ON THE SCHWINGER-DYSON EQUATIONS
FOR A VERTEX MODEL COUPLED TO
2D GRAVITY

Al. Kavalov

FB Physik, Universität Kaiserslautern, 67663 Kaiserslautern, Germany

Abstract

We consider a two matrix model with gaussian interaction involving the term \( \text{tr}ABA \), which is quartic in angular variables. It describes a vertex model (in particular case - of F-model type) on the lattice of fluctuating geometry and is the simplest representative of the class of matrix models describing coupling to two-dimensional gravity of general vertex models. This class includes most of physically interesting matrix models, such as lattice gauge theories and matrix models describing extrinsic curvature strings. We show that the system of loop (Schwinger-Dyson) equations of the model decouples in the planar limit and allows one to find closed equations for arbitrary correlators, including the ones involving angular variables. This provides a solution of the model in the planar limit. We write down the equations for the two-point function and the eigenvalue density and sketch the calculation of perturbative corrections to the free case.

1E-mail: kavalov@physik.uni-kl.de
2Permanent address: Yerevan Physics Institute, Alikhanyan Brothers St. 2, Yerevan, 375036, Armenia
1 Introduction

The matrix model approach to 2D gravity (see [1, 2] for a review) has proven to be very useful and illuminating, even providing a method of addressing the non-perturbative issues of string theory [3]. It suffers, however, a severe limitation, since all matrix models solved up to the present time correspond to strings embedded in spacetimes of unphysical dimensions $c \leq 1$. The models with $c > 1$ are investigated either perturbatively [4], or numerically [5]. The same $c = 1$ - barrier appears in the continuum approach to non-critical strings too, making them inappropriate for description of systems like gauge theories [6, 7]. Its physical origin is, at the present time, not very clear. There are indications that it reflects an intrinsic instability of random surfaces with respect to crumpling, is possibly connected to the existence of the tachionic mode of the bosonic string, and can be understood in terms of condensation of spikes [1, 3, 4, 10]. These observations suggest that the corresponding phase transition may be controllable in a model of surfaces with the action depending on the extrinsic curvature [11, 10]. Being initially simpler, the matrix model formulation of such models could shed a new light on the problem and even suggest a way out of it, were it not for the technical obstacle to solving interesting multi-matrix models which consists in inapplicability of the large-N saddle point technique to the systems with the number of degrees of freedom of order $N^2$ ($N$ being the size of the matrix). The latter is not the case for one-matrix models, where the action does not depend on the angular part of the matrix and one deals with a system of $N$ eigenvalues. For the next case, the simplest two-matrix model, Itzykson and Zuber were able to circumvent the problem by integrating over the angles explicitly [12]. The beautiful formula of Itzykson and Zuber allows for the solution of multi-matrix models with the tree-like structure of the target graph. A direct further generalisation of this approach to target spaces containing cycles would invoke calculation of the correlators of Itzykson-Zuber integral, for which only rather complicated expressions are known [8] (see, however, the interesting work of ref. [21]). Given the importance of the matrix models, it is therefore of some interest to look for models which are tractable by methods not involving the elimination of angular variables, such as the method of loop (Schwinger-Dyson) equations (see [14] for a review). The class of such models includes, of course, the ones solvable by previous techniques, in which case solving the loop equations sometimes allows the calculation of previously unknown quantities, such as the correlators involving angular variables [13, 14, 15]. The interesting question is whether it is possible to address in this way any new models. In the present letter we describe a step in this direction.
We define a gaussian 2 matrix model describing coupling to gravity of a vertex model (in particular case - of F-model type, [19]). The action of this model contains a term \( trABAB \) which prevents one from using the approach of Itzykson and Zuber. However, the infinite system of planar loop equations of the model can be reduced to a closed set, thus giving an analytical, though not very explicit, solution of the model and allowing one to find, in principle, all planar correlators, including the ones involving the angular variables. This provides a new example of a situation where loop equations are more effective than the “standard” approach.

