On Contractible Banach Algebras of Operators

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Abstract

In this short note, we give some results concerning contractible (super-amenable) Banach algebras: It is shown that any contractible central Banach algebra is local (i.e. has a unique maximal ideal), and any symmetrically contractible Banach algebra has a normalized trace with values in its center. It is shown that for a Banach subalgebra $A \subseteq \mathcal{L}(X)$ of bounded linear operators on infinite-dimensional Banach space $X$, which contains the ideal of finite-rank operators (e.g. $A = \mathcal{L}(X)$), the image of any diagonal of $A$ under the canonical algebra-morphism $\mathcal{L}(X) \otimes^\gamma \mathcal{L}(X) \to \mathcal{L}(X \otimes^\gamma X)$ is zero.

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1 Introduction

A Banach algebra $A$ is called contractible (super-amenable) if every bounded derivation from $A$ into any Banach $A$-bimodule, is inner [6, 8]. Contractibility is one of the many variants of the concept of amenability (for Banach algebras) originally introduced by Johnson [4]. (For various notions of amenability see [5, 6, 8].) It is known that any finite-dimensional contractible Banach algebra is a finite direct sum of full matrix algebras [8, Theorem 4.1.4]. The only known contractible Banach algebras are of this form. Indeed, it is a longstanding question whether every contractible Banach algebra is finite-dimensional [2, 6 Problem 15, page 224]. Also, the following special case of this question ([2, 6 Problem 16, page 224]) has not been answered yet: does for any Banach space $X$, the contractibility of the Banach algebra $\mathcal{L}(X) = \mathcal{L}(X, X)$ of all bounded linear operators on $X$, imply that $X$ is finite-dimensional? (For information on these questions see [8, §4.1 and page 196].) We must remark that the chance that there exist infinite-dimensional contractible Banach algebras is not very small: For a long time it was a common belief that for infinite-dimensional Banach spaces $X$, $\mathcal{L}(X)$ cannot be amenable (a weaker version of contractibility). But, in 2009, Argyros and Haydon [1] found out a specific
infinite-dimensional Banach space $E$ that has Scaler-Plus-Compact property, and as it was pointed out by Dales, $\mathcal{L}(E)$ become an amenable algebra. (See for more details [7].)

The aim of this short note, is to report some properties of contractible Banach algebras (of operators): In §§23 it is shown that any contractible central Banach algebra is local (i.e. has a unique maximal ideal), and symmetrically contractible Banach algebras have center-valued normalized traces. In [11] we show for any infinite-dimensional Banach space $X$, and any contractible Banach subalgebra $A$ of $\mathcal{L}(X)$ which contains the ideal of finite-rank operators, the image of any diagonal of $A$, under the canonical algebra-morphism,

$$A \otimes \gamma A \hookrightarrow \mathcal{L}(X) \otimes \gamma \mathcal{L}(X) \to \mathcal{L}(X \otimes \gamma X),$$

is zero. (For terminology see below.)

For preliminaries on contractibility, we refer the reader to Runde’s books [10, 8]. The topological dual of a Banach space $X$ is denoted by $X^*$. The completed projective tensor product of Banach spaces $X, Y$ is denoted by $X \otimes \gamma Y$. The projective norm is denoted by $\| \cdot \|_\gamma$. The Banach space of bounded linear operators from $X$ into $Y$ is denoted by $\mathcal{L}(X,Y)$. For a Banach algebra $A$, $\mathcal{Z}(A)$ denotes the center of $A$. If $X,Y$ are unital Banach left modules over a unital Banach algebra $A$, then $\mathcal{A} \mathcal{L}(X,Y) \subset \mathcal{L}(X,Y)$ denotes the closed linear subspace of bounded module-morphisms from $X$ into $Y$. For Banach algebras $A, B$, the Banach space $A \otimes \gamma B$ is a Banach algebra with the multiplication given by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ for $a, a' \in A, b,b' \in B$. The diagonal mapping $\Delta : A \otimes \gamma A \to A$ for $A$ is the unique bounded linear operator defined by $a \otimes b \mapsto ab$. Note that $A \otimes \gamma A$ is canonically a Banach $A$-bimodule with module-operations given by $c(a \otimes b) := (ca \otimes b)$ and $(a \otimes b)c := (a \otimes bc)$, and $\Delta$ is a bimodule-morphism. A diagonal for a unital Banach algebra $A$ is an element $M \in A \otimes \gamma A$ satisfying

