Renormalization group evolution of neutrino masses and mixing
in the Type-III seesaw mechanism

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Abstract

We consider the standard model extended by heavy right handed fermions transforming as triplets under SU(2)_L, which generate neutrino masses through the Type-III seesaw mechanism. At energies below their respective mass scales, the heavy fields get sequentially decoupled to give an effective dimension-5 operator. Above their mass thresholds, these fields also participate in the renormalization of the wavefunctions, masses and coupling constants. We compute the renormalization group evolution of the effective neutrino mass matrix in this model, with particular emphasis on the threshold effects. The evolution equations are obtained in a basis of neutrino parameters where all the quantities are well-defined everywhere, including at θ_{13} = 0. We also point out the important role of the threshold effects and Majorana phases in the evolution of mixing angles through illustrative examples.

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I. INTRODUCTION

In the last decade, results from solar, atmospheric, accelerator and reactor experiments looking for neutrino flavour oscillations have succeeded in establishing that at least two of the neutrinos are massive and there is mixing between different flavors \[1\]. The present best-fit values of the mass squared differences and mixing angles determined from analyses of global data on neutrino oscillation are \[2\]

\[
\Delta m_{21}^2 = 7.65^{+0.69}_{-0.60} \times 10^{-5} \text{ eV}^2, \quad |\Delta m_{31}^2| = 2.40^{+0.35}_{-0.33} \times 10^{-3} \text{ eV}^2 ,
\]

\[
\sin^2 \theta_{12} = 0.30^{+0.07}_{-0.05} , \quad \sin^2 \theta_{23} = 0.50^{+0.17}_{-0.14} , \quad \sin^2 \theta_{13} = 0.01^{+0.046}_{-0.01} ,
\]

where \(\Delta m_{ij}^2 \equiv m_i^2 - m_j^2\) are the mass squared differences and \(\theta_{ij}\) the mixing angles. The relative position of the third mass eigenstate \(m_3\) with respect to the other two is unknown, though the solar neutrino data give \(\Delta m_{21}^2 > 0\). This results in two possible orderings of the neutrino masses: normal \((m_1 < m_2 < m_3)\) and inverted \((m_3 < m_1 < m_2)\).

One of the most distinctive features emerging out of the above results is the occurrence of two large and one small mixing angles which is rather different from the quark sector where all three mixing angles are small. The absolute masses of neutrinos are also orders of magnitude smaller than those of quarks and charged leptons, the current bound from cosmology on the sum of neutrino masses being \(\sum m_i \lesssim 1.5 \text{ eV} \[3\]. The most favored mechanisms to generate such small neutrino masses and nontrivial mixings are the so called seesaw mechanisms which need the introduction of one or more heavy fields. At energies below their mass scales, the heavy fields get integrated out giving rise to an effective dimension-5 operator \[4\]

\[
\mathcal{L}_{5} = \kappa_5 l_L l_L \phi \phi ,
\]

where \(l_L\) and \(\phi\) are respectively the lepton and Higgs doublets belonging to the standard model (SM). Here \(\kappa_5\) is the effective coupling which has inverse mass dimension and can be expressed in terms of a dimensionless coupling \(a_5\) as \(\kappa_5 = a_5/\Lambda\), with \(\Lambda\) some high energy scale. In this picture the SM serves as an effective theory valid up to the mass scale \(\Lambda\), which can be taken to be the mass of the lightest of the heavy fields. Such an operator violates lepton number by two units and hence gives rise to Majorana masses for neutrinos: \(m_\nu \sim \kappa_5 v^2\), where \(v\) is the vacuum expectation value of the Higgs field \(\phi\) after spontaneous symmetry breaking. Taking \(v \sim 246 \text{ GeV}\), a neutrino mass of \(\sim 0.05 \text{ eV}\) implies \(\Lambda \sim 10^{15} \text{ GeV}\) if \(a_5 \sim 1\).
There are four possible ways to form a dimension-5 gauge singlet term out of the two lepton doublets and two Higgs doublets: (i) each $l_L$-$\phi$ pair forms a fermion singlet, (ii) each of the $l_L$-$l_L$ and $\phi$-$\phi$ pair forms a scalar triplet, (iii) each $l_L$-$\phi$ pair forms a fermion triplet, and (iv) each of the $l_L$-$l_L$ and $\phi$-$\phi$ pair forms a scalar singlet. Case (i) can arise from the tree level exchange of a right handed fermion singlet and this corresponds to the Type-I seesaw mechanism [5]. Case (ii) arises when the heavy particle is a Higgs triplet giving rise to the Type-II seesaw mechanism [6, 7]. For case (iii) the exchanged particle should be a right-handed fermion triplet, which corresponds to generating neutrino mass through the Type-III seesaw mechanism [8]. The last scenario gives terms of the form $\nu_L^C e_L$ which cannot generate a neutrino mass.

Type-III seesaw mechanism mediated by heavy fermion triplets transforming in the adjoint representation has been considered earlier in [8, 9]. Very recently there has been a renewed interest in these type of models. The smallness of neutrino masses usually implies the mass of the heavy particle to be high $\sim 10^{11−15}$ GeV. However, it is also possible to assume that one or more of the triplets have masses near the TeV scale, making it possible to search for their signatures at the LHC [10, 11, 12, 13]. In such models, the Yukawa couplings need to be small to suppress the neutrino mass. Lepton flavour violating decays in the context of Type-III seesaw models have also been considered in [14]. Recently it has also been suggested that the neutral member of the triplet can serve as the dark matter and can be instrumental in generating small neutrino mass radiatively [15].

The possibility of being able to add one triplet fermion per family without creating anomalies was one of the consequences of a general analysis in [16] which discussed adding an extra U(1) gauge group to the SM. Possible ways of adding fermion triplets in an anomaly-free manner have been explored [17], with some specific models studied in [18]. A possible origin of such an extra U(1) gauge group has been proposed in [19]. Fermions in the adjoint representation fit naturally into the 24-dimensional representation of SU(5), and can rectify the two main problems encountered in SU(5) Grand Unified Theory (GUT) models, viz. generation of neutrino masses and gauge coupling unification [10, 11, 20, 21]. The latter requirement constrains the fermionic triplets to be of mass below TeV for $M_{\text{GUT}} \sim 10^{16}$ GeV, making the model testable at the LHC. Leptogenesis mediated by triplet fermions has been explored in [22]. Additional fermions transforming as triplet representations in the context of left-right symmetric model have been studied in [23]. Minimal supersymmetric
standard model extended by triplet fermions has recently been considered in [24].

Whether the exchanged particle at the high scale is a singlet fermion (Type-I seesaw) or a triplet fermion (Type-III seesaw), the light neutrino mass matrix is given as $m_D^T M_R^{-1} m_D$. Here $m_D$ is the Dirac mass matrix coupling the left handed neutrinos with the right handed heavy fields, and $M_R$ is the Majorana mass matrix for the right handed fields. Thus the generation of the light neutrino mass matrix is similar in the Type-I and Type-III seesaw mechanisms, both of which are fermion mediated. Since the neutrino mass is generated at the high scale while the neutrino masses and mixings are measured experimentally at a low scale, the renormalization group (RG) evolution effects need to be included. These radiative corrections in Type-I and Type-III seesaw are different, since the heavy fermions couple differently to the other particles in the theory. We note that below the mass scale of the lightest of the heavy particles, the effect of all heavy degrees of freedom are integrated out and the effective mass operators in these scenarios become identical.

The effect of RG induced quantum corrections on leptonic masses and mixings have been studied extensively in the literature [25, 26, 27, 28, 29, 30, 31, 32]. These effects can have interesting consequences such as the generation of large mixing angles [33, 34, 35, 36, 37, 38], small mass splittings for degenerate neutrinos [39, 40, 41, 42, 43, 44, 45, 46, 47], or radiative generation of $\theta_{13}$ starting from a zero value at the high scale [48, 49, 50, 51]. RG induced deviations from various high scale symmetries like tri-bimaximal mixing scenario [52, 53, 54] or quark-lepton complimentarity [54, 55, 56, 57] and correlations with low scale observables have been explored. Such effects can have significant contributions from the threshold corrections [58, 59, 60]. The RG evolution of the neutrino mass operator in the SM and the Minimal Supersymmetric Standard Model (MSSM) in the context of Type-I seesaw [59, 61, 62] and Type-II seesaw [63, 64] have been studied in the literature. In the context of Type-III seesaw with degenerate heavy fermions, the impact of the RG evolution on the vacuum stability and perturbativity bounds of the Higgs Boson has been explored in [65].

In this work we study the RG evolution in the SM in the context of the Type-III seesaw model with nondegenerate heavy fermions. Our model consists of the SM with additional massive fermion triplets $\Sigma$ with masses $\sim M_i$, ($i = 1, 2, \cdots, M_r$) such that $M_1 < M_2 < \cdots < M_r$. Below the mass scale $M_1$ all the triplets will be decoupled from the model and the RG evolution will be according to the SM. The triplets will manifest themselves at this low scale in the form of an effective operator $\kappa$ obtained by integrating out the heavy
fields. For energy scales above $M_1$, the effect of the heavy fermions will come into play successively and above $M_r$ all the three triplets will contribute to the RG running. We evaluate the contributions of these fermion triplets to the wavefunction, mass and coupling constant renormalization of the SM fields and of the triplet fields themselves. We obtain the $\beta$-functions for RG evolution of the Yukawa couplings, the Higgs self-coupling, the Majorana mass matrix of the fermion triplets, the effective vertex $\kappa$ and the gauge couplings, including the extra contribution due to the additional triplets wherever applicable. We obtain analytic expressions for the runnings of the masses, mixing angles and phases in a basis where all the quantities are well-defined at every point in the parameter space including $\theta_{13} = 0 \ [51]$. We also solve the RG equations numerically and present some illustrative examples of running of masses and mixing angles. We analyze the effect of the seesaw thresholds and Majorana phases and check if such a scheme can generate masses and mixing angles consistent with the current bounds.

The plan of the paper is as follows. In Sec. II we outline the basic features of the Type-III seesaw model including extra SU(2)$_L$-triplet fermions, and describe how the effective neutrino mass operator can arise by this mechanism. In Sec. III we describe how to include the varying mass thresholds of the heavy particles in the analysis, discuss the renormalization of the SM extended with heavy triplets, and give the expressions of the $\beta$ functions including the effect of the extra triplets. In Sec. IV we detail the changes in the RG equations of the effective neutrino mass operator due to the inclusion of the extra fermion triplets. In Sec. V we numerically demonstrate the modifications in the RG equations of the neutrino masses and mixing angles. We summarize our results in Sec. VI.

