Supersymmetric $D3$ brane and $\mathcal{N}=4$ SYM actions in plane wave backgrounds

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Abstract

The explicit (all-order in fermions) form of the kappa-symmetric $D3$ brane probe action was previously found in the two maximally supersymmetric type IIB vacua: flat space and $AdS_5 \times S^5$. Here we present the form of the action in the third maximally supersymmetric type IIB background: gravitational plane wave supported by constant null 5-form strength. We study $D3$ brane action in both covariant and light cone kappa symmetry gauges. Like the fundamental string action, the $D3$ brane action takes a simple form once written in the light cone kappa-symmetry gauge. We also consider the $\mathcal{N}=4$ SYM theory in 4d plane wave background. Since some (super)symmetries of plane wave SYM action are friendly to (super)symmetries of the type IIB superstring in plane wave Ramond-Ramond background we suggest this SYM model may be useful in the context of AdS/CFT duality. We develop the Hamiltonian light cone gauge formulation for this theory.
1 Introduction

Recently, new maximally supersymmetric solution of IIB supergravity with Ramond-Ramond flux was found [1]. It turns out that the light cone gauge Green-Schwarz superstring action on this background is quadratic in both bosonic and fermionic superstring 2d fields, and therefore, this model can be explicitly quantized [2]. On the other hand, in [3] it was proposed that this superstring in plane wave background corresponds to a certain (large R-charge) sector of the $\mathcal{N}=4$ SYM theory. Given that the plane wave superstring model can be quantized explicitly [2, 4] one can study the duality correspondence between string states and gauge theory operators at the string-mode level [3]. This new duality [3] (which can be understood also from the point of view of semiclassical approximation to the original AdS/CFT setting [5, 6], see for review [7]) turned out to be very fruitful. It renewed interest in various aspects of string/gauge theory correspondence and triggered many investigations both on the gauge theory and string theory sides. On the string theory side many new interesting more general solutions to IIA and IIB supergravities were found and corresponding world-sheet superstring actions were constructed [8]. Remarkably simple structure of plane wave background admits also to construct string superfield theory [9, 10, 11] in plane wave background hence giving first example of string field theory on curved background\(^1\). On the gauge theory side various matrix models techniques were developed and applied to the study of new duality [20, 21, 22].

According to the idea of string/gauge theory duality each state of string theory can be associated to some operator of gauge theory [23]. For the case of the type IIB superstring on $AdS_5 \times S^5$ Ramond-Ramond background the dual theory should be $\mathcal{N}=4$ supersymmetric Yang-Mill theory (SYM) “living” at the boundary of $AdS_5$ [24]. This boundary can be taken to be flat Minkowski space time or any space-time which is obtainable from the Minkowski space-time by a conformal transformation. All such theories have the same global conformal supersymmetries generated by the $psu(2,2|4)$ superalgebra which on the string theory side is realized as the algebra of super-isometries of the AdS+RR background. Thus in the case of the AdS string we have the same superalgebra in the bulk and at the boundary.

Now let us turn to the plane-wave version (or “limit”) of the superstring - gauge theory duality. Here we still deal with (a sector of) the $\mathcal{N}=4$ SYM, (as it is this theory that we want to study), i.e. on the gauge theory side we have again $psu(2,2|4)$ as defining superconformal algebra. In contrast to the original AdS string case, however, this superalgebra no longer coincides with the algebra of global symmetries of the plane wave background, i.e. the plane wave superstring and the $\mathcal{N}=4$ SYM have different algebras of global symmetries.\(^2\) One may argue that this is not surprising since we should restrict our attention only to a particular sector of (large R-charge) states of SYM theory on which a smaller symmetry may be acting.

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1Surprisingly the plane wave superstring field theory can be successfully used for a study of a new string/gauge theory duality [12]. Discussion of Penrose limit and plane wave duality for $AdS_5 \times T^{1,1}$ and various orbifolds like $AdS_5 \times S^5/Z_N$ may be found in [13],[14]. Non-supersymmetric $\mathcal{N}=0$ theories and theories with $\mathcal{N}=1,2$ supersymmetries were studied [15] and [16]. Plane wave duality for $ads_3 \times s^3$ and for various gauge theories in six dimensions were investigated in [17],[18]. Study of strings in RR plane wave background at finite temperature is given in [19].

2Indeed, the symmetry superalgebra of plane wave superstring is obtainable from $psu(2,2|4)$ superalgebra via a contraction procedure [25]. Interesting discussion of this contraction procedure at the level of oscillator construction may be found in [26].
Still, it is reasonable, and this is what we are going to do in this paper, to look for a subset of symmetries in the plane wave superstring and SYM theories which can be matched. Here by matching the symmetries we simply mean a coincidence of commutation relations of the symmetry generators on the string theory side and the appropriate generators on the SYM side. It seems very likely that in order to increase the number of matching symmetries as much as possible we should consider the SYM theory not in flat but in a 4d plane wave background. Another attractive feature of plane wave SYM is a discrete spectrum of the light cone energy operator. As is well known spectrum of the light cone energy operator of plane wave superstring is also discrete [4]. Therefore is it natural to expect that it is the plane wave SYM theory that is most appropriate for establishing a precise correspondence with plane wave superstring\(^3\). This is our motivation for the study of \(\mathcal{N} = 4\) SYM theory in plane wave background.

Another closely related theme of this paper is the structure of D3 brane action in plane wave Ramond-Ramond background. Our interest in the form of the D3 brane action is due to the following reasons. First, as is well known, the action for a D3 brane probe in plane wave background respects the same symmetries as the plane wave superstring, and after imposing appropriate kappa symmetry gauge and static gauges to be discussed below the quadratic part of the D3 brane action describes the abelian \(\mathcal{N} = 4\) SYM multiplet propagating in 4d plane wave background. In this quadratic approximation some of the original symmetries of the 10d plane wave become broken. In other words, the study of the D3 brane action suggests a natural way how to formulate the plane wave SYM theory.

The second reason for our interest in D3 brane action is related to the desire to apply the supercoset method developed in [28, 29] (see also [30]) to the study of the D3 brane dynamics. The low-energy action of a probe D-brane in a curved type II supergravity background is given by a superspace generalization of the sum of the Born-Infeld action and a Wess-Zumino type term (similar in form to the curved superspace version of the Green-Schwarz string action [31]). While the formal expression for such an action in a generic on-shell type II supergravity background is known [32, 33, 34], its explicit component form is hard to determine (explicit solution of superspace constraints is not known in general). Resorting to expansion in powers of fermions it is cumbersome to go much far beyond terms quadratic in the fermions (see [35, 36]). There are, however, a few exceptional cases where there is a lot of symmetry allowing one to explicitly determine the full structure of the supervielbeins and field strengths, and thus the form of the D-brane actions to all orders in the fermions. The two previously studied cases are the maximally supersymmetric flat space [33] and the AdS\(_5\) × S\(_5\) background [29]. Here we shall find the explicit form of the D3 brane action\(^4\) in the third maximally supersymmetric type IIB background - the symmetric gravitational wave supported by the constant null 5-form strength [1]. Note that alternative way to derive plane wave D3 brane action is to consider AdS supersymmetric D3 brane action [29] in the Penrose limit. Discussion of this approach for the bosonic part of various Dp brane actions may be found in [37]. We do not use this approach for the study of supersymmetric D3 brane action because it

\(^3\)4d plane wave background can be obtained via Penrose limit from the space-time \(R \times S^3\) which is used for a study of AdS holography. Therefore one can expect that the 4d plane wave background is most appropriate for the study holographical issues of the new duality. Various discussions of plane wave holography may be found in [27].

\(^4\)We shall consider primarily the D3 brane action though similar actions can be found for all other Dp branes of type IIB theory.
does not make explicit use of the basic (super)symmetries of the problem.

As in the case of the $D3$ brane in $AdS_5 \times S^5$ [29], we shall find the explicit form of the $D3$ brane action using the supercoset approach [28]. It was already used in [2] in order to find the complete form of the fundamental GS string in the plane wave background.\(^5\)

We shall then see that like the fundamental string action [2], the $D3$ brane action takes quite simple form when written in the light cone kappa symmetry gauge.\(^6\)

We shall first determine the form of the action for a generic embedding of a $D3$ brane in the plane wave background (i.e. we shall not make a particular choice of a static gauge). Then we will be able to see explicitly in principle which $D3$ brane orientations preserve some parts of supersymmetry (following, e.g., the general approach of [45] and the analysis [46] of the BPS brane states in the $AdS_5 \times S^5$ case). A related discussion of supersymmetry of particular $Dp$ brane probe orientations in plane wave background appeared in [47, 48].\(^7\)

In Section 2 we find supersymmetric and $\kappa$ invariant action of $D3$ brane in plane wave Ramond-Ramond background. Most of this section follows closely the same strategy as was used in the $AdS_5 \times S^5$ case in [29].

In Section 3 we study covariant $\kappa$ symmetry gauge and static gauge fixed $D3$ brane action. We discuss broken and unbroken symmetries of such brane action.

In Section 4 we discuss light cone kappa symmetry gauge fixed $D3$ brane action.

In Section 5 we study $psu(2,2|4)$ superalgebra in various bases. We introduce notion of plane wave basis of this superalgebra and discuss interrelation of this basis with the conventional Lorentz basis.

In Section 6 we discuss the covariant and gauge invariant formulation of $\mathcal{N} = 4$ SYM in plane wave background. We study realization of 32 plane wave supersymmetries of the $psu(2,2|4)$ superalgebra in the covariant formulation of SYM theory.

In Section 7 we study the Hamiltonian light cone gauge formulation of plane wave SYM. We demonstrate that in contrast to covariant and gauge invariant formulation the 3-point and 4-point light cone gauge vertices of plane wave SYM theory takes exactly the same form as the ones of SYM in flat Minkowski space-time.

Section 8 is devoted to a study of realization of global symmetries of $psu(2,2|4)$ superalgebra on the physical fields. We find field theoretical realization of Noether (super)charges as generators of the $psu(2,2|4)$ superalgebra.

In Section 9 we discuss transformation rules of physical fields in the framework of light cone formulation.

Section 10 summarizes our conclusions and suggests directions for future research.

Our notation and conventions are explained in Appendices A and B. Appendix C

\(^5\)The supercoset construction turned out to be very effective and fruitful and was used to construct $AdS(3)$ and $AdS(2)$ superstring actions [38], [39], $D1$ and $D5$ brane actions in various super $AdS$+RR backgrounds [40], and various $AdS$ supermembrane actions [41]-[43]. Dual $D3$ $AdS$ brane action in the framework of supercoset construction were discussed in [44].

\(^6\)Our expression for the $D3$ brane action in the fermionic light cone gauge may be of interest also in the flat-space limit, as it has simpler structure than the corresponding covariant kappa symmetry gauge action in [33].

\(^7\)Microscopic open-string approaches to construction of supersymmetric $Dp$ branes in the plane wave background were developed in [49] (see also [50] for the corresponding discussion of branes in 11d case). Study of $Dp$ brane interactions in the framework of microscopic approach may be found in [51]. Derivation of conformal operators which are dual to open strings states ending on $D5$ branes may be found in [52]. Interesting discussion of realization of $Dp$ brane dynamics in plane wave background is given in [53].
contains some basic relations for Cartan 1-forms on the coset superspace. Appendix D contains some additional details about transformation rules of physical fields of plane wave SYM and generalization to plane wave massless higher spin fields.

2 General form of D3 brane action

The D3 brane action depends on the coset superspace coordinates \( X = (x^\alpha, \Theta) \) and vector field strength \( F_{ab} = \partial_a A_b - \partial_b A_a \). As in [32, 33, 34], it is given by the sum of the BI and WZ terms

\[
S = \int d^4 \sigma \mathcal{L}, \quad \mathcal{L} = \mathcal{L}_{BI} + \mathcal{L}_{WZ},
\]

where (we set the 3-brane tension to be 1)

\[
\mathcal{L}_{BI} = -\sqrt{-\det(G_{ab} + F_{ab})},
\]

\[
\mathcal{L}_{WZ} = d^{-1} H_5.
\]

The induced world-volume metric \( G_{ab} \) is \((a, b) = (0, 1, 2, 9)\)

\[
G_{ab} = L^\mu_a L^\mu_b, \quad L^\mu (X(\sigma)) = d\sigma^a L^\mu_a,
\]

where \( L^\mu \) are Cartan 1-forms (see Appendix C for definition of Cartan 1-forms). The supersymmetric extension \( F = \frac{1}{2} F_{ab} d\sigma^a \wedge d\sigma^b \) of the world-volume gauge field strength 2-form \( dA \) is found to be

\[
F = dA + 2i \int_0^1 dt \ L^\mu_t \wedge \Theta \gamma^\mu \tau_3 L_t,
\]

where \( L^\mu_t(x, \Theta) \equiv L^\mu(x, t\Theta), \quad L^\alpha_t(x, \Theta) \equiv L^\alpha(x, t\Theta) \) (we suppressed the spinor indices \( \alpha \) in (2.5)). The \( \Theta \)-dependent correction term in (2.5) given by the integral over the auxiliary parameter \( t \) is exactly the same 2-form as in the string action [2] (see also [28, 29]). This representation corresponds to the specific choice of coset representative made above. Note that while \( F \) is not expressible in terms of Cartan forms only, its exterior derivative is

\[
dF = i L \wedge \hat{L} \wedge \tau_3 L, \quad \hat{L}_{\alpha \beta} \equiv L^\mu_{\alpha \beta} \gamma^\mu.
\]

This important formula can be proved my making use of the Maurer-Cartan equations (C.4) – (C.5) and equations (C.13)–(C.16) from Appendix C. As a result, \( dF \) is manifestly invariant under supersymmetry, and then so is \( F \), provided one defines appropriately the transformation of \( A \) to cancel the exact variation of the second (string WZ) term in (2.5) (cf. [32, 33]).

As in flat space [32, 33], the super-invariance of \( S_{WZ} \) follows from supersymmetry of the closed 5-form \( H_5 \). We shall determine the supersymmetric \( H_5 \) from the requirement of

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8For fermionic coordinates we assume the convention for hermitian conjugation and permutations rules \((\theta_1 \theta_2)^\dagger = \theta_2^\dagger \theta_1^\dagger, \quad \theta_1 \theta_2 = -\theta_2 \theta_1\), while for fermionic Cartan 1-forms we adopt \((L_1 \wedge L_2)^\dagger = -L_2^\dagger \wedge L_1^\dagger, \quad L_1 \wedge L_2 = L_2 \wedge L_1\). For bosonic Cartan 1-forms we adopt the convention \( B_1 \wedge B_2 = -B_2 \wedge B_1\).

9Discussion of alternative representation for WZ part of GS action may be found in [54]. Various alternative covariant formulations of plane wave superstring action are given in [55].

10Note that this is the same transformation that is needed to make the superstring action [2] defined on a disc and coupled to \( A \) at the boundary invariant under supersymmetry.
**κ-symmetry of the full action $S$ which fixes this 5-form uniquely. The $\kappa$-transformations are defined by (see (C.9))**

\[
\hat{\delta}_\kappa x^\mu = 0, \quad \hat{\delta}_\kappa \Theta = K,
\]

where the transformation parameter satisfies the constraint

\[
\Gamma K = K, \quad \Gamma^2 = 1.
\]

Here $\Gamma$ is given by

\[
\Gamma = \epsilon_{a_1 \ldots a_4} \sqrt{-\det(G_{ab} + F_{ab})} \left( \frac{1}{4!} \gamma_{a_1 \ldots a_4} \tau_2 + \frac{1}{4} \gamma_{a_1 a_2} F_{a_3 a_4} \tau_1 + \frac{1}{8} F_{a_1 a_2} F_{a_3 a_4} \tau_2 \right),
\]

and we use the notation

\[
\gamma_{a_1 \ldots a_n} \equiv \hat{L}_{[a_1} \ldots \hat{L}_{a_n]}, \quad \hat{L}_a \equiv L_a^\mu \gamma^\mu.
\]

The corresponding variation of the metric $G_{ab}$ is

\[
\delta_\kappa G_{ab} = -2i \hat{\delta}_\kappa \Theta \hat{L}_a \hat{L}_b + \hat{L}_b \hat{L}_a,
\]

while the variation of $F$ is given by

\[
\delta_\kappa F = 2i \hat{\delta}_\kappa \Theta \hat{L}_a \tau_3 L_a - \hat{L}_a \tau_3 L_a,
\]

Then the $D3$ brane action $S$ in (3.1) is $\kappa$-invariant provided the 5-form $H_5$ is given by

\[
H_5 = i \Theta \wedge \left( \frac{1}{6} \hat{L} \wedge \hat{L} \wedge \hat{L} \wedge \tau_3 L + F \wedge \hat{L} \tau_1 \right) \wedge L
\]

\[
+ \frac{f}{6} \left( \epsilon^{i_1 \ldots i_4} L^+ \wedge L^{i_1} \wedge \ldots \wedge L^{i_4} + \epsilon^{i_1 \ldots i_4} L^+ \wedge L^{i_1} \wedge \ldots \wedge L^{i_4} \right).
\]

It is possible to check using Maurer-Cartan equations (see Appendix C) and well-known Fierz identity\(^{11}\)

\[
L \wedge \tilde{\gamma}^\mu \tau_2 L \wedge L \wedge \tilde{\gamma}^\nu L = 2L \wedge \tilde{\gamma}^\mu \tau_1 L \wedge L \wedge \tilde{\gamma}^\nu \tau_3 L
\]

that $H_5$ is closed, i.e. the equation (2.3) is consistent and thus determines $L_{\text{WZ}}$.

The important fact is that $H_5$ is expressed in terms of the Cartan 1-forms and super-invariant $F$ only. This implies that $H_5$ is invariant under space-time supersymmetry. Then from (2.3) we conclude that $\delta_{\text{susy}}(d^{-1}H_5)$ is exact, so that the WZ term (2.3), like the BI term (2.2), is supersymmetry-invariant.

To put the fermionic part of the WZ term in the action in a more explicit form let us make a rescaling $\Theta \rightarrow t \Theta$ and define

\[
H_{5t} \equiv H_5|_{\Theta \rightarrow t \Theta}, \quad F_t \equiv F|_{\Theta \rightarrow t \Theta}.
\]

Since $L(x, tt' \Theta) = L_t(x, t' \Theta) = L_{tt'}(x, \Theta)$ one can show that (cf. (2.5),(2.6))

\[
F_t = dA + 2i \int_0^t dt' \Theta \hat{L}_{t'} \wedge \tau_3 L_{t'}, \quad \partial_t F_t = 2i \Theta \hat{L}_t \wedge \tau_3 L_t.
\]

\(^{11}\)Throughout this paper symmetrization and anti-symmetrization rules are defined with normalization $(ab) = \frac{1}{2}(ab + ba)$, $[ab] = \frac{1}{2}(ab - ba)$. 
Then using the defining equations for the Cartan 1-forms (C.13)-(C.16) one finds from (2.13) the following differential equation

\[ \partial_t H_{5t} = d \left[ 2i \left( \frac{1}{6} \Theta \hat{L}_{t} \wedge \hat{L}_{t} \wedge \hat{L}_{t} \wedge \tau_2 \mathbf{L}_{t} + \Theta \hat{L}_{t} \wedge F_{t} \wedge \tau_1 \mathbf{L}_{t} \right) \right], \]  

(2.17)

which determines the \( \Theta \)-dependence of \( H_{5t} \). With the initial condition

\[ (H_{5t})_{t=0} = H_5|_{\Theta=0} = H_5^{(base)} = \frac{f}{6} e^+ \wedge \left( e^{i_1 \cdots i_4} e^{i_1} \wedge \cdots \wedge e^{i_4} + e^{i_1 \cdots i_4} e^{i_1} \wedge \cdots \wedge e^{i_4} \right), \]

(2.18)

where \( e^\mu \) are (pull-backs of) the vielbein forms of plane wave background. The explicit form of the \( \Theta \)-independent part \( L_{WZ}^{(base)} = d^{-1} H_{5}^{(base)} \) depends on a particular choice of coordinates on plane wave background. Thus the \( L_{WZ} \) term in (2.3) can be written as

\[ L_{WZ} = 2i \int_0^1 dt \left( \frac{1}{6} \Theta \hat{L}_{t} \wedge \hat{L}_{t} \wedge \tau_2 \mathbf{L}_{t} + \Theta \hat{L}_{t} \wedge F_{t} \wedge \tau_1 \mathbf{L}_{t} \right) + L_{WZ}^{(base)}. \]

(2.19)

Using (2.16) and expansion of Cartan 1-forms in terms of \( \Theta \) (see Appendix C and [2]) one can then find the expansion of \( L_{WZ} \) in powers of \( \Theta \).

The only non-trivial background fields in plane wave vacuum are the space-time metric and the self-dual RR 5-form. The bosonic parts of the last two terms in \( H_5 \) (2.13) represent, indeed, the standard bosonic couplings of the \( D3 \) brane to the 5-form background. The action we have obtained contains also the fermionic terms required to make this coupling supersymmetric and \( \kappa \)-invariant.

We started with the BI action expressed in terms of the Cartan 1-forms and the 2-form in (2.5) [2] as implied by the structure of the plane wave space or the basic symmetry superalgebra. We then fixed the form of \( H_5 \) from the requirement of \( \kappa \)-symmetry of the full action. As in the AdS_5 \times S^5 case [29], the fact that we have reproduced the bosonic part of the self-dual 5-form is in agreement with the result of [32, 34] that the \( D3 \) brane action is \( \kappa \)-symmetric only in a background which is a solution of type IIB supergravity.

