Continuous Controlled K-Frame for Hilbert $C^*$-Modules

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ABSTRACT. In this paper, we introduce and we study the concept of Continuous Controlled K-Frame for Hilbert $C^*$-Modules which are generalizations of discrete Controlled K-Frames.

1. Introduction and preliminaries

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer [9] in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper [7] by Daubechies, Grossman and Meyer, frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames [11]. Frames have been used in signal processing, image processing, data compression and sampling theory. The concept of a generalization of frames to a family indexed by some locally compact space endowed with a Radon measure was proposed by G. Kaiser [14] and independently by Ali, Antoine and Gazeau [5]. These frames are known as continuous frames. Gabardo and Han in [10] called these frames associated with measurable spaces, Askari-Hemmat, Delghani and Radjabalipour in [3] called them generalized frames and in mathematical physics they are referred to as coherent states [5]. In 2012, L. Gavruta [12] introduced the notion of K-frames in Hilbert space to study the atomic systems with respect to a bounded linear operator K. Controlled frames in Hilbert spaces have been introduced by P. Balazs [4] to improve the numerical efficiency of iterative algorithms for inverting the frame operator. Rahimi [17] defined the concept of controlled K-frames in Hilbert spaces and showed that controlled K-frames are equivalent to K-frames due to which the controlled operator C can be used as preconditions in applications. Controlled frames in $C^*$-modules were introduced by Rashidi and Rahimi [15], and the authors showed that they share many useful properties with their corresponding notions in a Hilbert space. We extended the results of frames in Hilbert spaces to Hilbert $C^*$-modules (see [13], [19], [21], [22], [23], [24], [25], [26], [27], [28], [29])

Motivated by the above literature, we introduce the notion of a continuous controlled K-frame in Hilbert $C^*$-modules.

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In the following we briefly recall the definitions and basic properties of \( C^* \)-algebra, Hilbert \( A \)-modules. Our references for \( C^* \)-algebras as \([6, 8]\). For a \( C^* \)-algebra \( A \) if \( a \in A \) is positive we write \( a \geq 0 \) and \( A^+ \) denotes the set of positive elements of \( A \).

**Definition 1.1.** \([18]\) Let \( A \) be a unital \( C^* \)-algebra and \( H \) be a left \( A \)-module, such that the linear structures of \( A \) and \( H \) are compatible. \( H \) is a pre-Hilbert \( A \)-module if \( H \) is equipped with an \( A \)-valued inner product \( \langle \cdot, \cdot \rangle_A : H \times H \to A \), such that is sesquilinear, positive definite and respects the module action. In the other words,

(i) \( \langle x, x \rangle_A \geq 0 \) for all \( x \in H \) and \( \langle x, x \rangle_A = 0 \) if and only if \( x = 0 \).
(ii) \( \langle ax + y, z \rangle_A = a \langle x, z \rangle_A + \langle y, z \rangle_A \) for all \( a \in A \) and \( x, y, z \in H \).
(iii) \( \langle y, y \rangle_A = \langle y, y \rangle_A^* \) for all \( x, y \in H \).

For \( x \in H \), we define \( ||x|| = ||\langle x, x \rangle_A||^{\frac{1}{2}} \). If \( H \) is complete with \( ||.|| \), it is called a Hilbert \( A \)-module or a Hilbert \( C^* \)-module over \( A \). For every \( a \) in \( C^* \)-algebra \( A \), we have \( |a| = (a^*a)^{\frac{1}{2}} \) and the \( A \)-valued norm on \( H \) is defined by \( |x| = \langle x, x \rangle_A^{\frac{1}{2}} \) for \( x \in H \).

Let \( H \) and \( K \) be two Hilbert \( A \)-modules, A map \( T : H \to K \) is said to be adjointable if there exists a map \( T^* : K \to H \) such that \( \langle Tx, y \rangle_A = \langle x, T^*y \rangle_A \) for all \( x \in H \) and \( y \in K \).

We reserve the notation \( \text{End}_A^*(H, K) \) for the set of all adjointable operators from \( H \) to \( K \) and \( \text{End}_A^*(H, H) \) is abbreviated to \( \text{End}_A^*(H) \).

**Lemma 1.2.** \([3]\) Let \( H \) and \( K \) two Hilbert \( A \)-modules and \( T \in \text{End}_A^*(H) \). Then the following statements are equivalent:

(i) \( T \) is surjective.
(ii) \( T^* \) is bounded below with respect to norm, i.e, there is \( m > 0 \) such that \( \|T^*x\| \geq m\|x\| \), \( x \in K \).
(iii) \( T^* \) is bounded below with respect to the inner product, i.e, there is \( m' > 0 \) such that, \( \langle T^*x, T^*x \rangle_A \geq m'\langle x, x \rangle_A \), \( x \in K \).

**Lemma 1.3.** \([18]\) Let \( H \) and \( K \) two Hilbert \( A \)-modules and \( T \in \text{End}_A^*(H) \). Then the following statements are equivalent,

(i) The operator \( T \) is bounded and \( A \)-linear.
(ii) There exist \( 0 \leq k \) such that \( \langle Tx, Tx \rangle_A \leq k\langle x, x \rangle_A \), \( x \in H \).

For the following theorem, \( R(T) \) denote the range of the operator \( T \).

