GLOBAL WELL-POSEDNESS, BLOW-UP AND STABILITY OF STANDING WAVES FOR SUPERCRITICAL NLS WITH ROTATION

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Abstract. We consider the focusing mass supercritical nonlinear Schrödinger equation with rotation

\[ iu_t = -\frac{1}{2}\Delta u + \frac{1}{2}V(x)u - |u|^{p-1}u + L_{\Omega}u, \quad (x, t) \in \mathbb{R}^N \times \mathbb{R}, \]

where \(N = 2\) or \(3\) and \(V(x)\) is an anisotropic harmonic potential. Here \(L_{\Omega}\) is the quantum mechanical angular momentum operator. We establish conditions for global existence and blow-up in the energy space. Moreover, we prove strong instability of standing waves under certain conditions on the rotation and the frequency of the wave. Finally, we construct orbitally stable standing waves solutions by considering a suitable local minimization problem. Those results are obtained for nonlinearities which are \(L^2\)-supercritical.

1. Introduction

Consider the focusing nonlinear Schrödinger equation with rotation

\[
\begin{aligned}
&iu_t = -\frac{1}{2}\Delta u + \frac{1}{2}V(x)u - |u|^{p-1}u + L_{\Omega}u, \quad (x, t) \in \mathbb{R}^N \times \mathbb{R}, \\
&u(x, 0) = u_0(x), \\
\end{aligned}
\]

(1.1)

where \(N = 2\) or \(3\), \(u : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}\) and \(1 < p < 2^*\). Here \(2^* = 1 + \frac{4}{N-2}\) if \(N = 3\), and \(2^* = \infty\) if \(N = 2\). The potential \(V(x)\) is assumed to be harmonic,

\[ V(x) = \sum_{j=1}^{N} \gamma_j^2 x_j^2, \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N, \quad \gamma_j \in \mathbb{R} \setminus \{0\}. \]

The parameters \(\gamma_j\) represent the harmonic trapping frequencies in each spatial direction. Through this paper we will assume that \(\gamma := \min_{1 \leq j \leq N} \{\gamma_j\} > 0\). The quantum mechanical angular momentum operator \(L_{\Omega}\) is expressed by \(L_{\Omega} := -\Omega \cdot L\), \(L := -ix \wedge \nabla\), where \(\Omega \in \mathbb{R}^3\) is the angular velocity vector. Notice that in \(N = 2\) the angular momentum operator takes the form:

\[ L_{\Omega} = -i|\Omega|(x_1 \partial_{x_2} - x_2 \partial_{x_1}), \]

where \(\Omega = (0, 0, |\Omega|) \in \mathbb{R}^3\). When the angular momentum operator \(L_{\Omega} = 0\), Eq. (1.1) is known as a model to describe the Bose-Einstein condensate under a magnetic trap. We refer the readers to [4,12,21,23] for more information. If \(L_{\Omega} \neq 0\), the model equation (1.1) describes the Bose Einstein condensate with rotation, which appears in a variety of physical settings such as the description of nonlinear waves and propagation of a laser beam in the optical fiber [11,22]. We refer the readers to [18] for a rigorous derivation in the stationary case of (1.1). Recently, the equation (1.1) has attracted attentions due to their significance in theory and applications,
see [1–3, 5, 6, 8, 14, 17] and references therein. Antonelli et al. in [2] proved existence
and uniqueness of the Cauchy problem. Moreover, they also showed the existence of
blow-up solutions in the $L^2$-critical and supercritical case (see also [3]). The issue of
stability of standing waves in the $L^2$-subcritical case have been investigated in [3].

Note that we can rewrite the equation (1.1) as

$$iu_t = \frac{1}{2}R_\Omega u - |u|^{p-1}u,$$

where the operator $R_\Omega := -\Delta + V(x) + 2L_\Omega$ admit a precise interpretation as self-
adjoint operator on $L^2(\mathbb{R}^N)$ associated with the quadratic form (see [20, Proposition
3.1])

$$t[u] := \|\nabla u\|_2^2 + \int_{\mathbb{R}^N} V(x)|u(x)|^2dx + 2l_\Omega(u)$$

defined on the domain

$$\text{dom}(t) = \Sigma := \{ u \in H^1(\mathbb{R}^N) : |x|u \in L^2(\mathbb{R}^N) \}.$$ 

Here $l_\Omega(u) := \langle L_\Omega u, u \rangle$ is the angular momentum. We observe that an integration
by parts shows that the angular momentum $l_\Omega(u)$ is always real valued. Formally,
the NLS (1.1) has the following two conserved quantities. The first conserved
quantity is the energy

$$E_\Omega(u) = \frac{1}{2}t[u] - \frac{2}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}dx.$$ 

The other conserved quantity is the mass

$$M(u) = \|u\|_2^2.$$ 

Notice that due to the appearance of the angular momentum term, the energy
functional $E_\Omega$ fails to be finite as well of class $C^1$ on $H^1(\mathbb{R}^N)$ (even if the potential
$V(x)$ is chosen to be identically zero). The local well-posedness for the Cauchy
problem (1.1) in the energy space $\Sigma$, equipped with the norm

$$\|u\|_{\Sigma}^2 = \int_{\mathbb{R}^N} \left( |\nabla u|^2 + |x|^2 |u|^2 + |u|^2 \right)dx,$$

can be proved using Strichartz estimates [2, Theorem 2.2]. More precisely, we have the
following result.

**Proposition 1.1.** Let $u_0 \in \Sigma$. Then there exists $T_+ \in (0, \infty]$ and a unique maximal
solution $u \in C([0, T_+), \Sigma)$ of the Cauchy problem (1.1) with $u(0) = u_0$. If $T_+ = \infty$,
then $u$ is called a global solution in positive time. If $T_+ < \infty$, then

$$\lim_{t \to T_+} \|\nabla u(t)\|_2^2 = \infty$$

and $u$ is called blows up in positive time. Moreover, the solution enjoys the conservation
of energy and mass i.e.,

$$E_\Omega(u(t)) = E_\Omega(u_0), \quad M(u(t)) = M(u_0) \quad \text{for every } t \in [0, T_+).$$

We note that the evolution of the angular momentum under the flow generated
by (1.1) is given by (see [2, Theorem 2.1.])

$$l_\Omega(u(t)) = l_\Omega(u_0) + \int_0^t \int_{\mathbb{R}^N} i|u(x,t)|^2(\Omega \cdot L)V(x)dx, \quad t \in [0, T_+).$$

By using a time-dependent change of coordinates and the conservation laws (1.2),
we have the global existence of Cauchy problem (1.1) in the $L^2$-subcritical case
$1 < p < 1 + \frac{4}{N}$ (see [2, Theorem 2.2] for more details). As observed in [2],
we have that in the $L^2$-supercritical case $1 + \frac{4}{N} < p < 2^*$ blow-up of the solution
may occur. In the super-critical case, the sharp thresholds of blow-up and global
existence become very interesting. In our first result, we establish sufficient and necessary conditions of global existence and blow-up in finite time for the rotational NLS (1.1) in the mass supercritical regime.

Remark 1.2. If the trapping frequencies are equal in each spatial direction, i.e., $\gamma = \gamma_j$ for all $j = 1, \ldots, N$, then we also have the conservation of the angular momentum $l_\Omega(u(t)) = l_\Omega(u_0)$ for every $t \in [0, T_+)$.

Remark 1.3. (i) As mentioned above, if the trapping frequencies are equal in each spatial direction, then we have that $l_\Omega(u(t)) = l_\Omega(u_0)$ for every $t \in [0, T_+)$.

(ii) Notice that if the nonlinearity is $L^2$-subcritical ($p < 1 + 4/N$), then $l \in \mathbb{R}$. Indeed, by [2, Theorem 2.1] we see that if $u(t)$ is the solution of (1.1), $u(t)$ exists globally and there exists $C > 0$ such that $\|xu(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 \leq C$ for all $t \in \mathbb{R}$. This implies by inequality (2.3) below that $|l_{\Omega}(u(t))|$ is uniformly bounded. Therefore, $l \in \mathbb{R}$.

(iii) In the $L^2$-supercritical case ($p > 1 + 4/N$), if $|\Omega| < \gamma$ and $\|u_0\|_{\Sigma}$ is small enough, a standard argument shows that there exists $C > 0$ such that $\|u(t)\|_{\Sigma} \leq C$ for every $t$ in the interval of existence. Thus, we can apply the local theory to extend the solution such that $\|u(t)\|_{\Sigma} \leq C$ for every $t \in \mathbb{R}$. Again, by inequality (2.3) below we infer that $l \in \mathbb{R}$.

For $p > 1 + 4/N$ (i.e. $0 < s_c < 1$) and $u_0 \in \Sigma$, if $l \in \mathbb{R}$ and $E_{\Omega}(u_0) \geq l$, we define the following subsets in $\Sigma$,

\[ K^+ = \{ u_0 \in \Sigma : (E_{\Omega}(u_0) - l)^{s_c} M(u_0)^{1-s_c} < E_{0,0}(Q)^{s_c} M(Q)^{1-s_c} \}, \]

\[ \|\nabla u_0\|_{L^2}^{s_c} \|u_0\|_{L^2}^{1-s_c} < \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c} \}, \]

and

\[ K^- = \{ u_0 \in \Sigma : (E_{\Omega}(u_0) - l)^{s_c} M(u_0)^{1-s_c} < E_{0,0}(Q)^{s_c} M(Q)^{1-s_c} \}, \]

\[ \|\nabla u_0\|_{L^2}^{s_c} \|u_0\|_{L^2}^{1-s_c} > \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c} \}, \]

where $Q$ denotes the unique positive and radially symmetric solution of (1.4) and $E_{0,0}(Q) = \frac{1}{2}\|\nabla Q\|_{L^2}^2 - \frac{2}{p+1} \|Q\|_{L^{p+1}}^p$. Notice that $K^+ \neq \emptyset$ (see Remark 1.5 below).

