Quantum Analysis of Jackiw and Teitelboim’s Model for 1+1 D Gravity and Topological Gauge Theory

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ABSTRACT

We study the BRST quantization of the 1+1 dimensional gravity model proposed by Jackiw and Teitelboim and also the topological gauge model which is equivalent to the gravity model at least classically. The gravity model quantized in the light-cone gauge is found to be a free theory with a nilpotent BRST charge. We show also that there exist twisted N=2 superconformal algebras in the Jackiw-Teitelboim’s model as well as in the topological gauge model. We discuss the quantum equivalence between the gravity theory and the topological gauge theory. It is shown that these theories are indeed equivalent to each other in the light-cone gauge.

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1. Introduction

It has been known for some time now that perturbative quantization fails in the Einstein theory for gravity due to non-renormalizability of the theory. On the other hand the non-perturbative canonical quantization is also quite difficult, because the constraint equations, i.e. Wheeler-DeWitt equations, are non-polynomial and highly complicated. Naive consideration leads us to these problems even for 2+1 dimensional pure gravity. However this is somehow surprising, since there is no local dynamical freedom in the 2+1 dimensional pure gravity.

E. Witten addressed such a contradiction and showed the 2+1 dimensional pure gravity in the Palatini formalism may be formulated as the so-called Chern-Simons topological gauge theory with a non-compact gauge group. Namely the action is given by

$$S = \frac{1}{2} \int d^3x \epsilon^{\mu \nu \rho} Tr \left( A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right),$$

where the components of the gauge fields $A_\mu$ should be identified with driein $e_\mu$ and spin-connections $\omega_\mu$. Then this theory turns out to be renormalizable as far as we perform the perturbation around $e_\mu = \omega_\mu = 0$, which is, however, a singular configuration as gravity. We may understand that such differences in the quantum behavior are caused by the different phase structures; namely the "unbroken phase" and the "broken phase", as Witten pointed out in the ref.[1]. Also in the canonical quantization, the constraint equations due to the gauge invariance are found to be polynomial ones, which are actually shown to be soluble, in sharp contrast to the Wheeler-DeWitt equations. Similar dramatic simplification in the canonical quantization has been discovered also for the 3+1 dimensional general relativity.

Once we admit that the gauge theory of the Chern-Simons type describes the 2+1 dimensional gravity, then the gravity seems to be very tractable and even soluble. However one may note slight differences in the ranges of the dynamical variables between those theories. In the Einstein general relativity, the inverse of
the metric variable \( g_{\mu \nu} \) appears, hence \( \det(g_{\mu \nu}) \equiv g \) cannot vanishes. Also the volume element \( \sqrt{-g} \) is demanded to be positive definite from the geometrical point of view. On the other hand the range of the gauge field is unrestricted. Besides the "volume" element which is given by \( \det(e^a_\mu) \equiv e \) in the topological gauge theory is not positive definite or the orientation of the space-time can be changed locally. Strictly speaking the positive definite volume element should be given by \( |e| = \sqrt{-g} \) instead of \( e \), where we used the relation \( g_{\mu \nu} = e^a_\mu e^b_\nu \eta_{ab} \).

1) If we naively change the element \( e \) to \( |e| \), then the topological features would disappear. Such differences in the ranges of the dynamical variables seem to become more important in quantum theories. It is because different quantum fluctuations are expected to give rise to different quantum theories generally, even if the classical trajectories are same. However this problem seems to be very subtle in the case of the 2+1 dimensional gravity, since there exists no physical quantum fluctuation. We may observe similar phenomena to appear in the Nambu-Goto string propagating in two dimensional space-time.\textsuperscript{[3]}

Here it may be natural to ask ourselves whether the Chern-Simons gauge theory is really equivalent to the 2+1 dimensional gravity in the quantum sense. One may also wonder if the 2+1 dimensional gravity still has any topological features like the Chern-Simons gauge theory. In order to explore such questions, however, it seems to be necessary to find out first the consistent quantum theory for 2+1 dimensional gravity. But this seems to be rather difficult to carry out. Therefore in this paper we would like to study 1+1 dimensional analogous models.

As is well known the Einstein action for pure gravity in two dimensional space-time is trivial, since it gives just the Euler number of the two dimensional surface. Polyakov proposed to define the gravitational theories in two dimensions as induced gravity by introducing matter fields.\textsuperscript{[4]} A lot of studies have been done on these models in various approaches. In the continuum approaches the light-cone gauge was found to be so powerful as to make it possible to solve the models exactly.\textsuperscript{[5,6]}

1) Our notation of the Minkowski metric is \( \eta_{\mu \nu} = \eta_{ab} = \text{diag}(+1, -1, -1) \).
About a year later David, Distler and Kawai\cite{7} have succeeded in the quantization in the conformal gauge, which is familiar in string theories. These discoveries have been generating much progress in the continuum two dimensional quantum gravity theories.

Here, however, we are going to consider another model proposed by Jackiw and Teitelboim\cite{8,9}

\[
S = \frac{1}{2} \int d^2 x \sqrt{-g} (R + 2\Lambda) \phi
\]  \hspace{1cm} (1.2)

instead of the induced gravity theories. Because it has been found that this action is actually equivalent (at least classically) to the so-called \(SO(2,1)\) topological gauge theory;\cite{10,11}

\[
S = \int d^2 \epsilon^{\mu\nu} Tr(\Phi F_{\mu\nu})
\]  \hspace{1cm} (1.3)

as will be shown in detail later. We may see that in fact the actions (1.2) and (1.3) are obtained by dimensional reduction from the Einstein action and the Chern-Simons action (1.1) in 2+1 dimensions. The questions that we are interested in here are the following. 1) Are the gravitational models defined by the actions (1.2) and (1.3) truly equivalent to each other in the quantum mechanical sense? 2) Are there any topological features appearing in the model (1.2), though the gauge theory (1.3) is indeed a so-called topological field theory? The answers to these questions may be useful to understand the relationship between the two kinds of formulations for the 2+1 dimensional quantum gravity.

In section 2 we are going to perform the BRST quantization of Jackiw-Teitelboim’s model in the light-cone gauge. We discuss also the quantization in the conformal gauge. The \(SO(2,1)(= SL(2, R))\) topological gauge theory is going to be quantized and it’s topological algebra will be given in section 3. In section 4 we will consider the quantum equivalence between these two models. Moreover, in section 5, it will be shown that actually topological structure is realized in Jackiw-Teitelboim’s model for 1+1 dimensional gravity as well as in the topological gauge theory.
2. Quantization of Jackiw-Teitelboim’s model

First let us write down the starting action of Jackiw-Teitelboim’s model again,

\[ S = \frac{1}{2} \int_{\Sigma} d^2x \sqrt{-g} (R + 2\Lambda)\phi. \]  \hfill (2.1)

Variation with respect to the scalar field \( \phi \) generates an equation of motion

\[ R + 2\Lambda = 0, \]  \hfill (2.2)

which means that the curvature is fixed to a constant. Hereafter we would like to assume that \( \Lambda > 0 \) and that the topology of the two dimensional surface \( \Sigma \) is fixed to \( R \times S^1 \), which is the same topology as the de Sitter space.

