DISJUNCTIVE QUANTUM LOGIC
IN DYNAMIC PERSPECTIVE

Bob Coecke

Free University of Brussels, Department of Mathematics,
Pleinlaan 2, B-1050 Brussels; bocoecke@vub.ac.be;

and

Imperial College of Science, Technology & Medicine, Theoretical Physics Group,
The Blackett Laboratory, South Kensington, London SW7 2BZ.

Current address:
University of Oxford, Computing Laboratory, Wolfson Building, Parks Road,
Oxford, OX1 3QD, UK; e-mail: coecke@comlab.ox.ac.uk.

Abstract

In Coecke (2002a) we proposed the intuitionistic or disjunctive representation of quantum logic, i.e., a representation of the property lattice of physical systems as a complete Heyting algebra of logical propositions on these properties, where this complete Heyting algebra goes equipped with an additional operation, the operational resolution, which identifies the properties within the logic of propositions. This representation has an important application “towards dynamic quantum logic”, namely in describing the temporal indeterministic propagation of actual properties of physical systems. This paper can as such be conceived as an addendum to “Quantum Logic in Intuitionistic Perspective” that discusses spin-off and thus provides an additional motivation. We derive a quantaloidal semantics for dynamic disjunctive quantum logic and illustrate it for the particular case of a perfect (quantum) measurement.

Key words: Quantum logic, dynamic logic, intuitionistic logic, property lattice, operational resolution, quantaloid.

1. INTRODUCTION

In Amira, Coecke and Stubbe (1998), Coecke and Stubbe (1999), Coecke (2000), Coecke, Moore and Stubbe (2001), Coecke and Smets (2000) and Sourbron (2000) steps have been taken towards a dynamic quantum logic, to great extend
inspired by the representation theorem for Schrödinger flows of Faure, Moore and Piron (1995), which itself incorporates the results of Faure and Frölicher (1993, 1994) on categorical representations of projective geometries; for previous attempts in that direction we refer to Pool (1968) and Daniel (1989). The crucial formal notion in this new approach is that of an operational resolution, a map that assigns to collections of either states or properties of a physical system the strongest property whose actuality is implied by that of each member in the collection: Indeterministic state transitions and property transitions are then exactly described by those maps between powersets of either the state space or the property lattice that preserve the operational resolution. Formally, this discussion takes place in the category of so-called quantaloids and quantaloid morphisms, i.e., the category of sup-lattice enriched categories. In this paper an ad hoc definition is given.

The notion of an operational resolution also emerged in a different context: If one represents the lattice of properties of a physical system in terms of logical propositions on these properties, the operational resolution comes in as an additional operation which identifies physical properties within this propositional logic, and which moreover establishes this representation as a true equivalence (Coecke 2002a, Section 4). However, the domain of the operational resolution in Coecke (2002a) is a restriction of the one introduced in Coecke and Stubbe (1999): it is not the powerset of the property lattice but its Bruns-Lakser distributive hull (Bruns and Lakser 1970), this since it is the latter that constitutes the logical propositions with respect to actuality of physical properties. The main message of this paper will as such be “imposing a refinement on the operational resolution in the capacity as the mathematical object that generates state and property transitions, guided by logical analysis of its domain”.

We refer to Coecke (2002a) for the preliminaries to this paper on states, properties, actuality of properties, Cartan maps and actuality sets; superposition states, superposition properties and superpositional faithfulness; atomistic lattices, complete lattices, Galois adjoints, complete Heyting algebras, Bruns-Lakser distributive ideals, Bruns-Lakser distributive hulls and distributive join dense closures; ortholattices, orthomodular lattices and Sasaki projections; for the latter we also refer to Piron (1976) and Kalmbach (1983). For a brief outline of ordinary and enriched category theory we refer to Borceux and Stubbe (2000), for a detailed one to Borceux (1994). The particular case of quantaloids is discussed in Rosenthal (1991).
2. TOWARDS A DYNAMIC QUANTUM LOGIC

In Amira, Coecke and Stubbe (1998) it was shown that the inducible state and property transitions on a physical system, the procedures that realize these transitions being called inductions, constitute a quantale, i.e., a complete lattice \((L, \lor)\) equipped with an additional associative operation \(-&- : L \times L \rightarrow L\) that distributes over suprema at both sides:

\[
\forall b \in B \ (a \& (\lor B)) = \lor (a \& b) \quad \land \quad (\lor A) \& b = \lor (a \& b).
\]

