Quantization of the Bianchi type-IX model in
supergravity with a cosmological constant

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ABSTRACT

Diagonal Bianchi type-IX models are studied in the quantum theory of $N = 1$ supergravity with a cosmological constant. It is shown, by imposing the supersymmetry and Lorentz quantum constraints, that there are no physical quantum states in this model. The $k = +1$ Friedmann model in supergravity with cosmological constant does admit quantum states. However, the Bianchi type-IX model provides a better guide to the behaviour of a generic state, since more gravitino modes are available to be excited. These results indicate that there may be no physical quantum states in the full theory of $N = 1$ supergravity with a non-zero cosmological constant.

PACS numbers: 04.60.+ n, 04.65.+ e, 98.80. $Hw$
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I. Introduction

Recently a number of quantum cosmological models have been studied in which the action is that of supergravity, with possible additional coupling to supermatter [1–11]. It is sufficient, in finding a physical state, to solve the Lorentz and supersymmetry constraints of the theory [12,13]. Because of the anti-commutation relations \([S_A, \bar{S}_{A'}] \sim \mathcal{H}_{AA'}\), the supersymmetry constraints \(S_A \Psi = 0, \bar{S}_{A'} \Psi = 0\) on a physical wave function \(\Psi\) imply the Hamiltonian constraint \(\mathcal{H}_{AA'} \Psi = 0\) [12,13].

In the case of the Bianchi-I model in \(N = 1\) supergravity with cosmological constant \(\Lambda = 0\) [8], only two quantum states appear. Using the factor ordering of [8], one state is \(h^{\frac{1}{4}}\) in the bosonic sector, where \(h = \det h_{ij}\) is the determinant of the three-metric, and the other state is \(h^{-\frac{1}{4}}\) in the sector filled with fermions. In the case of Bianchi IX with \(\Lambda = 0\), there are again two states, of the form \(\exp(\pm I/\hbar)\) where \(I\) is a certain Euclidean action, one in the empty and one in the filled fermionic sector [9,14]. When the usual choice of spinors constant in the standard basis is made for the gravitino field, the bosonic state \(\exp(-I/\hbar)\) is the wormhole state [9,15]. With a different choice, one obtains the Hartle–Hawking state [14,16]. Similar states were found for \(N = 1\) supergravity in the more general Bianchi models of class \(A\) [10]. [Supersymmetry (as well as other considerations) forbids mini-superspace models of class \(B\).] It was also found in the general theory of quantized \(N = 1\) supergravity with \(\Lambda = 0\) that there are two bosonic states of the form \(\exp(-I/\hbar)\), where \(I\) is the wormhole or the Hartle–Hawking classical action [17]. [There may be many other bosonic states.] There are also two states of the form \(\exp(I/\hbar)\) in the filled sector.

It is of interest to extend these results, by studying more general locally supersymmetric actions, initially in Bianchi models. Possibly the simplest such generalization is the addition of a cosmological constant in \(N = 1\) supergravity [18]. It was found that in
the Bianchi-I case there are no physical states for $N = 1$ supergravity with a $\Lambda$-term [11]. The Bianchi-IX model with $\Lambda$-term and with $N = 1$ supersymmetry in one dimension was studied by Graham [4]. Here we treat the Bianchi-IX model with $\Lambda$-term with the full $N = 4$ supersymmetry in one dimension. We shall see that there are again no physical quantum states. The calculations are described in Sec. II. We also treat briefly in Sec. III the spherical $k = +1$ Friedmann model, and find that there is a two-parameter family of solutions of the quantum constraints with a $\Lambda$-term. Nevertheless, as will be seen, the Bianchi-IX model provides a better guide to the generic result, since more spin-$\frac{3}{2}$ modes are available to be excited in the Bianchi-IX model, while the form of the fermionic fields needed for supersymmetry in the $k = +1$ Friedmann model is very restrictive [6]. Sec. IV contains the Conclusion.

