Decomposition of linear metric perturbations on generic background spacetime

— Toward higher-order general-relativistic gauge-invariant perturbation theory —

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The decomposition of the linear-order metric perturbation is discussed in the context of the higher-order gauge-invariant perturbation theory. We show that the linear order metric perturbation is decomposed into gauge-invariant and gauge-variant parts on the general background spacetime which admits ADM decomposition. This decomposition was an important premise of the general framework of the higher order gauge-invariant perturbation theory proposed in the papers [K. Nakamura, Prog. Theor. Phys. 110 (2003), 723; ibid. 113 (2005), 481]. This implies that we can develop the higher-order gauge-invariant perturbation theory on generic background spacetime. Remaining issues to complete the general-framework of the higher-order gauge-invariant perturbation theories are also discussed.

§1. Introduction

Perturbation theories are powerful techniques in many area of physics and the developments of perturbation theories lead physically fruitful results and interpretations of natural phenomena.

In physics, physicists want to describe realistic situations in a compact manner. Exact solutions in a theory for physical situations are candidates which can describe realistic natural phenomena. However, in many theories of physics, realistic situations are too complicated and often difficult to describe by an exact solution of a theory. This difficulty may be due to the fact that exact solutions only describe special cases even if the theory is appropriate to describe the natural phenomena, or may be due to the lack of the applicability of the theory itself. Even in the case where an exact solution of a theory well describes a physical situation, the properties of the physical system will not be completely described only through the exact solution. In natural phenomena, there always exist “fluctuations”. In this case, perturbative treatments of the theory is a powerful tools and physicists investigate perturbative approach within the theory to clarify the properties of fluctuations.

General relativity is a theory in which the construction of exact solutions is not so easy. Although there are many exact solutions to the Einstein equation\(^1\) these are often too idealized. Of course, there are some exact solutions to the Einstein equation which well-describe our universe, or gravitational field of stars and black holes. These exact solutions by itself do not describe the fluctuations around these exact solutions. To describe these fluctuations, we have to consider the perturbations around these
exact solutions. Therefore, general relativistic linear perturbation theory is a useful technique to investigate the properties of fluctuations around exact solutions.\textsuperscript{2}

On the other hand, higher-order general-relativistic perturbations also have very wide applications. In these applications, second-order cosmological perturbations are topical subject\textsuperscript{3–7} due to the precise measurements in recent cosmology.\textsuperscript{5} Higher-order black hole perturbations are also discussed in some literature.\textsuperscript{9} Moreover, as a special example of higher-order perturbation theory, there are researches on perturbations of a spherical star\textsuperscript{10} motivated by the researches on the oscillatory behaviors of a rotating neutron star. Thus, there are many physical situations to which general relativistic higher-order perturbation theory should be applied.

As well-known, general relativity is based on the concept of general covariance. Intuitively speaking, the principle of general covariance states that there is no preferred coordinate system in nature, though the notion of general covariance is mathematically included in the definition of a spacetime manifold in a trivial way. This is based on the philosophy that coordinate systems are originally chosen by us, and that natural phenomena have nothing to do with our coordinate system. Due to this general covariance, the “gauge degree of freedom”, which is an unphysical degree of freedom of perturbations, arises in general-relativistic perturbations. To obtain physically results, we have to fix this gauge degrees of freedom or to extract some invariant quantities of perturbations. This situation becomes more complicated in higher-order perturbation theory. In some linear perturbation theories on some background spacetimes, there are so-called gauge-invariant perturbation theories. In these theories, one may treat only variables which are independent of gauge degree of freedom without any gauge fixing. Therefore, it is worthwhile to investigate higher-order gauge-invariant perturbation theory from a general point of view to avoid gauge issues.

According to these motivation, the general framework of higher-order general-relativistic gauge-invariant perturbation theory has been discussed in some papers\textsuperscript{11), 12) by the present author. We refer these works as KN2003\textsuperscript{11) and KN2005.\textsuperscript{12) Although these development of higher-order perturbation theory was originally motivated by the research on the oscillatory behavior of a self-gravitating Nambu-Goto membrane,\textsuperscript{13) these works are applicable to cosmological perturbations and we clarified the gauge-invariance of the second-order perturbations of the Einstein equations.\textsuperscript{7), 14), 15) In this paper, we refer these works as KN2007\textsuperscript{7) and KN2009.\textsuperscript{14)

In KN2003\textsuperscript{11} we proposed the procedure to find gauge-invariant variables for higher-order perturbations on a generic background spacetime. This proposal is based on the single assumption that we have already known the procedure to find gauge-invariant variables for the linear-order metric perturbation. Under the same assumption, we summarize some formulae for the second-order perturbations of the curvatures and energy-momentum tensor for the matter fields in KN2005\textsuperscript{12} and KN2009\textsuperscript{13} In KN2007\textsuperscript{7} we develop the second-order gauge-invariant cosmological perturbation theory after confirming that the above assumption is correct in the case of cosmological perturbations. Through these works, we find that our general framework of higher-order gauge-invariant perturbation theory is well-defined except for the above assumption for linear-order metric perturbations. Therefore, we pro-
posed the above assumption as a conjecture in KN2009. If this conjecture is true, higher-order general-relativistic gauge-invariant perturbation theory is completely formulated on generic background spacetime and has very wide applications.

The main purpose of this paper is to give a proof of this conjecture using the premise that the background spacetime admits ADM decomposition. Although some special modes are excluded in the proof in this paper, we may say that the above conjecture is almost correct for linear-order perturbations on generic background spacetimes. This paper is the complete version of our previous short letter.

The organization of this paper is as follows. In §2, we review the general framework of the second-order gauge-invariant perturbation theory developed in KN2003 and KN2005 with some additional explanations. In the context of this general framework, the above conjecture is also declared as Conjecture 2.1 in this section. In §3, we give a proof of Conjecture 2.1 From pedagogical point of view, we consider three different situations of the geometry of the background spacetime in terms of ADM decomposition. The first situation is trivial (§3.1), in which we may choose the unit lapse function \( \alpha = 1 \), the vanishing shift vector \( \beta^i = 0 \), and the vanishing extrinsic curvature \( K_{ij} = 0 \). Through this trivial case, we give the essential outline of the proof in more generic situations. The second situation is the case where \( \alpha = 1, \beta^i = 0, \) but \( K_{ij} \neq 0 \) (§3.2). Through this second case, we show a technical issue to prove the conjecture 2.1 in terms of ADM decomposition. The final situations is most generic case, in which \( \alpha \neq 1, \beta^i \neq 0, \) \( K_{ij} \neq 0 \) (§3.3). The calculations in this case is complicated. However the essential outline of the proof is same as in the first trivial case, and the essential technique using in the proof is already given in the second case. We also note that we assume that the existence of Green functions for two elliptic differential operators in these proofs. The comparison with the proof in the case of cosmological perturbations shown in KN2007 is discussed in §4. The final section is devoted to summary and discussion.

We employ the notation of KN2003 and KN2005 and use abstract index notation. We also employ natural units in which Newton’s gravitational constant is denoted by \( G \) and the velocity of light satisfies \( c = 1 \).

§2. General framework of the higher-order gauge-invariant perturbation theory

In this section, we review the general framework of the gauge-invariant perturbation theory developed in KN2003 and KN2005 to emphasize that Conjecture 2.1 is the important premise of our general framework. In §2.1 we review the basic understanding of the gauge degree of freedom in general relativistic perturbation theory based on the work of Stewart et al. and Bruni et al. When we consider perturbations in any theory with general covariance, we have to exclude these gauge degrees of freedom in the perturbations. To accomplish this, gauge-invariant variables of perturbations are useful, and these are regarded as physically meaningful quantities. In §2.2 we review the procedure for finding gauge-invariant variables of perturbations, which was developed in KN2003. After the introduction of gauge-invariant variables, in §2.3 we review the general issue of the gauge-invariant formulation for
the second-order perturbation of the Einstein equation developed in KN2005. We emphasize that the ingredients of this section do not depend on the details of the background spacetime, if the decomposition conjecture for the linear-order metric perturbation is correct.

2.1. Gauge degree of freedom in perturbation theory

2.1.1. Basic idea

Here, we explain the concept of gauge in general relativistic perturbation theory. To explain this, we first point out that, in any perturbation theory, we always treat two spacetime manifolds. One is the physical spacetime \( \mathcal{M} \), which we attempt to describe in terms of perturbations, and the other is the background spacetime \( \mathcal{M}_0 \), which is a fictitious manifold prepared for perturbative analyses by hand. We emphasize that these two spacetime manifolds \( \mathcal{M} \) and \( \mathcal{M}_0 \) are distinct. Let us denote the physical spacetime by \( (\mathcal{M}, \bar{g}_{ab}) \) and the background spacetime by \( (\mathcal{M}_0, g_{ab}) \), where \( \bar{g}_{ab} \) is the metric on \( \mathcal{M} \), and \( g_{ab} \) is the metric on \( \mathcal{M}_0 \). Further, we formally denote the spacetime metric and the other physical tensor fields on the physical spacetime by \( Q \) and its background value on the background spacetime by \( Q_0 \).

Second, in any perturbation theories, we always write equations for the perturbation of the physical variable \( Q \) in the form

\[
Q("p") = Q_0(p) + \delta Q(p).
\]

(2.1)

Usually, this equation is simply regarded as a relation between the physical variable \( Q \) and its background value \( Q_0 \), or as the definition of the deviation \( \delta Q \) of the physical variable \( Q \) from its background value \( Q_0 \). However, Eq. (2.1) has deeper implications. Keeping in our mind the above fact that we always treat two different spacetimes, \( (\mathcal{M}, \bar{g}_{ab}) \) and \( (\mathcal{M}_0, g_{ab}) \), in perturbation theory, Eq. (2.1) is a rather curious equation in the following sense: The variable on the left-hand side of Eq. (2.1) is a variable on the physical spacetime \( \mathcal{M} \), while the variables on the right-hand side of Eq. (2.1) are variables on the background spacetime, \( \mathcal{M}_0 \). Hence, Eq. (2.1) gives a relation between variables on two different manifolds.

Further, we point out the fact that, through Eq. (2.1), we have implicitly identified points in two different manifolds \( (\mathcal{M}, \bar{g}_{ab}) \) and \( (\mathcal{M}_0, g_{ab}) \). More specifically, \( Q("p") \) on the left-hand side of Eq. (2.1) is a field on \( \mathcal{M} \), and "\( p \)" \( \in \mathcal{M} \). Similarly, we should regard the background value \( Q_0(p) \) of \( Q("p") \) and its deviation \( \delta Q(p) \) of \( Q("p") \) from \( Q_0(p) \), which are on the right-hand side of Eq. (2.1), as fields on \( \mathcal{M}_0 \), and \( p \in \mathcal{M}_0 \). Because Eq. (2.1) is regarded as an equation for field variables, it implicitly states that the points "\( p \)" \( \in \mathcal{M} \) and \( p \in \mathcal{M}_0 \) are same. Therefore, through Eq. (2.1), we implicitly assume the existence of a map \( \mathcal{M}_0 \to \mathcal{M} : p \in \mathcal{M}_0 \mapsto "p" \in \mathcal{M} \), which is called a gauge choice in perturbation theory.

Further, we have to note that the correspondence between points on \( \mathcal{M}_0 \) and \( \mathcal{M} \), which is established by such a relation as Eq. (2.1), is not unique to the perturbation theory with general covariance. Rather, Eq. (2.1) involves the degree of freedom corresponding to the choice of the map \( \chi : \mathcal{M}_0 \to \mathcal{M} \). This is called the gauge degree of freedom in general relativistic perturbation theory. Such a degree of freedom always exists in perturbations of a theory with general covariance.
eral covariance” intuitively means that there is no preferred coordinate system in the theory. If general covariance is not imposed on the theory, there is a preferred coordinate system in our nature, and we naturally introduce this coordinate system onto both \( M_0 \) and \( M \). Then, through this preferred coordinate system, we can choose the identification map \( \mathcal{X} \). However, due to general covariance, there is no such coordinate system in general relativity, and we have no guiding principle to choose the identification map \( \mathcal{X} \). Actually, we may identify \( p \in M \) with \( q \in M_0 \) instead of \( p \in M_0 \). In the above understanding of the concept of “gauge” in general-relativistic perturbation theory, a gauge transformation is simply a change of the identification map \( \mathcal{X} \).

These are the basic ideas necessary to understand \textit{gauge degree of freedom} in the general relativistic perturbation theory proposed by Stewart and Walker.\(^{21}\) This understanding has been developed by Bruni et al.,\(^4\) and by the present author.\(^{11,12}\)

2.1.2. Formulation of perturbations

To formulate the above understanding in more detail, we introduce an infinitesimal parameter \( \lambda \) for the perturbation. Further, we consider the \((n+1) + 1\)-dimensional manifold \( N = M \times \mathbb{R} \), where \( n+1 = \dim M \) and \( \lambda \in \mathbb{R} \). The background spacetime \( M_0 = N|_{\lambda=0} \) and the physical spacetime \( M = M_\lambda = N|_{\mathbb{R}=\lambda} \) are also submanifolds embedded in the extended manifold \( N \). Each point on \( N \) is identified by a pair, \( (p, \lambda) \), where \( p \in M_\lambda \), and each point in the background spacetime \( M_0 \) in \( N \) is identified by \( \lambda = 0 \).

Through this construction, the manifold \( N \) is foliated by \((n+1)\)-dimensional submanifolds \( M_\lambda \) of each \( \lambda \), and these are diffeomorphic to the physical spacetime \( M \) and the background spacetime \( M_0 \). The manifold \( N \) has a natural differentiable structure consisting of the direct product of \( M \) and \( \mathbb{R} \). Further, the perturbed spacetimes \( M_\lambda \) for each \( \lambda \) must have the same differential structure with this construction. In other words, we require that perturbations be continuous in the sense that \((M, g_{ab})\) and \((M_0, g_{ab})\) are connected by a continuous curve within the extended manifold \( N \). Hence, the changes of the differential structure resulting from the perturbation, for example the formation of singularities, are excluded from our consideration.

Let us consider the set of field equations

\[ E[Q_\lambda] = 0 \quad (2.2) \]

on the physical spacetime \( M_\lambda \) for the physical variables \( Q_\lambda \) on \( M_\lambda \). The field equation \((2.2)\) formally represents the Einstein equation for the metric on \( M_\lambda \) and the equations for matter fields on \( M_\lambda \). If a tensor field \( Q_\lambda \) is given on each \( M_\lambda \), \( Q_\lambda \) is automatically extended to a tensor field on \( N \) by \( Q(p, \lambda) := Q_\lambda(p) \), where \( p \in M_\lambda \). In this extension, the field equation \((2.2)\) is regarded as an equation on the extended manifold \( N \). Thus, we have extended an arbitrary tensor field and the field equations \((2.2)\) on each \( M_\lambda \) to those on the extended manifold \( N \).

Tensor fields on \( N \) obtained through the above construction are necessarily “tangent” to each \( M_\lambda \), i.e., their normal component to each \( M_\lambda \) identically vanishes. To consider the basis of the tangent space of \( N \), we introduce the normal form and
its dual, which are normal to each $\mathcal{M}_\lambda$ in $\mathcal{N}$. These are denoted by $(d\lambda)_a$ and $(\partial/\partial \lambda)^a$, respectively, and they satisfy $(d\lambda)_a (\partial/\partial \lambda)^a = 1$. The form $(d\lambda)_a$ and its dual, $(\partial/\partial \lambda)^a$, are normal to any tensor field extended from the tangent space on each $\mathcal{M}_\lambda$ through the above construction. The set consisting of $(d\lambda)_a$, $(\partial/\partial \lambda)^a$, and the basis of the tangent space on each $\mathcal{M}_\lambda$ is regarded as the basis of the tangent space of $\mathcal{N}$.

To define the perturbation of an arbitrary tensor field $Q$, we compare $Q$ on the physical spacetime $\mathcal{M}_\lambda$ with $Q_0$ on the background spacetime, and it is necessary to identify the points of $\mathcal{M}_\lambda$ with those of $\mathcal{M}_0$. This point identification map is the so-called gauge choice in the context of perturbation theories, as mentioned above. The gauge choice is made by assigning a diffeomorphism $\mathcal{X}_\lambda : \mathcal{N} \rightarrow \mathcal{N}$ such that $\mathcal{X}_\lambda : \mathcal{M}_0 \rightarrow \mathcal{M}_\lambda$. Following the paper of Bruni et al., we introduce a gauge choice $\mathcal{X}_\lambda$ as an exponential map on $\mathcal{N}$, for simplicity. We denote the generator of this exponential map by $\lambda_x\eta^a$. This generator $\lambda_x\eta^a$ is decomposed by the basis on the tangent space of $\mathcal{N}$ which are constructed above. The arbitrariness of the gauge choice $\mathcal{X}_\lambda$ is represented by the tangential component of $\lambda_x\eta^a$ to $\mathcal{M}_\lambda$.

