Atiyah–Bott index on stratified manifolds

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Abstract

We define Atiyah–Bott index on stratified manifolds and express it in topological terms. By way of example, we compute this index for geometric operators on manifolds with edges.

1 Introduction

The paper deals with elliptic theory on stratified manifolds. The symbol of a pseudodifferential operator on a stratified manifold is a collection of symbols on the strata. The symbol on the stratum of maximal dimension, called the interior symbol, is of particular importance, since it is a scalar function on the cotangent bundle, while the symbols on the lower-dimensional strata are operator-valued.

An operator with elliptic interior symbol defines an element in the $K$-group of operators with zero interior symbol. This element, which we call the Atiyah–Bott index, has the following properties.

• It is determined by the interior symbol of the operator and is a homotopy invariant of the interior symbol.

• For a smooth manifold, this invariant coincides with the Fredholm index.

• The Atiyah–Bott index is the obstruction to making an operator with invertible interior symbol invertible by adding lower-order terms.

• For a manifold with boundary, this index coincides with the Atiyah–Bott obstruction (see [4]) to the existence of well-posed (Fredholm) boundary conditions for an elliptic operator in a bounded domain.

$^*$Research supported in part by RFBR grants Nos. 05-01-00982 and 06-01-00098 and DFG grant 436 RUS 113/849/0-1$^*$°K-theory and noncommutative geometry of stratified manifolds.$^*$

$^1$Similar invariants were studied in other situations, e.g., in [1] [2] [3].
The main result of this paper is a general formula expressing the Atiyah–Bott index in topological terms. We also compute the range of the index mapping, i.e., the $K$-group of operators with zero interior symbol.

This research was carried out during our stay at the Institute for Analysis, Hannover University (Germany). We are grateful to Professor E. Schrohe and other members of the university staff for their kind hospitality.

2 Atiyah–Bott index

First, we recall the key properties of pseudodifferential operators on stratified manifolds. For detailed exposition, e.g., see [5, 6, 7].

**Stratified manifolds.** Let $\mathcal{M}$ be a compact stratified manifold in the sense of [7]. Recall that $\mathcal{M}$ has a decreasing filtration

$$
\mathcal{M} = \mathcal{M}_0 \supset \mathcal{M}_1 \supset \mathcal{M}_2 \ldots \supset \mathcal{M}_N \supset \emptyset
$$

of length $N$ by closed subsets $\mathcal{M}_j$ such that the complement $\mathcal{M}_j \setminus \mathcal{M}_{j+1}$ (an open stratum) is homeomorphic to the interior $M^\circ_j$ of a compact manifold $M_j$ with corners (the blowup of $\mathcal{M}_j$). Let us denote the blowup of $\mathcal{M}$ by $\tilde{M}$. In addition, any $x \in \mathcal{M}_j \setminus \mathcal{M}_{j+1}$ has a neighborhood homeomorphic to the product

$$
U_x \times K_{\Omega_j},
$$

where $U_x$ is a neighborhood of $x$ in $M^\circ_j$ and

$$
K_{\Omega_j} = [0,1) \times \Omega_j / \{0\} \times \Omega_j
$$

is a cone whose base $\Omega_j$ is a stratified manifold with filtration of length $< N$. In particular, $\mathcal{M}_N$ is a smooth manifold.

**Pseudodifferential operators on stratified manifolds.** Let $\Psi(\mathcal{M})$ be the algebra of pseudodifferential operators of order zero on $\mathcal{M}$ acting in the space $L^2(\mathcal{M})$ of complex-valued functions. (For the definition of the algebra and of the measure defining the $L^2$-space, we refer the reader to [8] or [5, 6].) The Calkin algebra (the algebra of symbols) is denoted by $\Sigma(\mathcal{M})$.