II. QUANTUM STATES FOR THE BIANCHI-IX MODEL WITH A $\Lambda$-TERM

Using two-component spinors [6,13], the action [18] is

$$S = \int d^4x \left[ \left( \frac{2\kappa^2}{3} \right)^{-1} (\det e) \left( R - 3g^2 \right) + \frac{1}{2} \epsilon^\mu\nu\rho\sigma \left( \bar{\psi}^{A'}_{\mu} e_{AA'} e_{B'}^\rho \psi^A_{\sigma} \right) + H.c. \right] - \frac{1}{2} g (\det e) \left( \psi^A_{\mu} e_{AB'} e_B e_{B'} \psi^{B'}_{\nu} + H.c. \right).$$

Here the tetrad is $e^a_\mu$ or equivalently $e^{AA'}_\mu$. The gravitino field $(\psi^A_{\mu}, \bar{\psi}^{A'}_{\mu})$ is an odd (anti-commuting) Grassmann quantity. The scalar curvature $R$ and the covariant derivative $D_\rho$ include torsion. We define $\kappa^2 = 8\pi$. Here $g$ is a constant, and the cosmological constant is $\Lambda = \frac{3}{2} g^2$.

There are two possible approaches to the quantization of this model. One possibility is to substitute the Bianchi-IX Ansatz for the geometry $e^{AA'}_\mu$ and gravitino field $(\psi^A_{\mu}, \bar{\psi}^{A'}_{\mu})$ into the action (2.1). The components $\psi^A_{\mu} e^{BB'}_\mu$ and $\bar{\psi}^{A'}_{\mu} e^{BB'}_\mu$ are required to be spa-
tially constant with respect to the standard triad [19] on the Bianchi-IX three-sphere. One finds that, in order for the form of the Ansatz to be left invariant by one-dimensional local supersymmetry transformations, possibly corrected by coordinate and Lorentz transformations [6], one must study the general non-diagonal Bianchi-IX model [19]. The reduced action could then be computed, leading to the Hamiltonian and supersymmetry constraints. Finally the supersymmetry constraints could be imposed on physical wave functions. They would be complicated because of the number of parameters needed to describe the off-diagonal model.

The other alternative, taken here, is to apply the supersymmetry constraints of the general theory at a diagonal Bianchi-IX geometry [9]. This is valid since the supersymmetry constraints are of first order in bosonic derivatives, and give expressions such as $\delta \Psi/\delta h_{im}(x)$ in terms of known quantities and $\Psi$. These equations can be evaluated at a diagonal Bianchi-IX geometry, parametrized by three radii $A, B, C$. One multiplies (e.g.) by $\delta h_{im}(x) = \partial h_{im}/\partial A$ and integrates $\int d^3x(\ldots)$ to obtain an equation for $\partial \Psi/\partial A$ in terms of known quantities. The need to consider off-diagonal metrics is thereby avoided.

The general classical supersymmetry constraints are, with the help of [13], seen to be

\[ S_{A'} = gh^{1/2}e^{A'\ i}n_{AB'}\psi_{B'_i} + e^{ijk}e_{AA'i}\ 3sD_j\psi^{A}k + \frac{1}{2}i\kappa^2\psi^{A}_{i}p_{AA'i}, \quad (2.2) \]

and the conjugate $S_A$. Here $n^{AA'}$ is the spinor version of the unit future-pointing normal $n^\mu$ to the surface $t = \text{const}$. It is a function of the $e^{AA'}_i$, defined by

\[ n^{AA'}e_{AA'i} = 0, \quad n^{AA'}n_{AA'} = 1. \quad (2.3) \]

In Eq. (2.2), $p_{AA'i}$ is the momentum conjugate to $e^{AA'}_i$. The expression $3sD_j$ denotes the three-dimensional covariant derivative without torsion. Since the components of $\psi^A_k$ are taken to be constant in the Bianchi-IX basis, one can replace $3sD_j\psi^A_k$ by $\omega^A_{Bj}\psi^B_k$, where $\omega^A_{Bj}$ gives the torsion-free connection [13].
The corresponding quantum constraints read, with the help of [13],

\[
\overline{S}_A \Psi = -i\hbar h^{\frac{1}{2}} e^A \cdot n_{AB'} D_{ji}^{BB'} \left( h^{\frac{1}{2}} \frac{\partial \Psi}{\partial \psi_{ji}^B} \right) + \epsilon^{ijk} e_{A'B'} \omega_{B'}^A \psi_{ji}^B \Psi - \frac{1}{2} \hbar \kappa^2 \psi^A_i \delta \Psi \delta e^{AA'}_i = 0 ,
\]

