Density of states at disorder-induced phase transitions in a multichannel Majorana wire

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An $N$-channel spinless $p$-wave superconducting wire is known to go through a series of $N$ topological phase transitions upon increasing the disorder strength. Here, we show that at each of those transitions the density of states shows a Dyson singularity $\nu(\varepsilon) \propto \varepsilon^{-1}|\ln \varepsilon|^{-3}$, whereas $\nu(\varepsilon) \propto \varepsilon^{\beta t -1}$ has a power-law singularity for small energies $\varepsilon$ away from the critical points. Using the concept of “superuniversality” [Gruzberg, Read, and Vishveshwara, Phys. Rev. B 71, 245124 (2005)], we are able to relate the exponent $\alpha$ to the wire’s transport properties at zero energy and, hence, to the mean free path $l$ and the superconducting coherence length $\xi$.

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Introduction. Though stable against moderate amounts of disorder, topological phases are typically susceptible to strong disorder. This is particularly true for topological phases in one and two dimensions, for which strong disorder eventually leads to a localization of all electronic states. However, there are examples in which the effect of disorder may not be simply the transition from the topological into a topologically trivial localized phase, but more diverse physics appears.

The most prominent such example is the quantum Hall effect where disorder is an essential element needed to stabilize the topological phase and, hence, to explain the quantization of the conductance [1]. In the context of time-reversal invariant topological insulators, a topologically trivial system may be driven into a nontrivial phase by disorder, as it happens for topological Anderson insulators [2–5]. Also, when disorder preserves certain symmetries on the average, the disorder itself may drive a topological insulator into a new type of topological phase, the so-called “statistical topological insulator” [6, 7]. Topological superconductors, finally, can display thermal metal [8] or glassy phases [9] or enter a topologically nontrivial phase upon increasing disorder strength [10].

An example where disorder leads to a particularly rich phase diagram is that of a multichannel spinless superconducting wire. In Ref. [11] the authors, together with Adagideli, showed that upon increasing the disorder strength such a wire goes through a series of topological phase transitions, alternating between states with and without a Majorana bound state at the wire’s end [11]. For a wire with $N$ transverse channels, there are $N$ such transitions, which take place at mean-free path

$$\tilde{\ell}_{\text{crit}}^{(n)} = \frac{n \xi}{N+1},\ n = 1, 2, \ldots, N,$$

where $\xi$ the superconducting coherence length [12].

Whereas Ref. [11] identified the location of the topological phase transitions, it did not discuss the system’s spectral and transport properties in the vicinity of the critical point. A theoretical framework in which this question can be addressed was provided by Gruzberg, Read, and Vishveshwara [15], who argued that there exists a “superuniversality”, according to which all disorder-induced critical points in (quasi-)one di-
metries than particle-hole symmetry, implying that the system is in symmetry class “D” according to the Cartan classification for large $L$ and weak disorder is known in the literature \[37\].

\[ \langle \log \tau_n \rangle = \frac{2nL}{(N+1)l}, \quad \text{var} \log \tau_n = \frac{4L}{(N+1)l}. \tag{5} \]

For a mean free path $\ell$ near the critical value $\ell_{\text{crit}} = n\xi/(N+1)$, the quasiparticle transmission is dominated by the $n$th transmission eigenvalue $\tau_n$. Using the parameterization

\[ T = 1/\cosh^2 z, \tag{6} \]

one finds

\[ \langle z \rangle = \left( \frac{\ell_{\text{crit}}}{\xi} - \frac{l}{\xi} \right) \frac{L}{l}, \quad \text{var} z = \frac{L}{(N+1)l}. \tag{7} \]

Indeed, at the critical disorder strength (and at the critical disorder strength only) quasiparticle wavefunctions are delocalized throughout the sample \[38\].

The mapping between the scattering matrices of the disordered wire with and without superconductivity that was used in Ref. \[11\] exists for zero energy only. For that reason, Ref. \[11\] could not access the density of states $\nu(\varepsilon)$ of the multichannel Majorana wire in the vicinity of the critical points. We now show how the density of states can be obtained from the transmission statistics of Eqs. \(6\) and \(7\) using the “superuniversality” argument of Ref. \[15\].

**Mapping to one-dimensional model with chiral symmetry.** According to the “superuniversality” argument of Gruzberg, Read, and Vishveshwara \[15\], the quasiparticle transmission distribution $T$ and the density of states $\nu(\varepsilon)$ in the vicinity of the critical point should be the same as that of a one-dimensional disordered wire in the chiral symmetry class. (In this respect, the three chiral classes BDI, AIII, and CII are interchangeable.) Such systems have been analyzed abundantly in the literature, see, e.g., Refs. \[13\] \[17\] \[18\] \[39\]–\[42\] and we here summarize the main results of relevance to the present problem.

