Finite simple exceptional groups of Lie type in which all the subgroups of odd index are pronormal

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Abstract A subgroup $H$ of a group $G$ is said to be pronormal in $G$ if $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$. In this paper we classify finite simple groups $E_6(q)$ and $2E_6(q)$ in which all the subgroups of odd index are pronormal. Thus, we complete a classification of finite simple exceptional groups of Lie type in which all the subgroups of odd index are pronormal.

Keywords: finite group, simple group, exceptional group of Lie type, pronormal subgroup, odd index, Sylow 2-subgroup.

1 Introduction

Throughout the paper we consider only finite groups, and thereby the term ”group” means ”finite group”.

According to P. Hall [7], a subgroup $H$ of a group $G$ is said to be pronormal in $G$ if $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

Some of well-known examples of pronormal subgroups are the following: normal subgroups, maximal subgroups, Sylow subgroups, Sylow subgroups of proper normal subgroups, Hall subgroups of solvable groups.

In 2012, E. Vdovin and the third author [20] proved that the Hall subgroups are pronormal in all simple groups and, basing on some analysis of the proof, they conjectured that all the subgroups of odd index are pronormal in all simple groups. This conjecture was disproved in [13, 14]. Precisely, if $q \equiv \pm 3 \pmod{8}$ and $n \not\in \{2^m, 2^m(2^{2k} + 1) \mid m, k \in \mathbb{N} \cup \{0\}\}$, then the simple symplectic group $PSp_{2n}(q)$ contains a non-pronormal subgroup of odd index. Thus, the problem of classification of simple nonabelian groups in which the subgroups of odd index are pronormal naturally arises.

In [12], we confirmed the conjecture for many families of simple groups. Namely, it was proved that the subgroups of odd index are pronormal in the following simple groups: $A_n$, where $n \geq 5$, sporadic groups, groups of Lie type over fields of characteristic 2, $PSL_{2m}(q)$, $PSU_{2m}(q)$, $PSp_{2n}(q)$, where $q \not\equiv \pm 3 \pmod{8}$, $P\Omega_{2n+1}(q)$, $P\Omega^{\pm}_{2n}(q)$, exceptional groups of Lie type not isomorphic to $E_6(q)$ or $2E_6(q)$.

Moreover, in [14, 15] we have proved that if $q \equiv \pm 3 \pmod{8}$ and $n \in \{2^m, 2^m(2^{2k} + 1) \mid m, k \in \mathbb{N} \cup \{0\}\}$, then all the subgroups of odd index are pronormal in the simple symplectic group $PSp_{2n}(q)$. So, we received the complete classification of simple symplectic groups in which all the subgroups of odd index are pronormal.

We use the following notation: $E_6(q) = E_6^+(q)$ and $2E_6(q) = E_6^-(q)$.

In this paper, we prove the following theorem.
Theorem. Let $G = E_6^\varepsilon(q)$, where $\varepsilon \in \{+, -\}$, $q = p^m$, and $p$ is a prime. All the subgroups of odd index are pronormal in $G$ if and only if the following statements hold:

1. $q \not\equiv \varepsilon \pmod{18}$;
2. if $\varepsilon = +$, then $m$ is a power of 2.

So, we complete a classification of simple exceptional groups of Lie type in which all the subgroups of odd index are pronormal and receive the following result.

Corollary. Let $G$ be a simple exceptional group of Lie type. Then $G$ contains a non-pronormal subgroup of odd index if and only if $G \cong E_6^\varepsilon(q)$, where $\varepsilon \in \{+, -\}$, $q = p^m$ for prime $p$, and one of the following statements holds:

1. $q \equiv \varepsilon \pmod{18}$;
2. $\varepsilon = +$ and $m$ is not a power of 2.

The problem of classification of simple nonabelian groups in which all the subgroups of odd index are pronormal is still open for the following simple groups: $PSL_n(q)$ and $PSU_n(q)$, where $q$ is odd and $n$ is not a power of 2.

Note that a more detailed survey of investigations on pronormality of subgroups of odd index in finite (not necessarily simple) groups could be found in survey papers [6, 16]. These surveys contain a number of recent results and a number of conjectures and open problems. In particular, a problem of classification of direct products of nonabelian simple groups in which the subgroups of odd index are pronormal is of interest.

2 Preliminaries

Our terminology and notation are mostly standard and could be found, for example, in [1, 2, 3, 4, 9].

As usual, given a set $\pi$ of primes, $\pi'$ stands for the set of all primes not in $\pi$. Also, if $n$ is a positive integer, then $n_\pi$ is the largest natural divisor of $n$ such that all prime divisors of $n_\pi$ are in $\pi$.

For a group $G$ and a subset $\pi$ of the set of all primes, $O_\pi(G)$ and $Z(G)$ denote the $\pi$-radical (the largest normal $\pi$-subgroup) and the center of $G$. Also, it is common to write $O(G)$ instead of $O_2(G)$.

We denote by $E(G)$ the layer of $G$, i.e. the subgroup of $G$ generated by all components (subnormal quasisimple subgroups) of $G$.

The symmetric group of degree $n$ is denoted by $Sym_n$.

Let $P$ be a $p$-group, where $p$ is a prime, and let $n$ be a positive integer number. Following [3, page 17], we put

$$\Omega_n(P) = \langle x \in P \mid |x| \leq p^n \rangle$$

and

$$\mathfrak{U}^n(P) = \langle x^{p^n} \mid x \in P \rangle.$$  

The rank of an abelian group $G$ is the least number of generators of $G$. It is well-known that the rank of a subgroup of an abelian group $A$ is less or equal to the rank of $A$.

Lemma 1. [3, Chapter 5, Theorems 2.3 and 2.4] Let $P$ be an abelian $p$-group, where $p$ is a prime, and let $A$ be a $p'$-subgroup from $\text{Aut}(P)$. Then

$$P = C_P(A) \times [A, P],$$

and if $A$ acts trivially on $\Omega_1(P)$, then $A = 1$. 

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Lemma 2. [20, Lemma 5] Suppose that $G$ is a group and $H \leq G$. Assume also that $H$ contains a Sylow subgroup $S$ of $G$. Then the following statements are equivalent:

1. $H$ is pronormal in $G$;
2. the subgroups $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in N_G(S)$.

