STABLE NLS SOLITONS IN A CUBIC-QUINTIC MEDIUM WITH A DELTA-FUNCTION POTENTIAL

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Abstract. We study the one-dimensional nonlinear Schrödinger equation with the cubic-quintic combination of attractive and repulsive nonlinearities, and a trapping potential represented by a delta-function. We determine all bound states with a positive soliton profile through explicit formulas and, using bifurcation theory, we describe their behavior with respect to the propagation constant. This information is used to prove their stability by means of the rigorous theory of orbital stability of Hamiltonian systems. The presence of the trapping potential gives rise to a regime where two stable bound states coexist, with different powers and same propagation constant.

1. Introduction

In this paper we study the one-dimensional nonlinear Schrödinger (NLS) equation with the cubic-quintic (CQ) combination of attractive and repulsive nonlinearities, and a trapping potential represented by a delta-function:

\[ i\psi_z = -\psi_{xx} - \epsilon \delta(x) \psi - 2|\psi|^2 \psi + |\psi|^4 \psi, \]

for complex \( \psi = \psi(x,z) \), and \( \epsilon > 0 \). The objective of the analysis is the existence and stability of (soliton-like) bound states, in the form of \( \psi(x,z) = e^{ikz}u(x) \), with \( u(x) > 0 \) satisfying the respective stationary equation:

\[ u'' - ku + \epsilon \delta(x)u + 2u^3 - u^5 = 0. \]

Here and henceforth, \( ' \) stands for differentiation with respect to \( x \). Denoting by \( \langle \cdot, \cdot \rangle \) the duality product between \( H^{-1}(\mathbb{R}) \) and \( H^1(\mathbb{R}) \) — and recalling that \( H^1(\mathbb{R}) \subset C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) —, the potential \( \delta \) appearing in the soliton equation (1.2) is the Dirac distribution at \( x = 0 \), defined by \( \langle \delta, u \rangle = u(0) \) for all \( u \in H^1(\mathbb{R}) \). In the context of (1.1), \( \delta \) is interpreted similarly (at each fixed \( z \)). Hence, (1.1) and (1.2) should be understood in the sense of distributions, even though the solutions will be smooth outside of \( x = 0 \).

Problem (1.1)–(1.2) belongs to a family of models featuring the competition between self-focusing cubic and defocusing quintic terms, that have drawn considerable attention in both the physical and the mathematical communities in recent years, see [3, 12, 13, 16, 20, 26, 27] and the references therein. This combination of nonlinearities is well known in optical media, including liquid waveguides [11] and speciality glasses [3]. Especially interesting are colloids containing metallic nanoparticles, where the CQ nonlinearity can be widely adjusted by selecting the radius of the suspended nanoparticles and the colloidal filling factor [10]. Remarkably, the one-dimensional NLS equation with the CQ nonlinearity admits completely stable exact soliton solutions [7, 22], although this equation is not integrable. The exact soliton solutions are available, and stable too, also in the case when both the cubic and quintic terms in the one-dimensional NLS equation have the self-focusing sign [21]. The rigorous stability analysis of one-dimensional NLS solitons with general double-power nonlinearities can be found in [17, 19]. Then, the effective linear potential term...
added to the NLS equation represents a trapping (waveguiding) structure for light beams, induced by an inhomogeneity of the local refractive index. In particular, the delta-function term adequately represents a narrow trap which is able to capture broad solitonic beams.

Existence and stability of bound states of one-dimensional NLS equations with a delta potential and a single power-law nonlinearity $|\psi|^{p-1}\psi$, $p > 1$, have been extensively discussed earlier. We refer the reader to [12, 16, 18] for more information about this. From the mathematical point of view, the presence of the delta-function potential has several interesting consequences. The range of values of the propagation constant for solutions in free space (i.e., with $\epsilon = 0$) is $k \in (0, \frac{4}{9})$ (the same as for the above-mentioned exact soliton solutions in free space [1, 22]), and in this case the bifurcation diagram for the bound states is very simple, see Fig. 6 in Section 5. Namely, the solutions can be parametrized by $k \in (0, \frac{4}{9})$, they bifurcate from $u = 0$ at $k = 0$, and their $L^2$ norm (i.e., the integral power of the beam in the optical models) diverges as $k \nearrow \frac{4}{9}$. This accounts for the saturation of the nonlinear refractive index for high-power beams in the CQ optical media. The presence of the potential gives rise to a fold bifurcation point located to the right of $k = \frac{3}{4}$. The bifurcating curve now starts off the trivial line at $k = \frac{2}{3}$, can be parametrized by $k$ up to $k = \frac{3}{4} + \frac{4}{9}$ where it ‘turns backwards’, and again blows up in $L^2(\mathbb{R})$, but now as $k \searrow \frac{4}{9}$. The respective bifurcation diagrams are displayed in Fig. 3 in Section 6 for various values of the coupling constant $\epsilon > 0$. This phenomenon was already observed in [13], where solitons in a cubic focusing–quintic defocusing medium with a square well potential were studied by means of numerical methods and the variational approximation. In the case of the delta-potential considered here, the fold bifurcation can be described by an exact analysis, as demonstrated in Section 3. Since the parametrization by $k$ breaks down at $k = \frac{3}{4}$, where the linearization of (1.2) becomes singular, we resort to the work of Crandall and Rabinowitz [9] which provides the right framework to deal with this situation.

An important remark at this stage is the multiplicity of positive solutions of (1.2) for $k \in (\frac{4}{9}, \frac{3}{4})$. In fact, the first step of our analysis, in Section 2, is the explicit determination of all positive solutions of (1.2), in terms of elementary functions; this is a noteworthy feature of the present model. Of course, the expressions obtained are somewhat cumbersome, yet we are able to extract important information from them, notably as regards the stability of the bound states of (1.1). We will thus show explicitly that, for each fixed $k \in (\frac{4}{9}, \frac{3}{4})$, there are exactly two positive solutions of (1.2), and that the corresponding bound states of (1.1) are both stable. This bistability phenomenon was previously observed numerically in [13] for the square well potential, see also [27]. In the present context, we can prove the stability rigorously. The fact that the ‘upper branch’ is stable, while the $L^2$ norm of the solutions is decreasing along it, appeared puzzling when it was first discovered in [13]. However, in the case of a delta-potential considered here, a careful analysis of the spectrum of the linearization of (1.2) reveals that it is strictly positive along the upper branch, and its stability then follows from the general theory of orbital stability in [14]. Along the ‘lower branch’, the linearized operator has one simple negative eigenvalue, and the rest of its spectrum is positive. In this case, the Vakhitov–Kolokolov (VK) stability criterion [22] (which requires that the $L^2$ norm is increasing in $k$) ensures the stability. Note that, for each fixed $k \in (\frac{4}{9}, \frac{3}{4}, \frac{2}{3})$, the positive solution of (1.2) is unique, and the corresponding bound state is also stable. Therefore, all positive solutions of (1.2) give rise to stable bound states of (1.1). The stability analysis is carried out in full detail in Section 5. The bistability of coexisting bound states with different powers and same propagation constant offers potential applications to optics in terms of switching and other elements of all-optical data processing [13].