**Theorem 1.4.** \([30]\) Let \( H \) be a Hilbert \( A \)-module over a \( C^* \)-algebra \( A \) and let \( T, S \) two operators for \( \text{End}_A^*(H) \). If \( R(S) \) is closed, then the following statements are equivalent:

(i) \( R(T) \subset R(S) \).
(ii) \( TT^* \leq \lambda^2 SS^* \) for some \( \lambda \geq 0 \).
(iii) There exists \( Q \in \text{End}_A^*(H) \) such that \( T = SQ \).
2. Continuous Controlled K-Frame for Hilbert $C^*$-Modules

Let $X$ be a Banach space, $(\Omega, \mu)$ a measure space, and $f : \Omega \to X$ a measurable function. Integral of the Banach-valued function $f$ has been defined by Bochner and others. Most properties of this integral are similar to those of the integral of real-valued functions. Since every $C^*$-algebra and Hilbert $C^*$-module is a Banach space thus we can use this integral and its properties.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $C^*$-modules, $\{\mathcal{K}_w : w \in \Omega\}$ is a family of subspaces of $\mathcal{K}$, and $End^*_A(\mathcal{H}, \mathcal{K}_w)$ is the collection of all adjointable $\mathcal{A}$-linear maps from $\mathcal{H}$ into $\mathcal{K}_w$. We define

$$\bigoplus_{w \in \Omega} \mathcal{K}_w = \{x = \{x_w\} \in \mathcal{K} : x_w \in \mathcal{K}_w, \int_{\Omega} \|x_w\|^2 d\mu(w) < \infty\}.$$  

For any $x = \{x_w : w \in \Omega\}$ and $y = \{y_w : w \in \Omega\}$, if the $\mathcal{A}$-valued inner product is defined by $\langle x, y \rangle_A = \int_{\Omega} \langle x_w, y_w \rangle_A d\mu(w)$, the norm is defined by $\|x\| = \|\langle x, x \rangle_A\|^{\frac{1}{2}}$. Therefore, $\bigoplus_{w \in \Omega} \mathcal{K}_w$ is a Hilbert $C^*$-module (see [14]).

Let $\mathcal{A}$ be a $C^*$-algebra, $l^2(\mathcal{A})$ is defined by,

$$l^2(\mathcal{A}) = \{\{a_w\} \subseteq \mathcal{A} : \int_{\Omega} \|a_w a^*_w d\mu(w)\|^2 < \infty\}.$$  

$l^2(\mathcal{A})$ is a Hilbert $C^*$-module (Hilbert $\mathcal{A}$-module) with pointwise operations and the inner product defined as,

$$\langle \{a_w\}, \{b_w\} \rangle_A = \int_{\Omega} a_w b^*_w d\mu(w), \{a_w\}, \{b_w\} \subseteq l^2(\mathcal{A}),$$  

and

$$\|\{a_w\}\| = \left(\int_{\Omega} \|a_w a^*_w d\mu(w)\|\right)^{\frac{1}{2}}.$$  

**Definition 2.1.** Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module over a unital $C^*$-algebra, and $K \in End^*_A(\mathcal{H})$. A mapping $F : \Omega \to \mathcal{H}$ is called a continuous K-Frame for $\mathcal{H}$ if:

- $F$ is weakly-measurable, ie, for any $f \in \mathcal{H}$, the map $w \to \langle f, F(w) \rangle_A$ is measurable on $\Omega$.

- There exist two strictly positive constants $A$ and $B$ such that

$$A \langle K^* f, K^* f \rangle_A \leq \int_{\Omega} \langle f, F(w) \rangle_A \langle F(w), f \rangle_A d\mu(w) \leq B \langle f, f \rangle_A, f \in \mathcal{H}.$$  

The elements $A$ and $B$ are called continuous K-frame bounds.

If $A = B$ we call this Continuous K-Frame a continuous tight K-Frame, and if $A = B = 1$ it is called a continuous Parseval K-Frame. If only the right-hand inequality of (2.1) is satisfied, we call $F$ a continuous bessel mapping with Bessel bound $B$.

Let $F$ be a continuous bessel mapping for Hilbert $C^*$-module $\mathcal{H}$ over $\mathcal{A}$. The operator $T : \mathcal{H} \to l^2(\mathcal{A})$ defined by,

$$Tf = \{\langle f, F(\omega) \rangle_A\}_{\omega \in \Omega},$$
is called the analysis operator. 
There adjoint operator \( T^* : l^2(\mathcal{A}) \to \mathcal{H} \) given by,

\[
T^*(\{a_\omega\}_{\omega \in \Omega}) = \int_\Omega a_\omega F(\omega) d\mu(\omega),
\]

is called the synthesis operator.
By composing \( T \) and \( T^* \), we obtain the continuous K-frame operator, \( S : \mathcal{H} \to \mathcal{H} \) defined by

\[
Sf = \int_\Omega \langle f, F(\omega) \rangle_{\mathcal{A}} F(\omega) d\mu(\omega).
\]

It’s clear to see that \( S \) is positive, bounded and selfadjoint (see [5]).

For the following definition we need to introduce, \( GL^+ (\mathcal{H}) \) be the set of all positive bounded linear invertible operators on \( \mathcal{H} \) with bounded inverse.

**Definition 2.2.** Let \( \mathcal{H} \) be a Hilbert \( \mathcal{A} \)-module over a unital \( C^* \)-algebra and \( K \in \text{End}_\mathcal{A}^*(\mathcal{H}) \), \( C \in GL^+(\mathcal{H}) \). A mapping \( F : \Omega \to \mathcal{H} \) is called a continuous \( C \)-controlled K-Frame in \( \mathcal{H} \) if:

- \( F \) is weakly-measurable, ie, for any \( f \in \mathcal{H} \), the map \( w \to \langle f, F(w) \rangle_{\mathcal{A}} \) is measurable on \( \Omega \).

- There exists two strictly positive constants \( A \) and \( B \) such that

\[
A \langle C^{1/2} K^* f, C^{1/2} K^* f \rangle_{\mathcal{A}} \leq \int_\Omega \langle f, F(w) \rangle_{\mathcal{A}} (CF(w), f)_{\mathcal{A}} d\mu(w) \leq B \langle f, f \rangle_{\mathcal{A}}, f \in \mathcal{H}.
\]

The elements \( A \) and \( B \) are called continuous C-controlled K-frame bounds.