Lemma 3. [12, Lemma 5] Suppose that $H$ and $M$ are subgroups of a group $G$ and $H \leq M$. Then

1. if $H$ is pronormal in $G$, then $H$ is pronormal in $M$;
2. if $S \leq H$ for some Sylow subgroup $S$ of $G$, $N_G(S) \leq M$, and $H$ is pronormal in $M$, then $H$ is pronormal in $G$.

We use the following notation. Fix a prime $p$. Let $\mathfrak{X}_p$ be the class of all the groups in which a Sylow $p$-supgroup is self-normalized, and let $\mathfrak{Y}_p$ be the class of all the groups in which all the subgroups, whose indices are coprime to $p$, are pronormal.

Lemma 4. [14, Lemma 2] Let $X \triangleleft Y$, $X \in \mathfrak{X}_p$, and $Y/X \in \mathfrak{X}_p$, then $Y \in \mathfrak{X}_p$.

Lemma 5. [5, Lemma 6] Let $N$ be a normal subgroup of a group $G$ such that $G/N \in \mathfrak{X}_p$, and let $H$ be a subgroup of $G$ whose index in $G$ is coprime to $p$. Then the following statements are equivalent:

1. $H$ is pronormal in $G$;
2. $H$ is pronormal in $HN$.

Lemma 6. [5, Theorem 1] Let $G$ be a group and $A$ be a normal subgroup of $G$ such that $A \in \mathfrak{Y}_p$ and $G/A \in \mathfrak{X}_p$. Let $T$ be a Sylow $p$-subgroup of $A$. Then the following statements are equivalent:

1. $G \in \mathfrak{X}_p$;
2. $N_G(T)/T \in \mathfrak{Y}_p$.

Lemma 7. [13, Theorem 1] Let $H$ and $V$ be subgroups of a group $G$ such that $V$ is an abelian normal subgroup of $G$ and $G = HV$. Then the following statements are equivalent:

1. $H$ is pronormal in $G$;
2. $U = N_U(H)[H,U]$ for each $H$-invariant subgroup $U$ of $V$.

Lemma 8. Let $q$ be a power of an odd prime. Let $\alpha$ be a generator of a Sylow 2-subgroup of the multiplicative group $\mathbb{F}_q^*$ of the field $\mathbb{F}_q$. Then the elements $1, \alpha, \alpha^2, \ldots, \alpha^{2^k-1}$ form a basis of $\mathbb{F}_q$ as a vector space over its subfield $\mathbb{F}_p$.

Proof. Note that $\dim_{\mathbb{F}_p} \mathbb{F}_q = 2$. Thus, it is sufficient to prove that $\alpha \notin \mathbb{F}_q$. We have

$$|\mathbb{F}_p^*| = (q + 1)(q - 1) \geq 2(q - 1) > (q - 1)^2 = |\mathbb{F}_q^*|.$$ 

So, a Sylow 2-subgroup of $\mathbb{F}_q^*$ is not contained in $\mathbb{F}_q$. \qed

Lemma 9. Let $p$ be an odd prime and $q = p^{2^k}$. Assume that $\alpha$ is a generator of a Sylow 2-subgroup of the multiplicative group $\mathbb{F}_q^*$ of the field $\mathbb{F}_q$. Then elements $1, \alpha, \alpha^2, \ldots, \alpha^{2^k-1}$ form a basis of $\mathbb{F}_q$ as a vector space over its subfield $\mathbb{F}_p$. 

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Proof. We use inductive reasonings with respect to \( k \).

The case \( k = 0 \) is trivial, and in the case \( k = 1 \) we refer to Lemma \[2\].

Suppose that \( k > 1 \) and \( q_0 = p^{2^k-1} \). Note that \( q_0 \) is a square, therefore, \( q_0 + 1 \equiv 2 \pmod{4} \). We have

\[
|F^*_q|_2 = |F^*_q|_2 = (q_0 + 1)(q_0 - 1) = 2(q_0 - 1) = 2|F^*_q|_2.
\]

Therefore, \( \alpha^2 \in F^*_q \). Moreover, \( \beta = \alpha^2 \) is a generator of a Sylow 2 subgroup of \( F^*_q \).

Now using the inductive hypothesis we receive that the elements

\[
1, \beta, \beta^2, \ldots, \beta^{2^k-1}
\]

form a basis of \( F^*_q \) as a vector space over its subfield \( F_p \). Using Lemma \[2\] we receive that

\[
1, \alpha
\]

form a basis of \( F_q \) over its subfield \( F_{q_0} \). Thus, the set

\[
1, \alpha, \beta = \alpha^2, \alpha \beta = \alpha^3, \beta^2 = \alpha^4, \alpha \beta^2 = \alpha^5, \ldots, \beta^{2^k-1} = \alpha^{2^k-2}, \alpha \beta^{2^k-1} = \alpha^{2^k-1}
\]

form a basis of \( F_q \) over its subfield \( F_{q_0} \).

\[\square\]

Lemma 10. \[2\] Lemma 11] Each subgroup of odd index is pronormal in a simple group of Lie type over a field of characteristic 2.

Lemma 11. Let \( G = E^\varepsilon_6(q) \), where \( \varepsilon \in \{+, -\} \) and \( q \) is odd, and let \( S \) be a Sylow 2-subgroup of \( G \). Then the following statements hold:

1. \( Z(S) \) is a cyclic subgroup.
2. If \( t \) is an involution from \( Z(S) \) and \( C = C_G(t) \), then

\[
L := E(C) \cong Spin^\varepsilon_{10}(q),
\]

\[
Z := C_G(L) \cong Z(q-1)/(3,q-1),
\]

\[
C/Z \cong \text{Inndiag}(L/Z(L)) \cong P\Omega^\varepsilon_8(q), Z(4,q-1).
\]

