PULLBACK DYNAMIC BEHAVIOR FOR A NON-AUTONOMOUS INCOMPRESSIBLE NON-NEWTONIAN FLUID

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Abstract. This paper studies the pullback asymptotic behavior of solutions for a non-autonomous incompressible non-Newtonian fluid on 2D bounded domains. We show existence of the pullback exponential attractor introduced by Langa, Miranville and Real [27], moreover, give existence of the global pullback attractor with finite fractal dimension and reveal the relationship between the global pullback attractor and the pullback exponential attractor. These results improve our previous associated results in papers [29, 40] for the non-Newtonian fluid.

1. Introduction. In this paper, the following non-autonomous incompressible non-Newtonian fluid on 2D domains is studied.

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla) u - \nabla \cdot \tau(e(u)) + \nabla p = f(x, t),
\]

\[
\text{div } u = \nabla \cdot u = 0, \quad x = (x_1, x_2) \in \Omega,
\]

where $\Omega \subset \mathbb{R}^2$ is open and bounded with smooth enough boundary. Equations (1)-(2) describe the motion of an isothermal incompressible viscous fluid, where the unknown vector function $u = u(x, t) = (u_1, u_2)$ denotes the velocity of the fluid, $f(x, t) = (f_1, f_2)$ is the time-dependent external force function, the scalar function $p$ represents the pressure, and $\tau(e(u)) = (\tau_{ij}(e(u)))_{2 \times 2}$, which is usually called the extra stress tensor of the fluid, is a matrix of order $2 \times 2$ defined as

\[
\tau_{ij}(e(u)) = 2\mu_0(\eta + |e(u)|^2)^{-\alpha/2}e_{ij}(u) - 2\mu_1\Delta e_{ij}(u), \quad i, j = 1, 2,
\]

where

\[
e_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),
\]

\[
|e(u)|^2 = \sum_{i,j=1}^2 |e_{ij}(u)|^2,
\]

and $\mu_0, \mu_1, \alpha, \eta$ are parameters associated to the fluid which generally depend on the temperature and pressure. In this paper we assume that $\mu_0, \mu_1, \eta$ are positive constants and $0 < \alpha < 1$.

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From the viewpoint of physics, (1)-(2) are supplemented by the initial and boundary conditions

$$u|_{t=\tau} = u_\tau, \ x \in \Omega, \ \tau \in \mathbb{R},$$

$$u = 0, \ \tau_{ijl}n_jn_l = 0, \ x \in \partial\Omega,$$

where \(\tau_{ijl} = 2\mu_1 \frac{\partial e_{ij}}{\partial x_l} (i, j, l = 1, 2)\) is the component of the first multipolar stress tensor, and \(n = (n_1, n_2)\) denotes the exterior unit normal to the boundary \(\partial\Omega\). The first condition in (5) represents the usual no-slip condition associated with a viscous fluid, while the second one expresses the fact that the first moments of the traction vanish on \(\partial\Omega\). It is a direct consequence of the principle of virtual work. We refer to [3, 4, 5, 7, 25, 30, 32] and the references therein for detailed physical background.

In equation (3) if \(\tau_{ij}(e(u))\) depends linearly on \(e_{ij}(u)\) then we say the corresponding fluid is a Newtonian one. Generally speaking, gases, water, motor oil, alcohols, and simple hydrocarbon compounds tend to be Newtonian fluids and their motions can be described by the Navier-Stokes equations. If the relation between \(\tau_{ij}(e(u))\) and \(e_{ij}(u)\) is nonlinear, then the fluid is called to be non-Newtonian. For instance, molten plastics, polymer solutions and paints tend to be non-Newtonian fluids. One can refer to [3, 4, 5, 7, 25, 30, 32] and the references therein for detailed physical significance. Factually, equations (1)-(3) were firstly formulated by Ladyzhenskaya as a modification to the Navier-Stokes equations when the gradient \(|\nabla u|\) of the velocity is relatively large ([25]). Clearly, equations (1)-(3) reduce into Navier-Stokes equations when \(\alpha = \mu_1 = 0\) and into Euler equations as \(\mu_1 = \mu_0 = 0\).

Due to the wide applications in the real world, there is a large literature on the mathematical theory of the non-Newtonian fluid (see [3, 4, 5, 7, 13, 14, 24, 25, 28, 29, 30, 32, 36, 37, 38, 39, 40, 41], etc). For example, [32] proved the well-posedness of solutions to the Cauchy problem, [13, 14] studied the decay rate of solutions to the Cauchy problem. [3] proved the well-posedness of solutions to the initial boundary value problem on bounded domains. [4] proved the existence and uniqueness of solutions on 2D unbounded channel like domains. [5, 36, 38] studied the long time forward dynamic behavior \((t \rightarrow \infty)\) by obtaining the existence, regularity and related properties of various of attractors for solutions to the initial boundary problem. Nowadays, the long time pullback dynamic behavior \((t \rightarrow -\infty)\) of lots of evolutionary equations have attracted many researchers's attention (see [10, 11, 12, 20, 21, 23, 29, 40]). For this non-Newtonian fluid, in [28, 29, 37, 39, 40, 41], they have studied the existence, regularity, properties of global pullback attractor and pullback \(D\) attractor. However, on one hand, these pullback attractors are generally not stable under perturbations and the rate of convergence to the attractor is unknown, on the other hand, the dimensions of the attractors are generally infinite.

The objective of the present paper is concerned on formulating the pullback exponential attractor being a family of compact and positively invariant sets with uniformly finite fractal dimension which pullback attracts bounded subsets of the phase space at a uniform exponential rate for the process corresponding to equations (1)-(5). Moreover, we give existence of the global pullback attractor with finite fractal dimension and reveal the relationship between the global pullback attractor and the pullback exponential attractor.

Attractor is an important concept in the study of infinite dimensional dynamical systems for it captures the long-time behavior of solutions. There are many works concerning this subject (see, e.g., [2, 9, 17, 22, 26, 33, 34, 35]). For instance,
2. Notations and preliminaries. In this section, we first remark the fundamental notations.

We denote by $\mathbb{R}$ the set of real numbers. We also use $c$ to represent the generic constant which may take different values in different places. Let $L^p(\Omega)$ and $W^{m,p}(\Omega)$ be the usual Lebesgue space and Sobolev space endowed with norms $\| \cdot \|_p$ and $\| \cdot \|_{m,p}$ respectively (see e.g. [1]), where

$$
|\varphi|_p := \left( \int_\Omega |\varphi|^p dx \right)^{1/p} \quad \text{and} \quad |\varphi|_{m,p} := \left( \sum_{|\beta| \leq m} \int_\Omega |D^\beta \varphi|^p dx \right)^{1/p}.
$$

Then $L^p(\Omega) := L^p(\Omega) \times L^p(\Omega)$ and $W^{m,p}(\Omega) := W^{m,p}(\Omega) \times W^{m,p}(\Omega)$ represent the 2D vector Lebesgue space and 2D vector Sobolev space with norm $\| \cdot \|_{L^p(\Omega)}$ and $\| \cdot \|_{m,p}$ defined respectively by

$$
\| \varphi \|_{L^p(\Omega)} := (\| \varphi_1 \|_p^p + \| \varphi_2 \|_p^p)^{1/p}, \quad \varphi_1, \varphi_2 \in L^p(\Omega);
$$

$$
\| \varphi \|_{m,p} := (\| \varphi_1 \|_{m,p}^p + \| \varphi_2 \|_{m,p}^p)^{1/p}, \quad \varphi_1, \varphi_2 \in W^{m,p}(\Omega).
$$

Especially, we denote $\| \cdot \| := \| \cdot \|_{L^2(\Omega)}$, $H^m(\Omega) := \mathbb{R}^{m,2}(\Omega)$ and $H^m_0(\Omega)$ by the closure of $\{ \varphi \in C^0_0(\Omega) \times C^0_0(\Omega) \}$ in $H^m(\Omega)$ norm. Also, we introduce the following spaces:

$$
\mathcal{V} := \{ \varphi \in C^\infty_0(\Omega) \times C^\infty_0(\Omega) : \varphi = (\varphi_1, \varphi_2), \nabla \cdot \varphi = 0 \}.
$$
According to the above notations, we further denote

\[ L^p(I; E) := \text{space of strongly measurable functions on the interval I, with values in a Banach space } E, \text{ endowed with norm } \| \varphi \|_{L^p(I; E)} := \left( \int_I \| \varphi \|^p_E dt \right)^{1/p}, \text{ for } 1 \leq p < \infty. \]

\[ C(I; E) := \text{space of continuous functions on the interval I, with values in the Banach space } E, \text{ endowed with the usual norm}. \]

\[ \text{dist}_E(Y, Z) := \sup_{y \in Y} \inf_{z \in Z} \text{dist}_E(y, z) \text{ means the Hausdorff semidistance from } Y \subset E \text{ to } Z \subset E \text{ in a Banach space } E. \]

\[ L^2_{\text{loc}}(\mathbb{R}; H) := \text{space of locally integrable functions from } \mathbb{R} \text{ to } H. \]

Then we introduce some operators associated to the non-Newtonian fluid and recall some properties of the operators, previous results for the non-Newtonian fluid.