II. THE TYPE-III SEESAW MODEL

We consider the Type-III seesaw model where three heavy fermions are added to each family of the SM. These fermions have zero weak hypercharge, i.e. they are singlets of the gauge group U(1)$_Y$ of the SM. However, under the SU(2)$_L$ gauge, they transform as a triplet in the adjoint representation. In the basis of the Pauli matrices $\{\sigma^1, \sigma^2, \sigma^3\}$, this triplet can be represented as

$$\Sigma_R = \begin{pmatrix} \Sigma^0_R/\sqrt{2} & \Sigma^+_R \\ \Sigma^-_R & -\Sigma^0_R/\sqrt{2} \end{pmatrix} = \frac{\Sigma^i_R \sigma^i}{\sqrt{2}},$$

(2)
where $\Sigma^\pm_R = (\Sigma^1_R \mp i\Sigma^2_R)/\sqrt{2}$. For the sake of simplicity of further calculations, we combine $\Sigma_R$ with its charge conjugate

$$\Sigma^C_R = \begin{pmatrix} \Sigma^0_R / \sqrt{2} & \Sigma^{-C}_R \\ \Sigma^{+C}_R & -\Sigma^0_R / \sqrt{2} \end{pmatrix} \equiv \frac{\Sigma^C_i \sigma^i}{\sqrt{2}} ,$$

and use the quantity $\Sigma$, defined as

$$\Sigma \equiv \Sigma_R + (\Sigma_R)^C .$$

Clearly, $\Sigma$ also transforms in the adjoint representation of $\text{SU}(2)_L$. Note that though formally $\Sigma = \Sigma^C$, the individual elements of $\Sigma$ are not all Majorana particles. While the diagonal elements of $\Sigma$ are indeed Majorana spinors which represent the neutral component of $\Sigma$, the off-diagonal elements are charged Dirac spinors.

### A. The Lagrangian

Introduction of the fermionic triplets $\Sigma$ will introduce new terms in the Lagrangian. The net Lagrangian is

$$\mathcal{L} = \mathcal{L}_{SM} + \mathcal{L}_\Sigma ,$$

where

$$\mathcal{L}_\Sigma = \mathcal{L}_{\Sigma,kin} + \mathcal{L}_{\Sigma,mass} + \mathcal{L}_{\Sigma,Yukawa} .$$

Here,

$$\mathcal{L}_{\Sigma,kin} = \text{Tr}[\bar{\Sigma}i\not\!D\Sigma] ,$$

$$\mathcal{L}_{\Sigma,mass} = \frac{1}{2} \text{Tr}[\bar{\Sigma}M_\Sigma\Sigma] ,$$

$$\mathcal{L}_{\Sigma,Yukawa} = -\bar{\Sigma}_L \sqrt{2} \Sigma_{\tilde{Y}L} \Sigma \tilde{\phi} - \phi^T \varepsilon^T \Sigma \sqrt{2} \Sigma \not\!Y_L ,$$

where

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is the completely anti-symmetric tensor in the $\text{SU}(2)_L$ space. Here we have not written the generation indices explicitly. $M_\Sigma$ is the Majorana mass matrix of the heavy fermion triplets.
and $Y_\Sigma$ is the Yukawa coupling. The SM fields $l_L$, $\phi$ and $\tilde{\phi}$ are $SU(2)_L$ doublets and can be written as

$$
l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}_{y=-1}, \quad \phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}_{y=1}, \quad \tilde{\phi} = \varepsilon\phi^* = \begin{pmatrix} \phi^0 \\ -\phi^- \end{pmatrix}_{y=-1}.
$$

Each member of the $SU(2)_L$ doublet $l_L$ is a 4-component Dirac spinor. Since the fermion triplet $\Sigma$ is in the adjoint representation of $SU(2)_L$, the covariant derivative of $\Sigma$ is defined as

$$
D_\mu \Sigma = \partial_\mu \Sigma + ig_2 [W_\mu, \Sigma],
$$

where $g_2$ is the $SU(2)_L$ gauge coupling.

All the Feynman diagrams for the new vertices involving the triplet fermionic field $\Sigma$ are given in the Appendix A. The Feynman diagrams for the SM particles are shown in the Appendix B.

B. The effective vertex

In the low energy limit of the extended standard model, we have an effective theory which will be described by the SM Lagrangian with the additional operators obtained by integrating out the heavy fermion triplets added to it. The lowest dimensional one of such operators is the dimension-5 operator\(^1\)

$$
L_\kappa = \kappa_{fg} \left( \overline{l_L^f} \sigma^i \varepsilon \phi \right) \left( \phi^T \sigma^i \varepsilon l_L^g \right) + \text{h.c.},
$$

\begin{equation}
= -\kappa_{fg} \left( \overline{l_L^g} \phi_a l_L^b \phi_d \right) \frac{1}{2} \left( \varepsilon_{ac} \varepsilon_{bd} + \varepsilon_{ab} \varepsilon_{cd} \right) + \text{h.c. ,}
\end{equation}

where $\kappa$ is a symmetric complex matrix with mass dimension $(-1)$. Generation indices $f, g \in \{1, 2, 3\}$ are shown explicitly and $a, b, c, d \in \{1, 2\}$ are the $SU(2)_L$ indices. In writing the last line we have used

$$
(\sigma^i)_{ab} (\sigma^i)_{cd} = 2\delta_{ad} \delta_{bc} - \delta_{ab} \delta_{cd}
\Rightarrow (\sigma^i \varepsilon)_{ba} (\sigma^i \varepsilon)_{dc} = 2\varepsilon_{da} \varepsilon_{bc} - \varepsilon_{ab} \varepsilon_{dc}.
$$

\(^1\) We use this form to emphasize the triplet nature of the $l_L-\phi$ pairs. Since all the dimension-5 operators are equivalent, we choose the normalization such that $\kappa_{fg}$ defined here matches that in [66].
FIG. 1: The effective vertex $\kappa$ at an energy $\mu \ll M_1$, after all the heavy fermions have been decoupled from the theory. $f, g \in \{1, 2, 3\}$ are the generation indices. The SU(2)$_L$ and generation indices for $\Sigma$ are not shown explicitly since they are summed over.

and utilizing the $\phi_d \leftrightarrow \phi_a$ symmetry, we can write

$$2\varepsilon_{da}\varepsilon_{bc} - \varepsilon_{ba}\varepsilon_{dc} = \frac{1}{2} (\varepsilon_{ab}\varepsilon_{de} + \varepsilon_{db}\varepsilon_{ac}) .$$

(16)

The relevant diagrams in the complete theory giving rise to the effective operators in the low energy limit are shown in Fig 1. The “shaded box” on the left hand side represents the effective low energy vertex $\kappa$, while $A_{(a)}$ and $A_{(b)}$ are the amplitudes of the diagrams labeled as $(a)$ and $(b)$ on the right hand side. The amplitudes are given by

$$A_{(a)} = i\mu^\epsilon \left( Y_{\Sigma}^T M^{-1}_\Sigma Y_{\Sigma} \right) f_g \left[ (\varepsilon^T \sigma^i)_{ab} (\varepsilon^T \sigma^i)_{cd} \right] P_L ,$$

(17)

$$A_{(b)} = i\mu^\epsilon \left( Y_{\Sigma}^T M^{-1}_\Sigma Y_{\Sigma} \right) f_g \left[ (\varepsilon^T \sigma^i)_{db} (\varepsilon^T \sigma^i)_{ca} \right] P_L ,$$

(18)

with $\epsilon = 4 - D$ where $D$ is the dimensionality that we introduce in order to use dimensional regularization. Note that $A_{(b)}$ is obtained from $A_{(a)}$ just by $d \leftrightarrow a$ interchange. Using Eq. (15) one finally gets

$$A_{(a)} + A_{(b)} = -i\mu^\epsilon \left( Y_{\Sigma}^T M^{-1}_\Sigma Y_{\Sigma} \right) f_g \left( \varepsilon_{ab}\varepsilon_{cd} + \varepsilon_{ac}\varepsilon_{bd} \right) P_L .$$

(19)

This is equal to the left hand side of Fig. 1 with the identification

$$\kappa = 2Y_{\Sigma}^T M^{-1}_\Sigma Y_{\Sigma} .$$

(20)

Equation (20) gives the Feynman rule for the low energy effective vertex $\kappa$, as shown in the Appendix A. From Eqs. (20) and (14), one gets the neutrino mass after spontaneous
symmetry breaking to be
\[ m_\nu = -\frac{v^2}{2} Y_\Sigma^T M_\Sigma^{-1} Y_\Sigma \] (21)
which is the Type-III seesaw relation. Here, \( v \) denotes the vacuum expectation value of the Higgs field.

III. RADIATIVE CORRECTIONS IN TYPE-III SEESAW

A. Sequential decoupling of heavy fermions

Let us consider the most general case when there are \( r \) triplets having masses \( M_1 < M_2 < \cdots < M_{r-1} < M_r \). Above the heaviest mass \( M_r \), all the \( r \)-triplets are coupled to the theory and will contribute to the neutrino mass through seesaw mechanism as
\[ m_\nu^{(r+1)} = -\frac{v^2}{2} Y_\Sigma^T (r+1)^{-1} Y_\Sigma (\mu > M_r) . \] (22)

Here, \( Y_\Sigma \) is a \([r \times n_F]\) dimensional matrix (\( n_F \) is the number of flavors, which is 3 in our case), \( M_\Sigma \) is a \([r \times r]\) matrix and \( m_\nu^{(r+1)} \) is a \([n_F \times n_F]\) dimensional matrix. Below the scale \( M_r \), the heaviest triplet decouples from the theory. Integrating out this degree of freedom gives rise to an effective operator \( \kappa^{(r)} \). The matching condition at \( \mu = M_r \) is
\[ \left. \kappa^{(r)} \right|_{M_r} = 2Y_\Sigma^T (M_r)^{-1} Y_\Sigma \left. \right|_{M_r} . \] (23)

This condition ensures the continuity of \( m_\nu \) at \( \mu = M_r \). In order to get the value of the threshold \( M_r \), we need to write the above matching condition in the basis where \( M_\Sigma = \text{diag}(M_1, M_2, \cdots, M_r) \). Here it is worth mentioning that the matching scale has to be found carefully since \( M_\Sigma \) itself runs with the energy scale, i.e. \( M_i = M_i(\mu) \). The threshold scale \( M_i \) is therefore to be understood as \( M_i(\mu = M_i) \).