The pull-back $\mathcal{X}_\lambda^* Q$, which is induced by the exponential map $\mathcal{X}_\lambda$, maps a tensor field $Q$ on $\mathcal{M}_\lambda$ to a tensor field $\mathcal{X}_\lambda^* Q$ on $\mathcal{M}_0$. In terms of this generator $\lambda_x\eta^a$, the pull-back $\mathcal{X}_\lambda^* Q$ is represented by the Taylor expansion

$$Q(r) = Q(\mathcal{X}_\lambda(p)) = \mathcal{X}_\lambda^* Q(p) = Q(p) + \lambda \mathcal{L}_x\eta Q|_p + \frac{1}{2} \lambda^2 \mathcal{L}_x^2\eta Q|_p + O(\lambda^3), \quad (2.3)$$

where $r = \mathcal{X}_\lambda(p) \in \mathcal{M}_\lambda$. Because $p \in \mathcal{M}_0$, we may regard the equation

$$\mathcal{X}_\lambda^* Q(p) = Q_0(p) + \lambda \mathcal{L}_x\eta Q|_{\mathcal{M}_0} (p) + \frac{1}{2} \lambda^2 \mathcal{L}_x^2\eta Q|_{\mathcal{M}_0} (p) + O(\lambda^3) \quad (2.4)$$

as an equation on the background spacetime $\mathcal{M}_0$, where $Q_0 = Q|_{\mathcal{M}_0}$ is the background value of the physical variable of $Q$. Once the definition of the pull-back of the gauge choice $\mathcal{X}_\lambda$ is given, the perturbations of a tensor field $Q$ under the gauge choice $\mathcal{X}_\lambda$ are simply defined by the evaluation of the expansion (2.4) on $\mathcal{M}_0$

$$\mathcal{X}_\lambda^* Q|_{\mathcal{M}_0} = Q_0 + \lambda^{(1)} x Q + \frac{1}{2} \lambda^2 \lambda^{(2)} x Q + O(\lambda^3), \quad (2.5)$$

i.e.,

$$\lambda^{(1)} x Q := \mathcal{L}_x\eta Q|_{\mathcal{M}_0}, \quad \lambda^{(2)} x Q := \mathcal{L}_x^2\eta Q|_{\mathcal{M}_0}. \quad (2.6)$$

We note that all variables in this definition are defined on $\mathcal{M}_0$.

2.1.3. Gauge transformation

Here, we consider two different gauge choices. Suppose that $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ are two exponential maps with the generators $\lambda_x\eta^a$ and $\gamma_y\eta^a$ on $\mathcal{N}$, respectively. The integral curves of each $\lambda_x\eta^a$ and $\gamma_y\eta^a$ in $\mathcal{N}$ are the orbits of the actions of the gauge choices $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$, respectively. Since we choose $\lambda_x\eta^a$ and $\gamma_y\eta^a$ so that these are transverse to each $\mathcal{M}_\lambda$ everywhere on $\mathcal{N}$, the integral curves of these vector fields intersect with
According to generic arguments concerning the Taylor expansion of the pull-back of the perturbative expansion of the pulled-back variables $X_\lambda^* Q_\lambda|_{\mathcal{M}_0}$ and $Y_\lambda^* Q_\lambda|_{\mathcal{M}_0}$:

$$X_\lambda^* Q_\lambda|_{\mathcal{M}_0} = Q_0 + \lambda \frac{1}{2} \lambda^2 X_\lambda^* Q + O(\lambda^3), \quad (2.7)$$

$$Y_\lambda^* Q_\lambda|_{\mathcal{M}_0} = Q_0 + \lambda \frac{1}{2} \lambda^2 Y_\lambda^* Q + O(\lambda^3), \quad (2.8)$$

Although these two representations of the perturbations are different from each other, these should be equivalent because of general covariance.

Now, we consider the gauge-transformation rules between two different gauge choices. In general, the representation $X^\lambda Q_\lambda$ on $\mathcal{M}_0$ of the perturbed variable $Q$ on $\mathcal{M}_0$ depends on the gauge choice $X_\lambda$. If we employ a different gauge choice, the representation of $Q_\lambda$ on $\mathcal{M}_0$ may change. Suppose that $X_\lambda$ and $Y_\lambda$ are two different gauge choices and the generators of these gauge choices are given by $X_\lambda^a$ and $Y_\lambda^a$, respectively. In this situation, the change of the gauge choice from $X_\lambda$ to $Y_\lambda$ is represented by the diffeomorphism

$$\Phi_\lambda := (X_\lambda)^{-1} \circ Y_\lambda. \quad (2.9)$$

This diffeomorphism $\Phi_\lambda$ is the map $\Phi_\lambda : \mathcal{M}_0 \to \mathcal{M}_0$ for each value of $\lambda \in \mathbb{R}$. The diffeomorphism $\Phi_\lambda$ does change the point identification, as expected from the understanding of the gauge choice discussed above. Therefore, the diffeomorphism $\Phi_\lambda$ is regarded as the gauge transformation $\Phi_\lambda : X_\lambda \rightarrow Y_\lambda$.

The gauge transformation $\Phi_\lambda$ induces a pull-back from the representation $X^\lambda Q_\lambda$ of the perturbed tensor field $Q$ in the gauge choice $X_\lambda$ to the representation $Y^\lambda Q_\lambda$ in the gauge choice $Y_\lambda$. Actually, the tensor fields $X^\lambda Q_\lambda$ and $Y^\lambda Q_\lambda$, which are defined on $\mathcal{M}_0$, are connected by the linear map $\Phi_\lambda^*$ as

$$Y^\lambda Q_\lambda = Y^\lambda_\lambda^* Q|_{\mathcal{M}_0} = (Y^\lambda_\lambda^* (X_\lambda^{-1})^* Q)|_{\mathcal{M}_0} = (X^{-1}_\lambda Y_\lambda^*)^* (X_\lambda^* Q)|_{\mathcal{M}_0} = \Phi_\lambda^* X^\lambda Q_\lambda. \quad (2.10)$$

According to generic arguments concerning the Taylor expansion of the pull-back of a tensor field on the same manifold, it should be expressed the gauge transformation $\Phi_\lambda^* X^\lambda Q_\lambda$ in the form

$$\Phi_\lambda^* X^\lambda Q = X^\lambda Q + \lambda \xi_1^\lambda X^\lambda Q + \frac{\lambda^2}{2} \{ \xi_1^\lambda + \xi_2^\lambda \} X^\lambda Q + O(\lambda^3), \quad (2.11)$$

where the vector fields $\xi_1^\lambda$ and $\xi_2^\lambda$ are the generators of the gauge transformation $\Phi_\lambda$.

Comparing the representation (2.11) and that in terms of the generators $X^\eta^a$ and $Y^\eta^a$ of the pull-back $Y^\lambda_\lambda^* (X_\lambda^{-1})^* X^\lambda Q = (\Phi_\lambda^* X^\lambda Q)$, we obtain explicit correspondence
between \( \{ \xi_1^a, \xi_2^a \} \) and \( \{ X_1^a, Y_1^a \} \) as follows:

\[
\xi_1^a = Y_1^a - X_1^a, \quad \xi_2^a = [Y_1^a, X_1^a]^a.
\]

Further, because the gauge transformation \( \Phi_\lambda \) is a map within the background space-time \( \mathcal{M}_0 \), the generator should consist of vector fields on \( \mathcal{M}_0 \).

We can now derive the relation between the perturbations in the two different gauges. Up to second order, these relations are derived by substituting (2.7) and (2.7) into (2.11):

\[
(1) Y^Q - (1) X^Q = L_{\xi_1} Q_0, \quad (2.13)
\]

\[
(2) Y^Q - (2) X^Q = 2L_{\xi_1} (1) X^Q + \left\{ L_{\xi_2} (1) X^Q + L_{\xi_2} (2) \right\} Q_0. \quad (2.14)
\]

Here, we comment on the generic formula for the Taylor expansion (2.11). In the case where we regard the pull-backs \( X_\lambda^a \) and \( Y_\lambda^a \) of the gauge choices are exponential maps, the product of two exponential maps is also written by the exponential of the infinite sum of the Lie derivative along infinitely many generators through Baker-Campbell-Hausdorff formula. These infinitely many generators are constructed by the commutators of \( X_\eta^a \) and \( Y_\eta^a \), which are regarded as higher-order derivatives of \( X_\eta^a \) and \( Y_\eta^a \) in \( \mathcal{N} \) and are regarded as the vector fields on \( \mathcal{N} \). The expression of the Taylor expansion is just the expression up to \( O(\lambda^3) \) of the Baker-Campbell-Hausdorff formula. Of course, we may generalize the gauge choice \( X_\lambda^a \) and \( Y_\lambda^a \) to more general class of diffeomorphism than the exponential map. Even in this case, the Taylor expansion (2.11) is correct. Since the gauge-transformation rules (2.13) and (2.14) are direct consequences of the Taylor expansion (2.11), these gauge-transformation rules are not changed even if we generalize the gauge choice \( X_\lambda^a \) and \( Y_\lambda^a \), and we may regard that two generator \( \xi_1^a \) and \( \xi_2^a \) in Eqs. (2.13) are independent of each other. Therefore, we may say that gauge-transformation rules (2.13) and (2.14) are most general gauge-transformation rules of the first and second order, respectively.

2.1.4. Gauge invariance

We next introduce the concept of gauge invariance. The gauge invariance considered in this paper is order by order gauge invariance proposed in KN2009. We call the \( k \)th-order perturbation \( (p)\chi^a Q \) is gauge invariant if

\[
(2.15)
\]

for any gauge choice \( X_\lambda^a \) and \( Y_\lambda^a \). Through this concept of order by order gauge invariance, we can decompose any perturbation of \( Q \) into the gauge-invariant and gauge-variant parts, as shown in KN2003. In terms of these gauge-invariant variables, we can develop the gauge-invariant perturbation theory. However, this development is based on a non-trivial conjecture, i.e., Conjecture 2.1 for the linear order metric perturbation as explained below.

2.2. Gauge-invariant variables

Inspecting the gauge-transformation rules (2.13) and (2.14), we define gauge-invariant variables for metric perturbations and for arbitrary matter fields. First, we
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consider the metric perturbation and expand the metric $\bar{g}_{ab}$ on $\mathcal{M}$, which is pulled back to $\mathcal{M}_0$ using a gauge choice $\lambda$, in the form given in (2.15),

$$\lambda \bar{g}_{ab} = g_{ab} + \lambda h_{ab} + \frac{\lambda^2}{2} \Lambda_{ab} + O(\lambda^3), \quad (2.16)$$

where $g_{ab}$ is the metric on $\mathcal{M}_0$. Of course, the expansion (2.16) of the metric depends entirely on the gauge choice $\lambda$. Nevertheless, henceforth, we do not explicitly express the index of the gauge choice $\lambda$ if there is no possibility of confusion.

Our starting point to construct gauge-invariant variables is the following conjecture for the linear-order metric perturbation $h_{ab}$ defined by Eq. (2.16):

**Conjecture 2.1.** If there is a tensor field $h_{ab}$ of the second rank, whose gauge transformation rule is

$$\gamma h_{ab} - \chi h_{ab} = \mathcal{L}_{\xi(1)} g_{ab}, \quad (2.17)$$

then there exist a tensor field $H_{ab}$ and a vector field $X^a$ such that $h_{ab}$ is decomposed as

$$h_{ab} =: H_{ab} + \mathcal{L}_X g_{ab}, \quad (2.18)$$

where $H_{ab}$ and $X^a$ are transformed as

$$\gamma H_{ab} - \chi H_{ab} = 0, \quad \gamma X^a - \chi X^a = \xi^a_{(1)} \quad (2.19)$$

under the gauge transformation (2.13), respectively.

In this conjecture, $H_{ab}$ is gauge-invariant in the sense as mentioned above, and we call $H_{ab}$ as gauge-invariant part of the linear-order metric perturbation $h_{ab}$. On the other hand, the vector field $X^a$ in Eq. (2.18) is gauge dependent, and we call $X^a$ as gauge-variant part of the metric perturbation $h_{ab}$.

The main purpose of this paper is to prove Conjecture 2.1 in some sense. In the case of the cosmological perturbations on a homogeneous and isotropic universe, we confirmed Conjecture 2.1 is correct except for some special modes of perturbations, and then we developed the second-order cosmological perturbation theory in a gauge-invariant manner. On the other hand, in the case of the perturbation theory on a generic background spacetime, this conjecture was highly non-trivial due to the non-trivial curvature of the background spacetime. We see this situation in detail in §3. However, before going to the proof of Conjecture 2.1 we explain how the higher-order gauge-invariant perturbation theory is developed based on this conjecture, here. Through this explanation, we emphasize the importance of Conjecture 2.1.

As shown in KN2003, the second-order metric perturbations $l_{ab}$ are decomposed as

$$l_{ab} =: \mathcal{L}_{ab} + 2 \mathcal{L}_X h_{ab} + \left( \mathcal{L}_Y - \mathcal{L}_X^2 \right) g_{ab}, \quad (2.20)$$

where $\mathcal{L}_{ab}$ and $Y^a$ are the gauge-invariant and gauge-variant parts of the second order metric perturbations, i.e.,

$$\gamma \mathcal{L}_{ab} - \chi \mathcal{L}_{ab} = 0, \quad \gamma Y^a - \chi Y^a = \xi^a_{(2)} + [\xi_{(1)}, X^a]. \quad (2.21)$$
Actually, using the gauge-variant part $X^a$ of the linear-order metric perturbation $h_{ab}$, we consider the tensor field $\hat{L}_{ab}$ defined by

$$\hat{L}_{ab} := l_{ab} - 2\mathcal{L}_X h_{ab} + \mathcal{L}_X^2 g_{ab}. \quad (2.22)$$

Through the gauge-transformation rules (2.14) and (2.19) for $l_{ab}$ and $X^a$, respectively, the gauge-transformation rule for this variable $\hat{L}_{ab}$ is given by

$$\gamma \hat{L}_{ab} - \chi \hat{L}_{ab} = \mathcal{L}_{\sigma} g_{ab}, \quad \sigma^a := \xi^a_{(2)} + [\xi_{(1)}, X]^a. \quad (2.23)$$

This is identical to the gauge-transformation rule (2.17) in Conjecture 2.1 and we may apply Conjecture 2.1 to the variable $\hat{L}_{ab}$. Then, $\hat{L}_{ab}$ can be decomposed as

$$\hat{L}_{ab} = \mathcal{L}_{ab} + \mathcal{L}_Y g_{ab}, \quad (2.24)$$

where the gauge-transformation rules for $\mathcal{L}_{ab}$ and $Y^a$ are given by Eqs. (2.21). Together with the definition (2.22) of the variable $\hat{L}_{ab}$, the decomposition (2.24) leads the decomposition (2.20) for the second-order metric perturbation $l_{ab}$.

Furthermore, as shown in KN2003, using the first- and second-order gauge-variant parts, $X^a$ and $Y^a$, of the metric perturbations, the gauge-invariant variables for an arbitrary tensor field $Q$ other than the metric are given by

$$(1) Q := (1) Q - \mathcal{L}_X Q_0, \quad (2.25)$$

$$(2) Q := (2) Q - 2\mathcal{L}_X (1) Q - \{ \mathcal{L}_Y - \mathcal{L}_X^2 \} Q_0. \quad (2.26)$$

It is straightforward to confirm that the variables $(p)Q$ defined by (2.25) and (2.26) are gauge invariant under the gauge-transformation rules (2.13) and (2.14), respectively.

Equations (2.25) and (2.26) have an important implication. To see this, we represent these equations as

$$(1) Q = (1) Q + \mathcal{L}_X Q_0, \quad (2.27)$$

$$(2) Q = (2) Q + 2\mathcal{L}_X (1) Q + \{ \mathcal{L}_Y - \mathcal{L}_X^2 \} Q_0. \quad (2.28)$$

These equations imply that any perturbation of first and second order can always be decomposed into gauge-invariant and gauge-variant parts as Eqs. (2.27) and (2.28), respectively.

