A $W_2^n$-Theory of Elliptic and Parabolic Partial Differential Systems in $C^1$ domains

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Abstract

In this paper second-order elliptic and parabolic partial differential systems are considered on $C^1$ domains. Existence and uniqueness results are obtained in terms of Sobolev spaces with weights so that we allow the derivatives of the solutions to blow up near the boundary. The coefficients of the systems are allowed to substantially oscillate or blow up near the boundary.

Keywords: Elliptic systems, Parabolic systems, Weighted Sobolev spaces.

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1 Introduction

In this article we are dealing with the Sobolev space theory of second-order parabolic and elliptic systems:

\begin{align*}
  u_t^k &= a^{ij}_{kr} u_{x^i x^j}^r + b^r_{kr} u^r_x + c^r_{kr} u^r + f^k, \quad t > 0, x \in \mathcal{O} \quad (1.1) \\
  a^{ij}_{kr} u_{x^i x^j}^r + b^r_{kr} u^r_x + c^r_{kr} u^r + f^k &= 0, \quad x \in \mathcal{O}, \quad (1.2)
\end{align*}

where $\mathcal{O}$ is a $C^1$ domain in $\mathbb{R}^d$, $i,j = 1,2,\ldots,d$ and $k,r = 1,2,\ldots,d_1$. We used summation notation on repeated indices $i,j,r$.

Since the boundary is not supposed to be regular enough, we have to look for solutions in function spaces with weights, allowing derivatives of our solutions to blow up near the boundary. In the framework of Hölder space such setting leads to investigating so-called intermediate (or interior) Schauder estimates, which originated in [2]. For results about these estimates the reader is referred to [2], [4], [5] (elliptic case) and [3], [15] (parabolic case).

Various Sobolev spaces with weights and their applications to partial differential equations have been investigated since long ago; we do not want even to try to collect all relevant references (some of them can be found in [1]). The reader can find a part of references related to the subject of this article in the papers [8], [11] and [16], the results of which are extensively used in what follows.

The main source of our interest in the Sobolev space theory of systems (1.1) and (1.2) comes from [8], [11], [12] and [16], where weighted Sobolev space theory is constructed for single equations.

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The goal of this article is to extend the results for single equations in [8, 11, 12] and [10] to the case of the systems. We prove the uniqueness and existence results of systems [13] and [12] in weighted Sobolev spaces under minimal regularity conditions on the coefficients. As in the articles referred above, our coefficients $a_{ij}^{kr}$ are allowed to substantially oscillate near the boundary, and the coefficients $b_{kr}, c_{kr}$ are allowed to be unbounded and blow up near the boundary. For instance, if $d = d_1 = 1$ and $O = (0, \infty)$, then we allow $a := a_{11}^{11}$ to behave near $x = 0$ like $2 + \cos |\ln x|^\alpha$, where $\alpha \in (0, 1)$ (see Remark 3.7).

However, unlike in those articles, we were able to obtain only $L_2$-estimates, instead of $L_p$-estimates. This is due to the difficulty caused by considering systems instead of single equations. For $L_p$-theory, $p > 2$, one must overcome tremendous mathematical difficulties rising in the general settings; one of the main difficulties in the case $p > 2$ is that the arguments we are using in the proofs of Lemma 4.3 and Lemma 4.4 below are not working when $p > 2$ since in this case we get extra terms which we simply cannot control.

The organization of the article is as follows. Section 2 handles the Cauchy problem. In section 3 we present our main results, Theorem 3.10 and Theorem 3.11. In section 4 we develop some auxiliary results. Theorem 3.10 and Theorem 3.11 are proved in section 5 and section 6, respectively.

As usual $\mathbb{R}^d$ stands for the Euclidean space of points $x = (x^1, \ldots, x^d)$, $B_r(x) = \{y \in \mathbb{R}^d : |x - y| < r\}$, $B_r = B_r(0), \mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x^1 > 0\}$. For $i = 1, \ldots, d$, multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \{0, 1, 2, \ldots\}$, and functions $u(x)$ we set

$$u_{x_i} = \frac{\partial u}{\partial x_i} = D_i u, \quad D^\alpha u = D_{\alpha_1}^{1} \cdots D_{\alpha_d}^{d} u, \quad |\alpha| = \alpha_1 + \cdots + \alpha_d.$$ 

If we write $c = c(\cdots)$, this means that the constant $c$ depends only on what are in parenthesis.

## 2 The system on $\mathbb{R}^d$

First we introduce some solvability results of linear systems defined on $\mathbb{R}^d$. These results will be used later for systems defined on the half space and bounded $C^1$ domains.

Let $C_0^\infty = C_0^\infty(\mathbb{R}^d; \mathbb{R}^{d_1})$ denote the set of all $\mathbb{R}^{d_1}$-valued infinitely differentiable functions with compact support in $\mathbb{R}^d$. By $\mathcal{D}$ we denote the space of $\mathbb{R}^d$-valued distributions on $C_0^\infty$; precisely, for $u \in \mathcal{D}$ and $\phi \in C_0^\infty$ we define $(u, \phi) \in \mathbb{R}^d$ with components $(u, \phi)^k = (u^k, \phi^k)$, $k = 1, \ldots, d_1$. Each $u^k$ is a usual $\mathbb{R}$-valued distribution defined on $C^\infty(\mathbb{R}^d; \mathbb{R})$.

We define $L_p = L_p(\mathbb{R}^d; \mathbb{R}^{d_1})$ as the space of all $\mathbb{R}^{d_1}$-valued functions $u = (u^1, \ldots, u^{d_1})$ satisfying

$$\|u\|_{L_p}^p := \sum_{k=1}^{d_1} \|u^k\|_{L_p}^p < \infty.$$ 

Let $p \in [2, \infty)$ and $\gamma \in (-\infty, \infty)$. We define the space of Bessel potential $H_\gamma^p = H_\gamma^p(\mathbb{R}^d; \mathbb{R}^{d_1})$ as the space of all distributions $u$ such that $(1 - \Delta)^{\gamma/2} u \in L_p$ where we define each component by

$$(1 - \Delta)^{\gamma/2} u^k = (1 - \Delta)^{\gamma/2} u^k.$$
and the norm is given by
\[ \|u\|_{H^\gamma_p} := \|(1 - \Delta)^{\gamma/2}u\|_{L^p}. \]
Then, \( H^\gamma_p \) is a Banach space with the given norm and \( C_0^\infty \) is dense in \( H^\gamma_p \). Note that \( H^\gamma_p \) are usual Sobolev spaces for \( \gamma = 0, 1, 2, \ldots \). It is well known that the first order differentiation operators, \( \partial_i : H^\gamma_p(\mathbb{R}^d; \mathbb{R}) \to H^{\gamma-1}_p(\mathbb{R}^d; \mathbb{R}) \) given by \( u \to u_{x_i} \) \((i = 1, 2, \ldots, d)\), are bounded. On the other hand, for \( u \in H^\gamma_p(\mathbb{R}^d) \), if \( \text{supp}(u) \subset (a, b) \times \mathbb{R}^{d-1} \) with \(-\infty < a < b < \infty\), we have
\[ \|u\|_{H^\gamma_p(\mathbb{R}^d)} \leq c(d, a, b)\|u_{x_1}\|_{H^{\gamma-1}_p(\mathbb{R}^d; \mathbb{R})} \] (2.1)
(see, for instance, Remark 1.13 in [11]).

For a fixed time \( T \), we define
\[ H^\gamma_p(T) := L^p((0, T], H^\gamma_p), \quad L^p_p(T) := H^0(T) \]
with the norm given by
\[ \|u\|_{p, H^\gamma_p(T)} = \int_0^T \|u(t)\|_{H^\gamma_p}\,dt. \]
Finally, we set \( U^\gamma_p = H^{\gamma-2/p}_p \).

**Definition 2.1.** For a \( \mathcal{D}\)-valued function \( u \in H^{\gamma+2}_p(T) \), we write \( u \in H^{\gamma+2}_p(T) \) if \( u \in H^{\gamma+2}_p(T) \), \( u(0, \cdot) \in U^{\gamma+2}_p \) and there exists \( f \in H^{\gamma}_p(T) \) such that, for any \( \phi \in C_0^\infty \), the equality
\[ (u(t, \cdot), \phi) = (u(0, \cdot), \phi) + \int_0^t (f(s, \cdot), \phi)\,ds \] holds for all \( t \leq T \). In this case, we say that \( u_t = f \) in the sense of distributions.

The norm in \( H^{\gamma+2}_p(T) \) is defined by
\[ \|u\|_{H^{\gamma+2}_p(T)} = \|u\|_{H^{\gamma+2}_p(T)} + \|u_t\|_{H^{\gamma}_p(T)} + \|u(0)\|_{U^{\gamma+2}_p}. \]

For any \( d_1 \times d_1 \) matrix \( C = (c_{kr}) \) we let
\[ |C| := \sqrt{\sum_{k,r} (c_{kr})^2}. \]
Set \( A^{ij} = (a^{ij}_{kr}) \). Throughout the article we assume the following.

**Assumption 2.2.** There exist constants \( \delta, K^j, L > 0 \) so that
(i) \[ \delta|\xi|^2 \leq \xi^i A^{ij} \xi_j \] (3.3)
holds for any \( t, x \), where \( \xi \) is any \( (\text{real}) \, d_1 \times d \) matrix, \( \xi_i \) is the \( i \)-th column of \( \xi \), and again the summations on \( i, j \) are understood.

(ii) \[ |A^{ij}| \leq K^j, \quad j = 1, 2, \ldots, d. \] (2.4)
Before we study system (1.1), we consider the following system of equations with constant coefficients:

\[ u_t^k = a_{kr}^i u_{r,x}^i + f^k, \quad u^k(0) = u_0^k, \]  

(2.5)

where \( i, j = 1, 2, \ldots, d \) and \( k, r = 1, 2, \ldots, d_1 \); recall that we are using summation notation on \( i, j, r \).

The following \( L_2 \)-theory (even \( L_p \)-theory) is not new and can be found, for instance, in [14]. However, we give a short and independent proof for the sake of completeness.

**Theorem 2.3.** Let \( a_{kr}^i = a_{kr}^i(t) \), independent of \( x \). Then for any \( f \in H_2^\gamma(T) \) and \( u_0 \in U_2^{\gamma+2} \), system (2.5) has a unique solution \( u \in H_2^{\gamma+2}(T) \), and for this solution

\[ \|u_{xx}\|_{H_2^\gamma(T)} \leq c \|f\|_{H_2^\gamma(T)} + c \|u_0\|_{U_2^{\gamma+2}}, \]  

(2.6)

\[ \|u\|_{H_2^{\gamma+2}(T)} \leq c e^{cT} (\|f\|_{H_2^\gamma(T)} + \|u_0\|_{U_2^{\gamma+2}}), \]  

(2.7)

where \( c = c(d, d_1, \gamma, \delta, K_j) \).

