ARITHMETIC REPRESENTATION GROWTH OF VIRTUALLY FREE GROUPS

FABIAN KORTHAUER

Abstract. We adapt methods from quiver representation theory and Hall algebra techniques to the counting of representations of virtually free groups over finite fields. This gives rise to the computation of the $E$-polynomials of $GL_d(\mathbb{C})$-character varieties of virtually free groups. As examples we discuss the representation theory of $D_\infty$, $PSL_2(\mathbb{Z})$, $SL_2(\mathbb{Z})$, $GL_2(\mathbb{Z})$ and $PGL_2(\mathbb{Z})$.

1 Introduction

Arithmetic representation growth deals with counting the number of representations of algebras over finite fields. More precisely it is the study of the following counting functions: For $\mathcal{A}$ a finite type $\mathbb{F}_q$-algebra and $d \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_{\geq 1}$ define

\[ r_{ss,\mathcal{A}}^d(q^\alpha) := \# ssim_d(\mathcal{A} \otimes_{\mathbb{F}_q} \mathbb{F}_q^{\alpha}) \]
\[ r_{sim,\mathcal{A}}^d(q^\alpha) := \# sim_d(\mathcal{A} \otimes_{\mathbb{F}_q} \mathbb{F}_q^{\alpha}) \]
\[ r_{absim,\mathcal{A}}^d(q^\alpha) := \# absim_d(\mathcal{A} \otimes_{\mathbb{F}_q} \mathbb{F}_q^{\alpha}) \]

Here we denote by $iso_d(\mathcal{B}) \supseteq ssim_d(\mathcal{B}) \supseteq sim_d(\mathcal{B}) \supseteq absim_d(\mathcal{B})$ for each $d$ the sets of isomorphism classes of all, of all semisimple, of all simple and of all absolutely simple left modules $\mathcal{M}$ over a $K$-algebra $\mathcal{B}$ of dimension $\dim_K(\mathcal{M}) = d$. Recall that a left $\mathcal{B}$-module $\mathcal{M}$ is called absolutely simple if it is simple and $\text{End}_\mathcal{B}(\mathcal{M}) = K$ or equivalently if $\mathcal{M} \otimes_K \overline{K}$ is simple for the algebraic closure $\overline{K} \supseteq K$.

(1) defines functions $r_{ss,\mathcal{A}}^d(q^\alpha)$, $r_{sim,\mathcal{A}}^d(q^\alpha)$, $r_{absim,\mathcal{A}}^d(q^\alpha)$ on all $q$-powers. We call these functions counting functions, as they count the semisimple, simple and absolutely simple modules/representations of $\mathcal{A}$ over $\mathbb{F}_q$ up to isomorphism. If the algebra $\mathcal{A}$ is understood, we will usually drop it from the notation.

The counting functions (1) have been studied by S. Mozgovoy and M. Reineke in the cases $\mathcal{A} = \mathbb{F}_q \tilde{Q}$ the path algebra of a finite quiver and $\mathcal{A} = \mathbb{F}_q[F_\alpha]$ the group algebra of a finitely generated free group. One of their main results is the following theorem.

**Theorem 1.1**[1] If $\mathcal{A}$ is the path algebra of a finite quiver or the group algebra of a finitely generated free group (over $\mathbb{F}_q$ respectively), then there are polynomials

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1. i.e. finitely generated as an $\mathbb{F}_q$-algebra
2. In the quiver case Mozgovoy-Reineke’s result was in the more general context of counting absolutely stable representation of a fixed dimension vector. We state it in a weaker form here for expository purposes.
3. see [Rein06 Thm. 6.2] and [M.R.15 Thm. 1.1]
We call such polynomials realizing the counting functions (1) counting polynomials. More important than the mere existence of counting polynomials is the fact that Mozgovoy-Reineke obtained certain generating formulas which enable the practical computation of the counting polynomials (in low dimensions).

The main goal of this paper is to generalize Theorem 1.1 as well as the above mentioned generating formulas to the case where \( A = \mathbb{F}_q[G] \) is the group algebra of a finitely generated virtually free group (see Theorem 5.3 below). Furthermore we will investigate a few structural properties of the counting polynomials and relate these to the geometry of GIT moduli spaces of representations. SageMath code designed by the author for the practical computation of the counting polynomials is provided as an ancillary file to this publication on the arXiv.

This paper is organized as follows: We start by recalling most of the relevant preknowledge on virtually free groups and algebraic geometry within Section 2. Afterwards we discuss some invariants like dimension vectors and the homological Euler form in the context of representations of virtually free groups in Section 3. In Section 4 we review some Hall algebra methods, before we discuss the main result 5.3 in Section 5. Section 6 is devoted to hands-on examples and we conclude in Section 7 with discussing a few structural properties of the counting polynomials.

Acknowledgements. Almost all results of this paper have been obtained as part of the author’s PhD under the supervision of M. Reineke and B. Klopsch and the author is deeply indebted to his supervisors for all their help, support, kindness and patience. Special thanks also go to L. Hennecart, S. Mozgovoy and S. Schröer for useful discussions on parts of the content of this paper. Moreover the author thanks the Ruhr-University Bochum and most of its members for being a welcoming scientific home during his PhD. During his PhD the author’s research was financed by the research training group GRK 2240: Algebro-Geometric Methods in Algebra, Arithmetic and Topology, which is funded by the Deutsche Forschungsgemeinschaft.

2 Preliminaries

In this section we summarize the preknowledge from group theory, representation theory and algebraic geometry needed within this paper. Except for the notion of suitable fields nothing in this section is original.

2.1 Virtually free groups and their group algebras. We start by recalling some group theoretic notions. Given three groups \( E, F, H \) as well as two injective group homomorphisms \( \iota : F \hookrightarrow H, \kappa : F \hookrightarrow E \) we denote their pushout in the category of groups by \( H\ast_F E \) and call it the amalgamated free product of \( H \) and \( E \) over \( F \). If we are given two embeddings \( \iota, \kappa : F \hookrightarrow H \) with the same codomain, we consider the induced embeddings

\[
\iota', \kappa' : F \hookrightarrow H \ast_{C_\infty} E , \quad \iota'(f) := \iota(f) , \quad \kappa'(f) := t^{-1} \kappa(f)t
\]
where $t$ denotes the generator of the infinite cyclic group $C_\infty$. We denote the coequalizer of

$$
\mathcal{F} \xrightarrow{\kappa'} \mathcal{H} \ast C_\infty
$$

by $\mathcal{H}^{\ast \kappa'}_\mathcal{F}$ and call it the HNN extension of $\mathcal{H}$ by $\mathcal{F}$. Even though our description of amalgamated free products and HNN extensions make sense if the homomorphisms $\iota$ and $\kappa$ are not injective, we will follow the usual convention in group theory and only consider the case where they are.

A group $\mathcal{G}$ is called virtually free if it contains a finite index subgroup $\mathcal{G}$ which itself is a free group. However, Bass-Serre theory provides an equivalent definition of virtually free groups which will be more useful for us. A finitely generated group $\mathcal{G}$ is virtually free if and only if it contains a finite index subgroup $\mathcal{G}$ which is virtually free.

This is a consequence of [K.S.71, Thm. 4].
and \( C_b \). Prominent examples of this class are the infinite dihedral group \( \mathbb{D}_\infty = \Gamma_{2,2} \) and \( \Gamma_{2,3} \) which is isomorphic to \( \text{PSL}_2(\mathbb{Z}) \). We may enlarge this class of examples by picking a common divisor \( c \) of \( a \) and \( b \) as well as embeddings \( C_c \hookrightarrow C_a, C_b \). This gives rise to the virtually free group \( C_a \ast_{C_c} C_b \). A prominent example here is \( C_4 \ast C_2 \) which is isomorphic to \( \text{SL}_2(\mathbb{Z}) \).

The arithmetic groups \( \text{GL}_2(\mathbb{Z}) \) and \( \text{PGL}_2(\mathbb{Z}) \) are virtually free as well — they arise as \( \mathbb{D}_4 \ast_{C_2 \times C_2} \mathbb{D}_6 \) and \( \mathbb{D}_2 \ast_{C_2} \mathbb{D}_3 \). To define the inclusions of \( C_2 \times C_2 \) and \( C_2 \) we consider the presentation

\[
\mathbb{D}_c = \langle s, t \mid s^2 = t^2 = 1 = (st)^c \rangle
\]

of the dihedral group. If \( c = 2a \) is even, we embed \( C_2 \times C_2 \) into \( \mathbb{D}_{2a} \) by sending the 2 generators to \( s \) and \( (st)^a \). For arbitrary \( c \) we embed \( C_2 \) into \( \mathbb{D}_c \) by sending the generator to \( s \).

Since every finite index subgroup of a virtually free group is again virtually free, all congruence subgroups of the four above mentioned arithmetic groups are virtually free as well. Another class of examples are of course the free groups: The free group \( F_a \) on \( a \) generators arises by taking \( J = a \) trivial HNN extensions of the trivial group, i.e. in terms of our description \((3)\) set \( I = 0 \) and all \( G_i \), \( G'_j \) to be the trivial group.

In [LeBr05, §2] L. Le Bruyn discusses an analogue of graphs of groups for algebras: For \( \mathcal{A}, \mathcal{C} \) two \( K \)-algebras and \( \iota, \kappa : \mathcal{C} \hookrightarrow \mathcal{A} \) injective \( K \)-algebra homomorphisms we consider the induced embeddings

\[
i', \kappa' : \mathcal{C} \hookrightarrow \mathcal{A} \ast_K K[t, t^{-1}] \quad , \quad \i'(f) := \i(f) \quad , \quad \kappa'(f) := t^{-1} \kappa(f) t
\]

where \( \ast_K \) denotes the coproduct of \( K \)-algebras. We define the HNN extension \( \mathcal{A} \ast_{\mathcal{C}}^{\iota, \kappa} \)

of \( \mathcal{A} \) by \( \mathcal{C} \) as the coequalizer of

\[
\mathcal{C} \xrightarrow{\i' \kappa'} \mathcal{A} \ast_K K[t, t^{-1}]
\]

We are mostly interested in HNN extensions of algebras, because they arise as group algebras of HNN extensions of groups: The group algebra functor \( K[-] \) is a left adjoint, hence, it preserves colimits. So for every HNN extension of groups we obtain an isomorphism

\[
K[\mathcal{H}^{i, \kappa}] \cong K[\mathcal{H}]^{i, \kappa}_{K[\mathcal{F}]}
\]

Moreover applying the functor \( K[-] \) to our decomposition \((3)\) we get a \( K \)-algebra isomorphism between \( K[G] \) and

\[
\text{(6)} \quad (\ldots ((K[G_0] \ast_{K[G'_1]} K[G'_1]) \ast_{K[G'_2]} K[G'_2]) \ldots) \ast_{K[G'_{I+1}]} \ast_{K[G'_{I+1}]} \cdot \cdot \cdot
\]

Analogous to [LeBr05, §2] we say that a \( K \)-algebra \( \mathcal{A} \) is the fundamental algebra of a finite graph of finite dimensional semisimple \( K \)-algebras if there are \( I, J \in \mathbb{N}_0 \), maps \( s, t : \{1, \ldots, I + J\} \to \{0, \ldots, I\} \) fulfilling \((4)\) as well as finite dimensional semisimple \( K \)-algebras \( \mathcal{A}_0, \ldots, \mathcal{A}_I \) and \( \mathcal{A}'_1, \ldots, \mathcal{A}'_{I+J} \) and \( K \)-algebra embeddings

\[
t_j : \mathcal{A}'_j \hookrightarrow \mathcal{A}_{s(j)} \quad , \quad \kappa_j : \mathcal{A}'_j \to \mathcal{A}_{t(j)}
\]

such that \( \mathcal{A} \) is isomorphic to

\[
\text{(7)} \quad (\ldots ((\mathcal{A}_0 \ast_{\mathcal{A}'_1} \mathcal{A}_1) \ast_{\mathcal{A}'_2} \mathcal{A}_2) \ldots) \ast_{\mathcal{A}'_{I+1}} \ast_{\mathcal{A}'_{I+1}} \cdot \cdot \cdot
\]

\footnote{We denote the induced algebra homomorphisms \( K[\iota], K[\kappa] : K[\mathcal{F}] \to K[\mathcal{H}] \) simply by \( \iota \) and \( \kappa \).}
The group algebra $K[G]$ of the finitely generated virtually free group $G$ is the fundamental algebra of a finite graph of finite dimensional semisimple $K$-algebras whenever $\text{char}(K)$ is not a prime number dividing the order of one of the finite groups $G_i$, $0 \leq i \leq I$, since $K[G]$ is isomorphic to $\text{(6)}$.

The main reason we are studying virtually free groups in this paper is the following fact: If $H$ is a finitely generated group and $K$ a field, then the group algebra $K[H]$ is (left) hereditary if and only if $H$ is virtually free and contains no elements of order $\text{char}(K)$.

Recall that a $K$-algebra $A$ is called formally smooth if its Hochschild cohomology $HH^a(A, -)$ vanishes in degree $a \geq 2$. This is equivalent to $A$ satisfying a lifting property along square-zero extensions of $K$-algebras. Every formally smooth $K$-algebra is left and right hereditary and the fundamental algebra of a finite graph of finite dimensional semisimple $K$-algebras is formally smooth. So a group algebra $K[H]$ of a finitely generated group $H$ is formally smooth if and only if it is hereditary, i.e. if and only if $H$ is virtually free and contains no elements of order $\text{char}(K)$.

For (parts of) the machinery of this paper to work it is crucial that $K[G]$ is formally smooth, i.e. $\text{char}(K)$ has to be zero or a suitable prime. To make things more convenient we will moreover assume that $K$ is large enough which brings us to the notion of suitable fields:

Let $C$ be a finite dimensional semisimple $K$-algebra. By Artin-Wedderburn theory we know that $C$ is (isomorphic to) a product of matrix algebras

$$M_{b_1}(D_1) \times \cdots \times M_{b_c}(D_c)$$

with $D_1, \ldots, D_c$ finite dimensional $K$-division algebras. We say that $C$ is completely split if all simple left $C$-modules are absolutely simple or equivalently if $D_\gamma \cong K$ for all $1 \leq \gamma \leq c$, i.e. $C$ is completely split if and only if it is of the form

$$K^{c_1} \times M_2(K)^{c_2} \times \cdots \times M_e(K)^{c_e}$$

for non-negative integers $e, c_1, \ldots, c_e \in \mathbb{N}_0$. Note that all left $C \otimes_K F$-modules for every field extension $F \supseteq K$ are defined over $C$. We say that a field $K$ is of suitable characteristic for the virtually free group $G$ if $G$ contains no elements of order $\text{char}(K)$. We call a field $K$ suitable for $G$ if it is perfect, of suitable characteristic and $K[F]$ is completely split for every finite subgroup $F \subseteq G$.

Note that being suitable is a relative notion — it depends on which virtually free group it refers to. The readers may convince themselves that every algebraic field extension of a suitable field is again suitable and (using that every finite subgroup

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8see [Dick79, Thm. 1]
9Since every group is isomorphic to its opposite, left and right hereditaryness are equivalent.
10see [Weib94, Prop. 9.3.3]
11use e.g. [Weib94, Lemma 9.1.9]
12see [LeBr05, Thm. 1]
13We are mostly interested in the case $C = K[F]$ for $F$ a finite group of order coprime to $\text{char}(K)$. 
of \( G \) is contained in a \( G_i \) up to conjugation) that every perfect field of suitable characteristic admits a finite extension that is suitable.

Within this paper we will study representations of finitely generated virtually free groups over suitable finite fields. However, all of our methods apply to the more general case of representations of the fundamental algebra \( \mathcal{G} \) of a finite graph of finite dimensional semisimple \( \mathbb{F}_q \)-algebras under the assumption that each of the semisimple algebras \( A_i \) and \( A'_j \) are completely split.

### 2.2 Geometric methods.

The algebro-geometric methods in this paper are written in the language of schemes. Since we will work almost entirely with (affine) schemes of finite type over a perfect field \( K \), those readers who are less comfortable with schemes may instead think of the associated \( K \)-varieties and see \( K \)-valued points as fixed points of the natural Galois action of \( \text{Aut}_K(\mathcal{G}) \), connected/irreducible components as orbits of the natural Galois action on the connected/irreducible components, etc. For this whole subsection fix a field \( K \).

Let \( C \) be a commutative ring and \( X \) be a \( C \)-scheme. For each commutative \( C \)-algebra \( B \), we will denote by \( X(B) \) the set of \( B \)-valued points of \( X \), i.e. the set of \( C \)-scheme morphisms \( \text{Spec}(B) \to X \). If \( C = K \) is a field, we also use the term rational points for the \( K \)-valued points \( X(K) \).

Now assume \( C \) is of finite type over \( \mathbb{Z} \) and \( X \) is separated and of finite type over \( C \). A polynomial \( P \in \mathbb{Z}[s] \) is called counting polynomial of \( X \) if for every ring homomorphism \( C \to \mathbb{F}_q \) to a finite field \( X(\mathbb{F}_q) = P(q) \). We say that \( X \) is polynomial count if \( X \) admits a counting polynomial.

**Example 2.1** The general linear group (scheme) \( \text{GL}_d \) is polynomial count, its counting polynomial is given by \( P_{\text{GL}_d} := \prod_{s=0}^{d-1} (s^d - s^k) \).