In our first result, we obtain a criteria between blow-up and global existence for (1.1) in terms of the energy, mass and $l$ given by (1.5).
Theorem 1.4. Let $1 + \frac{4}{N} < p < 2^*$ (i.e., $0 < s_c < 1$), $u_0 \in \Sigma$ and let $u \in C([0, T_+), \Sigma)$ be the corresponding solution of (1.1) with initial data $u_0$.

(i) If $l = -\infty$, then there exists a sequence of times $\{t_n\}$ such that $t_n \to T_+$ and
\[
\lim_{t_n \to T_+} \|\nabla u(t_n)\|_2^2 = \infty.
\]

(ii) Assume that $l \in \mathbb{R}$ and $E_{\Omega}(u_0) \geq l$. Then one of the following two cases holds:

1. If $u_0 \in K^+$, then the corresponding solution $u(t)$ exists globally.
2. If $u_0 \in K^-$, the solution blows-up in finite time.

Moreover, the sets $K^\pm$ are invariant by the flow of the equation (1.1).

(iii) Assume that $l \in \mathbb{R}$ and $E_{\Omega}(u_0) < l$. Then the solution $u(t)$ blows up at finite time in $\Sigma$. In addition, for every $t$ in the existence time we have
\[
\|\nabla u(t)\|_2 \geq \left( \frac{(p - 1)N}{4} \right)^{1/(4 - N)} \left( \frac{\|Q\|_2}{\|u_0\|_2} \right)^{\frac{1}{4 - N}} \|\nabla Q\|_2.
\]

For the standard Schrödinger equation, the sharp thresholds of global existence and blow-up have been extensively studied during the past decades (see [9, 10, 15] and references therein). To prove the Theorem 1.4 we follow the arguments developed in Holmer and Roudenko [15, 16], where they proved similar results for the $L^2$-supercritical NLS with zero potential.

Remark 1.5. (i) The set $K^+$ is not empty for $|\Omega| < \gamma$. Indeed, if $\|u_0\|_\Sigma$ is small enough, by Remark 1.3 (iii) we have that $l \in \mathbb{R}$. Moreover, by the energy conservation and the Gagliardo-Nirenberg inequality (see (2.1)) we see that
\[
E(u_0) - l_{\Omega}(u(t)) \geq X(t) - CX(t)^{N(p - 1)/4},
\]
where $C > 0$ and $X(t) = \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{p}\|Vu(t)\|_2^2$. Since $p > 1 + 4/N$, taking $\|u_0\|_\Sigma$ small enough we infer that $E(u_0) - l_{\Omega}(u(t)) \geq 0$. This implies that $E(u_0) - l \geq 0$.

In conclusion, there exists $\varepsilon > 0$ such that if $\|u_0\|_\Sigma < \varepsilon$, then $u_0 \in K^+$.

(ii) We can extend the Theorem 1.4 to the case of potentials $V \in C^\infty(\mathbb{R}^N)$ such that $V \geq 0$ and $\partial^\alpha V \in L^\infty(\mathbb{R}^N)$ for all multi-indices $\alpha \in \mathbb{N}^N$ with $|\alpha| \geq 2$. Indeed, the proof of Theorem 1.4 works after obvious modifications. Notice also that in this case if $(\Omega \cdot L)V(x) \geq 0$ (see 1.3), then we have that $l_{\Omega}(u(t)) \geq l_{\Omega}(u_0)$, for all $t \in [0, T_+)$, i.e., $l = l_{\Omega}(u_0)$. As a consequence of this fact, we see that if $E_{\Omega}(u_0) - l_{\Omega}(u_0)$ is small enough and $\|u_0\|_2^2$ is sufficiently large, then $u_0 \in K^-$. Similarly, if $\|u_0\|_2^2$ is small enough then $u_0 \in K^+$.

Remark 1.6. (i) Notice that if $l = -\infty$, then by Theorem 1.4 one of the following two statements is true:

1. The solution blows-up in finite time, i.e, $T_+ < \infty$ and $\lim_{t \to T_+} \|\nabla u(t)\|_2^2 = \infty$.
2. The solution grows-up in time, i.e, $T_+ = \infty$ and there exists a sequence $t_n \to \infty$ such that $\lim_{t_n \to \infty} \|\nabla u(t_n)\|_2^2 = \infty$.

(ii) We observe that in the mass supercritical regime $1 + \frac{4}{N} < p < 2^*$, if $E_{\Omega}(u_0) \geq l$, then the condition $\|\nabla u_0\|_2^2 \|u_0\|_2^{1 - s_c} < \|\nabla Q\|_2^2 \|Q\|_2^{1 - s_c}$ is sharp for global existence except for the threshold level $\|\nabla u_0\|_2^2 \|u_0\|_2^{1 - s_c} = \|\nabla Q\|_2^2 \|Q\|_2^{1 - s_c}$.

In the second part of this paper, we study the stability and instability of standing waves. Throughout this paper, we call a standing wave a solution of (1.1) with the form $u(x, t) = e^{i\omega t} \varphi_\omega(x)$, where $\omega \in \mathbb{R}$ is a frequency and $\varphi_\omega$ satisfying the following nonlinear elliptic problem
\[
\begin{cases}
-\Delta \varphi + \omega \varphi + V(x) \varphi - 2|\varphi|^{p-1} \varphi + 2L_{\Omega} \varphi = 0, \\
\varphi \in \Sigma \setminus \{0\}.
\end{cases}
\]
For $\gamma = \min_{1 \leq j \leq N} \{ \gamma_j \} > 0$, it is well known that operator $R_\Omega$ has a purely discrete spectrum (see [19, Theorem 2.2] for more details). Thus, we define
\[
\lambda_0 := -\inf \left\{ \left\| \nabla u \right\|_2^2 + \int_{\mathbb{R}^N} V(x)|u(x)|^2\,dx + 2\ell_\Omega(u) : u \in \Sigma, \left\| u \right\|_2^2 = 1 \right\}.
\] (1.7)

Now, for $\omega > \lambda_0$, we denote the set of non-trivial solutions of (1.6) by
\[
\mathcal{A}_\omega = \{ \varphi \in \Sigma \setminus \{0\} : \mathcal{S}'_\omega(\varphi) = 0 \}.
\]
Moreover, we define the following functionals of class $C^2$:
\[
S_\omega(u) = \frac{1}{2} \int_{\mathbb{R}^N} |u|^2\,dx - \frac{\omega}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}\,dx,
\]
\[
I_\omega(u) = t[|u| + \omega] \int_{\mathbb{R}^N} |u|^2\,dx - 2 \int_{\mathbb{R}^N} |u|^{p+1}\,dx,
\]
\[
P(u) = \frac{1}{2} \int_{\mathbb{R}^N} \nabla u \cdot \nabla u\,dx - \frac{1}{2} \omega \int_{\mathbb{R}^N} |u|^2\,dx - \frac{N(p-1)}{2(p+1)} \int_{\mathbb{R}^N} |u|^{p+1}\,dx.
\]

We observed that the elliptic equation (1.6) can be written as $S'(\varphi) = 0$. A ground states for (1.6) is a function $\phi \in \mathcal{A}_\omega$ that minimizes $S_\omega$ over the set $\mathcal{A}_\omega$. The set of ground states is denoted by $\mathcal{G}_\omega$ and
\[
\mathcal{G}_\omega = \{ \varphi \in \mathcal{A}_\omega : S_\omega(\varphi) \leq S_\omega(v) \text{ for all } v \in \mathcal{A}_\omega \}.
\]

In the following result, we prove that the set of ground states $\mathcal{G}_\omega$ is not empty.

**Proposition 1.7.** Let $|\Omega| < \gamma$, $\omega > \lambda_0$ and $1 < p < 2^*$. Then the set of ground states $\mathcal{G}_\omega$ is not empty. Moreover, we have the following variational characterization
\[
\mathcal{G}_\omega = \{ \varphi \in \Sigma : S_\omega(\varphi) = d(\omega) \text{ and } I_\omega(u) = 0 \},
\]
where
\[
d(\omega) = \inf \{ S_\omega(u) : u \in \Sigma \setminus \{0\}, I_\omega(u) = 0 \}.
\]

Next we need the following definition.

**Definition 1.8.** We say that the set $\mathcal{M} \subset \Sigma$ is $\Sigma$-stable under the flow generated by (1.1) if, for $\varepsilon > 0$, there exists $\delta > 0$ such that for any initial data $u_0$ satisfying
\[
\inf_{v \in \mathcal{M}} \left\| u_0 - v \right\|_\Sigma < \delta,
\]
then the corresponding solution $u(t)$ of (1.1) with $u(t) = u_0$ exists for all $t \in \mathbb{R}$ and satisfies
\[
\inf_{v \in \mathcal{M}} \left\| u(t) - v \right\|_\Sigma < \varepsilon.
\]

Otherwise, $\mathcal{M}$ is said to be unstable. We say that the standing wave $u(x,t) = e^{it\varphi}(x)$ of (1.1) is stable in $\Sigma$ if $\mathcal{O}_\omega$ is stable and $u(x,t) = e^{it\varphi}(x)$ is unstable if $\mathcal{O}_\omega$ is unstable, where $\mathcal{O}_\omega = \{ e^{it\varphi} : \theta \in \mathbb{R} \}$.

