Manipulating matter-rogue waves and breathers in Bose-Einstein condensates

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We construct higher order rogue wave solutions and breather profiles for the quasi one-dimensional Gross-Pitaevskii equation with a time-dependent interatomic interaction and external trap through the similarity transformation technique. We consider three different forms of traps, namely (i) time-independent expulsive trap, (ii) time-dependent trap and (iii) time-dependent periodic trap. Our results show that when we change a parameter appearing in the time-independent or time-dependent trap the second and third-order rogue waves transform into the first-order like rogue wave. We also analyze the density profiles of breather solutions. Here also we show that the shape of the breathers changes when we tune the strength of trap parameter. Our results may help to manage rogue waves experimentally in a BEC system.

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I. INTRODUCTION

During the past few years considerable interest has been shown on exploring localized nonlinear waves in the variable coefficient nonlinear Schrödinger (NLS) equation and its generalizations [1–3]. The motivation comes from the fact that NLS equation and its variants appear in several branches/topics of physics, including nonlinear optics [4–5] and Bose-Einstein condensates (BECs) [6–8], etc. Focusing our attention on BECs alone, it is well known that the Gross-Pitaevskii (GP) equation governs the evolution of macroscopic wave function at ultralow temperatures [6, 9]. In particular, for cigar shaped BECs, it has been shown that the GP equation can be reduced to the 1D variable coefficient NLS equation [10–12]

\[ i\psi_t + \frac{1}{2}\psi_{xx} + R(t)|\psi|^2\psi + \frac{1}{2}\beta(t)x^2\psi = 0, \tag{1} \]

where \(\psi(x, t)\) is the condensate wave function, \(t\) is the dimensionless normalized time, \(x\) is the dimensionless normalized coordinate in the axial direction, \(R(t)\) represents the effective scattering length and \(\beta(t)\) is the axial trap frequency.

A simple and straightforward way of exploring the localized/periodic structures of (1) is by transforming it into a constant coefficient NLS equation through a suitable transformation. From the known solutions of the latter equation the solutions of the former equation can be identified. Using this procedure a class of solutions, in particular various soliton solutions, have been identified for the model [1] [11–13] [14]. The characteristic features associated with these nonautonomous solitons, in the context of BECs, have also been analyzed in detail, see for example Refs. [2] [11] [15] [16] and references therein. However only fewer attempts have been made to identify and analyze the so-called rogue wave (RW) solutions and breather solutions of (1) [17–22]. To the best of our knowledge neither higher-order RW solutions (with certain free parameters) nor the higher order breather solutions of (1) has been taken-up for study. We aim to undertake this task since higher order RW solutions and breather profiles of NLS equation are of contemporary interest [23–25].

RWs are waves which suddenly appear in the oceans that can reach amplitudes more than twice the value of significant wave height [26, 27]. A well known description of RW is that it appears from nowhere and disappears without a trace [28]. These waves may arise from the instability of a certain class of initial conditions that tends to grow exponentially and thus have the possibility of increasing up to very high amplitudes, due to modulation instability [29]. Recently efforts have been made to explain the RW excitation through a nonlinear process. It has been found that the NLS equation can describe many dynamical features of the RW. Certain kinds of exact solutions of NLS equations have been considered to describe possible mechanism for the formation of RWs such as Peregrine soliton [23], time periodic breather or Ma soliton (MS) and space periodic breather or Akhmediev breather (AB). As far as BEC is concerned the sudden increase of peaks in the condensate clouds is very similar to the nature of appearance of high peaks in the open ocean. Motivated by this observation, in this work, we construct the aforementioned localized and periodic solutions of (1). Besides constructing these two family of solutions we also investigate how the effective scattering length and the strength of trap parameter modify these localized structures. Indeed we find that the Feshbach resonance management [7, 32, 33] and the tunable atomic trap frequency in BECs provide us a powerful tool for manipulating the RWs.

Having stated the motivation now we proceed to construct a transformation that transforms Eq. (1) to the NLS equation. Following the standard procedure [11] [30] we find the similarity transformation should be of the form

\[ \psi(x, t) = r_0 \sqrt{R(t)} U(T, X) \]

\[ \times \exp \left[ i \left( b r_0^2 R_x - \frac{R_t}{R} x^2 - \frac{b^2 r_0^4}{2} \int R^2(t) dt \right) \right], \tag{2} \]
where
\[ X(x,t) = r_0 R(t) x - b r_0^3 \int R^2(t) dt, \quad T(t) = r_0^2 \int R^2(t) dt, \]
and \( U(X,T) \) is the solution of standard NLS equation, that is
\[ iU_t + \frac{1}{2} U_{xx} + |U|^2 U = 0. \]

In the above, \( b, r_0 \) are arbitrary constants and the modulational functions \( R(t) \) and \( \beta(t)^2 \) should satisfy the following condition
\[ \frac{d}{dt} \left( \frac{R_t}{R} \right) - \left( \frac{R_t}{R} \right)^2 + \beta(t)^2 = 0, \]
which is a Riccati type equation with dependent variable \( (R_t/R) \) and independent variable \( t \). Regardless of what \( R(t) \) is, as long as the condition \( (5) \) is satisfied, the GP equation is integrable \[11, 30]. \ We also note here that the Painlevé singularity structure analysis performed on Eq. (1) confirms the same restriction \( (5) \) on the system coefficients \[31]. \ We note that the solution (2) provides an equilibrium solution into a multi-peaked solution for (1) and investigate their characteristics when we alter the trap parameter. Finally, in Sec. V, we present a summary of the results and conclusions.

