A Girsanov-type formula for a class of anticipative transforms of Brownian motion associated with exponential functionals

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Abstract

In this paper, with the help of a result by Matsumoto–Yor (2000), we prove a Girsanov-type formula for a class of anticipative transforms of Brownian motion which possesses exponential functionals as anticipating factors. Our result unifies existing formulas in earlier works. As an application, we also consider the law of Brownian motion perturbed by a positive weight of a fairly wide class, and prove its invariance under an anticipative transformation associated with the perturbation. In the course of our exploration, a disintegration formula for the Wiener measure related to exponential functionals plays a key role.

1 Introduction and main results

A Girsanov-type formula, or a change of measure formula, for anticipative transforms of Brownian motion has been studied by a number of authors, especially in the framework of Malliavin calculus; see, e.g., [2, 6, 11, 12, 13, 14] and references therein. In the formula, the density with respect to the underlying Wiener measure is given by the product of two factors, one of which is a stochastic exponential in which the Itô integral is replaced by the Skorokhod integral, and the other is a Carleman–Fredholm determinant; in some specific settings, further factorizations of these two factors have also been investigated. As far as we know, there seem to be not so many concrete examples in which the corresponding densities, in particular, Carleman–Fredholm determinants, are explicitly calculated.

In this paper, with the help of a result by Matsumoto–Yor [8] on exponential functionals of Brownian motion, we introduce a class of anticipative transformations under

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which we can obtain a Girsanov-type formula in an explicit form; we expect that the result provides a number of examples in Malliavin calculus in which we are able to calculate Carleman–Fredholm determinants explicitly. We also apply it to derive the distributional invariance of Brownian motion perturbed by a positive weight of a wide class, which is described in terms of an anticipative transformation determined from the perturbation. As will be seen below, properties of a certain class of anticipative transformations investigated in [4], as well as a disintegration formula for the Wiener measure, play an essential role in the course of our exploration; see Lemma 2.1 and Proposition 2.1.

To state the main results of the paper, we prepare some of the notation. Let $B = \{B_t\}_{t \geq 0}$ be a one-dimensional standard Brownian motion. Let $C([0, \infty); \mathbb{R})$ be the space of continuous functions $\phi : [0, \infty) \to \mathbb{R}$, on which we define the transformation

$$A_t(\phi) := \int_0^t e^{2\phi_s} \, ds, \quad t \geq 0;$$

(1.1)

with slight abuse of notation, we will simply write $A_t$ for $A_t(B) = A_t(B)$. This exponential additive functional $A_t$, $t \geq 0$, which is the quadratic variation of the geometric Brownian motion $e^{B_t}$, $t \geq 0$, appears in a number of areas in probability theory such as mathematical finance and diffusion processes in random environments, and is known for its close relationship with planar Brownian motion (or two-dimensional Bessel process); see the detailed surveys [9, 10] by Matsumoto and Yor. Following the notation in [3], we also define

$$Z_t(\phi) := e^{-\phi_t} A_t(\phi), \quad t \geq 0,$$

(1.2)

for $\phi \in C([0, \infty); \mathbb{R})$, and denote $Z_t(B)$ by $Z_t$ for simplicity, too. Given $t > 0$, we restrict the transformation $A$ to the space $C([0, t]; \mathbb{R})$ of real-valued continuous functions over $[0, t]$, and recall from [4] the family $\{T_z\}_{z \in \mathbb{R}}$ of anticipative path transformations on $C([0, t]; \mathbb{R})$ defined by

$$T_z(\phi)(s) \equiv T_z^t(\phi)(s) := \phi_s - \log \left\{ 1 + \frac{A_s(\phi)}{A_t(\phi)} (e^z - 1) \right\}, \quad 0 \leq s \leq t,$$

(1.3)

for $\phi \in C([0, t]; \mathbb{R})$. In what follows, with $t > 0$ fixed, we suppress the superscript $t$ from the notation and suppose that each $T_z$ acts on $C([0, t]; \mathbb{R})$. We denote by $C([0, t]; \mathbb{R}^2)$ the space of $\mathbb{R}^2$-valued continuous functions over $[0, t]$. One of the main results of the paper is then stated as

**Theorem 1.1.** Let $h \equiv h(t, \cdot) : C([0, t]; \mathbb{R}) \to \mathbb{R}$ be a measurable function such that, for every $\phi \in C([0, t]; \mathbb{R})$, the function $h_\phi : \mathbb{R} \to \mathbb{R}$ defined by

$$h_\phi(\xi) := h(T_{\phi_t-\xi}(\phi)), \quad \xi \in \mathbb{R},$$


is of class \( C^1 \) and strictly monotone. Then, for every nonnegative measurable function \( F \) on \( C([0, t]; \mathbb{R}^2) \), we have

\[
\mathbb{E} \left[ F\left(T_{B_t - h(B)}(B), B\right) \exp \left\{ -\frac{\cosh h(B)}{Z_t} + \frac{\cosh B_t}{Z_t} \right\} \right] = \mathbb{E} \left[ F\left(B, T_{B_t - h^{-1}_B(B)}(B)\right)\right].
\]

(1.4)

where, for every \( \phi \in C([0, t]; \mathbb{R}) \), \( h^{-1}_\phi \) denotes the inverse function of \( h_\phi \).

In the above statement, as well as in the sequel, we equip \( C([0, t]; \mathbb{R}) \) with topology of uniform convergence, and we say that a real-valued function on this space is measurable if it is Borel-measurable with respect to the topology. The same remark also applies to \( C([0, t]; \mathbb{R}^2) \).

As listed below, existing formulas may be obtained from the above theorem with particular choices of the function \( h_\phi \), \( \phi \in C([0, t]; \mathbb{R}) \): given \( \alpha, x \geq 0 \) and \( z \in \mathbb{R} \),

- [3, Theorem 1.1] corresponds to \( h_\phi = \phi_t - \log \left\{ 1 + \alpha Z_t(\phi) \right\} \);
- [3, Theorem 1.5] corresponds to \( h_\phi = -\log \left\{ e^{-\phi_t} + \alpha Z_t(\phi) \right\} \);
- [4, Theorem 1.2] corresponds to \( h_\phi = \phi_t - z \);
- [5, Theorem 1.1] corresponds to \( h_\phi = -\phi_t \);
- the former relation in [5, Proposition 5.3] corresponds to
  \( h_\phi = \log \left\{ e^{-\phi_t} + 2xZ_t(\phi) \right\} \);
- the latter relation in [5, Proposition 5.3] corresponds to
  \( h_\phi = -\log \left\{ e^{\phi_t} + 2xZ_t(\phi) \right\} \).

In fact, the first two formulas are recovered with slight generalization. We also remark that the path transformation corresponding to the second case is a non-anticipative one, but the resulting formula is not the one that follows from Girsanov’s formula; details will be found in Subsection 4.1.

As an application of Theorem 1.1, we also prove

**Corollary 1.1.** Let \( \Lambda \equiv \Lambda(t, \cdot) \) be a positive continuous function on \( C([0, t]; \mathbb{R}) \) such that

\[
\int_{\mathbb{R}} \Lambda(\xi) \left( T_{\phi_t - \xi(\phi)} \right) \exp \left\{ -\frac{\cosh \xi}{Z_t(\phi)} \right\} < \infty \quad \text{for all} \quad \phi \in C([0, t]; \mathbb{R}).
\]

(1.5)

Then, for every nonnegative measurable function \( F \) on \( C([0, t]; \mathbb{R}^2) \), we have

\[
\mathbb{E} \left[ F\left(T_{B_t - h_\Lambda(B_t,B)}(B), B\right)\Lambda(B)\right] = \mathbb{E} \left[ F\left(B, T_{B_t - h_\Lambda(B_t,B)}(B)\right)\Lambda(B)\right],
\]

(1.6)
where \( h_A \equiv h_A(t, \cdot, \cdot) \) is defined through

\[
\int_{-\infty}^{h_A(\xi, \phi)} dx \Lambda(T_{\phi_t-x}(\phi)) \exp \left\{ \frac{-\cosh x}{Z_t(\phi)} \right\} = \int_{\xi}^{\infty} dx \Lambda(T_{\phi_t-x}(\phi)) \exp \left\{ \frac{-\cosh x}{Z_t(\phi)} \right\},
\]

for \( \xi \in \mathbb{R} \) and \( \phi \in C([0, t]; \mathbb{R}) \).

