Two-dimensional intra-band solitons in lattice potentials with local defects and self-focusing nonlinearity

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It is commonly known that stable bright solitons in periodic potentials, which represent gratings in photonics/plasmonics, or optical lattices in quantum gases, exist either in the spectral semi-infinite gap (SIG) or in finite bandgaps. Using numerical methods, we demonstrate that, under the action of the cubic self-focusing nonlinearity, defects in the form of “holes” in two-dimensional (2D) lattices support continuous families of 2D solitons embedded into the first two Bloch bands of the respective linear spectrum, where solitons normally do not exist. The two families of the embedded defect solitons (EDSs) are found to be continuously linked by the branch of gap defect solitons (GDSs) populating the first finite bandgap. Further, the EDS branch traversing the first band links the GDS family with the branch of regular defect-supported solitons populating the SIG. Thus, we construct a continuous chain of regular, embedded, and gap-mode solitons (“superfamily”) threading the entire bandgap structure considered here. The EDSs are stable in the first Bloch band, and partly stable in the second. They exist with the norm exceeding a minimum value, hence they do not originate from linear defect modes. Further, we demonstrate that double, triple and quadruple lattice defects support stable dipole-mode solitons and vortices.

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I. INTRODUCTION

A powerful toolbox for controlling the dynamics of photonic, plasmonic, and atomic waves is based on the use of periodic potentials, which can be induced by photonic lattices and crystals in optics [1], by gratings built into plasmonic waveguides (see original works [2–13] and review [14]), and by optical lattices (OLs) in Bose-Einstein condensates (BECs) [15–17] or degenerate Fermi gases [18]. Acting in the combination with material nonlinearity, periodic potentials give rise to various species of solitons. These include ordinary ones, existing in the semi-infinite gap (SIG) of the corresponding linear spectrum, and gap solitons populating finite bandgaps, see reviews [16, 17, 19, 20] and book [21]. Extended localized states, in the form of truncated nonlinear Bloch waves, which appear in finite bandgaps, have been observed and modeled too [22–24].

On the other hand, a specific species of solitary waves is known in the form of embedded solitons, which exist, counter-intuitively, inside spectral Bloch bands populated by radiation modes [25–39]. Although embedded solitons cannot exist, generically, in continuous families, due to the resonance with radiation waves, isolated embedded modes are possible, for which the rate of the decay into radiation vanishes. Under more specific conditions, continuous families of embedded solitons were constructed too [34–37]. Very recently, the existence of solitons of this type was demonstrated in a model combining the quadratic \(\chi^{(2)}\) nonlinearity and a complex lattice potential, whose imaginary part is subject to the condition of the \(PT\) symmetry, representing symmetrically placed and mutually balanced local gain and loss [38]. While embedded solitons have been studied in some detail in basic one-dimensional (1D) models, it is still an open question to find them in Bloch bands of 2D periodic potentials (stable intra-band solitons were reported in a 2D potential which combines a periodic lattice in one direction and a harmonic-oscillator trap acting perpendicular to the lattice [39]).

In addition to the studies of ideal lattices, it has been demonstrated that lattice defects may also be used in optical devices, such as integrated circuits [40–42], microcavities [43], and defect-mode lasers [44]. Point and line defects help trapping and guiding light flows in structured photonic media [45–53]. In particular, the bandgap guidance of light by defects in 1D and 2D photonic lattices has been realized in Refs. [54–56]. Defect-affected transmission of light was also considered in 2D photonic quasicrystals [57, 58]. The dynamics of solitons under the action of defects in discrete nonlinear-Schrödinger lattices has been studied too [59–64]. The dynamics of BEC trapped in OL potentials may also be strongly affected by defects [65–72].

The above-mentioned settings are based on the cubic nonlinearity. Solitons supported by local defects in the form of the localized \(\chi^{(2)}\) nonlinearity have been studied in Refs. [73–75]. Very recently, solitons pinned to local PT-symmetric defects, inserted into the 1D medium with the uniform \(\chi^{(2)}\) nonlinearity, were considered too [76]. On the other hand, a specific manifestation of the transmission of light [77–84] and matter waves [85–88] controlled by random sets of defects is the Anderson localization in the transverse plane, which does not require
the existence of linear defect modes in photonic crystals suggested by previously published results demonstrating
lated in Section II. The search for such soliton modes is
formed for the local control of local nonlinearity in BEC (via the Feshbach resonance \cite{92, 93}), and in photonic
ystals \cite{94, 95}, which allows one to design desirable nonlinear defects.

