GAUGE EQUIVALENCE AND THE INVERSE SPECTRAL PROBLEM FOR THE MAGNETIC SCHröDINGER OPERATOR ON THE TORUS

G. ESKIN AND J. RALSTON,
DEPARTMENT OF MATHEMATICS, UCLA,
LOS ANGELES, CA 90095-1555, USA

In memory of Mark Iosifovich Vishik

Abstract. We study the inverse spectral problem for the Schrödinger operator \( H \) on the two-dimensional torus with even magnetic field \( B(x) \) and even electric potential \( V(x) \). V. Guillemin [11] proved that the spectrum of \( H \) determines \( B(x) \) and \( V(x) \). A simple proof of Guillemin’s results was given by the authors in [3]. In the present paper we consider gauge equivalent classes of magnetic potentials and give conditions which imply that the gauge equivalence class and the spectrum of \( H \) determine the magnetic field and the electric potential. We also show that generically the spectrum and the magnetic field determine the “extended” gauge equivalence class of the magnetic potential. The proof is a modification of the proof in [3] with some corrections and clarifications.

1. Introduction

Let \( L = \{m_1 e_1 + m_2 e_2 : m = (m_1, m_2) \in \mathbb{Z}^2\} \) be a lattice in \( \mathbb{R}^2 \). Here \( \{e_1, e_2\} \) is a basis in \( \mathbb{R}^2 \). We assume that the lattice \( L \) has the following property:

\[
(1.1) \quad \text{For } d, d' \in L, \text{ if } |d| = |d'|, \text{ then } d' = \pm d.
\]

Let \( L^* = \{ \delta \in \mathbb{R}^2 : \delta \cdot d \in \mathbb{Z} \text{ for all } d \in L \} \) be the dual lattice. We consider a Schrödinger operator of the form

\[
(1.2) \quad H = \left( -i \frac{\partial}{\partial x_1} - A_1(x) \right)^2 + \left( -i \frac{\partial}{\partial x_1} - A_2(x) \right)^2 + V(x), \quad x \in \mathbb{R}^2,
\]

where \( A(x) = (A_1(x), A_2(x)) \) is the magnetic potential and \( V(x) \) is the electric potential.

Let \( B(x) \) be the magnetic field,

\[
(1.3) \quad B(x) = \text{curl } A(x) = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}.
\]
We assume that $B(x)$ and $V(x)$ are periodic, i.e.
$$B(x + d) = B(x), \quad V(x + d) = V(x), \quad \forall d \in L,$$
i.e. $B(x)$ and $V(x)$ are smooth functions on $\mathbb{T}^2 = \mathbb{R}^2/L$. We also assume that $B(x)$ and $V(x)$ are even, i.e. $B(-x) = B(x)$ and $V(-x) = V(x)$.

Denote by $G(\mathbb{T}^2)$ the gauge group of complex-valued functions $g(x) \in C^\infty(\mathbb{T}^2)$ such that $|g(x)| = 1$. Any $g(x) \in G(\mathbb{T}^2)$ has the following form
$$g(x) = \exp(2\pi i \delta \cdot x + i \varphi(x)),$$
where $\delta \in L^*$ and $\varphi(x)$ is periodic, $\varphi(x + d) = \varphi(x)$, $\forall d \in L$. The operator of multiplication by $g(x)$ transforms the equation $Hu = \lambda u$ to the equation
$$H' u'(x) = \lambda u'(x),$$
where $H'$ has the form (1.2) with $A(x)$ replaced by $A'(x)$.

Let (1.6) be the Fourier series expansion of $B(x)$. We assume that the coefficient
$$b_0 = |D|^{-1} \int_D B(x) dx,$$
is not zero. Here $D$ is a fundamental domain for the lattice $L$ given by
$$D = \{t_1 e_1 + t_2 e_2, \ |t_j| \leq \frac{1}{2}, j = 1, 2\}$$
and $|D|$ is the area of $D$. Note that if $x \in D$, then $-x \in D$.

