Extended chiral group and scalar diquarks

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Abstract

We introduce extended chiral transformation, which depends both on pseudoscalar and diquark fields as parameters and determine its group structure. Assuming soft symmetry breaking in diquark sector, bosonisation of a quasi-Goldstone $ud$-diquark is performed. In the chiral limit the $ud$-diquark mass is defined by the gluon condensate, $m_{ud} \approx 300\text{MeV}$. The diquark charge radius is $\langle r_{ud}^2 \rangle^{1/2} \approx 0.5\text{fm}$. We consider also the flavour triplet of scalar diquarks ($ud$), ($us$) and ($ds$) together with pseudoscalar mesons and calculate diquark masses and decay constants in terms of meson parameters and the gluon condensate.
1 Introduction.

Diquarks, which were introduced almost three decades ago [1], became now efficient tool for studying various processes in hadron physics (see e.g. [2,3] and reviews [4]). The diquark model was analysed from various points of view. However, the complete picture is still lacking.

It was suggested by Dosch et al [3] that wave function of pion and $ud$-diquark are the same at the origin. It was also shown [3] that the diquark decay constant following from this suggestion is very close to that estimated from the QCD sum rules.

In this paper we propose to go a bit further: we suppose that the similarity of wave functions of pion and diquark is due to common origin as parameters of a certain anomalous transformation which does not preserve the measure of the quark path integral. While at the classical level the chiral symmetry is broken by quark mass, the extended chiral ($E\chi$) symmetry is broken by quark mass and gluon fields. $E\chi$-group is $U(2N)$ for $N$ internal degrees of freedom, $N = N_c N_f$. Non-anomalous (measure preserving) generators span the Lie algebra of $O(2N)$, anomalous generators belong to the coset $U(2N)/O(2N)$. Anomalous generators describe chiral rotations and transformations with diquark variables (“diquark” rotations), non-anomalous part consists out of gauge transformations and combined chiral “diquark” rotations [5].

We assume that $E\chi$-symmetry breaking due to quark masses and gluon fields is soft in the sense that the action for bosonised diquark fields can be obtained by integrating corresponding $E\chi$-anomaly. Colorless chiral fields after bosonisation give rise to Goldstone particles – pseudoscalar mesons. We suggest that at low energies bosonised diquark parameters of $E\chi$-transformations with quantum numbers of lightest $J^P = 0^+$ $ud$-diquark can be treated as a Goldstone-like particle. Therefore, in bosonisation we restrict ourselves to the case of $E\chi$-transformations with $ud$-diquark fields. The $E\chi$-group in this case is $SU(4)$, non-anomalous transformations are just gauge transformations $SU(3) \times U(1)$ and the diquark Goldstone degrees of freedom belongs to $C P^3 = SU(4)/SU(3) \times U(1)$ [5].

Our analysis shows that the $ud$-diquark introduced a la Goldstone becomes massless in the limit of vanishing gluon condensate and current quark masses. Furthermore, we calculate the diquark mass and charge radius for the actual value of gluon condensate. The obtained values fall into the region allowed in other models [4]. Note, that our approach is a direct generalization
of chiral bosonisation scheme [6] for the case of new anomalous transformation. We introduce no new parameters.

In the section four we investigate $E\chi$ - action for $N_f = 3$ taking into account quark masses [7]. We obtain expressions which determine masses and decay constants of scalar diquarks ($ud$), ($us$) and ($ds$). We also derive relations connecting masses and decay constants of scalar diquarks and pseudoscalar mesons $\pi$ and $K$. By these calculations we demonstrate advantages of $E\chi$ group. Though diquarks are not physical particles, their masses and decay constants should be considered at the same level as quark masses. Applications to nucleon and scattering will be considered in a separate paper.

2 Group structure of $E\chi$-transformations.

It was demonstrated [8] that in order to consider quark-antiquark and quark-quark composites on equal footings one should introduce eight-component spinors $\Psi$ constructed from ordinary Dirac spinors $\psi$

$$\Psi = \begin{pmatrix} \psi \\ \bar{\psi}^T \end{pmatrix}$$  \hspace{1cm} (1)

The quark lagrangian can be rewritten in the form

$$\mathcal{L} = \frac{1}{2} \Psi^T \hat{F} \Psi, \hspace{1cm} \hat{F} = \begin{pmatrix} C\Phi & -D^T \\ D & \Phi^C \end{pmatrix} \hspace{1cm} F = -F^T$$  \hspace{1cm} (2)

where $D$ is the Dirac operator $D = i\gamma^\mu (\partial_\mu + v_\mu + \gamma_5 a_\mu)$, $^{\text{"a}T\text{"m}}$ means transposition and $\Phi = \gamma^\mu (\phi_{5\mu} + \gamma_5 \phi_{\mu})$, $\Phi = \gamma_0 \Phi^+ \gamma_0$. We have introduced various external fields $v_\mu$, $a_\mu$, $\phi_{\mu}$ and $\phi_{5\mu}$ generating both $\bar{\psi}\psi$ and $\psi\bar{\psi}$ composites. $C$ is charge conjugation matrix. The quark path integral becomes

