Controlled Quantum State Transfer in a Spin Chain

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I. INTRODUCTION

Great effort is being devoted to studies of spin chains as promising “quantum wires” for quantum information transfer. With spin chains, a quantum state can be transferred without requiring an interface between the communication channel and a quantum computer [1], i.e., quantum information can be transferred and processed with the same hardware. Spin chains also allow for quantum computing with an always-on interaction [2, 3], even in the presence of a global control field [3]. The latest experimental progress on fabrication and characterization of atomic spin chains was reported in Ref. [4]. Spin chain Hamiltonians may be also realized by atomic gas in an optical lattice.

Perfect state transfer in spin chains might occur under special circumstances [5, 6, 7]. However, in the general case, dispersion effects often degrade the transmission fidelity and improving the fidelity becomes a central issue. Notably, it has been proven that the transmission fidelity can be significantly improved if the receiver stores the received signal in a large quantum memory before decoding [8]. Another general approach to high-fidelity quantum state transfer advocates the use of quantum wavepackets to encode the quantum state of a qubit [9, 10]. This approach is important because dispersion of wavepackets can be insignificant. In particular, Osborne and Linden [9] have shown that high transmission fidelity can be achieved by exploiting, if attainable, a Gaussian wavepacket whose shape is well preserved. The slow dispersion of a wavepacket can be further suppressed by applying a static parabolic (hence global) magnetic field [11].

In the context of the wavepacket approach to quantum state transfer, we focus below on two questions: (1) how can one create spin wavepackets with certain desired features, and (2) how can one control the motion of a quantum wavepacket in a spin chain so that the packet can be stopped at an arbitrary time, held, and then restarted later, without loss of quantum information. Such type of controlled quantum state transfer, if possible, should be a highly valuable tool in a variety of situations, e.g., cases in which the information receivers need additional waiting time to repair a quantum memory, or to prepare for a time window of high transmission fidelity. The importance of stoppable quantum state transfer may be also appreciated by noting the analogy to the potential impact of the stopping of light [12] in quantum information science. Further, a working scenario for the stopping and perfect relaunching of quantum state evolution of a spin chain should be also of considerable interest in the context of perfect quantum state reconstruction and perfect quantum state storage in systems of interacting qubits [13].

In this paper we first show that by optimizing a particular transport property using quantum superposition states comprising only a few spins (e.g., four or five), wavepacket pairs with some highly desired features emerge automatically from the ensuing dynamics. We then demonstrate that by applying a sequence of pulsed parabolic magnetic fields one can manipulate these wavepackets, stopping them and later relaunching the travelling wavepackets without individually addressing the spins. As shown below, the stopping, followed by relaunching, can in principle perfectly preserve the quantum information being transferred. This is made possible by taking advantage of powerful relationships between controlling spin dynamics and controlling quantum diffusion dynamics in a paradigm of quantum chaos.

This paper is organized as follows. In Sec. II we introduce a mapping between a Heisenberg spin chain kicked by a parabolic magnetic field and a paradigm in quantum chaos [14, 15]. In Sec. III we propose a conceptually simple approach to the creation of spin wavepacket pairs moving along the spin chain with slow dispersion and other desired features. The key result of this work is in Sec. IV, where stopping and relaunching spin wavepackets are studied both numerically and analytically. Section V concludes this paper.
II. HEISENBERG SPIN CHAIN IN A PULSED MAGNETIC FIELD AND THE DETLA-KICKED ROTOR

Consider then an open-ended Heisenberg chain of N spins in a constant magnetic field B and subject to a parabolic δ-pulsed magnetic field. The Hamiltonian is given by

\[ H = -\frac{J}{2} \sum_{n=1}^{N-1} \sigma_n \cdot \sigma_{n+1} - B \sum_{n=1}^{N} \sigma_n^z \]
\[ + \sum_{j=1}^{N} \delta(t - jT_0) \sum_{n=1}^{N} \sigma_n^z C_j \frac{(n - n_0)^2}{2}, \]