In this work we report numerical analysis of 2D solitons in lattice potentials with local defects, under the action
of the self-focusing nonlinearity. The model is formulated in Section II. The search for such soliton modes is
suggested by previously published results demonstrating the existence of linear defect modes in photonic crystals
\cite{40-53}, including 2D modes \cite{96}, as well as in ultrasonic crystals \cite{97} and solid-state lattices (see Refs. \cite{98-101}
and references therein), although the solitons that we present below are not actually related to linear defect
modes. In Section III, we report numerical results which demonstrate the existence of a new type of embedded solitons, alias intraband solitons, viz., continuous families of 2D localized modes (unlike isolated solutions found previously in typical embedded-soliton models), which are embedded into the first and second Bloch bands of the linear spectrum, in terms of the propagation constant, and pinned to the defect, as concerns the spatial location of the modes. We name them embedded defect solitons (EDSs). It is relevant to stress that, for modes pinned to local lattice defects, it makes sense to identify the location of their propagation constant with respect to the bandgap spectrum of the infinite uniform OL, as the latter determines the spectrum of radiation waves into which the localized mode may decay.

As mentioned above, the EDSs do not emerge from a continuation of linear localized defect modes, as they exist with the norm exceeding a finite minimum value (and the propagation constant of linear defect modes cannot be located inside a Bloch band). By means of systematic direct simulations, the EDSs are found to be fully stable in the first band and in a part of the second. Together with the regular defect solitons populating the SIG, and gap defect solitons (GDSs) nested in the first finite bandgap (in the spectral slot between the first and second Bloch bands), which are pinned to the same defect, the intra-band EDSs form a continuous “superfamily” threading the entire band structure. The stability of the GDSs supported by the self-attractive nonlinearity is a remarkable finding, as the previous analysis of gap solitons in defect-free 2D lattices was usually restricted to the case of the self-repulsion, assuming that they would be unstable in the opposite case \cite{19, 102, 103} (gap solitons bifurcating under the action of the self-attractive nonlinearity from edges of the adjacent Bloch bands into the first finite bandgap were constructed in Ref. \cite{104}, but their stability was not studied there; in fact, it was recently found that, while 2D fundamental gap solitons are completely unstable in the first finite bandgap under the self-attractive nonlinearity, a family of dipole-mode gap solitons are stable in a part of the first bandgap in the same case, provided that the depth of the lattice potential exceeds a certain threshold value \cite{105}).

Further, in Section IV we demonstrate the existence of stable EDS and GDS modes with the dipole structure, in the lattice with double and triple local defects, and of vortices supported by quadruple defects. The paper is concluded by Section V.

II. THE MODEL

The analysis is based on the 2D nonlinear Schrödinger/Gross-Pitaevskii equation for the mean-field BEC wave function, or the amplitude of the electromagnetic wave propagating in a photonic lattice, \(\psi(x, y, z)\),

\[
i\psi_z = -(1/2)\nabla^2 \psi - (1 - \delta(x, y))V_{OL} \psi - |\psi|^2 \psi \quad (1)
\]

(in BEC, propagation distance \(z\) is replaced by time \(t\), with the self-attractive nonlinearity and \(\nabla^2 = \partial_x^2 + \partial_y^2\). In the absence of the defect, the lattice potential with depth \(2\varepsilon\) and period normalized to be \(\pi\) is taken in the usual form, \(V_{OL} = \varepsilon[\cos(2x) + \cos(2y)]\). The defect is a “hole” of radius \(r_0 = 1.2\) in the potential, which is accounted for by \(\delta(x, y) = 1\) at \(x^2 + y^2 < r_0^2\), and \(\delta(x, y) = 0\) at \(x^2 + y^2 \geq r_0^2\) in Eq. (1). Generic results can be adequately demonstrated for this shape of the defect with \(\varepsilon = 2\), which is fixed below (in fact, fully stable soliton families could be found in the interval of \(1.1 \leq r_0 \leq 1.3\), as well as for different shapes of the “hole”). The corresponding contour plots of the OL potential with the solitary or compound defects are displayed in Fig. 1.

Stationary solutions to Eq. (1) with propagation constant \(-\mu\) (or chemical potential \(\mu\), in terms of BEC)
are sought for as $\psi = \phi(x, y) \exp(-i\mu z)$, with function $\phi(x, y)$ obeying the stationary equation,

$$
\mu \phi = -(1/2)\nabla^2 \phi - [1 - \delta(x, y)] V_{OL}(x, y) \phi - |\phi|^2 \phi. \tag{2}
$$

While fundamental and dipole-mode solitons are described by real solutions for $\phi(x, y)$, vortices (that can be supported by quadruple defects, see below) naturally correspond to complex ones.

