To my wife

On Cyclotomic Polynomials

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Abstract. In this note, we present a more general proof that cyclotomic polynomials are irreducible over \( \mathbb{Q} \) and other number fields that meet certain conditions. The proof provides a new perspective that ties together well-known results, as well as some new consequences.

A. INTRODUCTION. For any integer \( n > 0 \), we define the \( n \)-th cyclotomic polynomial \( \Phi_n(X) \) as \( \prod (X - \zeta) \), with the product running over all primitive \( n \)-th roots of unity \( \zeta \). It is easy to see that \( \Phi_n(X) \) is a monic polynomial in \( \mathbb{Z}[X] \), because its coefficients are rational numbers that are integral over \( \mathbb{Z} \).

It is also well-known that \( \Phi_n(X) \) is irreducible over \( \mathbb{Q} \). That fact was first proven by Gauss for the case when \( n \) is prime. Many other distinguished mathematicians have also come up with proofs for the irreducibility of \( \Phi_n(X) \) over \( \mathbb{Q} \), including Schönemann and Eisenstein (for \( n \) prime), Kronecker, Dedekind, Landau, Schur, and van der Waerden (for general \( n \)). Dedekind himself gave three proofs for the irreducibility of \( \Phi_n(X) \) over \( \mathbb{Q} \), suggesting that Dedekind regarded the irreducibility of cyclotomic polynomials to be important. In fact, there is a connection that can be drawn between the irreducibility of cyclotomic polynomials and the Artin reciprocity homomorphism at the foundation of class field theory.

In this note, we provide a more general proof for the irreducibility of \( \Phi_n(X) \) over \( \mathbb{Q} \) and other number fields that meet certain conditions. The proof provides a new

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perspective that ties together well-known results, as well as some new consequences, including a necessary condition for the algebraic solution by radicals of certain irreducible polynomials.

B. STATEMENT AND PROOF. Let $K$ be a finite extension of $\mathbb{Q}$ (a number field) with ring of integers $A$. The ring $A$ is a Dedekind domain and any integral ideal $pA$ generated by a rational prime $p$ has a unique prime ideal factorization $\prod m_i$ (with possible repeated factors), where each $m_i$ is a maximal ideal in $A$ with $m_i \cap \mathbb{Z} = p\mathbb{Z}$. The residue field $A/m_i$ is an extension of the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and the degree of that residue extension is known as the inertial degree or residue degree of $m_i$ over the prime $p$ or over $\mathbb{Q}$.

Of particular interest to us are the rational primes $p$ such that one of the maximal ideals $m_i$ in the prime ideal factorization of $pA$ has residue degree one over $p$, that is to say the residue field $A/m_i$ is isomorphic to $\mathbb{F}_p$. These rational primes do not have a particular name in the literature, but for convenient reference we will refer to them in this note as primes that are semi-split in $K$. (Rational primes $p$ such that all maximal ideals in the factorization of $pA$ have residue degree one are said to split totally or completely in $K$)

If $K = \mathbb{Q}(\alpha)$ with an integral element $\alpha$ whose minimal polynomial is $f(X) \in \mathbb{Z}[X]$, then with a finite number of exceptions, the rational primes $p$ semi-split in $K$ are precisely the rational primes $p$ such that $f(X) \mod p$ has a root in $\mathbb{Z}/p\mathbb{Z}$, i.e., the prime factors $p$ of the numbers $f(k)$ as $k$ runs through $\mathbb{Z}$.

**Theorem 1:** Let $K$ be a number field. Suppose that for each residue class $a \mod n$ in a generating set for the group $(\mathbb{Z}/n\mathbb{Z})^*$ of invertible residue classes modulo $n$, we can find a rational prime $p$ such that $p \equiv a \pmod{n}$ and $p$ is semi-split in $K$. Then the $n^{th}$ cyclotomic polynomial $\Phi_n(X)$ is irreducible over $K$.

**Proof.** Note that each irreducible polynomial in $K[X]$ corresponds to an orbit in a given separable closure of $K$ under the action of the absolute Galois group of $K$. All elements in that orbit are the roots of the irreducible polynomial.
If $\Phi_n(X)$ is not irreducible over $K$, then there must be two different roots of $\Phi_n(X)$, say $u$ and $v$, that belong to two distinct Galois orbits over $K$. The integral elements $u$ and $v$ must then have distinct minimal polynomials $g$ and $h$ over $K$ whose product divides $\Phi_n(X)$ in $A[X]$.

