Quantum Kerr oscillators’ evolution in phase space: Wigner current, symmetries, shear suppression and special states

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The creation of quantum coherences requires a system to be anharmonic. The simplest such continuous 1D quantum system is the Kerr oscillator. It has a number of interesting symmetries we derive. Its quantum dynamics is best studied in phase space, using Wigner’s distribution $W$ and the associated Wigner phase space current $J$. Expressions for the continuity equation governing its time evolution are derived in terms of $J$ and it is shown that $J$ for Kerr oscillators follows circles in phase space. Using $J$ we also show that the evolution’s classical shear in phase space is quantum suppressed by an effective ‘viscosity’. Quantifying this shear suppression provides measures to contrast classical with quantum evolution and allows us to identify special quantum states.

I. INTRODUCTION

The formation of quantum coherences is of central importance in the study of quantum systems and their dynamics.

Here we consider closed 1D Kerr-type oscillators. These are anharmonic and can therefore create coherences [1]. Additionally, their dynamics has circular symmetry in phase space. This makes them the simplest continuous system to create coherences. Therefore, they are particularly suited to help us understand aspects of non-classical effects in quantum dynamics.

Wigner’s distribution $W$ [2, 3] is the closest quantum analog [3–7] of the classical phase space distribution $\rho$. In continuous 1D systems the creation of quantum coherences is represented by the creation of negative regions of the Wigner distribution [4–6, 8, 9]. The formation of such negative regions in the Wigner distribution is easily monitored numerically.

The evolution of $W$ is governed by the associated Wigner phase space current $J$ (strictly speaking $J$ is a probability current density). Generally, phase space-based approaches are suitable for comparison of quantum with classical dynamics [4, 10, 11]. Specifically, $J$ allows us to adopt a geometric approach [1, 5, 11–13] to studying quantum dynamics.

We introduce Kerr oscillators, their Wigner distribution $W$, and their associated Wigner current $J$ in section II. In Section III we show that there are no trajectories and no phase space flow for anharmonic systems such as Kerr oscillators. In Section IV we investigate how pulses in phase space smear out classical spirals [Fig. 1 (b)]. We find that pulses in phase space steepen and lengthen dynamically. This analysis is aided by the system’s circular symmetry and the fact that the probability on circles in phase space is conserved. In Section V we show that using Wigner current $J$’s effective ‘viscosity’ [11] allows us to contrast classical with quantum dynamics and pick out special quantum states.

II. WIGNER DISTRIBUTIONS AND WIGNER CURRENT OF KERR OSCILLATORS

A one-dimensional system’s Wigner distribution $W_{\rho}(x, p, t)$ [2, 3] ($x$ denotes position, $p$ the associated momentum, and $t$ time), for a quantum state described by a density matrix $\hat{\rho}$, is defined as the Fourier transform of its off-diagonal coherences $\rho(x + y, x - y, t)$ (parameterized by the shift $y$)

$$W_{\rho}(x, p, t) = \frac{1}{\pi \hbar} \int_{-\infty}^{\infty} dy \, \langle x + y | \hat{\rho}(t) | x - y \rangle \, e^{-\frac{y^2}{\hbar^2}}$$

where $\hbar = \hbar/(2\pi)$ is Planck’s constant. By construction $W$ is normalized and non-local (through $y$). Unlike $\hat{\rho}$, $W$ is always real-valued but, generically, $W$ features negativities [2]. Since $W_{\rho}$ is $\hat{\rho}$’s Fourier transform, $W$ and $\hat{\rho}$ are isomorphic to each other, allowing us to describe all aspects of the quantum system’s state and its dynamics using the Wigner representation of quantum theory [14].
A. Time evolution of the Wigner distribution

For conservative Kerr systems the time development of $W$ is given by the Moyal-bracket $\{\{.,.\}\}$ \( [14, 15] \)
\[
\partial_t W(x, p, t) = \{\{H, W\}\}
\]
\[
\equiv \frac{2}{\hbar} \iota \{H(x, p) \sin \left( \frac{\hbar}{2} (\partial_x \hat{p} - \partial_p \hat{x}) \right) \} W(x, p, t). \tag{3}
\]
Here, $\partial_x = \frac{\partial}{\partial x}$, etc.; the arrows over the derivatives indicate whether they act on (point towards) Hamiltonian or Wigner distribution.

