CAUSAL SITES AS QUANTUM GEOMETRY

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Abstract. We propose a structure called a causal site to use as a setting for quantum geometry, replacing the underlying point set. The structure has an interesting categorical form, and a natural “tangent 2-bundle,” analogous to the tangent bundle of a smooth manifold. Examples with reasonable finiteness conditions have an intrinsic geometry, which can approximate classical solutions to general relativity. We propose an approach to quantization of causal sites as well.

1. Introduction and Physical Motivation

This paper is part of a program to found quantum gravity in relational topology; more precisely, to replace point set topology with a special type of category as the underlying structure on which to put geometrodynamics.

Physically, the idea is that what we actually observe are interactions between bounded regions of spacetime. These could be either material systems or regions of empty space whose causal effects can be directly or indirectly distinguished by material systems. There should be a direct mathematical description of the flow of information between the regions, and points should appear only relative to an observer, as minimal distinguished regions. We will propose a specific axiomatic structure for a type of category which would contain the regions as objects and the relationships between them as different types of morphisms. The hope is that this would lead to a description of quantum physics free from ultraviolet divergences, by eliminating the underlying point set continuum.

Since categories as generalizations of topological spaces are well known in mathematics, where they are referred to as sites [Artin 1962 Mac Lane et al 1992], we are calling our new structures causal sites. Strictly speaking, a site is a category together with a structure called a “Grothendieck topology,” which is the analog of a topology for an ordinary space. We will explain below that our axioms also involve a structure very similar to a Grothendieck topology.

This paper will mainly discuss the “topological” side of the problem, i.e. the structure of the causal sites themselves. We discover that, unlike manifolds, causal sites with suitable finiteness conditions have an intrinsic geometry. Thus the distinction between topology and geometry is bridged over in our new picture. Examples are described which reproduce the classical solutions to general relativity above the Planck scale. We briefly consider the possibility of directly quantizing causal sites, thus directly producing a quantum theory of general relativity.

We will then briefly consider the appropriate classes of presheaves over causal sites, namely unitary and bisimplicial prestacks. We hope this can serve as a bridge between the topological aspects of causal site theory and the problem of constructing quantum physics.
over them. We will make some remarks as to how to construct a model for quantum gravity at the end.

A very important feature of the topology of causal sites is that they have a tangent 2-bundle, which is analogous to the tangent bundle of a manifold. We believe this will serve as a setting for applications to quantum geometry and physics.

The structure of causal sites is a synthesis of two constructions, one well known from homotopy theory and algebraic geometry, the other familiar in the relativity literature. A site [Artin 1962] is a category thought of as a generalization of the lattice of open sets of a topological space, with a distinguished family of covers for objects. (We warn the reader that our construction does not fully conform to the accepted definition of a site, since the covers in it do not satisfy the axioms of a Grothendieck topology, as explained below.) The morphisms of the category represent an abstract version of containment. Doing sheaf theory over such generalized spaces is an important part of modern mathematics.

On the other hand, a causal set [Sorkin 2003, Hawkins et al 2003], or partially ordered set, is a discrete point set approximating the causal structure of a spacetime manifold.

Now, since up to this point nobody has tried to use sites as a foundation for relativity, the natural structure that occurs when we combine the two ideas has never been considered. We will show that it is surprisingly rich and elegant.

Since causal sets are well known as models for general relativity, let us mention some of their similarities and differences with causal sites. Causal sites with a suitable finiteness condition have an intrinsic metric structure, as mentioned above. It is defined by counting the length of a maximal causal chain. This is quite similar to the intrinsic metric of a discrete causal set [Sorkin 2003]. We believe the greater flexibility of the families of regions in a causal site will mean they give much better approximations to the geometry of spacetimes than causal sets do. There can be infinitely many regions intermediate between two given ones in a causal site, and yet causal paths can have a maximal finite length, as we show in examples below. While random causal sets which approximate Minkowski space are believed to reproduce Lorentz invariance in the infinite volume limit, examples of causal sites which approximate Minkowski space have an invariance under germs of the Lorentz group on bounded regions as well. As models for general relativity, causal sites can be studied in many of the same ways as causal sets. Mathematically, causal sites are a “categorification” of causal sets.

The fundamental mathematical observation is that both the containment structure of a site and the causal structure of a causal set can be described as partial orders. As we will explain in Section 5, these two partial orders give rise to a natural bisimplicial set which we call the elementary classifying space of the causal site. From this point of view, certain collections of regions can be viewed as products of simplices, and these products are glued together in a well-defined way.

We also construct another bisimplicial set, which we call the physical classifying space. This turns out to be a “special” bisimplicial set, a type of bisimplicial set which corresponds to a weak 2-category [Tamsamani 1995]. This means that causal sites fall into a class of structure which is already studied, and that the family of simplicial presheaves over it is well understood.

We will argue at the end that this suggests an approach to doing quantum general relativity over causal sites, by putting state sum models on their tangent 2-bundles.

This paper is meant to open a number of lines of research. The contents are as follows: in Section 2 we present the axioms for a causal site and some simple consequences. In Section 3 we show how a causal site with suitable finiteness conditions can have an intrinsic geometry. In Section 4, we briefly discuss quantization of sites and in Section 5 we discuss
the simplicial and bisimplicial structures inherent in a causal site. In Section 6, we translate this into a bicategorical structure. Sections 7 and 8 discuss the structure of the tangent spaces and tangent bundle, while Section 9 discusses the general analog of a bundle for a causal site, namely a prestack. In Sections 10 and 11 we discuss how state sum models could emerge in a causal site. Section 12 contains conclusions and outlook.

1.1. Categorical background. We freely make use of terminology and ideas from category theory. In particular, we assume basic familiarity with weak 2-categories, which are also called bicategories. Good references for this topic are [Leinster 2004, Ch. 1], [Leinster 1998] and [Bénabou 1967]. We also give more specific references as needed.

2. The Axiomatic Structure of Causal Sites

We are now going to axiomatize the structure of regions in a causal spacetime.

Example 2.1. Let \( M \) be a Lorentzian manifold with no closed timelike curves and a global time orientation. For points \( p \) and \( r \) in \( M \), write \( p \ll r \) if there is a future-directed timelike curve from \( p \) to \( r \), and let \( D(p, r) \) be the set of all points \( q \) with \( p \ll q \ll r \). We call \( D(p, r) \) a diamond, and we say that a subset \( A \) of \( M \) is bounded if it is contained in a finite union of diamonds. For \( A \) and \( B \) bounded regions, write \( A \subseteq B \) when \( A \) is a subset of \( B \), and write \( A \prec B \) when every point in region \( A \) is in the causal past of every point in region \( B \). That is, for each \( a \) in \( A \) and \( b \) in \( B \), \( a \ll b \).

The motivation for the above definition of “bounded” is the following. If a subset \( A \) has compact closure, then it is bounded in the above sense. And if the manifold \( M \) is globally hyperbolic, then the converse holds. So for globally hyperbolic manifolds, bounded is equivalent to compact closure. However, in general, our completion axiom below is only satisfied if one allows regions without compact closure.

Below we list some properties that this set of regions has. We then want to consider more general systems satisfying the axioms. We believe the interesting examples will actually have fewer regions than the example above coming from a Lorentzian manifold.