A prototype of the disordered wire with chiral symmetry in one dimension is described by the Hamiltonian \[39\]

\[ H_{\text{chiral}} = -v_F p x + w(x) \sigma_x, \tag{8} \]

where $v_F$ is the Fermi velocity and $w$ is a random potential with mean $\langle w(x) \rangle = (\hbar v_F \alpha)/(2l)$ and variance $\langle w(x) w(x') \rangle = (\hbar^2 v_F^2/4l) \delta(x-x')$. The parameter $\alpha$ measures the distance to the critical point; $l$ is the mean free path in this system. In the vicinity of the critical point, the transmission $T = 1/\cosh^2 z$ of such a disordered one-dimensional wire of length $L$, coupled to ideal source and drain leads has a distribution given by

\[ \langle z \rangle = \alpha \frac{L}{2l}, \quad \text{var} z = \frac{L}{l}. \tag{9} \]

The density of states $\nu(\varepsilon)$ has a singularity at zero energy, which is best described through the integrated density of states

\[ N(\varepsilon) = \int_0^\varepsilon \nu(\varepsilon). \tag{10} \]
Extending the zero-\(\alpha\) calculation of Ref. [32] to the case of nonzero \(\alpha\), we find the integrated density of states (see Supplementary Material for details)

\[
N(\varepsilon) = \frac{L}{2\pi|K_{\pi/2}(2i\varepsilon_F/h)|^2}, \tag{11}
\]

where \(K_\pi(x)\) is the Bessel function of the second kind. For \(\alpha = 0\) Eq. (11) reproduces the Dyson singularity \(\nu(\varepsilon) \propto 1/|\varepsilon|\ln(\varepsilon_F/|\varepsilon|)|\), whereas for nonzero \(\alpha\) one has the asymptotic dependence \(\nu(\varepsilon) \propto |\varepsilon|^{\alpha-1}\). Near the critical point \(\alpha = 0\) Eq. (11) is to be preferred over the asymptotic power-law dependence, because it applies to a much wider range of energies than the simple asymptotic power law \(\nu(\varepsilon) \propto |\varepsilon|^{\alpha-1}\).

Comparing Eqs. (7) and (9), one immediately identifies

\[
\alpha_n = \frac{2(N + 1)}{\xi} \left( (\frac{\pi}{2}) - l \right), \quad \tilde{\alpha} = (N + 1)l, \tag{12}
\]

as the dimensionless distance to the \(n\)th critical point for the disordered multichannel Majorana wire, and the equivalent mean free path in the model [33], respectively. The density of states and transmission statistics are governed by the dimensionless distance to the closest critical point,

\[
|\alpha| = \min_{n=1}^{N} |\alpha_n|. \tag{13}
\]

**Numerics.** We now compare our predictions to numerical simulations of a disordered multichannel Majorana wire. For technical reasons, we first present numerical calculations for a slight variation of the model (2), in which the Majorana wire is represented by \(N\) coupled one-dimensional channels with Hamiltonian

\[
H_{mn} = \delta_{mn} \left( -\frac{\hbar^2}{2m} \partial_x^2 - \mu \right) \sigma_z - i\Delta'_n \partial_x \sigma_x + u_{mn}(x) \sigma_z, \tag{14}
\]

with a disorder term \(u_{mn}(x)\) that has zero mean and variance

\[
\langle u_{ij}(x) u_{kl}(x') \rangle = \frac{(\hbar v_F)^2}{L(N + 1)} \delta(x - x') \left( \delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl} \right), \tag{15}
\]

\(l\) being the mean free path. The technical advantage of Eq. (14) is that the normal-state distribution (5) of the transmission eigenvalues also holds up to moderately strong disorder strengths, so that numerical calculations can be performed for (comparatively) smaller system sizes. The Hamiltonian (14) anticommutes with \(\sigma_x\), i.e., it is in symmetry class BDI.

In order to determine the density of states, we couple one end of the \(N\)-channel wire to an ideal lead, keeping the other end closed. Following the method of Ref. [34] we calculate the wire’s scattering matrix \(S(\varepsilon, L)\) as a function of the length \(L\) of the disordered wire. The integrated density of states \(N(\varepsilon)\) can be obtained by numerically integrating the relation

\[
\frac{\partial N(\varepsilon)}{\partial L} = \frac{1}{2\pi} \text{Im} \frac{\partial}{\partial L} \log \det S(\varepsilon). \tag{16}
\]

The integrated density of states obtained this way can be fitted to the functional form (11), which allows us to obtain the dimensionless parameter \(\alpha\) as a function of the disorder strength. Results for Majorana wires with \(N = 1\) and \(N = 3\) are shown in Fig. 1. The agreement is excellent and holds throughout the entire range of disorder strengths, including points far away from the critical disorder strengths.

