Generalisations of the Camassa-Holm equation

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Abstract

We classify generalised Camassa-Holm type equations which possess infinite hierarchies of higher symmetries. We show that the obtained equations can be treated as negative flows of integrable quasi-linear scalar evolution equations of orders 2, 3 and 5. We present the corresponding Lax representations or linearisation transformations for these equations. Some of the obtained equations seem to be new.

1 Introduction

In recent years there has been a growing interest in integrable non-evolutionary partial differential equations of the form

\[(1 - D_x^2) u_t = F(u, u_x, u_{xx}, u_{xxx}, \ldots), \quad u = u(x,t), \quad D_x = \frac{\partial}{\partial x}. \quad (1)\]

Here \( F \) is some function of \( u \) and its derivatives with respect to \( x \). The most celebrated example of this type of equations is the Camassa–Holm equation \([1]\):

\[(1 - D_x^2) u_t = 3uu_x - 2u_x u_{xx} - uu_{xxx}.\]

Another equivalent form of the Camassa–Holm equation is

\[m_t = 2mu_x + um_x, \quad m = u - u_{xx}.\]

The Camassa–Holm equation is integrable by the inverse scattering transform. It possesses an infinite hierarchy of local conservation laws, bi-Hamiltonian structure and other remarkable properties of integrable equations. Despite its non-evolutionary form the Camassa–Holm equation possesses an infinite hierarchy of local higher symmetries - indeed this equation can be viewed as an inverse flow of the equation \( u_\tau = (4 - D_x^2)D_x(u - u_{xx})^{-4} \). Furthermore, the Camassa–Holm equation can be reduced via a reciprocal transformation to the first negative of the Korteweg–de Vries hierarchy (see also \([2]\)). The Camassa–Holm equation possesses multi-phase peakon solutions (peaked soliton solutions with discontinuous derivatives at the peaks).

Until 2002 the Camassa-Holm equation was the only known integrable example of the type \((1)\), which possesses peakon solutions, when Degasperis and Procesi isolated another equation

\[(1 - D_x^2) u_t = 4uu_x - 3u_x u_{xx} - uu_{xxx},\]

or in a different form

\[m_t = 3mu_x + um_x, \quad m = u - u_{xx},\]

which was also found to be integrable by the inverse scattering transform \([3]\). The Degasperis–Procesi equation also possesses infinitely many conservation laws, bi-Hamiltonian structure etc. It also possesses an infinite hierarchy of local higher symmetries and can be seen as a non-local symmetry of a local evolutionary equation \( u_\tau = (4 - D_x^2)D_x(u - u_{xx})^{-4} \). In fact, the Degasperis–Procesi equation can be reduced via a reciprocal transformation to the first negative flow of the Kaup-Kupershmidt hierarchy \([4]\).

One may ask questions: are there other integrable equations of the form \((1)\) and is it possible to classify all integrable equations of this type? The answer to both questions is positive.

The first classification result of equations of type \((1)\) was obtained in \([6]\) using the perturbative symmetry approach in the symbolic representation. In the symmetry approach the existence of infinite hierarchies of higher symmetries is adopted as a definition of integrability. The conditions of existence of higher symmetries are very restrictive and result in algorithmic and efficient integrability test. In particular, the following result was proved in \([6]\):
Theorem 1. If equation
\[ m_t = bm u_x + um_x, \quad m = u - u_{xx}, \quad b \in C \setminus \{0\} \]
possesses an infinite hierarchy of (quasi-) local higher symmetries then \( b = 2, 3 \).

Obviously, the case \( b = 2 \) corresponds to the Camassa–Holm equation, while \( b = 3 \) gives the Degasperis–Procesi equation.

In this paper we extend the classification result of \[6\] and apply the perturbative symmetry approach to isolate and classify more general class of integrable equations of the form \[1\]. We assume that function \( F \) in the right hand side is a homogeneous differential polynomial over \( C \), quadratic or cubic in \( u \) and its \( x \)-derivatives. The obtained list comprises of 28 equations (see section 3) and some of these equations seem to be new to the best of our knowledge. The list includes an equation of the form
\[
(1 - D_x^2)u_t = u^2u_{xxx} + 3uu_xu_{xx} - 4u_xu.
\]
Integrability and multipeakon solutions of this equation have been recently studied in \[7\] and \[11\]. For all the obtained equations we present their first non-trivial higher symmetries. We also give Lax representations or linearisation transformations for most of the equations. We show that all the obtained equations can be treated as negative flows of integrable quasi-linear scalar evolution equations of orders 2, 3 or 5. The classification results of the latter ones can be found in \[10\].

2 Integrability test

In this section we briefly remind the basic definitions and notations of the perturbative symmetry approach (for details see \[5\], \[12\]). We also present the integrability test \[6\], which we apply to isolate integrable generalisations of the Camassa–Holm equation.

2.1 Symmetries and approximate symmetries

In what follows we shall consider the Camassa–Holm type equation \[1\] with the right hand side being a differential polynomial over \( C \).

Let \( R \) be a ring of differential polynomials in \( u, u_x, u_{xx}, \ldots \) over \( C \). We shall adopt a notation
\[
u_i \equiv D^i_x(u).
\]
We shall often omit subscript 0 at \( u_0 \) and write \( u \) instead of \( u_0 \).

