Duality and helicity: a symplectic viewpoint

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Abstract

The theorem which says that helicity is the conserved quantity associated with the duality symmetry of the vacuum Maxwell equations is proved by viewing electromagnetism as an infinite dimensional symplectic system. In fact, it is shown that helicity is the moment map of duality acting as an SO(2) group of canonical transformations on the symplectic space of all solutions of the vacuum Maxwell equations.

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1. INTRODUCTION

The usual electromagnetic action in the vacuum,\(^1\)

\[ S = -\frac{1}{4} \int_M F_{\mu\nu} F^{\mu\nu} \, d^4x, \]  \hspace{1cm} (1.1)
suffers from well-known nevertheless inconvenient defects, namely the non-invariance of the Lagrange density under various symmetry transformations and the consequent non-symmetric form of its energy-momentum tensor, requiring to resort to various “improvements” [1, 2] \(^2\). In particular, while the vacuum Maxwell equations are invariant w.r.t. duality transformations,

\[ F \mapsto \hat{F} = \cos \theta F + \sin \theta \star (F), \]  \hspace{1cm} (1.2)

for any real \( \theta \) (where \( F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \) and \( \star (F) = \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} dx^\mu \wedge dx^\nu \) is the Hodge dual electromagnetic field strength), the Lagrange density in (1.1) is not invariant. The apparent contradiction can be resolved by observing that a duality rotation (1.2) changes the Lagrange density by a mere surface term. It is therefore a symmetry of the action [2, 3] and generates therefore, according to the Noether theorem, a conserved quantity identified here as the optical helicity [4]. The proof given in [4] is rather laborious, though, due to the complicated behavior of the vector potential and the subsequent use of the Hertz vector — a rather subtle, non-gauge-invariant tool. The treatment in [3] is also quite involved.

Another proposition [2, 5, 7] is to embed the Maxwell theory into a manifestly duality-symmetric one for which Noether’s theorem yields a seemingly different expression, namely,

\[ \chi_{CS} = \frac{1}{2} \int_{\mathbb{R}^3} (A \cdot B - C \cdot E) \, d^3r \]  \hspace{1cm} (1.3)

à la Chern-Simons, where \( A \) and \( C \) are vector potentials for the magnetic and the electric fields, \( \nabla \times A = B \) and \( \nabla \times C = -E \), respectively. It is worth noting that the second term in Eq. # (14) of [4] and, respectively, in Eq. # (2.9) of [3], both represent the vector potential for the dual field strength — a fact not recognized by none of these authors. See [2, 6, 7] for comprehensive presentations.

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\(^1\) Integration is performed over Minkowski spacetime, \( M \), endowed with metric \( g = g_{\mu\nu} \, dx^\mu \, dx^\nu \) of signature \((+,-,-,-)\). Let us stress that we will content ourselves with a special relativistic treatment of duality, although our main results spelled out in the next sections clearly hold true (with minor modifications) in a fixed gravitational background.

\(^2\) We refer to, e.g., [13] for a geometric standpoint associated with the principle of general covariance, enabling us to circumvent these difficulties.
In the first term in (1.3) we recognize the (magnetic) helicity, \( \chi_{\text{mag}} = \frac{1}{2} \int A \cdot B \, d^3r \) widely studied in (magneto)hydrodynamics \[8\], where it measures the winding of magnetic lines of force and/or fluid vortex lines, respectively. It is worth stressing that the magnetic helicity alone is not a constant of the motion in general, and the clue leading to (1.3) is that its non-conservation,

\[
\frac{d}{dt} \chi_{\text{mag}} = - \int_{\mathbb{R}^3} E \cdot B \, d^3r,
\]

is precisely compensated by that of the second term \[6\]. A remarkable fact is that (1.3) combines two Chern-Simons invariants \[9\], for both the electromagnetic and its dual field.

Duality and helicity have attracted considerable recent attention, namely in optics \[2, 7, 10\] and in heavy ion physics \[11\]. Our own interest stems from studying the helicity of semiclassical chiral particles \[12\].

In this Note we explain the duality and helicity from yet another viewpoint, which bypasses Lagrangians and gauge fixing altogether. Our clue is to view the set of solutions of electromagnetism as (an infinite-dimensional) symplectic space \[14–16\].

2. ELECTROMAGNETISM IN THE SYMPLECTIC FRAMEWORK

In the framework of Hamiltonian mechanics \[14\] one works with manifolds endowed with a closed two-form \( \omega \). If \( \dim \ker(\omega) \) has constant but nonzero dimension, \( \omega \) is called presymplectic; if its kernel is zero dimensional, it is called symplectic. In the physical applications we have in mind, we start with a manifold such that \((V, \omega)\) is presymplectic and is referred to as an “evolution space”, where the dynamics takes place. The characteristic leaves which integrate \( \ker(\omega) \) are identified with the motions of the system. The quotient of \( V \) by the characteristic foliation of \( \omega \), namely \( \mathcal{M} = V / \ker(\omega) \), is therefore endowed with a symplectic two-form \( \Omega \), whose pull-back to \( V \) is \( \omega \). Then \((\mathcal{M}, \Omega)\) is what has been called the “space of motions” in \[14\]. Crnković and Witten \[15\] call it the “true phase space”.