3. \( N_G(S) = S \times R \), where \( R = O(C) \cong Z(q-1)/(3,q-1) \).
4. If \( \varepsilon = - \), then \( C \) is a maximal subgroup of \( G \).
5. If \( \varepsilon = + \), then \( C \) is contained exactly in two parabolic maximal subgroups \( P_1 \) and \( P_2 \) of \( G \); moreover, for each \( i \in \{1, 2\} \), the unipotent radical of \( P_i \) is an elementary abelian subgroup of order \( q^{16} \), \( C \) is a Levi complement of \( P_i \), and \( P_1 \) and \( P_2 \) are conjugate in \( G \) by a graph automorphism of \( G \) which normalizes \( C \).
6. \( N_G(S) \) is contained in a maximal subgroup \( W \) of \( G \) such that

\[
E := E(W) \cong Spin^+_8(q) \cong 2^2 P\Omega^+_8(q),
\]

\[
C_W(E) \cong (q-1)^2/(3,q-1), \quad Z(W) = 1,
\]

\[
W/EC_W(E) \cong Sym_4, \quad \text{and} \quad R < O(W).
\]
(7) If \( q \equiv \varepsilon 1 \pmod{4} \), then \( N^\varepsilon _G(S) \) is contained in a maximal subgroup \( N^\varepsilon \) of \( G \) and 
\[
N^\varepsilon \cong (q - \varepsilon 1)^6/(3, q - \varepsilon 1). \text{Aut}(U_4(2)).
\]

(8) If \( q = p^m \), where \( p \) is a prime, then for any odd prime divisor \( r \) of \( m \), \( G \) has a field 
automorphism \( \varphi \) of order \( r \) such that 
\[
G_0 = G_0(r) = C_G(\varphi)
\]
is maximal in \( G \), \( G_0 \) contains a subgroup conjugate in \( G \) to \( S \), and 
\[
E(G_0) \cong E_6^\varepsilon (q_0), \quad \text{where} \quad q_0^r = q, \quad \text{and} \quad G_0 \leq \text{Inndiag}(E(G_0)).
\]

(9) Each maximal subgroup of odd index of \( G \) is conjugate in \( G \) to one of the following 
subgroups: \( W, G_0(r), P_1 \) or \( P_2 \) if \( \varepsilon = + \), \( C \) if \( \varepsilon = - \), \( N^\varepsilon \) if \( q \equiv \varepsilon 1 \pmod{4} \).

**Proof.** Proof of this follows directly from \([11, \text{Theorems 5–7, Lemma 1.3}], \([18, \text{Theorem, Paragraph 2}], \([19, \text{Theorem, Tables 5.1 and 5.2}], \([4, \text{Theorem 4.5.1, Proposition 4.9.1}].

**Lemma 12.** Assume that \( q \equiv \varepsilon 1 \pmod{4} \) and \( M = N^\varepsilon \) is a maximal subgroup of odd index 
in \( G = E_6^\varepsilon (q) \) which is the normalizer in \( G \) of a maximal torus 
\[
T = T^\varepsilon \cong (q - \varepsilon 1)^6/(3, q - \varepsilon 1)
\]
from Statement (7) of Lemma \([11].

Let \( \sim : M \to M/T \) be the natural epimorphism, \( S \) be a Sylow 2-subgroup of \( M \) (and of \( G \)), and \( t \) be a unique involution from \( Z(S) \). Put 
\[
V = \Omega_1(O_2(T)).
\]

Then the following statements hold:

(1) \( \overline{M} \cong \text{Aut}(U_4(2)) \cong GO_6^-(2). \)

(2) \( C_M(V) = T \) and \( \overline{M} \) acts faithfully on \( V \).

(3) \( V \) is an absolute irreducible \( \mathbb{F}_2 \overline{M} \)-module isomorphic to the natural \( \mathbb{F}_2 \text{GO}_6^- \)-module of dimension 6.

(4) Maximal subgroups of \( \overline{M} \) containing \( \overline{S} \) are exhausted by subgroups \( \overline{Q}_1 \) and \( \overline{Q}_2 \), where 
\( \overline{Q}_1 \) is the stabilizer in \( \overline{M} \) of a totally singular subspace \( \langle t \rangle \) of dimension 1 from \( V \), \( \overline{Q}_2 \) is the 
stabilizer in \( \overline{M} \) of a totally singular subspace \( Y \) of dimension 2 from \( V \), \( t \in Y \), 
\[
\overline{Q}_1 \cong 2^4 : \text{Sym}_5,
\]
\[
\overline{Q}_2 \cong (SL_2(3) \circ SL_2(3)).2^2.
\]

(5) If \( r \) is a prime divisor of \( |T| \) and \( r > 3 \), then the subgroup 
\[
T_r = \Omega_1(O_r(T))
\]
is an elementary abelian \( r \)-group of rank 6, and \( \overline{M} \) acts faithfully and absolutely irreducible 
on \( T_r \) (as on a vector space).
Proof. Statement (1) follows directly from Statement (7) of Lemma 11 and the following isomorphisms \( \overline{M} \cong \text{Aut}(U_4(2)) \cong GO_6^-(2) \) (see [2]).

Let us prove Statement (2). Note that \( V \) is a characteristic subgroup of \( T \). Therefore, \( V \leq M \) and \( C_M(V) \leq M \). In particular, \( V \leq S \), therefore a unique involution \( t \) from \( Z(S) \) belongs to \( V \).

Let us prove that \( C_M(V) = T \). The inclusion \( T \leq C_M(V) \) follows from the fact that \( T \) is abelian. Now we have that

\[
\overline{C_M(V)} \leq \overline{M} \cong \text{Aut}(U_4(2)).
\]

Therefore, if \( C_M(V) > T \), then the structure of the group \( \text{Aut}(U_4(2)) \) implies that

\[
|M : C_M(V)| \leq 2 \quad \text{and} \quad M = C_M(V)S \leq C_G(t).
\]

A contradiction to the maximality of \( M \) and Lemma 11. Thus, \( \overline{M} \) acts faithfully on \( V \).

Let us prove Statement (3). Let \( k \) be the algebraic closure of the field \( \mathbb{F}_2 \). Each composition factor of a faithful \( k\overline{M} \)-module \( k \otimes V \) has dimension at most \( \dim V = 6 \). In view of [3], the group \( \overline{M} \cong \text{Aut}(U_4(2)) \) has up to equivalence a unique faithful irreducible representation over \( k \) whose dimension is at most 6. Moreover, this dimension is equal to 6 and the natural representation of \( GO_6^-(2) \) is absolutely irreducible. Therefore, \( \overline{M} \) acts irreducible on \( V \), and the group \( V \) as a \( \mathbb{F}_2\overline{M} \)-module is isomorphic to the natural \( \mathbb{F}_2GO_6^-(2) \)-module of dimension 6.

