QUALITATIVE ANALYSIS ON AN SIS EPIDEMIC REACTION-DIFFUSION MODEL WITH MASS ACTION INFECTION MECHANISM AND SPONTANEOUS INFECTION IN A HETEROGENEOUS ENVIRONMENT

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Abstract. In the recent paper [29], a susceptible-infected-susceptible (SIS) epidemic reaction-diffusion model with a mass action infection mechanism and linear birth-death growth with no flux boundary condition was studied. It has been recognized that spontaneous infection is an important factor in disease epidemics, in addition to disease transmission [43]. In this paper, we investigate the SIS model in [29] with spontaneous infection. We establish the global boundedness and uniform persistence in the general heterogeneous environment, and derive the global stability of the unique constant endemic equilibrium in the homogeneous environment case. Moreover, we analyze the asymptotic behavior of the endemic equilibrium when the movement (migration) rate of the susceptible or infected population tends to zero. Compared to the case that there is no spontaneous infection, our study suggests that spontaneous infection can enhance persistence of infectious disease, and hence the disease becomes more threatening.

1. Introduction. As we all know, mathematical models have been an increasingly important tool in understanding the dynamics behavior of the epidemic disease [5, 8, 15, 19, 20, 25] and references therein. Especially, various kinds of susceptible-infected-susceptible (SIS) epidemic reaction-diffusion models have been extensively investigated by epidemiologists and mathematicians, one may refer to [9, 13, 16, 23, 30, 41, 44].

Recently, Wu and coauthors in [12, 45] studied the following SIS epidemic reaction-diffusion model with a mass action infection mechanism:

\[
\begin{align*}
S_t - d_S \Delta S &= -\beta(x)SI + \gamma(x)I, & x \in \Omega, t > 0, \\
I_t - d_I \Delta I &= \beta(x)SI - \gamma(x)I, & x \in \Omega, t > 0, \\
\frac{\partial S}{\partial \nu} &= \frac{\partial I}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\
S(x, 0) &= S_0(x) \geq 0, I(x, 0) = I_0(x) \geq, \neq 0, & x \in \Omega.
\end{align*}
\]

(1)

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Here, $\Omega$ is a bounded domain in $\mathbb{R}^m$ ($m \geq 1$) with the smooth boundary $\partial \Omega$. $S(x, t)$ and $I(x, t)$ represent the density of susceptible and infected individuals at location $x$ and time $t$, respectively. The positive diffusion coefficients $d_S$ and $d_I$, respectively, represent the migration/movement rate of the susceptible and infected populations. The functions $\beta$ and $\gamma$ are positive and Hölder continuous functions on $\overline{\Omega}$ and stand for the rates of disease transmission and recovery at location $x$, respectively. In addition, the Neumann boundary conditions imply that no population travels through the boundary $\partial \Omega$.

For system (1), Deng and Wu in [12] defined a basic reproduction number $R_0$ for the model. Then they proved that there exists a unique endemic equilibrium if $R_0 > 1$, and showed the global attractivity of the disease-free equilibrium (DFE) and the endemic equilibrium (EE) for two cases. If the disease transmission and recovery rates are constants or the diffusion rate of the susceptible individuals is equal to the diffusion rate of the infected individuals, they further showed that the DFE is globally attractive if $R_0 \leq 1$, while the EE is globally attractive if $R_0 > 1$.

In [45], Wu and Zou established the asymptotical behavior of the EE as one of the diffusive coefficients $d_S$, $d_I$ is small or large, and their results reveal some interesting phenomena of spatial distribution of epidemic disease.

System (1) does not take the birth/death effect of susceptible or infected individuals into consideration and thus the total population is conserved in the sense that

$$\int_{\Omega} [S(x, t) + I(x, t)] dx = \int_{\Omega} [S_0(x) + I_0(x)] dx, \quad \forall t > 0.$$ 

However, it is quite natural to consider the situation that susceptible individuals are subject to a recruitment (source) term modeling their birth and death rate, especially a linear one; see, for instance, [27, 28]. Thus, Li, Peng and Wang [29] studied the following reaction-diffusion epidemic system with varying total population and environmental heterogeneity:

\[
\begin{align*}
S_t - d_S \Delta S &= \Lambda(x) - S - \beta(x)SI + \gamma(x)I, & x \in \Omega, \ t > 0, \\
I_t - d_I \Delta I &= \beta(x)SI - [\alpha(x) + \gamma(x)]I, & x \in \Omega, \ t > 0, \\
\left. \frac{\partial S}{\partial \nu} \right|_{\partial \Omega} &= \left. \frac{\partial I}{\partial \nu} \right|_{\partial \Omega} = 0, & x \in \partial \Omega, \ t > 0, \\
S(x, 0) &= S_0(x) \geq 0, \quad I(x, 0) = I_0(x) \geq 0, \quad \forall x \in \Omega.
\end{align*}
\]

(2)

The recruitment $\Lambda(x) - S$ indicates that the susceptible individuals are subject to a linear growth and $\alpha(x)$ represents the infected death rate, and we assume that $\Lambda$ and $\alpha$ are positive Hölder continuous functions. The functions $\beta$ and $\gamma$ in system (2) have the same implications in (1).

For system (2), [29] established the stability of the DFE, uniform persistence property and the global stability of the EE in homogeneous environment, and investigated the asymptotic profile of EE (when it exists) in heterogeneous environment as the diffusion rate of susceptible or infected individuals is small.

Most existing SIS reaction-diffusion models of the infectious disease have concentrated on the transmission by having a contact between an infected and a susceptible individual [2, 3, 10, 11, 26, 38, 39]. As mentioned by Hill et al in [21, 22], social ‘infection’, however, can also be arised due to spontaneous factors other than transmission. Therefore, as in [43], they extend the SIS model to incorporate the effect of spontaneous infection by adding a term whereby uninfected individuals become infected at a rate $\eta$. With such a consideration, we are led to investigate the following
system:
\[
\begin{aligned}
S_t - d_S \Delta S &= \Lambda(x) - S - \beta(x)SI - \eta(x)S + \gamma(x)I, \\
I_t - d_I \Delta I &= \beta(x)SI + \eta(x)S - [\alpha(x) + \gamma(x)]I, \\
\frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} &= 0, \\
S(x, 0) &= S_0(x) \geq 0, I(x, 0) = I_0(x) \geq 0, x \in \Omega.
\end{aligned}
\]

In system (3), the spontaneous infection rate \(\eta\) is assumed to depend on spatial location \(x\), is Hölder continuous positive function on \(\Omega\). The function \(\eta\) models the propagation of the disease due to imported cases of the infection; such an import can be explained by brief contacts with individuals outside of the population. Interested readers can refer to [22, 24, 35]; there are also many different interpretations for \(\eta\); see, for example, [4, 21, 42, 46].

