HARDY AND LITTLEWOOD THEOREMS AND THE BERGMAN DISTANCE

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Dedicated to the memory of Professor Miroslav Pavlović

Abstract. We obtain non-Euclidean versions of classical theorems due to Hardy and Littlewood concerning smoothness of the boundary function of an analytic mapping on the unit disk with an appropriate growth condition.

Résumé. Nous obtenons des versions non euclidiennes des théorèmes classiques dus à Hardy et Littlewood concernant la régularité de la fonction frontière d’une fonction analytique sur le disque unité avec une condition de croissance appropriée.

1. Introduction

For the theory of Hardy spaces $H^p$ on the unit disk $D$ we refer to the classical Duren book [2] and the recent Pavlović monograph [8]. Recall that the space $H^\infty$ contains bounded analytic mappings on $D$. The space $H^p$, $p \in [1, \infty)$, consists of analytic mappings $f$ on $D$ such that $m_p(r, |f|)$ remains bounded in $r \in (0, 1)$, where for a continuous and nonnegative function $g$ on the unit disk we have denoted

$$m_p(r, g) = \left( \int_{-\pi}^{\pi} g(\rho e^{it})^p dt \right)^{\frac{1}{p}}.$$

It is well known that an analytic mapping $f \in H^p$, $p \in [1, \infty]$, has radial boundary value

$$f_b(t) = f(e^{it}) = \lim_{r \to 1} f(re^{it}) \text{ for a.e. } t \in [-\pi, \pi].$$

Moreover, the boundary function $f_b$ belongs to the Lebesgue space $L^p[-\pi, \pi]$, and the mapping $f$ is the Poisson extension of $f_b$, i.e.,

$$f(z) = P[f_b(t)](z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} f_b(t) dt, \quad z = re^{i\theta} \in \mathbb{D}.$$

The Lipschitz class $\Lambda_\alpha$, $\alpha \in (0, 1]$, consists of $2\pi$-periodic functions $\phi$ on $\mathbb{R}$ such that

$$\sup_{|s-t|<h} |\phi(t) - \phi(s)| = O(h^\alpha), \quad h \to 0.$$

The mean Lipschitz class $\Lambda_p^\alpha$, $\alpha \in (0, 1]$, $p \in [1, \infty)$, contains $2\pi$-periodic functions $\phi$ defined a.e. on $\mathbb{R}$ such that

$$\sup_{s \in (0, h)} \left( \int_{-\pi}^{\pi} |\phi(t+s) - \phi(t)|^p dt \right)^{\frac{1}{p}} = O(h^\alpha), \quad h \to 0.$$

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The following two propositions are classical results due to Hardy and Littlewood [3]. We refer to the fifth chapter of [2].

**Proposition 1.1.** An analytic mapping \( f \) on \( \mathbb{D} \) satisfies \( |f'(z)| = O(1 - |z|)^{\alpha - 1} \), \( |z| \to 1 \), \( \alpha \in (0, 1] \), if and only if it has continuous extension on \( \mathbb{D} \) and \( f_0 \in \Lambda_\alpha \).

**Proposition 1.2.** An analytic mapping \( f \) on \( \mathbb{D} \) satisfies \( m_p(r, |f'|) = O(1 - r)^{\alpha - 1} \), \( r \to 1 \), \( \alpha \in (0, 1] \), \( p \in [1, \infty) \), if and only if \( f \in H^p \) and \( f_0 \in \Lambda_\alpha^p \).

In 1982, Yamashita gave the hyperbolic counterparts of Proposition 1.1 and Proposition 1.2; see [9, Theorem 1 and Theorem 2]. Instead of the growth of \( |f'| \) his theorems concern the growth of the hyperbolic derivative

\[ f^*(z) = \frac{|f'(z)|}{1 - |f(z)|^2}, \quad z \in \mathbb{D}, \]

of an analytic mapping \( f : \mathbb{D} \to \mathbb{D} \). On the other hand, instead of the standard Lipschitz classes, the Yamashita results contain their hyperbolic analogues.

The aim of this article is to consider the Yamashita approach in a more general than the hyperbolic setting on the unit disk. We consider analytic mappings of \( \mathbb{D} \) into a domain in \( \mathbb{C} \) equipped with a metric with certain properties. Particularly, this domain may be bounded and equipped with the Bergman metric.

### 2. Preliminaries

We say that a function \( \rho \) is a metric on a domain \( D \subseteq \mathbb{C} \) if it is positive and continuous on \( D \). The \( \rho \)-distance on \( D \) is

\[ d(\zeta, \eta) = \inf_{\gamma} \int_{\gamma} \rho(\zeta)|d\zeta| = \int_a^b \rho(\gamma(t))|\gamma'(t)|dt, \quad \zeta, \eta \in D, \]

where the infimum is taken over all piecewise \( C^1 \)-smooth curves \( \gamma : [a, b] \to D \) with endpoints at \( \zeta \) and \( \eta \), i.e., \( \gamma(a) = \zeta \) and \( \gamma(b) = \eta \).

It may be shown that

\[ \lim_{\eta \to \zeta} \frac{d(\zeta, \eta)}{|\zeta - \eta|} = \rho(\zeta), \quad \zeta \in D. \]

For this result we refer to [5, 6].

For the notion of Bergman metric we refer to the 4th chapter in the Krantz book [4], but for the sake of completeness we mention below some facts.

For a bounded domain \( D \) let

\[ A^2(D) = \{ f : f \text{ is analytic mapping on } D \text{ and } \int_D |f|^2dA \text{ is finite} \}, \]

be the Bergman space; \( dA \) is the area measure. In \( A^2(D) \) one may introduce the inner product

\[ \langle f, g \rangle_{A^2(D)} = \int_{\Omega} f(z)g(z)dA(z), \quad f, g \in A^2(D). \]

With this inner product \( A^2(D) \) is the Hilbert space.

