Fine properties and a notion of quasicontinuity for BV functions on metric spaces *

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Abstract

On a metric space equipped with a doubling measure supporting a Poincaré inequality, we show that given a BV function, discarding a set of small 1-capacity makes the function continuous outside its jump set and “one-sidedly” continuous in its jump set. We show that such a property implies, in particular, that the measure theoretic boundary of a set of finite perimeter separates the measure theoretic interior of the set from its measure theoretic exterior, both in the sense of the subspace topology outside sets of small 1-capacity, and in the sense of 1-almost every curve.

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1 Introduction

Sobolev functions in Euclidean spaces are known to be quasicontinuous. This result holds also in the metric setting: if the measure on the metric space is doubling and supports a $(1, 1)$-Poincaré inequality, then for every Newton-Sobolev function $u \in N^{1,1}(X)$ there exists an open set $G \subset X$ of small 1-capacity such that the restriction $u|_{X \setminus G}$ is continuous, see for example [7]. For $p > 1$ one can even remove the requirement that the metric space support a $(1, p)$-Poincaré inequality. This follows from the fact that Lipschitz functions are dense in $N^{1,p}(X)$, see [2], together with the fact that density of Lipschitz functions implies quasicontinuity of $N^{1,p}$-functions.

Such a quasicontinuity property fails for functions of bounded variation, or BV functions. From [16, Theorem 4.3, Theorem 5.1] (see also [18, 25, 23]) we know that a set has small 1-capacity if and only if its codimension 1 Hausdorff content $\mathcal{H}_R$, for any fixed $R > 0$, is small. However, BV functions can have jump sets with $\mathcal{H}_R$-measure bounded away from 0, and it is not possible to enclose such sets within sets of small 1-capacity.

It is known that a BV function coincides with a Lipschitz function outside sets of small measure, see e.g. [12, p. 252] and [24, Proposition 4.3]. For spaces $BV_k(\mathbb{R}^n)$ of higher order BV functions, with $k \in \mathbb{N}$, Lusin-type approximations by means of differentiable functions outside sets of small 1-capacity are given in [9, Theorem 6.2]. However, even in the Euclidean setting, little appears to be known about the behavior of (first-order) BV functions outside sets of small 1-capacity. The goal of the current paper is to show a weak notion of quasicontinuity for BV functions, involving continuity outside the jump set and “one-sided” continuity up to the jump set.

In what follows, $X$ is a metric space equipped with a metric $d$ and a doubling Borel regular outer measure $\mu$ that supports $(1, 1)$-Poincaré inequality. Definitions and notation will be discussed systematically in Section [2]. The jump set of a function $u \in BV(X)$ is defined as

$$S_u := \{x \in X : u^\wedge(x) < u^\vee(x)\},$$

where $u^\wedge(x)$ and $u^\vee(x)$ are the lower and upper approximate limits of $u$ defined as

$$u^\wedge(x) := \sup \left\{ t \in \mathbb{R} : \lim_{r \to 0^+} \frac{\mu(B(x, r) \cap \{u < t\})}{\mu(B(x, r))} = 0 \right\}.$$
and
\[ u^\vee(x) := \inf \left\{ t \in \mathbb{R} : \lim_{r \to 0^+} \frac{\mu(B(x, r) \cap \{u > t\})}{\mu(B(x, r))} = 0 \right\}. \]

It was shown in [4, Theorem 5.3] that \( \mathcal{H} \) is a \( \sigma \)-finite measure on \( S_u \).

Furthermore, from [1, Theorem 5.4] we know that there is a number \( 0 < \gamma \leq 1/2 \) such that if \( E \subset X \) is a set of finite perimeter (that is, \( \chi_E \in \text{BV}(X) \)), then the perimeter measure \( P(E, \cdot) \) is carried on the set \( \Sigma_\gamma E \), which is the collection of points \( x \in X \) for which
\[ \gamma \leq \liminf_{r \to 0^+} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} \leq \limsup_{r \to 0^+} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} \leq 1 - \gamma. \]

Classical results on BV functions in the Euclidean setting can be formulated in terms of the approximate limits \( u^\wedge \) and \( u^\vee \), but in the general metric setting we need to consider a larger number of jump values. The reason for this will be illustrated in Example 5.1. Given \( u \in \text{BV}(X) \), we define the functions (jump values) \( u^l \), \( l = 1, \ldots, n := \lfloor 1/\gamma \rfloor \) (with \( \gamma \) as above), by
\[ u^1 := u^\wedge, \quad u^n := u^\vee, \quad \text{and for } l = 2, 3, \ldots, n - 1, \quad u^l(x) := \sup \left\{ t \in \mathbb{R} : \lim_{r \to 0^+} \frac{\mu(B(x, r) \cap \{u^l-1(x) + \delta < u < t\})}{\mu(B(x, r))} = 0 \quad \forall \delta > 0 \right\} \]
provided \( u^{l-1}(x) < u^\vee(x) \), and otherwise, we set \( u^l(x) = u^\vee(x) \). We have \( u^\wedge = u^1 \leq \ldots \leq u^n = u^\vee \). We also define \( \tilde{u} := (u^\wedge + u^\vee)/2 \). Note that if \( x \in X \setminus S_u \), then \( u^l(x) = \ldots = u^n(x) \).

The following theorem, which is the main result of this paper, introduces a notion of quasicontinuity for BV functions.

**Theorem 1.1.** Let \( u \in \text{BV}(X) \) and let \( \varepsilon > 0 \). Then there exists an open set \( G \subset X \) with \( \text{Cap}_1(G) < \varepsilon \) such that if \( y_k \to x \) with \( y_k, x \in X \setminus G \), then
\[ \min_{l_2 \in \{1, \ldots, n\}} \left| u^{l_1}(y_k) - u^{l_2}(x) \right| \to 0 \]
for each \( l_1 = 1, \ldots, n \).

In particular, \( \tilde{u}|_{X \setminus G} \) is continuous at every \( x \in X \setminus (S_u \cup G) \). The proof of Theorem 1.1 is given in two parts; in Proposition 4.7 we prove continuity outside the jump set, and in Proposition 5.4 we prove “one-sided” continuity up to the jump set. In proving the “one-sided” continuity, we show
that if \( x \in S_u \setminus G \), then \( X \) can be partitioned into at most \( n^2 \) number of sets \((u^1)^{-1}(A^k(x))\), defined in (5.1), such that when the sequence \( y_k \) lies in \((u^1)^{-1}(A^k(x)) \setminus G \) and converges to \( x \), we must have \( u^1(y_k) \to u^{k2}(x) \).

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## 2 Background

In this section we introduce the necessary definitions and assumptions.

Throughout the paper, \((X, d, \mu)\) is a complete metric space equipped with a Borel regular outer measure \(\mu\) satisfying a doubling property, that is, there is a constant \(C_d \geq 1\) such that

\[
0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty
\]

for every ball \(B = B(x, r)\) with center \(x \in X\) and radius \(r > 0\). Given a ball \(B = B(x, r)\) and \(\tau > 0\), we denote by \(\tau B\) the ball \(B(x, \tau r)\). In a metric space, a ball does not necessarily have a unique center and radius, but whenever we use the above abbreviation we will consider balls whose center and radii have been pre-specified, and so no ambiguity arises.

By iterating the doubling condition, we obtain that there are constants \(C \geq 1\) and \(Q > 0\) such that

\[
\frac{\mu(B(y, r))}{\mu(B(x, R))} \geq C^{-1} \left( \frac{r}{R} \right)^Q
\]

(2.1)

for every \(0 < r \leq R\) and \(y \in B(x, R)\). The choice \(Q = \log_2(C_d)\) works, but a smaller value of \(Q\) might satisfy the above condition as well.

In general, \(C \geq 1\) will denote a generic constant whose particular value is not important for the purposes of this paper, and might differ between each occurrence. When we want to specify that a constant \(C\) depends on the parameters \(a, b, \ldots\), we write \(C = C(a, b, \ldots)\). Unless otherwise specified, all constants only depend on the doubling constant \(C_d\) and the constants \(C_P, \lambda\) associated with the Poincaré inequality defined below.
Given \( x \in X \) and \( A_1, A_2 \subset X \), we set
\[
\text{dist}(x, A_1) := \inf \{ d(x, y) : y \in A_1 \}, \quad \text{dist}(A_1, A_2) := \inf \{ d(z, A_1) : z \in A_2 \}.
\]

A complete metric space with a doubling measure is proper, that is, closed and bounded sets are compact. Since \( X \) is proper, for any open set \( \Omega \subset X \) we define \( \text{Lip}_{\text{loc}}(\Omega) \) to be the space of functions that are Lipschitz in every \( \Omega' \Subset \Omega \). Here \( \Omega' \Subset \Omega \) means that \( \Omega' \) is open and that \( \overline{\Omega'} \) is a compact subset of \( \Omega \). We define other local spaces similarly.

For any set \( A \subset X \) and \( 0 < R < \infty \), the restricted spherical Hausdorff content of codimension 1 is defined as
\[
H_R(A) := \inf \left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), \ r_i \leq R \right\}.
\]
We define the above also for \( R = \infty \) by requiring \( r_i < \infty \). The codimension 1 Hausdorff measure of a set \( A \subset X \) is given by
\[
\mathcal{H}(A) := \lim_{R \to 0^+} \mathcal{H}_R(A).
\]

The measure theoretic boundary \( \partial^* E \) of a set \( E \subset X \) is the set of all points \( x \in X \) at which both \( E \) and its complement have positive upper density, i.e.
\[
\limsup_{r \to 0^+} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > 0 \quad \text{and} \quad \limsup_{r \to 0^+} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > 0.
\]

A curve is a rectifiable continuous mapping from a compact interval into \( X \). The length of a curve \( \gamma \) is denoted by \( \ell_\gamma \). We will assume every curve to be parametrized by arc-length, which can always be done (see e.g. [15, Theorem 3.2] or [5]). A nonnegative Borel function \( g \) on \( X \) is an upper gradient of an extended real-valued function \( u \) on \( X \) if for all curves \( \gamma \) on \( X \), we have
\[
|u(x) - u(y)| \leq \int_\gamma g \, ds,
\]
where \( x \) and \( y \) are the end points of \( \gamma \). We interpret \( |u(x) - u(y)| = \infty \) whenever at least one of \( |u(x)|, |u(y)| \) is infinite. Upper gradients were originally introduced in [21].
Let $\Gamma$ be a family of curves, and let $1 \leq p < \infty$. The $p$-modulus of $\Gamma$ is defined as

$$\operatorname{Mod}_p(\Gamma) := \inf \int_X \rho^p \, d\mu$$

where the infimum is taken over all nonnegative Borel functions $\rho$ such that $\int_\gamma \rho \, ds \geq 1$ for every $\gamma \in \Gamma$. If a property fails only for a curve family with $p$-modulus zero, we say that it holds for $p$-almost every (a.e.) curve. If $g$ is a nonnegative $\mu$-measurable function on $X$ and (2.2) holds for $p$-almost every curve, then $g$ is a $p$-weak upper gradient of $u$.

We consider the following norm

$$\|u\|_{N^{1,p}(X)} := \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},$$

with the infimum taken over all upper gradients $g$ of $u$. The substitute for the Sobolev space $W^{1,p}(\mathbb{R}^n)$ in the metric setting is the following Newton-Sobolev space

$$N^{1,p}(X) := \{ u : \|u\|_{N^{1,p}(X)} < \infty \} / \sim,$$

where the equivalence relation $\sim$ is given by $u \sim v$ if and only if

$$\|u - v\|_{N^{1,p}(X)} = 0.$$

Similarly, we can define $N^{1,p}(\Omega)$ for any open set $\Omega \subset X$. For more on Newton-Sobolev spaces, we refer to [32, 20, 6].

Next we recall the definition and basic properties of functions of bounded variation on metric spaces, see [31]. See also e.g. [3, 14, 34] for the classical theory in the Euclidean setting. For $u \in L^1_{\text{loc}}(X)$, we define the total variation of $u$ on $X$ to be

$$\|Du\|(X) := \inf \left\{ \liminf_{i \to \infty} \int_X g_{u_i} \, d\mu : u_i \in \operatorname{Lip}_{\text{loc}}(X), u_i \to u \text{ in } L^1_{\text{loc}}(X) \right\},$$

where each $g_{u_i}$ is an upper gradient of $u_i$. We say that a function $u \in L^1(X)$ is of bounded variation, and denote $u \in \operatorname{BV}(X)$, if $\|Du\|(X) < \infty$. A measurable set $E \subset X$ is said to be of finite perimeter if $\|D\chi_E\|(X) < \infty$. The perimeter of $E$ in $X$ is denoted by

$$P(E, X) := \|D\chi_E\|(X).$$

By replacing $X$ with an open set $\Omega \subset X$ in the definition of the total variation, we can define $\|Du\|(\Omega)$. The BV norm is given by

$$\|u\|_{\operatorname{BV}(\Omega)} := \|u\|_{L^1(\Omega)} + \|Du\|(\Omega).$$
It was shown in [31, Theorem 3.4] that for \( u \in \text{BV}(X) \), \( \|Du\| \) is the restriction to the class of open sets of a finite Radon measure defined on the class of all subsets of \( X \). This outer measure is obtained from the map \( \Omega \mapsto \|Du\|_\Omega \) on open sets \( \Omega \subset X \) via the standard Carathéodory construction. Thus, for an arbitrary set \( A \subset X \),

\[
\|Du\|(A) := \inf \{ \|Du\|(U) : A \subset U \subset X \text{ with } U \text{ open} \}.
\]

Similarly, if \( u \in \text{BV}(\Omega) \), then \( \|Du\|(\cdot) \) is a finite Radon measure on \( \Omega \).

We have the following coarea formula from [31, Proposition 4.2]: if \( F \subset X \) is a Borel set and \( u \in \text{BV}(X) \), then

\[
\|Du\|(F) = \int_{-\infty}^{\infty} P(\{u > t\}, F) \, dt. \tag{2.3}
\]

In particular, the map \( t \mapsto P(\{u > t\}, F) \) is Lebesgue measurable on \( \mathbb{R} \).

We will assume that \( X \) supports a \((1,1)\)-Poincaré inequality, meaning that there are constants \( C_P > 0 \) and \( \lambda \geq 1 \) such that for every ball \( B(x, r) \), for every locally integrable function \( u \) on \( X \), and for every upper gradient \( g \) of \( u \), we have

\[
\int_{B(x, r)} |u - u_{B(x, r)}| \, d\mu \leq C_P r \int_{B(x, \lambda r)} g \, d\mu,
\]

where

\[
u_{B(x, r)} := \int_{B(x, r)} u \, d\mu := \frac{1}{\mu(B(x, r))} \int_{B(x, r)} u \, d\mu.
\]

By applying the Poincaré inequality to approximating Lipschitz functions in the definition of the total variation, we get the following \((1,1)\)-Poincaré inequality for BV functions. There exists a constant \( C \) such that for every ball \( B(x, r) \) and every \( u \in L^1_{\text{loc}}(X) \), we have

\[
\int_{B(x, r)} |u - u_{B(x, r)}| \, d\mu \leq Cr \frac{\|Du\|(B(x, \lambda r))}{\mu(B(x, \lambda r))}. \tag{2.4}
\]

Sets of measure zero play a fundamental role in the theory of \( L^p \) spaces. In potential theory sets of measure zero can be too large to be discarded; a finer measure of the smallness of a set is needed. For \( 1 \leq p < \infty \), the \( p \)-capacity of a set \( A \subset X \) is given by

\[
\text{Cap}_p(A) := \inf \|u\|_{N^1, p(X)}, \tag{2.5}
\]
where the infimum is taken over all functions \( u \in N^{1,p}(X) \) such that \( u \geq 1 \) in a neighborhood of \( A \); we can further restrict the class of functions \( u \) by requiring that \( 0 \leq u \leq 1 \) on \( X \). It follows from [16, Theorem 4.3, Theorem 5.1] that \( \text{Cap}_1(E) = 0 \) if and only if \( \mathcal{H}(E) = 0 \).