The next ingredient is a Lie group \( G \) of canonical transformations, i.e., of diffeomorphisms of \( V \) preserving the two-form \( \omega \). Denote by \( \mathfrak{g} \) the Lie algebra of \( G \), and by \( Z_V \) the infinitesimal action (fundamental vector field) on \( V \) associated with \( Z \in \mathfrak{g} \).

We thus have \( L_{Z_V} \omega = 0 \) so that \( \omega(Z_V, \cdot) \) is a closed one-form for all \( Z \in \mathfrak{g} \). We now say that \( J : V \to \mathfrak{g}^* \) is a moment map for \((V, \omega, G)\) if the stronger condition

\[
\omega(Z_V, \cdot) = -d(J \cdot Z)
\]

(2.1)
holds for all $Z \in \mathfrak{g}$.

If the equations of motion are given by $\ker(\omega)$, as it happens in the mechanics of finite dimensional systems [14] and, as we will prove below, also for Maxwell’s electromagnetism, then $J$ clearly descends to the space of motions, $\mathcal{M} = \mathcal{V}/\ker(\omega)$, as the Noetherian quantity associated with the symmetry group $G$: indeed $J \cdot Z$ is a constant of the motion for all $Z \in \mathfrak{g}$.

Below we boldly extend this framework to the infinite dimensional “manifold” $\mathcal{M}$ which consists of all solutions of the vacuum Maxwell equations modulo gauge transformations we endow with a symplectic structure.

Let us show how all this comes about. Our first aim is to translate the usual variational approach into a symplectic language. The actual physical variable is the potential one-form $A = A_\mu \, dx^\mu$ locally defined by $F = dA$. Then the variation of the action (1.1) with respect to a variation $\delta A = \delta A_\mu \, dx^\mu$ of the 4-potential is

$$\delta S = \int_M \left[ \partial_\nu (F^{\mu\nu} \delta A_\mu) + (\partial_\mu F^{\mu\nu}) \delta A_\nu \right] d^4x.$$  \hspace{1cm} (2.2)

Assuming that the fields drop off sufficiently rapidly at infinity — or that the variations $\delta A$ have compact support — the surface term can be dropped, allowing us to deduce the vacuum Maxwell equations $\partial_{[\mu} F_{\nu\rho]} = 0$ and $\partial_\mu F^{\mu\nu} = 0$, also written as

$$dF = 0 \quad \text{and} \quad d\star(F) = 0.$$  \hspace{1cm} (2.3)

Denote by $\mathcal{V}$ the space of one-forms $A$ of Minkowski space $M$ whose associated field strength, $F = dA$, is a solution of (2.3). We contend that $\mathcal{V}$, which can be thought of as an infinite-dimensional manifold (affine space), is an “evolution space” for the Maxwell theory.

Firstly, a variation of a solution, $\delta A$, is a “tangent vector” to $\mathcal{V}$ at $A \in \mathcal{V}$ if $A + \delta A$ is still a solution of the field equations which vanishes at spatial infinity (as $A$ does). Since the associated field strength is $F + \delta F$, where $\delta F = d(\delta A)$, it follows that $\delta F$ also satisfies the Maxwell equations, $d(\delta F) = 0$ and $d\star(\delta F) = 0$.

Now, adapting Souriau’s procedure in [14], Sec. 7, to field theory, we define a symplectic form on the space of all solutions of the linear system (2.3). To this end, we consider the

\begin{itemize}
  \item For each point $x$ of $\mathcal{V}$, the quantity $J(x)$ belongs to the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$, and contracting with $Z \in \mathfrak{g}$ yields a function $x \mapsto J(x) \cdot Z$ on $\mathcal{V}$.
  \item A rigorous treatment of this infinite-dimensional differentiable structure would require the use of, e.g.,
    \begin{itemize}
      \item diffeology [17], especially when dealing with differential forms on this “diffeological space”.
      \item One-forms and vector fields are identified by lifting and lowering indices using the Minkowski metric.
    \end{itemize}
\end{itemize}
action (1.1) by integrating over the domain \( M' = [t_0, t_1] \times \Sigma \subset M \) defined by a Cauchy 3-surface \( \Sigma \) with arbitrary dates \( t_0 \) and \( t_1 \neq t_0 \), where \( t \) is some given time-function. When \( F \) is a solution of the Maxwell equations, the variation vanishes, \( \delta S = 0 \), and therefore Eq. (2.2) boils down to

