Existence of the Solution
for the ’t Hooft-Polyakov Monopole

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Abstract

In this paper we give a mathematical proof of the existence of the time independent and spherically symmetric solution to the ’t Hooft-Polyakov model of magnetic monopole by using 2D-shooting method.

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1. Introduction

The existence problem of the magnetic monopole has fascinated physicists since Dirac’s classic work [1] before fifty years ago. In 1974, 't Hooft [2] and Polyakov [3] proposed a model for a magnetic monopole which arises as a static solution of the classical equations for the $SU(2)$ Yang-Mills field coupled to an $SU(2)$ Higgs field. The model has been extended by Julia and Zee [4], and Cho and Maison [5]. In this paper, we discuss the existence and asymptotics for the solution of 't Hooft-Polyakov monopole. Prasad and Sommerfield [6] found exact solutions of the field equation for a special case ($\lambda = 0$). But so far to our knowledge the boundary value problem for the equation of motion with $\lambda > 0$ was just studied numerically. And the existence of the 't Hooft-Polyakov monopole is just convinced by numerical computations. In this paper we give a mathematical proof for the existence of the 't Hooft-Polyakov magnetic monopole.

The Lagrangian of the 't Hooft-Polyakov model [2] [3] is

\[ \mathcal{L} = -\frac{1}{4}F_{\mu \nu}^a F_{\mu \nu}^a - \frac{1}{2}D_\mu \phi_a D_\mu \phi_a + \frac{1}{2}\mu^2 \phi_a \phi_a - \frac{1}{4}\lambda (\phi_a \phi_a)^2, \]

where

\[ F_{\mu \nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + e\epsilon_{abc}A_\mu^b A_\nu^c, \]

\[ D_\mu \phi_a = \partial_\mu \phi_a + e\epsilon_{abc}A_\mu^b \phi_c. \]

The Wu-Yang [7]-'t Hooft-Polyakov Ansatz is to seek a solution of the equations of motion (which can be derived from the Lagrangian, see for example [3]) in the form

\[ A_i^a = \epsilon_{aib} \frac{x_j}{e r^2} (1 - K(r)), \]

\[ A_0^a = 0, \]

\[ \phi_a = \frac{x_a}{e r^2} H(r). \]

where $K(r), H(r)$ satisfy the equations

\[ r^2 K'' = K(K^2 + H^2 - 1), \]

\[ r^2 H'' = 2HK^2 + \frac{\lambda}{\epsilon^2}(H^3 - e^2 \rho_0^2 r^2 H), \]

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where \( \rho_0 = \mu/\sqrt{\lambda} \). Let’s put

\[ e = g_0, K(r) = f(r), H(r) = g_0 r \rho(r), \]  

then the equations (7), (8) reduce to

\[ f'' - f^2 - \frac{1}{r^2} f = g_0^2 \rho^2 f, \]  
\[ \rho'' + \frac{2}{r} \rho' - \frac{2f^2}{r^2} \rho = \lambda (\rho^2 - \rho_0^2) \rho, \]

with the boundary conditions [5]

\[ f(0) = 1, \quad \rho(0) = 0, \]  
\[ f(\infty) = 0, \quad \rho(\infty) = \rho_0. \]  

Because the exact solution was already found for \( \lambda = 0 \) [3], we just discuss the problem for \( \lambda > 0 \) in this paper. To solve this boundary value problem, we consider the asymptotics as \( r \to 0 \),

\[ f(r) \sim 1 - \alpha r^2, \]  
\[ \rho(r) \sim \beta r, \]

where \( \alpha, \beta > 0 \). We are going to use topological 2D-shooting method [8], [9] to prove that there exist such \( \alpha, \beta \) such that the corresponding \( f, \rho \) satisfy \( f(\infty) = 0, \rho(\infty) = \rho_0 \). That is we want to prove the following theorem in this paper.

**Theorem 1** For \( \lambda > 0, \rho_0 > 0, g_0 > 0 \), there is a solution \((f, \rho)\) to equations (10), (11) satisfying the boundary conditions (12), and

\[ 0 < f < 1, \quad 0 < \rho < \rho_0, \]  
\[ f' < 0, \quad \rho' > 0, \]  

for \( 0 < r < \infty \).

The plan of this paper is as follows. In sec. 2, we consider the asymptotic behaviour of the solution to the eqs. (10), (11). Then we discuss the behaviours of \( f(r) \) as \( \alpha \) small or large for \( \beta \) in a finite interval. We show that \( f' \) cross 0 when \( \alpha \) small in sec. 3, and \( f \) cross 0 when \( \alpha \) large in sec. 4. In sec. 5 we talk about the possibilities of the behaviours for \( \rho \) when \( f \) stays between 0 and 1. While \( f \) stays between 0 and 1, we show that when \( \beta \) is small \( \rho \) crosses 0 in sec. 6, and as \( \beta \) large \( \rho \) crosses \( \rho_0 \) in sec. 7. Finally by the topological lemma (McLeod and Serrin [8]) we show that the solution to the boundary value problem (10), (11) and (12) exists.
2. Asymptotics of Solution at the Origin

Lemma 1 For any α, β, there is a unique solution \((f, \rho)\) to the equations (10), (11), such that

\[
\begin{align*}
    f &\sim 1 - \alpha r^2, \\
    \rho &\sim \beta r,
\end{align*}
\]

as \(r \to 0\).