In the energy range \( M_{r-1} < \mu < M_r \), the effective mass of the neutrinos will be given as
\[ m_\nu^{(r)} = -\frac{v^2}{4} \left( \kappa^{(r)} + 2Y_\Sigma^T (M_r)^{-1} Y_\Sigma \right) . \] (24)

The first term in Eq. (24) is the contribution of the integrated out triplet of mass \( M_r \) through the effective operator \( \kappa^{(r)} \). The second term represents the contribution of the remaining \((r-1)\) heavy fermion triplets, which are still coupled to the theory, through the seesaw mechanism. \( \tilde{M}_\Sigma \) is now a \([(r-1) \times (r-1)]\) matrix while \( \tilde{Y}_\Sigma \) is a \([(r-1) \times n_F]\) dimensional matrix.
The matching condition at $\mu = M_{r-1}$ is

$$\begin{align*}
\left. \frac{^{(r-1)}\kappa}{M_{r-1}} \right|_{M_{r-1}} &= \left. \frac{^{(r)}\kappa}{M_{r-1}} \right|_{M_{r-1}} + 2^{^{(r)}\Sigma^T} (M_{r-1})^{-1} \left. \frac{^{(r)}\Sigma}{M_{r-1}} \right|_{M_{r-1}}.
\end{align*}$$

(25)

Generalizing the above sequence, we can say that if we consider the intermediate energy region between the $(n-1)^{th}$ and the $n^{th}$ threshold, i.e. $M_n > \mu > M_{n-1}$, then all the heavy triplets from masses $M_r$ down to $M_n$ have been decoupled. In this region the Yukawa matrix $^{(n)}\Sigma$ will be $[(n - 1) \times n_F]$ dimensional and will be given as

$$Y_\Sigma \to \begin{pmatrix}
(y_\Sigma)_{1,1} & \cdots & (y_\Sigma)_{1,n_F} \\
0 & \vdots & 0 \\
(y_\Sigma)_{n-1,1} & \cdots & (y_\Sigma)_{n-1,n_F} \\
0 & \vdots & 0
\end{pmatrix} = \left. \frac{^{(n)}\Sigma}{M_n-M_r} \right|_{M_n-M_r}$$

heavy triplets with masses $M_n-M_r$ integrated out.

$^{(n)}\Sigma$ will be $[(n - 1) \times (n - 1)]$ dimensional. In this energy range the effective neutrino mass matrix will be

$$\Pi_\nu^{(n)} = -\frac{\nu^2}{4}\left(\frac{^{(n)}\Sigma + ^{^{(n)}Q}}{^{(n)}M} \right),$$

(27)

with

$$^{^{(n)}Q} \equiv Y_\Sigma^{(n)} \Sigma^{-1} Y_\Sigma^{(n)},$$

(28)

while the matching condition at $\mu = M_n$ is given by Eq. (25) with $r$ replaced by $(n + 1)$. For $\mu < M_1$, all the heavy triplets will get decoupled and thus only $^{(1)}\kappa$ will contribute, which is the low energy effective neutrino mass operator.

**B. Dimensional regularization and renormalization**

Now we consider the radiative corrections to the fields, masses and couplings in our model, on the lines of that performed in [29, 66] in the context of Type-I seesaw. The wavefunction renormalizations are defined as

$$\psi_B^f = \left( Z_\psi^{\frac{1}{2}} \right)_{fg} \psi^g,$$

(29)
where $\psi \in \{l_L, q_L, e_R, u_R, d_R\}$. We denote the renormalized quantities as $X$ and the corresponding bare fields as $X_B$. For the fermion triplets

$$\Sigma_B = \left( Z_\Sigma^{\frac{1}{2}} \right)_{fg} \Sigma^g .$$

(30)

For the doublet Higgs

$$\phi_B = Z_\phi^\frac{1}{2} \phi ,$$

(31)

whereas

$$A_B = Z_A^\frac{1}{2} A$$

(32)

for the gauge bosons where $A \in \{ B, W^i, G_A \}$. For the Faddeev-Popov ghosts one has

$$c_B = Z_c^\frac{1}{2} c ,$$

(33)

however the ghosts will not appear in the RG evolution of the relevant quantities at one loop level. We introduce the abbreviation

$$\delta Z_X = Z_X - 1 ,$$

(34)

where $Z_X$ denotes the renormalization constant of any of the relevant quantities $X$.

We will use the dimensional regularization and the minimal subtraction scheme for renormalization. In this renormalization formalism, the counter terms are defined such that they only cancel out the divergent parts. Thus the renormalization constants are of the form

$$Z_X = 1 + \sum_{k \geq 1} \delta Z_{X,k} \frac{1}{\epsilon^k} ,$$

(35)

where the $\delta Z_{X,k}$ are independent of $\epsilon$. In our scenario, at the one loop level, the renormalization constants are proportional to $1/\epsilon$. The final results of course will be independent of the particular regularization as well as the renormalization scheme used for the calculations.

The diagrams contributing to the renormalization constants of the different quantities are all shown explicitly in Appendix C. The renormalization constants of different quantities are given by

\[
\delta Z_\phi = -\frac{1}{16\pi^2} \left( 2T - \frac{3}{10} (3 - \xi_1) g_1^2 - \frac{3}{2} (3 - \xi_2) g_2^2 \right) \frac{1}{\epsilon} ,
\]

(36)

\[
\delta Z_{l_L} = -\frac{1}{16\pi^2} \left( Y_{\phi} Y_{e}^\dagger + 3 Y_{\Sigma}^\dagger Y_{\Sigma} + \frac{3}{10} \xi_1 g_1^2 + \frac{3}{2} \xi_2 g_2^2 \right) \frac{1}{\epsilon} ,
\]

(37)

\[
\delta Z_{e_R} = -\frac{1}{16\pi^2} \left( 2 Y_{\phi} Y_{e}^\dagger + \frac{6}{5} \xi_1 g_1^2 \right) \frac{1}{\epsilon} ,
\]

(38)

\[
\delta Z_{\Sigma} = -\frac{1}{16\pi^2} \left[ \left( 2 Y_{\Sigma}^\dagger Y_{\Sigma} + 4 \xi_2 g_2^2 \right) P_R + \left( 2 Y_{\Sigma}^\dagger Y_{\Sigma} \right)^* + 4 \xi_2 g_2^2 \right] \frac{1}{\epsilon} .
\]

(39)
where we have used the $R_\xi$ gauge, and the GUT normalization of the gauge couplings $^2$.

The Yukawa couplings are renormalized as

$$
\delta Z_{Y_e} = -\frac{1}{16\pi^2}\left(-6Y_{\Sigma}^{\dagger}Y_{\Sigma} + \frac{9}{10}(2 + \xi_1)g_1^2 + \frac{3}{2}\xi_2g_2^2\right)\frac{1}{\epsilon},
$$

$$
\delta Z_{Y_\Sigma} = -\frac{1}{16\pi^2}\left(2Y_{e}^{\dagger}Y_{e} - \frac{3}{10}\xi_1g_1^2 - \frac{1}{2}(12 + 7\xi_2)g_2^2\right)\frac{1}{\epsilon},
$$

while the Majorana neutrino mass matrix gets renormalized as

$$
\delta Z_{M\Sigma} = -\frac{1}{16\pi^2}(12 + 4\xi_2)g_2^2\frac{1}{\epsilon}.
$$

The addition of the right handed fermion triplets to the SM will contribute one extra diagram to the renormalization of the Higgs self-coupling $\lambda$, as shown in the diagram (G1) of the Appendix C. This contribution will be$^3$

$$
\delta Z_{\lambda}\big|_{\text{new}} = -\frac{5i}{4\pi^2}\text{Tr}\left[Y_{\Sigma}^{\dagger}Y_{\Sigma}Y_{\Sigma}^{\dagger}Y_{\Sigma}\right] (\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd})\frac{1}{\epsilon}.
$$

Finally for the effective vertex $^{(n)}\kappa$, the renormalization constant is

$$
\delta^{(n)}\kappa = -\frac{1}{16\pi^2}\left[2^{(n)}\kappa Y_{e}^{\dagger}Y_{e} + 2\left(Y_{e}^{\dagger}Y_{e}\right)^{T^{(n)}\kappa} - \kappa \left(\frac{3}{2} - \xi_1\right)g_1^{2^{(n)}\kappa} - \left(\frac{3}{2} - 3\xi_2\right)g_2^{2^{(n)}\kappa}\right]\frac{1}{\epsilon}.
$$

We observe that there is no contribution from the fermion triplet $\Sigma$ in the loop, which means that $\delta^{(n)}\kappa$ will not directly depend on the fermion triplets still coupled to the theory. However, during RG evolution an indirect dependence will creep in via the other couplings.

### C. Calculation of the $\beta$ functions

To calculate the $\beta$ functions for the RG evolution of the Yukawa couplings, Majorana mass matrix, the effective vertex $\kappa$ and other relevant quantities, we consider the relations

$^2$ In $^{65}$ the contributions of fermion triplets to some of the above renormalization constants are calculated in the context of SM extended with these fields. Their conventions of field normalizations are different and hence the results may differ up to numerical constants in certain cases. However, their Eq. (19) for $\delta Y_{\nu}$, which is the same quantity as our $\delta Z_{Y_\Sigma}$ in Eq. (41), is missing the $Y_{e}^{\dagger}Y_{e}$ term. The source of this term is the diagram labelled as (F2) in Appendix C. The extra contribution to $\delta Z_{Y_{e}}$ from the fermion triplets has also not been calculated in $^{67}$.