Given a set \( E \subset X \) of finite perimeter, for \( \mathcal{H}\text{-a.e.} \ x \in \partial^*E \) we have

\[
\gamma \leq \liminf_{r \to 0^+} \frac{\mu(E \cap B(x,r))}{\mu(B(x,r))} \leq \limsup_{r \to 0^+} \frac{\mu(E \cap B(x,r))}{\mu(B(x,r))} \leq 1 - \gamma \tag{2.6}
\]

where \( \gamma \in (0, 1/2] \) only depends on the doubling constant and the constants in the Poincaré inequality, see [1, Theorem 5.4]. We denote the set of all such points by \( \Sigma\gamma E \).

For a Borel set \( F \subset X \) and a set \( E \subset X \) of finite perimeter, we know that

\[
\|D\chi_E\|_F = \int_{\partial^*E \cap F} \theta_E d\mathcal{H}, \tag{2.7}
\]

where \( \partial^*E \) is the measure-theoretic boundary of \( E \) and \( \theta_E : X \to [\alpha, C_d] \), with \( \alpha = \alpha(C_d, C_P, \lambda) > 0 \), see [1, Theorem 5.3] and [4, Theorem 4.6].

The jump set of \( u \in BV(X) \) is the set

\[
S_u := \{ x \in X : u^\wedge(x) < u^\vee(x) \},
\]

where \( u^\wedge(x) \) and \( u^\vee(x) \) are the lower and upper approximate limits of \( u \) defined respectively by

\[
u^\wedge(x) := \sup \left\{ t \in \mathbb{R} : \lim_{r \to 0^+} \frac{\mu(B(x,r) \cap \{ u < t \})}{\mu(B(x,r))} = 0 \right\} \tag{2.8}
\]

and

\[
u^\vee(x) := \inf \left\{ t \in \mathbb{R} : \lim_{r \to 0^+} \frac{\mu(B(x,r) \cap \{ u > t \})}{\mu(B(x,r))} = 0 \right\} \tag{2.9}
\]

We also define the functions \( u^l, l = 1, \ldots, n = \lfloor 1/\gamma \rfloor \), as follows: \( u^1 := u^\wedge \), \( u^n := u^\vee \), and for \( l = 2, \ldots, n - 1 \) we define inductively

\[
u^l(x) := \sup \left\{ t \in \mathbb{R} : \lim_{r \to 0^+} \frac{\mu(B(x,r) \cap \{ u^{l-1}(x) + \delta < u < t \})}{\mu(B(x,r))} = 0 \quad \forall \delta > 0 \right\} \tag{2.10}
\]

provided \( u^{l-1}(x) < u^\vee(x) \), and otherwise, we set \( u^l(x) = u^\vee(x) \). It can be shown that each \( u^l \) is a Borel function, and \( u^\wedge = u^1 \leq \ldots \leq u^n = u^\vee \).
Given the definition of the BV norm, we understand BV functions to be $\mu$-equivalence classes. To consider questions of continuity, we need to consider the pointwise representatives $u^l$, $l = 1, \ldots, n$. We also use the standard representative $\tilde{u} := (u^\wedge + u^\vee)/2$.

By [4, Theorem 5.3], the variation measure of a BV function can be decomposed into the absolutely continuous and singular part, and the latter into the Cantor and jump part, as follows. Given an open set $\Omega \subset X$ and $u \in BV(\Omega)$, we have

$$
\|Du\|_\Omega = \|Du\|^a_\Omega + \|Du\|^s_\Omega = \|Du\|^a_\Omega + \|Du\|^c_\Omega + \|Du\|^j_\Omega
$$

(2.11)

where $a \in L^1(\Omega)$ is the density of the absolutely continuous part and the functions $\theta_{\{u>t\}}$ are as in (2.7).

For $R > 0$, the restricted maximal function of a function $v \in L^1_{\text{loc}}(X)$ is given by

$$
M_Rv(x) := \sup_{0 < r \leq R} \int_{B(x,r)} |v| \, d\mu, \quad x \in X.
$$

The following result will be used a few times. See [23, Lemma 4.3, Remark 4.9] for a proof. While [23] makes the extra assumption $\mu(X) = \infty$, use of this assumption can be avoided by considering $H_R$ instead of $H_\infty$.

**Lemma 2.1.** There exists $C = C(C_d, C_P, \lambda, R)$ such that for every $u \in BV(X)$ and $t > 0$,

$$
\text{Cap}_1(\{M_Ru \geq t\}) \leq \frac{C\|u\|_{BV(X)}}{t}.
$$

## 3 Discrete convolutions

In this section we discuss functions in $BV(U)$ with zero boundary values on $\partial U$, and methods of “mollifying” BV functions in open sets. For a proof of the following theorem, see [29, Theorem 6.1] or [22].
**Theorem 3.1.** Let $U \subset X$ be an open set and $u \in \text{BV}(U)$. Assume that $\mathcal{H}(\partial U) < \infty$. If
\[\lim_{r \to 0^+} \frac{1}{\mu(B(x, r))} \int_{B(x, r) \cap U} |u| \, d\mu = 0\] (3.1)
for $\mathcal{H}$-a.e. $x \in \partial U$, then the zero extension of $u$ into the whole space $X$, denoted by $\hat{u}$, is in $\text{BV}(X)$ with $\|D\hat{u}\|(X \setminus U) = 0$.

The following technical lemma can be proved by a simple covering argument.

**Lemma 3.2 ([29, Lemma 6.4]).** Let $U \subset X$ be an open set, let $\nu$ be a finite Radon measure on $U$, and define
\[A := \left\{ x \in \partial U : \limsup_{r \to 0^+} r \frac{\nu(B(x, r) \cap U)}{\mu(B(x, r))} > 0 \right\} .\]
Then $\mathcal{H}(A) = 0$.

In most of the paper, we will work with Whitney type coverings of open sets. For the construction of such coverings and their properties, see e.g. [8, Theorem 3.1] (such coverings were originally introduced in the Euclidean setting by Whitney in [33, Section 8, page 67], and subsequently extended to more general settings in [11, Theorem III.1.3] and [30, Lemma 2.9]).

Given any open set $U \subset X$ and a scale $R > 0$, we can choose a Whitney type covering $\{B_j = B(x_j, r_j)\}_{j=1}^\infty$ of $U$ such that

1. for each $j \in \mathbb{N}$,
\[r_j = \min \left\{ \frac{\text{dist}(x_j, X \setminus U)}{40\lambda} , R \right\} ,\] (3.2)

2. for each $k \in \mathbb{N}$, the ball $10\lambda B_k$ meets at most $C_0 = C_0(C_d, \lambda)$ balls $10\lambda B_j$ (that is, a bounded overlap property holds),

3. if $10\lambda B_j$ meets $10\lambda B_k$, then $r_j \leq 2r_k$.

Given such a covering of $U$, we can take a partition of unity $\{\phi_j\}_{j=1}^\infty$ subordinate to this cover, such that $0 \leq \phi_j \leq 1$, each $\phi_j$ is a $C/r_j$-Lipschitz function, and $\text{supp}(\phi_j) \subset 2B_j$ for each $j \in \mathbb{N}$ (see e.g. [8, Theorem 3.4]).
Finally, we can define a discrete convolution \( v \) of any \( u \in L^1_{\text{loc}}(U) \) with respect to the Whitney type covering by

\[
v := \sum_{j=1}^{\infty} u_{B_j} \phi_j.
\]

In general, \( v \) is locally Lipschitz in \( U \), and hence belongs to \( L^1_{\text{loc}}(U) \). If \( u \in L^1(U) \), then \( v \in L^1(U) \).

The goal of the next proposition is to show that the discrete convolution \( v \) of \( u \) has the same boundary values as \( u \), i.e. that \( v - u \) has zero boundary values in the sense of Theorem 3.1.

**Proposition 3.3.** Let \( U \subset X \) be an open set, \( R > 0 \), and \( u \in \text{BV}(U) \). Let \( v \in \text{Lip}_{\text{loc}}(U) \) be the discrete convolution of \( u \) with respect to a Whitney type covering \( \{B_j = B(x_j, r_j)\}_{j=1}^{\infty} \) of \( U \) at scale \( R \). Then

\[
\lim_{r \to 0^+} \frac{1}{\mu(B(x, r))} \int_{B(x,r) \cap U} |v - u| \, d\mu = 0
\]

for \( \mathcal{H} \)-a.e. \( x \in \partial U \).

This proposition was previously given in [29, Proposition 6.5], but we include the proof here as well since it is simple enough and makes the exposition more self-contained.

**Proof.** Fix \( x \in \partial U \) and \( r > 0 \). Denote by \( I_r \) the set of indices \( j \in \mathbb{N} \) for which \( 2B_j \cap B(x, r) \neq \emptyset \). Note that from (3.2) and the fact that \( \lambda \geq 1 \) it follows that

\[
r_j \leq \frac{\text{dist}(2B_j, X \setminus U)}{38\lambda} \leq \frac{r}{38\lambda}
\]

for every \( j \in I_r \). Because \( \sum_{j \in \mathbb{N}} \phi_j = \chi_U \), we have (in fact, the following holds
for any $x \in X \setminus U$)

$$\int_{B(x,r) \cap U} |u - v| \, d\mu = \int_{B(x,r) \cap U} \left| \sum_{j \in I_r} u \phi_j - \sum_{j \in I_r} u_{B_j} \phi_j \right| \, d\mu$$

$$\leq \int_{B(x,r) \cap U} \sum_{j \in I_r} |\phi_j(u - u_{B_j})| \, d\mu$$

$$\leq \sum_{j \in I_r} \int_{2B_j} |u - u_{B_j}| \, d\mu$$

$$\leq \sum_{j \in I_r} \left( \int_{2B_j} |u - u_{2B_j}| \, d\mu + \int_{2B_j} |u_{2B_j} - u_{B_j}| \, d\mu \right) \quad (3.4)$$

$$\leq 2C_d \sum_{j \in I_r} \int_{2B_j} |u - u_{2B_j}| \, d\mu$$

$$\leq 4C_d C_P \sum_{j \in I_r} r_j \|Du\|(2\lambda B_j)$$

$$\leq C r \|Du\|(B(x,2r) \cap U).$$

In the above, we used the fact that $X$ supports a $(1,1)$-Poincaré inequality, and in the last inequality we used the fact that $2\lambda B_j \subset U \cap B(x,2r)$ for all $j \in I_r$, as well as the bounded overlap of the dilated Whitney balls $2\lambda B_j$. Thus by Lemma 3.2 we have

$$\lim_{r \to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r) \cap U} |u - v| \, d\mu = 0$$

for $\mathcal{H}$-a.e. $x \in \partial U$. \hfill $\square$

Let $U \subset X$ be an open set, $R > 0$, and as above, let $v$ be the discrete convolution of a function $u \in BV(U)$ with respect to a Whitney type covering $\{B_j\}_{j \in \mathbb{N}}$ of $U$ at scale $R$. Then $v$ has an upper gradient

$$g = C \sum_{j=1}^{\infty} \chi_{B_j} \frac{\|Du\|(5\lambda B_j)}{\mu(B_j)} \quad (3.5)$$

in $U$ (with $C$ depending, as usual, only on the doubling constant and the constants in the Poincaré inequality), see e.g. the proof of [24, Proposition 4.1]. From the proof of this result it also follows that in a small ball comparable
to the size of a Whitney ball, say $B = B(x, \min\{\text{dist}(x, X \setminus U)/20\lambda, R\})$, $v$ is Lipschitz with constant, say,
\[
C \frac{\|Du\|(B(x, \min\{\text{dist}(x, X \setminus U)/4, 5\lambda R\}))}{\mu(B)}.
\] (3.6)

Also, if $V_\epsilon \subset U$, $\epsilon > 0$, is any family of open subsets of $U$ and every $v_\epsilon$ is a discrete convolution of a function $u \in L^1(U)$ with respect to a Whitney type covering of $V_\epsilon$ at scale $\epsilon > 0$, then
\[
\lim_{\epsilon \to 0^+} \|v_\epsilon - u\|_{L^1(V_\epsilon)} = 0,
\] (3.7)
as seen by the discussion in the proof of [19, Lemma 5.3].

It is often useful to be able to “mollify” BV functions in small open sets where e.g. a certain part of the variation measure lives. Combining the above discussion on discrete convolutions with Theorem 3.1 and Proposition 3.3, we obtain the following result on such mollifications.

**Corollary 3.4.** Let $U \subset X$ be an open set, and let $u \in \text{BV}(U)$. Assume that $\mathcal{H}(\partial U) < \infty$. Let each $v_i \in \text{Lip}_{\text{loc}}(U)$ be the discrete convolution of $u$ with respect to a Whitney type covering of $U$ at scale $1/i$, $i \in \mathbb{N}$. Then $v_i \to u$ in $L^1(U)$, $\|Dv_i\|(U) \leq C\|Du\|(U)$, and the functions
\[
h_i := \begin{cases} v_i - u & \text{in } U, \\ 0 & \text{in } X \setminus U \end{cases}
\]
satisfy $h_i \in \text{BV}(X)$ and $\|Dh_i\|(X \setminus U) = 0$.

In the above we require that the boundary of $U$ has finite $\mathcal{H}$-measure. However, for the proof of the main theorem of this paper, we need “mollifications” on arbitrary open sets. In the following, we extend Corollary 3.4 to all open sets. Recall that $\tilde{u} := (u^\wedge + u^\vee)/2$, where the lower and upper approximate limits $u^\wedge$, $u^\vee$ were defined in (2.8) and (2.9).

**Theorem 3.5.** Let $U \subset X$ be an open set, $u \in \text{BV}(U)$, and $\kappa > 0$. Then there exists a function $w \in \text{BV}(U)$ satisfying the following: $\tilde{w} \in N^{1,1}(U) \cap \text{Lip}_{\text{loc}}(U)$ with an upper gradient $g$ satisfying $\|g\|_{L^1(U)} \leq C\|Du\|(U)$; $\|w - u\|_{L^1(U)} \leq \kappa$; and the function
\[
h := \begin{cases} w - u & \text{in } U, \\ 0 & \text{in } X \setminus U \end{cases}
\]
satisfies $h \in \text{BV}(X)$ and $\|Dh\|(X \setminus U) = 0$. 