The Hamiltonian of anharmonic single-mode oscillators of the Kerr type has the form
\[
\hat{H}_\Lambda = \left( \frac{\hat{p}^2}{2M} + \frac{k}{2} \hat{q}^2 \right) + \Lambda^2 \left( \frac{\hat{p}^2}{2M} + \frac{k}{2} \hat{q}^2 \right)^2, \tag{4}
\]
with the oscillator mass $M$ and spring constant $k$. Such Hamiltonians describe electromagnetic fields subjected to Kerr non-linearities $\chi^{(3)}$ (here $\Lambda^2 \propto \chi^{(3)}$) \([16–19]\). This system is fully solvable since wave functions of the harmonic oscillators are solutions to the Kerr Hamiltonian with eigen-energies $E_n = \hbar \sqrt{\frac{k}{M} ((n + \frac{1}{2}) + \Lambda^2 (n + \frac{1}{2})^2)}$.

Its quantum recurrence time is
\[
T_\Lambda = \frac{\pi}{|\Lambda|^2}. \tag{5}
\]

Following Wigner \([2]\), we cast expression (3) in the form of the phase space continuity equation
\[
\partial_t W + \nabla \cdot J = 0 , \tag{6}
\]
where $\nabla = \left( \partial_x \partial_p \right)$ is the gradient, and $J = \left( J_x J_p \right)$ denotes the Wigner current in phase space \([12]\). $J$ is the quantum analog \([20, 21]\) of the classical phase space current $j = \rho v$ \([22]\) which transports the classical probability density $\rho(x, p, t)$ according to Liouville’s continuity equation $\partial_t \rho = -\nabla \cdot j$.

$J$ reveals details \([12, 13]\) about quantum systems’ phase space dynamics previously thought inaccessible due to the supposed ‘blurring’ by Heisenberg’s uncertainty principle.

From now on we will consider $M = k = 1$ only. Then \[for a derivation see Eqns. (22) and (23) in the Appendix VI\], with $r = (x, p) = r(\cos \theta, \sin \theta)$, $r = \sqrt{x^2 + p^2}$ and $\Delta = \partial_x^2 + \partial_p^2$, $J$ can be written as
\[
J = \left( \begin{array}{c} \frac{p}{x} \\ -x \end{array} \right) \left[ 1 + \Lambda^2 \left( x^2 + p^2 - \frac{\hbar^2}{4} \Delta \right) \right] W = \left( \begin{array}{c} r \sin \theta \\ -r \cos \theta \end{array} \right) \left[ 1 + \Lambda^2 \left( r^2 - \frac{\hbar^2}{4} (\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2) \right) \right] W. \tag{7}
\]

$J$ is tangent to circles concentric with the origin of phase space. This circular symmetry allows us to consider an approximation of the dynamics on such individual circles, an observation we make use of below.

For future reference we split $J$ into its classical $j$ and quantum terms $J^Q$
\[
J = j + J^Q = Wv - \left( \begin{array}{c} \frac{p}{x} \\ -x \end{array} \right) \left( \frac{\hbar^2 \Lambda^2}{4} \Delta \right) W, \tag{8}
\]

here $v = \left( \begin{array}{c} -p \\ x \end{array} \right) (1 + \Lambda^2 r^2)$ is the classical phase space velocity. The quantum terms $J^Q$ are only present for anharmonic potentials \([1]\), which is why only anharmonic potentials create coherences. Harmonic systems’ phase space dynamics follows $v$ and is classical, see Refs. \([1, 5]\).