Among other approaches, one can use abstract Hilbert space methods to prove the existence of the so called Bergman kernel \( K_D : D \times D \to \mathbb{C} \) which has the following properties:

1. \( K(z, w) = \overline{K(w, z)} \), \( (z, w) \in D \times D \);
2. \( K_D^*(w) = K(z, w) \in A^2(D) \) for every \( z \in D \);
(3) the reproducing property: for every $f \in A^2(D)$ holds
\[ f(z) = \langle K_D^2, f \rangle _{A^2(D)} = \int_D K_D(z, w)f(w)dA(z), \quad z \in D. \]

The Hilbert space $A^2(D)$ is separable, so it has a countable base $\{ \phi_j : j \in \mathbb{N} \}$. The Bergman kernel may be represented as
\[ K(z, w) = \sum_{j=1}^{\infty} \phi_j(z)\overline{\phi_j(w)}, \quad z, w \in D. \]

This possibility gives opportunity to show that the continuous function
\[ \rho_D(\zeta) = \sqrt{\frac{\partial^2}{\partial \zeta \partial \zeta} \log K_D(\zeta, \zeta)}, \quad \zeta \in D. \]
is positive on $D$. Therefore, it is a metric on $D$ which is called the Bergman metric.

It is well known that the Bergman metric on the unit disk coincides up to a multiplicative constant with the hyperbolic metric on $\mathbb{D}$ which is given by
\[ \rho(\zeta) = \frac{1}{1 - |\zeta|^2}, \quad \zeta \in \mathbb{D}. \]
The hyperbolic distance on $\mathbb{D}$ is explicitly given by
\[ \sigma(\zeta, \eta) = \frac{1}{2} \log \frac{|1 - \overline{\zeta} \eta| + |\zeta - \eta|}{|1 - \zeta \overline{\eta}| - |\zeta - \eta|}, \quad \zeta, \eta \in \mathbb{D}. \]

Introduce now the generalized Lipschitz classes – non-Euclidean analogues to $\Lambda_\alpha$ and $\Lambda_p^\alpha$. Let $\rho$ be a metric on a domain $D$ and let $d = d_\rho$ be the $\rho$-distance on the same domain. We denote by $\rho\Lambda_\alpha$, $\alpha \in (0, 1]$, the class of all $2\pi$-periodic functions $\phi$ on $\mathbb{R}$ with values in $D$ such that
\[ \sup_{|s-t| < h} d(\phi(t), \phi(s)) = O(h^\alpha), \quad h \to 0. \]

By $\rho\Lambda_\alpha^p$, $\alpha \in (0, 1]$, $p \in [1, \infty)$, we denote the class of $2\pi$-periodic function defined a.e. on $\mathbb{R}$ with values in the domain $D$ such that
\[ \sup_{s \in (0, h)} \left( \int_0^{2\pi} d(\phi(t + s), \phi(t))^{p} dt \right)^{\frac{1}{p}} = O(h^\alpha), \quad h \to 0. \]

Note that, if $p < p'$, then $\rho\Lambda_\alpha \subseteq \rho\Lambda_\alpha^{p'} \subseteq \rho\Lambda_\alpha^p$, $\alpha \in (0, 1]$, $p, p' \in [1, \infty)$. If $\rho$ is the hyperbolic metric on $\mathbb{D}$, we have the hyperbolic Lipschitz classes $\sigma\Lambda_\alpha$ and $\sigma\Lambda_\alpha^p$.

Let $\Omega$ be a domain in $\mathbb{C}$ or in $\mathbb{R}^1$. The $\rho$-derivative of a mapping $f : \Omega \to D$ at $z \in \Omega$ is defined by
\[ f^*(z) = \lim_{w \to z} \frac{d(f(z), f(w))}{|z - w|}. \]
If $f : \Omega \to D$ is analytic (differentiable) mapping at $z_o \in \Omega$, then the $\rho$-derivative of $f$ at $z_o$ may be expressed in the following way
\[ f^*(z_o) = \rho(f(z_o))|f'(z_o)|. \]
Indeed, if $f'(z_o) \neq 0$, then $f(z) \neq f(z_o)$, if $z$ is in a sufficiently small neighborhood of $z_o$; namely, if $f(z_n) = f(z_o)$ for a sequence $z_n \to z_o$, then
\[ f'(z_o) = \lim_{n \to \infty} \frac{f(z_n) - f(z_o)}{z_n - z_o} = 0. \]
It follows
\[ f^*(z_0) = \lim_{z \to z_0} \frac{d(f(z), f(z_0))}{|z - z_0|} = \lim_{z \to z_0} \frac{d(f(z), f(z_0))}{f(z) - f(z_0)} \lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = \rho(f(z_0))|f'(z_0)|. \]

Now, let \( f'(z_0) = 0 \), and let \( r \) be a positive number such that the closed disk \( \overline{D}(f(z_0), r) \) is contained in \( D \). Denote \( m = \max_{z \in \overline{D}(f(z_0), r)} \rho(\zeta) \). For \( z \) sufficiently close to \( z_0 \) such that \( |f(z) - f(z_0)| \leq r \) we have
\[ d(f(z), f(z_0)) \leq m|f(z) - f(z_0)|. \]

It follows
\[ f^*(z_0) = \lim_{z \to z_0} \frac{d(f(z), f(z_0))}{|z - z_0|} \leq m \lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = 0, \]
which proves this case.

Let us note that for an analytic mapping \( f : D \to \mathbb{D} \), and for the hyperbolic metric on \( D \), \( f^*(z) \) is the hyperbolic derivative of \( f \) at \( z \in \mathbb{D} \).

3. Two generalizations of the Yamashita result

We will state the Yamashita results [9, Theorem 1 and Theorem 2] in the following proposition.

