Weakly linearly Lindelöf
monotonically normal spaces
are Lindelöf

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Abstract

We call a space $X$ weakly linearly Lindelöf if for any family $\mathcal{U}$ of non-empty open subsets of $X$ of regular uncountable cardinality $\kappa$, there exists a point $x \in X$ such that every neighborhood of $x$ meets $\kappa$-many elements of $\mathcal{U}$. We also introduce the concept of almost discretely Lindelöf spaces as the ones in which every discrete subspace can be covered by a Lindelöf subspace. We prove that, in addition to linearly Lindelöf spaces, both weakly Lindelöf spaces and almost discretely Lindelöf spaces are weakly linearly Lindelöf.

The main result of the paper is formulated in the title. It implies, among other things, that every weakly Lindelöf monotonically normal space is Lindelöf; this result seems to be new even for linearly ordered topological spaces.

We show that, under the hypothesis $2^\omega < \omega_\omega$, if the co-diagonal $\Delta^c_X = (X \times X) \setminus \Delta_X$ of a space $X$ is discretely Lindelöf, then $X$ is Lindelöf and has a weaker second countable topology; here $\Delta_X = \{(x, x) : x \in X\}$ is the diagonal of the space $X$. Moreover, the discrete Lindelöfness of $\Delta^c_X$ together with the Lindelöf $\Sigma$-property of $X$ imply that $X$ has a countable network.

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1 Introduction

The closures of discrete sets determine quite a few topological properties of a space $X$. For example, if $D$ is compact for any discrete $D \subset X$, then $X$ is compact [17]. If $D$ is linearly (hereditarily) Lindelöf for each discrete subset $D \subset X$, then $X$ is linearly (hereditarily) Lindelöf as well [2]. In case when $X$ is compact, countable character of the closures of all discrete subsets of $X$ implies that $\chi(X) \leq \omega$; this was proved in [2].

If $P$ is a topological property, it is said that a space $X$ is discretely $P$ if $D$ has $P$ for any discrete set $D \subset X$. Thus, every discretely compact space is compact. However, it is an open problem of Arhangel’skii [4] whether every discretely Lindelöf space is Lindelöf. It is easy to see that a linearly Lindelöf space $X$ must be Lindelöf if $l(X) < \omega\omega$. Besides, it is a result of Arhangel’skii and Buzyakova [5] that any discretely Lindelöf space of countable tightness is Lindelöf.

Since there is still a possibility that not all discretely Lindelöf spaces are Lindelöf, a natural line of research is to find out in which classes discrete Lindelöfness implies Lindelöfness and to try to prove for discretely Lindelöf spaces the classical results known for Lindelöf ones. In this spirit, it was proved in [22] that every discretely Lindelöf monotonically normal space is Lindelöf. If $X$ is a Tychonoff space and $\Delta_X = \{(x,x) : x \in X\}$ is its diagonal, then the Lindelöf property of the set $\Delta^c_X = (X \times X) \setminus \Delta_X$ implies that $X$ is Lindelöf if $i\nu(X) \leq \omega$, i.e., $X$ has a weaker second countable topology (see [3, Theorem 2.1.8]). Clearly, it would be interesting to prove the same for spaces $X$ such that $\Delta^c_X$ is discretely Lindelöf. The respective open questions were formulated in [1] and [7]. It is also worth mentioning that it is an open question (attributed in [16] to Arhangel’skii and Buzyakova) whether linear Lindelöfness of $\Delta^c_X$ for a compact $X$ implies that $X$ is metrizable.

Burke and Tkachuk established in [7] that for any countably compact space $X$, discrete Lindelöfness of $\Delta^c_X$ implies that $X$ is compact and metrizable. It was asked in [1] whether the same is true if the space $X$ is pseudocompact and $\Delta^c_X$ is discretely $\sigma$-compact. In this paper we show that discrete $\sigma$-compactness of $\Delta^c_X$ implies $hl(X \times X) \leq \omega$: it is easy to deduce from this fact that the answers to Questions 5.8 and 5.9 of the paper [1] are positive. We show that, under $2^\omega < \omega\omega$, any Tychonoff space $X$ such that $\Delta^c_X$ is discretely Lindelöf must be Lindelöf and has countable $i$-weight. Besides it is true in ZFC that discrete Lindelöfness of $\Delta^c_X$ implies that $X$ has a small diagonal. In particular, if $X$ is a Lindelöf $\Sigma$-space and $\Delta^c_X$ is discretely Lindelöf, then the space $X$ is cosmic; this answers Problem 4.6 from the paper [7].