II. CHARACTERISTICS OF ROGUE WAVES

To begin with, we consider the case in which the trap frequency is a constant, that is \( \beta(t)^2 = \text{constant} = \beta_0^2 \). A time-independent trap frequency implies that the frequency does not change with time and space. We then consider the trap frequency to be time-dependent and investigate the associated rogue wave solutions.

A. Time-independent trap

Substituting \( \beta(t)^2 = \beta_0^2 \) in the integrability condition \[11\], we find that the time-dependent interaction term should be of the form \( R(t) = \text{sech}(\beta_0 t + \delta) \), where \( \delta \) is an integration constant. Plugging this expression in (2), we find
\[ \psi_j(x,t) = r_0 \sqrt{\text{sech}(\beta_0 t + \delta)} U_j(X,T) \eta(x,t), \]
where
\[ \eta(x,t) = \exp \left\{ i \left[ c_1 r_0^2 \text{sech}(\beta_0 t + \delta) x + \frac{(\beta_0^2 x^2 - c_1^2 r_0^4) \tanh(\beta_0 t + \delta)}{2 \beta_0} \right] \right\}, \]
and \( U_j(X,T) \)'s, \( j = 1, 2, 3 \), are first, second and third-order RW solutions of NLS Eq. (4) whose explicit expressions are given in the Appendix (vide Eqs. (A2), (A4) and (A5)).

In Fig. 1, the first, second, and third columns represent the density profiles of first, second, and third-order RW solutions obtained from (6). In this figure we present the formation of RW in cigar-shaped BECs. By tuning the parameter \( r_0 \) from 0.5 to 1.2 in (6), we can visualize the formation and manipulation of RWs. For example, when we increase the amplitude parameter \( r_0 \) smoothly, we can observe the formation of crests and troughs as well as increasing amplitude of an isolated wave. At \( r_0 = 1.2 \) one can visualize a large amplitude wave which is localized both in space and time which in turn confirms the formation of RWs in BECs. The formation of second and third-order RWs is also demonstrated in Figs. (1h) and (1l) for the same parameter value \( r_0 \). From this figure we infer that the time-dependent nonlinear interaction between atoms induces density fluctuations over the condensates which get more and more localized both in space and in time as we increase the order of the RW.
Next we demonstrate how these localized structures vary with respect to the trap parameter $\beta_0$. Fig. 2 displays the first-order RW for the same nonlinearity management parameter $R(t)$ and the external trap frequency $\beta(t)^2 = \beta_0^2$. The nature of first-order RW for $\beta_0 = 0.1$ is depicted in Fig. 2(a). When we increase the strength of trap parameter the first-order RW gets more and more localized which is shown in Figs. 2(b) and 2(c) respectively. Fig. 3 displays the second-order RW for the same nonlinearity management parameter as a function of $\beta_0$. In Fig. 3(a) we display the second-order RW for $\beta_0 = 0.1$.

When the strength of the parameter $\beta_0$ is increased to 1.2, the wave subcrests start to get stretched as shown in Fig. 3(b). The wave subcrests get suppressed more and more when we increase the value of $\beta_0$ and finally the second-order RW attains a new structure as given in Fig. 3(c) for $\beta_0 = 5.0$. The resultant structure looks almost like a first-order RW, see Fig. 2(c). We also observe similar effects in the third-order RW case as well when we increase the value of $\beta_0$. Fig. 4 demonstrates the changes in third-order RW when we vary the interaction strength. At $\beta_0 = 1.5$, the third-order RW transforms into the second-order RW like structure as shown in Fig. 4(b). When we increase the value of the parameter $\beta_0$ to 5.0, one obtains a first-order like RW which is displayed in Fig. 4(c). The aforementioned results reveal that when we increase the strength of trap parameter $\beta_0$, the second and third-order RWs get more localized in time and delocalized in space, approaching the structure of a first-order RW. Thus one can control the RWs by decreasing the strength of trap frequencies.