Following the argument by Fukuzumi and Ohta [13], we can show a sufficient condition for the instability of standing waves in the mass supercritical regime.

**Theorem 1.9.** Let $|\Omega| < \gamma$, $\omega > \lambda_0$, $1 + \frac{4}{N} < p < 2^*$ and $\phi_\omega \in \mathcal{G}_\omega$. Assume that $\partial_t^2 E(\phi_\omega)|_{s=1} < 0$, where $\phi_\omega^*(x) = s^\frac{2}{p} \phi_\omega(sx)$. Then the standing wave $e^{it\varphi}(x)$ of (1.1) is unstable in $\Sigma$.

Under some conditions on the rotation $|\Omega|$ and frequency $\omega$, it is possible to show that $\partial_t^2 E(\phi_\omega)|_{s=1} < 0$. Notice that, since the standing wave $e^{it\varphi}(x)$ of (1.1) with $\Omega = 0$ is strongly unstable in $\Sigma$ when $p > 1 + \frac{4}{N}$ and $\omega$ is sufficiently large (see [12]), we expect that the standing wave solution $e^{it\varphi}(x)$ of (1.1) with $|\Omega| \ll \gamma$ can also be unstable in $\Sigma$ when $p > 1 + \frac{4}{N}$ and $\omega$ is sufficiently large. Indeed, we have the result.
Corollary 1.10. Let $1 + \frac{4}{N} < p < 2^*$ and $\phi_\omega \in \mathcal{G}_\omega$. There exists $\varepsilon > 0$ such that if $|\Omega| \leq \varepsilon \gamma$, then there is a sequence $\{\omega_n\}_{n=1}^{\infty}$ such that the standing wave $e^{it\phi_{\omega_n}(x)}$ of (1.1) is unstable. Moreover, $\omega_n \to \infty$ as $n \to \infty$.

Remark 1.11. We observe that under the conditions of Theorem 1.9, if the trapping frequencies are equal ($\gamma = \gamma_j$, $j = 1, \ldots N$), then thanks to the conservation of the angular momentum it is possible to show that the standing wave $e^{it\phi_{\omega}(x)}$ of (1.1) is strongly unstable in $\Sigma$. Indeed, the proof follows from exactly the same argument in Ohta [21, Theorem 1]. In particular, we infer that the standing wave $e^{it\phi_{\omega_n}(x)}$ in Corollary 1.10 is strongly unstable in $\Sigma$ (see the proof of Corollary 1.10 and Lemma 4.1 below).

Now, we focus on the stability of standing waves in the mass supercritical regime $p > 1 + \frac{4}{N}$. The more common approach to construct orbitally stable standing waves to (1.1) is to consider the following constrained minimization problems

$$J_q = \inf \{ E_\Omega(u), \; u \in \Sigma, \; \|u\|_2^2 = q \}.$$ 

In the mass subcritical case $p < 1 + \frac{4}{N}$ it is possible to show that $J_q > -\infty$ and any minimizing sequence of $J_q$ is relatively compact in $\Sigma$ (see [3]). In particular, this implies that the set of minimizers of $J_q$ is $\Sigma$-stable under the flow generated by (1.1).

On the other hand, in the mass supercritical case $p > 1 + \frac{4}{N}$, we have $J_q = -\infty$. Indeed, we set $u_\mu(x) := \mu^{\frac{2}{p}} u(\mu x)$ where $u \in \Sigma$ with $\|u\|_2^2 = q$. It is not difficult to show that $\|u_\mu\|_2^2 = \|u\|_2^2$, $I_\mu(u_\mu) = I_\mu(u)$ and

$$E_\Omega(u_\mu) = \frac{\mu^2}{2} \|\nabla u\|_2^2 + \mu^{\frac{p}{2}} \|V u\|_2^2 - \frac{\mu^{\frac{p}{p+1}}}{p+1} \|u\|_{p+1}^{p+1} + 2l_\Omega(u).$$

Since $p > 1 + \frac{4}{N}$, we infer that $E_\Omega(u_\mu) \to -\infty$ as $\mu$ goes to $+\infty$, and therefore $J_q = -\infty$. To overcome this difficulty, we consider a local minimization problem. Following [7], for $|\Omega| < \gamma$, we define the following subsets:

$$D_q := \{ u \in \Sigma : \|u\|_2^2 = q \},$$
$$B_r := \{ u \in \Sigma : \|u\|_H^2 \leq r \},$$

where $\| \cdot \|_H$ denotes the norm (see Section 3)

$$\|u\|_H^2 := \|\nabla u\|_2^2 + \int_{\mathbb{R}^N} V(x) |u(x)|^2 dx + 2l_\Omega(u).$$

Moreover, for a fixed $q > 0$ and $r > 0$, we set the following local variational problem

$$J_q^r = \inf \{ E_\Omega(u), \; u \in D_q \cap B_r \}.$$ (1.8)

Using the Gagliardo-Nirenberg inequality, it is not difficult to show that if $D_q \cap B_r \neq \emptyset$, then the variational problem $J_q^r$ is well defined; that is, $J_q^r > -\infty$ (see proof of Lemma 5.1 below). Let us denote the set of nontrivial solutions of (1.8) by

$$\mathcal{G}_q^r := \{ v \in D_q \cap B_r : \; v \text{ is a minimizer of } (1.8) \}.$$ 

The following result shows that, in the mass supercritical regime, the set $\mathcal{G}_q^r$ is not empty.

Theorem 1.12. Let $|\Omega| < \gamma$ and $1 + \frac{4}{N} < p < 2^*$. For any $r > 0$ there exists $q_0 > 0$ such that for all $q < q_0$ we have:

(i) Any minimizing sequence for (1.8) is precompact in $\Sigma$.

(ii) For every $\varphi \in \mathcal{G}_q^r$ there exists a Lagrange multiplier $\omega \in \mathbb{R}$ such that the stationary problem (1.6) is satisfied with the estimates

$$\lambda_0 < \omega \leq \lambda_0 (1 - C q^{\frac{p-1}{p} - 1}).$$
Note that from the above theorem, $\omega \to \lambda_0$ as $q \to 0$. Moreover, if $\varphi \in G_q^r$, then there exists $\omega > \lambda_0$ such that $\varphi$ is a solution of stationary problem (1.6). In particular, $u(x, t) = e^{2\pi it}\varphi(x)$ is a standing wave solution to (1.1).

We have the following stability result for the set $G_q^r$.

**Corollary 1.13.** If $|\Omega| < \gamma$, then for any fixed $r > 0$ and $q < q_0$ given in the Theorem 1.12 we have that the set $G_q^r$ is $\Sigma$-stable with respect to (1.1).

We remark that nothing is known about orbital stability of standing waves in the supercritical case when $|\Omega| > \gamma$. The study of the stability seems highly non-trivial; see the discussion presented after formula (1.6) in [3] for more details.

This paper is organized as follows. In Section 2 we prove our global existence/blow-up result stated in Theorem 1.4. In Section 3 we prove, by variational techniques, the existence of ground states (Proposition 1.7). In Section 4, we analyze the instability of the standing waves in Corollary 1.10. Finally, Section 5 is devoted to the proof of Theorem 1.12 and Corollary 1.13.

## 2. Conditions for Global existence and blow-up

In this section, we prove Theorem 1.4. First we recall the sharp Gagliardo-Nirenberg inequality [9],

$$\|u\|_{p+1}^{p+1} \leq c_{GN}\|\nabla u\|_2^\frac{N(p-1)}{2} \|u\|_2^{p+1-\frac{N(p-1)}{2}}, \tag{2.1}$$

where the sharp constant $c_{GN} > 0$ is explicitly given by

$$c_{GN} = \left(\frac{N(p-1)}{2(p+1) - N(p-1)}\right)^\frac{p+1}{p+1-N(p-1)} \frac{(p+1)}{N(p-1)\|Q\|_{p+1}^{p-1}}.$$

Next we recall the standard virial identity related to (1.1) (see [2]).

**Lemma 2.1.** Let $u_0 \in \Sigma$ and $u(x, t)$ the corresponding solution of Cauchy problem (1.1) on $[0, T)$, where $T$ is the maximum time of existence. We put $J(t) := \int_{\mathbb{R}^N} |x|^2|u(x, t)|^2 \, dx$. Then we have for all $t \in [0, T)$

$$J'(t) = 2 Im \int_{\mathbb{R}^N} (\nabla u(x, t) \cdot x) \overline{\varphi}(x, t) \, dx$$

and

$$J''(t) = 2 \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - 2 \int_{\mathbb{R}^N} V(x)|u(x, t)|^2 \, dx - 2 N \left(\frac{p-1}{p+1}\right) \int_{\mathbb{R}^N} |u(x, t)|^{p+1} \, dx.$$

Note that we can compute the virial identity in terms of $E_{\Omega}(u)$ and $l_{\Omega}(u)$. In indeed, a simple computation shows

$$J''(t) = \left(\frac{4-N(p-1)}{2}\right)\|\nabla u(t)\|_2^2 - \left(\frac{N(p-1)+4}{2}\right) \int_{\mathbb{R}^N} V(x)|u(x)|^2 \, dx$$

$$+ N(p-1)(E_{\Omega}(u(t)) - l_{\Omega}(u(t))). \tag{2.2}$$

We will frequently use the following inequality

$$|l_{\Omega}(\psi)| \leq \frac{1}{2a} |\Omega|^2\|\varphi\|_2^2 + \frac{a}{2}\|\nabla \varphi\|_2^2. \tag{2.3}$$

The proof of Theorem 1.4 is based on the following result.

