PSEUDO-RC-INJECTIVE MODULES
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ABSTRACT
The main purpose of this work is to introduce and study the concept of pseudo-rc-injective module which is a proper generalization of rc-injective and pseudo-injective modules. Numerous properties and characterizations have been obtained. Some known results on pseudo-injective and rc-injective modules generalized to pseudo-rc-injective. Rationally extending modules and semisimple modules have been characterized in terms of pseud-rc-injective modules. We explain the relationships of pseudo-rc-injective with some notions such as Co-Hopfian, directly finite modules.

Indexing terms/Keywords
Pseudo-injective modules; rc-injective modules; rc-quasi-injective; rationally closedsubmodules;pseudo-rc-injective modules; pseudo-c-injective modules; co-Hopfian modules.

Academic Discipline And Sub-Disciplines
Mathematic: algebra.

SUBJECT CLASSIFICATION
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1 INTRODUCTION

Throughout, R represent an associative ring with identity and all R-modules are unitary right modules.

Let M and N be two R-modules, N is called pseudo –M- injective if for every submodule A of M, any R-monomorphism f: A → N can be extended to an R-homomorphism α: M → N. An R-module M is called pseudo–injective, if it is pseudo N-injective. A ring R is said to be pseudo-injective ring, if R is pseudo-injective module (see [5] and [14]).

A submodule k of an R-module M is called rationally closed in M (denoted by K ≤rc M) if N has no proper rational extension in M [1]. Clearly, every closed submodule is rationally closed submodule and hence every direct summand is rationally closed, but the converse may not be true (see [1],[6],[9]).

M. S. Abbas and M. S. Nayeef in [3] introduce the concept of rc-injectivity. Let M1 and M2 be R-modules. Then M2 is called M1-rc-injective if for every homomorphism f: H → M2, where H is a ring rationally closed submodule of M1, can be extended to an R-module My: M1 → M2. An R-module M is called rc-injective, if M is N-rc-injective, for every R-module N. An R-module M is called rc-injective if M ≤rc M.

In [15], an R-module N is called pseudo–M-c-injective if for any homomorphism from a closed submodule of M to N can be extended to an R-module from M in to N, for every R-module M is called rationally extending or (RCS-module), if each submodule of M is rational in a direct summand. This is equivalent to saying that every rationally closed module of M is direct summand. It is clear that every rationally extending R-module is extending [1]. An R-module M is called to be Hopfian (Co-Hopfian), if every surjective (injective) endomorphism f: M → M is an isomorphism [16]. An R-module M is called directly finite if it is not isomorphic to a proper direct summand of M [10]. An R-module M is said to be monomorphic, if each submodule of M is rational [17].

2 Pseudo-rc-injective Modules

We start with the following definition.

Definition 2.1 Let M and N be two R-modules. Then N is pseudo M-rationally closed-injective (briefly pseudo M-rc-injective) if for every rationally closed submodule H of M, any R-homomorphism ψ: H → N can be extended to an R-homomorphism β: M → N. An R-module M is called pseudo-rc-injective, if N is pseudo M-rc-injective. A ring R is called self pseudo-rc-injective if it is a pseudo-R-rc-injective.

Remarks 2.2 (1) Every pseudo-injective module is rc-pseudo-injective. The converse may not be true in general, as following example let M = Z as Z-module. Then, clearly M is rc-injective, but Z is not pseudo-injective module. This shows that pseudo-rc-injective modules are a proper generalization of pseudo-injective.

(2) Clearly every rc-injective is pseudo-rc-injective. The converse may not be true in general. For example, [7, lemma 2], let M be an R-module whose lattice of submodules is

![Diagram]

Where N1 is not isomorphic to N2, and the endomorphism rings of N are isomorphic to Z/Z2 where i=1,2. S. Jain and S. Singh in [7] show that, M is pseudo-injective and hence by (1), M is pseudo-rc-injective which is not rc-quasi-injective, since N1 ⊕ N2 is rationally closed submodule of M and the natural projection of N1 ⊕ N2 onto Ni(i=1,2) can not be extended to an R-endomorphism of M [7]. Therefore, M is not rc-injective module. This shows that pseudo-rc-injective modules are a proper generalization of rc-injective modules.