Definition 2.2. A causal site is a set of “regions” with two binary relations denoted \( \subseteq \) and \( \prec \) satisfying the axioms below. If \( A \subseteq B \) we say that \( A \) is a subset of \( B \) or that \( B \) contains \( A \). If \( A \prec B \) we say that \( A \) precedes \( B \).

1. \( \subseteq \) is a partial order on the set of regions. This means that for all regions \( A \), \( B \) and \( C \):
   a) \( A \subseteq B \) and \( B \subseteq C \) implies \( A \subseteq C \);
   b) \( A \subseteq A \);
   c) \( A \subseteq B \) and \( B \subseteq A \) implies \( A = B \).

2. The partial order \( \subseteq \) has a minimum element \( \phi \). This means that \( \phi \) is contained in every region \( A \). This uniquely determines \( \phi \), and \( \phi \) is called the empty region.

3. The partial order \( \subseteq \) has unions. This means that for all regions \( A \) and \( B \), there exists a region \( A \cup B \) such that:
   a) \( A \subseteq A \cup B \) and \( B \subseteq A \cup B \);
   b) if \( A \subseteq C \) and \( B \subseteq C \) then \( A \cup B \subseteq C \).

These requirements uniquely determine \( A \cup B \), and the binary operation \( \cup \) is automatically associative and commutative.
4. \( \prec \) induces a strict partial order on the non-empty regions. This means that for all non-empty regions \( A, B \) and \( C \):
   a) \( A \prec B \) and \( B \prec C \) implies \( A \prec C \);
   b) \( A \) does not precede \( A \).

5. For all regions \( A, B \) and \( C \), \( A \subseteq B \) and \( B \prec C \) implies \( A \prec C \).

6. For all regions \( A, B \) and \( C \), \( A \subseteq B \) and \( C \prec B \) implies \( C \prec A \).

7. For all regions \( A, B \) and \( C \), \( A \prec C \) and \( B \prec C \) implies \( A \cup B \prec C \).

8. For all regions \( A \) and \( B \), there exists a region \( B_A \) such that:
   a) \( B_A \prec A \) and \( B_A \subseteq B \);
   b) if \( D \prec A \) and \( D \subseteq B \) then \( D \subseteq B_A \).

These requirements uniquely determine \( B_A \), and \( B_A \) is called the cutting of \( A \) by \( B \).

Note that \( B_A \) can be empty.

9. If \( A \) and \( C \) are non-empty regions such that \( A \prec C \) and there exists a \( D \) with \( A \prec D \prec C \), then there exists a \( B \) complete with respect to \( A \prec C \). The definition of “complete” is below (Definition 2.5).

**Definition 2.3.** Regions \( A \) and \( B \) in a causal site are disjoint if the only region which is contained in both \( A \) and \( B \) is the empty region. More generally, a set of regions is disjoint if each pair of regions it contains is disjoint.

Note that if \( A \prec B \) then \( A \) and \( B \) are disjoint.

**Definition 2.4.** Suppose \( S \) and \( T \) are sets of disjoint regions. If every region in \( S \) contains some region in \( T \), we say that \( T \) is a refinement of \( S \). That is, \( T \) is obtained from \( S \) by shrinking some regions and adding new regions.

A causal path \( P \) is a sequence \( A_1 \prec A_2 \prec \cdots \prec A_m \) of non-empty regions. If \( A \prec A_1 \) and \( A_m \prec B \), we say that \( P \) is a causal path from \( A \) to \( B \) and has length \( m + 1 \). There is exactly one causal path of length 1.

A refinement of the causal path \( A_1 \prec A_2 \prec \cdots \prec A_m \) is a causal path \( C_1 \prec C_2 \prec \cdots \prec C_n \) such that \( \{C_j\} \) is a refinement of \( \{A_i\} \).

If \( P \) is a causal path from \( A \) to \( B \), and \( A' \prec A \), then \( P \) is also a causal path from \( A' \) to \( B \). Analogous statements can be made when \( A' \subseteq A \), \( B \prec B' \) and \( B' \subseteq B \). See Section 8 for more details.

**Definition 2.5.** If \( A \prec B \prec C \), we say that \( B \) is complete for the causal pair \( A \prec C \) if every causal path from \( A \) to \( C \) can be refined to a causal path from \( A \) to \( C \) one of whose members is contained in \( B \). \( B \) is called a completion of \( A \prec C \).

See Figure 1 for an example of a completion. The final axiom of a causal site requires that completions exist. But note that they are rarely unique.

There are some elementary consequences whose proofs we leave to the reader.

**Proposition 2.6.** The following are true in any causal site.

1. For every region \( A \), \( \phi \prec A \) and \( A \prec \phi \).

2. \( B_A = B \) iff \( B \prec A \).

3. \( B \subseteq C \) implies \( B_A \subseteq C_A \).
4. \( B_A \cup C_A \subseteq (B \cup C)_A \). (The reverse inclusion fails for some of our examples.)

5. The collection of regions of a causal site which precede a region \( B \) forms another causal site. When \( B \) is non-empty, this causal site is called the \textbf{local site of} \( B \).

\textbf{Example 2.7.} Let \( M \) be as in Example 2.1. Then the bounded regions in \( M \), with the relations \( \subseteq \) and \( \prec \) defined earlier, satisfy the axioms.

We'll check the non-trivial axioms. For the cutting axiom (Axiom 8), let \( B_A \) be the set of all points \( p \) such that \( \{ p \} \prec A \) and \( p \in B \). This is bounded (contained in a finite union of diamonds) since \( B \) is.

For the completion axiom (Axiom 9 and Definition 2.4), a completion of \( A \prec C \) can be taken to be the set of all points \( p \) such that \( A \prec \{ p \} \prec C \). This is bounded because it is a subset of the diamond \( D(a,c) \) for any points \( a \in A \) and \( c \in C \). (In general, this completion may not have compact closure.)

The remaining axioms are straightforward.

\textbf{Example 2.8.} Let \( (M, \leq) \) be any poset (partially ordered set) and let \( \mathcal{P}(M) \) be the set of all subsets of \( M \). We define \( \subseteq \) to be the usual subset relation, and for \( A \) and \( B \) subsets of \( M \), we say that \( A \prec B \) iff \( \forall a \in A \forall b \in B \ a < b \). (Note that we use the strict inequality \( a < b \).) Then \( \mathcal{P}(M), \subseteq \), and \( \prec \) form a causal site. A completion of \( A \prec B \) can be taken to be \( \{ m \in M \mid \forall a \in A \forall b \in B \ a < m < b \} \).

\textbf{Example 2.9.} Let \( (M, \leq) \) be a poset such that \( \{ n \in M \mid m \leq n \leq m' \} \) is finite for each \( m \) and \( m' \) in \( M \). (Such a poset is called \textbf{locally finite} or a \textbf{causal set} [Sorkin 2003].) Let \( \mathcal{P}'(M) \) be the set of finite subsets of \( M \). Then \( \mathcal{P}'(M) \) is a causal site using relations defined as in the previous example. The hypothesis on \( M \) ensures that completions exist.