If \( A = B \) we call this continuous C-controlled K-Frame a continuous tight C-Controlled K-Frame, and if \( A = B = 1 \) it is called a continuous Parseval C-Controlled K-Frame. If only the right-hand inequality of (2.2) is satisfied, we call \( F \) a continuous C-controlled bessel mapping with Bessel bound \( B \).

**Example 2.3.**

\[
H = \mathcal{A} = l^2(\mathbb{C})
\]

\[
= \left\{ \{a_n\}_{n=1}^{\infty} \subset \mathbb{C} / \sum_{n=1}^{\infty} |a_n|^2 < +\infty \right\}.
\]

\( \mathcal{A} \) is recognized as a Hilbert \( \mathcal{A} \)-Module with the \( \mathcal{A} \)-inner product

\[
< \{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}>_{\mathcal{A}} = \{a_n \overline{b_n}\}_{n=1}^{\infty}.
\]

Consider now the borned linear operator

\[
C : \{a_n\}_{n=1}^{\infty} \to \{\alpha a_n\}_{n=1}^{\infty},
\]

where \( \alpha \in \mathbb{R}_+^* \). Then \( C \) is positive invertible and

\[
C^{-1}(\{a_n\}_{n=1}^{\infty}) = \{\alpha^{-1} a_n\}_{n=1}^{\infty}.
\]
Let $(\Omega, \mu)$ the measure space where $\Omega = [0, 1]$ and $\mu$ is the Lebesgue measure and let

$$F : \Omega \rightarrow H \quad w \mapsto F_w = \{\frac{w_n}{n}\}_{n=1}^\infty.$$ 

In the author hand, consider the projection

$$K : H \rightarrow H \quad \{a_n\}_{n=1}^\infty \mapsto (a_1, \ldots, a_r, 0, \ldots)$$

where $r$ is an integer ($r \geq 2$).

It's clear that $K^* = K$ and for each $f = \{a_n\}_{n=1}^\infty \in H = l^2(\mathbb{C})$, one has

$$\int_{\Omega} < f, F_w >_A < CF_w, f >_A d\mu(w) = \int_{[0,1]} \left\{ \frac{w^2}{n^2} |a_n|^2 \right\}_{n=1}^\infty . \left\{ \frac{w}{n} a_n \right\}_{n=1}^\infty d\mu(w)$$

$$= \alpha \left\{ \frac{|a_n|^2}{n^2} \right\}_{n=1}^\infty .$$

Hence

$$\int_{\Omega} < f, F_w >_A < CF_w, f >_A d\mu(w) \leq \frac{\alpha \pi^2}{18} < \{a_n\}_{n=1}^\infty , \{a_n\}_{n=1}^\infty >_A .$$

Furthermore,

$$< CK^* f, K^* f >_A = < (\alpha a_1, \ldots, \alpha a_r, 0, \ldots), (a_1, \ldots, a_r, 0, \ldots) >_A$$

$$= (\alpha |a_1|^2, \ldots, \alpha |a_r|^2, 0, \ldots).$$

Then for $A = \frac{1}{3r^2}$, one obtain

$$\frac{\alpha}{3r^2} (|a_1|^2, \ldots, |a_r|^2, 0, \ldots) \leq \left\{ \frac{\alpha |a_n|^2}{3 \frac{n^2}{n^2}} \right\}_{n=1}^\infty .$$

The conclusion is

$$\frac{1}{3r^2} < C^{1/2} K^* f, C^{1/2} K^* f >_A \leq \int_{\Omega} < f, F_w >_A < CF_w, f >_A d\mu(w) \leq \frac{\alpha \pi^2}{18} < f, f >_A$$

Let $F$ be a continuous C-controlled Bessel mapping for Hilbert $C^*$-module $\mathcal{H}$ over $\mathcal{A}$.

We define the operator frame

$$S_C : \mathcal{H} \rightarrow \mathcal{H} \quad \text{by},$$

$$S_C f = \int_{\Omega} < f, F(\omega) >_A CF(\omega) d\mu(\omega).$$
Remark 2.4. From definition of $S$ and $S_C$, we have, $S_C = CS$. Using [16], $S_C$ is $\mathcal{A}$-linear and bounded. Thus, it is adjointable. Since $\langle S_Cx, x \rangle_\mathcal{A} \geq 0$, for any $x \in \mathcal{H}$, it result, again from [16], that $S_C$ is positive and selfadjoint.

Theorem 2.5. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module, $K \in \text{End}^*_\mathcal{A}(\mathcal{H})$, and $C \in \text{GL}^+(\mathcal{H})$. Let $F : \Omega \rightarrow \mathcal{H}$ a map. Suppose that $C K = K C$, $R(C^{\frac{1}{2}}) \subset R(K^*C^{\frac{1}{2}})$ with $R(K^*C^{\frac{1}{2}})$ is closed. Then $F$ is a continuous $C$-controlled $K$-frame for $\mathcal{H}$ if and only if there exist two constants $0 < A, B < \infty$ such that :

$$A\|C^{\frac{1}{2}}K^*f\|^2 \leq \int_\Omega \langle f, F(w) \rangle_\mathcal{A} \langle CF(w), f \rangle_\mathcal{A} d\mu(w) \leq B\|f\|^2, f \in \mathcal{H}.$$ (2.3)

Proof. ($\Rightarrow$) obvious.