Let \((\cdot, \cdot)\) be the inner product in \(L^2(\Omega)\) or \(H\), and \(\langle \cdot, \cdot \rangle\) be the dual pairing between \(V\) and \(V'\). Then, we introduce the following three operators:

\[ \langle Au, v \rangle := \sum_{i,j,k=1}^{2} \int_\Omega \frac{\partial e_{ij}(u)}{\partial x_k} \frac{\partial e_{ij}(v)}{\partial x_k} dx, \forall u, v \in V, \]

\[ b(u, v, w) := \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx, \forall u, v, w \in V, \]

\[ \langle N(u), v \rangle := \sum_{i,j=1}^{2} \int_\Omega \mu(u) e_{ij}(u) e_{ij}(v) dx, \forall u, v \in V, \]

where \(\mu(u) := 2\mu_0(\eta + |e(u)|^2)^{-\alpha/2}\). For any \(u \in V\), we set

\[ \langle B(u), w \rangle := b(u, u, w), \forall w \in V. \]

Excluding the pressure \(p\), we can express the weak version of equations (1)-(5) in the sense of \(D'(\tau, +\infty; V')\) (see [4, 5, 29, 37, 38, 40]),

\[ \frac{\partial u}{\partial t} + 2\mu_1 Au + B(u) + N(u) = f(t), \]

\[ u(\tau, x) = u_0, \] (7)

Some useful estimates and properties for the operators \(A, b(\cdot, \cdot, \cdot), B(\cdot)\) and \(N(\cdot)\) have been established in the works [4, 5, 36, 37, 38]. For completeness, we recall them as following.

**Lemma 2.1.** (1) There are some positive constants \(c_1, c\) depending only on \(\Omega\) such that

\[ \frac{c_1}{2} \| u \|^2_V \leq \langle Au, u \rangle \leq \| u \|^2_V, \forall u \in V, \] (8)

\[ |\langle Au, v \rangle| \leq c \| u \|_V \| v \|_V, \forall u, v \in V, \] (9)
Lemma 2.3. \( u \) holds in the sense of dynamical systems in the following (see [8, 10, 11, 12, 27]). Denote by \( P \) back exponential attractor and the global pullback attractor for non-autonomous every process on \( X \).

Definition 2.4. It is said that a biparametric family of maps \( \{U(t, \tau)\}_{t \geq \tau} \) is a process on \( X \), which satisfies the following:

- \( U(t; \tau) : X \to X, \ t \geq \tau; \)
- \( U(\tau; \tau) = \text{identity}; \)
- \( U(t; \tau) = U(t; s)U(s; \tau), \ t \geq s \geq \tau. \)

Moreover, if for any \( t \geq \tau, U(t; \tau) \) is continuous on \( X \), then \( \{U(t, \tau)\}_{t \geq \tau} \) is a continuous process on \( X \).

Definition 2.5. A family of sets \( \hat{M} = \{\mathcal{M}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X) \) is called a pullback exponential attractor for the process \( \{U(t, \tau)\}_{t \geq \tau} \) on \( X \) if it has the following properties:

- Compactness: for any \( t \in \mathbb{R}, \mathcal{M}(t) \) is a nonempty compact subset of \( X \);
- Positive invariance: \( U(t, \tau)\mathcal{M}(\tau) \subset \mathcal{M}(t), \ \forall t \geq \tau; \)
- Pullback exponentially attracting: for any \( t \in \mathbb{R} \), bounded subset \( D \subset X \), some \( c > 0 \)
  \[
  \lim_{s \to -\infty} e^{cs}\text{dist}_X(U(t, t - s)D, \mathcal{M}(t)) = 0.
  \]
- Uniformly bounded fractal dimension:
  \[
  \sup_{t \in \mathbb{R}} \text{dim}(\mathcal{M}(t), X) := \sup_{t \in \mathbb{R}} \lim_{\epsilon \to 0} \frac{\log N_\epsilon(\mathcal{M}(t), X)}{\log \frac{1}{\epsilon}} < \infty,
  \]
where $\mathcal{N}_c(\mathcal{M}(t), X)$ is the minimal number of cubes with length $\epsilon$ in $X$ which are necessary to cover $\mathcal{M}(t)$.

**Definition 2.6.** A family of sets $\hat{A} = \{A(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is called a global pullback attractor for the process $\{U(t, \tau)\}_{t \geq \tau}$ on $X$ if it has the following properties:

- **Compactness:** for any $t \in \mathbb{R}$, $A(t)$ is a nonempty compact subset of $X$;
- **Invariance:** $U(t, \tau)A(\tau) = A(t)$, $\forall \tau \geq \tau$;
- **Pullback attracting:** for any $t \in \mathbb{R}$, bounded subset $D \subset X$
  \[ \lim_{\tau \to -\infty} \text{dist}_X (U(t, \tau)D, A(t)) = 0, \]
- **Minimality:** the family of sets $\hat{A}$ is minimal in the sense that if $\hat{O} = \{O(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is another family of closed sets such that
  \[ \lim_{\tau \to -\infty} \text{dist}_X (U(t, \tau)D, O(t)) = 0, \quad \text{for any bounded subset } D \subset X, \]
  then $A(t) \subset O(t)$ for $t \in \mathbb{R}$.

3. **A priori estimates.** In this section, we give several estimates for weak solutions of (6)-(7). Let $u$ be the weak solution of system (6)-(7) in Lemma 2.3, then

(i) $u$ satisfies the “energy equality”, for $u_0 \in H$,

\begin{equation}
\frac{1}{2} \frac{d}{dt} \|u\|^2 + 2\mu_1 \langle Au, u \rangle + \langle B(u), u \rangle + \langle N(u), u \rangle = \langle f(t), u \rangle. \tag{14}
\end{equation}

(ii) $u$ satisfies the “enstrophy equality”, for $u_0 \in H$,

\begin{equation}
\frac{1}{2} \frac{d}{dt} \|Au\|^2 + 2\mu_1 \|Au\|^2 + \langle B(u), Au \rangle + \langle N(u), Au \rangle = \langle f(t), Au \rangle. \tag{15}
\end{equation}

(iii) From Lemma 2.3, the maps defined by

\[ U(t, \tau) : u_0 \mapsto U(t, \tau)u_0 = u(t; \tau, u_0), \quad \tau \leq t, \quad u_0 \in H \text{ or } u_0 \in V, \tag{16} \]

generate a process $\{U(t, \tau)\}_{t \geq \tau}$ in $H$ and $V$, respectively. Moreover, we give a weaker assumption of $f(x, t)$.

**Assumption 3.1.** Assume $f \in L^2_{\text{loc}}(\mathbb{R}; H)$ and

\begin{equation}
M_f(t) := \sup_{\tau \leq t} \int_{t-1}^{t} \|f(\theta)\|^2 d\theta < \infty \quad \text{for all } t \in \mathbb{R}. \tag{17}
\end{equation}

Then we have the following estimates of solutions for (6)-(7).

**Lemma 3.1.** Suppose that Assumption 3.1 holds, $D \subset H$ is a bounded subset, then for any $t_0 \in \mathbb{R}, u_0 \in D, t \leq t_0$, the following inequalities hold,

\begin{equation}
\|U(t, \tau)u_0\|^2 \leq \rho_1(t_0), \quad \forall \tau \leq t - 2(c_1\mu_1)^{-1}\ln|D|; \tag{18}
\end{equation}

\begin{equation}
\int_{t-1}^{t} \|U(\theta, \tau)u_0\|^2 d\theta \leq \rho_2(t_0), \quad \forall \tau \leq t - 1 - 2(c_1\mu_1)^{-1}\ln|D|; \tag{19}
\end{equation}

\begin{equation}
\|U(t, \tau)u_0\|^2 \leq \rho_3(t_0), \quad \forall \tau \leq t - 1 - 2(c_1\mu_1)^{-1}\ln|D|; \tag{20}
\end{equation}

\begin{equation}
\int_{t-1}^{t} \|AU(\theta, \tau)u_0\|^2 d\theta \leq \rho_4(t_0), \quad \forall \tau \leq t - 2 - 2(c_1\mu_1)^{-1}\ln|D|, \tag{21}
\end{equation}

\[ \int_{t-1}^{t} \|B(\theta, \tau)u_0\|^2 d\theta \leq \rho_5(t_0), \quad \forall \tau \leq t - 2(c_1\mu_1)^{-1}\ln|D|, \tag{22}\]
where
\[\begin{align*}
\rho_1(t_0) &= 1 + cMf(t_0), \\
\rho_2(t_0) &= c(\rho_1(t_0) + Mf(t_0)), \\
\rho_3(t_0) &= c(\rho_1(t_0) + Mf(t_0)) \exp(\rho_1(t_0) + Mf(t_0)), \\
\rho_4(t_0) &= c(\rho_2(t_0) + \rho_3(t_0) + \rho_2(t_0)\rho_3(t_0) + Mf(t_0)),
\end{align*}\]
where \(|D| := \max(1, \sup_{v \in D} \|v\|)\).

