Distribution of cycles for one-dimensional random dynamical systems

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Periodic points of dynamical systems

$X$: space
$T$: $X \rightarrow X$ map

$$T^n(x) := \underbrace{T \circ T \circ \cdots \circ T}_{n \text{ times composition}}(x) \quad (n = 0, 1, 2, \ldots)$$

**PROBLEM:** Describe the structure of the orbit $\{T^n(x)\}_{n=0}^{\infty}$ for a majority of initial conditions $x$.

$\text{Fix}(T^n) := \{x \in X : T^n(x) = x\} \quad (n = 1, 2, \ldots)$. For $x \in \text{Fix}(T^n)$, the set $\{x, T(x), \ldots, T^{n-1}(x)\}$ is called a periodic orbit. The point $x$ is called a **periodic point of period $n$**.

$\text{Fix}(T^n) \ni x \rightarrow T(x) \rightarrow T^2(x) \rightarrow \cdots \rightarrow T^n(x) = x$.

- approximations of dynamical objects by periodic orbits (invariant set, invariant measure, pressure, ...)
- dynamical zeta function, Livschitz’s theorem, ...
Distribution of periodic points

\( X: \text{ space} \)
\( T: X \rightarrow X: \text{ map} \)
\( \varphi: X \rightarrow \mathbb{R} \text{ potential (weight function)} \)

\( S_n \varphi := \sum_{k=0}^{n-1} \varphi \circ T^k. \)

\( \nu_{n,\varphi} := \frac{1}{Z_n(\varphi)} \sum_{x \in \text{Fix}(T^n)} \exp(S_n \varphi(x)) \delta_x \)

where

\( Z_n(\varphi) := \sum_{x \in \text{Fix}(T^n)} \exp(S_n \varphi(x)) . \)

**Theorem 1 (Bowen 1975)**

Let \( X \) be a topologically mixing subshift of finite type and \( T: X \rightarrow X \) the left shift. For any Hölder continuous function \( \varphi: X \rightarrow \mathbb{R} \), the sequence \( \{\nu_{n,\varphi}\}_{n=1}^{\infty} \) converges to the equilibrium state for the potential \( \varphi \).

**How Theorem 1 can be extended to random dynamical systems?**
 Independently Identically Distributed random dynamical systems

$2 \leq N < \infty$ integer, $T_i : X \to X$ $(1 \leq i \leq N)$ maps $p = (p_1, \ldots, p_N)$: probability vector with $\prod_{i=1}^{N} p_i \neq 0$. We consider an i.i.d. random dynamical system in which $T_i$ is chosen with probability $p_i$ at each step.

$\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
$\Omega := \{1, 2, \ldots, N\}^\mathbb{N}$ sample space

For a sample path $\omega = (\omega_n)_{n\in\mathbb{N}} \in \Omega$ and $n \in \mathbb{N}$, consider a random composition

$$T^n_\omega := T_{\omega_n} \circ T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_1}.$$ 

For convenience, define $T^0_\omega$ to be the identity map on $X$. 

$\{T^n_\omega(x)\}_{n\in\mathbb{N}_0}$ is called a random orbit with initial condition $x$. 
A random cycle is an element of the set

$$\bigcup_{n \in \mathbb{N}} \bigcup_{\omega \in \Omega} \text{Fix}(T_\omega^n),$$

where

$$\text{Fix}(T_\omega^n) := \{x \in X: T_\omega^n(x) = x\}.$$ 

$x \in \text{Fix}(T^n)$ implies that the orbit $\{T^n(x)\}_{n \in \mathbb{N}_0}$ is finite as a set, whereas $x \in \text{Fix}(T_\omega^n)$ does not imply the finiteness of the random orbit $\{T^n_\omega(x)\}_{n \in \mathbb{N}_0}$ as a set. Indeed, we have

$$T^{n+1}_\omega(x) = T_{\omega_{n+1}} \circ T_{\omega_n} \circ \cdots \circ T_{\omega_2} \circ T_{\omega_1}(x)$$

$$= T_{\omega_{n+1}}(x),$$

which may not be contained in the set

$$\{x, T_{\omega_1}(x), T_{\omega_2} \circ T_{\omega_1}(x), \ldots, T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_1}(x)\}.$$
\[ \bigcup_{n \in \mathbb{N}} \bigcup_{\omega \in \Omega} \text{Fix}(T^n_\omega), \]

- **Samplewise (Quenched):** Fix \( \omega \in \Omega \), and ask behaviors of
  \[ \text{Fix}(T^n_\omega) \]
  as \( n \to \infty \).