Note that counting polynomials are unique and that the reduction \( X_{\text{red}} \) of a polynomial count scheme \( X \) is again polynomial count with the same counting polynomial. We will need the following two facts on polynomial count schemes:

(i) If \( P \in \mathbb{Q}(s) \) is a rational function and \( \# X(\mathbb{F}_q) = P(q) \) for each homomorphism \( C \to \mathbb{F}_q \), then \( P \) lies in the subring \( \mathbb{Z}[s] \) (and is a counting polynomial of \( X \)).

(ii) If \( C \) is a subring of \( \mathbb{C} \) and \( X \) is a polynomial count \( C \)-scheme with counting polynomial \( P \), then \( P(xy) \in \mathbb{Z}[x, y] \) is the E-polynomial\(^{14}\) of the analytification \( X(\mathbb{C}) = (X \times_C \text{Spec}(\mathbb{C}))^{\text{an}} \) and \( P(1) \) is the Euler characteristic of \( X(\mathbb{C}) \).

For proofs of the above facts see e.g. [H.R.08 Appendix] and [Rein06 §6].

The most important schemes we discuss in this paper arise from the representation spaces of algebras, whose construction we will now recall. Recall that the functor \( M_d : \text{CAlg}_K \to \text{Alg}_K \) which sends \( C \) to the matrix algebra \( M_d(C) \) is a right adjoint. We denote its left adjoint by \( R_d \). For \( A \) a finite type \( K \)-algebra we call \( \text{Rep}_d(A) := \text{Spec}(R_d(A)) \) the \( d \)-th representation space of \( A \). \( \text{Rep}_d(A) \) is an affine finite type \( K \)-scheme admitting a natural bijection

\[
\text{Rep}_d(A)(C) \cong \text{Alg}_K(A, M_d(C))
\]

\(^{14}\)For the definition of E-polynomials see e.g. [H.R.08 Appendix]. The notion of E-polynomials only occurs as an application/motivation, readers only interested in the counting of representations over finite fields may ignore it.
for every commutative $K$-algebra $C$. Denote the image of $x \in \text{Rep}_d(\mathcal{A})(C)$ under (9) by $\rho_x$. The right hand side of (9) admits a natural $\text{GL}_d(C)$-action via conjugation for each $C$, hence, the general linear group (scheme) $\text{GL}_{d,K}$ acts on $\text{Rep}_d(\mathcal{A})$ in terms of a $K$-scheme morphism $\sigma : \text{GL}_{d,K} \times_K \text{Rep}_d(\mathcal{A}) \to \text{Rep}_d(\mathcal{A})$.

For a group scheme action we have two notions of the orbit of a point: If $x$ is a $C$-valued point of $\text{Rep}_d(\mathcal{A})$, then we have the orbit $\text{GL}_d(C).x \subseteq \text{Rep}_d(\mathcal{A})(C)$ — which we call the set-theoretic orbit of $x$ — as well as the algebro-geometric orbit $\mathcal{O}_x$. The latter is defined as the image of the orbit map

$$\vartheta_x := (\sigma \circ (\text{id}_{\text{GL}_{d,K}} \times x), \text{pr}_2) : \text{GL}_{d,K} \times_K \text{Spec}(C) \to X \times_K \text{Spec}(C)$$

If $x \in \text{Rep}_d(\mathcal{A})(F)$ is an $F$-valued point for $F \supseteq K$ a finite field extension, then $\mathcal{O}_x \subseteq \text{Rep}_d(\mathcal{A})$ is locally closed and we may consider it as a reduced locally closed subscheme. The two notions of orbits are closely related to each other: If $x \in \text{Rep}_d(\mathcal{A})(K)$ is a $K$-valued point, $F \supseteq K$ a field extension and $x'$ the $F$-valued point associated to $x$ via pulling it back along $\text{Spec}(F) \to \text{Spec}(K)$, then $\mathcal{O}_x(F) = \text{GL}_d(F).x'$.

The purpose of the action on $\text{Rep}_d(\mathcal{A})$ is that $K$-valued points $x, y \in \text{Rep}_d(\mathcal{A})(K)$ are in the same (set-theoretic) orbit if and only if the representations $\rho_x$ and $\rho_y$ are isomorphic, i.e. there is a natural bijection $\text{iso}_d(\mathcal{A}) \cong \text{Rep}_d(\mathcal{A})(K)/\text{GL}_{d,K}(K)$.

Within this paper we will usually not distinguish strictly between the set-theoretic orbit $\text{GL}_d(K).x$ of a $K$-valued point $x$, the isomorphism class of the corresponding representation $\rho_x : \mathcal{A} \to \text{M}_d(K)$ and the isomorphism class of its associated left module, which we denote by $\text{M}_x$. Given a left $\mathcal{A}$-module $\mathcal{M}$ we denote the $K$-valued point corresponding to $\mathcal{M}$ under a given choice of basis by $x_{\mathcal{M}}$. If $\mathcal{M}$ is the left module associated to the point $x$, we will also sometimes denote the algebro-geometric orbit $\mathcal{O}_x$ by $\mathcal{O}_{\mathcal{M}}$.

Similar to the algebro-geometric orbits one defines the stabilizer $S(x)$ of a $C$-valued point $x \in \text{Rep}_d(\mathcal{A})(C)$ geometrically as the fibre product defined by the pullback square

$$\begin{array}{ccc}
S(x) & \longrightarrow & \text{Spec}(C) \\
\downarrow & & \downarrow \text{(x,id)} \\
\text{GL}_{d,K} \times_K \text{Spec}(C) & \longrightarrow & \text{Rep}_d(\mathcal{A}) \times_K \text{Spec}(C)
\end{array}$$

$S(x)$ is a closed $C$-subgroup scheme of $\text{GL}_{d,K} \times_K \text{Spec}(C)$. Its $B$-valued points for any commutative $C$-algebra $B$ are given by

$$(10) \quad S(x)(B) \cong \{g \in \text{GL}_d(B) \mid g.x' = x'\} = \text{Aut}_{A \otimes_K B}(M_x \otimes_C B)$$

where $x' \in \text{Rep}_d(\mathcal{A})(B)$ is the $B$-valued point associated to $x$. So $S(x)(B)$ is nothing but the (set-theoretic) stabilizer subgroup in $\text{GL}_d(B)$ of the point $x'$.

If $\varphi : \mathcal{A} \to \mathcal{B}$ is a homomorphism of finite type $K$-algebras, then functoriality gives us an induced $K$-scheme morphism $\text{Rep}_d(\mathcal{B}) \to \text{Rep}_d(\mathcal{A})$ for each $d$ which realizes the restriction of scalars functor geometrically. We denote this morphism by $\varphi^*$. In fact $\text{Rep}_d(-)$ is a left adjoint functor from (finite type) $K$-algebras

\[\text{In general the inclusion } \subseteq \text{ is wrong for group scheme actions. For representation spaces it holds, because representations of an algebra have no twisted forms, i.e. if } \mathcal{M}, \mathcal{N} \text{ are left } \mathcal{A}\text{-modules and } \mathcal{M} \otimes_K F \cong \mathcal{N} \otimes_K F \text{ for some field extension } F \supseteq K, \text{ then } \mathcal{M} \text{ and } \mathcal{N} \text{ are already isomorphic.}\]
to the opposite category of affine (finite type) $K$-schemes, hence, it maps colimits of (finite type) $K$-algebras to limits of affine (finite type) $K$-schemes. For example $\text{Rep}_d(\mathcal{A} \times_K \mathcal{B}) \cong \text{Rep}_d(\mathcal{A}) \times_K \text{Rep}_d(\mathcal{B})$ for $\mathcal{A}, \mathcal{B}$ two finite type $K$-algebras. Moreover \cite{Kiri16} shows that

$$\text{Rep}_d(\mathcal{A} \otimes_K F) \cong \text{Rep}_d(\mathcal{A}) \times_K \text{Spec}(F)$$

are naturally isomorphic $F$-schemes for all field extensions $F \supseteq K$.

The geometry of representation spaces and their orbits play a substantial role within this paper. We will frequently make use of the following fundamental facts:\footnote{See \cite{Kiri16} \S 2.3} for (v) and (vi) in the case $\mathcal{A} = \mathbb{C}Q$ the path algebra of a quiver. The proofs hold in our setting without significant change.

(iii) If $\mathcal{A}$ is formally smooth, then $\text{Rep}_d(\mathcal{A})$ is a regular scheme.\footnote{$\text{Rep}_d(\mathcal{A})$ is formally smooth over $K$ in the sense of \cite{Stacks} Tag 02H0 because of the lifting property of $\mathcal{A}$. So by \cite{Stacks} Tags 02H6 it is smooth over $K$, hence, regular by \cite{Stacks} Tag 056S.}

(iv) If $x \in \text{Rep}_d(\mathcal{A})(F)$ is an $F$-valued point for a field extension $F \supseteq K$, then $\mathcal{O}_x$ is irreducible and in particular connected.\footnote{It is the continuous image of the general linear group $\text{GL}_{d,F}$.}

(v) If $K$ is perfect and $0 \rightarrow \mathcal{N} \rightarrow \mathcal{W} \rightarrow \mathcal{M} \rightarrow 0$ a short exact sequence of finite dimensional left $\mathcal{A}$-modules, then $\mathcal{O}_{\mathcal{N} \oplus \mathcal{M}} \subseteq \mathcal{O}_{\mathcal{W}}$.\footnote{see e.g. \cite{M.F.K.94} \S 1.2, Thm. 1.1]

(vi) If $K$ is perfect, then an orbit $\mathcal{O}_\mathcal{M} \subseteq \text{Rep}_d(\mathcal{A})$ is closed if and only if the corresponding left $\mathcal{A}$-module $\mathcal{M}$ is semisimple.

Since the isomorphism classes of representations of $\mathcal{A}$ are parametrized by orbits of representation spaces, it is natural to define moduli spaces of representations of $\mathcal{A}$ in terms of quotients of representation space. We denote the GIT quotient\footnote{More generally it would be sufficient to require that $K$ is perfect and has trivial Brauer group.} of $\text{Rep}_d(\mathcal{A})$ by

$$M(\mathcal{A}, d) := \text{Rep}_d(\mathcal{A})//\text{GL}_{d,K} = \text{Spec}(\mathcal{R}_d(\mathcal{A})^{\text{GL}_{d,K}})$$

If the field $K$ is finite or algebraically closed then there is a natural bijection $M(\mathcal{A}, d)(F) \cong \text{ssim}_d(\mathcal{A} \otimes_K F)$ for every algebraic field extension $F \supseteq K$. So for such $K$ we call $M(\mathcal{A}, d)$ the (GIT) moduli space of $d$-dimensional semisimple representations of $\mathcal{A}$. It contains a (possibly empty) open subscheme $M^{\text{absim}}(\mathcal{A}, d)$ which for $K$ as above admits a natural bijection $M^{\text{absim}}(\mathcal{A}, d)(F) \cong \text{absim}_d(\mathcal{A} \otimes_K F)$ for every algebraic field extension $F \supseteq K$. Accordingly we call $M^{\text{absim}}(\mathcal{A}, d)$ the (GIT) moduli space of $d$-dimensional absolutely simple representations of $\mathcal{A}$.

Note that for $K = \mathbb{F}_q$ a finite field the counting functions $r^{\text{ssim}, \mathcal{A}}_d$ and $r^{\text{absim}, \mathcal{A}}_d$ count the rational points of these GIT moduli spaces. So whenever counting polynomials $P^{\text{ssim}}_d$ and $P^{\text{absim}}_d$ as in Theorem \ref{thm:counting} exist, they are in fact counting polynomials of these GIT moduli spaces.

Now assume that $\mathcal{A} = \mathcal{A}' \otimes \mathbb{F}_q := \mathcal{A}' \otimes_{\mathbb{Z}} \mathbb{F}_q$ is defined over $\mathbb{Z}$ by a finite type $\mathbb{Z}$-algebra $\mathcal{A}'$.\footnote{For example $\mathcal{A}' = \mathbb{Z}[G]$ for $G$ a finitely generated group or $\mathcal{A}' = \mathbb{Z}\bar{Q}$ for $\bar{Q}$ a finite quiver.} One can define representation spaces and GIT moduli spaces of $\mathcal{A}'$ as $\mathbb{Z}$-schemes using Seshadri's generalization of geometric invariant theory.\footnote{see \cite{Sesh77}} In this way we obtain $\mathbb{Z}$-schemes $M(\mathcal{A}', d)$ and $M^{\text{absim}}(\mathcal{A}', d)$ such that for $F = \mathbb{C}$ and...
F = ℤₚ for all primes p in an open subset of Spec (ℤ) we havë²³

\[ M(\mathcal{A}', d) \times \text{Spec } (F) \cong M(\mathcal{A}' \otimes F, d), \quad M_{\text{absim}}(\mathcal{A}', d) \times \text{Spec } (F) \cong M_{\text{absim}}(\mathcal{A}' \otimes F, d) \]

Hence, using (i) from Subsection 2.2 we see that if there are rational functions \( R_d, R_{d_{\text{absim}}} \in \mathbb{Q}(s) \) which satisfy (2), they must already belong to \( \mathbb{Z}[s] \). Furthermore using (ii) from Subsection 2.2 we see that whenever the counting polynomials exist, the E-polynomials of \( M(\mathcal{A}' \otimes \mathbb{C}, d)_{\text{an}} \) and \( M_{\text{absim}}(\mathcal{A}' \otimes \mathbb{C}, d)_{\text{an}} \) are given by \( R_d(xy) \) and \( R_{d_{\text{absim}}}(xy) \).

When \( \mathcal{A}' = \mathbb{Z}[\mathcal{G}] \) is the group algebra of a finitely generated group \( \mathcal{G} \), (the analytification of) the moduli space \( M(\mathbb{C}[\mathcal{G}], d) \) is also called the \( \text{GL}_d(\mathbb{C}) \)-character variety of \( \mathcal{G} \) and denoted by \( \mathcal{X}_G(\text{GL}_d(\mathbb{C})) \). Since our methods enable us to compute the counting polynomials explicitly (e.g. using the accompanying SageMath code), we obtain a new approach to determine the E-polynomials of \( \text{GL}_d(\mathbb{C}) \)-character varieties of virtually free groups.

We now want to recall the construction of associated fibre spaces.²⁴ Let \( G \) be a linear algebraic group over \( K \), \( H \subseteq G \) a closed subgroup and \( X \) an affine \( K \)-scheme endowed with an \( H \)-action. We define an induced \( H \)-action on \( G \times_K X \) via the natural transformation

\[ H(C) \times G(C) \times X(C) \rightarrow G(C) \times X(C), \quad h.(g, x) := (gh^{-1}, h.x) \]

for any commutative \( K \)-algebra \( C \). This is a free action and its respective quotient \( G \times_K X / H \) is called the associated \( G \)-fibre space.

If \( (g, x) \in G(C) \times X(C) \) is a \( C \)-valued point, we denote its image in \( (G \times^H X)(C) \) by \( g \ast x \). We have a natural morphism \( X \rightarrow G \times^H X \) given by \( x \mapsto 1 \ast x \). If \( Y \) is a \( K \)-scheme with \( G \)-action and \( \varphi : X \rightarrow Y \) is an \( H \)-equivariant morphism, then we obtain a unique \( G \)-equivariant morphism \( \varphi' \) such that \( \varphi \) factorizes as

(11) \[ X \rightarrow G \times^H X \xrightarrow{\varphi'} Y \]

For \( G \) and \( H \) (geometrically) irreducible we have that \( G \times^H X \) is irreducible/connected if and only if \( X \) is irreducible/connected. Moreover we have the following useful lemma.

**Lemma 2.2**²⁵ Let \( \varphi : Y \rightarrow G/H \) be a \( G \)-equivariant morphism. If \( e \in G/H(K) \) is the \( K \)-point lying under the unit of \( G \) and \( X := \varphi^{-1}(e) \) with inclusion map \( \iota : X \rightarrow Y \), then the induced map \( \iota' : G \times^H X \rightarrow Y \) from (11) is a \( G \)-equivariant isomorphism.

We also want to recall the notion of special algebraic groups. A linear algebraic group \( G \) over \( K \) is **special** if the quotient map \( \pi : X \rightarrow X/G \) for any affine finite type \( K \)-scheme \( X \) with a free \( G \)-action is Zariski-locally a trivial bundle, i.e. there is an open covering \( X/G = \bigcup_{\alpha} U_{\alpha} \) such that for each \( \alpha \) there is a \( G \)-equivariant

²³ see [C.V.04, Appendix B, Thm. B.3]
²⁴ in fact we can more generally compute the E-polynomials of the connected components of the character varieties individually.
²⁵ Most of the facts we are collecting here can e.g. be found in [Serr58].
²⁶ In the case of free actions we will denote the GIT quotient with a single \( / \) instead of \( \underline{\ } \).
²⁷ see [Slod80] §II.3.7, Lemma 4]
Example 2.3 Let $\mathcal{C}$ be a finite dimensional semisimple $K$-algebra which is completely split, i.e. $\mathcal{C}$ is of the form (8). Denote its (absolutely) simple left modules by $\mathcal{L}_0, \ldots, \mathcal{L}_{c-1}$. Each finite dimensional left $\mathcal{C}$-module $M$ is semisimple, i.e. $M \cong \bigoplus_{i=0}^{c-1} M_{m(i)}$, for some $m \in \mathbb{N}_0^c$. Using the bijection (10) and $\text{End}_A(\mathcal{L}_i) = K$ we obtain an isomorphism $S(x_M) \rightarrow S(x_M) \cong \bigoplus_{i=0}^{c-1} GL_{m(i)}(K)$ for every commutative $K$-algebra $B$. Hence, $S(x_M) \cong GL_{m(0)}(K) \times \ldots \times GL_{m(c-1)}(K)$ which is a special linear algebraic group.

