A Dynamical Bogomolov Property

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Abstract

A field $F$ is said to have the Bogomolov Property related to a height function $h$ if $h(\alpha)$ is either 0 or bounded from below by a positive constant for all $\alpha \in F$. In this paper we prove that the maximal algebraic extension of a number field $K$, which is unramified at a place $v \mid p$, has the Bogomolov Property related to all canonical heights coming from a Lattès map related to a Tate elliptic curve. To prove this algebraical statement we use analytic methods on the related Berkovich spaces.

In the whole paper $h$ is the standard logarithmic height, $p$ a prime number, $K$ a number field and $K^{nr,v}$ the maximal algebraic extension of $K$, which is unramified at the place $v \mid p$.

Definition 1.1. Let $\varphi : \mathbb{P}^n \to \mathbb{P}^n$ be a morphism of degree $d \geq 2$. The canonical height associated to $\varphi$ is the unique height function

$$\hat{h}_\varphi : \mathbb{P}^n \to \mathbb{R}$$

with the properties

$$\hat{h}_\varphi(P) = h(P) + O(1) \quad \text{and} \quad \hat{h}_\varphi(\varphi(P)) = d\hat{h}_\varphi(P).$$

See [Si07], 3.20, for a proof of the existence and uniqueness of this height. In [Si07], 3.22, it is shown that

$$\hat{h}_\varphi(P) = 0 \iff P \text{ is a preperiodic point of } \varphi.$$

We will write $\text{PrePer}(\varphi)$ for the set of all preperiodic points of $\varphi$. 
Definition 1.2. Let $E$ be an elliptic curve over a field $F$ of characteristic 0, $\Gamma$ a non trivial subgroup of $\text{Aut}(E)$, $\varphi$ an endomorphism of $E$ and $\pi : E \to E/\Gamma \cong \mathbb{P}^1_F$ a finite covering. A morphism $f : \mathbb{P}^1_F \to \mathbb{P}^1_F$ is called Lattès map related to $E$, if we have a commutative diagram

Here $T$ is an isomorphism of $E/\Gamma$ and $\mathbb{P}^1_F$.

Remark 1.3. This definition is independent of the choice of the isomorphism $T$. A change of the isomorphism is equivalent to a change of the coordinate on $\mathbb{P}^1_F$. So it would just change the representation of the map $f$ but not the map itself. This allows us to consider just the reduced Lattès diagram

with a finite covering $\pi : E \to \mathbb{P}^1_F$.

If $f$ is a Lattès map for a Tate elliptic curve $E$, then we will prove that $K^{nr,v}$ has the Bogomolov Property related to $\hat{h}_f$. In a slightly different context the Bogomolov Property for $K^{nr,v}$ is already known. We have the following result.

Theorem 1.4 (Gubler). Let $A$ be an abelian variety over $K$, which is totally degenerate at $v$. Further let $L$ be an ample even line bundle and $K'$ be a finite extension of $K^{nr,v}$. Then there is a $\varepsilon > 0$, such that $\hat{h}_L(P) \geq \varepsilon$ for all non-torsion points $P \in A(K')$.

Proof: See [Gu07], Corollary 6.7. \qed

In this theorem $\hat{h}_L$ stands for the canonical height, also called Néron-Tate height, associated to the line bundle $L$. For details on this height we refer to [BG], Chapter 9.2, and [La], Chapter 5.

One might expect the Bogomolov Property of $K^{nr,v}$ related to $\hat{h}_f$, for a Lattès map $f$ associated to a Tate elliptic curve, to be a direct consequence of theorem 1.4.
Let \( A \) be a Tate elliptic curve \( y^2 + xy = x^3 + a_4x + a_6 \) over \( K \), which is totally degenerate at \( v \). Notice that the Tate elliptic curves are precisely those with total degeneration at \( v \). Let \( \pi \) be the projection on the \( x \)-coordinate, \( \varphi = [m] \), \( m \in \mathbb{Z} \), be the multiplication by \( m \) map and \( f \) the corresponding Lattès map. (Later we will show that this is the general case.) The multiplication by \([-1]\) on \( A \) is given by \([-1](x, y) = (x, -x - y)\) (for example [BG], Proposition 8.3.8), so we can deduce that the line bundle \( L := \pi^*O(1) \) is even. By 1.1, 1.2 and the universal property of \( \hat{h}_L \) we get the equation

\[
\hat{h}_f \circ \pi = \hat{h}_L.
\] (1)