(3) Obviously, every pseudo-rc-injective is pseudo M-c-injective. The converse is not true in general. For example, consider the two Z-modules M = Z/9Z and N = Z/3Z it is clear that N is pseudo M-c-injective but N is not pseudo M-rc-injective. This shows that pseudo-rc-injective modules are stronger than of rc-injective modules.

Proof: Let H < N, clearly H is rationally closed submodule of M, and define α: H → N by α(0) = 0, α(3) = 1, α(6) = 2. Obvious, α is Z-monomorphism. Now, suppose that N is pseudo M-rc-injective then there is β: M → N and β(1) = n for some n ∈ N. Hence β(3) = 3β(1) = 3n and hence 3n = β(3) = α(3) = 1, implies 3n = 1, a contradiction, this shows that, N is not pseudo M-rc-injective.

(4) For a non-singular R-module M, if N is pseudo M-c-injective then N is pseudo M-rc-injective.

(5) Every monomorphic R-module is pseudo-rc-injective.

(6) An R-isomorphic pseudo-rc-injective module is pseudo-rc-injective.
So, by above we obtain the following implications for modules.

- Injective → quasi-injective ⇔ pseudo-injective ⇔ pseudo-rc-injective ⇒ pseudo-c-injective.
- Rc-injective⇒ rc-quasi-injective ⇒ pseudo-rc-injective⇒ pseudo-c-injective.

In the following result we show that, for a uniform R-module the concepts of the rc-injective modules and pseudo-rc-injective are equivalents.

**Theorem 2.3** Let M be a uniform R-module. M is rc-injective if and only if M is a pseudo-rc-injective module.

**Proof:** (⇒) Obviously.

(⇐) Suppose that M is a pseudo-rc-injective, let K be a rationally closed submodule of M and α : K → M be R-homomorphism. Since M is uniform module, either α or K ↪ α is a R-monomorphism. First, if α is R-monomorphism, then by pseudo-rc-injectivity of M, there exists R-homomorphism g : M → M such that g ◦ i_k = α. Finally, if K ↪ α is R-monomorphism, then by pseudo-rc-injectivity of M, there exists g : M → M such that g ◦ i_k = I_k - α hence I_k - g = α. Therefore M is rc-injective.

**Proposition 2.4** Let N_1 and N_2 be two R-modules and N = N_1 ⊕ N_2. Then N_2 is pseudo N_1-rc-injective if and only if for every (rationally closed) submodule A of N such that A ∩ N_2 = 0 and π_1(A) rationally closed submodule of N_1 (where π_1 is a projection map from N onto N_1), there exists a submodule A’ of N such that A ≤ A’ and N = A’ ⊕ N_2.

**Proof:** Similar to proving [3, proposition (2.3)].

Some general properties of pseudo-rc-injectivity are given in the following results.

**Proposition 2.5** Let M and N_i (i ∈ I) be R-modules. Then \( \prod_{i∈I} N_i \) is pseudo M-rc-injective if and only if N_i is pseudo M-rc-injective, for every i ∈ I.

**Proof:** Follows from the definition and injections and projections associated with the direct product.

The following corollary is immediately from proposition (2.5).

**Corollary 2.6** Let M and N_i be R-modules where i ∈ I and I is a finite index set, if \( \bigoplus_{i∈I} N_i \) is pseudo M-rc-injective, ∀i ∈ I, then \( N_i \) is pseudo- M-rc-injective. In particular every direct summand of pseudo-rc-injective R-module is pseudo-rc-injective.

**Proposition 2.7** Let M and N be R-modules. If M is pseudo N-rc-injective, then M is pseudo A-rc-injective for every rationally closed submodule A of N.

**Proof:** Let A ≤ rc N and let K ≤ rc A, f: K → N be R-monomorphism. Then, by [2, Lemma (3.2)] we obtain, (K ≤ rc N, hence by pseudo N-rc-injectivity of M, there exists a R-homomorphism h: N → M such that h ◦ i_k = f where i_k: K ↪ A and i_k: A → N are inclusion maps. Let \( \varphi = h ◦ i_k \). Clearly, \( \varphi \) is R-homomorphism, and \( \varphi = h ◦ i_k = h ◦ i_k ◦ i_k = f \) Then \( \varphi \) is extends f. Therefore, M is A-rc-injective.

In [15] was proved the following: Suppose that R is a commutative domain. Let c be a non-zero non-unit element of R. Then R-module R@R/xR is not pseudo-c-injective. From this result and remark (2.2)(3), we conclude the following proposition for pseudo-rc-injective modules.