While the axioms for a causal site were modelled on the example where regions are bounded subsets of a causally well-behaved Lorentzian manifold, and \( A \prec B \) iff every element of \( A \) is in the causal past of every element of \( B \), we can use the structure we have to define a weaker relation.

\textbf{Definition 2.10.} For regions \( A \) and \( B \), write \( A \preceq B \) if there exist non-empty subsets \( A' \subseteq A \) and \( B' \subseteq B \) with \( A' \prec B' \) or \( A' = B' \). We say that \( A \) \textbf{weakly precedes} \( B \).

In the example of a Lorentzian manifold (Example 2.1), \( A \preceq B \) iff some point of \( A \) is in the causal past of some point of \( B \).

Note that \( \preceq \) is not transitive. That is, \( A \preceq B \) and \( B \preceq C \) does not imply \( A \preceq C \).

\textbf{Definition 2.11.} For regions \( A \) and \( B \) with \( A \prec B \), we say that \( C \) is \textbf{strongly complete} for \( A \) and \( B \) if for all non-empty \( A' \subseteq A \) and \( B' \subseteq B \), any causal path from \( A' \) to \( B' \) can refined to a causal path from \( A' \) to \( B' \) one of whose members is contained in \( C \).

Note that we do not require that \( A \prec C \prec B \), since a strong completion will rarely satisfy this. A strong completion is “wider” than an ordinary completion. See Figure 1.

\textbf{Definition 2.12.} A region \( B \) is a \textbf{cover} of a region \( A \) if \( B \prec A \) and if every causal path ending in \( A \) can be refined to a causal path ending in \( A \) one of whose elements is contained in \( B \).

A finite set of regions \( \{ B_i \} \) is a \textbf{cover} of \( A \) if the union of the cover is a cover of \( A \).

\textbf{Remark 2.13.} We can regard a causal site as a category whose objects are the regions by saying that there is one morphism from \( A \) to \( B \) if \( A \prec B \) or \( A = B \), and no morphisms otherwise. The collection of covers defined above has the feel of being a Grothendieck topology [Mac Lane et al 1992]. But even if pullbacks exist in this category (they do, for example, in the case of Example 2.1), the pullback of a cover is not necessarily a cover.
Below we shall define a 2-category whose 1-arrows are all composable strings of arrows in this category, i.e. causal paths.

**Definition 2.14.** A causal site is **Noetherian** if every chain of strict descending regions $A_1 \supset A_2 \supset A_3 \supset \cdots$ is finite. It is **locally Noetherian** if every local site is Noetherian.

Noetherian implies locally Noetherian.

The causal site $\mathcal{P}'(M)$ described in Example 2.9 is always Noetherian. If $M = \mathbb{N}$ with the usual order, then the causal site $\mathcal{P}(M)$ described in Example 2.8 is locally Noetherian but not Noetherian.

**Definition 2.15.** A non-empty region $A$ in a causal site is an **absolute point** if it contains no subregions besides $\phi$ and $A$. If $A$ and $B$ are non-empty regions in a causal site, we say that $A$ is a **relative point** for $B$ if $A \prec B$ and for any $C$ with $C \prec B$ either $A = A_C$ (i.e. $A \prec C$) or $A_C = \phi$.

If $A$ is an absolute point and $A \prec B$, then $A$ is a relative point for $B$. And if $A$ is a relative point for $B$, then any non-empty subregion of $A$ is also a relative point for $B$.

In Example 2.1, an absolute point is just a point in the usual sense, and every relative point is an absolute point.

In Examples 2.8 and 2.9, the absolute points are just the elements of the poset, but the relative points can be larger. For example, let $Q$ be the poset $\{a, b, b', c\}$ with $a < b < c$, $a < b' < c$ and with $b$ and $b'$ unrelated. Then in the causal site $\mathcal{P}(Q)$, the subset $B = \{b, b'\}$ is a relative point for $C = \{c\}$. It is a union of the absolute points $\{b\}$ and $\{b'\}$.

We believe that because of the theorems that only a finite dimensional space of information can flow across horizons in general relativity Planagan et al 2000 there should be causal sites associated to Lorentzian manifolds which have interesting examples of relative points. The information contained in the relative position of a finite set of regions should exhaust what an external observer can see.

**Definition 2.16.** If $A$ and $B$ are regions in a causal site, we say that $A$ is **amply pointed** with respect to $B$ if $A \prec B$ and if $C, D \subseteq A$ then either $C$ or $D$ contains a relative point for $B$ which the other does not, or $C = D$. A causal site is **amply pointed** if all causal pairs in it are.
Definition 2.17. A causal site which is locally Noetherian and amply pointed is a **grained world**.

In addition to the examples coming from causal sets, we believe that the family of Noetherian causal sites described in Section 3 produces grained worlds whenever there are no pairs of points connected by more than one Planck-scale geodesic. This still needs proof.

Although one can think of the regions in a causal site as like the bounded subsets of a causal spacetime, we believe the interesting cases will be very different from such classical examples. The Noetherian property discussed above is an approach to imposing a Planck scale cutoff on the structure of a causal site.

3. The Intrinsic Geometry of Causal Sites

It turns out that a causal site can contain more information than just the causal structure of a manifold. The reason is that a causal site may have a fundamental graininess which sets a length and time scale. Physically, this graininess is expected to occur at the Planck scale, and serves as a measuring rod or clock. Heuristically, a measurement at a smaller scale would result in the formation of black hole, so the maximum possible number of successive measurements along a timelike path gives its duration in Planck units.

Definition 3.1. A causal site is **causally finite** if for any two regions \( A \prec B \), the length of any causal path from \( A \) to \( B \) is bounded above by a constant \( M_{A,B} \).

We now want to interpret the least upper bound for the length of a causal path between two causally related regions as the discretized timelike separation between them in Planck units. This is quite similar to the notion of length in a discrete causal set [Sorkin 2003].

We now give examples to show that the resulting geometry can be quite interesting.

**Example 3.2.** Consider Minkowski space. Recall from Example 2.1 that a **diamond** is a region \( D(p, r) = \{ q \mid p \ll q \ll r \} \). Define a **fundamental diamond** to be a region \( D(p, r) \) where the proper time from \( p \) to \( r \) is 1. Consider the set of bounded regions which are unions of fundamental diamonds. Define \( \subseteq \) and \( \prec \) as in Example 2.1. This forms a causal site.

Let’s check the non-trivial axioms. For the cutting axiom (Axiom 8), let \( B \cup A \) be the union of all fundamental diamonds \( D \) such that \( D \prec A \) and \( D \subseteq B \). This is bounded (contained in a finite union of diamonds) since \( B \) is. Note that it can happen that \( B \cup A \) is empty even though there are points of \( B \) which are in the past of \( A \).

For the completion axiom (Axioms 9 and Definition 2.5), a completion of \( A \prec C \) can be taken to be the union of all fundamental diamonds \( D \) such that \( A \prec D \prec C \). This is bounded because it is a subset of the diamond \( D(a, c) \) for any points \( a \in A \) and \( c \in C \).

The remaining axioms are straightforward.

Figure 2 gives two examples of fundamental diamonds. The timelike vectors shown have proper length 1.

The following theorem explains how this causal site captures the geometry of Minkowski space.

