Exact Partition Function and Boundary State of 2-D Massive Ising Field Theory with Boundary Magnetic Field

R. Chatterjee
Department of Physics and Astronomy
Rutgers University
P.O.Box 849, Piscataway, NJ 08855-0849

Abstract

We compute the exact partition function, the universal ground state degeneracy and boundary state of the 2-D Ising model with boundary magnetic field at off-critical temperatures. The model has a domain that exhibits states localized near the boundaries. We study this domain of boundary bound state and derive exact expressions for the “g function” and boundary state for all temperatures and boundary magnetic fields. In the massless limit we recover the boundary renormalization group flow between the conformally invariant free and fixed boundary conditions.

1 Introduction

Quantum field theory (QFT) in the presence of boundaries plays an important role in our understanding of a wide range of physical phenomena. Examples include various surface critical behaviors [1], impurity effects (e.g. the Kondo problem) [2, 3], junctions in quantum wires [4], dissipative quantum mechanics [5, 6], etc. It is also important in open string theory [7, 8] where the open string states are created by vertex operators built from boundary operators in the boundary conformal field theory.

An important concept in boundary QFT is the boundary state. In Langrangian formulation the boundary condition may be expressed either in terms of a boundary action or as

1Email: robin@physics.rutgers.edu
a relevant perturbation of a conformal boundary condition. On the space-time of a cylinder of length $L$ and circle length $R$ with coordinate $x$ running along the length and coordinate $y$ running along the circle (Fig.1), the action may be written as:

$$
A = \int_{D} dx dy \mathcal{L}_{\text{bulk}} + \int_{B_l} dy \Phi_{B_l}(y) + \int_{B_r} dy \Phi_{B_r}(y)
$$

where $\Phi_{B_l}$ and $\Phi_{B_r}$ are boundary fields on the left and right boundaries respectively. In Hamiltonian formulation there are two alternative approaches: (1) In the first one chooses $x$ to be the space coordinate and $y$ the Euclidean time coordinate. The Hilbert space of states $\mathcal{H}_{L}^{B_lB_r}$ associated with a $y = \text{constant}$ time slice must satisfy boundary conditions $B_l$ and $B_r$ at the left and right boundaries respectively. For example, in the $L \to \infty$ limit the states in $\mathcal{H}_{L}^{B_lB_r}$ may be classified as asymptotic scattering states. Particles scatter with each other and off the boundaries. If the scattering is factorizable (admitting boundary Yang Baxter symmetry [9]) then, relative to any boundary (and again in the limit of large $L$), one may choose either a basis of asymptotic “in” states or of asymptotic “out” states to span $\mathcal{H}_{L}^{B_lB_r}$. The two bases are related by the factorizable scattering matrix. And the momenta of these asymptotic states are constrained by the boundary conditions in this Hamiltonian picture. The partition function is expressed as:

$$
Z = \text{Trace}_{\mathcal{H}_{L}^{B_lB_r}} (e^{-RH_L})
$$

where $H_L$ is the Hamiltonian, and an n-point correlation function of local operators $O_1(x_1, y_1), \ldots, O_n(x_n, y_n)$ is given by:

$$
\frac{1}{Z} \text{Trace}_{\mathcal{H}_{L}^{B_lB_r}} (T_y (O_1(x_1, y_1) \ldots O_n(x_n, y_n) e^{-RH_L}))
$$

where $T_y$ implies “$y$–ordering”. (2) In the other approach $x$ is considered to be the Euclidean time coordinate and $y$ the space coordinate. The Hilbert space $\mathcal{H}_R$ coincides with that of the theory with periodic (or antiperiodic) boundary conditions (i.e. the theory in finite space without boundary). The boundary conditions now appear as the initial state $|B_l>$ at time $x = 0$ evolving to the final state $|B_r>$ at time $x = L$. These states are called boundary states and encode all information about the respective boundary conditions. The partition function is now expressed as:

$$
Z = \langle B_r | e^{-LHR} | B_l \rangle
$$

where $H_R$ is the Hamiltonian in this formulation. An n-point function is expressed as:

$$
\frac{\langle B_r | T_x O_1(x_1, y_1) \ldots O_n(x_n, y_n) | B_l \rangle}{\langle B_r | B_l \rangle}
$$

where $T_x$ now implies “$x$–ordering”.

The two dimensional Ising model exhibits a rich variety of phenomena many of whose characteristics can be calculated exactly. It has often formed the basis of our theoretical
understanding of phase transitions and critical phenomena, the renormalization group and general quantum field theory. The model on a manifold with boundaries and with a boundary magnetic field has several interesting features. The boundary model is integrable and its exact boundary scattering matrix is known for all temperatures [9]. At critical temperature, i.e. in the massless case, the model describes a renormalization group (RG) flow from the free to the fixed conformal boundary conditions [4, 5] and exhibits a monotonically decreasing universal “g function” [4] akin to Zamolodchikov’s c function [12] in the bulk. This g function and the full boundary state were computed exactly in [13]. They have also been computed in [14] using the thermodynamic Bethe ansatz (TBA) approach for boundary theories [15, 16].

In general, the boundary TBA approach yields only the ratios of the g factors due to some difficulties mentioned in [16]. In this paper, we compute the exact partition function, ground state degeneracy factor g and boundary state for the massive case using the approach in [13].

2 2-D Ising Model with Boundary Magnetic Field

The 2-D Ising model with boundary magnetic field has been a subject of investigation for quite some time. In [17, 18] (see also [19]) the model was studied on the lattice and the boundary contribution to the free energy and the boundary spin correlation function were computed (see also [20]). The free (i.e. boundary spins are free) and fixed (i.e. all boundary spins are fixed to +1 or −1) limits of the boundary condition were studied in the context of boundary conformal Field theory and the boundary spin operator was identified in [10, 11]. At critical temperature, the boundary RG flow from the free down to the fixed conformal boundary conditions was identified in [2]. The model was studied in the context of boundary integrable field theory in [9]. In [21] the local magnetization was computed exactly in the massless theory using operator product expansions and were found to be hypergeometric functions. Differential equations for the local energy and magnetization have been derived recently in [22]. The exact partition function and boundary state were computed for the massless theory in [13, 14].

