Diffusion theory of many circle swimmers

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Particles following circular trajectory widely exist in nature. Both lifeless circle particle and self-propelled circling microorganisms can be described by the same diffusion equation. We derive the diffusion equation of many circle swimmers from helical trajectory and Newton’s law. The driven force includes both deterministic force and stochastic force. One special diffusion phenomena for Circle particle is that the density gradient and velocity gradient in one direction can induce a transverse flow in the corresponding perpendicular direction. This transverse flow will drive the particles to concentrate on a perpendicular direction to its original diffusion direction. Vortex in large scale is very easy to form due to the transverse flow. When many particles are trapped by the turbulent flow, there would appear a density bump upon the conventional exponential decay of non-circling particles. This non-equilibrium diffusion phenomena is originated from the mathematical circular trajectory itself, it preserves for different biological or physical system. An experimental evidence of this transverse flow showed up in a sperm diffusion experiment[1].

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I. INTRODUCTION

In classical physics, there exist many familiar examples for particles moving along closed orbital, such as planet in solar system, electrically charged particle in magnetic field. Many microorganisms swim along a helical trajectory until they reach a boundary, then they began circling near the boundary. The bacterium *Escherichia coli* swims in clockwise circular motion near a solid boundary[2][3][4][5]. Many animal spermatozoa swims in circle when it is confined in a planar surface, the circling motion has a strongly preferred handedness[6][7]. Artificial circle swimmers are constructed too, such as red blood cell attached by a chain of colloidal magnetic particles[8], double faced nanorodes with different polarity on its two sides[9].

Different self-propelled circle swimmers have different swimming mechanisms. The circling bacterium *E.coli* near a solid boundary is driven by a rotating helical flagella, theoretical study suggested that the hydrodynamic interaction between flagella and the boundary provide the driven force[2]. The sperm is also driven by flagella, but the beating pattern of the flagella is different from that of *E.coli*. The sperm flagella is theoretically modeled as left-right asymmetric beating ribbon, molecular motors inside the sperm trigger this beating[10]. The over damped magnetic nanorod without flagella can also move in circles, however it is driven by a rotating external field[11]. The dynamic mechanism for a circling rod particle swimming in confined channel is modeled as a Brownian circle swimmers[12]. Both geometric and stochastic theory are proposed to study the motion of a single sperm[13][14].

Recently an experiment was conducted to study the diffusion phenomena of millions of *Arbacia punctulata* spermatozoa in seawater, jelly coat solution and resact solution[1]. The experiment observation[1] is performed in a rectangular migration channel with sperm reservoir.
and chemoattractant attached to its two ends (Fig. 1 (a)). The diffusion image shows the sperm liken to concentrate one side of the channel (Fig. 1 (b)). Sperms move faster near the boundary of the channel.

Motivated by this experiment, we developed a general theory on the diffusion phenomena of many circle particles. We also derived the complete equation of motion for a single circle particle, one can apply this equation to describe a Brownian rod by including the length and the angle of rod. One can also apply this equation of motion to a single sperm by assuming an effective force of chemical stimulus. We shall start from the mathematical description of a circling trajectory and Newton’s law of motion. The effective driven force is of the most general formulation. One can set it as a deterministic force or a stochastic force for the convenience of physics, the formulation of equations are all the same. The diffusion equation for many circle swimmers is established on the conservation equation of total particle number. This diffusion equation for the circles’ center can be viewed as a member of the large family of Fokker-Plank equations, but the diffusion coefficients and drifting coefficients in our equation are not arbitrary functions, they are under strong constraints and are determined by physical parameters. If we incorporate the local coordinates of the circle particles into the diffusion equation, it becomes a sophisticate differential equation defined by two independent coordinates system, which is much more complicate than any conventional Fokker-Plank equations. We studied a simple case of constant driven force. We found a transverse flow perpendicular to the density gradient. The direction of this transverse flow depends on the direction of angular velocity. This transverse flow can help us to understand the diffusion phenomena of *Arbacia punctulata* spermatozoa in seawater, jelly coat solution and resact solution [1].

**II. THE CENTRIPETAL FORCE EXPRESSED BY NONZERO ANGULAR VELOCITY**

This section is for the self-content of the manuscript, detail formulas are present to show that there is no assumption to introduce the centripetal force from a circular trajectory, and no assumption is involved to separate the motion of the center and the motion of the particle itself.

Newton’s inertial law tell us, in the absence of force, a classical particle either is at rest or moves in a straight line with constant speed. If a particle draws a circular trajectory, there must exist some effective centripetal force to drive it moving in that way. No matter how complex the mechanism might be to generated this force, the ultimate result of centripetal force is making a particle moving along circles. A nonzero angular velocity is a summary of the central pedal force. For specific physical system, the expression of angular velocity depends on specific sources of interactions. In the following we will show how to express the centripetal force from a circular trajectory.

We first consider a point particle moving along a circular trajectory around the origin. The circular trajectory is described by \( r(t) = r_x(t) + ir_y(t) = r_0(t)e^{i\omega t} \). The coordination of the point-particle is

\[
x(t) = r_0(t)\cos\omega t, \quad y(t) = r_0(t)\sin\omega t, \quad x^2 + y^2 = r_0^2(t).
\]

\( r_0(t) \) is the radius of the circle. If \( r_0(t) = at^2, a > 0 \), we get an expanding spiral. If \( r_0(t) = at^2, a < 0 \), the trajectory is shrinking spiral. If \( r_0(t) = r_0 + (a\sin(kt) + b\cos(kt)) \), we get a fluctuating curve whose central line is a circle. We can also take \( r_0(t) \) as a random variable, i.e., \( r_0(t) = r_0 + b\xi(t) \) with \( \xi(t) \) as a random variable, then one will get a randomly fluctuating circle. For the simplest case, \( r_0(t) = r_0 = const \), the velocity of the particle is

\[
\dot{x} = -r_0\omega \sin(\omega t), \quad \dot{y} = +r_0\omega \cos(\omega t),
\]

one can verify the two components of the velocity, \( \dot{x} \) and \( \dot{y} \) satisfy

\[
\dot{x}^2 + \dot{y}^2 = r_0^2\omega^2.
\]

The two components of the velocity of a circling particle are not independent. The acceleration of the particle is given by the second derivative,

\[
\ddot{x} = -\dot{y}\omega, \quad \ddot{y} = +\dot{x}\omega.
\]
as $\omega \neq 0$. We summarize Eq. 4 as,

$$m\ddot{r} = -m\dot{r} \times \vec{\omega}, \quad \vec{\omega} = \omega \epsilon_z, \quad \epsilon_z = \epsilon_x \times \epsilon_y.$$  (5)

For the self-propel particle, such as bacteria or sperm, the centripetal force comes from interaction between the beating flagella and thick liquid where the bacteria lives. If the bacteria or sperm is in vacuum alone, no matter how hard its flagella is beating, it will not move along circles. The motion of a bacteria or sperm in vacuum alone depends on its initial velocity, if it is static in the beginning, it will not move at all even though its flagella is beating, if it has initial velocity, it will move in a straight line with constant speed, this is inertion motion which has nothing to do with the beating of flagella. For a circling electron, the effective driven potential could be magnetic field, angular frequency is given by $\omega = eB/mc$, $B$ is magnetic field, $e$ is electric charge. We put those details aside, and summarize all those effective driven force into a non-zero angular velocity $\omega$.

If the center of the circle is not fixed and drifts around at low speed, the trajectory of the particle is a helical curve, it can be described by drifting circles. We introduce $\vec{R}(t) = \vec{R}_x(t) + i\vec{R}_y(t)$ to express the instantaneous center of the circle. The relative position of the particle with respect to the center is denoted as $r(t) = r_x(t) + ir_y(t) = r_0(t)e^{i\omega t}$, $r_0(t)$ is the relative distance between a particle and the center of circle. The complete position coordination of the circle swimmer is

$$\vec{q}(t) = x(t) + iy(t) = R(t) + r(t),$$  (6)

$$x(t) = R_x(t) + r_x(t), \quad y(t) = R_y(t) + r_y(t), \quad r_x(t) = r_0(t) \cos(\omega t), \quad r_y(t) = r_0(t) \sin(\omega t).$$  (7)

This coordination provide us a convenient mathematical description for an arbitrary helical curve in two dimensional space. The shape of the curve depends on the function of center coordination $\vec{R}(t) = R_x(t) + iR_y(t)$ and the function of radius $r_0(t)$. For example, if the center accelerates in X-direction with a constant acceleration $a$, and accelerates in Y-direction with acceleration $b$, the center coordination functions are $R_x(t) = \frac{1}{2}a t^2$, $R_y(t) = \frac{1}{2}b t^2$. In the mean time, if we assume the radius of the circle is fluctuating periodically, $r_0(t) = r_0 + \cos(kt)/100$, the equation for a drifting helical curve is

$$x(t) = \frac{1}{2}a t^2 + \left[ r_0 + \frac{\cos(kt)}{100} \right] \cos \omega t,$$

$$y(t) = \frac{1}{2}b t^2 + \left[ r_0 + \frac{\cos(kt)}{100} \right] \sin \omega t.$$  (8)

From physics point of view, the tangent vector of the curve is the velocity of a particle which move along the curve. The acceleration can be expressed as the curvature of the curve. The time parameter $t$ can be viewed as an arc length parameter of the curve. The first derivative of curve $q(t)$ with respect to $t$ give us the velocity,

$$\dot{x} = \dot{R}_x(t) - v_0(t) \sin(\omega t) + \dot{r}_0(t) \cos(\omega t),$$

$$\dot{y} = \dot{R}_y(t) + v_0(t) \cos(\omega t) + \dot{r}_0(t) \sin(\omega t).$$  (9)

$\dot{R}_x(t)$ and $\dot{R}_y(t)$ are the drifting velocity of the center. $v_0(t) = \dot{r}_0(t)/\omega$ is the speed of circling motion around the center. The mathematical definition of circle swimmer requires that the circling speed of swimmer is much larger than the speed of its drifting circle center, i.e., $v_0(t) \gg \dot{r}_0(t), \alpha = x, y$. For the opposite case, $v_0(t) \ll \dot{r}_0(t)$, the trajectory of the particle is no longer drifting circles, and becomes an arbitrary curve. In that case, one can choose differential geometry to construct an equivalent theoretical description. The second derivative of $q(t)$ with respect to $t$ reads

$$\ddot{x} = \ddot{R}_x(t) - \dot{y} \omega + \dot{r}_0(t) \cos(\omega t) - \dot{r}_0(t) \omega \sin(\omega t),$$

$$\ddot{y} = \ddot{R}_y(t) + \dot{x} \omega - \dot{r}_0(t) \omega \cos(\omega t) + \dot{r}_0(t) \omega \cos(\omega t).$$  (10)

