On the Hamming Auto- and Cross-correlation Functions of a Class of Frequency Hopping Sequences of Length $p^n$

Minglong QI$^{(a)}$, Member, Shengwu XIONG$^1$, and Jingling YUAN$^1$, Nonmembers

SUMMARY In this paper, a new class of frequency hopping sequences (FHSs) of length $p^n$ is constructed by using Ding-Helleseth generalized cyclotomic classes of order 2, of which the Hamming auto- and cross-correlation functions are investigated (for the Hamming cross-correlation, only the case $p \equiv 3 \pmod{4}$ is considered). It is shown that the set of the constructed FHSs is optimal with respect to the average Hamming correlation functions.

key words: frequency hopping sequences, Hamming cross-correlation function, Ding-Helleseth generalized cyclotomic classes.

1. Introduction

Let $F = \{f_0, f_1, \cdots, f_{p^n-1}\}$ be a set of $m$ elements called
the alphabet of available frequencies. A sequence with $\nu$ elements
taken from $F$ is said to be a frequency hopping sequence (FHS) over $F$ of length $\nu$. Let $X, Y$ be two FHSs taken from a set with $M$ FHSs, $S$, i.e., $X = (X(t))_{t=0}^{\nu-1}$, $Y = (Y(t))_{t=0}^{\nu-1}$ where $X(t), Y(t) \in F$, $0 \leq t \leq \nu - 1$. Define the periodic Hamming cross-correlation function between $X$ and $Y$ as the following equation:

$$H(X, Y : \tau) = \sum_{t=0}^{\nu-1} h[X(t + \tau), Y(t)], 0 \leq \tau < \nu \quad (1)$$

where $h[X(t + \tau), Y(t)] = 1$ if $X(t + \tau) = Y(t)$, and 0 otherwise. The subscript $\tau$ in (1) is performed modulo $\nu$.

Let $Y = X$ in (1), then $H(X, X : \tau)$ with $0 < \tau < \nu$ is called the Hamming autocorrelation function of $X$, denoted by $H(X : \tau)$.

If the FHSs set, $S$, is explicitly enumerated as $S = \{X_0, X_1, \cdots, X_{M-1}\}$, then we use $H(i, j : \tau)$ to denote the Hamming cross-correlation function between $X_i$ and $X_j$, and $H(i : \tau)$ to denote the Hamming autocorrelation function of $X_i$, where $0 \leq i, j < M$.

We need some maximum parameters on the FHSs in order to describe two important theoretical bounds described in the sequel. Let $X, Y \in S$. Define

$$H(X) = \max_{0 \leq \tau < \nu} (H(X : \tau)),$$

$$H(X, Y) = \max_{0 \leq \tau < \nu} [H(X, Y : \tau)],$$

$$H(S) = \max_{X \in S} [\max_{X \neq Y} (H(X, Y))].$$

In [1], Lempel and Greenberg gave the first theoretical bound on $H(X)$, called the Lempel-Greenberg bound on an FHS.

**Lemma 1.1** (The Lempel-Greenberg bound [1]). For any FHS $X$ of length $\nu$ over an alphabet of size $m$, we have

$$H(X) \geq \left\lceil \frac{(\nu - b)(\nu - b - m)}{m(n - 1)} \right\rceil$$

where $b$ is the least nonnegative residue of $\nu$ modulo $m$, and $\lceil r \rceil$ denotes the least integer no less than $r$, a real number.

The following result due to Fuji-Hara et al [3] may be used to check the Lempel-Greenberg bound:

**Corollary 1.1** ([3]). For any FHS $X$ of length $\nu$ over an alphabet of size $m$,

$$H(X) \geq \begin{cases} a, & \text{if } \nu \neq m \\ 0, & \text{if } \nu = m \end{cases}$$

where $\nu = am + b$, $0 \leq b < m$.

**Definition 1.1.** An FHS $X \in S$ is said to be optimal if $X$ is such that the equality in Lemma 1.1 is met.

In [2], Peng and Fan established a bound on $H(S)$, resumed in the following lemma:

**Lemma 1.2** (The Peng-Fan bounds [2]). Let $S$ be a set of $M$ FHSs of length $\nu$ over an alphabet of size $m$, and $l = \lfloor \nu / m \rfloor$, where $\lfloor r \rfloor$ denotes the integral part of $r$. Then,

$$H(S) \geq \left\lceil \frac{(\nu M - m)^{\lfloor v / m \rfloor}}{(\nu - 1)m} \right\rceil$$

and

$$H(S) \geq \left\lceil \frac{2IvM - (l + 1)IM}{(\nu - 1)M} \right\rceil.$$  

**Definition 1.2.** The FHS set $S$ is said to be optimal if it meets one of the equalities of the Peng-Fan bounds in Lemma 1.2.