The symbol $\sigma(D)$ of an operator $D$ on $\mathcal{M}$ is a collection

$$
\sigma(D) = (\sigma_0(D), \sigma_1(D), \ldots, \sigma_N(D))
$$

of symbols on the strata, where the symbol $\sigma_j(D)$ is defined on the cosphere bundle $S^*M_j$ of the blowup of the corresponding stratum. The symbol $\sigma_0(D)$ on the stratum $\mathcal{M} \setminus \mathcal{M}_1$ of maximal dimension is a scalar function called the interior symbol, and the remaining components of the symbol are operator-valued functions.
Definition of Atiyah–Bott index. Let
\[ \sigma_0 : \Psi(M) \longrightarrow C(S^*M) \]
be the mapping taking each operator to its interior symbol. This mapping is surjective. Consider the short exact sequence
\[ 0 \longrightarrow J \longrightarrow \Psi(M) \xrightarrow{\sigma_0} C(S^*M) \longrightarrow 0 \]
(1)
of \( C^* \)-algebras, where \( J \subset \Psi(M) \) is the ideal of operators with zero interior symbol.

The boundary mapping
\[ \delta : K_*(C(S^*M)) \longrightarrow K_{*+1}(J) \]
(2)
induced in \( K \)-theory by the exact sequence (1) is called the Atiyah–Bott index.

Let us explain why the mapping (2) is called an index. If \( M \) is a closed smooth manifold, then \( M = M \). (The blowup coincides with the original manifold.) Moreover, the kernel of \( \sigma_0 \) coincides with the ideal \( \mathcal{K} \) of compact operators, and the nontrivial boundary mapping
\[ \delta : K_1(C(S^*M)) \longrightarrow K_0(\mathcal{K}) \cong \mathbb{Z} \]
is given by the Fredholm index2

It turns out that the Atiyah–Bott index is the obstruction to making an operator with invertible interior symbol invertible by adding operators with zero interior symbol (cf. [3]).

**Theorem 2.1.** Let \( A \) be a matrix pseudodifferential operator with invertible interior symbol \( \sigma_0(A) \) on a stratified manifold \( M \). A necessary and sufficient condition that there exists an operator \( R \) with zero interior symbol such that \( A + R \) is invertible is that the Atiyah–Bott index \( \delta[\sigma_0(A)] \in K_0(J) \) is zero.

The proof is given in the Appendix.

### 3 Main theorem

**Analytic \( K \)-homology.** Let us recall several facts about analytic \( K \)-homology. (Detailed exposition and further references can be found in [9], Chapter 5.)

Let \( M \) be a compact stratified manifold. By \( D(M) \) we denote the algebra of local operators on \( L^2(M) \), i.e., operators compactly commuting with multiplications by

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2Indeed, an element of \( K_1(C(S^*M)) \) is determined by an invertible matrix with entries in \( C(S^*M) \). Consider the matrix as the symbol of some operator; then the mapping takes (the equivalence class of) the matrix to the Fredholm index of that operator.
continuous functions on $\mathcal{M}$. If $\mathcal{M}'$ is a closed subspace, then by $\mathcal{D}(\mathcal{M}, \mathcal{M}') \subset \mathcal{D}(\mathcal{M})$ we denote the ideal of \textit{locally compact operators}. By definition, these are operators whose composition with multiplications by continuous functions vanishing on $\mathcal{M}'$ is compact.

The $K$-homology groups of $\mathcal{M}$, $\mathcal{M}'$, and $\mathcal{M} \setminus \mathcal{M}'$ can be defined as

$$
K_*(\mathcal{M}) \simeq K_{*-1}\left(\mathcal{D}(\mathcal{M})/\mathcal{K}\right), \quad K_*(\mathcal{M}') \simeq K_{*-1}\left(\mathcal{D}(\mathcal{M}, \mathcal{M}')/\mathcal{K}\right), \quad (3)
$$

$$
K_*(\mathcal{M} \setminus \mathcal{M}') \simeq K_{*-1}\left(\mathcal{D}(\mathcal{M})/\mathcal{D}(\mathcal{M}, \mathcal{M}')\right). \quad (4)
$$

Moreover the $K$-homology sequence for the pair $\mathcal{M}' \subset \mathcal{M}$

$$
\ldots \to K_*(\mathcal{M}') \to K_*(\mathcal{M}) \to K_*(\mathcal{M} \setminus \mathcal{M}') \to K_{*-1}(\mathcal{M}') \to \ldots
$$

coincides with the $K$-theory sequence for the short exact sequence

$$
0 \to \mathcal{D}(\mathcal{M}, \mathcal{M}')/\mathcal{K} \to \mathcal{D}(\mathcal{M})/\mathcal{K} \to \mathcal{D}(\mathcal{M})/\mathcal{D}(\mathcal{M}, \mathcal{M}') \to 0
$$

of $C^*$-algebras.