III. NO TRAJECTORIES OR FLOW IN QUANTUM PHASE SPACE

Inspired by classical mechanics, there have been several attempts to treat quantum phase space evolution as a flow along trajectories \([5]\). Such attempts are ill-fated \([5]\) as we will explain now. They use the formal factorization $J = Ww$ to define a ‘quantum phase space velocity’ $w = J/W$, then the continuity equation (6) assumes the form \([5, 23, 24]\)
\[
\partial_t W + w \cdot \nabla W + W \nabla \cdot w = 0 . \tag{9}
\]
Here the convective term $w \cdot \nabla W$ describes the transport that carries $W$ along with the current (following fieldlines in phase space) without changing its values. In contrast, the current divergence term $W \nabla \cdot w$ changes values of $W$. This is best seen by formally rearranging Eq. (9) for the total derivative
\[
\frac{dW}{dt} = \partial_t W + w \cdot \nabla W = -W \nabla \cdot w . \tag{10}
\]
Treating a continuity equation in this form is known as its \emph{Lagrangian decomposition}. This decomposition has to be treated with extreme caution, since it essentially splits the well behaved and finite term $\nabla \cdot J$ into the two individually singular terms $w \cdot \nabla W$ and $W \nabla \cdot w$. Some implications are discussed below.
For the Kerr system this total derivative is
\[
\frac{dW}{dt} = -\frac{\Lambda^2 \hbar^2}{4} \left[ p \left( \frac{\partial_x W}{W} - \partial_x \right) - x \left( \frac{\partial_p W}{W} - \partial_p \right) \right] \Delta W = -\frac{\Lambda^2 \hbar^2}{4} W \partial_\theta \left( \frac{\Delta W}{W} \right),
\]
(11)
and the convective transport term in Eq. (10) is
\[
w \cdot \nabla W = \left( \Lambda^2 \left[ -r^2 + \frac{\hbar^2}{4W} \Delta W \right] - 1 \right) \partial_\theta W.
\]
(12)
Since the divergence \( \nabla \cdot w \) is non-zero, the quantum evolution does not preserve phase space volumes [1, 5, 15].

One could still describe quantum evolution by phase space transport if the magnitude of this divergence were finite across the entire phase space [5]. Indeed, modelling quantum phase space dynamics through such transport along trajectories has been attempted many times; in this context it has been considered an undesirable feature of \( w \) that it is a singular quantity when \( W \) is zero (see Ref. [5] for details). But zeros in \( W \) are unavoidable [25]:

The singularities in \( \nabla \cdot w \) are a fundamental and necessary feature to create negative regions in \( W \) and thus to create quantum coherences. Such singularities are not a flaw. A velocity field \( w \) with positive divergence that is bounded from above, \( B > \nabla \cdot w > 0 \), will by itself not be able to generate negativities. The associated expansion of phase space volumes can only reduce the initial value \( W(0) > 0 \) of a density towards zero, since Eq. (10) implies [5, 23]
\[
W(t)_{\text{comoving}} > W(0) \exp(-Bt) > 0
\]
(13)
for all times. Trahan and Wyatt noticed this and concluded that “the sign of the density riding along the trajectory cannot change” [23].

But this interpretation is incorrect. When \( W = 0 \) the velocity \( w \) and its divergence is singular, Eq. (11) cannot be integrated since \( w \)’s singularities render integrals and associated bounds such as (13) ill-defined [5]. Therefore, in anharmonic quantum systems neither trajectories nor transport along flow lines exist [5] (References [21] and [12], refer to Wigner ‘flow’ but were written before this was realized).

Because of the singular volume changes associated with Eq. (11), we feel the quantum Liouville equation (6) should be called Wigner’s continuity equation instead.

We are forced to conclude that a trajectory-based approach to quantum phase space evolution creates contradictions such as singular \( w \) and singular phase space volume changes. This highlights the stark differences between classical and quantum dynamics in an illuminating manner. The singularities in \( w \) and phase space volume changes are needed to violate inequality (13) thus allowing for the creation of quantum coherences and negative regions in \( W \) [1, 5].

IV. PULSES IN QUANTUM PHASE SPACE

In the classical case the probability (of \( \rho \)) on a classical trajectory of a conservative system is conserved over time. It can be checked that the probability (of \( W \)) on a classical trajectory is not conserved for typical anharmonic quantum systems.

