One-loop self-energy correction to the 1s and 2s hyperfine splitting in H-like systems

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The one-loop self-energy correction to the hyperfine splitting of the 1s and 2s levels in H-like low-$Z$ atoms is evaluated to all orders in $Z\alpha$. The results are compared to perturbative calculations. The residual higher-order contribution is evaluated. Implications to the specific difference of the hyperfine structure intervals $8\Delta\nu_2 - \Delta\nu_1$ in He$^+$ are investigated.

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INTRODUCTION

Hyperfine splitting of the ground state in hydrogen was measured already thirty years ago up to utmost precision of about $6 \times 10^{-13}$ [1,2]. For many years this measurement remained the most accurate experiment in modern physics. Unfortunately, the theoretical situation still remains much less satisfactory in this case, due to uncertainties in the proton-size and proton-polarizability corrections. At present, accuracy of the theoretical prediction is on the level of several ppm, and further progress is going to be rather slow. This fact limits perspectives for testing higher-order QED effects in comparison of theory and experiment for the ground-state hyperfine splitting.

It is possible partly to avoid problems connected with the proton structure, if we study the specific difference of the 1s and 2s hyperfine structure intervals ($\Delta\nu_1$ and $\Delta\nu_2$), $D_{21} = 8\Delta\nu_2 - \Delta\nu_1$. The most accurate measurements of the 1s $^3$ and 2s $^4$ hyperfine structure of $^3$He$^+$ yield

$$D_{21}(^3\text{He}^+) = 1 189.979(71) \text{ kHz}.$$
This difference is much less sensitive to the proton structure than $\Delta \nu_1$ and $\Delta \nu_2$ separately and, therefore, provides better possibilities to test higher-order QED effects by comparison with the experiment.

In the present investigation, we perform a nonperturbative (in the parameter $Z\alpha$) calculation of the one-loop self-energy correction for the specific difference of the $1s$ and $2s$ hyperfine structure intervals $D_{21}$ for H-like low-$Z$ atoms. We check consistency of our results with perturbative calculations and evaluate the unknown higher-order contribution to $D_{21}$ for hydrogen and He$^+$.  

I. SELF-ENERGY CORRECTION TO THE HYPERFINE SPLITTING

In the nonrecoil, point-nucleus limit, the hyperfine splitting of the $ns$ state of a H-like atom can be conveniently represented as

$$\Delta \nu_n = \frac{E_F}{n^3} \left[ B_n + \frac{\alpha}{\pi} D^{(2)}_n(Z\alpha) + \left(\frac{\alpha}{\pi}\right)^2 D^{(4)}_n(Z\alpha) + \ldots \right],$$

(1)

where $E_F$ is the non-relativistic Fermi energy, $B_n$ is the relativistic factor, and the functions $D^{(2r)}_n(Z\alpha)$ represent $r$-loop radiative corrections. In this work we are interested in the one-loop self-energy contribution to the hyperfine splitting, $D^{(2,SE)}_n(Z\alpha)$. This correction can be written as a double expansion in $Z\alpha$ and $\ln(Z\alpha)$,

$$D^{(2,SE)}_n(Z\alpha) = a_{00} + (Z\alpha)a_{10} + (Z\alpha)^2 \left\{ a_{22} \ln^2(Z\alpha) + a_{21} \ln(Z\alpha) + a_{20} + (Z\alpha) \left[ a_{31} \ln(Z\alpha) + F^{SE}_n(Z\alpha) \right] \right\},$$

(2)

where the coefficients $a_{ij}$ are listed in Table I, and the function $F^{SE}_n(Z\alpha)$ contains a constant entering in order $(Z\alpha)^3$ plus all higher-order contributions.