\textbf{Main theorem.} Pseudodifferential operators on stratified manifolds are local (see [6]). On the other hand, operators in $\mathcal{J}$ are locally compact with respect to the subspace $\mathcal{M}_1 \subset \mathcal{M}$; i.e., their compositions with functions vanishing on $\mathcal{M}_1$ are compact. Thus, we have the commutative diagram

$$
\begin{array}{ccc}
0 & \to & J \\
i^*_{\mathcal{M}_1} & \downarrow & \downarrow \downarrow \downarrow \downarrow \ni^*_{\mathcal{M}_1} \\
0 & \to & \mathcal{D}(\mathcal{M}, \mathcal{M}_1) & \to & \mathcal{D}(\mathcal{M}) & \to & \mathcal{D}(\mathcal{M})/\mathcal{D}(\mathcal{M}, \mathcal{M}_1) & \to & 0.
\end{array} \quad (5)
$$

In view of the isomorphisms (3) and (4), the vertical mappings in (5) induce homomorphisms of $K$-groups into $K$-homology groups:

$$
i^*_{\mathcal{M}_1}: K_{*-1}(J) \to K_{*-1}\left(\mathcal{D}(\mathcal{M}, \mathcal{M}_1)/\mathcal{K}\right) \simeq K_*(\mathcal{M}_1),
$$

$$
i^*_{\mathcal{M}_1}: K_*(C(S^*M)) \to K_*\left(\mathcal{D}(\mathcal{M})/\mathcal{D}(\mathcal{M}, \mathcal{M}_1)\right) \simeq K_{*-1}(\mathcal{M} \setminus \mathcal{M}_1).
$$

\textbf{Theorem 3.1.} If $\mathcal{M}$ has no closed smooth components, then the diagram

$$
\begin{array}{ccc}
K_*(C(S^*M)) & \delta & K_{*-1}(J) \\
i^*_{\mathcal{M}_1} & & \ni^*_{\mathcal{M}_1} \\
K_{*-1}(\mathcal{M} \setminus \mathcal{M}_1) & \partial & \tilde{K}_*(\mathcal{M}_1)
\end{array}
$$

commutes. Moreover, $i^*_{\mathcal{M}_1}$ is an isomorphism. Here
\[ \tilde{K}_*(\mathcal{M}_1) \simeq \ker \{ K_*(\mathcal{M}_1) \to K_*(pt) \} \] is the reduced \( K \)-homology group.

\( \partial \) is the boundary mapping in exact \( K \)-homology sequence of the pair \( \mathcal{M}_1 \subset \mathcal{M} \).

**Proof.** 1. Consider the commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & J & \to & \Psi & \to & C(S^*M) & \to & 0 \\
& & j & \downarrow & & \downarrow & & & \\
0 & \to & J/K & \to & \Sigma & \to & C(S^*M) & \to & 0,
\end{array}
\]

where \( \Psi \) is the algebra of pseudodifferential operators, \( \Sigma = \Psi/K \) is the algebra of symbols, and \( j \) is the natural projection. The boundary mappings corresponding to the upper and lower rows of the diagram are compatible. Since \( j \) induces a monomorphism in \( K \)-theory (see the lemma below), it suffices to compute the boundary mapping corresponding to the lower exact sequence.