(2.4)

\[
S_A \Psi = gh^{\frac{1}{2}} e^A \cdot n_{BA'} \psi_i^B \Psi - i\hbar \omega_{B_i}^A \left( h^{\frac{1}{2}} \frac{\partial \Psi}{\partial \psi_{B_i}} \right) + \frac{1}{2} i\hbar^2 \kappa^2 D_{ji}^{BA'} \left( h^{\frac{1}{2}} \frac{\partial \Psi}{\partial \psi_{ji}^B} \right) \delta \Psi \delta e^{AA'}_i = 0 .
\]

(2.5)

Here

\[
D_{ji}^{BA'} = -2i\hbar^{-\frac{1}{2}} e^{BB'} \epsilon_{CB'} \cdot n_{CA'} ,
\]

(2.6)

and \( \partial / \partial \psi_{B_j} \) denotes the left derivative [20]. We have made the replacement \( \delta \Psi / \delta \psi_{B_j} \rightarrow h^{\frac{1}{2}} \partial \Psi / \partial \psi_{B_j} \). The \( h^{\frac{1}{2}} \) factor ensures that each term has the correct weight in the equations.

One can also check that this replacement gives the correct supersymmetry constraints in the \( k = +1 \) Friedmann model (without \( \Lambda \)-term), where the model was quantized using the alternative approach via a supersymmetric Ansatz [6].

In addition to the supersymmetry constraints, a physical state \( \Psi \) must also obey the Lorentz constraints

\[
J_{AB}^i \Psi = 0 , \quad J_{A'B'}^i \Psi = 0 .
\]

(2.7)

These imply that \( \Psi \) is formed from the three-metric \( h_{ij} \) and from scalar invariants in the gravitino field. To specify this, note the decomposition [11] of \( \psi_{BB'}^A = e_{BB'}^i \psi_i^A \):

\[
\psi_{BB'}^A = -2 \epsilon_{BB'}^C \gamma_{ABC} + \frac{2}{3} (\beta_{AB} n_{BB'} + \beta_{B} n_{AB'}) - 2 \epsilon_{AB} n_{B'B} \beta_C ,
\]

(2.8)

where \( \gamma_{ABC} = \gamma_{(ABC)} \) is totally symmetric and \( \epsilon_{AB} \) is the alternating spinor. The general Lorentz-invariant wave function is a polynomial of sixth degree in Grassmann variables:
\[
\Psi\left( e^{AA_i} \psi^A_i \right) = \Psi_0 (h_{ij}) + (\beta_A \beta^A) \Psi_{21} (h_{ij}) + (\gamma_{ABC} \gamma^{ABC}) \Psi_{22} (h_{ij}) \\
+ (\beta_A \beta^A) (\gamma_{BCD} \gamma^{BCD}) \Psi_{41} (h_{ij}) + (\gamma_{ABC} \gamma^{ABC})^2 \Psi_{42} (h_{ij}) \\
+ (\beta_A \beta^A) (\gamma_{BCD} \gamma^{BCD})^2 \Psi_{6} (h_{ij}) .
\]

(2.9)

As described in [11], any other Lorentz-invariant fermionic polynomials can be written in terms of these.

We now proceed to solve the supersymmetry and Lorentz constraints. The diagonal Bianchi-IX three-metric is given in terms of the three radii \( A, B, C \) by

\[
h_{ij} = A^2 E^1_i E^1_j + B^2 E^2_i E^2_j + C^2 E^3_i E^3_j ,
\]

(2.10)

where \( E^1_i, E^2_i, E^3_i \) are a basis of unit left-invariant one-forms on the three-sphere [19]. In the calculation, we shall repeatedly need the expression, formed from the connection:

\[
\omega_{AB} n^A_{\ B'} e^{BB'} = \frac{i}{4} \left( \frac{C}{AB} + \frac{B}{CA} - \frac{A}{BC} \right) E^1_i E^1_j + \frac{i}{4} \left( \frac{A}{BC} + \frac{C}{AB} - \frac{B}{CA} \right) E^2_i E^2_j + \frac{i}{4} \left( \frac{B}{CA} + \frac{A}{BC} - \frac{C}{AB} \right) E^3_i E^3_j
\]

(2.11)

This can be derived from the expressions for \( \omega_{AB} \) given in [9,13].

