Interaction effects in Aharonov-Bohm-Kondo Rings

Yashar Komijani\textsuperscript{1} and \textsuperscript{2} Ryosuke Yoshih and Ian Affleck\textsuperscript{1}

\textsuperscript{1}Department of Physics and Astronomy and Quantum Materials Institute, University of British Columbia, Vancouver, B.C., Canada, V6T 1Z1
\textsuperscript{2}Yukawa Institute for Theoretical Physics, Kyoto University, Kitashirakawa Oiwake-Chō, Kyoto 606-8502, Japan
\textsuperscript{3}Research and Education Center for Natural Sciences, Keio University, 4-1-1 Hiyoshi, Kanagawa 223-8521, Japan

(Dated: May 11, 2014)

We study the conductance through an Aharonov-Bohm ring, containing a quantum dot in the Kondo regime in one arm, at finite temperature and arbitrary electronic density. We develop a general method for this calculation based on changing basis to the screening and non-screening channels. We show that an unusual term appears in the conductance, involving the connected 4-point Green’s function of the conduction electrons. However, this term and terms quadratic in the T-matrix can be eliminated at sufficiently low temperatures, leading to an expression for the conductance linear in the Kondo T-matrix. Explicit results are given for temperatures high compared to the Kondo temperature.

I. INTRODUCTION

The Kondo effect \[12\] is the entanglement of an impurity spin with conduction electrons as the temperature is lowered below the Kondo temperature \(T_K\) and the effective Kondo coupling renormalizes to large values. The Kondo effect continues to fascinate, especially since its experimental realization in gated semiconductor quantum dots \[33\]. One interesting and highly non-trivial extension of the basic Kondo model involves an Aharonov-Bohm (AB) ring with a Kondo impurity in the upper arm and interference with a lower “reference arm” which we refer to as an Aharonov-Bohm-Kondo (ABK) ring. Early works on this topic \[5-15\] were largely focussed on the \(T = 0\) behaviour, using Nozières local Fermi liquid theory (FLT) \[16\] and Numerical Renormalization Group (NRG) \[17\] techniques. A conclusion of these works was that \(T_K\) depends strongly on the magnetic flux through the ring. As shown in \[15\] and \[24\] Kondo scattering has both elastic and inelastic components which exhibit interesting variations with energy scale. Since inelastic scattering is generally known to destroy interference effects, a correspondingly rich dependence of the conductance of an ABK ring on temperature and flux is expected. Recently, the finite temperature behaviour of the ABK ring was also considered \[15\] and \[22\]. \[22\] studied the role of inelastic scattering on the visibility of AB oscillations through a large open ring with an embedded quantum dot. Various assumptions were made in this work including the idealized notion of a large ‘open ring’ which relate the conductance to the scattering cross section. Here we avoid these assumptions, calculating the full conductance including contributions from multiple traversals of the ring, using the Kubo formula.

We consider the short ABK ring introduced in \[10\] with tunneling amplitudes \(t_L\) and \(t_R\) between a quantum dot and left and right leads along with a direct tunneling amplitude \(t'\) between the leads. [See Fig. \[1\].] Unlike most previous work focusing on zero temperature, we study the temperature regime \(T \gg T_K\). In this regime we expect renormalization group improved perturbation theory in the Kondo coupling, \(J\), to be valid. Following earlier works \[23\] \[5\] \[14\] \[15\] we develop a general method to study this system based on changing to a basis of channels of conduction electrons, \((\psi, \phi)\) where \(\psi\) interacts with the quantum dot and \(\phi\) does not. This involves first transforming to the scattering state basis in the case \(t_L = t_R = 0\) and the transmission is only through the reference arm. Although the resulting Kondo coupling to \(\psi\) is a complicated function of all parameters in the model, including the flux, this transformation has the advantage that one can then invoke known universal results on the standard single channel Kondo model. In particular, \[10\] stated a formula for the conductance where all interaction effects were contained in the single-electron T-matrix for the corresponding single-channel Kondo model. The frequency and temperature dependence of \(T(\omega,T)\) has been well studied using renormalization group improved perturbation theory, FLT, NRG and other methods. Starting from the Kubo formula, we show that the conductance can be written in terms of a ‘transmission probability’ function which has a disconnected two-point function part (of zeroth, first and second order in the T-matrix) and a connected four-point function part \(G^C = \langle \psi \psi | \psi \psi \rangle \). However, using a suitable formulation of the Kubo formula, it is possible to eliminate both the connected term and the term quadratic in the T-matrix at temperatures small compared to the band width, which could still be large compared to \(T_K\), resulting in an expression for the conductance linear in the T-matrix. We relate this result to results of Meir and Wingreen \[24\] \[25\] showing that the conductance through a rather general interacting central region can be expressed in terms of the T-matrix. Similar formulas have been obtained in the past using Keldysh technique \[6\] \[10\]. Here we rederive them using Kubo formalism, generalize them to finite temperature and arbitrary density in the leads and examine critically when they are valid.
In Sec. [II] we introduce our model in more detail and show how we can transform the ABK ring model to a single channel Anderson or Kondo model. In Sec. [III] the Kubo formula for the linear conductance is expressed in the \((\psi, \phi)\) basis, containing terms involving the T-matrix as well as \(G^C\) and we present our formula for the conductance in terms of the T-matrix and connected Green’s function. In Sec. [IV] we perform a perturbative calculation of both connected and disconnected parts to second order in the Kondo coupling and give an explicit formula and graphs for the flux-dependent conductance at high temperature \(T \gg T_K\). In Sec. [V] we discuss the exact or approximate elimination of the connected Green’s function from the conductance, using both Kubo and Keldysh formalisms and explain why the Meir-Wingreen approach does not generally allow for an exact elimination of the connected part although it does for some simpler models including special cases of the ABK ring. We also present a formula, Eq. (5.33), for the conductance, containing only terms of zeroth and first order in the T-matrix, which should be valid at temperatures small compared to the band width. Sec. [VI] contains our conclusions and a discussion of open questions. Appendices [A] and [B] provide details related to the conductance calculations in the paper. In Appendix [C] the non-interacting limit of the ABK ring is discussed using Landauer, Fisher-Lee [20] and Keldysh techniques and the role of inelastic scattering is commented on.

II. THE MODELS: SCREENING AND NON-SCREENING CHANNELS

In this paper we consider interaction effects in a small Aharonov-Bohm ring with a quantum dot embedded in the upper arm. This system is modeled by a tight-binding Hamiltonian in which a direct link between first sites of left and right chains plays the role of the reference arm. Admittedly, transport experiments are usually performed on rings that are much larger than the one considered here, but as many references have studied this model, we choose to discuss it in order to illustrate various interesting features of the calculation. Moreover, the advantage of this model is that a direct solution of the non-interacting case is relatively simple and provides us with the possibility to confirm our Kubo calculations with various cross checks using both Keldysh and Landauer techniques. Once the method is established, generalization to larger rings and/or continuum models is straightforward.

The Hamiltonian is given by \(H = H_0 + H_T + H_d\) which consist of a non-interacting part

\[
H_0 = -t \left[ \sum_{n=-\infty}^{\infty} \sum_{\sigma} c_n^\dagger c_{n+1}^\sigma + h.c. \right] - t' (c_{-1}^\dagger c_1^\sigma + h.c.)
\]

composed of two semi-infinite leads, with the hopping parameter \(t\), coupled together with an amplitude \(t'\) which plays the role of the reference arm. A sum over spin indices is implied. The only interacting part of the model involves the quantum dot described by

\[
H_d = t_d d^\dagger d + U n_d d^\dagger d.
\]

In the absence of the dot and the reference arm, the electron wave-functions at \(n = \pm 1\) are of the form \(\sin(ka)\) where \(a\) is the lattice spacing, resulting in an energy-dependence of the tunneling amplitudes. (Some previous works [10,15] have neglected this energy-dependence in order to simplify the calculations). Unless explicitly mentioned, we assume \(a = 1\) and drop it from the formulas.

We are interested in the case \(t^2_L, t^2_R \ll Ut\), so that the dimensionless Kondo coupling \(\nu J \propto (t^2_L + t^2_R)/(Ut) \ll 1\) and universal behaviour characteristic of the Kondo effect may occur. On the other hand, we do not assume \(t'\) is small compared to \(t\) since this may not be the case in experiments and is not necessary to see manifestations of the Kondo effect.

A. Screening and non-screening channels

Transformation of the ABK ring model to a single-impurity Anderson model using scattering states has been introduced in [14,15]. The Hamiltonian without the dot can be diagonalized with a pair of degenerate scattering states \(H_0 |\epsilon_k\rangle = \epsilon_k |\epsilon_k\rangle\) and \(H_0 |\delta_k\rangle = \epsilon_k |\delta_k\rangle\) for \(k \in (0, \pi)\) which span the Hilbert space and are chosen to be orthogonal. \(H_0\) has parity symmetry and these states are even and odd eigenstates of the parity operator, i.e. their wave-functions satisfy \(\chi_{\epsilon/\delta}(−n) = \chi_{\epsilon/\delta}(n)\) and \(\chi_{\epsilon/\delta}(−n) = −\chi_{\epsilon/\delta}(n)\) and for \(n > 0\) can be written as

\[
\chi_{\epsilon/\delta}(n) = 2e^{i\delta^\pm_n} \sin(nk + \delta^\pm_k).
\]

For the present problem, the phase shifts have been calculated in [14] and satisfy the relation

\[
\sin \delta^\pm_k = \pm \tau \sin(k + \delta^\pm_k).
\]
where \( \tau = t'/t \). Of particular interest are the even and odd scattering wave-functions at \( n = 1 \), \( \Gamma_{ek} \equiv \chi_{ek}(1) \) and \( \Gamma_{ok} \equiv \chi_{ok}(1) \) given by

\[
\Gamma_{(e/o)k} = 2e^{i\delta_k^\pm} \sin(k + \delta_k^\pm) = \pm 2e^{i\delta_k^\pm} \sin \frac{\delta_k^\pm}{1 + \tau e^{i\delta_k^\pm}}.
\]

We can use these wave-functions to define a new set of annihilation operators \( q_{ek} \) and \( q_{ok} \) in terms of which the position-space operators are given by

\[
c_n = \frac{1}{\sqrt{2}} \int_0^\pi \frac{dk}{2\pi} (\chi_{ek}(n) q_{ek} + \chi_{ok}(n) q_{ok}),
\]

and the tunneling Hamiltonian becomes

\[
H_T^- = -\frac{1}{\sqrt{2}} \int_0^\pi \frac{dk}{2\pi} \left[ (t_{de} \Gamma_{ek} q_{ek}^\dagger - t_{do} \Gamma_{ok} q_{ok}) d + h.c. \right],
\]

in which the dot is coupled to even/odd scattering states by the flux-dependent amplitudes \( t_{de} \) and \( t_{do} \)

\[
t_{de} = t_L e^{-\varphi/2} + t_R e^{\varphi/2},
\]

\[
t_{do} = t_L e^{-\varphi/2} - t_R e^{\varphi/2}.
\]

The operators \( q_{ek} \) and \( q_{ok} \) are normalized so as to satisfy the anti-commutation relations

\[
\left\{ q_{ek}, q_{ek}^\dagger; q_{ok}, q_{ok}^\dagger \right\} = 2\pi \delta(k_1 - k_2).
\]

Another unitary transformation to screening \( \psi \) and non-screening \( \phi \) channels:

\[
\Psi_k = \begin{pmatrix} \psi_k \\ \phi_k \end{pmatrix} = U_k \begin{pmatrix} q_{ek} \\ q_{ok} \end{pmatrix}
\]

where

\[
U_k = \frac{1}{\sqrt{2V_k}} \begin{pmatrix} t_{de} \Gamma_{ek} - t_{do} \Gamma_{ok} \\ t_{do} \Gamma_{ek}^* - t_{de} \Gamma_{ok}^* \end{pmatrix},
\]

with the normalization factor \( V_k > 0 \) given by

\[
V_k = 2 \sin k \sqrt{t_L^2 + t_R^2} \sqrt{1 + \gamma^2 + 2\gamma\cos k \cos \phi}.
\]

maps the problem to a single channel coupled to an Anderson impurity \( H = H_0 + H_T + H_d \) with

\[
H_0 = \int_0^\pi \frac{dk}{2\pi} \epsilon_k \begin{pmatrix} \psi_k^\dagger \psi_k + \phi_k^\dagger \phi_k \end{pmatrix},
\]

\[
H_T = -\int_0^\pi \frac{dk}{2\pi} V_k \begin{pmatrix} \psi_k^\dagger d + d^\dagger \psi_k \end{pmatrix},
\]

where the non-screening channel \( \phi \) is decoupled from the dot. Here \( \epsilon_k = -2t \cos k \) is the dispersion relation of free electrons and the parameter

\[
\gamma = \frac{2t_L t_R}{t_L^2 + t_R^2}.
\]

characterizes the coupling asymmetry of the dot. Therefore, the problem of the AB ring with an embedded quantum dot is mapped to a single-impurity Anderson model with a generally energy-dependent and flux-dependent hybridization parameter, \( V_k \).

### B. Kondo model

An approximation to the Anderson model is given by the Kondo model \( [1] \). One can perform the Schrieffer-Wolff transformation \( [27] \) to arrive at

\[
H = \int \frac{dk}{2\pi} \epsilon_k \psi_k^\dagger \psi_k + \frac{1}{2} \phi_k^\dagger \phi_k
\]

\[
+ \int \frac{dkdk'}{(2\pi)^2} (J_{kk'} \psi_k^\dagger \psi_{k'}^\dagger + J_{kk'} \psi_k \psi_{k'}),
\]

Here \( \phi \) is the impurity spin and the couplings \( J_{kk'} \) and \( K_{kk'} \) are given by \( [27] \)

\[
J_{kk'} = V_k(j_k + j_{k'})V_{k'}, \quad K_{kk'} = V_k(\kappa_k + \kappa_{k'})V_{k'/2},
\]

where we have defined

\[
(j, \kappa)_k = \frac{1}{\epsilon_k - \epsilon_d} \pm \frac{1}{U + \epsilon_d - \epsilon_k}.
\]

The Kondo model is usually defined with a reduced band width \( D \ll t \) so the momentum dependence of the couplings can be ignored for the small ABK ring considered in this paper. Transport properties are usually given in terms of the diagonal coupling \( J_{kk} \propto V^2 \) which contains the first harmonic of the dimensionless flux, \( \phi \).

### III. CONDUCTANCE FROM KUBO FORMULA

The current operator may be written as

\[
I = -\frac{e}{2} \frac{d}{dt} \Delta N,
\]

where

\[
\Delta N \equiv N_R - N_L
\]

and

\[
N_{R/L} = \sum_{n=\pm 1} e_n^\dagger e_n.
\]

By adding a perturbation to the Hamiltonian, \( eV(t)\Delta N/2 \), with \( V(t) = V_0 \cos \Omega t \), the Kubo formula gives the DC conductance.

\[
G = \frac{e^2}{h} \lim_{\Omega \to 0} 2\pi i \Omega G'(\Omega)
\]

where \( G'(\Omega) \) is the retarded Green’s function of \( \Delta N/2 \):

\[
G'(\Omega) \equiv -\frac{i}{4} \int_0^\infty dt e^{i(\Omega + \eta)t} \langle [\Delta N(t), \Delta N(0)] \rangle.
\]

(\( \eta \) is an infinitesimal positive convergence factor.) Alternatively, the same formula for the conductance may be obtained by applying a vector potential between the quantum dot and sites \( \pm 1 \) and between sites \( +1 \) and
−1. In some cases it will be convenient to obtain $G′(\Omega)$ by analytic continuation from the imaginary time, time-ordered Green’s function:

$$G′(i\omega_p) \equiv -\frac{1}{4} \int_0^\beta d\tau e^{i\omega_p \tau} \langle T_\tau \Delta N(\tau)\Delta N(0) \rangle \quad (3.6)$$

where $\beta \equiv \hbar/T$ and $\omega_p \equiv 2\pi p/\beta$.