$$Z_\psi = \int \mathcal{D}\Psi \exp i \int d^4x \mathcal{L} = (\det \hat{G})^{1/2}$$

$$\hat{G} = \begin{pmatrix} D & \Phi \\ \Phi & D_c \end{pmatrix}, \hspace{1cm} D_c = C^{-1}D^TC,$$

$$\hat{F} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & C^{-1} \end{pmatrix} \hat{G} \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}$$  \hspace{1cm} (3)

where the operator $\hat{G}$ is $\gamma_0$-hermitian. A similar construction was considered by Ball [9] for Majorana spinors.
The lagrangian (2) is invariant under the following transformations

\[ \delta \Psi = -\Omega \Psi, \quad \Omega = \begin{pmatrix} \alpha + \gamma_5 \chi & (\xi + \gamma_5 \omega) \gamma_5 \alpha^* - \gamma_5 \chi^* \\ -\xi^* + \gamma_5 \omega^* \end{pmatrix} \]

(4)

provided external fields in the operators \( \hat{G} \) and \( \hat{F} \) transform according to the rules

\begin{align*}
\hat{F} \to \hat{F}' &= \exp \Omega^T \hat{F} \exp \Omega \\
\hat{G} \to \hat{G}' &= \exp(-\Xi + \gamma_5 \Theta) \hat{G} \exp(\Xi + \gamma_5 \Theta)
\end{align*}

(5)

\[ \Xi = \begin{pmatrix} \alpha & \xi \\ \xi^* & \alpha^* \end{pmatrix} \quad \Theta = \begin{pmatrix} \chi & \omega \\ -\omega^* & -\chi^* \end{pmatrix}. \]

The matrices \( \alpha \) and \( \chi \) are antihermitian, \( \xi \) is antisymmetric and \( \omega \) is symmetric in internal indices. The transformations (4) do not destroy the structure (1) of the eight-component spinor \( \Psi \). These transformations can be absorbed in transformations of background fields.

Due to the noninvariance of the measure, only part of the transformations (4) do not change the path integral (3). These are the transformations generated by \( \Xi \). The generators \( \Theta \) lead to quantum anomalies. The operators \( \alpha \) generate gauge transformations, \( \chi \) describe chiral rotations, the anomalous transformations \( \omega \) include fields with diquark quantum numbers, the generators \( \xi \) are needed for closure of the algebra.

The matrix commutator

\[ [\Xi(\alpha_1, \chi_1) + \gamma_5 \Theta(\xi_1, \omega_1), \Xi(\alpha_2, \chi_2) + \gamma_5 \Theta(\xi_2, \omega_2)] = \Xi(\alpha_3, \chi_3) + \gamma_5 \Theta(\xi_3, \omega_3) \]

(6)

induces the following Lie structure

\begin{align*}
\alpha_3 &= [\alpha_1, \alpha_2] + [\chi_1, \chi_2] + \xi_1 \xi_2 - \xi_2 \xi_1^* - \xi_1 \omega_2^* + \omega_2 \omega_1^*, \\
\chi_3 &= [\alpha_1, \chi_2] - [\alpha_2, \chi_1] - \xi_1 \omega_2^* - \xi_2 \omega_1^* + \xi_2 \omega_1^* + \omega_1 \xi_2^*, \\
\xi_3 &= \alpha_1 \xi_2 + \xi_1 \alpha_2^* - \alpha_2 \xi_1 - \xi_2 \alpha_1^* + \chi_1 \omega_2 - \omega_1 \chi_2^* - \chi_2 \omega_1 + \omega_2 \chi_1^*, \\
\omega_3 &= \alpha_1 \omega_2 + \chi_1 \xi_2 - \xi_1 \chi_2^* + \omega_1 \alpha_2^* - \alpha_2 \omega_1 - \chi_2 \xi_1 + \xi_2 \omega_1^* - \omega_2 \xi_1^*. \quad (7)
\end{align*}

One can verify that the composition laws (8) are induced also by the matrix commutator without \( \gamma_5 \)

\[ [\Xi(\alpha_1, \chi_1) + \Theta(\xi_1, \omega_1), \Xi(\alpha_2, \chi_2) + \Theta(\xi_2, \omega_2)] = \Xi(\alpha_3, \chi_3) + \Theta(\xi_3, \omega_3) \]

(8)
This means that the Lie algebras (6) and (8) are isomorphic. For the case of
$N$ internal degrees of freedom, $N = N_c N_f$, and maximally extended algebra
(i.e. when $\alpha, \chi, \xi$ and $\omega$ are all matrixes satisfying the above mentioned
hermiticity and symmetry properties), the $\Xi + \Omega$ form the space of hermitian
matrices $2N \times 2N$. Hence the algebra (8) is $U(2N)$. The generators $\Xi$
preserve symmetric non-degenerate bilinear form $O$

$$O = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Xi O + O \Xi^T = 0.$$  \hspace{1cm} (9)

Consequently the non-anomalous generators $\Xi$ span the Lie algebra of $O(2N)$
and the anomalous generators belong to the coset $U(2N)/O(2N)$. The generators $\alpha$
and $\omega$ preserve symplectic form

$$\Sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$  \hspace{1cm} (10)

and thus span the Lie algebra of the subgroup $Sp(N)$. The generators obviously form $U(N)$.

