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DSR as an explanation of cosmological structure

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Abstract

Deformed special relativity (DSR) is one of the possible realizations of a varying speed of light (VSL). It deforms the usual quadratic dispersion relations so that the speed of light becomes energy dependent, with preferred frames avoided by postulating a nonlinear representation of the Lorentz group. The theory may be used to induce a varying speed of sound capable of generating (near) scale-invariant density fluctuations, as discussed in a recent letter. We identify the nonlinear representation of the Lorentz group that leads to scale invariance, finding a universal result. We also examine the higher order field theory that could be set up to represent it.

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1. Introduction

In a recent letter [1], we proposed a mechanism for producing scale-invariant density fluctuations of appropriate amplitude based on a decaying speed of sound. The mechanism is quite general, and can be implemented using a variety of methods. The examples of $\kappa$-essence [2, 3] and varying speed of light (VSL) [4–6] were given, and other possibilities, such as non-adiabatic hydrodynamical matter, were considered. But in [1], we emphasized the generality of the proposal and deliberately chose not to marry it to any specific model. This attitude was reversed in [7], where we initiated model-building appealing to bimetric VSL theories. Here, we propose yet another concrete VSL realization of scale-invariant fluctuations, this time based on theories that rub shoulders with the phenomenology of quantum gravity.

In spite of the strong reactions they elicit [8–10], VSL theories may be nothing more offensive than non-trivial realizations of the Lorentz group. Two such approaches stand out: bimetric VSL and deformed special relativity (DSR). In the former, the Lorentz group is realized by different metrics for matter and for gravity, leading to different speeds for photons (and other massless matter particles) and gravitons [11]. In the latter, the dispersion relations are deformed from their usual quadratic form, so that the speed of all massless particles becomes energy dependent [13, 14]. To prevent the introduction of a preferred frame one then chooses a suitable nonlinear representation of the Lorentz group [15, 16]. Other interpretations are possible—and of importance—but they will not be relevant in this paper [17–20].
It is not evident that DSR may be used to implement the varying speed mechanism of [1], where the propagation speed $c_s$ is envisaged to vary with time, not with energy. In bimetric theories [7] the speed of light varies in time, and this is passed on to all matter propagation speeds, including the speed of acoustic oscillations, $c_s$. But in DSR the speed of light is energy or wavelength dependent: not time dependent. Yet DSR can indeed be used to implement a time-varying $c_s$ 'by proxy', in an expanding universe. If we focus on a fixed comoving mode (as done in the usual perturbations’ calculation) we see its physical size stretch in time. Its ‘energy scale’ therefore changes in time, and so, under DSR, the mode is effectively subject to a time varying speed of light and consequently of sound.

The cosmological redshift acts to convert a frequency-dependent speed of light into a time-dependent speed of light. This idea was already recognized in [21], in a different guise. If the speed of massless particles increases with energy, then the fact that the ‘average particle’ has a higher energy in the early universe means that the ‘ambient’ speed of light is also higher.

2. Very basic DSR

We start with a quick review of DSR, cast in a formalism that can be used here. It is possible that alternative formulations of DSR [19, 20] (and even of theories which break Lorentz symmetry) may be plugged into the fluctuations’ calculation that follows, but this has not been checked.

Let us consider a deformed dispersion relation (DDR) of the form

$$E^2 f_1^2 - k^2 f_2^2 = m^2, \quad (1)$$

where $f_1$ and $f_2$ may be general functions of $E$ and $k$. Then, the group speed of light (or any other massless particles), $c = \frac{dE}{dp}$, becomes energy dependent. Expression (1) is not invariant under the linear Lorentz transformations. However, it is possible to find a nonlinear representation of the Lorentz group which leaves it invariant and prevents the introduction of a preferred frame, so that relativity between inertial observers is preserved. This is done by considering the map

$$U \circ (E, p) = (E f_1, p f_2), \quad (2)$$

and then changing the representation according to

$$K^i = U^{-1} L^i_k U, \quad (3)$$

where $L^i_{ab} = p_a \frac{\partial}{\partial p^i} - p_b \frac{\partial}{\partial p^i}$ are the standard Lorentz generators. Exponentiation then gives us a set of nonlinear transformations for which (1) is invariant.