### A. Kubo formula in terms of Green’s functions of screening and non-screening channels

$\Delta N$ can be written in the scattering basis as

$$\Delta N = \int_0^{\pi} \frac{dk_1 dk_2}{(2\pi)^2} \left( q^+_k, q^+_q \right) \hat{A}_{k_1 k_2} \left( q_{k_2}, q_{q k_2} \right), \quad (3.7)$$

where the matrices $\hat{A}_{k_1 k_2}$ are given by

$$\hat{A}_{k_1 k_2} = \frac{1}{2} \sum_{n=1}^\infty -1 \left( \chi^+_{k_2}(n) \chi_{k_1}(n) - \chi_{k_2}(n) \chi^+_{k_1}(n) \right). \quad (3.8)$$

The off-diagonal matrix elements of $\hat{A}_{k_1 k_2}$ are partial overlaps of even and odd scattering wave-functions in the left/right leads. A direct summation of (3.8) using (2.4) and (2.5) and after introducing appropriate convergence factors yields

$$\hat{A}_{k_1 k_2} = 2\pi \delta(k_1 - k_2) \tau_x + g^R_{k_1}(\epsilon_k) \Gamma_{k_1 k_2}, \quad (3.9)$$

where

$$g^R_{k_1}(\epsilon) \equiv g_k(\epsilon \pm i\eta) = \frac{1}{\epsilon - \epsilon_k \pm i\eta}, \quad (3.10)$$

are the free retarded/advanced Green’s functions,

$$\Gamma_{k_1 k_2} \equiv \begin{pmatrix} 0 & f_{k_1 k_2} \\ -f^*_{k_2 k_1} & 0 \end{pmatrix}, \quad (3.11)$$

and

$$f_{k_1 k_2} = 2t' \Gamma^+_{k_1 k_2} \Gamma^*_{k_2 k_1}. \quad (3.12)$$

We refer to the first term in the left equation of (3.9) as the contact term and the second term as the overlap term. Here and in the following $\tau_x, \tau_y$ and $\tau_z$ are Pauli matrices in the $\psi - \phi$ basis and $\tau_0 \equiv \frac{1}{2}(\mathbb{1} + \tau_z)$ is the projection operator onto the $\psi$ state. Using $g^R_{k_1}(\epsilon_k) = -g^\Lambda_{k_1}(\epsilon_k)$, it can be seen that

$$\hat{A}_{k_1 k_2} = \hat{A}^\dagger_{k_2 k_1}. \quad (3.13)$$

We will be mainly interested in the diagonal elements of $f$ which can be written as

$$\pi \nu_k f_{kk} = e^{-i(\delta^+_k - \delta^-_k)} \sin(\delta^+_k - \delta^-_k) \quad (3.14)$$

where $\nu_k$ is the density of states per unit length, per spin, per channel,

$$\nu_k = \frac{1}{4\pi t \sin k}. \quad (3.15)$$

$\Delta N$ can be expressed in terms of screening and non-screening channels

$$\Delta N(t) = \int_0^{\pi} \frac{dk_1 dk_2}{(2\pi)^2} \Psi^\dagger_k(t) \nu_{k_1} \phi_{k_2}^\ast(t), \quad (3.16)$$

in which the matrix $M_{k_1 k_2}$ is defined as

$$M_{k_1 k_2} \equiv U_k \hat{A}_{k_1 k_2} U_k^\dagger \quad (3.17)$$

where $U_k$ is defined in Eq. (2.12). It follows immediately from Eq. (3.13) that:

$$M_{k_1 k_2} = M^\dagger_{k_2 k_1}. \quad (3.18)$$

a property which we will use below and which implies that $\Delta N$ is Hermitean.

### B. Conductance formula

Inserting Eq. (3.16) into Eq. (3.5) we obtain an exact expression for the Green’s function of $\Delta N$, and hence the conductance, in terms of retarded Green’s functions of the single channel Anderson or Kondo model and the non-interacting Green’s function of the non-screening field $\phi_k$. Obtaining the retarded Green’s function of $\Delta N$ from the analytic continuation of the time-ordered imaginary time Green’s function, the corresponding Feynman diagrams are drawn in Fig. (2). Both 2-point and connected 4-point Green’s functions occur.

1. **Disconnected part**

The disconnected part of $G’$ is most conveniently dealt with in real time domain where it is denoted by $G’^D$. Using Wick’s theorem it can be written:

$$G’^D(\Omega) = -\frac{i}{2} \int_0^{\pi} \frac{dk_1 dk_2 dq_1 dq_2}{(2\pi)^4} \int_0^\infty dt e^{i(\Omega + i\nu)t} Tr \left[ M_{k_1 k_2} G^>_{k_2 q_1}(t) M_{q_1 q_2} G^<_{q_2 k_1}(-t) - M_{q_1 q_2} G^<_{q_2 k_1}(-t) M_{k_1 k_2} G^>_{k_2 q_1}(t) \right] \quad (3.19)$$
where a factor of 2 from summation over spin indices is taken into account. Here we used the fact that

\[ G^<_{kq_{1}\alpha\beta}(t) = \delta_{\alpha\beta}G^<_{kq_{1}}(t) \quad (3.20) \]

due the SU(2) symmetry of the model. Here the Green’s functions \( G^<_{kq}(t) \) and \( G^<_{qk}(t) \) are diagonal matrices in the \( \psi - \phi \) space,

\[ G^<_{qk}(t) = -i \begin{pmatrix} \langle \psi_k(t)\psi_k(0) \rangle & 0 \\ 0 & \langle \phi_k(t)\phi_k(0) \rangle \end{pmatrix} \quad (3.21) \]

and

\[ G^<_{qk}(t) \equiv +i \begin{pmatrix} \langle \psi_k(0)\psi_k(t) \rangle & 0 \\ 0 & \langle \phi_k(0)\phi_k(t) \rangle \end{pmatrix}. \quad (3.22) \]

We can write these Green’s functions in the Fourier domain, do the time integral and use the general equilibrium identities \( G^<_{qk}(\omega) = -2i f(\omega) \text{Im} G^R_{qk}(\omega) \) and \( \text{Re} G^<_{qk}(\omega) = 2i(1 - f(\omega)) \text{Im} G^R_{qk}(\omega) \), where \( G^R_{qk}(\omega) \) is the matrix retarded single electron Green’s function and \( f(\omega) = (1 + e^{-\beta(\omega - \mu)})^{-1} \) is the Fermi distribution. Here we used the fact that \( G^R_{qk}(\omega) = G^{R}_{qk}(\omega) \) for the Anderson model, as can be seen from Eqs. (3.27) and (3.28). We thus obtain

\[ G'(\Omega) = -2i \int \frac{d\omega d\omega'}{(2\pi)^2} [f(\omega) - f(\omega')] \int_0^n \frac{dk_1dk_2dq_1dq_2}{(2\pi)^4} \text{Tr} [M'_{k_1k_2} \text{Im} G^R_{kq_{1}q_{2}}(\omega')M_{q_{1}q_{2}}G^R_{q_{2}k_1}(\omega)] \int_0^\infty dt e^{i(\omega - \omega' + \Omega + i\eta)t} \]

\[ = 2 \int \frac{d\omega d\omega'}{(2\pi)^2} [f(\omega) - f(\omega')] \int_0^n \frac{dk_1dk_2dq_1dq_2}{(2\pi)^4} \text{Tr} [M'_{k_1k_2} \text{Im} G^R_{kq_{1}q_{2}}(\omega')M_{q_{1}q_{2}}\text{Im} G^R_{q_{2}k_1}(\omega)] \]

The momentum integral can be seen to be real using \( \uparrow_{k_1k_2} = \uparrow_{k_2k_1}^\dagger \) and \( \text{Im} G^R_{qk}(\omega) = \text{Im} G^R_{qk}(\omega) \). Here we focus on the real part of the conductance; only the imaginary part of \( G' \) contributes to it. Thus, we only need

\[ \text{Im} G'(\Omega) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega [f(\omega) - f(\omega + \Omega)] \int_0^n \frac{dk_1dk_2dq_1dq_2}{(2\pi)^4} \text{Tr} [M'_{k_1k_2} \text{Im} G^R_{kq_{1}q_{2}}(\omega + \Omega)M_{q_{1}q_{2}}\text{Im} G^R_{q_{2}k_1}(\omega)]. \quad (3.24) \]

Inserting Eq. (3.24) into Eq. (3.4), gives the disconnected part of the DC conductance

\[ G^D = \frac{2e^2}{\hbar} \int_{-\infty}^{\infty} d\omega [-f'(\omega)] |T^D(\omega)| \quad (3.25) \]

where the disconnected part of the “transmission probability” is defined as

\[ T^D(\omega) \equiv \lim_{\Omega \to 0} \frac{\Omega^2}{2} \int_0^n \frac{dk_1dk_2dq_1dq_2}{(2\pi)^4} \text{Tr} [M'_{k_1k_2} \text{Im} G^R_{kq_{1}q_{2}}(\omega + \Omega)M_{q_{1}q_{2}}\text{Im} G^R_{q_{2}k_1}(\omega)]. \quad (3.26) \]

The retarded Green’s function can be written in terms of the T-matrix of the Anderson or Kondo model, \( T_{kq_{1}}(\omega) \):

\[ G^R_{kq_{1}}(\omega) = 2\pi\delta(k_2 - q_1)g^R_{k_2}(\omega)\mathbb{1} + \tau_\psi g^R_{k_2}(\omega)T_{kq_{1}}(\omega)g^R_{q_1}(\omega). \quad (3.27) \]

(Note that the \( \tau_\psi \) projection matrix implies that the second term is only present for the screening channel, \( \psi \).)
Also note that $G_{k_2 q_2, \alpha \beta}^R$, $g_{k_2 q_2, \alpha}$, and $T_{k_2 q_1, \alpha \beta}$ are all $\propto \delta_{\alpha \beta}$ due to the SU(2) symmetry of the model. We are suppressing all spin indices.) For the Anderson model of Eq. (2.14) this T-matrix is related to the retarded Green’s function of the dot $G_{dd}^R(\omega)$ via

$$T_{k_2 q_1}(\omega) = V_{k_2} G_{dd}^R(\omega) V_{q_1}. \tag{3.28}$$

We see that $G^D$ is a sum of terms of zeroth, first and second order in the T-matrix. The T-matrix is a smooth function of frequency and the needed divergences as $\Omega \to 0$ of the momentum integral in Eq. (3.26) arise from the singular behaviour of $A_{q_1 q_2}$ in Eq. (3.9), $\propto \delta_{q_1 q_2}$ by Eq. (3.17), and from the factors of $g^R$ in Eq. (3.27). We find that, after taking the limit $\Omega \to 0$, $\mathcal{T}(\omega)$ is only non-zero for $\omega$ inside the band, $|\omega| < 2t$. Therefore, it is convenient to write the integration variable $\omega$ in Eq. (3.25) as $\epsilon_p$:

$$G^D = \frac{2e^2}{\hbar} \int_{-2t}^{2t} d\epsilon_p [ -f'(\epsilon_p) ] |T^D(\epsilon_p)|. \tag{3.29}$$

We show in Appendix (B) that the transmission probability for the disconnected part of the conductance may be written in terms of the diagonal on-shell T-matrix of the Anderson or Kondo model, and the density of states as,

$$T^D(\epsilon_p) = \mathcal{T}_0(\epsilon_p) \quad + \quad \mathcal{Z}_R(\epsilon_p) \text{Re} \{ -\pi \nu_p T_{pp}(\epsilon_p) \} \quad + \quad \mathcal{Z}_I(\epsilon_p) \text{Im} \{ -\pi \nu_p T_{pp}(\epsilon_p) \} \quad + \quad \mathcal{Z}_2(\epsilon_p) \mid -\pi \nu_p T_{pp}(\epsilon_p) \mid^2 \tag{3.30}$$

where

$$\mathcal{T}_0(\epsilon_p) = \frac{4 \tau^2 \sin^2 p}{\left(1 + \tau^2 \right)^2 - 4 \tau^2 \cos^2 p}, \tag{3.31}$$

$$\mathcal{Z}_R(\epsilon_p) = \frac{4 \tau \cos p + 2 \gamma (1 + \tau^2) \cos \varphi \sqrt{\mathcal{T}_0[1 - \mathcal{T}_0]}}, \tag{3.32}$$

$$\mathcal{Z}_I(\epsilon_p) = 1 - 2 \mathcal{T}_0(\epsilon_p), \tag{3.33}$$

$$\mathcal{Z}_2(\epsilon_p) = \frac{- (1 - \tau^2)^2 (1 - \gamma^2) + 4 \gamma^2 \tau^2 \sin^2 p \sin^2 \varphi}{[1 + \tau^2 + 2 \gamma \tau \cos p \cos \varphi]^2}. \tag{3.34}$$

A non-trivial check of Eqs. (3.29)-(3.34) is the non-interacting ABK ring, $U = 0$. In this case the connected 4-point Green’s function vanishes, so $\mathcal{T}^D(\epsilon_p)$ gives the entire transmission probability. In App. (C) we derive Eq. (3.29) from the Landauer formalism and confirm that Eqs. (3.30)-(3.34) give the correct transmission probability.

2. Connected part

The connected contribution to $G'(i\omega_p)$ is given by

$$G'^C(\omega_p) = \frac{1}{4} \int_0^{\pi} dk_1 dk_2 dq_1 dq_2 M_{k_1 k_2}^{11} M_{q_1 q_2}^{11} g_{k_1 k_2 q_1 q_2}^C(i\omega_p), \tag{3.35}$$

which is a functional of the connected four-point Green’s function $G_{k_1 k_2 q_1 q_2}^C(i\omega_p)$ defined as

$$G_{k_1 k_2 q_1 q_2}^C(i\omega_p) = - \int_0^\beta e^{i\omega_p \tau} d\tau \sum_{\sigma \sigma'} \langle T_{\tau} \psi^\dagger_{k_1 \sigma} (\tau) \psi_{k_2 \sigma} (\tau) \psi^\dagger_{q_1 \sigma'} (0) \psi_{q_2 \sigma'} (0) \rangle_C. \tag{3.36}$$

The subscript and the superscript $C$ both refer to the connected part. Using equation-of-motion techniques and as is clear from Fig. 5, the connected part of the four-point function can be written in imaginary time domain as

$$G_{k_1 k_2 q_1 q_2}^C(\tau) = \int_0^\beta d\tau_1 d\tau_2 d\tau_3 d\tau_4 g_{k_1}(\tau - \tau_4) g_{k_2}(0 - \tau_4) G_{k_1 k_2 q_1 q_2}^{amp}(\tau_1, \tau_2, \tau_3, \tau_4) g_{q_1}(\tau_1 - \tau) g_{q_2}(\tau_3 - 0) \tag{3.37}$$

in terms of the amputated function which is proportional to the connected four-point function of the $d$ electrons

$$G_{k_1 k_2 q_1 q_2}^{amp}(\tau_1, \tau_2, \tau_3, \tau_4) = V_{k_1} V_{k_2} V_{q_1} V_{q_2} G_{dd}^C(\tau_1, \tau_2, \tau_3, \tau_4) \tag{3.38}$$

where

$$g_{k_1}^C(\tau_1, \tau_2, \tau_3, \tau_4) \equiv \sum_{\sigma \sigma'} - \langle T_{\tau} d^\dagger_{\sigma}(\tau_1) d_{\sigma}(\tau_2) d^\dagger_{\sigma'}(\tau_3) d_{\sigma'}(\tau_4) \rangle_C. \tag{3.39}$$

Here

$$g_k(\tau) \equiv [f(\tau) - \theta(\tau)] e^{-\tau |\tau|}. \tag{3.40}$$

In Fourier-domain
\[ G_{k_1k_2q_1q_2}^{C}(i\omega_p) = \frac{V_{k_1}V_{k_2}V_{q_1}V_{q_2}}{\beta^2} \sum_{mn} g_{k_1}(i\varpi_m + i\omega_p)g_{q_2}(i\varpi_n - i\omega_p)G_d^{C}(i\varpi_m, i\varpi_m + i\omega_p, i\varpi_n, i\varpi_n - i\omega_p)g_{k_2}(i\varpi_m)g_{q_1}(i\varpi_n). \]

Here \( \varpi_m \) and \( \varpi_n \) are fermionic and \( \omega_p \) is a bosonic Matsubara frequency. Analytic continuation to real frequencies, \( \omega_p \to \Omega + i\eta \) gives the connected part of the retarded four-point function to be plugged into Eq. (3.45).