**Proposition 2.8** For a commutative domain R. Let x be a non-zero non-unit element of R. The R-module R@R/xR is not pseudo rc-injective.

Now, we investigate more properties of pseudo rc-injectivity.

The R-module M_1 and M_2 are relatively (mutually) pseudo-rc-injective if M_i is pseudo M_j -rc – injective for every i, j ∈ {1,2}, i ≠ j.

The following result is generalization of [5, Theorem (2.2)].

**Theorem 2.9** If M@N is a pseudo-rc-injective module, then M and N are mutually rc-injective.

**Proof:** Suppose that M@N is a pseudo-rc-injective module. Let B be a rationally closed submodule of N and α: B → M be an R-homomorphism. Define \( \varphi : B → M @ N \) by \( \varphi(b) = (α(b), b) \) for all b ∈ B, it is clear that \( \varphi \) is an R-homomorphism. Since N is isomorphic to a direct summand of M@N, then by remark (2.2)(3) and proposition(2.7), we have M@N is pseudo-rc N-injective, thus, there exists an R-homomorphism \( f: N → M @ N \) such that \( f = f ◦ i_N \) where \( i_N : B → N \) be the inclusion map. Let
\( \pi_i : M \oplus N \to M \) be natural projection of \( M \oplus N \) onto \( M \). We have \( \pi_i \circ g = \pi_i \circ f \circ \iota_p \) and hence \( \alpha = \pi_i \circ f \circ \iota_p \), thus \( \pi_i \circ f : N \to M \) is \( R \)-homomorphism extending \( \alpha \). This show that \( M \) is \( N \)-rc-injective. As same way we can prove that \( N \) is \( M \)-rc-injective.

Corollary 2.10 If \( \bigoplus_{\in I} M_i \) is a pseudo \( \text{-}rc \) – injective, then \( M_i \) is a \( M_j \)-rc-injective for all distinct \( i, j \in I \).

Corollary 2.11 For any positive integer \( n \geq 2 \), if \( M^n \) is pseudo \( rc \)-injective, then \( M \) is \( rc \)-quasi-injective.

The following example shows that the direct sum of two pseudo-rc-injective is not pseudo-rc-injective in general. For a prim \( p \), let \( M_1 = \mathbb{Z} \) and \( M_2 = \mathbb{Z}/p^2\mathbb{Z} \), be a right \( \mathbb{Z} \)-modules. Since \( M_1 \) and \( M_2 \) are monomorphic then \( M_1 \) and \( M_2 \) are pseudo-rc-injective. But, by proposition (2.8), we have \( M_1 \oplus M_2 \) is not pseudo-rc-injective module.

Now, we consider the sufficient condition for a direct sum of two pseudo-rc-injective modules to be pseudo-rc-injective.

Theorem 2.12 The direct sum of any two pseudo-rc-injective modules is pseudo-rc-injective if and only if every pseudo-rc-injective module is injective.

Proof: Let \( M \) be a pseudo-rc-injective module, and \( E(M) \) its injective hull of \( M \). By hypothesis, we have \( M \oplus E(M) \) is pseudo-rc-injective. Let \( i_M : M \to M \oplus E(M) \) be a natural monomorphism, then there exists an \( R \)-homomorphism \( \alpha : M \oplus E(M) \to \alpha(M) \oplus E(M) \) such that \( i_M = \alpha \circ i_M \), where \( i_M \) is the identity of \( M \) and \( \pi_M \) is a projection map from \( M \oplus E(M) \) onto \( M \). Therefore \( i_M = g \circ i_M \), where \( g = \pi_M \circ \alpha \circ i_M \). Thus by \( [8, \text{Corollary (3.4.10)}] \), we obtain \( E(M) = M \oplus \ker g \). Since \( M \cap \ker g = 0 \) and \( M \leq E(M) \), \( \ker g = 0 \) and hence \( M = E(M) \). This shows that \( M \) is injective module. The other direction is obvious.

Recall that an \( R \)-module \( M \) is a multiplication if, each submodule of \( M \) has the form \( IM \) for some ideal \( I \) of \( R \) [9].

Proposition 2.13 Every rationally closed submodule of multiplication pseudo-rc-injective \( R \)-module is pseudo-rc-injective.

