Geodesic Deviation Equation in GR equivalent theory of $f(T)$ gravity

F. Darabi\textsuperscript{1} M. Mousavi\textsuperscript{1} and K. Atazadeh\textsuperscript{2}

Department of Physics, Azarbaijan Shahid Madani University, Tabriz, 53714-161 Iran

(Dated: January 5, 2015)

In this work, we show that it is possible to study the GR equivalent notion of geodesic deviation in $f(T)$ gravity, in spite of the fact that in teleparallel gravity there is no notion of geodesics, and the torsion is responsible for the appearance of gravitational interaction. In this regard, we obtain the GR equivalent of $f(T)$ gravity whose equations are in the modified gravity form such as $f(R)$ gravity. Then, we obtain the GDE within the context of this modified gravity. In this way, the obtained geodesic deviation equation will correspond to the $f(T)$ gravity. Eventually, we extend the calculations to obtain the modification of Matting relation.

PACS numbers: 04.50.Kd, 04.20.Jb, 04.20.Cv, 95.

I. INTRODUCTION

The fundamental equation of Einstein geometrodynamics and other metric theories of gravity is the Geodesic Deviation Equation (GDE)\textsuperscript{[1]}. It connects the spacetime curvature described by the Riemann tensor with a measurable physical quantity, namely the relative acceleration between two nearby test particles. This equation describes the tendency of free falling particles to approach or recede from one another while moving under the influence of a spatially varying gravitational field. Actually the presence of this kind of tidal force will cause the trajectories to bend towards or away from each other which produces relative acceleration\textsuperscript{[2]–[3]}. Moreover, the important Raychaudhuri equation and Mattig relation may be obtained by considering GDE equation for timelike and null congruences.

One extend gravity theory beyond general relativity is teleparallel gravity (TG). The birth of this gravity theory refers back to 1928\textsuperscript{[9]}. At that time Einstein was trying to redefine the unification of gravity and electromagnetism by introducing the notion of tetrad (vierbin) field together with the suggestion of absolute parallelism. In this theory the metric $g_{\mu \nu}$ is not the dynamical object, instead we have a set of tetrad fields $e_\alpha(x^\mu)$, and instead of the well-known torsionless Levi-Civita connection of GR theory, we work with a Weitzenböck connection to introduce the covariant derivative\textsuperscript{[13]}. Furthermore, the role of curvature scalar in GR is played by torsion scalar $T$ in the teleparallel gravity.

Recently, in order to account for the observed late time cosmic acceleration, the teleparallel Lagrangian density described by the torsion scalar $T$ has been promoted to a function of $f(T)$\textsuperscript{[5–8]}. The GDE has been studied in $f(R)$ gravity theory\textsuperscript{[10], [11]}, so it is appealing to study the GDE in the context of $f(T)$ gravity, too. However, there are conceptual differences between GR and TG. In GR which is fundamentally based on the weak equivalence principle, curvature is used to geometrize the gravitational interaction and the spinless particles follow the curvature of spacetime. In other words, the concept of force is replaced by geometry and the particle trajectories are determined by geodesics, rather than the force equation. In TG, on the other hand, the torsion is responsible for the appearance of gravitational interaction as a real force. Hence, there is no notion of geodesics in TG. In spite of this conceptual difference, one can show that the teleparallel description of the gravitational interaction is completely equivalent to that of general relativity\textsuperscript{[12]}. Therefore, it is possible to cast the force equation in TG into the form of a geodesic equation in GR and obtain the corresponding GDE in TG.

In the present work, we use another approach to obtain the GDE in $f(T)$ gravity. In this regard, first we use the method introduced in Refs\textsuperscript{[16–19]} to obtain the GR equivalent of $f(T)$ gravity where the field equations are in the modified gravity form, such as $f(R)$ gravity. Then, we can benefit of the approach followed in\textsuperscript{[10]} to obtain the GDE within the context of $f(R)$ gravity. In this way, we actually obtain the GDE within the context of $f(T)$ gravity.

\textsuperscript{1}Electronic address: f.darabi@azaruniv.ac.ir
\textsuperscript{2}Electronic address: mousavi@azaruniv.ac.ir
\textsuperscript{3}Electronic address: atazadeh@azaruniv.ac.ir
II. FIELD EQUATIONS IN f(T) GRAVITY