Observe that, for every fixed \( \phi \in C([0, t]; \mathbb{R}) \), by (1.5) and the positivity of \( \Lambda \), the function \( h_A(\cdot, \phi) \) is strictly decreasing and satisfies

\[
\lim_{\xi \to -\infty} h_A(\xi, \phi) = \infty, \quad \lim_{\xi \to \infty} h_A(\xi, \phi) = -\infty.
\]

If we denote by \( T_A \) the path transformation as in (1.6), namely, if we set

\[
T_A(\phi)(s) := T_{\phi_t-h_A(\phi_t, \phi)}(\phi)(s), \quad 0 \leq s \leq t,
\]

for \( \phi \in C([0, t]; \mathbb{R}) \), then the above corollary also indicates that \( T_A \) is an involution: \( T_A \circ T_A = \text{Id} \), which is indeed the case as will be noted in Remark 3.2. Here \( \text{Id} \) is the identity map on \( C([0, t]; \mathbb{R}) \).

For every \( \mu \in \mathbb{R} \), we denote by \( B^{(\mu)} = \{ B_s^{(\mu)} := B_s + \mu s \}_{s \geq 0} \) the Brownian motion with drift \( \mu \), to which we will also associate the two processes \( \{ A_s^{(\mu)} \}_{s \geq 0} \) and \( \{ Z_s^{(\mu)} \}_{s \geq 0} \) in such a way that

\[
A_s^{(\mu)} := A_s(B^{(\mu)}), \quad Z_s^{(\mu)} := Z_s(B^{(\mu)});
\]

when \( \mu = 0 \), we suppress it from the notation as already introduced above. If we apply Corollary 1.1 to the function \( A \) of the form

\[
A(\phi) = \exp \left( \mu \phi_t - \frac{\mu^2 t}{2} \right), \quad \phi \in C([0, t]; \mathbb{R}),
\]

then, by the Cameron–Martin formula, we see that the law of \( \{ B_s^{(\mu)} \}_{0 \leq s \leq t} \) is invariant under \( T_A \) with the above choice of \( A \), which extends [5, Theorem 1.1] to the case of Brownian motion with drift; see Subsection 4.2 for more details as well as other examples.

The rest of the paper is organized as follows. In Section 2, we state and prove the two key assertions, namely Lemma 2.1 and Proposition 2.1 as mentioned earlier, with which we prove Theorem 1.1 and Corollary 1.1 in Section 3. We devote Section 4 to examples that are obtained by applying Theorem 1.1 and Corollary 1.1.

### 2 Preliminaries

In this section, we state and prove Lemma 2.1 and Proposition 2.1. We keep \( t > 0 \) fixed and begin with properties of \( T_z, z \in \mathbb{R} \), investigated in [4]. Observe that the two
transformation $A$ and $Z$ defined respectively by (1.1) and (1.2) are related via

$$\frac{d}{ds} A_s(\phi) = -\left\{ \frac{1}{Z_s(\phi)} \right\}^2, \quad s > 0, \phi \in C([0, \infty); \mathbb{R}),$$  
(2.1)

and hence, restricted to $C([0, t]; \mathbb{R})$,

$$\frac{1}{A_s(\phi)} = \int_s^t \frac{du}{\{Z_u(\phi)\}^2} + \frac{e^{-\phi_t}}{Z_t(\phi)}, \quad 0 < s \leq t,$$  
(2.2)

for every $\phi \in C([0, t]; \mathbb{R})$, because of the relation $A_t(\phi) = e^{\phi_t} Z_t(\phi)$.

**Lemma 2.1 ([4, Proposition 2.1]).** The transformations $T_z, z \in \mathbb{R}$, have the following properties.

(i) For every $z \in \mathbb{R}$ and $\phi \in C([0, t]; \mathbb{R})$, $T_z(\phi)(t) = \phi_t - z$.

(ii) For every $z \in \mathbb{R}$ and $\phi \in C([0, t]; \mathbb{R})$,

$$\frac{1}{A_s(T_z(\phi))} = \frac{1}{A_s(\phi)} + \frac{e^z - 1}{A_t(\phi)}, \quad 0 < s \leq t;$$  
(2.3)

in particular, $A_t(T_z(\phi)) = e^{-z} A_t(\phi)$.

(iii) $Z \circ T_z = Z$ for any $z \in \mathbb{R}$.

(iv) (Semigroup property) $T_z \circ T_{z'} = T_{z+z'}$ for any $z, z' \in \mathbb{R}$; in particular,

$$T_z \circ T_{-z} = T_0 = \text{Id} \quad \text{for any } z \in \mathbb{R},$$

where $\text{Id}$ is the identity map on $C([0, t]; \mathbb{R})$ as referred to in Section 7.

We give below a proof of the above lemma for the reader’s convenience.

**Proof of Lemma 2.1.** (i) By definition,

$$T_z(\phi)(t) = \phi_t - \log \{1 + (e^z - 1)\} = \phi_t - z.$$

(ii) For the case $z = 0$ is obvious, we let $z \neq 0$ and compute, for every $0 \leq s \leq t$,

$$A_s(T_z(\phi)) = \int_0^s du \frac{e^{2\phi_u}}{\left\{1 + \frac{A_u(\phi)}{A_t(\phi)}(e^z - 1)\right\}^2}$$

$$= \frac{A_t(\phi)}{e^z - 1} \left\{1 - \frac{1}{1 + \frac{A_s(\phi)}{A_t(\phi)}(e^z - 1)}\right\}$$

$$= \frac{A_t(\phi)}{1 + \frac{A_s(\phi)}{A_t(\phi)}(e^z - 1)},$$
which entails (2.3).

(iii) In view of relation (2.1), taking the derivative with respect to $s$ on each side of (2.3) yields
\[
\{Z_s(T_z(\phi))\}^{-2} = \{Z_s(\phi)\}^{-2}, \quad 0 < s \leq t,
\]
from which the claim follows by the positivity of $Z$.

(iv) By noting that
\[
\phi_s = \frac{1}{2} \log \frac{d}{ds} A_s(\phi), \quad s > 0, \quad \phi \in C([0, \infty); \mathbb{R}),
\]
it suffices to show that, for each $\phi \in C([0, t]; \mathbb{R})$,
\[
A_s((T_z \circ T_{z'})(\phi)) = A_s(T_{z+z'}(\phi)), \quad 0 \leq s \leq t. \tag{2.4}
\]
To this end, first observe that relation (2.3) may be rewritten as
\[
\frac{1}{A_s(T_z(\phi))} = \int_s^t \frac{du}{\{Z_u(\phi)\}^2} + \frac{e^{-\phi_t + z}}{Z_t(\phi)}, \quad 0 < s \leq t, \tag{2.5}
\]
thanks to (2.2). For every $0 < s \leq t$, we apply property (iii) to (2.2) successively to see that
\[
\frac{1}{A_s((T_z \circ T_{z'})(\phi))} = \int_s^t \frac{du}{\{Z_u(\phi)\}^2} + \frac{e^{-(T_z \circ T_{z'})(\phi)(t)}}{Z_t(\phi)},
\]
in which repeated use of property (i) yields
\[
-(T_z \circ T_{z'})(\phi)(t) = -T_{z'}(\phi)(t) + z
= -\phi_t + z' + z.
\]
This proves (2.4) in view of (2.5).