First consider the \( \overline{\nabla}_A \Psi = 0 \) constraint at the level \( \psi^1 \) in powers of fermions. One obtains

\[
\frac{3}{16} h g h^2 e_{BA'} \psi^B_i \Psi_{21} + e^{jki} e_{AA'j} \omega^A_{Bk} \psi^B_{ij} \Psi_0 + \hbar k^2 e_{BA'} \psi^B_i \frac{\delta \Psi_0}{\delta h_{ij}} = 0 .
\]

(2.12)

Since this holds for all \( \psi^B_i \), one can conclude

\[
\frac{3}{16} h g h^2 e_{BA'} \psi^B_i \Psi_{21} + e^{jki} e_{AA'j} \omega^A_{Bk} \Psi_0 + \hbar k^2 e_{BA'j} \frac{\delta \Psi_0}{\delta h_{ij}} = 0 .
\]

(2.13)
Now multiply this equation by $e^{BA'm}$, giving

$$\frac{-3}{16} h g h^i m n^i h^{1/2} \Psi_{21} + \epsilon^{jki} e_{AA'j} e^{BA'm} \omega^A_{Bk} \Psi_0 - h \kappa^2 \frac{\delta \Psi_0}{\delta h_{im}} = 0 . \quad (2.14)$$

The second term can be simplified using [6]

$$e_{AA'j} e^{BA'}_m = -\frac{1}{2} h^i j m \epsilon^A_i B + i \epsilon_j mn h^{1/2} n_{AA'} e^{BA'n} . \quad (2.15)$$

One then notes, as above, that by taking a variation among the Bianchi-IX metrics, such as

$$\delta h_{ij} = \frac{\partial}{\partial A} = 2 A E^1_i E^1_j , \quad (2.16)$$

multiplying by $\delta \Psi_0 / \delta h_{ij}$ and integrating over the three-geometry, one obtains $\partial \Psi_0 / \partial A$. Putting this information together one obtains the constraint

$$h \kappa^2 \frac{\partial \Psi_0}{\partial A} + 16 \pi^2 A \Psi_0 + 6 \pi^2 h g BC \Psi_{21} = 0 , \quad (2.17)$$

and two others given by cyclic permutation of $ABC$.

Next we consider the $S_A \Psi = 0$ constraint at order $\psi^1$. One uses the relations

$$\partial (\beta_A \beta^A) / \partial \psi^B_i = -n_A B' e_{BB'}^i \beta^A \quad \text{and} \quad \partial (\gamma_{ADC} \gamma^{ADC}) / \partial \psi^B_i = -2 \gamma_{BDC} n^{CC'} e^D_{C'} i ,$$

and writes out $\beta^A$ and $\gamma_{BDC}$ in terms of $e^{EE'}_j$ and $\psi^E_j$. Proceeding by analogy with the previous calculation above, one again ‘divides out’ by $\psi^B_j$ to obtain
\[
gh^{\frac{1}{2}} e_A^{A'j} n_{BA'} \Psi_0 - \frac{1}{4} i \hbar \omega_A^C i \hbar^{\frac{1}{2}} e_{CB'}^j e_B^{B'j} \Psi_21 \\
- \left( \frac{1}{3} i \hbar \omega_{AB} h^{\frac{1}{2}} e_{DA'}^j e^{DA'i} \right) \Psi_{22} \\
+ \frac{2}{3} i \hbar \omega_A^E i \hbar^{\frac{1}{2}} e_{EA'}^j e_B^{A'i} \right) \Psi_{22} \\
+ \frac{1}{4} \hbar^2 \kappa^2 e_C B_i n_{C} A' e_B^{B'j} e_{AA'm} \delta \Psi_{21} \delta h_{im} \\
-2 \hbar^2 \kappa^2 \left( - \frac{2}{3} \delta_i^j n_B A' + \frac{1}{3} e_B^{C'} i n_{CA'} e_{CC'}^j \right) e_{AA'm} \delta \Psi_{22} \delta h_{im} = 0. \tag{2.18}
\]