In principle, any transformation of $\Theta$ could be related to a Goldstone
particle, whose dynamics is governed by quantum anomalies. However in
realistic models most of the symmetries (4) are broken already at classical
level by the presence of quark masses and gluon fields, and the vector field
prescribed by the transformation rules (8). Only colorless chiral fields $\chi$
are definitely interpreted as pseudoscalar mesons. Some other states were also
considered in literature [8]. We suggest, that at certain energy scale the fields $\omega$
with quantum numbers of lightest $J^P = 0^+ ud$-diquarks can also generate
Goldstone-like particles. In what follows we shall restrict ourselves to the
transformations

$$\omega = (1/f_\omega) \omega_c (i\sigma_2)_{jk} \epsilon_{abc}$$  \hspace{1cm} (11)

corresponding to $0^+ ud$-diquarks where $j, k$ are flavor and $a, b, c$ are color
indices.

In this special case transformations close in a smaller group. To see this
one should exclude the $i\sigma_2$ in the same way, as it was done previously with $\gamma_5$, and
use the commutation relations (4). One can obtain that after removing
$i\sigma_2$ and $\gamma_5$ the algebra becomes formally equivalent to that generated by the
$\alpha$ and $\xi$ operators in the case $N = 3$. Hence, the complete group is $O(6) \sim$
SU(4), and the non-anomalous transformations, that are now represented by \( \alpha \) generators, belong to \( U(3) \sim SU(3) \times U(1) \). The anomalous (Goldstone) diquark degrees of freedom belong to the complex projective space \( CP^3 = SU(4)/SU(3) \times U(1) \). The same result could be obtained in a straightforward but tedious way by computing matrix commutators in an appropriate basis.

As a consistency check we shall demonstrate that the diquark mass vanishes for zero gluon condensate and zero current quark masses. We shall also compute the diquark mass and charge radius for actual value of gluon condensate.

### 3 The diquark bosonisation. Gluon condensate as a source of diquark mass.

To define the diquark parameters we should regularize the quark path integral. We also need a method of extracting a non-invariant part of the path integral corresponding to anomalous transformations.

To reduce possible regularization dependence [11] we shall use exactly the same scheme [6] which was developed for chiral bosonisation and generalized [8] for the presence of diquark variables. Since the parameters of this scheme were defined through chiral dynamics, we will be able to compare our results for diquark with pion physics directly.

The basic object is the quark path integral over low scale region

\[
Z^L_\psi = \left( \det \{ \hat{G} \theta (1 - (\hat{G} - M^2) / \Lambda^2) \} \right)^{1/2}
\]

\[
\theta(x) = \int_{-\infty}^{\infty} dt \frac{\exp (ixt)}{2\pi i (t - i\theta)}.
\]  

(12)

The parameters \( \Lambda \) and \( M \) are defined below. The functional (12) can be represented in the form

\[
Z^L_\psi = (Z^L_\psi Z^{-1}_{\text{inv}}) Z_{\text{inv}}
\]

\[
Z^{-1}_{\text{inv}} = \int \mathcal{D}\Theta (Z^L_\psi(\Theta))^{-1}
\]  

(13)

where we integrate over anomalous transformations \( \Theta \), \( \mathcal{D}\Theta \) is invariant measure on the corresponding coset space and \( (Z^L_\psi(\Theta)) \) is the path integral (12).
with background fields transformed as in eq. (5). The $Z_{\text{inv}}$ does not depend on degrees of freedom described by $\Theta$. Hence all information over $\Theta$-noninvariant processes is contained in $(Z^L_\psi Z^{-1}_{\text{inv}})$ and the effective action for $\Theta$ can be defined as

$$ (Z^L_\psi Z^{-1}_{\text{inv}}) = \int \mathcal{D}\Theta \exp(iW_{\text{eff}}(\Theta)) $$

(14)

The effective action is obtained by integration of the corresponding anomaly $\mathcal{A}(x)$

$$ \mathcal{A}(x; \Theta) = \frac{1}{i} \delta \ln Z^L_\psi(\Theta) \delta \Theta $$

$$ W_{\text{eff}}(\Theta) = -\int d^4x \int_0^1 ds A(x; s\Theta) \Theta(x) $$

(15)

Previously [5] this method was applied to $\pi$-mesons.