Instead of using the torsionless Levi-Civita connection in General Relativity, we use the curvatureless Weitzenböck connection in Teleparallelism [13], whose non-null torsion $T^\rho_{\mu\nu}$ and contorsion $K^\rho_{\mu\nu}$ are defined respectively by

$$T^\rho_{\mu\nu} = \hat{\Gamma}^\rho_{\nu\mu} - \hat{\Gamma}^\rho_{\mu\nu} = e^A_\rho \left( \partial_\mu e^A_\nu - \partial_\nu e^A_\mu \right),$$

$$K^\rho_{\mu\nu} = \hat{\Gamma}^\rho_{\nu\mu} - \Gamma^\rho_{\mu\nu} = \frac{1}{2} (T^\rho_{\mu\nu} + T^\rho_{\nu\mu} - T^\rho_{\mu\nu}),$$

where $\Gamma^\rho_{\mu\nu}$ is the Levi-Civita connection. Moreover, instead of the Ricci scalar $R$ for the Lagrangian density in general relativity, the teleparallel Lagrangian density is described by the torsion scalar $T$ as follows

$$T = S^\rho_{\mu\nu} T^\rho_{\mu\nu},$$

where

$$S^\rho_{\mu\nu} = \frac{1}{2} \left( K^\mu_{\nu\rho} + \delta^\mu_{\rho} T^{\alpha\nu} \alpha - \delta^\nu_{\rho} T^{\alpha\mu} \alpha \right).$$

The modified teleparallel action for $f(T)$ gravity is given by [14]

$$S = \int d^4x |e| f(T) + \int d^4x |e| L_M,$$

where $|e| = \det (e^A_\mu) = \sqrt{-g}$ and the units have been chosen so that $16\pi G = c = 1$. Varying the action in equation (1) with respect to the vierbein vector field $e^A_\mu$, we obtain the equation [15]

$$\frac{1}{e} \partial_\mu (e S^\mu_{\nu\rho}) f_T(T) - e^A_\rho T^\rho_{\mu\nu} S^\mu_{\nu\rho} f_T(T) + S^\mu_{\nu\rho} \partial_\mu (T) f_{TT}(T) + \frac{1}{4} e^\nu_{\rho} f = \theta^\nu_{\rho},$$

where a subscript $T$ denotes differentiation with respect to $T$ and $\theta^\nu_{\rho}$ is the matter energy-momentum tensor.

On the other hand, from the relation between the Weitzenböck connection and the Levi-Civita connection given by equation (2), one can write the Riemann tensor for the Levi-Civita connection in the form

$$R^\rho_{\mu\lambda\nu} = \partial_\lambda \Gamma^\rho_{\nu\mu} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\sigma\lambda} \Gamma^\sigma_{\nu\mu} - \Gamma^\rho_{\sigma\nu} \Gamma^\sigma_{\mu\lambda},$$

whose associated Ricci tensor can then be written as

$$R_{\mu\nu} = \nabla_\nu K^\rho_{\mu\rho} - \nabla_\rho K^\rho_{\nu\mu} + K^\rho_{\sigma\nu} K^\sigma_{\mu\rho} - K^\rho_{\sigma\rho} K^\sigma_{\mu\nu}.$$  

Now, by using $K^\rho_{\mu\nu}$ given by equation (2) along with the relations $K^{(\mu\nu)}\sigma = T^{(\mu}\sigma\nu) = S^{\mu(\nu\sigma)} = 0$ and considering that $S^\rho_{\mu\nu} = 2K^\rho_{\mu\nu} = -2T^\rho_{\mu\nu}$ one has [10, 13]

$$R_{\mu\nu} = -\nabla^\rho S_{\nu\rho\mu} - g_{\mu\nu} \nabla^\sigma T^{\sigma\rho\mu} - S^{\rho\sigma}_{\mu\nu} K_{\sigma\rho\mu},$$

and thus obtain

$$G_{\mu\nu} = -\frac{1}{2} g_{\mu\nu} T = -\nabla^\sigma S_{\nu\rho\mu} - S^{\rho\sigma}_{\mu\nu} K_{\rho\sigma\mu},$$

where $G_{\mu\nu} = R_{\mu\nu} - (1/2) g_{\mu\nu} R$ is the Einstein tensor.

Finally, by using equation (10), the field equations for $f(T)$ gravity in terms of GR quantities, namely equation (6), can be rewritten in the following form [16, 19]

$$f_T G_{\mu\nu} + \frac{1}{2} \left( f(T) - T f_T \right) g_{\mu\nu} + B_{\mu\nu} f_{TT}(T) = \theta_{\mu\nu},$$

where $f_T = \frac{df(T)}{dT}$, $f_{TT}(T) = \frac{df_T}{dT}$, $B_{\mu\nu} = S_{\nu\rho\sigma} \nabla_\sigma T$ and $\theta_{\mu\nu}$ is the matter energy-momentum tensor. Now, this equation is in the form of field equation in modified gravity, such as $f(R)$ gravity.
III. GEODESIC DEVIATION EQUATION IN GR

Here we start with a little discussion about Geodesic Deviation Equation in general relativity. Imagine \( C_1 \) and \( C_2 \) are two adjacent geodesics with an affine parameter \( \nu \) on 2-surface \( S \) (see Figure 1). Two vector fields are \( V^\alpha = \frac{dx^\alpha}{d\nu} \) and \( \eta^\alpha = \frac{ds^\alpha}{ds} \) which they are tangents to the geodesics. In total we describe these geodesics with \( x^\alpha(\nu, s) \).

