Analogue Casimir force from a prethermal quasi-condensate of light

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We show that weakly incoherent optical beams propagating in a Kerr medium exhibit a universal algebraic coherence after a short propagation time, mimicking the quasi-long-range order of ultracold quantum Bose gases in two dimensions. If two plates are inserted in the medium, this optical quasi-condensate gives rise to a long-range Casimir-like force, attractive at large distances and repulsive at short distances.

In its original version, the Casimir force stems from the confinement of the quantum fluctuations of the electromagnetic field: two objects placed in vacuum modify the electromagnetic ground-state energy, which in turn induces an attractive interaction between them [1, 2]. Beyond this traditional scenario, it was quickly realized that fluctuation-driven forces may arise whenever objects are immersed in a fluctuating environment, which may or may not be made of photons [3-5]. Such forces were investigated, e.g., for impurities embedded in interacting quantum gases of massive particles [6-14]. This problem is especially interesting in low dimensions, where interacting Bose gases spontaneously form quasi-condensates, whose quantum fluctuations exhibit long-range correlations [13-18]. Since, in a Casimir-like scenario, the range of these correlations controls the range of the force, quasi-condensates constitute excellent candidates for the generation of a sizeable interaction between objects. In this context, special attention was paid to one-dimensional Bose gases at equilibrium [13, 14], where algebraic correlations give rise to long-range Casimir-like forces.

While the notion of condensation seems, at first sight, restricted to massive ultracold gases, many theoretical and experimental efforts have been recently undertaken to describe and observe Bose condensation of light. After seminal observations in polariton systems [19, 20], room-temperature condensates of light were achieved in dye-filled optical microcavities where the confined photons acquire an effective mass and thermalize via their interactions with the dye molecules [21]. Another strategy to thermalize massive photons consists in letting an optical beam propagate in a cavityless, nonlinear Kerr medium. In the paraxial approximation, the propagation is governed by a nonlinear Schrödinger equation where the optical axis plays the role of time and the nonlinearity the role of photon interactions [22, 23]. The beam thus behaves as a fluid of light [24], which may thermalize at long enough propagation time [25, 26]. Another fundamental interest of this setup lies in its two-dimensional nature. This implies that, if condensation cannot exist without cavity, quasi-condensation is on the other hand possible. To our knowledge however, quasi-condensates of light have not yet been considered nor observed experimentally. Regarding the interaction between fluids of light and matter, the drag forces experienced by an obstacle have been investigated theoretically [28], and a recent experiment showed evidence for the suppression of such forces in a photorefractive material [29]. This phenomenon was interpreted as the onset of superfluidity, a concept later validated via measurements of the Bogoliubov dispersion of photons in atomic vapors [30, 31]. Casimir-like forces in Kerr media, on the other hand, little been addressed so far.

In this Letter, we theoretically show that weakly incoherent optical beams propagating in a Kerr medium over a short time exhibit a universal, algebraic coherence, mimicking ultracold Bose quasi-condensates in two dimensions. If two objects are immersed in the medium, this long-range coherence leads to an enhanced, long-range Casimir-like force between them. Our analysis is based on a natural extension of an experimental setup recently used to measure drag forces on dielectric obstacles [29], and illustrated in Fig. 1. A monochromatic light beam carrying weak transverse spatial fluctuations is let propagate in a Kerr medium in which two plates, parallel to the optical axis $z$, are embedded. Due to the photon interactions pertinent to the nonlinear medium, the initial small fluctuations get amplified and, after a short propagation time, the beam reaches a quasi-stationary...
prethermal state \cite{32,36}. For low enough initial fluctuations, we find that this state exhibits long-range correlations in the transverse plane \((x, y)\), triggering an unconventional Casimir-like pressure which decays algebraically with the plate separation \(L\).

Before addressing the complete problem in Fig. 1, let us forget the plates for a moment and consider a monochromatic, plane-wave optical beam impinging on a homogeneous, semi-infinite Kerr material at \(z = 0\). We write the electric field at any point \((r_\perp, z)\) as \(E(r, z, t) = R[\Psi(r_\perp, z)e^{i k_0 z - \omega t}]e_y\), where \(\omega\) is the carrier frequency, \(k_0 = \omega/c\), and \(e_y\) is a unit polarization vector along the \(y\) axis. In the paraxial approximation, the complex field envelope \(\Psi(r, z)\) obeys the two-dimensional nonlinear Schrödinger equation \cite{22,23}

