Inclusion Properties of Orlicz Spaces and Weak Orlicz Spaces Generated by Concave Functions

M Taqiyuddin and A A Masta
Department of Mathematics Education, Universitas Pendidikan Indonesia, Jl. Dr. Setiabudi no. 229, Bandung 40154, Indonesia
*taqi94@hotmail.com

Abstract. In this paper, we introduce Orlicz spaces generated by concave functions and establish some inclusion properties between them. In addition, we also obtain similar result for weak Orlicz spaces. One of the keys to prove our result is by estimating characteristic function of open balls in $\mathbb{R}^n$.

1. Introduction and preliminaries

There is an ongoing discussion on Orlicz spaces over past decades. This space was firstly introduced by Z W Birnbaum and W Orlicz in 1931 [1] and the discussion is still going on. As one of them, in 1989, Maligandra [2] studied some inclusion properties on Orlicz spaces. In 2016, Masta et al. [3] also reprove some inclusion properties of Orlicz spaces and weak Orlicz spaces.

Furthermore, several observations of the inclusion properties of these spaces have been done by other researchers. Osançlıoğlu [4] has proved sufficient and necessary conditions for inclusion relation between two weighted Orlicz spaces. Related results about inclusion properties of Orlicz spaces can be found in [5-6]. These results have attracted us to observe further within the inclusion properties of these spaces and their weak type.

It is a convenient way to associate Orlicz spaces with convex functions (see for example [3,4,7]). However, recently, some researchers associated Orlicz spaces with concave function. In particular, Zhou et al. [8] discussed the weak Martingale Orlicz-Karamata-Hardy spaces associated with concave functions, and Zhang and Zhang [9] studied the weak Orlicz spaces associated with concave functions. Motivated by this trend, we have decided to study the inclusion properties of Orlicz spaces and weak Orlicz spaces generated by concave functions.

The rest of paper is organized as follows. In this section, we will recall and give some definitions and some lemmas which will be used in our proofs. More specifically, we will give the definitions of Orlicz spaces and weak Orlicz spaces generated by concave functions in this section. Then, we will present our results in section 2. The inclusion properties will be discussed there.

Throughout this paper, the letter $C$ will be used for constants that may change from line to line, while constants with subscripts, such as $C_1$ do not change in different lines.

Let $G$ be the set of all functions $\Phi : [0, \infty) \mapsto [0, \infty)$ be an increasing, bijective, continuous, and concave function satisfying $\Phi(0) = 0$, $\lim_{t \to \infty} \Phi(t) = 0$, and $\lim_{t \to 0} \Phi(t) = \infty$. We denote $\Phi_1 \preceq \Phi_2$ for $\Phi_1, \Phi_2 \in G$ if there is a constant $C > 0$ such that $\Phi_1(t) \leq \Phi_2(Ct)$ for all $t \geq 0$. For $\Phi \in G$, the Orlicz space generated by concave functions $L_\Phi(\mathbb{R}^n)$ is defined as follows
\[ L_\Phi(\mathbb{R}^n) = \left\{ f \in \mathbb{R}^n : \int_{\mathbb{R}^n} \Phi(b|f(x)|) \, dx < \infty \right\}, \]

for some \( b > 0 \), and its quasi-norm is
\[
\|f\|_{L_\Phi(\mathbb{R}^n)} := \inf \left\{ b > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{b}\right) \, dx \leq 1 \right\}. 
\]

By using similar argument on the Orlicz spaces with convex function (see [1,2,4,7,10]) we can prove that the Orlicz space generated by concave functions \( L_\Phi(\mathbb{R}^n) \) is a quasi-Banach space with respect to the quasi-norm \( \|f\|_{L_\Phi(\mathbb{R}^n)} \). Note that, if \( \Phi(t) = t^p \) for some \( 0 < p < 1 \), then \( L_\Phi(\mathbb{R}^n) = L_p(\mathbb{R}^n) \).

On the other hand, for \( \Phi \in G \), the weak Orlicz spaces generated by concave functions \( wL_\Phi(\mathbb{R}^n) \) is defined as follows
\[ wL_\Phi(\mathbb{R}^n) = \left\{ f \in \mathbb{R}^n : \|f\|_{wL_\Phi} < \infty \right\}, \]

where \( \|f\|_{wL_\Phi(\mathbb{R}^n)} := \inf \left\{ b > 0 : \sup_{t > 0} \Phi(t) \left\{ x \in \mathbb{R}^n : \frac{|f(x)|}{b} > t \right\} \leq 1 \right\} \). Weak Orlicz space generated by concave functions \( wL_\Phi(\mathbb{R}^n) \) is also a quasi-Banach space with respect to the quasi-norm \( \|f\|_{L_\Phi(\mathbb{R}^n)} \) (see [9]). For \( \Phi(t) = t^p \) (0 < \( p < 1 \)), \( wL_\Phi(\mathbb{R}^n) = wL_p(\mathbb{R}^n) \) is the weak Lebesgue space.

