Homeostasis is an active, self-regulating process by which an unstable state is stabilized against external perturbations. Such processes are ubiquitous in nature, but energetic resources needed for their existence were not studied systematically. Here we undertake such a study using the mechanical model of inverted pendulum, where its upper (normally unstable) state is stabilized due to a fast controlling degree of freedom and due to friction. It is shown that the stabilization itself does not need constant dissipation of energy. There is only a transient dissipation of energy related to the relaxation to the stable state. The stabilization is achieved not due to a constant dissipation of energy, but due to the energy initially stored in the controlling degree of freedom. In particular, the stored energy is needed for ensuring stability against multiple perturbations, a notion of stability that differs from the usual asymptotic stability against a single perturbation.

I. INTRODUCTION

Homeostasis is one of central concepts in biology and physiology [1–4] and is defined as stability of certain extensive relevant parameters (concentrations, coordinates etc) with respect to a class of external perturbations. For cases of biological relevance, this stability is achieved by active means, viz. via specific engines or controllers [3]. No stability will exist without their continuous action. This is an important point that distinguishes homeostasis from standard examples of physical stability that are achieved by passive means.

The general explanation for why homeostasis is needed is that it offers energetically cheaper realizations of various physiological functions [1]. This makes important to ask about its own energetic costs: how much energy is to be dissipated for realizing homeostasis? The energy cost problem was actively studied for the case of adaptation [11], where the stability is required with respect to external changes of intensive variables; e.g. temperature, chemical potential etc [5–8, 11]. Adaptation should be distinguished from homeostasis [12]. Adaptation is about stability with respect to intensive variables (e.g. temperature). It relates to structural changes in the system, while the homeostasis need not. Hence, adaptation is essentially restricted by the Le Chatelier-Braun principle, whereas homeostasis is not [6, 12, 13]. The existing approaches to energy cost of adaptation show that the energy is to be dissipated continuously if an adaptive state is maintained [5–8]. In that sense adaptation is similar to proof-reading and motor transport, biological processes that are essentially non-equilibrium and demand constant dissipation of energy; see [9] for a review.

Hence the energy cost of homeostasis—i.e. stabilizing the unstable state by active means—is an open problem. Its study is complicated by the fact that realistic models of homeostasis come from system biology [1, 11], i.e. they frequently do not have a physically consistent form that allows to ask questions about energy balance, let alone its dissipation. Given this situation, we decided to start studying the energy cost problem by addressing the simplest, but non-trivial model that demonstrates active homeostasis. This is the driven non-linear pendulum, whose upper (normally unstable state) can be stabilized by a sufficiently fast external force. Such models were first studied by Stephenon [14], and then by Kapitza [15, 16]; see [17] for a review. They still produce new physical results [18, 19]. The inverted pendulum and related models are also actively studied in animal locomotion [20], control theory [21] and robotics [22]. Thus the family of inverted pendulums models is foundational both for physics and control theory.

To study the energy cost problem we shall first make the driven pendulum completely autonomous via replacing the external field by a controller degree of freedom. This ensures finite energies and a complete accounting of all relevant degrees of freedom; see section II. The model is approximately solved in section III, where we show that the asymptotic stability of the inverted pendulum is determined (among other factors) by the energy stored in the

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1 To emphasize this aspect, “homeo” in homeostasis means “similar”, and not “same” [2]. Ref. [3] proposed a term homeokinetics for this.
2 For example, relaxation of an oscillator to its lowest energy state under friction is a passive process that does not need energy resources, though an energy sink (friction) is necessary.
3 The terminology here is messy; e.g. the famous model by Ashby is called homeostat [10], while it is certainly about adaptation.
controller. This generalizes the stability criterion presented in [15–17]. In section IV we confirm the analytic bound, and also show that the asymptotic stability of the (originally unstable) inverted state demands moderate energy costs during the transient relaxation process. Once the relaxation is over, the pendulum automatically decouples from the controller; hence no continuous dissipation of energy is present (or needed). The coupling emerges on-line together with an external perturbation.