2.3. Second-order gauge-invariant perturbation theory

When we consider the first- and the second-order perturbations of the Einstein equation, we have to consider the perturbative expansion of the Einstein tensor and the energy momentum tensor. Now, we consider the perturbative expansion of the Einstein tensor on $\mathcal{M}_\lambda$ as

$$\bar{G}^{\ a \ b} = G^{\ a \ b} + \lambda (1) G^{\ a \ b} + \frac{1}{2} \lambda^2 (2) G^{\ a \ b} + O(\lambda^3). \quad (2.29)$$

As shown in KN2005, the first- and the second-order perturbation of the Einstein tensor are given by

$$(1) G^{\ a \ b} = (1) G^{\ a \ b} [\mathcal{H}] + \mathcal{L}_X G^{\ a \ b}, \quad (2.30)$$

$$(2) G^{\ a \ b} = (1) G^{\ a \ b} [\mathcal{L}] + (2) G^{\ a \ b} [\mathcal{H}, \mathcal{H}] + 2\mathcal{L}_X (1) G^{\ a \ b} + \{ \mathcal{L}_Y - \mathcal{L}_X^2 \} G^{\ a \ b}. \quad (2.31)$$
where

\[ (1) G^b_a [A] := (1) \Sigma^b_a [A] - \frac{1}{2} \delta^b_a (1) \Sigma^c [A], \quad (2.32) \]

\[ (1) \Sigma^b_a [A] := -2 \nabla^b_a H^{cd} [A] - A^{bc} R_{ac}, \quad (2.33) \]

\[ (2) G^b_a [A, B] := (2) \Sigma^b_a [A, B] - \frac{1}{2} \delta^b_a (2) \Sigma^c [A, B], \quad (2.34) \]

\[ (2) \Sigma^b_a [A, B] := 2 R_{ad} B_c^b (A_{d}^{ac} + 2 H^c_{da} A_{e}^b e [B] + 2 H^{de} [B] H_{d}^{b} e [A] \]

\[ + 2 A_{e}^{d} \nabla_{a} H_{b}^{e} [B] + 2 B_{e}^{d} \nabla_{a} H_{b}^{e} [A] \]

\[ + 2 A_{c}^{b} \nabla_{a} H_{d}^{c d} [B] + 2 B_{c}^{b} \nabla_{a} H_{d}^{c d} [A], \quad (2.35) \]

and

\[ H_{ab}^{c} [A] := \nabla_{(a} A_{b)}^{c} - \frac{1}{2} \nabla^{c} A_{ab}, \quad (2.36) \]

\[ H_{abc} [A] := g_{cd} H_{ab}^{d} [A], \quad H^{bc} [A] := g^{bd} H_{ad}^{c} [A], \quad (2.37) \]

We note that \( (1) G^b_a [*] \) and \( (2) G^b_a [*] \) in Eqs. (2.30) and (2.31) are the gauge-invariant parts of the perturbative Einstein tensors, and Eqs. (2.30) and (2.31) have the same forms as Eqs. (2.27) and (2.28), respectively.

We also note that \( (1) G^b_a [*] \) and \( (2) G^b_a [*] \) defined by Eqs. (2.32)–(2.35) satisfy the identities

\[ \nabla_{a} (1) G^b_a [A] = -H_{ca}^{a} [A] G^{b}_{c} + H_{ba}^{c} [A] G^{a}_{c}, \quad (2.38) \]

\[ \nabla_{a} (2) G^b_a [A, B] = -H_{ca}^{a} [A] (1) G^{b}_{c} [B] - H_{ca}^{a} [B] (1) G^{b}_{c} [A] \]

\[ + H_{ba}^{e} [A] (1) G_{e}^{a} [B] + H_{ba}^{e} [B] (1) G_{e}^{a} [A] \]

\[ - \left( H_{bad} [B] A^{de} + H_{BAD} [B] B^{de} \right) G^{a}_{c} \]

\[ + \left( H_{caB} [B] A^{ad} + H_{cad} [A] B^{ad} \right) G^{b}_{c}, \quad (2.39) \]

for arbitrary tensor fields \( A_{ab} \) and \( B_{ab} \), respectively. We can directly confirm these identities without specifying arbitrary tensors \( A_{ab} \) and \( B_{ab} \) of the second rank, respectively. These identities (2.38) and (2.39) guarantee the first- and second-order perturbations of the Bianchi identity \( \nabla_{b} G^{b}_{a} = 0 \). This implies that our general framework of the second-order gauge-invariant perturbation theory is self-consistent.

On the other hand, the energy momentum tensor on \( \mathcal{M}_\lambda \) is also expanded as

\[ T^b_a = T^b_a + \lambda (1) T^b_a + \frac{1}{2} \lambda^2 (2) T^b_a + O(\lambda^3). \quad (2.40) \]

According to Eqs. (2.27) and (2.28), we can also decompose the first- and the second-order perturbations of the energy momentum tensor \( (1) T^b_a \) and \( (2) T^b_a \) as

\[ (1) T^b_a = (1) T^b_a + \mathcal{E} X T^b_a, \quad (2.41) \]

\[ (2) T^b_a = (2) T^b_a + 2 \mathcal{E} X (1) T^b_a + \{ \mathcal{E} Y - \mathcal{E}^2 \} T^b_a. \quad (2.42) \]
These decompositions are confirmed in the case of a perfect fluid, an imperfect fluid, and a scalar field in KN2009. Furthermore, in KN2009 we also showed that equations of motion for the matter field, which are derived from the divergence of the energy-momentum tensors, are also decomposed into gauge-invariant and gauge-variant parts as Eqs. (2.27) and (2.28). Therefore, we may say that the decomposition formulae (2.27) and (2.28) are universal.

Imposing order by order Einstein equations
\[ G_{\alpha\beta} = 8\pi G_{a}^{b}, \quad (1)G_{\alpha\beta} = 8\pi(1)T_{a}^{b}, \quad (2)G_{\alpha\beta} = 8\pi(2)T_{a}^{b}, \] (2.43)
the first- and the second-order perturbation of the Einstein equations are automatically given in gauge-invariant form as
\[ (1)G_{\alpha\beta} [H] = 8\pi G^{(1)}T_{a}^{b}, \quad (1)G_{\alpha\beta} [L] + (2)G_{\alpha\beta} [H, H] = 8\pi G^{(2)}T_{a}^{b}. \] (2.44)
Furthermore, in KN2009 we also showed that the equations of motion for matter fields, are automatically given in gauge-invariant form. Thus, we may say that any equation of order by order is automatically gauge-invariant and we do not have to consider the gauge degree of freedom at least in the level where we concentrate only on the equations of the general relativistic system.

We can also expect that the similar structure of equations of the systems will be maintained in the any order perturbations and our general framework be applicable to any order general-relativistic perturbations. Actually, decomposition formulae for the third-order perturbations in two-parameter case which correspond to Eqs. (2.27) and (2.28) are given in KN2003. Therefore, similar development is possible for the third-order perturbations. Since we could not find any difficulties to extend higher-order perturbations except for the necessity of long cumbersome calculations, we can construct any order perturbation theory in gauge-invariant manner, recursively.

We have to emphasize that the above general framework of the higher-order gauge-invariant perturbation theory are independent of the explicit form of the background metric \(g_{ab}\), except for Conjecture 2.4 and are valid not only in cosmological perturbation case but also the other generic situations if Conjecture 2.4 is true. This implies that if we prove Conjecture 2.4 for the generic background spacetime, the above general framework is applicable to perturbation theories on any background spacetime. This is the reason why we proposed Conjecture 2.4 in KN2009.

Thus, Conjecture 2.4 is the important premise of our general framework of higher-order gauge-invariant perturbation theory. In the next section, we give a proof of Conjecture 2.4 on the generic background spacetime which admits ADM decomposition (see Appendix A).

\section{3. Decomposition of the linear-order metric perturbation}

Now, we give a proof of Conjecture 2.4 on general background spacetimes which admit ADM decomposition (see Appendix A). Therefore, the background spacetime \(\mathcal{M}_{0}\) considered here is \(n + 1\)-dimensional spacetime which is described by the direct product \(\mathbb{R} \times \Sigma\). Here, \(\mathbb{R}\) is a time direction and \(\Sigma\) is the spacelike hypersurface with
\textbf{Decomposition of linear metric perturbations on generic background spacetime}

\( \dim \Sigma = n \) embedded in \( \mathcal{M}_0 \). This means that \( \mathcal{M}_0 \) is foliated by the one-parameter family of spacelike hypersurface \( \Sigma(t) \), \( t \in \mathbb{R} \) is a time function. The metric on \( \mathcal{M}_0 \) is given as Eq. (A.23), i.e.,

\[
g_{ab} = -\alpha^2(dt)_a(dt)_b + q_{ij}(dx^i + \beta^i dt)_a(dx^j + \beta^j dt)_b, \tag{3-1}\]

where \( \alpha \) is the lapse function, \( \beta^i \) is the shift vector, and \( g_{ab} = q_{ij}(dx^i)_a(dx^j)_b \) is the metric on \( \Sigma(t) \). The inverse of Eq. (3.1) is given by Eq. (A.21) in the Appendix A.

To consider the decomposition (2.18) of \( h_{ab} \), first, we consider the components of the metric \( h_{ab} \) as

\[
h_{ab} = h_{tt}(dt)_a(dt)_b + 2h_{ti}(dt)(a(dx^i)_b) + h_{ij}(dx^i)_a(dx^j)_b. \tag{3.2}\]

The components \( h_{tt}, h_{ti}, \) and \( h_{ij} \) are regarded as a scalar function, a vector field, and a tensor field on the spacelike hypersurface \( \Sigma(t) \), respectively. From the gauge-transformation rule (2.17), the components \( \{h_{tt}, h_{ti}, h_{ij}\} \) are transformed as

\[
yh_{tt} - \chi h_{tt} = 2\partial_t \xi_t - \frac{2}{\alpha} (\partial_t \alpha + \beta^i D_i \alpha - \beta^j \beta^i K_{ij}) \xi_t - 2 \left( \beta^i \beta^j K_{kj} - \beta^j \partial_t \alpha + \alpha q^{ij} \partial_t \beta_j \right) \xi_i - \alpha^2 D_i \alpha - \alpha \beta_k D^i \beta_k - \beta^i \beta^j D_j \alpha \right) \xi_i, \tag{3.3}\]

\[
yh_{ti} - \chi h_{ti} = \partial_t \xi_i + D_i \xi_t - \frac{2}{\alpha} (D_i \alpha - \beta^j K_{ij}) \xi_t - 2 \left( -\alpha^2 K^j_i + \beta^j \beta^i K_k i - \beta^j D_i \alpha + \alpha D_i \beta^j \right) \xi_j, \tag{3.4}\]

\[
yh_{ij} - \chi h_{ij} = 2D_i(\xi_j) + \frac{2}{\alpha} K_{ij} \xi_t - \frac{2}{\alpha} \beta^k K_{ij} \xi_k, \tag{3.5}\]

where \( K_{ij} \) is the extrinsic curvature of \( \Sigma \) defined Eq. (A.36) in Appendix A and \( D_i \) is the covariant derivative associate with the metric \( q_{ij} \) \( (D_i q_{jk} = 0) \).

Apparently, the gauge-transformation rules (3.3)–(3.5) have the complicated form and it seems difficult to find the decomposition as (2.18). Therefore, we consider the proof of the decomposition (2.18) from the simpler situations of the background spacetime. We consider the three cases. In §3.1, the case where \( \alpha = 1, \beta^i = 0, \) and \( K_{ij} = 0 \) is considered. In this simplest case, we may consider the non-trivial intrinsic curvature of \( \Sigma \). From the view point of a proof of Conjecture 2.1 this case is trivial. However, we find the outline of the proof of Conjecture 2.1 through this case. Second, in §3.2 we consider the case where \( \alpha = 1, \beta^i = 0, \) and \( K_{ij} \neq 0 \). This case includes not only many homogeneous background spacetimes but also the Schwarzschild spacetime. Furthermore, we note that the most non-trivial technical part of the proof of Conjecture 2.1 is given in this case. Finally, in §3.3 we consider the most general case where \( \alpha \neq 1, \beta^i \neq 0, \) and \( K_{ij} \neq 0 \) for completion.

3.1. \( \alpha = 1, \beta^i = 0, \) and \( K_{ij} = 0 \) case

Here, we consider \( \mathcal{M}_0 \) satisfies the conditions

\[
\alpha = 1, \quad \beta^i = 0, \quad K_{ij} = 0. \tag{3.6}\]
In this case, the gauge-transformation rules (3.10)–(3.12) are given by

\begin{align}
\gamma h_{tt} - \chi h_{tt} &= 2\partial_l \xi_t, \quad (3.7) \\
\gamma h_{ti} - \chi h_{ti} &= \partial_l \xi_i + D_i \xi_t, \quad (3.8) \\
\gamma h_{ij} - \chi h_{ij} &= 2D_i(\xi_j).
\end{align}

To prove Conjecture 2.1, we consider the decomposition of the symmetric tensor field on \( \Sigma \) reviewed in Appendix B

\begin{align}
h_{ij} &= D_i h_{(V)L} + h_{(V)i}, \quad D^j h_{(V)i} = 0, \quad (3.10) \\
h_{ij} &= \frac{1}{n} q_{ij} h_{(L)} + h_{(T)ij}, \quad q^{ij} h_{(T)ij} = 0, \quad (3.11) \\
h_{(T)ij} &= L h_{(TV)ij} + h_{(TT)ij}, \quad D^j h_{(TT)ij} = 0, \quad (3.12) \\
h_{(TV)i} &= D_i h_{(TV)L} + h_{(TV)V)i}, \quad D^j h_{(TV)V)i} = 0, \quad (3.13)
\end{align}

where \((L h_{(TV)})_{ij}\) is defined by [see Eq. (B.3) in Appendix B]

\[ (L h_{(TV)})_{ij} := D_i h_{(TV)j} + D_j h_{(TV)i} - \frac{2}{n} q_{ij} D^j h_{(TV)i}. \quad (3.14) \]

To derive gauge-transformation rules for \( \{h_{tt}, h_{(V)L}, h_{(V)i}, h_{(L)}, h_{(TV)i}, h_{(TT)ij}\}, \) or \( \{h_{tt}, h_{(V)L}, h_{(V)i}, h_{(L)}, h_{(TV)L}, h_{(TV)V)i}, h_{(TT)ij}\}, \) we decompose the generator \( \xi_i \) as

\[ \xi_i = D_i(\xi_L) + \xi_{(V)i}, \quad D^j \xi_{(V)i} = 0. \quad (3.15) \]

Through the decomposition (3.10) and (3.15), the gauge-transformation rule is given by

\begin{align}
\gamma h_{ti} - \chi h_{ti} &= D_i \left( \gamma h_{(V)L} - \chi h_{(V)L}\right) + \gamma h_{(V)i} - \chi h_{(V)i} \\
&= D_i \left( \partial_l \xi_L + \xi_i \right) + \partial_l h_{(V)i}, \quad (3.16)
\end{align}

where we used \( \partial_l D_i f = D_i \partial_l f \) for an arbitrary scalar function \( f \). Taking the divergence of Eq. (3.16), we see that

\[ \Delta \left( \gamma h_{(V)L} - \chi h_{(V)L} \right) = \Delta \left( \partial_l \xi_L + \xi_i \right) + D^j \partial_l \xi_{(V)i}. \quad (3.17) \]

Since \( K_{ij} = 0, \alpha = 1, \) and \( \beta_i = 0 \) in our case, Eq. (A-37) yields

\[ K_{ij} = -\frac{1}{2} \partial_l q_{ij} = 0. \quad (3.18) \]

From Eqs. (A-35), (3.18), and the property \( D^j \partial_l \xi_{(V)i} = 0, \) we can easily verify that

\[ D^j \partial_l \xi_{(V)i} = \partial_l D^j \xi_{(V)i} = 0. \quad (3.19) \]

Then the gauge-transformation rule (3.17) is given by

\[ \Delta \left( \gamma h_{(V)L} - \chi h_{(V)L} \right) = \Delta \left( \partial_l \xi_L + \xi_i \right). \quad (3.20) \]
Here, we assume the existence of the Green function of the Laplacian $\Delta := D^i D_i$ and ignore the mode which belongs to the kernel of the Laplacian $\Delta$. Then, we obtain

$$y h_{(VL)} - \chi h_{(VL)} = \partial_t \xi_{(L)} + \xi_t. \tag{3.21}$$

Substituting Eq. (3.21) into Eq. (3.16), we obtain

$$y h_{(V)i} - \chi h_{(V)i} = \partial_t \xi_{(V)i}. \tag{3.22}$$

Next, through Eq. (3.11), the trace part of Eq. (3.9) is given by

$$q_{ij} y h_{ij} - q_{ij} X h_{ij} = y h_{(L)} - \chi h_{(L)} = 2D^i \xi_i = 2\Delta \xi_{(L)}, \tag{3.23}$$

where we used Eq. (3.15). Then, through the decomposition (3.12), the gauge transformation rule of the traceless part $h(T)_{ij}$ is given by

$$y h_{(T)ij} - \chi h_{(T)ij} = y (L h_{(TV)})_{ij} - \chi (L h_{(TV)})_{ij} + y h_{(TT)ij} - \chi h_{(TT)ij} = (L \xi)_{ij}. \tag{3.24}$$

Taking the divergence of Eq. (3.24), we obtain

$$D^l (y h_{(TV)l} - \chi h_{(TV)l} - \xi_l) = 0, \tag{3.25}$$

where the derivative operator $D^{ij}$ is defined by

$$D^{ij} := q^{ij} \Delta + \left(1 - \frac{2}{n}\right) D^i D^j + R^{ij}, \tag{3.26}$$

and its properties are discussed in Appendix B. Here, we assume the existence of the Green function of the derivative operator $D^{ij}$ and ignore the modes which belong to the kernel of the derivative operator $D^{ij}$. Then, we obtain

$$y h_{(TV)i} - \chi h_{(TV)i} = \xi_i. \tag{3.27}$$

Substituting Eq. (3.27) into Eq. (3.24), we obtain

$$y h_{(TT)ij} - \chi h_{(TT)ij} = 0. \tag{3.28}$$

Moreover, through the decomposition (3.13), we obtain

$$y h_{(TVL)} - \chi h_{(TVL)} = \xi_{(L)}, \tag{3.29}$$

$$y h_{(TVV)i} - \chi h_{(TVV)i} = \xi_{(V)i}. \tag{3.30}$$

Here, we have also ignored the mode which belongs the kernel of $\Delta$.