Remark 2.1. If we denote by $R$ the operation of time reversal on $C([0, t]; \mathbb{R})$:
\[
R(\phi)(s) := \phi_{t-s} - \phi_t, \quad 0 \leq s \leq t, \quad \phi \in C([0, t]; \mathbb{R}), \tag{2.6}
\]
then it also holds that
\[
R \circ T_z = T_{-z} \circ R \tag{2.7}
\]
for any $z \in \mathbb{R}$; see [4, Proposition 2.1(v)].

The following may be regarded as a disintegration formula for the Wiener measure on $C([0, t]; \mathbb{R})$ in terms of the transformations $T_z, z \in \mathbb{R}$, which seems to be new to our knowledge, and is of interest in its own right.
Proposition 2.1. For every nonnegative measurable function $F$ on $C([0,t];\mathbb{R})$, one has

$$
\mathbb{E}[F(B)] = \mathbb{E}
\left[
\int_{\mathbb{R}} \frac{d\xi}{2K_0(1/Z_t(\phi))} F(T_{\phi_t-\xi}(\phi)) \exp\left\{-\frac{cosh \xi}{Z_t(\phi)}\right\} \bigg|_{\phi=B}
\right],
$$

(2.8)

where $K_0$ is the modified Bessel function of the third kind (or the Macdonald function) of order 0.

For the modified Bessel functions, refer to, e.g., [7, Section 5.7]. The above proposition is a consequence of the next two lemmas, the first one of which is a restatement of [8, Proposition 1.7] in the case of Brownian motion without drift. We denote by $\{Z_s\}_{s \geq 0}$ the natural filtration of the process $Z = \{Z_s\}_{s \geq 0}$, where, as mentioned in Section 1, $Z_s$ refers to $Z_s(B)$ for notational simplicity.

Lemma 2.2. For every nonnegative measurable function $f$ on $\mathbb{R}$, we have, a.s.,

$$
\mathbb{E}[f(B_t) \mid Z_t] = \int_{\mathbb{R}} \frac{d\xi}{2K_0(1/\zeta)} f(\xi) \exp\left(-\frac{cosh \xi}{\zeta}\right) \bigg|_{\zeta=Z_t}.
$$

Notice that, by the nonnegativity of $f$, the conditional expectation on the left-hand side in the above equation is well-defined in view of the conditional monotone convergence theorem, regardless of whether $f(B_t)$ is integrable or not; this convention will also be applied later.

Lemma 2.3. For every $\phi \in C([0,t];\mathbb{R})$ and $\xi \in \mathbb{R}$, it holds that

$$
\phi_s = -\log \left\{ Z_s(\phi) \int_s^t \frac{du}{(Z_u(\phi))^2} + \frac{Z_s(\phi)}{Z_t(\phi)} e^{-\phi_t} \right\},
$$

(2.9)

$$
T_{\phi_t-\xi}(\phi)(s) = -\log \left\{ Z_s(\phi) \int_s^t \frac{du}{(Z_u(\phi))^2} + \frac{Z_s(\phi)}{Z_t(\phi)} e^{-\xi} \right\},
$$

(2.10)

for all $0 < s \leq t$.

Remark 2.2. By tending $s \downarrow 0$, both of the right-hand sides of (2.9) and (2.10) converge to $\phi_0$ by the continuity of $\phi$ and $T_{\phi_t-\xi}(\phi)$; note that $T_{\phi_t-\xi}(\phi)(0) = \phi_0$ by definition.

The former relation in Lemma 2.3 indicates that, given $\phi \in C([0,t];\mathbb{R})$, we can reconstruct it from $Z(\phi)$ and the terminal value $\phi_t$. The latter reveals that, in the disintegration formula (2.8), the integrand with respect to $\xi$ in the right-hand side is determined only from $Z(\phi)$ and $\xi$. Notice that, for every $\phi \in C([0,t];\mathbb{R})$ and $\xi \in \mathbb{R},$

$$
T_{\phi_t-\xi}(\phi)(t) = \xi, \quad Z_s(T_{\phi_t-\xi}(\phi)) = Z_s(\phi) \quad \text{for all } 0 \leq s \leq t,
$$

(2.11)

by properties (i) and (iii) in Lemma 2.1.
Proof of Lemma 2.3. As for (2.9), by relation (2.2) and the definition (1.2) of the transformation $Z$,

$$Z_s(\phi) \int_s^t \frac{du}{\{Z_u(\phi)\}^2} + \frac{Z_s(\phi)}{Z_t(\phi)} e^{-\phi_t} = \frac{Z_s(\phi)}{A_s(\phi)} e^{-\phi_s},$$

due to (2.10), we apply (2.9) to $T_{\phi_t - \xi}(\phi) \in C([0,t];\mathbb{R})$ to obtain the desired expression owing to (2.11).

We are in a position to prove Proposition 2.1.

Proof of Proposition 2.1. In view of Lemma 2.3, it suffices to prove the assertion in the case that $F$ is of the form $F(\phi) = f(\phi_t)G(Z(\phi))$, $\phi \in C([0,t];\mathbb{R})$, with $f$ a nonnegative measurable function on $\mathbb{R}$ and $G$ a nonnegative measurable function on $C([0,t];\mathbb{R})$; then, by approximation, the formula extends to any nonnegative measurable function of $(\phi_t, Z(\phi))$, $\phi \in C([0,t];\mathbb{R})$. To this end, by Lemma 2.2 we have, conditionally on $Z_t$,

$$E[F(B)] = E[G(Z)E[f(B_t) \mid Z_t]]$$

$$= E\left[G(Z) \int_{\mathbb{R}} \frac{d\xi}{2K_0(1/\xi)} f(\xi) \exp\left(-\frac{\cosh \xi}{\xi}\right)\right]_{\xi = Z_t}.$$

Thanks to (2.11), the last expression agrees with the right-hand side of (2.8) because, with the above choice of $F$,

$$F(T_{\phi_t - \xi}(\phi)) = f(\xi)G(Z(\phi))$$

for every $\phi \in C([0,t];\mathbb{R})$ and $\xi \in \mathbb{R}$.

3 Proofs of Theorem 1.1 and Corollary 1.1

In this section, we prove Theorem 1.1 and Corollary 1.1

3.1 Proof of Theorem 1.1

In this subsection, we prove Theorem 1.1.

Proof of Theorem 1.1. First observe that, for every $\phi \in C([0,t];\mathbb{R})$ and for every $\xi, z \in \mathbb{R}$,

$$T_{\psi_t - b(\psi)}(\psi)|_{\psi = T_{\phi_t - \xi}(\phi)} = T_{\phi_t - b_0(\xi)}(\phi),$$

and

$$h_{T_{\phi}(\phi)}(\xi) = h_\phi(\xi).$$
As for (3.1), by the fact that \( T_{\phi_t - \xi}(\phi)(t) = \xi \) as seen in (2.11), and by the definition of the function \( h_{\phi} \), the left-hand side of (3.1) is equal to

\[
T_{\xi - h_{\phi}(\xi)}(T_{\phi_t - \xi}(\phi)),
\]

which coincides with the right-hand side by Lemma 2.1(iv). In view of the definition of \( h_{\phi} \), the latter observation (3.2) is verified in the same way, by noting that

\[
T_{T z}(\phi)(t) - \xi(\phi)(t) = T_{\phi_t - \xi}(\phi),
\]

which may also be seen as a consequence of Lemma 2.1(iii) and the fact that, as (2.10) indicates, the dependence of \( T_{\phi_t - \xi}(\phi) \) on \( \phi \) is through \( Z(\phi) \). Then, by the above two observations and Proposition 2.1, the left-hand side of (1.4) is written as

\[
E \left[ \int_{\mathbb{R}} \frac{d\xi}{2K_0(1/Z_t(\phi))} F(T_{\phi_t - h_{\phi}(\xi)}(\phi), T_{\phi_t - \xi}(\phi) \exp \left\{ -\frac{\cosh h_{\phi}(\xi)}{Z_t(\phi)} \right\} \right]_{\phi = B},
\]

in which, by changing the variables with \( \eta = h_{\phi}(\xi) \), the integral with respect to \( \xi \) is equal to

\[
\int_{h_{\phi}(\mathbb{R})} \frac{d\eta}{2K_0(1/Z_t(\phi))} F(T_{\phi_t - h_{\phi}(\eta)}(\phi), T_{\phi_t - h_{\phi}^{-1}(\eta)}(\phi) \exp \left\{ -\frac{\cosh \eta}{Z_t(\phi)} \right\} \right].
\]

Therefore, by using Proposition 2.1 again, the above expectation coincides with the right-hand side of (1.4).

