Non–vanishing for Koszul cohomology of curves

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Abstract

We study the relationship between rank $p + 2$ Koszul classes and rank 2 vector bundles on a smooth curve. We show that every rank $p + 2$ Koszul class is obtained from a rank 2 vector bundle and give an explicit nonvanishing theorem for Koszul classes arising in this way.

1 Introduction

Let $X$ be a smooth complex projective variety. The geometry of $X$ is reflected in the behaviour of the Koszul cohomology groups $K_{p,q}(X,L)$ introduced by Green [4], more specifically the vanishing/nonvanishing of certain Koszul cohomology groups. The fundamental result in this direction is the nonvanishing theorem of Green–Lazarsfeld [5]. This theorem states that if a line bundle $L$ admits a decomposition $L = L_1 \otimes L_2$ with $r_i = h^0(X, L_i) - 1 \geq 1$ ($i = 1, 2$) then $K_{r_1+r_2-1,1}(X, L) \neq 0$. Voisin [9, (1.1)] has given a different proof of this result under the hypothesis that $L_1$ and $L_2$ are globally generated.

The aim of this note is to give a more geometric approach to this type of problems. The starting point is the following construction due to Voisin. Given a rank two vector bundle $E$ on $X$ with determinant $L$, Voisin [11, (2.22)] defined a homomorphism

$$\varphi : S^p H^0(X, E) \otimes \bigwedge^{p+2} H^0(X, E) \rightarrow \bigwedge^p H^0(X, L) \otimes H^0(X, L).$$

By [11, Lemma 5], this homomorphism produces elements of $K_{p,1}(X, L)$. If we take $E = L_1 \oplus L_2$, we get back the classes constructed by Green and Lazarsfeld. As one of the referees pointed out to us, Koh and Stillman [7] had generalised the Green–Lazarsfeld construction before from a different point of view.

Recall that the rank of a Koszul class $\gamma \in K_{p,1}(X, L)$ is the minimal dimension of a linear subspace $W \subset H^0(X, L)$ such that $\gamma$ is represented by an element in $\bigwedge^p W \otimes H^0(X, L)$; cf. [6, Definition 2.2]. (Note that the subspace $W$ is uniquely determined if $p \geq 2$.) By definition, the Koszul classes constructed in this paper are of rank $p + 2$ if the vector bundle $E$ is indecomposable.

Section 3 contains the main results of this paper. We first give a necessary and sufficient condition for nonvanishing of Koszul classes on smooth curves obtained from rank 2 vector bundles (Theorem 3.1). This result generalises the nonvanishing
the determinant map. Given $t$, be the image of this element under the map $d$ by

$$
\{11\} \text{, using coordinates. If we}
$$

Suppose that \( \dim (U) = p + 2 \) with \( p \geq 1 \), and put \( W = d_t(U) \cong U \). The restriction of \( d \) to \( \Lambda^2 U \) defines a map \( \Lambda^2 U \to V \), which we can view as an element of

$$
\Lambda^2 U^\vee \otimes V \cong \Lambda^p U \otimes V.
$$

Let

$$
\gamma \in \Lambda^p W \otimes V \subset \Lambda^p V \otimes V
$$

be the image of this element under the map \( d_t \).

Following Voisin [11 (2.22)], we prove that \( \gamma \) defines a Koszul class in \( K_{p,1}(X,L) \). To this end, we make the previous construction explicit using coordinates. If we choose a basis \( \{e_1,\ldots,e_{p+3}\} \) of \( (t) \oplus U \subset H^0(X,E) \) such that \( e_1 = t \), we have

$$
\gamma = \sum_{i<j} (-1)^{i+j}d(t \wedge e_2) \wedge \ldots \wedge d(t \wedge e_i) \wedge \ldots
$$

$$
\ldots \wedge d(t \wedge e_j) \ldots \wedge d(t \wedge e_{p+3}) \otimes d(e_i \wedge e_j).
$$