**B. Time-dependent trap**

Next we consider the time-dependent trap frequency in the form $\beta^2(t) = \left(\beta_0^2/2\right)[1 - \tanh(\beta_0 t/2)]$. For this choice, the relation fixes the interatomic interaction term to be of the form $R(t) = 1 + \tanh(\beta_0 t/2)$. The first, second, and third-order RW solutions for this trap frequency and the strength of interatomic interaction are
where

$$\psi_3(x, t) = r_0 \sqrt{1 + \tanh\left(\frac{\beta_0 t}{2}\right)} U_j(X, T) \eta(x, t), \quad j = 1, 2, 3,$$

\( (8) \)

The qualitative nature of the first, second, and third-order RWs and their corresponding contour plots for \( R(t) = 1 + \tanh(\beta_0 t/2) \) and \( \beta(t)^2 = \beta_0^2/2 [1 - \tanh(\beta_0 t/2)] \) turn out to be qualitatively the same as in the previous case (Fig. 1) and so we do not display the outcome here separately.

In Fig. 5, we depict the first-order RW for these choices of \( R(t) \) and \( \beta(t) \). When \( \beta_0 = 0.1 \) the first-order RW is as shown in Fig. 5(a). By altering the value of trap parameter \( \beta_0 \) to 1.0, the structure of first-order RW is as shown in Fig. 5(b). From the corresponding contour plot (Fig. 5(c)) we see that the upper side of RW gets diverged, while the lower side of RW gets suppressed in the plane wave background. When \( \beta_0 = 5.0 \) the modified structure of first-order RW is given in Fig. 5(e), where this feature is even more prominent. The density profiles of the corresponding second-order RW is presented in Fig. 6. When we tune the parameter \( \beta_0 \) from 0.1 upwards the wave subcrests start to get stretched. From the contour plots we can observe that the stretches occur on one side of the RW only. When we increase the value of \( \beta_0 \) further the second-order RW gets modified to a first-order RW like structure which is demonstrated in Fig. 6(c). A similar transition has also been observed in the third-order RW case as well which is illustrated in Fig. 7. The third-order RW acquires a new structure as shown in Fig. 7(b) when we increase the value of the parameter \( \beta_0 \) from 0.1 to 1.0. At \( \beta_0 = 5.0 \), we observe that the third-order RW acquires a further modified structure which is displayed in Fig. 7(c). Note the similarity in the central part with that of the first-order RW as given in Fig. 2.
In the third choice we consider the time-dependent periodic trap frequency to be of the form $\beta(t)^2 = 4\beta_0^2 [1 + 2 \tan^2 (2\beta_0 t)]$ so that the strength of time-dependent periodic interatomic interaction turns out to be $R(t) = \cos(2\beta_0 t)$. Substituting these two expressions in (2), we find

$$\psi_j(x,t) = r_0 \sqrt{\cos(2\beta_0 t)} U_j(X,T) \eta(x,t), \quad j = 1, 2, 3,$$

where

$$\eta(x,t) = \exp \left\{ i \left[ \beta_0 \tan(2\beta_0 t)x^2 + c_1 r_0^2 \cos(2\beta_0 t)x ight. ight.$$

$$- c_2^2 r_0^2 (4\beta_0 t + \sin(r\beta_0 t)) \left. \right] \right\}. \quad (10)$$

Here $U_1(X,T)$, $U_2(X,T)$, and $U_3(X,T)$ are again the first, second and third-order RW solutions of NLS equation (vide Eqs. (A2), (A4), and (A5)).

In Fig. 8 we present the density profiles of first, second, and third-order RWs (first row) and their corresponding contour plots (second row) for the strength of time-dependent interatomic interaction $R(t) = \cos(2\beta_0 t)$ and time-dependent periodic trap frequency $\beta(t)^2 = 4\beta_0^2 [1 + 2 \tan^2 (2\beta_0 t)]$. Fixing the value of $\beta_0$ at 2.5, Fig. 8 shows the appearance of RWs in the periodic wave background for the above forms of $R(t)$ and $\beta^2(t)$. Further we have also checked that the behaviour of the RWs as a function of $\beta_0$ follows the same qualitative picture discussed above for the other forms of $R(t)$ and $\beta^2(t)$.

### III. Characteristics of Triplet RWs

Very recently it has been shown that one can also introduce certain free parameters in the higher order RW solutions which allow one to split the symmetric form solution into a multi-peaked solution and that by varying these free parameters one can extract certain novel patterns of RWs [21]. Motivated by this, in the following, we consider the second- and third-order RW solutions with suitable free parameters and analyze how the RW patterns change with respect to these free parameters when we vary the strength of nonlinearity and trap parameter. To begin with, we confine our attention to the second-order RW solution. In this case, we have the following modified expressions for $G_2$, $H_2$ and $D_2$ in Eq. (A3), that is

$$G_2 = 12(3 - 16X^4 - 24X^2(4T^2 + 1) - 48lX - 80T^4 - 72T^2 - 48mT),$$

$$H_2 = 24 \left\{ T [15 - 16X^4 + 24X^2 - 48lX - 8(1 - 4X^2)T^2 - 16T^4] + 6m(1 - 4T^2 + 4X^2) \right\},$$

$$D_2 = 64X^6 + 48X^4(4T^2 + 1) + 12X^2(3 - 4T^2)^2 + 64T^6 + 432T^4 + 396T^2 + 9 + 48m \left\{ 18m + T(9 + 4T^2 - 12X^2) \right\} + 48l \left\{ 18l + X(3 + 12T^2 - 4X^2) \right\}. \quad (11)$$