**Lemma 3.2.** The natural projection \( j: J \to J/K \) induces isomorphisms

\[ K_0(J) \simeq K_0(J/K), \quad K_1(J) = \ker (\text{ind}: K_1(J/K) \to \mathbb{Z}). \]

**Proof.** The boundary mapping in the \( K \)-theory exact sequence of the pair \( J \to J/K \) is the index mapping

\[ K_1(J/K) \to K_0(K) = \mathbb{Z}. \]

It is surjective. The exact sequence of pair \( K \subset J \) gives the desired isomorphism. \( \square \)

2. Consider the commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & J/K & \to & \Sigma & \to & C(S^*M) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & D(\mathcal{M},\mathcal{M}_1)/\mathcal{K} & \to & D(\mathcal{M})/\mathcal{K} & \to & D(\mathcal{M})/D(\mathcal{M},\mathcal{M}_1) & \to & 0.
\end{array}
\]

As we mentioned earlier, the sequence of \( K \)-groups induced by the lower row in (6) is isomorphic to the \( K \)-homology exact sequence of the pair \( \mathcal{M}_1 \subset \mathcal{M} \).

Since \( \mathcal{M} \) is assumed to have no closed smooth components, there exists a nonsingular vector field (a section \( M \to S^*M \) of the cosphere bundle) on the blowup \( M \). It follows that \( K_*(\Sigma) \) and \( K_*(C(S^*M)) \) have direct summands \( K_*(C(M)) \), which are mapped isomorphically onto each other by \( \sigma_0: \Sigma \to C(S^*M) \). Hence (6) gives the diagram

\[
\begin{array}{cccccccc}
K_*(J/K) & \to & K_*(\Sigma)/K_*(C(M)) & \to & K_*(C(S^*M))/K_*(C(M)) & \to & K_{*+1}(J/K) & \to \\
& i_*^{\mathcal{M}_1} \downarrow & i_*^{\mathcal{M}} \downarrow & i_*^{M^\circ} \downarrow & i_*^{\mathcal{M}_1} \downarrow & \\
K_{*+1}(\mathcal{M}_1) & \to & K_{*+1}(\mathcal{M}) & \to & K_{*+1}(\mathcal{M} \setminus \mathcal{M}_1) & \overset{\partial}{\to} & K_*(\mathcal{M}_1) & \to \\
\end{array}
\]
of $K$-groups with exact rows.

We proved in [7] that $i^M_*$ and $i^{M^o}_*$ are isomorphisms. Hence $i^{M_1}_*$ is an isomorphism as well by the five lemma. Now the commutative rightmost square in (7), together with Lemma 3.2 gives the desired commutative diagram

$$
\begin{array}{ccc}
K_*(C(S^* M)) & \xrightarrow{\delta} & K_{*+1}(J) \\
\downarrow i^M_0 & & \downarrow i^{M_1}_* \\
K_{*+1}(\mathcal{M} \setminus \mathcal{M}_1) & \xrightarrow{\partial} & \tilde{K}_*(\mathcal{M}_1)
\end{array}
$$

where $i^{M_1}_*$ is an isomorphism.

The proof of Theorem 3.1 is complete. \qed

**Cohomological index formula.** To be definite, we write out a (co)homological formula for an element $[a] \in K_1(C(S^* M))$ represented by an invertible symbol $a$ on $S^* M$. Consider the $K$-homology element

$$
i^{M_1}_* \delta [a] \in K_1(\mathcal{M}_1).
$$

(8)

The Chern character of this element is a rational homology class on $\mathcal{M}_1$. Let us evaluate the pairing of this class with an arbitrary cohomology class on $\mathcal{M}_1$.

To this end, let $Y \subset M$ be a compact smooth manifold of dimension $\dim M - 1$, homeomorphic to the boundary $\partial M = M \setminus M^o$. (The smooth structure is obtained by smoothing the corners.) Next, let

$$
\pi : Y \to \mathcal{M}_1
$$

be the projection onto the singularity set $\mathcal{M}_1$.

**Corollary 3.3.** For any $[a] \in K_1(C(S^* M))$ and $x \in H^{odd}(\mathcal{M}_1)$, one has

$$
\langle \ch(i_*^{M_1} \delta [a]), x \rangle = \langle \ch [a]_{|Y} \cdot \Td(T^* Y \otimes \mathbb{C}) \pi^* x, [T^* Y \times \mathbb{R}] \rangle,
$$

(9)

where $\ch(i_*^{M_1} \delta [a]) \in H_{odd}(\mathcal{M}_1, \mathbb{Q})$ is the homological Chern character and $[a]_{|Y} \in K_0^c(T^* Y \times \mathbb{R})$ is the restriction of interior symbol $a$ to $Y$.