Let $B$ be the ring of integers in the extension $K(u, v)$ of $K$. For each maximal ideal $m$ of $A$ such that $\Phi_n(X) \mod m$ is separable, consider any maximal ideal $\mathfrak{m}$ of $B$ in the prime ideal factorization of $mB$. The images $u \mod \mathfrak{m}$ and $v \mod \mathfrak{m}$ in the residue field $B/\mathfrak{m}$ must belong to different Galois orbits over $A/m$, because they are the roots of polynomials $g \mod m$ and $h \mod m$ whose product divides the separable polynomial $\Phi_n(X) \mod m$.

Because both $u$ and $v$ are primitive $n^{th}$ roots of unity, there is an invertible residue class $a \mod n$ such that $u^a = v$. For our purpose, we can assume that $a \mod n$ belongs to the given generating set for the group $(\mathbb{Z}/n\mathbb{Z})^*$. Indeed, if both $u$ and $u^a$ belong to the same Galois orbit over $K$ regardless of which primitive $n^{th}$ root of unity $u$ we choose and which residue class $a \mod n$ in the generating set we choose, then all the primitive $n^{th}$ roots of unity must be in the same Galois orbit over $K$, contrary to our initial assumption.

By hypothesis, we can find a rational prime $p$ that is semi-split in $K$ and congruent to $a \mod n$. This rational prime $p$ does not divide $n$, and hence the polynomial $X^n - 1$ is separable over any field of characteristic $p$. In particular, for each maximal ideal $m$ in $A$ such that $m \cap \mathbb{Z} = p\mathbb{Z}$, the polynomial $X^n - 1$ is separable over the residue field $A/m$. That means $\Phi_n(X)$ is also separable over $A/m$. In light of our foregoing discussion, if $\mathfrak{m}$ is any maximal ideal of $B$ in the prime ideal factorization of $mB$, then the elements $u \mod \mathfrak{m}$ and $v \mod \mathfrak{m}$ will be in distinct Galois orbits over $A/m$.

The congruence condition on $p$ means that $v$ is equal to the $p^{th}$ power of $u$, so that the element $v \mod \mathfrak{m}$ is the transform of $u \mod \mathfrak{m}$ under the Frobenius automorphism which raises each element in a field of characteristic $p$ to the $p^{th}$ power. Note that each element algebraic over $\mathbb{F}_p$ and its $p^{th}$ power must belong to the same Galois orbit over $\mathbb{F}_p$, because the Frobenius automorphism is a generator of any finite Galois group over $\mathbb{F}_p$. 
By the semi-split condition for \( p \), we can choose a maximal ideal \( m \) of \( A \) sitting over \( p \) such that \( A/m \) is isomorphic to \( \mathbb{F}_p \). That means \( u \mod \varphi \) and \( v \mod \varphi \) must belong to the same Galois orbit over \( A/m \). However, we saw earlier that \( u \mod \varphi \) and \( v \mod \varphi \) must be in different Galois orbits over \( A/m \), in part because \( u \) and \( v \) are assumed to be in different Galois orbits over \( K \). This contradiction shows that all the roots of \( \Phi_d(X) \) must be in one Galois orbit over \( K \). Therefore \( \Phi_d(X) \) must be irreducible over \( K \). □

C. SOME CLASSICAL RESULTS. What are the number fields \( K \) that satisfy the conditions of the theorem above? Certainly \( \mathbb{Q} \) would do. The semi-split condition is automatic for any rational prime when \( K = \mathbb{Q} \), and the Dirichlet’s theorem on primes in arithmetic progressions tells us that there are infinitely many primes in each invertible residue class \( a \mod n \). That is more than what we need for Theorem 1, which requires at most only one semi-split rational prime for each invertible residue class modulo \( n \). Accordingly, \( \Phi_d(X) \) is irreducible over \( \mathbb{Q} \). Alternatively, we can also do without the Dirichlet’s theorem by observing that the rational primes that do not divide \( n \) give us a generating set of invertible residue classes modulo \( n \).

For any integer \( m \) relatively prime to \( n \), i.e., \( \gcd(m, n) = 1 \), the cyclotomic extension \( \mathbb{Q}(\zeta_m) \), where \( \zeta_m = \exp(2\pi i/m) \), would also meet the conditions of the theorem. The rational primes \( p \equiv 1 \mod m \) are semi-split in \( \mathbb{Q}(\zeta_m) \) because the residue field \( \mathbb{Z}/p\mathbb{Z} \) contains \( m \)th roots of unity. The Chinese remainder theorem and Dirichlet’s theorem combine to tell us that there are always rational primes \( p \) (indeed infinitely many) that satisfy both the congruence \( p \equiv 1 \mod m \) and the congruence \( p \equiv a \mod n \) for each invertible residue class \( a \mod n \). Each such rational prime would be semi-split in \( \mathbb{Q}(\zeta_m) \) and would also belong to the residue class \( a \mod n \). Accordingly, the \( n \)th cyclotomic polynomial \( \Phi_n(X) \) is irreducible over \( \mathbb{Q}(\zeta_m) \).