We can assume that \( V \) is a space of dimension 6 over the field of order 2 with a non-degenerate quadratic form \( Q \) of type "-" and \( \overline{M} \) is the group of all non-degenerate linear transformations of \( V \) stabilizing \( Q \). This fact and the list of all maximal subgroups of the group \( GO_6^-(2) \) (see, for example, [2]) imply Statement (4).

Let us prove Statement (5). The structure of \( T \) implies that \( T_\sigma \) is an elementary abelian \( r \)-group of rank 6 and \( C_M(T_\sigma) \) is a normal subgroup of \( M \) containing \( T \). Suppose that \( T < C_M(T_\sigma) \). Then, as in the proof of Statement (2), we have \( |M : C_M(T_\sigma)| \leq 2 \) and \( M = C_M(T_\sigma)S \) for any Sylow 2-subgroup \( S \) of \( M \). In view of Statement (7) of Lemma 11, we can assume that \( N_G(S) < M \) and \( N_G(S) = S \times R \), where \( R = O(C_G(\Omega_1(S))) \cong \mathbb{Z}_{(q-1)/2}/(3,q-1) \).

Since \( N_{\overline{M}}(S) = \overline{S} \) (see [2]), \( R \) is contained in \( T \) and, therefore, \( R \cap T_\sigma = \Omega_1(O_\sigma(R)) \cong \mathbb{Z}_r \).

But then the normalizer \( N_G(R \cap T_\sigma) \) contains both subgroups \( S \) and \( C_M(T_\sigma) \). Therefore, \( N_G(R \cap T_\sigma) \) contains \( M = C_M(T_\sigma)S \). On the other hand, Lemma 11(2) implies that \( N_G(R \cap T_\sigma) = N_G(\Omega_1(O_\sigma(R))) \) contains the subgroup \( C = C_G(t) \). A contradiction to Lemma 11 because the maximality of subgroups \( M \) and \( C \) in \( G \) implies that the subgroup \( R \cap T_\sigma \) is normal in the simple group \( G \). Thus, \( \overline{M} \) acts faithfully on \( T_\sigma \).

Tables of ordinary (see [2]) and 5-modular Brauer (see [3]) irreducible characters of the group \( \text{Aut}(U_4(2)) \) imply that the group \( \overline{M} \) has up to equivalence a unique faithful irreducible representation of degree at most 6 over the algebraic closure of the field \( \mathbb{F}_r \), and its dimension is exactly 6. As in the case of characteristic 2 (see above), this fact entails that \( \overline{M} \) acts irreducible on \( T_\sigma \).

\( \square \)
3 Examples of non-pronormal subgroups of odd index in groups $E^\pm_6(q)$

Proposition 1. Let $G = E^\varepsilon_6(q)$, where $\varepsilon \in \{+, -\}$ and $q$ is odd. If 9 divides $q - \varepsilon \cdot 1$, then $G$ contains a non-pronormal subgroup of odd index.

Proof. We use notation from Lemma $\mathbb{I}$. Let us consider a maximal subgroup $W$ of $G$ from Statement (6) of Lemma $\mathbb{I}$ whose index in $G$ is odd. Let $P$ be a Sylow 3-subgroup of $O(W)$. Assume that 9 divides $q - \varepsilon \cdot 1$. Then we have

$$P \cong \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n-1}$$

for some integer $n > 1$. Put

$$V = \Omega_1(P).$$

Thus, $V$ is an elementary abelian 3-group of rank 2.

Let us prove that $W$ contains a subgroup $H$ of odd index such that $H$ is not pronormal in $HV$. It is easy to see that in this case $H$ is a subgroup of odd index in $G$ which is not pronormal in $G$.

In view of Lemma $\mathbb{II}$ we have

$$W/EC_W(E) \cong Sym_4.$$ 

Let $U$ be the complete preimage of $O_2(W/EC_W(E))$ in $W$ with respect to the natural homomorphism $W \to W/EC_W(E)$. Then we have $U \unlhd W$ and $W/U \cong Sym_3$. Let us fix a Sylow 2-subgroup $S$ of $W$. Let $T = U \cap S$ and $N = N_W(T)$. It is clear that $T$ is a Sylow 2-subgroup of $U$. Therefore, using the Frattini argument we receive that $W = NU$. Now we have $S < N$ and

$$N/(N \cap U) \cong NU/U = W/U \cong Sym_3.$$ 

It implies that there exists a 3-element $x$ from $N$ whose image in $N/(N \cap U)$ generates $O_3(N/(N \cap U))$.

Put

$$H = \langle S, x \rangle.$$ 

It is clear that $H$ is a subgroup of odd index both in $W$ and in $G$ because $S \leq H$. Let us prove that $H$ is not pronormal in $HV$.

More precisely, we are going to prove that

- $[H, V]$ coincides with $\mathcal{U}^{n-1}(P)$ (note that $|\mathcal{U}^{n-1}(P)| = 3$), therefore, $[H, V] < V$, and

- $N_V(H) \leq [H, V]$.

Then Lemma $\mathbb{IV}$ implies that $H$ is not pronormal in $HV$.

Remind that in view of Lemma $\mathbb{III}$ the subgroup $W$ contains $N_G(S) = S \times R$, where $R$ is a cyclic subgroup from $O(W) \leq Z(EC_W(E))$ whose order is divisible by 3 (because 9 divides $q - \varepsilon \cdot 1$). In particular, $1 < V \cap R < V$ and $|V \cap R| = 3$. Further,

$$C_V(H) \leq C_V(S) = V \cap R.$$
Let us prove that $C_V(H) = 1$. Since

$$EC_W(E) \leq U \leq EC_W(E)S,$$

we have $R \leq Z(U)$. Moreover, we have

$$W = HU$$

by the definition of $H$. If $C_V(H) \neq 1$, then $C_V(H) = V \cap R$ and

$$1 < V \cap R \leq Z(W) = 1,$$

see Lemma 11. A contradiction.

We claim that $U = C_W(V) \subseteq W$. Indeed, $C_W(V \cap R)$ contains the subgroup $SEC_W(E)$ which has index 3 in $W$. Taking into account that $Z(W) = 1$, we have $C_W(V \cap R) = SEC_W(E)$. This implies that

$$C_W(V) \leq C_W(V \cap R) = SEC_W(E).$$

Moreover, $C_W(V) \subseteq W$ and we have that $C_W(V) / EC_W(E)$ is a normal 2-subgroup in $W / EC_W(E) \cong Sym_4$. Therefore, either $C_W(V) = U$ or $C_W(V) = EC_W(E)$. Since $W / C_W(V)$ is isomorphic to a subgroup of $Aut(V) \cong GL_2(3)$ which does not contain subgroups isomorphic to $Sym_4$, we conclude that $C_W(V) = U$.