As in [11] one shows that the image of the \( \gamma \) by the Koszul differential

$$
\delta : \Lambda^p V \otimes H^0(X,L) \to \Lambda^{p-1} V \otimes S^2 H^0(X,L)
$$

equals

$$
\sum_{i<j<k} (-1)^{i+j+k}d(t \wedge e_2) \wedge \ldots \wedge d(t \wedge e_i) \wedge \ldots \wedge d(t \wedge e_j) \wedge \ldots \wedge d(t \wedge e_k) \ldots \wedge d(t \wedge e_{p+3})
$$

$$
\otimes\{d(t \wedge e_i)d(e_j \wedge e_k) - d(t \wedge e_j)d(e_i \wedge e_k) + d(t \wedge e_k)d(e_i \wedge e_j)\}.
$$

Lemma 2.1 (Voisin) Given four elements \( w_1, w_2, w_3, w \in H^0(X,E) \) we have the relation

$$
d(w \wedge w_1)d(w_2 \wedge w_3) - d(w \wedge w_2)d(w_1 \wedge w_3) + d(w \wedge w_3)d(w_1 \wedge w_2) = 0
$$

in \( H^0(X,L^2) \).

Proof: See [11] Lemma 5.

\[ \square \]
The previous lemma shows that $\gamma$ belongs to the kernel of the Koszul differential

$$\delta_X : \wedge^p V \otimes H^0(X, L) \rightarrow \wedge^{p-1} V \otimes H^0(X, L^2).$$

Hence $\gamma$ defines a Koszul class $[\gamma] \in K_{p,1}(X, L, W) \subseteq K_{p,1}(X, L)$. Clearly the given class only depends on $t$ and $W$; we write $[\gamma] = \gamma(W, t)$.

### 2.2 The method of Green–Lazarsfeld

Let $L_1, L_2$ be two line bundles on a smooth projective variety $X$ such that $r_i = h^0(X, L_i) - 1 \geq 1$ ($i = 1, 2$). Write $L_i = M_i + F_i$ with $M_i$ the mobile part and $F_i$ the fixed part. Let $B$ be the divisorial part of $F_1 \cap F_2$. It is possible to choose $s_i \in H^0(X, L_i)$ such that $V(s_1, s_2) = B \cup Z$ with $\text{codim } Z \geq 2$. Set $L = L_1 \otimes L_2$, and put $t = (s_1, s_2) \in H^0(X, L_1 \oplus L_2)$, $W = \text{im}(d_t) \subset H^0(X, L(-B))$. By construction $h^0(X, O_X(B)) = 1$, hence $\dim W = r_1 + r_2 + 1$. By the previous discussion, we obtain a Koszul class $\gamma(W, t) \in K_{r_1 + r_2 - 1, 1}(X, L)$. We call such classes Green–Lazarsfeld classes.

Note that the rank of a Green–Lazarsfeld class is either $p + 1$ or $p + 2$. Classes of rank $p + 1$ are of scrollar type; see e.g. [8] or [6, Corollary 5.2].

**Definition 2.2** Given a nonnegative integer $k \geq 0$, let $K_{k,1}(X, L)_{GL} \subseteq K_{k,1}(X, L)$ be the subspace generated by Green–Lazarsfeld classes for all decompositions $L = L_1 \otimes L_2$ with $k = r_1 + r_2 - 1$, ($r_1 \geq 1$, $r_2 \geq 1$).

### 2.3 The method of Koh–Stillman

Voisin’s method produces syzygies of rank $\leq p + 2$. As we have seen in the previous subsection, rank $p + 1$ syzygies are Green–Lazarsfeld syzygies of scrollar type. Rank $p + 2$ syzygies can be obtained in the following way. Suppose that $L$ is a globally generated line bundle on a projective variety $X$, and let $[\gamma] \in K_{p,1}(X, L)$ be a nonzero class represented by an element $\gamma \in \wedge^p W \otimes V$ with $\dim W = p + 2$. We view $\gamma$ as an element in $\wedge^2 W^\vee \otimes V \cong \text{Hom}(\wedge^2 W, V)$. Following [6] Proof of Theorem 6.1 we consider the map

