Charge Superselection Sectors for QCD on the Lattice

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Abstract

We study quantum chromodynamics (QCD) on a finite lattice $\Lambda$ in the Hamiltonian approach. First, we present the field algebra $\mathfrak{A}_\Lambda$ as comprising a gluonic part, with basic building block being the crossed product $C^*$-algebra $C(G) \otimes \alpha G$, and a fermionic (CAR-algebra) part generated by the quark fields. By classical arguments, $\mathfrak{A}_\Lambda$ has a unique (up to unitary equivalence) irreducible representation. Next, the algebra $O^i_\Lambda$ of internal observables is defined as the algebra of gauge invariant fields, satisfying the Gauss law. In order to take into account correlations of field degrees of freedom inside $\Lambda$ with the “rest of the world”, we have to extend $O^i_\Lambda$ by tensorizing with the algebra of gauge invariant operators at infinity. This way we construct the full observable algebra $\mathfrak{D}_\Lambda$. It is proved that its irreducible representations are labelled by $\mathbb{Z}_3$-valued boundary flux distributions. Then, it is shown that there exist unitary operators (charge carrying fields), which intertwine between irreducible sectors leading to a classification of irreducible representations in terms of the $\mathbb{Z}_3$-valued global boundary flux. By the global Gauss law, these 3 inequivalent charge superselection sectors can be labeled in terms of the global colour charge (triality) carried by quark fields. Finally, $\mathfrak{D}_\Lambda$ is discussed in terms of generators and relations.
1 Introduction

In a series of papers, we have started to analyze the non-perturbative structure of gauge theories, with the final aim being the formulation and investigation of gauge models purely in terms of observables. To start with one should clarify basic structures like that of the field algebra, the observable algebra and the superselection structure of the Hilbert space of physical states. It is well-known that the standard Doplicher-Haag-Roberts theory [1], [2] for models, which do not contain massless particles, does not apply here. Nonetheless, there are interesting partial results within the framework of general quantum field theory both for quantum electrodynamics (QED) and for non-abelian models, see [3], [4], [5] and [6].

To approach the problem in a rigorous way, we put the system on a finite (regular cubic) lattice and formulate the model within the Hamiltonian approach. For basic notions concerning lattice gauge theories (including fermions) we refer to [7] and references therein. Within the finite lattice context, we have analyzed the structure of the observable algebra both for spinorial and scalar QED and we have shown that the physical Hilbert space decomposes into a direct sum of superselection sectors labelled by the total electric charge, see [8], [9] and [10]. Finally, of course, one wants to construct the continuum limit. In full generality, this is an extremely complicated problem of constructive field theory. However, the results obtained until now suggest that there is some hope, to control the thermodynamic limit, see [8] for a heuristic discussion. We also mention that for simple toy models, these problems can be solved, see [11].

In [12] we have started to investigate quantum chromodynamics (QCD) within the above framework. In particular, we have analyzed the global Gauss law and the notion of global colour charge (triality). Comparing with QED, the notion of global charge in QCD is much more complicated, according to the fact that the local Gauss law is neither built from gauge invariant operators nor is it linear. We have shown that one can extract from the local Gauss equation of QCD a gauge invariant, additive law for operators with eigenvalues in \( \mathbb{Z}_3 \) (the center of \( SU(3) \)). This implies – as in QED – a gauge invariant conservation law: The global \( \mathbb{Z}_3 \)-valued colour charge is equal to a \( \mathbb{Z}_3 \)-valued gauge invariant quantity obtained from the color electric flux at infinity.

We stress that within lattice gauge theories, the notion of colour charge is already implicitly contained in a paper by Kogut and Susskind, see [13]. In particular, Mack [14] used it to propose a certain (heuristic) scheme of colour screening and quark confinement, based upon a dynamical Higgs mechanism with Higgs fields built from gluons. For similar ideas we also refer to papers by ‘t Hooft, see [15] and references therein. This concept was also used in a paper by Borgs and Seiler [16], where the confinement problem for Yang-Mills theories with static quark sources at nonzero temperature was discussed.

In the present paper, we continue to investigate lattice QCD as initiated in [12]. Our starting point is the notion of the algebra \( \mathfrak{A}_A \) of field operators, see Section 2. After discussing local gauge transformations, the Gauss law and boundary data, we show
unique representations of $A$. In Section 3, the algebra of internal observables $\mathcal{O}_\Lambda$ is defined as the algebra of gauge invariant fields, satisfying the Gauss law. We show that its irreducible representations are labeled by distributions of colour electric fluxes running through the boundary to infinity. It is remarkable that for the very classification of irreducible representations the abstract characterization of $\mathcal{O}_\Lambda$ as the subalgebra, invariant under the group of local gauge transformations, factorized with respect to the ideal generated by the Gauss law, is sufficient. This is due to the fact that here we work within the compact formulation, which implies that we “stay within” the representation space of the field algebra. This remark does not apply to QED in the non-compact formulation.

In chapter 4, we explain that in order to implement a strategy to construct the thermodynamic limit via an inductive (resp. projective) limit procedure for observable algebras (resp. state spaces), we have to extend both the algebra of internal observables $\mathcal{O}_\Lambda$ (by adding certain “external observables”) and the Hilbert space $H_\Lambda$ (by tensorizing with the Hilbert space of tensors at infinity). These external observables enable us to take into account the correlations between the field degrees of freedom contained in $\Lambda$ and the “rest of the world”. The algebra obtained this way is called full observable algebra and is denoted by $\mathcal{O}_\Lambda$. Within this context, we show that different Hilbert space sectors, corresponding to different boundary flux distributions, carry equivalent representations of $\mathcal{O}_\Lambda$ if and only if their global $\mathbb{Z}_3$-valued flux is the same, with intertwiners given by charge carrying fields. This reduces the irreducible representations of $\mathcal{O}_\Lambda$ to three inequivalent sectors, labeled by the $\mathbb{Z}_3$-valued global colour electric flux. By the global Gauss law, this flux coincides with the global colour charge (triality) carried by the quark fields.

Finally, in Section 5, $\mathcal{O}_\Lambda$ is discussed in terms of generators and relations. We start with presenting a set of genuine invariants generating $\mathcal{O}_\Lambda$, which is, however, highly redundant. In the remainder of this section, we use some gauge fixing methods to reduce this set. In this context, a couple of delicate questions arises – all in some sense related to the Gribov problem and to the fact that the underlying classical configuration space has a complicated stratified structure with respect to the gauge group action. A more complete treatment of $\mathcal{O}_\Lambda$ as an algebra presented by generators and relations will be given in two separate papers, see [17] and [18].

Finally, we mention that we have made some attempts to formulate continuum gauge models in terms of observables within the functional integral approach, see [19], [20] and further references therein.

## 2 The Field Algebra

### 2.1 Basic definitions

We consider QCD in the Hamiltonian framework on a finite regular cubic lattice $\Lambda \subset \mathbb{Z}^3$, with $\mathbb{Z}^3$ being the infinite regular lattice in 3 dimensions. We denote the lattice boundary
by $\partial \Lambda$ and the set of oriented, $i$-dimensional elements of $\Lambda$, respectively $\partial \Lambda$, by $\Lambda^i$, respectively $\partial \Lambda^i$, where $i = 0, 1, 2, 3$. Such elements are (in increasing order of $i$) called sites, links, plaquettes and cubes. Moreover, we denote the set of external links connecting boundary sites of $\Lambda$ with “the rest of the world” by $\Lambda^1_\infty$ and the set of endpoints of external links at infinity by $\Lambda^0_\infty$. For the purposes of this paper, we may assume that for each boundary site there is exactly one link with infinity. Then, external links are labeled by boundary sites and we can denote them by $(x, \infty)$ with $x \in \partial \Lambda^0$. The set of non-oriented $i$-dimensional elements will be denoted by $|\Lambda|^i$. If, for instance, $(x, y) \in \Lambda^1$ is an oriented link, then by $|(x, y)| \in |\Lambda|^1$ we mean the corresponding non-oriented link. The same notation applies to $\partial \Lambda^1$ and $\Lambda^1_\infty$.

The basic fields of lattice QCD are quarks living at lattice sites and gluons living on links, including links connecting the lattice under consideration with “infinity”. The field algebra is thus, by definition, the $C^*$-tensor product of fermionic and bosonic algebras:

$$\mathfrak{A}_\Lambda := \mathfrak{F}_\Lambda \otimes \mathfrak{B}_\Lambda,$$

with

$$\mathfrak{F}_\Lambda := \bigotimes_{x \in \Lambda^0} \mathfrak{F}_x \quad (2.2)$$

and

$$\mathfrak{B}_\Lambda := \mathfrak{B}^i_\Lambda \otimes \mathfrak{B}^b_\Lambda = \bigotimes_{|(x, y)| \in |\Lambda|^1} \mathfrak{B}_{|(x, y)|} \otimes \bigotimes_{x \in \partial \Lambda^0} \mathfrak{B}_{|(x, \infty)|}. \quad (2.3)$$

Here, $\mathfrak{B}^i_\Lambda$ and $\mathfrak{B}^b_\Lambda$ are the internal and boundary bosonic algebras respectively. We impose locality of the lattice quantum fields by postulating that the algebras corresponding to different elements of $\Lambda$ (anti-)commute with each other.

**Remark:**

The bosonic boundary data represent nontrivial colour electric flux through the boundary, which – as will be seen later – allows for non-trivial colour charge. As will be shown, nontrivial boundary flux is necessary for taking into account correlations of field degrees of freedom inside $\Lambda$ with the “rest of the world”. Even if, for some reasons, the global charge of the Universe vanishes, there is no reason to assume that an arbitrary finite part $\Lambda$ is also neutral.

The fermionic field algebra $\mathfrak{F}_x$ associated with a lattice site $x$ is the algebra of canonical anticommutation relations (CAR) of quarks at $x$. The quark field generators are denoted by

$$\Lambda^0 \ni x \rightarrow \psi^a A(x) \in \mathfrak{F}_x,$$

where $a$ stands for bispinorial and (possibly) flavour degrees of freedom and $A = 1, 2, 3$ is the colour index corresponding to the fundamental representation of the gauge group $G = SU(3)$. (In what follows, writing $G$ we have in mind $SU(3)$, but essentially our discussion can be extended to arbitrary compact groups and their representations.) The
conjugate quark field is denoted by $\psi^*_{aA}(x)$, where we raise and lower indices by the help of the canonical hermitian metric tensor $g_{AB}$ in $\mathbb{C}^3$ and the canonical skew-symmetric structure $\epsilon_{ab}$ in the spinor space. The only nontrivial canonical anti-commutation relations for generators of $\mathfrak{g}_x$ read:

\[
[\psi^*_{aA}(x), \psi^b_{bB}(x)]_+ = \delta^B_A \delta^a_b .
\] (2.5)

The bosonic field algebra $\mathfrak{B}_{(x,y)}$ associated with the non-oriented link $|(x,y)|$, (where $y$ also stands for $\infty$), is given in terms of its isomorphic copies $\mathfrak{B}_{(x,y)}$ and $\mathfrak{B}_{(y,x)}$, corresponding to the two orientations of the link $(x,y)$. The algebra $\mathfrak{B}_{(x,y)}$ is generated by matrix elements of the gluonic gauge potential on the link $(x,y)$,

\[
\Lambda^1 \ni (x,y) \rightarrow U^A B(x,y) \in \mathfrak{C}_{(x,y)} ,
\] (2.6)

with $\mathfrak{C}_{(x,y)} \cong C(G)$ being the commutative $C^*$-algebra of continuous functions on $G$ and $A, B = 1, 2, 3$ denoting colour indices, and by colour electric fields, spanning the Lie algebra $\mathfrak{g}_{(x,y)} \cong su(3)$. Choosing an orthonormal basis $\{t_i\}$, $i = 1, \ldots, 8$, of $su(3)$ we denote by $\{E_i(x,y)\}$ the corresponding basis of $\mathfrak{g}_{(x,y)}$:

\[
\Lambda^1 \ni (x,y) \rightarrow E_i(x,y) := t_i \in \mathfrak{g}_{(x,y)} .
\] (2.7)

These elements generate, in the sense of Woronowicz [21], the $C^*$-algebra $\mathfrak{P}_{(x,y)} \cong C^*(G)$, see [22, 23] for a definition of the group $C^*$-algebra $C^*(G)$.

Observe that $G$ acts on $C(G)$ naturally by the left regular representation,

\[
L_g(u)(g') := u(g^{-1}g') , u \in C(G) .
\] (2.8)

Differentiating this relation, we get an action of $e \in su(3)$ on $u \in C^\infty(G)$ by the corresponding right invariant vector field $e^R$. Thus, we have a natural commutator between generators of $\mathfrak{P}_{(x,y)}$ and smooth elements of $\mathfrak{C}_{(x,y)}$:

\[
i [e, u] := e^R(u) .
\] (2.9)

To summarize, we have a $C^*$-dynamical system $\left(\mathfrak{C}_{(x,y)}, G, \alpha\right)$, with automorphism $\alpha$ given by the left action [22, 23]. The field algebra $\mathfrak{B}_{(x,y)}$ is, by definition, the corresponding crossed product $C^*$-algebra,

\[
\mathfrak{B}_{(x,y)} := \mathfrak{C}_{(x,y)} \otimes_\alpha G ,
\] (2.10)

see [22, 24] for these notions.

Remarks:

1. The crossed product algebra $C(G) \otimes_\alpha G$ is a $C^*$-algebra without unit, defined as the completion of $L^1(G, C(G))$ in the sup-norm taken over all Hilbert space representations of $L^1(G, C(G))$. It can be viewed as a skew tensor product of $C(G)$ with
$C^*(G)$ in the following sense: For each $u \in C(G)$ and $f \in L^1(G)$ denote by $u \otimes f$ the element of $L^1(G, C(G))$ given by $(u \otimes f)(g) := uf(g)$. Then, the linear span of such elements, with $f$ taken from a dense subset of $L^1(G)$, is dense in $L^1(G, C(G))$. Note that $C^*(G) \subset C(G) \otimes_{\alpha} G$, with the canonical injection given by

$$f \to 1 \otimes f.$$ 

As already noted, the group algebra $C^*(G)$ is a $C^*$-algebra generated by unbounded elements in the sense of Woronowicz, see [21]. Consequently, $C(G) \otimes_{\alpha} G$ is of this type, too. We stress that the $su(3)$-generators $e$ of $C^*(G)$ do not belong to the algebra, but are only affiliated in the $C^*$-sense.

