Acceleration disturbances due to local gravity gradients in ASTROD I

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Abstract. The Astrodynamical Space Test of Relativity using Optical Devices (ASTROD) mission consists of three spacecraft in separate solar orbits and carries out laser interferometric ranging. ASTROD aims at testing relativistic gravity, measuring the solar system and detecting gravitational waves. Because of the larger arm length, the sensitivity of ASTROD to gravitational waves is estimated to be about 30 times better than Laser Interferometer Space Antenna (LISA) in the frequency range lower than about 0.1 mHz. ASTROD I is a simple version of ASTROD, employing one spacecraft in a solar orbit. It is the first step for ASTROD and serves as a technology demonstration mission for ASTROD. In addition, several scientific results are expected in the ASTROD I experiment. The allowed acceleration noise level of ASTROD I is $10^{-13}$ m s$^{-2}$ Hz$^{-1/2}$ at the frequency of 0.1 mHz. In this paper, we focus on local gravity gradient noise that could be one of the largest acceleration disturbances in the ASTROD I experiment. We have carried out gravitational modelling for the current test-mass design and simplified configurations of ASTROD I by using an analytical method and the Monte Carlo method. Our analyses can be applied to figure out the optimal designs of the test mass and the constructing materials of the spacecraft, and the configuration of compensation mass to reduce local gravity gradients.

1. Introduction
A gravitational mission, Astrodynamical Space Test of Relativity using Optical Devices (ASTROD) [1, 2], was proposed to test relativistic gravity, measure the solar-system parameters with high precision and detect gravitational waves from massive black holes and galactic binary stars. The concept of ASTROD is to put two spacecraft in separate solar orbits and carry out laser interferometric ranging with a spacecraft near the L1/L2 point. A simple version of ASTROD, ASTROD I, has been studied as the first step to ASTROD. ASTROD I employs one spacecraft in a solar orbit and carries out interferometric ranging and pulse ranging with ground stations [3].

The acceleration disturbance goal of ASTROD I is $10^{-13}$ m s$^{-2}$ Hz$^{-1/2}$ at frequency $\nu$ of 0.1 mHz. Assuming a 10 ps timing accuracy and the acceleration noise of $10^{-13}$ m s$^{-2}$ Hz$^{-1/2}$ at frequency of about 0.1 mHz, a simulation for 400 days (350-750 days after launch) showed that ASTROD I could determine the relativistic parameters $\gamma$ and $\beta$, and the solar quadrupole parameter $J_2$ to levels of $10^{-7}$, $10^{-7}$ and $10^{-8}$, respectively [4]. In order to achieve the acceleration disturbance goal, a drag-free control system using capacitive sensors will be employed.

A preliminary overview of sources and magnitude of acceleration disturbances for ASTROD I
is given by Shiomi and Ni [5]. According to their estimates assuming simple models, local gravity gradients can be one of the largest contributions to acceleration disturbances in ASTROD I. Therefore, an elaborate gravitational modelling seems necessary.

The sources of acceleration disturbances due to local gravity gradients can be classified into two categories. One is thermal deformation of the spacecraft and the payload, mainly caused by solar radiation. Inherent temperature fluctuations in solar radiation result in unwanted acceleration. Because composing materials of the spacecraft and payload vary in thermal expansion coefficient, there would be complicated relative positional changes inside of the spacecraft. Elaborate thermal modelling works are required for a complete analysis. However, we will not discuss this in this paper. The other is positional fluctuations of the test mass. Even when there is no deformation in the spacecraft and payload, positional fluctuations of the test mass produce unwanted acceleration.

Xu and Ni did a preliminary work of gravitational modelling for the ASTROD mission [6]. They calculated gravitational interaction between a test mass and a cylindrical reference mass (a hollow cylinder with two end disks) by using the expression derived for the shape design of STEP (Satellite Test of the Equivalence Principle [7]) test-masses [8]. They evaluated the magnitude of gravitational acceleration caused when the test mass, located at the centre of the reference mass, was shifted along the axial axis of the reference mass. The expression they used was obtained for the analyses of cylindrical bodies with homogeneous density. We use more general expressions (section 3), applicable to arbitrary shapes, and consider the gravitational acceleration between the ASTROD I test mass and cylindrical bodies (section 5), and rectangular parallelepiped objects (section 6) in this paper. Also, we carry out the Monte Carlo simulation (section 4) to estimate the gravitational acceleration due to positional fluctuations of the ASTROD I test mass.