$$\Delta(M) = 1, \quad (c \otimes 1)M = M(1 \otimes c), \quad (c \in A).$$

It is well-known that a Banach algebra is contractible iff it is unital and has a diagonal: Suppose that $A$ is contractible. Let $E$ denote the Banach $A$-bimodule with the underlying Banach space $A$ and, left and right module-operations $ax := ax$ and $xa := 0$ for $a \in A, x \in E$. Then $\text{id} : A \to E$ is a derivation and hence inner. Thus $A$ has a right unit. Similarly, it is proved that $A$ has a left unit and hence $A$ is unital. Now, consider the derivation $D : A \to \ker(\Delta)$ defined by $a \mapsto (1 \otimes a) - (a \otimes 1)$. $D$ must be inner and thus there is $N \in \ker(\Delta)$ with the property $aN - Na = D(a)$; hence $M := N + (1 \otimes 1)$ is a diagonal for $A$. Conversely, suppose that $M$ is a diagonal for $A$. If $D : A \to X$ is a bounded derivation, then it can be checked that for the element $z := \sum_{n=1}^\infty a_n D(b_n)$ of $X$ we have $D(a) = az - za$. Thus $D$ is inner.

2 Contractible Central Banach Algebras Are Local

The content of the following lemma is known in literatures:
Lemma 2.1. Let $A$ be a contractible Banach algebra and let $E, F$ be unital Banach left $A$-modules. Then any diagonal for $A$ gives rise to a bounded projection $\Phi = \Phi_{E,F}$ from $\mathcal{L}(E,F)$ onto $\mathcal{L}_A(E,F)$.

Proof. Let $M$ be a diagonal for $A$ of the form $\Phi_{M}$. For any $T \in \mathcal{L}(E,F)$ let

$$\Phi(T) : E \to F, \quad x \mapsto \sum_{n=1}^{\infty} a_n T(b_n x), \quad (x \in E).$$

Then it can be checked that $\Phi(T)$ is well-defined and belongs to $\mathcal{L}_A(E,F)$. Also, it is easily verified that $T \mapsto \Phi(T)$ is a bounded linear projection. \hfill \Box

Proposition 2.2. Let $A$ be a contractible Banach algebra. Then any diagonal for $A$ gives rise to a canonical bounded linear operator $\Psi : A \to \mathcal{Z}(A)$ with $\Psi(1) = 1$.

Proof. Consider $A$ as a Banach left $A$-module in the canonical fashion. For any $c \in A$, let $\ell_c : A \to A$ denote the left multiplication operator by $c$. By the notations of Lemma 2.1,

$$\Phi_{A,A}(\ell_c) : A \to A, \quad x \mapsto \sum_{n=1}^{\infty} a_n cb_n x$$

is a left module-morphism and hence there is a $\tilde{c} \in A$ such that $\Phi_{A,A}(\ell_c) = r_{\tilde{c}}$ where $r_{\tilde{c}} : A \to A$ denotes the right multiplication operator by $\tilde{c}$. It is clear that $\tilde{c} = \sum_{n=1}^{\infty} a_n cb_n$ and $\tilde{c} \in \mathcal{Z}(A)$. We let $\Psi$ to be defined by $c \mapsto \tilde{c}$. \hfill \Box

Proposition 2.3. Let $A$ be a contractible central Banach algebra. Then any diagonal for $A$ gives rise to a canonical bounded linear functional $\psi \in A^*$ with $\psi(1) = 1$.

Proof. We have $\mathcal{Z}(A) = C1$. With the notations of Proposition 2.2 $\psi$ is defined by

$$\Psi(c) = \sum_{n=1}^{\infty} a_n cb_n = \psi(c)1, \quad (c \in A).$$

\hfill \Box

The following theorem is the main result of this section.

Theorem 2.4. Let $A$ be a contractible central Banach algebra. Then $A$ has a unique maximal (two-sided) ideal $\mathcal{M}_A$.