Now we see that the Bogomolov Property of \( K^{nr,v} \) related to \( \hat{h}_f \) would be a direct consequence of Theorem 1.4 if \([K^{nr,v}(\pi^{-1}(\mathbb{P}_K^1(K^{nr,v}))) : K^{nr,v}] \) is finite. But in general this is not the case. A counterexample for \( K = \mathbb{Q} \) is the Tate elliptic curve \( y^2 + xy = x^3 + 6 \) when we choose \( p = 3 \). To see this one needs a lot of algebraic number theory, but the proof will not be shown here. After this short justification we will prove our main theorem.

**Theorem 1.5.** Let \( f \) be a Lattès map related to a Tate elliptic curve over \( K^{nr,v} \) and let \( \hat{h}_f \) be the canonical height associated to \( f \). Then \( K^{nr,v} \) has the Bogomolov Property related to \( \hat{h}_f \). In other words:

There is a \( C > 0 \), such that for all \( P \in K^{nr,v} \setminus \text{PrePer}(f) \) we have \( \hat{h}_f(P) > C \).

Moreover there are just finitely many points \( a \in K^{nr,v} \) with \( \hat{h}_f(a) = 0 \).

Let \( w \mid v \) be an absolute value on \( K^{nr,v} \). By \( \mathbb{K} \) we denote the completion of the algebraic closure of the completion of \( K^{nr,v} \) by \( w \). We denote the unique extension of \( w \) to \( \mathbb{K} \) also by \( w \). Notice that \( \mathbb{K} \) is algebraically closed ([BGR], Proposition 3.4.3). Let \( E : y^2 + xy = x^3 + a_4x + a_6 \) be a Tate elliptic curve over \( \mathbb{K} \) and \( \pi : E \to \mathbb{P}_\mathbb{K}^1 \) be a finite covering. Further we take a Lattès map \( f \) with the commutative diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\varphi} & E \\
\downarrow{\pi} & & \downarrow{\pi} \\
\mathbb{P}_\mathbb{K}^1 & \xrightarrow{f} & \mathbb{P}_\mathbb{K}^1 \\
\end{array}
\]

where \( \varphi \) is the multiplication by \( m \in \mathbb{Z} \) with \( |m| \geq 2 \). The GAGA-functor on Berkovich spaces (see [Ber], Chapter 3.4) leads us to the commutative diagram
By $E^{an}$ and $(\mathbb{P}^1_K)^{an}$ we denote the Berkovich spaces related to $E$, respectively $\mathbb{P}^1_K$, and by $\varphi^{an}$, $\pi^{an}$ and $f^{an}$ we denote the analytification (in the sense of Berkovich) of the respective map.

We will work with the valuation function on $(\mathbb{P}^1_K)^{an}$

$$\text{val} : (\mathbb{P}^1_K)^{an} \to \mathbb{R} \cup \{\pm \infty\} ; \ y \mapsto -\log |X|_y,$$

where $X$ is the variable in the ring of polynomials $\mathbb{K}[X]$. As an analytic group $E$ is isomorphic to $\mathbb{K}^*/q^Z$ for a $q \in \mathbb{K}$ with $|q|_w < 1$. So there is a canonical valuation function $\text{val}$ on $E^{an} = (G_m)^{an}/q^Z$

$$\text{val} : (G_m)^{an}/q^Z \to \mathbb{R}/w(q)Z ; \ \bar{y} \mapsto -\log |X|_{\bar{y}}.$$

Obviously we have $\text{val}(E^{an}) = \mathbb{R}/w(q)Z$.

**Proposition 1.6.** $E$ has no complex multiplication and the map $\pi : E \to \mathbb{P}^1_K$ is, after a suitable coordinate transformation, explicitly given by $(x, y) \mapsto x$.