2. The configuration of the ASTROD I spacecraft
The ASTROD I spacecraft has a cylindrical shape with diameter 2.5 m and height 2 m. Its surface is covered with solar panels. The cylindrical axis is perpendicular to the orbit plane and a telescope on board is set to point toward a ground laser station. The total mass of the spacecraft is about 350 kg and that of payload is 100-120 kg (see [9] and [10] for more detailed descriptions of the configuration). The orbit distance from the Sun varies from about 0.5 AU to 1 AU (see figure 2 of [9] for a detailed description).

The test mass ($M_{TM} = 1.75$ kg) is a rectangular parallelepiped ($50 \times 50 \times 35$ mm$^3$) made from Au-Pt alloy with density of $2 \times 10^4$ kg m$^{-3}$. The test mass is located at the centre of the spacecraft. The six sides of the test mass are surrounded by electrodes mounted on the housing for capacitive sensing. The gap between each side of the test mass and the opposing electrode is 2 mm.

3. Acceleration of a test mass
The gravitational potential energy of a test mass (density distribution $\rho_t(x')$ and volume $v_t$) in a gravitational field produced by a source mass (density distribution $\rho_s(x)$ and volume $v_s$) can be written by:

$$ V = -4\pi G \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} q_{lm} Q_{lm} $$

where

$$ q_{lm} = \int_{v_t} \rho_t(x') r'^l Y^*_{lm} (\theta', \phi') d^3x' $$

$$ Q_{lm} = \int_{v_s} \rho_s(x) r^{-(l+1)} Y_{lm} (\theta, \phi) d^3x $$

(1)
Inner gravitational multipole moments and outer gravitational multipole moments, $q_{lm}$ and $Q_{lm}$, represent the mass distribution of the test mass and the source mass, respectively. $G$ ($=6.67 \times 10^{-11}$ N m$^2$ kg$^{-2}$) is the gravitational constant.

The force between the test mass and the source mass in the sensitive axis (say, the $x$-axis) can be obtained by shifting the multipole moments of the test mass along the axis by $dX'$. This method was used by Speake to obtain the $z$-component of force for STEP test masses [11]. A detailed description of the expression for STEP is given in section 3.2 of [12].

Using the formula by D’Urso and Adelberger (equation (10) of [13]), the leading order term of the shifted multipole moments is given by:

$$q_{LM} = \mp \frac{1}{2} \sqrt{(2l+3)(l \pm m + 1)(l \pm m + 2)} q_{lm} dX'$$  \hspace{1cm} (4)

for $L = l+1$ and $M = m \pm 1$. With equation (1), the $x$-component of the force is given by:

$$F_x = -4\pi G \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left\{ \sqrt{\frac{(l+m+1)(l+m+2)}{2(2l+1)(2l+3)}} q_{lm} Q_{l+1,m+1} - \sqrt{\frac{(l-m+1)(l-m+2)}{2(2l+1)(2l+3)}} q_{lm} Q_{l+1,m-1} \right\}$$  \hspace{1cm} (5)

From equation (4), one can see that a positional fluctuation $X_p$ of the test mass produces the first leading terms of $q_{lm}$, which are proportional to $q_{00}X_p$. From equation (5), one can see that these terms couple to $Q_{2,m \pm 1}$ and produce unwanted acceleration. It should be noted that the magnitude of the unwanted acceleration is independent of the shapes of the test mass to the first order, but is dependent on the mass distribution of the spacecraft ($Q_{2,m \pm 1}$). This indicates that the mass distribution of the spacecraft has to be designed carefully.

We apply these formulae to estimate gravitational acceleration of the ASTROD I test mass.

4. Monte Carlo integration

The $x$-component of force of a test mass can also be obtained by calculating the following formula:

$$F_x = G \int_{v_s} \int_{v_t} dx^3 \int dx'^3 \rho_t(x') \rho_s(x) (x - x') \frac{1}{|x - x'|^3}$$  \hspace{1cm} (6)

We have carried out the integration over the volumes by the Monte Carlo method [14]. A pair of random points, one is in the region of $v_t$ and the other is in the region of a box that includes $v_s$, was generated at least $10^8$ times in each simulation. The integrand was estimated and added up every time when the random point generated in the box was inside of the region of $v_s$. The same simulation was repeated at least 70 times. The average and the standard deviation of the results were estimated.