The symmetric energy momentum tensor is given by the functional derivative

\[ T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}, \]  \hfill (2.3)

which will play an important role in the quantum analysis later on. After some straightforward calculations we may obtain

\[ T_{\mu\nu} = \left( \frac{1}{2} R\phi - \nabla^\lambda \nabla_\lambda \phi + \Lambda \phi \right) g_{\mu\nu} + \nabla_\mu \nabla_\nu \phi - R_{\mu\nu} \phi. \]  \hfill (2.4)

The action is obviously invariant under the general coordinate transformation,

\[ \delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \]
\[ \delta \phi = \xi^\lambda \partial_\lambda \phi. \]  \hfill (2.5)

2.1 Quantization in the light-cone gauge
We first perform the BRST quantization by using the light-cone gauge, which are given by $g_- = 0$ and $g_- = 1$. It will be convenient to define also $g_+ = 2h_+$. Here $\pm$ denote the light-cone coordinates defined by $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$. It should be noted here that $\sqrt{-g}$ is fixed to be 1 in this gauge. The Ricci tensor and the scalar curvature are written down in terms of $h_+$ as

\begin{align*}
R_+ &= 2h_+ \partial_+^2 h_+,
R_- = R_+ = \partial_-^2 h_+,
R_- &= 0,
R &= 2\partial_-^2 h_+.
\end{align*}

(2.6)

We may obtain the energy-momentum tensor in the light-cone gauge by substituting the gauge conditions into (2.4) as

\begin{align*}
T_- &= \partial_-^2 \phi,
T_- &= 2h_+ T_- + T_+^N,
T_+ &= (2h_+)^2 T_- + 2h_+ T_- + T_+^N + T_+^N,
\end{align*}

where we defined

\begin{align*}
T_+^N &= -(\partial_+ - \partial_- h_+) \partial_- \phi + \Lambda \phi,
T_+^N &= (\partial_+ + \partial_- h_+ - 2h_+ \partial_-) \partial_+ \phi - \partial_+ h_+ \partial_- \phi.
\end{align*}

(2.8)

We note here that $T_-$ can be set to zero by using one of the equations of motion, $\partial_-^2 \phi = 0$. Therefore the new tensors may be expressed as

\begin{align*}
T_+ &= T_+^N,
T_+ &= T_+ - 2h_+ T_-.
\end{align*}

(2.9)

Such structure has been already observed in the induced gravity theory.\cite{6}
Following the standard BRST procedure, the quantum action in this gauge will be given as

\[ S = \int d^2x \left\{ \phi \left( \partial^2 h_{++} + \Lambda \right) + 2b_{++} \partial_- c^+ + b \left( \partial_+ c^+ + \partial_- c^- + 2h_{++} \partial_- c^+ \right) \right\}, \]  

(2.10)

by taking account of the BRST transformations,

\[ \delta^B g_{--} = 2\partial_- c^+, \]
\[ \delta^B g_{+-} = \partial_+ c^+ + \partial_- c^- + 2h_{++} \partial_- c^+. \]  

(2.11)

Here we introduced the ghost fields \( c^+ \) and \( c^- \) for the general coordinate transformation (2.5), and also the anti-ghost fields \( b_{++} \) and \( b \). If we define a new anti-ghost by \( b'_{++} \equiv 2(b_{++} + h_{++} b) \), then the action (2.10) is reduced to a free one,

\[ S = \int d^2x \left\{ \phi \left( \partial^2 h_{++} + \Lambda \right) + b'_{++} \partial_- c^+ + b \left( \partial_+ c^+ + \partial_- c^- \right) \right\}. \]  

(2.12)

This quantum action is actually invariant under the following BRST transformations;

\[ \delta^B h_{++} = \partial_+ c^- + c^+ \partial_+ h_{++} + c^- \partial_- h_{++} + 2\partial_+ c^+ h_{++}, \]
\[ \delta^B \phi = c^+ \partial_+ \phi + c^- \partial_- \phi, \]
\[ \delta^B c^+ = c^+ \partial_+ c^+ + c^- \partial_- c^+, \]
\[ \delta^B c^- = c^+ \partial_+ c^- + c^- \partial_- c^-, \]
\[ \delta^B b'_{++} = T^N_{++} + c^+ \partial_+ b'_{++} + c^- \partial_- b'_{++} + 2\partial_+ c^+ b'_{++}, \]
\[ \delta^B b = T^N_{+-} + c^+ \partial_+ b + c^- \partial_- b. \]  

(2.13)

Once we know these BRST transformations, then it is easy to find the corresponding BRST currents and they are found to be

\[ J^{BRST}_+ = c^+ \left( T^N_{++} + \partial_+ c^+ b'_{++} \right) + c^- \left( T^N_{+-} + \partial_+ c b \right), \]
\[ J^{BRST}_- = c^+ T^N_{+-}. \]  

(2.14)

We may easily show that they satisfy \( \partial_- J^{BRST}_+ = \partial_+ J^{BRST}_- = 0 \) by using the equations of motion.
Let us mention here the residual symmetry after imposing the light-cone gauge. The parameters for the residual transformation should satisfy

\[
\delta g_{-} = 2 \partial_- \xi^+ = 0, \\
\delta g_+ = \partial_+ \xi^+ + \partial_- \xi^- + 2 h_{++} \partial_- \xi^+ = 0.
\]

These equations are solved as

\[
\xi^+ = \hat{\xi}^+(x^+), \\
\xi^- = \hat{\xi}^-(x^+) - x^- \partial_+ \hat{\xi}^+(x^+).
\]

Therefore the transformation for the residual symmetry is given by the parameters \(\hat{\xi}^+(x^+)\) and \(\hat{\xi}^-(x^+)\). The corresponding conserved currents \(\hat{T}^{grav}_{++}\) and \(\hat{T}^{grav}_{+-}\), which satisfy \(\partial_- \hat{T}^{grav}_{+-} = \partial_- \hat{T}^{grav}_{++} = 0\), are found to be

\[
\hat{T}^{grav}_{++} = T^N_{++}, \\
\hat{T}^{grav}_{+-} = T^N_{+-} + x^- \partial_+ T^N_{++},
\]

where \(T^N_{++}\) and \(T^N_{+-}\) are defined in (2.8).

Now we are in a position to perform the quantization of the action (2.12). However it would be more convenient to redefine the ghost variables and the anti-ghost variables as

\[
\hat{c}^+ \equiv c^+,
\]

\[
\hat{c}^- \equiv c^- + x^- \partial_+ c^+,
\]

\[
\hat{b}^{++} \equiv b^{++} + x^- \partial_+ b,
\]

\[
\hat{b} \equiv b.
\]

Then the action (2.12) turns out to be a rather simple one;

\[
S = \int d^2x \left\{ \phi \left( \partial^2 h_{++} + \Lambda \right) + \hat{b}^{++} \partial_- \hat{c}^+ + \hat{b} \partial_- \hat{c}^- \right\}.
\]

The BRST currents given in (2.14) are also rewritten in terms of the new ghost
variables into
\[ J^{BRST}_+ = c^+ \hat{T}^{grav}_++ c^- \hat{T}^{grav}_- \]
\[ + \hat{c}^+ \partial_+ c^+ \hat{b}_++ \hat{c}^- \partial_+ \hat{b} - \partial_+ \left( x^- \hat{c}^+ \hat{T}^{grav}_+- \right), \]
\[ J^{BRST}_- = c^+ \hat{T}^{grav}_-, \]
where the total divergence appearing in $J^{BRST}_+$ is irrelevant for the BRST charge.