The model we consider belongs to a very general class of matrix models describing vertex models coupled to gravity. This class includes most of the physically interesting matrix models, among them lattice gauge theories and the matrix models describing the extrinsic curvature strings. In view of the above, the latter property seems to be particularly interesting and is our primary motivation for introducing and considering this class of models. We comment on this connection in the next section, after having introduced our model which is, unfortunately, only the simplest representative of the class and corresponds to \( c = 1 \). We hope, however, that it shares some formal features with its more interesting relatives. It is interesting by itself, too, as a novel \( c = 1 \) two-matrix model.

In the last section we use the orthogonal decomposition of the matrices (used recently in the works [15, 16, 17]) to derive a system of loop equations written in the variables, allowing for its decoupling. We obtain a Fredholm equation of the second kind for the eigenvalue density and a more general non-singular nonlinear integral equation of Hammerstein’s type for the general two-point function \( < trA^nB^m > \). This equations have unique solutions in terms of Fredholm series. Analogous equations exist for arbitrary planar correlators of the model. The equations we obtain are rather complicated, but well suited for perturbation expansion. We sketch the calculation in the first order.

## 2 A model

Consider a system of two hermitian matrices with the partition function defined by

\[
Z = \int dAdBe^{-NS},
\]  

(1)
and the quadratic action

\[ S = \frac{1}{2} tr \rho (A^2 + B^2) - 2 c tr AB + \frac{\lambda}{2} tr ABAB + \frac{\kappa}{2} tr A^2 B^2. \]  

(2)

This action leads to a perturbation theory involving two types of propagators,

\[ P(A, A) = P(B, B) = \frac{r}{r^2 - c^2}, \]  

(3)

\[ P(A, B) = \frac{c}{r^2 - c^2}, \]  

(4)

and two types of vertices, with the couplings \( \lambda \) and \( \kappa \). One can imagine labeling, for each vertex of the given diagram, every A-type (B-type) leg by an outgoing (incoming) arrow. One then obtains a configuration of a vertex model, which is very similar to the familiar F version of the 6-vertex model \[19\], which has a critical regime with \( c = 1 \). The difference in the present case is that the propagators of the type (3) allow for configurations with directions of the arrows not preserved along the link. This may be viewed as some deformation of the F-model. One can hope to recover the pure F-model continuing the final expressions to the point \( r = 0 \). Note, however, that this limit is singular from the point of view of the original integral (1). Alternatively, one can put from the very beginning \( c = 0 \). This will leave the diagrams, containing only the “wrong” propagators (3). Than one can change locally the direction of all arrows for half of the vertexes to show that there exists a configuration of the F-model corresponding to each diagram. For some diagrams this procedure will meet, however, global obstructions. Note also, that from the vertex model point of view the terms \( A^4 \) and \( B^4 \) would correspond, if added to the action, to deforming the 6-vertex model to a non-critical 8-vertex model \[19\]. A way to couple the F-model to gravity is to use the complex matrices which give rise naturally to the oriented propagators (see \[2\]). This corresponds to taking \( c = 0 \) and adding the quartic terms with the coefficients chosen to ensure the charge conservation. Such a model arises also as a reduction of the generalised Weingarten model and is solved in general, and, in the planar case, in the presence of a vortex term, in the works \[20, 21\].

The model (2) belongs to a very general class of matrix models described by the actions

\[ S = \frac{1}{2} tr l_{\alpha \beta} \Phi^\alpha \Phi^\beta + V^{(3)}_{\alpha \beta \gamma} \Phi^\alpha \Phi^\beta \Phi^\gamma + \ldots \]  

