Cut-norms and spectra of matrices

Vladimir Nikiforov

Department of Mathematical Sciences, University of Memphis, Memphis TN 38152

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Abstract

One of the aims of this paper is to solve an open problem of Lovász about relations between graph spectra and cut-distance. The paper starts with several inequalities between two versions of the cut-norm and the two largest singular values of arbitrary complex matrices, extending, in particular, the well-known graph-theoretical Expander Mixing Lemma and giving a hitherto unknown converse of it.

Next, cut-distance is defined for Hermitian matrices, and, separately, for arbitrary complex matrices; using these extensions, we give upper bounds on the difference of corresponding eigenvalues and singular values of two matrices, thus solving the problem of Lovász.

Finally, we deduce a spectral sampling theorem, which informally states that almost all principal submatrices of a real symmetric matrix are spectrally similar to it.

Keywords: Cut-norm; cut-distance; operator norm; singular values; spectral sampling.

1 Introduction

In 1997, Frieze and Kannan [11] introduced and studied the cut-norm of matrices; ever since then this parameter kept getting new attention. It has been extended and used for multidimensional matrices in [2] and its algorithmic aspects have been studied in [3]. More recently, starting with the cut-norm, Lovász and his coauthors in [15, 16, 5, 6] defined a measure of similarity between graphs, which they called the cut-distance and used to investigate the asymptotics of sequences of dense graphs.

It turned out that the cut-distance is related to many fundamental graph parameters. In particular, in [6], among many other things, it was proved that if two graphs are close in cut-distance, then they are close spectrally. Yet, since this specific result did not produce explicit inequalities, Lovász [17] raised the problem to find the best upper bound on the spectral difference in terms of the cut-distance of graphs.

One of the aims of this paper is to solve this problem and extend it to arbitrary matrices. To this end, we start by establishing several tight inequalities between two versions of the cut-norm and the two largest singular values of arbitrary complex matrices. As first-hand applications of these inequalities we extend the well-known graph-theoretical Expander Mixing Lemma and its converse.
In particular, we obtain a new converse of the Expander Mixing Lemma, which is simpler than those in [7, 8, 9].

Next, we extend the concept of cut-distance to Hermitian matrices, and, separately, to arbitrary complex matrices; using these extensions, we give upper bounds on the difference of corresponding eigenvalues and corresponding singular values of two matrices.

As an application we deduce a spectral sampling theorem, which informally states that almost all principal submatrices of a real symmetric matrix are spectrally similar to it. These result complements results of [18] and [10].

The rest of the paper is organized as follows: in the remaining subsections of the introduction we state our main results together with some discussions. All proofs are collected in Section 2. At the end, some open question are raised.

1.1 Notation and definitions

First we introduce some notation and conventions. For undefined matrix notation we refer the reader to [12]. We write:

- $\mathcal{M}_{m,n}$ for the class of all complex matrices of size $m \times n$;
- $\mathcal{H}_n$ for the class of all Hermitian matrices of size $n$;
- $\mathcal{P}_n$ for the class of all permutation matrices of size $n$;
- $J_{m,n}$ for the all ones matrix of size $m \times n$, and set $J_n = J_{n,n}$.

- $\langle x, y \rangle$ for the standard inner product in $\mathbb{C}^n$;
- $y \otimes x$ for the $m \times n$ matrix $[y_i x_j]$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$.

Given a matrix $A = [a_{ij}] \in \mathcal{M}_{m,n}$, we write:

- $|A|_{\infty}$ for $\max_{i,j} |a_{ij}|$;
- $\|A\|_F$ for the Frobenius norm $\sqrt{\sum_{i,j} |a_{ij}|^2}$;
- $\|A\|_2$ for the operator norm of the linear map $A : \mathbb{C}^n \to \mathbb{C}^m$;
- $\sigma_1(A) \geq \cdots \geq \sigma_m(A)$ for the singular values of $A$;
- $\mu_1(A) \geq \cdots \geq \mu_m(A)$ for the eigenvalues of $A$ if $A$ is Hermitian;
- $\Sigma(A)$ for the sum of the entries of $A$;
- $\rho(A)$ for $\Sigma(A)/mn$;
- $A^*$ for the conjugate transpose of $A$;
- $A[X,Y]$ for the submatrix of all $a_{ij}$ with $i \in X, j \in Y$, where $X \subset [m], Y \subset [n]$.

**Definition 1** Following [11], for every $A \in \mathcal{M}_{m,n}$, define the cut-norm $\|A\|_{\square}$ of $A$ by

$$\|A\|_{\square} = \max_{X \subset [m], Y \subset [n]} \frac{1}{mn} |\Sigma(A[X,Y])|.$$

**Definition 2** A similar, yet distinct norm $\|A\|_{\square}$ can be defined by

$$\|A\|_{\square} = \max_{X \subset [m], Y \subset [n], X \neq \emptyset} \frac{1}{\sqrt{|X||Y|}} |\Sigma(A[X,Y])|.$$
The norm $\|A\|_{\square}$ is implicit in numerous papers related to the second singular value and to expansion of graphs: specifically, when $A$ is the adjacency matrix of a graph, the value $\|A - \rho(A) J_n\|_{\square}$ appeared first as the $\alpha$-parameter in Thomason [20, 21]; it was developed further by Alon, Chung and Spencer [1, 4], and more recently it was studied in [7, 8, 9].

Note that neither $\|A\|_{\Box}$ nor $\|A\|_{\square}$ are sub-multiplicative; therefore, they are not matrix norms in the strict sense as defined, say, in [12], Ch. 5.

1.2 Bounds on cut-norms

The following two upper bounds on $\sigma_1(A)$ in terms of $\|A\|_{\Box}$ and $\|A\|_{\square}$ are the cornerstones of our investigation.