As an example, that means the 8th-cyclotomic polynomial \( \Phi_8(X) = X^4 + 1 \) is irreducible over any cyclotomic extension \( \mathbb{Q}(\zeta_m) \) for odd integers \( m \).

If \( \Phi_n(X) \) is irreducible over a number field \( K \) and \( u \) is a root of \( \Phi_n(X) \), then \( K(u) \) is a Galois extension of \( K \) whose degree equal to the degree of \( \Phi_n(X) \). For any extension \( L \) of \( K \),
it is a straightforward consequence of Galois theory that $\Phi_n(X)$ is irreducible over $L$ if and only if $[L(u): L] = [K(u): K]$ or $L$ and $K(u)$ are linearly disjoint over $K$, which is the case if and only if $K(u) \cap L = K$ because $K(u)$ is a Galois extension of $K$.

In particular, because $\Phi_n(X)$ is irreducible over $\mathbb{Q}$ (based on the foregoing or other proof), it follows that $\Phi_n(X)$ is irreducible over $\mathbb{Q}(\zeta_m)$ if and only if $\mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$. Because we already established that $\Phi_n(X)$ is irreducible over $\mathbb{Q}(\zeta_m)$ when $m$ and $n$ are relatively prime integers, it follows as a consequence that $\mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$. Conversely, if we can show by means of ramification theory or other results that $\mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$ when $m$ and $n$ are relatively prime, then that also implies $\Phi_n(X)$ is irreducible over $\mathbb{Q}(\zeta_m)$.

**Proposition 2.** If $n$ is an odd integer, then any quadratic field $\mathbb{Q}(\sqrt{m})$, where $m$ is a square-free integer relatively prime to $n$, also meets the conditions of Theorem 1.

**Proof.** We want to show that for each invertible residue class $a$ modulo $n$, we can find at least a rational prime $p$ such that $p \equiv a \pmod{n}$ and $p$ is semi-split in $\mathbb{Q}(\sqrt{m})$. Because the quadratic field $\mathbb{Q}(\sqrt{m})$ is a Galois extension of $\mathbb{Q}$, $p$ is semi-split in $\mathbb{Q}(\sqrt{m})$ if and only if the ideal generated by $p$ in the ring of integers of $\mathbb{Q}(\sqrt{m})$ is the product of two maximal ideals (which may be the same) of residue degree one over $\mathbb{Q}$. In other words, $p$ must either be ramified, or split completely in $\mathbb{Q}(\sqrt{m})$.

With the possible exception of $p = 2$ when $m \equiv 1 \pmod{4}$, all the rational primes that are semi-split in $\mathbb{Q}(\sqrt{m})$ are precisely the primes $p$ such that $m$ is a quadratic residue mod $p$. Also, we know that there is a primitive Dirichlet character $\chi$ taking values 1 or $-1$, such that $\chi(p) = 1$ for an odd prime number $p$ exactly when $m$ is a quadratic residue modulo $p$. The conductor $d$ of this Dirichlet character $\chi$ is equal to $|m|$ if $m \equiv 1 \pmod{4}$, and to $4|m|$ if $m \equiv 2$ or $3 \pmod{4}$. See, e.g., [1] at Theorem 3.7 of chapter 1. (This result is essentially equivalent to the law of quadratic reciprocity.)

Accordingly, whether or not an odd prime number $p$ is semi-split in $\mathbb{Q}(\sqrt{m})$ depends entirely on the residue class of $p$ modulo the conductor $d$. Because the conductor $d$ is relatively prime to $n$, we can always find by the Chinese remainder theorem an invertible
residue class \( \ell \) modulo \( nd \) such that \( \ell \equiv a \ (mod \ n) \) where \( a \) is any invertible residue class modulo \( n \), and \( \ell \equiv b \ (mod \ d) \) where \( b \) is an invertible residue class modulo \( d \) with \( \chi(b) = 1 \). Dirichlet’s theorem tells us that there are always rational primes in such an invertible residue class \( \ell \) modulo \( nd \). Any rational prime \( p \) in that residue class would satisfy our conditions. ■