Note here that the collection of all inductions that can be effectuated on a particular physical system include both measurements and evolution, the latter to be understood as “let the system evolve”. To fix ideas, let us consider the example of a perfect measurement induction \(e_{PM}\) as it is outlined in Coecke and Smets (2000), based on the notion of perfect measurement in Piron (1976). \(^{1}\) Given a system described by a complete orthomodular lattice \(L\), e.g., a classical system or a quantum system, then actuality of a property \(a \in L\) in a measurement with as eigenproperties \(b\) and \(b'\) guarantees actuality of either

\[
\varphi_{b}(a) := b \land (a \lor b') \quad \lor \quad \varphi_{b'}(a) := b' \land (a \lor b),
\]

i.e., the Sasaki projection of \(a\) on either \(b\) or \(b'\). \(^{2}\) Note that in the case that one of the two alternatives turns out to be 0 the outcome is determined (since 0 is impossible). If this lattice is moreover atomistic and satisfies the covering law, what is still the case both for classical and quantum systems (Piron 1976), then, provided that the states are encoded as atoms, \(e_{PM}\) imposes a change of the initial state to either

\[
\varphi_{b}(p) := b \land (p \lor b') \quad \lor \quad \varphi_{b'}(p) := b' \land (p \lor b),
\]

again provided that none of the alternatives yields 0, in which case we only consider the non-0 outcome. To \(e_{PM}\) we can as such attribute a map \(^{3}\)

\[
ed_{PM} : \mathcal{P}(\Sigma) \rightarrow \mathcal{P}(\Sigma) : T \mapsto \{b \land (p \lor b'), b' \land (p \lor b) \mid p \in T\} \setminus \{0\}
\]

\(^{1}\)These perfect measurements encode in quantum logical terms the measurements in orthodox quantum theory represented by self-adjoint operators with a binary spectrum.

\(^{2}\)Recall here, as mentioned in Footnote 8 of Coecke (2002a), that we consider a quantum measurement as an external action on the system that changes its state, and as such also its actual properties.

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that assigns the possible outcome states whenever the system is initially in a
state in $T$. For the more general case of a property transition the situation
is however somewhat more complicated to describe. Although at first sight one
could go for a union preserving map between powersets of the property lattice
as it is done in Coecke and Smets (2000), saying that actuality of $a$

guarantees actuality of either $\varphi_b(a)$ or $\varphi_{b'}(a)$ is indeed somewhat ambiguous. Whenever
$\varphi_b(a)$ is actual also any $c \geq \varphi_b(a)$ is actual, so one might additionally want to
elucidate something in the sense of “one focuses on maximally strong possible
outcomes”, whatever this might mean. But then again, taking for example an
atomistic property lattice where states are encoded as atoms, actuality of $\varphi_b(a)$
also guarantees that some state in $\{ p \in \Sigma | p \leq \varphi_b(a) \}$ is actual, what illustrates
that the above elucidation is indeed sloppy. The key to solve this problem is
exactly the logic of actuality sets proposed in Coecke (2002a), a presentation of
the logical propositions on properties of a physical system in terms of actuality,
which indicates that, under the assumption of superpositional faithfulness we
have to define

$$\hat{e}_{PM} : DI(L) \to DI(L) : A \mapsto C\left( \{ b \wedge (a \vee b'), b' \wedge (a \vee b) \mid a \in A \} \right)$$

as the map that describes propagation of actuality sets in a perfect measurement,
recalling here that $C$ stands for the composite of the implicative and disjunctive

closure, i.e.,

$$C(A) := \bigvee_{L} B \mid B \subseteq \downarrow[A] \cap D(L) .$$

Turning back the collection of all inductions that can be effectuated on a par-
ticular physical system, this collection being denoted as $\check{E}$, we obtain as such
two quantales

$$\check{E} := \{ \check{e} : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma) \mid e \in \mathcal{E} \}$$

$$\check{E}_{DI} := \{ \check{e} : DI(L) \to DI(L) \mid e \in \mathcal{E} \}$$

respectively expressing the inducible state and property transitions for this sys-
tem, provided the system is not destroyed. The suprema in these quantales are

calculated pointwisely with respect to the suprema of the codomain, the quan-
tale multiplication coincides with composition of maps, and closure of these
quantales under these operations is to be understood in terms of inductions re-
spectively as “arbitrary choice on effectuation” and “consecutive effectuation”