We also introduce the classes of almost discretely Lindelöf spaces and weakly linearly Lindelöf spaces. It turns out that these classes have nice properties; besides, any weakly linearly Lindelöf and monotonically normal space is Lindelöf. This result, which we consider to be interesting in itself, seems to be new even for weakly Lindelöf linearly ordered topological spaces.
2 Notation and terminology

All spaces are assumed to be $T_1$. Given a space $X$, the family $\tau(X)$ is its topology and $\tau^*(X) = \tau(X) \setminus \{\emptyset\}$; besides, $\tau(x, X) = \{U \in \tau(X) : x \in U\}$ for any $x \in X$; if $A \subseteq X$ then $\tau(A, X) = \{U \in \tau(X) : A \subseteq U\}$. All ordinals are identified with the set of their predecessors and are assumed to carry the order topology. We denote by $\tau$ the cardinal $2^{\omega}$, by $\mathbb{D}$ the set $\{0, 1\}$ with the discrete topology and $\mathbb{N} = \omega \setminus \{0\}$. If $X$ is a space then $\Delta_X = \{(x, x) : x \in X\} \subseteq X \times X$ is the diagonal of $X$. A space $X$ is said to have a small diagonal if for any uncountable set $A \subseteq \Delta_X = (X \times X) \setminus \Delta_X$, there exists an uncountable $B \subseteq A$ such that $\overline{B} \cap \Delta_X = \emptyset$.

The cardinal $l(X) = \min\{\kappa : \text{every open cover of } X \text{ has a subcover of cardinality at most } \kappa\}$ is called the \textit{Lindelöf number} of $X$ and $hl(X) = \sup\{l(Y) : Y \subseteq X\}$ is the hereditary Lindelöf number of $X$. A space $X$ is called \textit{Lindelöf} if $l(X) \leq \omega$. If for every open cover $\mathcal{U}$ of a space $X$ there exists a countable $\mathcal{U}' \subseteq \mathcal{U}$ such that $\bigcup \mathcal{U}'$ is dense in $X$, then the space $X$ is called weakly Lindelöf. A space $X$ is called \textit{generalized ordered space} or \textit{simply GO space} if $X$ is homeomorphic to a subspace of a linearly ordered space.

We say that a family $\mathcal{F}$ of subsets of a space $X$ is a \textit{network modulo a cover $\mathcal{C}$} if for any $C \in \mathcal{C}$ and $U \in \tau(C, X)$ there exists $F \in \mathcal{F}$ such that $C \subseteq F < U$. A Tychonoff space $X$ is \textit{Lindelöf} $\Sigma$ (or has the \textit{Lindelöf $\Sigma$-property}) if there exists a countable family $\mathcal{F}$ of subsets of $X$ such that $\mathcal{F}$ is a network modulo a compact cover $\mathcal{C}$ of the space $X$. A space $X$ is called \textit{monotonically normal} if it admits an operator $O$ (called the \textit{monotone normality operator}) that assigns to any point $x \in X$ and any $U \in \tau(x, X)$ a set $O(x, U) \in \tau(x, X)$ such that $O(x, U) \subseteq U$ and for any points $x, y \in X$ and sets $U, V \in \tau(X)$ such that $x \in U$ and $y \in V$, it follows from $O(x, U) \cap O(y, V) \neq \emptyset$ that $x \in U$ or $y \in U$.

As usual, we denote by $d(X)$ the minimal cardinality of a dense subset of $X$ and $hd(X) = \sup\{d(Y) : Y \subseteq X\}$. The minimal cardinality of a local base at a point $x \in X$ is called the \textit{character} of $X$ at $x$; it is denoted by $\chi(x, X)$ and $\chi(X) = \sup\{\chi(x, X) : x \in X\}$. If $X$ is a space and $x \in X$ then let $\psi(x, X) = \min\{|U| : U \subseteq \tau(X) \text{ and } \bigcap U = \{x\}\}$ and $\psi(x) = \sup\{\psi(x, X) : x \in X\}$; the cardinal $\psi(X)$ is called the \textit{pseudocharacter} of the space $X$. Given an infinite cardinal $\kappa$ we say that $t(X) \leq \kappa$ if, for any $A \subseteq X$ and $x \in \overline{A}$ there exists a set $B \subseteq A$ such that $|B| \leq \kappa$ and $x \in B$. For a Tychonoff space $X$, the cardinal $iw(X) = \min\{\kappa : \text{the space } X \text{ has a weaker Tychonoff topology of weight } \kappa\}$ is called the \textit{i-weight} of $X$. The cardinal $c(X) = \sup\{|U| : U \subseteq \tau^*(X) \text{ is disjoint}\}$ is the \textit{Souslin number} of $X$; the spaces whose Souslin number is countable are said to have the \textit{Souslin property}.

Given a space $X$, a family $\mathcal{N}$ of subsets of $X$ is a network of $X$ if for every $U \in \tau(X)$ there exists a family $\mathcal{N}' \subseteq \mathcal{N}$ such that $U = \bigcup \mathcal{N}'$. Furthermore, $nw(X) = \min\{|\mathcal{N}'| : \mathcal{N}' \text{ is a network in } X\}$. The cardinal $nw(X)$ is the \textit{network weight} of $X$; the spaces with a countable network are called \textit{cosmic}. If $\kappa$ is an infinite cardinal, then a space $X$ is said to be $\kappa$-monolithic if $nw(\overline{A}) \leq \kappa$ for any set $A \subseteq X$ such that $|A| \leq \kappa$. For a set $A \subseteq X$, we say that $x \in X$ is a \textit{complete accumulation point of $A$} if $|U \cap A| = |A|$ for every $U \in \tau(x, X)$. 