Apart from the Hamming auto- and cross-correlation functions presented so far, the average Hamming correlation functions are important as well to indicate the performance of the FHSs set, $S$. We at first define two overall numbers of the Hamming auto- and cross-correlation function as follows:

$$\text{average Hamming autocorrelation}\, \quad \text{average Hamming cross-correlation}$$
Lemma 1.3

\[ N_a(S) = \sum_{X \in S} \sum_{\tau=0}^{\nu-1} H(X : \tau), \]
\[ N_c(S) = \frac{1}{2} \sum_{X,Y \in S} \sum_{\tau=0}^{\nu-1} H(X,Y : \tau) \]

From above two overall numbers can be defined the average Hamming auto- and cross-correlation functions.

\[ A_a(S) = \frac{N_a(S)}{M(\nu - 1)} \]
\[ A_c(S) = \frac{2N_c(S)}{\nu M (M - 1)} \]

We recall \( M \) is the number of the FHSs in the set \( S \), \( \nu \) is the length of each such FHS, and \( m \) is the size of the frequency alphabet set \( F \). In a context not confused we write \( A_a \) instead of \( A_a(S) \), \( A_c \) instead of \( A_c(S) \). In [4], the authors gave a theoretical bound on \( A_a \) and \( A_c \) that relates other parameters \( M, \nu \), and \( m \) together.

**Lemma 1.3 ([4]).**

\[ \frac{A_a}{\nu (M - 1)} + \frac{A_c}{\nu - 1} \geq \frac{\nu M - m}{m(\nu - 1)(M - 1)} \] (2)

**Definition 1.3.** The FHSs set \( S \) is said to be optimal (AH Optimal) with respect to the average Hamming auto- and cross-correlation functions if it is such that the equality in Lemma 1.3 is met.

It is difficult and tedious to check if or not an FHS set is AH Optimal if we start up by computing explicitly \( A_a \) and \( A_c \) and then substitute them into (2). There is an indirect and efficient way to verify the AH Optimality. We begin by introducing the concept of an uniformly distributed FHS set [5], [6].

**Definition 1.4.** Let the symbols used here be that defined so far. The FHSs set \( S \) is said to be uniformly distributed FHSs set if \( N_S(f) \) is a constant for any \( f \in F \) where

\[ N_S(f) = \sum_{X \in S} N_X(f) \]

and

\[ N_X(f) = |\{0 \leq t \leq \nu - 1 : X(t) = f\}|. \]

Next theorem is the criterion to check if or not an FHSs set is AH Optimal.

**Theorem 1.1 ([5],[6]).** The FHSs set \( S \) is AH Optimal if only if it is uniformly distributed.

Frequency-hopping sequences (FHSs) play an important role in communication systems such as frequency-hopping code-division multiple-access (FH-CDMA) systems, multi-user radar and sonar systems, etc. [7]. So, the construction of the FHSs with the optimal Hamming properties mentioned so far is an important research topics. There are several algebraic and combinatorial constructions in the literature [3], [8]–[20].

In this paper, we construct the FHSs of length \( p^v \) using Ding-Helleseth generalized cyclotomy [22], show that the FHSs set is AH Optimal by the criterion stated in Theorem 1.1, and compute out explicitly the Hamming auto- and cross-correlation function. The rest of the paper is structured as follows: in Sect. 2, it is briefly introduced Ding-Helleseth cyclotomy, based upon which the FHSs of length \( p^v \) are constructed, and their AH Optimal property is established. In Sect 3 and Sect. 4, it is given the formulae of the Hamming auto- and cross-correlation function of these FHSs. At the end of Sect. 4, we give an application to the case the length of the FHSs is equal to \( p^v \), and finally in Sect. 5, some concluding remarks are presented.

2. Ding-Helleseth Cyclotomy and Construction of the FHSs of Length \( p^v \)

Let \( n \geq 2 \) be an integer and \( Z^n \) be the set of all invertible elements of the additive group modulo \( n \), \( Z_n \). For any partition of \( Z^n = \bigcup_{i=0}^{d-1} D_i \), where \( D_0 \) is a subgroup of \( Z^n \), if there exist \( d \) elements \( g_1, g_2, \cdots , g_{d-1} \), of \( Z^n \), such that \( D_i = g_i D_0 \), then \( D_i \) is called a generalized cyclotomic class of order \( d \).

In [22], Ding and Helleseth introduced a generalized cyclotomy with respect to \( n = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t} \), where \( p_1, p_2, \cdots , p_t \) are \( ts \) distinct odd primes, and \( e_1, e_2, \cdots , e_t \) are \( ts \) positive integers. Their initial aim was to extend Whitman generalized cyclotomy of \( pq \) [21], and construct balanced binary sequences for the use in cryptography.

Let \( p \) be an odd prime. It is known that if \( g \) is a primitive root modulo \( p^2 \), then \( g \) is also a primitive root modulo \( p^k \), \( k \geq 1 \). By the Euler totient function, the order of \( g \) modulo \( p^k \) is equal to \( p^k - p^{k-1} \). Let \( D_0^{(p^k)} = \langle g^2 \rangle \) be the cyclic group generated by \( g^2 \) modulo \( p^k \), and \( D_1^{(p^k)} = g \langle g^2 \rangle \) be the cost of \( D_0^{(p^k)} \) by \( g \). It is clear that both \( D_0^{(p^k)} \) and \( D_1^{(p^k)} \) are the Ding-Helleseth generalized cyclotomic class of order 2 which give a partition of the multiplicative group modulo \( p^k \), \( Z^{p^k} \). The additive group modulo \( p^k \) can be decomposed into the union of \( D_0^{(p^k)} \)’s and \( D_1^{(p^k)} \)’s, \( 1 \leq k \leq n \) [22, Lemma 12]:

\[ Z^{p^n} \setminus \{0\} = \bigcup_{k=1}^{n} p^{-k} D_0^{(p^k)} \cup \bigcup_{k=1}^{n} p^{-k} D_1^{(p^k)}. \]