\[
i \partial_z \Psi(r_\perp, z) = \left[-\frac{1}{2k_0} \nabla_\perp^2 + g|\Psi(r_\perp, z)|^2\right] \Psi(r_\perp, z), \tag{1}
\]

where \(g\) controls the strength of the Kerr nonlinearity, assumed to be defocusing, \(g > 0\). Suppose now that the incident beam is prepared as a superposition of a uniform background of intensity \(I_0\) and a spatially fluctuating speckle field \(\phi(r_\perp)\), \(\Psi(r_\perp, z = 0) = \sqrt{I_0} + \epsilon \phi(r_\perp)\). We describe the latter as a complex, Gaussian random function of two-point correlation \(\langle \phi(r_\perp) \phi^*(r_\perp + \Delta r) \rangle = I_0 \gamma(\Delta r)\), where the brackets refer to statistical averaging. For definiteness, in the following we consider a Gaussian correlation \(\gamma(\Delta r) \equiv \exp(-\Delta r^2/4\xi^2)\), with correlation length \(\xi\) \cite{37}. The main results of the Letter are, however, independent of this specific choice. From now on, we also mainly focus on the limit \(\epsilon \ll 1\) of a weakly incoherent field. This, indeed, corresponds to the most interesting configuration where the incident beam mimics a noninteracting, low-temperature Bose gas undergoing an interaction quench upon entering the nonlinear medium.

The coherence properties of the beam in the material are encoded in the coherence function \(g_1(\Delta r, z) \equiv \langle \Psi(r_\perp, z)\Psi^*(r_\perp + \Delta r, z) \rangle\) \cite{37}. We have first calculated \(g_1\) by numerically propagating the initial state \(\Psi(r_\perp, z = 0)\) with Eq. (1) using a split-step method. For the simulations we choose a coherence length such that the ratio \(\xi/\sigma\) of the healing length \(\xi \equiv 1/\sqrt{4gI_0 k_0}\) to the speckle correlation length is small, a condition required for the nonlinearity to have a significant impact on the beam evolution. The results are shown in Fig. 2 against \(\Delta r/\sigma\) (dots) for increasing values of \(z/z_{\text{NL}}\) where \(z_{\text{NL}} \equiv 1/2gI_0\) is the nonlinear length. At \(z = 0\) (upper black dots), the result coincides with \(g_1(\Delta r, z = 0) = I_0 [1 + e^2 \gamma(\Delta r)]\), which describes the initial superposition of a fully coherent of plane-wave signal with a small incoherent component on the top (solid black curve). This structure changes dramatically when \(z \neq 0\). After a fast, transient evolution over a few tens of \(z_{\text{NL}}\), the overall coherence drops but the short-range component \(I_0 e^{2\gamma(\Delta r)}\) is converted into a long-range, algebraic correlation, as shown in the inset Fig. 2. Once this regime has been reached, \(g_1\) also varies rather weakly with \(z\) over a spatial range set by the Lieb-Robinson bound \(\Delta r = 2c_s z\), where \(c_s \equiv \sqrt{gI_0/k_0}\) is the speed of sound. This phenomenon, known as prethermalization, describes a quasi-stationary regime where the beam behaves as a quasi-thermalized, weakly interacting fluid \cite{32,36}. Since \(\epsilon \ll 1\), the associated effective temperature is typically low and the fluid is similar to a quasi-condensate, mimicking the well-known quasi-long-range order of ultracold quantum Bose gases in two dimensions \cite{15,18}. Out of the “light cone”, i.e. for \(\Delta r > 2c_s z\), long-range correlations have not yet the time to establish and \(g_1\) reaches a plateau reminiscent of the coherent component of the initial beam.

Theoretically, this behavior is well captured by a time-dependent Bogoliubov description. This approach has been previously used to describe the out-of-equilibrium dynamics of quenched, weakly interacting quantum gases \cite{38,39}. Here we adapt it to a classical light beam evolving from the initial state \(\Psi(r_\perp, z = 0)\) onwards. Since the beam propagates in an effective two-dimensional space (\(z\) playing the role of a propagation time), its phase fluctuations are large. This requires to make use of a density-phase formalism \cite{18}, as detailed in the Supplemental Material (SM) \cite{40}. The result for \(g_1\) is:

\[
g_1(\Delta r, z) = I \exp \left\{ -e^2 \int \frac{d^2q}{(2\pi)^2} (1 - \cos q \cdot \Delta r) \right\} \times \left[ 1 + \frac{(2gI_0)^2}{2k^2(q)} \sin^2 k(q)z \right], \tag{2}
\]