**Proposition 3.1.** Let \( f : \mathbb{D} \to \mathbb{D} \) be an analytic mapping, and let \( f^* \) be the hyperbolic derivative of \( f \). Then we have
\[ f^*(z) = O(1 - |z|)^{\alpha - 1}, |z| \to 1, \quad \alpha \in (0, 1], \]
if and only if \( f_b \in \sigma\Lambda_\alpha \), and
\[ m_p(r, f^*) = O(1 - r)^{\alpha - 1}, r \to 1, \quad \alpha \in (0, 1], p \in [1, \infty), \]
if and only if \( f_b \in \sigma\Lambda_\alpha^p \).

Our main results are given in the fourth and fifth section. Here we will state their consequences for bounded domains with the Bergman metric and for Dini-smooth Jordan domains with the quasi-hyperbolic metric which both extend Proposition 3.1. The Bergman metric on the unit disk \( \mathbb{D} \) coincides up to a positive constant with the hyperbolic metric on this domain. Therefore, the result given below indeed generalize the Yamashita one.

As a direct consequence of Theorem 4.1, Theorem 4.2, and Theorem 5.3 we have the following theorem.

**Theorem 3.2.** Let \( D \subseteq \mathbb{C} \) be a bounded domain equipped with the Bergman metric \( \rho_D \). For an analytic mapping \( f : \mathbb{D} \to D \) let
\[ f^*(z) = \rho_D(f(z))|f'(z)|, \quad z \in \mathbb{D}, \]
be the \( \rho_D \)-derivative of \( f \). Then
\[ f^*(z) = O(1 - |z|)^{\alpha - 1}, |z| \to 1, \quad \alpha \in (0, 1], \]
if and only if \( f_b \in \rho_D\Lambda_\alpha \), and
\[ m_p(r, f^*) = O(1 - r)^{\alpha - 1}, r \to 1, \quad \alpha \in (0, 1], p \in [1, \infty), \]
if and only if \( f_b \in \rho_D\Lambda_\alpha^p \).
A $C^1$-smooth Jordan curve $\gamma$ is called Dini-smooth if, considered as a function of the arc length over the segment $[0, l]$, where $l$ is the length of $\gamma$, satisfies
\[ |\gamma'(s) - \gamma'(t)| \leq \omega(|s - t|), \quad s, t \in [0, l], \]
and $\omega(x)$ is an increasing function such that
\[ \int_0^l \frac{\omega(s)}{s} \]
is finite.

**Theorem 3.3.** Let $D$ be a Jordan domain with Dini-smooth boundary with the quasi-hyperbolic metric
\[ \rho(\zeta) = \frac{1}{d(\zeta)}, \quad \zeta \in D, \]
where $d(\zeta)$ is the Euclidean distance of $\zeta$ from $\partial D$. For an analytic mapping $f : D \to D$ let
\[ f_q^*(z) = \frac{|f'(z)|}{d(z)}, \quad z \in \mathbb{D} \]
be the quasi-hyperbolic derivative of $f$. We have
\[ f_q^*(z) = \mathcal{O}(1 - |z|)^{\alpha - 1}, \quad |z| \to 1, \quad \alpha \in (0, 1], \]
if and only if $f_b \in \rho^{\Lambda}$, and
\[ m_p(r, f_q^*) = \mathcal{O}(1 - r)^{\alpha - 1}, \quad r \to 1, \quad \alpha \in (0, 1], p \in [1, \infty), \]
if and only if $f_b \in \rho^{\Lambda_p}$.

**Proof.** Let the quasi-hyperbolic distance on the domain $D \subseteq \mathbb{C}$ be denoted $d_q$, and denote by $d_\beta$ the Bergman distance on the same domain. It is possible to estimate the Bergman metric via the quasi-hyperbolic metric on a Jordan domain with Dini-smooth. Quite recently Nikolov and Trybula [7] proved: There exists a constant $c > 1$ such that
\[ \sqrt{2} \log \left( 1 + \frac{|\zeta - \eta|}{c \sqrt{d(\zeta)d(\eta)}} \right) \leq d_\beta(\zeta, \eta) \leq \sqrt{2} \log \left( 1 + \frac{c|\zeta - \eta|}{\sqrt{d(\zeta)d(\eta)}} \right), \quad \zeta, \eta \in D. \]
This forces the following relation between the Bergman and the quasi-hyperbolic metric (if we divide each side above by $|\zeta - \eta|$ and let $\eta \to \zeta$):
\[ (3.1) \quad \frac{c^{-1}\sqrt{2}}{d(\zeta)} \leq \rho_D(\zeta) \leq \frac{c\sqrt{2}}{d(\zeta)}, \quad \zeta \in D. \]
It follows
\[ c^{-1}\sqrt{2}d_q(\zeta, \eta) \leq d_\beta(\zeta, \eta) \leq c\sqrt{2}d_q(\zeta, \eta), \quad \zeta, \eta \in D. \]
Therefore, we have the coincidence of the classes $\rho_D\Lambda_\alpha = \rho\Lambda_\alpha$, $\alpha \in (0, 1]$, and $\rho_D\Lambda^p_\alpha = \rho\Lambda^p_\alpha$, $\alpha \in (0, 1]$, $p \in [1, \infty)$. Clearly, the inequality (3.1) implies the relation between the derivatives:
\[ c^{-1}\sqrt{2}f_q^*(z) \leq f^*(z) \leq c\sqrt{2}f_q^*(z), \quad z \in \mathbb{D}, \]
where $f^*$ is the $\rho_D$-derivative. The conclusion of this theorem now follows from Theorem 3.2. \qed
4. Growth of the derivative and the boundary function

In the theorems which follows we assume that $D$ is a domain equipped with a metric $\rho$ and the corresponding distance $d = d_\rho$, such that the following two conditions are satisfied:

1. $C_\rho := \inf_{\zeta \in D} \rho(\zeta) > 0$;
2. $d(\zeta, \eta) \to \infty$, if $\zeta \to \partial D$; $\eta \in D$ is arbitrary.