The ring \( R \) is a differential ring with a derivation
\[
D_x = \sum_{i \geq 0} u_{i+1} \frac{\partial}{\partial u_i}.
\]
The ring has a natural gradation with respect to degrees of non-linearity in \( u \) and its \( x \)-derivatives:
\[
R = \bigoplus_{i \geq 0} R_i, \quad R_i = \{ f(u, u_1, \ldots, u_k) \in R \mid f(\lambda u, \lambda u_1, \ldots, \lambda u_k) = \lambda^i f(u, u_1, \ldots, u_k) \}, \quad \lambda \in C.
\]
The space \( R_0 = C, R_1 \) is a space of linear polynomials in \( u, u_1, \ldots, R_2 \) is a space of quadratic polynomials etc. It is convenient to introduce a notion of “little oh” as
\[
f = o(R_p) \iff f \in \bigoplus_{i > p} R_i.
\]

Let us denote by \( R_+ \) a differential ring without a unit
\[
R_+ = \bigoplus_{i > 0} R_i.
\]

Suppose that the right hand side of the equation \[1\] \( F \in R_+ \). We can formally rewrite equation \[1\] as an evolutionary equation as
\[
u_t = \Delta(F), \quad \Delta = (1 - D_x^2)^{-1}.
\]
Symmetries and conservation laws of this equation, if they exist, may also contain operator \( \Delta \) in their structure and therefore we need an extension of the differential ring \( R_+ \) with the operator \( \Delta \). The construction of such extension was first suggested in \[6\] for the evolutionary 2 + 1 dimensional equations. For the Camassa–Holm type equations it was first applied in \[6\]. Namely, let us construct a sequence of spaces \( R^0_+, R^1_+, \ldots \) as follows:
\[
R^0_+ = R_+, \quad R^1_+ = R^0_+ \cup \Delta(R^0_+), \quad R^{n+1}_+ = R^n_+ \cup \Delta(R^n_+).
\]
The subscript \( n \) in \( \mathcal{R}_n^+ \) is the “nesting depth” of the operator \( \Delta \). The extension construction is compatible with the natural gradation:

\[
\mathcal{R}_n^+ = \bigoplus_{i>0} \mathcal{R}_{i+n}^+ \quad \mathcal{R}_n = \{ f[u] \in \mathcal{R}_n^+ \mid f[\lambda u] = \lambda^i f[u] \}, \quad \lambda \in \mathbb{C}.
\]

It is clear that \( \Delta(F) \) in the equation (2) belongs to \( \mathcal{R}_1^+ \). The symmetries of the equation may belong to \( \mathcal{R}_n^+ \) for some appropriate \( k \geq 0 \) and we introduce the following definition of a symmetry:

**Definition 1.** A function \( G \in \mathcal{R}_k^+ \), \( k \geq 0 \) is called a generator of a symmetry of equation (2) if a differential equation

\[
u_t = G
\]

is compatible with the equation (2): \( G_t - F_r = 0 \).

We adopt the following definition of integrability:

**Definition 2.** Equation (2) is integrable if it possesses an infinite hierarchy of symmetries.

In addition to the definition of a symmetry we also introduce a definition of an approximate symmetry:

**Definition 3.** A function \( G \in \mathcal{R}_k^+ \), \( k \geq 0 \) is called a generator of an approximate symmetry of degree \( p \) of equation (2) if \( G_t - F_r = o(\mathcal{R}_p^+) \).

Any equation

\[
u_t = \Delta(F) = \Delta(F_1) + \Delta(F_2) + \cdots + \Delta(F_k), \quad F_k \in \mathcal{R}_k
\]

possesses an infinite hierarchy of approximate symmetries of degree 1 – these are symmetries of its linear part \( \nu_t = \Delta(F_1) \). The condition of existence of approximate symmetries of degree 2 imposes strong restrictions on the equation. However an equation may possess infinitely many approximate symmetries of degree 2, but fail to possess approximate symmetries of degree 3. On the other hand an integrable equation possesses infinitely many approximate symmetries of any degree. The degree of approximate symmetry can be viewed as a measure to the integrability. In many cases the existence of approximate symmetries of sufficiently large degree implies integrability.

In order to derive the conditions of existence of symmetries and approximate symmetries it is convenient to introduce the symbolic representation of the ring \( \mathcal{R}_+ \) and its extension.

### 2.2 Symbolic representation

We start by introducing the symbolic representation \( \hat{\mathcal{R}}_+ \) of \( \mathcal{R}_+ \). We first introduce the symbolic representation of spaces \( \mathcal{R}_k, \ k = 1, 2, \ldots \):

1) To a linear monomial \( u_i \in \mathcal{R}_1 \) we put into correspondence a symbol

\[
u_i \rightarrow \hat{u}_i \xi_i.
\]

2) To a quadratic monomial \( u_i u_j \in \mathcal{R}_2 \) we put into correspondence a symbol

\[
u_i\nu_j \rightarrow \hat{u}_i^2 \xi_i^2 + \hat{u}_i \xi_i \xi_j + \hat{u}_j \xi_i \xi_j.
\]

3) We represent a generic \( u_i^{n_0} u_j^{n_1} \cdots u_k^{n_h} \in \mathcal{R}_n \), \( n = n_0 + n_1 + \cdots + n_h \) by a symbol

\[
u_i^{n_0} \nu_j^{n_1} \cdots \nu_k^{n_h} \rightarrow \hat{u}^n \langle \xi_i^{n_0} \xi_j^{n_1} \cdots \xi_k^{n_h} \rangle,
\]

where brackets \( \langle \rangle \) denote a symmetrisation operation:

\[
\langle f(\xi_1, \ldots, \xi_n) \rangle = \sum_{\sigma \in S_n} f(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(n)}).
\]

We define addition, multiplication and derivation as follows. Let \( f \in \mathcal{R}_i, \ g \in \mathcal{R}_j \) be two monomials and their symbolic representation is given by \( f \rightarrow \hat{u}^i a(\xi_1, \ldots, \xi_i), \ g \rightarrow \hat{u}^j b(\xi_1, \ldots, \xi_j) \). Then

\[
f + g \rightarrow \hat{u}^i a(\xi_1, \ldots, \xi_i) + \hat{u}^j b(\xi_1, \ldots, \xi_j)
\]

and

\[
f \cdot g \rightarrow \hat{u}^{i+j} \langle a(\xi_1, \ldots, \xi_i) b(\xi_{i+1}, \ldots, \xi_{i+j}) \rangle.
\]

In particular, if \( i = j \) then \( f + g \rightarrow \hat{u}^i \langle a(\xi_1, \ldots, \xi_i) + b(\xi_1, \ldots, \xi_i) \rangle \).