$^3$ Note that Ref. $^{65}$ gives this quantity ($\delta \lambda$ in their Eq. (20)) to be of the form $\text{Tr}(Y_{\Sigma}^{\dagger}Y_{\Sigma})$. However, the additional contribution to the Higgs quartic coupling $\delta Z_{\lambda}$ should be of the form $\text{Tr}(Y_{\Sigma}^{\dagger}Y_{\Sigma}Y_{\Sigma}^{\dagger}Y_{\Sigma})$, since it comes from the diagram (G1) in Appendix C.
between the bare \((X_B)\) and the corresponding renormalized \((X)\) quantities given by

\[
Z_{\Sigma}^{T^{\frac{1}{2}}} M_{\Sigma_B} Z_{\Sigma}^{\frac{1}{2}} = Z_{M_{\Sigma}} M_{\Sigma},
\]

(45)

\[
Z_{\Sigma R}^{\frac{1}{2}} Y_{\Sigma_B} Z_{\phi}^{\frac{1}{2}} Z_{l_{l}}^{\frac{1}{2}} = \mu^{2} Y_{\Sigma} Z_{Y_{\Sigma}},
\]

(46)

\[
Z_{e_{R}^{e_{e}}}^{\frac{1}{2}} Z_{\Sigma}^{\frac{1}{2}} Z_{\phi}^{\frac{1}{2}} Z_{l_{l}}^{\frac{1}{2}} = \mu^{2} Y_{\phi} Z_{Y_{\phi}},
\]

(47)

\[
Z_{l_{l}}^{T^{\frac{1}{2}}} Z_{\phi}^{\frac{1}{2}} Z_{\Sigma}^{\frac{1}{2}} Z_{l_{l}}^{\frac{1}{2}} = \mu^{2} (\kappa + \delta \kappa),
\]

(48)

where \(Z_{\Sigma R} = P_{R} Z_{\Sigma}\). We further use the functional differentiation method as in [66] to find the \(\beta\) functions for the Yukawa couplings as

\[
16\pi^{2} \beta_{Y_{e}} = Y_{e} \left( \frac{3}{2} Y_{e}^{\dag} Y_{e} + \frac{15}{2} Y_{\Sigma} Y_{e} + T - \frac{9}{4} g_{1}^{2} - \frac{9}{4} g_{2}^{2} \right),
\]

(49)

\[
16\pi^{2} \beta_{Y_{\Sigma}} = Y_{\Sigma} \left( \frac{5}{2} Y_{e}^{\dag} Y_{e} + \frac{5}{2} Y_{\Sigma} Y_{\Sigma} + T - \frac{9}{20} g_{1}^{2} - \frac{33}{4} g_{2}^{2} \right),
\]

(50)

\[
16\pi^{2} \beta_{Y_{u}} = Y_{u} \left( \frac{3}{2} Y_{u}^{\dag} Y_{u} - \frac{3}{2} Y_{d}^{\dag} Y_{d} + T - \frac{17}{20} g_{1}^{2} - \frac{9}{4} g_{2}^{2} - 8 g_{3}^{2} \right),
\]

(51)

\[
16\pi^{2} \beta_{Y_{d}} = Y_{d} \left( \frac{3}{2} Y_{d}^{\dag} Y_{d} - \frac{3}{2} Y_{u}^{\dag} Y_{u} + T - \frac{1}{4} g_{1}^{2} - \frac{9}{4} g_{2}^{2} - 8 g_{3}^{2} \right).
\]

(52)

Here

\[
T = \text{Tr} \left[ Y_{e}^{\dag} Y_{e} + \frac{15}{2} Y_{\Sigma} Y_{e} + Y_{u}^{\dag} Y_{u} + \frac{3}{2} Y_{d}^{\dag} Y_{d} \right],
\]

(53)

and \(\beta_{X} \equiv \mu (dX/d\mu)\). Note that \(Y_{\Sigma}\) is given in Eq. (26), with \((n - 1)\) the number of heavy fermion triplets still coupled to the theory.

Since the fermion triplets have non-zero SU(2)\(_{L}\) charge, they couple to the \(W\) bosons and hence will affect the RG evolution of the gauge coupling \(g_{2}\) via

\[
16\pi^{2} \beta_{g_{2}} = b_{2} g_{2}^{3},
\]

(54)

where

\[
b_{2} = -\frac{19}{6} + \frac{4(n - 1)}{3}.
\]

(55)

Note that if the number of heavy fermion triplets is \(\leq 2\), the value of \(b_{2}\) is always negative. On the other hand, if the number is \(\geq 3\), then \(b_{2}\) becomes positive above the mass scale \(M_{3}\). Adding fermion triplets shifts the \(g_{1}-g_{2}\) intersection to higher energy scales, and the \(g_{2}-g_{3}\) intersection to lower energy scales, as can be seen from Fig. 2. The exact situation would depend on the values of \(M_{i}\).
FIG. 2: The solid (red) line and the dashed (green) lines show the energy scale variations of $g_1$ and $g_3$ respectively in the SM, which is unaffected in Type-III seesaw. The dotted (blue) line gives the SM running of $g_2$, while dot-dashed (magenta), dot-dot-dashed (sky) and densely dotted (black) lines show the running if there were one, two or three fermion triplets respectively.

The RG evolution of $\lambda$ is given by

$$\frac{16\pi^2}{C_5} \beta_\lambda = 6\lambda^2 - 3\lambda \left( \frac{3}{5} g_1^2 + 3g_2^2 \right) + 3g_1^4 + \frac{3}{2} \left( \frac{3}{5} g_1^2 + g_2^2 \right)^2 + 4\lambda T - 8 \text{Tr}[Y^t_e Y_e Y^t_e Y_e + 3Y^t_u Y_u Y^t_u Y_u + 3Y^t_d Y_d Y^t_d Y_d] - 20 \text{Tr}[Y^t_{\Sigma} Y_{\Sigma} Y_{\Sigma} Y_{\Sigma}] - 12g_2^2 M_\Sigma,$$  \hspace{1cm} (56)

As it is evident from Eq. (56), the last term is the new contribution to the $\beta$-function from the heavy triplets still coupled to the theory.

The RG evolution of the Majorana mass matrix of the heavy triplet fermions is given by

$$\frac{16\pi^2}{C_5} \beta_{M_\Sigma} = \text{Tr} \left[ \begin{pmatrix} (n)_{\Sigma} \\ (n)_{\Sigma} \end{pmatrix} P_L + \left( (n)_{\Sigma} \right)^* P_R \right] M_\Sigma + \text{Tr} \left[ \begin{pmatrix} (n)_{\Sigma} \\ (n)_{\Sigma} \end{pmatrix}^* P_L + \left( (n)_{\Sigma} \right)^* P_R \right] - 12g_2^2 M_\Sigma,$$  \hspace{1cm} (57)

where it is always possible to separate the components of different chirality to get the left-
chiral part as

\[
16\pi^2 \beta_{M\Sigma} = \left( Y_{\Sigma} Y_{\Sigma}^\dagger \right) M_{\Sigma} + M_{\Sigma} \left( Y_{\Sigma} Y_{\Sigma}^\dagger \right)^T - 12g_2^2 M_{\Sigma} ,
\]

since \( P_L + P_R = \mathbb{I} \). Thus all the \( \beta \)-functions are gauge-independent, as they should be. The anomalous dimension of \( M_{\Sigma} \) is

\[
-16\pi^2 \gamma^{(0)}_{M\Sigma} = M_{\Sigma}^{-1} \left[ \left( Y_{\Sigma} Y_{\Sigma}^\dagger \right) P_L + \left( Y_{\Sigma} Y_{\Sigma}^\dagger \right)^* P_R \right] M_{\Sigma} + \left[ \left( Y_{\Sigma} Y_{\Sigma}^\dagger \right)^* P_L + \left( Y_{\Sigma} Y_{\Sigma}^\dagger \right) P_R \right] - 12g_2^2 .
\]

Similar to the left-chiral component of \( \beta_{M\Sigma} \) in Eq. (58), the left-chiral component of \( \gamma^{(0)}_{M\Sigma} \) is

\[
-16\pi^2 \gamma^{(0)}_{M\Sigma} = M_{\Sigma}^{-1} \left( Y_{\Sigma} Y_{\Sigma}^\dagger \right) M_{\Sigma} + \left( Y_{\Sigma} Y_{\Sigma}^\dagger \right)^* - 12g_2^2 .
\]

As seen from Eq. (27), the RG evolution of the light neutrino mass matrix \( m_{\nu} \) is controlled by the evolutions of both \( \kappa \) and \( Q \), which are given by

\[
16\pi^2 \beta_{\kappa} = \alpha_{\kappa} \kappa + P_{\kappa}^{T(n)} \kappa + \kappa P_{\kappa} , \\
16\pi^2 \beta_{Q} = \alpha_{Q} Q + P_{Q}^{(n)} Q + Q P_{Q} ,
\]

with

\[
P_{\kappa} = \frac{3}{2} Y_{\Sigma} Y_{\Sigma}^\dagger - \frac{3}{2} Y_e Y_e ; \quad \alpha_{\kappa} = 2T + \lambda - 3g_2^2 ,
\]

\[
P_{Q} = \frac{3}{2} Y_{\Sigma} Y_{\Sigma}^\dagger + \frac{5}{2} Y_e Y_e ; \quad \alpha_{Q} = 2T - \frac{9}{10} g_1^2 - \frac{9}{2} g_2^2 .
\]

IV. RG RUNNING OF NEUTRINO MASSES AND MIXING ANGLES

To derive the RG evolution for the neutrino masses and mixings we follow the standard procedure \[29, 61\]. At any energy scale \( \mu \), the neutrino mass matrix \( m_{\nu} \) can be diagonalized by a unitary transformation via

\[
U_{\nu}(\mu)^T m_{\nu}(\mu) U_{\nu}(\mu) = \text{diag}(m_1(\mu), m_2(\mu), m_3(\mu)) .
\]

In a basis where \( Y_e \) is diagonal, the neutrino mixing matrix is given as

\[
U_{PMNS} = U_{\nu} ,
\]
where $U_{PMNS}$ is the Pontecorvo-Maki-Nakagawa-Sakata neutrino mixing matrix \[67, 68\]. From Eqs. (49) it is seen that above and between the thresholds, off-diagonal terms will be generated in $Y_e$ even if we start with a diagonal $Y_e$ at the high scale, due to the $Y_e^\dagger Y_\Sigma$ terms. These terms will give additional contributions to the evolution of different parameters. In the presence of $Y_e$ with off-diagonal entries, the neutrino mixing matrix will be given as

$$U_{PMNS} = U_e^\dagger U_\nu,$$

(67)

where $U_e$ is the unitary matrix that diagonalizes $Y_e^\dagger Y_e$ by a unitary transformation. $U_{PMNS}$ is parameterized as \[67, 68\]

$$U_{PMNS} = \text{diag}(e^{i\delta_e}, e^{i\delta_\mu}, e^{i\delta_\tau}). U \cdot \text{diag}(e^{-i\phi_1}, e^{-i\phi_2}, 1),$$

(68)

with

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{13}s_{12}e^{i\delta} & c_{23}c_{13} \end{pmatrix}. \quad (69)$$

Here $c_{ij}$ and $s_{ij}$ are the cosines and sines respectively of the mixing angle $\theta_{ij}$, $\delta$ is the Dirac CP violating phase, $\phi_i$ are the Majorana phases. The “flavor” phases $\delta_e$, $\delta_\mu$ and $\delta_\tau$ do not play any role in the phenomenology of neutrino mixing.

