CONTINUANTS, RUN LENGTHS, AND BARRY’S MODIFIED PASCAL TRIANGLE

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Abstract. We show that the \( n \)’th diagonal sum of Barry’s modified Pascal triangle can be described as the continuant of the run lengths of the binary representation of \( n \). We also obtain an explicit description for the row sums.

1. Introduction

In 2006 in the On-Line Encyclopedia of Integer Sequences (OEIS) [8], sequence A119326, Paul Barry introduced a modified Pascal triangle, defined for integers \( 0 \leq k \leq n \), as follows:

\[
T(n, k) = \sum_{0 \leq j \leq n-k \atop 2 \mid j} \binom{k}{j} \binom{n-k}{j}.
\]

The first few rows of this triangle are as follows:

1

1 1

1 1 1 1

1 1 2 1 1

1 1 4 4 1 1

1 1 7 10 7 1 1

1 1 11 19 19 11 1 1

1 1 16 31 38 31 16 1 1

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Similarly, one can consider $T(n, k) \mod 2$, whose terms are given by sequence A114213:

\[
\begin{array}{cccccccccc}
1 & & & & & & & \\
1 & 1 & 1 & & & & & \\
1 & 1 & 1 & 1 & & & & & \\
1 & 1 & 0 & 1 & 1 & & & & & \\
1 & 1 & 0 & 0 & 1 & 1 & & & & & \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & & \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \\
\end{array}
\]

Sequences A114212 and A114214, respectively, are the row sums and diagonal sums of this latter triangle. We denote them by $r(n)$ and $d(n)$, respectively:

\[
\begin{align*}
r(n) &= \sum_{k=0}^{n} (T(n, k) \mod 2) \\
d(n) &= \sum_{k=0}^{\lfloor n/2 \rfloor} (T(n - k, k) \mod 2).
\end{align*}
\]

In May 2016, the first author observed, empirically, a connection between $d(n)$ and the binary representation of $n$. In this note we prove this connection, and also prove a formula for $r(n)$. The connection involves Stern’s “diatomic sequence” $s(n)$, defined by $s(0) = 0$, $s(1) = 1$, $s(2n) = s(n)$, and $s(2n + 1) = s(n) + s(n + 1)$; see [9].

2. The diagonal sums

Let the binary representation of $n$ be denoted by $\sum_{i=0}^{j} \varepsilon_i(n)2^i$. We show that the diagonal sum $d(n)$ can be expressed in terms of this representation. Given a string $s$ of 0’s and 1’s, we consider its run lengths: the lengths of maximal blocks of consecutive identical elements. For example, if $s = 111000011111$, then the run lengths of $s$ are $(3, 4, 5)$.

If $m$ is a sequence of positive integers, we may associate an integer with it via the continued fraction expansion: if $m = (m_0, \ldots, m_k)$, we say that the continuant of $m$ is the numerator of the continued fraction $[m_0; m_1, \ldots, m_k]$ (see [3, Ch. 34, §4]).

**Theorem 2.1.** Let $n \geq 0$ be an integer and let $m$ be the sequence of run lengths of the binary representation of $n$. Then $d(n)$ equals the continuant of $m$.

We will use Lucas’ famous congruence for binomial coefficients [7, p. 230]: if $p$ is a prime number and $n = (n_\nu \cdots n_0)p$ and $k = (k_\nu \cdots k_0)p$, then

\[
\binom{n}{k} \equiv \left(\frac{n_\nu}{k_\nu} \cdots \frac{n_0}{k_0}\right) \pmod{p}.
\]

This implies that $\binom{n}{k}$ is not divisible by $p$ if and only if $k_i \leq n_i$ for all $i$. Moreover, it follows that the number of odd binomial coefficients $\binom{n}{k}$ equals $2^{s_2(n)}$, where $s_2$ is the binary sum-of-digits function [4].

We prove the following statement, which reduces the problem to divisibility by 2 of binomial coefficients. We will derive Theorem 2.1 from it in a moment.

**Proposition 2.2.** Let $n$ and $k$ be nonnegative integers such that $k \leq n$. If $2 \mid n + k$, then $T(n, k) \equiv \binom{n}{k} \pmod{2}$. Otherwise, $T(n, k) \equiv \binom{n-1}{k} \pmod{2}$.
By Proposition 2.2 and Eq. (2) we therefore have

\[ \binom{n}{j}^j \equiv \binom{n + k}{k} \quad (\mod 2). \]

By Lucas’ congruence the left-hand side is congruent to

\[ \sum_{j=0}^{n} \binom{n}{j}^j \equiv \sum_{j=0}^{n} \binom{n \land k}{j} \equiv 2^{s_2(n \land k)} \quad (\mod 2), \]

where \( n \land k \) is the integer whose binary digits satisfy \( \varepsilon_i(n \land k) = \min(\varepsilon_i(n), \varepsilon_i(k)) \). This expression is odd if and only if \( s_2(n \land k) = 0 \), which is the case if and only if the binary representations of \( n \) and \( k \) are disjoint. To handle the right-hand side of Eq. (2), we note that \( \binom{n + k}{k} \) is odd if \( n \land k = 1 \). On the other hand, if the binary representations of \( n \) and \( k \) are not disjoint, then the condition \( \varepsilon_i(k) \leq \varepsilon_i(n + k) \) is violated for \( i = \min\{j : \varepsilon_j(n) = 1, \varepsilon_j(k) = 1\} \); therefore \( \binom{n + k}{k} \) is even. This proves the first assertion.