(1)

where \( \sigma \equiv (\sigma^x, \sigma^y, \sigma^z) \) are the Pauli matrices, \( J \) is the nearest-neighbor spin-spin interaction constant, \( C_j \) and \( n_0 \) are the coefficient and minimum location of the parabolic kicking field, and \( T_0 \) is the kicking period. Below, we denote the \( n = 1 \) (\( n = N \)) spin as the left (right) end of the chain. The constant field \( B \) lifts the system degeneracy and the dynamics is restricted to a subspace with fixed total polarization \( S_z \) defined as

\[ S_z = \sum_{n=1}^{N} \sigma_n^z. \]

(2)

Throughout this work we consider only the subspace of \( S_z = 1 \).

Let \( |m \rangle \) be one of the basis states, with the \( m \)th spin up and all other spins down. The propagator for the time period \([jT_0 - 0^+; (j + 1)T_0 - 0^+]\) is \( \hat{V}(2jT_0)\hat{U}(C_j) \). Here \( \hat{U}(C_j) \) represents the action due to the delta pulse, with

\[ \langle m|\hat{V}(2jT_0)|n \rangle = \exp[-i(C_j/2)(n - n_0)^2]\delta_{mn}. \]

(3)

The term \( \hat{V} \) stems from the evolution inherent in the Heisenberg interaction. An important recent study \cite{14} has shown that, in the \( N \to +\infty \) limit (and apart from some irrelevant phase)

\[ \langle m|\hat{V}(2jT_0)|n \rangle \approx i^{(m-n)} J_{(m-n)}(2jT_0), \]

(4)

where \( J_{(m-n)} \) is an ordinary Bessel function. The analytical behavior of \( \hat{U}(C_j) \) and \( \hat{V}(2jT_0) \) is therefore completely in parallel with that associated with the propagator of the δ-kicked rotor (DKR) (the best known model in quantum chaos \cite{16}) with Hamiltonian

\[ H_{DKR} = (\hat{P} - P_0)^2/2 - K \cos(\theta) \sum_{j} \delta(t - j). \]

(5)

Indeed, in the representation of the basis states \( |m \rangle \equiv \cos(m\theta)/\sqrt{\pi} \) and for an effective Planck constant \( \hbar \), the DKR propagator takes the familiar form \( \hat{v}(k)\hat{u}(\hbar) \), with

\[ \langle m|\hat{v}(k)|n \rangle = \exp[-i(\hbar/2)(n - \tilde{n}_0)^2]\delta_{mn}, \]

(6)

and

\[ \langle m|\hat{v}(k)|n \rangle \approx i^{(m-n)} J_{(m-n)}(k) \]

(7)

with \( k = K/\hbar \). Comparing these two systems, it is clear that upon the mapping

\[ |m \rangle \leftrightarrow |m \rangle, \]
\[ 2jT_0 \leftrightarrow k, \]
\[ n_0 \leftrightarrow P_0/\hbar, \]
\[ C_j \leftrightarrow \hbar, \]

(8-11)

the many-body spin chain dynamics is mapped to that of DKR \cite{14, 15, 17}, \( i.e. \), the motion of a spin wavepacket along the spin chain is mapped to DKR quantum diffusion dynamics in its \( m \)-space. Hence we can, whenever possible, shed light on the former by considering aspects of the latter, \( e.g. \), quantum resonance, Kolmogorov-Arnold-Moser (KAM) curves in phase space, etc. More significantly for this work, as shown below, it allows us to use tools from the control of quantum DKR dynamics \cite{18, 19, 20} to manipulate states in the spin chain. Further, this mapping between spin-chain and DKR allows us to go beyond the parameter regime confined by the true DKR (discussed below).