Proof. Let $\mathcal{M}_A := \text{closed linear span of } \{ c \in A : c \text{ belongs to a proper ideal of } A \}$. It is clear that $\mathcal{M}_A$ is an ideal of $A$ which contains every proper ideal of $A$. With $\psi$ as in Proposition 2.3, for any $c \in A$ which is contained in a proper ideal $J$ of $A$, we must have $\psi(c) = 0$, because otherwise we must have $1 = \psi(c)^{-1} \sum_{n=1}^{\infty} a_n cb_n \in J$, a contradiction. Thus $\mathcal{M}_A \subseteq \ker(\psi)$ and hence $\mathcal{M}_A$ is a proper ideal of $A$. \hfill \Box

A closed linear subspace $F$ of a Banach space $E$ is called topologically complemented if there is a closed linear subspace $F'$ of $E$ such that $E = F \oplus F'$. In this case $F'$ is called a topological complement for $F$. $F$ is topologically complemented in $E$ iff there is a bounded linear projection from $E$ onto $F$. The following lemmas are very well-known. For the sake of completeness we bring their proofs.
Lemma 2.5. Let $A$ be a contractible Banach algebra and let $E$ be a unital Banach left $A$-module. Suppose that $F \subset E$ is a closed submodule which is (as a Banach space) topologically complemented in $E$. Then $F$ has a topological complement in $E$ which is also a closed submodule.

Proof. Let $p$ be a bounded linear projection from $E$ onto $F$. By Lemma 2.1, $\Phi_{E,F}(p)$ is a module-morphism from $E$ into $F$. It is easily verified that $\Phi_{E,F}(p)$ is also a projection from $E$ onto $F$. Thus $\ker \Phi_{E,F}(p)$ is the desired complement for $F$. $\square$

If $A, A'$ are contractible Banach algebras with diagonals $M, M'$ of the forms as in (1), then $\sum_{n,m=1}^{\infty} a_n \otimes a'_m \otimes b_n \otimes b'_m$ is a diagonal for $A \otimes A'$. Also, $\sum_{n=1}^{\infty} b_n \otimes a_n$ is a diagonal for $A^{\text{op}}$, the opposite algebra of $A$. Thus if $A$ is contractible then $A \otimes A^{\text{op}}$ is contractible.

Lemma 2.6. The analogue of Lemma 2.5 is satisfied for bimodules: Let $A$ be a contractible Banach algebra and $E$ a unital Banach $A$-bimodule. If $F$ is a closed sub-bimodule of $E$ which is topologically complemented, then it has a complement in $E$ which is also a sub-bimodule.

Proof. Any unital Banach $A$-bimodule $E$ may be considered as unital Banach left $A \otimes A^{\text{op}}$-module with module operation given by $(a \otimes b)x := axb$ ($a \in A, b \in A^{\text{op}}, x \in E$). In this fashion, any $A$-bimodule-morphism is a left $A \otimes A^{\text{op}}$-module-morphism. The converses of this facts are also satisfied. Now, the desired result follows from Lemma 2.5. $\square$

Theorem 2.7. Let $A$ be a contractible central Banach algebra. Then $M_A$ (as a closed subspace) is topologically complemented in $A$ iff $M_A = 0$.

Proof. Suppose that $M_A$ is topologically complemented in $A$. Then by Lemma 2.6 there is an ideal $J$ in $A$ such that $M_A \oplus J = A$. Since $M_A$ is the only maximal ideal of $A$ we must have $J = A$. The proof is complete. $\square$

The following result is directly concluded from Theorems 2.4 and 2.7.

Corollary 2.8. Let $X$ be an infinite-dimensional Banach space. If $\mathcal{L}(X)$ is contractible then it has a unique maximal ideal. Moreover, this ideal is not topologically complemented.

3 A Trace on Symmetrically Contractible Algebras

A contractible Banach algebra $A$ is called symmetrically contractible if $A$ has a symmetric diagonal; that is, a diagonal $M$ satisfying $\mathcal{F}_A(M) = M$ where $\mathcal{F}_A : A \otimes A \to A \otimes A$ denotes flip i.e., the unique bounded linear mapping defined by $(a \otimes b) \mapsto (b \otimes a)$. The matrix algebra $M_n$ is symmetrically contractible. Indeed, it is well-known that $M_n$ has the unique diagonal $n^{-1} \sum_{i,j=1}^{n} \delta_{ij} \otimes \delta_{ji}$ where $\delta_{ij}$‘s denote the standard basis of $M_n$. Thus any finite-dimensional contractible Banach algebra is symmetrically contractible.

Theorem 3.1. Let $A$ be a symmetrically contractible Banach algebra. Then any symmetric diagonal of $A$ gives rise to a bounded normalized $\mathbb{Z}(A)$-valued trace for $A$. 