Note that this RW solution now contains two free parameters, namely $l$ and $m$. Substituting the above expressions in (9), (8) and (7), for $j = 2$, we obtain the second-order RW solution to the GP Eq. (1). When $l = m = 0$, this solution coincides with the one given earlier (vide Eq. (A4)) which contains one largest crest and four subcrests with two deepest troughs. When $l$ and $m$ are not equal to 0, the second-order RW splits into three first-order RWs. These waves emerge in a triangular fashion (a triplet pattern). The parameters $l$
and \( m \) describe the relative positions of the first-order RWs in the triplet. The three first-order RWs form a triangular pattern with 120 degrees of angular separation between them. In Figs. 9(a)-(c) we display the triplet pattern for \( R(t) = \text{sech}(\beta l t + \delta) \) and \( \beta l t = \beta_0^2 \) when \( l = 15 \) and \( m = 25 \). The triplet RW pattern for \( \beta_0 = 0.1 \), is shown in Fig. 9(a). When we increase the parameter \( \beta_0 \) to 0.3 we observe that the triplet pattern gets widened and starts to collapse as shown in Fig. 9(b). It gets fully collapsed when we increase the \( \beta_0 \) value further which is demonstrated in Fig. 9(c). Figs. 9(d)-(g) represent the triplet pattern for \( R(t) = 1 + \tanh(\beta_0 t/2) \) and \( \beta(2t)^2 = \beta_0^2/2 [1 - \tanh(\beta_0 t/2)] \). The formation of triplet RWs is shown in Fig. 9(d) when \( \beta_0 = 0.1 \). When we increase the value \( \beta_0 \) to 0.3 one of the single RWs in the triplet pattern vanishes which is illustrated in Fig. 9(e). By increasing the parameter \( \beta_0 \) further we observe that two first-order RWs get widened and finally disappears as shown in Fig. 9(f). Figs. 9(g)-(i) represent the triplet pattern for \( R(t) = \cos(2\beta_0 t) \) and \( \beta(2t)^2 = 4\beta_0^2/1 + 2\tan^2(2\beta_0 t) \). The form of the triplet pattern for \( \beta_0 = 0.1 \) is displayed in Fig. 9(g). Here also when we increase the value of \( \beta_0 \) we observe the collapse of the triplet pattern in the periodic wave background (vide Fig. 9(i)).

![Fig. 9](image-url)

FIG. 9. (Color online) Triplet RWs. (a)-(c) \( R(t) = \text{sech}(\beta l t + \delta) \) and \( \beta(2t)^2 = \beta_0^2 \), (d)-(f) \( R(t) = 1 + \tanh(\beta_0 t/2) \) and \( \beta(2t)^2 = \beta_0^2/2 [1 - \tanh(\beta_0 t/2)] \), and (g)-(i) \( R(t) = \cos(2\beta_0 t) \) and \( \beta(2t)^2 = 4\beta_0^2/1 + 2\tan^2(2\beta_0 t) \). (a), (d) and (g) \( \beta_0 = 0.1 \), (b), (e) and (h) \( \beta_0 = 0.3 \) and (c), (f) and (i) \( \beta_0 = 1.0 \). The other parameters are \( l = 15 \), \( m = 25 \), \( r_0 = 1.0 \), \( c_1 = 0.01 \) and \( \delta = 0.01 \).

We then move on to investigate the structure of third-order RW solution with four free parameters, namely \( l, m, g \) and \( e \). The third-order RW solution with four free parameters is much lengthier than the one without free parameters and so we do not give the explicit expression here and analyze the results only graphically. Here also we analyze the solution with respect to the free parameters. When \( l = m = g = e = 0 \), we have the classical third-order RW solution which is shown in Fig. 2(c). It has one largest crest and six subcrests with two deepest troughs. For non-zero values of \( l, m, g \) and \( e \), the third-order RW splits into six separated first-order RWs. When we increase the value of free parameters, the six first-order RWs take new positions. The sextet pattern is displayed in Figs. 10(a)-(c) for \( R(t) = \text{sech}(\beta_0 t + \delta) \) and \( \beta(2t)^2 = \beta_0^2 \) when \( l = 10, m = 20, g = 500 \) and \( h = 500 \). For \( \beta_0 = 0.1 \) the set of six first-order RWs is shown in Fig. 10(a). When we increase the value of \( \beta_0 \) to 0.25, three peaks disappear and the remaining peaks start to bend as shown in Fig. 10(b). When we increase the value of \( \beta_0 \) further the RWs bend in the plane wave background as given in Fig. 10(c). Figs. 10(d)-(f) represent six first-order RWs for time-dependent nonlinearity strength \( R(t) = 1 + \tanh(\beta_0 t/2) \) and the time-dependent external trap frequency \( \beta(2t)^2 = \beta_0^2/2 [1 - \tanh(\beta_0 t/2)] \). The formation of six first-order RWs at \( \beta_0 = 0.1 \) is shown in Fig. 10(d). When we increase the strength of the parameter \( \beta_0 \) to 0.25, one of six first-order RWs vanishes as seen in Fig. 10(e). If we increase the value \( \beta_0 \) further three out of six peaks bend in the plane wave background and eventually collapse which is shown in Fig. 10(f). Figs. 10(g)-(i) represent the sextet pattern for \( R(t) = \cos(2\beta_0 t) \) and \( \beta(2t)^2 = 4\beta_0^2/1 + 2\tan^2(2\beta_0 t) \). The form of the six first-order RWs for \( \beta_0 = 0.05 \) is displayed in Fig. 10(g) and for further increase in \( \beta_0 \), the modified structure in the periodic wave background are as shown in Figs. 10(h) and 10(i).