**Theorem 3** Let $A \in \mathcal{M}_{m,n}$. If $A$ is real, then

$$\sigma_1(A) \leq 2 \sqrt{|A|_{\infty} \|A\|_{\Box} mn};$$

(1)

if $A$ is complex, then

$$\sigma_1(A) \leq 4 \sqrt{|A|_{\infty} \|A\|_{\square} mn}$$

(2)

and

$$\sigma_1(A) \leq C \|A\|_{\Box} \sqrt{\log m \log n}$$

(3)

for some positive $C < 10^5$. Inequalities (1), (2) and (3) are tight up to constant factors.

Inequalities (1) and (3) can be inverted to some extent. Indeed, Schur’s identity $\sigma_1(A) = \|A\|_2$ (19) implies that

$$\sigma_1(A) \geq \|A\|_{\Box},$$

(4)

and consequently, $\sigma_1(A) \geq \|A\|_{\square} \sqrt{mn}$.

In turn, inequality (4) implies an extension of the Expander Mixing Lemma, including its bipartite version which is implicit in Gowers [14], Lemma 2.9. For convenience, we restate this graph-theoretical result:

*Let $G$ be a bipartite graph with vertex classes $U$ and $V$, and let $A$ be its biadjacency matrix. Suppose that $G$ is semiregular, i.e., vertices belonging to the same vertex class have the same degree. Then

$$\sigma_2(A) \geq \max_{S \subseteq U, R \subseteq V, S \neq \emptyset} \frac{1}{\sqrt{|S||R|}} \left| e(S, R) - \frac{e(U, V)}{|U||V|} |S||R| \right|.$$*

Here $e(X, Y)$ stands for the number of edges $uv$ such that $u \in X, v \in Y$. The theorem below extends the Expander Mixing Lemma to any matrices. Note that for nonnegative matrices essentially the same result has been obtained by Butler in [9], Theorem 1.

For $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, $y = (y_1, \ldots, y_m) \in \mathbb{C}^m$, write $y \otimes x$ for the $m \times n$ matrix $[y_i x_j]$.
**Theorem 4** Let $A \in \mathcal{M}_{m,n}$ and let $x \in \mathbb{C}^n$, $y \in \mathbb{C}^m$ be two unit vectors such that $\sigma_1 (A) = \langle Ax, y \rangle$. Then

$$\sigma_2 (A) \geq \| A - \sigma_1 (A) y \otimes x \| \Box.$$  

In particular, if $A$ is a nonnegative matrix with equal row sums and equal column sums, then

$$\sigma_2 (A) \geq \| A - \rho (A) J_{m,n} \| \Box.$$  

Along this line, inequalities (1) and (3) imply the following upper bounds on $\sigma_2$.

**Theorem 5** For every $A \in \mathcal{M}_{m,n}$ we have

$$\sigma_2 (A) \leq 4 \sqrt{\| A - \rho (A) J_{m,n} \| \Box mn} \tag{5}$$

and

$$\sigma_2 (A) < C \| A - \rho (A) J_{m,n} \| \Box \sqrt{\log m \log n} \tag{6}$$

for some positive $C < 10^5$. Inequalities (3) and (4) are tight up to constant factors.

In the above general matrix setup, inequality (6) is new, but for Hermitian matrices it is known from [8]. For regular graphs somewhat better results were obtained by Bilu and Linial [7], and for nonnegative matrices, by Butler [9]. On the other hand, inequality (5) is entirely new; while it seems less subtle than (6), it is much easier to use.

### 1.3 Extending the cut-distance to complex matrices

Following the general idea of cut-distance for graphs, we shall define cut-distance for arbitrary matrices. Note that, in fact, Lovász and his coauthors have defined the cut-distance for real measurable functions $f : [0, 1]^2 \rightarrow \mathbb{R}$, in particular for real symmetric matrices. For Hermitian matrices we follow their footprints, but for arbitrary complex matrices we make a necessary adjustment, producing in fact a slightly different version of the cut-distance, even for graphs.

**The cut-distance of Hermitian matrices**

Given $A = [a_{ij}] \in \mathcal{H}_n$ and integer $p \geq 1$, let

$$A^{(p)} = A \otimes J_p,$$

where $\otimes$ denotes the Kronecker product. Thus, $A^{(p)}$ is obtained by replacing each entry $a_{ij}$ with the matrix $a_{ij}J_p$. Note that $A^{(p)} \in \mathcal{H}_{np}$. Now, for every $A, B \in \mathcal{H}_n$, define $\hat{\delta}_{\Box} (A, B)$ as

$$\hat{\delta}_{\Box} (A, B) = \min \left\{ \| A - PBP^{-1} \| \Box : P \in \mathcal{P}_n \right\}.$$ 

Finally, extend the function $\hat{\delta}_{\Box} (A, B)$ to matrices of different sizes as follows: for every $A \in \mathcal{H}_n$ and $B \in \mathcal{H}_m$, define the cut-distance $\delta_{\Box} (A, B)$ as

$$\delta_{\Box} (A, B) = \lim_{k \to \infty} \hat{\delta}_{\Box} (A^{(km)}, B^{(kn)}).$$
It is not immediate, but is rather simple to see that the limit above exists, and moreover,
\[ \delta (A, B) = \inf_k \delta_k \left( A^{(km)}, B^{(kn)} \right). \]

Note also that the function \( \delta (A, B) \) is symmetric and satisfies the triangle inequality
\[ \delta (A, B) \leq \delta (A, C) + \delta (C, B) \]
for all Hermitian matrices \( A, B, C \). However, \( \delta_k \left( A^{(p)}, A^{(q)} \right) = 0 \), and so, \( \delta (\cdot, \cdot) \) is not a true metric, but only a pre-metric.