We would also like to comment on the important role symbolic computer calculations (using Mathematica) and numerical simulations played in our analysis. Mathematica was a powerful tool to compute exact formulas that were too involved to be dealt with manually. This transpires both in the calculation of solutions in the regime $k \in (\frac{4}{9}, \frac{3}{4})$ in Section 2 and in the stability analysis of Section 5. On the other hand, numerical experiments were very useful at early stages of this work, in order to understand the behavior of solutions, before their explicit representations had been found. We used the so-called ‘continuous normalized gradient flow’ (CNGF), which was studied and implemented in [1] in the context of the NLS equation with a cubic nonlinearity. The excellent agreement between the numerical and the exact solutions (see Fig. 6) demonstrates the effectiveness of this scheme in the context of (1.2). The CNGF method being based on constraint minimization (see Section 6), this also suggests that the positive solutions of (1.2) should admit a variational characterization. Our analytical approach allows us to describe the spectral and stability
properties of the bound states of (1.1) without resorting to such a variational characterization. This would however present an interest on its own; see for instance (12) for results in this direction in the case of a delta-potential combined with a single power nonlinearity.

Lastly, it is relevant to mention that recent numerical and analytical considerations have demonstrated that the same delta-like attractive potential may effectively stabilize trapped solitons in the NLS equation with a combination of defocusing cubic and focusing quintic terms (the signs opposite to those dealt with in the present work) (28). In the free-space version of the latter equation, all solitons are completely unstable.

2. Explicit solutions

We first establish some elementary properties of positive solutions of (1.2).

Proposition 1. Let \( k > 0 \) and \( u \in H^1(\mathbb{R}) \) be a non-negative non-trivial (i.e., \( u \geq 0 \) but \( u \not\equiv 0 \)) solution of (1.2). Then \( u \) satisfies:

(i) \( u'' - ku + 2u^3 - u^5 = 0, \) \( x \neq 0; \)
(ii) \( u \in C^r(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R}), \) \( r = 1, 2; \)
(iii) \( u'(0^\pm) = \mp (\epsilon/2) u(0); \)
(iv) \( u \) is even on \( \mathbb{R}; \)
(v) \( u(x), u'(x) \to 0 \) as \( |x| \to \infty. \)

Proof. Properties (i)–(iii) and (v) can be established following the proofs of Lemmas 3.1 and 3.2 in (12). As for (iv), observing that \( w(x) := u(x) - u(-x) \) satisfies the initial value problem

\[
\begin{align*}
w''(x) - kw(x) + 2a(x)w(x) - b(x)w(x) &= 0, \quad w(0) = w'(0) = 0,
\end{align*}
\]

where

\[
a(x) = u^2(x) + u(x)u(-x) + u^2(-x), \quad b(x) = u^4(x) + u^3(x)u(-x) + u^2(x)u^2(-x) + u(x)u^3(-x) + u^4(-x),
\]

we conclude by Cauchy’s theorem that \( w(x) \equiv 0 \), which proves the claim. \( \Box \)

It turns out that all solutions described by Proposition 1 are in fact positive, that is, satisfy \( u(x) > 0 \) for all \( x \in \mathbb{R} \). Furthermore — and this is quite remarkable — they can all be expressed in terms of elementary functions. As will be seen shortly, this is especially striking in the range of the propagation constant \( k > 3/4 \) which is not allowed in free space (i.e., when \( \epsilon = 0 \)) (12, 22). We will now determine all positive solutions of (1.2).

First, multiplying (1.2) by \( u' \) and integrating from \( x > 0 \) to \( x < 0 \), respectively from \( -\infty \) to \( x < 0 \), we get

\[
(u')^2(x) - ku^2(x) + u^4(x) - (1/3)u_0^2(x) = 0, \quad x \neq 0.
\]

In particular, taking the limit \( x \to 0^\pm \) and using Proposition 1 (iii) yields, assuming that \( u(0) \neq 0 \),

\[
u^4(0) - 3u^2(0) + 3(k - \epsilon^2/4) = 0,
\]

the solutions of which are

\[
u^2_{\pm, k, \epsilon}(0) = \frac{3}{2} \left( 1 \pm \sqrt{1 - \frac{3}{4} (k - \frac{\epsilon^2}{4})} \right).
\]

Note that both \( \nu^2_{\pm, k, \epsilon}(0) \) exist and are positive if and only if

\[
\frac{\epsilon^2}{4} < k \leq \frac{3}{4} + \frac{\epsilon^2}{4}.
\]

Next, with a view of further integrating (2.1), we express \( u' \) as

\[
u'(x) = \pm u(x) \sqrt{\frac{3}{4} u^2(x) - u^2(x) + k}, \quad x \neq 0
\]

(recall we seek solutions with \( u > 0 \)). The positivity condition for (2.5) to hold reads \( u^2(x) \in (0, \tilde{u}^2_+) \cup (\tilde{u}^2_+, \infty) \), where

\[
\tilde{u}^2_+ = \frac{3}{2} \left( 1 \pm \sqrt{1 - \frac{4\epsilon^2}{27}} \right).
\]
Since positive solutions satisfying (2.5) are even and strictly decreasing in \(x > 0\), the continuity and the decay of \(u\) at infinity allow only for
\[
0 \leq u^2(0) = \frac{3}{2} \left(1 - \sqrt{1 - \frac{4k}{3}}\right), \quad \frac{3}{4} < k < \frac{3}{2},
\] (2.7)
If \(k > 3/4\), (2.5) is well defined without further restriction on \(u(0)\), and condition (2.7) is void. (The nature of the degeneracy at \(k = 3/4\) will become more apparent later.) In view of (2.3), (2.4) and (2.7), we identify two different regimes:

(A) \(\frac{3}{4} < k < \frac{3}{2}\): there is only one soliton, \(u_{-k, \epsilon}(0)\), corresponding to \(u^2_{-k, \epsilon}(0)\);
(B) \(\frac{3}{4} < k < \frac{3}{2} + \frac{\epsilon^2}{4}\): there are two different solitons, \(u_{\pm, k, \epsilon}(0)\), corresponding respectively to \(u^2_{\pm, k, \epsilon}(0)\).

(See the bifurcation diagrams for various values of \(\epsilon\) in Section 3.)

Notice, in particular, that regime (A) is void if \(\epsilon \geq \sqrt{3}\), so we will suppose \(0 < \epsilon < \sqrt{3}\) from now on. Also, we already see from the above analysis that a fold bifurcation occurs at \(k = \frac{3}{4} + \frac{\epsilon^2}{4}\) where two distinct solutions merge and disappear (there is no soliton for \(k > k_0\)).

From (2.5) and the previous discussion, any positive solution of (1.2) with \(\frac{3}{4} < k < \frac{3}{2} + \frac{\epsilon^2}{4}\) decaying at infinity satisfies
\[
u'(x) = -\text{sgn}(x) u(x) \sqrt{\frac{1}{4} u^4(x) - u^2(x) + k}, \quad x \neq 0.
\] (2.8)
In particular, \(u\) is even, \(u'(x) < 0\) for \(x > 0\), and \(\lim_{x \to \infty} u'(x)/u(x) = -\sqrt{k}\), so \(u(x)\) decays like \(e^{-\sqrt{k} |x|}\) as \(|x| \to \infty\).