Define

\[ D_0^{(k)} = p^{-k} D_0^{(p^k)} , \quad D_1^{(k)} = p^{-k} D_1^{(p^k)} , \]
\[ C_0 = D_0^{(k)} \cup \{0\}, \quad C_1 = D_1^{(k)} , \]
\[ \cdots \]
\[ C_{2(k-1)} = D_0^{(k)}, \quad C_{2(k-1)+1} = D_1^{(k)} , \]
\[ \cdots \]
\[ C_{2(n-1)} = D_0^{(n)}, \quad C_{2(n-1)+1} = D_1^{(n)} , \] (3)
where $1 \leq k \leq n$. It is clear that with respect to $Z_p^n$, there are $m = 2n$'s such sets: $C_0, C_1, \cdots , C_{2n-1}$. $C_i \cap C_j = \emptyset$ for $0 \leq i \neq j \leq 2n - 1$ where $\emptyset$ denotes the empty set, and $Z_p^n = \bigcup_{i=0}^{2n-1} C_i$.

Next, we describe the construction of the FHSs of length $p^n$, based on the Ding-Helleseth generalized cyclotomic classes of order 2 with respect to $p^n$, where $n \geq 3$ and $p$ is an odd prime.

Let $X = (X(t))_{t=1}^v$ be an FHS of length $v$ over the frequency alphabet set $\mathcal{F}$ of size $m$. The support of $f \in \mathcal{F}$ in the sequence $X$ is defined by

$$\text{support}_X(f) = \{ t | X(t) = f, 0 \leq t < v \}.$$ \textbf{Construct. 2.1.} Let $\mathcal{S} = \{ X_0, X_1, \cdots , X_{2n-1} \}$ be a set of FHSs of length $p^n$. Let $X_i, 0 \leq i \leq 2n - 1$, be such that

$$\text{support}_X(f) = \{ t | X(t) = f, 0 \leq t < v - 1 \}.$$ \textbf{Define the generalized cyclotomic number of order two $\mu_k = \lambda_k$} \textbf{where}\n
$$\mu_k = \lambda_k = \{ \tau \in D_j^k : \tau \neq 0 \}.$$ \textbf{where the subscript $i + j$ is reduced modulo $m = 2n$.}

It is obvious that Construct. 2.1 has the frequency alphabet set $\mathcal{F} = \{ f | 0 \leq f \leq 2n - 1 \}$, and the family size is equal to $2n$ as well. We have the following results:

\textbf{Theorem 2.1.} The FHSs Set, $\mathcal{S}$, constructed from Construct. 2.1, is uniformly distributed.

\textbf{Proof.} Let $j \in \mathcal{F}$. From Definition 1.4 and Construct. 2.1,

$$N_\mathcal{S}(j) = \sum_{i=0}^{2n-1} N_\mathcal{X}(j) = \sum_{i=0}^{2n-1} |C_{i+j}| = \sum_{k=0}^{2n-1} |C_k| = p^n.$$ So, for each $j \in \mathcal{F}$, $N_\mathcal{S}(j)$ is constant. By Definition 1.4, the FHSs Set $\mathcal{S}$ is uniformly distributed. \hfill $\Box$

\textbf{Theorem 2.2.} The FHSs Set, $\mathcal{S}$, constructed from Construct. 2.1, is $AH$ Optimal.

\textbf{Proof.} By Theorem 1.1 and 2.1. \hfill $\Box$

Define the generalized cyclotomic number of order two modulo $p^k$ [22] as below

$$(i, j)_{p^k} = \left| \left( D_i^{p^k} + 1 \right) \cap D_j^{p^k} \right|$$

where $i, j = 0, 1$ and $1 \leq k \leq n$. The formulae to compute above generalized cyclotomic numbers are given by the following equations [22]:

If $p \equiv 1$ (mod 4), then

$$(0, 0)_{p^k} = \frac{p^{k-1}(p - 5)}{4},$$

$$(0, 1)_{p^k} = (1, 0)_{p^k} = (1, 1)_{p^k} = \frac{p^{k-1}(p - 1)}{4}.$$ \hfill (4)

If $p \equiv 3$ (mod 4), then

$$(0, 1)_{p^k} = \frac{p^{k-1}(p - 1)}{4},$$

$$(0, 0)_{p^k} = (1, 0)_{p^k} = (1, 1)_{p^k} = \frac{p^{k-1}(p - 3)}{4}.$$ \hfill (5)

In order to establish explicitly the Hamming auto-correlation function of the FHSs of length $p^n$ in the sequel, we now define two types of distance functions:

$$\Delta_{\mathcal{A}_k}(i : \tau) = \left| \left( D_i^{(k)} + \tau \right) \right|.$$ \hfill (6)

where $i, j = 0, 1, 0 \leq \tau < p^n$, and $1 \leq l, k \leq n$.