DSR has provided an excellent bridge between phenomenology and quantum gravity. The existence of an invariant length or energy scale is the central connection, and this can easily be represented by the singular points of the transformation $U$. For this reason such theories are also called ‘doubly special’ relativity (fortunately leading to the same acronym). However, one may look at it more generally. For a recent review the reader can consult [29].

A number of issues arise at once. The theory is defined in momentum space, and once linearity is lost duals no longer decouple and mimic each other. The introduction of an energy-dependent spacetime metric may then be necessary, the so-called rainbow metric [22]. But other constructions are possible [23–26]. Also the field theory realization of DSR remains an open issue [19, 20]. Higher order derivative (HOD) theories will be advocated here as a realization of DSR [27], not so much because we love them, but because they permit a direct realization of the varying speed of sound mechanism, which we stress is the central topic in this paper.

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HOD theories have been much maligned, not always fairly. They are usually blamed for hiding ghosts, but the argument is seldom understood, often not even known to those who wield it. Many pathologies used to smear HOD theories only occur if a theory with infinitely many derivatives is truncated [32, 33]. Also the threat of pathologies does not even arise if the HOD behaviour is reserved to spatial derivatives, then all that happens is that more data are required at spatial infinity for the problem to be well posed. This is definitely the case for the HOD theories considered in this paper.

Note that various DSRs and HOD theories may be made to correspond to a given DDR, particularly in the massless limit. The expressions for $f_1$ and $f_2$ in (1) could be changed (and they could be seen as different functions of $E$ and $k$) leading to an algebraically equivalent DDR but a different $U$ map, and so a different DSR and HOD field theory.

3. An adaptation of the decaying speed of sound mechanism

It is straightforward to adapt the calculation in [1] to the present, slightly different context (this was already done, with a different motivation, by [30, 31]). Here we predict ‘from first principles’, by means of DSR, deformed dispersion relations for the fluctuations. Thus, their speed of sound is wavelength dependent. But since the wavelength of each mode increases due to cosmic expansion, its speed of sound is time dependent. This time dependence comes about by proxy—due to the indirect effect of expansion—but it has equivalent effects in terms of structure formation.

Let us consider DDRs in the form (1) with $f_1 = 1$ and $f_2 = g(\lambda k)$, where $k$ is a wavevector, and $\lambda$ is a parameter with units of $M^{-1}$. We can then take the massless limit $m = 0$, and adapt to an expanding universe by introducing comoving $\omega$ and $k$, with $g \to g(\lambda k/a)$. Thus,

$$\omega = k g(\lambda k/a),$$ (4)

and if we consider an asymptotic regime where $g$ behaves as the power-law $g(x) \propto x^\gamma$ we find

$$c = \frac{d\omega}{dk} = (\gamma + 1) \frac{\omega}{k} \propto \left(\frac{\lambda k}{a}\right)^\gamma.$$

Since $a \propto \eta^{1/3}$ with $\epsilon = \frac{1}{2}(1 + w)$, where $w = p/\rho$ is the equation of state, we can make contact with the law, $c \propto \eta^{-\alpha}$, used in [1], with

$$\alpha = \frac{\gamma}{\epsilon - 1}.$$ (6)

But note that $c$ is now $k$ dependent too, an important difference with respect to [1].

Whether we employ a hydrodynamical fluid or a scalar field the density fluctuations are described by a modified harmonic oscillator equation. In terms of the curvature perturbation $\zeta = -v/z$, its equation takes the form [3, 34, 35]

$$v'' + \left[\omega^2 - \frac{z''}{z}\right] v = 0,$$ (7)

but now $z \propto a$, as explained in [30] (cf [1]). Here we can write $\omega = ck$ for convenience, because for the models considered $\omega/k$ and $d\omega/dk$ (the phase and group speeds) are the same up to a factor of order one.