$$\gamma' : \wedge^2 (C \oplus W) = W \oplus \wedge^2 W \rightarrow V$$

defined by taking the direct sum of $\gamma$ and the inclusion $W \hookrightarrow V$. If we choose a generator $e_1$ for the first summand and a basis $\{e_2, \ldots, e_{p+3}\}$ for $W$, we obtain a skew–symmetric $(p + 3) \times (p + 3)$ matrix $A$ by setting

$$a_{ij} = \gamma'(e_i \wedge e_j).$$

By construction, the inclusion $W \rightarrow V$ corresponds to the map $\gamma'(e_1 \wedge -)$. This allows us to identify $a_{1j}$ and $e_j$, $2 \leq j \leq p + 3$. Let $\alpha$ be the image of $\gamma$ under the Koszul differential

$$\delta : \wedge^p V \otimes V \rightarrow \wedge^{p-1} V \otimes S^2 V.$$
Writing this out, we obtain
\[ \alpha = \sum_{i<j<k} (-1)^i j k a_{12} \wedge \ldots \wedge a_{1,i} \wedge \ldots \wedge a_{1,j} \wedge \ldots \wedge a_{1,k} \ldots \wedge a_{1,p+3} \otimes \text{Pf}_{1ijk}(A). \] (3)

As the elements \( \{a_{12}, \ldots, a_{1,p+3}\} = \{e_2, \ldots, e_{p+3}\} \) are linearly independent, this expression is nonzero if and only if at least one of the Pfaffians \( \text{Pf}_{1ijk}(A) \) is nonzero. Furthermore, since \( \alpha \) maps to zero in \( \bigwedge^{p-1} V \otimes H^0(X, L^2) \) the Pfaffians \( \text{Pf}_{1ijk}(A) \) have to vanish on the image of \( X \).

The preceding discussion shows that every rank \( p + 2 \) syzygy arises from a skew-symmetric \( (p+3) \times (p+3) \) matrix \( A \) such that

(i) the elements \( \{a_{12}, \ldots, a_{1,p+3}\} \) are linearly independent;
(ii) there exists a nonzero Pfaffian \( \text{Pf}_{1ijk}(A) \);
(iii) the Pfaffians \( \text{Pf}_{1ijk}(A) \) vanish on the image of \( X \) in \( \mathbb{P}(V^\vee) \).

This is exactly the method used by Koh and Stillman to produce syzygies; see [7, Lemma 1.3].

Remark 2.3 In the geometric setting of subsection 2.1 let \( Y \) be the image of \( X \) in \( \mathbb{P}(V^\vee) \). The expression (2) shows that the canonical isomorphism
\[ K_{p,1}(X, L) \cong K_{p-1,2}(\mathbb{P}^r, I_Y, \mathcal{O}_F(1)) \]
maps the class \( \gamma(W, t) \) to the element \( \alpha \) defined in (3). Moreover, if \( d \) does not vanish on decomposable elements then \( \gamma(W, t) \neq 0 \). Indeed, this condition is satisfied if and only if the matrix \( A \) has no generalised zero; cf. [7, Definition (1.1)]. One then applies [loc. cit., Remark p. 122].

3 Main results

Theorem 3.1 Let \( X \) be a smooth curve, let \( L \) be a base-point free line bundle on \( X \) and let \( W \subset H^0(X, L) \) be a linear subspace. Put \( B = \text{Bs}(W) \), and let \( t \) be a section of \( H^0(X, \mathcal{O}_X(B)) \) vanishing on \( B \). Consider an extension
\[ 0 \rightarrow \mathcal{O}_X(B) \rightarrow E \rightarrow L(-B) \rightarrow 0 \] (4)
such that
\[ W \subset (\ker H^0(X, L(-B)) \xrightarrow{d} H^1(X, \mathcal{O}_X(B))). \]
Then the Koszul class \( \gamma(W, t) \) defined in section 2.1 is nonzero if and only if the extension (4) is non-split.
Proof: The proof proceeds in several steps. We use the notation of section 2.1.

Step 1. Suppose that the extension (4) splits. In this case, one readily verifies that $d$ vanishes identically on $\bigwedge^2 U$. The formula (1) then shows that $\gamma(W,t) = 0$.