Let us briefly discuss global plane wave supersymmetries and \( \kappa \) symmetries of \( D3 \) brane action. The supersymmetry transformations of (super)coordinates brane fields \( x^\mathbf{L} \), \( \Theta \), \( A = A_\mu d\sigma^\mu \) can be presented as Taylor series expansion in fermionic field \( \Theta \) which terminates at terms \( \Theta^{32} \) in general. The leading terms of this expansion we will need below are fixed to be

\[ \delta_{\text{susy}} x^\mathbf{L} = i e^\mathbf{L}_\mu (\delta_{\text{susy}} \Theta) \bar{\gamma}^\mu \Theta + O(\Theta^3), \]

(2.20)

\[ \delta_{\text{susy}} \Theta = \epsilon(x) + O(\Theta^2), \]

(2.21)

\[ \delta_{\text{susy}} A = i e^\mu (\delta_{\text{susy}} \Theta) \bar{\gamma}^\mu \tau_3 \Theta + O(\Theta^3), \]

(2.22)

where the Killing spinor \( \epsilon(x) \) is given by

\[ \epsilon(x) = U \epsilon_0, \quad U = \exp \left( -\frac{f}{2} \nu^I \Pi \gamma^+ \bar{\gamma}^I \tau_2 \right) \exp \left( -\frac{f}{2} x^+ \Pi \gamma^+ \bar{\gamma}^+ \tau_2 \right). \]

(2.23)

The supersymmetry transformations for (super)coordinates \( x^\mathbf{L} \) and \( \Theta \) (2.20),(2.21) are fixed via standard coset construction. The supersymmetry transformations for brane world-volume field \( A \) (2.22) is chosen then so that the generalized field strength \( F \) (2.5) be invariant with respect to supersymmetry transformations

\[ \delta_{\text{susy}} F = 0. \]

(2.24)
The \( \kappa \) transformations given in \((2.7)\) can be expressed in terms of conventional variation of super(coordinates) \( \delta_{\kappa}x^\nu \) and \( \delta_{\kappa}\Theta \) by using formulas \((C.9)\) and the expressions for Cartan 1-forms given in \((C.17)\)

\[
\delta_{\kappa}x^\nu = -ie^\mu_H(\delta_{\kappa}\Theta)\bar{\gamma}^\mu\Theta + O(\Theta^3), \tag{2.25}
\]

\[
\delta_{\kappa}\Theta = K + O(\Theta^2), \tag{2.26}
\]

\[
\delta_{\kappa}A = -ie^\mu(\delta_{\kappa}\Theta)\bar{\gamma}^\mu\tau_3\Theta + O(\Theta^3). \tag{2.27}
\]

Here the kappa-transformation of world-volume field \( A \) is chosen so that the kappa-transformation of generalized field strength \( F \) \((2.5)\) takes the form given in \((2.12)\).

3 Gauge fixed \( D3 \) brane action

\( D3 \) brane action can be simplified by fixing local fermionic kappa and world-volume diffeomorphism symmetries. Various possibilities to fix these symmetries can be divided into two classes - covariant and noncovariant gauges. In this Section we find the form of the \( D3 \) brane action in the plane wave background with R-R 5-form flux in the covariant gauge. Our discussion of the covariant gauge fixing closely repeats the same steps as in ref.[33], where the case of flat space-time was treated.

In flat space the \( D3 \) brane gauge fixing procedure consists of the two stages:

(I) fermionic covariant gauge choice, i.e., fixing the \( \kappa \)-symmetry by \( \theta^1 = 0 \)

(II) bosonic covariant gauge choice, i.e., fixing the world-volume diffeomorphism symmetry by \( x^a(\sigma) = \sigma^{a12} \).

3.1 \( \kappa \) symmetry covariant gauge fixed action

Our fermionic \( \kappa \)-symmetry covariant gauge is the same as in flat \( D3 \) brane \( \theta^1 = 0 \). One usually imposes the \( \kappa \)-symmetry covariant gauge by starting with the explicit representation for the \( D3 \) brane Lagrangian in terms of \( \theta^s \). However it is convenient to first impose the covariant gauge at the level of the Cartan forms \( L^\mu, L^a \) and then to use them in \((2.1)\). In what follows we adopt this strategy.\(^{13}\)

By applying argument similar to the ones in [33] we impose the following covariant kappa symmetry gauge

\[
\theta^1 = 0, \quad \theta^2 = \lambda, \quad i.e. \quad \Theta = \begin{pmatrix} 0 \\ \lambda \end{pmatrix} \tag{3.1}
\]

Because the \( D3 \) brane action \((2.1)\) is expressible in terms of Cartan 1-forms we should simply to evaluate the Cartan 1-forms in this gauge. This can be done straightforwardly by using representation for the Cartan 1-forms given in Appendix C (see \((C.17)-(C.19)\)) and plugging there the expression for \( \Theta \) given in \((3.1)\). This leads to the following Cartan 1-forms

\[
\begin{align*}
L &= \frac{\sinh m}{m} D\Theta, \\
L^\mu &= e^\mu - 2i\Theta \bar{\gamma}^\mu \cosh m - \frac{1}{m^2} D\Theta,
\end{align*} \tag{3.2}
\]

\(^{12}\)Discussions of alternative covariant bosonic gauge choices may be found in [56].

\(^{13}\)This strategy was first successfully used in [57],[30] while deriving \( \kappa \)-symmetry fixed action for long superstring in \( AdS_5 \times S^5 \) and for \( \kappa \)-symmetry fixed light cone \( AdS \) superstring in [4]. Application of this method to plane wave superstring may be found in [2].
where covariant derivative is simplified as compared with (C.18) and is given by
\[
\mathcal{D}\Theta = \begin{pmatrix}
\frac{\tau}{2}\epsilon^\mu\Pi\gamma^+\bar{\gamma}^\mu\lambda \\
D^\lambda
\end{pmatrix}, \quad D^\lambda \equiv d + \frac{1}{4}\omega^{\mu\nu}\gamma^{\mu\nu},
\tag{3.3}
\]
while the matrix \(\mathbf{m}\) is
\[
\mathbf{m}^2 = \begin{pmatrix}
0 & -\mathrm{if}(\Pi\gamma^+\bar{\gamma}^\mu\lambda)^\alpha(\lambda\bar{\gamma}^\nu)_\beta \\
\mathrm{if}(\gamma^+\bar{\gamma}^\mu\lambda)^\alpha(\lambda\bar{\gamma}^\nu)_\beta & 0
\end{pmatrix}.
\tag{3.4}
\]
In contrast to the matrix \(\mathcal{M}\) (C.19) which enters definition of general Cartan 1-form the gauge fixed matrix \(\mathbf{m}\) turns out to be off-diagonal. This considerably simplify structure of gauge fixed action. In order to enter definition of action we evaluate gauge fixed 2-form \(F\) which is given by \(F = F^{(2)}\)
\[
F = dA - 2i\int_0^t dt' \lambda\hat{L}_{t'} \land L^2_{t'},
\tag{3.5}
\]
where \(L^2\) is the second component of \(L\) (see (3.2),(C.2)). The expressions above given define BI part the action (2.2). Gauge fixed WZ part of the action can be obtained from (2.19)
\[
\mathcal{L}_{\text{WZ}} = 2i\int_0^1 dt \left( -\frac{1}{6}\lambda \hat{L}_t \land \hat{L}_{t'} L_t^2 + \lambda \hat{L}_t \land F_t \land L_t^2 \right) + \mathcal{L}_{\text{WZ}}^{(\text{bose})}.
\tag{3.6}
\]
Above given expression provides possibility to find expansion of \(D3\) brane action in terms of fermionic field. Note however that though the gauge fixed action is much more simpler than the covariant one there is no natural way to integrate out parameter \(t\) in expressions for \(F_t\) and \(\mathcal{L}_{\text{WZ}}\) to bring action to completely explicit form\(^{14}\). Total Lagrangian can presented as Taylor series in field strength \(F_{ab}\) and the fermionic field \(\lambda\)
\[
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + O(F^3\lambda^2,F\lambda^4,\lambda^6),
\tag{3.7}
\]
where
\[
\mathcal{L}_0 = -\sqrt{g} + \mathcal{L}_{\text{WZ}}^{(\text{bose})},
\tag{3.8}
\]
\[
\mathcal{L}_2 = \sqrt{g}\left( -\frac{1}{4}F_{ab}F_{ab} + i\epsilon^{\mu\lambda}\lambda\bar{\gamma}^\mu D_d^\lambda \right) - \frac{\mathrm{if}}{12}e^{abcd}e^\mu_\alpha e^\nu_\beta e^\rho_\gamma e^\sigma_\delta (\lambda\bar{\gamma}^{\mu\rho\gamma\delta} \Pi \gamma^+ \bar{\gamma}^\sigma \lambda),
\tag{3.9}
\]
\[
\mathcal{L}_3 = i\sqrt{g}F_{ab}e^\mu_\alpha \lambda\bar{\gamma}^\mu D_b^\lambda + \frac{\mathrm{if}}{4}e^{abcd}F_{ab}e^\mu_\alpha (\lambda\bar{\gamma}^\mu \Pi \gamma^+ \bar{\gamma}^\nu \lambda),
\tag{3.10}
\]
\[
\mathcal{L}_4 = \sqrt{g}\left( \frac{1}{8}F_{ab}F_{bc}F_{cd}F_{da} - \frac{1}{32}(F_{ab}F_{ab})^2 \right)
- iF^{abc}F_{c}^\alpha e^\mu_\alpha \lambda\bar{\gamma}^\mu D_b^\lambda + \frac{1}{4}F_{bc}^{\alpha\beta}e^{\mu\lambda}\lambda\bar{\gamma}^\mu D_a^\lambda
+ \frac{1}{2}\lambda\bar{\gamma}^\mu D_b^\alpha \lambda\lambda\bar{\gamma}^\nu D_a^\lambda + \frac{1}{2}(\epsilon^{\mu\lambda}e^\mu_\alpha \lambda\bar{\gamma}^\mu D_a^\lambda)^2 - e^\mu_\alpha e^\nu_\beta \lambda\bar{\gamma}^\mu D_b^\lambda \lambda\bar{\gamma}^\nu D_a^\lambda
+ f_{eabcd}e^\mu_\alpha e^\nu_\beta e^\rho_\gamma e^\sigma_\delta \left( \frac{1}{8}e^\sigma_\delta \lambda\bar{\gamma}^\rho D_b^\lambda - \frac{1}{72}e^\rho_\delta \lambda\bar{\gamma}^\sigma D_d^\alpha \right)(\lambda\bar{\gamma}^{\mu\rho\gamma\delta} \Pi \gamma^+ \bar{\gamma}^\sigma \lambda)
+ \frac{f}{4}e^{abcd}e^\mu_\alpha e^\nu_\beta e^\rho_\gamma e^\sigma_\delta \lambda\bar{\gamma}^\rho D_d^\lambda (\lambda\bar{\gamma}^{\mu\rho\gamma\delta} \Pi \gamma^+ \bar{\gamma}^\sigma \lambda) + \sqrt{g}\frac{\tau^2}{24} \lambda\bar{\gamma}^+ \mu \lambda \lambda\bar{\gamma}^{+\mu} \lambda.
\tag{3.11}
\]
\(^{14}\)This minor problem can be by passed in light cone gauge.
In these formulas $g_{ab}$ is a bosonic body of the induced world-volume metric $G_{ab}$ (2.4)
\[
g_{ab} = e^\mu_a e^\mu_b, \quad e^\mu = d\sigma^\mu e^\mu_a, \quad g \equiv -\text{det} g_{ab}. \tag{3.12}
\]
The above given representation for $D3$ brane action is valid for arbitrary coordinate on plane wave background. The explicit form of the bosonic body of WZ part of the action, $\mathcal{L}^{\text{bose}}_{\text{WZ}}$, depends on a particular choice of coordinates on plane wave background. We chose the coordinate frame in which vielbeins $e^\mu$ take the form
\[
e^+ = dx^+, \quad e^I = dx^I, \quad e^- = dx^- - \frac{f^2}{2} x^I x^J dx^I, \tag{3.13}
\]
For this particular choice of coordinates we get the following representation for $\mathcal{L}^{\text{bose}}_{\text{WZ}}$
\[
\mathcal{L}^{\text{bose}}_{\text{WZ}} = -f^6 \epsilon^{abcd} \partial_a x^+ \partial_b x^- + (\epsilon^{ijkl} x^i \partial_b x^j \partial_c x^k \partial_d x^l + \epsilon^{ij'}k'v' x^i \partial_b x^{j'} \partial_c x^{k'} \partial_d x^{l'}), \tag{3.14}
\]
while the induced metric tensor takes the form
\[
g_{ab} \equiv 2\partial_{(a} x^+ \partial_{b)} x^- - f^2 x^I x^J \partial_a x^+ \partial_b x^- + \partial_a x^I \partial_b x^I. \tag{3.15}
\]
In what follows we assume this particular choice of the coordinates. We normalize the Levi-Civita symbols to be $\epsilon^{0129} = \epsilon^{1234} = \epsilon^{5678} = 1$. Note that as $f \to 0$ the expansion for $\mathcal{L}$ shown in (3.8)-(3.11) reduces to the one in the flat space time (see [33],[58]).

### 3.2 Static gauge fixed $D3$ brane action and its symmetries

Our static gauge being the same as in flat space leads to the action of self-interacting abelian $N = 4$ SYM in $4d$ plane wave background. In this section we study realization of bosonic symmetries of static gauge fixed brane action. We demonstrate that only isometry symmetries of $4d$ plane wave background plus R-symmetries $SO(2) \times SO'(4)$ are realized linearly while the remaining bosonic symmetries are realized non-linearly. In this respect the situation is similar to the one in flat space. Because number of isometry symmetries in $4d$ plane wave background is equal to seven we note that number of linearly realized space-time symmetries is equal to number of linearly realized R-symmetries\footnote{This is not the case in flat space. After imposing the static gauge in the $D3$ brane action the Poincaré symmetries of ten-dimensional Minkowski space-time reduces to the 10 linearly realized Poincaré symmetries of $4d$ Minkowski space, 15 linearly realized R-symmetries of $SO(6)$ plus certain non-linearly realized symmetries. Thus in the flat space the number of linearly realized space-time symmetries is not equal to the number of linearly realized R-symmetries.}.\n
Making field redefinitions and introducing conventional notation for six scalar fields
\[
x^m(\sigma) \to -x^\underline{m}(\sigma), \quad \phi^M(\sigma) \equiv -x^M(\sigma), \quad m = 0, 1, 2, 9; \quad M = 3, \ldots, 8, \tag{3.16}
\]
we impose the standard static gauge
\[
x^m(\sigma) = \delta^m_a \sigma^a, \quad a = 0, 1, 2, 9, \tag{3.17}
\]
where $\delta^m_a = 1(0)$ for $\underline{m} = a(\underline{m} \neq a)$. Introducing light cone coordinates on $D3$ brane
\begin{align}
\sigma^\pm &= \frac{1}{\sqrt{2}}(\sigma^9 \pm \sigma^9), \quad \sigma^i, \quad \hat{i} = 1, 2, \\
\end{align} 

and inserting the static gauge into (3.15) we get the gauge fixed induced metric\(^{16}\)

\begin{equation}
g_{m\bar{n}} = g_{m\bar{n}}^{(pw)} - f^2 \phi^M \phi^M \delta^+ \delta^+ + \partial_m \phi^M \partial_n \phi^M, 
\end{equation}

where \(g_{m\bar{n}}^{(pw)}\) is a metric tensor of 4d plane wave background

\begin{equation}
g_{m\bar{n}}^{(pw)} d\sigma^m d\sigma^n = 2 d\sigma^+ d\sigma^- - f^2 \delta^i \delta^i d\sigma^+ + d\sigma^i d\sigma^i. 
\end{equation}

Taking into account the formula (3.19) we see that if we will treat \(g^{(pw)}\) as background field and make Taylor series expansion with respect to remaining terms, \(f^2 \delta^2\) and \((\partial \phi)^2\), then the static gauge fixed D3 brane action will be manifestly invariant with respect to isometry symmetries of 4d plane wave background. It is natural to expect then that these isometry symmetries are realized linearly. To demonstrate this point explicitly we consider transformations of brane fields with respect to original plane wave symmetries in ten dimensions. This is to say we start with global plane wave transformations supplemented with local diffeomorphism transformation

\begin{equation}
\delta x^\mu = \xi^G G^\mu + \epsilon^a \partial_a x^\mu,
\end{equation}

where \(\xi^G G^\mu\) is Killing vector corresponding to plane wave global transformation generated by element of algebra denoted by \(G\). Representation of the Killing vectors in term of differential operators \(G = \xi^G G^\mu \partial_\mu\) is given in (B.10)-(B.13). From the requirement that the transformation (3.21) maintains the static gauge (3.17), \(\delta x^m = 0\), we fix parameter of compensating transformation

\begin{equation}
\epsilon^m = -\xi^G m. \end{equation}

Plugging this into (3.21) we get transformation rules of six scalar fields

\begin{equation}
\delta G x^M = \xi^G M - \xi^G m \partial_m x^M.
\end{equation}

Making use of concrete representation for Killing vectors plane wave background (B.10)-(B.13) we get the following transformations with respect to transverse translations and Lorentz boosts

\begin{align}
\delta_{(a^I P^I)} x^M &= \cos f x^+ a^M - \left(\cos f x^+ a^I \partial_i + f \sin f x^+ (a^I x^I) \partial^+\right) x^M, \\
\delta_{(b^I J^I)M} &= \sin f x^+ b^M - \left(\frac{\sin f x^+}{f} b^I \partial_i - \cos f x^+ (b^I x^I) \partial^+\right) x^M,
\end{align}

where \(a^I\) and \(b^I\) are parameters of the appropriate transformations. Plugging the static gauge (3.17) into transformation rules (3.24),(3.25) we see that the transformations generated by \(P^i\) and \(J^i\)

\(^{16}\)After choice of static gauge the indices \(m, n\) are used for target space vectors while indices \(a, b\) are used for tangent space vectors. These vectors are related as \(A^M = e^m_a A^a\), where \(e^m_a\) is inverse to the basis of one forms \(e^a = e^m_a dx^m, e^m_a e^b_m = \delta^a_b\). The basis of \(e^a\) is specified to be \(e^+ = dx^+, e^i = dx^i, e^- = dx^- - (f^2/2) \sigma^i \sigma^i dx^+\).
\[ \delta_{P^i} \phi^M = (\cos f \sigma^\dagger \partial_i + f \sin f \sigma^\dagger \sigma^\dagger \partial^+ \phi^M, \] (3.26)
\[ \delta_{J^{i\dagger}} \phi^M = (\frac{\sin f \sigma^\dagger}{f} \partial_i - \cos f \sigma^\dagger \sigma^\dagger \partial^+ \phi^M, \] (3.27)

are realized linearly and coincide with transformation of isometry algebra of 4d plane wave background. It is easy to check that transformations generated by translations in light cone directions \( P^\pm \) and \( SO(2) \times SO(2) \times SO'(4) \) rotations generated by \( J^{12}, J^{34}, J^{i\dagger j} \) which are obtainable from (3.21), are also realized linearly. The generators \( P^\pm, P^i, J^{i\dagger} \) and \( J^{12} \) form algebra of isometry symmetries of 4d plane wave space-time while the generators \( J^{34}, J^{i\dagger j} \) are responsible for R-symmetries, which are \( SO(2) \times SO'(4) \) rotations.

The transformations of remaining six translations \( P^N \) and six Lorentz boosts \( J^{+N} \) take the form
\[ \delta_{a^N} P_N \phi^M = \cos f \sigma^\dagger a^M - f \sin f \sigma^\dagger a^N \phi^N \partial^+ \phi^M, \] (3.28)
\[ \delta_{b^N} J^{+N} \phi^M = \frac{\sin f \sigma^\dagger}{f} b^M + \cos f \sigma^\dagger b^N \phi^N \partial^+ \phi^M, \] (3.29)

and these transformations are obviously broken and realized non-linearly.

Broken and unbroken (super)symmetries of kappa symmetry and static gauge fixed D3 brane action are collected in Table 1.

TABLE 1: Broken and unbroken (super)symmetries of kappa symmetry and static gauge fixed D3 brane action

| Generators of plane wave superalgebra | Generators of unbroken symmetries | Generators of broken symmetries |
|---------------------------------------|-----------------------------------|---------------------------------|
| \( P^+, P^- \)                        | \( P^+, P^- \)                    | \( P^M, M = 3, \ldots, 8 \)    |
| \( P^I, I = 1, \ldots, 8 \)           | \( P^\dagger, \dagger i = 1, 2 \)| \( J^{+M}, M = 3, \ldots, 8 \) |
| \( J^{+I}, I = 1, \ldots, 8 \)        | \( J^{+\dagger}, \dagger i = 1, 2 \)|                                |
| \( J^{ij}, i, j = 1, \ldots, 4 \)     | \( J^{12}, J^{34} \)              | \( J^{13}, J^{14}, J^{23}, J^{24} \) |
| \( J^{i\dagger j}, i\dagger, j = 5, \ldots, 8 \) | \( J^{i\dagger j}, i\dagger, j = 5, \ldots, 8 \) |                          |
| \( Q^{1}_\alpha, Q^{2}_\alpha, \alpha = 1, \ldots, 16 \) | \( Q^1 \zeta_0 + Q^2 \)         | \( Q^1 \zeta_0 - Q^2 \)        |

It is instructive to find commutation relations of generators of unbroken symmetries. These commutation relations can be obtained from the ones of the plane wave superalgebra given in (B.1)-(B.8). Bosonic generators form symmetries of 4d plane wave space-time and R-symmetries. All that is required is to find (anti)commutators involving unbroken supercharge
\[ Q = \frac{1}{\sqrt{2}}(Q^1 \zeta_0 + Q^2), \quad \zeta_0 \equiv \gamma^{-12}. \] (3.30)

11
Commutation relations of bosonic generators with this supercharge take the form

\[ [J^{ij}, Q_\alpha^\pm] = \frac{1}{2} Q_\beta^\pm (\gamma^{ij})^\beta_\alpha, \quad [J^{i'i'}, Q_\alpha] = \frac{1}{2} Q_\beta^\pm (\gamma^{i'i'})^\beta_\alpha, \quad [J^{+i}, Q_\alpha] = \frac{1}{2} Q_\beta^+(\gamma^{+i})^\beta_\alpha, \]

\[ [P^a, Q_\alpha] = -\frac{f}{2} Q_\beta (\gamma^{+34a})^\beta_\alpha, \tag{3.31} \]

where we have to keep just the \( \text{so}(2) \oplus \text{so}(2) \) part of \( J^{ij} \) given by \( J^{12}, J^{34} \). The anticommutator of supercharges corresponding to unbroken supersymmetries is given by

\[ \{Q_\alpha, Q_\beta\} = -2i \gamma^{+\alpha_\beta} P^a + 2i f \gamma^{+\alpha_\beta} J^{+i} \]

\[ + 2i f (\gamma^{+i})^\alpha_\beta J^{12} - 2i f (\gamma^{+i})^\alpha_\beta J^{34} - if (\gamma^{+i})^\alpha_\beta J^{i'i'}. \tag{3.32} \]

These (anti)commutation relations tell us that generators of unbroken symmetries form some subsuperalgebra of the original plane wave superalgebra. We demonstrated that the unbroken bosonic symmetries are realized linearly. As to the unbroken supersymmetries it is not obvious that they can also be realized linearly by appropriate choice of parametrization of fermionic fields. For the case of static gauge fixed \( D9 \) brane in flat space-time it is known that these supersymmetries are realized non-linearly [59, 60]. Because \( D3 \) brane has the same amount of unbroken supersymmetries it seems highly likely that the \( D3 \) brane unbroken supersymmetries are also realized non-linearly.