Note that (1) is satisfied if the metric $\rho$ has the stronger property: $\rho(\zeta) \to \infty$, if $\zeta \to \partial D$ (the boundary is in the topology of $\mathbb{C}$). For example this stronger condition is satisfied by the Bergman metric on a bounded domain. Also, it is known that the Bergman metric has the second property (see [4]).

The first condition immediately implies that

\[(4.1) \quad d(\zeta, \eta) \geq C_\rho |\zeta - \eta|, \quad \zeta, \eta \in D,\]

which forces the inclusions $\rho \Lambda_\alpha \subseteq \Lambda_\alpha$, $\rho \Lambda^p_\alpha \subseteq \Lambda^p_\alpha$, $\alpha \in (0, 1]$, $p \in [1, \infty)$.

The $\rho$-Hardy space $\rho H^p$, $p \in [1, \infty)$, is defined as a space of analytic mappings $f : \mathbb{D} \to D$ such that

\[
\|f\|_{\rho H^p} = \sup_{r \in (0, 1)} \left( \int_{-\pi}^{\pi} d(f(re^{it}), f(0))^p dt \right)^{\frac{1}{p}}
\]

is finite. The property (4.1) implies the inclusion $\rho H^p \subseteq H^p$, $p \in [1, \infty)$. Indeed, for $f \in \rho H^p$ and $r \in (0, 1)$ we have

\[
\left( \int_{-\pi}^{\pi} |f(re^{it}) - f(0)|^p dt \right)^{\frac{1}{p}} \leq C_\rho^{-1} \|f\|_{\rho H^p},
\]

which throughout some elementary inequalities easily implies that $f \in H^p$, since

\[
m_p(|f|, r) \leq 4\pi |f(0)| + 2C_\rho^{-1} \|f\|_{\rho H^p}.
\]

The main results of this section are stated in the following two theorems. Note that if we take $\rho \equiv 1$ and $D = \mathbb{C}$ (then $d_\rho$ is the Euclidean distance) Theorem 4.1 became Proposition 1.1 and Theorem 4.2 became one part of Proposition 1.2.

**Theorem 4.1.** An analytic mapping $f : \mathbb{D} \to D$ satisfies the condition

\[f^*(z) = O(1 - |z|)^{\alpha-1}, \quad |z| \to 1, \quad \alpha \in (0, 1].\]

if and only if it has continuous extension on $\overline{\mathbb{D}}$ and $f_b \in \rho \Lambda_\alpha$.

**Theorem 4.2.** If an analytic mapping $f : \mathbb{D} \to D$ satisfies

\[(4.2) \quad m_p(r, f^*) = O(1 - r)^{\alpha-1}, \quad r \to 1, \quad \alpha \in (0, 1], p \in [1, \infty)\]

then $f \in \rho H^p$ and $f_b \in \rho \Lambda^p_\alpha$.

The following fact will be used several times in the proofs of these theorems: For an analytic mapping $f : \mathbb{D} \to D$ we have

\[d(f(z), f(w)) \leq \int_\gamma f^*(\zeta)|d\zeta|,
\]

where $\gamma : [a, b] \to \mathbb{D}$ is a piecewise $C^1$-smooth curve which joins $z$ and $w$. Indeed, since $f \circ \gamma : [a, b] \to D$ is a piecewise $C^1$-smooth curve which connects $f(z)$ and
f(w), we have
\[ d(f(z), f(w)) \leq \int_{f \circ \gamma} \rho(\eta)|d\eta| = \int_a^b \rho(f(\gamma(t)))|f'(\gamma(t))||\gamma'(t)|dt \]
\[ = \int_\gamma \rho(f(\zeta))|f'(\zeta)||d\zeta| = \int_\gamma f^*(\zeta)|d\zeta|, \]
which we aimed to prove.

**Proof of Theorem 4.1.** Let us first assume that \( f^*(z) = O(1 - |z|)^{\alpha-1}, |z| \to 1. \)
Since \( f^*(z) = \rho(f(z))|f'(z)| \geq C_\rho|f'(z)|, \ z \in \mathbb{D}, \)
we also have \( |f'(z)| = O(1 - |z|)^{\alpha-1}, |z| \to 1. \) By Proposition 1.1, we conclude that \( f \) has continuous extension on \( \mathbb{D}, \) and moreover \( f_b \in \Lambda_\alpha. \)
It remains to show that \( f_b \in \rho \Lambda_\alpha. \)

Let \( C \) be a constant such that \( f^*(z) \leq C(1 - |z|)^{\alpha-1}, z \in \mathbb{D}. \)
For \( r \in (0, 1) \) and \( t \in [-\pi, \pi] \) we have
\[ d(f(re^{it}), f(0)) \leq \int_0^r f^*(xe^{it})dx \leq \int_0^1 f^*(xe^{it})dx \]
\[ \leq C \int_0^1 (1-x)^{\alpha-1}dx = \frac{C}{\alpha}. \]
Having in mind our assumption on the distance \( d, \) it follows that
\[ f_b(t) = \lim_{r \to 1} f(re^{it}) \in D, \quad t \in [-\pi, \pi]. \]