As is known, the continuum theory of the Ising model is described by free Majorana fermions. On a manifold D with boundaries B_i, i = 1...n, the model is described by the action:

\[
\mathcal{A} = \frac{1}{2\pi} \int_D dx \, dy \left[ \psi \partial_z \psi + \bar{\psi} \partial_{\bar{z}} \bar{\psi} + im\psi\bar{\psi} \right] + \sum_{j=1}^n \left\{ \int_{B_j} dt \left[ -\frac{i}{4\pi} \psi\partial_t \psi + \frac{1}{2} a\bar{a} \right] + ih \int_{B_j} dt \, a(t)(\frac{1}{2}\partial_t \psi + \frac{1}{2}\partial_t \bar{\psi})(t) \right\}
\]

6

Here z = x + iy and \( \bar{z} = x - iy \) are the usual complex coordinates on the manifold D; z = Z_j(t) and \( \bar{z} = \bar{Z}_j(t) \), with t being a real parameter, parametrize each boundary B_j; e_j(t) = \( \frac{dZ_j}{dt} \), e_j(t) = \( \frac{d\bar{Z}_j}{dt} \) are components of the tangent vector (e_j, \( \bar{e}_j \)) to the jth boundary with e_j(t)\( \bar{e}_j(t) = 1 \); \( \psi(x, y) \), \( \bar{\psi}(x, y) \) are the free Majorana fermion fields and m is the mass; a(t) = is a fermionic boundary field (see [11, 21]), anticommuting with \( \psi \) and \( \bar{\psi} \), with the two
point function:
\[
\langle a(t) \, a(t') \rangle_{\text{free}} = \frac{1}{2} \text{sign}(t - t')
\]  
and \( \dot{a} \equiv \frac{da}{dt} \); \( h \) is the appropriately rescaled external boundary magnetic field with the dimension of \([\text{length}]^{\frac{1}{2}}\). The integrand in the last term in (3) is the boundary spin operator \( \sigma_B \) (see [3, 9, 10]):
\[
\sigma_B(t) = a(t)(e_{j}^\frac{1}{2} \psi + \bar{e}_{j}^\frac{1}{2} \bar{\psi})(t)
\]  
The first two terms in (3) comprise the bulk action and the fermion mass \( m \) is related to the temperature by
\[
m \sim (T_c - T)
\]  
As is known, the field theories corresponding to the ordered \((T < T_c)\) and disordered \((T > T_c)\) phases are related by the duality transformation \( \psi \rightarrow \psi, \bar{\psi} \rightarrow -\bar{\psi} \) and are equivalent. We will assume the field theory for the low temperature phase i.e. \( m > 0 \) throughout this paper.

From (6) \( \psi, \bar{\psi} \) satisfy:
\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \psi + im\bar{\psi} = 0
\]  
\[
\left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \bar{\psi} - im\psi = 0
\]  
in the bulk and
\[
\left( \frac{d}{dt} + i\lambda \right) \psi_j(t) = \left( \frac{d}{dt} - i\lambda \right) \bar{\psi}_j(t)
\]  
at each boundary \( \mathcal{B}_j \), where
\[
\lambda = 4\pi h^2
\]  
and
\[
\psi_j(t) = e_{j}^\frac{1}{2}(t)\psi(Z_j(t)) \quad , \quad \bar{\psi}_j(t) = \bar{e}_{j}^\frac{1}{2}(t)\bar{\psi}(\bar{Z}_j(t))
\]  
For our cylindrical geometry of Fig.1 boundary conditions (12) take the form:
\[
\left( \frac{d}{dy} + i\lambda \right) \bar{\omega} \psi(0, -y) = \left( \frac{d}{dy} - i\lambda \right) \omega \bar{\psi}(0, -y) \quad \text{at the left boundary } x = 0, \quad \text{and}
\]  
\[
\left( \frac{d}{dy} + i\lambda \right) \omega \psi(L, y) = \left( \frac{d}{dy} - i\lambda \right) \bar{\omega} \bar{\psi}(L, y) \quad \text{at the right boundary } x = L
\]  
where \( \omega = e^{\frac{\pi}{4}} \) and \( \bar{\omega} = e^{-\frac{\pi}{4}} \).

In the Hamiltonian picture (which from now on we call the “L channel”) where \( x \) plays the role of Euclidean time and \( y \) is the spatial coordinate, \( \psi, \bar{\psi} \) admit the following decompositions in terms of plane waves in the Neveu Schwarz (NS) (where \( \psi, \bar{\psi} \) are antiperiodic along \( y \)) and Ramond (R) (where \( \psi, \bar{\psi} \) are periodic along \( y \)) sectors:
\[
\psi(x, y) = \sum_{\theta_n} \left[ a(\theta_n) \omega e^{\frac{\theta_n}{2} m (\cosh \theta_n x + i \sinh \theta_n y)} + a^\dagger(\theta_n) \bar{\omega} e^{\frac{\theta_n}{2} m (\cosh \theta_n x - i \sinh \theta_n y)} \right]
\]  

4
\[ \tilde{\psi}(x, y) = \sum_{\theta_n} \left[ a(\theta_n) \tilde{\omega} e^{-\frac{\theta_n}{2} e^{-m(\cosh \theta_n x + i \sinh \theta_n y)}} + a^\dagger(\theta_n) \omega e^{-\frac{\theta_n}{2} e^{m(\cosh \theta_n x + i \sinh \theta_n y)}} \right] \] (18)

where

\[ \sinh \theta_n = \frac{2\pi}{mR} (n + \frac{1}{2}), \quad n \in \mathbb{Z} \quad \text{in the NS sector} \] (19)

and

\[ \sinh \theta_n = \frac{2\pi}{mR} n, \quad n \in \mathbb{Z} \quad \text{in the R sector} \] (20)

In (17), (18) \( a(\theta_n) \) and \( a^\dagger(\theta_n) \) are the annihilation and creation operators for a Fermi particle moving along the circle with momentum \( m \sinh \theta_n \) and energy \( m \cosh \theta_n \), \( \theta_n \) being its rapidity, and satisfying standard anticommutation relations. A boundary state \( |B_n\rangle \) at \( x = 0 \) satisfies

\[ \left[ a(-\theta_n) - i \tanh \frac{\theta_n}{2} \left( \frac{Q - \cosh \theta_n}{Q + \cosh \theta_n} \right) a^\dagger(\theta_n) \right] |B_n\rangle = 0 \] (21)

where

\[ Q = \left( \frac{\lambda}{m} - 1 \right) = \left( \frac{4\pi h^2}{m} - 1 \right) \] (22)

and a similar equation holds for a boundary state at \( x = L \). These are obtained by imposing (15), (16) on (17) and (18). Notice that the factor in the second term in (21) is precisely the boundary scattering matrix in the cross channel \( \mathcal{K}(\theta_n) = \mathcal{R}(\frac{4\pi h^2}{m} - \theta_n) \) discussed in [9].