III. GENERATION OF SPIN WAVEPACKETS

In the context of the wavepacket approach to quantum state transfer, we now consider the first issue on spin wavepacket generation \cite{21}. Given a small number of basis states that could be used for encoding the state of a qubit, what initial superposition states should be exploited to induce the creation of quantum wavepackets with slow dispersion? Here this interesting question is considered in the absence of an external field, where the system propagator is given by \( \hat{V}(2jT_0) \). Remarkably, the associated DKR analogy now becomes a case of quantum resonance with \( \hbar = 4\pi \), with a propagator analogously given by

\[ \hat{v}(k = 2jT_0) = \exp[i(2jT_0)\cos(\theta)]. \]

(12)

Using this connection, the issue becomes to find initial superposition states within a given small subspace, such that the evolving quantum state remains well localized. At first glance this “localization” requirement seems too demanding because the main feature of quantum resonance dynamics is ballistic diffusion in the DKR \( m \)-space. However, as we have discovered, this can still be obtained by maximizing a diffusion rate of DKR. Qualitatively speaking, for superposition states maximizing a quadratic diffusion rate introduced below, the ensuing dynamics will push outwards as much as possible the excitation profile in the \( m \)-space, thus generating two well-separated wavepackets with almost zero amplitude in between.
FIG. 1: Emergence of a wavepacket pair in a Heisenberg chain of 201 spins, shown with the projection probability of the many-body wavefunction onto basis states $|m\rangle$. The initial condition is a superposition state $|\psi_{m_0}\rangle$ given by Eq. (15), for $m_0 = 101$ in (a) and $m_0 = 30$ in (b). The system wavefunction then evolves, with its shape given by the solid lines at time $t_1$ with $2Jt_1 = 15$, and by the dashed lines after an additional period $t_2$, with $2Jt_2 = 30$ in (a) and $2Jt_2 = 45$ in (b). The arrows show the travel direction of the wavepackets.

Quantitatively, let us first define the diffusion rate operator as

$$\hat{D} = \lim_{t\to\infty} \frac{\hat{E}(t) - \hat{E}(0)}{t^2},$$

where $\hat{E}(t)$ is the energy operator for the free rotor in the Heisenberg representation. For the quantum resonance case considered here one obtains

$$\hat{D} = A \sin^2(\theta),$$

where $A$ is a constant. Note that $\hat{D}$ only couples states $|m\rangle$ of the same parity. Consider now a sample case where a superposition state

$$|\psi_{m_0}\rangle = \sum_{n=-2}^{n=2} \beta_{2n} |m_0 + 2n\rangle$$

is exploited to encode a qubit state, i.e., only five basis states are used here. The state with the largest diffusion rate, denoted $D$, is simply given by the eigenfunction of $\hat{D}$ in the subspace of $|m_0 + 2n\rangle$ ($-2 \leq n \leq 2$) with the largest eigenvalue. In particular, if these basis states do not involve state $|0\rangle$, then the maximized $D$ is attained if $\beta_0 = 0.577$, $\beta_{-2} = \beta_{+2} = 0.5$, $\beta_{-4} = \beta_{+4} = 0.289$.

The significance of such an initial superposition state with maximized $D$ is demonstrated in Fig. 1(a), with $m_0 = 101$ for a 201-spin chain. In particular, a well-separated wavepacket pair is seen to quickly emerge, and its dispersion after its emergence is impressively slow in the absence of any external static fields. Note that, unlike the accelerator mode approach proposed in Ref. [14], the wavepacket pair is created by the system dynamics itself. Note also that the excitation amplitudes between the two wavepackets are surprisingly small. Because the total polarization here is fixed, a well-separated wavepacket pair directly indicate quantum entanglement between well-separated parts of the spin chain, and their travel in opposite directions distributes information or entanglement to both ends of the spin chain. Further results (not shown) indicate that in the case of minimized $D$ or an arbitrarily chosen initial state, the system dynamics generically creates a quickly delocalizing state (note also that even in chaotic cases different initial superposition states may also lead to dramatic differences in the ensuing quantum diffusion dynamics [22]). These further demonstrate the important role of an initial superposition state with a maximized diffusion rate in encoding the quantum information. Certainly, if more basis states are allowed in encoding the quantum information, then wavepacket pairs with even slower dispersion can be created with the same approach.