### 3.3 \( D3 \) brane friendly form of abelian \( \mathcal{N} = 4 \) SYM in plane wave background

Now we study the \( D3 \) brane action in quadratic approximation in fields. Looking at the part of \( D3 \) brane action given by \( \mathcal{L}_0 \) and \( \mathcal{L}_2 \) (see (3.8),(3.9)) one can expect appearance of some mass like terms for fermionic and bosonic scalar fields. Therefore in quadratic approximation in fields the \( D3 \) brane action reduces to the standard action of free abelian \( \mathcal{N} = 4 \) SYM in plane wave background plus some mass like terms for both the fermionic and bosonic scalar fields. It is instructive to understand structure of these mass-like terms. This is what we are doing in this section.

In static gauge this part of \( D3 \) brane action is obtainable from \( \mathcal{L}_0, \mathcal{L}_2 \) given in (3.8), (3.9). Introducing the notation

\[ \psi \equiv \sqrt{2}\lambda, \quad Z \equiv \frac{1}{\sqrt{2}}(\phi^3 + i\phi^4), \quad \bar{Z} \equiv \frac{1}{\sqrt{2}}(\phi^3 - i\phi^4), \tag{3.33} \]

we get the following Lagrangian for abelian spin one field \( A_m \equiv \delta^a_m A_a \), four real-valued scalars \( \phi^i \), one complex-valued scalar \( Z \), and the sixteen component real-valued one-half spin fermionic field \( \psi \)

\[ \mathcal{L} = \mathcal{L}_{st} + \Delta \mathcal{L}, \tag{3.34} \]

where \( \mathcal{L}_{st} \) stands for the standard Lagrangian of abelian \( \mathcal{N} = 4 \) SYM theory in 4d plane wave background

\[ \mathcal{L}_{st} = -\frac{1}{4} F^{mn} F_{mn} - \frac{1}{2} g_{mn} \partial_m \phi^i \partial_n \phi^i - g_{mn} \partial_m Z \partial_n Z - \frac{i}{2} \psi \gamma^m D_m \psi, \tag{3.35} \]
\[ D^m_l = \partial_m - \frac{f^2}{2} \sigma^i \gamma^{+i} \delta^+_{\gamma}, \quad \gamma^m = e^m_a \gamma^a, \]  

(3.36)

while $\Delta \mathcal{L}$ describes unusual mass-like terms

\[ \Delta \mathcal{L} = -2if(\bar{Z} \partial^+ Z - Z \partial^+ \bar{Z}) - \frac{i}{2} f \psi \bar{\gamma}^{34} \psi. \]  

(3.37)

Note that $\mathcal{L}_{st}$ is obtainable from the kinetic part of $D3$ brane action (2.2) while the mass-like terms $\Delta \mathcal{L}$ are coming from WZ part of brane action (2.3). Thus modulo unusual mass-like terms for the complex-valued scalar fields $Z$ and fermionic field $\psi$ we get the Lagrangian for abelian $\mathcal{N} = 4$ SYM theory in 4d plane wave background. It turns out that these unusual mass-like terms depend on field redefinitions. Indeed making use of the field redefinition

\[ Z \rightarrow e^{i(w+2)f} Z, \quad \psi \rightarrow e^{-w+2\frac{f}{4}} \gamma^{34} \psi, \]  

(3.38)

where $w$ is a constant parameter we get the Lagrangian (3.34) with the following mass terms

\[ \Delta \mathcal{L} = iwf(\bar{Z} \partial^+ Z - Z \partial^+ \bar{Z}) + iwf \frac{4}{\gamma^{34}} \psi, \quad \partial^+ \equiv \partial/\partial \sigma^-. \]  

(3.39)

Thus we see that mass-like terms depend on the parameter of fields redefinition $w$. A scheme in which $\Delta \mathcal{L}$ takes the form given in (3.39) we shall refer to as $w$-scheme.

The Lagrangian given in (3.35),(3.37) was derived directly from the Lagrangian of the $D3$ brane and it corresponds to $w = -2$ scheme. Therefore this scheme we shall refer to as $D3$ brane friendly scheme. Another scheme, which we shall refer to as conventional scheme, does not involve unusual mass-like terms. This conventional scheme is achieved by setting $w = 0$. From the transformation given in (3.38) it is clear that Hamiltonian of abelian plane wave SYM in arbitrary $w$ scheme is given by

\[ P^w_\gamma = P^- + f(w + 2)J^{34}, \]  

(3.40)

where $P^-$ is the Hamiltonian of plane wave SYM taken to be in $D3$ friendly $w = -2$ scheme (see (3.35),(3.37)). Peculiar properties of various schemes, $D3$ brane friendly and conventional ones, can be understood by study of supersymmetry. Let us discuss therefore supersymmetry transformations of abelian plane wave SYM.

Because the 4d plane wave metric is conformally flat the abelian $\mathcal{N} = 4$ plane wave SYM is invariant with respect to 30 bosonic and 32 supersymmetries. Only fourteen bosonic and sixteen supersymmetry transformations of plane wave abelian SYM can be derived from unbroken symmetries of $D3$ brane action. The fourteen unbroken $D3$ brane symmetries which are visible in plane wave SYM are shown in Table 1. According this Table the sixteen supersymmetries of abelian SYM related with unbroken symmetries of $D3$ brane are generated by supercharge $Q$ (3.30). Taking into account representation for the plane wave SYM Hamiltonian $P^w_\gamma$ (3.40) and the commutators (3.31) we get the following commutators between $w$-scheme Hamiltonian $P^w_\gamma$ and kinematical and dynamical supercharges

\[ [P^w_\gamma, Q^w_\alpha] = \frac{w}{2} fQ^w_\beta (\gamma^{34})^w_\alpha, \quad [P^w_\gamma, Q^-_\alpha] = \frac{2 + w}{2} fQ^-_\beta (\gamma^{34})^w_\alpha, \]  

(3.41)

\[ \text{Table 1}^{17} \quad \text{Remaining } D3 \text{ brane 16 bosonic and 16 super symmetries being broken already in static gauge fixed } D3 \text{ brane action become to be contracted when we restrict } D3 \text{ brane action to the action of abelian SYM.} \]
where kinematical, $Q^+$, and dynamical, $Q^-$, supercharges are defined in standard way

$$Q^+ = \frac{1}{2} \tilde{\gamma} - \gamma^+ Q, \quad Q^- = \frac{1}{2} \tilde{\gamma}^+ \gamma^- Q.$$  \hspace{1cm} (3.42)

From the commutators given (3.41) we see that in $D3$ brane friendly scheme, $w = -2$, the Hamiltonian of plane wave SYM is commuting with dynamical charges and does not commute with kinematical supercharges. Because supermultiplet of SYM theory is built out with the help of kinematical supercharges this implies that lowest energy values for fields of SYM, $A_m, Z$ and $\psi$, are different\(^{18}\). On the other hand in conventional scheme, $w = 0$, the Hamiltonian is commuting with kinematical supercharge and does not commute with dynamical supercharge. This implies that in conventional scheme all fields of SYM have the same lowest energy values.

To finish discussion of plane wave SYM we write down an explicit form of sixteen supersymmetry transformations of abelian SYM which are obtainable from the unbroken supersymmetries of $D3$ brane\(^{19}\)

\[
\delta A_m = i e^a_m \gamma^a \hat{\epsilon}, \quad \delta \phi^M = i \psi \gamma^M \hat{\epsilon},
\]

\[
\delta \psi = \left( \frac{1}{2} \gamma^{mn} F_{mn} - \gamma^M \gamma^m \partial_m \phi^M \right) \hat{\epsilon} - f \phi^i \gamma^i \gamma^+ \gamma^3 \hat{\epsilon} - f(w + 1)(Z \gamma^Z + \bar{Z} \gamma^Z) \gamma^+ \gamma^3 \hat{\epsilon}, \quad \gamma^Z = (\gamma^Z)^*, \quad (3.44)
\]

where parameter of transformation $\hat{\epsilon}$, which is the Killing spinor, satisfies the equation

\[
D^i \hat{\epsilon} = \frac{f}{2} (e^a \gamma^a \gamma^+ \gamma^3 + we^+ \gamma^3) \hat{\epsilon}, \quad D^i = d\sigma^m D_m^i. \quad (3.45)
\]

Explicit solution to this equation is fixed to be

\[
\hat{\epsilon} = \exp \left( \frac{w+2}{2} f \gamma^3 x^+ \right) \exp \left( -\frac{f}{2} x^i \gamma^+ \gamma^3 \hat{\gamma}^i \right) \exp \left( -\frac{f}{2} x^+ \gamma^+ \gamma^3 \hat{\gamma}^- \right) \epsilon_0, \quad (3.46)
\]

where $\epsilon_0$ is a sixteen component real-valued constant fermionic parameter. The w-scheme Lagrangian which is invariant with respect to the transformations (3.43), (3.44) is given by (3.34),(3.39).

### 3.4 Supersymmetries of gauge fixed brane action

In this section we investigate realization of supersymmetries of kappa gauge and static gauge fixed $D3$ brane action. We would like to learn which supersymmetries are unbroken and which ones become broken. To investigate this problem we study supersymmetry transformations of brane fields. In general all original supersymmetry transformations of brane fermionic fields (see (2.21)) look like Goldstone type, i.e. they shift the fermionic field by constant parameter. It turns out however that it is possible to construct some combination of original brane transformations which do not shift the fermionic field by

\(^{18}\)Alternative way to achieve this conclusion is simply to solve equations of motions.

\(^{19}\)Note that the supersymmetry transformations obtainable from $D3$ brane correspond to $w = -2$ scheme. Transformations rules (3.43),(3.44) can be derived then by going from this $w = -2$ scheme to the arbitrary scheme via transformations (3.38).
constant parameter and these transformations correspond to unbroken symmetries. All the remaining transformations are Goldstone type and they are associated with broken symmetries.

The plane wave supersymmetries of covariant kappa symmetry gauge and static gauge fixed brane action can be found following standard procedure [33]. Introducing notation $\Xi$ for brane fields $(x^{\nu}, \Theta, A_a)$ we start with supersymmetry transformations supplemented by local $\kappa$-symmetry transformations

$$\delta \Xi = \delta_{susy} \Xi + \delta_\kappa \Xi. \quad (3.47)$$

The parameter of $\kappa$ transformations is fixed by the requirement that the complete transformation (3.47) maintains the $\kappa$-symmetry gauge (3.1)

$$\delta_{susy} \theta^1 + \delta_\kappa \theta^1 = 0. \quad (3.48)$$

Representing the parameters of $\kappa$-transformation (2.7), supersymmetry transformation $\epsilon(x)$ (2.23) and the matrix $\Gamma$ (2.9) as

$$K^\alpha = \begin{pmatrix} \kappa_1^{1\alpha} \\ \kappa_2^{2\alpha} \end{pmatrix}, \quad \epsilon = \begin{pmatrix} \epsilon_1^{1\alpha} \\ \epsilon_2^{2\alpha} \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0 & \zeta \\ \tilde{\zeta} & 0 \end{pmatrix}, \quad (3.49)$$

we find that the relations (2.8) imply

$$\zeta \tilde{\zeta} = \bar{\zeta} \zeta = 1, \quad \kappa_1 = \zeta \kappa_2, \quad \kappa_2 = \tilde{\zeta} \kappa_1. \quad (3.51)$$

Taking into account supersymmetry and $\kappa$ transformations for $\Theta$ (2.21),(2.26) we get from (3.48) solution to $\kappa_1$: $\kappa_1 = -\epsilon^1$. In view of (3.51) we conclude that both the local $\kappa$ parameters, $\kappa_1$ and $\kappa_2$, are expressible in terms of global parameter of supersymmetry transformations

$$\kappa_1 = -\epsilon^1, \quad \kappa_2 = -\tilde{\zeta} \epsilon^1. \quad (3.52)$$

Plugging solution to $\kappa$ parameters into (3.47) and taking into account transformations (2.20)-(2.22),(2.25)-(2.27) and notation (3.1) we get the following supersymmetry transformations of brane fields

$$\delta A = -ie^\mu (\epsilon^2 + \tilde{\zeta} \epsilon^1) \gamma^\mu \lambda, \quad (3.53)$$

$$\delta x^{\nu} = ie^\mu (\epsilon^2 + \tilde{\zeta} \epsilon^1) \gamma^\mu, \quad (3.54)$$

$$\delta \lambda = \epsilon^2 - \tilde{\zeta} \epsilon^1. \quad (3.55)$$

From the transformation rule given in (3.55) it is clear that the brane fermionic field transforms like Goldstone field in general. Note however that this does not imply that all supersymmetries are broken. It turns out that sixteen supersymmetries are still to be
unbroken. Let us demonstrate this point explicitly. Representing the constant parameter \( \epsilon_0 \) (2.23)

\[
\epsilon_0 = \begin{pmatrix} \epsilon_1^0 \\ \epsilon_2^0 \end{pmatrix},
\]

we get from (2.23) the following representation for the components of the Killing spinor \( \epsilon(x) \)

\[
\epsilon^1(x) = \frac{1}{2}(\gamma^{-}\bar{\gamma}^{+} + \gamma^{+}\bar{\gamma}^{-})\cos fx^+\epsilon_0^1 - \frac{1}{2}\Pi\gamma^+(fx^I\bar{\gamma}^I + \bar{\gamma}^- \sin fx^+)\epsilon_0^2,
\]

\[
\epsilon^2(x) = \frac{1}{2}(\gamma^{-}\bar{\gamma}^{+} + \gamma^{+}\bar{\gamma}^{-})\cos fx^+\epsilon_0^2 + \frac{1}{2}\Pi\gamma^+(fx^I\bar{\gamma}^I + \bar{\gamma}^- \sin fx^+)\epsilon_0^1.
\]

Now we restrict the 32 supersymmetries associated with parameters \( \epsilon_1^0, \epsilon_2^0 \) to the 16 supersymmetries by imposing the constraint on the parameters of transformations

\[
\epsilon_{0r}^1 = \frac{1}{2}\zeta_0 \eta_0, \quad \epsilon_{0r}^2 = \frac{1}{2}\eta_0, \quad \zeta_0 \equiv \gamma^{-12},
\]

where \( \eta_0 \) is a real-valued sixteen component constant spinor and suffix ‘r’ is used to indicate the fact we restrict 32 parameters \( \epsilon_1^0, \epsilon_2^0 \) (3.57),(3.58) to 16 independent parameters \( \eta_0 \). Now if we plug these \( \epsilon_{0r}^1, \epsilon_{0r}^2 \) in expressions for \( \epsilon^1(x), \epsilon^2(x) \) (3.57),(3.58) and insert such restricted values of \( \epsilon^1(x), \epsilon^2(x) \) on r.h.s of (3.55) then we learn that linear field independent term proportional to \( \eta_0 \) cancels out. This means that 16 supersymmetries associated with parameter \( \eta_0 \) are unbroken. Evaluating

\[
\epsilon_{0r}^1 Q^1 + \epsilon_{0r}^2 Q^2 = \frac{1}{2}\eta_0 (Q^1\zeta_0 + Q^2)
\]

we conclude that sixteen supersymmetries generated by the supercharges \( Q^1\zeta_0 + Q^2 \) are unbroken while the remaining supersymmetries sixteen generated by the supercharges \( Q^1\zeta_0 - Q^2 \) are broken and realized non-linearly.

### 3.5 Supersymmetries of static gauge fixed D3 brane action via supercurrents

In this section we investigate realization of D3 brane supersymmetries by exploring formalism of supercurrent. Concerning the broken and unbroken supersymmetries we arrive at the same conclusions of previous section, i.e. a study of this section is supplementary to the one above given. We evaluate the supercurrents and divide them into two class. The supercurrents which do not involve terms linear in fermionic fields are responsible for unbroken supersymmetries. Remaining supercurrents involve terms linear in fermionic fields and these supercurrents are responsible for broken supersymmetries (Goldstone type supersymmetries). In order to investigate these issues it is sufficient to restrict an attention to the terms up to the second order in the fermionic field \( \Theta \). The corresponding part of gauge invariant D3 brane Lagrangian is given by (see (2.1))

\[
\mathcal{L}^{(2)} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4,
\]
\[ \mathcal{L}_1 = -\sqrt{g} + i\sqrt{g} \theta \gamma^a \mathcal{D}_a \theta, \]

\[ \mathcal{L}_2 = \sqrt{g} \left( -\frac{1}{4} F_{ab} F_{ab} - i F_{ab} \theta \gamma_a \tau_3 \mathcal{D}_b \theta \right), \]

\[ \mathcal{L}_3 = \frac{i}{6} \epsilon^{abcd} \theta \gamma_{abc} \tau_2 \mathcal{D}_d \theta + \mathcal{L}_{WZ}^{(bose)} \]

\[ \mathcal{L}_4 = \frac{i}{2} \epsilon^{abcd} F_{ab} \theta \gamma_c \tau_1 \mathcal{D}_d \theta, \quad \gamma^a_{\alpha \beta} \equiv \gamma^\mu_{\alpha \beta} \epsilon^a_\mu, \]

where the \( \mathcal{L}_{WZ}^{(bose)} \) is given in (3.14) and the covariant Killing spinor derivative \( \mathcal{D} = d\sigma^a \mathcal{D}_a \) is given in (C.18). The remaining notation is given in (3.12), (3.13). Supersymmetry transformations given in (2.20)-(2.22) lead to a conserved current\textsuperscript{20} \( Q^a, \partial_a Q^a = 0, \)

\[ Q^a = V \left( \sqrt{g} (\gamma^a + F_{ab} \gamma_b \tau_3) - \frac{1}{6} \epsilon_{abcd} \gamma_{bcd} \tau_2 - \frac{1}{2} \epsilon_{abcd} F_{bc} \gamma_d \tau_1 \right) \theta, \]

where the matrix \( V = V_{a}^{\beta} \) is given by

\[ V = \exp \left( \frac{f}{2} \tau_2 x^+ \gamma^- \Pi \gamma^+ \right) \exp \left( \frac{f}{2} \tau_2 x I \gamma^ I \Pi \gamma^+ \right) . \]

Note that \( V_{a}^{\beta} \equiv U_{a}^{\beta} \) (see (2.23)). Now we should impose both the bosonic static gauge and fermionic covariant kappa symmetry gauge. To this end we represent the supercurrent \( Q^a \) (3.66) as follows

\[ Q^a = \begin{pmatrix} Q^{1a} \\ Q^{2a} \end{pmatrix}, \quad Q^a = V \hat{Q}^a, \]

and evaluate the supercurrents \( \hat{Q}^a \) taken to be in covariant kappa symmetry gauge (3.1)

\[ \hat{Q}^{1a} = -\frac{1}{6} \epsilon_{abcd} \gamma_{bcd} \lambda - \frac{1}{2} \epsilon_{abcd} \hat{\gamma}_b F_{cd} \lambda, \quad \hat{Q}^{2a} = \gamma^a \lambda - F_{ab} \hat{\gamma}_b \lambda. \]

Plugging these expressions into (3.68) and by acting with matrix \( V \) we find the following components of supercurrent \( Q^a \):

\[ Q^{1a} = \frac{1}{2} (\gamma^+ \gamma^- + \gamma^- \gamma^+ \cos f x^+) (-\frac{1}{6} \epsilon_{abcd} \hat{\gamma}_{bcd} - \frac{1}{2} \epsilon_{abcd} \hat{\gamma}_b F_{cd} ) \lambda \]

\[ + \frac{1}{2} (f \gamma^I x^I + \gamma^- \sin f x^+) \Pi \gamma^+ (\hat{\gamma}^a - F_{ab} \hat{\gamma}_b) \lambda, \]

\[ Q^{2a} = \frac{1}{2} (\gamma^+ \gamma^- + \gamma^- \gamma^+ \cos f x^+) (\hat{\gamma}^a - F_{ab} \hat{\gamma}_b) \lambda \]

\[ + \frac{1}{2} (f \gamma^I x^I + \gamma^- \sin f x^+) \Pi \gamma^+ (\frac{1}{6} \epsilon_{abcd} \hat{\gamma}_{bcd} + \frac{1}{2} \epsilon_{abcd} \hat{\gamma}_b F_{cd} ) \lambda. \]

These supercurrents \( Q^{1a} \) and \( Q^{2a} \) taken to be in static gauge involve unwanted terms which are linear in fermionic field and do not depend on the remaining fields. These are terms that generate Goldstone transformation. Motivated by desire to cancel these

\textsuperscript{20}The current (3.66) can be found by using standard Noether method based on localization of the parameters of associated global transformation \( \epsilon_0 \) (2.23). Replacing \( \epsilon_0 \) by function of world-volume coordinates \( \sigma^a \) the variation of Lagrangian by module of total derivatives is found to be \( \delta \mathcal{L} = -2i(\partial_a \epsilon_0) Q^a. \)
unwanted terms we look for the linear combination of the supercurrents $Q^{1a}$ and $Q^{2a}$ which cancels out unwanted terms. As expected from analysis of previous Section such a combination does exist and is given by $Q^{1a}ζ_0 + Q^{2a}$. Making use of field redefinitions (3.16) this desired combination of supercurrents takes the form\(^{21}\)

$$-(Q^{1m}ζ_0 + Q^{2m}) = \frac{1}{2} \exp\left(\frac{f}{2} σ^+ γ^- γ^{+34}\right) \exp\left(\frac{f}{2} σ^i γ^i γ^{+34}\right) F_{μν} γ^{μν} γ^m λ \pm \phi^M γ^M γ^{+34} γ^m λ.$$  

(3.72)

To transform this supercurrent into conventional $w = 0$ scheme which is friendly to $\mathcal{N} = 4$ SYM theory to be studied below we use the transformation (3.38) with $w = 0$ (see also (3.33)) and we get finally

$$-(Q^{1m}ζ_0 + Q^{2m}) = \exp\left(-\frac{f}{2} σ^+ γ^{+34} γ^-\right) \exp\left(-\frac{f}{2} σ^i γ^i γ^{+34}\right) \left(\frac{1}{2} F_{μν} γ^{μν} \pm \phi^M γ^M γ^{+34} γ^m λ\right).$$  

(3.73)

### 4 D3 brane action in the kappa symmetry light cone gauge

Fixing fermionic kappa symmetry by using light cone gauge simplifies considerably the structure of D3 brane action. In order to discuss light cone gauge fixed action we find it convenient to represent covariant action it terms of complex spinors. Complex frame fermionic coordinates $θ^a, \bar{θ}^α = (θ^α)\dagger$ and Cartan 1-forms $L_α, \hat{L}_α = (L^α)\dagger$ are related with 2-vector notation $Θ$ and $L$ used in previous sections

$$θ = \frac{1}{\sqrt{2}}(θ^1 + iθ^2), \quad L = \frac{1}{\sqrt{2}}(L^1 + iL^2),$$  

(4.1)

where the components $θ^1, θ^2$ and $L^1, L^2$ are related with $Θ$ and $L$ as in (C.2). In this notation the 2-form field $F$ takes the form\(^{22}\)

$$F = F_{t=1}, \quad F_t = dA + (2i \int_0^t dt' θ\hat{L}_{t'} \wedge L_{t'} + h.c.),$$  

(4.2)

while the defining $H_5$ form can be cast into the form

$$H_5 = \frac{1}{3} \hat{L} \wedge \hat{L} \wedge \hat{L} \wedge L \wedge F \wedge (L \wedge \hat{L} \wedge L - \hat{L} \wedge L \wedge \hat{L})$$

$$+ \frac{f}{6} (ε^{i_1...i_4} L^+ \wedge L^{i_1} \wedge ... \wedge L^{i_4} + ε^{i_1...i_4} L^+ \wedge L'^{i_1} \wedge ... \wedge L'^{i_4}),$$  

(4.3)

which leads to the following WZ part of the Lagrangian $\mathcal{L}_{WZ} = d^{-1}H_5$

$$\mathcal{L}_{WZ} = \int_0^1 dt \left(\frac{1}{3} \bar{θ}^+ \hat{L}_t \wedge \hat{L}_t \wedge L_t + 2F_t \wedge θ\hat{L}_t \wedge L_t + h.c.\right) \pm \mathcal{L}_{WZ}^{(base)}.$$

(4.4)

Fermionic kappa symmetry light cone gauge is defined as

$$\hat{γ}^+ θ = \hat{γ}^+ \bar{θ} = 0.$$  

(4.5)

\(^{21}\)Here we introduce $Q^m = δ^m_a Q^a$ and collect $A^m$ and $φ^M$ into 10d vector $Λ^m$. Details of 10d notation we use may be found in Sect. 6.