**Proof.** Since the Chern character is a rational isomorphism of $K$-theory and homology, it suffices to evaluate the pairing with an element $x$ of the form $x = \ch y$, where $y \in K_1(\mathcal{M}_1)$.

By Theorem 3.1

$$
\langle \ch(i_*^{M_1} \delta [a]), \ch y \rangle = \langle i_*^{M_1} \delta [a], y \rangle = \langle \partial i_*^{M_0} [a], y \rangle.
$$
It follows from the properties of the boundary mapping $\partial$ in $K$-homology that
\[
\langle \partial \cdot i_*^{M^0} [a], y \rangle = \langle \pi_* \cdot \partial_0 \cdot i_*^{M^0} [a], y \rangle,
\]
where $\partial_0 : K_0(M \setminus M_1) \to K_1(Y)$ is the boundary mapping for the pair $\partial M \subset M$.

We now transfer the computation of the pairing to $Y$:
\[
\langle \pi_* \partial_0 i_*^{M^0} [a], y \rangle = \langle \partial_0 i_*^{M^0} [a], \pi_* y \rangle.
\]
By applying the Atiyah–Singer formula on $Y$, we obtain
\[
\langle \partial_0 i_*^{M^0} [a], \pi_* y \rangle = \langle \text{ch} [a]_{|Y}, \text{Td}(T^*Y \otimes \mathbb{C}) \text{ch} \pi_* y, [T^*Y \times \mathbb{R}] \rangle.
\]
The right-hand side coincides with the desired cohomological expression (9), since $x = \text{ch} y$.

\section{Examples}

In this section, we compute the Atiyah–Bott index for geometric operators on stratified manifolds with stratification of length one. Such manifolds are called \textit{manifolds with edges}. Geometrically, a manifold with edges is obtained as follows. Take a smooth manifold $M$ whose boundary $\partial M$ is fibered over a smooth base $X$, $\pi : \partial M \to X$. Then identify the points in each fiber of $\pi$. What is obtained is a manifold $\mathcal{M}$ with edge $X$. The blowup of $\mathcal{M}$ is just $M$, and the manifold $Y$ in Corollary \ref{corollary} is diffeomorphic to the boundary $\partial M$.

The Atiyah–Bott index is zero for the Beltrami–Laplace and Euler operators, since the restrictions of their principal symbols to $Y$ give rise to zero elements in $K_0^c(T^*Y \times \mathbb{R})$.

Let us (rationally) compute the Atiyah–Bott index of the Dirac operator.

Suppose that $M$, $X$, and $\pi$ are equipped with spin structures. Next, assume that the induced spin structure on the total space $\partial M$ of the bundle $\pi$ is compatible with the spin structure on $\partial M$. Let
\[
\mathcal{D} : S_+(M^0) \longrightarrow S_-(M^0)
\]
be the Dirac operator on $M^0$. (We assume that the dimension of $M$ is even.)

\begin{proposition}
For the Dirac operator $\mathcal{D}$ on a manifold $\mathcal{M}$ with edge $X$, the homology class
\[
\text{ch}(i_*^{X} \delta[\sigma_0(\mathcal{D})]) \in H_*(X) \otimes \mathbb{Q}
\]
is Poincaré dual to the cohomology class
\[
A(X) \cdot \pi_*(A(\Omega)),
\]
\end{proposition}
where $A(X)$ and $A(\Omega)$ are the $A$-classes of the base and of the fibers of $\pi$, respectively, and $\pi_*: H^*(Y) \to H^{*-\dim \Omega}(X)$ stands for integration over the fibers.

Proof. Choosing a connection in $\pi$, we obtain a decomposition

$$T^*M|_Y \simeq \pi^*T^*X \oplus (T^*\Omega \oplus \mathbb{R})$$

of the restriction of the cotangent bundle to $Y$ into horizontal and vertical components. With regard to this decomposition, the symbol of the Dirac operator over $Y$ is the tensor product of the pull-back of the symbol of the Dirac operator on the base by the family of Dirac operators in the fibers. This gives a factorization

$$[\sigma(D)|_Y] = (\pi^*[\sigma(D_X)])[\sigma(D_{\Omega \times \mathbb{R}})],$$

of the corresponding element in $K$-theory, where $[\sigma_0(D_X)] \in K_c^{\dim X}(T^*X)$ and $[\sigma(D_{\Omega \times \mathbb{R}})] \in K_c^{\dim \Omega + 1}(T^*\Omega \oplus \mathbb{R})$.