For modes to start oscillating and then freeze out (i.e. for the horizon problem to be solved), we need the term in $\omega$ in equation (7) to dominate first. For an expanding universe with $w > -1/3$ (for which $\eta$ is positive and increases from zero) this requires $\alpha > 1$, that is

$$\gamma > \gamma_0 = \epsilon - 1 = \frac{1 + 3w}{2}.$$ (8)

This should be seen as the condition zero for our calculation to make sense.
As in [1], equation (7) can be transformed into a Bessel equation, with a boundary condition obtained in the WKB limit. Under (8) modes start inside the horizon (set by \(ck \eta \sim 1\)), so that we can ignore the term in \(z''/z\) and find the appropriately normalized WKB solution

\[
v \sim \frac{e^{ik\int c d\eta}}{\sqrt{ck}} \sim \frac{e^{-i\beta ck \eta}}{\sqrt{ck}}, \tag{9}\]

where \(\beta = 1/(\alpha - 1) > 0\). The full solution to equation (7) then becomes

\[
v = \sqrt{\beta} \eta (A J_\nu(\beta ck \eta) + B J_{-\nu}(\beta ck \eta)), \tag{10}\]

where \(A\) and \(B\) are \(k\)-independent numbers of order 1. The order \(\nu\) is given by

\[
\nu = \frac{3 - \epsilon}{2(\gamma - \epsilon + 1)}, \tag{11}\]

if \(-1/3 < w < 1\) (minus this expression if \(w > 1\)).

The spectrum left outside the horizon can now be found. Since \(c \eta\) is a decreasing function of time, the negative order solution is the growing mode, so that asymptotically we have

\[
v \sim \frac{\sqrt{\beta} \eta}{(ck \eta)^\nu}. \tag{12}\]

A further adaptation of [1] is required, because \(c\) in this expression is \(k\) dependent. Since \(\zeta = -v/z\), and the spectral index is defined from \(k^3 \zeta^2 = A^2 k^{n_s - 1}\), we have

\[
n_s - 1 = \frac{\epsilon(\gamma - 2)}{\gamma - \epsilon + 1}. \tag{13}\]

Scale invariance therefore requires \(\gamma = 2\) for all equations of state which satisfy \(-1/3 < w < 1\).\(^2\) This also complies with (8) for all \(w\) in this range. We can make the spectrum as red as we want (with \(\gamma_0 < \gamma < 2\)), but the bluest it gets is \(n_s = 1 + \epsilon\), for \(\gamma \to \infty\).

As in [1], the amplitude of the fluctuations may be found by refining the DDRs to \(g = 1 + (\lambda k)^2\), enforcing the correct low-energy limit. The scale \(\lambda\) is then responsible for the amplitude \(A\) in \(k^3 \zeta^2 = A^2 k^{n_s - 1}\). Straightforward algebra leads to \(A \sim 1/(\lambda M_{Pl})\), implying that \(\lambda \sim 10^5 L_{Pl}\).

It is somewhat surprising that scale invariance is associated with a universal law, with details such as the background equation of state \(w\) falling out of the result (see [1, 7] for similar miracles). However, here this is only true if the sub-horizon normalization is chosen to match a vacuum quantum state. Should it be a thermal state, as discussed in [1], we have

\[
n_s - 1 = \frac{(\epsilon - 1)(\gamma - 1) - 2}{\gamma - \epsilon + 1},. \tag{14}\]

so that scale invariance now requires

\[
\gamma = \frac{\epsilon + 1}{\epsilon - 1} = \frac{5 + 3w}{1 + 3w}, \tag{15}\]

\(^1\) This expression cannot be simply read off from [1] because here \(z \propto a\), not \(z \propto a/c\).

