Reaction-free trajectories of a classical point charge

Michael Ibison

Institute for Advanced Studies at Austin
4030 West Braker Lane, Suite 300, Austin, TX 78759

Abstract

Though it is well-known that 1+1 D hyperbolic motion in space and time is reaction-free, it is not widely acknowledged that a classical point charge can also execute a curved path through space, i.e. in 2+1 D, without incurring a reaction force. This note catalogs the full family of reaction-free trajectories, giving a geometrical interpretation by which means the curved path possibility is easily related to the better known case of hyperbolic motion in 1+1 D.
I. INTRODUCTION

Though it is widely accepted that uniform acceleration of a charge in 1+1D experiences no reaction, there has been no published account of the full catalog of reaction-free trajectories in two space dimensions and one time dimension. Only Rohrlich appears to have given an expression for the complete family of such trajectories, though his subsequent analysis and discussion are restricted to the well-known special case of uniform acceleration in just 1+1 D. The purpose of this note is to fill in the details, with special emphasis on the space-time geometry. We avoid getting involved in the continuing debate on whether or not a charged particle radiates whilst in reaction-free motion, for which the reader is referred to the literature.

II. THE ABRAHAM - VON LAUE VECTOR

In Heaviside units with \( c = 1 \), the Lorentz-Abraham-Dirac equation for a classical charge of mass \( m \) is

\[
ma = f_{\text{ext}} + \Gamma \tag{1}
\]

where \( f_{\text{ext}} \) is the external force 4-vector, and \( \Gamma \) is the von Laue - or Abraham - 4-vector given by

\[
\Gamma = \frac{2e^2}{3} \left( \frac{da}{d\tau} + a^2 u \right), \tag{2}
\]

where \( a \) and \( u \) are, respectively, the proper acceleration and velocity 4-vectors and where - introducing the Lorentz scalar product symbol and metric used hereafter - \( a^2 \equiv a \circ a = a^\mu a_\mu = a_0^2 - a^2 \). We include Abraham in the authorship of (1) in accord with the position taken by Rohrlich. The first of the two terms in \( \Gamma \) is the Schott term - also called the ‘acceleration reaction force’ by von Laue. The second term is called the radiation reaction. Noting that

\[
\frac{d}{d\tau} (u \circ a) = 0 = a^2 + u \circ \frac{da}{d\tau}, \tag{3}
\]

the Abraham - von Laue vector can also be written

\[
\Gamma^\mu = \frac{2e^2}{3} (\eta^{\mu\nu} - u^\mu u^\nu) \frac{da_\nu}{d\tau}. \tag{4}
\]

In this form it is clear that \( u \circ \Gamma = 0 \), a result demanded of any four vector supplement to the relativistic equation of motion \( m_e a = f_{\text{ext}} \), because, if for example the external force is
the Lorentz force, then \( u \circ a = u \circ f_{\text{ext}} = 0 \).

The Abraham - von Laue vector can be derived from the action of the retarded EM fields of a charged sphere upon itself in the limit that the radius of the sphere goes to zero\(^6,12\). Obviously, though they are responsible for singular self-energy, the retarded fields of a uniformly moving charge can produce no self-force, from which it follows that the Abraham - von Laue vector stands for the retarded self-force of a charge in non-uniform motion. It is to be noted that because the self-force can be temporarily non-zero even when the proper acceleration is zero - \( a^2 u = 0 \), \( da/d\tau \neq 0 \) - non-uniform motion responsible for a non-vanishing von Laue vector does not exclusively imply acceleration. By contrast, in the reaction-free case under consideration here, the self-force is zero even though the charge is accelerating: \( \Gamma = 0 \), \( a^2 \neq 0 \).

Historically it was held that \( \Gamma = 0 \) implies that the charge is not radiating\(^2,10\). But later analysis\(^1,3,4,7\) and commentary\(^5,6\) decided in favor of the presence of radiation whenever there is acceleration - independent of the value of \( \Gamma \). In this case the Lorentz-invariant generalization of the radiated power is given by the relativistic Lamor formula, \( P = -2e^2 a^2 / 3 \), and if \( \Gamma = 0 \) then the charge produces radiation with no net reaction back upon the source. A more recent exchange\(^8,9,15,16\), however, has re-opened the issue, turning on the definition of uniform acceleration ‘for all time’. In that case, in order to maintain that \( \Gamma = 0 \) implies no radiation, one would have to modify the relativistic Lamor formula somehow, a non-relativistic example of which has been given by Peierls\(^11\).