**Proof.** Inserting (8), (11) and non-negativeness of the term \(\langle N(u), u \rangle\) into the energy equality (14), it follows
\[\frac{1}{2} \frac{d}{d\theta} \|u(\theta)\|^2 + 2c_1 \mu_1 \|u(\theta)\|_V^2 \leq \langle f(\theta), u(\theta) \rangle.\] (22)
Multiplying both sides of (22) with \(e^{c_1 \mu_1 \theta}\) and using the Cauchy inequality give
\[
\begin{align*}
\frac{d}{d\theta} \left( e^{c_1 \mu_1 \theta} \|u(\theta)\|^2 \right) + 4c_1 \mu_1 e^{c_1 \mu_1 \theta} \|u(\theta)\|_V^2 &\leq 2e^{c_1 \mu_1 \theta} \|f(\theta)\| \|u(\theta)\| + c_1 \mu_1 e^{c_1 \mu_1 \theta} \|u(\theta)\|^2 \\
&\leq 2c_1 \mu_1 e^{c_1 \mu_1 \theta} \|u(\theta)\|^2 + c e^{c_1 \mu_1 \theta} \|f(\theta)\|^2 \\
&\leq 2c_1 \mu_1 e^{c_1 \mu_1 \theta} \|u(\theta)\|^2 + c e^{c_1 \mu_1 \theta} \|f(\theta)\|^2.\end{align*}
\] (23)
Integrating (23) with respect to \(\theta\), we obtain
\[\|u(t)\|^2 \leq e^{-c_1 \mu_1 (t-\tau)} \|u_0\|^2 + ce^{-c_1 \mu_1 t} \int_\tau^t e^{c_1 \mu_1 \theta} \|f(\theta)\|^2 \, d\theta.\] (24)
It follows by (17) in Assumption 3.1,
\[
e^{-c_1 \mu_1 t} \int_\tau^t e^{c_1 \mu_1 \theta} \|f(\theta)\|^2 \, d\theta \\
\leq e^{-c_1 \mu_1 t} \int_{-\infty}^{t} e^{c_1 \mu_1 \theta} \|f(\theta)\|^2 \, d\theta = e^{-c_1 \mu_1 t} \sum_{n=0}^{\infty} \int_{t-(n+1)}^{t-n} e^{c_1 \mu_1 \theta} \|f(\theta)\|^2 \, d\theta \\
\leq e^{-c_1 \mu_1 t} \sum_{n=0}^{\infty} e^{c_1 \mu_1 (t-n)} \int_{t-(n+1)}^{t-n} \|f(\theta)\|^2 \, d\theta \leq c M_f(t)
\]
and (24) that for all \(t \geq \tau\),
\[\|u(t)\|^2 \leq e^{-c_1 \mu_1 (t-\tau)} \|u_0\|^2 + c M_f(t).\] (25)
From (25), we can easily deduce that
\[\|u(t)\|^2 \leq \rho_1(t_0)\]
for all \(t_0, \tau \leq t - 2(c_1 \mu_1)^{-1} \ln|D|, u_0 \in D\), where \(\rho_1(t_0)\) is given in Lemma 3.1.

By (22) and the Cauchy inequality, we get
\[
\frac{d}{d\theta} \|u(\theta)\|^2 + 4c_1 \mu_1 \|u(\theta)\|_V^2 \leq 2\langle f(\theta), u(\theta) \rangle \leq 2 \|f(\theta)\| \|u(\theta)\|_V \\
\leq c_1 \mu_1 \|u(\theta)\|_V^2 + c \|f(\theta)\|^2.
\] (26)
Integrating (26) with respect to $\theta$, we obtain for $t - 1 > \tau$, 
\[
3c_1\mu_1 \int_{t-1}^{t} \|u(\theta)\|^2 d\theta \leq \|u(t-1)\|^2 + c \int_{t-1}^{t} \|f(\theta)\|^2 d\theta.
\] (27)

Therefore, by (17) in Assumption 3.1 and (18), it follows
\[
\int_{t-1}^{t} \|u(\theta)\|^2 d\theta \leq \rho_2(t_0)
\]
for $t \leq t_0$, $\tau \leq t - 1 - 2(c_1\mu_1)^{-1}\ln|D|$, $u_0 \in D$, where $\rho_2(t_0)$ is given in Lemma 3.1.

For the enstrophy equality (15), by the Cauchy inequality, we obtain
\[
2|(f(\theta), A\mu(\theta))| \leq c\|f(\theta)\|^2 + 2\mu_1\|A\mu(\theta)\|^2.
\] (28)

Using the Hölder inequality, the Gagliardo-Nirenberg inequality and the Young inequality, we have
\[
\|\langle B(u(\theta)), A\mu(\theta) \rangle \| \leq \|u(\theta)\|_{L^4(\Omega)} \|\nabla u(\theta)\|_{L^4(\Omega)} \|A\mu(\theta)\|
\leq c\|u(\theta)\|^{1/2}\|\nabla u(\theta)\|^{1/2}\|u(\theta)\|^{1/4}\|\Delta u(\theta)\|^{3/4}\|A\mu(\theta)\|
\leq c\|u(\theta)\|^{3/4}\|u(\theta)\|^{1/2}\|\nabla u(\theta)\|^{3/4}\|A\mu(\theta)\|
\leq \frac{1}{4}\mu_1\|A\mu(\theta)\|^2 + c\|u(\theta)\|^4.
\] (29)

For the term $\langle N(u(\theta)), A\mu(\theta) \rangle$, since $u \in L^2(\tau + \epsilon, T; D(A))$, we have that
\[
\langle N(u(\theta)), A\mu(\theta) \rangle = -\int_{\Omega} \{\nabla \cdot (\mu(u(\theta))e(u(\theta)))\} \cdot A\mu(\theta)dx.
\] (30)

To estimate (30), we set
\[
F(S) = 2\mu_0(\epsilon + |S|^2)^{-\alpha/2}S,
\]
where
\[
S = \left(\begin{array}{cc}s_1 & s_2 \\
s_3 & s_4\end{array}\right) \in M_{2 \times 2}, \quad |S|^2 = \sum_{i=1}^{4} s_i^2, \quad s_i \in \mathbb{R}, \quad i = 1, 2, 3, 4,
\]
and $M_{2 \times 2}$ means the set of all the matrices of order $2 \times 2$. By some computations (see [37, 38]) we see that the first order and second order Fréchet derivatives of $F(S)$ satisfy
\[
\|DF(S)\| + \|D^2F(S)\| \leq c, \quad \forall S \in M_{2 \times 2},
\] (31)

where $c$ is a positive constant depending only on $\mu_0, \epsilon$ and $\alpha$. For any $S_1, S_2 \in M_{2 \times 2}$, we have
\[
F(S_2) - F(S_1) = \int_{0}^{1} DF(S_1 + \tau(S_2 - S_1))(S_2 - S_1) d\tau.
\]

Taking $S_1 = e(u) = (e_{ij}(u(\theta)))$, $S_2 = e(0) = (e_{ij}(0))$, applying the integration by parts first and (31) and Young inequality, we have
Applying the Gronwall inequality to (35), we have for all \( \tau < t \)

\[
\|N(u(\theta), Au(\theta))\| = \left| - \int_{\Omega} \{
\nabla \cdot [F(e(u(\theta))) - F(e(0))]\} \cdot Au(\theta) dx \right|
\leq c(\|\nabla u(\theta)\| + \|\Delta u(\theta)\|)\|Au(\theta)\|
\leq c(\|u(\theta)\|_V + \|u(\theta)\|_V)\|Au(\theta)\|
\leq \frac{\mu_1}{4}\|Au(\theta)\|^2 + c\|u(\theta)\|_V^2.
\]

(32)

Inserting (28), (29) and (32) into the enstrophy equality (15) and by (8), we obtain

\[
\frac{d}{d\theta}(Au(\theta), u(\theta)) + \mu_1\|Au(\theta)\|^2
\leq c\|f(\theta)\|^2 + c\|u(\theta)\|_V^2 + c\|u(\theta)\|_V^2\langle Au(\theta), u(\theta)\rangle.
\]

(33)

Setting

\[
H(\theta) = \langle Au(\theta), u(\theta)\rangle, \quad K(\theta) = c\|u(\theta)\|_V^2, \quad I(\theta) = c\|f(\theta)\|^2 + c\|u(\theta)\|_V^2.
\]

Then (34) gives

\[
\frac{d}{d\theta}H(\theta) \leq K(\theta)H(\theta) + I(\theta).
\]

(35)

Applying the Gronwall inequality to (35), we have for all \( \tau < t - 1 \leq s \leq t \) that

\[
H(t) \leq (H(s) + \int_{t-1}^{t} I(\theta)d\theta) \exp(\int_{t-1}^{t} K(\theta)d\theta)
= \langle Au(s), u(s)\rangle + \int_{t-1}^{t} I(\theta)d\theta \exp(\int_{t-1}^{t} K(\theta)d\theta).
\]

(36)

Integrating (36) with respect to \( s \) between \( t - 1 \) and \( t \), we obtain

\[
H(t) \leq (\int_{t-1}^{t} \langle Au(s), u(s)\rangle ds + \int_{t-1}^{t} I(\theta)d\theta) \exp(\int_{t-1}^{t} K(\theta)d\theta).
\]

(37)

It then follows from (8) and (27) that

\[
\int_{t-1}^{t} \langle Au(s), u(s)\rangle ds + \int_{t-1}^{t} I(\theta)d\theta \leq c \int_{t-1}^{t} \|u(\theta)\|_V^2 d\theta + c \int_{t-1}^{t} \|f(\theta)\|^2 d\theta
\leq c(\|u(t - 1)\|^2 + \int_{t-1}^{t} \|f(\theta)\|^2 d\theta).
\]

(38)

Similarly, we have

\[
\int_{t-1}^{t} K(\theta)d\theta = \int_{t-1}^{t} c\|u(\theta)\|_V^2 d\theta \leq c(\|u(t - 1)\|^2 + \int_{t-1}^{t} \|f(\theta)\|^2 d\theta).
\]

(39)
By (8) and (37)- (39), we obtain
\[
\|u(t)\|^2_V \leq \frac{1}{c_1} H(t) \leq c(\|u(t-1)\|^2 + \int_{t-1}^{t} \|f(\theta)\|^2 d\theta)
\]
\[
\times \exp(c(\|u(t-1)\|^2 + \int_{t-1}^{t} \|f(\theta)\|^2 d\theta)).
\] (40)

Therefore, by (18) and (40), we get
\[
\|u(t)\|^2_V \leq \rho_3(t_0)
\]
for \( t \leq t_0, \tau \leq t - 2(c_1\mu_1)^{-1} \ln |D|, u_0 \in D, \) where \( \rho_3(t_0) \) is given in Lemma 3.1.

