Subexponential tail equivalence of the stationary queue length distributions of BMAP/GI/1 queues with and without retrials

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Abstract

The main contribution of this paper is to prove the subexponential tail equivalence of the stationary queue length distributions in the BMAP/GI/1 queues with and without retrials. We first present a stochastic-decomposition-like result of the stationary queue length in the BMAP/GI/1 retrial queue, which is an extension of the stochastic decomposition of the stationary queue length in the M\(^X\)/GI/1 retrial queue. The stochastic-decomposition-like result shows that the stationary queue length distribution in the BMAP/GI/1 retrial queue is decomposed into two parts: the stationary conditional queue length distribution given that the server is idle; and a certain matrix sequence associated with the stationary queue length distribution in the corresponding standard BMAP/GI/1 queue (without retrials). Using the stochastic-decomposition-like result and matrix analytic methods, we prove the subexponential tail equivalence of the stationary queue length distributions in the BMAP/GI/1 queues with and without retrials. This tail equivalence result does not necessarily require that the size of an arriving batch is light-tailed, unlike Yamamuro’s result for the M\(^X\)/GI/1 retrial queue (Queueing Syst. 70:187–205, 2012). As a by-product, the key lemma to the proof of the main theorem presents a subexponential asymptotic formula for the stationary distribution of a level-dependent M/G/1-type Markov chain, which is the first reported result on the subexponential asymptotics of level-dependent block-structured Markov chains.

Keywords: BMAP/GI/1 retrial queue; Subexponential asymptotics; Tail equivalence; Stochastic decomposition; Queue length distribution; level-dependent M/G/1-type Markov chain

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1 Introduction

Retrial queues are queueing models such that a customer finding all the servers busy on arrival joins the virtual waiting line (called orbit) and retries to occupy an idle server after a random time (this process is repeated until the customer finds an idle server and occupies it). Many

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researchers have studied retrial queues for more than a half of century since the early studies, e.g., [7, 25]. However, the analytical results of retrial queues are less extensive than those of standard (work-conserving and non-preemptive) queueing models without retrials. In particular, exact (that is, not approximate) solutions have been derived for a few simple models such as M/M/c (c = 1, 2, 3, 4) retrial queues (see [10, 13, 36, 37]). For detailed overview, see the survey papers [9, 45] and the books [3, 10]. See also the bibliographies [1, 2] and the references therein.

Recently, the asymptotic analysis has been a hot topic in the study of retrial queues. Liu and Zhao [28] and Kim et al. [22] study the light-tailed asymptotics of the stationary queue length distribution in the M/M/c retrial queue. These results are extended to an M/M/c retrial queue with non-persistent customers [18, 27]. Kim et al. [20] study the light-tailed asymptotics of the stationary queue length distribution in an M/GI/1 retrial queue with exponential retrials, which is generalized to the Markovian arrival case by Kim et al. [19].

As for the subexponential asymptotics, there are a few studies. Before reviewing them, we introduce the subexponential class of distributions and related ones.

**Definition 1.1**

(i) The nonnegative random variable U and its distribution $F_U$ are said to be heavy-tailed (denoted by $U \in \mathcal{H}$ and $F_U \in \mathcal{H}$) if $\lim_{x \to \infty} e^{\varepsilon x} P(U > x) = \infty$ for all $\varepsilon > 0$.

(ii) The nonnegative random variable U and its distribution $F_U$ are said to be long-tailed (denoted by $U \in \mathcal{L}$ and $F_U \in \mathcal{L}$) if $P(U > x) > 0$ for all $x \geq 0$ and

$$\lim_{x \to \infty} \frac{P(U > x + y)}{P(U > x)} = 1 \quad \text{for some (thus all) } y > 0.$$

(iii) The nonnegative random variable U and its distribution $F_U$ are said to be subexponential (denoted by $U \in \mathcal{S}$ and $F_U \in \mathcal{S}$) if $P(U > x) > 0$ for all $x \geq 0$ and

$$\lim_{x \to \infty} \frac{P(U_1 + U_2 > x)}{P(U > x)} = 2,$$

where $U_i$’s ($i = 1, 2, \ldots$) are independent copies of U.

(iv) The nonnegative random variable U and its distribution $F_U$ belong to class $\mathcal{R}(-\alpha)$ ($\alpha \geq 0$) if $P(U > x)$ is regularly varying with index $-\alpha$, i.e.,

$$\lim_{x \to \infty} \frac{P(U > vx)}{P(U > x)} = v^{-\alpha} \quad \text{for all } v > 0.$$

It is known that $U_{\alpha \geq 0} \mathcal{R}(-\alpha) \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{H}$. In particular, class $\mathcal{S}$ is the largest tractable subclass of heavy-tailed distributions, and it includes heavy-tailed Weibull, lognormal, Burr, loggamma distributions, and Pareto distributions, etc. For further details, see [11, 12, 40].

We now review the literature on the subexponential asymptotics of retrial queues. Kim et al. [21] consider an M/GI/1 retrial queue with exponential retrials and the service time distribution in $\mathcal{R}(-\beta)$, where $\beta > 1$. For this retrial queue, the authors show that the waiting time
distribution belongs to class $\mathcal{R}(-\beta+1)$. Shang et al. [39] also consider the M/GI/1 retrial queue with exponential retrials, and they prove the subexponential tail equivalence of the stationary queue length distributions in the M/GI/1 queues with and without retrials. In order to specify this tail equivalence result, we denote by $L^{(\mu)}$ the stationary queue length in the M/GI/1 retrial queue with exponential retrial rate $\mu$, and denote by $L^{(\infty)}$ the stationary queue length in the corresponding standard M/GI/1 queue (it is shown that $L^{(\mu)}$ converges to $L^{(\infty)}$ in distribution as $\mu \to \infty$; see Theorem 1.8 in [10]). In this setting, Shang et al. [39]'s result is stated as follows: If $L^{(\infty)} \in \mathcal{S}$, then

$$P(L^{(\mu)} > x) \sim P(L^{(\infty)} > x),$$

(1.1)

where $f(x) \sim g(x)$ represents $\lim_{x \to \infty} f(x)/g(x) = 1$. Note here that $L^{(\infty)} \in \mathcal{S}$ and (1.1) imply $L^{(\mu)} \in \mathcal{S}$ (see, e.g., [40, Proposition 2.8]). Yamamuro [44] extends the tail equivalence (1.1) to the batch arrival model, i.e., M$^X$/GI/1 retrial queue with exponential retrials, though the batch size distribution is assumed to be light-tailed.

This paper considers a BMAP/GI/1 retrial queue with exponential retrials, where BMAP represents batch Markovian arrival process [30]. The main contribution of this paper is to prove the subexponential tail equivalence of the stationary queue length distributions in the BMAP/GI/1 queues with and without retrials, which is an extension of Yamamuro [44]'s result.

To prove the main result of this paper, we first present a stochastic-decomposition-like result of the stationary queue length, which is a generalization of the stochastic decomposition for the M$^X$/GI/1 retrial queue with exponential retrials [44]. The stochastic-decomposition-like result shows that the stationary queue length distribution in a BMAP/GI/1 retrial queue with exponential retrials is decomposed into two parts. The first part is the stationary conditional queue length distribution given that the server is idle. On the other hand, the second part itself does not have a probabilistic interpretation. However, pre-multiplying the second part by a certain probability vector, we have the stationary queue length distribution in the corresponding standard BMAP/GI/1 queue (without retrials).

Next we prove the main theorem on the subexponential tail equivalence by combining the stochastic-decomposition-like result with matrix analytic methods [14, 26, 35]. The key to the proof of the main theorem is to discuss the tail asymptotics of the stationary conditional queue length distribution given that the server is idle, which is reduced, by change of measure, to the subexponential asymptotics of a level-dependent M/G/1-type Markov chain with asymptotic level-independence. To the best of our knowledge, there are no studies on the subexponential asymptotics of level-dependent block-structured Markov chains. In addition, the main theorem of this paper is proved without Yamamuro [44]'s assumption mentioned above, i.e., the light-tailedness of the batch size distribution.

The rest of this paper is divided into three sections. Section 2 introduces basic definitions, notation and preliminary results. Section 3 presents the main theorem. Section 4 is devoted to the proof of a lemma, which is key to prove the main theorem.
2 Preliminary

2.1 Basic definitions and notation

Let $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ and $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Let $\mathbf{e}$ and $\mathbf{I}$ denote the column vector of ones and the identity matrix, respectively, with appropriate dimensions according to the context. The superscript “t” represents the transpose operator for vectors and matrices. The notation $[\cdot]_{i,j}$ (resp. $[\cdot]_i$) represents the $(i,j)$th (resp. $i$th) element of the matrix (resp. vector) in the square brackets.

For any matrix $\mathbf{M}$, let $|\mathbf{M}|$ denote a matrix obtained by taking the absolute value of each element of $\mathbf{M}$, i.e., $[|\mathbf{M}|]_{i,j} = |[\mathbf{M}]_{i,j}|$. For any matrix sequence $\{\mathbf{M}(k); k \in \mathbb{Z}\}$, we define $\overline{\mathbf{M}}(k)$ and $\overline{\overline{\mathbf{M}}}(k)$ ($k \in \mathbb{Z}$) as

$$
\overline{\mathbf{M}}(k) = \sum_{l=k+1}^{\infty} \mathbf{M}(l), \quad \overline{\overline{\mathbf{M}}}(k) = \sum_{l=k+1}^{\infty} \overline{\mathbf{M}}(l),
$$

respectively. We then define the convolution of two matrix sequences $\{\mathbf{M}(k); k \in \mathbb{Z}\}$ and $\{\mathbf{N}(k); k \in \mathbb{Z}\}$ as follows:

$$
\mathbf{M} \ast \mathbf{N}(k) = \sum_{l \in \mathbb{Z}} \mathbf{M}(k-l)\mathbf{N}(l), \quad k \in \mathbb{Z},
$$

where the product $\mathbf{M}(k_1)\mathbf{N}(k_2)$ is well-defined for all $k_1, k_2 \in \mathbb{Z}$. For any square matrix sequence $\{\mathbf{M}(k); k \in \mathbb{Z}\}$, we also define $\{\mathbf{M}^{*m}(k); k \in \mathbb{Z}\}$ ($m \in \mathbb{N}$) as the $m$-fold convolution of $\{\mathbf{M}(k)\}$ with itself, i.e.,

$$
\mathbf{M}^{*m}(k) = \sum_{l \in \mathbb{Z}} \mathbf{M}^{*(m-1)}(k-l)\mathbf{M}(l), \quad k \in \mathbb{Z},
$$

where $\mathbf{M}^{*0}(0) = \mathbf{I}$ and $\mathbf{M}^{*0}(k) = \mathbf{O}$ for $k \in \mathbb{Z} \setminus \{0\}$. In addition, for two matrix-valued functions $\mathbf{M}_1(\cdot)$ and $\mathbf{M}_2(\cdot)$ with the same dimension, the notation $\mathbf{M}_1(x) \overset{\sim}{\sim} \mathbf{M}_2(x)$ represents $[\mathbf{M}_1(x)]_{i,j} \overset{\sim}{\sim} [\mathbf{M}_2(x)]_{i,j}$, i.e.,

$$
\lim_{x \to \infty} \frac{[\mathbf{M}_1(x)]_{i,j}}{[\mathbf{M}_2(x)]_{i,j}} = 1 \quad \text{for all } i \text{'s and } j \text{'s}.
$$

The above definitions and notation for matrices are applied to vectors and scalars in an appropriate manner.

2.2 Subexponential asymptotics for BMAP/GI/1 queue without retrials

We first introduce the BMAP. Behind the BAMP, there exists a continuous-time Markov chain with a finite state space $\mathbb{M} := \{1, 2, \ldots, M\}$, which is called the background Markov chain (or the underlying Markov chain). Let $\{J(t); t \geq 0\}$ denote the background Markov chain of
the BMAP. Let \( N(t) (t \geq 0) \) denote the total number of arrivals in time interval \( (0, t] \), where \( N(0) = 0 \) is assumed.

For simplicity, we denote by \( E_{i0} \) an appropriate real-valued function on \([0, \infty)\) such that \( \lim_{x \to 0^+} E_{i0}(x)/x = 0 \). It then follows by definition (see, e.g., [30]) that the stochastic process \( \{(N(t), J(t)); t \geq 0\} \) evolves as follows:

\[
P(N(t + \Delta t) - N(t) = k, J(t) = j \mid J(0) = i) = \begin{cases} 
1 + [C]_{i,i} \Delta t + E_{i0}(\Delta t), & k = 0, i = j \in \mathbb{M}, \\
[C]_{i,j} \Delta t + E_{i0}(\Delta t), & k = 0, i \neq j, i, j \in \mathbb{M}, \\
[D(k)]_{i,j} \Delta t + E_{i0}(\Delta t), & k \in \mathbb{N}, i, j \in \mathbb{M},
\end{cases}
\]  

(2.1)

where \( D(k) \) \((k \in \mathbb{N})\) is an \( M \times M \) nonnegative matrix and \( C \) is an \( M \times M \) matrix such that \([C]_{i,i} < 0 \) \((i \in \mathbb{M})\), \([C]_{i,j} \geq 0 \) \((i \neq j, i, j \in \mathbb{M})\) and \((C + \sum_{k=1}^{\infty} D(k))e = 0\). The BMAP characterized in (2.1) is denoted by BMAP \( \{C, D(k); k \in \mathbb{N}\} \).

Let \( \hat{D}(z) = \sum_{k=1}^{\infty} z^k D(k) \). From (2.1), we then have

\[
E[z^{N(t)} \cdot \mathbb{I}(J(t) = j) \mid J(0) = i] = [e^{(C+\hat{D}(z))t}]_{i,j},
\]

(2.2)

where \( \mathbb{I}(\cdot) \) denotes an indicator function that takes value of one if the statement in the parentheses is true; and takes value of zero otherwise. Note here that the infinitesimal generator of the background Markov chain \( \{J(t); t \geq 0\} \) is given by \( C + D \), where \( D = \sum_{k=1}^{\infty} D(k) \).

