Identifiability and Estimation of Possibly Non-Invertible SVARMA Models: A New Parametrisation

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Proposed Running Head
Possibly Non-Invertible SVARMA in WHF

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Abstract

This paper deals with parameterisation, identifiability, and maximum likelihood (ML) estimation of possibly non-invertible structural vector autoregressive moving average (SVARMA) models driven by independent and non-Gaussian shocks. We introduce a new parameterisation of the MA polynomial matrix based on the Wiener-Hopf factorisation (WHF) and show that the model is identified in this parametrisation for a generic set in the parameter space (when certain just-identifying restrictions are imposed). When the SVARMA model is driven by Gaussian errors, neither the static shock transmission matrix, nor the location of the determinantal zeros of the MA polynomial matrix can be identified without imposing further identifying restrictions on the parameters. We characterise the classes of observational equivalence with respect to second moment information at different stages of the modelling process. Subsequently, cross-sectional and temporal independence and non-Gaussianity of the shocks is used to solve these identifiability problems and identify the true root location of the MA polynomial matrix as well as the static shock transmission matrix (up to permutation and scaling). Typically imposed identifying restrictions on the shock transmission matrix as well as on the determinantal root location are made testable. Furthermore, we provide low level conditions for asymptotic normality of the ML estimator. The estimation procedure is illustrated with various examples from the economic literature and implemented as R-package.

Keywords: Non-invertibility, structural vector autoregressive moving-average models, non-Gaussianity, Identifiability

JEL classification: C32, C51, E52
1 Introduction

Tracing out the response of variables of interest with respect to underlying economic shocks is part of almost every macroeconometric analysis. The main tool for generating this so-called impulse response function (IRF) is the structural vector autoregressive (SVAR) model. In this article, we will point out the deficiencies of SVAR models and suggest a superior alternative: possibly non-invertible SVARMA models.

If the error terms driving the economy are Gaussian or (cross-sectionally) uncorrelated (as opposed to independent), one has to resort to identifying restrictions (obtained from economic theory) in order to conclude on the underlying shocks driving the economy and with respect to which we want to analyse system responses. An immense body of literature has therefore been dedicated to devise (mainly story-driven) identification strategies for the static shock transmission matrix in SVARs (Kilian and Lütkepohl, 2017, Chapter 4). Recently, Lanne et al. (2017) and Gouriéroux et al. (2017) have shown that structural vector autoregressive (SVAR) models driven by independent non-Gaussian components are identified up to scaling and permutations which makes the typically imposed identifying restrictions testable.

In particular, infinitely many linear combinations of shocks generating the same second moments are reduced to a finite set of linear combinations generating the same distributional outcome. It is thus possible to employ a data-driven approach instead of a story-telling approach.

Deficiencies of IRF Analysis with SVAR Models. While these data-driven SVAR identification and estimation strategies are a step forward, two deficiencies of SVAR models remain. First, it is known that complex dynamics are better approximated and described by SVARMA models (Hannan and Deistler, 2012). Especially in macroeconometrics, where data is sometimes available only at quarterly instances, it is of paramount importance to use parsimoniously parameterised models (like e.g. SVARMA models) for which the IRF and variance decompositions can be obtained straightforwardly. Poskitt (2016), Poskitt and Yao (2017), Raghavan et al. (2016), Athanasopoulos and Vahid (2008a), and Athanasopoulos and Vahid (2008b) provide ample evidence and make a strong point for using VARMA models instead of VAR models for econometric analysis. Second, SVAR models exclude a priori the existence of determinantal MA roots. This is especially problematic in structural economic environments where economic agents have more information available than outside observers (corresponding the determinantal MA roots inside the unit circle (Hansen and Sargent, 1991, page 86)). While the literature on SVAR models is abundant, the contributions regarding possibly non-invertible SVARMA models are easier to keep track, see Gouriéroux et al. (2019) and references therein.

The Dynamic Identifiability Problem. Extending the approach in Lanne et al. (2017) to invertible SVARMA models creates a well-understood source of possible non-identifiability in terms of possible non-coprimeness of the AR and MA matrix polynomials, the static identifiability problem concerning the
static shock transmission matrix remains the same. However, when allowing for possibly non-invertible
SVARMA models, a different and more difficult identifiability problem appears. The difficulty is due to
the fact that (multivariate) spectral factorisation techniques are necessary to understand the structure
of observational equivalence with the same second moment information. The recent contribution
Gouriéroux et al. (2019) provides an overview of estimation strategies (mainly in the case of one MA lag)
and applications in macroeconomics and finance, and they apply the results by Chan and Ho (2004);
Chan et al. (2006) on unique representation of multivariate linear processes to derive identifiability
the possibly non-invertible SVARMA model. We focus here on a general treatment of the whole
model class, provide a new parametrisation for the MA polynomial, show that this parametrisation is
identifiable under different non-Gaussianity assumptions and (just-identifying) parameter restrictions,
and provide low-level conditions on the true shock densities such that the ML estimator is asymptotically
normal. Moreover, we characterise the classes of observational equivalence in terms of second moment
information at different stages of the modelling process, i.e. from rational spectral density to spectral
factors (or equivalently the IRF), from spectral factor to AR and MA polynomial and static shock
transmission matrix, and finally from the MA matrix polynomial to the (without further assumptions
in general non-unique) WHF factorisation.

Consequences of Dynamic Non-Identifiability for IRF. To illustrate the importance of identify-
ing the root location correctly, consider the example given in Gouriéroux et al. (2019) who refer to
Lippi and Reichlin (1993). Notice that when the true model for productivity is given as \( y_t = \varepsilon_t + b\varepsilon_{t-1} \),
where \((\varepsilon_t)\) is an i.i.d. shock to productivity with variance equal to one, and such that the largest impact
of a productivity shock is delayed, i.e. \( b > 1 \), we cannot reconstruct these shocks from present and
past observed data (thence the term “non-invertibility”). Moreover, it is easy to see that the process
\( x_t = \eta_t + \frac{1}{b}\eta_{t-1} \) where \((\eta_t)\) is a white noise process with variance \(b^2\) has the same autocovariance
function (and spectral density) as \((y_t)\).

Software for possibly non-invertible VARMA estimation. One (perceived) disadvantage of VARMA
models is increased complexity of the estimation procedure compared to VAR models. Two rebuttals are
in order. First, there are many sophisticated (e.g. non-linear threshold) VAR models whose estimation
is arguably more involved than the one of VARMA models. Second, there are many stable and openly
available software implementations which should put the complexities of estimation of VAR and VARMA
models on equal footing. Examples for implementations in the R software environment R Core Team
(2019) are Scherrer and Funovits (2020b), Tsay (2013), Tsay and Wood (2018) and Gilbert (2015),
see also Scherrer and Deistler (2019) for a comparison and further comments on the latter packages,
and in MATLAB Gomez (2015, 2016). This article is accompanied by an R-package which imple-
ments the developed methods and contains various worked examples from the economic literature in

\[ \text{It can be downloaded from } \text{https://github.com/bfunovits/} \]
vignettes. The package builds on Scherrer and Funovits (2020b,a) which is under development. In order to provide a stable package with focus on this article, the necessary routines are extracted from Scherrer and Funovits (2020b,a) such that unnecessary dependencies are avoided.

Outline. The rest of the paper is structured as follows. In section 2, the SVARMA model and the WHF parametrisation are introduced and the latter is shown to be unique under certain parameter restrictions. In section 3, the identifiability problem is analysed and the classes of observational equivalence with respect to second moment information are characterised. Moreover, the (static and dynamic) identifiability result is stated and proved, and an identification scheme for selecting a particular signed permutation is presented. In section 4, the maximum likelihood (ML) estimator is derived and shown to be consistent and asymptotically normal. Detailed illustrations are contained in the associated R-package. The Appendix contains results on zeros and poles at infinity of rational matrices, details on the (non-) uniqueness of the WHF, and derivations regarding asymptotic normality of the ML estimator.

Notation. We use $z$ as a complex variable as well as the backward shift operator on a stochastic process, i.e. $z (y_t)_{t\in\mathbb{Z}} = (y_{t-1})_{t\in\mathbb{Z}}$ and define $i = \sqrt{-1}$. The transpose of an $(m \times n)$-dimensional matrix $A$ is represented by $A'$. For the sub-matrix of $A$ consisting of rows $m_1$ to $m_2$, $0 \leq m_1 \leq m_2 \leq m$, we write $A[\overline{m_1:m_2}]$ and analogously $A[\overline{\bullet:n_1:n_2}]$ for the sub-matrix of $A$ consisting of columns $n_1$ to $n_2$, $0 \leq n_1 \leq n_2 \leq n$. The column-wise vectorisation of $A \in \mathbb{R}^{m \times n}$ is denoted by $\text{vec}(A) \in \mathbb{R}^{mn \times 1}$ and for a square matrix $B \in \mathbb{R}^{n \times n}$ we denote with $\text{vecd}^0(B) \in \mathbb{R}^{n(n-1)}$ the vectorisation where the diagonal elements of $B$ are left out. The $n$-dimensional identity matrix is denoted by $I_n$, an $n$-dimensional diagonal matrix with diagonal elements $(a_1, \ldots, a_n)$ is denoted by diag $(a_1, \ldots, a_n)$, and the inequality $" > 0"$ means positive definiteness in the context of matrices. The column vector $\iota_i$ has a one at positions $i$ and zeros everywhere else. The expectation of a random variable with respect to a given probability space is denoted by $E (\cdot)$. Convergence in probability and in distribution are denoted by $p-\rightarrow$ and $d-\rightarrow$, respectively. Partial derivatives $\left. \frac{\partial f(x)}{\partial x} \right|_{x=x_0}$ of a real-valued function $f(x)$ evaluated at a point $x_0 \in \mathbb{R}^k$ are denoted by $f_x(x_0)$ and considered columns.
2 Model

We start from an $n$-dimensional VARMA system

$$
(I_n - a_1 z - \cdots - a_p z^p) y_t = (I_n + b_1 z + \cdots + b_q z^q) B \varepsilon_t, \quad a_i, b_i \in \mathbb{R}^{n \times n}.
$$

(1)

The shocks $(\varepsilon_t)_{t \in \mathbb{Z}}$ driving the system are identically and independently distributed (i.i.d.) across time, have zero mean, and diagonal covariance matrix $\Sigma^2$ with positive diagonal elements $\sigma_i^2$, whose positive square root is in turn denoted by $\sigma_i$. To simplify presentation, we also introduce the column vector $\sigma = (\sigma_1, \ldots, \sigma_n)'$ and $\Sigma = \text{diag} (\sigma_1, \ldots, \sigma_n)$, as well as $x_{t-1} = (y_{t-1}, \ldots, y_{t-p})$ and $w_{t-1} = (\varepsilon_{t-1} B', \ldots, \varepsilon_{t-q} B')$ such that equation (2) can be written as

$$
y_t = (a_1, \ldots, a_p) x_{t-1} + (b_1, \ldots, b_q) w_{t-1} + B \varepsilon_t.
$$

We assume that the stability condition

$$
\det (a(z)) \neq 0, \ |z| \leq 1,
$$

(2)

holds, and that there are no determinantal zeros of $b(z)$ on the unit circle, i.e.

$$
\det (b(z)) \neq 0, \ |z| = 1
$$

(3)

hold, and that $B$ is invertible and has ones on its diagonal. Furthermore, we assume that the polynomial matrices $a(z)$ and $b(z)$ are left-coprime, that $a_p$ and $b_q$ are non-zero, and that $(a_p, b_q)$ is of full rank.

Remark 1. An assumption similar to the full rank assumption on $(a_p, b_q)$ seems to be missing in Gouriéroux et al. (2019). While assuming coprimeness reduces the equivalence class of SVARMA models $(a(z), b(z)B)$ that generate the same transfer function $k(z) = a(z)^{-1} b(z) B$, it is not sufficient to guarantee that the equivalence class is a singleton. For example, if $u_1 (a_p, b_q) = 0$ and $u_1 \neq 0$, then for $\tilde{u}(z) = I_n + u_1 z$ the pair $(\tilde{u}(z) a(z), \tilde{u}(z) b(z) B)$ is another realisation of the transfer function $k(z) = (\tilde{u}(z) a(z))^{-1} (\tilde{u}(z) b(z) B)$ which satisfies all requirements on the parameter space.

The stationary solution $(y_t)_{t \in \mathbb{Z}}$ of the system (1) is called an ARMA process.

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2 Determinantal zeros of $b(z)$ correspond to unit canonical correlations between the future $(y_t, y_{t+1}, \ldots)$ and the past $(y_{t-1}, y_{t-2}, \ldots)$ of a stationary stochastic process (Hannan and Poskitt, 1984). Therefore, it seems reasonable to exclude this case from analysis.

3 Two matrix polynomials are called left-coprime if $(a(z), b(z))$ is of full row rank for all $z \in \mathbb{C}$. For equivalent definitions see Hannan and Deistler (2012) Lemma 2.2.1 on page 40.

4 The stability, coprimeness, and full-rank assumptions on the parameters in $a(z)$ and $b(z)$ could be relaxed. The full rank assumption on $(a_p, b_q)$ is over-identifying in the sense that some rational transfer function cannot be parameterized by any VARMA($p,q$) system which satisfies this assumption, see Hannan (1971) or Hannan and Deistler (2012), Chapter 2.7 on page 77. To solve this problem, one could consider the parameter space where the column degrees of $(a(z), b(z))$ are fixed to be $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ as in Deistler (1983) or Hannan and Deistler (2012, Chapter 2.7). Be that as it may, we impose slightly stronger assumptions to strike a balance between notational complexity and generality, and to focus on the essential part of this contribution. Using non-Gaussianity to reduce the equivalence class of stable SVARMA models which generate the same second moments.
2.1 Parametrisation using the Wiener-Hopf Factorisation

The following parametrisation of the MA polynomial matrix \( b(z) \) is useful for gaining structural insights into the behaviour of the system and for deriving asymptotic properties and analytic expressions for the score, the information matrix, and the Hessian of the ML estimator. Every \( b(z) = I_n + b_1 z + \cdots + b_q z^q \) without zeros on the unit circle can be represented as a product of a backward, a shift, and a forward factorization mentioned above, we will discuss different definitions of finite and infinite zeros and poles (in a more general setting) in the Appendix.

Regarding the point at infinity, \( R \) generally, \( \lim_{|z| \to \infty} R(z) = 0 \) if and only if \( \det \left( R(z_0) \right) = 0 \). More generally, \( R(z) \) has a zero at \( z_0 \) if and only if \( R(z)^{-1} \) has a pole at \( z_0 \).

Existence of the WHF. The factorisation of \( b(z) \) into \( (p(z), s(z), f(z)) \) is known as the Wiener-Hopf factorisation (WHF), (Clancey and Gohberg, 1981, Chapter I), (Gohberg et al., 2003), see also Onatski (2006), Al-Sadoon (2018), Al-Sadoon and Zwiernik (2019) for its use in rational expectations models. The WHF exists in more general cases than required for the representation of \( b(z) \) described above: Every rational matrix function without determinantal zeros on the unit circle admits a WHF.
(Clancey and Gohberg 1981, Chapter I). In particular, a (general) polynomial matrix $M(z)$ could be singular when evaluated at $z = 0$. Consequently, starting from an MA polynomial in WHF can be considered slightly more general than starting from an MA matrix polynomial $b(z)$ with $b(0) = I_n$.