For the converse, we suppose that $0 < A, B < \infty$ such that :

$$A\|C^{\frac{1}{2}}K^*f\|^2 \leq \int_\Omega \langle f, F(w) \rangle_\mathcal{A} \langle CF(w), f \rangle_\mathcal{A} d\mu(w) \leq B\|f\|^2, f \in \mathcal{H}.$$ We have,

$$\| \int_\Omega \langle f, F(w) \rangle_\mathcal{A} \langle CF(w), f \rangle_\mathcal{A} d\mu(w) \| = \| \langle S_C f, f \rangle_\mathcal{A} \|
= \| \langle CS f, f \rangle_\mathcal{A} \|
= \| \langle (CS)^{\frac{1}{2}} f, (CS)^{\frac{1}{2}} f \rangle_\mathcal{A} \|
= \| (CS)^{\frac{1}{2}} f \|^2.$$ Since, $R(C^{\frac{1}{2}}) \subset R(K^*C^{\frac{1}{2}})$ with $R(K^*C^{\frac{1}{2}})$ is closed, then by theorem [1.4], there exists $0 \leq m$ such that,

$$(C^{\frac{1}{2}})(C^{\frac{1}{2}})^* \leq m(K^*C^{\frac{1}{2}})(K^*C^{\frac{1}{2}})^*.$$ Thus,

$$\langle (C^{\frac{1}{2}})(C^{\frac{1}{2}})^* f, f \rangle_\mathcal{A} \leq m \langle (K^*C^{\frac{1}{2}})(K^*C^{\frac{1}{2}})^* f, f \rangle_\mathcal{A}.$$ Consequently,

$$\|C^{\frac{1}{2}} f\|^2 \leq m\|K^*C^{\frac{1}{2}} f\|^2.$$ Then,

$$A\|C^{\frac{1}{2}} f\|^2 \leq A m\|K^*C^{\frac{1}{2}} f\|^2 \leq m\|(CS)^{\frac{1}{2}} f\|^2.$$ Hence,

$$\frac{A}{m}\|C^{\frac{1}{2}} f\|^2 \leq \|(CS)^{\frac{1}{2}} f\|^2.$$ So,

$$\sqrt{\frac{A}{m}}\|C^{\frac{1}{2}} f\| \leq \|(CS)^{\frac{1}{2}} f\|.$$ (2.4)

From lemma[1.2] we have,

$$\sqrt{\frac{A}{m}}\langle C^{\frac{1}{2}} f, C^{\frac{1}{2}} f \rangle_\mathcal{A} \leq \langle C^{\frac{1}{2}} S^{\frac{1}{2}} f, C^{\frac{1}{2}} S^{\frac{1}{2}} f \rangle_\mathcal{A}.$$
Then,
\[ \langle C^{\frac{1}{2}} f, C^{\frac{1}{2}} f \rangle_A \leq \sqrt{\frac{m}{A}} \langle CS f, f \rangle_A. \]
So,
\[ \langle C^{\frac{1}{2}} f, C^{\frac{1}{2}} f \rangle_A \leq \sqrt{\frac{m}{A}} \langle S_C f, f \rangle_A. \]
One the deduce
\[ \langle C^{\frac{1}{2}} K^* f, C^{\frac{1}{2}} K^* f \rangle_A \leq \| K^* \|^2 \langle C^{\frac{1}{2}} f, C^{\frac{1}{2}} f \rangle_A \leq \| K^* \|^2 \sqrt{\frac{m}{A}} \langle S_C f, f \rangle_A. \]
Hence,
\[ \frac{1}{\| K^* \|^2} \sqrt{\frac{A}{m}} \langle C^{\frac{1}{2}} K^* f, C^{\frac{1}{2}} K^* f \rangle_A \leq \langle S_C f, f \rangle_A. \]
Since \( S_C \) is positive, selfadjoint and bounded \( A \)-linear map, we can write
\[ \langle S_C^{\frac{1}{2}} f, S_C^{\frac{1}{2}} f \rangle_A = \langle S_C f, f \rangle_A = \int_\Omega \langle f, F(w) \rangle_A \langle CF(w), f \rangle_A d\mu(w). \]
From lemma 1.3 there exists \( D > 0 \) such that,
\[ \langle S_C^{\frac{1}{2}} f, S_C^{\frac{1}{2}} f \rangle_A \leq D \langle f, f \rangle_A, \]
then by (2.5) and (2.6), we conclude that \( F \) is a continuous \( C \)-controlled \( K \)-frame in Hilbert \( C^* \)-module \( H \) with frame bounds \( \frac{1}{\| K^* \|^2} \sqrt{\frac{A}{m}} \) and \( D \).

\[ \boxed{\text{Lemma 2.6. Let } C \in GL^+(H). \text{ Suppose } CS_C = S_C C \text{ and } R(S_C^{\frac{1}{2}}) \subset R((CS_C)^{\frac{1}{2}}) \text{ with } R((CS_C)^{\frac{1}{2}}) \text{ is closed. Then } \| S_C^{\frac{1}{2}} f \|^2 \leq \lambda \| (CS_C)^{\frac{1}{2}} f \|^2 \text{ for some } \lambda \geq 0.} \]

\[ \text{Proof. By theorem 1.4, there exists some } \lambda > 0 \text{ such that,} \]
\[ (S_C^{\frac{1}{2}})(S_C^{\frac{1}{2}})^* \leq \lambda (CS_C^{\frac{1}{2}})(CS_C^{\frac{1}{2}})^*. \]
Hence,
\[ \langle (S_C^{\frac{1}{2}})(S_C^{\frac{1}{2}})^* f, f \rangle_A \leq \lambda \langle (CS_C^{\frac{1}{2}})(CS_C^{\frac{1}{2}})^* f, f \rangle_A. \]
So,
\[ \| S_C^{\frac{1}{2}} f \|^2 \leq \lambda \| (CS_C^{\frac{1}{2}}) f \|^2, f \in H. \]