Let \( \tilde{d} \) be the half of the distance between \( \{ f_b(t) : t \in [-\pi, \pi] \} \) and \( \partial D \) (in the case \( D = \mathbb{C}, \) take \( \tilde{d} = 1). \) Denote
\[ M = \max_{\zeta \in \bigcup_{t \in [-\pi, \pi]} \mathbb{D}(f_b(t), \tilde{d})} \rho(\zeta). \]
Let \( h > 0 \) be such that \( Ch^\alpha < \tilde{d}. \) Because of uniform continuity of \( f_b, \) for sufficiently small \( h \) we have: \( |t-s| < h \Rightarrow |f_b(t) - f_b(s)| \leq Ch^\alpha < \tilde{d}. \) It follows
\[ d(f_b(t), f_b(s)) = \inf_\gamma \int_\gamma \rho(\zeta)|d\zeta| \leq \int_{[f_b(t), f_b(s)]} \rho(\zeta)|d\zeta| \]
\[ \leq M|f_b(t) - f_b(s)| \leq MCh^\alpha \]
(let us say that \( \gamma \subseteq D \) is among all piecewise \( C^1 \)-smooth curves that join \( f_b(t) \) and \( f_b(s), \) and \( [f_b(t), f_b(s)] \) is the segment \( \{ f_b(t) + \lambda(f_b(s) - f_b(t)), \lambda \in [0, 1]\} \)). This proves that \( f_b \in \rho \Lambda_\alpha. \)

In order to prove the converse, assume that \( f \) has continuous extension on \( \mathbb{D} \) with values in \( D, \) and \( f_b \in \rho \Lambda_\alpha. \) Then \( f_b \in \Lambda_\alpha, \) so we may apply the Hardy and Littlewood result given in Proposition 1.1. We conclude that there exists a constant \( B \) such that \( |f'(z)| \leq B(1 - |z|)^{\alpha-1}, z \in \mathbb{D}. \) Since \( f_b(t) = f(e^{it}) \) is continuous, let \( m = \max_{z \in \mathbb{D}} \rho(f_b(z)). \) By uniform continuity of \( \rho \circ f \) on \( \mathbb{D}, \) if \( r \) is sufficiently close to 1, we have \( \rho(f(z)) \leq 2m, \ z \in \mathbb{D} \setminus r \mathbb{D}. \) It follows that
\[ f^*(z) = \rho(f(z))|f'(z)| \leq 2m C(1 - |z|)^{\alpha-1}, \quad |z| > r, \]
i.e., \( f^*(z) = O(1 - |z|)^{\alpha-1}, \ |z| \to 1, \) which we aimed to prove. \( \square \)
Proof of Theorem 4.2. We shall first show that \( f \in \rho H^p \) which implies that \( f_b(t) \) is finite for a.e. \( t \in [-\pi, \pi] \). Then we prove that \( f_b(t) \in D \) for a.e. \( t \in [-\pi, \pi] \).

Assume that for a constant \( C \) there holds

\[
m_p(r, f^*) \leq C(1-r)^{\alpha-1}, \quad r \in (0, 1).
\]

Since

\[
d(f(re^{it}), f(0)) \leq \int_0^r f^*(xe^{it})dx,
\]

applying the Minkowski inequality, we obtain

\[
\left( \int_{-\pi}^{\pi} d(f(re^{it}), f(0))^p dt \right)^{\frac{1}{p}} \leq \left( \int_{-\pi}^{\pi} \left( \int_0^r f^*(xe^{it})dx \right)^p dt \right)^{\frac{1}{p}}
\]

\[
\leq \int_0^r \left( \int_{-\pi}^{\pi} f^*(xe^{it})^p dt \right)^{\frac{1}{p}} dx
\]

\[
\leq \int_0^r m_p(x, f^*)dx
\]

\[
\leq C \int_0^1 (1-x)^{\alpha-1}dx
\]

\[
= C \frac{1}{\alpha}.
\]

Therefore, \( f \in \rho H^p \). By the Fatou theorem and the above inequality we conclude that

\[
\int_{-\pi}^{\pi} \liminf_{r \to 1} d(f(re^{it}), f(0))^p dt
\]

is finite, which means that \( \liminf_{r \to 1} d(f(re^{it}), f(0)) \) is finite for a.e. \( t \in [-\pi, \pi] \).

By our assumption on the distance \( d \), we must have \( f_b(t) \in D \) for a.e. \( t \in [-\pi, \pi] \).

We now proceed to show that \( f_b \in \rho \Lambda^p \). For \( r \in (0, 1) \), \( t \in [-\pi, \pi] \), and sufficiently small \( s \in (0, \frac{r}{2}) \), let us consider the following piecewise \( C^1 \)-smooth curve contained in the unit disk:

\[
\gamma(x) = \begin{cases} 
(r-x)e^{it}, & 0 \leq x \leq s; \\
(r-s)e^{i\lambda(x)}, & s \leq x \leq r-s; \\
x e^{i(t+s)}, & r-s \leq x \leq r,
\end{cases}
\]

where

\[
\lambda(x) = \frac{s}{r-2s}(x-s) + t
\]

is the affine mapping that maps \([s, r-s]\) onto \([t, t+s]\). For this path we have

\[
d(f(re^{i(t+s)}), f(re^{it})) \leq \int_0^s f^*(z)|dz| = - \int_0^s f^*((r-x)e^{it})dx \\
+ \int_s^{r-s} f^*((r-s)e^{i\lambda(x)})(r-s)\lambda'(x)dx \\
+ \int_{r-s}^r f^*(xe^{i(t+s)})dx.
\]

Let \( \chi_A \) be the characteristic function of the set \( A = \{e^{i\lambda} : t \leq \lambda \leq t+s\} \subseteq \partial D \).