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The rest of our terminology is standard and follows [8]; the survey [11] and the book [12] can be consulted for definitions and properties of cardinal invariants.

3 The results

We start by giving some results that involve assumptions of discrete Lindelöfness type on the co-diagonal. Our main aim is to show that it is consistent with ZFC that discrete Lindelöfness of the co-diagonal of a Tychonoff space \( X \) implies the Lindelöfness of \( X \) and the existence of a weaker second countable Tychonoff topology on \( X \).

3.1 Proposition. If \( X \) is a Tychonoff space and \( \Delta^c_X \) is discretely \( \sigma \)-compact, then \( hl(X \times X) = \omega \).

Proof. If \( K \) is a compact subset of \( X \), then the space \( \Delta^c_K \) is discretely \( \sigma \)-compact being a closed subset of \( \Delta^c_X \). Therefore \( K \) is metrizable by Proposition 3.3 of [7]. This shows that all compact subsets in \( X \) and hence in \( X \times X \) are metrizable. If \( D \subset \Delta^c_X \) is discrete, then \( \overline{D} \) is the union of countably many metrizable compact subsets (the bar denotes the closure in \( \Delta^c_X \)) and hence \( hl(\Delta^c_X) \leq \omega \). It follows from Proposition 2.1 of [2] that \( hl(\Delta^c_X) \leq \omega \). Finally, observe that \( hl(X \times X) = hl(\Delta^c_X) \leq \omega \).

The following corollary solves Questions 5.8 and 5.9 from the paper [1].

3.2 Corollary. Suppose that \( X \) is a Tychonoff space and \( \Delta^c_X \) is discretely \( \sigma \)-compact.

(a) If \( X \) is pseudocompact, then it is compact and metrizable.

(b) If \( X \) is a Lindelöf \( \Sigma \)-space, then it is cosmic.

Proof. Apply Proposition 3.1 to see that in both cases, \( hl(X \times X) \leq \omega \) and hence \( iw(X) \leq \omega \) (see [3, Theorem 2.1.8]). For the case (a) this implies that \( X \) is compact and metrizable by Problem 140 of the book [19]. For the case (b) we can conclude that \( nw(X) \leq \omega \) applying [18, Theorem 2].

3.3 Theorem. Suppose that \( X \) is a Hausdorff space such that \( \Delta^c_X \) is discretely Lindelöf. Then \( hl(X) \leq \mathfrak{c} \) and hence \( |X| \leq 2^\mathfrak{c} \).

Proof. Take any discrete subspace \( D \subset X \) and consider the set \( F = \overline{D} \). Since \( \Delta^c_F \) is a closed subspace of \( \Delta^c_X \), it must be discretely Lindelöf. It is straightforward to check that the set \( (D \times D) \setminus \Delta_F \) is discrete and dense in \( \Delta^c_F \), hence \( \Delta^c_F \) is Lindelöf and so is \( F \). Using the Lindelöf property of \( \Delta^c_F \) it is easy to find a family \( \mathcal{U} = \{U_n, V_n : n \in \omega\} \) of open subsets of \( F \) such that \( U_n \cap V_n = \emptyset \) for every \( n \in \omega \) and \( \bigcup \{U_n \times V_n : n \in \omega\} = \Delta^c_F \). It is immediate that the family \( \mathcal{U} \) is (even \( T_2 \))-separating, so \( |\mathcal{U}| \leq \mathfrak{c} \) by [11, Theorem 3.7(a)]. This means that \( |\overline{D}| \leq \mathfrak{c} \) holds for any discrete \( D \subset X \) and, consequently, \( hl(X) \leq \mathfrak{c} \). Indeed, this is immediate from the fact that \( hl(X) \) is also the supremum of the sizes.
of all right separated (i.e. scattered) subspaces of $X$. Then $|X| \leq 2^{hl(X)} \leq 2^\epsilon$ follows.

The following corollary gives a consistent answer to Problem 4.5 from the paper [7].

3.4 Corollary. Assume that $\epsilon < \omega_\omega$ and $X$ is a regular space such that its co-diagonal $\varDelta^c_X$ is discretely Lindelöf. Then actually $\varDelta^c_X$ is Lindelöf and hence $l(X \times X) = iw(X) = \omega$.

Proof. It follows from Theorem 3.3 that $l(\varDelta^c_X) \leq hl(\varDelta^c_X) \leq \epsilon < \omega_\omega$. Every discretely Lindelöf space $Z$ is Lindelöf if $l(Z) < \omega_\omega$, so $\varDelta^c_X$ is Lindelöf. It is immediate from this that $l(X \times X) = l(\varDelta^c_X) = \omega$. But then $X$ is a regular Lindelöf space with a $G_\delta$-diagonal, which implies $iw(X) = \omega$ by [3, Theorem 2.1.8].

Next we introduce and study a couple of weakenings of the discretely Lindelöf property.

