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Abstract

We define a tau function for a generic Riemann-Hilbert problem posed on a union of non-intersecting smooth closed curves with jump matrices analytic in their neighborhood. The tau function depends on parameters of the jumps and is expressed as the Fredholm determinant of an integral operator with block integrable kernel constructed in terms of elementary parametrices. Its logarithmic derivatives with respect to parameters are given by contour integrals involving these parametrices and the solution of the Riemann-Hilbert problem. In the case of one circle, the tau function coincides with Widom’s determinant arising in the asymptotics of block Toeplitz matrices. Our construction gives the Jimbo-Miwa-Ueno tau function for Riemann-Hilbert problems of isomonodromic origin (Painlevé VI, V, III, Garnier system, etc) and the Sato-Segal-Wilson tau function for integrable hierarchies such as Gelfand-Dickey and Drinfeld-Sokolov.

1 Introduction

Tau functions play a central role in the theory of integrable equations, both in fields of isospectral and isomonodromic deformations. They had been introduced in the 80s by the Kyoto school, with the explicitly stated aim\textsuperscript{4} [JMU] to construct a generalization of the theta functions appearing, since Riemann\textsuperscript{4} [Rie], as particular solutions of some non–linear equations\textsuperscript{4}.

In the theory of isomonodromic deformations, tau functions are constructed starting from a certain differential 1-form $\omega_{JMU}$ defined on the space of the deformation parameters [JMU]. Under the hypothesis that the parameters are of isomonodromic type, the form $\omega_{JMU}$ is closed and the tau function $\tau_{JMU}$ is defined (locally and up to a multiplicative constant) by the formula

$$d\ln \tau_{JMU} := \omega_{JMU},$$

(1.1)

where $d$ denotes the total differential.

Quite differently, on the side of isospectral deformations, Sato\textsuperscript{5} defined the tau function starting from his interpretation of the KP hierarchy in terms of the geometry of Grassmannian manifolds. Namely, to each solution of the KP hierarchy, one can associate a point $W$ in an infinite dimensional Grassmannian, and the related tau function is nothing but the formal series

$$\tau_W := \sum_{\gamma \in \Sigma} s_{\gamma} W_{\gamma},$$

(1.2)

where $\Sigma$ is the set of partitions, $\{s_{\gamma}\}$ are the Schur polynomials and $\{W_{\gamma}\}$ is the set of the Plücker coordinates of $W$. In\textsuperscript{5} [SW], Segal and Wilson provided an analytic version of Sato’s theory, where formal series are replaced by $L^2$ functions, and rewrote the tau function as the (analytically well-defined) Fredholm determinant of a certain projection operator.

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\textsuperscript{4}See also the work of S. Kovalevskaya\textsuperscript{4} and the more recent ones on finite-gap integration; e.g. [DMN, Mat] and references therein.
Since the 80s, many generalizations of both definitions had been constructed; giving a complete account of the literature on the subject is out of the scope of this introduction. The generalizations touch different branches of mathematics as diverse as the representation theory of infinite-dimensional Lie algebras [Kac], Frobenius manifolds [DZ], instanton counting [Nek, NO], Riemann-Hilbert boundary value problems [Ber1, ILP] and topological recursion [Lyn], to name few of them. The reasons of such a flourishing literature, from our point of view, are to be found on the side of applications: while the several different definitions of tau functions could seem very abstract, the explicit computation of some of them are important for a growing mathematical community working on e.g. random matrix theory, statistical models, algebraic and symplectic enumerative geometry.

The aim of this paper is to show that, at least for a very wide array of examples touching both the worlds of isomonodromic deformations, tau functions coincide with a pretty simple object whose introduction by Widom goes back to 1976 [W2], before the very first seminal papers of the Kyoto school. Namely, they are the (Szegő-)Widom constants associated to matrix–valued symbols \( J \) is a circle centered at the origin. Recall that, given a symbol \( J(z) = \sum_{k \in \mathbb{Z}} J_k z^k \), the associated \( n \)-th block Toeplitz matrix is defined by \( T_n(J) := (J_{k-l})_{k,l=1}^n \). The asymptotics of \( T_n(J) \) had been extensively studied in the operator theory literature, very often with motivations coming from applications in statistical mechanics such as, for instance, the Ising model [DIK]. In particular, a celebrated theorem of Widom [W2] states that, under certain analytical conditions on the symbol \( J \),

\[
\lim_{n \to \infty} G(J)^{-n} \det T_n(J) = \tau(J).
\]

Here \( G(J) = \exp \frac{1}{2\pi i} \oint_{\mathcal{C}} z^{-1} \ln \det J(z) \) and \( \tau(J) \), which is known nowadays as the Widom constant, is the Fredholm determinant (the notations are explained in the next section)

\[
\tau(J) = \det_{H_+}(\Pi_+, J^{-1} \Pi_+, J).
\] (1.3)

Indeed, this is a highly non-trivial extension of the celebrated strong Szegő theorem [Sz], treating the case of scalar Toeplitz determinants.

On the isospectral side, the coincidence between the Widom constant and the Sato-Segal-Wilson tau function had been established, for the so-called Gelfand-Dickey hierarchies, by one of the authors in [Caf], and successively extended to the Drinfeld-Sokolov hierarchies associated to an arbitrary Kac-Moody algebra in [CW1]. Based on this identification, effective computations had been carried out in [CW2, CDD] for topological and polynomial tau functions.

In this work we show that, quite surprisingly, the recent results of [GL16, GL17], inspired by the isomonodromy/CFT/gauge theory correspondence, lead to a Fredholm determinant representation of the isomonodromic tau functions which is ultimately the same as given in (1.3). This implies, in particular, that the combinatorial expansion of the Sato’s tau function and the much more recent series representations for the isomonodromic tau functions of Painlevé VI, V and III equations [GL12, GL13] (at 0), which originated from the AGT correspondence, are both of the same nature. Namely, they are nothing but the expansions of the Fredholm determinant [L3], their terms being products of Plücker coordinates of subspaces in the Sato-Segal-Wilson Grassmannian.

It turns out to be very fruitful to consider the symbol \( J(z) \) as a jump matrix for a pair of Riemann-Hilbert problems (RHPs) on the circle \( \mathcal{C} \). To construct Fredholm determinant and series representations for the tau functions of more general isomonodromic problems (e.g. the Garnier system), one needs to consider RHPs set on a union of non-intersecting ovals. In the present paper, we show how the definition [L3] of \( \tau(J) \) can be generalized in this case and prove a formula for the log-differential of the appropriate extension of the Widom’s determinant with respect to parameters of the jumps, which leads to its identification with the Jimbo-Miwa-Ueno tau function for RHPs of isomonodromic origin.

We expect that the identification between the Widom constants and isomonodromic tau functions will lead to an effective way to compute the latter in the so far unsolved problems. These include, in particular, the construction of explicit asymptotic expansions of irregular type (at \( \infty \)) for Painlevé I–V transcendents, cf [BLMST, Nag]. Furthermore, the results of [BSh, JNS] on \( q \)-Painlevé III and \( q \)-Painlevé VI equation give a hope that our approach may also be adapted to the \( q \)-difference setting.
Starting from the foundational work \cite{Brezin}, the standard scheme of asymptotic analysis of Painlevé transcendents \cite{FIKN} is to construct an approximate solution of the appropriate RHP from solutions of “elementary” RHPs (parametrices), and then to extract the asymptotics of Painlevé functions from this approximation. The main ideological shift of our approach is that it gives an exact Fredholm determinant expression for the tau functions in terms of parametrices, which define the relevant integrable kernels. The Fredholm determinant yields, with a relatively little effort, a complete asymptotic expansion of the tau function. The solution of the RHP (exact or approximate) is not needed at all, even though it can also be expressed via the resolvent of the appropriate integral operator.

Let us now briefly describe the organization of the paper. After introducing basic notations and recalling relevant results in Subsection 2.1, we show that the Widom’s determinant $\tau [J]$ admits a combinatorial series expansion whose individual terms are indexed by tuples of Young diagrams and are given by products of minors of Cauchy-Plemelj operators. In Subsection 2.3 we explain how $\tau [J]$ appears in the isomonodromy theory considering the example of Fuchsian systems with 4 regular singular points and systems with 2 irregular singularities; relations to previously known results on integrable hierarchies are also discussed. Section 3 is devoted to Riemann-Hilbert problems posed on a union of non-intersecting smooth closed curves. Specifically, we propose an extension of $\tau$ to Riemann-Hilbert problems posed on a union of non-intersecting smooth closed curves. They ask to find GL$(N, C)$ such that $\text{det} H,$ well-defined. The dual RHP is solvable iff the operator $P := \Pi_+ J^{-1}$ is invertible on $H_+$, in which case its inverse is given by $P^{-1} = \Psi_+^{-1} \Pi_+ \Psi_+$. Likewise, the direct RHP is solvable iff the operator $Q := \Pi_- J$ is invertible, the inverse being equal to $Q^{-1} = \Psi_-^{-1} \Pi_- \Psi_-$. If the direct or dual RHP is not solvable, then either $P$ or $Q$ has a nontrivial kernel and $\tau [J]$ clearly vanishes.

Suppose that $J (z)$ admits a direct factorization \eqref{2.1a}. Define two Cauchy-Plemelj operators on $H$,

$$ a_H = \Psi_+ \Pi_+ \Psi_+^{-1} - \Pi_+ , \quad d_H = \Psi_- \Pi_- \Psi_-^{-1} - \Pi_- . $$

2 One-circle case

2.1 Widom formulas

Let $\mathcal{C} \subset \mathbb{C}^1$ be an anticlockwise oriented circle centered at the origin, and let $f^{[+]}$ and $f^{[-]}$ denote its interior and exterior. Pick a loop $J : \mathcal{C} \to \text{GL}(N, C)$ that can be analytically continued into a fixed annulus $\mathcal{A} > \mathcal{C}$ and such that $\text{det} J (z)$ has no winding along $\mathcal{C}$. We are going to associate to the pair $(\mathcal{C}, J)$ two Riemann-Hilbert problems (RHPs). They ask to find $\text{GL}(N, C)$ matrix functions $\Psi_\pm (z), \Psi_\pm (z)$ analytic in $f^{[\pm]}$ whose boundary values on $\mathcal{C}$ satisfy

\begin{align}
\text{direct RHP:} \quad J (z) &= \Psi_+ (z)^{-1} \Psi_+ (z) , \quad (2.1a) \\
\text{dual RHP:} \quad J (z) &= \Psi_+ (z) \Psi_- (z)^{-1} . \quad (2.1b)
\end{align}

It is a classical fact that $J (z)$ admits Birkhoff factorizations

\begin{equation}
J (z) = Y_+ (z)^{-1} z^D Y_+ (z) = \tilde{Y}_+ (z) \tilde{z}^{\tilde{D}} \tilde{Y}_- (z)^{-1} , \quad (2.2)
\end{equation}

where $Y_\pm (z), \tilde{Y}_\pm (z)$ can be continued to analytic functions in $f^{[\pm]} \cup \mathcal{A}$, and $D = \text{diag}(d_1, \ldots, d_N), \tilde{D} = \text{diag}(\bar{d}_1, \ldots, \bar{d}_N)$ with all $d_k, \bar{d}_k \in \mathbb{Z}$ such that $\sum_{k=1}^N d_k = \sum_{k=1}^N \bar{d}_k = 0$. The sets $\{d_k\}$ and $\{\bar{d}_k\}$ of partial indices are uniquely determined by $J$. The direct (dual) RHP is solvable iff $D = 0$ (resp. $\tilde{D} = 0$).

Introduce the Hilbert space $H = L^2 (\mathcal{C}, C^N)$. Its elements will be regarded as column vector functions. This space can be decomposed as $H = H_+ \oplus H_-$, where the functions from $H_+$ (and $H_-$) continue analytically inside $\mathcal{C}$ (resp. outside $\mathcal{C}$ and vanish at $\infty$). We denote by $\Pi_\pm$ the projections on $H_\pm$ along $H_\mp$.