Given $B(x)$, we let
$$A(x) = A^0(x) + a_0 + \sum_{\beta \in (L^* \setminus 0)} a_{\beta} e^{2\pi i \beta \cdot x},$$
where $a_0 = (a_{01}, a_{02})$ is a constant,
$$A^0(x) = \frac{b_0}{2} (-x_2, x_1)$$
and
$$a_{\beta} = b_{\beta} (2\pi i)^{-1} (\beta_1^2 + \beta_2^2)^{-1} (-\beta_2, \beta_1).$$
Note that \( \text{curl} \ A = B(x) \).

Let \( A'(x) \) be any magnetic potential that is gauge equivalent to \( A(x) \). Since \( \text{curl} \ \nabla \phi = 0 \),

\[
B'(x) = B(x),
\]

where \( B'(x) = \text{curl} \ A'(x) \). Therefore \( A'(x) \) can be also represented in the form \((1.9)\) with \( a'_0 \) not necessarily equal to \( a_0 \). For the gauge equivalence of \( A'(x) \) and \( A(x) \), in addition to the equality of the magnetic fields, one needs (cf. \((1.5)\))

\[(1.12) \quad a'_0 - a_0 = 2\pi \delta, \]

for some \( \delta \in L^* \). The main question in this paper is: For the class of magnetic potentials considered here, to what extent do the spectra of magnetic Schrödinger operators and the gauge equivalence classes of magnetic potentials determine the magnetic and electric fields?

In §2 we will describe the domain on which \( H \) is a self-adjoint operator with compact resolvent and hence has a discrete spectrum. Here we will describe the gauge equivalence classes of the magnetic potential assuming that the magnetic field \( B(x) \) is fixed.

Let \( \gamma_j, j = 1, 2, \) be the basis of the homology group of the torus, given by \( \gamma_j = \{te_j, 0 \leq t \leq 1\} \). Let

\[
\alpha_j = \int_{\gamma_j} a_0 \cdot dx = a_0 \cdot e_j,
\]

where \( a_0 \) is the constant vector in \((1.9)\). For any \( d = m_1 e_1 + m_2 e_2 \in L \) we have \( a_0 \cdot d = m_1 \alpha_1 + m_2 \alpha_2 \), i.e. knowing \( \{\alpha_1, \alpha_2\} \) determines \( a_0 \cdot d \) for any \( d \in L \).

Let \( A'(x) \) be a magnetic potential of the form \((1.9)\) with \( a_0 \) replaced by \( a'_0 \). Define

\[
\alpha'_j = \int_{\gamma_j} a'_0 \cdot dx = a'_0 \cdot e_j.
\]

Let \( \{e_1^*, e_2^*\} \) be the basis in \( L^* \) dual to \( \{e_1, e_2\} \), i.e. \( e_j \cdot e_k^* = \delta_{jk} \).

The potentials \( A(x) \) and \( A'(x) \) are gauge equivalent if and only if \( \text{curl} \ A = \text{curl} \ A' \) and (cf. \((1.12)\))

\[(1.13) \quad \alpha'_j - \alpha_j = (a'_0 - a_0) \cdot e_j = 2\pi \delta \cdot e_j, \quad j = 1, 2,
\]

for some \( \delta \in L^* \). Short equivalent forms of \((1.13)\) are

\[
e^{i\alpha_j} = e^{i\alpha'_j}, \quad j = 1, 2,
\]
and

\[(1.14) \quad e^{ia_0 \cdot d} = e^{ia'_0 \cdot d} \quad \text{for any} \quad d = n_1 e_1 + n_2 e_2 \in L.\]

Changing \(x\) to \(-x\) we get the operator \(H'\) which is just \(H\) with \(a_0\) changed to \(a'_0 = -a_0\). Note that \(H\) and \(H'\) have the same spectrum but their magnetic potentials are not gauge equivalent when \(a_0 \neq 0\). Since we are looking for consequences of isospectrality, we introduce a weaker notion of gauge equivalence, namely

\[(1.15) \quad \cos a_0 \cdot d = \cos a'_0 \cdot d, \quad \forall d \in L.\]

The condition \((1.15)\) is equivalent to \(\cos \alpha_j = \cos \alpha'_j, \quad j = 1, 2.\) Since

\[
\cos \alpha_j - \cos \alpha'_j = 2 \sin(\frac{\alpha_j - \alpha'_j}{2}) \sin(\frac{\alpha_j + \alpha'_j}{2}),
\]