5. Cylindrical Spacecraft and the ASTROD I test mass

We consider the gravitational acceleration between the ASTROD I test mass, located at the centre of the spacecraft, and the spacecraft. The test mass is a rectangular parallelepiped. Assuming a uniform density for the test mass, the following terms of gravitational multipole moments of the test mass (with the origin at the centre of mass of the test mass) are null because of the geometrical symmetries: $l$-odd terms, $m$-odd terms and terms with $m = 2,6,10,14,\ldots$. Therefore, the leading terms of the gravitational multipole moments of the test mass are $q_{00}$, $q_{20}$, $q_{40}$, $q_{4,\pm 4}$ and so on. When there is a positional fluctuation of $X_p$ along the sensitive axis, $q_{00}$ and $q_{20}$ produce the leading terms $\tilde{q}_{1,\pm 1}$ and $\tilde{q}_{3,\pm 1}$, respectively. From equation (5), $\tilde{q}_{1,\pm 1}$ couple to $Q_{20}$ and $Q_{2,\pm 2}$, and $\tilde{q}_{3,\pm 1}$ couple to $Q_{40}$ and $Q_{4,\pm 2}$. $Q_{2,\pm 2}$ and $Q_{4,\pm 2}$ are zero for homogeneous
cylindrical bodies (with the origin at their centre of mass) because of its geometrical symmetry. Therefore, the acceleration of the test mass is given by

\[ a_x = \frac{8\pi G}{MTM} \left\{ \frac{1}{\sqrt[3]{3}} q_{11} Q_{20} + \frac{1}{\sqrt[2]{21}} q_{31} Q_{40} + ... \right\} \tag{7} \]

where the relations, \( q_{11} = -q_{1,-1} \) and \( q_{31} = -q_{3,-1} \), are used. By substituting the following relations, obtained from equation (4), into equation (7),

\[ q_{11} = -\sqrt{\frac{3}{2}} q_{10} X_p, \quad q_{31} = -\sqrt{2 \pi} q_{20} X_p \tag{8} \]

we obtain,

\[ a_x = -\frac{4\pi G}{MTM} \sqrt{\frac{T}{5}} (q_{00} Q_{20} + 2 q_{20} Q_{40} + ...) X_p \equiv -K_x X_p \tag{9} \]

where \( K_x \) is a coupling constant.

We assume that the outer dimensions of the spacecraft is 2.0 m long by a diameter of 2.5 m and the thickness is 5 mm, and it has a uniform density of 2300 kg m\(^{-3}\); the mass of the spacecraft is 292 kg. For this spacecraft, \( Q_{20} = 3.31 \) kg m\(^{-3}\) and \( Q_{40} = 4.07 \) kg m\(^{-5}\). For the ASTROD I test mass, \( q_{00} = \frac{MTM}{\sqrt{4\pi}} = 0.493 \) kg and \( q_{20} = -1.17 \times 10^{-4} \) kg m\(^2\). Therefore, from equation (9), \( K_x \approx 3.5 \times 10^{-10} \) s\(^{-2}\). The first term of equation (9) is larger than the second term by more than three orders of magnitude. Therefore, in order to reduce the magnitude of this gravitational coupling, \( Q_{20} \) has to be minimized.

To shorten the simulation time and decrease the uncertainty, we have carried out the Monte Carlo integration for a smaller cylinder (its inner radius, outer radius, outer length and thickness of each end disk are 45 mm, 50 mm, 100 mm and 26 mm, respectively). The density of the cylinder is assumed to be 3000 kg m\(^{-3}\). \( 10^6 \) random points were generated in the region of the test mass and the region of a box \((100 \times 100 \times 100 \) mm\(^3\)), in which the cylinder can fit. The simulation was repeated 70 times. From the average of the 70 results, we have obtained \( a_x = -2.85 \times 10^{-10} \) m s\(^{-2}\) when the test mass was shifted by 1 mm along the \( x \)-axis. The standard deviation of the 70 results was 1.4 \%. For this cylinder, \( Q_{20} = 2.80 \times 10^3 \) kg m\(^{-3}\) and \( Q_{40} = 2.71 \times 10^5 \) kg m\(^{-5}\). Therefore, using equation (9), we obtain \( a_x = -2.8 \times 10^{-10} \) m s\(^{-2}\) for \( X_p = 1 \) mm. The ratio of the magnitude of the first and second terms in equation (9) is \( \approx 4.5 \% \) in this configuration.