Now consider the function $D^{(2,SE)}_{21} = D^{(2,SE)}_2 - D^{(2,SE)}_1$ which corresponds to the specific difference of the $1s$ and $2s$ hyperfine structure intervals, $8\Delta \nu_2 - \Delta \nu_1$. The coefficients $a_{00}$, $a_{10}$, $a_{22}$, and $a_{31}$ are state-independent, and, therefore, the function $D^{(2,SE)}_{21}$ can be written as
\[ D_{21}^{(2),\text{SE}}(Z\alpha) = (Z\alpha)^2 \left[ a_{21} \ln(Z\alpha) + a_{20} + (Z\alpha)F_{21}^{\text{SE}}(Z\alpha) \right] . \]  

The coefficients \( a_{21} \) and \( a_{20} \) for this difference were investigated in Refs. [8,9] and recently re-evaluated in Ref. [14] (see also references therein),

\[ a_{21}(2s) - a_{21}(1s) = \frac{16}{3} \ln 2 - 7 , \]

\[ a_{20}(2s) - a_{20}(1s) = -5.221233(3) . \]

While the traditional approach to the calculation of QED effects in loosely bound systems is based on the perturbative expansion in the parameter \( Z\alpha \), it turns out that complexity of calculations grows very rapidly in higher orders. As a result, the exact numerical treatment of radiative corrections to all orders in \( Z\alpha \) becomes rather competitive. Despite the fact that nonperturbative calculations of the self-energy corrections for light atoms are impeded by large numerical cancellations, several accurate results complete in \( Z\alpha \) have been obtained for hydrogen during last few years [15–18]. Among them we mention the calculation by Blundell, Cheng and Sapirstein [15] in which the function \( F_1^{\text{SE}}(Z\alpha) \) was determined numerically to all orders in \( Z\alpha \). For hydrogen, their evaluation yields \( F_1^{\text{SE}}(1\alpha) = -12.0(2.0) \).

**II. NON-PERTURBATIVE EVALUATION OF THE SELF-ENERGY CORRECTION**

The self-energy correction to the ground-state hyperfine splitting in various H-like ions was calculated by a number of authors [19–23,15,24]. We note that the first two evaluations turned out to be not completely correct. In our first calculation [19] we employed the partial-wave renormalization procedure. As was found later (see, e.g., [25]), this procedure delivers a certain spurious term due to the non-covariant nature of the regularization. This spurious term yields an additional contribution of about 3\% for bismuth (\( Z = 83 \)). In the second evaluation [20], a certain term in the vertex part was omitted, which contributes on the level of about 1\% for bismuth (see the related discussion in Ref. [22]). The self-energy correction to the 2\( s \) hyperfine splitting was calculated in our previous studies [22,26] for several medium- and high-\( Z \) ions and, for \( Z = 83 \), also in Ref. [27]. In the present investigation, we extend our
calculations to the low-$Z$ region and increase numerical accuracy, in order to get information about terms of order $\alpha(Z\alpha)^3$ and higher.

The self-energy correction to the hyperfine splitting can be conveniently represented as a perturbation of the first-order self-energy correction by the hyperfine-splitting interaction

$$V_{\text{hfs}}(r) = \frac{|e| \alpha \cdot [\mu \times r]}{4\pi r^3},$$

where $\alpha$ denotes the Dirac matrices, and $\mu$ is the nuclear magnetic moment. Taken to first order in $V_{\text{hfs}}$, this perturbation gives rise to the modification of the wave functions, the electron propagator, and the binding energy. We refer to the corresponding contributions as the irreducible part ($\Delta E_{\text{ir}}$), the vertex part, and the reducible part, respectively. The irreducible contribution can be expressed as a non-diagonal matrix element of the first-order self-energy operator. For its evaluation we use the numerical scheme described in detail in Ref. [28]. We note that an expression for a wave function modified by the magnetic field can be derived analytically both in configuration and in momentum space using virial relations for the Dirac equation [29]. Corresponding formulas for the $1s$ and $2s$ wave functions modified by the magnetic field can be found in Ref. [30].