The quantum Kerr system is an exception as its evolution preserves probability on rings around the origin:
\[
\oint \delta \theta \partial_\theta W = -\oint \delta \theta \nabla \cdot J = 0,
\]
(14)
since \( \nabla \cdot J = r \partial_\theta (|v(r)| - \Lambda^2 \frac{\hbar^2}{4} \Delta |W|) \). Additionally to the circular symmetry displayed in Eq. (7), this probability conservation on circles is the primary reason why considering the Kerr dynamics on circles is suitable.

The classical velocity profile \( v(r) \) leads to the formation of fine detail in the classical evolution: in the case of a Gaussian initial state, the state becomes wrapped into a single tightly wound spiral, see Fig. 1 (b). The quantum evolution shows this tendency of spiral wrapping as well, but while the formation of fine detail is suppressed through “viscous” behaviour (see Section V), negativities of the Wigner distribution emerge.

To study this in more detail, consider \( W \) on a ring of radius \( r \), as displayed in Fig. 2.

The quantum ‘cross-talk’ terms \( \partial_t^2 + \frac{1}{r} \partial_r \) in Eq. (7) couple the current on adjacent rings. We can cast these terms aside if we may assume that the Wigner distribution’s azimuthal curvature \( \partial_\theta^2 W \) is much greater than its radial curvature and gradient. Making this assumption

FIG. 2. (Color online) Time evolution of \( W(\theta) \) on a ring with fixed radius \( r = 1.0 \) for initial coherent state \( |\alpha\rangle \) = \( |7/12\rangle \) over time \( t = 0 \) to \( rT_{\chi}/4 = 4\pi = 12.56 \) [\( \Lambda = \frac{1}{4} \)]. The darker and thinner the curves, the more time has elapsed. The curves move clockwise on the ring, towards increasing values of ‘–\( \theta \)’. The quantum evolution leads to a speed-up over the classical evolution (the classical phase angle \( v \) is subtracted). Additionally, under quantum evolution the pulse widens and steepens at the front, this triggers the formation of oscillations with negative regions in front of the pulse which eventually catch up with the main pulse from ‘behind’.

temporarily, the velocity on a ring is approximately
\[ w(r, \theta) \approx r \left[ 1 + \Lambda^2 \left( r^2 - \frac{k}{4r^2} \partial_\theta^2 W \right) \right]. \quad (15) \]

This approximation is obviously poor when \( W \approx 0 \), but Eq. (15) is still useful for the discussion that follows.

In Figs. 2-4 the full evolution is portrayed, not its approximate behaviour of Eq. (15). The axis ‘-\( \theta \)' is chosen in Figs. 2-4 since classical evolution proceeds clockwise, in the direction of negative values of \( \theta \).

The effect of the \( \theta \)-curvature term, retained in Eq. (15), is primarily twofold: for a Wigner distribution on a circle, forming a hump, the hump’s leading and trailing edges, having positive curvature, get delayed. Conversely, the lower curvature term, retained in Eq. (15),

\[ \Lambda - \left[ \frac{1}{16} - \frac{1}{3} \right] \partial_\theta^2 W \]

over time \( t \), thus no terms suppress the effects of the angular velocity gradients, and, as time progresses, non-singular probability distributions in phase space get sheared into ever finer filaments [see Fig. 1 (b)].

V. J’S VISCOITY AND SPECIAL STATES

In the preceding Section IV we discussed motion on a ring. Here we consider cross talk between motion on neighbouring rings.

Over time classical Hamiltonian phase space flow shears \( \rho \) since \( \nabla \) creates non-zero gradients of its angular velocity across energy shells. This flow is inviscid as \( \nabla \) is independent of \( \rho \), thus no terms suppress the effects of the angular velocity gradients, and, as time progresses, non-singular probability distributions in phase space get sheared into ever finer filaments [see Fig. 1 (b)].

The associated classical phase space shear has been derived in Ref. [11] as
\[ s(x, p; H) = \partial_x \partial_p \left( -\nabla \times \nu \right) = \partial_x \partial_p \left( \partial_p v_x - \partial_x v_p \right). \quad (16) \]

Here the directional derivative across energy shells \( \partial_x \partial_p \), is formed from the normalized gradient \( \hat{\nabla}_H = \nabla H/|\nabla H| \) of the Hamiltonian \( H \). Because of the Kerr system’s circular symmetry \( \hat{\nabla}_H = \partial_r \).