In the cross channel (henceforth we call it the “R channel”) in which \( y \) is the Euclidean time and \( x \) is the space coordinate, the mode expansions for \( \psi, \tilde{\psi} \) are:

\[ \psi(x, y) = \sum_{\theta_l} \left[ i b(\theta_l) e^{-\frac{\theta_l}{2} e^{m(i \sinh \theta_l x + \cosh \theta_l y)}} + b^\dagger(\theta_l) e^{-\frac{\theta_l}{2} e^{-m(i \sinh \theta_l x + \cosh \theta_l y)}} \right] \] (23)

\[ \tilde{\psi}(x, y) = \sum_{\theta_l} \left[ -i b(\theta_l) e^\frac{\theta_l}{2} e^{m(i \sinh \theta_l x + \cosh \theta_l y)} + b^\dagger(\theta_l) e^\frac{\theta_l}{2} e^{-m(i \sinh \theta_l x + \cosh \theta_l y)} \right] \] (24)

for \( y \in [0, R) \) and

\[ \psi(x, R) = -\psi(x, 0) \quad \text{and} \quad \tilde{\psi}(x, R) = -\tilde{\psi}(x, 0) \quad \text{in the NS sector}, \] (25)

\[ \psi(x, R) = \psi(x, 0) \quad \text{and} \quad \tilde{\psi}(x, R) = \tilde{\psi}(x, 0) \quad \text{in the R sector} \] (26)

Here \( b(\theta_l), b^\dagger(\theta_l) \) destroy and create fermions (in the R channel) which now move along \( x \) with rapidity \( \theta_l \). These fermions scatter off the boundaries with the scattering matrix \( \mathcal{R}(\theta) \) given by [9]:

\[ b^\dagger(-\theta) = \mathcal{R}(\theta) b^\dagger(\theta); \quad \mathcal{R}(\theta) = i \tanh \left( \frac{i\pi}{4} - \frac{\theta}{2} \right) \frac{Q + i \sinh \theta}{Q - i \sinh \theta} \] (27)

(28)

which can be obtained from (15), (23) and (24). The rapidities \( \theta_l \) in (23), (24) are solutions of:

\[ 1 + X = 0, \quad \text{where} \quad X = e^{2imL \sinh \theta \tanh^2 \left( \frac{i\pi}{4} - \frac{\theta}{2} \right)} \left( \frac{Q - i \sinh \theta}{Q + i \sinh \theta} \right)^2 \] (29)
which we get from (16), (23), (24) and (28). We notice that although $\theta_l = 0$ is a solution of (29) it is unphysical since $R(0) = -1$ implying that the wave function for this zero mode vanishes. In the next section we start with the partition function (2) as a trace over asymptotic states (23), (24), (29) and make explicit transformations so as to express it as a sum over states in the cross channel. In the large $L$ limit this will yield us the boundary state.

3 Partition Function

As mentioned in the earlier section, the spinor fields $\psi, \bar{\psi}$ can be either periodic or antiperiodic along the circle of the cylinder. We have been calling these the Ramond (R) and the Neveu Schwarz (NS) sectors respectively. In the R channel the states associated with the $y = 0$ and $y = R$ lines are identical in the Ramond case while for the Neveu Schwarz case they differ by a phase of $\pi$. The partition function (2) in this channel generalizes to $Z_+$ and $Z_-$ in the Neveu Schwarz and Ramond sectors respectively, where:

$$Z_\pm = \text{Trace}_{R^+_L, R^-_L} \left( (\pm 1)^{\mathcal{F}} e^{-RH_L} \right),$$

$\mathcal{F}$ being the fermion number operator. For our boundary Ising model (6) the partition function (30) is given by:

$$Z_\pm = e^{-RE_L} \prod_{\theta_l > 0} (1 \pm e^{-mR \cosh \theta_l})$$

where $E_L$ is the ground state energy and $\theta_l$'s are solutions of (28). The free energy $F_\pm$ is given by:

$$-RF_\pm = -RE_L + \sum_{\theta_l > 0} \log(1 \pm e^{-mR \cosh \theta_l})$$

which can be written as (as in [13]):

$$-RF_\pm = -RE_L + \frac{1}{2\pi i} \int_C d\theta \frac{X'}{1 + X} \log(1 \pm e^{-mR \cosh \theta})$$

where $X' = \frac{dX}{d\theta}$ and the contour $C$ in the complex $\theta$ plane is shown in Fig.2 corresponding to the case when $Q > 1$. We note here that $X(-\theta) = 1/X(\theta)$ and $X(i\pi - \theta) = X(\theta)$ The positions of zeroes and poles of $1 + X$ in the complex $\theta$ plane fall into three distinct domains:

(i) $Q > 1$: In this case all zeroes of $1 + X$ lie on the real axis and they cluster closer together further from the origin and with increasing $L$. There are no poles in the upper half plane in the physical strip. The picture is shown in Fig.2.

(ii) $0 < Q < 1$: In this domain also all zeroes are on the real axis as in (i). There is a pole on the negative imaginary axis at $-i \sin^{-1} Q$ in addition to the ubiquitous pole at $-i\pi/2$ but none in the physical strip of the upper half plane. The analytic structure is shown in Fig.3.

(iii) $Q < 0$: This domain is perhaps more interesting. There is a pole on the positive imaginary axis at $\theta = -i \sin^{-1} Q$ and possible zeroes on the imaginary axis as well. The pole
corresponds to a boundary bound state in the R channel. We will discuss this domain in greater detail later in the section.