**Definition 2.1.** The tau function of the RHPs defined by $(\mathcal{C}, J)$ is defined as Fredholm determinant

\begin{equation}
\tau [J] = \det_{H_+} (\Pi_+ J^{-1} \Pi_+) . \quad (2.3)
\end{equation}

The operator $\Pi_+ J^{-1} \Pi_+$ is known to be a trace class perturbation of the identity on $H_+$, which makes the determinant \eqref{2.3} well-defined. The dual RHP is solvable iff the operator $P := \Pi_+ J^{-1}$ is invertible on $H_+$, in which case its inverse is given by $P^{-1} = \Psi_+^{-1} \Pi_+ \Psi_+^{-1}$. Likewise, the direct RHP is solvable iff the operator $Q := \Pi_- J$ is invertible, the inverse being equal to $Q^{-1} = \Psi_-^{-1} \Pi_- \Psi_-$. If the direct or dual RHP is not solvable, then either $P$ or $Q$ has a nontrivial kernel and $\tau [J]$ clearly vanishes.
They can be explicitly written as integral operators

\[
(a_{HG})(z) = \frac{1}{2\pi i} \oint_{\mathcal{E}} a(z, z') \, g(z') \, dz', \quad (d_{HG})(z) = \frac{1}{2\pi i} \oint_{\mathcal{E}} d(z, z') \, g(z') \, dz',
\]

where

\[
a(z, z') = \frac{1 - \Psi_+(z) \Psi_+^{-1}(z')}{z - z'}, \quad d(z, z') = \frac{\Psi_-(z) \Psi_-^{-1}(z') - 1}{z - z'}.
\]

The integral kernels \(a(z, z')\) and \(d(z, z')\) have integrable form, are not singular on the diagonal \(z = z'\) and extend to analytic functions on \(\mathcal{A} \times \mathcal{A}\). Since \(\text{im} \, a_H \subseteq H_+ \subseteq \ker a_H, \text{im} \, d_H \subseteq H_- \subseteq \ker d_H\), it is convenient to consider the restrictions

\[
a = a_H|_{H_-} : H_- \to H_+,
\]

\[
d = d_H|_{H_-} : H_+ \to H_-.
\]

**Lemma 2.2.** If the direct RHP \((2.1a)\) is solvable, then \(\tau(J)\) admits Fredholm determinant representation

\[
\tau(J) = \det_H (I + L), \quad L = \begin{pmatrix} 0 & a \\ d & 0 \end{pmatrix} \in \text{End}(H_+ \oplus H_-),
\]

where integral operators \(a\) and \(d\) have block integrable kernels defined by \((2.4)\).  

**Proof.** We have

\[
\det_H (I + L) = \det_H (I - \text{ad}) = \det_H (I - \Psi_+ \Psi_+^{-1} \Pi_- \Psi_-^{-1}) = \det_H (I - \Pi_+ J^{-1} \Pi_- J),
\]

where the last equality is obtained by conjugating the action on \(H_+\) by multiplication by \(\Psi_+\). Replacing \(\Pi_- = 1 - \Pi_+\) in the last expression, we obtain the determinant \((2.3)\).

Of course, an analog of the representation \((2.3)\) can also be written for the dual factorization \((2.1b)\). However, from the point of view of applications it is convenient to consider the direct factorization as given; the relevant matrix functions \(\Psi_\pm\) define the jump \(J\). The Fredholm determinant \((2.3)\) then yields an explicit representation for \(\tau(J)\), whereas the dual factorization remains to be found.

Let us now recall a formula for derivatives of \(\tau[J]\) with respect to parameters of the jump matrix. It appeared as an intermediate result in the work of Widom [W1] in 1974 and was rediscovered in [IK] more than 30 years later. As we explain below, this result is a precursor of the Jimbo-Miwa-Ueno definition of the isomonodromic tau function [MU].

**Theorem 2.3.** Consider a smooth family of \(\text{GL}(N, \mathbb{C})\)-loops \((z, t) \mapsto J(z, t)\) which depend on an additional parameter \(t\) and admit both factorizations \((2.1)\). Then

\[
\partial_t \ln \tau(J) = \frac{1}{2\pi i} \oint_{\mathcal{E}} \text{Tr} \{ J^{-1} \partial_t J [\partial_z \Psi_+ \Psi_+^{-1} + \partial_z \Psi_- \Psi_-^{-1}] \} \, dz.
\]

**Proof.** Using previously defined operators \(P = \Pi_+ J^{-1}, \ Q = \Pi_+ J\) as well as their inverses \(P^{-1} = \Psi_+ \Pi_+ \Psi_-^{-1}, \ Q^{-1} = \Psi_+^{-1} \Pi_+ \Psi_-\), we may write

\[
\partial_t \ln \tau(J) = \partial_t \ln \det_H, \quad P = \text{Tr}_H \left( \partial_t P \, P^{-1} + Q^{-1} \partial_t Q \right) = \text{Tr}_H \left( -\Pi_+ J^{-1} \partial_t J J^{-1} \Pi_+ \Psi_+ \Psi_-^{-1} \right) \partial_t \Pi_+ J. \tag{2.7}
\]

Let us simplify this expression. First observe that since \(\Pi_+ \Psi_+ \Pi_+ = \Psi_+ \Pi_+ \Psi_-\), \(\Pi_+ \Psi_- \Pi_- = 0\), the expression under trace (recall that this is an operator on \(H_+\)) can be rewritten as

\[
-\Pi_+ J^{-1} \partial_t J \Psi_+ \Pi_+ \Psi_-^{-1} + \Psi_+^{-1} \Pi_+ \Psi_+ J^{-1} \partial_t J = \Pi_+ J^{-1} \partial_t J (\Psi_+ \Pi_+ \Psi_-^{-1} - \Pi_-) + (\Psi_+^{-1} \Pi_+ \Psi_+ - \Pi_+) J^{-1} \partial_t J.
\]

The operators \(a'_{H} = \Psi_+^{-1} \Pi_+ \Psi_-^{-1} - \Pi_-\), \(d_{H} = \Psi_+ \Pi_+ \Psi_-^{-1} - \Pi_+\) on \(H\) have the properties \(\text{im} \, a'_{H} \subseteq H_+, \ H_- \subseteq \ker d_{H}\). This allows to extend the trace in \((2.7)\) to the whole space \(H\) and to rewrite it as

\[
\partial_t \ln \tau(J) = \text{Tr}_H \left( J^{-1} \partial_t J \{ a'_{H} + d_{H} \} \right). \tag{2.8}
\]
Since $J^{-1} \partial J$ is a multiplication operator, to compute the last trace it suffices to know expressions for the kernels $a'(z, z'), \tilde{d}(z, z')$ of the integral operators $a_H, \tilde{d}_H$ along the diagonal $z = z'$. From the respective counterparts of the formulas (2.4) it follows that

$$a'(z, z) = \Psi_+(z)^{-1} \partial_z \Psi_+(z), \quad \tilde{d}(z, z) = \partial_z \tilde{\Psi}_-(z) \tilde{\Psi}_-(z)^{-1}.$$ 

In combination with (2.6), this yields the statement of the theorem. □

**Remark 2.4.** Theorem 2.3 and its proof above clearly remain valid if we replace the circle $\gamma$ of [W1][H1K] by any simple closed curve and consider the loops $I$ that continue to its tubular neighborhood. Notice that the right side of (2.6) is closely related to the Malgrange-Bertola 1-form [Mal][Ber1]

$$\omega_{\text{MB}}(\delta) := \frac{1}{2\pi i} \int_{\gamma} \text{Tr} \left\{ J^{-1} \partial J \Psi_+^{-1} \partial_z \Psi_+ \right\} dz,$$

where $\delta$ is an arbitrary vector field on the space of parameters of $J$. The curvature of the latter form does not necessarily vanish. Its analog defined by (2.6) is on the other hand closed by construction, i.e. the contributions to curvature from the direct and dual factorization counterbalance each other. An attempt to relate the tau function defined by $\omega_{\text{MB}}$ (in those cases where $\omega_{\text{MB}}$ is closed) to Fredholm determinants with integrable kernels was made in [Ber2]. It corresponds to decomposition of the initial RHP into a sequence of auxiliary RHPs with simpler matrix structure of the jumps, but does not seem to us to be directly related to our construction.

### 2.2 Combinatorial expansion

In this subsection we briefly outline some of the results on expansions of the Fredholm determinant $\tau[J]$. While the previous works [GL16][GL17] focus on RHPs of isomonodromic origin, the combinatorial structure remains the same for generic jump $J$.

#### 2.2.1 Maya and Young diagrams

Let $Z' = Z + \frac{1}{2}$ be the half-integer lattice, $Z'_\pm = Z'_\geq 0$, and let $\text{Conf}(Z') = \{0, 1\}^{Z'}$ be the set of all finite subsets of $Z'$. The elements $X \in \text{Conf}(Z')$ determine the positions of particles $p_X := X \cap Z'_+$ and holes $h_X := X \cap Z'_-$, thereby defining point configurations on $Z'$. A configuration $X$ may be alternatively represented by

- A Maya diagram $m_X$ obtained by drawing filled circles at sites $(Z'_+ \setminus p_X) \cup h_X$ and empty circles at $p_X \cup (Z'_- \setminus h_X)$, see Fig. 1. The charge of $m_X$ is defined as $Q_X = |p_X| - |h_X|$. The set of all Maya diagrams will be denoted by $\mathcal{M}$.

- A charged partition $(Y_X, Q_X) \in \mathcal{Y} \times Z$ where $\mathcal{Y}$ denotes the set of partitions $Y \equiv \{Y_1 \geq Y_2 \geq \ldots \geq 0\}$, with all $Y_k \in Z_{\geq 0}$. The partitions are identified with Young diagrams in the usual way. The Maya diagram corresponding to a charged partition $(Y_X, Q_X)$ can be described by the positions of empty circles, given by $\{Y_k - k + \frac{1}{2} + Q_X\}_{k=1}^{\infty}$, cf Fig. 1.

Let $L \in C^{X \times X}$ be a matrix indexed by a discrete set $X$. The latter can be infinite, in which case $L$ is required to be a trace class operator on $L^2(X)$. The determinant $\det(I + L)$ can be expressed as the sum of principal minors enumerated by all possible subsets of $X$:

$$\det(I + L) = \sum_{Q \in \{0, 1\}^X} \det L_Q,$$

where $L_Q$ is the restriction of $L$ to rows and columns $Q$. □

In order to apply this formula to the determinant (2.5), rewrite the integral operators $a$ and $d$ in the Fourier basis. Their kernels (2.4) may be expressed as

$$a(z, z') = \sum_{p, q \in \mathbb{Z}_+} a_{p, q} z^{-\frac{1}{2} + p} z'^{-\frac{1}{2} + q}, \quad d(z, z') = \sum_{p, q \in \mathbb{Z}_+} d_{p, q} z^{-\frac{1}{2} - q} z'^{-\frac{1}{2} - p},$$

where

$$a_{p, q} := \frac{1}{\pi i} \int_{\gamma} \text{Tr} \left\{ J^{-1} \partial J \Psi_+^{-1} \partial_z \Psi_+ \right\} dz,$$

$$d_{p, q} := \frac{1}{\pi i} \int_{\gamma} \text{Tr} \left\{ J^{-1} \partial J \tilde{\Psi}_-^{-1} \partial_z \tilde{\Psi}_- \right\} dz.$$
where the coefficients $a_{pq}^d$ are themselves $N \times N$ matrices whose elements we write as $a_{pq}^d = \delta_{pq}^{\alpha} d_{p}^{\beta}$. The "color" indices $\alpha, \beta = 1, \ldots, N$ correspond to $GL(N, \mathbb{C})$-matrix structure of the RHP defined by the loop $J$. The principal minors of $L$ in (2.5) are therefore labeled by $N$-tuples of Maya diagrams

$$m = (m_1, \ldots, m_N) = (p, h) \in \mathbb{M}^N,$$

$$p = p_1 \cup \ldots \cup p_N, \quad h = h_1 \cup \ldots \cup h_N.$$

Here $p_\alpha \in \{0, 1\}^{Z|J|}$, $h_\alpha \in \{0, 1\}^{Z|J|}$ denote the positions of particles and holes of color $\alpha \in \{1, \ldots, N\}$. The minors with $|p| \neq |h|$ clearly vanish, cf Fig. 2. We may thus restrict the summation to $N$-tuples of Maya diagrams of zero total charge,

$$\tau [J] = \sum_{m \in \mathbb{M}^N : |p| = |h|} Z_{m}^{[+]} Z_{m}^{[-]},$$

(2.11)

$$Z_{m}^{[+]} = \det a_{p}^{h}, \quad Z_{m}^{[-]} = (-1)^{|p|} \det a_{p}^{h}.$$

The matrices $a_{p}^{h}, a_{p}^{h} \in \text{Mat}_{|p| \times |p|}(\mathbb{C})$ correspond to the upper-right and lower-left block in the principal minor in Fig. 2. Using the identification of Maya diagrams and charged partitions described above, the individual contributions to (2.11) may also be labeled by an $N$-tuple of partitions $Q \in \mathbb{Y}^N$ and an integer charge vector $Q \in \mathbb{Q}_{N-1}$ from the $A_{N-1}$ root lattice

$$\Omega_{N-1} = \left\{(Q_1, \ldots, Q_N) \in \mathbb{Z}^N \mid \sum_{a=1}^{N} Q_a = 0 \right\}.$$

Adapting the notation, the combinatorial expansion (2.11) may then be written as

$$\tau [J] = \sum_{Q \in \Omega_{N-1}} \sum_{Y \in \mathbb{Y}^N} Z_{Y Q}^{[+]} Z_{Y Q}^{[-]},$$

(2.12)

The structure of this series coincides with that of the dual Nekrasov-Okounkov partition functions introduced in [NO]; in fact, in some cases these partition functions can be obtained as specializations of (2.12).

**Remark 2.5.** If $L$ is such that all principal minors $\det L_{2j}$ in (2.9) are non-negative, then $\text{Prob}(Q) := \frac{\det L_{Q}}{\det (1 + L)}$ may be interpreted as a probability measure for a random point process on $X$ called the $L$-ensemble. This

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**Figure 1:** Correspondence between Maya and Young diagrams. The positions of particles and holes are $p_X = \{\frac{1}{2}, \frac{5}{2}, \frac{13}{2}\}$ and $h_X = \{-\frac{21}{2}, -\frac{15}{2}, -\frac{9}{2}, -\frac{3}{2}, -\frac{1}{2}\}$. The charge $Q_X = -3$ corresponds to signed distance between the vertical axis and left boundary of the profile of $Y_X$. 