The reducible and the vertex part are conveniently evaluated together. Both of them are ultraviolet divergent. In order to isolate the divergences in a covariant way, we separate terms in which bound electron propagators are replaced with free propagators. The sum of these terms for the reducible and the vertex part is denoted by $\Delta E_{\text{vr}}^{(0)}$. It is evaluated in momentum space, where the ultraviolet divergences can be treated in a standard way. The remaining part of the reducible and the vertex contribution is referred to as $\Delta E_{\text{vr}}^{(1+)}$. This part does not contain any ultraviolet divergences and can be evaluated in configuration space. We note that both the reducible and the vertex part of $\Delta E_{\text{vr}}^{(1+)}$ possess some infrared divergences that cancel each other when considered together.

Generally, our scheme of the numerical evaluation is rather similar to that used in Ref. [15]. Here we address several features of our calculation which distinguish it from previous studies. First, our treatment of infrared divergences in $\Delta E_{\text{vr}}^{(1+)}$ is essentially different from that in Ref. [17]. These divergences can be easily separated when bound electron propagators
are written in the spectral representation,

$$S_F(\varepsilon, \mathbf{x}, \mathbf{y}) = \sum_n \frac{\psi(\mathbf{x}) \overline{\psi}(\mathbf{y})}{\varepsilon - \varepsilon_n (1 - i0)}. \quad (7)$$

As can easily be shown, the infrared divergence arises only when the energy $\varepsilon_n$ in the spectral representation of all internal electron propagators coincides with the energy of the initial state $\varepsilon_a$. In that case, the integration over the energy of the virtual photon can be carried out analytically, and a finite result is achieved explicitly in the sum of the reducible and the vertex contribution. So, we divide $\Delta E_{vr}^{(1+)}$ into two parts. The first one is the finite expression derived from the infrared-divergent part, and the next is given by the point-by-point difference of the original expression and its divergent part, which is a regular function of the virtual-photon energy.

Some additional complications arise for the self-energy correction to the 2s hyperfine splitting, as compared to that for the ground state. For the 2s state, the integration over the virtual-photon energy involves the integrand with an additional pole in the first quadrant of the complex energy plane. This pole arises from the virtual 1s state which is more deeply bound than the initial state. When the standard Wick rotation is applied, the whole contribution is separated into an integral along the imaginary axis and a pole term. This separation leads to a significant numerical cancellation in the low-$Z$ region. An additional problem in this case is the presence of a pole due to the 2p$_{3/2}$ state near the integration contour. This pole is separated from the contour only by relativistic effects that nearly vanish for very low $Z$. We avoid both these problems by performing the integration along the contour $C$ in the complex energy plane, as shown in Fig. [I].

The crucial point of the evaluation, which defines numerical accuracy of the result, is the infinite sum over the angular-momentum quantum number of intermediate states, $|\kappa|$. To reduce the uncertainty due to the termination of the summation, we extrapolate it to infinity. The extrapolation was carried out using two different schemes. First, we use least-squares fitting of several last expansion terms to polynomial in $1/|\kappa|$. Next, we approximate last terms of the expansion by $1/|\kappa|^\alpha$, where the exponent $\alpha$ is fitted. Convergence of different fits as a function of the truncation parameter $|\kappa_{\text{max}}|$ was studied, and the most stable fit was
Naturally, a reasonable extrapolation of the truncated summation is possible only if expansion terms exhibit a clear asymptotic behavior. A problem appears for the irreducible contribution, in which the expansion terms change their sign at a certain point $|\kappa| = |\kappa_0|$. It is clear that when $|\kappa_0|$ comes close to the truncation point $|\kappa_{\text{max}}|$, no reliable extrapolation is possible. While this problem is present for the ground state if $Z$ is very small, it appears much more immensely for the $2s$ state. We found that by a minor modification of the renormalization procedure it is possible to eliminate this unfortunate feature of the partial-wave expansion of the irreducible contribution.