We have the following lemmas related to $\Delta_{\mathcal{A}_k}(i : \tau)$ and $\Delta_{\mathcal{A}_k}(i, j : \tau)$:

\textbf{Lemma 2.1.} 1. If $p \equiv 1$ (mod 4), then

$$\Delta_{\mathcal{A}_k}(i : \tau) = \begin{cases} 1, & \text{if } \tau \in D_j^k \text{ and } j = i \\ 0, & \text{otherwise} \end{cases}$$

2. If $p \equiv 3$ (mod 4), then

$$\Delta_{\mathcal{A}_k}(i, j : \tau) = \begin{cases} 1, & \text{if } \tau \in D_j^k \text{ and } j \neq i \\ 0, & \text{otherwise} \end{cases}$$

\textbf{Proof.} Proof for $\Delta_{\mathcal{A}_k}(0 : \tau)$ is already given in [23, Lemma 1]. Proof for $\Delta_{\mathcal{A}_k}(1 : \tau)$ is similar, so omitted. \hfill $\Box$

To compute $\Delta_{\mathcal{A}_k}(i, j : \tau)$, three cases $l < k, l = k$ and $l > k$, are distinguished.

\textbf{Lemma 2.2.} 1. $l < k$. In this case, $\Delta_{\mathcal{A}_k}(0, 0 : \tau) = \Delta_{\mathcal{A}_k}(1, 0 : \tau)$, and $\Delta_{\mathcal{A}_k}(0, 1 : \tau) = \Delta_{\mathcal{A}_k}(1, 1 : \tau)$.

a. $p \equiv 1$ (mod 4),

$$\Delta_{\mathcal{A}_k}(0, 0 : \tau) = \begin{cases} \frac{1}{4}(p^l - p^{l-1}), & \text{if } \tau \in D_0^k \\ 0, & \text{otherwise} \end{cases}$$

$$\Delta_{\mathcal{A}_k}(1, 1 : \tau) = \begin{cases} \frac{1}{4}(p^l - p^{l-1}), & \text{if } \tau \in D_1^k \\ 0, & \text{otherwise} \end{cases}$$

b. $p \equiv 3$ (mod 4),

$$\Delta_{\mathcal{A}_k}(0, 0 : \tau) = \begin{cases} \frac{1}{4}(p^l - p^{l-1}), & \text{if } \tau \in D_0^k \\ 0, & \text{otherwise} \end{cases}$$

$$\Delta_{\mathcal{A}_k}(1, 1 : \tau) = \begin{cases} \frac{1}{4}(p^l - p^{l-1}), & \text{if } \tau \in D_1^k \\ 0, & \text{otherwise} \end{cases}$$

2. $l = k$.

$$\Delta_{\mathcal{A}_k}(0, 0 : \tau) = \begin{cases} (0, 0)_{p^k}, & \text{if } \tau \in D_0^k \\ (1, 1)_{p^k}, & \text{if } \tau \in D_1^k \\ 0, & \text{otherwise} \end{cases}$$

$$\Delta_{\mathcal{A}_k}(0, u : \tau) = \begin{cases} \frac{1}{4}(p^l - p^{l-1}), & \text{if } \tau \in D_0^k \cup D_1^k \\ 0, & \text{otherwise} \end{cases}$$

and $u < k$.
3. Hamming Auto-correlation Function of the FHSs of Length $p^n$

**Theorem 3.1.** Let $X_i \in \mathcal{S}$ be an FHS generated by Construct. 2.1, then its Hamming auto-correlation function can be determined according to two cases:

1. If $p \equiv 1 \pmod{4}$, then $H(i : \tau) =$

   $$
   \begin{align*}
   &\frac{1}{2}(2p^n - p + 1), & \text{if } \tau \in D_1^{(k)} \\
   &\frac{1}{2}(2p^n - p - 3), & \text{if } \tau \in D_1^{(k)} \\
   &\frac{1}{2}(2p^n - p^k - 3p^{k-1}), & \text{if } \tau \in D_0^{(k)} \cup D_1^{(k)} \\
   \end{align*}
   $$

   and $2 \leq k \leq n$

2. If $p \equiv 3 \pmod{4}$, then $H(i : \tau) =$

   $$
   \begin{align*}
   &\frac{1}{2}(2p^n - p - 1), & \text{if } \tau \in D_0^{(k)} \cup D_1^{(k)} \\
   &\frac{1}{2}(2p^n - p^k - 3p^{k-1}), & \text{if } \tau \in D_0^{(k)} \cup D_1^{(k)} \\
   \end{align*}
   $$

   and $2 \leq k \leq n$

**Proof.** It is clear that the number of the FHSs of length $p^n$, constructed from Construct. 2.1, is equal to $2n$. Let $m = 2n$, and $X_i$ be an FHS where $0 \leq i \leq m - 1$. Then, the Hamming auto-correlation function of $X_i$, $H(i : \tau)$ with $1 \leq \tau < p^n$, can be computed as follows:

$$
H(i : \tau) = \sum_{i=0}^{n-1} |C_i \cap (C_i + \tau)|
= |C_0 \cap (C_0 + \tau)| + \sum_{i=1}^{n-1} |C_i \cap (C_i + \tau)|
= |D_0^{(1)} \cap (D_0^{(1)} + \tau)| + |D_1^{(1)} \cap \tau|
+ \sum_{k=1}^{n} |D_0^{(k)} \cap (D_0^{(k)} + \tau)|
+ \sum_{k=1}^{n} |D_1^{(k)} \cap (D_1^{(k)} + \tau)|
= |D_0^{(1)} \cap \tau| + \Delta_{1,1}(0 : \tau)
+ \sum_{k=1}^{n} \Delta_{k,1}(0 : \tau) + \sum_{k=1}^{n} \Delta_{k,1}(1 : \tau)
$$

(7)