\(^2\) For \(w = 1\) we have \(n_s = 4\) for all \(\gamma\); for \(w > 1\) the answer is more complex.
and the condition for scale invariance becomes $w$ dependent. In the range under study ($-1/3 < w < 1$) it requires $\gamma > 2$, for example, for a radiation background it requires $\gamma = 3$.

4. The associated DSR

What can we learn about DSR from this calculation? Foremost we have constrained the dispersion relations of the fluctuations. Specifically, if vacuum quantum fluctuations are responsible for the structure of the universe, then for all $w$ the DDRs should be of the form

$$\omega^2 - k^2(1 + (\lambda k)^2)^2 = m^2,$$

with $\lambda \sim 10^5 L_{Pl}$. This is not a truncation: quite the opposite. We are probing the $\lambda k \gg 1$ regime, so the term in $(\lambda k)^4$ should be the highest power in the dispersion relation. Different lower powers are admissible, for example, the DDRs could equally be $\omega^2 - k^2(1 + \lambda k)^4 = m^2$.

The speed of light profile could be $c = 1 + (\lambda k)^2$ or $c = (1 + \lambda k)^2$ with the same effects for structure formation, but terms in $(\lambda k)^3$ or $(\lambda k)^6$ in $c(k)$ are excluded. However if thermal (as opposed to quantum vacuum) fluctuations are behind the structure of the universe more general DDRs become possible.

These DDRs may be incorporated into a variety of DSRs or other similar such constructions and the assumptions of the calculation are not totally insensitive to the specific realization. Since structure formation only probes the massless limit, only the ratio of $f_1$ and $f_2$ is constrained. $f_1$ and $f_2$ can then be seen as functions of $k$, $E$ or both. These algebraic rearrangements do not affect the DDRs themselves, but do affect the DSR that contains them [16]. They are also reflected in the associated field theory [27, 33].

Crucially in our calculation, the field theory should be such that the deformations only affect spatial gradients, i.e. derivatives $D_\mu = (g_{\mu\nu} - n_\mu n_\nu)\nabla^\nu$ where to zeroth order $n_\mu$ points along the cosmological time. Then, the background evolution in a perturbed expanding universe is not affected by the deformation, also $z \propto a$ in the perturbation calculations (see [30] for details). We stress that had we chosen a DSR with a HOD theory deformed in time derivatives none of this would be true. The Friedman equations would then be modified, the constant $w$ assumption could not be guaranteed, and the usual background scaling solutions could not be used. As discussed in [30], the linearized perturbation equation (7) would also acquire extra terms (or seen in another way, $z$ would no longer be proportional to $a$). All of this would render the calculation very difficult (no longer simply solved in terms of Bessel functions). We shall therefore assume a DSR with an associated HOD theory that only deforms the spatial gradients.

For example, we could take a DSR generated by

$$U \circ (E, p) = (E, p\sqrt{(1 + (\lambda p)^2)}).$$

Its associated HOD field theory satisfies the Klein–Gordon equation:

$$[\partial_0^2 - \partial_i^2(1 + (\lambda \partial_i)^2) + m^2]\phi = 0.$$

A coupling to gravity could be chosen such that this became

$$[\nabla_\mu \nabla^\mu + \lambda^4(D_\mu D^\mu)^2 + m^2]\phi = 0$$

(see [22, 36] for possible couplings of DSR and gravity). The conditions of our calculation (that the deformation is purely spatial) are then satisfied. It is possible that other frameworks for DSR and its coupling to gravity satisfy this condition.