\(\cos \alpha_j = \cos \alpha'_j\) implies that either \(\alpha_j - \alpha'_j\) or \(\alpha_j + \alpha'_j\) is an integer multiple of \(2\pi\). Thus there are two choices for each \(j\): \(e^{ia_0 \cdot e_j} = e^{ia'_0 \cdot e_j}\) or \(e^{ia_0 \cdot e_j} = e^{-ia_0 \cdot e_j}\). We will say that \(a'_0\) and \(a_0\) belong to the same “extended gauge equivalence class” if \((1.15)\) holds. Thus for every extended gauge equivalence class of magnetic potentials, there are four choices of \(a_0\), including \(a'_0 = a_0\) and \(a'_0 = -a_0\), giving distinct gauge equivalence classes when \(a_0 \neq 0\).

Our first result gives conditions for the spectrum of \(H\) and gauge equivalence class of \(A\) to determine the fields:

**Theorem 1.1.** Let \(B(x), V(x)\) be periodic and even smooth functions, and assume \(L\) satisfies the condition \((1.1)\). Suppose

\[(1.16) \quad \int_D B(x)dx = 2\pi.\]

Consider the spectrum of the Schrödinger operator \(H\) with \(A(x)\) having the form \((1.3)\). Suppose that

\[(1.17) \quad |B(x) - b_0| < |b_0|,\]

where \(b_0 = 2\pi/|D|\). Then the spectrum of \(H\) determines uniquely \(B(x)\) and \(V(x)\) assuming that \(\cos a_0 \cdot d, \quad d \in L,\) is given and

\[(1.18) \quad \cos a_0 \cdot d \neq 0 \quad \text{for all} \quad d \in L.\]

**Theorem 1.2.** Assume that the conditions \((1.1)\), \((1.16)\), \((1.17)\) hold, and that the spectrum of \(H\) and the magnetic field \(B(x)\) are given. If \(B(x)\) satisfies a generic condition (stated in the proof), then \(\cos a_0 \cdot d\)
is determined for all $d \in L$, i.e. the extended gauge equivalence class of $A(x)$ is determined by the spectrum and the magnetic field.

It follows from Theorem 1.2 that if $B(x), V(x)$ are fixed and the extended gauge equivalence classes of $A(x)$ and $A'(x)$ are different, i.e. if $\cos a_0 \cdot d \neq \cos a'_0 \cdot d$ for some $d \in L$, then the corresponding operators $H$ and $H'$ have different spectra. This confirms the Aharonov-Bohm effect stating that different gauge equivalence classes have a different quantum mechanical effects, for example, the corresponding Schrödinger operators have different spectra (cf. [5]).

The case when $B(x)$ and $V(x)$ are even and $a_0 = 0$ (cf. (1.9)) was proven in an important paper of Guillemin [11]. In [3] we reproved the result of [6] by a different and simpler method.

The method of the present paper is a modification of the method of [3] with some clarifications and corrections.

We mention briefly some related results on the inverse spectral problems in two and higher dimensions: The magnetic Schrödinger operator on $\mathbb{T}^2$ with $\int_D B(x)dx = 0$ and $A(x)$ periodic was studied in [1]. In [10] Gordon et al. generalized [11] to the case of $n$-dimensional tori. Guillemin and Kazhdan [13] studied the inverse spectral problem for negatively curved manifolds. Guillemin [12] studied the inverse spectral problem on $S^2$. Zeldich [18] solved the inverse spectral problem for analytic bi-axisymmetric plane domains. In [6] the inverse spectral problems on the torus for the Schrödinger operator $-\Delta + q(x)$ were studied. See also [4], [8], [16].

Gordon [7] and Gordon-Schuth [9] gave many interesting examples of isospectral manifolds which were not isometric.