Proof Set

\[ s = \log r, f(r) = 1 + p(s), \rho(r) = q(s). \]

Then eqs. (10), (11) are reduced to

\[
\begin{align*}
    p'' - p' - 2p &= 3p^2 + p^3 + g_0e^{2s}q^2p, \\
    q'' + q' - 2q &= 2(2p + p^2)q + \lambda e^{2s}(q^2 - \rho_0^2)q,
\end{align*}
\]

which is at least formally equivalent to the integral equations

\[
\begin{align*}
    p(s) &= -\alpha e^{2s} + \frac{1}{3} \int_{-\infty}^{s} (e^{2(s-\sigma)} - e^{-(s-\sigma)})(3p^2 + p^3 + g_0^2e^{2\sigma}q^2p) \, d\sigma, \\
    q(s) &= \beta e^{s} + \frac{1}{2} \int_{-\infty}^{s} (e^{s-\sigma} - e^{-2(s-\sigma)})(2(2p + p^2)q + \lambda e^{2\sigma}(q^2 - \rho_0^2)q) \, d\sigma.
\end{align*}
\]

Let

\[ p = e^{2s} \phi, q = e^{s} \psi. \]

The eqs. (3) and (4) become

\[
\begin{align*}
    \phi &= -\alpha + \frac{1}{3} \int_{-\infty}^{s} (e^{2\sigma} - e^{-3s+5\sigma})(3\phi^2 + e^{2\sigma} \phi^3 + g_0^2e^{2\sigma}q^2\phi) \, d\sigma, \\
    \psi &= \beta + \frac{1}{2} \int_{-\infty}^{s} (e^{2\sigma} - e^{-3s+5\sigma})(2(2\phi + e^{2\sigma} \phi^2)\psi + \lambda (e^{2\sigma} \psi^2 - \rho_0^2)\psi) \, d\sigma,
\end{align*}
\]

which can be solved by iteration by setting

\[
\begin{align*}
    \phi_0 &= -\alpha, \psi_0 = \beta, \\
    \phi_{n+1}(s) &= -\alpha + \frac{1}{3} \int_{-\infty}^{s} (e^{2\sigma} - e^{-3s+5\sigma})(3\phi_n^2 + e^{2\sigma} \phi_n^3 + g_0^2e^{2\sigma}q^2\phi_n) \, d\sigma, \\
    \psi_{n+1}(s) &= \beta + \frac{1}{2} \int_{-\infty}^{s} (e^{2\sigma} - e^{-3s+5\sigma})(2(2\phi_n + e^{2\sigma} \phi_n^2)\psi_n + \lambda (e^{2\sigma} \psi_n^2 - \rho_0^2)\psi_n) \, d\sigma.
\end{align*}
\]
for \( n \geq 0 \).

We choose constant numbers \( K, S \) by

\[
K = \max(2|\alpha|, 2|\beta|, 3), \tag{10}
\]

\[
e^{2S} = \max(M_1, M_2, M_3, M_4), \tag{11}
\]

where

\[
M_1 = \frac{1}{2}(2(2K + K^2) + \lambda(K^2 + \rho_0^2)),
\]

\[
M_2 = \frac{1}{3}(6K + 2K^2 + 2g_0^2K^2),
\]

\[
M_3 = (8 + 3\lambda)K^2 + \lambda\rho_0^2,
\]

\[
M_4 = 4K + 3(1 + g_0^2)K^2.
\]

We claim that for \( s \in (-\infty, -S] \), there are

\[
|\phi_n|, |\psi_n| \leq K, \tag{12}
\]

\[
|\phi_{n+1} - \phi_n| \leq \frac{M}{3n+1}e^{2s}, \tag{13}
\]

\[
|\psi_{n+1} - \psi_n| \leq \frac{M}{3n+1}e^{2s}, \tag{14}
\]

for \( n \geq 0 \), where

\[
M = \max(4K^3 + \lambda(K^2 + \rho_0^2)K, (2 + g_0^2)K^3). \tag{15}
\]

This can be proved by induction. We skip the details here. Hence \( \{(\phi_n, \psi_n)\}_{n=0}^{\infty} \) is convergent. The uniqueness is similar.