2. Formula (2.9) is a natural generalization of the Heisenberg commutation relation $[p,q] = -i$, describing the abelian case $G = \mathbb{R}^1$. It corresponds to “canonical quantization” over the phase space

$$T^*(G) \cong \mathfrak{g}^* \times G,$$

with $\mathfrak{g} = su(3)$ being the Lie algebra of $G$ and $\mathfrak{g}^*$ being the dual space. Quantization applies to functions on $\mathfrak{g}^* \times G$, hence, in particular, to elements of $\mathfrak{g}$. Thus, from the purely algebraic point of view, one would then define $\mathcal{P}_{(x,y)}$ as the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$, see [12], yielding for the bosonic field algebra $\mathcal{B}_{(x,y)}$ the following crossed product structure of Hopf algebras:

$$C^\infty(G) \otimes_{\alpha} \mathcal{U}(\mathfrak{g}).$$

This is an example of a Heisenberg double of Hopf algebras, see of [25, 26]. This choice has, however, substantial drawbacks related to the fact that the operators assigned to the Lie algebra elements $e$ are necessarily unbounded. This is why we choose the functional-analytic framework above, where all observables are bounded operators in a Hilbert space. We call the algebra (2.10) the algebra of generalized canonical commutation relations (CCR) over the group $G$.

3. Within the above framework, one can prove a generalization of the classical uniqueness theorem by von Neumann [27], stating that any irreducible representation of the above CCR-algebra is equivalent to the generalized Schrödinger representation, acting on the Hilbert space $L^2(G)$ (with respect to the Haar measure). This will be shown in Subsection 2.2.

The transformation law of elements of $\mathcal{B}_{(x,y)}$ under the change of the link orientation is derived from the fact that the (classical) $G$-valued parallel transporter $g(x,y)$ on $(x,y)$ transforms to $g^{-1}(x,y)$ under the change of orientation. This transformation lifts naturally to an isomorphism

$$\mathcal{I}_{(x,y)} : \mathcal{B}_{(x,y)} \to \mathcal{B}_{(y,x)}$$

(2.11)
of field algebras, defined by:

\[ I_{(x,y)}(f) := \tilde{f}, \quad I_{(x,y)}(e) := \tilde{e}, \]

where \( \tilde{f}(g) := f(g^{-1}) \) and \( \tilde{e} \) is the left invariant vector field on \( G \), generated by \(-e\). The bosonic field algebra \( \mathfrak{B}_{(x,y)} \) is obtained from \( \mathfrak{B}_{(x,y)} \) and \( \mathfrak{B}_{(y,x)} \) by identifying them via \( I_{(x,y)} \).

Now, we give a full list of relations satisfied by generators of \( \mathfrak{B}_{(x,y)} \). Being entries of the fundamental representation of \( SU(3) \), the generators of \( \mathfrak{C}_{(x,y)} \) have to fulfil the following conditions:

\[
(U_B^A(x,y))^* U_A^C(x,y) = \delta_B^C \mathbf{1}, \quad \epsilon_{ABC} U_D^A(x,y) U_I^B(x,y) U_J^C(x,y) = \epsilon_{DEF} \mathbf{1}.
\]

In what follows, we will use the traceless matrix

\[ E^A_B(x,y) := \sum_i E_i(x,y) t^A_i, \]

built from generators of \( \mathfrak{P}_{(x,y)} \). Its entries obviously fulfil

\[ (E^A_B(x,y))^* = E_B^A(x,y). \]

The transformation law (2.12) of these objects under the change of the link orientation is given by the following relations:

\[
U_B^A(y,x) = \tilde{U}_B^A(x,y) = (U_B^A(x,y))^*, \quad E_B^A(y,x) = \tilde{E}_B^A(x,y) = -U_D^A(y,x) U_C^B(x,y) E_D^C(x,y).
\]

The \( su(3) \)-commutation relations read

\[
[E_B^A(x,y), E_D^C(u,z)] = \delta_{xy} \delta_{yz} \left( \delta_B^C E_D^A(x,y) - \delta_A^D E_B^C(x,y) \right),
\]

(formula (2.12) implies that all the components \( E_B^A(x,y) \) commute with all the components \( E_D^C(y,x) \), because the left invariant and the right invariant fields on the group commute). The canonical commutation relations (2.9) take the following form:

\[
i [E_B^A(x,y), U_D^C(u,z)] = +\delta_{xy} \delta_{yz} \left( \delta_B^C U_D^A(x,y) - \frac{1}{3} \delta_A^D U_B^C(x,y) \right) - \delta_{xz} \delta_{yu} \left( \delta_A^D U_C^B(y,x) - \frac{1}{3} \delta_B^C U_A^D(y,x) \right).
\]

To summarize, the field algebra \( \mathfrak{A}_A \), given by (2.11) – (2.13), is a \( C^* \)-algebra, generated by elements

\[
\{ \psi^{aA}(x), \psi^{*}_{aA}(x), U_B^A(x,y), E_B^A(x,y) \},
\]

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fulfilling relations (2.13), (2.14), (2.16), (2.17) and (2.18), together with canonical (anti-)commutation relations (2.5), (2.19) and (2.20).

**Remark:** According to formula (2.18), the transformation of the colour electric field $E$ from $(x, y)$ to $(y, x)$ consists of two steps:
1) the parallel transport from point $x$ to $y$ by means of the two parallel transporters $U$, each of them acting appropriately on the two indices of $E$
2) the change of the sign due to the change of the orientation.

In what follows, we always treat $E(x, y)$ as being attached to the site $x$.

### 2.2 Uniqueness of irreducible representations

Here, we prove uniqueness of the generalized canonical commutation relations as announced in Subsection 2.1.

We use the one-one-correspondence between non-degenerate representations of crossed products and covariant representations of $C^*$-dynamical systems, see [22]. Thus, let us consider the following covariant representation of $(C(G), G, \alpha)$ on $L^2(G)$, (with respect to the Haar measure):

1. Take the representation $\pi$ of the commutative $C^*$-algebra $C(G)$ given by multiplication with elements of $C(G)$. This is, obviously, a representation by bounded operators on $L^2(G)$.

2. Consider the left regular (unitary) representation $\hat{\pi}$ of $G$ on $L^2(G)$,

$$ (\hat{\pi}(g)h)(g') := h(g^{-1}g') , h \in L^2(G). \quad (2.22) $$

We calculate

$$ (\hat{\pi}(g) \circ \pi(u) \circ \hat{\pi}(g^{-1})) (h)(g') = ((\pi(u) \circ \hat{\pi}(g^{-1})) (h)) (g^{-1}g') = u(g^{-1}g') \cdot (\hat{\pi}(g^{-1})(h)) (g^{-1}g') = u(g^{-1}g') \cdot h(g') = \pi(\alpha_g(u)) (h)(g'). $$

This yields

$$ \hat{\pi}(g) \circ \pi(u) \circ \hat{\pi}(g^{-1}) = \pi(\alpha_g(u)) , \quad (2.23) $$

showing that the pair $(\pi, \hat{\pi})$ defines a covariant representation of $(C(G), G, \alpha)$ on $L^2(G)$, indeed.

The corresponding non-degenerate representation of $C(G) \otimes_{\alpha} G$ is called left regular representation. In the physical context, we call it generalized Schrödinger representation.

Observe that differentiating relation (2.23) yields the generalized canonical commutation relations (2.9).
Now, consider an arbitrary covariant representation \((\rho, \hat{\rho})\) of the the \(C^*\)-dynamical system \((C(G), G, \alpha)\), with \(\rho\) being a nondegenerate representation of \(C(G)\) on a Hilbert space \(H\) and \(\hat{\rho}\) being a strongly continuous unitary representation of \(G\) on \(H\). By the Gelfand-Najmark theorem for commutative \(C^*\)-algebras, we have a spectral measure \(dE\) on \(G\), such that

\[
\rho(u) = \int u(g) \, dE(g), \quad (2.24)
\]

for \(u \in C(G)\), and by covariance, the pair \((\rho, \hat{\rho})\) also fulfils (2.23). Thus, we get

\[
\hat{\rho}(g) \circ dE(g') \circ \hat{\rho}(g^{-1}) = dE(gg'). \quad (2.25)
\]

We conclude that the spectral measure \(dE\) defines a transitive system of imprimitivity for the representation \(\hat{\rho}\) of \(G\) based on the group manifold \(G\). Then, the imprimitivity theorem, see [28],[29], yields the following

**Theorem 2.1.** Any (non-degenerate) irreducible representation of \(C(G) \otimes \alpha G\) is equivalent to the generalized Schrödinger representation.

**Remarks:**

1. Disregarding the \(C^*\)-context, Theorem 2.1 is a classical result of Mackey, see [29] and references therein.

2. Within the \(C^*\)-context, there is a formulation of the commutation relations (2.23) for an arbitrary locally compact group in terms of the pentagon equation, which generalizes to quantum groups [30].

The following statement is a simple consequence of Theorem 7.7.12 in [22].

**Lemma 2.2.** For any compact Lie group, the generalized Schrödinger representation defines the following isomorphism of \(C^*\)-algebras:

\[
C(G) \otimes \alpha G \cong \mathfrak{K}(L^2(G)),
\]

where \(\mathfrak{K}(L^2(G))\) denotes the algebra of compact operators on \(L^2(G)\).

Now, we take the tensor product of generalized Schrödinger representations over all links:

\[
\bigotimes_{(x,y) \in \Lambda^1} L^2(C_{(x,y)}) \bigotimes_{x \in \partial \Lambda^0} L^2(C_{(x,\infty)}) \cong L^2(C), \quad (2.26)
\]

with

\[
C := \prod_{(x,y) \in \Lambda^1} C_{(x,y)} \prod_{x \in \partial \Lambda^0} C_{(x,\infty)}
\]
and each of the spaces $C_{(x,y)}$ being diffeomorphic to the group space of $G$.

This is, by Theorem 2.1, the unique representation space of the gluonic field algebra $\mathfrak{B}_\Lambda$. Moreover, using the classical uniqueness theorem for CAR-representations by Jordan and Wigner \[31\], any representation of fermionic fields is equivalent to the fermionic Fock representation. Consequently, using Lemma 2.2 we get the following

**Corollary 2.3.** The field algebra $\mathfrak{A}_\Lambda$ can be identified with the algebra $\mathcal{R}(H_\Lambda)$ of compact operators on the Hilbert space

$$H_\Lambda = \mathcal{F}(\mathbb{C}^{12N}) \otimes L^2(C),$$

with $\mathcal{F}(\mathbb{C}^{12N})$ denoting the fermionic Fock space generated by $12N$ anti-commuting pairs of quark fields.

The subspace $\mathcal{F}(\mathbb{C}^{12N})$ is spanned by vectors

$$\psi^*_{a_1A_1}(x_1) \ldots \psi^*_{a_nA_n}(x_n)|0>,$$

obtained from the fermionic Fock vacuum by the action of quark creation operators. Consequently, any element of $H_\Lambda$ is a linear combination of these fermionic vectors with coefficients being $L^2$-functions depending on gluonic potentials $U$.

### 2.3 Gauge transformations and local Gauss law

The group $G_\Lambda$ of local gauge transformations related to the lattice $\Lambda$ consists of mappings

$$\Lambda^0 \ni x \rightarrow g(x) \in G,$$

which represent internal gauge transformations, and of gauge transformations at infinity,

$$\Lambda^0_{\infty} \ni z \rightarrow g(z) \in G.$$

Thus,

$$G_\Lambda := G^i_\Lambda \times G^\infty_\Lambda = \prod_{x \in \Lambda^0} G_x \prod_{z \in \Lambda^0_{\infty}} G_z,$$

with $G_y \cong SU(3)$, for every $y$. We denote the corresponding Lie algebra by

$$\mathfrak{g}_\Lambda := \mathfrak{g}^i_\Lambda \oplus \mathfrak{g}^\infty_\Lambda = \bigoplus_{x \in \Lambda^0} \mathfrak{g}_x \bigoplus_{z \in \Lambda^0_{\infty}} \mathfrak{g}_z,$$

with $\mathfrak{g}_y \cong su(3)$, for every $y$.

The group $G_\Lambda$ acts on the classical configuration space $C$ as follows:

$$C_{(x,y)} \ni g(x,y) \rightarrow g(x)g(x,y)g(y)^{-1} \in C_{(x,y)}.$$
with \( g(x) \in G_x \) and \( g(y) \in G_y \). This action lifts naturally to functions on \( \mathcal{C} \). Moreover, we have an action of \( G_x \) on itself by inner automorphisms. This yields an action of \( G_\Lambda \) by automorphisms on each \( C^* \)-dynamical system \( (\mathcal{C}_{(x,y)}, G, \alpha) \) and, therefore, on the gluonic field algebra \( \mathfrak{B}_\Lambda \). For generators of \( \mathfrak{B}_{(x,y)} \subset \mathfrak{B}_\Lambda \), this action is given by

\[
U^A_B(x,y) \rightarrow g^A_C(x)U_C^D(x,y)(g^{-1})_B^D(y), \tag{2.32}
\]

\[
E^A_B(x,y) \rightarrow g^A_C(x)E_C^D(x,y)(g^{-1})_B^D(x), \tag{2.33}
\]

with \( y \) standing also for \( \infty \). Fermionic generators transform under the fundamental representation:

\[
\psi^{aA}(x) \rightarrow g^A_B(x)\psi^{aB}(x). \tag{2.34}
\]

To summarize, the group of local gauge transformation \( G_\Lambda \) acts on the field algebra \( \mathfrak{A}_\Lambda \) in a natural way by automorphisms.