3.5 Definition. A space $X$ will be called almost discretely Lindelöf if for any discrete set $D \subset X$, there exists a Lindelöf set $L \subset X$ such that $D \subset L$.

The following proposition lists a few basic properties of this concept. Its proof is straightforward and so is left to the reader.

3.6 Proposition. (a) if $X$ is almost discretely Lindelöf, then $ext(X) \leq \omega$.
(b) Any discretely Lindelöf space is almost discretely Lindelöf.
(c) Any space of countable spread is almost discretely Lindelöf.
(d) Any continuous image of an almost discretely Lindelöf space is almost discretely Lindelöf.

3.7 Theorem. If $X$ is an almost discretely Lindelöf Hausdorff space such that $\psi(X) \leq \omega$ and $t(X) \leq \omega$, then $|X| \leq 2^\epsilon$.

Proof. If $D \subset X$ is discrete, then there exists a Lindelöf subspace $L$ of the space $X$ such that $D \subset L$. But then it follows from $\psi(L) \cdot t(L) \leq \psi(X) \cdot t(X) \leq \omega$ that $|L| \leq c$ (see [3, Theorem 1.1.10]) and hence $|D| \leq |L| \leq c$. Thus we have shown that $s(X) \leq c$. But then we may conclude $|X| \leq 2^{s(X)} \cdot s(X) \leq 2^\epsilon$ applying say 2.15 (a) of [12].

Coupling the above proof with the argument we used in the proof of Theorem 3.3 we get the following result.

3.8 Corollary. If $X$ is an almost discretely Lindelöf Hausdorff space of character $\chi(X) \leq \omega$, then $hl(X) \leq \epsilon$ (and hence $|X| \leq 2^\epsilon$).

We do not know if the upper bound $2^\epsilon$ for the cardinality can be improved to $\epsilon$ in the above two results.

The second new concept we introduce is the weakly linearly Lindelöf property that figures in the title of our paper. As we shall see, it is actually a weakening
of the previously treated almost discretely Lindelöf property. Our aim is to prove that every monotonically normal and weakly linearly Lindelöf space is Lindelöf, as is stated in the title. We think, however, that this new concept is also interesting in itself.

3.9 Definition. We say that a space $X$ is weakly linearly Lindelöf if for any family $U \subset \tau^*(X)$ such that $\kappa = |U|$ is an uncountable regular cardinal, we can find a point $x \in X$ such that every $V \in \tau(x,X)$ intersects $\kappa$-many elements of $U$. Such a point $x$ is called a complete accumulation point of $U$.

The following result implies that in the definition of weak linear Lindelöfness we could have restricted ourselves to disjoint families $U$ of open sets.

3.10 Proposition. Let $X$ be any space and assume that $U \subset \tau^*(X)$ is such that $\kappa = |U|$ is a regular cardinal, moreover $U$ has no complete accumulation point. Then there is a disjoint family $V \subset \tau^*(X)$ with $\kappa = |V|$ such that $V$ has no complete accumulation point either.

Proof. We are going to define, by transfinite recursion on $\alpha < \kappa$, sets $U_\alpha \in U$ and $V_\alpha \in \tau^*(X)$ with $V_\alpha \subset U_\alpha$ such that $\alpha \neq \beta$ implies both $U_\alpha \neq U_\beta$ and $V_\alpha \cap V_\beta \neq \emptyset$. Clearly, then $V = \{V_\alpha : \alpha < \kappa\}$ is as required.

So, assume that $\alpha < \kappa$ and for every $\beta < \alpha$ we have defined $U_\beta \in U$ and $V_\beta \in \tau^*(X)$ with $V_\beta \subset U_\beta$ with the additional property that

$$\{|U \in U : U \cap V_\beta \neq \emptyset\}| < \kappa.$$ 

The regularity of $\kappa$ then implies that we can choose $U_\alpha \in U$ that is disjoint from $V_\beta$ for all $\beta < \alpha$. But no point of $U_\alpha$ is a complete accumulation point of $U$ by our assumption, hence we may clearly find a non-empty open $V_\alpha \subset U_\alpha$ for which $\{|U \in U : U \cap V_\alpha \neq \emptyset\}| < \kappa$. Clearly, $V_\alpha$ is disjoint from $V_\beta$ and hence $U_\alpha \neq U_\beta$ for all $\beta < \alpha$. This shows that our inductive procedure can be completed. 

The proofs of the following two propositions are straightforward and hence are left to the reader.

3.11 Proposition. Suppose that $X$ is a weakly linearly Lindelöf space. Then

(a) any locally countable family $U \subset \tau^*(X)$ is countable;
(b) collectionwise normality of $X$ implies $\text{ext}(X) \leq \omega$;
(c) every continuous image of $X$ is weakly linearly Lindelöf;
(d) every regular closed subspace of $X$ is weakly linearly Lindelöf;
(e) if $X$ is a dense subspace in a space $Y$, then $Y$ is weakly linearly Lindelöf;
(f) if $K$ is compact and Hausdorff, then $X \times K$ is weakly linearly Lindelöf;
(g) every perfect irreducible preimage of a weakly linearly Lindelöf space is weakly linearly Lindelöf.