**Uniqueness of the WHF.** While the WHF is not unique, the non-uniqueness can be tamed with reasonable effort for the cases relevant to us. The relevant cases are the ones where the first $k$ partial indices are equal to $\kappa + 1$ and the last $(n - k)$ ones are equal to $\kappa$. We will denote this by $(\kappa, k)$, $0 \leq \kappa \leq q$ and $k \in \{0, \ldots, n-1\}$. In the case $(\kappa, 0)$, the WHF is essentially unique in the sense that the equivalence class of WHFs for $b(z)$ is parametrised by the set of non-singular matrices of dimension $(n \times n)$. In particular, requiring that $p(0) = I_n$ results in a unique WHF of $b(z)$. In the case $(\kappa, k)$, $k \neq 0$, the equivalence class of WHFs for $b(z)$ is parametrised by the block upper triangular unimodular matrices for which $u_{[k+1:n,1:k]}(z) = 0$, the diagonal blocks are constant, and the degree of $u_{[1:k,k+1:n]}(z)$ is at most one. Generically, one can choose a canonical representative of a simple form by restricting certain parameters to zero and one (which is easily implementable). For the construction of this canonical representative we refer to the Appendix.

**Generic WHF.** The reason for considering the above cases as the relevant ones is the following. It is a generic property of the parameter space (for which $\det (b(z)) \neq 0$ for $|z| = 1$ holds and which is endowed with the relative topology of the $qn^2$-dimensional Euclidean space) for the MA polynomial matrix $b(z)$ that the difference between the largest and the smallest partial index is smaller than two (Gohberg and Krein 1960), (Gohberg et al. 2003 Section 1.5), (Al-Sadoon, 2018 Supplementary Appendix).

We summarise this in

**Theorem 1.** Every matrix polynomial $b(z) = I_n + b_1 z + \cdots + b_q z^q$ without determinantal zeros on the unit circle and whose parameter space is the open subset $\bigcup_{|z_0| = 1} \{ (b_1, \ldots, b_q) \in \mathbb{R}^{n \times q} \mid \det (b(z_0)) \neq 0 \}$ can generically be factorised as $b(z) = p(z) s(z) f(z)$ where $p(z)$ has no zeros or poles inside or on the unit circle, $s(z) = \text{diag}(z^{\kappa+1}, \ldots, z^{\kappa+1}, z^{\kappa}, \ldots, z^{\kappa})$ with $(\kappa, k)$ such that $0 \leq \kappa \leq q$ and $k \in \{0, \ldots, n-1\}$ and $f(z)$ has no zeros or poles outside or on the unit circle. There are $n \cdot \kappa + k$ zeros inside the unit circle and $\deg (\det (b(z))) = (n \cdot \kappa + k)$ zeros outside the unit circle. In the case $k = 0$, $p(z) = I_n + p_1 z + \cdots + p_{q-\kappa} z^{q-\kappa}$ and $f(z) = f_0 + f_1 z^{-1} + \cdots + I_n z^{-n}$. In the case $k \neq 0$, we have that $\deg (p_{[k+1:n]}(z)) = q - \kappa - 1$, $\deg (p_{[1:k+1]}(z)) = q - \kappa$, $p_0 = \left( \begin{array}{c} I_n \\ 0 \end{array} \right)$, $p_{1,12} = 0_{\kappa(n-k)}$, and that $f(z) = f_0 + f_1 z^{-1} + \cdots + f_{\kappa+1} z^{-\kappa+1}$ where $f_{\kappa+1, \cdot}^{(1)}$.

For derivations, we refer to the Appendix. In the Appendix, we also discuss the relation of the rep-
representation/ factorisation of the AR polynomial matrix in \cite{LanneSaikkonen2013} to the WHF as well as its generality, since it is sometimes considered “very restrictive” \cite[pages 124 and 125]{GouriourexJasiak2017}.

**Factorisation involving polynomials in $z$.** For obtaining formulae in connection with asymptotic behaviour of the maximum likelihood estimator, it is advantageous to consider the factorisation $b(z) = p(z)g(z)$, where $g(z) := s(z)f(z)$. Important properties of $g(z)$ are the non-singularity of the zero-lag coefficient, the non-singular row-end matrix (see the Appendix for a definition), and the row degrees of $\kappa + 1$ of the first $k$ rows, and $\kappa$ of the last $n - k$ rows. Furthermore, notice that $g_0$ is the identity in the case $(\kappa, 0)$. 
3 Identifiability Analysis

We follow Rothenberg (1971) to define identifiability of parametric models. The external characteristic of the stationary solution \((y_t)_{t \in \mathbb{Z}}\) of (1) is the probability distribution function (or a subset of corresponding moments). A particular system (1) is described by the parameters of (1) which satisfy assumptions (2) and (3) as well as the coprimeness assumption, the full rank assumption and the assumptions on \(B\) and \(\Sigma^2\). The model is then characterised by the set of all a priori possible systems which we will call internal characteristics. Two systems of the form (1) are called observationally equivalent if they imply the same external characteristics of \((y_t)_{t \in \mathbb{Z}}\). A system is identifiable if there is no other observationally equivalent system. The identifiability problem is concerned with the existence of an injective function from the internal characteristics to the external characteristics, see Deistler and Seifert (1978) for a more detailed discussion.

3.1 Characterisation of Non-Identifiability from Second Moments

The classical (non-)identifiability issues where the external characteristics are described by the second moments of \((y_t)_{t \in \mathbb{Z}}\) are best understood in terms of the spectral density of the stationary solution of (1). The spectral density, i.e. the Fourier transform of the autocovariance function \(\gamma(s) = \mathbb{E}(y_t y_{t-s}^\prime)\), \(s \in \mathbb{Z}\), of \((y_t)_{t \in \mathbb{Z}}\), is

\[
f(z) = a(z)^{-1} b(z) B \Sigma^2 B' b'(\frac{1}{z}) a'(\frac{1}{z})^{-1},
\]

evaluated at \(z = e^{-i\lambda}, \lambda \in [-\pi, \pi]\).

The Dynamic Identifiability Problem. Starting identifiability from this rational spectral density, it is well known (Rozanov, 1967, Theorem 10.1, page 47), (Hannan, 1970, Theorem II.10' page 66 and Theorem III.1 on page 129), (Baggio and Ferante, 2016), that there exists a canonical rational spectral factor \(l(z)\) without zeros or poles inside or on the unit circle such that \(f(z) = l(z) l'(\frac{1}{z})\). This canonical spectral factor is unique up to orthogonal post-multiplication. In order to focus on the non-uniqueness implied by different pole and zero locations, we will for now abstract from the “static” non-uniqueness of spectral factors implied by orthogonal post-multiplication by requiring that the coefficient pertaining to power zero of \(z\) in the respective spectral factor is lower-triangular with positive diagonal elements.

When allowing for spectral factors with unrestricted zero and pole location, there exists, in general, infinitely many rational all-pass filters \(V(z)\), which satisfy \(V(z) V'(\frac{1}{z}) = I_n\) (Alpay and Gohberg, 1988, page 207), such that \(f(z) = [l(z) V(z)] V' (\frac{1}{z}) l'(\frac{1}{z}) = \tilde{l}(z) \tilde{l}' (\frac{1}{z})\) holds. Requiring that the spectral factors with arbitrary pole and zero location be minimal, Baggio and Ferrante (2019) have recently...
shown that the finite set of all minimal spectral factors $\tilde{l}(z)$ of $f(z)$ can be obtained by right-multiplying the divisors\(^{11}\) of a particular rational all-pass filter $T(z)$ on the canonical spectral factor $l(z)$. We may obtain $T(z) = l(z)^{-1}j(z)$ from the canonical spectral factor $l(z)$ (without zeros and poles inside or on the unit circle) and another “extremal” spectral factor $j(z)$ which has no zeros and poles outside or on the unit circle. Since $l(z)l'(z) = j(z)j'(z)$, it is clear that $T(z)$ is indeed all-pass. Moreover, the all-pass filter $T(z)$ may be represented as the product of orthogonal matrices and so-called Blaschke matrices of the form \(\begin{pmatrix} I_r & 0_{n \times (n-r)} \\ 0_{r \times n} & -1/(I_r + z^{-1}z) \end{pmatrix}\), see Hannan (1970, page 65), Lippi and Reichlin (1994, Theorem 1, page 311), and Alpay and Gohberg (1988, Theorem 3.12, page 208), which immediately provides the (finitely many) all-pass divisors of $T(z)$ which in turn generate a finite number of minimal spectral factors with different zero and pole locations.

**The Static Identifiability Problem.** Let us now turn to the static identifiability problem and notice that the dynamic and static identifiability problem cannot be treated independently. Indeed, for transfer functions $k(z) = a(z)^{-1}b(z)B$ satisfying the assumptions of section 2 it holds that $k_0k'_0$ is maximal when all zeros of $b(z)$ are outside the unit circle (Rozanov, 1967, Theorem 4.2, page 60). This is a consequence of the fact that the Blaschke factor $b_0(z) = \frac{1-\alpha z}{z-\alpha}$ which mirrors a zero at $\alpha$ with $|\alpha| > 1$ inside the unit circle has absolute value smaller than one when evaluated at $z = 0$. Intuitively, this is due to the fact that whenever there are zeros of the MA polynomial matrix inside the unit circle, the information space of the agents is strictly larger than the information space of the outside observer (Hansen and Sargent, 1991, page 86).

Assuming that we know the true zero and pole locations in $k(z) = a(z)^{-1}b(z)B$, it can be shown that any other minimal spectral factor $\tilde{k}(z)$ with the same zero and pole locations can be obtained by orthogonal right transformation of $k(z)$, i.e. $\tilde{k}(z) = k(z)Q$ where $Q$ is an orthogonal matrix.\(^{12}\) Continuing with the parametrisation that we discussed in section 2 i.e. $a_0 = I_n$, $b_0 = I_n$, the static shock transmission matrix has ones on its diagonal, and $\Sigma$ contains the (positive) variances of the economic shocks, we will now conclude the discussion of static observational equivalence in terms of second moments. Transforming the pair $(B, \Sigma)$ with an orthogonal matrix $Q$ to \(\left(B\Sigma Q \tilde{\Sigma}_1^{-1}, \Sigma_1\right)\), where $\tilde{\Sigma}_1$ is a diagonal matrix such that the diagonal elements of $B$ are equal to one and $\Sigma_1$ is the same matrix but with positive elements only, generates the same spectral density because $B_1\Sigma_2B_1' = B\Sigma_2B'$ where $B_1 = B\Sigma Q \tilde{\Sigma}_1^{-1}$. Hence, the class of observational equivalence is at least $\frac{n(n-1)}{2}$-dimensional.

We will show in the next section that under two different sets of assumptions on the joint distribution of the components of the inputs $(\varepsilon_t)$ to \(\Sigma\), $(a(z), b(z))$ are unique and $(B, \Sigma)$ are unique up to signed factors that are obtained by post-multiplying the canonical spectral factor by all-pass filters which do not cancel any zero or pole of $l(z)$ and which correspond to what Lippi and Reichlin (1994) call “non-basic representations”.

\(^{11}\)The rational matrices $T_1(z)$ and $T_2(z)$ are respectively left all-pass divisor and right all-pass divisor of the rational all-pass filter $T(z)$ if $T(z) = T_1(z)T_2(z)$ holds and there are no (finite or infinite) pole or zero cancellations between $T_1(z)$ and $T_2(z)$.

\(^{12}\)A square matrix is orthogonal if $QQ^* = Q^*Q = I_n$. 


permutation, i.e. the orthogonal matrix above is replaced by a signed permutation.

### 3.2 Identifiability using Higher Order Information

In this section, we will first provide some intuition as to how non-Gaussianity and higher order information may help identifying, on the one hand, the orthogonal matrix and the static shock transmission matrix (up to signed permutations) and, on the other hand, the dynamic all-pass filter which "rotates" the canonical spectral factor to the true the zero and pole location. Subsequently, we will use these insights to prove show under which conditions our model is identifiable. Finally, we discuss advantages and disadvantages of various rules for choosing a particular permutation and scaling.

In order to strengthen intuition as to how non-Gaussianity and independence help reducing the size of the class of observational equivalence, consider the following example featuring two identically and independently uniformly distributed random variables. Rotating these two variables 45 degrees (with rotation matrix \( \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \)) leads to marginal distributions which are "more Gaussian" (e.g. measured by the absolute value of the excess kurtosis) than the original variables. This suggests that searching for linear combinations that lead to “maximally non-Gaussian” variables might pin down a rotation. Similarly, the all-pass filters described in the previous section can be interpreted as “dynamic rotations”. Rather than taking linear combinations of the components at one point in time, special linear combinations of the whole stochastic process are considered. In the dynamic setting, we are thus searching for the “dynamic rotation” which transforms uncorrelated inputs (which one obtains from any spectral factor) to independent underlying economic shocks.

The (non-) uniqueness of the infinite MA representation of multivariate linear processes driven by non-Gaussian inputs is well understood in the literature and analysed, e.g., in Chan et al. (2006); Chan and Ho (2004). These insights are used in Lanne and Saikkonen (2013) and Gouriéroux et al. (2019) to show to which extent their respective non-causal and non-invertible models are identified. Interestingly, the dynamic identifiability result of Chan and Ho (2004) builds in the same way on Chapter 5 in Kagan et al. (1973) as the static identifiability problem described in Lanne et al. (2017) builds on Theorem 3.1.1 on page 89 in Kagan et al. (1973). In both cases, higher order information is included in the guise of the characteristic function of the whole process or the components at one point in time, respectively.

We now introduce the first of two possible assumptions on the joint distributions of the components of \( \varepsilon_t \) that is sufficient for identifiability of model (1).

**Assumption 1** (Non-zero cumulant). The components of \( \varepsilon_t \) are mutually independent (but not necessarily identically distributed). Each component has a non-zero cumulant of order \( r \geq 3 \) and finite moments up to order \( \tau \), where \( \tau \) is an even integer and strictly larger than \( r \).
The requirement that a cumulant of order at least three be non-zero excludes the Gaussian distribution. Before proving the general result, we would like to illustrate how the class of observational equivalence is reduced from orthogonal matrices to (signed) permutations, i.e. how to solve the static identifiability problem. The dynamic identifiability problem is solved similarly by using the frequency domain equivalent: The (higher order) Fourier transformation of the auto-cumulant functions.

The following lemma allows to conclude from the independence of the sums of independent variables on the distribution of the underlying summands. In particular, it is useful to conclude on the coefficients pertaining to the summands if one makes additional assumptions on the distribution of the summands.

**Lemma 1** (Kagan et al. (1973), Theorem 3.1.1). Let \( X_1, \ldots, X_n \) be independent (not necessarily identically distributed) random variables, and define \( Y_1 = \sum_{i=1}^{n} a_i X_i \) and \( Y_2 = \sum_{i=1}^{n} b_i X_i \) where \( a_i \) and \( b_i \) are constants. If \( Y_1 \) and \( Y_2 \) are independent, then the random variables \( X_j \) for which \( a_j b_j \neq 0 \) are all normally distributed.