**Theorem 3.3.** Let \( \gamma \) be a finite timelike geodesic segment in Minkowski space, starting at the point \( s \) and ending at \( t \). In the causal site of Example 3.2, consider the causal paths \( A_1 \prec \cdots \prec A_m \) with \( s \in A_1 \) and \( t \in A_m \). Then the proper length of \( \gamma \), rounded up to an integer, is equal to the least upper bound of the lengths of these causal paths.

The key element of the proof is the observation that using fundamental diamonds in any other rest frame than the one determined by the geodesic produces a shorter chain. This
is clear from the above picture. In physical applications, we would use units in which the
Planck time is 1.

To handle curved spacetimes, we need to introduce the concept of stable causality. A
Lorentzian manifold \( M \) is **stably causal** if it has no closed timelike curves, and every small
perturbation of the metric \( g \) also has this property. This can be expressed by saying that
there exists a metric \( h \) whose light cones are wider than those of \( g \) and such that \( h \) has
no closed timelike curves. This condition is more physical that simply requiring no closed
timelike curves, since measurements have only finite accuracy.

It is a non-trivial result [Hawking et al 1973, Prop. 6.4.9] that stable causality is equiv-
alent to the existence of a global time function. In particular, a stably causal spacetime is
automatically time orientable.

Suppose that \( M \) is stably causal and we have chosen a time orientation. It is shown
in [Penrose 1972] that since \( M \) is stably causal, the diamonds \( D(p, r) \) determine the topology
of \( M \).

**Example 3.4.** Generalizing the previous example, let \( M \) be a stably causal Lorentzian
manifold. Define a **fundamental diamond** to be a region \( D(p, r) \) such that there is a
future-directed timelike geodesic of length 1 from \( p \) to \( r \). Consider the set of bounded
regions which are unions of fundamental diamonds. Define \( \subseteq \) and \( \prec \) as in Example 2.1.
This forms a causal site. The proof is the same as Example 3.2.

The following conjecture explains how these causal sites capture the large-scale geometric
information of the manifold.

**Conjecture 3.5.** Let \( M \) be a stably causal Lorentzian manifold whose sectional curvatures
are much less than 1, and let \( \gamma \) be a timelike curve from \( s \) to \( t \) in \( M \) whose radius of
curvature and length are large. In the causal site of Example 3.4 consider the causal paths
\( A_1 \prec \cdots \prec A_m \) such that each region \( A_i \) intersects \( \gamma \). Then the proper length of \( \gamma \) is well
approximated by the least upper bound of the lengths of these causal paths, with an error
which is small compared to the length of \( \gamma \).

Of course, the precise statement of the error of the approximation in terms of the cur-
vatures will require a more delicate analysis.
We believe this causal site reflects classical geometry more accurately than a causal set can. This is because we have enough relative points to adjust to any direction to get the best fit.

It is well known that the lengths of timelike curves (clock times), are enough to describe the geometry of a Lorentzian manifold completely. Thus our conjecture implies that any solution to general relativity whose sectional curvature is small compared to the Planck scale, with any type of matter whatsoever, can be approximated by the intrinsic geometry of a causally finite causal site with accuracy on the order of the Planck scale.

The causal sites discussed in this section have a fundamental graininess, but are nevertheless not Noetherian or locally Noetherian. It is not hard to invent modifications of our examples which would have stronger finiteness conditions and still recover interesting geometrodynamics. For example, fix a discrete closed subspace $L$ of Minkowski space and a discrete closed subspace $S$ of the hyperboloid of unit timelike vectors. Now consider bounded regions $R$ in Minkowski space which are unions of fundamental diamonds $D(l - s/2, l + s/2)$ centered on points $l$ of $L$ and pointing in directions $s$ in $S$. Since $R$ is bounded, it contains only finitely many points in $L$. Moreover, in any sufficiently boosted coordinate system, it would be too contracted to contain a fundamental diamond. Thus it only contains finitely many fundamental diamonds whose directions are contained in $S$ and which are centered on points of $L$. Therefore this collection of regions forms a Noetherian causal site which reproduces the spacetime geometry of Minkowski space at a large scale. Such a model is computationally accessible, and is a good candidate for quantization, as discussed below.

The discrete subspace $L$ of Minkowski space could be chosen to be a random sprinkling of points, or a lattice. In a general manifold, $L$ would have to be an irregular collection of points. If the manifold is globally hyperbolic, then one again obtains a Noetherian causal site.

Note that the use of fundamental diamonds is not crucial to any of the discussion in this section. One could use other shapes, such as “fundamental cylinders,” which are regions formed by working in normal coordinates and taking the cartesian product of a timelike interval of length 1 with a ball of radius 1 in the hyperplane orthogonal to the chosen timelike direction.

As yet, we do not know how to impose Einstein’s equation on a causal site purely intrinsically.

4. **Quantum Sites**

A causal site can be thought of as a large number of answers to questions, either telling us one region is inside another, or that one region can observe another.

The answers to these questions can be grouped together in certain good examples and reinterpreted as describing physically interesting geometries.

It is therefore natural to reinterpret the statements that define a causal site as quantum observables and attempt to extract a quantum geometry from them. This could be a new avenue of attack on the problem of quantizing gravity.

It is not hard to see how to begin such a program. We could tensor together a finite dimensional Hilbert space for each pair of regions, and construct operators for containment and causal relatedness on each. The interesting part would be to see if suitable commutation relations could be found on the operators to reproduce Einstein’s equation in the classical limit, for suitable families of causal sites. We have not yet investigated this.
5. **Simplicial Structures and Causal Sites**

We first remind the reader of the relationship between partial orders and simplicial sets. A good reference for simplicial sets is [Goerss et al 1999].

Roughly speaking, a simplicial set is a set of abstract simplices (points, edges, triangles, tetrahedra, etc.) with the faces of the simplices of dimension $n$ identified with simplices of dimension $n - 1$, thus gluing the simplices together into a combinatorial model of a space. The mathematically natural definition of a simplex includes an ordering of its vertices, and includes “degenerate” simplices in which some vertices are repeated. If $X$ is a simplicial set, we write $X_n$ for the set of $n$-simplices.

Associated to a given partially ordered set is a simplicial set which contains all of the information. An $n$-simplex of this simplicial set is a weakly ascending chain of length $n + 1$. The face maps come from omitting one member of the chain, and the degeneracies from repeating one member.

Since the regions of a causal site have two partial orders on them, they have a natural description as the vertices of a bisimplicial set. Roughly speaking, a bisimplicial set is a collection of abstract cartesian products of pairs of simplices, with attachments along face maps corresponding to both simplicial factors separately. The product of a triangle with a tetrahedron, for example, has three faces which are edge $\times$ tetrahedron, and four which were triangle $\times$ triangle. If $X$ is a bisimplicial set, we write $X_{m,n}$ for the set of $(m,n)$-bisimplices.

Let us give an explicit description of the bisimplicial set of a causal site. An $(m,n)$-bisimplex is a family $A_{i,j}$ of regions in the site, $i = 0, \ldots, m$, $j = 0, \ldots, n$, such that if $a < b$ then $A_{a,j} \prec A_{b,j}$ or $A_{a,j} = A_{b,j}$ and if $c < d$ then $A_{i,c} \subseteq A_{i,d}$. The two types of face maps come from omitting one value of $i$ or $j$, and the two types of degeneracies from repeating items in the sequences.