Next, we recall some lemmas which will be used in our proofs.

**Lemma 1.0.** Let \( G, a \in \mathbb{R}^n \), and \( r > 0 \) arbitrary. Then
\[
\|X_{B(a,r)}\|_{L_\Phi(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}\left(\frac{1}{|B(a,r)|}\right)},
\]

where \( |B(a,r)| \) denotes the volume of open ball \( B(a,r) \).

**Proof.**

Let \( a \in \mathbb{R}^n \) and \( r > 0 \). Observe that \( \|X_{B(a,r)}\|_{L_\Phi(\mathbb{R}^n)} = \inf \left\{ b > 0 : \Phi\left(\frac{1}{b}\right) \leq \frac{1}{|B(a,r)|} \right\} = \inf A \), where
\[
A = \left\{ b > 0 : \Phi\left(\frac{1}{b}\right) \leq \frac{1}{|B(a,r)|} \right\}. 
\]
Meanwhile, \( \Phi^{-1}\left(\frac{1}{|B(a,r)|}\right) = \inf \left\{ r \geq 0 : \Phi(r) > \frac{1}{|B(a,r)|} \right\} = \inf E \), where \( E = \left\{ r \geq 0 : \Phi(r) > \frac{1}{|B(a,r)|} \right\} \). Choosing \( b = \frac{1}{\Phi^{-1}\left(\frac{1}{|B(a,r)|}\right)} \), we have \( |B(a,r)|\Phi\left(\frac{1}{b}\right) \leq 1 \). So we have \( \|X_{B(a,r)}\|_{L_\Phi(\mathbb{R}^n)} \leq \frac{1}{\Phi^{-1}\left(\frac{1}{|B(a,r)|}\right)} \). Now, suppose that \( \|X_{B(a,r)}\|_{L_\Phi(\mathbb{R}^n)} < \frac{1}{\Phi^{-1}\left(\frac{1}{|B(a,r)|}\right)} \) or \( \Phi^{-1}\left(\frac{1}{|B(a,r)|}\right) > \|X_{B(a,r)}\|_{L_\Phi(\mathbb{R}^n)} \). By using the definition of \( \Phi^{-1}\left(\frac{1}{|B(a,r)|}\right) \), we can find \( r_1 \in E \) such that
\[
\Phi^{-1}\left(\frac{1}{|B(a,r)|}\right) < r_1 \leq \frac{1}{\|X_{B(a,r)}\|_{L_\Phi(\mathbb{R}^n)}}. 
\]
Since \( r_1 \in E \) we obtain \( \frac{1}{r_1} \notin A \), so we have \( \frac{1}{r_1} \leq \|X_{B(a,r)}\|_{L_\Phi(\mathbb{R}^n)} \). This contradicts the fact that \( r_1 \leq \frac{1}{\|X_{B(a,r)}\|_{L_\Phi(\mathbb{R}^n)}} \). Hence we must have
\[
\|X_{B(a,r)}\|_{L_\Phi(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}\left(\frac{1}{|B(a,r)|}\right)}. 
\]

Furthermore, we have similar result on weak Orlicz space generated by concave function presented in the following Lemma.

**Lemma 1.1.**[9] Let \( G, a \in \mathbb{R}^n \), and \( r > 0 \) is arbitrary. Then we have
\[
\|X_{B(a,r)}\|_{wL_\Phi(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}\left(\frac{1}{|B(a,r)|}\right)},
\]

where \( |B(a,r)| \) denotes the volume of open ball \( B(a,r) \).
Lemma 1.2. [9] Let $G$, $1 \leq C < \infty$, and $0 \leq t < \infty$. Then $\Phi(Ct) \leq C\Phi(t)$.