It is also desirable to be able to create a well-separated wavepacket pair that transfers information to a common end of the spin chain. For example, if two wavepackets with identical shape can be created, then one of them may be analogous to a “backup” copy as the other is being transferred and received first. Note that this possibility is not in violation of the quantum no-cloning theorem, because here the two wavepackets do not independently describe the quantum state of the involved spins. Rather, the two wavepackets describe the entanglement between two particular sections of the spin chain.

The creation of such a wavepacket pair is achieved here by going beyond the kicked rotor perspective and exploiting the boundary effect associated with the spin chain. That is, we apply the above scenario, but with the initial encoding state $|\psi_{m_0}\rangle$ located at $m_0 < N/2$, and with the requirement that no information receiver presents at the left ($n = 1$) end. To be more specific, consider a sample result shown in Fig. 1(b), with $m_0 = 30$. The wavepacket creation dynamics in the early stage is seen to be analogous to the case of Fig. 1(a). Sometime later, the generated wavepacket moving to the left hits the boundary and gets reflected. As demonstrated in Fig. 1(b), this then creates a pair of wavepackets where both of the members of the pair are moving to the right, with their shape indistinguishable from one another, with almost zero excitation in-between, and a peak-to-peak distance of $2(m_0 - 1)$ spins.

IV. STOPPING AND RELANCHEING WAVEPACKET PROPAGATION

In this section, taking the wavepackets obtained in the previous section as examples, we shall consider an independent issue, namely, stopping and relaunching the quantum state transfer along a spin chain. A parabolic magnetic field [see Eq. (1)] is proposed as the control field with a simple global feature, and we aim to achieve our control objective with an always-on Heisenberg interaction.

Thanks to the DKR analogy discussed above, the prob-
classical control mechanism, does not offer a satisfactory means of stopping the wavepacket. To improve the control one must compensate for the quantum phases that are accumulated during the stopping process. This phase accumulation is due to the spin-spin interaction as well as the kicking field.

Consider then an important observation made in our previous work on the quantum control of DKR dynamics \cite{18, 20}, i.e.,

$$
\langle m | \hat{v}(k) | n \rangle = \frac{1}{\pi} \int_0^{2\pi} \cos(m\theta) \exp[ik\cos(\theta)] \cos(n\theta) d\theta
$$

The first line in Eq. (16) holds by definition, the second line becomes obvious if the integration variable $\theta$ is changed to $\theta + \pi$, and the last line is obtained by use of the definition of $\hat{u}(\hat{h})$. Equation (16) hence proves

$$
\hat{v}(k) = \hat{u}(2\pi)\hat{v}(-k)\hat{u}(2\pi).
$$

Returning to a finite spin chain system, this result indicates that

$$
\hat{V}(2JT_0) \approx \hat{U}(2\pi)\hat{V}(-2JT_0)\hat{U}(2\pi).
$$

That is, the sign of the intrinsic interaction constant $J$ can be effectively reversed if we apply two parabolic $\delta$-kicks of particular strength. As such, it becomes possible to compensate for the quantum phase evolution inherent in the spin chain. As to the quantum phases induced by the kicking field, they can also be compensated for by considering kicking fields with the sign of $C_j$ reversed.

Given these considerations we present in Table I an explicitly designed pulse sequence for the stopping of arbitrary wavepackets for a period of $2M$ kicks. $C$ is a constant discussed in the text. Note that some system parameters used here are beyond what is allowed in a true kicked rotor system.

| $j$ | $M$ | $M + 1$ | $(M + 1, 2M)$ | $2M + 1$ |
|-----|-----|---------|--------------|----------|
| $C_j$ | $C/2 + 2\pi$ | $C$ | $2\pi$ | $-C$ | $-C/2$ |

The $j$-dependence of $C_j$ [see Eq. (1)] in an explicitly designed pulse sequence for the stopping of arbitrary wavepackets for a period of $2M$ kicks. $C$ is a constant discussed in the text. Note that some system parameters used here are beyond what is allowed in a true kicked rotor system.