- **Sample-averaged (Annealed):** Ask behaviors of
  \[ \bigcup_{\omega \in \Omega} \text{Fix}(T^n_\omega) \]
  as \( n \to \infty \).
Some results/facts on random cycles

- Dynamical zeta functions defined by random cycles were considered by Ruelle (1990), Buzzi (2002).
- A dynamical zeta function defined by random cycles of certain random matrices cannot be extended holomorphically beyond its disk of holomorphy, almost surely. (Buzzi (2002))
- Distribution of $\bigcup_{\omega \in \Omega} \text{Fix}(T^n_\omega)$ as $n \to \infty$ for Ruelle expanding maps. (Carvalho/Rodrigues/Varandas (2017))
- Growth of $\# \text{Fix}(T^n_\omega)$ as $n \to \infty$ (Asaoka/Shinohara/Turaev (2017)) for random interval maps systems with expansion/contraction
$X$: compact interval

A **fully branched map** on $X$ is a map $T: \bigcup_{a \in \mathcal{A}} J_a \to X$ where $\mathcal{A} \subset \mathbb{N}$ with $2 \leq \# \mathcal{A} < \infty$, and $(J_a)_{a \in \mathcal{A}}$ is a collection of pairwise disjoint subintervals of $X$ such that:

- $X = \bigcup_{a \in \mathcal{A}} J_a$;
- for each $a \in \mathcal{A}$, the restriction of $T$ to $J_a$ extends to a $C^2$ diffeomorphism on $\text{cl}(J_a)$;
- for each $a \in \mathcal{A}$, $\text{cl}(T(J_a)) = X$.

A fully branched map $T$ on $X$ is **uniformly expanding** if there exists a constant $\gamma > 1$ such that $\inf_{x \in J_a} |(T|_{J_a})'x| \geq \gamma$ for any $a \in \mathcal{A}$. 
I.i.d. random dynamical system

\( T_1, \ldots, T_N, 1 \leq N < \infty \) fully branched uniformly expanding maps on \( X \) (do not assume a common Markov partition).

\( \Omega := \{1, 2, \ldots, N\}^\mathbb{N}, \ p = (p_1, \ldots, p_N) \) probability vector with \( \prod_{i=1}^N p_i > 0 \), \( m_p \) : Bernoulli measure on \( \Omega \) determined by \( p \).

For a sample \( \omega = (\omega_n)_{n \in \mathbb{N}} \in \Omega \) and \( n \geq 1 \),

\[ T^n_\omega(x) := T_{\omega_n} \circ T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_1}(x), \quad T^0_\omega(x) = x. \]

By Pelikan’s theorem (1984), \( \exists! \) a Borel probability measure \( \lambda_p \) on \( X \) s.t. \( \lambda_p \ll \text{Leb} \) and \( \lambda_p = \sum_{i=1}^N p_i \lambda_p \circ T_i^{-1} \). From the random ergodic theorem, For \( m_p \)-a.e. \( \omega \in \Omega \) and any \( \phi : X \to \mathbb{R} \) continuous,

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(T^k_\omega(x)) = \int \phi d\lambda_p \quad \text{for } \lambda_p\text{-a.e. } x \in X, \]

namely, for \( \lambda_p\text{-a.e. } x \in X, \)

\[ \delta^\omega_n := \frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^k_\omega(x)} \to \lambda_p \quad \text{in the weak* topology as } n \to \infty. \]
Random cycle measures

For $\omega \in \Omega$, define a samplewise random cycle measure $\xi_n^\omega$ on $X$ by

$$\xi_n^\omega = \frac{1}{Z_{\omega,n}} \sum_{x \in \text{Fix}(T_n^\omega)} |(T_n^\omega)' x|^{-1} \delta_x^n \quad (n = 1, 2, \ldots),$$

where $(T_n^\omega)' x := \prod_{i=1}^n (T_\omega i)' (T_i^{-1}(x))$ and $Z_{\omega,n} := \sum_{x \in \text{Fix}(T_n^\omega)} |(T_n^\omega)' x|^{-1}$ is the normalizing constant.