One replaces the free spinor indices $AB$ by the spatial index $n$ on multiplying by $n^A D e^{BD'n}$, giving

\[
- \frac{1}{2} gh^{\frac{1}{2}} h^{jn} \Psi_0 \\
+ \frac{1}{8} i \hbar h^{\frac{1}{2}} \left( h^{ij} \omega_{AB} n_{B'} e^{BB'n} - h^{in} \omega_{AB} n_{A'} e^{BB'}^j + \right. \\
+ h^{jn} \omega_{AB} n_{A'} e^{BB'i} \\
+ \frac{1}{3} i \hbar h^{\frac{1}{2}} \left( 2 h^{ij} \omega_{AB} n_{A'} e^{BB'n} + h^{in} \omega_{AB} n_{A'} e^{BB'}^j \right. \\
- h^{jn} \omega_{AB} n_{A'} e^{BB'i} \\
+ \frac{1}{16} \hbar^2 \kappa^2 \left( \delta_i^j \delta_m^n - \delta_i^n \delta_m^j + h_{im} h^{jn} \right) \delta \Psi_{21} \delta h_{im} \\
- \frac{1}{3} \hbar^2 \kappa^2 \left( 2 \delta_i^j \delta_m^n + \delta_i^n \delta_m^j - h_{im} h^{jn} \right) \delta \Psi_{22} \delta h_{im} = 0. \tag{2.19}
\]
Multiplying by different choices \( \delta h_{im} = \partial h_{im}/\partial A \) etc. and integrating over the manifold, one finds the constraints

\[
\frac{1}{16} h^2 \kappa^2 A^{-1} \left( A \frac{\partial \Psi_{21}}{\partial A} + B \frac{\partial \Psi_{21}}{\partial B} + C \frac{\partial \Psi_{21}}{\partial C} \right) - \frac{1}{3} h \kappa^2 \left[ 3 \frac{\partial \Psi_{22}}{\partial A} - A^{-1} \left( A \frac{\partial \Psi_{22}}{\partial A} + B \frac{\partial \Psi_{22}}{\partial B} + C \frac{\partial \Psi_{22}}{\partial C} \right) \right] - 16 \pi^2 g B C \Psi_0 - \pi^2 h C \left( A \frac{A}{B C} + B \frac{B}{C A} + C \frac{C}{A B} \right) \Psi_{21} + \frac{1}{3} \left( 16 \pi^2 \right) h B C \left( \frac{2 A}{B C} - \frac{B}{C A} - \frac{C}{A B} \right) \Psi_{22} = 0 .
\] (2.20)

and two more equations given by cyclic permutation of \( ABC \).

Now consider the \( \overline{\nabla}_A \Psi = 0 \) constraint at order \( \psi^3 \). It will turn out that we need go no further than this. The constraint can be written as

\[
\frac{1}{2} h \hbar \bar{h} e_{B A i}^B e_{B B}^j \beta_C \left( \gamma_{CDE} \gamma_{DEF} \right) \Psi_{41}
+ \epsilon^{ijk} e_{AA'j} \omega_{B j}^B \left( \beta_{C} \gamma_{CDE} \right) \Psi_{21} + \left( \beta_{C} \gamma_{CDE} \right) \Psi_{22}
- \frac{1}{2} h^2 \kappa^2 \psi_A^i \left[ \beta_{C} \gamma_{CDE} \right] = 0 .
\] (2.21)

The terms \( \psi_{B k}^B \) and \( \psi_{A i}^A \) in the last two lines can be rewritten in terms of \( \beta_A \) and \( \gamma_{FGH} \), using Eq. (2.8). Then one can set separately to zero the coefficient of \( \beta_{C} \left( \gamma_{DEF} \gamma_{DEF} \right) \), the symmetrized coefficient of \( \gamma_{DEF} \) \( \beta_{C} \beta_{C} \) and the symmetrized coefficient of \( \gamma_{FGH} \) \( \gamma_{CDF} \gamma_{CDEF} \). These three equations give

\[
\frac{3}{4} h \hbar \bar{h} n^C A_i \Psi_{41} - \frac{8}{3} \epsilon^{ijk} e_{AA'j} \omega_{B j}^B n^C e_{CC}^C \Psi_{21} + \frac{4}{3} h \kappa^2 n^A B_i e_{CB} e_{CB}^i \delta \Psi_{22} = 0 ,
\] (2.22)

