Quantum-limited Euler angle measurements using anticoherent states

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(Dated: June 8, 2018)

Many protocols require precise rotation measurements. We here present a general class of states that surpass the shot noise limit for measuring rotations around arbitrary axes. We then derive a quantum Cramér-Rao bound for simultaneously estimating all three parameters of a rotation (e.g., the Euler angles), and discuss states that achieve Heisenberg-limited sensitivities for all parameters; the bound is saturated by “anticoherent” states. We elucidate geometrical and mathematical techniques for finding new anticoherent states, which have garnered much attention in recent years. Finally, we discuss the potential for divergences in multiparameter estimation due to singularities in spherical coordinate systems. These results are useful for numerous tasks in quantum metrology and quantum communication.

I. INTRODUCTION

Estimating rotations is a highly relevant problem. Rotation measurements have applications in mathematics, physics, and beyond, ranging from geodesy [1] to physiology [2] (see Ref. [3] for a recent review). The problem of estimating rotations around a known axis is well-understood [4-7], while rotation measurements around unknown axes comprise a nascent endeavour [8, 10]. Measurement precision can be enhanced using special quantum input states in the known-axis case [9, 10]; here we investigate quantum enhancements for simultaneously estimating rotation angles and rotation axes.

Single parameter estimation has a long history [13]. One of the most famous examples is interferometry, in which the parameter in question is a phase imparted on a beam of light, which can be used to measure biomolecules [11, 12], gravitational waves [15, 16], and many things in between [17]. Classical states of light are limited in their measurement precision by shot noise arising from photon statistics, leading to precisions bounded from below by $1/\sqrt{N}$, where $N$ is the number of photons involved in the measurement [18]. However, this is not a fundamental limit; cleverly designed schemes can take advantage of quantum correlations between photons to achieve the so-called Heisenberg limit, in which measurement precisions scale as $1/N$ [17, 18]. In this work we seek similar quantum advantages for simultaneously estimating three parameters.

A suitably designed quantum mechanical setup can indeed enhance the simultaneous estimation of multiple parameters [19]. For measuring phases imparted by either commuting or non-commuting operators, enhancements can be on the order of the number of parameters being estimated [19, 20]. A common technique for finding these enhancements involves the quantum Cramér-Rao bound, which bounds the covariance matrix between parameters being estimated by the inverse of the quantum Fisher information matrix (QFIM). The quantum Cramér-Rao bound optimizes the covariance over all possible measurement techniques, and the QFIM depends on the chosen input state [21]; therefore, an important task is finding states that maximize the QFIM.

One area in which the QFIM has been studied is for reference frame alignment. Consider two parties who want to share some spatial information; to do so, they must know each other’s coordinate system. Estimating the rotation required to align two coordinate systems has been studied, and it was found that “anticoherent” states maximize the QFIM [10]. This result is highly insightful for measurements of rotations about unknown axes.

Anticoherent states are those whose polarization vectors vanish, and whose higher order polarization moments are isotropic [22]. They are the furthest states from perfectly polarized states of light [23], with both classical and quantum notions of polarization vanishing for anticoherent states [22, 24]. Because polarized light behaves more classically than unpolarized light, anticoherent states are in some sense the least classical quantum states [22, 25, 27].

Anticoherent states have numerous mathematical and physical applications, relating to old problems of distributing points around a sphere [28, 29] and new challenges such as maximizing quantum entanglement [24, 25, 30] or other notions of nonclassicality [31, 32]. The conditions for a state to be anticoherent have become more clear in recent years [33, 34]; our mathematical and geometrical formulations of the conditions yield simple ways of finding new anticoherent states: one must simply find states with two independent rotational symmetries. Some of these states have already been created experimentally using light’s orbital angular momentum degrees of freedom [37]. The states can be readily used for optimizing estimates of rotation parameters.

A generic rotation in three dimensions is characterized by three parameters. These parameters can be the two angular coordinates of the rotation axis as well as the angle rotated around that axis, or any of the sets of three Euler (or Tait-Bryan) angles [35]. In this work we optimize estimates of the Euler angles; were the rotation axis to be known a priori, one could simply use single param-
eter estimation techniques. Nonetheless, any set of three parameters can be obtained from any other triplet. We find that, regardless of parametrization, there exist angles for which the measurement precision diverges. This is related to the so-called “hairy ball theorem.” The multiparameter technique outperforms classical shot noise scaling everywhere, even in this diverging regime.

This work is arranged as follows. In Section II we give a background to the problem of estimating quantum mechanical rotations, specifically in the language of polarimetry, and we briefly discuss single parameter estimation for rotations about a known axis in Section III. In Section IV we introduce the quantum Fisher information matrix and the quantum Cramér-Rao bound. We use these to derive a bound on the covariance matrix of rotation parameters as well as combinations of single parameter estimation schemes. Section V contains a discussion of the anticoherent states that are used to optimize the Fisher information. There, we mention earlier results for identifying 2-anticoherent states, and provide intuitive alternative methods for identifying more general classes of such states. We give concluding remarks in Section VI.