(Distribution of random cycles). Does the sequence $\{\xi_n^\omega\}_{n=1}^\infty$ converge? If so, what is the limit measure?

$\mathcal{M}(X)$: the space of Borel probability measures on $X$.
For $\omega \in \Omega$, define a samplewise random cycle measure $\tilde{\xi}_n^\omega$ on $\mathcal{M}(X)$ by

$$\tilde{\xi}_n^\omega = \frac{1}{Z_{\omega,n}} \sum_{x \in \text{Fix}(T_n^\omega)} |(T_n^\omega)' x|^{-1} \delta_{\delta_x^n} \quad (n = 1, 2, \ldots),$$

where $\delta_{\delta_x^n}$ is the unit point mass at $\delta_x^n$. 
Theorem A

Let $2 \leq N < \infty$, and let $T_1, \ldots, T_N$ be fully branched uniformly expanding maps on $X$. Let $p = (p_1, \ldots, p_N)$ be a probability vector with $\prod_{i=1}^N p_i > 0$. For $m_p$-almost every $\omega \in \Omega$, the sequence $\{\tilde{\xi}_{\omega}^n\}_{n=1}^\infty$ of samplewise random cycle measures on $\mathcal{M}(X)$ converges to the unit point mass at $\lambda_p$ in the weak* topology.

i.e., for any continuous function $\bar{\varphi}: \mathcal{M}(X) \to \mathbb{R}$, $\int \bar{\varphi} d\tilde{\xi}_n^\omega \to \bar{\varphi}(\lambda_p)$.

Corollary 1

For $m_p$-almost every $\omega \in \Omega$, the sequence $(\xi_{\omega}^n)_{n=1}^\infty$ converges in the weak* topology to $\lambda_p$ as $n \to \infty$.

i.e., for any continuous function $\varphi: X \to \mathbb{R}$, $\int \varphi d\xi_n^\omega \to \int \varphi d\lambda_p$.

Proof of Corollary 1.

Given $\varphi: X \to \mathbb{R}$, apply Theorem A to the continuous function $\nu \in \mathcal{M}(X) \mapsto \int \varphi d\nu \in \mathbb{R}$.
Corollary 2 (Inspired by Olsen (2003))

Let $T_1, \ldots, T_N$ and $p = (p_1, \ldots, p_N)$ be as in Theorem A.

(a) If $\varphi, \psi : X \to \mathbb{R}$ are continuous, then for $m_p$-almost every $\omega \in \Omega$,

$$
\lim_{n \to \infty} \frac{1}{Z_{\omega,n}} \sum_{x \in \text{Fix}(T^n_\omega)} |(T^n_\omega)'(x)|^{-1} \frac{1}{n^2} \sum_{k=0}^{n-1} \varphi(T^k_\omega(x)) \sum_{k=0}^{n-1} \psi(T^k_\omega(x)) = \int \varphi d\lambda_p \int \psi d\lambda_p.
$$

(b) If $\varphi : X \to \mathbb{R}, \psi : X \to \mathbb{R}$ are continuous with $\inf \psi > 0$, then for $m_p$-almost every $\omega \in \Omega$,

$$
\lim_{n \to \infty} \frac{1}{Z_{\omega,n}} \sum_{x \in \text{Fix}(T^n_\omega)} |(T^n_\omega)'(x)|^{-1} \frac{1}{n^2} \sum_{k=0}^{n-1} \varphi(T^k_\omega(x)) \sum_{k=0}^{n-1} \psi(T^k_\omega(x)) = \frac{\int \varphi d\lambda_p}{\int \psi d\lambda_p}.
$$
Corollary 2 (Continued)