**Proof.** By Theorem 5.1 in [10], for each \( k \), the equation

\[ u_t^k = \delta \Delta u^k + f^k, \quad u^k(0) = u_0^k \]

has a solution \( u^k \in H_2^{\gamma+2}(T) \). For \( \lambda \in [0, 1] \) define \( A_{\lambda}^{ij} := (1 - \lambda) A^{ij} + \delta_{ij} \lambda \delta I \). Then

\[ |A_{\lambda}^{ij}| \leq |A^{ij}|, \quad \delta|\xi|^2 \leq \sum_{i,j} \xi^* A_{\lambda}^{ij} \xi \]

with any \( d_1 \times d \)-matrix \( \xi \). Thus having the method of continuity in mind, we only prove that (2.6) and (2.7) hold given that a solution \( u \) already exists.

**Step 1.** Assume \( \gamma = 0 \). Applying the chain rule \( d|u^k|^2 = 2u^k du^k \) for each \( k \),

\[ |u^k(t)|^2 = |u_0^k|^2 + \int_0^t 2u^k(u_{kr}^i u_{r,x}^i + f^k) ds, \quad t > 0. \]  

(2.8)

By integrating with respect to \( x \) and using integrating by parts,

\[ \int_{\mathbb{R}^d} |u(t)|^2 dx = 2 \int_0^t \int_{\mathbb{R}^d} \sum_{i,j} (u_{x}^i)^* A_{\lambda}^{ij} u_{x}^i dx ds \]

\[ = \int_{\mathbb{R}^d} |u_0|^2 dx + \int_0^t \int_{\mathbb{R}^d} 2u^* f dx ds. \]  

(2.9)

Hence, it follows that

\[ \int_{\mathbb{R}^d} |u(t)|^2 dx \leq 2 \int_0^t \int_{\mathbb{R}^d} |u|^2 dx ds + \int_0^t \int_{\mathbb{R}^d} |f|^2 dx ds. \]  

(2.10)
Similarly, for \( v = u_{x^n} \) with any \( n = 1, 2, \ldots, d \), we get (see (2.9))
\[
\int_{\mathbb{R}^d} |v(t)|^2 \, dx + 2\delta \int_0^t \int_{\mathbb{R}^d} |v_x|^2 \, dx \, ds \\
\leq \int_{\mathbb{R}^d} |(u_0)_{x^n}|^2 \, dx + \int_0^t \int_{\mathbb{R}^d} -2u_{x^n} \, f \, dx \, ds \\
\leq \|u_0\|_{L^2}^2 + \varepsilon \|u_{x^n}\|_{L^2}^2 + c\|f\|_{L^2}^2.
\] (2.11)

Choosing small \( \varepsilon \) and considering all \( n \), we have (2.6). Now, (2.11), (2.10) and Gronwall’s inequality easily lead to (2.7).

**Step 2.** Let \( \gamma \neq 0 \). The results of this case easily follow from the fact that \( (1 - \Delta)^{\mu/2} : H_p^\gamma \to H_{p}^{\gamma - \mu} \) is an isometry for any \( \gamma, \mu \in \mathbb{R} \) when \( p \in (1, \infty) \); indeed, \( u \in \mathcal{H}^{\gamma+2}_2(T) \) is a solution of (2.5) if and only if \( v := (1 - \Delta)^{\gamma/2}u \in \mathcal{H}^{\gamma}_2(T) \) is a solution of (2.5) with \( (1 - \Delta)^{\gamma/2}f, (1 - \Delta)^{\gamma/2}u_0 \) in place of \( f, u_0 \) respectively. Moreover, for instance,
\[
\|u\|_{\mathcal{H}^{\gamma+2}_2(T)} = \|v\|_{\mathcal{H}^{\gamma}_2(T)} \leq ce^{cT} \left( \|(1 - \Delta)^{\gamma/2}f\|_{L^2(T)} + \|(1 - \Delta)^{\gamma/2}u_0\|_{L^2(T)} \right) \\
= ce^{cT} \left( \|f\|_{\mathcal{H}^{\gamma}_2(T)} + \|u_0\|_{L^2(T)} \right).
\]
The theorem is proved.

\[\square\]

Theorem (2.3) is extended to the systems with variable coefficients in the followings.

Fix \( \mu > 0 \). For \( \gamma \in \mathbb{R} \) define \( |\gamma|_+ = |\gamma| \) if \( |\gamma| = 0, 1, 2, \ldots \) and \( |\gamma|_+ = |\gamma| + \mu \) otherwise. Also define
\[
B^{\gamma}_+ = \begin{cases} 
B(\mathbb{R}) & : \gamma = 0 \\
C^{\gamma+1,1}(\mathbb{R}) & : |\gamma| = 1, 2, \ldots \\
C^{\gamma+\mu}(\mathbb{R}) & : \text{otherwise},
\end{cases}
\]
where \( B \) is the space of bounded functions, and \( C^{\gamma+1,1} \) and \( C^{\gamma+\mu} \) are usual Hölder spaces.

Consider the system with variable coefficients:
\[
u_i^k = a_{kr}^{ij}(x, t) + b_{kr}^i u_x^r + c_{kr} u_r^r + f^k, \quad u^k(0) = u_0^k. \tag{2.12}
\]

**Theorem 2.4.** Assume that the coefficients \( a_{kr}^{ij} \) are uniformly continuous in \( x \), that is, for any \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \) so that for any \( t > 0, i, j, k, r \),
\[
|a_{kr}^{ij}(t, x) - a_{kr}^{ij}(t, y)| < \varepsilon, \quad \text{if} \quad |x - y| < \delta.
\]

Also, assume for any \( t > 0, i, j, k, r \),
\[
|a_{kr}^{ij}(t, \cdot)|_{\gamma|_+} + |b_{kr}^i(\omega, t, \cdot)|_{\gamma|_+} + |c_{kr}(\omega, t, \cdot)|_{\gamma|_+} < L.
\]

Then for any \( f \in \mathcal{H}^{\gamma}_2(T) \) and \( u_0 \in U^{\gamma+2}_2 \), system (2.12) has a unique solution \( u \in \mathcal{H}^{\gamma+2}_2(T) \), and for this solution we have
\[
\|u\|_{\mathcal{H}^{\gamma+2}_2(T)} \leq ce^{cT}(\|f\|_{\mathcal{H}^{\gamma}_2(T)} + \|u_0\|_{L^2(T)}),
\]
where \( c = c(d, d_1, \gamma, \delta, K^l, L) \).
Proof. This is an easy extension of Theorem 2.3 and can be proved by repeating the proof of Theorem 5.1 in [10], where the theorem is proved when \(d_1 = 1\). We leave the details to the reader. \(\square\)

3 The system on \(O \subset \mathbb{R}^d\)

Assumption 3.1. The domain \(O\) is of class \(C^1\). In other words, for any \(x_0 \in \partial O\), there exist constants \(r_0, K_0 \in (0, \infty)\) and a one-to-one continuously differentiable mapping \(\Psi\) of \(B_{r_0}(x_0)\) onto a domain \(J \subset \mathbb{R}^d\) such that

(i) \(J_+ := \Psi(B_{r_0}(x_0) \cap O) \subset \mathbb{R}^d_+\) and \(\Psi(x_0) = 0\);

(ii) \(\Psi(B_{r_0}(x_0) \cap \partial O) = J \cap \{ y \in \mathbb{R}^d : y_1 = 0 \}\);

(iii) \(\|\Psi\|_{C^1(B_{r_0}(x_0))} \leq K_0\) and \(|\Psi^{-1}(y_1) - \Psi^{-1}(y_2)| \leq K_0|y_1 - y_2|\) for any \(y_i \in J\);

(iv) \(\Psi_x\) is uniformly continuous in for \(B_{r_0}(x_0)\).

To proceed further we introduce some well known results from [4] and [8] (also, see [13] for details).

Lemma 3.2. Let the domain \(O\) be of class \(C^1\). Then

(i) there is a bounded real-valued function \(\psi\) defined in \(\bar{O}\) such that the functions \(\psi(x)\) and \(\rho(x) := \text{dist}(x, \partial O)\) are comparable in the part of a neighborhood of \(\partial O\) lying in \(O\). In other words, if \(\rho(x)\) is sufficiently small, say \(\rho(x) \leq 1\), then \(N^{-1}\rho(x) \leq \psi(x) \leq N\rho(x)\) with some constant \(N\) independent of \(x\),

(ii) for any multi-index \(\alpha\),

\[\sup_O \psi^{\alpha}(x)|D^\alpha \psi_x(x)| < \infty. \tag{3.1}\]

To describe the assumptions of \(f\) we use the Banach spaces introduced in [8] and [16]. Let \(\zeta \in C_0^\infty(\mathbb{R}_+)\) be a function satisfying

\[\sum_{n=\infty}^{\infty} \zeta(e^{n+x}) > c > 0, \quad \forall x \in \mathbb{R}, \tag{3.2}\]

where \(c\) is a constant. Note that any nonnegative function \(\zeta, \zeta > 0\) on \([1, e]\), satisfies (3.2). For \(x \in O\) and \(n \in \mathbb{Z} = \{0, \pm 1, \ldots\}\) define

\[\zeta_n(x) = \zeta(e^{n}\psi(x)).\]

Then we have \(\sum_n \zeta_n \geq c\) in \(O\) and

\[\zeta_n \in C_0^\infty(O), \quad |D^m \zeta_n(x)| \leq N(m)e^{mn}.\]

For \(\theta, \gamma \in \mathbb{R}\), let \(H^\gamma_{p,\theta}(O)\) be the set of all distributions \(u = (u^1, u^2, \ldots u^d)\) on \(O\) such that

\[\|u\|_{H^\gamma_{p,\theta}(O)}^p := \sum_{n \in \mathbb{Z}} e^{\gamma \theta} \|\zeta_n(e^{n}\cdot) u(e^{n}\cdot)\|_{H^\gamma_{p,\theta}}^p < \infty. \tag{3.3}\]
It is known (see, for instance, [16]) that up to equivalent norms the space \( H^\gamma_{p,\theta}(\mathcal{O}) \) is independent of the choice of \( \zeta \) and \( \psi \). Moreover if \( \gamma = n \) is a non-negative integer then

\[
\|u\|_{P,H^{\gamma}_{p,\theta}(\mathcal{O})} \sim \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \int_\mathcal{O} |\psi^k D^\alpha u(x)|^p \psi^{\theta-d}(x) \, dx. \tag{3.4}
\]

Denote \( \rho(x, y) = \rho(x) \wedge \rho(y) \) and \( \psi(x, y) = \psi(x) \wedge \psi(y) \). For \( n \in \mathbb{Z}, \mu \in (0, 1) \) and \( k = 0, 1, 2, \ldots \), define

\[
|u|_C = \sup_{\mathcal{O}} |u(x)|, \quad |u|_{C^\mu} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\mu}.
\]

\[
[u]^{(n)}_k = [u]^{(n)}_{k,\mathcal{O}} = \sup_{x \in \mathcal{O}} |\psi^{k+n}(x)| D^\beta u(x)|, \quad \beta = 0, 1, \ldots, \nu.
\]

\[
[u]^{(n)}_{k+[\mu]} = [u]^{(n)}_{k+[\mu],\mathcal{O}} = \sup_{x,y \in \mathcal{O} \atop |\beta|= k} \psi^{k+n+\mu}(x,y) \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\mu}.
\]

In case \( \mathcal{O} = \mathbb{R}_+ \), we also define the norm \( |u|^{(n)}_* = |u|^{(n)}_{k,\mathbb{R}_+} \) by using \( \rho(x) = x^1 \) and \( \rho(x) \wedge \rho(y) \) in place of \( \psi(x) \) and \( \psi(x, y) \) respectively in (3.4) and (3.5).