Figure 2 displays the linear spectrum of Eq. (1), which has been produced by a numerical solution of the linearized version of Eq. (2). It is seen that, for given values of the parameters, a growing number of defect-induced isolated eigenvalues appear in the second finite bandgap with the increase in the size of the defect (single $\rightarrow$ double $\rightarrow$ triple), while the first bandgap remains unaffected. Below, we focus not on quite obvious quasi-linear defect modes corresponding to these isolated eigenvalues, but rather on localized states which have no linear limit, but may be created by the defects in the first finite bandgap, as well as in the two Bloch bands adjacent to it. These states were constructed as numerical solutions of Eq. (2) by means of the Newton’s method in the domain of size $30 \times 30$, covered by a grid of $192 \times 192$ points. The initial guess was the isotropic Gaussian, $\phi(x, y) = A \exp[-(x^2 + y^2)/(2W^2)]$, with amplitude $A$, width $W$, and total power (or number of atoms, in the BEC), $N = \int \int \phi^2(x, y) \, dx \, dy = \pi (AW)^2$. The stability of the solutions was subsequently tested in direct simulations of perturbed evolution in the framework of Eq. (1), using the same numerical domain with absorbing boundary conditions.

III. RESULTS OF THE ANALYSIS FOR THE SINGLE DEFECT

As shown in Fig. 3, the numerical solution reveals the existence of continuous families of localized modes (fundamental EDSs), pinned to the single defect [the one shown in Fig. 1(a)], inside the first and second Bloch bands. The EDS families are continuously linked to the family of GDSs, attached to the same defect, whose propagation constant belongs to the first finite bandgap. The EDS family in the first Bloch band starts exactly at its boundary with the SIG, where it is linked to the family of stable regular solitons pinned to the defect. The solitons of the latter type are not presented here in detail, as their existence and properties are quite obvious.

It is observed that the combined family of the pinned solitons in Fig. 3 exists above a finite minimum value of the total power, $N_{\text{min}} = 1.56$, which exactly corresponds to the boundary between the first Bloch band and the first finite bandgap. This means, as said above, that the family of the localized modes supported by the defect does not emerge as a nonlinear continuation of any linear defect mode, as the latter would correspond to $N \rightarrow 0$.

Direct simulations demonstrates that the EDS family is completely stable in the first Bloch band, while the EDS becomes unstable in the second band beyond a certain critical point. It is worthy to note that the stability of the intra-band solitons supported by the defect in the first Bloch band agrees with the Vakhitov-Kolokolov (VK) criterion, which is relevant to localized modes in the case of the self-focusing nonlinearity $|106|$, $\partial \mu / \partial N < 0$, while the EDS branch features $\partial \mu / \partial N > 0$ in the second band, where the modes are stable in a limited region (strictly speaking, this may be a region of a very weak instability, which is, nevertheless, tantamount to stabil-
FIG. 4. (Color online) Examples of the intra-band (embedded) defect solitons, EDSs, found in the first (a) and second (b,c) Bloch bands, which correspond to points A and B,C, respectively, in Fig. 3. The shapes of the modes are shown by means of 3D images (left) and contour plots (right). The mode in panels (a) and (b) are stable, while the one in (c) is unstable.

ity in terms of a possible experiment). Of course, there is no direct proof of the applicability of the VK criterion to the present model.

Typical examples of stable EDSs in the first and second Bloch bands, found at points marked (A) and (B) in Fig. 3, along with an example of an unstable EDS corresponding to point C, are displayed in Fig. 4. Throughout the first band, the shape of the solitons is very similar to that displayed in Fig. 4(a). On the other hand, the localized modes feature a more complex structure, with pronounced side peaks, in the second band [see Figs. 4(b,c)], in accordance with the fact that the shape of Bloch modes is more complex too in the same band. In fact, the extended tail of the soliton shown in Fig. 4(c) initiates the onset of the instability of this soliton. In direct simulations, unstable modes decay into radiation (not shown here in detail).

To the best of our knowledge, the current model presents the first example of nonlinear localized modes found inside Bloch bands of the spectrum induced by the 2D lattice (in Ref. [39], which was dealing with 2D solitons of the embedded type, the lattice was one-dimensional). In fact, the EDS family threading the first Bloch band plays the role of a missing link between the two well-known types of solitons existing in the lattice potential, viz., regular ones populating the SIG, and gap solitons in the first finite bandgap (in the present case, they all are pinned to the single defect).