### 3. The Solutions for Small \( \alpha \)

**Lemma 2** For any \( \beta_2 \geq \beta_1 > 0 \), when \( \beta \in [\beta_1, \beta_2] \), there exists \( \alpha_1 > 0 \), such that if \( \alpha \in (0, \alpha_1] \), there is \( r^- > 0 \), so that

\[
f'(r^-) > 0, \tag{1}
\]

\[
f(r) > 0, \text{ for } 0 < r \leq r^- . \tag{2}
\]
Proof  We make a scaling
\[ t = \frac{r}{\sqrt{\alpha}}, \quad f(r) = 1 + \alpha^2 p(t), \quad \rho(r) = \sqrt{\alpha} q(t). \tag{3} \]
Then the equations become
\[
\begin{align*}
  p'' &- \frac{2p + \alpha^2 p^2}{t^2}(1 + \alpha^2 p) = g_0^2 q^2(1 + \alpha^2 p), \\
  q'' &+ \frac{2}{t} q' - \frac{2(1 + \alpha^2 p)^2}{t^2} q = \lambda \alpha (\alpha q^2 - \rho_0^2)q,
\end{align*}
\tag{4,5}
\]
with the asymptotics at the origin
\[
\begin{align*}
  p(t) &\sim -t^2, \\
  q(t) &\sim \beta t,
\end{align*}
\tag{6,7}
as \( t \to 0 \).
Letting \( \alpha = 0 \), the problem is reduced to
\[
\begin{align*}
  P(t) &\sim -t^2, Q(t) \sim \beta t, t \to 0, \\
  P'' &- \frac{2P}{t^2} = g_0^2 Q^2, \\
  Q'' &+ \frac{2}{t} Q' - \frac{2}{t^2} Q = 0.
\end{align*}
\tag{8,9,10}
\]
There is a unique solution for this problem. It’s not difficult to see that the solution is
\[
\begin{align*}
  P(t) &= -t^2 + \frac{g_0^2 \beta^2}{10} t^4, \\
  Q(t) &= \beta t,
\end{align*}
\tag{11,12}
and then
\[
  P'(t) = 2t(-1 + \frac{g_0^2 \beta^2}{5} t^2). \tag{13}
\]
For \( \beta \in [\beta_1, \beta_2] \), choose small \( \epsilon^- > 0 \), and
\[
  t_0 = \frac{\sqrt{5}}{g_0 \beta_1} + \epsilon^-, \tag{14}
\]
such that
\[ P'(t_0) > 0, \]
\[ |P(t)| \leq t_0^2 + \frac{g_0^2 \beta^2}{10} t_0^4, 0 \leq t \leq t_0. \]  
(15)

Now for the solution \( p, q \) to the equations (4), (5) with the conditions (6) and (7), since \( p, p', q \) are continuous in \( r, \alpha, \beta \), by uniform continuity in compact set, there exists \( \alpha_1 > 0 \) satisfying
\[ 2\alpha_1^2 < \left( t_0^2 + \frac{g_0^2 \beta^2}{10} t_0^4 \right)^{-1}, \]
such that if \( \alpha \in (0, \alpha_1], \beta \in [\beta_1, \beta_2] \),
\[ p'(t_0) > 0, \]
\[ |p(t)| \leq 2 \left( t_0^2 + \frac{g_0^2 \beta^2}{10} t_0^4 \right), 0 \leq t \leq t_0, \]  
(16)

which implies that for \( \alpha \in (0, \alpha_1] \)
\[ f'(r^-) = \alpha^{3/2} p' \left( \frac{r^-}{\sqrt{\alpha}} \right) > 0, \]
\[ f(r) > 1 - \alpha^2 |p(t)| \]
\[ \geq 1 - 2\alpha_1^2 \left( t_0^2 + \frac{g_0^2 \beta^2}{10} t_0^4 \right) > 0, \quad 0 < r \leq r^-, \]
where \( r^- = \sqrt{\alpha} t_0 \). So the lemma is proved.

4. The Solutions for Large \( \alpha \)

**Lemma 3** For any \( \beta_2 \geq \beta_1 \geq 0 \), when \( \beta \in [\beta_1, \beta_2] \), there is a large \( \alpha_2 > 0 \), such that if \( \alpha \in [\alpha_2, \infty) \), there exists \( r^+ > 0 \), so that
\[ f(r^+) < 0, \]
\[ f'(r) < 0, \quad 0 < r \leq r^+. \]  
(1)
(2)

**Proof** We make another scaling
\[ t = \sqrt{\alpha} r, f(r) = 1 - \psi(t), \rho(t) = \phi(t). \]  
(3)
Then the equations become

\[ \psi'' - \frac{\psi(1 - \psi)(2 - \psi)}{t^2} = -\frac{1}{\alpha} g_0^2 \phi^2(1 - \psi), \quad (4) \]

\[ \phi'' + \frac{2}{t} \phi' - \frac{2(1 - \psi)^2}{t^2} \phi = \frac{1}{\alpha \lambda(\phi^2 - \rho_0^2)} \phi, \quad (5) \]

with the asymptotics as \( t \to 0 \)

\[ \psi(t) \sim t^2, \]

\[ \phi(t) \sim \frac{1}{\sqrt{\alpha t}}. \]

Then as \( \alpha \to \infty \), \( \phi \to 0 \) uniformly on compact intervals in \( t \), while \( \psi \) tends, also uniformly on compact intervals in \( t \), to the solution \( \Psi \) of

\[ \Psi'' = \Psi(1 - \Psi)(2 - \Psi) \]

\[ \Psi(t) \sim t^2, t \to 0. \]