The cut-distance of arbitrary matrices

The matrix setup allows an easy modification of \( \delta (\cdot, \cdot) \) for arbitrary complex matrices. Given \( A = [a_{ij}] \in \mathcal{M}_{m,n} \) and two positive integers \( p, q \), let
\[ A^{(p,q)} = A \otimes J_{p,q}. \]

Note that \( A^{(p,q)} \in \mathcal{M}_{mp,nq} \). Now, for every \( A, B \in \mathcal{M}_{m,n} \), define \( \delta_k (A, B) \) as
\[ \delta_k (A, B) = \min \left\{ \| A - PBQ \|_k : P \in \mathcal{P}_m, Q \in \mathcal{P}_n \right\} \]
Finally, extend the function \( \delta_k (A, B) \) to matrices of different sizes as follows: for every \( A \in \mathcal{M}_{m,n} \) and \( B \in \mathcal{M}_{r,s} \), define \( \delta_k (A, B) \) as
\[ \delta_k (A, B) = \lim_{k \to \infty} \delta_k \left( A^{(kr,ks)}, B^{(km,kn)} \right). \]

As in the case of Hermitian matrices, the above limit exists and we have
\[ \delta (A, B) = \inf_k \delta_k \left( A^{(kr,ks)}, B^{(km,kn)} \right). \]

Also, the function \( \delta (\cdot, \cdot) \) is symmetric and satisfies the triangle inequality, but is only a pre-metric.

Note that now, for Hermitian matrices we have two cut-distances: \( \delta_k (\cdot, \cdot) \) and \( \delta (\cdot, \cdot) \). It is not difficult to prove that
\[ \delta (A, B) \leq \delta_k (A, B) \leq 2 \delta (A, B) \]
for every two Hermitian matrices \( A \) and \( B \).

1.4 The spectral difference of matrices

Having inequality (1) and the definition of \( \delta (\cdot, \cdot) \) in hand, we can bound the difference of corresponding eigenvalues of two Hermitian matrices \( A \) and \( B \) in terms of \( \delta_k (A, B) \). The main difficulties here come from the fact that \( A \) and \( B \) can be of different size and consequently have a different number of eigenvalues. But even when \( A \) and \( B \) are of the same size, there may be complications due to a huge difference in the number of their positive eigenvalues. Thus, the theorem below gives two conclusions from the same premise: one when eigenvalue signs are taken into account (clauses ii.a and ii.b), and one when they are not (clause i.).
Theorem 6 Let \( n \geq m \geq 1 \), and let \( A \in \mathcal{H}_n \), \( B \in \mathcal{H}_m \) satisfy \( |A|_\infty = |B|_\infty = 1 \). Then

(i) for every \( i = 1, \ldots, \lceil m/2 \rceil \), we have

\[
\left| \frac{\mu_i(A)}{n} - \frac{\mu_i(B)}{m} \right| \leq \frac{1}{\sqrt{n/2}} + \frac{1}{\sqrt{m/2}} + 6\delta \square (A, B)^{1/2}
\]

\[
\left| \frac{\mu_{n-i+1}(A)}{n} - \frac{\mu_{m-i+1}(B)}{m} \right| \leq \frac{1}{\sqrt{n/2}} + \frac{1}{\sqrt{m/2}} + 6\delta \square (A, B)^{1/2}
\]

(ii.a) if \( \mu_i(A) \geq 0 \) and \( \mu_i(B) \geq 0 \), then,

\[
\left| \frac{\mu_i(A)}{n} - \frac{\mu_i(B)}{m} \right| \leq 6\delta \square (A, B)^{1/2}
\]

(ii.b) if \( \mu_i(A) \leq 0 \) and \( \mu_i(B) \leq 0 \), then

\[
\left| \frac{\mu_{n-i+1}(A)}{n} - \frac{\mu_{m-i+1}(B)}{m} \right| \leq 6\delta \square (A, B)^{1/2}
\]

Since in clause (i) of the above theorem eigenvalue signs are not taken into account, the undesired term \((n/2)^{-1/2} + (m/2)^{-1/2}\) appears in the right-hand side. In general, this term seems unavoidable: indeed, taking \( A = B^{(k)} \), we have \( \delta \square (A, B) = 0 \), but the difference

\[
\left| \frac{\mu_i(B^{(k)})}{mk} - \frac{\mu_i(B)}{m} \right|
\]

can be as large as \( m^{-1/2}/2 \), say when \( B \) is a Paley graph of sufficiently large order \( m \) and \( i = \lceil m/2 \rceil + 1 \).

Fortunately, for singular values, everything goes smoothly.

Theorem 7 Let \( A \in \mathcal{M}_{m,n} \) and \( B \in \mathcal{M}_{r,s} \) satisfy \( |A|_\infty = |B|_\infty = 1 \). Then for every \( i = 1, \ldots, \min(m, n, r, s) \), we have

\[
\left| \frac{\sigma_i(A)}{\sqrt{mn}} - \frac{\sigma_i(B)}{\sqrt{rs}} \right| \leq 6\delta \square (A, B)^{1/2}
\]

Remark. If the matrices in Theorems 6 and 7 are real, the coefficient 6 in the right-hand side can be replaced by 3.

1.5 Matrix sampling

Alon, de la Vega, Kannan and Karpinski [2] came up with a powerful matrix sampling result, further improved by Borgs, Chayes, Lovász, Sós, and Vesztergombi in [5], Theorem 2.9; for convenience we restate it in a slightly weaker form:
Let \( n \geq k \geq 1 \) and let \( A \) be a real symmetric matrix of size \( n \). Let \( B = A[X,X] \), where \( X \) is a uniformly random subset of \([n]\) of size \( k \). Then
\[
\delta_{\Box}(A,B) < 10|A|_\infty (\log_2 k)^{-1/2}.
\]
with probability at least \( 1 - \exp\left(-k^2/(2 \log_2 k)\right) \).