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[Diagnosis: The image contains a diagram illustrating the correspondence between Maya and Young diagrams, with labels indicating positions of particles and holes, and a mathematical expression for the combinatorial expansion involving Maya diagrams and charge vectors. The text discusses the structure of the series and provides a remark about the interpretation of the $L$-ensemble probability measure.]
Explicit formulae for the inverses of \( P \) and \( Q \), already used in the proof of Theorem 2.3, then allow to express \( K \in \End(H_+ \oplus H_-) \) in terms of solutions of the direct and dual RHPs,

\[
K = \begin{pmatrix}
\Pi_+ \Psi_+ & \Psi_- \Pi_- \Psi^{-1}_- \Pi_+ & -\Pi_+ \Psi_+ & \Psi_- \Pi_- \Psi^{-1}_- \Pi_+ \\
\Pi_- \Psi_+ & \Psi_- \Pi_- \Psi^{-1}_- \Pi_+ & -\Pi_- \Psi_+ & \Psi_- \Pi_- \Psi^{-1}_- \Pi_+
\end{pmatrix}.
\]

Denote by \( \chi_\mathcal{J} \) the indicator function of any subset \( \mathcal{J} \subseteq \mathcal{Z}^\prime \). It is worth observing that the component \( K_{++} := \chi_{\mathcal{Z}^N} K \chi_{\mathcal{Z}^N} \), i.e. the term in the upper-left corner of the matrix representation above, is nothing but the Fredholm operator appearing in the celebrated Borodin-Okounkov formula [BO], in its matrix version [BW]. Hence, the (gap) probability of finding no particles of any colour in the sites \( \mathcal{J}_n \) is given by

\[
\text{Prob}(p \cap \mathcal{J}_n^N = \emptyset) = \det \left( \frac{1}{2} - \chi_{\mathcal{J}_n} K_{++} \chi_{\mathcal{J}_n} \right) = \frac{\det T_n \left( J^{-1} \right)}{\tau(J)}, \tag{2.13}
\]

where the last equality (valid under assumption that \( \det J(z) \) has geometric mean 1) is precisely the content of the Borodin–Okounkov formula.

### 2.2.2 Grassmannian interpretation

It is possible to give an interpretation of the formulas above in the setting of the Sato-Segal-Wilson theory of infinite-dimensional Grassmannians. We will start with the analytic theory [SW] and then comment on the relation with Sato’s formal definition of tau function [Sato]. Consider the point \( W := \Psi_- H_+ \) in the Segal-Wilson Grassmannian \( \Gr(H) \). The subspace \( W \) is spanned by the columns of the (rectangular) matrix

\[
G^{-1} := \begin{pmatrix}
\Pi_+ \Psi_+ \Pi_+ \\
\Pi_- \Psi_- \Pi_+
\end{pmatrix}.
\]

This is a frame for the point \( W \). More generally, a frame for \( W \) will be a rectangular matrix \((w_+, w_-)^T\) whose columns span \( W \), and the frame will be called admissible if \( w_- \neq 0 \) is of trace class on \( H_+ \). Of course, in general \( G^{-1} \) will not be admissible. Nevertheless, since \( \Pi_+ \Psi_+ \) is invertible, there is a canonical way to transform \( G^{-1} \) into an admissible frame by right multiplication:

\[
G^{-1} \mapsto G^{-1}(\Pi_+ \Psi_+ \Pi_+)^{-1} = \begin{pmatrix}
\frac{1}{2} & -d \\
\Pi_- \Psi_- \Pi_+ \Psi^{-1}_+ \Pi_+
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} \\
-d
\end{pmatrix}.
\]

\footnotesize{For comparison, one should use that \( K_{++} = \Pi_+ \Psi_+ \Pi_+ \Psi^{-1}_- \Psi^{-1}_- \Pi_+ \), and compare the factorizations of the jump \( \phi \) in [BW] with those of \( J^{-1} \) in the present paper.

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In other words, \( -d \) is the map whose graph is equal to \( W \). Now, we can act on \( W \) with multiplication by \( \Psi_+^{-1} \), and the Segal-Wilson tau function \( \tau_W(\Psi_+) \) is defined by the formula

\[
\tau_W(\Psi_+) = \sigma(\Psi_+^{-1} W),
\]

(2.14)

where \( \sigma \) is the canonical global section of the determinant line bundle \( \text{Det}^* \) over \( \text{Gr}(H) \), and the action of \( \Psi_+^{-1} \) on \( \text{Gr}(H) \) is extended to \( \text{Det}^* \). The reader is referred to [SW] for the details. What is important here is that, since the operator of multiplication by \( \Psi_+^{-1} \) has block form of type

\[
\Psi_+^{-1} = \begin{pmatrix}
\Pi_+ \Psi_+^{-1} \Pi_+ & \Pi_+ \Psi_+^{-1} \Pi_-
\Pi_+ \Psi_+^{-1} \Pi_+ & \Pi_+ \Psi_+^{-1} \Pi_-
\end{pmatrix},
\]

the tau function is given by the Fredholm determinant, see [SW] formula (3.5)],

\[
\tau_W(\Psi_+) = \det_{H_+} \left( 1 - \Pi_+ \Psi_+ \Psi_+^{-1} \Pi_+ \Psi_+^{-1} \Pi_- \right) = \det_{H_+} (1 - \text{ad}),
\]

(2.15)

so that finally we have \( \tau_W(\Psi_+) = \tau(J) \).

If we are willing to work with formal series instead of analytic functions, there is no reason to restrict to admissible frames and, in Sato’s style [Sato], one can simply define the tau function as

\[
\tau_W(\Psi_+) = \det_{H_+} \left( G^{[1]} G^{-[1]} \right),
\]

(2.16)

where \( G^{[1]} \) and \( G^{-[1]} \) are given, respectively, by the matrices associated to \( \Pi_+ \Psi_+^{-1} \) and \( \Pi_+ \Psi_-^{-1} : \)

\[
\Pi_+ \Psi_+^{-1} = G^{[1]} := \begin{pmatrix}
\Pi_+ \Psi_+^{-1} \Pi_+ & \Pi_+ \Psi_+^{-1} \Pi_-
\Pi_+ \Psi_+^{-1} \Pi_+ & \Pi_+ \Psi_+^{-1} \Pi_-
\end{pmatrix},
\]

\[
\Pi_+ \Psi_-^{-1} = G^{-[1]} := \begin{pmatrix}
\Pi_+ \Psi_-^{-1} \Pi_+ & \Pi_+ \Psi_-^{-1} \Pi_-
\Pi_+ \Psi_-^{-1} \Pi_+ & \Pi_+ \Psi_-^{-1} \Pi_-
\end{pmatrix}.
\]

Since \( \Pi_+ \Psi_+^{-1} \Pi_+ \) and \( \Pi_+ \Psi_-^{-1} \Pi_+ \) are respectively upper and lower triangular with the identity on the main diagonal, the two definitions (2.15) and (2.16) are (formally!) the same. Nevertheless, \( G^{[1]} G^{-[1]} - 1 \) is not a trace class operator, therefore the determinant in (2.16) is to be understood as the limit of the determinant whose size goes to infinity. Indeed, this is nothing but \( \det_{y \to \infty} \left[ f^{-1} \right] \), and the equality between (2.15) and (2.16) is simply a rephrasing of the Widom’s theorem stated in the introduction. The way to compute the latter is through the Cauchy-Binet formula. Namely, for any \( N \)-tuple \( m \) of Maya diagrams of zero total charge, define the associated Plücker coordinates \( G_m^{[1]} \), the determinants of the square matrices obtained by choosing the columns/lines of the matrix \( G_m^{[1]} \) in correspondence with the filled circles in \( m \). Then \( \tau_W(\Psi_+) \) is given by

\[
\tau_W(\Psi_+) = \sum_{m \in m^d} G_m^{[1]} G_m^{-[1]}.
\]

(2.17)

It is natural to wonder what is the relation between the two expansions (2.17) and (2.11). The answer is that they are actually the same, since \( G_m^{[1]} = Z_m^{[1]} \). This identity (cf Proposition 2.1 in [EH]) is indeed the main step in the proof of the so-called Giambelli identity, relating Plücker coordinates associated to an arbitrary Young diagram to the hooked ones. We conclude this subsection by noticing that the integral formula (2.4) for the so-called affine coordinates \( a, \ -d \) already appeared, in the context of Gelfand-Dickey equations, in [BD].

2.2.3 Matrix elements

The reader might wonder when matrix elements \( a, \frac{d}{\nu}, \frac{d}{\nu} \) become effectively computable. In applications to integrable hierarchies, the entries of \( a \) are universal (i.e. they do not depend on the solution but just on the hierarchy) and are given explicitly in terms of the elementary Schur polynomials, while \( d \) determines the point in the Grassmannian corresponding to the given solution, see Subsection 2.3.3 below for more details. Another typical situation where such calculation is possible occurs in the context of monodromy preserving deformations. Suppose that \( \Psi_\pm(z) \) satisfy

\[
\partial_z \Psi_\pm(z) = \Psi_\pm(z) A_\pm(z) + z^{-1} \Lambda_\pm(z) \Psi_\pm(z),
\]

(2.18)

with \( A_\pm(z) \) rational in \( z \) and \( \Lambda_\pm(z) \) polynomial in \( z^{\pm 1} \). The latter condition holds in a number of examples where \( \Psi_\pm(z) \) are related to fundamental matrix solutions of linear systems. It should be seen as an analog
of the conditions used in \cite{TW} to derive nonlinear PDEs satisfied by Fredholm determinants of certain scalar integrable kernels.

Introduce the operator \( \mathcal{L}_0 = z \partial_z + z' \partial_{z'} + 1 \). Since \( \mathcal{L}_0 \frac{1}{z} = 0 \), we have

\[
\mathcal{L}_0 a_\pm (z, z') = \pm \Psi_\pm (z) A_\pm (z, z') \Psi_\pm (z') - 1 + A_\pm (z) a_\pm (z, z') - a_\pm (z, z') A_\pm (z, z'),
\]

(2.19)

where \( a_+ (z, z'), a_- (z, z') = d (z, z') \) and

\[
A_\pm (z, z') = \frac{z' A_\pm (z') - z A_\pm (z)}{z - z'} , \quad A_\pm (z, z') = \frac{A_\pm (z') - A_\pm (z)}{z - z'} .
\]

There exist \( M_\pm \in \mathbb{Z}_{\geq 0} \) such that \( A_\pm (z, z') = \sum_{m=1}^{M_\pm} q_m (z) \phi_m (z') \), where \( \phi_m \) and \( \phi_m \) are column and row \( N \)-vectors. On the other hand, applying \( \mathcal{L}_0 \) directly to Fourier expansions (2.10), one has

\[
\mathcal{L}_0 \Phi (z, z') = \sum_{p, q \in \mathbb{Z}_+^*} (p + q) \left( \phi_{\pm 1} (z) \phi_{\pm 1} (z') \right)^{-1} \phi_{\pm 1} (z) \phi_{\pm 1} (z')^{1} - \phi_{\pm 1} (z) \phi_{\pm 1} (z')^{1} .
\]

(2.20)

Comparing this expression with (2.19), we obtain a system of linear equations that determine \( a_- \), \( d_- \) in terms of Fourier modes of \( \Psi_\pm (z) \phi_m (z) \), \( \phi_m (z) \Psi_\pm (z)^{-1} \) and the coefficients of \( A_\pm (z) \) in \( \text{Mat}_{N \times N} (\mathbb{C}) \), \( \left[ z^{\pm 1} \right] \).

The simplest nontrivial situation corresponds to \( M_0 = M_- = 1 \) and \( \Lambda_\pm \) given by constant diagonal matrices. It occurs, in particular, for generic (non-logarithmic) solutions of Painlevé VI, V and III. In these cases, (2.19) and (2.20) imply that

\[
\sum_{p, q \in \mathbb{Z}_+^*} (p + q - a_{\pm 1}) \phi_{\pm 1} (z) \phi_{\pm 1} (z') = \Psi_\pm (z) \phi_\pm (z) \phi_\pm (z')^{1} - \Psi_\pm (z) \phi_\pm (z) \phi_\pm (z')^{1} .
\]

The modes \( a_- \), \( d_- \) are therefore given by Cauchy matrices which in turn implies that the determinants \( Z_{Y, Q} \) have nice factorized expressions.

### 2.3 Applications

In this subsection, we demonstrate the relation between our Definition 2.1 and tau functions of certain classes of isomonodromic systems and integrable hierarchies. The general strategy is to reduce the associated linear problem to a RHP on a circle and make use of Widom’s differentiation formula (Theorem 2.3).