\(^{22}\)Complex notation counterpart of relation (2.16) is $dF = iL \wedge \hat{L} \wedge L + h.c.$
In this gauge the Cartan 1-forms can easily be evaluated (by transforming result of Appendix C into complex form or by using Ref. [2]) and they are given by

\[ L^+ = e^+, \quad L^I = e^I, \quad L^- = e^- - i(\bar{\theta}\gamma^- d\theta + \theta\gamma^- d\bar{\theta}) - 2fe^+\bar{\theta}\gamma^- \Pi \theta, \quad (4.6) \]

\[ L = d\theta - ife^+\Pi \theta, \quad (4.7) \]

where we use coordinates of plane wave background in which vielbeins \( e^\mu \) take the form given in (3.13). By using these kappa symmetry gauge fixed Cartan 1-form we find immediately from (2.4) (4.2) the corresponding \( G_{ab} \) and \( F_{ab} \) terms

\[ G_{ab} = g_{ab} - 2i\partial_a x^+ (\bar{\theta}\gamma^- \partial_b \theta + \theta\gamma^- \partial_b \bar{\theta}) - 4f\bar{\theta}\gamma^- \Pi \theta \partial_a x^+ \partial_b x^+, \quad (4.8) \]

\[ F_{ab} = F_{ab} + 2i\partial_a x^+ (\theta\gamma^- \partial_b \theta + \bar{\theta}\gamma^- \partial_b \bar{\theta}), \quad (4.9) \]

where the bosonic body \( g_{ab} \) is given by (3.15). As compared to covariant gauge the light cone gauge fixed Cartan 1-forms, the metric \( G_{ab} \), and 2-form F do not involve terms higher than second order in the fermionic coordinates \( \theta \) and due to this we are able to write down explicit representations for both the kinetic and WZ parts of gauge fixed D3 brane Lagrangian. Plugging these expressions into the Lagrangian (and making a redefinition \( x^\mu \rightarrow -x^\mu \)) we get (after simple integration over \( t \) in \( \mathcal{L}_{WZ} \)) the corresponding BI and WZ parts

\[ \mathcal{L}_{BI} = -\sqrt{-\det(g_{ab} + F_{ab} + M_{ab})}, \quad (4.10) \]

\[ \mathcal{L}_{WZ} = -\epsilon^{abcd} \partial_a x^+ (\partial_b x^I \partial_c x^J \bar{\theta}\gamma^{-1} \partial_d \theta + \frac{1}{2} F_{bc}(\theta\gamma^- \partial_d \theta - \bar{\theta}\gamma^- \partial_d \bar{\theta})) + \frac{f}{6} \epsilon^{abcd} \partial_a x^+ (\epsilon^{ijkl} x^i \partial_b x^j \partial_c x^k \partial_d x^l + \epsilon^{ij'k'l'} x^i x^j x^k x^l - \epsilon^{ij'k'l'} x^i x^j x^k x^l), \quad (4.11) \]

where the matrix \( M_{ab} \) describes the \( \theta \)-dependent part of the BI Lagrangian

\[ M_{ab} \equiv 2i\partial_b x^+ (\bar{\theta}\gamma^- \partial_a \theta + \theta\gamma^- \partial_a \bar{\theta}) - 4f\bar{\theta}\gamma^- \Pi \theta \partial_a x^+ \partial_b x^+. \quad (4.12) \]

First terms in expansion of \( \mathcal{L} \) are (modulo cosmological term)

\[ \frac{1}{\sqrt{g}} \mathcal{L}_{BI} = -\frac{1}{4} F_{ab} \bar{F}^{ab} - i(g^{ab} - F^{ab}) \partial_a x^+ (\bar{\theta}\gamma^- \partial_b \theta + \theta\gamma^- \partial_b \bar{\theta}) + 2f g^{ab} \partial_a x^+ \partial_b x^+ \bar{\theta}\gamma^- \Pi \theta, \quad (4.13) \]

where \( g \) is defined as in (3.12).

Light cone gauge fixing the world-volume diffeomorphism symmetries can be made by following standard procedure (see for instance [61]). This subject is beyond scope of this paper.

5 \textit{psu}(2,2|4) superalgebra in various bases

Now we turn to a study of plane wave SYM theory. Since our approach is based essentially on realization of supersymmetries generated by \textit{psu}(2,2|4) superalgebra we start with discussion of (anti)commutation relations of this superalgebra in various bases.
psu(2, 2|4) superalgebra is more familiar in so(3, 1) ⊕ su(4) ≃ sl(2, C) ⊕ su(4) or so(4, 1) ⊕ so(5) basis. The former basis is convenient in study of \( N = 4, 4d \) superconformal symmetry of SYM theory while the latter basis introduced in \[28\] is preferable for discussion of the covariant GS action in AdS\(_5\) × S\(5\) Ramond-Ramond background \[28\].

Note that all these bases respect Lorentz symmetries generated by so(3, 1) algebra. The Lorentz symmetries however are broken in plane wave background and therefore these bases are not convenient for discussion of the 4d plane wave \( N = 4 \) SYM. It turns out that more suitable and convenient basis for study plane wave SYM is the one in which a proper plane wave generator of time translation \( P^- \) and dilatation generator \( D \) are realized as diagonal elements of psu(2, 2|4) superalgebra. This basis will be referred to as plane wave basis.

### 5.1 psu(2, 2|4) superalgebra in Lorentz basis

Since we use extensively interrelations between Lorentz and plane wave bases let us start with discussion of psu(2, 2|4) superalgebra in Lorentz basis. As is well known bosonic part of psu(2, 2|4) superalgebra is the algebra of conformal transformations so(4, 2) plus the algebra of \( R \)-symmetries so(6). The so(4, 2) algebra being realized in flat space-time consists of translation generators \( P^m \), dilatation generator \( D \), generators of so(3, 1) rotations \( J^{mn} \), generators of so(6) rotations \( J^{MN} \) and conformal boosts generators \( K^m \). The fermionic part of psu(2, 2|4) superalgebra consists of sixteen generators of Poincaré supertranslations \( Q_a \) and sixteen generators of superconformal translations \( S^\alpha \). Commutators between bosonic generators of superalgebra are given by\(^{25}\)

\[
[D, P^m] = -P^m, \quad [D, K^m] = K^m, \quad [P^m, K^n] = \eta^{mn}D - J^{mn}, \quad (5.1)
\]

\[
[P^m, J^{mn'}] = \eta^{mn}P^{n'} - \eta^{mn'}P^n, \quad [K^m, J^{mn'}] = \eta^{mn}K^{n'} - \eta^{mn'}K^n, \quad (5.2)
\]

\[
[J^{mn}, J^{m'n'}] = \delta^{nm}J^{m'n'} + 3 \text{ terms}, \quad [J^{MN}, J^{M'N'}] = \delta^{NM}J^{M'N'} + 3 \text{ terms}. \quad (5.3)
\]

Commutators between bosonic generators and the fermionic ones are

\[
[S, P^m] = -\gamma^mQ, \quad [Q, K^m] = \frac{1}{2}\gamma^mS, \quad (5.4)
\]

\[
[D, Q] = -\frac{1}{2}Q, \quad [Q, J^{mn}] = \frac{1}{2}\gamma^{mn}Q, \quad [Q, J^{MN}] = \frac{1}{2}\gamma^{MN}Q, \quad (5.5)
\]

\[
[D, S] = \frac{1}{2}S, \quad [S, J^{mn}] = \frac{1}{2}\gamma^{mn}S, \quad [S, J^{MN}] = \frac{1}{2}\gamma^{MN}S. \quad (5.6)
\]

Anticommutators are fixed to be

\[
\{Q, Q\} = -2i\gamma^mP^m, \quad \{S, S\} = -4i\gamma^mK^m, \quad (5.7)
\]

\[
\{S^\alpha, Q_\beta\} = -2i\delta^\alpha_\betaD - i(\gamma^{mn})^\alpha_\betaJ^{mn} + i(\gamma^{MN})^\alpha_\betaJ^{MN}. \quad (5.8)
\]

\(^{25}\)Study of covariant AdS\(_5\) × S\(5\) Green-Schwarz superstring action in su(2, 2) ⊕ su(4) and sl(2, C) × su(4) bases may be found in \[62\] and \[63\] respectively.

\(^{24}\)Here and below we use bold face notation for generators of psu(2, 24) superalgebra taken in Lorentz notation. This notation should not be confused with the one for 2-vectors we used in previous Sections.

\(^{23}\)m, n, m', n' = 0, 1, 2, 9; M, N, M', N' = 3, 4, . . . , 8. In light cone frame \( m, n, m', n' = +, -, 1, 2 \) and the only non-vanishing components of flat metric tensor \( \eta^{mn} \) are given by \( \eta^{++} = 1, \eta^{ij} = \delta^{ij} \).
All bosonic generators $G = (P^m, K^m, D, J^{mn}, J^{MN})$ are taken to be anti-hermitian $G^\dagger = -G$, while fermionic generators are considered to be hermitian

$$Q^i_\alpha = Q_\alpha, \quad S^\alpha \dagger = S^\alpha.$$  \hspace{1cm} (5.9)

In flat space the symmetries generated by translations $P^m$, and Lorentz rotations $J^{mn}$ are realized as isometry symmetries of Minkowski space-time while $D$ and $K^m$ are responsible for proper conformal transformations.

### 5.2 $psu(2, 2|4)$ superalgebra in plane wave basis

First of all let us following [64] to discuss a manifest representation for generators of conformal symmetries of 4d plane wave geometry. This representation will be used throughout this paper. Since for plane wave background the Weyl tensor vanishes, the conformal algebra in question is isomorphic to the $so(4, 2)$ algebra. Below we establish the isomorphism explicitly, i.e., we map generators of $so(4, 2)$ taken to be in the plane wave basis to generators of $so(4, 2)$ taken to be in the Lorentz basis. Note that because Lorentz symmetries are broken in plane wave background the symmetries of $so(4, 2)$ algebra are realized in a different way in plane wave space-time as compared to ones in the Minkowski space-time. In plane wave background only seven generators of the $so(4, 2)$ algebra are realized as isometry symmetries while the remaining eight generators of the $so(4, 2)$ algebra are realized as proper conformal symmetries. This is to be compared with Minkowski space where the 10 generators of $so(4, 2)$ algebra, four translations $P^m$ and six Lorentz boosts $J^{mn}$, are realized as isometry symmetries while the remaining 5 generators of the $so(4, 2)$ algebra, dilatation $D$ and conformal boosts $K^m$, are realized as proper conformal transformations.

Our study is based on usage of concrete parametrization of 4d plane wave background in which the line element takes the form

$$ds^2 = 2dx^+dx^- - \ell^2 x^i x^i dx^+ dx^+ + dx^i dx^i, \quad \hat{i} = 1, 2.$$  \hspace{1cm} (5.10)

We start with discussion of the algebra of isometry symmetries which is subalgebra of conformal algebra. The algebra of isometry symmetries leaves the line element (5.10) to be form-invariant and is generated by translations $P^+$, $P^-$, $P^\hat{i}$, by Lorentz boosts $J^{+\hat{i}}$ and by $so(2)$ rotations $J^{\hat{i}\hat{j}} = \epsilon^{\hat{i}\hat{j}} J^{12}$, i.e. the dimension of this algebra is equal to seven. These generators satisfy the well known commutation relations

$$[P^-, P^\hat{i}] = -\ell^2 J^{+\hat{i}}, \quad [P^\hat{i}, J^{+\hat{j}}] = -\delta^{\hat{i}\hat{j}} P^+, \quad [P^-, J^{+\hat{i}}] = P^\hat{i},$$

$$[P^\hat{i}, J^{\hat{k}\hat{j}}] = \delta^{\hat{i}}\hat{j} P^{\hat{k}} - \delta^{\hat{i}}\hat{k} P^{\hat{j}}, \quad [J^{+\hat{i}}, J^{\hat{j}\hat{k}}] = \delta^{\hat{i}}\hat{j} J^{+\hat{k}} - \delta^{\hat{i}}\hat{k} J^{+\hat{j}}.$$  \hspace{1cm} (5.11)

Sometimes in what follows instead of generators $P^\hat{i}$, $J^{+\hat{i}}$ we prefer to use complex frame generators $T^\hat{i}$, $\bar{T}^\hat{i}$ defined by

$$T^\hat{i} = P^\hat{i} - i f J^{+\hat{i}}, \quad \bar{T}^\hat{i} = P^\hat{i} + i f J^{+\hat{i}},$$  \hspace{1cm} (5.13)

which are related by hermitian conjugation rule $T^{\hat{i}} = -\bar{T}^{\hat{i}}$. Thus the isometry symmetry algebra is representable by the following seven generators

$$P^-, \quad P^+ \quad T^\hat{i}, \quad \bar{T}^\hat{i}, \quad J^{12}, \quad \text{isometry generators}.$$  \hspace{1cm} (5.14)
To find concrete representation for these generators we solve the equations for the Killing vectors of isometry transformations

$$D_m \xi_n + D_n \xi_m = 0. \quad (5.15)$$

Well known representation for these Killing vectors in terms of differential operators $G = \xi^m \partial_m$ is given by

$$P^\pm = \partial^\pm, \quad T^i = e^{-ifx^+}(\partial^i + if x^i \partial^+), \quad J^{ij} = x^i \partial^j - x^j \partial^i. \quad (5.16)$$

Because these seven generators form the isometry algebra the remaining eight generators of the conformal algebra so($4,2$) are responsible for proper conformal transformations, i.e. they scale the line element (5.10). To find these proper conformal generators we are solving the equations for conformal Killing vectors in 4d plane wave background (5.10)

$$D_m \xi_n + D_n \xi_m = \frac{1}{2}g_{mn}D_k \xi^k. \quad (5.17)$$

Solution to these equations can be described by eight Killing vectors denoted by

$$D, \quad C, \quad \bar{C}, \quad C^i, \quad \bar{C}^i, \quad K^-, \quad \text{proper conformal generators}, \quad (5.18)$$

where the complex-valued generators $C^i, \bar{C}^i, C, \bar{C}$ are related by hermitian conjugation rule $\bar{C}^i = -C^{i\dagger}, C = -C^{\dagger i}$. Representation of these proper conformal generators in terms of differential operators $G = \xi^m \partial_m$ is fixed to be [64]

$$D = 2x^- \partial^+ + x \partial, \quad (5.19)$$

$$C = e^{-2ifx^+}(\partial^- - if x \partial + f^2 x^2 \partial^+), \quad (5.20)$$

$$C^i = e^{-ifx^+}(x^- \partial^i - x^i \partial^- + if(-\frac{1}{2} x^2 \partial^i + x^i (x^- \partial^+ + x \partial)) - \frac{f^2}{2} x^2 x^i \partial^+), \quad (5.21)$$

$$K^- = -\frac{1}{2} x^2 \partial^- + x^- (x^- \partial^+ + x \partial) - \frac{f^2}{4} (x^2)^2 \partial^+. \quad (5.22)$$

Here and below we use the conventions

$$x^2 \equiv x^i x^i, \quad x \partial \equiv x^i \partial^i, \quad \partial^2 \equiv \partial^i \partial^i. \quad (5.23)$$

Note that generators $P^+, J^{ij}$ (5.16), $D$ (5.19) and $K^-$ (5.22) do not depend on evolution parameter $x^+$, i.e. they are commuting with plane wave time translation generator $P^- (5.15)$. It is easy to see also that generators $T^i, C$ (5.20) and $C^i$ (5.21) are eigenvectors of $P^- = \partial^-$. Straightforward inspection of generators (5.15),(5.19)-(5.22) demonstrates that all these generators are also eigenvectors of $D$ and transform in scalar or vector representations of $J^{12}$.

As mentioned before the isometry generators (5.15) and proper conformal generators (5.18) form plane wave basis of the so($4,2$) algebra. Their commutation relations are given in (5.37)-(5.42). It is instructive to relate generators (5.15)-(5.18) with generators of so($4,2$) algebra taken in Lorentz basis $P^m, K^m, D, J^{mn}$ which satisfy well known recognizable commutation relations (5.1)-(5.3). This can be done in a rather straightforward way by comparing commutation relations of generators in Lorentz basis (5.1)-(5.3) with
the commutation relations in plane wave basis (5.37)-(5.42). Doing this we find an interrelation between generators of the conformal algebra in Lorentz basis and the ones in plane wave basis

\[ P^+ = P^- + f^2 K^+ , \quad P^+ = P^- , \quad J^{12} = J^{12}, \tag{5.24} \]

\[ T^i = P^i - i f J^+ , \quad \bar{T}^i = P^i + i f J^+ , \tag{5.25} \]

\[ D = D - J^+ , \tag{5.26} \]

\[ C = P^+ - f^2 K^+ - i f (D + J^+ ) , \quad \bar{C} = P^- - f^2 K^+ + i f (D + J^- ) , \tag{5.27} \]

\[ C^i = J^{-i} + i f K^i , \quad \bar{C}^i = J^{-i} - i f K^i , \tag{5.28} \]

\[ K^- = K^- . \tag{5.29} \]

Note that the relations given in (5.25) and definition of \( T^i \) (5.13) imply matching the transverse translations \( P^i \) and \( J^+ \) Lorentz boosts \( J^{1+} \):

\[ P^i = P^i , \quad J^{1+} = J^{1+} . \tag{5.30} \]

As for \( R \)-symmetries \( so(6) \), these symmetries are obviously unbroken and therefore one has the simple correspondence

\[ J^{MN} = J^{MN} . \tag{5.31} \]

Now let us turn to supercharges. What is required is to find the interrelation of supercharges in plane wave and Lorentz bases. As in bosonic case the supercharges in plane wave basis should, by definition, be eigenvectors of plane wave time translation generator \( P^- \) and plane wave dilatation generator \( D \). Because we have already established interrelation of two bases in bosonic body of superalgebra the most straightforward way to establish of interrelation of supercharges in two bases is to find linear combinations of Lorentz basis supercharges \( Q, S \) so that this combinations be eigenvectors of plane wave time translation generator \( P^- \) and plane wave dilatation generator \( D \). Because the interrelation between bosonic generators is known we can establish similar interrelation between supercharges in a rather straightforward way. Doing this we found that plane wave basis supercharges can be collected into two sorts of supercharges which we denote by \( \Omega \) and \( S^- \). This is to say that the requirement these supercharges be eigenvectors of generators \( P^- \) and \( D \) fixes the following representation for sixteen component complex-valued supercharge \( \Omega \) and sixteen component real-valued supercharge \( S^- \)

\[ \Omega = Q - \frac{if}{2} \bar{\gamma}^+ S^+ , \quad S^- = S^- . \tag{5.32} \]

Here and below we use decomposition of \( S^\alpha \) into plus and minus parts

\[ S = S^+ + S^- , \quad S^+ = \frac{1}{2} \gamma^- \bar{\gamma}^+ S , \quad S^- = \frac{1}{2} \gamma^+ \bar{\gamma}^- S , \tag{5.33} \]

while for \( Q_\alpha \) similar decomposition takes the form

\[ Q = Q^+ + Q^- , \quad Q^+ = \frac{1}{2} \bar{\gamma}^- \bar{\gamma}^+ Q , \quad Q^- = \frac{1}{2} \bar{\gamma}^+ \bar{\gamma}^- Q . \tag{5.34} \]
The supercharges \( \Omega \) and \( S^- \) satisfy the algebraic constraints. Decomposing the supercharge \( \Omega \) as in (5.34)

\[
\Omega = \Omega^+ + \Omega^- ,
\]
we have the constraints

\[
\Omega^{+\dagger} = \Omega^+ , \quad \bar{\gamma}^+ S^- = 0 .
\]

These constraints are derivable from hermitian properties for generators \( Q, S \) given in (5.9) and definitions given in (5.33). The first constraint in (5.36) tells us that though the supercharge \( \Omega \) is complex-valued its \( \Omega^+ \) part is real-valued. This implies that the supercharge \( \Omega \) has 24 real-valued components. The second constraint in (5.36) tells us that there are only 8 real-valued components of \( S^- \). Thus the total number of real-valued supercharges is equal to 32 as it should be.