To get the behaviour of \( \Psi \), we make a transformation

\[ s = \log t. \]

Then the equation is reduced to

\[ \Psi_{ss} = \Psi_s + 2\Psi - 3\Psi^2 + \Psi^3, -\infty < s < \infty, \]

\[ \Psi(s) \sim e^{2s}, s \to -\infty. \]

Multiplying this equation by \( d\Psi/ds \) and integrating, we arrive at

\[ \frac{1}{2} \Psi_s^2 = \Psi^2(1 - \frac{\Psi}{2})^2 + \int_{-\infty}^{s} \Psi_s^2 d\sigma. \]

This makes clear that \( d\Psi/ds \) does not vanish and \( \Psi \) becomes unbounded and certainly crosses 1, while \( d\Psi/ds \) keeps positive at least before the crossing.

By the same argument as in lemma 1, we see that for \( \beta \in [\beta_1, \beta_2] \), there is \( \alpha_2 > 0 \), such that if \( \alpha \geq \alpha_2 \), there exists \( r^+ = r^+(\alpha) \), so that the lemma holds.
5. Argument for $\alpha \notin S^-_\beta \cup S^+_\beta$

For any $\beta > 0$, define

$$S^-_\beta = \{\alpha > 0 | f' \text{ cross } 0 \text{ before } f \text{ cross } 0\},$$
$$S^+_\beta = \{\alpha > 0 | f \text{ cross } 0 \text{ before } f' \text{ cross } 0\}.$$

By Lemma 2,3, $S^-_\beta$ and $S^+_\beta$ are not empty and disjoint. By implicit function theorem, it’s not hard to prove that $S^-_\beta, S^+_\beta$ are open sets, so that $(0, \infty) \setminus (S^-_\beta \cup S^+_\beta)$ is not empty and closed set. For $\alpha \in (0, \infty) \setminus (S^-_\beta \cup S^+_\beta)$, we simply denote it as $\alpha \notin (S^-_\beta \cup S^+_\beta)$. By eq. (10) we see that if $f = 0, f' = 0$ at the same time, then the $f$ is identically zero, which is impossible. So we have proved the following lemma.

**Lemma 4** If $\alpha \notin (S^-_\beta \cup S^+_\beta)$, then

$$0 < f < 1, f' \leq 0,$$

for $0 < r < \infty$.

**Lemma 5** For $\beta > 0, \alpha \notin S^-_\beta \cup S^+_\beta$, there are three possibilities for $\rho$,

(A) $\rho'$ cross 0 at some point $r = r_0$, while $0 < \rho < \rho_0$, for $0 < r \leq r_0$.

(B) $\rho$ cross $\rho_0$.

(C) $0 < \rho < \rho_0, \rho' \geq 0$, for $0 < r < \infty$, and

$$\rho(\infty) = \rho_0, \quad (1)$$
$$f(\infty) = 0. \quad (2)$$

**Proof** Because $\rho'(0) = \beta > 0$, if $\rho$ does not cross $\rho_0$ (case (B)), then either $\rho'$ crosses 0 at some point $r = r_0$, while $0 < \rho < \rho_0$, for $0 < r \leq r_0$(case(A)) , or $\rho' \geq 0, 0 < \rho < \rho_0$, for $0 < r < \infty$(case(C)) . There is no possibility that when $\rho$ does not cross $\rho_0$, $\rho = \rho_0$ at some point. In fact, if $\rho = \rho_0$ and $\rho' = 0$ at the same time, then by eq. (11) we have $\rho'' > 0$ at this point, since $\beta > 0, \alpha \notin S^-_\beta \cup S^+_\beta$. Then $\rho$ crosses $\rho_0$, which is a contradiction.

For case (C), we have

$$f(0) = 1, 0 < f < 1, f' \leq 0, f'(\infty) = 0, f(\infty) = a, \quad (3)$$
$$\rho(0) = 0, 0 < \rho < \rho_0, \rho' \geq 0, \rho'(\infty) = 0, \rho(\infty) = b, \quad (4)$$
for some $a \in [0, 1), b \in (0, \rho_0]$, where the second and third parts of (3), (4) are for $0 < r < \infty$. We want to show $a = 0, b = \rho_0$.

Suppose $b < \rho_0$. Choose $r_1 > 0$, so that $b/2 \leq \rho$, for $r_1 \leq r < \infty$. By eq. (11), there is

\[
(r^2 \rho')' = \rho + \lambda r^2 \left(\rho^2 - \rho_0^2\right) \rho \\
\leq b - \lambda r^2 \left(\rho_0^2 - b^2\right) \frac{b}{2},
\]

for $r_1 \leq r < \infty$. Integrating from $r_1$ to $r$, and dividing $r^2$ both sides, finally we get

\[
\rho'(r) \leq \frac{r_1^2}{r^2} \rho'(r_1) + b \left(\frac{1}{r} - \frac{r_1}{r^2}\right) - \frac{\lambda b}{6} \left(\rho_0^2 - b^2\right) \left(r - \frac{r_1^3}{r^3}\right) \\
\rightarrow -\infty, \text{ as } r \rightarrow \infty.
\]

This is a contradiction. So $b = \rho_0$.