In the following, Lemma 1 is used to conclude on the columns of \( M \) in \( \varepsilon_t = M \varepsilon_t^* \), where \( M = B^{-1}B^* \), where both \( \varepsilon_t \) and \( \varepsilon_t^* \) are assumed to be (cross-sectionally) independent and non-Gaussian. The components of \( \varepsilon_t \) correspond to \( Y_1, Y_2 \), the components of \( \varepsilon_t^* \) correspond to \( X_1, \ldots, X_n \). E.g., for component 1 and 2 of \( \varepsilon_t \) we have \( \varepsilon_{1,t} = (m_{11}, \ldots, m_{1n}) \varepsilon_t^* \) and \( \varepsilon_{2,t} = (m_{21}, \ldots, m_{2n}) \varepsilon_t^* \). If any pair of coefficients \((m_{1k}, m_{2k})\) satisfies \( m_{1k} m_{2k} \neq 0 \), then the corresponding component \( \varepsilon_{k,t}^* \) is Gaussian according to the Lemma. By Assumption 1, at most one component of \( \varepsilon_t^* \) is allowed to have a Gaussian marginal distribution. It follows that there cannot be another pair \((m_{1l}, m_{2l}), l \neq k, \) that satisfies \( m_{1l} m_{2l} \neq 0 \). In particular, there is (at most) one non-zero coefficient in the scalar product \( (m_{1\bullet}, m_{2\bullet}) = m_{1k} m_{2k} \neq 0, \) where \( m_{i\bullet} \) denotes the \( i \)-th row of \( M \). If \( (m_{1\bullet}, m_{2\bullet}) = m_{1k} m_{2k} \neq 0, \) we obtain a contradiction to the assumption that \( E(\varepsilon_{1,t} \varepsilon_{2,t}^*) = 0 \) because from the fact that one (exactly one) component \( \varepsilon_{k,t}^* \) is Gaussian and \( \varepsilon_{i,t} = m_{i\bullet} (\varepsilon_{1,t}^* \cdots \varepsilon_{n,t}^*) \), we obtain that \( E(\varepsilon_{1,t} \varepsilon_{2,t}) = m_{1\bullet} \varepsilon_{2,t}^* = d_{k,t} m_{1k} m_{2k} \neq 0 \). It thus follows that all pairs \((m_{1k}, m_{2k})\) satisfy \( m_{1k} m_{2k} = 0 \). Since this argument holds for all pairs in \( \varepsilon_{1,t}, \ldots, \varepsilon_{n,t}, \) it follows that every column contains at most one non-zero element. Finally, non-singularity implies that every column contains exactly one non-zero element.

The second set of assumptions on the joint distribution of the economic shocks is summarised in

**Assumption 2** (Identically distributed components). The components of \( \varepsilon_t \) are independent, identically distributed, and non-Gaussian.

Note that in Chan and Ho (2004, Theorem 3, page 8), the authors do not require that the components of \( \varepsilon_t \) be non-Gaussian but only that they be independent and identically distributed. The non-Gaussianity follows in their case from assuming that the observed output process be non-Gaussian.

Finally, let us state the result on identifiability of our model.
Theorem 2. Under Assumption 1 or 2, and the assumptions outlined below equation (1), the parameters \((a(z), p(z)s(z)f(z), B, \Sigma)\) in model (1) are identifiable up to signed permutations of \(B\).

The proof is a straightforward application of Chan and Ho (2004); Chan et al. (2006) and using the fact that the shifts are identified, see Gouriéroux et al. (2019, Appendix B, page 34f.).

3.3 Static Identification Scheme: Choosing a Unique Permutation and Scaling

In this section, we describe how to pick one particular permutation and scaling from the class of observational equivalence described in the previous section. In order to do this, we describe different identification schemes, i.e. rules for choosing a particular permutation and scaling of the matrix \(B\).

We start by repeating two identification schemes presented in Lanne et al. (2017) (which are in turn based on Ilmonen and Paindaveine (2011) and Hallin and Mehta (2015)). The first identification scheme, which is convenient for deriving asymptotic properties and which we refer to as identification scheme A, consists in firstly scaling all columns of \(B\) such that their norm is equal to one, secondly, permutating the columns such that the absolute value of each diagonal element is larger than the absolute value of all elements in the same row with a higher column index, and finally scaling all columns of \(B\) such that the diagonal elements are equal to one. The second identification scheme consists of the same first two steps but instead of scaling the columns in the last step such that their diagonal elements are equal to one, it is required that the diagonal elements are positive. Sometimes, the second identification scheme turns out to be more flexible, for example when testing hypotheses involving diagonal elements. Regarding the derivation of asymptotic properties, however, one would need to maximise the constrained (log-)likelihood function where the restrictions that the columns of \(B\) have length one are taken into account. Given that in the case \((\kappa, k), k \neq 0\), one needs to impose (non-overidentifying) restrictions on the parameters in the WHF, (non-overidentifying) restrictions on the parameters in an otherwise unconstrained static shock transmission matrix \(B\) do not add further burden on the researcher.

It is important to realise that the transformations used in the identification schemes described above, exist not on the whole parameter space but only on a topologically large set in the parameter set. For details, see Proposition 2 in Lanne et al. (2017) including an example of a matrix or which the above identification schemes are not defined. The third identification scheme, similar to the one in Chen and Bickel (2005) on page 3626, does not exclude any non-singular matrix \(B\) and is defined by the

\[\text{Note that in the derivation of the ML estimator, we impose only that the diagonal elements of } B \text{ be equal to one. Thus, the restrictions, in general, do not suffice to pin down the particular permutation and scaling for } B. \text{ However, the fact that the observationally equivalent points in the parameter space are discrete ensures the existence of a consistent root, i.e. the solution of the first order conditions obtained from taking derivatives of the standardized log-likelihood function. Should the gradient descent algorithm return a } B \text{ matrix which does not satisfy the identification scheme, it can be easily transformed such that the identification scheme is satisfied. The companion R-package to this article transforms the } B \text{ matrix such that all restrictions described here are satisfied.}\]
following transformations. Firstly, the columns of $B$ are scaled to have norm equal to one. Secondly, in each column, the element with largest absolute value is made positive. Finally, the columns are ordered according to $\prec$ such that $c \prec d$ for two columns $c, d$ of $B$ if and only if there exists a $k \in \{1, \ldots, n\}$ such that $c_k < d_k$ and $c_j = d_j$ for all $j \in \{1, \ldots, k-1\}$.

Now that we have firstly obtained a discrete set of observationally equivalent SVARMA systems and secondly provided different rules to select a unique representative, we may proceed to local ML estimation of the true underlying parameter.
4 Maximum Likelihood Estimation

In this section, we treat local ML estimation of (1) in the parametrisation derived in Theorem 1. In particular, we show that the ML estimator (MLE) is asymptotically normal.

Whereas the essential part of this article is the identifiability analysis and the implied non-singularity of the information matrix of the MLE when (1) is parametrised (including zero-, one-, and equality restrictions on the polynomial matrices) with the WHF, the asymptotic theory is standard. Except for the fact that we consider here the multivariate case, it is identical to the asymptotic analysis in Lii and Rosenblatt (1992) or here. The multivariate matrix calculus and the treatment of the components’ densities is similar to Lanne et al. (2017). The derivations of the score and second order partial derivatives, the information matrix, and the Hessian are straightforward but tedious. The scores and essential differences to the derivations in Lii and Rosenblatt (1992) and Lanne et al. (2017) are summarised in the Appendix. More detail regarding the implementation can be found in the documentation of the associated R-package which can be downloaded from https://github.com/bfunovits/

4.1 Parameter Space and Log-Likelihood Function

We first describe the parameter space over which we optimise the log-likelihood function. Second, we make assumptions on the densities of the components of $\varepsilon_t$. This allows us to provide explicit expressions for the individual contributions to the standardised log-likelihood function and its partial derivatives.

For given integer valued parameters $(p, q, (\kappa, k))$, we vectorise the system parameters, i.e. the ones in $(a(z), p(z), f(z))$, in column-major order. This order is chosen because firstly the ML estimation is implemented in R (R Core Team, 2019), whose storage order is column-major, and secondly it builds on the packages RLDM and rational matrices whose objects lend themselves to vectorising in the described way. The AR parameters are vectorised as $\tau_1 = vec(a_1, \ldots, a_p)$, the “stable” MA parameters for $(\kappa, 0)$ as $\tau_2 = vec(p_1, \ldots, p_{q-k})$ and for $(\kappa, k)$, $k \neq 0$, as

$$
\tau_2 = vec \left( \begin{pmatrix}
I_k & 0_{k \times (n-k)} \\
p_{0,21} & I_{n-k}
\end{pmatrix}
\right)
\begin{pmatrix}
p_{1,11} & 0_{k \times (n-k)} & \cdots & p_{k-1,1} & p_{k,[\kappa+1:n]} \\
p_{1,21} & p_{1,22} & & & \\
\end{pmatrix}
$$

[14] These authors in turn refer to Lehmann (1983, page 430). However, the proof in Lehmann (1983) requires assumptions on the third order partial derivatives of the individual contributions to the log-likelihood function (rather than the second order partial derivatives) while this is not necessary in Lii and Rosenblatt (1992) or here.

[15] A parametrisation where the columns of $(a(z), b(z))$ are reordered as

$$
vec \left( [a_1, \ddots, a_p, b_1, \ddots, b_q] \right)
$$

is advocated in Hannan and Deistler (2012, page 133) for the invertible VARMA case because it leads to comparably simple formulae for the covariance of the asymptotic distribution. However, in our case it is more difficult (if not impossible) to obtain an elegant integral representation. Therefore, we opt for a form in which the partial derivatives are easier to obtain.

[16] The abbreviation RLDM stands for Rational Linear Dynamic Models.
It turns out that it is more convenient to parametrise the “unstable” MA parameters in

\[ g(z) = s(z)f(z) \]

\[ = \left( f_{n+1,[1:k,\bullet]} + f_{n+1,[k+1:n,\bullet]} \right) z + \cdots + \left( f_{1,[1:k,\bullet]} + f_{0,[1:k,\bullet]} \right) z^{\kappa+1} + \left( f_{0,[1:k,\bullet]} \right) z^{\kappa+1} \]

rather than the ones in \( f(z) \) directly. Of course, they are in a one-to-one relation and can be easily obtained from each other, whenever necessary. Note that none of the parameters in \( g_0 \) are free because there are equality restrictions between \( p_0 \) and \( g_0 \) (in the case \( (\kappa, k) \neq 0 \)). The parameters in \( g(z) \) are vectorised, in the case \( (\kappa, k) \neq 0 \), as

\[ \tau_3 = vec \left( \begin{pmatrix} I_k & 0_{k \times (n-k)} \\ -p_{0,21} & I_{n-k} \end{pmatrix} \right) \begin{bmatrix} g_1, \ldots, g_{\kappa} \end{bmatrix} \]

and as \( \tau_3 = vec (g_1, \ldots, g_{\kappa}) \) when \( k = 0 \).

**Restrictions on system parameters.** Obviously, not all parameters in \( \tau' = (\tau_1', \tau_2', \tau_3') \) are free. There are \( n(n-1) + kn \) zero-restrictions and \( n \) one-restrictions in \( \tau_2 \), \( kn + (n-k)^2 - n + (n-k)n \) zero restrictions, and \( n \) one-restrictions in \( \tau_3 \), and \( k(n-k) \) restrictions between the parameters in \( \tau_2 \) and \( \tau_1 \), as described in Theorem 1. We represent these restrictions in the implicit form (Gouriéroux and Monfort, 1989) as \( R\tau = r \) where \( R \) is of full row rank and of dimension \( 3n^2 \times n_\tau \), where \( n_\tau = n^2 (p+q+3) \). Note, however, that when implementing this estimation method, it is more convenient to write them in the explicit form.

**The parameter space in detail.** The (free) parameters pertaining the the underlying economic shocks are vectorised and summarised in

**Assumption 3.** The true parameter value \( \theta_0 \) belongs to the permissible parameter space \( \Theta = \Theta_\tau \times \Theta_\beta \times \Theta_\sigma \times \Theta_\lambda = \Theta_\tau \times \Theta_\gamma \), where

1. \( \Theta_\tau \) with \( \Theta_\tau \subseteq \mathbb{R}^{n^2(p+q)} \) is such that conditions (2), (3), the coprimeness assumption and the full rank assumption on \( (a_p, b_q) \) are satisfied, and
2. \( \Theta_\beta = vec^{\oplus} (B) = \{ \beta \in \mathbb{R}^{n(n-1)} | \beta = vec^{\oplus} (B) \text{ for some } B \in B \} \). The vector \( \beta \) collects the off-diagonal elements of \( B \).
3. For the scalings, \( \Theta_\sigma = \mathbb{R}_+^n \) holds, and
4. For the additional parameters appearing in the component densities, we have $\Theta_\lambda = \Theta_{\lambda_1} \times \cdots \times \Theta_{\lambda_n} \subseteq \mathbb{R}^d$ with $\Theta_{\lambda_i} \subseteq \mathbb{R}^{d_i}$ open for every $i \in \{1, \ldots, n\}$ and $d = d_1 + \cdots + d_n$.

We also introduce the non-singleton compact and convex subset $\Theta_0 = \Theta_{0,\tau} \times \Theta_{0,\gamma}$ of the interior of $\Theta$ which contains the true parameter value $\theta_0$.

**The component densities.** Regarding the component densities of the i.i.d. shock process $(\varepsilon_t)$, we have

**Assumption 4.** For each $i \in \{1, \ldots, n\}$ the distribution of the error term $\varepsilon_{i,t}$ has a (Lebesgue) density $f_{i,\sigma}(x; \lambda_i) = \sigma_i^{-1} f_i(\sigma_i^{-1} x; \lambda_i)$ which may also depend on a parameter vector $\lambda_i \in \mathbb{R}^{d_i}$.

Thus, the individual contributions in the (standardised) log-likelihood function

$$L_T(\theta) = \frac{1}{T} \sum_{t=1}^{T} l_t(\varepsilon_t(\theta), \theta)$$

are

$$l_t(\varepsilon_t(\theta), \theta) = \sum_{i=1}^{n} \log \left[ f_i \left( \sigma_i^{-1} \varepsilon_t(\theta) \right) \right] - \log \left[ \det B(\beta) \right] - \log \left[ \det f_0(\theta) \right] - \sum_{i=1}^{n} \log (\sigma_i),$$

where $u_t(\theta) = a_r(z) y_t + (1 - p_r(z) s_r(z) f_r(z)) B(\beta) \varepsilon_t(\theta)$ and $i_t$ is the unit column-vector with a one at the $i$-th position.