The acceleration may be split into two parts: the acceleration of the center, the acceleration of circling motion around the center. We multiply the mass on both sides of Eq. 10 to demonstrate Newton’s law,

$$m\ddot{x} = m[\ddot{R}_x(t) + \dot{r}_0(t) \cos(\omega t)] - m[\dot{y} \omega - \dot{r}_0(t) \omega \sin(\omega t)],$$

$$m\ddot{y} = m[\ddot{R}_y(t) + \dot{r}_0(t) \sin(\omega t)] + m[\dot{x} \omega - \dot{r}_0(t) \omega \cos(\omega t)].$$  (11)

The absolute acceleration of the center is

$$a_x = [\ddot{R}_x(t) + \dot{r}_0(t) \cos(\omega t)],$$

$$a_y = [\ddot{R}_y(t) + \dot{r}_0(t) \sin(\omega t)],$$

$$V_x = [\dot{x} - \dot{R}_x(t) + \dot{r}_0(t) \cos(\omega t)],$$

$$V_y = [\dot{y} - \dot{R}_y(t) + \dot{r}_0(t) \sin(\omega t)].$$  (12)

The acceleration of a circling particle can be expressed into vector formulation,

$$m\ddot{\vec{z}}(t) = m\ddot{a} - m\dot{\vec{V}} \times \vec{\omega}, \quad \vec{\omega} = \omega \epsilon_z.$$  (13)

The non-zero angular frequency $\omega$ summarized the effective centripetal force for the circling motion. The centripetal force $m\dot{\vec{V}} \times \vec{\omega}$ is consumed by three motions, one is circular motion with respect to the center, the second consumer is the motion of the center, the third consumer is the expanding or shrinking of the radius. $m\ddot{a}$ summarized the other external effective driven force $F_{external}$,

$$F_{external} = m\ddot{a}, \quad F_{centripetal} = -m\dot{\vec{V}} \times \vec{\omega}.$$  (14)

As long as we have a non-zero angular frequency $\omega$, the effective driven forces from different sources are summarized into $\omega$. For an electron moving in magnetic field,
the non-zero $\omega$ means the existence of magnetic field. For a self-propelled particle, such as bacteria or sperm, the non-zero $\omega$ summarized the effective centripetal force generated by the interaction between beating flagella and environment.

III. THE DIFFUSION EQUATION OF CIRCLING PARTICLES IN TWO DIMENSIONS

A. random fluctuation upon deterministic trajectory

Circling motion is a kind of deterministic dynamics driven by a deterministic force. When a particle is surrounded by many thermally fluctuating molecules, the constant collision between the particle and other molecules would cause stochastic deviations from the deterministic path. This stochastic fluctuations is driven by a random noise force. The amplitude of stochastic fluctuation $\xi(t)$ should be much smaller than the radii of the circle so that people can observe a fluctuating circle trajectory. If the fluctuating amplitude is much larger than the radii of the circle, one would just see some random trajectory instead of circular trajectory. So we take the deterministic path as dominant trajectory, and introduce a small amplitude of random fluctuation $\xi^3 = \xi_x^3(t) + \xi_y^3(t)$,

$$
\begin{align*}
x(t) &= R_x(t) + r_0(t) \cos(\omega t) + \xi^3_x(t), \\
y(t) &= R_y(t) + r_0(t) \sin(\omega t) + \xi^3_y(t).
\end{align*}
$$

(15)

The random fluctuation $\xi(t)$ can be viewed as the fluctuation of the center coordination,

$$
\begin{align*}
x(t) &= [R_x(t) + \xi^3_x(t)] + r_0(t) \cos(\omega t), \\
y(t) &= [R_y(t) + \xi^3_y(t)] + r_0(t) \sin(\omega t),
\end{align*}
$$

(16)

$\xi(t)$ can also be taken as the fluctuation of the radii of the circle,

$$
\begin{align*}
x(t) &= R_x(t) + [r_0(t) + \xi^3_x(t)] \cos(\omega t), \\
y(t) &= R_y(t) + [r_0(t) + \xi^3_y(t)] \sin(\omega t), \\
\xi_x(t) &\ll r_0(t), \quad \xi_y(t) \ll r_0(t).
\end{align*}
$$

(17)

The angular frequency is not always constant for some self-propel particles. So we can introduce a random variable $\xi(t)$ into the deterministic angular frequency,

$$
\begin{align*}
x(t) &= R_x(t) + r_0(t) \cos([\omega(t) + \xi_x(t)]t), \\
y(t) &= R_y(t) + r_0(t) \sin([\omega(t) + \xi_y(t)]t).
\end{align*}
$$

(18)

One can choose different coordination for different physical systems. If one can fix the center and the radii of the circle at the same time, the fluctuation may be combined into the angular frequency. If the center and radius of the circle is not fixed, for example, a randomly swimming bacteria, it is more convenient to summarize the small stochastic motion by fluctuating center or fluctuating radius. For the most complex case, we can combine these three types of fluctuation together. The random fluctuation of the center coordinates and angular velocity are independent. One can choose a convenient description for different cases.

We shall consider the diffusion phenomena of millions of circles in the following, each of them obeys an equation of motion,

$$
\begin{align*}
\dot{x} &= \dot{R}_x(t) - v_0(t) \sin(\omega t) + \dot{r}_0(t) \cos(\omega t), \\
\dot{y} &= \dot{R}_y(t) + v_0(t) \cos(\omega t) + \dot{r}_0(t) \sin(\omega t).
\end{align*}
$$

(19)

where $\omega = \omega + \xi^3(t)$, the angular frequency is fluctuating around a central frequency. The solution of Eq. (19) is a drifting circle.

B. Diffusion equation of circle swimmers in liquid with low Reynold number

When we talk about diffusion phenomena, it usually means the evolution of density distribution for a large amount of particles in an area with or without boundary. Density is defined by the number of particles in one unit area. Circle particle has two microscopic scales, one is the size of the particle itself, the other is the size of the circle. We can define density as the number of particles in an unit scale of the particle itself. If we fix the center of the circle, the density function for one particle is a sharp peak circling along circular trajectory. The average of these peaks in a longer time scale just sit at the center of the circle. We will not see diffusion phenomena unless the center of the circle moves. So we should choose an appropriate unit scale to get an appropriate definition of density function for diffusion. The unit spatial scale is the radius of the circle, the unit time scale is the periodicity of the circling motion,

$$
\Delta x \approx 2r_0, \quad \Delta y \approx 2r_0, \quad \Delta t \approx T_0,
$$

(20)

where $T_0$ measure how much time is spent for a particle to draw a circle. We can either project the mass of particle onto the center of the circle, or fill it homogeneously in the unit disk $\pi r^2_0$. We assume that several particles are allowed to swim along the same circle. This is to meet the experiment observation of sperm swimming that two or more sperms can swim together[8]. We assume there can be at most $s$ particles in one circle. The biggest number of circles in a finite area $S$ is $l = S/\pi r^2_0$. We can define density as the number of particles in one circle.

If the step size of experiment measurement is much larger than the radius of the circle, we can choose a larger unit scale

$$
\Delta x > 2r_0, \quad \Delta y > 2r_0, \quad \Delta t > T_0,
$$

(21)
The density is defined by the number of particles in unit box $\Delta x \Delta y \Delta t$. In an thermal environment, the phases of circling motion for different particles are usually randomly distributed. The average of these relative circling velocity with respect to the centers of the circles is zero. The internal motion of the particle inside the unit box can be neglect for diffusion phenomena. However the centripetal force induced by the circular motion will be consumed by the center.

We separate the motion between the circling and the drifting of centers, the circling motion is described by a deterministic equation
\begin{align}
\dot{r}_x &= -v_0(t) \sin(\omega t) + \dot{r}_0(t) \cos(\omega t), \\
\dot{r}_y &= +v_0(t) \cos(\omega t) + \dot{r}_0(t) \sin(\omega t). \quad (22)
\end{align}

The drifting motion of the center is determined by diffusion dynamics which does not concern the motion of particle within the scales of a circle. In the diffusive dynamics, we only need to study the dynamics of the center,
\begin{align}
\dot{R}_x(t), \quad \dot{R}_y(t). \quad (23)
\end{align}

We can combine the motions of the center and the small circling motion together to get the equation of motion for every particle, but for diffusion phenomena, it will not make much difference.

When a swimmer is placed into a collection of thousands of swimmers which has inhomogeneous density distribution, it has larger probability of entering into the low density region, but less probability of moving into the high density region. Because there are more free spaces in low density region, and less free space in the high density region. The density gradient acts as an effective driven force to drive a swimmer from high density region to low density region. We denote it as diffusive driven force $F_{\text{diffusion}} \propto -\langle \nabla R N_s \rangle / N_s$.

Physical circle swimmers have a finite body volume and an effective circling volume. If a swimmer tries to push itself into a crowd of swimmers, it has to expel the molecules and other swimmers in that area, it will inevitably encounter friction resistance. The faster it moves forward, the stronger resistance it will receive, so we take the resistance as a negative force proportional to velocity, $F_{\text{friction}} \propto -\eta \dot{V}$.

According to equation of motion (12) and (13), we add the external driven force $F_{\text{external}}$ and centripetal force $F_{\text{centripetal}}$ upon the center. We are dealing with millions of particles, one reasonable choice of elementary unit is to take a small unit box of many particles. We neglect the internal motion of individual particle inside the box. The unit box is viewed as a whole to interact with other boxes, the equation of motion for a square of particles is given by Newton’s law,
\begin{align}
m N_s \frac{\partial \dot{V}}{\partial t} = -m N_s \dot{V} \times \dot{\omega} + N_s \ddot{F} - D \nabla N_s - m N_s \eta \dot{V}. \quad (24)
\end{align}

$\eta$ indicates the friction coefficient. In fact, if we divide both the two sides of Eq. (24) with the density $N_s$, it leads to an effective equation of motion for single particle,
\begin{align}
m \ddot{r} = -m \ddot{r} \times \dot{\omega} + \ddot{F} - D \nabla N_s - m \eta \dot{r}. \quad (25)
\end{align}

We can equally split the unit box’s centripetal force, external force $F$ and friction force $m N_s \eta V$ into $N_s$ pieces, each particle inside share one piece, then we will get Eq. (25) for single particle in the absence of a special term, the effective diffusion force
\begin{align}
f_{\text{diffusion}} = -D \nabla N_s. \quad (26)
\end{align}

Density gradient is a conception of many particles, for a single particle, density gradient does not exist. This diffusion force term only exist when the particle is placed in a many particle system. The external driven force $\ddot{F}$ may include all different kinds of forces, deterministic or stochastic, i.e.,
\begin{align}
F_x &= f_x^d - \partial_{x} V(R) - \frac{f_x^\xi(t)}{m}, \\
F_y &= f_y^d - \partial_{y} V(R) - \frac{f_y^\xi(t)}{m}. \quad (27)
\end{align}

$V(R)$ is a trapping potential, $f^d$ is a deterministic driven force, $f_x^\xi(t) = m \xi_x(t)$ and $f_y^\xi(t) = m \xi_y(t)$ is white noise random force. If we identify $\ddot{F}$ as white noise random force, Eq. (27) becomes a generalized Langevin equation with a special diffusion driven force $f_{\text{diffusion}}$. Equation of motion (25) includes two coupled equations,
\begin{align}
m \ddot{R}_x &= -m \omega \ddot{R}_y - D_x \frac{\partial R_x N_s}{N_s} - m \eta \dot{R}_x + F_x, \\
m \ddot{R}_y &= m \omega \ddot{R}_x - D_y \frac{\partial R_y N_s}{N_s} - m \eta \dot{R}_y + F_y. 
\end{align}

Here the diffusion coefficient $D_x$ and $D_y$ are state dependent, $D_x = D_x(x, y)$ and $D_y = D_y(x, y)$.