The second integral above may be estimated applying the Hölder inequality in the
Applying now the elementary inequality $(\alpha \beta \gamma \leq \beta \gamma + \gamma \beta)$, which results from setting $\beta = \alpha + \gamma$, we derive

$$\int_\pi^{r-s} f^\ast((r-s)e^{i\lambda})\lambda'(x)dx = \int_\pi^{r-s} f^\ast((r-s)e^{i\lambda})d\lambda$$

$$= \int_{-\pi}^\pi f^\ast((r-s)e^{i\lambda})\chi_A(\lambda)d\lambda$$

$$\leq \left( \int_{-\pi}^\pi f^\ast((r-s)e^{i\lambda})^p d\lambda \right)^{\frac{1}{p}} \left( \int_{-\pi}^\pi \chi_A(\lambda)^q d\lambda \right)^{\frac{1}{q}}$$

$$= m_p(r-s, f^\ast) \cdot s,$$

which results in

$$d(f(re^{i(t+s)}), f(re^{it})) \leq \int_{r-s}^r f^\ast(xe^{it})dx + \int_{r-s}^r f^\ast(xe^{i(t+s)})dx$$

$$+ s(r-s)m_p(r-s, f^\ast).$$

Applying now the elementary inequality $(\alpha + \beta + \gamma)^p \leq 3^{p-1}(\alpha^p + \beta^p + \gamma^p)$, where $\alpha$, $\beta$, $\gamma$ are non-negative numbers, then integrating against $t$ over $[-\pi, \pi]$, and finally applying the inequality $(\alpha + \beta + \gamma)^\frac{p}{q} \leq \alpha^\frac{p}{q} + \beta^\frac{p}{q} + \gamma^\frac{p}{q}$, we obtain

$$\left( \int_{-\pi}^\pi d(f(re^{i(t+s)}), f(re^{it}))^p dt \right)^{\frac{1}{p}} \leq 3^{\frac{1}{p}-\frac{1}{q}} \left( \int_{-\pi}^\pi \left( \int_{r-s}^r f^\ast(xe^{it})dx \right)^p dt \right)^{\frac{1}{p}}$$

$$+ 3^{\frac{1}{p}-\frac{1}{q}} \left( \int_{-\pi}^\pi \left( \int_{r-s}^r f^\ast(xe^{i(t+s)})dx \right)^p dt \right)^{\frac{1}{p}}$$

$$+ 3^{\frac{1}{p}-\frac{1}{q}} s(r-s)m_p(r-s, f^\ast) \cdot (2\pi)^{\frac{1}{p}}.$$

Applying the Minkowski inequality on the first and the second integral on the right side of the last inequality, we obtain

$$\left( \int_{-\pi}^\pi d(f(re^{i(t+s)}), f(re^{it}))^p dt \right)^{\frac{1}{p}} \leq 6 \int_{r-s}^r m_p(x, f^\ast)dx + 27s(r-s)m_p(r-s, f^\ast).$$

Now, having in mind (4.3), we derive

$$\left( \int_{-\pi}^\pi d(f(re^{i(t+s)}), f(re^{it}))^p dt \right)^{\frac{1}{p}} \leq \frac{6C}{\alpha} ((1-(r-s))^{\alpha} - (1-r)^{\alpha})$$

$$+ 27Cs(r-s)(1-(r-s))^{\alpha-1}.$$

If we let $r \to 1$, by applying the Fatou lemma on the term on the left side, we obtain

$$\left( \int_{-\pi}^\pi d(f_b(t+s), f_b(t))^p dt \right)^{\frac{1}{p}} \leq \frac{6C}{\alpha} s^\alpha + 27C(1-s)s^\alpha \leq \frac{33C}{\alpha} s^\alpha.$$

Therefore,

$$\sup_{s \in (0, h)} \left( \int_{-\pi}^\pi d(f_b(t+s), f_b(t))^p dt \right)^{\frac{1}{p}} \leq \frac{33C}{\alpha} h^\alpha,$$

which means that $f_b \in \rho\Lambda^p_\alpha$. \qed
5. Hardy and Littlewood theorems and invariant metrics

**Definition 5.1.** We say that a domain $D$ is $\mathcal{F}$-transitive, $\mathcal{F} \subseteq \text{Aut}(D)$, if there exists a compact set $K \subseteq D$ such that for every $\zeta \in D$ there exists $\varphi = \varphi_{\zeta} \in \mathcal{F}$ with $\varphi(\zeta) \in K$.

**Definition 5.2.** We say that a metric $\rho$ on a domain $D$ is $\mathcal{F}$-invariant, $\mathcal{F} \subseteq \text{Aut}(D)$, if the corresponding distance $d = d_{\rho}$ satisfies

\[
    d(\varphi(\zeta), \varphi(\eta)) = d(\zeta, \eta), \quad \zeta, \eta \in D
\]

for every $\varphi \in \mathcal{F}$.

Note that if a metric $\rho$ on a domain $D$ is $\mathcal{F}$-invariant, then we have

\[
    |\varphi'(\zeta)| = \frac{\rho(\zeta)}{\rho(\varphi(\zeta))}, \quad \zeta \in D
\]

for every $\varphi \in \mathcal{F}$. Indeed, for different $\zeta$ and $\eta$ the relation (5.1) may be rewritten as

\[
    \frac{d(\varphi(\zeta), \varphi(\eta))}{|\varphi(\zeta) - \varphi(\eta)|} = \frac{\varphi(\zeta) - \varphi(\eta)}{|\zeta - \eta|} = 1.
\]

If we let $\eta \to \zeta$ above, we obtain

\[
    \rho(\varphi(\zeta))|\varphi'(\zeta)|\rho(\zeta)^{-1} = 1,
\]

which gives (5.2).

The following result is the converse of Theorem 4.2 but in a special case.

**Theorem 5.3.** Assume that a domain $D$ is $\mathcal{F}$-transitive, $\mathcal{F} \subseteq \text{Aut}(D)$, and let $\rho$ be a metric on $D$ which is $\mathcal{F}$-invariant. If for an analytic mapping $f : \mathbb{D} \to D$ we have $f \in \rho\mathbb{H}^p$, $p \in [1, \infty)$, and $f_b \in \rho\Lambda^p_\alpha$, $\alpha \in (0,1]$, then

\[
    m_\rho(r, f^*) = \mathcal{O}(1 - r)^{\alpha - 1}, \quad r \to 1.
\]

Clearly, every simply-connected domain $D \subseteq \mathbb{C}$ with at least two boundary points is $\text{Aut}(D)$-transitive. If the domain $D$ is bounded, it is well known that the Bergman metric is $\text{Aut}(D)$-invariant. Hence, the preceding theorem gives the converse part of Theorem 5.2.