Now (2.8) is a first order ODE with separated variables, which can be integrated explicitly. Alternatively, the solutions in regime (A) are easily constructed by applying some surgery to the known explicit solitons in free space, given in \([1, 22]\) as
\[
u_{-k, 0}(x) = \sqrt{\frac{2k}{1 + \sqrt{1 - \frac{4k}{3}} \cosh (2\sqrt{k}x)}}, \quad 0 < k < \frac{3}{4}.
\] (2.9)
The corresponding solutions pinned to the delta-potential with \(\epsilon > 0\) are obtained as
\[
u_{-k, \epsilon}(x) = \sqrt{\frac{2k}{1 + \sqrt{1 - \frac{4k}{3}} \cosh (2\sqrt{k}(|x| + \xi))}}, \quad \frac{3}{4} < k < \frac{3}{2},
\] (2.10)
where \(\xi = \xi(k, \epsilon)\) is determined by the jump condition in Proposition 1 (iii), which yields
\[
\frac{\sinh(2\sqrt{k} \xi)}{1 + \sqrt{1 - \frac{4k}{3}} \cosh (2\sqrt{k} \xi)} = \epsilon \frac{1}{2\sqrt{k} \sqrt{1 - \frac{4k}{3}}}.
\]
It is not difficult to check that this equation has a unique solution \(\xi \in \mathbb{R}\) if \(k > \frac{\epsilon^2}{4}\). In fact this solution can be computed explicitly:
\[
ee^{2\sqrt{k} \xi} = \epsilon + \epsilon \sqrt{1 + (\frac{4k}{3} - 1)(1 - \frac{4k}{3})} \frac{1}{2(\sqrt{k} - \frac{\epsilon}{2}) \sqrt{1 - \frac{4k}{3}}}.
\]
Thus, the solutions in (2.10) take the form of
\[
u_{-k, \epsilon}(x) = \sqrt{\frac{2k}{1 + \epsilon + \epsilon \sqrt{1 + (\frac{4k}{3} - 1)(1 - \frac{4k}{3})} e^{2\sqrt{k} |x|} + \frac{(1 - \frac{4k}{3})(\sqrt{k} - \epsilon/2)}{\epsilon + \epsilon \sqrt{1 + (\frac{4k}{3} - 1)(1 - \frac{4k}{3})} e^{2\sqrt{k} |x|}}}, \quad \frac{3}{4} < k < \frac{3}{2}.
\] (2.11)

1Note that \(\hat{u}_2^2 = \psi_{\text{sol}}(0, 0)^2\) in [3].
For $k = 3/4$, a similar procedure applied to the ‘front soliton’ given in Eq. (11) of [3] yields a solution
\[
    u_{f, \epsilon}(x) = \sqrt{\frac{3}{2}} \left[ 1 + \frac{\epsilon}{\sqrt{3} - \epsilon} e^{\sqrt{3}|x|} \right]^{-1/2}.
\]  
(2.12)

As can be seen in Section 3 by comparing the bifurcation diagrams for solutions in free space to those with $\epsilon > 0$, solutions with $k > 3/4$ only exist in the presence of the delta-potential. In other words, regime (B) above is void for $\epsilon = 0$. Therefore, no free-space solutions are available that could be pinned to the delta-potential by the same sort of surgery as above, and one has to integrate the equation manually. We integrate (2.8) using an Euler substitution, which yields
\[
    u_{\pm, k, \epsilon}(x) = \frac{k}{2} \sqrt{a \sqrt{k}(|x| - c) + e^{-\sqrt{k}(|x| - c)}} \left( (2\sqrt{\frac{k}{3}} + 1)e^{\sqrt{k}(|x| - c)} - (2\sqrt{\frac{k}{3}} - 1)e^{-\sqrt{k}(|x| - c)} \right), \quad \frac{3}{4} < k < \frac{3}{4} + \frac{\epsilon^2}{4},
\]  
(2.13)
where the integration constant $c = c_{\pm, k, \epsilon} \in \mathbb{R}$ can be determined from (2.8). The expressions for the integration constants are somewhat cumbersome. They can be computed using Mathematica, which yields
\[
    e^{\sqrt{k}c_{-k, \epsilon}} = \sqrt{\frac{3 - \sqrt{3} \sqrt{3 + \epsilon^2} - 4k + 2\epsilon \sqrt{k} - 4k}{3 + \sqrt{3} \sqrt{3 + \epsilon^2} - 4k + 2\sqrt{k} \sqrt{3} + 2k - 4k}}
\]  
and
\[
    e^{\sqrt{k}c_{+k, \epsilon}} = \sqrt{\frac{3 - \sqrt{3} \sqrt{3 + \epsilon^2} - 4k + 2\epsilon \sqrt{k} + 4k}{3 + \sqrt{3} \sqrt{3 + \epsilon^2} - 4k - 2\sqrt{k} \sqrt{3} - 2k + 4k}}
\]

Hence the explicit form (2.13) is not very convenient to work with, but we shall see in Section 5 that some information can nevertheless be extracted from it. However, for given values of the parameters, the exact form of the solutions may be useful, especially in numerical calculations. For instance, at the fold bifurcation point, where $k = \frac{3}{4} + \frac{\epsilon^2}{4}$, the solution takes the more tractable form:
\[
    u_\epsilon(x) = \sqrt{\frac{3}{2}} \left[ \frac{3 + \epsilon^2}{3 + \epsilon^2 \cosh(\sqrt{3 + \epsilon^2}|x|) + \epsilon \sqrt{3 + \epsilon^2} \sinh(\sqrt{3 + \epsilon^2}|x|)} \right].
\]

**Remark 1.** It can also be checked that, as $\epsilon \to 0$, the solutions in (2.11) converge to the corresponding free-space solitons in (2.10). It will be seen in the proof of Lemma 2 that, in fact, they can be extended to a holomorphic family of functions parametrized by $\epsilon$ in a complex domain containing zero.

### 3. The bifurcation analysis

In this section we will embed the above explicit solutions in a bifurcation-theoretic framework, suitable to the rigorous stability analysis which will be carried out in Section 5. We will prove the following result.

**Theorem 1.** Let $\epsilon \in (0, \sqrt{3})$. The solutions $(k, u)$ of (1.2) obtained in (2.11) - (2.13) form a smooth curve in $\mathbb{R} \times H^1(\mathbb{R})$, which bifurcates from the trivial solution $u \equiv 0$ at $k = \frac{3}{4}$, consists of the solutions $u_{-k, \epsilon}$ up to $k = \frac{3}{4} + \frac{\epsilon^2}{4}$, where it has a turning point, and then consists of the solutions $u_{+, k, \epsilon}$ and becomes unbounded as $k \searrow 3/4$. More precisely,
\[
    \lim_{k \searrow 3/4} \|u_{-k, \epsilon}\|_{H^1} = 0 \quad \text{and} \quad \lim_{k \searrow 3/4} \|u_{+, k, \epsilon}\|_{L^2} = \infty.
\]

A good mental picture of Theorem 1 can be grasped from the bifurcation diagrams in Section 6 where $\|u_{+, k, \epsilon}\|_{L^2}$ is plotted against $k$, for various values of the coupling constant $\epsilon > 0$.