Let $x \in H^{odd}(X)$ be a cohomology class. Let us compute the pairing

$$\left< \text{ch}(i^X\delta[\sigma_0(D)]), x \right>.$$  

By Corollary 3.3, this number is equal to

$$\left< \text{ch}[\sigma_0(D)]_Y \text{Td}(T^*Y \otimes \mathbb{C})\pi^*x, [T^*Y \times \mathbb{R}] \right>.$$  

Using the factorization of $[\sigma_0(D)]|_Y$, we see that this expression is equal to

$$\left< \text{ch}[\sigma(D_X)] \text{ch}[\sigma(D_{\Omega \times \mathbb{R}})] \text{Td}(T^*X \otimes \mathbb{C}) \text{Td}(T^*\Omega \otimes \mathbb{C})\pi^*x, [T^*Y \times \mathbb{R}] \right> = \left< \text{ch}[\sigma(D_X)]\pi_*(\text{ch}[\sigma(D_{\Omega \times \mathbb{R}})] \text{Td}(T^*\Omega \otimes \mathbb{C})) \text{Td}(T^*X \otimes \mathbb{C})x, [T^*X] \right> = \left< A(X)\pi_*(A(\Omega))x, [X] \right>.$$  

(At the last step, we have used the standard transition from cohomology classes on the cotangent bundle to cohomology classes on the manifolds themselves by using the Thom isomorphism.)

Thus,

$$\left< \text{ch}(i^X\delta[\sigma_0(D)]), x \right> = \left< A(X)\pi_*(A(\Omega))x, [X] \right>$$

for all $x \in H^{odd}(X)$. It follows that the classes $\text{ch}(i^X\delta[\sigma_0(D)])$ and $A(X)\pi_*(A(\Omega))$ are Poincaré dual.

A similar computation of the Atiyah–Bott index can be carried out for the signature operator. In this case, one should replace the $A$-classes by the $L$-classes.
Appendix. Proof of Theorem 2.1

Auxiliary lemmas. The following two lemmas are standard.

Lemma 4.2. Let $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ be a continuous family of Fredholm operators acting in spaces of sections of infinite-dimensional Hilbert bundles over a locally compact base $X$. Suppose that $\mathcal{A}$ is invertible at infinity. For the existence of a continuous family of finite rank operators $R(x), x \in X,$ vanishing at infinity such that the family $\mathcal{A} + R$ is everywhere invertible, it is necessary and sufficient that

$$\text{ind} \ A = 0 \in K_0^c(X).$$

Lemma 4.3. Let $\mathcal{H}$ be an infinite-dimensional Hilbert bundle over a locally compact base $X$. Then each element of $K_1(C_0(X, \mathcal{K}(\mathcal{H})))$ has a representative that is an invertible element of the unital algebra $C_0(X, \mathcal{K}(\mathcal{H}))^+.$

Proof of Theorem 2.1. If there exists an $R$ with the desired properties, then $\mathcal{A} + R$ is invertible, $[\mathcal{A} + R] \in K_1(\Psi(M))$, and $(\sigma_0)_* [\mathcal{A} + R] = [\sigma_0(\mathcal{A})].$ We conclude that $\delta[\sigma_0(\mathcal{A})] = \delta((\sigma_0)_* [\mathcal{A} + R]) = 0$, since the sequence

$$\ldots \to K_1(\Psi) \xrightarrow{(\sigma_0)_*} K_1(C(S^*M)) \xrightarrow{\delta} K_0(J) \to \ldots$$

is exact.

Let us prove the converse. To this end, let us recall several properties of pseudodifferential operators on stratified manifolds (see [5, 6]).