III. THE REACTION-FREE TRAJECTORIES

The reaction-free condition that there be no retarded self-force acting back upon the charge is that the Abraham - von Laue vector is zero:

\[
\frac{da}{d\tau} + a^2 u = 0. \tag{5}
\]

Because \( a \circ u = 0 \) the Lorentz scalar product with the acceleration gives

\[
a \circ \frac{da}{d\tau} = 0 \Rightarrow a^2 = -k^2 \tag{6}
\]

where \( k \) is a real constant. The sign follows because \( a \) is space-like, which follows in turn from \( a \circ u = 0 \) and that \( u \) is time-like. With this (5) implies that

\[
\frac{d^2 u}{d\tau^2} = k^2 u, \tag{7}
\]
the general solution of which can be written

\[ x = r + \left( p \cosh (k\tau) + q \sinh (k\tau) \right) / k \quad (8) \]

where \( p, q \) and \( r \) are constant 4-vectors, which is the result given by Rohrlich\(^{12}\). \( p \) and \( q \) are not entirely arbitrary, but must be chosen to satisfy (6):

\[ (p \cosh (k\tau) + q \sinh (k\tau))^2 = -1. \quad (9) \]

Since this must be true for all \( t \) it follows that

\[ p^2 = -1, \quad q^2 = 1, \quad p \cdot q = 0. \quad (10) \]

With these, (8) then gives that \( u^2 = 1 \), as required. Eq. (8) with Eq. (10) gives the full family of reaction-free trajectories for the classical charged particle.

In (8) there are 4 constant degrees of freedom in each of \( p, q \) and \( r \), and one degree of freedom in \( k \). Eq. (10) removes 3 degrees of freedom, which leaves a total of 10. This is one more than to be expected from the solution in ordinary time of the three second order differential equations implicit in (5). The extra degree of freedom is in the arbitrariness of the origin of \( \tau \), which need not have any relation to the origin of \( t \). That arbitrariness will be employed to require that at \( t = r_0 \) at \( \tau = 0 \). With this, (8) gives that \( p_0 = 0 \), whereupon (10) gives

\[ p = (0, \hat{p}), \quad q = \left( \sqrt{1 + q^2}, q \right), \quad \hat{p} \cdot q = 0 \quad (11) \]

where the sign of \( q_0 \) has been chosen so that \( \tau \) is an increasing function of \( t \).

Since \( \mathbf{p} \) and \( \mathbf{q} \) are orthogonal, it is convenient to suppose that the space axes have been oriented so that \( \mathbf{p} \) and \( \mathbf{q} \) are parallel to the \( x_1 \) and \( x_2 \) axes respectively. Let us suppose also that the space and time axes have been located so that \( r = 0 \). Then (8) becomes

\[ x \equiv (t, x_1, x_2, x_3) = \left( \sqrt{1 + q_2^2 \sinh (k\tau)}, \cosh (k\tau), q_2 \sinh (k\tau), 0 \right) / k \quad (12) \]

where \( q_2 \) is an ordinary signed scalar. Defining a new constant

\[ v_b \equiv q_2 / \sqrt{1 + q_2^2} \quad (13) \]

(where evidently \( |v_b| < 1 \) ) one obtains

\[ x = \frac{1}{k} \left( \frac{\sinh (k\tau)}{\sqrt{1 - v_b^2}}, \cosh (k\tau), \frac{v_b \sinh (k\tau)}{\sqrt{1 - v_b^2}}, 0 \right) . \quad (14) \]
Eliminating the proper time one has that the reaction-free trajectory is either of the branches of
\[ x_1 = \sqrt{(1-v_b^2) t^2 + 1/k^2}, \quad x_2 = v_b t. \] (15)