Below we collect some other properties of spaces \( H^\gamma_{p,\theta}(\mathcal{O}) \).

**Lemma 3.3. (17)** Let \( d - 1 < \theta < d - 1 + p \).

(i) Assume that \( \gamma - d/p = m + \nu \) for some \( m = 0, 1, \ldots \) and \( \nu \in (0, 1] \). Then for any \( u \in H^\gamma_{p,\theta}(\mathcal{O}) \) and \( i \in \{0, 1, \ldots, m\} \), we have

\[
|\psi_i^{m+\nu} D^i u|_C + |\psi^{m+\nu} D^m u|_{C^\nu} \leq c\|u\|_{H^\gamma_{p,\theta}(\mathcal{O})}.
\]

(ii) Let \( \alpha \in \mathbb{R} \), then \( \psi^\alpha H^\gamma_{p,\theta+\alpha p}(\mathcal{O}) = H^\gamma_{p,\theta}(\mathcal{O}) \),

\[
\|u\|_{H^\gamma_{p,\theta}(\mathcal{O})} \leq c\|\psi^{-\alpha} u\|_{H^\gamma_{p,\theta+\alpha p}(\mathcal{O})} \leq c\|u\|_{H^\gamma_{p,\theta}(\mathcal{O})}.
\]

(iii) There is a constant \( c = c(d, p, \gamma, \theta) \) so that

\[
\|af\|_{H^\gamma_{p,\theta}(\mathcal{O})} \leq c|a|_{[\gamma,1]} f|_{H^\gamma_{p,\theta}(\mathcal{O})}.
\]

(iv) \( \psi D, D \psi : H^\gamma_{p,\theta}(\mathcal{O}) \rightarrow H^{\gamma-1}_{p,\theta}(\mathcal{O}) \) are bounded linear operators, and

\[
\|u\|_{H^\gamma_{p,\theta}(\mathcal{O})} \leq c\|u\|_{H^{\gamma-1}_{p,\theta}(\mathcal{O})} + c\|\psi D u\|_{H^{\gamma-1}_{p,\theta}(\mathcal{O})} \leq c\|u\|_{H^\gamma_{p,\theta}(\mathcal{O})},
\]

\[
\|u\|_{H^\gamma_{p,\theta}(\mathcal{O})} \leq c\|u\|_{H^{\gamma-1}_{p,\theta}(\mathcal{O})} + c\|D \psi u\|_{H^{\gamma-1}_{p,\theta}(\mathcal{O})} \leq c\|u\|_{H^\gamma_{p,\theta}(\mathcal{O})}.
\]

Denote

\[
\mathbb{H}^\gamma_{p,\theta}(\mathcal{O}, T) = L_p([0, T], H^\gamma_{p,\theta}(\mathcal{O})), \quad L_{p,\theta}(\mathcal{O}, T) = \mathbb{L}^0_{p,\theta}(\mathcal{O}, T),
\]

\[
U^\gamma_{p,\theta}(\mathcal{O}) = \psi^{1-2/p} H^{\gamma-2/p}_{p,\theta}(\mathcal{O}).
\]
Definition 3.4. We write \( u \in \mathcal{H}^{\gamma+2}_p(O, T) \) if \( u = (u^1, \cdots, u^{d_1}) \in \psi H^{\gamma+2}_p(O, T) \), \( u(0, \cdot) \in U^{\gamma+2}_p(O) \) and for some \( f \in \psi^{-1}H^{\gamma}_p(O, T) \), it holds that \( u_t = f \) in the sense of distributions. The norm in \( \mathcal{H}^{\gamma+2}_p(O, T) \) is introduced by

\[
\|u\|_{\mathcal{H}^{\gamma+2}_p(O, T)} = \|\psi^{-1}u\|_{\psi H^{\gamma+2}_p(O, T)} + \|\psi u_t\|_{\psi H^\gamma(O, T)} + \|u(0, \cdot)\|_{U^{\gamma+2}_p(O)}.
\]

The following result is due to N.V. Krylov (see [9] and [10]).

Lemma 3.5. Let \( \|u\|_{C^{\gamma+1}_p(O)} \leq c \|u\|_{\mathcal{H}^{\gamma+2}_p(O, T)} \).

In particular, for any \( t \leq T \),

\[
\|u\|_{C^{\gamma+1}_p(O, t)} \leq c \int_0^t \|u\|_{\mathcal{H}^{\gamma+2}_p(O, s)} ds.
\]

Assumption 3.6. (i) The functions \( a^{ij}_{kr}(t, \cdot) \) are point-wise continuous in \( O \), that is, for any \( \varepsilon > 0, x \in O \) there exists \( \delta = \delta(\varepsilon, x) \) so that

\[ |a^{ij}_{kr}(t, x) - a^{ij}_{kr}(t, y)| < \varepsilon \]

whenever \( y \in O \) and \( |x - y| < \delta \).

(ii) There is control on the behavior of \( a^{ij}_{kr}, b^{ij}_{kr} \) and \( c_{kr} \) near \( \partial O \), namely,

\[
\lim_{\rho(x) \to 0} \sup_{x \in O} \sup_{|y| \leq \rho(x)} |a^{ij}_{kr}(t, x) - a^{ij}_{kr}(t, y)| = 0. \tag{3.7}
\]

\[
\lim_{\rho(x) \to 0} \sup_{x \in O} [\rho(x)|b^{ij}_{kr}(t, x)| + \rho^2(x)|c_{kr}(t, x)|] = 0. \tag{3.8}
\]

(iii) For any \( t > 0 \),

\[ |a^{ij}_{kr}(t, \cdot)|_{\gamma_1}^{(0)} + |b^{ij}_{kr}(t, \cdot)|_{\gamma_1}^{(1)} + |c_{kr}(t, \cdot)|_{\gamma_1}^{(2)} \leq L. \]

Remark 3.7. It is easy to see that (3.7) is much weaker than uniform continuity condition. For instance, if \( \delta \in (0, 1), d = d_1 = 1, \) and \( O = \mathbb{R}_+ \), then the function \( a(x) \) equal to \( 2 + \sin(|\ln x|^d) \) for \( 0 < x \leq 1/2 \) satisfies (3.7). Indeed, if \( x, y > 0 \) and \( |x - y| \leq x \land y \), then

\[ |a(x) - a(y)| = |x - y| |a'(\xi)|, \]

where \( \xi \) lies between \( x \) and \( y \). In addition, \( |x - y| \leq x \land y \leq \xi \leq 2(x \land y) \), and \( |\ln x|^d \leq |\ln 2(x \land y)|^{d-1} \to 0 \) as \( x \land y \to 0 \).

Also observe that (3.8) allows the coefficients \( b^{ij}_{kr} \) and \( c_{kr} \) to blow up near the boundary at a certain rate.
Now, for each \(i, j\), we define the symmetric part \(S^{ij}\) and the diagonal part \(S^{ij}_d\) of \(A^{ij}\) as follows:
\[
S^{ij} = (s^{ij}_{kr}) := (A^{ij} + (A^{ij})^*)/2, \quad S^{ij}_d = (s^{ij}_{d,kr}) := (\delta_{kr}a^{ij}_{kr}) = (\delta_{kr}s^{ij}_{kr}).
\]
Also define
\[
H^{ij} := A^{ij} - (A^{ij})^*, \quad S^{ij}_o = S^{ij} - S^{ij}_d.
\]
Assume there exist constants \(\alpha, \bar{\alpha}, \beta_1, \ldots, \beta_d \in [0, \infty)\) so that
\[
|H^{ij}| \leq \beta^i \quad \forall j = 1, 2, \ldots, d, \quad |S^{ij}_o| \leq \alpha,
\]
\[
\xi^i S^{ij}_o \xi_j \leq \bar{\alpha} |\xi|^2,
\]
for any \((\text{real})\) \(d_1 \times d_\text{matrix} \xi\). Here \(\xi_i\) is the \(i\)th column of \(\xi\), and again the summations on \(i, j\) are understood. Denote
\[
K := \sqrt{\sum_j (K^j)^2}, \quad \beta = \sqrt{\sum_j (\beta^j)^2}.
\]

**Assumption 3.8.** One of the following four conditions is satisfied:
\[
\theta \in \left(d - \frac{\delta}{2K - \delta}, \ d + \frac{\delta}{2K + \delta}\right); \quad (3.11)
\]
\[
\theta \in (d - 1, d], \quad 2\delta(d + 1 - \theta)^2 - 2(d + 1 - \theta)(d - \theta)\beta - 4(d - \theta)(d + 1 - \theta)K^1 > 0; \quad (3.12)
\]
\[
\theta \in (d - 1, d], \quad (\delta - \bar{\alpha}) - \frac{(d - \theta)}{(d + 1 - \theta)}(2\delta - \beta - 2\alpha) > 0; \quad (3.13)
\]
\[
\theta \in [d, d + 1), \quad 8(d + 1 - \theta)\delta^2 - (\theta - d)\beta^2 > 0. \quad (3.14)
\]

**Remark 3.9.** (i) If \(A^{ij}\) are symmetric, i.e., \(\beta = 0\), then (3.12) combined with (3.13) is \(\theta \in (d - \frac{\delta}{2K - \delta}, d + 1)\) which is weaker than (3.11).

(ii) If \(A^{ij}\) are diagonal matrices, that is if \(\alpha = \beta^i = 0\), then (3.12) combined with (3.14) is \(\theta \in (d - 1, d + 1)\). This is the case when the system is not correlated.

(iii) We also mention that if \(\theta \in (d - 1, d + 1)\) then Theorem 3.10 is false even for the heat equation \(u_t = \Delta u + f\) (see [11]).

Here are the main results of this article. The proofs of the theorems will be given in section 5 and section 6 after we develop some auxiliary results on \(\mathbb{R}^d_+\) in section 4.