The notion of asymptotic stability is however not sufficient for characterizing the homeostasis, because there will never be a single external perturbation: it is more realistic to consider a sequence of well-separated perturbations within the attraction basin. For an actively stabilized system (such as the inverted pendulum) this notion does not reduce to the asymptotic stability. Indeed, in section IV we show that when the back-reaction from the pendulum to the controller is moderate (or large), the unstable state becomes asymptotically stable with a definite attraction basin, but it is not stable with respect to multiple well-separated perturbations. To achieve this stability the back-reaction to the controller has to be sufficiently small. Moreover, if the back-reaction is sizable, the asymptotic stability is replaced by a metastable stabilization that has a finite (though possibly long) life-time: the controller slowly but steadily dissipates energy for supporting the metastable state. Once this energy is lower than a certain threshold, the metastable state suddenly decays with dissipating away all the energy. Our results are summarized in the last section, where we also discuss drawbacks of this model.

II. PENDULUM WITH MOVING END

Consider a pendulum moving on \((x, y)\) plane in a homogeneous gravity field \(g\); see Fig. 1. The pendulum is a material point with mass \(m\) fixed at one end of a rigid rod with length \(l\). Let the coordinates and velocities of the mass be

\[
\begin{align*}
  x &= l \sin \varphi, \quad \dot{x} \equiv \frac{dx}{dt} = l \cos \varphi \cdot \dot{\varphi}, \\
  y &= \xi_0 - l \cos \varphi, \quad \dot{y} \equiv \frac{dy}{dt} = \dot{\xi}_0 + l \sin \varphi \cdot \dot{\varphi},
\end{align*}
\]

where \(\xi_0\) refers to vertical (along \(y\)-axes) motion of the opposite end of the road. The Lagrangian of the autonomous system with coordinates \((x, y, \xi)\) reads

\[
L = \frac{m}{2} [\dot{x}^2 + \dot{y}^2] - mgy + \frac{\mu}{2} \dot{\xi}_0^2 - \frac{k}{2} \xi_0^2, \tag{3}
\]

\(\text{This is certainly the case for a passively stabilized system, e.g. a harmonic oscillator under friction: if external perturbations are separated by the relaxation time, they are independent from each other.}\)
where \( \mu \) is the mass of \( \xi_0 \), and we assumed a harmonic potential \( k\xi_0^2 \) for \( \xi_0 \). Putting (1, 2) into (3), denoting \( \xi = \xi_0 + \frac{mg}{k} \), and dropping a constant term from Lagrangian we find

\[
L = \frac{m l^2}{2} \dot{\varphi}^2 + \frac{\mu + m}{2} \dot{\xi}^2 + m\ddot{\xi}\dot{\varphi}\sin\varphi + mgl\cos\varphi - \frac{k}{2}\xi^2,
\]

which implies Lagrangian equations of motion \( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} \) with coordinates \( q = (\varphi, \xi) \) and velocities \( \dot{q} = (\dot{\varphi}, \dot{\xi}) \):

\[
\ddot{\varphi} + \frac{g}{l}\sin\varphi = -\frac{1}{l}\sin\varphi \cdot \dddot{\xi},
\]

\[
\ddot{\xi} + \omega^2\xi = -\epsilon \left[ \cos \varphi \cdot \dot{\varphi}^2 + \sin \varphi \cdot \dddot{\varphi} \right],
\]

\[
\omega^2 \equiv \frac{k}{(\mu + m)}, \quad \epsilon \equiv \frac{l}{1 + \frac{\mu}{m}},
\]

where \( \omega \) is the frequency of \( \xi \), while \( \epsilon \) characterizes the back-reaction of \( x \) on \( \xi \). Note from (3, 6) that whenever \( \mu \gg m \), i.e. the controller is much heavier than the pendulum, we can neglect the left-hand-side of (6), and revert to the usual (non-autonomous) driven pendulum. This, however, does not suffice for the full understanding of energy costs, which arise due to the very back-reaction of \( x \) on \( \xi \).

Eqs. (5, 6) are deduced from the time-independent Lagrangian (4); hence they are conservative and reversible. The conserved energy related to (4) reads:

\[
E = \frac{ml^2}{2} \dot{\varphi}^2 + \frac{\mu + m}{2} \dot{\xi}^2 + m\ddot{\xi}\dot{\varphi}\sin\varphi - mgl\cos\varphi + \frac{k}{2}\xi^2.
\]

We shall now add a friction with parameter \( \gamma > 0 \) to (5) writing it as

\[
\ddot{\varphi} + \frac{g}{l}\sin\varphi + \gamma \dot{\varphi} = -\frac{1}{l}\sin\varphi \cdot \dddot{\xi}.
\]

As seen below, this friction is a means of stabilizing the motion of \( \varphi \). We do not add a friction to the controller degree of freedom \( \xi \), since this will reach no constructive goal besides providing an additional channel for losing energy. We shall also mostly neglect random noises acting on \( \varphi \), i.e. we do not study Langevin equations. The influence of a random noise is discussed in section IV B.