The sign convention using the negative curl in \( s \) in Eq. (16) is designed to yield a positive sign for clockwise orientated fields since this is the prevailing direction of the classical velocity field \( \nu \). This choice yields \( s > 0 \) for hard potentials [potentials for which the magnitude of the force increases with increasing amplitude, i.e., \( \Lambda^2 > 0 \)], since they induce clockwise shear, see Fig. 1 (b). \( s > 0 \) for harmonic oscillators [i.e., \( \Lambda = 0 \)], and \( s < 0 \) for soft potentials [for which the magnitude of the force decreases with increasing amplitude, i.e., \( \Lambda^2 < 0 \)] since they induce anti-clockwise shear. The reaction of quantum dynamics to classical shear \( s \) has to reside in \( J^Q \) of Eq. (8). To extract it we form the vorticity of \( J^Q \) [11]
\[ \delta(x, p; t; H) = -\nabla \times J^Q = \partial_p J^Q_x - \partial_x J^Q_p. \quad (17) \]

\( \delta \)'s sign distribution shows a pronounced polarization pattern, see Fig. 5.

Specifically, for a system with clockwise shear Fig. 5 (b) illustrates that \( \delta(H_{\Lambda_+}) \) [with \( \Lambda^2_+ = +(1/4)^2 \)] tends to be positive on the inside [towards the origin] and negative on the outside of the positive main ridge of \( W \) [see inset of Fig. 5 (a)]. Because of this, the outside is being slowed down while the inside speeds up. This polarized distribution of \( \delta \) therefore counteracts the classical shear \( \{s_{H_{\Lambda_+}} > 0\} \) and can suppress it altogether [11]. The same applies to other positive regions of \( W \), whereas for its negative regions the current \( J \) tends to be inverted [12, 13] inverting \( \delta \)'s polarization pattern, see Ref. [11] and Fig. 5 (b).

When the same state \( W \) is governed by a Hamiltonian \( H_{\Lambda_-} \) with anti-clockwise shear [11] [i.e. \( \Lambda^2_- < 0 \)] \( \delta(H_{\Lambda_-}) \) tends to be the sign-inverted form of \( \delta(H_{\Lambda_+}) \) (for
Kerr systems we find $\delta(H_{\Lambda_+}) = -\delta(H_{\Lambda_-})$ if $|\Lambda_+| = |\Lambda_-|$. This is illustrated in Fig. 5 (c), where $\Lambda^2 = -(1/4)^2$ is negative, whereas in Fig. 5 (b) $\Lambda^2 = +(1/4)^2$ is positive.

The distribution of $\delta$’s polarization can be picked up with the directional derivative $\partial_{\omega} \delta(t; H) = \partial_\omega \delta(t; H)$. This we multiply with $W$, because negative regions of $W$ invert the current $J$ [12], and because we want to weight it with the local contribution of the state. The resulting local measure for weighted shear polarization is [11]

$$\Pi(t; H) = \langle\langle \pi(t; H) \rangle\rangle = \int_{-\infty}^{\infty} dx dp \pi(x, p, t; H).$$

Fig. 6 illustrates that $\Pi(t)$ initially drops and after a while levels off.

![FIG. 5. Polarization of the vorticity $\delta$ and inversion of this polarization. (a), the Wigner distribution $W$ of a Gaussian initial state centered on $x = -4$, $p = 0$ and evolved to $t = 40$ using $H_{\Lambda_+} = H_{1/4}$. Its contours, for emphasis the zero contour is shown as black-green dashed lines, are also employed in (b) and (c). The inset for $W$ in (a) is reproduced showing the effects of, (b), clockwise shear $[\delta(H_{\Lambda_+})]$ and, (c), anti-clockwise shear $[\delta(H_{\Lambda_-})]$. Comparing (b) with (c) demonstrates polarization inversion of $\delta$ associated with shear inversion of the system, here $\Lambda^2 = +(1/4)^2 = -\Lambda^2$.](image)

We emphasize that the levelling-off behaviour of $\Pi(t)$ is in marked contrast to the classical case: for long enough times, in simple bound state classical systems non-singular states $\rho(t)$ get stretched out linearly [11] into ever finer threads, see Fig. 1 (b), therefore $\langle \delta_{r}(-\nabla \times j) \rangle \propto t$ [11]. The quantum evolution counteracts this classical shear resulting in values of the shear suppression $\Pi$ which are opposite in sign to those of $s$ [11] (for the Kerr system $s = \text{sign}[(\Lambda^2)]$).