In light of Proposition 2, the \( n^{th} \) cyclotomic polynomial \( \Phi_n(X) \) is irreducible over any quadratic field \( \mathbb{Q}(\sqrt{m}) \) if \( n \) is odd and \( m \) is relatively prime to \( n \). In that case, we must have \( \mathbb{Q}(\sqrt{m}) \cap \mathbb{Q}(\zeta_n) = \mathbb{Q} \). Alternatively, by using ramification theory, we can show that when \( n \) is odd and \( m \) is relatively prime to \( n \), we must have \( \mathbb{Q}(\sqrt{m}) \cap \mathbb{Q}(\zeta_n) = \mathbb{Q} \), which would imply that the \( n^{th} \) cyclotomic polynomial \( \Phi_n(X) \) is irreducible over \( \mathbb{Q}(\sqrt{m}) \).

**Proposition 3.** (a) Let \( p \) be a prime number \( \equiv 3 \ (mod \ 4) \). The \( p^{th} \) cyclotomic polynomial is irreducible over the quadratic field \( \mathbb{Q}(\sqrt{p}) \), i.e., the quadratic field \( \mathbb{Q}(\sqrt{p}) \) is linearly disjoint from the cyclotomic extension \( \mathbb{Q}(\zeta_p) \).

(b) Let \( p \) be a prime number \( \equiv 1 \ (mod \ 4) \). The \( p^{th} \) cyclotomic polynomial is irreducible over the quadratic field \( \mathbb{Q}(\sqrt{-p}) \), i.e., the quadratic field \( \mathbb{Q}(\sqrt{-p}) \) is linearly disjoint from the cyclotomic extension \( \mathbb{Q}(\zeta_p) \).

**Proof.** Nowadays, this proposition can readily be proved using discriminants. However, as an illustration of Theorem 1, we will show how the law of quadratic reciprocity implies this proposition.

(a) We will show that if \( p \) is a prime number \( \equiv 3 \ (mod \ 4) \), then the quadratic field \( \mathbb{Q}(\sqrt{p}) \) meets the conditions of Theorem 1 with respect to the \( p^{th} \) cyclotomic polynomial \( \Phi_p(X) \). The rational primes that are semi-split in \( \mathbb{Q}(\sqrt{p}) \) are 2, \( p \) and the odd prime numbers \( q \) such that \( p \) is a quadratic residue modulo \( q \). By the law of quadratic reciprocity and the fact that \( p \) is a prime number \( \equiv 3 \ (mod \ 4) \), we have

\[
(p \mid q) = (-1)^{(q - 1)/2} (q \mid p),
\]

where we write \((p \mid q)\) and \((q \mid p)\) for the Legendre symbols of quadratic residues.
For each invertible residue class $a \mod p$, the Chinese remainder theorem and Dirichlet’s theorem together assure us that we can find (infinitely many) primes $q$ such that:

- $q \equiv 1 \pmod{4}$ and $q \equiv a \pmod{p}$ if $a \mod p$ is a quadratic residue.
- $q \equiv 3 \pmod{4}$ and $q \equiv a \pmod{p}$ if $a \mod p$ is a quadratic non-residue.

Each such prime $q$ would give us $(p \mid q) = 1$, i.e. $p$ is a quadratic residue modulo $q$. All these primes $q$ would be semi-split in $Q(\sqrt{p})$. Accordingly, there are semi-split primes $q$ in each invertible residue class $a \mod p$, and the quadratic field $Q(\sqrt{p})$ meets the conditions of Theorem 1.

(b) If $p$ is a prime number $\equiv 1 \pmod{4}$, then $-p \equiv 3 \pmod{4}$. The rational primes that are semi-split in $Q(\sqrt{-p})$ are 2, $p$ and the odd prime numbers $q$ such that $-p$ is a quadratic residue modulo $q$, i.e., $(-p \mid q) = 1$.

By the law of quadratic reciprocity and the fact that $p$ is a prime number $\equiv 1 \pmod{4}$, we have $(-p \mid q) = (-1)^{(q-1)/2} (p \mid q) = (-1)^{(q-1)/2} (q \mid p)$.

As in the proof for (a), we can find prime numbers $q$ in each invertible residue class $a \mod p$ such that $(-p \mid q) = (-1)^{(q-1)/2} (q \mid p) = 1$. These prime numbers would be semi-split in $Q(\sqrt{-p})$. The quadratic field $Q(\sqrt{-p})$ meets the conditions of Theorem 1 with respect to the $p^{th}$ cyclotomic polynomial $\Phi_p(X)$, and therefore is linearly disjoint from the cyclotomic extension $Q(\zeta_p)$. ■

C. SOME OTHER CONSEQUENCES. From Theorem 1 and the foregoing discussion, we have the following basic result, which we summarize in the form of a proposition for convenient reference.