\section*{IV. SPACE-TIME GEOMETRY OF THE TRAJECTORY}

Eq. (15) describes the curve that is the intersection of the space-time plane
\[ x_2 = v_b t, \quad |v_b| < 1 \] (16)
with the hyperboloid
\[ x_1^2 + x_2^2 = t^2 + 1/k^2. \] (17)
That is, equation (15), for various $v_b$ and $k$, are sections of a space-time hyperboloid. Consequently the trajectories are hyperbolas in the sectioning plane - plane curves in two space dimensions and one time dimension. The space-time hyperboloid (17) is always oriented along the time axis, asymptotic to a 45° cone. It can be arbitrarily located in space and time, and arbitrarily oriented in 3D space. It can have any waist size (given by $1/k$). The sectioning space-time plane is arbitrary except for the constraint that the angle between its normal and the time axis, modulo 180°, must be greater than 45°. This guarantees that it cuts the hyperbola along a hyperbolic space-time path - it cannot cut the hyperboloid along an elliptical path. A particular case is depicted in Fig. 1: the shaded plane is (16) with $v_b = 0.75$, and the wire-frame surface is the hyperboloid (17) with $k = 1$.

From the considerations above it follows that the space projection of the hyperbolic path is entirely arbitrary. That is, a charge following any hyperbolic path in 2 space dimensions can be rendered reaction-free - provided the speed on the path is chosen in accordance with (8). For example, the spatial projection of (15), i.e. the path through space without regard to the time, is the hyperbola
\[ x_1^2 - (1/v_b^2 - 1) x_2^2 = 1/k^2. \] (18)
This hyperbola (18) is oriented along the $x_1$ axis and has major axis of length $1/k$ and asymptotes to the lines $x_2 = \pm v_b x_1 / \sqrt{1 - v_b^2}$.

In the particular case that the sectional plane contains the time-axis, $x_2 = 0 \Leftrightarrow v_b = 0$, one has from (16) and (17) that
\[ x_2 = 0, \quad x_1 = \sqrt{t^2 + 1/k^2} \] (19)
for the projection of the trajectory onto the $t, x_1$ axes. The corresponding space-only projection is then either of the semi-infinite straight lines $x_1 > 1/k$, $x_2 = 0$, or $x_1 < -1/k$, $x_2 = 0$. This is the traditional case (hereafter: ‘special case’) of a 1+1 D space-time hyperbola considered in the literature and is shown in Fig. 2.

Viewing the special case trajectory from a moving frame travelling at speed $v_b$ in the negative $x_2$ direction and referenced by primed coordinates, one has

$$t = \frac{t' - v_b x'_2}{\sqrt{1 - v_b^2}}, \quad x_1 = x'_1, \quad x_2 = \frac{x'_2 - v_b t'}{\sqrt{1 - v_b^2}},$$  

(20)

In the new coordinates (19) becomes

$$x'_1 = \sqrt{(t' - v_b x'_2)^2 - \frac{1}{k^2}} = \sqrt{(1 - v_b^2) t'^2 + 1/k^2}, \quad x'_2 = v_b t'$$  

(21)

which agrees with (15). Hence it is clear that the novel degrees of freedom in the ‘non-special’ space-time trajectory (i.e., apart from the obvious freedoms of spatial orientation and space-time location) can be obtained from boosts of the special case hyperbolic space-time curve. This correspondence justifies the choice of the symbol $v_b$, introduced in (13); initially regarded as one of the arbitrary constants in the solution of (5), it turns out to be the velocity of the boosted viewpoint of the special case. Additionally, one sees (retrospectively) that $q_2$ in (13) is the proper speed of the boosted viewpoint.

Rather than boosts of the special space-time curve, an alternative way to generate the family of reaction-free curves is to boost the special case surfaces - the hyperboloid and the sectioning plane - themselves. The hyperboloid (17) is a special case of an invariant space-time surface under Lorentz boosts:

$$x_1^2 + x_2^2 = t^2 + 1/k^2 \rightarrow x'^1_1 + x'^2_2 = t'^2 + 1/k^2.$$  

(22)

The plane $x_2 = 0$ is not an invariant surface, but transforms under boosts like

$$x_2 = 0 \rightarrow x'_2 = v_b t'.$$  

(23)

With reference to Fig. 1, the result now follows immediately that the family of reaction-free curves is generated by rotating the sectioning plane whilst leaving the hyperboloid unchanged.
A. Uniform motion

In the limit that the speed of the boosted viewpoint is \( v_b = 1 \), (16) gives that the plane is inclined at 45° with respect to the time axis and Eq. (17) then gives that \( x_1 = \pm 1/k \). These reaction-free trajectories are the two parallel straight line null rays in \( x_2, t \) located at \( x_1 = 1/k \) and \( x_1 = -1/k \). Through reorientation and relocation of the axes and variation of the arbitrary value of \( k \), the geometry can generate every possible pair of parallel null rays, a particular example of which is given in Fig. 3.