For the second assertion, we use Lucas’ congruence again: for \( 2 \nmid j \) and \( 2 \nmid m \) we have \( \binom{m}{j} \equiv \binom{m + 1}{j} \) (\mod 2). Since \( 2 \nmid n - k \), we obtain \( \binom{n - k}{j} \equiv \binom{n - 1 - k}{j} \) (\mod 2). Moreover, by \( 2 \nmid n - k \) the ranges of summation in \( T(n, k) \) and \( T(n - 1, k) \) are the same. \( \square \)

From this proposition we obtain in particular the identity

\[ d(2n) = d(2n + 1). \]

Carlitz [2] proved that Stern’s diatomic sequence \( s(n) \) satisfies

\[ s(n + 1) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n - k}{k} \quad (\mod 2). \]

By Proposition 2.2 and Eq. (2) we therefore have

\[ d(2n) = d(2n + 1) = s(2n + 1). \]

It is well-known [5, 6] that if \( m = (m_0, \ldots, m_k) \) is the sequence of run-lengths of the binary representation of \( n \) and \( n \) is odd, then \( s(n) \) is the continuant of \( m \). Therefore \( d(n) \) is the continuant of \( m \). In order to complete the proof of the conjecture, we have to show that the same is true for even \( n \). By Eq. (3) it is sufficient to prove the following lemma.

**Lemma 2.3.** If \( n \) is even, then the continuant of the sequence of run-lengths of the binary representation of \( n \) is equal to the continuant corresponding to \( n + 1 \).

**Proof.** Let \( n = 1^{m_0}0^{m_1} \cdots 1^{m_{k-1}}0^{m_k} \). We distinguish between two cases. If \( m_k = 1 \), then \( n + 1 = (1^{m_0}0^{m_1} \cdots 0^{m_k-2}1^{m_{k-1}+1}) \) and the statement follows from the identity \( [m_0; m_1, \ldots, m_{k-1}, 1] = [m_0; m_1, \ldots, m_{k-1} + 1] \). If \( m_k \geq 2 \), then \( n + 1 = (1^{m_0}0^{m_1} \cdots 0^{m_{k-2}}1^{m_{k-1}}0^{m_k-1}) \) and the statement follows from \( [m_0; m_1, \ldots, m_k] = [m_0; m_1, \ldots, m_{k-1}, m_k - 1, 1] \). \( \square \)

**Remark.** The sequence \( (d(n))_{n \geq 0} \) is a 2-regular sequence [1], as it satisfies the equalities

\[
\begin{align*}
d(2n + 1) &= d(2n) \\
d(4n + 2) &= 3d(2n) - d(4n) \\
d(8n) &= -d(2n) + 2d(4n) \\
d(8n + 4) &= 4d(2n) - d(4n).
\end{align*}
\]
3. The row sums

We will prove

**Theorem 3.1.**

\[ r(n) = \begin{cases} 
2^{s_2(n)}, & \text{if } n \text{ odd;} \\
2^{s_2(n)} + 2^{s_2(n-2)}, & \text{if } n \text{ even.}
\end{cases} \]

A similar characterization was stated, without proof or attribution, in the notes to A114212 of the OEIS.

**Proof.** From Proposition 2.2 we get, for integers \( n \geq k \geq 0 \), that

\[ T(2n, 2k) \equiv T(2n + 1, 2k) \equiv T(2n + 1, 2k + 1) \equiv \binom{n}{k} \pmod{2}; \]

\[ T(2n, 2k + 1) \equiv \binom{n-1}{k} \pmod{2}. \]

Then

\[
\begin{align*}
r(2m) &= \sum_{k=0}^{2m} (T(2m, k) \mod 2) \\
&= \sum_{k=0}^{m} (T(2m, 2k) \mod 2) + \sum_{k=0}^{m-1} (T(2m, 2k + 1) \mod 2) \\
&= \sum_{k=0}^{m} \left( \binom{m}{k} \mod 2 \right) + \sum_{k=0}^{m-1} \left( \binom{m-1}{k} \mod 2 \right) \\
&= 2^{s_2(m)} + 2^{s_2(m-1)} \\
&= 2^{s_2(2m)} + 2^{s_2(2m-2)}.
\end{align*}
\]

Similarly,

\[
\begin{align*}
r(2m + 1) &= \sum_{k=0}^{2m+1} (T(2m + 1, k) \mod 2) \\
&= \sum_{k=0}^{m} (T(2m + 1, 2k) \mod 2) + \sum_{k=0}^{m} (T(2m + 1, 2k + 1) \mod 2) \\
&= \sum_{k=0}^{m} \left( \binom{m}{k} \mod 2 \right) + \sum_{k=0}^{m} \left( \binom{m}{k} \mod 2 \right) \\
&= 2^{s_2(m)} + 2^{s_2(m)} \\
&= 2^{s_2(2m+1)}.
\end{align*}
\]

□

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