It is merely a technicality to avoid the restriction “provided the system is not destroyed”; for details on this matter we refer to Coecke, Moore and Stubbe (2001).
(Amira, Coecke and Stubbe 1998, Coecke and Stubbe 1999). Once at this point, one might prefer to express $\hat{\mathcal{E}}$ rather in terms of the distributive hull $H$ of $L$ than in terms of actuality sets, i.e., in terms of propagation of logical propositions on properties rather than in terms of propagation of actuality sets:

$$\hat{\mathcal{E}} := \{ \hat{e} : H \to H \mid e \in \mathcal{E} \} ,$$

recalling from Bruns and Lakser (1970) that the inclusion $i : L \hookrightarrow H$ satisfies

$$L \xrightarrow{i} H \approx \downarrow[L] \hookrightarrow \mathcal{D}\mathcal{I}(L)$$

where the isomorphism between $\mathcal{D}\mathcal{I}(L)$ and $H$ is established via $A \mapsto \bigvee H A$ and that between $\downarrow[L]$ and $L$ via $A \mapsto \bigvee L A$.

It makes however sense to generalize the above to maps where the codomain is different from the domain, and this for two reasons: (i) It allows to describe “change of system”, where we conceive a system as being defined exactly by its set of distinct (with respect to the corresponding actual properties) possible realizations, i.e., by its set of states; (ii) Besides temporal propagation it also allows to encode entanglement, or any form of interaction including separation, in terms of “mutual induction of properties” (Coecke 2000). Therefore we will extend our framework from quantales to quantaloids, i.e., categories enriched in sup-lattices. Explicitly, a quantaloid $\mathcal{Q}$ is a category in which all the morphism sets $\mathcal{Q}(A, B)$ are complete lattices with the ordering such that the by $f : A \to B$ induced morphism actions $f \circ -$ : $\mathcal{Q}(B, C) \to \mathcal{Q}(A, C)$ and $- \circ f : \mathcal{Q}(C, A) \to \mathcal{Q}(C, B)$ preserve suprema. Quantaloid morphisms are those functors that preserve suprema when restricted to morphism sets (Rosenthal 1991). Quantales with multiplicative unit are then exactly one-object quantaloids. The unit in our setting is provided by the induction “freeze” with obvious significance. Denoting the quantaloid of complete lattices and sup-morphisms as Sup we then have that $\hat{\mathcal{E}} \hookrightarrow \text{Sup}(\mathcal{P}(\Sigma_1), \mathcal{P}(\Sigma_2))$ and $\hat{\mathcal{E}}_{\mathcal{D}\mathcal{I}} \cong \hat{\mathcal{E}} \hookrightarrow \text{Sup}(H_1, H_2) \cong \text{Sup}(\mathcal{D}\mathcal{I}(L_1), \mathcal{D}\mathcal{I}(L_2))$ are functorial sup-inclusions, where functorial is to be understood in the sense that any composition of inductions encodes as Sup-composition. However, as we will see below, there is an additional feature to this inclusion.
Clearly, there is a strong analogy of the above with the non-commutative geometric logic or observational semantics of Abramsky and Vickers (1993) and Resende (2000) that has been developed in order to describe sequences of interaction with and observation of computational devices. However, as for example mentioned in Resende (2000 §3.1), the observational semantics proposed in Abramsky and Vickers (1993) is not applicable to quantum processes, this in particular since in quantum processes both the suprema in the property lattice and disjunctions of properties are essential. We will now show how this implements formally within the above setting.

We will refer by propagation of strongest actual properties with respect to an induction $e$ to the map $\bar{e} : L_1 \rightarrow L_2$ that assigns to a property $a \in L_1$ the strongest property $b \in L_2$ of which actuality after effectuating $e$ is guaranteed by actuality of $a$ before effectuating $e$. Following Faure, Moore and Piron (1995), Coecke (2000) and Coecke, Moore and Stubbe (2001), the map that describes propagation of strongest actual properties preserves suprema. This follows from the fact that propagation is adjoint to causal assignment (Coecke 2000, Coecke, Moore and Stubbe 2001). Roughly, the argument goes as follows: By conjunctivity of infima in property lattices it follows that assignment of weakest causes of actuality preserves all non-empty infima, and as such, it induces a unique left Galois adjoint on the upper pointed extensions of the involved property lattices (Coecke and Moore 2000, Coecke, Moore and Stubbe 2001). One then verifies that this left Galois adjoint expresses propagation of strongest actual properties.