The particular case that \( k = 0 \) requires special treatment: From either (6) or (7) one has that there is no acceleration. Eq. (7) then generates single trajectories with arbitrary velocity; rectilinear motion is reaction-free.

V. EXTERNAL FORCES CAUSING NO REACTION

In the case that there is no reaction, all that remains of the Lorentz-Abraham-Dirac equation (1) is that \( f_{ext} = ma \) where \( f_{ext} = \gamma (F \cdot v, F) \) is the proper 4-force, and \( F \) is the ordinary, e.g. Lorentz, force. In the coordinate system located so that \( r = 0 \) one has \( \mathbf{a} = k^2 \mathbf{x} \), and therefore from (14)

\[
\mathbf{F} = ma/\gamma = mk^2 \mathbf{x}/\gamma ,
\]

where

\[
\gamma = \sqrt{1 + \mathbf{u}^2} = \sqrt{1 + \sinh^2 (k\tau) + \frac{v_b^2 \cosh^2 (k\tau)}{1 - v_b^2}} = \gamma_b \cosh (k\tau)
\]

(25)

where \( \gamma_b = 1/\sqrt{1 - v_b^2} \). Note that the proper acceleration is not constant. Using this and (14) the components of force required to produce reaction-free motion are found to be

\[
F_1 = mk/\gamma_b, \quad F_2 = v_b mk \tanh (k\tau) = \frac{v_b mk^2 t}{\sqrt{\gamma_b^2 + k^2 t^2}} ,
\]

(26)

where the ordinary time form of the last expression may be obtained from the 0th component of \( \mathbf{x} \) as given in (14).

Clearly the special case \( v_b = 0 \) requires only a constant ordinary force, for example a uniform electric field. If \( v_b \neq 0 \) - corresponding to reaction-free trajectory that is hyperbolic in space - the force is still constant along the major axis (\( x_1 \)-axis), whereas an additional transverse force is required that is odd in time (and therefore in the direction of the minor
axis). This component of force tends to the constant value $F_2 \to v_bmk$ as $|x_2|, |t| \to \infty$. It may at first seem surprising that a transverse component of force is necessary, since the component of velocity of the charge in that direction is just the constant $v_b$, and therefore the ordinary acceleration in the direction of $x_2$ is zero. However the proper acceleration in that direction is not zero; one has

$$\frac{d^2x}{d\tau^2} = \gamma \frac{d}{dt} \left( \gamma \frac{dx}{dt} \right) = \gamma^2 \frac{d^2x}{dt^2} + \frac{1}{2} \frac{d\gamma^2}{dt} \frac{dx}{dt}$$  \hspace{1cm} (27)$$

from which one observes that the proper acceleration in any fixed direction can be driven, via the term $d\gamma^2/dt$, by speed changes exclusively in other, orthogonal, directions.

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* Electronic address: ibison@earthtech.org

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FIG. 1: A reaction-free space-time trajectory in 1+1 D, depicted as the intersection of a plane with a hyperboloid. The single heavy straight line is the time axis. The two heavy curved lines are the two branches of the reaction-free hyperbolic space-time path given by Eq. 15. They can be regarded as the special case reaction-free space-time trajectories of Fig. 2 viewed from a frame moving at speed $v_b = 0.75$ in a direction normal to the shaded plane of that figure.
FIG. 2: The traditional - special case - reaction-free space-time trajectory in 1+1 D, depicted as the intersection of a plane containing the $t$-axis with a hyperboloid.
FIG. 3: Parallel null rays resulting from a sectioning plane inclined at 45° to the time axis. They can be regarded as the special case trajectories of Fig. 2 viewed from a frame moving at light speed in a direction normal to the shaded plane of that figure.