Lemma 11 implies that $C_W(P) = U$.

Let $\overrightarrow{w} : W \to W/U$ be the canonical epimorphism. Remind that $H$ acts on $V$ and on $P$ by conjugations. These actions induce faithful actions of $\overrightarrow{W} = W \cong Sym_3$ on $V$ and on $P$, respectively. Let $I = \{ \overrightarrow{t}_i \mid i = 1, 2, 3 \}$ be the set of all the involution from $\overrightarrow{W}$. Without loss of generality, we can assume that $\overrightarrow{S} = \langle \overrightarrow{t}_1 \rangle$. Moreover, the group $G = \langle \overrightarrow{t}_i \rangle$ acts transitively on $I$ by conjugations. The group $\overrightarrow{W}$ is generated by any two different involutions from $I$.

Using Lemma 11 we conclude that

$$P = C_P(\overrightarrow{t}_1) \times [P, \overrightarrow{t}_1].$$

The group $C_P(\overrightarrow{t}_1) = C_P(S)$ coincides with a Sylow 3-subgroup of $R$. Therefore, $C_P(\overrightarrow{t}_1) = C_P(S)$ is the cyclic group of order $3^{n-1}$. This and decomposition (11) imply that

$$[P, \overrightarrow{t}_1] \cong \mathbb{Z}_{3^n}.$$

Therefore, $[P, \overrightarrow{t}_1] \cap \overrightarrow{U}^{n-1}(P) \neq 1$ and, since $|\overrightarrow{U}^{n-1}(P)| = 3$, we have

$$\overrightarrow{U}^{n-1}(P) \leq [P, \overrightarrow{t}_1].$$

Further, $\overrightarrow{U}^{n-1}(P) \leq V$, and the involution $\overrightarrow{t}_1$ inverts each element from $[P, \overrightarrow{t}_1]$, therefore, $\overrightarrow{U}^{n-1}(P) \leq [V, \overrightarrow{t}_1]$.

Taking into account that

$$C_V(\overrightarrow{t}_1) = C_V(S) = V \cap R \neq 1,$$

we conclude that $\overrightarrow{U}^{n-1}(P) = [V, \overrightarrow{t}_1]$. Finally, $\overrightarrow{U}^{n-1}(P)$ is a characteristic subgroup of $W$, and, since $\overrightarrow{H} = \overrightarrow{W}$ acts transitively on the set $I$, we have

$$\overrightarrow{U}^{n-1}(P) = [V, \overrightarrow{t}_1].$$
for any $i \in \{1, 2, 3\}$. Since $V$ is abelian and $\mathcal{H} = \langle \overline{t}_i \mid i = 1, 2, 3 \rangle$, we conclude that

$$[H, V] = [V, \mathcal{H}] = \prod_{i=1}^{3} [V, \overline{t}_i] = \mathcal{O}^{n-1}(P),$$

as claimed.

Let us prove that $N_V(H) \leq [H, V]$.

Note that for any $i \in \{1, 2, 3\}$ we have $[H, V] = [V, \overline{t}_i]$ and

$$V = C_V(\overline{t}_i) \times [V, \overline{t}_i] = C_V(\overline{t}_i) \times [H, V].$$

Moreover,

$$C_V(\overline{t}_i) \cap C_V(\overline{t}_j) = C_V(\overline{t}_i \overline{t}_j) = C_V(\mathcal{H}) = 1$$

for $1 \leq i < j \leq 3$, and the number of distinct subgroups of order 3 in $V$, distinct from $[H, V]$, equals to $(9-3)/2 = 3$. Therefore, these subgroups are exhausted by the centralizers $C_V(\overline{t}_i)$, and the group $\mathcal{H}$ acts on these subgroups transitively.

If we suppose that $N_V(H) \not\leq [H, V]$, then $N_V(H)$ contains $C_V(\overline{t}_i)$ for some $i$. But $N_V(H)$ is invariant under $\mathcal{H}$, therefore contains $C_V(\overline{t}_i)$ for each $i \in \{1, 2, 3\}$.

Now the subgroup $C_V(\overline{t}_1) = V \cap R$ of order 3 acts on the set of all Sylow 2-subgroups of $H$, therefore, acts on the set $I$ of order 3. But $C_V(\overline{t}_1)$ has a fixed point $\overline{t}_1$ on $I$, therefore, the action of $C_V(\overline{t}_1)$ on $I$ is trivial. In particular, $C_V(\overline{t}_1) \leq C_V(\overline{t}_2)$. A contradiction to the facts that $C_V(\overline{t}_i) \cap C_V(\overline{t}_j) = 1$ if $i \neq j$ and $C_V(\overline{t}_1)$ has order 3.

\[ \square \]

Proposition 2. Let $q_0$ be a power of an odd prime $p$ and $q = q_0^r$ for an odd prime $r$. Then the group $G = E_6(q)$ contains a non-pronomal subgroup of odd index.

Proof. We use notation from Lemma [11].

Let $\varphi$ be the canonical field automorphism of order $r$ of $G$ and $G_0 = C_G(\varphi)$. Then in view of Lemma [11] and [3] Proposition 4.9.1, we have

1. $|G : G_0|$ is odd;
2. $G_0 = E(G_0) \cong E_6(q_0)$ and $G_0$ is isomorphic to a subgroup of $\text{Inndiag}(E_6(q_0))$;
3. $G_0$ considered as a Chevalley group contains a parabolic subgroup $P_0$ of odd index such that $P_0$ is contained in a parabolic subgroup $P \in \{P_1, P_2\}$ of $G$, where $P_1$ and $P_2$ are subgroups from Lemma [11] and $P$ is invariant under $\varphi$. Moreover, $P_0 = C_P(\varphi)$ and the unipotent radical $O_p(P_0)$ of $P_0$ is contained in $O_P(P)$.