We assume the particle swims in a thick liquid with low Reynolds number, the friction on the center is much larger than the inertial term of the center, the motion of the center is over damped. Then we set the inertial term as zero,
\begin{align}
m \ddot{R}_x = 0, \quad m \ddot{R}_y = 0. \quad (28)
\end{align}

Noticing here it is the center’s inertial term, the motion of the particle itself is not over damped since it has self-driven force. This self-driven force comes from the intrinsic beating pattern of the flagella. They were born to swim in circles. In the smaller scale inside the circle, the particle generates an effective driven force by its internal motors, this effective driven force canceled the friction force along the circle, in the meantime generated an effective centripetal force to make itself follow a circular trajectory. Part of this effective driven force is consumed
by the center through the term $\mathbf{m} \dot{\mathbf{R}} \times \mathbf{\omega}$. The formulation of velocity is derived by solving Eq. (28),
\[
\dot{R}_x = -D_{xy} \frac{\partial R_y N_s}{N_s} - D_{xx} \frac{\partial R_x N_s}{N_s} + \sigma_{xx} F_x + \sigma_{xy} F_y, \\
\dot{R}_y = -D_{yx} \frac{\partial R_x N_s}{N_s} - D_{yy} \frac{\partial R_y N_s}{N_s} + \sigma_{yy} F_y + \sigma_{yx} F_x.
\]
(29)

where the diffusion tensor $D$ and drifting tensor $\sigma$ are
\[
D_{xx} = \frac{D \eta}{m A^2}, \quad D_{xy} = -\frac{D \eta}{m A^2}, \quad D_{yx} = \frac{D \eta}{m A^2}, \quad D_{yy} = \frac{D \eta}{m A^2}, \\
\sigma_{xx} = \frac{\eta}{m A^2}, \quad \sigma_{xy} = -\frac{\omega}{m A^2}, \quad \sigma_{yy} = \frac{\eta}{m A^2}, \quad \sigma_{yx} = \frac{\omega}{m A^2}.
\]
(30)

$A^2 = \eta^2 + \omega^2$ is a normalization factor. We express Eq. (29) by a brief formula,
\[
\dot{R}_i = \sum_j \sigma_{ij} F_j(x, y) - \sum_j D_{ij} (x, y) \frac{\partial N_s}{N_s},
\]
(31)

Combining the over damped center equation and the deterministic circular motion, we get the complete equation of motion
\[
\dot{x} = -D_{xy} \frac{\partial R_y N_s}{N_s} - D_{xx} \frac{\partial R_x N_s}{N_s} + \sigma_{xx} F_x + \sigma_{xy} F_y, \\
- r_0 (t) \omega \sin(\omega t) + \dot{r}_0 (t) \cos(\omega t), \\
\dot{y} = -D_{yx} \frac{\partial R_x N_s}{N_s} - D_{yy} \frac{\partial R_y N_s}{N_s} + \sigma_{yy} F_y + \sigma_{yx} F_x, \\
+ r_0 (t) \omega \cos(\omega t) + \dot{r}_0 (t) \sin(\omega t).
\]
(32)

The solution of this equation is a drifting circle. This is the dynamics of a single particle. When we considering the diffusion phenomena of millions of particles, we care about how many particles is distributed in certain area at time $t$. The unit scale used to measure the diffusion is usually larger than the radius of the circle. The circling phase of the particles inside the unit box are random, so we can neglect the deterministic circling part, and only take into account of the motion of the center. The total number of swimmers flow into an unit box must equal to the total number of particles flowing out of the unit box. The particle flow is $J_0 = N_s \mathbf{V}$. The differential formulation of the mass conservation equation of particles is
\[
\partial_t N_s + \nabla R \cdot J_0 = 0.
\]
(33)

Substituting Eq. (29) into the mass conservation equation, we derived the diffusion equation,
\[
\frac{\partial N_s}{\partial t} + \sum_{ij} [\partial_t [\sigma_{ij} F_j(x, y) N_s] - \partial N_s] = 0.
\]
(34)

The friction coefficient $\eta$ may be state dependent when the particle is in a complex environment, so we introduce the friction function $\eta = \eta(x, y)$ to meet more general cases. The drift tensor $\sigma = \sigma(x, y)$ is state dependent as well. We introduce the drift function,
\[
T_i(x, y) = \sum_j [\sigma_{ij} (x, y) F_j(x, y)],
\]
(35)

to express the diffusion equation (34) into the formulation of Fokker-Planck equation,
\[
\frac{\partial N_s}{\partial t} + \sum_{ij} [\partial_t [T_i(x, y) N_s] - \partial N_s] = 0.
\]
(36)

If the diffusion tensor is homogeneous in space, they are constant without any dependence of space time variables. Then we get a simpler diffusion equation, its explicit formulation is,
\[
\frac{\partial N_s}{\partial t} - [D_{xy} + D_{yx}] \partial_{R_x} \partial_{R_y} N_s - D_{xx} \partial_{R_x}^2 N_s - D_{yy} \partial_{R_y}^2 N_s + [\sigma_{yy} F_y + \sigma_{yx} F_x] \partial_{R_x} N_s - [\sigma_{yy} \partial_{R_y} F_y + \sigma_{yx} \partial_{R_x} F_x] N_s + [\sigma_{xx} \partial_{R_x} F_x + \sigma_{xy} \partial_{R_y} F_y] N_s = 0.
\]
(37)

One simplification of this complex diffusion equation is to set the diffusion coefficient along X-direction and Y-direction as the equal number, $D_x = D_y = D$. Then the diffusion tensor becomes simpler,
\[
D_r = D_{xx} = D_{xy} = \frac{D \eta}{m A^2}, \quad D_{yx} = -D_{xy} = \frac{D \eta}{m A^2}.
\]
(38)

C. Transverse force flow induced by velocity gradient

One special character of circle particle is that the two components of its velocity are not independent. If its average velocity increase in X-direction, its average speed in Y-direction will increase as well so that it can draw a circle, otherwise, its trajectory just becomes ellipse, or any other irregular geometry. Thus if there exist velocity gradient, a transverse flow will be induced on the boundary between fast swimmers and slow swimmers. The velocity field for circle particle is expressed by density function and external driven field. We define the transverse flow induced by velocity gradient as
\[
\mathbf{J} = -\nabla \mathbf{V} \times \mathbf{\omega},
\]
(39)

where $\mathbf{\omega} = \omega \mathbf{e}_z$ for two dimensions, $\mathbf{e}_z$ denotes the normal vector of $X - Y$ plane. $\omega < 0$ means the swimmer
is rotating in clockwise direction, \( \omega > 0 \) means the circle swimmer rotates in counterclockwise direction. In physics, the derivative of velocity is acceleration. So the vector flow \( \vec{J} \) is actually the effective driven force field. For the velocity field Eq. (29), the transverse force flow is

\[
J_y = \omega \partial_{Rz}[\sigma_{xx}F_x] + \omega \partial_{Rz}[\sigma_{xy}F_y], \\
+ \omega \partial_{Ry}[D_{xy}\partial_{Rx}\ln N_s] + \omega \partial_{Ry}[D_{xx}\partial_{Rx}\ln N_s] \\
J_x = \omega \partial_{Ry}[\sigma_{yy}F_y] + \omega \partial_{Ry}[\sigma_{yx}F_x]. \\
+ \omega \partial_{Rx}[D_{yx}\partial_{Ry}\ln N_s] + \omega \partial_{Rx}[D_{yy}\partial_{Ry}\ln N_s]. \tag{40}
\]

When the circle swimmer switch its circling handedness, i.e., from clockwise to counterclockwise, or vice verse, the transverse flow switch its direction as well. The switching of circling handedness is simply flipping sign of the angular velocity, \( \omega \rightarrow -\omega \), or \( -\omega \rightarrow \omega \). According to Eq. (40), the direction of flow \( J_y \) will flip to \(-J_y\), so does \( J_x \). If the velocity gradient only exist in X-direction, the transverse flow \( \vec{J} \) is along Y-axis (Fig. 3 (a)). \(-\omega \) and \( \omega \) leads the transverse force flow to opposite direction. We can predict an observable experiment phenomena according to the flow field equation \( \vec{J} = (J_x, J_y) \). We mix two groups of circle swimmers with opposite circling handedness together, and let them diffuse in a rectangular channel. The particles will splits into two separated groups during diffusion. All the particles in one group have the same circling handedness. The two groups are moving in opposite direction, one goes to positive Y-direction, the other goes to negative Y-direction (Fig 3 (b)).

**FIG. 3:** (a) The transverse flow induced by velocity gradient. (b) Swimmers with opposite circling handedness will split during diffusion.