(5)

where \( V^{(n)}_{\alpha \beta \gamma} \ldots \) (with \( V^{(2)} = l \)) are invariant with respect to cyclic transposition of indices. In diagrammatic expansion of this model \( l_{\alpha \beta} \) ( = inverse of \( l_{\alpha \beta} \)) is the propagator of the matrix field \( \Phi \) and \( V^{(n)} \) with \( n > 2 \) represent the vertexes. From the point of view of

\[ 1 \]I thank S. Dalley for calling my attention to these works.
the lattice dual to the given diagram one has a statistical model with “spins” \( \alpha \) living on the links of a triangulation of some surface, and the statistical weight obtained by prescribing the factor \( V^{(n)}_{\alpha\beta\gamma...} \) to each face (triangle for \( n = 3 \)) and \( l^{\alpha\beta} \) to each pair of the links identified in forming the surface. This action was first written down by Bachas and Petropoulos in connection to topological models on a two-dimensional lattice [22]. The spins \( \alpha \) take values in some set \( G \). The familiar matrix models are the particular case with \( V^{(0)}_{\alpha\beta\gamma...} \sim \delta_{\alpha\beta}\delta_{\alpha\gamma} \ldots, (n > 2) \). One can take

\[
[\Phi^\alpha]^\dagger = \Phi^{\alpha\dagger}
\]  

(6)

with the dagger on the left hand side denoting the hermitian conjugation and on the r.h.s. - some involution on \( G \). For \( \alpha^{\dagger} = \alpha \) one has a hermitian matrix model.

Let us show that the matrix models of this type describe, for special choice of \( G \), the extrinsic curvature strings. To see the idea take \( \alpha \) to be a pair \((x, n)\) where \( x \) is a coordinate in 3-dimensional space and \( n \) is a unit 3-vector, and arrange the couplings to have

\[
S = \frac{1}{2} \sum_{x, n_{1}, n_{2}} tr\Phi(x, n_{1})\Phi(x, n_{2})l_{n_{1}n_{2}} + \sum_{x, n} trV(\Phi(x, n))
\]

\[
+ \sum_{x_{1}, x_{2}, x_{3}, n} tr\Phi(x_{1}, n)\Phi(x_{2}, n)\Phi(x_{3}, n)\delta(n \sim x_{12} \times x_{23})e^{-Area(123)}. \quad (7)
\]

In the perturbation expansion of this model each link of the triangulated surface dual to a diagram is labeled by a position \( x \) in the embedding space of the string and there is a unit vector \( n \) ascribed to each face of the triangulation. The \( \delta \)-function in the cubic term ensures that \( n \) is a unit normal to the triangle (123), and one may take \( l^{n_{1}n_{2}} = exp(-n_{12}^2) \) to have this mutual orientation-dependent factor associated with each neighbouring pair of triangles. The \( Area(123) \) denotes the area of the triangle (123), while the potential term represents the couplings which add to the intrinsic area of the surface leaving the extrinsic one unchanged, and thus describe the fluctuating intrinsic metrics (absent in the usual continuum space formulation of the extrinsic curvature strings [11]).

This action, though illustrating how one can hope to obtain the matrix model description of extrinsic curvature string, is not the “final” correct version of the extrinsic curvature matrix model. One still has to modify it by taking the matrices to be complex and depending on some additional unit vector. For the discrete embedding space this corresponds to matrices living on the oriented links of the lattice and interacting along the plaquettes in the usual Wilson way; the index \( n \) will describes in this case the orientation of the plaquette. Identifying the matrices with different \( n \) one gets the Weingarten
model [23] (see [21]). Generalization to higher dimensions is obvious. This connection
of such matrix models to extrinsic curvature strings is our primary motivation for their
introduction and investigation.

Coming back to the model (2), let us note that due to the presence of the interaction
term \( trABAB \) the target graph of the model contains cycles preventing one from using the
approach of Itzykson and Zuber. In spite of this, it is possible to proceed in calculating
arbitrary correlators of the model. We will show below that the system of the planar loop
equations of the model reduces to a finite set.

3 Loop equations

It turns out to be convenient to write the loop equations in terms of the variables defined
via a decomposition

\[
M = \sum_{\alpha} m_{\alpha\alpha} P_{\alpha\alpha},
\]

where \( m_{\alpha\alpha} \) are the eigenvalues and \( P_{\alpha\alpha} \) the projectors on the corresponding eigenvector.