\[ \boxed{\text{Theorem 2.7. Let } F : \Omega \to H \text{ a map and } C \in GL^+(H). \text{ Suppose } CS_C = S_C C \text{ and } R(S_C^{\frac{1}{2}}) \subset R((CS_C)^{\frac{1}{2}}) \text{ with } R((CS_C)^{\frac{1}{2}}) \text{ is closed. Then } F \text{ is a continuous } C \text{-controlled Bessel mapping with bound } B \text{ if and only if } U : l^2(A) \to H \text{ defined by } U(\{a_w\}_{w \in \Omega}) = \int_\Omega a_w CF(w)d\mu(w) \text{ is well defined bounded with } \| U \| \leq \sqrt{B} \| C^{\frac{1}{2}} \|.} \]
Proof. Assume that $F$ is a continuous $C$-controlled Bessel with bound $B$. Hence, 
\[
\left\| \int_{\Omega} \langle f, F(w) \rangle_A (CF(w), f) \, d\mu(w) \right\| \leq B \| f \|^2, \quad f \in \mathcal{H}.
\]
So, 
\[
\left\| \langle S_C f, f \rangle_A \right\| \leq B \| f \|^2.
\]
In the beginning, we show that $U$ is well defined. For each $\{a_w\}_{\omega \in \Omega} \in l^2(\mathcal{A})$, 
\[
\|U(\{a_w\}_{\omega \in \Omega})\|^2 = \sup_{f \in \mathcal{H}, \|f\|=1} \| \langle U(\{a_w\}_{\omega \in \Omega}), f \rangle_A \|^2
\]
\[
= \sup_{f \in \mathcal{H}, \|f\|=1} \left\| \int_{\Omega} a_w CF(w) \, d\mu(w), f \right\|^2_A
\]
\[
= \sup_{f \in \mathcal{H}, \|f\|=1} \left\| \int_{\Omega} a_w (CF(w), f) \, d\mu(w) \right\|^2
\]
\[
\leq \sup_{f \in \mathcal{H}, \|f\|=1} \left\| \int_{\Omega} \langle f, CF(w) \rangle_A (CF(w), f) \, d\mu(w) \right\| \cdot \int_{\Omega} a_w a_w^* \, d\mu(w)
\]
\[
= \sup_{f \in \mathcal{H}, \|f\|=1} \left\| \langle CS_C f, f \rangle_A \right\| \cdot \int_{\Omega} a_w a_w^* \, d\mu(w)
\]
\[
= \sup_{f \in \mathcal{H}, \|f\|=1} \left\| (CS_C)^{\frac{1}{2}} f, (CS_C)^{\frac{1}{2}} f \right\|_A \cdot \| \{a_w\}_{\omega \in \Omega} \|^2
\]
\[
\leq \sup_{f \in \mathcal{H}, \|f\|=1} \left\| (C)^{\frac{1}{2}} \right\|^2 \| (S_C f)^{\frac{1}{2}} \|^2 \| \{a_w\}_{\omega \in \Omega} \|^2
\]
\[
\leq B \| (C)^{\frac{1}{2}} \|^2 \| \{a_w\}_{\omega \in \Omega} \|^2.
\]
Then, 
\[
\|U\| \leq \sqrt{B} \| (C)^{\frac{1}{2}} \|.
\]
Hence $U$ is well defined and bounded. Now, suppose that $U$ is well defined, and 
\[
\|U\| \leq \sqrt{B} \| (C)^{\frac{1}{2}} \|.
\]
For any $f \in \mathcal{H}$ and $\{a_w\}_{\omega \in \Omega} \in l^2(\mathcal{A})$, we have, 
\[
\langle f, U(\{a_w\}_{\omega \in \Omega}) \rangle_A = \langle f, \int_{\Omega} a_w CF(w) \, d\mu(w) \rangle_A
\]
\[
= \int_{\Omega} \langle a_w^* f, F(w) \rangle_A \, d\mu(w)
\]
\[
= \int_{\Omega} \langle C f, F(w) \rangle_A a_w^* \, d\mu(w)
\]
\[
= \langle \{C f, F(w)\}_A \rangle_{\omega \in \Omega}, \{a_w\}_{\omega \in \Omega} \rangle_A.
\]
Then, $U$ has an adjoint, and 
\[
U^* f = \{C f, F(w)\}_A \rangle_{\omega \in \Omega}.
\]
Also,
\[
\|U\|^2 = \sup_{\{(a_w)_{w \in \Omega}\} = 1} \|U(\{a_w\}_{w \in \Omega})\|^2 = \sup_{\{(a_w)_{w \in \Omega}\} = 1, \|f\| = 1} \|\langle U(\{a_w\}_{w \in \Omega}), f \rangle_A\|^2 = \sup_{\|(a_w)_{w \in \Omega}\| = 1, \|f\| = 1} \|\{a_w\}_{w \in \Omega}, U^*f\rangle_A\|^2 = \sup_{\|f\| = 1} \|U^*f\|^2 = \|U^*\|^2.
\]

So,
\[
\|U^*f\|^2 = \|\langle U^*f, U^*f \rangle_A\| = \|\langle UU^*f, f \rangle_A\| = \|\langle CS_Cf, f \rangle_A\|.
\]

Then,
\[
(2.7) \quad \|U^*f\|^2 = \|\langle CS_C \rangle \frac{1}{2}f\|^2 \leq B\|\langle C \rangle \frac{1}{2}f\|^2.
\]

From lemma 2.6 we have,
\[
\|\langle S_C \rangle \frac{1}{2}f\|^2 \leq \lambda\|\langle CS_C \rangle \frac{1}{2}f\|^2,
\]
for some \(\lambda > 0\).