(c) If $\pi_1, \pi_2: X \to \mathbb{R}$ are continuous and $g: \mathbb{R} \to \mathbb{R}$ is bounded continuous, then for $m_p$-almost every $\omega \in \Omega$ we have

$$
\lim_{n \to \infty} \frac{1}{Z_{\omega,n}} \sum_{x \in \text{Fix}(T^n_\omega)} |(T^n_\omega)'(x)|^{-1} \frac{1}{n^2} \times
$$

$$
\sum_{k_1, k_2 = 0}^{n-1} g(\pi_1(T^{k_1}_\omega(x)) + \pi_2(T^{k_2}_\omega(x)))
$$

$$
= \int g d(\lambda_p \circ \pi_1^{-1} \otimes \lambda_p \circ \pi_2^{-1}),
$$

where $\otimes$ denotes the convolution.

Proof of Corollary 2.

Apply Theorem A to the continuous functions

$v \in \mathcal{M}(X) \mapsto \int \varphi d\nu \int \psi d\nu$, $v \in \mathcal{M}(X) \mapsto \int \varphi d\nu / \int \psi d\nu$, $v \in \mathcal{M}(X) \mapsto \int gd(v \circ \pi_1^{-1} \otimes v \circ \pi_2^{-1})$ respectively.
By Riesz’s representation theorem, for each $p = (p_1, \ldots, p_N)$ and $n \in \mathbb{N}$, there exists a Borel probability measure $\tilde{\eta}_{p,n}$ on $\mathcal{M}(X)$ s.t.

$$
\int \tilde{\varphi} d\tilde{\eta}_{p,n} = \int dm_p(\omega) \int \tilde{\varphi} d\tilde{\xi}_n^\omega \quad \text{for any continuous } \tilde{\varphi} : \mathcal{M}(X) \rightarrow \mathbb{R}.
$$

Also, there exists a Borel probability measure $\eta_{p,n}$ on $X$ s.t.

$$
\int \varphi d\eta_{p,n} = \int dm_p(\omega) \int \varphi d\xi_n^\omega \quad \text{for any continuous } \varphi : X \rightarrow \mathbb{R}.
$$

**Corollary 3**

Let $T_1, \ldots, T_N$ and $p = (p_1, \ldots, p_N)$ be as in Theorem A. Then $(\tilde{\eta}_{p,n})_{n=1}^\infty$ converges to $\delta_{\lambda_p}$ in the weak$^*$ topology as $n \rightarrow \infty$ and $(\eta_{p,n})_{n=1}^\infty$ converges to $\lambda_p$ in the weak$^*$ topology as $n \rightarrow \infty$. 
Distribution of random cycles: sample-averaged result

For $\omega \in \Omega$ and $n \in \mathbb{N}$, write $T_{\omega_1 \cdots \omega_n} = T^n_\omega$ and $\delta_{x}^{\omega_1 \cdots \omega_n} = \delta_{x}^n$.

For $p = (p_1, \ldots, p_N)$, $n \in \mathbb{N}$ and $\omega_1 \cdots \omega_n \in \{1, \ldots, N\}^n$, put

$$Q_p(\omega_1 \cdots \omega_n) := \prod_{i=1}^{N} p_i^{\#\{1 \leq k \leq n: \omega_k = i\}}.$$

Define an averaged random cycle measure on $X$ by

$$\kappa_{p,n} :=\left( \frac{\sum_{\omega_1 \cdots \omega_n \in \{1, \ldots, N\}^n} Q_p(\omega_1 \cdots \omega_n) \sum_{x \in \text{Fix}(T_{\omega_1 \cdots \omega_n})} |(T_{\omega_1 \cdots \omega_n})'x|^{-1} \delta_{x}^{\omega_1 \cdots \omega_n}}{\text{normalize}} \right),$$

and define an averaged random cycle measure on $\mathcal{M}(X)$ by

$$\tilde{\kappa}_{p,n} :=\left( \frac{\sum_{\omega_1 \cdots \omega_n \in \{1, \ldots, N\}^n} Q_p(\omega_1 \cdots \omega_n) \sum_{x \in \text{Fix}(T_{\omega_1 \cdots \omega_n})} |(T_{\omega_1 \cdots \omega_n})'x|^{-1} \delta_{x}^{\omega_1 \cdots \omega_n}}{\text{normalize}} \right).$$
Theorem B