The latter satisfy the conditions

\[
P_{\alpha\alpha} P_{\beta\beta} = \delta_{\alpha\beta} P_{\alpha\alpha}, \quad tr P_{\alpha\alpha} = 1, \quad \sum_{\alpha} P_{\alpha\alpha} = 1.
\]

We will denote below the eigenvalues of \( A \) (resp. \( B \)) by \( x_\alpha \) (resp. \( y_\beta \)) and the corresponding
projectors as \( A_\alpha \) and \( B_\beta \). Another useful notation is

\[
P_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n} = tr A_{\alpha_1} B_{\beta_1} A_{\alpha_2} B_{\beta_2} \cdots A_{\alpha_n} B_{\beta_n},
\]

and also

\[
P_{n\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_k\beta_k} = \sum_{\alpha_1,\beta_1} x_{\alpha_1} y_{\beta_1} A_{\alpha_1} B_{\beta_1} A_{\alpha_2} B_{\beta_2} \cdots A_{\alpha_k} B_{\beta_k}.
\]

To avoid the possible confusion let us keep in mind that the numbers in the sequence of in-
dices of the objects like \( P_{12\alpha_2\cdots\alpha_n\beta_n} \) will always denote the power of the matrix in the corre-
spending position in the trace in \( (\Pi) \), while the Greek letters will be used for number-
ing the projectors. The same will apply to the continuum notation \( P(1, 2, x_3, y_4, \cdots, x_n, y_n) \)
with continuous variables \( x_i \) (\( y_i \)) taking the places of the discrete indexes \( \alpha_i \) (\( \beta_i \)). The
variables \( (8) \) were recently used by Alfaro to solve the 2-matrix model and to write down
the loop equations for D-dimensional matrix models [16, 17].

The equations we need follow from identities of type

\[
\int \frac{\partial}{\partial A^i_j}(A_{\alpha_1} B_{\beta_1} A_{\alpha_2} B_{\beta_2} \cdots A_{\alpha_n} B_{\beta_n})_i^j e^{-NS} = 0.
\]
In particular, one finds for $n = 2$
\[
[r x_{\alpha} - c y_{\beta} + \frac{\kappa}{2} x_{\alpha} y_{\beta}^2] P_{\alpha \beta} - \frac{1}{N} \sum_{\gamma \neq \alpha} \frac{1}{x_{\gamma \alpha}} (P_{\alpha \beta} + P_{\gamma \beta}) - \frac{1}{N} \sum_{\gamma \neq \alpha} \frac{1}{x_{\gamma \alpha}} (P_{\alpha \beta} + P_{\gamma \beta}) = 0. \quad (13)
\]

Summing this expression over $\alpha_2$ and using (4) one obtains the relation for $n = 1$:

\[
[r x_{\alpha} - c y_{\beta} + \frac{\kappa}{2} x_{\alpha} y_{\beta}^2] P_{\alpha \beta} - \frac{1}{N} \sum_{\gamma \neq \alpha} \frac{1}{x_{\gamma \alpha}} (P_{\alpha \beta} + P_{\gamma \beta}) + \lambda y_{\beta} P_{\alpha \beta 1} + \frac{\kappa}{2} P_{\alpha \beta 12} = 0. \quad (14)
\]

An analogous set of equations is obtained by taking in (12) the derivative \( \frac{\partial}{\partial \gamma} \) instead of \( \frac{\partial}{\partial \alpha} \). This new equations can be taken into account by requiring $P_{\alpha_1 \beta_2 \ldots \alpha_n \beta_n}$ to be invariant under the cyclic transposition of indexes.