We assume that \( C + D \) is irreducible, and then define \( \pi > 0 \) as the unique stationary probability vector of \( C + D \). We also define \( \lambda \) as the mean arrival rate, i.e.,

\[
\lambda = \pi \sum_{k=1}^{\infty} kD(k)e = \pi \sum_{k=0}^{\infty} D(k)e.
\]

(2.3)

To exclude trivial cases, we assume \( \lambda > 0 \), which implies that

\[
D(k_0) \geq O, \neq O \quad \text{for some } k_0 \in \mathbb{N}.
\]

(2.4)

Next we describe a standard BMAP/GI/1 queue, i.e., BMAP/GI/1 queue without retrials. The system has a single server and a buffer of infinite capacity. Customers arrive at the system according to BMAP \( \{C, D(k); k \in \mathbb{N}\} \). If customers arriving in a batch find the server idle, then one of them immediately occupies the server and the others join the waiting line; otherwise all of them join the waiting line. We assume that the service times of customers are independent of BMAP \( \{C, D(k); k \in \mathbb{N}\} \) and independent and identically distributed (i.i.d.) according to a general distribution function \( H \) on \([0, \infty)\) with mean \( h \in (0, \infty) \).

We define \( \rho \) as the traffic intensity, i.e.,

\[
\rho = \lambda h.
\]

(2.5)

We also define \( A(k) \) \((k \in \mathbb{Z}_+)\) as an \( M \times M \) matrix such that

\[
[A(k)]_{i,j} = P(N(T) = k, J(T) = j \mid J(0) = i), \quad i, j \in \mathbb{M},
\]
where $T$ denotes a generic random variable for i.i.d. service times with distribution function $H$. It follows from (2.2) that

$$
\hat{A}(z) := \sum_{k=0}^{\infty} z^k A(k) = \int_{0}^{\infty} e^{(C+D)x} \, dH(x).
$$

Note here that $A := \hat{A}(1) = \int_{0}^{\infty} e^{(C+D)x} \, dH(x) > O$ and $Ae = e$ because $C + D$ is an irreducible infinitesimal generator. Thus $A$ has the unique stationary probability vector, which is equal to $\pi$. Note also that

$$
\rho = \pi \sum_{k=1}^{\infty} k A(k) e.
$$

Throughout this paper, we assume $\rho < 1$, which is the stability condition for the standard BMAP/GI/1 queue (see [29]). We then summarize the results on the subexponential asymptotics of the stationary queue length distribution in the standard BMAP/GI/1 queue.

Let $x(k)$ ($k \in \mathbb{Z}_+$) denote a $1 \times M$ vector whose $i$th element $[x(k)]_i$ represents the stationary joint probability that the queue length in the standard BMAP/GI/1 queue is equal to $k$ and the background Markov chain is in state $i$. According to [41], $\{x(k); k \in \mathbb{Z}_+\}$ is equivalent to the stationary distribution of the following $M/G/1$-type Markov chain:

$$
P_{M/G/1} := \begin{pmatrix}
A(0) & A(1) & A(2) & A(3) & \cdots \\
A(0) & A(1) & A(2) & A(3) & \cdots \\
O & A(0) & A(1) & A(2) & \cdots \\
O & O & A(0) & A(1) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

To characterize $\{x(k)\}$, we introduce some matrices. Let $G$ denote the minimal nonnegative solution of

$$
G = \sum_{k=0}^{\infty} A(k) G^k.
$$

Since $A$ is irreducible, $G$ is stochastic under the stability condition $\rho < 1$ (see Theorem 2.3.1 in [35]). We can also show that $G > O$ (see page 382 of [34]). Thus $G$ has the unique and positive stationary probability vector, denoted by $g > 0$. Further it is known [41] that

$$
\hat{x}(z) = (1 - \rho)g(z - 1) \left( zI - \hat{A}(z) \right)^{-1} \hat{A}(z), \quad |z| < 1.
$$

**Remark 2.1** Since $G > O$, $G$ is an irreducible and aperiodic stochastic matrix. Thus $G$ has a simple eigenvalue $\gamma_1 = 1$ (called Perron-Frobenius eigenvalue) and the other eigenvalues $\gamma_2, \gamma_3, \cdots, \gamma_M$ are located inside the unit circle in the complex plane, i.e., $1 > |\gamma_2| \geq |\gamma_3| \geq \cdots \geq |\gamma_M|$ (see, e.g., [16 Theorem 8.4.4]). In addition, $\lim_{m \to \infty} G^m = eg > O$ (see, e.g., [16 Theorem 8.2.8]).
Let $R(0) = \mathbf{O}$ and $R(k)$ ($k \in \mathbb{N}$) denote
\[
R(k) = \sum_{m=0}^{\infty} A(k + m + 1) G^m (I - U(0))^{-1}, \quad k \in \mathbb{N},
\]
where
\[
U(0) = \sum_{m=0}^{\infty} A(m + 1) G^m.
\]
The matrices $R(k), G$ and $U(0)$ satisfy the following equation (called $RG$-factorization; see [47, Theorem 14]):
\[
zI - \hat{A}(z) = (I - \hat{R}(z)) (I - U(0))(zI - G), \tag{2.9}
\]
where $\hat{R}(z) = \sum_{k=1}^{\infty} z^k R(k)$. From (2.9), we have
\[
\pi = (1 - \rho) g(I - U(0))^{-1}(I - R)^{-1},
\]
where $R = \hat{R}(1)$ (see [42, Lemma 14]). Further substituting (2.9) into (2.8) yields
\[
\hat{x}(z) = (1 - \rho) g (I - U(0))^{-1} \left( I - \hat{R}(z) \right)^{-1} \hat{A}(z). \tag{2.10}
\]

We now make the following assumption:

**Assumption 2.1** There exists some $\mathbb{Z}_+^+$-valued random variable $Y \in S$ such that for some nonzero vector $e^A$,
\[
\lim_{k \to \infty} \frac{\overline{A}(k)e}{P(Y > k)} = e^A \geq 0, \neq 0. \tag{2.11}
\]

Under Assumption 2.1, we have the following result.

**Proposition 2.1 (Corollary 4.1 in [32])** If Assumption 2.1 holds, then
\[
\lim_{k \to \infty} \frac{\overline{x}(k)}{P(Y > k)} = \frac{\pi e^A}{1 - \rho} \cdot \pi.
\]

Finally, we present a sufficient condition Assumption 2.1. To this end, we define $T_e$ as a random variable that is independent of BMAP $\{C, D(k); k \in \mathbb{N}\}$ and is distributed with
\[
P(T_e \leq x) = \frac{1}{E[T]} \int_{0}^{x} P(T > y)dy = \frac{1}{h} \int_{0}^{x} H(y)dy, \quad x \geq 0,
\]
which is called the residual service time or the equilibrium random variable of the service time $T$.

**Proposition 2.2** Suppose that (i) $\sqrt{T_e} \in \mathcal{L}$; and (ii) $\lim_{k \to \infty} e^\delta \sqrt{k} D(k) < \infty$ for some $\delta > 0$. If $T_e \in S$, then Assumption 2.1 holds for $Y = \lambda T_e$ and $e^A = \rho e$. 

Remark 2.2 According to Proposition 2.2, Assumption 2.1 does not necessarily require that \( \{D(k)\} \) is light-tailed. Some other sufficient conditions for Assumption 2.1 are presented in Section 4 in [32].

Proof of Proposition 2.2 Let \( A_e(k) \ (k \in \mathbb{Z}_+) \) denote an \( M \times M \) matrix such that
\[
[A_e(k)]_{i,j} = \mathbb{P}(N(T_e) = k, J(T_e) = j \mid J(0) = i), \quad i, j \in \mathbb{M}.
\]
It then follows from Lemma 4.1 in [32] that
\[
\overline{A}(k)e = h \cdot A_e \ast D(k)e = h \cdot \sum_{l=0}^{k} A_e(l)D(k-l)e. \tag{2.12}
\]

It also follows from Corollary B.1 in [32] that under the conditions (i) and (ii),
\[
\mathbb{P}(N(T_e) > k \mid J(0) = i) \sim k \cdot \mathbb{P}(\lambda T_e > k). \tag{2.13}
\]

Thus following the proof of Lemma 3.1 in [34] (see Appendix D therein), we have for \( i, j \in \mathbb{M} \),
\[
[\overline{A}_e(k)]_{i,j} = \mathbb{P}(N(T_e) > k, J(T_e) = j \mid J(0) = i) \sim [\pi]_j \mathbb{P}(\lambda T_e > k). \tag{2.14}
\]

Applying (2.13), (2.14) and Proposition B.2(iii) to (2.12), we have
\[
\overline{A}(k)e \sim k \cdot he \pi \sum_{k=0}^{\infty} D(k)e \cdot \mathbb{P}(\lambda T_e > k) = \rho e \cdot \mathbb{P}(\lambda T_e > k),
\]
where the last equality is due to (2.3) and (2.5). As a result, Assumption 2.1 holds for \( Y = \lambda T_e \) and \( c^A = \rho e \).

3 Main results

In this section, we first provide some basic results on the BMAP/GI/1 retrial queue (subsection 3.1). We then show a stochastic-decomposition-like result of the stationary queue length in the BMAP/GI/1 retrial queue (subsection 3.2). Combining the stochastic-decomposition-like result with matrix analytic methods, we prove the subexponential tail equivalence of the stationary queue length distributions in the BMAP/GI/1 queues with and without retrials (subsection 3.3).
3.1 BMAP/GI/1 retrial queue

We begin with the description of the BMAP/GI/1 retrial queue. Customers arrive at a single-server system with no buffer according to BMAP \( \{C, D(k); k \in \mathbb{N}\} \). Such customers are called primary customers. If primary customers arriving in a batch find the server idle, then one of them immediately occupies the server and the others join the orbit (virtual waiting line); otherwise all of them join the orbit. The customers in the orbit are called retrial customers.

We assume that the service times of retrial customers are i.i.d. according to an exponential distribution function \( H \) on \([0, \infty)\) with mean \( h \in (0, \infty)\).

We now consider the queue length process in the BMAP/GI/1 retrial queue. As in subsection 2.2 let \( J(t) (t \geq 0) \) denote the state of the background Markov chain at time \( t \). Let \( Q^{(\mu)}(t) (t \geq 0) \) denote the number of retrial customers in the orbit at time \( t \). Further let \( S^{(\mu)}(t) (t \geq 0) \) denote the number of customers in the server at time \( t \). Clearly, \( S^{(\mu)}(t) \in \{0, 1\} \) for all \( t \geq 0 \) and \( \{L^{(\mu)}(t) := Q^{(\mu)}(t) + S^{(\mu)}(t); t \geq 0\} \) is the queue length, i.e., the total number of customers in the server and orbit.

By definition, the process \( \{(S^{(\mu)}(t), Q^{(\mu)}(t), J(t)); t \geq 0\} \) is a semi-regenerative process (see [6, Chapter 10, Section 6]) such that regenerative points are service completion instants, i.e., time points at each of which the service of a customer is completed. Let \( 0 \leq \tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots \) denote service completion instants. It then follows that \( S^{(\mu)}(\tau_n) = 0 \) for all \( n \in \mathbb{Z}_+ \) and \( \{(Q^{(\mu)}(\tau_n), J(\tau_n), \tau_n); n \in \mathbb{Z}_+\} \) is a Markov renewal process (see [6, Chapter 10, Section 1]).

**Remark 3.1** We have a Markov chain by observing \( \{(S^{(\mu)}(t), Q^{(\mu)}(t), J(t)); t \geq 0\} \) at service beginning instants, i.e., time points at each of which the service of a customer starts. Thus service beginning instants can be regenerative points of \( \{(S^{(\mu)}(t), Q^{(\mu)}(t), J(t)); t \geq 0\} \).

Recall here that the diagonal elements of \( C \) are negative and thus

\[
P(N(x) = 0, J(x) = i | J(0) = i) = [e^{Cx}]_{i,i} > 0, \quad \forall x > 0, \quad \forall i \in \mathbb{M}. \tag{3.1}
\]

Note also that \( e^{(C+D)x} > O \) for all \( x > 0 \) due to the irreducibility of \( C + D \). It then follows from (2.4) that there exists some \( k_0 \in \mathbb{N} \) such that for any \( m \in \mathbb{N} \),

\[
\int_0^x dx_m \prod_{j=0}^{m-1} \int_0^{x_j} dx_{m-j} e^{(C+D)x} D(k_0) \cdot e^{(C+D)(x_{m-1} - x_j)} D(k_0) \cdot e^{(C+D)(x_{m-2} - x_{m-1})} D(k_0) \cdot e^{(C+D)(x_{m-3} - x_{m-2})} D(k_0) \cdot \cdots > O, \quad \forall x > 0,
\]

and thus

\[
P(N(x) \geq mk_0, J(x) = j | J(0) = i) > 0, \quad \forall x > 0, \quad \forall (i, j) \in \mathbb{M}^2. \tag{3.2}
\]
It follows from (3.1) and (3.2) that the embedded Markov chain \( \{(Q^{(\mu)}(\tau_n), J(\tau_n))\} \) is irreducible. Further the Markov renewal process \( \{(Q^{(\mu)}(\tau_n), J(\tau_n), \tau_n)\} \) is aperiodic due to the Markov property of \( \{(N(t), J(t))\} \).

It should be noted that for all \( k \in \mathbb{Z}_+ \) and \( i \in \mathbb{M} \),

\[
d_i(k) := E[\tau_{n+1} | Q^{(\mu)}(\tau_n) = k, J(\tau_n) = i] \\
\leq \frac{1}{1 - \rho} + h < \infty.
\] (3.3)

Therefore, if the embedded Markov chain \( \{(Q^{(\mu)}(\tau_n), J(\tau_n))\} \) is positive recurrent, then for any initial state, the semi-regenerative process \( \{(S^{(\mu)}(t), Q^{(\mu)}(t), J(t))\} \) has the same limiting distribution (see [6, Chapter 10, Theorem 6.12]). In addition, if \( \rho < 1 \), then the embedded Markov chain \( \{(Q^{(\mu)}(\tau_n), J(\tau_n))\} \) is positive recurrent (see, e.g., [8, Theorem 3]). As a result, if \( \rho < 1 \), then \( \{(S^{(\mu)}(t), Q^{(\mu)}(t), J(t)); t \geq 0\} \) is stable (i.e., its limiting distribution exists; see [29]) and its limiting distribution is independent of initial conditions.

On the other hand, if \( \rho \geq 1 \), then the standard BMAP/GI/1 queue (without retrials) is unstable [29]. Thus following the proof of Theorem 2 in [15], we can prove that \( \rho < 1 \) is a necessary condition for the stability of the BMAP/GI/1 retrial queue.

The above discussion is summarized in the following:

**Lemma 3.1** \( \{(S^{(\mu)}(t), Q^{(\mu)}(t), J(t)); t \geq 0\} \) is stable and its limiting distribution is independent of initial conditions if and only if \( \rho < 1 \).