2. The singularities of the wave trace

We introduce the “magnetic translation operators” (cf. [17])

\begin{equation}
T_j u(x) = e^{-iA^0(e_j) \cdot x} u(x + e_j), \quad j = 1, 2,
\end{equation}

where $A^0(x)$ is from (1.10). These operators are required to commute with each other and with $H$. This implies that

\begin{equation}
A^0(e_1) \cdot e_2 = -A^0(e_2) \cdot e_1 = \pi l,
\end{equation}

where $l$ is an integer. Using (1.7), (1.10) we get that (2.2) is equivalent to

\begin{equation}
\int_D B(x) dx_1 dx_2 = 2\pi l.
\end{equation}
Later we shall assume that \( l = 1 \). Having that \( T_1, T_2 \) and \( H \) commute we denote by \( D_0 \) the subspace of the Sobolev space \( H^2(\mathbb{R}^2) \) consisting of \( u(x) \in H^2(\mathbb{R}^2) \) such that \( T_j u = u, j = 1, 2 \). Then the operator \( H \) is self-adjoint in \( L_2(D) \) on the restriction of \( D_0 \) to the fundamental domain \( D \). We shall denote this operator by \( H_D \).

Let \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \) be the spectrum of \( H_D \) and let \( E_D(x, y, t) \) be the fundamental solution for the wave equation on \( \mathbb{R}^2/L \). Then the wave trace formula gives the equality as distributions in \( t \)

\[
\sum_{j=1}^{\infty} \cos t \sqrt{\lambda_j} = \int_D E_D(x, x, t)dx.
\]

The distribution \( E_D(x, y, t) \) is defined as follows: Let \( E(x, y, t) \) be the fundamental solution for the wave equation on \( \mathbb{R}^2 \):

\[
\frac{\partial^2 E(x, y, t)}{\partial t^2} + HE(x, y, t) = 0, \quad x \in \mathbb{R}^2, y \in \mathbb{R}^2,
\]

\[
E(x, y, 0) = 0, \quad \frac{\partial E(x, y, 0)}{\partial t} = \delta(x - y), \quad x \in \mathbb{R}^2, y \in \mathbb{R}^2.
\]

Then

\[
E_D(x, y, t) = \sum_{(m,n) \in \mathbb{Z}^2} T_1^m T_2^n E(x, y, t).
\]

Note that

\[
T_1^m T_2^n E(x, y, t) = e^{-iA_0(d) x} E(x + d, y, t),
\]

where \( d = mc_1 + nc_2 \). We used in (2.7) that \( A_0(e_j) \cdot e_j = 0 \). Since \( E(x, y, t) \) is singular only when \( |x-y|^2 = t^2 \) and since the condition (1.1) holds, the singularities of the trace (2.4) at \( t = |d| \), \( d = m_1 c_1 + m_2 c_2 \) come only from two terms

\[
\int_D (T_1^m T_2^n E(x, x, t) + T_{-m}^{-n} T_2^{-n} E(x, x, t))dx.
\]

To compute the singularities in (2.8) we will use as in [3] and [4] the Hadamard-Hörmander parametrix (cf. [14], [15]). We have

\[
E(x, y, t) = \frac{\partial}{\partial t} (E_+(x, y, t) - E_+(x, y, -t)),
\]

where \( E_+(x, y, t) \) is the forward fundamental solution:

\[
\left( \frac{\partial^2}{\partial t^2} + H \right) E_+(x, y, t) = \delta(t) \delta(x - y), \quad E_+(x, y, t) = 0 \text{ for } t < 0.
\]
It follows from [14] that

\begin{equation}
E_+(x, y, t) = m_0(x, y) \frac{1}{2\pi} (t^2 - |x - y|^2)^{-\frac{1}{2}}_+ \\
+ m_1(x, y) 2^{-2\pi^{-1}}(t^2 - (x - y)^2)^{-\frac{1}{2}}_+ + O((t^2 - (x - y)^2)^{\frac{3}{2}}),
\end{equation}

where

\begin{equation}
m_0(x, y) = \exp \left( i \int_0^1 (x - y) \cdot A(y + s(x - y)) ds \right),
\end{equation}

\begin{equation}
m_1(x, y) = -m_0(x, y) \left( \int_0^1 V(y + s(x - y)) ds + b(x, y) \right),
\end{equation}

where

\begin{equation}
b(x, y) = \left[ \left(-i \frac{\partial}{\partial x} - A(x) \right)^2 m_0 \right] m_0^{-1}(x, y).
\end{equation}