In the limit of $N \to \infty$ we introduce the usual rescaled variables $x_i = x_{\alpha_i}, i = \frac{\alpha_i}{N}$, $y_j = y_{\beta_j}, j = \frac{\beta_j}{N}$ and assume that the eigenvalue density $\rho(x) = \frac{dx}{x}$ (the same for both matrices) has a finite support $[-a, a]$. One obtains, for instance, for eq.(14):

\[
[r x - c y + \frac{\kappa}{2} x y^2 + \int_a^{-a} \frac{u(z) dz}{x - z} P(x, y) + \int_{-a}^{a} \frac{u(z) dz}{z - x} P(x, y)] + \lambda y P(x, y, 1, 1) + \frac{\kappa}{2} P(x, y, 1, 2) = 0, \quad (15)
\]

with the density of eigenvalues and the correlators normalized by the conditions following from (3)

\[
\int_{-a}^{a} \rho(x) dx = 1, \quad (16)
\]
\[
\int_{-a}^{a} \rho(x) P(x, y) dx = 1, \quad (17)
\]
\[
\int_{-a}^{a} \rho(x) P(x, y, 1, 1) dx = y P(1, y), \quad \text{etc.} \quad (18)
\]

The symbol $\int$ denotes the principal value of the integral. The equation (13) is rewritten analogously as

\[
[r x_{1} - c y_{2} + \frac{\kappa}{2} x_{1} y_{2}^2 + \int_{-a}^{a} \frac{u(z) dz}{x_{1} - z} P(x_{1}, y_{1}, x_{2}, y_{2}) + \int_{-a}^{a} \frac{u(z) dz}{z - x_{1}} P(z, y_{1}, x_{2}, y_{2}) + P(x_{1}, y_{1}) \int_{-a}^{a} \frac{u(z) dz}{z - x_{1}} P(z, y_{2}) [\frac{\delta(x_{1} - x_{2})}{\rho(x_{1})} - \frac{\delta(z - x_{2})}{\rho(z)}] + \lambda y P(x_{1}, y_{1}, x_{2}, y_{2}, 1, 1) + \frac{\kappa}{2} P(x_{1}, y_{1}, x_{2}, y_{2}, 1, 2) = 0. \quad (19)
\]

Taken for arbitrary $n$, (12) thus provides an infinite set of linear relations between the correlators of the system. Our aim is to show, that eqs.(13) and (19) reduce, in fact,
to a closed system. To see this, note that multiplying both sides of eq. (15) by $\rho(x)$ and integrating over $x$ one obtains an algebraic equation for the correlator $P(x, 1)$ which gives

$$P(x, 1) = \frac{cx}{r + (\lambda + \kappa)x^2}. \quad (20)$$

A similar thing occurs when one integrates over $x_1$ in eq. (19). One obtains the equation

$$[r + \lambda y_1 y_2 + \frac{\kappa}{2}(y_1^2 + y_2^2)]P(1, y_1, x_2, y_2) =$$

$$c\frac{\delta(y_1 - y_2)}{\rho(y_1)} P(x_2, y_2) - \int_{-a}^{a} \frac{dz \rho(z)}{z - x} [P(x, y_1)P(z, y_2) + P(z, y_1)P(x, y_2)], \quad (21)$$

which expresses the 4-point correlator $P(1, y_1, x_2, y_2)$ in terms of 2-point functions. From here one finds easily

$$P(x, y, 1, 1) = \frac{cy^2}{r + (\lambda + \kappa)y^2} P(x, y) -$$

$$\int_{-a}^{a} \frac{d\zeta \rho(\zeta)\zeta^2}{r + \lambda \zeta y + \frac{\kappa}{2}(\zeta^2 + y^2)} \int_{-a}^{a} \frac{\rho(z)dz}{z - x} [P(x, \zeta)P(z, y) + P(x, y)P(z, \zeta)], \quad (22)$$

and

$$P(x, y, 1, 2) = \frac{cy^3}{r + (\lambda + \kappa)y^2} P(x, y) -$$

$$\int_{-a}^{a} \frac{d\zeta \rho(\zeta)\zeta^2}{r + \lambda \zeta y + \frac{\kappa}{2}(\zeta^2 + y^2)} \int_{-a}^{a} \frac{\rho(z)dz}{z - x} [P(x, \zeta)P(z, y) + P(x, y)P(z, \zeta)]. \quad (23)$$

Substituting these into eq. (15) gives a non-linear singular integral equation involving two unknown functions, $\rho(x)$ and $P(x, y)$. The second equation is provided by the symmetry condition $P(x, y) = P(y, x)$.