Even within this framework other $U$ are possible. For example, with

$$U \circ (E, p) = (E, p(1 + (\lambda p)^2)),$$
one gets the nonlinear representation of Lorentz transformations:

\[
\begin{align*}
E' &= \gamma [E - v p_x (1 + (\lambda p)^2)], \\
p'_{x} (1 + (\lambda p')^2) &= \gamma [p_x (1 + (\lambda p)^2) - v E], \\
p'_{y} (1 + (\lambda p')^2) &= p_y (1 + (\lambda p)^2), \\
p'_{z} (1 + (\lambda p')^2) &= p_z (1 + (\lambda p)^2).
\end{align*}
\] (21)

If we restrict ourselves to transformations along the direction of motion, in the limit \(\lambda p \gg 1\) these may be written out explicitly as

\[
\begin{align*}
E' &= \gamma (E - v p_x (\lambda p)^2), \\
p' &= [\gamma (p^2 - v E \lambda^{-2})]^{1/3}.
\end{align*}
\] (22)

The HOD field theory associated with it has Klein–Gordon equation:

\[
\begin{align*}
\partial_0^2 - \partial_i^2 (1 + (\lambda \partial_i)^2)^{2/3} + m^2 \phi &= 0,
\end{align*}
\] (23)

which is still within the requirements of our calculation.

But not all DSRs will do, even if they incorporate (16). For example, we could have taken

\[
\begin{align*}
E^2 - k^2 (1 + (\lambda E)^2)^{1/3} &= m^2, \\
E &= \gamma (E - v k_x (1 + (\lambda E)^2)^{1/3}), \\
E' &= \gamma (E - v k_x (1 + (\lambda E)^2)^{1/3}), \\
E' &= \gamma (E - v k_x (1 + (\lambda E)^2)^{1/3}), \\
E' &= \gamma (E - v k_x (1 + (\lambda E)^2)^{1/3}),
\end{align*}
\] (24)

which in the massless limit is equivalent to (16). It is realized by the nonlinear representation

\[
\begin{align*}
E' &= \gamma (E - v k_x (1 + (\lambda E)^2)^{1/3}), \\
k_x' &= \frac{\gamma (k_x (1 + (\lambda E)^2)^{1/3} - v E)}{(1 + \gamma^2 \lambda^2 (E - v k_x (1 + (\lambda E)^2)^{1/3})^2)^{1/3}}, \\
k_y' &= \frac{k_y}{(1 + \gamma^2 \lambda^2 (E - v k_x (1 + (\lambda E)^2)^{1/3})^2)^{1/3}}, \\
k_z' &= \frac{k_z}{(1 + \gamma^2 \lambda^2 (E - v k_x (1 + (\lambda E)^2)^{1/3})^2)^{1/3}}.
\end{align*}
\] (25)

and the modified Klein–Gordon theory

\[
\begin{align*}
-\partial_0^2 + \partial_i^2 (1 - \lambda \partial_i^2)^{2/3} + m^2 \phi &= 0.
\end{align*}
\] (26)

This is no longer consistent with the assumptions of the calculation, i.e., complete decoupling between deformation and time derivatives.

None of these DSR theories is ‘doubly special’ in the sense that it has an energy or momentum scale which is invariant under Lorentz transformations. This could be easily implemented by choosing DDRs of the form

\[
\begin{align*}
\frac{E^2 - p^2 (1 + (\lambda p)^2)^2}{1 - (L p^2 E)^2} = m^2.
\end{align*}
\] (27)

Although the DDRs are the correct ones the ensuing HOD field theory does not comply with the assumptions of the calculation. It is possible, however, that a modified calculation could be carried out and lead to scale invariance even for these theories.

### 5. Conclusions

If the speed of light, seen as a function of the wavelength, has a pole of degree 2 at the origin, then the simplest adaptation of the varying speed of sound mechanism for structure formation leads to scale-invariant fluctuations. Additional minimal technical assumptions on DSR and its coupling to gravity have to be made (realized, for example, using [22, 27]). Other
interpretations of DSR may or may not comply with the assumptions of the calculation, and we encourage their proponents to carry out this work. Here, for definiteness, we embedded the required DDRs into nonlinear representations of the Lorentz group [16], HOD field theories [27] and the rainbow metric [22], but it may well be that this is not strictly necessary.