\[
2 \epsilon^{ijk} e_{AA'j} \omega_{B j}^B n^D B e_{CB}^k \Psi_{21} - h \kappa^2 n^D B_i e_{CB}^i \delta \Psi_{21} = 0 .
\]
\[ + (BCD \rightarrow CDB) + (BCD \rightarrow DBC) = 0 , \] 
(2.23)

and Eq. (2.23) with \( \Psi_{21} \) replaced by \( \Psi_{22} \). Contracting Eq. (2.22) with \( n_{C}A' \) and integrating over the three-surface gives

\[
\frac{3}{4} (16\pi^2) \hbar g A B C \Psi_{41} + \frac{2}{3} (16\pi^2) (A^2 + B^2 + C^2) \Psi_{22} \\
+ \frac{2}{3} \hbar \kappa^2 \left( A \frac{\partial \Psi_{22}}{\partial A} + B \frac{\partial \Psi_{22}}{\partial B} + C \frac{\partial \Psi_{22}}{\partial C} \right) = 0 .
\] 
(2.24)

Contracting Eq. (2.23) with \( e^{BA'e_{CC'}e_D^{C'N}} \), multiplying by \( \delta h_{\ell n} = \partial h_{\ell n}/\partial A \) and integrating gives

\[
3\hbar \kappa^2 \frac{\partial \Psi_{21}}{\partial A} - \hbar \kappa^2 A^{-1} \left( A \frac{\partial \Psi_{21}}{\partial A} + B \frac{\partial \Psi_{21}}{\partial B} + C \frac{\partial \Psi_{21}}{\partial C} \right) \\
- 16\pi^2 B C \left( \frac{C}{A B} + \frac{B}{C A} - 2 \frac{A}{B C} \right) = 0 ,
\] 
(2.25)

and two more equations given by permuting \( ABC \) cyclically. The equation (2.25) also holds with \( \Psi_{21} \) replaced by \( \Psi_{22} \).

There is a duality between wave functions \( \Psi \left( e^{AA'}, \psi_i^A \right) \) and wave functions \( \tilde{\Psi} \left( e^{AA'}, \tilde{\psi_i}^A \right) \), given by a fermionic Fourier transform [13]. The \( S_A \) and \( S_{A'} \) operators interchange rôles under this transformation, and the rôles of \( \Psi_0 \) and \( \Psi_6 \), \( \Psi_{21} \) and \( \Psi_{42} \), and \( \Psi_{22} \) and \( \Psi_{41} \) are interchanged. We shall proceed by showing that \( \Psi_{22} \), \( \Psi_{21} \) and \( \Psi_0 \) must vanish for \( g \neq 0 \) (or \( \Lambda \neq 0 \)), and hence by the duality the entire wave function must be zero.

Consider first the equation (2.25) and its permutations for \( \Psi_{21} \) and \( \Psi_{22} \). One can check that these are equivalent to

\[
\hbar \kappa^2 \left( A \frac{\partial \Psi_{21}}{\partial A} - B \frac{\partial \Psi_{21}}{\partial B} \right) = 16\pi^2 \left( B^2 - A^2 \right) \Psi_{21}
\] 
(2.26)
and cyclic permutations. One can then integrate Eq. (2.26) along a characteristic $AB = \text{const.}, C = \text{const.}$, using the parametric description $A = w_1 e^\tau, B = w_2 e^{-\tau}$, to obtain

$$\Psi_{21} = h_1(AB, C) \exp \left[ -\frac{8\pi^2}{\hbar \kappa^2} \left( A^2 + B^2 \right) \right]. \tag{2.27}$$

The general solution of

$$\hbar \kappa^2 \left( B \frac{\partial \Psi_{21}}{\partial B} - C \frac{\partial \Psi_{21}}{\partial C} \right) = 16\pi^2 (C^2 - B^2) \Psi_{21} \tag{2.28}$$

is similarly

$$\Psi_{21} = h_2(BC, A) \exp \left[ -\frac{8\pi^2}{\hbar \kappa^2} \left( B^2 + C^2 \right) \right]. \tag{2.29}$$

Eqs. (2.27) and (2.29) are only consistent if $\Psi_{21}$ has the form

$$\Psi_{21} = F(ABC) \exp \left[ -\frac{8\pi^2}{\hbar \kappa^2} \left( A^2 + B^2 + C^2 \right) \right]. \tag{2.30}$$