II. ROTATIONS AND POLARIZATION

We begin by considering polarization states of light; these are mathematically equivalent to any two-mode quantum states. The two modes are associated with operators $\hat{a}$ and $\hat{b}$ satisfying bosonic commutation relations $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$, $\hat{a}_i \in \{\hat{a}, \hat{b}\}$, such that a general state can be written as

$$|\psi\rangle = \sum_{m,n} c_{m,n} |m,n\rangle,$$

where $|m,n\rangle \equiv \hat{a}^\dagger m \hat{b}^\dagger n |\text{vac}\rangle / \sqrt{m!n!}$. With these operators in hand we can define angular momentum operators

$$\hat{S}_0 = (\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}) / 2,$$
$$\hat{S}_1 = (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}) / 2,$$
$$\hat{S}_2 = (\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}) / 2,$$
$$\hat{S}_3 = -i (\hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a}) / 2$$

satisfying the usual $su(2)$ algebra

$$[\hat{S}_\mu, \hat{S}_\nu] = i (1 - \delta_{\mu 0}) (1 - \delta_{\nu 0}) \sum_{j=1}^3 \epsilon_{\mu \nu j} \hat{S}_j$$
$$\hat{S}_1^2 + \hat{S}_2^2 + \hat{S}_3^2 = \hat{S}_0 (\hat{S}_0 + 1).$$

Here, $\hat{S}_0$ counts the total number of quanta in the system, and is mathematically equivalent to the total spin operator. If, for example, $\hat{a}$ and $\hat{b}$ represent annihilation operators for two orthogonal polarizations of light, then we can associate the operators in Eq. (2) with quantum Stokes operators, whose expectation values are the classical Stokes parameters (up to a normalization factor $\Re$).

The Stokes parameters contain all of the polarization information of classical states of light. A state is unpolarized if the vector $\hat{S} \equiv \langle \hat{S}_1, \hat{S}_2, \hat{S}_3 \rangle$ obeys $\langle \hat{S} \rangle = 0$. Quantum mechanically, there may still be some polarization information in these classically unpolarized states $|\psi\rangle$. This prompted the definition of “$t$-anticoherent” states as those for which $\langle (\hat{S} \cdot \hat{n})^k \rangle = e_k$ for all positive integers $k \leq t$ and all unit vectors $\hat{n}$.

Stokes operators furnish the $SU(2)$ rotation operators

$$\hat{R}(\omega, \hat{n}) = \exp i \omega \hat{S} \cdot \hat{n},$$

which rotate Stokes vectors $\langle \hat{S} \rangle$ by angles $\omega$ about axes $\hat{n}$. For example, the rotation

$$\hat{R}_{\theta,\phi} = \exp (-i \xi \hat{a}^\dagger \hat{b} + i \xi \hat{b}^\dagger \hat{a}),$$

generates the transformation

$$\hat{R}_{\theta,\phi} \left( \begin{array}{c} \hat{a} \\ \hat{b} \end{array} \right) \hat{R}^\dagger_{\theta,\phi} = \left( \begin{array}{cc} \cos \theta & e^{-i\phi} \sin \theta \\ -e^{i\phi} \sin \theta & \cos \theta \end{array} \right) \left( \begin{array}{c} \hat{a} \\ \hat{b} \end{array} \right),$$

which is mathematically equivalent to an interferometer.

The goal of this paper is to identify ways of measuring the parameters of these rotation operators $\hat{R}$. The rotation operators $\hat{R}$ can be parametrized in many equivalent ways; in Section IV we switch to the Euler angle representation for $\hat{R}$.

III. SINGLE PARAMETER ESTIMATION

For a known axis $\hat{n}$, one can try to optimize measurements of the rotation angle $\omega$ around that axis. One method of estimating $\omega$ is by measuring the projection $\hat{P} = |\langle \psi | \hat{\omega} | \psi \rangle|^2$ of an initially prepared pure state $|\psi\rangle$ onto the rotated state $\hat{R} |\psi\rangle$:

$$\langle \hat{P} \rangle = \langle (\langle \psi | \hat{R}(\omega, \hat{n}) | \psi \rangle)^2 \rangle,$$

where expectation values are henceforth taken with respect to the rotated state. For small angles $\omega$, one can expand the exponential in Eq. (1) to find $\langle \hat{P} \rangle = 1 - \omega^2 \text{Var} [\hat{S} \cdot \hat{n}] + O (\omega^4)$. This can be used to calculate the variance of the estimated angle

$$\text{Var} [\omega] = \frac{\text{Var} [\hat{P}]}{\langle \partial^2 (\hat{P}) / \partial \omega^2 \rangle} \approx \frac{1}{4 \text{Var} [\hat{S} \cdot \hat{n}]}.$$
Coherent state inputs with average photon number $N$, such as $e^{-N/2} \sum_{m=0}^{\infty} \frac{N^m}{\sqrt{m!}} |m,0\rangle$, have $4 \text{Var} \left[ \hat{S} \cdot \mathbf{n} \right] = N$. These classical states can at best achieve the shot noise precision $\Delta \omega \equiv \sqrt{\text{Var} [\omega]} = 1/\sqrt{N}$. In comparison, the NOON states $|\psi\rangle = \left| N,0 \right> + 0, N \rangle$ satisfy $4 \text{Var} \left[ \hat{S} \cdot \left( \sin \theta \cos \phi , \sin \theta \sin \phi , \cos \theta \right) \right] = N^2 \cos^2 \theta + N \sin^2 \theta$, and so can achieve Heisenberg-limited precisions $\Delta \omega = 1/N$ for rotations around a unit vector $\mathbf{n}$ aligned with the $z$-axis. This is an important example of the fact that particular input states can provide quantum-enhanced sensitivities in parameter estimation.