Now we can write down commutation relations of the \( psu(2,2|4) \) superalgebra in plane wave basis. Because in this basis all generators are eigenvectors of \( P^- \) and \( D \) it is convenient to introduce a notion of \( q(P^-) \) and \( q(D) \) charges. Given generator \( G \) its \( q_G(P^-) \) and \( q_G(D) \) charges are defined by commutators

\[
[P^-, G] = -i q_G(P^-) G , \quad [D, G] = q_G(D) G .
\]

These charges for generators of the \( psu(2,2|4) \) superalgebra are given in Table 2.

**TABLE 2**: \( q(P^-) \) and \( q(D) \) charges of generators of \( psu(2,2|4) \) superalgebra

| Generators | \( P^+ \) | \( T^i \) | \( \bar{T}^i \) | \( K^- \) | \( C^i \) | \( \bar{C}^i \) | \( C \) | \( \bar{C} \) | \( \Omega^- \) | \( \bar{\Omega}^- \) | \( \Omega^+ \) | \( S^- \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( q_G(P^-) \) charge | 0 | 1 | -1 | 0 | 1 | -1 | 2 | -2 | 1 | -1 | 0 | 0 |
| \( q_G(D) \) charge | -2 | -1 | -1 | 2 | 1 | 1 | 0 | 0 | 0 | 0 | -1 | 1 |

Note that the \( q \)-charges of the generators \( J^{ij} \) and \( J^{MN} \) are equal to zero. Remaining (anti)commutators can be collected in several groups.

Commutators between elements of isometry algebra are

\[
[T^i, T^j] = 2i \delta^{ij} P^+ , \quad [T^i, J^{jk}] = \delta^{ij} T^k - \delta^{ik} T^j .
\]

Commutators between proper conformal generators are given by

\[
[C^i, C^j] = 2i \delta^{ij} K^- , \quad [\bar{C}, C] = -4i f P^- , \quad [\bar{C}^i, C] = -2i f C^i .
\]

Commutators between generators of isometry algebra and proper conformal generators take the form

\[
[P^+, K^-] = D , \quad [P^+, C^i] = T^i , \quad [T^i, K^-] = C^i ,
\]

\[
[T^i, C^j] = -\delta^{ij} C , \quad [T^i, \bar{C}] = 2i f T^i ,
\]

\[
[\bar{C}^i, T^j] = \delta^{ij} (P^- + if) + 2i f J^{ij} .
\]
Commutators between isometry algebra and supercharges are

\[ [T^i, \Omega^-] = i\gamma^i \Omega^+, \quad [T^i, S^-] = \gamma^i \Omega^-, \quad [P^+, S^-] = \gamma^+ \Omega^+, \quad (5.43) \]

\[ [\Omega^\pm, J^{ij}] = \frac{1}{2} \hat{\gamma}^{ij} \Omega^\pm, \quad [S^-, J^{ij}] = \frac{1}{2} \hat{\gamma}^{ij} S^-. \]

Commutators between proper conformal generators and supercharges are

\[ [K^-, \Omega^+] = -\frac{1}{2} \hat{\gamma}^- S^-, \quad [C^i, \Omega^+] = -\frac{1}{2} \hat{\gamma}^- i \Omega^-, \quad [\hat{C}^i, \Omega^-] = -i\hat{\gamma}^i S^-, \quad (5.44) \]

\[ [C, \hat{\Omega}^-] = 2i\hat{\Omega}^-. \quad (5.45) \]

In view of relations (5.31) commutators of \( so(6) \) algebra and commutators between generators of the \( so(6) \) and supercharges in plane wave basis take the same form as in the Lorentz basis (see last commutators in (5.3), (5.5),(5.6)).

Anticommutators between supercharges are

\[ \{\Omega^+, \Omega^+\} = -2i\hat{\gamma}^+ P^+, \quad \{\Omega^-, \Omega^-\} = -2i\hat{\gamma}^+ C, \quad \{S^-, S^-\} = -4i\gamma^+ K^-, \quad (5.46) \]

\[ \{\hat{\Omega}^-, \Omega^-\} = -2i\hat{\gamma}^+ P^- + f\gamma^{+ij} J^{ij} - f\gamma^{+MN} J^{MN}, \quad (5.47) \]

\[ \{\Omega^-, \Omega^+\} = -i\hat{\gamma}^+ \gamma^- \hat{\gamma}^i T^i, \quad \{\Omega^-, S^-\} = 2i\hat{\gamma}^+ i C^i, \quad (5.48) \]

\[ \{S^-, \Omega^+\} = -i\gamma^+ \gamma^- D - \frac{1}{2} \gamma^+ \gamma^- \gamma^{ij} J^{ij} + \frac{1}{2} \gamma^+ \gamma^- \gamma^{MN} J^{MN}. \quad (5.49) \]

Modulo (anti)commutators obtainable from the ones above-given by applying hermitian conjugation, the remaining (anti)commutators are equal to zero.

Now let us match the symmetries generated by the \( psu(2,2|4) \) superalgebra and the ones generated by superalgebra of plane wave superstring (see Appendix B). Because these superalgebras do not coincide we can match only some subset of generators of these superalgebras. Let \( G_{str} \) be generators of plane wave superstring superalgebra with (anti)commutators given in Appendix B. Making use of Lorentz basis of the \( psu(2,2|4) \) superalgebra we find the following interrelations of the generators of the \( psu(2,2|4) \) and plane wave superstring superalgebra

\[ P_{str}^- + 2f J_{str}^{34} = P^- + f^2 K^+, \quad P_{str}^i = P^i, \quad P_{str}^+ = P^+, \quad (5.50) \]

\[ J_{str}^{+i} = J^{+i}, \quad J_{str}^{12} = J^{12}, \quad J_{str}^{34} = J^{34}, \quad J_{str}^{ij} = J^{ij}, \quad (5.51) \]

\[ \frac{1}{\sqrt{2}} (Q_{str}^1 \gamma^{-+12} + Q_{str}^2) = Q - \frac{f}{2} \gamma^{+34} S^+. \quad (5.52) \]

Thus one can match explicitly 14 bosonic and 16 super symmetries of plane wave SYM and corresponding 14 bosonic and 16 super- symmetries of plane wave superstring theory26.

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26Interesting discussion of realization of conformal and Heisenberg algebras in plane wave CFT correspondence may be found in [65].
6 N=4 SYM in plane wave background

Our goal is to develop light cone formulation of \(N = 4\) super YM theory in 4d plane wave background. The most convenient way to do this is to start with covariant and gauge invariant formulation. As compared to flat space the covariant formulation of plane wave SYM becomes to be more complicated as in this case we deal with theory in curved background\(^{27}\). Light cone gauge formulation leads to dramatic simplification. A remarkable fact we demonstrate below is that the 3-point and 4-point interaction vertices of light cone gauge action of plane wave SYM take the same form as the ones in the flat space. The only difference of plane wave SYM as compared to the one in flat space is an appearance of mass-like terms in the quadratic part of the action.

Following our strategy we start with covariant and gauge invariant action of SYM theory defined on the 4d plane wave background with line element given in (5.10) To simplify our presentation of the 4d SYM theory we use the standard trick of 10d notation. We use 10d space-time which is direct product of 4d plane wave background (5.10) and flat \(R^6\) Euclidean space. In this 10d space-time we introduce the 10d target space vector \(A_\mu, \mu = 0, 1, \ldots, 9\). This vector is splitted into 4d plane wave target space vector \(A_m, m = 0, 1, 2, 9\), and \(SO(6)\) vector \(A_M, M = 3, \ldots, 8\). The fermionic partner of \(A_\mu\) is a sixteen component spinor field \(\psi^a\) which in the 32 component notation corresponds to positive chirality Majorana-Weyl spinor. Both the \(A_\mu\) and \(\psi\) are transforming in adjoint representation of certain gauge group. They can be decomposed into generators of Lie algebra and are assumed to be anti-hermitian

\[
A_\mu = A_\mu^a t_a, \quad \psi = \psi^a t_a, \quad A_\mu^\dagger = -A_\mu, \quad \psi^\dagger = -\psi.
\]

The Lagrangian of SYM in plane wave background is given by\(^{28}\)

\[
\mathcal{L} = \hat{\text{Tr}} \left[ -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{i}{2} \bar{\psi} \gamma^\mu D_\mu \psi \right],
\]

where the gauge and Lorentz covariant derivative \(D_\mu\) and the field strength \(F_{\mu\nu}\) are defined to be

\[
D_\mu \psi = D^\nu_\mu \psi + [A_\mu, \psi], \quad D^\nu_\mu = \partial_\mu - \frac{i f^2}{2} x^i \gamma^+ \gamma^i \delta^\nu_m, \quad D^\nu_\mu = \partial_\mu,
\]

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].
\]

In (6.2) and below the \(\hat{\text{Tr}}\) denotes minus trace

\[
\hat{\text{Tr}} Y \equiv -\text{Tr} Y.
\]

\(^{27}\)Discussion of SYM theory in various curved backgrounds may be found in [66]. Plane wave SYM does not fall in the cases considered in [66]. Detailed study of the Hamiltonian formulation of SYM in \(R \times S^3\) background may be found in [67]. Relevance of a curved background for SYM was discussed (in the original AdS/CFT context) in [68].

\(^{28}\)Note that because Richi scalar of 4d plane wave metric (5.10) is equal to zero the action (6.2) is indeed describes conformal scalar fields. The target space \(\gamma^\mu\)-matrices are defined in terms of tangent space ones as \(\gamma^\mu = e^\mu_\nu \gamma^\nu\), where \(e^\mu_\nu\) is inverse to the basis of one forms \(e^\mu = e^\mu_\nu dx^\nu, e^\mu_\nu e^\nu_\mu = \delta^\mu_\nu\). The basis of \(e^\mu\) is specified in (6.7). Notation \(\gamma^+, \gamma^-, \gamma^I\) is used only for tangent space \(\gamma\)-matrices defined in (A.3).
All fields are assumed to be independent of six coordinates \( x^M \)

\[
\partial^M A_\mu = 0, \quad \partial^M \psi = 0. \tag{6.6}
\]

10d target space indices are contracted by metric tensor \( g_{\mu \nu}, A_\mu = g_{\mu \nu} A_\nu \), which describes 4d plane wave space-time \((5.10) \times \) flat \( R^6 \) Euclidean space. 10d tangent space vectors \( A^\mu, \mu = 0, 1, \ldots, 9 \), are defined in terms the target space ones as \( A^\mu = e^\mu_\nu A_\nu \) where basis of one-forms \( e^\mu = e^\mu_\nu dx^\nu \) is specified to be

\[
e^+ = dx^+, \quad e^- = dx^- - \frac{f^2}{2} x^i x^i dx^+, \quad e^I = dx^I, \quad I = 1, \ldots, 8, \tag{6.7}
\]

so that we have

\[
g_{\mu \nu} = \left( \begin{array}{cc} g_{mn} & 0 \\ 0 & \delta_{MN} \end{array} \right), \quad g_{\mu \nu} = e^\mu_\nu e_\mu, \quad g_{nn} = e^2_{nm} e_{nm}, \tag{6.8}
\]

\( \mu = (m, M); \mu = (m, M); \nu = (m, n) = 0, 1, 2, 9; m, n = 0, 1, 2, 9; M, N = 3, \ldots, 8^n \).

The Lagrangian (6.2) is invariant with respect to the local gauge transformations and the global \( psu(2, 2|4) \) superalgebra transformations. All these transformations are realized on the space of covariant fields \( A_\mu, \psi \). Below we will be interested in transformations of physical fields under action of \( psu(2, 2|4) \) superalgebra. These transformations can be derived in a most straightforward way by starting with transformations of the covariant fields \( A_\mu, \psi \). To fix our notation let us briefly discuss the latter transformations.

(i) **Local gauge transformations** take the standard well known form

\[
\delta A_\mu = \partial_\mu \alpha + [A_\mu, \alpha], \quad \delta \psi = [\psi, \alpha]. \tag{6.9}
\]

(ii) **Global so(4, 2) conformal transformations.** The transformations of fields under action of the so(4, 2) algebra are given by the relations

\[
\delta^{cov} A^m = \xi^n \partial^m A^n - A^n \partial^m \xi^n + \frac{1}{2} \sqrt{g} (\partial_\mu \sqrt{g} \xi^\mu) A^m, \tag{6.10}
\]

\[
\delta^{cov} \phi^M = \xi^n \partial^M \phi^n + \frac{1}{4} \sqrt{g} (\partial_\mu \sqrt{g} \xi^\mu) \phi^M, \tag{6.11}
\]

\[
\delta^{cov} \psi = (\xi^n D^L_m + \frac{1}{4} \gamma^m_n \partial_m \xi^n + \frac{3}{8} \sqrt{g} (\partial_\mu \sqrt{g} \xi^\mu)) \psi, \tag{6.12}
\]

where \( \xi^\mu \) are Killing vectors of the so(4, 2) algebra transformations. In plane wave basis these vectors are given by relations \((5.16), (5.19)-(5.22)\).

(iii) **Global R-symmetries generated by so(6) algebra** take the standard form.

(iv) **Global plane wave supersymmetry transformations of the psu(2, 2|4) superalgebra.** In order to find these transformations it is convenient to start with transformation of fields under action of supercharges taken to be in the Lorentz basis, i.e. basis of \( Q \) and \( S \) supercharges. Transformation of fields in such a basis are well known and are given by

\[
\delta^{cov} A^\mu = i \psi \gamma^\mu \epsilon_Q, \quad \delta^{cov} \psi = \frac{1}{2} F^{\mu \nu} \gamma_{\mu \nu} \epsilon_Q - 2 \phi^M \gamma^M \epsilon_S, \tag{6.13}
\]

\[^{29}\text{In light cone frame} \, _\mu^\nu = +, -, 1, \ldots, 8; \, \mu, \nu = +, -, 1, \ldots, 8; \, m, n = +, -, 1, 2; \, m, n = +, -, 1, 2.\]
where $\epsilon_Q$ and $\epsilon_S$ are appropriate real-valued sixteen component Killing spinors which are still to fixed. These Killing spinors can be found from the general formula

$$\epsilon_Q^\alpha Q_\alpha + \epsilon_S^\alpha S_\alpha = g_x(\epsilon_{0Q} Q + \epsilon_{0S} S)g_x^{-1},$$  

(6.14)

where bosonic coset representative $g_x$ is defined by the relation

$$g_x = \exp(x^i P_i + x^- P^+) \exp(x^+ P^-),$$  

(6.15)

and $\epsilon_{0Q}$ and $\epsilon_{0S}$ are constant sixteen component fermionic parameters. Note that the Killing spinors above defined satisfy the equations

$$D_{Lm}^\mu \epsilon_Q = \gamma_m^\mu \epsilon_S,$$  

$$D_{\bar{m}}^\mu \epsilon_S = -\frac{f^2}{2} \delta_{\bar{m}}^\mu \gamma^+ \epsilon_Q,$$

(6.16)

which should be supplemented by initial conditions

$$\epsilon_Q |_{x^\infty = 0} = \epsilon_{0Q}, \quad \epsilon_S |_{x^\infty = 0} = \epsilon_{0S}. $$

(6.17)

Explicit representation for Killing spinors can be derived then either from these equations or from formula (6.14)

$$\epsilon_Q = \frac{1}{2}(\gamma^+ \gamma^- + \gamma^- \gamma^+ \cos fx^+) \epsilon_{0Q} + (x^- \gamma^+ + x^i \gamma^i) \epsilon_S + \gamma^- \sin fx^+ \epsilon_{0S},$$  

(6.18)

$$\epsilon_S = \frac{1}{2}(\gamma^- \gamma^+ + \gamma^+ \gamma^- \cos fx^+) \epsilon_{0S} - \frac{f}{2} \gamma^+ \sin fx^+ \epsilon_{0Q},$$

(6.19)

Making use equations for Killing spinors (6.16) one can make sure that the action (6.2) is indeed invariant with respect to transformations given in (6.13). Thus there are 32 Killing spinors which are responsible for 32 super(conformal) symmetries.

By using interrelation between the Lorentz basis supercharges $Q, S$ and plane wave basis supercharges $\Omega, S^-$ (5.32) we can bring Lorentz basis supersymmetry transformations (6.13) to the plane wave basis supersymmetry transformations

$$\delta_{\Omega} A^\mu = i \psi \gamma^\mu \epsilon_\Omega,$$  

$$\delta_{s^-} A^\mu = i \psi \gamma^\mu (x^- \gamma^+ + x^i \gamma^i) \epsilon_{s^-},$$

(6.20)

$$\delta_{\Omega} \psi = \frac{1}{2} F^{\mu\nu} \gamma^{\mu\nu} \epsilon_\Omega + if^M \gamma^M \gamma^+ \epsilon_\Omega, $$

$$\delta_{s^-} \psi = \frac{1}{2} F^{\mu\nu} \gamma^{\mu\nu} (x^- \gamma^+ + x^i \gamma^i) \epsilon_{s^-} - 2f^M \gamma^M \epsilon_{s^-},$$

(6.21)

(6.22)

where the plane wave basis Killing spinors $\epsilon_\Omega$ corresponding to the supercharge $\Omega$ is given by

$$\epsilon_\Omega = \exp(-i\frac{f}{2} x^i \gamma^i \gamma^+) \exp(-i\frac{f}{2} x^+ \gamma^- \gamma^+) \epsilon_{0\Omega}. $$

(6.23)

The constant sixteen component spinors $\epsilon_{0\Omega}^\alpha$, which is complex-valued, and $(\epsilon_{0s^-})_\alpha$, which real-valued, subject to the constraints implemented by the ones for the supercharges $\Omega, S^-$ (5.36)

$$(\epsilon_{0\Omega})^\dagger = \epsilon_{0\Omega}, \quad \gamma^- \epsilon_{0s^-} = 0.$$  

(6.24)

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30The formula (6.14) can be used by exploiting the relations (5.24),(5.30) and commutation relations given in (5.4)-(5.8).
7 Hamiltonian light cone gauge dynamics of plane wave SYM

Following the standard procedure we impose light cone gauge

\[ A_\perp = 0, \]  

and plug this gauge into the covariant Lagrangian (6.2). Because such gauge fixed Lagrangian describes non-propagating field \( A_+ \) we find that the equation of motions for \( A_+ \) leads to the constraint

\[ \partial^+ A^\perp = -\partial^+ A_\parallel - \int \frac{1}{\partial^+}[A^I, \partial^+ A^I] + i \frac{1}{\partial^+}(\psi^\oplus \bar{\gamma}^+ \psi^\oplus), \]  

which allows us to express non-physical field \( 31 A_\perp = A_\parallel \) in terms of physical bosonic field \( A^I = (A^I, A^M = \phi^M) \) and physical fermionic field \( \psi^\oplus \), where

\[ \psi = \psi^\oplus + \psi^\ominus, \quad \psi^\oplus = \frac{1}{2} \gamma^- \bar{\gamma}^+ \psi, \quad \psi^\ominus = \frac{1}{2} \gamma^+ \bar{\gamma}^- \psi. \]  

Fermionic field \( \psi^\ominus \) also turns out to be non-propagating and therefore we find that equation of motion for \( \psi^\ominus \) leads to the constraint which allows us to express the \( \psi^\ominus \) in terms of physical field \( \psi^\oplus \)

\[ \psi^\ominus = -\frac{1}{2\partial^+} \gamma^+ \bar{\gamma}^I D^I \psi^\oplus, \quad D^I \psi^\oplus \equiv \partial^I \psi^\oplus + [A^I, \psi^\oplus]. \]  

Plugging solution for \( A^\perp \) and \( \psi^\ominus \) into Lagrangian (6.2) we get light cone gauge Lagrangian which can be presented as follows

\[ \mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4, \]  

where \( \mathcal{L}_2 \) describes quadratic part of Lagrangian while \( \mathcal{L}_3 \) and \( \mathcal{L}_4 \) describe 3-point and 4-point interaction vertices respectively. The quadratic part \( \mathcal{L}_2 \) is given by

\[ \mathcal{L}_2 = \hat{\text{Tr}} \frac{1}{2} A^I \Box A^I - \frac{1}{4} \psi^\oplus \bar{\gamma}^+ \Box \psi^\oplus, \]  

where the covariant D’Alembertian operator in 4d plane wave background (5.10) defined by standard relation \( \Box = \frac{1}{\sqrt{g}} \partial_m \sqrt{g} g^{mn} \partial_n \) is given by

\[ \Box = 2 \partial^+ \partial^- + \partial^j \partial^j + \Gamma^2 x^j \partial^+ \partial^+ . \]  

The 3-point and 4-point interaction vertices are given by

\[ \mathcal{L}_3 = \hat{\text{Tr}} - [A^I, A^J] \partial^I A^J - \partial^I A^J \frac{1}{\partial^+} [A^I, \partial^+ A^J] \]  

\[ + i \partial^I A^J \frac{1}{\partial^+} (\psi^\oplus \bar{\gamma}^+ \psi^\oplus) + i \frac{1}{4} [A^I, \psi^\oplus] \bar{\gamma}^+ \bar{\gamma}^I \gamma^J \partial^I \psi^\oplus + i \frac{1}{4} \partial^I \psi^\oplus \bar{\gamma}^+ \bar{\gamma}^I \gamma^J [A^I, \psi^\oplus], \]

\[ ^{31}\text{Note that after imposing light cone gauge (7.1) the relations between covariant and contr-variant vectors are simplified } A_\perp = A_\parallel = 0, A_\perp = g_{+\perp} A_{\perp} = A^- . \]
\( \mathcal{L}_4 = \frac{1}{4}[A^I, A^J]^2 - \frac{1}{2}(\frac{1}{\partial^+}[A^I, \partial^+ A^J])^2 \\
+ \frac{i}{\partial^+}(\psi^{\oplus} \gamma^+ \psi^{\oplus}) \frac{1}{\partial^+}[A^I, \partial^+ A^J] + \frac{i}{4}[A^I, \psi^{\oplus}] \frac{\gamma^I \gamma^J}{\partial^+}[A^J, \psi^{\oplus}] \\
+ \frac{1}{2}(\frac{1}{\partial^+}(\psi^{\oplus} \gamma^+ \psi^{\oplus}))^2. \) (7.9)

From these expressions it is easily seen that dimensionful constant \( f \) does not appear in the interaction vertices \( \mathcal{L}_3 \) and \( \mathcal{L}_4 \). In other words, the light cone gauge vertices of SYM in plane wave background take the same form as the ones in flat Minkowski space. This implies that many interesting investigations carried out previously in the literature for the case of light cone gauge SYM theory in flat space can be extended in relatively straightforward way to the case of SYM theory in plane wave background. All that is required is to take into account the new modified kinetic term \( \mathcal{L}_2 \) (7.6). Note that this extension is not straightforward because there are subtleties related to the mass-like terms proportional in the covariant D'Alembertian operator (see \( f^2 \)-term in (7.7)). These mass terms lead to discretization of the spectrum of the energy operator and this should be taken into account upon quantization.