Starting with \( [V, \eta] = 0 \) which leads to \( \nabla_V \nabla_\nu \eta^\alpha = \nabla_V \nabla_\eta V^\alpha \) and using \( \nabla_X \nabla_\nu Z^\alpha - \nabla_\nu \nabla_X Z^\alpha - \nabla_{[X,Y]} Z^\alpha = R^\alpha_{\beta\gamma\delta} Z^\beta X^\gamma Y^\delta \) in which \( Y^\alpha = \eta^\alpha \) and \( X^\alpha = Z^\alpha = V^\alpha \), we can obtain the GDE as follows

\[
\frac{D^2 \eta^\alpha}{D\nu^2} = -R^\alpha_{\beta\gamma\delta} V^\beta \eta^\gamma V^\delta,
\]

where all indices on the manifold run over 0, 1, 2, 3, and \( e^\mu_A \) form the tangent vector on the tangent space over which the metric \( \eta_{AB} \) is defined.

As an introduction, here we review briefly the results of finding GDE in GR. We take the standard model line element (FLRW universe) including \( a(t) \) and \( k \), respectively as scale factor and spatial curvature of the universe, as

\[
ds^2 = -dt^2 + a^2(t) \left[ dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right].
\]

whose Weyl tensor is zero. We also take the energy momentum tensor in the form of a perfect fluid

\[
\Theta_{\mu\nu} = (\rho + p)u_\alpha u_\beta + p g_{\alpha\beta},
\]

where \( \rho \) is the energy density and \( p \) is the pressure. The trace of energy momentum tensor is

\[
\Theta = 3\rho - \rho.
\]

Now, we are supposed to calculate \( R \) and \( R_{\mu\nu} \) by looking at the standard form of the Einstein field equations in GR (with cosmological constant)

\[
R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa \Theta_{\mu\nu}.
\]

The Ricci scalar and Ricci tensor are obtained as follows

\[
R = \kappa(\rho - 3p) + 4\Lambda,
\]

\[
R_{\mu\nu} = \kappa(\rho + p)u_\alpha u_\beta + \frac{1}{2}[\kappa(\rho - p) + 2\Lambda] g_{\mu\nu}.
\]

Considering these expressions, the right side of GDE is found as

\[
R^\alpha_{\beta\gamma\delta} V^\beta \eta^\gamma V^\delta = \left[ \frac{1}{3}(\kappa \rho + \Lambda) \epsilon + \frac{1}{2} \kappa(\rho + p) E^2 \right] \eta^\alpha,
\]

where \( \epsilon = V^\alpha V_\alpha \) and \( E = -V_\alpha u_\alpha \). This equation is known as Pirani equation [2]. A great deal of important results from this equation have been obtained including some solutions for spacelike, timelike and null congruences [20].

IV. GEODESICS DEVIATION EQUATION IN GR EQUIVALENT OF \( f(T) \) GRAVITY

In this section, we shall obtain the GDE in GR equivalent theory of \( f(T) \) gravity. Before going through the details of calculations we extract \( R \) by taking the trace of (11) which results in

\[
R = \frac{1}{f’T} \left[ 2f(T) - 2T f(T) + f’T S^\mu_{\mu\nu} \nabla^\nu T - \kappa \Theta \right].
\]
The Riemann tensor is written as \[ R_{\mu\nu} = \frac{1}{f_T} \left[ \frac{3}{2} g_{\mu\nu} (f - T f_T + f_{T T} S_{\mu\rho} \nabla^\rho T - \kappa \Theta) - f_{T T} S_{\nu\mu} \nabla^\rho T + \kappa \Theta_{\mu\nu} \right]. \] \( \text{FIG. 1: Geodesic Deviation...} \)

Inserting this Ricci scalar into (11) we obtain

\[ R_{\alpha\beta\gamma\delta} = \frac{1}{2 (g_{\alpha\gamma} R_{\delta\beta} - g_{\alpha\delta} R_{\gamma\beta} + g_{\beta\delta} R_{\gamma\alpha} - g_{\gamma\delta} R_{\alpha\beta}) - \frac{R}{6} (g_{\alpha\gamma} g_{\delta\beta} - g_{\alpha\delta} g_{\gamma\beta}) + C_{\alpha\beta\gamma\delta}, \]

where \( C_{\alpha\beta\gamma\delta} \) is the Weyl tensor. Using this expression results in

\[ R_{\alpha\beta\gamma\delta} = \frac{1}{2 f_T} \left[ \kappa (g_{\alpha\gamma} \Theta_{\delta\beta} - g_{\alpha\beta} \Theta_{\gamma\delta} + g_{\beta\delta} \Theta_{\gamma\alpha} - g_{\gamma\delta} \Theta_{\alpha\beta}) + (f - \kappa \Theta - T f_T + S_{\mu\rho} \nabla^\rho T f_{T T})(g_{\alpha\gamma} g_{\delta\beta} - g_{\alpha\delta} g_{\gamma\beta}) \right] + \frac{1}{6 f_T} \left[ 2 f - 2 T f_T + f_{T T} S_{\mu\rho} \nabla^\rho T - \kappa \Theta \right] (g_{\alpha\gamma} g_{\delta\beta} - g_{\alpha\delta} g_{\gamma\beta}), \]

where

\[ D_{\mu\nu} = - S_{\nu\mu} \nabla^\rho T \partial_\tau. \]