To prove Theorem 1 first observe that Eq. (1.2) can be formulated as
\[
    F_{\epsilon}(k, u) = 0,
\]  
(3.1)

3The positivity of the expressions under the square roots can be checked by plotting their graphs (as functions of $\epsilon$ and $k$) in Mathematica.
More precisely, following the proof of \cite[Lemma 10]{16}, \( D uF_\epsilon(k,u) \) can be interpreted as a self-adjoint operator acting in \( L^2(\mathbb{R}) \), with domain
\[
\mathcal{D}_\epsilon = \left\{ v \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}) : v'(0^+) - v'(0^-) = -\epsilon v(0) \right\},
\]
defined by
\[
D uF_\epsilon(k,u) v = v'' - kv + \epsilon \delta(x)v + [6 - 5u^2]u^2v, \quad v \in H^1(\mathbb{R}).
\]

More precisely, following the proof of \cite[Lemma 10]{16}, \( D uF_\epsilon(k,u) \) can be interpreted as a self-adjoint operator acting in \( L^2(\mathbb{R}) \), with domain
\[
\mathcal{D}_\epsilon = \left\{ v \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}) : v'(0^+) - v'(0^-) = -\epsilon v(0) \right\},
\]
defined by
\[
D uF_\epsilon(k,u) v = v'' - kv + [6 - 5u^2]u^2v, \quad v \in \mathcal{D}_\epsilon.
\]
Using the explicit formulas for the solutions obtained in Section 2 (in particular their uniform exponential decay), it can be shown that
\[
S_{-\epsilon} = \{(k,u_{-\epsilon,k,\epsilon}) : k \in (\frac{\epsilon^2}{4},\infty)\} \quad \text{and} \quad S_{+\epsilon} = \{(k,u_{+\epsilon,k,\epsilon}) : k \in (\frac{\epsilon^4}{4},\infty)\}
\]
define two continuous curves in \( \mathbb{R} \times H^1(\mathbb{R}) \). In the remainder of the paper, we will obtain much more information about these sets. It will be convenient to call \( S_{-\epsilon} \) the lower curve and \( S_{+\epsilon} \) the upper curve.

**Proposition 2.** The sets \( S_{\epsilon,\pm} \) are smooth curves of non-degenerate solutions of \((3.1)\), in the sense that \( D uF_\epsilon(k,u) \) is non-singular along \( S_{-\epsilon} \) and \( S_{+\epsilon} \). Furthermore, \( S_{-\epsilon} \) bifurcates from the point \((\frac{\epsilon^2}{4},0)\) in \( \mathbb{R} \times H^1(\mathbb{R}) \), and meets \( S_{+\epsilon} \) at the point \((\pm \frac{\epsilon^2}{4},\pi_\epsilon)\), where \( D uF_\epsilon(k,u) \) becomes singular.

**Proof.** First, it is easily seen that
\[
\ker D uF_\epsilon(\frac{\epsilon^2}{4},0) = \text{span} \left\{ e^{-\frac{\epsilon^2}{4}|x|} \right\},
\]
so that zero is a simple eigenvalue of \( D uF_\epsilon(\frac{\epsilon^2}{4},0) \). It then follows from standard bifurcation theory that \( S_{-\epsilon} \) bifurcates from \((\frac{\epsilon^2}{4},0)\). More precisely, the Crandall-Rabinowitz theorem \cite[Theorem 1.7]{8} yields the existence of a unique local continuous curve of solutions bifurcating from the line of trivial solutions \( \{(k,0) : k \in \mathbb{R}\} \) in \( \mathbb{R} \times H^1(\mathbb{R}) \) at the point \((\frac{\epsilon^2}{4},0)\). Since our explicit solutions all belong to \( H^1(\mathbb{R}) \), they coincide with the Crandall-Rabinowitz curve in a neighborhood of \((\frac{\epsilon^2}{4},0)\) in \( \mathbb{R} \times H^1(\mathbb{R}) \).

The smoothness of the curves \( S_{-\epsilon} \) and \( S_{+\epsilon} \) follows from the implicit function theorem in \( \mathbb{R} \times H^1(\mathbb{R}) \), provided that \( D uF_\epsilon(k,u) : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R}) \) is non-singular along the solution curves, which is given by Lemma 1 below. \(\square\)

In view of the more detailed spectral analysis that will be carried out later, and in order to follow the usual sign convention of the spectral theory of Schrödinger operators, it is convenient to introduce the self-adjoint operators
\[
T_{\pm,k,\epsilon} : \mathcal{D}_\epsilon \subset L^2(\mathbb{R}) \to L^2(\mathbb{R}),
\]
\[
T_{\pm,k,\epsilon} v := -D uF_\epsilon(k,u_{\pm,k,\epsilon}) v = -v'' + kv - [6 - 5u_{\pm,k,\epsilon}^2(x)]u_{\pm,k,\epsilon}^2(x)v.
\]

Again, note that \( T_{\pm,k,\epsilon} \) can be seen as an operator acting between \( H^1(\mathbb{R}) \) and \( H^{-1}(\mathbb{R}) \), by interpreting the right-hand side of \((3.3)\) as a distribution.

**Lemma 1.** The linearized operator \((3.3)\) satisfies:

(i) \( T_{-,k,\epsilon} : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R}) \) is an isomorphism for all \( k \in (\frac{\epsilon^4}{4},\infty) \);

(ii) \( T_{+,k,\epsilon} : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R}) \) is an isomorphism for all \( k \in (\frac{\epsilon^2}{4},\infty) \);

(iii) \( D uF_\epsilon(\pm \frac{\epsilon^2}{4},\pi_\epsilon) \) is singular with
\[
\ker D uF_\epsilon(\pm \frac{\epsilon^2}{4},\pi_\epsilon) = \text{span} \{ \eta_\epsilon \}, \quad \eta_\epsilon = |\pi_\epsilon'|.
\]

Furthermore, since \( \eta_\epsilon > 0 \), zero is the principal eigenvalue of \( D uF_\epsilon(\pm \frac{\epsilon^2}{4},\pi_\epsilon) \).
Proof. A first important remark is that each operator \( T_{\pm,k,\epsilon} \) is a compact perturbation of \(-\frac{\partial^2}{\partial x^2} + k : H^1(\mathbb{R}) \rightarrow H^{-1}(\mathbb{R})\), the latter being an isomorphism for all \( k > 0 \). It then follows from standard spectral theory (see, e.g., [13, 24]) that the spectrum of \( T_{\pm,k,\epsilon} \) consists of a finite number of isolated eigenvalues (of finite multiplicity) lying below a continuous part \([k, \infty)\). Furthermore, \( T_{\pm,k,\epsilon} : H^1(\mathbb{R}) \rightarrow H^{-1}(\mathbb{R}) \) is an isomorphism if and only if

\[ \ker T_{\pm,k,\epsilon} = \{0\}. \tag{3.5} \]

We will now show that (3.5) holds for all \( \frac{2}{4} < k < \frac{\tau}{k} \). If \( T_{\pm,k,\epsilon}v = 0 \) then \( v \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}) \) satisfies

\[ -v'' + kv - [6 - 5u_{\pm,k,\epsilon}^2(x)]u_{\pm,k,\epsilon}^2(x)v = 0, \quad x \neq 0, \tag{3.6} \]

\[ v'(0^+) - v'(0^-) = -\alpha v(0). \tag{3.7} \]

Applying Theorem 3.3 of [2] to (3.6) separately on \((-\infty, 0)\) and \((0, \infty)\), and using the continuity of \( v \), there exists a constant \( \alpha \in \mathbb{R} \) such that \( v = \alpha |u_{\pm,k,\epsilon}'| \). Hence,

\[ v(0) = -\alpha u_{\pm,k,\epsilon}'(0^-) = \alpha u_{\pm,k,\epsilon}'(0^+) = -\frac{\alpha}{2} u_{\pm,k,\epsilon}(0), \]

and since

\[ u''_{\pm,k,\epsilon}(0^-) = u''_{\pm,k,\epsilon}(0^+) = ku_{\pm,k,\epsilon}(0) - 2u_{\pm,k,\epsilon}^3(0) + u_{\pm,k,\epsilon}^5(0), \]

it follows from (3.7) that

\[ 4[k - 2u_{\pm,k,\epsilon}^2(0) + u_{\pm,k,\epsilon}^4(0)] = \alpha^2. \]