Let us examine the cancellation of ultraviolet divergences in the sum of the zero- and the one-potential term for the first-order self-energy correction (for the detailed discussion of the renormalization procedure we refer to [28]). The free self-energy operator $\Sigma^{(0)}(p)$ can be represented as

$$\Sigma^{(0)}(p) - \delta m = -\frac{\alpha}{4\pi} \Delta_\epsilon (p_\mu \gamma^\mu - m) + \Sigma_R^{(0)}(p), \quad (8)$$

where $\delta m$ denotes the mass counterterm, $\Delta_\epsilon$ is a divergent constant, $p$ is the 4-momentum. In the zero-potential term, the time component of $p$ is fixed at the physical energy of the initial state, $p_0 = \epsilon_a$. The renormalized free self-energy operator $\Sigma_R^{(0)}(p)$ is finite. Analogously, we can write the time component of the free vertex operator as

$$\Gamma^0(p,p') = \frac{\alpha}{4\pi} \Delta_\epsilon \gamma^0 + \Gamma_R^0(p,p'), \quad (9)$$

where $\Gamma_R^0(p,p')$ is finite. Assuming that the operators act on the Dirac wave functions, the divergences cancel each other due to the Dirac equation

$$(p_\mu \gamma^\mu - m)\psi_a(p) = \int \frac{dp'}{(2\pi)^3} \gamma^0 V_C(p-p')\psi_a(p'). \quad (10)$$

In the one-potential term, the time components of the 4-momenta $p$ and $p'$ in the vertex operator are normally fixed at the physical energy, $p_0 = p'_0 = \epsilon_a$. However, as can be seen from Eq. (8), the ultraviolet divergent part of the vertex operator does not depend on them. Therefore, we can formally regard $\epsilon = p_0 = p'_0$ as a free parameter. In this case the one-potential term loses its transparent physical meaning but still contains the same ultraviolet...
divergent part. Of course, care should be taken calculating the many-potential term in this case. In our approach (see Ref. [28]), the many-potential contribution is calculated as the point-by-point difference of the unrenormalized expression and the zero- and one-potential terms in configuration space. Therefore, it is sufficient simply to shift the energy in the one-potential part from its physical value, $\varepsilon_a \rightarrow \varepsilon$.

We found that terms of the partial-wave expansion of the irreducible part do not change their sign if the parameter $\varepsilon$ is chosen as $\varepsilon = (\varepsilon_a + m)/2$. In that case, the expansion terms are monotonically decreasing and can be reasonably fitted. A similar procedure can be applied to the computation of $\Delta E_{\nu}(1+\nu)$. In our implementation, it was used as an additional cross-check of the numerical evaluation.

III. NUMERICAL RESULTS AND DISCUSSION

The numerical calculation was carried out in the Feynman gauge, for the point nuclear model. The partial-wave summation was terminated typically at $|\kappa_{\text{max}}| = 40 \div 50$. The tail of the series was estimated by fitting as described above. Various contributions to the functions $D_{1}^{(2),\text{SE}}(Z\alpha)$ and $D_{2}^{(2),\text{SE}}(Z\alpha)$ are listed in Tables I and II respectively. The quoted errors originate mainly from the partial-wave extrapolation. The estimated accuracy is essentially worse for the 2s state than for the 1s state due to a larger contribution of high partial waves. For the ground state, our results are in reasonable agreement with the previous calculation by Blundell and co-workers [15]. However, we note a small systematic deviation of our results from those of Ref. [15]. For the 2s state, the present evaluation agrees well with our previous calculation [22] of this correction in the medium- and high-Z region. For $Z = 49$ we found $D_{2}^{(2),\text{SE}}(Z\alpha) = -3.555(1)$, as compared to the point-nucleus result of $-3.553(18)$ [22]. The present results for the difference $D_{21}^{(2),\text{SE}}(Z\alpha)$ show to be consistent with perturbative calculations. Fitting our data to the form given by Eq. (3) with undetermined coefficients, we found $-3.3(6)$ for $a_{21}$ and $-5.1(1.0)$ for $a_{20}$, which analytically are $-3.303$ and $-5.221$, respectively.