Now, integrating (33) with respect to \( \theta \), then by (8), we have
\[
\int_{t-1}^{t} \|Au(\theta)\|^2 d\theta \leq c\|u(t-1)\|^2_\nu + c \int_{t-1}^{t} \|f(\theta)\|^2 d\theta
\]
\[
+ c(1 + \sup_{\theta \in [t-1,t]} \|u(\theta)\|^2_\nu) \int_{t-1}^{t} \|u(\theta)\|^2_\nu d\theta.
\]

Therefore, by (19) and (20), we obtain
\[
\int_{t-1}^{t} \|Au(\theta)\|^2 d\theta \leq \rho_4(t_0)
\]
for all \( t \leq t_0, \tau \leq t - 2(c_1\mu_1)^{-1} \ln |D|, u_0 \in D, \) where \( \rho_4(t_0) \) is given in Lemma 3.1.

Moreover, we give the estimates for the difference of two solutions of (6)-(7) for different initial data.

**Lemma 3.2.** Let \( f \in L^2_{\text{loc}}(\mathbb{R}; H) \) and two initial data \( u_{0_i} \in V, i = 1, 2, \) and denote by \( u_i(t) = U(t, \tau)u_{0_i}, i = 1, 2, \) the corresponding solutions of (6)-(7). Then, for all \( \tau \leq t, \) it holds
\[
\|U(t, \tau)u_{0_1} - u_{0_1}\|^2 \leq c \left( \|u_{0_1}\|^2_\nu(t - \tau) + \|u_{0_1}\|^2_\nu \|u_{0_1}\|^2(t - \tau) + \int_{\tau}^{t} \|f(\theta)\|^2 d\theta \right)
\]
\[
\times \exp\left( c\|u_{0_1}\|^2_\nu(t - \tau) \right).
\] (41)

For the difference of two solutions of (6)-(7), we have the following estimates
\[
\|U(t, \tau)u_{0_1} - U(t, \tau)u_{0_2}\|^2
\]
\[
\leq \|u_{0_1} - u_{0_2}\|^2 \exp\left( c(\|u_{0_1}\|^2 + \int_{\tau}^{t} \|f(\theta)\|^2 d\theta) \right),
\] (42)
\[
\|U(t, \tau)u_{0_1} - U(t, \tau)u_{0_2}\|^2_\nu
\]
\[
\leq c\|u_{0_1} - u_{0_2}\|^2_\nu \exp\left( c(t - \tau) + c \sum_{i=1}^{2} \|u_{0_i}\|^2 + \int_{\tau}^{t} \|f(\theta)\|^2 d\theta \right).
\] (43)
By the Cauchy inequality and (9), we have

\[
(t - \tau)\|U(t, \tau)u_{01} - U(t, \tau)u_{02}\|_V^2
\leq c\|u_{01} - u_{02}\|^2 \left(1 + \|u_{01}\|^2 + \int \frac{\|f(\theta)\|^2 d\theta}{\tau} \exp(c\|u_{01}\|^2 + \int \frac{\|f(\theta)\|^2 d\theta}{\tau})\right)
\times \exp \left(c(t - \tau) + \sum_{i=1}^2 (\|u_{01}\|^2 + \int \frac{\|f(\theta)\|^2 d\theta}{\tau})\right).
\] 

(44)

**Proof.** Firstly, we prove (41) holds and set \(w(t) := u(t) - u_0\). From (6), it holds in the sense of \(D'(\tau, T; V')\),

\[
\frac{\partial(w(\theta))}{\partial \theta} + 2\mu_1 A(w(\theta)) + B(u(\theta)) + N(u(\theta)) = f(\theta) - 2\mu_1 Au_0.
\]

(45)

Taking inner product with \(w(\theta)\) in (45) and by (8), we obtain

\[
\frac{1}{2} \frac{d}{d\theta}\|w(\theta)\|^2 + 2c_1\|w(\theta)\|^2_V
\leq (f(\theta), w(\theta)) - 2\mu_1 \langle Au_0, w(\theta) \rangle - \langle B(u(\theta)), w(\theta) \rangle - \langle N(u(\theta)), w(\theta) \rangle.
\]

(46)

By the Cauchy inequality and (9), we have

\[
|\langle f(\theta), w \rangle| \leq \|f(\theta)\| \|w\|_V \leq c\|f(\theta)\|^2 + \frac{1}{4} c_1\mu_1 \|w(\theta)\|^2_V,
\]

(47)

\[
|\mu_1 \langle Au_0, w(\theta) \rangle| \leq c\mu_1 \|u_0\|_V \|w(\theta)\|_V \leq c\|u_0\|^2 + \frac{1}{4} c_1\mu_1 \|w(\theta)\|^2_V.
\]

(48)

By (11), we deduce

\[
\langle B(u(\theta)), w(\theta) \rangle = b(u(\theta), u(\theta), u(\theta) - u_0).
\]

(49)

By (10) and the Cauchy inequality, it follows

\[
|b(w(\theta), u_0, w(\theta))| \leq c\|w\|_V \|w\|_V \|u_0\|_V \leq c\|u_0\|_V^2 + \frac{c_1\mu_1}{4} \|w\|^2_V,
\]

(50)

\[
|b(u_0, \theta, w(\theta))| \leq c\|u_0\|_V \|w\|_V \|w\|_V \leq c\|u_0\|_V^2 + \frac{c_1\mu_1}{4} \|w(\theta)\|^2_V.
\]

(51)

By (12), we get \(\langle N(u) - N(u_0), w(\theta) \rangle \geq 0\), from which one deduces

\[
\langle N(u(\theta)), w(\theta) \rangle \geq \langle N(u_0), w(\theta) \rangle,
\]

(52)

which implies from (12) and the Cauchy inequality that

\[
-\langle N(u(\theta)), w(\theta) \rangle \leq -\langle N(u_0), w(\theta) \rangle \leq |\langle N(u_0), w(\theta) \rangle|
\leq c\|u_0\|_V \|w(\theta)\|_V \leq c\|u_0\|^2 + \frac{1}{2} c_1\mu_1 \|w(\theta)\|^2_V.
\]

(53)

It follows from (46)-(53) that

\[
\frac{d}{d\theta}\|w(\theta)\|^2 \leq c\|u_0\|^2 \|w(\theta)\|^2 + c\|u_0\|^2 \|w(\theta)\|^2 + c\|u_0\|^2 \|w(\theta)\|^2 + c\|f(\theta)\|^2.
\]
which implies, by the Gronwall inequality, that
\[ \|w(t)\|^2 \leq c \left( \|u_0\|^2_{V'} (t - \tau) + \|u_0\|^2 \|u_0\|^2 (t - \tau) + \int_\tau^t \|f(\theta)\|^2 d\theta \right) \]
\[ \times \exp \left( c\|u_0\|^2_{V'} (t - \tau) \right) \]
which implies (41) holds.

Then we prove (42), (43) and (44) hold and set \( w(t) := u_1(t) - u_2(t) \). From (6), \( w(\theta) \) satisfies the following equation in the sense of \( \mathcal{D}'(\tau; V') \),
\[ \frac{\partial w(\theta)}{\partial \theta} + 2\mu_1 A w + B(u_1) - B(u_2) + N(u_1) - N(u_2) = 0. \]  
(54)

Taking inner product in (54) with \( w(\theta) \) gives
\[ \int_\tau^t \frac{d}{d\theta} \|w(\theta)\|^2 + 2\mu_1 \langle A w(\theta), w(\theta) \rangle + \langle B(u_1(\theta)) - B(u_2(\theta)), w(\theta) \rangle + \langle N(u_1(\theta)) - N(u_2(\theta)), w(\theta) \rangle = 0. \]
(55)

By (11), (10) and the Cauchy inequality, we deduce
\[ \|B(u_1(\theta)) - B(u_2(\theta), w(\theta))\| = |b(w, u_1, w) - b(w, u_2, w)| \]
\[ = |b(w, u_1, w) + b(u_2, u_1, w) - b(u_2, u_2, w)| \]
\[ = |b(w(\theta), u_1(\theta), w(\theta))| \]
\[ + \leq c\|w(\theta)\|\|w(\theta)\|_{V'} \|u_1(\theta)\|_{V'} \]
\[ \leq c_1 \mu_1 \|w(\theta)\|^2_{V'} + c\|w(\theta)\|^2 \|u_1(\theta)\|^2_{V'}. \]
(56)

By (12), we deduce
\[ \langle N(u_1(\theta)) - N(u_2(\theta)), w(\theta) \rangle \geq 0. \]
(57)

It then follows from (55)-(57) and (8), we have
\[ \frac{d}{d\theta} \|w(\theta)\|^2 + 2\mu_1 \|w(\theta)\|^2_{V'} \leq c\|w(\theta)\|^2 \|u_1(\theta)\|^2_{V'}, \]
(58)

which implies, by the Gronwall inequality, that
\[ \|w(t)\|^2 \leq \|w(\tau)\|^2 \exp \left( c \int_\tau^t \|u_1(\theta)\|^2_{V'} d\theta \right) \]
(59)

From (59) and (27), we obtain
\[ \|w(t)\|^2 \leq \|u_0_1 - u_0_2\|^2 \exp \left( c\|u_0_1\|^2 + \int_\tau^t \|f(\theta)\|^2 d\theta \right) \]

which implies (42) holds.