First let us consider the gravitational part of the action (2.19). The equations of motion
\[ \partial^2_- h_{++} + \Lambda = 0, \]
\[ \partial^2_- \phi = 0 \] (2.21)
are readily solved as
\[ h_{++}(x^+, x^-) = \hat{h}_{++}(x^+) + x^- \hat{h}_+(x^+) - \frac{1}{2}(x^-)^2 \Lambda, \]
\[ \phi(x^+, x^-) = \hat{\phi}(x^+) + x^- \hat{\phi}^+(x^+). \] (2.22)

On the other hand the equal time canonical commutation relations
\[ \left[ h_{++}(x^1), \pi_h(y^1) \right] = i\delta(x^1 - y^1), \]
\[ \left[ \phi(x^1), \pi_\phi(y^1) \right] = i\delta(x^1 - y^1), \] (2.23)
where \( \pi_h \) and \( \pi_\phi \) are the canonical conjugate momentums of \( h_{++} \) and \( \phi \) respectively, tell us that the component fields defined by (2.22) satisfy
\[ \left[ \hat{\phi}_+(x^1), \hat{h}_{++}(y^1) \right] = i\delta(x^1 - y^1), \]
\[ \left[ \hat{h}_+(x^1), \hat{\phi}(y^1) \right] = i\delta(x^1 - y^1). \] (2.24)

Since we have seen that these component fields are independent of \( x^- \), the operator product expansions (O.P.E.'s) between these fields may be easily found to be
\[ \hat{\phi}^+(x^+) \hat{h}_{++}(y^+) \sim \frac{1}{x^+ - y^+}, \]
\[ \hat{h}_+(x^+) \hat{\phi}(y^+) \sim \frac{1}{x^+ - y^+}, \] (2.25)
where we ignored the common irrelevant factor \( \frac{1}{2} \) in the right-hand sides. In a
similar way the O.P.E.’s between the ghost fields are given by

\[
\hat{c}^+(x^+)\hat{b}^+(y^+) \sim \frac{1}{x^+ - y^+}, \tag{2.26}
\]

\[
\hat{c}^-(x^+)\hat{b}(y^+) \sim \frac{1}{x^+ - y^+}.
\]

Now it is an important observation for the consistency to see whether the quantum BRST charge is nilpotent or not. For the purpose of verifying the BRST nilpotency, first we should derive the quantum algebra between the total "energy-momentum tensor" \( \hat{T}_{\text{tot}}^{++} \) and \( \hat{T}_{\text{tot}}^{+-} \). The gravitational parts of these conserved currents are given explicitly by

\[
\hat{T}_{\text{grav}}^{+-} (x^+) = \Lambda \hat{\phi} - \partial_+ \hat{\phi}^+ + \hat{\phi}^+ \hat{h}_+,
\]

\[
\hat{T}_{\text{grav}}^{++} (x^+) = \left( \partial_+^2 \hat{\phi} + \partial_+ \hat{\phi} \hat{h}_+ \right) - \left( \hat{\phi}^+ \partial_+ \hat{h}^{++} + 2 \partial_+ \hat{\phi}^+ \hat{h}^{++} \right), \tag{2.27}
\]

\( \equiv \hat{T}_{++}^{c=2} + \hat{T}_{++}^{c=26}. \)

Then the quantum algebra between these currents can be evaluated by using the O.P.E.’s in (2.25) and are found to be in O.P.E. forms

\[
\hat{T}_{++}^{\text{grav}} (x^+) \hat{T}^{\text{grav}}_{++} (y^+) \sim 0,
\]

\[
\hat{T}_{++}^{\text{grav}} (x^+) \hat{T}^{\text{grav}}_{+-} (y^+) \sim \left( \frac{2 + 26}{x^+ - y^+} \right)^2 + \frac{2 \hat{T}_{++}^{\text{grav}} (y^+)}{x^+ - y^+} + \frac{\partial_+ \hat{T}_{++}^{\text{grav}} (y^+)}{x^+ - y^+}, \tag{2.28}
\]

\[
\hat{T}_{++}^{\text{grav}} (x^+) \hat{T}^{\text{grav}}_{++} (y^+) \sim \frac{\partial_+ \hat{T}_{++}^{\text{grav}} (y^+)}{x^+ - y^+}.
\]

Therefore we see that the Virasoro algebra of the gravitational part carries the central charge of 28 and that \( \hat{T}_{++}^{\text{grav}} \) is a commuting current with spin 0. The ghost parts of the "energy-momentum tensor" from the action (2.19) can be derived as the conserved currents for the residual symmetry, and are given by

\[
\hat{T}_{++}^{\text{gh}} (x^+) = \hat{c}^+ \partial_+ \hat{b},
\]

\[
\hat{T}_{++}^{\text{gh}} (x^+) = -\hat{c}^+ \partial_+ \hat{b} + \left( \hat{c}^+ \partial_+ \hat{b}^{++} + 2 \partial_+ \hat{c}^+ \hat{b}^{++} \right), \tag{2.29}
\]

\( = \hat{T}_{++}^{c=-2} + \hat{T}_{++}^{c=-26}. \)

By using the O.P.E.’s (2.26) we may see \( \hat{T}_{++}^{\text{gh}} \) and \( \hat{T}_{++}^{\text{gh}} \) satisfy the similar algebra
to (2.27):

\[ \hat{T}_{++}^{gh}(x^+) \hat{T}_{++}^{gh}(y^+) \sim 0, \]
\[ \hat{T}_{++}^{gh}(x^+) \hat{T}_{++}^{gh}(y^+) \sim -\frac{(2 + 26)}{4} + \frac{2 \hat{T}_{++}^{gh}(y^+)}{(x^+ - y^+)^2} + \frac{\partial_+ \hat{T}_{++}^{gh}(y^+)}{x^+ - y^+}, \tag{2.30} \]
\[ \hat{T}_{++}^{gh}(x^+) \hat{T}_{+-}^{gh}(y^+) \sim \frac{\partial_+ \hat{T}_{+-}^{gh}(y^+)}{x^+ - y^+}, \]

from which the contribution of the ghost part to the central charge is read off to be \(-28\). Therefore there appears no anomaly in the Virasoro algebra of the total energy-momentum tensor \( \hat{T}^{tot} = \hat{T}^{grav} + \hat{T}^{gh} \);

\[ \hat{T}_{++}^{tot}(x^+) \hat{T}_{++}^{tot}(y^+) \sim \frac{2 \hat{T}_{++}^{tot}(y^+)}{(x^+ - y^+)^2} + \frac{\partial_+ \hat{T}_{++}^{tot}(y^+)}{x^+ - y^+}, \tag{2.31} \]

Consequently we can indeed verify the nilpotency of the BRST charge defined by (2.19):

\[ \left( Q_{BRST}^{+} \right)^2 = \left( Q_{BRST}^{-} \right)^2 = 0. \tag{2.32} \]

Actually this may be expected before the calculations, because the anomaly does not seem to appear without any physical freedoms in the local dynamics.

2.2 Quantization in the conformal gauge

The conformal gauge, which is very familiar to string physicists, is defined by \( g_{\mu\nu} = e^{2\varphi} \eta_{\mu\nu} \) or

\[ g_{++} = g_{--} = 0, \]
\[ g_{+-} = g_{-+} = e^{\varphi}, \tag{2.33} \]

where we note that the volume form \( \sqrt{-g} = e^{\varphi} \) is kept positive definite without restricting the range of the conformal mode \( \varphi \). The good point of this gauge is that the so-called conformal symmetry, which includes the global Lorentz symmetry, is maintained as the residual symmetry. However this gauge will bring some troubles to proceed the quantization, especially in the case of non-zero cosmological constant, as is seen later on.
By inserting the gauge conditions (2.33) into (2.4) the energy-momentum tensor becomes

\[
T_{++}^{\text{grav}} = \nabla_+ \nabla_+ \phi = \partial_+^2 \phi - \partial_+ \varphi \partial_+ \phi, \\
T_{--}^{\text{grav}} = \nabla_- \nabla_- \phi = \partial_-^2 \phi - \partial_- \varphi \partial_- \phi, \\
T_{+-}^{\text{grav}} = -\partial_+ \partial_- \phi + \Lambda \phi e^\varphi.
\]

(2.34)

The quantum action in the conformal gauge also is obtained through the usual BRST procedure and

\[
S = \int d^2 x \left\{ \phi (-\partial_+ \partial_- \varphi + \Lambda e^\varphi) + b_{++} \partial_- c^+ + b_{--} \partial_+ c^- \right\}.
\]

(2.35)

It should be noted that there is an interaction term unless \( \Lambda = 0 \) in contrast to the action (2.12) in the light-cone gauge. The equations of motion of the gravitational part are readily derived as

\[
\partial_+ \partial_- \varphi = \Lambda e^\varphi, \\
\partial_+ \partial_- \phi = \Lambda \phi e^\varphi.
\]