An important dynamical information is contained in parameters $\Lambda$ and $M$. These parameters define a low-energy region of the model [6]. The quark and the gluon condensates are related to $\Lambda$ and $M$ in the following way

$$ C_g = \frac{3N_c}{2\pi^2}(6\Lambda^2M^2 - \Lambda^4 - M^4), \quad C_q = \frac{N_c}{2\pi^2}(\Lambda^2M - \frac{1}{3}M^3). $$

(16)

It was found also that these parameters are related to the pion decay constant by the formula

$$ f_\pi^2 = \frac{N_c}{2\pi^2}(\Lambda^2 - M^2) $$

(17)

We will not report here details of computations of $W_{\text{eff}}(\omega)$ for the case $\Theta = \omega$, where $\omega$ is given by (13). They can be performed in same manner as in the papers [5,6]. Neglecting all external fields except vector gauge fields

$$ v_\mu = -iQA_\mu + \frac{\lambda^a}{2i}G^a_\mu, \quad Q = \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix} $$

(18)

where $A_\mu$ is electromagnetic field, $G^a_\mu$ are gluons and taking zero current quark masses we obtain in quadratic order of $\omega$

$$ W_{\text{eff}}(\omega) = \frac{1}{192\pi^2 f_\omega^2} \text{tr} \left\{ 6(\Lambda^2-M^2)[D_\mu,\omega^*][D^\mu,\omega] \\
+ [D_\mu, [D^\mu, \omega^*]] [D_\nu, [D^\nu, \omega]] + 2[D_\mu, F^{\mu\nu}](\omega[D_\nu, \omega^*] + [D_\nu, \omega] \omega^*) \\
+ (F^{\mu\nu} F_{\mu\nu} \omega^* - F^{\mu\nu} \omega F_{\mu\nu}^T \omega^*) \right\} $$

(19)
where \([D_\mu, \omega] = (\partial_\mu \omega) + v_\mu \omega + \omega v_\mu^T, [D_\mu, \omega^*] = (\partial_\mu \omega^*) - v_\mu^T \omega^* - \omega^* v_\mu\) and \(F_{\mu\nu} = (\partial_\mu v_\nu) - (\partial_\nu v_\mu) + [v_\mu, v_\nu].\) From (19) we see that the mass of ud-diquark \(\omega\) is defined by the gluon condensate \(\langle G^2_{\mu\nu} \rangle (N_c = 3)\)

\[
M^2_\omega = -2\pi^2 f^2 + \sqrt{4\pi^4 f^4 + \frac{\langle G^2_{\mu\nu} \rangle}{12}}
\]

and vanishes when \(\langle G^2_{\mu\nu} \rangle \rightarrow 0.\) For derivation of (20) we used

\[
\langle G^a_{\mu\nu} G^b_{\mu\nu} \rangle = \frac{1}{8} \delta^{ab} \langle G^2_{\mu\nu} \rangle
\]

For \(\langle G^2_{\mu\nu} \rangle = (365 MeV)^4\) we get \(M_\omega \approx 300 MeV.\) The correction of this evaluation due to quark masses is provided by \(M^2_\omega(m_q \neq 0) = M^2_\omega(m_q = 0) + m^2_{\pi}.\) This gives \(M^2_\omega(m_q \neq 0) \approx 340 MeV,\) which falls into the region allowed in the other models [4], though lies close to the lower boundary. \(f_\omega\) is defined by requirement that the residue of the diquark propagator at \(k^2 = M^2_\omega\) is unity,

\[
f^2_\omega = \frac{\Lambda^2 - M^2}{2\pi^2} + \frac{1}{6\pi^2} M^2_\omega
\]

The coefficient before the term \(\partial^2 A^\mu(\omega^*(\partial_\mu \omega) - (\partial_\mu \omega^*)\omega)\) allows us to evaluate the mean square radius of diquark charge distribution

\[
\langle r^2 \rangle^{1/2} \approx 0.5 fm
\]

This value is also compatible with other data [4] for diquark effective radius.

Our desire to describe diquarks as a quasi-particle similar to \(\pi\) meson has more than aesthetic grounds. This model allows to explain relatively low mass of the scalar diquark and include diquark variables in framework of current algebra and chiral perturbative theory. As far as we were able to verify, this suggestion does not lead to any contradictions. We obtained quite sensible results for diquark mass and charge radius. The model has no free parameters. All this indicates that broken \(E_\chi\)-symmetry deserves further investigations.

4 \(E_\chi\) – action for \(SU(3)\) flavour structure.

It was demonstrated in the section one that in order to consider quark-antiquark and quark-quark composites on an equal footing one should introduce an eight-component spinor \(\Psi\) constructed from ordinary Dirac spinors...
\[ \psi \]

\[ \Psi = \begin{pmatrix} \psi \\ \bar{\psi}^T \end{pmatrix} \]

In this section we consider $SU(3)$ flavour structure of the fermion field and transformations of the form:

\[ \delta \Psi = \Theta \Gamma_5 \Psi, \quad \Theta = \begin{pmatrix} \chi & C\omega \\ C\omega^* & \chi^T \end{pmatrix}, \] (24)

where $\Gamma_5$ is a block-diagonal matrix constructed from the $\gamma_5$ matrixes, $\chi$ contains pseudoscalar meson states ($J^P = 0^-$) and $\omega$ contains scalar diquark states ($J^P = 0^+$)

\[ \chi = \chi(1), \quad \omega = \omega(3). \] (25)

The representations of the colour group $SU(3)$ are indicated explicitly. Thus, $\chi(1)$ contains only flavour octet (the singlet component connected with $U_A(1)$–current is excluded as it is in the chiral theory), $\omega(3)$ is antitriplet $(\bar{3}_c \otimes \bar{3}_f)$.