Let us first consider domains (i) and (ii) where all solutions of (29) are real and $1 + X$ has no pole in the physical strip.

### 3.1 $Q > 0$: Domain of No Boundary Bound State

Here we treat domains (i) and (ii) together. First we compute $E_L$:

$$E_L = -\frac{1}{2} \sum_{\theta_i > 0} m \cosh \theta_i = -\frac{1}{4\pi i} \int_{C} d\theta \frac{X'}{1 + X} m \cosh \theta$$  \hspace{1cm} (34)

We can rewrite the integral on the right above as:

$$E_L = -\frac{1}{8\pi i} \left( \int_{C_+} d\theta + \int_{C_-} d\theta - \int_{C_0} d\theta \right) \frac{X'}{1 + X} m \cosh \theta$$  \hspace{1cm} (35)

where the contours $C_+$, $C_-$ and $C_0$ are shown in Fig.4 and Fig.5. The last integral in (35) is:

$$\frac{m}{8\pi i} \int_{C_0} d\theta \frac{X'}{1 + X} \cosh \theta = \frac{m}{4}$$  \hspace{1cm} (36)

We rewrite the $C_-$ integral in (35) as:

$$\frac{m}{8\pi i} \int_{C_-} d\theta \frac{X'}{1 + X} \cosh \theta = -\frac{m}{8\pi i} \int_{C_-} d\theta \frac{X' \cosh \theta}{X^2(1 + \frac{1}{X})} + \frac{m}{8\pi i} \int_{C_0} d\theta \frac{X'}{X} \cosh \theta$$  \hspace{1cm} (37)

After changing variable $\theta \to -\theta$ in the first term above and expanding the second term we obtain:

$$E_L = \frac{m}{4} - \frac{1}{4\pi i} \int_{C_+} d\theta \frac{X'}{1 + X} \cosh \theta - \frac{m^2 L}{2\pi} \int_{0}^{\infty} d\theta \cosh^2 \theta - \frac{m}{2\pi} \int_{0}^{\infty} d\theta \left[ 1 + \frac{2Q \cosh^2 \theta}{Q^2 + \sinh^2 \theta} \right]$$  \hspace{1cm} (38)

The third term in (38) contains the nonuniversal bulk intensive free energy which we set to zero. We recognize the fourth term to be the boundary intensive free energy and denote it by $\varepsilon_\lambda$:

$$2\varepsilon_\lambda = -\frac{m}{2\pi} \int_{0}^{\infty} d\theta \left[ 1 + \frac{2Q \cosh^2 \theta}{Q^2 + \sinh^2 \theta} \right]$$  \hspace{1cm} (39)

We now consider the second term in (38). We shift the contour up by $\frac{i\pi}{2}$ (see Fig.6) (note that we can do this trivially since the integrand has no singularity in this region), change variable $\theta \to (\frac{i\pi}{2} - \theta)$ and integrate by parts to obtain:

$$\frac{1}{4\pi i} \int_{C_+} d\theta \frac{X'}{1 + X} \cosh \theta = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\theta \cosh \theta \log(1 + Y_L(\theta))$$  \hspace{1cm} (40)
where

\[ Y_L(\theta) = X\left(\frac{i\pi}{2} - \theta\right) = e^{-2mL \cosh \theta \tanh^2 \frac{\theta}{2} \left(\frac{Q - \cosh \theta}{Q + \cosh \theta}\right)^2} \]  

(41)

Finally we combine (38), (39), (40) to obtain:

\[ E_L = \frac{m}{4} + 2\varepsilon \lambda - \frac{m}{4\pi} \int_{-\infty}^{\infty} d\theta \cosh \theta \log(1 + Y_L(\theta)) \]  

(42)

Now let us compute the second term in (33). As we did for \( E_L \), we rewrite the integral as:

\[ -RF_\pm = -RE_L + \frac{1}{4\pi i} \left( \int_{C_+} d\theta + \int_{C_-} d\theta - \int_{C_0} d\theta \right) \frac{X'}{1 + X} \log(1 \pm Y_R) \]  

(43)

where

\[ Y_R = e^{-mR \cosh \theta} \]  

(44)

The last integral is trivial:

\[ \frac{1}{4\pi i} \int_{C_0} d\theta \frac{X'}{1 + X} \log(1 \pm Y_R) = \frac{1}{2} \log(1 \pm e^{-mR}) \]  

(45)

Once again, in the integral along contour \( C_- \), we rewrite in the integrand:

\[ \frac{X'}{1 + X} = -\frac{X'}{X^2(1 + \frac{1}{X})} + \frac{X'}{X} \]  

(46)

Then we transform variable \( \theta \rightarrow -\theta \) as we did in (38) and get:

\[ \frac{1}{4\pi i} \int_{C_-} d\theta \frac{X'}{1 + X} \log(1 \pm Y_R) = \frac{1}{4\pi i} \int_{C_+} d\theta \frac{X'}{1 + X} \log(1 \pm Y_R) \]

\[ + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta mL \cosh \theta \log(1 \pm Y_R) + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta \left[ \frac{1}{\cosh \theta} + \frac{2Q \cosh \theta}{Q^2 + \sinh^2 \theta} \right] \log(1 \pm Y_R) \]  

(47)

We recognize the second term in (47) to be the Casimir energy:

\[ E_R^{(\pm)} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta mL \cosh \theta \log(1 \pm e^{-mR \cosh \theta}) \]  

(48)

where the \( \pm \) signs, as usual, correspond to the NS and R sectors. Next, for the first term in (48), we apply a Wick rotation of the momentum axis in the complex plane by shifting the contour \( C_+ \) in the \( \theta \) plane up by \( i\pi/2 \) (see Fig.6). This enables us to go to the cross channel. Then we change variable \( \theta \rightarrow (\theta - \frac{i\pi}{2}) \) and get:

\[ \frac{1}{2\pi i} \int_{C_+} d\theta \frac{X'}{1 + X} \log(1 \pm Y_R) = -\frac{1}{2\pi i} \int_{C'_+} d\theta \frac{Y'_L}{1 + Y_L} \log(1 \pm X_R) \]  