As concerns the family of GDSs found inside the first finite bandgap, which links the intra-band EDS families populating the two adjacent Bloch bands (see Fig. 3), our numerical tests have demonstrated that these solitons, being pinned to the defect, are stable only in the case of the self-focusing nonlinearity [as set in Eq. (1)], on the contrary to gap solitons in perfect lattices, which are usually assumed to be stable solely in the case of the self-defocusing [19, 102]. The case of the self-defocusing nonlinearity was explored too, and it was concluded that the pinned GDSs are unstable in that case (not displayed here in detail). The latter finding may be explained by the fact that the effective mass of the gap soliton supported by the self-defocusing nonlinearity is negative, thus reversing the character of the soliton-defect interaction, and making unstable the bound state of the gap soliton pinned to the attractive defect. In line with this argument, additional analysis demonstrates that at the local defect of the opposite sign, in comparison with the one considered here, gives rise to stable pinned states of gap solitons under the self-defocusing nonlinearity, and unstable states in the case of the self-focusing. The negative mass of gap solitons explains a number of other counter-intuitive dynamical effects featured by these modes [107-110]. Numerical results also demonstrate that the self-defocusing nonlinearity does not give rise to solitons that would be embedded into Bloch bands and pinned to the defect.

A typical example of the stable pinned GDS, found in the first finite bandgap in the case of the self-focusing nonlinearity, which corresponds to point D in Fig. 3, is presented in Fig. 5. Finally, our numerical simulations suggest that stable GDSs pinned to the defect do not exist in the relatively narrow second finite bandgap (see Fig. 2), in agreement with the general trend of gap solitons to be unstable in the second bandgap [16, 17, 21].
IV. RESULTS FOR MULTIPLE DEFECTS: STABLE DIOLES AND VORTICES

We have extended the above analysis for the double, triple, and quadruple lattice defects, which are shown in Fig. 1(b,c,d). While the solitary defect may only support fundamental solitons of both the EDS and GDS types, as shown above, the double defects readily give rise to stable dipole-mode bound states of two solitons with opposite signs. Examples of such modes, found in the first Bloch band and in the first finite bandgap (i.e., of the EDS and GDS types, respectively), are shown in Fig. 6.

The triangular defects [see Fig. 1(c)] also support stable triangularly shaped dipole modes, see examples displayed in Fig. 7. Further, the quadruple defects [see Fig. 1(d)] create stable solitary vortices, characteristic examples of which are shown in Fig. 8. As is typical for vortices supported by lattice potentials [19, 91, 111, 112], they are built as rhombic sets of four intensity peaks, with a nearly empty site in the center (therefore this type of vortices is often called onsite-centered), and phase shifts $\pi/2$ between the peaks, which corresponds to the global phase circulation of $2\pi$ (i.e., topological charge 1). As indicated in Fig. 9, direct simulations demonstrate that the pinned vortices are unstable in the second Bloch band, being stable elsewhere.

V. CONCLUSION

In this work, we have studied localized states which can be supported by single, double, triple, and quadruple defects of 2D lattice potentials in the self-focusing medium. The model describes lattices in nonlinear photonic media and BEC trapped in the OL. The first result is that the single defect supports families of intra-band (embedded) solitons, in the first and second Bloch bands. These modes exist above the minimum value of the norm, i.e., they do not emerge from any linear defect mode. Direct simulations demonstrate stability of the family in the first Bloch band, and in a part of the second band. The branch of the stable pinned solitons embedded into the first band links families of regular and gap solitons.
hosted by the semi-infinite and first finite bandgaps, respectively, and pinned to the same defect. The family of the gap solitons, pinned to the defect, is stable, despite the fact that the nonlinearity is self-attractive, while it is usually assumed that 2D fundamental gap solitons may be stable only under the self-repulsion (in uniform lattices). Finally, it has been found that the double and triple defects support stable dipole solitons, in the first Bloch band and in the first finite bandgap alike, and, similarly, the quadruple defect creates stable vortex solitons.

This work can be naturally extended in other directions—in particular, to the model with a rectilinear defect, as an elongated version of the double one. Search for stable vortex solitons with higher topological charges, and, possibly, “supervortices” (vortex rings built of compact localized vortices) [113, 114] may be relevant too.

On the other hand, it may be interesting to consider defects in complex lattices, whose imaginary part is subject to the condition of the $\mathcal{PT}$ symmetry. 2D gap solitons in uniform $\mathcal{PT}$-symmetric lattices were found recently, but they are unstable [115]. Stable 1D solitons pinned to defects in $\mathcal{PT}$-symmetric lattices, embedded into the medium with the self-defocusing cubic nonlinearity, were reported very recently in Ref. [116]. In this connection, it is also relevant to mention that 1D and 2D gap solitons can be readily made stable in a more general Ginzburg-Landau system, which does not impose the balance condition on the gain and dissipation [117, 118].

VI. ACKNOWLEDGMENTS

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