### 4.2 Low-Level Integrability Assumptions and Asymptotic Normality

The expressions for the partial derivatives of the individual contributions to the standardised log-likelihood function are given as

$$\frac{\partial l_t(\theta)}{\partial \tau_1} = -x_{b,t-1}(\theta) B'(\beta)^{-1} \Sigma^{-1} e_{x,t}(\theta)$$

$$\frac{\partial l_t(\theta)}{\partial \tau_2} = - \left( [f(z)^{-1} s(z)^{-1} p(z)^{-1}] \left[ u'_{g,t-1}(\theta) \right] \otimes I_n \right)' B'(\beta)^{-1} \Sigma^{-1} e_{x,t}(\theta)$$

$$\frac{\partial l_t(\theta)}{\partial \tau_3} = - \left( [f(z)^{-1} s(z)^{-1} p(z)^{-1}] \left[ u'_{p,t-1}(\theta) \right] \otimes I_n \right)' B'(\beta)^{-1} \Sigma^{-1} e_{x,t}(\theta) - \frac{\partial vec(f_0)}{\partial \tau_3} vec(f_0^{-1})$$

$$\frac{\partial l_t(\theta)}{\partial \beta} = -H' \sum_{i=1}^{q} \left( B(\beta)^{-1} u_{t-1}(\theta) \otimes B'(\beta)^{-1} \Sigma^{-1} e_{x,t}(\theta) \right) - H' \left( B(\beta)^{-1} u_t(\theta) \otimes B'(\beta)^{-1} \Sigma^{-1} e_{x,t}(\theta) \right) - H' vec \left( B'(\beta)^{-1} \right)$$

$$\frac{\partial l_t(\theta)}{\partial \sigma} = -\Sigma^{-2} \left( e_{x,t}(\theta) \otimes \varepsilon_t(\theta) + \sigma \right)$$

$$\frac{\partial l_t(\theta)}{\partial \lambda} = e_{\lambda,t}(\theta)$$
where \( x'_{b,t-1} = [f(z)^{-1}z^{-p}p(z)^{-1}] [x'_{t-1} \otimes I_n] \), \( x'_{t-1} = (y'_{t-1}, \ldots, y'_{t-p}) \).

\[
\begin{align*}
\mathbf{w}'_{b,t-1} &= (g(z)u_{1,t-1}(\theta), \ldots, g(z)u_{n,t-1}(\theta)) \cdot |g(z)u_{1,t-(q-\kappa)}(\theta), \ldots, g(z)u_{n,t-(q-\kappa)}(\theta)|, \\
\mathbf{w}'_{p,t-1} &= (p(z)u_{1,t-1}(\theta), \ldots, p(z)u_{n,t-1}(\theta)) \cdot |p(z)u_{1,t-(q-\kappa)}(\theta), \ldots, p(z)u_{n,t-(q-\kappa)}(\theta)|,
\end{align*}
\]

The matrix \( H \in \mathbb{R}^{n^2 \times n(n-1)} \) consisting of zeros and ones is implicitly defined by \( \text{vec}(B(\beta)) = H\beta + \text{vec}(I_n) \) for \( B \in B \).

The other main differences in the partial derivatives of the log-likelihood function compared to the invertible Gaussian case are the appearance of \( \text{vec}(f) \) and \( \text{vec}(g) \), the term \( \log|\det[f_0]| \), and the fact that the expressions

\[
\begin{align*}
e_{i,x,t}(\theta) &= \frac{\partial}{\partial x} \log [f_i \left( \sigma_{\iota}^{-1}i,B(\beta)^{-1}u_t(\theta); \lambda_i \right)] = \frac{f_i \left( \sigma_{\iota}^{-1}i,B(\beta)^{-1}u_t(\theta); \lambda_i \right)}{f_i \left( \sigma_{\iota}^{-1}i,B(\beta)^{-1}u_t(\theta); \lambda_i \right)}, \\
e_{i,\lambda,t}(\theta) &= \frac{\partial}{\partial \lambda_i} \log [f_i \left( \sigma_{\iota}^{-1}i,B(\beta)^{-1}u_t(\theta); \lambda_i \right)] = \frac{f_i \left( \sigma_{\iota}^{-1}i,B(\beta)^{-1}u_t(\theta); \lambda_i \right)}{f_i \left( \sigma_{\iota}^{-1}i,B(\beta)^{-1}u_t(\theta); \lambda_i \right)},
\end{align*}
\]

with \( f_i(x; \lambda_i) = \frac{\partial}{\partial x} f_i(x; \lambda_i) \) and \( f_i, \lambda(x; \lambda_i) = \frac{\partial}{\partial \lambda_i} f_i(x; \lambda_i) \) do not simplify as in the Gaussian case (compare the terms \( \tilde{f} \) and \( \tilde{J} \) in Rosenblatt (2000, Chapter 8)). Evaluated at the truth, i.e., \( \theta = \theta_0 \), we have that \( e_{i,t}(\theta_0) = e_{i,t} \) and

\[
\begin{align*}
e_{i,x,t} &= e_{i,x,t}(\theta_0) = \frac{\partial}{\partial x} \log [f_i \left( \sigma_{\iota}^{-1}i,B(\beta)^{-1}u_t(\pi); \lambda_i \right)] \bigg|_{\theta = \theta_0} = \frac{f_i \left( \sigma_{\iota}^{-1}i,B(\beta)^{-1}u_t(\pi); \lambda_i,0 \right)}{f_i \left( \sigma_{\iota}^{-1}i,B(\beta)^{-1}u_t(\pi); \lambda_i,0 \right)}.
\end{align*}
\]

The following assumptions are similar to Lii and Rosenblatt (1992); Lanne et al. (2017).

**Assumption 5.** The following conditions hold for \( i \in \{1, \ldots, n\} \).

1. For all \( x \in \mathbb{R} \) and all \( \lambda_i \in \Theta_{0,\lambda_i} \), \( f_i(x; \lambda_i) > 0 \) and \( f_i(x; \lambda_i) \) is twice continuously differentiable with respect to \( (x; \lambda_i) \).

2. The function \( f_i(x; \lambda_i,0) \) is integrable with respect to \( x \), i.e., \( \int |f_i(x; \lambda_i,0)| \, dx < \infty \).

3. For all \( x \in \mathbb{R} \)

\[
\begin{align*}
x^2 \frac{f_i^2(x; \lambda_i)}{f_i^2(x; \lambda_i)} + \frac{\|f_i,\lambda_i(x; \lambda_i)\|^2}{f_i^2(x; \lambda_i)}
\end{align*}
\]

are dominated by \( c_1 (1 + |x|^2) \) with \( c_1, c_2 \geq 0 \) and \( \int |x|^2 f_i(x; \lambda_i,0) \, dx < \infty \).

4. \( \int \sup_{\lambda_i \in \Theta_{0,\lambda_i}} \|f_i,\lambda_i(x; \lambda_i,0)\| \, dx < \infty \).

and

**Assumption 6.** The following conditions hold for \( i \in \{1, \ldots, n\} \).
1. The functions \( f_{i,x}(x; \lambda_0) \) and \( f_{i,x}(x; \lambda_0) \) are integrable with respect to \( x \), i.e.,
\[
\int |f_{i,x}(x; \lambda_0)| \, dx < \infty \quad \text{and} \quad \int \|f_{i,x}(x; \lambda_0)\| \, dx < \infty.
\]

2. \( \sup_{\lambda_i \in \Theta, \lambda_i} \|f_{i,x}(x; \lambda_0)\| \, dx < \infty \)

3. For all \( x \in \mathbb{R} \) and all \( \lambda_i \in \Theta_0, \lambda_i \),
\[
\frac{f_{i,x}(x; \lambda_0)}{f_i(x; \lambda_0)} \quad \text{and} \quad \left| \frac{f_{i,x}(x; \lambda_0)}{f_i(x; \lambda_0)} \right|
\]
are dominated by \( a_0 (1 + |x|^{a_1}) \),
\[
\left\| \frac{f_{i,x}(x; \lambda_0)}{f_i(x; \lambda_0)} \right\| \quad \text{and} \quad \left\| \frac{f_{i,x}(x; \lambda_0)}{f_i(x; \lambda_0)} \right\|
\]
are dominated by \( a_0 (1 + |x|^{a_2}) \),
\[
\left\| \frac{f_{i,x}(x; \lambda_0)}{f_i(x; \lambda_0)} \right\|^2 \quad \text{and} \quad \left\| \frac{f_{i,x}(x; \lambda_0)}{f_i(x; \lambda_0)} \right\|^2
\]
are dominated by \( a_0 (1 + |x|^{a_3}) \), with \( a_0, a_1, a_2, a_3 \geq 0 \) such that \( \int \left( |x|^{2+a_1} + |x|^{1+a_2} + |x|^{a_3} \right) f_i(x; \lambda_0) \, dx < \infty \).

In combination, these assumptions allow to prove, in the same way as in Li and Rosenblatt (1992),

**Theorem 3.** Under Assumptions 3, 1, 6, and one of Assumption 4 or 5, there exists a sequence of maximisers \( \hat{\theta}_T \) of (4) such that \( \sqrt{T} (\hat{\theta}_T - \theta_0) \) converges in distribution to \( \mathcal{N} (0, S) \), where

\[
S = \begin{pmatrix} I_0 & R^t \\ R & 0 \end{pmatrix}^{-1} \begin{pmatrix} I_0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_0 & R^t \\ R & 0 \end{pmatrix}^{-1}
\]

and \( I_0 = \mathbb{E} \left[ l_{\theta,t} (\theta_0) l_{\theta,t}' (\theta_0) \right] \).

5 Illustrations

We illustrate the estimation procedure by estimating the two equation system of Blanchard and Quah (1989), the three equation monetary model involving (log-deviation from the steady state of) the unemployment gap, the inflation rate, and the Federal Funds rate, and the four equation model where we include additionally the Kansas City Financial Condition Index (KCFCI). The analyses are available in the vignettes of the associated R-package which can be downloaded with the command
remotes::install_github("bfunovits/svarmawhf", auth_token = "__", build_vignettes = TRUE).
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7 Conclusion

In this article, we introduced a new parametrisation for stable and possibly non-invertible SVARMA models (1) driven by independent and non-Gaussian shocks. Every MA polynomial with no determinantal zeros on the unit circle can be factorised in the way described. We showed that the model in this parametrisation is (under certain affine restrictions) identifiable up to permutation and scaling of the static shock transmission matrix. These results generalise the SVAR results in Lanne et al. (2017) to the possibly non-invertible SVARMA case. Moreover, we provide a computationally feasible method for estimating possibly non-invertible SVARMA models. Illustrations can be found in the vignette of the associated R-package, downloadable from [https://github.com/bfunovits/](https://github.com/bfunovits/).

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A Zeros and Poles at Infinity

We first define for univariate rational functions zeros and poles at infinity. Zeros at infinity are important for the correct specification of the parameter space (which is mainly relevant for the multivariate case). The importance of poles at infinity relies mainly in the fact that the number of finite and infinite poles must always be equal to the number of finite and infinite zeros.

Then, we do the same for matrices whose entries are polynomials or rational functions where we additionally discuss the Smith-McMillan form for obtaining finite zeros and poles, column-reduced and row-reduced matrices, and two different ways for obtaining the zeros and poles at infinity (one via Möbius transformations, the other via valuation theory).

A.1 Univariate Rational Functions

Pole at Infinity. A rational function \( r(z) \) has a pole at infinity of degree \( n \) if and only if \( \lim_{z\to\infty} \frac{r(z)}{z^n} \) exists and is a non-zero number.

Example 1. A polynomial \( p(z) = p_0 + p_1 z + \cdots + p_d z^d \) with \( p_d \neq 0 \) has \( d \) poles at infinity. When dividing this polynomial by \( z^d \), one obtains a function without zeros at infinity because for \( r(z) = \frac{p(z)}{z^d} = p_0 z^{-d} + p_1 z^{-d+1} + \cdots + p_d \) the limit \( \lim_{z\to\infty} r(z) \) is the non-zero number \( p_d \).

Zero at Infinity and Valuation at Infinity. A rational function \( r(z) = \frac{n(z)}{d(z)} \), where \( n(z) \) and \( d(z) \) are polynomials with \( \deg(n(z)) = p_n \) and \( \deg(d(z)) = p_d \), has a zero at infinity if \( p_d > p_n \). In this case, the degree of this zero at infinity is equal to \( p_d - p_n \). One could rewrite the rational function by dividing \( n(z) \) and \( d(z) \) by their respective highest degrees to obtain

\[
r(z) = \frac{z^{p_n} \left( n_0 z^{-p_n} + n_1 z^{-p_n+1} + \cdots + n_{p_n} \right)}{z^{p_d} \left( d_0 z^{-p_d} + d_1 z^{-p_d+1} + \cdots + d_{p_d} z^{-p_d} \right)} = z^{p_d-p_n} \frac{\left( n_0 z^{-p_n} + n_1 z^{-p_n+1} + \cdots + n_{p_n} \right)}{\left( d_0 z^{-p_d} + d_1 z^{-p_d+1} + \cdots + d_{p_d} z^{-p_d} \right)}
\]

from which the degree of a pole \( (p_n > p_d) \) or zero \( (p_d > p_n) \) at infinity can be easily obtained. We will also define the valuation of \( r(z) \) at infinity as \( v_\infty (r(z)) = p_d - p_n \), i.e. the degree of the denominator of \( r(z) \) minus the degree of the numerator of \( r(z) \).

Thus, a pole at infinity implies a negative valuation at infinity, and a zero at infinity implies a positive valuation at infinity. The concept of valuations will be important when characterising zeros and poles at infinity of rational matrix functions.

Defining Zeros at Infinity with respect to the Parameter Space. Notice that in the definition given above, zeros at infinity appear only in conjunction with rational functions. There are, however, other definitions for zeros at infinity which are relevant for polynomials and which additionally consider an appropriate parameter space. Let the \((d+1)\)-dimensional tuple of complex numbers \((c_0, c_1, \ldots, c_d)\) be the parameter space for the polynomial \( c(z) = c_0 + c_1 z + \cdots + c_d z^d \). The number of zeros at infinity of such a \( c(z) \) is equal to \( d - \deg(c(z)) \). If, for example, \( c_d = 0 \) and \( c_{d-1} \neq 0 \), then \( c(z) \) is said to have one zero at infinity. As an example, consider the polynomial \( c + bz + az^2 \) and its associated parameter space \((c, b, a)\). The roots for \( a \neq 0 \) are equal to \( z_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \). When considering the limit for \( a \) going to zero, it is easy to see (applying the rule of l’Hôpital) that \( z_+ \) converges to \( -\frac{b}{2a} \) and that \( z_- \) is unbounded. For a more formal statement, see Theorem 4.1.2 on page 371 in Hinrichsen and Pritchard (2005).
Confusing Notation for Backward Shift in System Theory. In system theory (Kailath, 1980; Anderson and Moore, 2005; Hinrichsen and Pritchard, 2005), it is common to use the complex variable $z$ as forward shift (rather than as backward shift as is commonly the case in econometrics and statistics). Intuitively (and with a certain amount of hand-waiving), this is due to analogy with continuous time systems, where the infinitesimal operator $dt$ corresponds to “small forward step in time”.