**Theorem 3.10.** Let \(\gamma \geq 0\) and \(\mathcal{O}\) be bounded. Also let Assumptions 2.2, 3.1, 3.6 and 3.8 hold. Then for any \(f \in \psi^{-1}H_{2,\theta}^\gamma(\mathcal{O}, T), u_0 \in U_{2,\theta}^{\gamma+2}(\mathcal{O})\), system (2.13) admits a unique solution \(u \in \mathcal{S}_{2,\theta}^{\gamma+2}(\mathcal{O}, T)\), and for this solution
\[
\|\psi^{-1} u\|_{H_{2,\theta}^{\gamma+2}(\mathcal{O}, T)} \leq ce^{CT} \left(\|\psi f\|_{H_{2,\theta}^{-\gamma}(\mathcal{O})} + \|u_0\|_{U_{2,\theta}^{\gamma+2}(\mathcal{O})}\right), \quad (3.15)
\]
where \(c = c(d, \delta, \theta, K, L)\).
Theorem 3.11. Let \( \gamma \geq 0 \) and \( \mathcal{O} \) be bounded. Assume \( a_{k,r}^{ij}, b_{k,r}^{ij}, c_{k,r} \) are independent of \( t \) and \( \lambda^k \) are sufficiently large constants (actually, any constants bigger than \( c \) from \( \sqrt{13} \)). Under the assumptions of Theorem 3.10, for any \( f \in \psi^{-1}H_{2,0}^{\gamma}(\mathcal{O}) \) there is a unique \( u \in \psi H_{2,0}^{\gamma+2}(\mathcal{O}) \) such that in \( \mathcal{O} \),
\[
a_{k,r}^{ij}u_{x_r}^{r} + b_{k,r}^{ij}u_{x_r}^{r} + c_{k,r}u^{r} - \lambda^k u^k + f^k = 0.
\]
Furthermore,
\[
\|\psi^{-1}u\|_{H_{2,0}^{\gamma+2}(\mathcal{O})} \leq N\|\psi f\|_{H_{2,0}^{\gamma}(\mathcal{O})},
\]
where the constant \( N \) is independent of \( f \).

Remark 3.12. Actually Theorem 3.10 and Theorem 3.11 hold even for \( \gamma < 0 \). Using results for the case \( \gamma \geq 0 \), repeat the arguments in the proof of Theorem 2.10 in [8], where the theorems are proved when \( d_1 = 1 \). We leave the details to the reader. Also by inspecting the proofs carefully one can check that the above two theorems hold true even if \( \mathcal{O} \) is not bounded.

4 Auxiliary results: some results on \( \mathbb{R}^d_+ \)

In this section we develop some results for the systems defined on \( \mathbb{R}^d_+ \). Here we use the Banach spaces \( H_{p,0}^{\gamma}(\mathbb{R}^d_+) \), \( \mathbb{H}_{p,0}^{\gamma}(T) \) and \( \mathcal{H}_{p,0}^{\gamma}(T) \) defined on \( \mathbb{R}^d_+ \). They are defined on the basis of (3.3) by formally taking \( \psi(x) = x^1 \), so that \( \zeta_n(e^n x) = \zeta(x) \) and
\[
\|u\|_{H_{p,0}^{\gamma}} := \sum_{n \in \mathbb{Z}} e^{n\theta}\|u(e^n \cdot)\|_{H_{p}^{\gamma}} < \infty.
\]
Observe that the spaces \( H_{p,0}^{\gamma}(\mathbb{R}^d_+) \) and \( \mathbb{H}_{p,0}^{\gamma}(T) \) are different since \( \psi \) is bounded. Actually for any nonnegative function \( \xi = \xi(x^1) \in C^\infty_0(\mathbb{R}) \) so that \( \xi = 1 \) near \( x^1 = 0 \) we have
\[
\|u\|_{H_{p,0}^{\gamma}(\mathbb{R}^d_+)} \sim \left( \|\xi u\|_{H_{p,0}^{\gamma}} + \|(1 - \xi)u\|_{H_{p,0}^{\gamma}} \right).
\]
Also, it is known (see [14]) that for any \( \eta \in C^\infty_0(\mathbb{R}^d_+) \),
\[
\sum_{n = -\infty}^{\infty} e^{n\theta}\|u(e^n \cdot)\|_{H_{p}^{\gamma}} \leq c \sum_{n = -\infty}^{\infty} e^{n\theta}\|u(e^n \cdot)\|_{H_{p}^{\gamma}},
\]
where \( c \) depends only on \( d, d_1, \gamma, \theta, p, \eta, \zeta \). Furthermore, if \( \gamma = n \) is a nonnegative integer then (see [3,4])
\[
\|u\|_{H_{p,0}^{\gamma}} \sim \sum_{k=0}^{n} \sum_{|\alpha| = k} \int_{\mathbb{R}^d_+} |\psi^k D^\alpha u(x)|_{H_{p}^{\gamma}} dx.
\]

Let \( M^\alpha \) be the operator of multiplying \( (x^1)^\alpha \) and \( M = M^1 \).

Lemma 4.1. The assertions (i)-(iv) in Lemma 3.3 hold true if one formally replaces \( H_{p,0}^{\gamma}(\mathcal{O}) \) and \( \psi \) by \( H_{p,0}^{\gamma} \) and \( M \), respectively.

We need the following three lemmas to prove the main result of this section.
Lemma 4.2. Let $a_{kr}^{ij} = a_{kr}^{ij}(t)$, independent of $x$. Assume that $f \in M^{-1}H^{\gamma,0}_{2,\theta}(T)$, $u(0) \in U^{\gamma+2}_{2,\theta}$ and $u \in M^{H^{\gamma+1}_{2,\theta}(T)}$ is a solution of system \[ (2.7) \] on $[0,T] \times \mathbb{R}_+$, then $u \in M^{H^{\gamma+2}_{2,\theta}(T)}$ and

$$
\|M^{-1}u\|_{H^{\gamma+2}_{2,\theta}(T)} \leq c\|M^{-1}u\|_{H^{\gamma,1}_{2,\theta}(T)} + c\|MF\|_{H^{\gamma}_{2,\theta}(T)} + c\|u(0)\|_{U^{\gamma+2}_{2,\theta}},
$$

(4.4)

where $c = c(d, d_1, \gamma, \theta, \delta, K, L)$.

Proof. By Lemma 4.1 and (2.1),

$$
\|M^{-1}u\|_{H^{\gamma+2}_{2,\theta}(T)} \leq c\sum_n e^{\gamma n\theta} \|u(t, e^n x)\|_{H^{\gamma+2}_{2,\theta}(T)} = c\sum_n e^{\gamma n\theta} \|u(e^{2n} t, e^n x)\|_{H^{\gamma+2}_{2,\theta}(e^{-2n} T)} \leq c\sum_n e^{\gamma n\theta} \|u(e^{2n} t, e^n x)\|_{H^{\gamma+2}_{2,\theta}(e^{-2n} T)}.
$$

Denote

$$
v_n(t, x) = u(e^{2n} t, e^n x)\zeta(x), \quad a_{n,kr}^{ij}(t) = a_{kr}^{ij}(e^{2n} t).
$$

Then since $v_n$ has compact support in $\mathbb{R}_d^+$, $v_n$ is in $H^{\gamma+1}_{2,\theta}(e^{-2n} T)$ and satisfies

$$
(v_n)_{t} = a_{n,kr}^{ij}(v_n^{k})_{x^i x^j} + f_n, \quad v_n(0, x) = \zeta(x)u_0^k(e^n x),
$$

where

$$
f_n = -2e^n a_{n,kr}^{ij}u^r_x(e^{2n} t, e^n x)\zeta_x e^{x^j} + a_{n,kr}^{ij}u^r(e^{2n} t, e^n x)\zeta_x e^{x^j} + e^{2n} f^k(e^{2n} t, e^n x)\zeta(x).
$$

By Theorem 2.2, $v_n$ is in $H^{\gamma+2}_{2,\theta}(e^{-2n} T)$ and

$$
\|(v_n)_{xx}\|_{H^{\gamma+2}_{2,\theta}(e^{-2n} T)} \leq c(d, d_1, \gamma, \theta, K, L)(\|f_n\|_{H^{\gamma+2}_{2,\theta}(e^{-2n} T)} + \|\zeta(x)u_0^k(e^n x)\|_{H^{\gamma+2}_{2,\theta}}).
$$

Thus by (4.2) and Lemma 4.1

$$
\sum_n e^{\gamma n\theta} \sum_n e^{\gamma n\theta} \|u(e^{2n} t, e^n x)\|_{H^{\gamma+2}_{2,\theta}(e^{-2n} T)} \leq c\sum_n e^{\gamma n\theta} \|u(t, e^n x)\|_{H^{\gamma}_{2,\theta}(T)} + c\sum_n e^{\gamma n\theta} \|u(t, e^n x)\|_{H^{\gamma}_{2,\theta}(T)} + c\sum_n e^{\gamma n\theta} \|u_0(t, e^n x)\|_{H^{\gamma+2}_{2,\theta}} \leq c\|u\|_{H^{\gamma}_{2,\theta}(T)} + c\|MF\|_{H^{\gamma}_{2,\theta}(T)} + c\|u_0\|_{U^{\gamma+2}_{2,\theta}}.
$$

The lemma is proved.

It follows from the above lemma that if $\gamma \geq 0$, then

$$
\|M^{-1}u\|_{H^{\gamma+2}_{2,\theta}(T)} \leq c\|M^{-1}u\|_{H^{\gamma}_{2,\theta}(T)} + c\|MF\|_{H^{\gamma}_{2,\theta}(T)} + c\|u_0\|_{U^{\gamma+2}_{2,\theta}}.
$$

Thus to get a priori estimate, we only need to estimate $\|M^{-1}u\|_{H^{\gamma}_{2,\theta}(T)}$ in terms of $f$ and $u_0$. 

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Lemma 4.3. Let $a_{kr}^{ij} = a_{kr}^{ij}(t)$, independent of $x$. Assume
\[ \theta \in \left( d - \frac{\delta}{2K - \delta}, \ d + \frac{\delta}{2K + \delta} \right) \] (4.5)
and $u \in M_{\mathbb{H}_2^1}(T)$ is a solution of (2.2) so that $u \in C([0, T], C^0_0((1/N, N) \times \{ x' : |x'| < N \}))$ for
some $N > 0$. Then we have
\[ \| M^{-1}u \|_{L^2_2, \theta(T)}^2 \leq c_0(\| Mf \|_{L^2_2, \theta(T)}^2 + \| u_0 \|_{U^1_{1, \theta}}^2), \] (4.6)
where $c_0 = c_0(d, \delta, \theta, K, L)$.