Let

\begin{equation}
I(d) = \int_D \exp \left[ -A^0(d) \cdot x + \int_0^1 (A(x + sd) \cdot d) ds \right] dx
\end{equation}

\begin{equation}
J(d) = \int_D \int_0^1 [(V(x+sd)+b(x+sd, x)] ds \exp \left[ -iA^0(d) \cdot x + i \int_0^1 (A(x+sd) \cdot d) ds \right] dx
\end{equation}

It follows from (2.8)-(2.13) that $I(d) + I(-d)$ and $J(d) + J(-d)$ are determined by the spectrum of $H$.

Using that $A^0(d) \cdot d = 0$ and that $A^0(d) \cdot x = -A^0(x) \cdot d$ we can rewrite (2.14) in the form

\begin{equation}
I(d) = \int_D \exp i \left[ 2A^0(x) \cdot d + a_0 \cdot d + \int_0^1 (A^1(x + sd) \cdot d) ds \right] dx,
\end{equation}

where

\begin{equation}
A^1(x) = \sum_{\beta \in L^* \setminus \{0\}} a_{\beta} e^{2\pi i \beta \cdot x},
\end{equation}

and $a_{\beta}$ are defined in (1.11).

Let $\{e_1^*, e_2^*\}$ be the basis in $L^*$ dual to the basis $\{e_1, e_2\}$, i.e.

\begin{equation}
e_j^* \cdot e_k = \delta_{jk}, \quad 1 \leq j, k \leq 2.
\end{equation}
We shall construct $e_j^*, j = 1, 2$, explicitly. Let $e_j^* = (-e_{j2}, e_{j1})$, Denote by $\Delta$ the determinant $\begin{vmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{vmatrix}$. We assume that $\Delta > 0$. Note that $\Delta$ is the area of the fundamental domain $D : \Delta = |D|$. Now we define

$$
e_1^* = -\frac{1}{\Delta} e_2^*, \quad e_2^* = \frac{1}{\Delta} e_1^*.
$$

We have $d = me_1 + ne_2 = k(m_0e_1 + n_0e_2)$, where $k \geq 1$, is an integer and $m_o, n_o$ have no common factors. Then

$$A^0(d) = \frac{b_0}{2} d_\perp = \frac{b_0 k}{2} (m_0e_1^{\perp} + n_0e_2^{\perp}) = \frac{b_0 k \Delta}{2} (m_0e_2^* - n_0e_1^*) = \frac{b_0 k \Delta}{2} \delta,
$$

where $\delta = -n_0e_1^* + m_0e_2^*$. Using (1.7), (2.3) we get

$$A^0(d) = \pi kl \delta.
$$

If $\beta \cdot d \neq 0$, then $\int_0^1 e^{2\pi is^\beta \cdot d} ds = 0$. When $\beta \cdot d = 0$, we have $\beta = p\delta$, $p \in \mathbb{Z} \setminus O$, $\delta = -n_0e_1^* + m_0e_2^*$. We shall compute the inner product $d \cdot a_\beta$, where $a_\beta$ is given by (1.11). Note that $d = k(m_0e_1 + n_0e_2)$, $\delta^{\perp} = -n_0(e_1^*)^{\perp} + m_0(e_2^*)^{\perp} = \frac{1}{\Delta} (n_0e_2 + m_0e_1)$ (cf. (2.19). Therefore

$$d \cdot a_\beta = \frac{kb_{p\delta} p|m_0e_1 + n_0e_2|^2}{\Delta 2\pi ip|\delta|^2} = \frac{kb_{p\delta} \Delta}{2\pi ip},
$$

since $|\delta|^2 = \frac{1}{\Delta} |m_0e_1 + n_0e_2|^2$. It follows from (1.7) and (2.3) that $\Delta = |D| = \frac{2\pi l}{b_0}$. Hence

$$d \cdot a_\beta = \frac{k b_{p\delta}}{ipb_0}.
$$

Therefore $I(d)$ has now the form

$$I(d) = e^{ia_0 \cdot d} \int_D \exp \left(2\pi i kl (x \cdot \delta) + A_\delta^1(\delta \cdot x)\right) dx,
$$

where

$$A_{\delta}^1(\delta \cdot x) = \sum_{p \neq 0} \frac{b_{p\delta}}{2\pi ipb_0} e^{2\pi ip(\delta \cdot x)}.
$$