By the formulae for the case $l = k$ in Lemma 2.2, it can be derived that

$$
\sum_{k=1}^{n} \Delta_{k,1}(0 : \tau) = \begin{cases} 
(0, 0)_{p^k} + \frac{1}{2}(p^n - p^k), & \text{if } \tau \in D_0^{(k)} \\
(1, 1)_{p^k} + \frac{1}{2}(p^n - p^k), & \text{if } \tau \in D_1^{(k)} 
\end{cases}
$$

(8)

and

$$
\sum_{k=1}^{n} \Delta_{k,1}(1 : \tau) = \begin{cases} 
(1, 1)_{p^k} + \frac{1}{2}(p^n - p^k), & \text{if } \tau \in D_0^{(k)} \\
(0, 0)_{p^k} + \frac{1}{2}(p^n - p^k), & \text{if } \tau \in D_1^{(k)} 
\end{cases}
$$

(9)

Taking account of the value of $\Delta_{1,1}(0 : \tau)$ from Lemma 2.1 and substituting (8)-(9) into the last equality of (7), we have

1. If $p \equiv 1 \pmod{4}$, then $H(i : \tau) =$

   $$
   \begin{align*}
   &2 + (0, 0)_{p^k} + (1, 1)_{p^k} + p^n - p, & \text{if } \tau \in D_0^{(k)} \\
   &(0, 0)_{p^k} + (1, 1)_{p^k} + p^n - p, & \text{if } \tau \in D_1^{(k)} \\
   \end{align*}
   $$

   and $2 \leq k \leq n$

(10)

2. If $p \equiv 3 \pmod{4}$, then $H(i : \tau) =$

   $$
   \begin{align*}
   &1 + (0, 0)_{p^k} + (1, 1)_{p^k} + p^n - p, & \text{if } \tau \in D_0^{(k)} \cap D_1^{(k)} \\
   &(0, 0)_{p^k} + (1, 1)_{p^k} + p^n - p, & \text{if } \tau \in D_0^{(k)} \cap D_1^{(k)} \\
   \end{align*}
   $$

   and $2 \leq k \leq n$

(11)

Now, by substituting the formulae of the generalized cyclochemical numbers given in (4)-(5) into (10)-(11) respectively,
we can establish the formulae in the actual Theorem. The proof is complete.

\[ \] 4. Hamming Cross-correlation Function of the FHSs of Length \( p^n \) for \( p \equiv 3 \pmod{4} \)

Throughout this section, suppose that \( p \) is an odd prime and \( p \equiv 3 \pmod{4} \). Let \( S \) be the FHS set constructed according to Construct. 2.1. \( X_i, X_j \in S \) be two distinct FHSs. Let \( \delta = (j - i) \pmod{m} \) where \( m = 2n \), \( \delta' = \frac{\delta}{2} \) if \( \delta \) even, and \( \delta' = \frac{\delta + 1}{2} \) if \( \delta \) odd. Without loss of generality, suppose \( j > i \). The Hamming cross-correlation function can be determined according to various cases of \( \delta ': \delta \) even, \( \delta \) odd and \( \delta' = 1 \), \( \delta \) odd and \( \delta' = n \), and \( \delta \) odd and \( 1 < \delta' < n \).

**Proposition 4.1.** Suppose that \( \delta \) is odd and \( 1 < \delta' < n \). Then,

1. For \( n \) even and \( 2\delta' = n \), \( H(i, i + \delta : \tau) = \)

\[
\begin{align*}
0, & \quad \text{if } \tau \in \{0\} \\
0, & \quad \text{if } \tau \in D_0^{(k)} \cup D_1^{(k)} \\
\frac{1}{2}(p + 1), & \quad \text{if } \tau \in D_0^{(k)} \\
0, & \quad \text{if } \tau \in D_0^{(k)} \\np, & \quad \text{if } \tau \in D_0^{(k+1)} \\
\frac{1}{2}p^{k-\delta' -2}(p^2 + 1)(p - 1), & \quad \text{if } \tau \in D_0^{(k)} \\
p^{k-\delta' -1}(p - 1), & \quad \text{if } \tau \in D_1^{(k)} \\
\end{align*}
\]

2. For \( n \) even and \( 2\delta' = n + 2 \), \( H(i, i + \delta : \tau) = \)

\[
\begin{align*}
0, & \quad \text{if } \tau \in \{0\} \\
0, & \quad \text{if } \tau \in D_0^{(k)} \cup D_1^{(k)} \\
0, & \quad \text{if } \tau \in D_0^{(k)} \\
\frac{1}{2}(p + 1), & \quad \text{if } \tau \in D_0^{(k+1)} \\
p, & \quad \text{if } \tau \in D_0^{(k)} \\
\frac{1}{2}p^{k-\delta' -2}(p^2 + 1)(p - 1), & \quad \text{if } \tau \in D_0^{(k)} \\
p^{k-\delta' -1}(p - 1), & \quad \text{if } \tau \in D_1^{(k)} \\
\end{align*}
\]

3. For \( n \) odd and \( 2\delta' = n + 1 \), \( H(i, i + \delta : \tau) = \)

\[
\begin{align*}
0, & \quad \text{if } \tau \in \{0\} \\
0, & \quad \text{if } \tau \in D_0^{(k)} \cup D_1^{(k)} \\
\frac{1}{2}(p + 1), & \quad \text{if } \tau \in D_0^{(k)} \\
\frac{1}{2}p^{k-\delta' -1}(p^2 + 1)(p - 1), & \quad \text{if } \tau \in D_1^{(k)} \\
\end{align*}
\]