Remark 3.1. Theorem 1.1 indicates that, for any \( \phi \in C([0, t]; \mathbb{R}) \) such that \( \phi_t \in h_{\phi}(\mathbb{R}) \), setting \( \psi \in C([0, t]; \mathbb{R}) \) by

\[
\psi_s := T_{\phi_t - h_{\phi}^{-1}(\phi_t)}(\phi)(s), \quad 0 \leq s \leq t,
\]

we have

\[
T_{\psi_t - h(\psi)}(\psi) = \phi,
\]

which is indeed the case as seen below. First note that \( \psi_t = h_{\phi}^{-1}(\phi_t) \) by Lemma 2.1(i). Moreover, by the definition of \( h_{\phi} \),

\[
h(\psi) = h_{\phi}(h_{\phi}^{-1}(\phi_t)) = \phi_t.
\]

Therefore we have

\[
T_{\psi_t - h(\psi)}(\psi) = T_{h_{\phi}^{-1}(\phi_t) - \phi_t} \left( T_{\psi_t - h_{\phi}^{-1}(\phi_t)}(\phi) \right),
\]

which is equal to \( \phi \) by Lemma 2.1(iv).
3.2 Proof of Corollary 1.1

We apply Theorem 1.1 to the function \( h(\phi) = h_A(\phi_t, \phi) \), \( \phi \in C([0, t]; \mathbb{R}) \), with \( h_A \) defined through (1.7). Notice that, for every \( \phi \in C([0, t]; \mathbb{R}) \), the associated function \( h_\phi : \mathbb{R} \to \mathbb{R} \) is given by

\[
h_\phi(\xi) = h_A(\xi, \phi), \quad \xi \in \mathbb{R}.
\]

Indeed, by the definition of \( h_\phi \) and Lemma 2.1(i),

\[
h_\phi(\xi) = h_A(T_{\phi_t-\xi}(\phi)(t), \phi)
\]

of which the integrand in the defining relation (1.7) is the same as that of \( h_A(\xi, \phi) \).

Namely, it holds that, by properties (i), (iii) and (iv) of Lemma 2.1,

\[
A \left( T_{\phi_t-\xi}(\phi)(t) \right) \exp \left\{ -\frac{\cosh x}{Z_t(\phi)} \right\} = A(T_{\phi_t-\phi}(\phi)) \exp \left\{ -\frac{\cosh x}{Z_t(\phi)} \right\}
\]

for any \( x \in \mathbb{R} \). By the continuity and positivity of \( A \), together with the observation given just below the statement of Corollary 1.1, it is clear that \( h_\phi(\mathbb{R}) = \mathbb{R} \). Moreover, because relation (1.7) is rewritten as

\[
\int_{-\infty}^{\xi} dx A(T_{\phi_t-\phi}(\phi)) \exp \left\{ -\frac{\cosh x}{Z_t(\phi)} \right\} = \int_{h_A(\xi, \phi)}^{\infty} dx A(T_{\phi_t-\phi}(\phi)) \exp \left\{ -\frac{\cosh x}{Z_t(\phi)} \right\}
\]

for any \( \xi \in \mathbb{R} \), we have

\[
h_A^{-1}(\xi, \phi) = h_A(\xi, \phi), \quad \xi \in \mathbb{R},
\]

where \( h_A^{-1} \) denotes the inverse function of \( h_A \) in the first variable.

Proof of Corollary 1.1. Taking \( h(\phi) = h_A(\phi_t, \phi) \), \( \phi \in C([0, t]; \mathbb{R}) \), in Theorem 1.1 we replace \( F \) by a function of the form

\[
F(\phi^1, \phi^2)A(\phi^1), \quad (\phi^1, \phi^2) \in C([0, t]; \mathbb{R}^2),
\]

where \( F \) is again a nonnegative measurable function on \( C([0, t]; \mathbb{R}^2) \). Then, with the notation \( T_A \) in (1.8), the left-hand side of (1.4) turns into

\[
\mathbb{E} \left[ F(T_A(B), B)A(T_A(B)) \exp \left\{ -\frac{\cosh h_A(B_t, B)}{Z_t} + \frac{\cosh B_t}{Z_t} \right\} \right],
\]
where the derivative of $h_A$ is taken with respect to the first variable. The above expectation is equal to $E\left[F\left(T_A(B), B\right)A(B)\right]$ since relation (1.7) entails that

$$A(\mathbb{T}_{\phi_t} - h_A(\xi, \phi)(\phi)) \exp \left\{ -\frac{\cosh h_A(\xi, \phi)}{Z_t(\phi)} \right\} h'_A(\xi, \phi) = -A(\mathbb{T}_{\phi_t} - \xi(\phi)) \exp \left\{ -\frac{\cosh \xi}{Z_t(\phi)} \right\}$$

for every $\xi \in \mathbb{R}$ and $\phi \in C([0, t]; \mathbb{R})$, and hence, by inserting $\xi = B_t$ and $\phi = B$,

$$A(T_A(B)) \exp \left\{ -\frac{\cosh h_A(B_t, B)}{Z_t} \right\} h'_A(B_t, B) = -A(\Lambda) \exp \left( -\frac{\cosh B_t}{Z_t} \right).$$

Here we used the fact that $T_0 = \text{Id}$ in the right-hand side of the last equality. On the other hand, with the above replacement of $F$ and in view of (3.4), the right-hand side of (1.4) becomes $E\left[F\left(B, T_A(B)\right)A(B)\right]$, verifying the claim. \hfill \Box

Remark 3.2. By virtue of (3.4), relation (3.3) reveals that $T_A$ is an involution.

4 Examples

In this section, we apply Theorem 1.1 and Corollary 1.1 to provide some examples. In what follows, $t > 0$ is fixed as above and, as in the proof of Corollary 1.1, the symbol $F$ refers to a generic nonnegative measurable function on $C([0, t]; \mathbb{R}^2)$ which may differ in different contexts.