(2.36)

The first equation is the so-called Liouville equation. The second one means \( T_{+-}^{\text{grav}} = 0 \), namely the presence of the conformal symmetry in the classical level. It is also a rather easy task to find the BRST transformations and they are given by

\[
\delta^B \varphi = \partial_+ c^+ + \partial_- c^- + c^+ \partial_+ \varphi + c^- \partial_- \varphi, \\
\delta^B \phi = c^+ \partial_+ \phi + c^- \partial_- \phi, \\
\delta^B c^+ = c^+ \partial_+ c^+ + c^- \partial_- c^+, \\
\delta^B c^- = c^+ \partial_+ c^- + c^- \partial_- c^-, \\
\delta^B b_{++} = T_{++}^{\text{grav}} + c^+ \partial_+ b_{++} + 2 \partial_+ c^+ b_{++} + c^- \partial_- b_{++}, \\
\delta^B b_{--} = T_{--}^{\text{grav}} + c^- \partial_- b_{--} + 2 \partial_- c^- b_{--} + c^+ \partial_+ b_{--}.
\]

(2.37)

By the Noether method using (2.37) we may derive the BRST currents:

\[
J_{\text{BRST}}^+ = c^+ \left( T_{++}^{\text{grav}} + \frac{1}{2} T_{++}^{\text{gh}} \right), \\
J_{\text{BRST}}^- = c^- \left( T_{--}^{\text{grav}} + \frac{1}{2} T_{--}^{\text{gh}} \right).
\]

(2.38)
where we introduced the energy-momentum tensor of the ghost part,

\[ T_{++}^{gh} = c^+ \partial_+ b_{++} + 2\partial_+ c^+ b_{++}, \]
\[ T_{--}^{gh} = c^- \partial_- b_{--} + 2\partial_- c^- b_{--}. \]  

(2.39)

Here we note the structure of the BRST currents are identical to one of the induced gravity or string theories in the conformal gauge. We can verify easily also the conservation of these BRST currents, \( \partial_- J_{+}^{BRST} = \partial_+ J_{-}^{BRST} = 0 \), by using the equations of motion (2.36). The total energy-momentum tensor is given by

\[ T_{++}^{tot} = T_{++}^{grav} + T_{++}^{gh}, \]
\[ T_{--}^{tot} = T_{--}^{grav} + T_{--}^{gh}, \]  

(2.40)

which are the generators of the conformal symmetry.

Now we would like to consider to quantize the action (2.35). As is seen from the equations of motion (2.36), this problem may be closely related to the quantization of the Liouville theory.\{12\}[13] From the action

\[ S = \int d^2 x \left\{ \frac{1}{2}(\partial_0 + \partial_1)\phi(\partial_0 - \partial_1)\varphi + \frac{1}{2}(\partial_0 - \partial_1)\phi(\partial_0 + \partial_1)\varphi + 2\Lambda\phi e^\varphi \\
+ b_{++}(\partial_0 - \partial_1)c^+ + b_{--}(\partial_0 + \partial_1)c^- \right\}, \]  

(2.41)

where we rescaled the variables slightly for the simplicity, the canonical conjugate momentums are given by

\[ \pi_\phi = \partial_0 \varphi, \]
\[ \pi_\varphi = \partial_0 \phi, \]
\[ \pi_{c^+} = b_{++}, \]
\[ \pi_{c^-} = b_{--}. \]  

(2.42)

On the other hand the energy-momentum tensor of the gravitational sector may
be expressed in terms of these canonical variables as

\[
T_{++}^{\text{grav}} = -\frac{1}{2} : (\pi_\phi + \partial_1 \varphi)(\pi_\varphi + \partial_1 \phi) : + \Lambda \phi e^\varphi + \partial_1 (\pi_\varphi + \partial_1 \phi),
\]

\[
T_{--}^{\text{grav}} = -\frac{1}{2} : (\pi_\phi - \partial_1 \varphi)(\pi_\varphi - \partial_1 \phi) : + \Lambda \phi e^\varphi - \partial_1 (\pi_\varphi - \partial_1 \phi),
\]

where :: denotes the normal ordering. By using the equal time commutation relations,

\[
[\phi(x^1), \pi_\phi(y^1)] = i \delta(x^1 - y^1),
\]

\[
[\varphi(x^1), \pi_\varphi(y^1)] = i \delta(x^1 - y^1),
\]

we may extract the short distance singularities from the product of two \(T_{++}^{\text{grav}}(x^1)\)'s. After some manipulations\(^{[12,13]}\) they are found to be

\[
T_{++}^{\text{grav}}(x^1)T_{++}^{\text{grav}}(y^1) \sim -\frac{26}{2} \left(\frac{1}{(x^1 - y^1)^4} + \frac{2 T_{++}^{\text{grav}}(y^1)}{(x^1 - y^1)^2} + \frac{\partial_1 T_{++}^{\text{grav}}(y^1)}{x^1 - y^1}\right)
- \Lambda \left[\frac{1}{(x^1 - y^1)^2} e^\varphi + \frac{1/2}{x^1 - y^1} \partial_1 \phi e^\varphi\right].
\]

Similarly we obtain

\[
T_{++}^{\text{gh}}(x^1)T_{++}^{\text{gh}}(y^1) \sim -\frac{26}{2} \left(\frac{1}{(x^1 - y^1)^4} + \frac{2 T_{++}^{\text{gh}}(y^1)}{(x^1 - y^1)^2} + \frac{\partial_1 T_{++}^{\text{gh}}(y^1)}{x^1 - y^1}\right)
- \Lambda \left[\frac{1}{(x^1 - y^1)^2} e^\varphi + \frac{1/2}{x^1 - y^1} \partial_1 \phi e^\varphi\right].
\]

for the energy-momentum tensor of the ghosts. Thus it seems that the anomalies do not cancel each other between the gravitational part and the ghost part, even if \(\Lambda = 0\), contrary to the results in the light-cone gauge (2.31). In the presence of the non-zero cosmological constant \(\Lambda\), the situation looks much worse. Indeed the short distance singularities proportional to \(\Lambda\) in (2.45) seem to prevent the Virasoro algebra from closing.

These anomalous results, however, are caused by the inconsistency of our quantization procedure. Actually it is necessary to take care of the renormalization
effects due to the non-trivial interactions. Besides we have to evaluate the Jacobian which is probably generated through the change of the dynamical variable $g_{+-} = e^\varphi$ to $\varphi$. Suppose, therefore, the energy-momentum tensor is improved to

$$
T_{++}^{\text{grav}} = -\frac{1}{2} \left( \partial_1 \varphi \right) \left( \partial_1 \varphi + \partial_1 \phi \right) : + \Lambda^{\text{ren}} \phi e^{\alpha \varphi} + \partial_1 (\pi \varphi + \partial_1 \phi) + \beta \partial_1 (\pi \phi + \partial_1 \varphi),
$$

(2.47)

where $\alpha$ and $\beta$ are unknown parameters to be determined by the consistency.$^{[13,7]}$

If we set the parameters to

$$
\alpha = 1,
\beta = \frac{1}{2},
$$

(2.48)

then the total energy-momentum tensor $T_{++}^{\text{tot}}$ indeed satisfies the anomaly free and closed algebra;

$$
T_{++}^{\text{tot}}(x^1) T_{++}^{\text{tot}}(y^1) \sim \frac{2}{(x^1 - y^1)^2} T_{++}^{\text{tot}}(y^1) + \frac{1}{x^1 - y^1} \partial_1 T_{++}^{\text{tot}}(y^1).
$$

(2.49)