Now suppose $a > 0$. Choose $r_2 > 0$ so that

\[
\frac{1 - a^2}{r^2} \leq \frac{g_0^2 \rho_0^2}{8}; \\
\rho \geq \rho_0/2,
\]

for $r_2 \leq r < \infty$. Then we have from eq. (10)

\[
f'' = f \left(\frac{f^2 - 1}{r^2} + g_0^2 \rho^2\right) \\
\geq f \left(\frac{a^2 - 1}{r^2} + \frac{g_0^2 \rho_0^2}{4}\right) \\
\geq f \frac{g_0^2 \rho_0^2}{8} \\
\geq \frac{ag_0^2 \rho_0^2}{8}.
\]

Integrating from $r_2$ to $r$, we get

\[
f'(r) \geq f'(r_2) + \frac{ag_0^2 \rho_0^2}{8} (r - r_2) \rightarrow \infty, \text{ as } r \rightarrow \infty.
\]

This is a contradiction. So $a = 0$. So the lemma is proved.
6. The Solutions for Small $\beta$

We make a transformation

$$t = \sqrt{\lambda p_0} r, f(r) = p(t), \rho(r) = \beta q(t).$$  \tag{1}$$

Then the equation and the asymptotics become

$$p'' - \frac{p^2 - 1}{t^2} p = \beta^2 g_1^2 q^2 p, \tag{2}$$

$$P q'' + \frac{2}{t} q' + (1 - \frac{2p^2}{t^2})q = \frac{\beta^2}{p_0^2} q^3, \tag{3}$$

$$p(t) \sim 1 - \frac{\alpha}{\lambda p_0^2} t^2, q(t) \sim \frac{t}{\sqrt{\lambda p_0}}, \text{ as } t \to 0, \tag{4}$$

where $g_1 = \lambda p_0^2$, and we will consider $\alpha \geq 0, \beta \geq 0$. The difference between this system and the system \((10), (11)\) and \((12)\) is only for the case $\beta = 0$, because when $\beta \neq 0$ these two systems are equivalent. But for $\beta = 0$ we can show by the same method that this problem has unique solution. For $\beta = 0, \alpha > 0$, by the same argument as in the proof of Lemma 3, we see that $p$ cross 0. And for $\beta = 0, \alpha = 0$, there is $p = 1$.

Let’s choose $\bar{\beta} > 0$. By Lemma 3 for $\beta \in [0, \bar{\beta}]$ there is $\bar{\alpha} > 0$, such that for any $\alpha \geq \bar{\alpha}$, we have $\alpha \in S^+_\beta$. Now let’s define in the $(\alpha, \beta)$ plane the sets

$$D = [0, \bar{\alpha}] \times [0, \bar{\beta}], \tag{5}$$

$$D^- = \{ (\alpha, \beta) \in D | p' \text{ cross 0 before } p \text{ cross 0} \}, \tag{6}$$

$$D^+ = \{ (\alpha, \beta) \in D | p \text{ cross 0 before } p' \text{ cross 0} \}, \tag{7}$$

$$l = \{0\} \times (0, \bar{\beta}], \tag{8}$$

$$D_0 = D \setminus (l \cup D^- \cup D^+). \tag{9}$$

We have that $D^-, D^+$ are open in $D$, not empty and disjoint. By Lemma 2, we see that $l \cup D^-$ is open in $D$, so $D_0$ is compact. Note that for any $\beta \in [0, \bar{\beta}]$, there is $D_0 \cap ([0, \bar{\alpha}] \times \{\beta\}) \neq \emptyset$. And for $\beta = 0$, only $(0, 0) \in D_0$.

Also let’s set

$$B = \{p| (p, q) \text{ is a solution to } (P) \text{ for } (\alpha, \beta) \in D_0 \}. \tag{10}$$

We see that for any $p \in B$, the properties in \((\Bar{3}), (\Bar{4})\) are not satisfied, and $p, p'$ can not vanish at the same time by uniqueness of the solution. So we get $0 < p(t) \leq 1$, for $0 < t < \infty$, if $p \in B$. 

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Now let’s restrict \((\alpha, \beta)\) in \(D_0\).

We have already seen that in \(D_0\), the problem (P) has a unique solution \((p, q)\) satisfying \(0 < p \leq 1\), and by eq. (2) we see that \(q\) is bounded in any finite interval for \(\beta > 0\). And for \(\beta = 0, \alpha = 0\), there are \(p = 1, q\) is expressed by \(J_{3/2}(t)\) (see (12) for \(p = 1\)). Thus we get the following result.