3.12 Proposition. If $X$ is a space such that $X = \bigcup_{n \in \omega} X_n$ and every $X_n$ is weakly linearly Lindelöf, then so is $X$. 

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3.13 Example. Under CH, Michael constructed in [14] an example of a regular Lindel"of space $X$ such that $X \times X$ is paracompact but not Lindel"of. As a consequence, there exists a discrete uncountable family of non-empty open subsets in $X \times X$. Applying Proposition 3.11 (b), we conclude that $X \times X$ is not weakly linearly Lindel"of. This shows that compact cannot be replaces by Lindel"of in Proposition 3.11 (f).

3.14 Theorem. (a) Every linearly Lindel"of space is weakly linearly Lindel"of; 
(b) every almost discretely Lindel"of space is weakly linearly Lindel"of; 
(c) every weakly Lindel"of space is weakly linearly Lindel"of.

Proof. (a) and (b) Assume that the space $X$ is linearly Lindel"of or almost discretely Lindel"of. Clearly, then every discrete subset of $X$ of uncountable regular cardinality has a complete accumulation point. Now take any disjoint family $U = \{U_\alpha : \alpha < \kappa\} \subset \tau^*(X)$ for some uncountable regular cardinal $\kappa$ and for every $\alpha < \kappa$ pick a point $x_\alpha \in U_\alpha$. Then the, clearly discrete, set $\{x_\alpha : \alpha < \kappa\}$ has a complete accumulation point $x$ by the above. It is obvious then that $x$ is also a complete accumulation point of the family $U$ and therefore, by Proposition 3.10, the proof is completed.

(c) Suppose that $X$ is a weakly Lindel"of space and $U \subset \tau^*(X)$ is a family of an uncountable regular cardinality $\kappa$ that has no complete accumulation point. For every $x \in X$ we can take a set $V_x \in \tau(x, X)$ such that the family $P_x = \{U \in U : U \cap V_x \neq \emptyset\}$ has cardinality less than $\kappa$. For the open cover $\{V_x : x \in X\}$ of the space $X$ we can find a countable set $B \subset X$ such that $\bigcup \{V_x : x \in B\}$ is dense in $X$. As an immediate consequence, $\bigcup \{P_x : x \in B\} = U$ which contradicts the regularity of $\kappa$ because $|P_x| < \kappa$ for all $x \in B$.

Our proof actually shows that in a weakly Lindel"of space $X$ every family $U \subset \tau^*(X)$ whose cardinality has uncountable cofinality admits a complete accumulation point.

Now we turn to presenting our main result formulated in the title of our paper. An earlier version of the result used the following lemma that, however, was replaced by the use of part (b) of Proposition 3.11. Still, we decided to keep it because we think it has some independent interest.

3.15 Lemma. Assume that $X$ is a collectionwise normal space and $Y$ is a dense subspace of $X$. Then $\text{ext}(X) \leq \text{ext}(Y)$.

Proof. Suppose that $\text{ext}(Y) = \kappa$ and $D \subset X$ is a closed discrete subspace such that $|D| = \kappa^+$. Then there exists a discrete family $U = \{U_d : d \in D\} \subset \tau(X)$ such that $d \in U_d$ for any $d \in D$. Pick a point $x(d) \in U_d \cap Y$ for every $d \in D$. Then $D' = \{x(d) : d \in D\}$ is a closed discrete subset of $Y$ with $|D'| = \kappa^+$ which is a contradiction.

The following example shows that we cannot replace the extent with the Lindel"of number in Lemma 3.15.
3.16 Example. If $X = \{ x \in D^{\omega_1} : |x^{-1}(1)| \leq \omega \}$ is the $\Sigma$-product in $D^{\omega_1}$, then $X$ is a collectionwise normal non-Lindelöf space (see Problem 102 of the book [21]) which has the dense $\sigma$-compact subspace $S = \{ x \in D^{\omega_1} : |x^{-1}(1)| < \omega \}$.

3.17 Theorem. Every monotonically normal and weakly linearly Lindelöf space is Lindelöf.

Proof. Let $X$ be a monotonically normal and weakly linearly Lindelöf space. Then $X$ is collectionwise normal, hence we can apply part (b) of Proposition 3.11 to conclude that $\text{ext}(X) \leq \omega$. But every paracompact space of countable extent is Lindelöf, so it suffices to prove that $X$ is paracompact.

If $X$ is not paracompact, then we can apply the celebrated characterization theorem of Balogh and Rudin (see [6, Theorem I]) to conclude that there exists a closed set $F \subset X$ homeomorphic to a stationary subset of some uncountable regular cardinal $\kappa$. The set $F$ being scattered, we can choose a discrete subspace $D \subset F$ such that $|D| = |F| = \kappa$.