The ($-$) sector also enjoys the similar improvement. Thus Jackiw-Teitelboim’s model may be quantized consistently also in the conformal gauge, if we use the energy-momentum tensor given by (2.47) and (2.48). However further investigations are needed to see whether the quantization using (2.47) really gives us the same results as the light-cone quantization. The quantization of the similar action has been examined by the perturbative approach also.$^{[14]}$
3. Quantization of the $SL(2, R)$ topological gauge theory

In this section we are going to perform the BRST quantization of the topological gauge theory with $SO(2, 1)$ gauge group;

$$S = \int d^2x e^{\mu \nu} Tr(\Phi F_{\mu \nu}), \quad (3.1)$$

where $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ is a field strength. The $SO(2, 1)$ or 2d de Sitter algebra is

$$[P_a, P_b] = \epsilon_{ab} J, \quad [J, P_a] = \epsilon_{ab} P_b, \quad (3.2)$$

where $a$ and $b$ are 0 or 1. For the later conveniences let us define new generators by $T_\pm \equiv \frac{1}{\sqrt{2}}(P_0 \pm P_1)$ and $T_0 \equiv J$. Then they form a $SL(2, R)$ algebra,

$$[T_0, T_\pm] = \pm T_\pm, \quad [T_+, T_-] = T_0. \quad (3.3)$$

If we expand the gauge field $A_\mu$ and the scalar field $\phi$ with respect to the $SL(2, R)$ generators as

$$A_\mu = \sqrt{\Lambda} e_\mu^+ T_+ + \sqrt{\Lambda} e_\mu^- T_- + \omega_\mu T_0, \quad \Phi = \frac{1}{\sqrt{\Lambda}} \phi^+ T_+ + \frac{1}{\sqrt{\Lambda}} \phi^- T_-, \quad (3.4)$$

then the action in terms of the component fields turns to be

$$S = \int d^2x \left\{ \phi^- \left( D_+ e_\nu^+ - D_- e_\nu^- \right) + \phi^+ \left( D_+ e_\nu^- - D_- e_\nu^+ \right) \right. + \left. \phi \left( \partial_+ \omega_- - \partial_- \omega_+ - \Lambda (e_\mu^+ e_\nu^- + e_\mu^- e_\nu^+) \right) \right\}, \quad (3.5)$$

where $D_\pm$ denote the covariant derivatives defined by $D_\mu e_\nu^\pm = (\partial_\mu \pm \omega_\mu)e_\nu^\pm$. The multipliers $\phi^\pm$ generate the so-called torsion-free conditions; $D_+ e_\nu^- - D_- e_\nu^+ = 0$. If we solve the spin connections $\omega_\pm$ in terms of the zweibeins $e_\pm$ by using these
torsion-free conditions, then indeed the action (3.5) is found to be reduced to
the gravitational action (2.1).\textsuperscript{[11]} However we need to change
\( e = e^+e^- - e^-e^+ \) to \( \sqrt{-g} = |e| \) to obtain (2.1). This ambiguity of the sign does not affect the equations
of motion, hence we may say that this topological gauge theory is equivalent to
Jackiw-Teitelboim’s model in the classical level. Hereafter let us rescale \( e^\pm \) and \( \phi^\pm \)
to absorb the factor \( \sqrt{\Lambda} \).

The action (3.1) is obviously invariant under the \( SL(2, R) \) gauge transforma-
tions which are

\begin{align}
\delta A_\mu &= \partial_\mu \epsilon + [A_\mu, \epsilon], \\
\delta \Phi &= [\Phi, \epsilon],
\end{align}

where we introduced the gauge parameter \( \epsilon = \epsilon^+T^- + \epsilon^-T^+ + \epsilon^0T^0 \). One may note
that this action is invariant also under the general coordinate transformations. In
practice, however, we do not have to concern this symmetry, because the general
coordinate transformations and the gauge transformations are "reducible" to each
other. Actually we will see that it is enough to fix only the gauge transformations
(3.6) in order to construct the quantum action.

3.1 Quantization in \( A^a_\mu = 0 \) gauge

We would like to consider first the BRST quantization by the gauge conditions
\( A^a_\mu \) or

\[ e^+_\alpha = e^-_\alpha = \omega_\alpha = 0. \]

(3.7)

Other gauges also will be examined in the next section. We note that the gauge
conditions (3.7) give a vanishing "volume form" \( e = e^+_\alpha e^-_\alpha - e^-_\alpha e^+_\alpha = 0 \), which is
a singular configuration as gravity. By introducing ghosts \( \bar{c} \)'s and anti-ghosts \( \bar{b} \)'s,

\textsuperscript{2)} In the case of \( \Lambda < 0 \) the gauge group should be \( SO(1, 2) \) (anti-de Sitter group) instead of
\( SO(2, 1) \). If \( \Lambda = 0 \), then we have to define the theory in the limit of \( \Lambda \to 0 \). Such cases are
also discussed in the ref.[11], though we are not going to mention in this paper.
the BRST invariant quantum action in the $A_0 = 0$ gauge is given by

$$S = \int d^2 x \left\{ -\phi^- \partial_- e_+^+ - \phi^+ \partial_- e_-^- - \phi \partial_- \omega_+ \\
+ \tilde{b}_+^+ \partial_- \tilde{e}^+ + \tilde{b}_+^- \partial_- \tilde{e}^- + \tilde{b}_+ \partial_- \tilde{c} \right\}.$$  \hspace{1cm} (3.8)

The BRST transformations are easily found to be the following,

$$\delta^B e_+^+ = \partial_+ \tilde{c}^+ + \omega_+ \tilde{e}^+ - e_+ \tilde{c},$$
$$\delta^B e_-^- = \partial_+ \tilde{c}^- - \omega_+ \tilde{e}^- + e_+ \tilde{c},$$
$$\delta^B \omega_+ = \partial_+ \tilde{c} + e_+ \tilde{c}^- - e_+ \tilde{c}^+,$$
$$\delta^B \phi^+ = \phi \tilde{c}^- - \phi^+ \tilde{c},$$
$$\delta^B \phi^- = \phi^- \tilde{c} - \phi^- \tilde{c},$$
$$\delta^B \phi = \phi^+ \tilde{c}^- - \phi^- \tilde{c}^+,$$
$$\delta^B \tilde{c}^+ = \tilde{c}^+ \tilde{c},$$
$$\delta^B \tilde{c}^- = \tilde{c} \tilde{c}^-,$$
$$\delta^B \tilde{c} = \tilde{c} \tilde{c}^+,$$
$$\delta^B \tilde{e}^+ = J^\text{tot}^+,$$
$$\delta^B \tilde{e}^- = J^\text{tot}^-,.$$

Here $J^\text{tot}^+ \equiv J^{\text{gauge}^+} + J^{\text{gh}^+}$ are the conserved currents for the residual $SL(2, R)$ symmetry, which are given explicitly by

$$J^{\text{gauge}^+} = -\left( \partial_+ + \omega_+ \right) \phi^+ + e_+ \phi,$$
$$J^{\text{gauge}^-} = -\left( \partial_+ - \omega_+ \right) \phi^- - e_+ \phi,$$
$$J^{\text{gauge}^0} = -\partial_+ \phi + e_+ \phi^+ - e_+ \phi^-,$$
$$J^{\text{gh}^+} = \tilde{b}_+^+ \tilde{c} - \tilde{b}_+ \tilde{c}^+,$$
$$J^{\text{gh}^-} = \tilde{b}_+ \tilde{c}^- - \tilde{b}_+^- \tilde{c}^-,$$
$$J^{\text{gh}^0} = \tilde{b}_+^- \tilde{c}^+ - \tilde{b}_+ \tilde{c}^-.$$

Note that there are no symmetry currents in the ($-$) sector. In terms of these
currents the BRST current may be expressed neatly as
\[
J_{BRST}^+ = c^+ \left( J_{\text{gauge}}^- + \frac{1}{2} J_{gh}^- \right) + c^- \left( J_{\text{gauge}}^+ + \frac{1}{2} J_{gh}^+ \right) + \tilde{c} \left( J_{\text{gauge}0}^+ + \frac{1}{2} J_{gh0}^+ \right).
\]