To a derivative of \( f \rightarrow \hat{u}^i a(\xi_1, \ldots, \xi_i) \) we put into correspondence

\[
D_{\xi_i}(f) \rightarrow \hat{u} \xi_i a(\xi_1, \ldots, \xi_i)(\xi_1 + \cdots + \xi_i).
\]

This concludes the construction of the symbolic representation \( \hat{\mathcal{R}}_+ \) of the differential ring \( \mathcal{R}_+ \).
We also introduce a notion of a pseudo-differential formal series in the symbolic representation. We reserve a special symbol \( \eta \) for the operator \( D_\eta \) in the symbolic representation with an action rule
\[
\eta(\tilde{u}^n a(\xi_1, \ldots, \xi_n)) = \tilde{u}^n a(\xi_1, \ldots, \xi_n)(\xi_1 + \cdots + \xi_n).
\]
Let \( fD^p \) and \( gD^q \), \( p, q \in \mathbb{Z} \) be two (pseudo)-Differential operators and suppose that \( f \rightarrow \tilde{u}^i a(\xi_1, \ldots, \xi_i) \), \( g \rightarrow \tilde{u}^j b(\xi_1, \ldots, \xi_j) \). Then for the symbolic representation of these operators we have
\[
fD^p \rightarrow \tilde{u}^i a(\xi_1, \ldots, \xi_i)\eta^p, \quad gD^q \rightarrow \tilde{u}^j b(\xi_1, \ldots, \xi_j)\eta^q.
\]
For the addition and composition of pseudo-differential operators in the symbolic representation we have
\[
fD^p + gD^q \rightarrow \tilde{u}^i a(\xi_1, \ldots, \xi_i)\eta^p + \tilde{u}^j b(\xi_1, \ldots, \xi_j)\eta^q
\]
\[
fD^p \circ gD^q \rightarrow \tilde{u}^{i+j} a(\xi_1, \ldots, \xi_i, \eta + \xi_{i+1} + \cdots + \xi_{i+j})b(\xi_{i+1}, \ldots, \xi_{i+j}, \eta)\eta^{i+j}.
\]
More generally we shall consider formal series in the form
\[
A = a_0(\eta) + \tilde{u}a_1(\xi_1, \eta) + \tilde{u}^2 a_2(\xi_1, \xi_2, \eta) + \tilde{u}^3 a_3(\xi_1, \xi_2, \xi_3, \eta) + \cdots
\]
where functions \( a_k(\xi_1, \ldots, \xi_k, \eta) \) are symmetric functions with respect to arguments \( \xi_1, \ldots, \xi_k \). The addition rule of such series is obvious while for composition of two monomials we have
\[
\tilde{u}^i a(\xi_1, \ldots, \xi_i, \eta) \circ \tilde{u}^j b(\xi_1, \ldots, \xi_j, \eta) = \tilde{u}^{i+j} a(\xi_1, \ldots, \xi_i, \eta + \xi_{i+1} + \cdots + \xi_{i+j})b(\xi_{i+1}, \ldots, \xi_{i+j}, \eta)
\]
where the symmetrisation operation is taken with respect to all arguments \( \xi_1, \ldots, \xi_{i+j} \), but not \( \eta \).

We introduce a notion of \textit{locality} of a pseudo-differential operator

\begin{definition}
Function \( a(\xi_1, \ldots, \xi_i, \eta) \) is called local if all coefficients \( a_j(\xi_1, \ldots, \xi_i) \) of its expansion in \( \eta \) at \( \eta \to \infty \)
\[
a(\xi_1, \ldots, \xi_i, \eta) = \sum_{j<s} a_j(\xi_1, \ldots, \xi_i)\eta^j
\]
are symmetric polynomials in variables \( \xi_1, \ldots, \xi_i \). Formal series \( [3] \) is called local if all functions \( a_j(\xi_1, \ldots, \xi_i), j = 1, 2, \ldots \) in \( [3] \) are local.

To construct the symbolic representation of the extension of the ring \( \mathcal{R}_k \) with the operator \( \Delta = (1-D_\eta)^{-1} \)

it is enough to note that the symbolic representation of the operator \( \Delta \) is
\[
\Delta \rightarrow (1-\eta^2)^{-1}.
\]
Indeed, if \( f \in \mathcal{R}_k \) and \( f \rightarrow \tilde{u}^k a(\xi_1, \ldots, \xi_k) \), then
\[
\Delta(f) \rightarrow \tilde{u}^k \frac{a(\xi_1, \ldots, \xi_k)}{1-(\xi_1 + \cdots + \xi_k)^2}.
\]
Using if necessary the addition and multiplication operations we thus can obtain the symbolic representation of any space \( \mathcal{R}^*_k \).

In addition to the notion of locality of a pseudo-differential series we also introduce a notion of \textit{quasi-locality}

\begin{definition}
A pseudo-differential operator
\[
\tilde{u}^n a(\xi_1, \ldots, \xi_n, \eta) = \sum_{i<s} \tilde{u}^n a_i(\xi_1, \ldots, \xi_n)\eta^i
\]
is called quasi-local if for all \( i < s \) \( \tilde{u}^n a_i(\xi_1, \ldots, \xi_n) \) are symbolic representations of some elements from \( \mathcal{R}^*_k \) for some \( k \geq 0 \). A formal series \( [3] \) is called quasi-local if all its terms are quasi-local.