Note that energy (8) governed by (6, 9) decays in time as it should:

\[
\frac{dE}{dt} = -ml^2\gamma \dot{\varphi}^2 \leq 0.
\]

As confirmed below, it is useful to separate the energy \( E \) in (8) into two contributions, those describing the motion of \( \xi \) and \( \varphi \):

\[
E = E_\varphi + E_\xi, \quad \quad E_\xi = \frac{\mu + m}{2} \dot{\xi}^2 + \frac{k}{2}\xi^2.
\]

III. SOLVING THE MODEL VIA SLOW AND FAST VARIABLES

To solve non-linear (6, 9), we shall apply both separation of time-scales and perturbation theory. We assume that \( \omega \) in (6) is a large parameter, i.e. \( \xi \) oscillates fastly. Next, we separate \( \varphi \) into slow \( \Phi \) and fast \( \zeta \) components [16]:

\[
\varphi = \Phi + \zeta, \quad \zeta \ll \Phi
\]

where \( \Phi \) is the slow part and \( \zeta \) is the fast part, which is also small compared with \( \Phi \), as verified below. Then from (6, 9, 12) we get after expanding over \( \zeta \) and keeping the first non-vanishing term only:

\[
\ddot{\Phi} + \ddot{\zeta} + \frac{g}{l}\sin\Phi + \zeta g l \cos\Phi + \gamma \dot{\Phi} + \gamma \dot{\zeta} = -\frac{1}{l}\sin\Phi \cdot \dddot{\xi} - \frac{1}{l}\cos\Phi \cdot \zeta \dddot{\xi},
\]

\[
\ddot{\xi} + \omega^2\xi = -\epsilon[\cos\Phi - \zeta \sin\Phi)(\Phi + \dot{\zeta})^2 + (\sin\Phi + \zeta \cos\Phi)(\dot{\Phi} + \dot{\zeta})].
\]
Assuming that $\gamma \gtrsim \frac{1}{\mu}$ we can equalize fast and big components in (8) and have:

$$\ddot{\zeta} + \gamma \dot{\zeta} = -\frac{1}{l} \sin \Phi \cdot \dot{\zeta}, \quad \ddot{\xi} + \omega^2 \xi = -\epsilon \sin \Phi \cdot \dot{\zeta},$$

(15)

$$\zeta(0) = 0,$$

(16)

$$\xi(0) = 0,$$

(17)

where initial condition (17) is imposed without loss of generality; cf. (12). Note that (15, 16) do not contain the contribution $\frac{g}{l} \cos \Phi$ coming from the potential $-mgl \cos \varphi$ [cf. (13)], since this contribution is not sufficiently fast, i.e. (15, 16) involve only time-derivatives of $\zeta$.

Eq. (15) can be integrated over the time. A constant of integration should be put to zero, since $\zeta$ and $\xi$ are oscillating in time with their time-average being zero. Hence the integration of (15) implies

$$\dot{\zeta}(0) = -\frac{\sin \Phi}{l} \xi(0).$$

(18)

Once (15, 16) are solved via the Laplace transform [see Appendix], we can average (13) by time:

$$\bar{\Phi} + \frac{g}{l} \sin \Phi + \gamma \dot{\Phi} + \frac{1}{l} \cos \Phi \cdot \bar{\zeta}(t) \bar{\xi}(t) = 0,$$

(19)

and get in (19) an effective (generally time-dependent) potential $\Pi(\Phi)$ [see Appendix]:

$$\ddot{\Phi} + \gamma \dot{\Phi} = -\partial_\Phi \Pi(\Phi).$$

(20)