Lemma 1.3. Suppose that $\Phi \in G$ and $\Phi^{-1}$ is defined as $\Phi^{-1}(t) = \inf\{ r \geq 0 : \Phi(r) > t \}$. Then, the following statements are true:

1. $\Phi(\Phi^{-1}(s)) \leq s \leq \Phi^{-1}(\Phi(s))$, for $0 \leq s < \infty$.
2. If, for some constants $C_1, C_2 > 0$, we have $\Phi_2^{-1}(s) \leq C_1 \Phi_1^{-1}(C_2 s)$, then $\Phi_1\left(\frac{t}{C_1}\right) \leq C_2 \Phi_2(t)$ for $t = \Phi_2^{-1}(s)$.

Proof. We leave the proof of Lemma 1.3 (1) to the reader. Now, take arbitrary $s > 0$, and set $t = \Phi_2^{-1}(s)$. Observe that by using Lemma 1.3. (1), we have

$$\Phi_1\left(\frac{t}{C_1}\right) = \Phi_1\left(\frac{\Phi_2^{-1}(s)}{C_1}\right) \leq \Phi_1\left(\Phi_1^{-1}(C_2 s)\right) \leq C_2 s$$

By $t = \Phi_2^{-1}(s)$ and the definition of $\Phi_2^{-1}(s) = \inf\{ r \geq 0 : \Phi_2(r) > s \}$ we get $\leq \Phi_2(t)$. Hence we can conclude that $\Phi_1\left(\frac{t}{C_1}\right) \leq C_2 \Phi_2(t)$. $\blacksquare$

2. Main results

In this section, we present two main results which are stated as Theorem 2.1 and Theorem 2.2. The following theorem contains a necessary and sufficient condition for the inclusion properties of Orlicz spaces generated by concave function.

Theorem 2.1. Let $\Phi_1, \Phi_2 \in G$. Then the following statements are equivalent:

1. $\Phi_1 \preceq \Phi_2$.
2. $L_{\Phi_1}(\mathbb{R}^n) \subseteq L_{\Phi_2}(\mathbb{R}^n)$.
3. There exist a constant $C > 0$ such that $\|f\|_{L_{\Phi_1}(\mathbb{R}^n)} \leq C\|f\|_{L_{\Phi_2}(\mathbb{R}^n)}$, for every $f \in L_{\Phi_2}(\mathbb{R}^n)$.

Proof. Suppose that (1) holds. Let $f \in L_{\Phi_2}(\mathbb{R}^n)$, then there exist $\alpha > 0$ such that

$$\int_{\mathbb{R}^n} \Phi_2(\alpha |f(\alpha)|) dx < \infty.$$

Hence, we have

$$\int_{\mathbb{R}^n} \Phi_1\left(\frac{\alpha}{C} |f(\alpha)|\right) dx \leq \int_{\mathbb{R}^n} \Phi_2\left(\frac{\alpha}{C} |f(\alpha)|\right) dx = \int_{\mathbb{R}^n} \Phi_2(\alpha |f(\alpha)|) dx < \infty.$$

This proves $L_{\Phi_2}(\mathbb{R}^n) \subseteq L_{\Phi_1}(\mathbb{R}^n)$.

Next, since $(L_{\Phi_2}(\mathbb{R}^n), L_{\Phi_1}(\mathbb{R}^n))$ is a quasi-Banach pair, we can prove (2) and (3) are equivalent by similar argument in [3].

Now assume that (3) holds. By Lemma 1.0., we have

$$\frac{1}{\Phi_1^{-1}\left(1 \left|\frac{1}{B(a,r)}\right|\right)} = \|X_B(a,r)\|_{L_{\Phi_1}(\mathbb{R}^n)} \leq C\|X_B(a,r)\|_{L_{\Phi_2}(\mathbb{R}^n)} = \frac{C}{\Phi_2^{-1}\left(1 \left|\frac{1}{B(a,r)}\right|\right)}.$$  

In other words, we have $C\Phi_1^{-1}\left(1 \left|\frac{1}{B(a,r)}\right|\right) \geq \Phi_2^{-1}\left(1 \left|\frac{1}{B(a,r)}\right|\right)$ for arbitrary $a \in \mathbb{R}^n$ and $b > 0$.

By Lemma 1.3, we have

$$\Phi_1\left(\frac{t_0}{C}\right) \leq \Phi_2(t_0)$$

for $t_0 = \Phi_2^{-1}\left(1 \left|\frac{1}{B(a,r)}\right|\right)$. Since $a \in \mathbb{R}^n$ and $b > 0$ are arbitrary, we can conclude that $\Phi_1(t) \leq C \Phi_2(Ct)$ for $t > 0$. $\blacksquare$
We also have obtained a necessary and sufficient condition for the inclusion properties of weak Orlicz spaces generated by concave function presented in the following theorem.