Finally we introduce a notion of a Frechet derivative in the symbolic representation: let \( f \in \mathcal{R}^*_k \), \( k > 0, n \geq 0 \) and its symbolic representation is given by \( f \rightarrow \tilde{u}^k a(\xi_1, \ldots, \xi_k) \). Then to the Frechet derivative \( f_* \) corresponds:
\[
f_* \rightarrow f_* = k\tilde{u}^{k-1} a(\xi_1, \ldots, \xi_{k-1}, \eta).
\]

### 2.3 Symmetries and approximate symmetries in the symbolic representation

Now we derive conditions of existence of symmetries and approximate symmetries of equation \( [2] \). We shall suppose that \( F \in \mathcal{R}_+ \) and thus we can rewrite equation \( [2] \) as
\[
u_i = \Delta(F) = \Delta(F_1) + \Delta(F_2) + \cdots + \Delta(F_k), \quad F_i \in \mathcal{R}_i, \ i = 1, 2, \ldots \quad (4)
\]
We write the symbolic representation of \( \Delta(F) \) as
\[
\Delta(F) \rightarrow F = \tilde{u}\omega(\xi_1) + \tilde{u}^2 a_1(\xi_1, \xi_2) + \cdots + \tilde{u}^k a_{k-1}(\xi_1, \ldots, \xi_k).
\]
\[
(5)
\]
By construction \(a_i(\xi_1, \ldots, \xi_{i+1})\), \(i = 1, \ldots, k-1\) are symmetric rational functions in \(\xi_1, \ldots, \xi_{i+1}\) of the form

\[a_i(\xi_1, \ldots, \xi_{i+1}) = \frac{b_i(\xi_1, \ldots, \xi_{i+1})}{1 - (\xi_1 + \cdots + \xi_{i+1})^2}\]

where symmetric polynomials \(b_i(\xi_1, \ldots, \xi_{i+1})\) are symbolic representations of differential polynomials \(F_{i+1}\), \(i = 1, 2, \ldots, k-1\). Similarly \(\omega(\xi_1) = \tilde{\omega}(\xi_1)/(1 - \xi_1^2)\) and \(\tilde{\omega}(\xi_1)\) a symbolic representation of \(F_1\). We shall suppose that \(F_1\) is such that \(\omega(\xi_1) \neq \text{const} \xi_1\).

Let \(G \in \mathcal{R}_+^n, n \geq 0\) is a symmetry of \((\mathcal{G})\). Without loss of generality we can suppose that

\[G = G_1 + G_2 + \ldots + G_m, \quad G_i \in \mathcal{R}_+^n, \quad i = 1, \ldots, m.\]

Let

\[G \to \tilde{u}\Omega(\xi_1) + \tilde{u}^2 A_1(\xi_1, \xi_2) + \cdots + \tilde{u}^k A_{m-1}(\xi_1, \ldots, \xi_m)\]

be a symbolic representation of \(G\), i.e. \(\tilde{u}^i A_{i-1}(\xi_1, \ldots, \xi_i)\), \(i = 1, \ldots, m-1\) are symbolic representations of \(G_i \in \mathcal{R}_+^n\) and thus are symmetric rational functions in \(\xi_1, \ldots, \xi_i\).

The following proposition holds:

**Proposition 1.** Function \(G \in \mathcal{R}_+^n, n \geq 0\) with the symbolic representation \((\mathfrak{G})\) is a generator of a symmetry of equation \((\mathcal{G})\) with the symbolic representation \((\mathfrak{G})\) if and only if

\[
A_1(\xi_1, \xi_2) = \frac{\Omega(\xi_1 + \xi_2) - \Omega(\xi_1) - \Omega(\xi_2)}{\omega(\xi_1 + \xi_2) - \omega(\xi_1) - \omega(\xi_2)} A_2(\xi_1, \xi_2),
\]

\[
A_m(\xi_1, \ldots, \xi_{m+1}) = \left. \frac{G^2(\xi_1, \ldots, \xi_{m+1})}{G(\xi_1, \ldots, \xi_{m+1})} \right|_{A_m(\xi_1, \ldots, \xi_{m+1})} + \frac{G^m(\xi_1, \ldots, \xi_{m+1})}{G(\xi_1, \ldots, \xi_{m+1})} \left( \sum_{j=1}^{m-1} \sum_{k=j+1}^{m+1} A_j(\xi_1, \ldots, \xi_j) \cdot A_{m-j}(\xi_{m-j+1}, \ldots, \xi_{m+1}) \right) -
\]

\[
- \sum_{j=1}^{m-1} \sum_{k=j+1}^{m+1} A_{m-j}(\xi_1, \ldots, \xi_{m-j}) \cdot A_j(\xi_{m-j+1}, \ldots, \xi_{m+1}) \]

where

\[G^m(\xi_1, \ldots, \xi_m) = \omega(\xi_m) - \sum_{n=1}^{m} \omega(\xi_n), \quad G^0(\xi_1, \ldots, \xi_m) = \omega(\xi_1) - \sum_{n=1}^{m} \omega(\xi_n)\]

and \(u^i A_{i-1}(\xi_1, \ldots, \xi_i)\) are symbolic representations of elements of \(\mathcal{R}_+^n\).

The proof follows from the compatibility conditions of equations \((\mathcal{G})\) and \(u^i G = G\) (for details see \((\mathfrak{G})\)).

Proposition 1 gives necessary and sufficient conditions of existence of an approximate symmetry of degree \(p\). Indeed, if for a given equation \((\mathcal{G})\) with the symbolic representation \((\mathfrak{G})\), \(u^i A_{i-1}(\xi_1, \ldots, \xi_i)\) are symbolic representations of elements of \(\mathcal{R}_+^n\) for all \(i = 1, 2, \ldots, p\) then \(G\) is an approximate symmetry of degree \(p\). Note that if \(G\) is a symmetry then it is completely determined by its linear part \(G_1\). From proposition 1 it follows that to characterise a hierarchy of symmetries it is sufficient to characterise a hierarchy of admissible linear terms.

However it is possible to derive the necessary conditions of existence of an infinite hierarchy of (approximate) symmetries without knowing the structure of admissible linear terms of the symmetries. To do so we introduce a notion of a formal recursion operator:

**Definition 6.** A quasi-local formal series

\[
\Lambda = \phi(\eta) + \tilde{u}_1 \phi_1(\xi_1, \eta) + \tilde{u}_2 \phi_2(\xi_1, \xi_2, \eta) + \tilde{u}_3 \phi_3(\xi_1, \xi_2, \xi_3, \eta) + \cdots
\]

is called a formal recursion operator for equation \((\mathcal{G})\) if it satisfies

\[
\Lambda \subset \hat{\mathfrak{F}} \circ \Lambda \circ \hat{\mathfrak{F}}
\]

where \(\hat{\mathfrak{F}}\) is a symbolic representation of a Frechet derivative of \(F\).