In this work, we consider $r = 3$ heavy fermion triplets, one for each generation. Then $Y_\Sigma$ is a $3 \times 3$ matrix at high scale and is identically zero for $\mu < M_1$. The RG evolution of the neutrino parameters is then controlled by

$$16\pi^2 \beta_{Y_e} = Y_e F + \alpha_e Y_e , \quad (70)$$

$$16\pi^2 \beta_{\pi_\nu} = P^T \pi_\nu + \pi_\nu P + \alpha_e \pi_\nu , \quad (71)$$

where

$$P = C_e Y_e^\dagger Y_e + C_\Sigma Y_\Sigma^\dagger Y_\Sigma , \quad (72)$$

$$F = D_e Y_e^\dagger Y_e + D_\Sigma Y_\Sigma^\dagger Y_\Sigma . \quad (73)$$

Eqs. (70) and (71) are essentially the same as the $\beta$-functions given in Eqs. (49), (61) and (62), which we rewrite in the above form for later discussions. For $\mu > M_3$ and $\mu < M_1$, the evolutions of $Y_e$ and $\pi_\nu$ can be written in simple analytic forms, using Table I. Note that
\[ C_e \quad C_\Sigma \quad D_e \quad D_\Sigma \quad \alpha_e \quad \alpha_\nu \]

\[ \begin{array}{cccccc}
\mu > M_3 & \frac{5}{2} & \frac{4}{5} & \frac{3}{2} & \frac{15}{2} & T - \frac{9}{16}g_1^2 - \frac{9}{2}g_2^2 \\
\mu < M_1 & -\frac{3}{2} & 0 & \frac{3}{2} & 0 & T - \frac{9}{16}g_1^2 - \frac{9}{2}g_2^2 + 2T + \lambda - 3g_2^2 \end{array} \]

TABLE I: Coefficients of the \( \beta \)-functions governing the running of neutrino masses and mixings in the energy regimes \( \mu > M_3 \) and \( \mu < M_1 \). The quantity \( T \) is defined in Eq. (53).

for \( \mu > M_3 \) the running of the neutrino masses will be governed by \( \beta_Q \) and so \( P \) in Eq. (72) is the same as \( P_Q \) as defined in Eq. (64). On the other hand, for \( \mu < M_1 \), we have \( P = P_\nu \) as given in Eq. (63). \( P \) and \( F \) are \( 3 \times 3 \) matrices, with the rows and columns representing generations. We denote the elements of \( P \) and \( F \) by \( P_{fg} \) and \( F_{fg} \). The coefficient of \( P_{fg} \) and \( F_{fg} \) in the running of \( Y_e \) and \( \pi_\nu \) can be read off directly from [61], since the structure of Eqs. (70) and (71) remain the same both in Type-I and Type-III seesaw. The values of \( P_{fg} \) and \( F_{fg} \) themselves will however be different because of different underlying theories. The values of the relevant coefficients in Type-III seesaw are shown in Table I.

If we consider the running equations in the basis \( \mathcal{P}_3 = \{m_i; \theta_{12}, \theta_{13}, \theta_{23}; \phi_i; \delta\} \), then both \( \delta \) and \( \dot{\delta} \) become ill-defined at \( \theta_{13} = 0 \) [29, 30] and as a consequence, \( \dot{\theta}_{13} \) also becomes ill-defined because of its \( \delta \) dependence. This is only an apparent singularity. One can get rid of it by imposing a particular value of \( \cot \delta \) at \( \theta_{13} = 0 \) [29, 30] or by using the basis \( \mathcal{P}_J = \{m_i; \theta_{12}, \theta_{23}, \theta_{13}^2; \phi_i; J_{CP}, J'_{CP}\} \), where the singularity does not appear at all [51]. Here \( J_{CP} \) and \( J'_{CP} \) are defined as

\[ J_{CP} \equiv \frac{1}{2}s_{12}s_{23}c_{23}s_{13}c_{13}^2 \sin \delta, \quad (74) \]
\[ J'_{CP} \equiv \frac{1}{2}s_{12}s_{23}c_{23}s_{13}c_{13}^2 \cos \delta. \quad (75) \]

In the limit \( \theta_{13} \to 0 \), \( J_{CP}, J'_{CP} \to 0 \). From the point of view of the experiments also, the Jarlskog invariant \( J_{CP} \) is the quantity which appears in the probability expressions for CP violation in neutrino oscillation experiments, and is therefore directly measurable. \( J'_{CP} \) is needed in order to have complete information on \( \delta \), since \( J_{CP} \) has no information on the sign of \( \cos \delta \). We also choose to write the RG evolution for \( \theta_{13}^2 \) instead of \( \theta_{13} \) as is traditionally done. This quantity turns out to have a smooth behaviour at \( \theta_{13} = 0 \). Moreover, since \( \theta_{13} \geq 0 \) by convention, the complete information about \( \theta_{13} \) lies within \( \theta_{13}^2 \). The information about the Dirac phase will be present in \( J_{CP}, J'_{CP} \).
basis since the quantities $\theta^{\pm}_{ij}$, defined as
\[
\begin{align*}
\theta^{\pm}_{ij} &= \frac{|m_3 \pm m_1 e^{2i\phi_1}|^2}{\Delta m_{\text{atm}}^2 (1 + \zeta)} , \quad \theta^{\pm}_{23} = \frac{|m_3 \pm m_2 e^{2i\phi_2}|^2}{\Delta m_{\text{atm}}^2} , \quad \theta^{\pm}_{12} = \frac{|m_2 e^{2i\phi_2} \pm m_1 e^{2i\phi_1}|^2}{\Delta m_{\text{sol}}^2} , \quad (76)
\end{align*}
\]
\[
\begin{align*}
\delta_{ij} &= \frac{m_1 m_3 \sin 2\phi_1}{\Delta m_{\text{atm}}^2 (1 + \zeta)} , \quad \delta_{23} = \frac{m_2 m_3 \sin 2\phi_2}{\Delta m_{\text{atm}}^2} , \quad \delta_{12} = \frac{m_1 m_2 \sin (2\phi_1 - 2\phi_2)}{\Delta m_{\text{sol}}^2} , \quad (77)
\end{align*}
\]
depend on the mass eigenvalues and Majorana phases only. However the running of $\theta^{2}_{13}$, as seen from the Table III depends on the quantities $\tilde{A}_{ij}^{\pm}, \tilde{B}_{ij}^{\pm}$ defined as
\[
\begin{align*}
\tilde{A}_{ij}^{\pm} &= \frac{4 (m_1^2 + m_3^2) J'_{\text{CP}} \pm 8 m_1 m_3 (J'_{\text{CP}} \cos 2\phi_1 + J_{\text{CP}} \sin 2\phi_1)}{a \Delta m_{\text{atm}}^2 (1 + \zeta)} , \quad (78)
\end{align*}
\]
\[
\begin{align*}
\tilde{A}_{23}^{\pm} &= \frac{4 (m_2^2 + m_3^2) J'_{\text{CP}} \pm 8 m_2 m_3 (J'_{\text{CP}} \cos 2\phi_2 + J_{\text{CP}} \sin 2\phi_2)}{a \Delta m_{\text{sol}}^2} ,\quad (79)
\end{align*}
\]
TABLE III: Coefficients of $P_{fg}$ in the RG evolution equations of the Jarlskog invariant $J_{CP}$, the quantity $J_{CP}' \equiv J_{CP} \cot \delta$, and the Majorana phase difference $(\phi_1 - \phi_2)$, in the limit $\theta_{13} \to 0$. The convention used here is $a \equiv s_{12} c_{12} s_{23} c_{23}$, and $J_{CP} \equiv (a/2)s_{13} c_{13}^2 \sin \delta$.

|          | $64\pi^2 J_{CP}/a$ | $64\pi^2 J_{CP}'/a$ | $32\pi^2 (\phi_1 - \phi_2)$ |
|----------|---------------------|----------------------|-----------------------------|
| $P_{11}$ | 0                   | 0                    | $-4S_{12} \cos 2\theta_{12}$ |
| $P_{22}$ | $-4aG_s^+$          | $2a(G_0^+ - 2G_c^-)$ | $4S_{12} c_{23}^2 \cos 2\theta_{12}$ |
| $P_{33}$ | $4aG_s^-$           | $-2a(G_0^+ - 2G_c^-)$ | $4S_{12} c_{23}^2 \cos 2\theta_{12}$ |
| Re $P_{21}$ | $4s_{23}G_s^+$   | $2s_{23}(G_0^+ + 2G_c^-)$ | $-8S_{12} c_{23} \cos 2\theta_{12} \cot 2\theta_{12}$ |
| Re $P_{31}$ | $4c_{23}G_s^+$   | $2c_{23}(G_0^+ + 2G_c^-)$ | $8S_{12} c_{23} \cos 2\theta_{12} \cot 2\theta_{12}$ |
| Re $P_{32}$ | $-2 \sin 2\theta_{12} \cos 2\theta_{23} G_0^-$ | $\sin 2\theta_{12} \cos 2\theta_{23}(G_0^+ - 2G_c^-)$ | $-4S_{12} \cos 2\theta_{12} \sin 2\theta_{23}$ |
| Im $P_{21}$ | $2s_{23}(G_0^+ - 2G_c^-)$ | $4s_{23}G_s^+$ | $-4Q_{12} c_{23} \cot 2\theta_{12}$ |
| Im $P_{31}$ | $2c_{23}(G_0^+ - 2G_c^-)$ | $4c_{23}G_s^+$ | $4Q_{12} s_{23} \cot 2\theta_{12}$ |
| Im $P_{32}$ | $\sin 2\theta_{12}(G_0^+ + 2G_c^-)$ | $-2 \sin 2\theta_{12} G_0^-$ | $0$ |

where $a \equiv s_{12} c_{12} s_{23} c_{23}$. Clearly these quantities depend on $J_{CP}$, $J_{CP}'$ in addition to the masses and Majorana phases. The coefficients for the RG evolution of $J_{CP}$ and $J_{CP}'$ are presented in Table III where the quantities $G_{0,c,s}^\pm$ are given by

\[
G_0^\pm = \frac{m_2^2 + m_3^2}{\Delta m_{atm}^2} \pm \frac{m_1^2 + m_3^2}{\Delta m_{atm}^2 (1 + \zeta)}, \quad (82)
\]
\[
G_s^\pm = \frac{m_1 m_3 \sin 2\phi_1}{\Delta m_{atm}^2 (1 + \zeta)} \pm \frac{m_2 m_3 \sin 2\phi_2}{\Delta m_{atm}^2}, \quad (83)
\]
\[
G_c^\pm = \frac{m_1 m_3 \cos 2\phi_1}{\Delta m_{atm}^2 (1 + \zeta)} \pm \frac{m_2 m_3 \cos 2\phi_2}{\Delta m_{atm}^2}. \quad (84)
\]

Thus all the the quantities appearing in the evolution equations (78) – (84) have finite well-defined limits for $\theta_{13} \to 0$ in the $P_f$ basis.