Using (2.7) we get,
\[
\|\langle S_C \rangle \frac{1}{2}f\|^2 \leq \lambda\|\langle CS_C \rangle \frac{1}{2}f\|^2 \leq \lambda B\|\langle C \rangle \frac{1}{2}f\|^2.
\]

Hence \(F\) is a continuous C-controlled Bessel mapping with Bessel bound \(\lambda B\|\langle C \rangle \frac{1}{2}f\|^2\).

\[\Box\]

**Proposition 2.8.** Let \(F\) be a continuous C-controlled K-frame for \(\mathcal{H}\) with bounds \(A\) and \(B\). Then:
\[ACKK^* I \leq SC \leq BI.\]

**Proof.** Suppose \(F\) is a continuous C-controlled K-frame with bounds \(A\) and \(B\). Then,
\[
A\langle C^\frac{1}{2}K^*f, C^\frac{1}{2}K^*f \rangle_A \leq \int_{\Omega} \langle f, F(w) \rangle_A \langle CF(w), f \rangle_A d\mu(w) \leq B\langle f, f \rangle_A.
\]

Hence,
\[
A\langle CKK^*f, f \rangle_A \leq \langle SCf, f \rangle_A \leq B\langle f, f \rangle_A.
\]

So,
\[ACKK^* I \leq SC \leq BI.\]

\[\Box\]

**Proposition 2.9.** Let \(F\) be a continuous C-controlled Bessel mapping for \(\mathcal{H}\), and \(C \in GL^+(\mathcal{H})\). Then \(F\) is a continuous C-controlled K-frame for \(\mathcal{H}\) if and only if there exists \(A > 0\) such that:
\[ACKK^* \leq CS.\]
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Proof. \((\Rightarrow)\) obvious. 
\((\Leftarrow)\) Assume that there exists \(A > 0\) such that: \(ACKK^* \leq CS\), then,
\[A \langle CKK^* f, f \rangle_A \leq \langle S f, f \rangle_A .\]
Hence,
\[A \langle C^{1/2} K^* f, C^{1/2} K^* f \rangle_A \leq \langle S f, f \rangle_A .\]
Therefore,
\[A \langle C^{1/2} K^* f, C^{1/2} K^* f \rangle_A \leq \int_\Omega \langle f, F(w) \rangle_A \langle CF(w), f \rangle_A d\mu(w) .\]
Hence \(F\) is a continuous \(C\)-controlled \(K\)-frame.

**Proposition 2.10.** Let \(C \in GL^+(\mathcal{H})\), \(K \in End_A^*(\mathcal{H})\) and \(F\) be a continuous \(C\)-controlled \(K\)-frame for \(\mathcal{H}\) with lower and upper frames bounds \(A\) and \(B\) respectively. Suppose \(KC = CK\) and \(R(C^{1/2}) \subset R(K^*C^{1/2})\) with \(R(K^*C^{1/2})\) is closed. Then \(F\) is a continuous \(K\)-frame for \(\mathcal{H}\) with lower and upper frames bounds \(A\|C^{-1/2}\|^{-2} \|C\|_2^{-2}\|f\|^2\) and \(B\|C^{1/2}\|^2\) respectively.

Proof. Assume that \(F\) is a continuous \(C\)-controlled \(K\)-frame with lower and upper frames bounds \(A\) and \(B\). From theorem \([2.5]\) we have:
\[A \|C^{1/2} K^* f\|^2 \leq \| f \|^2, f \in \mathcal{H} .\]
Then,
\[A \|K^* f\|^2 = A \|C^{-1/2} C^{1/2} K^* f\|^2 \leq A \|C^{-1/2}\|^2 \|C^{1/2} K^* f\|^2 \leq \|C^{-1/2}\|^2 \int_\Omega \langle f, F(w) \rangle_A \langle CF(w), f \rangle_A d\mu(w) .\]
So,
\[(2.8)\]
\[A \|K^* f\|^2 \leq \|C^{1/2}\|^2 \|S f, f \|_A .\]
Moreover,
\[\langle S f, f \rangle_A = \langle CS f, f \rangle_A \]
\[= \langle (CS)^{1/2} f, (CS)^{1/2} f \rangle_A \]
\[= \| (CS)^{1/2} f \|^2 \leq \| (C)^{1/2} \|^2 \| (S)^{1/2} f \|^2 \]
\[= \| (C)^{1/2} \|^2 \langle (S)^{1/2} f, (S)^{1/2} f \rangle_A \]
\[= \| (C)^{1/2} \|^2 \langle S f, f \rangle_A ,\]
then,
\[(2.9)\]
\[\langle S f, f \rangle_A \leq \| (C)^{1/2} \|^2 \langle S f, f \rangle_A .\]
From (2.8) and (2.9), we have,
\[ A\|K^*f\|^2 \leq \|C^{-\frac{1}{2}}\|(C)^{\frac{1}{2}}\|Sf, f\|_A \]
\[ = \|C^{-\frac{1}{2}}\|(C)^{\frac{1}{2}}\|\int \langle f, F(w)\rangle_A\langle F(w), f\rangle_A d\mu(w). \]
Hence,
\[\|C^{-\frac{1}{2}}\|^{-2}\|(C)^{\frac{1}{2}}\|^{-2}A\|K^*f\|^2 \leq \int \langle f, F(w)\rangle_A\langle F(w), f\rangle_A d\mu(w).\]
Moreover,
\[ \| \int \langle f, F(w)\rangle_A\langle F(w), f\rangle_A d\mu(w) \| = \|\langle Sf, f\rangle_A \| \]
\[ = \|\langle (C^{-1}CS) f, f\rangle_A \| \]
\[ = \|\langle (C^{-1}CS)^{\frac{1}{2}} f, (C^{-1}CS)^{\frac{1}{2}} f\rangle_A \| \]
\[ = \|\langle (C^{-1}CS)^{\frac{1}{2}} f \|^{2} \]
\[ \leq \|C^{-\frac{1}{2}}\|^2\|(C^2)^{\frac{1}{2}} f \|^{2} \]
\[ = \|C^{-\frac{1}{2}}\|^2 \|(CS)^{\frac{1}{2}} f, (CS)^{\frac{1}{2}} f\rangle_A \]
\[ = \|C^{-\frac{1}{2}}\|^2 \|(CSf, f\rangle_A \]
\[ \leq \|C^{-\frac{1}{2}}\|^2 B \|f\|^2. \]

Then F is a continuous K-frame for H with lower and upper frames bounds
\[ A\|C^{-\frac{1}{2}}\|^{-2}\|(C)^{\frac{1}{2}}\|^{-2} \text{ and } B\|C^{-\frac{1}{2}}\|^2. \]

Proposition 2.11. Let \( C \in GL^+(H) \) and \( K \in End_A(H) \). We Suppose that \( KC = CK \), \( R(C^{\frac{1}{2}}) \subset R(K^*C^{\frac{1}{2}}) \) with \( R(K^*C^{\frac{1}{2}}) \) is closed and F is a continuous K-frame for H with lower and upper frames bounds A and B respectively.
Then F is continuous C-controlled K-frame for H with lower and upper frames bounds A and \( ||C|| S|| \).