**D. Vortex generated during diffusion**

The sea urchins spermatozoa confined in planar surface are randomly swimming at low density. When the sperm density is increased to about 2500 spermatozoa per mm\(^2\), spermatozoa self-organized into dynamic vortex array with local hexagonal order\(^6\). Every vortex contains 10 \( \pm \) 2 sperm swimming with the same handedness. This kind of vortex is the smallest vortex. It is of the same size as one circle. We can classify this small vortex by vorticity which is defined by

\[
\Gamma_r = \vec{\nabla}_r \times \vec{V}_r = \partial_{r_x} \dot{y}_r - \partial_{r_y} \dot{x}_r = -2\omega, \tag{41}
\]

where the velocity field is

\[
\dot{x}_r = -r_0\omega \sin(\omega t) = -\omega y_r, \\
\dot{y}_r = +r_0\omega \cos(\omega t) = \omega x_r. \tag{42}
\]

This vorticity \( \Gamma_r \) quantifies the vortex observed in sperm experiment\(^3\). If we go to larger scale, the non-zero angular velocity upon the centers of those small circles will also generate vortex in larger scale. The vorticity of the center’s velocity field is,

\[
\Gamma_V = \vec{\nabla} \times \vec{V} = \partial_{Rx} \dot{R}_y - \partial_{Ry} \dot{R}_x, \tag{43}
\]

here the velocity field of the center is given by Eq. (29). We consider the special case that the diffusion tensor are state-independent, and the diffusion coefficient in X-direction equals the diffusion coefficient in Y-direction, \( D_x = D_y \), then the diffusion tensor and drifting tensor satisfy the relations,

\[
D_{xx} = D_{yy}, \quad D_{xy} = -D_{yx}, \\
\sigma_{xx} = \sigma_{yy}, \quad \sigma_{xy} = -\sigma_{yx}. \tag{44}
\]

Substituting Eq. (29) into Eq. (43), we get

\[
\Gamma_V = \sigma_{xx}(\partial_{Rz}F_y - \partial_{Ry}F_x) + \sigma_{yx}(\partial_{Rz}F_x + \partial_{Ry}F_y) \\
+ D_{xy}(\partial_{Ry}\partial_{Rx}\ln N_s + \partial_{Rx}\partial_{Rx}\ln N_s). \tag{45}
\]

According to the Green-function theory,

\[
\partial_{Ry}\partial_{Rx}\ln N_s + \partial_{Rx}\partial_{Rx}\ln N_s = 2\pi\delta(N_s), \tag{46}
\]

we can simplify \( \Gamma_V \) further,

\[
\Gamma_V = \sigma_{xx}(\partial_{Rz}F_y - \partial_{Ry}F_x) + \sigma_{yx}(\partial_{Rz}F_x + \partial_{Ry}F_y) \\
+ D_{xy}2\pi\delta(N_s). \tag{47}
\]

The particles are circling around a common center, they are pulled away from the center by centrifugal force, so the density vanishes at the center. We can technically split the density function into two components, and express the density function as \( N_s = \sqrt{N_x^2 + N_y^2} \). Then we can identify every vortex with a topological number by applying Duan’s topological current theory\(^16\). It should be noticed that there are many vortex field here.

More over, the transverse force flow induced by velocity gradient may also have vortex configuration. We measure this kind of vortex by

\[
\Gamma_J = \vec{\nabla} \times \vec{J} = \partial_{Rx}J_y - \partial_{Ry}J_x, \tag{48}
\]

\( \Gamma_J \) defines a higher order of vortex. Both \( \Gamma_J \) and \( \Gamma_V \) are determined by the density distribution and external field. When the density distribution evolute following diffusion equation\(^{67} \), the vortex configuration will change correspondingly. The vortex generated by the external field is controlled by external potential.
E. The fluctuation of diffusion coefficient

The diffusion tensor depends on friction coefficient, diffusion coefficient and angular velocity. The angular velocity of a circling particle confined in two dimensions has only one component \( \omega_z \). Bacteria swim in circles near solid surface. Experiment found the angular frequency of bacteria’s circling has strong fluctuations\[13\]. The angular frequency actually measure the strength of centripetal force. If we fix the center of the circle, a fluctuating angular frequency just make the particle move faster or slower along the circle, the center is static. If we release the center, it will consumed part of the centripetal force. So the diffusion coefficient obtained above strongly depend on the fluctuations. We introduce a fluctuating vector \( \xi_z(t) \) into the angular velocity vector

\[
\omega = \omega_z^0(t) + \xi(t).
\]

For the state independent diffusion coefficient, the diffusion tensor with angular fluctuation is,

\[
D_{xx} = \frac{D_x \eta}{m(\omega(t) + \xi(t))^2 + \eta^2},
\]

\[
D_{yx} = \frac{D_z \omega(t) + \xi(t)}{m(\omega(t) + \xi(t))^2 + \eta^2}.
\]

If the fluctuation reduced the angular velocity, both diffusion coefficient \( D_{xx} \) and \( D_{xy} \) will increase(Fig. 4), but the \( D_{yx} \) will reach climax first(Fig. 5). If the fluctuation increase the angular velocity, both the diffusion coefficients \( D_{xx} \) and \( D_{xy} \) will decrease, but \( D_{xy} \) decreases much slower than \( D_{xx} \). If we take the ratio between \( D_{xx} \) and \( D_{xy} \),

\[
\frac{D_{xx}}{D_{yx}} = \frac{\eta}{\omega(t) + \xi(t)},
\]

one can see clearly that the increase of frequency will make the off-diagonal diffusion stronger. As the total number of particle must be conserved, the increase of off-diagonal diffusion will decrease the diagonal diffusion. When the friction coefficient \( \eta \) is state dependent, i.e., \( \eta = \eta(x, y, t) \), the behavior of diffusion coefficient is determined by the complete coefficient Eq. (50).

IV. THE DIFFUSION EQUATION FOR \( D_x = D_y \)

We first study a special case of diffusion equation (44). The diffusion tensor and drifting tensor are state-independent. We take the diffusion coefficient \( D_x \) as an equal value of \( D_y \),

\[
D_x = D_y = D.
\]

Then the off-diagonal diffusion coefficients and drifting coefficients are,

\[
\sigma_{yx} = -\sigma_{xy} = \frac{\omega}{mA^2}, \quad D_{yx} = - D_{xy} = \frac{D_\omega}{mA^2}.
\]

![FIG. 4: The diagonal diffusion coefficients (\( D_{xx}=D_{yy} \)](image)

The external driven force is constant both in X-direction and in Y-direction,

\[
F_x = C_x, \quad F_y = C_y.
\]

If \( C_y \neq 0 \), the particle would have a drifting velocity in Y-direction. For a non-circling particle, if \( F_y = 0 \), the density center will not drift in Y-direction. But for circle particles, even if \( F_y = 0 \), the density center still has a drifting velocity in Y-direction. To see this more clearly, we take a further step of simplification,

\[
F_x = C_x \neq 0, \quad F_y = 0.
\]

There is no external force in Y-direction now. For the circle particle, the angular frequency \( \omega \) is not zero. If the particle is circling is in counterclockwise direction, \( \omega > 0 \). If the circling is in clockwise direction, \( \omega < 0 \). The diffusion equation (44) now becomes,

\[
\frac{\partial N_s}{\partial t} = D_{xx} \frac{\partial^2}{\partial x^2} N_s + D_{yy} \frac{\partial^2}{\partial y^2} N_s - b \frac{\partial}{\partial y} N_s - a \frac{\partial}{\partial x} N_s.
\]

The drifting velocity is \( a = \sigma_{yx} F_x \), \( b = \sigma_{xy} F_x \). The drift of density center along Y-direction is driven by off-diagonal diffusion tensor \( \sigma_{yx} \). Since there is no external driven force in Y-direction, this drifting velocity is a special character of circle particles.

The analytic solution of diffusion Eq. (50) with infinite
boundary condition is

\[ N_s = \frac{1}{4\pi D_r t} \exp\left[-\frac{(R_x - at)^2}{4D_r t} - \frac{(R_y - bt)^2}{4D_r t}\right]. \tag{57} \]

The center of sperm density is drifting under the action of external driven force. We integrated the density along Y-axis and derived the density evolution along X-axis, \( N_s(x, t) = \int_0^h N_s dy \), where \( h \) is the width of channel along Y-direction. If the drifting velocity is much larger than the diffusion coefficient, we would observe apparent drifting of the density center (as shown in Fig. 6(a)). When the diffusion coefficient is much larger than the drifting, one would see a rapid decay in beginning, then there appears a plateau due to the drifting (Fig. 6(b)). We integrated \( X \) out to keep the \( N_s \) as a function of \((y, t)\), \( N_s(y, t) = \int_0^h N_s dx \). One can see the drifting velocity of density center in Y-direction is \( b \). When particles are diffusing in X-direction, their density distribution in Y-direction is not homogeneous, they concentrates to either positive Y-direction or negative Y-direction.

There exist a typical vortex configuration for the simple case of \( D_{xx} = D_{yy} = D_r \) (Fig. 7). For the analytic solution of diffusion equation (56), we can get an explicit formulation of the velocity field \( \vec{V}_R = (V_x, V_y) \). The vorticity of this vortex is given by the vorticity Eq. (58). Substituting the density function Eq. (57) into the velocity Eq. (24), then we can express the vorticity Eq. (43) explicitly as

\[ \Gamma_V = D_{xy} \partial_{R_y} \partial_{R_y} L(R) + \partial_{R_x} \partial_{R_x} L(R). \tag{58} \]

where the function \( L(R) \) is given by Eq. (59). \( \Gamma_J \) quantifies the vortex in the space of higher order derivative of density distribution.

If the particles are confined in a finite rectangular region, the solution of diffusion Eq. (56) do not have explicit expression. We have to do numerical computation to study the density evolution \( N_s(x, t) = \int_0^h N_s dy \). We take a rectangular region covered by \( 0 \leq x \leq 10, 0 \leq y \leq 5 \). The diagonal diffusion coefficient are \( D_{xx} = D_{yy} = D_r = 1 \). The density evolution for a small drifting velocity \( a = b = 1 \) is an exponential decay in the beginning (Fig. 8). As the diffusion goes on, the exponential decaying curve is approaching to a straight line. Further more, we computed the density evolution for larger drift velocity \( a = b = 18 \). The density distribution curve first transform from an exponential decaying curve to a curve with plateau, and then approaches to a linear distribution (Fig. 8).

\[ \Gamma_J = D_{xy} \omega [\partial_{R_x} \partial_{R_y} L(R) - \partial_{R_y} \partial_{R_x} L(R)] + D_{xx} \omega [\partial_{R_x}^3 L(R) - \partial_{R_y}^3 L(R)]. \tag{60} \]
cross-diffusion term, \(D\). An example to demonstrate the difference between \(\omega\) with respect to the diffusion equation. We performed a numerical computation of diffusion equation (56) in a rectangular region. The drifting speed is very large, \(a=b=18\).

V. THE DIFFUSION EQUATION FOR \(D_x \neq D_y\)

When \(D_x \neq D_y\), the diffusion equation will include a cross-diffusion term, \(D_{yx} \partial R_x \partial R_y N_s\). If the angular velocity of the particles is zero, \(\omega = 0\), this cross-diffusion term vanishes. The density distribution for \(\omega > 0\) and \(\omega < 0\) are different. The numerical evolution of diffusion equation shows, the density moves to opposite directions with respect to \(\omega > 0\) and \(\omega < 0\). We study a special case as an example to demonstrate the difference between \(\omega > 0\) and \(\omega < 0\). The diffusion coefficients and drifting tensor are state independent. The external force is constant, \(F_x = C_x, F_y = C_y\).

The diffusion equation for \(D_x \neq D_y\) is,

\[
\frac{\partial N_s}{\partial t} = D_{xx} \partial_{R_x}^2 N_s + D_{yy} \partial_{R_y}^2 N_s + (D_{yx} + D_{xy}) \partial_{R_x} \partial_{R_y} N_s - b \partial_{R_y} N_s - a \partial_{R_x} N_s. \tag{62}
\]

The drifting velocity is \(a = \sigma_{xx} C_x + \sigma_{xy} C_y, b = \sigma_{yy} C_y + \sigma_{yx} C_x\).