**Lemma 2.2.** Under the conditions of the Theorem 1.4 the following statements hold. Assume that

$$(E_{\Omega}(u_0) - l)^{a_2} M(u_0)^{1-s_2} < E_{0,0}(Q)^{a_2} M(Q)^{1-s_2},$$

$$E_{\Omega}(u_0) - l \geq 0, \tag{2.4}$$
(i) If
\[
\|\nabla u_0\|_2^2 \|u_0\|_2^{1-s_c} < \|\nabla Q\|_2^2 \|Q\|_2^{1-s_c}
\]
then \(u(t)\) is a global solution and for every \(t \in \mathbb{R}\)
\[
\|\nabla u(t)\|_2^2 \|u_0\|_2^{1-s_c} < \|\nabla Q\|_2^2 \|Q\|_2^{1-s_c}
\]
(2.5)

(ii) If
\[
\|\nabla u_0\|_2^2 \|u_0\|_2^{1-s_c} > \|\nabla Q\|_2^2 \|Q\|_2^{1-s_c}
\]
then the solution \(u(t)\) blows up at finite time. Moreover, we also have
\[
\|\nabla u(t)\|_2^2 \|u_0\|_2^{1-s_c} > \|\nabla Q\|_2^2 \|Q\|_2^{1-s_c}
\]
for every \(t\) in the existence time.
(iii) If, in place of (2.4) and (2.7), we assume
\[
E_\Omega(u_0) - l < 0,
\]
then the solution \(u(t)\) blows up at finite time in \(\Sigma\). Moreover, for every \(t\) in the existence time we have
\[
\|\nabla u(t)\|_2 \geq \left(\frac{(p-1)N}{4}\right)^{\frac{1}{N-1}} \left(\frac{\|Q\|_2}{\|u_0\|_2}\right)^{\frac{1-s_c}{s_c}} \|\nabla Q\|_2.
\]

Proof. The proof is inspired by the one of Theorem 2.1 in [15] (see also [16]). Let \(u(t)\) be the corresponding solution of (1.1) with initial data \(u_0\). By the sharp Gagliardo-Nirenberg inequality (2.1) we get
\[
E_\Omega(u) - l_\Omega(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u(x)|^2dx - \frac{2}{p+1}\|u\|_{p+1}^{p+1}
\geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{2c_{GN}}{p+1}\|\nabla u\|_2^{\frac{N(p-1)}{2}}\|u_0\|_2^{\frac{p+1-N(p-1)}{2}}.
\]

Define the function \(f(x) = \frac{1}{2}x^2 - \beta_{p,N}x^{\frac{2}{p}}(p-1)\), where \(\beta_{p,N} = \frac{2c_{GN}}{p+1}\|u_0\|_2^{p+1-N(p-1)}\). Notice that \(\text{deg}(f) \geq 2\) and
\[
f'(x) = x - \frac{N}{2}(p-1)\beta_{p,N}x^{\frac{1}{p}-1}
\]
\[
= x\left(1 - \frac{N}{2}(p-1)\beta_{p,N}x^{(p-1)s_c}\right).
\]

A simple computation shows that \(f'(x) = 0\) when \(x_0 = 0\) and
\[
x_1 = \left(\frac{2}{N\beta_{p,N}(p-1)}\right)^{\frac{p}{N(p-1)-1}} = \left(\frac{p+1}{N(p-1)c_{GN}}\right)^{\frac{1}{s_c(p-1)}}\|u_0\|_2^{-\frac{(1-s_c)}{s_c}}.
\]

Notice that \(f\) has a local minimum at \(x_0\) and a local maximum at \(x_1\), with maximum value \(f(x_1) = \frac{2}{N}x_1^2\). Now, it is not difficult to show that
\[
\left(\frac{p+1}{N(p-1)c_{GN}}\right)^{\frac{1}{s_c(p-1)}} = \left(\frac{2N(p-1)}{2(p+1) - N(p-1)}\right)^{\frac{1}{s_c}}\|Q\|_2^{\frac{1}{s_c}}.
\]

Moreover, by the Pohozaev identities we infer that
\[
\|\nabla Q\|_2^\frac{1-s_c}{s_c} = \left(\frac{2N(p-1)}{2(p+1) - N(p-1)}\right)^{\frac{1}{s_c}}\|Q\|_2^{\frac{1}{s_c}},
\]

\[
E_{0,0}(Q)M(Q)^{\frac{(1-s_c)}{s_c}} = \frac{s_c}{N} \left(\frac{2N(p-1)}{2(p+1) - N(p-1)}\right) \|Q\|_2^{\frac{1}{s_c}}.
\]
In particular, since \( f(x_1) = \frac{n}{N} x_1^2 \), using the condition (2.4) we see that

\[
E_{\Omega}(u_0) - l < E_{0,0}(Q)M(Q)^{(\frac{1-x_0}{x_0})} \|u_0\|_2^{2(\frac{1-x_0}{x_0})} \leq \frac{8c}{N} \left( \frac{2N(p-1)}{2(p+1) - N(p-1)} \right) \|Q\|_2^\frac{1}{2} \|u_0\|_2^{2(\frac{1-x_0}{x_0})} = f(x_1).
\]  

(2.9)

Since \( E_{\Omega}(u(t)) \) is independent of \( t \), we infer that

\[
f(\| \nabla u(t) \|_2) \leq E_{\Omega}(u(t)) - l_{\Omega}(u(t)) \leq E_{\Omega}(u_0) - l < f(x_1). \quad \text{(2.10)}
\]

On the other hand, using the condition (2.5) we infer that

\[
\| \nabla u_0 \|_2 < \| \nabla Q \|_2 \|Q\|_2\| \nabla u_0 \|_2^{\frac{1}{2}} \|u_0\|_2^{\frac{1}{2}} = \left( \frac{2N(p-1)}{2(p+1) - N(p-1)} \right)^\frac{1}{2} \|Q\|_2^\frac{1}{2} \|u_0\|_2^{\frac{1}{2}} = x_1.
\]

Therefore, by the continuity of \( \| \nabla u(t) \|_2 \) in \( t \), and considering (2.10) we deduce that \( \| \nabla u(t) \|_2 < x_1 \) for any \( t \) as long as the solutions exists. Denote by \( I \) the maximal interval of existence of the solution \( u \). Since \( \| \nabla u(t) \|_2 \leq K \) for any \( t \in I \), from the energy conservation and (2.1) we get

\[
\frac{1}{2} \| \nabla u(t) \|_2^2 + \frac{1}{2} \| V u(t) \|_2^2 = E_{\Omega}(u_0) - l_{\Omega}(u(t)) + \frac{2}{p+1} \| u(t) \|_{p+1}^{p+1} \leq E_{\Omega}(u_0) - l + C \| \nabla u(t) \|_2^{\frac{N(p-1)}{2}} \| u(t) \|_2^{p+1 - \frac{N(p-1)}{2}} \leq E_{\Omega}(u_0) - l + CK \| u_0 \|_2^{p+1 - \frac{N(p-1)}{2}}.
\]

for every \( t \in I \). Thus, \( \| u(t) \|_2^2 \) is bounded for all time \( t \in I \). Then we infer that the solution exists globally in time. Next, we turn our attention to the proof of part (ii). Suppose by contradiction that the corresponding solution \( u(t) \) of (1.1) with \( u(0) = u_0 \) satisfies the hypothesis (2.4)-(2.7) exists globally. Notice that by the condition (2.7), we have \( \| \nabla u_0 \|_2 > x_1 \). Now applying the condition (2.4), it is clear that there exists \( \delta_1 > 0 \) such that

\[
(E_{\Omega}(u_0) - l)M(u_0)^{\frac{1-x_0}{x_0}} < (1 - \delta_1)E_{0,0}(Q)M(Q)^{\frac{1-x_0}{x_0}}.
\]

We deduce from (2.9)-(2.10) that

\[
f(\| \nabla u(t) \|_2) \leq E_{\Omega}(u(t)) - l_{\Omega}(u(t)) \leq E_{\Omega}(u_0) - l < (1 - \delta_1)f(x_1).
\]

Therefore, by the continuity of \( \| \nabla u(t) \|_2 \) in \( t \) and (2.7), there exists \( \delta_2 > 0 \) such that \( \| \nabla u(t) \|_2^2 \geq x_1^2 + \delta_2 \) for any \( t \geq 0 \). Thus, using the relation (2.9) and multiplying
the viral identity (2.2) by $M[u_0] \frac{1}{1-x}$ we obtain for $t > 0$,
\[
M(u_0) \frac{1}{1-x} J''(t) = N(p-1) \left( E_{\Omega}(u_0) - l_2(u(t)) \right) M[u_0] \frac{1}{1-x}
- \left( \frac{N(p-1)-4}{2} \right) \|\nabla u(t)\|^2 M[u_0]^{\frac{1}{1-x} - \frac{4}{2}}

- \left( \frac{N(p-1)+4}{2} \right) \int_{\mathbb{R}^N} V(x)|u(x)|^2 dx M[u_0]^{\frac{1}{1-x} - \frac{4}{2}}
\]

< $N(p-1) \left( E_{\Omega}(u_0) - l \right) M[u_0]^{\frac{1}{1-x} - \frac{4}{2}}$

- \left( \frac{N(p-1)-4}{2} \right) \delta_2 M[u_0]^{\frac{1}{1-x} - \frac{4}{2}}
\]

= \left( \frac{N(p-1)-4}{2} \right) \delta_2 M[u_0]^{\frac{1}{1-x} - \frac{4}{2}}. \quad (2.11)

Since $p > 1 + \frac{2}{N}$, integrating (2.11) twice and taking $t$ large, the right-hand side of (2.11) becomes negative, which is a contradiction. Thus, the maximum existence time is finite.