Taking inner product in (54) with \( A w(\theta) \) gives
\[ \frac{1}{2} \frac{d}{d\theta} \langle A w(\theta), w(\theta) \rangle + 2\mu_1 \|A w(\theta)\|^2 + \langle B(u_1(\theta)) - B(u_2(\theta)), A w(\theta) \rangle \]
\[ = -\langle N(u_1(\theta)) - N(u_2(\theta)), A w(\theta) \rangle. \]
(60)

By (11), the Hölder inequality, the embedding \( H^2(\Omega) \hookrightarrow L^\infty(\Omega) \) and the Cauchy inequality, we get
\[
|\langle B(u_1(\theta)) - B(u_2(\theta)), Aw(\theta)\rangle|
\leq |b(u_1(\theta), u_1(\theta), Aw(\theta))| - b(u_2(\theta), u_2(\theta), Aw(\theta))|
= |b(u_1(\theta) - u_2(\theta), u_1(\theta), Aw(\theta))| + b(u_2(\theta), u_1(\theta), Aw(\theta))| - b(u_2, u_2, Aw)|
= |b(w(\theta), u_1(\theta), Aw(\theta))| + b(u_2(\theta), w(\theta), Aw(\theta))|
\leq c\|w(\theta)\|_{L^\infty(\Omega)}\|\nabla u_1(\theta)\|_{V}^2 + c\|v_2(\theta)\|_{L^\infty(\Omega)}\|\nabla w(\theta)\|_{V}^2
\leq c\|w(\theta)\|_{V}^2 + \mu_1\|Aw(\theta)\|^2.
\] (61)

Similar to the derivations of (32), we take $S_1 = \phi(u_1(\theta))$, $S_2 = \phi(u_2(\theta))$, we have
\[
|\langle N(u_1(\theta)) - N(u_2(\theta)), Aw(\theta)\rangle|
\leq c\|\nabla (u_1 - u_2)\| + \|\Delta (u_1 - u_2)\|_{V}^2
\leq c\|w(\theta)\|V\|Aw(\theta)\| \leq c\|w(\theta)\|V^2 + \frac{\mu_1\|Aw(\theta)\|^2}{2}.
\] (62)

It follows from (60)-(62) and (8) that
\[
\frac{d}{d\theta} (Aw(\theta), w(\theta)) \leq c(1 + \|u_1(\theta)\|_V^2 + \|u_2(\theta)\|_V^2)\|w(\theta)\|_V^2
\leq c(1 + \|u_1(\theta)\|_V^2 + \|u_2(\theta)\|_V^2) (Aw(\theta), w(\theta)),
\]
which implies, by the Gronwall inequality and (27), that
\[
\langle Aw(t), w(t)\rangle
\leq \langle Aw(\tau), w(\tau)\rangle \exp \left( \int_{\tau}^{t} c(1 + \|u_1(\theta)\|_V^2 + \|u_2(\theta)\|_V^2) d\theta \right)
\leq \|u_{01} - u_{02}\|_V^2 \exp \left( c(t - \tau) + c \sum_{i=1}^{2} \|u_{0i}\|^2 + \int_{\tau}^{t} \|f(\theta)\|^2 d\theta \right)
\]
which implies (43) from (8).

Without loss of generality, we let $\tau = 0$. Taking inner product in (54) with $\theta Aw(\theta)$, we obtain
\[
\langle \frac{\partial w}{\partial \theta}, \theta Aw \rangle + 2\mu_1 \theta \langle Aw, Aw \rangle + \langle B(u_1) - B(u_2), \theta Aw \rangle + \langle N(u_1) - N(u_2), \theta Aw \rangle
= \frac{1}{2} \frac{d}{d\theta} \|\theta (Aw, w)\|^2 - \frac{1}{2} \langle Aw, w \rangle + 2\mu_1 \theta \|Aw\|^2 + \theta (B(u_1) - B(u_2), Aw)
+ \theta (N(u_1) - N(u_2), Aw) = 0.
\] (63)

Similar to the estimate (61), we obtain
\[
\theta |B(u_1) - B(u_2), Aw| \leq c\theta (\|u_1\|_V^2 + \|u_2\|_V^2)\|w\|_V^2 + \frac{1}{2} \mu_1 \theta \|Aw\|^2.
\] (64)

Similar also to the estimate (62), we obtain
\[
\theta |N(u_1) - N(u_2), Aw| \leq c\theta \|w\|_V \|Aw\| \leq c\theta \|w\|_V^2 + \frac{1}{2} \mu_1 \theta \|Aw\|^2.
\] (65)
It follows from (63)-(65) and (8) that
\[
\frac{d}{d\theta} \langle Aw, w \rangle \leq \langle Aw, w \rangle + c\theta(1 + \|u_1\|_V^2 + \|u_2\|_V^2)\|w\|_V^2
\]
which implies
\[
\frac{d}{d\theta} H(\theta) \leq K(\theta)H(\theta) + I(\theta),
\]
where
\[
H(\theta) = \theta \langle Aw(\theta), w(\theta) \rangle, \quad K(\theta) = c(1 + \|u_1(\theta)\|_V^2 + \|u_2(\theta)\|_V^2), \quad I(\theta) = \|w(\theta)\|_V^2.
\]
Applying the Gronwall inequality to (66), we get
\[
H(t) \leq (H(0) + \int_0^t I(\theta)d\theta) \exp(\int_0^t K(\theta)d\theta) = \int_0^t I(\theta)d\theta \exp(\int_0^t K(\theta)d\theta).
\]
On one hand,
\[
\int_0^t I(\theta)d\theta = \int_0^t \|w(\theta)\|_V^2d\theta.
\]
Integrating (58) in the interval \([0, t]\), we obtain
\[
\int_0^t \|w(\theta)\|_V^2d\theta \leq c(\|u_1 - u_2\|^2 + \int_0^t \|w(\theta)\|_V^2\|u_1(\theta)\|_V^2d\theta).
\]
By (42) and (27), we deduce
\[
\int_0^t \|w(\theta)\|^2\|u_1(\theta)\|_V^2d\theta
\]
\[
\leq \|u_1 - u_2\|^2 \exp(c(\|u_0\|^2 + \int_0^t \|f(\theta)\|_V^2d\theta)) \int_0^t \|u_1(\theta)\|_V^2d\theta
\]
\[
\leq c(\|u_1 - u_2\|^2(\|u_0\|^2 + \int_0^t \|f(\theta)\|_V^2d\theta) \exp(c(\|u_0\|^2 + \int_0^t \|f(\theta)\|_V^2d\theta))).
\]
From (68)-(70), we obtain
\[
\int_0^t I(\theta)d\theta \leq c\|u_1 - u_2\|^2 \times
\]
\[
\left(1 + (\|u_1\|^2 + \int_0^t \|f(\theta)\|_V^2d\theta) \exp(c(\|u_0\|^2 + \int_0^t \|f(\theta)\|_V^2d\theta))\right).
\]
On the other hand, by (27), we get
\[
\int_0^t K(\theta)d\theta = c \int_0^t (1 + \|u_1(\theta)\|_V^2 + \|u_2(\theta)\|_V^2)d\theta
\]
\[
\leq c(t + \sum_{i=1}^{2} (\|u_{0i}\|^2 + \int_{0}^{t} \|f(\theta)\|^2 d\theta)).
\]

Therefore, inserting (71) and (72) into (67) and by (8), we get
\[
t\|w(t)\|^2 \leq \frac{1}{c_1} H(t)
\]
\[
\leq c\|u_{01} - u_{02}\|^2 \left(1 + (\|u_{01}\|^2 + \int_{0}^{t} \|f(\theta)\|^2 d\theta) \exp(c(\|u_{01}\|^2 + \int_{0}^{t} \|f(\theta)\|^2 d\theta))\right)
\]
\[
\times \exp\left(ct + c\sum_{i=1}^{2} (\|u_{0i}\|^2 + \int_{0}^{t} \|f(\theta)\|^2 d\theta)\right)
\]
which implies (44) holds. \(\square\)

From the inequalities (43) and (44) in Lemma 3.2, we immediately gain

**Lemma 3.3.** Assume \(f \in L^2_{\text{loc}}(\mathbb{R}; H), u_{01}, u_{02} \in V\). Then there exists a positive function \(L = L(r_1, r_2, r_3, r_4)\), depending on \(\Omega\), but not on \(f\), such that \(L \in C^\infty(\mathbb{R}^4)\), is increasing in each of the four variables \(r_i (i = 1, 2, 3, 4)\) and satisfies for all \(\tau \leq t\),
\[
\|U(t, \tau)u_{01} - U(t, \tau)u_{02}\|_V
\]
\[
\leq L(\|u_{01}\|, \|u_{02}\|, t - \tau, \int_{\tau}^{t} \|f(\theta)\|^2 d\theta)\|u_{01} - u_{02}\|_V.
\]
\[
\sqrt{t - \tau} \|U(t, \tau)u_{01} - U(t, \tau)u_{02}\|_V
\]
\[
\leq L(\|u_{01}\|, \|u_{02}\|, t - \tau, \int_{\tau}^{t} \|f(\theta)\|^2 d\theta)\|u_{01} - u_{02}\|. \quad (74)
\]

4. **Pullback dynamic behavior for the non-Newtonian fluid.** This section is to construct the pullback exponential attractor and the pullback attractor for the non-Newtonian fluid. Following the theoretical framework in [8, 27], it is natural to verify the following hypotheses.