Even if one starts with diagonal $Y_e$ (i.e. $Y_e = {\text{diag}}(y_e, y_\mu, y_\tau)$) at the high scale, non-zero off-diagonal elements of $Y_e$ will be generated through Eqs. (70) – (73) since $Y^{(n)}\Sigma^{(n)}$ is not diagonal. These off-diagonal elements will give additional contributions to the running of masses and mixing above and between the thresholds through $F$ and $\alpha_e$. Since $\alpha_e$ is flavor
\[
\begin{array}{cccccccc}
16\pi^2 \dot{\theta}_{12} & 16\pi^2 \dot{\theta}_{13} & 16\pi^2 \dot{\theta}_{23} & 16\pi^2 \dot{J}_{\text{CP}} & 16\pi^2 \dot{J}'_{\text{CP}} & 16\pi^2 \dot{U}_e & 16\pi^2 \dot{\phi}_1 & 16\pi^2 \dot{\phi}_2 \\
F_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{Re } F_{21} & -c_{23} & -4s_{23}J'_{\text{CP}}/a & 0 & 0 & -s_{23}a/2 & 0 & 0 \\
\text{Re } F_{31} & s_{23} & -4c_{23}J'_{\text{CP}}/a & 0 & 0 & -c_{23}a/2 & 0 & 0 \\
\text{Re } F_{32} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\text{Im } F_{21} & 0 & -4s_{23}J_{\text{CP}}/a & 0 & -s_{23}a/2 & 0 & c_{23}c_{12}/s_{12} & -c_{23}s_{12}/c_{12} \\
\text{Im } F_{31} & 0 & -4c_{23}J_{\text{CP}}/a & 0 & -c_{23}a/2 & 0 & -s_{23}c_{12}/s_{12} & s_{23}s_{12}/c_{12} \\
\text{Im } F_{32} & 0 & 0 & 0 & 0 & 0 & -1/(c_{23}s_{23}) & -1/(c_{23}s_{23}) \\
\end{array}
\]

TABLE IV: Coefficients of \( F_{fg} \) in the RG evolution equations of all the angles \( (\theta_{12}, \theta_{13}^2, \theta_{23}) \), \( J_{\text{CP}}, J'_{\text{CP}} \) and the Majorana phases \( \phi_i \) in the limit \( \theta_{13} \to 0 \). The convention used here is \( a \equiv s_{12}c_{12}s_{23}c_{23} \), and \( J_{\text{CP}} \equiv (a/2)s_{13}c_{13}^2 \sin \delta \). We neglect \( y_e \) and \( y_\mu \) compared to \( y_\tau \), and take vanishing flavor phases.

diagonal, it will contribute to the running of \( y_e, y_\mu \) and \( y_\tau \), while off-diagonal components of \( F \) will contribute additional terms in the \( \beta \)-functions of angles and phases, as tabulated in Table IV. These contributions will just get added to the \( P_{fg} \) contribution for the evolution of the quantities in Tables III, III, IV. Note that the \( F_{fg} \) coefficients are \( \lesssim \mathcal{O}(1) \), whereas the \( P_{fg} \) coefficients are \( \gtrsim \mathcal{O}(m_i^2/\Delta m^2_{\text{atm}}) \). Since the running is significant only when \( m_i^2 \gg \Delta m^2_{\text{atm}} \), in almost all the region of interest \( P_{fg} \) contributions dominate over the \( F_{fg} \) contribution.

Note that the analytical expressions obtained in Eq. (76) onwards, and those given in the tables, are valid only in the two extreme regions \( \mu > M_3 \) and \( \mu < M_1 \). For the intermediate energy scales, \( m_\nu \) will receive contributions from both \( \kappa \) and \( Q \). In the SM these two quantities have non-identical evolutions, as seen from Eqs. (61) and (62), and therefore the net evolution of \( Y_e \) and \( m_\nu \) is rather complicated. We perform it numerically in the next section.
V. ILLUSTRATIVE EXAMPLES OF RG RUNNING OF MASSES AND MIXING

In this section we numerically calculate the RG evolution of the masses and mixing parameters within the Type-III seesaw model including the impact of running between the thresholds. This analysis is done by imposing suitable matching conditions \((25)\) at the thresholds. For illustration, we start at \(\mu_0 = 10^{16} \text{ GeV}\) and choose the basis in which \(Y_e\) is diagonal, so that \(U_{PMNS} = U_\nu\). We further choose \(U_\nu\) at this high scale to be the bimaximal mixing matrix \(U_{\nu,\text{bimax}}\) \([39, 69]\), i.e. \(\theta_{12} = \theta_{23} = \pi/4\) and \(\theta_{13} = 0\). This scenario is clearly inconsistent with the current data in the absence of RG evolution. We shall check if the radiative corrections to the masses and mixing angles can make it consistent with the data at the low scale.

If the low energy theory in the complete energy range \(\mu < \mu_0\) is the SM, then \(\theta_{12}\) decreases as the energy scale decreases, however the running is not sufficient to achieve compatibility with the low energy data. If the low energy theory is the MSSM, then \(\theta_{12}\) increases with decreasing energy scale \([56]\), so that compatibility with the data is not possible. However, it has been shown in \([59, 70, 71]\) in the context of Type-I seesaw mechanism, that the inclusion of threshold effects can make the mixing angle \(\theta_{12}\) decrease substantially as we go to lower energy scale and can give the correct values consistent with the Large Mixing Angle (LMA) solution. In this section we study the evolution from bi-maximal mixing at high scale in the context of Type-III seesaw scenario, including the seesaw threshold effects.

We write the neutrino mass matrix as

\[
\mathbf{m}_\nu = U_{\nu,\text{bimax}}^\dagger \text{diag}(m_1, m_2, m_3) U_{\nu,\text{bimax}}^\dagger ,
\]

with \(\delta_e = \delta_\mu = \delta_\tau = 0\) at the high scale. Given the masses of the three fermion triplets and the light neutrino masses at the high scale, one can determine a \(Y_\Sigma\) at the high scale\(^4\) that satisfies the seesaw relation \(\mathbf{m}_\nu = -(v^2/2)Y_\Sigma^T \mathbf{M}_\Sigma^{-1} Y_\Sigma\). We then evolve the parameters using the analysis of Sec IV.

Among the neutrino mixing angles, \(\theta_{12}\) is expected to be the most sensitive to RG effects. Table II shows that \(\dot{\theta}_{12}\) is proportional to \(Q_{12}^+\) and \(S_{12}\), which are in turn proportional to \(\mathbf{m}_\nu\). The solution for \(Y_\Sigma\) need not be unique, however any one of the solutions would suffice for the illustration. For practicality, we first choose an “trial” \(Y_\Sigma\), calculate the corresponding \(\mathbf{M}_\Sigma\) from the seesaw relation, and then apply the basis transformation that makes \(\mathbf{M}_\Sigma\) diagonal and takes the “trial” \(Y_\Sigma\) to its final form.
FIG. 3: RG evolution of mixing angles and mass squared differences, starting from bimaximal mixing at $\mu_0 = 10^{16}$ GeV, for normal mass ordering and hierarchical neutrino masses. The left panels represent the scenario where the Majorana phases vanish at $\mu_0$. The right panel shows a representative case of nonzero Majorana phases ($\phi_1 = 89.0^\circ, \phi_2 = 0.4^\circ$) at $\mu_0$. The values of parameters at the high scale have been chosen such that the $\Delta m^2$'s and $g_2$ at the low scale are reproduced.

$(m_i^2/\Delta m^2_{\odot})$ as can be seen from Eqs. (76) and (77). For the other angles $\theta_{ij}$, the corresponding quantities $Q^+_{ij}$ and $S_{ij}$ are proportional to $(m_i^2/\Delta m^2_{\text{atm}})$, so the evolution of these angles is smaller. The direction of $\theta_{12}$ evolution depends on the details of the Yukawa coupling matrix and masses of the heavy fermions.

Since the values of Majorana phases at the low scale are completely unknown, we first consider the case where $\phi_1 = \phi_2 = 0$. In this case the CP violation will remain zero at all energy scales. The left panels of Fig. 3 show the running of mixing angles and mass squared differences for the normal mass ordering in this scenario. It is observed that $\theta_{12}$ in the
intermediate energy region changes more rapidly than in the extreme regions, however this change is in the opposite direction to what is required. As a result, bimaximal mixing at the high scale is not compatible with the low energy data in our model when the Majorana phases vanish. With nonzero Majorana phases, however, it is possible to achieve compatibility with the low scale data, as can be seen from the right panels of the figure.

The lower panels of Fig. 3 show the evolution of \( m_0 \), the lowest mass scale, and the two mass squared differences. As can be observed, the running of masses is quite substantial in Type-III seesaw, as compared to the SM, the MSSM [30], or the Type-I seesaw [59]. Most of this running occurs in the intermediate energy range \( M_1 < \mu < M_3 \), where threshold effects play a crucial role in enhancing the running. Note that the values of \( m_0 \) required to cause substantial running of mixing angles is quite small: in the case of vanishing (non vanishing) Majorana phases, we have taken \( m_0 = 0.04(0.01) \) eV at \( \mu = \mu_0 \). Thus, even at extremely small \( m_0 \), substantial running of neutrino parameters can be present in the Type-III seesaw.

The example of the bimaximal mixing discussed above was just for illustration. However, it brings out certain salient features of the RG running in Type-III seesaw scenario. The running of neutrino masses can be quite substantial here in the intermediate energy range. Moreover, threshold effects can enhance the extent of running of mixing angles, as well as the direction of the evolution, similar to the Type-I seesaw scenario [59]. Majorana phases are also seen to play an important role in determining the extent and the direction of RG running of neutrino mixing parameters.

In Fig. 4, we illustrate the RG evolution of parameters when the neutrino masses are quasi-degenerate. We have taken the parameter values at the high scale to achieve compatibility with the low scale data, without imposing any special symmetry. However in order to bring out certain salient features of the RG evolution that are independent of the threshold effects, we have chosen a small \( \theta_{13} \) value, \( |\phi_1 - \phi_2| \approx \pi/2 \), and \( Y_2^\dagger Y_\Sigma \) to be almost diagonal in the charged lepton basis, with hierarchical eigenvalues. These conditions ensure that \( P_{21} \) and \( P_{31} \) are small, and \( S_{12} \) vanishes, so that from Table III the evolution of \( (\phi_1 - \phi_2) \) is extremely small. Thus \( |\phi_1 - \phi_2| \) is expected to stay close to \( \pi/2 \) even after evolution, which is verified by the figure. Moreover, combined with \( m_1 \approx m_2 \), the choice \( |\phi_1 - \phi_2| \approx \pi/2 \) makes \( Q^+_{12} \) extremely small, thus restricting the \( \theta_{12} \) evolution.

It is observed that the running of \( \theta_{23} \) is now large, owing to \( m_0^2/\Delta m^2_{\text{atm}} \approx 1 \). This makes it possible to mimic maximal mixing accidentally, even if the mixing generated at the high
FIG. 4: RG evolution of mixing angles, mass squared differences, and CP violating phases, for quasi-degenerate neutrino masses and normal mass ordering. The values of parameters at the high scale have been chosen such that the $\Delta m^2$'s and $g_2$ at the low scale are reproduced. Note that for the Majorana phases $\phi_i$, the regions $(0^\circ - 180^\circ)$ and $(180^\circ - 360^\circ)$ should be identified with each other.