Let us now rewrite this system resolving the singular part of the equations. We introduce the functions

$$H(x) = \int_{-a}^{a} \frac{u(z)dz}{z - x}, \quad (24)$$

$$\phi(x; y) = \int_{-a}^{a} \frac{u(z)dz}{z - x} P(z, y), \quad (25)$$

and

$$G(x; y) = rx - cy - \frac{\kappa}{2}xy^2 + H(x). \quad (26)$$

The variable $x$ in these functions takes values on the complex plane cut along the interval $[-a, a]$, while $y$ is real and belongs to $[-a, a]$. One has, for $x \in [-a, a],$

$$Disc_\phi(x; y) \equiv \phi(x + i0; y) - \phi(x - i0; y) = 2\pi i \rho(x) P(x, y) \quad (27)$$
\[ Cont_x \phi(x; y) \equiv \phi(x + i0; y) + \phi(x - i0; y) = 2 \int_{-a}^{a} \frac{\rho(z)dz}{z - x} P(z, y). \]  

(28)

Using these relations, in (15), (22) and (23) one can easily show that both sides of the equation (15) are given by discontinuities of some functions. One obtains

\[
\begin{align*}
Disc_x G(x; y) \phi(x; y) &= -Disc_x (\lambda + \frac{\kappa}{2}) \frac{cy^3}{r + (\lambda + \kappa) y^2} \phi(x, y) \\
+ Disc_x \phi(x, y) \int_{-a}^{a} d\zeta \rho(\zeta) \frac{\lambda \zeta y + \frac{\kappa}{2} \zeta^2}{r + \lambda \zeta y + \frac{\kappa}{2} (\zeta^2 + y^2)} \phi(x, \zeta),
\end{align*}
\]

(29)

which implies that

\[
\begin{align*}
G(x; y) \phi(x; y) + (\lambda + \frac{\kappa}{2}) \frac{cy^3}{r + (\lambda + \kappa) y^2} \phi(x, y) \\
- \phi(x, y) \int_{-a}^{a} d\zeta \rho(\zeta) \frac{\lambda \zeta y + \frac{\kappa}{2} \zeta^2}{r + \lambda \zeta y + \frac{\kappa}{2} (\zeta^2 + y^2)} \phi(x, \zeta) &= Pol.(x),
\end{align*}
\]

(30)

where \( Pol.(x) \) is some polynomial of \( x \). Taking \( x \to \infty \) and using the known behaviour of \( H(x) \) and \( \phi(x, y) \) in this limit, one finds \( Pol.(x) = r - \frac{\kappa}{2} y^2 \) and, after some algebra, (30) reaches its final form

\[
\phi(x; y) K \phi(x; y) + [x - \frac{cy}{r + (\lambda + \kappa) y^2}] \phi(x; y) + 1 = 0
\]

(31)

with

\[
K \phi(x; y) = \int_{-a}^{a} d\zeta \rho(\zeta) \frac{\phi(x, \zeta)}{r + \lambda \zeta y + \frac{\kappa}{2} (\zeta^2 + y^2)}. 
\]

(32)

This is a non-linear integral equation of Hammerstein’s type which defines the function \( \rho(x) \phi(x, y) \). Another equation, expressing the symmetricity of \( P(x, y) \) can be written as \( \Phi(x, y) = \Phi(y, x) \) with

\[
\Phi(x, y) = \int_{-a}^{a} \frac{\rho(\zeta) d\zeta}{\zeta - y} \phi(x, \zeta).
\]

(33)

Taking here the limit of \( y \to \infty \) one obtains the simpler condition

\[
H(x) = \int_{-a}^{a} d\zeta \rho(\zeta) \phi(x, z) \\
= - \int_{-a}^{a} \frac{dz \rho(z)}{x - \frac{cz}{r + (\lambda + \kappa) z^2}} + \int_{-a}^{a} \frac{d\zeta \rho(\zeta) \phi(x, \zeta)}{r + \lambda \zeta y + \frac{\kappa}{2} (\zeta^2 + y^2)}. 
\]

(34)

Equations (31) and (33) allow one to find the eigenvalue density \( \rho(x) \) and the two-point function \( P(x, y) \), and provide the decoupling of the system (12). The existence of solutions for (31) and (33) follows from the theorems of the theory of integral equations, but we do not know, at the present moment, how to find them explicitly. The same theorems
say a lot about the properties of the solutions. Leaving the complete investigation of the system (31, 33) to the future, let us have a brief look on its perturbative properties.