What practical advantages could this mechanism have over inflation? Foremost there is no need for reheating. The DSR behaviour studied in this paper concerns regular matter, with the high energies experienced by the early universe triggering VSL behaviour. As the universe expands and cools, this unusual behaviour disappears, leaving the universe filled with standard radiation engaged in ‘business as usual’, without the need for a ‘decay’ or reheating. In other words, there is no esoteric matter here, merely regular matter behaving in an esoteric way.

Beyond this obvious practical advantage, we believe that the main novelty of the scenario proposed here is that it connects better with theories of phenomenology of quantum gravity, such as the DSR arena.

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References

[1] Magueijo J 2008 Phys. Rev. Lett. 100 231302
[2] Armendariz-Picon C, Damour T and Mukhanov V 1999 Phys. Lett. B 458 209
[3] Garriga J and Mukhanov V 1999 Phys. Lett. B 458 219–25
[4] Magueijo J 2003 Rep. Prog. Phys. 66 2025
[5] Moffat J W 1993 Int. J. Mod. Phys. D 2 351
[6] Albrecht A and Magueijo J 1999 Phys. Rev. D 59 043516
[7] Magueijo J 2008 arXiv:0807.1689
[8] Bruneton J 2007 Phys. Rev. D 75 085013
[9] Ellis G F R 2007 arXiv:astro-ph/070375
[10] Magueijo J and Moffat J 2007 arXiv:0705.4507
[11] Clayton M A and Moffat J W 2001 Phys. Lett. B 506 177–86
[12] Drummond I 2001 Phys. Rev. D 63 043503
[13] Amelino-Camelia G 2002 Nature 418 34–5
[14] Kowalski-Glikman J 2001 Phys. Lett. A 286 391–4
[15] Magueijo J and Smolin L 2002 Phys. Rev. Lett. 88 190403
[16] Magueijo J and Smolin L 2003 Phys. Rev. D 67 044017
[17] Kowalski-Glikman J 2002 Phys. Lett. B 547 291–6
[18] Freidel L, Kowalski-Glikman J and Nowak S 2007 Phys. Lett. B 648 70–5 (arXiv:0706.3658)
[19] Agostini A et al 2007 Mod. Phys. Lett. A 22 1779–86
[20] Alexander S and Magueijo J 2001 arXiv:hep-th/0104093
[21] Magueijo J and Smolin L 2004 Class. Quantum Grav. 21 1725–36
[22] Smolin L ‘DSR from the semi-classical limit of quantum gravity’ at press
[23] Hossenfelder S 2007 Phys. Lett. B 649 310–6
[24] Kimberly D, Magueijo J and Medeiros J 2004 Phys. Rev. D 70 084007
[25] Mignemi S 2005 Phys. Rev. D 72 087703
[26] Magueijo J 2006 Phys. Rev. D 73 124020
[27] Hossenfelder S et al 2003 Phys. Lett. B 575 85
[28] Amelino-Camelia G 2008 arXiv:0806.0339
[30] Armendariz-Picon C and Lim E 2003 J. Cosmol. Astropart. Phys. JCAP12(2003)002
Armendariz-Picon C 2006 J. Cosmol. Astropart. Phys. JCAP10(2006)010
[31] Piao Y 2007 Phys. Rev. D 75 063517
[32] Barnaby N and Kamran N 2008 J. High Energy Phys. JHEP02(2008)008
[33] Hossenfelder S 2008 Class. Quantum Grav. 25 038003
[34] Mukhanov S, Feldman H and Brandenberger R 1992 Phys. Rept. 215 203–333
Mukhanov V 2005 Physical Foundations of Cosmology (Cambridge: Cambridge University Press)
[35] Lidsey J et al 1997 Rev. Mod. Phys. 69 373–410
[36] Girelli F, Liberati S and Sindoni L 2007 Phys. Rev. D 75 064015
Amelino-Camelia G 2001 Phys. Lett. B 510 255–63