Let $2 \leq N < \infty$, and let $T_1, \ldots, T_N$ be fully branched uniformly expanding maps on a compact interval $X$. Let $p = (p_1, \ldots, p_N)$ be a probability vector with $\prod_{i=1}^{N} p_i > 0$. The sequence $\{\tilde{\kappa}_{p,n}\}_{n=1}^{\infty}$ of sample-averaged random cycle measures converges to the unit point mass at $\lambda_p$ in the weak* topology.

i.e., for any continuous function $\tilde{\varphi}: \mathcal{M}(X) \rightarrow \mathbb{R}$,
$$\int \tilde{\varphi} d\tilde{\kappa}_{p,n} \rightarrow \tilde{\varphi}(\lambda_p).$$

Corollary 4

Let $T_1, \ldots, T_N$ and $p = (p_1, \ldots, p_N)$ be as in Theorem B. The sequence $\{\kappa_{p,n}\}_{n=1}^{\infty}$ of sample-averaged random cycle measures converges to $\lambda_p$ in the weak* topology.

i.e., for any continuous function $\varphi: X \rightarrow \mathbb{R}$,
$$\int \varphi d\kappa_{p,n} \rightarrow \int \varphi d\lambda_p.$$
Consider a skew product map

\[ R: (\omega, x) \in \Omega \times X \mapsto (\theta \omega, T_{\omega_1} x) \in \Omega \times X, \]

where \( \theta: \Omega \to \Omega \) denotes the left shift \( (\theta \omega)_k = \omega_{k+1} \).

Key observation: \( x \in \text{Fix}(R^n) \implies (\omega', x) \in \text{Fix}(R^n) \), where \( \omega' \) is the repetition of \( \omega_1 \omega_2 \cdots \omega_n \) and

\[ \text{Fix}(R^n) = \{(\omega, x) \in \Omega \times X: R^n(\omega, x) = (\omega, x)\}. \]

1. (Level-2) large deviation principle on periodic points of \( R \) (Kifer (1994))

2. Conversion to samplewise large deviations (adapt Aimino/Nicol/Vaienti (2015))

3. Project to the original space \( X \).
\( \mathcal{M}(\Omega \times X) \): the space of Borel probability measures on \( \Omega \times X \). For \((\omega, x) \in \Omega \times X\) and \(n \geq 1\), let \( \delta_n^{(\omega, x)} = (1/n) \sum_{k=0}^{n-1} \delta_{R^k(\omega, x)} \). Define a Borel probability measure \( \tilde{\mu}_n \) on \( \mathcal{M}(\Omega \times X) \) by

\[
\tilde{\mu}_n := \frac{1}{\text{normalize}} \sum_{(\omega, x) \in \text{Fix}(R^n)} Q_p(\omega_1 \cdots \omega_n) |(T^n_\omega)'x|^{-1} \delta_n^{(\omega, x)},
\]

where \( \delta_n^{(\omega, x)} \) is the unit point mass at \( \delta_n^{(\omega, x)} \).

**Proposition 1 (Kifer (1994) Large Deviation Principle)**

There exists a lower semicontinuous function \( I: \mathcal{M}(\Omega \times X) \rightarrow [0, \infty] \) such that: (a) \( I(\mu) = 0 \) iff \( \mu = m_p \times \lambda_p \); (b) for any Borel set \( B \subset \mathcal{M}(\Omega \times X) \),

\[
- \inf_{\text{int}B} I \leq \lim_{n \to \infty} \inf \frac{1}{n} \log \tilde{\mu}_n(\text{int}B) \leq \lim_{n \to \infty} \sup \frac{1}{n} \log \tilde{\mu}_n(\text{cl}B) \leq - \inf_{\text{cl}B} I.
\]
For each \( \omega \in \Omega \) and \( n \geq 1 \), define a Borel probability measure \( \tilde{\mu}_n^\omega \) on \( \mathcal{M}(\Omega \times X) \) by

\[
\tilde{\mu}_n^\omega := \frac{1}{Z_{\omega,n}} \sum_{x \in \text{Fix}(T^n_\omega)} |(T^n_\omega)' x|^{-1} \delta_{\delta_n(\omega,x)}.
\]