In view of this theorem, we can use Theorems 6 to derive a spectral sampling theorem for real symmetric matrices. There is a rich literature dedicated to this topic, see, e.g., the references of [18]; we shall mention only two recent milestones: Chatterjee and Ledoux [10] proved that almost all principal submatrices of a Hermitian matrix \( A \) have empirical eigenvalue distribution close to the expected eigenvalue distribution. Prior to that, Rudelson and Vershynin [18] have obtained more precise results, but only for the singular values of special submatrices. Here we take an intermediate approach. We prove a sampling result about principal submatrices of real symmetric matrices, bounding all eigenvalues of the sample submatrix, but not attempting the level of precision as in [18]. In addition, our methods are much simpler than the methods of [10] and [18].

**Theorem 8** Let \( n \geq k \geq 1 \) and \( A \) be a real symmetric matrix of size \( n \). Let \( B = A[X,X] \), where \( X \) is a uniformly random subset of \([n]\) of size \( k \). Then with probability at least \( 1 - \exp\left(-k^2/(2 \log_2 k)\right) \), for every \( i = 1, \ldots, k \), we have
\[
(i) \text{ if } \mu_i(B) \geq 0, \text{ then } \left| \frac{\mu_i(A)}{n} - \frac{\mu_i(B)}{k} \right| < 30(\log_2 k)^{-1/4};
\]
\[
(ii) \text{ if } \mu_i(B) \leq 0, \text{ then } \left| \frac{\mu_{n-k+i}(A)}{n} - \frac{\mu_i(B)}{k} \right| < 30(\log_2 k)^{-1/4}.
\]

**2 Proofs**

**2.1 Proof of Theorem 3**

For the proof of inequality (1) we need a standard lemma that can be traced back to [11]. We prove it here for convenience.

**Lemma 9** Let \( A \in \mathcal{M}_{m,n}, x \in \mathbb{R}^n, y \in \mathbb{R}^m \). Then
\[
|\langle Ax, y \rangle| \leq 4 \|x\|_\infty \|y\|_\infty \|A\|_\Box mn.
\]

If \( x \in \mathbb{C}^n, y \in \mathbb{C}^m \), then
\[
|\langle Ax, y \rangle| \leq 16 \|x\|_\infty \|y\|_\infty \|A\|_\Box mn.
\]

7
Proof Assume for simplicity that \( \|x\|_\infty = \|y\|_\infty = 1 \). We shall prove first (7). Since \( \langle Au, v \rangle \) maps the cube \([-1, 1]^{m+n}\) linearly in each coordinate of \( u \) and \( v \), max \( |\langle Au, v \rangle| \) is attained for some \( u = (u'_1, \ldots, u'_n) \in \{-1, 1\}^n \), and \( v = (v'_1, \ldots, v'_n) \in \{-1, 1\}^m \). Set
\[
R^+ = \{ x : u'_x = 1 \}, \quad R^- = \{ x : u'_x = -1 \}, \\
C^+ = \{ x : u'_x = 1 \}, \quad C^- = \{ x : u'_x = -1 \}.
\]

Now we see that
\[
|\langle Au, v \rangle| = | \Sigma (A [R^+, C^+]) + \Sigma (A [R^-, C^-]) - \Sigma (A [R^+, C^-]) - \Sigma (A [R^-, C^+]) |
\leq | \Sigma (A [R^+, C^+]) | + | \Sigma (A [R^-, C^-]) | + | \Sigma (A [R^+, C^-]) | + | \Sigma (A [R^-, C^+]) |
\leq 4 \| A \| \square mn,
\]
completing the proof of (7).

To prove (8), suppose that \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_m) \), and set
\[
x_0 = (\text{Re} x_1, \ldots, \text{Re} x_n), \quad x_1 = (\text{Im} x_1, \ldots, \text{Im} x_n), \\
y_0 = (\text{Re} y_1, \ldots, \text{Re} y_m), \quad y_1 = (\text{Im} y_1, \ldots, \text{Im} y_m).
\]

We have
\[
|\langle Ax, y \rangle| = |\langle Ax_0, y_0 \rangle - \langle Ax_0, y_1 \rangle i + \langle Ax_1, y_0 \rangle i + \langle Ax_1, y_1 \rangle |
\leq |\langle Ax_0, y_0 \rangle| + |\langle Ax_0, y_1 \rangle| + |\langle Ax_1, y_0 \rangle| + |\langle Ax_1, y_1 \rangle|.
\]

Since \( x_0, x_1, y_0, y_1 \) are real, inequality (8) follows from (7). \( \square \)

For the proof of (3) we need the following lemma, proved in [8].

Lemma 10 Let \( p \geq 1 \), \( n \geq 1 \) and \( 0 < \varepsilon < 1 \). Then for every \( x = (x_1, \ldots, x_n) \in \mathbb{C}^n \) with \( \|x\| = 1 \), there is a vector \( y = (y_1, \ldots, y_n) \in \mathbb{C}^n \) such that \( y_i \) take no more than
\[
\left\lceil \frac{8\pi}{\varepsilon} \right\rceil \left\lceil \frac{4}{\varepsilon} \log \frac{4n}{\varepsilon} \right\rceil
\]
values and \( \|x - y\| \leq \varepsilon \).

Proof of inequalities (2) and (11) We shall prove first (2). Let \( A = [a_{ij}] \), \( A^* = [a^*_{ij}] \), \( A^* A = [b_{ij}] \).