Moreover, starting from an initial Gaussian state, the magnitude $|\Pi(t)|$ initially grows the more the evolution stretches out the state into finer structures. Eventually quantum shear suppression stops classical shear from creating finer structures in phase space [11]: $|\Pi(t)|$ levels off.

In other words, the quantum evolution is effectively ‘viscous’. This ‘viscosity’ is the mechanism by which quantum evolution enforces that $W$ can typically not form structures below the size scale identified by Zurek [4]. Therefore $\Pi(t)$ settles when the state has formed structures at the Zurek scale. This can e.g. be quantified by monitoring the phase-spatial frequency content of $W$ as a function of time, for details see [11].

Yet, quantum evolution is not truly viscous, it allows for revivals. Interestingly, these are picked up by the deviation of $\Pi(t)$ from the local time average. For the Kerr system, the special states for which this deviation is largest are (fractional) revival states [19, 27], see Fig. 6.

We emphasize that such revival states are traditionally picked up through the overlap of the evolved state with a suitably chosen reference state (such as a Gaussian initial state) [27], instead, our measure $\Pi(t)$ does not depend on the state, it makes it more versatile then the use of wave function overlaps.

![FIG. 6. Smoothed $\Pi(t)$ picks out special states. Deviations of $\Pi(t)$ from the settled value ($\approx -115$) single out special states: the evolution shows recurrence of the initial state at time $T_{\Lambda_+} = 16 \pi \approx 50.3$ ($\Lambda_+ = 1/2$). Pronounced peaks and troughs at intermediate times identify fractional revival states [28] with special $n$-fold symmetries.](image)
circular symmetry, graphs of as smooth as those for $\Pi(t)$ obtained in Fig. 6 require frequency filtering \[11\].

Additionally to the symmetries identified above, also in this regard are Kerr oscillators the simplest possible continuous quantum systems that alter quantum coherences.

To conclude, quantum dynamics that generates coherences in continuous systems is most easily studied in phase space and using Kerr systems, since these have special symmetries. The two new symmetries we have identified are circular phase space current $J$, Eq. (7), and probability conservation for $W$ on rings, Eq. (14). These imply the absence of high-frequency components in $\Pi(\sigma)$, see Fig. 7: $J_\sigma$ since the parameterised by the interpolation parameter $\sigma$ assigned to $\Lambda = \lambda$. Here we keep the two parameters $\Lambda$ and $\lambda$ distinct to allow us to tune the system's non-linearities independently and help with keeping track of terms in the derivation of the form of $J$.

The square brackets bracket the terms arising from the Kerr Hamiltonian’s anharmonic part whereas the terms $\{ \{ H, W \} \} = \{ \{ \lambda k x p^2 / 2M + \lambda k^2 x^2 / 2M, W \} \}$ contain the classical Hamiltonian current terms, the round brackets the terms $\{ \{ H, W \} \} = \{ \{ \lambda k x p^2 / 2M + \lambda k^2 x^2 / 2M, W \} \}$ of $\lambda k x p^2 / 2M$. Here we keep the two parameters $\Lambda$ and $\lambda$ distinct to allow us to tune the system’s non-linearities independently and help with keeping track of terms in the derivation of the form of $J$.