Step 2. If $\gamma(W,t) = 0$ there exists a linear map $h : U \to \mathbb{C}$ such that
\[
d(u_1 \wedge u_2) = h(u_2)d_t(u_1) - h(u_1)d_t(u_2)
\]
for all $u_1, u_2 \in U$.

Indeed, suppose that there exists a nonzero element $\tilde{\gamma} \in \bigwedge^{p+1} W \cong W^\vee$ such that $\gamma$ is the image of $\tilde{\gamma}$ under the Koszul differential. Then $\gamma$ coincides with the composition of maps
\[
\bigwedge^2 W \to W \otimes W \xrightarrow{\tilde{\gamma} \otimes \text{id}} W \hookrightarrow V.
\]
Since
\[
d(u_1 \wedge u_2) = \gamma(d_t(u_1) \wedge d_t(u_2)) = \tilde{\gamma}(d_t(u_2))d_t(u_1) - \tilde{\gamma}(d_t(u_1))d_t(u_2),
\]
condition (5) is satisfied with $h = \tilde{\gamma}d_t : U \to \mathbb{C}$.

Step 3. Let $u_1, u_2 \in U$ be two sections such that $d_t(u_1)$ and $d_t(u_2)$ generate $L(−B)$. If $d(u_1 \wedge u_2) = 0$, the extension (4) splits.

To prove this assertion, put $s_i = d_t(u_i)$ ($i = 1, 2$) and consider the commutative diagram
\[
\begin{array}{ccccccc}
0 & \to & \mathcal{O}_X(B) & \to & E & \to & L(−B) & \to & 0 \\
& & \uparrow_{\text{ev}_1} & & & & \uparrow_{\text{ev}_2} & & \\
0 & \to & \langle u_1, u_2 \rangle \otimes \mathcal{O}_X & \to & \langle s_1, s_2 \rangle \otimes \mathcal{O}_X & \to & 0.
\end{array}
\]

Put $M = \ker(\text{ev}_1)$, and note that $\ker(\text{ev}_2) \cong L^{-1}(B)$ since $\text{ev}_2$ is surjective. By the Snake Lemma we obtain an exact sequence
\[
0 \to M \to L^{-1}(B) \to \mathcal{O}_X(B) \to \text{coker} (\text{ev}_1) \to 0.
\]

Note that
\[
d(u_1 \wedge u_2) = 0 \iff \text{rank } \text{im}(\langle u_1, u_2 \rangle \otimes \mathcal{O}_X \to E) = 1 \iff \text{rank } M = 1
\]
where the first equivalence follows from [10] p. 380. If $d(u_1 \wedge u_2) = 0$ the above exact sequence shows that $M \cong L^{-1}(B)$, hence the isomorphism $\langle u_1, u_2 \rangle \otimes \mathcal{O}_X \sim \langle s_1, s_2 \rangle \otimes \mathcal{O}_X$ induces an isomorphism $\text{im}(\text{ev}_1) \cong L(−B)$. The inverse of this isomorphism provides a splitting of the extension (4).

Step 4. Suppose that $\gamma(W,t) = 0$. Then there exists a linear map $h : U \to \mathbb{C}$ as in Step 2. Consider the morphism
\[
\pi : X \to \mathbb{P}(W^\vee)
\]
In particular there exists $i$. Hence there exists an injective homomorphism $\varphi$ and this is impossible since $L$ is globally generated, since $K_{p,1}(X,L(-Bs(L))) \cong K_{p,1}(X,L)$. 

Remark 3.2 In the statement of Theorem 3.1 it is not necessary to suppose that $L$ is globally generated, since $K_{p,1}(X,L(-Bs(L))) \cong K_{p,1}(X,L)$.

Theorem 3.3 (Green–Lazarsfeld) Let $X$ be a smooth curve, and let $L$ be a line bundle on $X$ that admits a decomposition $L = L_1 \otimes L_2$ with $r_i = \dim |L_i| \geq 1$ for $i = 1, 2$. Then $K_{r_1+r_2-1,1}(X,L) \neq 0$.