IV. CHARACTERISTICS OF BREATHERS

In the previous two sections we have analyzed how the RW profiles get modified by the variations of the distributed coefficients present in the variable coefficients NLS Eq. (1). In this section we analyze how the breather structures get modified in the condensates when we vary the strength of external trap parameter.

To begin with we consider the first-order breather solution of NLS Equation, which is given by [25]

\[
U_1(X,T) = \left\{ \frac{m^2 \cos^2[\alpha(T - T_1)] + 2i m v \sinh[\alpha(T - T_1)]}{2 \cos[\alpha(T - T_1)] - 2v \cos[m(X - X_1)]} - 1 \right\} \times \exp(it),
\]

(12)

where the parameters \( m \) and \( v \) are expressed in terms of a complex eigenvalue (say \( \lambda \)), that is \( m = 2\sqrt{1 + \lambda^2} \) and \( v = \text{Im}(\lambda) \), and \( X_1 \) and \( T_1 \) serve as coordinate shifts from the origin. The parameter \( \alpha = m v \) in (12) is the growth rate of modulation instability. Substituting this breather solution of NLS equation into (6), (8) and (9), we study the underlying dynamics of (1).
The other parameters are $\beta_0$, the breather structure gets changed and a new structure against a breather background is obtained. To visualize this we fix the value of $\beta_0$ to be 0.8. For this value a stretching occurs in space in the case of AB (Fig. 12(a)) whereas in the case of Ma breather (Fig. 12(b)) the stretching occurs in time. Figs. 12(c)-(d) illustrate AB and Ma breather profiles for the time-dependent nonlinearity coefficient $R(t) = 1 + \tanh (\beta_0 t/2)$ and time-dependent trap frequency $\beta(t)^2 = (\beta_0^2/2) [1 - \tanh (\beta_0 t/2)]$. Here also we tune the strength of trap parameter $\beta_0$ to 0.8 and observe that stretching occurs over space in AB which is depicted in Fig. 12(c) and the Ma breather gets more localized and when $t \leq 0$ the breather profile completely disappears as shown in Fig. 12(d). Our results reveal the fact that when we tune the parameter $\beta_0$ in the obtained breather solution, the breather gets a modified structure corresponding to a distortion of the breather profile. Figs. 12(e)-(f) represent the AB and Ma breathers.

For illustration, we consider the case $R(t) = \operatorname{sech} (\beta_0 t + \delta)$ and $\beta(t)^2 = \beta_0^2$, and plot the outcome in Fig. 11. The first row in Fig. 11 represents an Akhme-

**FIG. 10.** (Color online) Sextet RWs. (a)-(c) $R(t) = \operatorname{sech} (\beta_0 t + \delta)$ and $\beta(t)^2 = \beta_0^2$, (d)-(f) $R(t) = 1 + \tanh (\beta_0 t/2)$ and $\beta(t)^2 = \beta_0^2 [1 - \tanh (\beta_0 t/2)]$. (a) and (g) $\beta_0 = 0.1$, (b), (e) and (h) $\beta_0 = 0.25$ and (c), (f) and (i) $\beta_0 = 1.0$. The other parameters are $l = 10$, $m = 20$, $g = 500$, $h = 500$, $r_0 = 1.0$, $c_1 = 0.01$ and $\delta = 0.01$.

**FIG. 11.** (Color online) (a) Akhmediev breather for $\lambda = 0.6i$, (b) Ma breather for $\lambda = 1.4i$ when $R(t) = \operatorname{sech} (\beta_0 t + \delta)$ and $\beta(t)^2 = \beta_0^2$, (c) and (d) are their corresponding contour plots. The other parameters are $r_0 = 1.0$, $\delta_0 = 0.01$, $c_1 = 0.01$, and $\delta = 0.01$.