We have also performed numerical calculations for the two-dimensional Hamiltonian (2) in a strip geometry. We choose the two pairing terms \(\Delta'_x\) and \(\Delta'_y\) to be equal. Since such a system is no longer in class BDI, we expect slight deviations in the quantitative estimates of the critical disorder strength and the dimensionless distance to the critical point. The numerical results for a wire with \(N = 2\) indeed show a slight deviation of the critical disorder strength at the second phase transition, although, within the accuracy of our numerical calculations, no deviation for the dimensionless distance \(\alpha\) can be discerned.
see Fig. 2.

**Conclusion.** We have investigated the density of states of a multichannel spinless superconducting wire, as it goes through a series of disorder-driven topological phase transitions. Using the concept of “superuniversality” of Gruzberg, Read, and Vishveshwar [15], we could establish a relation between the known quasiparticle transmission statistics at zero energy and the singular contribution to the density of states at finite energies. A comparison with a numerical solution of the problem in excellent agreement with these analytical results. Our results are a powerful demonstration of the concept of superuniversality, showing that in one dimension, as well as in quasi one dimension, the scaling relations for the density of states remain valid across boundaries between symmetry classes.

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**Appendix.** To the best of our knowledge, Eq. (11) is not known in the literature, although it can be derived rather quickly by adapting existing calculations of the density of states in a wire with chiral symmetry at the critical point \( \alpha = 0 \). Here we take Ref. [42] as our starting point, where the density of states was calculated from the stationary distribution \( \rho(x) \) of the reflection eigenvalue \( R = \tanh^2(x) \) of a wire with Hamiltonian [8], evaluated at the imaginary energy \( \varepsilon = -i\omega \), \( \omega > 0 \), and in the limit of a large wire length \( L \). In Ref. [42] this distribution is found as the stationary solution of the Fokker-Planck equation

\[
\frac{\partial \rho(x)}{\partial L} = \frac{\partial}{\partial x} \left[ \frac{\omega}{v_F} \sinh 2x + \frac{1}{2J} \frac{\partial}{\partial x} \right] \rho(x),
\]

where \( J \) is a Jacobian which, for the case of a one-dimensional wire with chiral symmetry takes the value \( J = 1 \) at the critical point \( \alpha = 0 \). Solving Eq. (17) gives the stationary solution

\[
\rho(x) = \frac{1}{Z(\alpha)} J e^{-\alpha \cosh 2x},
\]

with \( \alpha = \omega l / v_F \) and \( Z(\alpha) \) a normalization factor. The key result of Ref. [42] is a general relation between the integrated density of states \( N(\varepsilon) \) and this normalization factor,

\[
N(\varepsilon) = \frac{L}{\pi l} \Im \frac{1}{a} \left[ \frac{\partial}{\partial a} \ln Z(a) \right]_{a \rightarrow i\varepsilon / v_F}.
\]

The calculation of Ref. [42] is easily generalized to the case \( \alpha \neq 0 \): Nonzero \( \alpha \) gives rise to a constant drift term in the Fokker-Planck equation (17) or, equivalently, an exponential factor in the Jacobian \( J \),

\[
J = e^{-\alpha x}.
\]

The stationary solution and the integrated density of states are then obtained in the same way as described above. One finds

\[
Z(a) = K_{a/2}(a),
\]

from which the result (11) follows directly.