(49)
where the contour $C'$ is shown in Fig. 7, $Y_L$ is given by (41) and

$$X_R(\theta) = Y_R(\frac{i\pi}{2} - \theta) = e^{-mR \sinh \theta}$$

(50)

After integrating by parts (49) yields:

$$\frac{1}{2\pi i} \int_{C_+} d\theta \frac{X'}{1 + X} \log(1 \pm Y_R) = \frac{1}{2\pi i} \int_{C'} d\theta \frac{X'}{X_R \pm 1} \log(1 + Y_L)$$

(51)

We rewrite the integral along $C'$ as a sum of integrals along $C'_+$ and $C'_-$ (see Fig. 7)

$$\frac{1}{2\pi i} \int_{C'} d\theta \frac{X'}{X_R \pm 1} \log(1 + Y_L) = \frac{1}{2\pi i} \left( \left[ \int_{C_+} d\theta + \int_{C_-'} d\theta \right] \frac{X'}{X_R \pm 1} \log(1 + Y_L) \right)$$

(52)

Our aim is to express the free energy as a sum over states in the cross (i.e. L) channel. So with hindsight we change variables $\theta \rightarrow -\theta$ in the $C'_-$ integral in (52) and obtain:

$$\frac{1}{2\pi i} \int_{C_-'} d\theta \frac{X'}{X_R \pm 1} \log(1 + Y_L) = \frac{1}{2\pi i} \left( \left[ \int_{C_+} d\theta + \int_{C_-'} d\theta \right] \frac{X'}{X_R \pm 1} \log(1 + Y_L) \right)$$

(53)

Thus (52) becomes:

$$\frac{1}{2\pi i} \int_{C'} d\theta \frac{X'}{1 + X} \log(1 \pm Y_R) = \frac{1}{2\pi i} \int_{C'} d\theta \frac{X'}{X_R \pm 1} \log(1 + Y_L)$$

$$- \frac{mR}{2\pi} \int_0^\infty d\theta \cosh \theta \log(1 + Y_L)$$

(54)

where $C'' = C'_+ - C'_-$ is shown in Fig. 8. The first term above evaluates to:

$$\frac{1}{2\pi i} \int_{C''} d\theta \frac{X'}{X_R \pm 1} \log(1 + Y_L)$$

$$= \begin{cases} 
\log \Sigma_+ = \sum_{l \geq 0} \log(1 + Y_L(\omega_l)) , & m \sinh \omega_l = \frac{2\pi}{R} , \quad l \in \mathbb{Z} + \frac{1}{2} \text{ for (+) } \\
\log \Sigma_- = \sum_{n \geq 0} \log(1 + Y_L(\omega_n)) , & m \sinh \omega_n = \frac{2\pi}{R} , \quad n \in \mathbb{Z} \text{ for (−) }
\end{cases}$$

(55)

Finally combining (42), (43), (45), (47), (48), (54) and (55) we obtain:

$$- RF_\pm = - LE_R^{(\pm)} - 2\varepsilon \lambda R + \log \Sigma_\pm - \frac{1}{2} \log(e^{\frac{mR}{2}} + e^{-\frac{mR}{2}})$$

$$+ \frac{1}{2\pi} \int_{-\infty}^\infty d\theta \left[ \frac{1}{\cosh \theta} + \frac{2Q \cosh \theta}{Q^2 + \sinh^2 \theta} \right] \log(1 \pm e^{-mR \cosh \theta})$$

(56)
We identify the last two terms as the universal ground state degeneracy:

\[
\log g_\pm(R) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\theta \left[ \frac{1}{\cosh \theta} + \frac{2Q \cosh \theta}{Q^2 + \sinh^2 \theta} \right] \log(1 \pm e^{-mR \cosh \theta}) \\
- \frac{1}{4} \log(e^{\frac{mR}{2}} \pm e^{-\frac{mR}{2}})
\]  

(57)

Thus from (56) and (57) the partition function is:

\[
Z_\pm = e^{-LE_{(\pm)} - 2\Sigma_\pm} \Sigma_\pm g_\pm^2(R)
\]  

(58)

Expressions (56) and (58) with (55) and (57) are exact. Notice that in the \( L \to \infty \) limit \( \Sigma_\pm \to 1 \).

Let us now look closely at the \( Q < 0 \) domain.

### 3.2 \( Q < 0 \) : Domain Containing Boundary Bound State

In the \( Q < 0 \) domain, i.e. for sufficiently weak boundary magnetic field, the boundary scattering matrix [9] has a pole in the physical strip at \( \theta = -i \sin^{-1} Q \). Correspondingly \( 1 + X \) has a double pole at this same \( \theta \). A careful analysis shows that, for sufficiently large distance \( L \) between the boundaries, there are also two zeroes pinching the pole from above and below and approaching it along the imaginary axis exponentially fast with increasing \( L \). There are also two more zeroes on the negative imaginary axis located symmetrically to the upper ones and approaching \( \theta = i \sin^{-1} Q \) at the same rate. There is however no pole there. These zeroes correspond to states (in the \( R \) channel) localized near the boundaries. In fact there are two such states because the particle can be bound near the left or the right boundary and the zero above the pole corresponds to a symmetric (S) superposition of states localized near the left and right boundaries while the zero below corresponds to their antisymmetric (A) superposition. Each boundary may be looked upon as an infinitely heavy particle sitting there with the light fermions scattering off it. At weak boundary magnetic fields they form bound states – the fermion gets trapped by the heavy boundary particle. In the limit of large \( L \) the S and A superposed states are degenerate (each bound fermion does not see the opposite boundary and their wave functions do not overlap) while, for finite \( L \), they split by an amount exponentially small in \( L \) due to the overlap between the left and right localized states (the bound fermion can now tunnel through the finite barrier and become bound to the other boundary particle). Moreover, when \( L \) decreases one zero each from above and below the real axis, which are closer to the origin, eventually leave the imaginary axis and go to the real axis joining their friends there. The picture for sufficiently large \( L \) is shown in Fig.8. As we will see our results will not depend on \( L \).