**Proposition 4.** If a number field $K$ meets the conditions of Theorem 1, then $K$ is linearly disjoint from $Q(\zeta_n)$, i.e., $K \cap Q(\zeta_n) = Q$. If a number field $K$ is not linearly disjoint from $Q(\zeta_n)$, i.e., $K \cap Q(\zeta_n) \neq Q$, then $\Phi_n(X)$ is reducible over $K$ and $K$ must fail the conditions of Theorem 1.
Using the basic result of Proposition 4, we can obtain a necessary condition for a number field $K$ to contain a nontrivial solvable extension of $\mathbb{Q}$.

**Theorem 5.** Let $K$ be a number field. For $K$ to contain a nontrivial solvable extension of $\mathbb{Q}$, it is necessary that we can find an integer $n$ and an invertible residue class $a \mod n$ that contains no rational prime number semi-split in $K$.

**Proof.** By the definition of solvable extensions, it follows that $K$ contains a nontrivial solvable extension of $\mathbb{Q}$ if and only if $K$ contains a nontrivial abelian extension of $\mathbb{Q}$. According to the Kronecker-Weber theorem, any abelian extension of $\mathbb{Q}$ is contained in some cyclotomic extension $\mathbb{Q}(\zeta_n)$. So $K$ contains a nontrivial abelian extension of $\mathbb{Q}$ if and only if $K \cap \mathbb{Q}(\zeta_n) \neq \mathbb{Q}$ for some $n$. As discussed above, that is equivalent to saying that some $\Phi_n(X)$ is reducible over $K$. In light of Theorem 1, for $\Phi_n(X)$ to be reducible over $K$, it is necessary that the number field $K$ does not meet the condition of Theorem 1 for $n$.

If a number field $K$ fails the condition of Theorem 1 for $n$, there must be an invertible residue class $a \mod n$ that contains no rational prime number semi-split in $K$. ■

**Proposition 6.** Let $K$ be a number field. If all but a finite number of rational primes in $\mathbb{Q}$ are semi-split in $K$, then the only solvable extension of $\mathbb{Q}$ contained in $K$ is $\mathbb{Q}$ itself.

**Proof.** Because each invertible residue class $a \mod n$ contains infinitely many prime numbers, the hypothesis of this proposition implies that there are infinitely many semi-split prime $p$ in each such residue class. In light of Theorem 5, $K$ cannot contain a nontrivial solvable extension of $\mathbb{Q}$. ■

The algebraic solution by radicals of polynomial equations of degree $> 4$ is not possible in general. The foregoing discussion in our note gives us the following necessary condition:

**Proposition 7.** Let $K = \mathbb{Q}(\alpha)$ be a Galois extension of $\mathbb{Q}$, with $\alpha$ an algebraic integer. If $f(X)$ is the minimal polynomial of $\alpha$, then a necessary condition for $f(X)$ to be solvable by
radicals is that we can find an integer $n$ and an invertible residue class $a \mod n$ that does not contain infinitely many prime factors $p$ of the numbers $f(k)$ as $k$ runs through $\mathbb{Z}$.

**Proof.** We know from Galois theory that in this situation, $f(X)$ is solvable by radicals if and only if the Galois extension $K = \mathbb{Q}(\alpha)$ is a solvable extension of $\mathbb{Q}$. By Theorem 5 above, for $K$ to be a solvable extension of $\mathbb{Q}$ it is necessary that we can find an integer $n$ and an invertible residue class $a \mod n$ that contains no rational prime number semi-split in $K$.

Because the algebraic integer $\alpha$ is a primitive element of the extension $K$, we know that except for the finitely many rational primes that divide the index of $\mathbb{Z}[[\alpha]]$ in the ring of integers of $K$, a rational prime $p$ is semi-split in $K$ if and only if $f(X) \mod p$ has a linear factor, i.e., $f(X) \mod p$ has a root in $\mathbb{Z}/p\mathbb{Z}$. Those prime numbers are prime factors of the numbers $f(k)$ as $k$ runs through $\mathbb{Z}$.

Accordingly, for $K = \mathbb{Q}(\alpha)$ to be a solvable extension of $\mathbb{Q}$, it is necessary that we can find an integer $n$ and an invertible residue class $a \mod n$ that does not contain infinitely many prime factors $p$ of the numbers $f(k)$ as $k$ runs through $\mathbb{Z}$. ■

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**REFERENCES**

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