If $H$ is a special linear algebraic group acting freely on an affine finite type $K$-scheme $X$ with quotient $X/H$, then the canonical injection

\[(12) \quad X(F)/H(F) \rightarrow (X/H)(F)\]

is bijective for every field extension $F \supseteq K$. This in particular applies to the case where $X/H$ is an associated fibre space $G \times^H X$.

3 Some invariants of virtually free groups

For the whole section fix a perfect field $K$.

3.1 Dimension vectors. We will now associate to every finite type $K$-algebra $A$ a commutative monoid $\mathcal{T}(A)$ and a monoid homomorphism $|.| : \mathcal{T}(A) \rightarrow \mathbb{N}_0$ which generalizes the dimension vector monoid from quiver representation theory.

For $d \in \mathbb{N}_0$ we denote by $\mathcal{T}_d(A)$ the set of connected components $Z \subseteq \text{Rep}_d(A)$ containing a rational point, i.e. $Z(K) \neq \emptyset$. As a set we define $\mathcal{T}(A)$ as the disjoint union

$$\mathcal{T}(A) := \bigsqcup_{d \geq 0} \mathcal{T}_d(A)$$

and we define the map $|.|$ via $|\mathcal{T}_d(A)| = d$. To define the monoid structure on $\mathcal{T}(A)$ we consider the direct sum map

$$\oplus_{c,d} : \text{Rep}_c(A) \times_K \text{Rep}_d(A) \rightarrow \text{Rep}_{c+d}(A)$$

For $(Z, Z') \in \mathcal{T}_c(A) \times \mathcal{T}_d(A)$ the product $Z \times_K Z'$ is connected, hence, there is a unique connected component $Z + Z' \in \mathcal{T}_{c+d}(A)$ containing $\oplus_{c,d}(Z \times_K Z')$.
The monoid $\mathcal{T}(\mathcal{A})$ has been studied in the past under other names like component semigroup. Since we want to emphasize the analogy to dimension vectors, we will refer to it as the \textit{dimension vector monoid} of $\mathcal{A}$ and call the elements of $\mathcal{T}_d(\mathcal{A})$ \textit{dimension vectors of total dimension $d$}. Usually we will think of dimension vectors as abstract monoid elements. Whenever we want to refer to the connected component (associated to) $m$ as a geometric object, we will denote it by $\text{Rep}_m(\mathcal{A})$.

Since the orbit $\mathbb{O}_M$ associated to a left $\mathcal{A}$-module $M$ is connected, it belongs to a unique connected component. Denote the corresponding dimension vector by $\dim(M)$.

The dimension vector monoid $\mathcal{T}(\mathcal{A})$ is contravariant functorial in $\mathcal{A}$: If $\varphi : \mathcal{A} \to \mathcal{B}$ is a $K$-algebra homomorphism and $m \in \mathcal{T}(\mathcal{B})$ a dimension vector, then denote by $\mathcal{T}(\varphi)(m)$ the dimension vector associated to the connected component which contains the image of $\text{Rep}_m(\mathcal{B})$ under $\varphi^*$. This defines a monoid homomorphism $\mathcal{T}(\varphi) : \mathcal{T}(\mathcal{B}) \to \mathcal{T}(\mathcal{A})$.

The homomorphism $\mathcal{T}(\varphi)$ induces maps $\mathcal{T}_d(\varphi) : \mathcal{T}_d(\mathcal{B}) \to \mathcal{T}_d(\mathcal{A})$ for all $d \in \mathbb{N}_0$, because restriction of scalars preserves the vector space dimension of modules. Since every bijective monoid homomorphism is an isomorphism, $\mathcal{T}(\varphi)$ is an isomorphism if and only if the map $\mathcal{T}_d(\varphi)$ is bijective for every $d$.

\textbf{Example 3.1}

a) Let $\mathcal{C}$ be a finite dimensional semisimple $K$-algebra. We assume that $\mathcal{C}$ is completely split, i.e. of the form $\bigoplus$. All left $\mathcal{C} \otimes_K F$-modules for every field extension $F \supseteq K$ are defined over $\mathcal{C}$. Therefore the (finitely many) orbits of the $K$-valued points cover $\text{Rep}_d(\mathcal{C})$ and all of them are connected and closed by (iv) and (vi) in Subsection 2.2. We deduce that the orbits of the $K$-valued points are nothing but the connected components and that the map $\text{iso}_d(\mathcal{C}) \to \mathcal{T}_d(\mathcal{C})$, $[\mathcal{M}] \mapsto \dim(M)$ is bijective for every $d \in \mathbb{N}_0$. Since there is precisely one simple left $\mathcal{C}$-module for every matrix algebra factor in $\bigoplus$, we have established a monoid isomorphism

$$\mathcal{T}(\mathcal{C}) \cong \mathbb{N}_0^{c_1+\ldots+c_e}$$

In fact, $\mathcal{T}(\mathcal{C})$ is nothing but the submonoid of the Grothendieck group $G_0(\mathcal{C})$ generated by the equivalence classes of the (absolutely) simples in this situation. If $m_\gamma \in \mathbb{N}_0^{c_1+\ldots+c_e}$ is the $\gamma$-th standard basis vector for $0 \leq \gamma < c_1 + \ldots + c_e$, then $|m_\gamma| = c_\epsilon$ for the unique $1 \leq \epsilon \leq e$ with $c_1 + \ldots + c_{\epsilon-1} < \gamma \leq c_1 + \ldots + c_\epsilon$.

b) Let $K\overrightarrow{Q}$ be the path algebra of a finite quiver $\overrightarrow{Q}$ with vertex set $v(\overrightarrow{Q})$. Applying (a) to the subalgebra $\mathcal{C}$ spanned by the paths of length zero, one can use Lemma 2.2 to establish a monoid isomorphism

$$\mathcal{T}(K\overrightarrow{Q}) \cong \mathcal{T}(\mathcal{C}) \cong \mathbb{N}_0^{v(\overrightarrow{Q})}$$

The following proposition gives a complete description of the dimension vector monoid of the group algebra $K[G]$ of a finitely generated virtually free group $G$ over a suitable field $K$.

\textbf{Proposition 3.2} Let $\mathcal{A}$ be a finite type $K$-algebra, $\mathcal{B}$ and $\mathcal{C}$ completely split finite dimensional semisimple $K$-algebras and $\varphi_1 : \mathcal{C} \to \mathcal{B}$, $\varphi_2$, $\varphi_3 : \mathcal{C} \to \mathcal{A}$ $K$-algebra homomorphisms.

\textsuperscript{33}see e.g. [LeBr05, §4]
a) Consider the $K$-algebra pushout $A\ast_C B$ given by $\varphi_1, \varphi_2$. The commutative square

$$
\begin{array}{ccc}
T(A \ast_C B) & \rightarrow & T(B) \\
\downarrow & & \downarrow \\
T(A) & \rightarrow & T(C)
\end{array}
$$

is a pullback square of commutative monoids.

b) If $\iota_A : A \rightarrow A \ast_K K[t, t^{-1}]$ is the canonical $K$-algebra embedding and 
$\Phi : A \ast_K K[t, t^{-1}] \rightarrow A \ast_K K[t, t^{-1}]$ the $K$-algebra automorphism $\Phi(f) := t^{-1} f t$, 
then $T(\iota_A)$ is an isomorphism and $T(\Phi) = \text{id}$. 

c) Consider the HNN extension $A\ast_{C_1^2} C_2$ given by $\varphi_2, \varphi_3$. The diagram

$$
\begin{array}{ccc}
T(A\ast_{C_1^2} C_2) & \rightarrow & T(A) \\
\downarrow & & \downarrow \\
T(C) & \rightarrow & T(C)
\end{array}
$$

is an equalizer diagram of commutative monoids.

d) If $\mathcal{G}$ is the finitely generated virtually free group given by $[\mathcal{G}]$ and $K$ is a suitable field for $\mathcal{G}$, then $T(K[\mathcal{G}])$ is given by

$$
\left\{(m_i) : \prod_{i=0}^I T(K[\mathcal{G}_i]) \quad \forall 1 \leq j \leq I : T(\iota_j)(m_{a(j)}) = T(\kappa_j)(m_{b(j)})\right\}
$$

\textbf{Proof:} About a): $[\mathcal{G}]$ induces a homomorphism $\theta : T(A \ast_C B) \rightarrow T(A) \times_{T(C)} T(B)$ with 

$$
T(A) \times_{T(C)} T(B) = \{(m, n) : T(A) \times T(B) \mid T(\varphi_2)(m) = T(\varphi_1)(n)\}
$$

$\theta$ is an isomorphism if and only if its restriction $\theta_d : T_d(A \ast_C B) \rightarrow T_d(A) \times_{T_d(C)} T_d(B)$ is bijective for every $d \in \mathbb{N}_0$. Denote the natural homomorphisms $A \rightarrow A \ast_C B$ and 
$B \rightarrow A \ast_C B$ by $\iota_A$ and $\iota_B$ and the connected components of $\text{Rep}_d(A)$ and $\text{Rep}_d(B)$ 
by $X_0, \ldots, X_a$ and $Y_0, \ldots, Y_b$ respectively. 

Since the contravariant functor $\text{Rep}_d(\ast)$ maps colimits to limits, we have a natural isomorphism 
$\text{Rep}_d(A\ast_C B) \cong \text{Rep}_d(A) \times_{\text{Rep}_d(C)} \text{Rep}_d(B)$. This yields a decomposition 

$$
\text{Rep}_d(A \ast_C B) = \bigcup_{0 \leq \alpha \leq a, \beta \leq b} X_{\alpha} \times_{\text{Rep}_d(C)} Y_{\beta}
$$

into open and closed subsets. Hence, each connected component $\text{Rep}_d(A \ast_C B)$ associated to a dimension vector $v \in T(A \ast_C B)$ lies in a unique $X_{\alpha} \times_{\text{Rep}_d(C)} Y_{\beta}$ and 
one checks that the homomorphism $\theta$ is given by $\theta_d(v) = (m_{\alpha}, n_{\beta})$ where $m_{\alpha} \in T(A)$ 
and $n_{\beta} \in T(B)$ are the dimension vectors (associated to) $X_{\alpha}$ and $Y_{\beta}$. We claim that 
$\text{Rep}_{m_{\alpha}}(A) \times_{\text{Rep}_d(C)} \text{Rep}_{n_{\beta}}(B) = X_{\alpha} \times_{\text{Rep}_d(C)} Y_{\beta}$ is connected and contains a rational 
point for each $(m_{\alpha}, n_{\beta}) \in T(A) \times_{T(C)} T(B)$ which proves that $\theta_d$ is bijective. 

Denote by $u := T(\varphi_2)(m_{\alpha}) = T(\varphi_1)(n_{\beta}) \in T(C)$ the dimension vector lying below 
$(m_{\alpha}, n_{\beta})$. By construction both $\text{Rep}_{m_{\alpha}}(A)$ and $\text{Rep}_{n_{\beta}}(B)$ map into the connected 
component $Z := \text{Rep}_d(C)$ and we obtain an isomorphism 
$X_{\alpha} \times_{\text{Rep}_d(C)} Y_{\beta} \cong X_{\alpha} \times_Z Y_{\beta}$. 
Since $m_{\alpha}$ and $n_{\beta}$ are dimension vectors, there are $K$-valued points $x \in \text{Rep}_{m_{\alpha}}(A)(K)$ 
and $y \in \text{Rep}_{n_{\beta}}(B)(K)$. We set $z := \varphi_2(x)$. 

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By Example 3.1(a) we have $Z = \mathcal{O}_z \cong \text{GL}_{d,K}/S(z)$ and $Y_\beta = \mathcal{O}_y \cong \text{GL}_{d,K}/S(y)$. Since $\mathcal{O}_z(K) = \text{GL}_d(K).z$ and $\varphi^*_g$ restricts to a $\text{GL}_{d,K}$-equivariant map $Y_\beta \to Z$, we may assume without loss of generality that $\varphi^*_1(y) = z = \varphi^*_2(x)$. So $X_\alpha \times Z Y_\beta(K)$ is non-empty.

Furthermore taking fibres we obtain a commutative diagram

\[\begin{array}{ccc}
X_\alpha \times Z Y_\beta & \xrightarrow{\overline{\tau}} & \text{GL}_{d,K}/S(y) \\
\downarrow & & \downarrow \\
X_\alpha & \xrightarrow{\overline{\varphi}_1} & \text{GL}_{d,K}/S(z) \quad \text{id} \\
\overline{\varphi}_2^{-1}(z) & \xrightarrow{\tau_B^{-1}(y)} & \text{Spec}(K) \\
\end{array}\]

where $\overline{\varphi}$ and $\tau_B^{-1}$ are given as the restrictions of $\varphi^*_2$ and $\tau_B$. The bottom, top and back squares of (16) are pullback squares, hence, the front square is too and we obtain an isomorphism

$\tau_B^{-1}(y) \cong \overline{\varphi}_1^{-1}(z)$

Applying Lemma 2.2 we obtain isomorphisms

$X_\alpha \cong \text{GL}_{d,K} \times S(z) \overline{\varphi}_1^{-1}(z)$, $X_\alpha \times Z Y_\beta \cong \text{GL}_{d,K} \times S(y) \tau_B^{-1}(y)$

So since $X_\alpha$ is connected by assumption, $\tau_B^{-1}(y) \cong \overline{\varphi}_1^{-1}(z)$ and $X_\alpha \times Z Y_\beta$ are connected too.

About b): Since $\text{Rep}_d(K[t, t^{-1}]) \cong \text{GL}_{d,K}$ and $\text{Rep}_d(K) \cong \text{Spec}(K)$ are connected, we have isomorphisms $\mathcal{T}(K[t, t^{-1}]) \cong \mathbb{N}_0 \cong \mathcal{T}(K)$ given by $|.|$ respectively. So $\mathcal{T}(\iota_A)$ is an isomorphism by part a).

Now if $(x, g) \in \text{Rep}_d(A)(K) \times \text{GL}_d(K) = \text{Rep}_d(A \ast_k K[t, t^{-1}](K)$ is a $K$-valued point, then $\Phi^*(x, g) = g^{-1}.(x, g) \in \mathcal{O}_{(x,g)}(K)$ . So $(x, g)$ and $\Phi^*(x, g)$ lie in the same connected component, because $\mathcal{O}_{(x,g)}$ is connected. This proves $\mathcal{T}(\Phi) = \text{id}$.

About c): Denote the natural projection $A \ast_k K[t, t^{-1}] \to A^{g_2 \ast_3}$ by $\pi$. By construction of the HNN extension we have $\pi \circ \iota_A \circ \varphi_2 = \pi \circ \Phi \circ \iota_A \circ \varphi_3$ . So using part b) we obtain that $\theta := \mathcal{T}(\iota_A) \circ \mathcal{T}(\pi) : \mathcal{T}(A^{g_2 \ast_3}) \to \mathcal{T}(A)$ factorizes over $\text{Eq}(\mathcal{T}(\varphi_2), \mathcal{T}(\varphi_3)) = \{ m \in \mathcal{T}(A) \mid \mathcal{T}(\varphi_2)(m) = \mathcal{T}(\varphi_3)(m) \} \subseteq \mathcal{T}(A)$.

We have to show that $\theta$ is an isomorphism.

Denote the connected components of $\text{Rep}_d(A)$ by $X_0, \ldots, X_a$. As for a) we obtain a natural isomorphism $\text{Rep}_d(A^{g_2 \ast_3}) \cong \text{Eq}((\iota_A \circ \varphi_2)^*, (\Phi \circ \iota_A \circ \varphi_3)^*)$ and a decomposition

$\text{Rep}_d(A^{g_2 \ast_3}) = \bigsqcup_{0 \leq \alpha \leq a} (\iota_A \circ \varphi_2)^{-1}(X_\alpha)$

$^{34}$For $\varphi^*_1(y) = g.z$ we may replace $y$ by $g^{-1}.y$. 

\[\text{Eq}(\mathcal{T}(\varphi_2), \mathcal{T}(\varphi_3)) = \{ m \in \mathcal{T}(A) \mid \mathcal{T}(\varphi_2)(m) = \mathcal{T}(\varphi_3)(m) \} \subseteq \mathcal{T}(A)\]
into open and closed subsets \((\iota_A \circ \pi^*)^{-1}(X_\alpha) = (\pi^*)^{-1}(X_\alpha \times_K \text{GL}_{d,K})\) and it remains to show that \((\pi^*)^{-1}(X_\alpha \times_K \text{GL}_{d,K})\) is connected and contains a rational point if \(X_\alpha = \text{Rep}_{m_\alpha}(A)\) corresponds to a dimension vector \(m_\alpha \in \text{Eq}(\mathcal{T}(\varphi_2), \mathcal{T}(\varphi_3))\).