Let $S$ be a Sylow 2-subgroup of $P_0$ and $V_0 = O_p(P_0)$. Let us put $H = SV_0$ and prove that $H$ is not pronormal in $P$. More precisely, if $V = O_p(P)$ is the unipotent radical of $P$, $C$ is a Levi complement in $P$ such that $S \leq C$, and $Z = Z(C)$, then $H$ is not pronormal in $ZSV$. Indeed, in view of Lemma [11], the subgroup $Z$ is isomorphic to a subgroup of the multiplicative group $\mathbb{F}_q^*$ and $|\mathbb{F}_q^* : Z|$ divides 3, in view of the Zsigmondy theorem [21]. $Z$ contains an element whose order does not divide $|\mathbb{F}_{q_0}^*| = q_0 - 1$ (otherwise

$$q_0^r - 1 = |\mathbb{F}_q^*| \text{ divides } 3|\mathbb{F}_{q_0}^*| = 3(q_0 - 1),$$

and we conclude that $H$ is not pronormal in $P$. Hence, $H$ is not pronormal in $G$. Therefore, $G$ contains a non-pronomal subgroup of odd index.
whence

$$q_0^{r-1} + q_0^{r-2} + \cdots + q_0 + 1 \text{ divides } 3$$

and $q_0 = 2$, a contradiction to the fact that $p$ is odd).

Let $g \in Z$ be an element whose order does not divide $q_0 - 1$. Since $Z$ is contained in a Cartan subgroup of $G$, the element $g$ corresponds to some character

$$\chi : \mathbb{Z}\Phi \rightarrow \mathbb{F}_q^*,$$

where $\Phi$ is a root system of type $E_6$.

and each root system $X_r = \{ x_r(\alpha) \mid \alpha \in \mathbb{F}_q \}$ for $r \in \Phi$ is invariant under $g$ with respect to the action

$$g^{-1}x_r(\alpha)g = x_r(\chi(r)^{-1}\alpha)$$

(see [1, 7.1, in particular, P. 100]). The element $g$ was chosen such a way that $\chi(r) \notin \mathbb{F}_q^*$ for some root $r \in \Phi$, where $X_r \leq V$ since $g$ belongs to the center of a Levi complement $C$.

Further,

$$V_0 = \langle x_s(\alpha) \mid X_s \leq V, \alpha \in \mathbb{F}_{q_0} \rangle,$$

and there exists an ordering of roots such that each element from $V$ and each element from $V_0$ has a unique representation as a product of root elements $x_s(\alpha)$ corresponding to distinct roots $s$ taken with respect to this ordering. Thus,

$$g^{-1}x_r(1)g = x_r(\chi(r)^{-1}) \in V_0^g \setminus V_0$$

and the subgroups $V_0$ and $V_0^g$ are distinct. But $g \in Z(C)$, therefore, $S^g = S$. Thus,

$$\langle H, H^g \rangle = \langle V_0, V_0^g, S \rangle.$$

If we suppose that the subgroups $H$ and $H^g$ are conjugate in the subgroup $\langle H, H^g \rangle$, then the subgroups $V_0 = O_p(H)$ and $V_0^g = O_p(H^g)$ are conjugate in $\langle H, H^g \rangle$. But the subgroups $V_0$ and $V_0^g$ are normal in $\langle H, H^g \rangle = \langle V_0, V_0^g, S \rangle$ (since both $V_0$ and $V_0^g$ are invariant under $S$ and centralize each other because they both are subgroups of $V$ which is abelian), therefore, $V_0$ and $V_0^g$ are not conjugate in $\langle H, H^g \rangle$. A contradiction.

\[\square\]

## 4 Proof of Theorem

Let $G = E_6^\varepsilon(q)$, where $\varepsilon \in \{+, -\}$, $q = p^m$, and $p$ is a prime.

Propositions [1] and [2] imply that if all the subgroups of odd index are pronormal in $G$, then $q \not\equiv 1 \pmod{18}$ and if $\varepsilon = +$, then $m$ is a power of 2.

Let us prove the converse.

Assume that $q \not\equiv 1 \pmod{18}$ and if $\varepsilon = +$, then $m$ is a power of 2. Let us prove that all the subgroups of odd index are pronormal in $G$.

In view of Lemma [11] we can assume that $q$ is odd. Let $H$ be a subgroup of odd index of $G$, $S$ be a Sylow 2-subgroup of $H$ (which is a 2-Sylow subgroup of $G$ at the same time), and $g$ be an element of odd order from $N_G(S)$. In view of Lemma [11] we have $N_G(S) = S \times R$, where $R$ is the cyclic group of order $(q - 1)/(3, q - 1)$, therefore, $g \in R \leq Z(N_G(S))$ and $|g|$ is not divisible by 3. In view of Lemma [2] to prove that $H$ is pronormal in $G$ it is enough to prove that $H$ and $H^g$ are conjugate in $K = \langle H, H^g \rangle$. Moreover, it is nothing to prove if $K = G$, therefore, we can assume that $K < G$. Thus, there exists a maximal subgroup $M$
of $G$ such that $K \leq M$. It is easy to see that $M$ is one of the maximal subgroups of odd index of $G$ that are listed in Lemma 11.

Let us note that we can assume that $g \in M$. Indeed, in view of Lemma 11, we have $\mathcal{N}_G(S) \leq M$ or $M = C_G(\varphi)$ for a field automorphism $\varphi$ of odd prime order $r$. In the latter, let us suppose that $g \notin C_G(\varphi)$. We have

$$x := [g^{-1}, \varphi] = g(g^{-1})^\varphi \neq 1.$$

Note that $x$ centralizes $H$. Indeed, if $h \in H$, then since

$$h, h^g \in K \leq M = C_G(\varphi) = C_G(\varphi^{-1}),$$

we have

$$h^\varphi = (h^g)^{\varphi^{-1}}g^{-1}\varphi = (h^g)^{-1}\varphi = h^\varphi = h.$$

Since $S \leq H$, we have that $x \in \mathcal{N}_G(S)$. But $g \in Z(\mathcal{N}_G(S))$, therefore,

$$g \in C_G(x) < G.$$

Let $M_0$ be a maximal subgroup of $G$ which contains $C_G(x)$. Since $H \leq C_G(x)$, we have that $M_0$ is a subgroup of odd index in $G$ containing $H, g$, and $K = \langle H, H^g \rangle$, as claimed.

Thus, in view of Lemma 3 to prove that all the subgroups of odd index are pronormal in $G$ it is enough to prove that all the subgroups of odd index are pronormal in any maximal subgroup $M$ of odd index of $G$. Let us consider these subgroups case by case with respect to Lemma 11. So, below we use notation from Lemma 11.