Proof. Assume that F is a continuous K-frame for H with lower and upper frames bounds A and B. Then we have:
\[ A\langle K^*f, K^*f\rangle_A \leq \int \langle f, F(w)\rangle_A\langle F(w), f\rangle_A d\mu(w) \leq B\langle f, f\rangle_A, \]
Since \( \langle K^*f, K^*f\rangle_A > 0 \) and \( \langle f, f\rangle_A > 0 \) then,
\[ A\|K^*f\|^2 \leq \| \int \langle f, F(w)\rangle_A\langle F(w), f\rangle_A d\mu(w) \| \leq B\|f\|^2. \]
Then for every \( f \in H, \)
\[ A\|C^{\frac{1}{2}}K^*f\|^2 = A\|K^*C^{\frac{1}{2}} f\|^2 \]
\[ \leq \| \int \langle C^{\frac{1}{2}} f, F(w)\rangle_A\langle F(w), C^{\frac{1}{2}} f\rangle_A d\mu(w) \|. \]
\[ A\|C^\frac{1}{2}K^*f\|^2 \leq \|\langle SCf, f\rangle_A\| \leq \|S\|\|C\|\|f\|^2. \]

By (2.11) and theorem 2.5, we conclude that F is continuous C-controlled K-frame for \( \mathcal{H} \) with lower and upper frames bounds \( A \) and \( \|C\|\|S\| \).

**Theorem 2.12.** Let \( C \in GL^+(\mathcal{H}) \), and F be a continuous C-controlled K-frame for \( \mathcal{H} \) with bounds A and B. Let \( M, K \in \text{End}_A(\mathcal{H}) \) such that \( R(M) \subseteq R(K) \), \( R(K) \) is closed and C commutes with \( M^* \) and \( K^* \). Then F is continuous C-controlled M-frame for \( \mathcal{H} \).

**Proof.** Assume that F be a continuous C-controlled K-frame for \( \mathcal{H} \) with bounds A and B, then,

(2.12) \[ A\|C^\frac{1}{2}K^*f\|^2 \leq \|\langle SCf, f\rangle_A\| \leq \|S\|\|C\|\|f\|^2. \]

Since \( R(M) \subseteq R(K) \), by theorem 1.4, there exists some \( 0 \leq \lambda \) such that \( MM^* \leq \lambda KK^* \).

Hence,

\[ \langle MM^*C^\frac{1}{2}f, C^\frac{1}{2}f\rangle_A \leq \lambda \langle KK^*C^\frac{1}{2}f, C^\frac{1}{2}f\rangle_A, \]

then,

\[ \frac{A}{\lambda} \langle MM^*C^\frac{1}{2}f, C^\frac{1}{2}f\rangle_A \leq A\langle KK^*C^\frac{1}{2}f, C^\frac{1}{2}f\rangle_A. \]

By (2.12), we have,

\[ \frac{A}{\lambda} \langle M^*C^\frac{1}{2}f, M^*C^\frac{1}{2}f\rangle_A \leq \int_{\Omega} \langle f, F(w)\rangle_A \langle CF(w), f\rangle_A d\mu(w) \leq B\langle f, f\rangle_A. \]

Then F is continuous C-controlled M-frame for \( \mathcal{H} \) with bounds \( \frac{A}{\lambda} \) and B. \( \square \)

The following results gives the invariance of a continuous C-controlled Bessel mapping by a adjointable operator.

**Proposition 2.13.** Let \( T \in \text{End}_A(\mathcal{H}) \) such that \( TC = CT \) and F be a continuous C-controlled Bessel mapping with bound D. Then TF is also a continuous C-controlled Bessel mapping with bound \( D\|T^*\| \).
Proof. Assume that $F$ is a continuous $C$-controlled Bessel mapping with bound $D$. Hence we have,
\[ \int_{\Omega} \langle f, F(w) \rangle_A \langle CF(w), f \rangle_A d\mu(w) \leq D \langle f, f \rangle_A, f \in H. \]
We have,
\[ \int_{\Omega} \langle f, TF(w) \rangle_A \langle CTF(w), f \rangle_A d\mu(w) = \int_{\Omega} \langle T^* f, F(w) \rangle_A \langle TCF(w), f \rangle_A d\mu(w) \]
\[ = \int_{\Omega} \langle T^* f, F(w) \rangle_A \langle CF(w), T^* f \rangle_A d\mu(w) \]
\[ \leq D \langle T^* f, T^* f \rangle_A \]
\[ \leq D \|T^*\|^2 \langle f, f \rangle_A. \]

The result holds. \hfill \square

Now, we study the invariance of a continuous $C$-controlled K-frame mapping by adjointable operator.

**Theorem 2.14.** Let $C \in GL^+(H)$, and $F$ be a continuous $C$-controlled K-frame for $H$ with bounds $A$ and $B$. If $T \in \text{End}^*_A(H)$ with closed range such that $R(K^* T^*)$ is closed and $C, K, T$ commute with each other. Then $TF$ is a continuous $C$-controlled K-frame for $R(T)$.