A. The diffusion equation for \(\omega > 0\)

We first study the diffusion phenomena of a positive angular frequency, \(\omega > 0\). Diffusion Eq. (62) has no analytic solution. We performed a numerical computation of the density evolution in the two dimensional plane of \(R_x - R_y\). In the beginning, the particles are distributed along Y-axes at \(R_x = 0\), the initial density is \(N_s = 10\). The diffusion coefficients are \(D_{xx} = 0.9\) and \(D_{yy} = 1\). We choose a large drifting velocity, \(b = 5\) and \(a = 15\). The off-diagonal diffusion coefficient is \(D_{yx} + D_{xy} = -1\) which corresponds to a positive angular frequency. One snapshot of the density evolution at \(t = 15\) is shown in Fig. 10 (a), the light color demonstrates the high density, the dark color shows the low density. The center of high density region shifts to positive \(R_y\)-direction (Fig. 10 (a)).

Later we increase the off-diagonal diffusion tensor and drift tensor to \(D_{yx} + D_{xy} = -2, b = 10\), but the diagonal diffusion tensor remains invariant. A snapshot of density evolution at \(t = 15\) shows the drift in \(Y\)-direction speeds up (Fig. 10 (b)). In \(X\)-direction, the density center does not reach as far as the diffusion in low angular frequency.

![FIG. 10: (a) The density distribution \(N_s(R_x, R_y)\) at time \(t = 15\) for the numerical evolution of diffusion equation (62). The bright color indicates high density, dark color indicates low density. The particles concentrate on the line \(R_x = 0\) at time \(t = 0\). Parameters are \(D_{yx} + D_{xy} = -1, \omega > 0, b = 5\) and \(a = 15\). (b) The density distribution \(N_s(R_x, R_y)\) at time \(t = 15\) for \(D_{yx} + D_{xy} = -2, b = 10, \omega > 0\) and \(a = 15\).](image-url)
FIG. 11: (a) The velocity field $\vec{V} = (V_x, V_y)$ corresponds to the density distribution of Fig. 10 (a), $\omega > 0$. (b) The corresponding vorticity distribution for the vortex configuration of the left panel. The light color indicates high vorticity, dark color indicates low vorticity. This figure is a snapshot at time $t = 15$ for the positive angular frequency case, $\omega > 0$. to free space, the radius of vortex is also expanding. The region covered by non zero vorticity becomes larger as diffusion goes on.

The evolution of the velocity at next step is determined by the acceleration of the particles. For circle particle, we can see where the particles is going to move from the transverse force flow $\vec{J} = (J_x, J_y)$. The transverse force flow $\vec{J}$ is determined by density distribution,

$$
J_y = D_{xy}\omega \partial_{R_x}\partial_{R_y}\ln N_s + D_{xx}\omega \partial_{R_x}\partial_{R_y}\ln N_s ,
$$

$$
J_x = D_{yx}\omega \partial_{R_y}\partial_{R_x}\ln N_s + D_{yy}\omega \partial_{R_y}\partial_{R_y}\ln N_s .
$$

(63)

The transverse flow distribution is constantly changing from one configuration to another during diffusion process. We computed the corresponding transverse force field $\vec{J} = (J_x, J_y)$ (Fig. 12) with respect to the density distribution of Fig. 10 (a). The flow field indicates the acceleration direction of particles. Fig. 12 shows the particle on the upper boundary flows straightly to positive X-direction, the particles in the lower half plane flows up to the upper half plane. In the high density region, there exist a turbulence region, vortex configuration come into being there. Compare the flow field (Fig. 12) with the corresponding density function (Fig. 10 (a)), we see that the turbulence region situated right at the high density region.

FIG. 12: The effective force flow field $\vec{J} = (J_x, J_y)$ corresponding to the density distribution of Fig. 10 (a), $\omega > 0$.

B. The diffusion phenomena for $\omega < 0$ under the same parameter setting as $\omega > 0$

In last section, We have studied the diffusion equation for $\omega > 0$. The angular velocity in the diffusion Eq. (62) is positive, that means the circle particles are circling in counterclockwise direction. If the angular velocity switch from $+\omega$ to $-\omega$, the off-diagonal diffusion tensor and drift tensor will switch their sign simultaneously, i.e.,

$$
D_{yx} \leftrightarrow -D_{yx}, \quad \sigma_{yx} \leftrightarrow -\sigma_{yx}. \quad (64)
$$

So the off-diagonal diffusion coefficient of cross diffusion term will flip a sign,

$$
(D_{yx} + D_{xy}) \Rightarrow -(D_{yx} + D_{xy}), \quad b \Rightarrow -b. \quad (65)
$$

The drift velocity of the density center will switch to negative Y-direction. The diffusion in X-direction is still in positive X-direction. For $(D_{yx} + D_{xy}) = 1$, $b=-5$, $a=15$, we performed the numerical evolution of Eq. (62), the density distribution at time $t = 15$ is shown in Fig. 13. The particle concentrated on lower half plane of Y-direction.

We computed the corresponding velocity field(Fig. 14 (a)) with respect to the density distribution of Fig. 13 (a)). one can see the vortex line shifted the lower half plane too, so does the turbulence region(Fig. 14 (a)). Numerical evolution of the corresponding vorticity distribution also shifted to the lower half-plane (Fig. 14 (b)). The corresponding transverse force flow $\vec{J} = (J_x, J_y)$ for the
density distribution Fig. 13 shows the particle flow in the upper half plane now point down to the lower half plane(Fig. 15). The most turbulent region now shifted to the lower half plane. The vortex sits right at the highest density region. Now we can come to a conclusion that, when many circle particles are diffusing across a rectangular channel along X-axis, circles with positive angular frequency drift to upper boundary(positive Y-direction), while circles with negative angular frequency drift to bottom boundary(negative Y-direction).

VI. DIFFUSION IN A CYLINDRICAL DENSITY GRADIENT

If initial density distribution of particles has cylindrical symmetry, it is more convenient to express the diffusion equation (34) by polar coordination system. Suppose the particles concentrate in a small disk with the origin as its center, then the gradient of density would has cylindrical symmetry. Newton’s equation for circle’s center in polar coordination is of the same form as that in Cartesian coordination,

\[
m \frac{d \vec{V}}{dt} = -m \vec{V} \times \vec{\omega} + \vec{F} - D \frac{\nabla N_s}{N_s} - m \eta \vec{V},
\]

Here the velocity is \( \vec{V} = (V_R, \ V_\theta) \). The two components of the velocity are

\[
V_R = \dot{R}, \quad V_\theta = R \ \dot{\theta}, \quad V_R \perp V_\theta.
\]

Newton’s law give us the same equation of motion,

\[
\begin{align*}
    m \dot{V}_\theta &= -m \omega V_R - \frac{D \theta \partial \theta N_s}{R} \frac{\nabla N_s}{N_s} - m \eta V_\theta + F_\theta, \\
    m \dot{V}_R &= m \omega V_\theta - \frac{D R \partial R N_s}{N_s} \frac{\nabla N_s}{N_s} - m \eta V_R + F_R,
\end{align*}
\]

FIG. 13: One snapshot of the numerical evolution of density distribution of equation (62) at time \( t = 15, \ \omega < 0 \). The bright color indicates high density, dark color indicates low density.

FIG. 14: (a) The velocity field \( \vec{V} = (V_x, V_y) \) for \( \omega < 0 \). (b) The corresponding vorticity distribution \( \Gamma V \) of diffusion equation (62) for \( \omega < 0 \).

FIG. 15: The transverse force field \( \vec{J} = (J_x, J_y) \) for \( \omega < 0 \). All the parameters are the same as that for Fig. 13.
where \( A \) is the diffusion coefficient along the circle. \( D_R \) is the diffusion coefficient along the radius. In the over damped case, we set \( m\dot{V}_0 = 0 \) and \( m\dot{V}_R = 0 \). Then we can solve the equation above to get explicit equation of velocity,

\[
\begin{align*}
\dot{R} \theta &= -D_{\theta R} \frac{\partial R N_s}{N_s} - D_{\theta \theta} 1 \frac{\partial \theta N_s}{N_s} + \sigma_{\theta \theta} F_0 + \sigma_{\theta R} F_R, \\
\dot{R} &= -D_{R \theta} \frac{\partial R N_s}{N_s} - D_{R R} \frac{\partial R N_s}{N_s} + \sigma_{R R} F_R + \sigma_{R \theta} (F_0)
\end{align*}
\]

Here the diffusion tensor \( D \) and drifting tensor \( \sigma \) are

\[
D_{\theta \theta} = \frac{D_{\theta \theta}}{mA^2}, \quad D_{\theta R} = -\frac{D_{\theta \theta}}{mA^2},
\]

\[
D_{R R} = \frac{D_{R R}}{mA^2}, \quad D_{R \theta} = \frac{D_{\theta \theta}}{mA^2},
\]

\[
\sigma_{\theta \theta} = \frac{\eta}{mA^2}, \quad \sigma_{\theta R} = -\frac{\omega}{mA^2},
\]

\[
\sigma_{R R} = \frac{\eta}{mA^2}, \quad \sigma_{R \theta} = \frac{\omega}{mA^2},
\]

where \( A^2 = \eta^2 + \omega^2 \) is a normalization factor. The equation of a circle in polar coordination is

\[
\begin{align*}
R_c &= R + r_0 \cos(\omega t - \theta), \\
\theta_c &= \theta + \arctan \left( \frac{r_0 \sin(\omega t - \theta)}{R + r_0 \cos(\omega t - \theta)} \right),
\end{align*}
\]

We can solve Eq. (68) to get the coordination of the circle’s center \((R, \theta)\), and substitute them into the position equation above, then we get a trajectory of drifting circles in polar coordination. Here we study the diffusion of many particles, it is the circle’s center that determines the position of particle in larger scale, so we can put the local circular motion aside.

We consider a cylindrical density concentration, i.e., \( \partial \theta N_s = 0 \), the density gradient only exist along the radius. Further more, we withdraw the external driven force, \( F_0 = 0 \), \( F_R = 0 \). The velocity under nonzero density gradient is,

\[
\begin{align*}
\dot{R} \theta &= D_{\theta R} \frac{\partial R N_s}{N_s}, \\
\dot{R} &= D_{R R} \frac{\partial R N_s}{N_s}
\end{align*}
\]

Now we see, besides the particle flow \( \dot{V}_R \) along the radius, there is a transverse flow \( \dot{V}_0 \) perpendicular to the radius. The transverse velocity is induced by the density gradient along the radius(Fig. 16 (a)).