As stated in subsection 2.2, the stability condition \( \rho < 1 \) is assumed. Thus we define \( p_0(k) \) and \( p_1(k) \) \( (k \in \mathbb{Z}_+) \) as \( 1 \times M \) vectors such that

\[
[p_0(k)]_i = \lim_{t \to \infty} P(S^{(\mu)}(t) = 0, Q^{(\mu)}(t) = k, J(t) = i), \quad i \in \mathbb{M},
\] (3.4)

\[
[p_1(k)]_i = \lim_{t \to \infty} P(S^{(\mu)}(t) = 1, Q^{(\mu)}(t) = k, J(t) = i), \quad i \in \mathbb{M},
\] (3.5)

respectively. We also define \( \hat{p}_0(z) \) and \( \hat{p}_1(z) \) as

\[
\hat{p}_0(z) = \sum_{k=0}^{\infty} z^k p_0(k), \quad \hat{p}_1(z) = \sum_{k=0}^{\infty} z^k p_1(k),
\]

respectively.

**Remark 3.2** Since the embedded Markov chain \( \{(Q^{(\mu)}(\tau_n), J(\tau_n))\}; n \in \mathbb{Z}_+ \} \) is irreducible and positive recurrent, it has the positive stationary probability vector. Thus we define \( \varphi(k) \) \( (k \in \mathbb{Z}_+) \) as a \( 1 \times M \) vector whose \( i \)th element \( [\varphi(k)]_i \) \( (i \in \mathbb{M}) \) represents the stationary probability that the embedded Markov chain \( \{(Q^{(\mu)}(\tau_n), J(\tau_n))\} \) is in state \( (k, i) \). It then follows from (3.3) and Theorem 6.12 in Chapter 10 of [6] that

\[
p_0(k) \geq \frac{\varphi(k) \int_0^\infty e^{Cx} dx}{\varphi d} = \frac{\varphi(k)(-C)^{-1}}{\varphi d} > 0, \quad \forall k \in \mathbb{Z}_+,
\]
where \( \mathbf{d} := (d_i(k))_{(k,i) \in \mathbb{Z}_+ \times \mathbb{M}} \) and \( \varphi := (\varphi(0), \varphi(1), \varphi(2), \ldots) \). Similarly, we can confirm that \( \mathbf{p}_1(k) > 0 \) for all \( k \in \mathbb{Z}_+ \), though we have to consider another embedded Markov chain of \( \{(S^{(\mu)}(t), Q^{(\mu)}(t), J(t))\} \) observed every time the service of a customer starts.

**Lemma 3.2** \( \hat{\mathbf{p}}_0(z) \) and \( \hat{\mathbf{p}}_1(z) \) satisfy the following equations:

\[
\mu \hat{\mathbf{p}}'_0(z) \left( z \mathbf{I} - \hat{\mathbf{A}}(z) \right) - \hat{\mathbf{p}}_0(z) \left( \mathbf{C} + z^{-1} \hat{\mathbf{D}}(z) \hat{\mathbf{A}}(z) \right), \quad |z| < 1, \quad (3.6)
\]

\[
\hat{\mathbf{p}}_1(z) \left( z \mathbf{I} - \hat{\mathbf{A}}(z) \right) = \hat{\mathbf{p}}_0(z) \left( \hat{\mathbf{A}}(z) - \mathbf{I} \right), \quad |z| < 1, \quad (3.7)
\]

where \( \hat{\mathbf{p}}'_0(z) = (d/dz) \hat{\mathbf{p}}_0(z) \).

**Proof.** This lemma can be proved in a similar way to that of Theorem 1 in [19]. However, we here provide a complete proof because the discussion in Section 4 uses some of the symbols introduced to prove this lemma.

We first prove (3.6). For this purpose, we consider a censored process \( \{(\tilde{Q}^{(\mu)}(t), \tilde{J}(t)); t \geq 0\} \) of \( \{(S^{(\mu)}(t), Q^{(\mu)}(t), J(t)); t \geq 0\} \), which is obtained by observing \( \{(S^{(\mu)}(t), Q^{(\mu)}(t), J(t))\} \) only when \( S^{(\mu)}(t) = 0 \). It is easy to see that \( \{(\tilde{Q}^{(\mu)}(t), \tilde{J}(t))\} \) is a Markov chain whose transition matrix is given by

\[
\tilde{\mathbf{T}} := \begin{pmatrix}
\tilde{T}_0(0) & \tilde{T}_0(1) & \tilde{T}_0(2) & \tilde{T}_0(3) & \cdots \\
\tilde{T}_1(-1) & \tilde{T}_1(0) & \tilde{T}_1(1) & \tilde{T}_1(2) & \cdots \\
O & \tilde{T}_2(-1) & \tilde{T}_2(0) & \tilde{T}_2(1) & \cdots \\
O & O & \tilde{T}_3(-1) & \tilde{T}_3(0) & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots
\end{pmatrix}, \quad (3.8)
\]

where

\[
\tilde{T}_n(-1) = n \mu \mathbf{A}(0), \quad n \in \mathbb{N}, \quad (3.9)
\]

\[
\tilde{T}_n(0) = n \mu \mathbf{A}(1) + \mathbf{D}(1) \mathbf{A}(0) + \mathbf{C} - n \mu \mathbf{I}, \quad n \in \mathbb{Z}_+, \quad (3.10)
\]

\[
\tilde{T}_n(k) = n \mu \mathbf{A}(k + 1) + \sum_{l=1}^{k+1} \mathbf{D}(l) \mathbf{A}(k - l + 1), \quad n \in \mathbb{Z}_+, \quad k \in \mathbb{N}. \quad (3.11)
\]

Recall here (see Remark 3.2) that the embedded Markov chain \( \{(Q^{(\mu)}(\tau_n), J(\tau_n))\} \) is irreducible and positive recurrent and thus the censored (continuous-time) Markov chain \( \{(\tilde{Q}^{(\mu)}(t), \tilde{J}(t))\} \) is irreducible and recurrent (see [5, Chapter 8, Definitions 5.1 and 5.2]). Therefore \( \tilde{\mathbf{T}} \) has a unique (up to a multiplicative factor) positive invariant measure (see [5, Chapter 8, Theorem 5.1]). Further since \( \lim_{n \to \infty} \|\tilde{T}_n(0)\| = \infty \), the unique positive invariant measure is normalized and thus there exists the unique probability vector \( \tilde{\mathbf{p}}_0 \) such that \( \tilde{\mathbf{p}}_0 \tilde{\mathbf{T}} = \mathbf{0} \) (see [5, Chapter 8, Theorem 5.1]).

On the other hand, Lemma 3.1 and Remark 3.2 imply that the censored Markov chain \( \{(\tilde{Q}^{(\mu)}(t), \tilde{J}(t))\} \) has the unique and positive limiting distribution (independent of initial conditions). As a result, \( \tilde{\mathbf{p}}_0 \) is the unique limiting distribution of the Markov chain \( \{(\tilde{Q}^{(\mu)}(t), \tilde{J}(t))\} \) (see [5, Chapter 8, Theorems 5.3 and 6.1]).
We partition \( \tilde{p}_0 \) as \((\tilde{p}_0(0), \tilde{p}_0(1), \ldots)\), where
\[
[\tilde{p}_0(k)]_i = \lim_{t \to \infty} P(\tilde{Q}^{(i)}(t) = k; \tilde{J}(t) = i) > 0, \quad i \in \mathbb{M}.
\]
By definition, there exists some constant \( c > 0 \) such that
\[
p_0(k) = c \tilde{p}_0(k), \quad k \in \mathbb{Z}_+.
\] (3.12)
Thus from (3.12), we have
\[
(p_0(0), p_0(1), \ldots) \tilde{T} = 0.
\] (3.13)
It follows from (3.8)–(3.11) and (3.13) that for \( k \in \mathbb{Z}_+ \),
\[
0 = \sum_{n=0}^{k+1} p_0(n) \tilde{T}_n(k - n)
\]
\[
= \sum_{n=0}^{k+1} p_0(n)n\mu A(k - n + 1) + \sum_{n=0}^{k+1} p_0(n) \sum_{l=1}^{k-n+1} D(l) A(k - n - l + 1)
\]
\[
+ p_0(k)(C - k\mu I),
\]
where the summation over the empty set is defined as zero. Multiplying both sides of the above equation by \( z^k \) and summing them for all \( k \in \mathbb{Z}_+ \), we obtain
\[
0 = \mu \sum_{n=1}^{\infty} nz^{n-1} p_0(n) \sum_{k=n-1}^{\infty} z^{k-n+1} A(k - n + 1)
\]
\[
+ \sum_{n=0}^{\infty} z^n p_0(n) \cdot z^{-1} \sum_{k=n}^{\infty} z^{k-n+1} \sum_{l=1}^{k-n+1} D(l) A(k - n - l + 1)
\]
\[
+ \sum_{k=0}^{\infty} z^k p_0(k) C - \mu z \sum_{k=1}^{\infty} k z^{k-1} p_0(k)
\]
\[
= \mu \tilde{p}_0'(z) \left( \tilde{A}(z) - zI \right) + \tilde{p}_0(z) \left( C + z^{-1} \tilde{D}(z) \tilde{A}(z) \right),
\]
which leads to (3.6).

Next we prove (3.7). Let \( r_i(k) \) \((k \in \mathbb{Z}_+, i \in \mathbb{M})\) denote the stationary probability that the number of retrial customers is equal to \( k \) and the background state is \( i \) immediately after the service of a customer starts, which is well-defined due to Lemma 3.1. Note here that the time-average number of customers in service is equal to the arrival rate \( \lambda \) of primary customers. It then follows that for \( k \in \mathbb{Z}_+ \),
\[
r(k) := (r_i(k))_{i \in \mathbb{M}} = \frac{1}{\lambda} \left( p_0(k+1)(k+1)\mu + \sum_{l=0}^{k} p_0(l) D(k - l + 1) \right),
\]
which yields
\[
\tilde{r}(z) := \sum_{k=0}^{\infty} z^k r(k) = \frac{1}{\lambda} \left( \mu \tilde{p}_0'(z) + z^{-1} \tilde{p}_0(z) \tilde{D}(z) \right).
\] (3.14)
Recall that \( \{(S^{(\mu)}(t), Q^{(\mu)}(t), J(t)); t \geq 0\} \) is a semi-regenerative process that is regenerated every time the service of a customer starts (see Remark 3.1). Since the mean regenerative cycle is equal to \( 1/\lambda \), it follows from Theorem 6.12 in Chapter 10 of [6] that
\[
\hat{p}_1(z) = \lambda \hat{r}(z) \int_0^\infty e^{(C + \hat{D}(z)x)} \overline{H}(x) dx,
\]
where \( \overline{H}(x) = 1 - H(x) \) for \( x \geq 0 \). Combining this with (3.14) yields
\[
\hat{p}_1(z) = \left( \mu \hat{p}_0(z) + z^{-1} \hat{p}_0(z) \hat{D}(z) \right) \cdot \int_0^\infty e^{(C + \hat{D}(z)x)} \overline{H}(x) dx
\]
\[
= \left( \mu \hat{p}_0(z) + z^{-1} \hat{p}_0(z) \hat{D}(z) \right) \cdot \left( C + \hat{D}(z) \right)^{-1} \left( \hat{A}(z) - I \right),
\]
for all \( 0 \leq |z| < 1 \). From (3.15) and (3.6), we have
\[
\hat{p}_1(z) \left( zI - \hat{A}(z) \right) = \left( \mu \hat{p}_0(z) + z^{-1} \hat{p}_0(z) \hat{D}(z) \right) \cdot \left( \mu \right) \cdot \left( C + \hat{D}(z) \right)^{-1} \left( \hat{A}(z) - I \right) \cdot \left( zI - \hat{A}(z) \right)
\]
\[
= \hat{p}_0(z) \left\{ \left( C + z^{-1} \hat{D}(z) \hat{A}(z) \right) + z^{-1} \hat{D}(z) \left( zI - \hat{A}(z) \right) \right\}
\]
\[
\times \left( C + \hat{D}(z) \right)^{-1} \left( \hat{A}(z) - I \right)
\]
\[
= \hat{p}_0(z) \left( \hat{A}(z) - I \right),
\]
where the second equality holds because \( C + \hat{D}(z) \) and \( \hat{A}(z) \) are commutative. \( \square \)

### 3.2 Stochastic-decomposition-like result

Let \( x^{(\mu)}(k) (k \in \mathbb{Z}_+) \) denote a \( 1 \times M \) vector such that
\[
[x^{(\mu)}(k)]_i = \lim_{t \to \infty} P(L^{(\mu)}(t) = k, J(t) = i), \quad i \in \mathbb{M},
\]
where \( L^{(\mu)}(t) = Q^{(\mu)}(t) + S^{(\mu)}(t) \). Further let \( \hat{x}^{(\mu)}(z) = \sum_{k=0}^\infty z^k x^{(\mu)}(k) \). By definition, we then have
\[
x^{(\mu)}(k) = \begin{cases} p_0(0), & k = 0, \\
p_0(k) + p_1(k - 1), & k \in \mathbb{N}, \end{cases}
\]
\[
\hat{x}^{(\mu)}(z) = \hat{p}_0(z) + z \hat{p}_1(z).
\]

The following lemma is an extension of the stochastic decomposition of the stationary queue length in the M\(^X\)/GI/1 retrial queue (see Proposition 1 in [44]).
Lemma 3.3 For \( \mu \in (0, \infty) \),
\[
\hat{x}^{(\mu)}(z) = \frac{\hat{p}_0(z)}{1 - \rho} \cdot \hat{X}(z), \quad |z| < 1, \tag{3.18}
\]
\[
\hat{g} \hat{X}(z) = \hat{x}(z), \quad |z| < 1, \tag{3.19}
\]
\[
\lim_{z \uparrow 1} \hat{X}(z) = e\pi, \tag{3.20}
\]
where \( \hat{X}(z) := \sum_{k=-\infty}^{\infty} z^k X(k) \) (\(|z| < 1\)) is defined as
\[
\hat{X}(z) = (1 - \rho)(z - 1) \left( zI - \hat{A}(z) \right)^{-1} \hat{A}(z). \tag{3.21}
\]

Proof. Applying (3.7) to (3.17) yields
\[
\hat{x}^{(\mu)}(z) = \hat{p}_0(z) \left\{ I + z \left( \hat{A}(z) - I \right) \left( zI - \hat{A}(z) \right)^{-1} \right\}
\]
\[
= \hat{p}_0(z)(z - 1) \hat{A}(z) \left( zI - \hat{A}(z) \right)^{-1}
\]
\[
= \hat{p}_0(z)(z - 1) \left( zI - \hat{A}(z) \right)^{-1} \hat{A}(z).
\]

Combining this with (3.21), we have (3.18). Further (3.19) follows from (2.8) and (3.21).