Let us introduce the $E\chi$-field according to (2)

\[ U = - \exp i\Theta \] (26)

Introducing scalar diquarks on the equal footing with pseudoscalar mesons in the field $U$ can be considered as an alternative for the method, based on composite quark-quark fields [12]. In the present approach states with quantum numbers $J^P = 0^+$ in quark-quark sector occupy the same place as states with $J^P = 0^-$ in quark-antiquark sector (the space part of the wave function is symmetrical, fermions are in $S$-wave). The distinctions appear at dynamical level by analysis of the effective action. In particular, diquark states are massive in the chiral limit.

Let us present $E\chi$-action (15) It would be convenient to write $W_{\text{eff}} = W + W'$ where $W$ is $SU(N_f)_L \otimes SU(N_f)_R$ invariant part of $E\chi$–action (zero order of quark masses) and part $W'$ comes from soft chiral $SU(N_f)_L \otimes SU(N_f)_R$ symmetry breaking (due to quark masses ).

The Lagrangian corresponding to the action $W$ has the standard form [6]

\[ \mathcal{L} = \frac{\Lambda^2 - M^2}{32\pi^2} \text{tr} \begin{pmatrix} D_\mu \mathcal{U} \end{pmatrix} \begin{pmatrix} D^\mu \mathcal{U} \end{pmatrix}^\dagger + \frac{1}{192\pi^2} \text{tr} \begin{pmatrix} D_\mu^2 \mathcal{U} \end{pmatrix} \begin{pmatrix} D_\mu^2 \mathcal{U} \end{pmatrix}^\dagger \\
+ \frac{1}{2} \begin{pmatrix} D_\mu \mathcal{U} \end{pmatrix} \begin{pmatrix} D_\mu \mathcal{U} \end{pmatrix}^\dagger \begin{pmatrix} D^\mu \mathcal{U} \end{pmatrix} \begin{pmatrix} D^\mu \mathcal{U} \end{pmatrix}^\dagger - \left( \begin{pmatrix} D_\mu \mathcal{U} \end{pmatrix} \begin{pmatrix} D^\mu \mathcal{U} \end{pmatrix} \right)^2 \]
\[
+2(D_\mu F^{\mu\nu}) \left( (D_\nu U) U^\dagger + (D_\nu U)^\dagger U \right) - \frac{1}{2} [F_{\mu\nu}, U][F^{\mu\nu}, U^\dagger] \right) \}
\]

(27)

The Wess - Zumino - Witten action will not be considered here. The action \( W' \) containing terms with quark masses is determined by Lagrangian

\[
\mathcal{L}' = -\frac{1}{8\pi^2 (b,c,f)} \left\{ \left( \Lambda^2 M - \frac{M^3}{3} \right) \hat{m}_q (U + U^\dagger) + \frac{\Lambda^2 - M^2}{4} (U \hat{m}_q U \hat{m}_q + \hat{m}_q U^\dagger \hat{m}_q U^\dagger) \right\} - \frac{M}{2} \hat{m}_q (D_\mu D^\mu U + D_\mu D^\mu U^\dagger) + O(\hat{m}_q^3),
\]

(28)

where matrix \( \hat{m}_q \) describes quark masses

\[
\hat{m}_q = 1_b \otimes 1_c \otimes \text{diag}(m_u, m_d, m_s).
\]

(29)

The covariant derivative \( D_\mu \) is \( (D_\mu)^* = (\partial_\mu)^* + [V_\mu, *] \), \( V_\mu \) and its field strength \( F_{\mu\nu} \) are block - diagonal matrixes depending only on gluon field \( G_\mu \) and its field strength correspondingly

\[
V_\mu = \begin{pmatrix} G_\mu & 0 \\ 0 & -G_\mu^T \end{pmatrix}, \quad F_{\mu\nu} = \begin{pmatrix} G_{\mu\nu} & 0 \\ 0 & -G_{\mu\nu}^T \end{pmatrix}.
\]

(30)

Let us note that the global flavour invariance \( SU(3)_L \otimes SU(3)_R \) of lagrangian [27] is realized as follows

\[
U \rightarrow \begin{pmatrix} \ell & 0 \\ 0 & r^{-1}T \end{pmatrix} U \begin{pmatrix} r & 0 \\ 0 & \ell^T \end{pmatrix}, \quad \ell \in SU(3)_L, \quad r \in SU(3)_R.
\]

(31)

It is easy to see that though field \( U \) has a complicated structure (when \( N_f = 3 \) the field \( U \) is an element of \( SU(18)/O(18) \) coset space (see section two)), global symmetries of lagrangian \( \mathcal{L} \) wider than \( SU(3)_L \otimes SU(3)_R \) are not possible in the presence of coloured gluon fields in the covariant derivative.