Light-cone gauge action of plane wave SYM theory

\[ S = \int dx^+ d^3x (\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4), \quad d^3x \equiv dx^- dx^1 dx^2, \] (7.10)

can be brought into the Hamiltonian form

\[ S = \int dx^+ d^3x \partial^+ \left( -\partial^+ A^I \partial^- A^I - \frac{i}{2} \psi^{\oplus} \gamma^+ \partial^- \psi^{\oplus} \right) + \int dx^+ P^-, \] (7.11)

with the Hamiltonian \( H = -P^- \)

\[ P^- = \int d^3x \mathcal{P}^- , \] (7.12)

where the Hamiltonian density \( \mathcal{P}^- \) is given by

\[ \mathcal{P}^- = \partial^+ \left( -\frac{1}{2} \partial_i A^I \partial_i A^J - \frac{f^2}{2} x^2 (\partial^+ A^I)^2 - \frac{i}{4} \psi^{\oplus} \gamma^+ \partial^+ \left( \partial^2 + f^2 x^2 \partial^+ \right) \psi^{\oplus} \right) + \mathcal{L}_3 + \mathcal{L}_4. \] (7.13)

Applying standard methods to the action (7.10) we find the well known canonical Poisson-Dirac brackets

\[ [A^I(x)^a, A^J(x')^b] \mid_{P.D. \ equal x^+} = -\frac{1}{2 \partial^+} \delta(x^- - x'^-) \delta^{(2)}(x - x') \delta^{IJ} \hat{1}^{aa'}, \] (7.14)

\[ \{\psi^{\oplus \alpha}(x)^a, \psi^{\oplus \beta}(x')^b\} \mid_{P.D. \ equal x^+} = -\frac{1}{2} (\gamma^-)^{\alpha \beta} \delta(x^- - x'^-) \delta^{(2)}(x - x') \hat{1}^{aa'}, \] (7.15)

where \( \hat{1}^{aa'} \) is a minus projector operator which we insert to respect the Lie algebra indices of the physical fields \( A^I \) and \( \psi^{\oplus} \). All that is required this operator should satisfy the relation

\[ \hat{1}^{ac} \partial^+ (tc^c t_b) = \delta^a_b. \] (7.16)

30
Equation of motions for the physical fields $A^I$ and $\psi^{\oplus}$ takes then the standard Hamiltonian form

$$\partial^- A^I = [A^I, P^-]_{P.D.}, \quad \partial^- \psi^{\oplus} = [\psi^{\oplus}, P^-]_{P.D.}.$$  \hspace{1cm} (7.17)

Here and below brackets $[...,...]_{P.D.}$ stand for Poisson-Dirac brackets evaluated for equal values of evolution parameter $x^+$. In (7.17) and below the suffix ‘equal $x^+$’ is implicit.

8 Global symmetries of $\mathcal{N} = 4$ SYM in plane wave background

In this section we discuss field theoretical realization of global supersymmetries of $\mathcal{N} = 4$ SYM theory in plane wave background which are generated by $psu(2,2|4)$ superalgebra. To do that we use the framework of Noether charges. The Noether charges play an important role in analysis of the symmetries of dynamical systems. The choice of the light cone gauge spoils manifest global symmetries, and in order to demonstrate that these global invariances are still present, one needs to find the Noether charges that generate them.

8.1 Bosonic Noether charges as generators of the $psu(2,2|4)$ superalgebra

We start our discussion with bosonic Noether charges. These charges can be found following the standard procedure. Let $T^{mn}$ be symmetric, conserved, and traceless energy-momentum tensor

$$D_m T^{mn} = 0, \quad T^{mm} = 0, \quad T^{mn} = T^{nm}. \hspace{1cm} (8.1)$$

For each Killing vector $\xi^G_m$ satisfying the equations (5.15) (or (5.17)) we can construct conserved current $G^m$:

$$G^m = T^{mn} \xi^G_n, \quad \partial_m (\sqrt{g} G^m) = 0. \hspace{1cm} (8.2)$$

Making use of equations for Killing vectors (5.15),(5.17) and relations for energy-momentum tensor given in (8.1) one can make sure that the currents (8.2) satisfy the conservation law. Note that we use the coordinates given by (5.10) in which $\sqrt{g} = 1$ and therefore the conservation law takes simplified form $\partial_m G^m = 0$. Appropriate bosonic charges $G$ take then the standard form

$$G = \int d^3x T^+_m \xi^G_m, \hspace{1cm} (8.3)$$

where the measure $d^3x$ is defined in (7.10). The energy momentum tensor for $\mathcal{N} = 4, 4d$ SYM (6.2) satisfying the requirements (8.1) is well known and is given by

$$T^{\mu\nu} = \hat{\text{Tr}} \left( F^{\mu\nu} F_{\rho\sigma}^\perp + i/4 \psi (\gamma^\mu D^\nu + \gamma^\nu D^\mu) \psi \right) + \Delta T^{\mu\nu} + g^{\mu\nu} L, \hspace{1cm} (8.4)$$

where ‘improving’ contribution given by
\[
\Delta T^{\mu \nu} = \frac{1}{6}(g^{\mu \nu} D^2_{\mu} - D^\mu D^\nu + R^{\mu \nu}) \hat{\text{Tr}} \phi^2, \quad \Delta T^{M N} = \Delta T^{M N} = 0, \quad (8.5)
\]

is to respect traceless condition (8.1). While writing the expression for \( T^{\mu \nu} \) (8.4) we take into account that Richi curvature scalar in plane wave background (5.10) is equal to zero \( R = 0 \). Note that the only non-zero component of Richi tensor is \( R^{-} = 2\phi^2 \).

To evaluate the charges (8.3) we have to determine expressions for \( T^{+ -} \), \( T^{+ I} \), \( T^{++} \).

To this end we plug light cone gauge (7.1) and solution for non-physical fields (7.2), (7.4) into expressions for energy-momentum tensor (8.4) and find32

\[
T^{+ -} = \mathcal{P}^+ - \frac{1}{6} \partial^+ \partial^+ \hat{\text{Tr}} \phi^2, \quad (8.6)
\]

\[
T^{+ I} = \mathcal{P}^I - \frac{1}{2} \partial^I \mathcal{M}^{IJ} - \frac{1}{2} \partial^+ (\mathcal{M}^{-I} - \mathcal{R}^{-I}) - \frac{1}{6} \delta_I^J \partial^+ \partial^+ \hat{\text{Tr}} \phi^2, \quad (8.7)
\]

\[
T^{++} = \mathcal{P}^+ + \frac{1}{2} \partial^J (\mathcal{M}^{-I} + \mathcal{R}^{-I}) + \frac{1}{6} (\Box - \partial^+ \partial^-) \hat{\text{Tr}} \phi^2, \quad (8.8)
\]

where D’Alembertian operator \( \Box \) is defined by (7.7) and we introduce momentum densities

\[
\mathcal{P}^+ \equiv \hat{\text{Tr}} \partial^+ A^J \partial^+ A^J + i\frac{1}{2} \psi^{\oplus} \gamma^+ \partial^+ \psi^{\oplus}, \quad (8.9)
\]

\[
\mathcal{P}^I \equiv \hat{\text{Tr}} \partial^+ A^J \partial^I A^J + i\frac{1}{2} \psi^{\oplus} \gamma^+ \partial^I \psi^{\oplus}, \quad (8.10)
\]

\[
\mathcal{P}^- = \hat{\text{Tr}} \left( \frac{1}{2} \partial^J A^I \partial^I A^J - f^2 A^2 (A^+ A^+)^2 - \frac{i}{2} \psi^{\oplus} \gamma^+ \frac{2}{\partial^+} \hat{\text{Tr}} \phi^2 \right) + \mathcal{L}_3 + \mathcal{L}_4, \quad (8.11)
\]

and spin densities33

\[
\mathcal{M}^{IJ} \equiv \hat{\text{Tr}} \partial^+ A^I \partial^+ A^J - \partial^+ A^I A^J + i\frac{1}{4} \psi^{\oplus} \gamma^{IJ} \psi^{\oplus}, \quad (8.12)
\]

\[
\mathcal{M}^{-I} \equiv \hat{\text{Tr}} - A^I \partial^+ A^J + A^J \partial^I A^J + i\frac{1}{4} \psi^{\oplus} \gamma^+ \gamma^J \frac{\partial^J}{\partial^+} \psi^{\oplus} - A^J \frac{1}{\partial^+} ([A^J, \partial^+ A^J] - i\psi^{\oplus} \gamma^+ \psi^{\oplus}). \quad (8.13)
\]

The expressions for 3-point and 4-point vertices \( \mathcal{L}_3, \mathcal{L}_3 \) which enter definition of Hamiltonian density \( \mathcal{P}^- \) (8.11),(7.13) are given in (7.8),(7.9). The density \( \mathcal{R}^{-I} \) which appears in (8.7),(8.8) is given by

\[
\mathcal{R}^{-I} \equiv \hat{\text{Tr}} A^I \frac{1}{\partial^+} ([A^J, \partial^+ A^I] - i\psi^{\oplus} \gamma^+ \psi^{\oplus}) - i\frac{1}{4} \psi^{\oplus} \gamma^+ \gamma^J \frac{\partial^J}{\partial^+} [A^I, \psi^{\oplus}]. \quad (8.14)
\]

32In formulas (8.6)-(8.14) we keep dependence on six coordinates \( x^M \). To get expressions corresponding to 4d SYM one needs to apply rules (6.6).

33The matrix \( \mathcal{M}^{IJ} \) should not be confused with the one given in (C.17),(C.19).
In contrast to the momentum and spin densities (8.9)-(8.13) the density $\mathcal{R}^{-1}$ does not contribute to charges.

Making use of general formula for charges (8.3), explicit representation for components of energy-momentum tensor (8.6)-(8.8), and for Killing vectors (5.16), (5.19)-(5.22) we can get explicit representation for charges in a rather straightforward way. For instance field theoretical representation for kinematical generators of isometry transformations, i.e. $P^+, T^i$, and $J^{ij}$ is given by

$$P^+ = \int d^3x \mathcal{P}^+,$$  
(8.15)

$$T^i = \int d^3x e^{-i\mathbf{x}^+} (\mathcal{P}^i + i\mathbf{x}^i\mathcal{P}^+) ,$$  
(8.16)

$$J^{ij} = \int d^3x (x^j\mathcal{P}^i - x^i\mathcal{P}^j + \mathcal{M}^{ij}).$$  
(8.17)

From these expressions it is easily seen that expressions for generators $P^+, J^{ij}$ coincide with the ones in flat Minkowski space-time. Field theoretical representation for generators of $R$-symmetries so(6) is given by

$$J^{MN} = \int d^3x \mathcal{P}_R \partial^+ \phi^M \phi^N - \partial^+ \phi^N \phi^M + \frac{i}{4} \psi^{[i} \mathcal{P}^{+M} \mathcal{P}^N] \psi_{j]} .$$  
(8.18)

This expression for $J^{MN}$ obviously coincides with the one in flat space-time.

Field theoretical representation for generators of conformal transformations, i.e. $D$, $C$, $C^i$ and $K^-$, is given by

$$D = \int d^3x (2x^-\mathcal{P}^+ + x^i\mathcal{P}^i) ,$$  
(8.19)

$$C = \int d^3x e^{-2ifx^+} (\mathcal{P}^- - ifx^i\mathcal{P}^i + f^2x^2\mathcal{P}^+) ,$$  
(8.20)

$$C^i = \int d^3x e^{-ifx^+} (x^-\mathcal{P}^i - x^i\mathcal{P}^- + i(f\frac{1}{2}x^2\mathcal{P}^i + x^i(x^-\mathcal{P}^+ + x^j\mathcal{P}^j))$$  
$$-\frac{f^2}{2}x^i\mathcal{P}^+ + \mathcal{M}^{-i} + if\mathcal{M}^{ij}x^j) ,$$  
(8.21)

$$K^- = \int d^3x (-\frac{1}{2}x^2\mathcal{P}^- + x^- (x^-\mathcal{P}^+ + x^i\mathcal{P}^i) - \frac{f^2}{4}(x^2)^2\mathcal{P}^+ + \mathcal{M}^{-i}x^i - \hat{\mathcal{T}} \phi^2) .$$  
(8.22)

Taking into account expressions for momentum densities (8.9),(8.10) it is easy to see that dilatation generator $D$ (8.19) is quadratic in fields, i.e. the $D$ is a kinematical generator. The remaining proper conformal generators $C$, $C^i$, and $K^-$ are realized non-linearly, i.e. they can be considered as dynamical generators.

With the definition of charges given in (8.15)-(8.22) and commutation relations for fields given in (7.14),(7.15) the transformation rules of the physical fields $A^I = (A^i, A^M = \phi^M)$ and $\psi^{[i}$ under action the conformal algebra so(4,2) and so(6) algebra are given by

$$\delta_G A^I = [A^I, G]_{P.D.} , \quad \delta_G \psi^{[i} = [\psi^{[i}, G]_{P.D.} .$$  
(8.23)

Discussion of some details of these transformations may be found in Appendix D.
8.2 Noether supercharges as generators of the superalgebra \( psu(2, 2|4) \)

In this section we describe field theoretical realization of supercharges of \( \mathcal{N} = 4 \) plane wave SYM theory. We start our discussion with description of supercurrents. As before it is convenient to start with study of supercurrents taken to be in Lorentz basis. Because in the Lorentz basis we deal with the supercharges \( Q, S \) we introduce corresponding supercurrents denoted by \( Q^m \) and \( S^m \). These supercurrents can be found by using standard procedure and explicit expression for them can be fixed by using the formula

\[
\epsilon_{oo} Q^m + \epsilon_{os} S^m = \hat{\text{Tr}} \frac{1}{2} \epsilon_Q \gamma^{\mu\nu} F^{\mu\nu} \gamma^m \psi + 2 \epsilon_s \phi^M \gamma^M \gamma^m \psi ,
\]

where \( \epsilon_{oo} \) and \( \epsilon_{os} \) are constant spinor while \( \epsilon_Q \) and \( \epsilon_s \) are the Killing spinor in 4d plane wave background. These Killing spinors satisfy the defining equations (6.16) and explicit solution to these equations is given in (6.18),(6.19). Making use of defining equations for Killing spinors (6.16) and equations of motion for fields of SYM one can check that super-currents (8.24) are indeed conserved

\[
\partial_m (\sqrt{g} Q^m) = 0 , \quad \partial_m (\sqrt{g} S^m) = 0 .
\]

Explicit expressions for supercurrents \( Q^m \) and \( S^m \) in terms of covariant fields can be obtained by plugging solution to Killing spinors into (8.24)

\[
Q^m = \hat{\text{Tr}} \frac{1}{4} \left( \gamma^- \gamma^+ + \gamma^+ \gamma^- \cos fx^+ - f \gamma^+ x^j \sin f x^+ \right) F^{\mu\nu} \gamma^{\mu\nu} \gamma^m \psi ,
\]

\[
S^m = \hat{\text{Tr}} \frac{1}{4} \left( \gamma^+ \gamma^- - \gamma^- \gamma^+ \cos fx^+ \right) \left( x^- \gamma^+ + x^j \gamma^j \right) F^{\mu\nu} \gamma^{\mu\nu} \gamma^m \psi + \gamma^- \frac{\sin fx^+}{2f} F^{\mu\nu} \gamma^{\mu\nu} \gamma^m \psi ,
\]

The expressions for supercharges \( Q \) and \( S \) can be found then from \( m = + \) components of the supercurrents (8.26),(8.27)

\[
Q = \int d^3 x \; Q^m \big|_{m=+} , \quad S = \int d^3 x \; S^m \big|_{m=+} .
\]

Taking into account the interrelation of Lorentz basis supercharges and the ones of plane wave basis (5.32) we find plane wave Noether supercharge

\[
\Omega = \int d^3 x \; \frac{1}{2} \text{Tr} \exp(-i f x^+ \gamma^+ \gamma^-) \exp(-i f x^j \gamma^j \gamma^+ \gamma^-) F^{\mu\nu} \gamma^{\mu\nu} \gamma^+ \psi .
\]

Dividing the supercharge \( \Omega \) into \( \Omega^+ \) and \( \Omega^- \) parts and working out an expression for the remaining supercharge \( S^- \) gives the following representation for the plane wave basis Noether supercharges in terms of the covariant fields

\[
\Omega^+ = \int d^3 x \; \frac{1}{4} \gamma^- \gamma^+ F^{\mu\nu} \gamma^{\mu\nu} \gamma^+ \psi ,
\]

\[
\Omega^- = \int d^3 x \; \frac{1}{4} \gamma^+ \gamma^- F^{\mu\nu} \gamma^{\mu\nu} \gamma^- \psi .
\]
\[ \Omega^- = \int d^3 x \, \text{Tr} \frac{1}{4} e^{-ifx^+} (\gamma^+\gamma^- - if\gamma^+ x^\gamma) F_{\mu\nu} \gamma^\mu \gamma^\nu \psi, \]  
(8.31)

\[ S^- = \int d^3 x \, \text{Tr} \frac{1}{4} \gamma^+ \gamma^- (x^- \gamma^+ + x^\gamma \gamma^+) F_{\mu\nu} \gamma^\mu \gamma^\nu \psi + 2\phi^M \gamma^M \gamma^+ \psi. \]  
(8.32)

In light cone frame these expressions take the form

\[ \Omega^+ = \int d^3 x \, \text{Tr} \left( -2F^{++} \gamma^I \psi^{\oplus} \right), \]  
(8.33)

\[ \Omega^- = \int d^3 x \, \text{Tr} e^{-ifx^+} \left( -F^{--} + \frac{1}{2} \gamma^{IJ} F^{IJ} + ifx^\gamma \gamma^J F^{+J} \right) \gamma^+ \psi^{\oplus}, \]  
(8.34)

\[ S^- = \int d^3 x \, \text{Tr} \left( 2x^- \gamma^I F^{+I} + x^\gamma \gamma^+ (F^{++} - \frac{1}{2} \gamma^{IJ} F^{IJ}) + 2\gamma^M \phi^M \right) \gamma^+ \psi^{\oplus}. \]  
(8.35)

Note that while writing these expressions we have not used light cone gauge.

Light cone gauge representation for above-given supercharges in terms of physical fields is obtainable then by using light cone gauge fixed field strengths. Such field strengths can be obtained by plugging light cone gauge (7.1) and solution to constraints (7.2) into covariant field strengths given in (6.4). Doing this we get the following light cone gauge fixed field strengths

\[ F^{+I} = \partial^+ A^I, \quad F^{IJ} = \partial^I A^J - \partial^J A^I + [A^I, A^J], \]  
(8.36)

\[ F^{++} = -\partial^+ A^I - \frac{1}{\partial^+} [A^I, \partial^+ A^I] + \frac{i}{\partial^+} (\psi^{\oplus} \gamma^+ \psi^{\oplus}). \]  
(8.37)

Making use of these formulas we get the following useful relation to be inserted into expression for supercharges \( \Omega^-, S^- \)

\[ -F^{++} + \frac{1}{2} \gamma^{IJ} F^{IJ} = \left( \partial^I A^J + \frac{1}{2} \partial^+ [A^I, \partial^+ A^J] \right) \gamma^I \gamma^J - \frac{i}{\partial^+} (\psi^{\oplus} \gamma^+ \psi^{\oplus}). \]  
(8.38)

In Section 7 we demonstrated that the light cone gauge vertices of SYM in plane wave background take the same form as the ones in flat Minkowski space-time. Now it can easily be checked that on the surface of initial data \( x^+ = 0 \) all bosonic and fermionic Noether charges of plane wave SYM coincide with the ones of flat Minkowski space-time. It is clear that underlying reason for these coincides is related with conformal invariance of these SYM theories.

### 9 Transformations of physical fields under action of \( psu(2, 2|4) \) superalgebra

Because the expressions for Noether charges are given entirely in terms of physical field \( A^I, A^M = \phi^M \) and \( \psi^{\oplus} \) transformations of these fields under action of \( psu(2, 2|4) \) superalgebra could be found in principle by using formulas (8.23) and Poisson-Dirac brackets for physical fields (7.14), (7.15). Here we discuss simpler method of deriving field transformations generated by \( psu(2, 2|4) \) algebra. Let us start our discussion with transformations
generated by the bosonic symmetries which are \(so(4, 2) \oplus so(6)\). Because realization of \(R\)-symmetries of \(so(6)\) takes the standard form we proceed to discussion of transformations generated by the conformal \(so(4, 2)\) algebra.