The right hand bottom panel of Fig. 4 shows the evolution of $m_\beta \equiv \sqrt{\sum_i |U_{ei}|^2 m_i^2}$, the effective neutrino mass measured in the Tritium beta decay experiments [72], as well as $m_{ee} \equiv |\sum_i U_{ei}^2 m_i|$, the effective neutrino Majorana mass in the neutrinoless double beta decay. Note that since $\theta_{13}$ is small, $m_1 \approx m_2 \approx m_0$, and since $|\phi_1 - \phi_2| \approx \pi/2$ in addition, we have $m_{ee} \approx m_0 \cos 2\theta_{12}$. Also in the quasi-degenerate case, the sum of neutrino masses that is restricted by cosmology is $\sum m_i \approx 3m_0$. The large running of these masses suggests that, even if the beta decay experiments were to bound $m_\beta$ to $\leq 0.3$ eV, or the neutrinoless double beta
beta decay experiments were to bound $m_{ee}$ to $\leq 0.1$ eV, or the cosmological observations were to restrict $m_0$ at the low scale to $\leq 0.3$ eV, the value of $m_0$ generated at the high scale can still be substantially larger.

It is thus observed that in Type-III seesaw, the RG evolution of masses, angles as well as CP violating phases can be significant between the thresholds even at low $m_0$ values. The reason behind this, as well as the exact dependence of the evolution on the mass thresholds and Majorana phases, needs to be studied in further detail for a better understanding of the allowed neutrino parameter space at high energies.

VI. SUMMARY AND CONCLUSIONS

In this paper we have studied the RG evolution of neutrino masses and mixing angles in the context of Type-III seesaw mechanism mediated by heavy fermions $\Sigma$ transforming as triplets under SU(2)$_L$. Tree level exchange of such particles gives rise to an effective operator $\kappa_5 l_L l_L \phi \phi$ below their lowest mass threshold. If one or more such triplets are present in the model, they affect the RG evolution of wavefunctions, masses and couplings. We compute these extra contributions using dimensional regularization and minimal subtraction scheme.

We calculate the beta functions for the Yukawa couplings $Y_e$, $Y_u$, $Y_d$ and $Y_{\Sigma}$, the SU(2)$_L$ gauge coupling $g_2$, the Higgs self-coupling $\lambda$, the heavy fermion triplet mass matrix $M_{\Sigma}$, and finally the light neutrino mass matrix $m_\nu$. We do our calculation in the $R_\xi$ gauge and show the gauge invariance explicitly by demonstrating that the terms containing $\xi$ are not present in the $\beta$-functions.

It is found that the presence of the triplets does not give rise to any additional diagram for the effective vertex $\kappa$. However, the presence of these fields is felt indirectly in the running of $\kappa$ through their contribution to the evolution of the other quantities. Since the fermion triplets couple to W bosons, the evolution of the SU(2)$_L$ gauge coupling $g_2$ is significantly affected, with more than two $\Sigma$ triplets changing the sign of the $\beta$ function for $g_2$. This may also have implications for the unification of gauge couplings. In turn, the masses of the $\Sigma$'s are also affected substantially due to the coupling with $g_2$.

We give the analytic expressions for the RG evolutions of the neutrino masses and mixing above the highest mass threshold and below the lowest one. We use a basis $P_J = \{m_i, \theta_{12}, \theta_{23}, \theta^2_{13}, \phi_i, J_{CP}, J'_{CP}\}$ instead of the commonly used basis $P_\delta = \{m_i, \theta_{12}, \theta_{23}, \theta^2_{13}, \phi_i, J_{CP}, J'_{CP}\}$.
\{m_i, \theta_{12}, \theta_{23}, \theta_{13}, \phi_i, \delta \}. The advantage of the \( P_J \) basis is that all the evolution equations are explicitly non-singular at all points in the parameter space including at \( \theta_{13} = 0 \) [51].

We consider the scenario with three triplets having non-degenerate masses and include the effect of successive decoupling of the heavy triplets at their respective mass thresholds by imposing suitable matching conditions at each threshold. We present illustrative examples of running of masses and mixings by numerical diagonalization of the effective neutrino mass matrix. Although the running of neutrino parameters is not very large in the SM, in our model the running can be large due to threshold effects of the heavy triplets. In particular we find that starting from bi-maximal mixing at a high scale it is possible to generate low scale values of masses and mixing angles for the normal hierarchical neutrino spectrum. However, this requires non-zero values of the Majorana phases. Indeed it is observed that threshold effects and Majorana phases can influence the evolution of the mixing angles significantly.

We show that even in the case of hierarchical neutrinos, the RG evolution of neutrino masses and mixing between the thresholds can be substantial in the Type-III seesaw scenario. Moreover for quasi-degenerate neutrinos, the large running of masses implies that the value of \( m_0 \) at the high scale can be quite large, even if the mass related measurements from the beta decay, neutrinoless double beta decay, or cosmology, restrict its value at the low scale.

In conclusion, this work studies threshold effects in the context of the Type-III seesaw mechanism. It is crucial for testing the viability of a high scale theory with low scale data. Indeed it is seen that theories that are excluded by the data in the absence of RG running can become viable once these effects are included. In order to determine the allowed neutrino parameter space at the high scale, a detailed exploration of the dependence of RG effects on various parameters is necessary. This is all the more important in view of the onset of the precision era in neutrino physics.

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**APPENDIX A: FEYNMAN RULES INVOLVING THE FERMION TRIPLET $\Sigma$**

In this appendix, we list the Feynmen rules involving the fermion triplets $\Sigma$. Following \cite{73}, we introduce the fermion flow arrow for the leptons, which is the gray arrow in the diagrams. The black arrows indicate the lepton number flow. However interactions involving $\Sigma$ may violt lepton numbers and thus the $\Sigma$ line does not carry any lepton flow arrow. For the lepton number conserving interactions, the two arrows are parallel for particles, and antiparallel for the charge-conjugate fields. The Feynman rules are also given for the effective operator in the low energy limit of the theory obtained by integrating out these heavy fermion triplets.

1. **Propagator**

\[
\frac{\Sigma^{gj}}{\Sigma^{fi}} = \frac{i(p + M_f)}{p^2 - M_f^2 + i\epsilon} \delta_{fg} \delta_{ij}
\]

2. **Yukawa interactions**

\[
\begin{align*}
\phi_a &= -i\mu^\epsilon/2 (Y_\Sigma^l)_{fg} (\sigma^i \varepsilon)_{ba} P_R \\
\phi_a &= -i\mu^\epsilon/2 (Y_\Sigma^l)_{fg} (\varepsilon^T \sigma^i)_{ba} P_L \\
\phi_a &= -i\mu^\epsilon/2 (Y_\Sigma^l)_{gf} (\varepsilon^T \sigma^i)_{ab} P_L \\
\phi_a &= -i\mu^\epsilon/2 (Y_\Sigma^l)_{gf} (\sigma^i \varepsilon)_{ab} P_R
\end{align*}
\]
3. Gauge boson interactions

\[ -i\mu \gamma^\mu (i\varepsilon^{ijk}) \]

4. Counterterms

\[ \Sigma^{fi} \Sigma^{fj} = i \left[ \rho (\delta Z_{\Sigma})_{fg} - (\delta Z_{M_\Sigma M_\Sigma})_{fg} \right] \delta_{ij} \]

\[ \phi_a = -i\mu^{\epsilon/2} (\delta Z_{\Sigma Y_\Sigma}^T)_{fg} (\varepsilon^T \sigma^i)_{ab} P_R \]

\[ \phi_a = -i\mu^{\epsilon/2} (\delta Z_{\Sigma Y_\Sigma})_{fg} (\varepsilon^T \sigma^i)_{ab} P_L \]

\[ \phi_a = -i\mu^{\epsilon/2} (\delta Z_{\Sigma Y_\Sigma}^T)_{fg} (\varepsilon^T \sigma^i)_{ab} P_L \]

\[ \phi_a = -i\mu^{\epsilon/2} (\delta Z_{\Sigma Y_\Sigma})_{fg} (\varepsilon^T \sigma^i)_{ab} P_R \]
5. Effective vertex $\kappa$

\[
\begin{align*}
\left(\kappa^f\right)_{Lb}^{Ld} &= i\mu\varepsilon^{1/2} \kappa_{fg} \frac{1}{3} (\varepsilon_{ab}\varepsilon_{cd} + \varepsilon_{ad}\varepsilon_{bc}) P_L \\
\left(\kappa^f\right)_{Lb}^{Ld} &= i\mu\varepsilon^{1/2} (\kappa^*)_{fg} \frac{1}{3} (\varepsilon_{ab}\varepsilon_{cd} + \varepsilon_{ad}\varepsilon_{bc}) P_R
\end{align*}
\]

6. Counterterms for $\kappa$

\[
\begin{align*}
\left(\kappa^f\right)_{Lb}^{Ld} &= i\mu\varepsilon^{1/2} (\delta\kappa)_{fg} \frac{1}{3} (\varepsilon_{ab}\varepsilon_{cd} + \varepsilon_{ad}\varepsilon_{bc}) P_L \\
\left(\kappa^f\right)_{Lb}^{Ld} &= i\mu\varepsilon^{1/2} (\delta\kappa^*)_{fg} \frac{1}{3} (\varepsilon_{ab}\varepsilon_{cd} + \varepsilon_{ad}\varepsilon_{bc}) P_R
\end{align*}
\]

**APPENDIX B: FEYNMAN RULES FOR THE SM FIELDS**

In this appendix, we list the Feynman rules involving the SM fields only, also given in [66], which are needed for our calculations. The directions of the arrows should be interpreted in the same way as stated at the beginning of Appendix [A].
1. Propagators

\[ \frac{i\not{p}}{p^2 + i\epsilon} \delta_{fg} \delta_{ab} \quad ; \quad \frac{i\not{p}}{p^2 + i\epsilon} \delta_{fg} \delta_{ab} \quad , \quad X \in \{u, d\} \]

\[ \frac{i\not{p}}{p^2 + i\epsilon} \delta_{fg} \delta_{ab} \quad ; \quad \frac{i\not{p}}{p^2 + i\epsilon} \delta_{fg} \delta_{ab} \quad , \quad X \in \{B, W^i\} \]

where \( \xi = \xi_1 \) for B boson and \( \xi = \xi_2 \) for W boson.