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Proof. The following coarea inequality is known to hold: if \( \omega \in \text{Lip}(X) \) is \( L \)-Lipschitz, then
\[
\int_\mathbb{R} \mathcal{H}(A \cap \omega^{-1}(t)) \, dt \leq CL\mu(A)
\]
(3.8)
for any Borel set \( A \subset X \), see [5, Proposition 3.1.5] or [13]. The proof in [5] deals with Hausdorff measures \( \mathcal{H}^{k-m} \), \( \mathcal{H}^m \) and \( \mathcal{H}^k \) instead of \( \mathcal{H} \), \( dt \) and \( \mu \); however, their proof of this result holds in our setting when we replace \( \mathcal{H}^k \) with \( \mu \), \( \mathcal{H}^{k-m} \) with \( \mathcal{H} \), and set \( m = 1 \). By considering the 1-Lipschitz functions
\[
\omega_1(y) := \text{dist}(y, X \setminus U) \quad \text{and} \quad \omega_2(y) := d(y, x)
\]
for a fixed \( x \in X \), we can pick open sets \( U_1 \subset U_2 \subset \cdots \subset U \), defined as
\[
U_i := \{ \omega_1 > \alpha_i \text{ and } \omega_2 < 1/\alpha_i \}
\]
for some strictly decreasing sequence \( \alpha_i \downarrow 0 \). Clearly \( U = \bigcup_{i \in \mathbb{N}} U_i \), and by a suitable choice of the sequence \( \alpha_i \), (3.8) gives \( \mathcal{H}(\partial U_i) < \infty \) for each \( i \in \mathbb{N} \).

Fix a scale \( R > 0 \). For each \( i \in \mathbb{N} \), define \( v_i \) to be the discrete convolution of \( u \) with respect to a Whitney type covering \( \{ B_j \}_{j \in \mathbb{N}} \) of \( U_i \), at scale \( R \). By (3.7) we can choose \( R \) to be small enough so that \( \| v_i - u \|_{L^1(U_i)} \leq \kappa \) for each \( i \in \mathbb{N} \). Each \( v_i \) has an upper gradient \( g_i \) in \( U_i \), defined in (3.5), with \( \| g_i \|_{L^1(U_i)} \leq C \| Du \|_{U_i} \).

By Corollary 3.4, the function
\[
h_i = \begin{cases} 
v_i - u & \text{in } U_i, \\
0 & \text{in } X \setminus U_i
\end{cases}
\]
satisfies \( h_i \in \text{BV}(X) \) and \( \| Dh_i \|_{X \setminus U_i} = 0 \). Hence
\[
\| Dh_i \|_{X} \leq \| Dv_i \|_{U_i} + \| Du \|_{U_i} \leq \| g_i \|_{L^1(U_i)} + \| Du \|_{U_i}
\]
\[
\leq C \| Du \|_{U_i} \leq C \| Du \|_{U}.
\]
By the weak compactness of \( \text{BV} \) functions, see [31, Theorem 3.7], a subsequence that we still denote by \( h_i \) converges in \( L^1_{\text{loc}}(X) \) to a function \( h \in \text{BV}(X) \) for which we clearly have \( h = 0 \) in \( X \setminus U \). Now let \( w := u + h \in \text{BV}(U) \). Since \( \| h_i \|_{L^1(U)} = \| v_i - u \|_{L^1(U_i)} \leq \kappa \) for each \( i \in \mathbb{N} \), it follows that
\[
\| w - u \|_{L^1(U)} = \| h \|_{L^1(U)} \leq \kappa.
\]
To prove that $\tilde{w} \in \text{Lip}_{\text{loc}}(U) \cap N^{1,1}(U)$, pick a Whitney type covering $\{B_j\}_{j \in \mathbb{N}}$ of $U$ at scale $R$, and fix a ball $B_j = B(x_j, r_j)$. For large enough $i_0 \in \mathbb{N}$,

$$B_j = B(x_j, \min\{\text{dist}(x_j, X \setminus U)/40\lambda, R\})$$

$$\subset B(x_j, \min\{\text{dist}(x_j, X \setminus U_{i_0})/20\lambda, R\}) =: B.$$

By (3.6) we know that in the ball $B$, each $v_i, i \geq i_0$, has Lipschitz constant at most

$$C \frac{\|Du\|(B(x_j, \min\{\text{dist}(x_j, X \setminus U_i)/4, 5\lambda R\}))}{\mu(B(x_j, \min\{\text{dist}(x_j, X \setminus U_{i_0})/20\lambda, R\}))} \leq C \frac{\|Du\|(10\lambda B_j)}{\mu(B_j)}.$$

In the last inequality we used the fact that $B_j \subset B$, so that $\mu(B_j) \leq \mu(B)$.

Now the $L^1$-limit $\tilde{w}$ of the sequence of functions $v_i$, $i \geq i_0$, has Lipschitz constant at most

$$\frac{\|Du\|(B(x, r))}{\mu(B(x, 2r))} \leq r \|Du\|((B(x, 2r) \cap U),$$

where the last inequality follows from (3.4). Then by Lemma 3.2, we have that $g := C \sum_{j \in \mathbb{N}} \chi_{B_j} \frac{\|Du\|(10\lambda B_j)}{\mu(B_j)}$ is an upper gradient of $\tilde{w}$ in $U$. By the bounded overlap of the dilated Whitney balls $10\lambda B_j$, $\|g\|_{L^1(U)} \leq C\|Du\|(U)$.

Now choose $x \in \partial U$ and $r > 0$. Since $h_i \to h$ in $L^1_{\text{loc}}(X)$ and thus in $L^1(B(x, r))$, we have

$$\int_{B(x, r) \cap U} |h| \, d\mu = \lim_{i \to \infty} \int_{B(x, r) \cap U} |h_i| \, d\mu = \lim_{i \to \infty} \int_{B(x, r) \cap U} |v_i - u| \, d\mu$$

$$\leq Cr \|Du\|(B(x, 2r) \cap U),$$

where the last inequality follows from (3.4). Then by Lemma 3.2 we have that

$$\lim_{r \to 0^+} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |h| \, d\mu = \lim_{r \to 0^+} \frac{1}{\mu(B(x, r))} \int_{B(x, r) \cap U} \frac{1}{\mu(B(x, r))} \int_{B(x, r) \cap U} |h| \, d\mu = 0 \quad (3.10)$$

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for $H$-a.e. $x \in \partial U$. For such $x$ and all $t \neq 0$, we conclude that $x \notin \partial^*\{h > t\}$. By the coarea formula (2.3) and (2.7), this implies

$$\|Dh\|((\partial U)) = \int_{\mathbb{R}} P(\{h > t\}, \partial U) dt \leq C \int_{\mathbb{R}} \mathcal{H}(\partial^*\{h > t\} \cap \partial U) dt = 0.$$ 

We conclude that $\|Dh\|(X \setminus U) = 0$. 

In this paper, we will only need the following corollary of Theorem 3.5.

**Corollary 3.6.** Let $U \subset \Omega \subset X$ be open sets, $u \in BV(\Omega)$, and $\kappa > 0$. Then there exists a function $w \in BV(\Omega)$ with $w = u$ in $\Omega \setminus U$ such that $\|w - u\|_{L^1(U)} \leq \kappa$, $\tilde{w}|_U \in N^{1,1}(U) \cap \text{Lip}_\text{loc}(U)$ with an upper gradient $g$ satisfying $\|g\|_{L^1(U)} \leq C\|Du\|(U)$, and

$$\|D(w - u)\|(\Omega \setminus U) = 0. \quad (3.11)$$

**Proof.** Let $w := u + h$, where $h \in BV(X)$ is given in Theorem 3.5. Then $w \in BV(\Omega)$, and the required properties of $w$ were shown in the theorem. 

We will also need the following consequence of Theorem 3.5. The proof will be similar to one given in [22].

**Proposition 3.7.** Let $\Omega \subset X$ be an open set, let $u \in BV(\Omega)$, and let $H \subset \Omega$ be a closed set such that $\tilde{u}|_H$ is continuous and

$$\int_{B(x,r)} |u - \tilde{u}(x)| \, d\mu \to 0 \quad \text{as } r \to 0 \quad (3.12)$$

locally uniformly in the set $H$. Let $w$ be the function given by Corollary 3.6 with $U = \Omega \setminus H$ and any $\kappa > 0$. Then $\tilde{w}$ is continuous in $\Omega$, and $\tilde{w}(x) = \tilde{u}(x)$ for all $x \in H$.

**Proof.** Observe that $\tilde{w}$ is continuous in $U = \Omega \setminus H$ by Corollary 3.6.

Let $R$ be the scale used in the construction of the Whitney type coverings of the sets $U_i$ in Theorem 3.5, corresponding to the given value of $\kappa$. Fix $x \in H$. If $x$ is in the interior of $H$, then $\tilde{w}$, which agrees with $\tilde{u}$ in the interior of $H$, is continuous at $x$. Now suppose that $x$ is not in the interior of $H$. Let $\delta \in (0, R)$ such that $B(x, 3\delta) \subset \Omega$. Consider a sequence $y_k \in B(x, \delta) \setminus H$ that converges to $x$. We note that for every $y_k$ there exists $x_k \in H$ for which $d(y_k, x_k) = \text{dist}(y_k, H)$. Since $d(y_k, x_k) \leq d(y_k, x) \to 0$ as $k \to \infty$, ...
it follows that \( d(x_k, x) \to 0 \). The latter, together with the assumption that \( \tilde{u}|_H \) is continuous, implies that \( \tilde{u}(x_k) \to \tilde{u}(x) \). So we only need to show that
\[
|\tilde{w}(y_k) - \tilde{u}(x_k)| \to 0 \text{ as } k \to \infty.
\]

Fix \( k \in \mathbb{N} \). For large enough \( i \in \mathbb{N} \), the sets \( U_i \) (defined in the proof of Theorem 3.5) satisfy
\[
U_i \supset \{ y \in B(x, 2\delta) : \text{dist}(y, H) > \text{dist}(y_k, H)/2 \} \ni y_k.
\] (3.13)

Fixing such \( i \), by the properties of the Whitney type covering \( \{ B^i_j \} \) of \( U_i \), for any \( 2B^i_j \ni y_k \) we have
\[
40\lambda r^i_j \leq \text{dist}(x^i_j, H) \leq 2r^i_j + \text{dist}(y_k, H) < 2r^i_j + \delta, \tag{3.14}
\]
and it follows that \( r^i_j < R \). Therefore
\[
r^i_j = \min \left\{ \frac{\text{dist}(x^i_j, X \setminus U_i)}{40\lambda}, R \right\} = \frac{\text{dist}(x^i_j, X \setminus U_i)}{40\lambda} \geq \frac{\text{dist}(y_k, X \setminus U_i) - 2r^i_j}{40\lambda},
\]
from which we see that
\[
r^i_j \geq \frac{\text{dist}(y_k, X \setminus U_i)}{50\lambda} \geq \frac{\text{dist}(y_k, H)}{100\lambda} = \frac{d(y_k, x_k)}{100\lambda}.
\]
Thus by the doubling property of \( \mu \), \( C \mu(2B^i_j) \geq \mu(B(x_k, 2d(x_k, y_k))) \) for \( C = C(C_d, \lambda) \). Furthermore, by the first two inequalities of (3.14),
\[
r^i_j \leq \frac{d(y_k, x_k)}{38\lambda},
\]
so that \( 2B^i_j \subset B(x_k, 2d(x_k, y_k)) \). Recall that \( w \) was defined in the proof of Theorem 3.5 as the limit of the discrete convolutions \( v_i \) of \( u \) in \( U_i \). Noting that \( k \) and \( i \) are fixed and that the summations below are over indices \( j \), we have
\[
|v_i(y_k) - \tilde{u}(x_k)| = \left| \sum_{y_k \in 2B^i_j} \phi^i_j(y_k)(u_{B^i_j} - \tilde{u}(x_k)) \right|
\leq C \sum_{y_k \in 2B^i_j} \int_{2B^i_j} |u - \tilde{u}(x_k)| \, d\mu
\leq C \sum_{y_k \in 2B^i_j} \int_{B(x_k, 2d(x_k, y_k))} |u - \tilde{u}(x_k)| \, d\mu
\leq CC_0 \int_{B(x_k, 2d(x_k, y_k))} |u - \tilde{u}(x_k)| \, d\mu,
\]

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where $C_0$ was the overlap constant of the Whitney balls. Letting $i \to \infty$, we get

$$|\tilde{w}(y_k) - \tilde{u}(x_k)| \leq C \int_{B(x_k, 2d(x_k, y_k))} |u - \tilde{u}(x_k)| \, d\mu,$$

which converges to 0 as $k \to \infty$, because the convergence in (3.12) was locally uniform. In total, $\tilde{w}(y_k) \to \tilde{u}(x)$ and then $\tilde{w}(x) = \tilde{u}(x)$ for every $x \in H$, so we have the desired conclusion. \hfill $\Box$

### 4 Proof of Theorem 1.1: outside the jump set

In this section we use the tools developed in the previous section to prove one part of the main theorem of this paper, Theorem 1.1. As a by-product, we obtain some approximation results for BV functions. First, we highlight some properties of the 1-capacity $\text{Cap}_1$ relevant to this paper — recall the definition from (2.5).

**Remark 4.1.** From [23, Lemma 3.4] it follows that $\text{Cap}_1(A) \leq 2C_d \mathcal{H}_1(A)$ for any $A \subset X$. On the other hand, by combining [16, Theorem 4.3] and the proof of [16, Theorem 5.1], we know that $\mathcal{H}_\varepsilon(A) \leq C(C_d, C_P, \lambda, \varepsilon) \text{Cap}_1(A)$ for any $A \subset X$ and $\varepsilon > 0$. Thus we could also control the size of the "exceptional set" $G$ in Theorem 1.1 and elsewhere by its $\mathcal{H}_\varepsilon$-measure, for arbitrarily small $\varepsilon > 0$. Finally, we note that $\text{Cap}_1$ is an outer capacity, meaning that

$$\text{Cap}_1(A) = \inf\{\text{Cap}_1(U) : U \supset A \text{ is open}\}$$

for any $A \subset X$, see e.g. [6, Theorem 5.31]. Thus in Theorem 1.1 and elsewhere we can always make the set $G$ open, even if its construction does not automatically make it as such.

A version of the following lemma was previously known for Newton-Sobolev functions (see e.g. [23]).

**Lemma 4.2.** Let $u_i, u \in \text{BV}(X)$ with $u_i \to u$ in $\text{BV}(X)$. Let $\varepsilon > 0$. Then there exists $F \subset X$ with $\text{Cap}_1(F) < \varepsilon$ such that, by picking a subsequence if necessary, $u_i^\wedge \to u^\wedge$ and $u_i^\vee \to u^\vee$ (and thus also $\tilde{u}_i \to \tilde{u}$) uniformly in $X \setminus F$. 