#### 2.3.1 Four regular singularities

Our basic example deals with a linear system with four Fuchsian singularities placed at \( 0, t, 1, \infty \). It is given by

\[
\partial_z \Phi = A (z) \Phi , \quad A (z) = \frac{A_0}{z} + \frac{A_t}{z - t} + \frac{A_\infty}{z - \infty} ,
\]

(2.21)

with \( A_{0, t, 1} \in \text{Mat}_{N \times N} (\mathbb{C}) \). Consider generic situation where \( A_0 \) and \( A_{\infty} := -A_0 - A_t - A_1 \) are diagonalizable. For \( a = 0, t, 1, \infty \), fix the diagonalizations \( A_a = G_a^{-1} \Theta_a G_a \) with diagonal \( \Theta_a \). Assume that the eigenvalues of \( A_a \) are distinct mod \( Z \). Then there exist unique fundamental matrix solutions \( \Phi^{(a)} (z) \) of (2.21), holomorphic on the universal covering of \( \mathbb{C} \setminus \{0, t, 1\} \) and such that

\[
\Phi^{(a)} (z) = \begin{cases} (a - z)^{\Theta_a} G^{(a)} (z) , & \text{for } a = 0, t, 1 , \\ (-z)^{-\Theta_\infty} G^{(\infty)} (z) , & \text{for } a = \infty , \end{cases}
\]

where \( G^{(a)} (z) \) is holomorphic and invertible in a finite open disk around \( z = a \) and satisfies the normalization condition \( G^{(a)} (a) = G_a \).
Further assume for notational simplicity that \(t \in (0, 1)\). The canonical solutions \(\Phi^{(0, \infty)}(z)\) analytically continue to single-valued matrix functions on the cut Riemann sphere \(\mathbb{CP}^1 \setminus \{\mathbb{R}_{<0}\}\). Similarly, \(\Phi^{(I)}(z)\) and \(\Phi^{(II)}(z)\) are naturally defined on \(\mathbb{CP}^1 \setminus ((-\infty, 0) \cup (t, \infty))\) and \(\mathbb{CP}^1 \setminus ((-\infty, t) \cup [1, \infty))\), respectively. Take an arbitrary fundamental solution \(\Phi(z)\), defined on \(\mathbb{CP}^1 \setminus \{\mathbb{R}_{<0}\}\). The connection matrices \(C_{\alpha, \epsilon} = \Phi(z) \Phi^{(\alpha)}(z)^{-1}\) with \(\epsilon = \text{sgn} \, \Im z\), are independent of \(z\). They satisfy the compatibility conditions

\[
\begin{align*}
C_{0, +} &= C_{0, -}, & C_{\infty, +} &= C_{\infty, -}, \\
M_0 &= C_{0, -} e^{2 \pi i \theta_0} C_{0, +}^{-1} = C_{t, +} - C_{t, -}, & M_{\infty} &= C_{1, -} e^{2 \pi i \theta_1} C_{1, +}^{-1} = C_{\infty, -} e^{-2 \pi i \theta_\infty} C_{\infty, +}, \\
M_0 M_I &= (M_1 M_\infty)^{-1} = C_{t, -} e^{2 \pi i \theta_1} C_{1, +}^{-1} = C_{1, -} C_{1, +}^{-1},
\end{align*}
\tag{2.22}
\]

where \(M_a\) denotes anticlockwise monodromy matrix of \(\Phi(z)\) around the Fuchsian singular point \(a \in (0, t, 1, \infty)\).

The connection matrices \(\{C_{\alpha, \epsilon}\}\) and exponents \(\{\Theta_\alpha\}\) of local monodromy constitute the monodromy data for the 4-point Fuchsian system (2.21).

Let us now explain how to transform (2.21) into a Riemann-Hilbert problem on a circle. This will be achieved in several steps:

1. Start with the contour \(\tilde{\Gamma}\) shown in Fig. 3a by solid black curves. Denote by \(D_a\) the disk around \(z = a\) bounded by \(\gamma_a\) and define

\[
\Psi(z) = \begin{cases} 
\Phi(z), & z \in D_a, \\
C_{(a)}(z), & z \in \mathbb{R}_{\geq 0} \cup \hat{D}_0 \cup \hat{D}_1 \cup \hat{D}_\infty.
\end{cases}
\]

Comparing with (2.11), we see that the matrix function \(\Psi(z)\) solves a dual RHP set on \(\hat{\Gamma}\) with the jumps indicated in Fig. 3b.

2. Next cancel the constant jump \((M_0 M_I)^{-1}\) on the real segment cut out by the dashed red circles \(\mathcal{C}_{\text{out}, \text{in}}\).

To this end, let us write \(M_0 M_I = e^{2 \pi i \Theta}\). There is a certain freedom in the choice of \(\Theta\); for example, in the generic situation where \(\Theta\) may be assumed diagonal, we may add to it any integer diagonal matrix. Denote by \(\mathcal{A}\) the open annulus bounded by \(\mathcal{C}_{\text{out}, \text{in}}\) and set

\[
\begin{align*}
\Psi(z) &= \begin{cases} 
(z)^{-\Theta} \Phi(z), & z \in \mathcal{A}, \\
\Phi(z), & z \notin \mathcal{A}.
\end{cases}
\end{align*}
\]

The dual RHP for \(\Psi(z)\) is set on the contour \(\tilde{\Gamma}\) indicated in Fig. 3b by solid black lines. The jump matrices associated to \(\mathcal{C}_{\text{out}}\) and \(\mathcal{C}_{\text{in}}\) are \((z)^{-\Theta}\); on the rest of the contour the jumps are the same as for \(\Psi(z)\).

3. The contour \(\tilde{\Gamma}\) has two connected components, \(\tilde{\Gamma}_-\) and \(\tilde{\Gamma}_+\), containing respectively \(\mathcal{C}_{\text{out}}\) and \(\mathcal{C}_{\text{in}}\). Choose \(\Theta\) so that \(\text{Tr} \, \Theta = \text{Tr} \, (\Theta_0 + \Theta_\infty) = -\text{Tr} \, (\Theta_1 + \Theta_\infty)\) (this choice still allows for \(\Omega_{N-1}\)-shifts). The RHPs obtained by restricting the initial contour to \(\tilde{\Gamma}_-\) or \(\tilde{\Gamma}_+\) while keeping the same jumps are then generically solvable. Their solutions are related to fundamental matrices \(\Phi_-(z)\) and \(\Phi_+(z)\) of 3-point Fuchsian systems whose singular points are \(0, t, \infty\). Let us denote these solutions by \(\Psi_-(z)\) and \(\Psi_+(z)\). The subscript reminds that these functions are analytic outside \(\mathcal{C}_{\text{out}}\) and inside \(\mathcal{C}_{\text{in}}\), respectively.

Consider an auxiliary circle \(\mathcal{C}\) inside \(\mathcal{A}\), indicated by dashed red line in Fig. 3b, and define

\[
\begin{align*}
\Psi(z) &= \begin{cases} 
\Psi_+(z) \Psi_-(z)^{-1} \Phi(z), & \text{outside } \mathcal{C}, \\
\Psi_-(z)^{-1} \Phi(z), & \text{inside } \mathcal{C}.
\end{cases}
\end{align*}
\tag{2.23}
\]

The matrix function \(\Psi(z)\) has no jumps except on \(\mathcal{C}\). The jump of the relevant dual RHP is written in the form of direct factorization,

\[
J(z) = \Psi_-(z)^{-1} \Psi_+(z),
\tag{2.24}
\]

cf. (2.1a). The problem of solving the 4-point Fuchsian system with a prescribed monodromy is therefore converted into a RHP for \(\Psi_+(z)\) on a single circle (Fig. 3c), with the jump matrix expressed in terms of 3-point solutions \(\Psi_+(z)\). The latter will be considered as known, even though their explicit expressions in higher rank \(N \geq 3\) are available only in a few special cases (rigid systems, etc).
Indeed, the analog of (2.21) for \( \Phi \) gives one more equation, \( \partial_z \Phi^- = -\Phi^- A^- (z - t)^{-1} \), which implies that the first term in (2.28) (given by the residue at \( z = \infty \) vanishes. The second term may be rewritten as

\[
\frac{1}{2\pi i} \int_{C} \text{Tr} \left( \frac{A^-}{z - t} \Phi^{-1} \Phi A(z) \Phi^{-1} \Phi_- \right) dz = \partial_t \ln \tau_{\text{JMU}} (t) - \frac{\text{Tr} A_0^- A_t^-}{t} + \text{res}_{z=t} \frac{\text{Tr} A_0^- \Phi^{-1} \Phi A_t \Phi^{-1} \Phi_-}{(z - t)^2}.
\]
Here, the last expression corresponds to the contribution with a 2nd order pole at \( z = t \) and the first two are the residues of the rest at simple poles \( z = t \) and \( z = 0 \). Since \( 2 \text{Tr} A^*_t A^*_t = \text{Tr} (\Theta^2 - \Theta_0^* - \Theta_t^2) \), it now suffices to show that the last expression vanishes to finish the proof. Indeed, since

\[
\Phi(z)^{-1} \Phi_-(z) = G_t^{-1} \left( \frac{\Phi(z)}{z - t} + O(z - t^2) \right) G_t \quad \text{as} \quad z \to t,
\]

with some \( g_t \in \text{Mat}_{N \times N}(C) \), the last contribution to (2.30) is equal to \( \text{Tr} G_t \left( G_t^{-1} A^*_t \right)^{-1}, G_t A_t G_t^{-1} \). But we have \( G_t^{-1} A^*_t \left( G_t^{-1} \right)^{-1} = G_t A_t G_t^{-1} = \Theta_t \), and the statement follows. \( \square \)

2.3.2 Two irregular singularities

Let us now consider a linear system with two irregular singularities at \( z = 0 \) and \( z = \infty \) of respective Poincaré ranks \( R_0, R_\infty \in \mathbb{Z}_{>0} \). The general form of such a system reads

\[
\partial_z \Phi = \Phi A(z), \quad A(z) = \sum_{k=-R_0}^{R_\infty} z^{k-1} A_k, \quad (2.31)
\]

where all \( \Theta_k^{(a)} \) are given by diagonal matrices. These matrices, together with the coefficients \( g_k^{(a)} \), are uniquely fixed by the linear system (2.31).

There exist unique formal fundamental solutions

\[
\Phi^\text{form}_a(z) = e^{\Theta^{(a)}(z)} \hat{\Phi}^{(a)}(z) G_a, \quad a = 0, \infty,
\]

with

\[
\hat{\Phi}^{(0)}(z) = \frac{1}{z} + \sum_{k=1}^{\infty} g_k^{(0)} z^{-k}, \quad \hat{\Phi}^{(\infty)}(z) = \frac{1}{z} + \sum_{k=1}^{\infty} g_k^{(\infty)} z^{-k},
\]

\[
\Theta^{(0)}(z) = \frac{1}{k} \sum_{k=-R_0}^{R_\infty} \Theta_k^{(0)} z^k + \Theta_0^{(0)} \ln z, \quad \Theta^{(\infty)}(z) = \sum_{k=1}^{R_\infty} \Theta_k^{(\infty)} \frac{1}{k} z^{-k} - \Theta_0^{(\infty)} \ln z,
\]

where \( R_\infty = R_0 + 2 \). Genuine canonical solutions \( \Phi_k^{(a)}(z) \) with \( k = 1, \ldots, 2R_\infty + 1 \) are asymptotic to \( \Phi^\text{form}_k(z) \) in \( 2R_\infty + 1 \) Stokes sectors \( S_k^{(a)} \) around \( z = a \), and are related by Stokes matrices \( S_k = \Phi_k^{(a)}(z) \Phi_k^{(a)}(z)^{-1} \) on their overlap.

1. Introduce a function \( \Psi(z) \) which coincides with the canonical solutions \( \Phi_k^{(a)}(z) \) inside the sectors \( \Omega_k^{(a)} \subset S_k^{(a)} \) schematically represented in Fig. 4. The rays therein belong to overlaps of adjacent Stokes sectors. The function \( \Psi(z) \) solves a dual RHP on the contour \( \bar{\Gamma} \), indicated in Fig. 4 by solid black curves. Besides the jumps on the rays (given by the Stokes matrices), one has constant jumps on different arcs of the connection circle. All of the latter can be expressed in terms of one connection matrix, e.g. \( E = \Phi_1^{(0)}(z) \Phi_1^{(\infty)}(z)^{-1} \). There is also an asymptotic condition \( e^{-\Theta^{(a)}(z)} \Psi(z) = O(1) \) as \( z \to a \) on \( \mathbb{C} \setminus \bar{\Gamma} \).