The form of this potential simplifies if we assume that $\beta = \frac{\sin^2 \Phi}{1 + \frac{\mu}{m}} \ll 1$, due to $\frac{\mu}{m} \gg 1$ (no back-reaction), and take the first non-vanishing term over $\beta$ (this approximation is supported when the slow variable $\Phi$ relaxes to $\pi$ or to $0$):

$$\Pi = \frac{g}{l} \cos \Phi - \frac{\omega^2}{\gamma^2 + \omega^2} \left(\ddot{\Phi}(0) + \frac{\omega^2 \xi^2(0)}{8l^2} \right) \cdot \cos 2\Phi.$$  

(21)

Now $\Phi = \pi$ is stable, i.e. $\partial_\Phi \Pi(\Phi)|_{\Phi=\pi} = 0$ and $\partial^2_\Phi \Pi(\Phi)|_{\Phi=\pi} > 0$, if:

$$\frac{\ddot{\Phi}(0) + \omega^2 \xi^2(0)}{2gl} > 1 + \frac{\gamma^2}{\omega^2}. $$

(22)

Note that larger values of $\omega$ expectedly increase the stability domain. However, larger values of the friction constant $\gamma$ decrease it. Hence, the friction plays a double role in this system, since the very relaxation of $\Phi(t)$ (e.g. $\Phi(t) \to \pi$) is achieved due to friction 5.

Eqs. (20, 21) imply that when $\Phi = \pi$ is a stable rest point, we get that the slow part $\Phi(t)$ of the angle variable $\phi(t)$ convergence due to the friction: $\Phi(t) \to \pi$, if $\Phi(0)$ is in the attraction basin 6 of $\Phi = \pi$. What happens then to the fast part $\zeta(t)$ of $\varphi(t)$? This is a convoluted question that we clarify below numerically. Our results in Appendix show that when the derivation (12–18) applies—i.e. both the time-scale separation and the perturbation hold—we get $\zeta(t) \to 0$ together with $\Phi(t) \to \pi$; see (A5). This is indeed observed numerically, as seen below. However, there is also a regime that is not described by (12–18), where $\Phi(t) \to \pi$ for sufficiently long, but finite times, and where $\zeta(t)$ stays non-zero; see below for details. Eventually, this fact leads to decaying of the $\Phi = \pi$ state, i.e. $\Phi = \pi$ turns out to be a metastable state.

Finally, note that the left-hand-side of (22) is just the initial dimensionless energy of $\zeta$; cf. (7, 11). We call this quantity the initial stored energy and remind again that (22) was obtained for vanishing back-reaction $\epsilon \to 0$.

5 Note that Ref. [23] studied the inverted pendulum with friction and deduced an effective potential that is akin to (21) (and even contains higher-order terms), but does not contain friction explicitly, since the latter was assumed to be small.

6 Note that the fact of stabilizing the unstable rest-point $\phi = \pi$ of the potential $-mgl \cos \varphi$ is due to the choice of this potential. Put differently, would we choose this potential as $-mgl (\varphi - \varphi_0)$, then the effective potential will still have the form $\frac{\omega^2}{\gamma^2 + \omega^2} \left(\ddot{\Phi}(0) + \omega^2 \xi^2(0)\right) \cos 2\Phi$; cf. (21). Recall that (15, 16) for fast variables—that eventually create the effective potential—do not contain the contribution coming from the potential. For the potential $-mgl (\varphi - \varphi_0)$, $\Phi = \pi$ is (for a general $\varphi_0$) not a stable rest-point.
IV. SCENARIOS OF (DE)STABILIZATION FOR THE INVERTED PENDULUM

A. Asymptotic stability vs. stability with respect to several perturbations

Fig. 2(a) shows solution of (6, 9) when condition (22) holds. It is seen that indeed the state $\varphi = \pi$ gets asymptotically stabilized, $\varphi(t) \to \pi$ at least when $\Phi(0)$ is sufficiently close to $\pi$, i.e. $\varphi(0)$ is in the attraction basin of $\pi$. This relaxation is accompanied by the energy dissipation. The decaying (dissipating) quantity here is mostly the stored energy $\frac{1}{m} [\dot{\xi}^2(t) + \omega^2 \xi^2(t)]$, i.e. the dimensionless energy related to $E_\xi$ in (11). Once $\varphi$ approaches $\pi$ sufficiently close, the coupling between $\xi$ and $\varphi$ is switched off; see (4). This means that the stored energy is not anymore dissipated and stays constant for subsequent times; see Fig. 2(a).