**Definition 5.1.** The bisimplicial set associated to a causal site above is called its **elementary classifying space**.

Note that if $X$ is any bisimplicial set, and we fix $m$, then there is a natural simplicial set $X_{m,\cdot}$ whose $n$-simplices are the $(m,n)$-bisimplices of $X$. We call $X_{m,\cdot}$ the **simplicial set of $m$-simplices**.

It is now an important observation that the elementary classifying space of a causal site satisfies the **Segal condition** [Leinster 2002, Defn. Ta], [Tamsamani 1995]. The Segal condition for a bisimplicial set has two parts. Translated into our situation, the first part states that the simplicial set of $m$-simplices is the subset of the product of $m$ copies of the simplicial set of 1-simplices, where the adjacent 0-simplex objects are equal. More explicitly, it says that for each $m$ and $n$, the natural map

$$X_{m,n} \rightarrow X_{m,1} \times X_{m,0} \cdots \times X_{m,0} \times X_{m,1}$$

must be a bijection. This is true for the elementary classifying space, essentially because giving a chain $A_{i,0} \subseteq \cdots \subseteq A_{i,n}$ is the same as giving $n$ chains $A_{i,0} \subseteq A_{i,1}$, $A_{i,1} \subseteq A_{i,2}$, $\ldots$, $A_{i,n-1} \subseteq A_{i,n}$.

The second part of the Segal condition for bisimplicial sets is more subtle in general. The first part of the Segal condition tells us that for each $m$, the simplicial set $X_{m,\cdot}$ can be regarded as a category. The second part of the Segal condition then requires that for each $m$, the natural map

$$X_{m,\cdot} \rightarrow X_{1,\cdot} \times X_{0,\cdot} \cdots \times X_{0,\cdot} \times X_{1,\cdot}$$
must be an equivalence of categories. In our case, something stronger is true: for each \(m\) and \(n\), the natural map

\[ X_{m,n} \to X_{1,n} \times X_{0,n} \times \cdots \times X_{0,n} \]

is a bijection. The reason this is true is similar to the reason that the first part of the Segal condition holds. And from this it follows that the categories above are equivalent.

Now let us describe the **Segal condition** for a simplicial set \(A\). It simply states that there is exactly one \(n\)-simplex for each chain of \(n\) 1-simplices with the second vertex of the \(i\)-th 1-simplex equal to the first vertex of the \((i+1)\)-st. That is, it states that the natural map

\[ A_n \to A_1 \times A_0 A_1 \times A_0 \]

is a bijection. This is true, for example, for the simplicial set associated to any partially ordered set. We will explain the importance of this condition below, when we study the relationship with category theory.

One point of all this is that we can completely capture the structure of a causal site as a combinatorially described space. This allows us to translate the problem of constructing physical theories on a causal site into the construction of presheaves over such a space, a problem which is well understood. Another is to join the concept of causal sites to the field of higher category theory.

Now we need to construct a modification of the elementary classifying space of a causal site. The modification will allow us to associate to a causal site a “special” bisimplicial set; see [Tamsamani 1995], where it is referred to as axiom C1, and [Leinster 2002, Defn. Ta].

**Definition 5.2.** A bisimplicial set \(X\) is **special** if the simplicial set \(X_{0,n}\) of 0-simplices is a disjoint union of points. In other words, all of the face and degeneracy maps between sets of the form \(X_{0,n}\) are bijections.

This condition is important in category theory, because it is necessary for a bisimplicial set to be the nerve of a weak 2-category. As we shall see, this condition also plays a natural role when we try to construct models for quantum physical systems.

Now we want to define a new version of the classifying space of a causal site.

**Definition 5.3.** The **physical classifying space** of a causal site is the bisimplicial set whose \((m,n)\)-bisimplices correspond to the following data:

a) a sequence \(A_0, \ldots, A_m\) of regions with \(A_i \prec A_{i+1}\);

b) for each \(i = 0, \ldots, m-1\) and \(j = 0, \ldots, n\) a causal path \(C_{ij}^j\) from \(A_i\) to \(A_{i+1}\);

such that for each \(i\) and \(j\), the causal path \(C_{ij}^j\) is a refinement of \(C_{ij}^{j+1}\). The two types of face maps are given by composition of rows and columns respectively.

In this definition, refinements of causal paths play a central role. The physical motivation for this is the goal of writing discrete analogs of Feynman path integrals, by summing effects of propagation along causal paths with the same beginning and end. Mathematically, we will see below that the \((m,n)\)-bisimplices of the physical classifying space fit together to form a weak 2-category, because the initial and final paths of a refinement share a common initial and final region.

We would like to formulate the idea that all the information of a causal site is contained in its physical classifying space. We therefore make the following tentative definition:

**Definition 5.4.** A causal site is **tractable** if its elementary and physical classifying spaces are homotopy equivalent.
We do not know which causal sites are tractable, or whether this will prove to be a useful way to formulate the idea that the physical classifying space “suffices”.

**Proposition 5.5.** The physical classifying space of a causal site is a special bisimplicial set satisfying the Segal condition.

The proof is similar to the argument given for the elementary classifying space, including the fact that a stronger form of the second part of the Segal condition holds.

Thus, as we explain below, the physically natural idea of focusing on the causal paths of a causal site supplies the missing mathematical ingredient to replace a spacetime with a weak 2-category. We think this is suggestive for our program of applying higher categorical ideas of topology to quantum physics.

### 6. The Weak 2-category Structure of a Causal Site

There is a very strong connection between simplicial and multisimplicial sets and higher categories. We begin with a standard theorem about (1-)categories [Leinster 2002, p. 34], [Tamsamani 1995].

**Theorem 6.1.** There is a one-to-one correspondence between categories and simplicial sets satisfying the Segal condition.

The connection can be described by taking the nerve of the category. This is a simplicial set with one $n$-simplex for each string of $n$ composable morphisms in the category, with face maps given by compositions of pairs of adjacent morphisms and degeneracies given by inserting identity maps.

This theorem has recently been extended from categories to 2-categories by Tamsamani [Leinster 2002, Defn. Ta], [Tamsamani 1995]: he proved that there is a natural one-to-one correspondence between special bisimplicial sets satisfying the Segal condition and weak 2-categories.

The connection proceeds by taking the “2-nerve” of the 2-category. This is the bisimplicial set with $(0,0)$-bisimplices the objects of the 2-category, $(1,0)$-bisimplices the 1-morphisms in the 2-category, $(1,1)$-bisimplices the 2-morphisms in the 2-category. A general $(m, n)$-bisimplex is a doubly indexed array of $mn$ 2-morphisms along with some additional data.

The analogy with the structure of the physical classifying space prompts the following definition:

**Definition 6.2.** If the causal path $A \prec P_1 \prec P_2 \prec \cdots \prec P_p \prec B$ is a refinement of the causal path $A \prec Q_1 \prec Q_2 \prec \cdots \prec Q_q \prec B$ then we say that there is a **chain inclusion** from the causal path $\{Q_j\}$ to the causal path $\{P_i\}$.