Proof: We define $s_1$, $s_2$, $t$, $W$, $B$ and $\gamma(W,t)$ as in section 2.2. Let $C$ be the base locus of $W$, seen as a subspace of $H^0(X,L(-B))$. We prove that $\gamma(W,t) \neq 0$. Suppose that $\gamma(W,t) = 0$. Consider the extension

$$0 \to \mathcal{O}_X(B) \to L_1 \oplus L_2 \to L(-B) \to 0.$$ 

Pulling back this extension along the injective homomorphism $L(-B-C) \to L(-B)$, we obtain an induced extension

$$0 \to \mathcal{O}_X(B) \to E \to L(-B-C) \to 0.$$ 

Applying Theorem 3.1 to the line bundle $L(-C)$, we find that this extension splits. Hence there exists an injective homomorphism

$$\mathcal{O}_X(B) \oplus L(-B-C) \to L_1 \oplus L_2.$$ 

In particular there exists $i \in \{1, 2\}$ such that $\text{Hom}(L(-B-C), L_i) \neq 0$. This implies that

$$r_i + 1 = h^0(X, L_i) \geq h^0(X, L(-B-C)) \geq \dim W = r_1 + r_2 + 1,$$

and this is impossible since $r_1 \geq 1$ and $r_2 \geq 1$. $\square$
Theorem 3.4 Let $X$ be a smooth curve, and let $\alpha \neq 0 \in K_{p,1}(X,L)$ be a Koszul class of rank $p + 2$ represented by an element of $\Lambda^p W \otimes H^0(X,L)$ with $\dim W = p + 2$. There exist a rank 2 vector bundle $E$ on $X$ and a section $t \in H^0(X,E)$ such that $\alpha = \gamma(W,t)$.

Proof: Put $T = \mathbb{C} \oplus W$, and choose a basis $\{e_1, \ldots, e_{p+3}\}$ of $T$ such that $t = e_1$ is the generator of the first summand. Writing $z_{ij} = e_i \wedge e_j$, we obtain a skew-symmetric matrix $Z = (z_{ij})$ and coordinates $(z_{ij})_{1 \leq i < j \leq p + 3}$ on $\mathbb{P}(\Lambda^2 T^\vee)$. Consider the Grassmannian $G = G(2, T)$ of 2-dimensional quotients of $T$. The ideal of $G$ under the Plücker embedding $G \subset \mathbb{P}(\Lambda^2 T^\vee)$ is generated by the $4 \times 4$ Pfaffians $\text{Pf}_{ijkl}(Z)$ of the matrix $Z$. Taking exterior powers in the exact sequence

$$0 \to \langle t \rangle \to T \to W \to 0$$

we obtain an exact sequence

$$0 \to \langle t \rangle \otimes W \to \Lambda^2 T \to \Lambda^2 W \to 0.$$

The linear subspace $\mathbb{P}(\Lambda^2 W^\vee) \subset \mathbb{P}(\Lambda^2 T^\vee)$ is defined by the vanishing of the linear forms $z_{1j}$, $j = 2, \ldots, p + 3$. A straightforward computation then shows that the ideal of the union

$$G(2, T) \cup \mathbb{P}(\Lambda^2 W^\vee) \subset \mathbb{P}(\Lambda^2 T^\vee)$$

is generated by the Pfaffians $\text{Pf}_{1ijk}(Z)$. The tautological exact sequence

$$0 \to S \to T \otimes O_G \to Q \to 0$$

induces an isomorphism $T \cong H^0(G, Q)$. Under this isomorphism, we have $G(2, W) = V(t)$. As in section 2.3 we associate to the Koszul class $\alpha$ a matrix $A = (a_{ij})$ of linear forms $A = (a_{ij})$ such that

(a) The linear forms in the first row of $A$ span $W$;
(b) There exists a nonzero $4 \times 4$ Pfaffian of $A$ involving the first row and column;
(c) The $4 \times 4$ Pfaffians involving the first row and column of $A$ vanish on the image of $X$ in $\mathbb{P}H^0(X, L)^\vee$.