We now argue that the connected part of the conductance can also be written:

\[ G^C = \frac{2e^2}{h} \int_{-\infty}^{\infty} d\omega [ -f'(\omega) ] T^C(\omega) \tag{3.41} \]

in terms of a connected part of the “transmission probability” \( T^C(\omega) \). This is an important result since it implies that the total conductance at temperatures small compared to the band width is determined by universal low energy properties of the system. It is also crucial for approximately eliminating the connected term from the conductance at low temperatures, as we show in Sec. [3]. To establish this result it is convenient to write \( G^C(i\omega_p) \) in terms of a partially amputated Green’s function, \( P(i\varpi_m, i\varpi_m + i\omega_p) \):

\[ G^C(i\omega_p) = \frac{1}{\beta} \sum_m P(i\varpi_m, i\varpi_m + i\omega_p) \tag{3.42} \]

where

\[ P(i\varpi_m, i\varpi_m + i\omega_p) \equiv \frac{1}{4} \int_0^{\pi} \frac{dk_1dk_2}{(2\pi)^2} \frac{M_{k_1k_2}^{(1)}}{M_{k_1k_2}^{(1)}} g_{k_1}(i\varpi_m + i\omega_p)V_{k_1}V_{k_2} \times \]

\[ \int_0^{\beta} d\tau_1d\tau_2 e^{i\varpi_m\tau_1 + i(\varpi_m + \omega_p)\tau_2} \langle T_{\tau_1}d^{\dagger}(\tau_1)d(\tau_2)\Delta N(0) \rangle_C. \tag{3.43} \]

(Here \( \eta' \) is a positive infinitesimal corresponding to the displacements of the integration lines.) Next, we consider the analytic continuation of \( G^C(i\omega_p) \) to real frequency:

\[ i\omega_p \to \Omega + i\eta, \quad G^C \to G', \tag{3.45} \]

where \( \eta \) is another positive infinitesimal, with \( \eta' \ll \eta < 1 \). Finally, from Eq. (3.44), we must multiply \( G'(\Omega) \) by a factor of \( \Omega \) and take \( \Omega \to 0 \). It can be seen that the integrals of the first and last terms in Eq. (3.44) remain finite in this \( \Omega \to 0 \) limit. This follows because all singularities of \( P(\omega + i\eta', \omega + \Omega + i\eta) \) are below the real \( \omega \) axis, at \( \omega = E - i\eta' \) or \( E - \Omega - i\eta \) where \( E \) is the difference of energies of two states of the system. This follows from the spectral decomposition:

\[ \int_0^{\beta} d\tau_1d\tau_2 e^{z_1\tau_1 + z_2\tau_2} \langle T_{\tau_1}d(\tau_1)d(\tau_2)\Delta N(0) \rangle \]

\[ = \frac{1}{2} \sum_{mnkp} \langle p | \Delta N | m \rangle \left[ \frac{\langle m|d^{\dagger}|n \rangle \langle n|d|p \rangle}{z_2 + E_n - E_p} \left( e^{-\beta E_p} - e^{-\beta E_m} \right) \left( \frac{z_2 - z_1 + E_m - E_p}{z_2 - z_1 + E_m - E_p} \right) - \frac{e^{-\beta E_n} + e^{-\beta E_m}}{z_1 + E_n - E_m} \right] \]

\[ + \frac{\langle m|d^{\dagger}|n \rangle \langle n|d|p \rangle}{z_2 + E_m - E_n} \left( e^{-\beta E_p} + e^{-\beta E_n} \right) \left( \frac{z_2 - z_1 + E_n - E_p}{z_2 - z_1 + E_n - E_p} \right) \]

\[ + \frac{\langle m|d^{\dagger}|n \rangle \langle n|d|p \rangle}{z_2 + E_m - E_n} \left( e^{-\beta E_p} + e^{-\beta E_n} \right) \left( \frac{z_2 - z_1 + E_n - E_p}{z_2 - z_1 + E_n - E_p} \right). \tag{3.46} \]
(Z is the partition function.) We see that, for $z_1 = \omega + i\eta'$, $z_2 = \omega + \Omega + i\eta$, all singularities occur below the real $\omega$ axis. [Actually we must subtract the disconnected part from Eq. (3.46) to get a representation of $P(z_1, z_2)$, but this also obeys the desired property as mentioned at the beginning of App. B3 using results from App. A.] It thus follows that we may deform the line integral in the $\omega$ plane a finite distance above the real axis so that the integral remains finite as $\Omega \to 0$. The same argument applies to $P(\omega - \Omega - i\eta, \omega - i\eta')$. On the other hand, the integrals of the second and third terms in Eq. (3.44) diverge as $\Omega \to 0$, and thus contribute to the conductance. This follows because $P(\omega - i\eta', \omega + \Omega + i\eta)$ has singularities both above and below the real $\omega$-axis which can pinch the integration contour as $\Omega \to 0$. Finally, shifting the integration variable $\omega - \omega + \Omega$, in the third term in Eq. (3.44), we may make the approximation, valid for $\Omega \to 0$:

$$G'(\Omega) \approx \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} [f(\omega + \Omega) - f(\omega)] P(\omega - i\eta_1, \omega + \Omega + i\eta_2)$$

for positive infinitesimals $\eta_1$ and $\eta_2$. For small $\Omega$ we may use

$$f(\omega + \Omega) - f(\omega) \approx \Omega f'(\omega).$$

(3.48)

From Eqs. (3.4) and (3.47) we then obtain the connected part of the transmission probability:

$$T^C(\omega) = \lim_{\Omega \to 0} \frac{\Omega^2}{8} P(\omega - i\eta_1, \omega + \Omega + i\eta_2) + c.c.$$  

(3.49)

where $P$ is defined in Eq. (3.43), after analytic continuation to real frequency. We again expect that $T^C(\omega)$ will only be non-zero at $\Omega \to 0$ for $\omega$ inside the energy band, $|\omega| < 2t$, so we may again replace the integration variable $\omega$ by $\epsilon_p$. Note that our analysis of the connected part of the transmission probability is less complete that of the disconnected part where we were able to explicitly take the limit $\Omega \to 0$ and express $T^D$ in terms of smooth functions. For the connected part, we have not so far been able to accomplish this. See however subsection IV.B.

We confirm that the connected part of the conductance can be written in terms of a transmission probability in our perturbation calculation in Sec. IV.

**IV. PERTURBATIVE CALCULATION**

In this section, we calculate the conductance perturbatively to order $J^2 \propto V_k^4 / U^2$ a result which should be valid for $T \gg T_K$.

**A. Disconnected Part: T-matrix of $\psi$ electrons**

The relevant Feynman diagrams for the (diagonal element of the retarded on-shell) T-matrix are shown in Fig. 4 and are given by

$$T_{pp}(\epsilon_p + i\eta) = K_{pp} + \int_{\epsilon_p - D}^{\epsilon_p + D} \nu_q d\epsilon_q \frac{K^2_{qp}}{\epsilon_p - \epsilon_q + i\eta}$$

(4.1)

$$+ \int_{\epsilon_p - D}^{\epsilon_p + D} \nu_q d\epsilon_q \left( \frac{3}{4} \pi \nu J^2 \delta_{\epsilon_p - \epsilon_q + i\eta} + O(J^3 \propto V^6) \right).$$

(4.2)

The factor of $3/16$ multiplying the third term comes from the correlation function of the impurity spin. At lowest order in $J$ we may use the free spin Green’s function:

$$\langle T_{\tau}\hat{S}_d^a(\tau_1)\hat{S}_d^b(\tau_2) \rangle = \frac{1}{4} \delta^{ab}$$

(4.3)

independent of $\tau_1$ and $\tau_2$, yielding:

$$\sum_{\beta} \langle T_{\tau}\hat{S}_d^a(\tau_1)\hat{S}_{\alpha\beta}(\tau_2)\rangle \langle \hat{S}_d^b(\tau_2)\hat{S}_{\beta}^c(\tau_1) \rangle = \frac{3}{4} \delta_{\alpha\gamma}.\]$$

(4.4)

The first two terms in Eq. (4.1) are potential scattering terms that satisfy the optical theorem

$$|(-\pi \nu_p T_{pp}(\epsilon_p)|^2 = \text{Im} \left[-\pi \nu_p T_{pp}(\epsilon_p)\right].\]$$

(4.5)

They depend on the position of the dot level $\epsilon_d$ and can be set to zero ($K_{pp} \approx 0$) by tuning the dot to the middle of two Coulomb resonances $\epsilon_d - \epsilon_F \approx -U/2$. The third term contains both real and imaginary parts. The real part depends on the details of the conduction band. This
term shows that the conductance is determined by the properties of the system not only at energies close to the Fermi energy but all energies over the full reduced band \( \pm D \); it introduces non-universalities that limit the predictive power of the Kondo model. Generally, for energies much smaller than the original band width \( D \ll t \) the total S-matrix can be written as a product from spin and charge sectors \[28\] which implies that the single-particle T-matrix takes the form \[29\]

\[-2\pi i T_{pp}^K(\epsilon_p, T) = -2\pi i T_{pp}^K(\epsilon_p, T)e^{2i\delta} + i(e^{2i\delta} - 1).
\]

(4.6)

Here \( T_{pp}^K(\epsilon_p) \) corresponds to the T-matrix of a particle-hole symmetric Kondo Hamiltonian (for example, the model considered here with \( \tau = 0 \) and \( p_F = \pi/2 \)) and is purely imaginary to order \( J^2 \). \( \delta \) corresponds to the total phase shift at the Fermi energy induced by all potential scattering sources that break particle-hole symmetry and is a complicated function of all the parameters of the model. Thus, the \( O(J^2) \) term in the real part of the T-matrix is expected to merely contribute to \( \delta \) at low temperatures. Although these potential scatterings contribute to the AB oscillations in the conductance, they do not have a strong dependence on energy or temperature and are not relevant for the Kondo physics. Therefore, they will be neglected, \( \delta \rightarrow 0 \), in the following. To order \( J^2 \) the rest is

\[ -\pi \nu_p T_{pp}^K(\epsilon_p) = i \frac{3\pi^2}{16} \nu_p J^2. \]

(4.7)

The vertex function is given by

\[ G^{\text{amp}}_{k_1 k_2 q_1 q_2}(\tau_1, \tau_2, \tau_3, \tau_4) = \frac{1}{4} J_{2q_2 k_1} J_{2q_1 k_1} \sum_{\alpha \beta} \left\langle T_T(\vec{S}_d(\tau_1), \vec{S}_d(\tau_2), \vec{S}_d(\tau_3)) \right\rangle \delta(\tau_1 - \tau_4) \delta(\tau_2 - \tau_3). \]

(4.8)

Using the result of Eq. (4.4) and Fourier transforming we get

\[ G^{\text{amp}}_{k_1 k_2 q_1 q_2}(i\omega_m, i\omega_m + i\omega_p, i\omega_m - i\omega_p) = \frac{3}{8} J_{2q_2 k_1} J_{2q_1 k_1} \delta(i\omega_m + i\omega_p - i\omega_m, 0). \]

(4.9)

Plugging this into Eq. (3.37) we have

\[ G^{C}_{k_1 k_2 q_1 q_2}(i\omega_p) = \frac{3}{8} J_{2q_2 k_1} J_{2q_1 k_1} \sum m g_{k_2}(i\omega_m + i\omega_p) g_{q_2}(i\omega_m) g_{k_1}(i\omega_m) g_{q_1}(i\omega_m + i\omega_p). \]

(4.10)

The vertex function does not depend on energy and we can express the summation over Matsubara frequencies as a contour integral and deform the contour as sketched in Fig. [3]. After analytic continuation, \( i\omega_p \rightarrow \Omega + i\eta \), we write the result as an integral over real frequency

\[ G^{C}_{k_1 k_2 q_1 q_2}(\Omega + i\eta) = \frac{3 J_{2q_2 k_1} J_{2q_1 k_1}}{16\pi i} \int_{-\infty}^{+\infty} d\omega \left\{ + f(\omega) \left[ g_{k_2}(\omega + \Omega) g_{q_2}(\omega) g_{k_1}(\omega) g_{q_1}(\omega + \Omega) \right] \\
- f(\omega) \left[ g_{k_2}(\omega + \Omega) g_{q_2}(\omega) g_{k_1}(\omega) g_{q_1}(\omega + \Omega) \right] \\
+ f(\omega) \left[ g_{k_2}(\omega) g_{q_2}(\omega - \Omega) g_{k_1}(\omega) g_{q_1}(\omega) \right] \\
- f(\omega) \left[ g_{k_2}(\omega) g_{q_2}(\omega - \Omega) g_{k_1}(\omega) g_{q_1}(\omega) \right]. \]

(4.11)

Note that this is a special case of the general result discussed in Sub-Section (III B 2). In this simple case the validity of this expression can be checked explicitly. Terms that contain all retarded or all advanced propagators do not

B. Connected Part

To order \( J^2 \) the relevant contribution is given by the Feynman diagram shown in Fig. [5].