The **inclusion 2-category** of a causal site is the 2-category whose objects are regions, 1-morphisms are causal paths, and 2-morphisms are chain inclusions.

The inclusion 2-category of a causal site connects regions, processes between regions, and inclusions of processes. These are the elements which would go into a description of an experiment. A 2-functor on this 2-category would then give us a mathematical language in which to associate calculations to an experiment, by associating concrete mathematical structures and relations to the objects and processes.

As we shall discuss below, the state sum models for quantum gravity have a natural formulation in terms of 1- or 2-categories. Any two 2-categories are connected by a 2-category of functors, natural transformations and modifications. This means the categorical structure we have described gives us a sort of calculus for constructing physical models over causal sites using the state sum models as local physical data.
To put things simply, one could easily feel that passing from point sets to regions led to a mathematical wilderness. The structure we have discovered has geometric, categorical and algebraic aspects which make available to us a large framework of definitions and theorems which can guide us to natural constructions of physical models. The natural definition of a physical model on a causal site is a type of functor on it. The definition could in principle be found without the categorical language, but it is a foolish cave explorer who throws away a light.

7. Bisimplicial patches

The geometrical and physical applications of smooth manifolds largely grow out of the fact that to every smooth manifold we can naturally associate a tangent bundle. This develops from the more elementary fact that every point of a manifold has a neighborhood which can be described as a space of a fixed and familiar kind. In this section we propose such a neighborhood structure. In the next, we show how to construct an analog of the tangent bundle, but a relational, or 2-categorical one.

Definition 7.1. If $A$ is a region in a causal site, the bisimplicial set of regions contained in it is its **bisimplicial patch**. This bisimplicial set sits naturally inside the elementary classifying space (Definition 5.1).

With this definition in place, it is possible to treat causal sites in a manner analogous to manifolds, by working on the local bisimplicial patches thought of as coordinate patches. Simplicial and bisimplicial sets have a homotopy theory equivalent to the homotopy theory of topological spaces. This subtle mathematical fact should allow us to use the combinatorics of the bisimplicial patch as an approximation to the local topology of a region.

Definition 7.2. Let $R \subseteq R'$ be regions. The **bisimplicial patch** of the complement is the bisimplicial set of regions $A$ which are contained in $R'$ and disjoint from $R$. The **relative homotopy type** of $R$ in $R'$ means the homotopy type of the pair $(X,Y)$ where $X$ is the bisimplicial patch of $R'$ and $Y$ is the bisimplicial patch of the complement of $R$ in $R'$.

Definition 7.3. A region $R$ is an **$n$-ball** if it is contained in a region $R'$ such that the relative homotopy type is that of an $n$-sphere.

A causal site is a **grained $n$-manifold** if every region is contained in a union of a finite number of $n$-balls.

8. The Tangent 2-Bundle

In a physical model on a causal site, we would like to think of information being transferred along causal paths. We will now describe a construction of a version of a tangent bundle for causal sites in which the regions contain just such information as can be observed along the causal paths, while the compositions and inclusions of causal paths have a natural action.

8.1. Causal paths. For $A \prec B$ we define Path$(A, B)$ to be the set of causal paths from $A$ to $B$. For any $A \prec B \prec C$ there is a natural composition map Path$(A, B) \times$ Path$(B, C) \rightarrow$ Path$(A, C)$ which sends $A \prec P_1 \prec \cdots \prec P_m \prec B$ and $B \prec Q_1 \prec \cdots \prec Q_n \prec C$ to $A \prec P_1 \prec \cdots \prec P_m \prec B \prec Q_1 \prec \cdots \prec Q_n \prec C$. The composition map is injective.

For a region $A$, define the **future cone of $A$** to be $A^\uparrow = \{ B \mid A \prec B \}$ and define the **future tangent space of $A$** to be $A^\uparrow = \{ A'^\uparrow \mid A' \subseteq A \}$, the poset of future cones of subregions of $A$, ordered by inclusion. The future cone $A^\uparrow$ is a minimum element of $A^\uparrow$. 
For a region $B$, define the past cone of $B$ to be $B\downarrow = \{A \mid A \prec B\}$ and define the past tangent space of $B$ to be $B\down\uparrow = \{B' \mid B' \subset B\}$, ordered by inclusion. The past cone $B\downarrow$ is a minimum element of $B\down\uparrow$.

The relevance of these cones is that if $A_1\uparrow = A_2\uparrow$, then $\text{Path}(A_1, B) = \text{Path}(A_2, B)$ for any $B$. (This is an equality rather than a bijection because a path from $A$ to $B$ doesn’t include the regions $A$ and $B$.) More generally, if $A_1\uparrow \subset A_2\uparrow$, then $\text{Path}(A_1, B) \subset \text{Path}(A_2, B)$ for any $B$. Similarly, $B_1\downarrow \subset B_2\downarrow$ implies that $\text{Path}(A, B_1) \subset \text{Path}(A, B_2)$ for any $A$. Writing $\text{Fut}$ for the poset $\{A\uparrow\}$ of all future cones and $\text{Past}$ for the poset $\{B\down\uparrow\}$ of all past cones, this says that $\text{Path}(-, -)$ is an order preserving map from the poset $\text{Fut} \times \text{Past}$ to the poset $\text{Inc}$ of sets and inclusions. In fancier language, $\text{Path}(-, -)$ is a functor.

This has various consequences. For example, if $A' \subset A$, then $A'\uparrow \supseteq A\uparrow$ and so $\text{Path}(A', B) \supseteq \text{Path}(A, B)$ for any $B$. And if $A' \prec A$, then $A'\uparrow \supseteq A\uparrow$ and so again $\text{Path}(A', B) \supseteq \text{Path}(A, B)$ for any $B$. Similar reasoning shows that if $B' \subset B$ or $B' \succ B$ (note the reversals!), then $B'\uparrow \supseteq B\uparrow$ and so $\text{Path}(A, B') \supseteq \text{Path}(A, B)$ for any $A$.

8.2. **Causal paths subordinate to a given path.** If we are given a causal path $P$ from $A$ to $B$, we can assign to any pair of subregions $A' \subset A$ and $B' \subset B$ the set $\text{Path}_P(A', B')$ of causal paths from $A'$ to $B'$ which are refinements of $P$. As above, this defines a functor, in this case from $A\uparrow \times B\down\uparrow$ to $\text{Inc}$.

We next describe a 2-category which is a natural target for this construction.

**Definition 8.1.** Define a weak 2-category $\mathcal{P}\mathcal{P}$ in the following way. The objects are pairs $(F, P)$ of posets with minimum elements $m_F$ and $m_P$. A 1-morphism from $(F_1, P_1)$ to $(F_2, P_2)$ is a functor from $F_1 \times P_2$ to $\text{Inc}$. The composite of $S : F_1 \times P_2 \to \text{Inc}$ with $T : F_2 \times P_3 \to \text{Inc}$ is the functor $TS : F_1 \times P_3 \to \text{Inc}$ defined by $TS(f, p) = S(f, m_{P_2}) \times T(m_{F_2}, p)$. If $S$ and $T$ are 1-morphisms from $(F_1, P_1)$ to $(F_2, P_2)$, i.e. functors from $F_1 \times P_2$ to $\text{Inc}$, a 2-morphism from $S$ to $T$ is a family of injections $S(f, p) \to T(f, p)$.