4. For \( 2\delta' < n \), let \( \epsilon = n - \delta' + 1 \). \( H(i, i + \delta : \tau) = \)

\[
\begin{align*}
0, & \quad \text{if } \tau \in \{0\} \\
0, & \quad \text{if } \tau \in D_0^{(k)} \cup D_1^{(k)} \\
\frac{1}{2}(p + 1), & \quad \text{if } \tau \in D_0^{(k)} \\
\frac{1}{2}p^{k-\delta' -2}(p^2 + 1)(p - 1), & \quad \text{if } \tau \in D_0^{(k)} \\
p^{k-\delta' -1}(p - 1), & \quad \text{if } \tau \in D_1^{(k)} \\
\end{align*}
\]

5. For \( 2\delta' > n + 2 \), let \( \epsilon = n - \delta' + 1 \). \( H(i, i + \delta : \tau) = \)

\[
\begin{align*}
0, & \quad \text{if } \tau \in \{0\} \\
0, & \quad \text{if } \tau \in D_0^{(k)} \cup D_1^{(k)} \\
\frac{1}{2}(p + 1), & \quad \text{if } \tau \in D_0^{(k)} \\
p, & \quad \text{if } \tau \in D_0^{(k)} \\
\frac{1}{2}p^{k-\delta' -2}(p^2 + 1)(p - 1), & \quad \text{if } \tau \in D_0^{(k)} \\
p^{k-\delta' -1}(p - 1), & \quad \text{if } \tau \in D_1^{(k)} \\
\end{align*}
\]

**Proof.** From Construct. 2.1 and (3), it is clear that

\[
\begin{align*}
H(i, i + \delta : \tau) & = \sum_{k=0}^{k_1-1} |C_k \cap (C_{k+\delta} + \tau)| \\
& = \sum_{k=1}^{k_1} |C_{2(k-1)} \cap (C_{2(k-1)+\delta} + \tau)| \\
& \quad + \sum_{k=1}^{k_1} |C_{2(k-1)+1} \cap (C_{2(k-1)+1+\delta} + \tau)| \\
\end{align*}
\]

\[ \]
In (12), \( C_0 \) occurs two times at \( k = 1 \) and \( k = \frac{(m-(\delta + 1))}{2} + 1 = n - \delta + 1 \), respectively. The corresponding items in the summation are \(|C_0 \cap (C_0 + \tau)| = |(D_1^{(0)} \cup \{0\}) \cap (D_1^{(0)}) + \tau|\) and \(|C_{m-\delta} \cap (C_0 + \tau)| = |D_1^{(n-\delta' -1)} \cap ((D_1^{(0)}) + \tau) \cup \{0\})\), respectively. From above analysis and (12), we pursue

\[
H(i, i + \delta : \tau) = \sum_{k=1}^{n-\delta' + 1} \Delta_{k, k+\delta'}(0, 1 : \tau) + \sum_{k=1}^{n-\delta} \Delta_{k, k+\delta} + \sum_{k=1}^{n-\delta' + 1} \Delta_{k, k+\delta'}(1, 0 : \tau) + |D_1^{(n-\delta' + 1)} \cap \{\tau\}|
\]

(13)

In the last equation of (13), each summation can be split into two parts whose generic items, \( \Delta_{l, k}(i, j : \tau) \), correspond to cases \( l < k \) and \( l > k \), respectively. From above analysis and (13), we have

\[
H(i, i + \delta : \tau) = \sum_{k=1}^{n-\delta' + 1} \Delta_{k, k+\delta'-1}(0, 1 : \tau) + \sum_{k=1}^{n-\delta} \Delta_{n-\delta' + k+1, k}(0, 1 : \tau)
\]

(14)

By Lemma 2.2, each summation in (14) can be explicitly written down:

\[
\sum_{k=1}^{n-\delta' + 1} \Delta_{k, k+\delta'-1}(0, 1 : \tau) = \begin{cases} 
\frac{1}{2}(p^{k+\delta' -1} - p^{k-\delta'}), & \text{if } \tau \in D_1^{(k)} \\
0, & \text{otherwise}
\end{cases}
\]

(15)

\[
\sum_{k=1}^{n-\delta} \Delta_{n-\delta' + k+1, k}(0, 1 : \tau) = \begin{cases} 
\frac{1}{2}(p^{k+\delta'-1} - p^{k+\delta'-2}), & \text{if } \tau \in D_1^{(k)} \\
0, & \text{otherwise}
\end{cases}
\]

(16)

\[
\sum_{k=1}^{n-\delta'} \Delta_{k, k+\delta'}(1, 0 : \tau) = \begin{cases} 
\frac{1}{2}(p^{k-\delta'} - p^{k-\delta'-1}), & \text{if } \tau \in D_1^{(k)} \\
0, & \text{otherwise}
\end{cases}
\]

(17)

Consider the lower bounds of \( k' \)'s in (15)-(18). It leads to five possibilities: \( n \) even and \( 2\delta' = n \), which indicates the lower bound of (17) meets the one of (18); \( n \) even and \( 2\delta' = n + 2 \), which indicates the lower bound of (15) meets the one of (16); \( n \) odd and \( 2\delta' = n + 1 \), which indicates the lower bound of (15) meets the one of (18); \( 2\delta' < n \), and \( 2\delta' > n + 2 \). Further analysis of those cases together with help of Lemma 2.1 is straightforward, so omitted.