Since, given an actuality set $A$, the strongest property that is actual with certainty is exactly $\bigvee A$, a role that is played for states by the operational resolution $R_\Sigma : \mathcal{P}(\Sigma) \rightarrow L$ sensu Coecke and Stubbe (1999) and discussed in the first paragraph of the introduction to this paper, it then follows that for any map in $\tilde{E}$ or $\tilde{E}_{DT}$ there exists a map $\bar{e} : L_1 \rightarrow L_2$ such that we respectively have commutation of

$$
\begin{align*}
L_1 \xrightarrow{\bar{e}} L_2 & \quad L_1 \xrightarrow{\bar{e}} L_2 \\
\mathcal{P}(\Sigma_1) \xrightarrow{\bar{e}} \mathcal{P}(\Sigma_2) & \quad \mathcal{D}I(L_1) \xrightarrow{\bar{e}} \mathcal{D}I(L_2)
\end{align*}
$$

One then verifies that this left Galois adjoint expresses propagation of strongest actual properties.

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4 The formal need to consider these upper pointed extensions formally implements the physical possibility of destruction of the system, and as such embodies a way to avoid the restriction “provided the system is not destroyed” mentioned in the previous footnote. A discussion concerning can also be found in Sourbron (2000).
which translates in terms of $\mathcal{E}$ as commutation of

\[
\begin{array}{cccc}
L_1 & \xrightarrow{\tilde{e}} & L_2 \\
\uparrow \pi_1 & & \uparrow \pi_2 \\
H_1 & \xrightarrow{\tilde{e}} & H_2
\end{array}
\]

where we slightly abused notation by restricting the codomain of the operational resolution. Note that when replacing in the above $\mathcal{D}\mathcal{I}(L)$ by $\mathcal{P}(L)$, i.e., requiring for a union preserving map $g : \mathcal{P}(L_1) \rightarrow \mathcal{P}(L_2)$ that there exists a map $f : L_1 \rightarrow L_2$ such that $\bigvee_2(g(-)) = f(\bigvee_1(-))$ does not assure existence of a map $h : \mathcal{D}\mathcal{I}(L_1) \rightarrow \mathcal{D}\mathcal{I}(L_2)$ such that we have commutation of

\[
\begin{array}{ccc}
\mathcal{D}\mathcal{I}(L_1) & \xrightarrow{h} & \mathcal{D}\mathcal{I}(L_2) \\
\uparrow \mathcal{C}_1 & & \uparrow \mathcal{C}_2 \\
\mathcal{P}(L_1) & \xrightarrow{g} & \mathcal{P}(L_2)
\end{array}
\]

(1)

It suffices to note that $\mathcal{D}\mathcal{I}(L)$-suprema and $\mathcal{P}(L)$-suprema don’t coincide. Therefore, the considerations made in this paper reveal this aspect as an additional feature of the maps in $\mathcal{E}$ on propagation of actuality sets besides the one imposed by preservation of suprema for propagation of strongest actual properties. In particular can all this be encoded as the factorization of quantaloid morphisms expressed in the following commutative diagram in the category of quantaloids:

\[
\begin{array}{cccc}
& & \text{Sup} & \\
G & \xhookrightarrow{\kappa_H} & \text{DCHeyt} & \xleftarrow{F} \text{PSup} \\
\mathcal{E} & \hookrightarrow & \end{array}
\]

where

- $\text{PSup}$ denotes the category of complete lattices $L$ with morphisms $g : \mathcal{P}(L_1) \rightarrow \mathcal{P}(L_2)$ that preserve unions and satisfy both eq.(1) and

  \[\bigvee_1(A) = \bigvee_1(B) \implies \bigvee_2(g(A)) = \bigvee_2(g(B));\]

- $\text{DCHeyt}$ denotes the category of complete Heyting algebras $H$ equipped with a disjunctive join dense closure $\mathcal{R} : H \rightarrow H$ with morphisms $h : H_1 \rightarrow H_2$ that preserve suprema and satisfy

  \[\mathcal{R}_1(A) = \mathcal{R}_1(B) \implies \mathcal{R}_2(h(A)) = \mathcal{R}_2(h(B));\]