Next, we prove (iii) of theorem. Since $E_{\Omega}(u_0) - l < 0$, by (2.2) we infer that the corresponding solution blows up in finite time. Now we will show that
\[
\|\nabla u(t)\|_2 \geq \left( \frac{(p-1)N}{4} \right) \left( \|Q\|_2 \right)^{\frac{1}{p+1} - \frac{4}{2}} \|Q\|_2.
\]

for every $t$ in the existence time. Indeed, using (2.1) and multiplying both sides of $E_{\Omega}(u)$ by $M(u)^{\frac{1}{1-x} - \frac{4}{2}}$ we infer that
\[
\left( E_{\Omega}(u) - l_2(u) \right) M(u)^{\frac{1}{1-x} - \frac{4}{2}} = \frac{1}{2} \left( \|\nabla u\|_2 \|u\|_2^{\frac{1}{1-x} - \frac{4}{2}} \right)^2 - \frac{2}{p+1} \|u\|_2^{2(\frac{1}{1-x} - \frac{4}{2})} \|u\|_2^p + 1
\]

\[
\geq \frac{1}{2} \left( \|\nabla u\|_2 \|u\|_2^{\frac{1}{1-x} - \frac{4}{2}} \right)^2 - \frac{2CGN}{p+1} \left( \|\nabla u\|_2 \|u\|_2^{\frac{1}{1-x} - \frac{4}{2}} \right)^{\frac{p-1}{2}}
\]

= h \left( \|\nabla u\|_2 \|u\|_2^{\frac{1}{1-x} - \frac{4}{2}} \right),
\]

where
\[
h(x) = \frac{1}{2} x^2 - \frac{2 CGN}{p+1} \frac{x^{(p-1)}}{x^{p+1}}, \quad \text{for } x \geq 0.
\]

A simple computation shows that $h$ is increasing on $(0, x_{max})$ and decreasing on $(x_{max}, \infty)$, where
\[
x_{max} = \left( \frac{2N(p-1)}{2(p+1) - N(p-1)} \right)^{\frac{1}{2}} \|Q\|_2 \|Q\|_2^{\frac{1}{1-x} - \frac{4}{2}} - \|\nabla Q\|_2 \|Q\|_2^{\frac{1}{1-x} - \frac{4}{2}}.
\]

Moreover, we have
\[
h(x_{max}) = \frac{2N(p-1)}{2(p+1) - N(p-1)} \|Q\|_2^2 = E_{\Omega}(Q) M(Q)^{\frac{1}{1-x} - \frac{4}{2}}.
\]

Notice that $h(x) > 0$ for small enough $x > 0$. It is not difficult to show that $h$ has a unique positive root, denoted by $x_r$,
\[
x_r = \left( \frac{(p-1)N}{4} \right) \|\nabla Q\|_2 \|Q\|_2^{\frac{1}{1-x} - \frac{4}{2}}.
\]
On the other hand, we deduce from the condition (2.8) that $h(||\nabla u(t)||_2 ||u_0||^{-\frac{4}{p-2}}) < 0$. Therefore, since $p > 1 + \frac{4}{N}$, this implies that

$$||\nabla u(t)||_2 ||u_0||^{-\frac{4}{p-2}} \geq \left(\frac{(p-1)N}{4}\right)^{\frac{1}{p-2}} ||\nabla Q||_2 ||Q||^{-\frac{4}{p-2}}$$

for every $t$ in the existence time, which completes the proof of Lemma 2.2. □

Now we give the proof of Theorem 1.4.

**Proof of Theorem 1.4.** Let $u \in C([0, T_+), \Sigma)$ be the solution of (1.1) with initial data $u_0$.

(i) Notice that by (2.3) with $a = 2$ we infer that

$$|l_{\Omega}(u(t))| \leq \frac{1}{4} |\Omega|^2 ||xu(t)||^2 + ||\nabla u(t)||^2,$$

for any $t$ as long the solution exists. Suppose that $l = -\infty$. Then there exists a sequence of times $\{t_n\}_{n=1}^\infty$ such that $\lim_{n \to \infty} |l_{\Omega}(u(t_n))| = \infty$. Thus, by the inequality above we see that

$$\lim_{n \to \infty} \left[\frac{1}{4} |\Omega|^2 ||xu(t_n)||^2 + ||\nabla u(t_n)||^2\right] = \infty.$$

Assume by contradiction that there exists $C > 0$ such that $||\nabla u(t_n)||^2 \leq C$ for all $n$. By conservation of energy and (2.3) we see that for $a > 0$,

$$\left(\frac{\gamma^2}{2} - \frac{1}{2a} |\Omega|^2\right) ||xu(t_n)||^2 \leq C + E_{\Omega}(u_0)$$

for all $n$, which is an absurd. Therefore, $\lim_{n \to \infty} ||\nabla u(t_n)||^2 = \infty$. By the local theory, after extracting a subsequence, we have that $t_n \to T_+$.

Statements (ii) and (iii) are an immediate consequence of Lemma 2.2 □

3. Existence and characterization of ground states

In this section we give the proof of the existence of ground states given in Proposition 1.7. We define

$$d(\omega) = \inf \{S_\omega(u) : u \in \Sigma \setminus \{0\}, L_\omega(u) = 0\},$$

$$M_\omega = \{\varphi \in \Sigma : S_\omega(\varphi) = d(\omega), L_\omega(u) = 0\}.$$ By using the fact that $\lambda_0$ the smallest eigenvalue of the Schrödinger operator $R_\Omega = -\Delta + V(x) + 2L_\Omega$ (see (1.7)), we infer that $t[u]$ is bounded from below and $t[u] \geq -\lambda_0 ||u||_2^2$. Notice that $\sqrt{t[u]} + \omega ||u||_2^2$ define a norm in the space $\Sigma$ for $\omega > \lambda_0$. We have the following result.

**Lemma 3.1.** Let $\omega > \lambda_0$. For $|\Omega| < \gamma$ we have the equivalence of norms

$$\sqrt{t[u]} + \omega ||u||_2^2 \cong ||u||_\Sigma.$$

**Proof.** By the Young’s inequality we infer that for any $a > 0$

$$|l_{\Omega}(v)| \leq \frac{a}{2} ||\nabla v||^2 + \frac{|\Omega|^2}{2a} ||xv||^2. \quad (3.1)$$

By using (3.1), a simple calculation shows that there exists $C_{max} > 0$ such that $\sqrt{t[u]} + \omega ||u||_2^2 \leq C_{max} ||u||_\Sigma$. On the other hand, suppose that $t[u_n] + \omega ||u_n||_2^2 \to 0$ as $n \to 0$. From (1.7) we infer that $||u_n||_2^2 \to 0$ as $n \to \infty$. This implies that $t[u_n] \to 0$. Now, since $|\Omega| < \gamma$, by (3.1) we see that there exists $C_{min} > 0$ such that

$$t[u_n] \geq C_{min} \{||\nabla u_n||^2 + ||xu_n||_2^2\}.$$

Therefore $||u_n||_2^2 \to 0$ as $n \to \infty$. The proof is complete. □
We claim that 

\[ u_{n+1} = C(t[u] + \omega \|u\|_2^2)^{p_{n+1}} \]

This implies that 

\[ \|u\|_{p+1} \leq C \|u\|_{p} \]

Taking the infimum, we get 

\[ d := \inf_{u \in M} X \cdot u \]

Lemma 3.2. If $\omega > \lambda_0$, then the quantity $d(\omega)$ is positive.

Proof. Let $u \in \Sigma$ be such that $I_\omega(u) = 0$. Since $I_\omega(u) = 0$, we infer that 

\[ \|u\|_{p+1} = C(t[u] + \omega \|u\|_2^2)^{p+1} \]

Therefore 

\[ S_\omega(u) = \frac{1}{2} I_\omega(u) + \frac{p-1}{p+1} \|u\|_{p+1} \]

Taking the infimum, we get $d(\omega) > 0$. The proof is complete. \[ \square \]

Lemma 3.3. Let $\omega > \lambda_0$. The set $M_\omega$ is non-empty.