Note that the composites $(TS)R$ and $T(SR)$ of 1-morphisms are in general not equal. Instead, there is a natural bijection between them. Thus $\mathcal{P}\mathcal{P}$ is a weak 2-category. This also explains why the 2-morphisms allow arbitrary injections rather than just inclusions.

The horizontal and vertical compositions of the 2-morphisms of $\mathcal{P}\mathcal{P}$ are given by cartesian product and composition of injections, respectively. The coherence of $\mathcal{P}\mathcal{P}$ is natural.

The discussion above can be summarized by saying that there is a weak 2-functor from the inclusion 2-category of our causal site to the weak 2-category $\mathcal{P}\mathcal{P}$. This can be thought of as a prestack. It sends a region $A$ to the pair $(A\uparrow, A\down\uparrow)$ of posets. It sends a causal path $P$ from $A$ to $B$ to the functor $\text{Path}_P(-, -)$ from $A\uparrow \times B\down\uparrow$ to $\text{Inc}$. And it sends a chain inclusion from $P'$ to $P$ (which means that $P$ and $P'$ are both causal paths from $A$ to $B$, and $P'$ is a refinement of $P$) to the natural family of inclusions $\text{Path}_{P'}(A', B') \subset \text{Path}_P(A', B')$. If $P$ is a causal path from $A$ to $B$ and $Q$ is a causal path from $B$ to $C$, there is a natural injection $\text{Path}_P(A', B) \times \text{Path}_Q(B, C') \to \text{Path}_{Q \circ P}(A', C')$, where $Q \circ P$ denotes the composite of $P$ and $Q$.

**Definition 8.2.** We call this weak 2-functor the **tangent 2-bundle** of the causal site.

The tangent 2-bundle contains information about how causal paths can link the observable tangent spaces together. Since we can also refine causal paths, expansions into discretized path sums will be possible by decomposing intermediate regions into unions and summing over the families of refined paths which occur.

We think the analogy between smooth manifolds and causal sites may be a good guide to applications. We use the tangent bundle as a setting in order to apply calculus to the geometry of manifolds. Similarly, the local simplicial or bisimplicial structures on a site may
enable us to apply the calculus of categorical state sums to quantum geometry on them. In constructing a quantum theory over a causal site, it should be possible to use the connecting complexes as discretized analogs of a Feynman path integral, and their 2-categorical structure should help constrain such a construction, allowing us to find interesting models just from the requirement of 2-functoriality.

9. Unitary Prestacks on a Causal Site

In the next few sections, we investigate a more conventional approach to constructing geometrical or physical models on causal sites, namely, putting suitable presheaves and prestacks over them.

We will begin with a preliminary investigation of the type of physical model which the mathematical structure of a causal site suggests. We will not attempt to construct specific physically realistic models in this paper.

The original idea of a site was motivated by the fact that a presheaf over a topological space $X$ is equivalent to a contravariant functor from the category of open subsets of $X$.

Since a bundle, including the tangent bundle of a manifold, can be regarded as a type of sheaf, we have a language for describing analogs of the basic structures underlying Yang-Mills theory and general relativity available for sites, although we will probably need stronger regularity assumptions to be able to create more precise analogs.

It is interesting that including a causal structure in our model, which traditionally is expressed by changing from a Riemannian to a Lorentzian metric, is expressed here by passage from a 1-category to a 2-category, a process which has been called categorification \[\text{Crane et al. 1994, Baez-Dolan 1998}.\]

This has the immediate consequence that rather than looking for models over a causal site within an ordinary category, such as the category of Hilbert spaces, we must turn to a 2-category, such as the category of 2-Hilbert spaces \[\text{Baez 1996}.\] Instead of a presheaf of Hilbert spaces, we work with a prestack of 2-Hilbert spaces. A prestack is a weak 2-functor from our causal site to a 2-category \[\text{Breen 1994}.\] In a prestack, a triangle of restriction morphisms only needs to commute up to a 2-morphism.

This means that rather than thinking of an assignment of a single Hilbert space to a region (something like the space of quantum states propagating through it), we assign to it a category of Hilbert spaces. Physically, we can think of this as related to the idea that the state space of a region has dimension related to the area of its boundary, so that if the geometry is itself a quantum variable, then the Hilbert space itself cannot be unique.

Indeed, if we tried to construct a physical theory over a causal site, we would quickly discover that we couldn't find a natural linear map to associate to containment, because of the nonlocal correlations in quantum theories. We would be trapped trying to map pure states to mixed states.

It actually provides an interesting new slant on the interpretation of quantum mechanics rewriting this process in terms of the different choices of Hilbert spaces in the categories corresponding to two regions, one of which contains the other.

Let us explain in a very simple situation, using the category 2-Vect \[\text{Kapranov et al. 1994} \] rather than 2-Hilb, how this would look. Objects in 2-Vect are $n$-vector spaces, i.e. the category of all $n$-tuples of vector spaces for some $n$. Arrows in 2-Vect are functors defined by tensoring by some column of vector spaces. Thus, a (contravariant) functor from a causal site to 2-Vect would find the vector spaces which could appear on a subregion being combined into tensor products of vector spaces associated to a larger region.
Now there is neither a natural map $A \to A \otimes B$ nor $A \otimes B \to A$, so there would be no way to represent inclusion as a map on individual Hilbert spaces. This is a mathematically elegant way to express the existence of nonlocal correlations in compound systems in quantum mechanics. The overall rule which tells us how the different Hilbert spaces associated to regions are related has an elegant functorial form, which cannot be recovered on the individual Hilbert spaces. In [Hawkins et al 2003], a similar problem was studied for models on causal sets, and it was discovered that maps between causally related points had to be expressed as completely positive maps on spaces of mixed states (i.e., Hermitian operators over the Hilbert spaces, rather than the Hilbert spaces themselves). The setting we are exploring, combining both causality and containment, should contain examples of this, but set in a broader algebraic context.

Now we can ask ourselves which of the models which have appeared in mathematical physics would be good candidates for extending to a causal site. An obvious choice would be 2-Yang Mills theory [Baez 2002, Girelli et al 2003], which is a theory which puts a nonlinear partial differential operator on a 2-form, by analogy with the Yang-Mills equation on a 1-form.

Since 2-Yang Mills theory has the interpretation of physics on the curvature of a gerbe, which is really a special type of stack of categories, it will have a direct generalization to causal sites.

10. A Brief Review of Categorical State Sum Models

Categorical state sums are models of quantum geometry on simplicial complexes [Crane et al 1994b, Barrett et al 1998, Crane et al 2003, Noui et al 2003, Crane et al 2001]. They are constructed out of unitary representations of Lie groups or quantum groups, intertwining maps of these representations, and in some cases, 2-intertwiners. The maps are composed in closed patterns related to the combinatorics of the simplicial complex, traced, and summed over. The result is a discrete version of a path integral for quantum geometry, with the unitary representations acting as Hilbert spaces on which geometric quantities are represented as operators.

For example, in the model of [Barrett et al 1998], we have a four-dimensional simplicial complex. The 2-simplices (triangles) are labelled with unitary representations of the Lorentz group. We think of these as quantizations of the oriented area elements a geometry would assign to the faces, because oriented area elements can be identified with elements of the Lie algebra of the Lorentz group.