Consider first the free case. Taking $\lambda = \kappa = 0$ one finds from the symmetricity condition (34)

$$ rH_0(x) = \int_{-a}^{a} \rho(z)dz $$

or

$$ H_0 \left( \frac{rx + H_0(x)}{c} \right) = \frac{c}{r} H_0(x). $$

This functional equation is similar to the one found in different contexts by Matitsyn [24] and by Boulatov [25]. It can be solved either by noticing directly that it implies

$$ rH_0(x)^2 + (r^2 - c^2)xH_0(x) = \text{const.} $$

and finding, for $x \to \infty$, the constant to be $c^2 - r^2$, or by taking $c \ll 1$ and performing successive approximations. The solution is given by

$$ H_0(x) = -\frac{r^2 - c^2}{2r} \left[ x - \sqrt{x^2 - \frac{4r}{r^2 - c^2}} \right] $$

and leads, of course, to Wigner’s semi-circle distribution

$$ \rho_0(x) = \frac{r^2 - c^2}{2\pi r} \sqrt{x^2 - \frac{4r}{r^2 - c^2}}. $$

The 2-point function $\phi$ is, in this limit, given by

$$ \phi_0(x; y) = -\frac{r}{rx - cy - H_0(x)}. $$

Coming back to eq. (31) note that in the limit of $x \to \infty$ the non-linear term in it is much smaller that the other two. Applying the successive approximations to

$$ \phi(x; y) = -\frac{1}{x - \frac{cy}{r + (\lambda + \kappa)y^2}} - \frac{1}{x - \frac{cy}{r + (\lambda + \kappa)y^2}} \phi(x; y)K\phi(x; y), $$

one obtains $\phi(x, y)$ as an expansion in $\frac{1}{x - \frac{cy}{r + (\lambda + \kappa)y^2}}$. The first term is given by

$$ \phi^{(0)}(x; y) = -\frac{1}{x - \frac{cy}{r + (\lambda + \kappa)y^2}}, $$

while the successive ones contain $\rho(x)$:

$$ \phi^{(1)}(x; y) = -\frac{1}{x - \frac{cy}{r + (\lambda + \kappa)y^2}} - \frac{1}{x - \frac{cy}{r + (\lambda + \kappa)y^2}} \phi^{(0)}(x; y)K\phi^{(0)}(x; y), \quad \text{etc.} $$