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Proof. For \( v \in \text{BV}(X) \), by Lemma 2.1 we know that

\[
\text{Cap}_1(\{ x \in X : M_1 v(x) > t \}) \leq \frac{C_1}{t} \| v \|_{\text{BV}(X)}
\]

for any \( t > 0 \), where \( C_1 \) is the constant from the lemma, corresponding to the choice \( R = 1 \). By the coarea formula (2.3), there exists a countable dense set \( T \subset \mathbb{R} \) such that for every \( s \in T \), \( P(\{ v > s \}, X) < \infty \). Recall the definition of \( \Sigma_{\gamma}E \) for sets \( E \subset X \) from (2.6). We set

\[
N := \bigcup_{s \in T} \partial^* \{ v > s \} \setminus \Sigma_{\gamma} \{ v > s \}.
\]

By (2.6) we know that \( H(N) = 0 \). For \( x \in X \setminus N \), if \( t > 0 \) and \( t < v^\vee(x) \), by the definition of the upper approximate limit we have that

\[
\limsup_{r \to 0^+} \frac{\mu(B(x, r) \cap \{ v > t \})}{\mu(B(x, r))} > 0.
\]

Then, since \( x \in X \setminus N \), for any \( s < v^\vee(x) \) with \( s \in T \) we have

\[
\liminf_{r \to 0^+} \frac{\mu(B(x, r) \cap \{ v > s \})}{\mu(B(x, r))} \geq \gamma.
\]

Thus for any \( v \in \text{BV}(X) \) we have that \( M_1 v(x) \geq \gamma v^\vee(x) \) for any \( x \in X \setminus N \), and so

\[
\text{Cap}_1(\{ x \in X : v^\vee(x) > t \}) = \text{Cap}_1(\{ x \in X \setminus N : v^\vee(x) > t \})
\leq \text{Cap}_1(\{ x \in X \setminus N : M_1 v(x) > \gamma t \})
\leq \frac{C_1}{\gamma t} \| v \|_{\text{BV}(X)} \tag{4.1}
\]

for any \( t > 0 \).

Now let \( u_i, u \) be as in the statement of the lemma. By picking a subsequence if necessary, we can assume that for each \( i \in \mathbb{N} \), \( \| u_i - u \|_{\text{BV}(X)} \leq 2^{-2i} \gamma / C_1 \). It is easy to check that we can write

\[
\{|u^\vee_i - u^\vee| > 2^{-i}\} \subset \{|u_i - u| > 2^{-i}\} \cup \{|u^\vee| = \infty\} \cup \{|u^\vee_i| = \infty\} =: F_i.
\]

By [24, Lemma 3.2] we know that \( \mathcal{H}(\{|u^\vee| = \infty\} \cup \{|u^\vee_i| = \infty\}) = 0 \), and then by (4.1), \( \text{Cap}_1(F_i) \leq 2^{-i} \) for each \( i \in \mathbb{N} \), so that for large enough \( k \in \mathbb{N} \) we have

\[
\text{Cap}_1\left( \bigcup_{i=k}^{\infty} F_i \right) < \varepsilon.
\]

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Note that \( u_i^\vee \to u^\vee \) uniformly in \( X \setminus \bigcup_{i=k}^{\infty} F_i \). Similarly we get \( u_i^\wedge \to u^\wedge \) uniformly in \( X \setminus \bigcup_{i=k}^{\infty} F_i \).

Recall that the jump set \( S_u \) of a BV function \( u \) is defined as the set where \( u^\wedge < u^\vee \).

**Lemma 4.3.** Let \( u \in \text{BV}(X) \) with \( \mathcal{H}(S_u) = 0 \). Then there exists a sequence \( w_i \in \text{BV}(X) \cap C(X) \) with \( w_i \to u \) in \( \text{BV}(X) \).

**Proof.** By [24, Theorem 3.5] we know that

\[
\lim_{r \to 0^+} \int_{B(x,r)} |u - \tilde{u}(x)| \, d\mu = 0 \tag{4.2}
\]

for \( \mathcal{H} \)-a.e. \( x \in X \), in particular for \( \|Du\| \)-a.e. \( x \in X \), as by (2.7) and the coarea formula (2.3), \( \|Du\| \) is always absolutely continuous with respect to \( \mathcal{H} \).

Note that \( \tilde{u} \) is a Borel function, and hence is measurable with respect to the Radon measure \( \|Du\| \). By Lusin’s theorem and Egorov’s theorem, we can pick compact sets \( H_i \subset X \) with \( \|Du\|(X \setminus H_i) < 1/i, \ i \in \mathbb{N} \), such that \( \tilde{u}|_{H_i} \) is continuous and the convergence in (4.2) as \( r \to 0 \) is uniform in \( H_i \).

For each \( i \in \mathbb{N} \), apply Corollary 3.6 with \( U = X \setminus H_i \) and \( \kappa = 1/i \), to obtain a function \( w_i \in \text{BV}(X) \) with \( w_i = u \) in \( H_i \). We have \( w_i \to u \) in \( L^1(X) \), and

\[
\|D(w_i - u)\|(X) = \|D(w_i - u)\|(X \setminus H_i)
\leq \|Dw_i\|(X \setminus H_i) + \|Du\|(X \setminus H_i)
\leq C\|Du\|(X \setminus H_i) + \|Du\|(X \setminus H_i)
\leq C/i
\]

for each \( i \in \mathbb{N} \), so that in fact \( w_i \to u \) in \( \text{BV}(X) \). By Proposition 3.7, each \( \tilde{w}_i \) is continuous in \( X \). \( \square \)

**Remark 4.4.** If \( u \in \text{BV}(X) \) and we have a sequence of continuous functions \( u_i \to u \) in \( \text{BV}(X) \), then \( \|Du_i\|(S_u) = 0 \) by the facts that \( S_u \) is \( \sigma \)-finite with respect to \( \mathcal{H} \) (by e.g. the decomposition (2.11)) and \( \|Du_i\|^2(X) = 0 \) for all \( i \in \mathbb{N} \), see [14, Theorem 5.3]. Thus

\[
\|Du\|(S_u) = \|D(u - u_i)\|(S_u) \leq \|D(u - u_i)\|(X) \to 0
\]

as \( i \to \infty \), so that \( \mathcal{H}(S_u) = 0 \). Hence the subspace \( \{u \in \text{BV}(X) : \mathcal{H}(S_u) = 0\} \) is the closure of \( \text{BV}(X) \cap C(X) \) in \( \text{BV}(X) \).
Now we turn to our first quasicontinuity result.

**Proposition 4.5.** Let \( u \in BV(X) \) with \( \mathcal{H}(S_u) = 0 \), and let \( \varepsilon > 0 \). Then there exists \( G \subset X \) with \( \text{Cap}_1(G) < \varepsilon \) such that \( \tilde{u}|_{X \setminus G} \) is continuous.

**Proof.** By Lemma 4.3, we can pick a sequence \( w_i \in BV(X) \cap C(X) \) with \( w_i \to u \) in \( BV(X) \). By Lemma 4.2, there exists, by passing to a subsequence if necessary, a set \( F \subset X \) with \( \text{Cap}_1(F) < \varepsilon \) such that \( w_i \to \tilde{u} \) uniformly in \( X \setminus F \). Thus \( \tilde{u}|_{X \setminus F} \) is continuous. \( \square \)

**Corollary 4.6.** Let \( \Omega \subset X \) be an open set and \( u \in BV(\Omega) \) with \( \mathcal{H}(S_u) = 0 \). Let \( \varepsilon > 0 \). Then there exists \( G \subset \Omega \) with \( \text{Cap}_1(G) < \varepsilon \) such that \( \tilde{u}|_{\Omega \setminus G} \) is continuous.

**Proof.** We denote

\[
\Omega_\delta := \{ x \in \Omega : \text{dist}(x, X \setminus \Omega) > \delta \}, \quad \delta > 0.
\]

For \( \delta > 0 \), take \( \eta_\delta \in \text{Lip}(X) \) with \( 0 \leq \eta_\delta \leq 1 \), \( \eta_\delta = 1 \) in \( \Omega_\delta \), and \( \eta_\delta = 0 \) outside \( \Omega_{\delta/2} \). Then clearly \( u\eta_\delta \in BV(X) \). By the previous proposition, for each \( i \in \mathbb{N} \) there exists \( G_i \subset \Omega \) with \( \text{Cap}_1(G_i) < 2^{-i}\varepsilon \) such that \( \tilde{u}\eta_1|_{X \setminus G_i} \) is continuous, and clearly \( \tilde{u}\eta_1/i \equiv \tilde{u}\eta_1/i = \tilde{u} \) in \( \Omega_1/i \). Define \( G := \bigcup_{i \in \mathbb{N}} G_i \). Then for each \( i \in \mathbb{N} \), \( \tilde{u}|_{\Omega_1/i \setminus G} \) is continuous, whence \( \tilde{u}|_{\Omega \setminus G} \) is continuous, and \( \text{Cap}_1(G) < \varepsilon \). \( \square \)

Now we can prove Theorem 1.1 for points outside the jump set of a BV function.

**Proposition 4.7.** Let \( u \in BV(X) \) and let \( \varepsilon > 0 \). Then there exists \( G \subset X \) with \( \text{Cap}_1(G) < \varepsilon \) such that whenever \( y_k \to x \) with \( y_k \in X \setminus G \) and \( x \in X \setminus (G \cup S_u) \), then \( u^\wedge(y_k) \to \tilde{u}(x) \) and \( u^\vee(y_k) \to \tilde{u}(x) \).

Note that the conclusion of the proposition is stronger than saying that \( \tilde{u}(y_k) \to \tilde{u}(x) \).

**Proof.** Since \( \|Du\| \) is a Radon measure and \( S_u \) is a Borel set, we can find compact sets \( H_i \subset S_u \) with \( \|Du\|(S_u \setminus H_i) < 1/i \) for each \( i \in \mathbb{N} \). For each \( i \in \mathbb{N} \), take an open set \( U_i \subset X \) with \( U_i \supset S_u \setminus H_i \) and \( \|Du\|(U_i) < 1/i \), and apply Corollary 3.6 with \( U = U_i \) and \( \kappa = 1/i \) to obtain a function.
\( w_i \in \text{BV}(X) \) with \( w_i = u \) in \( X \setminus U_i \). We have \( w_i \to u \) in \( L^1(X) \), and by (3.11),

\[
\|D(w_i - u)(X) = \|D(w_i - u)(U_i) \leq \|Dw_i\|(U_i) + \|Du\|(U_i) \leq C\|Du\|(U_i) \leq C/i
\]

for each \( i \in \mathbb{N} \), so in fact \( w_i \to u \) in \( \text{BV}(X) \). By Corollary 3.6 for each \( i \in \mathbb{N} \), \( \tilde{w}_i \) is continuous in \( U_i \) and hence has no jump part there; therefore by (3.11),

\[
\|Dw_i\|^j(U_i) + \|Du\|^j(X \setminus (H_i \cup U_i)) \leq \|Dw_i\|^j(U_i) + \|Du\|^j(X \setminus S_u) = 0,
\]

so the jump set of \( w_i \) satisfies \( \mathcal{H}(S_{w_i} \setminus H_i) = 0 \). Thus by Corollary 4.6 applied to the open set that is \( X \setminus H_i \), there exists \( G_i \subset X \) with \( \text{Cap}_1(G_i) < 2^{-i-1}\varepsilon \) such that \( \tilde{w}_i|_{X \setminus (H_i \cup G_i)} \) is continuous. Since also \( \text{Cap}_1(S_{w_i} \setminus H_i) = 0 \), we can assume that \( G_i \supset S_{w_i} \setminus H_i \), so that \( \tilde{w}_i = w_i^\wedge = w_i^\vee \) in \( X \setminus (H_i \cup G_i) \). Let \( G_\infty := \bigcup_{i \in \mathbb{N}} G_i \). Since \( w_i \to u \) in \( \text{BV}(X) \), by Lemma 4.2 and by picking a subsequence if necessary, there exists \( F \subset X \) with \( \text{Cap}_1(F) < \varepsilon/2 \) such that \( w_i^\wedge \to u^\wedge \) and \( w_i^\vee \to u^\vee \) uniformly in \( X \setminus F \). For \( G := F \cup G_\infty \), clearly \( \text{Cap}_1(G) < \varepsilon \).

Finally, let \( y_k \to x \) with \( y_k \in X \setminus G \) and \( x \in X \setminus (S_u \cup G) \). Note that since each \( H_i \subset S_u \) is compact, for each \( i \in \mathbb{N} \) we necessarily have \( y_k \in X \setminus H_i \) for large enough \( k \), so for these indices, \( \tilde{w}_i(y_k) = w_i^\wedge(y_k) = w_i^\vee(y_k) \). For some sequence of nonnegative numbers \( \alpha_i \to 0 \), we have

\[
|w_i^\wedge(z) - u^\wedge(z)| \leq \alpha_i \quad \text{and} \quad |w_i^\vee(z) - u^\vee(z)| \leq \alpha_i
\]

for all \( z \in X \setminus F \), \( i \in \mathbb{N} \). Thus

\[
\limsup_{k \to \infty} |u^\wedge(y_k) - \tilde{u}(x)| \\
\leq \limsup_{k \to \infty} (|u^\wedge(y_k) - w_i^\wedge(y_k)| + |w_i^\wedge(y_k) - \tilde{w}_i(x)| + |\tilde{w}_i(x) - \tilde{u}(x)|) \\
\leq \limsup_{k \to \infty} (\alpha_i + |w_i^\wedge(y_k) - \tilde{w}_i(x)| + \alpha_i) \\
= \limsup_{k \to \infty} |\tilde{w}_i(y_k) - \tilde{w}_i(x)| + 2\alpha_i \\
= 2\alpha_i
\]

by the continuity of \( \tilde{w}_i|_{X \setminus (H_i \cup G_i)} \). Letting \( i \to \infty \) completes the proof for \( u^\wedge \). For \( u^\vee \), the proof is the similar. This completes the proof of the proposition. \( \square \)
5 Proof of Theorem 1.1: within the jump set

In this section we complete the proof of Theorem 1.1. First we consider a generalization of the Lebesgue differentiation theorem for the jump set of a BV function. Recall the definition of the number $Q > 0$ from (2.1). We know from [28, Theorem 4.3] that for $u \in \text{BV}(X)$ and $\mathcal{H}$-a.e. $x \in S_u$, there exist $t_1, t_2 \in (u^\wedge(x), u^\vee(x))$ such that

$$\lim_{r \to 0^+} \int_{B(x,r) \cap \{u < t_1\}} |u - u^\wedge(x)|^{Q/(Q-1)} d\mu = 0$$

and

$$\lim_{r \to 0^+} \int_{B(x,r) \cap \{u > t_2\}} |u - u^\vee(x)|^{Q/(Q-1)} d\mu = 0.$$ 

We cannot in general pick $t_1, t_2$ freely from the interval $(u^\wedge(x), u^\vee(x))$, as we can in the Euclidean setting, as demonstrated by the following example.

**Example 5.1.** Consider the one-dimensional space

$$X := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0 \text{ or } x_2 = 0\}$$

consisting of the two coordinate axes. Equip this space with the Euclidean metric inherited from $\mathbb{R}^2$, and the 1-dimensional Hausdorff measure. This measure is doubling and supports a $(1, 1)$-Poincaré inequality. Let

$$u := \chi_{\{x_1 > 0\}} + 2\chi_{\{x_2 > 0\}} + 3\chi_{\{x_1 < 0\}} + 4\chi_{\{x_2 < 0\}}.$$ 

For brevity, denote the origin $(0, 0)$ by 0. Now $S_u = \{0\}$ with $\mathcal{H}(\{0\}) = 2$, and $(u^\wedge(0), u^\vee(0)) = (1, 4)$. However, we cannot choose $t_1$ to be larger than 2, nor $t_2$ to be smaller than 3. This demonstrates that in a metric space, a BV function can, in a measure theoretic sense, take more than 2 values all along its jump set $S_u$.

Higher-dimensional example spaces can be obtained by simply taking Cartesian products of $X$ with e.g. $[0, 1]$.