It is easy to check that, for \( x \in \Lambda^0 \), the above automorphisms are generated by the following derivations of the field algebra:

\[
\mathcal{G}^A_B(x) := \rho^A_B(x) - \sum_{y \leftrightarrow x} E^A_B(x,y), \tag{2.35}
\]

where \( y \leftrightarrow x \) means that the sum is taken over all nearest neighbours \( y \) of \( x \) (with \( y \) also standing for \( \infty \)), and where

\[
\rho^A_B(x) = \sum_a \left( \psi^{aA}_x(x)\psi^{aB}_x(x) - \frac{1}{3} \delta^A_B \psi^{aC}_x(x)\psi^{aC}_x(x) \right) \tag{2.36}
\]

is the local matter charge density, fulfilling \( \rho^A_A(x) = 0 \). Observe that both \( \rho^A_A(x) \) and \( \rho^A_B(x) \) satisfy the \( su(3) \)-commutation relations separately and that the set \( \{\mathcal{G}^A_B(x)\} \) of generators spans the Lie algebra \( \mathfrak{g}_\Lambda \).

The local Gauss law at \( x \in \Lambda^0 \) reads

\[
\sum_{y \leftrightarrow x} E^A_B(x,y) = \rho^A_B(x), \tag{2.37}
\]

meaning that the gauge generator \( \mathcal{G}^A_B(x) \) defined by formula \( \mathcal{G}^A_B(x) \) vanishes. Observe that for every \( x \in \partial \Lambda^0 \), the corresponding boundary flux \( E^A_B(x,\infty) \) enters the Gauss law. All the Gauss laws at boundary points can thus be easily “solved” by expressing the boundary fluxes in terms of internal fields.

For \( z \in \Lambda^0_\infty \), the generator \( \mathcal{G}^A_B(z) \) reduces to the boundary flux \( E^A_B(z,\infty) \). Non-vanishing of this flux means gauge dependence of the quantum state under the action of \( G_z \subset G^\infty_\Lambda \). Neglecting these boundary fluxes means neglecting the possibility that a non trivial colour charge occurs \( \mathcal{G}^A_B(z,\infty) \). As will be discussed in Subsection 4.1, such a “truncated theory” is not useful as a discrete approximation of the continuum theory.
(The continuum limit should be constructed in terms of an inductive (resp. projective) limit of observable algebras (resp. quantum states). In this context, “external fluxes” represent the necessary link between any two intersecting lattices belonging to a whole sequence of lattices.)

Remark: We stress that the Gauss law (2.37) is the lattice counterpart of the “covariant divergence law”

\[ D_k E^k \equiv \partial_k E^k + [A_k, E^k] = \rho \]

in the continuum theory. There, the volume integration yields on the left hand side a standard boundary flux term (by applying Stokes theorem) and an additional volume integral contribution corresponding to the \([A_k, E^k]\)-term. In our lattice formulation, the volume integration corresponds to summation over all local Gauss laws (2.37). This yields a sum over boundary terms living on external links \((x, \infty)\) and a volume contribution equal to \(E(x, y) + E(y, x)\) on each lattice link. This term mimics the term \([A_k, E^k]\) of the continuum theory, (e. g. it vanishes only if the parallel transporter \(U(x, y)\) is trivial, which corresponds to the case \(A_k = 0\) in the continuum theory).

3 The Algebra of Internal Observables

3.1 Basic definitions

Physical observables, internal relative to \(\Lambda\), are, by definition, gauge invariant fields respecting the Gauss law. Hence, we have to take the subalgebra

\[ \mathfrak{A}^{G} \subset \mathfrak{A}_\Lambda \]

of \(G_\Lambda\)-invariant elements of the field algebra \(\mathfrak{A}_\Lambda\). This means, in particular, that observables have to commute with all gauge generators \(G^A_B(x)\).

Moreover, we have to impose all relations inherited from the local Gauss laws at all lattice sites (not including sites at infinity) as defining relations of the observable algebra. This means that the generators of \(G^i_\Lambda\) have to vanish in all possible gauge-invariant algebraic combinations. Hence, imposing (2.37) at the algebraic (representation-independent) level means that we require vanishing of the ideal \(\mathfrak{I}^i_\Lambda \cap \mathfrak{A}^{G}\), with \(\mathfrak{I}^i_\Lambda\) being the ideal generated by \(g^i_\Lambda\). Thus, the algebra \(\mathfrak{O}^i_\Lambda\) of internal observables is obtained from \(\mathfrak{A}^{G}\) by factorizing with respect to this ideal.

Definition 3.1. The algebra of internal observables relative to \(\Lambda\) is defined as

\[ \mathfrak{O}^i_\Lambda = \mathfrak{A}^{G}/\{\mathfrak{I}^i_\Lambda \cap \mathfrak{A}^{G}\}, \quad (3.1) \]

where \(\mathfrak{A}^{G} \subset \mathfrak{A}_\Lambda\) is the subalgebra of \(G_\Lambda\)-invariant elements of \(\mathfrak{A}_\Lambda\) and \(\mathfrak{I}^i_\Lambda \subset \mathfrak{A}_\Lambda\) is the ideal generated by \(g^i_\Lambda\).
Remarks:
i) The above ideal \( \mathfrak{I}_\Lambda \) is generated by unbounded elements in the sense of Woronowicz. It is obtained by multiplying its generators (elements of \( \mathfrak{g}_\Lambda \)) from both sides by elements of \( \mathfrak{A}_\Lambda \) belonging to their common dense domain, e.g. the so called smooth elements (corresponding to \( C^\infty \)-functions on \( G \)).
ii) The notion of an observable in the above sense is somewhat narrow, e.g. only compact operator functions built from Wilson loops belong to \( \mathfrak{O}_\Lambda \). This is due to the fact that the discussion of irreducible representations of the generalized commutation relations for bosonic fields as discussed in Section 2 necessarily leads to the algebra \( \mathfrak{K}(H_\Lambda) \) of compact operators on \( H_\Lambda \). Moreover, it is only this algebra, for which the notion of “being generated by unbounded elements” makes sense. If one gives up these basic structures, one can extend the field algebra, for instance, to the algebra of all bounded operators on \( H_\Lambda \) (the multiplier algebra of \( \mathfrak{K}(H_\Lambda) \)) and take the commutant of \( G_\Lambda \) there.

3.2 Classification of irreducible representations

By Corollary 2.3 we can identify \( \mathfrak{A}_\Lambda \) with the algebra of compact operators acting on the Hilbert space \( H_\Lambda \), given by (2.27). Under this identification, we have a unitary representation of the gauge group \( G_\Lambda \) on \( H_\Lambda \) and the subalgebra \( \mathfrak{A}^{G_\Lambda} \) can be viewed as the commutant \( (G_\Lambda)' \) of this representation in \( \mathfrak{K}(H_\Lambda) \).

Consider the closed subspace \( \mathcal{H}_\Lambda \subset H_\Lambda \) consisting of vectors, which are invariant with respect to internal gauge transformations,
\[
\mathcal{H}_\Lambda := \{ h \in H_\Lambda \mid G_\Lambda^i h = h \}.
\]

**Theorem 3.2.** The algebra of internal observables is canonically isomorphic with the algebra of those compact operators on the Hilbert space \( \mathcal{H}_\Lambda \), which commute with the action of the group \( G_\Lambda^\infty \):
\[
\mathfrak{O}_\Lambda^i \cong \mathfrak{K}(\mathcal{H}_\Lambda) \cap (G_\Lambda^\infty)'.
\]

**Proof:** Consider the direct sum decomposition
\[
H_\Lambda = \mathcal{H}_\Lambda \oplus \mathcal{H}_\Lambda^\perp,
\]
with \( \mathcal{H}_\Lambda^\perp \) denoting the orthogonal complement of \( \mathcal{H}_\Lambda \). Since the actions of \( G_\Lambda^i \) and \( G_\Lambda^\infty \) commute, \( \mathcal{H}_\Lambda \) is invariant under the action of \( G_\Lambda^\infty \) and, thus, under the full gauge group \( G_\Lambda \). Consequently, by unitarity of \( G_\Lambda \), \( \mathcal{H}_\Lambda^\perp \) is invariant, too. This implies the following block-diagonal structure of elements of \( G_\Lambda \) with respect to the decomposition (3.4):
\[
\begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix},
\]

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with \( A \) and \( B \) denoting unitary operators on \( \mathcal{H}_\Lambda \) and \( \mathcal{H}_\Lambda^\perp \), respectively. It can be easily shown that
\[
(G_\Lambda)' = \left\{ \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \in \mathcal{R}(\mathcal{H}_\Lambda) : [A,C] = 0 = [B,D], \text{ for all } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in G_\Lambda \right\}. \tag{3.5}
\]
Indeed, an arbitrary element \( \begin{pmatrix} C & E \\ F & D \end{pmatrix} \) belongs to \( (G_\Lambda)' \) iff
\[
AC = CA, \ AE = EB, \ BF = FA, \ BD = DB,
\]
for any \( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in G_\Lambda \). For \( \begin{pmatrix} h \\ 0 \end{pmatrix} \in \mathcal{H}_\Lambda \) we have \( \begin{pmatrix} 0 \\ Fh \end{pmatrix} \in \mathcal{H}_\Lambda^\perp \). On the other hand, any element of \( G_\Lambda^i \) has the form \( \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \). Thus,
\[
\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 \\ Fh \end{pmatrix} = \begin{pmatrix} 0 \\ BFh \end{pmatrix} = \begin{pmatrix} 0 \\ Fh \end{pmatrix},
\]
for all elements of \( G_\Lambda^i \), yielding \( \begin{pmatrix} 0 \\ Fh \end{pmatrix} \in \mathcal{H}_\Lambda \). Thus, \( Fh = 0 \), for all \( h \in \mathcal{H}_\Lambda \), implying \( F = 0 \). In an analogous way one shows \( E = 0 \). This gives formula \((3.5)\).

We decompose \( \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \). Since the restriction of a compact operator to a closed subspace is compact, we have \( \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{R}(\mathcal{H}_\Lambda) \). Moreover,
\[
\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \in \mathcal{J}_\Lambda^i. \text{ This yields the direct sum decomposition}
\]
\[
(G_\Lambda)' = (\mathcal{R}(\mathcal{H}_\Lambda) \cap (G_\Lambda)') \oplus (\mathcal{J}_\Lambda^i \cap (G_\Lambda)') \tag{3.6}
\]

Consequently, the algebra of internal observables \( \mathcal{D}_\Lambda = (G_\Lambda)'/\{\mathcal{J}_\Lambda \cap (G_\Lambda)\} \) is represented by the direct sum complement
\[
\mathcal{R}(\mathcal{H}_\Lambda) \cap (G_\Lambda)' = \mathcal{R}(\mathcal{H}_\Lambda) \cap (G_\Lambda^i)' \cap (G_\Lambda^\infty)'\]
in \( (G_\Lambda)' \). Finally, by \((3.5)\) we have that arbitrary elements of \( (G_\Lambda^i)' \) have the form
\[
\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}, \text{ with } C \in \mathcal{R}(\mathcal{H}_\Lambda) \text{ and } [D,B] = 0, \text{ for any unitary } B \text{ acting on } \mathcal{H}_\Lambda^\perp. \text{ Thus,}
\]
\( \mathcal{R}(\mathcal{H}_\Lambda) \subset (G_\Lambda^i)' \), yielding the isomorphism \((3.3)\).

The restriction of the unitary action of \( G_\Lambda^\infty \) to the subspace \( \mathcal{H}_\Lambda \) is not irreducible. Thus, \( \mathcal{H}_\Lambda \) splits into the direct sum of irreducible subspaces. Each irreducible representation of
$G_z, z \in \Lambda_\infty^0$, is labeled by its highest weight $(m, n)$ and is equivalent to the corresponding tensor representation in the space $S^m_n(C^3)$ of $m$-contravariant, $n$-covariant, traceless and totally symmetric tensors over $C^3$, endowed with the natural scalar product induced by the scalar product on $C^3$. Therefore, irreducible representations of $G^\infty_\Lambda$ are labeled by sequences of highest weights

$$(m, n) = (m_{z_1}, \ldots, m_{z_M}; n_{z_1}, \ldots, n_{z_M}),$$

(3.7)

where $(z_1, \ldots, z_M)$ label the lattice sites at infinity. These representations are equivalent to tensor products of representations in spaces $S^{m_{z_i}}_{n_{z_i}}(C^3)$. Let us denote by $H_{\Lambda}^{(m,n)}$ the sum of all the irreducible subspaces with respect to the action of $G^\infty_\Lambda$, which carry the same type $(m, n)$. Then we have

$$H_{\Lambda} = \bigoplus H_{\Lambda}^{(m,n)}.$$

(3.8)

Obviously, every subspace of type $(m, n)$ is invariant under the action of the observable algebra,

$$O_\Lambda H_{\Lambda}^{(m,n)} \subset H_{\Lambda}^{(m,n)}.$$

This yields the following

**Corollary 3.3.** The irreducible representations of $O_\Lambda$ are labelled by highest weight representations $(m, n)$ of $G^\infty_\Lambda$. For any $(m, n)$, the corresponding irreducible representation of $O_\Lambda$ coincides with the algebra of those compact operators on $H_{\Lambda}^{(m,n)}$, which commute with the action of the group $G^\infty_\Lambda$.

We call the pair $(m, n)$ the boundary flux distribution carried by the gluonic field. In the next subsection, it will become obvious that all distributions $(m, n)$ occur.