FIG. 5: The amputated connected function to order \( J^2 \). Note that in contrast to some previous works \[30\], here we trace over the impurity spin.
contribute to the DC conductance. So we drop the first and the last lines and shift the integration variable by \( \Omega \) in the third line to obtain, after Taylor expanding \( f(\omega - \Omega) \),

\[
G^C_{k_1k_2q_1q_2}(\Omega + i\eta) \to \frac{3J_{q_2k_1}J_{k_2q_1}}{16\pi i} \int_{-\infty}^{+\infty} d\omega \Omega f'(\omega) \left[ g^*_p(\omega)g_p(\omega + \Omega)g_{q_1}(\omega + \Omega)g^*_{q_2}(\omega) \right].
\]

(4.12)

Inserting this into Eq. (3.35) we can write the connected part of the conductance as

\[
G^C = \frac{-e^2}{4h} \lim_{\Omega \to 0} \int d\epsilon_p f'(\epsilon_p) \int_0^\pi \frac{dk_1dk_2dq_1dq_2}{(2\pi)^4} \frac{3J_{q_2k_1}J_{k_2q_1}}{8V_kV_qV_{q_1}V_{q_2}} I_C(k_1, k_2; \epsilon_p, \Omega)I'_C(q_1, q_2; \epsilon_p, \Omega)
\]

(4.13)

where the two functions \( I_C \) and \( I'_C \) are defined as

\[
I_C(k_1, k_2; \epsilon_p, \Omega) = TV_kTV_qM_{k_1k_2}^\psi g^*_k(\epsilon_p)g_k(\epsilon_p + \Omega), \quad I'_C(q_1, q_2; \epsilon_p, \Omega) = TV_qTV_{q_2}M_{q_1q_2}^\psi g_{q_1}(\epsilon_p + \Omega)g^*_{q_2}(\epsilon_p).
\]

(4.14)

Taking the limit \( \Omega \to 0 \) of such functions is explained in Appendix (A). Indeed, these functions are very similar to the matrices \( 1_D^2 \) and \( I_D^2 \) and the needed propagator-product identities are similar to those of Eqs. (A12) and (A13) apart from some extra factors of \( 2(2\pi)^2 \delta(p - k_1)\delta(p - k_2) \) and \( 2(2\pi)^2 \delta(p - q_1)\delta(p - q_2) \). Therefore the connected four-point contribution to the conductance can be written in terms of a transmission probability

\[
G^C = \frac{-e^2}{4h} \int d\epsilon_p [-f'(\epsilon_p)] T^C(\epsilon_p),
\]

(4.15)

where

\[
T^C(\epsilon_p) = Z_2(\epsilon_p) \frac{3\pi^2}{16} \nu_p J_{pp}^2
\]

(4.16)

and \( Z_2(\epsilon_p) \) is given in Eq. (3.34). Note that the disconnected and connected parts of the transmission probability are the same order of magnitude.

C. total conductance

The total conductance to order \( O(J^2) \) is given by combining the connected and disconnected parts:

\[
G = \int d\epsilon_p [-f'(\epsilon_p)] T_0(\epsilon_p) + \Delta G,
\]

(4.17)

where the corrections to the conductance due to the presence of the dot \( \Delta G \) are equal to

\[
\Delta G = \frac{-e^2}{h} \int d\epsilon_p [-f'(\epsilon_p)] [Z_2(\epsilon_p) + Z_I(\epsilon_p)] \frac{3\pi^2}{16} (\nu_p J_{pp}^2)^2,
\]

(4.18)

where \( Z_2(\epsilon_p) \) and \( Z_I(\epsilon_p) \), proportional to connected and disconnected parts, are given in Eqs. (3.34) and (3.33), respectively. The Kondo coupling factor is equal to

\[
\frac{3\pi^2}{16} (\nu_p J_{pp}^2)^2 = 12 \sin^2 \frac{p}{2} \left( \frac{\tau^2 + \tau^2}{\tau} \right)^2 \left( 1 + \tau^2 + 2\gamma \tau \cos \phi \cos \phi \right)^2
\]

(4.19)

At temperatures small compared to the band width but large compared to \( T_K \), considered here, the integration over \( f'(\epsilon_p) \) does not introduce a considerable thermal smearing and we can replace \( \epsilon_p \) with \( \epsilon_F \) in the rest of the integrand, yielding

\[
\Delta G = \frac{2e^2}{h} \left[ Z_2(\epsilon_F) + Z_I(\epsilon_F) \right] \frac{3\pi^2}{16} (\nu_p J_{ppp})^2.
\]

(4.20)

The flux-dependence of the conductance has two origins. One is through the flux-dependence of the Kondo coupling \( J \) which affects the flux dependence of the Kondo temperature \( T_K \). The other source of flux dependence is through \( Z_2 \).

Of course, Eq. (4.18) is just the result of perturbation theory to \( O(J^2) \). We expect that higher order terms will renormalize the Kondo coupling, giving it a temperature dependence:

\[
J_{ppp}(T) = J_{ppp} + \nu_p J_{ppp}^2 \ln(D/T) + \cdots,
\]

(4.21)

where \( D \) is of order the band width, \( t \). The simple form of this renormalized coupling comes from the fact that for the small ring considered here, \( J_{kp} \) is a slowly varying function of energy on scales of order the band width. The effective Kondo coupling thus grows large at the Kondo temperature

\[
T_K \approx D e^{-1/(\nu_p J_{ppp})}.
\]

(4.22)

Thus our perturbative result should only be reliable at \( T \gg T_K \). In this regime we may write:

\[
\nu_p J_{ppp}(T) \approx \frac{1}{\ln(T/T_K)}.
\]

(4.23)

In the opposite limit, \( T \ll T_K \) we can use the results of \( T \) based on Fermi liquid theory and NRG.

Fig. 3 shows \( \Delta G versus \phi \) for \( \gamma = 1 \) and various values of \( \tau = t'/t \), electron density \( 2p_F/\pi \) and \( (t^2 + t^2_R)/(Ut) \) at \( T_K \ll T \ll t \). We have adjusted the value of \( (t^2 + t^2_R)/(Ut) \), as we adjust \( p_F \) and \( \tau \), so that \( \nu_p J_{ppp} \) has the maximum value .217 for each curve (occurring for
φ = 0), small enough we hope for perturbation theory to be valid. This corresponds to the condition
\[ \sin p_F \cdot \frac{t_L^2 + t_R^2}{Ut} \approx \frac{1 + \tau^2 + 2\tau \cos p_F}{(1 + \tau^2)^2 - 4\tau^2 \cos^2 p_F} = \pi \times 0.217 \approx 0.0852 \]  

At \( p_F = \pi/2 \), the flux-dependence of the Kondo coupling, \( J_{k,ke} \) vanishes and the AB oscillations originate from the flux-dependence of the coefficient \( Z_2(\epsilon_L) \) in Eq. (3.34) which only contains the second harmonic of \( \phi \) at this density. This can be seen in Fig. (6a), while Figs. (b,c) show these corrections for lower densities, i.e. \( p_F = \pi/3 \) and \( \pi/6 \). We need to emphasize that the corrections \( \Delta G \) shown are only the part containing the imaginary part of the T-matrix which is universal. At low densities, the flux-dependence of the Kondo coupling becomes important and higher harmonics create a plateau-like feature in the conductance as a function of \( \phi \).

Note that for symmetric coupling (\( \gamma = 1 \)) and at zero flux (\( \phi = 0 \)), we have \( Z_2 = 0 \) and the sign of Kondo-type conductance correction is set by the sign of \( Z_2 \) as \( Z_2(\epsilon_L) \approx 1 - 2T_0 \). Thus, introducing the dot leads to an enhancement (suppression) of the conductance for \( T_0 < 0.5 \) (\( T_0 > 0.5 \)). It can be shown that this is a rather general criterion for parity-symmetric networks with an embedded quantum dot [31] and it might be related to similar scenarios in transport through molecular junctions with vibrational modes [32]. For the present model, the transition happens at \( \tau \approx 0.414 \) at half-filling (\( p_F = \pi/2 \)) as can be seen in Fig. (6b).

V. ELIMINATING THE CONNECTED 4 POINT FUNCTION FROM THE CONDUCTANCE

A well-known result of Meir and Wingreen [23] based on Keldysh formalism, shows that for quite general models of interacting quantum dots connected to non-interacting leads, the conductance can be expressed in terms of the T-matrix. [10] and [15] also assumed this. Thus it is perhaps surprising that our formula for the conductance includes a contribution from a connected 4-point Green’s function of the Anderson or Kondo model. The Meir-Wingreen argument is based on the fact that the source-drain voltage could be applied with an arbitrary asymmetry parameter, \( y \), between left and right sides of the quantum dot and the same current should result. In the linear response regime, considered here, Keldysh and Kubo formalisms should yield identical results. In subsection (V A) we recast the Meir-Wingreen argument in Kubo formalism and show that it does straightforwardly allow for exact elimination of the connected part in special cases: for no reference arm, \( t' = 0 \), for parity symmetry, \( t_L = t_R \), \( \varphi = 0 \) or for \( t_L \) or \( t_R = 0 \). In subsection (V B) we use the fact, established in Sec. (III), that both disconnected and connected parts of the conductance can be written as integrals over energy, \( \epsilon_p \), of \( f'(\epsilon_p) \) multiplied by “transmission probabilities”, \( T^D(\epsilon_p) \) and \( T^C(\epsilon_p) \). Let us denote the disconnected/connected term in the transmission probability, when the Kubo formula is used with \( \Delta N/2 \) replaced by \( N_0 \) in Eq. (3.5), by \( T^{D/C}(\epsilon_p) \). We show that if the total transmission probability \( T^{D/C}(\epsilon_p) + T^{D}(\epsilon_p) \), is assumed to be independent of \( y \) at all \( \epsilon_p \), then the connected part can be eliminated and the conductance expressed as a sum of terms of zeroth and first order in the T-matrix only. We show that this strong assumption holds in lowest order perturbation theory. However, it appears unlikely that it holds exactly. In subsection (V C) we show that the connected part can be approximately eliminated, for temperatures small compared to the band width, using only the \( y \)-independence of the total conductance, the energy integral of \( f'(\epsilon_p)[T^{D/C}(\epsilon_p) + T^{D}(\epsilon_p)] \). However, while this elimination is possible for the short ABK ring considered here, we argue that it would fail for a long ABK ring of length \( L \) except at extremely small temperatures, less than the finite size gap of the ring, \( \approx v_F/L \). In the final subsection we apply Keldysh formalism to the ABK ring and again show that the Meir-Wingreen argu-
ment does not apply exactly. We show that it may apply approximately, for temperatures small compared to the band width (and finite size gap) subject to a plausible assumption about a non-equilibrium Green’s function.

A. Meir-Wingreen argument recast in Kubo formalism

In Sec. (III), we derived the Kubo formula by adding an infinitesimal perturbation $eV(t)\Delta N/2 \equiv eV(t)(N_R - N_L)/2$ to the Hamiltonian. (Here
\[ V(t) = V_0 \cos \Omega t \] and we eventually take the limit $\Omega \to 0$.) In the linear response regime, it should be equivalent to apply the voltage asymmetrically, adding $eV(t)N_y$ to the Hamiltonian, where
\[ N_y \equiv yN_R - (1-y)N_L. \] (5.2)

For the Kondo model, $N_R + N_L$ is the total charge and commutes with the Hamiltonian and with $\Delta N$. Therefore it is easily proven that the Kubo formula with $\Delta N/2$ replaced by $N_y$ gives the same conductance as the Kubo formula with $\Delta N/2$. We also expect this to be true for the Anderson model in the parameter range where charge fluctuations of the quantum dot can be ignored at low energies. Applying the source-drain voltage asymmetrically, $eV(t)N_y$, is equivalent to applying an asymmetric vector potential to the links between the quantum dot and sites $\pm 1$ (and between sites 1 and $-1$). To apply Kubo formalism for arbitrary $y$, the simplest approach is to define the current as $I = dN_y/dt$ and measure it in linear response to the perturbation $eV(t)N_y$.

As Meir and Wingreen observed, it may be possible for some models to choose a convenient value of the parameter $y$ so that the connected part is exactly eliminated and the conductance calculation is thus simplified. Unfortunately, that does not appear to work for the ABK ring for general values of the parameters. Let us see why that is so. We now calculate the conductance via the Kubo formula, Eq. (3.4), with $\Delta N/2$ replaced by $N_y$, for an arbitrary real parameter $y$, in Eq. (3.5). We next express $N_y$ in the screening, non-screening basis:
\[ N_y(t) = \frac{1}{2} \int_0^\pi \frac{dk_1dk_2}{(2\pi)^2} \Psi^\dagger_{k_1}(t) \Psi^\dagger_{k_2}(t). \] (5.4)

where $\Psi^\dagger_{k_1,k_2}$ is again expressed in terms of the unitary matrix $U_k$ (which is independent of $y$) and a matrix $A^y_{k_1,k_2}$ by
\[ A^y_{k_1,k_2} = \|_{k_1}A^y_{k_2} U_{k_2}. \] (5.5)

The matrix $A^y_{k_1,k_2}$ is only modified by a shift of the “contact term” proportional to the unit matrix:
\[ A^y_{k_1,k_2} = A^{1/2}_{k_1,k_2} + (2y - 1)2\pi \delta(k_1 - k_2) \| \] (5.6)

where $A^{1/2}_{k_1,k_2}$ is the quantity simply denoted as $A_{k_1,k_2}$ in Eq. (3.9). The connected part of the conductance is again given by Eq. (3.35) with $\psi^y_{k_1,k_2}$ replaced by $\psi^y_{k_1,k_2}$. Thus the connected part of the conductance will be eliminated if it is possible to choose the real parameter $y$ so that
\[ \psi^y_{k_1,k_2} = 0 \] (5.7)

for all $k_1$ and $k_2$. Unfortunately, since $\psi^y_{k_1,k_2}$ generally depends non-trivially on $k_1$ and $k_2$, this is usually not possible.

An exception occurs for the case of no reference arm, $\tau = 0$. Then the matrix $U_k$ becomes independent of $k$, the overlap term in $A^y_{k_1,k_2}$ vanishes and $\psi^y_{k_1,k_2}$ simplifies to:
\[ \psi^y_{k_1,k_2} = 2\pi \delta(k_1 - k_2)(y - 1)2\pi \delta(k_1 - k_2). \] (5.8)

(Here we assume, without loss of generality, that $t_L > t_R$.) Thus $\psi^y_{k_1,k_2} = 0$ for all $k_1$ and $k_2$ when we choose:
\[ y = \frac{1}{2} \left(1 + \sqrt{1 - \gamma^2}\right) = \frac{t_L^2}{t_L^2 + t_R^2}. \] (5.9)

The vanishing of $\psi^y_{k_1,k_2}$ also implies that the disconnected term quadratic in the $T$-matrix vanishes. We extend the calculation of the $Z$ coefficients to general $y$ in App. (B4). Eqs. (3.32), (B20) and (B21) then give
\[ Z^y_R = Z^y_L = 0, \quad Z^y_\gamma = \gamma^2 \] (5.10)

for the special value of $y$ in Eq. (5.9). Thus the conductance can be written:
\[ G = \frac{2e^2}{h} \gamma^2 \int dp \left[-f'(\epsilon_p)\right] \text{Im} \left[-\pi \nu_p T(\epsilon_p)\right]. \] (5.11)

This result has been already obtained in [24, 34, 35, 22]. Another way of understanding what is so much simpler about the case of no reference arm is to observe that, even for the symmetric case $y = 1/2$, the term in $\Delta N$ quadratic in $\psi$ has the simple form:
\[ \Delta N_{\psi \psi} = -\sqrt{1 - \gamma^2} \int_0^\pi \frac{dk}{2\pi} \psi^\dagger_k \psi_k \equiv -\sqrt{1 - \gamma^2} N_{\psi}. \] (5.12)

$N_{\psi}$, the total number of screening electrons, is an exactly conserved quantity in the Kondo model and approximately conserved at low energies in the Anderson model in the regime where charge fluctuations of the dot can be ignored. It then follows that the total contribution to $G^4(\Omega)$, defined in Eq. (3.3) which is quartic in $\psi$ is:
\[ G^4(\Omega) \equiv -\frac{i(1 - \gamma^2)}{4} \int_0^\infty dt e^{i(\Omega + i\gamma)t} \langle [N_{\psi}(t), N_{\psi}(0)] \rangle = 0 \] (5.13)
Then Eqs. (3.32) to (3.34) reduce to:

\[ t y \]

Now \( U \gamma \) is non-zero due to the contribution of multi-particle final states. Rather, the conservation of \( N \) becomes the identity matrix since only the even \( k \)-values of \( \psi \) are conserved quantities.