Proof. Let $\{u_n\}_{n=1}^\infty$ be a minimizing sequence of $d(\omega)$. Since $S_\omega(u_n) = \frac{1}{2} I_\omega(u_n) + \frac{p-1}{p+1} \|u\|_{p+1} \to d(\omega)$ as $n$ goes to $\infty$, we infer that $\|u_n\|_{p+1}$ is bounded. Thus, from $I_\omega(u_n) = 0$ we obtain that $t[u_n] + \omega \|u_n\|_2^2$ is bounded in $\Sigma$. Therefore, there exists $u \in \Sigma$ such that, up to sequence, $u_n \to u$ weakly in $\Sigma$ and 

\[ t[u_n] + \omega \|u_n\|_2^2 \leq \inf \{ \inf \{t[u] + \omega \|u\|_2^2 \} \} \]

Now, since $\Sigma \hookrightarrow L^{p+1}$ is compact for $1 \leq p < 2^*$, we have $u_n \to u_0$ strongly in $L^{p+1}$. By (3.2), this implies 

\[ I_\omega(u_0) \leq \lim \inf \{ t[u_n] + \omega \|u_n\|_2^2 - 2 \|u_n\|_{p+1} \} = \lim \inf \{ I_\omega(u_n) = 0 \} \]

and we also have 

\[ d(\omega) = \lim_{n \to \infty} S_\omega(u_n) = \lim_{n \to \infty} \frac{p-1}{p+1} \|u_n\|_{p+1} = \frac{p-1}{p+1} \|u\|_{p+1} \]

We claim that $u_0 \in M_\omega$. To show this we only need to show that $I_\omega(u_0) = 0$. To see this, suppose that $I_\omega(u_0) < 0$. For $\kappa > 0$ we see that 

\[ \kappa^2 I_\omega(\kappa u_0) = t[u_0] + \omega \|u_0\|_2^2 - 2 \kappa^{p-1} \|u_0\|_{p+1} \]

A simple calculation shows that the only solution to the equation $\kappa^2 I_\omega(\kappa u_0) = 0$ is 

\[ \kappa = \left( \frac{t[u_0] + \omega \|u_0\|_2^2}{2 \|u_0\|_{p+1}} \right)^{\frac{p}{p-1}} \]

Notice that $0 < \kappa < 1$. Now, since $I_\omega(\kappa u_0) = 0$, by definition of $d(\omega)$ we see that 

\[ \frac{p-1}{p+1} \|u_0\|_{p+1} = \frac{p-1}{p+1} \|u_0\|_{p+1} = \frac{p-1}{p+1} \|u_0\|_{p+1} \]

which is a contradiction. Therefore $d(\omega) = S_\omega(u_0)$ and $I_\omega(u_0) = 0$; that is, $u_0 \in M_\omega$. This completes the proof of the lemma. \[ \square \]

Lemma 3.4. If $\omega > \lambda_0$, then $\mathcal{G}_\omega \subset M_\omega$.

Proof. First we show that we have $M_\omega \subset \mathcal{G}_\omega$. Let $\varphi \in M_\omega$. Then there exists a Lagrange multiplier $\kappa \in \mathbb{R}$ such that $S'_\omega(\varphi) = \kappa I'_\omega(\varphi)$. A simple calculation shows that 

\[ 0 = I_\omega(\varphi) = \langle S'_\omega(\varphi), \varphi \rangle = \kappa \langle I'_\omega(\varphi), \varphi \rangle \]

Moreover, since $t[\varphi] + \omega \|\varphi\|_2^2 = 2 \|\varphi\|_{p+1}^2$, we get 

\[ \langle I'_\omega(\varphi), \varphi \rangle = 2t[\varphi] + 2 \omega \|\varphi\|_2^2 - 2(p + 1) \|\varphi\|_{p+1}^2 = -2(p - 1) \|\varphi\|_{p+1}^2 < 0. \]
Therefore, $\kappa = 0$. This implies that $\varphi$ satisfies the stationary problem (1.6); that is, $\varphi \in \mathcal{A}_\omega$. Next if $v \in \mathcal{A}_\omega$, then $I_\omega(v) = \langle S_\omega(v), v \rangle = 0$ and, since $\varphi \in \mathcal{M}_\omega$, we infer that $S_\omega(\varphi) \leq S_\omega(v)$; that is,

$$S_\omega(\varphi) \leq S_\omega(v) \quad \text{for all } v \in \mathcal{A}_\omega.$$

Hence $\varphi \in \mathcal{G}_\omega$ and $\mathcal{M}_\omega \subset \mathcal{G}_\omega$. In particular, $\mathcal{G}_\omega$ is not-empty. On the other hand, let $\varphi \in \mathcal{G}_\omega$. Since $\mathcal{G}_\omega \subset \mathcal{A}_\omega$ we see that $I_\omega(\varphi) = 0$. Moreover, by using the fact that $\mathcal{M}_\omega \subset \mathcal{G}_\omega$, we infer that $d(\omega) = S_\omega(\varphi)$. Thus, $\varphi \in \mathcal{M}_\omega$. The proof is complete. □

**Proof of Proposition 1.7.** The proof of Proposition 1.7 is a consequence of the Lemmas 3.2, 3.3 and 3.4. □

### 4. Instability of standing waves

Concerning the sufficient condition for instability, i.e., Theorem 1.9, the proof follows from exactly the same argument in [13, Proposition 1.1] and we omit the details. In order to prove Collorary 1.10 we establish some notation and a lemma. We follow closely the approach of Fukuizumi and Ohta [13]. Let $\phi_\omega \in \mathcal{G}_\omega$. We set the rescaled function

$$\phi_\omega(x) = \omega^{\frac{3}{2}} \tilde{\phi}_\omega(\sqrt{\omega} x) \quad \text{for } \omega > 0. \quad (4.1)$$

It is not difficult to prove that $\tilde{\phi}_\omega(x)$ satisfies the elliptic equation

$$-\Delta \varphi + \varphi + \omega^{-2}V(x)\varphi - 2|\varphi|^{p-1}\varphi + 2\omega^{-1}I_\Omega \varphi = 0, \quad x \in \mathbb{R}^N.$$ 

Moreover, we have

$$\int_{\mathbb{R}^N} V(x)|\phi_\omega(x)|^2 dx + 2I_\Omega(\phi_\omega) = \omega^{-2} \int_{\mathbb{R}^N} V(x)|\tilde{\phi}_\omega(x)|^2 dx + 2\omega^{-1}I_\Omega(\tilde{\phi}_\omega). \quad (4.2)$$

**Lemma 4.1.** Assume $|\Omega| \ll \gamma$ and $1 + \frac{4}{N} < p < 2^*$. Let $\tilde{\phi}_\omega(x)$ be the rescaled function given in (4.1). Then there exists a sequence $\{\omega_n\}_{n=1}^{\infty}$ such that $\omega_n \to \infty$ as $n \to \infty$, and

$$\lim_{n \to \infty} \left\{ \omega_n^{-2} \int_{\mathbb{R}^N} V(x)|\tilde{\phi}_{\omega_n}(x)|^2 dx + 2\omega_n^{-1}I_\Omega(\tilde{\phi}_{\omega_n}) \right\} \leq 0.$$

**Proof.** Let $Q$ be the unique positive ground state for (1.4). It is well known that

$$||Q||_{p+1}^{p+1} = \{ ||v||_{p+1}^{p+1} : v \in \Sigma \setminus \{0\} : I_{0,1}(v) \leq 0 \},$$

where

$$I_{0,1}(v) = \int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} |v|^2 dx - 2 \int_{\mathbb{R}^N} |v|^{p+1} dx.$$ 

By Proposition 1.7, it is not difficult to show that (see proof of Lemma 3.1 in [13])

$$||\phi_\omega||_{p+1}^{p+1} = \{ ||v||_{p+1}^{p+1} : v \in \Sigma \setminus \{0\} : I_{\omega}(v) \leq 0 \}.$$ 

Thus, by using (4.1) we infer that

$$||\tilde{\phi}_\omega||_{p+1}^{p+1} = \{ ||v||_{p+1}^{p+1} : v \in \Sigma \setminus \{0\} : \tilde{I}_{\omega}(v) \leq 0 \},$$

where

$$\tilde{I}_{\omega}(v) = \int_{\mathbb{R}^N} |\nabla v|^2 dx + \omega^{-2} \int_{\mathbb{R}^N} V(x)|v(x)|^2 dx + \omega^{-2} \int_{\mathbb{R}^N} |v|^{p+1} dx + 2\omega^{-1}I_\Omega(v).$$ 

Note that for every $\kappa > 1$, there exists $\omega(\kappa) > 0$ such that $\tilde{I}_\omega(\kappa Q) < 0$ holds for all $\omega \in (\omega(\kappa), \infty)$. Indeed, since $Q$ is radial, it follows that