### 3.3 Explicit description of irreducible representations

Now we give an explicit description of the above irreducible representations, using the explicit form (2.27) of $H_{\Lambda}$. Any element of $H_{\Lambda}$ is a linear combination of fermionic vectors (2.28) with coefficients being $L^2$-functions depending on gluonic potentials $U$. The invariant subspace $H_{\Lambda} \subset H_{\Lambda}$ is spanned by vectors from $H_{\Lambda}$, which are scalars with respect to $G^\infty_\Lambda$. This means that for every $x \in \Lambda^0_\infty$, all the colour indices $(A_1, \ldots, A_n)$ of the fermionic state (2.28) must be saturated with the upper indices of either $U_A(x, y)$ or the canonical tensor $\epsilon^{ABC}$. After such contractions, we are – in general – left with free indices at infinity points $z_i \in \Lambda_\infty^0$. Finally, such a vector can be multiplied by gauge invariant functions of gluonic potentials $U$. The general form of such functions is as follows:

$$f([U]) = f(\text{Tr}(U_{\gamma_1}), \ldots, \text{Tr}(U_{\gamma_n})), $$

where $f$ is a function of $n$ scalar variables, each $\gamma = (x_1, x_2, \ldots, x_m)$ is an arbitrary closed lattice path and $U_\gamma$ is the corresponding parallel transporter along $\gamma$,

$$U^A_{\gamma B} = U^A_{\gamma_1 C_2}(x_1, x_2) U^C_{\gamma_2 C_3}(x_2, x_3) \ldots U^C_{\gamma_m B}(x_{m-1}, x_m).$$

(3.9)
Let us denote the result of these operations by

$$\Psi = \left( \Psi_{...A_1...A_{m_1i},...}, B_{1...B_{n_1i},...} \right). \quad (3.10)$$

This is a collection of $G^i_\Lambda$-invariant vectors belonging to $H_\Lambda$, labeled by tensor indices assigned to boundary points $z_i \in \Lambda^0_\infty$, i.e. an $H_\Lambda$-valued tensor over

$$\mathbb{C}^{3M} = \bigoplus_{z_i \in \Lambda^0_\infty} \mathbb{C}^3_{z_i}.$$ 

Linear combinations of those elements span the invariant subspace $H_\Lambda$, with coefficients built from products of tensors

$$t(z_i) = \left( t_{A_1...A_{m_1i}}(z_i) \right) \in \mathbb{T}^{m_1i}_{m_1z_i}(\mathbb{C}^3). \quad (3.11)$$

The resulting vector belonging to $H_\Lambda$ is a scalar obtained by contraction of (3.10) with these tensors:

$$\Psi = \Psi(t(z_1), \ldots, t(z_M)) = t_{B_1...B_{n_1i}}(z_i) \ldots \Psi_{...A_1...A_{m_1i},...} \ldots B_{1...B_{n_1i},...}. \quad (3.12)$$

Each of the irreducible components $H_\Lambda^{(m,n)}$ is composed of combinations (3.12), for which all the tensors $t(z_i)$ are irreducible (symmetric, traceless), i.e. where $t(z_i) \in S^{m_1i}_{m_1z_i}(\mathbb{C}^3) \subset \mathbb{T}^{m_1i}_{m_1z_i}(\mathbb{C}^3)$. If $t$’s are not irreducible, (3.12) is a sum of irreducible components belonging to different weights $(m, n)$, according to the decomposition of tensors $t(z_i)$ into the sum of products of irreducible (symmetric, traceless) tensors with canonical tensors $\delta^A_B$, $\epsilon^{ABC}$ and $\epsilon_{ABC}$.

### 4 The Full Algebra of Observables

#### 4.1 Motivation and basic definitions

One of the main perspectives of this work is the construction of the thermodynamical limit of finite lattice QCD. In [3] we have, in the context of finite lattice QED, outlined a strategy based upon an inductive (resp. projective) limit procedure for observable algebras (resp. state spaces). In what follows, we will argue that in order to implement this strategy, we have to extend both the algebra of observables $O^i_\Lambda$ (by adding certain “external observables”) and the Hilbert space $H_\Lambda$ (by tensorizing with the Hilbert space of tensors at infinity). Then, each collection \( \left( \Psi_{...A_1...A_{m_1i},...}, B_{1...B_{n_1i},...} \right) \) of $G^i_\Lambda$-invariant $H_\Lambda$-vectors labelled by free indices at infinity points (see (3.10)) will constitute a physical state.

Thus, let us consider two lattices $\Lambda_1$ and $\Lambda_2$ which are disjoint ($\Lambda_1 \cap \Lambda_2 = \emptyset$) and have a common wall such that their sum $\tilde{\Lambda} = \Lambda_1 \cup \Lambda_2$ is also a cubic lattice. If $x \in \Lambda_1$
and $y \in \Lambda_2$ are adjacent points in $\tilde{\Lambda}$, then we identify their infinities. This joint infinity $z$ may be visualised e.g. as the middle point of the connecting link $(x, y)$. The parallel transporter on $(x, y)$ is defined by

$$U^A_B(x, y) := U^A_C(x, z) U^C_B(z, y).$$

Observables internal relative to $\Lambda_1$ (respectively $\Lambda_2$) are built, among others, from parallel transporters $U_\gamma$ along lattice paths $\gamma$, which are completely contained in $\Lambda_1$ (respectively $\Lambda_2$). On the other hand, there exist observables internal relative to $\tilde{\Lambda}$, built from parallel transporters along paths crossing the set of joint infinity points. Such observables describe correlations between phenomena occurring in the two disjoint regions $\Lambda_1$ and $\Lambda_2$. As examples, consider the $\tilde{\Lambda}$-internal observables

$$J^{ab}_\gamma(x, y) := \psi^* A (x) U^A_B \psi^B (y),$$

with $\gamma$ being a path from $x \in \Lambda_1$ to $y \in \Lambda_2$, or observables

$$U_\gamma := U^A_\gamma A,$$

with $\gamma$ being a closed path lying partially in $\Lambda_1$ and partially in $\Lambda_2$. (These are operators belonging to the multiplier algebra of the observable algebra $O^{i}_{\tilde{\Lambda}}$ and, whence, may be called observables in a wider sense only – see Remark ii) on page 14). According to the above mentioned inductive limit procedure, the observable algebra related to $\tilde{\Lambda}$ should be constructed from observables related to $\Lambda_1$ and $\Lambda_2$. But, in order to construct observables related to $\tilde{\Lambda}$ of the above type (describing correlations), we have to admit fields “having free tensor indices at infinity”, like $\psi^* A (x) U^A C (x, z)$ and $U^C_B (z, y) \psi^B (y)$, with $z$ being a joint infinity point. Quantities of this type are usually called “charge carrying fields”, they were first introduced by Mandelstam (see [32]).

Observe that charge carrying fields do not act neither on $H_{\Lambda_1}$ nor on $H_{\Lambda_2}$. They carry the fundamental (respectively its contragredient) representation of $SU(3)$ associated with the corresponding point at infinity. Thus, we have to extend the Hilbert space $H_{\Lambda}$ by tensorising it with the Hilbert space $T_{\infty}$ generated by the fundamental and its contragredient representations of $SU(3)$ associated with all points at infinity:

$$T_{\infty} := \bigotimes_{z \in \Lambda_0^{\infty}} T(z), \quad T(z) := \bigoplus_{(m,n)} T^m_n (z).$$

(4.2)

Here, $T^m_n (z)$ denotes the space of all – not necessarily irreducible – $m$-contravariant, $n$-covariant tensors over $C^2_3$. This way, we are led to consider the Hilbert space $T_{\infty} \otimes H_{\Lambda}$. The action of the gauge group $G_{\Lambda}^{\infty}$ extends in a natural way from $H_{\Lambda}$ to this tensor product:

$$T(g) (t \otimes \Psi) := t \otimes (g \cdot \Psi), \quad t \otimes \Psi \in T_{\infty} \otimes H_{\Lambda}, \quad g \in G_{\Lambda}^{\infty}. \quad (4.3)$$

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On the other hand, the natural action of $G^\infty_{\Lambda}$ on $T_\infty$ can be also extended to this product:

$$R(g)(t \otimes \Psi) := (g \cdot t) \otimes \Psi.$$  \hspace{1cm} (4.4)

It is clear that elements of $T_\infty \otimes H_\Lambda$ may be represented as “wave functions with free boundary indices”, i.e. objects of type \((3.10)\). Indeed, such tensors can be naturally viewed as anti-linear mappings

$$T_\infty \ni s \mapsto (t \otimes \Psi)(s) := (s|t)\Psi \in H_\Lambda ,$$  \hspace{1cm} (4.5)

where $(\cdot|\cdot)$ denotes the scalar product in $T_\infty$. Obviously, wave functions \((3.10)\) are mappings of this type, too.

The above extension of the Hilbert space is necessary if we want to construct the thermodynamical limit of the theory by the above mentioned projective limit procedure. According to this procedure, physical states related to $\Lambda_1$ are obtained from physical states related to $\check{\Lambda}$ by applying a projection operator $P_{\Lambda_1,\check{\Lambda}}$, which consists in averaging over the degrees of freedom located in $\Lambda_2$. More precisely, consider a wave function $\tilde{\psi} \in H_{\check{\Lambda}}$, given by \((3.12)\), take the corresponding projector $|\tilde{\psi} \rangle <\tilde{\psi}|$, split all parallel transporters on links joining $\Lambda_1$ and $\Lambda_2$ according to \((4.1)\), and integrate over all degrees of freedom related to $\Lambda_2$ (including those located on external links of $\Lambda_2$). The result is a mixed state related to $\Lambda_1$, which can be represented as a mixture of pure states, each of them being a $H_{\Lambda_1}$-valued tensor $\Psi$ with respect to joint infinities of $\Lambda_1$ and $\Lambda_2$. In other words, the averaging procedure produces, in general, free indices on the common boundary between $\Lambda_1$ and $\Lambda_2$. To be consistent, we must admit such free indices from the very beginning.

From the above discussion we see that tensors $t(z)$ occurring in elements $t \otimes \Psi \in T_\infty \otimes H_\Lambda$ are not \textit{a priori} given c-number quantities, but quantum averages over external (i.e. contained in $\Lambda_2$) degrees of freedom.

There is, however, an additional requirement, which we impose on physical states of the system: the averaging procedure described above should be compatible with gauge transformations. Assume that we average a state $\tilde{\psi} \in H_{\check{\Lambda}}$ over $\Lambda_2$, with the parallel transporters on links joining $\Lambda_1$ and $\Lambda_2$ split according to \((4.1)\). Any gauge transformation $g$ at a common infinity point $z$ between $\Lambda_1$ and $\Lambda_2$, acting on $\tilde{\psi}$ can be either implemented by the action of $G^\infty_{\Lambda_1}$ or $G^\infty_{\Lambda_2}$. After averaging over $\Lambda_2$, the action of the gauge group $G^\infty_{\Lambda_1}$ is of course still represented by $T(g)$, whereas the action of $G^\infty_{\Lambda_2}$ reduces to $R(g)$ representing gauge transformations in “the rest of the world”. But, compatibility of averaging with gauging means that the result of this gauge transformation should not depend upon its implementation. Hence, we postulate

$$T(g)\Psi = R(g)\Psi.$$  \hspace{1cm} (4.6)

As a result of our discussion, we define the physical Hilbert space as

$$H_\Lambda := \{\Psi \in T_\infty \otimes H_\Lambda \mid T(g)\Psi = R(g)\Psi , \text{ for any } g \in G^\infty_{\Lambda}\} ,$$  \hspace{1cm} (4.7)
with gauge transformation being, according to the above discussion, represented by \( T \). The property (4.6) is obviously not shared by elements, which have partially contracted indices at infinity or indices which do not come from the bosonic wave functions \( U(x, \infty) \). To illustrate this, we consider the following examples:

\[
\Psi_{\ldots, A_1 \ldots A_m} \Psi_{\ldots, B_1 \ldots B_n} \Psi_{\ldots, A_{m+1}} \quad \text{or} \quad \Psi_{\ldots, B_1 \ldots B_n} \Psi_{\ldots, A_{m+1}} (z), \quad \text{with} \quad r(z) \in T_1(z).
\]

(4.8)

Thus, (4.6) is fulfilled precisely by elements of type (3.10) having all indices free. Admitting objects of type \( r \), which would live at joint infinity points \( z \), would mean admitting additional degrees of freedom, relevant for the description of physical states on the lattice \( \tilde{\Lambda} = \Lambda_1 \cup \Lambda_2 \). In that case, these joint infinity points could not be removed from \( \tilde{\Lambda} \).

Using (4.2), we have

\[
H_\Lambda = \bigoplus_{(m,n)} H_\Lambda^{(m,n)}.
\]

(4.9)

Here, \( H_\Lambda^{(m,n)} \) denotes the intersection of \( H_\Lambda \) with \( T_\infty^{(m,n)} \otimes H_\Lambda \), where

\[
T_\infty^{(m,n)} := \bigotimes_{z \in \Lambda_0^0} T_{n_z}^m(z)
\]

(4.10)

is the subspace of tensorial type \((m, n_z)\) at each \( z \in \Lambda_0^0 \). Note that, contrary to (3.8), (4.9) is not a decomposition into irreducible components.

Next, observe that the scalar products on \( H_\Lambda \) and on \( T_\infty \) induce a natural scalar product on \( H_\Lambda \). Using the representation (3.10), it is given by:

\[
(\Psi | \Phi)_{H_\Lambda} = \left( \Psi_{\ldots, A_1 \ldots A_{m_z} \ldots} \Phi_{\ldots, B_1 \ldots B_{n_z} \ldots} \right)_{H_\Lambda} \times
\]

\[
\times \ldots g_{A_1 C_1} \ldots g_{A_{m_z} C_{m_z}} \ldots g_{B_1 D_1} \ldots g_{B_{m_z} D_{m_z}} \ldots.
\]

(4.11)

Tensors with different valences (i.e. having a different number of indices) are, by definition, orthogonal.

We define the full algebra of observables related to \( \Lambda \) as the \( C^* \)-algebra of gauge invariant compact operators acting on \( H_\Lambda \),

\[
\mathcal{O}_\Lambda := \mathcal{F}(H_\Lambda) \cap (G^\infty_\Lambda)'.
\]

(4.12)

Consider the algebra \( \mathcal{O}_\Lambda^\infty \) of those compact operators acting on \( T_\infty \), which are invariant with respect to the action of \( G^\infty_\Lambda \). By classical invariant theory, this algebra is generated (in the sense of Woronowicz) by operations of tensorizing or contracting with \( SU(3) \)-invariant tensors \( \delta^A_B \), \( \epsilon^{ABC} \) and \( \epsilon_{ABC} \), and by projection operators \( P^{(m,n)} \) onto \( T_\infty^{(m,n)} \subset T_\infty \).