Potentials of the form (2.23) are called “directional potentials” in [6]. Note that

$$\frac{d}{ds} A_{\delta}(s) = \frac{1}{b_0} B_{\delta}(s),
$$

where

$$B_{\delta}(s) = \sum_{p \neq 0} b_{p\delta} e^{ips}.\)
is a directional potential for $B(x)$. From here on we assume that $l = 1$ and set $d_0 = \frac{1}{k} d = m_0 e_1 + n_0 e_2$.

Choose $\delta' \in L^*$ so that $(\delta, \delta')$ is a basis in $L^*$ and let $(\gamma, \gamma')$ be the basis in $L$ dual to $(\delta, \delta') \in L^*$. We let $D'$ be the fundamental domain for $L$ with respect to the basis $\{\gamma, \gamma'\}$ in the form $\{s \gamma + s' \gamma', -\frac{1}{2} \leq s, s' \leq \frac{1}{2}\}$, and continue to let $D$ be the fundamental domain for $L$ from (1.8). Setting $x = s \gamma + s' \gamma'$, we $x \cdot \delta = s, x \cdot \delta' = s'$. Since the image of $D$ is $D'$, using this change of variables we get

\[(2.25)\]

\[
\int_D \exp 2\pi ik((x \cdot \delta) + A_3(x \cdot \delta)) dx = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \left[\exp 2\pi ik(s + A_3^1(s))\right] c_0 ds ds',
\]

where $c_0 = \left|\frac{\partial(x_1, x_2)}{\partial(s, s')}\right|$ is the Jacobian. Note that $c_0$ is the area of a fundamental domain for $L$. Therefore, integrating in $s'$ we get

\[
I(d) = c_0 \int_{-1/2}^{1/2} \exp \left[2\pi ik \left(s + \frac{a_0 \cdot d_0}{2\pi} + A_3^1(s)\right)\right] ds.
\]

When $a_0 = 0$ and $A_3^1(s)$ is an odd function of $s$ then $I(-d) = I(d)$ and the computations are simplified. When $a_0 \neq 0$ the spectral invariant is $I(d) + I(-d)$ and we get, changing $k$ to $-k$:

\[(2.26)\]

\[
I(d) + I(-d) = 2c_0 \int_{-1/2}^{1/2} \cos \left[2\pi k \left(s + \frac{a_0 \cdot d_0}{2\pi} + A_3^1(s)\right)\right] ds
\]

Since \[
\int_{-1/2}^{1/2} e^{2\pi i(x+sd) \cdot \delta'} ds = 0 \text{ if } d \cdot \delta' \neq 0 \text{ and it is equal } e^{2\pi i x \cdot \delta} \text{ when } \delta \cdot d = 0,
\]
the directional potential $B_{\delta}$ is given by

\[
B_{\delta}(x) = \int_{-1/2}^{1/2} (B(x + sd) - b_0) ds.
\]

Hence, by the assumption $|B(x) - b_0| < |b_0|$

\[(2.27)\]

\[
\max |B_{\delta}(x)| \leq \int_{-1/2}^{1/2} \max |B(x) - b_0| ds < |b_0|.
\]

Letting

\[(2.28)\]

\[y = s + A_3^1(s),\]
we have
\[
\frac{dy}{ds} = 1 + \frac{d}{ds} A_\delta^1(x) = 1 + \frac{B_\delta(s)}{b_0} > 0.
\]

Therefore the inverse function \( s = s(y), \ y \in \mathbb{R}^1 \), is defined. Since \( y = y(s) \) is odd, the inverse function \( s = s(y) \) is also odd. Since \( y(s+1) = y(s) + 1 \) we have \( s(y+1) = s(y) + 1 \). Differentiating in \( y \) we get \( s'(y+1) = s'(y) \), i.e. \( s'(y) \) is periodic of period 1. The function \( s'(y) \) is even since \( s(y) \) is odd. Let \( e(y) = s(y) - y \). Since \( s(y+1) = s(y) + 1 \) we get that \( e(y+1) = e(y) \), i.e. \( e(y) \) is periodic and odd. Note that
\[
s'(y) = 1 + e'(y).
\]