The Wigner distribution of the Kerr oscillator obeys the phase space continuity equation \[18, 29, 30\]

$$\partial_t W(x, p, t) = \{ \{ H, W \} \} = \frac{2}{\hbar} H(x, p) \sin \left( \frac{\hbar}{2}(\partial_x \partial_p - \partial_p \partial_x) \right) W(x, p, t)$$

$$= \left( -\lambda^2 \frac{\hbar^2 k^2 p^2 \partial_x^2}{4M^2} + \frac{\lambda^2 k^2 x^2}{4} \partial_p^2 - \left\{ \lambda \frac{\hbar k x p^2}{M} + \lambda \frac{k^2 x^2}{2M} \right\} \partial_p \right) + \frac{\hbar^2 k^2}{4M} \partial_x \partial_p^2$$

$$- \lambda \frac{\hbar k^2 x \partial_p \partial_x^2}{4M} + \left\{ \lambda \frac{k x^2 p}{M} + \lambda \frac{k^2 x^2}{2M} \right\} \partial_x + \frac{p}{M} \partial_x - k x \partial_p \right) W(x, p, t).$$

The square brackets bracket the terms arising from the Kerr Hamiltonian’s anharmonic part whereas the terms $\{ \{ H, W \} \} = \{ \{ \lambda k x p^2 / 2M, W \} \}$ of $\lambda k x p^2 / 2M$ stem from the harmonic oscillator contribution $p^2 / (2M) + k x^2 / 2$.

The associated Wigner current components \[6\] are

$$J_x = \left[ \frac{\hbar^2}{4M^2} \partial_x \partial_p^2 - \lambda \frac{k x^2}{4M} \partial_x \partial_p^2 \right] W(x, p, t)$$

and

$$J_p = \left[ \frac{\hbar^2}{4M^2} \partial_x \partial_p^2 + \lambda \frac{k x^2}{4M} \partial_x \partial_p^2 \right] - \left\{ \lambda \frac{k x^2}{2M} + \lambda \frac{k^2 x^2}{2M} \right\} \partial_x \right) W(x, p, t).$$

The curly brackets in Eqs. \(22\) and \(23\) contain the classical Hamiltonian current terms, the round brackets the quantum terms.

To justify this assignment, note that the first term in $J_p$ is of the form $\frac{\hbar^2}{4M^2} \partial_x \partial_p^2 W$ \[2, 12\] and thus has to be assigned to $J_p$, while the first term of $J_x$ is its ‘partner’ term for the position case. What remains somewhat ambiguous is whether the second terms in $\{ \{ H, W \} \}$ and $\{ \{ H, W \} \}$ have been assigned correctly. To highlight this ambiguity consider

$$J_x^{(\sigma)} = J_x + \sigma \lambda \frac{\hbar^2 k^2}{4M} \left[ x \partial_p \partial_x + p \partial_x \partial_p \right] W(x, p, t)$$

and

$$J_p^{(\sigma)} = J_p - \sigma \lambda \frac{\hbar^2 k^2}{4M} \left[ x \partial_p \partial_x + p \partial_x \partial_p \right] W(x, p, t),$$

parameterised by the interpolation parameter $\sigma$ with $0 \leq \sigma \leq 1$. This interpolation fulfils the continuity equation \[6\] since the $\sigma$-dependent terms are divergence-free for $0 \leq \sigma \leq 1$.

To remove the ambiguity we can use Wigner current plots. We notice that the field plots of $J^{(\sigma=0)}$ do not ‘make sense’, see Fig. 7: $J^{(\sigma=0)}$ of Eqs. \(22\) and \(23\), or \(7\) is the correct Wigner current expression.
FIG. 7. (Color online) **Wigner distribution, incorrect and correct Wigner current pattern for state** \((|0\rangle + |1\rangle) / \sqrt{2}\). With \(\Lambda = \lambda\) the dynamics of this superposition state is isomorphic to that of the harmonic oscillator, except for an extra phase due to the Kerr oscillator’s different energy spectrum. The incorrect expression \(J^{(\sigma=1)}\) for the current (middle panel) does not respect this isomorphism, it breaks the system’s circular symmetry and is therefore discarded. The correct expression \(J^{(\sigma=0)}\) for the current is depicted in the right panel. The region represented by green coloring is that where \(W < 0\), this leads to current inversion [12]. For the Kerr system the only point of stagnation [12] of the current is the coordinate-origin. When the current stagnates elsewhere in phase space, it forms *lines of stagnation* [13].

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