Interestingly, below, are exact equalities for the Kondo model and approximate ones for the Anderson model. By comparing the disconnected part at \( y = 1/2 \), and \( \tau = 0 \), determined by Eqs. (3.30) to (3.34), with the exact conductance given in Eq. (5.11), we see that the connected part of the conductance, for the symmetric case \( y = 1/2 \), is given by:

\[
G^{C} = -\frac{1 - \gamma^2}{4} \lim_{\Omega \to 0} \int_{0}^{\infty} dt e^{i(\Omega + im)t} \langle [N_{\psi}(t), N_{\psi}(0)] \rangle_{C} \\
= -(1 - \gamma^2) \frac{2e^2}{h} \int dp [-f'(\epsilon)] \Delta T_{pp}(\epsilon_p) \tag{5.14}
\]

where \( \Delta T_{pp}(\epsilon_p) \) is defined in Eq. (B26). An important check on this result is that for the non-interacting case, where \( G^{C} \) must be zero, \( \Delta T_{pp} \) vanishes due to the optical theorem. (For the interacting case \( \Delta T_{pp} \) is generally non-zero due to the contribution of multi-particle final states to the optical theorem.) Unfortunately, for the ABK ring, \( \Delta N_{\psi} \) is generally not a conserved quantity.

Another special case where the connected part can be eliminated is with parity symmetric \( \gamma = 1, \varphi = 0 \). Now \( U_k \) becomes the identity matrix since only the even channel couples to the impurity. Thus:

\[
\mathcal{M}_{k_1k_2}^{y_{11}} = 2\pi \delta(k_1 - k_2)(2y - 1) \tag{5.15}
\]

which is zero for the parity symmetric choice \( y = 1/2 \). Then Eqs. (3.32) to (3.34) reduce to:

\[
Z_R(\epsilon_p) = \frac{4\tau \cos p + (1 + \tau^2)}{1 + \tau^2 + 2\tau \cos p} \sqrt{\mathcal{T}_0[1 - \mathcal{T}_0]}, \tag{5.16}
\]

\[
Z_L(\epsilon_p) = 1 - 2\mathcal{T}_0(\epsilon_p), \tag{5.17}
\]

\[
Z_2(\epsilon_p) = 0. \tag{5.18}
\]

Yet another special case is when \( t_R \) or \( t_L = 0 \). Then we find:

\[
\mathcal{M}_{k_1k_2}^{y_{11}} = 2\pi \delta(k_1 - k_2) \left[ 2y - 1 - \frac{1 - \tau^2}{1 + \tau^2} \right] \tag{5.19}
\]

which is zero for

\[
y = \frac{1}{1 + \tau^2}. \tag{5.20}
\]

Eqs. (3.32), (B20) and (B21) then give, for this value of \( y \):

\[
Z_R(\epsilon_p) = \frac{4\tau \cos p}{1 + \tau^2} \sqrt{\mathcal{T}_0(\epsilon_p)[1 - \mathcal{T}_0(\epsilon_p)]},
\]

\[
Z_L(\epsilon_p) = 1 - 2\mathcal{T}_0(\epsilon_p) - \left( \frac{1 - \tau^2}{1 + \tau^2} \right)^2
\]

\[
Z_2(\epsilon_p) = 0. \tag{5.21}
\]

In general, the crucial function \( \mathcal{M}_{k_1k_2}^{y_{11}} \) has the form:

\[
M_{k_1k_2}^{y_{11}} = \left[ 2y - 1 + m_{k_1k_2} \right] 2\pi \delta(k_1 - k_2)
+ 2it \sin k_1 m_{k_1k_2} \mathcal{R}(\epsilon_k)
\]

where the functions \( m_{k_1k_2} \) and \( m_{k_1k_2}^{pp} \), in the diagonal case \( k_1 = k_2 = p \), are given in Eqs. (B13) and (B16) respectively. Clearly, for general values of \( \tau, \gamma \) and \( \varphi \), we cannot make \( M_{k_1k_2}^{y_{11}} \) zero for all \( k_1 \) and \( k_2 \) for any choice of \( y \).

### B. Using the “transmission probability” expression for the connected and disconnected parts of the conductance

In Sec. (III) we showed that both disconnected and connected parts of the conductance can be written in the form:

\[
G^{D/C} = \frac{2e^2}{h} \int dp [-f'(\epsilon_p)] T^{D/C}(\epsilon_p) \tag{5.23}
\]

defining disconnected and connected parts of an energy-dependent “transmission probability”. It is convenient to define the imaginary frequency Green’s function:
The connected part of the transmission probability can now be written:

\[
\mathcal{T}^{\nu C}(\omega) = \lim_{\Omega \to 0} \frac{\Omega^2}{4} P^{\nu}(\omega - i\eta_1, \omega + \Omega + i\eta_2) + c.c.
\]

\[
= \lim_{\Omega \to 0} \frac{\Omega^2}{16} \int_0^\pi \frac{dk_1 dk_2}{(2\pi)^2} M^{p11}_{k_1 k_2} g^A_{k_1}(\omega) g^R_{k_2}(\omega + \Omega) \cdot Q^y_{k_1 k_2}(\omega - i\eta_1, \omega + \Omega + i\eta_2) + c.c.
\]

where \( Q^y_{k_1 k_2}(\omega - i\eta_1, \omega + \Omega + i\eta_2) \) is the analytic continuation of \( Q^y_{k_1 k_2}(i\omega_m, i\omega_m + i\omega_p) \) to real frequencies. Using the expression for \( M^{p11}_{k_1 k_2} \) in Eq. (5.22) and the propagator product identities of App. (A) we see that

\[
\lim_{\Omega \to 0} \Omega M^{p11}_{k_1 k_2} g^A_{k_1}(\epsilon_p) g^R_{k_2}(\epsilon_p + \Omega) = 2\pi\nu_p [2y - 1 + m^{\nu}_{pp} + m^{o}_{pp}] \delta(k_1 - k_2) \delta(k_1 - p).
\]

We expect an additional factor of \( 1/\Omega \) to arise from \( Q^y_{k_1 k_2}(\omega - i\eta_1, \omega + \Omega + i\eta_2) \) as \( \Omega \to 0 \). Thus we may write:

\[
\mathcal{T}^{\nu C}(\epsilon_p) = -[2y - 1 + m^{\nu}_{pp} + m^{o}_{pp}] F_y(\epsilon_p)
\]  

(5.28)

where

\[
F_y(\epsilon_p) \equiv \frac{\pi\nu_p}{8} \lim_{\Omega \to 0} \Omega Q^y_{pp}(\epsilon_p - i\eta_1, \epsilon_p + \Omega + i\eta_2) + c.c.
\]

Due to the \( N_y \) operator in \( Q^y \), \( F_y(\epsilon_p) \) is a sum of terms of zeroth and first order in \( y \). Despite the simplifications resulting from using the transmission probability expression for the conductance, it appears that no choice of \( y \) will make the connected part vanish. Nonetheless, the fact that the total conductance must be independent of \( y \) implies some relationship between \( \mathcal{T}^{\nu C}(\epsilon_p) \) and \( \mathcal{T}^{\nu D}(\epsilon_p) \). This implied relationship involves the integral of these functions. However it is interesting to consider the consequences of the stronger assumption that \( \mathcal{T}^{\nu C}(\epsilon_p) + \mathcal{T}^{\nu D}(\epsilon_p) \) is independent of \( y \) for all energies. Using the expression for \( \mathcal{T}^{\nu D}(\epsilon_p) \) in Eqs. (B21) to (B26), we may write:

\[
\mathcal{T}^{\nu C}(\epsilon_p) + \mathcal{T}^{\nu D}(\epsilon_p) = \mathcal{T}(\epsilon_p) + (2y - 1 + m^{\nu}_{pp} + m^{o}_{pp})[(2y - 1 + m^{\nu}_{pp})\Delta T_{pp} - F_y(\epsilon_p)]
\]

(5.29)

\( \Delta T_{pp} \), given in Eq. (B26), measures violations of the optical theorem when the \( T \)-matrix is restricted to the single particle sector. As mentioned above, \( F_y(\epsilon_p) \) is a sum of terms of zeroth and first order in \( y \). We see that \( y \)-independence of \( \mathcal{T}^{\nu C}(\epsilon_p) + \mathcal{T}^{\nu D}(\epsilon_p) \) would require:

\[
F_y(\epsilon_p) = (2y - 1 + m^{\nu}_{pp})\Delta T_{pp}
\]

(5.30)

Then, the total transmission probability becomes \( \mathcal{T}(\epsilon_p) \), given in Eq. (B25). It can be seen that our perturbative calculation is actually consistent with this stronger assumption. In this case

\[
F_y(\epsilon_p) = \frac{\pi\nu_p}{2} \Im \left[ \lim_{\Omega \to 0} \Omega \int_0^\pi dq_1 dq_2 \frac{3}{16} J_{q_1 q_2} \left( M^{p11}_{q_1 q_2} g^A_{q_1}(\epsilon_p + \Omega) g^R_{q_2}(\epsilon_p) \right) \right]
\]

\[
= [2y - 1 + m^{\nu}_{pp}] \frac{3\pi^2}{16} \nu_p^2 J_{pp}^2 = [2y - 1 + m^{\nu}_{pp}]\Delta T_{pp}
\]

(5.31)

where the fact that \( T_{pp} \) is \( O(J^2) \) was used in the last step so that \( \Delta T \) reduces to \( \Im[-\pi\nu_p T_{pp}(\epsilon_p)] \), whose perturbative value is given in Eq. (4.7). However, it seems unlikely that this stronger assumption will survive higher orders of perturbation theory so we now proceed without making it.

**C. Approximate elimination of connected term**

Since the connected part of the conductance is given in terms of \( \mathcal{T}^{\nu C}(\epsilon_p) \) by Eq. (3.41), we see that, for temperatures small compared to the band width, \( G^{\nu C} \) will approximately vanish provided we choose \( y \) so that \( \mathcal{T}^{\nu C}(\epsilon_p) \) vanishes at the Fermi energy, \( \epsilon_p = \epsilon_F \). This choice is:

\[
y = \frac{1}{2}[1 - m^{\nu}_{pp} - m^{o}_{pp}].
\]

(5.32)
Since $m^\alpha_{g\nu} + m^\gamma_{g\nu}$ is a smooth function of $\epsilon_p$, as can be seen from Eqs. [16] and [15], we may calculate the leading contribution of $G^{\nu C}$ to the conductance, for this choice of $y$, by the Sommerfeld expansion, giving a suppression factor of order $(T/t)^2$ where $T$ is the temperature and $4t$ the band width. We see from Eq. [20] that $Z_2^\nu(\epsilon_F)$ vanishes for the special value of $y$, Eq. (3.32) which makes the connected part approximately vanish. The reason for this can be seen in App. [3].

The T-matrix contains sharp features on the scale of $T/m$ at $T$ that within second order perturbation theory in Kondo at temperatures small compared to the band width. Note that the T-matrix of a single channel Kondo or Anderson model on an ABK ring can be expressed entirely in terms of the connected part of the conductance will be necessary, except at extremely low temperatures.

Thus we may write the conductance for $T \ll t$ as a linear function of the T-matrix:

$$G \approx \frac{2e^2}{h} \int d\epsilon_p [-f'(\epsilon_p)] \{T_0(\epsilon_F)$$

$$+ 2R(\epsilon_F) \text{Re} [-\pi\nu_p T_{pp}(\epsilon_p)]$$

$$+ 2f'(\epsilon_F) \text{Im} [-\pi\nu_p T_{pp}(\epsilon_p)] \} \quad (5.33)$$

where $T_0(\epsilon_p)$, $Z_R(\epsilon_p)$ and $Z_L(\epsilon_p)$ are given in Eqs. (3.31), (3.32) and (B22) respectively. Eq. (5.33), along with our formulas for the coefficients, is one of the main results of this paper. It shows that the conductance through the small ABK ring can be expressed entirely in terms of the T-matrix of a single channel Kondo or Anderson model at temperatures small compared to the band width. Note that within second order perturbation theory in Kondo coupling (valid at $T \gg T_K$), the T-matrix is also smooth and can be approximated by its value at the Fermi and taken out of the integral. However, at lower temperatures the T-matrix contains sharp features on the scale of $T_K$ and the thermal averaging is relevant.

It is interesting to compare this result to [10] which also gives a formula for the conductance of the short ABK ring as a sum of terms of zeroth and first order in the T-matrix. A precise agreement cannot be expected since [10] assumes energy independent tunneling parameters and expresses the result in terms of parameters at the Fermi surface only. We find precise agreement at half-filling, $p_F = \pi/2$, only.

Note that our argument depends crucially on the fact that $m^\nu_{pp}$ and $m^{\gamma}_{pp}$ defined in Eqs. [16] and [15], are smooth functions of $\epsilon_p$; the energy scale over which they vary significantly is the band width, $4t$. However, we expect this not to be the characteristic energy scale for a large ring of length $L$. The problem is that a small energy scale enters the calculation, the finite size gap $\propto v_F/L$. We then expect the analogue of $m^\nu_{pp} + m^{\gamma}_{pp}$ to vary on this scale, making the approximate elimination of $G^{\nu C}$ only possible for $T \ll v_F/L$ which is much less than the band width for a ring much larger than a lattice constant. Thus, we may expect that a calculation of the connected part of the conductance will be necessary, except at extremely low temperatures.