Proof: Let \( A \) be a rationally closed submodule of a rationally closed submodule \( H \) of \( M \) and let \( f : A \to H \) be an \( R \)-monomorphism. Since \( H \) is a rationally closed of \( M \), it follows that by \([2, \text{Lemma (3.2)}]\), \( A \) is also a rationally closed submodule of \( H \). Since \( M \) is pseudo-rc-injective, then there exist an \( R \)-homomorphism \( \phi : M \to M \) that extends \( f \). Since \( M \) is multiplication module, we have \( H = MI \) for some ideal \( I \) of \( R \). Thus \( \phi|_H = \phi(H) = \phi(MI) = \phi(M)I \leq MI = H \). This show that \( H \) is pseudo-rc-injective.

In the following part we give characterizations of known \( R \)-modules in terms of pseudo-rc-injectivity.

We start with the following results which are given a characterization of rationally extending modules. Firstly, the following lemma is needed.

Lemma 2.14 Let \( A \) be rationally closed submodule of \( R \)-module \( M \). If \( A \) is pseudo \( M \)-rc-injective, then \( A \) is a direct summand of \( M \).

Proof: Since \( A \) is a pseudo \( M \)-rc-injective \( R \)-module, there exists an \( R \)-homomorphism \( f : M \to A \). That extends \( M \) is a direct summand of \( M \).

Proposition 2.15 An \( R \)-module \( M \) is rationally extending if and only if every \( R \)-module is pseudo \( M \)-rc-injective.

Proof: (\( \Rightarrow \)). It is similarly to prove \([3, \text{Proposition (2.4)}]\). (\( \Leftarrow \)). Follow from lemma (2.14).

Note that, by proposition (2.15), every rationally extending \( R \)-module is pseudo-rc-injective. But the converse is not true in general. As in the following example: consider the \( \mathbb{Z} \)-module \( M = \mathbb{Z}/p^2\mathbb{Z} \) where \( p \) is prime number. It is clear that, \( M \) is pseudo-rc-injective (in fact, \( M \) is \( rc \)-injective). Obviously, \( A = \langle P \rangle \) is rationally closed submodule of \( M \) but \( A \) is not direct summand of \( M \). Thus \( M \) is not rationally extending.

Theorem 2.16 For an \( R \)-module \( M \), the following statements are equivalent:

(1) \( M \) is rationally extending;
(2) Every \( R \)-module is an \( M \)-rc-injective;
(3) Every \( R \)-module is pseudo \( M \)-rc-injective;
(4) Every rationally closed submodule of \( M \) is an \( M \)-rc-injective;
(5) Every rationally closed submodule of \( M \) is a pseudo \( M \)-rc-injective.

Proof: (1) \( \Leftrightarrow \) (2) Follows from \([3, \text{Proposition (2.4)}]\).
(2) $\Rightarrow$ (4). Clear.

(4) $\Rightarrow$ (1). It is follows from lemma (2.14).

Now, (1) $\Rightarrow$ (3). It is follows from proposition (2.15).

(3) $\Rightarrow$ (5). It is obvious.

(5) $\Rightarrow$ (1). It is follows from lemma (2.14).

An $R$-module $M$ is directly finite if and only if $f \circ g = I_M$ implies that $g \circ f = I_M$ for all $f, g \in \text{End}(M)$ [10, proposition (1.25)]. The $Z$-module $Z$ is directly finite, but it is not co-Hopfian. In the following proposition we show that the co-Hopfian and directly finite $R$-modules are equivalent under pseudo-rc-injective property.

**Proposition 2.17** A pseudo-rc-injective $R$-module $M$ is directly finite if and only if it is co-Hopfian.

**Proof:** Let $\varphi$ be an injective map belong to $\text{End}(M)$ and $I$ is identity $R$-homomorphism from $M$ to $M$. By pseudo-rc-injectivity of $M$, there exists an $R$-homomorphism $\beta: M \to M$ such that $\beta \circ \varphi = I_M$. Since $M$ is directly finite, we have $\varphi \circ \beta = I_M$ which is shows that $\varphi$ is an $R$-automorphism. Therefore, $M$ is co-Hopfian. The other direction it is clear. □

The following corollary is immediately from proposition (2.17).

**Corollary 2.18** An rc-injective $R$-module $M$ is directly finite if and only if it is Co–Hopfian. □

Since every indecomposable module is directly finite then by proposition (2.17), we obtain the following corollary.