The steady state (equilibrium) problem corresponding to (3) satisfies the following elliptic system
\[
\begin{aligned}
-s_t - d_s \Delta s &= \Lambda(x) - s - \beta(x)sI - \eta(x)s + \gamma(x)i, \\
i_t - d_i \Delta i &= \beta(x)si + \eta(x)s - [\alpha(x) + \gamma(x)]i, \\
\frac{\partial s}{\partial \nu} = \frac{\partial i}{\partial \nu} &= 0, \\
s(x, 0) &= S_0(x) \geq 0, i(x, 0) = I_0(x) \geq 0, x \in \Omega.
\end{aligned}
\]

Here \(S(x)\) and \(I(x)\) denote the density of susceptible and infected populations, respectively, at location \(x\) at equilibrium. Generally, we’re only interest in the solution of (4) \((S, I)\) satisfying \(S \geq 0\) and \(I \geq 0\) on \(\Omega\). It is easy to see that (4) has no such solution \((S, 0)\). That is to say, the DFE of (3) is not exist. If the EE \((S, I) \in C^2(\Omega) \times C^2(\Omega)\) of (3) exists, then \(S(x) > 0\) and \(I(x) > 0\) for all \(x \in \Omega\) by applying the strong maximum principle and Hopf lemma for elliptic equations [17].

Now, the heterogeneity of the spatial environment and the movement of the population also play an increasingly important role in the dynamic behavior of the epidemic disease. Hence, this paper is aims to study the uniform persistence of the unique EE and the global stability of EE when the spatial environment is homogeneous, as well as the asymptotic behavior of the EE respects to small movement (migration) rate of the susceptible and infected individuals in spatially heterogenous environment. The theoretical results in this paper imply that the spontaneous infections can make the epidemic disease more threatening compared to the case that there is no spontaneous infections.

We would like to mention that Tong and the first named author [43] dealt with the SIS epidemic reaction-diffusion model with frequency-dependent infection mechanism, linear birth-death growth and spontaneous infection. Their theoretical findings also suggest that spontaneous infection can enhance persistence of infectious diseases.

This paper is organized as follows. In section 2, we derive the global boundedness and uniform persistence of system (3). Section 3 is devoted to the global stability of EE in homogenous environment when the parameters \(\Lambda, \alpha, \beta, \eta, \gamma\) are positive constants. In section 4, we establish the asymptotic behavior of EE as the diffusive coefficient \(d_s\) or \(d_i\) goes to zero.

2. Global boundedness and uniform persistence of (3). In this section, we aim to establish the global boundedness and the uniform persistence property of solution to (3). Our main result reads as follows.
Theorem 2.1. The unique solution \((S, I)\) of the system (3) satisfies
\[
S(x, t) \geq M_S \text{ and } I(x, t) \geq M_I, \forall x \in \overline{\Omega}, \ t > 0,
\]
where
\[
M_S = \min\{M, \min_{x \in \Omega} S_0(x)\}, \ M = \min \left\{ \frac{\min_{x \in \Omega} \Lambda(x)}{1 + \max_{x \in \Omega} \eta(x)}, \frac{\min_{x \in \Omega} \gamma(x)}{\max_{x \in \Omega} \beta(x)} \right\}
\]
and
\[
M_I = \min \left\{ \frac{M_S \min_{x \in \Omega} \eta(x)}{\max_{x \in \Omega} \alpha(x) + \max_{x \in \Omega} \gamma(x)}, \min_{x \in \Omega} I_0(x) \right\}.
\]
Moreover, there exist positive constants \(C_1, C_2\), which are independent of initial data \((S_0, I_0)\), such that
\[
C_1 \leq S(x, t), \ I(x, t) \leq C_2, \ \forall x \in \overline{\Omega}, \ t \geq T,
\]
for some large time \(T > 0\). In particular, this implies that system (3) admits at least one EE.

Proof. Using the first equation in (3), we obtain
\[
\frac{\partial S}{\partial t} - d_S \Delta S = \Lambda(x) - S - \beta(x)SI - \eta(x)S + \gamma(x)I
\]
\[
= \Lambda(x) + \gamma(x)I - [1 + \beta(x)I + \eta(x)]S
\]
\[
= [1 + \beta(x)I + \eta(x)] \left[ \frac{\Lambda(x) + \gamma(x)I}{1 + \beta(x)I + \eta(x)} - S \right]
\]
\[
\leq [1 + \beta(x)I + \eta(x)] \left[ \max_{x \in \Omega} \Lambda(x) \cdot I \right]
\]
\[
\leq [1 + \beta(x)I + \eta(x)] \left[ \max_{x \in \Omega} \Lambda(x) \right]
\]
for all \( t > 0, x \in \Omega \). Let \( M_1 = \max \left\{ \frac{\max_{x \in \Omega} \Lambda(x)}{1 + \min_{x \in \Omega} \eta(x)}, \frac{\max_{x \in \Omega} \gamma(x)}{\min_{x \in \Omega} \beta(x)} \right\} \). We consider the following problem
\[
\begin{aligned}
& u_t - d_S \Delta u = [1 + \beta(x)I + \eta(x)](M_1 - u), \quad x \in \Omega, \ t > 0, \\
& \frac{\partial u}{\partial n} = 0, \quad x \in \partial \Omega, \ t > 0, \\
& u(x, 0) = \max_{x \in \Omega} S_0(x) \geq 0, \quad x \in \Omega,
\end{aligned}
\]
which has a unique solution denoted by \( u \). It is clear that \( \max_{x \in \Omega} \left\{ M_1, \max_{x \in \Omega} S_0(x) \right\} \) is a supersolution to (5). We use the standard comparison principle for parabolic equations to deduce that
\[
u(x, t) \leq \max_{x \in \Omega} \left\{ M_1, \max_{x \in \Omega} S_0(x) \right\}, \ \forall x \in \overline{\Omega}, \ t > 0.
\]

Note that \( S \) is a subsolution to (5). It then follows from the parabolic comparison principle that
\[
S(x, t) \leq u(x, t) \leq \max_{x \in \Omega} \left\{ M_1, \max_{x \in \Omega} S_0(x) \right\}, \ \forall x \in \overline{\Omega}, \ t > 0.
\]
Indeed, by elementary calculation, we observe that
\[ w(t) = M_1 + \max_{x \in \Omega} S_0(x) e^{-t}, \quad t \geq 0 \]
is a supersolution to (5). As a result, it follows that
\[ S(x,t) \leq u(x,t) \leq w(t), \quad \forall x \in \overline{\Omega}, \ t > 0. \]
In particular, this yields
\[ \lim_{t \to \infty} \sup_{t \to \infty} S(x,t) \leq \lim_{t \to \infty} w(t) = M_1 \text{ uniformly for } x \in \overline{\Omega}. \]
Thus, we can find a large time \( T_1 > 0 \) such that
\[ S(x,t) \leq 2M_1, \quad \forall x \in \overline{\Omega}, \ t \geq T_1. \quad (6) \]

Similarly, we can get
\[
\frac{\partial S}{\partial t} - d_S \Delta S \geq [1 + \beta(x) I + \eta(x)](M - S), \quad \forall t > 0, \ x \in \Omega,
\]
where \( M \) is given in Theorem 2.1. Then \( S \) is a supersolution to the following problem
\[
\begin{cases}
v_t - d_S \Delta v = [1 + \beta(x) I + \eta(x)](M - v), & x \in \Omega, t > 0, \\
\frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\
v(x,0) = \min_{x \in \overline{\Omega}} S_0(x) \geq 0, & x \in \Omega.
\end{cases} \quad (7)
\]
Denoted by \( v \) be the unique solution of (7). Obviously, \( \min_{x \in \overline{\Omega}} \{ M, \min_{x \in \overline{\Omega}} S_0(x) \} \) is a subsolution to (7). It follows from the comparison principle for parabolic equations that
\[ v(x,t) \geq \min_{x \in \overline{\Omega}} \{ M, \min_{x \in \overline{\Omega}} S_0(x) \}, \quad \forall x \in \overline{\Omega}, \ t > 0. \]
Hence, it holds
\[ S(x,t) \geq v(x,t) \geq \min_{x \in \overline{\Omega}} \{ M, \min_{x \in \overline{\Omega}} S_0(x) \}, \quad \forall x \in \overline{\Omega}, \ t > 0. \]