**CP** (Existence of a continuous process) The solution operator \(U(t, \tau)\) of non-Newtonian fluid defined by
\[
U(t, \tau) : u_0 \mapsto U(t, \tau)u_0 = u(t; \tau, u_0), \quad \tau \leq t, \quad u_0 \in V,
\]
generate a continuous process \(\{U(t, \tau)\}_{t \geq \tau}\) in \(V\).

**PAS** (Existence of a fixed pullback absorbing set) For any \(U(t, \tau) \in U_{t_0}\), where
\[
U_{t_0} := \{U(t, \tau) : \tau \leq t \leq t_0\},
\]
there exists a bounded and closed set \(B \subset V\) satisfying for any bounded \(D \subset V\), there exists a time \(s_D > 0\) such that for all \(s > s_D\),
\[
U(t, t - s)D \subset B.
\]

**SP** (A smooth property) For any \(U(t, \tau) \in U_{t_0}\), there exist \(\tau_0 > 0, \sigma > 0, K > 0\) such that
\[
U(t, t - \tau_0)(\mathcal{O}_\sigma(B)) \subset B
\]
and
\[ \|U(t, t - \tau_0)v_1 - U(t, t - \tau_0)v_2\|_V \leq K\|v_1 - v_2\|, \forall v_1, v_2 \in \mathcal{O}_\sigma(B), \]
where \( \mathcal{O}_\sigma(B) = \{v \in V : \inf_{w \in B} \|v - w\|_V < \sigma\} \).

(H1) Past continuity w.r.t. time: For any \( U(t, \tau) \in U_{t_0} \) and \( t \leq t_0 \), there exist some positive constants \( C_0, \varepsilon_0 \) and \( \gamma \) such that \( \varepsilon_0 \leq \tau_0 \) for all \( \tau_0 \leq r \leq 2\tau_0 \), \( 0 \leq s \leq \varepsilon_0 \) and \( v \in \mathcal{O}_1(B) \),
\[ \|U(t, t - r)v - U(t - s, t - r - s)v\|_V \leq C_0|s|^{\gamma}. \]

(H2) Past continuity w.r.t. initial data: For any \( U(t, \tau) \in U_{t_0} \) and \( t \leq t_0 \), there exist positive constants \( C_B \) such that for all \( v, w \in B \), for any \( 0 \leq r \leq 2\tau_0 \),
\[ \|U(t, t - r)v - U(t, t - r)w\|_V \leq C_B\|v - w\|. \]

(H3) Past continuity w.r.t. final time: For any \( U(t, \tau) \in U_{t_0} \) and \( t \leq t_0 \), there exist positive constants \( C_0' \) and \( \gamma' \) such that for all \( \tau_0 \leq r \leq 2\tau_0 \), \( 0 \leq s \leq \varepsilon_0 \) and \( v \in B \),
\[ \|U(t, t - r)v - U(t - s, t - r)v\|_V \leq C_0'|s|^{\gamma'}. \]

(H4) Future continuity w.r.t. initial data: For any \( U(t, \tau) \in U_{t_0} \) and \( t > t_0 \) and \( D_1, D_2 \) bounded subsets of \( V \), there exists a positive constant \( L(t, D_1, D_2) \) such that
\[ \|U(t, t_0)v - U(t, t_0)w\|_V \leq L(t, D_1, D_2)\|v - w\|_V, \text{ for all } v \in D_1, w \in D_2. \]

Next, we will verify the above hypotheses step by step. Hypothesis (CP) is proved using following Lemma.

**Lemma 4.1.** Suppose that Assumption 3.1 holds, then the solution operator \( U(t, \tau) \) of non-Newtonian fluid defined by
\[ U(t, \tau) : u_0 \mapsto U(t, \tau)u_0 = u(t; \tau, u_0), \tau \leq t, u_0 \in V, \]
generate a continuous process \( \{U(t, \tau)\}_{t \geq \tau} \) in \( V \).

**Proof.** It is the immediate result of (16) and (43). \( \square \)

Hypothesis (PAS) is proved using the following Lemma.

**Lemma 4.2.** Suppose that Assumption 3.1 holds, then for any \( U(t, \tau) \in U_{t_0} \), there exists a bounded and closed set \( B \subset V \) satisfying for any bounded \( D \subset V \), there exists a time \( s_D > 0 \) such that for all \( s > s_D \),
\[ U(t, t - s)D \subset B. \]

**Proof.** It follows from (20) in Lemma 3.1 that
\[ B := \{v \in V : \|v\|_V \leq r_5^{1/2}(t_0)\} \]
and
\[ s_D := 1 + 2(c_1\mu_1)^{-1}\ln|D|. \]

Hypothesis (SP) is proved using the following Lemma.
Lemma 4.3. Suppose that Assumption 3.1 holds, then for any \( U(t, \tau) \in U_{t_0} \), there exist \( \tau_0 > 0 \), \( \sigma > 0 \), \( K > 0 \) such that
\[
U(t, t - \tau_0)(\mathcal{O}_\sigma(B)) \subset B
\]
and
\[
\|U(t, t - \tau_0)v_1 - U(t, t - \tau_0)v_2\|_V \leq K\|v_1 - v_2\|, \quad \forall v_1, v_2 \in \mathcal{O}_\sigma(B),
\]
where \( \mathcal{O}_\sigma(B) = \{ v \in V : \inf_{w \in B} \|v - w\|_V < \sigma \} \).

Proof. From (20) in Lemma 3.1 and (74) in Lemma 3.3, we take \( \sigma = 1 \),
\[
\tau_0 = 3 + 2(c_1 \mu_1)^{-1} \ln \left(1 + \rho_3^{1/2}(t_0)\right)
\]
and
\[
K = L \left(1 + \rho_3^{1/2}(t_0), 1 + \rho_3^{1/2}(t_0), \tau_0, [\tau_0 + 1]M_f(t_0)\right),
\]
such that the lemma is valid. \( \Box \)

Hypothesis (H2) is proved using the following Lemma.

Lemma 4.4. Suppose that Assumption 3.1 holds, then for any \( U(t, \tau) \in U_{t_0} \) and \( t \leq t_0 \), there exists some positive constant \( C_B \) such that for all \( v, w \in B \), for any \( 0 \leq r \leq 2 \tau_0 \),
\[
\|U(t, t - r)v - U(t, t - r)w\|_V \leq C_B\|v - w\|_V.
\]

Proof. By (73), for all \( t \in \mathbb{R} \), \( r \geq 0 \), \( v, w \in V \), we have
\[
\|U(t, t - r)v - U(t, t - r)w\|_V \leq L(\|v\|, \|w\|, r, \int_{t-r}^{t} \|f(\theta)\|^2 d\theta)\|v - w\|_V.
\]
In particular, for all \( r \in [0, 2 \tau_0] \), \( v, w \in B \), we gain by Lemma 4.2 and the fact that the function \( L \) is increasing in all its entries,
\[
\|U(t, t - r)v - U(t, t - r)w\|_V \leq L(\rho_3^{1/2}(t_0), \rho_3^{1/2}(t_0), 2 \tau_0, [2 \tau_0 + 1]M_f(t_0))\|v - w\|_V,
\]
which implies \( U(t, \tau) \) satisfies the Hypothesis (H2) with
\[
C_B = L(\rho_3^{1/2}(t_0), \rho_3^{1/2}(t_0), 2 \tau_0, [2 \tau_0 + 1]M_f(t_0)).
\]
\( \Box \)

Hypothesis (H4) is proved using the following Lemma.

Lemma 4.5. Suppose that Assumption 3.1 holds, then for any \( U(t, \tau) \in U_{t_0} \) and \( t > t_0 \) and \( D_1, D_2 \) bounded subsets of \( V \), there exists a positive constant \( L(t, D_1, D_2) \) such that
\[
\|U(t, t_0)v - U(t, t_0)w\|_V \leq L(t, D_1, D_2)\|v - w\|_V, \quad \text{for all } v \in D_1, w \in D_2.
\]

Proof. It is obtained immediately from (73) in Lemma 3.3. \( \Box \)

The remainder is to prove the Hypotheses (H1) and (H3). It is necessary to improve the assumption on \( f \) and get a higher regularity estimate of \( U(t, \tau) \) by the Giga-Sohr argument (see [19, 27]).
**Assumption 4.1.** \( f(t,x) \in L^2_{\text{loc}}(\mathbb{R}; H) \) and there exists some \( t_0 \in \mathbb{R} \), for any \( q > 2 \), such that

\[
M_{f,q}(t_0) := \sup_{r \leq t_0} \int_{r-1}^{r} \|f(s)\|^q ds < \infty.
\]

**Remark 1.** It is obvious that the Assumption 4.1 implies the Assumption 3.1.

**Lemma 4.6.** Assume that Assumption 4.1 holds, then for any \( U(t,\tau) \in U_{t_0} \), there exists a positive constant \( c_6 \) independent of \( t,\tau \) such that for any \( u_0 \in \mathcal{O}_1(B) \),

\[
\|U(t,\tau)u_0\|_{H^{\mu(q-1)/q}(\Omega)} \leq c_6,
\]

for all \( t \leq t_0 - 1, \tau \leq t - \tau_0 \).