For parameters of Fig. 2(a) the point $\varphi = \pi$ is asymptotically stable with a well-defined attraction basin; i.e. it is stable with respect to a single perturbation, which refers to the initial state $\varphi(0) = 0.93\pi$, $\dot{\varphi}(0) = 0$, and specific initial state $(\xi(0), \dot{\xi}(0))$ of $\xi$; see Fig. 2(a). In the context of homeostasis it is necessary to generalize the notion of asymptotic stability, because it is unrealistic to consider only a single perturbation. Let us assume that after $\varphi(t)$ relaxed to $\pi$, we apply at some random time $\tau$ larger than the relaxation time yet another (second) perturbation $\varphi = \pi \to \varphi(0) = \varphi(\tau)$ within the same attraction basin, i.e. $\varphi(0) = 0.93$ for parameters of Fig. 2(b). The initial state of $\xi$ is then reset as $(\xi(\tau), \dot{\xi}(\tau))$. Hence we now run the dynamics anew with initial states $\varphi(0) = 0.93\pi$, $\dot{\varphi}(0) = 0$, $\xi(\tau)$ and $\dot{\xi}(\tau)$.

It appears that for parameters of Fig. 2(a), $\varphi = \pi$ is unstable after the second perturbation. The issue here is that the back-reaction $\epsilon$ is sufficiently large, hence the second perturbation alters the controlling degree of freedom $\xi(t)$ (via resetting its initial state) and also dries out its stored energy, which already decreased after the first perturbation. If for parameters of Fig. 2(a) we decrease $\epsilon$ from 0.8 to 0.1, $\varphi$ becomes stable to many well-separated (in the above sense) perturbations coming at random times. One reason for this is that the stored energy decrease within one perturbation is smaller. Another reason is that the motion of $\xi$ becomes more stable with respect to resetting its initial conditions.

![FIG. 2: (a) $\varphi$ (black, upper curve) and $\xi$ (blue, oscillating curve) are numerical solutions of (6, 9) versus time $t$; $\eta = \frac{\epsilon}{\mu + \gamma}$ (red curve in the middle) is the scaled energy that includes the stored energy $\frac{1}{m} [\dot{\xi}^2 + \omega^2 \xi^2]$; cf. (8, 22). Parameters are: $\epsilon = 0.8$, $\xi(0) = 2$ (\xi(0) = 0), $\varphi(0) = 0.93\pi$ (\dot\varphi(0) = 0), $\omega = 4$, $\gamma = 3$, $m = g = l = 1$. It is seen that $\varphi$ quickly stabilizes at $\varphi = \pi$, which is normally an unstable state. The stabilization process takes a relatively small amount of energy. After the stabilization $\varphi$ and $\xi$ decouple; $\xi$ continues oscillating, and the scaled energy $\eta$ is then constant in time. The stability condition (22) holds. The difference between its LHS and RHS is 0.4375. (b) The same parameters as in (a), but now the backreaction parameter $\epsilon = 0.9$ is slightly larger. We also display $\eta = \frac{\epsilon}{\mu + \gamma} \frac{\dot{\xi}^2}{m} - \frac{\omega^2}{2} \xi^2$ (green curve, third from top), where $\eta$ is the (scaled) energy related to the angle variable only; cf. (8). Now there is rather long ($t \sim 150$) period of metastability, accompanied by a slow dissipation of energy. After this the stability is lost: $\varphi$ quickly relaxes to the minimum $\varphi = 0$ of the potential, $\xi$ looses all its energy and eventually stops moving (i.e. $\xi(t) \to 0$). The whole stored energy is dissipated away. The physical reason of scenario in (b) is that fast oscillations around $\varphi = \pi$ do not disappear, i.e. they persist in the metastable state, continuously dissipate energy [cf. (10)], and once the initial energy decreases sufficiently, the $\varphi$ and $\xi$ (relatively) suddenly move to the global energy minima $\varphi = \dot{\varphi} = \xi = \dot{\xi} = 0$. We emphasize that it is the energy $\frac{1}{m} \dot{\xi}^2 + \frac{\omega^2}{2} \xi^2$ stored in $\xi$ that decays in time. To confirm this, (b) also shows the scaled energy $\zeta_\varphi$ related to $\varphi$. It stays constant in time for the whole metastability period; see the green curve. A similar scenario happens, when the initial condition $\varphi(0)$ is out of the attraction basin of $\varphi = \pi$. However, here $\varphi(t)$ never reaches $\pi$.}
Above we assumed multiple, strong and well-separated (in time) perturbation. Another way of implementing multiple perturbations is to include an external noise in (9). Let us add to the right-hand-side of (9) a white, Gaussian random noise:

\[ \sigma f(t), \quad \langle f(t) \rangle = 0, \quad \langle f(t)f(t') \rangle = \delta(t-t'), \quad (23) \]

where \( \sigma \) is the noise intensity. The modified (9) becomes then the Langevin equation (for \( \varphi(t) \)). In contrast to strong and well-separated (in time) perturbations, (23) allows uncorrelated, densely located perturbations that are (most probably) weak if \( \sigma \) is small.

Now Fig. 3(a) shows that a weak-random noise monotonously dries out the stored energy, and hence the stability of \( \varphi = \pi \) is lost after a sufficiently long time. Fig. 3(b) shows that the same scenario hold for a stronger noise, though in somewhat blurred form and for a shorter life-time of the metastable state. For parameters of Fig. 3(a) and \( \sigma = 0 \), \( \varphi = \pi \) is asymptotically stable with respect to 10 – 11 strong, well-separated perturbations.

C. Metastability

For parameters of Fig. 2(a), if the back-reaction \( \epsilon \) is larger than 0.8 another interesting scenario takes place: the notion of asymptotic stability with respect a single perturbation is lost and is replaced by metastability; see Fig. 2(b).

Now small oscillations of \( \varphi(t) \) around \( \Phi = \pi \) persist and do not decay in time; cf. (12). According to (10), these oscillations slowly drain out the initial stored energy \( \frac{1}{2}\epsilon \xi^2(0) + \omega^2 \xi(0) \), and when it gets sufficiently low, the metastable state \( \Phi = \pi \) suddenly decays to the global energy minimum \( \varphi = \xi = \xi = 0 \); see Fig. 2(b). During this sudden decay the whole stored energy is dissipated away. Note that in the metastability time-window the energy \( E_\varphi \) related to \( \varphi \) stays constant; see (11) and Fig. 2(b).

The transition between regimes in Fig. 2(a) and Fig. 2(b), i.e. between the truly stable and the metastable state takes place for a critical value of the backreaction parameter \( \epsilon_c \). For parameters of Fig. 2(a) and 2(b) we have \( \epsilon_c \approx 0.86075 \). We checked numerically that the life-time of the metastable state can be very large for \( \epsilon \) approaching \( \epsilon_c \) from above. Hence we conjecture that this time can be arbitrary large for \( \epsilon \to \epsilon_c + 0 \).

We emphasize that the transition between regimes in Fig. 2(a) and Fig. 2(b) is described for fixed initial conditions of \( \varphi \): \( \varphi(0) = 0.93 \pi \) (\( \dot{\varphi}(0) = 0 \)), i.e. for a fixed attraction basin of the stabilized state. If these initial conditions are changed by making \( \varphi(0) \) closer to the stability point \( \pi \) (i.e. the attraction basin is shrunk), then the transition from the stable to metastable regime takes place at a larger value of \( \epsilon \), or does not take place at all. Likewise, if the attraction basin is enlarged, the transition taken place for smaller \( \epsilon \). For example, if \( \varphi(0) = 0.92 \pi \) (\( \dot{\varphi}(0) = 0 \)), then the transition takes place at \( \epsilon_c \approx 0.6536 \), while for \( \varphi(0) = 0.935 \pi \) (\( \dot{\varphi}(0) = 0 \)), the solution is stable for all \( \epsilon < 1 \), which is the physical range of \( \epsilon \) for parameters of Fig. 2(a) and Fig. 2(b); cf. (7). Note that the stabilization with the largest attraction basin demands vanishing values of \( \epsilon \). In particular, for parameters of Fig. 2(a), no stabilization of the \( \varphi = \pi \) state occurs for \( |\dot{\varphi}(0)| < 0.64105 \pi \) (\( \dot{\varphi}(0) = 0 \)), while for \( |\dot{\varphi}(0)| \geq 0.64105 \pi \) the stabilization demands \( \epsilon \to 0 \).

Note that the metastability in Figs. 3(a) and 3(b) is different from that in Fig. 2(b), because in the latter case the metastability is due to a strong back-reaction, and not permanently acting perturbations.