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Comparing (43) to the expansion
\[ \phi(x; y) = -\frac{1}{x} - \frac{P(1, y)}{x^2} - \frac{P(2, y)}{x^3} - \cdots, \]  
(44)
one finds the correlators \( P(n, y) \) in terms of \( \rho \). In particular,
\[ P(2, y) = -\frac{c^2 x}{r + (\lambda + \kappa)x^2} + \int_{-a}^{a} \frac{d\zeta \rho(\zeta)}{r + \lambda \zeta y + \frac{\kappa}{2}(\zeta^2 + y^2)}. \]  
(45)
Note an exact relation following from (43):
\[ \phi(x; y; c) = \phi(x - \frac{cy}{r + (\lambda + \kappa)y^2}; y; c = 0). \]  
(46)
Finally, multiply both sides of eq. (15) by \( \rho(y) \) and integrate over \( y \in [-a, a] \). One obtains
\[ \int_{-a}^{a} \frac{\rho(z)}{z - x} = -\frac{1}{2} \left[ rx - cP(x, 1) + \kappa x P(x, 2) + \lambda P(x, 1, 1, 1) \right]. \]  
(47)
Using (22) to find
\[ P(x, 1, 1, 1) = \frac{c^2 x^3}{r + (\lambda + \kappa)x^2} + \int_{-a}^{a} \frac{d\zeta \rho(\zeta)\zeta}{r + \lambda \zeta x + \frac{\kappa}{2}(\zeta^2 + x^2)}, \]  
(48)
as well as (43) and (20), one arrives at the equation containing \( \rho(x) \) as the only unknown:
\[ \int_{-a}^{a} \frac{\rho(z)}{z - x} = \frac{1}{2} f(x), \]  
(49)
where
\[ f(x) = -rx - \frac{rc^2 x}{r + (\lambda + \kappa)x^2} + \int_{-a}^{a} \frac{d\zeta \rho(\zeta)\zeta}{r + \lambda \zeta x + \frac{\kappa}{2}(\zeta^2 + x^2)}. \]  
(50)
The same equation could, of course, be derived directly by integrating in (1) over one of the matrices. This would lead to an effective action involving ‘non-local’ interactions of the type \((trA^n)^m\). Such matrix models were considered in the works of ref.[26].

Once again, one can resolve the singular part of (43) to reduce it to
\[ H(x) = r \frac{\sqrt{a^2 - x^2}}{2\pi i} \int_{-a}^{a} \frac{dz}{z - x} \frac{1}{\sqrt{a^2 - z^2}} \left[ -z + \frac{c^2 z}{r + (\lambda + \kappa)z^2} \right] \]
\[ -\frac{\sqrt{a^2 - x^2}}{2\pi i} \int_{-a}^{a} \frac{dz}{z - x} \frac{1}{\sqrt{a^2 - z^2}} \int_{-a}^{a} \frac{d\zeta \rho(\zeta)\zeta}{r + \lambda \zeta z + \frac{\kappa}{2}(\zeta^2 + z^2)}. \]  
(51)
The first integral in this equation can be calculated easily by rewriting it as a contour integral around the cut and inflating the contour to touch the singularities at \( z = x, \pm i\sqrt{r/(\lambda + \kappa)} \) and \( \infty \).
Taking the discontinuity of both sides of (51) one obtains a Fredholm equation of second kind for \( \rho(x) \). One can then perform the successive approximations once again to obtain the solution in terms of Fredholm series. Alternatively, one can use (51) directly to calculate \( H(x) \) as power series in \( \lambda \) and \( \kappa \). Consider, as a simple example, the case of \( c = \kappa = 0 \). One finds from (51), in the lowest order

\[
H(x) = -\frac{1}{2} \left[ r - \left( \frac{\lambda}{\rho} \right)^2 \rho_2 \right] \left[ x - \sqrt{x^2 - a^2} \right],
\]

where the coefficient \( \rho_2 = \int_{-a}^{a} dz \rho(z) z^2 \), as well as \( a^2 \), have to be determined from comparison of (52) to the expansion

\[
\rho(x) = -\frac{1}{x} - \frac{\rho_2}{x^3} - \frac{\rho_4}{x^5} - \cdots
\]

One finds, after some algebra,

\[
\rho_2 = \frac{a^2}{4}, \quad a^2 = \frac{4}{r} \left[ 1 + \left( \frac{\lambda}{\rho} \right)^2 \frac{1}{r^2} \right].
\]

It is not difficult to write down the corrections of first two orders for general \( c \). The formulae are, however, not very illuminating.

4 Comments

This concludes our preliminary investigation of the model (2). Evidently, there is still much to be done, as the progress achieved in the present work is mostly of technical nature. We feel however that it is of some interest, as a demonstration of usefulness of the loop equations for calculations of the correlators involving angular variables for the vertex model on the lattice of fluctuating geometry which was not considered previously. Our interest in the model (2) was triggered by the observation that it contains an interaction of the same type as the matrix models describing extrinsic curvature strings. The latter models can lead to important physical insights. Further work is in progress.

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