Finally we prove (3.20). Let \( \sigma_i(z) \)'s (\( z > 0, i = 1, 2, \ldots, M \)) denote the eigenvalues of \( \hat{A}(z) \) such that \( |\sigma_1(z)| \geq |\sigma_2(z)| \geq \cdots \geq |\sigma_M(z)| \). Note here that since \( A = \hat{A}(1) \) is irreducible, so is \( \hat{A}(z) \) for \( 0 < z < r_A \), where \( r_A \) is the convergence radius of \( \sum_{k=0}^\infty z^k A(k) \). Thus \( \sigma_1(z) \) is the Perron-Frobenius eigenvalue and \( \sigma_1(z) > |\sigma_2(z)| \geq \cdots \geq |\sigma_M(z)| \). In addition, from \( \sigma_1(1) = 1, \pi A = \pi \) and \( A e = e \), we have
\[
\sigma'_1(1) := \frac{d}{dz} \sigma_1(z) \bigg|_{z=1} = \pi \sum_{k=1}^\infty k A(k) e = \rho.
\]

Therefore, following the proof of Lemma 3.3 in [23], we can show that
\[
\lim_{z \uparrow 1} (z - 1) \left( zI - \hat{A}(z) \right)^{-1} = \frac{e\pi}{1 - \rho}.
\]
Applying this to (3.21) yields (3.20). \( \square \)

We conclude this subsection with some remarks on the coefficient matrices \( X(k) \) (\( k \in \mathbb{Z} \)) of the power series expansion of \( \hat{X}(z) \) in (3.21).

Combining (3.21) with (2.9), we have for \(|z| < 1\),
\[
\hat{X}(z) = (1 - 1/z)(I - G/z)^{-1}
\]
\[
\times (1 - \rho)(I - U(0))^{-1} \left( I - \hat{R}(z) \right)^{-1} \hat{A}(z)
\]
\[
= \hat{X}_1(z) \hat{X}_2(z), \tag{3.22}
\]
Tail equivalence for BMAP/GI/1 queues with/without retrials

where \( \hat{X}_1(z) := \sum_{k=0}^{\infty} z^{-k} X_1(k) \) and \( \hat{X}_2(z) := \sum_{k=0}^{\infty} z^{k} X_2(k) \) are given by

\[
\hat{X}_1(z) = (1 - 1/z)(I - G/z)^{-1}, \quad |z| < 1, \tag{3.23}
\]

\[
\hat{X}_2(z) = (1 - \rho)(I - U(0))^{-1}\left( I - \hat{R}(z) \right)^{-1} \hat{A}(z), \quad |z| \leq 1. \tag{3.24}
\]

From (3.22) and (3.23), we have

\[
X(k) = \sum_{m=\text{max}(-k,0)}^{\infty} X_1(m) X_2(k + m), \quad k \in \mathbb{Z}, \tag{3.25}
\]

\[
X_1(k) = \begin{cases} 
I, & k = 0, \\
G^k - G^{k-1}, & k \in \mathbb{N}. 
\end{cases} \tag{3.26}
\]

Substituting (3.26) into (3.25) yields

\[
X(k) = X_2(k) + \sum_{m=1}^{\infty} \left( G^m - G^{m-1} \right) X_2(k + m), \quad k \in \mathbb{Z}_+. \tag{3.27}
\]

Pre-multiplying both sides of (3.27) by \( g \) and using (3.19), we have

\[
gX(k) = gX_2(k) = x(k), \quad k \in \mathbb{Z}_+. \tag{3.28}
\]

Equation (3.24) implies that \( X_2(k) \geq O \) for all \( k \in \mathbb{Z}_+ \). On the other hand, (3.27) shows that \( X(k) (k \in \mathbb{Z}) \) itself may not be nonnegative. It should be noted that if background state space \( \mathbb{M} = \{1\} \), i.e., the BMAP/GI/1 retrial queue is reduced to the M\(^X\)/GI/1 retrial queue, then \( g = 1 \) and thus (3.28) yields

\[
X(k) = X_2(k) = x(k), \quad k \in \mathbb{Z}_+,
\]

which shows that \( \{X(k)\} \) and \( \{X_2(k)\} \) are equivalent to the stationary queue length distribution in the M\(^X\)/GI/1 retrial queue.

### 3.3 Subexponential tail equivalence

In this subsection, we present the main theorem. To this end, we provide three lemmas.

**Lemma 3.4** \( X(-k) = O(\gamma^k)ee^t \) for some \( \gamma \in (0, 1) \), where \( f(x) = O(g(x)) \) represents \( \lim \sup_{x \to \infty} |f(x)/g(x)| < \infty \).

**Proof.** According to Remark 2.1, we fix \( \gamma \in (0, 1) \) such that

\[
G^k = eg + O(\gamma^k)ee^t, \quad k \in \mathbb{Z}_+. \tag{3.29}
\]

From (3.26), we then have

\[
X_1(k) = O(\gamma^k)ee^t, \quad k \in \mathbb{Z}_+. \tag{3.30}
\]
From (3.24), we also have $\sum_{l=0}^{\infty} \gamma^l X_2(l) < \infty$. Therefore substituting (3.30) to (3.25) yields for $k \in \mathbb{N}$,

$$\gamma^{-k} X(-k) = \gamma^{-k} \sum_{m=k}^{\infty} X_1(m) X_2(-k + m)$$

$$= e \varepsilon^t \sum_{m=k}^{\infty} O(\gamma^{-k+m}) X_2(-k + m) < \infty,$$

which completes the proof. \qed

**Lemma 3.5** If Assumption 2.1 holds, then

$$\overline{X}(k) \overset{k}{\sim} e \pi(k) \overset{k}{\sim} \pi \varepsilon A \frac{\pi e^A}{1 - \rho} \cdot P(Y > k). \tag{3.31}$$

**Proof.** From (3.27), we have

$$\overline{X}(k) = \overline{X}_2(k) + \sum_{m=1}^{\infty} (G^m - G^{m-1}) \overline{X}_2(k + m)$$

$$= \sum_{m=0}^{\infty} G^m \{ \overline{X}_2(k + m) - \overline{X}_2(k + m + 1) \}$$

$$= \sum_{m=0}^{\infty} G^m X_2(k + m + 1) \geq O, \quad k \in \mathbb{Z}_+. \tag{3.32}$$

Recall here that $\lim_{m \to \infty} G^m = e g$ (see Remark 2.1). Thus for any $\varepsilon > 0$, there exists some $m_0 := m_0(\varepsilon) \in \mathbb{N}$ such that for all $m \geq m_0$,

$$(1 - \varepsilon) e g \leq G^m \leq (1 + \varepsilon) e g. \tag{3.33}$$

Applying (3.33) to (3.32) and using (3.28), we obtain

$$\liminf_{k \to \infty} \frac{\overline{X}(k)}{P(Y > k)} \geq \liminf_{k \to \infty} \sum_{m=m_0}^{\infty} G^m \frac{X_2(k + m + 1)}{P(Y > k)}$$

$$\geq (1 - \varepsilon) e \cdot \liminf_{k \to \infty} \frac{\overline{\pi}(k + m_0)}{P(Y > k)}. \tag{3.34}$$

Similarly,

$$\limsup_{k \to \infty} \frac{\overline{X}(k)}{P(Y > k)} \leq (1 + \varepsilon) e \cdot \limsup_{k \to \infty} \frac{\overline{\pi}(k + m_0)}{P(Y > k)}$$

$$+ \sum_{m=0}^{m_0-1} G^m \limsup_{k \to \infty} \frac{X_2(k + m + 1)}{P(Y > k)}. \tag{3.35}$$
Since \( g > 0 \), there exists some constant \( K > 0 \) such that \( G^m \leq Keg \) for all \( m \in \mathbb{Z}_+ \). Therefore from (3.35), we have

\[
\limsup_{k \to \infty} \frac{X(k)}{P(Y > k)} \leq (1 + \varepsilon) e \cdot \limsup_{k \to \infty} \frac{\overline{F}(k + m_0)}{P(Y > k)} + Ke \cdot \sum_{m=0}^{m_0-1} \limsup_{k \to \infty} \frac{\overline{F}(k + m) - \overline{F}(k + m + 1)}{P(Y > k)}.
\] (3.36)

It follows from Assumption 2.1 and \( Y \in S \subset L \) that for any fixed \( m \in \mathbb{Z}_+ \),

\[
\lim_{k \to \infty} \frac{\overline{F}(k + m)}{P(Y > k)} = \lim_{k \to \infty} \frac{\overline{F}(k + m + 1)}{P(Y > k)} = \frac{\pi e^A}{1 - \rho}. \tag{3.37}
\]

Substituting (3.37) into (3.34) and (3.36) and letting \( \varepsilon \downarrow 0 \), we obtain

\[
\lim_{k \to \infty} \frac{X(k)}{P(Y > k)} = \frac{\pi e^A}{1 - \rho} e. \tag{3.39}
\]

Combining this with Proposition 2.1 yields (3.31).

\[\square\]

Lemma 3.6 If Assumption 2.1 holds, then \( \lim_{k \to \infty} \frac{\hat{p}_0(k)}{P(Y > k)} = 0 \).

Lemma 3.6 is key to the proof of the main theorem (Theorem 3.1 below). We postpone, however, the proof of this lemma until the next section because the proof is somewhat long and technical.

The main theorem of this paper is as follows:

**Theorem 3.1** If Assumption 2.1 holds, then \( \overline{F}^{(\mu)}(k) \sim \overline{F}(k) \).

**Proof.** From (3.18), we have

\[
(1 - \rho)\overline{F}^{(\mu)}(k) = \hat{p}_0(1)\overline{F}(k) + \sum_{m=-\infty}^{k-m} \hat{p}_0(k - m)\overline{F}(m)
\]

\[
= \hat{p}_0(1)\overline{F}(k) + \sum_{m=1}^{\infty} \hat{p}_0(k + m)\overline{F}(-m) + \sum_{m=0}^{k} \hat{p}_0(k - m)\overline{F}(m). \tag{3.38}
\]

Lemma 3.5 implies that

\[
\lim_{k \to \infty} \hat{p}_0(1)\frac{\overline{F}(k)}{P(Y > k)} = \hat{p}_0(1)e \cdot \frac{\pi e^A}{1 - \rho}. \tag{3.39}
\]

It follows from (3.18), (3.20) and \( \overline{F}^{(\mu)}(1) = 1 \) that

\[
\hat{p}_0(1)e = 1 - \rho. \tag{3.40}
\]
Substituting (3.40) into (3.39) yields
\[
\lim_{k \to \infty} \hat{p}_0(1) \frac{X(k)}{\mathbf{P}(Y > k)} = \pi c^A \cdot \pi. \tag{3.41}
\]

Further since \(\{\hat{p}_0(k); k \in \mathbb{Z}_+\}\) is nonincreasing, Lemmas 3.4 and 3.6 imply that
\[
\limsup_{k \to \infty} \left| \sum_{m=1}^{\infty} \frac{\hat{p}_0(k + m) X(-m)}{\mathbf{P}(Y > k)} \right| \leq \limsup_{k \to \infty} \frac{\hat{p}_0(k)}{\mathbf{P}(Y > k)} \sum_{m=1}^{\infty} |X(-m)| = 0. \tag{3.42}
\]

Applying (3.41) and (3.42) to (3.38), we have
\[
\lim_{k \to \infty} \frac{x(\mu)(k)}{\mathbf{P}(Y > k)} = \frac{\pi c^A}{1 - \rho} \pi + \frac{1}{1 - \rho} \sum_{m=0}^{k} \frac{\hat{p}_0(k - m) X(m)}{\mathbf{P}(Y > k)}. \tag{3.43}
\]

Therefore to complete the proof, it suffices to show that
\[
\limsup_{k \to \infty} \left| \sum_{m=0}^{k} \frac{\hat{p}_0(k - m) X(m)}{\mathbf{P}(Y > k)} \right| = O.
\]

In what follows, we prove this equation.

According to (3.29) and \(g > 0\), there exist some \(K > 0\) and \(\gamma \in (0, 1)\) such that
\[
|G^m - G^{m-1}| \leq K \gamma^m eg, \quad \forall m \in \mathbb{N},
\]
\[
X_2(k) \leq K eg X_2(k), \quad \forall k \in \mathbb{Z}_+.
\]

Substituting these inequalities into (3.27) and using (3.28) yield for \(k \in \mathbb{Z}_+\),
\[
|X(k)| \leq Ke \left( gX_2(k) + \sum_{m=1}^{\infty} \gamma^m gX_2(k + m) \right)
\]
\[
= Ke \left( x(k) + \sum_{m=1}^{\infty} \gamma^m x(k + m) \right) =: X^+(k).
\]

Since \(\{\bar{x}(k); k \in \mathbb{Z}_+\}\) is nonincreasing, it follows from Proposition 2.1 that
\[
\sum_{m=1}^{\infty} \gamma^m \frac{\bar{x}(k + m)}{\mathbf{P}(Y > k)} \leq \sup_{k \in \mathbb{Z}_+} \frac{\bar{x}(k)}{\mathbf{P}(Y > k)} \gamma < \infty.
\]

Thus using the dominated convergence theorem, Proposition 2.1 and \(Y \in \mathcal{S} \subset \mathcal{L}\), we obtain
\[
\lim_{k \to \infty} \frac{X^+(k)}{\mathbf{P}(Y > k)} = Ke \left( \lim_{k \to \infty} \frac{\bar{x}(k)}{\mathbf{P}(Y > k)} + \sum_{m=1}^{\infty} \gamma^m \lim_{k \to \infty} \frac{\bar{x}(k + m)}{\mathbf{P}(Y > k)} \right)
\]
\[
= K \left( 1 + \frac{\gamma}{1 - \rho} \right) \frac{\pi c^A}{1 - \rho} e\pi = \frac{K - \pi c^A}{1 - \gamma - 1 - \rho} e\pi < \infty. \tag{3.43}
\]
Combining this with Lemma 3.6 and Proposition B.2, we have

\[ \lim_{k \to \infty} \frac{p_0 \cdot X^+(k)}{P(Y > k)} = \frac{K}{1 - \gamma} \frac{\pi c^A}{1 - \rho} \hat{p}_0(1) e = \frac{K}{1 - \gamma} \pi c^A \cdot \pi, \]  

(3.44)

where we use (3.40) in the second equality. Note here that

\[ \frac{p_0 \cdot X^+(k)}{P(Y > k)} = \frac{\hat{p}_0(1)X^+(k)}{P(Y > k)} + \sum_{m=0}^{k} \frac{p_0(k - m)X^+(m)}{P(Y > k)}, \]

where the first term converges to the right hand side of (3.44) as \( k \to \infty \), due to (3.43). Therefore

\[ \lim_{k \to \infty} \sum_{m=0}^{k} \frac{p_0(k - m)X^+(m)}{P(Y > k)} = 0, \]

which leads to

\[ \limsup_{k \to \infty} \left| \sum_{m=0}^{k} \frac{p_0(k - m)X(m)}{P(Y > k)} \right| \leq \limsup_{k \to \infty} \sum_{m=0}^{k} \frac{p_0(k - m)X^+(m)}{P(Y > k)} = 0. \]

\[ \blacklozenge \]

As mentioned in the introduction, Yamamuro [44] proves the subexponential tail equivalence of the queue length distributions in the M^X/GI/1 retrial queues with and without retrials, under the assumption that the batch size distribution is light-tailed. On the other hand, Proposition 2.2 shows that Assumption 2.1 and thus Theorem 3.1 do not necessarily require that \( \{D(k)\} \) is light-tailed.