## D. Keldysh approach

The presence of the connected four-point function in the conductance (even if it can be approximately eliminated) is surprising given that, according to [10] and following the Keldysh approach of Meir and Wingreen [24], the conductance can be expressed entirely in terms of the retarded two-point function $G_{dd}^R$. In this sub-section, we calculate the conductance using Keldysh approach and point out that generally symmetrization fails and the equilibrium two-point Green’s functions are not sufficient for determining the conductance. However, similar to the discussion of Kubo section, at temperatures small compared to the band width the non-equilibrium Green’s functions can be approximately eliminated. Similar calculations have been reported previously in [5, 10, 33]. Going back to the Anderson model and defining $N_L = \sum_{n>0} c^\dagger_{-n} c_{-n}$, we can write the current in the left lead as

$$I_L = \frac{ie}{\hbar} \left[ \langle \hat{N}_L, \hat{H} \rangle \right]$$

$$= -\frac{2e}{\hbar} \text{Re} \left[ G_{dd}^L(0) + tG_{RL}^L(0) \right]$$

where $G_{dd}^L(\omega)$ is expressed in terms of non-equilibrium equal-time Green’s functions defined as $G_{dd}^L(t) = i\langle \hat{c}^\dagger_{-1}(t) \hat{d}(0) \rangle$ and $G_{RL}^L(t) = i\langle \hat{c}^\dagger_{-1}(t) \hat{c}_{-1}(0) \rangle$ involving the dot and the first sites of the left and right leads. Here $L$ and $R$ indices represent sites -1 and 1 respectively. We denote non-equilibrium operators with a hat and corresponding Green’s functions with Sans-serif font. Replacing equal-time Green’s functions with a frequency integral over the Fourier transform of corresponding unequal-time Green’s functions we get

$$I_L = \int_{-\infty}^{+\infty} d\omega I_L(\omega), \quad I_L(\omega) = -\frac{2e}{\hbar} \text{Re} \left[ W^<_L(\omega) \right]$$

where

$$W^<_L(\omega) \equiv 2tL e^{i\omega/2} G_{dd}^L(\omega) + 2t' G_{RL}^L(\omega).$$

In the non-interacting case, $I_L(\omega)$ can be interpreted as the contribution to the current from electrons of energy $\omega$. Following Meir-Wingreen the mixed functions $G_{dd}$ and $G_{RL}$ are related to the Green’s function of the dot. For that purpose, the Green’s functions are generalized to complex times on Keldysh contour $C_K$ and we use the equation of motion to obtain

$$G_{dd}(\tau, \tau') = -t_L e^{-i\epsilon/2} \int_{C_K} d\tau_1 G_{dd}(\tau, \tau_1) g_{LL}(\tau_1, \tau')$$

$$- t' \int_{C_K} d\tau_1 G_{dd}(\tau, \tau_1) g_{LL}(\tau_1, \tau').$$

Here $G_{dd}(\tau, \tau') \equiv (i\tau C_{d}(\tau) c^\dagger_{-1}(\tau'))$ is the contour-ordered mixed Green’s function and $g_{LL}(\tau, \tau')$ is the Green’s function of the first site of the decoupled left
lead in equilibrium with its own electrochemical potential \(\mu_L\). Going to real time and taking the Fourier transform we can represent this equation in Keldysh space by the matrix equation \([37]\)

\[
\dot{G}_{dd}(\omega) = -t_L e^{-i\varphi/2} \hat{G}_{dd}(\omega) g_L - t' \hat{G}_{dd}(\omega) g_{LL}(\omega).
\]

Here the \(\hat{G}(\omega)\) and \(g(\omega)\)'s are 2 \times 2 matrices in Keldysh space whose structure is

\[
\hat{G}(\omega) = \begin{pmatrix} G^R(\omega) & G^K(\omega) \\ 0 & G^A(\omega) \end{pmatrix},
\]

with \(G^K(\omega) = G^>(\omega) + G^<\omega)\). Similarly, for the other Green's functions, suppressing energy-dependences, we can write

\[
\dot{G}_{RR} = -t_R e^{i\varphi/2} \hat{G}_{dd}(\omega) \hat{G}_{RR} - t' \hat{G}_{dd}(\omega) \hat{G}_{RL},
\]

\[
\dot{G}_{LL} = \hat{G}_{LL} - t_L e^{i\varphi/2} \hat{G}_{dd}(\omega) \hat{G}_{ll} - t' \hat{G}_{dd}(\omega) \hat{G}_{RL},
\]

\[
\dot{G}_{RL} = -t_R e^{-i\varphi/2} \hat{G}_{dd}(\omega) \hat{G}_{RL} - t' \hat{G}_{dd}(\omega) \hat{G}_{LL}.
\]

These equations can be alternatively derived starting from a representation of the Hamiltonian in momentum space. After some algebra we arrive at the matrix equation

\[
\dot{X}_L \dot{W}_L \dot{X}_L = -2t^2 g_{RL} \dot{X}_L + 2 \hat{P}_L \hat{G}_{dd} \dot{Q}_L
\]

where \(\dot{X}_L(\omega), \dot{P}_L(\omega), \) and \(\dot{Q}_L(\omega)\) are given by

\[
\dot{X}_L = \hat{X}_L = \hat{I} - t^2 g_{R} \hat{R}
\]

\[
\dot{P}_L = \hat{P}_L = \hat{I} t e^{i\varphi/2} - \hat{R} + t' \hat{R} e^{-i\varphi/2},
\]

\[
\dot{Q}_L = -t_L e^{-i\varphi/2} \hat{R} + t' \hat{R} e^{i\varphi/2} \hat{R}.
\]

The real part of the lesser component of \(\hat{W}_L(\omega)\) gives the energy-resolved current from the left lead \(I_L(\omega)\)

\[
\frac{\hbar}{2e} I_L(\omega, \Delta\mu) = z_{0}^{KL}(\omega, \Delta\mu) + z_{R}^{KL}(\omega, \Delta\mu) \text{Re} [G_{dd}^{R}(\omega)]
\]

\[
+ z_{I}^{KL}(\omega, \Delta\mu) \text{Im} [G_{dd}^{R}(\omega)]
\]

\[
+ z_{<}^{KL}(\omega, \Delta\mu) [iG_{dd}^{<}(\omega)].
\]

The subscripts of the \(z^{KL}\) coefficients are related to the corresponding correlation function of the dot and their superscript means they are obtained from Keldysh technique and related to the left lead. The current in the right lead is obtained from \(L \leftrightarrow R\) and \(\varphi \leftrightarrow -\varphi\) substitution. Assuming a symmetric applied bias, the bias dependence of the \(z\) coefficients are caused by \(f_{L/R}(\omega) = f(\omega) \pm \frac{1}{2} \Delta\mu f'(\omega) + \cdots\) inside \(g_L(\omega)\) and \(g_R(\omega)\) and are indicated explicitly, but the Green's functions also have an implied bias-dependence. Using Eq. (5.43), it can be shown that the \(z\) coefficients have a Taylor series in the bias \(\Delta\mu\) of the form

\[
\begin{align*}
 z_{0}^{KL}(\omega, \Delta\mu) &= \Delta\mu [f'(\omega) T_0(\omega)] + \cdots, \\
 z_{R}^{KL}(\omega, \Delta\mu) &= z_{0}^{KL}(\omega, 0) + \Delta\mu [f'(\omega)] T_{R}^{KL}(\omega) + \cdots, \\
 z_{I}^{KL}(\omega, \Delta\mu) &= \Delta\mu [f'(\omega)] T_{I}^{KL}(\omega) + \cdots,
\end{align*}
\]

(5.46)

defining the coefficients \(T_{R}^{KL}(\omega)\), whereas \(z_{<}^{KL}(\omega, \Delta\mu) = z_{<}^{KL}(\omega)\) is independent of the bias. The equilibrium components of the first two terms are zero, \(z_{0}^{KL}(\omega, 0) = z_{I}^{KL}(\omega, 0) = 0\). But \(z_{I}^{KL}(\omega, 0)\) is nonzero and satisfies \([24]\)

\[
z_{I}^{KL}(\omega, 0) = -2 f(\omega) z_{<}^{KL}(\omega).
\]

(5.47)

which by the equilibrium condition \(iG_{dd}^{<}(\omega) = 2 f(\omega) \text{Im} [G_{dd}^{R}(\omega)]\) ensures that the expectation-value of the current operator defined by Eq. (5.34) is indeed zero in equilibrium. The function \(T_0(\omega)\) is the same back-ground transmission we had in Kubo calculations. Using Eqs. (5.46), (5.47) we can write the current at energy \(\omega = \epsilon_p\) as

\[
\frac{\hbar}{2e} I_L(\epsilon_p, \Delta\mu) = \Delta\mu [f'(\omega)] T_0(\epsilon_p)
\]

\[
+ z_{R}^{KL}(\epsilon_p) \text{Re} [G_{dd}^{R}(\epsilon_p)]
\]

\[
+ z_{I}^{KL}(\epsilon_p) \text{Im} [G_{dd}^{R}(\epsilon_p)]
\]

\[
+ z_{<}^{KL}(\epsilon_p) \Pi(\epsilon_p, \Delta\mu),
\]

(5.48)

where

\[
\Pi(\epsilon_p, \Delta\mu) = iG_{dd}^{<}(\epsilon_p) - 2 f(\epsilon_p) \text{Im} [G_{dd}^{R}(\epsilon_p)].
\]

(5.49)

The first four lines of Eq. (5.48) contains two-point equilibrium Green’s function of the dot whereas the last line, written in terms of \(\Pi(\omega, \Delta\mu)\) contains non-equilibrium Green’s functions and is more complicated to compute. Since \(z_{<}^{KL}(\omega)\) is nonzero, in order to get the linear-response current in general one needs to do a first-order perturbative-in-bias expansion of the non-equilibrium functions in \(\Pi(\omega, \Delta\mu)\), which leads to both connected four-point and disconnected two-point contributions related to the terms proportional to \(Z_2\) in the Kubo framework.

The Meir-Wingreen approach \([24]\) uses the fact that the DC current satisfies \(\langle I_L + I_R \rangle = 0\) to symmetrize the current between left and right leads in order to eliminate the non-equilibrium Green’s function of the dot \(G_{dd}^{R}(\omega)\). However, such a procedure fails for the present problem as already noticed by Dinu et al. \([28]\). This can be seen most easily by taking the three sites –1, d and 1 as the central sites of the device and noticing that the coupling matrices introduced by Meir-Wingreen \([24]\) do not satisfy their “proportional coupling” condition.
Equivalently, one can attempt to find a parameter $y$ for which the symmetrized current $I_y = yI_R - (1-y)I_L$ is not a functional of $\Pi(\omega, \Delta \mu)$. Generally we can use

$$y(\epsilon_F) \equiv \frac{z_{KL}(\epsilon_F)}{z_{KL}(\epsilon_F) + z_{KR}(\epsilon_F)}$$

$$= \left(\frac{\tau_1^2 + \tau_2^2}{1 + \tau^2}\right) + \left(\frac{\tau_2}{1 + 4\epsilon_{\Omega}^2}\right) \cos \phi$$

(5.50)

to eliminate the non-equilibrium functions at the Fermi energy. This is precisely the same condition on $y$ obtained in Eq. (5.32) using the Kubo approach. This function is independent of energy if and only if at least one of the three parameters $t_L$, $t_R$ or $\tau = t'/t$ is zero or for the case when $t_L = t_R$ and $\varphi = 0$, indicating that the total symmetrized current does not contain $\Pi(\epsilon_F, \Delta \mu)$ in these special cases. These are precisely the special cases discussed above using the Kubo approach. The energy dependence of $y(\epsilon_F)$ for general parameters indicates that non-trivial symmetrization requires $I_L(\omega) + I_R(\omega)$ to be zero (conservation of energy-resolved currents) which is not the case in interacting systems at finite temperature. However, it is expected that the function $\Pi(\omega, \Delta \mu)$ contains a derivative of the Fermi distribution function, i.e. we can write

$$\Pi(\epsilon_F, \Delta \mu) = \Delta \mu [-f'(\epsilon_F)]\Pi'(\epsilon_F) + O(\Delta \mu^2)$$

(5.51)

Therefore, for temperatures smaller than the band width, the slowly varying parameter $y(\epsilon_F)$ is constant within $I_L$ the energy integral range and can be taken out of the integral and again an approximate symmetrization can be used to eliminate the difficult function $\Pi'(\epsilon_F)$ by setting its coefficient approximately equal to zero. The parameter $Z_{\mathcal{R}}(\epsilon_F)$ is the same for both leads and is unaffected by symmetrization and it is $-\pi \nu_p V_{\mathcal{F}}^2$ times the corresponding parameter in the Kubo calculations. The parameters $Z_{\mathcal{R}}^{KL}(\epsilon_F)$ and $Z_{\mathcal{R}}^{KR}(\epsilon_F)$ are, however, different but the symmetrized parameter $y(\epsilon_F)Z_{\mathcal{R}}^{KR}(\epsilon_F) + [1 - y(\epsilon_F)]Z_{\mathcal{R}}^{KL}(\epsilon_F)$ is equal $-\pi \nu_p V_{\mathcal{F}}^2$ times the parameter $Z_{\mathcal{R}}'(\epsilon_F)$ obtained in the Kubo section at $\epsilon_F = \epsilon_F$.

VI. CONCLUSIONS

We have studied transport properties of a small Aharonov-Bohm ring with an embedded quantum dot in one of its arms. The DC conductance is calculated using the Kubo formula and it is shown that there is a contribution which involves a connected four-point function. We have shown that for $T$ small compared to the band width, this term and terms quadratic in the T-matrix can be eliminated, leaving a formula for the conductance linear in the T-matrix. This is a useful result because rather precise results exist on the T-matrix for a wide range of temperatures and frequencies, using renormalization group improved perturbation theory, Nozières Fermi liquid theory, numerical renormalization group and other methods. We have calculated the conductance perturbatively in the Kondo coupling, a result that should be valid for $T \gg T_K$.

A natural question to ask is whether our $O(J^2)$ results are consistent with the observation that Kondo scattering is largely inelastic at $T \gg T_K$ [18]. This observation simply follows from the fact that the (single-particle) T-matrix of the Kondo model starts with an imaginary term of order $J^2$. Then the optical theorem:

$$-i(\hat{T} - \hat{T}^\dagger) = \hat{T}^\dagger \hat{T}$$

(6.1)

is badly violated if the sum over intermediate states, inserted between $\hat{T}^\dagger$ and $\hat{T}$ is restricted to the single-particle sector [18]. The full flux dependence of the conductance through an ABK ring is complicated. Using our approach, it arises partly from the flux dependence of the coupling of the quantum dot to the screening channel, which introduces a flux dependence of the Kondo coupling and hence the Kondo temperature. Further flux dependence arises from the $Z_{\mathcal{R}}(\epsilon_F)$ and $Z_{\mathcal{R}}'(\epsilon_F)$ coefficients given in Eqs. (5.32) and (5.22) relating the conductance to the real and imaginary parts of the T-matrix. (At higher temperatures, where the connected part must be included, the flux dependence becomes even more complicated.) The conductance is quadratic in the Kondo coupling, in perturbation theory, while being first order in the potential scattering. Thus, the absence of a term linear in the Kondo coupling leads to a reduction of flux dependence at high $T$ and can be “explained” by the fact that the scattering is purely inelastic in that limit [22].

We leave the extension of our results to lower temperature and to larger rings for future work [31]. As discussed in Sec. (7), for a large ring of length $L$ the connected term in the conductance can only be safely eliminated at temperatures below the finite size energy level spacing $(T \ll \nu_p/L)$. Thus, a thorough treatment will probably require calculation of the novel connected 4-point Green’s function at lower temperatures. Moreover, the relation between the degree of flux dependence of the conductance and the degree of inelastic scattering in general also remains an open question.

Acknowledgement

We thank M. Eto and Z. Shi for stimulating discussions. This work was supported by NSERC and CIFAR. Y. K. gratefully acknowledges financial support from the Swiss National Science Foundation. R. Y. is the Yukawa Fellow and this work is partially supported by Yukawa Memorial Foundation.

Appendix A: Propagator product identities

In this Appendix we show explicitly how the limit $\Omega \rightarrow 0$ is taken in various equations in this paper.
The results presented in this Appendix also support our argument that the first and last terms in Eq. (3.44) can be dropped as \( \Omega \to 0 \). We start by considering

\[
\lim_{\Omega \to 0} 1 \frac{\Omega - 2i\eta}{(\omega - \epsilon_k + i\eta)(\omega - \epsilon_k - i\eta)} = \frac{1}{\omega - \epsilon_k + i\eta} - \frac{1}{\omega - \epsilon_k - i\eta} = \frac{\Omega - 2i\eta}{(\omega - \epsilon_k + i\eta)(\omega - \epsilon_k - i\eta)} \tag{A1}
\]

Thus

\[
\lim_{\Omega \to 0} \Omega g_k^R(\omega) g_k^A(\omega + \Omega) = g_k^R(\omega) - g_k^A(\omega) = -2i\delta(\omega - \epsilon_k). \tag{A2}
\]

On the other hand, the same reasoning gives

\[
\lim_{\Omega \to 0} \Omega g_k^R(\omega) g_k^R(\omega + \Omega) = 0. \tag{A3}
\]

These results can be checked by doing the \( \omega \) integral.

\[
\int_{-\infty}^{\infty} d\omega \frac{1}{(\omega + \Omega - i\eta)(\omega + \Omega + i\eta)} = \frac{2\pi i}{-\Omega}. \tag{A4}
\]

The integral is non-zero since the poles are on opposite sides of the real \( \omega \) axis. On the other hand the integral of \( g_k^R(\omega) g_k^R(\omega + \Omega) \) is zero because both poles are on the same side.