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Proof. Let $M$ be a symmetric diagonal for $A$ of the form (1). We saw in Proposition 2.2 that the assignment $c \mapsto \sum_{n=1}^{\infty} a_n c b_n$ defines a bounded linear mapping $\Psi : A \to \mathcal{Z}(A)$ with $\Psi(1) = 1$. For every $c, c' \in A$ we have $\sum_{n=1}^{\infty} b_n \otimes a_n c c' = \sum_{n=1}^{\infty} c b_n \otimes a_n c'$ and hence $\sum_{n=1}^{\infty} a_n c c' \otimes b_n = \sum_{n=1}^{\infty} a_n c' \otimes c b_n$. Thus we have

$$\Psi(cc') = \Delta(\sum_{n=1}^{\infty} a_n c c' \otimes b_n) = \Delta(\sum_{n=1}^{\infty} a_n c' \otimes c b_n) = \Psi(c'c).$$

Theorem 3.2. Let $A$ be a symmetrically contractible central Banach algebra. Then any symmetric diagonal of $A$ gives rise to a normalized trace $\psi \in A^*$. 

Proof. We let $\psi$ to be defined by $\Psi(c) = \psi(c)1$ where $\Psi$ is given by Theorem 3.1.

For matrix algebra $M_n$, the unique diagonal of $M_n$ gives rise to the ordinary trace. We have the following direct corollary of Theorem 3.2.

Corollary 3.3. Let $X$ be a Banach space. If $\mathcal{L}(X)$ is symmetrically contractible then it has a normalized trace.

4 A Null-Property of Diagonals for $\mathcal{L}(X)$

Let $X$ be a Banach space. Consider the unique bounded linear operator

$$\Upsilon : \mathcal{L}(X) \otimes^{\gamma} \mathcal{L}(X) \to \mathcal{L}(X \otimes^{\gamma} X),$$

defined by

$$[\Upsilon(T \otimes S)](x \otimes y) = T(x) \otimes S(y), \quad (T, S \in \mathcal{L}(X), x, y \in X).$$

Then $\Upsilon$ is an algebra-morphism between Banach algebras. We denote the image under $\Upsilon$ of any element $N \in \mathcal{L}(X) \otimes^{\gamma} \mathcal{L}(X)$, by $N^{op}$. It follows from properties of projective tensor product, that $\|\Upsilon\| = 1$ and hence $\|N^{op}\| \leq \|N\|_\gamma$. Note that, in general, $\Upsilon$ is not one-to-one. (This fact can be concluded from the fact that the canonical mapping from $X^* \otimes^{\gamma} X^*$ onto the space of nuclear bilinear functionals on $X \times X$ is not necessarily one-to-one; see [9, §2.6].)

Proposition 4.1. Let $\Lambda \in \mathcal{L}(X \otimes^{\gamma} X)$ be such that for every one-rank operator $T \in \mathcal{L}(X)$,

$$(T \otimes 1)^{\text{op}} \Lambda = \Lambda (1 \otimes T)^{\text{op}}.$$

Then there is a unique operator $\Gamma$ in $\mathcal{L}(X)$ such that $\Lambda = (1 \otimes \Gamma)^{\text{op}} \mathcal{F}_X$.

Proof. Let $y$ be a nonzero vector in $X$, and let $f \in X^*$ be such that $f(y) = 1$. Let $T \in \mathcal{L}(X)$ to be defined by $x \mapsto f(x)y$. For every $x \in X$ we have

$$(T \otimes 1)^{\text{op}} \Lambda(x \otimes y) = \Lambda(x \otimes y). \quad (2)$$
$X$ has the decomposition $<y> \oplus \ker(f)$ where $<y>$ denotes the subspace generated by $y$. There exist $z \in \ker(f) \otimes \gamma X$ and $w \in X$ such that $\Lambda(x \otimes y) = y \otimes w + z$. It follows from (2) that $\Lambda(x \otimes y) = y \otimes w$. Since the mapping $x \mapsto \Lambda(x \otimes y)$ is linear and bounded, there is $\Gamma_y \in \mathcal{L}(X)$ such that $\Lambda(x \otimes y) = y \otimes \Gamma_y(x)$. Now, suppose that $y, y'$ in $X$ are linearly independent. We have

\[
\Lambda(x \otimes (y + y')) = y \otimes \Gamma_y(x) + y' \otimes \Gamma_{y'}(x),
\]

\[
\Lambda(x \otimes (y + y')) = (y + y') \otimes \Gamma_{y+y'}(x).
\]

Thus $\Gamma_y = \Gamma_{y'}$. Also, it can be checked that for every nonzero scalar $\lambda$ we have $\Gamma_{\lambda y} = \Gamma_y$. Thus there exists $\Gamma \in \mathcal{L}(X)$ such that $\Lambda(x \otimes y) = y \otimes \Gamma(x)$ for every $x, y \in X$. The proof is complete.