\[
0 = \int_M \partial_\nu (F^{\mu\nu} \delta A_\mu) \, d^4x = \int_{\Sigma_1} \star (F(\delta A)) - \int_{\Sigma_0} \star (F(\delta A)),
\]

where \( \Sigma_i = \{ t_i \} \times \Sigma \) for \( i = 0, 1 \), implying that the integral does not depend on the choice of \( t_0 \) and \( t_1 \); the one-form\(^6\)

\[
\alpha(\delta A) = \int_{\Sigma} \star (F(\delta A)) = - \int_{\Sigma} \star (F') \wedge \delta A \quad (2.5)
\]

is therefore well-defined; it is the Cartan one-form. The expression (2.5) represents the flux of the vector field \( F(\delta A) = (F^{\mu\nu} \delta A_\mu) \partial_\nu \) across the Cauchy surface \( \Sigma \). Calculating the exterior derivative, \( \omega = d\alpha \), via \( d\alpha(\delta A, \delta' A) = \delta (\alpha(\delta' A)) - \delta'(\alpha(\delta A)) - \alpha(\delta, \delta' A) \), we find

\[
\omega(\delta A, \delta' A) = \int_{\Sigma} \delta A \wedge \star (\delta' F) - \delta' A \wedge \star (\delta F). \quad (2.6)
\]

The two-form (2.6) corresponds exactly to that given by Eq. (23) in [15].

From this point on, we do not use any Lagrangian; the starting point of all our subsequent investigations will be the two-form (2.6).

Let us now show that \( (\mathcal{V}, \omega) \) becomes a formal presymplectic space. To that end, let us compute its characteristic distribution. We thus must determine the kernel of \( \omega \), i.e., all variations \( \delta A \) of a solution \( A \in \mathcal{V} \) such that \( \omega(\delta A, \delta' A) = 0 \) for all \( \delta' A \), subject to the constraint \( \delta'(d\star (F)) = 0 \) to comply with the field equations. Using a Lagrange multiplier, \( f \), we look for all solutions \( \delta A \) of

\[
\int_{\Sigma} \delta A \wedge \star (\delta' F) - \delta' A \wedge \star (\delta F) = - \int_{\Sigma} f \alpha(\delta' (d\star (F))) = \int_{\Sigma} df \wedge \star (\delta F) \quad (2.7)
\]

for all compactly supported variations \( \delta' A \). Eq. (2.7) readily yields that the kernel is indeed given by all gauge transformations,

\[
\delta A \in \ker(\omega) \iff \delta A = df \quad (2.8)
\]

\(^6\) In a coordinate system where the metric is \( g = dt^2 - dx^2 \) and \( \Sigma \) given by \( t = \text{const} \), Eq. (2.5) reads

\[
\alpha(\delta A) = \int F^{\mu\nu} \delta A_\mu \partial_\nu t \, d^3x. \quad (2.4)
\]
for some smooth function $f$. (Note that we duly have $\delta F = 0$.) Then, the leaves of the characteristic distribution $\ker(\omega)$ are identified to the orbits of the electromagnetic gauge group $\mathcal{J}$ generated by smooth functions $\varphi$ of $M$, which acts on $\mathcal{V}$ according to $A \mapsto A + d\varphi$. At last, the quotient

$$\mathcal{M} = \mathcal{V} / \mathcal{J}$$

is the the “space of motions” of electromagnetism; it is identified with the space of all vector potentials which are solutions of the free Maxwell equations modulo gauge transformations, to which $\omega$ projects as the canonical symplectic two-form $\Omega$.

3. DUALITY SYMMETRY

Let us now consider duality rotations (1.2) which form, as said before, a manifest symmetry group for the free Maxwell equations.\footnote{The field equations being linear, any real linear transformation $\hat{F} = aF + b \ast (F) & \ast(\hat{F}) = cF + d \ast (F)$, with $ad - bc \neq 0$, permutes the solutions of (2.3). Now, the Hodge star defines a complex structure on the 2-dimensional space spanned by $F$ and $\ast(F)$, since $\ast^2 = -1$. Restricting our considerations to transformations that preserve the “star” $\ast$, i.e., to $\text{Sp}(1, \mathbb{R}) \cong \text{SL}(2, \mathbb{R})$, an easy calculation shows that $c = -b$ and $d = a$, implying $a^2 + b^2 = 1$; hence $a = \cos \theta$ and $b = \sin \theta$ as in Eq. (1.2).} Using our symplectic language, we claim that the two-form $\omega$ in (2.6) is invariant under the (1.2), implemented on the potentials as

$$\hat{A} = \cos \theta A + \sin \theta C, \quad \hat{C} = \cos \theta C - \sin \theta A$$