If we do not specify a measurement scheme, the quantum Cramér-Rao bound tells us that $\Delta \omega \geq 1/\sqrt{I}$, for quantum Fisher information [41]

$$I = 4 \left[ \partial_{\omega} \left( \langle \psi | \hat{R}^\dagger \right) \partial_{\omega} \left( \hat{R} \langle \psi \rangle \right) - \left| \langle \psi | \hat{R}^\dagger \partial_{\omega} \hat{R} \langle \psi \rangle \right| ^2 \right].$$

(9)

The above measurement scheme of projecting onto the initial state saturates the Cramér-Rao bound, which is always possible for single parameter estimation [3] [19] [26]. The Fisher information together with the quantum Cramér-Rao bound can thus be used as a way of determining input states that will achieve optimal sensitivities for a particular transformation.

It is clear that states with isotropic $\text{Var} \left[ \hat{S} \cdot \mathbf{n} \right] = \mathcal{O}(N^2)$ are useful for estimating rotations about arbitrary, known axes $\mathbf{n}$. These states will achieve Heisenberg-scaling precisions regardless of the rotation axis $\mathbf{n}$, albeit with slightly less sensitivity than the NOON states rotating about the $z$-axis. The 2-anticoherent states have $\langle \hat{S} \rangle = 0$ and $\langle (\hat{S} \cdot \mathbf{n})^2 \rangle = \langle \hat{S}_0 (\hat{S}_0 + 1) \rangle / 3$, which allows the variances to scale quadratically with the number of quanta in the initial states. Recent experiments have used these states to achieve Heisenberg-scaling sensitivities for rotations around a variety of rotation axes [37]. We presently investigate these states in the context of measuring rotations whose axes are not known a priori.

IV. MULTIPARAMETER ESTIMATION

In this section we investigate the most sensitive techniques for measuring changes in the Euler angles of a rotation.

A. Quantum Fisher information matrix for Euler angle estimation

The QFIM for pure states has components [45]

$$[I_{\theta}]_{l,m} = \langle \psi_{\theta} | L_{l} L_{m} + L_{m} L_{l} | \psi_{\theta} \rangle$$

(10)

for symmetric logarithmic derivatives

$$L_i = 2 \left( \langle \partial_{\theta_i} \psi_{\theta} | \psi_{\theta} \rangle + \langle \psi_{\theta} | \partial_{\theta_i} \psi_{\theta} \rangle \right).$$

(11)

Here $|\psi_{\theta}\rangle = \hat{R} (\theta) |\psi_0\rangle$, for input state $|\psi_0\rangle$ and some triplet of rotation parameters $\theta$. The operator $L_{\omega}$ is relatively easy to compute given $\hat{R} = \exp \left( i \hat{S} \cdot \mathbf{n} \right)$, because in taking the derivative $d_\theta \hat{R}$ one need not consider operator ordering, but the other components are more difficult; hence, we switch to Euler angle parametrizations.

Following the notation in Ref. [40] discussing the SU(2) representation of beam splitters, we parametrize our rotation operators by

$$\hat{B} = e^{-i \Phi \hat{S}_z} e^{-i \Theta \hat{S}_y} e^{-i \Psi \hat{S}_z}.$$  

(12)

Our goal is to estimate the parameters $(\Phi, \Theta, \Psi)$. For this we must evaluate the symmetric logarithmic derivatives defining the quantum Fisher information, which rely on derivatives of $\hat{B}$ with respect to each of the three angles:

$$\frac{d\hat{B}}{d\Phi} = -i \hat{S}_z \hat{B} \equiv -i \hat{H} \hat{B}$$

$$\frac{d\hat{B}}{d\Theta} = -i \left( e^{-i \Phi \hat{S}_z} \hat{S}_y e^{i \Phi \hat{S}_z} \right) \hat{B} = -i \left( -\sin \Phi \hat{S}_x + \cos \Phi \hat{S}_y \right) \hat{B} \equiv -i \hat{H}_\Theta \hat{B}$$

$$\frac{d\hat{B}}{d\Psi} = -i \left( \hat{B} \hat{S}_y \hat{B}^\dagger \right) \hat{B} = -i \left( \sin \Theta \cos \Phi \hat{S}_z + \sin \Theta \sin \Phi \hat{S}_y + \cos \Theta \hat{S}_z \right) \hat{B} \equiv -i \hat{H}_\Psi \hat{B}.$$  

(13)

Then we find that $[I_{\theta}]_{l,m}$ takes the form

$$[I_{\theta}]_{l,m} = 4 \text{Cov} \left[ \hat{H}_l, \hat{H}_m \right],$$

(14)

where we use $\text{Cov} [X, Y] = \langle XY + YX \rangle - \langle X \rangle \langle Y \rangle$ and expectation value are taken with respect to the rotated state $|\psi_{\theta}\rangle$. All that remains is to find states $|\psi_{\theta}\rangle$ that maximize the amount of information in this matrix.