Proof. As in the proof of Theorem 2.3 applying the chain rule $d|u_k|^2 = 2u_k du_k$ for each $k$, we have
\[ |u_k(t)|^2 = |u_0|^2 + \int_0^t 2u_k(a_{kr}^{ij} u_{kr}^{ij} + f^k) \, ds \]
where the summations on $i, j, r$ are understood. Denote $c = \theta - d$. For each $k$, we have
\[ 0 \leq \int_{\mathbb{R}^4} |u_k(T, x)|^2 (x^1)^c \, dx \]
\[ = \int_{\mathbb{R}^4} |u_k(0, x)|^2 (x^1)^c \, dx \]
\[ + 2 \int_0^T \int_{\mathbb{R}^4} a_{kr}^{ij} u_k u_{kr}^{ij} (x^1)^c \, dxds + 2 \int_0^T \int_{\mathbb{R}^4} (M^{-1}u_k)(Mf^k)(x^1)^c \, dxds. \] (4.7)

Note that, by integration by parts, the second term in (4.7) is
\[ \int_0^T \int_{\mathbb{R}^4} \left[ -2a_{kr}^{ij} u_k^{ij} - 2c(a_{kr}^{ij} u_{kr}^{ij})(M^{-1}u_k)^2 \right] (x^1)^c \, dxds \] (4.8)
\[ \leq \int_0^T \int_{\mathbb{R}^4} -2a_{kr}^{ij} u_k^{ij} u_{kr}^{ij} (x^1)^c \, dxds + |c| \left( \kappa \| u_k \|_{L^2_2, \theta(T)}^2 + K^2 \kappa^{-1} \| M^{-1}u_k \|_{L^2_2, \theta(T)}^2 \right), \]
for each $\kappa > 0$, because for any vectors $v, w \in \mathbb{R}^n$ and $\kappa > 0$,
\[ | \langle A^{ij}v, w \rangle | \leq |A^{ij}v||w| \leq K^2 |v||w| \leq \frac{1}{2}( |v|^2 + \kappa^{-1}(K^2)^2 |w|^2). \]

By summing up the terms in (4.7) over $k$ and rearranging the terms, we get
\[ 2 \int_0^T \int_{\mathbb{R}^4} u_k^{ij} A^{ij} u_{kr}^{ij} (x^1)^c \, dxds \]
\[ \leq |c| \left( \kappa \| u_k \|_{L^2_2, \theta(T)}^2 + K^2 \kappa^{-1} \| M^{-1}u \|_{L^2_2, \theta(T)}^2 \right) + \varepsilon \| M^{-1}u \|_{L^2_2, \theta(T)}^2 \] (4.9)
\[ + \varepsilon |c| \| Mf \|_{L^2_2, \theta(T)}^2 + \| u(0) \|_{U^1_{1, \theta}}^2, \] (4.10)
where $\kappa, \varepsilon > 0$ will be decided below. Assumption 2.2(i), inequality (4.10) and the inequality
\[ \| M^{-1}u \|_{L^2_2, \theta}^2 \leq \frac{4}{(d + 1 - \theta)^2} \| u_0 \|_{U^1_{1, \theta}}^2 \] (4.11)
(see Corollary 6.2 in [11]) lead us to
\[
2\delta \|u_x\|_{L^2_{\theta}(T)}^2 - |c| \left( \kappa + \frac{4K^2}{\kappa(d + 1 - \theta)^2} \right) \|u_x\|_{L^2_{\theta}(T)}^2 \\
\leq c\varepsilon \|u_x\|_{L^2_{\theta}(T)}^2 + c(\varepsilon)\|Mf\|_{L^2_{\theta}(T)}^2 + \|u(0)\|_{L^2_{\theta}}^2.
\]

Now it is enough to take \( \kappa = 2K/(d + 1 - \theta) \) and observe that \( 4.5 \) is equivalent to the condition
\[
2\delta - |c| \left( \kappa + \frac{4K}{\kappa(d + 1 - \theta)^2} \right) = 2\delta - \frac{4|c|K}{d + 1 - \theta} > 0.
\]
Choosing a small \( \varepsilon = \varepsilon(d, \delta, \theta, K, L) \), the lemma is proved. \( \square \)

**Lemma 4.4.** Let \( a_{kr}^{ij} = a_{cr}^{ij}(t) \). Suppose either
\[
\theta \in (d - 1, d], \quad 2d(d + 1 - \theta)^2 - 2(d + 1 - \theta)(d - \theta)\beta - 4(d - \theta)(d + 1 - \theta)K^1 > 0 \quad (4.12)
\]
or
\[
\theta \in (d - 1, d], \quad (\delta - \bar{\alpha}) - \frac{(d - \theta)}{(d + 1 - \theta)(2\delta - \beta - 2\alpha) > 0; \quad (4.13)
\]
or
\[
\theta \in [d, d + 1], \quad 8(d + 1 - \theta)\beta - (\theta - d)\beta^2 > 0. \quad (4.14)
\]
Let \( u \in M^{1,0}_{\theta}(T) \) be a solution of \( 2.5 \) so that \( u \in C\left([0,T], C^0_0((1/N, N) \times \{x' : |x'| < N\})\right) \) for some \( N > 0 \). Then the assertion of Lemma 4.3 holds.

**Proof.** 1. Denote \( S^{ij} = \langle s_{kr}^{ij}\rangle = \frac{1}{2}(A^{ij} + (A^{ij})^*) \) as the symmetric part of \( A^{ij} \). Then \( A^{ij} = S^{ij} + \frac{1}{2}H^{ij} \), and for any \( \xi \in \mathbb{R}^d \) we notice that \( \xi^* A^{ij} \xi = \xi^* S^{ij} \xi \). Let \( c := \theta - d \). Note that, by integration by parts,
\[
\int_{\mathbb{R}^d_+} u^* S^{11} u_x^c(x^1)dx = -c - \frac{1}{2} \int_{\mathbb{R}^d_+} u^* S^{11} u(x^1)dx = -c - \frac{1}{2} \int_{\mathbb{R}^d_+} u^* A^{11} u(x^1)dx
\]
and hence
\[
-2c \int_{\mathbb{R}^d_+} u^* A^{11} u_x^c(x^1)dx = -2c \int_{\mathbb{R}^d_+} u^* S^{11} u_x^c(x^1)dx - c \int_{\mathbb{R}^d_+} u^* H^{11} u_x^c(x^1)dx = c(c - 1) \int_{\mathbb{R}^d_+} u^* A^{11} u(x^1)dx - c \int_{\mathbb{R}^d_+} u^* H^{11} u_x^c(x^1)dx.
\]
Moreover, another usage of integration by parts gives us
\[
\int_{\mathbb{R}^d_+} u^* S^{1j} u_x^c(x^1)dx = -\int_{\mathbb{R}^d_+} u^* S^{1j} u_x^c(x^1)dx = -\int_{\mathbb{R}^d_+} u^* (S^{1j})^* u_x^c(x^1)dx
\]
for \( j \neq 1 \), meaning that \( \int_{\mathbb{R}^d_+} u^* S^{1j} u_x^c(x^1)dx = 0 \) and
\[
-2c \int_{\mathbb{R}^d_+} u^* A^{1j} u_x^c(x^1)dx = -c \int_{\mathbb{R}^d_+} u^* H^{1j} u_x^c(x^1)dx.
\]
We gather the above terms to get

\[-2c \int_{\mathbb{R}^d_+} (a_{k_1}^{ij} u_x^{k_1}) (x^1)^{c-1} dx = c(c-1) \int_{\mathbb{R}^d_+} u^* A^{11} u(x^1)^{c-2} dx - c \int_{\mathbb{R}^d_+} u^* H^{1j} u_x (x^1)^{c-1} dx,\]

where the summation on \( j \) includes \( j = 1 \).

Now, as in the proof of Lemma 4.3, we have

\[
2\delta \| u_x \|_2^2 \leq 2 \int_0^T \int_{\mathbb{R}^d_+} u_x^* A^{11} u_x (x^1)^c dx ds \\
\leq \int_{\mathbb{R}^d} |u^k(0, x)|^2 x^c dx + c(c-1) \int_0^T \int_{\mathbb{R}^d} a_{k_1}^{ij} (M^{-1} u^k)(M^{-1} u^r)(x^1)^c dx ds - c \int_0^T \int_{\mathbb{R}^d} (h_{k_1}^{ij} u_x^r)(M^{-1} u^k)(x^1)^c dx ds \\
+ 2 \int_0^T \int_{\mathbb{R}^d} (M^{-1} u^k)(M f^k)(x^1)^c dx ds. 
\]

(4.15)

Note that the first and last terms in the right hand side of (4.15) are bounded by

\[\varepsilon \| M^{-1} u \|_{2,\theta}^2 (T) + c(\varepsilon) \| M f \|_{L,\theta}^2 (T) + \| u(0) \|_{V,\theta}^2.\]

2. If \( c(c-1) \geq 0 \), hence \( \theta \in (d-1, d] \), then

\[c(c-1) \int_0^T \int_{\mathbb{R}^d} a_{k_1}^{ij} (M^{-1} u^k)(M^{-1} u^r)(x^1)^c dx ds \leq c(c-1) K^1 \| M^{-1} u \|_{2,\theta}^2 (T). \]

Also,

\[\left| -c \int_0^T \int_{\mathbb{R}^d} (h_{k_1}^{ij} u_x^r)(M^{-1} u^k)(x^1)^c dx ds \right| \leq \frac{1}{2} c \left( \kappa \| u_x \|_{2,\theta}^2 (T) + \kappa^{-1} \beta^2 \| M^{-1} u \|_{2,\theta}^2 (T) \right) \leq \frac{1}{2} c \left( \kappa + \frac{4 \beta^2}{\kappa(d+1-\theta)^2} \right) \| u_x \|_{2,\theta}^2 (T) \]

for any \( \kappa > 0 \). To minimize this we take \( \kappa = 2\beta/(d+1-\theta) \), then

\[\left| -c \int_0^T \int_{\mathbb{R}^d} (h_{k_1}^{ij} u_x^r)(M^{-1} u^k)(x^1)^c dx ds \right| \leq \frac{2\beta(d-\theta)}{(d+1-\theta)} \| u_x \|_{2,\theta}^2 (T). \]

(4.16)

Thus we deduce

\[\left( 2\delta - \frac{2\beta(d-\theta)}{(d+1-\theta)} - \frac{4}{(d+1-\theta)^2} c(c-1) K^1 \right) \| u_x \|_{2,\theta}^2 (T) \leq c \varepsilon \| u_x \|_{2,\theta}^2 (T) + c(\varepsilon) \| M f \|_{L,\theta}^2 (T) + \| u(0) \|_{V,\theta}^2. \]