After the change of variables \( s = s(y) \) in (2.26), we have
\[
I(d) + I(-d) = 2c_0 \int_{-1/2}^{1/2} \cos(2\pi ky + ka_0 \cdot d_0) s'(y) dy.
\]

Since \( \cos(2\pi ky + ka_0 \cdot d_0) s'(y) \) is periodic and \( A_\delta^1(1/2) = A_\delta^1(-1/2) \) we get
(2.29) \[
I(d) + I(-d) = 2c_0 \int_{-1/2}^{1/2} \cos(2\pi ky + ka_0 \cdot d_0) s'(y) dy.
\]

Since \( \sin 2\pi ky s'(y) \) is an odd function on \((-1/2, 1/2)\),
\[
2c_0 \int_{-1/2}^{1/2} (\sin ka_0 \cdot d_0)(\sin 2\pi ky) s'(y) dy = 0,
\]
and this implies
(2.30) \[
I(d) + I(-d) = 2c_0 \int_{-1/2}^{1/2} (\cos ka_0 \cdot d_0)(\cos 2\pi ky) s'(y) dy
\]

As an even smooth function, \( s'(y) \) is a sum of its Fourier cosine series on \((-1/2, 1/2)\). Suppose \( \cos(ka_0 \cdot d_0) \) is known and nonzero for all \( k \geq 1 \). Then we know the Fourier cosine coefficients for \( k \geq 1 \). This uniquely determines \( s'(y) \) up to a constant. Therefore \( s'(y) = C + s_1(y) \), where \[
\int_{-1/2}^{1/2} s_1(y) dy = 0.
\]
Since \( s(y) = y + e(y) \), where \( e(y) \) is periodic and odd we get \( s'(y) = 1 + e'(y) \). Knowing \( s'(y) \) we can find \( s(y) \) and subsequently \( A_\delta^1(s) \) from the knowledge of the spectrum of \( H \) and
\( \cos(ka_0 \cdot d_0), \ k \geq 1 \). Repeating the same arguments for any \( d \in L \) we can recover \( A^1(x) \).

Now we can recover \( V(x) \) assuming that \( \int_D V(x) dx = 0 \). One can check from (2.14′) that \( b(x, y) \) does not depend on \( a_0 \); the only term in \( m_0(x, y) \) (cf. (2.12)) that contains \( a_0 \) has the form \( e^{i a_0 \cdot (x - y)} \). Therefore \( -i \frac{\partial}{\partial x} m_0(x, y) = (a_0 + c_1(x, y))m_0(x, y) \) where \( c_1(x, y) \) is independent of \( a_0 \). Hence \((-i \frac{\partial}{\partial x} - A)m_0(x, y) = (a_0 + c_1(x, y) - (a_0 + \frac{b_0}{2} x^\perp + A^1(x))m_0(x, y) \) will not contain \( a_0 \). Therefore \( b(x, y) \) is known once we know \( A^1(x) \).

Since \( \int_0^1 V(x + sd) ds = V_\delta(x) \), we have

\[
(2.31) \quad J(d) = J_1(d) + J_2(d),
\]

where

\[
(2.32) \quad J_1(d) = \int_D V_\delta(x \cdot \delta) \exp[2\pi ik(x \cdot \delta + \frac{a_0 \cdot d_0}{2\pi} + A^1_\delta(x \cdot \delta))] dx,
\]

and \( J_2(d) \) is the term containing \( b(x, y) \), i.e. \( J_2(d) \) is known. Therefore \( J_1(d) + J_1(-d) \) is determined by the spectrum assuming that \( \cos(ka_0 \cdot d) \) is known.