More formally, it is due to the definitions of the $z$-transform of a discrete time signal $(y_t)_{t \in \mathbb{N}}$ as a formal power series $\mathcal{L}(y_t)(z) = \sum_{t=0}^{\infty} y_t z^{-t}$ (Hinrichsen and Pritchard, 2005, page 735ff.) and the Laplace transform of a continuous time signal $(y_t)_{t \in \mathbb{R}_{>0}}$ as, without considering well-definedness of the integral, $\mathcal{L}(y_t)(s) = \int_0^{\infty} y_t e^{-st} dt$ (Hinrichsen and Pritchard, 2005, page 739ff.). In particular, the $z$-transform of the discrete time signal of $(y_{t-1})$ corresponds to the one of $(y_t)$ multiplied by $z^{-1}$, see Proposition A.3.6.(i) on page 737 in Hinrichsen and Pritchard (2005).

A.2 Matrices whose Elements are Polynomials or Rational Functions

In addition to the definitions given in the main text which are only given for square rational matrices whose determinant is not identically zero, we will here provide definitions for a (possibly rank-deficient) rational matrix $R(z)$ of dimension $(n \times q)$. The Smith-McMillan form of $R(z)$ is used to define finite poles and zeros. The zeros and poles at infinity are characterised directly via valuation and also with the Smith-McMillan form of $R(\frac{1}{z})$.

These definitions provide more insight into the structure of the rational matrix in terms of multiplicities and the dimension of the kernel.

Definition of the Smith-McMillan form. The Smith-McMillan form is a canonical form for rational matrices and is based on the Smith form which is the equivalent canonical form for polynomial matrices. The Smith form (SF) of a polynomial matrix $P(z)$ of dimensions $(n \times q)$ is obtained from elementary row and column polynomial matrix transformations, i.e. by left- and right-multiplication with so-called unimodular matrices (Gohberg et al., 2009, Chapter S1.1, pages 313ff.). The Smith-McMillan form (SMF), in turn, is obtained by first obtaining the smallest common multiple $s(z)$ of the denominators of the elements of $R(z)$ and subsequently performing the Smith form of $s(z)R(z)$. Eventually, the Smith-McMillan form (Hannan and Deistler, 2012, page 53), (Kailath, 1980, Section 6.5.2, page 443) of a rational matrix $R(z)$ of dimensions $(n \times q)$ and rank $s$ is equal to $R(z) = u(z)\Lambda(z)v(z)$ where $u(z)$ and $v(z)$ are unimodular matrices (i.e. polynomial matrices with non-zero constant determinant), and $\Lambda(z)$ is a diagonal matrix in which only the first $s$ elements $\lambda_i(z) = \frac{n_i(z)}{d_i(z)}$ are non-zero, and $n_i(z), d_i(z)$ are relatively prime monic (i.e. the coefficient pertaining to the highest power is one) polynomials. Furthermore, it is required that $n_i(z)$ divides $n_{i+1}(z)$, and that $d_{i+1}(z)$ divides $d_i(z)$.

Definition of finite zeros and poles of a rational matrix. The finite poles of $R(z)$ are the zeros of the denominator polynomials $d_i(z)$. Note that it is possible that $R(z)$ has both a pole and a zero at $z_0$.

The finite zeros of $R(z)$ are the zeros of the numerator polynomials $n_i(z)$. Note that if (for square matrices) the determinant of $R(z)$ has a zero of multiplicity 2 at $z_0$, the rank deficiency of $R(z)$ can be one, the “usual” case when the parameters of $R(z)$ are

17 A unimodular matrix is a square polynomial matrix with non-zero constant determinant.
18 If the parameters of a square $n$-dimensional $R(z)$ are unrestricted, “usually” $n_1(z) = \cdots = n_{n-1}(z) = 1$ and $d_n(z) = \cdots = d_2(z) = 1$ hold such that $n_k(z) = \prod_{i} (z - z_i)$ and $d_i(z) = \prod_{j} (z - p_j)$.
unrestricted, or two. In the square case, a definition involving the determinant can only make sense in the non-singular case, i.e. when \( \det(R(z)) \) is not identically zero. Still, it is possible that a zero in one \( d_i(z) \) at \( z_0 \) may cancel out a zero in one \( n_j(z), \ j \neq i \), at \( z_0 \) in the determinant.

**Definition of zeros and poles at infinity of a rational matrix with the Smith-McMillan form.** The zeros and poles of \( R(z) \) at infinity are the zeros and poles of \( R \left( \frac{az + b}{cz + d} \right) \) at \( z = -\frac{d}{c} \) where \( c \neq 0 \) and \( ad - bc \neq 0 \). Often, \( c = b = 1 \) and \( a = d = 0 \) are chosen. In that case, the zeros and poles of \( R(z) \) at infinity are obtained as the zeros and poles of the numerator and denominator polynomials of the Smith-McMillan form of \( R \left( \frac{1}{z} \right) \) at \( z = 0 \). Note that this definition using the Smith-McMillan form does not require to be precise about the considered parameter space (as was the case for univariate polynomials).

**Definition of zeros and poles at infinity of a rational matrix via valuation.** Obviously, it is possible to rewrite the Smith-McMillan form as a product of diagonal matrices \( M_\alpha(z) \) which has only (finite) zeros and poles at \( \alpha \) and one non-square matrix \( S \) consisting of zeros and ones such that \( \Lambda(z) = S \prod_{\alpha=1}^{\infty} M_\alpha(z) \). The matrices \( M_\alpha(z) \) can be obtained directly by calculating the minors of \( R(z) \) as will be described below. Importantly, this is also possibly for \( M_\infty(z) \) which contains the zero and pole structure at infinity. In this way, one may circumvent the calculation of the Smith-McMillan form of \( R \left( \frac{1}{z} \right) \).

Identically to the valuation at infinity of a univariate rational function, we define the valuation at \( \alpha \in \mathbb{C} \) as the integer \( v \) in \( r(z) = (z-\alpha)^v \frac{p(z)}{q(z)} \) where \( p(z) \) and \( q(z) \) are polynomials without common factors and do not have \( (z-\alpha) \) as factor. We will denote the valuation of \( r(z) \) at \( \alpha \) as \( v_\alpha(r(z)) \).

For treating the multivariate case, we need to consider minors\(^{19}\) of dimensions \((i \times i), \ i \in \{1, \ldots, \min(n, q)\}\). The \( i \)-th valuation of \( R(z) \) at \( \alpha \in \mathbb{C} \), denoted as \( v_\alpha^{(i)}(r(z)) \), is obtained as the minimal degree of all \((i \times i)\) minors of \( R(z) \) where the valuation of a polynomial that is identically zero is equal to infinity. The degrees of the diagonal elements in \( M_\alpha(z) \) are obtained as \( v_\alpha^{(1)}(r(z)), v_\alpha^{(2)}(r(z)), \ldots, v_\alpha^{(s)}(r(z)) = v_\alpha^{(s-1)}(r(z)). \)

The degrees of \( M_\infty(z) \) can thus be obtained as \( v^{(1)}_\infty(r(z)), v^{(2)}_\infty(r(z)), \ldots, v^{(s)}_\infty(r(z)) = v^{(s-1)}_\infty(r(z)). \)

**Zeros at infinity of unimodular matrices.** While for univariate polynomials, introducing zeros at infinity seem to be a bit artificial, they have an immediate interpretation for unimodular matrices. For example, the unimodular matrix \( t(z) = \begin{pmatrix} 1 & 1 \\ 0 & z \end{pmatrix} \) has a zero at infinity because the Smith-McMillan form of \( t \left( \frac{1}{z} \right) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \) is\(^{20}\) \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Thus, the unimodular matrix \( u(z) \) has a zero and a pole at infinity. Equivalently, the zeros and poles at infinity can be obtained via its valuations. Since \( v_{(1)}(t(z)) = -1 \) and \( v_{(2)}(t(z)) = 0, \) we obtain that the degrees in \( M_\infty(z) \) are equal to \((-1, 1)\).

### A.3 Row- and Column-Reduced Polynomial Matrices

The degree of a polynomial matrix is defined as the maximum of the degrees of its elements. Likewise, the degree of row \( i \) is defined as the maximum of the degrees of the polynomials in row \( i \). The rows of the row-end-matrix are the coefficients pertaining to the row in the SCM. There are \( \binom{n}{i} \binom{i}{j} \) different \((i \times i)\)-minors of an \((n \times q)\)-dimensional matrix. First multiply \( t \left( \frac{1}{z} \right) \) by the SCM and obtain that \( z \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{z} & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). Dividing both sides by \( z \), one results in the Smith-McMillan form.
row degree of the respective row. A polynomial matrix is called row-reduced, if its row-end-matrix is of full rank. For example, the polynomial matrix \( \begin{bmatrix} z^n & 0 \\ 0 & z^m \end{bmatrix} \) is row-reduced, while \( \begin{bmatrix} z & z^2 \\ 0 & z \end{bmatrix} \) is not. Sometimes it is useful to write a (square) polynomial matrix \( P(z) \) with row-end-matrix denoted as \( P_{hr} \) and row degrees \((p_1, \ldots, p_n)\) as \( P(z) = \begin{pmatrix} z^{p_1} \\ \vdots \\ z^{p_n} \end{pmatrix} \) \( P_{hr} + M(z) \) where the degree of \( M_{i,q}(z) \) is smaller than \( p_i \). The same applies to the columns of a polynomial matrix to obtain the column-end-matrix.

### B The Wiener-Hopf Factorisation

In this section, we construct the (left-) WHF of \( b(z) \) using the SF, see also Al-Sadoon (2018). We start from the matrix polynomial \( b(z) = I + b_1 z + \cdots + b_q z^q \) and obtain

\[
\begin{aligned}
\tilde{f}(z) &= \tilde{p}(z) \Lambda_p(z) \tilde{f}(z) \\
\tilde{p}(z) &= \tilde{p}(z) w(z)^{-1} \\
\tilde{f}(z) &= \tilde{f}(z) w(z) \tilde{f}(z)
\end{aligned}
\]

where \( \Lambda_p(z) \) has only zeros outside the unit circle, and \( \Lambda_f(z) \) has only zeros inside the unit circle, and \( w(z) \) is a unimodular matrix which row-reduces \( \tilde{f}(z) \), see Wolovich (1974, Theorem 2.5.7, page 28), Kailath (1980, page 386), Geurts and Praagman (1996).

Subsequently, we permute the rows of \( \tilde{f}(z) \) such that the row degrees \( \kappa_i \), the inequalities \( \kappa_1 \geq \cdots \geq \kappa_n \) hold and we extract the highest degree of each row to obtain the partial indices

\[
\begin{aligned}
\tilde{f}(z) &= \tilde{p}(z) \Lambda_p(z) \tilde{f}(z) \\
\tilde{p}(z) &= \tilde{p}(z) P^{\star} \\
\tilde{f}(z) &= \tilde{f}(z) P \tilde{f}(z)
\end{aligned}
\]

where \( \Lambda_p(z) \) has only zeros outside the unit circle, and \( \Lambda_f(z) \) has only zeros inside the unit circle, and \( w(z) \) is a unimodular matrix which row-reduces \( \tilde{f}(z) \), see Wolovich (1974, Theorem 2.5.7, page 28), Kailath (1980, page 386), Geurts and Praagman (1996).

Note that \( f(z) \) does not have poles at infinity since its degree is zero and that it does not have zeros at infinity because \( f(z) \) evaluated at \( z = 0 \) is by construction of full rank.

### B.1 Non-Uniqueness of the WHF and Degrees of the Factors

It is shown in Clancey and Gohberg (1981, Theorem 1.1.2, page 11) that for \( (\kappa, k) \), the equivalence class of WHFs is described by the block upper triangular unimodular matrices for which \( u_{[k+1:n,1:k]}(z) = 0 \), the diagonal blocks are constant, and the degree of \( u_{[1,k,k+1:n]}(z) \) is at most one. More specifically, we have that \( \tilde{p}(z) = p(z) u(z), \ s(z) = s(z), \ \tilde{f}(z) = s(z)^{-1} u(z)^{-1} s(z) f(z) \). Note that \( v(z) = s(z)^{-1} u(z)^{-1} s(z) f(z) \) is of the form \( v(z) = v_0 + \begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix} z^{-1} \) and that this transformation does not change the
row degrees of \( f(z) = f_0 + f_1 z^{-1} + \cdots + (f_{k, [1:k, \bullet]} \cdot z^{-k}) + (f_{k+1, [1:k, \bullet]} \cdot z^{-k-1}) \) or \( g(z) := s(z) f(z) = (f_{k+1, [1:k, \bullet]} \cdot z^{-k-1} + (f_{k, [k+1:n, \bullet]} \cdot z^{-k+1}) \).

Moreover, it follows from the row-reducedness of \( f(z) \) and \( s(z) f(z) \), together with the predictable degree property (Kailath, 1980, Theorem 6.3-13, page 387) that the first \( k \) columns of \( p(z) \) have degree smaller than or equal to \( q - \kappa - 1 \) and the last \( n - k \) columns of \( p(z) \) have degree smaller than or equal to \( q - \kappa \). Therefore, the transformation \( u(z) = u_0 + \binom{0}{\tilde{u}_1} z \) does not change the highest column degrees of \( p(z) \).

Last, note that due to the fact that \( b(0) = I_n \), it holds that \( p_0^{-1} = (f_{k+1, [1:k, \bullet]} \cdot f_{k, [k+1:n, \bullet]} \cdot I) \).

### B.2 Canonical Representative for \((\kappa, k) \cdot k \neq 0\)

We will now construct a canonical WHF by choosing \( u(z) \) and setting certain parameters in \( p_0 \) and \( p_1 \) of \( p(z) = p_0 + p_1 z + \cdots + p_{12} z^{12} \) equal to zero and one.

First, we will determine \( u_0 \) in \( u(z) = u_0 + \binom{\tilde{u}_1}{0} z \). Let us partition the matrix \( p_0 = \begin{pmatrix} p_{0,11} & p_{0,12} \\ p_{0,21} & p_{0,22} \end{pmatrix} \) and assume that \( p_{0,11} \) is invertible. Then, right-multiplying \( p(z) \) with

\[
\begin{pmatrix} p_{0,11} & p_{0,12} \\ p_{0,21} & p_{0,22} \end{pmatrix} u_0 = \begin{pmatrix} I_k & 0 \\ 0 & I_{n-k} \end{pmatrix} \begin{pmatrix} -p_{0,12} & 0 \\ 0 & -p_{0,22} \end{pmatrix} \begin{pmatrix} I_k & 0 \\ 0 & I_{n-k} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ (p_{0,22} - p_{0,21} p_{0,11}^{-1} p_{0,12})^{-1} \end{pmatrix} \]

we obtain

\[
\begin{pmatrix} p_{0,11} & p_{0,12} \\ p_{0,21} & p_{0,22} \end{pmatrix} u_0 = \begin{pmatrix} I_k & 0 \\ 0 & I_{n-k} \end{pmatrix} \begin{pmatrix} I_k & 0 \\ 0 & (p_{0,22} - p_{0,21} p_{0,11}^{-1} p_{0,12})^{-1} \end{pmatrix} \]

\[
= \begin{pmatrix} I_k & 0 \\ 0 & I_{n-k} \end{pmatrix} \begin{pmatrix} p_{0,21} p_{0,11}^{-1} & 0 \\ 0 & (p_{0,22} - p_{0,21} p_{0,11}^{-1} p_{0,12})^{-1} \end{pmatrix} \]

Last, we may choose \( \tilde{u}_1 \) such that \( p_{1,12} = 0 \).