\[
\sum_{k=1}^{n-\delta'} \Delta_{n-\delta+1, k}(1, 0 : \tau) = \begin{cases} 
\frac{1}{2}(p^{k+\delta'-1} - p^{k+\delta'-n}), & \text{if } \tau \in D_1^{(k)} \\
0, & \text{otherwise}
\end{cases}
\]

(18)

**Proposition 4.2.** Suppose that \( \delta \) is odd. Then,

1. for \( \delta' = 1 \), \( H(i, i + 1 : \tau) = \)

\[
\begin{cases} 
0, & \text{if } \tau = 0 \\
\frac{1}{2}(p + 1), & \text{if } \tau \in D_0^{(1)} \cup D_0^{(1)} \\
\frac{1}{2}p^{k-1}(p - 3), & \text{if } \tau \in D_0^{(1)} \\
\frac{1}{2}p^{k-2}(p^2 + 3p - 2), & \text{if } \tau \in D_1^{(k)} \\
\frac{1}{2}(p^{n-2}(p^2 + 3p - 2) + 2p + 2), & \text{if } \tau \in D_1^{(n)}
\end{cases}
\]

(19)

2. For \( \delta' = n \), \( H(i, i + 2n - 1 : \tau) = \)

\[
\begin{cases} 
0, & \text{if } \tau = 0 \\
\frac{1}{2}(p + 1), & \text{if } \tau \in D_0^{(1)} \cup D_1^{(1)} \\
\frac{1}{2}p^{k-1}(p^2 + 3p - 2), & \text{if } \tau \in D_0^{(k)} \\
\frac{1}{2}p^{k-2}(p^2 + 3p - 2), & \text{if } \tau \in D_0^{(k)} \\
\frac{1}{2}(p^{n-2}(p^2 + 3p - 2) + 2p + 2), & \text{if } \tau \in D_1^{(n)}
\end{cases}
\]

(20)

**Proof.** From Construct. 2.1 and (3), we can obtain

\[
H(i, i + 1 : \tau) = \Delta_{i,1}(1 : \tau) + |D_1^{(n)} \cap \{\tau\}|
\]

(21)

(H(2) - H(3) - H(4) - H(5), respectively).

Further analysis is similar to the proof of Proposition 4.1, so omitted. \( \square \)
Proposition 4.3. Suppose that $\delta$ is even. Then,

1. For $n$ even and $2\delta' = n$, $H(i, i + \delta : \tau) =$

\[
\begin{cases}
0, & \text{if } \tau \in \{0\} \\
0, & \text{if } \tau \in D_0^{(k)} \cup D_1^{(k)} \text{ and } 1 \leq k < \delta' + 1 \\
p, & \text{if } \tau \in D_{\delta'}^{(k+1)} \cup D_1^{(k+1)} \\
p^k-\delta'-1(p-1), & \text{if } \tau \in D_0^{(k)} \cup D_1^{(k)} \text{ and } \delta' + 1 < k \leq n
\end{cases}
\]

2. For $2\delta' < n$, let $\epsilon = n - \delta' + 1$. Then, $H(i, i + \delta : \tau) =$

\[
\begin{cases}
0, & \text{if } \tau \in \{0\} \\
0, & \text{if } \tau \in D_0^{(k)} \cup D_1^{(k)} \text{ and } 1 \leq k < \delta' + 1 \\
\frac{1}{2}(p - 1), & \text{if } \tau \in D_0^{(k+1)} \text{ and } \delta' + 1 < k < \epsilon \\
\frac{1}{2}(p + 1), & \text{if } \tau \in D_1^{(k+1)} \\
p^{k-\delta'}(p-1), & \text{if } \tau \in D_0^{(k)} \cup D_1^{(k)} \text{ and } \epsilon < k \leq n
\end{cases}
\]

3. For $2\delta' > n$, let $\epsilon = n - \delta' + 1$. $H(i, i + \delta : \tau) =$

\[
\begin{cases}
0, & \text{if } \tau \in \{0\} \\
0, & \text{if } \tau \in D_0^{(k)} \cup D_1^{(k)} \text{ and } 1 \leq k < \epsilon \\
\frac{1}{2}(p + 1), & \text{if } \tau \in D_0^{(k+1)} \text{ and } \delta' + 1 < k < \epsilon \\
\frac{1}{2}(p - 1), & \text{if } \tau \in D_1^{(k+1)} \\
p^{k-\epsilon}(p-1), & \text{if } \tau \in D_0^{(k)} \cup D_1^{(k)} \text{ and } \epsilon < k \leq n
\end{cases}
\]

Proof. From Construct. 2.1 and (3), we can obtain

\[
H(i, i + \delta : \tau) = \Delta_{\delta'+1}(0 : \tau) + \left| D_0^{(n-\delta'+1)} \cap \{\tau\} \right|
\]

\[
+ \sum_{k=1}^{n} \Delta_{\delta',k}(0, 0 : \tau) + \sum_{k=1}^{n} \Delta_{\delta',k+1}(1, 1 : \tau).
\]

Further analysis is similar to the proof of Proposition 4.1, so omitted.