2. We would now like to cancel the jumps inside the open annulus \( \mathcal{A} \) bounded by the circles \( \mathcal{C}_{\text{out}}, \text{in} \) indicated in Fig. 4 by dashed red lines. Pick any fundamental matrix solution \( \hat{\Psi}(z) \) of (2.31), e.g. \( \hat{\Psi}_1^{(0)}(z) \). Let \( M_0 \in \text{SL}(N, C) \) be its anticlockwise monodromy around \( z = 0 \). This matrix is determined up to conjugation by the Stokes matrices. Write \( M_0 = e^{2\pi i \Theta} \) choosing \( \Theta \) so that \( \text{Tr} \Theta = 0 \). Define

\[
\hat{\Psi}(z) = \begin{cases} (-z)^{-\Theta} \hat{\Phi}(z), & z \in \mathcal{A}, \\ \hat{\Psi}(z), & z \notin \mathcal{A}. \end{cases}
\]

The dual RHP for \( \hat{\Psi}(z) \) is posed on the contour \( \bar{\Gamma} \) indicated in Fig. 4 by solid black lines.
3. As before in (2.23), it now suffices to divide $\hat{\Psi}(z)$ inside and outside of an auxiliary circle $\mathcal{C}$ by the solutions $\Psi_-(z)$ and $\Psi_+(z)$ of the auxiliary dual RHPs set on the two connected components $\Gamma_-$ and $\Gamma_+$ of $\hat{\Gamma}$, containing $\mathcal{C}_{\text{out}}$ and $\mathcal{C}_{\text{in}}$. They are related to the solutions $\Phi_-(z)$ and $\Phi_+(z)$ of two auxiliary linear systems. The first one has an irregular singular point of Poincaré rank $R_0$ at $z=0$ and a regular singularity at $z=\infty$. In the second, there is a regular singular point at $z=0$ and an irregular singularity of Poincaré rank $R_\infty$ at $z=\infty$. We thereby obtain a dual RHP for a function $\hat{\Psi}(z)$, with the jump

$$J(z) = \Psi_-(z)^{-1}\Psi_+(z) = \Phi_-(z)^{-1}\Phi_+(z),$$

on $\mathcal{C}$ written in the form of a direct factorization. Similarly to the above, we make the identifications $\Psi_{\pm}(z) = (-z)^{-R_\pm}\Phi_{\pm}(z)$, $\Psi_{\pm}(z) = \Phi_{\pm}(z)^{-1}\Phi(z)$.

The set $\mathcal{F}$ of isomonodromic times consists of the diagonal elements of $\Theta_{\pm}(0)$ with $k \neq 0$. We accordingly decompose it as $\mathcal{F} = \mathcal{F}^{(0)} \cup \mathcal{F}^{(\infty)}$. The Jimbo-Miwa-Ueno tau function of the system (2.31) is defined by the closed 1-form [JMU eq. (1.23)]

$$d_{\mathcal{F}} \ln \tau_{\text{JMU}}(\mathcal{F}) = -\sum_{a=0,\infty} \text{res}_{z=a} \text{Tr} \left( \partial_z \hat{\Phi}^{(a)}(z) \hat{\Phi}^{(a)}(z)^{-1} d_{\mathcal{F}}(a) \Theta^{(a)}(z) \right).$$

(2.32)

Let us now make contact between this formula and the construction given in Definition 2.1.

**Corollary 2.7.** Let $\tau_{\text{JMU}}^{(0)}(\mathcal{F}^{(0)})$ and $\tau_{\text{JMU}}^{(\infty)}(\mathcal{F}^{(\infty)})$ be the Jimbo-Miwa-Ueno tau functions of auxiliary linear systems for $\Phi_-(z)$ and $\Phi_+(z)$, and $\tau[J]$ be the Fredholm determinant defined by (2.4)–(2.5). Then

$$\tau[J] = \left[ \tau_{\text{JMU}}^{(0)}(\mathcal{F}^{(0)}) \tau_{\text{JMU}}^{(\infty)}(\mathcal{F}^{(\infty)}) \right]^{-1} \tau_{\text{JMU}}(\mathcal{F}).$$

(2.33)

**Proof.** We choose again the normalization in which $\Phi_+(z) = (-z)^{R_\pm}$ as $z \to 0$ and $\Phi_-(z) = (-z)^{R_\pm}$ as $z \to \infty$. This implies that $\Phi_-(z)$ is independent of $\mathcal{F}^{(\infty)}$ and $\Phi_+(z)$ independent of $\mathcal{F}^{(0)}$. From the Widom’s differentiation formula then follows an analog of the equation (2.27),

$$d_{\mathcal{F}^{(0)}} \ln \tau[J] = \frac{1}{2\pi i} \int_{\mathcal{C}} \text{Tr} \left( d_{\mathcal{F}^{(0)}} \Phi_- \Phi_-^{-1} \frac{\Theta}{z} - d_{\mathcal{F}^{(0)}} \Phi_- \Phi_-^{-1} \frac{\partial_z \Phi_-^{-1}}{z} \right) \, dz,$$

(2.34)

and a similar formula for $d_{\mathcal{F}^{(\infty)}} \ln \tau[J]$. The first term in the integrand of (2.34) is analytic outside $\mathcal{C}$ and the corresponding integral reduces to the residue at $z=\infty$. The isomonodromy equation for $\Phi_-$ has the form...
\[ d_{\mathcal{F}} \ln \tau [J] = - \res_{z=0} \Tr \left( d_{\mathcal{F}} \Phi \cdot \Phi^{-1} \partial_z \Phi \Phi^{-1} \right) = - \res_{z=0} \Tr \left( d_{\mathcal{F}} \Phi \cdot \Phi^{-1} \partial_z \Phi \partial_z \Phi^{-1} \left( e^{\Theta(0)} + e^{-\Theta(0)} d_{\mathcal{F}} \Phi \cdot \Phi^{-1} \partial_z \Phi \partial_z \Phi^{-1} \right) \right) = - \res_{z=0} \Tr \left( U^{-}(z) A^{-}(z) - d_{\mathcal{F}} \Theta(0) \partial_z \Phi^{-1} + d_{\mathcal{F}} \Phi(0) \partial_z \Phi^{-1} \right), \]

where \( A^{-}(z) = \Phi^{-1} \partial_z \Phi \). Since \( A^{-}(z) = \sum_{k=-R_0}^{0} z^{k-1} A_k^{-} \), the residue of the first term vanishes, while the second and third yield \( -d_{\mathcal{F}} \ln \tau_{\text{IMU}}(0) + d_{\mathcal{F}} \ln \tau_{\text{IMU}} \). Similarly computing the differential \( d_{\mathcal{F}} \ln \tau [J] \), we arrive at the expression (2.33).

We have thus shown that \( \tau [J] \) coincides with \( \tau_{\text{IMU}}(\mathcal{F}) \) up to more elementary factors depending separately on \( \mathcal{F}(0) \) and \( \mathcal{F}(\infty) \). These normalization factors are the tau functions of the auxiliary linear systems arising upon “decorated pants decomposition” of the Riemann sphere with 2 irregular punctures into two spheres with 1 irregular and 1 regular puncture. Schematically,

\[ \tau_{\text{IMU}}^0 = \tau_{\text{IMU}}(0) \tau_{\text{IMU}}(0) \det \begin{pmatrix} 1 & a \\ d & 1 \end{pmatrix}. \tag{2.35a} \]

The regular holes here correspond to Fuchsian singularities and cusps represent anti-Stokes directions. The prefactor \( e^{\Theta(0)} + e^{-\Theta(0)} d_{\mathcal{F}} \Phi \cdot \Phi^{-1} \partial_z \Phi \partial_z \Phi^{-1} \) in (2.26) has a similar interpretation: it represents the isomonodromic tau function of the auxiliary Fuchsian system for \( \Phi \) having singular points at 0, \( t \) and \( \infty \), while the tau function of the auxiliary 3-point system for \( \Phi \) is just a constant, so that

\[ \tau_{\text{IMU}}^1 = \tau_{\text{IMU}}(1) \tau_{\text{IMU}}(0) \det \begin{pmatrix} 1 & a \\ d & 1 \end{pmatrix}. \tag{2.35b} \]

The idea to associate Riemann surfaces with cusped boundaries to monodromy manifolds of isomonodromic systems in rank \( N = 2 \) first appeared in [CMJ]. Our results illustrate the use of the corresponding pictures in the analytic setting.

### 2.3.3 Integrable hierarchies

As a second example, we will show how to apply our definition of tau function to the study of integrable hierarchies. The results outlined in this section are not new (see [CSW1], [CSW2], [CDJ]); the aim is to describe them in a way that makes the comparison with the case of isomonodromic deformations more transparent. To start with, consider a differential operator of fixed degree \( N \)

\[ L := D^N + u_{N-2} D^{N-2} + \ldots + u_0, \tag{2.36} \]

where we denoted \( D := \partial_x \), and the coefficients \( u_0, \ldots, u_{N-2} \) depend on \( x \) and some additional parameters we are now going to describe. The isospectral deformations of \( L \) are described by the Lax system

\[ \begin{cases} L \phi = z \phi, \\ \partial_j \phi = \left( L^{j/N} \right)_+ \phi, \quad j \neq 0 \mod N. \end{cases} \tag{2.37} \]

giving rise to the Gelfand-Dickey equations for the variables \( \{ u_0(x,t), \ldots, u_{N-2}(x,t) \} \), written in the Lax form as the compatibility conditions of the system (2.37).
Here and below \( t \) denotes the collection of all the deformation parameters \( t := \{ t_1, \ldots, t_{N-1}, t_{N+1}, \ldots \} \).

Converting the equations (2.37) into a 1st order system of size \( N \), we get

\[
\begin{align*}
\partial_x \Phi &= \Phi \begin{pmatrix} 0 & 0 & \ldots & 0 & z - u_0 \\ 1 & 0 & \ldots & 0 & -u_1 \\ 0 & 1 & \ldots & 0 & -u_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & 1 & 0 \end{pmatrix}, \\
\partial_{t_j} \Phi &= \Phi M_{t_j}, \quad j \not\equiv 0 \mod N.
\end{align*}
\]  

(2.39)

where the matrices \( M_j \) are completely determined by the coefficients of \( (L/H) \). We fix uniquely a fundamental solution \( \Phi(x,t;z) \) by imposing the asymptotics at infinity

\[
\Phi(x,t;z) = e^{x A + \sum_{j \not\equiv 0 \mod N} t_j A_j} \left[ 1 + O(z^{-1}) \right] = e^{x A + \sum_{j \not\equiv 0 \mod N} t_j A_j} \Phi_+(x,t;z), \quad z \to \infty,
\]  

(2.40)

where

\[
\Lambda := \begin{pmatrix}
0 & 0 & \ldots & 0 & z \\
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & \ldots & 1 & 0
\end{pmatrix}
\]

Indeed, in some cases \( \Phi \) will be a genuine analytic function in a neighborhood of \( \mathcal{C} \), and the corresponding solution belongs to the Segal–Wilson Grassmannian. Otherwise, one can still consider \( \Phi \) just as a formal series, and in this case the solution belongs to the Sato Grassmannian. We now define our direct RHP by imposing

\[
J(x,t;z) := \Phi_+^{-1}(0,0;z) e^{x A + \sum_{j \not\equiv 0 \mod N} t_j A_j} =: \Psi_+^{-1}(z) \Psi_+(x,t;z).
\]  

(2.41)

In [Caf, CW1] it has been proven that the related tau function defined in Definition 2.1 coincides with the one defined by Segal and Wilson. Indeed, one can act with the matrix-valued series \( \Psi_- \) on the subspace \( H_- \) and obtaining a subspace \( W := H_+ \cdot \Psi_- \) in \( \text{Gr}(H) \). The operator \( -d : H_+ \to H_- \) is nothing but the operator whose graph gives the subspace \( W \) (i.e. the operator \( A \) in the notation of [SW]) and the formulas to be compared are [SW] eq. (3.5) and

\[
\tau[J] = \det_{H_+} (I + L) = \det_{H_-} (I - \text{ad}).
\]

Concerning the combinatorial expansion in the Subsection 2.2 we start by observing that the matrix \( G^{[+]} \) does not depend on the particular solution to be studied, and can be computed explicitly. It reads

\[
G^{[+]} = \begin{pmatrix}
\ldots & \ldots & \ldots & \ldots \\
\ldots & 1 & s_1 & s_2 & s_3 & s_4 & s_5 & \ldots \\
\ldots & 0 & 1 & s_1 & s_2 & s_3 & s_4 & \ldots \\
\ldots & 0 & 0 & 1 & s_1 & s_2 & s_3 & \ldots
\end{pmatrix}
\]  

(2.42)

where the elementary Schur polynomials \( s_j = s_j(t) \) are defined by the relation

\[
\sum_{j \geq 0} s_j(t) z^j = \exp \left( \sum_{j \not\equiv 0 \mod N} t_j z^j \right).
\]  

(2.43)

In particular, its minors give, by definition (see for instance [Mac]), the Schur polynomials associated to an arbitrary partition. Equivalently, one can compute the same Schur polynomial through the principal minor of a, and the equivalence between the two approaches is given by the Giambelli identity [EH]. As we mentioned before, the graph of the function \( -d \) determines the point \( W \) associated to the particular solution we want to study. This means that its minors give the Plücker coordinates of \( W \), which are equivalently computed as the minors of \( G^{[+]} \), again via the Giambelli identity. Hence, the combinatorial formula (2.17), in this case, is the standard tau function expansion in terms of Schur polynomials and Plücker coordinates.

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Note that, in this case, any loop $\Psi_- (z)$ with the prescribed asymptotics $\Psi - O(z^{-1})$ determines a solution of the hierarchy, so that it does not make much sense to ask about the general form of $-d$. On the other hand, one can wonder how the already known solutions of the Gelfand-Dickey equations are obtained through our procedure, and this had been answered already for quite a few families:

- Suppose that $\Psi_- (z) = e^{X(z)}$, with $X(z)$ a nilpotent element of $s_N[z^{-1}]$. Then there is just a finite number of non-zero minors of $d$. In this case, the combinatorial expansion (2.17) becomes a finite sum and the tau function will be a polynomial obtained by particular linear combinations of Schur polynomials, see [CDD]. These solutions are much studied in the literature, as their zeros evolve according to the (rational) Calogero–Mosser hierarchy (see [Wii] and references therein).