Case 1: $M = G_0 = C_G(\varphi)$, where $\varphi$ is a field automorphism of odd prime order $r$ of the group $G$.

It is known that any two elements of order $r$ from the coset $G\varphi$ of the group $\text{Aut}(G)$ by $\text{Inn}(G)$ (which is identified to $G$), are conjugate by an inner-diagonal automorphism (see [4, Proposition 4.9.1]). Therefore centralizers of these elements in $G$ are isomorphic. Let us identify $\varphi$ to an automorphism $\varphi : \mathbb{F}_q \to \mathbb{F}_q$ of the field $\mathbb{F}_q$ such that for each root element $x_r(\alpha)$ of a non-twisted group of Lie type $E_6$ (either this group coincides with $G$ or $G$ was constructed via this group), the following equalities hold

$$x_r(\alpha)^\varphi = x_r(\overline{\alpha}) = x_r(\alpha^\varphi)$$

(See [12, 12.2, in particular, P. 200]). Let $\mathbb{F}_{q_0}$ be the subfield of $\mathbb{F}_q$ which consists of fixed points of $\varphi$, thus, $q = q_0^r$. Let us prove that $G_0 \cong E_6^{\varphi}(q_0)$ and then use inductive reasonings to prove that all the subgroups of odd index are pronormal in $G_0$. It is enough to prove that

$$|G_0/E(G_0)| = 1.$$

Suppose for the contradiction that $|G_0/E(G_0)| \neq 1$. In view of [4, Proposition 4.9.1] the group $E(G_0) \cong E_6^{\varphi}(q_0)$ has an outer diagonal automorphism, therefore,

$$q_0 \equiv \varepsilon 1 \pmod{3}.$$

Remind that 9 does not divide the number

$$q - \varepsilon 1 = (q_0 - \varepsilon 1) \left( \sum_{i=0}^{r-1} (\varepsilon 1)^i q_0^{-i-1} \right).$$

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therefore a Sylow 3-subgroup of the multiplicative group of the field $\mathbb{F}_q$ is contained in the subfield $\mathbb{F}_{q_0}$ which consists of fixed points of $\varphi$. Consider an extended Cartan subgroup $\hat{H}$ of $G$, which is invariant under $\varphi$, and let $\hat{G} = \hat{H}G$ be the group of inner-diagonal automorphisms of $G$. Since $|\hat{G} : G| = 3$, there exists a 3-element $\delta$ from $\hat{H}$ such that $\hat{G} = \langle G, \delta \rangle$. Since $\varphi$ centralizes a Sylow 3-subgroup of the multiplicative group of the field $\mathbb{F}_q$, we have that $\varphi$ considered as a field automorphism of $G$ and of $\hat{G}$ centralizes a Sylow 3-subgroup of an abelian group $\hat{H}$. In particular, $\varphi$ centralizes $\delta$.

It is known (see [4, Proposition 4.9.1]) that $C_{\hat{G}}(\varphi)$ is isomorphic to the group of inner-diagonal automorphisms of $G'_0 \cong E_6(q_0)$. Since $\delta \in C_{\hat{G}}(\varphi)$, we have $\hat{G} = GC_{\hat{G}}(\varphi)$, therefore,

$$C_{\hat{G}}(\varphi)/C_G(\varphi) \cong \hat{G}/G \cong \mathbb{Z}_3.$$ 

Thus, $G_0 = C_G(\varphi)$ coincides with a unique normal subgroup of index 3 of $C_{\hat{G}}(\varphi)$, therefore, $G_0 \cong E_6(q_0)$.

C a s e 2: $M = C = C(t)$ is the centralizer in $G$ of an involution $t$ from the center of a Sylow 2-subgroup $S$ of $G$.

Note that $C$ is a maximal subgroup of $G$ only if $\varepsilon = -$. We omit this condition in our reasonings because if $\varepsilon = +$, then we need to prove that the subgroups of odd index are pronormal in $C = C(t)$ to consider cases 4 and 5 below.

If $H$ is a subgroup of $C$ containing $S$ and $g$ is an element of odd order from $N_G(S)$, then $g \in Z(C)$ in view of Lemma [11] and, therefore, $H = H^g$. Thus, $H$ is pronormal in $C$ in view of Lemma [2].

C a s e 3: $M = W$.

Remind that $g \in R$ and $R$ is a $\{2, 3\}$-subgroup of $O(W)$, where $O(W)$ is a normal abelian subgroup of $W$ which is contained in $C_W(E)$. Thus, in this case it is sufficient to prove that $H$ is pronormal in $O_{\{2, 3\}}(W)H$. Let us prove that for each $H$-invariant subgroup $U$ of $O_{\{2, 3\}}(W)$ the inclusion (really, the equality) $U \leq N_U(H)[H, U]$ holds, then the pronormality of $H$ in $O_{\{2, 3\}}(W)H$ will follow from Lemma [7]. Since $U$ is a direct product of its Sylow subgroups, and each of these Sylow subgroups is $H$-invariant, it is sufficient to consider a case when $U$ is an $r$-group for some prime $r > 3$.

In view of Lemma [11] $O(W)$ is contained in the center of $EC_W(E)$ and

$$W/EC_W(E) \cong \text{Sym}_4.$$ 

So that, $U$ is contained in the center of $EC_W(E)$. It follows that $\overline{H} := H/C_H(U)$ is a $\{2, 3\}$-group. In view of Lemma [11] we have

$$U = C_U(\overline{H})[\overline{H}, U] = C_U(H)[H, U] \leq N_U(H)[H, U].$$

C a s e 4: $M = N^\varepsilon$, where $q \equiv \varepsilon 1 \pmod{4}$.

In view of Lemma [11] $M$ contains an abelian normal subgroup $T$ of the form $(q - \varepsilon 1)^6/(3, q - \varepsilon 1)$ such that $M/T$ is isomorphic to $\text{Aut}(U_4(2)) \cong GO_6(2)$. Let $\overline{M}$ be the canonical epimorphism $M \rightarrow M/T$. It is known (see [2]) that a Sylow 2-subgroup of the group $\overline{M}$ coincides with its normalizer in $M$. Since $q \neq \varepsilon 1 \pmod{18}$, we have that $R$ is contained in $O_{\{2, 3\}}(T)$. Thus, in view of Lemmas [2] [5] and [7] it is sufficient to prove that $H$ is pronormal in $HO_{\{2, 3\}}(T)$.