After rising \( \alpha \) index in the Riemann tensor and contracting with \( V^\beta \eta^\gamma V^\delta \) we will have

\[ R_{\alpha\beta\gamma\delta} V^\beta \eta^\gamma V^\delta = \]

\[ = \frac{1}{2 f_T} \left[ \kappa (\delta^\gamma_\beta \Theta_{\delta\beta} - \delta^\gamma_\delta \Theta_{\beta\gamma} + g_{\beta\delta} \Theta^\gamma_\alpha - g_{\gamma\delta} \Theta^\gamma_\beta) + (f - \kappa \Theta - T f_T + S_{\mu\rho} \nabla^\rho T f_{T T})(\delta^\gamma_\delta g_{\beta\gamma} - \delta^\gamma_\beta g_{\delta\gamma}) \right] + \frac{1}{6 f_T} \left[ 2 f - 2 T f_T + f_{T T} S_{\mu\rho} \nabla^\rho T - \kappa \Theta \right] \times (g_{\alpha\gamma} g_{\delta\beta} - g_{\alpha\delta} g_{\gamma\beta}) V^\beta \eta^\gamma V^\delta. \]

Now, we can use this equation to obtain the GDE in GR equivalent of \( f(T) \) gravity model by using FLRW metric. Obviously, our final result will be acceptable provided that the GR limit of this model be checked.

**A. FLRW universe**

By considering the FLRW metric in the equation (25) we will find

\[ R_{\alpha\beta\gamma\delta} = \frac{1}{2 f_T} \left[ \kappa (\rho + p)(g_{\alpha\gamma} u^\gamma u_\beta - g_{\alpha\delta} u_\gamma u_\beta + g_{\beta\delta} u_\alpha u_\gamma - g_{\gamma\delta} u_\alpha) + \right. \]

\[ \left. (g_{\alpha\gamma} g_{\delta\beta} - g_{\alpha\delta} g_{\gamma\beta}) \left( \frac{2 \kappa \rho}{3} - T f_T / 3 + \frac{2}{3} S_{\mu\rho} \nabla^\rho T f_{T T} + \frac{f}{3} \right) + (g_{\alpha\gamma} D_{\delta\beta} - g_{\alpha\delta} D_{\gamma\beta} + g_{\beta\delta} D_{\gamma\alpha} - g_{\gamma\delta} D_{\alpha\beta}) f_T \right]. \]
Under the condition of vector field normalization, we have $V^\alpha V_\alpha = \epsilon$ and

$$R_{\alpha\beta\gamma\delta}V^\beta V^\delta = \frac{1}{2f_T}[(\kappa(\rho + p)(u_\beta V^\beta)^2 - 2(u_\beta V^\beta)\bar{V}_{(\alpha}u_{\gamma)} + \epsilon u_\alpha u_\gamma) + \left(\frac{2\kappa}{3} - \frac{T f_T}{3} + \frac{2}{3}S^\mu_{\mu\rho}\nabla^\rho T f_{TT} + \frac{f}{3}\right)(\epsilon g_{\alpha\gamma} - V_\alpha V_\gamma) + [(g_{\alpha\gamma}D_\delta - g_{\alpha\delta}D_\gamma + g_{\beta\delta}D_{\gamma\alpha} - g_{\beta\gamma}D_{\alpha\delta})f_T]V^\beta V^\delta].$$

Again we rise the first index then contract with $\eta^\gamma$

$$R^\alpha_{\beta\gamma\delta}V^\beta V^\gamma V^\delta = \frac{1}{2f_T}[\kappa(\rho + p)((u_\beta V^\beta)^2\eta^\alpha - (u_\beta V^\beta)V^\alpha(u_\gamma\eta^\gamma) - (u_\beta V^\beta)u^\alpha(V_\eta^\gamma) + \epsilon u_\alpha u_\gamma\eta^\gamma) + \left(\frac{2\kappa}{3} - \frac{T f_T}{3} + \frac{2}{3}S^\mu_{\mu\rho}\nabla^\rho T f_{TT} + \frac{f}{3}\right)(\epsilon \eta^\alpha - V_\alpha(V_\eta^\gamma)) + [(\delta^\alpha_{\gamma\delta}D_{\beta\delta} - \delta^\alpha_{\delta\beta}D_{\gamma\delta} + g_{\beta\delta}D^\alpha_\gamma - g_{\beta\gamma}D^\alpha_\delta)f_T]V^\beta V^\delta \eta^\gamma].$$