So now we include the contributions from the extra solutions to (29) in the integrals in (34) and (43). We have:

\[
E_L = -\frac{1}{4\pi i} \left[ \int_C d\theta + \int_{\sigma_1} d\theta + \int_{\sigma_2} d\theta \right] \frac{X'}{1+X} m \cosh \theta
\]  

(59)
and
\[-RF_\pm = -RE_L + \frac{1}{2\pi i} \left[ \int_C d\theta + \int_{\sigma_1} d\theta + \int_{\sigma_2} d\theta \right] \frac{X'}{1 + X} \log(1 \pm e^{-mR \cosh \theta}) \tag{60}\]

where \(\sigma_1\) and \(\sigma_2\) are small closed contours enclosing the zeroes below and above the pole respectively in the upper half plane. We repeat the same steps as before in computing the integrals along contour \(C\). However, when we shift the integration contour \(C_+\) up by \(\frac{i\pi}{2}\) in obtaining (40) and (49) we have to avoid the singularities of \(\frac{X'}{1 + X}\) in the integrand at the two zeroes and the (double) pole of \((1 + X)\) (see Fig.9). In the process we pick up contributions from each of these singularities (see Fig.10):

\[
\frac{1}{2\pi i} \int_{C_+} d\theta X' \left\{ \int_C d\theta - \int_{\sigma_1} d\theta - \int_{\sigma_2} d\theta - \int_{\sigma_{pole}} d\theta \right\} \frac{X'}{1 + X} \tag{61}\]

Here \(C_+\) is the same as in (35) and (43). The last two integrals cancel exactly with corresponding integrals in (59) and (60) and we are left with the integral along \(\sigma_{pole}\) only. Thus the extra contribution, in effect, comes only from the pole, irrespective of the number of zeroes on the imaginary axis (of course we have to include all these zeroes in (59), to start with). In the light of the above discussion, our result is therefore independent of \(L\) as long as we include a \(\sigma\) contour for each zero (of \(1 + X\)) in the physical strip. Thus we have:

\[
\frac{1}{4\pi i} \int_{\sigma_{pole}} d\theta X' m \cosh \theta = -mR \cos u \tag{62}\]

where
\[u = -\sin^{-1} Q, \quad Q < 0 \tag{63}\]

Expression (62) gives the bound state contribution to the boundary energy. For \(Q < 0\) the intensive boundary energy (let us now denote it by \(\varepsilon_{\lambda}^b\)) is therefore:

\[
2\varepsilon_{\lambda}^b = -\frac{m}{2\pi} \int_0^\infty d\theta \left[ 1 + \frac{2Q \cosh^2 \theta}{Q^2 + \sinh^2 \theta} \right] - mR \cos u, \quad Q < 0 \tag{64}\]

And we have:

\[
-\frac{1}{2\pi i} \int_{\sigma_{pole}} d\theta X' \log(1 \pm e^{-mR \cosh \theta}) = 2 \log(1 \pm e^{-mR \cos u}) \tag{65}\]

implying that now in the presence of boundary bound state in the \(R\) channel, the \(L\) channel ground state degeneracy function (which we denote by \(g_{bL}^b\) for \(Q < 0\)) is:

\[
g_{bL}(R) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\theta \left[ \frac{1}{\cosh \theta} + \frac{2Q \cosh \theta}{Q^2 + \sinh^2 \theta} \right] \log(1 \pm e^{-mR \cosh \theta})
- \frac{1}{4} \log(e^{\frac{mR}{2}} \pm e^{-\frac{mR}{2}}) + \log(1 \pm e^{-mR \cos u}), \quad Q < 0 \tag{66}\]
In an alternative interpretation, we can consider $g^b_\pm$ as an analytic continuation of $g_\pm$ given in (57) with the integral in (57) now being along the contour $C^{(Q)}$ shown in Fig.11 instead of along the real axis. The contour surrounds the two poles of the integrand at $\pm i \sin^{-1} Q$. Notice that the term in $\cdots$ in the integrand is precisely the derivative of the phase shift from boundary scattering and appears also in the boundary thermodynamic Bethe ansatz approach \cite{16}. $g_\pm$ defined this way covers the whole domain of $Q$ and we treat this as our general definition for $g_\pm$. Denoting this by $g^{(Q)}_\pm(R)$, we have:

$$g^{(Q)}_\pm(R) = \frac{1}{4\pi} \int_{C^{(Q)}} d\theta \left[ \frac{1}{\cosh \theta} + \frac{2Q \cosh \theta}{Q^2 + \sinh^2 \theta} \right] \log(1 \pm e^{-mR \cosh \theta}) - \frac{1}{4} \log \left( e^{mR} \mp e^{-mR} \right)$$

(67)

$$g^{(Q)}_\pm(R) = \begin{cases} 
  g_\pm(R), & Q \geq 0 \\
  g^b_\pm(R), & Q < 0 
\end{cases}$$

(68)

where the contour $C^{(Q)}$ is shown in Fig.11. In the same way, we define the intensive boundary energy $\varepsilon^{(Q)}_\lambda$ for all $Q$:

$$2\varepsilon^{(Q)}_\lambda = -\frac{m}{4\pi} \int_{C^{(Q)}} d\theta \left[ 1 + \frac{2Q \cosh^2 \theta}{Q^2 + \sinh^2 \theta} \right]$$

(69)

$$\varepsilon^{(Q)}_\lambda = \begin{cases} 
  \varepsilon_\lambda, & Q \geq 0 \\
  \varepsilon^b_\lambda, & Q < 0 
\end{cases}$$

(70)

The partition function in the whole domain of $Q$ is therefore given by:

$$Z^{(Q)}_\pm = e^{-LE^{(\pm)}_R - 2R \varepsilon^{(Q)}_\lambda} \Sigma_\pm (g^{(Q)}_\pm(R))^2$$

(71)

We will now compute the boundary state in the next section.