In summary, we have obtained the gauge-transformation rule of variables $\{h_T, h_{(VL)}, h_{(TV)i}, h_{(TT)ij}\}$ or $\{h_T, h_{(VL)L}, h_{(VL)i}, h_{(TV)L}, h_{(TV)i}, h_{(TT)L}, h_{(TT)i}\}$ as

$$y h_T - \chi h_T = 2\partial_t \xi_t, \tag{3.31}$$

$$y h_{(VL)} - \chi h_{(VL)} = \partial_t \xi_{(L)} + \xi_t. \tag{3.32}$$
Now, we construct gauge-invariant variables. First, the gauge-transformation rule \(3.37\) shows \(\hat{h}_{ij}^{TT} \) is gauge invariant by itself:

\[
\chi_{ij} := \hat{h}_{ij}^{TT}.
\]  

(3.38)

From Eq. \(3.12\), this variable \(\chi_{ij}\) satisfy the transverse-traceless condition

\[
D_i \chi_{ij} = 0 = q_{ij} \chi_{ij}.
\]

(3.40)

Next, we consider scalar modes. First, from the gauge-transformation rules \(3.34\) and \(3.35\), we see that the variable \(\Psi\) defined by

\[
-2n\psi := h_{L} - 2\Delta h_{LV}
\]

(3.41)

is gauge invariant. Actually, the gauge-transformation rule for \(\psi\) is given by

\[
y\psi - \chi\psi = - \frac{1}{2n} (y h_{L} - 2\Delta y h_{LV}) + \frac{1}{2n} (x h_{L} - 2\Delta x h_{LV}) = 0.
\]

(3.42)

To define another gauge-invariant variables for scalar modes, we consider the gauge-transformation rule \(3.32\) and \(3.35\). We find that the variable \(\hat{X}_t\) defined by

\[
\hat{X}_t := h_{VL} - \partial_t h_{TVL}
\]

(3.43)

is transformed as

\[
y\hat{X}_t - \chi\hat{X}_t = \xi_t.
\]

(3.44)

Using this variable \(\hat{X}_t\), we define the gauge-invariant combination as

\[
-2\Phi := h_{tt} - 2\partial_t \hat{X}_t.
\]

(3.45)

Actually, we can easily check the variable \(\Phi\) is gauge invariant under the gauge-transformation rules \(3.31\) and \(3.41\).
In terms of the gauge-invariant variables defined by Eqs. (3.38), (3.39), (3.41), and (3.45), and the variable \( \hat{X}_t \) defined by Eq. (3.43), the original set \( \{h_{tt}, h_{ti}, h_{ij}\} \) of the components of the linear metric perturbation is given by

\[
\begin{align*}
  h_{tt} &= -2\Phi + 2\partial_t \hat{X}_t, \\
  h_{ti} &= \nu_i + \partial_t h_{(TV)i} + D_i \hat{X}_t, \\
  h_{ij} &= -2q_{ij}\Psi + \chi_{ij} + D_i h_{(TV)j} + D_j h_{(TV)i}.
\end{align*}
\]  

(3.46)

(3.47)

(3.48)

On the other hand, the components of Eq. (2.18) with \( X_a =: X_t(dt)_a + X_i(dx^i)_a \) are given by

\[
\begin{align*}
  h_{tt} &= \mathcal{H}_{tt} + 2\partial_t X_t, \\
  h_{ti} &= \mathcal{H}_{ti} + \partial_t X_i + D_i X_t, \\
  h_{ij} &= \mathcal{H}_{ij} + D_i X_j + D_j X_i.
\end{align*}
\]  

(3.50)

(3.51)

(3.52)

Since the variable \( \mathcal{H}_{tt}, \mathcal{H}_{ti}, \) and \( \mathcal{H}_{ij} \) are gauge-invariant and the gauge-transformation rules for the variable \( X_t \) and \( X_i \) are given by

\[
\begin{align*}
  yX_t - \chi X_t &= \xi_t, \\
  yX_i - \chi X_i &= \xi_i,
\end{align*}
\]  

(3.53)

respectively, we may naturally identify the variables as follows:

\[
\begin{align*}
  \mathcal{H}_{tt} &= -2\Phi, & \mathcal{H}_{ti} &= \nu_i, & \mathcal{H}_{ij} &= -2q_{ij}\Psi + \chi_{ij}, \\
  X_t &= \hat{X}_t, & X_i &= h_{(TV)i}.
\end{align*}
\]  

(3.54)

(3.55)

The gauge-transformation rules (3.27) [or equivalently Eqs. (3.35) and (3.36)] and (3.44) support these identifications. Thus, Eqs. (3.46)–(3.48) show that the linear-order metric perturbation \( h_{ab} \) is decomposed into gauge-invariant and gauge-variant parts as Eq. (2.18) in the case for the background spacetime with \( \alpha = 1, \beta^i = 0 \), and \( K_{ij} = 0 \).

3.2. \( \alpha = 1, \beta^i = 0, \text{ but } K_{ij} \neq 0 \) case

Here, we also consider the case where the background metric is described by Eq. (3.1) with \( \alpha = 1, \beta^i = 0 \), but the background spacetime has the non-trivial extrinsic curvature \( K_{ij} \neq 0 \). In this case, the extrinsic curvature \( K_{ij} \) is proportional to the time derivative of the metric \( q_{ij} \) on \( \Sigma \)

\[
K_{ij} = -\frac{1}{2} \partial_t q_{ij}
\]  

(3.56)

from Eq. (A.37) in Appendix A and gauge-transformation rules (3.3)–(3.5) for the components \( \{h_{tt}, h_{ti}, h_{ij}\} \) are given by

\[
\begin{align*}
  yh_{tt} - \chi h_{tt} &= 2\partial_t \xi_t, \\
  yh_{ti} - \chi h_{ti} &= \partial_t \xi_i + D_i \xi_t + 2K^j \xi_j, \\
  yh_{ij} - \chi h_{ij} &= 2D_i(\xi_j) + 2K_{ij} \xi_t.
\end{align*}
\]  

(3.57)

(3.58)

(3.59)
Inspecting gauge-transformation rules \((3.58)\)–\((3.59)\), we first define a new symmetric tensor field \(\hat{h}_{ab}\) whose components are defined by

\[
\hat{H}_{ti} := h_{tt}, \quad \hat{H}_{ti} := h_{ti}, \quad \hat{H}_{ij} := h_{ij} - 2K_{ij}\hat{X}_t. \tag{3.60}
\]

Here, we assume the existence of the variable \(\hat{X}_t\) whose gauge-transformation rule is given by

\[
y\hat{X}_t - \chi\hat{X}_t = \xi_t. \tag{3.61}
\]

The existence of the variable \(\hat{X}_t\) is confirmed later soon. Similar technique is given by T. S. Pereira et al.\(^{22}\) in the perturbations on Bianchi type I cosmology. Now, the components \(\hat{H}_{ti}\) and \(\hat{H}_{ij}\) are regarded as a vector and a symmetric tensor on \(\Sigma(t)\), respectively. Then, we may apply the decomposition in Appendix B to \(\hat{H}_{ti}\) and \(\hat{H}_{ij}\):

\[
\hat{H}_{ti} = D_i h_{(VL)} + h_{(V)i}, \quad D^i h_{(V)i} = 0, \tag{3.62}
\]

\[
\hat{H}_{ij} = \frac{1}{n} q_{ij} h_{(L)} + h_{(T)ij}, \quad q^{ij} h_{(T)ij} = 0, \tag{3.63}
\]

\[
h_{(T)ij} = (L h_{(TV)})_{ij} + h_{(TT)ij}, \quad D^i h_{(TT)ij} = 0, \tag{3.64}
\]

\[
h_{(TV)i} = D_i h_{(TVL)} + h_{(TVV)i}, \quad D^i h_{(TVV)i} = 0. \tag{3.65}
\]

Through Eqs. \((3.62)\) and \((3.60)\), the gauge-transformation rule \((3.58)\) is given by

\[
D_i \left( y h_{(VL)} - \chi h_{(VL)} \right) + y h_{(V)i} - \chi h_{(V)i} = \partial_t \xi_t + D_t \xi_t + 2K^j_i \xi_j. \tag{3.66}
\]

Taking the divergence of this gauge-transformation rule \((3.66)\), we obtain

\[
\Delta \left( y h_{(VL)} - \chi h_{(VL)} \right) = D^i \partial_i \xi_t + \Delta \xi_t + 2D^i \left( K^j_i \xi_j \right), \tag{3.67}
\]

where we used the divergenceless property of the variable \(h_{(V)i}\).

Now, we evaluate the term \(D^i \partial_i \xi_t\) in Eq. \((3.67)\). First, we note that the time-derivative of the intrinsic metric \(\partial_t q_{ij}\) is given by Eq. \((3.56)\), i.e.,

\[
\partial_t q_{ij} = -2K_{ij}. \tag{3.68}
\]

Keeping the relation \((3.68)\) in our mind, we can evaluate \(D_i \partial_i \xi_j\) as

\[
D_i \partial_i \xi_j = \partial_t D_i \xi_j - \xi_k \left( D_i K^k_j + D_j K^k_i - D^k K_{ij} \right). \tag{3.69}
\]

From Eq. \((3.69)\), we further evaluate the term \(D^i \partial_i \xi_t\) as

\[
D^i \partial_i \xi_t = \partial_t \left( D^i \xi_t \right) + \xi_k D^k K - 2D_i \left( K^j_i \xi_j \right). \tag{3.70}
\]

Through Eq. \((3.70)\), the gauge-transformation rule \((3.67)\) is given by

\[
\Delta \left( y h_{(VL)} - \chi h_{(VL)} \right) = \partial_t \left( D^i \xi_i \right) + \Delta \xi_t + \xi_k D^k K. \tag{3.71}
\]

Here again, it is convenient to introduce the decomposition of \(\xi_t\) as Eq. \((3.65)\), i.e.,

\[
\xi_t = D_t \xi_{(L)} + \xi_{(V)i}, \quad D^i \xi_{(V)i} = 0. \tag{3.72}
\]
Applying the decomposition formulae (3.65) and (3.72), the gauge-transformation rule for the variable \( \hat{h}_{(V)L} \) is given as

\[
\Delta (\gamma h_{(V)L} - \chi h_{(V)L}) = \Delta \partial_i \xi (L) + 2D_i (K^{ij}D_j \xi (L)) + D^k K \xi (V)_k. \tag{3.73}
\]

Since we ignore the modes which belong to the kernel of the operator \( \Delta \), we obtain the gauge-transformation rule for the variable \( h_{(V)L} \) as

\[
\gamma h_{(V)L} - \chi h_{(V)L} = \xi_t + \partial_i \xi (L) + \Delta^{-1} \left[ 2D_i \left( K^{ij}D_j \xi (L) \right) + D^k K \xi (V)_k \right]. \tag{3.74}
\]

Substituting Eq. (3.74) into Eq. (3.66), we obtain the gauge-transformation rule for the variable \( h_{(V)i} \) as follows:

\[
\gamma h_{(V)i} - \chi h_{(V)i} = \partial_i \xi (V)_i + 2K^{ij}D_j \xi (L) + 2K^j \xi (V)_j
\]

\[
-D_i \Delta^{-1} \left[ 2D_i \left( K^{ij}D_j \xi (L) \right) + D^k K \xi (V)_k \right]. \tag{3.75}
\]

The divergenceless property of Eq. (3.75) can be easily checked through Eq. (3.70).

From Eqs. (3.59) and (3.61), the gauge transformation rule for \( \hat{h}_{ij} \) is given as

\[
\gamma \hat{h}_{ij} - \chi \hat{h}_{ij} = \left( \gamma h_{ij} - 2K_{ij} \gamma \hat{X}_t \right) - \left( \chi h_{ij} - 2K_{ij} \chi \hat{X}_t \right) = 2D_{ij} \xi (t). \tag{3.76}
\]

In terms of the decomposition (3.63)–(3.65), the gauge-transformation rules for the variables \( h_{(L)} \) and \( h_{(T)ij} \) is derived from

\[
1 \frac{1}{n} q_{ij} (\gamma h_{(L)} - \chi h_{(L)}) + (\gamma h_{(T)ij} - \chi h_{(T)ij}) = 2D_{ij} \xi (t). \tag{3.77}
\]

The trace part of the gauge transformation rule (3.77) is given by

\[
\gamma h_{(L)} - \chi h_{(L)} = 2D^i \xi (t), \tag{3.78}
\]

and the traceless part of the gauge transformation rule (3.77) is given by

\[
\gamma h_{(T)ij} - \chi h_{(T)ij} = (L \xi)_{ij}. \tag{3.79}
\]

Applying Eq. (3.64), the gauge-transformation rules (3.79) are given by

\[
(L \left( \gamma h (TV) - \chi h (TV) \right))_{ij} + (\gamma h (TT)_{ij} - \chi h (TT)_{ij}) = (L \xi)_{ij}. \tag{3.80}
\]

Taking the divergence of Eq. (3.80), we obtain

\[
\mathcal{D}^i \left[ \gamma h (TV)_i - \chi h (TV)_i - \xi (t) \right] = 0. \tag{3.81}
\]

Since we ignore the modes which belong to the kernel of the elliptic derivative operator \( \mathcal{D}^i \) in this paper, we obtain

\[
\gamma h (TV)_i - \chi h (TV)_i = \xi (t). \tag{3.82}
\]

Applying the decomposition formulae (3.66) and (3.72), the gauge-transformation rules for the variable \( \hat{h}_{(TV)L} \) and \( \hat{h}_{(TV)V} \) are given by

\[
\gamma h_{(TV)L} - \chi h_{(TV)L} = \xi (t)L, \tag{3.83}
\]

\[
\gamma h_{(TV)V} - \chi h_{(TV)V} = \xi (t)V. \tag{3.84}
\]
since we ignore the modes which belong to the kernel of $\Delta$ in this paper. Further, substituting Eq. (3.82) into Eq. (3.80), we obtain the gauge-transformation rule for $h_{(TT)ij}$ as follows

$$\gamma h_{(TT)ij} - \chi h_{(TT)ij} = 0. \quad (3.85)$$

Thus, the gauge transformation rule for the variables $h_{tt}$, $h_{(VL)}$, $h_{(V)i}$, $h_{(L)}$, $h_{(TV)i}$, $(h_{(TV)L})$ and $h_{(TVV)i}$, and $h_{(TT)ij}$ are summarized as:

$$\gamma h_{tt} - \chi h_{tt} = 2\partial_t \xi_t, \quad (3.86)$$
$$\gamma h_{VL} - \chi h_{VL} = \partial_t \xi_L + \xi_t + \Delta^{-1} \left[ 2D_i \left( K_{ij} D_j \xi_L \right) + D^k K \xi_{V(k)} \right] \quad (3.87)$$
$$\gamma h_{V(i)} - \chi h_{V(i)} = \partial_t \xi_{V(i)} + 2K_{ij} D_j \xi_L + 2K_{ij} \xi_{V(j)}$$
$$- D_i \Delta^{-1} \left[ 2D_i \left( K_{ij} D_j \xi_L \right) + D^k K \xi_{V(k)} \right], \quad (3.88)$$
$$\gamma h_{(L)} - \chi h_{(L)} = 2D_i \xi_t, \quad (3.89)$$
$$\gamma h_{(TV)i} - \chi h_{(TV)i} = \xi_t, \quad (3.90)$$
$$\gamma h_{(TT)ij} - \chi h_{(TT)ij} = 0. \quad (3.91)$$

The equation (3.90) is equivalent to the gauge-transformation rules

$$\gamma h_{(TVL)} - \chi h_{(TVL)} = \xi_{(L)}, \quad (3.92)$$
$$\gamma h_{(TVV)i} - \chi h_{(TVV)i} = \xi_{(V)i}. \quad (3.93)$$

Now, we construct gauge-invariant variables. First, Eq. (3.91) shows that the variable $h_{(TT)ij}$ is itself gauge invariant. Therefore, we define the transverse-traceless gauge-invariant tensor as

$$\chi_{ij} := h_{(TT)ij}. \quad (3.94)$$

Second, through Eqs. (3.88), (3.92), and (3.93), we can define the gauge-invariant variable for vector mode as

$$\nu_i = h_{(V)i} - \partial_t h_{(TVV)i} - 2K_{ij} D_j h_{(TVL)} + h_{(TVV)i}$$
$$+ D_i \Delta^{-1} \left[ 2D_i \left( K_{ij} D_j h_{(TVL)} \right) + D^k K h_{(TVV)k} \right]. \quad (3.95)$$

Actually, it is straightforward to confirm that the variable $\nu_i$ defined by Eq. (3.95) is gauge invariant. Further, we can also confirm the divergenceless property of the variable $\nu_i$, i.e., $D^i \nu_i = 0$ through the definition (3.95) and the formula (3.70).