- $F : \text{PSup} \rightarrow \text{DCHeyt} : L \mapsto (\mathcal{D}\mathcal{I}(L), \mathcal{R}_{\mathcal{D}\mathcal{I}(L)}), g(-) \mapsto \mathcal{C}_2(g(-));$
\[ G: \text{DCHeyt} \to \text{Sup} : (H, \mathcal{R}) \mapsto \mathcal{R}(H), \ h(-) \mapsto \mathcal{R}_2(h(-)); \]

\[ H: \text{PSup} \to \text{Sup} : L \mapsto L, \ g(-) \mapsto \bigvee_2 \big( g(\downarrow(-)) \big). \]

Note that the object correspondences of \( F \) and \( G \) are indeed those of Coecke (2002a §3), Definition 1, and in particular that \( G: \text{DCHeyt} \to \text{Sup} \) is a full quantaloidal morphism but not an equivalence. This fact will constitute the core of the discussion below.

4. DISCUSSION

Recalling that we mentioned in Coecke (2002a) that “via physical and logical considerations we rediscover a purely mathematical result by Bruns and Lakser (1970) on injective hulls of meet-semilattices”, it is then via these considerations on state transitions and property transitions as the morphisms equipping operational resolutions that the different underlying motivations in Bruns and Lakser (1970) and Coecke (2002a) reveal themselves explicitly in a formal way. Indeed, in Coecke (2002a) we skipped any consideration on morphisms by only considering object equivalences, which only requires specification of isomorphisms. Obviously, there are different canonical ways to extend such an object equivalence categorically, e.g., via a pointwise lift of the \emph{chosen} morphisms of the complete lattices to the corresponding complete Heyting algebras of distributive ideals where as well \emph{meet}-morphisms, \emph{inf}-morphisms and \emph{sup}-morphisms are candidate morphisms for the complete lattices. From this perspective, in Bruns and Lakser (1970) the choice of morphisms, i.e., \emph{meet}-morphisms, is such that it establishes distributive hulls as injective hulls. It was moreover noted in Harding (1999) that when defining \emph{distributive morphisms} as those maps that preserve finite meets, distributive suprema and the underlying sets that have distributive suprema, then \textbf{Frame} (Johnstone 1982) is a full monoreflective subcategory of the category of meet-semilattices and distributive morphisms, with as reflector a functor that assigns an object to its distributive hull. However, in view of the application of distributive hulls “towards dynamic quantum logic”, we canonically obtain a full but not faithful quantaloidal correspondence between complete lattices and complete Heyting algebras equipped with an operational resolution, this as a consequence of our manifestly different choice of morphisms. These considerations obviously indicate many new open problems, as do the open questions posed in Coecke (2002a) when restated in the presence

\[ ^5 \text{Obviously provided that the chosen morphisms of the complete lattices are } \text{Set}-\text{concrete.} \]
of this dynamical setting. We do mention at this point that recently we constructed a dynamic logic that realizes the semantics presented in this paper as a true logic with forward and backward implication and corresponding tensors — see Coecke (2002b) and Coecke and Smets (2001) for a general presentation and Smets (2001) for a detailed development of the atomistic case (i.e., Boolean propositions) — and, that has the representation presented in Coecke (2002a) as statical limit. Implementation of this framework for the particular case of quantum measurements represented by projectors acting on the underlying Hilbert space can be found in Coecke and Smets (2001). In that case we obtain a family of implicative hooks labeled by properties. It is argued there that the transition from either classical or constructive/intuitionistic logic to quantum logic entails besides the introduction of an additional unary connective operational resolution the shift from a binary connective implication to a ternary connective where two of the arguments have an ontological connotation and the third, the new one, an empirical. This second aspect of the shift from classical or constructive/intuitionistic to quantum will then be the one that requires orthomodularity of the underlying lattice of properties as a crucial feature.

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REFERENCES

ABRAMSKY, S. and VICKERS, S. (1993) ‘Quantales, Observational Logic and Process Semantics’, Mathematical Structures in Computer Science 3, 161.

AMIRA, H., COECKE, B. and STUBBE, I. (1998) ‘How Quantales Emerge by Introducing Induction within the Operational Approach’, Helvetica Physica Acta 71, 554.