**Proof.** By (20) in Lemma 3.1, for \( \tau \leq t - 3 - 2(c_1\mu_1)^{-1} \ln |\mathcal{O}_1(B)|, u_0 \in \mathcal{O}_1(B) \), we have

\[
\|u(\theta)\|^2_N \leq \rho_3(t_0), \quad \forall t - 2 \leq \theta \leq t.
\]

From the estimate (29), we deduce that

\[
\|B(u(t))\| \leq \|u(t)\|_{L^2(\Omega)} \|\nabla u(t)\|_{L^2(\Omega)} \leq c\|u(t)\|^2_N.
\]

From the estimate (32), we also obtain

\[
\|N(u(t))\| \leq \epsilon(\|u(t)\| + \|\Delta u(t)\|) \leq c\|u(t)\|_V.
\]

By (76), (77), (75) and Assumption 4.1, we deduce that there exists a positive constant \( c_3 \), independent of \( t,\tau \) and \( u_0 \), such that the function

\[
h(\theta) = -B(u(\theta)) - N(u(\theta)) + f(\theta)
\]

satisfies

\[
\int_{t-2}^{t} \|h(\theta)\|^q d\theta \leq c_3.
\]

Consider the function \( v(\theta) = (\theta - t + 2)u(\theta) \). It is clear that \( v \) satisfies

\[
\begin{align*}
\frac{dv}{d\theta}(\theta) + 2\mu_1 Av(\theta) &= (\theta - t + 2)h(\theta) + u(\theta), \quad \theta > t - 2, \\
v(t - 2) &= 0.
\end{align*}
\]

Applying the abstract theorem [19, Theorem 2.1] to the elliptic operator \( A = P\Delta^2 \) \( \text{in fact, the abstract theorem has been applied successfully to the Stokes operator } P\Delta \text{ in [19], we use the similar argument as [19] to verify the conditions of [19, Theorem 2.1] for the operator } A = P\Delta^2 \text{ by the classical spectral analysis [6, Theorem II.6.6] or [35, pp.56], here we omit it} \), we deduce that there exists a positive constant \( c_4 \), independent of \( t,\tau \) and \( u_0 \), such that

\[
\int_{t-2}^{t} \|\frac{dv}{d\theta}(\theta)\|^q + \|Av(\theta)\|^q d\theta \leq c_4 \int_{t-2}^{t} \|h(\theta)\|^q + \|u(\theta)\|^q d\theta.
\]

Since

\[
\int_{t-2}^{t} (\theta - t + 2)^\gamma \|\frac{dv}{d\theta}(\theta)\|^q d\theta \leq 2^\gamma \int_{t-2}^{t} \|\frac{dv}{d\theta}(\theta)\|^q + \|u(\theta)\|^q d\theta,
\]
by (79), we have
\[
\int_{t-2}^{t} (\theta - t + 2)^q \| \frac{d}{d\theta} u(\theta) \|^q + (\theta - t + 2)^q \| Au(\theta) \|^q d\theta
\leq 2^q (c_4 + 1) \int_{t-2}^{t} \| h(\theta) \|^q + \| u(\theta) \|^q d\theta. \tag{80}
\]

By (75), (78) and (80), we deduce that there exists a positive constant \( c_5 \), independent of \( t, \tau \) and \( u_0 \), such that
\[
\int_{t-1}^{t} \frac{d}{d\theta} u(\theta) \left\| + \| Au(\theta) \right\| d\theta \leq \int_{t-1}^{t} (\theta - t + 2)^q \frac{d}{d\theta} u(t) \|/q + (\theta - t + 2)^q \| Au(\theta) \|^2 d\theta \leq c_5. \tag{81}
\]

Then by (81) and the equivalence of \( \| Au \| \) and \( \| u \|_{H^4(\Omega)} \) when \( u \in D(A) \), we deduce
\[
\frac{d}{d\theta} u(\theta) \in L^q(t-1, t; H), \quad u(\theta) \in L^q(t-1, t; D(A)).
\]

Since \( 2 < 4(q - 1)/q < 4 \), then \( D(A) \) is embedded compactly in \( H^{4(q - 1)/q}(\Omega) \) and \( H^{4(q - 1)/q}(\Omega) \) is embedded compactly in \( H \). Therefore, by the embedding theorem [9, Theorem 1.4, pp.32], we get
\[
u(t \in L^q(t - 1, t; H^{4(q - 1)/q}(\Omega)),
\]

which implies that \( u(t - 1) \in H^{4(q - 1)/q}(\Omega) \). By (81), we deduce there exists a positive constant \( c_6 \), independent of \( t, \tau \) and \( u_0 \), such that
\[
\| u(t - 1) \|_{H^{4(q - 1)/q}(\Omega)} \leq c_6,
\]

which implies
\[
\| U(t, \tau) u_0 \|_{H^{4(q - 1)/q}(\Omega)} \leq c_6
\]
for all \( t \leq t_0 - 1, \tau \leq \tau_0 \). \( \square \)

Hypotheses (H1) and (H3) are proved using the following Lemma.

**Lemma 4.7.** Assume the Assumption 4.1 holds, then for any \( t \leq t_0 \), there exist some positive constants \( C_0, \varepsilon_0 \) and \( \gamma \) such that \( \varepsilon_0 \leq \tau_0 \) for all \( \tau_0 \leq r \leq 2\tau_0 \), \( 0 \leq s \leq \varepsilon_0 \) and \( v \in C_1(B), \)
\[
\| U(t, t - r) v - U(t - s, t - r - s) v \| V \leq C_0 |s|^{\gamma}.
\]

Moreover, there exist positive constants \( C'_0 \) and \( \gamma' \) such that for all \( \tau_0 \leq r \leq 2\tau_0 \), \( 0 \leq s \leq \varepsilon_0 \) and \( v \in B, \)
\[
\| U(t, t - r) v - U(t - s, t - r - s) v \| V \leq C'_0 |s|^{\gamma'}.
\]

**Proof.** One gets
\[
\| U(t, t - r) v - U(t - s, t - r - s) v \| \leq J_1 + J_2, \tag{82}
\]
where
\[
J_1 = \| U(t, t - r) v - U(t - s, t - r) v \|,
J_2 = \| U(t - s, t - r) v - U(t - s, t - r - s) v \|.
\]
Next, we give the estimates of $J_1$ and $J_2$. For any $s \geq 0, t - s \geq \tau$, we have

$$u(t) - u(t-s) = \int_{t-s}^{t} \frac{d}{d\theta} u(\theta) d\theta$$

$$= \int_{t-s}^{t} (-2\mu_1 Au(\theta) - B(u(\theta)) - N(u(\theta)) + f(\theta)) d\theta. \quad (83)$$

By (8), we have

$$c_1\|u\|_V^2 \leq (Au, u) = (Au, u) \leq \|Au\| \|u\| \leq \|Au\| \|u\|_V, \forall u \in D(A),$$

which implies

$$c_1\|u\|_V \leq \|Au\|, \quad \forall u \in D(A). \quad (84)$$

Inserting (76) and (77) into (83) and by (20), (84), we obtain

$$\|U(t, \tau)u_0 - U(t-s, \tau)u_0\| = \|u(t) - u(t-s)\| = \| \int_{t-s}^{t} \frac{d}{d\theta} u(\theta) d\theta \|$$

$$\leq c \int_{t-s}^{t} \|Au(\theta)\| d\theta + \int_{t-s}^{t} \|B(u(\theta))\| d\theta + \int_{t-s}^{t} \|N(\theta)\| d\theta + \int_{t-s}^{t} \|f(\theta)\| d\theta$$

$$\leq c \int_{t-s}^{t} \|Au(\theta)\| d\theta + c \int_{t-s}^{t} \|u(\theta)\|_V^2 d\theta + c \int_{t-s}^{t} \|u(\theta)\|_V d\theta + \int_{t-s}^{t} \|f(\theta)\| d\theta$$

$$\leq c(1 + \sqrt{p_3(t_0)}) \left( \int_{t-s}^{t} \|Au(\theta)\|^2 d\theta \right)^{1/2} + \left( \int_{t-s}^{t} \|f(\theta)\|^2 d\theta \right)^{1/2} s^{1/2},$$

which implies, by (17) and (21), that there exists a positive constant $C_1$, independent of $t, s, \tau$ and $u_0$, such that for all $0 \leq s \leq 1, \tau \leq t - 2 - 2(\mu_1 t_0)^{-1} \ln|O_1(B)|, u_0 \in O_1(B)$,

$$\|U(t, \tau)u_0 - U(t-s, \tau)u_0\| \leq C_1 s^{1/2}. \quad (85)$$

Then by (85), we deduce for all $0 \leq s \leq 1, r \geq \tau_0, v \in O_1(B)$,

$$J_1 = \|U(t, t-r)u - U(t-s, t-r)u\| \leq C_1 s^{1/2}. \quad (86)$$

By (42), for all $0 \leq s \leq 1, \tau_0 \leq r \leq 2\tau_0, v \in V$, we have

$$J_2 = \|U(t-s, t-r)v - U(t-s, t-s, r)v\|$$

$$= \|U(t-s, t-r)v - U(t-s, t-r)U(t-r, t-s, r)v\|$$

$$\leq \|v - U(t-r, t-r, s)v\| \exp(c(v)^2 + \int_{t-r}^{t-s} \|f(\theta)^2 d\theta\))$$