- More generally, if $\Psi_- (z)$ has a truncated expansion (i.e. just a finite number of non-zero Fourier coefficients, say up to $n$) then, using a result obtained by Widom [W1], one can compute the Szegő-Widom constant associated to matrix-valued algebraic symbols had been developed for the first time in [IIK, IMM]. The idea to use it to compute the Szegő-Widom constant of $f^{-1}$ as the (finite-size) determinant $\det T_n [f]$. Such solutions are the ones associated to rational curves with singularities (see [SW]), i.e. they correspond to multi-solitons, see [Cal].

- Consider a Riemann surface of type (symmetric $N$-covering)

$$\lambda^N = \prod_{j=1}^{Nk+1} (z - a_j) = P(z),$$

and define $N$ functions $\{w_1 (z), \ldots, w_N (z)\}$ by

$$w_i (z) := \left( \frac{P(z)}{z} \right)^i \frac{1}{\prod_{j=1}^{i-1} (z - a_j)}, \quad i = 1, \ldots, N.$$  

The point in the Grassmannian generated by $\Psi_- (z) := \text{diag}(w_1 (z), \ldots, w_N (z))$ belongs to the so-called Krichever locus $\text{SW}$. It corresponds to an algebro-geometric solution of the Gelfand-Dickey hierarchy, whose spectral curve is exactly (2.44). The Schur expansion for these tau functions had been studied in great detail in [EH]. A way to compute the tau function in this case is using the Widom’s differentiation formula (2.6). Indeed, here one can reduce the dual RHP (2.1b) to a RHP with constant jumps on the intervals $[\alpha_i, \alpha_j] \cup \cdots \cup [\alpha_{Nk-1}, \alpha_{Nk}] \cup [\alpha_{Nk+1}, \infty)$, and solve it explicitly in terms of theta functions associated to the curve (2.44), as described in [Cal] for the case $N = 2$ (i.e. for hyper-elliptic curves). The procedure is a classical one, used already in the 80s by the Saint Petersburg school in the context of the so-called finite-gap integration (see for instance [Mat] and references therein). The idea to use it to compute the Szegő-Widom constant associated to matrix-valued algebraic symbols had been developed for the first time in [IIK, IMM].

- As an example of solutions not belonging to the Segal-Wilson Grassmannian, we consider topological solutions, uniquely fixed by the equations of the hierarchy and the additional string equation

$$\left( \sum_{i \neq \text{mod} N} \left( i + N \frac{t_i + N \delta_{i,j}}{N} \right) \frac{\delta}{\delta t_i} + \frac{1}{2N} \sum_{i+j = N} i j t_i t_j \right) r(t) = 0.$$  

(2.46)

For $N = 2$, this is the celebrated Witten-Kontsevich tau function [Kon], and for $N > 2$ this is the Frobenius potential, to all genera, of the singularity of type $A_{N-1}$. Finding the element $\Psi_- (z)$ defining the relevant jump (2.41) corresponds to finding the point in the Sato-Segal-Wilson Grassmannian associated to the given solution of the hierarchy. This had been achieved, in the 90s, by Kac and Schwarz [KS], who proved that $W = H \cdot \Psi_-$ is uniquely fixed by the two conditions\[^6\]

$$zW \subseteq W, \quad \mathfrak{R}_N W \subseteq W,$$  

(2.47)

where $\mathfrak{R}_N$ is the differential operator

$$\mathfrak{R}_N = \partial_z - \lambda + (Nz)^{-1} \rho,$$  

(2.48)

\[^6\]Note that here we are working on the vector-valued Grassmannian, while in the cited paper the two conditions are (equivalently) stated in the scalar Grassmannian and the Kac-Schwarz operator, in particular, is a scalar one.
and ρ is a diagonal traceless matrix whose coefficients depend on the Cartan matrix of the Lie algebra \( sl_N \), see for instance [CW2] eq. (3.24). Another way of stating the second equation in (2.47) is to say that \( \Psi_\gamma(z) \) satisfies the so-called reduced string equation

\[
\mathcal{R}_N \Psi_\gamma = \Psi_- (\Psi_-^{-1} \Lambda \Psi_\gamma)_+, \quad (2.49)
\]

and it had been proven in [CW2] that there exists a unique solution \( \Psi_- (z) = e^{X(z)} \), with \( X(z) \) an element in the loop (sub)algebra \( sl_N[[z^{-1}]] \). This solution is in general just a formal series, exactly as the corresponding tau function, whose coefficients give intersection numbers on the Deligne-Mumford moduli space of stable curves. Note that (2.49) is of the same form as (2.18). This is not surprising, as the connection between isomonodromic deformations and string equations goes back to the work [Mo] (see also [ACV]), where this connection is established using the Kac-Schwarz operators described above).

In order to simplify the notations, in this subsection we only considered the case of Gelfand-Dickey hierarchies. Nevertheless, most of the results described above apply as well to the more general case of the Drinfeld-Sokolov hierarchies associated to arbitrary (affine) Kac-Moody algebras [DS]. The idea is to consider direct RHPs as the one in (2.41) above, but with \( \Psi_- \) an element of the form \( \Psi_- (z) = e^{X(z)} \), with \( X(z) \in \mathfrak{g}[[z^{-1}]] \), and \( \mathfrak{g} \) an arbitrary (finite-dimensional) Lie algebra. The element \( \Psi_+ \), on the other hand, will be replaced by

\[
\Psi_+ (x, t, z) = e^{-x E_1 - \sum_{j \in E_+} t_j A_j},
\]

where \( E_+ \) is the set of the (positive) exponents of the Kac-Moody algebra \( \mathfrak{g}[[z, z^{-1}]] \oplus \mathbb{C} c \), and \( \{ A_{j, j} \in E_+ \} \) is (half of) the Heisenberg sub-algebra associated to an arbitrary gradation of the algebra. Polynomial and topological solutions of these hierarchies had been treated, using this formalism, in [DDD, CW2]. It would be interesting (but technically involved, because of the size of matrix representations) to study algebro-geometric solutions associated to arbitrary Drinfeld-Sokolov hierarchies.

## 3 General contour

The Definition 2.1 of τ [J] makes sense if we replace the circle \( \mathcal{C} \) by a finite collection \( \Gamma = \bigcup_{i=1}^{M} \mathcal{C}_a \) of non-intersecting smooth closed curves which we sometimes call ovals. However, defining the jump of the relevant dual RHP in the form of direct factorization is no longer natural from the point of view of applications, which makes the Fredholm determinant representation (2.5) and the differentiation formula (2.6) less useful in this setting. What we would like to have instead are the formulae of the same type but expressed in terms of the direct factorization of the individual jumps on each curve \( \mathcal{C}_a \).

The existence of such expressions is suggested by the recent work [GL16] by two of the authors, where Fredholm determinant and combinatorial series representations were obtained for the tau function of the \( n \)-point Fuchsian system — including, in particular, the tau function of the Garnier system \( \mathfrak{g}_{n-3} \). The paper [GL16] deals with the special case where the contour is given by a collection of concentric circles coming from a “linear” pants decomposition of \( \mathbb{C} P^1 \setminus \{ n \) points\}. Our aim here is to extend these results to RHPs with more general jumps on arbitrary configurations of ovals such as the one represented in Fig. 5c.

### 3.1 Notations and setting of the RHP

The complement of \( \Gamma \) in \( \mathbb{C} P^1 \) has \( M + 1 \) connected components, which will be called faces (or pants). It admits a unique, up to permutation, 2-coloring by colors \( \{+, -\} \) which will be fixed once and for all. Denote by \( F_\pm \) the set of faces of color \( \pm \), by \( C := \{ \mathcal{C}_1, \ldots, \mathcal{C}_M \} \) the set of all ovals and by \( C_f \) the set of boundary curves of the face \( f \). Let \( \phi_\pm : C \to F_\pm \) be the map assigning to each curve \( \mathcal{C} \in C \) the unique face in \( F_\pm \) having \( \mathcal{C} \) among the boundary components.

The coloring allows to choose a convenient orientation of \( \Gamma \) for which all faces of color \( + \) (or \( - \)) are located on the positive (resp. negative) side of their boundary curves. We denote by \( \varphi_+ (\mathcal{C}) \) and \( \varphi_- (\mathcal{C}) \) the closure of interior and exterior of the curve \( \mathcal{C} \) with respect to the above orientation. Clearly \( \phi_\pm (\mathcal{C}) \subset \varphi_\pm (\mathcal{C}) \), but in general \( \varphi_\pm (\mathcal{C}) \) can contain faces of different colors. Assign to every boundary \( \mathcal{C} \in C \) a pair of functions \( \Psi_{\mathcal{C}, \pm} : \mathcal{C} \to \text{GL}(N, \mathbb{C}) \) that continue analytically to \( \varphi_\pm (\mathcal{C}) \). Also, to every \( \mathcal{C} \in C \) we assign the space of functions
do not necessarily belong to Riemann-Hilbert problem whose integral kernels have integrable form and are given by direct factorization,

\[ \phi \quad J \]

That is, we want to find an analytic invertible matrix function \( \bar{\Psi} \) such that \( \bar{\Psi} H \subseteq H \), \( \bar{\Psi} C \). In particular, for \( \phi \in C \), \( C \) we have \( \det(\bar{\Psi} C, C) = \Lambda(\bar{\Psi} C, C) \). It may be decomposed as \( H = H_+ + H_- \) and consider \( \bar{\Psi} C, C \). In particular, \( \bar{\Psi} C, C \) have \( \Pi \phi \), \( \Pi \phi \) whose boundary values on \( \Gamma \) satisfy \( \bar{\Psi} \Psi^{-1} \).

Denote by \( \Pi \phi \), the projections on \( H_\pm \) along \( H_\pm \) and consider

\[ H = H_+ \oplus H_- \quad H_\pm = \bigoplus_{\phi \in C} H_\phi. \]

For \( \phi, \phi' \in C \), \( \phi \in \phi_a \) \( \phi' \) define the operators \( \Pi \phi \), \( \Pi \phi' \) \( H \phi \rightarrow H \phi' \) such that \( \Pi \phi \phi' \phi g \) is the restriction to \( \phi \) of the analytic continuation of \( \Pi \phi' \phi g \) to \( \phi_a \). In particular, for \( \phi \in C \phi_1 \) \( \phi \neq \phi' \) we have \( \Pi \phi \phi' \phi = \Pi \phi' \phi \).

**Definition 3.1.** The tau function \( \tau[J] \) associated to the above RHP is defined as the Fredholm determinant

\[ \tau[J] = \det_H (1 + L) \quad L = \begin{pmatrix} 0 & A_{-+} \\ A_{+-} & 0 \end{pmatrix} \in \text{End}(H_+ \oplus H_-), \quad (3.1) \]

where the operators \( A_{\pm \mp} : H_\pm \rightarrow H_\pm \) are defined by

\[ A_{\pm \mp} = \sum_{\phi \in C_\phi} A_{\phi, \pm \mp} \]

\[ A_{\phi, \pm \mp} = \Psi^{-1} \Pi \phi \phi g - \delta_{\phi, \phi'} \Pi \phi' \phi. \]

One can consider elementary summands \( A_{\phi, \pm \mp} \) as integral operators on \( H \),

\[ (A_{\phi, \pm \mp} g) (z) = \frac{1}{2\pi i} \oint_{\Gamma} A_{\phi, \pm \mp} (z, z') g (z') dz', \]

whose integral kernels have integrable form and are given by

\[ A_{\phi, \pm \mp} (z, z') = \pm \chi_{\phi} (z) \Psi^{-1}_{\phi, \pm}(z') \frac{1 - \delta_{\phi, \phi'}}{z - z'} \chi_{\phi'} (z'), \]

where

\[ \chi_{\phi} (z) = \Psi_{\phi, +} (z) \Psi^{-1}_{\phi, -}(z') \frac{1 - \delta_{\phi, \phi'}}{z - z'} \]

\[ \chi_{\phi'} (z') = \Psi_{\phi', +} (z) \Psi^{-1}_{\phi', -}(z') \frac{1 - \delta_{\phi, \phi'}}{z' - z}. \]

**Figure 5:** “Sicilian” RH contour for 6-point Fuchsian systems.
with \( \chi_C(z) \) denoting the indicator function of \( C \). We leave it as an exercise for the reader to verify that \( \text{im} A_{\pm, \pm} \subseteq \ker A_{\pm, \pm} \). The above expressions provide a many-oval counterpart of the one-circle formula [2,3]. The operators \( A_{\pm, \pm} \) and \( A_{\pm, \mp} \) are analogs of the operators \( a \) and \( d \). They can be regarded as matrices whose operator entries are labeled by pairs of curves in \( C \); these entries are non-zero only if the relevant curves bound the same face.