4 Boundary State

We consider the low temperature phase $T < T_c$ of the Ising system as we have been doing. As we discussed earlier the boundary state belongs to the $L$ channel Hilbert space $\mathcal{H}_R$. In this space there are two vacuum states $|0\rangle$ and $|\sigma\rangle$ lying in the Neveu-Schwarz and Ramond sectors respectively. In the infinite volume (large $R$) limit the allowed momenta are continuous. The ground states $|0\rangle$ and $|\sigma\rangle$ become degenerate. They are respectively the symmetric and antisymmetric combinations of the vacuum states $|0_+\rangle$ and $|0_-\rangle$ which are the broken $Z_2$ symmetry ground states and correspond to all spins pointing up and all spins pointing down respectively:

$$|0\rangle = \frac{1}{\sqrt{2}} (|0_+\rangle + |0_-\rangle), \quad |\sigma\rangle = \frac{1}{\sqrt{2}} (|0_+\rangle - |0_-\rangle)$$

(72)
If \( \Pi \) is the overall spin flip operator, we have \( \Pi \ket{0} = \ket{0} \) and \( \Pi \ket{\sigma} = -\ket{\sigma} \). Obviously \( Z_+ \) and \( Z_- \) represent contributions from the \( \Pi = 1 \) and \( \Pi = -1 \) respectively. We expect that:

\[
Z_\pm^{(Q)} = Z_{hh} \pm Z_{h-h}
\]

where

\[
Z_{hh} = \bra{B_h} e^{-L H_R} \ket{B_h} \quad \text{and} \quad Z_{h-h} = \bra{B_h} e^{-L H_R} \ket{B_{-h}}
\]

implying that

\[
Z_{hh} = \frac{1}{2} \left( Z_+^{(Q)} + Z_-^{(Q)} \right) \quad \text{and} \quad Z_{h-h} = \frac{1}{2} \left( Z_+^{(Q)} - Z_-^{(Q)} \right)
\]

For large \( L \) we have:

\[
Z_{hh} = e^{-LE_R^{(+)} \left( \bra{0} B_h \ket{} \right)^2} + e^{-LE_R^{(-)} \left( \bra{\sigma} B_h \ket{} \right)^2}
\]

\[
Z_{h-h} = e^{-LE_R^{(+)} \left( \bra{0} B_{-h} \ket{} \right)^2} - e^{-LE_R^{(-)} \left( \bra{\sigma} B_{-h} \ket{} \right)^2}
\]

where we have used \( \bra{0} B_{-h} \ket{} = \bra{0} B_h \ket{} \) and \( \bra{\sigma} B_{-h} \ket{} = -\bra{\sigma} B_h \ket{} \) and we have chosen the phase of \( \ket{0} \) to be such that \( \bra{0} B_h \ket{} \) and \( \bra{\sigma} B_h \ket{} \) are real (see [2]). Thus, from (75) – (77) it follows that:

\[
\bra{0} B_h \ket{} = \frac{\sqrt{2}}{1 + e^{-\frac{2\pi}{mR} j \theta}}
\]

\[
\bra{\sigma} B_h \ket{} = \frac{\sqrt{2}}{1 + e^{-\frac{2\pi}{mR} j \theta}}
\]

As was shown in [3], the presence of an infinite set of integrals of motion for a system with boundary makes the boundary state take a simple form with contributions from pairs of particles of opposite momenta (and possibly a zero mode) only. Following [3] and as was done in [4] for the critical case, we write down the boundary state from (17), (18) and (78):

\[
B_{\pm h} = \frac{1}{\sqrt{2}} e^{-R \varepsilon_\lambda^{(Q)}(R)} \left[ g_+^{(Q)}(R) \exp \left\{ \sum_{n=0}^{\infty} K(\theta_{n+\frac{1}{2}}) a^\dagger(\theta_{n+\frac{1}{2}}) a(-\theta_{n+\frac{1}{2}}) \right\} \ket{0} \right. \\
\left. \quad \pm g_-^{(Q)}(R) \exp \left\{ \sum_{n=1}^{\infty} K(\theta_n) a^\dagger(\theta_n) a(-\theta_n) \right\} \ket{\sigma} \right]
\]

where \( \sinh \theta_j = \frac{2\pi}{mR} j \), \( g_{\pm}^{(Q)}(R) \) and \( \varepsilon_\lambda^{(Q)} \) are given by (37) and (38) respectively, and from (28),

\[
K(\theta) = R \left( \frac{i\pi}{2} - \theta \right) = i \tanh \left( \frac{Q - \cosh \theta}{Q + \cosh \theta} \right)
\]

5 Asymptotic Limits

\( m R \to 0 \)

We can take the (bulk) ultraviolet limit \( m R \to 0 \) in two ways: (i) we can either make \( m \to 0 \) i.e. increase the temperature \( T \to T_c \) on the same cylinder, or (ii) we can reduce the size
R of the system $R \to 0$ while maintaining the same temperature so that the system size becomes much smaller compared to its correlation length.

Let us discuss case (i) first. As $m \to 0$, $Q$ increases, the boundary bound state, if initially present, disappears and $g_{\pm}^{(Q)} = g_{\pm}$. The theory now describes a renormalization group flow in the space of boundary interactions interpolating between the free and the fixed conformal boundary conditions \[ \mathcal{E}, \mathcal{Q} \]. The massless particles group into right and left movers which do not see each other. We have to obtain the $mR \to 0$ limits of $\mathcal{R}(\theta)$ and $\mathcal{K}(\theta)$ carefully. We do this by shifting the rapidity occurring in $\mathcal{R}(\theta)$ by infinite amounts: $\theta \to \log \frac{2}{mR} + \theta$ for the right movers and $\theta \to -\log \frac{2}{mR} - \theta$ for the left movers. For the continuation of $\mathcal{R}$ in the cross channel, i.e. $\mathcal{K}(\theta)$, however, we must shift $\theta \to -\log \frac{2}{mR} - \theta$ for right movers and $\theta \to \log \frac{2}{mR} + \theta$ for the left movers instead to obtain the correct $\mathcal{K}(k) = \mathcal{K}(ik)$, $k$ being the momentum of the massless fermions (e.g. for right movers $k = \frac{1}{R} e^{i\theta}$). Moreover, in the $mR \to 0$ limit the integral in (57) falls into three regions: (i) $\theta_0 \ll \log \frac{2}{mR}$, i.e. $h^2 R \ll 1$ where $\theta_0 = \sinh^{-1} \sqrt{Q^2 - 2}$ is an extremum of $\Phi(\theta) = \frac{2Q \cosh \theta}{\alpha^2 + \sinh^2 \theta}$, (ii) $\theta_0 \sim \log \frac{2}{mR}$, i.e. $h^2 R \sim 1$ and (iii) $\theta_0 \gg \log \frac{2}{mR}$, i.e. $h^2 R \gg 1$. It is easy to see from (57) that in region (ii) the following hold:

$$g_+(R) = \frac{\sqrt{2\pi}}{\Gamma(\alpha + \frac{1}{2})} \left( \frac{\alpha}{e} \right)^\alpha$$

$$g_-(R) = \frac{2^{1/4}\sqrt{\pi \alpha}}{\Gamma(\alpha + 1)} \left( \frac{\alpha}{e} \right)^\alpha$$

where $\alpha = 2h^2 R$.