We first check using the universal property of the equalizer \(\text{Rep}_d(A_{\pi^2,\pi^3})\) that a \(K\)-valued point \((x, g) \in X_\alpha(K) \times \text{GL}_d(K) = (X_\alpha \times_K \text{GL}_{d,K})(K)\) lies in the image of the closed embedding \(\pi^*\) if and only if

\[
(17) \quad \rho_x \circ \varphi_2 = (\iota_A \circ \varphi_2)^*(x, g) = (\Phi \circ \iota_A \circ \varphi_3)^*(x, g) = g^{-1}.(\rho_x \circ \varphi_3)
\]

The rational points associated to \(\rho_x \circ \varphi_2\) and \(\rho_x \circ \varphi_3\) belong to the same connected component \(Z \subseteq \text{Rep}_d(C)\), because we assumed \(m_\alpha \in \text{Eq}(\mathcal{T}(\varphi_2), \mathcal{T}(\varphi_3))\). Again using Example 3.1(a) we know that \(Z = \bigoplus_x \cong \text{GL}_{d,K}/S(z)\) for some \(z \in \text{Rep}_d(C)(K)\). Hence, there is a \(g \in \text{GL}_d(K)\) satisfying \(\rho_x \circ \varphi_2 = g^{-1}.(\rho_x \circ \varphi_3)\) which yields a rational point in \((\pi^*)^{-1}(X_\alpha \times_K \text{GL}_{d,K})\).

Now denote the restriction of \((\pi \circ \iota_A \circ \varphi_2)^*\) to \((\pi^*)^{-1}(X_\alpha \times_K \text{GL}_{d,K})\) by \(\psi\). The criterion (17) yields that \(\psi^{-1}(z) \cong (\varphi_2^*)^{-1}(z) \times_K S(z)\). So as \(S(z)\) is geometrically irreducible by Example 2.3 \(\psi^{-1}(z)\) is connected if and only if \((\varphi_2^*)^{-1}(z)\) is \(\S \) \(\S \) We now again use Lemma 2.2 to obtain isomorphisms

\[
X_\alpha \cong \text{GL}_{d,K} \times_{S(z)} (\varphi_2^*)^{-1}(z), \quad (\pi^*)^{-1}(X_\alpha \times_K \text{GL}_{d,K}) \cong \text{GL}_{d,K} \times_{S(z)} \psi^{-1}(z)
\]

So \((\pi^*)^{-1}(X_\alpha \times_K \text{GL}_{d,K})\) is connected, because the connected component \(X_\alpha\) is.

About (d): The claim follows from the decomposition [6] of \(K[G]\) and Example 3.1(a) by repeatedly applying part (a) and (c) above. \(\square\)

As a corollary from (15) we obtain that the dimension vector monoid \(\mathcal{T}(K[G])\) does not depend on the choice of the suitable field \(K\). First let \(\mathcal{H}\) be a finite group and \(\mathcal{F} \subseteq \mathcal{H}\) be a subgroup. Using well-known arguments from the representation theory of finite groups and Example 3.1(a) one first shows that the monoids \(\mathcal{T}(K[\mathcal{F}])\) and \(\mathcal{T}(K[\mathcal{H}])\) as well as the homomorphism \(\mathcal{T}(K[\mathcal{H}]) \rightarrow \mathcal{T}(K[\mathcal{F}])\) do not depend on \(K\). So since \(\mathcal{T}(K[G])\) is the limit of a diagram of monoids which itself does not depend on \(K\), \(\mathcal{T}(K[G])\) does not depend on \(K\) as well. We will therefore drop \(K\) from the notation and simply write \(\mathcal{T}(G)\).

We conclude our current discussion of dimension vectors with a few general remarks. However, the readers may feel free to skip forward to Section 6 for some hands-on examples at this point. We first note another immediate consequence of the isomorphism (15): \(\mathcal{T}(G)\) is equipped with a canonical embedding into the free commutative monoid \(\bigoplus_i \mathcal{T}(G_i)\). This does not imply that \(\mathcal{T}(G)\) is free itself, but is a huge restriction on the class of monoids arising as \(\mathcal{T}(G)\).

Moreover \(\mathcal{T}(G)\) comes with a canonical homomorphism \(\mathcal{T}(G) \rightarrow \mathcal{T}(G_i)\) for each \(0 \leq i \leq I\) and a canonical homomorphism \(\mathcal{T}(G) \rightarrow \mathcal{T}(G_j')\) for each \(1 \leq j \leq I + J\).

We say that \(m \in \mathcal{T}(G)\) lies over \(m_i \in \mathcal{T}(G_i)\) for \(0 \leq i \leq I\) and \(u_j \in \mathcal{T}(G_j')\) for \(1 \leq j \leq I + J\) if these are the images of \(m\) under these canonical homomorphisms. Note that these images uniquely determine \(m\) due to the isomorphism (15).

For \(c \in \mathbb{N}_{\geq 1}\) and \(m \in \mathcal{T}(G)\) we write \(c|m\) if there is an \(n \in \mathcal{T}(G)\) fulfilling \(m = c.n = n + \ldots + n\). Such an \(n\) is necessarily unique and we denote it by

\(35\text{ see } \text{Stacks}, \text{Tag 0385]\n\(36\text{ see } \text{Serr}, \text{§14.6, 15.1} \& \text{Prop. 43 in §15.5]}

\[ m/c := n \, . \] Moreover \{ c \in \mathbb{N}_{\geq 1} \mid c|m \} \] is a finite set — this as well as the uniqueness of \( m/c \) are immediate consequences of the embedding \( \mathcal{T}(\mathcal{G}) \hookrightarrow \prod_i \mathcal{T}(\mathcal{G}_i) \). We denote
\[ \gcd(m) := \max\{ c \in \mathbb{N}_{\geq 1} \mid c|m \} = \operatorname{lcm}\{ c \in \mathbb{N}_{\geq 1} \mid c|m \} \]
Another important property of dimension vectors is that they are additive on short exact sequences, i.e. for every short sequence \( 0 \to \mathcal{N} \to \mathcal{W} \to \mathcal{M} \to 0 \) of left modules over a finite type \( K \)-algebra \( \mathcal{A} \) we have \( \dim(\mathcal{W}) = \dim(\mathcal{N}) + \dim(\mathcal{M}) \). If the sequence splits, this is true by definition of \( \mathcal{T}(\mathcal{A}) \). For the general case one uses iv) and (v) in Subsection 2.2.

3.2 Homological Euler form. We now want to discuss another object which again has a well-known analogue in quiver representation theory: the \textit{(homological) Euler form}. For \( \mathcal{A} \) a (left hereditary) \( K \)-algebra and finite dimensional left \( \mathcal{A} \)-modules \( \mathcal{M}, \mathcal{N} \) we define
\[ \langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{A}} := \dim_K(\operatorname{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})) - \dim_K(\operatorname{Ext}^1_{\mathcal{A}}(\mathcal{M}, \mathcal{N})) \]
Example 3.3 Let \( \mathcal{C} \) be a completely split finite dimensional semisimple \( K \)-algebra. Recall that \( \mathcal{C} \) may be written as (8). \( \mathcal{C} \) admits precisely \( c := c_1 + \ldots + c_e \) pairwise non-isomorphic (absolutely) simple modules — choose a representative \( L_\gamma \) for each isomorphism class. For two arbitrary finite dimensional left \( \mathcal{C} \)-modules \( \mathcal{M} = \bigoplus_{\gamma=0}^{c-1} L_\gamma \otimes m(\gamma) \) and \( \mathcal{N} = \bigoplus_{\gamma=0}^{c-1} L_\gamma \otimes n(\gamma) \) we compute the homological Euler form
\[ \langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{C}} = \dim_K(\operatorname{Hom}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})) = \sum_{\gamma=0}^{c-1} m(\gamma)n(\gamma) \]
by using Schur’s Lemma for the absolutely simple modules \( L_\gamma \). For \( m := \dim(\mathcal{M}) \) and \( n := \dim(\mathcal{N}) \) we also introduce the notion \( (m, n)_{\mathcal{C}} := \langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{C}} \) which is well-defined, since \( \dim : \operatorname{iso}_d(\mathcal{C}) \to T_d(\mathcal{C}) \) is bijective by Example 3.1(a).

As for the dimension vector monoid we now want to compute the homological Euler form of the group algebra of a finitely generated virtually free group \( \mathcal{G} \) over a suitable field.

Proposition 3.4 Let \( \mathcal{A} \) and \( \mathcal{B} \) be left hereditary finite type \( K \)-algebras, \( \mathcal{C} \) a finite dimensional semisimple \( K \)-algebra and \( \varphi_1 : \mathcal{C} \to \mathcal{B} \), \( \varphi_2, \varphi_3 : \mathcal{C} \to \mathcal{A} \) \( K \)-algebra homomorphisms.

a) Consider the pushout given by \( \varphi_1, \varphi_2 \) and let \( \mathcal{M}, \mathcal{N} \) be finite dimensional left \( \mathcal{A}_*\mathcal{C} \mathcal{B} \)-modules. The homological Euler form of \( \mathcal{A}_*\mathcal{C} \mathcal{B} \) is given by
\[ \langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{A}_*\mathcal{C} \mathcal{B}} = \langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{A}} + \langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{B}} - \langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{C}} \]
b) Consider the HNN extension \( \mathcal{A}_*\mathcal{C}^{\varphi_2, \varphi_3} \) and let \( \mathcal{M}, \mathcal{N} \) be finite dimensional left \( \mathcal{A}_*\mathcal{C}^{\varphi_2, \varphi_3} \)-modules. The homological Euler form of \( \mathcal{A}_*\mathcal{C}^{\varphi_2, \varphi_3} \) is given by
\[ \langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{A}_*\mathcal{C}^{\varphi_2, \varphi_3}} = \langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{A}} - \langle \mathcal{M}, \mathcal{N} \rangle_{\mathcal{C}} \]
c) If $\mathcal{G}$ is the finitely generated virtually free group given by (1) and $K$ is suitable for $\mathcal{G}$, then $\langle -, - \rangle_{K[\mathcal{G}]}$ is given by

\[
\langle \mathcal{M}, \mathcal{N} \rangle_{K[\mathcal{G}]} = \sum_{i=0}^{I} (m_i, n_i)_{K[\mathcal{G}]} - \sum_{j=1}^{I+J} (u_j, v_j)_{K[\mathcal{G}']}
\]

where $\dim(\mathcal{M})$ is the dimension vector lying over $m_i \in \mathcal{T}(\mathcal{G}_i)$ for $0 \leq i \leq I$ and $u_j \in \mathcal{T}(\mathcal{G}_j')$ for $1 \leq j \leq I+J$ and $\dim(\mathcal{N})$ is lying over $n_i \in \mathcal{T}(\mathcal{G}_i)$ and $v_j \in \mathcal{T}(\mathcal{G}_j')$ respectively.

**Proof:** Let $\mathcal{D}$ be a finite type $K$-algebra and $\mathcal{W}$ a $K$-linear $(\mathcal{D}, \mathcal{D})$-bimodule. We consider the $K$-linear map $\eta : \mathcal{W} \rightarrow \text{Der}_K(\mathcal{D}, \mathcal{W})$ which sends $w \in \mathcal{W}$ to its inner derivation $\eta(w) = (f \mapsto f \cdot w - w \cdot f)$ and obtain an exact sequence

\[
0 \rightarrow \ker(\eta) \rightarrow \mathcal{W} \xrightarrow{\eta} \text{Der}_K(\mathcal{D}, \mathcal{W}) \rightarrow \text{coker}(\eta) \rightarrow 0
\]

For the bimodule $\mathcal{W} = \text{Hom}_K(\mathcal{M}, \mathcal{N})$ we obtain $\ker(\eta) = \text{Hom}_D(\mathcal{M}, \mathcal{N})$ and $\text{coker}(\eta) \cong \text{Ext}_D^1(\mathcal{M}, \mathcal{N})$. Hence, (20) yields

\[
\langle \mathcal{M}, \mathcal{N} \rangle_D = \dim_K(\mathcal{M}) \cdot \dim_K(\mathcal{N}) - \dim_K \text{Der}_K(\mathcal{D}, \text{Hom}_K(\mathcal{M}, \mathcal{N}))
\]

So we can reformulate the claimed identity a) as

\[
\dim_K \text{Der}_K(\mathcal{D}, \mathcal{W}) = \dim_K \text{Der}_K(\mathcal{A}, \mathcal{W}) + \dim_K \text{Der}_K(\mathcal{B}, \mathcal{W}) - \dim_K \text{Der}_K(\mathcal{C}, \mathcal{W})
\]

for $\mathcal{D} = \mathcal{A} \ast_C \mathcal{B}$ and $\mathcal{W} = \text{Hom}_K(\mathcal{M}, \mathcal{N})$ and the claimed identity b) takes the form

\[
\dim_K \text{Der}_K(\mathcal{D}, \mathcal{W}) = \dim_K \text{Der}_K(\mathcal{A}, \mathcal{W}) + \dim_K(\mathcal{W}) - \dim_K \text{Der}_K(\mathcal{C}, \mathcal{W})
\]

for $\mathcal{D} = \mathcal{A} \ast_C^{\varphi_2, \varphi_3}$ and $\mathcal{W} = \text{Hom}_K(\mathcal{M}, \mathcal{N})$.

For the identity a) we use that $\text{Der}_K(\mathcal{D}, \mathcal{W})$ is the $K$-vector space pullback induced by $\varphi_1^*$ and $\varphi_2^*$, hence, $\text{Der}_K(\mathcal{D}, \mathcal{W})$ is the kernel of the map

\[
(\varphi_2^*, -\varphi_1^*) : \text{Der}_K(\mathcal{A}, \mathcal{W}) \oplus \text{Der}_K(\mathcal{B}, \mathcal{W}) \rightarrow \text{Der}_K(\mathcal{C}, \mathcal{W})
\]

which is surjective, because $\mathcal{C}$ is separable, i.e. every derivation of $\mathcal{C}$ is inner.

The identity b) is proven similarly:

\[
\text{Der}_K(\mathcal{D}, \mathcal{W}) \xrightarrow{\pi^*} \text{Der}_K(\mathcal{A} \ast_K K[t, t^{-1}], \mathcal{W}) \xrightarrow{(\varphi_2^*)^*} \text{Der}_K(\mathcal{C}, \mathcal{W})
\]

is an equalizer diagram of vector spaces, i.e. $\text{Der}_K(\mathcal{D}, \mathcal{W})$ is the kernel of the map

\[
\text{Der}_K(\mathcal{A} \ast_K K[t, t^{-1}], \mathcal{W}) \xrightarrow{(\varphi_2^*)^* - (\varphi_1^*)^*} \text{Der}_K(\mathcal{C}, \mathcal{W})
\]

which is surjective as well, because $\mathcal{C}$ is a separable $K$-algebra. This proves b), since we have an isomorphism

\[
\text{Der}_K(\mathcal{A} \ast_K K[t, t^{-1}], \mathcal{W}) \cong \text{Der}_K(\mathcal{A}, \mathcal{W}) \oplus \mathcal{W}, \delta \mapsto (\delta \circ \iota_A, \delta(t))
\]

The proof of c) is now just a repeated application of the parts a) and b).
Since the righthand side of the formula (19) only depends on the dimension vectors $\dim(M)$ and $\dim(N)$ and is $\mathbb{N}_0$-linear in both arguments, Proposition 3.4 yields that the homological Euler form induces a well-defined $\mathbb{N}_0$-bilinear map

$$\langle -, - \rangle_{K[G]} : T(G) \times T(G) \to \mathbb{Z}$$

Furthermore we see from formula (19) that $\langle -, - \rangle_{K[G]}$ does not depend on $K$. So we will simply denote it by $\langle -, - \rangle_G$. Moreover (19) combined with Example 3.3 shows that $\langle -, - \rangle_G$ is symmetric.

As before we postpone explicit examples to Section 6 but the readers may feel free to skip forward to it now.

3.3 Counting representation spaces. Now assume $K = \mathbb{F}_q$ is finite. We want to show in this subsection that the connected components $\text{Rep}_m(\mathbb{F}_q[G])$, $m \in T(G)$ are polynomial count if $\mathbb{F}_q$ is suitable for $G$. As before we start with the case of semisimple algebras.

Example 3.5 Let $C$ be a completely split finite dimensional semisimple $\mathbb{F}_q$-algebra with (absolutely) simple left modules $L_0, \ldots, L_{c-1}$. If $M$ is a finite dimensional left $C$-module of dimension vector $m = \dim(M) = \sum_\gamma m(\gamma) \cdot \dim(L_\gamma)$ we know from Example 3.1(a) that $\text{Rep}_m(C) = \bigotimes M \cong GL_{m(K)}$. Since $S(x_M) \cong \prod_{\alpha} GL_{m(\alpha),\mathbb{F}_q}$ is special (see Example 2.3), we obtain

$$\# \text{Rep}_m(C)(\mathbb{F}_q^\alpha) = \# GL_{m(\alpha),\mathbb{F}_q}/S(x_M)(\mathbb{F}_q^\alpha) = \frac{P_{GL_{m(\gamma)}}}{\prod_{\gamma=1}^{c-1} P_{GL_{m(\gamma)}}}(q^\alpha)$$

Using (i) in Subsection 2.2 we see that the rational function $P_m^C := P_{GL_{m(\gamma)}}/\prod_{\gamma=1}^{c-1} P_{GL_{m(\gamma)}}$ is in fact a counting polynomial for $\text{Rep}_m(C)$.

Similar to $T(G)$ and $\langle -, - \rangle_G$ we give a full description of the counting polynomials of $\text{Rep}_m(\mathbb{F}_q[G])$.