**Corollary 2.19** If $M$ is an indecomposable pseudo-rc-injective module then $M$ is a Co-Hopfian. □

In [33] was proved that every Hopfian $R$-module is directly finite. Thus the following result follows from proposition (2.17).

**Corollary 2.20** If $M$ is a pseudo-rc-injective and Hopfian $R$-module. Then $M$ is a Co-Hopfian. □

For any an $R$-module $M$ we consider the following definition.

**Definition 2.21** An $R$-module $M$ said to be complete rationally closed module (briefly CRC module), if each submodule of $M$ is a rationally closed. It is clear that every semisimple module is CRC module, but the converse is not true in general.

For example $Z_4$ as $Z$-module is CRC module, but not semisimple since $<2>$ is not direct summand of $Z_4$.

An $R$-module $M$ is said to be satisfies (C_2)-condition, if for each submodule of $M$ which is isomorphic to a direct summand of $M$, then it is a direct summand of $M$. Recall that an $R$-module $M$ is said to satisfy the generalized C_2-condition (or GC_2) if, any $N \leq M$ and $N \cong M$, $N$ is a summand of $M$.

The following result is a generalization of [5, Theorem (2.6)].

**Proposition 2.22** Every pseudo-rc-injective CRC module satisfies C_2 (and hence GC_2).

**Proof:** Let $M$ be a pseudo-rc-injective CRC module, let $H \leq M$ and $K \leq M$ such that $H$ is isomorphic to $K$ with $H \cong K$. Since $M$ is a pseudo-rc- injective then by corollary (2.6), we obtain $H$ is a pseudo-rc-injective. But $H \cong K$ thus, by remark (2.2)(9), $K$ is a pseudo-rc-injective. By assumption, we have $K$ is rationally closed sub module of $M$. Thus, by Lemma (2.14), we get $K \leq M$. Hence $M$ satisfies C_2. The last fact follows easily. □

Although the $Z$-module $M = Z$ is a pseudo-rc-injective, but it is not satisfies C_2, since there is a submodule $H = nZ$ (where $n \geq 2$) of which is isomorphic to $M$ but it is not a direct summand in $M$. This shows that the CRC property of the module in proposition (2.22) cannot be dropped.

In [4], an $R$-module $M$ is called direct-injective, if given any direct summand $K$ of $M$, an injection map $j_K: K \to M$ and every $R$-monomorphism $\alpha: K \to M$, there is an $R$-endomorphism $\beta$ of $M$ such that $\beta \circ j_K = j_K$.

In [11, Theorem (7.13)], it was proved that, an $R$-module $M$ is a direct-injective if and only if $M$ satisfies (C_2)-condition. Thus by proposition (2.22) we can conclude the following result.

**Proposition 2.23** Every pseudo-rc-injective CRC module is direct-injective. □

In [13, p.32], recall that a right $R$-module $M$ is called divisible, if for each $m \in M$ and for each $r \in R$ which is not left zero-divisor, there exist $m^* \in M$ such that $m = m^*r$. In [4] was proved that every direct-injective $R$-module is divisible. Thus we have the following corollary which follows from proposition (2.23).

**Corollary 2.24** Every pseudo-rc-injective CRC module is divisible.
Recall that an $R$-module $M$ is self-similar if, every submodule of $M$ is isomorphic to $M$. The $Z$-module $Z$ is both self-similar and pseudo-rc-injective module but it is not semisimple and CRC module. Also, $Z_4$ as $Z$-module is pseudo-rc-injective CRC module but it is not self-similar module. Note that from above examples the concepts CRC-modules and self-similar modules are completely different.

In the following result we show that the pseudo-rc-injective and semisimple $R$-modules are equivalent under self-similar CRC modules.

**Theorem 2.25** Let $M$ is a self-similar CRC module. Then the following statements are equivalent:

(i) $M$ is semisimple module;

(ii) $M$ is pseudo-rc-injective.

**Proof:** (i) $\Rightarrow$ (ii). Clear.

(ii) $\Rightarrow$ (i). Let $K$ be any submodule of $M$, then by self-similarity of $M$, we have $K$ is isomorphic to $M$. Since $M$ is pseudo-rc-injective CRC module thus, by proposition (2.22), $M$ satisfy GC$_2$-condition. So, $K$ is a direct summand of $M$, therefore, $M$ is semisimple module.

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