### 4 Proof of a Key Lemma (Lemma 3.6)

To facilitate the discussion, we apply a change of measure to \( p_0 : = (p_0(0), p_0(1), p_0(2), \ldots). \)

Let \( q : = (q(0), q(1), q(2), \ldots) \) denote a probability vector

\[ q(k) = \frac{\max(k, 1)p_0(k)}{\sum_{l=0}^{\infty} \max(l, 1)p_0(l)e} > 0, \quad k \in \mathbb{Z}_+, \]

(4.1)

where the positivity of \( q(k) \) follows from Remark 3.2. We then have

\[ q = \frac{p_0 \Delta^{-1}}{p_0 \Delta^{-1} e}, \]

(4.2)
where \( \Delta \) is a diagonal matrix such that

\[
\Delta = \begin{pmatrix}
I & 0 & 0 & 0 & 0 & \cdots \\
0 & I & 0 & 0 & 0 & \cdots \\
0 & 0 & \frac{1}{2}I & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{1}{3}I & 0 & \cdots \\
0 & 0 & 0 & 0 & \frac{1}{4}I & \ddots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

It follows from (4.2) and \( p_0 \tilde{T} = 0 \) (see (3.13)) that \( q > 0 \) is the stationary probability vector of the following infinitesimal generator:

\[
\tilde{T} := \Delta \tilde{T} = \begin{pmatrix}
\tilde{T}_0(0) & \tilde{T}_0(1) & \tilde{T}_0(2) & \tilde{T}_0(3) & \cdots \\
\tilde{T}_1(-1) & \tilde{T}_1(0) & \tilde{T}_1(1) & \tilde{T}_1(2) & \cdots \\
O & \tilde{T}_2(-1) & \tilde{T}_2(0) & \tilde{T}_2(1) & \cdots \\
O & O & \tilde{T}_3(-1) & \tilde{T}_3(0) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

where

\[
\tilde{T}_0(0) = C + D(1)A(0),
\]

\[
\tilde{T}_0(k) = \sum_{l=1}^{k+1} D(l)A(k-l+1), \quad k \in \mathbb{N},
\]

and for \( n = 1 \),

\[
\tilde{T}_n(-1) = \mu A(0),
\]

\[
\tilde{T}_n(0) = -\mu I + \mu A(1) + \frac{1}{n} \{ C + D(1)A(0) \},
\]

\[
\tilde{T}_n(k) = \mu A(k+1) + \frac{1}{n} \sum_{l=1}^{k+1} D(l)A(k-l+1), \quad k \in \mathbb{N}.
\]

For convenience, we uniformize the transition rate matrix \( \tilde{T} \) as follows:

\[
\tilde{P} := I + \frac{\tilde{T}}{\mu + \theta},
\]

where \( \theta = \max_{i \in \mathcal{M}} |[C]_{i,i}| \). From (4.3)–(4.8), we have

\[
\tilde{P} = \begin{pmatrix}
\tilde{A}_0(0) & \tilde{A}_0(1) & \tilde{A}_0(2) & \tilde{A}_0(3) & \cdots \\
\tilde{A}_1(-1) & \tilde{A}_1(0) & \tilde{A}_1(1) & \tilde{A}_1(2) & \cdots \\
O & \tilde{A}_2(-1) & \tilde{A}_2(0) & \tilde{A}_2(1) & \cdots \\
O & O & \tilde{A}_3(-1) & \tilde{A}_3(0) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]
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where

\[
\mathbf{A}_0(0) = \mathbf{I} + \frac{1}{\mu + \theta} \{ \mathbf{C} + \mathbf{D}(1) \mathbf{A}(0) \},
\]

\[
\mathbf{A}_0(k) = \frac{1}{\mu + \theta} \sum_{l=1}^{k+1} \mathbf{D}(l) \mathbf{A}(k-l+1), \quad k \in \mathbb{N},
\]  

(4.11)

and for \( n \in \mathbb{N} \),

\[
\mathbf{A}_n(-1) = \frac{\mu}{\mu + \theta} \mathbf{A}(0),
\]  

(4.12)

\[
\mathbf{A}_n(0) = \frac{1}{\mu + \theta} \left[ \theta \mathbf{I} + \mu \mathbf{A}(1) + \frac{1}{n} \{ \mathbf{C} + \mathbf{D}(1) \mathbf{A}(0) \} \right],
\]  

(4.13)

\[
\mathbf{A}_n(k) = \frac{1}{\mu + \theta} \left[ \mu \mathbf{A}(k+1) + \frac{1}{n} \sum_{l=1}^{k+1} \mathbf{D}(l) \mathbf{A}(k-l+1) \right], \quad k \in \mathbb{N}.
\]  

(4.14)

Let \( \mathbf{A}(k) (k \geq -1) \) denote

\[
\mathbf{A}(-1) = \mathbf{A}_n(-1) = \frac{\mu}{\mu + \theta} \mathbf{A}(0),
\]  

(4.15)

\[
\mathbf{A}(0) = \frac{\theta}{\mu + \theta} \mathbf{I} + \frac{\mu}{\mu + \theta} \mathbf{A}(1),
\]  

(4.16)

\[
\mathbf{A}(k) = \frac{\mu}{\mu + \theta} \mathbf{A}(k+1), \quad k \in \mathbb{N}.
\]  

(4.17)

It then follows from (4.12)–(4.17) that

\[
\lim_{n \to \infty} \sum_{k=-1}^{\infty} |\mathbf{A}_n(k) - \mathbf{A}(k)| \leq \frac{1}{\mu + \theta} \lim_{n \to \infty} \frac{1}{n} (|\mathbf{C}| + \mathbf{D} \mathbf{A}) = O,
\]  

(4.18)

and thus

\[
\lim_{n \to \infty} \mathbf{A}_n(k) = \mathbf{A}(k), \quad \text{uniformly all } k \geq -1.
\]  

(4.19)

By definition, \( \mathbf{q} = (\mathbf{q}(0), \mathbf{q}(1), \mathbf{q}(2), \ldots) > 0 \) is the stationary probability vector of \( \mathbf{P} \).

Note here that (4.3) and (4.9) yield

\[
\mathbf{P} = \mathbf{I} + \frac{\Delta \mathbf{T}}{\mu + \theta}.
\]

Note also that \( \mathbf{T} \) has the unique stationary probability vector \( \mathbf{p}_0 = c^{-1} \mathbf{p}_0 \) (this equality is due to (3.12)). Therefore \( \mathbf{q} \) is the unique and positive stationary probability vector of \( \mathbf{P} \), which implies that \( \mathbf{P} \) is irreducible and positive recurrent (see [5] Chapter 3, Theorem 3.1).

As shown in (4.10) and (4.19), \( \mathbf{P} \) is a level-dependent M/G/1-type stochastic matrix with asymptotic level-independence. Utilizing this special structure of \( \mathbf{P} \), we can prove Lemma 4.1 below.
Lemma 4.1 Suppose Assumption 2.1 holds. The following then hold.

(i) \( \limsup_{k \to \infty} \frac{\overline{q}(k)}{P(Y > k)} \) is finite; and

(ii) \( \lim_{k \to \infty} \frac{q(k)}{P(Y > k)} = c\pi \) for some \( c \in (0, \infty) \) if \( \limsup_{k \to \infty} \frac{D(k)e}{P(Y > k)} \) exists.

From (4.1), we have

\[
p_0(k) = p_0 \Delta^{-1} e \cdot \frac{q(k)}{k}, \quad k \in \mathbb{N},
\]

which leads to

\[
\overline{p}_0(k) = p_0 \Delta^{-1} e \cdot \sum_{l=k+1}^{\infty} \frac{q(l)}{l} \leq p_0 \Delta^{-1} e \cdot \frac{\overline{q}(k)}{k}, \quad k \in \mathbb{N}.
\]

Thus Lemma 3.6 is immediate from statement (i) of Lemma 4.1.

Remark 4.1 Although statement (ii) of Lemma 4.1 is not necessary for Lemma 3.6, the statement is, as far as we know, the first reported result on the subexponential asymptotics of level-dependent structured Markov chain. In addition, from statement (ii), we can guess that under Assumption 2.1 and additional conditions, the following locally subexponential asymptotic formula holds:

\[
q(k) \sim c\pi \cdot P(Y = k).
\]

This can be proved by extending the results on the locally subexponential asymptotics of level-independent GI/G/1-type Markov chains (see Section 4 in [24]) to a level-dependent M/G/1-type Markov chain with asymptotic level-independence. If (4.21) holds, then (4.20) yields

\[
p_0(k) \sim c' \pi \cdot k^{-1} P(Y = k) \quad \text{for some constant } c' > 0.
\]

Further if the stronger conditions in Proposition 2.2 are assumed instead of Assumption 2.1 (of course, other additional conditions are needed for the locally subexponential asymptotics), then

\[
p_0(k) \sim c'_h \pi \cdot k^{-1} P(\lambda T > k),
\]

where \( T \) is the service time.

We need several technical lemmas to prove Lemma 4.1. In the rest of this section, we present the technical lemmas and then give the proof of Lemma 4.1 at the end of this section.

We begin with the following lemma.

Lemma 4.2 (i) The diagonal elements of \( A(0) \) and \( \tilde{A}(-1) \) are positive; and (ii) for all \( k \in \mathbb{Z}_+ \), \( \sum_{l=k}^{\infty} A(l) > O \) and \( \sum_{l=k-1}^{\infty} \tilde{A}(l) > O \).
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Proof. It suffices to prove statements (i) and (ii) for \( \{A(k)\} \) due to (4.15)–(4.17). It follows from (3.1) and (3.2) that

\[
[A(0)]_{i,i} = \left[ \int_0^\infty e^{Cx} dH(x) \right]_{i,i} > 0, \quad \forall i \in \mathbb{M},
\]

and that there exists some \( k_0 \in \mathbb{N} \) such that for all \( m \in \mathbb{N} \) and \((i, j) \in \mathbb{M}^2\),

\[
\left[ \sum_{l=mk_0}^\infty A(l) \right]_{i,j} = \int_0^\infty dH(x)P(N(x) \geq mk_0, J(x) = j \mid J(0) = i) > 0,
\]

which completes the proof. \( \square \)

For further discussion, we introduce some symbols. We define \( \ddot{P}_n \) \((n \in \mathbb{N})\) as a submatrix of \( \ddot{P} \) in (4.10) such that

\[
\ddot{P}_n = \begin{pmatrix}
\ddot{A}_n(0) & \ddot{A}_n(1) & \ddot{A}_n(2) & \ddot{A}_n(3) & \cdots \\
\ddot{A}_{n+1}(-1) & \ddot{A}_{n+1}(0) & \ddot{A}_{n+1}(1) & \ddot{A}_{n+1}(2) & \cdots \\
O & \ddot{A}_{n+2}(-1) & \ddot{A}_{n+2}(0) & \ddot{A}_{n+2}(1) & \cdots \\
O & O & \ddot{A}_{n+3}(-1) & \ddot{A}_{n+3}(0) & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots
\end{pmatrix} \quad (4.22)
\]

It follows from (4.18) and (4.22) that

\[
\lim_{n \to \infty} \ddot{P}_n = \begin{pmatrix}
\ddot{A}(0) & \ddot{A}(1) & \ddot{A}(2) & \ddot{A}(3) & \cdots \\
\ddot{A}(-1) & \ddot{A}(0) & \ddot{A}(1) & \ddot{A}(2) & \cdots \\
O & \ddot{A}(-1) & \ddot{A}(0) & \ddot{A}(1) & \cdots \\
O & O & \ddot{A}(-1) & \ddot{A}(0) & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots
\end{pmatrix} =: \ddot{P}_\infty, \quad (4.23)
\]

where the convergence is uniform over all the elements.

Recall that \( \ddot{P} \) is irreducible and positive recurrent and thus the set of states \( \{(m, i); m \geq n, i \in \mathbb{M}\} \) is not closed for any \( n \in \mathbb{N} \). Therefore for any \( n \in \mathbb{N} \), there exists the minimal nonnegative inverse of \( I - \ddot{P}_n \) (see, e.g., [38, Corollary 2 of Lemma 5.4]), which is denoted by \((I - \ddot{P}_n)^{-1}\) and given by

\[
(I - \ddot{P}_n)^{-1} = \sum_{m=0}^\infty (\ddot{P}_n)^m.
\]

Using the inverse \((I - \ddot{P}_n)^{-1}\), we define some matrices, which play a role in matrix analytic methods. Let \( \ddot{N}_n(0, 0) \) denote the \( M \times M \) northwest corner of \((I - \ddot{P}_n)^{-1}\), i.e.,

\[
\ddot{N}_n(0, 0) = \sum_{m=0}^\infty \ddot{P}_n^{(m)}(0, 0), \quad (4.24)
\]
where \( \tilde{P}_n^{(m)}(0, 0) \) is the \( M \times M \) northwest corner of \( (\tilde{P}_n)^m \). Further let \( \tilde{G}_n (n \in \mathbb{N}) \) and \( \tilde{U}_n(0) (n \in \mathbb{N}) \) and \( \tilde{R}_n(k) (n \in \mathbb{Z}_+, k \in \mathbb{N}) \) denote

\[
\tilde{G}_n = \tilde{N}_n(0, 0)\tilde{A}_n(-1) = \sum_{m=0}^{\infty} \tilde{P}_n^{(m)}(0, 0)\tilde{A}_n(-1),
\]

\[
\tilde{U}_n(0) = \sum_{k=0}^{\infty} \tilde{A}_n(k) \prod_{l=n+k}^{n+1} \tilde{G}_l, \quad (4.26)
\]

\[
\tilde{R}_n(k) = \sum_{m=0}^{\infty} \tilde{A}_n(k + m) \left( \prod_{l=n+k+m}^{n+k+1} \tilde{G}_l \right) \tilde{N}_{n+k}(0, 0),
\]

respectively, where for \( \nu, \eta \in \mathbb{N} \),

\[
\prod_{l=\nu}^{\eta} \tilde{G}_l = \left\{ \begin{array}{cl} I, & \nu < \eta, \\ \tilde{G}_\nu \tilde{G}_{\nu-1} \cdots \tilde{G}_\eta, & \nu \geq \eta. \end{array} \right.
\]

In order to interpret the matrices \( \tilde{N}_n(0, 0), \tilde{G}_n, \tilde{U}_n(0) \) and \( \tilde{R}_n(k) \), we consider a discrete-time Markov chain \( \{ (\tilde{L}_m, \tilde{J}_m); m \in \mathbb{Z}_+ \} \) with state space \( \mathbb{Z}_+ \times \mathbb{M} \) and transition matrix \( \tilde{P} \). For simplicity, we also define \( \mathbb{L}(n) (n \in \mathbb{Z}_+) \) as the set of states \( \{(n, i); i \in \mathbb{M}\} \). In this setting, the interpretation of \( \tilde{N}_n(0, 0), \tilde{G}_n, \tilde{U}_n(0) \) and \( \tilde{R}_n(k) \) is as follows (see [46]):

(i) \( [\tilde{N}_n(0, 0)]_{i,j} \) represents the conditional expected number of visits to state \( (n, j) \) before entering \( \bigcup_{\nu=0}^{n-1} \mathbb{L}(\nu) \) given that \( \{(\tilde{L}_m, \tilde{J}_m)\} \) starts with state \( (n, i) \).