Next, we consider the gauge invariant variables for scalar modes. Before doing this, we first construct the variable $\hat{X}_i$ in Eq. (3.60). Inspecting gauge-transformation rules (3.87), (3.92), and (3.93), we consider the combination

$$\hat{X}_i := h_{(VL)} - \partial_t h_{(TVL)} - \Delta^{-1} \left[ 2D_i \left( K_{ij} D_j h_{(TVL)} \right) + D^k K h_{(TVV)k} \right]. \quad (3.96)$$

The variable $\hat{X}_i$ defined by Eq. (3.96) does satisfy the gauge-transformation rule (3.61). We also define the variable $\hat{X}_i$ by

$$\hat{X}_i := h_{(TV)i} = D_i h_{(TVL)} + h_{(TVV)i}. \quad (3.97)$$
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From this gauge-transformation rule (3.90), or equivalently Eqs. (3.92) and (3.94), the gauge-transformation rule for the variable $\hat{X}_i$ defined by Eq. (3.97) is given by

$$y\hat{X}_i - X\hat{X}_i = \xi_i.$$  

(3.98)

Now, we define the gauge invariant variables for the scalar mode. First, inspecting gauge-transformation rules (3.61) and (3.86), we define the variable $\Psi$ by

$$-2\Phi := h_{tt} - 2\partial_t \hat{X}_t.$$  

(3.99)

Actually, we can easily confirm that the variable $\Phi$ is gauge invariant. Second, inspecting gauge-transformation rules (3.89) and (3.98), we define the gauge-invariant variable $\Psi$ by

$$-2n\Psi := h(L) - 2D^i\hat{X}_i.$$  

(3.100)

In terms of the gauge-invariant variables $\chi_{ij}$, $\nu_i$, $\Phi$, and $\Psi$, which are defined by Eq. (3.94), (3.95), (3.99), and (3.100), respectively, and the gauge-variant variables $\hat{X}_t$ and $\hat{X}_i$ which are defined by Eqs. (3.96) and (3.97), respectively, the original set $\{h_{tt}, h_{ti}, h_{ij}\}$ of the components of the linear metric perturbation is given by

$$h_{tt} = -2\Phi + 2\partial_t \hat{X}_t,$$  

(3.101)

$$h_{ti} = \nu_i + D_i \hat{X}_t + \partial_i \hat{X}_i + 2K_i \hat{X}_j,$$  

(3.102)

$$h_{ij} = -2\Psi q_{ij} + \chi_{ij} + D_i \hat{X}_j + D_j \hat{X}_i + 2K_{ij} \hat{X}_t.$$  

(3.103)

On the other hand, we consider the decomposition formula (2.18). The components of the expression (2.18) in the situation of this subsection with

$$X_a =: X_i(dt)_a + X_i(dx^i)_a$$  

(3.104)

are given by

$$h_{tt} = \mathcal{H}_{tt} + 2\partial_t X_t,$$  

(3.105)

$$h_{ti} = \mathcal{H}_{ti} + \partial_t X_i + D_t X_i + 2K_i X_j,$$  

(3.106)

$$h_{ij} = \mathcal{H}_{ij} + D_i X_j + D_j X_i + 2K_{ij} X_t.$$  

(3.107)

Comparing Eqs. (3.101)–(3.103) and Eqs. (3.105)–(3.107), we easily see a natural choice of the components of the gauge-invariant part $\mathcal{H}_{ab}$ and the components of the gauge-variant parts $X_a$ are given by

$$\mathcal{H}_{tt} = -2\Phi, \quad \mathcal{H}_{ti} = \nu_i, \quad \mathcal{H}_{ij} = -2\Psi q_{ij} + \chi_{ij},$$  

(3.108)

and

$$X_t = \hat{X}_t, \quad X_i = \hat{X}_i.$$  

(3.109)

Of course, we may add the Killing vectors associated with the metric $q_{ab}$ to the definition of $X_a$. The gauge-transformation rules (3.61) and (3.95) support these identifications. Thus, Eqs. (3.108) and (3.109) show that the linear-order metric perturbation $h_{ab}$ is also decomposed into gauge-invariant and gauge-variant parts as Eq. (2.18) even in the case for the background spacetime with $\alpha = 1$, $\beta^i = 0$, but $K_{ij} \neq 0$. These results are already reported in the previous letter\textsuperscript{10} by the present author.
3.3. The case for arbitrary $\alpha$, $\beta_i$, and $K_{ij}$

Now, we consider the most generic case of the metric \( (3.11 \text{a}) \) where $\alpha \neq 1$, $\beta^i \neq 0$, and $K_{ij} \neq 0$. Considering the components of the metric perturbation $h_{ab}$ as Eq. \( (3.2 \text{a}) \), the gauge-transformation rules for these components are given by Eqs. \( (3.3 \text{a}–3.5) \).

To do this, we first assume that the existence of the variables $\hat{X}_t$ and $\hat{X}_i$ whose gauge-transformation rules are given by

$$y \hat{X}_t - x \hat{X}_t = \xi_t,$$  \hspace{1cm} (3.110)

$$y \hat{X}_i - x \hat{X}_i = \xi_i.$$  \hspace{1cm} (3.111)

This assumption is confirmed through the construction of the gauge-invariant variables for the linear-order metric perturbation below. Inspecting gauge-transformation rules \( (3.3 \text{a}–3.5) \), we define the symmetric tensor field $\hat{H}_{ab}$ whose components are given by

$$\hat{H}_{tt} := h_{tt} + \frac{2}{\alpha} \left( \partial_t \alpha + \beta^i D_i \alpha - \beta^i \beta^j K_{ij} \right) \hat{X}_t$$
$$+ \frac{2}{\alpha} \left( \beta^i \beta^j \beta^k K_{kj} - \beta^i \partial_t \alpha + \alpha q^i j \partial_i \beta_j \right. $$
$$\left. + \alpha^2 D^i \alpha - \alpha \beta^k D^i \beta_k - \beta^i \beta^j D_j \alpha \right) \hat{X}_t,$$ \hspace{1cm} (3.112)

$$\hat{H}_{ti} := h_{ti} + \frac{2}{\alpha} \left( D_i \alpha - \beta^j K_{ij} \right) \hat{X}_t$$
$$+ \frac{2}{\alpha} \left( -\alpha^2 K^j_i + \beta^j \beta^k K_{ki} - \beta^j D_i \alpha + \alpha D_i \beta^j \right) \hat{X}_t,$$ \hspace{1cm} (3.113)

$$\hat{H}_{ij} := h_{ij} - \frac{2}{\alpha} K_{ij} \hat{X}_t + \frac{2}{\alpha} \beta^k K_{ij} \hat{X}_k.$$ \hspace{1cm} (3.114)

The gauge transformation rules \( (3.3 \text{a}–3.5) \) and our assumptions \( (3.110 \text{a} \text{ and b}) \) give the gauge-transformation rules of the components of $\hat{H}_{ab}$ as follows:

$$\chi \hat{H}_{tt} - x \hat{H}_{tt} = 2 \partial_t \xi_t,$$ \hspace{1cm} (3.115)

$$\chi \hat{H}_{ti} - x \hat{H}_{ti} = \partial_t \xi_i + D_i \xi_t,$$ \hspace{1cm} (3.116)

$$\chi \hat{H}_{ij} - x \hat{H}_{ij} = 2 D_i \xi_j.$$ \hspace{1cm} (3.117)

Since the components $\hat{H}_{tt}$ and $\hat{H}_{ij}$ are regarded as a vector and a symmetric tensor on $\Sigma(t)$, respectively, we may apply the decomposition reviewed in Appendix B to $\hat{H}_{tt}$ and $\hat{H}_{ij}$:

$$\hat{H}_{tt} = D_t H_{(V)L} + h_{(V)i}, \hspace{1cm} D^i h_{(V)i} = 0,$$ \hspace{1cm} (3.118)

$$\hat{H}_{ij} = \frac{1}{n} g_{ij} h_{(L)} + h_{(T)ij}, \hspace{1cm} \phi^j h_{(T)ij} = 0,$$ \hspace{1cm} (3.119)

$$h_{(T)ij} = (L h_{(TV)})_{ij} + h_{(TT)ij}, \hspace{1cm} D^i h_{(TT)ij} = 0,$$ \hspace{1cm} (3.120)

$$h_{(TV)i} = D_i h_{(TV)0} + h_{(TV)i}, \hspace{1cm} D^i h_{(TV)i} = 0.$$ \hspace{1cm} (3.121)

The gauge-transformation rules \( (3.110 \text{a} \text{ and b}) \) gives the gauge-transformation rules for the variables $h_{(V)L}$, $h_{(V)i}$, $h_{(L)}$, $h_{(T)ij}$, $h_{(TV)i}$ (or equivalently $h_{(TV)0}$ and $h_{(TV)i}$), and $h_{(TT)ij}$ as in the previous subsection.
First, we consider the gauge-transformation rule (3.116) in terms of the decomposition (3.118):

\[ \gamma \mathcal{H}_i - \lambda \mathcal{H}_i = D_i \left( \gamma h_{(VL)} - \lambda h_{(VL)} \right) + (\gamma h_{(V)i} - \lambda h_{(V)i}) = \partial_t \xi_i + D_i \xi_i. \quad (3.122) \]

Taking the divergence of this gauge-transformation rule and through the property

\[ D \partial_i = \partial_i D \]

as Eq. (3.72), i.e.,

\[ \partial_i q_{ij} = -2\alpha K_{ij} + 2D_i(\beta j) \quad (3.124) \]

from Eq. (A.37) in Appendix A. Keep this equation in our mind, we consider the derivative \( D_j \partial_i \xi_i \) as in the previous subsection. This is given by

\[ D_j \partial_i \xi_i = \partial_t D_j \xi_i - D_t \left( \alpha K_{ij} - D_{(k} \beta j) \right) \xi^k \\
- D_j \left( \alpha K_{ki} - D_{(k} \beta j) \right) \xi^k + D_k \left( \alpha K_{ij} - D_{(i} \beta j) \right) \xi^k. \quad (3.125) \]

From Eq. (3.125), we can easily derive \( D^i \partial_i \xi_i \) as

\[ D^i \partial_i \xi_i = \partial_t D^i \xi_i - 2 \left( \alpha K^{ij} - D^{(ij \beta j)} \right) D_j \xi_i \\
- 2D_t \left( \alpha K^{ij} - D^{(ij \beta j)} \right) \xi_i + D^i \left( \alpha K - D^j \beta j \right) \xi_i. \quad (3.126) \]

Further, we consider the decomposition of the component \( \xi_i \) as Eq. (3.72), i.e.,

\[ \xi_i = D_t \xi_{(L)} + \xi_{(V)i}, \quad D^i \xi_{(V)i} = 0. \quad (3.127) \]

Through the decomposition (3.127) and the formula (3.129), the gauge-transformation rule (3.123) is given by

\[ \Delta \left( \gamma h_{(VL)} - \lambda h_{(VL)} - \xi_t \right) = \Delta \partial_t \xi_{(L)} - 2 \left( \alpha K^{ij} - D^{(ij \beta j)} \right) D_j \xi_{(V)i} \\
- 2D_t \left( \alpha K^{ij} - D^{(ij \beta j)} \right) \xi_{(V)i} + D^i \left( \alpha K - D^j \beta j \right) \xi_{(V)i}, \quad (3.128) \]

where we have used Eq. (3.126) twice. Ignoring the mode which belongs to the kernel of the derivative operator \( \Delta \), we obtain

\[ \gamma h_{(VL)} - \lambda h_{(VL)} = \xi_t + \partial_t \xi_{(L)} \\
+ \Delta^{-1} \left[ -2 \left( \alpha K^{ij} - D^{(ij \beta j)} \right) D_j \xi_{(V)i} + D^i \left( \alpha K - D^j \beta j \right) \xi_{(V)i} - 2D_t \left( \alpha K^{ij} - D^{(ij \beta j)} \right) \xi_{(V)i} \right]. \quad (3.129) \]
Substituting Eq. (3.129) into Eq. (3.122) we obtain

\[ y h_{(V)i} - y h_{(V)i} = \partial_t \xi_{(V)i} - D_i \Delta^{-1} \left[ -2 \left( \alpha K^{ij} - D^{(i} \beta^{j)} \right) D_j \xi_{(V)i} \right. \]
\[ + \left. \left\{ D^l \left( \alpha K - D^j \beta_j \right) - 2D_i \left( \alpha K^{li} - D^{(l} \beta^{i)} \right) D_j \xi_{(V)i} \right\} \right]. \tag{3.130} \]

The gauge-transformation rules for \( h_L \) and \( h_{(T)ij} \) are given from Eq. (3.117). Since we consider the decomposition (3.119), the gauge-transformation rule (3.117) is given by

\[ y \hat{H}_{ij} - x \hat{H}_{ij} = \frac{1}{n} q_{ij} \left( y h_{(L)} - x h_{(L)} \right) + \left( y h_{(T)ij} - x h_{(T)ij} \right) = 2D_i \xi_{ij}. \tag{3.131} \]

Taking the trace of Eq. (3.131), we obtain

\[ y h_{(L)} - x h_{(L)} = 2D^j \xi_i. \tag{3.132} \]

The traceless part of Eq. (3.131) is given by

\[ y h_{(T)ij} - x h_{(T)ij} = (L \xi)_{ij}. \tag{3.133} \]

Note that the variable \( h_{(T)ij} \) is also decomposed as Eq. (3.120) and the gauge-transformation rules for the variable \( h_{(T)ij} \) is given by

\[ y h_{(T)ij} - x h_{(T)ij} = \left( L \left( y h_{TV} - x h_{TV} \right) \right)_{ij} + y h_{(TT)ij} - x h_{(TT)ij} = (L \xi)_{ij}. \tag{3.134} \]

Taking the divergence of Eq. (3.134), we obtain

\[ \mathcal{D}^l \left( y h_{TVl} - x h_{TVl} - \xi_l \right) = 0. \tag{3.135} \]

Since we ignore the modes which belong to the kernel of \( \mathcal{D}^l \), we obtain

\[ y h_{TVl} - x h_{TVl} = \xi_l. \tag{3.136} \]

Through the decomposition formula (3.121) and (3.127), we easily derive

\[ y h_{(TVL)} - x h_{(TVL)} = \xi_{(L)}, \tag{3.137} \]
\[ y h_{(TVV)} - x h_{(TVV)} = \xi_{(V)}, \tag{3.138} \]

where we ignore the mode which belong to the kernel of \( \Delta \). Substituting Eq. (3.136) into (3.131), we obtain

\[ y h_{(TT)ij} - x h_{(TT)ij} = 0. \tag{3.139} \]