**Remark 3.2.** One may wonder whether it is also possible to obtain an analog of the Widom’s determinant [2,3], i.e. not to use direct factorization and express \( \tau |J| \) solely in terms of the jumps. The answer is positive and can be obtained by conjugation of \( L \) by the multiplication operator \( (\bigoplus_{\epsilon \in C} \Psi_\epsilon, +) \circ (\bigoplus_{\epsilon \in C} \Psi_\epsilon, -) \): one can equivalently write

\[
\tau |J| = \det_H (1 + L), \quad L = \begin{pmatrix} 0 & A_{\pm, \mp} \\ A_{\pm, \mp} & 0 \end{pmatrix},
\]

\[
A_{\pm, \mp} = \sum_{f \in F_{\pm}} \sum_{\epsilon, \epsilon' \in C_f} (\Pi_{\epsilon-\epsilon', \mp} \tilde{f}_\epsilon^{\mp} - \delta_{\epsilon, \epsilon'} \tilde{f}_\epsilon^{\mp}) \Pi_{\epsilon', \mp}.
\]

(3.2)

### 3.2 Differentiation formula

Let us now establish the many-oval counterpart of the differentiation formula given in Theorem [2,3]. The first step is the calculation of the inverse \((1 + L)^{-1}\). To this end we first rewrite \( 1 + L \) in the form

\[
1 + L = \left( \sum_{f \in F_{\pm}} \sum_{\epsilon, \epsilon' \in C_f} \Psi_{\epsilon, \mp} \Pi_{\epsilon-\epsilon', \pm} \Psi_{\epsilon', \pm}^{-1} \sum_{f \in F_{\pm}} \sum_{\epsilon, \epsilon' \in C_f} \Psi_{\epsilon, \pm} \Pi_{\epsilon-\epsilon', \mp} \Psi_{\epsilon', \pm}^{-1} \right),
\]

(3.3)

where the first and second column correspond to the action of \( 1 + L \) on \( H_+ \) and \( H_- \). Let us note that

\[
\Gamma = \bigcup_{f \in F_{\pm}} \bigcup_{\epsilon \in C_f} C_{\epsilon}.
\]

It is useful to interpret the contributions \( \mathcal{P}_{f|f|}\left( <_{\pm} \right) := \sum_{\epsilon, \epsilon' \in C_f} \Psi_{\epsilon, \mp} \Pi_{\epsilon-\epsilon', \pm} \Psi_{\epsilon', \pm}^{-1} \) of individual faces \( f \in F_{\pm} \) to the above sums as integral operators acting from \( \bigoplus_{\epsilon \in C_f} H_\pm (C_{\epsilon}) \) to \( \bigoplus_{\epsilon \in C_f} H (C_{\epsilon}) \) by

\[
\left( \mathcal{P}_{f|f|}\left( <_{\pm} \right) \right) (z) = \pm \sum_{\epsilon, \epsilon' \in C_f} \frac{1}{2\pi i} \int_{C_{\epsilon'}} \frac{\chi_{C}(z) \Psi_{\epsilon, \mp}(z) \Psi_{\epsilon', \mp}(z')^{-1} g_{\epsilon'}(z')}{z' - z} \, dz'.
\]

(3.4)

The contour of integration is deformed to the face \( \phi_{+}(C_{\epsilon}) \) (i.e. outside the face \( f \)) whenever it becomes necessary to interpret the singular factor \( 1/(z' - z) \).

Next we construct in a similar fashion the operators \( \mathcal{P}_{\pm} : H \rightarrow H_{\pm} \) defined by

\[
\left( \mathcal{P}_{\pm} g \right) (z) = \pm \sum_{\epsilon \in C_{\epsilon}} \frac{1}{2\pi i} \int_{C_{\epsilon}} \frac{\chi_{C}(z) \Psi_{\epsilon, \mp}(z) \Psi_{\epsilon, \mp}^{-1}(z') \Psi_{\epsilon', \mp}(z')^{-1} g_{\epsilon'}(z')}{z' - z} \, dz'.
\]

(3.5)

The convention for the contour is the same, i.e. it is pushed away slightly to the negative (positive) faces for \( \mathcal{P}_{+} \) (resp. \( \mathcal{P}_{-} \). In contrast to the face operators \( \mathcal{P}_{f|f|} \), the operators \( \mathcal{P}_{\pm} \) involve the solution \( \Psi_{\epsilon} \) of the dual RHP. Constructing this solution is essentially equivalent to the calculation of the resolvent \((1 + L)^{-1}\) thanks to the following lemma.

**Lemma 3.3.** If the dual RHP is solvable, then \((1 + L)^{-1} = \left( \mathcal{P}_{\pm} \right) \).

**Proof.** Let us compute, for instance, the action of the “++” component of the product \( \left( \mathcal{P}_{\pm} \right) (1 + L) \). Given \( g_+ \in H_+ \), it reads

\[
\left( \mathcal{P}_{\pm} \sum_{f \in F_{\pm}} \sum_{\epsilon, \epsilon' \in C_f} \Psi_{\epsilon, \mp} \Pi_{\epsilon-\epsilon', \pm} \Psi_{\epsilon', \pm}^{-1} g_+ \right) (z) = \]

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= \sum_{\epsilon, \epsilon' \in C} \sum_{\epsilon \in \omega_{\epsilon, \epsilon'}} \Pi_{\epsilon, +} \left( \frac{1}{2\pi i} \oint_{e} \frac{\chi_{\epsilon}(z) \Psi_{\epsilon, +}(z) \Psi_{\epsilon', -}(z) \Psi_{\epsilon', -}(z')^{-1} \Psi_{\epsilon', +}(z')^{-1} g_{\epsilon', -}(z') d z'}{(z' - z)(z'' - z') \zeta} \right).

Recall that in this expression, the contour of integration w.r.t. \( z'' \) is slightly deformed to the face \( \phi_{+}(\epsilon'') \), and the contour of integration w.r.t. \( z' \) is slightly deformed to the face \( \phi_{-}(\epsilon') \). Therefore the integration contour in

\[ \sum_{\epsilon'' \in C} \oint_{\epsilon''} (z'')^{-1} d z' \]

can be collapsed through the face \( \phi_{-}(\epsilon'') \) and the corresponding integral vanishes. The vanishing of the "--" component may be shown in a similar way using in addition that, by definition of \( \Psi \), we have \( \Psi_{\epsilon, -} \Psi_{\epsilon, +} = \Psi_{\epsilon, +} \Psi_{\epsilon, -} \) and shrinking the contours through positive faces.

For the "++" component, the analog of the above is

\[ \left( \mathcal{P}_{2} + \sum_{j \in \mathcal{F}_{2}} \sum_{\epsilon, \epsilon' \in \mathcal{C}_{j}} \Psi_{\epsilon, -} \Pi_{\epsilon, -} \epsilon', + \Psi_{\epsilon', -}^{-1} g_{j}(z) \right) = \sum_{\epsilon, \epsilon' \in C} \sum_{\epsilon' \in \omega_{\epsilon, \epsilon'}} \Pi_{\epsilon, +} \left( \frac{1}{2\pi i} \oint_{e} \frac{\chi_{\epsilon}(z) \Psi_{\epsilon, +}(z) \Psi_{\epsilon', -}(z) \Psi_{\epsilon', -}(z')^{-1} \Psi_{\epsilon', +}(z')^{-1} g_{\epsilon', -}(z') d z'}{(z' - z)(z'' - z') \zeta} \right). \]

The contours of integration w.r.t. \( z' \) and \( z'' \) are deformed to \( \phi_{-}(\epsilon') \) and \( \phi_{-}(\epsilon'') \), and the former is located to the positive side of the latter on the coinciding faces. Collapsing the contour in the integral through the face \( \phi_{+}(\epsilon'') \), we eventually pick up a residue at \( z' = z \), equal to \( \frac{2\pi i \Psi_{\epsilon, +}(z)^{-1}}{z'' - z} \), if \( \epsilon \in C_{\phi_{+}(\epsilon'')} \).

Using this, reorganize the previous expression as

\[ \sum_{\epsilon \in C} \sum_{\epsilon' \in \omega_{\epsilon, \epsilon'}} \Pi_{\epsilon, +} \left( \frac{1}{2\pi i} \oint_{e} \frac{\chi_{\epsilon}(z) \Psi_{\epsilon, -}(z) \Psi_{\epsilon', -}(z')^{-1} g_{\epsilon', -}(z') d z'}{z'' - z} \right). \]

For \( \epsilon \neq \epsilon'' \), the integral defines a function of \( z \) that analytically continues to \( \varphi_{+}(\epsilon'') \supset \phi_{-}(\epsilon') \) and therefore vanishes under the action of \( \Pi_{\epsilon, +} \). There remains a sum

\[ \sum_{\epsilon \in C} \Pi_{\epsilon, +} \left( \frac{1}{2\pi i} \oint_{e} \frac{\chi_{\epsilon}(z) \Psi_{\epsilon, -}(z) \Psi_{\epsilon', -}(z')^{-1} g_{\epsilon', -}(z') d z'}{z'' - z} \right), \]

where \( \epsilon \) is slightly deformed to \( \phi_{-}(\epsilon) \) as to avoid the singularity at \( z'' = z \). Deforming it instead to \( \phi_{+}(\epsilon) \) we obtain a function of \( z \) annihilated by \( \Pi_{\epsilon, +} \) at the expense of picking up the residue at \( z'' = z \), equal to \( g_{\epsilon, +} \). This ultimately yields the expected result \( \sum_{\epsilon \in C} \Pi_{\epsilon, +} \chi_{\epsilon} g_{\epsilon, +} = g_{+} \). The calculation for the "--" component is completely analogous.

**Theorem 3.4.** Suppose that the functions \( \Psi_{\epsilon, \pm}(z) \) appearing in the direct factorization of individual jumps \( J_{\epsilon}(z) \) smoothly depend on an additional parameter \( t \). If the solution \( \Psi_{\pm}(z) \) of the dual RHP exists and smoothly depends on \( t \), then

\[ \partial_{t} \ln \tau [J] = \sum_{\epsilon \in C} \frac{1}{2\pi i} \oint_{e} \mathrm{Tr} \left( \Pi_{\epsilon, +} f^{-1} J_{\epsilon} \left( \partial_{t} \Psi_{\epsilon, -} \Psi_{\epsilon, -}^{-1} + \Psi_{\epsilon, +}^{-1} \partial_{t} \Psi_{\epsilon, +} \right) \right) d z. \]  

(3.6)

**Proof.** We are going to mimick the proof of Theorem 2.3. Note e.g. that the operators \( A_{\pm} \) in (3.1), or more precisely their conjugates \( A_{\pm}^{-1} \) in (3.2), are analogs of the operators \( \Pi_{\pm} f^{-1} \Pi_{\pm} \) and \( \Pi_{\pm} \Pi_{\pm} \). The main difference here is that it becomes convenient to write various projection operators as explicit contour integrals.
Differentiating the determinant yields

$$\partial_t \ln |J| = \text{Tr}_H \left( (1 + L)^{-1} \partial_t L \right) = \text{Tr}_H \left( \mathcal{P}_{z^+} - |H_+ \partial_z A_{+} + \mathcal{P}_{z^-} + |H_- \partial_z A_{-} \right),$$

where the last equality is obtained using that im \( A_{\pm} \) $\subseteq H_\pm$. Moreover, thanks to the property that $H_\pm \subseteq \ker A_{\pm}$, the first projector in the definition (3.5) of $\mathcal{P}_{z^\pm}$ may be omitted. The computation of traces then reduces to residue calculation. Indeed, we have

$$\text{Tr}_H \mathcal{P}_{z^-} \partial_z A_{-} = \sum_{\phi' \in \mathcal{C}_{\phi^+} \setminus \mathcal{C}_{\phi^-}} \sum_{\epsilon \in \mathcal{C}_{\phi^+} \setminus \mathcal{C}_{\phi^-}} \frac{1}{(2\pi i)^2} \oint_{\epsilon'} \oint_{\epsilon''} \text{Tr} \left\{ \chi_{\epsilon''} (z) \Psi_{\epsilon',+} (z) \overline{\Psi}_{\epsilon',-} (z')^{-1} \Psi_{\epsilon'',+} (z')^{-1} \times \frac{\partial_1 \left( \Psi_{\epsilon'',+} (z') \overline{\Psi}_{\epsilon'',+} (z'')^{-1} \right)}{z'' - z'} \bigg|_{z'' = z} \right\} dz' dz =$$

$$= \sum_{\epsilon \in \mathcal{C}} \sum_{\phi' \in \mathcal{C}_{\phi^+} \setminus \mathcal{C}_{\phi^-}} \frac{1}{2\pi i} \oint_{\epsilon} \text{Tr} \left\{ \partial_1 Z_{\epsilon',-} \overline{Z}_{\epsilon',-} \partial_1 Z_{\epsilon',+} \right\} dz . \quad (3.7a)$$

Recall that the contours $\epsilon'$ of integration with respect to $z'$ are slightly pushed to positive faces $\phi_+$ (of $\epsilon'$) according to the definition of $\mathcal{P}_{z^-}$. The contribution of the 1st term under trace is readily computed by collapsing the contours $\epsilon'$ through negative faces and is given by (minus) the residue at $z' = z$,

$$\frac{1}{2\pi i} \oint_{\epsilon} \text{Tr} \left\{ \partial_1 Z_{\epsilon',-} \overline{Z}_{\epsilon',-} \partial_1 Z_{\epsilon',+} \right\} \partial_1 Z_{\epsilon',-} dz . \quad (3.7b)$$

To compute in a similar way the 2nd term under trace, rearrange the sum $\sum_{\epsilon \in \mathcal{C}} \sum_{\phi' \in \mathcal{C}_{\phi^+} \setminus \mathcal{C}_{\phi^-}}$ as $\sum_{\epsilon \in \mathcal{C}} \sum_{\phi' \in \mathcal{C}_{\phi^+} \setminus \mathcal{C}_{\phi^-}}$. Shrink $\epsilon$ afterwards the contours $\epsilon'$ through negative faces we meet no poles and therefore the corresponding integrals sum up to zero.