For a more complex density gradient distribution, the particle number conservation equation still holds,

\[
\frac{\partial N_s}{\partial t} + \frac{\partial}{\partial R} (N_s V_R) + \frac{1}{R} \frac{\partial}{\partial \theta} (N_s V_\theta) = 0.
\]

Substituting the velocity equation (68) into the conservation equation above, we get the diffusion equation in polar coordination,

\[
\begin{align*}
\frac{\partial N_s}{\partial t} &= \frac{\partial}{\partial R} [D_{R R} \frac{1}{R} \frac{\partial R N_s}{N_s} + D_{R \theta} \frac{\partial R N_s}{N_s}]
+ \frac{\partial}{\partial \theta} [\sigma_{\theta \theta} N_s F_0 + \sigma_{\theta R} N_s F_R] \\
&- \frac{1}{R} \frac{\partial}{\partial \theta} [\sigma_{R \theta} N_s F_R + \sigma_{R \theta} N_s F_0]
+ \frac{1}{R} \frac{\partial}{\partial \theta} [\sigma_{R R} N_s F_R + \sigma_{R \theta} N_s F_0]
= 0.
\end{align*}
\]

The drift tensor and diffusion tensor are state dependent,

\[
\sigma_{ij} = \sigma_{ij}(R, \theta), \quad D_{ij} = D_{ij}(R, \theta), \quad i, j = R \text{ or } \theta.
\]

Eq. (73) is an equivalent formulation of the diffusion equation (44) in polar coordination.

We first consider a special case that the drift tensor and diffusion tensor are both constant. Then for a simple external driven force,

\[
F_0 = 0, \quad F_R = CR,
\]

the diffusion equation (73) becomes,

\[
\frac{\partial N_s}{\partial t} = D_{R R} \frac{\partial^2 N_s}{\partial R^2} + D_{\theta \theta} \frac{1}{R^2} \frac{\partial^2 N_s}{\partial \theta^2}
+ \left[ D_{R \theta} + D_{R \theta} \right] \frac{1}{R} \frac{\partial N_s}{\partial \theta R} - D_{R \theta} \frac{1}{R} \frac{\partial N_s}{\partial \theta^2}
- \sigma_{R R} C \frac{\partial N_s}{\partial R} - \sigma_{R \theta} C R \frac{\partial N_s}{\partial \theta} - \sigma_{R \theta} C N_s.
\]

We first computed the special case that \( C=0 \), this means there is no external force. The friction parameter is \( \eta = 1 \). The normalization factor \( mA^2 = 1 \). Angular frequency is \( \omega = 10 \). The radial diffusion constant is \( D_R = 1 \). Angular diffusion coefficient \( D_\theta = 0.9 \). The numerical value of diffusion tensors are \( D_{R R} = 1, D_{\theta \theta} = 0.9, D_{R \theta} + D_{R \theta} = -1, \) \( D_{R \theta} = 9 \). The drift tensor are \( \sigma_{\theta \theta} = \sigma_{R R} = 1, \sigma_{R \theta} = -\sigma_{R R} = 10 \). The particles concentrate on a small half circle with radius \( R = 0.1 \) from 0 to \( \pi \). The initial value of density on the small circle is 1. The numerical evolution of the density distribution at time \( t = 9 \) is shown in Fig. 16, the density center shifted to \( \pi \). In an external force field with rotational symmetry around origin, the trajectory of circle particle during diffusion is not simply along the radius direction, but has
transverse shift perpendicular to the radius. In the region covered by a half disc, the particle concentrating on the radius of \( \theta = \pi \) is much more than that on the radius of \( \theta = 0 \).

We computed the velocity field distribution, one can see clearly the transverse flow perpendicular to radius(Fig. 19 (a)). One special character of circle particle flow is there exist a transverse flow induced by velocity gradient. In the polar coordination, this transverse flow field is given by

\[
J_R = \frac{1}{R} \partial_\theta V_R = \frac{1}{R} \partial_\theta (D_{RR} \frac{1}{R} \frac{\partial N_s}{N_s} + D_{R\theta} \frac{\partial R}{R} \frac{\partial N_s}{N_s})
+ \frac{1}{R} \partial_\theta (\sigma_{RR} R + \sigma_{R\theta} F_\theta),
\]

\[
J_\theta = -\partial_R V_\theta = \partial_R (D_{\theta R} \frac{\partial N_s}{N_s} + D_{\theta\theta} \frac{1}{R} \frac{\partial N_s}{N_s})
- \partial_R (\sigma_{RR} F_R + \sigma_{R\theta} F_\theta).
\] (77)

The derivative of velocity is acceleration. The flow field \( \vec{J} \) is actually the off-diagonal acceleration. The corresponding flow field \( \vec{J} \) with respect to the velocity field distribution Fig. 19 (a) is showed in Fig. 19 (a). The effective force flow field \( \vec{J} \) has large projection perpendicular to radius.

FIG. 17: The density distribution in \( R-\theta \) plane at time \( t = 9 \). The slop of the external force is \( C = 0 \), i.e., no external force.

We now propose an experiment to illustrate the transverse flow of diffusing circle particles in a disk region. First, we collect millions of circle particles, and confine them on a very small disk covered the origin. Second, one can place two blocking wall along the radius, one is placed along \( \theta = 0 \), the other is placed along \( \theta = \pi \). Then, we withdraw the confining potential in the small disk region, and set the particles free. If there is no blocking walls, the density wave is just expanding circles with the origin as their common center. There is a transverse flow \( V_\theta \) perpendicular to the radius, but we can not see it from density distribution, because the flow also has rotational symmetry. The blocking wall break the radial symmetry, the flow is cut off when it hits the wall, thus we will observe the accumulation of particle on one side of wall, and only a few of particles on the other side of the wall. Fig. 16 (b) showed the accumulation phenomena for particle circling in clockwise direction. If the particle is circling in the counterclockwise direction, the particle will accumulate in the opposite side of the wall. If there exist two kinds of particles, one is circling in clockwise direction, the other is circling in counter clock wise direction, the clockwise particle will accumulate in the region \( \theta = (-\alpha, 0) \) and \( \theta = (\pi - \alpha, \pi) \), \( \alpha > 0 \). The counterclockwise particle will accumulate in the \( \theta = (0, \beta) \) and \( \theta = (\pi, \pi + \beta) \), \( \beta > 0 \).

The strength of external field determines whether we can observe the transverse flow or not. In the discussions above, we are studying the case of \( C = 0 \), i.e., no external field is applied. Now we take \( C = 1/8 \), there is a small external field along the radius with slop \( C = 1/8 \). We numerically computed the density distribution by keeping the same value of other parameters as that for \( C = 0 \). The amplitude of the density center on \( \theta = \pi \) is shrinking(Fig. 19 (a)). We continue to increase the slop of the external field to , the density distributed in the block zone \((0, \pi)\) becomes almost homogeneous(Fig. 19 (b)). For \( C = 2 \), the external force along the radius is so strong that the effective transverse force can be neglected. Thus for a strong external force along radius, it is hard to observe an apparent bias distribution of density. But the transverse flow still exist.

VII. THREE DIMENSIONAL DIFFUSION EQUATION OF MANY CIRCLING PARTICLES

Microorganisms usually swim in three dimensional space, they follow a helical trajectory. Their angular velocity fluctuates according to external stimulus. A classical electron or ion moving in three dimensions follows a helical trajectory. The center line of the helical path
is magnetic field line. We will study a box of circling particles in three dimensions (Fig. 20). The particle is swimming in thick liquid with low Reynolds number. We focus on the most general case, and do not distinguish the self-propelled particles or passive swimming particles.

We take the helical path as the combination of a local circling motion and a motion of the circle’s center. The local circling motion is confined in a sphere with radius $r_0$. The direction of the circling motion is characterized by an angular velocity $\omega(t)$. The circles center can move around in the whole space. They are steered by external driven force of external stimulus. In two dimensions, the angular velocity has only one component. The particle is constantly rotating in $X-Y$ plane. In three dimensions, the particle may rotates in different directions, the angular velocity can be projected to $X-Y-Z$ three directions,

$$\omega(t) = \omega_x(t)e_x + \omega_y(t)e_y + \omega_z(t)e_z,$$

$$\omega^2(t) = \omega_x^2(t) + \omega_y^2(t) + \omega_z^2(t).$$

If the center of the circle is placed at the origin, and the angular velocity vector is $\vec{\omega}(t)$, the instantaneous position of the particle can be expressed as

$$r_x(t) = \frac{r_0}{\omega_x^2}[\omega_x\omega_x - \omega_x\omega_z \cos(\omega t) - \omega_y\omega_z \sin(\omega t)],$$

$$r_y(t) = \frac{r_0}{\omega_x}[\omega_y\omega_x - \omega_x\omega_y \cos(\omega t) + \omega_x\omega_y \sin(\omega t)],$$

$$r_z(t) = \frac{r_0}{\omega_x^2}[\omega_x\omega_x + (\omega_y^2 + \omega_z^2) \cos(\omega t)].$$

In two dimension, the particle is rotating around $Z-axis$, so the other two component of the angular velocity is zero, $\omega_x = \omega_y = 0$. Then we have $\omega = \omega_z$, the position equation (79) reduced exactly to the circle in two dimensions

$$r_x(t) = r_0(t) \cos \omega t, \quad r_y(t) = r_0(t) \sin \omega t.$$  (80)

We place the center of the circle to an arbitrary curve $\vec{R}(t)$, this curve is the trajectory of the center of the circle (Fig. 19). We assume the angular velocity $\vec{\omega}(t)$ is parallel to the tangent vector of the $\vec{R}(t)$. Then position of the circling particle is the solution of equation,

$$\dot{x}(t) = \dot{R}_x + \dot{r}_x(t),$$

$$\dot{y}(t) = \dot{R}_y + \dot{r}_y(t),$$

$$\dot{z}(t) = \dot{R}_z + \dot{r}_z(t).$$

Following the same process as we model the diffusion phenomena in two dimensions, we took the unit step size to measure diffusion larger than the radius of the circle. Then we consider the equation of motion for the center line. We place the center of the circle to an arbitrary curve $\vec{R}(t)$, this curve is the trajectory of the center of the circle. We assume the angular velocity $\vec{\omega}(t)$ is parallel to the tangent vector of the $\vec{R}(t)$. Then position of the circling particle is the solution of equation,

$$m\frac{\partial\vec{V}_R}{\partial t} = -m\vec{V}_R \times \vec{\omega} + \vec{F} - D \frac{\nabla N_s}{N_s} - m\eta\vec{V}_R,$$  (82)

except that here every vector has three components. This equation is composed of three coupled equations,

$$m\ddot{R}_x = -m(\dot{R}_y\omega_z - \dot{R}_z\omega_y) - D_x \frac{\partial R_x N_s}{N_s} - m\eta\dot{R}_x + F_x,$$

$$m\ddot{R}_y = -m(\dot{R}_z\omega_x - \dot{R}_x\omega_z) - D_y \frac{\partial R_y N_s}{N_s} - m\eta\dot{R}_y + F_y,$$

$$m\ddot{R}_z = -m(\dot{R}_x\omega_y - \dot{R}_y\omega_x) - D_z \frac{\partial R_z N_s}{N_s} - m\eta\dot{R}_z + F_z.$$  

Again we assume the swim over damped, then we set the inertial terms as zero,

$$m\ddot{R}_x = 0, \quad m\ddot{R}_y = 0, \quad m\ddot{R}_z = 0.$$  

References:

1. *Reference Name* (20XX). *Title*, *Publisher*, *Page Numbers*.