In summary, we have obtained the gauge-transformation rules for the variables \( \hat{H}_t, h_{(V)L}, h_{(V)i}, h_{(L)}, h_{(T)ij}, h_{(TV)L}, h_{(TVV)i}, \) and \( h_{(TT)ij} \) as follows:

\[ y \hat{H}_t - x \hat{H}_t = 2\partial_t \xi_t, \tag{3.140} \]
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\[ Y_{h}(V_{L}) - X_{h}(V_{L}) = \xi_{t} + \partial_{t} \xi_{(L)} + \Delta^{-1} \left[ -2 \left( \alpha K^{ij} - D^{(i} \beta^{j)} \right) D_{j} \xi_{(V)_{i}} \right. \]

\[ \left. + \left\{ D^{l} \left( \alpha K - D^{l} \beta_{l} \right) - 2 D_{i} \left( \alpha K^{li} - D^{l} \beta_{i} \right) \right\} \xi_{(V)_{i}} \right] \]  \quad (3.141)

\[ Y_{h}(V_{i}) - X_{h}(V_{i}) = \partial_{t} \xi_{(V)_{i}} - D_{i} \Delta^{-1} \left[ -2 \left( \alpha K^{kj} - D^{(k} \beta^{j)} \right) D_{j} \xi_{(V)_{k}} \right. \]

\[ \left. + \left\{ D^{l} \left( \alpha K - D^{l} \beta_{k} \right) \right\} \xi_{(V)_{k}} \right] \]  \quad (3.142)

\[ Y_{h}(L) - X_{h}(L) = 2 D^{i} \xi_{i} \]  \quad (3.143)

\[ Y_{h}(T)_{ij} - X_{h}(T)_{ij} = (L \xi)_{ij} \]  \quad (3.144)

\[ Y_{h}(T_{V})_{i} - X_{h}(T_{V})_{i} = \xi_{t} \]  \quad (3.145)

\[ Y_{h}(T_{V})_{i} - X_{h}(T_{V})_{i} = \xi_{(V)_{i}} \]  \quad (3.146)

\[ Y_{h}(T_{V}V)_{i} - X_{h}(T_{V}V)_{i} = \xi_{(T_{V})_{i}} \]  \quad (3.147)

\[ Y_{h}(T T)_{ij} - X_{h}(T T)_{ij} = 0 \]  \quad (3.148)

Here, we note that the gauge transformation rule (3.144) coincides with the gauge transformation rule (3.111) for the variable $\hat{X}_{i}$. Then, we may identify the variable $\hat{X}_{i}$ with $h_{(T_{V})_{i}}$:

\[ \hat{X}_{i} := h_{(T_{V})_{i}} \]  \quad (3.149)

Thus, we have confirmed the existence of the variable $\hat{X}_{i}$. Next, we show the existence of the variable $\hat{X}_{t}$ whose gauge-transformation rule is given by Eq. (3.110). To do this, we consider the gauge transformation rules (3.141), (3.146), and (3.147). Inspecting these gauge transformation rules, we find the definition of $\hat{X}_{t}$ as

\[ \hat{X}_{t} := h_{(V_{L})} - \partial_{t} h_{(T_{V}L)} - \Delta^{-1} \left[ -2 \left( \alpha K^{ij} - D^{(i} \beta^{j)} \right) D_{j} h_{(T_{V}V)_{i}} \right. \]

\[ \left. + \left\{ D^{l} \left( \alpha K - D^{l} \beta_{i} \right) - 2 D_{i} \left( \alpha K^{li} - D^{l} \beta_{i} \right) \right\} h_{(T_{V}V)_{i}} \right] \]  \quad (3.150)

Actually, the gauge transformation rule for $\hat{X}_{t}$ defined by Eq. (3.150) is given by Eq. (3.110). This is desired property for the variable $\hat{X}_{t}$. Thus, we have confirmed the existence of the variables $\hat{X}_{t}$ and $\hat{X}_{i}$ which was assumed in the definitions (3.112)–(3.114) of the components of the tensor field $\hat{H}_{ab}$.

Now, we construct gauge invariant variables for the linear-order metric perturbation. First, the gauge transformation rule (3.148) shows that $h_{(T T)_{ij}}$ is gauge invariant by itself and we define the gauge-invariant transverse-traceless tensor by

\[ \chi_{ij} := h_{(T T)_{ij}} \]  \quad (3.151)
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Inspecting the gauge-transformation rules \( (3.142) \) and \( (3.147) \), we define the vector mode \( \nu_i \) by

\[
\nu_i := h(V)_i - \partial_t h(TVV)_i + D_i \Delta^{-1} \left[ -2 \left( \alpha K^{kj} - D^{(k} \beta^{j)} \right) D_j h(TVV)_k 
+ \left\{ \alpha K - D^k \beta_k \right\} 
- 2D_k \left( \alpha K^{ik} - D^{(i} \beta^{k)} \right) \right] h(TVV)_i .
\]

(3.152)

Actually, we can easily confirm that the variable \( \nu_i \) is gauge invariant, i.e., \( \gamma \nu_i - \chi \nu_i = 0 \). Through the divergenceless property \( D_i h(V)_i = 0 \) for the variable \( h(V)_i \), we easily derive

\[
D^i \nu_i = -D^i \partial_t h(TVV)_i + \left( -2 \alpha K^{kj} + 2 D^{(k} \beta^{j)} \right) D_j h(TVV)_k 
+ \left\{ -2D_k \left( \alpha K^{ik} - D^{(i} \beta^{k)} \right) + D^i \left( \alpha K - D^k \beta_k \right) \right\} h(TVV)_i .
\]

(3.153)

Further, through the formula \( (3.126) \) for the variable \( h(TVV)_i \) and the divergenceless property of the variable \( h(TVV)_i \), we easily see the divergenceless property \( D^i \nu_i = 0 \).

Next, we consider the scalar modes. First, inspecting gauge-transformation rules \( (3.110) \) and \( (3.115) \), we define the scalar variable \( \Phi \) by

\[
-2 \Phi := \hat{H}_{tt} - 2 \partial_t \hat{X}_i .
\]

(3.154)

Actually, we can easily confirm that this variable \( \Phi \) is gauge invariant. Inspecting the gauge-transformation rules \( (3.111) \) and \( (3.132) \), we define another gauge-invariant variable \( \Psi \) by

\[
-2 \Psi := h(L)_i - 2D^i \hat{X}_i .
\]

(3.155)

We can easily confirm the gauge invariance of the variable \( \Psi \) through the gauge-transformation rules \( (3.111) \) and \( (3.132) \).

In summary, we have defined gauge invariant variables as follows:

\[
-2 \Phi := \hat{H}_{tt} - 2 \partial_t \hat{X}_i ,
-2 \Psi := h(L)_i - 2D^i \hat{X}_i ,
\nu_i := h(V)_i - \partial_t h(TVV)_i 
+ D_i \Delta^{-1} \left[ -2 \left( \alpha K^{kj} - D^{(k} \beta^{j)} \right) D_j h(TVV)_k 
+ \left\{ D^i \left( \alpha K - D^k \beta_k \right) 
- 2D_k \left( \alpha K^{ik} - D^{(i} \beta^{k)} \right) \right\} h(TVV)_i \right] ,
\chi_{ij} := h(TT)_{ij} .
\]

(3.156)

(3.157)

(3.158)

(3.159)

In terms of these gauge-invariant variables and the variables \( \hat{X}_i \) and \( \hat{X}_i \), which are defined by Eqs. \( (3.150) \) and \( (3.149) \), respectively, the original components \( \{ h_{tt}, h_{ti}, \)
Decomposition of linear metric perturbations on generic background spacetime

\( h_{ij} \) of the metric perturbation \( h_{ab} \) is given by

\[
h_{tt} = -2\Phi + 2\partial_t \hat{X}_t - \frac{2}{\alpha} \left( \partial_t \alpha + \beta^i D_i \alpha - \beta^j \partial_j \beta^i \right) \hat{X}_t \\
- \frac{2}{\alpha} \left( \beta^i \beta^k \beta^j K_{kj} - \beta^i \partial_t \alpha + \alpha q^j \partial_t \beta_j \\
+ \alpha^2 D^j \alpha - \alpha \beta^k D^j \beta_k - \beta^j \beta^i D^j \alpha \right) \hat{X}_t,
\]

(3.160)

\[
h_{ti} = \nu_i + D_i \hat{X}_t + \partial_t \hat{X}_i - \frac{2}{\alpha} \left( D_i \alpha - \beta^j K_{ij} \right) \hat{X}_t \\
- \frac{2}{\alpha} \left( -\alpha^2 K^j_i + \beta^j \beta^k K_{ki} - \beta^j D_i \alpha + \alpha D_i \beta^j \right) \hat{X}_j,
\]

(3.161)

\[
h_{ij} = -2\Psi q_{ij} + \chi_{ij} + D_i \hat{X}_j + D_j \hat{X}_i + \frac{2}{\alpha} K_{ij} \hat{X}_t - \frac{2}{\alpha} \beta^k K_{ij} \hat{X}_k.
\]

(3.162)

On the other hand, the component representations of the decomposition formula (2.18) with

\[
X_a =: X_t(dt)_a + X_i(dx^i)_a
\]

are given by

\[
h_{tt} = \mathcal{H}_{tt} + 2\partial_t X_t - \frac{2}{\alpha} \left( \partial_t \alpha + \beta^i D_i \alpha - \beta^k \partial_k \beta^i \right) X_t \\
- \frac{2}{\alpha} \left( \beta^i \beta^k \beta^j K_{kj} - \beta^i \partial_t \alpha + \alpha q^j \partial_t \beta_j \\
+ \alpha^2 D^j \alpha - \alpha \beta^k D^j \beta_k - \beta^j \beta^i D^j \alpha \right) X_i,
\]

(3.164)

\[
h_{ti} = \mathcal{H}_{ti} + \partial_t X_i + D_i X_t - \frac{2}{\alpha} \left( D_i \alpha - \beta^j K_{ij} \right) X_t \\
- \frac{2}{\alpha} \left( -\alpha^2 K^j_i + \beta^j \beta^k K_{ki} - \beta^j D_i \alpha + \alpha D_i \beta^j \right) X_j,
\]

(3.165)

\[
h_{ij} = \mathcal{H}_{ij} + D_i X_j + D_j X_i + \frac{2}{\alpha} K_{ij} X_t - \frac{2}{\alpha} \beta^k K_{ij} X_k.
\]

(3.166)

Comparing Eqs. (3.164)–(3.166) with Eqs. (3.160)–(3.162), we may identify the components of the gauge-invariant variables \( \mathcal{H}_{ab} \) so that

\[
\mathcal{H}_{tt} := -2\Phi, \quad \mathcal{H}_{ti} := \nu_t, \quad \mathcal{H}_{ij} := -2\Psi q_{ij} + \chi_{ij}
\]

(3.167)

and the components of the gauge-variant variables \( X_a \) so that

\[
X_t := \hat{X}_t, \quad X_i := \hat{X}_i.
\]

(3.168)

Thus, the decomposition formula (2.18) is correct for the linear-order perturbation on a generic background spacetime, if we assume the existence of two Green function of the derivative operators \( \Delta := D^i D_i \) and \( \mathcal{D}^j \) which is defined by Eq. (3.26). In other words, in the above proof, we ignore the modes which belong to the kernel of these derivative operators \( \Delta \) and \( \mathcal{D}^j \). To take these modes into account, the different treatments are necessary.
§4. Comparison with the FRW background case

In this section, we consider the comparison with the case where the background spacetime $\mathcal{M}_0$ is a homogeneous and isotropic universe which is discussed in KN2007\cite{KN2007}. This case corresponds to the case $\alpha = 1$, $\beta^i = 0$ and $K_{ij} = -Hq_{ij}$, where $H = \partial_\tau a/a$ and $a$ is the scale factor of the universe.

In the paper KN2007\cite{KN2007} we consider the decomposition of the components $h_{ti}$ and $h_{ij}$ of the metric perturbation $h_{ab}$ as

\begin{align}
  h_{ti} &= \tilde{D}_i\tilde{h}_{(V)L} + \tilde{h}_{(V)i}, \quad \tilde{D}^j\tilde{h}_{(V)j} = 0, \\
  h_{ij} &= a^2\tilde{h}_{(L)\gamma ij} + a^2\tilde{h}_{(T)ij}, \quad \gamma^{ij}\tilde{h}_{(T)ij} = 0, \\
  \tilde{h}_{(T)ij} &= \left(\tilde{D}_i\tilde{D}_j - \frac{1}{n}\gamma_{ij}\tilde{\Delta}\right)\tilde{h}_{(TL)} + 2\tilde{D}_i\tilde{h}_{(TV)j} + \tilde{h}_{(TT)ij}, \\
  \tilde{D}^j\tilde{h}_{(TV)j} &= 0, \quad \tilde{D}^j\tilde{h}_{(TT)ij} = 0,
\end{align}

where $q_{ij} = a^2\gamma_{ij}$, $\gamma_{ij}$ is the metric on a maximally symmetric space, $\tilde{D}_i$ is the covariant derivative associated with the metric $\gamma_{ij}$, and $\tilde{\Delta} := \tilde{D}^i\tilde{D}_i$. This decomposition is slightly different from the decomposition (3.62)–(3.65) with the definition (3.60) of the variable $\tilde{H}_{ab}$. Furthermore, as noted in KN2007\cite{KN2007} there should exist Green functions of the derivative operators $\tilde{\Delta}$, $\tilde{\Delta} + 2K$, and $\tilde{\Delta} + 3K$ to guarantee the one to one correspondence of the set $\{h_{ti}, h_{ij}\}$ and $\{\tilde{h}_{ti}, \tilde{h}_{ij}\}$. In this section, we briefly discuss these correspondence.

First, we note that the decomposition (4.1) of the component $h_{ti}$ of the metric perturbation $h_{ab}$ is equivalent to (3.62). Although the tiny difference between Eq. (4.1) and (3.62) is in the definition of the covariant derivatives $D_i$ (associated with the metric $q_{ij}$) and $\tilde{D}_i$ (associated with the metric $\gamma_{ij} := (1/a^2)q_{ij}$), we may say that $h_{(V)L}$ and $\tilde{h}_{(V)L}$ in Eq. (4.1) are identical with $h_{(V)L}$ and $\tilde{h}_{(V)L}$ in Eq. (3.62), respectively.

We also note that the trace parts of these two decompositions are almost equivalent. Actually, since the extrinsic curvature $K_{ij}$ on the background $\Sigma$ is proportional to the intrinsic metric $q_{ij}$ in this case, we easily see that

\begin{align}
  q^{ij}h_{ij} &= q^{ij}\tilde{H}_{ij} + 2q^{ij}K_{ij}\tilde{X}_i \\
  &= h_{(L)} - 2nH\left(h_{(TV)L} - \partial_\tau h_{(TV)L}\right) + 2Hh_{(TV)L},
\end{align}

where we used Eq. (3.66). On the other hand, the trace part of $h_{ij}$ given by Eq. (4.2) is $\tilde{n}\tilde{h}_{(L)}$. Thus, the variable $\tilde{h}_{(L)}$ in Eq. (4.2) corresponds to the variables in this paper as

\begin{align}
  \tilde{h}_{(L)} &= \frac{1}{n}h_{(L)} - 2H\left(h_{(TV)L} - \partial_\tau h_{(TV)L}\right) + 2Hh_{(TV)L}.
\end{align}
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Since the extrinsic curvature $K_{ij}$ in Eq. (3.60) is proportional to the intrinsic metric $q_{ij}$ in this case, the main difference between decompositions (3.63)–(3.65) and (4.2)–(4.4) are in the traceless part. The traceless part $\tilde{h}_{(T)ij}$ in Eq. (4.3) is also given by

$$\tilde{h}_{(T)ij} = \left(\tilde{D}_i \tilde{D}_j - \frac{1}{n} \gamma_{ij} \tilde{\Delta}\right) \tilde{h}_{(TL)} + 2 \tilde{D}_i (\tilde{D}_j \tilde{h}_{(TV)ij}) + \tilde{h}_{(TT)ij}$$

$$= a^2 D_i \left(\frac{1}{2} D_j \tilde{h}_{(TL)} + \tilde{h}_{(TV)ij}\right) + a^2 D_j \left(\frac{1}{2} D_i \tilde{h}_{(TL)} + \tilde{h}_{(TV)ij}\right) - \frac{2}{n} q_{ij} a^2 D^k \left(\frac{1}{2} D_k \tilde{h}_{(TL)} + \tilde{h}_{(TV)ij}\right) + a^2 \tilde{h}_{(TT)ij},$$

(4.7)

where we used

$$0 = \tilde{D}^k \tilde{h}_{(TV)ik} = a^2 D^k \tilde{h}_{(TV)ik}.\quad (4.8)$$

Comparing (3.64), we obtain the correspondence of the variables

$$h_{(TV)i} = \frac{1}{2} a^2 D_j \tilde{h}_{(TL)} + a^2 \tilde{h}_{(TV)ij}, \quad h_{(TT)ij} = a^2 \tilde{h}_{(TT)ij}.$$

(4.9)

$$h_{(TV)L} = \frac{1}{2} a^2 \tilde{h}_{(TL)}, \quad h_{(TVV)i} = a^2 \tilde{h}_{(TV)i}.$$

(4.10)

Therefore, in the case of the homogeneous isotropic universe, the decomposition (3.62)–(3.65) is equivalent to the decomposition (4.1)–(4.4).

However, in the case of the generic background spacetime, the decomposition (4.1)–(4.4) is ill-defined. Actually, if we regard that the decomposition (4.1)–(4.4) is that for the generic background spacetime, we cannot separate $\tilde{h}_{(TL)}$ and $\tilde{h}_{(TV)ij}$ due to the non-trivial curvature terms of the background $\mathcal{M}_0$ as pointed out by Deser (24). These curvature terms come from the commutation relation between the covariant derivative $D_i$ and the derivative operator $\tilde{D}^i$. This is why we apply the decomposition (3.62)–(3.65) instead of (4.1)–(4.4).