(3.1)

where $A$ and $C$ are (local) 4-potentials for the field and its dual, $F = dA$ and $\ast(F) = dC$. Note that $A$ and $C$ here are not independent since their field strengths are each other’s duals. Using the properties of the Hodge star operation, $\ast$, one shows indeed that

$$\omega(\delta \hat{A}, \delta' \hat{A}) = \omega(\delta A, \delta' A)$$

(3.2)

for all variations $\delta A$ and $\delta' A$ compatible with the constraints (2.3). This proves that the duality transformation (1.2), implemented as above is a canonical transformation of the evolution space, $(\mathcal{V}, \omega)$, and therefore also of the space of motions, $(\mathcal{M}, \Omega)$.

We now turn to the moment map of duality symmetry. The infinitesimal duality action on $\mathcal{V}$ is given by $\delta_\varepsilon A = \varepsilon C$ and $\delta_\varepsilon C = -\varepsilon A$, where $\varepsilon \in \mathbb{R}$. A straightforward calculation
then shows that, for all $\delta' A$ compatible with the constraints (2.3), we have

$$\omega(\delta \varepsilon A, \delta' A) = \int_{\Sigma} \{ \delta'(\ast(F)) \wedge \varepsilon C + \varepsilon F \wedge \delta' A\} = \frac{1}{2} \varepsilon \delta' \int_{\Sigma} \{ C \wedge \ast(F) + A \wedge F \}$$

(3.3)

since $\delta' A \wedge F \equiv \frac{1}{2} \delta'(A \wedge F)$ and, likewise, $\delta' C \wedge \ast(F) \equiv \frac{1}{2} \delta'(C \wedge \ast(F))$ – modulo an exact three-form. It follows that we do actually have a moment map $J : V \rightarrow \mathbb{R}$, i.e., such that

$$\omega(\delta \varepsilon A, \delta' A) = -\delta'(J(A)\varepsilon)$$

for the duality group acting on $(V, \omega)$, and thus on the space of motions of all solutions of the Maxwell equations, namely

$$J(A) = -\frac{1}{2} \int_{\Sigma} A \wedge dA + C \wedge dC,$$

(3.4)

which is indeed the geometric form of of the helicity, (1.3). The conservation of (3.4) can also be checked directly: the two Chern-Simons three-forms are both the anti-derivatives of the same Pontriagin density, but with opposite signs,

$$d(A \wedge F) = F \wedge F = -\ast(F) \wedge \ast(F) = -d(C \wedge \ast(F)).$$

(3.5)

Let us consider two Cauchy surfaces $\Sigma_0$ and $\Sigma_1$ with dates $t = t_0$ and $t = t_1$ and view them as the boundaries of a four-volume $V$. The integral of the four-form $-\frac{1}{2}d(A \wedge F + C \wedge \ast(F))$ on $V$ vanishes in view of (3.5), proving that the fluxes across $\Sigma_0$ and $\Sigma_1$ are equal, and that the moment map $J$ in (3.4) is therefore independent of $\Sigma$.

The equivalence of (3.4) with the optical formula in the literature which says that the optical helicity is in fact the difference of the left- and right-handed photons,

$$\chi_O = N_L - N_R,$$

(3.6)

can be shown along the lines followed in [4, 6].

Here we just mention an alternative yet incomplete approach: the general form (1.3) was narrowly missed by Rañada [5], who did correctly identify both terms — without adding them however, and considering only the special case $E \cdot B = 0$, when both terms are separately conserved. cf. (1.4). Under such condition he could show that the two integrals are indeed the degrees, $N_L$ and $N_R$, of suitable Hopf maps $S^3 \rightarrow S^2$, confirming (3.6) in such a case. Extension of this approach to the general case is under investigation.

4. CONCLUSION

In this “variation on a themes”-type Note we re-derive, using the symplectic framework in infinite dimensions, the helicity formula (3.4), equivalent to the one (1.3) proposed in the
literature. Unlike for previous authors [2, 4, 6], our derivation is gauge-invariant, as it did not require any choice of gauge.

We note also that our two-form (2.6) is manifestly duality-invariant, whereas the Cartan one-form $\alpha$ in (2.5) is clearly not, as it follows from the non-invariance of the standard Maxwell Lagrangian (1.1). This highlights the advantage of using the presymplectic Maxwell two-form (2.6) to deal with symmetries, and in particular with duality.

The situation is reminiscent of what happens for a Dirac monopole, for which no manifestly radially symmetric vector potential and thus no symmetric Lagrangian or Cartan one-form can exist, whereas the two-form which represents the field strength resp. the dynamics is perfectly rotationally invariant [18].

We would also mention that this formula can also be obtained using the Pauli-Lubanski approach [19], also followed in [12].

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