For every \( j \in [n] \), set
\[
c_j = (a_{1j}, a_{2j}, \ldots, a_{mj}).
\]

Select \( i \in [n] \) so that \( \sum_{j \in [n]} |b_{ij}| \) is maximal. It is well-known that
\[
\sigma_1^2(A) = \mu_1(A^* A) \leq \sum_{j \in [n]} |b_{ij}|.
\]
Note also that
\[b_{ij} = \sum_{k \in [m]} a^*_k a_{kj} = \sum_{k \in [m]} a_{kj} \bar{a}_{ki} = \langle c_j, c_i \rangle\]

For every \(j \in [n]\), set
\[x_j = \begin{cases} \|\langle c_j, c_i \rangle\| / \langle c_j, c_i \rangle & \text{if } \langle c_j, c_i \rangle \neq 0 \\ 0 & \text{if } \langle c_j, c_i \rangle = 0 \end{cases}\]
and let \(x = (x_1, \ldots, x_n)\). Also for every \(k \in [m]\), set \(y_k = a_{ki}\) and let \(y = (y_1, \ldots, y_m)\). Note that
\[
\sum_{j \in [n]} |b_{ij}| = \sum_{j \in [n]} \langle c_j, c_i \rangle x_j = \sum_{j \in [n]} \sum_{k \in [m]} a_{ki} a_{kj} x_j = \sum_{k \in [m]} \sum_{j \in [n]} a_{ki} a_{kj} x_j = \langle A x, y \rangle.
\]
Since \(|y|_\infty \leq 1\), in view of (8), we see that
\[
\langle A x, y \rangle \leq 16 \|A\|_\square \|x\|_\infty \|y\|_\infty = 16 \|A\|_\square \|A\|_\infty,
\]
completing the proof of inequality (2).

Inequality (11) follows likewise, using (7) instead of (8).

To prove that inequality (11) is tight, define a square symmetric matrix \(A = [a_{ij}]\) of size \(2n + 1\), by letting
\[a_{1i} = a_{i1} = \begin{cases} 1 & \text{if } 2 \leq i \leq n + 1, \\ -1 & \text{if } n + 1 < i \leq 2n + 1 \end{cases}\]
and let all other entries of \(A\) to be 0. We easily find that \(\sigma_1 (A) = \sqrt{2n}\). Also \(\|A\|_\square (2n + 1)^2 = 2n\), for if \(X \subset [2n + 1]\), \(Y \subset [2n + 1]\) are such that \(|\Sigma (A [X, Y])|\) is maximal, the only contributions to \(|\Sigma (A [X, Y])|\) come from the first row and the first column, and each of them can be at most \(n\). Thus, we have
\[\sigma^2_1 (A) = 2n = |A|_\infty \|A\|_\square (2n + 1)^2,\]
and so inequality (11) is tight up to a factor of 2 and inequality (2) is tight up to a factor of 4. \(\square\)

**Proof of inequality (3)** By Schur's identity \(\sigma_1 (A) = \|A\|_2\), there exists unit vectors \(x = (x_1, \ldots, x_n) \in \mathbb{C}^n\) and \(y = (y_1, \ldots, y_m) \in \mathbb{C}^m\) such that
\[
\sigma_1 (A) = \langle A x, y \rangle.
\]
Applying Lemma 10 with \(\varepsilon = 1/3\), we can find vectors
\[x' = (x'_1, \ldots, x'_n) = \ y' = (y'_1, \ldots, y'_m)\]
such that \(x'_i\) take \(p\) distinct values \(\alpha_1, \ldots, \alpha_p\) and \(y'_i\) take \(q\) distinct values \(\beta_1, \ldots, \beta_q\), and
\[
\|x - x'\| < 1/3, \\
\|y - y'\| < 1/3, \\
p \leq 12 \lceil 24\pi \rceil \lceil \log 12n \rceil = 912 \lceil \log 12n \rceil, \tag{9} \\
q \leq 12 \lceil 24\pi \rceil \lceil \log 12m \rceil = 912 \lceil \log 12m \rceil. \tag{10}
\]
Hence, in view of (11), (9) and (10), we see that

$$N_i = \{ u : x'_u = \alpha_i \},$$
$$M_j = \{ u : y'_u = \beta_j \}.$$  

Clearly, $$N \cup \cdots \cup N_p$$ and $$M \cup \cdots \cup M_q$$ are partitions of $$[n]$$ and $$[m]$$.

Our first goal is to prove that

$$\sigma_1 (A) \leq \frac{9}{2} |\langle Ax', y' \rangle|.$$  \hspace{1cm} (11)

Indeed, we have

$$|\langle Ax, y \rangle - \langle Ax', y' \rangle| = |\langle Ax, y \rangle - \langle Ax', y \rangle + \langle Ax', y \rangle - \langle Ax', y' \rangle|$$
$$\leq |\langle A (x - x'), y \rangle| + |\langle Ax', y - y' \rangle|$$
$$\leq \sigma_1 (A) (\|x - x'||y\| + \|x'||y - y'\|)$$
$$\leq \sigma_1 (A) (\|x - x'||y\| + (\|x\| + \|x - x'\|) \|y - y'\|)$$
$$\leq \sigma_1 (A) \left(2 + \frac{1}{3}\right) \frac{1}{3} \leq \frac{7}{9} \sigma_1 (A),$$

implying that

$$|\sigma_1 (A) - |\langle Ax', y' \rangle| \leq \frac{7}{9} \sigma_1 (A).$$

and inequality (11) follows.