The following statement holds:

**Theorem 2.** If equation \((\mathcal{G})\) possesses an infinite hierarchy of higher symmetries then it possesses a formal recursion operator \((\Lambda)\) with \(\phi(\eta) = \eta\).

The proof of the theorem can be found in \((\mathfrak{G})\).

Equation \(\Lambda = \hat{\mathfrak{F}} \circ \Lambda \circ \hat{\mathfrak{F}}\) can be resolved in terms of functions \(\phi_i(\xi_1, \ldots, \xi, \eta)\):

**Proposition 2.** Let \(\phi(\eta)\) be an arbitrary function and formal series

\[
\Lambda = \phi(\eta) + \tilde{u}_1 \phi_1(\xi_1, \eta) + \tilde{u}_2 \phi_2(\xi_1, \xi_2, \eta) + \tilde{u}_3 \phi_3(\xi_1, \xi_2, \xi_3, \eta) + \cdots
\]
be a solution of equation (8), then its coefficients \( \phi_m(\xi_1, \ldots, \xi_m, \eta) \) can be found recursively

\[
\phi_1(\xi_1, \eta) = \frac{2(\phi(\eta + \xi_1) - \phi(\eta))}{G'(\xi_1, \eta)} a_1(\xi_1, \eta)
\]

\[
\phi_m(\xi_1, \ldots, \xi_m, \eta) = \frac{1}{G'(\xi_1, \ldots, \xi_m, \eta)} \left( (m + 1)(\phi(\eta + \xi_1 + \ldots + \xi_m) - \phi(\eta))a_m(\xi_1, \ldots, \xi_m, \eta) + \sum_{n=1}^{m-1} \phi_n(\xi_1, \ldots, \xi_n, \eta) a_m(\xi_1, \ldots, \xi_m, \eta) + \sum_{i=1}^{m-1} \xi_i a_m(\xi_{n+1}, \ldots, \xi_m, \eta) \right).
\]

The proof can be found in [6].

Theorem 3 and proposition 2 suggest the following integrability test for equation (4):

- Compute the symbolic representation of equation (9) and calculate the first few coefficients \( \phi_i(\xi_1, \ldots, \xi_i, \eta) \), \( i = 1, 2, \ldots; \)
- Check the quasi-locality conditions

In the next section we apply this test to isolate and classify integrable generalisations of the Camassa–Holm equation.

## 3 Lists of generalised Camassa–Holm type equations

In this section we present the classification results of Camassa–Holm type equations with quadratic and cubic non-linearity. We consider the following three ansätze for equation (4):

(1 - \( \epsilon^2 D_x^2 \))u_t = c_1 u_x + \epsilon \left( c_2 u_{xx} + c_3 u_x^2 \right) + \epsilon^2 \left[ c_4 u_{xxx} + c_5 u_x u_{xx} + c_6 u_x^3 \right] + \epsilon^3 \left[ c_6 u_{xxxx} + c_7 u_x u_{xxx} + c_8 u_x^2 u_{xx} \right] + \epsilon^4 \left[ c_9 u_{xxxxx} + c_{10} u_x u_{xxxx} + c_{11} u_x^2 u_{xxx} \right]. \tag{9}

(1 - \( \epsilon^2 D_x^2 \))u_t = c_1 u_x + \epsilon^3 \left[ c_3 u_x u_{xxx} + c_4 u_x^2 u_{xx} \right] + \epsilon^4 \left[ c_5 u_x u_{xxxx} + c_6 u_x u_{xxx} \right] \tag{10}

and

(1 - \( \epsilon^2 D_x^2 \))u_t = c_1 u_x^2 + \epsilon \left[ c_2 u_x^2 u_{xx} + c_3 u_x u_{x}^2 \right] + \epsilon^2 \left[ c_4 u_x^2 u_{xxx} + c_5 u_x u_{xx}^2 + c_6 u_x^3 \right] + \epsilon^3 \left[ c_6 u_x^2 u_{xxxx} + c_7 u_x u_{xxx}^2 + c_8 u_x^2 u_{xx} \right] + \epsilon^4 \left[ c_{11} u_x^2 u_{xxxxx} + c_{12} u_x u_{xxxx}^2 + c_{13} u_x u_{xxx}^2 + c_{14} u_x^2 u_{xxx} + c_{15} u_x^2 u_{xx} \right]. \tag{11}

Here \( \epsilon \) and \( c_i \) are complex parameters and \( \epsilon \neq 0 \). The right hand sides of equations (9), (10) and (11) are homogeneous differential polynomials of weights 1, 2 and 1 respectively if we assume that weight of \( u_i \) is \( i \) and weight of \( \epsilon \) equals \(-1\).

### 3.1 Equations with quadratic nonlinearity

**Theorem 3.** Suppose that at least one of the following equations is not satisfied:

\[
c_2 = 0, \quad c_6 = 0, \quad c_9 = 0, \quad c_{11} + c_{14} = 0.
\]