Now, in light of the second equation in (3), we have
\[
\frac{\partial I}{\partial t} - d_I \Delta I = \beta(x) S I + \eta(x) S - [\alpha(x) + \gamma(x)] I \\
\geq \eta(x) S - [\alpha(x) + \gamma(x)] I \\
\geq [\alpha(x) + \gamma(x)] \left[ \frac{\min_{x \in \overline{\Omega}} \eta(x) \cdot S}{\max_{x \in \overline{\Omega}} \alpha(x) + \max_{x \in \overline{\Omega}} \gamma(x) - I} \right] \\
\geq [\alpha(x) + \gamma(x)] \left[ \frac{\min_{x \in \overline{\Omega}} \eta(x)}{\max_{x \in \overline{\Omega}} \alpha(x) + \max_{x \in \overline{\Omega}} \gamma(x) - I} \right]
\]
for all \( x \in \overline{\Omega}, t > 0 \). Arguing similarly as before, one immediately sees that
\[ I(x,t) \geq \min \left\{ \min_{x \in \overline{\Omega}} \min_{x \in \overline{\Omega}} \eta(x), \ \frac{\min_{x \in \overline{\Omega}} \eta(x)}{\max_{x \in \overline{\Omega}} \alpha(x) + \max_{x \in \overline{\Omega}} \gamma(x)} \right\}, \quad \forall x \in \overline{\Omega}, \ t > 0. \]

Next we want to establish the upper bound of \( I \). To this end, let us define
\[ \Theta(t) = \int_{\Omega} [S(x,t) + I(x,t)] dx. \]
Then, making use of (3), we obtain
\[ \Theta'(t) = \int_{\Omega} \Lambda dx - \int_{\Omega} (S + \alpha I) dx \]
\[ \leq \int_{\Omega} \Lambda dx - \min \{ 1, \min_{x \in \Omega} \alpha \} \Theta, \quad \forall t > 0, \]
from which, it is easily seen that
\[ \lim_{t \to \infty} \Theta(t) \leq \int_{\Omega} \Lambda dx - \min \{ 1, \min_{x \in \Omega} \alpha \}. \]

By the definition of \( \Theta \), we infer that there exists a large \( T_2 > 0 \) such that
\[ \int_{\Omega} [S(x,t) + I(x,t)] dx \leq 2 \int_{\Omega} \Lambda(x) dx \min \{ 1, \min_{x \in \Omega} \alpha \}, \quad \forall t \geq T_2. \]

Now, applying \([18, \text{exercise 4 of section 3.5}] \) (or \([1, \text{theorem 3.1}] \)) to the second equation in system (3), we can assert that there is a positive constant \( K \), which is independent of initial data \( (S_0, I_0) \), such that
\[ \max_{x \in \Omega} I(x,t) \leq K, \quad \forall t \geq T_3, \]
for some \( T_3 \geq \max \{ T_1, T_2 \} \).

In view of (7), \( S \) is a supersolution to the following problem
\[ Z_t = M - [1 + K \max_{x \in \Omega} \beta(x) + \max_{x \in \Omega} \eta(x)] Z, \quad t > T_3, \]
\[ Z(T_3) = \min_{x \in \Omega} S(x, T_3) > 0. \]

As a consequence, we obtain
\[ \liminf_{t \to \infty} S(x, t) \geq \lim_{t \to \infty} Z(t) = \frac{M}{1 + K \max_{x \in \Omega} \beta(x) + \max_{x \in \Omega} \eta(x)} > 0 \]
uniformly for \( x \in \Omega \). This implies that
\[ S(x,t) \geq \frac{M}{2 [1 + K \max_{x \in \Omega} \beta(x)] + \max_{x \in \Omega} \eta(x)} =: N > 0, \quad \forall x \in \Omega, t \geq T_4, \]
for some \( T_4 \geq T_3 \).

Due to (10), we notice that \( I \) is a supersolution to the following problem
\[ W_t = N \min_{x \in \Omega} \eta(x) - [\max_{x \in \Omega} \alpha(x) + \max_{x \in \Omega} \gamma(x)] W, \quad t > T_4, \]
\[ W(T_4) = \min_{x \in \Omega} I(x, T_4) > 0. \]
As above, from (11) we can deduce that
\[ I(x,t) \geq \frac{N \min_{x \in \Omega} \eta(x)}{2 \max_{x \in \Omega} \alpha(x) + \max_{x \in \Omega} \gamma(x)} > 0, \quad \forall x \in \Omega, t \geq T_5, \]
for some \( T_5 \geq T_4 \).

In view of (6), (9), (10) and (12), one can apply the abstract dynamical system theory developed in \([47, \text{theorem 2.3}] \) (see also \([34, \text{theorem 4.5}] \)) to conclude that system (3) admits at least one EE.

The proof is now complete. \( \square \)
Biologically, Theorem 2.1 implies that the epidemic disease will always persist for large time; moreover, if the initial data \((S_0, I_0)\) satisfies \(S_0(x), I_0(x) > 0\) for all \(x \in \Omega\), the disease persists uniformly for all time \(t \geq 0\).

3. **Global stability of the EE in the homogeneous environment.** This section is concerned with the global stability of the endemic equilibrium (EE) of (3) in the case of homogeneous environment, that is, the parameters \(\Lambda, \alpha, \beta, \eta, \gamma\) are assumed to be positive constants.

By the basic calculation, it is easy to see that (3) has a unique constant EE, denoted by \((\bar{S}, \bar{I})\), where

\[
\bar{S} = \frac{\beta \Lambda + \alpha \eta + \alpha + \gamma - \sqrt{(\beta \Lambda - \alpha \eta - \alpha - \gamma)^2 + 4 \alpha \beta \eta \Lambda}}{2 \beta},
\]

\[
\bar{I} = \frac{\beta \Lambda - \alpha \eta - \alpha - \gamma + \sqrt{(\beta \Lambda - \alpha \eta - \alpha - \gamma)^2 + 4 \alpha \beta \eta \Lambda}}{2 \alpha \beta}.
\]

In the sequel, we will construct a suitable Lyapunov functional to derive the global stability of the EE \((\bar{S}, \bar{I})\).

Before stating the main result, let us recall the following Lemma, which can be referred to [40, Lemma 4.3]

**Lemma 3.1.** Let \(a\) and \(b\) be positive constants. Assume that \(\phi, \psi \in C^1([a, \infty))\), \(\psi \geq 0\), and \(\phi\) is bounded from below in \([a, \infty)\). If \(\phi'(t) \leq -b \psi(t)\) and \(\psi'(t) \leq K\) on \([a, \infty)\) for some positive constant \(K\), then \(\lim_{t \to \infty} \psi(t) = 0\).

Indeed, we are able to deduce the following result.

**Theorem 3.2.** Assume that the parameters \(\Lambda, \alpha, \beta, \eta, \gamma\) are positive constants, and \(d_S = d_I = 1\). Then the EE \((\bar{S}, \bar{I})\) is globally stable.