Now, define the matrix $$C = [c_{ij}] \in M_{p,q}$$ by

$$c_{ij} = \frac{1}{\sqrt{|M_i| \cdot |N_j|}} \sum_{u \in M_i} \sum_{v \in N_j} a_{uv}.$$  

For every $$i \in [p], j \in [q]$$, set $$s_i = \sqrt{|N_i| \alpha_i}, t_j = \sqrt{|M_j| \beta_j}$$ and let $$t = (t_1, ..., t_q), s = (s_1, ..., s_p).$$

Clearly $$\|s\| = \|x'\|$$ and $$\|t\| = \|y'\|$$. Also, we see that

$$|\langle Ax', y' \rangle| = \left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x'_j y'_i \right| = \left| \sum_{j=1}^{p} \sum_{i=1}^{q} a_{ij} x'_j y'_i \right|$$
$$= \left| \sum_{j=1}^{p} \sum_{i=1}^{q} c_{ij} s_i \overline{t_j} \right| \leq \sigma_1 (C) \|s\| \|t\| = \sigma_1 (C) \|x'\| \|y'\|$$
$$\leq \sigma_1 (C) \left(\|x\| + \frac{1}{3}\right) \left(\|y\| + \frac{1}{3}\right) = \frac{16}{9} \sigma_1 (C)$$

Hence, in view of (11), (9) and (10), we see that

$$\sigma_1 (A) \leq 8 \sigma_1 (C) \leq \|C\|_F \leq 8 \sqrt{pq} \max_{i,j \in [m]} |c_{ij}|$$
$$\leq 8 \cdot 912 \sqrt{\log 12n \log 12m \|A\|_F}.$$
To complete the proof of (3) assume that \( n \geq 2 \) and \( m \geq 2 \) and observe that
\[
8 \cdot 912 \sqrt{\log 12n \log 12m} \leq 8 \cdot 912 \sqrt{(10 \ln 12 + 1) \log n \log m} < 10^5 \sqrt{\log n \log m}.
\]

To prove that inequality (3) is tight up to a constant factor, define a square symmetric matrix \( A = [a_{ij}] \) of size \( n \) by letting \( a_{ij} = (ij)^{-1/2} \). Set \( s_n = \sum_{i=1}^{n} 1/i \). It is easy to see that the vector \( x = (x_1, \ldots, x_n) \), where \( x_i = (is_n)^{-1/2} \) is of length 1 and thus satisfies,
\[
\sigma_1 (A) \geq \langle Ax, x \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{ij} s_n = s_n > \log n.
\]
On the other hand, let \( X \subset [n] \), \( Y \subset [n] \) be such that
\[
\left\| A \right\|_{\Box} = \frac{1}{\sqrt{|X||Y|}} \left| \Sigma (A [X,Y]) \right|
\]
Clearly \( X = [p] \), \( Y = [q] \) for some \( p, q \in [n] \). We thus have
\[
\frac{1}{\sqrt{|X||Y|}} \left| \Sigma (A [X,Y]) \right| = \frac{1}{\sqrt{pq}} \sum_{i=1}^{p} \sum_{j=1}^{q} \frac{1}{\sqrt{ij}} < \frac{4\sqrt{pq}}{\sqrt{pq}} = 4.
\]
Hence,
\[
\sigma_1 (A) > \log n > \frac{1}{4} \log n > \frac{1}{4} \left\| A \right\|_{\Box} \log n,
\]
and so, inequality (3) is tight up to a constant factor. \( \square \)

### 2.2 Proofs of Theorems 4 and 5

**Proof of Theorem 4** The proof is essentially a tautology of the singular value decomposition theorem (see, e.g., [12], Ch. 7). Let

\[
A = \sigma_1 (A) \overline{y} \otimes x + \sum_{i=2}^{m} \sigma_i (A) \overline{y_i} \otimes x_i
\]

be a singular value decomposition of \( A \), where \( y, y_2, \ldots, y_m \in \mathbb{C}^m \) are unit orthogonal left singular vectors and \( x, x_2, \ldots, x_m \in \mathbb{C}^n \) are unit orthogonal right singular vectors to \( \sigma_1 (A), \sigma_2 (A), \ldots, \sigma_m (A) \). Hence,
\[
A - \sigma_1 (A) \overline{y} \otimes x = \sum_{i=2}^{m} \sigma_i (A) \overline{y_i} \otimes x_i,
\]
and so, \( \sigma_2 (A) = \sigma_1 (A - \sigma_1 (A) \overline{y} \otimes x) \). Now (3) implies that
\[
\sigma_2 (A) = \sigma_1 (A - \sigma_1 (A) \overline{y} \otimes x) \geq \left\| A - \sigma_1 (A) x \otimes y \right\|_{\Box}.
\]
If $A$ is nonnegative and its row sums are equal and its column sums are equal, then we can choose $x = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)$, $y = \left(\frac{1}{\sqrt{m}}, \ldots, \frac{1}{\sqrt{m}}\right)$ and so,

$$\sigma_1(A) y \otimes x = \rho(A) J_{m,n},$$

completing the proof.

**Proof of Theorem 5** Weyl’s inequalities for singular values (see, e.g., [13], Theorem 3.3.16) state that

$$\sigma_2(X + Y) \leq \sigma_1(X) + \sigma_2(Y).$$

Setting $Y = \rho(A) J_{m,n}$, $X = A - Y$, and noting that $\sigma_2(Y) = 0$, inequalities [2] and [3] follow from [1] and [3] respectively.

To see that inequality [3] is tight, define the square symmetric matrix $B = [b_{ij}]$ of size $n$ by letting $b_{ij} = (ij)^{-1/2}$, and set

$$A = \left(\begin{array}{cc} J_n + B & J_n - B \\ J_n - B & J_n + B \end{array}\right).$$

Clearly $\rho(A) = 1$, and so

$$A - \rho(A) J_{2n} = \left(\begin{array}{cc} B & -B \\ -B & B \end{array}\right).$$

As shown in the proof of Theorem 3, $\|B\|\leq 4$, and so

$$\|A - \rho(A) J_{2n}\| \leq 4 \|B\| < 16.$$ 

On the other hand, $\sigma_1(A) = \mu_1(A) = 2n$, and the all ones vector is an eigenvector to $\mu_1(A)$. It is easy to check that the $2n$-vector

$$(1, 2^{-1/2}, \ldots, n^{-1/2}, -1, -2^{-1/2}, \ldots, -n^{-1/2}),$$

is an eigenvector of $A$ to the eigenvalue $2 \sum_i^n 1/i$. Hence,

$$\sigma_2(A) \geq 2 \sum_{i=1}^n \frac{1}{i} \geq 2 \log n \geq \frac{1}{8} \|A - \rho(A) J_{2n}\| \sqrt{\log n \log n},$$

and so [3] is tight up to a constant factor.