**Proposition 2 (Samplewise large deviations upper bound)**

For \( m_p \)-almost every \( \omega \in \Omega \) and any closed subset \( C \) of \( \mathcal{M}(\Omega \times X) \), we have

\[
\limsup_{n \to \infty} \frac{1}{n} \log \tilde{\mu}_n^\omega(C) \leq -\inf_C l.
\]

For our purpose, there is no need for a lower bound.

Idea of proof of Proposition 2: Adapt the trick of conversion (sample-averaged \( \to \) samplewise) by Aimino/Nicol/Vaienti (2015) to periodic points (random cycles).
Since $\mathcal{M}(\Omega \times X)$ is metrizable, it is separable. So, enough to show that for each closed set $C$, $\exists$ a Borel set $\Omega_C \subset \Omega$ s.t. for $m_p$-a.e. $\omega \in \Omega_C$,

$$\limsup_{n \to \infty} \frac{1}{n} \log \tilde{\mu}_n^\omega(C) \leq - \inf_C l.$$ 

We may assume $0 < \inf_C l < \infty$. There is a uniform constant $K > 0$ such that

$$\tilde{\mu}_n(C) = \frac{1}{\text{normalize}} \sum_{\substack{(\omega,x) \in \text{Fix}(R^n) \\ \delta_n^{(\omega,x)} \in C}} Q_p(\omega_1, \ldots, \omega_n) |(T^n_\omega)'x|^{-1}$$

$$= \int \tilde{\mu}_n^\omega(C) \left( Z_{\omega,n} \bigg/ \int Z_{\omega',n} \, dm_p(\omega') \right) \, dm_p(\omega)$$

$$\geq K \int \tilde{\mu}_n^\omega(C) \, dm_p(\omega).$$

Key: $Z_{\omega,n}$ is bounded away from 0 and $+\infty$ uniformly on $\omega$ and $n$. 

Conversion to samplewise level-2 large deviations
For $\epsilon \in (0, 1)$ and $n \geq 1$, set

$$
\Omega_{\epsilon,n} = \left\{ \omega \in \Omega : \tilde{\mu}_n^\omega(C) \geq \exp \left( -n(1 - \epsilon) \inf_C I \right) \right\}.
$$

By Markov's inequality,

$$
m_p(\Omega_{\epsilon,n}) \leq \exp \left( n(1 - \epsilon) \inf_C I \right) \int \tilde{\mu}_n^\omega(C) dm_p(\omega)
$$

$$
\leq K^{-1} \exp \left( n(1 - \epsilon) \inf_C I \right) \tilde{\mu}_n(C).
$$

By Proposition, $\tilde{\mu}_n(C)$ decays exponentially as $n \to \infty$, so $m_p(\Omega_{\epsilon,n})$ decays exponentially as $n \to \infty$. By Borel-Cantelli's lemma,

$$
\# \{ n \in \mathbb{N} : \tilde{\mu}_n^\omega(C) \geq \exp(-n(1 - \epsilon) \inf_C I) \} < \infty
$$

for $m_p$-almost every $\omega \in \Omega$. Since $\epsilon$ is arbitrary, we obtain the desired upper bound for $m_p$-almost every $\omega \in \Omega$. 

\qed
Some possible extensions of the main results

- maps with non-full branches (Dajani/de Vries (2005))
- maps with neutral fixed points (Liverani/Saussol/Vaienti (1999) etc.)
Some possible extensions of the main results

- maps with infinitely many branches (Kalle/Kempton/Verbitskiy (2017) etc.)

Figure: graphs of the Gauss and Rényi transformations

Thank you for your attention.