**Proof.** We take \(d_S = d_I = 1\) without loss of generality. Let us first define

\[
V(S, I) = \frac{1}{2} \left[(S - \bar{S}) + (I - \bar{I})\right]^2 + \frac{\alpha + 1}{\beta} \left[I - \left(\frac{\eta}{\beta} + \bar{I}\right)\right] \ln(\beta I + \eta)].
\]

For convenience, let

\[
g_1(S, I) = \Lambda - S - \beta SI - \eta S + \gamma I, \quad g_2(S, I) = \beta SI + \eta S - (\alpha + \gamma)I.
\]

Clearly, we have

\[
\frac{\partial V}{\partial S}(S, I) = (S - \bar{S}) + (I - \bar{I})
\]

and

\[
\frac{\partial V}{\partial I}(S, I) = (S - \bar{S}) + (I - \bar{I}) + \frac{\alpha + 1}{\beta I + \eta} (I - \bar{I})
\]

\[
= \frac{\partial V}{\partial S}(S, I) + \frac{\alpha + 1}{\beta I + \eta} (I - \bar{I}). \tag{13}
\]

Hence, direct computation shows

\[
\frac{\partial V}{\partial S}(S, I)g_1(S, I) + \frac{\partial V}{\partial I}(S, I)g_2(S, I)
\]

\[
= [(S - \bar{S}) + (I - \bar{I})]g_1(S, I) + [(S - \bar{S}) + (I - \bar{I}) + \frac{\alpha + 1}{\beta I + \eta} (I - \bar{I})]g_2(S, I)
\]
= [(S - \bar{S}) + (I - \bar{I})][g_1(S, I) + g_2(S, I)] + \frac{\alpha + 1}{\beta I + \eta}(I - \bar{I})g_2(S, I)
= [(S - \bar{S}) + (I - \bar{I})][\Lambda - S - \alpha I] + \frac{\alpha + 1}{\beta I + \eta}(I - \bar{I})[\beta S I + \eta S - (\alpha + \gamma)I]
= [(S - \bar{S}) + (I - \bar{I})][(S - \bar{S}) + \alpha I - \alpha I] + \frac{\alpha + 1}{\beta I + \eta}(I - \bar{I})[\beta S I + \eta S - (\alpha + \gamma)I - \beta \bar{S} I - \eta \bar{S} + (\alpha + \gamma)\bar{I}],

where the fact
\Lambda = \bar{S} + \alpha I \quad \text{and} \quad \beta \bar{S} I + \eta \bar{S} - (\alpha + \gamma)\bar{I} = 0

was used above. For sake of convenience, we set
\[ f_1(S, I) = [(S - \bar{S}) + (I - \bar{I})](S - \bar{S} + \alpha I - \alpha I), \]

and
\[ f_2(S, I) = \frac{\alpha + 1}{\beta I + \eta}(I - \bar{I})[\beta S I - \beta S \bar{I} + \beta \bar{S} I - \beta \bar{S} \bar{I}] \]

Then it follows that
\[ f_1(S, I) = [(S - \bar{S}) + (I - \bar{I})][-(S - \bar{S}) - \alpha(I - \bar{I})] \]
\[ = -(S - \bar{S})^2 - \alpha(I - \bar{I})^2 - (\alpha + 1)(S - \bar{S})(I - \bar{I}), \]

and
\[ f_2(S, I) = \frac{\alpha + 1}{\beta I + \eta}(I - \bar{I})[\beta S I - \beta S \bar{I} + \beta \bar{S} I - \beta \bar{S} \bar{I}] \]
\[ + \frac{\alpha + 1}{\beta I + \eta}(\beta I + \eta)(S - \bar{S})(I - \bar{I}) \]
\[ = \frac{\alpha + 1}{\beta I + \eta}[\beta S - (\alpha + \gamma)](I - \bar{I})^2 + \frac{\alpha + 1}{\beta I + \eta}(\beta I + \eta)(S - \bar{S})(I - \bar{I}) \]
\[ = \frac{\alpha + 1}{\beta I + \eta}[\beta S - (\alpha + \gamma)](I - \bar{I})^2 + \frac{\alpha + 1}{\beta I + \eta}(\beta I + \eta)(S - \bar{S})(I - \bar{I}) \]
\[ = \frac{\alpha + 1}{\beta I + \eta}[\beta S - (\alpha + \gamma)](I - \bar{I})^2 + \frac{\alpha + 1}{\beta I + \eta}(\beta I + \eta)(S - \bar{S})(I - \bar{I}) \]
\[ = \frac{\alpha + 1}{\beta I + \eta}[\beta S - (\alpha + \gamma)](I - \bar{I})^2 + \frac{\alpha + 1}{\beta I + \eta}(\beta I + \eta)(S - \bar{S})(I - \bar{I}) \]
\[ = \frac{\alpha + 1}{\beta I + \eta}[\beta S - (\alpha + \gamma)](I - \bar{I})^2 + \frac{\alpha + 1}{\beta I + \eta}(\beta I + \eta)(S - \bar{S})(I - \bar{I}). \]
Therefore, we have

\[ f_1(S, I) + f_2(S, I) = \frac{\partial V}{\partial S}(S, I)g_1(S, I) + \frac{\partial V}{\partial I}(S, I)g_2(S, I) \]

\[ = -(S - \bar{S})^2 - (\alpha + 1)(S - \bar{S})(I - \bar{I}) - \alpha(I - \bar{I})^2 \]
\[ + \frac{\alpha + 1}{\beta I + \eta}[\beta\bar{S} - (\alpha + \gamma)](I - \bar{I})^2 + (\alpha + 1)(S - \bar{S})(I - \bar{I}) \]
\[ = -(S - \bar{S})^2 - \alpha(I - \bar{I})^2 + \frac{\alpha + 1}{\beta I + \eta}[\beta\bar{S} - (\alpha + \gamma)](I - \bar{I})^2. \]

We also observe that

\[ \beta\bar{S} - (\alpha + \gamma) = -\frac{\eta\bar{S}}{I} < 0. \]

Thus

\[ \frac{\partial V}{\partial S}(S, I)g_1(S, I) + \frac{\partial V}{\partial I}(S, I)g_2(S, I) \leq 0. \] (14)

For any solution \((S, I)\) of (3), we then define

\[ L(t) = \int_{\Omega} V(S(x, t), I(x, t)) \, dx, \quad \forall t > 0. \]

In view of (13), we have

\[ L'(t) = \int_{\Omega} \left[ \frac{\partial V}{\partial S}(S, I) \frac{\partial S}{\partial t}(x, t) + \frac{\partial V}{\partial I}(S, I) \cdot \frac{\partial I}{\partial t}(x, t) \right] \, dx \]
\[ = \int_{\Omega} \frac{\partial V}{\partial S}(S, I)[\Delta S + g_1(S, I)] \, dx + \int_{\Omega} \frac{\partial V}{\partial I}(S, I)[\Delta I + g_2(S, I)] \, dx \]
\[ = \int_{\Omega} \frac{\partial V}{\partial S}(S, I)[\Delta S + g_1(S, I)] \, dx + \int_{\Omega} \frac{\partial V}{\partial I}(S, I)[\Delta I + \frac{\alpha + 1}{\beta I + \eta}(I - \bar{I})] \, dx \]
\[ + \int_{\Omega} \frac{\partial V}{\partial I}(S, I)g_2(S, I) \, dx \]
\[ = \int_{\Omega} \frac{\partial V}{\partial S}(S, I)[\Delta S + \Delta I] \, dx + \int_{\Omega} \frac{\alpha + 1}{\beta I + \eta}(I - \bar{I}) \Delta I \, dx \]
\[ + \int_{\Omega} \left[ \frac{\partial V}{\partial S}(S, I)g_1(S, I) + \frac{\partial V}{\partial I}(S, I)g_2(S, I) \right] \, dx. \]