Combining this with (3.5) yields

\[ 1 - \frac{4}{3} \left( k - \frac{\epsilon^2}{4} \right) = \pm \sqrt{1 - \frac{4}{3} \left( k - \frac{\epsilon^2}{4} \right)}. \tag{3.8} \]

The ‘+’ sign in (3.8) corresponds to \( u_{-k,\epsilon} \) and yields \( k = \frac{\epsilon^2}{4} \) or \( k = \frac{3}{4} + \frac{\epsilon^2}{4} \), from which (i) and (iii) follow. The ‘−’ sign in (3.8) corresponds to \( u_{+k,\epsilon} \) and yields \( k = \frac{3}{4} + \frac{\epsilon^2}{4} \), so (ii) must hold. The lemma is proved.

Even though the linearized operator \( D_u F_{\epsilon}(k, u) \) becomes singular at \((\overline{k}, \overline{\pi}_\epsilon)\), we have the following result.

**Proposition 3.** The set

\[ S := S_{-,\epsilon} \cup \{(\overline{k}, \overline{\pi}_\epsilon)\} \cup S_{+,\epsilon} \tag{3.9} \]

is a smooth curve in \( \mathbb{R} \times H^1(\mathbb{R}) \).

To prove Proposition 3 we will use a theorem of Crandall and Rabinowitz, which enables us to reparametrize the bifurcation curve around the point \((\overline{k}, \overline{\pi}_\epsilon)\), where the parametrization by \( k \) breaks down. For the reader’s convenience we reproduce this result here.

**Theorem 2** (Theorem 3.2 of [4]). Let \((k_0, u_0) \in \mathbb{R} \times X\) where \( X \) is a Banach space and let \( F \) be a continuously differentiable mapping of an open neighborhood of \((k_0, u_0)\) into another Banach space \( Y \). Suppose that \( \ker D_u F(k_0, u_0) = \text{span}\{\eta_0\} \) is one-dimensional, that \( \text{codim rge} D_u F(k_0, u_0) = 1 \), and that \( D_k F(k_0, u_0) \notin \text{rge} D_u F(k_0, u_0) \). If \( Z \) is a complement of \( \text{span}\{\eta_0\} \) in \( X \), then the solutions of \( F(k, u) = F(k_0, u_0) \) near \((k_0, u_0)\) form a curve \((k(s), u(s)) = (k_0 + \tau(s), u_0 + s\eta_0 + z(s)), \) where \( s \rightarrow (\tau(s), z(s)) \in \mathbb{R} \times Z \) is a continuously differentiable function near \( s = 0 \), and \( \tau(0) = \tau'(0) = 0, \) \( z(0) = z'(0) = 0 \). Here, the ‘dot’ denotes differentiation with respect to \( s \).

**Proof.** Apply the implicit function theorem to the function \( f : \mathbb{R} \times \mathbb{R} \times Z \rightarrow Y \) defined by

\[ f(s, \tau, z) = F(k_0 + \tau, u_0 + s\eta_0 + z) \]

at the point \((s, \tau, z) = (0, 0, 0)\). \( \square \)
Proof of Proposition 3. Firstly, since the operator 
\[ D_u F_\epsilon(k, u_\epsilon) \] is self-adjoint, it follows from (3.4) that 
\[ \text{codim rge } D_u F_\epsilon(k, u_\epsilon) = \dim \ker D_u F_\epsilon(k, u_\epsilon) = 1. \]
Furthermore, the range of 
\[ D_u F_\epsilon(k, u_\epsilon) \] is characterized by 
\[ \text{rge } D_u F_\epsilon(k, u_\epsilon) = \{ v \in L^2(\mathbb{R}) : \int_\mathbb{R} v \eta_\epsilon \, dx = 0 \}. \]
Next, we need to check that 
\[ D_k F_\epsilon(k, u_\epsilon) \not\in \text{rge } D_u F_\epsilon(k, u_\epsilon). \]
But this is clear, as 
\[ D_k F_\epsilon(k, u_\epsilon) = -u_\epsilon \]
and 
\[ \int_\mathbb{R} u_\epsilon \eta_\epsilon \, dx = 2 \int_0^\infty u_\epsilon \eta'_\epsilon \, dx = -u_\epsilon^2(0) < 0. \]
It then follows from Theorem 2 that the solutions of (3.1) in a neighborhood of 
\[ (k, u_\epsilon) \] form a smooth curve, 
\[ \{ (k_s, u_s) : s \in (-\varepsilon, \varepsilon) \} \subset \mathbb{R} \times H^1(\mathbb{R}) \] (for some 
\[ \varepsilon > 0 \] ) (3.10)
such that, at \( s = 0, \)
\[ k_0 = k, \quad \dot{k}_0 = 0, \quad u_0 = u_\epsilon, \quad \dot{u}_0 = \eta_\epsilon. \] (3.11)
Consequently, the lower and upper curves \( S_{-\varepsilon} \) and \( S_{+\varepsilon} \) meet smoothly at the turning point \( (k, u_\epsilon). \) □

4. Spectral properties

Let \( n(T_{\pm,k,\epsilon}) \) denote the number of negative eigenvalues of the self-adjoint operator \( T_{\pm,k,\epsilon}. \) We will see in Section 5 that, for a given solution \( u_{\pm,k,\epsilon}, \) this number plays an important role as regards the dynamical stability (with respect to (1.1)) of this solution.

Proposition 4. The spectrum of the linear operator \( T_{\pm,k,\epsilon} : D_\epsilon \subset L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) consists of a finite number of simple isolated eigenvalues and a continuous part \([k, \infty)\).

For all \( k \in (\frac{1}{4}, k), \)
\[ n(T_{-k,\epsilon}) = 1 \]
and
\[ n(T_{+k,\epsilon}) = 0 \]
Proof. As noted earlier in the proof of Lemma 1, the basic structure of the spectrum of \( T_{+\pm k, \epsilon} \) follows from standard spectral theory, see for instance [13, 24]. For the simplicity of eigenvalues, suppose that \( u, v \in \mathcal{D}_\epsilon \) are eigenfunctions of \( T_{+\pm k, \epsilon} \) corresponding to an eigenvalue \( \lambda < k \). Then \( (uv'' - u'v)' = uv'' - u'v = 0 \) on \( \mathbb{R} \setminus \{0\} \), therefore there exists a constant \( C \in \mathbb{R} \) such that \( uv'' - u'v = C \) on \( \mathbb{R} \setminus \{0\} \). However, \( \lim_{|x| \to \infty} uv'' - u'v = 0 \), as \( u, v \in H^2(\mathbb{R} \setminus \{0\}) \). Hence \( C = 0 \) and \( u, v \) are linearly dependent.