(ii) \( [\tilde{G}_n]_{i,j} \) represents the conditional probability that the first passage time to \( \mathbb{L}(n-1) \) ends with state \( (n-1, j) \) given that \( \{(\tilde{L}_m, \tilde{J}_m)\} \) starts with state \( (n, i) \). Note that during the first passage time to \( \mathbb{L}(n-1) \) from \( \mathbb{L}_m, \{(\tilde{L}_m, \tilde{J}_m)\} \) does not visit any state in \( \bigcup_{\nu=0}^{n-2} \mathbb{L}(\nu) \) because it is skip-free to the left (see, e.g., [26, Chapter 13]).

(iii) \( [\tilde{U}_n(0)]_{i,j} \) represents the conditional probability that the first passage time to \( \bigcup_{\nu=0}^{n} \mathbb{L}(\nu) \) ends with state \( (n, j) \) given that \( \{(\tilde{L}_m, \tilde{J}_m)\} \) starts with state \( (n, i) \).

(iv) \( [\tilde{R}_n(k)]_{i,j} \) represents the conditional expected number of visits to state \( (n+k, j) \) before entering \( \bigcup_{\nu=0}^{n+k-1} \mathbb{L}(\nu) \) given that \( \{(\tilde{L}_m, \tilde{J}_m)\} \) starts with state \( (n, i) \).

**Lemma 4.3**

(i) For all \( n \in \mathbb{N} \), \( \tilde{G}_n \) is stochastic matrix; and

(ii) there exists some \( \xi \in (0, 1) \) such that

\[
\sup_{n \in \mathbb{N}} \tilde{U}_n(0)e \leq \xi e, \quad \sup_{n \in \mathbb{N}} \tilde{N}_n(0, 0)e \leq \frac{1}{1 - \xi} e,
\]

\[
\sup_{n \in \mathbb{Z}_+} \sum_{k=1}^{\infty} \tilde{R}_n(k)e \leq \frac{1}{1 - \xi} \frac{1}{\mu + \theta} \left[ \mu \tilde{A}'(1)e + D \tilde{A}'(1)e + \tilde{D}'(1)e \right],
\]

\[
\text{(4.28)}
\]
where \( \hat{A}'(z) = (d/dz) \hat{A}(z) \) and \( \hat{D}'(z) = (d/dz) \hat{D}(z) \).

**Proof.** Note that \( \hat{P} \) and thus \( \{(\hat{L}_m, \hat{J}_m)\} \) are irreducible and positive recurrent. Note also that \( \{(\hat{L}_m, \hat{J}_m)\} \) is skip-free to the left. Therefore the probabilistic interpretation of \( \hat{G}_n \) implies that statement (i) is true.

Next we prove statement (ii). From (4.26), (4.15) and \( \hat{G}_n e = e \), we obtain

\[
\hat{U}_n(0)e = \sum_{k=0}^{\infty} \hat{A}_n(k)e = e - \hat{A}(-1)e < e, \quad \forall n \in \mathbb{N},
\]  
(4.29)

where the last inequality holds due to statement (i) of Lemma 4.2. According to (4.29), there exists some \( \xi \in (0, 1) \) such that

\[
\hat{U}_n(0)e \leq \xi e, \quad \forall n \in \mathbb{N}.
\]  
(4.30)

In addition, the interpretation of \( \hat{N}_n(0,0) \) and \( \hat{U}_n(0) \) implies that

\[
\hat{N}_n(0,0) = \sum_{m=0}^{\infty} \left( \hat{U}_n(0) \right)^m = \left( I - \hat{U}_n(0) \right)^{-1}.
\]  
(4.31)

Substituting (4.30) into (4.31) yields

\[
\hat{N}_n(0,0)e = \left( I - \hat{U}_n(0) \right)^{-1}e \leq \frac{1}{1 - \xi}e, \quad \forall n \in \mathbb{N}.
\]  
(4.32)

It remains to prove (4.28). From (4.27) and (4.32), we have for \( n \in \mathbb{Z}_+ \),

\[
\sum_{k=1}^{\infty} \hat{R}_n(k)e \leq \frac{1}{1 - \xi} \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \hat{A}_n(k + m)e = \frac{1}{1 - \xi} \sum_{m=0}^{\infty} \hat{A}_n(m)e.
\]

From (4.11) and (4.14), we also have for \( n \in \mathbb{Z}_+ \),

\[
\sum_{m=0}^{\infty} \hat{A}_n(m)e \leq \sum_{m=0}^{\infty} \hat{A}_1(m)e = \sum_{k=0}^{\infty} k \hat{A}_1(k)e \leq \sum_{k=0}^{\infty} (k + 1) \hat{A}_1(k)e
\]

\[
= \frac{1}{\mu + \theta} \sum_{k=1}^{\infty} k \left[ \mu A(k) + \sum_{l=1}^{k} D(l) A(k - l) \right] e
\]

\[
= \frac{1}{\mu + \theta} \left[ \mu \hat{A}'(1)e + \hat{D}(1)e + \hat{D}'(1)e \right],
\]

which is finite due to (2.3), (2.7) and \( \rho = \lambda h < 1 \). As a result, (4.28) holds. \( \square \)

Using Lemma 4.3 and the dominated convergence theorem, we obtain the following:
Lemma 4.4 Let \( \tilde{P}_\infty^{(m)}(0,0) = \lim_{n \to \infty} \tilde{P}_n^{(m)}(0,0) \) for \( m \in \mathbb{Z}_+ \), which is the \( M \times M \) northwest corner of \((\tilde{P}_\infty)^m\). We then have

\[
\lim_{n \to \infty} \tilde{G}_n = \sum_{m=0}^{\infty} \tilde{P}_\infty^{(m)}(0,0) \tilde{A}(-1) =: \tilde{G} > O, \tag{4.33}
\]

\[
\lim_{n \to \infty} \tilde{U}_n(0) = \sum_{k=0}^{\infty} \tilde{A}(k) \tilde{G}^k =: \tilde{U}(0) > O, \tag{4.34}
\]

\[
\lim_{n \to \infty} \tilde{N}_n(0,0) = \left( I - \tilde{U}(0) \right)^{-1} > O, \tag{4.35}
\]

and for \( k \in \mathbb{N} \),

\[
\lim_{n \to \infty} \tilde{R}_n(k) = \sum_{m=0}^{\infty} \tilde{A}(k+m) \tilde{G}^m \left( I - \tilde{U}(0) \right)^{-1} =: \tilde{R}(k) > O. \tag{4.36}
\]

Before the proof of Remark 4.4 we give a remark on \( \tilde{G} \) and \( \tilde{R}(k) \).

Remark 4.2 Consider an M/G/1-type stochastic matrix:

\[
\tilde{P}_{M/G/1} = \begin{pmatrix}
\tilde{B}(0) & \tilde{A}(1) & \tilde{A}(2) & \tilde{A}(3) & \cdots \\
\tilde{A}(-1) & \tilde{A}(0) & \tilde{A}(1) & \tilde{A}(2) & \cdots \\
O & \tilde{A}(-1) & \tilde{A}(0) & \tilde{A}(1) & \cdots \\
O & O & \tilde{A}(-1) & \tilde{A}(0) & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots
\end{pmatrix}, \tag{4.37}
\]

where \( \tilde{B}(0) = \tilde{A}(-1) + \tilde{A}(0) \). From (4.23), we have

\[
\tilde{P}_{M/G/1} = \begin{pmatrix}
\tilde{B}(0) & \tilde{A}(1) & \tilde{A}(2) & \tilde{A}(3) & \cdots \\
\tilde{A}(-1) & \tilde{A}(0) & \tilde{A}(1) & \tilde{A}(2) & \cdots \\
O & \tilde{A}(-1) & \tilde{A}(0) & \tilde{A}(1) & \cdots \\
O & O & \tilde{A}(-1) & \tilde{A}(0) & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots
\end{pmatrix} = \tilde{P}_\infty.
\]

Thus (4.33) and (4.36) imply that \( \tilde{G} \) and \( \tilde{R}(k) \) are the \( G \)- and \( R \)-matrices of the M/G/1-type stochastic matrix \( \tilde{P}_{M/G/1} \) (see [47]). It follows from (4.15)–(4.17) and statement (ii) of Lemma 4.2 that

\[
\tilde{A} := \sum_{k=1}^{\infty} \tilde{A}(k) = \frac{\theta}{\mu + \theta} I + \frac{\mu}{\mu + \theta} A > O,
\]

which shows that \( \tilde{A} \) is an irreducible stochastic matrix and has the same stationary probability vector \( \pi \) as \( A \). Further combining (4.15)–(4.17) with (2.7) and \( \rho < 1 \), we have

\[
\tilde{\rho} := \pi \sum_{k=0}^{\infty} (k+1) \tilde{A}(k)e = \frac{\theta}{\mu + \theta} + \frac{\mu}{\mu + \theta} \pi \sum_{k=1}^{\infty} kA(k)e = \frac{\rho \mu + \theta}{\mu + \theta} < 1.
\]
Tail equivalence for BMAP/GI/1 queues with/without retrials

Note here that \( \tilde{P}_{M/G/1} \) is irreducible due to Lemma 4.2. These facts imply that the irreducible stochastic matrix \( \tilde{P}_{M/G/1} \) is positive recurrent (see, e.g., [4, Chapter XI, Proposition 3.1]). In addition, \( \tilde{G} \) is stochastic (see [35, Theorem 2.3.1]) and the spectral radius of \( \tilde{R} := \sum_{k=1}^{\infty} \tilde{R}(k) \) is less than one (see [47, Theorem 23]).

Proof of Lemma 4.4. Using the dominated convergence theorem, we take the limit of (4.24)–(4.27) as \( n \to \infty \) and obtain (4.33)–(4.36). Therefore it remains to prove the positivity of the limiting matrices.

We note that \( \tilde{G} \) is the \( G \)-matrix of the M/G/1-type stochastic matrix \( \tilde{P}_{M/G/1} \) (see Remark 4.2) and thus it is the unique accumulation point of the following sequence \( \{ \tilde{G}_\nu \} \) (see [35, Chapter 2]):

\[
\tilde{G}_0 = O, \quad \tilde{G}_\nu = \tilde{A}(-1) + \sum_{k=0}^{\infty} \tilde{A}(k) \left( \tilde{G}_{\nu-1} \right)^{k+1} \quad \text{for } \nu \in \mathbb{N},
\]

which leads to

\[
\tilde{G} \geq \tilde{A}(-1) + \sum_{k=0}^{\infty} \tilde{A}(k) \left( \tilde{A}(-1) \right)^{k+1}.
\] (4.38)

Lemma 4.2 shows that the diagonal elements of \( \tilde{A}(-1) \) are all positive and \( \sum_{k=0}^{\infty} \tilde{A}(k) > O \). Thus from (4.38) and (4.34), we have

\[
\tilde{G} > O, \quad \tilde{U}(0) = \sum_{k=0}^{\infty} \tilde{A}(k) \tilde{G}^k > O, \quad (I - \tilde{U}(0))^{-1} \geq \tilde{U}(0) > O.
\]

Finally, the positivity of \( \tilde{R}(k) \) in (4.36) follows from \( \tilde{G}(I - \tilde{U}(0))^{-1} > O \) and statement (ii) of Lemma 4.2. \( \square \)

Lemma 4.4 and Remark 4.2 show that \( \tilde{G} \) is an irreducible stochastic matrix. Thus \( \tilde{G} \) has the unique and positive stationary probability vector, which is denoted by \( \tilde{g} > 0 \) hereafter.

Lemma 4.5 below shows a relationship between \( \tilde{g} \) and \( \pi \). We can readily prove this lemma by using Remark 4.2 and following the proof of Lemma 14 in [42]. Thus we omit the proof.

Lemma 4.5

\[
\pi = (1 - \hat{\rho}) \tilde{g} \left( I - \tilde{U}(0) \right)^{-1} \left( I - \tilde{R} \right)^{-1},
\] (4.39)

where \( \tilde{R} = \sum_{k=1}^{\infty} \tilde{R}(k) \) and \( \hat{\rho} = (\rho\mu + \theta)/(\mu + \theta) \).

Recall that \( q = (q(0), q(1), q(2), \ldots) \) is the stationary probability vector of \( \tilde{P} \) in (4.10). According to Theorems 2.1 and 2.6 in [46], \( q = (q(0), q(1), q(2), \ldots) \) can be characterized as follows:

\[
q(k) = \sum_{n=0}^{k-1} q(n) \tilde{R}_n(k - n), \quad k \in \mathbb{N}.
\] (4.40)

Therefore we discuss the asymptotics for \( \{ \overline{q}(k) \} \) through \( \{ \tilde{R}_n(k) \} \), which requires some preparations.
Lemma 4.6 If Assumption 2.1 holds, then (i) \( c^D := \limsup_{k \to \infty} \overline{D}(k)e/P(Y > k) \) is finite; and (ii) \( \lim_{k \to \infty} \overline{A}(k)/P(Y > k) = O \) and \( \lim_{k \to \infty} \overline{D}(k)/P(Y > k) = O \).