Let $S_0$ be a Sylow 2-subgroup of $T$. Put $V = \Omega_1(S_0)$. In view of Lemma [12] we have that $V$ is an elementary abelian 2-group of rank 6, and a unique involution $t$ from $Z(S)$ belongs to $V$. Lemma [12] implies that one of the following cases appears:
(4a) $\overline{H} \leq Q_1$, where $Q_1$ is the stabilizer in $M$ of an isotropic subspace $\langle t \rangle$ of dimension 1 from $V$, and 
\[ Q_1 \cong 2^4 : Sym_5; \]

(4b) $\overline{H} \leq Q_2$, where $Q_2$ is the stabilizer in $M$ of an isotropic subspace $Y$ of dimension 2 from $V$, and 
\[ \overline{Q}_2 \cong (SL_2(3) \circ SL_2(3)).2^2; \]

(4c) $\overline{H} = M$.

If Case (4a) appears, then we reduce to Case 2 considered above.
If Case (4b) appears, then $\overline{H}$ is a $\{2, 3\}$-group and, as in Case 3, we have 
\[ U = C_U(\overline{H})[\overline{H}, U] = C_U(H)[H, U] \leq N_U(H)[H, U]. \]

Assume that Case (4c) appears. Let us prove that $U = N_U(H)[H, U]$ for each $H$-invariant subgroup $U$ of $O_{(2,3)}(T)$ and use Lemma [7]. As in Case 3, we can assume that $U$ is an $r$-group for some prime $r > 3$ from $O_{(2,3)}(T)$, whence the equality $U = [H, U] = N_U(H)[H, U]$ obviously follows. Indeed, in view of Lemmas [11] and [12] the following statements hold: $C_{O(T)}(Q_1) = R$, $C_{O(T)}(Q_2) \leq R$, and $M = \langle Q_1, Q_2 \rangle$. But in view of Lemma [12] the group $M$ acts irreducible on $\Omega \langle O_r(T) \rangle$ for each prime divisor $r$ of the number $|O_{(2,3)}(T)|$, therefore, $C_{O_r(T)}(Q_2) = 1$ in view of Lemma [14]. Since $Q_2$ is a $\{2, 3\}$-group, in view of Lemma [14] we have $U = C_U(Q_2) \times [Q_2, U] = [Q_2, U] \leq [M, U]$. Thus, $U = [\overline{H}, U]$.

To complete the proof it remains to consider the following case.

Case 5: $\varepsilon = +$ and $M$ is conjugate to a subgroup $P \in \{P_1, P_2\}$ from Lemma [11].

Let us use Lemma [6]. Let $V$ be the unipotent radical of $P$. Sylow 2-subgroups of $P/VZ \cong C/Z$ are self-normalized (see [10] and Lemma [4]). A Sylow 2-subgroup $T$ of $Z$ coincides with a Sylow 2-subgroup of the group $VZ$, and $N_P(T)$ coincides with a Levi complement $C$ of $P$. Using Case 2 considered above we conclude that all the subgroups of odd index are pronormal in $C$. In view of Lemma [6] it is sufficient to prove that all the subgroups of odd index are pronormal in $VZ$.

Let $H_1$ be a subgroup of odd index in $VZ$. Consider a subgroup $U = H_1 \cap V = O_p(H_1)$. In view of the Schur-Zassenhaus theorem, we have $H_1 = XU$, where $X$ is a Hall $p'$-subgroup of $H_1$, and in view of the Hall theorem, we can assume that $X \leq Z$ since $Z$ is a Hall $p'$-subgroup of $VZ$ and $VZ$ is solvable. In particular, $X$ contain a Sylow 2-subgroup $T$ of $Z$. Let us prove that $U$ is normal in $VZ$. It is easy to see that $U$ is normal in $V$ since $V$ is abelian. Thus, in is sufficient to prove that $U$ is $Z$-invariant, i.e. $u^z \in U$ for each $u \in U$ and each $z \in Z$.

In view of [17, Table 3], $C$ acts on $V$ by conjugation and induces on $V$ a faithful irreducible $F_qC$-module. In view of the Clifford theorem [3, Theorem 3.4.1], $F_qZ$-module $V$ is completely reducible and is equal to a direct product of irreducible $F_qZ$-modules which are pairwise conjugate in $C$. Since $F_q$ is the decomposition field of the cyclic group $Z$ whose order divides $q - 1$, the group $Z$ acts on $V$ with scalar action, i.e. for any element $z$ from $Z$ there exists an element $\beta \in F_q^*$ such that $v^z = \beta v$ for each $v \in V$.

The subgroup $U$ defined above could be considered as a subspace of $V$ considering as a vector space over the prime subfield $F_p$ of the field $F_q$.

Remind that the degree of $F_q$ over $F_p$ is $2^k$. Let $x$ be a generator of a Sylow 2-subgroup of $Z$ and $v^x = \alpha v$ for all $v \in V$. Since $V$ is a faithful module and $|Z| = (q - 1)/(3, q - 1)$, the
element $\alpha$ is a generator of a Sylow 2-subgroup of the group $\mathbb{F}_q^*$, and, in view of Lemma 9, elements

$$1, \alpha, \alpha^2, \ldots, \alpha^{2^k-1}$$

form a basis of the field $\mathbb{F}_q$ as a vector space over $\mathbb{F}_p$.

Take an arbitrary element $u \in U$. Since $x \in T \leq H_1$, we have

$$\alpha^i u = u^{x^i} \in U.$$  

If for an arbitrary element $z \in Z$ the equality $u^z = \beta u$ holds, where $\beta \in \mathbb{F}_q^*$, then for some $\lambda_i \in \mathbb{F}_p$ we have

$$\beta = \sum \lambda_i \alpha^i \quad \text{and} \quad u^z = \beta u = \sum \lambda_i (\alpha^i u) = \sum \lambda_i u^{x^i} \in U.$$  

Take $g \in N_{VZ}(T) = Z$. Then $g$ normalizes $X$ since $Z$ is abelian, and $g$ normalizes $U$ in view of the fact proved above. Then $g$ normalizes $H_1 = XU$, and in view of Lemma 2, the subgroup $H_1$ is pronormal in $VZ$.

The proof of Theorem is complete.

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