By considering $E = -V_\alpha u^\alpha$ and $\eta_\alpha u^\alpha = \eta_\alpha V^\alpha = 0$, equation (28) converts into

$$R^\alpha_{\beta\gamma\delta}V^\beta V^\gamma V^\delta = \frac{1}{2f_T}[\kappa(\rho + p)E^2 + \epsilon \left(\frac{2\kappa}{3} - \frac{T f_T}{3} + \frac{2}{3}S^\mu_{\mu\rho}\nabla^\rho T f_{TT} + \frac{f}{3}\right)\eta^\alpha + \frac{1}{2f_T}\left[(\delta^\alpha_{\gamma\delta}D_{\beta\delta} - \delta^\alpha_{\delta\beta}D_{\gamma\delta} + g_{\beta\delta}D^\alpha_\gamma - g_{\beta\gamma}D^\alpha_\delta)f_T\right]V^\beta V^\delta \eta^\gamma].$$

According to the second section, the expression for the Torsion scalar for the flat FLRW metric reduces to

$$T = -6H^2,$$

where $H = \dot{a}$ is the Hubble parameter. Apparently, the Torsion scalar is thoroughly time-dependant, thus we are supposed to be concerned just about time derivatives of $T$ in $D_{\mu\nu}$.

The vector field normalization implies that $V_\alpha V^\alpha = \epsilon$ and also we have $E = -V_\alpha u^\alpha$, $\eta_\alpha u^\alpha = \eta_\alpha V^\alpha = 0$, $\eta_0 u^0 = 0$. Not only these mentioned conditions but also the non-vanishing components of $S$ tensor will be used to extract the final result for the action of operator $D_{\mu\nu}$ on $f_T$. Therefore, after cumbersome calculations we obtain

$$\frac{1}{2f_T}\left((\delta^\alpha_{\gamma\delta}D_{\beta\delta} - \delta^\alpha_{\delta\beta}D_{\gamma\delta} + g_{\beta\delta}D^\alpha_\gamma - g_{\beta\gamma}D^\alpha_\delta)f_T\right) V^\beta V^\delta \eta^\gamma = \frac{-\epsilon}{2f_T}S^\alpha_{a0}\nabla^0 T f_{TT} \eta^\alpha = \frac{-6\epsilon}{f_T}H^2 \dot{H} f_{TT} \eta^\alpha,$$

where $S^1_{10} = S^2_{20} = S^3_{30} = H(t)$. Thus, we can write the reduced expression for $R^\alpha_{\beta\gamma\delta}V^\beta V^\gamma V^\delta$ as follows

$$R^\alpha_{\beta\gamma\delta}V^\beta V^\gamma V^\delta = \frac{1}{2f_T}[\kappa(\rho + p)E^2 + \epsilon \left(\frac{K\rho}{3} + 24H^2 \dot{H} f_{TT} - \frac{f}{6}\right)\eta^\alpha],$$

which is the generalized Pirani equation. Now, we can write the GDE in $f(T)$ gravity model

$$\frac{D^2 \eta^\alpha}{Dv^2} = -\frac{1}{2f_T}[\kappa(\rho + p)E^2 + \epsilon \left(\frac{K\rho}{3} + 24H^2 \dot{H} f_{TT} - \frac{f}{6}\right)\eta^\alpha].$$

Apparently, as a result of homogeneity and isotropy of FLRW metric, in this equation only the magnitude of the deviation vector $\eta^\alpha$ is changed along the geodesics. Whereas in anisotropic universe, like Bianchi I, we can also infer a change in the direction of the deviation vector, as described in [21].
B. Fundamental observers

Here, we are going to limit ourselves to the fundamental observers. In this particular case, we interpret \( V^\alpha \) and \( \nu \) (the affine parameter) as the four-velocity of the fluid \( u^\alpha \) and \( t \) (the proper time), respectively. Since we are treating with temporal geodesics we have \( \epsilon = -1 \) and also we fix the vector field normalization by \( E = 1 \) which leads to

\[
R^\alpha_{\beta\gamma\delta} u^\beta \eta^\gamma u^\delta = \frac{1}{2f_T} \left[ \frac{2\kappa \rho}{3} + \kappa p - 24H^2 \dot{H} f_{TT} + \frac{f}{6} \right] \eta^\alpha. \tag{34}
\]

We know that if \( \eta^\alpha = \ell e^\alpha \), where \( e^\alpha \) is parallel propagated along \( t \), then the isotropy results in

\[
\frac{De^\alpha}{Dt} = 0, \tag{35}
\]

from which we have

\[
\frac{D^2 \eta^\alpha}{Dt^2} = \frac{d^2 \ell}{dt^2} e^\alpha. \tag{36}
\]