Proof. From (2.6), we have
\[
\sum_{k=0}^{\infty} z^k A(k) = \sum_{m=0}^{\infty} \int_{0}^{\infty} e^{-\theta x} (\theta x)^m \frac{m!}{m!} dH(x) \cdot \left[I + \theta^{-1} \left(C + \hat{D}(z)\right)\right]^m \geq \int_{0}^{\infty} e^{-\theta x} (\theta x) dH(x) \cdot \left[I + \theta^{-1} \left(C + \hat{D}(z)\right)\right],
\]
which leads to
\[
A(k) \geq \zeta \cdot D(k), \quad k \in \mathbb{N}, \quad (4.41)
\]
where \( \zeta = \int_{0}^{\infty} xe^{-\theta x} dH(x) \in (0, \infty) \) due to \( h = \int_{0}^{\infty} xdH(x) \in (0, \infty) \). Therefore (4.41) and Assumption 2.1 show that statement (i) is true. Further, Assumption 2.1 implies that
\[
\lim_{k \to \infty} A(k)e/P(Y > k) \leq \lim_{k \to \infty} \frac{\overline{A}(k-1)ee^t - \overline{A}(k)ee^t}{P(Y > k)} = O.
\]
Combining this and (4.41) yields \( \lim_{k \to \infty} \overline{D}(k)/P(Y > k) = O. \)

It follows from (4.11) and (4.14) that
\[
\tilde{A}_n(k) = \frac{\min(n, 1)\mu}{\mu + \theta} A(k + 1) + \frac{D \ast A(k + 1)}{\max(n, 1)(\mu + \theta)}, \quad n \in \mathbb{Z}_+, \quad k \in \mathbb{N}, \quad (4.42)
\]
where \( D(0) = O \) is defined for convenience. Using (4.42), we show the asymptotics of \( \{\overline{A}_n(k)\} \) and \( \{\tilde{A}_n(k)\} \).

Lemma 4.7 Suppose that Assumption 2.1 is satisfied, then the following hold:

(i) For \( n \in \mathbb{Z}_+ \),
\[
\lim_{k \to \infty} \frac{\overline{A}_n(k)}{P(Y > k)} = O, \quad (4.43)
\]
\[
\limsup_{k \to \infty} \frac{\overline{A}_n(k)e}{P(Y > k)} \leq \frac{\mu}{\mu + \theta} \left(\min(n, 1)c^A + \frac{c^D + Dc^A}{\max(n, 1)\mu}\right) =: \frac{\mu}{\mu + \theta} c^A_n, \quad (4.44)
\]
where \( \sup_{n \in \mathbb{Z}_+} c^A_n \) is finite and \( c^A_n \) is nonzero for all \( n \in \mathbb{N} \) (but \( c^A_0 \) can be a zero vector).

(ii) If \( \lim_{k \to \infty} \overline{D}(k)e/P(Y > k) = c^D \), then
\[
\lim_{k \to \infty} \frac{\overline{A}_n(k)e}{P(Y > k)} = \frac{\mu}{\mu + \theta} c^A_n, \quad n \in \mathbb{Z}_+. \quad (4.45)
\]
Proof. From (4.42), we have
\[
\bar{A}_n(k) = \min(n,1)\mu A(k+1) + D \bar{A}(k+1) \max(n,1)(\mu + \theta), \quad (n,k) \in \mathbb{Z}_+^2, \tag{4.46}
\]

\[
\bar{A}_n(k)e = \min(n,1)\mu A(k+1)e
\]
\[
+ D(k+1)e + D \bar{A}(k+1)e \max(n,1)(\mu + \theta), \quad (n,k) \in \mathbb{Z}_+^2, \tag{4.47}
\]

where we use \( D \bar{A}(k) = D(k)A + D \bar{A}(k) \) in (4.47). It follows from statement (ii) of Lemma 4.6 and Proposition B.2 that
\[
\lim_{k \to \infty} \frac{D \bar{A}(k+1)}{P(Y > k)} = O. \tag{4.48}
\]

Applying (4.48) and \( \lim_{k \to \infty} \bar{A}(k+1)/P(Y > k) = O \) to (4.46) yields (4.43).

Using Assumption 2.1, Lemma 4.6 and Proposition B.2 we obtain
\[
\limsup_{k \to \infty} \frac{D(k+1)e + D \bar{A}(k+1)e}{P(Y > k)} \leq c^D + Dc^A. \tag{4.49}
\]

Further, if \( \lim_{k \to \infty} \bar{D}(k)e/P(Y > k) = c^D \),
\[
\lim_{k \to \infty} \frac{\bar{D}(k+1)e + D \bar{A}(k+1)e}{P(Y > k)} = c^D + Dc^A. \tag{4.50}
\]

Applying (4.49) and Assumption 2.1 to (4.47), we have (4.44). Similarly, if \( \lim_{k \to \infty} \bar{D}(k)e/P(Y > k) = c^D \), we have (4.45), though we use (4.50) instead of (4.49). The statement on \( \{c^A_n\} \) follows from the definition of \( \{c^A_n\} \) and \( c^A \geq 0, \neq 0 \). \( \square \)

Lemma 4.8 If Assumption 2.1 is satisfied, then the following hold:

(i) The limit
\[
\lim_{k \to \infty} \frac{\bar{R}(k)}{P(Y > k)} = \frac{\mu}{\mu + \theta} c^A g \left( I - \bar{U}(0) \right)^{-1} =: C^R \tag{4.51}
\]
exists, and \( C^R \) has no zero columns.

(ii) For \( n \in \mathbb{Z}_+ \),
\[
\limsup_{k \to \infty} \frac{\bar{R}_n(k)}{P(Y > k)} \leq \frac{\mu}{\mu + \theta} c^A_n g(I - \bar{U}(0))^{-1} =: C^R_n, \tag{4.52}
\]

where \( \sup_{n \in \mathbb{Z}_+} C^R_n \) is finite and \( C^R_n \) has no zero columns for all \( n \in \mathbb{N} \) (but \( C^R_0 \) can be a zero matrix).
(iii) If \( \lim_{k \to \infty} \overline{D}(k) e / \mathbb{P}(Y > k) = c^D \), then

\[
\lim_{k \to \infty} \frac{\overline{R}_n(k)}{\mathbb{P}(Y > k)} = C_{n, \infty}, \quad n \in \mathbb{Z}^+.
\]  

(4.53)

**Proof.** See Appendix A.1.

Lemma 4.9 Let \( \Gamma(k) \ (k \in \mathbb{Z}^+) \) denote

\[
\Gamma(k) = \sum_{m=0}^{\infty} D \ast A(k + m + 1) \tilde{G}^m (I - \tilde{U}(0))^{-1}, \quad k \in \mathbb{Z}^+.
\]  

(4.54)

The following hold:

(i) For any \( \varepsilon > 0 \), there exists some \( n_0 := n_0(\varepsilon) \in \mathbb{N} \) such that for all \( n \geq n_0 \),

\[
(1 - \varepsilon) \tilde{R}(k) \leq \tilde{R}_n(k) \leq (1 + \varepsilon) \left\{ \tilde{R}(k) + \varepsilon \Gamma(k) \right\}, \quad k \in \mathbb{N}.
\]  

(4.55)

(ii) If Assumption 2.1 holds, then

\[
\limsup_{k \to \infty} \frac{T(k)}{\mathbb{P}(Y > k)} \leq (Dc^A + c^D) \tilde{g} \left( I - \tilde{U}(0) \right)^{-1} =: C^\Gamma.
\]  

(4.56)

**Proof.** See Appendix A.2.

We are now ready to prove Lemma 4.1.

**Proof of Lemma 4.1.** For \( \varepsilon > 0 \), we fix \( n_0 := n_0(\varepsilon) \) for which statement (i) of Lemma 4.9 holds. We then define \( s^+_\varepsilon = (s^+_\varepsilon(0), s^+_\varepsilon(1), s^+_\varepsilon(2), \ldots) \) and \( s^-\varepsilon = (s^-\varepsilon(0), s^-\varepsilon(1), s^-\varepsilon(2), \ldots) \) as follows:

\[
s^+_\varepsilon(0) = s^-\varepsilon(0) = (q(0), q(1), \ldots, q(n_0)),
\]  

(4.57)

and for \( k \in \mathbb{N} \),

\[
s^+_\varepsilon(k) = s^+_\varepsilon(0) \tilde{R}_{(0,n_0)}(k) + (1 + \varepsilon) \sum_{n=1}^{k-1} s^+_\varepsilon(n) \left( \tilde{R}(k-n) + \varepsilon \Gamma(k-n) \right),
\]  

(4.58)

\[
s^-\varepsilon(k) = s^-\varepsilon(0) \tilde{R}_{(0,n_0)}(k) + (1 - \varepsilon) \sum_{n=1}^{k-1} s^-\varepsilon(n) \tilde{R}(k-n),
\]  

(4.59)

where

\[
\tilde{R}_{(0,n_0)}(k) = \begin{pmatrix}
\tilde{R}_0(k + n_0) \\
\tilde{R}_1(k + n_0 - 1) \\
\vdots \\
\tilde{R}_{n_0}(k)
\end{pmatrix}, \quad k \in \mathbb{N}.
\]  

(4.60)
For convenience, let \( \tilde{R}_{0,n_0}(0) = O \) and \( \tilde{R}(0) = O \). Let \( \tilde{R} = \sum_{k=0}^{\infty} \tilde{R}(k) \) and \( \Gamma = \sum_{k=0}^{\infty} \Gamma(k) \). Recall here that the spectral radius of \( \tilde{R} \) is less than one (see Remark 4.2) and thus so is that of \( (1 - \varepsilon) \tilde{R} \). Further for any sufficiently small \( \varepsilon > 0 \), the spectral radius of \( (1 + \varepsilon)(\tilde{R} + \varepsilon \Gamma) \) is less than one (see, e.g., Theorem 8.1.18 in [16]). We fix \( \varepsilon > 0 \) to be such a small value.

Following the proof of Theorem 1 in [43], we can readily show that

\[
\begin{align*}
\mathbf{e}_k = \mathbf{e}_k^+(0) \tilde{R}_{0,n_0} \sum_{m=0}^{\infty} (1 + \varepsilon)^m (\tilde{R} + \varepsilon \Gamma)^m(k) =: \mathbf{e}_k^+(k), \quad k \in \mathbb{N},
\end{align*}
\]

where \( \{(\tilde{R} + \varepsilon \Gamma)^m(k); k \in \mathbb{Z}_+\} \) is the \( m \)-fold convolution of \( \{\tilde{R}(k) + \varepsilon \Gamma(k); k \in \mathbb{Z}_+\} \) itself.

It follows from statement (i) of Lemma 4.8, statement (ii) of Lemma 4.9 and Proposition B.1 that

\[
\begin{align*}
\limsup_{k \to \infty} \sum_{m=0}^{\infty} \frac{(1 + \varepsilon)^m (\tilde{R} + \varepsilon \Gamma)^m(k)}{p(Y > k)} \leq & \left\{ I - (1 + \varepsilon)(\tilde{R} + \varepsilon \Gamma) \right\}^{-1} (1 + \varepsilon)(C^R + \varepsilon C^\Gamma) \\
& \times \left\{ I - (1 + \varepsilon)(\tilde{R} + \varepsilon \Gamma) \right\}^{-1}.
\end{align*}
\]

Further statement (ii) of Lemma 4.8 yields

\[
\begin{align*}
\limsup_{k \to \infty} \frac{\tilde{R}_{0,n_0}(k)}{p(Y > k)} \leq \left( \begin{array}{c}
C^R_{0,n_0} \\
C^R_{1,n_0} \\
\vdots \\
C^R_{n_0}
\end{array} \right) =: C^R_{(0,n_0)} \neq O.
\end{align*}
\]

Applying Proposition B.2 to (4.61) and using (4.62) and (4.63), we obtain

\[
\begin{align*}
\limsup_{k \to \infty} \frac{\mathbf{e}_k}{p(Y > k)} \leq \mathbf{e}_k^+(0) C^R_{(0,n_0)} \left\{ I - (1 + \varepsilon)(\tilde{R} + \varepsilon \Gamma) \right\}^{-1} \\
+ \mathbf{e}_k^+(0) \sum_{n=1}^{\infty} \tilde{R}_{0,n_0}(n) \left\{ I - (1 + \varepsilon)(\tilde{R} + \varepsilon \Gamma) \right\}^{-1} \\
\times (1 + \varepsilon)(C^R + \varepsilon C^\Gamma) \left\{ I - (1 + \varepsilon)(\tilde{R} + \varepsilon \Gamma) \right\}^{-1}.
\end{align*}
\]

Recall here that \( n_0 \to \infty \) as \( \varepsilon \downarrow 0 \) (see Lemmas 4.4 and 4.9). Recall also that \( \sup_{n \in \mathbb{Z}_+} \sum_{k=1}^{\infty} \tilde{R}_n(k) e \) is finite (see Lemma 4.3). Thus using (4.57), (4.60) and the dominated convergence theorem,
we have

\[ \lim_{\varepsilon \downarrow 0} s_\varepsilon^+(0) \sum_{n=1}^{\infty} \tilde{R}_{(0,n_0)}(n) = \lim_{n_0 \to \infty} \sum_{n=1}^{n_0} \sum_{l=0}^{n_0} q(l) \tilde{R}_l(n_0 - l) = \sum_{l=0}^{\infty} q(l) \lim_{n_0 \to \infty} \tilde{R}_l(n_0 - l) = 0. \]

Therefore letting \( \varepsilon \downarrow 0 \) in (4.64) and using (4.57) and (4.60) yield

\[ \lim_{\varepsilon \downarrow 0} \limsup_{k \to \infty} s_\varepsilon^+(k) \leq \sum_{n=0}^{\infty} q(n) c_n^A \frac{\mu}{\mu + \theta} \frac{\pi}{1 - \bar{\rho}}. \] (4.65)

It also follows from (4.39) and the definition of \( C_n^R \) (see (4.52)) that

\[ C_n^R \left( I - \bar{R} \right)^{-1} = \frac{\mu}{\mu + \theta} \frac{c_n^A \pi}{1 - \bar{\rho}}. \]

Substituting this equation into (4.65), we obtain

\[ \lim_{\varepsilon \downarrow 0} \limsup_{k \to \infty} \frac{\mathbf{s}_\varepsilon^+(k)}{\mathbf{P}(Y > k)} \leq \sum_{n=0}^{\infty} q(n) c_n^A \cdot \frac{\mu}{\mu + \theta} \frac{\pi}{1 - \bar{\rho}}. \] (4.66)

It is proved later that

\[ s_-^-(k) \leq q(k + n_0) \leq s_+^+(k), \quad k \in \mathbb{N}. \] (4.67)

Combining (4.67) with (4.66) and using \( Y \in S \subset L \), we obtain

\[ \limsup_{k \to \infty} \frac{\bar{q}(k)}{\mathbf{P}(Y > k)} \leq \sum_{n=0}^{\infty} q(n) c_n^A \cdot \frac{\mu}{\mu + \theta} \frac{\pi}{1 - \bar{\rho}}. \] (4.68)

Note here that \( q(n) > 0 \) for all \( n \in \mathbb{Z}_+ \) (see (4.1)) and that \( \sup_{n \in \mathbb{Z}_+} c_n^A \) is finite and \( c_n^A \geq 0, \neq 0 \) for all \( n \in \mathbb{N} \) (see statement (i) of Lemma 4.7). As a result,

\[ 0 < \sum_{n=0}^{\infty} q(n) c_n^A < \infty, \]

which completes the proof of statement (i).