4.1 Examples of Theorem 1.1

In all of the examples below, we consider a specific case that $h : C([0, t]; \mathbb{R}) \to \mathbb{R}$ is of the form

$$h(\phi) = k(\phi_t, Z_t(\phi)), \quad \phi \in C([0, t]; \mathbb{R}),$$

where $k \equiv k(t, \cdot, \cdot)$ is a measurable function on $\mathbb{R} \times (0, \infty)$ such that, for every $\zeta > 0$, the function

$$\mathbb{R} \ni \xi \mapsto k(\xi, \zeta)$$

is of class $C^1$ and strictly monotone. Note that, in view of (2.11), we have $h(\phi)(\xi) = k(\xi, Z_t(\phi))$ for every $\phi \in C([0, t]; \mathbb{R})$ and $\xi \in \mathbb{R}$. We start with restating (1.3) under the above setting assuming that

$$|k'(\xi, \zeta)| > 0 \quad \text{for all } (\xi, \zeta) \in \mathbb{R} \times (0, \infty).$$
Here and in what follows, the derivative, as well as the inverse, is taken with respect to the first variable. In (4.1), we replace $F$ by a function of the form

$$F(\phi^1, \phi^2) \exp \left\{ \frac{\cosh k(\phi^2, Z_t(\phi^2)) - \cosh \phi_t^2}{Z_t(\phi^2)} - \frac{1}{|k'(\phi_t^2, Z_t(\phi^2))|} \right\}, \quad (\phi^1, \phi^2) \in C([0, t]; \mathbb{R}^2).$$

Then, since, denoting $\phi^2 = \mathbb{T}_{B_t-k^{-1}(B_t, Z_t)}(B)$, we have

$$\phi_t^2 = k^{-1}(B_t, Z_t) \quad \text{and} \quad Z_t(\phi^2) = Z_t \quad (4.1)$$

by (i) and (iii) of Lemma 2.1, relation (4.1) turns into

$$E\left[F\left(\mathbb{T}_{B_t-k(B_t, Z_t)}(B), B\right)\right] = E\left[F\left(B, \mathbb{T}_{B_t-k^{-1}(B_t, Z_t)}(B)\right) \exp \left\{ \frac{\cosh B_t - \cosh k^{-1}(B_t, Z_t)}{Z_t} \right\} \times |(k^{-1})'(B_t, Z_t)|; \ (B_t, Z_t) \in D_k \right],$$

where a measurable set $D_k \subset \mathbb{R} \times (0, \infty)$ is defined by

$$D_k := \{(\xi, \zeta) \in \mathbb{R} \times (0, \infty); \xi \in k(\mathbb{R}, \zeta)\},$$

with $k(\mathbb{R}, \zeta)$ the image of $\mathbb{R}$ under $k(\cdot, \zeta)$ for every $\zeta > 0$.

Example 4.1. Given $\alpha \geq 0$, let $k(\xi, \zeta) = \xi - \log(1 + \alpha \zeta)$. Then $D_k = \mathbb{R} \times (0, \infty)$ and $k^{-1}(\xi, \zeta) = \xi + \log(1 + \alpha \zeta)$. Noting that

$$\frac{\cosh \xi - \cosh k^{-1}(\xi, \zeta)}{\zeta} = \frac{\alpha}{2} \left( \frac{e^{-\xi}}{1 + \alpha \zeta} - e^\xi \right)$$

for every $\xi \in \mathbb{R}$ and $\zeta > 0$, we have, from (1.2),

$$E\left[F\left(\mathbb{T}_{\log(1+\alpha Z_t)}(B), B\right)\right] = E\left[F\left(B, \mathbb{T}_{-\log(1+\alpha Z_t)}(B)\right) \exp \left\{ \frac{\alpha}{2} \left( \frac{e^{-B_t}}{1 + \alpha Z_t} - e^{B_t} \right) \right\} \right].$$

For every fixed $\mu \in \mathbb{R}$, we further replace $F$ by

$$F(\phi^1, \phi^2)e^{\mu \phi_t^2 - \mu^2 t/2}, \quad (\phi^1, \phi^2) \in C([0, t]; \mathbb{R}^2). \quad (4.3)$$

Then, noting that, as to the right-hand side, $\exp\{\mathbb{T}_{-\log(1+\alpha Z_t)}(B)(t)\} = e^{B_t}(1 + \alpha Z_t)$ in view of the former relation in (4.1), we obtain, by the Cameron–Martin formula,

$$E\left[F\left(\mathbb{T}_{\log(1+\alpha Z_t^{(\mu)})}(B^{(\mu)}), B^{(\mu)}\right)\right] = E\left[F\left(B^{(\mu)}, \mathbb{T}_{-\log(1+\alpha Z_t^{(\mu)})}(B^{(\mu)})\right) \exp \left\{ \frac{\alpha}{2} \left( \frac{e^{-B_t^{(\mu)}}}{1 + \alpha Z_t^{(\mu)}} - e^{B_t^{(\mu)}} \right) \right\} \{1 + \alpha Z_t^{(\mu)}\}^{\mu} \right].$$
When $F$ is independent of the second variable $\phi^2$, the above relation is Theorem 1.1 in [3] since, by the definition (1.3) of $\{T_z\}_{z \in \mathbb{R}}$, 

$$T_{\log(1+\alpha Z_t^{(\mu)})}(B^{(\mu)})(s) = B_s^{(\mu)} - \log \left\{ 1 + \alpha e^{-B_s^{(\mu)}} A_s^{(\mu)} \right\}, \quad 0 \leq s \leq t,$$

where we have used the relation $Z_t^{(\mu)} = e^{-B_t^{(\mu)}} A_t^{(\mu)}$ by the definition (1.2) of the transformation $Z$.

**Example 4.2.** Given $\alpha \geq 0$, let $k(\xi, \zeta) = - \log(e^{-\xi} + \alpha \zeta)$. Then 

$$D_k = \left\{ (\xi, \zeta) \in \mathbb{R} \times (0, \infty); 1/(e^\xi \zeta) > \alpha \right\},$$

on which we have $k^{-1}(\xi, \zeta) = - \log(e^{-\xi} - \alpha \zeta)$ and 

$$\cosh \xi - \cosh k^{-1}(\xi, \zeta) = \frac{\alpha}{2} \left( 1 - \frac{e^{2\xi}}{1 - \alpha e^\xi \zeta} \right),$$

as well as $(k^{-1})'(\xi, \zeta) = 1/(1 - \alpha e^\xi \zeta)$. Therefore, by (4.2), 

$$\mathbb{E} \left[ F(T_{\log(1+\alpha A_t^{(\mu)})}(B^{(\mu)}), B^{(\mu)}) \right] \\
= \mathbb{E} \left[ F(B^{(\mu)}, T_{\log(1-\alpha A_t^{(\mu)})}(B^{(\mu)})) \exp \left\{ \frac{\alpha}{2} \left( 1 - \frac{e^{2B_t^{(\mu)}}}{1 - \alpha A_t^{(\mu)}} \right) \right\} \frac{1}{1 + \alpha A_t^{(\mu)}}; \frac{1}{A_t^{(\mu)}} > \alpha \right],$$

where we have used the relation $A_t = e^{B_t} Z_t$. For every fixed $\mu \in \mathbb{R}$, we replace $F$ by a function of the form (4.3) to deduce further that, by the Cameron–Martin formula,

$$\mathbb{E} \left[ F(T_{\log(1+\alpha A_t^{(\mu)})}(B^{(\mu)}), B^{(\mu)}) \right] \\
= \mathbb{E} \left[ F(B^{(\mu)}, T_{\log(1-\alpha A_t^{(\mu)})}(B^{(\mu)})) \exp \left\{ \frac{\alpha}{2} \left( 1 - \frac{e^{2B_t^{(\mu)}}}{1 - \alpha A_t^{(\mu)}} \right) \right\} \frac{1}{1 - \alpha A_t^{(\mu)}}; \frac{1}{A_t^{(\mu)}} > \alpha \right],$$

noting that $\exp \{ T_{\log(1-\alpha A_t^{(\mu)})}(B)(t) \} = e^{B_t}/(1-\alpha A_t)$ by Lemma 2.1(i) as to the right-hand side. Since, by the definition (1.3) of $\{T_z\}_{z \in \mathbb{R}}$, the transformation of the form 

$$T_{\log(1+\alpha A_t^{(\mu)})}(\phi), \quad \phi \in C([0, t]; \mathbb{R}),$$

is expressed as 

$$\phi_s - \log \{ 1 + \alpha A_s(\phi) \}, \quad 0 \leq s \leq t,$$

the last displayed relation extends [3 Theorem 1.5] particularly to the case that $\mu$ is allowed to take negative values.
Remark 4.1. Although the transformation (4.5) is non-anticipative, it is clear that relation (4.4) is not the one that follows from Girsanov’s formula, for which we also refer to [3, Remark 1.1].