By using (12) and (34) we can write

\[
\frac{d^2 \ell}{dt^2} = -\frac{1}{2f_T} \left[ \frac{2\kappa \rho}{3} + \kappa p - 24H^2 \dot{H} f_{TT} + \frac{f}{6} \right] \ell, \tag{37}
\]

which for the particular case \( \ell = a(t) \) leads to

\[
\frac{\ddot{a}}{a} = \frac{1}{f_T} \left[ -\frac{\kappa \rho}{3} - \frac{\kappa p}{2} + 12H^2 \dot{H} f_{TT} - \frac{f}{12} \right]. \tag{38}
\]

This equation is nothing but a special case of the generalized Raychaudhuri equation. It is worth to mention here that the above generalized Raychaudhuri equation can be obtained by the standard forms of the modified Friedmann equations in \( f(T) \) gravity model for flat universe \[15\].

\[
H^2 = \frac{\kappa}{3} \left( \rho + \frac{1}{2\kappa} (2Tf_T - f - T) \right), \tag{39}
\]

\[
2\dot{H} + 3H^2 = -\kappa p - 4\dot{H} f_{TT} - 2\dot{H} f_T + 2\dot{H} + T f_T - \frac{f}{2} - \frac{T}{2}. \tag{40}
\]

The consistency between the modified Friedmann equations in \( f(T) \) gravity for flat universe \[15\] and the generalized Raychaudhuri equation for flat universe \[33\] confirms the fact that the approach followed here is a correct one.

C. Null vector fields

In this section, we consider the null past directed vector fields, namely \( V^\alpha = k^\alpha, k_\alpha k^\alpha = 0 \), for which the equation \[32\] reduces to

\[
R^\alpha_{\beta\gamma\delta} k^\beta \eta^\gamma k^\delta = \frac{1}{2f_T} \kappa (\rho + p) E^2 \eta^\alpha. \tag{41}
\]

Actually, this is Ricci focusing in \( f(T) \) gravity as is explained in the following. By considering \( \eta^\alpha = \eta e^\alpha \), \( e_\alpha e^\alpha = 1 \), \( e_\alpha u^\alpha = e_\alpha k^\alpha = 0 \) and also writing an aligned base parallel propagated \( \frac{De^\alpha}{Dv} = k^\beta \nabla_\beta e^\alpha = 0 \), we obtain the null GDE \[33\] new form as follows

\[
\frac{d^2 \eta}{dv^2} = -\frac{1}{2f_T} \kappa (\rho + p) E^2 \eta. \tag{42}
\]
According to the GR studied in [2] all classes of past-directed null geodesics experience focusing if we have $\kappa(\rho + p) > 0$. Therefore, in a particular case with equation of state $p = -\rho$ (cosmological constant) we can not recognize any focusing effect. Obviously [2] shows the focusing condition for $f(T)$ gravity model provided that

$$\frac{\kappa(\rho + p)}{f_T} > 0. \quad (43)$$

Now, we have an expression [2] which can be written in terms of the redshift parameter $z$. To do this, we may write

$$\frac{d}{d\nu} = \frac{dz}{d\nu} \frac{d}{dz}, \quad (44)$$

which results in

$$\frac{d^2}{d\nu^2} = \left(\frac{d\nu}{dz}\right)^{-2} \left[ - \left(\frac{d\nu}{dz}\right)^{-1} \frac{d^2\nu}{dz^2} \frac{d}{dz} \right]. \quad (45)$$

Let us consider the null geodesics for which we have

$$(1 + z) = \frac{a_0}{a} = \frac{E}{E_0} \rightarrow \frac{dz}{1 + z} = - \frac{da}{a}. \quad (46)$$

Choosing $a_0 = 1$ (the current day value of the scale factor), leads to the following result for the past-directed case

$$dz = (1 + z) \frac{1}{a} \frac{da}{d\nu} = (1 + z) \frac{\dot{a}}{a} E d\nu = E_0 H (1 + z)^2 d\nu. \quad (47)$$

Thus we obtain

$$\frac{d\nu}{dz} = \frac{1}{E_0 H (1 + z)^3}, \quad (48)$$

and so

$$\frac{d^2\nu}{dz^2} = - \frac{1}{E_0 H (1 + z)^3} \left[ \frac{1}{H} (1 + z) \frac{dH}{dz} + 2 \right], \quad (49)$$

where

$$\frac{dH}{dz} = \frac{d\nu}{dz} \frac{dt}{d\nu} \frac{dH}{dt} = - \frac{1}{H(1 + z)} \frac{dH}{dt}. \quad (50)$$

where use has been made of $\frac{dt}{d\nu} = E = E_0 (1 + z)$. From the definition of Hubble parameter we can write