B. Quantum Fisher information matrix for optimum input states

We assume that the input states have exactly $N$ quanta, as these will always perform at least as well as superposition states with various numbers of quanta [10]. It was shown that the best states for estimating the three components of a reference frame have $\langle \hat{S} \rangle = 0$ and $\langle \hat{S}_i \hat{S}_j \rangle = \delta_{ij} \frac{N}{2} \left( \frac{N}{2} + 1 \right)$ [10]; these are the 2-anticoherent states [22]. We calculate the QFIM for 2-anticoherent states with $N$ quanta, in the $(\Phi, \Theta, \Psi)$ basis:

$$[I_{\theta}] = \frac{N (N + 2)}{3} \begin{pmatrix} 1 & 0 & \cos \Theta \\ 0 & 1 & 0 \\ \cos \Theta & 0 & 1 \end{pmatrix}. $$

(15)
The quantum Cramér-Rao bound says that the covariance matrix for the parameters \((\Phi, \Theta, \Psi)\) satisfies the inequality
\[
\text{Cov}[\theta] \geq I_0^{-1} = \frac{3}{N(N+2)} \begin{pmatrix}
\frac{1}{\sin^2 \Theta} & 0 & -\cos \Theta \\
0 & 1 & 0 \\
-\cos \Theta & 0 & \frac{1}{\sin^2 \Theta}
\end{pmatrix};
\]
this bound can be saturated for pure states with real symmetric logarithmic derivatives \([3, 19, 26]\), which is always the case here.

This result gives great scaling with \(N\). The actual measurement precisions have very small bounds for \(\sin \Theta \approx 1\), but are worse when \(\sin \Theta \approx 0\), which can be expected from the chosen Euler angle parametrization. This is because, for \(\Theta = 0\), the beam splitter simply acts as \(\hat{B} = e^{-i(\Phi + \Psi) \hat{S}_z}\), and so one would only ever be able to estimate \(\Theta\) and the sum \(\Phi + \Psi\). Alternatively, we can see this divergence by considering the difference in parameters \(\Phi - \Psi\). The eigenvector \((1, 0, -1)\) of the QFIM given in Eq. \((15)\) corresponding to the difference \(\Phi - \Psi\) has eigenvalue proportional to \(1 - \cos \Theta\), which vanishes at \(\Theta = 0\). Our estimation scheme does well everywhere other than at this angle.

For any \(\Theta > 0\) we thus get Heisenberg scaling in the variance of estimating \(\Phi, \Theta,\) and \(\Psi\) simultaneously:
\[
\text{Var}[\Phi] + \text{Var}[\Theta] + \text{Var}[\Psi] \geq \text{Tr}[I_0^{-1}]
= \frac{3}{N(N+2)} \left(1 + \frac{2}{\sin^2 \Theta}\right).
\]
Eq. \((17)\) shows that quantum enhancements can be achieved in the simultaneous estimation of all three rotation parameters.

### C. Comparison to single parameter estimation and other rotation parametrizations

We can compare our result in Eq. \((17)\) to the best possible single parameter estimation techniques, as well as parametrizations other than \((\Phi, \Theta, \Psi)\).

The optimum single parameter estimation techniques use NOON states. A scheme that uses three NOON states to measure rotations around three axes \(z, u_1,\) and \(u_2\), with \(N/3\) particles in each state, only yields the quantum Cramér-Rao bound (see Appendix \(\text{A}\)).

Even for the best choice of \(u_1\) and \(u_2\), our multiparameter scheme outperforms this single parameter scheme by a factor \(d + 2d/N\), where \(d = 3\) is the number of parameters being estimated. This is because one can only use \(N/d\) quanta per measurement in the single parameter scheme, and is similar to the \(O(d)\) enhancements found by Ref. \([19, 20]\) in using \(d\)-mode schemes to simultaneously estimate \(d\) parameters.

However, we are leery of combining the Cramér-Rao bounds for single parameters when multiple parameters are unknown. The off-diagonal elements of the QFIM are what determine its singularities; it is unfair to compare to single parameter estimation schemes when the evolution mixes the modes onto which the parameters are being imparted. The multiparameter technique outlined above is thus the best that one can do for simultaneously estimating \(\Phi, \Theta,\) and \(\Psi\).

We further note that every parametrization of the rotation parameters has divergences in the trace of the covariance matrix for particular angles. For example, if we use the rotation operators \(\hat{U}(\alpha, \beta, \gamma) = e^{-i \alpha \hat{S}_z} e^{-i \beta \hat{S}_x} e^{-i \gamma \hat{S}_z}\) in our multiparameter scheme, we find that
\[
\text{Var}[\alpha] + \text{Var}[\beta] + \text{Var}[\gamma] \geq \frac{3}{N(N+2)} \left(1 + \frac{2}{\sin^2 (2\alpha)} \right); \tag{19}
\]
whose divergence at \(\sin (2\alpha) = \cos (\beta)/\sin^2 (\beta/2)\) (see Appendix \(\text{B}\) for further discussion of these coordinate singularities). This relates to the assertion of Brouwer’s fixed-point theorem (the “hairy-ball theorem”) that non-vanishing continuous tangent vector fields on \(S^2\) do not exist \([17]\), which implies that a function mapping a fixed unit vector \(\mathbf{n}_0\) to orthogonal basis vectors cannot be continuous. The QFIM being singular is a signature of the discontinuity present in every choice of mapping.