V. SUMMARY

The purpose of this work is to understand energy costs of active homeostasis: a process that stabilizes an unstable state due to an active controlling process. Biological and physiological discussions imply that homeostasis is needed for controlling (and providing advantages for) metabolic processes in organisms [1–4]. This makes necessary to ask about the proper energy costs of the homeostasis itself. That such costs can be substantial is known from biology, e.g. humming birds (colibri)—for which energy saving is crucial—fall at night into a torpor state that is different from the normal sleep. In this state several homeostatic mechanisms—including internal energy regulation—are ceased. Thereby birds are able to save a substantial amount of energy: up to 60% of the normal usage [24].

It is however clear that a microscopic understanding of the homeostasis energy cost problem cannot start from the (extremely complex!) organismal level. To this end, one needs plausible models with a well-established history of physical [14–19] and control-theoretic [20–22] applications. Here we studied the inverted pendulum model, where the upper (normally unstable) state is activated by fast motion of controlling degrees of freedom. Usually, this degree of freedom is taken as an external field. But here we modeled it as an explicit degree of freedom, because we want to
study an autonomous system with a full control of energy and its dissipation. Our main results are summarized as follows.

The unstable state of the pendulum can be asymptotically stabilized—with a finite attraction basin—without permanent energy dissipation, because the controller-pendulum coupling is automatically switched off once the pendulum is stabilized. Here there is only a transient dissipation of a small amount of energy related to the stabilization. This regime is reached when both the backreaction of the pendulum to the controller is sufficiently small and the controller oscillates sufficiently fast, i.e. it does have a stored energy.

In the context of the active homeostasis, the notion of asymptotic stability is not enough: we need to study stability with respect to multiple perturbations. We implemented two scenarios for multiple perturbations: strong, widely separated in time perturbations and a weak white noise acting on the target degree of freedom. An asymptotically stable state may not be stable with respect to several perturbations. The latter type of stability is achieved only if the back-reaction to the controller is small.

When the back-reaction is even larger the very notion of asymptotic stability is lost and is replaced by metastability. Now the stabilization is temporary (metastable), because small oscillations around the stabilized state do not decay. They dry out the stored energy of the controller, and once it is lower than some threshold the metastable state decays. In this case, we do get that a constant dissipation rate is needed for supporting the metastable state.

Thus the energy stored in the controlling degree of freedom is the main resource of active homeostasis. In that respect it is similar to the energy stored in the living organism, one of major concepts in biological thermodynamics [25–32]. It also relates to adaptation energy introduced in physiology [11]; cf. [12] for a critical discussion. The stored energy is phenomenologically employed in Dynamic Energy Budget Theory (DEBT) [31, 32] and applied for estimating metabolic flows of concrete organisms.

This concept is not yet well-formalized, but some of its qualitative features are known. The stored energy is not the usual free energy, since the latter is present in equilibrium as well. From the viewpoint of a modern thermodynamics, the notion of the stored energy resembles the energy kept at certain negative temperature, because it is capable of doing work in a cyclic process [33]. Now for the inverted pendulum the stored energy is mechanical (not chemical) and relates to an oscillating degree of freedom, but it is also capable of doing a cyclic work. The inverted pendulum does demonstrates how stored energy relates to stabilizing unstable states. However, the model is imitational: it assumes, but does not explain why specifically the stabilization via an active process is needed.

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![FIG. 3: (a) $\phi$ (blue, piecewise constant) is the numerical solution of (6, 9) versus time $t$, where to the RHS of (9) we added an additive white noise $f(t)$ given by (23). Here $\eta = \frac{\phi \cdot \omega}{(m + \rho)}$ (orange, decaying curve) is the scaled energy; cf. (8). Parameters are: $\sigma = 0.1$, $\epsilon = 0.4$, $\hat{\xi}(0) = 4$ ($\bar{\xi}(0) = 0$), $\phi(0) = 0.93\pi$ ($\hat{\phi}(0) = 0$), $\omega = 4$, $\gamma = 3$, $m = g = l = 1$. It is seen that $\phi$ is stabilized around $\pi$ in a metastable state whose life-time is $\approx 500$. During this life-time the energy $\eta$ slowly decays, till it is below some critical value, and then the metastable state suddenly decays. (b) The same as in (a), but for a stronger noise $\sigma = 1.25$.](image-url)
Appendix A: Solution of Eqs. (15, 16)