**Example 5.2.** Closely related to this issue are the locality conditions discussed in [4] and [28]. We say that $X$ supports the strong locality condition if for every pair of sets $E_1 \subset E_2 \subset X$ of finite perimeter, we have

$$\lim_{r \to 0^+} \frac{\mu(B(x, r) \cap (E_2 \setminus E_1))}{\mu(B(x, r))} = 0$$

for brevity, denote the origin $(0, 0)$ by 0. Now $S_u = \{0\}$ with $\mathcal{H}(\{0\}) = 2$, and $(u^\wedge(0), u^\vee(0)) = (1, 4)$. However, we cannot choose $t_1$ to be larger than 2, nor $t_2$ to be smaller than 3. This demonstrates that in a metric space, a BV function can, in a measure theoretic sense, take more than 2 values all along its jump set $S_u$.

Higher-dimensional example spaces can be obtained by simply taking Cartesian products of $X$ with e.g. $[0, 1]$.
for $\mathcal{H}$-a.e. $x \in \partial^* E_1 \cap \partial^* E_2$. Following [4], we also say that $X$ supports the locality condition if for every pair of sets $E_1 \subset E_2 \subset X$ of finite perimeter, we have $\theta_{E_1}(x) = \theta_{E_2}(x)$ for $\mathcal{H}$-a.e. $x \in \partial^* E_1 \cap \partial^* E_2$ (the function $\theta_E$ was defined in (2.7)). In [4, Proposition 6.2], the authors show that the strong locality condition implies the locality condition. In [28, Theorem 4.10] it was shown that if the space supports the strong locality condition, then every pair $t_1, t_2$ from the interval $(u^\wedge(x), u^\vee(x))$ satisfies the two equations from the beginning of this section. However, either locality condition can fail in a metric space, even one with a doubling measure supporting a Poincaré inequality. Consider the space from Example 5.1. The sets

$$E_1 := \{x_1 > 0\}, \quad E_2 := \{x_1 > 0\} \cup \{x_2 > 0\}$$

are easily seen to be of finite perimeter, and $\partial^* E_1 = \partial^* E_2 = \{0\}$, that is, the measure theoretic boundaries only contain the origin. We have $\mathcal{H}(\{0\}) = 2$. The strong locality condition fails at the origin, since

$$\lim_{r \to 0^+} \frac{\mu(B(0, r) \cap (E_2 \setminus E_1))}{\mu(B(0, r))} = \lim_{r \to 0^+} \frac{\mu(B(0, r) \cap \{x_2 > 0\})}{\mu(B(0, r))} = \frac{1}{4}.$$

In addition, we see that $P(E_1, X) = 1$, since we can take approximating Lipschitz functions with support in $\{x_1 > 0\}$. But this does not work for $E_2$, and so we get $P(E_2, X) = 2$. On the other hand, obviously $\mathcal{H}(\partial^* E_1) = \mathcal{H}(\partial^* E_2)$, because both sets consist of the same point. Thus $\theta_{E_1}(0) = 1/2$ but $\theta_{E_2}(0) = 1$, and the locality condition fails as well.

Recall the definition of $\gamma > 0$ from (2.3), the definition $n = \lceil 1/\gamma \rceil$, and the definition of the functions $u^l$ (defined also below) for $u \in BV(X)$ from (2.10). Denote by $n(x)$ the number of distinct values $u^l(x)$, $l \in \{1, \ldots, n\}$. Also, for $u \in BV(X)$, $x \in X$, and $\delta > 0$, we denote

$$A^\delta_l(x) := [u^{l-1}(x) + \delta, u^{l+1}(x) - \delta], \quad l = 2, \ldots, n(x) - 1,$$

$$A^\delta_1(x) := (-\infty, u^2(x) - \delta], \quad A^\delta_{n(x)}(x) := [u^{n(x)-1}(x) + \delta, \infty).$$

(5.1)

**Theorem 5.3.** Let $u \in BV(X)$. Then for $\mathcal{H}$-a.e. $x \in S_u$, the following two properties hold: $-\infty < u^1(x) < \ldots < u^{n(x)}(x) < \infty$, and

$$\lim_{r \to 0^+} \int_{B(x, r) \cap \{u \in A^\delta_l(x)\}} |u - u^l(x)|^{\rho/(\rho-1)} \, d\mu = 0 \quad (5.2)$$

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there is a countable dense set

\[ \{ u^j(x), j = 1, \ldots, n(x) - 1 \} \]

For \( l = 2, \ldots, n(x) - 1 \), we can in fact replace \( Q/(Q - 1) \) with any \( q > 0 \).

**Proof.** This is a generalization of results in [28]. Denote, for brevity, the super-level sets of \( u \) by \( E_t := \{ u > t \}, t \in \mathbb{R} \). By the coarea formula (2.3), there is a countable dense set \( T \subset \mathbb{R} \) such that for every \( t \in T \), the set \( E_t \) is of finite perimeter. Let

\[ N := \bigcup_{t \in T} \partial^* E_t \setminus \Sigma E_t \]

and

\[ \widetilde{N} := \bigcup_{s,t \in T: s < t} \partial^* (E_s \setminus E_t) \setminus \Sigma (E_s \setminus E_t) \]

Recalling (2.6), and since the sets \( E_s \setminus E_t, s, t \in T \), are also of finite perimeter by [31] Proposition 4.7, we have \( \mathcal{H}(N \cup \widetilde{N}) = 0 \).

Fix \( x \in S_u \setminus (N \cup \widetilde{N}) \). By discarding another \( \mathcal{H} \)-negligible set, we can assume that \( u^\vee(x), u^\wedge(x) \) are finite, see [24] Lemma 3.2. Set \( u^l(x) = u^\wedge(x) \), and define inductively for \( l = 2, \ldots, n - 1 = \lfloor 1/\gamma \rfloor - 1 \)

\[ u^l(x) := \sup \left\{ t \in \mathbb{R} : \lim_{r \to 0^+} \frac{\mu(B(x, r) \cap \{ u^{l-1}(x) + \varepsilon < u < t \})}{\mu(B(x, r))} = 0 \quad \forall \varepsilon > 0 \right\} \]

provided \( u^{l-1}(x) < u^l(x) \), and otherwise set \( u^l(x) := u^\vee(x) \). We also set \( u^n(x) := u^\vee(x) \). Fix \( l \) and suppose that \( u^l(x) < u^\vee(x) \). We can find \( t_i \in T \) with \( u^l(x) < t_i < u^\vee(x) \) for each \( i \in \mathbb{N} \) such that \( t_i \setminus u^l(x) \) as \( i \to \infty \). Then whenever \( \partial^* \{ t_{i+1} \leq u < t_i \} \) has density 1 at \( x \) or \( x \in \partial^* \{ t_{i+1} \leq u < t_i \} \), we must have

\[ \liminf_{r \to 0^+} \frac{\mu(\{ t_{i+1} \leq u < t_i \} \cap B(x, r))}{\mu(B(x, r))} \geq \gamma. \]

By the choice of \( n \), this can happen only for at most \( n \) number of indices \( i \) (because the sets \( \{ t_{i+1} \leq u < t_i \} \) are pairwise disjoint). It follows that for sufficiently large \( i \), the sets \( \{ t_{i+1} \leq u < t_i \} \) have density 0 at \( x \). Thus if \( u^l(x) < u^\vee(x) \), necessarily \( u^{l+1}(x) > u^l(x) \). Thus \( u^n(x)(x) = u^n(x) \).

By the definition of the functions \( u^l \), we have

\[ \limsup_{r \to 0^+} \frac{\mu(B(x, r) \cap \{ u^l(x) - \varepsilon < u < u^l(x) + \varepsilon \})}{\mu(B(x, r))} > 0 \]
for every \( \varepsilon > 0 \) and all \( l = 1, 2, \ldots, n \). Since \( x \notin \tilde{N} \), we have in fact

\[
\liminf_{r \to 0^+} \frac{\mu(B(x, r) \cap \{u^l(x) - \varepsilon < u < u^l(x) + \varepsilon\})}{\mu(B(x, r))} \geq \gamma. 
\]

(5.3)

Now, if for

\[
\alpha := \sup \left\{ t \in \mathbb{R} : \lim_{r \to 0^+} \frac{\mu(B(x, r) \cap \{u^{n(x)}(x) - t < u < t\})}{\mu(B(x, r))} = 0 \quad \forall \varepsilon > 0 \right\}
\]

we have \( \alpha < u^{n(x)}(x) = u^n(x) \), then necessarily \( n(x) = n \), and as above, we can conclude that for every \( \varepsilon > 0 \), the set \( \{\alpha - \varepsilon < u < \alpha + \varepsilon\} \) has lower density at least \( \gamma \) at \( x \). Moreover, as the sets \( \{\alpha - \varepsilon < u < \alpha + \varepsilon\} \) and \( \{u^l(x) - \varepsilon < u < u^l(x) + \varepsilon\}, l = 1, \ldots, n \) are all disjoint for small enough \( \varepsilon \), this contradicts the definition \( n = \lfloor 1/\gamma \rfloor \). Thus \( \alpha = u^{n(x)}(x) \).

For \( l = 1, \ldots, n(x) \), we note that by the definition of the numbers \( u^l(x) \) and the fact that \( \alpha = u^{n(x)}(x) \), the set

\[
\{u \in A_1^l(x) \} \setminus \{u^l(x) - \varepsilon < u < u^l(x) + \varepsilon\}
\]

has density 0 at \( x \) for any \( \varepsilon > 0 \), and this together with (5.3) implies for any \( l = 2, \ldots, n(x) - 1 \) and \( q > 0 \) that

\[
\lim_{r \to 0^+} \frac{1}{\mu(B(x, r))} \int_{B(x,r) \cap \{u \in A_1^l(x)\}} |u - u^l(x)|^q \, d\mu = 0.
\]

By combining this with (5.3), we get (5.2). The cases \( l = 1 \) and \( l = n(x) \) require additional computations, since we integrate over sets where \( u \) may be unbounded, but these cases were already covered in [28, Theorem 4.3].

Thus we have a rather complete measure theoretic description of the behavior of a BV function in its jump set: at \( \mathcal{H} \)-almost every point \( x \in S_u \), the space \( X \) can be partitioned into at most \( \lfloor 1/\gamma \rfloor \) sets such that in each set, \( u \) converges in a Lebesgue point sense to a real number in the interval \([u^\wedge(x), u^\vee(x)]\). Note that in Example 5.1, we have \( \gamma = 1/4 \).

**Proposition 5.4.** Let \( u \in \text{BV}(X) \) and let \( \varepsilon > 0 \). Then there exists \( G \subset X \) with \( \text{Cap}_1(G) < \varepsilon \) such that if \( y_k \to x \) with \( y_k \in X \setminus G \) and \( x \in S_u \setminus G \), then

\[
\min_{l_2 \in \{1, \ldots, n\}} |u^{l_1}(y_k) - u^{l_2}(x)| \to 0
\]

(5.4)

for every \( l_1 = 1, \ldots, n \).
Proof. For \( x \in X \), set
\[
\delta(x) := \min\{u^{l+1}(x) - u^l(x), \ l = 1, \ldots, n(x) - 1\}/2.
\]
We divide the proof into two steps.

**Step 1.** First assume that we have a compact set \( H \subset S_u \) where \( n(x) \) is constant, the functions \(-\infty < u^1 < \ldots < u^{n(x)} < \infty\) are continuous, and
\[
\frac{1}{\mu(B(x, r))} \int_{B(x, r) \cap \{u \in A^l(x) / \delta(x)\}} |u - u^l(x)| \, d\mu \to 0 \quad \text{as } r \to 0 \quad (5.5)
\]
uniformly in the set \( H \) for every \( l = 1, \ldots, n(x) \).

We will demonstrate that there is a set \( \tilde{G} \subset X \) with \( \text{Cap}_{1}(\tilde{G}) < \varepsilon \) such that whenever \( y_k \to x \) with \( y_k \in X \setminus (H \cup \tilde{G}) \), \( x \in H \), and \( u^{l_1}(y_k) \in A^l_2(x) \) for given \( l_1 \in \{1, \ldots, n\} \) and \( l_2 \in \{1, \ldots, n(x)\} \), then
\[
\lim_{k \to \infty} |u^{l_1}(y_k) - u^{l_2}(x)| = 0. \quad (5.6)
\]
In other words, we have continuity up to the jump set as long as we approach it from a specific "side", more precisely a specific level set of \( u \).

For \( p \in \mathbb{N} \), let
\[
A_p := \{ x \in X : 2^{-p-1} \leq \text{dist}(x, H) < 2^{-p} \}.
\]
Since \( \|Du\|(X) \) is finite and the sets \( A_p \) are pairwise disjoint, we have
\[
\sum_{p \in \mathbb{N}} \|Du\|(A_p) < \infty.
\]
It follows that for each \( j \in \mathbb{N} \) there exists \( N_j \in \mathbb{N} \) such that
\[
\sum_{p \geq N_j} \|Du\|(A_p) \leq 4^{-j}\varepsilon.
\]
We can choose \( j \mapsto N_j \) to be strictly increasing. We set \( a_p := 2^{-j} \) for \( N_j < p \leq N_{j+1} \), so that \( a_p \to 0 \) as \( p \to \infty \). Now
\[
\sum_{p \geq N_{l+1}} \frac{\|Du\|(A_{p-1})}{a_p} = \sum_{j \in \mathbb{N}} \sum_{p=N_j+1}^{N_{j+1}} 2^j \|Du\|(A_{p-1}) \leq \sum_{j \in \mathbb{N}} 2^{-j}\varepsilon = \varepsilon. \quad (5.7)
\]
Let
\[ G_p := \left\{ z \in A_p : \exists 0 < r_z < 2^{-p-2}/\lambda \text{ s.t. } \int_{B(z,r_z)} |u - u_{B(z,r_z)}| \, d\mu > a_p \right\}. \]

Pick \( p \geq 2 \) and take a cover \( \{B(z, \lambda r_z)\}_{z \in G_p} \) of \( G_p \). By the 5-covering theorem, we can select a countable disjoint subcollection \( \{\lambda B_j = B(z_j, \lambda r_j)\}_{j \in \mathbb{N}} \) such that the balls \( 5\lambda B_j \) cover \( G_p \). For each \( j \in \mathbb{N} \), we have by the Poincaré inequality
\[
a_p < \int_{B_j} |u - u_{B_j}| \, d\mu \leq Cr_j \frac{\|Du\| (\lambda B_j)}{\mu(B_j)}.
\]

Since all the radii necessarily satisfy \( 5\lambda r_j \leq 1 \),
\[
\text{Cap}_1(G_p) \leq C\mathcal{H}_1(G_p) \leq C \sum_{j \in \mathbb{N}} \frac{\mu(5\lambda B_j)}{5\lambda r_j} \leq C \sum_{j \in \mathbb{N}} \frac{\mu(B_j)}{r_j} \leq C \sum_{j \in \mathbb{N}} \frac{\|Du\| (\lambda B_j)}{a_p}
\]
\[
\leq \frac{C}{a_p} \|Du\| (A_{p-1} \cup A_p \cup A_{p+1}).
\]

In the last inequality we used the fact that the balls \( \lambda B_j \) are disjoint. Defining \( G := \bigcup_{p \geq N_1+1} G_p \), we have by (5.7)
\[
\text{Cap}_1(G) \leq \sum_{p \geq N_1+1} \text{Cap}_1(G_p) \leq C \sum_{p \geq N_1+1} \frac{\|Du\| (A_{p-1})}{a_p} \leq C\varepsilon.
\]

We need to prove an analog of Proposition 4.7 this time not for \( \tilde{u} \) but for the functions \( u^j \). For each \( m \in \mathbb{N} \), set \( W_m := \bigcup_{p=m}^{\infty} A_p \), and apply Corollary 3.6 with \( U = W_m \) and \( \kappa = \kappa_m \setminus 0 \) to obtain a function \( w^m \in \text{BV}(X) \). By the proof of Theorem 4.4, we can assume that the scale of the corresponding Whitney type coverings is fixed with \( R = 1 \). Fix \( m \geq N_1 + 1 \).