**FIG. 12.** (Color online) Stretching of breathers. (a), (c) and (e) AB with an eigenvalue $\lambda = 0.6i$, (b), (d) and (f) Ma breather with $\lambda = 1.4i$. (a)-(d) $\beta_0 = 0.8$, (e)-(f) $\beta_0 = 1.5$. The other parameters are $r_0 = 1.0$, $c_1 = 0.01$ and $\delta = 0.01$. 
in the periodic background for $R(t) = \cos(2\beta_0 t)$ and
\[ \beta(t)^2 = 4\beta_0^2 [1 + 2\tan^2(2\beta_0 t)] \]
when $\beta_0 = 1.5$.

Next, we proceed to construct two-breather solutions of (1) and analyze how these solutions get distorted by the variations of modulation parameters. The two-breather solution of NLS equation is given by

\[
\phi_2(X, T) = \left[ (-1)^j + \frac{\tilde{G}_2(X, T) + i\tilde{H}_2(X, T)}{\tilde{D}_2(X, T)} \right] \exp(iT),
\]

(13)

where $\tilde{G}_2$, $\tilde{H}_2$, and $\tilde{D}_2$ are given by

\[
\tilde{G}_2 = -(k_1^2 - k_2^2) \left[ \frac{k_2^2 \delta_2}{k_2} \cosh(\delta_1 T_{s1}) \cos(k_2 X_{s2}) \right. \\
- \frac{k_2^2 \delta_1}{k_1} \cosh(\delta_2 T_{s2}) \cos(k_1 X_{s1}) \\
- (k_1^2 - k_2^2) \cosh(\delta_1 T_{s1}) \cosh(\delta_2 T_{s2}) \right],
\]

(14a)

\[
\tilde{H}_2 = -2(k_1^2 - k_2^2) \left[ \frac{\delta_1 \delta_2}{k_1} \sinh(\delta_1 T_{s1}) \cos(k_2 X_{s2}) \\
- \frac{\delta_1 \delta_2}{k_2} \sinh(\delta_2 T_{s2}) \cos(k_1 X_{s1}) \\
- \delta_1 \sinh(\delta_1 T_{s1}) \cosh(\delta_2 T_{s2}) \\
+ \delta_2 \sinh(\delta_2 T_{s2}) \cosh(\delta_1 T_{s1}) \right],
\]

(14b)

\[
\tilde{D}_2 = 2(k_1^2 + k_2^2) \frac{\delta_1 \delta_2}{k_1 k_2} \cosh(k_1 X_{s1}) \cos(k_2 X_{s2}) \\
+ 4\delta_1 \delta_2 \sin(k_1 X_{s1}) \sin(k_2 X_{s2}) \\
+ \sinh(\delta_1 T_{s1}) \sinh(\delta_2 T_{s2}) \\
- (2k_1^2 - k_2^2) \frac{\delta_1}{k_1} \cosh(k_1 X_{s1}) \cos(\delta_2 T_{s2}) \\
- 2(k_1^2 - k_2^2) \frac{\delta_2}{k_2} \cos(k_1 X_{s1}) \cosh(\delta_2 T_{s2}) \\
- \frac{\delta_2}{k_2} \cosh(k_2 X_{s2}) \cosh(\delta_1 T_{s1}) \right],
\]

(14c)

where the modulation frequencies, $k_j = 2\sqrt{1 + \lambda_j^2}$, $j = 1, 2$, are described by the (imaginary) eigenvalues $\lambda_j$. In the above expressions, $X_j$, $T_j$, $j = 1, 2$, represent the shifted point of origin, $\delta_j = k_j \sqrt{4 - k_j^2}/2$ is the instability growth rate of each component and $X_{s1} = X - X_j$ and $T_{s1} = T - T_j$ are shifted variables.

With two purely imaginary eigenvalues, $\lambda_j$, $j = 1, 2$, the solution (13) is capable of describing a variety of possible second-order breather structures. The solution includes ABs, Ma solitons and the intersection of AB and Ma breathers for certain combination of eigenvalues. For example, when the imaginary parts of both the eigenvalues $\Im(\lambda_j)$, $j = 1, 2$, lie between 0 and 1, we obtain the ABs. On the other hand when both of them are greater than one ($\Im(\lambda_1) > 1$) we obtain the Ma breathers and the mixed possibility, that is one of the eigenvalues is less than one ($\Im(\lambda_1) < 1$) and the other eigenvalue ($\Im(\lambda_2) > 1$) is greater than one, we obtain the intersection of AB and Ma breathers.