Define from (7)

\[ \beta \equiv \frac{\epsilon \sin^2 \Phi}{l} = \frac{\sin^2 \Phi}{1 + \frac{\mu}{m}} < 1, \quad (A1) \]

and solve (15, 16) via the Laplace transform as:

\[ \hat{\xi}(s) = \frac{(1 - \beta)a(s)}{(s^2 + \omega^2)(s + \gamma) - \beta \omega^2}, \quad \hat{\zeta}(s) = -\frac{\sin \Phi}{l} \frac{(1 - \beta)b(s)}{(s^2 + \omega^2)(s + \gamma) - \beta \omega^2}, \quad (A2) \]

\[ a(s) = (s + \gamma)\hat{\xi}(0) + s(s + \frac{\gamma}{1 - \beta})\hat{\xi}(0), \quad b(s) = s\hat{\xi}(0) - \hat{\xi}(0)\frac{\omega^2}{1 - \beta} \quad (A3) \]

where in addition to initial condition (17) we also employed (18).

The inverse Laplace transform taken from (A2) reads

\[ \xi(t) = \frac{e^{s_1 t}a(s_1)}{(s_2 - s_1)(s_3 - s_1)} + \frac{e^{s_2 t}a(s_2)}{(s_1 - s_2)(s_3 - s_2)} + \frac{e^{s_3 t}a(s_3)}{(s_1 - s_3)(s_2 - s_3)} \quad (A4) \]

\[ \zeta(t) = -\frac{\sin \Phi}{l} \left[ \frac{e^{s_1 t}b(s_1)}{(s_2 - s_1)(s_3 - s_1)} + \frac{e^{s_2 t}b(s_2)}{(s_1 - s_2)(s_3 - s_2)} + \frac{e^{s_3 t}b(s_3)}{(s_1 - s_3)(s_2 - s_3)} \right] \quad (A5) \]

where \( s_1, s_2 \) and \( s_3 \) solve

\[ s^3 + \frac{\gamma s^2}{1 - \beta} + \frac{\omega^2 s}{1 - \beta} + \frac{\omega^2 \gamma}{1 - \beta} = (s - s_1)(s - s_2)(s - s_3) = 0. \quad (A6) \]

Without loss of generality we take the following parametrization:

\[ s_1 = -\Gamma, \quad s_2 = -\tilde{\gamma} + i\tilde{\omega}, \quad s_3 = -\tilde{\gamma} - i\tilde{\omega}, \quad \Gamma > 0, \quad \tilde{\gamma} \geq 0. \quad (A7) \]

Now we calculate \( \frac{\zeta(t)}{\xi(t)} \). Hence when taking the product \( \zeta(t)\hat{\xi}(t) \) all oscillating terms with frequency \( \tilde{\omega} \) are to be neglected:

\[ \frac{\zeta(t)}{\xi(t)} = -\frac{\sin \Phi}{l} \left( \frac{s_1^2 a(s_1) b(s_1) e^{-2\Gamma t}}{(\gamma - 1)^2 + \omega^2} + \frac{s_2^2 a(s_2) b(s_2) + s_3^2 a(s_3) b(s_3) e^{-2\tilde{\gamma} t}}{4\tilde{\omega}^2(\gamma - 1)^2 + \omega^2} \right) \quad (A9) \]

If in (A1), \( \beta \to 0 \) due to \( \mu/m \gg 1 \), we can keep in (A6) only the simplest non-zero order putting there \( \beta = 0 \). This approximation is supported if the slow variable \( \Phi \) tends to \( \pi \), i.e., \( \beta \) gets additional smallness. Now (A6) will read \( (s + \gamma)(s^2 + \omega^2) = 0 \), i.e., we get via (A7):

\[ \Gamma = \gamma, \quad \tilde{\gamma} = 0, \quad \tilde{\omega} = \omega. \quad (A10) \]

This means that the term \( \propto e^{-2\Gamma t} \) in (A9) can be neglected, and we have

\[ \frac{\zeta(t)}{\xi(t)} = \frac{\sin \Phi}{l} \frac{\xi^2(0) + \omega^2 \xi^2(0)}{\omega^2} \quad (A11) \]

Eq. (A11) leads us to (19, 21).