This and (4.11) yield a priori (4.6), since (4.12) is equivalent to

\[2\delta - \frac{2\beta(d-\theta)}{(d+1-\theta)} - \frac{4}{(d+1-\theta)^2} c(c-1) K^1 > 0.\]
3. Again assume \( c(c - 1) \geq 0 \). By (4.13) and (4.16),
\[
2 \int_0^T \int_{\mathbb{R}^d_+} u_x^* \left( S_d^{ij} + S_0^{ij} \right) u_{x^*} (x^1)^c dx ds
\leq \int_{\mathbb{R}^d_+} |u^k(0, x)|^2 x^c dx
\]
\[+ c(c - 1) \int_0^T \int_{\mathbb{R}^d_+} (s_{d,kr}^{11} + s_{kr}^{11}) (M^{-1}u^k)(M^{-1}u^r)(x^1)^c dx ds
\]
\[+ \frac{2d(d - \theta)}{(d + 1 - \theta)} \|u_x\|^2_{L_{2,\theta}} + \varepsilon \|M^{-1}u\|^2_{L_{2,\theta}} + c \|Mf\|^2_{L_{2,\theta}}.
\]
By Corollary 6.2 of [11], for each \( t \),
\[
c(c - 1) \int_{\mathbb{R}^d_+} s_{d,kr}^{11} (M^{-1}u^k)(M^{-1}u^r)(x^1)^c dx \leq \frac{4d(d - \theta)}{(d + 1 - \theta)} \int_{\mathbb{R}^d_+} u_x^*, S_d^{ij} u_{x^*} (x^1)^c dx.
\]
By assumptions,
\[
2 \int_{\mathbb{R}^d_+} u_x^*, S_d^{ij} u_{x^*} (x^1)^c dx \leq 2\alpha \int_{\mathbb{R}^d_+} |u_x|^2 (x^1)^c dx,
\]
\[
c(c - 1) \int_{\mathbb{R}^d_+} s_{d,kr}^{11} |M^{-1}u^k| |M^{-1}u^r|(x^1)^c dx \leq \alpha c(c - 1) \int_{\mathbb{R}^d_+} |M^{-1}u|^2 (x^1)^c dx
\]
\[\leq \frac{4\alpha(d - \theta)}{(d + 1 - \theta)} \int_{\mathbb{R}^d_+} |u_x|^2 (x^1)^c dx.
\]
It follows
\[
\left[ (\delta - \alpha) - \frac{(d - \theta)}{(d + 1 - \theta)} (2\delta - \beta - 2\alpha) \right] \|u_x\|^2_{L_{2,\theta}(T)} \leq \varepsilon \|u_x\|^2_{L_{2,\theta}(T)} + c \|Mf\|^2_{L_{2,\theta}(T)} + |u_0|^2_{U_{2,\theta}}.
\]
This, (4.13) and (4.11) lead to the a priori estimate.

4. If \( c(c - 1) \leq 0 \), hence \( \theta \in [d, d + 1) \), then
\[
c(c - 1) \int_0^T \int_{\mathbb{R}^d_+} a_{kr}^{11} (M^{-1}u^k)(M^{-1}u^r)(x^1)^c dx ds \leq \delta c(c - 1) \|M^{-1}u\|^2_{L_{2,\theta}(T)};
\]
for this we consider a \( d_1 \times d \) matrix consisting of \( M^{-1}u \) as the first column and zeros for the rest, and apply the assumption 2.3. Next, as before, we have
\[
-c \int_0^T \int_{\mathbb{R}^d_+} (h_{kr}^{ij} u_{x^j})(M^{-1}u^k)(x^1)^c dx ds \leq \frac{1}{2} \left( \kappa \|u_x\|^2_{L_{2,\theta}(T)} + \kappa^{-1} \beta^2 \|M^{-1}u\|^2_{L_{2,\theta}(T)} \right)
\]
and hence from (4.15) it follows
\[
2\delta \|u_x\|^2_{L_{2,\theta}(T)} - \frac{1}{2} c \left( \kappa \|u_x\|^2_{L_{2,\theta}(T)} + \kappa^{-1} \beta^2 \|M^{-1}u\|^2_{L_{2,\theta}(T)} \right) - \delta c(c - 1) \|M^{-1}u\|^2_{L_{2,\theta}(T)}
\]
\[\leq \varepsilon \|u_x\|^2_{L_{2,\theta}(T)} + c(\varepsilon) \|Mf\|^2_{L_{2,\theta}(T)} + |u(0)|^2_{U_{2,\theta}}.
\]
As we take
\[
\kappa = \frac{\beta^2}{2\delta(1 - c)}
\]
the terms with \( \|M^{-1}u\|^2_{L_{2,\theta}(T)} \) in the left hand side of (4.17) are canceled. Now, (4.14), which is equivalent to \( 2\delta - \frac{\varepsilon \beta^2}{c(1 - c)} > 0 \) gives us a priori estimate (4.16). The lemma is proved. \( \square \)
Lemma 4.7. Let constants \( a_{kl}^{ij} = a_{kl}^{ij}(t) \). Assume that one of (4.9), (4.12), (4.13) and (4.14) holds. Then for any \( f \in M^{-1} H^2_{2, \theta}(T) \) and \( u_0 \in U^{\gamma + 2}_{2, \theta} \), system (2.12) admits a unique solution \( u \in \mathcal{D}^{\gamma + 2}_{2, \theta}(T) \), and for this solution

\[
||M^{-1}u||_{\mathcal{D}^{\gamma + 2}_{2, \theta}(T)} \leq c||Mf||_{H^2_{2, \theta}(T)} + c||u_0||_{U^{\gamma + 2}_{2, \theta}}.
\]

where \( c = c(d, \delta, \theta, K, L) \).

Proof. 1. By Theorem 3.3 in [12], for each \( k \), the equation

\[
u_k = \delta \Delta u^k + f^k, \quad u^k(0) = u_0^k
\]

has a solution \( u^k \in \mathcal{D}^{\gamma + 2}_{2, \theta}(T) \). As in the proof of Theorem 2.3 we only need to show that estimate (4.18) holds given that a solution already exists.

2. By Theorem 2.9 in [12], for any nonnegative integer \( n \geq \gamma + 2 \), the set

\[
\mathcal{D}^n_{2, \theta}(T) \cap \bigcup_{N=1}^{\infty} C([0,T], C_0((1/N, N) \times \{x' : |x'| < N\}))
\]

is everywhere dense in \( \mathcal{D}^{\gamma + 2}_{p, \theta}(T) \) and we may assume that \( u \) is sufficiently smooth in \( x \) and vanishes near the boundary. Thus a priori estimate (4.18) follows from Lemma 3.2, Lemma 4.3 and Lemma 4.4. The theorem is proved.

Here is the main result of this section.

Theorem 4.6. Let \( \gamma \geq 0 \) and Assumption 3.8 hold. Assume that for each \( t \)

\[
|a_{kl}^{ij}(t, \cdot)|_{\gamma_t} + |b_{kl}^{ij}(t, \cdot)|_{\gamma_t} + |c_{kl}(t, \cdot)|_{\gamma_t} \leq L
\]

and

\[
|a_{kl}^{ij}(t, x) - a_{kl}^{ij}(t, y)| + |M^2 c_{kl}(t, x)| < \kappa
\]

for all \( x, y \in \mathbb{R}^d \) with \( |x - y| \leq x^1 \land y^1 \). Then there exists \( \kappa_0 = \kappa_0(d, \theta, \delta, K, L) \) so that if \( \kappa \leq \kappa_0 \), then for any \( f \in M^{-1} H^2_{2, \theta}(T) \), and \( u_0 \in U^{\gamma + 2}_{2, \theta} \), system (2.12) admits a unique solution \( u \in \mathcal{D}^{\gamma + 2}_{2, \theta}(T) \), and furthermore

\[
||u||_{\mathcal{D}^{\gamma + 2}_{2, \theta}(T)} \leq c||Mf||_{H^2_{2, \theta}(T)} + c||u_0||_{U^{\gamma + 2}_{2, \theta}}
\]

where \( c = c(d, d_1, \delta, \theta, K, L) \).

To prove Theorem 4.6 we use the following lemmas taken from [8].

Lemma 4.7. Let constants \( C, \delta \in (0, \infty) \), a function \( u \in H^2_{p, \theta} \), and \( q \) be the smallest integer such that \( |\gamma| + 2 \leq q \).

(i) Let \( \eta_n \in C^\infty(\mathbb{R}^d_+) \), \( n = 1, 2, ... \), satisfy

\[
\sum_n M^{[\alpha]} |D^\alpha \eta_n| \leq C \quad \text{in} \quad \mathbb{R}^d_+
\]

(ii) Let \( \eta_n \in C^\infty(\mathbb{R}^d) \), \( n = 1, 2, ... \), satisfy

\[
\sum_n M^{[\alpha]} |D^\alpha \eta_n| \leq C \quad \text{in} \quad \mathbb{R}^d.
\]
for any multi-index $\alpha$ such that $0 \leq |\alpha| \leq q$. Then
\[
\sum_n \|\eta_n u\|_{H^s_{p,\theta}}^p \leq N C_p \|u\|_{H^s_{p,\theta}}^p,
\]
where the constant $N$ is independent of $u$, $\theta$, and $C$.

(ii) If in addition to the condition in (i)
\[
\sum_n \eta_n^2 \geq \delta \quad \text{on } \mathbb{R}^d,
\]
then
\[
\|u\|_{H^s_{p,\theta}}^p \leq N \sum_n \|\eta_n u\|_{H^s_{p,\theta}}^p,
\]
where the constant $N$ is independent of $u$ and $\theta$.

The reason the first inequality in (4.23) below is written for $\eta_4^n$ (not for $\eta_2^n$) as in the above lemma is to have the possibility to apply Lemma 4.7 to $\eta_2^n$. Also observe that obviously $\sum a^2 \leq (\sum |a|)^2$.

**Lemma 4.8.** For each $\varepsilon > 0$ and $q = 1, 2, \ldots$ there exist non-negative functions $\eta_n \in C^\infty_0(\mathbb{R}^d)$, $n = 1, 2, \ldots$ such that (i) on $\mathbb{R}^d_{+}$ for each multi-index $\alpha$ with $1 \leq |\alpha| \leq q$ we have
\[
\sum_n \eta_n^4 \geq 1, \quad \sum_n \eta_n = N(d), \quad \sum_n M^{[\alpha]}|D^{\alpha}\eta_n| \leq \varepsilon; \quad (4.23)
\]

(ii) for any $n$ and $x, y \in \text{supp} \eta_n$ we have $|x - y| \leq N(x^1 \wedge y^1)$, where $N = N(d, q, \varepsilon) \in [1, \infty)$.

**Lemma 4.9.** Let $p \in (1, \infty)$, $\gamma, \theta \in \mathbb{R}$. Then there exists a constant $N = N(\gamma, |\gamma|, p, d)$ such that if $f \in H^\gamma_{p,\theta}$ and $a$ is a function with finite norm $|a|^{(0)*}_{|\gamma|_{+}, \mathbb{R}^d_{+}}$, then
\[
\|af\|_{H^\gamma_{p,\theta}} \leq N|a|^{(0)*}_{|\gamma|_{+}, \mathbb{R}^d_{+}} \|f\|_{H^\gamma_{p,\theta}}, \quad (4.24)
\]

In addition,

(i) if $\gamma = 0, 1, 2, \ldots$, then
\[
\|af\|_{H^\gamma_{p,\theta}} \leq N \sup_{\mathbb{R}^d_{+}} |a| \|f\|_{H^\gamma_{p,\theta}} + N_0 \|f\|_{H^{\gamma-1}_{p,\theta}} \sup_{\mathbb{R}^d_{+}} |M^{[\alpha]}|D^{\alpha}a|, \quad (4.25)
\]
where $N_0 = 0$ if $\gamma = 0$, and $N_0 = N_0(\gamma, d) > 0$ otherwise.