**Theorem 2.2.** Let $\Phi_1, \Phi_2 \in G$. Then the following statements are equivalent:

1. $\Phi_1 \preccurlyeq \Phi_2$.
2. $wL_{\Phi_2}(\mathbb{R}^n) \subseteq wL_{\Phi_1}(\mathbb{R}^n)$.
3. There exists a constant $C > 0$ such that $\|f\|_{wL_{\Phi_1}(\mathbb{R}^n)} \leq C\|f\|_{wL_{\Phi_2}(\mathbb{R}^n)}$ for every $f \in wL_{\Phi_2}(\mathbb{R}^n)$.

**Proof.**

Suppose that (1) holds. Let $f \in wL_{\Phi_2}(\mathbb{R}^n)$,

$$A_{\Phi_1} = \left\{ b > 0 : \sup_{t>0} \Phi_1(t) \left|\left\{ x \in \mathbb{R}^n : \frac{|f(x)|}{b} > t \right\}\right| \leq 1 \right\},$$

and

$$A_{\Phi_2} = \left\{ b > 0 : \sup_{t>0} \Phi_2(Ct) \left|\left\{ x \in \mathbb{R}^n : \frac{|f(x)|}{b} > t \right\}\right| \leq 1 \right\},$$

for $s = Ct$.

Take an arbitrary $t > 0$ and $b \in A_{\Phi_2}$, observe that

$$\Phi_1(t) \left|\left\{ x \in \mathbb{R}^n : \frac{|f(x)|}{b} > t \right\}\right| \leq \Phi_2(Ct) \left|\left\{ x \in \mathbb{R}^n : \frac{|f(x)|}{b} > t \right\}\right| = \Phi_2(s) \left|\left\{ x \in \mathbb{R}^n : \frac{C|f(x)|}{b} > s \right\}\right| \leq 1.$$

Hence, we have

$$\|f\|_{wL_{\Phi_1}(\mathbb{R}^n)} = \inf A_{\Phi_1} \leq \inf A_{\Phi_2} = C\|f\|_{wL_{\Phi_2}(\mathbb{R}^n)},$$

which proves that $wL_{\Phi_2}(\mathbb{R}^n) \subseteq wL_{\Phi_1}(\mathbb{R}^n)$.

Next, by using similar argument in the Theorem 2.1., we can get the equivalence of (2) and (3).

Assume that (3) holds. By using Lemma 1.1., we have

$$\frac{1}{\Phi_1^{-1} \left( \frac{1}{|B(a,r)|} \right)} = \| \chi_{B(a,r)} \|_{wL_{\Phi_1}(\mathbb{R}^n)} \leq C\| \chi_{B(a,r)} \|_{wL_{\Phi_2}(\mathbb{R}^n)} = \frac{C}{\Phi_2^{-1} \left( \frac{1}{|B(a,r)|} \right)}.$$ 

In the other words, we have $C\Phi_1^{-1} \left( \frac{1}{|B(a,r)|} \right) \geq \Phi_2^{-1} \left( \frac{1}{|B(a,r)|} \right)$ for arbitrary $a \in \mathbb{R}^n$ and $b > 0$.

By Lemma 1.3., we have

$$\Phi_1 \left( \frac{t_0}{C} \right) \leq \Phi_2(t_0)$$

for $t_0 = \Phi_2^{-1} \left( \frac{1}{|B(a,r)|} \right)$. Since $a \in \mathbb{R}^n$ and $b > 0$ are arbitrary, we can conclude that

$$\Phi_1(t) \leq \Phi_2(Ct)$$

for $t > 0$. ■

3. **Concluding remarks**

We have established the sufficient and necessary conditions for inclusion relations among Orlicz spaces and weak Orlicz spaces generated by concave function. Combining the results on Theorem 2.1 and Theorem 2.2, we can state that the inclusion relation between Orlicz spaces generated by concave functions is equivalent to that between their weak type. Furthermore, it follows from Theorem 2.1 and Theorem 2.2 that there cannot be an inclusion relation between $L_{p_1}(\mathbb{R}^n)$ and $L_{p_2}(\mathbb{R}^n)$ nor $wL_{p_1}(\mathbb{R}^n)$ and $wL_{p_2}(\mathbb{R}^n)$ for distinct values of $p_1$ and $p_2$. 
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