The domain $\mathbb{C}$ is $\{\varphi_b(z) = z + b : b \in \mathbb{C}\}$-transitive; for the role of the compact set one may take any one-point set. Note that this domain is also $\{\varphi_{n,m}(z) = z + m + in : m, n \in \mathbb{Z}\}$-transitive; here once should take $[-1,1] \times [-1,1]$ for the compact set. If we take $\rho \equiv 1$ on $\mathbb{C}$, then $d_\rho$ is the Euclidean distance which is $\{\varphi_b(z) = z + b : b \in \mathbb{C}\}$-invariant. Therefore, in this spacial case Theorem 5.3 is the converse part of the Hardy and Littlewood theorem given in Proposition 1.2.

**Proof of Theorem 5.3.** Let $K$ be a compact set as in Definition 5.1 and denote

\[
    C_K = \max_{\zeta \in K} \rho(\zeta).
\]

Let $z \in \mathbb{D}$ and select a conformal mapping $\varphi = \varphi_z \in \mathcal{F}$ (this mapping depends on $z$) such that $\varphi(f(z)) \in K$. In the sequel we shall consider the analytic mapping $g = g_z = \varphi \circ f : \mathbb{D} \to D$. It is easy to see (by the $\mathcal{F}$-invariance of the distance $d$) that $g \in \rho\mathbb{H}^p$ and $g_b \in \rho\Lambda^p_\alpha$. 
Using (5.2) we obtain

\[ f^*(z) = \rho(f(z))|f'(z)| = \rho(\varphi(f(z))) \frac{\rho(f(z))}{\rho(\varphi(f(z)))}|f'(z)| \]

\[ = \rho(\varphi(f(z)))|\varphi'(f(z))||f'(z)| \leq C_K(|\varphi \circ f)'(z)| = C_K|g'(z)|. \]

Therefore, we have proved

(5.3) \[ f^*(z) \leq C_K|g'(z)|, \quad z \in D. \]

Let \( z = re^{i\theta}, r \in (0, 1), \theta \in [-\pi, \pi] \). We are going to show that the \( \rho \)-derivative of \( f \) satisfies the appropriate growth condition. We separate the following two cases:

(i) \( \alpha \in (0, 1) \): here we need to prove that \( m_\rho(r, f^*) = O(1 - r)^{\alpha - 1}, r \to 1 \), and (ii) \( \alpha = 1 \), where we have to show that \( m_\rho(r, f^*) \) is bounded in \( r \in (0, 1) \).

(i) By the Cauchy integral theorem applied to \( g - g_\theta(\theta) \), from (5.3) we obtain

\[
f^*(re^{i\theta}) \leq C_K g'(re^{i\theta}) = C_K \left| \int_{-\pi}^{\pi} \frac{g_\theta(t) - g_\theta(\theta)}{(e^{it} - re^{i\theta})^{1/2}} \frac{dt}{2\pi} \right| \leq C_K \int_{-\pi}^{\pi} \frac{|g_\theta(t) - g_\theta(\theta)|}{|e^{it} - re^{i\theta}|^{1/2}} \frac{dt}{2\pi} \]

\[
\leq C_K C_\rho^{-1} \int_{-\pi}^{\pi} \frac{d(g_\theta(t), g_\theta(\theta))}{1 - 2r \cos(t - \theta) + r^2} \frac{dt}{2\pi} = C_K C_\rho^{-1} \int_{-\pi}^{\pi} \frac{d(g_\theta(t + \theta), g_\theta(\theta))}{1 - 2r \cos t + r^2} dt
\]

for a.e. \( \theta \in [-\pi, \pi] \). Since the distance \( d \) is invariant with respect to the conformal mapping \( \varphi \), for \( s \in (0, 1) \) we have

\[ d(g(se^{i(t+\theta)}), g(se^{i\theta})) = d(\varphi(f(se^{i(t+\theta)})), \varphi(f(se^{i\theta}))) = d(f(se^{i(t+\theta)}), f(se^{i\theta})). \]

Taking the limit as \( s \to 1 \), it follows

\[ d(g_\theta(t + \theta), g_\theta(\theta)) = d(f_\theta(t + \theta), f_\theta(\theta)) \text{ for a.e. } t \in [-\pi, \pi]. \]

Therefore, we may conclude that the following inequality holds

\[
f^*(re^{i\theta}) \leq C_K C_\rho^{-1} \int_{-\pi}^{\pi} \frac{d(f_\theta(t + \theta), f_\theta(\theta))}{1 - 2r \cos t + r^2} dt.
\]

Now, applying the Minkowski inequality, we obtain

\[
\left( \int_{-\pi}^{\pi} (f^*(re^{i\theta}))^p dt \right)^{\frac{1}{p}} \leq \left( \int_{-\pi}^{\pi} \left( \int_{-\pi}^{\pi} \frac{d(f_\theta(t + \theta), f_\theta(\theta))}{1 - 2r \cos t + r^2} dt \right)^p dt \right)^{\frac{1}{p}} \leq C_K C_\rho^{-1} \int_{-\pi}^{\pi} \left( \int_{-\pi}^{\pi} \frac{d(f_\theta(t + \theta), f_\theta(\theta))}{1 - 2r \cos t + r^2} dt \right)^p dt.
\]

Since \( f_\theta \in \rho \Lambda_\alpha \), there exist a constant \( C_\alpha \) such that

\[
\left( \int_{-\pi}^{\pi} d(f_\theta(\theta + t), f(\theta))^p d\theta \right)^{\frac{1}{p}} dt \leq C_\alpha |t|^\alpha.
\]