**Proof:** Let $j(E)$ be the $j$-invariant of $E$. Then we have the equation

$$|j(E)|_w = |q|^{-1}_w > 1.$$

Since $w$ is an extension of a $p$-adic absolute value, $j(E)$ is no algebraic integer. That shows the first statement (see [Si99], Theorem II.6.1). Further we know $\text{Aut}(E) = \{id, [-1]\}$ (see [Si07], Proposition 6.26). So there is just one quotient curve of $E$ which is different from $E$ itself, namely $E/\text{Aut}(E) \cong \mathbb{P}^1_K$. The function field of $E/\text{Aut}(E)$ is as the fixed field $\mathbb{K}(x, y)^{\text{Aut}(E)}$ given by $\mathbb{K}(x, y^2 + xy) = \mathbb{K}(x)$. Therefore the projection $\pi : E \to E/\text{Aut}(E)$ is given by $(x, y) \mapsto x$. \qed

Again we take the even ample line bundle $L := \pi^* \mathcal{O}(1)$. We have $\deg(f) = \deg(\varphi) = m^2$ (see [Si07], Theorem 6.51). This leads us to an isomorphism $\Phi : \mathcal{O}(1)^{m^2} \to f^* \mathcal{O}(1)$. $L$ is even and so the theorem of the cube tells us $\varphi^* L \cong L^{m^2}$. We choose the isomorphism $\Psi := \pi^* \Phi$. Notice that, by the commutativity of the Lattès diagram, we have $\varphi^* L = \varphi^* \pi^* \mathcal{O}(1) = \pi^* f^* \mathcal{O}(1)$. There are unique metrics $\|\cdot\|_f$ and $\|\cdot\|_{\varphi}$ on
\( \mathcal{O}(1) \) respectively \( L \) (to be more formal: on the analytifications of these line bundles) with the properties

\[
(\Phi_{\text{an}})^* (f_{\text{an}}^*)^\|_f = \|_f m^2 \quad \text{and} \quad (\Psi_{\text{an}})^* (\varphi_{\text{an}}^*)^\|_\varphi = \|_\varphi m^2.
\]

As usual we denote by \( \Psi_{\text{an}} \) and \( \Phi_{\text{an}} \) the analytifications (in the sense of Berkovich) of \( \Psi \) and \( \Phi \). For the existence of these metrics and further information we refer to [Zh], [Gu98] (especially Theorem 7.12) and [Gu10]. Just using the definitions of the different maps we get

\[
(\Psi_{\text{an}})^* (\varphi_{\text{an}}^*) (\pi_{\text{an}}^*)^\|_f = ((\pi_{\text{an}}^*)^\|_f)^m.
\]

This implies the equation

\[
(\pi_{\text{an}})^* \|_f = \|_\varphi.
\]  

(2)

Now we have a look at the canonical measures, also called Chambert-Loir measures, of the arithmetical dynamical systems \( (E, \varphi, L) \) and \( (\mathbb{P}^1_K, f, \mathcal{O}(1)) \). We denote these measures by \( \mu_{\varphi} = c_1(L, \|_\varphi) \) and \( \mu_f = c_1(\mathcal{O}(1), \|_f) \). For details, we refer to [Ch] and [Gu10]. Using the projection formula (for example [Gu07a], Corollary 3.9 b)) and \([2]\) we deduce

\[
(\pi_{\text{an}})^* \mu_{\varphi} = \text{deg}(\pi) \mu_f.
\]

This leads us to \( \text{supp}(\pi_{\text{an}})^* \mu_{\varphi}) = \text{supp}(\mu_f) \). As \( \mu_{\varphi} \) is a positive measure we get

\[
\pi_{\text{an}}(\text{supp}(\mu_{\varphi})) = \text{supp}(\pi_{\text{an}})^* \mu_{\varphi}) = \text{supp}(\mu_f).
\]  

(3)

**Remark 1.7.** Every disk \( (a, r) \) around \( a \in \mathbb{K} \) with radius \( r \in \mathbb{R}^+ \) gives us a multiplicative seminorm on the Tate-algebra \( \mathbb{K}\{X, qX^{-1}\} \), and hence a point of \( E^{\text{an}} \). Explicitly \( |\cdot|_{(a, r)} \) is given by

\[
| \sum_{n \in \mathbb{Z}} a_n X^n |_{(a, r)} = \left| \sum_{n \in \mathbb{Z}} b_n (X - a)^n \right|_{(a, r)} = \max_{n \in \mathbb{Z}} |b_n|_w r^n.
\]

The subdomain of \( E^{\text{an}} \) consisting of all points \((0, r)\), with \( |q|_w < r < 1 \) is called the skeleton of \( E \). We denote the skeleton of \( E \) by \( S(E) \). It is easy to see, that \( \text{val} \) maps \( S(E) \) homeomorphic onto \( \mathbb{R}/w(\mathbb{Q})\mathbb{Z} \). For the general theory of skeletons we refer to [Ber], Chapter 6.5, and for more information on our special case we refer to [Gu10], Example 7.2.