Finally, we consider the correspondence of the special modes which we ignore in this paper and KN2007 (7) Trivially, the above operator $\tilde{\Delta} := \tilde{D}^i \tilde{D}_i$ corresponds to the Laplacian $\Delta$ in this paper. The above derivative operator $\tilde{\Delta} + (n - 1)K$ corresponds to the derivative operator $\tilde{D}^{ij}$. In the case of the maximally symmetric $n$-space, the Riemann curvature and Ricci curvature are given by

$$(n) R_{ijkl} = 2K q_{[k[l} q_{j]l]} = 2K q_{[k[l} q_{j]l]}, \quad R_{ik} = q^{jl(n)} R_{ijkl} = (n - 1)K q_{ik}.$$  (4.11)

In this case, the derivative operator $\tilde{D}^{ij}$ defined by Eq. (3.26) is given by

$$\tilde{D}^{ij} = q^{ij} (\Delta + (n - 1)K) + \left(1 - \frac{2}{n}\right) D^i D^j.$$  (4.12)

When the operator $\tilde{D}^{ij}$ acts on an arbitrary transverse vector field $v_i (D^i v_i = 0)$, we easily see that

$$\tilde{D}^{ij} v_j = (\Delta + (n - 1)K) v^i.$$  (4.13)
Finally, we point out that the above derivative operator \( \Delta + nK \) appears in the case where the derivative operator \( D^j \) acts on the gradient \( D_l f \) of an arbitrary scalar function \( f \). Actually, we easily see that

\[
D^{jl} D_l f = 2 \frac{n-1}{n} D^j (\Delta + nK) f. \tag{4.14}
\]

In the case of maximally symmetric \( n \)-space, curvature tensors are given by Eqs. (4.11) and the derivative operator \( D^{jl} D_l \) is given by

\[
D^{jl} D_l f = 2 \frac{n-1}{n} D^j (\Delta + nK) f. \tag{4.15}
\]

When we solve the equation

\[
D^{jl} D_l f = g^j, \tag{4.16}
\]

we have to use the Green function \( \Delta \) and \( \Delta + nK \). These are the reason for the fact that the Green functions \( \Delta^{-1} \), \( (\Delta + (n-1)K)^{-1} \), and \( (\Delta + nK)^{-1} \) were necessary to guarantee the one-to-one correspondence between the components \( \{h_{ti}, h_{ij}\} \) and \( \{h_{(VL)i}, h_{(V)i}, h_{(T)L)}, h_{(T)V)i}, h_{(TT)i}\} \) in Eqs. (4.11–4.14). In other words, we may say that the special modes belong to the kernel of the derivative operators \( \Delta \) and \( D^{ij} \) which are ignored in this paper are equivalent to the special modes which belong to the kernel of the derivative operators \( \Delta, \Delta + (n-1)K, \) and \( \Delta + nK \) which are ignored in the paper KN2007.

§5. Summary and discussions

In summary, after reviewing the general framework of the higher-order gauge-invariant perturbation theory in general relativity, we prove Conjecture 2.1 for generic background spacetime which admits ADM decomposition. In this proof, we assumed the existence of Green functions of the elliptic derivative operators \( \Delta \) and \( D^{ij} \). Roughly speaking, Conjecture 2.1 states that we know the procedure to decompose the linear-order metric perturbation \( h_{ab} \) into its gauge-invariant part \( H_{ab} \) and gauge-variant part \( X_a \). In the cosmological perturbation case, this conjecture is confirmed and the second-order cosmological perturbation theory was developed in our series of papers. However, as reviewed in §2, Conjecture 2.1 was the only non-trivial part when we consider the general framework of gauge-invariant perturbation theory on generic background spacetimes. Although there may exist many approaches to prove Conjecture 2.1 in this paper, we just proposed a proof for generic background spacetimes.

As noted above, in our proof, we assume the existence of the Green functions for the elliptic derivative operators \( \Delta \) and \( D^{ij} \). This assumption implies that we have ignored the modes which belong to the kernel of these derivative operators. Within the arguments in this paper, there is no information for the treatment of these mode. To discuss these modes, different treatments of perturbations are necessary. We call this problem as zero-mode problem. The situation is similar to the cosmological
perturbation case as noted in §4 and zero-mode problem exists even in the cosmological perturbation case. In the cosmological perturbation case, zero-mode means the modes which belong to the kernel of the derivative operator $\Delta$, $\Delta + (n - 1)K$, and $\Delta + nK$, where $K$ is the curvature constant of the maximally symmetric space in cosmology and $n$ is the dimension of this maximally symmetric space.

This zero-mode problem in cosmological perturbations also corresponds to the $l = 0$ and $l = 1$ mode problem in perturbation theory on spherically symmetric background spacetimes. In the perturbation theory on spherically symmetric background spacetimes, we consider the similar decomposition to Eqs. (4.2)–(4.4) and the indices $i, j, ...$ in these equations correspond to the indices of the components of a tensor field on $S^2$. Since $S^2$ is a 2-dimensional maximally symmetric space with the positive curvature, we may regard $n = 2$ and $K = 1$. Then, the above three derivative operators are given by $\Delta$, $\Delta + 1$, and $\Delta + 2$. Since the eigenvalue of the Laplacian $\Delta$ on $S^2$ is given by $\Delta = -(l + 1)$, we may say that the modes with $l = 0$ and $l = 1$ belong to the kernel of the derivative operator $\Delta$, $\Delta + (n - 1)K$, and $\Delta + nK$. Therefore, we may say that the problem concerning about the modes with $l = 0$ and $l = 1$ in the perturbations on spherically symmetric background spacetime is the same problem as the zero-mode problem mentioned above.

Thus, the arguments in this paper shows that zero-mode problem generally appears in many perturbation theories in general relativity we have seen that the appearance of this zero-mode problem from general point of view. To resolve this zero-mode problem, carefully discussions on domains of functions for perturbations will be necessary. We leave this zero-mode problem as a future work.

Although we should take care of the zero-mode problem, we have almost completed the general framework of the higher-order gauge-invariant perturbation theory in general relativity. The proof of Conjecture 2.1 shown in this paper gives rise to the possibility of the application of our general framework for the higher-order gauge-invariant perturbation theory not only to cosmological perturbations but also to perturbations of black hole spacetimes or perturbations of general relativistic stars. Therefore, we may say that the wide applications of our gauge-invariant perturbation theory are opened due to the discussions in this paper. We also leave these development of gauge-invariant perturbation theories for these background spacetimes as future works.

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Appendix A

--- ADM decomposition ---

Here, we briefly review the ADM decomposition.
We consider the $n + 1$-dimensional spacetime $(M, g_{ab})$. The topology of $M$ is given by $M = \mathbb{R}^1 \times \Sigma$, where $\Sigma$ is the $n$-dimensional manifold. This means that the entire $M$ is foliated by the one-parameter family of the manifolds $\Sigma_t$ where $t$ is the parameter along $\mathbb{R}^1$ in $\mathbb{R}^1 \times \Sigma$. Here, we note that it is not necessary to impose that the entire spacetime $M$ is decomposed into $\mathbb{R}^1 \times \Sigma$ in the global sense. However, in this section, we impose that there exists a one-parameter family of $n$-dimensional submanifolds $\Sigma_t$ in a $n + 1$-dimensional manifold $M$, for simplicity.

In general relativity, $t$ is regarded as the time function on $M$ and the decomposition of $M$ into $\mathbb{R}^1 \times \Sigma$ is regarded as the $n + 1$-decomposition of the spacetime $M$ into the space $\Sigma$ and time $\mathbb{R}^1$. From the viewpoint of this decomposition, we can describe the $M$ by the time-evolution of $\Sigma$, i.e., the geometry of $M$ is described in terms of the geometry of $\Sigma$ which is embedded in $M$. Any geometrical quantities on $M$ is given in terms of the geometry of $\Sigma$ and its “time-evolution”.

First, we consider the $n + 1$-decomposition of the metric $g_{ab}$ on $M$. Let $t^a$ be a vector field on $M$ satisfying

$$
t^a \nabla_a t = 1,
$$

i.e.,

$$
t^a = \left( \frac{\partial}{\partial t} \right)^a.
$$

Let us call the direction along which the function $t$ increases as the future direction and $t^a$ defined by (A.1) is called future-directed, the direction along which the function $t$ decreases as the past direction.

On the other hand, let us denote the unit normal to $\Sigma_t$ by $n^a$, which is hypersurface orthogonal. The normalization condition for $n^a$ is given by

$$
n^a n_a = -1.
$$

The metric $g_{ab}$ on $M$ induces a metric $q_{ab}$ on $\Sigma_t$. On each $\Sigma_t$, $q_{ab}$ is given by

$$
q_{ab} := g_{ab} + n_a n_b.
$$

The overall signature of $n^a$ is chosen so that $n^a$ is future-directed. Then there is a positive function $\alpha$ so that

$$
n^a \nabla_a t = \frac{1}{\alpha}.
$$

This positive function $\alpha$ is called the lapse function with respect to $t^a$. Further, we impose the $n_a$ is the hypersurface orthogonal to the hypersurfaces $\Sigma_t$, which implies that $n_a \propto \nabla_a t$. Due to the normalization condition (A.3), we easily see that

$$
n_a = -\alpha \nabla_a t = -\alpha (dt)_a.
$$

From Eq. (A.6), we decompose the vector field $t^a$ into its normal and tangential parts to $\Sigma_t$

$$
\alpha = -t^a n_a, \quad \beta_a := q_{ab} t^b.
$$
where $\beta_a$ is called the shift vector with respect to $t^a$. Equivalently, the vector field $t^a$ is decomposed as

$$t^a = t^b \delta^a_b = \beta^a + \alpha n^a.$$  \hfill (A.8)

From Eq. (A.1) and the definition (A.5) of the lapse function $\alpha$, we easily see that

$$1 = t^a \nabla_a t = \alpha n^a \nabla_a t + \beta^a \nabla_a t = 1 + \beta^a \nabla_a t,$$  \hfill (A.9)

which yield

$$\beta^a \nabla_a t = 0,$$  \hfill (A.10)

i.e., $\beta^a$ has no component along $(\partial / \partial t)^a$. Further, due to the normalization (A.3) of the vector $n_a$, the lapse function $\alpha$ is also given by

$$\alpha^2 = -\frac{1}{\bar{g}^{ab} (\nabla_a t) (\nabla_b t)}.$$  \hfill (A.11)

The decomposition (A.8) of the vector field $t^a = (\partial / \partial t)^a$ is also yields

$$n^a = \frac{1}{\alpha} [t^a - \beta^a].$$  \hfill (A.12)

Introducing the spatial coordinate so that $(t, x^i)$ is the spacetime coordinate, i.e., the coordinate basis of the tangent space on the spacetime is the set

$$\{(dt)_a, (dx^i)_a\}, \quad \left\{ \left( \frac{\partial}{\partial t} \right)^a, \left( \frac{\partial}{\partial x^i} \right)^a \right\},$$  \hfill (A.13)

$\beta^a$ is given by

$$\beta^a = \beta^i \left( \frac{\partial}{\partial x^i} \right)^a,$$  \hfill (A.14)

because of Eq. (A.10). Then, (A.12) is given in terms of the coordinate system

$$n^a = \frac{1}{\alpha} \left[ (\partial / \partial t)^a - \beta^i \left( \frac{\partial}{\partial x^i} \right)^a \right].$$  \hfill (A.15)

From

$$\nabla_a t = \delta_a^b \nabla_b t = (q_a^c - n_a n^c) \nabla_c t q_a^b \nabla_b t - n_a \frac{1}{\alpha},$$  \hfill (A.16)

and (A.6), we can also see that

$$q_a^b \nabla_b t = 0.$$  \hfill (A.17)

This implies

$$q^{ab} \nabla_b t = q^{ba} \nabla_b t = 0,$$  \hfill (A.18)

which yields $q^{ab}$ as no component along $(\partial / \partial t)^a$. This means that the induced inverse metric $q^{ab}$ has the following component representation

$$q^{ab} = q^{ij} \left( \frac{\partial}{\partial x^i} \right)^a \left( \frac{\partial}{\partial x^j} \right)^b.$$  \hfill (A.19)
Together with the component representation (A.12) and (A.19), we obtain the spacetime inverse metric \( g^{ab} \) in terms of the coordinate system \((t, x^i)\):

\[
g^{ab} = -n_a n_b + q^{ab}, \quad (A.20)
\]

\[
e = \frac{\epsilon}{\alpha} \left\{ \left( \frac{\partial}{\partial t} \right)^a - \beta^i \left( \frac{\partial}{\partial x^i} \right)^a \right\} \left\{ \left( \frac{\partial}{\partial t} \right)^b - \beta^j \left( \frac{\partial}{\partial x^j} \right)^b \right\} + q^{ij} \left( \frac{\partial}{\partial x^i} \right)^a \left( \frac{\partial}{\partial x^j} \right)^b. \quad (A.21)
\]

Through the relation

\[
g_{ab} g^{bc} = \delta^c_a, \quad (A.22)
\]

the straightforward calculation leads the coordinate representation of the metric \( g_{ab} \), which is given by

\[
g_{ab} = -\alpha^2 (dt)_a (dt)_b + q_{ij} (dx^i + \beta^i dt)_a (dx^j + \beta^j dt)_b, \quad (A.23)
\]

where \((dt, dx^i)\) is the coordinate basis on \(\mathcal{M}\) (more precisely, on an open set \(\mathcal{U} \subset \mathcal{M}\)), and \(q_{ij}\) is the inverse matrix of \(q^{ij}\) in Eq. (A.21), i.e.,

\[
q_{ij} q^{jk} =: q_i^k = \delta_i^k \quad (A.24)
\]

and \(\delta_i^k\) is the \(n\)-dimensional Kronecker's delta. In terms of the coordinate basis (A.23), the unit normal vector \(n^a\) is given by

\[
n_a = -\alpha (dt)_a, \quad (A.25)
\]

\[
n^a = g^{ab} n_b = \frac{1}{\alpha} \left( \frac{\partial}{\partial t} - \beta^i \frac{\partial}{\partial x^i} \right)^a. \quad (A.26)
\]

Here, we summarize the components of the spacetime metric \(g_{ab}\) and inverse metric \(g^{ab}\) on \(\mathcal{M}\) as follows:

\[
g_{tt} = -\alpha^2 + q_{ij} \beta^i \beta^j, \quad g_{ti} = g_{it} = q_{ij} \beta^j = \beta_i, \quad g_{ij} = q_{ij}, \quad (A.27)
\]

\[
g^{tt} = -\frac{1}{\alpha^2}, \quad g^{ti} = g^{it} = \frac{1}{\alpha} \beta^i, \quad g^{ij} = q^{ij} - \frac{1}{\alpha^2} \beta^i \beta^j. \quad (A.28)
\]

Next, we consider the connection between the covariant derivative \(\nabla_a\) on \((\mathcal{M}, g_{ab})\) and the covariant derivative \(D_a\) on \((\Sigma, q_{ab})\). This correspondence is given from the Christoffel symbol \(\Gamma^i_{jk}\) associated with the metric \(g_{ab}\) in the coordinate system (A.23):

\[
\Gamma^t_{tt} = \frac{1}{\alpha} \partial_t \alpha + \frac{1}{\alpha} \beta^i D_t \alpha + \frac{1}{2\alpha^2} \beta^k \beta^j \left( \partial_t q_{ij} - 2D_{(i}\beta_{j)} \right), \quad (A.29)
\]

\[
\Gamma^t_{ti} = \frac{1}{\alpha} D_t \alpha + \frac{1}{2\alpha^2} \beta^j \left( \partial_t q_{ij} - 2D_{(i}\beta_{j)} \right), \quad (A.30)
\]

\[
\Gamma^i_{ij} = \frac{1}{\alpha} \beta^j \left( \partial_t q_{ij} - 2D_{(i}\beta_{j)} \right), \quad (A.31)
\]

\[
\Gamma^i_{tt} = \frac{1}{2\alpha^2} \beta^j \beta^k \beta^l \left( \partial_t q_{kjl} - 2D_{(k} \beta_{l)} \right) - \frac{1}{\alpha} \beta^i \partial_t \alpha + q^{ij} \partial_t \beta_j.
\]
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\[ + \alpha D^\alpha \alpha - \beta^k D^i \beta_k - \frac{1}{\alpha} \beta^i \beta^j D_j \alpha, \]  
(A.32)

\[ \Gamma^i_{jt} = \frac{1}{2} q^{jk} \left\{ \partial_t q_{kj} - 2D_{(k} \beta_{j)} \right\} - \frac{1}{2\alpha^2} \beta^j \beta^k \left\{ \partial_t q_{kj} - 2D_{(k} \beta_{j)} \right\} \]
\[ - \frac{1}{\alpha} \beta^i \partial_t \alpha + D_j \beta^t, \]  
(A.33)

\[ \Gamma^i_{jk} = -\frac{1}{2\alpha^2} \beta^i \left\{ \partial_t q_{kj} - 2D_{(k} \beta_{j)} \right\} + (n)\Gamma^i_{jk}, \]  
(A.34)

where \((n)\Gamma^i_{jk}\) is the Christoffel symbol associated with the metric \(q_{ij}\):

\[ (n)\Gamma^i_{jk} = \frac{1}{2} q^{ij} \left( \partial_j q_{ik} + \partial_k q_{ij} - \partial_i q_{jk} \right). \]  
(A.35)