In order to find global transformations of physical fields we start with corresponding global transformations of covariant fields supplemented with local gauge transformations

\[
\delta_G A^\mu = \delta^{\text{cov}} A^\mu + D^\mu \alpha^G, \quad \delta_G \psi = \delta^{\text{cov}} \psi + [\psi, \alpha^G],
\]

where global transformations of the covariant fields \(A^\mu = (A^\mu, \phi^M)\) and \(\psi\) are given in (6.10)-(6.12). As usual the parameter of gauge transformation \(\alpha^G\) corresponding to global transformation generated by Noether charge \(G\) is fixed by the requirement that the transformation (9.1) maintains the light cone gauge

\[
\delta_G A^+ = 0.
\]

From this equation we can get solution to parameters of compensating gauge transformations. It turns out that parameters of compensating gauge transformations for the isometry transformations generated by \(P^+, T^i, J^{ij}\) and three proper conformal transformations generated by \(D\) and \(C\) are equal to zero

\[
\alpha^{P^+} = 0, \quad \alpha^{T^i} = 0, \quad \alpha^{J^{ij}} = 0, \quad \alpha^D = 0, \quad \alpha^C = 0,
\]

while the parameters corresponding to \(C^i\) and \(K^-\) are given by

\[
\alpha^{C^i} = -e^{-ifx^+} \frac{1}{\partial^+} A^i, \quad \alpha^K^- = -\frac{1}{\partial^+} x^i A^i.
\]

Plugging the solutions parameters into (9.1) we can get desired transformations for physical fields. The transformations corresponding to isometry generators \(P^+, T^i, J^{ij}\) are given in (D.5)-(D.8) while the transformations corresponding to proper conformal generators \(D, C, C^i, K^-\) are listed below.

**Proper conformal transformation of the physical spin 1 field \(A^i\):**

\[
\begin{align*}
\delta_D A^i & = (\xi^D \partial + 1) A^i, \\
\delta_C A^i & = (\xi^C \partial - ife^{-2ifx^+}) A^i, \\
\delta_{C^j} A^i & = (\xi^{C^j} \partial + ifx^j e^{-ifx^+}) A^i - e^{-ifx^+} \delta^{ij} A^- \\
& + ife^{-ifx^+} (x^k A^k \delta^{ij} - x^i A^j) - e^{-ifx^+} D^i \frac{1}{\partial^+} A^j, \\
\delta_{K^-} A^i & = (\xi^{K^-} \partial + x^-) A^i - x^i A^- - D^i \frac{1}{\partial^+} (x^j A^j).
\end{align*}
\]

**Proper conformal transformation of the scalar fields \(A^M = \phi^M\):**

\[
\begin{align*}
\delta_D \phi^M & = (\xi^D \partial + 1) \phi^M, \\
\delta_C \phi^M & = (\xi^C \partial - ife^{-2ifx^+}) \phi^M, \\
\delta_{C^i} \phi^M & = (\xi^{C^j} \partial + ifx^j e^{-ifx^+}) \phi^M - e^{-ifx^+} [\phi^M, \frac{1}{\partial^+} A^i], \\
\delta_{K^-} \phi^M & = (\xi^{K^-} \partial + x^-) \phi^M - [\phi^M, \frac{1}{\partial^+} x^i A^i].
\end{align*}
\]
Proper conformal transformation of the physical spin 1/2 field $\psi^\oplus$:

$$\delta_D \psi^\oplus = (\xi^D \partial + 2) \psi^\oplus,$$

(9.13)

$$\delta_C \psi^\oplus = (\xi^C \partial - i e^{-2ifx^+}) \psi^\oplus,$$

(9.14)

$$\delta_{c^i} \psi^\oplus = (\xi^{c^i} \partial + e^{-ifx^+} (\frac{3i}{2} f x^i + i f_\gamma \gamma^i x^j)) \psi^\oplus + \frac{1}{2} e^{-ifx^+} \gamma^{-i} \psi^\oplus + [\psi^\oplus, \alpha^{c^i}],$$

(9.15)

$$\delta_{K^-} \psi^\oplus = (\xi^{K^-} \partial + 2x^-) \psi^\oplus + \frac{1}{2} \gamma^{-i} x^i \psi^\oplus + [\psi^\oplus, \alpha^{K^-}].$$

(9.16)

Solutions to the non-physical fields $A^-$, $\psi^\ominus$ to be inserted in these expressions are given by (7.2), (7.4), while the expressions for Killing vectors in terms of differential operators $\xi^G \partial \equiv \xi^G \partial_\mu$ are given in (5.19)-(5.22). Replacing in above given transformations rules the time derivatives of physical fields by the commutator with Hamiltonian (7.17) we get famous off-shell light cone realization of global symmetries.

### 9.1 Supersymmetry transformations of physical fields

Supersymmetry transformations of physical fields can be fixed by using the same procedure as in previous section. As before we start with general transformation rules (9.1) where the supersymmetry transformations of covariant fields on r.h.s. are given by (6.20)-(6.22). Before to proceed to transformations of physical fields let us cast the transformations of covariant fields into more convenient form. In light cone formalism it is convenient to divide explicitly the supersymmetry transformations into the ones generated by supercharges $\Omega^+$ and $\Omega^-$. This can be made in a straightforward way by using formulas for $\Omega$ transformations given in (6.20)-(6.22). In transparent form supersymmetry transformations in basis formed by supercharges $\Omega^\pm$, $S^-$ take then the form

$$\delta^\text{cov} A^\mu = i \psi^{\gamma_\mu} \epsilon_{\alpha q^+},$$

(9.17)

$$\delta^\text{cov} A^\mu = i e^{-ifx^+} \psi_{\gamma^\mu} (1 + i \frac{f}{2} x^i \gamma^i) \epsilon_{\alpha q^-},$$

(9.18)

$$\delta^\text{cov} A^\mu = i \psi_{\gamma^\mu} (x^- \gamma^+ + x^i \gamma^i) \epsilon_{s^-},$$

(9.19)

$$\delta^\text{cov} \psi = \frac{1}{2} F^{\mu\nu} \gamma^\mu \epsilon_{\alpha q^+},$$

(9.20)

$$\delta^\text{cov} \psi = \frac{1}{2} e^{-ifx^+} F^{\mu\nu} \gamma^\mu (1 + i \frac{f}{2} x^i \gamma^i) \epsilon_{\alpha q^-} + i e^{-ifx^+} \phi^M \gamma^M \gamma^+ \epsilon_{\alpha q^-},$$

(9.21)

$$\delta^\text{cov} \psi = \frac{1}{2} F^{\mu\nu} \gamma^\mu (x^- \gamma^+ + x^i \gamma^i) \epsilon_{s^-} - 2 \phi^M \gamma^M \epsilon_{s^-},$$

(9.22)

where the constant parameters of transformations satisfy the constraints

$$\gamma^+ \epsilon_{\alpha q^+} = 0, \quad \gamma^- \epsilon_{\alpha q^-} = 0, \quad \gamma^- \epsilon_{s^-} = 0.$$

(9.23)

Now we plug these transformations rules on r.h.s. of (9.1) and from the requirement (9.2) we find solution to the parameters of compensating gauge transformations

$$\alpha^{\Omega^+} = 0, \quad \alpha^{\Omega^-} = -i \frac{e^{-ifx^+} \psi^{\gamma^+} \gamma^+ \epsilon_{\alpha q^-}}{\partial^+}, \quad \alpha^S^- = -i \frac{\psi^{\gamma^+} \gamma^+ x^i \epsilon_{s^-}}{\partial^+}.$$

(9.24)
Plugging these compensating parameters into (9.1) we get the desired supersymmetry transformations of the physical fields \( A^I = (A^i, A^M = \phi^M) \) and \( \psi^\oplus \):

\[
\begin{align*}
\delta_{\alpha^+} A^I &= i \psi^\oplus \gamma^I \epsilon_{\alpha^+}, \\
\delta_{\alpha^-} A^I &= i e^{-i x^+} (\psi^\oplus \gamma^I + \frac{if}{2} \gamma^I \gamma^J \gamma^k \phi^M) \epsilon_{\alpha^-} + D^I \alpha^- , \\
\delta_{\alpha^-} A^I &= i (x^- \psi^\oplus \gamma^I + x^+ \psi^\oplus \gamma^I \gamma^j) \epsilon_{\alpha^-} + D^I \alpha^- , \\
\delta_{\alpha^+} \psi^\oplus &= \gamma^{-I} F^{+I} \epsilon_{\alpha^+}, \\
\delta_{\alpha^-} \psi^\oplus &= e^{-i x^+} \left( F^{+} + \frac{1}{2} \gamma^{IJ} F^{IJ} - i f \gamma^{J} \gamma^I x^I \right) \epsilon_{\alpha^-} + [\psi^\oplus, \alpha^-], \\
\delta_{\alpha^-} \psi^\oplus &= \left( (F^{+} + \frac{1}{2} \gamma^{IJ} F^{IJ}) x^k \gamma^k - 2 x^- F^{+} \gamma^J - 2 \gamma^M \phi^M \right) \epsilon_{\alpha^-} + [\psi^\oplus, \alpha^-].
\end{align*}
\]

The expressions for the non-physical field \( \psi^\ominus \) and gauge fixed field strengths to be inserted in these formulas may be found in (7.4),(8.36),(8.37).

### 10 Conclusions

To summarize, we have found the supersymmetric action for a \( D3 \) brane probe propagating in plane wave Ramond-Ramond background. The action is given by (2.1)–(2.5) with the closed 5-form defining the WZ term given in (2.13). This action is world-volume reparametrisation invariant and \( \kappa \)-invariant. Its advantage is that it is manifestly invariant under the symmetries of plane wave vacuum: 30 bosonic isometries and 32 supersymmetries. It does not have a particularly simple form when written in terms of the coordinates \( (x, \theta) \), even using the closed expressions for the Cartan 1-forms in terms of \( \theta \) (see Appendix C).

The action can be put in a more explicit form by imposing various kappa symmetry gauges. For instance to establish a connection to the abelian \( \mathcal{N} = 4 \) SYM theory as discussed in the Introduction we

(i) fix the \( \kappa \)-symmetry gauge in a way that simplifies the fermionic part of the action.
(ii) fix the static gauge so that the \( D3 \) probe is oriented parallel to the \( D3 \) source.

After fixing the local symmetry gauges only the bosonic seven isometry symmetries of 4d plane wave background, the seven R-symmetries \( SO(2) \times SO'(4) \), and 16 supersymmetries of the original symmetries remain unbroken, while the remaining 16 bosonic and 16 supersymmetries are broken and realized non-linearly. Interesting fact is that the number of unbroken space-time symmetries (seven isometry symmetries of 4d plane wave background) is equal to the number of unbroken R-symmetries (seven symmetries of \( SO(2) \times SO'(4) \)). Note also that the number of broken bosonic symmetries is equal to the number of broken supersymmetries.

We developed the Hamiltonian light cone gauge formulation of plane wave SYM and demonstrated that in contrast to covariant and gauge invariant formulation the 3-point and 4-point light cone gauge vertices of plane wave SYM theory takes exactly the same form as the ones of SYM in flat Minkowski space-time.
The results presented here should have a number of interesting applications and generalizations, some of which are:

1. In Refs. [20, 69] various improvements of the original BMN operators [3] were suggested. We expect that study based on analysis of conformal properties of plane wave SYM which we performed in this papers together with ideas and approaches of Refs. [70, 71] should allow us to derive BMN operators from first principles and fix their precise form.

2. In this paper we develop light cone gauge formulation of plane wave SYM using field component formulation. As in flat space [72] the plane wave SYM could also be reformulated in terms of unconstrained light cone superfield. It would be interesting then to apply such superfield formulation to the study of BMN conjecture [3] along the line of Ref. [22].

3. The plane wave background (5.10) can be obtained from the $R \times S^3$ space-time via the Penrose limit. Indeed consider the line element of $R \times S^3$

$$ds^2 = -dt^2 + R^2(\cos^2 \theta d\psi^2 + 2 d\psi^2 + \sin^2 \theta d\varphi^2),$$

where $R$ is a radius of $S^3$. Introducing the coordinates $x^\pm, x^3$

$$\begin{align*}
t &= \sqrt{2} f x^+ - \frac{x^-}{2\sqrt{2} f R}, \\
\psi &= \sqrt{2} f x^+ + \frac{x^-}{2\sqrt{2} f R}, \\
\theta &= \frac{|x|}{R},
\end{align*}$$

$|x| = (x^i x^i)^{1/2}$, and taking the Penrose limit $R \to \infty$ we get the line element of plane wave background (5.10). Making use of relations

$$\begin{align*}
\partial_t &= -iH = -i\frac{\Delta}{R}, \\
\partial_{x^+} &= -iH_{t.c.} = -iE_{t.c.}, \\
\partial_{x^-} &= ip^+, \\
\partial_\psi &= iJ,
\end{align*}$$

we find then the relations between light cone energy $E_{t.c.}$ and momentum $p^+$ of (states) operators of plane wave SYM and conformal dimension $\Delta$ and angular momentum $J$ of (states) operators of SYM in $R \times S^3$ space-time

$$\begin{align*}
E_{t.c.} &= \sqrt{2} f(\Delta - J), \\
p^+ &= \frac{\Delta + J}{2\sqrt{2} f R}.
\end{align*}$$

Taking into account these relations and interrelations of $\Delta$ and $J$ with plane wave superstring energy and momentum $p^+$ found in [3] it seems highly likely that new duality [3] can be re-formulated as interrelations between light cone energies and $p^+$ momenta of (states) operators of $4d$ plane wave SYM and (states) operators of $10d$ plane wave superstring theory

$$E_{t.c.}(pw \text{ SYM}) \sim E_{t.c.}(pw \text{ string}), \quad p^+(pw \text{ SYM}) \sim p^+(pw \text{ string}).$$

It is desirable to understand these interrelations better.

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Appendix A  Notation and conventions

We use the following conventions for the indices:

\[
\begin{align*}
\mu, \nu, \rho &= 0, 1, \ldots, 9 & \text{so}(9, 1) \text{ vector indices (tangent space indices)} \\
\mu, \nu, \rho &= 0, 1, \ldots, 9 & \text{Sects. 2-4: coordinate indices of } 10d \text{ plane wave background} \\
\mu, \nu, \rho &= 0, 1, \ldots, 9 & \text{Sects. 5-9: coordinate indices of } 4d \text{ plane wave } \times R^6 \text{ background} \\
I, J, K, L &= 1, \ldots, 8 & \text{so}(8) \text{ vector indices (tangent space indices)} \\
M, N &= 3, \ldots, 8 & \text{so}(6) \text{ vector indices (tangent space indices)} \\
i, j, k, l &= 1, \ldots, 4 & \text{so}(4) \text{ vector indices (tangent space indices)} \\
i', j', k', l' &= 5, \ldots, 8 & \text{so'}(4) \text{ vector indices (tangent space indices)} \\
\hat{i}, \hat{j}, \hat{k} &= 1, 2 & \text{so}(2) \text{ vector indices (tangent space indices)} \\
a, b &= 0, 1, 2, 9 & 4d \text{ D3 brane world-volume coordinate indices} \\
m, n &= 0, 1, 2, 9 & 4d \text{ plane wave space-time coordinate indices} \\
\alpha, \beta, \gamma &= 1, \ldots, 16 & \text{so}(9, 1) \text{ spinor indices in chiral representation} \\
\mathcal{I}, \mathcal{J} &= 1, 2 & \text{labels of the two real MW spinors}
\end{align*}
\]

We suppress the flat space metric tensor \( \eta_{\mu\nu} = (-, +, \ldots, +) \) in scalar products, i.e. \( X^\mu Y^\nu \equiv \eta_{\mu\nu} X^\mu Y^\nu \). We decompose \( x^2 \) into the light cone and transverse coordinates: \( x^2 = (x^+, x^-, x^I), \) \( x^I = (x^i, x^i') \), where \( x^\pm \equiv \frac{1}{\sqrt{2}}(x^0 \pm x^9) \). The scalar products of tangent space vectors are decomposed as

\[
X^\mu Y^\mu = X^+ Y^- + X^- Y^+ + X^I Y^I, \quad X^I Y^I = X^i Y^i + X^{i'} Y^{i'}. \quad (A.1)
\]

We use the chiral representation for the \( 32 \times 32 \) Dirac matrices \( \Gamma^\mu \) in terms of the \( 16 \times 16 \) matrices \( \gamma^\mu \)

\[
\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \bar{\gamma}^\mu & 0 \end{pmatrix},
\]

\[
\gamma^\mu \bar{\gamma}^\nu + \gamma^\nu \bar{\gamma}^\mu = 2\eta^\mu\nu, \quad \gamma^\mu = (\gamma^\mu)^{\alpha\beta}, \quad \bar{\gamma}^\mu = \bar{\gamma}^{\mu*}, \quad (A.3)
\]

\[
\gamma^\mu = (1, \gamma^I, \gamma^9), \quad \bar{\gamma}^\mu = (-1, \gamma^I, \gamma^9), \quad \alpha, \beta = 1, \ldots, 16. \quad (A.4)
\]

We adopt the Majorana representation for \( \Gamma \)-matrices, \( C = \Gamma^0 \), which implies that all \( \gamma^\mu \) matrices are real and symmetric, \( \gamma^\mu_{\alpha\beta} = \gamma^\mu_{\beta\alpha} \), \( (\gamma^\mu_{\alpha\beta})^* = \gamma^\mu_{\alpha\beta} \). As in [2] \( \gamma^{\mu_1 \ldots \mu_k} \) are the antisymmetrized products of \( k \) gamma matrices, e.g., \( (\gamma^{\mu\nu})_{\alpha\beta} \equiv \frac{1}{2}(\gamma^{\mu\nu})^0_{\alpha\beta} - (\mu \leftrightarrow \nu) \), \( (\bar{\gamma}^{\mu\nu})_{\alpha\beta} \equiv \frac{1}{2}(\bar{\gamma}^{\mu\nu})^0_{\alpha\beta} - (\mu \leftrightarrow \nu) \), \( (\gamma^{\mu\nu\rho})^{\alpha\beta} \equiv \frac{1}{6}(\gamma^{\mu\nu\rho})^0_{\alpha\beta} \pm 5 \) terms. Note that \( (\gamma^{\mu\nu\rho})^{\alpha\beta} \) are antisymmetric in \( \alpha, \beta \) while \( (\gamma^{\mu_1 \ldots \mu_k})_{\alpha\beta} \) is symmetric. We assume the normalization

\[
\Gamma_{11} \equiv \Gamma^0 \ldots \Gamma^0 = \begin{pmatrix} I_{16} & 0 \\ 0 & -I_{16} \end{pmatrix}, \quad \gamma^0 \gamma^1 \ldots \gamma^8 \gamma^9 = I_{16}. \quad (A.5)
\]

We use the following definitions \( \Pi^\alpha_{\mu \beta} \equiv (\gamma^1 \gamma^5 \gamma^3 \gamma^4)^{\alpha}_{\beta} \), \( (\Pi')^\alpha_{\mu \beta} \equiv (\gamma^5 \gamma^6 \gamma^7 \gamma^8)^{\alpha}_{\beta} \). Because of the relation \( \gamma_0 \gamma^9 = \gamma^+ \) the normalization condition (A.5) takes the form \( \gamma^+ \Pi \Pi' = 1 \).
The 32-component positive chirality spinor $\theta$ and the negative chirality spinor $Q$ are decomposed in terms of the 16-component spinors as
\[
\theta = \begin{pmatrix} \theta^\alpha \\ 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 \\ Q_\alpha \end{pmatrix}.
\] (A.6)

The complex Weyl spinor $\theta$ is related to the two real Majorana-Weyl spinors $\theta^1$ and $\theta^2$ by
\[
\theta = \frac{1}{\sqrt 2} (\theta^1 + i \theta^2), \quad \bar{\theta} = \frac{1}{\sqrt 2} (\theta^1 - i \theta^2).
\] (A.7)

The short-hand notation like $\bar{\theta} \gamma^\mu \theta$ and $\gamma^\mu \theta$ stand for $\bar{\theta} \gamma^\mu a_\alpha \gamma^\beta \theta$ and $\bar{\theta} \gamma^\mu a_\alpha \theta$ respectively.

**Appendix B  Plane wave Ramond-Ramond superalgebra**

Plane wave Ramond-Ramond superalgebra contains ten translation generators $P^\mu$ (in light cone frame $P^+, P^-, P^I, I = 1, \ldots, 8$), eight Lorentz boosts $J^{+I}$, six generators of $SO(4)$ rotations, $J^{ij}, i, j = 1, \ldots, 4$, six generators of $SO'(4)$ rotations, $J^{ij'}, i', j' = 5, \ldots, 8$ and two sixteen component real-valued spinor $\bar{Q}^I_\alpha, I = 1, 2, \alpha = 1, \ldots, 16$. In the 32 spinor component notation the $Q^I_\alpha$ correspond to two negative chirality Majorana-Weyl spinors (see (A.6)). Commutation relations between the even generators are given by\(^{34}\)

\[
[P^-, P^I] = -i^2 J^{+I},
\] (B.1)
\[
[P^I, J^{+J}] = -\delta^{IJ} P^+, \quad [P^-, J^{+I}] = P^I,
\] (B.2)
\[
[J^{i+}, J^{j+}] = \delta^{ij} J^{k+} - \delta^{ik} J^{j+}, \quad [J^{i+}, J^{j+}] = \delta^{ij} J^{k+} - \delta^{ik} J^{j+},
\] (B.3)
\[
[J^{ij}, J^{kl}] = \delta^{ik} J^{jl} + 3 \text{ terms}, \quad [J^{ij'}, J^{k'l'}] = \delta^{ij'} J^{k'l'} + \delta^{ik'} J^{jl'} + 3 \text{ terms},
\] (B.5)

where $f$ is a dimensionful parameter.