(3.11)

In this case the quantization is rather simple. The non-trivial O.P.E.’s between the fields are also readily found to be
\[
e^\pm_+(x^+) \phi^\mp(y^+) \sim \frac{1}{x^+ - y^+};
\]
\[
\omega^\pm_+(x^+) \phi(y^+) \sim \frac{1}{x^+ - y^+};
\]
\[
c^\pm_+(x^+) \tilde{b}^\mp_+(y^+) \sim \frac{1}{x^+ - y^+};
\]
\[
\tilde{c}(x^+) \tilde{b}_+(y^+) \sim \frac{1}{x^+ - y^+}.
\]

(3.12)

By using these O.P.E.’s (3.12) we may easily see that the currents \( J_{\text{gauge}}^+ \) and \( J_{gh}^+ \) indeed form the following \( SL(2, \mathbb{R}) \) Kac-Moody algebras;
\[
J_{\text{gauge}}^+(x^+), J_{\text{gauge}}^-(y^+) \sim \frac{-2}{(x^+ - y^+)^2} + \frac{J_{\text{gauge}0}^+(y^+)}{x^+ - y^+},
\]
\[
J_{\text{gauge}0}^+(x^+), J_{\text{gauge}0}^-(y^+) \sim \frac{-2}{(x^+ - y^+)^2},
\]
\[
J_{\text{gauge}0}^+(x^+), J_{\text{gauge}}^\pm(y^+) \sim \pm \frac{J_{\text{gauge}^\pm}(y^+)}{x^+ - y^+},
\]

(3.13)

and
\[
J_{gh}^+(x^+), J_{gh}^-(y^+) \sim \frac{2}{(x^+ - y^+)^2} + \frac{J_{gh0}^+(y^+)}{x^+ - y^+},
\]
\[
J_{gh0}^+(x^+), J_{gh0}^-(y^+) \sim \frac{2}{(x^+ - y^+)^2},
\]
\[
J_{gh0}^+(x^+), J_{gh}^\pm(y^+) \sim \pm \frac{J_{gh}^\pm(y^+)}{x^+ - y^+}.
\]

(3.14)

Here we note that the Schwinger terms in the gauge sector and the ghost sector
cancel each other. Thus it is verified that there appears no anomaly in the Kac-Moody algebra of the total current $J_{\text{tot}}^+$, as is expected. The nilpotency of the BRST charge $(Q_{BRST}^\cdot)^2 = 0$ follows from these calculations immediately.

In the last part of this section we would like to discuss the topological algebra, which is expected to exist in this topological gauge theory. It is known that two dimensional topological field theories may be characterized commonly by the twisted $N = 2$ superconformal algebras (SCA’s). Indeed it is found that this topological gauge model also has a twisted $N = 2$ SCA. The set of generators of the twisted $N = 2$ SCA consists of the energy-momentum tensor $T_{++}$, a spin 1 super-current $G_+$, a spin 2 super-current $\bar{G}_{++}$ and a $U(1)$ current $I_+$. If we define them as

\begin{align*}
T_{++} &\equiv \partial_+ \phi^+ e^-_+ + \partial_+ \phi^- e^+_+ + \partial_+ \phi \omega_+ + \partial_+ \bar{c}^+ b^-_+ + \partial_+ \bar{c}^- b^+_+ + \partial_+ \bar{b}^+_+ , \\
G_+ &\equiv J^{BRST}_+ , \\
\bar{G}_{++} &\equiv \bar{b}^+_+ e^-_+ + \bar{b}^-_+ e^+_+ + \bar{b}^+_+ \omega_+ , \\
I_+ &\equiv J^{gh}_+ = -\bar{b}^+_+ \bar{c}^-_+ - \bar{b}^-_+ \bar{c}^+_+ - \bar{b}^+_+ \bar{c}_+ , \tag{3.15}
\end{align*}

where $J^{BRST}_+$ is the BRST current given in (3.11) and $J^{gh}_+$ is the ghost number current, then these currents are found to form the following algebra;

---

3) It should be noted that the energy-momentum tensor $T_{++}$ in (3.15) is not the one by Sugawara construction from the Kac-Moody currents. Such topological models also have been considered.\[16\]
This is nothing but the $N = 2$ twisted SCA with the central charge $\hat{c} = 3$. Now it would be natural to wonder if such a twisted SCA is realized also in Jackiw-Teitelboim’s model which was examined extensively in the last section. This will be considered in section 5.

4. On the quantum equivalence

In the last section we have seen that both of Jackiw-Teitelboim’s model and the $SL(2, R)$ topological gauge model can be consistently quantized in the BRST formulation. What we are interested in now is the quantum equivalence between those two models. In order to see the equivalence it would be one way to examine the physical spaces which are determined by the BRST-cohomologies. However the BRST currents obtained in (2.20) and in (3.11) look rather different in the structure from each other. The BRST current for Jackiw-Teitelboim’s model is based on the energy-momentum tensor $\tilde{T}^{grav}_{++}$ and the $U(1)$ current $\tilde{T}^{grav}_{+-}$. On the other hand the BRST current for the $SL(2, R)$ topological gauge model is based
on the $SL(2, R)$ Kac-Moody currents $J_{gauge}$. Thus it would be far from obvious whether these two BRST charges really give identical physical spaces. Conversely if they are truly equivalent to each other, then it would suggest some relations linking the Virasoro algebra and the $SL(2, R)$ Kac-Moody algebra. Actually it has been already known that the gauged Wess-Zumino-Novikov-Witten (WZNW) model with $SL(2, R)$ gauge group is reduced to the so-called gravitational WZNW model by means of the Hamiltonian reduction. In this respect it seems to be very interesting to clarify the equivalence in the algebraic point of view, though we are not going to touch on this subject in this paper.

4.1 The topological gauge theory in the "light-cone" gauge

In this section we would like to consider the equivalence from a different point of view without concerning the physical states themselves. We should remember that the gauge conditions imposed to the gravity theory and to the gauge theory differed from each other. Especially it should be noted that the $A_x^a = 0$ gauge (3.7) seems to be improper, if we want to regard the topological gauge theory as the gravitational theory. In order to compare the quantum theories started from the actions (2.1) and (3.1) directly, therefore, we shall examine the "light-cone" gauge which is defined here by

$$
e^+_+ = e^-_+ = 1, \\
e^+_+ = 0, \\
e^-_+ = h_{++},$$

(4.1)

to also the $SL(2, R)$ topological gauge theory. These gauge conditions give us $e = e^+_+e^-_+e^+_+e^-_+ = 1$, which should be compared with the previous gauge (3.7). The light-cone gauge $g_{--} = 0$, $g_{+-} = 1$ and $g_{++} = 2h_{++}$ also follow from these conditions.

