Parallel Batch-Dynamic $k$-Core Decomposition

Julian Shun (MIT CSAIL)

Joint work with Quanquan Liu, Jessica Shi, Shangdi Yu, and Laxman Dhulipala
Project Presentations

- Final project presentations in class on 5/14
  - Problem and motivation
  - Prior work
  - Your technical contributions
  - Challenges encountered
  - Experimental results
- 5 minutes per person (10 minutes for groups of 2)
- ~1 minute Q&A

- Please send your slides to ishun@mit.edu by 9am on 5/14
- We will play all slides from the same computer to minimize transition time
- Project report due 5/14
Graphs are rapidly changing (500M tweets/day, 547K new websites/day)

Graphs are becoming very large

Size

3.5 billion vertices
128 billion edges

Largest publicly available graph

272 billion vertices
5.9 trillion edges

Proprietary graph

> 100 billion vertices
6 trillion edges

Proprietary graph
Parallelism and Dynamic Algorithms for High Performance

• Take advantage of parallel machines

• Design dynamic algorithms to avoid unnecessary work on updates
Parallel Batch-Dynamic Algorithms

- Process updates in batches, and use parallelism within each batch

A batch of edge insertions/deletions

Current graph + Current statistics

Updated graph + Updated statistics

- pink: Insertion
- gray: Deletion
Our Parallel Batch-Dynamic Algorithms

- \( k \)-core decomposition
- Clique counting
- Low out-degree orientation
- Maximal matching
- Graph coloring
- Minimum spanning forest
- Single-linkage clustering
- Closest pair

Theory

- \( O(n \log n) \)
- \( O(n) \)
- \( O(\log n) \)

Practice

Quanquan C. Liu, Jessica Shi, Shangdi Yu, Laxman Dhulipala, Julian Shun, “Parallel Batch-Dynamic Algorithms for \( k \)-Core Decomposition and Related Graph Problems,” SPAA 2022
Related Work on Parallel Batch-Dynamic Algorithms

- Triangle Counting [EB10, MBG17]
- Euler Tour Trees [TDB19]
- Connected Components [FL94, MGB13, AABD19]
- Rake-Compress Trees [AABDW20]
- Incremental Minimum Spanning Trees [ABT20]
$k$-Core Decomposition
**k-Core Decomposition**

*k*-core: maximal connected subgraph of G such that all vertices have induced degree \( \geq k \)

Coreness(\( v \)): largest value of \( k \) where \( v \) participates in the \( k \)-core

Coreness(\( v \)) = 3

Goal: compute coreness for all vertices
Approximate $k$-Core Decomposition

$k$-core: maximal connected subgraph of $G$ such that all vertices have induced degree $\geq k$

c-Approx-Coreness($v$): value within multiplicative $c$ factor of Coreness($v$)
Applications of $k$-core Decomposition

- Graph clustering
- Community detection
- Graph visualization
- Protein network analysis
- Approximating network centrality
Work-Span Model

- **Work** = number of operations
- **Span** = length of longest sequential dependence

\[ \text{Running time} \leq \frac{\text{Work}}{\text{#processors}} + O(\text{Span}) \]

- Goal: Design **low-span** parallel algorithms that are **work-efficient** (work asymptotically matches that of the best sequential algorithm)
Our Results for $k$-core Decomposition

- Our algorithm dynamically maintains a $(2 + \epsilon)$-approximation for coreness of every vertex.

- A batch of $B$ updates takes $O(B \log^2 n)$ amortized work and $O(\log^2 n \log \log n)$ span with high probability.

- Our algorithm is work-efficient, matching the work of the state-of-the-art sequential algorithm by Sun et al.

- Our algorithm is based on a parallel level data structure.
Sequential Level Data Structures for Dynamic Problems

- Maximal Matching [Baswana-Gupta-Sen ‘18, Solomon ‘16]
- \((\Delta + 1)\)-Coloring [Bhattacharya-Chakrabarty-Henzinger-Nanongkai ‘18, Bhattacharya-Grandoni-Kulkarni-Liu-Solomon ‘19]
- Clustering [Wulff-Nilsen ‘12]
- Low out-degree orientation [Solomon-Wein ‘20, Henzinger-Neumann-Weiss ‘20]
- Densest subgraph [Bhattacharya-Henzinger-Nanongkai-Tsourakakis ‘15]
Sequential Level Data Structure (LDS)