Clearly, the set $D$ has no complete accumulation point in $F$ and hence in $X$. Thus, for any point $x \in X$ we may pick an open neighborhood $W_x \in \tau(x,X)$ such that $|D \cap W_x| < \kappa$. It follows from the hereditary collectionwise normality of $X$ that we can find a set $V_d \in \tau(d,X)$ for every $d \in D$ such that $V_d \cap (F \setminus D) = \emptyset$ and the family $\{ V_d : d \in D \}$ is disjoint.

Fix a monotone normality operator $O$ for $X$ and consider the family $\mathcal{U} = \{ O(d,V_d) : d \in D \} \subset \tau^*(X)$. Then $\mathcal{U}$ is disjoint and its cardinality is equal to $\kappa$. If $x \in V_d$ for some $d \in D$, then $V_d$ trivially witnesses that $x$ is not a complete accumulation point of $\mathcal{U}$. If, on the other hand, $x \in X \setminus \bigcup \{ V_d : d \in D \}$, then $O(x,W_x) \cap O(d,V_d) \neq \emptyset$ must imply that $d \in W_x$, consequently we have

$$\{ d \in D : O(x,W_x) \cap O(d,V_d) \neq \emptyset \} \subset D \cap W_x.$$  

But then, as $|D \cap W_x| < \kappa$, we conclude that $x$ is not a complete accumulation point of $\mathcal{U}$. As a consequence, the family $\mathcal{U}$ has no complete accumulation point in $X$, contradicting the weak linear Lindelöfness of the space $X$ and thus completing the proof.  

From Theorems 3.14 and 3.17 we immediately get the following.

3.18 Corollary. Suppose that $X$ is a monotonically normal space possessing any one of the following properties: weak Lindelöfness, linear Lindelöfness, or almost discrete Lindelöfness. Then $X$ is Lindelöf.

Moreover, from part (e) of Proposition 3.11 we may obtain the following statement that is formally stronger than Theorem 3.17.

3.19 Corollary. If a monotonically normal space has a dense weakly linearly Lindelöf subspace, then it is Lindelöf.

It is well-known that spaces of countable cellularity, i.e. spaces $X$ with $c(X) = \omega$, are examples of weakly Lindelöf spaces that are not necessarily
Lindelöf. However, we have $c(X) = hl(X)$ for every monotonically normal space $X$ (see e.g., [9, Theorem A]), consequently Theorem 3.17 says nothing new for monotonically normal spaces of countable cellularity. However, Corollary 3.18 seems to give us new information even for linearly ordered spaces:

**3.20 Corollary.** Every weakly Lindelöf GO space is Lindelöf.

**3.21 Corollary.** If a GO space $X$ has a dense linearly Lindelöf subspace, then $X$ is Lindelöf.

**3.22 Example.** Any countably compact but non-compact S-space is an example of an almost discretely Lindelöf space that is not linearly Lindelöf. Such examples are the HFD space constructed from CH in [13] that is hereditarily collectionwise normal and Ostaszewski's space constructed in [15] from Jensen's Axiom ♣ that is even perfectly normal (and hence first countable). While both of these examples have countable tightness, we recall that every discretely Lindelöf space of countable tightness must be Lindelöf by [5, Corollary 3.5]. These examples also show that monotone normality cannot be weakened essentially in Theorem 3.17.

We do not know whether an example of an almost discretely Lindelöf but not linearly Lindelöf space can be given in ZFC.

To conclude our paper, we now present a couple of results that involve co-diagonals and the almost discrete Lindelöfness property.

**3.23 Theorem.** Suppose that $X$ is a regular space and its co-diagonal $\Delta^c_X$ has the following properties:

- (a) The closure of any countable discrete subset of $\Delta^c_X$ in $\Delta^c_X$ is Lindelöf;
- (b) $\Delta^c_X$ is almost discretely Lindelöf.

Then $X$ has small diagonal.

**Proof.** Consider any set $A \subset \Delta^c_X$ with $|A| = \omega_1$. We can assume that

$(*)$ $A$ has no complete accumulation points in $\Delta^c_X$.

Indeed, if $z \in \Delta^c_X$ is a complete accumulation point of $A$ in $\Delta^c_X$, then we may choose a set $U \in \tau(z, X \times X)$ such that $\overline{U} \cap \Delta_X = \emptyset$. (The closure is taken in $X \times X$.) But then $B = \overline{U} \cap A \subset A$ is uncountable and $B \cap \Delta_X = \emptyset$.

Take any point $x_0 \in A$ and a set $U_0 \in \tau(x_0, X \times X)$ such that $\overline{U_0} \cap \Delta_X = \emptyset$ and $U_0 \cap A$ is countable. Proceeding by induction assume that $\alpha < \omega_1$ and we have a set $D_{\alpha} = \{x_\beta : \beta < \alpha\} \subset \Delta^c_X$ and a family $\{U_\beta : \beta < \alpha\} \subset \tau(X \times X)$ with the following properties:

- (4) $x_\beta \in U_\beta \cap A$, the set $\overline{U_\beta} \cap A$ is countable and $\overline{U_\beta} \cap \Delta_X = \emptyset$ for every $\beta < \alpha$;
- (5) $x_\beta \notin Q_\beta = \bigcup\{U_\gamma : \gamma < \beta\} \cup \{x_\gamma : \gamma < \beta\}$ for each $\beta < \alpha$.