Consider a sequence \( y_k \to x \) with \( y_k \in X \setminus (H \cup G) \) and \( x \in H \), such that for a fixed \( l_2 \in \{1, \ldots, n(x)\} \), \( w^m(y_k) \in A_{l^2}^{(x)/2}(x) \) for each \( k \in \mathbb{N} \). For each \( y_k \) let \( x_k \in H \) such that \( d(y_k, x_k) = \text{dist}(y_k, H) \). Clearly \( d(y_k, x_k) \to 0 \) as \( k \to \infty \), and thus also \( d(x_k, x) \to 0 \), whence \( u^{l^2}(x_k) \to u^{l^2}(x) \). Thus we need to show that \( |w^m(y_k) - u^{l^2}(x_k)| \to 0 \) as \( k \to \infty \).

Define \( B_k := B(y_k, \text{dist}(y_k, H)/4\lambda) \) for each \( k \in \mathbb{N} \), and then fix \( y_k \in W_{m+2} \). According to the proof of Theorem 3.5, \( w^m = \lim_{i \to \infty} w_i \) for discrete convolutions
\[
w_i = \sum_{j \in \mathbb{N}} u_{B_j} \phi^i_j
\]
defined in open sets \( U_i \subset W_m, i \in \mathbb{N} \), at scale \( R = 1 \). For large enough \( i \in \mathbb{N} \) so that
\[
U_i \supset W_{m+1} \cap \{ z \in X : \text{dist}(z, H) \geq \text{dist}(y_k, H)/2 \},
\]
we have for all \( 2B_j \ni y_k \) that \( B_j \subset B_k \) with radii comparable to \( \text{dist}(y_k, H) \). Thus
\[
|w_i(y_k) - u_{B_k}| \leq \sum_{j \in \mathbb{N}} |\phi^j(y_k)||u_{B_j} - u_{B_k}|
\]
\[
= \sum_{j \in \mathbb{N}, 2B_j \ni y_k} |\phi^j(y_k)||u_{B_j} - u_{B_k}|
\]
\[
\leq C_0 \int_{B_j} |u - u_{B_k}| d\mu
\]
\[
\leq C \int_{B_k} |u - u_{B_k}| d\mu.
\]
By taking the limit \( i \to \infty \), we get
\[
|w^m(y_k) - u_{B_k}| \leq C \int_{B_k} |u - u_{B_k}| d\mu \leq Ca_p, \quad (5.8)
\]
where \( p \in \mathbb{N} \) is such that \( y_k \in A_p \setminus G_p \). As \( k \to \infty \) we have \( p \to \infty \), and so \( a_p \to 0 \). Hence \( u_{B_k} \in A_{l_2}^{\delta(x)/3}(x) \) for large \( k \), and
\[
\frac{\mu(B_k \cap \{ u \notin A_{l_2}^{\delta(x)/4}(x) \})}{\mu(B_k)} \leq \frac{12}{\delta(x)} \int_{B_k} |u - u_{B_k}| d\mu \to 0 \quad (5.9)
\]
as \( k \to \infty \). Therefore
\[
|u_{B_k \cap \{ u \in A_{l_2}^{\delta(x)/4}(x) \}} - u_{B_k}| \leq \int_{B_k \cap \{ u \in A_{l_2}^{\delta(x)/4}(x) \}} |u - u_{B_k}| d\mu
\]
\[
\leq \frac{\mu(B_k)}{\mu(B_k \cap \{ u \in A_{l_2}^{\delta(x)/4}(x) \})} \int_{B_k} |u - u_{B_k}| d\mu \quad (5.10)
\]
\( \to 0 \)
as \( k \to \infty \). Now we can estimate
\[
|w^m(y_k) - u^{l_2}(x_k)| \leq |w^m(y_k) - u_{B_k}|
\]
\[
+ |u_{B_k} - u_{B_k \cap \{ u \in A_{l_2}^{\delta(x)/4}(x) \}}| + |u_{B_k \cap \{ u \in A_{l_2}^{\delta(x)/4}(x) \}} - u^{l_2}(x_k)|.
\]
Here the first term converges to 0 as \( k \to \infty \) by (3.8), and the second term converges to 0 by (5.10). For large enough \( k \), by (5.9) we have \( \mu(B_k \cap \{ u \in A_{l_2}^{\delta(x)/4}(x) \}) \geq 1/2 \), so that also \( C\mu(B_k \cap \{ u \in A_{l_2}^{\delta(x)/4}(x) \}) \geq \mu(B(x_k, 2d(y_k, x_k))) \), and by the continuity of the functions \( u^l \) in \( H \) we have \( |u^l(x_k) - u^l(x)| < \delta(x)/10 \) for all \( l = 1, \ldots, n \). Thus the third term is at most

\[
\int_{B_k \cap \{ u \in A_{l_2}^{\delta(x)/4}(x) \}} |u - u^l_2(x_k)| \, d\mu \\
\leq \frac{C}{\mu(B(x_k, 2d(y_k, x_k)))} \int_{B(x_k, 2d(y_k, x_k)) \cap \{ u \in A_{l_2}^{\delta(x)/4}(x) \}} |u - u^l_2(x_k)| \, d\mu \\
\leq \frac{C}{\mu(B(x_k, 2d(y_k, x_k)))} \int_{B(x_k, 2d(y_k, x_k)) \cap \{ u \in A_{l_2}^{\delta(x)/8}(x) \}} |u - u^l_2(x_k)| \, d\mu,
\]

which converges to 0 by (5.5). It follows that \( |w^m(y_k) - u^l_2(x_k)| \to 0 \) as \( k \to \infty \), and since we had \( u^l_2(x_k) \to u^l_2(x) \), we have \( w^m(y_k) \to u^l_2(x) \) as \( k \to \infty \).

By Corollary 3.6 we know that \( w^m \to u \) in \( BV(X) \) as \( m \to \infty \), and so by Lemma 4.2 and by picking a subsequence, if necessary, there exists \( F \subset X \) with \( \text{Cap}_1(F) < \varepsilon \) such that for some sequence \( \alpha_m \downarrow 0 \), \( |(w^m)^\wedge - u^\wedge| \leq \alpha_m \) and \( |(w^m)^\vee - u^\vee| \leq \alpha_m \) in \( X \setminus F \) for any \( m \in \mathbb{N} \). But \( (w^m)^\wedge = (w^m)^\vee = \tilde{w}^m \) in \( W_m \), and so

\[
|\tilde{w}^m - u^l| \leq \alpha_m \quad (5.11)
\]

in \( W_m \setminus F \) for any \( l = 1, \ldots, n \) and \( m \in \mathbb{N} \). Take a sequence \( y_k \to x \) with \( y_k \in X \setminus (F \cup G \cup H) \), \( x \in H \), and \( u^l_1(y_k) \in A_{l_2}^{\delta(x)}(x) \) for given \( l_1 \in \{1, \ldots, n\} \) and \( l_2 \in \{1, \ldots, n(x)\} \). Then for sufficiently large \( m \in \mathbb{N} \), by (5.11) we have \( \tilde{w}^m(y_k) \in A_{l_2}^{\delta(x)/2}(x) \) for \( k \) large enough such that \( y_k \in W_m \), so that

\[
\limsup_{k \to \infty} |u^{l_1}(y_k) - u^{l_2}(x)| \\
\leq \limsup_{k \to \infty} |u^{l_1}(y_k) - \tilde{w}^m(y_k)| + \limsup_{k \to \infty} |\tilde{w}^m(y_k) - u^{l_2}(x)| \\
\leq \alpha_m.
\]

Thus we have (5.6).

**Step 2.** Now we consider the general case. Partition the Borel set \( S_u \) into sets \( S_p \), \( p = 1, \ldots, n \), in which \( n(x) = p \) for all \( x \in S_p \). Since

\[
S_p = \{ u^1 < \ldots < u^{p-1} = u^p < u^\vee \} \cup \{ u^1 < \ldots < u^{p-1} < u^p = u^\vee \},
\]

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each $S_p$ is a Borel set. (Note that the set $\{u^1 < \ldots < u^{p-1} = u^p < u^\vee\}$ is of 1-capacity zero, by the proof of Theorem 5.3.)

For each $i \in \mathbb{N}$, pick compact sets $K_p^i \subset S_p$ such that for $H_i := \bigcup_{p=1}^n K_p^i$ we have $\|Du\|(S_u \setminus H_i) < 2^{-i}\varepsilon$. By Lusin’s theorem, we can assume that each $u^l$ is continuous in $H_i$, and by Theorem 5.3 and Egorov’s theorem we can assume that for every $x \in H_i$, $-\infty < u^1(x) < \ldots < u^{n(x)}(x) < \infty$ with

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r) \cap \{u \in A^\delta_i(x)/\delta(x)\}} |u - u^l(x)| \, d\mu \to 0 \quad \text{as } r \to 0$$

(5.12)

uniformly in $H_i$ for every $l = 1, \ldots, n(x)$.

For any of the sets $K_p^i$, we are now in the situation described in Step 1. Therefore for each $i \in \mathbb{N}$ there is a set $G_i$ with $\operatorname{Cap}_1(G_i) < 2^{-i}\varepsilon$ such that we have the following. Let $y_k \to x$ with $y_k \in X \setminus (K_p^i \cup G_i)$, $x \in K_p^i$, and $u^{l_1}(y_k) \in A^\delta_i(x)$ for some $l_1 \in \{1, \ldots, n\}$ and $l_2 \in \{1, \ldots, n(x) = p\}$. Then

$$u^{l_1}(y_k) \to u^{l_2}(x)$$

(5.13)

by Step 1. Moreover,

$$\|Du\| \left( S_u \setminus \bigcup_{i \in \mathbb{N}} H_i \right) = 0,$$

so that by (2.11),

$$\mathcal{H} \left( S_u \setminus \bigcup_{i \in \mathbb{N}} H_i \right) = 0.$$

Define

$$G := \bigcup_{i \in \mathbb{N}} G_i \cup \left( S_u \setminus \bigcup_{i \in \mathbb{N}} H_i \right) \cup \{|u^\wedge| = \infty\} \cup \{|u^\vee| = \infty\}.$$

Then $\operatorname{Cap}_1(G) < \varepsilon$. Let $y_k \to x$ with $y_k \in X \setminus G$, $x \in S_u \setminus G$, and $u^{l_1}(y_k) \in A^\delta_{l_2}(x)$ for some $l_1 \in \{1, \ldots, n\}$ and $l_2 \in \{1, \ldots, n(x)\}$. Note that $x \in H_i$ for some $i \in \mathbb{N}$. If $y_k \in H_i$, then by the continuity of the functions $u^l$ in $H_i$ we have $u^{l_1}(y_k) \to u^{l_1}(x)$, and since $u^{l_1}(y_k) \in A^\delta_{l_2}(x)$, we necessarily have $l_1 = l_2$ and thus $u^{l_1}(y_k) \to u^{l_2}(x)$. On the other hand, if $y_k \in X \setminus H_i$, then $u^{l_1}(y_k) \to u^{l_2}(x)$ by (5.13).
This immediately implies (5.4), since we have
\[ \bigcup_{l_2=1}^{n(x)} A_{l_2}^{\delta(x)}(x) = \mathbb{R} \]
at every \( x \in X \setminus G \).

By combining Proposition 4.7 and Proposition 5.4 with the fact that \( \text{Cap}_1 \) is an outer capacity as noted in Remark 4.1, Theorem 1.1 is proved.

**Example 5.5.** It is not true that by discarding a suitable set of small capacity \( G \), we would have that \( u^l|_{S_u \setminus G} \) is continuous for each \( l \). Consider \( X = \mathbb{R} \) with the Euclidean distance and the 1-dimensional Lebesgue measure, and set
\[ u := \chi_{[-1,0]} + \sum_{i \in \mathbb{N}} 2^{-i} \chi_{(2^{-i-1},2^{-i}]} \]
Then \( u^\vee(2^{-i-1}) = 2^{-i} \not\to 1 = u^\vee(0) \) as \( i \to \infty \). Moreover, the 1-capacity of every point is 2, so the only set of 1-capacity smaller than 2 is the empty set.

### 6 Application to sets of finite perimeter

In this section we will discuss the implications of Theorem 1.1 for sets of finite perimeter. Federer’s structure theorem states that a set \( E \subset \mathbb{R}^n \) is of finite perimeter if and only if \( \mathcal{H}(\partial^* E) \) is finite, see [13, Section 4.5.11]. In a complete metric space \( X \) with a doubling measure that supports a \((1,1)\)-Poincaré inequality, the “only if” direction has been shown by Ambrosio, see (2.7). The “if” direction was shown for a certain class of metric measure spaces in [27], but remains open in general. As part of the proof of the “if” direction it is usually shown that the collection of lines parallel to the coordinate axes in \( \mathbb{R}^n \), which pass from the measure theoretic interior of \( E \) to the measure theoretic exterior of \( E \) but do not intersect \( \partial^* E \), must have 1-modulus zero, see for example the proof in [12, p. 222]. In this section we will prove a similar result in the metric setting, provided we know that \( E \subset X \) is of finite perimeter. We also give a partial converse, namely that if \( E \) is a \( \mu \)-measurable set with \( \mathcal{H}(\partial^* E) \) finite and the 1-modulus of curves intersecting both the measure theoretic interior of \( E \) and the measure theoretic exterior of \( E \) without intersecting \( \partial^* E \) in between is zero, then \( E \) is of finite perimeter. (A related partial generalization was previously considered in [26].)
The measure theoretic interior $\mathcal{I}(E)$ and the measure theoretic exterior $\mathcal{E}(E)$ of a $\mu$-measurable set $E \subset X$ are defined as follows:

$$\mathcal{I}(E) := \left\{ x \in X : \lim_{r \to 0^+} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} = 1 \right\}$$

and

$$\mathcal{E}(E) := \left\{ x \in X : \lim_{r \to 0^+} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} = 0 \right\}.$$ 

Clearly $\partial^* E = X \setminus (\mathcal{I}(E) \cup \mathcal{E}(E))$. Let $u = \chi_E$. Observe that $x \in \mathcal{I}(E)$ means that $u^\vee(x) = u^\wedge(x) = 1$, $x \in \mathcal{E}(E)$ means that $u^\vee(x) = u^\wedge(x) = 0$, and $x \in \partial^* E$ means that $u^\vee(x) = 1$ and $u^\wedge(x) = 0$, i.e. $x \in S_u$.

First we note that some sets of finite perimeter, such as the enlarged rationals, can exhibit bizarre behavior that demonstrates the necessity of excluding a set $G$ in Theorem 1.1.