Proposition 3.6 Let $A$ be a finite type $\mathbb{F}_q$-algebra, $B$ and $C$ completely split finite dimensional semisimple $\mathbb{F}_q$-algebras and $\varphi_1 : C \to B$, $\varphi_2, \varphi_3 : C \to A$ homomorphisms of $\mathbb{F}_q$-algebras. For $d \in \mathbb{N}_0$ fix dimension vectors $m \in T_d(A)$, $n \in T_d(B)$ and $u \in T_d(C)$.

a) Consider the pushout $A \ast_C B$ given by $\varphi_1, \varphi_2$ and assume that $(m, n)$ is a dimension vector in $T(A) \times T(C) T(B) = T(A \ast_C B)$ lying over $u$. If $\text{Rep}_m(A)$ admits a counting polynomial $P_m^A$, then the rational function $P_{(m,n)}^{A \ast_C B} := P_m^A P_n^B/P_u^C$ is a counting polynomial for $\text{Rep}_{(m,n)}(A \ast_C B)$.

b) Consider the HNN extension $A \ast_C^{\varphi_2, \varphi_3}$ and assume that $m$ is an element of the equalizer $\text{Eq}(T(\varphi_2), T(\varphi_3)) = T(A \ast_C^{\varphi_2, \varphi_3})$ lying over $u$. If $\text{Rep}_m(A)$ admits a counting polynomial $P_m^A$, then the rational function $P_{m}^{A \ast_C^{\varphi_2, \varphi_3}} := P_m^A P_{\text{GL}_d}/P_u^C$ is a counting polynomial for $\text{Rep}_m(A \ast_C^{\varphi_2, \varphi_3})$.

c) If $G$ is the finitely generated virtually free group given by (3) and $\mathbb{F}_q$ is suitable for $G$, then

$$P_G^m := P_{\text{GL}_d} \prod_{i=0}^{I_0} P_{m_i}^G \prod_{j=1}^{I_+} P_{\alpha_j}^G = P_{\text{GL}_d} \prod_{j=1}^{I_+} \prod_{\gamma} P_{\text{GL}_{m_j(\gamma)}} \prod_{i=0}^{I_0} \prod_{\beta} P_{\text{GL}_{m_i(\beta)}}$$
is a counting polynomial for $\text{Rep}_m(\mathbb{F}_q[\mathcal{G}])$ where $m \in \mathcal{T}(\mathcal{G})$ is the dimension vector lying over $m_i \in \mathcal{T}(\mathcal{G}_i)$ for $0 \leq i \leq I$ and over $u_j \in \mathcal{T}(\mathcal{G}_j')$ for $1 \leq j \leq I + J$.

Proof: About (a): As in the proof of Proposition 3.2(a) we may express $\text{Rep}_m(\mathcal{A})$ and $\text{Rep}_{(m,n)}(\mathcal{A} \ast \mathcal{C} \mathcal{B})$ as associated fibre spaces

$$\text{Rep}_m(\mathcal{A}) \cong \text{GL}_{d,F_q} \times^{S(z)} Y, \quad \text{Rep}_{(m,n)}(\mathcal{A} \ast \mathcal{C} \mathcal{B}) \cong \text{GL}_{d,F_q} \times^{S(y)} Y$$

for $S(y) \subseteq S(z) \subseteq \text{GL}_{d,K}$ the stabilizers of points $y \in \text{Rep}_n(\mathcal{B})(\mathbb{F}_q)$, $z \in \text{Rep}_u(\mathcal{C})(\mathbb{F}_q)$ and $Y$ an affine finite type $\mathbb{F}_q$-scheme with $S(z)$-action. Since $S(y)$ and $S(z)$ are special, we may use (12) to obtain

$$\# \text{Rep}_m(\mathcal{A} \ast \mathcal{C} \mathcal{B})(\mathbb{F}_q) = \frac{P_{\text{GL}_d}(q^a)}{\#S(y)(\mathbb{F}_q)} \#Y(\mathbb{F}_q) = \frac{P_{\text{GL}_d}(q^a)}{\#S(y)(\mathbb{F}_q)} \frac{#S(z)(\mathbb{F}_q)}{P_{\text{GL}_d}(q^a)} P_{m}(q^a)$$

Using Example 3.5 this proves part (a).

About (b): As above we use the proof of Proposition 3.2(c) to obtain

$$\text{Rep}_m(\mathcal{A}) \cong \text{GL}_{d,F_q} \times^{S(z)} Y, \quad \text{Rep}_m(\mathcal{A} \ast \mathcal{C}^{(2),\varphi_z}) \cong \text{GL}_{d,F_q} \times^{S(z)} (Y \times \mathbb{F}_q, S(z))$$

for $S(z) \subseteq \text{GL}_{d,F_q}$ the stabilizer of a point $z \in \text{Rep}_u(\mathcal{C})(\mathbb{F}_q)$ and we calculate

$$\# \text{Rep}_m(\mathcal{A} \ast \mathcal{C}^{(2),\varphi_z})(\mathbb{F}_q) = \frac{P_{\text{GL}_d}(q^a)}{\#S(y)(\mathbb{F}_q)} \#Y(\mathbb{F}_q) = \frac{P_{\text{GL}_d}(q^a)}{\#S(y)(\mathbb{F}_q)} \frac{#S(z)(\mathbb{F}_q)}{P_{\text{GL}_d}(q^a)} P_{m}(q^a)$$

About (c): We obtain $P^G_m$ by repeatedly applying part (a) and (b) above to our decomposition (6) of $K[\mathcal{G}]$ (note that $J$ is the number of HNN-extensions involved in (6)). The second expression comes from Example 3.5 by cancelling out the $P_{\text{GL}_d}$ occurring in the numerator and denominator of the fraction.

The formula (21) in particular shows that the polynomials $P^G_m$ are independent of the choice of a finite suitable field for $\mathcal{G}$.

### 4 Hall algebra methods

Consider the field $\mathbb{Q}(s)$ of rational functions in the variable $s$ as well as its subring

$$\mathbb{Q}[s]_{(s-q)} = \{ f/q \in \mathbb{Q}(s) \mid Q(q) \neq 0 \}$$

The $\mathbb{Q}$-algebra homomorphisms $\mathbb{Q}(s) \leftrightarrow \mathbb{Q}[s]_{(s-q)} \xrightarrow{ev_q} \mathbb{Q}$ induce homomorphisms of $\mathcal{T}(\mathcal{G})$-graded $\mathbb{Q}$-algebras

$$\mathbb{Q}(s)[\mathcal{T}(\mathcal{G})] \leftrightarrow \mathbb{Q}[s]_{(s-q)}[\mathcal{T}(\mathcal{G})] \xrightarrow{ev_q} \mathbb{Q}[\mathcal{T}(\mathcal{G})]$$

The homomorphism $|.| : \mathcal{T}(\mathcal{G}) \rightarrow \mathbb{N}_0$ endows every $\mathcal{T}(\mathcal{G})$-graded algebra $\mathcal{C}$ with an $\mathbb{N}_0$-grading where $\mathcal{C}_d$ is spanned by all homogeneous elements with degree in $\mathcal{T}(\mathcal{G})$. So we may endow $\mathcal{C}$ as every $\mathbb{N}_0$-graded algebra with a topology by taking the ideals $\bigoplus_{d \geq 0} \mathcal{C}_d$ as a neighbourhood basis of 0 and form the completion $\widehat{\mathcal{C}} = \prod_{d \geq 0} \mathcal{C}_d$ of $\mathcal{C}$ with respect to it. We have the following facts on completions of $(graded)$ algebras:

- (vii) An element $(f_\delta)_{\delta \geq 0} \in \widehat{\mathcal{C}}$ is a multiplicative unit if and only if $f_0 \in \mathcal{C}_0$ is a unit.
- (viii) Every graded homomorphism $\mathcal{C} \rightarrow \mathcal{C}'$ between $\mathcal{T}(\mathcal{G})$-graded algebras extends uniquely to a continuous algebra homomorphism $\widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{C}}'$. 


So by taking completions of \ref{22} we obtain continuous $\mathbb{Q}$-algebra homomorphisms
\begin{equation}
\mathbb{Q}(s)[\mathcal{T}(\mathcal{G})] \leftarrow \mathbb{Q}[s]_{(s-q)}[\mathcal{T}(\mathcal{G})] ^{ev_q} \rightarrow \mathbb{Q}[\mathcal{T}(\mathcal{G})]
\end{equation}
We now define a second multiplication on the monoid algebras considered above: The so called twisted multiplication on $\mathbb{Q}[\mathcal{T}(\mathcal{G})]$ is given by bilinear extension of
\[ t^m \ast t^n := q^{-(m,n)\mathcal{G}} t^{m+n} \]
Analogously we define $t^m \ast t^n := s^{-(m,n)\mathcal{G}} t^{m+n}$ on $\mathbb{Q}(s)[\mathcal{T}(\mathcal{G})]$ and $\mathbb{Q}[s]_{(s-q)}[\mathcal{T}(\mathcal{G})] \ .$ We denote the resulting $\mathcal{T}(\mathcal{G})$-graded $\mathbb{Q}$-algebras by $\mathbb{Q}^\mathcal{T}[\mathcal{T}(\mathcal{G})] \ , \ \mathbb{Q}(s)^\mathcal{T}[\mathcal{T}(\mathcal{G})]$ and $\mathbb{Q}[s]_{(s-q)}^\mathcal{T}[\mathcal{T}(\mathcal{G})] \ .$ As for the monoid algebras we have $\mathcal{T}(\mathcal{G})$-graded $\mathbb{Q}$-algebra homomorphisms analogous to \ref{22} and continuous $\mathbb{Q}$-algebra homomorphisms like \ref{23} between their twisted versions.

The twisted monoid algebras are in fact isomorphic to their untwisted counterparts.
To construct explicit isomorphisms between them we need a monoid homomorphism $\mathbb{Y} : \mathcal{T}(\mathcal{G}) \rightarrow \mathbb{Z}$ which satisfies
\[ \langle m, n \rangle \mathcal{G} \equiv \mathbb{Y}(m) \mod (2) \ \forall \ m \in \mathcal{T}(\mathcal{G}) \]
We construct a distinguished $\mathbb{Y}$ to show existence, but everything that follows does not depend on this choice. For a finite group $\mathcal{F}$ we have an identification $\mathcal{T}(\mathcal{F}) \cong \mathbb{N}_0^c$ and may take $\mathbb{Y}(m) := \sum_{\gamma=0}^{c-1} m(\gamma) \ .$ For the general case of $\mathcal{G}$ we can mimic our computation of the Euler form and define
\[ \mathbb{Y}(m) := \sum_{i=0}^{l} \sum_{\gamma} m_i(\gamma) - \sum_{j=1}^{l+j} \sum_{\gamma} u_j(\gamma) \]
where $m \in \mathcal{T}(\mathcal{G})$ is the dimension vector lying over $m_i \in \mathbb{N}_0^{c_i} \cong \mathcal{T}(\mathcal{G}_i) \text{ and over } u_j \in \mathbb{N}_0^{c_j} \cong \mathcal{T}(\mathcal{G}_j) \ .$ We may now define a $\mathbb{Q}$-vector space isomorphism
\[ \mathcal{S} : \mathbb{Q}(s)^\mathcal{T}[\mathcal{T}(\mathcal{G})] \rightarrow \mathbb{Q}[\mathcal{T}(\mathcal{G})] \ , \ \mathcal{S}(t^m) := q^{\frac{1}{2}(\langle m, m \rangle \mathcal{G} - \mathbb{Y}(m))} t^m \]
and isomorphisms $\mathbb{Q}(s)^\mathcal{T}[\mathcal{T}(\mathcal{G})] \cong \mathbb{Q}(s)[\mathcal{T}(\mathcal{G})] \ , \ \mathbb{Q}[s]_{(s-q)}^\mathcal{T}[\mathcal{T}(\mathcal{G})] \cong \mathbb{Q}[s]_{(s-q)}[\mathcal{T}(\mathcal{G})] \ \text{via } \mathcal{S}(t^m) := s^{\frac{1}{2}(\langle m, m \rangle \mathcal{G} - \mathbb{Y}(m))} t^m \ .$ We call each of the maps $\mathcal{S}$ shift operator. By construction the shift operators preserve the $\mathcal{T}(\mathcal{G})$-grading and using that $\langle -, - \rangle \mathcal{G}$ is symmetric one can deduce that they are isomorphisms of graded algebras. Hence, they extend uniquely to continuous algebra homomorphisms between the completed monoid algebras.

Shift operators like $\mathcal{S}$ have already appeared in Mozgovoy-Reineke’s treatment of the free group case in \cite{M.R.15}. To get rid of the correction form $\mathbb{Y}$ we could also define a shift operator by $\mathcal{S}'(t^m) := q^{\frac{1}{2}(\langle m, m \rangle \mathcal{G})} t^m \ .$ This would mean however that we have to work with $\mathbb{Q}^{\sqrt{q}}$ instead of $\mathbb{Q} \ , \ \mathbb{Q}^{\sqrt{q}}(\sqrt{s})$ instead of $\mathbb{Q}(s)$ etc.

Now fix a finite field $K = \mathbb{F}_q$ which is suitable for $\mathcal{G} \ .$ We briefly recall the construction of the finitary Hall algebra $\mathbb{H}(\mathcal{A})$ of a finite type $\mathbb{F}_q$-algebra $\mathcal{A} :$

Denote by $\text{iso}(\mathcal{A}) := \bigsqcup_{d \geq 0} \text{iso}_d(\mathcal{A})$ the set of all isomorphism classes of finite dimensional left $\mathcal{A}$-modules\footnote{Analogously we denote by $\text{ssim}(\mathcal{A}) , \text{sim}(\mathcal{A})$ and $\text{absim}(\mathcal{A})$ the sets of all isomorphism classes of semisimple, simple and absolutely simple modules respectively.} \cite{H} $\mathbb{H}(\mathcal{A})$ is defined as the free $\mathbb{Q}$-vector space on the
A basis $\text{iso}(A)$ with multiplication given by $[\mathcal{M}] \cdot [\mathcal{N}] = \sum_{[\mathcal{W}]} F_{\mathcal{M},\mathcal{N}}^{\mathcal{W}} [\mathcal{W}]$ via structure coefficients

$$F_{\mathcal{M},\mathcal{N}}^{\mathcal{W}} := \#\{ \mathcal{L} \subseteq \mathcal{W} \text{ left } A\text{-submodule} \mid \mathcal{L} \cong \mathcal{N}, \mathcal{W}/\mathcal{L} \cong \mathcal{M} \}$$

Since dimension vectors are additive on short exact sequences, $H(A)$ is $\mathcal{T}(A)$-graded $- H_{m}(A)$ is the $\mathbb{Q}$-linear span of $\{ [\mathcal{M}] \in \text{iso}(A) \mid \dim(\mathcal{M}) = m \}$. In particular the homomorphism $\mid . \mid : \mathcal{T}(A) \to \mathbb{N}_{0}$ induces an $\mathbb{N}_{0}$-grading $H(A) = \bigoplus_{\delta \geq 0} H_{\delta}(A)$. As for the monoid algebras (22) we may complete $H(A)$ with respect to this $\mathbb{N}_{0}$-grading. Denote the completed finitary Hall algebra by $H(\langle A \rangle)$.

We consider the element $\varepsilon := \sum_{[\mathcal{M}] \in \text{iso}(A)} [\mathcal{M}] \in H(\langle A \rangle)$ which is a multiplicative unit by (vii) from above. It was shown by M. Reineke in [Rein06, Lemma 3.4] that the coefficients $e_{\mathcal{M}}$ of the inverse $\varepsilon^{-1} = \sum_{[\mathcal{M}]} e_{\mathcal{M}} [\mathcal{M}]$ are given by

$$\prod_{[\mathcal{L}] \in \text{sim}(A)} (-1)^{a_{\mathcal{L}}} \# \text{End}_{A}(\mathcal{L})^{\varepsilon(\mathcal{L}^{-1}/2)} \quad \text{if } \mathcal{M} = \bigoplus_{[\mathcal{L}] \in \text{sim}(A)} \mathcal{L}^{\varepsilon a_{\mathcal{L}}} \text{ semisimple}$$

$$0 \quad \text{if } \mathcal{M} \text{ not semisimple}$$

For $A = \mathbb{F}_{q}[G]$ we have a homomorphism of $\mathcal{T}(G)$-graded $\mathbb{Q}$-algebras

$$\int : H(\mathbb{F}_{q}[G]) \to \mathbb{Q}^{q\text{-tw}}[\mathcal{T}(G)]$$

analogously to [Rein06, Lemma 3.3] \footnote{Reineke’s proof of the analogous construction for quiver representations in [Rein06 §3] holds without any changes. It only relies on the fact that $\mathbb{F}_{q}[G]$ is (left) hereditary and would work for any left hereditary algebra for which the Euler form factorizes over a proper analogue of the dimension vector monoid $\mathcal{T}(KQ)$ (like $\mathcal{T}(G)$ in our case).} \footnote{This mostly means that the lift should not depend on the specific prime power $q$.} The map $f$ is called a Hall algebra integral. By (viii) from above it extends uniquely to a continuous $\mathbb{Q}$-algebra homomorphism between the completions.