6. A rectangular-parallelepiped box and the ASTROD-I test mass

We consider the gravitational interaction between the test mass and a rectangular-parallelepiped box. We assume that the test mass is enclosed in the box. The gravitational acceleration between them has the similar relation with equation (9) when the \( x \) and \( y \) dimensions of the box are the same; \( q_{2,\pm 2} \) and \( q_{4,\pm 2} \) are also zero in this geometry.

This configuration is similar to the electrode box for the capacitive sensing and the test mass. The gaps between the opposing sides of the test mass and the electrode box are 2 mm. Therefore, the inner dimensions of the electrode box are \( 54 \times 54 \times 39 \) mm\(^3\). The distance between the electrode box and the test mass is so close that we need to consider higher terms to estimate the gravitational acceleration between them. For a simplicity, we consider a box larger than the electrode box: the inner dimensions are \( 100 \times 100 \times 70 \) mm\(^3\). We assume that the thickness of each wall of the box is 5 mm and it has a uniform density of 10280 kg m\(^{-3}\). For this box, the acceleration of the test mass is given by

\[ a_x = -\frac{4\pi G}{MTM} \left\{ \frac{1}{\sqrt[5]{5}} q_{00} Q_{20} + \frac{2}{\sqrt[5]{5}} q_{20} Q_{40} + \frac{5}{\sqrt[13]{13}} q_{40} Q_{60} + 2 \sqrt{\frac{15}{13}} q_{44} Q_{64} + ... \right\} X_p \tag{10} \]

\[ \approx -5.6 \times 10^{-8} X_p \tag{11} \]
where \( q_{40} = -2.63 \times 10^{-8} \text{ kg m}^4 \), \( q_{44} = -1.14 \times 10^{-7} \text{ kg m}^4 \), \( Q_{20} = 5.93 \times 10^2 \text{ kg m}^{-3} \), \( Q_{40} = 2.28 \times 10^5 \text{ kg m}^{-5} \), \( Q_{60} = -4.57 \times 10^8 \text{ kg m}^{-7} \) and \( Q_{64} = 2.47 \times 10^7 \text{ kg m}^{-7} \). In equation (10), the first term is dominant. To reduce the acceleration disturbance, \( Q_{20} \) has to be minimized.

We have carried out the Monte Carlo method for this configuration. We have generated \( 2 \times 10^9 \) random points in the regions of the test mass and a box whose dimensions are identical to the outer dimensions of the box (\( 110 \times 110 \times 80 \text{ mm}^3 \)). This simulation was repeated 70 times. From the average of the 70 results, we obtain \( a_x = -5.60 \times 10^{-11} \text{ m s}^{-2} \) when the test mass was shifted by 1 mm along the \( x \)-axis. The standard deviation of the 70 results was 0.7 %.

We have carried out the Monte Carlo simulation for the electrode box with the inner dimensions of \( 54 \times 54 \times 39 \text{ mm}^3 \). We assume that the thickness of each wall of the electrode box is 5 mm and the density is \( 10280 \text{ kg m}^{-3} \). \( 2 \times 10^9 \) random points were generated in the region of the test mass and the region of a box whose dimensions are identical to the outer dimensions of the electrode box (\( 64 \times 64 \times 49 \text{ mm}^3 \)). This simulation was repeated 90 times and the average of the 90 results has resulted in \( a_x = -3.93 \times 10^{-11} \text{ m s}^{-2} \) when the test mass was shifted by 1 mm along the \( x \)-axis. The standard deviation of the 90 results was 1.9 % of the average. This result of the simulations indicates that the magnitude of the gravitational stiffness is \( \approx 4 \times 10^{-8} \text{ s}^{-2} \). This is approximately same as the value of gravitational stiffness we currently use for the estimation of acceleration disturbances in ASTROD I and as the gravity gradient uncertainty used in current error estimates for LISA spurious accelerations [15]. For this configuration, contributions from the leading terms in equation (10) are comparable. To reduce the gravitational coupling, each moment has to be minimized.