If the set $Q_\alpha = \bigcup\{U_\gamma : \gamma < \alpha\} \cup \{x_\gamma : \gamma < \alpha\}$ covers $A$, then the induction procedure stops. If not, then we pick a point $x_\alpha \in A \setminus Q_\alpha$ and choose a set $U_\alpha \in \tau(x_\alpha, X \times X)$ such that $\overline{U_\alpha} \cap \Delta_X = \emptyset$ and $\overline{U_\alpha} \cap A$ is countable. It is clear that conditions (4) and (5) are satisfied for all $\beta \leq \alpha$. 

9
Let us observe first that every set $D_\alpha$ constructed above is discrete because for any ordinal $\beta < \alpha$ we have

$$U_\beta \cap \left( X \times X \setminus \{x_\gamma : \gamma < \beta\} \right) \cap D_\alpha = \{x_\beta\}.$$ 

Now, assume first that our inductive procedure ended at some step $\alpha < \omega_1$, i.e. $A \subset Q_\alpha$. Then $A \cap \bigcup\{U_\gamma : \gamma < \alpha\}$ is countable, hence this implies that $A \cap \overline{D}_\alpha$ is uncountable. But by condition (a) $\overline{D}_\alpha \cap \Delta_X^c$ is Lindelöf, hence the set $A$ must have a complete accumulation point in $\overline{D}_\alpha \cap \Delta_X^c$, contradicting ($*$).

If on the other hand our procedure lasts $\omega_1$-many steps, then we have constructed an uncountable discrete set $D = \{x_\alpha : \alpha < \omega_1\} \subset A$. Then, by condition (b), there exists a Lindelöf set $L \subset \Delta_X^c$ such that $D \subset L$ and so the uncountable set $D \subset A$ must have a complete accumulation point in $L$ and hence in $\Delta_X^c$. But this again contradicts ($*$). \hfill \Box

3.24 Corollary. If $X$ is a regular space and its co-diagonal $\Delta_X^c$ is discretely Lindelöf, then $X$ has small diagonal.

3.25 Corollary. Suppose that $X$ is an $\omega$-monolithic regular space such that $\Delta_X^c$ is almost discretely Lindelöf. Then $X$ has small diagonal.

The following statement gives a positive answer to Problem 4.6 from the paper [7] and Question 5.8 from [1].

3.26 Corollary. If $X$ is a Lindelöf $\Sigma$-space such that $\Delta_X^c$ is discretely Lindelöf, then $X$ has a countable network.

Proof. Given a compact $K \subset X$ observe that $\Delta_K^c$ is discretely Lindelöf being a closed subspace of $\Delta_X^c$. By [7, Proposition 3.3], the space $K$ is metrizable. Since all compact subsets of $X$ are metrizable and $X$ has small diagonal by Corollary 3.24, we can apply [10, Theorem 2.1] to conclude that $X$ has a countable network. \hfill \Box

3.27 Corollary. Suppose that $X$ is an $\omega$-monolithic compact Hausdorff space of countable tightness. If $\Delta_X^c$ is almost discretely Lindelöf, then $X$ is metrizable.

In particular, if $X$ is a Corson compact space and $\Delta_X^c$ is almost discretely Lindelöf, then $X$ is metrizable.

Proof. It follows from Theorem 3.23 that the space $X$ has small diagonal. But any compact $\omega$-monolithic space of countable tightness with a small diagonal is metrizable by [20, Problem 296]. \hfill \Box

3.28 Example. In [23], Kunen and de la Vega constructed under CH a non-metrizable compact Hausdorff space $X$ such that $X^n$ is hereditarily separable for all $n \in \mathbb{N}$. In particular, then $\Delta_X^c$ is hereditarily separable and hence almost discretely Lindelöf. Thus, almost discrete Lindelöfness of $\Delta_X^c$ does not imply the metrizability of a compact space $X$, at least consistently.

It is easy to see that for the Kunen–de la Vega example $X$ we have $w(X) = \omega_1$. Since every compact space having small diagonal and weight $\omega_1$ is metrizable, the space $X$ cannot have small diagonal (see Problem 295 of the book.
Therefore, under CH, the almost discrete Lindelöfness of $\Delta^c_X$ does not imply that $X$ has small diagonal for a compact Hausdorff space $X$.

We do not know whether there exists in ZFC a non-metrizable compact space $X$ whose co-diagonal $\Delta^c_X$ is almost discretely Lindelöf.

4 Open questions

Since it is still not known whether every discretely Lindelöf space is Lindelöf, it is important to find out in which situations discrete Lindelöfness implies Lindelöfness or some weaker property. The list of respective open questions is given below.

4.1 Question. Is every almost discretely Lindelöf space weakly Lindelöf?

4.2 Question. Is every discretely Lindelöf space weakly Lindelöf?

4.3 Question. Suppose that $X$ is a space such that $\Delta^c_X$ is discretely $\sigma$-compact. Is it true that $X$ is cosmic?