In addition, integrating by parts, we have

\[ \int_{\Omega} \frac{\partial V}{\partial S}(S, I)[\Delta S + \Delta I] \, dx = \int_{\Omega} [(S - \bar{S}) + (I - \bar{I})](\Delta S + \Delta I) \, dx \]
\[ = -\int_{\Omega} |\nabla S|^2 + 2\nabla S \nabla I + |\nabla I|^2 \, dx \]
\[ = -\int_{\Omega} |\nabla(S + I)|^2 \, dx \]
\[ \leq 0, \quad \forall t > 0, \] (15)

and

\[ \int_{\Omega} \frac{\alpha + 1}{\beta I + \eta}(I - \bar{I}) \Delta I \, dx = -(\alpha + 1) \int_{\Omega} \frac{\beta I + \eta}{(\beta I + \eta)^2} |\nabla I|^2 \, dx \leq 0 \] (16)

for all \(t > 0\).
From (14), (15) and (16), it follows that
\[
\mathcal{L}'(t) \leq 0, \quad \forall t > 0.
\]

On the other hand, by Theorem 2.1,\[ \|S(\cdot, t)\|_{L^\infty} \text{ and } \|I(\cdot, t)\|_{L^\infty} \text{ are all bounded.}
\]
Consequently, using [7, Theorem A2], we find
\[
\|S(\cdot, t)\|_{C^1(\Omega)} + \|I(\cdot, t)\|_{C^1(\overline{\Omega})} \leq C_0, \quad \forall t \geq 1,
\]
for some positive constant \(C_0\).

Now, in Lemma 3.1, we set
\[
\phi(t) = \mathcal{L}(t)
\]
and
\[
\psi(t) = \int_\Omega (S - \bar{S})^2 + \alpha(I - \bar{I})^2 - \frac{\alpha + 1}{\beta I + \eta} [\beta \bar{S} - (\alpha + \gamma)](I - \bar{I})^2 dx.
\]
Thus,
\[
\psi(t) \geq 0, \quad \forall t > 0,
\]
and due to (17),
\[
\psi'(t) \leq C, \quad \forall t \geq 1,
\]
for some constant \(C > 0\). Hence, from Lemma 3.1 it follows that
\[
(S(\cdot, t), I(\cdot, t)) \to (\bar{S}, \bar{I}) \quad \text{in } [L^2(\Omega)]^2, \quad \text{as } t \to \infty.
\]
(18)

One also notices that (17) implies that \(\{(S(\cdot, t), I(\cdot, t))\}_{t \geq 1}\) is compact in \(C(\overline{\Omega})\). By virtue of (18), we can conclude that
\[
(S(x, t), I(x, t)) \to (\bar{S}, \bar{I}) \quad \text{uniformly for } x \in \overline{\Omega}, \quad \text{as } t \to \infty.
\]
The proof is complete. \(\square\)

4. **Asymptotic behavior of the EE.** In this section, we consider the asymptotic behavior of the EE of system (4) as the diffusion rate \(d_S\) or \(d_I\) tends to zero.

First of all, we recall a simple fact, which can be found in [36, Lemma 3.1](or see [33]).

**Lemma 4.1.** Assume that \(w \in C^2(\overline{\Omega})\) and \(\frac{\partial w}{\partial \nu} = 0\) on \(\partial \Omega\), then we have
\[
(i) \quad \text{If } w \text{ has a local maximum at } x_0 \in \overline{\Omega}, \text{ then } \nabla w(x_0) = 0 \text{ and } \Delta w(x_0) \leq 0;
\]
\[
(ii) \quad \text{If } w \text{ has a local minimum at } y_0 \in \overline{\Omega}, \text{ then } \nabla w(y_0) = 0 \text{ and } \Delta w(y_0) \geq 0.
\]

4.1. **The case of \(d_S \to 0\).** In this subsection, we are interested in the asymptotic profiles of any positive solution of (4) as \(d_S \to 0\) while \(d_I > 0\) is fixed. Our result is stated as follows.

**Theorem 4.2.** Fix \(d_I > 0\), and let \(d_S \to 0\). Then any positive solution \((S, I)\) of (4) satisfies (up to a subsequence of \(d_S \to 0\))
\[
(S, I) \to (\Phi_S, \Phi_I) \quad \text{uniformly on } \overline{\Omega},
\]
where
\[
\Phi_S(x) = \frac{\Lambda(x) + \gamma(x)\Phi_I(x)}{1 + \beta(x)\Phi_I(x) + \eta(x)},
\]
and \(\Phi_I(x)\) is a positive solution to
\[
\begin{aligned}
-d_I \Delta \Phi_I &= \beta(x)\Phi_S \Phi_I + \eta(x)\Phi_S - [\alpha(x) + \gamma(x)]\Phi_I, \quad x \in \Omega; \\
\frac{\partial \Phi_I}{\partial \nu} &= 0, \quad x \in \partial \Omega.
\end{aligned}
\]
Proof. Our proof is divided into four steps.

Step 1. The estimate for $S$. Assume that there exists $x_0 \in \Omega$, $S(x)$ has a maximum value at $x = x_0$, that is, $S(x_0) = \max S(x)$. By Lemma 4.1 as applied to the first equation of (4), we find that $\Delta S(x_0) \leq 0$. Thus, it holds

$$\Lambda(x_0) - S(x_0) - \beta(x_0) S(x_0) I(x_0) - \eta(x_0) S(x_0) + \gamma(x_0) I(x_0) \geq 0,$$

or

$$\max_{x \in \Omega} \Lambda(x) \geq \Lambda(x_0) \geq [1 + \eta(x_0)] S(x_0) + [\beta(x_0) S(x_0) - \gamma(x_0)] I(x_0).$$

There are going to be two cases. If $\beta(x_0) S(x_0) - \gamma(x_0) \leq 0$, then

$$\max_{x \in \Omega} S(x) = S(x_0) \leq \frac{\gamma(x_0)}{\beta(x_0)} \leq \max_{x \in \Omega} \frac{\lambda(x)}{\beta(x)}.$$

If $\beta(x_0) S(x_0) - \gamma(x_0) > 0$, then

$$\max_{x \in \Omega} S(x) = S(x_0) \leq \frac{\Lambda(x_0)}{1 + \eta(x_0)} \leq \max_{x \in \Omega} \frac{\lambda(x)}{1 + \eta(x)}.$$

As a consequence, we obtain

$$S(x) \leq \max_{x \in \Omega} S(x) \leq \max \left\{ \max_{x \in \Omega} \frac{\lambda(x)}{\beta(x)} \max_{x \in \Omega} \frac{\lambda(x)}{1 + \eta(x)} \right\}, \quad (20)$$

Similarly, we assume that $S(x_1) = \min_{x \in \Omega} S(x)$, then use Lemma 4.1 again to infer that

$$\Lambda(x_1) - S(x_1) - \beta(x_1) S(x_1) I(x_1) - \eta(x_1) S(x_1) + \gamma(x_1) I(x_1) \leq 0,$$

equivalently,

$$\frac{\Lambda(x_1) + \gamma(x_1) I(x_1)}{1 + \beta(x_1) I(x_1) + \eta(x_1)} \leq S(x_1).$$