Let us calculate masses and decay constants of 3 scalar diquarks. Only these fields can form colourless baryon states with a third quark. For analysis of diquark parameters we consider decay constants and masses of pseudoscalar mesons as parameters known from an experiment and express dynamical characteristics of scalar diquarks in terms of the gluon condensate and these parameters.
Let us suppose for simplicity that \( m_u = m_d =: m \), i.e.

\[
\hat{m}_q = 1_b \otimes 1_c \otimes \text{diag}(m, m, m)
\]

in (28). The matrix of pseudoscalar mesons is given by

\[
\chi_{1c} = \begin{pmatrix}
    \pi^0 & \pi^+ & K^+ \\
    \sqrt{2}f_\pi & f_\pi & f_K \\
    \frac{\pi^-}{f_\pi} & \frac{-\pi^0}{f_\pi} & \frac{K^0}{f_K} \\
    \frac{K^-}{f_K} & \frac{K^0}{f_K} & 0
\end{pmatrix}
\]

where \( f_\pi \) and \( f_K \) are decay constants of \( \pi \)-meson and \( K \)-meson correspondingly, \( f_\pi \approx 132 \text{Mev}, f_K \approx 165 \text{Mev} \). The matrix \( \omega^{(3)} \) in (23) containing fields of 3 scalar diquarks has the following form:

\[
(\omega^{(3)})^{ab}_{ij} = \epsilon^{abc} \epsilon_{ijk} \cdot \frac{\omega^c_{\omega_k}}{f_{\omega_k}},
\]

where \( a, b, c \) are colour indexes, \( i, j, k \) are flavour indexes and \( \omega^c_1, \omega^c_2, \omega^c_3 \) are fields of \((ud), (us), (ds)\) diquarks, \( f_{\omega_1}, f_{\omega_2}, f_{\omega_3} \) are corresponding diquark decay constants. Owing to residual \( SU(2)_f \) invariance (32) \( f_{\omega_1} = f_{\omega_2} \equiv f_{\omega_{1,2}} \).

Consider the propagators \( S(p^2) \) (in the momentum representation). The presence of the “tachyonic” term \((D^2U)(D^2U^\dagger)\) in (27) leads to appearance of terms proportional to \( p^4 \). Let us introduce a real parameter \( \alpha \in [0, 1] \) replacing \((D^2U)(D^2U^\dagger)\) by \( \alpha(D^2U)(D^2U^\dagger) \) to facilitate discussing of the “tachyonic” term. The solution of the equation

\[
S^{-1}(p^2) = 0
\]

gives us the mass. The physical mass \( m_{\text{phys}} \) is the solution \( p^2 = m^2_{\text{phys}} > 0 \). The other solution \( p^2 = -m^2_{\text{tach}} < 0 \) gives non-physical “tachyonic” pole of the propagator \( m_{\text{tach}} \rightarrow \infty \) when \( \alpha \rightarrow 0 \). The decay constant is determined by the condition

\[
\text{res}_{p^2=m^2_{\text{phys}}} S(p^2) = 1.
\]

We consider first the meson sector. The inverse propagator of \( \pi \)-meson has the form

\[
S^{-1}_{\pi}(p^2) = \frac{\alpha}{4\pi^2 f_\pi} p^4 + \frac{A_\pi}{f_\pi^2} p^2 - \frac{B_\pi}{f_\pi^2},
\]
where \( A_\pi \) and \( B_\pi \) are constants of dimensions two and four correspondingly. The propagator of \( K \)-meson has also the form (37) with constants \( f_K, A_K, B_K \) instead of \( f_\pi, A_\pi, B_\pi \). Masses and decay constants of \( \pi \) and \( K \) mesons can be expressed in the terms of constants \( A_\pi, B_\pi \) and \( A_K, B_K \) with the help of (35) and (36). Inversion of these expressions gives:

\[
A_\pi = f_\pi^2 \left( 1 - \alpha \frac{m_\pi^2}{2\pi^2 f_\pi^2} \right), \quad B_\pi = m_\pi^2 f_\pi^2 \left( 1 - \alpha \frac{m_\pi^2}{4\pi^2 f_\pi^2} \right),
\]

\[
A_K = f_K^2 \left( 1 - \alpha \frac{m_K^2}{2\pi^2 f_K^2} \right), \quad B_K = m_K^2 f_K^2 \left( 1 - \alpha \frac{m_K^2}{4\pi^2 f_K^2} \right).
\]

(38)