### B.3 The Factorisation in Lanne and Saikkonen (2013)

Here, we point out differences and similarities of the WHF to the factorisation of the AR matrix polynomial in Lanne and Saikkonen (2013). It is of the form \( \tilde{a}(z) = \Pi(z) \Phi \left( z^{-1} \right) = (I - \Pi_1 z - \cdots - \Pi_r z^r) \left( I - \Phi_1 z^{-1} - \cdots - \Phi_s z^{-s} \right) \) where both \( \det(\Pi(z)) \) and
det (Φ(z)) have no zeros inside or on the unit circle. Thus, Φ (\(\frac{1}{z}\)) corresponds (roughly) to our \(f(z)\).

Let us start by noting that if \(s > 0\), there are negative powers in \(a(z)\). This is due to the fact that \textcite{lanne2013} do not start from a polynomial matrix but directly from the factorisation. Moreover, the coefficient matrices pertaining to \(z^{-s}, z^0, z^r\) are \(\Phi_s, I_n + \sum_{k=1}^{\min(s,r)} \Pi_k \Phi_h, \Pi_r\), none of which is assumed to be of full rank.

In order to make the comparison easier, we introduce ‘pseudo partial indices’ such that \(a(z) = \Pi(z)z^s\Phi (\frac{1}{z}) = (I - \Pi_1 z - \cdots - \Pi_r z^r) z^s\) and compare it to the WHF of a polynomial matrix of the same form as the MA polynomial in the main text but with partial indices \(\kappa = \kappa\), i.e. \(b(z) = p(z)z^\nu f(z)\) where \(\deg(p(z)) = q - \kappa\) and \(\deg(f(z)) = \kappa\), \(b(0) = I_n\), and \(\det(b(z)) \neq 0\) for \(|z| = 1\).

First, note that in the case of constant partial indices it is possible to normalise \(p_0\) to \(I_n\) and that the condition \(b(0) = I_n\) implies that \(f_\kappa = I_n\). Thus, the normalisation of \(\Pi(z)\) seems reasonable in this context, while the normalisation of \(\Phi (\frac{1}{z})\) is more difficult to bring into line with the WHF (and the generic existence of a factorization of the kind in \textcite{lanne2013}). Second, while \(s(z)f(z)\) is row-reduced and \(f(\infty)\) is of full rank as well, the row-end-matrix of \(\Phi(z)\) is not necessarily of full rank. However, the row-end-matrix of \(z^s\Phi (\frac{1}{z})\) is by definition of the factorisation equal to the identity matrix and therefore of full rank. Last, and even though this is entangled with \(\Phi(z)\) not being of full row rank, the “pseudo partial indices” are restricted to be identical.

C Analytic Formulae and Asymptotic Derivations

Under the assumptions of Section 4, we will fill here the missing pieces and technicalities regarding the asymptotic behaviour of the MLE. In particular, we will discuss the representation of the WHF with (finite sections of) Toeplitz operators \textcite{bottcher2009}, and derive the partial derivatives of the log-likelihood function from which the conditions for the asymptotic theory can be easily verified.

C.1 Notation and (Toeplitz) System Representations

The individual contribution at time \(t\) to the (standardised) log-likelihood function, i.e. equation (5), is here repeated as

\[
l_t (\theta) = \sum_{i=1}^{n} \log \left[ f_i \left( \sigma_i^{-1} \varepsilon_{i,t} (\theta) ; \lambda_i \right) \right] - \log \left[ |\det (f_0)| \right] - \log \left[ |\det [B (\beta)]| \right] - \sum_{i=1}^{n} \log (\sigma_i),
\]

where \(\varepsilon_{i,t} (\theta) = i_t' B (\beta)^{-1} u_t (\theta)\) and \(u_t (\theta) = m(z; \theta)^{-1} y_t\) with \(m(z; \theta) = a(z; \theta)^{-1} p(z; \theta) s(z; \theta) f(z; \theta)\) such that \(m(z; \theta)B(\beta) = k(z; \theta)\).

Derivatives of the component densities. For the first partial derivatives of \(l_t (\theta)\), the expressions

\[
\frac{\partial}{\partial x} \log \left[ f_i \left( \sigma_i^{-1} i_t' B (\beta)^{-1} u_t (\theta) ; \lambda_i \right) \right] = \frac{f_i \left( \sigma_i^{-1} i_t' B (\beta)^{-1} u_t (\theta) ; \lambda_i \right)}{f_i \left( \sigma_i^{-1} i_t' B (\beta)^{-1} u_t (\theta) ; \lambda_i \right)}.
\]
where \( \epsilon_i \) is the unit vector which is one at position \( i \) and zero otherwise, and

\[
e_{i,\lambda,t}(\theta) = \frac{\partial}{\partial \lambda_i} \log \left[ f_i \left( \sigma_i^{-1} \epsilon_i' B (\beta)^{-1} u_t (\theta) : \lambda_i \right) \right] = \frac{f_{i,\lambda} \left( \sigma_i^{-1} \epsilon_i' B (\beta)^{-1} u_t (\theta) : \lambda_i \right)}{f_i \left( \sigma_i^{-1} \epsilon_i' B (\beta)^{-1} u_t (\theta) : \lambda_i \right)},
\]

where \( f_{i,x} (x; \lambda_i) = \frac{\partial}{\partial x} f_i (x; \lambda_i) \) and \( f_{i,\lambda} (x; \lambda_i) = \frac{\partial}{\partial \lambda} f_i (x; \lambda_i) \) will be used extensively. The corresponding versions for all components are \( e_{x,t} (\theta) = (e_{1,x,t} (\theta), \ldots, e_{n,x,t} (\theta))' \) of dimension \( n \) and \( e_{\lambda,t} (\theta) = (e_{1,\lambda_1,t} (\theta), \ldots, e_{n,\lambda_n,t} (\theta))' \) of dimension \( d = d_1 + \cdots + d_n \). The notation \( \frac{\partial e_{i,\lambda,t}(\theta)}{\partial x} := \frac{\partial e_{i,\lambda,t}(\theta)}{\partial x} \big|_{\theta = \theta_0} \) is used to denote the derivative evaluated at a particular point.

### Two different ways to express the partial derivatives of \( u_t (\theta) \)

The observations may be represented at one particular point in time or as a system containing all observations \( (y_T, \ldots, y_1) \) as well as starting values \( (y_0, \ldots, y_{1-p}) \). The starting values for the process \( (u_t) \) are set to zero, i.e. \( (u_0, \ldots, u_{1-q}) = 0 \). For simplicity, we also set the starting values \( (y_0, \ldots, y_{1-p}) \) equal to zero. If clarity of presentation is not affected, we use \( x_{t-1} = (y_{t-1}, \ldots, y_{1-p})' \) of dimension \( np \) and \( u_{t-1}^{(q)} = (u_{t-1} (\theta), \ldots, u_{t-q} (\theta))' \) of dimension \( nq \) as shorthand notation.

#### One point in time.
For one particular point in time, we have

\[
u_t (\theta) = y_t - (a_1, \ldots, a_p) \begin{pmatrix} y_{t-1} \\ \vdots \\ y_{t-p} \end{pmatrix} - (b_1, \ldots, b_q) \begin{pmatrix} u_{t-1} (\theta) \\ \vdots \\ u_{t-q} (\theta) \end{pmatrix}
\]

for \( t \in \{1, \ldots, T\} \).

#### System representation.
All observations can be written as

\[
(y_T \cdots y_1 - a_1 (y_{T-1} \cdots y_0) - \cdots - a_p (y_{T-p} \cdots y_{1-p}) = (u_T (\theta) \cdots u_1 (\theta)) + b_1 (u_{T-1} (\theta) \cdots u_{0} (\theta)) + \cdots + b_q (u_{T-q} (\theta) \cdots u_{1-q} (\theta)).
\]

Defining the matrix

\[
L = \begin{pmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & 1 & \vdots \\
\vdots & 0 & \ddots & \ddots \\
0 & \cdots & \cdots & 1 \\
\end{pmatrix} \in \mathbb{R}^{T \times T}
\]
corresponding to the (non-invertible) lag operator on $\mathbb{N}$ such that

$$L \begin{pmatrix} u_T(\theta) \\ u_{T-1}(\theta) \\ \vdots \\ u_1(\theta) \end{pmatrix} = \begin{pmatrix} u_{T-1}(\theta) \\ u_{T-2}(\theta) \\ \vdots \\ 0_{1 \times n} \end{pmatrix},$$

equation (7) can be written as

$$(y_T \cdots y_1) - a_1(y_{T-1} \cdots y_0) - \cdots - a_p(y_{T-p} \cdots y_{1-p}) = (u_T(\theta) \cdots u_1(\theta)) + b_1(u_T(\theta) \cdots u_1(\theta)) L + \cdots + b_q(u_T(\theta) \cdots u_1(\theta)) (L^q)^T$$

Vectorizing equation (7) leads to

$$\begin{array}{c}
\text{vec}(y_T \cdots y_1) - a_1 \text{vec}(y_{T-1} \cdots y_0) - \cdots - a_p \text{vec}(y_{T-p} \cdots y_{1-p}) \\
= \text{vec}((u_T(\theta) \cdots u_1(\theta))) + \\
L \begin{pmatrix} u_T(\theta) \\ u_{T-1}(\theta) \\ \vdots \\ u_1(\theta) \end{pmatrix} \otimes I_1, L \begin{pmatrix} u_T(\theta) \\ u_{T-1}(\theta) \\ \vdots \\ u_1(\theta) \end{pmatrix} \otimes I_2, \ldots, \\
L \begin{pmatrix} u_T(\theta) \\ u_{T-1}(\theta) \\ \vdots \\ u_1(\theta) \end{pmatrix} \otimes I_n \\
= \left[I_{Tn} + \sum_{i=1}^q (L^i \otimes b_i) \right] \text{vec}(u_T(\theta) \cdots u_1(\theta))
\end{array}$$

where the vectorisation formula $\text{vec}(ABC) = (C^T \otimes A) \text{vec}(B)$ has been applied to $\{[I_n] [a_j] [(y_{T-j} \cdots y_{1-j})]\}$ on the left-hand-side and to $\{[b_j] [(u_T(\theta) \cdots u_1(\theta))] [(L^j)^T]\}$ and $\{[I_n] [b_j] [(u_T(\theta) \cdots u_1(\theta))] [(L^j)^T]\}$ on the right-hand-side of equation (7).

By using the (conditional maximum likelihood) assumption that $(y_0, \ldots, y_{1-p})$ be zero, we can also vectors the left-hand-side of equation (7) as

$$\text{vec}((y_T \cdots y_1) - a_1(y_T \cdots y_1) L - \cdots - a_p(y_T \cdots y_1) (L^p)^T) = \text{vec}(y_T \cdots y_1) - \sum_{j=1}^p (L^j \otimes a_j) \text{vec}(y_T \cdots y_1)$$

in order to obtain

$$B \begin{pmatrix} u_T(\theta) \\ \vdots \\ u_1(\theta) \end{pmatrix} = A \begin{pmatrix} y_T \\ \vdots \\ y_1 \end{pmatrix}$$
where

\[ A = \left[ I_{Tn} - \sum_{i=1}^{p} (L^i \otimes a_i) \right] = \begin{pmatrix}
I_n - a_1 & \cdots & -a_p & 0 & \cdots & 0 \\
0 & I_n - a_1 & \cdots & -a_p & 0 & \cdots \\
0 & 0 & I_n - a_1 & \cdots & -a_p & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & I_n - a_1 & \cdots & -a_p \\
0 & \cdots & 0 & \cdots & 0 & I_n
\end{pmatrix} \in \mathbb{R}^{Tn \times Tn} \]

and

\[ B = \left[ I_{Tn} + \sum_{i=1}^{q} (L^i \otimes b_i) \right] = \begin{pmatrix}
I_n & b_1 & \cdots & b_q & 0 & \cdots & 0 \\
0 & I_n & b_1 & \cdots & b_q & 0 & \cdots \\
0 & 0 & I_n & b_1 & \cdots & b_q & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & I_n & b_1 & \cdots & b_q \\
0 & \cdots & 0 & \cdots & 0 & I_n
\end{pmatrix} \in \mathbb{R}^{Tn \times Tn} \]

**WHF as (finite sections of) Toeplitz operators.** Similar to, e.g., Böttcher and Grudsky (2005, Chapter 1), we represent the WHF\(^{21}\) of \( b(z) = p(z)s(z)f(z) \) in terms of finite section of the corresponding Toeplitz operator. We have for \((\kappa, 0)\) that

\[ \begin{pmatrix}
I_n & b_1 & \cdots & b_q & 0 & \cdots & 0 \\
0 & I_n & b_1 & \cdots & b_q & 0 & \cdots \\
0 & 0 & I_n & b_1 & \cdots & b_q & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & I_n & b_1 & \cdots & b_q \\
0 & \cdots & 0 & \cdots & 0 & I_n
\end{pmatrix} = \begin{pmatrix}
I_n & p_1 & \cdots & p_{q-n} & 0 & \cdots & 0 \\
0 & I_n & p_1 & \cdots & p_{q-n} & 0 & \cdots \\
0 & 0 & I_n & p_1 & \cdots & p_{q-n} & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & I_n & p_1 & \cdots & p_{q-n} \\
0 & \cdots & 0 & \cdots & 0 & I_n
\end{pmatrix} = \begin{pmatrix}
I_k & 0 \\
0 & I_{n-k}
\end{pmatrix} \]

and for \((\kappa, k)\) with \( k \neq 0 \), the matrix \( S \) is replaced with \( (L \otimes I_n)^{\kappa} \cdot [(L \otimes S_{1,k}) (I_T \otimes S_{2,k})] \), where \( S_{1,k} = \begin{pmatrix} I_k & 0 \\ 0 & 0_{n-k} \end{pmatrix} \) and \( S_{2,k} = \begin{pmatrix} 0_k & 0 \\ 0 & I_{n-k} \end{pmatrix} \), and some matrices in \( B \) and \( F \) are adjusted and have zero-, one- and equality-restrictions. The matrices \( B \) and \( F \) are invertible if and only if the matrix on the diagonal is invertible. For \( B \) this is obvious, for \( F \) it holds by construction of the WHF.

\(^{21}\)Notice that we consider a left-WHF in contrast to the right-WHF analysed in Böttcher and Grudsky (2005, page 6). Therefore, the results in Böttcher and Grudsky (2005) are sometimes not directly transferable. Moreover, these authors treat the univariate case. Be that as it may, the multivariate generalisation is (for our requirements) obvious.
Moreover, it also holds that for growing sample size $T$, the inverses of $P$ and $F$ exist in the sense that the operator norms that are induced by the $l_\infty$ and the $l_1$ norm, i.e. the maximum row-sum and maximum column-sum norm, are finite. The same does not hold for $B$: While $B$ is invertible for every finite sample size $T$, the norm of the inverses diverges to infinity for sample size going to infinity! See [Böttcher and Grudsky (2005), Chapter 1.6] for a more precise statement.