- Described by Bhattacharya, Henzinger, Nanongkai, Tsourakakis [2015] and Henzinger, Neumann, Wiese [2020]

- Vertices partitioned into levels

- Maintain invariants per vertex, which give upper/lower bounds on roughly its number of “up-neighbors” (neighbors at around its level and above)

- We prove that levels translate to coreness estimates
Sequential Level Data Structure (LDS)

Vertices partitioned into levels

$O(\log^2 n)$

$\# \text{up-neighbors: } > 2.1(1 + \delta)^i$

= edge insertion
Sequential Level Data Structure (LDS)

Vertices partitioned into levels

$O(\log^2 n)$

# up-neighbors: $> 2.1(1 + \delta)^i$
Sequential Level Data Structure (LDS)

Vertices partitioned into levels

$O(\log^2 n)$

# up-neighbors: $< (1 + \delta)^i$

= edge deletion
Sequential Level Data Structure (LDS)

Vertices partitioned into levels

$O(\log^2 n)$

# up-neighbors: $< (1 + \delta)^i$
Difficulties with Parallelization

Large sequential dependencies

Large span
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Difficulties with Parallelization

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Large span
Difficulties with Parallelization

- Large sequential dependencies
- Only processes one update at a time
- Large span
Our Parallel Batch-Dynamic Level Data Structure (PLDS)
Our Parallel Batch-Dynamic Level Data Structure (PLDS)

- Vertices only need to move down, and never up

Only the lower bound invariant is ever violated.
Our Parallel Batch-Dynamic Level Data Structure (PLDS)

For vertices incident to updated edges, calculate desire-level (dl): closest level that satisfies invariants.

Only the lower bound invariant is ever violated.
Our Parallel Batch-Dynamic Level Data Structure (PLDS)

Deletions

For vertices incident to updated edges, calculate desire-level \((dl)\): closest level that satisfies invariants.

Iterate from bottommost level to top level and move vertices to desire-level.

Only the lower bound invariant is ever violated.
Our Parallel Batch-Dynamic Level Data Structure (PLDS)

For vertices incident to updated edges, calculate desire-level (dl): closest level that satisfies invariants

To achieve parallelism (low span), we need to move all vertices together for each desire-level

Only the lower bound invariant is ever violated.

Iterate from bottommost level to top level and move vertices to desire-level
Our Parallel Batch-Dynamic Level Data Structure (PLDS)

For vertices incident to updated edges, calculate *desire-level* \((dl)\): closest level that satisfies invariants.

Deletions

Only the lower bound invariant is ever violated.

Iterate from bottommost level to top level and move vertices to desire-level.
Our Parallel Batch-Dynamic Level Data Structure (PLDS)

For vertices incident to updated edges, calculate desire-level (dl): closest level that satisfies invariants

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Only the lower bound invariant is ever violated.

Iterate from bottommost level to top level and move vertices to desire-level.
For vertices incident to updated edges, calculate desire-level \((dl)\): closest level that satisfies invariants.

Our Parallel Batch-Dynamic Level Data Structure (PLDS)

Deletions

Iterate from bottommost level to top level and move vertices to desire-level.

Only the lower bound invariant is ever violated.
Our Parallel Batch-Dynamic Level Data Structure (PLDS)

For vertices incident to updated edges, calculate desire-level $(dl)$: closest level that satisfies invariants

Iterate from bottommost level to top level and move vertices to desire-level

Only the lower bound invariant is ever violated.
Our Parallel Batch-Dynamic Level Data Structure (PLDS)

For vertices incident to updated edges, calculate desir-e-level (dl): closest level that satisfies invariants

Iterate from bottommost level to top level and move vertices to desire-level

Only the lower bound invariant is ever violated.
Our Parallel Batch-Dynamic Level Data Structure (PLDS)

- For vertices incident to updated edges, calculate *desire-level* ($dl$): closest level that satisfies invariants.

- Iterate from bottommost level to top level and move vertices to desire-level.

- Only the lower bound invariant is ever violated.