Example 4.3. Given $z \in \mathbb{R}$, let $k(\xi, \zeta) = \xi - z$. Then $D_k = \mathbb{R} \times (0, \infty)$ and $k^{-1}(\xi, \zeta) = \xi + z$, whence, by (4.2),

$$
\mathbb{E} \left[ F(T_z(B), B) \right] = \mathbb{E} \left[ F(B, T_{-z}(B)) \exp \left\{ \cosh B_t \frac{\cosh(B_t + z)}{Z_t} - \cosh(B_t + z) \right\} \right],
$$

which recovers [4, Theorem 1.2]. The above relation is consistent with the property $T_z \circ T_{-z} = \text{Id}$ in Lemma 2.1(iv).

We use the notation in [5] to denote

$$
T(\phi)(s) := T_{2\phi_1}(\phi)(s), \quad 0 \leq s \leq t,
$$

for $\phi \in C([0, t]; \mathbb{R})$.

Example 4.4. In this example, we let $k(\xi, \zeta) = -\xi$ in (4.2) to see that

$$
\mathbb{E} \left[ F(T_{2B_t}(B), B) \right] = \mathbb{E} \left[ F(B, T_{2B_t}(B)) \right],
$$

that is, with the notation recalled above, we have

$$
\left\{ (T(B)(s), B_s) \right\}_{0 \leq s \leq t} \overset{(d)}{=} \left\{ (B_s, T(B)(s)) \right\}_{0 \leq s \leq t},
$$

which is [5, Theorem 1.1] or, more precisely, [5, Corollary 1.1] with $\mu = 0$ therein. In particular, the Wiener measure on $C([0, t]; \mathbb{R})$ is invariant under $T$. A generalization of (4.5) to the case of Brownian motion with drift or other diffusion processes is given in Subsection 4.2 as an application of Corollary 1.1. For properties of $T$ such as $T \circ T = \text{Id}$ and the compatibility with the time-reversal operator $R$ defined in (2.6) that follows from (2.7), we refer to [3, Proposition 2.1].

Example 4.5. Given $x \geq 0$, we consider the following two cases:

(i) $k(\xi, \zeta) = \log(e^{-\xi} + 2x\zeta)$;

(ii) $k(\xi, \zeta) = -\log(e^\xi + 2x\zeta)$.

(i) In this case,

$$
D_k = \left\{ (\xi, \zeta) \in \mathbb{R} \times (0, \infty); e^\xi/(2\zeta) > x \right\},
$$

on which we have $k^{-1}(\xi, \zeta) = -\log(e^\xi - 2x\zeta)$,

$$
\frac{\cosh \xi}{\zeta} - \frac{\cosh k^{-1}(\xi, \zeta)}{\zeta} = x - \frac{x}{e^{2\xi} - 2x e^\xi \zeta},
$$
and \((k^{-1})'(\xi, \zeta) = -e^\xi/(e^\xi - 2x\zeta)\). Therefore, by [1.2] and by recalling the relation 
\(A_t = e^{B_t} Z_t,\)
\[
\mathbb{E}\left[F\left(T_{\log(e^{2B_t}/(1+2xA_t))}(B), B\right)\right] \\
= \mathbb{E}\left[F(B, T_{\log(e^{2B_t}-2xA_t)}(B)) \exp \left(x - \frac{x}{e^{2B_t} - 2xA_t} \right) \frac{e^{2B_t}}{e^{2B_t} - 2xA_t}; e^{2B_t}/2A_t > x\right], \tag{4.9}
\]
which is the former relation in [5, Proposition 5.3].

(ii) In the case \(k(\xi, \zeta) = -\log(e^\xi + 2x\zeta),\)
\[
D_k = \{ (\xi, \zeta) \in \mathbb{R} \times (0, \infty); 1/(2e^\xi) > x \},
\]
on which we have \(k^{-1}(\xi, \zeta) = \log(e^{-\xi} - 2x\zeta)\) and
\[
\frac{\cosh \xi}{\zeta} - \frac{\cosh k(\xi, \zeta)}{\zeta} = x - \frac{x e^{2\xi}}{1 - 2xe^\xi},
\]
as well as \((k^{-1})'(\xi, \zeta) = 1/(1 - 2xe^\xi)\). Therefore, by [1.2],
\[
\mathbb{E}\left[F\left(T_{\log(e^{2B_t}+2xA_t)}(B), B\right)\right] \\
= \mathbb{E}\left[F(B, T_{\log(e^{2B_t}/(1-2xA_t))}(B)) \exp \left(x - \frac{xe^{2B_t}}{1 - 2xA_t} \right) \frac{1}{1 - 2xA_t}; \frac{1}{2A_t} > x\right], \tag{4.10}
\]
which is the latter relation in [5, Proposition 5.3].

**Remark 4.2.** With \(\alpha = 2x\) in [1.4], the above three relations [4.4], [4.9] and [4.10] are equivalent and related via the identity [1.8] in law. For instance, if, in [4.4], we replace \(F\) by a function of the form
\[
F(\phi^1, \mathcal{T}(\phi^2)), \quad (\phi^1, \phi^2) \in C([0, t]; \mathbb{R}^2),
\]
then the left-hand side turns into
\[
\mathbb{E}\left[F\left(T_{\log(1+\alpha A_t)}(B), \mathcal{T}(B)\right)\right] = \mathbb{E}\left[F\left(T_{\log(1+\alpha A_t(\mathcal{T}(B)))}(\mathcal{T}(B)), B\right)\right]
\]
owing to [1.8], which agrees with the left-hand side of [1.10] because, by the definition (4.6) of \(\mathcal{T}\),
\[
T_{\log(1+\alpha A_t(\mathcal{T}(B)))}(\mathcal{T}(B)) = T_{\log(1+\alpha e^{-2B_tA_t})}(T_{2B_t}(B))
= T_{\log(1+\alpha e^{-2B_tA_t})+2B_t}(B),
\]
thanks to (ii) and (iv) of Lemma 2.1. On the other hand, as for the right-hand side, note that, by the definition of \(\mathcal{T}\) and properties (i) and (iv) of Lemma 2.1,
\[
\mathcal{T}(T_{\log(1-\alpha A_t)}(B)) = T_{2B_t-2\log(1-\alpha A_t)}(T_{\log(1-\alpha A_t)}(B))
= T_{2B_t-\log(1-\alpha A_t)}(B)
\]
on the event that \(A_t < 1/\alpha\), and hence, with the above replacement of \(F\) and \(\alpha = 2x\), the right-hand side of [4.4] agrees with that of [4.10]. Other implications between these three relations may be verified in a similar manner (cf. [5] Remarks 5.3 and 5.4).
4.2 Examples of Corollary 1.1

In this subsection, we explore several examples of Corollary 1.1. We mainly focus on a specific case that $\Lambda : C([0, t]; \mathbb{R}) \to (0, \infty)$ is of the form

$$\Lambda(\phi) = \lambda(\phi_t, Z_t(\phi)), \quad \phi \in C([0, t]; \mathbb{R}),$$

with $\lambda \equiv \lambda(t, \cdot, \cdot) : \mathbb{R} \times (0, \infty) \to (0, \infty)$ a continuous function satisfying

$$\int_{\mathbb{R}} d\xi \lambda(\xi, \zeta) \exp\left(-\frac{\cosh \xi}{\zeta}\right) < \infty \quad \text{for all } \zeta > 0.$$

Then, because of the fact that, for every $\xi \in \mathbb{R}$ and $\phi \in C([0, t]; \mathbb{R})$,

$$\Lambda(\mathcal{T}_{\phi_t-\xi}(\phi)) = \lambda(\xi, Z_t(\phi))$$

in view of (2.11), Corollary 1.1 is restated in such a way that, for every nonnegative measurable function $F$ on $C([0, t]; \mathbb{R}^2)$,

$$E[F(\mathcal{T}_{B_t-h_\lambda(B_t, Z_t)}(B), B)\lambda(B_t, Z_t)] = E[F(B, \mathcal{T}_{B_t-h_\lambda(B_t, Z_t)}(B))\lambda(B_t, Z_t)],$$

(4.11)

where, with slight abuse of notation, $h_\lambda$ is defined through

$$\int_{-\infty}^{h_\lambda(\xi, \zeta)} dx \lambda(x, \zeta) \exp\left(-\frac{\cosh x}{\zeta}\right) = \int_{\xi}^{\infty} dx \lambda(x, \zeta) \exp\left(-\frac{\cosh x}{\zeta}\right)$$

(4.12)

for $\xi \in \mathbb{R}$ and $\zeta > 0$. We see from (3.4) that

$$h_\lambda^{-1}(\xi, \zeta) = h_\lambda(\xi, \zeta)$$

(4.13)

for all $\xi \in \mathbb{R}$ and $\zeta > 0$.