**2.3 Proof of Theorem 6**

The following two facts are derived by straightforward methods.

**Proposition 11** Let $A \in \mathcal{H}_n$ and $k \geq 2$. Then the eigenvalues of $A^{(k)}$ are $k\mu_1(A), \ldots, k\mu_n(A)$ together with $(k - 1)n$ additional 0’s.
Proposition 12 Let $A \in \mathcal{M}_{m,n}$ and $p, q \geq 2$. Then the singular values of $A^{(p,q)}$ are
\[ \sqrt{pq} \sigma_1 (A), \ldots, \sqrt{pq} \sigma_m (A) \]
and the rest are zeroes.

For the proof of Theorem 6 we shall show that the extremal $k$ eigenvalues of $A^{(k)}$ are roughly proportional to the corresponding eigenvalues of $A$.

Lemma 13 Let $k \geq 2$. Then for every $i = 1, \ldots, n$,
\[ 0 \leq \frac{\mu_i (A^{(k)})}{kn} - \frac{\mu_i (A)}{n} \leq \frac{\| A \|_F}{n\sqrt{n-i+1}} \quad (12) \]
\[ 0 \geq \frac{\mu_{n-i+1} (A^{(k)})}{tn} - \frac{\mu_{n-i+1} (A)}{n} \geq - \frac{\| A \|_F}{n\sqrt{n-i+1}} \quad (13) \]

Proof We shall prove only (12); inequality (13) follows likewise, applying (12) to $-A$. Note that Proposition 12 implies that $A^{(k)}$ and $A$ have the same number of positive eigenvalues. If $\mu_i (A^{(k)}) > 0$, then $\mu_i (A) > 0$ and $\mu_i (A^{(k)}) = k \mu_i (A)$, so (12) holds. If $\mu_i (A^{(k)}) \leq 0$, then $\mu_i (A) \leq 0$ and so
\[ 0 \geq \mu_i (A) \geq \cdots \geq \mu_n (A). \]
Hence, inequality (12) follows from
\[ (n-i+1) \mu_i^2 (A) \leq \sum_{j=i}^{n} \mu_j^2 (A) \leq \| A \|_F^2. \]

Proof of Theorem 6 Let $k$ be a positive integer. By the definition of $\hat{\delta}_{\square} (\cdot, \cdot)$, there is a permutation matrix $P \in P_{mnk}$ such that
\[ \hat{\delta}_{\square} (A^{(mk)}, B^{(nk)}) = \| A^{(mk)} - PB^{(nk)} P^{-1} \|_{\square}. \]
Referring to [13], Theorem 3.3.16, we have
\[ | \mu_i (A^{(mk)}) - \mu_i (B^{(nk)}) | = | \mu_i (A^{(mk)}) - \mu_i (PB^{(nk)} P^{-1}) | \leq \sigma_1 (A^{(mk)} - PB^{(nk)} P^{-1}). \]
Now, inequality (2) implies that
\[ | \mu_i (A^{(mk)}) - \mu_i (B^{(nk)}) | \leq \sigma_1 (A^{(mk)} - PB^{(nk)} P^{-1}) \leq 4 \sqrt{2} \| A^{(mk)} - PB^{(nk)} P^{-1} \|_{\square mnk} \leq 6 \sqrt{\hat{\delta}_{\square} (A^{(mk)}, B^{(nk)}) mnk}. \]
To prove (i), note that the triangle inequality and Lemma 13 imply that
\[
\left| \frac{\mu_i(A^n)}{m} - \frac{\mu_i(B^n)}{m} \right| \leq \frac{1}{\sqrt{n-i+1}} + \frac{1}{\sqrt{m-i+1}} + 6\sqrt{\delta_{\square}(A^n, B^n)}.
\]
Letting \( k \) tend to infinity and passing to limits in the above inequality, we obtain
\[
\left| \frac{\mu_i(A)}{n} - \frac{\mu_i(B)}{m} \right| \leq \frac{1}{\sqrt{n-i+1}} + \frac{1}{\sqrt{m-i+1}} + 6\sqrt{\delta_{\square}(A, B)}.
\]
Hence, for every \( i = 1, \ldots, \left\lfloor m/2 \right\rfloor \),
\[
\left| \frac{\mu_i(A)}{n} - \frac{\mu_i(B)}{m} \right| \leq \frac{1}{\sqrt{n/2}} + \frac{1}{\sqrt{m/2}} + 6\sqrt{\delta_{\square}(A, B)},
\]
\[
\left| \frac{\mu_{n-i+1}(A)}{n} - \frac{\mu_{m-i+1}(B)}{m} \right| \leq \frac{1}{\sqrt{n/2}} + \frac{1}{\sqrt{m/2}} + 6\sqrt{\delta_{\square}(A, B)}.
\]
Now let us prove (ii.a). Suppose that \( \mu_i(A) \geq 0 \) and \( \mu_i(B) \geq 0 \). Then using Proposition 11 and 14, we find that
\[
\left| \frac{\mu_i(A)}{n} - \frac{\mu_i(B)}{m} \right| = \frac{1}{mn} \left| \mu_i(A^{(mk)}) - \mu_i(B^{(nk)}) \right| \leq 6\sqrt{\delta_{\square}(A^{(mk)}, B^{(nk)})}.
\]
Letting \( k \) tend to infinity and passing to limits (ii.a) follows. The clause (ii.b) follows by a similar argument. \( \square \)