Our aim is the higher-order contributions $F_{n}^{\text{SE}}(1\alpha)$ and $F_{n}^{\text{SE}}(2\alpha)$. In order to get them,
we have to subtract the known terms of the \( Z\alpha \) expansion from our numerical results. Of course, this subtraction leads to a severe loss of accuracy, especially for very low \( Z \). At the present level of numerical precision, the direct determination of the higher-order contribution for \( Z = 1 \) and 2 is not possible. Still, we can evaluate this contribution for higher \( Z \) (where cancellations are smaller) and then extrapolate it to \( Z = 1 \) and 2, as it was done in Ref. [13] for the 1s state. Aiming this, we tabulate the functions \( F_{21}^{\text{SE}}(Z\alpha) \) and \( F_{21}^{\text{SE}}(Z\alpha) \) in a wide region of \( Z \) starting from \( Z = 4 \), as presented in Table IV. We note that for fitting purposes the difference \( F_{21}^{\text{SE}}(Z\alpha) \) is more appropriate than the function \( F_{2}^{\text{SE}}(Z\alpha) \). In this difference a number of higher-order contributions cancel each other (e.g., the cubed logarithm of order \( \alpha(Z\alpha)^4 \)), and some others have smaller coefficients.

We start our analysis with the function \( F_{21}^{\text{SE}}(Z\alpha) \). The corresponding results are plotted in Fig. 2. We note that the function varies smoothly in a wide range of \( Z \), which again indicates that our results are consistent with the first two terms of Eq. (3). Surprisingly enough, the general behaviour of the higher-order contribution is rather close to linear in \( Z \), even for relatively large nuclear charge numbers of about 30. For very low \( Z \), we observe a systematic deviation of our results from the linear behavior, which is, however, well within the given error bars. We interpret this as a small systematic error in the partial-wave extrapolation of \( D_{21}^{(2)\text{SE}}(Z\alpha) \) divided by the factor \( (Z\alpha)^3 \) when converting to \( F_{21}^{\text{SE}}(Z\alpha) \). A least-squares fit to the numerical results yields \( F_{21}^{\text{SE}}(1\alpha) = 6.5(8) \) and \( F_{21}^{\text{SE}}(2\alpha) = 6.3(6) \). We estimate the uncertainty of the fitting procedure by studying sensitivity of the result to the form of the fit, and assuming that the leading logarithm in the order \( \alpha(Z\alpha)^4 \) enters with the coefficient of about unity. For the introduction into the fitting procedure we refer to [31], where fitting of nonperturbative results for the one-loop self-energy correction to the Lamb shift is discussed in detail.

Next, we examine the function \( F_{1}^{\text{SE}}(Z\alpha) \). The obtained results are plotted in Fig. 3, together with the data from Ref. [13]. A least-squares fit to our results yields \( F_{1}^{\text{SE}}(1\alpha) = -14.3(1.1) \) and \( F_{1}^{\text{SE}}(2\alpha) = -14.5(7) \). The error of the fit was estimated assuming that the leading logarithm of order \( \alpha(Z\alpha)^4 \) enters with the coefficient of about unity. Our fitting result for \( F_{1}^{\text{SE}}(1\alpha) \) agrees with the previous calculation of Ref. [13] which yields \(-12.0(2.0)\).
In the present investigation we evaluated the higher-order contribution to the one-loop self-energy correction to the 1s and 2s hyperfine splitting of H-like ions. The calculation was carried out to all orders in \(Z\alpha\). The higher-order correction to the specific difference of the hyperfine structure intervals \(D_{21} = 8\Delta\nu_2 - \Delta\nu_1\) is found to be 0.394(38) kHz for \(^3\text{He}^+\), 0.0083(10) kHz for H, and 0.0019(2) kHz for D. For helium, our result should be compared with the experimental value \(D_{21} = 1\,189.979(71)\) kHz \[3,4\]. We see that the higher-order contribution evaluated in this work is rather significant for comparison of theory and experiment. Still, it should be noted that there is a number of other corrections contributing at the same level. For the latest analysis of theoretical predictions for the hyperfine structure of hydrogen and He\(^+\) atoms we refer to \[14\].