**Proposition 4.1.** The full observable algebra can be characterized as follows:

\[
\mathcal{O}_\Lambda \cong \mathcal{O}_\Lambda^i \otimes \mathcal{O}_\Lambda^\infty.
\]

(4.13)
**Proof:** Any compact gauge invariant operator $A$ acting on $H_{\Lambda}$ can be extended to a gauge invariant operator on the whole tensor product $T_{\infty} \otimes H_{\Lambda}$, using contractions and tensor products with boundary tensors $r$ in $T_{\infty}$, see formula (4.8). More precisely, we put:

$$A(C(r \otimes \Psi)) := C(r \otimes A(\Psi)),$$

for any tensor $r \in T_{\infty}$ and any contraction operator $C$ (the result of the contraction on the right-hand-side vanishes by definition if a corresponding index is missing in $A\Psi$). This way we have proved that

$$\mathcal{R}(H_{\Lambda}) \cap (G_{\Lambda}^{\infty})' \cong \mathcal{R}(T_{\infty} \otimes H_{\Lambda}) \cap (G_{\Lambda}^{\infty})'.$$

But we have

$$\mathcal{R}(T_{\infty} \otimes H_{\Lambda}) \cong \mathcal{R}(T_{\infty}) \otimes \mathcal{R}(H_{\Lambda}),$$

and, consequently,

$$\mathcal{R}(H_{\Lambda}) \cap (G_{\Lambda}^{\infty})' \cong \mathcal{R}(T_{\infty}) \otimes (\mathcal{R}(H_{\Lambda}) \cap (G_{\Lambda}^{\infty})') \cong \mathcal{R}(T_{\infty}) \otimes \mathcal{D}_{\Lambda}.$$

Taking again the intersection with $(G_{\Lambda}^{\infty})'$ and implementing it by the representation $R$ yields the thesis.

By abuse of language, elements of $\mathcal{D}_{\Lambda}^{\infty}$ can be called ,,external observables”. We also recall once again that, according to Remark ii) on page 14, gauge invariant (not necessarily compact) operators acting on $H_{\Lambda}$ can be called ,,generalized observables” or observables in a broader sense.

Adopting the point of view that $t$ occurring in $t \otimes \Psi$ represents the “quantum averages over the external field degrees of freedom”, one can argue that the only trace of the action of external observables, which may be seen from $\Lambda$, are external gauge invariant operators on $T_{\infty}$. Hence, formula (4.13) could be taken as an axiomatic definition of the full observable algebra. The results of the following subsection show that this approach is equivalent to the one used in the present section.

### 4.2 Classification of irreducible representations

Obviously, $T_{\infty}$ is not irreducible with respect to the action of $\mathcal{D}_{\Lambda}^{\infty}$. If $t(z) \in T_{n}^{m}(z)$, then its image under this action is a sum of components belonging to $T_{n}^{k}(z)$, with $(k, l)$ fulfilling

$$(m - n) \mod 3 = (k - l) \mod 3.$$

We see that the $\mathbb{Z}_{3}$-valued flux

$$\Phi(z) := (m_{z} - n_{z}) \mod 3$$

(4.18)
through each external link \((x,z), \ z \in \Lambda_0^\infty\), is conserved under the action of \(\mathfrak{S}_\Lambda^\infty\). Let us denote the sequence of \(\mathbb{Z}_3\)-valued fluxes assigned to all boundary points by

\[
\Phi := (\Phi(z_1), \Phi(z_2), \ldots).
\]

In what follows, we call \(\Phi\) boundary flux distribution. Consequently, we define the subspace \(H^\Phi_\Lambda \subset H_\Lambda\) as the space spanned by those tensors (3.10) which fulfill condition \(\Phi(z_i) = (m_{z_i} - n_{z_i}) \mod 3\). In other words, we put

\[
H^\Phi_\Lambda := \bigoplus_{m(z_i) - n(z_i) \mod 3 = \Phi(z_i)} H^{(m,n)}_\Lambda,
\]

with \(H^{(m,n)}_\Lambda\) given by decomposition (4.19). Obviously, these subspaces are invariant under the action of the full observable algebra \(\mathfrak{S}_\Lambda\) and \(\mathfrak{S}_\Lambda\) acts irreducibly on each of them. Moreover, the physical Hilbert space \(H_\Lambda\) splits into the direct sum of them:

\[
H_\Lambda = \bigoplus_\Phi H^\Phi_\Lambda.
\]

We obviously have

**Lemma 4.2.** The spaces \(H^\Phi_\Lambda\) provide all the irreducible representations of the algebra of observables \(\mathfrak{S}_\Lambda\).

We denote the irreducible component of \(\mathfrak{S}_\Lambda\), acting on \(H^\Phi_\Lambda\), by \(\mathfrak{S}^\Phi_\Lambda\).

Now, we define the global flux associated with a given boundary flux distributions \(\Phi\) putting

\[
\Phi_\partial_\Lambda := \left(\sum_{z \in \Lambda_0^\infty} \Phi(z)\right) \mod 3.
\]

Let us denote the total number of gluonic and antiguonic flux lines running through the boundary by

\[
m := \sum_{z_i \in \Lambda_0^\infty} m(z_i),
\]

\[
n := \sum_{z_i \in \Lambda_0^\infty} n(z_i).
\]

Then, we get

\[
\Phi_\partial_\Lambda = (m - n) \mod 3.
\]

**Lemma 4.3.** The irreducible representations of \(\mathfrak{S}_\Lambda\) in \(H^\Phi_\Lambda\) and in \(H^{\Phi'}_\Lambda\) are unitarily equivalent, if and only if \(\Phi\) and \(\Phi'\) carry the same global flux,

\[
\Phi_\partial_\Lambda = \Phi'_\partial_\Lambda.
\]
Proof: Suppose that we are given a pair \((\Phi, \Phi')\) such that \(\Phi_{\partial \Lambda} = \Phi'_{\partial \Lambda}\). Then, \(H^\Phi_{\Lambda}\) is given by (4.19) and, similarly,

\[ H^\Phi_{\Lambda} = \bigoplus_{m'(z_i) - n'(z_i) \mod 3 = \Phi'(z_i)} H^\Phi_{\Lambda}^{(m', n')} \]

For \(\Phi_{\partial \Lambda} = \Phi'_{\partial \Lambda}\), formula (4.21) implies that we can choose a pair of labels \((m_0, n_0)\) and \((m'_0, n'_0)\) such that \(m_0 = m'_0\), \(n_0 = n'_0\). Acting with tensorial operators \(\xi \in \mathfrak{O}_\Lambda\) (tensorising and contracting with canonical tensors \(\delta^A_B\), \(\epsilon^{ABC}\)) on \(H^\Phi_{\Lambda}^{(m_0, n_0)}\) (respectively \(H^\Phi_{\Lambda}^{(m'_0, n'_0)}\)), we may pass to any other subspace \(H^\Phi_{\Lambda}^{(m, n)}\) of \(H^\Phi_{\Lambda}\) (respectively any other subspace \(H^\Phi_{\Lambda}^{(m', n')}\) of \(H^\Phi_{\Lambda}'\)). This way we construct a bijection between the two sets \((m, n)\) and \((m', n')\) corresponding to \(\Phi\) and \(\Phi'\), preserving the total number of gluonic and antigluonic lines, \(m = m'\) and \(n = n'\).

We shall construct an intertwining operator between representations on \(H^\Phi_{\Lambda}\) and \(H^\Phi'_{\Lambda}\). For this purpose, we first define a sequence of isometric isomorphisms of Hilbert spaces,

\[ U_{(m', n')(m, n)} : H^\Phi_{\Lambda}^{(m, n)} \rightarrow H^\Phi_{\Lambda}^{(m', n')} \]

corresponding to the above bijection, as follows. We choose two finite families \(\{\beta_a\}\) and \(\{\gamma_b\}\) of lattice paths in \(\Lambda\), fulfilling the following conditions:

- their starting points \(x_a\) and \(y_b\), together with their end points \(z_a\) and \(w_b\) belong to the boundary \(\partial \Lambda\),
- for every \(x \in \Lambda_0^0\) there are exactly \(\Delta n(x) := n(x) - n'(x)\) paths \(\beta\) starting from \(x\) if \(\Delta n(x) > 0\) and zero otherwise,
- for every \(x \in \Lambda_0^0\) there are exactly \(\Delta m(x) := m(x) - m'(x)\) paths \(\gamma\) ending at \(x\) if \(\Delta m(x) > 0\) and zero otherwise,
- for every \(x \in \Lambda_0^0\) there are exactly \(-\Delta n(x)\) paths \(\beta\) ending at \(x\) if \(\Delta n(x) < 0\) and zero otherwise,
- for every \(x \in \Lambda_0^0\) there are exactly \(-\Delta m(x)\) paths \(\gamma\) starting from \(x\) if \(\Delta m(x) < 0\) and zero otherwise.

Now, the action of the operator \(U_{(m', n')(m, n)}\) on a vector \(\Psi \in H^\Phi_{\Lambda}^{(m, n)}\) is defined as follows. We multiply the tensor (3.11) by all parallel transporters \(U^ {B_a}_{\beta_a, A_a}\) and \(U^ {B_b}_{\gamma_b, A_b}\), see formula (3.9). Then, we contract all the subsequent upper indices \(B_a\) with the corresponding subsequent lower indices \(B_i\) of \(\Psi\) at the starting points of the curves \(\beta\) and all the subsequent lower indices \(A_b\) with the corresponding upper indices \(A_i\) of \(\Psi\) at the end points of the curves \(\gamma\). It is easy to see that the inverse (adjoint) operator consists in multiplying \(\Psi\) by the same transporters, but in contracting indices \(A_a\) with the corresponding subsequent
upper indices $A_i$ of $\Psi$ at the starting points of the curves $\beta$ and all the subsequent indices $B_i$ of $\Psi$ at the end points of the curves $\gamma$. This implies that

$$U^*_{(m',n')(m,n)}U_{(m',n')(m,n)} = \text{id} = U_{(m',n')(m,n)}U^*_{(m',n')(m,n)} \ .$$

Organizing the operators $U_{(m',n')(m,n)}$ into a block matrix, we get an isometric isomorphism denoted by

$$U_{\Phi'\Phi} : H^\Phi_A \rightarrow H^{\Phi'}_A \ .$$

Next, observe that the action

$$U^*_{\Phi'\Phi} \delta^\Phi_A U_{\Phi'\Phi} : H^\Phi_A \rightarrow H^\Phi_A \ ,$$

defines an irreducible representation of $\delta^\Phi_A$ on $H^\Phi_A$. By Lemma [12] this representation must be unitarily equivalent to $\delta^\Phi_A$, i.e. there exists a unitary (intertwining) operator

$$S : H^\Phi_A \rightarrow H^\Phi_A \ ,$$

such that

$$S^*U^*_{\Phi'\Phi} \delta^\Phi_A U_{\Phi'\Phi}S = \delta^\Phi_A \ .$$

It remains to show that equivalent representations yield equal global fluxes: If $H^\Phi_A$ and $H^{\Phi'}_A$ carry equivalent representations, then there exists an intertwiner $V_{\Phi'\Phi}$, such that

$$\Phi'_{\partial\Lambda} = V_{\Phi'\Phi} \Phi_{\partial\Lambda} V^{-1}_{\Phi'\Phi} \ .$$

But, according to formula [11,22], $\Phi_{\partial\Lambda}$ is a scalar on every irreducible representation space of $\delta^\Phi_A$. Thus, it commutes with $V_{\Phi'\Phi}$ yielding $\Phi'_{\partial\Lambda} = \Phi_{\partial\Lambda}$. \hfill $\Box$

**Theorem 4.4.** There are three inequivalent representations of $\delta^\Phi_A$ labeled by values of the global flux $\Phi_{\partial\Lambda}$. Consequently, the space $H^\Phi_A$ splits into the sum of three eigenspaces of $\Phi_{\partial\Lambda}$

$$H^\Phi_A = \bigoplus_{\lambda=-1,0,1} H^\lambda_A \ .$$

Each of the spaces $H^\lambda_A$ is a sum of superselection sectors $H^\Phi_A$ corresponding to all possible distributions $\Phi$ of the global flux $\lambda$. They carry equivalent representations of $\delta^\Phi_A$.

Anticipating the final result, we call the spaces $H^\lambda_A$ “charge superselection sectors”, in contrast to the “boundary-flux-distribution superselection sectors” $H^\Phi_A$. (For an analogous discussion in continuum QED see [33].)
4.3 Global colour charge and superselection structure

Here we show that, according to the global Gauss law, irreducible representations of \( \mathcal{D}_\Lambda \) are, alternatively, labeled by global colour charge (triality), which is carried by the quark fields. We briefly recall the notion of triality and derive the global Gauss law, for details see [12].

Consider any integrable representation \( F \) of the Lie algebra \( \mathfrak{su}(3) \) on a Hilbert space \( H \), i.e. a collection of operators \( F^{AB} \) in \( H \), fulfilling

\[
[F^{AB}, F^{CD}] = \delta^C_B F^{AD} - \delta^A_D F^{CB}.
\]

By (2.19), (2.36) and (2.5), the operators \( E^{AB}(x,y) \) and \( \rho^A_B(x) \), occurring on both sides of the local Gauss law, are of this type. Integrability means that for each \( F \) there exists a unitary representation \( \mathcal{G} \ni g \rightarrow \bar{\mathcal{F}}(g) \in \mathcal{B}(H) \) of the group \( \mathcal{G} \) such that \( F \) is its derivative.

If \( F_1 \) and \( F_2 \) are two commuting (integrable) representations of \( \mathfrak{su}(3) \), then so is \( F_1 + F_2 \).

Such a collection of operators is an operator domain in the sense of Woronowicz, see [34].