$$\dot{H} = \frac{\ddot{a}}{a} - H^2. \quad (51)$$

Using (48), $\dot{H}$ becomes

$$\dot{H} = \frac{1}{f_T} \left[ \frac{-\kappa \rho}{3} - \frac{\kappa p}{2} + 12 \dot{H} H^2 f_T - \frac{f}{12} \right], \quad (52)$$

thus

$$\frac{d^2\nu}{dz^2} = - \frac{3}{E_0 H (1 + z)^3} \left[ 1 + \frac{1}{3H^2 f_T} \left( \frac{\kappa \rho}{3} + \frac{\kappa p}{2} - 12 \dot{H} H^2 f_T + \frac{f}{12} \right) \right]. \quad (53)$$
Putting this result in (55), we obtain
\[
\frac{d^2 \eta}{dz^2} = E_0 H(1+z)^2 \left[ \frac{d^2 \eta}{dz^2} + \frac{3}{(1+z)} \left[ 1 + \frac{1}{3H^2} \left( \frac{\kappa \rho}{3} + \frac{\kappa p}{2} - 12 \dot{H} H^2 f_{TT} + \frac{f}{12} \right) \right] \frac{d \eta}{dz} \right].
\]  
Finally, using (52) the null GDE takes the following form
\[
\frac{d^2 \eta}{dz^2} + \frac{3}{(1+z)} \left[ 1 + \frac{1}{3H^2} \left( \frac{\kappa \rho}{3} + \frac{\kappa p}{2} - 12 \dot{H} H^2 f_{TT} + \frac{f}{12} \right) \right] \frac{d \eta}{dz} + \frac{\kappa(\rho + p)}{2H^2(1+z)^2 f_T} \eta = 0.
\]  
Matter and radiation contributions to \( \rho \) and \( p \) can be written respectively as
\[
\kappa \rho = 3H_0^2 \Omega_{m0}(1+z)^3 + 3H_0^2 \Omega_{r0}(1+z)^4, \quad \kappa p = H_0^2 \Omega_{r0}(1+z)^4,
\]  
where we have used \( p_m = 0 \) and \( p_r = \frac{4}{3} \rho_r \). By using equation (56) the null GDE equation (55) can be written as
\[
\frac{d^2 \eta}{dz^2} + P(H,T,z) \frac{d \eta}{dz} + Q(H,T,z) \eta = 0,
\]  
where
\[
P(H,T,z) = \frac{\Omega_{m0}(1+z)^3 + \frac{4}{3} \Omega_{r0}(1+z)^4 + \frac{f}{12H^2} + (3f_T + \frac{T f_{TT}}{H})[\Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{DE}]}{f_T(1+z)[\Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{DE}]},
\]  
\[
Q(H,T,z) = \frac{3\Omega_{m0}(1+z) + 4\Omega_{r0}(1+z)^2}{2f_T[\Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{DE}]},
\]  
in which we have applied the following new form of (39)
\[
H^2 = H_0^2[\Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{DE}],
\]  
where \( \Omega_{DE} \) has been defined as
\[
\Omega_{DE} = \frac{1}{H_0^2} \left( \frac{T f_T}{3} - \frac{f + T}{6} \right) \equiv \Omega_{\Lambda}.
\]  
Note that in solving equation (57), we must use (39). In order to check for the agreement of the above results with those of GR, we take the particular case \( f(T) = T - 2\Lambda \). As a result of this choice we have \( f_T = 1 \) and \( f_{TT} = 0 \). Furthermore \( \Omega_{DE} \) reduces to
\[
\Omega_{DE} = \frac{1}{H_0^2} \left( \frac{T - 2\Lambda}{3} - \frac{T - 2\Lambda + T}{6} \right) = \frac{\Lambda}{3H_0^2} \equiv \Omega_{\Lambda}.
\]  
which can be written as the Friedmann equation in GR
\[
H^2 = H_0^2[\Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{\Lambda}],
\]  
Hence we find the reduced expressions \( P \) and \( Q \) as follows
\[
P(z) = \frac{7\Omega_{m0}(1+z)^3 + 4\Omega_{r0}(1+z)^4 + 2\Omega_{\Lambda}}{(1+z)[\Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{\Lambda}]},
\]  
\[
Q(z) = \frac{3\Omega_{m0}(1+z) + 4\Omega_{r0}(1+z)^2}{2[\Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{\Lambda}]}.
\]  
Eventually, the GDE for null vector fields becomes
\[
\frac{d^2 \eta}{dz^2} + \frac{\frac{7}{2} \Omega m_0 (1+z)^3 + 4 \Omega r_0 (1+z)^4 + 2 \Omega_A}{(1+z)[\Omega m_0 (1+z)^3 + \Omega r_0 (1+z)^4 + \Omega_A]} \frac{d\eta}{dz} + \frac{3 \Omega m_0 (1+z) + 4 \Omega r_0 (1+z)^2}{2(\Omega m_0 (1+z)^3 + \Omega r_0 (1+z)^4 + \Omega_A)} \eta = 0. \tag{66}
\]