**Proof of Theorem 7** The proof is a straightforward modification of the proof of Theorem 6. Let \( k \) be a positive integer. By the definition of \( \delta_{\square} \), there exist permutation matrices \( P \in \mathcal{P}_{m_{nk}} \) and \( Q \in \mathcal{P}_{rsk} \) such that
\[
\delta_{\square}(A^{(rk, sk)}, B^{(mk, nk)}) = \| A^{(rk, sk)} - PB^{(mk, nk)}Q \|_{\square}.
\]
Since \( \sigma_i(B^{(mk, nk)}) = \sigma_i(PB^{(mk, nk)}Q) \), referring to the inequality
\[
|\sigma_i(X) - \sigma_i(Y)| \leq \sigma_1(X - Y),
\]
(see, e.g., [13], Theorem 3.3.16), we obtain
\[
|\sigma_i(A^{(rk, sk)}) - \sigma_i(B^{(mk, nk)})| \leq |\sigma_i(A^{(rk, sk)}) - \sigma_i(PB^{(mk, nk)}Q)| \leq \sigma_1(A^{(rk, sk)} - PB^{(mk, nk)}Q).
\]

Now, inequality (2) implies that
\[
\left| \sigma_i \left( A^{(rk,sk)} \right) - \sigma_i \left( B^{(mk,nk)} \right) \right| \leq \sigma_1 \left( A^{(rk,sk)} - PB^{(mk,nk)}Q \right) \\
\leq 4k \sqrt{2} \| A^{(rk,sk)} - PB^{(mk,nk)}Q \|_{mnrs} \\
\leq 6k \sqrt{\delta_{\Box} \left( A^{(rk,sk)}, B^{(mk,nk)} \right) mnrs},
\]
and so,
\[
\frac{1}{k \sqrt{mnrs}} \left| \sigma_i \left( A^{(rk,sk)} \right) - \sigma_i \left( B^{(mk,nk)} \right) \right| \leq 6 \sqrt{\delta_{\Box} \left( A^{(rk,sk)}, B^{(mk,nk)} \right)}. 
\]

Finally, using Proposition [12] we find that
\[
\left| \frac{\sigma_i (A)}{\sqrt{mn}} - \frac{\sigma_i (B)}{\sqrt{rs}} \right| = \frac{1}{k \sqrt{mnrs}} \left| \sigma_i \left( A^{(rk,sk)} \right) - \sigma_i \left( B^{(mk,nk)} \right) \right| \leq 6 \sqrt{\delta_{\Box} \left( A^{(rk,sk)}, B^{(mk,nk)} \right)}. 
\]

Letting \( k \) tend to infinity and passing to limits, the proof is completed. \( \square \)

### 2.4 Proof of Theorem 8

**Proof of Theorem 8** Let \( X \) be a uniformly random subset of \([n]\) of size \( k \). Let \( B = A [X, X] \). The result Borgs et al. implies that
\[
\delta_{\Box} (A, B) \leq 10 |A|_\infty (\log_2 k)^{-1/2}
\]
with probability at least \( 1 - \exp \left( -k^2 / (2 \log_2 k) \right) \).

By Cauchy’s Interlacing theorem, for every \( i = 1, \ldots, k, \)
\[
\mu_i (A) \geq \mu_i (B) \geq \mu_{n-i+1} (A). 
\]
Hence, if \( \mu_i (B) \geq 0 \), then \( \mu_i (A) \geq 0 \), and clause (ii.a) of Theorem 8 implies that
\[
\left| \frac{\mu_i (A)}{n} - \frac{\mu_i (B)}{k} \right| < 3\delta_{\Box} (A, B)^{1/2} \leq 30 |A|_\infty (\log_2 k)^{-1/4}.
\]

If \( \mu_i (B) < 0 \), then \( \mu_{n-i+1} (A) < 0 \), and clause (ii.b) of Theorem 8 implies that
\[
\left| \frac{\mu_{n-k+i} (G)}{n} - \frac{\mu_i (H)}{k} \right| \leq 3\delta_{\Box} (A, B)^{1/2} \leq 30 |A|_\infty (\log_2 k)^{-1/4},
\]
completing the proof of Theorem 8. \( \square \)
Concluding remarks

1. Note that the norm $\|A\|_\square$ seems subtler than $\|A\|_\Box$. Yet while $\|A\|_\Box$ was used in a successful version of Szemerédi’s Regularity Lemma ([11]), $\|A\|_\square$ has never been studied explicitly in this respect. A natural question arises: what type of Regularity Lemma one can prove using $\|A\|_\square$.

2. Let $A \in \mathcal{M}_{m,n}$ and $B \in \mathcal{M}_{p,q}$. We studied the spectral difference of $A$ and $B$ in the form

$$\max_{1 \leq i \leq \min(p,q,m,n)} \left| \frac{\sigma_i(A)}{\sqrt{mn}} - \frac{\sigma_i(B)}{\sqrt{pq}} \right|$$

On the other hand, the vector of the singular values of a matrix $A$ becomes a unit vector when divided by $\|A\|_F$. Thus, it seems more appropriate to study

$$\max_{1 \leq i \leq \min(p,q,m,n)} \left| \frac{\sigma_i(A)}{\|A\|_F} - \frac{\sigma_i(B)}{\|B\|_F} \right| .$$

Can Theorem 7 be modified accordingly? Similar modifications seem possible for the eigenvalues of Hermitian matrices.

3. Some, but not all, of our results can be extended for complex graphons, i.e., measurable functions $f : [0,1]^2 \to \mathbb{C}$. One way of doing this is approximation by step functions with finitely many steps. We leave these extensions to interested readers.

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