One can analogously prove that

$$\text{Tr}_H \mathcal{P}_{z^+} \partial_z A_{+} = -\frac{1}{2\pi i} \sum_{\epsilon \in \mathcal{C}} \oint_{\epsilon} \text{Tr} \left\{ \partial_2 Z_{\epsilon,+} \overline{Z}_{\epsilon,+} \partial_2 Z_{\epsilon,+} \right\} \partial_2 Z_{\epsilon,+} dz . \quad (3.7b)$$

It remains to show that the sum of (3.7a) and (3.7b) coincides with the right side of (3.6). To this end, note that

$$\text{Tr} J_{\epsilon'}^{-1} \partial_1 J_{\epsilon'} \left[ \partial_2 Z_{\epsilon',-} \overline{Z}_{\epsilon',-} \partial_2 Z_{\epsilon',+} \right] =$$

$$= \text{Tr} \left\{ \partial_1 Z_{\epsilon',-} \overline{Z}_{\epsilon',-} \partial_1 Z_{\epsilon',-} \right\} = \text{Tr} \left\{ \partial_1 Z_{\epsilon',+} \overline{Z}_{\epsilon',+} - \partial_1 Z_{\epsilon',-} \overline{Z}_{\epsilon',+} \right\} \overline{Z}_{\epsilon',+} = \left\{ \partial_1 Z_{\epsilon',-} \overline{Z}_{\epsilon',-} \partial_1 Z_{\epsilon',+} \right\} =$$

where to obtain the first equality, we use that $J_{\epsilon'} = \Psi_{\epsilon',+} \overline{\Psi}_{\epsilon',+}$; the second equality is obtained by replacing $\Psi_{\epsilon',+} = \Psi_{\epsilon',-} \overline{\Psi}_{\epsilon',-}$ \( \mathcal{P}_{z^+} \) $\mathcal{P}_{z^-} \) in the 4th term under trace. In the last expression, the 1st and 4th term reproduce [3.7a] and [3.7b]. The 2nd and 3rd term are given by the boundary values of functions analytic in $\phi_+$ (of $\epsilon'$) and $\phi_-$ (of $\epsilon'$), therefore the corresponding integrals vanish.

Remark 3.5. We conclude this subsection by mentioning the work of Palmer [Pal] on the tau function of the massive Euclidean Dirac operator in the presence of Aharonov-Bohm fluxes. While it may seem unrelated to the present paper, it is the adaptation of the localization ideas of [Pal] to the chiral case which led to [GL16]. Here we made a substantial further improvement by getting rid of artificial doubling of the RHP contours, generalizing the results to arbitrary oval configurations and providing a concise definition of $\tau (J)$. It might be interesting to understand whether it is possible to go backwards and apply our results, in particular, series expansions of Subsection 2.2 to the study of correlation functions of twist fields in free massive QFTs.
3.3 Jimbo-Miwa-Ueno differential

In this subsection, we explain how to recover from Theorem 3.4 the Jimbo-Miwa-Ueno definition [MU] of the isomonodromic tau function for systems of linear differential equations with rational coefficients. Let us start with a Fuchsian system with \( n \) regular singular points \( a_0 = 0, a_1, \ldots, a_{n-2}, a_{n-1} = \infty \) on \( \mathbb{C}P^1 \),

\[
\partial_z \Phi = \Phi A(z), \quad A(z) = \sum_{k=0}^{n-2} \frac{A_k}{z - a_k}, \quad A_k \in \text{Mat}_{N \times N}(\mathbb{C}). \tag{3.8}
\]

For simplicity it is assumed that \( a_1, \ldots, a_{n-2} \in \mathbb{R}_{>0} \) and that the singularities are ordered as \( a_1 < \cdots < a_{n-2} \). The fundamental solution \( \Phi(z) \) can then be considered as a single-valued analytic function on \( \mathbb{C}\setminus \mathbb{R}_{\geq 0} \). Similarly to Subsection 2.3.1 we also assume that \( A_0, \ldots, A_{n-2}, A_{n-1} := -\sum_{k=0}^{n-2} A_k \) are diagonalizable as \( A_k = G_k^{-1} \Theta_k G_k \) and have non-resonant eigenvalues, so that in the neighborhood of each singular point we have

\[
\Phi(z) = C_{k_0} (a_k - z)^{\Theta_k} G_k(z), \quad c = \text{sgn} \Im z,
\]

where \( G_k(z) \) are holomorphic invertible and normalized so that \( G_k'(\infty) = G_k(\infty) = 1 \) (for \( a_{n-1} = \infty \) the formula above should be modified in the obvious manner). The connection matrices \( \{C_{k+1}\} \) satisfy the compatibility conditions analogous to (2.22) and, together with local monodromy exponents \( \{\Theta_k\} \), encode the monodromy representation of \( \pi_1(\mathbb{C}P^1 \setminus \{n \text{ points}\}) \) associated to \( \Phi(z) \).

Different pants decompositions of the \( n \)-punctured sphere give rise to different RHPs associated with the linear system (3.3) and distinct Fredholm determinant representations of the tau functions, adapted to analysis of different asymptotic regimes. Since at this point we only want to give an example of an \( n > 4 \) analog of the relation (2.26), let us pick the simplest “linear” pants decomposition leading to a RHP set on a collection of \( n-3 \) circles \( \mathcal{C}_1, \ldots, \mathcal{C}_{n-3} \) decomposing \( \mathbb{C}P^1 \) into \( n-2 \) faces \( f^{[1]}, \ldots, f^{[n-2]} \) as shown in Fig. 6. By convention, the faces \( f^{[2k+1]} \) and \( f^{[2k]} \) will be of color \( + \) and \( - \), respectively.

Let \( M_k \in \text{GL}(N, \mathbb{C}) \) \((k = 0, \ldots, n-1)\) denote the monodromy of \( \Phi(z) \) along the contour starting on the negative real axis and going around \( a_k \) counterclockwise. These monodromies satisfy the cyclic relation \( M_0 M_1 \cdots M_{n-1} = I \). It will be convenient for us to consider the products \( M_{0-k} := M_0 \cdots M_k \) and suppose that they can be diagonalized as

\[
M_{0-k} = S_k^{-1} e^{2\pi i \mathcal{E}_k} S_k, \quad k = 0, \ldots, n-2,
\]

where the eigenvalues of diagonal matrices \( \mathcal{E}_k \) are assumed to be pairwise distinct mod \( \mathbb{Z} \). It may also be assumed that \( \text{Tr} \mathcal{E}_k = \sum_{j=0}^{k} \text{Tr} \Theta_j \) and that \( \mathcal{E}_0 = \Theta_0, \mathcal{E}_{n-2} = -\Theta_{n-1} \).

One may assign to an arbitrary collection of non-intersecting ovals a tree graph with vertices given by faces in \( F_+ \cup F_- \) and the edges given by their common boundaries. The contour shown in Fig. 6b leads to a linear graph whereas e.g. the contour in Fig. 5a yields a star-shaped graph with 4 vertices.

Figure 6: RH contour and coloring by \{+,-\} associated to linear pants decomposition for Fuchsian systems.
Denote by $\Phi^{[k]}(z)$ ($k = 1, \ldots, n-2$) the solution of 3-point Fuchsian system associated to the face $f^{[k]}$ (cf Subsection 23.3) which has regular singularities at 0, $a_k$ and $\infty$ characterized by monodromies $M_{0-k-1}$, $M_k$ and $M_{0-k}$. The local behavior of this solution near the singular points is given by

$$
\Phi^{[k]}(z) = \begin{cases}
S_{k-1} - (z)^{-2k-1} G^{[k]}_0(z), & z \to 0, \\
C_k (a_k - z)^{\epsilon_k} G^{[k]}_k(z), & z \to a_k, \\
S_k (z)^{-2} G^{[k]}_{\infty}(z), & z \to \infty,
\end{cases}
$$

where $G^{[k]}_{0,\infty}(z)$ are holomorphic invertible in the respective neighborhoods of 0, $a_k$, $\infty$. The 3-point solutions define the jumps on $\mathcal{C}_1 \cup \ldots \cup \mathcal{C}_{n-3}$: in the notation of the previous subsection, we have

$$
\Psi^{[k]}_{\epsilon_k+1}(z) = G^{[2k]}_0(z) = (z)^{-2k-1} S^{-1}_{2k-1} \Phi^{[2k]}(z),
$$

$$
\Psi^{[k]}_{\epsilon_k-1}(z) = G^{[2k-1]}_0(z) = (z)^{-2k-1} S^{-1}_{2k-1} \Phi^{[2k-1]}(z),
$$

$$
\Psi^{[k]}_{\epsilon_k+}(z) = G^{[2k]}_\infty(z) = (z)^{-2k} S^{-1}_{2k} \Phi^{[2k]}(z),
$$

$$
\Psi^{[k]}_{\epsilon_k-}(z) = G^{[2k+1]}_0(z) = (z)^{-2k-1} S^{-1}_{2k} \Phi^{[2k+1]}(z),
$$

and $\Psi(z) = \Phi^{[k]}(z)^{-1} \Phi(z)$ for $z \in f^{[k]}$.

Substitute these expressions into differentiation formula (3.6) choosing therein $t = a_k$. The only circles that contribute to the sum are $\mathcal{C}_{\epsilon_k-1}$ and $\mathcal{C}_{\epsilon_k}$ (for the others $\beta, J = 0$). Moreover, for instance, for odd $k$ the only $\Psi^{[k]}_{\epsilon_k}$ depending on $a_k$ are $\Psi^{[k]}_{\epsilon_k-}$ and $\Psi^{[k]}_{\epsilon_k-}$, so that in this case

$$
\partial_{a_k} \ln \tau [J] = - \frac{1}{2 \pi i} \oint_{\mathcal{C}} \left\{ \partial_z \Psi^{[k]}_{\epsilon_k-1}, \Psi^{[k]}_{\epsilon_k-1}, \Psi^{[k]}_{\epsilon_k-1}, \partial_{a_k} \Psi^{[k]}_{\epsilon_k-1} \right\} \, dz + \oint_{\mathcal{C}} \partial_z \psi^{[k]}_{\epsilon_k+}, \psi^{[k]}_{\epsilon_k+}, \psi^{[k]}_{\epsilon_k+}, \partial_{a_k} \psi^{[k]}_{\epsilon_k+} \bigg| \, dz = - \text{res}_{z=a_k} \left\{ \partial_z \left( G^{[k]}_{\epsilon_k-1} G^{[k]}_{\epsilon_k-1} \right) \right\} = \text{res}_{z=a_k} \left\{ \partial_z \left( G^{[k]}_{\epsilon_k-1} G^{[k]}_{\epsilon_k-1} \frac{\Theta_k}{z-a_k} \right) \right\} = \frac{\partial_{a_k} \tau_{\text{JMU}} - \partial_{a_k} \tau_{\text{JMU}}^{[k]}}{a_k},
$$

where $\tau_{\text{JMU}}$ is the tau function of the $n$-point system (3.8) and $\tau_{\text{JMU}}^{[k]} = \text{const} \cdot a_k^{\frac{1}{2} \text{Tr}[S^{-2} - S^{-2} - S^{-2}]}$ is the tau function of the 3-point Fuchsian system for $\Phi^{[k]}$. The transition from the 2nd to the 3rd line is obtained using that the integrands continue to the same meromorphic function on $f^{[k]}$ with the only pole at $z = a_k$. The 4th line follows from (3.9) and the 5th from the fact that $G^{[k]}_{\epsilon_k-1} G^{[k]}_{\epsilon_k-1}$ and $\partial_{g_k} G^{[k]}_{\epsilon_k-}$ are holomorphic in $f^{[k]}$. The final equality follows from the Jimbo-Miwa-Ueno definition of the tau function, cf [JMU] eq. (1.23).

We have thus shown that for Fuchsian systems and linear pants decomposition the tau function defined by the Fredholm determinant (3.1) coincides with

$$
\tau [J] = \frac{\tau_{\text{JMU}}(a_1, \ldots, a_{n-2})}{\prod_{k=1}^{n-2} \tau_{\text{JMU}}^{[k]}(a_k)},
$$

(3.10)

When the singular points $a_0, \ldots, a_{n-1}$ are irregular, one can obtain a similar identification by decomposing the original RHP into e.g. an $n$-point Fuchsian one (with regular singularities at $a_0, \ldots, a_{n-1}$) and $n$ two-point RHPs with one regular and one irregular singularity.

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