On the other hand, it follows from expressions for effective Lagrangians (27) and (28) that

\[
A_\pi = \frac{3}{2\pi^2} \left( \Lambda^2 - M^2 + 2Mm \right),
\]

\[
B_\pi = \frac{3}{2\pi^2} \left\{ 4 \left( \Lambda^2 M - \frac{M^3}{3} \right) m + 4 \left( \Lambda^2 - M^2 \right) m^2 + \mathcal{O}(m^3) \right\},
\]

\[
A_K = \frac{3}{2\pi^2} \left( \Lambda^2 - M^2 + M(m + m_s) \right),
\]

\[
B_K = \frac{3}{2\pi^2} \left\{ 2 \left( \Lambda^2 M - \frac{M^3}{3} \right) (m + m_s) + \left( \Lambda^2 - M^2 \right) (m + m_s)^2 + \mathcal{O}((m + m_s)^3) \right\}.
\]

(39)

We remind that meson masses and decay constants of mesons are supposed to be given parameters. Thus, the relations (38) define the constants \( A_\pi, B_\pi, A_K, B_K \). The expressions (39) are helpful when considering the diquark sector.

The inverse propagators of \((ds), (us)\) and \((ud)\) diquark fields \( \omega_1^c, \omega_2^c \) and \( \omega_3^c \) correspondingly can be presented in the form similar to (37)

\[
S_{\omega_1,2}^{-1}(p^2) = \frac{\alpha}{6\pi^2 f_{\omega_{1,2}}^2} p^4 + \frac{A_{\omega_{1,2}}}{f_{\omega_{1,2}}^2} p^2 - \frac{B_{\omega_{1,2}}}{f_{\omega_{1,2}}^2},
\]

\[
S_{\omega_3}^{-1}(p^2) = \frac{\alpha}{6\pi^2 f_{\omega_3}^2} p^4 + \frac{A_{\omega_3}}{f_{\omega_3}^2} p^2 - \frac{B_{\omega_3}}{f_{\omega_3}^2}.
\]

(40)
(the propagators of $\omega_1$ and $\omega_2$ are identical owing to residual $SU(2)_f$ symmetry \((32)\)). For masses and decay constants of diquarks in the terms of $A_{\omega_1,2}$, $B_{\omega_1,2}$, $A_{\omega_3}$, $B_{\omega_3}$ we obtain

\[
m_{\omega_1,2}^2 = \frac{3\pi^2 A_{\omega_1,2}}{\alpha} \left( \sqrt{1 + \frac{2\alpha B_{\omega_1,2}}{3\pi^2 A_{\omega_1,2}} - 1} \right), \quad f_{\omega_1,2}^2 = A_{\omega_1,2} \sqrt{1 + \frac{2\alpha B_{\omega_1,2}}{3\pi^2 A_{\omega_1,2}}},
\]

\[
m_{\omega_3}^2 = \frac{3\pi^2 A_{\omega_3}}{\alpha} \left( \sqrt{1 + \frac{2\alpha B_{\omega_3}}{3\pi^2 A_{\omega_3}} - 1} \right), \quad f_{\omega_3}^2 = A_{\omega_3} \sqrt{1 + \frac{2\alpha B_{\omega_3}}{3\pi^2 A_{\omega_3}}},
\]

\[(41)\]

The constants $A_{\omega_1,2}$, $B_{\omega_1,2}$, $A_{\omega_3}$, $B_{\omega_3}$, in their turn can be represented as a series of quark masses. The constants $B_{\omega_1,2}$, $B_{\omega_3}$ contain also the terms depending on the gluon condensate. Comparison of expressions for the constants $A_{\omega_1,2}$, $B_{\omega_1,2}$, $A_{\omega_3}$, $B_{\omega_3}$ with analogous expressions for the constants of the meson sector $A_{\pi}$, $B_{\pi}$, $A_{K}$, $B_{K}$ \((39)\) leads to the following identities:

\[
A_{\omega_1,2} = \frac{3}{2} A_K, \quad B_{\omega_1,2} = \frac{C_g}{18} + \frac{3}{2} B_K,
\]

\[
A_{\omega_3} = \frac{3}{2} A_{\pi}, \quad B_{\omega_3} = \frac{C_g}{18} + \frac{3}{2} B_{\pi},
\]

\[(42)\]

where $C_g$ is the gluon condensate. The constants $A_{\pi}$, $B_{\pi}$, $A_{K}$, $B_{K}$ are expressed in the terms of dynamical characteristics of mesons by formulae \((38)\). Substituting \((38)\) in \((42)\) and using expressions \((11)\) for $A_{\omega_1,2}$, $B_{\omega_1,2}$, $A_{\omega_3}$, $B_{\omega_3}$ in \((11)\) we finally have

\[
m_{\omega_1,2}^2 = \frac{2\pi^2}{\alpha} f_K \left( \sqrt{1 + \frac{\alpha C_g}{12\pi^2 f_K^4} - 1 + \frac{\alpha m_{K}^2}{2\pi^2 f_K^2}} \right), \quad f_{\omega_1,2}^2 = \frac{2}{3} f_K \sqrt{1 + \frac{\alpha C_g}{12\pi^2 f_K^4}},
\]

\[
m_{\omega_3}^2 = \frac{2\pi^2}{\alpha} f_{\pi} \left( \sqrt{1 + \frac{\alpha C_g}{12\pi^2 f_{\pi}^4} - 1 + \frac{\alpha m_{\pi}^2}{2\pi^2 f_{\pi}^2}} \right), \quad f_{\omega_3}^2 = \frac{2}{3} f_{\pi} \sqrt{1 + \frac{\alpha C_g}{12\pi^2 f_{\pi}^4}}.
\]

\[(43)\]

The expressions \((43)\) are the main result of the paper \([7]\).