Since $\frac{\Lambda(x_1) + \gamma(x_1) I(x_1)}{1 + \beta(x_1) I(x_1) + \eta(x_1)}$ has a positive lower bound (independent of $I$ and $x_1$), we can assert that there is a positive constant $\kappa > 0$, independent of $d_S$ and $d_I$, such that

$$\frac{\Lambda(x_1) + \gamma(x_1) I(x_1)}{1 + \beta(x_1) I(x_1) + \eta(x_1)} \geq \kappa.$$

This then gives

$$S(x) \geq \min_{x \in \Omega} S(x) = S(x_1) \geq \kappa > 0. \quad (21)$$

Step 2. The estimate for $I$. Set $I(x_2) = \min_{x \in \Omega} I(x)$. Then an application of Lemma 4.1 yields

$$\beta(x_2) S(x_2) I(x_2) + \eta(x_2) S(x_2) - [\alpha(x_2) + \gamma(x_2)] I(x_2) \leq 0,$$

or

$$\eta(x_2) S(x_2) \leq \beta(x_2) S(x_2) I(x_2) + \eta(x_2) S(x_2) \leq [\alpha(x_2) + \gamma(x_2)] I(x_2).$$

Hence, due to (21), it holds

$$I(x_2) \geq \frac{\eta(x_2) S(x_2)}{\alpha(x_2) + \gamma(x_2)} \geq \frac{\kappa \eta(x_2)}{\alpha(x_2) + \gamma(x_2)},$$

which in turn gives

$$I(x) \geq \min_{x \in \Omega} I(x) = I(x_2) \geq \frac{\kappa \min_{x \in \Omega} \eta(x)}{\max_{x \in \Omega} [\alpha(x) + \gamma(x)]} \geq C_0. \quad (22)$$
for some positive constant $C_0$, which does not depend on $d_s$, $d_I > 0$.

We next want to obtain the upper bound of $I$. For our purpose, integrating both equations of system (4) over $\Omega$, we obtain

$$\int_{\Omega} [\Lambda(x) - S(x) - \beta(x)S(x)I(x) - \eta(x)S(x) + \gamma(x)I(x)] dx = 0,$$

$$\int_{\Omega} [\beta(x)S(x)I(x) + \eta(x)S(x) - (\alpha(x) + \gamma(x))I(x)] dx = 0.$$

These two identities give

$$\min_{x \in \Omega} \alpha(x) \int_{\Omega} I(x) dx \leq \int_{\Omega} \alpha(x)I(x) dx + \int_{\Omega} S(x) dx,$$

$$= \int_{\Omega} \Lambda(x) dx \leq |\Omega| \max_{x \in \Omega} \Lambda(x), \quad (23)$$

and

$$\min_{x \in \Pi} \beta(x) \int_{\Omega} S(x)I(x) dx \leq \int_{\Omega} \beta(x)S(x)I(x) dx,$$

$$= \int_{\Omega} \alpha(x) + \gamma(x) |I(x) dx - \int_{\Omega} \eta(x)S(x) dx.$$

Furthermore, it is clear that

$$\int_{\Omega} [\alpha(x) + \gamma(x)] I(x) dx - \int_{\Omega} \eta(x)S(x) dx,$$

$$\leq \int_{\Omega} [\alpha(x) + \gamma(x)] I(x) dx \leq \max_{x \in \Pi} [\alpha(x) + \gamma(x)] \int_{\Omega} I(x) dx.$$

Thus

$$\min_{x \in \Pi} \beta(x) \int_{\Omega} S(x)I(x) dx \leq \max_{x \in \Pi} [\alpha(x) + \gamma(x)] \int_{\Omega} I(x) dx. \quad (24)$$

The identity (23) implies the following

$$\int_{\Omega} I(x) dx \leq |\Omega| \frac{\max_{x \in \Pi} \Lambda(x)}{\min_{x \in \Pi} \alpha(x)}. \quad (25)$$

By (24) and (25), it holds

$$\int_{\Omega} S(x)I(x) dx \leq |\Omega| \frac{\max_{x \in \Pi} \Lambda(x) \max_{x \in \Pi} (\alpha(x) + \gamma(x))}{\min_{x \in \Pi} \alpha(x) \min_{x \in \Pi} \beta(x)}. \quad (26)$$

(Not that (26) will be used in the proof of Theorem 4.3 below.)

Now we rewrite the second equation of (4) as

$$\begin{cases} -\Delta I = \frac{1}{d_I} [\beta S + \eta(x)S_T - (\alpha(x) + \gamma(x))] I, & x \in \Omega, \\ \frac{\partial I}{\partial n} = 0, & x \in \partial \Omega. \quad (27) \end{cases}$$

Due to (20) and (22), it is clear that

$$\| \frac{1}{d_I} [\beta S + \eta(x)S_T - (\alpha + \gamma)] \|_{L^{\infty}(\Omega)} \leq C.$$

Hereafter, the positive constant $C$ may be different from place to place, but is independent of $d_s > 0$. Thus, by the Harnack-type inequality for elliptic equations (see [31, 32] or [37, Lemma 2.2]) and (25), we have

$$I(x) \leq \max_{x \in \Pi} I(x) \leq C \min_{x \in \Pi} I(x) \leq C \frac{1}{|\Omega|} \int_{\Omega} I dx \leq C. \quad (28)$$
Step 3. Convergence of $I$ as $d_S \to 0$. Notice that $I$ satisfies
\[
\begin{cases}
-d_n \Delta I + (\alpha(x) + \gamma(x)) I = \beta(x) S(x) I(x) + \eta(x) S(x) := h(x), & x \in \Omega, \\
\frac{\partial I}{\partial \nu} = 0, & x \in \partial \Omega.
\end{cases}
\]
According to (20) and (28), we have
\[
\|h\|_{L^p(\Omega)} = \|\beta S I + \eta S\|_{L^p(\Omega)} \leq C, \quad \forall p > 1.
\]
Using the standard $L^p$-estimate for elliptic equations [17], one can conclude that
\[
\|I\|_{W^{2,p}(\Omega)} \leq C\|h\|_{L^p(\Omega)} \leq C.
\]
Taking $p > \frac{m}{2}$ and using the Sobolev embedding theorem, we obtain
\[
\|I\|_{C^{1+s}(\Omega)} \leq \|I\|_{W^{2,p}(\Omega)} \leq C,
\]
for some $0 < \theta < 1$.

Thus, $\{I\}_{0 < d_S \leq 1}$ is compact in $C^1(\overline{\Omega})$. Consequently, there is a sequence of $d_S$, denoted by $d_n := d_S n$, satisfying $d_n \to 0$ as $n \to \infty$, and a corresponding positive solution $(S_n, I_n)$, such that
\[
I_n \to \Phi_I \text{ in } C^1(\overline{\Omega}), \quad \text{as } n \to \infty,
\]
where $\Phi_I > 0$ in $C^1(\overline{\Omega})$ due to (22).