2. *Reference Name* (20XX). *Title*, *Publisher*, *Page Numbers*.

3. *Reference Name* (20XX). *Title*, *Publisher*, *Page Numbers*.

**Figures:**

**Fig. 19:** (a) The velocity field distribution in $R-\theta$ plane at time $t = 9$. The slope of the external force is $C = 0$. (b) The corresponding transverse force flow distribution in $R-\theta$ plane at time $t = 9$. The slope of the external force is $C = 0$.

**Fig. 20:** (a) The circling particles swimming in three dimensions. (b) The decomposition of the motion of a circling particle in three dimensions. The local circling motion is around $\omega$. The motion of the circle’s center is along the dark thick curve. $\omega$ is parallel to the tangent vector of the center line (the dark thick curve).
The solution of the center velocity is

\[
\dot{R}_x = -D_{xz} \frac{\partial_R N_s}{N_s} - D_{xy} \frac{\partial_R N_s}{N_s} - D_{xy} \frac{\partial_R N_s}{N_s} \\
+ \sigma_{xx} F_x + \sigma_{xy} F_y + \sigma_{xz} F_z,
\]

\[
\dot{R}_y = -D_{y} \frac{\partial_R N_s}{N_s} - \sigma_{yy} F_y + \sigma_{yz} F_z,
\]

\[
\dot{R}_z = -D_{zz} \frac{\partial_R N_s}{N_s} - D_{zy} \frac{\partial_R N_s}{N_s} - D_{zz} \frac{\partial_R N_s}{N_s} \\
+ \sigma_{xz} F_x + \sigma_{yy} F_y + \sigma_{zz} F_z.
\] (84)

This equation governs the motion of the center. The complete equation for a helical path is Eq. (81). The explicit formulation for the ultimate equation of a helix can be derived by substituting the center equation (84) and the circling equation (80) into Eq. (81). When the circling velocity \( \dot{r} \gg \dot{R} \), the solution of Eq. (81) is a helical curve.

Both the diffusion tensor and drift tensor of the center Eq. (84) are \( 3 \times 3 \) matrix, the elements of the matrix are

\[
D_{xx} = D_{x} \sigma_{xx}, \quad \sigma_{xx} = \frac{1}{C_0} (\eta^2 + m \omega_z^2),
\]

\[
D_{xy} = D_{y} \sigma_{xy}, \quad \sigma_{xy} = \frac{1}{C_0} (-\eta \omega_z + \omega_x \omega_y),
\]

\[
D_{yy} = D_{y} \sigma_{yy}, \quad \sigma_{yy} = \frac{1}{C_0} (\eta^2 + \omega_y^2),
\]

\[
D_{y} = D_{z} \sigma_{yz}, \quad \sigma_{yz} = \frac{1}{C_0} (-\eta \omega_x + \omega_y \omega_z),
\]

\[
D_{zz} = D_{z} \sigma_{zz}, \quad \sigma_{zz} = \frac{1}{C_0} (\eta^2 + \omega_z^2),
\] (85)

and the normalization factor is

\[
C_0^2 = m \eta (\eta^2 + m \omega_x^2 + m \omega_y^2 + m \omega_z^2). \] (86)

The drifting tensor \( \sigma \) is the sum of a symmetric matrix \( \sigma^1 \) and antisymmetric matrix \( \sigma^2 \),

\[
\sigma = \sigma^1 + \sigma^2,
\] (87)

\[
\sigma^1 = \frac{1}{C_0^2} \begin{pmatrix} \eta^2 + m \omega_x^2 & \omega_x \omega_y & \omega_x \omega_z \\ \omega_y \omega_x & \eta^2 + m \omega_y^2 & \omega_y \omega_z \\ \omega_z \omega_x & \omega_z \omega_y & \eta^2 + m \omega_z^2 \end{pmatrix},
\]

\[
\sigma^2 = \frac{1}{C_0^2} \begin{pmatrix} 0 & -\eta \omega_y & \eta \omega_z \\ -\frac{\eta \omega_z}{\omega_y} & 0 & -\frac{\eta \omega_x}{\omega_y} \\ \frac{\eta \omega_y}{\omega_x} & -\frac{\eta \omega_x}{\omega_y} & 0 \end{pmatrix}.
\]

Following the same calculation for two dimensional diffusion theory, we can get the diffusion equation in three dimensions by substituting the velocity in to the mass conservation equation

\[
\partial_t N_s + \nabla_R \cdot N_s \vec{V} = 0.
\] (88)

If the friction coefficient \( \eta = \eta(R_x, R_y, R_z) \), drift tensor \( \sigma_{ij} = \sigma(R_x, R_y, R_z) \) and diffusion tensor \( D_i = D_i(R_x, R_y, R_z) \) are all state-dependent, we will get a complicate three dimensional diffusion equation,

\[
\frac{\partial N_s}{\partial t} + \sum_{ij} \frac{\partial}{\partial R_i} \left[ \sigma_{ij} F_j N_s \right] - \sum_{ij} \frac{\partial}{\partial R_i} \left[ D_{ij} \frac{\partial}{\partial R_j} N_s \right] = 0.
\] (89)

For the special case that, the friction coefficient \( \eta \), drift tensor \( \sigma_{ij} \) and diffusion tensor \( D_i \) are constants, we get an explicit expansion of the diffusion equation,

\[
\frac{\partial N_s}{\partial t} - \left[ D_{xy} + D_{yz} \right] \frac{\partial R_y}{\partial R_x} N_s - D_{xz} \frac{\partial R_z}{\partial R_x} N_s = 0.
\] (90)

One can choose different coordinate system to reformulate the three dimensional diffusion equation for the convenience of physics, especially when the physical system has rotational symmetry. If the initial distribution of particle has spherical symmetry, then we can assume the density gradient also has spherical symmetry. Then we can express Eq. (89) by polar coordination. The density gradient distribution looks like a spherical balloon expanding away from its center, but there are transverse flows distributed on the surfaces of the balloon. There is a topological constrain on the distribution of the flows on the surface of a sphere, it states there must exist two singular points at which the flows contradict one another. So the flow on the spherical surface like to fuse into the third flow along radius.

**VIII. THEORETICAL UNDERSTANDING OF THE SPERM DIFFUSION EXPERIMENT**

In the experiment of sperm diffusion[1], the sperms are kept in a sperm reservoir in the beginning. The sperm reservoir is situated at one end of a rectangular migration channel. On the other end of the rectangular migration channel is a chemoattractant reservoir. By observing the
diffusion phenomena, they studied how the sperm behaves under the action of chemoattractant [1]. Some other earlier experiments showed that the chemoattractant acts as some effective attractive action to sperms by chemical stimulus. This chemoattractive force change the velocity of sperm [17] [18] and makes them move to eggs. The experimental data of Ref. [1] shows that sperm like to concentrate on one side of the channel(Fig. 1(a)) and the sperm density concentration illustrated a plateau along the length of channel, the plateau is near the reservoir source of sperm(around $x_2$ in Fig. 1). When chemoattractant spreads into the solution, the sperm will receive stimulus, and reorient its swimming direction. It is believed that the sperm will bent their circling direction to the source of chemoattractant [12] [19]. When a sperm is swimming near a boundary, it swims in a circle.

The sperm used in experiment [1] is Arbacia punctulata spermatozoa. The size of the sperm head is about 5 $\mu$m, but its flagellum has a length of 30-60 $\mu$m. In three dimensions, when there is no external stimulus, the trajectory of Arbacia punctulata spermatozoa is a helical curve. The center line of the helical curve is almost straight [19]. When chemoattractant spreads into the solution, the sperm will receive stimulus, and reorient its swimming direction. It is believed that the sperm will bent their circling direction to the source of chemoattractant [12] [19]. When a sperm is swimming near a boundary, it swims in a circle.

![Figure 21](image1.png)

**FIG. 21:** A cross section of the migration channel of experiment [1]. The length along $X$-axes is 7000-9000 $\mu$m. The width in $Y$-axes is 500 $\mu$m, and the height in $Z$-axes is 100 $\mu$m. The radius of the sperm circle is 30 $\mu$m.

The rectangular migration channel in experiment [1] has a length of 7000-9000 $\mu$m, its width is 500 $\mu$m, and the height of the channel is 100 $\mu$m. This channel is a three dimensional channel for one sperm, but for many sperms, it can be well approximated by a two dimensional channel. The radius of the sperm circle is fluctuating around 30 $\mu$m. So one sperm need at least a small square of $60 \times 60 \mu m^2$ to swim [7] [17] [18]. The rectangular channel of experiment [1] has two surfaces: the top surface and the bottom surface. The distance between these two surfaces is 100 $\mu$m. A sperm head is 5 $\mu$m, so there are 90 $\mu$m free distance left for the sperm to swim between the two surfaces. When the sperm is right in the middle, the distance between the sperm and one of the two surfaces reaches maximal, 45 $\mu$m. This distance is larger than the radius of the sperm circle. So the angular velocity of one sperm can rotate from $\omega_z$ to $\omega_x$ or to $\omega_y$. But when we consider two sperms, one is on top of another. If the two sperms try to rotate their angular velocity without any conflicting, the height of the channel must be at least 120 $\mu$m. The fluctuation of angular velocity for one sperm is suppressed by the presence of other sperms. The sperm experiment [1] study the diffusion of millions of sperms instead of one. The sperms in the top surface will swim in circles. The thickness of the top layer is about 5 $\mu$m, which is the size of one sperm head. The top layer behaves like a boundary for the sperms below, so the sperms in the second layer will also swim in circles. We can theoretically get at most 20 layers of circling sperms. In reality, it is impossible pile up alive sperms like bricks. The sperms are not restricted in the top surface, they are fluctuating up and down all the time. So a rough estimate of the effective thickness of the one layer is 10 $\mu$m. If the density is high enough, we can have 10 layers.

When the density is only high enough for sperms to form three layers, the two dimensional diffusion theory is still a good approximation. Suppose there are 3 layers on top of each other within 100 $\mu$m. The effective thickness of one layer is 10 $\mu$m, then we have 70 $\mu$m free distance. Suppose the distance between the nearest neighboring layers are equal, the distance is just 70/3 ≈ 23 $\mu$m. Recall that minimal square of one sperm circle is $60 \times 60 \mu m^2$, the sperms do not enough space to switch its angular velocity from $\omega_z$ to ($\omega_x$, $\omega_y$). In most cases, the the majority of the sperms would swim in circle. The plane of the circling is parallel to the top surface. The two dimensional diffusion theory captured the most significant dynamics. The fluctuation in the third dimension is a small perturbation. If the density is very low, we should use many body theory and take into account of the interaction between sperms. The diffusion theory is not such an accurate theory for few-body system.

![Figure 22](image2.png)

**FIG. 22:** The density distribution $N_x(R_x, R_y, t = 15)$ of diffusion Eq. (92) for $C = 2$. The light color indicates high density, dark color indicates low density.