Having in mind the inequality

\[ 1 - 2r \cos t + r^2 = (1 - r)^2 + 4r \sin^2 \frac{t}{2} \geq (1 - r)^2 + \frac{4r t^2}{\pi}, \]

we can conclude that...
it follows
\[
\left( \int_{-\pi}^{\pi} f^*(re^{\theta})^p d\theta \right)^{\frac{1}{p}} \leq \frac{C_K C_{\rho}^{-1} C_{\alpha}}{2\pi} \int_{-\pi}^{\pi} \left| t^\alpha \right| \frac{1}{1 - 2r \cos t + r^2} dt
\]
\[
= \frac{C_K C_{\rho}^{-1} C_{\alpha}}{\pi} \int_0^\pi \frac{t^\alpha dt}{1 - 2r \cos t + r^2}
\]
\[
\leq \frac{C_K C_{\rho}^{-1} C_{\alpha}}{\pi} \int_0^\pi \frac{t^\alpha dt}{\left( 1 - r \right)^{\frac{\alpha+1}{2}} + \frac{4r^2}{\pi^2}} \quad (u = \frac{2\sqrt{r}}{\pi} t)
\]
\[
= \frac{C_K C_{\rho}^{-1} C_{\alpha}}{\pi} \left( \frac{\pi}{2\sqrt{r}} \right)^{\alpha+1} (1 - r)^{\alpha-1} \int_0^\pi \frac{u^\alpha du}{1 + u^2}
\]
\[
\leq \frac{C_K C_{\rho}^{-1} C_{\alpha}}{\pi} \left( \frac{\pi}{2\sqrt{r}} \right)^{\alpha+1} (1 - r)^{\alpha-1} \int_0^\infty \frac{u^\alpha du}{1 + u^2}
\]
\[
= \frac{C_K C_{\rho}^{-1} C_{\alpha}}{\pi} \left( \frac{\pi}{2\sqrt{r}} \right)^{\alpha+1} (1 - r)^{\alpha-1} \frac{1}{2 \cos(\frac{\pi}{2})},
\]
which proves that \( m_p(r, f^*) = O(1 - r)^{\alpha-1}, \ r \to 1. \)

(ii) Here we shall first note that \( f_b \) is absolutely continuous. Since \( f_b \in \rho \Lambda_p^1 \subseteq \rho \Lambda_p^1 \subseteq \Lambda_p^1 \), by the Hardy and Littlewood result [2, Lemma 1, p. 72] it follows that \( f_b \) coincides a.e. with a function of bounded variation on \([\pi, \pi]\). Now, by [2, Theorem 3.11 and Theorem 3.10] it follows that \( f_b \) is absolutely continuous and that \( f' \) belongs to \( H^1 \). Moreover, by the same theorems, we have the following connection between the derivatives
\[ f'_b(t) = ie^{it} f'(e^{it}) \] for a.e. \( t \in [-\pi, \pi] \).

In the same way, because \( g \in \rho H^p \) and \( g_b \in \rho \Lambda_p^1 \), we conclude that \( g_b \) is also absolutely continuous and \( g' \in H^1 \).

Since \( f_b \) and \( g_b \) are absolutely continuous, and \( f_b \in D \) for a.e. \( t \in [-\pi, \pi] \), we may conclude that \( g_b(t) = (\varphi \circ f_b)(t) \) and \( g'_b(t) = \varphi'(f_b(t)) f'_b(t) \) for almost every \( t \in [-\pi, \pi] \).

Since \( f_b \) belongs to the class \( \rho \Lambda_p^1 \), there exists a constant \( C \) such that
\[ \int_{-\pi}^{\pi} \frac{d(f_b(t + h), f_b(t))^p}{|h|^p} dt \leq C \]
for \( h \) sufficiently close to 0. If we let \( h \to 0 \) above, by the Fatou theorem we obtain
\[ (5.4) \quad \int_{-\pi}^{\pi} f^*_b(t)^p dt \leq C, \]
i.e., \( f^*_b \in L^p[-\pi, \pi] \). For a.e. \( t \in [-\pi, \pi] \) we have
\[ |g'_b(t)| = |\varphi'(f_b(t))| f'_b(t) = \frac{\rho(f_b(t))}{\rho(\varphi(f_b(t)))} f'_b(t) = \frac{f'_b(t)}{\rho(\varphi(f_b(t)))} \]
since \( \rho \geq C_{\rho} \) on \( D \).

We have \( g'(w) = P[g'(e^{it})](w) = P[-ie^{-it}g'_b(t)](w) \) (recall that \( g \in H^1 \)). Applying now the Jensen inequality and the inequality above: \( |g'_b(t)| \leq C_{\rho}^{-1} f'_b(t) \), we obtain
\[
|g'(w)|^p \leq |P[-ie^{-it}g'_b(t)](w)|^p \leq P[|g'_b|^p](w) \leq C_{\rho}^{-p} P[|f^*_b|^p](w), \quad w \in D.
\]
Particularly, for \( w = z \), having in mind the inequality (5.3), we derive

\[
(f^* (z))^p \leq C_K^p |g'(z)|^p \leq C_K^p C_{\rho}^{-p} P[f^*_b(t)^p](z).
\]

Applying the Fubini theorem we obtain

\[
\int_{-\pi}^{\pi} f^*(re^{i\theta})^p d\theta \leq C_K^p C_{\rho}^{-p} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r \cos(t - \theta) + r^2} d\theta \right) f^*_b(t)^p dt.
\]

Since the inner integral is equal to \( 2\pi \), the preceding inequality may be rewritten as

\[
m_p(r, f^*) \leq C_K C_{\rho}^{-1} \| f^*_b \|_p,
\]

which implies that \( m_p(r, f^*) \) is bounded in \( r \in (0, 1) \). \( \square \)

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