Next, (1.1) follows from a perturbation analysis similar to the proof of Lemma 12 in [16]. The idea is first to observe that the property holds when \( \epsilon = 0 \). Indeed, in this case the kernel of the linearization at the free-space soliton (2.4) is spanned by its derivative \( u'_{-k,0} \), which has a unique zero at \( x = 0 \), where it changes sign. Therefore, by Sturm’s oscillation theory, zero is the second eigenvalue of the linearization, and so \( n(T_{-k,0}) = 1 \). With this information at hand, we then use perturbation theory to make sure that the second eigenvalue becomes positive for small values of \( \epsilon > 0 \). A first step in this direction is the smoothness of the family of operators \( T_{-k, \epsilon} \) at \( \epsilon = 0 \).

Lemma 2. For any \( \rho > 0 \) small enough, there exists an open connected neighborhood \( \Omega \) of the real-line segment \([0, 2\sqrt{k} - \rho]\) in \( \mathbb{C} \), such that \( \{T_{-k, \epsilon} : \epsilon \in \Omega \} \) is a holomorphic family of operators.

Proof. The result follows in a similar way to Lemma 13 in [16], and is based on the notion of holomorphic family of unbounded operators of type (B) in the sense of Kato, see Theorem 4.2 in chapter VII of [15]. The argument boils down to checking that, for each fixed \( x \in \mathbb{R} \), the mapping \( \epsilon \mapsto u_{-k, \epsilon}(x) \) is holomorphic on a suitable domain, independent of \( x \). That this domain can be taken as stated in Lemma 2 follows by a careful inspection of (2.11), and the observation that the function \( \epsilon \mapsto \sqrt{\epsilon^2 + (4k - \epsilon^2)(1 - 4k/3)} \) is holomorphic on the strip \( \{ \epsilon \in \mathbb{C} : \text{Im} \epsilon \in (-\sqrt{3 - 4k}, \sqrt{3 - 4k}) \} \).

Thanks to Lemma 2, perturbation theory [13] yields two holomorphic mappings,

\[
\Omega \ni \epsilon \mapsto \lambda_\epsilon \in \mathbb{R}, \quad \Omega \ni \epsilon \mapsto w_\epsilon \in \mathcal{D}_\epsilon
\]

such that \( \lambda_0 = 0 \), \( w_0 = u'_{-k,0} \), where \( \lambda_\epsilon \) and \( w_\epsilon \) are, respectively, the second eigenvalue and eigenvector of \( T_{-k, \epsilon} \):

\[
T_{-k, \epsilon}w_\epsilon = \lambda_\epsilon w_\epsilon, \quad \epsilon \in \Omega. \tag{4.3}
\]

We will now show that \( \lambda_0 > 0 \), which implies that the second eigenvalue of \( T_{-k, \epsilon} \) is positive for small \( \epsilon > 0 \). We use the ‘dot’ here to denote differentiation with respect to \( \epsilon \), \( \lambda_0 > 0 \) being the derivative of \( \lambda \) with respect to \( \epsilon \) at \( \epsilon = 0 \), and similarly for other quantities below. Differentiating (4.3) with respect to \( \epsilon \) at \( \epsilon = 0 \) yields

\[
-\dot{w}_0'' + kw_0 - 4[3 - 5u_0^2]u_0\dot{u}_0u'_0 - [6 - 5u_0^2]u_0^2\dot{u}_0 = \dot{\lambda}_0\dot{w}_0,
\]

where we have put \( u_0 \equiv u_{-k,0} \) and \( \dot{u}_0 \equiv \dot{u}_{-k,0} \) to simplify the notation. Observe that \( w_\epsilon \in \mathcal{D}_\epsilon \Rightarrow \dot{w}_0 \in \mathcal{D}_0 = H^2(\mathbb{R}) \).

Now, differentiating (1.2) with respect to \( \epsilon \) at \( \epsilon = 0 \) shows that

\[
T_0\dot{u}_0 := -u_0'' + ku_0 - [6 - 5u_0^2]u_0^2\dot{u}_0 = \delta(x)u_0.
\]

Multiplying (4.4) by \( u'_0 \), integrating by parts and using \( T_0u'_0 = 0 \) then yields

\[
\dot{\lambda}_0 = \frac{4\int_{\mathbb{R}}[5u_0^2 - 3]u_0\dot{u}_0(u'_0)^2}{\int_{\mathbb{R}}(u'_0)^2} \tag{4.6}
\]

Furthermore, straightforward calculations show that

\[
4[5u_0^2 - 3]u_0(u'_0)^2 = T_0(-ku_0 + 2u_0^3 - u_0^5) = T_0(-u_0''),
\]

and it follows by (4.5) that

\[
4\int_{\mathbb{R}}[5u_0^2 - 3]u_0(u'_0)^2\dot{u}_0 = (T_0(-u_0''), \dot{u}_0)_{L^2} = (-u_0'', T_0\dot{u}_0)_{L^2} = -u_0(0)'u(0) > 0,
\]

showing that \( \dot{\lambda}_0 \) is indeed positive. This implies that (4.1) holds for \( \epsilon > 0 \) small enough. To complete the proof of (1.1), we invoke the continuous dependence of the first two eigenvalues of \( T_{-k, \epsilon} \) on \( \epsilon \in [0, 2\sqrt{k}] \) (given by Lemma 2), and the fact that the eigenvalues cannot cross zero unless \( \epsilon = 2\sqrt{k} \) (Lemma 1 (i)).
We now turn to the proof of (4.2). By (4.1), the first eigenvalue of $T_{-k,\epsilon}$ is negative, for all $k \in (\frac{2}{\epsilon^2}, \frac{1}{k})$. Since $\ker T_{\pm k,\epsilon} \neq \{0\}$ by Lemma 1, we only need to show that the first eigenvalue of $T_{\pm k,\epsilon}$ crosses zero at $(k, u_\epsilon)$. Using the parametrization (3.10), and denoting by $\mu_s$ the first eigenvalue along the curve, this amounts to showing that $\dot{\mu}_0 \neq 0$. The first eigenvalue and eigenfunction $\mu_s, v_s$ satisfy $\mu_0 = 0$, $v_0 = \eta_\epsilon$, and $v_s \in D_\epsilon$, $-v''_s + kv_s - [6u^2_s - 5u^4_s]v_s = \mu_sv_s$, $s \in (-\epsilon, \epsilon)$.

In view of (3.11), differentiating with respect to $s$ and letting $s = 0$ yields

$$-\dot{v}'_0 + k\dot{v}_0 - 4[3 - 5\pi^2_\epsilon]\pi^3_\epsilon \eta^3_\epsilon - [6 - 5\pi^2_\epsilon]\pi^2_\epsilon \dot{v}_0 = \dot{\mu}_0 \eta_\epsilon. \tag{4.7}$$

Multiplying both sides of (4.7) by $\eta_\epsilon$, integrating by parts and using $D_u F_\epsilon(k, \pi_\epsilon)\eta_\epsilon = 0$ yields

$$\dot{\mu}_0 = \frac{4 \int_{\mathbb{R}} [5\pi^2_\epsilon - 3] \pi_\epsilon \eta^3_\epsilon}{\int_{\mathbb{R}} \pi^3_\epsilon \eta^3_\epsilon}. \tag{4.8}$$

We were not able to find an analytical argument showing that $f(\epsilon) := \int_{\mathbb{R}} [5\pi^2_\epsilon - 3] \pi_\epsilon \eta^3_\epsilon = 2 \int_{0}^{\infty} [5\pi^2_\epsilon - 3] \pi_\epsilon |\pi_\epsilon'|^3 \neq 0$.