**Step 4.** Convergence of $S$. Notice that for any $n \geq 1$, $S_n$ solves
\[
\begin{cases}
-d_n \Delta S_n = \Lambda(x) - S_n - \beta(x) S_n I_n - \eta(x) S_n + \gamma(x) I_n, & x \in \Omega, \\
\frac{\partial S_n}{\partial \nu} = 0, & x \in \partial \Omega.
\end{cases}
\]
In view of (29), for any small $\varepsilon > 0$, it holds
\[
0 < \Phi_I(x) - \varepsilon \leq I_n(x) \leq \Phi_I(x) + \varepsilon, \quad \forall x \in \overline{\Omega},
\]
for all large $n$. Hence, taking sufficiently large $n$, we observe that on $\overline{\Omega}$,
\[
\begin{align*}
\Lambda - S_n - \beta S_n I_n - \eta S_n + \gamma I_n & \geq \Lambda - S_n - \beta S_n (\Phi_I + \varepsilon) - \eta S_n + \gamma (\Phi_I - \varepsilon), \\
\Lambda - S_n - \beta S_n I_n - \eta S_n + \gamma I_n & \leq \Lambda - S_n - \beta S_n (\Phi_I - \varepsilon) - \eta S_n + \gamma (\Phi_I + \varepsilon).
\end{align*}
\]
Next, for fixed large $n$, we consider the following auxiliary problem
\[
\begin{cases}
-d_n \Delta Z_n = \Lambda(x) + \gamma(x) (\Phi_I(x) - \varepsilon) - [1 + \beta(x) (\Phi_I(x) + \varepsilon) + \eta(x)] Z_n, & x \in \Omega, \\
\frac{\partial Z_n}{\partial \nu} = 0, & x \in \partial \Omega.
\end{cases}
\]
It can be easily shown that (30) has a unique positive solution, denoted by $Z_n$. Furthermore, using the similar proof of [14, Lemma 2.4], we can show that
\[
Z_n \to \frac{\Lambda + \gamma (\Phi_I - \varepsilon)}{1 + \beta(\Phi_I + \varepsilon) + \eta} \text{ uniformly on } \overline{\Omega}, \quad \text{as } n \to \infty.
\]
It is clear that a supersolution of (30) is $S_n$ and a sufficiently small positive constant is a subsolution. Together with the uniqueness of positive solution to (30), the standard sub-supersolution iteration theory yields that $S_n$ satisfies
\[
\lim_{n \to \infty} \inf S_n \geq \lim_{n \to \infty} Z_n = \frac{\Lambda + \gamma (\Phi_I - \varepsilon)}{1 + \beta(\Phi_I + \varepsilon) + \eta} \text{ uniformly on } \overline{\Omega}.
\]
The same reasoning as above shows that the following problem
\[
\begin{aligned}
-d_d \Delta Z &= \Lambda(x) + \gamma(x)(\Phi_I(x) + \varepsilon) \\
\frac{\partial Z}{\partial n} &= 0, \\
-\gamma(x)[1 + \beta(x)(\Phi_I(x) - \varepsilon) + \eta(x)]Z, & \quad x \in \Omega, \\
& \quad x \in \partial \Omega
\end{aligned}
\]
has a unique solution, denoted by $Z_n$ and a subsolution of (32) is $S_n$, and
\[
\limsup_{n \to \infty} S_n \leq \lim_{n \to \infty} Z_n = \frac{\Lambda + \gamma(\Phi_I + \varepsilon)}{1 + \beta(\Phi_I - \varepsilon) + \eta} \quad \text{uniformly on } \overline{\Omega}, \text{ as } n \to \infty. \tag{33}
\]
Due to the arbitrariness of small $\varepsilon > 0$, thanks to (31) and (33), we have
\[
S_n(x) \to \Phi_S(x) := \frac{\Lambda(x) + \gamma(x)\Phi_I(x)}{1 + \beta(x)\Phi_I(x) + \eta(x)} \quad \text{uniformly for } x \in \overline{\Omega}, \text{ as } n \to \infty.
\]
Because $I_\eta$ fulfills (27), it can be easily seen that $\Phi_I$ satisfies (19).

Therefore, the proof is complete. \qed

4.2. The case of $d_I \to 0$. This subsection concerns the asymptotic behaviour of EE as $d_I \to 0$ and $d_S$ is fixed.

First of all, we deal with one space dimension case, that is $m = 1$. Without loss of generality, we take $\Omega = (0, 1)$. Then we can state

**Theorem 4.3.** Assume that $\Omega = (0, 1)$, and let $d_I \to 0$. Then every positive solution $(S, I)$ of (4) (up to a subsequence of $d_I \to 0$) satisfies that there exists $S^* \in C([0, 1])$ with $S^* > 0$ in $C([0, 1])$, such that $S \to S^*$ uniformly on $[0, 1]$, and $\int_0^1 I dx \to I^*$ for some positive constant $I^*$.

**Proof.** We observe that (20), (21), (22), (25) and (26) remain true for all $d_I > 0$. Furthermore, according to (22), we also have
\[
\int_0^1 I dx \geq \int_0^1 C dx = C. \tag{34}
\]
Here and below, $C$ is a positive constant, which is independent of $d_I > 0$, but allows to be different.

Clearly, $S$ satisfies the following problem
\[
\begin{aligned}
-d_S S'(x) + [1 + \eta(x)] S(x) &= \Lambda(x) - \beta(x) S(x) I(x) + \gamma(x) I(x), & \quad x \in (0, 1), \\
S'(0) &= S'(1) = 0.
\end{aligned}
\]

By (25) and (26), one has
\[
\|\Lambda - \beta SI + \gamma I\|_{L^1((0, 1))} \leq C.
\]
Thus, by the elliptic $L^1$-theory which was established in [6] as applied to (35), we infer
\[
\|S\|_{W^{1,p}((0, 1))} \leq C, \quad \forall p > 1.
\]
If $p$ is properly large, for some $\theta \in (0, 1)$, owing to the Sobolev embedding theorem, we obtain
\[
\|S\|_{C^\theta((0, 1))} \leq \|S\|_{W^{1,p}((0, 1))} \leq C.
\]
Therefore, there is a subsequence of $d_I \to 0$, labelled as $d_{I,n} := d_{I,n}$, with $d_{I,n} \to 0$ as $n \to \infty$, such that the corresponding positive solution $(S_{I,n})$ of (4) satisfies
\[
S_{I,n} \to S^* \quad \text{in } C([0, 1]) \quad \text{as } n \to \infty,
\]
where $S^* > 0$ on $[0,1]$ due to (21). In light of (25), by passing to a further subsequence of $d_n$, if necessary, we obtain that

$$\int_0^1 I_n(x)dx \to I^*, \quad \text{as } n \to \infty,$$

where $I^* > 0$ due to (34).