The divergences are also related to the inability to probe three parameters with equal sensitivities using two mode states \([45]\). Creating and controlling \(N\)-particle states with more than two modes adds an additional experimental challenge, as the rotation operators must leave any extra modes unchanged while maintaining entanglement between all of the modes. We restrict our attention to the readily-available two mode states, such
as photon polarization states, even though there will be particular combinations of rotation parameters for which the QFIM is singular.

V. FINDING 2-ANTICOHERENT STATES

The states that optimize the QFIM for estimating Euler angles are 2-anticoherent states: classically unpolarized states with isotropic variances in their Stokes operators. These subsume the t-anticoherent states, which also have isotropic higher order moments of the Stokes operators, and have been studied by a number of authors in various contexts.

There have been efforts over the years to find simple ways of characterizing 2-anticoherent states, on which we focus here. The original requirement for an N-qubit state

\[ |\psi^{(N)}\rangle = \sum_{m=0}^{N} c_m |m, N - m\rangle \] (20)

to be 2-anticoherent can be written as

\[ S = 0, \quad S = \frac{N(N+2)}{12} I, \] (21)

where we define the vector \( S = \langle \hat{S} \rangle \) and the Hermitian tensor \( S \) with components \( S_{ij} = \langle \hat{S}_i \hat{S}_j \rangle \) [22]. To generate such states, the usual approach is via the Majorana representation [49]. A deep conjecture relating the Majorana representation of anticoherent states to spherical designs was proposed in 2010 [33], but counterexamples were elucidated shortly thereafter [34]. A fruitful new approach has recently come to light [27, 35, 50]: we comment briefly on some of these approaches and show an elegant, geometrical method for finding 2-anticoherent states that has not yet been elucidated.

A. Mathematical scheme

Similar to the method in Ref. [35], we present simple mathematical criteria for finding 2-anticoherent states. 2-anticoherent states must have

\[ \sum_{m=0}^{N} |c_m|^2 m^2 = \frac{N(2N+1)}{6} \],

due to \( \langle \hat{S}_z \rangle = \langle \hat{S}_z^2 \rangle = 0 \), in addition to the usual normalization \( \sum_m |c_m|^2 = 1 \). The other conditions can all be satisfied if we impose the additional requirement that \( c_m c_{m+1} = c_m c_{m+2} = 0 \) for all \( m \) (i.e., the spacing between each nonzero \( c_m \) should be at least 2 values of \( m \)). This yields a set of three equations for the real parameters \( |c_m|^2 \), which can be solved analytically or numerically for a given choice of nonzero \( \{c_m\} \) (see especially Ref. [35] for interesting numerical results).

As an example, we give an analytical solution for systems with four nonzero coefficients \( c_{N/4}, c_{N/2}, c_{3N/4}, \) and \( c_N \) (we choose this example because there are no 2-anticoherent states listed in Refs. [10, 35] with exactly four nonzero coefficients). We find the infinite family of states

\[ |\psi^{(N)}_4\rangle = c |N, 0\rangle + e^{i\phi_1} \sqrt{\frac{2(2+N)}{3N} - 3c^2} |3N/4, N/4\rangle + e^{i\phi_2} \sqrt{3c^2 - \frac{8+N}{3N}} |N/2, N/2\rangle + e^{i\phi_3} \sqrt{\frac{2(2+N)}{3N} - c^2} |N/4, 3N/4\rangle, \] (23)

for arbitrary \( c \in \left( \frac{8+N}{3N}, \frac{4+2N}{3N} \right) \) and \( N \geq 12 \). The states \( |\psi^{(N)}_4\rangle \), and other easy-to-find 2-anticoherent states, can thus be used to achieve Heisenberg scaling of \( O(1/N) \) in the precision of simultaneously estimating all three Euler angles of a rotation.

B. Geometrical scheme

There are important geometrical properties of 2-anticoherent states that can be used to find new such states without solving systems of linear equations. These make use of the Majorana representation.

1. Majorana representation

The Majorana representation is a geometrical interpretation of N-particle two-mode pure states, uniquely mapping each state onto the positions of \( N \) points on a sphere. The state given by Eq. (20) can be uniquely rewritten up to a global phase as

\[ |\psi^{(N)}\rangle = \frac{1}{\sqrt{N}} \prod_{k=1}^{N} \hat{a}^\dagger_{\theta_k, \phi_k} |\text{vac}\rangle, \] (24)

where \( \mathcal{N} \) is a normalization factor and \( \hat{a}^\dagger_{\theta_k, \phi_k} = \cos \frac{\theta_k}{2} \hat{a}^\dagger + e^{i\phi_k} \sin \frac{\theta_k}{2} \hat{b}^\dagger \) [32, 51]. The coefficients \( \{c_m\} \) from Eq. (20)
are determined by
\[
c_m = \sqrt{\frac{m! (N - m)!}{N}} \prod_{k=1}^{m} \cos \frac{\theta_k}{2} \prod_{k=m+1}^{N} e^{i\phi_k} \sin \frac{\theta_k}{2}. \tag{25}
\]

The Majorana representation then defines the one-to-one mapping from the state \(|\psi^{(N)}\rangle\) to the \(N\) indistinguishable points \(\{(\theta_m, \phi_m)\}\) on the unit sphere (known as the Poincaré sphere in the context of photons) \cite{50,51,52}.