Example 6.1. Let $\{q_i\}_{i \in \mathbb{N}}$ be an enumeration of $\mathbb{Q} \times \mathbb{Q} \subset \mathbb{R}^2$, and define

$$E := \bigcup_{i \in \mathbb{N}} B(q_i, 2^{-i}).$$

Clearly $\mathcal{L}^2(E) \leq 2\pi$, and $\chi_E = \lim_j \chi_{E_j}$, where $E_j := \bigcup_{i=1}^j B(q_i, 2^{-i})$, the limit occurring in $L^1(\mathbb{R}^2)$. Since $P(E_j, \mathbb{R}^2) \leq 2\pi \sum_{i=1}^j 2^{-i}$, we have $P(E, \mathbb{R}^2) < \infty$, so that also $\mathcal{H}(\partial^* E) < \infty$. However, $\partial^* E = \mathbb{R}^2 \setminus E$. Thus, denoting $u := \chi_E$, for every Lebesgue point $x \in \mathbb{R}^2 \setminus E$ there exists a sequence $y_k \to x$ with $y_k \in E$ such that

$$u^\wedge(y_k) = u^\vee(y_k) = 1 \not\rightarrow 0 = u^\wedge(x) = u^\vee(x),$$

so that the conclusion of Theorem 1.1 fails with the choice $G = \emptyset$. On the other hand, given $\varepsilon > 0$, by choosing $G := \bigcup_{i=k}^\infty \overline{B(q_i, 2^{-i})}$ (or a slightly larger open set) with large enough $k$ we have that the conclusion of Theorem 1.1 holds.

Denote the 2-dimensional Lebesgue measure by $\mathcal{L}^2$. For every Lebesgue point $x \in \mathbb{R}^2 \setminus E$ and every $r > 0$ we have

$$0 < \frac{\mathcal{L}^2(B(x, r) \cap E)}{\mathcal{L}^2(B(x, r))} < 1,$$

and so $P(E, B(x, \lambda r)) > 0$ by the Poincaré inequality (2.4). Now by (2.7) we must have $\mathcal{H}(\partial^* E \cap B(x, \lambda r)) > 0$, and so $\partial^* E = \mathbb{R}^2 \setminus E$. 

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This example demonstrates that the measure theoretic boundary of a set of finite perimeter need not be closed, that it can be much smaller than the topological boundary, and that the conclusion of Theorem 1.1 can fail in a very large set if we choose $G = \emptyset$. However, from Theorem $1.1$, by removing a suitable set $G$ of small capacity, both the topological and measure theoretic boundaries of a set of finite perimeter become very reasonably behaved. For $A, E \subset X$, let us denote by $\partial_A E$ the boundary of $E \cap A$ in the subspace topology of $A$.

**Proposition 6.2.** Let $E \subset X$ be a set of finite perimeter. For $\varepsilon > 0$ let $G \subset X$ be an open set provided by Theorem $1.1$, with $\text{Cap}_1(G) < \varepsilon$. Then

$$ \partial_{X \setminus G} I(\varepsilon) \subset \partial^* E \setminus G, \tag{6.1} $$

and both $\partial_{X \setminus G} I(\varepsilon)$ and $\partial^* E \setminus G$ are closed subsets of $X$.

**Proof.** If $x \in \partial_{X \setminus G} I(\varepsilon)$, there are sequences $y_i$ in $I(\varepsilon) \setminus G$ converging to $x$ and $z_i$ in $X \setminus (I(\varepsilon) \cup G)$ also converging to $x$. Set $u := \chi_E$. Then $u^\wedge(y_i) = 1$ and $u^\wedge(z_i) = 0$ (note that we have either $z_i \in E$ or $z_i \in \partial^* E$). Thus by Theorem $1.1$, we must have $u^\wedge(x) = 0$ and $u^\vee(x) = 1$, that is, $x \in \partial^* E$.

Now we show that $\partial^* E \setminus G$ is closed in $X \setminus G$. If $x_i \in \partial^* E \setminus G$ with $x_i \rightarrow x \in X \setminus G$, then $u^\wedge(x_i) = 0$ and $u^\vee(x_i) = 1$ for all $i \in \mathbb{N}$, so again by Theorem $1.1$ we have $u^\wedge(x) = 0$ and $u^\vee(x) = 0$. Since $G$ is open, the sets $\partial_{X \setminus G} I(\varepsilon)$ and $\partial^* E \setminus G$ are closed also in $X$. □

**Lemma 6.3.** For $i \in \mathbb{N}$, let $G_i \subset X$ be a nested sequence of sets (that is, $G_{i+1} \subset G_i$) with $\text{Cap}_1(G_i) < 2^{-i}$. Let $\hat{\Gamma}$ be the family of non-constant curves that intersect each $G_i$. Then $\text{Mod}_1(\hat{\Gamma}) = 0$.

**Proof.** We will use the following observation in this proof. By [6, Theorem 1.56], every function in $N^{1,1}(X)$ is absolutely continuous on 1almost every curve in $X$.

For each $i \in \mathbb{N}$, take $u_i \in N^{1,1}(X)$ such that $0 \leq u_i \leq 1$ on $X$, $u_i \geq 1$ in $G_i$, and $\|u_i\|_{N^{1,1}(X)} < 2^{-i}$. The sequence $\{\sum_{i=1}^j u_i\}_{j \in \mathbb{N}}$ is a Cauchy sequence in $N^{1,1}(X)$, and converges therefore to $u := \sum_{i \in \mathbb{N}} u_i \in N^{1,1}(X)$ (a proof of the fact that $N^{1,1}(X)$ is a Banach space can be found in [32, 6]). Because for each $i$ we have $G_{i+1} \subset G_i$, we know that $u$ is not bounded on any of the curves in $\hat{\Gamma}$, and it follows that $u$ is not absolutely continuous on any of those curves. Now by the observation above, the desired conclusion follows. □
As a consequence of Proposition 6.2 and Lemma 6.3, we have the following analog of the result used in the proof of Federer’s theorem, in the metric setting.

**Corollary 6.4.** Let \( E \subset X \) be of finite perimeter. Let \( \Gamma \) be the collection of curves \( \gamma \) in \( X \) such that there exist \( t_0, t_1 \in [0, \ell_\gamma] \) with \( t_0 < t_1 \) and either

1. \( \gamma(t_0) \in \mathcal{I}(E), \gamma(t_1) \in \mathcal{E}(E) \) and \( \gamma([t_0, t_1]) \cap \partial^* E \) is empty, or

2. \( \gamma(t_0) \in \mathcal{E}(E), \gamma(t_1) \in \mathcal{I}(E) \) and \( \gamma([t_0, t_1]) \cap \partial^* E \) is empty.

Then \( \text{Mod}_1(\Gamma) = 0 \).

**Proof.** Let \( u := \chi_E \). For each \( i \in \mathbb{N} \), let \( G_i \) be an open set with \( \text{Cap}_1(G_i) < 2^{-i} \), given by Theorem 1.1. By replacing each \( G_i \) with \( \bigcap_{j=1}^i G_j \), if necessary, we may assume that for each \( i \in \mathbb{N} \), \( G_{i+1} \subset G_i \). Let \( \gamma \in \Gamma \). If there exists \( i \in \mathbb{N} \) such that \( \gamma \) does not intersect \( G_i \), then necessarily \( \gamma([t_0, t_1]) \cap \partial^* E \neq \emptyset \) according to (6.1). We conclude that \( \gamma \) intersects each set \( G_i \), that is, \( \Gamma \subset \hat{\Gamma} \), and \( \text{Mod}_1(\hat{\Gamma}) = 0 \) by Lemma 6.3. \( \square \)

Now we prove the following result that partially generalizes Federer’s structure theorem to the metric setting.

**Theorem 6.5.** Let \( E \subset X \) be bounded and \( \mu \)-measurable. Then \( E \) is of finite perimeter if and only if \( \mathcal{H}(\partial^* E) \) is finite and \( E \) satisfies the conclusion of Corollary 6.4.

**Proof.** One part of the claim follows directly from Corollary 6.4. Thus it suffices to prove that if \( E \) satisfies the conclusion of Corollary 6.4 and \( \mathcal{H}(\partial^* E) < \infty \), then \( E \) is of finite perimeter. To do so, it suffices to find an \( L^1 \)-approximation of \( \chi_E \) with \( L^1 \)-bounded weak upper gradients.

Since \( \mathcal{H}(\partial^* E) < \infty \), for each \( \varepsilon > 0 \) we can find a cover of \( \partial^* E \) by balls \( B_i = B(x_i, r_i), i \in \mathbb{N} \), with radius no more than \( \varepsilon \), such that

\[
\sum_{i \in \mathbb{N}} \frac{\mu(B_i)}{r_i} \leq \mathcal{H}(\partial^* E) + \varepsilon.
\]

For each ball \( B_i \) in the cover, we fix a \( 1/r_i \)-Lipschitz function \( u_i \) such that \( 0 \leq u_i \leq 1 \) on \( X \), \( u_i = 1 \) on \( B_i \), and the support of \( u_i \) is contained in \( 2B_i \). Now let

\[
u_{\varepsilon}(x) := \begin{cases} 1 & \text{if } x \in \mathcal{I}(E), \\ \min\{1, \sum_{i \in \mathbb{N}} u_i(x)\} & \text{otherwise.} \end{cases}
\]
Furthermore, let \( v_\varepsilon(x) := \min\{1, \sum_{i \in \mathbb{N}} u_i(x)\} \). Note that because \( E \) is bounded, \( u_\varepsilon \in L^1(X) \). Set
\[
g_\varepsilon := \sum_{i \in \mathbb{N}} \frac{1}{r_i} \chi_{2B_i}.
\]
Clearly \( g_\varepsilon \) is an upper gradient of \( v_\varepsilon \). We will show that \( g_\varepsilon \) is an upper gradient of \( u_\varepsilon \) as well. Take a curve \( \gamma \notin \Gamma \) with end points \( x, y \), where \( \Gamma \) was defined in Corollary 6.4. If \( x, y \in X \setminus \mathcal{I}(E) \), then
\[
|u_\varepsilon(x) - u_\varepsilon(y)| = |v_\varepsilon(x) - v_\varepsilon(y)| \leq \int_{\gamma} g_\varepsilon \, ds.
\]
If the end points \( x, y \) both lie in \( \mathcal{I}(E) \), then \( u_\varepsilon(x) = u_\varepsilon(y) \), and hence the upper gradient inequality
\[
|u_\varepsilon(x) - u_\varepsilon(y)| \leq \int_{\gamma} g_\varepsilon \, ds \tag{6.2}
\]
is satisfied. If \( x \in \mathcal{I}(E) \) and \( y \in X \setminus \mathcal{I}(E) \cup \bigcup_{i \in \mathbb{N}} 2B_i \subset \mathcal{E}(E) \), then \( |u_\varepsilon(x) - u_\varepsilon(y)| = 1 \), and since \( \gamma \notin \Gamma \), the curve \( \gamma \) intersects \( \partial^* E \), and so it intersects \( B_j \) for some \( j \) and also intersects the complement of \( 2B_j \). Thus
\[
\int_{\gamma} g_\varepsilon \, ds \geq \frac{1}{r_j} \int_{\gamma} \chi_{2B_j} \, ds \geq \frac{r_j}{r_j} \geq 1.
\]
So again the pair \( u_\varepsilon, g_\varepsilon \) satisfies the upper gradient inequality (6.2).

Finally, if \( x \in \mathcal{I}(E) \) and \( y \in \bigcup_{i \in \mathbb{N}} 2B_i \setminus \mathcal{I}(E) \), again since \( \gamma \notin \Gamma \), there is some \( t_0 \in [0, \ell_\gamma] \) such that \( \gamma(t_0) \in \partial^* E \), and thus \( \gamma(t_0) \in B_j \) for some \( j \in \mathbb{N} \). Note that \( u_\varepsilon(x) = u_\varepsilon(\gamma(0)) = 1 \), \( u_\varepsilon(\gamma(t_0)) = v_\varepsilon(\gamma(t_0)) = 1 \), and \( u_\varepsilon(y) = v_\varepsilon(y) \). It follows that
\[
|u_\varepsilon(x) - u_\varepsilon(y)| \leq |u_\varepsilon(\gamma(0)) - u_\varepsilon(\gamma(t_0))| + |u_\varepsilon(\gamma(t_0)) - u_\varepsilon(\gamma(\ell_\gamma))| \leq \int_{\gamma} g_\varepsilon \, ds.
\]
Thus in all cases the pair \( u_\varepsilon, g_\varepsilon \) satisfies the upper gradient inequality for 1-almost every curve in \( X \). Furthermore,
\[
\int_X g_\varepsilon \, d\mu \leq \sum_{i \in \mathbb{N}} \frac{\mu(2B_i)}{r_i} \leq C_d \sum_{i \in \mathbb{N}} \frac{\mu(B_i)}{r_i} \leq C_d (\mathcal{H}(\partial^* E) + \varepsilon) < \infty.
\]
It follows that for $0 < \varepsilon \leq 1$, $u_\varepsilon \in N^{1,1}(X)$ with 1-weak upper gradients $g_\varepsilon$ with a bounded $L^1$-norm. Moreover,

$$
\int_X |u_\varepsilon - \chi_E| \, d\mu \leq \int_X \chi_{\bigcup_{i \in N} 2B_i} \, d\mu \leq \varepsilon \sum_{i \in N} \frac{\mu(2B_i)}{r_i} \\
\leq \varepsilon (\mathcal{H}(\partial^* E) + 1) \to 0
$$

as $\varepsilon \to 0$. It follows that $u_\varepsilon \to \chi_E$ in $L^1(X)$, and thus $\chi_E \in \text{BV}(X)$, that is, $E$ is of finite perimeter.

## 7 Strong quasicontinuity

It is known that if the measure on a metric measure space $X$ is doubling and supports a $(1,p)$-Poincaré inequality for some $1 \leq p < \infty$, then Lipschitz functions are dense in $N^{1,p}(X)$, see for example [20, Theorem 8.2.1]. Similarly, Lemma 4.3 shows that continuous functions are dense in the space of BV functions with a $\mathcal{H}$-negligible jump set. On the other hand, from Proposition 4.5 we know that the restrictions of BV functions with a $\mathcal{H}$-negligible jump set outside sets of small capacity are continuous, just like the restrictions of Newton-Sobolev functions.

The concept of strong quasicontinuity essentially combines these two results: it involves a Lusin-type approximation of a function $u$ by a continuous function that approximates $u$ simultaneously in the BV (or Newton-Sobolev) norm and outside a set of small capacity. In [22, Theorem 7.1] such a Lusin-type approximation result for Newton-Sobolev functions was given. Here we show strong quasicontinuity for BV functions with a $\mathcal{H}$-negligible jump set. Note that such BV functions need not be in the Newton-Sobolev class, since the Cantor part of their variation measure need not be zero.

**Lemma 7.1.** Let $u \in \text{BV}(X)$ with $\mathcal{H}(S_u) = 0$, and let $\varepsilon > 0$. Then there exists $G \subset X$ with $\text{Cap}_1(G) < \varepsilon$ such that

$$
\lim_{r \to 0^+} \frac{\|Du\|(B(x,r))}{\mu(B(x,r))} \to 0 \quad \text{as } r \to 0
$$

uniformly in $X \setminus G$.