Fig. 13 displays the evolution of two-breather solution of (1) for $R(t) = \sech(\beta_0 t + \delta)$ and $\beta(t)^2 = \beta_0^2 = (0.01)^2$ with imaginary eigenvalues. To obtain the ABs from (13) we consider both the eigenvalues...
$\lambda_1$ and $\lambda_2$ to be less than $i$ ($\lambda_1 = 0.55i$ and $\lambda_2 = 0.75i$). One AB developing with a time delay after another is shown in Fig. 13(a), while in Fig. 13(b) we present the case where there is no such time delay. When we take both the eigenvalues $\lambda_1$ and $\lambda_2$ to be $1.3i$ and $1.4i$, we obtain two Ma breather solutions. Similarly the evolution of the two Ma breathers with and without spatial delay is shown in Figs. 13(d) and 13(e), respectively. We also observe that the distance between the Ma breathers increases when we set both the eigenvalues to be nearly equal, say for example $\lambda_1 = 1.3i$ and $\lambda_2 = 1.31i$ which is not displayed here. When we take the eigenvalues as $\lambda_1 = 0.5i$ and $\lambda_2 = 1.3i$, the AB intersects with Ma breather which is displayed in Fig. 13(f). When we tune the strength of the trap parameter $\beta_0$ to 0.15, both the ABs get widened in the plane wave background which is demonstrated in Fig. 14(a). When $\beta_0 = 0.5$, in the intersection of AB and Ma breathers we observe that AB gets a bending structure while Ma breather fully disappears in the plane wave background which is shown in Fig. 14(b).

In Fig. 14(c) we note that both the Ma breathers develop a bending structure in the plane wave background when $\beta_0 = 0.2$. Figs. 14(d)-(f) display the evolution of two-breather solution of (1) for $R(t) = 1 + \tanh (\beta_0 t/2)$ and $\beta(t)^2 = \beta_0^2 / 2 [1 - \tanh (\beta_0 t/2)]$ with the imaginary eigenvalues. When $\beta_0 = 0.15$ one of the ABs get stretched which is not shown here and on further increase of the value of $\beta_0$ to 1.5, one of the ABs gets annihilated and the other AB bends in the plane wave background which is demonstrated in Fig. 14(d). When $\beta_0 = 0.5$ the intersection of AB-Ma breathers is as shown in Fig. 14(e). In Fig. 14(f) we observe that both the Ma breathers get a bending structure in the plane wave background. Figs. 14(g)-(i) show the evolution of the two-breather solution of (1) for $R(t) = \cos (2\beta_0 t)$ and $\beta(t)^2 = 4\beta_0^2 [1 + 2 \tan^2 (2\beta_0 t)]$ with the imaginary eigenvalues. Here also when we increase the value of $\beta_0$ we observe the collapse of the two-breather solution in the periodic wave background.

V. CONCLUSION

In this work, we have constructed higher order RW solutions with and without free parameters for the quasi one-dimensional GP equation with time-dependent interatomic interaction and external trap through the similarity transformation technique. By mapping the variable coefficients NLS equation onto constant coefficient NLS equation we have derived these solutions. We have shown that the mapping can be done when the external trap and the nonlinearly interatomic interaction of atoms satisfy a constraint. From the known higher order RW and breather solutions of the constant coefficient NLS equation, we have derived the solutions of (1). In our analysis, we have considered the harmonic trap frequencies in three different forms, namely (i) time-independent explosive trap, (ii) time-dependent (non-periodic) trap, and (iii) time-dependent periodic trap and correspondingly fixed the effective scattering length. We then studied the characteristics of the constructed rogue waves in detail. We have observed that the second- and third-order rogue waves transform to first-order rogue wave-like structures when a parameter appearing in a harmonic trap (time-independent and time-dependent traps) is varied. We have then analyzed the characteristics of triplet and sextet pattern of matter RWs for (1). We have also constructed one-breather and two breather solutions of (1). We have investigated how these periodic localized waves change in the plane wave background when we tune the trap parameter in the obtained breather solutions. Our results may provide possibilities to manipulate rogue waves experimentally in a BEC system.

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Appendix A

In the absence of trap and the nonlinearity strength $R(t)$ is equal to one, (1) reduces to the standard NLS equation (4). Several localized and periodic structures of standard NLS are documented in the literature [24, 25, 35]. Eq. (4) admits $N$th order RW solution. We present the RW solution of NLS equation in the following form [24],

$$U_j(X, T) = \left[ (-1)^j + \frac{G_j(X, T) + iT H_j(X, T)}{D_j(X, T)} \right] \exp (iT),$$

where $j = 1, 2, \ldots, G_j, H_j$ and $D_j$ are polynomials in the variables $T$ and $X$.

For the first-order ($j = 1$) RW solution $G_1 = 4$, $H_1 = 8$ and $D_1 = 1 + 4X^2 + 4T^2$. From (A1) we get $U_1 = \left( \frac{4}{\exp (1 + 4X^2 + 4T^2)} - 1 \right) \exp (iT)$. For convenience we multiply this expression by $-1 = \exp (i\pi)$ and consider the solution in the form

$$U_1(X, T) = \left( 1 - \frac{4}{\exp (1 + 4X^2 + 4T^2)} \right) \exp (iT).$$

(A2)
We use only this form of expression in our analysis. For the second-order $(j = 2)$ RW solution, the function $G_2$, $H_2$, and $D_2$ turn out to be \cite{24}

\begin{align*}
G_2 &= \frac{3}{8} - 3X^2 - 2X^4 - 9T^2 - 10T^4 - 12X^2T^2, \\
H_2 &= \frac{15}{4} + 6X^2 - 4X^4 - 2T^2 - 4T^4 - 8X^2T^2, \\
D_2 &= \frac{1}{8} \left( \frac{3}{4} + 9X^2 + 4X^4 + \frac{16}{3}X^6 + 33T^2 + 36T^4 \\
&+ \frac{16}{3}T^6 - 24X^2T^2 + 16X^4T^2 + 16X^2T^4 \right).
\end{align*}