After fixing the gauge transformations (3.6) by the gauge conditions (4.1), we
obtain the quantum action

\[ S = \int d^2x \left\{ \phi^+ (-\omega_+ - \partial_- h_{++} + \omega_- h_{++}) \right. \\
- \phi^- \omega_- + \phi (\partial_+ \omega_+ - \partial_- \omega_+ - 1) + \tilde{b}_{++} (\partial_- \tilde{c}^+ + \omega_- \tilde{c}^+) \right. \\
+ \tilde{b}_{--} (\partial_+ \tilde{c}^+ + \omega_+ \tilde{c}^+ - \tilde{c}) + \tilde{b}_{+-} (\partial_-' \tilde{c} - \omega_- \tilde{c} + \tilde{c}) \left. \right\} \]  

(4.2)

through the BRST procedure. The BRST transformations leaving this action invariant are found to be

\[ \delta^B h_{++} = \partial_+ \tilde{c}^- - \omega_+ \tilde{c}^- + h_{++} \tilde{c}, \]
\[ \delta^B \omega_+ = \partial_+ \tilde{c} + \tilde{c}^- - h_{++} \tilde{c}^+, \]
\[ \delta^B \omega_- = \partial_- \tilde{c} - \tilde{c}^+, \]
\[ \delta^B \phi^+ = \phi \tilde{c}^- - \phi \tilde{c}^+, \]
\[ \delta^B \phi^- = \phi \tilde{c} - \phi \tilde{c}^-, \]
\[ \delta^B \phi = \phi \tilde{c}^- - \phi \tilde{c}^+, \]
\[ \delta^B \tilde{c}^+ = \tilde{c}^+ \tilde{c}, \]
\[ \delta^B \tilde{c}^- = \tilde{c} \tilde{c}^-, \]
\[ \delta^B \tilde{c} = \tilde{c} \tilde{c}^+ , \]
\[ \delta^B \tilde{b}_{++} = -\partial_+ \phi^- + \omega_+ \phi^- - \phi h_{++} + \tilde{b}_{++} \tilde{c}, \]
\[ \delta^B \tilde{b}_{--} = -\partial_- \phi^- - \omega_- \phi^- + \phi - \tilde{b}_{--} \tilde{c}, \]
\[ \delta^B \tilde{b}_{+-} = -\partial_+ \phi^+ - \omega_+ \phi^+ + \phi + \tilde{b}_{+-} \tilde{c}. \]  

(4.3)

However, if we redefine the variables as
\[ \phi'^- \equiv \phi^- + \partial_+ \phi - h_{++} \phi^+ + \tilde{b}_{+-} \tilde{c}^- - \tilde{b}_{++} \tilde{c}^+, \]
\[ \phi'^+ \equiv \phi^+ - \partial_- \phi - \tilde{b}_{--} \tilde{c}^+, \]
\[ \omega'_+ \equiv \omega_+ + \partial_- h_{++}, \]
\[ \tilde{b}' \equiv \tilde{b}_{--} - \tilde{b}_{+-}, \]
\[ \tilde{c}' \equiv \tilde{c} + \partial_- \tilde{c}^-, \]
\[ \tilde{b} \equiv \tilde{b}_{++}, \]
\[ \tilde{c}'^- \equiv \tilde{c}^- - h_{++} \tilde{c}^+, \]
\[ \tilde{b}'_{++} \equiv \tilde{b}_{++} + h_{++} \tilde{b}, \]

where it should be noted that no Jacobians appear through these redefinition, then the action (4.2) turns out to be

\[ S = \int d^2 x \left\{ \phi \left( \partial_-^2 h_{++} + 1 \right) + \tilde{b}'_{++} \partial_- \tilde{c}^+ + \tilde{b} \left( \partial_+ \tilde{c}^+ + \partial_- \tilde{c}'^- \right) \right. \]
\[ \left. - \phi'^- \omega'_- - \phi'^+ \omega'_+ + \tilde{b}' \tilde{c}' \right\}. \]

Here we may take away the fields \( \phi'^\pm, \omega'_\pm, \tilde{b}' \) and \( \tilde{c}' \), since they are non-dynamical and are completely decoupled from the others. Taking account of this we see the action (4.5) is just same as the quantum action (2.12) for Jackiw-Teitelboim’s model in the light-cone gauge.

Next we need to examine also the BRST charge which determines the physical space. Through the redefinition of the variables (4.4) the BRST transformations (4.3) will be changed into the followings,
\[ \delta^B h_{++} = \partial_+ c^+ + c^+ \partial_+ h_{++} + c^- \partial_- h_{++} + 2h_{++} \partial_+ c^+ - h_{++} \left( \partial_+ c^+ + \partial_- c^- + h_{++} \partial_- c^+ \right), \]

\[ \delta^B \phi = \tilde{c}^+ \partial_+ \phi + \tilde{c}^- \partial_- \phi, \]

\[ \delta^B \tilde{c}^+ = \tilde{c}^+ \partial_+ \tilde{c}^+ + \tilde{c}^- \partial_- \tilde{c}^+ - \tilde{c}^+ \left( \partial_+ \tilde{c}^+ + \partial_- \tilde{c}^- + h_{++} \partial_- \tilde{c}^+ \right) - \tilde{c}^- \partial_- \tilde{c}^+, \]

\[ \delta^B \tilde{c}^- = \tilde{c}^+ \partial_+ \tilde{c}^- + \tilde{c}^- \partial_- \tilde{c}^- = \tilde{c}^+ \partial_+ \tilde{c}^+ + \tilde{c}^- \partial_- \tilde{c}^- + \tilde{c}^+ \left( \partial_+ \tilde{c}^+ + \partial_- \tilde{c}^- + h_{++} \partial_- \tilde{c}^+ \right) + h_{++} \tilde{c}^- \partial_- \tilde{c}^+, \]

\[ \delta^B \tilde{b}^+_{++} = T^N_{++} + T^{N,gh}_{++} + (h_{++})^2 \partial^2 \phi \]

\[ + h_{++} \left( \tilde{c}^- + h_{++} \tilde{c}^+ \right) \partial_- \tilde{b} - \left( \tilde{c}^- + h_{++} \tilde{c}^+ \right) \left( \partial_+ \tilde{b} + \partial_- \tilde{b}'_{++} \right), \]

\[ \delta^B \tilde{b} = T^N_{+-} + T^{N,gh}_{+-} + h_{+-} \partial^2 \phi + h_{+-} \tilde{c}^+ \partial_- \tilde{b} - \tilde{c}^+ \left( \partial_+ \tilde{b} + \partial_- \tilde{b}'_{++} \right), \]

where \( \tilde{T}_{++} \) and \( \tilde{T}_{+-} \) are the same combinations of the fields as those defined by (2.27) and (2.29). \( T_{++}^{N,gh} \) and \( T_{+-}^{N,gh} \) are newly defined as

\[ T_{++}^{N,gh} = \tilde{c}^+ \partial_+ \tilde{b}'_{++} + 2 \partial_+ \tilde{c}^+ \tilde{b}'_{++} + \tilde{c}^- \partial_- \tilde{b}'_{++}, \]

\[ T_{+-}^{N,gh} = \tilde{c}^+ \partial_+ \tilde{b} + \tilde{c}^- \partial_- \tilde{b}. \]

If these are compared with the BRST transformations obtained previously in (2.13), then we notice the disagreement. However the deviation from (2.13) are found to disappear on-shell. We may suppose that this on-shell equivalence between the BRST transformations (4.6) and (2.13) is caused by the reducibility between the gauge transformations and the general coordinate transformations. Anyhow we may easily show that the BRST currents corresponding to the BRST transformations (4.6) just coincide with the former ones;

\[ J^{BRST}_+ = c^+ \left( T^N_{++} + \frac{1}{2} T^{N,gh}_{++} \right) + c^- \left( T^N_{+-} + \frac{1}{2} T^{N,gh}_{+-} \right), \]

\[ J^{BRST}_- = c^+ T^N_{+-}. \]

Thus not only the quantum action but also the BRST charge are identical to those of the Jackiw-Teitelboim’s model completely. Therefore we may conclude that
the two models are actually equivalent in the quantum sense as far as we use the light-cone gauge.

4.2 The gauge a la Polyakov

It is said generally that the physical spectrum is independent of the choice of the gauge condition. If we can apply this argument to our case, then it would mean that we have already proven the quantum equivalence between the two models, since we have seen the equivalence in the light-cone gauge. However this seems to be too naive. Because the global modes in the physical spectrum could depend on the gauge choice, though the local dynamics is, of course, gauge independent. on the other hand Jackiw-Teitelboim’s model as well as the topological gauge model is free from any local freedom. Therefore the physical spaces may be spanned by only the global modes. Thus, in turn, the equivalence between the light-cone gauge and the \( A^-_a = 0 \) gauge seems to be important to explore.