However, numerical computation of this integral (Fig. 1) clearly shows that it is positive for all values of $\epsilon \in (0, \sqrt{3})$,

![Figure 1. The graph of $f(\epsilon)$](image)

which concludes the proof. \hfill \Box

5. Stability

We consider the stability of the bound states

$$\psi_{\pm k,\epsilon}(x, z) = e^{ikz} u_{\pm k,\epsilon}(x) \tag{5.1}$$

with respect to perturbations of the initial soliton profile, $u_{\pm k,\epsilon}$, in $H^1(\mathbb{R})$. Let us first remark that the Cauchy problem associated with (1.1) is locally well posed in $H^1(\mathbb{R})$, see 3. That is, for any initial profile $\psi(\cdot, 0) \in H^1(\mathbb{R})$, there exists a unique continuous map $z \mapsto \psi(x, z) \in H^1(\mathbb{R})$, defined on a maximal $z$-interval $[0, z_0)$, where $z_0$ depends on the initial data $\psi(\cdot, 0)$. If $z_0 = +\infty$, the solution $\psi(x, z)$ is said to be global 3.

We will now define precisely what we mean by the stability of the bound state solutions of (1.1). It is well known that, due to the $U(1)$-invariance of (1.1), the appropriate notion of stability in this context is that of orbital stability.

3This is clearly the case for the bound states (5.1).
Theorem 3. Let \( S \) be defined by \( (3.9) \), then, for all \((u, k, \varepsilon) \) in \( S \), \( \psi(x, z) = e^{ikz}u(x) \) is an orbitally stable solution of \((1.1)\).

Proof. We first address the stability of the solutions belonging to the pieces of curve \( S_{\pm, \varepsilon} \). In view of Proposition 3, the stability of \( S_{\pm, \varepsilon} \) immediately follows from statement (I) above, whereas the stability of \( S_{-, \varepsilon} \) will follow from (II) if we prove that the function

\[
(\frac{2}{3}, \overline{\varepsilon}) \ni k \mapsto \|u_{-, k, \varepsilon}\|_{L^2}
\]

is strictly increasing. Firstly, for \( k \in (\frac{2}{3}, \overline{\varepsilon}) \), we find using Mathematica that \(4\)

\[
\frac{d}{dk} \|u_{-, k, \varepsilon}\|_{L^2}^2 = -\frac{2\sqrt{4k - 3}}{4k - 3} \quad \text{and} \quad \frac{d}{dk} \|u_{+, k, \varepsilon}\|_{L^2}^2 = -\frac{2\sqrt{4k - 3}}{4k - 3}.
\]

It follows that \( \|u_{-, k, \varepsilon}\|_{L^2} \) is indeed increasing, while \( \|u_{+, k, \varepsilon}\|_{L^2} \) is decreasing, for \( k \in (\frac{2}{3}, \overline{\varepsilon}) \). We also observe explicitly here that \( \lim_{k \searrow \frac{2}{3}} \frac{d}{dk} \|u_{+, k, \varepsilon}\|_{L^2}^2 = -\infty \), which is consistent with Theorem 1.

\[\text{The monotonicity condition in \((1.2)\) seems to have first been formulated by Vakhitov and Kolokolov in \((29)\), and so is often referred to as the 'Vakhitov-Kolokolov condition' (VK condition for short). It is also sometimes called the 'slope condition'.}\]

\[\text{It turns out that the expressions for} \: \frac{d}{dk} \|u_{\pm, k, \varepsilon}\|_{L^2}^2 \text{are much simpler than those for} \: \|u_{\pm, k, \varepsilon}\|_{L^2}^2 \text{in the regime} \: k \in (\frac{2}{3}, \overline{\varepsilon}).\]
For $k < 3/4$, a straightforward calculation using (2.11) shows that
\[
\|u_{-k,\epsilon}\|_{L^2}^2 = \sqrt{3} \log \varphi_\epsilon(k) \quad \text{where} \quad \varphi_\epsilon(k) := \frac{\sqrt{3} + \sqrt{3k^2 + (4k - \epsilon^2)(3 - 4k)} + (\sqrt{3} + 2\sqrt{k})(2\sqrt{k} - \epsilon)}{\sqrt{3} + \sqrt{3k^2 + (4k - \epsilon^2)(3 - 4k)}} \quad (5.5)
\]
Differentiation then yields
\[
\frac{d}{dk} \varphi_\epsilon(k) = 8\sqrt{k} \frac{\sqrt{3} \sqrt{3k^2 + (4k - \epsilon^2)(3 - 4k)} + 2\sqrt{k}(3 + \epsilon^2 - 2\epsilon\sqrt{k})}{\sqrt{3k^2 + (4k - \epsilon^2)(3 - 4k)}(3 + \epsilon^2 + 2\sqrt{k}(3 - 2\sqrt{k})(2\sqrt{k} - \epsilon))}, \quad (5.6)
\]
where we observe that
\[
k < \frac{3}{4} \implies 3 + \epsilon^2 - 2\epsilon\sqrt{k} > (\sqrt{3} - \epsilon)^2 + \sqrt{3\epsilon} > 0.
\]
Therefore, $\frac{d}{dk} \varphi_\epsilon(k) > 0$, so $\|u_{-k,\epsilon}\|_{L^2}$ is also increasing for all $k \in (\frac{3}{4}, \frac{3}{2})$. We have thus proved that the curves $S_{\pm, \epsilon}$ are both stable.

The stability of the solution at the fold does not fall within the scope of statements (I) and (II) above, as $D_u F_\epsilon(k, \varpi)$ has zero as an eigenvalue. However, we can use stable close-by orbits of solutions belonging to $S_{\pm, \epsilon}$ in order to prove the stability of $(k, \varpi)$. Given $\epsilon > 0$, we will show that there exists $\delta > 0$ such that, for any solution $\psi(x, z)$ of (1.1), there holds
\[
\|\psi(\cdot, 0) - \varpi\|_{H^1} \leq \delta \implies \text{dist}(\psi(\cdot, z), \Theta(\varpi)) \leq \epsilon, \quad z \geq 0.
\]
First observe that, by compactness, there exists $\delta_0 > 0$ such that
\[
\|\psi(\cdot, 0) - u_{k,\epsilon}\|_{H^1} \leq \delta_0 \implies \text{dist}(\psi(\cdot, z), \Theta(u_{k,\epsilon})) \leq \epsilon/2, \quad z \geq 0, \quad (5.7)
\]
for all $(k, u_{k,\epsilon}) \in S_{-\epsilon} \cup S_{+\epsilon}$ satisfying $\|u_{k,\epsilon} - \varpi\|_{H^1} \leq \epsilon/2$. Now choose a point $(k_0, u_{k_0,\epsilon}) \in S_{-\epsilon} \cup S_{+\epsilon}$ such that
\[
\|u_{k_0,\epsilon} - \varpi\|_{H^1} \leq \min\{\epsilon/2, \delta_0/2\}. \quad (5.8)
\]
For any solution $\psi(x, z)$ of (1.1) with $\|\psi(\cdot, 0) - \varpi\|_{H^1} \leq \delta := \delta_0/2$, one has
\[
\|\psi(\cdot, 0) - u_{k_0,\epsilon}\|_{H^1} \leq \|\psi(\cdot, 0) - \varpi\|_{H^1} + \|u_{k_0,\epsilon} - \varpi\|_{H^1} \leq \delta_0.
\]
It then follows from (5.7) and (5.8) that, for all $z \geq 0$,
\[
\text{dist}(\psi(\cdot, z), \Theta(\varpi)) \leq \text{dist}(\psi(\cdot, z), \Theta(u_{k_0,\epsilon})) + \text{dist}(\Theta(u_{k_0,\epsilon}), \Theta(\varpi)) \leq \text{dist}(\psi(\cdot, z), \Theta(u_{k_0,\epsilon})) + \|u_{k_0,\epsilon} - \varpi\|_{H^1} \leq \epsilon.
\]
The proof is complete.