(ii) if $\gamma$ is not integer, then
\[
\|af\|_{H^\gamma_{p,\theta}} \leq N(\sup_{\mathbb{R}^d_{+}} |a|^{s}_{|\gamma|_{+}})^s (|a|^{(0)*}_{|\gamma|_{+}})^{1-s} \|f\|_{H^\gamma_{p,\theta}}, \quad (4.26)
\]
where $s := 1 - \frac{|\gamma|}{|\gamma|_{+}} > 0$.

**Proof of Theorem 4.6**
We closely follow the proof of Theorem 2.16 of [7]. As usual, for simplicity, we assume $u_0 = 0$. Also having the method of continuity in mind, we convince ourselves that to prove the theorem
it suffices to show that there exist \( \kappa_0 \) such that the a priori estimate (4.19) holds given that the solution already exists and \( \kappa \leq \kappa_0 \). We divide the proof into two cases. This is because if \( \gamma \) is an integer we use (4.29), and otherwise we use (4.30).

**Case 1:** \( \gamma = 0 \) or \( \gamma \) is not integer. Take the least integer \( q \geq |\gamma| + 4 \). Also take an \( \varepsilon \in (0,1) \) to be specified later and take a sequence of functions \( \eta_n, n = 1, 2, \ldots, \) from Lemma 4.8 corresponding to \( \varepsilon, q \). Then by Lemma 4.7, we have

\[
\| M^{-1}u \|_{H_{2,0}^{s+2}(T)}^2 \leq N \sum_{n=1}^{\infty} \| M^{-1}u\eta_n^2 \|_{H_{2,0}^{s+2}(T)}^2.
\] (4.27)

For any \( n \) let \( x_n \) be a point in \( \text{supp} \eta_n \) and \( a_{ij}^n (t) = a_{ij}^n (t, x_n) \). From (2.12), we easily have

\[
(a^k \eta_n^a)_t = a_{ij}^n (t) u_{x_j}^n + M^{-1} f_n^k,
\]

where

\[
f_n^k = (a_{ij}^n - a_{ij}^n (x_j^n, \eta_{in}^k, \eta_{in}^k, \eta_{in}^k, \eta_{in}^k)) M u_{x_j}^n + 2 a_{ij}^n \eta_{in}^k M u_{x_j}^n - a_{ij}^n \eta_{in}^k M^{-1} u^2 \eta_{in}^k M^2 u_{x_j}^n - a_{ij}^n \eta_{in}^k M u_{x_j}^n + a_{ij}^n \eta_{in}^k M^2 c_k r M^{-1} u^r + M f_n^k \eta_{in}^k.
\]

By Theorem 4.5, for each \( n \), we have

\[
\| M^{-1}u\eta_n^2 \|_{H_{2,0}^{s+2}(T)}^2 \leq N \| f_n \|_{H_{2,0}^{s}(T)}^2
\] (4.28)

and by (4.20),

\[
\| (a_{ij}^n - a_{ij}^n (x_j^n, \eta_{in}^k, \eta_{in}^k, \eta_{in}^k, \eta_{in}^k)) M u_{x_j}^n \|_{H_{p,0}^{q}} \leq N \| \eta_n M u_{x_j}^n \|_{H_{p,0}^{r}} \sup_{t,x} \| (a_{ij}^n - a_{ij}^n (x_j^n, \eta_{in}^k, \eta_{in}^k, \eta_{in}^k, \eta_{in}^k)) \|_{H_{p,0}^{r}},
\] (4.29)

where \( s = 1 \) if \( \gamma = 0 \), and \( s = 1 - \frac{2}{\gamma} > 0 \) otherwise.

By Lemma 4.8(ii), for each \( n \) and \( x, y \in \text{supp} \eta_n \) we have \( |x - y| \leq N(\varepsilon)(x^1 \wedge y^1) \), where \( N(\varepsilon) = N(d, q, \varepsilon) \), and we can easily find not more than \( N(\varepsilon) + 2 \leq 3N(\varepsilon) \) points \( x_i \) lying on the straight segment connecting \( x \) and \( y \) and including \( x \) and \( y \), such that \( |x_i - x_{i+1}| \leq x^1_i \wedge x^1_{i+1} \). It follows from our assumptions

\[
\sup_{t,x} \| (a_{ij}^n - a_{ij}^n (x_j^n, \eta_{in}^k, \eta_{in}^k, \eta_{in}^k, \eta_{in}^k)) \|_{H_{p,0}^{r}} \leq 3N(\varepsilon) \kappa.
\]

We substitute this to (4.29) and get

\[
\| (a_{ij}^n - a_{ij}^n (x_j^n, \eta_{in}^k, \eta_{in}^k, \eta_{in}^k, \eta_{in}^k)) \|_{H_{p,0}^{q}} \leq 3N(\varepsilon) \kappa^2 \| \eta_n M u_{x_j}^n \|_{H_{p,0}^{r}}.
\]

Similarly,

\[
\| \eta_n^2 M b_k r_{x_j} u^r_{x_j} \|_{H_{2,0}^{s}(T)} + \| \eta_n^2 M^2 c_k r M^{-1} u^r \|_{H_{2,0}^{s}(T)} \leq 3N(\varepsilon) \kappa^2 (\| \eta_n M u_{x_j}^n \|_{H_{p,0}^{r}(T)} + \| \eta_n M^{-1} u \|_{H_{p,0}^{r}(T)}).
\]

Coming back to (4.25) and (4.26) and using Lemma 4.7, we conclude

\[
\| M^{-1}u \|_{H_{2,0}^{s+2}(T)}^2 \leq 3N(\varepsilon) \kappa^2 \| M u_{x_j}^n \|_{H_{2,0}^{s}(T)}^2 + \| u_{x_j}^n \|_{H_{2,0}^{s}(T)}^2 + \| M^{-1}u \|_{H_{2,0}^{s}(T)}^2
\]

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where
\[ C = \sup_{x \in \mathbb{R}^d} \sup_{|\alpha| \leq q-1} \sum_{n=1}^{\infty} M^{\alpha_j}(|D^n(M(\eta^2_{n,2})_x)| + |D^n(M^2(\eta^2_{n,2})_{xx})|). \]

By construction, we have \( C \leq N\varepsilon \). Furthermore (see, Lemma 4.1)
\[ \|u_x\|_{H^{\gamma+1}_{2,\theta}} \leq N\|M^{-1}u\|_{H^{\gamma+2}_{2,\theta}}, \quad \|Mu_{xx}\|_{H^{\gamma}_{2,\theta}} \leq N\|M^{-1}u\|_{H^{\gamma+2}_{2,\theta}}. \] (4.31)

Hence (4.30) yields
\[ \|M^{-1}u\|_{H^{\gamma+2}_{2,\theta}(T)} \leq N_1(N(\varepsilon)\kappa^{2s} + \varepsilon^2)\|M^{-1}u\|_{H^{\gamma+2}_{2,\theta}(T)} + N(\|Mf\|_{H^\gamma_{2,\theta}(T)}). \]
Finally to get the a priori estimate, it’s enough to choose first \( \varepsilon \) and then \( \kappa_0 \), so that \( N_1(N(\varepsilon)\kappa^{2s} + \varepsilon^2) \leq 1/2 \) for \( \kappa \leq \kappa_0 \).

**Case 2**: \( \gamma \in \{1, 2, \ldots\} \). Proceed as in Case 1 with \( \varepsilon = 1 \) and arrive at (4.28) which is
\[ \|M^{-1}u\|_{H^{\gamma+2}_{2,\theta}(T)} \leq N\|f\|_{L^2_{2,\theta}(T)}. \]

Now we use (4.23) to get
\[ \|\{a^{ij}_{kr} - a^{ij}_{k,rn}\}u_{x^i,x^j}\|_{H^{\gamma-2}_{2,\theta}(T)} \leq N\kappa\|\eta_nMu_{xx}\|_{H^{\gamma}_{2,\theta}(T)} + N\|\eta_nMu_{xx}\|_{H^{\gamma}_{2,\theta}(T)}. \]

From this point by following the arguments in case 1, one easily gets
\[ \|M^{-1}u\|_{H^{\gamma+2}_{2,\theta}(T)} \leq N_1\kappa\|M^{-1}u\|_{H^{\gamma+2}_{2,\theta}(T)} + N_2\|M^{-1}u\|_{H^{\gamma+1}_{2,\theta}(T)} + N\|Mf\|_{H^\gamma_{2,\theta}(T)}. \] (4.32)

This and the embedding inequality
\[ \|M^{-1}u\|_{H^{\gamma+1}_{2,\theta}} \leq \frac{1}{2N_2}\|M^{-1}u\|_{H^{\gamma+2}_{2,\theta}} + N(N_2, \gamma)\|M^{-1}u\|_{H^{\gamma}_{2,\theta}} \]
yield
\[ \|M^{-1}u\|_{H^{\gamma+2}_{2,\theta}(T)} \leq 2N_1\kappa\|M^{-1}u\|_{H^{\gamma+2}_{2,\theta}(T)} + N\|M^{-1}u\|_{H^\gamma_{2,\theta}(T)} + N\|Mf\|_{H^\gamma_{2,\theta}(T)}. \] (4.33)

Now take \( \kappa_0 \) is from Case 1 for \( \gamma = 0 \), then it is enough to assume \( \kappa \leq \kappa_0 \wedge 1/(4N_1) \), because by the result of Case 1,
\[ \|M^{-1}u\|_{H^\gamma_{2,\theta}(T)} \leq N\|Mf\|_{L^2_{2,\theta}(T)}. \]

The theorem is proved.

**5 Proof of Theorem 3.10**

By Theorem 2.10 in [13], for each \( k, f^k \in \psi^{-1}H^\gamma_{2,\theta}(O, T) \) and \( u_0^k \in U^\gamma+2_{2,\theta}(O) \), the equation
\[ u_i^k = \Delta u^k + f^k, \quad u^k(0) = u_0^k(0) \]
has a unique solution \( u \in \mathcal{H}_{2,\theta}^{\gamma,2}(T) \), and furthermore
\[
\|\psi^{-1} u^k\|_{\mathcal{H}_{2,\theta}^{\gamma,2}(\mathcal{O}, T)} \leq c\|\psi f^k\|_{\mathcal{H}_{2,\theta}^{\gamma,2}(\mathcal{O}, T)} + c\|u_0^k\|_{\mathcal{H}_{2,\theta}^{\gamma,2}(\mathcal{O})}.
\]

To prove the theorem we only need to prove that (3.15) holds given that a solution \( u \in \mathcal{H}_{2,\theta}^{\gamma,2}(\mathcal{O}, T) \) already exists. As usual we assume \( u_0 = 0 \). Let \( x_0 \in \partial \mathcal{O} \) and \( \Psi \) be a function from Assumption 3.1. In [8] it is shown that \( \Psi \) can be chosen in such a way that for any non-negative integer \( n \)
\[
|\Psi_x|_{n, B_{r_0}(x_0) \cap \mathcal{O}} + |\Psi_x^{-1}|_{n, J_+} < N(n) < \infty
\] (5.1)
and
\[
\rho(x)\Psi_x(x) \to 0 \quad \text{as} \quad x \in B_{r_0}(x_0) \cap \mathcal{O}, \text{and} \quad \rho(x) \to 0,
\] (5.2)
where the constants \( N(n) \) and the convergence in (5.2) are independent of \( x_0 \).