**Lemma 6** For \((\alpha, \beta) \in D_0\), the problem (P) has a unique solution and for any finite \(\bar{t} > 0\), \(q(t)\) is uniformly bounded for \((t, \alpha, \beta) \in [0, \bar{t}] \times D_0\), and \(0 < p \leq 1\) for \((t, \alpha, \beta) \in [0, \infty) \times D_0\).

Next, we want to find two linearly independent solutions of the equation

\[
Q'' + \frac{2}{t} Q' + (1 - \frac{2p^2}{t^2}) Q = 0. \tag{11}
\]

Let

\[
Q(t) = \frac{1}{\sqrt{t}} y(t),
\]

which reduces the equation (11) to

\[
y'' + \frac{1}{t} y' + \left(1 - \frac{\nu^2}{t^2} + \frac{2(1 - p^2(t))}{t^2} \right) y = 0, \tag{12}
\]

where \(\nu = 3/2\).

Consider the Bessel functions

\[
J_{3/2}(t) = \left(\frac{2}{\pi t}\right)^{1/2} \left(\frac{\sin t}{t} - \cos t\right), \tag{13}
\]

\[
J_{-3/2}(t) = -\left(\frac{2}{\pi t}\right)^{1/2} \left(\frac{\cos t}{t} + \sin t\right). \tag{14}
\]

Let \(t_0\) be the first positive zero of \(J_{3/2}(t)\). Choose \(t_1 > t_0\), so that \(J_{3/2}(0) = J_{3/2}(t_0) = 0, J_{3/2}(t) > 0\) for \(t \in (0, t_0)\), and \(J_{3/2}(t) < 0\) for \(t \in (t_0, t_1)\).

**Lemma 7** For each \((\alpha, \beta) \in D_0, p \in B\), there are two linearly independent solutions \(y_p^{(1)}(t), y_p^{(2)}(t)\) to the equation (12) uniquely determined by the asymptotics

\[
y_p^{(1)}(t) \sim J_{3/2}(t) \sim \frac{1}{3} \left(\frac{2}{\pi}\right)^{1/2} t^{3/2}, \tag{15}
\]

\[
y_p^{(2)}(t) \sim J_{-3/2}(c_0 t) \sim -\left(\frac{2}{\pi}\right)^{1/2} (c_0 t)^{-3/2}, \tag{16}
\]
as \( t \to 0 \), where

\[
c_0 = \sqrt{1 + \frac{4\alpha}{\lambda \rho_0^2}}.
\]

There is singularity only for \( y_p^{(2)} \) at the origin.

**Proof** First of all we have the Wronskian

\[
W \left( J_{3/2}, J_{-3/2} \right) = \frac{2}{\pi t}.
\]

Define \( y_p^{(1)} \) for \( 0 < t < \infty \) by the integral equation

\[
y_p^{(1)}(t) = J_{3/2}(t) + \pi J_{-3/2}(t) \int_0^t \sigma J_{3/2}(\sigma) \frac{p^2(\sigma) - 1}{\sigma^2} y_p^{(1)}(\sigma) d\sigma
\]

\[
- \pi J_{3/2}(t) \int_0^t \sigma J_{-3/2}(\sigma) \frac{p^2(\sigma) - 1}{\sigma^2} y_p^{(1)}(\sigma) d\sigma.
\]

By iteration method one can show that \( y_p^{(1)}(t) \) is uniquely defined without singularity.

Now let’s consider how to define \( y_p^{(2)}(t) \). By eq. (6.2) it’s not hard to see that

\[
p(t) = 1 - \frac{\alpha}{\lambda \rho_0^2} t^2 + O(t^4),
\]

as \( t \to 0 \). Let

\[
s = c_0 t,
y(t) = z(s)
\]

\[
R(s) = c_0^{-2} \left( \frac{4\alpha}{\lambda \rho_0^2} - \frac{2(1 - p^2(t))}{t^2} \right).
\]

By simple calculation we see that

\[
R(s) = O(s^2),
\]

as \( s \to 0 \). So now eq. (6.11) becomes

\[
z'' + \frac{1}{s} z' + \left( 1 - \frac{\nu^2}{s^2} - R(s) \right) z = 0.
\]
Define $z(s)$ by
\[
z(s) = J_{-3/2}(s) + \frac{\pi}{2} J_{-3/2}(s) \int_0^s \sigma J_{3/2}(\sigma) R(\sigma) z(\sigma) d\sigma - \frac{\pi}{2} J_{3/2}(s) \int_0^s \sigma J_{-3/2}(\sigma) R(\sigma) z(\sigma) d\sigma,
\]
\[
z(s) \sim J_{-3/2}(s), \quad \text{as } s \to 0.
\]

Now let $y_p^{(2)}(t) = z(s)$, and then the lemma is done.