The masses and decay constants of scalar diquarks are expressed in \((43)\) in terms of masses and decay constants of pseudoscalar mesons and the gluon condensate. The parameter $\alpha$ displays an influence of the “tachyonic” term. Its taking into account corresponds to $\alpha = 1$ according to \((27)\).
Though relations (42) were obtained by comparison series up to \( O(m_q^2) \) it is easy to see that these relations and consequently formulae (43) are valid in all orders. Indeed, the identities (42) are result of introducing scalar diquarks in the effective low-energy theory together with pseudoscalar mesons by means of \( E\chi \)-field. The specific structure of \( E\chi \)-field induces a co-ordinated influence of quark masses on dynamical parameters of fields that is expressed in particular as identities (42) and in the end relations (43).

The expressions (43) allow to estimate dynamical parameters of scalar diquarks. For values \( f_\pi = 132 \text{ Mev}, m_\pi = 139 \text{ Mev}, f_K = 165 \text{ Mev}, m_K = 495 \text{ Mev} \) and \( C_g = (365 \div 405 \text{ Mev})^4 \) the formulae with included “tachyonic” term give \( m_{\omega_3} = 310 \div 360 \text{ Mev}, f_{\omega_3} = 120 \div 125 \text{ Mev}, m_{\omega_{1,2}} = 545 \div 570 \text{ Mev}, f_{\omega_{1,2}} = 140 \div 145 \text{ Mev} \).

Using (43) it is easy to relate masses and decay constants of mesons and diquarks. Excluding the gluon condensate from (43) gives the following expressions:

\[
\frac{m_{\omega_{1,2}}^2 - m_K^2}{m_{\omega_3}^2 - m_\pi^2} = \frac{f_\pi^2 + (\alpha/4\pi^2)(m_{\omega_3}^2 + m_\pi^2)}{f_K^2 + (\alpha/4\pi^2)(m_{\omega_{1,2}}^2 + m_K^2)}
\]

\[
f_{\omega_3}^2 = \frac{2}{3} f_\pi^2 + \frac{\alpha}{3\pi^2} (m_{\omega_3}^2 - m_\pi^2), \quad f_{\omega_{1,2}}^2 = \frac{2}{3} f_K^2 + \frac{\alpha}{3\pi^2} (m_{\omega_{1,2}} - m_K^2).
\]

The second and the third relations in (44) can also be considered as an alternative presentations for decay constants of scalar diquarks. In the limit when \( \alpha = 0 \) and the tachyonic term is absent (44) is especially simple:

\[
f_\pi^2 m_{\omega_3}^2 - f_K^2 m_{\omega_{1,2}}^2 = f_\pi^2 m_\pi^2 - f_K^2 m_K^2, \quad f_{\omega_{1,2}}^2 = \frac{2}{3} f_K^2, \quad f_{\omega_3}^2 = \frac{2}{3} f_\pi^2.
\]

The last relation for \( f_{\omega_3} \) has already been obtained [3] from a calculation of colour degrees of freedom. In [3] the constants \( g_\pi \) and \( g_\omega \) are determined as

\[
g_\pi = \langle 0 | \bar{u}\gamma_5 d | \pi^+ \rangle = \sqrt{2} g_\omega \delta_{ad} = \langle 0 | \varepsilon^{abc} u_b^T C \gamma_5 d_c | \omega_d \rangle
\]
and the relation $g_\pi = g_\omega$ was obtained. We can also obtain $g_\pi$ and $g_\omega$ considering an interaction of the chiral field with an external pseudoscalar field $P$. As a result we have

$$
\frac{g_\pi}{m_\pi^2} \frac{1}{m_u + m_d} + \frac{1}{f_\pi} \frac{3M}{2\pi^2} - \frac{\alpha}{f_\pi} \frac{m_\pi^4}{4\pi^2} \frac{1}{m_u + m_d} \quad (47)
$$

For $g_\omega$ the expression is

$$
\frac{f_\pi}{f_\omega} \sqrt{\frac{2}{3}} g_\pi = g_\omega \quad (48)
$$

that agrees with result of [3].

In conclusion we would emphasize that formulae (43) for masses and decay constants of scalar diquarks, as well as relations (45) are based only on the assumption that both scalar diquarks and pseudoscalar mesons belong to the same chiral field of the extended chiral group $E_\chi$. Our estimates for dynamical characteristics of scalar diquarks give reasonable numerical values [2],[13] and this is an argument in favour of describing scalar diquarks by the effective $E_\chi$ chiral theory. Consequently, $E_\chi$-action can be used in low-energy diquark-meson physics.

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