The other important propagator product identity involves 3 propagators, \( \lim_{\Omega \to 0} \Omega g_k^A(\omega) g_k^R(\omega + \Omega) g_k^R(\epsilon_k) \).

Now we use:

\[
\lim_{\Omega \to 0} \Omega g_k^A(\omega) g_k^R(\omega + \Omega) g_k^R(\epsilon_k) = (2\pi)^2 \delta(\omega - \epsilon_k_1) \delta(\epsilon_k_1 - \epsilon_k_2). \tag{A6}
\]

Again this result can be checked by doing the \( \Omega \) integral:

\[
\int_{-\infty}^{\infty} d\omega \frac{1}{(\omega - \epsilon_1 - i\eta)(\omega + \Omega - \epsilon_2 + i\eta)(\epsilon_2 - \epsilon_1 + i\eta)} = -\frac{\Omega}{2\pi i} \frac{1}{(\epsilon_1 - \epsilon_2 + \Omega + 2i\eta)(\epsilon_1 - \epsilon_2 - i\eta)}. \tag{A7}
\]

The result of the previous paragraph tells us that the limit \( \Omega \to 0 \) of the right hand side is \( (2\pi)^2 \delta(\epsilon_1 - \epsilon_2) \), consistent with Eq. (A6). Note that, again, it is crucial which side of the real axis the poles are on. It can be seen from Eq. (A7) that complex conjugating any one of the propagators gives zero as \( \Omega \to 0 \). It is also important to note that

\[
g_k^R(\epsilon_k_2) = -g_k^A(\epsilon_k_1), \tag{A8}
\]

allowing the identity in Eq. (A6) to be written in another equivalent way.

Appendix B: Disconnected contribution to the transmission probability

Here we present the details that lead to Eqs. (3.30)-\(3.34\) for the disconnected part of the transmission probability, starting from Eqs. (3.26) and (3.27). From Eqs. (3.27) we obtain

\[
-2i \text{Im} G_{q_2 k_1}^{R}(\epsilon_p) = (2\pi)^2 \delta(q_2 - k_1) \delta(\epsilon_p - \epsilon_k_1) \mathbb{1} + \tau_p \left( g_{q_2}^R(\epsilon_p) V_{q_2} g_{d_d}^R(\epsilon_p) \right) V_{k_1} g_{k_1}^R(\epsilon_p) - g_{q_2}^A(\epsilon_p) V_{q_2} g_{d_d}^A(\epsilon_p) V_{k_1} g_{k_1}^A(\epsilon_p) \right). \tag{B1}
\]

We denote the first term of this expression by a subscript \( B \). The second by \( R \) and the third by \( A \). Plugging these into Eq. (3.26) we get 9 terms for the disconnected part of the transmission probability \( T^D \), which we label \( T_{0R}^D, T_{0A}^D, \ldots, T_{AA}^D \). In the following we write

\[
\epsilon_p' \equiv -2t \cos p' \equiv \epsilon_p + \Omega, \tag{B2}
\]

defining a (positive) momentum \( p' \) in terms of \( p \) and \( \Omega \).

1. Background transmission probability

The background transmission probability \( T_0(\epsilon_p) \equiv T_{00}^D(\epsilon_p) \) is obtained by choosing the delta-function term
from Eq. (B1), in both factors of $\text{Im} G^R$ in Eq. (3.26)
which leads to

$$T_0(\epsilon_p) = \frac{1}{8} \lim_{\Omega \to 0} \Omega^2 (2\pi)^2 \nu_p \nu_{p'} \frac{1}{\Omega + i0 + \Omega + i0} \text{Tr} [F_{pp'}F_{p'p}]$$

$$= \frac{4\pi^2 \sin^2 p}{(1 + \tau^2)^2 - 4\pi^2 \cos^2 p}$$  \hspace{1cm} (B3)

and is a Landauer-type formula for the transmission probability through the reference arm. This part of the conductance comes from the free part of the Green’s functions and the non-diagonal (in momentum) part of $\Delta N$ corresponding to the overlap (second) term in $\Delta_{A_{1}k_{2}}$ in Eq. (3.9). The contact term of $\Delta_{pp'}$ does not contribute since $p \neq p'$, $\Omega > 0$ until the end of the calculation.

2. Terms linear in T-matrix

The terms in the transmission probability linear in $G_{dd}$ are $T_{D1} = T_{D1}^R + T_{D1}^R + T_{D1}^A + T_{D1}^R$ and we only need to calculate the first two as they are the complex conjugate of the second two. Using Eqs. (5.5) and (B1), the first term is

$$T_{D1}^R(\epsilon_p) = \frac{1}{8} \lim_{\Omega \to 0} 2\pi \Omega^2 \nu_p \int_0^\pi \frac{dk_1 dq_2}{2\pi} G_{dd}^R(\epsilon_p) \text{Tr} \left[U_{k_1, A_{1}p'} U_{p'q_2}^\dagger g_{q_2}(\epsilon_p) V_{q_2}^\dagger V_{k_1} g_{k_1}(\epsilon_p)\right].$$  \hspace{1cm} (B4)

Using the cyclic property of the trace, this can be rearranged to give

$$T_{D1}^R(\epsilon_p) = \frac{\pi \nu_p}{4} \lim_{\Omega \to 0} \text{Tr} \left[I_{D1}(\epsilon_p, \Omega) \tau_0 I_{D1}(\epsilon_p, \Omega)\right] iG_{dd}^R(\epsilon_p),$$

where the matrices $I_{D1}$ and $I_{D1}'$ are defined as

$$I_{D1}(\epsilon_p, \Omega) = \Omega \int_0^\pi \frac{dq_2}{2\pi} g_{q_2}(\epsilon_p) \Delta_{p'q_2}^\dagger,$$

$$I_{D1}'(\epsilon_p, \Omega) = \Omega \int_0^\pi \frac{dk_1}{2\pi} V_{k_1} g_{k_1}(\epsilon_p) U_{k_1} A_{1}p.'$$

Using the definition of Eq. (3.9), the first one of these can be written as

$$I_{D1}(\epsilon_p, \Omega) = -\tau_x V_{p'}^\dagger,$$

$$I_{D1}'(\epsilon_p, \Omega) = -\Omega \int_0^\pi \frac{dq_2}{2\pi} g_{q_2}(\epsilon_p) A_{1}p_{q_2}^\dagger V_{q_2}^\dagger.$$  \hspace{1cm} (B7)

Using the propagator product identity of Eq. (A2), we obtain

$$I_{D1}(\epsilon_p, \Omega) \rightarrow \left( -\tau_x + 2\pi i\nu_p F_{pp}\right) V_{p'}^\dagger.$$  \hspace{1cm} (B8)

Similarly,

$$I_{D1}'(\epsilon_p, \Omega) \rightarrow -V_{p'} \tau_x.$$  \hspace{1cm} (B9)

The reason the $F$-term is present in Eq. (B8) but absent in Eq. (B9) is that $I_{D1}$ includes a term $\propto g_{q_2}^R(\epsilon_p) g_{q_2}(\epsilon_p + \Omega)$ whereas $I_{D1}'$ includes a term $\propto g_{k_1}^R(\epsilon_p) g_{k_1}(\epsilon_p + \Omega)$. In terms of $I_{D1}$ and $I_{D1}'$, we can write the second contribution to $T_{D1}^R$ as

$$T_{D1}^R(\epsilon_p) = \frac{\pi \nu_p}{4} \lim_{\Omega \to 0} iG_{dd}^R(\epsilon_p) \text{Tr} \left[I_{D1}(\epsilon_p, \Omega, -\Omega) \tau_0 I_{D1}(\epsilon_p + \Omega, -\Omega)\right].$$  \hspace{1cm} (B10)

The DC limit of the matrices $\|_{D1}(\epsilon_p + \Omega, -\Omega)$ and $\|_{D1}'(\epsilon_p + \Omega, -\Omega)$ is the same as before with an overall minus sign for each one and therefore $T_{D1} = T_{D1}$. Inserting these results into the trace in (B5), the final result is given in Eqs. (3.32) and (3.33).

3. Terms quadratic in T-matrix

The terms in the transmission probability quadratic in $G_{dd}$ are $T_{D2} = T_{D2}^R + T_{D2}^R + T_{D2}^A + T_{D2}^R$. It can be shown that in the DC limit the terms $T_{D2}^R$ and $T_{D2}^A$, which are proportional to $(G^{R}_{dd})^2$ and $(G^{A}_{dd})^2$, are zero because the propagator products involved in these terms do not produce any $\Omega^2$-divergence. $T_{D2}^R$ and $T_{D2}^R$ are the complex conjugate of each other and it suffices to calculate the first one which is

$$T_{D2}^R(\epsilon_p) = \frac{\pi \nu_p}{4} \lim_{\Omega \to 0} \text{Tr} \left[I_{D2}(\epsilon_p, \Omega) I_{D2}(\epsilon_p, \Omega) G_{dd}^R(\epsilon_p + \Omega) G_{dd}^R(\epsilon_p),\right]$$

where the functions $I_{D2}$ and $I_{D2}'$ are defined as

$$I_{D2}(\epsilon_p, \Omega) = \Omega \int_0^\pi \frac{dk_1 dq_2}{2\pi} V_{k_1} V_{q_2} \left\{ M_{k_1k_2}^{I} g_{k_1}^R(\epsilon_p + \Omega) g_{k_2}^R(\epsilon_p)\right\},$$

$$I_{D2}'(\epsilon_p, \Omega) = \Omega \int_0^\pi \frac{dq_1 dq_2}{2\pi} V_{q_1} V_{q_2} \left\{ M_{q_1q_2}^{I} g_{q_1}^R(\epsilon_p + \Omega) g_{q_2}^R(\epsilon_p)\right\}.$$  \hspace{1cm} (B11)

Separating the contact and overlap terms of $M_{k_1k_2}^{I}$, defined in Eqs. (3.9)-(5.5) we have

$$M_{k_1k_2}^{I} = m_{k_1k_2}^c 2\pi \delta(k_1 - k_2) + 2\pi \sin k_1 m_{k_1k_2}^e,$$

$$A_{1}p_{k_1k_2}^c 2\pi \delta(k_1 - k_2) + 2\pi \sin k_1 m_{k_1k_2}^e,$$  \hspace{1cm} (B13)

A factor of $2\pi \sin k_1$ is included for later convenience. The diagonal parameters $m_{pp}^{e}$ and $m_{pp}^{c}$ can be calculated from Eqs. (3.9) and (5.5) are equal to

$$m_{pp}^{c} = \frac{|\Gamma_{ep}\Gamma_{op}|}{V_{p}^{2}} \frac{\text{Re} \left[t_{dc}^{*} e^{-i\delta_{p}\delta_{p}^{*}}\right]}{1 + \tau^2 + 2\pi \gamma \sin \varphi \sin p},$$

and

$$m_{pp}^{e} = \frac{|\Gamma_{ep}\Gamma_{op}|}{V_{p}^{2}} \frac{\text{Im} \left[t_{dc}^{*} e^{-i\delta_{p}\delta_{p}^{*}}\right]}{1 + \tau^2 + 2\pi \gamma \sin \varphi \sin p}.$$  \hspace{1cm} (B15)
Using the results of App. [A] to take the limit $\Omega \to 0$, we obtain

$$I_{D2}(\epsilon_p, \Omega) \to (m_{pp}^c + m_{pp}^o) 2\pi i v_p V_p^2$$
$$I'_{D2}(\epsilon_p, \Omega) \to m_{pp}^c 2\pi i v_p V_p^2.$$  \hspace{1cm} (B17)

Inserting these results into Eq. (B11), gives Eq. (3.34).

4. Extension to general $y$

Here we extend our results for the disconnected part of the transmission probability to the case where the source-drain voltage is applied asymmetrically, multiplying $N_y(t)$, defined in Eq. (5.2). This has the effect of modifying $\mathbb{A}_{k_1k_2}^y$ as indicated in Eq. (5.6). This has no effect on the background transmission amplitude since it gets no contribution from the contact term in $\mathbb{A}_{k_1k_2}^y$. In the calculation of terms linear in the $T$-matrix the matrices $\mathbb{D}_1$ and $\mathbb{D}'_1$ both get shifted:

$$\mathbb{D}_1 \to \mathbb{D}_1 + (1 - 2y)V_p U_p^\dagger$$
$$\mathbb{D}'_1 \to \mathbb{D}'_1 + (1 - 2y)V_p U_p.$$  \hspace{1cm} (B18)

In the calculation of the terms quadratic in the $T$ the quantity $m_{pp}^c$ appearing in $M_{k_1k_2}^{1\dagger}$, $\mathbb{D}_1$ and $\mathbb{D}'_1$ gets shifted

$$m_{pp}^c \to (2y - 1) + m_{pp}^c.$$  \hspace{1cm} (B19)

The background transmission probability, $T_0(\epsilon_p)$ and $Z_R$ (the coefficient of $\text{Re}[T]$), are independent of $y$. However

$$Z_1^y(\epsilon_p) = 1 - 2T_0(\epsilon_p) + (2y - 1)^2$$
$$\quad \quad \quad - \frac{2\sqrt{1 - \gamma^2(1 - \tau^2)(2y - 1)}}{1 + \tau^2 + 2\gamma \tau \cos p \cos \varphi}$$  \hspace{1cm} (B20)

and

$$Z_2^y(\epsilon_p) = -\frac{(1 - \tau^2)(1 - \gamma^2) + 4\gamma^2 \tau^2 \sin^2 p \sin^2 \varphi}{[1 + \tau^2 + 2\gamma \tau \cos p \cos \varphi]^2}$$
$$\quad \quad \quad - (2y - 1)^2 + \frac{2\sqrt{1 - \gamma^2(1 - \tau^2)(2y - 1)}}{1 + \tau^2 + 2\gamma \tau \cos p \cos \varphi}$$
$$\quad \quad \quad = -[2y - 1 + m_{pp}^c][2y - 1 + m_{pp}^c + m_{pp}^c]$$  \hspace{1cm} (B21)

where $m_{pp}^o$ is defined in Eq. (B16). Note that the sum of the two coefficients

$$Z_1^y(\epsilon_k) = Z_1^y(\epsilon_k) + Z_2^y(\epsilon_k)$$
$$= 1 - 2T_0(\epsilon_p) + \frac{4\gamma^2 \tau^2 \sin^2 p \sin^2 \varphi - (1 - \tau^2)^2(1 - \gamma^2)}{[1 + \tau^2 + 2\gamma \tau \cos p \cos \varphi]^2}$$  \hspace{1cm} (B22)

is independent of $y$, implying that we may write

$$T^{yD}(\epsilon_p) = T(\epsilon_p) + \Delta T^{yD}$$  \hspace{1cm} (B23)

where

$$\Delta T^{yD}(\epsilon_p) = -Z_1^y(\epsilon_p) \Delta T_{pp}(\epsilon_p)$$  \hspace{1cm} (B24)

and we have separated the transmission probability into a $y$-independent part:

$$T(\epsilon_p) = T_0(\epsilon_p) + Z_R(\epsilon_p) \text{Re}[-\pi \nu_p T_{pp}(\epsilon_p)]$$
$$+ Z_1'(\epsilon_p) \text{Im}[-\pi \nu_p T_{pp}(\epsilon_p)]$$  \hspace{1cm} (B25)

and a part which depends on $y$ through $Z_2^y(\epsilon_p)$, the factor multiplying this coefficient is

$$\Delta T_{pp}(\epsilon_p) = \text{Im}[-\pi \nu_p T_{pp}(\epsilon_p)] - |\pi \nu_p T_{pp}(\epsilon_p)|^2.$$  \hspace{1cm} (B26)

The function $\Delta T_{pp}(\epsilon_p)$ is equal to deviations of the single-particle sector of the $T$-matrix from the optical theorem [Eq. (4.5)] and therefore it quantifies the interaction and is zero for non-interacting systems.