**Proposition 1.8.** With the same notations as above, we have

\[
\text{supp}(\mu_{\varphi}) = S(E).
\]
Proof: See [Gu10], Corollary 7.3. □

To prove Theorem 1.5 we assume that \( \hat{h}_f \) has no positive lower bound on \( K_{nr,v} \setminus \text{PrePer}(f) \). Then there are elements \( \{P_n\}_{n \in \mathbb{N}} \) in \( K_{nr,v} \setminus \text{PrePer}(f) \), such that

\[
\lim_{n \to \infty} \hat{h}_f(P_n) \to 0.
\]

We will show that this contradicts (3) and Proposition 1.8. In our setting, Yuan’s equidistribution Theorem states the following.

**Theorem 1.9** (Yuan). The Galoisorbits of \( \{P_n\} \) are equidistributed in \( (\mathbb{P}^1_K)^{an} \). This means:

\[
\mu_f = \lim_{n \to \infty} \frac{1}{|O(P_n)|} \sum_{P' \in O(P_n)} \delta_{P'},
\]

where \( O(P_n) \) denotes the Galoisorbit of \( P_n \) and \( \delta_{P'} \) the Dirac-measure at \( P' \).

**Proof:** See [Yu], Theorem 3.1. □

**Corollary 1.10.** If there is a sequence \( \{P_n\}_{n \in \mathbb{N}} \) as above, then

\[
\text{supp}(\mu_f) \subseteq \text{val}^{-1}(\frac{1}{e_v|_p} \mathbb{Z} \cup \{\pm \infty\}),
\]

where \( e_v|_p \) is the ramification index of \( v \) over \( p \).

**Proof:** Let \( y \in (\mathbb{P}^1_K)^{an} \) with \( \text{val}(y) \notin \frac{1}{e_v|_p} \mathbb{Z} \cup \{\pm \infty\} \). Choose an open neighbourhood \( I \) of \( \text{val}(y) \), such that \( I \) doesn’t contain an element of \( \frac{1}{e_v|_p} \mathbb{Z} \). The value group of \( w \) on \( K_{nr,v} \) is \( \frac{1}{e_v|_p} \mathbb{Z} \) and \( \text{val} \) is continuous, so the open neighbourhood \( U_y := \text{val}^{-1}(I) \) of \( y \) doesn’t contain a rational point of \( (\mathbb{P}^1_K)^{an}(K_{nr,v}) \). With (4) we get

\[
\mu_f(U_y) = \lim_{n \to \infty} \frac{1}{|O(P_n)|} \sum_{P' \in O(P_n)} \delta_{P'}(U_y) = 0.
\]

So \( y \) is no point of \( \text{supp}(\mu_f) \). This proves the Corollary. □

With Proposition 1.8 and (3) we conclude

\[
\pi^{an}(S(E)) \subseteq \text{val}^{-1}(\frac{1}{e_v|_p} \mathbb{Z} \cup \{\pm \infty\}).
\]

(5)

As \( \pi^{an} \) and \( \text{val} \) are continuous, \( \frac{1}{e_v|_p} \mathbb{Z} \cup \{\pm \infty\} \) is discrete and \( S(E) \) is not, this is very likely to be impossible. But to prove this we need a better understanding of the map \( \pi^{an} \).
In rigid geometry, Tate has described the isomorphism between $\mathbb{K}^*/q^\mathbb{Z}$ and $E^\text{an}(\mathbb{K})$. The $x$ and $y$ coordinate in $E^\text{an}(\mathbb{K})$ of an element $\zeta \in \mathbb{K}^*/q^\mathbb{Z}$ are explicitly given by

$$x(\zeta) = \sum_{n=-\infty}^{\infty} \frac{q^n \zeta}{(1-q^n \zeta)^2} - 2 \sum_{n=1}^{\infty} \frac{nq^n}{(1-q^n)},$$

$$y(\zeta) = \sum_{n=-\infty}^{\infty} \frac{q^{2n} \zeta^2}{(1-q^n \zeta)^2} + \sum_{n=1}^{\infty} \frac{nq^n}{(1-q^n)}.$$

For a proof and further information on this isomorphism we refer to [Si99], V.3 and V.4.