$$\leq \|v - U(t-r, t-r, s)v\| \exp(c(v)^2 + [2\tau_0 + 1]M_f(t_0))). \quad (87)$$
Now, by (41), we get
\[
\|v - U(t - r, t - r - s)v\|
\leq c \left( \|v\|_s^2 + \|v\|_q^2 + \|v\|_{r,t}^2 + \int_{t-r-s}^{t-r} \|f(\theta)\|^2 d\theta \right)^{1/2} \exp \left( c \|v\|_q^2 s \right). \tag{88}
\]
It is obvious that, for $0 \leq s \leq 1$ and $t \leq t_0$,
\[
\int_{t-r-s}^{t-r} \|f(\theta)\|^2 d\theta \leq \left( \int_{t-r-s}^{t-r} \|f(\theta)\|^q d\theta \right)^{2/q} \frac{1}{s^{(q-2)/q}} \leq \left( M_{f,q}(t_0) \right)^{2/q} \frac{1}{s^{(q-2)/q}}. \tag{89}
\]
Inserting (89) into (88) and using the fact $s^{1/2} \leq s^{(q-2)/(2q)}$ for all $0 \leq s \leq 1$, we obtain for all $0 \leq s \leq 1, \tau_0 \leq r \leq 2\tau_0, v \in V$,
\[
\|v - U(t - r, t - r - s)v\|
\leq c \left( \|v\|_q + \|v\|_q \|v\| + \left( M_{f,q}(t_0) \right)^{1/q} \right) \exp \left( c \|v\|_q^2 s \right) \frac{1}{s^{(q-2)/(2q)}}. \tag{90}
\]
Inserting (90) into (87) and taking into account (18) and (20), we obtain there exists a positive constant $C_2$ such that for $0 \leq s \leq 1, r \geq \tau_0, v \in \mathcal{O}_1(B)$,
\[
J_2 = \|U(t - s, t - r)v - U(t - s, t - s - r)v\| \leq C_2 s^{(q-2)/(2q)}. \tag{91}
\]
Inserting (86), (91) into (82), we deduce there exists a positive constant $C_3$ such that for all $0 \leq s \leq 1, \tau_0 \leq r \leq 2\tau_0, v \in \mathcal{O}_1(B)$,
\[
\|U(t - s, t - r)v - U(t - s, t - s - r)v\| \leq C_3 s^{(q-2)/(2q)}. \tag{92}
\]
To end the proof, we only need to change the norm $\| \cdot \|$ to the norm $\| \cdot \|_V$ at the left in inequalities (86) and (92). Thanks to the Lemma 4.6, we can do it by exploiting the Sobolev interpolation inequality. From the interpolation inequality
\[
\| v \|_V \leq C_4 \left( \| v \|_q^{p/(1+p)} \| v \|_{H^{2+2\gamma/3}(\Omega)}^{1/(1+p)} \right), \tag{93}
\]
Taking $p = 1 - \frac{2}{q}$ and using (92) and (93), we obtain for all $t \leq t_0 - 1, 0 \leq s \leq 1, \tau_0 \leq r \leq 2\tau_0, v \in \mathcal{O}_1(B)$,
\[
\|U(t - s, t - r)v - U(t - s, t - s - r)v\|_V \leq C_4 C_3^{(q-2)/(2q-2)} (2c_6)^{q/(2q-2)} s^{(q-2)/(4q(q-1))}
\]
which implies $U(t, \tau)$ satisfies the first part of Lemma 4.7 with
\[
\varepsilon_0 = 1,
\gamma = (q-2)^2/(4q(q-1)),
C_0 = C_4 C_3^{(q-2)/(2q-2)} (2c_6)^{q/(2q-2)}
\]
and $t_0 - 1$ instead of $t_0$. Analogously, from (86) and (93), we obtain for all $t \leq t_0 - 1, 0 \leq s \leq 1, \tau_0 \leq r \leq 2\tau_0, u_0 \in \mathcal{O}_1(B)$,
\[
\|U(t - s, t - r)v - U(t - s, t - r)v\|_V \leq C_4 C_4^{(q-2)/(2q-2)} (2c_6)^{q/(2q-2)} s^{(q-2)/(4q(q-1))}
\]
which implies $U(t, \tau)$ satisfies the second part of Lemma 4.7 with
\[
\gamma' = (q-2)/(4q(q-1)),
C'_0 = C_4 C_4^{(q-2)/(2q-2)} (2c_6)^{q/(2q-2)}
\]
Lemma 4.8. Assume Assumption 4.1 holds, $U_{\alpha}$ is a process on $V$, then the family
\[ \hat{\mathcal{M}} = \{ \mathcal{M}_U(t) : t \in \mathbb{R} \} \]
defined by
\[ \mathcal{M}_U(t) := \bigcup_{s \in [0,\tau_0]} U(t, t-s-\tau_0) \mathcal{M}_U(t-s-\tau_0) \text{ for all } t \leq t_0 - 1. \]
\[ \hat{\mathcal{M}}_U(t) := \begin{cases} \mathcal{M}_U(t), & t \leq t_0 - 1, \\ U(t, t_0 - 1) \mathcal{M}_U(t_0 - 1), & t > t_0 - 1. \end{cases} \]
satisfies:
(1) $U(t, \tau) \hat{\mathcal{M}}_U(\tau) \subset \hat{\mathcal{M}}_U(t)$ for all $\tau \leq t$.
(2) $\hat{\mathcal{M}}_{T_-U}(t) = \hat{\mathcal{M}}_U(t - \tau)$ for all $\tau \geq 0$ and any $t \leq t_0 - 1$, and
$\hat{\mathcal{M}}_{T_-U}(t) \subset \hat{\mathcal{M}}_U(t - \tau)$ for all $\tau \geq 0$ and any $t > t_0 - 1$.
where $T_-U(t, s) := U(t - \tau, s - \tau)$.
(3) For any $D \subset V$ bounded, $\hat{\mathcal{M}}$ satisfies for all $\tau \geq s_D$ and any $t \leq t_0 - 1$,
\[ \text{dist}_V(U(t, t - \tau)D, \hat{\mathcal{M}}_U(t)) \leq \hat{C}e^{\tilde{\alpha}sD}e^{-\tilde{\alpha}\tau} \]
And for all $\tau \geq s_D + t - t_0 + 1$ and any $t > t_0 - 1$,
\[ \text{dist}_V(U(t, t - \tau)D, \hat{\mathcal{M}}_U(t)) \leq L(t, B, \hat{\mathcal{M}}_U(t_0 - 1)) \hat{C}_1e^{\tilde{\alpha}(s_D + t - t_0 + 1)}e^{-\tilde{\alpha}\tau} \]
The constants $\hat{C}, \hat{C}_1, \tilde{\alpha}$ are positive constants only depending on $\Omega, \mu_0, \mu_1$ and $M_f(t_0)$.
(4) $\hat{\mathcal{M}}_U(t)$ is a compact subset of $V$, with finite fractal dimension, for all $t \in \mathbb{R}$, and more precisely,
\[ \begin{cases} \text{dim}(\hat{\mathcal{M}}_U(t), V) \leq \hat{C}, & \text{for any } t \leq t_0 - 1, \\ \text{dim}(\hat{\mathcal{M}}_U(t), V) \leq \frac{\hat{C}}{L(t, \hat{\mathcal{M}}_U(t_0 - 1), \hat{\mathcal{M}}_U(t_0 - 1))}, & \text{for any } t > t_0 - 1. \end{cases} \]
The constant $\hat{C}$ only depends on $\Omega, \mu_0, \mu_1, M_f(t_0)$ and $M_{f,q}(t_0), q$.

Now let us give our main results as follows.

Theorem 4.9. Suppose Assumption 4.1 holds, then the process \{\text{U}(t, \tau)\}_{t \geq \tau} has a pullback exponential attractor $\hat{\mathcal{M}} = \{ \hat{\mathcal{M}}_U(t) : t \in \mathbb{R} \}$.

Proof. According to the definition 2.5, the Theorem 4.9 follows directly from Lemma 4.8.

Theorem 4.10. Suppose Assumption 4.1 holds, then the process \{\text{U}(t, \tau)\}_{t \geq \tau} has a global pullback attractor $\hat{A} = \{ \hat{A}_U(t) : t \in \mathbb{R} \}$ with finite fractal dimension. Furthermore,
\[ \hat{A}_U(t) \subset \hat{\mathcal{M}}_U(t), \text{ for all } t \in \mathbb{R}. \]

Proof. The existence of global pullback attractor
\[ \hat{A}_U(t) := \bigcup_{D \subset V \text{ bounded}} \Lambda(D, t), \quad \Lambda(D, t) := \bigcap_{s \leq t} \left( \bigcup_{\tau \leq s} U(t, \tau)D \right), \]
for the process \( \{U(t, \tau)\}_{t \geq \tau} \) is based on the definitions 2.5, 2.6 and Theorem 4.9. Further, by comparing definition 2.5 and definition 2.6, one has

\[ \mathcal{A}_U(t) \subset \mathcal{M}_U(t) \quad \text{for all } t \in \mathbb{R}. \]

In particular, as a consequence of Theorem 4.9, it holds that

\[
\begin{cases}
\dim(\mathcal{A}_U(t), V) \leq \bar{C}, & \text{for any } t \leq t_0 - 1, \\
\dim(\mathcal{A}_U(t), V) \leq \frac{\bar{C}}{L(t, \mathcal{M}_U(t_0 - 1), \mathcal{M}_U(t_0 - 1))}, & \text{for any } t > t_0 - 1.
\end{cases}
\]

Therefore, Theorem 4.10 is completed.

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