- Each vertex moves only once, unlike in sequential LDS.
We set the coreness estimate of a vertex to be 

\[(1 + \delta)^{\max\left( \lfloor \frac{\text{level}(v) + 1}{4 \log_{1+\delta} n} \rfloor - 1, 0 \right)}\]

- Exponent is roughly the group number
- Higher vertices have higher coreness estimates
- This gives a \((2 + \epsilon)\)-approximation
- Getting better than a 2-approximation is P-complete
- Automatically get \((4 + \epsilon)\)-approximation to densest subgraph value
Implementation Details

• Designed an optimized multicore implementation
• Used parallel primitives and data structures from the Graph Based Benchmark Suite [Dhulipala et al. ‘20]
• Maintain concurrent hash tables for each vertex \( v \)
  • One for storing neighbors on levels \( \geq \text{level}(v) \)
  • One for storing neighbors on every level \( i \) in \([0, \text{level}(v)-1]\)
• Moving vertices around in the PLDS requires carefully updating these hash tables for work-efficiency
Complexity Analysis

- \( O(\log^2 n) \) levels
  - \( O(\log \log n) \) span per level to calculate desire-levels using doubling search
  - \( O(\log^* n) \) span with high probability for hash table operations
- **Total span:** \( O(\log^2 n \log \log n) \)

- \( O(B \log^2 n) \) amortized work is based on potential argument
  - Uses very similar analysis to Bhattacharya, Henzinger, Nanongkai, Tsourakakis [2015]
  - Vertices and edges store potential based on their levels in PLDS, which is used to pay for the cost of moving vertices around
  - We need to map parallel operations to an equivalent set of sequential operations
Experiments
Experimental Setup

• c2-standard-60 Google Cloud instances
  • 30 cores with two-way hyper-threading
  • 236 GB memory

• m1-megamem-96 Google Cloud instances
  • 48 cores with two-way hyperthreading
  • 1433.6 GB memory

• 3 different types of batches:
  • All batches of insertions
  • All batches of deletions
  • Mixed batches of both insertions and deletions
Runtimes/Accuracy vs. State-of-the-Art Algorithms

**PLDS:** our algorithm
**PLDSOpt:** optimized PLDS

**Hua et al.:** parallel, exact, dynamic algorithm
**Sun et al.:** sequential, approx., dynamic algorithm

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PLDSOpt: 19–544x speedup over Sun et al.

DBLP

PLDSOpt: 2.5–25x speedup over Hua et al.

LJ

4.8M vertices, 85.7M edges
Scalability vs. # Hyper-threads

PLDS: our algorithm
PLDSOpt: optimized PLDS

Hua et al.: parallel, exact, dynamic algorithm
Sun et al.: sequential, approx., dynamic algorithm

- Self-relative parallel speedups
  - PLDSOpt: 33x, PLDS: 26x, Hua: 3.6x
  - PLDSOpt is faster than all of the other algorithms at 4 or more cores
Runtime vs. Batch Size

**PLDS**: our algorithm

**PLDSOpt**: optimized PLDS

**Hua et al.**: parallel, exact, dynamic algorithm

(Sun et al. does not have a batch method)

- PLDSOpt achieves 2.5-115x speedup over Hua et al.
Runtime vs. Static Algorithms

- Parallel exact $k$-core decomposition [Dhulipala, Blelloch, Shun 2018]
- Parallel $(2 + \epsilon)$-approximate $k$-core decomposition

- We achieve speedups for all but the smallest graphs
- Speedups of up to 122x for Twitter (1.2B edges) and Friendster (1.8B edges)
Conclusion

• Theoretically-efficient and practical batch-dynamic \( k \)-core decomposition algorithm

• Using our PLDS, we designed parallel batch-dynamic algorithms for several other problems:
  • Low out-degree orientation
  • Maximal matching
  • Clique counting
  • Graph coloring

• Source code available at https://github.com/qqliu/batch-dynamic-kcore-decomposition

Quanquan C. Liu, Jessica Shi, Shangdi Yu, Laxman Dhulipala, Julian Shun, “Parallel Batch-Dynamic Algorithms for \( k \)-Core Decomposition and Related Graph Problems,” SPAA 2022
Course Summary

• Congratulations on making it through all the lectures!
• Lots of exciting research going on in algorithm and performance engineering
• Look out for relevant seminars
  • CSAIL seminars mailing list: seminars@csail.mit.edu
• Relevant conferences: SPAA, PPoPP, ACDA, ALENEX, ESA, SEA, PODC, IPDPS, SC, VLDB, SIGMOD, and more