BRUNS, G. and LAKSER, H. (1970) ‘Injective Hulls of Semilattices’, Canadian Mathematical Bulletin 13, 115.

Preprints and postscript files of published papers by the current author can be downloaded at http://www.vub.ac.be/CLEA/Bob/Coecke.html.
Borceux, F. (1994) *Handbook of Categorical Algebra I & II*, Cambridge University Press.

Borceux, F. and Stubbe, I. (2000) ‘Short Introduction to Enriched Categories’, In: B. Coecke, D.J. Moore and A. Wilce, (Eds.), *Current Research in Operational Quantum Logic: Algebras, Categories and Languages*, pp.167–194, Kluwer Academic Publishers.

Coecke, B. (2000) ‘Structural Characterization of Compoundness’, *International Journal of Theoretical Physics* **39**, 581; arXiv: quant-ph/0008054.

Coecke, B. (2002a) ‘Quantum Logic in Intuitionistic Perspective’, *Studia Logica* **70**, 411; arXiv: math.LO/0011208.

Coecke, B. (2002b) ‘Do we have to Retain Cartesian Closedness in the Topos-Approaches to Quantum Theory, and, Quantum Gravity ?’, Preprint.

Coecke, B. and Moore, D.J. (2000) ‘Operational Galois Adjunctions’, In: B. Coecke, D.J. Moore and A. Wilce, (Eds.), *Current Research in Operational Quantum Logic: Algebras, Categories and Languages*, pp.195–218, Kluwer Academic Publishers; arXiv: quant-ph/0008021.

Coecke, B., Moore, D.J. and Stubbe, I. (2001) ‘Quantaloids Describing Causation and Propagation for Physical Properties’, *Foundations of Physics Letters* **14**, 133; arXiv: quant-ph/0009100.

Coecke, B. and Smets, S. (2000) ‘A Logical Description for Perfect Measurements’, *International Journal of Theoretical Physics* **39**, 591; arXiv: quant-ph/0008017.

Coecke, B. and Smets, S. (2001) ‘The Sasaki-Hook is not a [Static] Implicative Connective but Induces a Backward [in Time] Dynamic One that Assigns Causes’, Paper submitted to *International Journal of Theoretical Physics* for the proceedings of IQSA V, Cesena, Italy, April 2001; arXiv:quant-ph/0111076.

Coecke, B. and Stubbe, I. (1999) ‘Operational Resolutions and State Transitions in a Categorical Setting’, *Foundations of Physics Letters* **12**, 29; arXiv: quant-ph/0008020.

Daniel, W. (1989) ‘Axiomatic Description of Irreversible and Reversible Evolution of a Physical System’, *Helvetica Physica Acta* **62**, 941.

Faure, Cl.-A. and Frôlicher, A. (1993) ‘Morphisms of Projective Geometries
and of Corresponding Lattices’, *Geometriae Dedicata* **47**, 25.

Faure, Cl.-A. and Frölicher, A. (1994) ‘Morphisms of Projective Geometries and Semilinear Maps’, *Geometriae Dedicata* **53**, 237.

Faure, Cl.-A., Moore, D.J. and Piron, C. (1995) ‘Deterministic Evolutions and Schrödinger Flows’, *Helvetia Physica Acta* **68**, 150.

Harding, J. (1999) Private communication.

Johnstone, P.T. (1982) *Stone Spaces*, Cambridge University Press.

Kalmbach, G. (1983) *Orthomodular Lattices*, Academic Press.

Piron, C. (1976) *Foundations of Quantum Physics*, W.A. Benjamin, Inc.

Pool, J.C.T. (1968) ‘Baer * -Semigroups and the Logic of Quantum Mechanics’, *Communications in Mathematical Physics* **9**, 118.

Resende, P. (2000) ‘Quantales and Observational Semantics’, In: B. Coecke, D.J. Moore and A. Wilce, (Eds.), *Current Research in Operational Quantum Logic: Algebras, Categories and Languages*, pp.263–288, Kluwer Academic Publishers.

Rosenthal, K.I. (1991) ‘Free Quantaloids’, *Journal of Pure and Applied Algebra* **77**, 67.

Smets, S. (2001): ‘The Logic of Physical Properties in Static and Dynamic Perspective’, PhD-thesis, Free University of Brussels.

Sourbron, S. (2000) ‘A Note on Causal Duality’, *Foundations of Physics Letters* **13**, 357.