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[1] J. T. Chalker in Topological Aspects of low dimensional systems, A. Comtet, T. Jolicour, S. Ouvry, and F. David (Springer, 1999).
[2] J. Li, R.-L. Chu, J. K. Jain, and S.-Q. Shen, Phys. Rev. Lett. 102, 136806 (2009).
[3] C. W. Groth, M. Wimmer, A. R. Akhmerov, J. Tworzydlo, and C. W. J. Beenakker, Phys. Rev. Lett. 103, 196805 (2009).
[4] H.-M. Guo, G. Rosenberg, G. Refael, and M. Franz, Phys. Rev. Lett. 105, 216601 (2010).
[5] H.-M. Guo, Phys. Rev. B 82, 115122 (2010).
[6] L. Fu and C. L. Kane, Phys. Rev. Lett. 109, 246605 (2012).
[7] I. C. Fulga, B. van Heck, J. M. Edge, and A. R. Akhmerov, Phys. Rev. B 89, 155424 (2014).
[8] I. C. Fulga, A. R. Akhmerov, J. Tworzydlo, T. Béri, and C. W. J. Beenakker, Phys. Rev. B 86, 054505 (2012).
[9] F. Crepin, G. Zarand and P. Simon, arXiv:1405.4283 (2014).
[10] I. Adagideli, M. Wimmer, and A. Teker Phys. Rev. B 89, 144506 (2014).
[11] M.-T. Rieder, P. W. Brouwer and I. Adagideli Phys. Rev. B 88, 060509(R) (2013).
[12] Multichannel disordered wires with chiral symmetry, but without superconducting pairing, are also known to exhibit series of topological phase transitions as a function of the disorder strength, see Refs. [13] and [14].
[13] P. W. Brouwer, C. Mudry, B. D. Simons and A. Altland Phys. Rev. Lett. 81, 862 (1998).
[14] P. W. Brouwer, C. Mudry, and A. Furusaki, Phys. Rev. Lett. 84, 2913 (2000).
[15] I. A. Gruzberg , N. Read and S. Vishveshwar , Phys. Rev. B 71, 245124 (2005).
[16] F. J. Dyson, Phys. Rev. 92, 1331 (1953).
[17] G. Theodorou and M. H. Cohen, Phys. Rev. B 13, 4597 (1976).
[18] T. P. Eggarter and R. Riedinger, Phys. Rev. B 18, 569 (1978).
[19] D. S. Fisher, Phys. Rev. B 50, 3799 (1994); 51, 6411 (1995).
[20] A. A. Ovchinnikov and N. S. E rikhman, Zh. Eksp. Teor. Fiz. 73, 650 (1977) [Sov. Phys. JETP 46, 340 (1977)].
[21] W. P. Su, J. R. Schrieffer, and A. J. Heeger Phys. Rev. Lett. 42, 1698 (1980).
[22] R. Jackiw and G. Semenoff Phys. Rev. Lett. 50, 439 (1983).
[23] M. J. Rice and E. J. Mele Phys. Rev. Lett. 49, 1455 (1982).
[24] O. Motrunich, K. Damle, and D. A. Huse, Phys. Rev. B 63, 224204 (2001).
[25] P. W. Brouwer, M. Duckheim, A. Romito and F. von Oppen, Phys. Rev. Lett. 107, 196804 (2011).
[26] L. Fu and C. L. Kane Phys. Rev. Lett. 100, 096407 (2008).
[27] R. M. Lutchyn, J. D. Sau, and S. Das Sarma, Phys. Rev. Lett. 105, 077001 (2010).
[28] Y. Oreg, G. Refael, and F. von Oppen, Phys. Rev. Lett. 105, 177002 (2010).
[29] A. Cook and M. Franz Phys. Rev. B 84, 201105(R) (2011).
[30] M. Duckheim and P. W. Brouwer, Phys. Rev. B 83, 054513 (2011).
[31] M.-T. Rieder, G. Kells, M. Duckheim, D. Meidan, and P. W. Brouwer, Phys. Rev. B 86, 125423 (2012).
[32] A. Altland and M. R. Zirnbauer, Phys. Rev. B 55, 1142 (1997).
[33] S. Ryu, A. Schnyder, A. Furusaki, and A. Ludwig, New J. Phys. 12, 065010 (2010).
[34] F. Evers and A. Mirlin, Rev. Mod. Phys. 80, 1355 (2008).
[35] S. Tewari and J. D. Sau, Phys. Rev. Lett 109, 150408 (2012).
[36] G. Kells, D. Meidan and P. W Brouwer, Phys. Rev. B 85, 060507(R) (2012).
[37] C. W. Beenakker Rev. Mod. Phys. 69, 731 (1997).
[38] A. R. Akhmerov, J. P. Dahlhaus, F. Hassler, M. Wimmer, and C. W. J. Beenakker, Phys. Rev. Lett. 106, 057001 (2011). 
[39] A. Comtet, J. Desbois, and C. Monthus, Ann. Phys. (NY) 239, 312 (1995).
[40] P. W. Brouwer, C. Mudry, A. Furusaki Physica E 9, 333 (2001).
[41] A. D. Stone and J. D. Joannopoulos, Phys. Rev. B 24, 3592 (1981).
[42] M. Titov, P. W Brouwer, A. Furusaki and C. Mudry, Phys. Rev. B 63, 235318 (2001). 
[43] P. W. Brouwer, M. Duckheim, A. Romito and F. von Oppen, Phys. Rev. B 84, 144526 (2011).