**Proof.** Given $\delta > 0$, let

$$
A := \left\{ x \in X : \limsup_{r \to 0^+} \frac{\|Du\|(B(x,r))}{\mu(B(x,r))} \geq \delta \right\}.
$$
By [5, Theorem 2.4.3], we know that \( \|Du\|(A) \geq \delta \mathcal{H}(A) \). Now by [4, Theorem 5.3], since \( \|Du\|(X) = 0 \), we have \( \|Du\|(F) = 0 \) for any \( F \) with \( \mathcal{H}(F) < \infty \), and so we must have \( \|Du\|(A) = \mathcal{H}(A) = 0 \). It follows that

\[
\mathcal{H} \left( \left\{ x \in X : \limsup_{r \to 0^+} r \frac{\|Du\|(B(x, r))}{\mu(B(x, r))} > 0 \right\} \right) = 0.
\]

By Egorov’s theorem, we can pick compact sets \( H_1 \subset H_2 \subset \ldots \) and radii \( \frac{1}{5} \geq r_1 \geq r_2 \geq \ldots > 0 \) such that \( \|Du\|(X \setminus H_i) < 2^{-i} \varepsilon \), and

\[
r \frac{\|Du\|(B(x, r))}{\mu(B(x, r))} \leq \frac{1}{C_d^2 i}
\]

for all \( x \in H_i \) and \( r \in (0, 2r_i] \). Then define for \( i \in \mathbb{N} \)

\[
G_i := \left\{ x \in X \setminus H_i : \exists r \in (0, r_i] \text{ s.t. } B(x, r) \subset X \setminus H_i \text{ and } r \frac{\|Du\|(B(x, r))}{\mu(B(x, r))} > \frac{1}{i} \right\}.
\]

Now we show that for all \( x \in X \setminus G_i \) and \( r \in (0, r_i] \),

\[
r \frac{\|Du\|(B(x, r))}{\mu(B(x, r))} \leq \frac{1}{i}.
\] (7.1)

The only case that needs to be checked is when \( x \in X \setminus (H_i \cup G_i) \) and \( B(x, r) \cap H_i \neq \emptyset \) for some \( r \in (0, r_i] \). Then for any point \( y \in B(x, r) \cap H_i \), we have

\[
r \frac{\|Du\|(B(x, r))}{\mu(B(x, r))} \leq C_d^2 r \frac{\|Du\|(B(y, 2r))}{\mu(B(y, 2r))} \leq \frac{1}{i}
\]

by the definition of the sets \( H_i \).

Fix \( i \in \mathbb{N} \). From the definition of \( G_i \) we get a covering \( \{B(x, r(x))\}_{x \in G_i} \) of \( G_i \), and by the 5-covering theorem, we obtain a countable collection of disjoint balls \( \{B(x_j, r_j)\}_{j \in \mathbb{N}} \) such that the balls \( B(x_j, 5r_j) \) cover \( G_i \). Thus

\[
\frac{1}{2C_d} \text{Cap}_1(G_i) \leq \mathcal{H}_1(G_i) \leq \sum_{j \in \mathbb{N}} \frac{\mu(B(x_j, 5r_j))}{5r_j} \leq C_d^3 \sum_{j \in \mathbb{N}} \frac{\mu(B(x_j, r_j))}{r_j} \leq C_d^3 \sum_{j \in \mathbb{N}} \|Du\|(B(x_j, r_j)) \leq C_d^3 \|Du\|(X \setminus H_i).
\]
Let $G := \bigcup_{i \in \mathbb{N}} G_i$, so that
\[
\text{Cap}_1(G) \leq \sum_{i \in \mathbb{N}} \text{Cap}_1(G_i) \leq C \sum_{i \in \mathbb{N}} i \|Du\| (X \setminus H_i) \leq C \sum_{i \in \mathbb{N}} i 2^{-i} \varepsilon \leq C \varepsilon.
\]
Moreover, by (7.1), for every $x \in X \setminus G$, $i \in \mathbb{N}$, and $r \in (0, r_i]$ we have
\[
r \frac{\|Du\| (B(x, r))}{\mu(B(x, r))} \leq \frac{1}{i}.
\]

**Proposition 7.2.** Let $u \in \text{BV}(X)$ with $\mathcal{H}(S_u) = 0$, and let $\varepsilon > 0$. Then there exists $G \subset X$ with $\text{Cap}_1(G) < \varepsilon$ such that
\[
\int_{B(x, r)} |u - \tilde{u}(x)| \, d\mu \to 0 \quad \text{as } r \to 0
\]
locally uniformly in the set $X \setminus G$.

**Proof.** Our proof largely follows corresponding proofs concerning Lebesgue points of Newton-Sobolev functions, see e.g. [23, Theorem 4.1]. First note that
\[
\begin{align*}
\int_{B(x, r)} |u - \tilde{u}(x)| \, d\mu &\leq \int_{B(x, r)} |u - u_B(x, r)| \, d\mu + |u_B(x, r) - \tilde{u}(x)| \\
&\leq Cr \frac{\|Du\| (B(x, \lambda r))}{\mu(B(x, \lambda r))} + |u_B(x, r) - \tilde{u}(x)|.
\end{align*}
\]
The first term converges uniformly to zero as $r \to 0$ outside a set $F$ with $\text{Cap}_1(F) < \varepsilon/2$ by Lemma 7.1. So we only need to consider the second term. By Lemma 4.3 there is a sequence $u_i \in \text{BV}(X) \cap C(X)$ with
\[
\|u_i - u\|_{\text{BV}(X)} \leq \frac{2^{-2i-2\gamma}}{C_1} \varepsilon,
\]
where $C_1$ is the constant from Lemma 2.1 corresponding to the choice $R = 1$. For $i \in \mathbb{N}$, let
\[
G_i := \{x \in X : \max\{|u_i(x) - \tilde{u}(x)|, M_1(u_i - u)(x)\} > 2^{-i}\}.
\]
By Lemma 2.1 and the proof of Lemma 4.2, \( \text{Cap}_1(G_i) \leq 2^{-i-1}\varepsilon \). Define 
\[ G := \bigcup_{i \in \mathbb{N}} G_i \cup F, \]
so that \( \text{Cap}_1(G) < \varepsilon \). Now for \( x \in X \setminus G \) and \( r \in (0,1] \),
\[ |u_{B(x,r)} - \tilde{u}(x)| \leq |u_{B(x,r)} - (u_i)_{B(x,r)}| + |(u_i)_{B(x,r)} - u_i(x)| + |u_i(x) - \tilde{u}(x)| \]
\[ \leq \mathcal{M}_1(u_i - u)(x) + |(u_i)_{B(x,r)} - u_i(x)| + |u_i(x) - \tilde{u}(x)| \]
\[ \leq 2^{-i} + |(u_i)_{B(x,r)} - u_i(x)| + 2^{-i} \]
\[ \leq 2^{-i+1} \int_{B(x,r)} |u_i - u_i(x)| \, d\mu. \]

Fix a ball \( B(z, \tilde{r}), \) and \( \delta > 0 \). Picking \( i \) sufficiently large, the first term above is less than \( \delta/2 \). Then the corresponding function \( u_i \) is, as a continuous function, locally uniformly continuous, so that it is uniformly continuous in \( B(z, \tilde{r} + 1) \). Thus we can pick \( r > 0 \) small enough that the second term is less than \( \delta/2 \) for every \( x \in B(z, \tilde{r}) \). Since \( \delta > 0 \) was arbitrary, this establishes local uniform convergence.

**Theorem 7.3.** Let \( u \in \text{BV}(X) \) with \( \mathcal{H}(S_u) = 0 \), and let \( \varepsilon > 0 \). Then there exists an open set \( G \subset X \) with \( \text{Cap}_1(G) < \varepsilon \), and \( w \in \text{BV}(X) \cap C(X) \) such that \( w = \tilde{u} \) in \( X \setminus G \) and \( \|w - u\|_{\text{BV}(X)} < \varepsilon \).

**Proof.** By Lusin’s and Egorov’s theorems, we can find an open set \( F \subset X \) with \( \|Du\|(F) < \varepsilon \) such that \( \tilde{u}|_{X \setminus F} \) is continuous and
\[ \int_{B(x,r)} |u - \tilde{u}(x)| \, d\mu \to 0 \quad \text{as } r \to 0 \quad (7.2) \]
uniformly in the set \( X \setminus F \). By Theorem 1.1 and Proposition 7.2 and the fact that \( \text{Cap}_1 \) is an outer capacity, we can find an open set \( \tilde{G} \subset X \) with \( \text{Cap}_1(\tilde{G}) < \varepsilon \) such that \( \tilde{u}|_{X \setminus \tilde{G}} \) is continuous and the convergence in (7.2) is locally uniform in the set \( X \setminus \tilde{G} \). Defining \( G := \tilde{G} \cap F \), we have \( \text{Cap}_1(G) < \varepsilon \) and \( \|Du\|(G) < \varepsilon \). Apply Corollary 3.6 with \( U = G \) and \( \kappa = \varepsilon \) to obtain a function \( w \in \text{BV}(X) \) with \( \|w - u\|_{\text{BV}(X)} \leq C\varepsilon \). Then by Proposition 3.7, \( w \in C(X) \) and \( w = \tilde{u} \) in \( X \setminus G \).

We say that \( X \) supports a **strong relative isoperimetric inequality** if for every \( \mu \)-measurable set \( E \subset X \), \( P(E, X) < \infty \) whenever \( \mathcal{H}(\partial^* E) < \infty \), see the discussion in Section 6 as well as [22] and [27] for more on this question. In [22, Theorem 7.1] the following Lusin-type approximation for Newton-Sobolev functions was given. The authors made the additional assumption
that the space supports a strong relative isoperimetric inequality, which we can now remove.

**Corollary 7.4.** Let \( 1 \leq p < \infty \), \( u \in N^{1,p}(X) \), and \( \varepsilon > 0 \). Then there exists an open set \( G \subset X \) and \( w \in N^{1,p}(X) \cap C(X) \) such that \( \text{Cap}_p(G) < \varepsilon \), \( w = \tilde{w} \) in \( X \setminus G \), and \( \|w - u\|_{N^{1,p}(X)} < \varepsilon \).

**Proof.** When \( p = 1 \), this is a special case of Theorem 7.3, since \( \|w - u\|_{N^{1,1}(X)} \leq C\|w - u\|_{BV(X)} \), see [17, Theorem 4.6]. The case \( 1 < p < \infty \) follows by suitably adapting Theorem 3.1 (see [22, Theorem 1.1]), Theorem 3.5, Proposition 3.7 (the same proof applies), and Proposition 7.2, combined with the \( p \)-quasicontinuity of \( u \in N^{1,p}(X) \).

In this section so far, we have only dealt with BV functions with a \( \mathcal{H} \)-negligible jump set. A strong version of our quasicontinuity-type result, Theorem 1.1, would be the following. Note that below we require (7.3) to hold everywhere, not just outside a set of small capacity.

**Open Problem.** Let \( u \in BV(X) \) and let \( \varepsilon > 0 \). Then there exists an open set \( G \subset X \) with \( \text{Cap}_1(G) < \varepsilon \), and \( w \in BV(X) \) such that \( w^l = u^l \) in \( X \setminus G \) for all \( l = 1, \ldots, n \), \( \|w - u\|_{BV(X)} < \varepsilon \), and whenever \( y_k \to x \in X \),

\[
\min_{l_2 \in \{1, \ldots, n\}} |w^{l_1}(y_k) - w^{l_2}(x)| \to 0 \tag{7.3}
\]

for each \( l_1 = 1, \ldots, n \).

Though we can pick a set \( G \) as in Theorem 1.1, it is not obvious how the function \( w \) should be defined in \( G \) to ensure that (7.3) holds. On the other hand, we do get the following Lusin-type approximation for general BV functions.

**Theorem 7.5.** Let \( u \in BV(X) \) and \( \varepsilon > 0 \). Then for any open set \( W \supset S_u \) there exists an open set \( V \supset W \) with \( \text{Cap}_1(V \setminus W) < \varepsilon \), and a function \( v \in BV(X) \cap C(X) \) with \( v = \tilde{u} \) in \( X \setminus V \) and

\[
\|v - u\|_{L^1(X)} \leq \varepsilon, \quad \|D(v - u)\|(X) \leq C\|Du\|^j(X) + \varepsilon \tag{7.4}
\]

For example, we can require \( W \) and hence \( V \) to have \( \mu \)-measure less than \( \varepsilon \). This theorem also gives better control of \( \|D(v - u)\|(X) \) than a Lusin-type approximation by a Lipschitz function given in [24, Proposition 4.3], but on the downside, we only get an approximation by a continuous function.
Proof. By making $W$ smaller, if necessary, we can assume that $\|Du\|(W) \leq \|Du\|(S_u) + \varepsilon$. Apply Corollary 3.6 with $U = W$ and $\kappa = \varepsilon/2$ to obtain a function $w \in BV(X)$ with $w = u$ in $X \setminus W$, $\|w - u\|_{L^1(X)} \leq \varepsilon/2$, and

$$\|D(w - u)\|(X) = \|D(w - u)\|(W) \leq C\|Du\|(W) \leq C\|Du\|(S_u) + C\varepsilon.$$ 

Note that by (3.10), we have in fact $w^\wedge = w^\vee = \tilde{u}$ in $X \setminus (W \cup \tilde{N})$, for some $\mathcal{H}$-negligible set $\tilde{N} \subset X$. By Remark 4.1 there exists an open set $N \supset \tilde{N}$ with $\text{Cap}_1(N) < \varepsilon/2$. Furthermore, $\mathcal{H}(S_w) = 0$, so that we can apply Theorem 7.3 to get an open set $G \subset X$ with $\text{Cap}_1(G) < \varepsilon/2$ and a function $v \in BV(X) \cap C(X)$ with $v = \tilde{w}$ in $X \setminus G$ and $\|v - w\|_{BV(X)} \leq \varepsilon/2$. Thus for $V := W \cup N \cup G$ we have $v = \tilde{u}$ in $X \setminus V$, $\|v - u\|_{L^1(X)} \leq \varepsilon$, and

$$\|D(v - u)\|(X) \leq C\|Du\|(S_u) + C\varepsilon = C\|Du\|_{1}(X) + C\varepsilon.$$ 

If $X$ supports a strong relative isoperimetric inequality, we can use the proposition below instead of Corollary 3.6 in the proof of Theorem 7.5 and then we will get (7.4) with the constant $C = 2 + \varepsilon$.

**Proposition 7.6** ([29, Corollary 6.7]). Let $U \subset X$ be an open set, and let $u \in BV(U)$. Assume either that the space supports a strong relative isoperimetric inequality, or that $\mathcal{H}(\partial U) < \infty$. Then there exist functions $\hat{v}_i \in \text{Lip}_{\text{loc}}(U)$, $i \in \mathbb{N}$, with $\hat{v}_i \to u$ in $L^1(U)$, $\|D\hat{v}_i\|(U) \to \|Du\|(U)$, and such that the functions

$$h_i := \begin{cases} 
\hat{v}_i - u & \text{in } U, \\
0 & \text{in } X \setminus U,
\end{cases}$$

satisfy $h_i \in BV(X)$ with $\|Dh_i\|(X \setminus U) = 0$.

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