Since the number operator \(\hat{S}_0\) commutes with the other operators of the algebra, total particle number \(N\) is conserved by SU(2) operations. Thus, rotation operators simply rotate all of the points \(\{(\theta_m, \phi_m)\}\) about the Poincaré sphere \cite{32,51,52}, conserving all quantum numbers other than \(m\) \cite{52}. To optimize estimates of the Euler angles, one seeks states that are highly sensitive to rotations. These optimal states, the 2-anticoherent states, are found to be states with highly symmetric Majorana representations \cite{10,32}.

2. **Platonic solids and the Majorana representation**

The problem of distributing points symmetrically about a sphere is not new. It has been studied in relation to mathematics \cite{29}, biology \cite{53}, and quantum entanglement \cite{24}. One of the earliest results for distributing points symmetrically uses the Platonic solids; the vertices of any of the five Platonic solids will be symmetrically spaced about a sphere circumscribing the solid.

States whose Majorana representations form a Platonic solid are always anticoherent to order 2 or higher \cite{10,22}. For example, the state
\[
|\psi\rangle = \frac{1}{\sqrt{3}} |4, 0\rangle + \sqrt{\frac{2}{3}} |1, 3\rangle = \left( \frac{\hat{a}_+^{14} + \hat{a}_+^{1b}^{i3}}{\sqrt{72}} \right) |\text{vac}\rangle, \tag{26}
\]
which is the most nonclassical state with \(N = 4\), a maximally entangled state of four qubits, and the most sensitive \(N = 4\) state for reference frame alignment, has vertices that form a tetrahedron, one of the Platonic solids \cite{20}. Moreover, states with \(m\) Majorana points at each of the vertices of a Platonic solid, like
\[
|\psi\rangle \propto \left( \frac{\hat{a}_+^{14} + \hat{a}_+^{1b}^{i3}}{\sqrt{72}} \right)^m |\text{vac}\rangle, \tag{27}
\]
are also anticoherent\(^1\) Platonic solids with \(m\) Majorana points at each vertex can thus be used to increase the

\(^1\)This little-known fact is mentioned in a footnote in Ref. \cite{24} without proof. We show why this is true in Section \[\text{V B 3}\]. In addition, there are 2-anticoherent states whose Majorana representations are not Platonic solids; we suspect that multiplying the degeneracies of the Majorana points for any 2-anticoherent state by an arbitrary integer will yield another 2-anticoherent state.
The proportionality constant in $S \propto I$ is fixed by Eq. [31] to yield the 2-anticoherence properties given in Eq. [21]:

$$S = 0, \quad S = \frac{N(N + 2)}{12} I.$$  \hspace{1cm} (30)$$

Any state with two independent rotational symmetries [Eq. (29)] will achieve Heisenberg scaling in estimating rotation parameters.

4. Applications of the geometrical condition

All of the Platonic solids have multiple discrete rotational symmetries, about axes defined by the lines from the centre of the sphere through any of the solids’ vertices or through the middle of any of the solids’ faces. These symmetries are a property of the geometry alone, and so states whose Majorana representations have $m$-fold degeneracies at each of the vertices of a Platonic solid are 2-anticoherent states.

Moreover, the duals of a polyhedron share its rotational symmetries, so any state whose Majorana representation features $m$-fold degeneracies at the vertices of a Platonic solid as well as $n$-fold degeneracies at the vertices of the Platonic solid’s dual is also 2-anticoherent. For example, a state whose Majorana constellation has $m$ points at each of the vertices of a cube and $n$ points at each of the vertices of the cube’s dual, an octahedron, is a 2-anticoherent state (see Fig. 1 for a similar example). Combinations of Platonic solids and their duals can be used to measure rotation parameters with high sensitivity.

Our criteria thus help motivate the Platonic solids as ideal states for measuring rotations. They also point to a much broader class of ideal states. One such extension is the class of Archimedean solids (Figs. 1-2). The 13 Archimedean solids all have discrete rotational symmetries along multiple independent axes.
For example, the truncated tetrahedron, an Archimedean solid, has the same rotational symmetries as the tetrahedron, with 12 vertices. Thus, one can form Majorana constellations made from any combination of $m$ points at each of the vertices a tetrahedron, $n$ points at the vertices of the tetrahedron’s dual tetrahedron, $i$ points at the vertices of the associated truncated tetrahedron, and $j$ points at the vertices of the truncated tetrahedron associated with the dual tetrahedron, to obtain a 2-anticoherent state with $N = 4n + 4m + 12i + 12j$ quanta (Fig. 1). Similar things can be done with all of the Platonic and Archimedean solids that share rotational symmetries.

This gives a broad class of states that achieve Heisenberg scaling in estimating the three angles of a rotation. Our symmetry property can be used as a simple geometrical method for generating new 2-anticoherent states.

VI. CONCLUSIONS

We have presented a thorough investigation of how to attain maximally sensitive measurements of rotations. We derived the quantum Fisher information matrix for Euler angle measurements, and established a quantum Cramér-Rao bound on the covariance between the Euler angle measurements, and established a quantum Fisher information matrix for Erwin angle measurements, offering enhancements over the shot noise limit. We showed that the corresponding matrix is always singular for particular combinations of rotation parameters. This is an important consideration in all rotation measurements.