$$\kappa^{-2} \tilde{I}_\omega(\kappa Q) = -2(\kappa^{p+1} - 1)||Q||_{p+1}^{p+1} + \omega^{-2} \int_{\mathbb{R}^N} V(x)|Q(x)|^2 dx. \quad (4.3)$$
Moreover, it is well known that $Q$ has an exponential decay at infinity, thus
\[
\lim_{\omega \to \infty} \omega^{-2} \int_{\mathbb{R}^N} V(x) |Q(x)|^2 \, dx = 0. \tag{4.4}
\]
Therefore, from (4.3) and (4.4), we infer that for every $\kappa > 1$, there exists $\omega(\kappa) > 0$ such that $\tilde{I}_\omega(\kappa Q) < 0$ holds for all $\omega \in (\omega(\kappa), \infty)$. In particular, this implies that
\[
\|\tilde{\phi}_\omega\|_{p+1}^{p+1} \leq \kappa^{p+1} \|Q\|_{p+1}^{p+1} \quad \text{for every } \omega \in (\omega(\kappa), \infty). \tag{4.5}
\]
On the other hand, for every $\kappa > 1$ we see that
\[
\kappa^{-2} I_{0,1}(\kappa \tilde{\phi}_\omega) = -2(\kappa^{p+1} - 1) \|\tilde{\phi}_\omega\|_{p+1}^{p+1} - \omega^{-2} \int_{\mathbb{R}^N} V(x) |\tilde{\phi}_\omega(x)|^2 \, dx - 2\omega^{-1} l_\Omega(\tilde{\phi}_\omega).
\]
Moreover, a simple calculation shows that (Pohozaev identity)
\[
\|\nabla \tilde{\phi}_\omega\|^2 - \omega^{-2} \int_{\mathbb{R}^N} V(x) |\tilde{\phi}_\omega(x)|^2 \, dx = \frac{N(p-1)}{p+1} \|\tilde{\phi}_\omega\|_{p+1}^{p+1}.
\tag{4.6}
\]
Notice also that
\[
\omega^{-2} \int_{\mathbb{R}^N} V(x) |\tilde{\phi}_\omega(x)|^2 \, dx \leq \frac{2\gamma^2}{\gamma^2 - |\Omega|^2} \|\tilde{\phi}_\omega\|_{p+1}^{p+1}. \tag{4.7}
\]
Indeed, since $\tilde{I}_\omega(\tilde{\phi}_\omega) = 0$, by inequality (2.3) with $a = \omega^{-1}$ we have
\[
\omega^{-2} \int_{\mathbb{R}^N} V(x) |\tilde{\phi}_\omega(x)|^2 \, dx \leq -\|\nabla \tilde{\phi}_\omega\|^2 + 2\|\tilde{\phi}_\omega\|_{p+1}^{p+1} + 2\omega^{-1} l_\Omega(\tilde{\phi}_\omega)
\]
\[
\leq 2\|\tilde{\phi}_\omega\|_{p+1}^{p+1} + \omega^{-2}|\Omega|^2 \|x \tilde{\phi}_\omega\|_{L^2}^2
\]
\[
\leq 2\|\tilde{\phi}_\omega\|_{p+1}^{p+1} + \omega^{-2}|\Omega|^2 \int_{\mathbb{R}^N} V(x) |\tilde{\phi}_\omega(x)|^2 \, dx.
\]
The inequality above implies (4.7). On the other hand, inequality (2.3) with $a = \omega|\Omega|^2/\gamma^2$ implies that
\[
- \omega^{-2} \int_{\mathbb{R}^N} V(x) |\tilde{\phi}_\omega(x)|^2 \, dx - 2\omega^{-1} l_\Omega(\tilde{\phi}_\omega) \leq \frac{|\Omega|^2}{\gamma^2} \|\nabla \tilde{\phi}_\omega\|^2. \tag{4.8}
\]
Thus, from (4.6), (4.7) and (4.8) we have for $\omega$ sufficiently large
\[
\kappa^{-2} I_{0,1}(\kappa \tilde{\phi}_\omega) \leq -2(\kappa^{p+1} - 1) \kappa^{p+1} \|Q\|_{p+1}^{p+1} \frac{|\Omega|^2}{\gamma^2} \left( \frac{N(p-1)}{p+1} + \frac{2\gamma^2}{\gamma^2 - |\Omega|^2} \right) \|Q\|_{p+1}^{p+1}. \tag{4.9}
\]
By (4.5), we infer that there exists a sequence $\{\omega_n\}_{n=1}^{\infty}$ such that $|\tilde{\phi}_{\omega_n}|_{p+1} \to \alpha$ as $n \to \infty$, where $0 < \alpha \leq \|Q\|_{p+1}$. We claim that $\alpha = \|Q\|_{p+1}$. Indeed, suppose that $\alpha = a \|Q\|_{p+1}$ with $0 < a < 1$. We define $\kappa > 1$ such that $\kappa^{p+1} a < 1$. From (4.9), we see that if $|\Omega| < \gamma$, then $I_{0,1}(\kappa \tilde{\phi}_{\omega_n}) \leq 0$. This implies that
\[
\|Q\|_{p+1}^{p+1} \leq \kappa^{p+1} \lim_{n \to \infty} \|\tilde{\phi}_{\omega_n}\|_{p+1}^{p+1} = \kappa^{p+1} a \|Q\|_{p+1}^{p+1} < \|Q\|_{p+1}^{p+1},
\]
which is a contradiction. Therefore $|\tilde{\phi}_{\omega_n}|_{p+1} \to \|Q\|_{p+1}^{p+1}$ as $n \to \infty$. Now it is not difficult to show that exists $b_n$ such that $I_{0,1}(b_n \tilde{\phi}_{\omega_n}) = 0$ for every $n \in \mathbb{N}$. Since $|\tilde{\phi}_{\omega_n}|_{p+1} \to \|Q\|_{p+1}^{p+1}$, we infer that $b_n \geq 1$. From $I_{0,1}(Q) = 0$ and $I_{0,1}(b_n \tilde{\phi}_{\omega_n}) = 0$, we obtain
\[
\lim_{n \to \infty} \left\{ \|\nabla \tilde{\phi}_{\omega_n}\|^2 + \|\tilde{\phi}_{\omega_n}\|^2 \right\} = 2 \lim_{n \to \infty} b_n^{-1} \|\tilde{\phi}_{\omega_n}\|_{p+1}^{p+1}
\]
\[
\geq 2 \left\{ \|\nabla Q\|^2 + \|Q\|^2 \right\}.
\]
Therefore, since \( \tilde{I}_\omega(\tilde{\phi}_n) = 0 \) for all \( n \in \mathbb{N} \), we get
\[
\lim_{n \to \infty} \left\{ \omega_n^{-2} \int_{\mathbb{R}^N} V(x)|\tilde{\phi}_n(x)|^2 \, dx + 2\omega_n^{-1} l_1(\tilde{\phi}_n) \right\} \leq 2\|Q\|_{p+1}^2 - \|Q\|_2^2 = 0.
\]
This completes the proof. \( \square \)

**Proof of Corollary 1.10.** Set \( \phi_n(x) := \psi_\omega(sx) \). Some straightforward computations reveal that
\[
E(\phi_n^*) = \frac{s^2}{2} \|\nabla \phi_n\|_2^2 + \frac{1}{s^2} \int_{\mathbb{R}^N} V(x)|\phi_n|_2^2 \, dx - 2s \frac{s^{2(p-1)}}{p+1} \|\phi_n\|_{p+1}^{p+1} + l_1(\phi_n).
\]
Since \( P(\phi_n) = \partial_s S_\omega(\phi_n^*) \big|_{s=1} = 0 \), it follows that
\[
\partial_s^2 E(\phi_n^*)|_{s=1} = 4 \int_{\mathbb{R}^N} V(x)|\phi_n(x)| \, dx - Np - 1 \left( \frac{4 - N(p-1)}{p+1} \right) \|\phi_n\|_{p+1}^{p+1} = (4.10)
\]
On the other hand, by Lemma 4.1 and Eq. (4.2), we see that
\[
\frac{\int_{\mathbb{R}^N} V(x)|\phi_n(x)|^2 \, dx}{\|\phi_n\|_{p+1}^2} + 2\frac{l_1(\phi_n)}{\|\phi_n\|_{p+1}^2} \leq 0 \quad \text{as} \quad n \to \infty. \quad (4.11)
\]
Moreover, by (4.9), we infer that if \( |\Omega| \ll \gamma \), then for \( \omega \) sufficiently large \( I_{0,1}(\kappa \tilde{\phi}_n) \leq 0 \). This implies that there exists \( \beta > 0 \) such that
\[
\|\nabla \phi_n\|_2^2 + \omega_n^{-1} \|\phi_n\|_2^2 \leq 2\beta \quad \text{as} \quad n \to \infty. \quad (4.12)
\]
In particular, \( \|\nabla \phi_n\|_2^2 \leq 2\beta \|\phi_n\|_{p+1}^{p+1} \) for sufficiently large \( n \). Moreover, from (2.3) with \( a = \gamma^2/4 \) we have
\[
\|l_1(\phi_n)\| \leq \frac{|\Omega|^2}{\gamma^2} \|\nabla \phi_n\|_2^2 + \frac{1}{4} \int_{\mathbb{R}^N} V(x)|\phi_n(x)|^2 \, dx. \quad (4.13)
\]
From (4.11), (4.12) and (4.13), we infer that for sufficiently large \( n \),
\[
\frac{\int_{\mathbb{R}^N} V(x)|\phi_n(x)|^2 \, dx}{\|\phi_n\|_{p+1}^{p+1}} \leq \frac{8\beta|\Omega|^2}{\gamma^2}. \quad (4.14)
\]
Therefore, by (4.10) and (4.14), we have that if \( |\Omega| \ll \gamma \), then \( \partial_s^2 E(\phi_n^*)|_{s=1} < 0 \). Thus, from Theorem 1.9 we have that the standing wave \( e^{i\omega t}\phi_n(x) \) of (1.1) is unstable. The proof is complete. \( \square \)

5. Stability of standing waves

This section is devoted to the proof of Theorem 1.12 and Corollary 1.13. The following is the key lemma for our proof.