In this subsection we would like to re-examine the \( SL(2, R) \) topological gauge theory in another type of gauge. The gauge conditions are rather similar to the \( A^-_a = 0 \) gauge and are given by

\[
\begin{align*}
  e^-_+ &= 1, \\
  e^+_+ &= \omega_- = 0. 
\end{align*}
\]  

(4.9)

This gauge has been considered by Polyakov\(^{[18]}\) so as to show the reduction from the gauged \( SL(2, R) \) WZNW model to the gravitational WZNW model. In this gauge the ”volume form” is given by \( e = e^+_+ \), therefore it may take any value irrespective to positive or negative unlike the gauges previously considered. Therefore the difference between \( e \) and \(|e|\) would become sensible.

Following the BRST procedure the quantum action in this gauge is found to be

\[
S = \int d^2 x \left\{ -\phi^+ (\omega_+ + \partial_- h_{++}) - \phi^- \partial_- e^+_+ + \phi (e^+_+ - \partial_- \omega_+) \\
+ \tilde{b}_+ \partial_- \tilde{c}^+ + \tilde{b}_- (\partial_- \tilde{c}^- + \tilde{c}) + \tilde{b}_+ (\partial_- \tilde{c} - \tilde{c}^+) \right\}. 
\]  

(4.10)
As is easily seen this action is invariant under the same BRST transformations given in (3.9). Also the BRST current is found to be identical to the BRST current obtained in the $A^a_\pm = 0$ gauge (3.11). Thus the quantum theory in Polyakov’s gauge looks very similar to the one in the $A^a_\pm = 0$ gauge. Actually we may show they are completely equivalent in the quantum sense by performing some field redefinitions.

However we can reduce the number of the dynamical variables in this gauge as follows. If we introduce new variables as

\[\begin{align*}
e^\prime_+ &= e^+ + \partial_+ \omega_+,
\omega_+ &= \omega_+ + \partial_+ h_++,
\phi^\prime &= \phi^+ + \partial^2_\omega^-, \\
\phi_+ &= \phi + \partial_+ \phi_-,
\tilde{c}^\prime &= \tilde{c}^+ - \partial_+ \tilde{c},
\tilde{c}_+ &= \tilde{c} + \partial_+ \tilde{c}^-,
\tilde{b}^\prime_+ &= \tilde{b}_+ + \partial_- \tilde{b}_++,
\tilde{b}^\prime_+ &= \tilde{b}_+ + \partial_2 \tilde{b}_++,
\end{align*}\]

then we may rewrite the action (4.10) into

\[S = \int d^2x \left\{ \phi^- \partial^3 h_++ - \tilde{b}_+ \partial_2 \tilde{c}^- \right\}, \tag{4.12}\]

where we have eliminated the non-dynamical fields and have replaced $e^-$ to $h_++$. After some calculation the BRST current also turns out to be

\[J^{BRST}_+ = \left( - \partial^2_\omega^- + \partial_- \tilde{c}^- \partial_+ - \tilde{c}^- \partial_2 \right) \left( J^{grav-}_+ + \frac{1}{2} J^{gh-}_+ \right), \tag{4.13}\]

where the currents $J^{grav-}_+$ and $J^{gh-}_+$ are given by

\[\begin{align*}
J^{grav-}_+ &= \partial_+ \phi^- + \partial_- h_+ \phi^- - h_+ \partial_+ \phi^- , \\
J^{gh-}_+ &= \tilde{b}_+ \partial_- \tilde{c}^- - \partial_- \tilde{b}_+ \tilde{c}^- .
\end{align*}\]

Now we should compare the action (4.12) and the BRST charge given by (4.12) with those obtained in the light-cone gauge (2.19) and (2.20). However we may
note that the equation of motion for the gravitational field \( h_{++} \)

\[
\partial^3 h_{++} = 0
\]  

(4.15)

has lost the information of the cosmological constant contrary to \( \partial^2 h_{++} + \Lambda = 0 \)
in (2.21). Therefore we may expect naively that the quantum theory in Polyakov’s
gauge or in the \( A^a_\perp = 0 \) gauge has a slightly different physical space from that in
the light-cone gauge. Of course our analysis is, however, far from complete to show
the inequivalence. Investigation in more detail on the BRST cohomologies will be
required.

5. Topological algebra in Jackiw-Teitelboim’s model

In section 3 we have found that the twisted N=2 SCA is indeed realized in the
\( SL(2, R) \) topological gauge theory quantized in the \( A^a_\perp = 0 \) gauge as was shown
in (3.16). However it seems to be hard to expect that Jackiw-Teitelboim’s model,
which has been shown to be equivalent to the \( SL(2, R) \) topological gauge theory
quantized in the light-cone gauge, also has such a twisted N=2 SCA. Because the
\( SL(2, R) \) symmetry seems to have been completely lost in the Jackiw-Teitelboim’s
model. Moreover \( \bar{G}_{++} \) defined in (3.15) is the generator for the spin-1 supersymmetry, which may be readily seen from the action (3.8). On the other hand there
seems to be no supersymmetry realized in the action (2.19).

However we may find out another kind of the twisted N=2 SCA in the light-
cone gauge. Define the generators for the twisted N=2 SCA in terms of the fields
introduced in (2.22) as

\[
\begin{align*}
T_{++} &= \hat{T}_{++}^{grav} + \hat{T}^{gh}, \\
G_+ &= \hat{c}^+ \hat{T}_{++}^{grav} + \hat{c}^+ \partial_+ \hat{c}^+ \hat{b}_{++} + \hat{c}^- \hat{c}^+ \partial_+ \hat{b} \\
&\quad + \partial_+ \left( \hat{b}^+ \hat{c}^- \right) - \partial_+ \left[ \hat{c}^+ \left( \frac{1}{2} \hat{h}_+ - \partial_+ \hat{\phi} \right) \right], \\
\bar{G}_{++} &= \hat{b}_{++}, \\
I_+ &= - \hat{b}_{++} \hat{c}^+ - \hat{b} \hat{c}^- - \hat{h}_{++} \hat{\phi}^+ + \left( \frac{1}{2} \hat{h}_+ - \partial_+ \hat{\phi} \right),
\end{align*}
\]  

(5.1)
where $\hat{T}^{grav}_{++}$ and $\hat{T}^{gh}_{++}$ are given in (2.27) and in (2.29). Then we can verify that these currents indeed form a twisted N=2 SCA like (3.16) but with the central charge $\hat{c} = 0$ by using the O.P.E.’s given in (2.25) and (2.26). Here it should be noted that the spin-1 fermionic current $G_+$ is not exactly same as the BRST current $J^{BRST}_+$ in (2.20). If, however, we set $\hat{T}^{grav}_{+-} = 0$, then the BRST current is reduced to $G_+$ up to the total divergent terms. Actually we may see that the physical states $|phys>$ satisfy $\hat{T}^{grav}_{+-}|phys> = 0$. Therefore $G_+$ may be play the same role as the BRST current effectively on the physical space. We also note that $I_+$ is not the ghost number current unlike $I_+$ in (3.15).

Recently Fujikawa and Suzuki\cite{19} have found that the two dimensional gravity coupled to $c = -2$ matter has also the twisted N=2 SCA with the central charge $\hat{c} = 0$. In this case also the spin-1 fermionic current is given by the BRST current up to total derivatives. Their analysis, however, has been done in the conformal gauge. It would be natural to expect that we may find out such a twisted N=2 SCA for the theory quantized in the light-cone gauge also. This is indeed the case and the SCA in the light-cone gauge will be found to have quite similar structure to the algebra formed by the generators in (5.1).\cite{20} The readers should refer the ref.\cite{20} for more details. It seems to be also interesting to examine the topological algebra for Jackiw-Teitelboim’s model quantized in the conformal gauge.

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