So now
\[
Q_p^{(2)}(t) = \frac{1}{\sqrt{t}} y_p^{(2)}(t), \quad j = 1, 2
\]
forms a basis of eq. (11) with the Wronskian
\[
W(Q_p^{(1)}, Q_p^{(2)}) = \gamma t^2,
\]
where
\[
\gamma = \lim_{t \to 0} t^2 W(Q_p^{(1)}, Q_p^{(2)}) = \frac{5}{3\pi c_0^{-3/2}}.
\]

Lemma 8
\[
m = \sup_{p \in B} \inf_{t \in [0, t_1]} y_p^{(1)}(t) < 0,
\]
and
\[
m_0 = \sup_{p \in B} \inf_{t \in [0, t_1]} Q_p^{(1)}(t) < 0.
\]

Proof Because $J_{3/2}(0) = 0 = J_{3/2}(t_0)$, and for any $p \in B, 0 < p \leq 1$, by Sturm comparison principle, there is a zero of $y_p^{(1)}(t)$ between 0 and $t_0$. Since $y_p^{(1)}(t)$ is not a trivial solution, by the uniqueness theorem, $(y_p^{(1)}(t))'(y_p^{(1)}(t))^\prime\prime$ can be zero at the same time. Thus $y_p^{(1)}(t)$ must cross 0 in $(0, t_1)$ for any $p \in B$, which implies
\[
\inf_{t \in [0, t_1]} y_p^{(1)}(t) < 0.
\]

So $m \leq 0$.

Suppose $m = 0$. Then there is a sequence $\{(\alpha^{(n)}, \beta^{(n)})\}_{n=1}^\infty \subset D_0$, such that
\[
\inf_{t \in [0, t_1]} y_p^{(1)}(t) \to 0,
\]
as $n \to \infty$, where $p^{(n)} = p(t, \alpha^{(n)}, \beta^{(n)})$. Because $D_0$ is compact, without loss of generality, assume

$$(\alpha^{(n)}, \beta^{(n)}) \to (\alpha^*, \beta^*) \in D_0,$$

as $n \to \infty$. By continuity, there is

$$\inf_{t \in [0, t_1]} y_p^{(1)}(t) = 0, \quad p^* = p(t, \alpha^*, \beta^*)$$

This is a contradiction because $(\alpha^*, \beta^*) \in D_0$, which implies $p^*$ satisfies (21). Thus $m < 0$. And if $m_0 \geq 0$, then $m \geq 0$. So we also have $m_0 < 0$.

**Lemma 9** There is a small $\beta_1 > 0$, such that for any $\beta \in (0, \beta_1]$, if $\alpha \notin S^-_\beta \cup S^+_\beta$, then (A) is satisfied.

**Proof** Suppose $(p, q)$ is a solution to (2), (3) and (4), then by the variation of parameter, $q$ satisfies the integral equation

$$q(t) = \mu Q_p^{(1)}(t) + \frac{\beta^2}{\gamma \rho_0^2} G(t), 0 \leq t \leq t_1, \quad (22)$$

where

$$G(t) = Q_p^{(2)}(t) \int_0^t s^2 Q_p^{(1)}(s) q(s)^3 \, ds - Q_p^{(1)}(t) \int_t^{t_1} s^2 Q_p^{(2)}(s) q(s)^3 \, ds, \quad (23)$$

and

$$\mu = \frac{3}{\rho_0 \sqrt{2\lambda}}.$$

By Lemma 6,7, we see that $G(t) = O(t^3)$, as $t \to 0$, and $G$ is uniformly bounded in $[0, t_1] \times D_0$. Choose $\bar{\beta}_1 < \bar{\beta}$, so that if $0 < \beta < \bar{\beta}_1$, there is

$$\left| \frac{\beta^2}{\gamma \rho_0^2} G(t) \right| < \frac{\mu |m_0|}{2},$$

which implies by (22) and by Lemma 8

$$\inf_{t \in [0, t_1]} q(t) < \inf_{t \in [0, t_1]} \left( \mu Q_p^{(1)}(t) + \frac{\mu |m_0|}{2} \right) < \frac{\mu m_0}{2} < 0,$$

which means $q$ crosses 0, since $q'(0) = 1$. And then $q'$ or $\rho'$ cross 0. Choose smaller positive $\beta_1 < \bar{\beta}_1$, such that $0 < \rho < \rho_0$ before $\rho'$ crosses 0. So the lemma is proved.
7. The Solutions for Large $\beta$

Lemma 10 There is a large $\beta_2 > 0$, such that if $\alpha \not\in S^-_\beta \cup S^+_\beta$, then (B) is satisfied.

Proof Recall the integral equation form we used in section 2,

$$\phi = -\alpha + \frac{1}{3} \int_{-\infty}^{s} (e^{2\sigma} - e^{-3s+5\sigma})(3\phi^2 + e^{2\sigma}\phi^3 + g_0^2 e^{2\sigma} \psi^2 \phi) \, d\sigma, \quad (1)$$

$$\psi = \beta + \frac{1}{2} \int_{-\infty}^{s} (e^{2\sigma} - e^{-3s+5\sigma})(2(2\phi + e^{2\sigma}\phi^2)\psi + \lambda(e^{2\sigma}\psi^2 - \rho_0^2)\psi) \, d\sigma, \quad (2)$$

where

$$s = \log r, \ f(r) = 1 + e^{2s}\phi(s), \ \rho(r) = e^s\psi(s). \quad (3)$$

Suppose there is a sequence $\{(\alpha^{(n)}, \beta^{(n)})\}_{n=1}^{\infty} \subset (0, \infty) \times (0, \infty)$ with $\beta^{(n)} \to \infty$, as $n \to \infty$, and $\alpha^{(n)} \not\in S^-_\beta \cup S^+_\beta$, such that (B) is not satisfied for each $n$. By Lemma 5 for each $n$, either (A) or (C) is satisfied. If (C) is satisfied, the theorem is proved. So we consider for each $n$, (A) is satisfied.