Finally, we discussed conditions for finding new 2-anticoherent states. We first mentioned a simple mathematical technique for finding 2-anticoherent states with arbitrarily high particle number $N$. Then, we showed that states whose Majorana representations have two independent rotational symmetries are 2-anticoherent. This yielded 2-anticoherent states whose Majorana representations corresponded to the vertices of the Platonic solids, the Archimedean solids, and degeneracies and combinations thereof. This geometrical technique is a powerful way of uncovering 2-anticoherent states. We conjecture that increasing the degeneracy of the Majorana representation of any 2-anticoherent state will yield a new 2-anticoherent state.

Using 2-anticoherent states to optimize rotation measurements has many applications. The quantum enhancements obtained by using 2-anticoherent states can naturally be used in polarimetry and ellipsometry, using light’s polarization degree of freedom, and can further be used in precision measurements of electric and magnetic fields, biological samples, and even components of quantum technologies. The exciting field of quantum-enhanced multiparameter estimation has many important ramifications for the near future.

ACKNOWLEDGMENTS

This work was supported by the Alexander Graham Bell Scholarship #504825. A.G. acknowledges insightful discussions with H. Ferretti.

Appendix A: Single parameter variances

In this appendix we derive a bound on the variances of three rotation parameters estimated using three optimal single-parameter estimation schemes.

We consider as usual the NOON states $|\psi_{\text{NOON}}\rangle = |N,0\rangle + |0,N\rangle \sqrt{2}$ and the rotation operators $\hat{B}(\Phi, \Theta, \Psi) = e^{-i\Phi S_z^x} e^{-i\Theta S_z^y} e^{-i\Psi S_z^z}$. We try to minimize the variance in estimating $\Phi$, $\Theta$, and $\Psi$ by using three NOON states, each with $N/3$ particles, aligned along various axes. Without loss of generality, we consider the states $|\psi_0(a,b)\rangle = \hat{B}(0, a, b) |\psi_{\text{NOON}}\rangle$, where $a = \tan^{-1}(u_y/u_x)$ and $b = \cos^{-1}(u_z)$ parametrize the unit vector $u = (u_x, u_y, u_z)$ around which the NOON state $|\psi_0(a,b)\rangle$ is most sensitive to measuring rotations.

The state $|\psi_{\text{NOON}}\rangle$ has

$$S = 0, \quad \hat{S} = \frac{1}{4} \begin{pmatrix} N & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & N^2 \end{pmatrix}. \quad (A1)$$

The quantum Fisher information is calculated as before with the $\hat{H}_I$ operators, but now we must take expectation values with respect to the states $|\psi_0(a,b)\rangle$. For this, we use the transformations

$$\hat{B}^\dagger (0, a, b) \hat{B} (0, a, b) = -\sin a \hat{S}_x + \cos a \hat{S}_z$$
$$\hat{B}^\dagger (0, a, b) \hat{B} (0, a, b) = \hat{S}_x \cos a (\sin b \cos \Phi - \cos b \sin \Phi) + \hat{S}_y (\cos b \cos \Phi + \sin b \sin \Phi) + \hat{S}_z \sin a (\sin b \cos \Phi - \cos b \sin \Phi)$$
$$\hat{B}^\dagger (0, a, b) \hat{B} (0, a, b) = \hat{S}_y (\sin \Theta \cos \Phi \cos a \sin b + \sin \Theta \sin \Phi \cos a \sin b - \cos \Theta \sin a)$$
$$+ \hat{S}_z (\sin \Theta \sin \Phi \cos b - \sin \Theta \cos \Phi \sin b) + \hat{S}_x (\sin \Theta \cos \Phi \sin a \cos b + \sin \Theta \sin \Phi \sin a \sin b + \cos \Theta \cos a). \quad (A2)$$
We are looking to optimize measurements of a single parameter, using $\text{Var} [\theta] \geq 1/4 \text{Var} [\hat{H}_I]$:

$$\text{Var} [\Phi] \geq \frac{1}{N^2 [u \cdot n_z]^2 + N [u \times n_z]^2}$$

$$\text{Var} [\Theta] \geq \frac{1}{N^2 [u \cdot n_{\Phi}]^2 + N [u \times n_{\Phi}]^2}$$

$$\text{Var} [\Psi] \geq \frac{1}{N^2 [u \cdot n_{\Theta,\Phi}]^2 + N [u \times n_{\Theta,\Phi}]^2},$$

for unit vectors

$$n_z = (0, 0, 1)$$

$$n_{\Phi} = (-\sin \Phi, \cos \Phi, 0)$$

$$n_{\Theta,\Phi} = (\sin \Theta \cos \Phi, \sin \Theta \sin \Phi, \cos \Theta)$$

$$u = (\sin a \cos b, \sin a \sin b, \cos a).$$

The variances are similar to those in the known-axis case, in which the denominators look like $N^2 \cos^2 \theta + N \sin^2 \theta$ for a rotation around axis $(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \phi)$.