Let $C$ be the base locus of the image of $A$. Replacing $L$ by $L(-C)$ if necessary ($W$ is obviously contained in the image of $A$) we can suppose that $C$ is empty, hence the matrix $A$ defines a morphism

$$\psi : X \to \mathbb{P}(\Lambda^2 T^\vee).$$

Condition (c) implies that the image $Y = \psi(X)$ is contained in the union $G(2, T) \cup \mathbb{P}(\Lambda^2 W^\vee)$, and condition (a) shows that $Y$ is not contained in $\mathbb{P}(\Lambda^2 W^\vee)$. As $Y$ is irreducible, this implies that $Y$ is contained in $G(2, T)$.

Put $E = \psi^*Q$. Twisting the exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_G \to \psi_*\mathcal{O}_X \to 0$$

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by the universal quotient bundle $Q$ and taking global sections, we obtain an exact sequence

$$0 \to H^0(G, Q \otimes \mathcal{T}_Y) \to H^0(G, Q) \xrightarrow{\psi^*} H^0(G, \psi_*\mathcal{O}_X \otimes Q) \cong H^0(X, E).$$

Condition (a) implies that $Y$ is not contained in $G(2, W) = G(2, T) \cap \mathbb{P}(\wedge^2 W^\vee)$, hence $t$ does not vanish identically on $X$ and defines a global section of $E$. The zero locus of this section is given by the equations $a_{12} = \ldots = a_{1,p+3} = 0$, hence it coincides with $B$. Consequently the line bundle $E$ is given by an extension

$$0 \to \mathcal{O}_X(B) \to E \to L(-B) \to 0. \quad (6)$$

Consider the commutative diagram

$$
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
H^0(G, \mathcal{O}_G) & \xrightarrow{\wedge t} & H^0(X, \mathcal{O}_X(B)) \\
\downarrow & & \downarrow \\
H^0(G, Q) & \xrightarrow{\psi^*} & H^0(X, E) \\
\downarrow & & \downarrow d_t \\
W & \xrightarrow{\gamma} & H^0(X, L(-B)).
\end{array}
$$

Note that $\ker i = W \cap H^0(G, \mathcal{O}_G(1) \otimes \mathcal{T}_Y) = 0$ by condition (a). As the map $H^0(G, Q) \to W$ is surjective, we find that $W$ is contained in the image of the map $d_t : H^0(X, E) \to H^0(X, L(-B))$. Hence the condition of Theorem 3.1 is satisfied. By condition (b) we have $\gamma(W, t) \neq 0$. Hence the extension (6) does not split by Theorem 3.1.

**Remark 3.5** The union $G(2, T) \cup \mathbb{P}(\wedge^2 W^\vee)$ is a generic syzygy scheme; see [6, Theorem 6.1]. In [loc. cit., Theorem 6.7] it was shown that a rank $p+2$ syzygy gives rise to a rank 2 vector bundle if $L$ is very ample and the ideal of $X$ is generated by quadrics.

The condition of Theorem 3.1 can be reinterpreted in terms of surjectivity of a natural multiplication map.

**Proposition 3.6** Let $X$ be a smooth curve, and let $W \subset H^0(X, L)$ be a linear subspace. We put $B = Bs(W)$ and view $W$ as a base–point free linear subspace of $H^0(X, L(-B))$. Let

$$\mu : W \otimes H^0(X, K_X(-B)) \to H^0(K_X \otimes L(-2B))$$

be the multiplication map. The following conditions are equivalent.

(i) The map $\mu$ is not surjective;
There exists a non-split extension

$$0 \to \mathcal{O}_X(B) \to E \to L(-B) \to 0$$

such that $W$ is contained in the kernel of the map $\delta : H^0(X, L(-B)) \to H^1(X, \mathcal{O}_X(B))$.

**Proof:** We first show that (i) implies (ii). Since $\mu$ is not surjective, there exists a hyperplane $H \subset H^0(X, K_X \otimes L(-B))$ that contains $\text{im}(\mu)$. Let $\eta$ be a linear functional defining $H$. Put $0 \neq \xi = \eta^\vee \in H^1(X, L^{-1}(B))$, and let

$$0 \to \mathcal{O}_X(B) \to E \to L(-B) \to 0$$

be the corresponding non-split extension. Given $w \in W$ and $v \in H^0(X, K_X(-B))$, the formula

$$\delta(w)(v) = (\eta \circ \mu)(w \otimes v)$$

(7)

shows that $W$ is contained in the kernel of $\delta$.