The Hamming cross-correlation function of six FHSs of length $p^3$ can be derived from Construct. 2.1 and Proposition 4.1-4.3:

Corollary 4.1. Let $X_1, X_j$ be two distinct FHSs of length $p^3$ constructed according to Construct. 2.1, then the Hamming cross-correlation function between $X_i$ and $X_j$, $H(i, j : \tau) = H(i, i + (j - i) \mod 6 : \tau)$, is given by the following equations:

1. Let $\delta = (j-i) \mod 6 = 1$, then $H(i, i + 1 : \tau) =$

\[
\begin{cases}
0, & \text{if } \tau = 0 \\
\frac{1}{2}(p + 1), & \text{if } \tau \in D_0^{(1)} \cup D_1^{(1)} \\
\frac{1}{2}(p - 3), & \text{if } \tau \in D_1^{(1)} \\
\frac{1}{2}(p^2 + 3p - 2), & \text{if } \tau \in D_0^{(2)} \\
\frac{1}{2}(p^2 - 3), & \text{if } \tau \in D_1^{(2)} \\
\frac{1}{2}(p^3 + 3p^2 + 2), & \text{if } \tau \in D_0^{(3)} \\
\frac{1}{2}(p^3 - 3), & \text{if } \tau \in D_1^{(3)}
\end{cases}
\]

2. Let $\delta = (j-i) \mod 6 = 5$, then $H(i, i + 5 : \tau) =$

\[
\begin{cases}
0, & \text{if } \tau = 0 \\
\frac{1}{2}(p + 1), & \text{if } \tau \in D_0^{(1)} \cup D_1^{(1)} \\
\frac{1}{2}(p^2 + 3p - 2), & \text{if } \tau \in D_0^{(2)} \\
\frac{1}{2}(p - 3), & \text{if } \tau \in D_1^{(2)} \\
\frac{1}{2}(p^2 + 3p^2 + 2), & \text{if } \tau \in D_0^{(3)} \\
\frac{1}{2}(p^3 - 3), & \text{if } \tau \in D_1^{(3)}
\end{cases}
\]

3. Let $\delta = (j-i) \mod 6 = 3$, then $H(i, i + 3 : \tau) =$

\[
\begin{cases}
0, & \text{if } \tau = 0 \\
0, & \text{if } \tau \in D_0^{(1)} \cup D_1^{(1)} \\
\frac{1}{2}(p + 1), & \text{if } \tau \in D_0^{(2)} \cup D_1^{(2)} \\
\frac{1}{2}(p - 1)(p - 1), & \text{if } \tau \in D_0^{(3)} \cup D_1^{(3)}
\end{cases}
\]

4. Let $\delta = (j-i) \mod 6 = 2$, then $H(i, i + 2 : \tau) =$

\[
\begin{cases}
0, & \text{if } \tau = 0 \\
0, & \text{if } \tau \in D_0^{(1)} \cup D_1^{(1)} \\
\frac{1}{2}(p - 1), & \text{if } \tau \in D_0^{(2)} \\
\frac{1}{2}(p + 1), & \text{if } \tau \in D_1^{(2)} \\
\frac{1}{2}(p^2 + 1), & \text{if } \tau \in D_0^{(3)} \\
\frac{1}{2}(p + 1)(p - 1), & \text{if } \tau \in D_1^{(3)}
\end{cases}
\]

5. Let $\delta = (j-i) \mod 6 = 4$, then $H(i, i + 4 : \tau) =$

\[
\begin{cases}
0, & \text{if } \tau = 0 \\
0, & \text{if } \tau \in D_0^{(1)} \cup D_1^{(1)} \\
\frac{1}{2}(p + 1), & \text{if } \tau \in D_0^{(2)} \\
\frac{1}{2}(p - 1), & \text{if } \tau \in D_1^{(2)} \\
\frac{1}{2}(p + 1)(p - 1), & \text{if } \tau \in D_0^{(3)} \\
\frac{1}{2}(p^2 + 1), & \text{if } \tau \in D_1^{(3)}
\end{cases}
\]

Proof. For each case of $\delta = (j-i) \mod 6$, substitute $n = 3$
in the corresponding Proposition from 4.1-4.3, and take care of the interval of $\tau$, that ranges from 0 to $p^3-1$.

**Theorem 4.1.** Let $X_i, X_j \in S$ be two FHSs generated by Construct. 2.1 with $i \neq j$. Then, their Hamming cross-correlation function is uniquely determined by the formulas given by Proposition 4.1-4.3.

**Proof.** Let $S$ be the FHS set constructed according to Construct. 2.1, $X_i, X_j \in S$ be two distinct FHSs. Let $\delta = (j - i) \pmod{m}$ where $m = 2n$, $\delta' = \frac{\delta}{2}$ if $\delta$ even, and $\delta' = \frac{\delta + 1}{2}$ if $\delta$ odd. It is clear that Proposition 4.1-4.3 cover all the possible cases $\delta$ may take.

**5. Conclusion**

In this paper, a new class of the frequency-hopping sequences (FHSs) of length $p^n$ is constructed based on Ding-Helleseth generalized cyclotomic classes of order two, of which the Hamming auto- and cross-correlation functions are established (for the Hamming cross-correlation, only the case $p \equiv 3 \pmod{4}$ is considered). It is shown that the constructed FHSs’ set is uniformly distributed, and optimal with respect to the average Hamming correlation functions.

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