We summarize the situation with the following commutative diagram:

$$\begin{align*}
\mathbb{Q}(s)^{\text{tw}}[\mathcal{T}(G)] &\xrightarrow{\gamma} \mathbb{Q}[s]^{\text{tw}}[\mathcal{T}(G)] \cong \mathbb{Q}[s][\mathcal{T}(G)] \\
\mathbb{Q}(s)[\mathcal{T}(G)] &\xrightarrow{\text{ev}_{q}} \mathbb{Q}[\mathcal{T}(G)]
\end{align*}$$

Most of the actual computations we are interested in happen in the ring $\mathbb{Q}(s)[\mathcal{T}(G)]$ while our knowledge of the representation theory of $\mathbb{F}_{q}[G]$ comes from the completed Hall algebra $H(\langle \mathbb{F}_{q}[G] \rangle)$. So the procedure is the following: First we observe an interesting identity in $H(\langle \mathbb{F}_{q}[G] \rangle)$, then we map it to $\mathbb{Q}[\mathcal{T}(G)]$ and check whether we can reasonably lift it along $\text{ev}_{q}$. Afterwards we can manipulate the obtained identity within $\mathbb{Q}(s)[\mathcal{T}(G)]$.

5 Counting polynomials

After introducing a lot of machinery we now come back to our original objective of counting functions and relate them to our machinery. Let $\mathbb{F}_{q}$ be a suitable finite field...
for $G$. For each dimension vector $m \in \mathcal{T}(G)$ define the refined counting functions

$$
\begin{align*}
\ r_{m}^{\text{absim}}(q^\alpha) &:= \#\{[\mathcal{M}] \in \text{absim}(\mathbb{F}_q[G]) \mid \dim(\mathcal{M}) = m\} \\
\ r_{m}^{\text{sim}}(q^\alpha) &:= \#\{[\mathcal{M}] \in \text{sim}(\mathbb{F}_q[G]) \mid \dim(\mathcal{M}) = m\} \\
\ r_{m}^{\text{ss}}(q^\alpha) &:= \#\{[\mathcal{M}] \in \text{ss}(\mathbb{F}_q[G]) \mid \dim(\mathcal{M}) = m\}
\end{align*}
$$

The refined counting functions $r_{m}^{\text{ss}}$ and $r_{m}^{\text{absim}}$ again count the rational points of GIT moduli spaces: All connected components $\text{Rep}_m(\mathbb{F}_q[G]) \subseteq \text{Rep}_{[m]}(\mathbb{F}_q[G])$ are $\text{GL}_{|\mathfrak{m}_G,\mathbb{F}_q}$-invariant, their GIT quotients $M(\mathbb{F}_q[G], m) := \text{Rep}_m(\mathbb{F}_q[G])//\text{GL}_{|\mathfrak{m}_G,\mathbb{F}_q}$ are the connected components of $M(\mathbb{F}_q[G], |m|)$. Moreover there is a $\text{GL}_{|\mathfrak{m}_G,\mathbb{F}_q}$-invariant open subscheme $\text{Rep}_{m}^{\text{absim}}(\mathbb{F}_q[G]) \subseteq \text{Rep}_m(\mathbb{F}_q[G])$ such that

$$
M^{\text{absim}}(\mathbb{F}_q[G], m) := \text{Rep}_m^{\text{absim}}(\mathbb{F}_q[G])//\text{GL}_{|\mathfrak{m}_G,\mathbb{F}_q} = M(\mathbb{F}_q[G], m) \cap M^{\text{absim}}(\mathbb{F}_q[G], |m|)
$$

The connected components of $M^{\text{absim}}(\mathbb{F}_q[G], d)$ are given by those $M^{\text{absim}}(\mathbb{F}_q[G], m)$, $m \in \mathcal{T}_d(G)$ which are non-empty. These GIT moduli spaces satisfy

$$
(27) \quad r_{m,c}^{\text{ss}}(q^\alpha) = \#M(\mathbb{F}_q[G], m)(\mathbb{F}_q^\alpha), \quad r_{m}^{\text{absim}}(q^\alpha) = \#M^{\text{absim}}(\mathbb{F}_q[G], m)(\mathbb{F}_q^\alpha)
$$

We can recover the original counting functions $\Pi$ from the refined ones via the formula $r_{d,xyz} = \sum_{|\mathfrak{m}|=d} r_{m}^{xyz}$. Moreover we define

$$
(28) \quad r_{m,c}^{\text{ss}}(q^\alpha) := \#\{[\mathcal{M}] \in \text{ss}(\mathbb{F}_q[G]) \mid \dim(\mathcal{M}) = m, \dim_{\mathbb{F}_q^\alpha}(\text{End}_{\mathbb{F}_q^\alpha}[G](\mathcal{M})) = c\}
$$

Analogously to [Rein06, §4] we obtain the identities

$$
(29) \quad r_{m,c}^{\text{absim}}(q^\alpha) = r_{m,c}^{\text{ss}}(q^\alpha), \quad r_{m,c}^{\text{ss}}(q^\alpha) = \begin{cases} \frac{1}{c} \sum_{\gamma \in \mathbb{N}} \mu(\gamma) r_{m/c,\gamma}^{\text{absim}}(q^{\gamma/c}) & \text{if } c|m \\ 0 & \text{else} \end{cases}
$$

Here $\mu : \mathbb{N} \to \{-1, 0, 1\}$ denotes the (classical) Möbius function. Since $\mathcal{T}(G)$ embeds into a free commutative monoid, $\mathbb{C}[\mathcal{T}(G)]$ can be embedded into a formal power series ring $\mathbb{C}[t_1, \ldots, t_\alpha]$, i.e. we may interprete the elements of $\mathbb{C}[\mathcal{T}(G)]$ as formal power series. Important examples are

$$
(30) \quad r_{xyz}(q^\alpha) := \sum_{m \in \mathcal{T}(G)} r_{m}^{xyz}(q^\alpha)t^m \in \mathbb{Q}[\mathcal{T}(G)]
$$

where $xyz \in \{\text{absim}, \text{sim}, \text{ss}\}$. Our goal is to lift the power series $r_{xyz}(q^\alpha)$ reasonably along $ev_{q^\alpha}$, the coefficients $R_{m}^{xyz}$ of such a lift $R^{xyz}$ will be the counting polynomials we are aiming for.

We now briefly recall the construction of plethystic exponentials and logarithms. First note that $\mathbb{Q}(s)[[\mathcal{T}(G)]]$ is a local ring with maximal ideal

$$
(31) \quad \mathfrak{m} := \left\{ \sum_{m \in \mathcal{T}(G)} f_m t^m \in \mathbb{Q}(s)[[\mathcal{T}(G)]] \mid f_0 = 0 \right\}
$$

which is open. The subset $1 + \mathfrak{m}$ is open as well and a topological group with respect to multiplication. $(\mathfrak{m}, +)$ and $(1 + \mathfrak{m}, \cdot)$ are isomorphic as topological groups,

The first identity holds just by the definition of absolutely simple modules, the second identity can be obtained from Galois descent and Möbius inversion.
mutually inverse continuous isomorphisms are given by
\[
m \xrightarrow{\log} 1 + m \quad \text{and} \quad \exp(f) := \sum_{\alpha \geq 0} \frac{f^\alpha}{\alpha!}, \quad \log(1 + f) := \sum_{\beta \geq 1} \frac{(-1)^{\beta+1}}{\beta} f^\beta
\]
Note that \(\exp\) and \(\log\) are equally well-defined for \(\mathbb{Q}[s][T(G)]\) and \(\mathbb{Q}[T(G)]\) and that they commute with the homomorphisms (23), e.g. \(\exp \circ \ev_q(f) = \ev_q \circ \exp(f)\) for each \(f \in m \cap \mathbb{Q}[s][T(G)]\).

For each \(a \in \mathbb{N} \geq 1\) we consider the Adams operation \(\psi_a: \mathbb{Q}(s)[T(G)] \rightarrow \mathbb{Q}(s)[T(G)]\),
\[
\psi_a(\sum m f^m) := \sum m (s^a)^t^m
\]
which is a continuous \(\mathbb{Q}\)-algebra homomorphism. They give rise to the mutually inverse continuous group automorphisms
\[
m \xrightarrow{\Psi^{-1}} m \quad \text{and} \quad \Psi(f) := \sum_{\alpha \geq 1} \frac{\psi_\alpha(f)}{\alpha}, \quad \Psi^{-1}(f) = \sum_{\beta \geq 1} \mu(\beta) \frac{\psi_\beta(f)}{\beta}
\]
The plethystic exponential and plethystic logarithm are defined by \(\text{Exp} := \exp \circ \Psi\) and \(\text{Log} := \Psi^{-1} \circ \log\). They are by definition mutually inverse continuous group isomorphisms, i.e. they in particular fulfill the usual functional equations
\[
\text{Exp}(f + g) = \text{Exp}(f) \text{Exp}(g) \quad \text{and} \quad \text{Log}(fg) = \text{Log}(f) + \text{Log}(g)
\]
Moreover the same identities hold for convergent infinite sums and products. \(\text{Exp}\) and \(\text{Log}\) can alternatively be defined on \(\mathbb{Q}((s))[T(G)]\) where \(\mathbb{Q}((s))\) denotes the field of formal Laurent series. By some calculations in \(\mathbb{Q}((s))[T(G)]\) one can prove\(^4\)
\[
(29) \quad \text{Exp}\left(\frac{1}{1 - sc^m}\right) = \left(\prod_{\alpha \geq 0} 1 - s^{c\alpha}.t^m\right)^{-1} = \sum_{b \geq 0} \left(\prod_{\beta = 1}^b (1 - s^{c\beta})\right)^{-1}.t^{b.m}
\]
Using the theorem of Krull-Remak-Schmidt for a product factorization of the power series \(r^{ss}(q)\) and the second formula in (28) one can prove the following lemma.

**Lemma 5.1** The power series
\[
E(q) := \sum_{m \in T(G), \beta \geq 1} \frac{1}{\beta} (q^\beta)^{t^\beta.m}
\]
is for each \(q\) convergent in \(\mathbb{Q}[T(G)]\) and satisfies \(\exp(E(q)) = r^{ss}(q)\).

See [Mozg07, Lemma 5] for the completely analogous proof in the case of absolutely indecomposables instead of absolutely simples. In Theorem 5.3 we will reformulate this lemma in terms of the plethystic exponential \(\text{Exp}\).

We are now ready to prove the existence of counting polynomials for the refined counting functions (28). We begin our proof with a lemma about the element \(\varepsilon^{-1} = \sum_{[M]} e_M[M]\) discussed at (24).

\(^4\)See e.g. (2) in [Mozg11, §2.3] for the calculation in the case \(c = 1\).
Lemma 5.2 Let $F_q$ be suitable for $G$. Denote by $\int : H(\mathbb{F}_q[G]) \to \mathbb{Q}^{\tau}-tw[\mathcal{T}(G)]$ the Hall algebra integral defined in (25). We consider $\int (\varepsilon^{-1}) \in \mathbb{Q}^{\tau}-tw[\mathcal{T}(G)]$ as an element of $\mathbb{Q}[\mathcal{T}(G)]$ within this lemma. This element satisfies

$$\log \left( \int (\varepsilon^{-1}) \right) = \sum_{m \in \mathcal{T}(G)} \sum_{\delta | m} \frac{1}{\delta(1-q^\delta)} r_{m/\delta}^{\text{absim}} (q^\delta) t^{\delta.m}$$

Proof: Using that the coefficients $e_M$ of $\varepsilon^{-1}$ are given by (24), one can calculate

$$\sum_{[M] \in \text{iso}(\mathbb{F}_q[G])} \frac{e_M}{\# \text{Aut}_{\mathbb{F}_q[G]}(M)} t^{\dim(M)} = \prod_{m \in \mathcal{T}(G), c|m} \left( \sum_{b \geq 0} \left( \prod_{\beta=1}^b (1-q^{c\beta}) \right)^{-1} r_{m,c}^{\text{sim}}(q)^{t^{b.m}} \right)$$

in $\mathbb{Q}[\mathcal{T}(G)]$. So we may apply (29) to obtain

$$\int (\varepsilon^{-1}) = \prod_{m \in \mathcal{T}(G), c|m} \left( \text{ev}_q \circ \text{Exp} \left( \frac{1}{1-s^c} t^{m} \right) \right)^{r_{m,c}^{\text{sim}}(q)}$$

By applying log and using that log and exp commute with $\text{ev}_q$ we deduce

$$\log \left( \int (\varepsilon^{-1}) \right) = \sum_{m \in \mathcal{T}(G), c|m} r_{m,c}^{\text{sim}}(q) \sum_{\beta \geq 1} \frac{1}{\beta(1-q^{c\beta})} t^{\beta.m}$$

Applying the second formula in (28) yields

$$\log \left( \int (\varepsilon^{-1}) \right) = \sum_{m \in \mathcal{T}(G), \beta \geq 1, c|m} \frac{1}{c\beta(1-q^{c\beta})} \sum_{\gamma|c} \mu(\gamma) r_{m,c}^{\text{absim}} (q^{\gamma}) t^{\beta.m}$$

The rest of the proof is done by substituting $n := m/c, \delta := c/\gamma, a := \beta \gamma$ and using that

$$\sum_{\gamma|a} \mu(\gamma) = \begin{cases} 1, & a = 1 \\ 0, & a > 1 \end{cases}$$

To formulate our main result below we define the power series

$$F := S \left( \sum_{m \in \mathcal{T}(G)} \frac{P^G_m}{P^{GL}_{\mathcal{T}(G)}} t^{m} \right) \in \mathbb{Q}[s-s^{-q^\alpha}][\mathcal{T}(G)]$$

Theorem 5.3 Let $F_q$ be suitable for the finitely generated virtually free group $G$. Define the power series

$$R_m^{\text{absim}} := (1-s) \log \left( S^{-1} \left( F^{-1} \right) \right), \quad R_m^{\text{ess}} := \text{Exp} \left( R_m^{\text{absim}} \right)$$

for $F$ as defined in (30) and denote their coefficients by $R_m^{\text{absim}}$ and $R_m^{\text{ess}}$ respectively. For each dimension vector $m \in \mathcal{T}(G)$ these coefficients satisfy $R_m^{\text{absim}}, K_m^{\text{ess}} \in \mathbb{Z}[s]$ and

$$\forall \alpha \geq 1 : R_m^{\text{absim}} (q^{\alpha}) = r_m^{\text{absim}} (q^{\alpha}), \quad R_m^{\text{ess}} (q^{\alpha}) = r_m^{\text{absim}} (q^{\alpha})$$

45 This computation is completely analogous to the proof of [M.R.09, Thm. 4.2].
Proof: By (i) in Subsection 2.2 and (27) it is sufficient to show $R_{m}^{\text{absim}} \in \mathbb{Q}[s_{(s-q^{\alpha})}]$ for each $\alpha$ and that $R_{m}^{\text{absim}}, R_{m}^{\text{ss}}$ fulfill (32). For each $\alpha \geq 1$ we consider the continuous $\mathbb{Q}$-algebra homomorphism

$$S \circ \int : \mathcal{H}(\mathbb{F}_{q^{\alpha}}[\mathcal{G}]) \to \mathbb{Q}[\mathcal{T}]=\mathcal{G}$$

Using that $\text{Aut} (\mathcal{M}_{x}) \cong S(x)(\mathbb{F}_{q^{\alpha}})$ by (10) and $\#(\mathcal{GL}_{d}(\mathbb{F}_{q^{\alpha}}),x) = R_{\mathcal{GL}_{d}[m]}(q^{\alpha})/\#S(x)(\mathbb{F}_{q^{\alpha}})$ for $x \in \text{Rep}_{m}(\mathbb{F}_{q^{\alpha}}[\mathcal{G}])$, we compute

$$\int (\varepsilon) = \frac{1}{\# \text{Aut} (\mathcal{M})} \sum_{m \in \mathcal{T} \cap G_{m}} \# \text{Rep}_{m}(\mathbb{F}_{q^{\alpha}}[\mathcal{G}]) = \frac{\# \text{Rep}_{m}(\mathbb{F}_{q^{\alpha}}[\mathcal{G}])}{R_{\mathcal{GL}_{d}[m]}(q^{\alpha})} = \text{ev}_{q^{\alpha}}(S^{-1}(F))$$

Hence, $\int (\varepsilon^{-1}) = \text{ev}_{q^{\alpha}}(S^{-1}(F^{-1}))$ for $\alpha \geq 1$. Since we have a power series $\int (\varepsilon^{-1}) \in \mathbb{Q}[\text{tw}][\mathcal{T}]=\mathcal{G}$ for each power $q^{\alpha}$, we consider the expression $\int (\varepsilon^{-1})$ as a function in $q^{\alpha}$-powers and denote its value in $q^{\alpha}$ by $\int (\varepsilon^{-1})_{q^{\alpha}}$.

Now define for $m \in \mathcal{T}$ and $\alpha \geq 1$

$$\Lambda_{m} := \sum_{\delta|m} \frac{1}{\delta(1-s^{\delta})} R_{m}^{\text{absim}}(s^{\delta}) \in \mathbb{Q} \setminus \{0\}, \quad \lambda_{m}(q^{\alpha}) := \sum_{\delta|m} \frac{1}{\delta(1-q^{\alpha \delta})} R_{m}^{\text{absim}}(q^{\alpha \delta}) \in \mathbb{Q}$$

By definition of $\Psi$ we have $\sum_{m} \lambda_{m} t^{m} = \Psi((1-s)^{-1} R_{m}^{\text{absim}}) = \log \circ S^{-1}(F^{-1})$. On the other hand we have

$$\sum_{m \in \mathcal{T} \cap G_{m}} \lambda_{m}(q^{\alpha}) t^{m} = \text{log} \left( \int (\varepsilon^{-1})_{q^{\alpha}} \right) = \text{ev}_{q^{\alpha}} \circ \log \circ S^{-1}(F^{-1})$$

by Lemma 5.2 where we use that log commutes with the evaluation homomorphism $\text{ev}_{q^{\alpha}}$. Hence, $\Lambda_{m}(q^{\alpha}) = \lambda_{m}(q^{\alpha})$ holds for all $m, \alpha$. Via induction on $\gcd(m)$ it can now be seen that $R_{m}^{\text{absim}} \in \mathbb{Q}[s_{(s-q^{\alpha})}]$ and $R_{m}^{\text{absim}}(q^{\alpha}) = R_{m}^{\text{absim}}(q^{\alpha})$ for all $\alpha \geq 1$.