Then if equation (9) possesses an infinite hierarchy of quasi-local higher symmetries then up to re-scaling \( x \to \alpha x, t \to \beta t, u \to \gamma u, \alpha, \beta, \gamma = \text{const} \) it is one of the list:
\[(1 - \epsilon^2 D_x^2)u_t = 3u_{xx} - 2\epsilon^2 u_x u_{xxx} - \epsilon^2 u_{xxxx}, \quad (13)\]
\[(1 - \epsilon^2 D_x^2)u_t = D_x (4 - \epsilon^2 D_x^2) u_x^2, \quad (14)\]
\[(1 - \epsilon^2 D_x^2)u_t = D_x [(4 - \epsilon^2 D_x^2)u_x]^2, \quad (15)\]
\[(1 - \epsilon^2 D_x^2)u_t = D_x (2 + \epsilon D_x) [(2 - \epsilon D_x)u_x]^2, \quad (16)\]
\[(1 - \epsilon^2 D_x^2)u_t = D_x (2 - \epsilon D_x)(1 + \epsilon D_x)u_x^2, \quad (17)\]
\[(1 - \epsilon^2 D_x^2)u_t = D_x (2 - \epsilon D_x) [(1 + \epsilon D_x)u_x]^2, \quad (18)\]
\[(1 - \epsilon^2 D_x^2)u_t = D_x (2 + \epsilon D_x) (1 + \epsilon D_x)u_x^2, \quad (19)\]
\[(1 - \epsilon^2 D_x^2)u_t = D_x (1 + \epsilon D_x) [(2 - \epsilon D_x)u_x]^2, \quad (20)\]
\[(1 - \epsilon^2 D_x^2)u_t = (1 - \epsilon^2 D_x) \left[ \frac{1}{2}(S(u)S(u_x)) - \frac{1}{4}(S(u_x))^2 - \frac{1}{2}\epsilon S(u)S(u_x) \right], \quad (21)\]
\[(1 - \epsilon^2 D_x^2)u_t = (1 - \epsilon^2 D_x) \left[ \xi S(u) - \frac{1}{2}(S(u))\xi (S(u)) - \frac{1}{2}\epsilon S(u)S(u_x) \right], \quad (22)\]

The condition that at least one of the equations in (12) is not satisfied insures that \(\omega(\xi) \neq \text{const} \xi_l\) in the corresponding symbolic representations of the equation. To prove the theorem it is sufficient to check the quasi-locality conditions of the first three terms of the corresponding formal recursion operator:

\[\Lambda = \eta + \omega_1 \xi + \omega_2 \phi_2 \xi + \omega_3 \phi_3 \xi + \omega_4 \phi_4 \xi.\]

We do not present here the explicit formulae for these functions as they are quite cumbersome. One can easily compute them using proposition 2.

**Theorem 4.** Suppose that at least one of the following equations is not satisfied:

\[c_2 = 0, \quad c_3 = 0, \quad c_7 = 0, \quad 2c_1 + c_3 = 0.\]

Then if equation (17) possesses an infinite hierarchy of quasi-local higher symmetries then up to re-scaling \(x \to ax, t \to \beta t, u \to \gamma u, \alpha, \beta, \gamma = \text{const}\) it is one of the list:

\[(1 - \epsilon^2 D_x^2)u_t = 3u_{xx} - 2\epsilon^2 u_x u_{xxx} - \epsilon^2 u_{xxxx}, \quad (23)\]
\[(1 - \epsilon^2 D_x^2)u_t = (4 - \epsilon^2 D_x^2) u_x^2, \quad (24)\]
\[(1 - \epsilon^2 D_x^2)u_t = [(4 - \epsilon^2 D_x^2)u_x]^2, \quad (25)\]
\[(1 - \epsilon^2 D_x^2)u_t = (2 + \epsilon D_x) [(2 - \epsilon D_x)u_x]^2, \quad (26)\]
\[(1 - \epsilon^2 D_x^2)u_t = (2 - \epsilon D_x)(1 + \epsilon D_x)u_x^2, \quad (27)\]
\[(1 - \epsilon^2 D_x^2)u_t = (2 - \epsilon D_x) [(1 + \epsilon D_x)u_x]^2, \quad (28)\]
\[(1 - \epsilon^2 D_x^2)u_t = [(2 - \epsilon D_x)(1 + \epsilon D_x)u_x]^2, \quad (29)\]
\[(1 - \epsilon^2 D_x^2)u_t = (1 + \epsilon D_x) [(2 - \epsilon D_x)u_x]^2. \quad (30)\]

To prove the theorem it is again necessary to check the quasi-locality conditions of the first three terms of the corresponding formal recursion operator.

Let us consider now some properties of equations (13)-(22) and (23)-(30). Camassa-Holm equation (15). The equation (15) is the Camassa-Holm equation. It can be rewritten as

\[m_t = 2mu_{xx} + um_x, \quad m = u - \epsilon^2 u_{xx}.\]
The equation (14) is the Degasperis–Procesi equation and it can be rewritten as
\[ u_t = D_x(u - \epsilon^2 u_{xx})^{-\frac{1}{2}}. \]

The Lax representation and the bi-Hamiltonian structure can be found in [14].

Degasperis–Procesi equation (13). The equation (13) is the Degasperis–Procesi equation and it can be rewritten as
\[ m_t = 6m u_x + 2mu_x, \quad m = (1 - \epsilon^2 D_x^2)u. \]

The Degasperis–Procesi equation also possesses an infinite hierarchy of local higher symmetries and the first non-trivial such a symmetry is
\[ u_t = (4 - \epsilon^2 D_x^2)D_x(u - \epsilon^2 u_{xx})^{-\frac{3}{2}}. \]

The bi-Hamiltonian structure and the Lax representation for the Degasperis–Procesi equation can be found in [5].
Equation (15). The first non-trivial symmetry of equation (15) is
\[ u_t = D_x [(4 - \epsilon^2 D_x^2) (1 - \epsilon^2 D_x^2)u]^{-\frac{3}{2}}. \]
Equation (15) can be rewritten as
\[ m_t = D_x (m + 3u)^2, \quad m = u - \epsilon^2 u_{xx}. \]

It is easy to see that the Degasperis–Procesi equation transforms into the equation (15) under the transformation
\[ u \to (4 - \epsilon^2 D_x^2)u. \]

The Lax representation for the equation (15) is
\[ \psi_x - \psi_{xxx} - \lambda (4m - \epsilon^2 m_{xx}) \psi = 0, \]
\[ \psi_t = \frac{2}{\lambda} \psi_{xx} + 2(m + 3u) \psi_x - 2(m_x + 3u_x + \frac{2}{3\lambda}) \psi. \]