4.4 Question. Suppose that $X$ is a pseudocompact space such that $\Delta^c_X$ is discretely Lindelöf. Is it true in ZFC that $X$ is compact and metrizable?

4.5 Question. Suppose that $X$ is an almost discretely Lindelöf first countable space. Then $|X| \leq 2^c$ but is it true that $|X| \leq c$?

4.6 Question. Is there a ZFC example of an almost discretely Lindelöf space that is not Lindelöf?

4.7 Question. Suppose that $\Delta^c_X$ is a discretely Lindelöf Tychonoff space. Is it true in ZFC that then $X$ is Lindelöf? and $iw(X) \leq \omega$?

4.8 Question. Suppose that $\Delta^c_X$ is a discretely Lindelöf Tychonoff space. Is it true in ZFC that $X$ has a $G_\delta$-diagonal?

4.9 Question. Suppose that $\Delta^c_X$ is discretely Lindelöf. Is it true in ZFC that $X$ is Lindelöf?

4.10 Question. Suppose that $\Delta^c_X$ is discretely Lindelöf. Is it true in ZFC that $|X| \leq c$?

4.11 Question. Suppose that $X$ is a monotonically normal space and $Y \subset X$ is dense in $X$. Is it true that $l(X) \leq l(Y)$?

4.12 Question. Suppose that $X$ is discretely Lindelöf and $K$ is a compact space. Must $X \times K$ be discretely Lindelöf?

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References

[1] O.T. Alas, L.R. Junqueira, V.V. Tkachuk, R.G. Wilson, Discrete reflexivity in squares, Houston J. Math., 42:2(2016), 659–673.

[2] O.T. Alas, V.V. Tkachuk, R.G. Wilson, Closures of discrete sets often reflect global properties, Topology Proc., 25(2000), 27–44.

[3] A.V. Arhangel’skii, The structure and classification of topological spaces and cardinal invariants (in Russian), Uspehi Matem. Nauk, 33:6(1978), 29–84.

[4] A.V. Arhangel’skii, A generic theorem in the theory of cardinal invariants of topological spaces, Comment. Math. Univ. Carolin., 36:2(1995), 303–325.

[5] A.V. Arhangel’skii, R.Z. Buzyakova, On linearly Lindelöf and strongly discretely Lindelöf spaces, Proc. Amer. Math. Soc., 127:8(1999), 2449–2458.

[6] Z. Balogh, M.E. Rudin, Monotone normality, Topology Appl., 47(1992), 115–127.

[7] D. Burke, V.V. Tkachuk, Discrete reflexivity and complements of the diagonal, Acta Math. Hungar., 139(2013), 120-133.

[8] R. Engelking, General Topology, Heldermann Verlag, Berlin, 1989.

[9] P.M. Gartside, Cardinal invariants of monotonically normal spaces, Topology Appl., 77:3(1997), 303–314.

[10] G. Gruenhage, Spaces having a small diagonal, Topology and Its Appl., 122(2002), 183–200.

[11] R.E. Hodel, Cardinal Functions I, Handbook of Set-Theoretic Topology, ed. by K. Kunen and J.E. Vaughan, North Holland, Amsterdam, 1984, 1–61.

[12] I. Juhász, Cardinal Functions in Topology — Ten Years Later, Mathematical Centre Tracts, 123, Amsterdam, 1980.

[13] I. Juhász, HFD and HFC type spaces, with applications, Top. Appl. 126 (2002), pp. 217–262.

[14] E.A. Michael, Paracompactness and the Lindelöf property in finite and countable Cartesian products, Compositio Math., 23:2(1971), 199–214.

[15] A.J. Ostaszewski, On countably compact, perfectly normal spaces, J. London Math. Soc. 14:2(1976), 505–516.

[16] E. Pearl, Linearly Lindelöf problems, Open Problems in Topology, II, Elsevier B.V., Amsterdam, 2007, 225–231.
[17] V.V. Tkachuk, *Spaces that are projective with respect to classes of mappings*, Trans. Moscow Math. Soc., 50(1988), 139–156.

[18] V.V. Tkachuk, *Lindelöf Σ-spaces: an omnipresent class*, RACSAM, Ser. A: Matematicas, 104:2(2010), 221-244.

[19] V.V. Tkachuk, *A C_p-theory Problem Book. Topological and Function Spaces*, Springer, New York, 2011.

[20] V.V. Tkachuk, *A C_p-theory Problem Book. Special Features of Function Spaces*, Springer, New York, 2014.

[21] V.V. Tkachuk, *A C_p-theory Problem Book. Compactness in Function Spaces*, Springer, New York, 2015.

[22] V.V. Tkachuk, R.G. Wilson, *Discrete reflexivity in GO spaces*, Glasnik Mat., 49(2014), 433–446.

[23] R. de la Vega, K. Kunen, *A compact homogeneous S-space*, Topol. Appl., 136(2004), 123–127.

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