Define \( r = r_0/K_0 \) and fix smooth functions \( \eta \in C_0^\infty(B_r), \varphi \in C_\infty(\mathbb{R}) \) such that \( 0 \leq \eta, \varphi \leq 1 \), and \( \eta = 1 \) in \( B_{r/2} \), \( \varphi(t) = 1 \) for \( t \leq -3 \), and \( \varphi(t) = 0 \) for \( t \geq -1 \). Observe that \( \Psi(B_{r_0}(x_0)) \) contains \( B_r \).

For \( n = 1, 2, \ldots, t > 0, x \in \mathbb{R}^d_+ \) introduce \( \varphi_n(x) = \varphi(n^{-1} \ln x^1) \),
\[
\hat{a}^{ij,n}(t, x) := \eta(x)\varphi_n(x) \left( \sum_{l,m=1}^d a^{lm}(t, \Psi^{-1}(x)) \cdot \partial_i \Psi^l(\Psi^{-1}(x)) \cdot \partial_m \Psi^j(\Psi^{-1}(x)) \right) + \delta^{ij}(1-\eta(x)\varphi_n(x)) I,
\]
\[
\hat{b}^{i,n}(t, x) := \eta(x)\varphi_n(x) \left[ \sum_{l,m} a^{lm}(t, \Psi^{-1}(x)) \cdot \partial_m \Psi^i(\Psi^{-1}(x)) + \sum_l b^l(t, \Psi^{-1}(x)) \cdot \partial_l \Psi^i(\Psi^{-1}(x)) \right],
\]
\[
\hat{c}^n(t, x) := \eta(x)\varphi_n(x)c(t, \Psi^{-1}(x)).
\]
Then by Assumption 3.6 (iii) and (5.1), one can show that there is a constant \( L' \) independent of \( n \) and \( x_0 \) such that
\[
|\hat{a}^{ij,n}(t, \cdot)|_{\gamma, a}^{(0)*} + |\hat{b}^{i,n}(t, \cdot)|_{\gamma, a}^{(1)*} + |\hat{c}^n(t, \cdot)|_{\gamma, a}^{(2)*} \leq L'.
\]

Take \( \kappa_0 \) from Theorem 4.6 corresponding to \( d, d_1, \theta, \delta, K \) and \( L' \). Observe that \( \varphi_n(x) = 0 \) for \( x^1 \geq e^{-n} \). Also (5.2) implies \( x^1|_{\mathcal{O}}(\Psi^{-1}(x)) \to 0 \) as \( x^1 \to 0 \). Using these facts and Assumption 3.6 (ii), one can find \( n > 0 \) independent of \( x_0 \) such that
\[
|\hat{a}_{kr}^{ij,n}(t, x) - \hat{a}^{ij,n}(t, y)| + (x^1)|\hat{b}_{kr}^i(t, x)| + x^2|\hat{c}_{kr}^n(t, x)| \leq \kappa_0,
\]
whenever \( t > 0, x, y \in \mathbb{R}^d_+ \) and \( |x - y| \leq x^1 \wedge y^1 \). Now we fix a \( \rho_0 < r_0 \) such that
\[
\Psi(B_{\rho_0}(x_0)) \subset B_{r/2} \cap \{ x : x^1 \leq e^{-3n} \}.
\]

Let \( \zeta \) be a smooth function with support in \( B_{\rho_0}(x_0) \) and denote \( v := (u\zeta)(\Psi^{-1}) \) and continue \( v \) as zero in \( \mathbb{R}^d_+ \setminus \Psi(B_{\rho_0}(x_0)) \). Since \( \eta\varphi_n = 1 \) on \( \Psi(B_{\rho_0}(x_0)) \), the function \( v \) satisfies
\[
v^{k}_r = \hat{a}^{ij,n}_{kr} v^{r}_{x^j x^i} + \hat{b}^{i,n}_{kr} v^{r}_{x^i} + \hat{c}^{n}_{kr} v^{r} + \hat{f}^{k}_{r}
\]
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where
\[ \tilde{f}^k = \tilde{f}^k(\Psi^{-1}), \quad \hat{f}^k = -2a_{ij}^k \partial_x^2 \zeta_{x^i} \zeta_{x^j} - a_{ij}^k \partial^r \zeta_{x^i x^j} - b_{ij}^k \partial^r \zeta_{x^i} + \zeta f^k. \]

Next we observe that by Lemma 3.2 and Theorem 3.2 in [10] (or see [8]) for any \( \nu, \alpha \in \mathbb{R} \) and \( h \in \psi^{-\alpha} H_{p, \theta}^\nu(\Omega) \) with support in \( B_{\rho_0}(x_0) \)

\[ \| \psi^\alpha h \|_{H_{p, \theta}^\nu(\Omega)} \sim \| M^\alpha h(\Psi^{-1}) \|_{H_{p, \theta}^\nu}. \quad (5.3) \]

Therefore we conclude that \( v \in \tilde{H}_{2, \theta}^{\gamma+2}(T) \), and by Theorem 4.6 we have, for any \( t \leq T \),

\[ \| M^{-1} v \|_{\tilde{H}_{2, \theta}^{\gamma+2}(t)} \leq N \| M \tilde{f} \|_{\tilde{H}_{2, \theta}^\nu(t)}. \]

By using \( 5.3 \) again we obtain

\[ \| \psi^{-1} u \zeta \|_{\tilde{H}_{2, \theta}^{\gamma+2}(\Omega, t)} \leq N \| a \zeta \psi u_x \|_{\tilde{H}_{2, \theta}^2(\Omega, t)} + N \| a \zeta \psi u \|_{\tilde{H}_{2, \theta}^2(\Omega, t)} + N \| \zeta \psi f \|_{\tilde{H}_{2, \theta}^2(\Omega, t)}. \]

Next, we easily check that

\[ | \zeta_x a(t, \cdot) |_{[\gamma]+}, \quad | \zeta x \psi a(t, \cdot) |_{[\gamma]+}, \quad | \zeta x \psi b(t, \cdot) |_{[\gamma]+} \]

are bounded on \([0, T]\), and conclude

\[ \| \psi^{-1} u \zeta \|_{\tilde{H}_{2, \theta}^{\gamma+2}(\Omega, t)} \leq N \| u \|_{\tilde{H}_{2, \theta}^\nu(\Omega, t)} + N \| u \|_{\tilde{H}_{2, \theta}^2(\Omega, t)} + N \| \psi f \|_{\tilde{H}_{2, \theta}^\nu(t)}. \]

Finally, to estimate the norm \( \| \psi^{-1} u \|_{\tilde{H}_{2, \theta}^{\gamma+2}(\Omega, t)} \), we introduce a partition of unity \( \zeta(i), \; i = 0, 1, 2, \ldots, M \) such that \( \zeta(i) \in C_0^\infty(\Omega) \) and \( \zeta(i) \in C_0^\infty(B_{\rho_0}(x_i)) \), \( x_i \in \partial \Omega \) for \( i \geq 1 \). Observe that since \( u \zeta(0) \) has compact in \( \Omega \), we get

\[ \| \psi^{-1} u \zeta(0) \|_{\tilde{H}_{2, \theta}^{\gamma+2}(\Omega, t)} \sim \| u \zeta(0) \|_{\tilde{H}_{2, \theta}^{\gamma+2}(t)}. \]

Thus we can estimate \( \| \psi^{-1} u \zeta(0) \|_{\tilde{H}_{2, \theta}^{\gamma+2}(\Omega, t)} \) using Theorem 2.4 and the other norms as above.

By summing up those estimates we get

\[ \| \psi^{-1} u \|_{\tilde{H}_{2, \theta}^{\gamma+2}(\Omega, t)} \leq N \| u \|_{\tilde{H}_{2, \theta}^\nu(\Omega, t)} + N \| u \|_{\tilde{H}_{2, \theta}^2(\Omega, t)} + N \| \psi f \|_{\tilde{H}_{2, \theta}^\nu(t)}. \]

Furthermore, we know that

\[ \| u \|_{\tilde{H}_{2, \theta}^\nu(\Omega)} \leq \| u \|_{\tilde{H}_{2, \theta}^{\gamma+1}(\Omega)}. \]

Therefore it follows

\[ \| u \|_{\tilde{H}_{2, \theta}^{\gamma+2}(\Omega, t)} \leq N \| u \|_{\tilde{H}_{2, \theta}^{\gamma+1}(\Omega, t)} + N \| \psi f \|_{\tilde{H}_{2, \theta}^\nu(t)} \leq N \int_0^t \| u \|_{\tilde{H}_{2, \theta}^{\gamma+2}(\Omega, s)} ds + N \| \psi f \|_{\tilde{H}_{2, \theta}^\nu(t)}, \]

where Lemma 3.5 is used for the second inequality. Now (5.13) follows from Gronwall’s inequality. The theorem is proved.
6 Proof of Theorem 3.11

Again we only show that a priori estimate (3.17) holds given that a solution \( u \in \psi H^{1+2}_{2,\theta}(\Omega) \) already exists. By (3.1) it follows that \( \psi \) is a point-wise multiplier in \( H^{\nu}_{p,\theta}(\Omega) \) for any \( \nu \) and \( p \). Thus

\[
\| u \|_{H^{1+2}_{2,\theta}(\Omega)} := \| u \|_{H^{1+1}_{2,\theta}(\Omega)} \leq c(\theta, \gamma) \| \psi^{-1} u \|_{H^{1+1}_{2,\theta}(\Omega)}.
\] (6.1)

Note that \( v^k := u^k e^{\lambda^k t} \) satisfies

\[
v^k_t = a_{kr}^i v^r x^i + b_{kr}^i u^r x^i + c_{kr} v^r + f^k e^{\lambda^k t}.
\]

By (6.10) and (6.11),

\[
g_1(T) \| \psi^{-1} u \|_{H^{1+2}_{2,\theta}(\Omega)} \leq c e^{cT} \left( \| \psi^{-1} u \|_{H^{1+2}_{2,\theta}(\Omega)} + g_2(T) \| \psi f \|_{H^{1}_{2,\theta}(\Omega)} \right),
\]

where

\[
g_1(T) = \left( \int_0^T e^{2t \min \{ \lambda^k \}} dt \right)^{1/2}, \quad g_2(T) = \left( \int_0^T e^{2t \max \{ \lambda^k \}} dt \right)^{1/2}.
\]

If \( \min \{ \lambda^k \} > \epsilon \), then the ratio \( c e^{cT}/g_1(T) \) tends to zero as \( T \to \infty \). Then after finding a \( T \) such that this ratio is less than \( 1/2 \) one gets (3.17). The theorem is proved.

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