We deduce the claim for $R_{m}^{\text{ss}}$ from Lemma 5.1. Since $R_{m}^{\text{absim}} \in \mathbb{Z}[s]$ for all $m$, we have that $\Psi(R_{m}^{\text{absim}}) \in \mathbb{Q}[s][\mathcal{T}]$. Hence, $R_{m}^{\text{ss}} = \exp \circ \Psi(R_{m}^{\text{absim}}) \in \mathbb{Q}[s][\mathcal{T}]$ too. Moreover one checks immediately that $E(q^{\alpha}) = \text{ev}_{q^{\alpha}} \circ \Psi(R_{m}^{\text{absim}})$ for all $\alpha$. So Lemma 5.1 shows that $\text{ev}_{q^{\alpha}}(R_{m}^{\text{ss}}) = \exp(E(q^{\alpha})) = R_{m}^{\text{ss}}(q^{\alpha})$ for all $\alpha \geq 1$.

Note that the counting polynomials are independent of the choice of the suitable field $\mathbb{F}_{q}$, because all objects involved in (30) and (31) are. As already stated at the end of Subsection 2.1 all statements in Theorem 5.3 hold in the more general setting of representations of the fundamental algebra (7) of a finite graph of finite dimensional semisimple $\mathbb{F}_{q}$-algebras for which each of the semisimple algebras $\mathcal{A}_{\alpha}$ and $\mathcal{A}_{\beta}$ are completely split.

The proof of the following corollary is immediate from Theorem 5.3 and (28).

**Corollary 5.4** Let $\mathbb{F}_{q}$ be suitable for $\mathcal{G}$. For $m \in \mathcal{T}$ and $c \geq 1$ define

$$R_{m,c}^{\text{sim}} := \begin{cases} \frac{1}{c} \sum_{\gamma | c} \mu(\gamma) R_{m/c}(s^{\gamma}) & \text{if } c|m \\ 0 & \text{else} \end{cases}$$

46For the definition of $\gcd(m)$ see (18) above.
and \( R_{m}^{\text{sim}} := \sum_{C} R_{m,c}^{\text{sim}} \). The polynomials \( R_{m,c}^{\text{sim}}, R_{m}^{\text{sim}} \in \mathbb{Q}[s] \) satisfy
\[
\forall \alpha \geq 1 : R_{m,c}^{\text{sim}}(q^\alpha) = \gamma_{m,c}^{\text{sim}}(q^\alpha) \quad , \quad R_{m}^{\text{sim}}(q^\alpha) = \gamma_{m}^{\text{sim}}(q^\alpha)
\]

6 Examples

6.1 Examples for section 3. In this section we want to provide explicit examples of the objects discussed within this paper. As all of these invariants associated to a virtually free group are derived from the invariants associated to its finite subgroups, we will start with applying the Examples 3.1(a), 3.3 and 3.5 to explicit finite groups.

In this section we want to provide explicit examples.

6.1 Examples for section 3.

Assume \( \mathcal{F} \) is a finite Abelian group of order \( \# \mathcal{F} = a \). Every (absolutely) simple representation of \( \mathcal{F} \) is of dimension 1. Hence, \( \mathbb{C}[\mathcal{F}] \cong \mathbb{C}^a \), \( \mathcal{T}(\mathcal{F}) \cong \mathbb{N}_0^a \) and \( |.| : \mathbb{N}_0^a \rightarrow \mathbb{N}_0 \) is given by \( |m| = \sum_{\alpha} m(\alpha) \). \( \langle -, - \rangle_{\mathcal{F}} \) and \( P_{m}^\mathcal{F} \) are given by
\[
\langle m, n \rangle_{\mathcal{F}} = \sum_{\alpha=0}^{a-1} m(\alpha)n(\alpha) \quad , \quad P_{m}^\mathcal{F} := P_{\mathbb{GL}|m|/\prod_{\alpha=0}^{a-1} P_{\mathcal{G}R_m(\gamma)}}
\]

More generally: If \( \mathcal{F} \) is any finite group, then \( \langle -, - \rangle_{\mathcal{F}} \) and \( P_{m}^\mathcal{F} \) are given by (33) where \( c \) is the number of generators of the free commutative monoid \( \mathcal{T}(\mathcal{F}) \).

Now consider the dihedral group \( \mathbb{D}_c = \langle s, t \mid s^2 = t^2 = 1 = (st)^c \rangle \) of order \( 2c \).

First consider the case \( c = 2a \) even: There are 4 (absolutely) simple representations of dimension 1 and \( a - 1 \) (absolutely) simple representations of dimension 2. Hence, \( \mathbb{C}[\mathbb{D}_{2a}] \cong \mathbb{C}^4 \times M_2(\mathbb{C})^{a-1} \), \( \mathcal{T}(\mathbb{D}_{2a}) = \mathbb{N}_0^{a+3} \) and \( |m| = \sum_{\gamma=0}^{3} m(\gamma) + 2 \sum_{\gamma=1}^{a+2} m(\gamma) \).

If \( c = 2a + 3 \) is odd, we have 2 (absolutely) simple representations of dimension 1 and \( a + 1 \) of dimension 2. So we have \( \mathbb{C}[\mathbb{D}_{2a+3}] \cong \mathbb{C}^2 \times M_2(\mathbb{C})^{a+1} \), \( \mathcal{T}(\mathbb{D}_{2a+3}) = \mathbb{N}_0^{a+3} \) and \( |m| = \sum_{\gamma=0}^{1} m(\gamma) + 2 \sum_{\gamma=2}^{a+2} m(\gamma) \).

We now consider the amalgamated free product \( C_a *_{C_c} C_b \). Denote the embeddings of \( C_c \) by \( \iota : C_c \hookrightarrow C_a \) and \( \kappa : C_c \hookrightarrow C_b \). For each (absolutely) simple representation of \( C_c \) there are \( a/c \) ones of \( C_c \) and \( b/c \) ones of \( C_b \) which are restricted to it. Hence, by Proposition 3.2 \( \mathcal{T}(C_a *_{C_c} C_b) \) is given by
\[
\mathbb{N}_0^a \times \mathbb{N}_0^c \mathbb{N}_0^b = \left\{ (m, n) \in \mathbb{N}_0^a \times \mathbb{N}_0^b \mid \forall 0 \leq \gamma < c : \sum_{\delta=0}^{a/c-1} m(\gamma + \delta c) = \sum_{\epsilon=0}^{b/c-1} n(\gamma + \epsilon c) \right\}
\]

with \( |(m, n)| = \sum_{\alpha} m(\alpha) = \sum_{\beta} n(\beta) \). By Proposition 3.4 the Euler form is given by
\[
\langle (m, n), (u, v) \rangle_{C_a *_{C_c} C_b} = \sum_{\alpha=0}^{a/c-1} m(\alpha)u(\alpha) + \sum_{\beta=0}^{b/c-1} n(\beta)v(\beta) - \sum_{\gamma=0}^{a/c-1} \sum_{\delta=0}^{b/c-1} \sum_{\epsilon=0}^{b/c-1} m(\gamma + \delta c)n(\gamma + \epsilon c)
\]

Note that by permuting the entries of \( \mathbb{N}_0^a \times \mathbb{N}_0^b \) we obtain a monoid isomorphism
\[
\mathbb{N}_0^a \times \mathbb{N}_0^c \mathbb{N}_0^b \cong \left( \mathbb{N}_0^{a/c} \times \mathbb{N}_0^{b/c} \right)^{c} \cong \mathcal{T}(C_{a/c} * C_{b/c})^{c}.
\]

Our last two examples in this subsection are the groups \( \mathbb{PGL}_2(\mathbb{Z}) \cong \mathbb{D}_2 *_{C_2} \mathbb{D}_3 \) and \( \mathbb{GL}_2(\mathbb{Z}) \cong \mathbb{D}_4 *_{C_2} \mathbb{C}_2 \mathbb{C}_2 \mathbb{D}_6 \). Using Proposition 3.2 and the computation of \( \mathcal{T}(\mathbb{D}_c) \) above, one can compute that \( \mathcal{T}(\mathbb{PGL}_2(\mathbb{Z})) \) is isomorphic to
\[
\left\{ (m, n) \in \mathbb{N}_0^4 \times \mathbb{N}_0^3 \mid m(0) + m(1) = n(0) + n(2), m(2) + m(3) = n(1) + n(2) \right\}
\]
with \(|(m,n)| = \sum m(\gamma) = n(0) + n(1) + 2n(2)|\) and that \(T(\text{GL}_2(\mathbb{Z}))\) is isomorphic to \(\{ (m,n) \in \mathbb{N}_0^5 \times \mathbb{N}_0^6 \mid (*) \}\) where \((*)\) are the three relations
\[
m(0) + m(1) = n(0) + n(1) + n(5) , \quad m(4) = n(4) , \quad m(2) + m(3) = n(2) + n(3) + n(5)
\]
and with \(|(m,n)| = 2m(4) + \sum_{\gamma=0}^{3} m(\gamma) = 2n(4) + 2n(5) + \sum_{\delta=0}^{3} n(\delta)
\].

### 6.2 Examples of counting polynomials

We now want to present some examples for the counting polynomials. A first trivial example are the counting polynomials of a finite group \(\mathcal{F}\): Here \(\dim : \text{iso}(\mathbb{C}[\mathcal{F}]) \rightarrow T(\mathcal{F})\) is bijective by Example 3.1. Hence, \(R_{m,\mathcal{F}}^{\text{absim}} = 1\) for all \(m \in T(\mathcal{F})\) and \(R_{m,\mathcal{F}}^{\text{absim}} = 1\) if the unique \([\mathcal{M}] \in \text{iso}(\mathbb{C}[\mathcal{F}])\) of \(\dim(\mathcal{M}) = m\) is (absolutely) simple and zero otherwise.

For \(\mathcal{G}\) an arbitrary finitely generated virtually free group given by \((3)\) one first needs to compute the free commutative monoids \(T(\mathcal{G}_i)\) and \(T(\mathcal{G}_j)\) as well as the homomorphisms \(T(\iota_j)\) and \(T(\iota_j)\) between them as we have done above for some examples, i.e. one has to classify the representation theory of these finite groups e.g. over \(\mathbb{C}\). The rest of the computation of the counting polynomials can be done by a computer, e.g. using the accompanying SageMath code. All of the examples below (and in fact many more) have been computed in this way.

First consider the group \(\mathcal{G}_c := C_{2c} \ast C_c C_2\). \(\mathcal{G}_c\) is a finite central extension of the infinite dihedral group \(\mathbb{D}_\infty = C_2 \ast C_2\). As discussed above its dimension vector monoid can be written as \(T(\mathcal{G}_c) \cong (\mathbb{N}_0^2 \times \mathbb{N}_0^2)^c\). For the dimension vector \(m = (m_0, \ldots, m_{c-1})\) we have
\[
R_{m,\mathcal{G}_c}^{\text{absim}} = \begin{cases} 1, & \text{if } |m| = 1 \\ s - 2, & \text{if } \exists \gamma \text{ s.t. } m_\gamma = (1,1,1,1) \& m_\delta = (0,0,0,0) \forall \delta \neq \gamma \\ 0, & \text{else} \end{cases}
\]

In particular all absolutely simple representations of \(\mathcal{G}_c\) over a suitable field occur in dimension 1 or 2. The group \(\mathcal{G}_c\) is among the few groups for which it is possible to determine all the polynomials \(R_{m,\mathcal{G}_c}^{\text{absim}}\) explicitly. In fact, we not only count but classify all absolutely simple representations of \(\mathcal{G}_c\) in Subsection 6.2 below.

We now consider \(\text{PSL}_2(\mathbb{Z}) \cong C_2 \ast C_3\) with \(T(\text{PSL}_2(\mathbb{Z})) \cong \mathbb{N}_0^2 \times \mathbb{N}_0^3\). For \(|m| \leq 4\) those \(R_{m,\text{PSL}_2(\mathbb{Z})}^{\text{absim}}\) which are non-zero are listed below.

| \((\text{m}_0, \text{m}_1, \text{m}_2, \text{m}_3)\) | \(R_{m,\text{PSL}_2(\mathbb{Z})}^{\text{absim}}\) | \((\text{m}_0, \text{m}_1, \text{m}_2, \text{m}_3)\) | \(R_{m,\text{PSL}_2(\mathbb{Z})}^{\text{absim}}\) |
|----------------|-----------------|----------------|-----------------|
| \((1,0,0,0)\)  | 1               | \((1,1,0,0)\)  | 1               |
| \((1,0,1,0)\)  | 1               | \((1,1,1,0)\)  | 1               |
| \((1,0,0,1)\)  | 1               | \((2,1,0,1)\)  | \(s^2 - 3s + 3\) |
| \((0,1,0,0)\)  | 1               | \((1,2,0,1)\)  | \(s^2 - 3s + 3\) |
| \((0,1,1,0)\)  | 1               | \((2,2,0,1)\)  | \(s^2 - 3s^2 + 5s - 4\) |
| \((1,1,0,1)\)  | 1               | \((2,2,1,1)\)  | \(s^2 - 3s^2 + 5s - 4\) |
| \((1,1,1,0)\)  | \(s - 2\)       | \((2,2,1,2)\)  | \(s^2 - 3s^2 + 5s - 4\) |

For \(|m| \leq 5\) all non-zero \(R_{m,\text{PSL}_2(\mathbb{Z})}^{\text{absim}}\) in a given total dimension \(|m|\) coincide. However, from total dimension \(|m| = 6\) on this fails as the following polynomials show.
The examples above suggest that there are symmetries on the set $\mathcal{T}_d(\text{PSL}_2(\mathbb{Z}))$ along which the counting polynomials stay the same. This is indeed the case for all of the groups $C_a \ast C_c C_b$ and we will discuss these symmetries in Section 7 below.

The polynomials $R_m^{\text{absim}, \text{SL}_2(\mathbb{Z})}$ are basically the same as those for $\text{PSL}_2(\mathbb{Z})$. More generally for $m = (m_0, \ldots, m_{r-1}) \in T(C_{a_0} \ast C_{b_0})^\infty \cong T(C_a \ast C_c C_b)$ we have

$$R_m^{\text{absim}, C_a \ast C_c C_b} = \begin{cases} R_m^\gamma, & \text{if } \exists \gamma \text{ s.t. } m_\delta = 0 \forall \delta \neq \gamma \\ 0, & \text{else} \end{cases}$$

However, the analogous statement for the counting polynomials $R_m^{\text{ss}, C_a \ast C_c C_b}$ is false.

### 6.3 Counting polynomials of character varieties.

We now want to give examples for the counting polynomials $R_d^{\text{ss}, G}$. Recall that these give the E-polynomials of the character varieties $X_G(\text{GL}_d(\mathbb{C})) = M(\mathbb{C}[G], d)$ as discussed in Subsection 2.2.

The highest $d$ for which the author has computed $R_d^{\text{ss}, \text{PSL}_2(\mathbb{Z})}$ so far is $d = 12$. $R_{12}^{\text{ss}, \text{PSL}_2(\mathbb{Z})}$ is given by $s^{25} + 3s^{24} + 18s^{23} + 38s^{22} + 67s^{21} + 48s^{20} - 49s^{19} - 210s^{18} - 186s^{17} + 329s^{16} + 738s^{15} - 1131s^{14} + 141s^{13} + 264s^{12} + 657s^{11} - 1067s^{10} + 542s^9 - 216s^8 + 753s^7 - 786s^6 + 508s^5 + 313s^4 - 224s^3 + 476s^2 - 143s + 215$.

| $d$ | $R_d^{\text{ss}, \text{PSL}_2(\mathbb{Z})}$ |
|-----|----------------------------------|
| 1   | 12                               |
| 3   | $4s^2 + 60s + 232$                |
| 5   | $2s^6 + 12s + 26$                 |
| 6   | $3s^5 + 5s^4 - 7s^3 + 9s - 6$     |
| 7   | $3s^4 - 3s^3 + 5s^2 - 7s^2 + 9s - 6$ |
| 8   | $s^3 - 3s^2 + 5s - 7s + 9s - 6$   |

| $d$ | $R_d^{\text{ss}, \text{PSL}_2(\mathbb{Z})}$ |
|-----|----------------------------------|
| 1   | 4                                |
| 3   | $8s + 28$                        |
| 5   | $20s^2 + 56s + 88$               |
| 6   | $s^3 + 8s^2 + 59s^2 + 101s + 147$ |
| 7   | $8s^3 + 36s^3 + 128s^2 + 156s + 212$ |
| 8   | $2s^6 + 6s^5 + 34s^4 + 96s^3 + 223s^2 + 242s + 323$ |
| 9   | $4s^4 + 16s^3 - 8s^2 + 148s^2 + 140s^4 + 400s^2 + 320s + 440$ |
| 10  | $s^6 + 8s^5 + 20s^4 + 23s^3 + 35s^2 + 306s^3 + 206s^2 + 647s^2 + 435s + 628$ |
6.4 Classification for $\mathcal{G}_c$. We will now classify all absolutely simple representations of $\mathcal{G}_c = C_{2c} \ast C_{2c} C_{2c}$ over a suitable ground field. This will in particular prove that the counting polynomials $R_{m}^{\text{absim}, \mathcal{G}_c}$ are given by (54). Recall that the dimension vector monoid of $\mathcal{G}_c$ is given by $T(\mathcal{G}_c) \cong (N_0^2 \times N_0 N_0)^c$.