In fact, in order to impose the necessary constraints on the geometry, we restrict ourselves to a class of representations called the balanced ones. We then label the tetrahedra with a sum over the balanced intertwining maps between the four representations, trace over each 4-simplex, multiply, then sum over all possible choices of representations.

There are a number of models of this type, differing by the dimension, choice of symmetry group, constraints, etc. These amount to different choices of the geometry to be quantized.

This type of model is closely related to the ideas discussed above describing simplicial sets as categories. They can be written as sums of functors from the category associated to the complex to the category of representations of the geometrical symmetry group or quantum group.

In short, the categorical state sum picture makes it natural to assign a quantum geometry to a simplicial complex.

In the context of causal sites, this can be thought of as a candidate for local quantum geometry on a tangent space, analogously to the use of inner products on tangent spaces in classical Riemannian geometry.
One of the reasons that manifold theory was so important to the development of relativity was that it gives a natural concrete expression for the equivalence principle: laws must be expressed in tensor form. The categorical underpinning of the structure of causal sites suggests a similar principle in quantum gravity: laws must be expressed in functorial form. This simple and attractive principle will cut down the possibilities greatly.

Given that we already have subcategories which express the relational world seen by an observer, it is also a natural expression of the idea behind the equivalence principle, if we take the point of view that coordinate systems are idealized observers.

11. State Sum Models on Causal Sites

The physically interesting examples of Section 4 all possess local symmetries. (Since a bounded region which is boosted too far no longer contains fundamental cylinders, a mathematical formalization of this would need to use actions of germs of the Lorentz or Poincare groups. We will not work this out in this paper.)

It therefore seems that the Hilbert spaces in a quantum site would group together in unitary representations of the Lorentz group.

Now one of the great challenges for our proposal is to find a way to derive Einstein’s equation as a classical limit for some formulation of the dynamics on a quantum site. Categorical state sum models are interesting, because the laws of combination of the representations produce a classical limit which implies Einstein’s equation. In other words, Einstein’s equation is actually determined by the symmetry of the model, expressed in the tensor category of unitary representations of the Lorentz group.

Now is it possible that we can introduce categorical state sum models into a suitable family of causal sites with local symmetries? Let us try to imagine how that could arise.

In order to define a class of causal sites which would be physically realistic near the Planck scale, we have defined “grained worlds” above as locally Noetherian, amply pointed causal sites. We think of this as a way to embed facts about the limited amount of information which can flow between regions in general relativity into the fabric of spacetime itself.

The combination of these two properties means that given a causally related pair of regions \( A \prec B \), we can find a finite set of relational points for \( B \) in \( A \) which in some sense exhaust the information which \( B \) can see in \( A \). This would mean that the relational tangent space would be adequately described by a simplicial set, assuming the set of points did not contain one another.

This suggests that the state sum model of \cite{Barrett1998}, or some similar model, might appear as a good approximation to the quantum theory of the part of a quantum site contained in a region.

This is probably a better setting for state sum models than direct application to a whole universe, since the finiteness of the simplicial set becomes better motivated.

Then, when we wanted to express the flow of information from one region, described by a state sum on a simplicial set, to another, along a given causal path, we would know that the map which described it would have to behave well when we included one causal path in another.

This includes the information flow law in a 2-categorical structure, which might be rich enough to allow us to determine it.

So a program suggests itself, to make a suitable set of assumptions on a prestack over a grained world, so that if we pass to a set of relational points the relative position information ends up containing all the transferred information, and takes the form of a state sum model.

At this point, we realize that our new point of view is making us ask a series of new questions about state sums which had no motivation in the past when they were thought of
as stand alone models. We need to express the relationship between two state sum models when the complex of one is embedded in the other, we need to work out expressions for causal flows of information between state sums, and we need to investigate in what sense they approximate points with quantum variables between them as they grow far apart. If nothing else, the causal sites proposal has widened our perspective on state sum models.

12. Conclusions and Future Directions

We now have defined a new class of mathematical objects, which are not point sets, but which can function as a topological foundation for classical general relativity. We have families of examples corresponding to classical spacetimes, which include descriptions of the geometrodynamics of classical spacetimes above the Planck scale, without the inclusion of a metric tensor as a mathematical datum separate from the underlying spacetime structure.

The representation of geometry by a causal site is analogous to the representation by a discrete causal set, but as we have discussed, it seems to give a much closer approximation. Our relative and absolute points should not be confused with ordinary points, either in manifolds or causal sets. They are not “atomic.” For example, a union of a finite family of fundamental diamonds in a Lorentzian manifold can contain an infinite number of other fundamental diamonds. Nevertheless, the causality relationship between them gives rise to a finiteness or discreteness in the structure. It therefore makes a bridge between continuous and discrete structures in a new way. The hypothesis of overlapping minimal regions seems more physically natural to us than the discrete point sets of causal sets or categorical state sum models. It also means that the continuous or discrete symmetry properties of a causal site can be more robust than a causal set. In the examples we have constructed, the symmetry with respect to the Lorentz group is not broken, because we have fundamental diamonds in all reference frames. In the theory of causal sets, by contrast, it is only possible to recover the symmetry on the average in the infinite volume of Minkowski space [Sorkin 2003]. The fact that we are able to produce a mathematical construction which combines a fundamental graininess with symmetry in local regions gives our construction added interest.

We think it will be interesting to try to find further families of examples, for instance to try to construct quantum n-manifolds neither related to causal sets nor directly derived from manifolds. Different classes of these might give new examples of “spacetime textures,” which could be explored as new settings for physical models. It may well be that the physical intuition expressed in the phrase “spacetime foam” can find better expression in our setting that when constrained by the necessity of considering only families of smooth manifolds.

Causality and time are integrated into the mathematical structure of a causal site in a striking fashion: spacetime appears as a categorification of space. The fact that causal sites can be considered either as bisimplicial sets or as 2-categories means we have a number of well-understood tools at our disposal for constructing examples.

In addition, the structure of a causal site seems to lend itself readily to quantization. We have not yet examined this in any detail.

We have, at this point, many more questions than answers. Can the geometric picture of Example 3.3 be generalized to more intrinsic models, i.e. models which have definitions not so directly dependent on a classical spacetime? Can it be quantized? Since it seems natural to approach quantization by beginning with the family of containments of a particular causal site, it will be easier to find semiclassical states in this approach than in other approaches to quantum gravity.

It would seem that the Hilbert spaces arising from quantization of a causal site with local symmetries would decompose into unitary representations of the Lorentz group. Do the state sum models arise in such a picture? We should remind ourselves that Einstein’s
equation is almost determined by symmetry considerations. Would a quantization of a causal site with Lorentz symmetry which was required to respect the symmetry be nearly determined as well?

We also have not yet studied the description of curvature in causal sites, nor searched for a geometric way to impose Einstein’s equation.

General relativity is intimately bound up with our ideas about spacetime and geometry. Changing the foundation of these creates a situation in which all the questions of classical and quantum relativity can be reexamined. The large family of accessible examples we have constructed means that a fairly broad program can be practically implemented.
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