In order to obtain the Mattig relation in GR \cite{22}, we have to fix \( \Omega = 0, \Omega r_0 + \Omega m_0 = 1 \) which lead to

\[
\frac{d^2 \eta}{dz^2} + \frac{\frac{7}{2} \Omega m_0 (1+z)^3 + 4 \Omega r_0 (1+z)^4}{(1+z)[\Omega m_0 (1+z)^3 + \Omega r_0 (1+z)^4]} \frac{d\eta}{dz} + \frac{3 \Omega m_0 (1+z)}{2(\Omega m_0 (1+z)^3 + \Omega r_0 (1+z)^4)} \eta = 0. \tag{67}
\]

So, the equation \cite{57} gives us an opportunity to generalize the Mattig relation in \( f(T) \) gravity. By considering the last result, we can infer the following expression for the observer area distance \( r_0(z) \) \cite{22}:

\[
r_0(z) = \sqrt{\frac{dA_0(z)}{d\Omega}} = \left| \frac{\eta(z')}{d\eta'/dz'|_{z'=0}} \right|, \tag{68}
\]

where \( A_0 \) is the area of the object and also \( \Omega \) is the solid angle. Equipped with the \( d/d\ell = E_0^{-1}(1+z)^{-1}d/d\nu = H(1+z)d/\nu = H(1+z)d/dz \) and setting the deviation at \( z = 0 \) to zero, clearly we have

\[
r_0(z) = \left| \frac{\eta(z)}{H(0)d\eta'/dz'|_{z'=0}} \right|, \tag{69}
\]

where \( H(0) \) is the evaluated modified Friedmann equation \cite{02} at \( z = 0 \).

To solve null GDE in \( f(T) \) gravity, for simplicity we assume \( \dot{T} = 0 \) which is equivalent to a constant \( T = -6H^2 = T_0 \). Thus, equation \cite{56} can be rewritten as follows

\[
\frac{d^2 \eta}{dz^2} + P(H, T_0, z) \frac{d\eta}{dz} + Q(H, T_0, z) \eta = 0, \tag{70}
\]

where

\[
P(H, T_0, z) = \frac{\Omega m_0 (1+z)^3 + \frac{3}{2} \Omega r_0 (1+z)^4 + \frac{f(T_0)}{12H^2} + 3f_T(T_0)[\Omega m_0 (1+z)^3 + \Omega r_0 (1+z)^4 + \Omega_{DE}(T_0)]}{f_T(T_0)(1+z)[\Omega m_0 (1+z)^3 + \Omega r_0 (1+z)^4 + \Omega_{DE}(T_0)]}, \tag{71}
\]

\[
Q(H, T_0, z) = \frac{3 \Omega m_0 (1+z) + 4 \Omega r_0 (1+z)^2}{2f_T(T_0)[\Omega m_0 (1+z)^3 + \Omega r_0 (1+z)^4 + \Omega_{DE}(T_0)]}, \tag{72}
\]

and

\[
\Omega_{DE}(T_0) = \frac{1}{H_0^2} \left( \frac{T_0 f_T(T_0)}{3} - \frac{(f(T_0) + T_0)}{6} \right). \tag{73}
\]

To solve equation \cite{70}, one can choose the functional form of the \( f(T_0) \) as follows

\[
f(T_0) = \alpha T_0 + \beta T_0^2, \tag{74}
\]

where \( \alpha \) and \( \beta \) are constants. We can solve equation \cite{70} numerically which results in \( \eta(z) \) plotted in Figure 2, also \( r_0(z) \) is plotted as a function of \( z \).

\[\text{V. CONCLUSIONS}\]

In this paper, we have considered Geodesic Deviation Equation(GDE) in the GR equivalent of \( f(T) \) gravity model. First, we have calculated the Ricci tensor and the Ricci scalar with the modified field equations in \( f(T) \) gravity theory. Then, in FLRW universe, the Geodesic Deviation Equation corresponding to these GR equivalent quantities of \( f(T) \) gravity is obtained. To show the consistency of our approach in constructing the GR equivalent of \( f(T) \) gravity, the generalized GDE and Pirani equations are recovered for \( f(T) = T - 2\Lambda \). We restrict our attention to extract the GDE for two special cases, namely the fundamental observers and past directed null vector fields. In these two cases we find the Raychaudhuri equation, the generalized Mattig relation and the diametral angular distance differential equation for \( f(T) \) gravity theory.
FIG. 2: Plot of the deviation vector magnitude η(z) (left) and plot of the observer area distance r_0(z) (right). The parameter values chosen are H_0 = 80 km/s/Mpc, Ω_m0 = 0.2, Ω_r0 = 0.1 and dη(z)/dz|_{z=0} = 0.1.

Acknowledgment

This work has been supported by a grant/research fund number 217/D/10976 from Azarbaijan Shahid Madani University.

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