The algebra $\Psi = \Psi(M)$ is solvable of length $(2N + 1)$ (e.g., see [5]), where $N$ is the length of the stratification of $M,$ with composition series

$$\Psi = \Psi_0 \supset \Psi_1 \supset \Psi_2 \supset \Psi_3 \supset \ldots \supset \Psi_{2N} \supset \Psi_{2N+1} = \mathcal{K} \supset \{0\}. \quad (10)$$

To describe the ideals $\Psi_j$ in more detail, recall that the symbol of an operator $D$ is the collection

$$\sigma(D) = (\sigma_0(D), \sigma_1(D), \ldots, \sigma_N(D))$$

of symbols on the strata. Here $\sigma_j(D)$ is a family of operators in $L^2(K_{\Omega_j})$, where $K_{\Omega_j}$ is cone with base $\Omega_j,$ parametrized by cosphere bundle $S^*M_j$; the conormal symbol (corresponding to the cone tips) of this family does not depend on the covariables in $S^*M_j$ and is a family, parametrized by $\mathbb{R} \times M_j$, of pseudodifferential operators on the base $\Omega_j$ of the cone.

Now the ideals in (10) can be described as follows. The ideal

$$\Psi_{2j+1} \subset \Psi_{2j}, \quad j \geq 0,$$
consists of operators with zero symbol $\sigma_j$ on the open stratum $\mathcal{M}_j^0$, and the ideal

$$\Psi_{2j} \subset \Psi_{2j-1}, \quad j \geq 1,$$

consists of operators with zero conormal symbol $\sigma_c(\sigma_j)$ of $\sigma_j$ on $\mathcal{M}_j^0$. Moreover, we have the isomorphisms

$$\Psi_{2j}/\Psi_{2j+1} \simeq C(S^*M_j, KL^2(K\Omega_j))$$

(the mapping is defined by the symbol $\sigma_j$) and

$$\Psi_{2j-1}/\Psi_{2j} \simeq C_0(\mathbb{R} \times M_j, KL^2(\Omega_j))$$

(the mapping is defined by the conormal symbol $\sigma_c(\sigma_j)$).

To construct an invertible perturbation of an operator with trivial Atiyah–Bott index, we use Proposition 4.4 below for $j = 1, 2, \ldots, 2N + 1$. As a result, we shall obtain an invertible operator with interior symbol equal to that of the original operator. This will end the proof of Theorem 2.1.

Let $A: L^2(\mathcal{M}, \mathbb{C}^n) \to L^2(\mathcal{M}, \mathbb{C}^n)$ be a matrix pseudodifferential operator. We say that it is invertible modulo the ideal $\Psi_j$ if there exists a matrix operator $B$ such that the compositions $AB$ and $BA$ are equal to identity modulo operators with matrix entries in $\Psi_j$.

**Proposition 4.4.** Let $A$ be a matrix pseudodifferential operator on $\mathcal{M}$ such that for some $j$ it is invertible modulo ideal $\Psi_j$ and

$$\delta_j[A] = 0,$$

where $\delta_j: K_1(\Psi/\Psi_j) \to K_0(\Psi_j)$ is the boundary map in $K$-theory for the pair $\Psi \to \Psi/\Psi_j$. Then there exists a pseudodifferential operator $\tilde{A}$ on $\mathcal{M}$ that is invertible modulo the ideal $\Psi_{j+1}$, is equal to $A$ modulo $\Psi_j$, and has zero index $\delta_{j+1}[\tilde{A}] = 0$ in $K_0(\Psi_{j+1})$.

**Proof.** Let $s_{2j} = \sigma_j$ and $s_{2j-1} = \sigma_c(\sigma_j)$. (In this notation, the ideal $\Psi_{k+1}$ in $\Psi_k$ is determined by the condition $s_k = 0$.)