We define an operator function on this domain, i.e. a mapping \( F \rightarrow \varphi(F) \), which satisfies

\[
\varphi(UF)U^{-1} = U \varphi(F)U^{-1}
\]

for any integrable representation \( F \) of \( \mathfrak{su}(3) \), consider the corresponding representation \( \bar{\mathcal{F}} \) of \( \mathcal{G} \). Its restriction to the center \( \mathcal{Z} \) of \( \mathcal{G} \) acts as a multiple of the identity on each irreducible subspace \( H_\alpha \) of \( \bar{\mathcal{F}} \),

\[
\bar{\mathcal{F}}(z)|_{H_\alpha} = \chi^\alpha_F(z) \cdot 1_{H_\alpha}, \quad z \in \mathcal{Z}.
\]

Obviously, \( \chi^\alpha_F \) is a character on \( \mathcal{Z} \) and, therefore, \((\chi^\alpha_F(z))^3 = 1\). We identify the group of characters on \( \mathcal{Z} = \{ \zeta \cdot 1_3 \mid \zeta^3 = 1 \} \), with the additive group \( \mathbb{Z}_3 \cong \{-1, 0, 1\} \) by assigning to any character \( \chi^\alpha_F \) a number \( k(\alpha) \in \{ -1, 0, 1 \} \) fulfilling

\[
\chi^\alpha_F(\zeta \cdot 1_3) = \zeta^{k(\alpha)}.
\]

Hence, there exists a \( \mathbb{Z}_3 \)-valued operator function \( F \rightarrow \varphi(F) \), defined by

\[
\begin{align*}
\zeta^{\varphi_{\alpha}(F)} &= \chi^\alpha_F(\zeta \cdot 1_3), \\
\varphi(F) &= \sum_{\alpha} \varphi_{\alpha}(F) 1_{H_\alpha}.
\end{align*}
\]

Since \( \chi^\alpha_F \) are characters, we have

\[
\varphi(F_1 + F_2) = \varphi(F_1) + \varphi(F_2),
\]

for \( F_1 \) and \( F_2 \) commuting. Now, using the equivalence of each irreducible representation \( \alpha \) of \( \mathcal{G} \) with highest weight \((m(\alpha), n(\alpha))\) with the tensor representation in the space \( \mathcal{T}^m(\alpha)_{n(\alpha)}(\mathbb{C}^3) \) of \( m(\alpha) \)-contravariant, \( n(\alpha) \)-covariant, completely symmetric and traceless tensors over \( \mathbb{C}^3 \), we get

\[
\chi^\alpha_F(z) = \zeta^{\varphi_{\alpha}(F)} = \zeta^{m(\alpha) - n(\alpha)},
\]

25
for $z = \zeta \cdot 1_3 \in \mathcal{Z}$. Thus, we have

$$\varphi_\alpha(F) = (m(\alpha) - n(\alpha)) \mod 3 \quad (4.32)$$

for every irreducible highest weight representation $(m(\alpha), n(\alpha))$. In [12] we have given an explicit construction of $\varphi(F)$ in terms of Casimir operators of $F$.

Applying $\varphi$ to the local Gauss law (2.37) and using additivity (4.30) we obtain a gauge invariant equation for operators with eigenvalues in $\mathbb{Z}_3$:

$$\sum_{y \leftrightarrow x} \varphi(E(x,y)) = \varphi(\rho(x)) \quad (4.33)$$

valid at every lattice site $x$. The quantity on the right hand side is the (gauge invariant) local colour charge density carried by the quark field.

Using the transformation law (2.12) for $E(x,y)$ under the change of the link orientation and additivity (4.30) of $\varphi$, we have for every lattice bond $(x,y)$:

$$\varphi(E(x,y)) + \varphi(E(y,x)) = \varphi(E(x,y)) + \varphi(\tilde{E}(x,y)) = \varphi(E(x,y) + \tilde{E}(x,y)) \quad (4.34)$$

because representations $E$ and $\tilde{E}$ commute. But the following identity holds:

$$\varphi(E(x,y) + \tilde{E}(x,y)) = 0 \quad (4.35)$$

because both representations have the same irreducible subspaces $H_\alpha$, with the values of $m(\alpha)$ and $n(\alpha)$ exchanged. (The identity follows also directly from formula (2.18). It implies that the second order Casimirs for $E$ and $\tilde{E}$ coincide: $K_2(E) = K_2(\tilde{E})$, whereas $K_3(E) = -K_3(\tilde{E})$, cf. [12]).

Now, we take the sum of equations (4.33) over all lattice sites $x \in \Lambda^0$. Due to the above identity, all terms on the left hand side cancel, except for contributions $\varphi(E(x,\infty)) \equiv \varphi(E(x,z))$, $z \in \Lambda^0_\infty$, coming from the boundary. The irreducible subspace $H^\Phi_\Lambda \subset H_\Lambda$ characterized by the flux distribution $\Phi$, see (1.19), is an eigenspace of the gauge invariant operator $\varphi(E(x,z))$, with eigenvalue $(m_z - n_z) \mod 3 = \Phi(z)$, according to formula (1.32). Thus, using (4.21) we obtain

$$\sum_{x \in \partial \Lambda^0} \varphi(E(x,\infty)) = \Phi_\Lambda \quad (4.36)$$

with $\Phi_\Lambda$ being the global $\mathbb{Z}_3$-valued boundary flux corresponding to the flux distribution $\Phi$. On the right hand side we get the (gauge invariant) global colour charge (triality), carried by the matter field

$$t_\Lambda := \sum_{x \in \Lambda^0} \varphi(\rho(x)) \quad (4.37)$$

Thus, the global Gauss law takes the following form:

$$\Phi_\Lambda = t_\Lambda \quad (4.38)$$
Both quantities appearing here take eigenvalues in the center $\mathbb{Z} \cong \mathbb{Z}_3$ of $G$.

Comparing with Theorem 4.4 the global Gauss law yields another, equivalent characterization of irreducible representations of the observable algebra:

**Corollary 4.5.** The inequivalent representations of $\mathcal{O}_\Lambda$ are labeled by eigenvalues of global colour charge $t_\Lambda$.

**Remark:**
To illustrate the triality concept observe that we can assign to single quark fields (being in the defining representation of $SU(3)$) triality $+1$ and, consequently to antiquarks $-1$, (or the other way around). A single lattice gluon field has, of course, triality 0. Now, imagine a state with $m$ quarks and $n$ antiquarks, located in an arbitrary way inside $\Lambda$. By additivity of triality, $t_\Lambda$ has eigenvalue $m - n \mod 3$ on this state. As discussed already, gauge invariance of states with respect to internal gauge transformations implies that all quark indices inside $\Lambda$ have to be contracted. Basically, this can be done by connecting quark-antiquark pairs inside $\Lambda$ with flux lines built from gluonic parallel transporters and by contracting with canonical tensors $\delta^A{}_B$, $\epsilon^{ABC}$ and $\epsilon_{ABC}$. On the other hand, some of the flux lines starting at a quark (or ending at an antiquark) inside can run through the boundary to end at an antiquark (or start from a quark) outside of $\Lambda$. By the global Gauss law, the total number of gluonic flux lines minus the number of antigluonic flux lines, calculated modulo 3, is equal to the eigenvalue of triality. But the external quark and antiquark fields are not taken into account by a theory on $\Lambda$. After averaging over external fields we are left with the action of external gauge invariant operators $\delta^A{}_B$, $\epsilon^{ABC}$ and $\epsilon_{ABC}$. The flux lines running through the boundary may by contracted at points $z \in \Lambda_0^\infty$ with these tensors, eventually leaving either none ($t_\Lambda = 0$), or one gluonic ($t_\Lambda = +1$) or one antigluonic ($t_\Lambda = -1$) non-contracted line.

## 5 Generators and Relations

In this section we wish to find a presentation of the observable algebra in terms of generators and relations, inherited from canonical commutation relations of fields and from the Gauss law. In order to formulate and to study field dynamics, we rather need a presentation in terms of a set of independent generators. Thus, we wish to solve the Gauss law relations explicitly, to end up with a reduced set of generators and their (anti)-commutation relations. We show how to implement this idea by a special gauge fixing procedure based upon the choice of a lattice tree. This procedure leaves us, however, with some discrete gauge freedom. Moreover, we restrict ourselves to the generic stratum of the gauge group action, disregarding all non-generic strata. But even there, our method works only on a dense subset. These two obstructions to global gauge fixing reflect the Gribov problem, which is well known in the continuum theory, see also \cite{36} for a discussion of this problem in the Ashtekar theory. Thus, following the gauge fixing idea leads to some delicate problems.
How to overcome these problems will be discussed in separate papers, see [17], [18]. Instead of trying to fix the gauge, one rather has to find a generating set of genuine invariants. Below we show that it is quite easy, to write down a highly redundant set of invariants, but it is very hard to reduce it. If we want to work with genuine invariants we are automatically forced to consider higher order monomials, built from basic bosonic and fermionic fields. These invariants inherit, of course, some (anti)-commutation relations, but the algebra generated by them does not close on the linear level. This way interesting new algebras occur. We refer to [35] for some first remarks on their structure. It turns out that algebras of similar types have been discussed in different areas of mathematical physics throughout the last decade, see the list of references in [35].

Since the generators of $\mathcal{O}_\Lambda^\infty$ have been already listed before, it remains to discuss $\mathcal{O}_\Lambda^i$ in terms of generators and relations.

### 5.1 Generators of $\mathcal{O}_\Lambda^i$

Below, we define a set of generators of $\mathcal{O}_\Lambda^i$ in terms of gauge-invariant combinations of the fields $(U, E, \psi, \psi^*)$. In the next subsections, we will systematically reduce the number of generators to a minimal set.

**Theorem 5.1.** The observable algebra $\mathcal{O}_\Lambda^i$ is generated by the following gauge invariant elements (together with their conjugates):

\[
U_\gamma \ := \ U^A_{\gamma A} \quad (5.1)
\]

\[
E_\gamma(x, y) \ := \ U^A_{\gamma B} E^B_A(x, y) \quad (5.2)
\]

\[
J^{ab}_\gamma(x, y) \ := \ \psi^a_A(x) U^A_{\gamma B} \psi^b_B(y) \quad (5.3)
\]

\[
W^{abc}_{\alpha\beta\gamma}(x, y, z) \ := \ \frac{i}{6} \epsilon_{ABC} U^A_{\alpha D} U^B_{\beta E} U^C_{\gamma F} \psi^a_D(x) \psi^b_E(y) \psi^c_F(z), \quad (5.4)
\]

with $\gamma$ denoting an arbitrary closed lattice path in formula (5.1), a closed lattice path starting and ending at $x$ in formula (5.2) and a path from $x$ to $y$ in formula (5.3). In formula (5.4), $\alpha$, $\beta$ and $\gamma$ are paths starting at some reference point $t$ and ending at $x$, $y$ and $z$, respectively. In formula (5.3), both $x$ and $y$ stand also for $\infty$.

For the proof see Appendix B.

Note that the observables $J^{ab}_\gamma$ and $W^{abc}_{\alpha\beta\gamma}$ represent hadronic matter of mesonic and baryonic type. They will play a basic role in future investigations towards a construction of an effective theory of interacting hadrons. Basically, the lattice hamiltonian can be expressed in terms of the above invariants. In particular, the kinetic energy $E^2$ of the gluonic field is given by second Casimirs, its potential energy $B^2$ by Wilson loops $U_\gamma$ and the matter field part is given in terms of $J$’s, (which, however, are related with $W$’s via non-linear constraints).
5.2 The reduction idea

The above generating set turns out to be highly redundant. There is a number of non-trivial relations between generators, inherited from the canonical (anti)-commutation relations and from the local Gauss laws. Below, we will show how to solve the local Gauss laws explicitly. This will be done by using a technique, based upon the choice of a lattice tree. This way we shall prove that $\mathcal{O}_\Lambda^i$ can be decomposed (in a tree-dependent way) into the tensor product of a gluonic and a matter field part. This presentation of $\mathcal{O}_\Lambda^i$ can be constructed in two steps:

1. First we fix a lattice point $x_0$ and impose gauge invariance with respect to the pointed gauge group at $x_0$, 

$$ G_\Lambda^0 = G_{\Lambda,0}^i \times G_\Lambda^\infty $$

with

$$ G_{\Lambda,0}^i = \prod_{x \neq x_0 \in \Lambda} G_x. $$

Moreover, we implement the Gauss laws at all points $\Lambda \ni x \neq x_0$, i.e. we factorize with respect to the ideal $\mathcal{I}_{\Lambda,0}^{i,0} \cap (G_{\Lambda,0}^0)'$, where $\mathcal{I}_{\Lambda,0}^{i,0}$ is generated by the Lie algebra $g_{\Lambda,0}^i \subset g_{\Lambda}^i$ of $G_{\Lambda,0}^i$. This gives the pointed algebra of internal observables:

$$ \mathcal{O}_{\Lambda,0}^i := (G_\Lambda^0)' / \{ \mathcal{I}_{\Lambda,0}^{i,0} \cap (G_{\Lambda,0}^0)' \}. $$

2. Next, we impose on $\mathcal{O}_{\Lambda,0}^{i,0}$ gauge invariance with respect to the residual gauge group $G_{x_0}$, and factorize with respect to the ideal $\mathcal{J}_{x_0} \subset \mathcal{O}_{\Lambda,0}^{i,0}$ generated by the local Gauss law at $x_0$.

Whereas the first step can be performed without any obstructions, in the second step, all the problems mentioned at the beginning of this chapter show up.

**Theorem 5.2.** The internal observable algebra can be viewed as follows:

$$ \mathcal{O}_\Lambda^i \cong \frac{(G_{x_0})'}{\{ \mathcal{J}_{x_0} \cap (G_{x_0})' \}}, $$

where $(G_{x_0})'$ and $\mathcal{J}_{x_0}$ are considered as subalgebras of $\mathcal{O}_{\Lambda,0}^{i,0}$.