where \(k(q) = \sqrt{q^2/2k_0 + q^2/2k_0 + 2gI_0}\) is the Bogoliubov dispersion relation, \(\gamma(q) = \int d^2q_\perp \gamma(\Delta r)e^{-iq \cdot \Delta r}\) is the speckle power spectrum, and \(I = I_0(1 + e^2) = 1\).
the three regions delineated by the plates into component, the difference of which defining the Casimir pressure. The radiation pressure is given by the diagonal component $\frac{\sigma}{\Delta r}$ of the stress tensor of the fluid in the nonlinear medium, stemming from the infrared divergence $1/k^2(q) \sim q^{-2}$ in Eq. (2). They yield the asymptotic law:

$$g_1(\Delta r, z \gg z_{NL}) \approx I \left( \frac{\sigma}{\Delta r} \right)^\alpha,$$

with an exponent $\alpha = c^2 \sigma^2/2g^2$. The algebraic law (3) is shown in the inset of Fig. 2 (dashed curve). It signals the formation of a quasi-condensate of light. Eq. (3) holds up to the light-cone bound $\Delta r = 2c_v z$. Out of the light cone, the coherence function saturates at $g_1 \approx I(\sigma/c_v)^2\alpha$.

The long-range coherence exhibited by optical beams in the prethermal regime makes the configuration of Fig. 4 promising for realizing of a sizeable Casimir-like force. To confirm this intuition, we now add the plates and explore the fluctuations-induced interaction between them. To calculate this interaction, we make use of a scattering approach to Casimir forces, in which the effect of the plates is described in terms of the components of the transmission coefficients, $S$ and $R$, and the reflection of field fluctuations in the absence of coupling, assuming unitarity only in the configuration of Fig. 1 the uniform mean-field component $\langle \Psi(z) \rangle$ does not yield any force. The first step of this approach consists in decomposing the field fluctuations in the three regions delineated by the plates into components moving forward and backward along the x axis, as shown in Fig. 1. In the two outer regions, we express the incoming field fluctuations as $\delta \Psi_{in}^+(r, z) = \int_{q \geq 0} d^2q/(2\pi)^2 \delta \Psi(q, z) e^{i q r}$, where the Fourier components $\delta \Psi(q, z) \equiv \Psi(q, z) - \langle \Psi(q, z) \rangle$ refer to the beam fluctuations in the absence of plates. The scattered fields then follow from $\delta \Psi^+_{out} = S \delta \Psi^+_{in}$ and $\delta \Psi^-_{out} = R \delta \Psi^-_{in}$. Calculation of the radiation pressure inside the cavity, finally, follows the same lines but now involves the elements of the scattering matrix $R$. The Casimir pressure $P = T_{xx}(\delta \Psi_{in}^+ + T_{xx}(\delta \Psi_{in}^-) - T_{xx}(\delta \Psi_{out}^+ - T_{xx}(\delta \Psi_{out}^-)$ then reads

$$P = \frac{2\epsilon_0 c R}{k_0} \int_{q > 0} \frac{d^2q}{(2\pi)^2} \left[ r^2(q)e^{2iq_x L} - 1 - r^2(q) \right] \left[ \delta \Psi(q, z) \right]^2,$$

where $r(q)$ denotes the reflection coefficient in the direction $q$, and $L$ is the plate separation. The Casimir pressure thus naturally appears as the noise (density plus current) spectrum of the fluid of light, weighted by the

![FIG. 3: Casimir pressure, Eq. (6), as a function of the plate separation, for $\epsilon = 0.1$. The pressure is attractive at large separation, where it exhibits an algebraic decay associated with the long-range coherence of the prethermal fluid of light, and repulsive at short separation. Its magnitude increases as the ratio of the healing length $\xi$ to the speckle correlation length $\sigma$ decreases, i.e. as the nonlinearity gets stronger. Inset: pressure for $\xi/\sigma = 0.1$ in double log scale, emphasizing the algebraic decay. The dashed line is Eq. (7).](image_url)
admittance of the cavity, summed over all possible scattering directions $q$. Eq. (5) can be further simplified by noting that, in the paraxial approximation, the fluctuations are essentially scattered at grazing incidence. It follows that $r^2(q) \simeq 1$ whatever the nature of the material the plates are made of. Eq. (5) can then be reformulated in position space as

$$P = -\frac{c_0}{2} \sum_{n=0}^{\infty} \left[ \frac{\partial^2 g_1(\Delta r, z)}{k_0 \partial \Delta x^2} + g_1(\Delta r, z) \right]_{\Delta x = 2L(n+1), \Delta y = 0}$$

where the sum runs over all resonance spatial frequencies of the cavity. In this relation, the first term involves the coherence function $g_1$. This term dominates at large separation $L \gg \sigma$, where the large phase fluctuations of the beam make $g_1$ long-range. We show in the SM that the current correlator $g_1(\Delta r, z) = 2 \Im(\delta \Psi(r, z) \partial_2 \delta \Psi(r + \Delta r, z))/k_0$, on the other hand, is essentially governed by the intensity fluctuations of the beam, which are typically small when $\epsilon \ll 1$. Its contribution is thus important at short scale $L \lesssim \sigma$ only.