**Corollary 4.2.** Let $M$ be an element of $\mathcal{L}(X) \otimes \gamma \mathcal{L}(X)$ that satisfies

\[
(T \otimes 1)M = M(1 \otimes T), \quad (T \in \mathcal{L}(X) \text{ of rank one}.
\]

Then there exists $\Gamma \in \mathcal{L}(X)$ such that $M^{op} = (1 \otimes \Gamma)^{op} \mathcal{F}_X$. Moreover, if $M$ is symmetric (i.e. $\mathcal{F}_{\mathcal{L}(X)}(M) = M$) then there exists a scalar $\lambda$ such that $M^{op} = \lambda \mathcal{F}_X$.

**Proof.** The first part follows directly from Proposition 4.1. Suppose that $M$ is symmetric. Then it follows from the identity $[\mathcal{F}_{\mathcal{L}(X)}(M)]^{op} = \mathcal{F}_X M^{op} \mathcal{F}_X$, that

\[
\mathcal{F}_X (1 \otimes \Gamma)^{op} = (1 \otimes \Gamma)^{op} \mathcal{F}_X.
\]

Thus for every $x, y \in X$ we have $\Gamma(y) \otimes x = y \otimes \Gamma(x)$. This means that $\Gamma$ is a scalar multiple of identity. The proof is complete.

Let $Y, Y', Z$ be finite-dimensional Banach spaces. Similar to the mapping $\Upsilon$ above, we denote by $\tilde{T} : N \mapsto N^{op}$ the unique bounded linear mapping given by

\[
\mathcal{L}(Y, Z) \otimes \gamma \mathcal{L}(Z, Y') \rightarrow \mathcal{L}(Y \otimes Y), \quad \mathcal{F}_X (y \otimes z) = \mathcal{F}_X (x \otimes S(z)).
\]

We know that this is a linear isomorphism.

**Lemma 4.3.** With the above assumptions, suppose that $\dim(Y) = \dim(Y')$. Suppose that $T : Y \rightarrow Y'$ is a linear isomorphism. For every finite-dimensional Banach space $Z$, let the linear mapping $\tilde{T}_Z$ be given by

\[
\tilde{T}_Z : Y \otimes \gamma Z \rightarrow Z \otimes \gamma Y', \quad (y \otimes z) \mapsto (z \otimes T(y)).
\]

There is a numerical positive constant $c$ such that $c$ is independent from $Z$ (independent from norm and dimension of $Z$) and such that:

\[
\|\Upsilon^{-1}(\tilde{T}_Z)\|_\gamma \geq c^{-1} \dim(Z).
\]
Proof. Suppose that \( y_1, \ldots, y_k \) and \( z_1, \ldots, z_m \) are vector basis respectively for \( Y \) and \( Z \), and let \( y'_i = T(y_i) \). Let the linear operators

\[
S_{ij} : Y \to Z, \quad S'_{ji} : Z \to Y',
\]

be given by

\[
S_{ij}(y_i) = z_j, S_{ij}(y_q) = 0, \quad (q \neq i), \quad S'_{ji}(z_j) = y'_i, S'_{ji}(z_q) = 0, \quad (q \neq j).
\]

Let \( N := S_{ij} \otimes S'_{ji} \). Then \( N^\text{op} = \tilde{T}_Z \) and hence \( \Upsilon^{-1}(\tilde{T}_Z) = N \).

Let \( \nu \) denote the linear functional on \( L(Y,Y') \) that associates to any operator \( Y \to Y' \), the normalized trace of its matrix in the bases \( y_1, \ldots, y_k \) and \( y'_1, \ldots, y'_k \) of \( Y \) and \( Y' \). Suppose that \( c \) denotes the functional-norm of \( \nu \). It is clear that \( c \neq 0 \). Consider the bilinear functional

\[
\mu : L(Y,Z) \times L(Z,Y') \to \mathbb{C}, \quad (P,Q) \mapsto \nu(QP).
\]

Then we have \( \|\mu\| \leq c \) and hence \( \|c^{-1}\mu\| \leq 1 \). Now, it follows from the properties of projective tensor-norm that

\[
\|N\|_\gamma \geq |c^{-1}\mu(N)| = c^{-1}m.
\]

\[\square\]

Theorem 4.4. Let \( X \) be an infinite-dimensional Banach space. Let \( M \in \mathcal{L}(X) \otimes^\gamma \mathcal{L}(X) \) be an element that satisfies (3). Then \( M \in \ker(\Upsilon) \). In other notation, \( M^\text{op} = 0 \).