**Proof:** Since $G_{x_0}$ commutes with $G_\Lambda^0$, we have $(G_{\Lambda}') = (G_{\Lambda,0}^0)' \cap (G_{x_0})'$. Moreover, the invariant subspaces $H_\Lambda^0$ and $H_{x_0}$ of $G_\Lambda^0$ and $G_{x_0}$ are both closed subspaces of $H_\Lambda$, whereas $\mathcal{J}_{\Lambda}^0$ and $\mathcal{J}_{x_0}$ are composed of operators vanishing on $H_\Lambda^0$ and $H_{x_0}$, respectively. Observe that

$$ H_\Lambda^0 \cap H_{x_0} = \mathcal{H}_\Lambda $$

and that the ideal

$$ \mathcal{J}_\Lambda = \mathcal{J}_\Lambda^0 \oplus \mathcal{J}_{x_0} $$
is composed of those operators which vanish on this intersection. Using completely anal-
ogous arguments as in the proof of Theorem (3.2), we obtain

\[ \mathcal{O}_\Lambda^{i,0} = \mathcal{R}(H^0) \cap (G_\Lambda^\infty)' . \] (5.9)

In the second step we have to factorize the commutant \((G_{x_0})'\) of \(G_{x_0}\) in \(\mathcal{O}_\Lambda^{i,0}\) with respect
to \(\mathcal{I}_{x_0} \cap (G_{x_0})'\). But by (5.9) we have

\[ (G_{x_0})' = (G_{x_0})' (\mathcal{R}(H^0)) \cap (G_\Lambda^\infty)' \]

where \((G_{x_0})' (\mathcal{R}(H^0))\) is the commutant taken in \(\mathcal{R}(H^0)\). Again, using similar arguments
as in the proof of Theorem (3.2), we get

\[ (G_{x_0})' / \{ \mathcal{I}_{x_0} \cap (G_{x_0})' \} \cong \mathcal{R}(H_\Lambda) \cap (G_\Lambda^\infty)' , \]

which is isomorphic to \(\mathcal{O}_\Lambda^i\), by formula (3.3).

5.3 Reduction with respect to pointed gauge transformations

As already mentioned, a convenient way to solve relations between generators is to choose
a tree, i.e. to assign a unique path connecting any pair of lattice sites. More precisely, a
tree is a pair \((x_0, \mathcal{T})\), where \(x_0\) is a distinguished lattice site (called root) and \(\mathcal{T}\) is a set of
lattice links such that for any lattice site \(x\) there is exactly one path from \(x\) to \(x_0\), with
links belonging to \(\mathcal{T}\). We denote this path by \(\beta(x)\). Consequently, for any pair \((x, y)\) of
lattice sites, there is a unique along tree path from \(x\) to \(y\), equal to \(\beta^{-1}(y) \circ \beta(x)\), where
\(\alpha \circ \beta\) denotes the composition of the two paths (a path obtained by first running through
\(\beta\) and next through \(\alpha\)) and \(\beta^{-1}\) denotes the path taken with the opposite orientation.
This does apply to sites at infinity also, because external links are treated as belonging
to the tree a priori.

To find an explicit set of generators of the pointed observable algebra \(\mathcal{O}_\Lambda^{i,0}\), given by
(5.9), we “parallel transport” all generators of the field algebra to the lattice root using
the above along tree paths. The transported generators feel only gauge transformations
at \(x_0\) and, therefore, are invariant with respect to \(G_\Lambda^i\). Hence, \((G_\Lambda^i)'\) is generated by:

1. \( \{ U_{\gamma}^A B, E_T^A (x, y), \psi^a_T A (x), \psi^{*a A} T (x) \} \) , \hspace{1cm} (5.10)

where \(\gamma\) is an arbitrary closed curve starting and ending at \(x_0\) and

\[ E_T^A (x, y) := U_{\beta(x)}^A C U_{\beta(x)}^{1A} E_B^C (x, y) , \hspace{1cm} (5.11) \]

\[ \psi^a_T (x) := U_{\beta(x)}^A B \psi^{a B} (x) , \hspace{1cm} (5.12) \]

with \(\beta(x)\) denoting the unique tree path from \(x\) to \(x_0\).
2. boundary fluxes:

\[ E^A_{TB}(x, \infty) := U^A_{\beta(x)C} U^D_{\beta(x)\beta^{-1}B} E^C_D(x, \infty). \]  

(5.13)

The generating set (5.10) is still enormously redundant. To reduce this redundancy, in a first step, we restrict the admissible paths to the form:

\[ \gamma(x, y) := \beta(x) \circ (x, y) \circ \beta^{-1}(y). \]  

(5.14)

We denote

\[ U^A_{TB}(x, y) := U^A_{\gamma(x,y)B}. \]

It is obvious that any \( U^A_{\gamma B} \) may be reconstructed from those quantities. Thus,

\[ \{ U^A_{TB}(x, y), E^A_{TB}(x, y), \psi^{\alpha A}_T(x), \psi^{\alpha A}_T(x) \} \]  

(5.15)

can be taken, together with boundary fluxes, as a set of generators of \( (G^0_A)' \). The bosonic and fermionic generators fulfill the same commutation relations as generators \( U^A_{TB}(x, y) \) and \( E^A_{TB}(x, y) \) (see Section 2). The local Gauss law can be easily rewritten:

\[ \rho^A_T(x) = \sum_{y \rightarrow x} E^A_{TB}(x, y), \]  

(5.16)

where \( \rho_T \) is given by (2.36) with \( \psi^{\alpha A}_T(x) \) replaced by \( \psi^{\alpha A}_T(x) \). However, a nontrivial commutator between \( E^A_{TB}(x, y) \) and the latter occurs.

Next, observe that for any on-tree-link \( (x, y) \in \mathcal{T} \) we have

\[ U^A_{TB}(x, y) = \delta^A_B. \]

Thus, the relevant information is carried by those \( U_T \)'s, which correspond to off-tree-links \( (x, y) \notin \mathcal{T} \). On the other hand, exactly those among the fields \( E_T \), which correspond to off-tree-links, may be chosen as independent generators. Indeed, the on-tree \( E_T \)'s can be calculated by solving the local Gauss law at all the points \( x \neq x_0 \). Observe that the off-tree \( E_T \)'s have trivial commutators with \( \psi_T \)'s. Thus, factorization of \( (G^0_A)' \) with respect to the local Gauss laws at all points \( x \neq x_0 \) consists in taking only independent internal generators, i.e. those among (5.15), which correspond to off-tree-links.

Let us denote the number of lattice sites by \( N \) and by \( L \) the number of links. Since the number of on-tree links is equal to \( N - 1 \), the number of off-tree links is equal to

\[ K = L - N + 1. \]

Enumerate these links putting \( (x_i, y_i) =: \ell_i \), where \( i = 1, \ldots, K \). We have thus the following set of independent generators of \( O^0_{\Lambda} \):

\[ \{ U^A_{TB}(\ell_i), E^A_{TB}(\ell_i), \psi^{\alpha A}_T(x), \psi^{\alpha A}_T(x) \} \]  

(5.17)
together with the boundary fluxes \((5.13)\). The latter commute with all generators \((5.17)\) and, after the final reduction with respect to \(G_{x_0}\), they will generate the center of the algebra.

Generators \((5.17)\) fulfill canonical (anti)-commutation relations, given by formulae \((2.19), (2.20)\) and \((2.5)\). They are all subject to gauge transformations at the tree root \(x_0\). Observe that, since bosonic and fermionic generators commute, we have
\[
\tilde{O}_i^\Lambda, \tilde{\Lambda} = \tilde{O}_i^\text{glu} T \otimes \tilde{\tilde{O}}_i^\text{mat} T \otimes O^b \Lambda,
\]
where \(\tilde{O}^\text{glu}_i T\) is the gluonic part generated by \(\{U^A_T(\ell), E^A_T(\ell)\}\) and \(\tilde{O}^\text{mat}_i T\) is the matter field part, generated by \(\{\psi^A_T(x), \psi^{A\dagger}_T(x)\}\). According to the above discussion, the gluonic algebra \(\tilde{O}^\text{glu}_T\) is isomorphic to the generalized CCR-algebra over the group \(G\), spanned by \(K\) pairs of generators, and \(\tilde{O}^\text{mat}_T\) is isomorphic to the CAR-algebra, generated by \(12N\) pairs of anticommuting elements. The subalgebra \(O^b \Lambda\) denotes the component generated by boundary fluxes \((5.13)\).

### 5.4 Removing the residual gauge freedom

In the first part of this paper we have mentioned that \(G\) can be, basically, an arbitrary compact Lie group. Here, we definitely consider \(G = SU(3)\) only. In what follows, we denote the \(K\)-fold cartesian product of \(G\) by \(G^K = G \times \cdots \times G\) and elements of \(G^K\) by \(g = (g_1, \ldots, g_K)\).

Let us denote the off-tree variables by
\[
E_i = E_T(\ell_i), \quad U_i = U_T(\ell_i), \quad i = 1, \ldots, K.
\]

The residual gauge group \(G_{x_0} \cong G\) acts on this set of variables by
\[
(E_i, U_i) \rightarrow (g E_i g^{-1}, g U_i g^{-1}),
\]
with \(g = g(x_0) \in G_{x_0}\). In what follows, we want to fix this residual gauge freedom. Thus, we have to consider the action of \(G\) on \(G^K\) by inner automorphisms
\[
G \times G^K \ni (h, (g_1, \ldots, g_K)) \mapsto (h g_1 h^{-1}, \ldots, h g_K h^{-1}) \in G^K.
\]

We wish to parameterize, by choosing a gauge, the space of equivalence classes of elements of \(G^K\) with respect to this group action, which by abuse of language, will be called \(\text{Ad}G\). Factorizing with respect to this action, we obtain the orbit space \(G^K/\text{Ad}G\). This is a complicated stratified set, which will be more deeply discussed in \cite{17}. Here, we restrict ourselves to the generic orbit type, respectively the generic stratum \(G^K_{\text{gen}}\), which is an open and dense submanifold in \(G^K\). An element \(g = (g_1, \ldots, g_K) \in G^K\) belongs to the generic stratum, iff its stabilizer is the center \(Z_3\) of \(G\). It is quite obvious that \(g\) belongs to the generic stratum, iff there does not exist any common eigenvector of the
matrices \((g_1, \ldots, g_K)\). Moreover, one can show \([17]\) that \(g\) belongs to the generic stratum, iff there exists a pair \((g_i, g_j)\) or a triple \((g_i, g_j, g_k)\) of elements not possessing any common eigenvector. Using arguments developed in \([36]\) one can prove that the bundle

\[
\pi: \mathbf{G}^K_{gen} \to \mathbf{G}^K_{gen}/\text{Ad}\mathbf{G}
\]

is non-trivial, for \(K \geq 2\). It can be considered as a principal fibre bundle with structure group \(G/\mathbb{Z}_3\). Moreover, one can find a system of local trivializations (respectively local sections) of this bundle, defined over a covering of \(\mathbf{G}^K_{gen}/\text{Ad}\mathbf{G}\) with open subsets, which are all dense with respect to the natural measure (the one induced by the Haar-measure).

Thus, let

\[
\mathbf{G}^K_{gen}/\text{Ad}\mathbf{G} \supset \mathcal{U} \ni [g] \mapsto s([g]) \equiv (s_1, \ldots, s_K)([g]) \in \mathbf{G}^K
\]

be one of these local sections, with \(\mathcal{U}\) being dense in \(\mathbf{G}^K_{gen}/\text{Ad}\mathbf{G}\). Since \(\text{Ad}\mathbf{G}\) acts (pointwise) on this section, we can fix the gauge by bringing \(s\) to a special form. Since pairs of group elements being in a non-generic position form a set of measure zero in \(G^2\), we can – without loss of generality – assume that \(s_{K-1}\) and \(s_K\) are in generic position on \(\mathcal{U}\). That means they have no common eigenvector. Thus, on this neighbourhood, we can fix the gauge in two steps: First, we diagonalize \(s_{K-1}\) and next we use the stabilizer of this diagonal element to bring \(s_K\) to a special form. Since \(s_{K-1}\) and \(s_K\) have no common eigenvector, this fixes the (remaining) stabilizer gauge completely, (up to \(\mathbb{Z}_3\)). Let us denote the function (which obviously depends only on \(s_{K-1}\) and \(s_K\)) implementing this gauge transformation by

\[
\pi^{-1}(\mathcal{U}) \ni (s_1, \ldots, s_K) \mapsto f(s_1, \ldots, s_K) = f(s_{K-1}, s_K) \in G
\]

and the local section after gauge fixing by

\[
\mathbf{G}^K_{gen}/\text{Ad}\mathbf{G} \supset \mathcal{U} \ni [g] \mapsto f([g]) \equiv (f_1, \ldots, f_K)([g]) \in \mathbf{G}^K,
\]

with

\[
f_i = f(s_{K-1}, s_K) \cdot s_i \cdot f(s_{K-1}, s_K)^{-1}, \quad i = 1, \ldots, K.
\]

The section \(f\) can be made explicit by using a system of local trivializations of \(G\) as an \(SU(2)\)-principal bundle over \(S^5\). We refer to \([17]\) for details and to Appendix A for one example of a local section of this bundle.

Suppose that we had started with another section \(\tilde{s}\), related to \(s\) by a gauge transformation given by \(g \in G\). Then, the function \(\tilde{f}(s_{K-1}, s_K) = f(s_{K-1}, s_K) \cdot g^{-1}\) yields the same section \(f\). Thus, \(f\) is equivariant with respect to gauge transformations,

\[
f(g \cdot s_{K-1} \cdot g^{-1}, g \cdot s_K \cdot g^{-1}) = f(s_{K-1}, s_K) \cdot g^{-1},
\]

and in this sense, we can consider the \(f_i\) as being “gauge invariant”. It is challenging to parameterize classes \([g]\) of gauge equivalent configurations more intrinsically, namely in terms of genuine invariants. In \([17]\) we will prove the following
Theorem 5.3. Any function on $G^2$ invariant with respect to the action by inner automorphisms

$$G \times G^2 \ni (h, (g_1, g_2)) \mapsto (h g_1 h^{-1}, h g_2 h^{-1}) \in G^2$$

can be expressed as a function in the following invariants and their complex conjugates:

$$T_1(g_1, g_2) := \text{tr}(g_1),$$
$$T_2(g_1, g_2) := \text{tr}(g_2),$$
$$T_3(g_1, g_2) := \text{tr}(g_1 g_2),$$
$$T_4(g_1, g_2) := \text{tr}(g_1 g_2^2),$$
$$T_5(g_1, g_2) := \text{tr}(g_1^2 g_2^2 g_1 g_2) - \text{tr}(g_1^2 g_2 g_1 g_2^2).$$

Moreover, there is one algebraic relation between those invariants such that for given values of $T_i, i = 1 \ldots 4$, there are at most two possible values of $T_5$.