Similarly

$$\Psi_{22} = G(ABC) \exp \left[ -\frac{8\pi^2}{\hbar \kappa^2} \left( A^2 + B^2 + C^2 \right) \right]. \tag{2.31}$$

Substituting Eqs. (2.30),(2.31) into Eq. (2.20), one obtains

$$16\pi^2 g\Psi_0 = -2\pi^2 h(ABC)^{-1} \left( A^2 + B^2 + C^2 \right) \left( \exp \right) F$$

$$+ \frac{3}{16} \hbar \kappa^2 (\exp) F' + \frac{2}{3} \left( 16\pi^2 \right) h(ABC)^{-1} \left( 2A^2 - B^2 - C^2 \right) \left( \exp \right) G \tag{2.32}$$

and cyclically, where

$$\exp = \exp \left[ -\frac{8\pi^2}{\hbar \kappa^2} \left( A^2 + B^2 + C^2 \right) \right]. \tag{2.33}$$

Now $\Psi_0$ should be invariant under permutations of $A, B, C$. Hence $G = 0$. I.e.

$$\Psi_{22} = 0. \tag{2.34}$$
The equation (2.32) and its cyclic permutations, with \( \Psi_{22} = 0 \), must be solved consistently with Eq. (2.17) and its cyclic permutations. Eliminating \( \Psi_0 \), one finds

\[
\frac{3h^3k^4}{16(16\pi^2g)} F'' - \frac{h^2k^2}{8g} \left( \frac{A^2 + B^2 + C^2}{ABC} \right) F' + 6\pi^2hgF - \frac{h^2k^2}{4g} \left( \frac{1}{B^2C^2} \right) F + \frac{h^2k^2}{8g} \left( \frac{(A^2 + B^2 + C^2)}{(ABC)^2} \right) F = 0 ,
\]

and cyclic permutations. Since \( F = F(ABC) \) is invariant under permutations, the \((BC)^{-2}F\) term and its permutations imply \( F = 0 \). Thus

\[
\Psi_{21} = 0 .
\]

Hence, using Eq. (2.32),

\[
\Psi_0 = 0 .
\]

Then we can argue using the duality mentioned earlier, to conclude that

\[
\Psi_{41} = \Psi_{42} = \Psi_6 = 0 .
\]

Hence there are no physical quantum states obeying the constraint equations in the diagonal Bianchi-IX model. This result will be discussed further in the Conclusion.

This shows that the Chern–Simons semi-classical wave function of Sano and Shiraishi [21] for \( N = 1 \) supergravity with \( \Lambda \)-term can only be an approximate, and not an exact state in the Bianchi-IX model. If it were exact, then one could make a Fourier transformation from the Ashtekar variables used in [21] to the variables \( A, B, C \) used here, to find a non-trivial solution.

**III THE \( k = +1 \) FRIEDMANN MODEL WITH \( \Lambda \)-TERM**

The \( k = +1 \) Friedmann model without a \( \Lambda \) term has been discussed in [2,6]. There are two linearly independent physical quantum states. One is bosonic and corresponds to
the wormhole state [15], the other is at quadratic order in fermions and corresponds to the Hartle–Hawking state [16]. In the Friedmann model with Λ term, the coupling between the different fermionic levels ‘mixes up’ this pattern [4].

In the Friedmann model, the wave function has the form [6]

\[ \Psi = \Psi_0(A) + (\beta_C \beta^C) \Psi_2(A) . \]  

As part of the Ansatz of [6], one requires \( \psi^A_i e^{AA'} \tilde{\psi}_{A'}^i \) and \( \tilde{\psi}^A_i = e^{AA'} \psi_A^i \); this is in order that the form of the one-dimensional Ansatz should be preserved under one-dimensional local supersymmetry, suitably modified by local coordinate and Lorentz transformations. Thus the gravitino field is truncated to spin \( \frac{1}{2} \). Note that \( \beta^A = \frac{3}{4} n^{AA'} \tilde{\psi}_{A'}^i \).