**Proof.** Assume that $F$ is a continuous $C$-controlled K-frame with bounds $A$ and $B$. Then,
\[ A \langle C^\frac{1}{2} K^* f, C^\frac{1}{2} K^* f \rangle_A \leq \int_{\Omega} \langle f, F(w) \rangle_A \langle CF(w), f \rangle_A \leq B \langle f, f \rangle_A, f \in H. \]
Since $T$ has a closed range, then $T$ has Moore-Penrose inverse $T^\dagger$ such that $TT^\dagger T = T$ and $T^\dagger TT^\dagger = T^\dagger$, so $TT^\dagger_{/R(T)} = I_{R(T)}$ and $(TT^\dagger)^* = I^* = I = TT^\dagger$.

We have,
\[ \langle K^* C^\frac{1}{2} f, K^* C^\frac{1}{2} f \rangle_A = \langle (TT^\dagger)^* K^* C^\frac{1}{2} f, (TT^\dagger)^* K^* C^\frac{1}{2} f \rangle_A \]
\[ = \langle (T^\dagger)^* T^* K^* C^\frac{1}{2} f, (T^\dagger)^* T^* K^* C^\frac{1}{2} f \rangle_A. \]

So,
\[ \langle K^* C^\frac{1}{2} f, K^* C^\frac{1}{2} f \rangle_A \leq \| (T^\dagger)^* \|^2 \langle T^* K^* C^\frac{1}{2} f, T^* K^* C^\frac{1}{2} f \rangle_A. \]

Therefore,
\[ \| (T^\dagger)^* \|^{-2} \langle K^* C^\frac{1}{2} f, K^* C^\frac{1}{2} f \rangle_A \leq \langle T^* K^* C^\frac{1}{2} f, T^* K^* C^\frac{1}{2} f \rangle_A. \]

Consequently, from theorem [1.4] and $R(T^* K^*) \subset R(K^* T^*)$, there exists some $\lambda \geq 0$ such that,
\[ \langle T^* K^* C^\frac{1}{2} f, T^* K^* C^\frac{1}{2} f \rangle_A \leq \lambda \langle K^* T^* C^\frac{1}{2} f, K^* T^* C^\frac{1}{2} f \rangle_A. \]
Hence, using (2.14) and (2.15) we have,
\[ \int_{\Omega} \langle f, TF(w) \rangle_\mathcal{A} \langle CTF(w), f \rangle_\mathcal{A} d\mu(w) = \int_{\Omega} \langle T^* f, F(w) \rangle_\mathcal{A} \langle CTF(w), f \rangle_\mathcal{A} d\mu(w) \]
\[ = \int_{\Omega} \langle T^* f, F(w) \rangle_\mathcal{A} \langle CF(w), T^* f \rangle_\mathcal{A} d\mu(w) \]
\[ \geq A \langle C^{\frac{1}{2}} K^* T^* f, C^{\frac{1}{2}} K^* T^* f \rangle_\mathcal{A} \]
\[ \geq \frac{A}{\lambda} \langle T^* C^{\frac{1}{2}} K^* f, T^* C^{\frac{1}{2}} K^* f \rangle_\mathcal{A}, \]
then,
\[ (2.16) \int_{\Omega} \langle f, TF(w) \rangle_\mathcal{A} \langle CTF(w), f \rangle_\mathcal{A} d\mu(w) \geq \frac{A}{\lambda} \| (T^*)^* \|^{-2} \langle C^{\frac{1}{2}} K^* f, C^{\frac{1}{2}} K^* f \rangle_\mathcal{A} \]

Using (2.16) and proposition 2.13, the result holds.

\[ \square \]

**Theorem 2.15.** Let \( C \in GL^1(\mathcal{H}) \) and \( F \) be a continuous \( C \)-controlled \( K \)-frame for \( \mathcal{H} \) with bounds \( A \) and \( B \).
If \( T \in \text{End}^*_A(\mathcal{H}) \) is an isometry such that \( R(T^* K^*) \subset R(K^* T^*) \) with \( R(K^* T^*) \) is closed and \( C, K, T \) commute with each other, then \( TF \) is a continuous \( C \)-controlled \( K \)-frame for \( \mathcal{H} \).

**Proof.** Using theorem 1.4, there exists some \( \lambda \geq 0 \) such that,
\[ \| T^* K^* C^{\frac{1}{2}} f \|^2 \leq \lambda \| K^* T^* C^{\frac{1}{2}} f \|^2. \]
Assume \( A \) the lower bound for the continuous \( C \)-controlled \( K \)-frame \( F \) and \( T \) is an isometry then,
\[ \frac{A}{\lambda} \| C^{\frac{1}{2}} K^* f \|^2 = \frac{A}{\lambda} \| T^* C^{\frac{1}{2}} K^* f \|^2 \]
\[ \leq A \| K^* T^* C^{\frac{1}{2}} f \|^2 \]
\[ = A \| C^{\frac{1}{2}} K^* T^* f \|^2 \]
\[ \leq \int_{\Omega} \langle T^* f, F(w) \rangle_\mathcal{A} \langle CF(w), T^* f \rangle_\mathcal{A} d\mu(w) \]
\[ = \int_{\Omega} \langle f, TF(w) \rangle_\mathcal{A} \langle TCF(w), f \rangle_\mathcal{A} d\mu(w), \]
then,
\[ (2.17) \frac{A}{\lambda} \| C^{\frac{1}{2}} K^* f \|^2 \leq \int_{\Omega} \langle f, TF(w) \rangle_\mathcal{A} \langle CTF(w), f \rangle_\mathcal{A} d\mu(w). \]

Hence, from proposition 2.13 and inequality (2.17), we conclude that \( TF \) is a continuous \( C \)-controlled \( K \)-frame for \( \mathcal{H} \) with bounds \( \frac{A}{\lambda} \) and \( B \| T^* \|^2 \).

\[ \square \]
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