By this Theorem, it follows that the entries of $f_i, i = K - 1, K$, and, therefore, the group elements $f_i$ themselves can be expressed in terms of the above set of invariants:

$$f_i = f_i(T_1(g_{K-1}, g_K), \ldots, T_5(g_{K-1}, g_K)), \quad i = K - 1, K.$$ 

Since the section $f$ parameterizes the gauge orbit space, it is clear that the remaining group elements $f_i, i = 1, \ldots, K - 2$, can be expressed as a function of traces (5.1), too.

To summarize, applying this special gauge-fixing to a gauge configuration $(U_1, \ldots, U_K)$, with generic $(U_{K-1}, U_K)$ corresponding to the pair $(x_{K-1}, y_{K-1}), (x_K, y_K)$ of off-tree-links, we obtain a local parameterization of its gauge equivalence class by $(u_1, \ldots, u_K)$ defined by

$$u_i := f(U_{K-1}, U_K) \cdot U_i \cdot (f(U_{K-1}, U_K))^{-1}. \quad (5.22)$$

We denote the indices of these matrices by $r, s, \ldots$, $u_i = \{u_i^r\}$, and, consequently, the matrix elements of $f$ by $f = \{f^r_A\}$. Then we have

$$u_i^r = f^r_A \cdot U_i^A \cdot (f^{-1})^B_s \quad (5.23)$$

and the gauge transformation (5.21) reads:

$$f^r_B \to f^r_A \cdot (g^{-1})^A_B, \quad (5.24)$$

(the colour indices $A, B, C, \ldots$ feel gauge transformations, whereas indices $r, s, \ldots$ – assuming the same values $1, 2, 3,$ – label “gauge-invariant quantities”).

By inspecting formula (A.1), we see that there are two independent degrees of freedom in the matrix $u_{K-1}$ and 6 independent degrees of freedom in the matrix $u_K$. They may be combined into 8 degrees of freedom of a single element of $G$ in the following way:

$$u_0 := u_{K-1} \cdot u_K \cdot (u_{K-1})^{-1}. \quad (5.25)$$
The above procedure is defined up to a discrete symmetry only. This symmetry arises from the action of the permutation group \( S_3 \subset \text{AdSU}(3) \) on entries of the diagonal matrix \( u_{K-1} \in T^2 \subset SU(3) \). To fix this \( S_3 \)-gauge freedom means choosing for each element of \( T^2/S_3 \) a unique representative on the torus \( T^2 \). Changing this representative by an even permutation does not change the image of the mapping \( (u_{K-1}, u_K) \mapsto u_0 \) given by \((5.25)\), which does not cover the entire group \( SU(3) \) but only “half of it”. Using also odd permutations we cover a dense subset of \( SU(3) \), but then the discrete symmetry \( u_0 \to \bar{u}_0 \) must be taken into account.

It can be shown that the decomposition of \( u_0 \) into the above product of elements of special form is unique (up to the discrete symmetry) and both \( u_{K-1} \) and \( u_{K} \) may be reconstructed from \( u_0 \). By the above discussion, we can consider the observable \( u_0 \) as \( SU(3) \)-valued, provided we keep the discrete symmetry \( u_0 \to \bar{u}_0 \). To summarize, we have shown that, locally, the full information carried by the fields \( U_i \) is encoded in \( K - 1 \) elements \( u_i, \, i = 0, \ldots, K - 2 \), of \( G \), modulo the discrete symmetry just described.

Analogously, we construct \( K \) gauge invariant generators

\[
e_i^*_s = f^r_A \cdot E^A_{iB} \cdot (f^{-1})_{Bs},
\]

which have to fulfill the residual Gauss law at \( x_0 \). To describe the unconstrained information carried by the fields \( e_i \), we divide the information contained in \( e_{K-1} \) and \( e_K \) (16 gauge invariant generators) into 8 independent generators encoded in the momentum \( e_0 \) canonically conjugate to \( u_0 \) and 8 other combinations of \( e_{K-1} \) and \( e_K \), which can be reconstructed from the global Gauss law at \( x_0 \). More precisely, at each point of the section \((5.24)\), we decompose the pair \((e_{K-1}, e_K)\) into a pair \((e_{K-1}, e_K)\) of vectors tangent to this section and a pair \((e_{K-1}, e_K)\) of vectors orthogonal to it. Here, orthogonality is of course meant in the sense of the natural scalar product induced by the Killing metric. The tangent components sum up to the momentum \( e_0 \) canonically conjugate to \( u_0 \). More precisely, \( e_0 \) is the image of \((e_{K-1}, e_K)\) under the tangent mapping of \((u_{K-1}, u_K) \mapsto u_0 \) given by \((5.25)\). By a simple calculation, we get:

\[
e_0 = (\text{Ad}_{u^{-1}_0} - 1) \circ \text{Ad}_{u_{K-1}} (e_{K-1}^\parallel) + \text{Ad}_{u_{K-1}} (e_K^\parallel),
\]

This formula is invertible and enables us to calculate uniquely both \( e_{K-1}^\parallel \) and \( e_K^\parallel \) once we know \( e_0 \). On the other hand, the sum \( e_{K-1} + e_K \) is given from the Gauss law. This enables us to calculate \((u_{K-1}, u_K, e_{K-1}, e_K)\) once we know \((u_0, e_0)\). We end up with \(2(K-1)\) independent generators \((e_i, u_i) \), \( i = 0, \ldots, K - 2 \), of the gluonic part of the observable algebra.

It is easy to show that these bosonic generators satisfy the generalized canonical commutation relations over \( G \):

\[
[e_i^r s, e_j^p q] = \delta_{ij} (\delta^p_s e_i^r q - \delta^r_q e_i^p s),
\]

\[
[e_i^r s, u_j^p q] = \delta_{ij} (\delta^p_s u_i^r q - \frac{1}{3} \delta^r_s u_i^p q),
\]

\[
[u_i^r s, u_j^p q] = 0.
\]
For the fermionic observables, we denote:

\[ \mathfrak{a}^{ar}(x) := f^r_A \psi^{aA}(x) = f^r_A U^A_{\beta(x)B} \psi^{aB}(x). \]  

(5.31)

Introducing the joint index \( k = (a, r, x) \), \( k = 1, \ldots, 12N \), we get,

\[ \mathfrak{a}_k := \mathfrak{a}^{ar}(x). \]  

(5.32)

Formally, these quantities fulfil the canonical anti-commutation relations

\[ [\mathfrak{a}_k, \mathfrak{a}_l]^+ = \delta^k_l, \]  

(5.33)

but again, an additional discrete symmetry has to be taken into account. This symmetry arises, because the section \( f \), given by (5.20), is defined only up to the stabilizer \( \mathbb{Z}_3 \) of the generic stratum. Observe that this ambiguity does not affect the bosonic quantities \( u \) and \( e \), because they are “quadratic” in \( f \).

Let us denote the bosonic (resp. fermionic) observable algebra part, obtained from \( \tilde{O}_{\text{glu}}^T \) (resp. \( \tilde{O}_{\text{mat}}^T \)) after fixing the residual gauge, by \( O_{\text{glu}}^T \) (resp. \( O_{\text{mat}}^T \)). Then we have

\[ \mathcal{O}_\Lambda = \mathcal{O}_{\text{glu}}^T \otimes \mathcal{O}_{\text{mat}}^T \otimes \mathcal{O}_\Lambda^b \otimes \mathcal{O}_\Lambda^\infty. \]  

(5.34)

The above discussion shows that locally (on a dense subset of the generic stratum) and up to discrete symmetries, \( \mathcal{O}_{\text{glu}}^T \) (resp. \( \mathcal{O}_{\text{mat}}^T \)) coincides with the algebra of generalized canonical commutation (resp. anti-commutation) relations for the reduced data \( (u_i, e_i) \), with \( i = 0, \ldots, K - 2 \), (resp. \( (a_k, a^{*k}) \), with \( k = 1, \ldots, 12N \)).

As already mentioned at the beginning of this section, a systematic study of \( \mathcal{O}_\Lambda \) as an algebra defined in terms of generators and relations on the level of genuine invariants will be presented in [17] and [18]. In particular, we will show that the fermionic part is generated by the following sesqui-linear and trilinear combinations of \( a_k \) and \( a^{*k} \):

\[ j_{\ell}^k = a^{*k} a_{\ell}, \]  

(5.35)

\[ w_{pqr} = a_p a_q a_r, \]  

(5.36)

\[ w^{*ijk} = a^{*k} a^{*j} a^{*i}. \]  

(5.37)

Similarly, to parameterize the bosonic part in terms of genuine invariants, one has to take – according to classical invariant theory – all trace-invariants, built from \( e \) and \( u \). This set is, however, highly redundant and it is a complicated task to find, for a fixed number \( K \), the full set of relations.
A Appendix: A Local Parameterization of \((SU(3) \times SU(3))_{\text{gen}}/\text{Ad}SU(3)\)

Consider the action of the group of inner automorphisms, here denoted by AdSU(3), on \(SU(3) \times SU(3)\). In Subsection 5.4 we have used an explicit local parameterization of the generic stratum of this action in terms of a bundle section. Such a section can be obtained as follows. Let \((g_1, g_2) \in SU(3) \times SU(3)\) be a pair of group elements lying in the generic stratum. First, we diagonalize \(g_1\). Since \(g_1\) is generic, the stabilizer of the Ad-action is isomorphic to \(U(1) \times U(1)\). Next, treating \(SU(3)\) as an \(SU(2)\)-principal bundle over \(S^5\), one can bring \(g_2\) to a special form using the \(U(1) \times U(1)\)-action. This yields a family of local sections defined on dense subsets over the generic stratum, corresponding to a family of local trivializations of the \(SU(2)\)-principal bundle \(SU(3) \to S^5\). For details we refer to [17].

As a local section of the above type one can choose:

\[
\begin{align*}
f_1 &= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \\
f_2 &= \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & d \\ 0 & -\bar{d} & \bar{c} \end{bmatrix} \end{align*}
\]

Here, \(\lambda_i\) are eigenvalues of \(g_1\), fulfilling
\[
|\lambda_1| = |\lambda_2| = |\lambda_3| = 1, \quad \lambda_1 \lambda_2 \lambda_3 = 1,
\]
The entries
\[
a, \delta \in \mathbb{C}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad b_1, b_2 \in \mathbb{R}_+,
\]
of the first factor in \(f_2\) fulfil
\[
|a|^2 + b_1^2 + b_2^2 = 1, \quad |\delta| = 1, \quad a = |a|\delta^{-2}
\]
and the lower diagonal block of the second factor is an \(SU(2)\)-matrix in the standard parameterization, with
\[
|c|^2 + |d|^2 = 1.
\]

B Appendix: Proof of Theorem 5.1

It is well-known that the only gauge invariant combinations built exclusively from the \(U\)'s are the Wilson-loops (5.1).

Any other invariant is built by contracting the colour indices of the fields \(E^A_B(x, y)\), \(U^A_B(x, y)\), \(\psi^{aA}(x)\) and \(\psi^*_{aA}(x)\). Consider such an invariant \(I\) and replace in its definition the above fields by their gauge invariant counterparts \(e, u, a\) and \(a^*\). In particular, the
missing on-tree quantities $e$ and $u$, are defined as combinations of the off-tree ones, using the Gauss law (for $e$) and the Bianchi identities (for $u$). Formally, the new invariant obtained this way coincides with $I$, because the factors $f$ and $f^{-1}$ coming from the definition of the quantities $e$, $u$, $a$ and $a^*$ disappear under contraction. Moreover, all the fermionic quantities $a$ and $a^*$ appearing in the invariant may be grouped to give quantities $j$, $w$ and $w^*$.

As already mentioned in Subsection 5.4, the invariant quantities $u$ can be expressed as (nonlinear) functions of traces of $U$ (invariants (5.1)). We show that also $e$, $j$, $w$ and $w^*$ can be expressed in terms of invariants (5.1) - (5.4), listed in Theorem 5.1. Invariants $e_i^s$ can be dealt with as follows: We contract them with 8 different $u$'s and use formulae (5.23) and (5.26) to obtain the following system of linear equations for the 8 independent components of $e_i^s$:

$$e_i^s u^{(1)}_r = E_1^s, \ldots, e_i^s u^{(8)}_r = E_8^s,$$

with the right-hand-sides all being invariants of type (5.2). Analogously, we can write down systems of linear equations of this type for the quantities $j$, $w$ and $w^*$. Solving these systems of linear equations, we obtain $e$, $u$, $j$, $w$ and $w^*$ as functions, linear with respect to invariants (5.2), (5.3), (5.4) and nonlinear with respect to invariants (5.1) listed in Theorem 5.1.

Hence, we have formally expressed $I$ as a combination $\tilde{I}$ of invariants listed in Theorem 5.1. In particular all Casimir operators, built from the electric fields $E$ may be expressed in terms of these generators.

The above formulae, expressing any gauge invariant field $I$ as a combination $\tilde{I}$ of the invariants listed in Theorem 5.1 were derived by the help of the gauge fixing section (5.20), which is not globally defined. Hence, equality $I = \tilde{I}$ holds on a dense subset of the configuration space only. But $I$ is a differential operator (with smooth coefficients) on the whole configuration space (the rank of such an operator is equal to its algebraic order with respect to variables $E$). The invariant $\tilde{I}$ is a differential operator of the same rank, but a priori its coefficients are well defined on a dense set of the configuration space only. But, if two such operators coincide on a dense set, they coincide everywhere.

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