Remark 2. One can deduce from (5.4) and (5.5)–(5.6) that
\[
\lim_{k \downarrow 3/4} \frac{d}{dk} \|u_{-k,\epsilon}\|_{L^2}^2 = \lim_{k \uparrow 3/4} \frac{d}{dk} \|u_{-k,\epsilon}\|_{L^2}^2 = \sqrt{3} \left( \frac{1}{\epsilon^2} + \frac{1}{3} \right),
\]
showing that the slopes calculated from the solutions with $k < 3/4$ and with $k > 3/4$ indeed match where the two portions of $S_{-\epsilon}$ meet.

6. NUMERICS

Hereafter we present a numerical method, which we used for computing solutions of Eq. (1.2). This was helpful to understand the behavior of solutions before we had found their explicit representations. The method is based on the continuous normalized gradient flow, which was studied and implemented in [1] in the context of the NLS equation with a cubic nonlinearity. The numerical scheme being variational, the excellent agreement we obtain with the exact solutions (Fig. 6) suggests that the positive solutions of (1.2) should admit a variational characterization.
6.1. The numerical scheme. We look for a minimizer of the energy
\[ E(u) = \frac{1}{2} \left\{ \|u_x\|_{L^2}^2 - \epsilon |u(0)|^2 - \|u\|_{L^4}^4 + \frac{1}{3} \|u\|_{L^6}^6 \right\}, \]
with a given mass constraint
\[ \|u\|_{L^2} = a > 0. \]  
(6.1)

The minimizer is then a solution of (1.2) which can be interpreted as a (nonlinear) eigenfunction with eigenvalue
\[ k = -\frac{\|u_x\|_{L^2}^2 + \epsilon |u(0)|^2 + 2\|u\|_{L^4}^4 - \|u\|_{L^6}^6}{\|u\|_{L^2}^2}. \]
(6.2)

In the physics literature, this method is known as imaginary time propagation \((z \to -it)\). 

Thus, in order to solve (1.2), we introduce the imaginary time and iterate in this time. After each time step, we renormalize the solution so as to maintain the constraint (6.2). The discretization of (6.1) is done by means of semi-implicit backward Euler central differences.

Let us consider the time sequence \(t_0 = t_1 < t_2 < \cdots < t_n\), with time step \(dt = t_n - t_{n-1}\), and space grid \(x_j = x_0 + jh_x\) with \(j = 0, 1, 2, \ldots, J\), where we solve the equation on \([x_0, x_J]\) with \(J\) grid points and the mesh size \(h_x = (x_J - x_0)/J\). The discrete solution is denoted by \(u^n_j = u(t^n, x_j)\) and \(j_0\) is the index for which \(x_{j_0} = 0\). At \(x_{j_0}\) we use the properties of \(u\) in Proposition 1 (iii), in the discrete form:
\[ u_{j_0+1} = \left(1 - \frac{h_x \cdot \epsilon}{2}\right) u^n_{j_0}, \quad u_{j_0-1} = \left(1 - \frac{h_x \cdot \epsilon}{2}\right) u^n_{j_0}. \]

On \([t_n, t_{n+1}]\) we solve:
\[
\begin{align*}
\frac{u^*_j - u^n_j}{dt} &= \frac{u^*_{j+1} - 2u^*_j + u^*_{j-1}}{h_x^2} + 2(u^n_j)^2 u^*_j - (u^n_j)^4 u^*_j \quad &\text{for } 0 \leq j < j_0 - 1 \text{ and } j_0 + 1 < j \leq J; \\
\frac{u^*_j - u^n_j}{dt} &= \frac{u^*_j (\frac{2}{2} - h_x \cdot \epsilon) - 2 + u^*_j}{h_x^2} + 2(u^n_j)^2 u^*_j - (u^n_j)^4 u^*_j \quad &\text{for } j = j_0 - 1; \\
\frac{u^*_j}{dt} &= \frac{u^n_{j-1}}{(1 - h_x \cdot \epsilon)} \quad &\text{for } j = j_0; \\
\frac{u^n_j - u^*_j}{dt} &= \frac{u^*_{j+1} + u^*_j (\frac{2}{2} - h_x \cdot \epsilon) - 2 + (u^n_j)^2 u^*_j - (u^n_j)^4 u^*_j}{h_x^2} \quad &\text{for } j = j_0 + 1; \\
u^*_j &= \frac{a \cdot u^*_j}{\|u^*\|_2} \quad &\text{for all } j.
\end{align*}
\]

6.2. Numerical simulations. In this section we compare the discretized solution \(u^n_j\) with the exact one for different values of the parameters. We solve the equation on \([-40, 40]\), with \(J = 3200\) grid points and time step \(dt = 10^{-4}\) (thus \(x_0 = -40, x_J = 40, j_0 = 1600\) and \(h_x = 1/10\)). For fixed \(\epsilon\) we draw the bifurcation diagram for the ‘mass’ of the exact solution (2.11)–(2.13), i.e., its norm \(\|u\|_{L^2}\), and pick up values \(a_1, a_2, \ldots, a_6\) of the mass, see Fig. 1. Then we calculate the discretized solution \(u^n_j\) with fixed mass \(a_l\) \((l = 1, 2, \ldots, 6)\), and compare it to the exact solution with the corresponding \(k\).
For $\epsilon = 0$, we plot $\|u\|_{L^2}$ against $k$, using the explicit solution $u_{-k,0}$ obtained in Section 2.

For $\epsilon = 0.1\sqrt{3}$, we plot $\|u\|_{L^2}$ against $k$, using the explicit solutions $u_{\pm k, \epsilon}$ obtained in Section 2.
Figure 4. For $\epsilon = 0.5\sqrt{3}$, we plot $\|u\|_{L^2}$ against $k$, using the explicit solutions $u_{\pm,k,\epsilon}$ obtained in Section 2.

Figure 5. For $\epsilon = 0.9\sqrt{3}$, we plot $\|u\|_{L^2}$ against $k$, using the explicit solutions $u_{\pm,k,\epsilon}$ obtained in Section 2.
Figure 6. For $\epsilon = 0.1\sqrt{3}$, in each of plots 1)–6) we compare the discrete solution $u^n(x_j)$ (*) to the exact solution $u_{\pm,k,\epsilon}(x)$ (solid line), corresponding to the points on the bifurcation curve in Fig. 3.
Figure 7. For $\epsilon = 0.5\sqrt{3}$, in each of plots 1)–6) we compare the discrete solution $u^n(x_j)$ (*) to the exact solution $u_{\pm,k,\epsilon}(x)$ (solid line), corresponding to the points on the bifurcation curve in Fig. 4.
Figure 8. For $\epsilon = 0.9\sqrt{3}$, in each of plots 1)–6) we compare the discrete solution $u^n(x_j)$ (**) to the exact solution $u_{\pm,k,\epsilon}(x)$ (solid line), corresponding to the points on the bifurcation curve in Fig. 5.
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