Thus $\pi^\text{an}$ is defined on rational points of $(G_1^m)^\text{an}/q^\mathbb{Z}$ by

$$\pi^\text{an}(\zeta) = \sum_{n=-\infty}^{\infty} \frac{q^n X}{(1-q^n X)^2} - 2 \sum_{n=1}^{\infty} \frac{nq^n}{(1-q^n)}.$$

As a morphism of strict $\mathbb{K}$-affinoid spaces $\pi^\text{an}$ is induced by a homomorphism $(\pi^\text{an})^\sharp: \mathbb{K}[X] \to \mathbb{K}\{X, qX^{-1}\}$ of the related $\mathbb{K}$-affinoid algebras (see [Ber], Chapter 2 and [Bo]). With Tate’s isomorphism we know

$$(\pi^\text{an})^\sharp(X) = \sum_{n=-\infty}^{\infty} \frac{q^n X}{(1-q^n X)^2} - 2 \sum_{n=1}^{\infty} \frac{nq^n}{(1-q^n)}.$$

Thus for any $f(X) \in \mathbb{K}[X]$ and any $y \in E^\text{an}$ we have

$$|f(X)|_{\pi^\text{an}(y)} = \left| f \left( \sum_{n=-\infty}^{\infty} \frac{q^n X}{(1-q^n X)^2} - 2 \sum_{n=1}^{\infty} \frac{nq^n}{(1-q^n)} \right) \right|_y.$$

In order to compute $\text{val}(\pi^\text{an}(0, r)) = -\log|X|_{\pi^\text{an}(0, r)}$ for an element $(0, r) \in S(E)$ we have to compute $|((\pi^\text{an})^\sharp(X))_{(0, r)}|$. It holds $|q^n X|_{(0, r)} = |q^n|_0 r < 1$ for all $n \geq 0$ and hence

$$\left| \frac{q^n X}{(1-q^n X)^2} \right|_{(0, r)} = |q^n X|_{(0, r)}$$

for all $n \geq 0$. Obviously we have also

$$r = |X|_{(0, r)} = |q^0 X|_{(0, r)} > |q^1 X|_{(0, r)} > \cdots$$

leading us to

$$\sum_{n=0}^{\infty} \frac{q^n X}{(1-q^n X)^2} \bigg|_{(0, r)} = r.$$  (6)
For all negative integers $n$, we have $|q^n X|_{(0, r)} > 1$, and hence

$$\left| \frac{q^n X}{(1 - q^n X)^2} \right|_{(0, r)} = \left| \frac{1}{q^n X} \right|_{(0, r)}$$

for all $n < 0$. With the trivial inequalities

$$\left| \frac{1}{q^{-1} X} \right|_{(0, r)} > \left| \frac{1}{q^{-2} X} \right|_{(0, r)} > \cdots$$

we conclude

$$\sum_{n=1}^{\infty} \frac{q^{-n} X}{(1 - q^{-n} X)^2} = \left| \frac{1}{q^{-1} X} \right|_{(0, r)} = |q|_w r^{-1}. \quad (7)$$

The equation

$$2 \sum_{n=1}^{\infty} \frac{nq^n}{(1 - q^n)} = 2 \sum_{n=1}^{\infty} \frac{nq^n}{(1 - q^n)} = 2|q|_w \quad (8)$$

similarly follows with elementary properties of non-archimedean absolute values. Since $(0, r)$ is an element of the skeleton, we know $|q|_w < r < 1$. So $\mathbf{[6], [7]}$ and $\mathbf{[8]}$ leads us to

$$|X|_{\pi^n(0, r)} = \left| \sum_{n=-\infty}^{\infty} \frac{q^n X}{(1 - q^n X)^2} - 2 \sum_{n=1}^{\infty} \frac{nq^n}{(1 - q^n)} \right|_{(0, r)} \leq \max\{r, |q|_w r^{-1}\}. \quad (9)$$