### 7. Discussion

We have estimated gravitational acceleration between the ASTROD I test mass and cylindrical bodies, and rectangular parallelepiped bodies by using the analytical method and the Monte Carlo method. The results obtained by the two methods were consistent.

From the estimations described above, one of the effective ways to reduce the gravitational couplings seems to reduce the magnitude of \( Q_{20} \). This can be done by choosing the geometries of constructing materials of the spacecraft. For a hollow cylindrical body with an inner radius \( A \), outer radius \( B \), outer length \( L \) and thickness of the end disks \( T \), \( Q_{20} \) is zero when they satisfy the relation: \( A^4 = 1 - 2B^2 \). For the spacecraft we have considered in section 5, when the thickness is 4 mm, the dominant term, proportional to \( q_{00}Q_{20} \), becomes zero. For a box, \( Q_{20} \) is null when it is a cube.

For constructing materials closer to the test mass, contributions from higher terms are comparable and they have to be reduced simultaneously. In practice, many parts of the mass distribution of the spacecraft are dictated by technical requirements. Therefore, the most practical way of reducing the magnitude of higher terms might be to have a test-mass shape design that minimizes the magnitude of \( q_{lm} \). In this regard, one of the ideal shapes for the test mass is a sphere with homogeneous density; it only has \( q_{00} \). If \( Q_{20} \) of materials close to the spherical test mass is designed to be zero, the gravitational stiffness would be reduced significantly. Other more realistic favoured shapes for the test mass may be a cube or a cylinder with the aspect ratio of \( l/a = \sqrt{3}/2 \), where \( a \) and \( l \) are the radius and half-length of the cylinder; their \( q_{20} \) is zero.

In summary, it can be said that, in order to minimize the gravitational couplings, the spacecraft mass distribution, \( Q_{2, m \pm 1} \), and higher gravitational moments of the test mass (\( q_{lm} \)) have to be minimized. However, the design of the test mass and spacecraft will be a compromise between the requirements from the shape design to minimize the gravitational couplings and other technical requirements. Therefore, an elaborate gravitational modelling for the final design of the test mass and spacecraft mass distribution will be necessary to ensure that the resultant gravitational coupling is sufficiently small. It might be impossible to model all the
mass distributions on board precisely; complicated shaped objects are difficult to model and some approximation might be necessary in the modelling. This limit on modelling needs to be considered in the estimation of uncertainty in the gravity gradients effects. If the resultant gravitational coupling were found too large, it needs to be adjusted by employing compensation mass to reduce the local gravity gradients. The analytical method provides the knowledge of which gravitational moments contribute significantly to the resultant gravitational coupling. It would help to figure out the optimal designs for the test mass and the constructing materials, and the configuration of mass compensation if necessary.

In practice, even if the test mass and the spacecraft mass distribution are designed perfectly to make the gravitational coupling sufficiently small, some imperfections, such as machining tolerances and density inhomogeneities, in the test mass and the composing materials of the spacecraft, could cause unwanted gravitational couplings. These practical issues also have to be studied carefully.

8. Conclusions
We have carried out gravitational modelling for the current test mass design and simplified configurations of ASTROD I by using the analytical method and the Monte Carlo simulation. In order to minimize the gravitational couplings, the spacecraft mass distribution, $Q_{2m±1}$, and higher gravitational moments of the test mass ($g_{lm}$) have to be minimized. The designs of the test mass and spacecraft mass distribution will be a compromise between the requirements from the shape designs to minimize the gravitational couplings and other technical requirements. To ensure that the resultant gravitational coupling is sufficiently small, an elaborate gravitational modelling for the final design of the test mass and spacecraft mass distribution is necessary. Our analyses can be applied to figure out the optimal designs of the test mass and the constructing materials of the spacecraft, and the configuration of compensation mass to reduce local gravity gradients.

Acknowledgements
The author thanks W.-T. Ni for his useful comments on this work and the manuscript, and A. Pulido Patón and H. Hatanaka for their comments on the manuscript. This work was funded by the National Science Council.

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