Commutation relations between the even and odd parts are
\[
[J^{ij}, Q^I_\alpha] = \frac{1}{2} Q^I_\beta (\gamma^{ij})^\beta_\alpha, \quad [J^{ij'}, Q^I_\alpha] = \frac{1}{2} Q^I_\beta (\gamma^{ij'})^\beta_\alpha,
\] (B.6)
\[
[J^{+I}, Q^I_\alpha] = \frac{1}{2} Q^I_\beta (\gamma^{+I})^\beta_\alpha, \quad [P^\mu, Q^I_\alpha] = -\frac{f}{2} \tau^{TJ} Q^I_\beta (\Pi \gamma^\mu \gamma^{+I})^\beta_\alpha,
\] (B.7)

The anticommutator takes the form
\[
\{Q^I_\alpha, Q^J_\beta\} = -2i \delta^{TJ} \gamma^\mu a_\alpha P^\mu - 2i f \tau^{TJ} \left( (\gamma^{ij})_{\alpha\beta} J^{+i} + (\gamma^{ij'})_{\alpha\beta} J^{+i'} \right) + \frac{f}{2} \tau^{TJ} \left( (\gamma^+ \gamma^{ij})_{\alpha\beta} J^{ij} + (\gamma^+ \gamma^{ij'})_{\alpha\beta} J^{ij'} \right),
\] (B.8)

\(^{34}\)This algebra was found in [1] as an algebra of isometry symmetries of plane wave Ramond-Ramond solution of IIB supergravity. Derivation of this superalgebra from the super-AdS$_5 \times S^5$ algebra by the Inoue-Wigner contractions may be found in [25]. Discussion of the corresponding plane wave solution of 11d supergravity is given in [73, 74]. Plane wave supermembrane and matrix theories were investigated in [75] and [76] (see also [77] for some related studies).
where $\tau_2$ is defined in (C.3). All the other commutators and anticommutators vanish. The bosonic generators are assumed to be anti-hermitian while the fermionic generators are hermitian, $(Q^T_\alpha)^\dagger = Q^T_\alpha$. The generators of the full transformation group $G$ of the plane wave RR superspace are $P^\mu$, $Q^T_\alpha$, $J^{ij}$, $J^{i'j'}$, $J^I$. The generators of the stability subgroup $H$ are $J^{ij}$, $J^{i'j'}$, $J^I$. The plane wave RR superspace is defined then as coset superspace $G/H$. The specific choice of the representative of supercoset which we use frequently in this paper is given by

$$G(x, \theta) = e^{x^+ P^-} e^{-x^- P^+} e^{i\alpha I Q^T_\alpha}. \quad (B.9)$$

Representation of the bosonic generators of plane wave superalgebra in terms of Killing vectors acting on the superspace $(x^+, x^-, x^I, \theta^{\alpha I})$ defined by (B.9) takes the form

$$P^\pm = \partial^\pm, \quad (B.10)$$

$$P^I = \cos fx^+ \partial^I + f \sin fx^+ x^I \partial^+ - \frac{f}{2} \sin fx^+ \partial_{\theta^I} (\gamma^+)^\alpha \beta \theta^{\beta I}, \quad (B.11)$$

$$J^{i+j} = \sin fx^+ \partial^I - \cos fx^+ x^I \partial^+ + \frac{1}{2} \cos fx^+ \partial_{\theta^I} (\gamma^+)^\alpha \beta \theta^{\beta I}, \quad (B.12)$$

$$J^{i+j} = x^I \partial^I - x^J \partial^J + \frac{1}{2} \partial_{\theta^{i+j}} (\gamma^{i+j})^\alpha \beta \theta^{\alpha I}, \quad (B.13)$$

where just the $so(4) \oplus so'(4)$ part of $J^{i+j}$ which is given by $J^{ij}$, $J^{i'j'}$ enters the plane wave superalgebra. Complete expressions for fermionic generators $Q^T_\alpha$ are complicated and not illuminating. Leading terms of their expansion in fermionic coordinates are given by

$$\epsilon_\alpha Q = \epsilon^{\alpha I} (x) \partial_{\theta^{i+j}} - i \epsilon^{\alpha I} \epsilon(x) \gamma^\nu \Theta \partial_{\theta^\nu} + O(\Theta^2 \partial_\Theta, \Theta^3), \quad (B.14)$$

where the Killing spinor $\epsilon(x)$ is defined by relation (2.23). In light cone gauge only the terms shown explicitly in (B.14) gives non-zero contribution to supercharge $Q^T_\alpha$. In above expressions the fermionic partial derivatives are defined to be $\partial_{\theta^{i+j}} = \partial/\partial \theta^{\alpha I}$ and we use the following conventions for bosonic partial derivatives

$$\partial^+ \equiv \partial_+ = \frac{\partial}{\partial x^+}, \quad \partial^- \equiv \partial_+ = \frac{\partial}{\partial x^-}, \quad \partial^I \equiv \partial_+ = \frac{\partial}{\partial x^I}. \quad (B.15)$$

Appendix C  Basic relations for Cartan forms on coset superspace

The left-invariant Cartan 1-forms $L^- = dX^\Delta L^-_\Delta$, $X^\Delta = (x^\mu, \theta^{\alpha I})$ are defined by

$$G^{-1}dG = L^\mu P^\mu + \frac{1}{2} L^\mu \nu J^\mu \nu + L^{\alpha I} Q^T_\alpha, \quad (C.1)$$

where $G = G(x, \theta)$ is a coset representative plane wave supergroup. $L^\mu$ are the 10-beins, $L^{\alpha I}$, $(L^{\alpha I})^\dagger = L^{\alpha I}$, are the two spinor 16-beins and $L^\mu \nu (L^\mu = 0, L^{i'j'} = 0)$ are the Cartan connections. The two 16 component Cartan 1-forms $L^{\alpha I}$ and fermionic coordinates $\theta^{\alpha I}$ are combined into 2-vectors

$$L^\alpha = \begin{pmatrix} L^{1\alpha} \\ L^{2\alpha} \end{pmatrix}, \quad \Theta^\alpha = \begin{pmatrix} \theta^{1\alpha} \\ \theta^{2\alpha} \end{pmatrix}. \quad (C.2)$$
Throughout this paper we use $2 \times 2$ matrices $\tau_1$, $\tau_2$, $\tau_3$ defined to be

$$
\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

(C.3)

Complex notation fermionic Cartan 1-forms $L^\alpha$, $\bar{L}^\alpha$ are defined as in (A.7). The Cartan 1-forms satisfy the Maurer-Cartan equations implied by the basic symmetry superalgebra

$$
dL^\mu = -L^{\mu \nu} \omega_{\nu} - iL \bar{\gamma}^\mu L, \tag{C.4}
$$

$$
dL = -\frac{1}{4} L^{\mu \nu} \gamma_{\mu \nu} \L + \frac{f}{2} L^\mu L^\nu \gamma^\mu \tau_2 \L . \tag{C.5}
$$

We note that we use the following sign conventions under permutations of Cartan 1-forms:

$$
L^\mu \wedge L^\nu = -L^\nu \wedge L^\mu, \quad L^\mu \wedge L^\alpha = -L^\alpha \wedge L^\mu, \quad L^\alpha \wedge L^\beta = L^\beta \wedge L^\alpha. \tag{C.6}
$$

It is often (while analysis of kappa-invariance and etc) useful to use the following expressions for the variations of Cartan 1-forms which are also implied by the structure of the basic symmetry superalgebra

$$
\delta L^\mu = d\hat{x}^\mu + L^\nu \hat{\omega}^\nu + L^{\mu \nu} \hat{x}^\nu + 2iL \bar{\gamma}^\mu \delta \Theta , \tag{C.7}
$$

$$
\delta L = d\hat{\Theta} + \frac{f}{2} L^\nu \gamma^\nu \tau_2 \delta \Theta + \frac{1}{4} L^{\mu \nu} \gamma^{\mu \nu} \delta \Theta 
- \frac{f}{2} \hat{x}^\mu \gamma^\mu \tau_2 L - \frac{1}{4} \hat{x}^{\mu \nu} \gamma^{\mu \nu} L . \tag{C.8}
$$

where

$$
\hat{x}^\mu \equiv \delta X^\mu L^\mu_A, \quad \hat{x}^{\mu \nu} \equiv \delta X^\mu L^{\mu \nu}_A, \quad \hat{\Theta} \alpha \equiv \delta X^\Theta L^\alpha_A . \tag{C.9}
$$

A specific choice of $G(x, \theta)$ which we use in this paper is

$$
G = g(x) e^{g_{\alpha} x^\bar{\alpha}}, \tag{C.10}
$$

where $g(x)$ is a bosonic body of coset representative of, i.e. $x = (x^\mu)$ provides a certain parametrization of plane wave background which may be kept arbitrary.\(^{35}\) Let us make the rescaling $\theta \rightarrow t\theta$ and introduce

$$
L^\mu_t(x, \theta) \equiv L^\mu(x, t\theta), \quad L^{\mu \nu}_t(x, t\theta) \equiv L^{\mu \nu}(x, t\theta), \quad L_t(x, \Theta) \equiv L(x, t\Theta) , \tag{C.11}
$$

with the initial condition

$$
L^\mu_{t=0} = e^\mu , \quad L^{\mu \nu}_{t=0} = \omega^{\mu \nu} , \quad L_{t=0} = 0 , \tag{C.12}
$$

where $e^\mu$, $\omega^{\mu \nu}$, $\omega^{+ \mu} = 0$, $\omega^{ij} = 0$, are the vielbeins and the Lorentz connections for plane wave RR background. Then the defining equations for the Cartan 1-forms are

$$
\partial_t L^\mu_t = d\Theta + \frac{1}{4} L^\mu_{\nu \rho} \gamma^{\nu \rho} \Theta + \frac{f}{2} L^\mu_t \gamma^{\rho \tau_2} \Theta , \tag{C.13}
$$

$$
\partial_t L^\nu_t = -2i\Theta \gamma^\nu L_t , \tag{C.14}
$$

$$
\partial_t L^{\nu \nu}_t = -2i\Theta \gamma^{\nu \nu} \Pi \tau_2 L_t , \quad \partial_t L^{\nu \rho}_t = -2i\Theta \gamma^{\nu \rho} \Pi \tau_2 L_t , \tag{C.15}
$$

$$
\partial_t L^{\nu \rho}_t = 2i\Theta \gamma^{\nu \rho} \Pi \tau_2 L_t , \quad \partial_t L^{\rho \nu}_t = 2i\Theta \gamma^{\rho \nu} \Pi \tau_2 L_t . \tag{C.16}
$$

\(^{35}\)The use of a concrete parametrization for $\theta$ is needed, however, to find the representation for the 2-form $F$ which enters the BI action (see below). As in the flat space case [33], $F$ cannot be expressed in terms of the Cartan forms only.
While the relations (C.1)–(C.8) are valid in an arbitrary parametrization of the coset superspace, the relations (C.13)–(C.16) apply only in the parametrization of (C.10). The equations (C.13)–(C.16) can be solved in a rather straightforward way

\[ L = \frac{\sinh M}{M} D\Theta, \quad L^\mu = e^\mu - 2i\Theta^\mu \frac{\cosh M - 1}{M^2} D\Theta, \]  \tag{C.17}

where covariant differential \( D\Theta \) and matrix \( M \) are defined to be

\[ D\Theta = \left( d + \frac{1}{4} e^{\mu\nu} \gamma^{\mu\nu} + \frac{f}{2} e^\mu \Pi \gamma^{\mu\tau_2} \right) \Theta, \]  \tag{C.18}

\[ (M^2)^{\mathcal{I}}\mathcal{J}_{\alpha \beta} = -i \left( (\Pi \gamma^{\tau_2} \theta_2)^{\alpha \mathcal{I}} (\theta_2 \tau_2 \bar{\gamma}^{\mathcal{I}} \Pi_2)^{\beta \mathcal{J}} + (\gamma^{i\mathcal{I}} \theta^\tau_2 \tau_2 \bar{\gamma}^{\mathcal{I}} \Pi_2)^{\alpha \beta} \right) \]

\[ + \frac{i}{2} \left( (\gamma^{i\mathcal{I}} \theta^\tau_2 \tau_2 \bar{\gamma}^{\mathcal{I}} \Pi_2)^{\alpha \beta} \right), \]  \tag{C.19}

Note that in many formal calculations it is more convenient to use directly the defining equations (C.13)–(C.16) rather than the explicit solution above given (in complex parametrization this solution was given in [2]).

**Appendix D** Action of conformal algebra symmetries on physical fields

In this Appendix we give more details about field theoretical realization of the conformal algebra generators and transformations rules of the SYM physical fields and extend these results to the case of plane wave massless arbitrary spin fields. Let us start our discussion with generators of isometry symmetries which are \( P^+, T^i, J^{ij} \). Plugging momentum and spin densities (8.9)-(8.13) into expressions (8.15)-(8.17) we get the following explicit representation for generators

\[ P^+ = \int d^3 x \hat{\Theta} \partial^+ A^I r_0(P^+) A^I + \frac{i}{2} \psi^\beta \gamma^+ r_0(P^+) \psi^\beta, \]  \tag{D.1}

\[ T^i = \int d^3 x \hat{\Theta} \partial^+ A^I r_0(T^i) A^I + \frac{i}{2} \psi^\beta \gamma^+ r_0(T^i) \psi^\beta, \]  \tag{D.2}

\[ J^{ij} = \int d^3 x \hat{\Theta} \partial^+ A^L r_0(J^{ij}) A^L + \frac{i}{2} \psi^\beta \gamma^+ r_0(J^{ij}) \psi^\beta \]

\[ + \partial^+ A^i A^j - \partial^+ A^j A^i + \frac{i}{4} \psi^\beta \gamma^{ij} \psi^\beta, \]  \tag{D.3}

where we use the following conventions for differential operators acting on the physical fields \( A^I, \psi^\beta \)

\[ r_0(P^+) = \partial^+, \quad r_0(T^i) = e^{-ix^+} (\partial^i + i x^\dot{i} \partial^+) , \quad r_0(J^{ij}) = x^\dot{i} \partial^j - x^\dot{j} \partial^i . \]  \tag{D.4}

Making use of commutation relations (7.14),(7.15) we get the following transformations of fields under action of isometry symmetries generated by \( P^+ \) and \( T^i \)

\[ [A^I, P^+] = r_0(P^+) A^I , \quad [\psi^\beta, P^+] = r_0(P^+) \psi^\beta, \]  \tag{D.5}
\[ [A^I, T^i] = r_0(T^i)A^I, \quad [\psi^\oplus, T^i] = r_0(T^i)\psi^\oplus. \quad (D.6) \]

Action of isometry \( so(2) \) rotations generated by \( J^{ij} \) takes the form
\[ [A^I, J^{ij}] = r_0(J^{ij})A^I + \delta^k_i A^j - \delta^i_j A^k, \quad [\phi^M, J^{ij}] = r_0(J^{ij})\phi^M, \quad (D.7) \]
\[ [\psi^\oplus, J^{ij}] = (x^i \partial^j - x^j \partial^i + \frac{1}{2} \gamma^{ij})\psi^\oplus. \quad (D.8) \]

Now we turn to discussion of the proper conformal generators and the transformations rules of the physical fields. Complete expressions for the generators and the transformations are given in (8.19)-(8.22) and (9.5)-(9.16). Because we are going to consider an arbitrary spin field we restrict ourselves to the linear transformations, i.e. to the ones generated by parts of the generators which are quadratic in fields. Introducing notation \( G_{(2)} \) for parts of generators which are quadratic in physical fields and plugging momentum and spin densities (8.9)-(8.13) into expressions (8.19)-(8.22) we get the following explicit representation for generators
\[ D_{(2)} = \int d^3x \, \text{Tr} \, \partial^+ A^I r_0(D) A^I + \frac{i}{2} \psi^\oplus \gamma^+ r_0(D) \psi^\oplus, \quad D_{(2)} = D, \quad (D.9) \]
\[ C_{(2)} = \int d^3x \, \text{Tr} \, \partial^+ A^I r_0(C) A^I + \frac{i}{2} \psi^\oplus \gamma^+ r_0(C) \psi^\oplus, \quad (D.10) \]
\[ \tilde{C}_{(2)} = \int d^3x \, \text{Tr} \left( \partial^+ A^I r_0(C^\dagger) A^I + \frac{i}{2} \psi^\oplus \gamma^+ r_0(C^\dagger) \psi^\oplus \right) + e^{-ifx^+} (M_{(2)}^i + \text{i} f M_{(2)}^i x^j), \quad (D.11) \]
\[ K_{(2)}^\dagger = \int d^3x \, \text{Tr} \left( \partial^+ A^I (r_0(K^-) - \frac{1}{2\partial^+}) A^I + \frac{i}{2} \psi^\oplus \gamma^+ r_0(K^-) \psi^\oplus - \phi^2 \right) + \mathcal{M}_{(2)}^i x^i, \quad (D.12) \]

where we use the following notation for differential operators acting on the physical fields
\[ r_0(D) = 2x^- \partial^+ + x \partial, \quad (D.13) \]
\[ r_0(C) = e^{-2ifx^+} \left( \frac{1}{2\partial^+} \partial^2 - if x \partial + \frac{f^2}{2} x^2 \partial^+ \right), \quad (D.14) \]
\[ r_0(C^\dagger) = e^{-ifx^+} \left( x^- \partial^i + \frac{1}{2\partial^+} x^i \partial^2 + if(-\frac{1}{2} x^2 \partial^i + x^i (x^- \partial^+ + x \partial)) \right), \quad (D.15) \]
\[ r_0(K^-) = \frac{1}{4\partial^+} x^2 \partial^2 + x^- (x^- \partial^+ + x \partial). \quad (D.16) \]

These differential operators are obtainable from the ones given in (5.19)-(5.22) my making the following substitution there
\[ \partial^- \to - \frac{1}{2\partial^+} \partial^2 - \frac{f^2}{2} x^2 \partial^+. \quad (D.17) \]

This substitution reflects simply the fact that we are using equations of motion in expressions for generators. Making use of the commutation relations (7.14),(7.15) gives the following transformation rules under action of plane wave dilatation operator
\[ [A^I, D] = (r_0(D) + 1) A^I, \quad [\psi^\oplus, D] = (r_0(D) + 2) \psi^\oplus. \quad (D.18) \]
Appearance of the unusual factor 2 in transformation of fermionic field under action plane wave dilatation generator \( D \) (D.18) can be understood from the relation (5.26).

Now we consider action of the remaining conformal transformations generated by \( C, C^i, \) and \( K^\pm \). We find that the transformations rules of the physical fields \( A^i, \psi^\oplus \) under action of the generator \( C\!(2) \) take the same form:

\[
[A^i, C\!(2)] = (r_0(C) - ife^{-2ifx^+})A^i, \quad [\psi^\oplus, C\!(2)] = (r_0(C) - ife^{-2ifx^+})\psi^\oplus. \tag{D.19}
\]

As to the transformations generated by \( \hat{C}\!(2), \) and \( K^-\!(2) \) they take different form

\[
[A^i, \hat{C}\!(2)] = (r_0(\hat{C}^i) + ifx^i e^{-ifx^+})A^i + e^{-ifx^+}(\frac{\partial^k}{\partial^+} + ifx^k)(\delta_{ij} A^k - \delta^{ik} A^j), \tag{D.20}
\]

\[
[\phi^M, \hat{C}\!(2)] = (r_0(\hat{C}^i) + ifx^i e^{-ifx^+})\phi^M, \tag{D.21}
\]

\[
[\psi^\oplus, \hat{C}\!(2)] = \left( r_0(\hat{C}^i) + \left( \frac{\partial^i}{2\partial^+} + \frac{3i}{2} f x^i \right) e^{-ifx^+} + \frac{1}{2} e^{-ifx^+} \left( \frac{\partial^k}{\partial^+} + ifx^k \right) \gamma^{ik} \right) \psi^\oplus, \tag{D.22}
\]

\[
[A^i, K^-\!(2)] = (r_0(K^-) + x^- - \frac{1}{\partial^+})A^i + x^i \frac{\partial^k}{\partial^+} (\delta^{ij} A^k - \delta^{ik} A^j), \tag{D.23}
\]

\[
[\phi^M, K^-\!(2)] = (r_0(K^-) + x^-)\phi^M, \tag{D.24}
\]

\[
[\psi^\oplus, K^-\!(2)] = \left( r_0(K^-) + 2x^- + \frac{1}{2\partial^+} x \partial + \frac{1}{2} \gamma^{ij} x^i \frac{\partial^j}{\partial^+} \right) \psi^\oplus. \tag{D.25}
\]

These transformations can be cast however into a unifying form. To this end we introduce a new fermionic field \( \tilde{\psi}^\oplus = (\partial^+)^{1/2} \tilde{\psi}^\oplus \) and get the following transformations

\[
[\tilde{\psi}^\oplus, C\!(2)] = \left( r_0(\hat{C}^i) + ifx^i e^{-ifx^+} + \frac{1}{2} e^{-ifx^+} \left( \frac{\partial^k}{\partial^+} + ifx^k \right) \gamma^{ik} \right) \tilde{\psi}^\oplus, \tag{D.26}
\]

\[
[\tilde{\psi}^\oplus, K^-\!(2)] = \left( r_0(K^-) + x^- - \frac{1}{4\partial^+} + \frac{1}{2} \gamma^{ij} x^i \frac{\partial^j}{\partial^+} \right) \tilde{\psi}^\oplus, \tag{D.27}
\]

which together with the transformations given in (D.18),(D.20),(D.21),(D.23),(D.24) can be generalized to arbitrary spin \( s \) plane wave massless field \( \Xi_s \) in a straightforward way

\[
[\Xi_s, D] = (r_0(D) + 1)\Xi_s, \tag{D.28}
\]

\[
[\Xi_s, C\!(2)] = (r_0(C^i) + ifx^i e^{-ifx^+})\Xi_s + e^{-ifx^+} \left( \frac{\partial^k}{\partial^+} + ifx^k \right) [\Xi_s, M^{ik}], \tag{D.29}
\]

\[
[\Xi_s, K^-\!(2)] = (r_0(K^-) + x^- - \frac{s^2}{\partial^+})\Xi_s + x^i \frac{\partial^j}{\partial^+} [\Xi_s, M^{ij}], \tag{D.30}
\]

The transformations rules for the physical fields of SYM theory \( \phi^M, A^i, \tilde{\psi}^\oplus \) are obtainable from these transformation by respective setting \( s = 0, \ s = 1, \ s = 1/2 \). The bracket \( [\Xi_s, M^{ij}] \) denotes transformation of spin \( s \) field \( \Xi_s \) under action of spin part of generator of \( so(2) \) algebra \( J^{ij} \). For instance for spin \( s = 0 \) scalar field \( [\phi, M^{ij}] = 0 \), while for spin \( s = 1/2 \) fermionic field \( [\tilde{\psi}^\oplus, M^{ij}] = \frac{1}{2} \gamma^{ij} \tilde{\psi}^\oplus \). Light cone gauge transformations rules of the spin \( s \) field \( \Xi_s \) under action of the generators \( P^+, T^i, C\!(2) \) takes the same form as for the fields of SYM theory (see (D.5),(D.6),(D.19)). Light cone gauge equations motion for the field \( \Xi_s \) also take a simplified form \( \Box \Xi_s = 0 \) (with \( \Box \) given in (7.7)) .

46
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