2. Yukawa interactions

\[ -i\mu \ell^{/2} (Y_e^T)_{gf} \delta_{ab} P_L \]

\[ -i\mu \ell^{/2} (Y_e^T)_{gf} \delta_{ab} P_L \]

\[ -i\mu \ell^{/2} (Y_e^T)_{gf} \delta_{ab} P_L \]

\[ -i\mu \ell^{/2} (Y_e^T)_{gf} \delta_{ab} P_L \]

Similar Feynman rules, as those in the left panel, are there for Yukawa interactions of \( q_L-u_R \)
and \( q_L-d_R \) with the Higgs \( \phi \) having coefficients \( Y_u \) and \( Y_d \) respectively.
3. Gauge boson – lepton interactions

\[ \frac{\bar{\nu}_R}{\nu_R} B = i\mu g_1 \gamma^\mu \delta_{gf} P_R \]

\[ \frac{\bar{\nu}_R}{\nu_R} B = -i\mu g_1 \gamma^\mu \delta_{fg} P_L \]

\[ \frac{\bar{e}_R}{e_R} W^i B = -i\mu \bar{\tau}_2 g_2 \gamma^\mu (\sigma^i)_{ab} \delta_{gf} P_L \]

\[ \frac{\bar{e}_R}{e_R} W^i B = i\mu \bar{\tau}_2 g_2 \gamma^\mu (\sigma^i)_{ba} \delta_{fg} P_R \]

\[ \frac{\bar{\nu}_L}{\nu_L} B = \frac{i}{2} \mu \bar{\tau}_2 g_2 \gamma^\mu \delta_{gf} \delta_{ab} P_L \]

\[ \frac{\bar{\nu}_L}{\nu_L} B = -\frac{i}{2} \mu \bar{\tau}_2 g_2 \gamma^\mu \delta_{fg} \delta_{ab} P_R \]

4. Gauge boson – Higgs interactions

\[ \frac{\phi_a}{\phi_b} B = -\frac{1}{2} \mu \bar{\tau}_1 g_1 (p_\mu + q_\mu) \delta_{ab} \]

\[ \frac{\phi_a}{\phi_b} W^i = -i\mu \bar{\tau}_2 g_2 (p_\mu + q_\mu) (\sigma^i)_{ba} \]

The vertices involving two Higgses and two gauge bosons are not shown since they do not appear explicitly in our analysis.

5. Higgs self-interaction

\[ \frac{\phi_a}{\phi_b} \phi_c \phi_d = -i\mu \lambda \frac{1}{2} (\delta_{ac} \delta_{bd} + \delta_{bc} \delta_{ad}) \]
6. Counterterms

\[ l^f_{La} \rightarrow l^g_{Lb} = \text{i} \gamma^\mu (\delta Z_{l_{L}})_{gf} P_L \delta_{ba} \]

\[ l^f_{La} \rightarrow l^g_{Lb} = -\text{i} \gamma^\mu (\delta Z_{l_{L}})_{fg} P_L \delta_{ba} \]

\[ e^f_R \rightarrow e^g_R = \text{i} \gamma^\mu (\delta Z_{e_R})_{gf} P_R \]

\[ e^f_R \rightarrow e^g_R = -\text{i} \gamma^\mu (\delta Z_{e_R})_{fg} P_R \]

\[ \phi_a \rightarrow \phi_b = \text{i} \left( \gamma^\mu \left( \frac{p^2 \delta Z_{\phi} - \delta m^2_\phi}{4} \right) \delta_{ba} \right) \]

APPENDIX C: CALCULATION OF RENORMALIZATION CONSTANTS

Here we show the Feynman diagrams contributing to the renormalization constants of different quantities. Note that for particles in the loop, we suppress the flavor as well as the SU(2)_L indices.

1. Doublet Higgs wavefunction and mass \((Z_\phi \text{ and } \delta m^2_\phi)\)

\[ A_1 + A_2 + \phi \phi \]

\[ \phi \phi \]

\[ A_3 + A_4 + \phi \phi \]

\[ \phi \phi \]

\[ A_5 + A_6 + \phi \phi \]

\[ \phi \phi \]

\[ A_7 + A_8 = \text{UV finite} \]

\[ \Rightarrow \delta Z_\phi = -\frac{1}{16\pi^2} \left( 2T - \frac{3}{10} (3 - \xi_1) g_1^2 - \frac{3}{2} (3 - \xi_2) g_2^2 \right) \frac{1}{\epsilon}, \]

and \[ \delta m^2_\phi = \frac{1}{16\pi^2} \left( 3\lambda m^2_\phi - \frac{3}{10} \xi_1 g_1^2 m^2_\phi - \frac{3}{2} \xi_2 g_2^2 m^2_\phi - 4 \text{ Tr}[3Y^1_{\Sigma} Y_{\Sigma}] M^2_{\Sigma} \right) \frac{1}{\epsilon}. \]
2. Left-handed lepton wavefunction \((Z_L^l)\)

\[
\begin{align*}
\delta Z_L &= -\frac{1}{16\pi^2} \left( Y^e\bar{Y}_e + 3Y^i\bar{Y}_\Sigma + \frac{3}{10}\xi_1 g_1^2 + \frac{3}{2}\xi_2 g_2^2 \right) \frac{1}{\epsilon}.
\end{align*}
\]

3. Wavefunction and mass of fermion triplet \((Z_\Sigma\) and \(Z_{M_\Sigma}\))

\[
\begin{align*}
\delta Z_\Sigma &= -\frac{1}{16\pi^2} \left[ \left( 2Y_\Sigma Y_\Sigma^\dagger + 4\xi_2 g_2^2 \right) P_R + \left( 2(Y_\Sigma Y_\Sigma^\dagger)^* + 4\xi_2 g_2^2 \right) P_L \right] \frac{1}{\epsilon},
\end{align*}
\]

and

\[
\begin{align*}
\delta Z_{M_\Sigma} &= -\frac{1}{16\pi^2} \left( 12 + 4\xi_2 \right) g_2^2 \frac{1}{\epsilon}.
\end{align*}
\]
4. Right-handed charged lepton wavefunction ($Z_{eR}$)

\[ e_R^g \equiv e_R^g + e_R^g + e_R^g + e_R^g + e_R^g + e_R^g + e_R^g = \text{UV finite} \]

\[ \Rightarrow \delta Z_{eR} = -\frac{1}{16\pi^2} \left( 2Y_eY_e^\dagger + \frac{6}{5} \xi_1 g_1^2 \right) \frac{1}{\epsilon} . \]

5. $l_L e_R^g \phi$ Yukawa vertex ($Z_{Ye}$)

\[ \Rightarrow \delta Z_{Ye} = -\frac{1}{16\pi^2} \left( -6Y_{e\Sigma}Y_{e\Sigma}^\dagger + \frac{9}{5} \left( 1 + \frac{1}{2} \xi_1 \right) g_1^2 + \frac{3}{2} \xi_2 g_2^2 \right) \frac{1}{\epsilon} . \]
6. $l_L \Sigma \phi$ Yukawa vertex ($Z_{Y\Sigma}$)

\[
\delta Z_{Y\Sigma} = -\frac{1}{16\pi^2} \left( 2Y_e^\dagger Y_e - \frac{3}{10} \xi_1 g_1^2 - \frac{1}{2} \left(12 + 7\xi_2\right) g_2^2 \right) \frac{1}{\epsilon}.
\]

7. The extra diagram contributing to $Z_{\lambda}$

\[
= -\frac{5i}{4\pi^2} \text{Tr} \left[ Y_{\Sigma}^\dagger Y_{\Sigma} Y_{\Sigma}^\dagger Y_{\Sigma} \right] (\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd}) \frac{1}{\epsilon} + \text{UV finite}.
\]
8. Calculation of $Z_\kappa$

$$
\begin{aligned}
\phi_a &\quad l_{Lc}^f \quad \phi_a &\quad l_{Lc}^f \\
\phi_d &\quad l_{Lb}^g \quad \phi_d &\quad l_{Lb}^g \\
\end{aligned}
\quad \equiv \quad H1

\begin{aligned}
\phi_d &\quad e_R &\quad l_{Lc}^g \\
l_L &\quad + \quad a \leftrightarrow d &\quad l_L \\
l_{Lb}^g &\quad \phi_d &\quad l_{Lb}^g \\
\end{aligned}
\quad + \quad H2

\begin{aligned}
\phi_d &\quad e_R &\quad l_{Lc}^g \\
l_L &\quad + \quad a \leftrightarrow d &\quad l_L \\
l_{Lb}^g &\quad \phi_d &\quad l_{Lb}^g \\
\end{aligned}
\quad + \quad H3

\begin{aligned}
\phi_d &\quad e_R &\quad l_{Lc}^g \\
l_L &\quad + \quad a \leftrightarrow d &\quad l_L \\
l_{Lb}^g &\quad \phi_d &\quad l_{Lb}^g \\
\end{aligned}
\quad + \quad H4

\begin{aligned}
\phi_d &\quad e_R &\quad l_{Lc}^g \\
l_L &\quad + \quad a \leftrightarrow d &\quad l_L \\
l_{Lb}^g &\quad \phi_d &\quad l_{Lb}^g \\
\end{aligned}
\quad + \quad H5

\begin{aligned}
\phi_d &\quad e_R &\quad l_{Lc}^g \\
l_L &\quad + \quad a \leftrightarrow d &\quad l_L \\
l_{Lb}^g &\quad \phi_d &\quad l_{Lb}^g \\
\end{aligned}
\quad + \quad H6

\begin{aligned}
\phi_d &\quad e_R &\quad l_{Lc}^g \\
l_L &\quad + \quad a \leftrightarrow d &\quad l_L \\
l_{Lb}^g &\quad \phi_d &\quad l_{Lb}^g \\
\end{aligned}
\quad + \quad H7

\begin{aligned}
\phi_d &\quad e_R &\quad l_{Lc}^g \\
l_L &\quad + \quad a \leftrightarrow d &\quad l_L \\
l_{Lb}^g &\quad \phi_d &\quad l_{Lb}^g \\
\end{aligned}
\quad + \quad H8

\begin{aligned}
\phi_d &\quad e_R &\quad l_{Lc}^g \\
l_L &\quad + \quad a \leftrightarrow d &\quad l_L \\
l_{Lb}^g &\quad \phi_d &\quad l_{Lb}^g \\
\end{aligned}
\quad + \quad H9

\quad = \quad \text{UV finite}

\Rightarrow \quad \delta_\kappa = -\frac{1}{16\pi^2} \left[ 2\kappa \left(Y_e^\dagger Y_e\right) + 2 \left(Y_e^\dagger Y_e\right)^T \kappa - \lambda_\kappa - \left(\frac{3}{2} - \xi_1\right) g_1^2 \kappa - \left(\frac{3}{2} - 3\xi_2\right) g_2^2 \kappa \right] \frac{1}{\epsilon}.

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