Proof. Suppose that \( \Gamma \in \mathcal{L}(X) \) is as in Corollary 4.2. Suppose that \( M^\text{op} \neq 0 \) and hence \( \Gamma \neq 0 \). Let \( y,y' \) be two nonzero vectors in \( X \) such that \( \Gamma(y) = y' \). Suppose that \( Y,Y' \) denote the one-dimensional subspaces of \( X \) generated respectively by \( y,y' \), and suppose that \( T : Y \to Y' \) is defined by \( T(y) = y' \). Let \( Z \) be an arbitrary finite-dimensional subspace of \( X \). Suppose that \( E_Y : Y \to X \) and \( E_Z : Z \to X \) denote the embedding-maps and \( P_Y : X \to Y' \) is an arbitrary continuous projection from \( X \) onto \( Y' \). By Kadec-Snobar’s Theorem [3, Theorem 6.28] we know that there exists a continuous projection \( P_Z : X \to Z \), from \( X \) onto \( Z \), such that \( \|P_Z\| < 1 + \sqrt{\dim(Z)} \). Let

\[
N := (P_Z \otimes P_Y)M(E_Y \otimes E_Z) \in \mathcal{L}(Y,Z) \otimes^\gamma \mathcal{L}(Z,Y').
\]

We have

\[
\|N\|_\gamma \leq \|P_Z\|\|P_Y\|\|M\|_\gamma, \quad N^\text{op} = \tilde{T}_Z,
\]

where \( \tilde{T}_Z \) is as in Lemma 4.3. Now, by Lemma 4.3 we have

\[
\frac{\dim(Z)}{c\|P_Y\|\left(1 + \sqrt{\dim(Z)}\right)} < \|M\|_\gamma.
\]

This implies that \( \|M\|_\gamma = \infty \), a contradiction. Thus we must have \( M^\text{op} = 0 \). \[\square\]

We have the following direct corollary of Theorem 4.4.
Corollary 4.5. Let $X$ be an infinite-dimensional Banach space. Let $A \subseteq \mathcal{L}(X)$ be a contractible Banach subalgebra such that contains the ideal of finite-rank operators. Then for any diagonal $M$ of $A$ we have $M^{\text{op}} = 0$.

Note that any Banach algebra $A$ as in Corollary 4.5 is central and hence by Theorem 2.4 has a unique maximal ideal.

Remark 4.6. Suppose that $\mathcal{L}(X)$ is contractible. By Corollary 4.5, to prove that $X$ is finite-dimensional, it is enough to prove that at least one of the diagonals of $\mathcal{L}(X)$ is invertible as a member of the Banach algebra $\mathcal{L}(X) \otimes \mathcal{L}(X)$. Note that for the unique diagonal $M$ of $M_n$ we have $n^2 M^2 = 1 \otimes 1$ in the Banach algebra $M_n \otimes M_n$.

Remark 4.7. Suppose that $X$ is an infinite dimensional Banach space for which the canonical mapping $\Upsilon$ is one-to-one. Then, by Corollary 4.5, any Banach subalgebra of $\mathcal{L}(X)$ containing the ideal of finite-rank operators, is not contractible.

Remark 4.8. Let $X$ be an infinite dimensional Banach space. If $A = \mathcal{L}(X)$ has at least two maximal ideals, then by Corollary 2.3 we know that $A$ is not contractible. Suppose that $A$ has only one maximal ideal $J$. To prove that $A$ is not contractible it is enough to show that the closer $\tilde{J}$ of the ideal $(J \otimes A) + (A \otimes J) \subset A \otimes A$ is a maximal ideal of $A \otimes A$: Indeed, if $A$ is contractible then $A \otimes A$ is contractible and since $A \otimes A$ is central (this fact can be checked by considering projections onto finite-dimensional subspaces of $X$ similar to the first part of the proof of Lemma 4.3) then it must have a unique maximal ideal. Thus we must have $\ker(\Upsilon) \subseteq \tilde{J}$ and hence for any diagonal $M$ of $A$, $M$ belongs to $\tilde{J}$. Therefore, we have $1 = \Delta(M) \in \Delta(\tilde{J}) \subset J$ that contradicts properness of $J$.

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