Making the change of variables \( x = s\gamma + s'\gamma' \) as in (2.25) and integrating in \( s' \) we get

\[
(2.33) \quad J_1(d) + J_1(-d) = 2c_0 \int_{-1/2}^{1/2} V_\delta(s) \cos 2\pi k(s + \frac{a_0 \cdot d_0}{2\pi} + A^1_\delta(s)) ds.
\]

Making the change of variables \( y = s + A^1_\delta(s) \) as in (2.29) we get

\[
(2.34) \quad J_1(d) + J_1(-d) = 2c_0 \int_{-1/2}^{1/2} V_\delta(s(y)) s'(y) \cos(2\pi ky + ka_0 \cdot d_0) dy,
\]

where \( s = s(y) \) is the inverse to \( y = s + A^1_\delta(s) \). Note that \( s'(y) \) is even periodic function of period 1, and \( V_\delta(s(y)) \) is also even periodic, since \( V(x) \) is even periodic and \( s(y) \) is an odd function satisfying \( s(y + 1) = s(y) + 1 \).
Since $V_δ(s(y))s′(y)$ is an even function, 
\[ \frac{1}{-1} \int V_δ(s(y))s′(y) \sin 2\pi k y dy = 0. \] Thus, as in (2.30) we have:

(2.35)

\[ J_1(d) + J_1(-d) = 2\epsilon_0 \cos(ka_0 \cdot d_0) \int_{-1/2}^{1/2} V_δ(s(y))s′(y) \cos 2\pi k y dy, \quad k \geq 1. \]

Knowing $J_1(d) + J_1(-d)$ and $\cos(ka_0 \cdot d_0)$ we know the Fourier cosine coefficients of the even function $V_δ(s(y))s′(y)$ for $k \geq 1$. Therefore we can determine $V_δ(s(y))s′(y)$ up to a constant

(2.36)

\[ V_δ(s(y))s′(y) = C + \epsilon_1(y), \]

where $\epsilon_1(y)$ is known and $\int_{-1/2}^{1/2} \epsilon_1(y) dy = 0$.

Integrating (2.36) we get

\[ C = \int_{-1/2}^{1/2} V_δ(s(y))s′(y) dy = \int_{-1/2+\epsilon(-1/2)}^{1/2+\epsilon(1/2)} V_δ(s) ds = \int_{-1/2}^{1/2} V_δ(s) ds = 0, \]

where $s(y) = y + \epsilon(y)$, and $\epsilon(y)$ is periodic with period 1. Thus we can recover $V_δ(s)$ for each $\delta \in L^*$ and therefore we can recover $V(x)$. This concludes the proof of Theorem 1.1.

Now we shall prove Theorem 1.2. We shall assume that the magnetic field is generic in the following sense: There are two directions $\delta_1$ and $\delta_2$ which form a basis for $L^*$ such that the directional fields $B_{\delta_1}(s)$ and $B_{\delta_2}(s)$ are not identically zero. In this case the functions $s_1(x)$ and $s_2(x)$ (cf. (2.28)) corresponding to $\delta_1$ and $\delta_2$, respectively, are not identically zero. We make the additional generic assumption that $a_{1j} \neq 0$, $j = 1, 2$, where $a_{kj}$ are the Fourier cosine coefficients of $s′_j(y), k \geq 0$. Then from the main relation (2.30) we can recover $\cos a_0 \cdot d_j, j = 1, 2$, where $\{d_1, d_2\}$ is the dual basis to $\{\delta_1, \delta_2\}$. Given $d \in L$, there are are integers $m$ and $n$ such that $d = md_1 + nd_2$. Hence

\[ \cos a_0 \cdot d = \text{Re}\{e^{i a_0 \cdot (md_1 + nd_2)}\} = \text{Re}\{(\cos(a_0 \cdot d_1) \pm i \sqrt{1 - (\cos(a_0 \cdot d_1)^2)} m (\cos(a_0 \cdot d_2) \pm i \sqrt{1 - (\cos(a_0 \cdot d_2)^2)^n})\}, \]

and, since the ±’s disappear when one takes the real part, $\cos(a_0 \cdot d)$ is determined by $\cos(a_0 \cdot d_1)$ and $\cos(a_0 \cdot d_2)$. Thus, $\cos a_0 \cdot d = \cos a_0' \cdot d$ for all $d \in L$ as in (cf. (1.15)). This proves Theorem 1.2.
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