If we choose now $(0, r) \in S(E)$ with $1 < r^2 < |q|_w$ and $\log r \not\in \frac{1}{e_{cl} |p|} \mathbb{Z}$, then the value in $\mathbf{[9]}$ is equal to $r$. So there is an element with

$$\text{val}(\pi^n((0, r))) = -\log |X|_{\pi^n(0, r)} = -\log r \not\in \frac{1}{e_{cl} |p|} \mathbb{Z}.$$

This contradicts $\mathbf{[5]}$. We have shown, that there are no elements $\{P_n\} \in K^{nr,v} \setminus \text{PrePer}(f)$ with $\hat{h}_f(P_n) \to 0$. The finiteness of points $a \in K^{nr,v}$ with $\hat{h}_f(a) = 0$ follows with the same proof, when we assume the existence of infinitely many pairwise distinct points $\{P_n\}_{n \in \mathbb{N}}$ with $\hat{h}_f(P_n) = 0$. This is equivalent to the finiteness of $\text{PrePer}(f) \cap K^{nr,v}$ and proves theorem 1.5. \hfill \Box

**Remark 1.11.** The finiteness statement in Theorem 1.5 doesn’t hold if we start with an elliptic curve with (potential) good reduction at $v$. In this case the criterion of Néron-Ogg-Shafarevich states, that infinitely many torsion points are unramified over $v$. This means, that there are infinitely many torsion points $P$ in $E(K^{nr,v})$. With $\mathbf{[1]}$ we get for these points $P \in E(K^{nr,v})$ the property $\hat{h}_f \circ \pi(P) = 0$. As $\pi(P)$ is obviously unramified over $v$ and $\deg(\pi) = 2$, we have infinitely many elements $a \in K^{nr,v}$ with $\hat{h}_f(a) = 0$. 
The next proposition shows, that also the Bogomolov Property in Theorem 1.5 in general fails if we start with an elliptic curve with potential good reduction at \( v \).

**Proposition 1.12.** Let \( E \) be an elliptic curve over \( K \) with potential good reduction at \( v \parallel p \). Let \( f \) be a Lättès map associated to \( E \) and \( [m] \). If \( p \nmid m \) then there is a finite extension \( K' | K \) endowed with a non-archimedean place \( w | v \) and a sequence \( \{q_n\}_{n \in \mathbb{N}_0} \) in \( K^{nr,w} \) with

\[
\hat{h}_f(q_n) \to 0.
\]

**Proof:** Choose \( K' | K \) endowed with a non-archimedean absolute value \( w | v \) such that \( E \) over \( K' \) has good reduction at \( w \) and there is at least one non-torsion point in \( E(K^{nr,w}) \). Let \( \mathcal{E} \) be the Néron-model of \( E \) over the valuation ring \( R_w \) of \( K' \). Since \( E \) has good reduction at \( w \) and \( p \nmid m \), we know that the map \( [m] : \mathcal{E} \to \mathcal{E} \) is étale (see [BLR], 7.3 Lemma 2b) and [Si99], Example IV.3.1.4). Especially \( [m] \) is unramified.

Now we choose a non-torsion point \( Q_0 \in E(K^{mr,w}) \), then \( q_0 := \pi(Q_0) \) is no preperiodic point of \( f \). Let \( Q_1 \in E(K^{mr,w}) \) be a pre-image of \( Q_0 \) under \([m] \). Using \( [m] : \mathcal{E} \to \mathcal{E} \) is unramified, we get \( Q_1 \in E(K^{mr,w}) \) ([BG], Proposition B.3.6).

By successive repetition of this, when we replace \( Q_n \) by \( Q_{n+1} \) in each step, we get a sequence \( \{Q_n\}_{n \in \mathbb{N}_0} \) in \( E(K^{mr,w}) \) with \([m]^nQ_n = Q_0 \). We set \( q_n := \pi(Q_n) \). By using the Lättès diagram we get \( f^n(q_n) = q_0 \) for all \( n \in \mathbb{N}_0 \). As \( Q_n \) is in \( E(K^{mr,w}) \), \( q_n \) is in \( K^{mr,w} \). So we have found a sequence \( \{q_n\}_{n \in \mathbb{N}_0} \) in \( K^{mr,w} \), such that

\[
\hat{h}_f(q_n) = \left( \frac{1}{m^2} \right)^n \hat{h}_f(q_0) \to 0
\]

(see Definition 1.1). This proves, that the Bogomolov Property cannot hold in this case. \( \square \)

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