Figure 2 displays the fate of the moving wavepacket pair shown in Fig. 1(a) (solid lines) after a kicking parabolic field is introduced. As is clearly seen in Fig. 2(a) and Fig. 2(b), the transfer of the wavepacket pair to both ends of the spin chain is stopped. This dynamical effect can also be understood as a type of quantum Zeno effect achieved by frequently applying (but far from infinitely fast) external pulses to a system.

However, although the wavepacket pair in Figs 2(a) and 2(b) has stopped moving, its internal structure is seen to be changing in a subtle manner. This indicates that evolution of the quantum phases characterizing the stopped wavepackets is still not frozen. This fact turns out to be disastrous when the kicking field is turned off in order to re-launch the state transfer. For example, Fig. 2(c) displays the wavefunction after the kicking field has been off for a period of $t_2$ with $2JT_2 = 30$: the background fluctuation is greatly increased, and the main peaks of the wavepacket pair do not move further.

Hence, using KAM curves alone, which is a purely theoretical question of how to stop and successfully restart the transport process in the DKR $m$-space. We do so below by exploiting control features of the DKR system. Two features are relevant, one classical and one quantal.

The classical basis of our control scenario arises by exploiting the phase space KAM curves of the underlying classical dynamics. That is, if the kicking magnetic field is sufficiently frequent that the chaoticity parameter $2JT_0C_j$ in the classical DKR is sufficiently small, the underlying classical dynamics will be mainly integrable and the associated KAM curves will present strong barriers to the quantum transport in the $m$-space. Because KAM curves will be almost everywhere, these classical structures can effectively stop the travel of arbitrary and unknown quantum wavepackets.

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The control scenario proposed in this work can also lead to other very interesting approaches to the manipulation of quantum entanglement dynamics of a spin chain. Here we briefly discuss three possibilities. First, by modifying the kicking field profile we can choose to stop only one component of a wavepacket pair, e.g., of the pair shown in Fig. 1(b) with dashed lines, thereby offering an interesting method of tuning the time delay between two wavepackets moving in the same direction. This then offers a means of controlling the distance between two entangled parts of the spin chain. Second, because the sign of the intrinsic spin-spin interaction constant $J$ can be effectively reversed if we apply $\delta$-kicks of particular strength, it can be easily shown that one can bounce back an arbitrary and unknown moving wavepacket to the sender at a time of our choosing. Third, by controlling the time delay between the two wavepackets and/or taking advantage of the feasibility of time-reversal, we may also recombine two localized wavepackets at a location different from that of the initial state. This recombination dynamics resembles that of a double-slit experiment, thereby generating interesting interference patterns along the spin chain. Such kind of interference patterns of spin excitations, and their fate under a variety of circumstances, may work as a novel interferometer for fundamental studies in quantum physics.

**V. CONCLUSION**

To conclude, based on a mapping between a kicked spin chain and the delta-kicked rotor system \cite{14}, we have shown that previous quantum control results in the delta-kicked rotor system \cite{13, 19, 20, 22} can be applied to the control of spin wavepacket propagation and hence the control of propagating quantum information encoded in wavepackets. Specifically, we have proposed a simple approach to wavepacket creation in a Heisenberg spin chain and demonstrated the possibility of stopping and relaunching information transfer without individually addressing spins or turning off spin-spin interactions. Several interesting applications of this work in manipulating the dynamics of a spin chain are also discussed. The results indicate that many insights from the quantum chaos research can be very useful for quantum information science. This work also adds more support to the use of spin chains as quantum wires, and might be useful in designing new quantum computation algorithms with an always-on qubit-qubit interaction \cite{2, 3}. Extensions to other types of spin chains are under consideration.

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