Appendix C: Non-interacting limit

In this Appendix we use Landauer and Fisher-Lee methods to calculate the conductance of the small ABK ring in the limit of a non-interacting quantum dot $U = 0$ and compare the result to the one obtained from the Kubo formula. The model is the one sketched in Fig. [1] and described by the Hamiltonian (2.1)-(2.3). The resulting Landauer formula for the conductance should be valid at any temperature.

1. Landauer formula

Starting from Hamiltonian (2.1)-(2.3) we can write down the Schrödinger equation $H\phi = \epsilon \phi$ and seek for a solution of the type

$$\phi_n = \begin{cases} 
  e^{ikn} + re^{-ikn} & n \leq -1 \\
  te^{ikn} & n \geq +1.
\end{cases}$$  \hspace{1cm} (C1)

We obtain
In order to check the consistency of this result with the Kubo calculations presented in the paper, we need to calculate the retarded Green’s function of the non-interacting dot.

2. Green’s function of the dot

The Green’s function of the dot is equal to

\[ G_{dd}^R(\epsilon_k) = \frac{1}{\epsilon_k + i\eta - \epsilon_d - \Sigma_{dd}^R(\epsilon_k)}. \]  

(C3)

At \( U = 0 \) we can use the screening basis and Eq. (2.13) to write

\[ \Sigma_{dd}^R(\epsilon_k) = \int_0^{\pi} dq \frac{V^2}{2\pi \epsilon_k - \epsilon_q + i\eta} \]

\[ = -\frac{2tLT_{LR}}{\tau} \cos \varphi e^{2ik} + \frac{(t^2_R + t^2_R)e^{ik}}{t(1 - \tau^2 e^{2ik})} \]  

(C4)

where the integral is calculated using the contour technique. In the absence of interactions, the optical theorem takes the form of Eq. (4.5). For the dot Green’s function using Eq. (3.28) this implies

\[ -\pi \nu V^2_0 \left| G_{dd}^R(\epsilon_k) \right|^2 = \text{Im} \left[ G_{dd}^R(\epsilon_k) \right] \]  

(C5)

or in terms of the dot self-energy

\[ -\pi \nu V^2_0 = \text{Im} \left[ \Sigma_{dd}^R(\epsilon_k) \right] \]

which is indeed satisfied for the non-interacting quantum dot as can be checked from Eq. (C4) and Eq. (2.13).

In non-interacting systems, the connected part of the transmission probability is absent (\( T^C(\omega) = 0 \)) and it follows from Eq. (4.5) that the transmission probability function of Eq. (3.30) is \( y \)-independent. It can be shown that by plugging in Eqs. (C3) and (C4) into the disconnected part of the transmission probability [Eq. (3.30)] we get

\[ T^D(\epsilon_k) = \left| t(\epsilon_k) \right|^2. \]  

(C6)

3. Fisher-Lee conductance

An alternative approach to calculating the conductance in the non-interacting limit is to use the Fisher-Lee relation [26] which is obtained from a different version of the Kubo formula in which \( y = 0 \), so the voltage is applied to the left lead only, but the measured current is

\[ t(\epsilon_k) = -\frac{2i \sin k(t_L t_R e^{-i\varphi} + t_L e^{-i\varphi} + t_L e^{i\varphi} + t_R e^{i\varphi}))}{-t(\epsilon_d - \epsilon_k)(1 - \tau^2 e^{2ik}) + 2t_L t_R \tau e^{2ik}\cos \varphi + (t^2_L + t^2_R)e^{ik}}. \]  

(C2)

\[ I = dN_R/dt. \]

The connected part is absent in this non-interacting system. In this approach, the conductance has the Landauer-type form

\[ G_{FL} = \frac{2e^2}{h} \int d\omega [-f'(\omega)] T_{FL}(\omega), \]  

(C7)

with \( T_{FL}(\omega) = |T_{FL}(\omega)|^2 \) and the transmission amplitude through the non-interacting system is given by

\[ t_{FL}(\epsilon) \equiv i\sqrt{\Gamma_L(\omega)\Gamma_R(\omega)} G_{RL}^R(\omega) \]

where the propagator \( G_{RL}^R(\omega) \) is defined by

\[ G_{RL}^R(\omega) = \int_0^\infty dt e^{i(\omega + i\eta)t} \left\langle -i \left\{ c_1(t), c_1^\dagger(0) \right\} \right\rangle. \]  

(C8)

\[ \Gamma_L(\epsilon_k) = \Gamma_R(\epsilon_k) = 2t \sin k \] are the coupling to (or equivalently the velocity in) the leads. In the following we use two methods to relate \( G_{RL}^R(\epsilon) \) to the Green’s function of the dot.

\[ a. \quad \text{Keldysh approach} \]

The main matrix equations are those discussed in Eqs. (5.38) and (5.42) above. There, \( L \) and \( R \) refer to the first site of left and right leads, respectively. These were derived using the equation of motion technique in the Keldysh space. But here we are only interested in the equilibrium retarded Green’s functions. So we take the retarded components of these equations in which \( g_{LL}^R(\epsilon_k) = g_{RL}^R(\epsilon_k) = -e^{ik}/t \) is the retarded Green’s function for the first site of a semi-infinite chain. From Eqs. (5.38) and (5.40) we get

\[ (1 - \tau^2 e^{2ik})G_{dd}^R(\epsilon_k) = G_{dd}^R(\epsilon_k) \frac{e^{ik}}{t}(t_L e^{-i\varphi/2} + t_R e^{i\varphi/2} + \tau e^{ik}) \]

(C9)

and from Eqs. (5.41) and (5.42) we have

\[ (1 - \tau^2 e^{2ik})G_{RL}^R(\epsilon_k) = \frac{-\tau e^{2ik}}{t} \]

\[ + \frac{e^{ik}}{t}(t_L e^{-i\varphi/2} + t_L e^{i\varphi/2} + \tau e^{ik})G_{LL}^R(\epsilon_k). \]  

(C10)

\[ G_{LR}(\epsilon_k) \] is obtained from this with \( L \leftrightarrow R \) and \( \varphi \leftrightarrow -\varphi \) substitution. Combining these and using Eq. (C8) we get

\[ t_{FL}(\epsilon_k) = -\frac{2i \tau \sin k e^{2ik}}{1 - \tau^2 e^{2ik}} + \frac{2i e^{2ik} \sin k(t_L e^{-i\varphi/2} + t_L e^{i\varphi/2} + \tau e^{ik})(t_L e^{-i\varphi/2} + t_L e^{i\varphi/2} + \tau e^{ik})}{t(1 - \tau^2 e^{2ik})^2} G_{dd}^R(\epsilon_k). \]  

(C11)
At $U = 0$, $G_{Rd}^R(\epsilon_k)$ is given by Eqs. (C3) - (C4). Inserting that in Eq. (C11) leads to

$$t_{FL}(\epsilon_k) = t(\epsilon_k) e^{2ik}$$  \hspace{1cm} (C12)

which is the Landauer result, Eq. (C2) up to a factor of $e^{2ik}$ due to the propagation from site -1 to site +1.

b. Screening and non-screening channels

In this section we express the transmission amplitude through the ABK ring in terms of the Green’s function of the dot using the $\psi$ and $\phi$ basis in the non-interacting limit. The advantage of this method compared to the Keldysh technique is that it can be readily generalized to large rings. We start by using Eq. (2.7) to express $c_{-1}$ and $c_1$ in terms of $\psi$ and $\phi$ fields

$$c_1(t) = \frac{1}{\sqrt{2}} \int_0^\pi \frac{dk}{2\pi} (\Gamma_{ek} \Gamma_{ok}) \bar{U}_k^\dagger \Psi_k(t),$$  

$$c_{-1}(0) = \frac{1}{\sqrt{2}} \int_0^\pi \frac{dq}{2\pi} \Psi_q(0) U_q \left( \frac{\Gamma_{eq}^2}{\Gamma_{eq}^2 - \Gamma_{oq}^2} \right).$$  

(C13)

To get the Fisher-Lee transmission amplitude $t_{FL}(\epsilon_p) = 2it \sin p G_{RFL}^R(\epsilon_p)$ we need to calculate the propagator

$$G_{RFL}^R(\epsilon_p) = \frac{1}{2} \text{Tr} \left[ \int_0^\pi \frac{dkdq}{(2\pi)^2} (\Gamma_{ek} \Gamma_{ok}) \times \right.$$

$$\left. \bar{U}_k^\dagger G_{kq}^R(\epsilon_p) U_q \left( \frac{\Gamma_{eq}^2}{\Gamma_{eq}^2 - \Gamma_{oq}^2} \right) \right].$$  

(C14)

The Green’s function matrix has the same form we encountered in Kubo calculations and can be written in terms of the T-matrix of screening and non-screening channels using Eq. (3.27). Plugging this into Eq. (C14) we get two terms which after doing the momentum integral and taking the trace can be used to write the transmission as

$$t_{FL}(\epsilon_p) = \frac{2it \sin p}{2} \left[ (\tilde{\Gamma}_{ep} - \tilde{\Gamma}_{op}) \right.$$

$$\left. + \frac{1}{2} \left( t_{de}^* \tilde{\Gamma}_{ep} - t_{do}^* \tilde{\Gamma}_{op} \right) (t_{de} \tilde{\Gamma}_{ep} + t_{do} \tilde{\Gamma}_{op}) G_{Rd}^R(\epsilon_p) \right].$$  

(C15)

Parameters $\tilde{\Gamma}_{ep}$ and $\tilde{\Gamma}_{op}$ containing the momentum integrals are given by

$$\tilde{\Gamma}_{(o/e)p} \equiv \int_0^\pi \frac{dk}{2\pi} \frac{|G(\epsilon_{o/e})|}{\epsilon_p - \epsilon_k + i\eta} = \frac{e^{ip}/t}{1 \pm \tau e^{ip}}$$  

(C16)

where we have used contour integration technique and expressions for $\Gamma_{ek}$ and $\Gamma_{ok}$ are given by Eq. (2.6). Substitution of this formula into Eq. (C15) leads to Eq. (C11) obtained before.

c. Consistency of the Kubo conductance with the Fisher-Lee formula at $U = 0$

Eq. (C11) can be used to write

$$\mathcal{T}_{FL}(\epsilon_p) \equiv \left| t_{FL}(\epsilon_p) \right|^2 = \mathcal{T}_{0}^{FL}(\epsilon_p) + 2\mathcal{Z}_{1}^{FL}(\epsilon_p) \text{Re} \left[ -\pi \nu_p V^2 G_{Rd}^R(\epsilon_p) \right]$$

$$+ \mathcal{Z}_{2}^{FL}(\epsilon_p) \text{Im} \left[ -\pi \nu_p V^2 G_{Rd}^R(\epsilon_p) \right]$$

(C17)

The coefficients $\mathcal{T}_0^{FL} = T_0$ and $\mathcal{Z}_1^{FL} = Z_1$ are the same as before [Eqs. (3.31) and (3.32)]. However, $\mathcal{Z}_2^{FL}$ are different from the previously obtained coefficients of the T-matrix [Eqs. (B20), (B21)]. and are given by

$$\mathcal{Z}_{1}^{FL}(\epsilon_p) = -\frac{4\tau \sin p}{[(1 + \tau^2)^2 - 4\tau^2 \cos^2 p]} \left[ \sin \gamma (1 + \tau^4) \sin \varphi + 2\gamma \tau^2 \sin(2p - \varphi) \right],$$  

(C18)

$$\mathcal{Z}_{2}^{FL}(\epsilon_p) = \frac{4\tau^2 + \gamma^2 (1 + \tau^4 + 2\tau^2 \cos 2(\varphi - p)) + 4\gamma \tau (1 + \tau^4) \cos(p - \varphi)}{[1 + \tau^2 + 2\gamma \tau \cos \varphi]^2}.$$  

(C19)

d. Connection with the inelastic part of the S-matrix

The expression for the total transmission probability (B25) is linear in the T-matrix and is valid approximately in the interacting case provided that the temperature is low enough so that the elimination of the connected part is justified as discussed in section (V). One the other hand, the Fisher-Lee conductance contains terms both linear and quadratic in the T-matrix and gives the total conductance only in the non-interacting systems. The equality (C20) can be used to express the difference between the total conductance $G$ in Eqs. (B25) and $G_{FL}$
as

\[ G - G_{FL} \approx \int d\epsilon_p [-f'(\epsilon_p)] \left( (Z'_{FL} - Z'_{FL}) \Im [-\pi \nu_p V^2 G_{dd}^R] \right. \\
- \left. Z_{2}^{FL} \left| -\pi \nu_p V^2 G_{dd}^R \right|^2 \right) \]

(C21)

\[ = \int d\epsilon_p [-f'(\epsilon_p)] Z_{2}^{FL}(\epsilon_p) \Delta T_{pp}(\epsilon_p) \quad (C22) \]

where we have used Eqs. (3.28) and definition of \( \Delta T_{pp}(\epsilon_p) \) in Eq. (B26). This form manifestly shows that the difference \( G - G_{FL} \) vanishes if the optical theorem for single-particle sector of T-matrix is obeyed. It is interesting to note that this difference is proportional to the inelastic part of the S-matrix, as defined by Zarand et al. \[18,19\].

---

\[ \text{komijani@phas.ubc.ca} \]

1 J. Kondo, Prog. Theor. Phys. 32, 37 (1964).
2 A. C. Hewson, The Kondo Problem to Heavy Fermions, Cambridge University Press, England, 1993.
3 D. Goldhaber-Gordon, H. Shtrikman, D. Mahalu, D. Abuscher-Magder, U. Meirav, M. A. Kastner, Nature 39, 156 (1998).
4 S. M. Cronenwett, A. C. Hewson, P. Kouwenhoven, Science 281, 540 (1998).
5 C. Bruder, R. Fazio and H. Schoeller, Phys. Rev. Lett. 76, 114 (1996).
6 J. König and Y. Gefen, Phys. Rev. Lett 86, 3855 (2001).
7 B. R. Bulka and P. Stefański, Phys. Rev. Lett. 86, 5128 (2001).
8 L. G. V. Dias da Silva, N. Sandler, P. Simon, K. Ingersent and S. E. Ulloa, Phys. Rev. Lett 102, 166806 (2009).
9 O. Entin-Wohlman, A. Aharony, Y. Imry and Y. Levinson, arXiv:cond-mat/0109328 (2001).
10 W. Hofstetter, J. König and H. Schoeller, Phys. Rev. Lett. 87, 156803-1 (2001).
11 O. Entin-Wohlman, A. Aharony and Y. Meir, Phys. Rev. B 71, 035333 (2005).
12 A. Aharony and O. Entin-Wohlman, Phys. Rev. B 72, 073311 (2005).
13 P. Simon, O. Entin-Wohlman and A. Aharony, Phys. Rev. B 72, 245313 (2005).
14 J. Malecki and I. Affleck, Phys. Rev. B 82, 165426 (2010).
15 R. Yoshii and M. Eto, Phys. Rev. B 83, 165310 (2011).
16 P. Nozières, J. Low Temp. Phys. 17, 31 (1974).
17 H. R. Krishna-murthy, J. W. Wilkins, K. G. Wilson, PRB 21, 1003 (1980).