Next we prove statement (ii) under the condition that \( \lim_{k \to \infty} \mathbf{D}(k)e = c^D \). As with (4.61), the following equation holds:

\[ s_\varepsilon^-(k) = s_\varepsilon^-(0) \tilde{R}_{(0,n_0)} \ast \sum_{m=0}^{\infty} (1 - \varepsilon)^m \tilde{R}_m^*(k), \quad k \in \mathbb{N}. \] (4.69)
It follows from statements (i) and (iii) of Lemma 4.8 and Proposition B.1 that

\[
\lim_{k \to \infty} \frac{\mathcal{R}_{(0,n_0)}(k)}{\mathbb{P}(Y > k)} = C^R(0,n_0),
\]

\[
\lim_{k \to \infty} \sum_{m=0}^{\infty} \frac{(1-\varepsilon)^m \mathcal{R}^{\ast m}(k)}{\mathbb{P}(Y > k)} = \left\{ I - (1-\varepsilon)\mathcal{R} \right\}^{-1} (1-\varepsilon)C^R
\]

\times \left\{ I - (1-\varepsilon)\mathcal{R} \right\}^{-1}.
\]

Using (4.69)–(4.71) and following the proof of statement (i), we can show that

\[
\liminf_{k \to \infty} q(k) \mathbb{P}(Y > k) \geq \lim_{\varepsilon \downarrow 0} \lim_{k \to \infty} s_{\varepsilon}^-(1) = \sum_{n=0}^{n_0} q(n) \mathcal{R}_n(n_0 + 1 - n) = q(n_0 + 1),
\]

which shows that (4.67) holds for \(k = 1\). Suppose that (4.67) holds for some \(k = k_* \in \mathbb{N}\). Substituting this inductive assumption and the right inequality in (4.55) into (4.58) with \(k = k_* + 1\) yields

\[
s_{\varepsilon}^+(k_* + 1) \geq s_{\varepsilon}^+(0) \mathcal{R}_{(0,n_0)}(k_* + 1) + (1+\varepsilon) \sum_{n=1}^{k_*} q(n + n_0) \left( \mathcal{R}(k_* + 1 - n) + \varepsilon \Gamma(k_* + 1 - n) \right)
\]

\[
\geq s_{\varepsilon}^+(0) \mathcal{R}_{(0,n_0)}(k_* + 1) + \sum_{n=1}^{k_*} q(n + n_0) \mathcal{R}_{n+n_0}(k_* + 1 - n)
\]

\[
= \sum_{n=0}^{n_0} q(n) \mathcal{R}_n(k_* + 1 + n_0 - n) + \sum_{n=1}^{k_*} q(n + n_0) \mathcal{R}_{n+n_0}(k_* + 1 - n)
\]

\[
= \sum_{n=0}^{n_0} q(n) \mathcal{R}_n(k_* + 1 + n_0 - n) = q(k_* + 1 + n_0),
\]

where the last equality is due to (4.40). As a result, the right inequality in (4.67) has been proved. The left one is proved in a similar way.
A Proofs

A.1 Proof of Lemma 4.8

It follows from Assumption 2.1 and (4.17) that
\[ \lim_{k \to \infty} \frac{\tilde{A}(k)}{\Pr(Y > k)} e = \frac{\mu}{\mu + \theta} c^A. \]

Using this equation and proceeding as in the proof of Lemma 3.2 in [32], we can show that
\[ \lim_{k \to \infty} \frac{\tilde{R}(k)}{\Pr(Y > k)} = \frac{\mu}{\mu + \theta} c^A \pi(I - \tilde{R}) \left(1 - \rho\right), \]
from which and Lemma 4.5 it follows that the limit in (4.51) exists. It also follows from \(\tilde{g} > 0\), \((I - \tilde{U}(0))^{-1} > O\) (see Lemma 4.4) and \(c^A \geq 0\), \(\neq 0\) (see Assumption 2.1) that \(C^R\) has no zero columns. Thus statement (i) holds.

Similarly we can prove statements (ii) and (iii), though we need additional steps. For completeness, we provide the proof of statements (ii) and (iii).

Lemma 4.4 implies that
\[ \lim_{k \to \infty} \left( \prod_{t=n+k+1}^{n+k+1} \tilde{G}_t \right) \tilde{N}_{n+k}(0,0) = \tilde{G}^m \left(I - \tilde{U}(0)\right)^{-1} \text{ uniformly over } m, n \in \mathbb{Z}_+. \] (A.1)

Fix \(\varepsilon > 0\) arbitrarily, which is independent of \(n\). It then follows from (A.1) and (4.27) that for all sufficiently large \(k\),
\[ \tilde{R}_n(k) \leq (1 + \varepsilon) \sum_{m=0}^{\infty} \tilde{A}_n(k + m) \tilde{G}^m \left(I - \tilde{U}(0)\right)^{-1}, \quad n \in \mathbb{Z}_+, \] (A.2)
\[ \tilde{R}_n(k) \geq (1 - \varepsilon) \sum_{m=0}^{\infty} \tilde{A}_n(k + m) \tilde{G}^m \left(I - \tilde{U}(0)\right)^{-1}, \quad n \in \mathbb{Z}_+. \] (A.3)

Recall here that \(\tilde{G}\) is a positive stochastic matrix with stationary probability vector \(\tilde{g}\) and thus \(\lim_{m \to \infty} \tilde{G}^m = e\tilde{g}\) (see, e.g., [16] Theorem 8.2.8]). Therefore there exists some \(m_1 := m_1(\varepsilon) \in \mathbb{N}\) such that for all \(m > m_1\),
\[ (1 - \varepsilon)e\tilde{g} \leq \tilde{G}^m \leq (1 + \varepsilon)e\tilde{g}. \] (A.4)
Substituting the right inequality in (A.4) into (A.2), we have for all sufficiently large $k$,
\[
\bar{R}_n(k) \leq (1 + \varepsilon) \sum_{m=0}^{m_1} \bar{A}_n(k + m) \bar{G}^m \left( I - \bar{U}(0) \right)^{-1} + (1 + \varepsilon)^2 \sum_{m=m_1+1}^{\infty} \bar{A}_n(k + m) e \bar{g} \left( I - \bar{U}(0) \right)^{-1} \leq (1 + \varepsilon) \sum_{m=0}^{m_1} \bar{A}_n(k + m) \bar{G}^m \left( I - \bar{U}(0) \right)^{-1} + (1 + \varepsilon)^2 \bar{A}_n(k + m_1) e \bar{g} \left( I - \bar{U}(0) \right)^{-1}. \tag{A.5}
\]

It also follows from (4.43) and $Y \in S \subset L$ that for any fixed $m \in \mathbb{N}$ and $n \in \mathbb{Z}_+$,
\[
\lim_{k \to \infty} \frac{\bar{A}_n(k + m)}{P(Y > k)} = \lim_{k \to \infty} \frac{\bar{A}_n(k + m)}{P(Y > k + m)} \frac{P(Y > k + m)}{P(Y > k)} = O. \tag{A.6}
\]

Applying (A.6), (4.44) and Proposition B.2 to (A.5), we obtain
\[
\limsup_{k \to \infty} \frac{\bar{R}_n(k)}{P(Y > k)} \leq (1 + \varepsilon)^2 \frac{\mu}{\mu + \theta} \bar{c}_n \bar{g} \left( I - \bar{U}(0) \right)^{-1} = (1 + \varepsilon)^2 \bar{C}_n^R. \tag{A.7}
\]

Letting $\varepsilon \downarrow 0$ in (A.7) yields (4.52). In addition, since $\sup_{n \in \mathbb{Z}_+} \bar{c}_n$ is finite (see Lemma 4.7), so is $\sup_{n \in \mathbb{Z}_+} \bar{C}_n^R$. In addition, $\bar{C}_n^R$ ($\forall n \in \mathbb{N}$) has no zero columns because $\bar{g} > 0$, $\left( I - \bar{U}(0) \right)^{-1} > O$ and $\bar{c}_n \geq 0$, $\bar{c}_n \neq 0$ for $n \in \mathbb{N}$.

Finally, we assume that $\lim_{k \to \infty} \bar{D}(k) e / P(Y > k) = e^D$. Using (A.3) and the left inequality in (A.4) (and following the proof of (4.52)), we can show that
\[
\liminf_{k \to \infty} \frac{\bar{R}_n(k)}{P(Y > k)} \geq \bar{C}_n^R.
\]

Combining this and (4.52), we have (4.53).

### A.2 Proof of Lemma 4.9

We estimate $\bar{R}_n(k)$ in (4.27) as a function of $n$. Similarly to (A.1), Lemma 4.4 implies that
\[
\lim_{n \to \infty} \left( \prod_{i=n+k+m}^{n+k+1} \tilde{G}^i \right) \tilde{N}_{n+k}(0, 0) = \tilde{G}^m \left( I - \tilde{U}(0) \right)^{-1},
\]
where the convergence is uniform over $(k, m) \in \mathbb{N} \times \mathbb{Z}_+$. Thus for any $\varepsilon > 0$, there exists some $n' := n'(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n'$,
\[
\bar{R}_n(k) \leq (1 + \varepsilon) \sum_{m=0}^{\infty} \bar{A}_n(k + m) \bar{G}^m \left( I - \bar{U}(0) \right)^{-1}, \quad k \in \mathbb{N}, \tag{A.8}
\]
\[
\bar{R}_n(k) \geq (1 - \varepsilon) \sum_{m=0}^{\infty} \bar{A}_n(k + m) \bar{G}^m \left( I - \bar{U}(0) \right)^{-1}, \quad k \in \mathbb{N}. \tag{A.9}
\]
It follows from (4.42) and (4.17) that for all \( n \geq \lceil 1/(\varepsilon (\mu + \theta)) \rceil \),
\[
\tilde{A}_n(k) \leq \tilde{A}(k) + \varepsilon D \ast A(k + 1), \quad k \in \mathbb{N},
\]  
(A.10)
and that for all \( n \in \mathbb{N} \),
\[
\tilde{A}_n(k) \geq \tilde{A}(k), \quad k \in \mathbb{N}.
\]  
(A.11)
Substituting (A.10) and (A.11) into (A.8) and (A.9) respectively and using (4.36) and (4.54),
we obtain for all \( n \geq n_0 := n_0(\varepsilon) = \max(n', [1/(\varepsilon (\mu + \theta))]) \),
\[
\tilde{R}_n(k) \leq (1 + \varepsilon) \left( \tilde{R}(k) + \varepsilon R(k) \right), \quad k \in \mathbb{N},
\]
\[
\tilde{R}_n(k) \geq (1 - \varepsilon) \tilde{R}(k), \quad k \in \mathbb{N},
\]
which show that statement (i) holds.

As for statement (ii), we can prove this by using (A.4), (4.48) and (4.49) and following
the proof of Lemma 3.2 in [32]. The proof of statement (ii) is also similar to that of Lemma 4.8
(see Appendix A.1). Therefore we omit the details.

**B  Convolution of Matrix Sequences with Subexponential Tails**

The following are basic asymptotic results on the convolution of matrix sequences associated
with subexponential tails.

**Proposition B.1** Suppose that \( \{M \}(k); k \in \mathbb{Z}_+ \) is a sequence of nonnegative square matrices
such that \( \sum_{n=0}^{\infty} M^n = (I - M)^{-1} < \infty \).

(i) If there exists some \( U \in \mathcal{S} \) such that
\[
\limsup_{k \to \infty} \frac{\bar{M}(k)}{P(U > k)} \leq \tilde{M},
\]
then
\[
\limsup_{k \to \infty} \frac{\sum_{n=0}^{\infty} M^m(k)}{P(U > k)} \leq (I - M)^{-1} \tilde{M} (I - M)^{-1}.
\]
(ii) Replacing “\( \limsup \)” and “\( \leq \)” by “\( \liminf \)” and “\( \geq \)”, respectively, in statement (i), we
have a true statement.

(iii) Replacing “\( \limsup \)” and “\( \leq \)” by “\( \lim \)” and “\( = \)”, respectively, in statement (i), we have
a true statement.

**Proposition B.2** Suppose that \( \{M \}(k); k \in \mathbb{Z}_+ \) and \( \{N \}(k); k \in \mathbb{Z}_+ \) are finite-dimensional
nonnegative matrix sequences such that their convolution is well-defined. Further suppose that
\( M := \sum_{k=0}^{\infty} M(k) \) and \( N := \sum_{k=0}^{\infty} N(k) \) are finite. Under these conditions, the following
hold:
Tail equivalence for BMAP/GI/1 queues with/without retrials

(i) If there exists some $U \in \mathcal{S}$ such that

$$\limsup_{k \to \infty} \frac{M(k)}{P(U > k)} \leq \tilde{M}, \quad \limsup_{k \to \infty} \frac{N(k)}{P(U > k)} \leq \tilde{N},$$

then

$$\limsup_{k \to \infty} \frac{M \ast N(k)}{P(U > k)} \leq \tilde{M}N + M\tilde{N}.$$

(ii) Replacing “$\limsup$” and “$\leq$” by “$\liminf$” and “$\geq$”, respectively, in statement (i), we have a true statement.

(iii) Replacing “$\limsup$” and “$\leq$” by “$\lim$” and “$=$”, respectively, in statement (i), we have a true statement.

Proof of Propositions B.1 and B.2  The first statements (on the limit superiors) of Propositions B.1 and B.2 are presented in Lemma A.12 in [33]. Following the proof of the lemma, we can readily prove the second statements (on the limit inferiors) of the two propositions. The third statements are immediate from the first and second ones, and they also presented in Lemma 6 in [17] and Proposition A.3 in [31].

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