Example 4.6. We consider the case where $\lambda(\cdot, \cdot)$ is an even function for every $\zeta > 0$, in which case we have

$$h_\lambda(\xi, \zeta) = -\xi, \quad \xi \in \mathbb{R},$$

since relation (4.12) is rewritten as

$$\int_{-\infty}^{h_\lambda(\xi, \zeta)} dx \lambda(x, \zeta) \exp\left(-\frac{\cosh x}{\zeta}\right) = \int_{-\infty}^{-\xi} dx \lambda(-x, \zeta) \exp\left(-\frac{\cosh x}{\zeta}\right) = \int_{-\infty}^{-\xi} dx \lambda(x, \zeta) \exp\left(-\frac{\cosh x}{\zeta}\right).$$

Therefore, with the notation in (4.6), relation (4.11) becomes

$$E[F(\mathcal{T}(B), B)\lambda(B_t, Z_t)] = E[F(B, \mathcal{T}(B))\lambda(B_t, Z_t)].$$

(4.14)
(1) On one hand, the last relation indicates the identity (4.8) in law, which is seen not only by taking $\lambda \equiv 1$ but also by replacing $F$ by a function of the form

$$F(\phi^1, \phi^2)/\lambda(\phi^1_t, Z_t(\phi^2)), \quad (\phi^1, \phi^2) \in C([0, t]; \mathbb{R});$$

indeed, with the above replacement, relation (4.14) turns into

$$\mathbb{E}[F(T(B), B)] = \mathbb{E}\left[ \frac{F(B, T(B))}{\lambda(B_t, Z_t)} \lambda(B_t, Z_t) \right]$$

where, for the first line, we have used the fact that, for every $\phi \in C([0, t]; \mathbb{R})$,

$$T(\phi)(t) = -\phi_t \quad \text{and} \quad Z_t(T(\phi)) = Z_t(\phi)$$

by the definition (4.6) of $T$ and properties (i) and (iii) of Lemma 2.1.

(2) On the other hand, one may also deduce from (4.14) the distributional invariance under the transformation $T$ of processes that differ from Brownian motion. As an illustration, we consider the following example: given $\mu \in \mathbb{R}$, let

$$\lambda(\xi, \zeta) = \cosh(\mu \xi)e^{-\mu^2 t/2}, \quad \xi \in \mathbb{R},$$

for every $\zeta > 0$. Then, by noting that the process

$$\cosh(\mu B_s)e^{-\mu^2 s/2}, \quad s \geq 0,$$

is a martingale with initial value 1, Girsanov’s formula entails that the law of the solution $X^{(\mu)} = \{X^{(\mu)}_s\}_{0 \leq s \leq t}$ to the stochastic differential equation (SDE)

$$dX_s = dB_s + \mu \tanh(\mu X_s) \, ds, \quad X_0 = 0,$$

is also invariant under $T$; more precisely,

$$(T(X^{(\mu)}), X^{(\mu)}) \overset{(d)}{=} (X^{(\mu)}, T(X^{(\mu)})).$$

The case $\mu = 0$ agrees with (4.8).

Example 4.7. Given $\mu \in \mathbb{R}$, let

$$\lambda(\xi, \zeta) = \exp\left(\mu \xi - \frac{\mu^2}{2} t \right), \quad \xi \in \mathbb{R},$$

for every $\zeta > 0$. Then, with the notation in (1.8), Corollary 1.1 entails that, by the Cameron–Martin formula,

$$\{\{T_A(B^{(\mu)})(s), B^{(\mu)}_s\}_{0 \leq s \leq t} \overset{(d)}{=} \{\{B^{(\mu)}_s, T(A(B^{(\mu)})(s))\}_{0 \leq s \leq t},$$

which extends (1.8) to the case of Brownian motion with drift, for the case $\mu = 0$ corresponds to the case $\lambda \equiv 1$ in Example 4.6.
Recall from [5, Corollary 1.1] that the laws of Brownian motions with opposite drifts are related via
\[ \{ (\mathcal{T}(B(-\mu))(s), B_s(-\mu)) \}_{0 \leq s \leq t} \overset{(d)}{=} \{ (B_s(\mu), \mathcal{T}(B(\mu))(s)) \}_{0 \leq s \leq t} \] (4.17)
for every \( \mu \in \mathbb{R} \), which is seen by replacing \( F \) in (4.17) by a function of the form
\[ F(\phi^1, \phi^2)e^{\mu\phi^1}, \quad (\phi^1, \phi^2) \in C([0, t]; \mathbb{R}^2). \]

The above example enables us to obtain another distributional relationship between \( B(\mu) \) and \( B(-\mu) \) as in the proposition below. We denote by \( k_\mu \) the function \( h_\lambda \) corresponding to (4.15), namely \( k_\mu \) is defined through
\[ \int_{-\infty}^{k_\mu(\xi, \zeta)} dx \, e^{\mu x} \exp \left( -\frac{\cosh x}{\zeta} \right) = \int_{\xi}^{\infty} dx \, e^{\mu x} \exp \left( -\frac{\cosh x}{\zeta} \right) \]
for \( \xi \in \mathbb{R} \) and \( \zeta > 0 \). It is readily seen that
\[ -k_{-\mu}(\xi, \zeta) = k_\mu(-\xi, \zeta) \] (4.18)
for all \( \xi \in \mathbb{R} \) and \( \zeta > 0 \). We denote by \( S_\mu \) the path transformation defined by
\[ S_\mu(\phi)(s) := T_{\phi_t + k_\mu(\phi_t, Z_t(\phi))}(\phi)(s), \quad 0 \leq s \leq t, \]
for \( \phi \in C([0, t]; \mathbb{R}) \). It then holds that
\[ S_\mu \circ S_{-\mu} = \text{Id.} \] (4.19)
Indeed, pick \( \psi \in C([0, t]; \mathbb{R}) \) arbitrarily and set \( \phi = S_{-\mu}(\psi) \). Then we have, by (i) and (iii) of Lemma 2.1
\[ \phi_t + k_\mu(\phi_t, Z_t(\phi)) = -k_{-\mu}(\psi_t, Z_t(\psi)) + k_\mu(-k_{-\mu}(\psi_t, Z_t(\psi)), Z_t(\psi)) \]
\[ = -k_{-\mu}(\psi_t, Z_t(\psi)) + k_\mu(k_{-\mu}(\psi_t, Z_t(\psi)), Z_t(\psi)) \]
\[ = -k_{-\mu}(\psi_t, Z_t(\psi)) - \psi_t, \]
where we have used (4.18) for the second line and applied (4.13) to \( k_\mu \) for the third. Therefore
\[ S_\mu(\phi) = T_{-k_{-\mu}(\psi_t, Z_t(\psi)) - \psi_t}(T_{\psi_t + k_{-\mu}(\psi_t, Z_t(\psi))}(\psi)), \]
which, by Lemma 2.1(iv), is equal to \( \psi \) as claimed.