For the converse, note that formula (7) implies that $\eta|_{\text{im} \mu} \equiv 0$. □

**Remark 3.7** If $B$ is a fixed divisor, the result of the previous Proposition follows from Green’s duality theorem [4, Corollary (2.c.10)]. Indeed,

$$\text{coker } \mu \cong K_{0,1}(X, K_X(-B), L(-B), W) \cong K_{p,1}(X, B, L(-B), W)^\vee$$

(8)

and since $h^0(X, \mathcal{O}_X(B)) = 1$ we have an injection

$$K_{p,1}(X, B, L(-B), W) \hookrightarrow K_{p,1}(X, L).$$

Theorem 3.4 shows that Voisin’s method may produce nontrivial Koszul classes that are not contained in the space $K_{p,1}(X, L)_{GL}$ spanned by Green–Lazarsfeld classes.

**Example 3.8** By [2, Theorem 3.6 and Theorem 4.3] there exists a smooth curve of genus 14 and Clifford index 5 whose Clifford index is computed by a unique line bundle $L$ such that $L^2 = K_X$. The line bundle $L$ embeds $X$ in $\mathbb{P}^4$ as a projectively normal curve of degree 13 which is not contained in any quadric of rank $\leq 4$, and the ideal of $X$ is generated by the $4 \times 4$ Pfaffians of a skew–symmetric matrix $(a_{ij})_{1 \leq i,j \leq 5}$ with

$$\deg(a_{ij}) = \begin{cases} 2 & \text{if } i = 1 \text{ or } j = 1 \\ 1 & \text{if } i \geq 2 \text{ and } j \geq 2 \end{cases}$$

such that the quadric $Q = a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34}$ has rank 5.

By [loc.cit.] the group $K_{1,1}(X, L)$ is generated by $[Q]$, hence $I_X$ contains no quadrics of rank $\leq 4$. If $K_{1,1}(X, L)$ contains a Green–Lazarsfeld class this class would be of scrollar type, since it necessarily comes from two pencils $|L_1|, |L_2|$. This is impossible, since classes of scrollar type give rise to quadrics of rank $\leq 4$.

The Koszul class $[Q] \in K_{1,1}(X, L)$ has rank 3, since it is represented by the linear subspace $W = \langle a_{23}, a_{24}, a_{25} \rangle$. Hence $[Q]$ comes from Voisin’s method by Theorem 3.4.
Remark 3.9 A more geometric description of a subspace $W$ representing $[Q]$ is the following. A smooth intersection of the quadric $V(Q) \subset \mathbb{P}H^0(X, L)^\vee$ with one of the cubic Pfaffians is a $K3$ surface in $\mathbb{P}H^0(X, L)^\vee$ containing a line $\ell$ which is disjoint from $X$ by [2] Prop. 4.1. The line $\ell$ corresponds to a 3-dimensional linear subspace $W \subset H^0(X, L)$, which is base-point-free since $\ell$ does not meet $X$.

One could ask whether the syzygies constructed in section 2.1 span $K_{p,1}(X, L)$. In principle it may be possible to obtain higher rank syzygies as linear combinations of rank $p + 2$ syzygies. However, if $K_{p,1}(X, L)$ is spanned by a single syzygy of rank $\geq p + 3$ this is not possible.

Example 3.10 (Eusen–Schreyer) Eusen and Schreyer [3, Theorem 1.7 (a)] have constructed a smooth curve $X \subset \mathbb{P}^5$ of genus 7 and Clifford index 3 embedded by the linear system $|K_X(-x)|$ such that $K_{2,1}(X, K_X(-x)) \cong \mathbb{C}$ is spanned by a syzygy $s_0$. The explicit expression for $s_0$ given on p.8 of [loc. cit.] shows that $s_0$ is a rank 5 syzygy. Hence $s_0$ cannot be obtained by the Green–Lazarsfeld construction or the method of section 2.1.

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