with regions (i) and (iii) above appearing as $h^2 R \to 0$ and $h^2 R \to \infty$ limits of (82) respectively:

$$g_+ = \sqrt{2} \quad \text{and} \quad g_- = 0 \quad \text{for} \quad h^2 R \to 0,$$

$$g_+ = 1 \quad \text{and} \quad g_- = 2^{-\frac{1}{2}} \quad \text{for} \quad h^2 R \to \infty.$$  

We then obtain the boundary state in the massless limit from (79), (82) and (80):  

$$|B_{\pm h}\rangle = e^{-R \epsilon/\lambda} \left( \frac{\alpha}{e} \right)^\alpha \sqrt{\pi} \left[ \frac{1}{\Gamma(\alpha + \frac{1}{2})} \exp \left\{ i \sum_{n=0}^{\infty} \frac{n + \frac{1}{2} - \alpha}{n + \frac{1}{2} + \alpha} a_n^{\dagger} a_n^{\dagger} \right\} |0\rangle \right.$$

$$\pm \frac{2^{1/4} \sqrt{\alpha}}{\Gamma(\alpha + 1)} \exp \left\{ i \sum_{n=1}^{\infty} \frac{n - \alpha}{n + \alpha} a_n^{\dagger} a_n \right\} |\sigma\rangle \left. \right]$$

where $a_j^{\dagger}, a_j$ are creation operators for right and left movers of momentum $k_j = \frac{2\pi}{R} j$. Expressions (82) and (80) were obtained directly in [13, 14] in the case of the critical Ising model. As was shown in [13], at the conformal limits, (80) yields for:

$$h^2 R \to 0:\quad$$

$$|B_{\text{free}}\rangle = \exp \left( i \sum_{n=1}^{\infty} a_n^{\dagger} a_n^{\dagger} (|0\rangle - |\varepsilon\rangle) \right.)$$
where \( |\epsilon\rangle \) is the energy density operator in the bulk conformal field theory and we used:

\[
a_{\frac{1}{2}}^\dagger a_{\frac{1}{2}}^\dagger |0\rangle = i |\epsilon\rangle
\]

and for \( \hbar^2 R \to \infty \):

\[
|B_{\text{fixed}\pm}\rangle = \frac{1}{\sqrt{2}} \exp(-i \sum_{n=1}^{\infty} a_{n+\frac{1}{2}}^\dagger a_{n+\frac{1}{2}}^\dagger) (|0\rangle + |\epsilon\rangle)
\]

\[
\pm \frac{1}{2^{1/4}} \exp(-i \sum_{n=1}^{\infty} a_n^\dagger a_n^\dagger) |\sigma\rangle
\]

Both (87) and (89) agree with Cardy’s results [10]

Now we consider the case (ii) way of taking the \( mR \to 0 \) limit, namely we let \( R \to 0 \) while keeping the temperature constant. For finite \( Q \) it is easy to see that as \( R \to 0 \) the system reaches the conformally invariant limit of free boundary conditions. We readily see that in this limit (67) yields \( g_+ = \sqrt{2} \) and \( g_- = 0 \) for both \( Q \geq 0 \) and \( Q < 0 \) domains (here we note that for \( Q < 0 \) the second term in the integral in (66) yields \( \log \frac{1}{\sqrt{2}} \) instead of \( \log \sqrt{2} \) that one obtains for \( Q \geq 0 \)) and expression (87) for the boundary state.

\( mR \to \infty \)

Now we consider the case \( mR \to \infty \) when the size of the system is much larger compared to its correlation length. From (67) we readily see that in this limit,

\[
g_+^{(Q)}(R) = g_-^{(Q)}(R) \sim e^{-\frac{mR}{8}} \]

From (67) and (79) it is easy to obtain:

\[
|B_{\pm h}\rangle = e^{-\frac{mR}{8}} \frac{1}{\sqrt{2}} (|0\rangle \pm |\sigma\rangle) + \text{excited states}
\]

as we expect since these are just the broken symmetry vacua \( |0_+\rangle \) and \( |0_-\rangle \) (72) discussed in the previous section.

6 Conclusion

Expressions (67) for the “g function” and (79) for the boundary state are the most important results of this paper. We have computed these quantities by starting with the partition function expressed as a sum over states (satisfying the boundary conditions) in one channel and making explicit transformations so as to express it as a sum over states in the cross channel. The domain \( Q < 0 \), in which there are states localized near the boundaries (in the \( \mathbf{R} \) channel), is particularly interesting. We found single expressions for the g function and the boundary state valid in all domains of \( Q \) by analytic continuation, whereby the contour
of integration is made to surround some relevant poles. It would be interesting to see if this could form a general prescription for treating boundary bound states in obtaining the $g$ function in general integrable quantum field theory with boundary, and it would be desirable to gain a proper physical understanding underlying this treatment.

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\[ B_{\text{left}} \quad \xrightarrow{x=0} \quad x \quad \xrightarrow{\leftarrow} \quad x=L \quad B_{\text{right}} \]

**Fig. 1**

\[ \theta \quad \text{Re} \theta \quad \text{Im} \theta \quad -\frac{\pi}{2} \quad \pi \]

- \( \bullet \) = zero
- \( \times \) = pole

**Fig. 2**

\[ C \quad (\cosh^{-1}Q, -\frac{\pi}{2}) \]

**Fig. 3**

\[ -\frac{\pi}{2} \quad -\sin^{-1}Q \]
