_2$, so the edge $OD$ contradicts to lemma 2. In the case (2) $COD$ belongs to $e_3$ and we have a contradiction to lemma 2 in either $OC$ or $OD$. In the case (3) $COD$ belongs to $e_2$. The point $A$ divides boundary of $e_2$ into two segments $\alpha$ and $\beta$. If one of them (let it be $\alpha$) contains both of boundary lines $COD$ and $COB$, then we could rotate boundary lines along $OA$ so that

1. we don’t change any cells except $e_1$ and $e_2$.
2. $\alpha$ and $\beta$ become boundary lines of different cells $e_1'$ and $e_2'$.

So $OB$ or $OD$ belongs to at least 3 cells $e_1'$ and $e_2'$ which contradicts to lemma 2. □

Now we introduce an operation of gluing an oriented spine $T$ with one two–dimensional cell into a spine $S \in P(G)$.

Let us cut an edge $e \in G$ and consider 3 cutted boundary lines along $e$: $l_1$, $l_2$ and $l_3$. After we cut $e$ we get pairs $(l_1', l_1'')$, $(l_2', l_2'')$ and $(l_3', l_3'')$ of endpoints of $l_1$, $l_2$ and $l_3$. Let us consider an oriented spine $T$ and cut an edge $f$ of singularity graph $T$. Let $m_1$, $m_2$ and $m_3$ boundary lines along $f$. After we cut $f$ we get pairs of endpoints $(m_i', m_i'')$ for $i = 1, 2, 3$. Suppose (it is a significant assumption)
that if we travel from \((m'_1, m'_2, m'_3)\) along cutted \(m_1, m_2, m_3\) we will reach \((m''_1, m''_2, m''_3)\) (in another order, so the case \(m'_1 \rightarrow m'_2, m'_3 \rightarrow m'_4, m''_1 \rightarrow m''_3\) is forbidden). We will call such edges \(f\) cutable.

Then we could glue \(l'_i\) to \(m'_i\) and \(l''_i\) to \(m''_i\) and obtain a new spine \(S'\). The number of two-dimensional cells of \(S\) equals to the number of two-dimensional cells of \(S'\).

Lemma 4. There exists an oriented spine \(T\) with 3 vertices and one two-dimensional cell, so that there exist cutable edge \(f\).

**Proof.** Let the singularity graph contains vertices \(A, B, C\), loops \(l_1, l_6\) in vertices \(A\) and \(C\) and a pair of multiple oriented edges \(l_2, l_3\) from \(B\) to \(A\) and a pair of multiple edges \(l_4, l_5\) from \(B\) to \(C\). Let the boundary of two-dimensional cell passes along

\[l_4, l_5, l_2, l_3, l_1, l_2, l_5, l_1, l_1, l_3, l_4, l_6, l_6, l_5,\]

where by \(l^-_i\) we mean that boundary line passes along \(l_i\) in opposite direction. Then the edge \(l_6\) is cutable (note that \(l_4\) and \(l_5\) are not cutable).

Lemma 5. Let a graph \(G\) be a singularity graph of an oriented spine. Let \(G\) has a loop in a vertex \(A\) and edges \(BA\) and \(CA\). Let \(l_1, l_2\) and \(l_3\) be boundary lines passing near edge \(BA\). Let \(m_1, m_2\) and \(m_3\) be boundary lines passing near edge \(CA\). Let \(m_1, m_2\) and \(m_3\) be of different order, so the case \(l_1 \rightarrow l_2, l_3 \rightarrow l_1\) is forbidden. We will call such edges \(l\) cutable.

**Proof.** It follows from the definition of oriented spine.

Theorem 1. For every graph \(G \in A_n\) there exist a special oriented spine with singularity graph \(G\) and with at most 2 two-dimensional cells.

**Proof.** Let \(S \in P(G)\) be a spine with maximal number \(t(S)\) among all spines in \(P(G)\). If \(S\) has more then 2 two-dimensional cells, then it is easy to notice that it is possible to make a rotation along an edge so that \(t(S)\) will increase.

Theorem 2. For \(n \geq 8\) we have \(|M_n| \geq |A_{n-3}|\).

**Proof.** Let us consider an arbitrary graph \(G \in A_{n-3}\). Then there exist a special oriented spine \(S\) on the graph \(G\) with at most 2 two-dimensional cells. If \(S\) has one two-dimensional cell then we glue it with spine \(T\) from lemma 4. So we get a special oriented spine with \(n\) vertices and one two-dimensional cell. If \(S\) has two two-dimensional cells we find an edge \(e\) which belongs to different cells and we glue into \(e\) three loops consequently to obtain a special oriented spine with \(n\) vertices and one two-dimensional cell by lemma 5.

**References**

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