One then proceeds as in Sec. II to derive the consequences of the \( \overrightarrow{S}_A \Psi = 0 \) and \( S_A \Psi = 0 \) constraints at level \( \psi^1 \), by writing down the general expression for a constraint and then evaluating it at a Friedmann geometry. Note that it is not equivalent to set \( A = B = C \) in Eqs. (2.17) and (2.20); the coefficients in the constraint equations are different. One then obtains

\[ h\kappa^2 \frac{d^2 \Psi_0}{dA^2} + 48\pi^2 A \Psi_0 + 18\pi^2 h g A^2 \Psi_2 = 0 \]  

and

\[ h^2 \kappa^2 \frac{d^2 \Psi_2}{dA^2} - 48\pi^2 h A \Psi_2 - 256\pi^2 g A^2 \Psi_0 = 0 . \]  

These give second-order equations, for example

\[ A \frac{d^2 \Psi_0}{dA^2} - 2 \frac{d\Psi_0}{dA} + \left[ - \frac{48\pi^2}{h\kappa^2} A - \frac{(48)^2\kappa^4 A^3 + 9 \times 512\pi^4 g^2}{h^2\kappa^4} A^5 \right] \Psi_0 = 0 . \]  

This has a regular singular point at \( A = 0 \), with indices \( \lambda = 0 \) and 3. There are two independent solutions, of the form

\[ \Psi_0 = a_0 + a_2 A^2 + a_4 A^4 + \ldots , \]

\[ \Psi_0 = A^3 \left( b_0 + b_2 A^2 + b_4 A^4 + \ldots \right) , \]
convergent for all $A$. They obey complicated recurrence relations, where (e.g.) $a_6$ is related to $a_4$, $a_2$ and $a_0$.

One can look for asymptotic solutions of the type $\Psi_0 \sim (B_0 + \hbar B_1 + \hbar^2 B_2 + \ldots) \exp(-I/\hbar)$, and finds

$$I = \pm \frac{\pi^2}{g^2} \left( 1 - 2g^2 A^2 \right)^{\frac{3}{2}}, \quad (3.6)$$

for $2g^2 A^2 < 1$. The minus sign in $I$ corresponds to taking the action of the classical Riemannian solution filling in smoothly inside the three-sphere, namely a portion of the four-sphere $S^4$ of constant positive curvature. This gives the Hartle–Hawking state [16]. For $A^2 > (1/2g^2)$, the Riemannian solution joins onto the Lorentzian solution [22]

$$\Psi \sim \cos \left\{ \frac{1}{\hbar} \left[ \frac{\pi^2 (2g^2 A^2 - 1)}{g^2} - \frac{\pi}{4} \right] \right\}, \quad (3.7)$$

which describes de Sitter space-time.

**IV CONCLUSION**

We have seen here that there are no physical quantum states for $N = 1$ supergravity with a $\Lambda$-term, in the diagonal Bianchi-IX model. The same result was found for non-diagonal Bianchi-I models in [11]. The physical states found in Sec. III for the $k = +1$ Friedmann model, where the degrees of freedom carried by the gravitino field are $\beta_A$, disappear when the further fermionic degrees of freedom $\gamma_{ABC}$ of the Bianchi-IX model are included.

One could also study this from the point of view of perturbation theory about the $k = +1$ Friedmann model. As well as the usual gravitational harmonics [23], gravitino harmonics can be used [24]. For example, the Bianchi-IX model with radii $A, B, C$ close together describes a particular type of ‘gravitational wave’ distortion of the Friedmann
model; similarly for the $\gamma_{ABC}$ of the Bianchi-IX model, which describes a particular ‘gravitino wave’ distortion. Quite generally, in perturbation theory [23,25] one expects to find a wave function which is a product of the background wave function $\Psi(A)$ times an infinite product of wave functions $\psi_n$ (perturbations) where $n$ labels the harmonics. And one further expects that the perturbation wave function corresponding to the Bianchi-IX modes must be zero, by a perturbative version of the argument of Sec. II. [It will be interesting to investigate this.] Hence the complete perturbative wave function should be zero; then physical states would be forbidden for a generic model of the gravitational and gravitino fields with $\Lambda$-term. This suggests that the full theory of $N = 1$ supergravity with a non-zero $\Lambda$-term should have no physical states.

ACKNOWLEDGEMENTS

A.D.Y.C. thanks the Croucher Foundation of Hong Kong for financial support. P.R.L.V.M. gratefully acknowledges the support of a Human Capital and Mobility grant from the European Union (Program ERB401GT930714).
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