Then we get

\[ U_2(X, T) = \left[ 1 + \frac{G_2 + iTH_2}{D_2} \right] \exp (iT). \tag{A4} \]

For the third-order $(j = 3)$ RW solution, we have

\[ U_3(X, T) = \left[ -1 + \frac{G_3 + iTH_3}{D_3} \right] \exp (iT), \tag{A5} \]

where

\[ G_3(X, T) = g_0 + (2T)^2g_2 + (2T)^4g_4 + (2T)^6g_6 \\
+ (2T)^8g_8 + (2T)^{10}g_{10}, \tag{A6} \]

with

\begin{align*}
g_0 &= 1 - (2X)^2 - \frac{2}{3}(2X)^4 + \frac{14}{45}(2X)^6 + \frac{2X}{45} + \frac{(2X)^{10}}{675}, \\
g_2 &= -3 - 20(2X)^2 + \frac{2}{3}(2X)^4 - \frac{4}{45}(2X)^6 + \frac{(2X)^8}{45}, \\
g_4 &= 2 \left[ \left( \frac{17}{3} \right) + 5(2X)^2 - \frac{(2X)^4}{3^2} + \frac{(2X)^6}{3^3} \right], \\
g_6 &= \frac{2}{45} \left[ 73 + 14(2X)^2 + \frac{7}{3}(2X)^4 \right], \\
g_8 &= \frac{1}{15} \left( 11 + (2X)^2 \right), \quad g_{10} = \frac{11}{675}, \tag{A7} \\
\end{align*}

and

\[ H_3(X, T) = h_0 + (2T)^2h_2 + (2T)^4h_4 + (2T)^6h_6 \\
+ (2T)^8h_8 + (2T)^{10}h_{10}. \tag{A8} \]

with

\begin{align*}
h_0 &= 2 \left[ 7 + 7(2X)^2 - 2(2X)^4 - \frac{2}{3^2}(2X)^6 - \frac{(2X)^8}{45} \right. \\
&+ \left. \frac{(2X)^{10}}{675} \right], \\
h_2 &= \frac{2}{3} \left[ -11 - 28(2X)^2 - 2(2X)^4 - \frac{28}{45}(2X)^6 + \frac{(2X)^8}{45} \right], \\
h_4 &= \frac{4}{15} \left[ -107 + 19(2X)^2 - \frac{7}{3}(2X)^4 + \frac{(2X)^6}{3^2} \right], \\
h_6 &= \frac{4}{45} \left[ -29(2X)^2 + \frac{(2X)^4}{3} \right], \\
h_8 &= \frac{2}{3^3} \left[ 1 + \frac{(2X)^2}{5} \right], \quad h_{10} = \frac{2}{675}. \tag{A9} \\
\end{align*}

The denominator is represented by

\begin{align*}
D_3(X, T) &= d_0 + (2T)^2d_2 + (2T)^4d_4 + (2T)^6d_6 + (2T)^8d_8 \\
&+ (2T)^{10}d_{10} + (2T)^{12}d_{12}, \tag{A10} \\
\end{align*}

where

\begin{align*}
d_0 &= \frac{1}{2^3} \left[ 1 + 6(2X)^2 + \frac{5}{3}(2X)^4 + \frac{52}{45}(2X)^6 + \frac{2X}{15} \right. \\
&+ \left. \frac{2}{675}(2X)^{10} + \frac{(2X)^{12}}{2025} \right], \\
d_2 &= 23 - 9(2X)^2 + \frac{10}{3}(2X)^4 + \frac{2}{15}(2X)^6 - \frac{(2X)^8}{45} \\
&+ \frac{(2X)^{10}}{675}, \\
d_4 &= 2 \left[ 71 + \frac{116}{3}(2X)^2 - \frac{2}{3}(2X)^4 - \frac{4}{45}(2X)^6 + \frac{(2X)^8}{135} \right], \\
d_6 &= \frac{32}{3} \left[ \frac{17}{3} + 5(2X)^2 + \frac{(2X)^4}{45} + \frac{(2X)^6}{135} \right], \\
d_8 &= \frac{32}{15} \left[ \frac{83}{3} + 2(2X)^2 + \frac{(2X)^4}{3^2} \right], \\
d_{10} &= \frac{28}{225} \left[ 7 + \frac{(2X)^2}{3} \right], \quad d_{12} = \frac{2^9}{2025}. \tag{A11} \\
\end{align*}
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