Equation (16). The first non-trivial symmetry of equation (16) is
\[ u_t = (2 + \epsilon D_x)D_x [(2 - \epsilon D_x)(u - \epsilon^2 u_{xx})]^{-\frac{3}{2}}. \]
The Degasperis–Procesi equation transforms into (16) under the change of variables
\[ u \to (2 - \epsilon D_x)u. \]

The Lax representation for the equation (16) is
\[ \psi_x - \psi_{xxx} - \lambda (2m - \epsilon m_x) \psi = 0, \quad m = u - \epsilon^2 u_{xx} \]
\[ \psi_t = \frac{2}{\lambda} \psi_{xx} + 2(2u - \epsilon u_x) \psi_x - 2(2u_x - \epsilon u_{xx} + \frac{2}{3\lambda}) \psi. \]

Note that the other transformation \( u \to (2 + \epsilon D_x)u \) of Degasperis-Procesi gives the equation \((1 - \epsilon^2 D_x^2)u_t = D_x (2 - \epsilon D_x) [(2 + \epsilon D_x)u]^{2}, \)

which transforms into (16) under the change \( x \to -x, \ t \to -t. \)
Equation (17). Equation (17) possesses a hierarchy of local higher symmetries and the first non-trivial one is
\[ u_t = D_x [(1 - \epsilon D_x)u]^{-1}. \]
The last equation is linearisable by the transformation
\[ x = -\epsilon \log(v(y, t)), \quad u = \frac{1}{\sqrt{\epsilon \log(v(y, t))}} \Rightarrow v_t = v y. \]
Equation (18). The higher symmetries of this equation are quasi-local and the first non-trivial one is
\[ (1 + \epsilon D_x)u_t = D_x [(1 - \epsilon^2 D_x^2)u]^{-1}. \]
However, the equation (18) can be rewritten as
\[ m_t = D_x (2 - \epsilon D_x) [(1 + \epsilon D_x)u]^2, \quad m = u - \epsilon^2 u_{xx} \]
and the latter equation possesses an infinite hierarchy of local higher symmetries in dynamical variable \( m. \)
One can easily check that the first such a symmetry is
\[ m_t = D_x (1 - \epsilon D_x)m^{-1}. \]
The last equation is linearisable by the transformation:
\[ x = -\epsilon \log(v(y, t)), \quad m = -\frac{1}{\sqrt{\epsilon \log(v(y, t))}} \Rightarrow v_t = v y. \]
Equations (13) and (18) are related by the transformation \( u \rightarrow (1 + \epsilon D_x)u \). It is clear that this transformation does not preserve the locality of higher symmetries of equation (17).

Equation (19). The first non-trivial higher symmetry of this equation is quasi-local

\[
(1 + \epsilon D_x)u_x = D_x \left[ (2 - \epsilon D_x)(u - \epsilon^2 u_{xx}) \right]^{-2}.
\]

However, equation (19) can be written as

\[
m_t = D_x \left[ (2 - \epsilon D_x)(1 + \epsilon D_x)u \right]^2, \quad m = u - \epsilon^2 u_{xx}
\]

and the latter equation possesses an infinite hierarchy of local higher symmetries and the first one reads as \( m_t = D_x(1 - \epsilon D_x) \left[ (2 - \epsilon D_x)m \right]^{-2} \). The Lax representation for equation (19) is not known yet.

Equation (20) is the first non-trivial higher symmetry of equation (20)

\[
u_x = D_x \left[ (2 - \epsilon D_x)(1 - \epsilon D_x)u \right]^{-2}.
\]

This equation possesses an infinite hierarchy of local higher symmetries. Note that equation (20) can be obtained from (20) by the transformation \( u \rightarrow (1 + \epsilon D_x)u \). The Lax representation for this equation is not known yet.

Equation (21) is a local second order linearisable evolutionary equation (10), while equation (22) transforms into (21) as \( u \rightarrow (1 + \epsilon D_x)u \).

Equations (23)-(30) can be obtained from equations (16)-(20) via the potentiating transformation \( u \rightarrow u_x \).

### 3.2 Equations with cubic nonlinearity

Now we consider equations with cubic non-linearity:

**Theorem 5.** Suppose that at least one of the following equations is not satisfied:

\[
c_2 = 0, \quad c_7 = 0, \quad c_{11} = 0, \quad c_1 + c_4 = 0.
\]

Then if equation (37) possesses an infinite hierarchy of quasi-local higher symmetries then up to re-scaling \( x \rightarrow ax, \ t \rightarrow \beta t, \ u \rightarrow \gamma u, \ \alpha, \beta, \gamma = \text{const} \) it is one of the list:

\[
(1 - \epsilon^2 D_x^3) u_t = \epsilon^2 u_x^2 u_{xxx} + 3\epsilon^2 u_x u_{xx} x - 4\epsilon u_x, \quad (31)
\]

\[
(1 - \epsilon^2 D_x^3) u_t = D_x \left( \epsilon^2 u_x^2 u_{xx} - \epsilon^4 u_x u_{xx} + \epsilon^2 u_{xx}^2 - u^3 \right), \quad (32)
\]

\[
(1 - \epsilon^2 D_x^3) u_t = \epsilon^4 u_x u_{xxx} + \epsilon^4 u_x^2 u_{xx} + 2\epsilon^3 u_x u_{xxx} + \epsilon^4 u_{xxx} + \epsilon^3 u_x^2 u_{xx} + \epsilon^2 u_{xxx} - \epsilon^2 u_x^3 \quad (33)
\]

\[
- \epsilon^2 u_{xxx} - 3\epsilon u_{xx}^2 - 2u^2 u_x,
\]

\[
(1 - \epsilon^2 D_x^3) u_t = (1 + \epsilon D_x) \left( \epsilon^2 u_x^2 + \epsilon u_{xx} - 2\epsilon u_x \right), \quad (34)
\]