Let us denote

$$f^{(n)} = f(r, \alpha^{(n)}, \beta^{(n)}), \ \rho^{(n)} = \rho(r, \alpha^{(n)}, \beta^{(n)}).$$

For each $n$, let $[0, r_n]$ be the maximal interval such that

$$|\rho^{(n)}| \leq \rho_0. \quad (4)$$

Let

$$\bar{r} = \inf_n r_n.$$

We want to show $\bar{r} > 0$(maybe $\infty$). If $\bar{r} = 0$, without loss of generality, assume $r_n \to 0$, as $n \to \infty$. Let

$$s_n = \log r_n, \ f^{(n)}(r) = 1 + e^{2s}\phi^{(n)}(s), \ \rho^{(n)}(r) = e^s\psi^{(n)}(s). \quad (5)$$

Then $\phi^{(n)}, \psi^{(n)}$ have uniform bounds in $(-\infty, s_n]$. But by (2), we have $\psi^{(n)}(s_n) \to +\infty$, as $n \to \infty$, which is a contradiction. So $\bar{r} > 0$.

Now choose $0 < \tilde{r} < \infty$ so that (2) is satisfied for $r \in [0, \tilde{r}]$ for all $n$. Then still by (2) we have that $\psi^{(n)}(s) \to +\infty$, as $n \to \infty$, for all $s \in (-\infty, \log(\tilde{r})]$. This contradiction implies that the lemma is proved.
8. Proof of the Theorem

By Lemma 9 there is $\beta_1 > 0$ so that if $\alpha \notin S_{\beta_1}^- \cup S_{\beta_1}^+$, (A) is satisfied. By Lemma 10 there is $\beta_2 > \beta_1 > 0$ so that if $\alpha \notin S_{\beta_2}^- \cup S_{\beta_2}^+$, (B) is satisfied. For $\beta \in [\beta_1, \beta_2]$, by Lemma 2,3, there are $0 < \alpha_1 < \alpha_2$, so that if $\alpha \in (0, \alpha_1]$, then $\alpha \in S_{\beta}^-$, and if $\alpha \in [\alpha_2, \infty)$, then $\alpha \in S_{\beta}^+$. Define

$$I = [\alpha_1, \alpha_2] \times [\beta_1, \beta_2].$$

Let

$$S^- = \{ (\alpha, \beta) \in I | f' \text{ cross 0 before } f \text{ cross 0} \},$$

$$S^+ = \{ (\alpha, \beta) \in I | f \text{ cross 0 before } f' \text{ cross 0} \}.$$

We have seen that $S^-, S^+$ are open, nonempty and disjoint.

By the topological lemma in [8] (McLeod and Serrin), there is a continuum $\Gamma$ connects $\beta = \beta_1$ and $\beta = \beta_2$. Define

$$\Omega_- = \{ (\alpha, \beta) \in \Gamma | (A) \text{ is satisfied} \},$$

$$\Omega_+ = \{ (\alpha, \beta) \in \Gamma | (B) \text{ is satisfied} \}.$$

By the choice of $\beta_1, \beta_2$, we see that $\Omega_-, \Omega_+$ are not empty and disjoint, and it’s easy to show that $\Omega_-, \Omega_+$ are open in $\Gamma$. Thus there exists $(\alpha^*, \beta^*) \in \Gamma \setminus (\Omega_- \cup \Omega_+)$, such that (C) is satisfied. So $f(r, \alpha^*, \beta^*), \rho(r, \alpha^*, \beta^*)$ is a solution to the boundary value problem, and satisfying the properties

$$0 < f < 1, f' \leq 0,$$

$$0 < \rho < \rho_0, \rho' \geq 0,$$

for $0 < r < \infty$.

Finally we need to show $f' < 0, \rho' > 0$.

Suppose $r_1$ is a positive zero of $f'$. Then by eq. (1.10), there is

$$f'''(r_1) = 2 \left( \frac{1-f^2}{r^3} + g_0^2 \rho' \right) f > 0,$$

which is a contradiction. So $f' < 0$.

Now suppose $r_2$ is a zero point of $\rho'$. Then because $\rho' \geq 0$, there is $\rho''(r_2) = 0$. By eq. (1.11), we have

$$\rho'''(r_2) = 4 \left( \frac{f f'}{r^2} - \frac{f^2}{r^3} \right) \rho < 0,$$

which is a contradiction. So the theorem is proved.
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