If one can reduce the height of the rectangular channel in experiment [1] from 100 $\mu$m to 5 ~ 10 $\mu$m, then the angular velocity in $X$ and $Y$ direction, $\omega_x$, $\omega_y$, will be greatly suppressed. The dominant component of angular velocity is $\omega_z$. In that case, the two dimensional
diffusion theory would be a good theoretical approximation. Suppose one can do the same experiment\cite{1} but using a very thin channel with height $5 \sim 10 \mu m$, then two dimensional diffusion theory will be a good approximation. Since there is a chemoreservoir attached on the other end of the channel, we assume the chemo-attractive force distribution is

$$F_x = C \ R_x, \quad F_y = 0.$$ \hfill (91)

This means the shorter distance a sperm stays away from chemo-attractant, the stronger chemo-attractive force it feels. Biologist found that almost all sperms are circling with same handedness\cite{6}. So we choose angular velocity $\omega_z > 0$. The diffusion equation for this external driven force is

$$\frac{\partial N_s}{\partial t} = D_{xx} \partial^2_{Rz} N_s + D_{yy} \partial^2_{Ry} N_s + (D_{yx} + D_{xy}) \partial_{Rz} \partial_{Ry} N_s - C \sigma_{yx} R_x \partial_{Ry} N_s - C \sigma_{xx} R_z \partial_{Ry} N_s - C \sigma_{xx} N_s.$$ \hfill (92)

We take the diagonal diffusion coefficient as $D_{xx} = 0.9$, $D_{yy} = 1$. The off-diagonal diffusion coefficient is $D_{yx} + D_{xy} = -1$. The off-diagonal drifting tensor is $\sigma_{yx} = 5$. The diagonal drifting tensor is $\sigma_{xx} = 2$. The slop of the external force is controlled by $C$. From Eq. (92), one can see if we change $C$, there are three terms that will change simultaneously. We first study the case of $C = 2$.

The numerical evolution of density distribution at time $t = 15$ is shown in Fig. 22. The density center shift to positive $Y$-direction. We integrate the density along $Y$-axis $N_s(x, t) = \int^h_0 N_s \, dy$, and plot the density evolution $N_s(x, t)$ along $X$-direction(Fig. 23). The center of the density distribution is around $R_x = 2$. The diffusion curve along $R_x$ is not a simple exponential decay. The density center can propagate as a pump.

![Graph showing density distribution](image)

**FIG. 23:** The density distribution of $N_s(R_x, t = 15) = \int^h_0 N_s \, dR_y$ for diffusion Eq. (92) at $C = 2$.

In fact, the experiment\cite{1} measured the density distribution for one snapshot of diffusion image. The experiment date of Ref. \cite{1} is collected from the diffusion image along the length of the migration channel. A point on the curve represents the average density of sperms in each small rectangular box along $Y$-axis(the gray box in Fig. 1(a)). The data of Ref. \cite{1} is fitted by one dimensional diffusion equation. The data shows apparent deviation from exponential decay around $x = 2$ in Fig. 1(a). They measured the diffusion of *Arbacia punctulata* spermatozoa in seawater, jelly coat solution and resact solution for different solutions. In all of these measurements, the high density dots deviating from exponential decay appears around $x = 2$ in Fig. 1(a). Thus these high density dots are likely not erroneously fluctuating from experiment.

The vortex generated during diffusion will reduce the speed of diffusion in $X$-direction, as particles like to flow around vortex core. The vortex flow results in the trapping of a large amount of particles in a small region. The bump in density distribution is a signal of vortex configuration. The velocity of particle is given by

$$\dot{R}_x = -D_{xy} \frac{\partial_{Ry} N_s}{N_s} - D_{yx} \frac{\partial_{Rx} N_s}{N_s} + \sigma_{xx} C \ R_x,$$

$$\dot{R}_y = -D_{yx} \frac{\partial_{Rx} N_s}{N_s} - D_{yy} \frac{\partial_{Ry} N_s}{N_s} + \sigma_{yy} C \ R_x.$$ \hfill (93)

We computed the velocity field distribution at time $t = 15$ in $R_x - R_y$ plane (Fig. 24), one can see a vortex line is generated in the high density region particles are flowing out of this vortex lines. These vortex lines are distributed along the upper boundary. The most condensed particle flow is around the vortex line, it flows to positive $R_y$-direction.

For the external force configuration here $F_x = C \ R_x, \quad F_y = 0$, we computed the transverse flow field $\vec{J}$ induced by velocity gradient,

$$J_y = D_{xy} \omega \partial_{Rz} \partial_{Ry} \ln N_s + D_{xx} \omega \partial_{Rx} \partial_{Ry} \ln N_s + \omega \sigma_{xx} C,$$

$$J_x = D_{yx} \omega \partial_{Rz} \partial_{Rx} \ln N_s + D_{yy} \omega \partial_{Ry} \partial_{Rx} \ln N_s.$$ \hfill (94)

The strong flow of $\vec{J}$ is distributed in the high density region, it flows to positive $R_y$ direction(Fig. 25). On the upper boundary, all flows point to positive $R_x$ direction. In the low density region, turbulent flow began come into being, one can see clearly a string of vortex. The flows around vortex snake into the flow near the upper boundary. Generally we now can understand why particles on the upper boundary swim faster. Experiment observed fast sperms on the upper boundary, they swim faster than others. From our numerical evolution, we see even the particle are all have the same angular frequency, they will generate a fast particle flow near the boundary. Therefore, it is most likely that the fast sperm is not really fast, they are just being pushed by other sperms, this make them stay in front of others.

The data of Ref. \cite{1} shows the highest density peak is right at the origin $R_x = 0$, which is also the position of the sperm reservoir. We reduced the slop of external field from $C = 2$ to $C = 1/2$, the numerical evolution of the diffusion equation showed a peak at the origin(Fig.
FIG. 24. The velocity field distribution $\vec{V} = (V_x, V_y)$ corresponding to the density distribution Fig. 22. The slope of the external force is $C = 2$.

FIG. 25. The effective force flow field $\vec{J} = (J_x, J_y)$ at time $t = 15$ corresponding to the density distribution Fig. 22. The slope of the external force is $C = 2$.

FIG. 26: (a) The density distribution $N_s(R_x, R_y, t = 15)$ of diffusion Eq. (92) for $C = 1/2$. The light color indicates high density, dark color indicates low density. (b) The density curve of $N_s(R_x, t = 15) = \int_0^h N_s dR_y$ for $C = 1/2$.

The height of the bump is lowered down, and becomes a plateau (Fig. 22 (b)). Comparing Fig. 22 (b) with the density distribution for $C = 2$ (Fig. 23), we found the tail of the bump for $C = 1/2$ is broadened. The highest density region focused on a small region in upper half plane (Fig. 26 (a)). Compare it with the density distribution for $C = 2$, one may see, for the case of $C = 2$, the high density region covered a larger region along $Y$-direction. This means the increase of external field along $X$-axes enhanced the trapping of particles in $Y$-direction. In other words, if we increase the external force in $X$-direction, the particles do not move much in $X$-direction, instead they flow to $Y$-direction. This diffusion phenomena is special for circle swimmers.

The data in Ref. [1] is for one snapshot of the whole diffusion movie. We do not have enough information to compare theory with experiment. As far as we can see from the measurement of the one dimensional density distribution, the the highest density peak at the origin remains in different measurement. The sperm reservoir is right at the origin. Biological observation suggest that sperms like to stay near a boundary. I guess the appearance of this high density peak is due to experiment devices. The sperms are attached by the wall of the channel in the beginning, they did not move too much in the whole period of diffusion. So this peak at the origin do not have dynamic effect.

Generally speaking, the density evolution of the circle particles diffusing across a rectangular channel is not a simple exponential decay, many nonlinear flow field are involved during diffusion. For a more detail study, we may take into account of the shape of the circling particle. The sperm head has a shape of rod which is beating asymmetrically within a small angle around the tangent vector of circular trajectory [2]. The beating head of a boundary swimmer may hit the wall before its beating amplitude reach maximal point. As a swimmer liken to slide along the lengthy direction of the head, the cut off of beating amplitude in $Y$-direction by the wall will enhanced its
motion in $R_X$—direction. If a rod is surrounded by many rods, there exist frequent collision between rods. We can use Boltzmann equation to study the effect of collisions in diffusion process. One approximation of the linearized Boltzmann equation is Fokker-Planck equation. Theoretical simulation showed that a number of sperms can swim together under hydrodynamic interactions between beating flagella. So fluid dynamic theory may help us to understand the dynamics of many circle particles.

IX. CONCLUSION

The diffusion phenomena of circle particle is different from non-circling particles. For non-circling particle, the projected trajectory to $X$-axes is independent of its projection to $Y$-axes. But for circle particle, the motion in $X$-direction is coupled to the motion in $Y$-direction. There exist strong correlation between the different diffusion directions of circle swimmer. This correlation exist not only in two dimensions, but also in three dimensions. Circling motion can be theoretically modeled by non-zero angular velocity. We just start from a helical trajectory, and expressed Newton’s law by local coordination of the circle and a global coordination of the circle’s center. We assume the motion of the circle’s center is over damped, so one can drop the inertial term in the equation of motion for the circle’s center. The diffusion equation of many circle particle is expanded by nontrivial diffusion tensor and drift tensor. One can actually reformulated this diffusion equation into Fokker-Planck equation. For a simple external driven force, the numerical evolution of the tensor diffusion suggests that the diffusion flow along density gradient in one direction can induce a transverse flow in its perpendicular direction. Recall that we did not distinguish weather the particle is self-propelled particle or not, this nonequilibrium phenomena holds both for lifeless particles and self-propelled particles. In a rectangular diffusion channel, the transverse flow will increase the density on the boundary, which in turn drive the swimmers near the boundary move faster. The vortex configuration both from the velocity vector field and the transverse flow vector field induced by velocity gradient. Thus the diffusion of circle particle is a highly nonlinear process. Comparing with the experiment measurement of the density distribution projected to $X$-direction, we can understand the bump in the density curve as a special phenomena for circle particles. If all circle swimmers have the same circling handedness, i.e., either clockwise or counterclockwise, when they are diffusing in a channel along $X$-direction, the density center either shift to positive $Y$-direction or shift to negative $Y$-direction, the density distribution along $Y$-direction is not symmetric. The sperm experiment indeed observed sperms liken to concentrate to one side of the channel, and sperms swim faster on the boundary. We analyzed the flow field distribution, found that the particle flow point to boundary. This can explain why sperms on the boundary swim faster, because they are pushed by other sperms. Further more, we predict one phenomena: when we mix circle swimmers with opposite circling handedness together, and let them diffuse across a rectangular channel in $X$-direction, they will split to two groups, one goes to positive $Y$-direction, the other goes to negative $Y$-direction. The swimmers in each group all have the same circling handedness. The classical circle swimmers can give us a vivid demonstration of plasma physics.

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