It is also convenient to introduce the extrinsic curvature

\[ K_{ab} = -q_a^c q_b^d \nabla_c q_{nd}. \]  
(A.36)

In the coordinate system, on which the metric is given by (A.23), this extrinsic curvature is given by

\[ K_{ij} = -\frac{1}{2\alpha} \left[ \frac{\partial}{\partial t} q_{ij} - D_i \beta_j - D_j \beta_i \right]. \]  
(A.37)

Through this component of the extrinsic curvature \(K_{ij}\), the above components (A.29)–(A.34) of the Christoffel symbols \(\Gamma^i_{bc}\) are given by

\[ \Gamma^t_{it} = \frac{1}{\alpha} \partial_t \alpha + \frac{1}{\alpha} \beta^i D_i \alpha - \frac{1}{\alpha} \beta^k \beta^i K_{ij}, \]  
(A.38)

\[ \Gamma^t_{it} = \frac{1}{\alpha} D_t \alpha - \frac{1}{\alpha} \beta^j K_{ij}, \]  
(A.39)

\[ \Gamma^i_{ij} = -\frac{1}{\alpha} K_{ij}, \]  
(A.40)

\[ \Gamma^t_{it} = \frac{1}{\alpha} \beta^j \beta^k K_{kj} - \frac{1}{\alpha} \beta^i \partial_t \alpha + q^{ij} \partial_t \beta_j \]
\[ + \alpha D^i \alpha - \beta^k D^i \beta_k - \frac{1}{\alpha} \beta^i \beta^j D_j \alpha, \]  
(A.41)

\[ \Gamma^i_{jt} = -\alpha K_{ij} + \frac{1}{\alpha} \beta^i \beta^k K_{kj} - \frac{1}{\alpha} \beta^j D_j \alpha + D_j \beta^i, \]  
(A.42)

\[ \Gamma^i_{jk} = \frac{1}{\alpha} \beta^i K_{kj} + (n)\Gamma^i_{jk}. \]  
(A.43)

Appendix B

Covariant orthogonal decomposition of symmetric tensors

Since the each order metric perturbation is regarded as a symmetric tensor on the background spacetime \((\mathcal{M}_0, g_{ab})\) through an appropriate gauge choice, the covariant decomposition of symmetric tensors is useful and actually used in the main text. Here, we review of the covariant decomposition of symmetric tensors of the second rank on an curved Riemannian manifold based on the work by York.\[23\]
In the generic curved Riemannian space \((\Sigma, q_{ab})\) (dim \(\Sigma = n\)), one can decompose an arbitrary vector or one-form into its transverse and longitudinal parts as

\[
A_a = A_i(dx^i)_a = (D_iA_{(L)} + A_{(V)i})(dx^i)_a, \quad D^iA_{(V)i} = 0, \tag{B.1}
\]

where \(D_i\) is the covariant derivative associated with the metric \(q_{ab} = q_{ij}(dx^i)_a(dx^j)_b\), \(A_{(L)}\) is called the longitudinal part or the scalar part and \(A_{(V)i}\) is called the transverse part or vector part of the vector field \(A_a\) on \((\Sigma, q_{ab})\), respectively.

Moreover, this decomposition is not only covariant with respect to arbitrary coordinate transformations, it is also orthogonal in the natural global scalar product. To clarify this orthogonality, York\(^{23}\) introduced the inner product for the vector fields on \(\Sigma\). This is, for any two vectors \(V^a\) and \(W^a\), we have

\[
\int_\Sigma \epsilon_q V^a W^b q_{ab}. \tag{B.2}
\]

where \(\epsilon_q\) denotes the volume element which makes the integral invariant and the integration extends over the entire manifold \((\Sigma, q_{ab})\). In terms of this inner product, the orthogonality of the vector fields \(V^a = D^aV_{(L)} := q^{ab}D_bV_{(L)}\) and \(W^a = q^{ab}V_{(V)b}\) with \(D^aV_{(V)a} = 0\) is given by

\[
\int_\Sigma \epsilon_q D_aV_{(L)}V_{(V)b}q^{ab} = \int_{\partial\Sigma} s_aV_{(L)}V_{(V)b}q^{ab} - \int_\Sigma \epsilon_q V_{(L)}D_aV_{(V)b}q^{ab}, \tag{B.3}
\]

where \(s_a\) is the volume element of the \((n - 1)\)-dimensional boundary \(\partial\Sigma\) of \(\Sigma\). Since the second term of Eq. (B.3) vanishes due to the condition \(D^aV_{(V)a} = 0\), the inner product \((V, W)\) vanishes if \(V_{(L)}\) and \(V_{(V)b}\) satisfy some appropriate boundary conditions at the boundary \(\partial\Sigma\) of \(\Sigma\) so that the first term of Eq. (B.1) vanishes. In this sense, the scalar part (the first term in Eq. (B.1)) and the vector part (the second term in Eq. (B.1)) orthogonal to each other. Geometrically, the decomposition of 1-forms, and more generally \(p\)-forms, leads via de Rham’s theorem to a characterization of topological invariants of \(\Sigma\) (i.e., Betti Numbers)\(^{26}\).

In this appendix, it is assumed that the \(n\)-dimensional space \(\Sigma\) is closed (compact manifolds without boundary) following York’s discussions. The choice of closed spaces is made for mathematical convenience but the decomposition discussed here is also valid for any other \(n\)-dimensional spaces \(\Sigma\) with the boundary \(\partial\Sigma\) with some appropriate boundary conditions at \(\partial\Sigma\). Through this assumption, in this appendix, we consider the TT-decomposition (transverse traceless decomposition) of a symmetric tensor \(\psi^{ab}\) on \(\Sigma\), which is defined by

\[
\psi^{ab} = \psi^{ab}_{TT} + \psi^{ab}_{L} + \psi^{ab}_{Tr}, \tag{B.4}
\]

where the longitudinal part is

\[
\psi^{ab}_{L} := D^aW^b + D^bW^a - \frac{2}{n}q^{ab}D_cW^c =: (LW)^{ab} \tag{B.5}
\]

and the trace part is

\[
\psi^{ab}_{Tr} := \frac{1}{n} \psi^{ab}_q, \quad \psi := q_{cd}\psi^{cd}. \tag{B.6}
\]
Let us suppose that both an arbitrary symmetric tensor field $\psi_{ab}$ and the metric $q_{ab}$ are $C^\infty$ tensor fields on $\Sigma$. First, we define $\psi_{TT}^{ab}$ in accordance with Eq. (B.4) by

$$\psi_{TT}^{ab} := \psi^{ab} - \frac{1}{n} q^{ab} - (LW)^{ab}. \quad \text{(B.7)}$$

We note that the tensor $\psi_{TT}^{ab}$ is traceless, i.e.,

$$q_{ab} \psi_{TT}^{ab} = 0 \quad \text{(B.8)}$$

by its construction (B.7). Further, we require the transversality on the tensor field $\psi_{TT}^{ab}$, i.e.,

$$D_b \psi_{TT}^{ab} = 0. \quad \text{(B.9)}$$

Equation (B.9) leads to a covariant equation of the vector field $W^a$ in Eq. (B.7) as

$$D_a (LW)^{ab} = D_a \left( \psi^{ab} - \frac{1}{n} q^{ab} \right). \quad \text{(B.10)}$$

The explicit expression of (B.10) is given by

$$D^{bc} W_c = D_a \left( \psi^{ab} - \frac{1}{n} q^{ab} \right), \quad \text{(B.11)}$$

where the derivative operator $D^{bc}$ is defined by

$$D^{bc} := q^{bc} \Delta + \left( 1 - \frac{2}{n} \right) D^b D^c + R^{bc}, \quad \Delta := D^a D_a. \quad \text{(B.12)}$$

The basic properties of Eq. (B.11) are also discussed by York. The operator $D^{ab}$ defined by Eq. (B.12) is linear and second order by its definition. As discussed by York, this operator is strongly elliptic, negative-definite, self-adjoint, and its “harmonic” functions are always orthogonal to the source (right-hand side) in Eq. (B.11). Here, “harmonic” functions of $D^{ab}$ means functions which belong the kernel of the operator $D^{ab}$. Moreover, he showed that Eq. (B.11) will always possess solutions $W^a$ which is unique up to conformal Killing vectors. Due to these situation, in this paper, we assume that the Green function $(D^{-1})_{ab}$ defined by

$$(D^{-1})_{ab} D^{bc} = 0 \quad \text{(B.13)}$$

exists through appropriate boundary conditions at the boundary $\partial \Sigma$ of $\Sigma$. Although York’s discussions are for the case of the closed space $\Sigma$, we review his discussions here. In this review, we explicitly write the boundary terms which are neglected by the closed boundary condition to keep the extendibility to non-closed $\Sigma$ case of discussions in our mind.

The ellipticity of an operator depends only upon its principal part, i.e., the highest derivatives acting on the unknown quantities which it contains. To see the ellipticity of an operator, we consider the replacement of the each derivative
operator $D_a$ occurring in its principal part by an arbitrary vector $V_a$. Through this replacement, the principal part of the operator defines a linear transformation $\sigma_v$. The operator is said to be elliptic if $\sigma_v$ is an isomorphism. In the present case,

$$[\sigma_v(D)]^{ab} = V^bV^a + q^{ab}V_cV^c.$$  \hfill (B.14)

Here, $\sigma_v$ operates on vector $X_a$ and defines a vector-space isomorphism when the determinant of $\sigma_v$ is non-vanishing for all non-vanishing $V^a$. The fact that $\det \sigma_v \neq 0$ here is verified, for example, by choosing $V^a = (\partial/\partial x^\mu)^a$ in a local Cartesian frame $\{x^\mu\}$. The operator is said to be strongly elliptic if all the eigenvalues of $\sigma_v$ are nonvanishing and have the same sign. This is easily checked and $D^{ab}$ is strongly elliptic.

To show that $D^{ab}$ is negative definite, we consider the inner product (B.2) of the vector field $\mathcal{D}W^a := D^{ab}W_b$ and $W^a$:

$$\int_\Sigma \varepsilon_{qab}W^aD^{bc}W_c = \int_\Sigma \varepsilon_{qab}W^a(D_c(LW)^{bc})$$

$$= \int_\Sigma \varepsilon_q \left( D_c \left( W_b(LW)^{bc} \right) - \frac{1}{2}(LW)^{bc}(LW)^{bc} \right)$$

$$= \int_\Sigma s_c W_b(LW)^{bc} - \frac{1}{2} \int_\Sigma \varepsilon_q(LW)^{bc}(LW)^{bc},$$  \hfill (B.15)

where we use the fact that the tensor $(LW)^{bc}$ is symmetric and traceless. Eq. (B.14) shows that the operator $D^{ab}$ has the negative eigenvalues in the case where the first term (boundary term) in Eq. (B.15) is neglected, unless $(LW)^{bc} = 0$. The self-adjointness of the operator $D^{ab}$ is follows from a similar argument in which one integrates by parts twice:

$$\int_\Sigma \varepsilon_{qab}V^a(\mathcal{D}W)^b = \int_\Sigma \varepsilon_{qab}V^a(D_c(LW)^{bc})$$

$$= \int_\Sigma \varepsilon_q \left[ D_c \left( q_{ab}V^a(LW)^{bc} \right) - \nabla_c V_b(LW)^{bc} \right]$$

$$= \int_\Sigma \varepsilon_q \left[ D_c \left( q_{ab}V^a(LW)^{bc} \right) - \frac{1}{2}(LV)^{bc}(LV)_{bc} \right]$$

$$= \int_\Sigma \varepsilon_q \left[ D_c \left( q_{ab}V^a(LW)^{bc} \right) - (LV)^{bc}D_cW_b \right]$$

$$= \int_\Sigma \varepsilon_q \left[ D_c \left( q_{ab}V^a(LW)^{bc} \right) - D_c \left( (LV)^{bc}W_b \right) + W_b \nabla_c (LV)^{bc} \right]$$

$$= \int_\Sigma s_c \left[ V_b(LW)^{bc} - (LV)^{bc}W_b \right] + \int_\Sigma \varepsilon_q W_b D^{bc}V_c$$  \hfill (B.16)

for any vectors $V$ and $W$, where we use the fact that the tensor $(LW)^{ab}$ and $(LV)^{ab}$ are symmetric and traceless. Eq. (B.16) shows that the operator $D^{ab}$ is self-adjoint if the first term (boundary term) in Eq. (B.16) is neglected.
When we can neglect the boundary terms in Eq. (B.15), the right-hand side of (B.15) can vanish only if \((LW)^{ab} = 0\). This means either \(W^a = 0\) or \(W^a\) is a conformal Killing vector (or Killing vector) of the metric \(q_{ab}\). The condition for a conformal Killing vector is, of course, not satisfied for an arbitrary metric but this is given by

\[
\mathcal{L}_W q_{ab} = \lambda q_{ab} \tag{B.17}
\]

for some scalar function \(\lambda\), where \(\mathcal{L}_W\) denotes the Lie derivative along \(W\). Taking the trace of both sides, we find

\[
\lambda = \frac{2}{n} \nabla_c W^c. \tag{B.18}
\]

Therefore, \(W^a\) is a conformal Killing vector if and only if

\[
\nabla^a W^b + \nabla^b W^a - \frac{2}{3} q^{ab} \nabla_c W^c \equiv (LW)^{ab} = 0. \tag{B.19}
\]

It follows that the only nontrivial solutions of \(D^{ab} W_b = 0\) are conformal Killing vectors if they exist. Hence the nontrivial “harmonic” functions of \(D^{ab}\) are conformal Killing vectors. We shall now show that even if these “harmonic” solutions exist, they are always orthogonal to the right-hand side of (B.10) and, hence, can cause no difficulties in solving equation (B.10) by an eigen function expansion.

Denote the conformal Killing vectors by \(W^a = C^a\), where by definition \((LC)^{ab}\) enters in the definition (B.7) of \(\psi_{TT}^{ab}\), conformal Killing vectors cannot affect \(\psi_{TT}^{ab}\).

The orthogonality of \(\psi_{TT}^{ab}\), \((LW)^{ab}\), and \(\frac{1}{n} q^{ab} \psi\) is easily demonstrated. We see readily that \(\frac{1}{n} q^{ab} \psi\) is pointwise orthogonal to \((LW)^{ab}\) and to \(\psi_{TT}^{ab}\), as \((LW)^{ab}\) and \(\psi_{TT}^{ab}\) are both trace-free. To show that \(\psi_{TT}^{ab}\) and \((LV)^{ab}\) are orthogonal for any vector \(V\) and any TT tensor, we have only to show that

\[
\int \Sigma \epsilon_{abcd} q_{ac} (LW)^{ab} \psi^{cd}_{TT} = \int \Sigma 2 D_q W_b \psi^{ab}_{TT}
\]
\[ \int_\Sigma \epsilon_q \left( D_a \left( 2W_b \psi^{ab}_{TT} \right) - 2W_b D_a \psi^{ab}_{TT} \right) = \int_{\partial \Sigma} s_a \left( 2W_b \psi^{ab}_{TT} \right) - \int_\Sigma \epsilon_q \left( 2W_b D_a \psi^{ab}_{TT} \right) = 0, \quad (B.21) \]

where we use the fact that the tensor \( \psi^{ab}_{TT} \) is symmetric, traceless, and transverse \((B.9)\). We also neglect the boundary term in Eq. \((B.21)\). Thus, we conclude that the decomposition defined by \((B.7)\) exists, is unique, and is orthogonal.

One can further decompose the vector \( W^a \) uniquely into its transverse and longitudinal parts with respect to the metric \( q_{ab} \). This splitting is orthogonal, as in Eq. \((B.1)\).

Since the above discussions are for closed spaces \( \Sigma \), careful discussions on the boundary terms which are neglected in the closed \( \Sigma \) is necessary if we extend the above arguments to non-closed \( \Sigma \) case. However, we do not go into these detailed issues. Instead, in the main text, we assume that the existence of the Green function of the derivative operator \( D^{ab} \) and use the transverse-traceless decomposition for an arbitrary symmetric tensor on \( \Sigma \) discussed here.

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