\[
(1 - \epsilon^2 D_x^3) u_t = (1 + \epsilon D_x) \left( 2\epsilon^3 u_x^2 u_{xx} - \epsilon^4 u_x u_{xx} - \epsilon^2 u_x^2 u_{xx} - \epsilon u_{xxx}^2 + 2u^2 u_x \right), \quad (35)
\]

\[
(1 - \epsilon^2 D_x^3) u_t = (1 - \epsilon^2 D_x^3) \left( \epsilon^2 u_x^2 u_{xxx} + \epsilon^2 u_x u_{xx} - \epsilon^2 u_{xx}^2 + \frac{4}{9}\epsilon^2 u_x^3 + \epsilon u_{xx}^2 \right), \quad c \in \mathbb{C}, \quad (36)
\]

\[
(1 - \epsilon^2 D_x^3) u_t = (1 - \epsilon^2 D_x^3) \left( \epsilon^2 u_x^2 u_{xxx} + \epsilon^2 u_x u_{xx} - \frac{2}{9}\epsilon^2 u_x^3 + \epsilon u_{xx}^2 \right), \quad c \in \mathbb{C}, \quad (37)
\]

\[
(1 - \epsilon^2 D_x^3) u_t = (1 - \epsilon^2 D_x^3) \left( \epsilon^2 u_x^2 u_{xxx} + \epsilon^2 u_x u_{xx} - \frac{2}{9}\epsilon^2 u_x^3 + 3\epsilon u_{xx}^2 u_x \right. + \left. \epsilon u_{xxx}^2 + 2\epsilon^2 u_x^2 u_x \right), \quad c \in \mathbb{C}, \quad (38)
\]

\[
(1 - \epsilon^2 D_x^3) u_t = (1 - \epsilon^2 D_x^3) \left( \epsilon^2 u_x^2 u_{xxx} + \frac{1}{9}\epsilon^2 u_x^3 + 3\epsilon u_{xx}^2 u_x + \epsilon u_{xxx}^2 + 2\epsilon^2 u_x^2 u_x \right), \quad c \in \mathbb{C}, \quad (39)
\]

\[
(1 - \epsilon^2 D_x^3) u_t = (1 - \epsilon^2 D_x^3) \left( \epsilon u_x^2 u_{xx} + \epsilon u_{xx}^2 \right), \quad c \in \mathbb{C}. \quad (40)
\]

The proof requires to check the quasi-locality conditions of the first three terms of the formal recursion operator.

Equation (31). The first local higher symmetry of this equation is

\[
u_x = m \left[ \frac{11}{9} \epsilon^2 (mm_{xxx} - 5m_x m_{xx}) + \frac{49}{9} \epsilon^2 m^2 m_x^3 - 4m - \frac{11}{9} \epsilon^2 m_x \right], \quad m = u - \epsilon^2 u_{xx}.
\]
Equation (31) can be rewritten as
\[ m_t = -(u^2 m_x + 3m u u_x), \quad m = u - \epsilon^2 u_{xx}. \]

The Lax representation for the equation (31) is
\[ \epsilon^3 \psi_{xxx} = \epsilon \psi_x + \lambda m^2 \psi + 2\frac{m}{m^2} \psi_x, \quad \psi_t = \frac{\epsilon}{\lambda} \psi_{xx} - \frac{\epsilon}{\lambda} \frac{m u_x + u m_x}{m^2} \psi_x - u^2 \psi_x. \]

Equation (31) has been recently studied in detail in [7], where the Lax representation in a different form was constructed. The authors of [7], [11] also obtained the bi-Hamiltonian structure and constructed the peakon solutions for equation (31), for which the positions and amplitudes of the peaks satisfy a finite-dimensional integrable Hamiltonian system.

Equation (32). The first local higher symmetry of equation (32) is
\[ u_x = -(u^2 m_x + 3m u u_x), \quad m = u - \epsilon^2 u_{xx}. \]

This equation was recently derived from shallow water theory in [8], where the Lax representation, and bi-Hamiltonian structure were presented and different types of solutions were constructed; however, an equivalent form of this equation was given by Fokas in [13].

Equation (33). The higher symmetries of this equation are quasi-local and the first one reads as
\[ (1 + \epsilon D_x) u_x = m^{-7} (\epsilon m m_{xx} - 3 \epsilon m_x^2 - 2mm_x), \quad m = u - \epsilon^2 u_{xx}. \]

Equation (34) can be rewritten as
\[ m_t = -(u^2 m_x + 3m u u_x), \quad m = u - \epsilon^2 u_{xx}. \]

The latter equation possesses an infinite hierarchy of local higher symmetries in \( m \). The first such symmetry is
\[ m_x = (1 - \epsilon D_x) m^{-7} (\epsilon m m_{xx} - 3 \epsilon m_x^2 - 2mm_x), \quad m = u - \epsilon^2 u_{xx}. \]

Equation (34) possesses an infinite hierarchy of local higher symmetries and the first non-trivial one is
\[ u_x = v^{-7} (\epsilon v v_{xx} - 3 \epsilon v^2_x - 2v v_x), \quad v = u - \epsilon u_x. \]

Equation (35). The first local higher symmetry of this equation is
\[ u_x = v^{-2} (v + \epsilon v_x)^{-1} - v^{-3}, \quad v = u - \epsilon u_x. \]

The latter equation is linearisable as it a second order integrable evolution equation (cf. equations (17), (18)).

Equations (37)-(40) correspond to local evolutionary equations of orders 3 and 2.

4 Conclusions

In this paper we have considered polynomial homogeneous generalisations of the Camassa–Holm type equation with quadratic and cubic nonlinearity. We have classified all equations of the form (31), (32), (33), which possess infinite hierarchies of (quasi)-local higher symmetries. We have shown that the obtained equations can be treated as non-local symmetries of local scalar evolution quasi-linear integrable equations of orders 2, 3 and 5.

Some of the obtained equations seem to be new and are likely to provide more examples of solution phenomena (peakons, compactons, other weak/non-classical solutions) that do not appear in local evolution equations [13]. The study of multi-phase solutions of these equations remains out of the scope of this paper.

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