**Proposition 4.1.** For every \( \mu \in \mathbb{R} \), we have
\[ \{ (S_{-\mu}(B(-\mu))(s), B_{s}(\mu)) \}_{0 \leq s \leq t} \overset{(d)}{=} \{ (B_s(\mu), S_\mu(B(\mu))(s)) \}_{0 \leq s \leq t}. \]
Proof. By (4.16), we have
\[
\{ (\mathcal{T}_A(B^\mu))(s), \mathcal{T}(B^\mu)(s) \}_0 \leq s \leq t \overset{(d)}{=} \{ (B^\mu_s, (\mathcal{T} \circ \mathcal{T}_A)(B^\mu)(s)) \}_0 \leq s \leq t.
\]
By (4.17), the left-hand side is identical in law with
\[
\{ ((\mathcal{T}_A \circ \mathcal{T})(B^{-\mu}))(s), B^\mu_s \}_0 \leq s \leq t.
\]
Therefore, in order to prove the proposition, it suffices to verify the following two relations:
\[
\mathcal{T}_A \circ \mathcal{T} = S_{-\mu}; \quad \mathcal{T} \circ \mathcal{T}_A = S_{\mu}.
\]
As for the former, for every \( \phi \in C([0, t]; \mathbb{R}) \), we have
\[
\mathcal{T}_A(\mathcal{T}(\phi)) = \mathbb{T}_{\mathcal{T}(\phi)(t)} - k_\mu(\mathcal{T}(\phi)(t), Z_t(\mathcal{T}(\phi))) \left( \mathcal{T}(\phi) \right)
\]
\[
= \mathbb{T}_{-\phi_t - k_\mu(-\phi_t, Z_t(\phi))} (\mathbb{T}_{2\phi_t}(\phi))
\]
\[
= \mathbb{T}_{\phi_t + k_\mu(\phi_t, Z_t(\phi))}(\phi),
\]
which is \( S_{-\mu}(\phi) \), where the second line follows from (i) and (iii) of Lemma 2.1 together with the definition (1.6) of \( \mathcal{T} \), and the third from (4.18). Since \( (\mathcal{T}_A \circ \mathcal{T})^{-1} = \mathcal{T} \circ \mathcal{T}_A \), we also obtain the latter thanks to (4.19).

We return to examples of Corollary 1.1.

Example 4.8. Given \( \mu \in \mathbb{R} \) and \( \alpha > 0 \), let
\[
\lambda(\xi, \zeta) = \frac{K_\mu(\alpha \xi)}{K_\mu(\alpha)} \exp \left( -\frac{\alpha^2}{2} \xi \zeta - \frac{\mu^2}{2} t \right), \quad \xi \in \mathbb{R}, \zeta > 0,
\]
where \( K_\mu \) is the modified Bessel function of the third kind (or the Macdonald function) of order \( \mu \) (see, e.g., [7, Section 5.7]). Then, with the corresponding \( \mathcal{T}_A \), we have, from (4.11),
\[
\mathbb{E} \left[ F(\mathcal{T}_A(B), B) \frac{K_\mu(\alpha e^{B_t})}{K_\mu(\alpha)} \exp \left( -\frac{\alpha^2}{2} A_t - \frac{\mu^2}{2} t \right) \right]
\]
\[
= \mathbb{E} \left[ F(B, \mathcal{T}_A(B)) \frac{K_\mu(\alpha e^{B_t})}{K_\mu(\alpha)} \exp \left( -\frac{\alpha^2}{2} A_t - \frac{\mu^2}{2} t \right) \right],
\]
noting \( e^{B_t} Z_t = A_t \). Notice that the process
\[
\frac{K_\mu(\alpha e^{B_s})}{K_\mu(\alpha)} \exp \left( -\frac{\alpha^2}{2} A_s - \frac{\mu^2}{2} s \right), \quad s \geq 0,
\]
is a martingale with initial value 1. Therefore, applying Girsanov’s formula, we see that the law of the solution \( X^{(\alpha, \mu)} = \{ X_s^{(\alpha, \mu)} \}_0 \leq s \leq t \) to the SDE
\[
dX_s = dB_s + \left\{ \mu - \alpha e^{X_s} \left( \frac{K_{\mu+1}}{K_\mu} \right) (\alpha e^{X_s}) \right\} ds, \quad X_0 = 0,
\]
(4.20)
is invariant under $T_\Lambda$; in fact,
\[(T_\Lambda(X^{(\alpha,\mu)}), X^{(\alpha,\mu)}) \overset{(d)}{=} (X^{(\alpha,\mu)}, T_\Lambda(X^{(\alpha,\mu)}))\).

The expression of the drift term in the above SDE is due to the relation
\[\frac{d}{dz}\{z^{-\mu}K_\mu(z)\} = -z^{-\mu}K_{\mu+1}(z)\]
(see, e.g., [4 Equation (5.7.9)]).

**Remark 4.3.** (1) Because of the fact that $K_\mu = K_{|\mu|}$ (see, e.g., [4 Equation (5.7.10)]),
the law of $X^{(\alpha,\mu)}$ is the same as that of $X^{(\alpha,|\mu|)}$, and so is the drift term of SDE (4.20),
which is indeed the case thanks to the recurrence relation
\[K_{\mu-1}(z) - K_{\mu+1}(z) = -2\frac{\mu}{z}K_\mu(z)\]
(see, e.g., [4 Equation (5.7.9)]).

(2) We see from [8 Theorem 1.5′] that the infinitesimal generator of the diffusion
process \{- log $Z_s^{(\mu)}\}_{s > 0}$, namely
\[B_s^{(\mu)} - \log A_s^{(\mu)}, \quad s > 0,\]
is the same as that of $X^{(1,\mu)}$.

The next example utilizes Corollary 1.1 in full generality.

**Example 4.9.** We deal with Ornstein–Uhlenbeck processes as an example; see, e.g., [11 pp. 140 and 141] for their precise description. For every $\alpha \in \mathbb{R}$, consider the function
\[A(\phi) = \exp\left(\frac{-\alpha^2}{2} \phi_t^2 + \frac{\alpha^2}{2} \int_0^t \phi_s^2 ds\right), \quad \phi \in C([0,t]; \mathbb{R}),\]
which fulfills the assumption of Corollary 1.1. The associated function $h_A : \mathbb{R} \times C([0,t]; \mathbb{R}) \to \mathbb{R}$ is defined through
\[\int_{-\infty}^{h_A(\xi,\phi)} dx \exp\left\{-\frac{\alpha^2}{2} x^2 - \frac{\alpha^2}{2} \int_0^t (T_{\phi_t-x}(\phi)(s))^2 ds - \frac{\cosh x}{Z_t(\phi)}\right\}\]
\[= \int_{\xi}^{\infty} dx \exp\left\{-\frac{\alpha^2}{2} x^2 - \frac{\alpha^2}{2} \int_0^t (T_{\phi_t-x}(\phi)(s))^2 ds - \frac{\cosh x}{Z_t(\phi)}\right\}\]
for $\xi \in \mathbb{R}$ and $\phi \in C([0,t]; \mathbb{R})$. Then, as a consequence of Corollary 1.1, the law of the Ornstein–Uhlenbeck process $X = \{X_s\}_{0 \leq s \leq t}$ described by the SDE
\[dX_s = dB_s - \alpha X_s ds, \quad X_0 = 0,\]
is invariant under the corresponding transformation $T_\Lambda$, or more precisely,
\[(T_\Lambda(X), X) \overset{(d)}{=} (X, T_\Lambda(X)).\]
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