**Proposition 6.1** Let $K$ be a suitable field for $\mathcal{G}_c.$\footnote{i.e. char($K$) does not divide $2c$ and $K$ contains a primitive $2c$-th root of unity} Denote its group of $2c$-th roots of unity by $\mu_{2c}(K)$. Consider the presentation $\mathcal{G}_c = \langle f, g \mid f^2 = g^2, f^{2c} = 1 \rangle$. In dimension 1 all representations $\rho : \mathcal{G}_c \to \text{GL}_1(K)$ are absolutely simple and pairwise non-isomorphic. They are given by the set $\{(x, y) \in \mu_{2c}(K) \mid x^2 = y^2\}$ via the bijection $\rho \mapsto (\rho(f), \rho(g))$.

All other absolutely simple representations $\rho$ of $\mathcal{G}_c$ have dimension 2 and their isomorphism classes are in bijection with the set

$$\{(\overline{x}, y) \in K^2 \mid \overline{x} \in \mu_{2c}(K)/(\{\pm 1\}, y \in K \setminus \{\pm x\}\}$$

where an explicit representative is given by

$$(\rho(f), \rho(g)) = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \begin{pmatrix} y & 1 \\ x^2 - y^2 & -y \end{pmatrix}$$

**Proof:** The case of dimension 1 is elementary. For dimension $d \geq 2$ we first note that $\rho(f)$ and $\rho(g)$ are diagonalizable with eigen values in $\mu_{2c}(K)$, because char($K$) is suitable.

Now take $h := f^2 = g^2$. As $h \in Z(\mathcal{G}_c)$ is in the center, $\rho(h) = z.1_d$ is a scalar matrix with $z \in \mu_c(K)$ if $\rho$ is absolutely simple. Denote the two square roots of $z$ by $\pm x$. By construction $\rho(f)$ and $\rho(g)$ have no eigen values except for $\pm x$. We assume without loss of generality that

$$\rho(f) = \begin{pmatrix} x & 1_d \\ 0 & -x.1_d \end{pmatrix}$$

with $d_1 = d - d_2 \neq 0, d$ and consider the action of $\text{GL}_{d_1}(K) \times \text{GL}_{d_2}(K) \cong S(\rho(f))$ on $\rho(g) = (\phi, \psi)$. Since $\rho$ is simple, we know that $t_1 M t_2^{-1}, t_2 N t_1^{-1} \neq 0$ for all $(t_1, t_2)$. For $d = 2$ we have $d_1 = d_2 = 1$ and may take $(t_1, t_2) = (1, M)$ to get $(t_1, t_2).\rho(g) = (L'_M, W'_N)$. Using $\text{Tr}(\rho(g)) = 0$ and $\rho(g)^2 = z.1_2$ we obtain that $y := L'_M = -W'N$ and $N' = x^2 - y^2$. This proves the claim for dimension 2.

Now assume $\rho$ were an absolutely simple representation of dimension $d \geq 3$. First we note that $d_1 = d_2$: Denote by $c_1$ and $c_2$ the multiplicities of the eigen values $\pm x$ for the matrix $\rho(g)$. As for $\rho(f)$ we have $0 < c_1, c_2 < d$. Since every simultaneous

| $d$ | $\rho_{\text{PGL}_2(2)}$ | $P_{n}^{\mathfrak{a}}$ |
|-----|-----------------|---------|
| 1   | 4               | 2       |
| 3   | $4s + 28$       | 4       |
| 5   | $8s^2 + 32s + 84$ | 6       |
| 7   | $4s^4 + 16s^3 + 44s^2 + 96s + 180$ |        |
| 8   | $s^3 + 5s^2 + 11s^1 + 64s^1 + 64s^0 + 152s + 253$ |        |
| 9   | $4s^4 + 12s^3 + 20s^2 + 80s^1 + 56s + 156s^0 + 156s + 188s + 324$ |        |
| 10  | $6s^4 + 22s^3 + 16s^2 + 56s + 256s - 256s + 212s + 426$ |        |
| 11  | $4s^4 + 20s^3 + 36s^2 + 72s + 72s^0 + 56s^0 + 100s + 148s^0 + 228s^0 + 372s + 524$ |        |
| 12  | $s^4 + 4s^3 + 19s^2 + 27s - 25s^0 - 15s - 268s^0 + 268s^0 + 303s^0 + 178s^0 + 60s^0 + 438s^0 + 420s + 659$ |        |
eigen vector of $\rho(f)$ and $\rho(g)$ would span a subrepresentation of $\rho$, the multiplicities have to fulfill
\begin{equation}
    c_\gamma + d_\delta \leq d \quad \forall 1 \leq \gamma, \delta \leq 2
\end{equation}
The inequalities (35) yield that $d = 2r$ is even and $r = c_1 = c_2 = d_1 = d_2$. Furthermore we may assume that $\rho(g)$ is of the form $\rho(g) = \left( \begin{smallmatrix} L & 0 \\ N & W \end{smallmatrix} \right)$. By standard linear algebra arguments we may find $(t_1, t_2) \in \text{GL}_d(K)^2$ s.t. $t_1 Mt_2^{-1} = \left( \begin{smallmatrix} 1_{rk(M)} & 0 \\ 0 & 0 \end{smallmatrix} \right)$ and it remains to show that $\text{rk}(M) = r$. We write $L = \left( \begin{smallmatrix} L_1 & L_2 \\ L_3 & L_4 \end{smallmatrix} \right)$ and $W = \left( \begin{smallmatrix} W_1 & W_2 \\ W_3 & W_4 \end{smallmatrix} \right)$ as block matrices with $L_1, W_1$ square matrices of size $\text{rk}(M)$. Using $\rho(g)^2 = z.1_d$ we see that $W_2 = 0$. This means that the last $r - \text{rk}(M)$ basis elements span a subrepresentation, so by simpleness of $\rho$ we have $r = \text{rk}(M)$.

By again using $\rho(g)^2 = z.1_d$ we may deduce $\rho(g) = \left( \begin{smallmatrix} L & L^{-2} \\ z.1_r - L^{-2} & -L \end{smallmatrix} \right)$. Now let $v \in K$ be an eigen vector of $L$ considered as a matrix over $K$. One checks easily that $\binom{v}{0}$ and $\binom{0}{v}$ span a two dimensional subrepresentation of the base extension $\rho \otimes_K K$ which contradicts our assumption that $\rho$ is absolutely simple. \qed

7 Structural properties

We now discuss some of the main structural properties of the counting polynomials: their degree and the symmetries occurring among them. We start with the degree.

Proposition 7.1 Let $m \in T(\mathcal{G})$ be an arbitrary dimension vector and $\mathbb{F}_q$ suitable for $\mathcal{G}$. The polynomial $R_{ss}^m$ is monic of degree $\dim M(\mathbb{F}_q[\mathcal{G}], m)$. If $R_{ss}^m \neq 0$, then $R_{absim}^m$ is monic too and of the same degree
\begin{equation}
    \dim M(\mathbb{F}_q[\mathcal{G}], m) = \dim M_{absim}(\mathbb{F}_q[\mathcal{G}], m) = 1 - \langle m, m \rangle_{\mathcal{G}}
\end{equation}

Proof: We first recall a well-known theorem about counting polynomials which is due to S. Lang and A. Weil: Let $X$ be a polynomial count $\mathbb{F}_q$-scheme. If $X$ is geometrically irreducible, then its counting polynomial is monic of degree $\dim (X) \geq 1$. This proves the claim on $R_{ss}^m$, since it is a counting polynomial of $M(\mathbb{F}_q[\mathcal{G}], m)$ which is geometrically irreducible, because the connected component $\text{Rep}_m(\mathbb{F}_q[\mathcal{G}])$ surjects onto $M(\mathbb{F}_q[\mathcal{G}], m) \cong M(\mathbb{F}_q[\mathcal{G}], m) \times_{\mathbb{F}_q} \text{Spec}(\mathbb{F}_q)$ and is irreducible by (iii) in Subsection 2.3.43

The claim on $R_{absim}^m$ is proven analogously by replacing $\text{Rep}_m(\mathbb{F}_q[\mathcal{G}])$ with its open subscheme $\text{Rep}_{absim}(\mathbb{F}_q[\mathcal{G}])$ and it remains to prove the two equations in (36): The first equation follows from $M_{absim}(\mathbb{F}_q[\mathcal{G}], m) \subseteq M(\mathbb{F}_q[\mathcal{G}], m)$ being open and non-empty if $R_{absim}^m \neq 0$. For the second equation we note that there is an induced $\text{PGL}_{m,\mathbb{F}_q}$-action on representation spaces that operates freely on $\text{Rep}_m(\mathbb{F}_q[\mathcal{G}])$ and that its quotient $\text{Rep}_{absim}(\mathbb{F}_q[\mathcal{G}]) / \text{PGL}_{m,\mathbb{F}_q}$ is isomorphic to $M_{absim}(\mathbb{F}_q[\mathcal{G}], m)$. Hence,
\begin{equation}
    \dim M_{absim}(\mathbb{F}_q[\mathcal{G}], m) = \dim \text{Rep}_{absim}(\mathbb{F}_q[\mathcal{G}]) - \dim \text{PGL}_{m,\mathbb{F}_q}
\end{equation}
Moreover we have $\dim \text{Rep}_{absim}(\mathbb{F}_q[\mathcal{G}]) = \dim \text{Rep}_m(\mathbb{F}_q[\mathcal{G}])$, because $\text{Rep}_m(\mathbb{F}_q[\mathcal{G}])$ is geometrically irreducible. So the second equation in (36) is equivalent to the identity $\deg P^\mathcal{G}_m = |m|^2 - \langle m, m \rangle_{\mathcal{G}}$ which can be verified using our general formula (21). \qed

48See [Poon17 Thm. 7.7.1]
49In particular we have $R_{ss}^m \neq 0$ as $M(\mathbb{F}_q[\mathcal{G}], m)(\mathbb{F}_q)$ is non-empty because $\text{Rep}_m(\mathbb{F}_q[\mathcal{G}])/(\mathbb{F}_q)$ is.
In Subsection 6.2 we have seen that the counting polynomials are invariant with respect to certain symmetries on the dimension vectors of some virtually free groups $G$. More specifically, there is a finite group $S_G$ acting on $K[G]$ for $K$ suitable and by functoriality on each $T_d(G)$, $d \in \mathbb{N}_0$ such that $R^\text{absim}_m = R^{\text{absim}}_n$ and $R^\text{ss}_m = R^\text{ss}_n$ if $m, n \in T_d(G)$ belong to the same $S_G$-orbit. We will now sketch the construction of this group and its action on $K[G]$.

Let $K$ be a field which is suitable for $G$. Hence, all of the finite dimensional group algebras in $K[G]$ are of the form

$$C \cong K^{c_1} \times M_2(K)^{c_2} \times \ldots \times M_\ell(K)^{c_\ell}$$

We construct the group $S_G$ and its actions iteratively and we start with the case of (group algebras of) finite groups: The symmetric group $S_{c_\gamma}$ acts naturally on $M_\beta(K)^{c_\gamma}$ via $\tau(M_1, \ldots, M_{c_\gamma}) = (M_{\tau(1)}, \ldots, M_{\tau(c_\gamma)})$ for each $1 \leq \gamma \leq c$, hence, $S_C := S_{c_1} \times \ldots \times S_{c_\ell}$ acts on $C$ via $K$-algebra automorphisms.

Now assume $A, B, C$ are finite groups acting via $K$-algebra automorphisms on $K$-algebras $A, B, C$ and assume we are given group homomorphisms $A, B \rightarrow C$ and injective $K$-algebra homomorphisms $C \hookrightarrow A, B$ which are $A$- and $B$-equivariant. Then $A \times_C B$ acts naturally on $A \ast_C B$ via $K$-algebra automorphisms.

Finally, assume we have group homomorphisms $\varphi, \theta : A \rightarrow C$ and $K$-algebra embeddings $\iota, \kappa : C \hookrightarrow A$ such that $\iota$ is $A$-equivariant with respect to $\varphi$ and $\kappa$ via $\theta$. Then $\text{Eq}(\varphi, \theta) \subseteq A$ acts naturally on $A \ast_C^\iota \kappa B$ via $K$-algebra automorphisms.

All of the discussion so far works without any assumptions on the involved algebras. However, to iteratively get an induced action on $K[G]$ from the actions on the group algebras $K[G_i]$ and $K[G'_j]$ we need group homomorphisms $S_{G_{s(j)}} \rightarrow S_{G'_j} \leftarrow S_{G_{t(j)}}$ for each $j$ such that the embeddings $K[G_{s(j)}] \hookrightarrow K[G'_j] \hookrightarrow K[G_{t(j)}]$ become $S_{G_{s(j)}}$- and $S_{G_{t(j)}}$-equivariant. If $G'_j$ is the trivial group for each $j$, this obstruction is trivial and we obtain an action of $\prod_{i=0}^{n} S_{G_i}$ on $K[G]$.

However, in general this is a non-trivial combinatorial task which is why we assume from now on that $G_i$ is Abelian for each $0 \leq i \leq I$. Hence, each of the $C$ above is of the form $K^c$, $c = \dim_K(C)$ with an action of the symmetric group $S_c$. We consider an injective $K$-algebra homomorphism $\iota : K^c \hookrightarrow K^b$ and denote by $e_\gamma', 0 \leq \gamma < c$ the $\gamma$-th standard basis vector of $K^c$ and by $e_\beta, 0 \leq \beta < b$ the $\beta$-th standard basis vector of $K^b$. Both $(e_\gamma')_\gamma$ and $(e_\beta)_\beta$ are systems of pairwise orthogonal central primitive idempotents. Hence, there is a partition

$$\Pi := \{0, 1, \ldots, b - 1\} = \bigsqcup_{\gamma=0}^{c-1} \Pi_\gamma$$

such that $\iota(e'_\gamma) = \sum_{\beta \in \Pi_\gamma} e_\beta$  

Recall that each (absolutely) simple $K^b$-module is isomorphic to precisely one of the modules $e_\beta.K^b$ and that the (absolutely) simple $K^c$-modules analogously are given by $e'_\gamma.K^c$. By construction of the partition (37)
we have \( \nu^* (e_\beta K^b) \cong e'_{\gamma} K^c \) for each \( \beta \in \mathbb{I}_y \). Now consider the subgroup 
\[
\overline{S}_b := \{ \tau \in S_b \mid \forall 0 \leq \gamma < c : \exists! 0 \leq \tau(\gamma) < c : \tau(\mathbb{I}_y) = \mathbb{I}_y(\gamma) \}
\]
i.e. the subgroup of those \( \tau \in S_b \) that preserve the partition (37). We obtain a group homomorphism \( \overline{S}_b \to S_a, \tau \mapsto \overline{\tau} \) with respect to which the \( K \)-algebra embedding \( \iota \) is \( \overline{S}_b \)-equivariant and applying the iterative process described above yields a finite group \( \overline{S}_G \) acting on \( K[\mathcal{G}] \) via \( K \)-algebra automorphisms.

By functoriality every group action on \( K[\mathcal{G}] \) via \( K \)-algebra automorphisms yields an induced action on \( M(F_q[\mathcal{G}], d) \), \( M^{\text{absim}}(F_q[\mathcal{G}], d) \) and \( T_d(\mathcal{G}) \) for each \( d \in \mathbb{N}_0 \). If \( m, n \in T_d(\mathcal{G}) \) lie in the same orbit, then we obtain isomorphisms
\[
M^{\text{absim}}(F_q[\mathcal{G}], m) \cong M^{\text{absim}}(F_q[\mathcal{G}], n), \quad M(F_q[\mathcal{G}], m) \cong M(F_q[\mathcal{G}], n)
\]
So in particular the counting polynomials for \( m \) and \( n \) coincide.

**Example 7.2** We consider the case of \( \mathcal{G} = C_a \ast C_c C_b \) for \( a, b \geq 2 \), \( c \) a common divisor of \( a, b \). The partitions (37) of \( \mathbb{I} := \{0, \ldots, a - 1\} \) and \( \mathbb{J} := \{0, 1, \ldots, b - 1\} \) are given by \( \mathbb{I}_y = \{ \alpha \mid \alpha \equiv \gamma \mod (c) \} \), \( \mathbb{J}_y = \{ \beta \mid \beta \equiv \gamma \mod (c) \} \). This determines the subgroups \( \overline{S}_a \subseteq S_a \) and \( \overline{S}_b \subseteq S_b \).

The action of \( S_G = \overline{S}_a \times S_c \overline{S}_b \) on \( T(C_a \ast C_c C_b) \cong \mathbb{N}_0^a \times \mathbb{N}_0^c \mathbb{N}_0^b \) coincides with the restriction of the natural \( S_a \times S_b \)-action on \( \mathbb{N}_0^a \times \mathbb{N}_0^b \).

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Mathematisches Institut, Heinrich-Heine-Universität, 40204 Düsseldorf, Germany

*Email address: Korthauer.maths@gmx.de*