The Casimir pressure (6), calculated with the Bogoliubov theory, is shown in Fig. 4 as a function of the plate separation $L$, in the prethermal regime $z \gg z_{NL}$ where it is essentially independent of $z$. Its most remarkable feature is the behavior at large separation, where the pressure is attractive and decays algebraically due to the long-range character of $g_1$ (10):

$$P(L \gg \sigma) \sim -\frac{c_0 I_0}{(k_0 \sigma)^2} \alpha(\alpha + 1) (\sigma/L)^{\alpha + 2}. \quad (7)$$

The asymptotic law (7) is compared with the exact formula (6) in the inset of Fig. 4. It is entirely governed by the exponent $\alpha = c^2\sigma^2/2\xi^2$, which can be either larger or smaller than 1 since both $\epsilon \ll 1$ and $\xi/\sigma \ll 1$. In particular, when $\alpha > 1$, the obtained pressure is much larger than the pressure $P \sim -c_0 I_0/(k_0 \sigma)^2 (2L)$ expected from a fully developed speckle, i.e. $\Psi(r, z) = \phi(r)$. Furthermore, the decay (7) is univerusal, in the sense that it only depends on the small set of parameters ($\xi, \sigma, \epsilon$), but not on the specific shape of $\gamma(q)$. At small separation $L \lesssim \sigma$, the pressure (6) becomes governed by the current correlator and turns repulsive, as seen in Fig. 4. Its $L$ dependence at such short scale is nonuniversal in general, i.e. it depends on the specific shape of the power spectrum $\gamma$. Fig. 4 and Eq. (4) also reveal that the overall magnitude of the pressure increases with decreasing $\xi/\sigma$. This result can be understood as follows. When $\xi \ll \sigma$, the speckle spectrum selects only the low (phonon-like) Bogoliubov modes $|q| \xi \ll 1$, responsible for the algebraic decay of the coherence function and a sizable Casimir force. In contrast, when $\xi/\sigma \gtrsim 1$ the speckle spectrum also captures particle-like modes $|q| \xi \gtrsim 1$. Since these modes describe purely non-interacting particles, their coherence function hardly evolves from its form at $z = 0$, which carries small fluctuations and therefore yields a small force.

We finally comment on the role of the parameter $\epsilon$, which controls the amount of fluctuations in the incident beam. At small $\epsilon$, the effective temperature of the prethermal regime is small, so that the fluid of light effectively behaves as a low-temperature interacting Bose gas in two dimensions, i.e., a quasi-condensate. By analogy, a larger $\epsilon$ will describe a gas of temperature typically above the quasi-condensation threshold, i.e., of exponentially small coherence [44]. This qualitative picture is confirmed by numerical simulations of $g_1$ shown in Fig. 4. As $\epsilon$ increases, the algebraic behavior of $g_1$ turns to an exponential decay, making the pressure (6) much weaker.

![FIG. 4: Coherence function versus $\Delta r/\sigma$ for increasing values of $\epsilon$ and fixed $\xi/\sigma = 0.158$ and $z/z_{NL} = 200$, obtained from the numerical resolution of Eq. (4). While the algebraic law (6) is well observed at small $\epsilon$ (dashed curves), a crossover to an exponential decay (solid line) appears at larger $\epsilon$.](image-url)

Let us conclude with an experimental order of magnitude. In atomic vapors illuminated slightly away from resonance, nonlinearities such that $z_{NL} \approx 1$ mm and $\xi \approx 10 \mu m$ can be obtained [27]. For a cell length $z = 7$ cm, this yields $z/z_{NL} \approx 70$ and $2c_0 z \approx 1.4$ mm for the Lieb-Robinson bound, much larger than the speckle correlation $\sigma$, usually on the order of a few tens of microns. A large window $2c_0 z/\sigma$ of two or three orders of magnitude can thus be realized, making the long-range behavior of $g_1$ observable under rather reasonable conditions.

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