While $S$ (corresponding to the backward shift in the Toeplitz representation) is not invertible, its Moore-Penrose pseudo-inverse $S^\dagger$ is equal to $(F \otimes I_T)$ where

$$
F = \begin{pmatrix} 0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1
\end{pmatrix}.
$$

The polynomial $b(z)$ can also be represented as $b(z) = f(z)g(z)$ where $g(z) = s(z)f(z)$. This will be useful when deriving analytic formulae for the score with respect to the system parameters. In this case, we have in the case $(\kappa, 0)$ that

$$
\begin{pmatrix}
i_n & b_1 & \cdots & b_q & 0 & \cdots & 0 \\
i_n & i_n & b_1 & \cdots & b_q & \cdots & 0 \\
i_0 & i_n & \cdots & 0 & b_q & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\
i_n & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & i_n & b_1 & \cdots & b_q
\end{pmatrix} = \begin{pmatrix}
i_n & p_1 & \cdots & p_{q-\kappa} & 0 & \cdots & 0 \\
i_n & i_n & b_1 & \cdots & b_q & \cdots & 0 \\
i_0 & i_n & \cdots & 0 & b_q & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\
i_n & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & i_n & b_1 & \cdots & b_q
\end{pmatrix}
\implies S = \mathcal{P}\mathcal{Q}
$$

In the case $(\kappa, k)$, there are some changes to the parameter matrices as described above and in the main text. Notice that similar to the finite sections of the Toeplitz operator corresponding to $b(z)$, the $(nT \times nT)$-dimensional matrix $\mathcal{G}$ is invertible for every $T$ but for $T$ going to infinity, the induced operator norms of the inverses diverge.

## C.2 Score of System Parameters

### C.2.1 System Parameters: Generalities

$$
\frac{\partial u_t(\theta)}{\partial \tau} = \frac{\partial}{\partial \tau} \left( \sum_{i=1}^{n} \log \left[ f_i \left( \sigma_i^{-1} \left( u_i^\prime (\theta) \right) \right) ; \lambda_i \right] \right) - \log \left( |\det (f_0)| \right) - \log \left( |\det [B(\beta)]| \right) - \frac{\sum_{i=1}^{n} \log (\sigma_i)}
$$

$$
= \frac{\partial}{\partial \tau} \left( \sum_{i=1}^{n} \log \left[ f_i \left( \sigma_i^{-1} \left( u_i^\prime (\theta) B^{-1} (\beta) u_i \right) \right) ; \lambda_i \right] \right) - \log (|\det (f_0)|)
$$

$$
= \sum_{i=1}^{n} \epsilon_{i,x,t}(\theta) \frac{\partial u_t(\theta)}{\partial \tau} \sigma_i^{-1} B^\prime (\beta^{-1} \epsilon_i) - \frac{\partial}{\partial \tau} \log (|\det (f_0)|)
$$

$$
= \frac{\partial u_t(\theta)}{\partial \tau} B^\prime (\beta^{-1} \epsilon_{i,x,t}) - \frac{1}{\det (f_0)} \frac{\partial \det (f_0)}{\partial \tau} \frac{\partial \vec{v}(f_0)}{\partial \tau} v(f_{0}^{-1})
$$

$$
= \frac{\partial u_t(\theta)}{\partial \tau} B^\prime (\beta^{-1} \epsilon_{i,x,t}) - \frac{\partial \vec{v}(f_0)}{\partial \tau} v(f_{0}^{-1})
$$

---

22 The Moore-Penrose pseudo-inverse $A^\dagger$ of a square matrix $A$ satisfies $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $(AA^\dagger)^\prime = AA^\dagger$, and $(A^\dagger A)^\prime = A^\dagger A$. 

A-10
For the derivative of the determinant, we have that \( \frac{\partial \det(Z)}{\partial z} = \text{vec} \left[ \text{adj} \left( Z \right) \right]^t \frac{\partial \text{vec}(Z)}{\partial z} \) = \( \det(Z) \text{vec} \left( Z^{t-1} \right)^t \frac{\partial \text{vec}(Z)}{\partial z} \) or equivalently \( \frac{\partial \det(Z)}{\partial z} = \det(Z) \frac{\partial \text{vec}(Z)^t}{\partial z} \text{vec} \left( Z^{t-1} \right) \), see \( \text{Seber} (2008) \) 17.26(c), page 361.

Partitioning of system parameters. Parameters pertaining to \( a(z) \), \( p(z) \), and \( f(z) \) are in \( \tau_1 \), \( \tau_2 \), \( \tau_3 \) respectively. Remember that \( u_t(\theta) = a_T(z)y_t + (I - p_T(z)s_T(z)f_T(z))B(\gamma)e_t(\theta) \)

C.2.2 AR Parameters

The derivative of \( u_t \) with respect to \( \tau_1 \) for one equation. We obtain from vectorising \([6] \) that

\[
\begin{align*}
\frac{\partial u_t(\theta)}{\partial \tau_1} = - (x_{t-1} \otimes I_n) - \left( \frac{\partial u'_{t-1}(\theta)}{\partial \tau_1}, \ldots, \frac{\partial u'_{t-q}(\theta)}{\partial \tau_1} \right) \begin{pmatrix} v_1 \\ \vdots \\ v_q \end{pmatrix}.
\end{align*}
\]

Transposition and differentiation lead to

\[
\begin{align*}
\frac{\partial u_t(\theta)}{\partial \tau_1} = - \frac{\partial u'_t(\theta)}{\partial \tau_1} = - (x_{t-1} \otimes I_n) - \left( \frac{\partial u'_{t-1}(\theta)}{\partial \tau_1}, \ldots, \frac{\partial u'_{t-q}(\theta)}{\partial \tau_1} \right) \begin{pmatrix} v_1 \\ \vdots \\ v_q \end{pmatrix}.
\end{align*}
\]

Finally, we may express \( \frac{\partial u_t(\theta)}{\partial \tau_1} \) using a lag polynomial, i.e.

\[
\begin{align*}
[p(z)z^n f(z)] \frac{\partial u_t(\theta)}{\partial \tau_1} = - \left( x'_{t-1} \otimes I_n \right) \quad \iff \quad \frac{\partial u_t(\theta)}{\partial \tau_1} = - \left[ f(z)^{-1} z^{-k} p(z)^{-1} \right] \left[ x'_{t-1} \otimes I_n \right].
\end{align*}
\]

\(^{23}\)It is irrelevant here that we use \( b(z) \) instead of its WHF because both derivatives are zero.
Note that the power series \( f(z)^{-1} = \sum_{j=0}^{\infty} h_j z^{-j} \) only depends on non-positive powers of \( z \) and that \( h_0 = f_0^{-1} \) is non-singular.

For convenience, we define the quantity

\[
x'_{b,t-1} = \left[ f(z)^{-1} e p(z)^{-1} \right] \left[ x'_{t-1} \otimes I_n \right]
\]

**Result for \( l_{\tau_1,t}(\theta) \) for one point in time.** This implies for the score that

\[
\frac{\partial l_t(\theta)}{\partial \tau_1} = -x_{b,t-1}(\theta) B'(\beta)^{-1} \Sigma^{-1} e_{x,t}(\theta)
\]

**The derivative of \( u_t \) with respect to \( \tau_1 \) for all points in time.** Rewriting equation (8) as

\[
\begin{pmatrix}
  y_T \\
  \vdots \\
  y_1
\end{pmatrix} = \left[ \begin{pmatrix}
  x'_{T-1} \\
  \vdots \\
  x'_0
\end{pmatrix} \otimes I_n \right] \text{vec}(a_1, \ldots, a_p) = \left[ I_{Tn} + \sum_{i=1}^{q} (L^i \otimes b_i) \right] \text{vec}(u_T(\theta) \cdots u_1(\theta)),
\]

transposing it and taking partial derivatives leads to

\[
\frac{\partial}{\partial \tau_1} \begin{pmatrix}
  u_T(\theta) & \cdots & u_1(\theta)
\end{pmatrix} = -\frac{\partial}{\partial \tau_1} \begin{pmatrix}
  \tau'_1 \left[ \begin{pmatrix}
  x_T & \cdots & x_1
\end{pmatrix} \otimes I_n \right] P'^{-1} (S')^\dagger F'^{-1}
\end{pmatrix} = - \left[ \begin{pmatrix}
  x_{T-1} & \cdots & x_0
\end{pmatrix} \otimes I_n \right] P'^{-1} (S')^\dagger F'^{-1}.
\]

Note that \( P'^{-1} \) is block-lower-triangular and \( F'^{-1} \) is block-upper-triangular. Their block diagonals correspond to the coefficients of the associated power series in the WHF, whose (matrix-) norms are decreasing at an exponential rate.

**Result for \( \frac{\partial L_t(\theta)}{\partial \tau_1} \).** The partial derivative of the standardized log-likelihood function with respect to \( \tau_1 \) is

\[
\frac{\partial L_t(\theta)}{\partial \tau_1} = \frac{1}{T} \sum_{i=1}^{T} l_{\tau_1,t}(\theta) = -\frac{1}{T} \left[ \begin{pmatrix}
  x_{T-1} & \cdots & x_0
\end{pmatrix} \otimes I_n \right] P'^{-1} (S')^\dagger F'^{-1} \begin{pmatrix}
  I_T \otimes \Sigma^{-1} B'(\beta)^{-1} & e_{x,T}(\theta) \\
  \vdots & e_{x,1}(\theta)
\end{pmatrix}.
\]
C.2.3 "Stable" MA Parameters

We consider the case \((\kappa, 0)\) for \(\tau_2\) such that the free parameters are in \(\tau_2 = \text{vec}(p_1, \ldots, p_{q-k})\). Taking the partial derivative with respect to \(\tau_2'\) of

\[
a(z) g_{t-1} = \left[p(z) g(z)\right] u_1(\theta)
\]

\[
= (I_n, p_1, \ldots, p_{q-k}) (I_{q-k+1} \otimes g(z))
\]

\[
= (I_n, p_1, \ldots, p_{q-k})
\]

\[
= v_t(\theta) + (p_1, \ldots, p_{q-k})
\]

and obtain

\[
\frac{\partial v_t(\theta)}{\partial \tau_2'} = - (p_1, \ldots, p_{q-k})
\]

\[
\iff p(z) \frac{\partial v_t(\theta)}{\partial \tau_2'} = - \left(\begin{array}{c} v_{t-1}(\theta) \\ \vdots \\ v_{t-(q-k)}(\theta) \end{array}\right) \otimes I_n.
\]

We define \(^{24}\)

\[
w_{g,t-1} = \left(\begin{array}{c} v_{t-1}(\theta) \\ \vdots \\ v_{t-(q-k)}(\theta) \end{array}\right) \otimes I_n.
\]

and obtain

\[
(I_n + b_1 z + \cdots + b_q z^q) \frac{\partial u_1(\theta)}{\partial \tau_2'} = - [w_{g,t-1}' (\theta) \otimes I_n]
\]

\[
\iff \frac{\partial u_t(\theta)}{\partial \tau_2'} = - f(z)^{-1} z^{-\kappa} p(z)^{-1} [w_{g,t-1}' (\theta) \otimes I_n].
\]

\(^{24}\)Note that \(w_{g,t-1}' = (g(z) u_{1,t-1}, \ldots, g(z) u_{n,t-1} | \ldots | g(z) u_{1,t-(q-k)}, \ldots, g(z) u_{n,t-(q-k)})\).
All Equations. Rewriting equation (3) as

\[
\begin{pmatrix}
  u_T(\theta) \\
  \vdots \\
  u_1(\theta)
\end{pmatrix}
= (I_nT - \frac{\partial}{\partial \tau} \mathcal{P} \mathcal{G})
\begin{pmatrix}
  u_T(\theta) \\
  \vdots \\
  u_1(\theta)
\end{pmatrix}
+ \mathcal{A}
\begin{pmatrix}
  y_T \\
  \vdots \\
  y_1
\end{pmatrix}
= \mathcal{P}
\begin{pmatrix}
  v_T(\theta) \\
  \vdots \\
  v_1(\theta)
\end{pmatrix}
+ \mathcal{A}
\begin{pmatrix}
  y_T \\
  \vdots \\
  y_1
\end{pmatrix}
\]

transposing and taking derivatives leads to

\[
\frac{\partial}{\partial \tau} \begin{pmatrix}
  v_T'(\theta) \\
  \vdots \\
  v_1'(\theta)
\end{pmatrix}
= \mathcal{B}' = \mathcal{F} \mathcal{G}' \mathcal{P}'
\]

which in turn is equivalent to

\[
\frac{\partial}{\partial \tau} \begin{pmatrix}
  v_T'(\theta) \\
  \vdots \\
  v_1'(\theta)
\end{pmatrix}
= \begin{pmatrix}
  v_{T-1}(\theta) & \cdots & v_0(\theta) \\
  \vdots & \ddots & \vdots \\
  v_{T-(q-\kappa)}(\theta) & \cdots & v_{1-(q-\kappa)}(\theta)
\end{pmatrix}
\bigotimes I_n
\]

Result for \( \frac{\partial L_t(\theta)}{\partial \tau_2} \). Finally, we obtain for the partial derivative of the standardised log-likelihood function with respect to \( \tau_2 \) that

\[
\frac{\partial L_t(\theta)}{\partial \tau_2} = \frac{1}{T} \sum_{i=1}^{T} l_{\tau_2,t}(\theta)
\]

\[
= \frac{1}{T} \left[ \begin{pmatrix}
  w_{g,T-1}(\theta) & \cdots & w_{g,0}(\theta)
\end{pmatrix} \bigotimes I_n \right] \mathcal{P}'^{-1} \mathcal{S}' \mathcal{F}'^{-1}
\begin{pmatrix}
  e_{x,T}(\theta) \\
  \vdots \\
  e_{x,1}(\theta)
\end{pmatrix}
\]
C.2.4 “Unstable” MA Parameters

Similarly, we consider the case \((\kappa, 0)\) for \(\tau_3\) such that the free parameters are in \(\tau_3 = vec\ (g_1, \ldots, g_n)\). Taking the partial derivative with respect to \(\tau_3\) of

\[
a(z) y_t = [p(z)g(z)]u_t(\theta)
\]

\[
= p(z) (I_n, g_1, \ldots, g_n) \begin{pmatrix} u_t(\theta) \\ u_{t-1}(\theta) \\ \vdots \\ u_{t-\kappa}(\theta) \end{pmatrix}
\]

\[
= p(z) (g_1, \ldots, g_n) \begin{pmatrix} u_{t-1}(\theta) \\ \vdots \\ u_{t-\kappa}(\theta) \end{pmatrix} + p(z)u_t
\]

and obtain

\[
-p(z) \frac{\partial u_t(\theta)}{\partial \tau_3} = \left( \begin{pmatrix} u_{t-1}(\theta) \\ \vdots \\ u_{t-\kappa}(\theta) \end{pmatrix} \otimes p(z) \right) \frac{\partial \tau_3}{\partial \tau_3} + p(z) (g_1, \ldots, g_n) \frac{\partial u_{t-\kappa}(\theta)}{\partial \tau_3}
\]

\[
\iff p(z) g(z) \frac{\partial u_t(\theta)}{\partial \tau_3} = -\left( \begin{pmatrix} u_{t-1}(\theta) \\ \vdots \\ u_{t-\kappa}(\theta) \end{pmatrix} \otimes p(z) \right).
\]