**Lemma 5.1.** Let \( |\Omega| < \gamma \) and \( 1 + \frac{4}{p} < p < 2^* \). For every \( r > 0 \), there exists \( q_0 = q_0(r) \), such that if \( q < q_0 \), then
\[
\inf \left\{ E_{\Omega}(u), \quad u \in D_q \cap B_{r/q/2} \right\} < \inf \left\{ E_{\Omega}(u), \quad u \in D_q \cap (B_r \setminus B_{r/q}) \right\} \quad (5.1)
\]

**Proof.** Notice that \( -\lambda_0 > 0 \). First, we show that \( D_q \cap B_r \) is not empty set iff \( q \leq \frac{r}{\lambda_0} \).
Indeed, let \( f \in L^2(\mathbb{R}^N) \) be the eigenfunction associated with the eigenvalue \( \lambda_0 \) given in (1.7) such that \( \|f\|_2^2 = 1 \) (the function \( f \) can be found in [19, Section 3]). Now we set \( \eta(x) := \sqrt{\frac{q}{r}} f(x) \). For \( q \leq \frac{r}{\lambda_0} \), we see that
\[
\|\eta\|_2^2 = q \quad \text{and} \quad \|\eta\|_H^2 = t[\eta] = -\lambda_0 \|\eta\|_2^2 \leq r.
\]
This implies that \( D_q \cap B_r \) is not-empty. Now, if \( u \in D_q \cap B_r \), it follows from (1.7),
\[
 r \geq \|u\|_{L^2} = t[u] \geq \lambda_0 q,
\]
that is \( q \leq \frac{r}{\lambda_0}. \) Next, we show the inequality (5.1). A simple computation shows that for all \( a > 0 \) (see proof of Lemma 3.1),
\[
\frac{1}{2} t[u] \geq \left( 1 - \frac{a}{2} \right) \|\nabla u\|_{L^2}^2 + \frac{1}{2} \left( \gamma^2 - \frac{|\Omega|^2}{a} \right) \|xu\|_{L^2}^2,
\]
Since \( |\Omega| < \gamma \), we infer that there exists a constant \( C \) such that \( \|\nabla u\|_{L^2} \leq C \|u\|_{L^2}. \) By Gagliardo-Nirenberg inequality we have
\[
\begin{cases}
 E_{\Omega}(u) \geq \frac{1}{2}\|u\|_{L^2}^2 - Cq^{\frac{p+1}{2}} - \frac{N(p-1)}{4} \|u\|_{L^2}^{\frac{N(p-1)}{2}}, \\
 E_{\Omega}(u) \leq \frac{1}{2}\|u\|_{L^2}^2 = \Phi_q(\|u\|_{L^2}),
\end{cases}
\]
where
\[
\begin{cases}
 \Gamma_q(t) = \frac{1}{2} t(1 - 2Cq^{\delta}t^\delta) \\
 \Phi_q(t) = \frac{1}{2} t
\end{cases}
\]
and
\[
\chi = \frac{1}{2} \left( p + 1 - \frac{N(p-1)}{2} \right) > 0, \quad \delta = \frac{N(p-1) - 4}{4} > 0.
\]
It is clear that to prove the inequality (5.1), we need only show that there exists \( 0 < q_0 = q_0(r) \ll 1 \) such that, for every \( q < q_0 \),
\[
\Phi_q(\gamma r/2) < \inf_{t \in (\gamma r, r)} \Gamma_q(t).
\]
Indeed, it is not difficult to show that there exists \( q_0 > 0 \), depending only on \( r \), \( N \) and \( p \) such that, if \( q < q_0 \), then \( \Gamma_q(t) \geq \frac{1}{4} t \) for \( t \in (0, r) \). This implies that
\[
\Phi_q(\gamma r/2) = \frac{1}{4} qr < \frac{1}{3} qr \leq \inf_{t \in (\gamma r, r)} \Gamma_q(t),
\]
and the proof of lemma is complete.

**Proof of Theorem 1.12.** Let \( \{u_n\} \) be a minimizing sequence for \( J'_{\omega} \). Then \( \|u_n\|_{L^2}^2 = q \) and \( \|u_n\|_{L^2}^2 \leq r \). Since \( \Sigma \hookrightarrow L^2(\mathbb{R}^N) \) is compact, there exists \( \varphi \in \Sigma \), such that \( u_n \rightharpoonup \varphi \) weakly in \( \Sigma \) and \( \|\varphi\|_{L^2}^2 = q \). Moreover, by the lower semi-continuity we have
\[
\|\nabla \varphi\|_{L^2}^2 + \int_{\mathbb{R}^N} V(x)|\varphi(x)|^2 dx + 2l_\Omega(\varphi)
\leq \liminf_{n \to \infty} \left\{ \|\nabla u_n\|_{L^2}^2 + \int_{\mathbb{R}^N} V(x)|u_n(x)|^2 dx + 2l_\Omega(u_n) \right\}.
\]
Thus \( \varphi \in D_q \cap B_r \). On the other hand, since \( u_n \rightharpoonup \varphi \) in \( L^2(\mathbb{R}^N) \), Gagliardo-Nirenberg inequality implies that \( u_n \rightharpoonup \varphi \) in \( L^{p+1}(\mathbb{R}^N) \). Again, from lower semi-continuity we infer \( E_{\Omega}(\varphi) \leq \liminf_{n \to \infty} E_{\Omega}(u_n) = J'_{\omega} \). Therefore, \( u \in G'_{\omega} \) and \( u_n \rightharpoonup \varphi \) in \( \Sigma(\mathbb{R}^N) \). In particular, \( l_\Omega(u_n) \rightharpoonup l_\Omega(\varphi) \) as \( n \to \infty \), which completes the proof of (i).

Now we prove (ii). From (5.1), we see that \( \varphi \) does not belong to the boundary of \( D_q \cap B_r \); that is, \( E_\Omega \) has a local minimum in \( \varphi \). We also notice that \( \varphi \in B_{\Omega q} \). Therefore, there exists a Lagrange multiplier \( \omega \in \mathbb{R} \) such that \( \varphi \) satisfies the stationary equation
\[
-\Delta \varphi + \omega \varphi + V(x)\varphi - 2|\varphi|^{p-1}\varphi + 2L_\Omega \varphi = 0.
\]
Notice that
\[
J'_{\omega} \leq E_{\Omega}(\eta) = \frac{1}{2}\|\eta\|_{L^2}^2 - \frac{2}{p+1}\|\eta\|_{L^{p+1}}^2 < -\frac{1}{2}\lambda_0 q.
\]
where \( \eta \in D_q \cap B_r \) is given in Lemma 5.1. This implies that
\[
\omega \| \varphi \|^2_2 = -2E_G(\varphi) + 2\frac{(p-1)}{(p+1)}\| \varphi \|^p_{p+1} + 2J_q^e + 2\frac{(p-1)}{p+1}\| \varphi \|^p_{p+1} > \lambda_0 q.
\]
Since \( \| \varphi \|^2_2 = q \), it follows that \( \omega > -\lambda_0 \). Moreover, from \( I_\omega(\varphi) = 0 \), we infer
\[
\omega \| \varphi \|^2_2 = -4[\varphi] + 2\| \varphi \|^p_{p+1} \\
\leq -\| \varphi \|^2_H + C\| \varphi \|^{N(p-1)}_H q^{\frac{p+1}{2}} - \| \varphi \|^2_H \\
= -\| \varphi \|^2_H \left( 1 - C\| \varphi \|^{N(p-1)}_H - \frac{p+1}{2} - \| \varphi \|^2_H \right) \\
\leq -\| \varphi \|^2_H \left( 1 - C(q)^{\frac{p+1}{2}} \right),
\]
Thus, by using the fact that \( \| \varphi \|^2_H \geq -\lambda_0 \| \varphi \|^2_2 \) we see that
\[
\omega \leq \lambda_0 \left( 1 - C(q)^{\frac{p+1}{2}} \right),
\]
and finishes the proof.

Now, we are able to prove the stability the set \( G^e_\sigma \) given in Corollary 1.13.

**Proof of Corollary 1.13.** We verify the statement by contradiction. Assume that there exist \( \epsilon > 0 \) and two sequences \( \{ u_{0,n} \} \subset \Sigma \) and \( \{ t_n \} \subset \mathbb{R} \) such that
\[
\inf_{\varphi \in G^e_\sigma} \| u_{0,n} - \varphi \|_\Sigma < \frac{1}{n} \quad (5.2)
\]
\[
\inf_{\varphi \in G^e_\sigma} \| u_n(t_n) - \varphi \|_\Sigma \geq \epsilon \quad \text{for every } n \in \mathbb{N}, \quad (5.3)
\]
where \( u_n(t) \) is the solution to (1.1) with initial datum \( u_{0,n} \). A standard argument shows that we can assume \( \| u_{0,n} \|^2_2 = q \). The conservation of mass and energy implies that
\[
\| u_{0,n} \|^2_2 = q \quad \text{for every } n, \quad (5.4)
\]
\[
E(u_{0,n})(t_n) = E(u_{0,n}) \to J^e_q \quad \text{as } n \to +\infty. \quad (5.5)
\]
We next claim that there exists a subsequence \( \{ u_{0,n_k}(t_{n_k}) \} \) of \( \{ u_n(t_n) \} \) such that \( u_{0,n_k}(t_{n_k}) \in D_q \cap B_r \). Indeed, from (5.4), we only need to show that \( \| u_{0,n_k}(t_{n_k}) \|^2_H \leq r \). Suppose, by contradiction, that there exists \( K \geq 1 \) such that \( \| u_{n_k}(t_{n_k}) \|^2_H > r \) for all \( n \geq K \). By continuity and (5.2), we infer that there exists \( t^{*}_n \in (0,t_n) \) such that \( \| u_{n_k}(t^{*}_n) \|^2_H = r \). Thus, \( u_{n_k}(t^{*}_n) \in D_q \cap B_r \), and from (5.5), we see that \( \{ u_{n_k}(t^{*}_n) \} \) is a minimizing sequence of \( J^e_q \). By Theorem 1.12, there exists \( \psi \in \Sigma \) such that \( \| \psi \|^2_2 = q \) and \( \| \psi \|^2_2 = r \), which is a contradiction with the fact that the critical point \( \psi \) does not belong to the boundary of \( D_q \cap B_r \) (see Lemma 5.1) and the claim follows immediately. On the other hand, using (5.5), we see that \( \{ u_{0,n}(t_{n_k}) \} \) is a minimizing sequence for \( J^e_q \). Again, by Theorem 1.12 there exists \( f \in G^e_\sigma \) such that, passing to a subsequence if necessary, \( \{ u_{0,n}(t_{n_k}) \} \) converges strongly to \( f \) in \( \Sigma \), which is a contradiction with (5.3). This completes the proof of Corollary.
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