The proof is complete. \hfill $\square$

From the proof of Theorem 4.3, it is easily seen that in order to derive the asymptotical behavior of the EE as $d_I \to 0$, the main mathematical difficulty is to establish the upper bound of $I$. In the following, under an extra assumption on the parameters $\Lambda(x), \alpha(x), \beta(x), \eta(x)$ and $\gamma(x)$, we will obtain the asymptotical behavior of the EE as $d_I \to 0$ in any space dimension. Precisely speaking, we are able to state

**Theorem 4.4.** Assume that

$$\max \left\{ \max_{x \in \Omega} \frac{\gamma(\xi)}{\beta(x)}, \max_{x \in \Omega} \frac{\Lambda(x)}{1 + \eta(x)} \right\} < \min_{x \in \Omega} \frac{\alpha(x) + \gamma(x)}{\beta(x)},$$

(36)

Fix $d_S > 0$ and let $d_I \to 0$. Thus, up to a subsequence of $d_I \to 0$, $(S,I) \to (\Psi_S, \Psi_I)$ uniformly on $\bar{\Omega}$ for every positive solution $(S,I)$ of (4), where

$$\Psi_I(x) = \frac{\eta(x)\Psi_S(x)}{\alpha(x) + \gamma(x) - \beta \Psi_S(x)},$$

and $\Psi_S$ is a positive solution of the problem

$$\left\{ \begin{array}{ll}
-d_S \Delta \Psi_S = \Lambda(x) - \Psi_S - \beta(x) \Psi_S \Psi_I - \eta(x) \Psi_S + \gamma(x) \Psi_I, & x \in \Omega, \\
\frac{\partial \Psi_S}{\partial n} = 0, & x \in \partial \Omega.
\end{array} \right.$$

(37)

Proof. We first notice that (20), (21) and (22) are still valid. Combined with (20) and (36), we get

$$S(x) \leq \max \left\{ \max_{x \in \Omega} \frac{\gamma(x)}{\beta(x)}, \max_{x \in \Omega} \frac{\Lambda(x)}{1 + \eta(x)} \right\} < \min_{x \in \Omega} \frac{\alpha(x) + \gamma(x)}{\beta(x)}, \quad \forall x \in \bar{\Omega}.$$  

(38)

Set $I(x_0) = \max_{x \in \Omega} I(x)$. Then an application of Lemma 4.1 yields

$$\beta(x_0)S(x_0)I(x_0) + \eta(x_0)S(x_0) - [\alpha(x_0) + \gamma(x_0)]I(x_0) \geq 0.$$  

By (38), we then have

$$I(x) \leq I(x_0) \leq \frac{\eta(x_0)S(x_0)}{\alpha(x_0) + \gamma(x_0) - \beta(x_0)S(x_0)} \leq C_0, \quad \forall x \in \bar{\Omega},$$

for some constant $C_0 > 0$, which does not depend on $d_I > 0$. Hence, this, together with (20), (21) and (22), this implies that we have at hand the positive upper and lower bounds of $S$ and $I$, which are independent of $d_I > 0$.

Next, we will determine the convergence of $S$ and $I$ as $d_I \to 0$. Since $S$ solves

$$\left\{ \begin{array}{ll}
-d_S \Delta S + (1 + \eta)S = \Lambda - \beta SI + \gamma I, & x \in \Omega, \\
\frac{\partial S}{\partial n} = 0, & x \in \partial \Omega,
\end{array} \right.$$

(39)

and

$$||\Lambda - \beta SI + \gamma I||_{L^p(\Omega)} \leq C, \quad \forall p > 1,$$
Using the standard $L^p$-estimate for elliptic equations, one can find that

$$\|S\|_{W^{2,p}(\Omega)} \leq C.$$  

If $p > \frac{m}{2}$ is properly large, one can use the Sobolev embedding theorem to assert that

$$\|S\|_{C^{1+\theta}(\overline{\Omega})} \leq \|S\|_{W^{2,p}(\Omega)} \leq C;$$

for some $0 < \theta < 1$. Thus, there exists a subsequence of $d_n \to 0$, denoted by $d_n := d_{I,n}$, satisfying $d_n \to 0$ as $n \to \infty$, and a corresponding positive solution $(S_n, I_n)$, such that

$$S_n \to \Psi_S \quad \text{in} \quad C^1(\overline{\Omega}), \quad \text{as} \quad n \to \infty,$$

where $\Psi_S > 0$ in $\overline{\Omega}$.

Similarly, $I_n$ solves

$$\begin{cases}
-d_n\Delta I_n = \beta(x)S_nI_n + \eta(x)S_n - [\alpha(x) + \gamma(x)]I_n, & x \in \Omega, \\
\frac{\partial I_n}{\partial \nu} = 0, & x \in \partial\Omega.
\end{cases}$$

In view of (40), for any small $\varepsilon > 0$ and all large $n$, it holds

$$0 < \Psi_S(x) - \varepsilon \leq S_n(x) \leq \Psi_S(x) + \varepsilon, \quad \forall x \in \overline{\Omega}.$$  

Thus, if $n$ is sufficiently large, on $\overline{\Omega}$ we have

$$\beta S_nI_n + \eta S_n - (\alpha + \gamma)I_n \leq \beta I_n(\Psi_S + \varepsilon) + \eta(\Psi_S + \varepsilon) - (\alpha + \gamma)I_n,$$

$$\beta S_nI_n + \eta S_n - (\alpha + \gamma)I_n \geq \beta I_n(\Psi_S - \varepsilon) + \eta(\Psi_S - \varepsilon) - (\alpha + \gamma)I_n.$$  

Clearly, for any fixed large $n$, $I_n$ is a subsolution of

$$\begin{cases}
-d_n\Delta \overline{Q} = [\beta(x)(\Psi_S(x) + \varepsilon) - (\alpha(x) + \gamma(x))]\overline{Q} + \eta(x)(\Psi_S(x) + \varepsilon), & x \in \Omega, \\
\frac{\partial \overline{Q}}{\partial \nu} = 0, & x \in \partial\Omega,
\end{cases}$$

and $I_n$ is a supersolution of

$$\begin{cases}
-d_n\Delta \underline{Q} = [\beta(x)(\Psi_S(x) - \varepsilon) - (\alpha(x) + \gamma(x))]\underline{Q} + \eta(x)(\Psi_S(x) - \varepsilon), & x \in \Omega, \\
\frac{\partial \underline{Q}}{\partial \nu} = 0, & x \in \partial\Omega.
\end{cases}$$

It is easily shown that (42) and (43) has a unique solution, denoted by $\overline{Q}_n$ and $\underline{Q}_n$, respectively. A simple sub-supersolution argument, combined with the uniqueness, guarantees that $\underline{Q}_n \leq I_n \leq \overline{Q}_n$ on $\overline{\Omega}$ for all large $n$. Furthermore, together with (38), by a similar argument to that of [14, Lemma 2.4], it can be proved that

$$\overline{Q}_n \to \frac{\eta(\Psi_S + \varepsilon)}{-\beta(\Psi_S + \varepsilon) + (\alpha + \gamma)} \quad \text{uniformly on} \quad \overline{\Omega}, \quad \text{as} \quad n \to \infty,$$

and

$$\underline{Q}_n \to \frac{\eta(\Psi_S - \varepsilon)}{-\beta(\Psi_S - \varepsilon) + (\alpha + \gamma)} \quad \text{uniformly on} \quad \overline{\Omega}, \quad \text{as} \quad n \to \infty.$$  

This then gives

$$\frac{\eta(\Psi_S - \varepsilon)}{-\beta(\Psi_S - \varepsilon) + (\alpha + \gamma)} \leq \liminf_{n \to \infty} I_n \leq \limsup_{n \to \infty} I_n \leq \frac{\eta(\Psi_S + \varepsilon)}{-\beta(\Psi_S + \varepsilon) + (\alpha + \gamma)}.$$
Thanks to the arbitrariness of small $\varepsilon > 0$, we have
\[ I_n \to \frac{\eta \Psi_S}{(\alpha + \gamma) - \beta \Psi_S} := \Psi_I \quad \text{uniformly on } \overline{\Omega}, \quad \text{as } n \to \infty, \quad (44) \]
and $\Psi_S$ is a positive solution of (37).

The proof is complete. □

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