We define

\[
u_{p,t-1}' = \begin{pmatrix} u_{t-1}(\theta) \\ \vdots \\ u_{t-\kappa}(\theta) \end{pmatrix} \otimes p(z).
\]

and obtain

\[
(I_n + b_1 z + \cdots + b_q z^q) \frac{\partial u_t(\theta)}{\partial \tau_3} = -\left[ \nu_{p,t-1}' (\theta) \otimes I_n \right]
\]

\[
\iff \frac{\partial u_t(\theta)}{\partial \tau_3} = -f(z)^{-1} z^{-\kappa} p(z)^{-1} \left[ \nu_{p,t-1}' (\theta) \otimes I_n \right],
\]

\[
= -f(z)^{-1} \left[ [u_{t+\kappa-1}' (\theta), \ldots, u_t'(\theta)] \otimes I_n \right]
\]

\[25\text{Similar to the partial derivative with respect to } \tau_2, \text{ we see that } \nu_{p,t-1}' = (p(z)u_{1,t-1}, \ldots, p(z)u_{n,t-1}) \cdots (p(z)u_{1,t-\kappa}, \ldots, p(z)u_{n,t-\kappa}).\]
All Equations. Rewriting equation (3) as

\[
\begin{align*}
\mathcal{G} \begin{pmatrix} w_T(\theta) \\ \vdots \\ u_1(\theta) \end{pmatrix} &= \mathcal{P}^{-1} \mathcal{A} \begin{pmatrix} y_T \\ \vdots \\ y_1 \end{pmatrix} \\
\iff \begin{pmatrix} w_T(\theta) \\ \vdots \\ u_1(\theta) \end{pmatrix} &= (I_H - \mathcal{G}) \begin{pmatrix} w_T(\theta) \\ \vdots \\ u_1(\theta) \end{pmatrix} - \mathcal{P}^{-1} \mathcal{A} \begin{pmatrix} y_T \\ \vdots \\ y_1 \end{pmatrix}
\end{align*}
\]

transposing and taking derivatives leads to

\[
\frac{\partial (w_T'(\theta), \ldots, u_1'(\theta))}{\partial \tau_3} \mathcal{G}' = - \begin{pmatrix} w_{T-1}(\theta) & \cdots & u_0(\theta) \\ \vdots & \ddots & \vdots \\ w_{T-\kappa}(\theta) & \cdots & u_{1-\kappa}(\theta) \end{pmatrix} \otimes I_n \mathcal{P}'
\]

which in turn is equivalent to

\[
\frac{\partial (w_T(\theta), \ldots, u_1'(\theta))}{\partial \tau_3} = - \begin{pmatrix} w_{T-1}(\theta) & \cdots & u_0(\theta) \\ \vdots & \ddots & \vdots \\ w_{T-\kappa}(\theta) & \cdots & u_{1-\kappa}(\theta) \end{pmatrix} \otimes I_n S^t \mathcal{F}^{t-1}.
\]

Result for $\frac{\partial L_2(\theta)}{\partial \tau_3}$. Finally, we obtain for the partial derivative of the standardized log-likelihood function with respect to $\tau_2$ that

\[
\frac{\partial L_2(\theta)}{\partial \tau_3} = \frac{1}{T} \sum_{i=1}^T l_{\tau_2,i}(\theta)
\]

\[
= -\frac{1}{T} \left[ \begin{pmatrix} w_{T-1}(\theta) & \cdots & u_0(\theta) \end{pmatrix} \otimes I_n \right] S^t \mathcal{F}^{t-1} \begin{pmatrix} I_T \otimes \Sigma^{-1} B'(\beta)^{-1} \end{pmatrix} \left[ \begin{pmatrix} e_{x,T}(\theta) \\ \vdots \\ e_{x,1}(\theta) \end{pmatrix} \right] - \frac{\partial vec(f_0)}{\partial \tau_3} vec(f_0^{t-1})
\]

where $w_{T-1}'(\theta) = (w_{T-1}'(\theta), \ldots, w_{T-\kappa}'(\theta))$. 
C.3 Score of Noise Parameters

C.3.1 Partial Derivative with respect to β

By taking the derivative of \( \mathbf{I} \), we obtain for \( \beta \in \mathbb{R}^{n(n-1)} \):

\[
\frac{\partial t_1(\theta)}{\partial \beta} = \sum_{i=1}^{n} e_{i,x,t}(\theta) \frac{\partial}{\partial \beta} \left( \frac{1}{2} u_i'(\theta) B(\beta)^{-1} e_i + \frac{1}{2} \text{vec} \left( i'_x B(\beta)^{-1} u_i(\theta) \right) \right) - \frac{1}{\det(B(\beta))} \frac{\partial \det(B(\beta))}{\partial \beta}
\]

where we used again that the derivative of the determinant is

\[
\frac{\partial \det(Z)}{\partial \beta} = \text{vec} \left[ \text{adj}(Z)' \right] \frac{\partial \text{vec}(Z)}{\partial \beta} = \det(Z) \frac{\partial \text{vec}(Z)}{\partial \beta}
\]

or equivalently \( \frac{\partial \det(Z)}{\partial \beta} = \det(Z) \frac{\partial \text{vec}(Z)}{\partial \beta} \) \cite[Seber, 2008, 17.26(c), page 361]{Seber}. Moreover, we have that \( \text{vec}(B(\beta)) = H \beta + \text{vec}(I_n) \) and thus \( \frac{\partial \text{vec}(B(\beta))}{\partial \beta} = H \).

We obtain from \cite[Seber, 2008]{Seber} 17.33(b), page 363, that

\[
\frac{\partial \text{vec}(F^{-1})}{\partial x'} = - (F^{-1} \otimes F^{-1}) \frac{\partial \text{vec}(F)}{\partial x'} \quad \text{and} \quad \frac{\partial \text{vec}(F^{-1})'}{\partial x} = \frac{\partial \text{vec}(F)'}{\partial x} (F^{-1} \otimes F'^{-1}),
\]

which leads to

\[
\frac{\partial t_1(\theta)}{\partial \beta} = \sum_{i=1}^{n} e_{i,x,t}(\theta) \frac{\partial}{\partial \beta} \left( u_i'(\theta) B(\beta)^{-1} e_i + \frac{1}{2} \text{vec} \left( i'_x B(\beta)^{-1} u_i(\theta) \right) \right) - \frac{1}{\det(B(\beta))} \frac{\partial \det(B(\beta))}{\partial \beta}
\]

\[
= \left( \frac{\partial u_i'(\theta)}{\partial \beta} \right) B(\beta)^{-1} \Sigma^{-1} e_{i,x,t}(\theta) \quad \text{and} \quad \frac{\partial \text{vec}(B(\beta)^{-1})}{\partial \beta} = \frac{\partial \text{vec}(B(\beta))}{\partial \beta} \frac{\partial B(\beta)}{\partial \beta}
\]

\[
= \left( \frac{\partial u_i'(\theta)}{\partial \beta} \right) B(\beta)^{-1} \Sigma^{-1} e_{i,x,t}(\theta) - H' \left( B(\beta)^{-1} \otimes B(\beta)^{-1} \right) \left( u_i(\theta) \otimes \Sigma^{-1} e_{i,x,t}(\theta) \right) - H' \text{vec} \left( B(\beta)^{-1} \right)
\]

\[
= \left( \frac{\partial u_i'(\theta)}{\partial \beta} \right) B(\beta)^{-1} \Sigma^{-1} e_{i,x,t}(\theta) - H' \left( B(\beta)^{-1} \otimes B(\beta)^{-1} \Sigma^{-1} e_{i,x,t}(\theta) \right) - H' \text{vec} \left( B(\beta)^{-1} \right)
\]

\[(9)
\]

The derivative of \( u_t \) with respect to \( \beta \) for one equation. From

\[
u_t(\theta) = y_t - (a_1, \ldots, a_p) \begin{pmatrix} y_{t-1} \\ \vdots \\ y_{t-p} \end{pmatrix} - (b_1, \ldots, b_q) \begin{pmatrix} u_{t-1}(\theta) \\ \vdots \\ u_{t-q}(\theta) \end{pmatrix}
\]

we obtain immediately

\[
\frac{\partial u_t'(\theta)}{\partial \beta} = - \left( \frac{\partial u_{t-1}'(\theta)}{\partial \beta} \otimes \cdots \otimes \frac{\partial u_{t-q}'(\theta)}{\partial \beta} \right) \begin{pmatrix} b_1' \\ \vdots \\ b_q' \end{pmatrix}
\]

\footnote{This result can be obtained by taking the derivative of \( FF^{-1} = I \) such that we obtain \( F \frac{\partial F^{-1}}{\partial x} + \frac{\partial F}{\partial x} F^{-1} = 0 \). Vectorization of \( \frac{\partial F^{-1}}{\partial x} \) gives the desired result, see \cite{Harville} 1997 page 366.}
Additionally, an explicit expression for the derivative of $u_t(\theta) = B(\beta) \varepsilon_t(\theta) = (\varepsilon'_t(\theta) \otimes I_n)$ vec ($B(\beta)$) with respect to $\beta$ can be found as $\frac{\partial u'_t(\theta)}{\partial \beta} = H' (\varepsilon_t(\theta) \otimes I_n)$ and subsequently combined with the quantity above. We thus obtain

\[
\frac{\partial u'_t(\theta)}{\partial \beta} = - H' \left[ (\varepsilon_{t-1}(\theta) \otimes I_n), \ldots, (\varepsilon_{t-q}(\theta) \otimes I_n) \right] \begin{pmatrix} b'_1 \\ \vdots \\ b'_q \end{pmatrix}
\]

\[
= - H' \left[ (\varepsilon_{t-1}(\theta), \ldots, \varepsilon_{t-q}(\theta)) \otimes I_n \right] \begin{pmatrix} b'_1 \\ \vdots \\ b'_q \end{pmatrix}
\]

\[
= - H' \sum_{i=1}^{q} (\varepsilon_{t-i}(\theta) \otimes b'_i) = - H' \sum_{i=1}^{q} \left( B(\beta)^{-1} u_{t-i}(\theta) \otimes b'_i \right).
\]

Result for $l_{\beta,t}(\theta)$ for one point in time. The above leads to

\[
\frac{\partial l_t(\theta)}{\partial \beta} = \left( \frac{\partial u'_t(\theta)}{\partial \beta} \right) B'(\beta)^{-1} \Sigma^{-1}_{x,t}(\theta) - H' \left( B(\beta)^{-1} u_t(\theta) \otimes B'(\beta)^{-1} \Sigma^{-1}_{x,t}(\theta) \right) - H' \text{vec} \left( B'(\beta)^{-1} \right)
\]

\[
= - \left( \frac{\partial u'_t(\theta)}{\partial \beta} \right) B'(\beta)^{-1} \Sigma^{-1}_{x,t}(\theta) - H' \left( B(\beta)^{-1} u_t(\theta) \otimes B'(\beta)^{-1} \Sigma^{-1}_{x,t}(\theta) \right) - H' \text{vec} \left( B'(\beta)^{-1} \right)
\]

\[
= - H' \sum_{i=1}^{q} \left( B(\beta)^{-1} u_{t-i}(\theta) \otimes b'_i \right) B'(\beta)^{-1} \Sigma^{-1}_{x,t}(\theta) - H' \left( B(\beta)^{-1} u_t(\theta) \otimes B'(\beta)^{-1} \Sigma^{-1}_{x,t}(\theta) \right) - H' \text{vec} \left( B'(\beta)^{-1} \right)
\]

\[
= - H' \left[ B(\beta)^{-1} \otimes \left( I_n \quad b'_1 \quad \ldots \quad b'_q \right) \right] \sum_{i=1}^{q} \left( \begin{pmatrix} u_t(\theta) \\ u_{t-1}(\theta) \\ \vdots \\ u_{t-q}(\theta) \end{pmatrix} \otimes \left( B'(\beta)^{-1} \Sigma^{-1}_{x,t}(\theta) \right) \right) - H' \text{vec} \left( B'(\beta)^{-1} \right)
\]

Result for $\frac{\partial L_t(\theta)}{\partial \beta}$. Finally, we obtain for the partial derivative of the standardised log-likelihood function with respect to $\beta$ that

\[
\frac{\partial L_t(\theta)}{\partial \beta} = \frac{1}{T} \sum_{t=1}^{T} l_{\beta,t}(\theta)
\]

\[
= - \frac{1}{T} H' \left[ B(\beta)^{-1} \otimes \left( I_n \quad b'_1 \quad \ldots \quad b'_q \right) \right] \sum_{i=1}^{q} \left( \begin{pmatrix} u_t(\theta) \\ u_{t-1}(\theta) \\ \vdots \\ u_{t-q}(\theta) \end{pmatrix} \otimes \left( B'(\beta)^{-1} \Sigma^{-1}_{x,t}(\theta) \right) \right) - H' \text{vec} \left( B'(\beta)^{-1} \right)
\]

C.3.2 Partial Derivative with respect to $\sigma$

Since the individual contribution to the (standardised) log-likelihood function is

\[
l_t(\theta) = \sum_{i=1}^{n} \log \left[ f_i \left( \sigma_i^{-1} e_t B(\beta)^{-1} u_t(\theta) ; \lambda_i \right) \right] - \log \left( \det \left[ B(\beta) \right] \right) - \sum_{i=1}^{n} \log \left( \sigma_i \right),
\]
we obtain that

\[
\frac{\partial}{\partial \sigma} l_t(\theta) = \sum_{i=1}^{n} e_{i,x,t}(\theta) \left( -\iota_i \sigma_i^{-2} \right) \iota_i B(\beta)^{-1} u_t(\theta) - \sum_{i=1}^{n} \iota_i \sigma_i^{-1} \\
= -\sum_{i=1}^{n} \sigma_i^{-2} \left( \iota_i \iota_i' \right) e_{i,x,t}(\theta) e_t(\theta) - \Sigma^{-2} \sigma \\
= -\Sigma^{-2} [e_{x,t}(\theta) \odot e_t(\theta) + \sigma]
\]

where \( \odot \) denotes element-wise multiplication. The partial derivative of \( l_t(\theta) \) with respect to \( \sigma \) is thus identical to the one derived in [Lanne et al. (2017)](#).

**Result for \( \frac{\partial L_t(\theta)}{\partial \sigma} \).** Finally, we obtain for the partial derivative of the standardised log-likelihood function with respect to \( \beta \) that

\[
\frac{\partial L_t(\theta)}{\partial \sigma} = \frac{1}{T} \sum_{t=1}^{T} l_{\sigma,t}(\theta) \\
= -\frac{1}{T} \Sigma^{-2} \left( \sum_{t=1}^{T} e_{x,t}(\theta) \odot e_t(\theta) \right) - \left( \begin{array}{c} \sigma_1^{-1} \\ \vdots \\ \sigma_n^{-1} \end{array} \right)
\]

**C.3.3 Partial Derivative with respect to \( \lambda \)**

Analogous to \( l_{\sigma,t}(\theta) \), the partial derivative of \( l_t(\theta) \) with respect to \( \lambda \) is identical to the one derived in [Lanne et al. (2017)](#), i.e. \( \frac{\partial}{\partial \lambda_i} l_t(\theta) = e_{i,\lambda_i,t} \) for all \( i \).