0. Consider the diagram

\[
\begin{array}{ccccccc}
K_1(\Psi_j/\Psi_{j+1}) & \longrightarrow & K_1(\Psi/\Psi_{j+1}) & \longrightarrow & K_1(\Psi/\Psi_j) & \longrightarrow & K_0(\Psi_j/\Psi_{j+1}) \\
\downarrow_{\delta_{j+1}} & & \downarrow_{\delta_j} & & \downarrow_{\delta_j} & & \downarrow_{\delta_j} \\
K_1(\Psi_j/\Psi_{j+1}) & \longrightarrow & K_0(\Psi_{j+1}) & \longrightarrow & K_0(\Psi_j) & \longrightarrow & K_0(\Psi_j/\Psi_{j+1})
\end{array}
\]

The middle columns are parts of the exact sequences for the pairs $\Psi \to \Psi/\Psi_{j+1}$ and $\Psi \to \Psi/\Psi_j$. The rows are the exact sequences of the pairs $\Psi/\Psi_{j+1} \to \Psi/\Psi_j$ and
Ψ_j → Ψ_j/Ψ_{j+1}. Since the boundary mapping is natural, it follows that the diagram commutes.

1. Suppose that δ_j[A] = 0. Then δ''[A] = 0. Hence the element
   \[ \text{ind } s_{j+1} = δ''[A] ∈ K_0(Ψ_j/Ψ_{j+1}) \]
is zero. (Indeed, s_{j+1} is a Fredholm family invertible at infinity, since s_j is invertible by assumption.) Thus, by Lemma 4.2, there exists a pseudodifferential operator \( B = A \mod Ψ_j \) on \( M \) invertible modulo Ψ_{j+1}.

2. Since \( γδ_{j+1}[B] = δ_j[A] = 0 \), we have \( δ_{j+1}[B] = δ'(z) \) for some \( z ∈ K_1(Ψ_j/Ψ_{j+1}) \).

3. For \( A \) we take the composition
   \[ \tilde{A} = BZ^{-1}, \]
where \( Z ∈ Ψ_j^+ \) is a pseudodifferential operator invertible modulo Ψ_{j+1} such that \([Z] = z\). (The existence of \( Z \) is guaranteed by Lemma 4.3)

   It is clear that \( \tilde{A} \) has the desired properties: it is invertible modulo Ψ_{j+1}, is equal to \( A \) modulo Ψ_j, and has zero index \( δ_{j+1}[\tilde{A}] = 0 \), since

   \[ δ_{j+1}[\tilde{A}] = δ_{j+1}[B] - δ_{j+1}[Z] = δ_{j+1}[B] - δ'(z) = 0 \]
by construction.

   The proof of Proposition 4.4 is complete.

\[ \square \]

References

[1] H. Upmeier. Toeplitz operators and index theory in several complex variables. In Operator Theory: Operator Algebras and Applications, Part 1 (Durham, NH, 1988), volume 51 of Proc. Sympos. Pure Math., 1990, pages 585–598, Providence, RI. AMS.

[2] B. Monthubert. Pseudodifferential calculus on manifolds with corners and groupoids. Proc. Amer. Math. Soc., 127 No. 10, 1999, 2871–2881.

[3] B. Monthubert and V. Nistor. A topological index theorem for manifolds with corners. arxiv: math.KT/0507601, 2005.

[4] M. F. Atiyah and R. Bott. The index problem for manifolds with boundary. In Bombay Colloquium on Differential Analysis, 1964, pages 175–186, Oxford. Oxford University Press.

[5] V. Nazaikinskii, A. Savin, and B. Sternin. Pseudodifferential operators on stratified manifolds I. Differential Equations, 43, No. 4, 2007, 536–549.
[6] V. Nazaikinskii, A. Savin, and B. Sternin. Pseudodifferential operators on stratified manifolds II. *Differential Equations*, **43**, No. 5, 2007, 704–716.

[7] V. Nazaikinskii, A. Savin, and B. Sternin. On the homotopy classification of elliptic operators on stratified manifolds. *Izvestiya. Mathematics*, **71**, 2007. (in print). Preliminary version: [arXiv:math/0608332](https://arxiv.org/abs/math/0608332).

[8] B. A. Plamenevsky and V. N. Senichkin. Representations of $C^*$-algebras of pseudodifferential operators on piecewise-smooth manifolds. *Algebra i Analiz*, **13**, No. 6, 2001, 124–174.

[9] N. Higson and J. Roe. *Analytic K-homology*. Oxford University Press, Oxford, 2000.