The unknown axis contributes the parameters $\Theta$ and $\Phi$, and one must choose combinations of $a$ and $b$ that optimize estimates of $\Phi$, $\Theta$, and $\Psi$. The best possible choice for estimating $\Phi$ is by taking $u = n_z$. Similarly, the best possible choices for estimating $\Theta$ and $\Psi$ use $u = n_{\Phi}$ and $u = n_{\Theta,\Phi}$, respectively; however, these axes cannot be known a priori.

Each scheme can only use $N/3$ particles, so the optimal combination of these three variances for various choices of $a$ and $b$ yields Eq. (A3) above, where we have chosen $n_{\Phi} \cdot u = \cos \theta_1$ and $n_{\Theta,\Phi} \cdot u = \cos \theta_2$. However, this idea of combining the single-parameter Cramér-Rao bounds for a multiparameter estimation technique, like in Refs. [19] [20], cannot be sufficient. If it were the case, we could simply take the diagonal components of Eq. (13) above, invert them, and achieve Heisenberg scaling precisions for $\Phi$, $\Theta$, and $\Psi$ regardless of rotation angle, which is impossible. The difference here is that our evolution mixes the modes onto which the parameters are being imparted, so one cannot truly subdivide the system and estimate a single parameter for each section while being ignorant of the other parameters. Only a true multiparameter estimation technique can succeed in our case.

**Appendix B: Divergences in every rotation angle parametrization**

We here discuss the fact that every rotation angle parametrization yields singular quantum Fisher information matrices for particular combinations of rotation angles.

We start by choosing three rotation parameters $a$, $b$, and $c$. Any triplet of rotation parameters can in principle be obtained from any other such triplet (e.g., by writing equating the rotation matrices for the various parametrizations and solving the resulting nonlinear equations), so we use our parametrization $\tilde{B}(\Phi, \Theta, \Psi) = e^{-i\Phi S_z} e^{-i\Theta S_x} e^{-i\Psi S_z}$ for variables

$$\Phi = \Phi(a, b, c), \quad \Theta = \Theta(a, b, c), \quad \Psi = \Psi(a, b, c).$$

Next, we formally compute the derivatives

$$d_k \tilde{B} = -i \left( \tilde{H}_\Phi d_k \Phi + \tilde{H}_\Theta d_k \Theta + \tilde{H}_\Psi d_k \Psi \right) \equiv -i \tilde{H}_k \tilde{B},$$

for $k \in \{a, b, c\}$ and $d_k \equiv d/dk$. In the $\tilde{\theta} = (a, b, c)$ parametrization, the quantum Fisher information matrix has components related to the $\theta = (\Phi, \Theta, \Psi)$ QFIM:

$$[\mathbf{I}_{\tilde{\theta}}]_{i,j} = (d_i \Phi \ d_j \Theta \ d_k \Psi) \mathbf{I}_{\theta} \left( \frac{d_j \Phi}{d_k \Psi} \right).$$

We recognize the Jacobian

$$\mathbf{J} = \begin{pmatrix} d_\Phi & d_\Phi & d_\Phi \\ d_\Theta & d_\Theta & d_\Theta \\ d_\Psi & d_\Psi & d_\Psi \end{pmatrix}$$

and the transformation $\mathbf{I}_{\tilde{\theta}} = \mathbf{J}^T \mathbf{I}_{\theta} \mathbf{J}$. We have already shown that the matrix $\mathbf{I}_{\theta}$ is singular at angle $\Theta = 0$, because $\text{Det} [\mathbf{I}_{\theta}] \propto \sin^2 \Theta$. The new matrix $\mathbf{I}_{\tilde{\theta}}$ is singular whenever $\mathbf{I}_{\theta}$ is singular, unless $\text{Det} [\mathbf{J}]$ diverges as $1/\sin \Theta$. This singularity is a coordinate singularity; there is no set of coordinates that can cover a sphere without such singularities. The best that can be done is to hope for a parametrization whose coordinate singularity occurs at a different set of coordinates.

The only possibility of $\mathbf{I}_{\tilde{\theta}}$ being invertible at $\Theta = 0$ is if $\text{Det} [\mathbf{J}] \propto 1/\sin \Theta$ for all values of $\Phi$ and $\Psi$ at that point. Namely, one would need $d_\Phi \Theta (d_\Phi d_\Phi \Psi - d_\Psi d_\Phi \Psi) + d_\Theta (d_\Phi d_\Phi \Psi - d_\Psi d_\Phi \Psi) + d_\Psi (d_\Phi d_\Phi \Psi - d_\Psi d_\Phi \Psi) \propto 1/\sin \Theta$ for all values of $\Phi$ and $\Psi$ as $\Theta \to 0$. This requires that $d_\Phi \Theta \propto d_\Theta \propto d_\Theta \propto 1/\sin \Theta$ as $\Theta \to 0$. If that were the case, then one would never be able to estimate the value of $\Theta$ from the three parameters $a$, $b$, and $c$ near $\Theta = 0$.

Still, if one indeed had $\text{Det} [\mathbf{J}] \propto 1/\sin \Theta$, we would expect $\text{Det} [\mathbf{J}] \to 0$ at a different value of $\theta$, because coordinate singularities are present regardless of parametrization. Since the rotation parameters are unknown a priori, it is impossible to definitively choose a parametrization that is guaranteed to be nonsingular for a given rotation measurement. Perhaps one could avoid the divergences by using $N/2$ particles in each of two separate rotation measurements whose singular coordinates do not coincide.