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physics/0009069.
TABLE I. Coefficients of the expansion of the function $D_{\alpha}^{(2,SE)}(Z\alpha)$, Eq. (2).

| $a_{00}(ns)$ | 1/2 |
|---------------|-----|
| $a_{10}(ns)$  | $(\ln 2 - \frac{13}{2})\pi$ |
| $a_{22}(ns)$  | $-8/3$ |
| $a_{21}(1s)$  | $(\frac{16}{5} \ln 2 - \frac{37}{36})$ |
| $a_{20}(1s)$  | 17.1227(11) |
| $a_{31}(ns)$  | $(-5 \ln 2 + \frac{191}{16})\pi$ |

TABLE II. Individual contributions to $D_{1}^{SE}(Z\alpha)$.

| $Z$ | $\Delta E_{\text{ir}}$ | $\Delta E_{\text{vi}}^{(0)}$ | $\Delta E_{\text{vi}}^{(1+)}$ | $D_{1}^{SE}(Z\alpha)$ | Ref. [15] |
|-----|-----------------|-----------------|-----------------|-----------------|-----------|
| 4   | -0.077320       | 2.392195        | -2.073870       | 0.241005(25)    | 0.24103(1) |
| 5   | -0.104961       | 2.298749        | -2.019760       | 0.174028(20)    | 0.17405(1) |
| 6   | -0.134332       | 2.210244        | -1.969100       | 0.106812(20)    | 0.10684(1) |
| 7   | -0.165112       | 2.126160        | -1.921577       | 0.039471(20)    | 0.03950(1) |
| 8   | -0.197073       | 2.046065        | -1.876933       | -0.027941(20)   | -0.02791(1) |
| 9   | -0.230047       | 1.969610        | -1.834936       | -0.095373(20)   | -0.09535(1) |
| 10  | -0.263906       | 1.896443        | -1.795397       | -0.162860(20)   | -0.16283(1) |
| 12  | -0.333923       | 1.759010        | -1.723003       | -0.297916(15)   |           |
| 14  | -0.406614       | 1.631950        | -1.658572       | -0.433236(15)   |           |
| 16  | -0.481714       | 1.513797        | -1.601149       | -0.569066(15)   |           |
| 18  | -0.559114       | 1.403346        | -1.549961       | -0.705729(15)   |           |
| 20  | -0.638817       | 1.299586        | -1.504357       | -0.843588(15)   | -0.84356(1) |
| 25  | -0.848878       | 1.064059        | -1.411445       | -1.196264(15)   | -1.19621(1) |
| 30  | -1.077047       | 0.854302        | -1.343905       | -1.566650(15)   |           |
TABLE III. Individual contributions to $D^\text{SE}_2(Z\alpha)$.

| $Z$ | $\Delta E_{\text{tr}}$ | $\Delta E_{\text{tr}}^{(0)}$ | $\Delta E_{\text{tr}}^{(1+)}$ | $D^\text{SE}_2(Z\alpha)$ |
|-----|-------------------|-----------------|----------------|-----------------|
| 4   | −0.06817          | 3.95047         | −3.63560       | 0.24670(12)     |
| 5   | −0.09171          | 3.85376         | −3.58009       | 0.18196(10)     |
| 6   | −0.11646          | 3.76182         | −3.52821       | 0.11714(10)     |
| 7   | −0.14219          | 3.67417         | −3.47970       | 0.05228(10)     |
| 8   | −0.16871          | 3.59045         | −3.43435       | −0.01261(10)    |
| 9   | −0.19588          | 3.51034         | −3.39197       | −0.07751(10)    |
| 10  | −0.22365          | 3.43355         | −3.35241       | −0.14251(10)    |
| 12  | −0.28070          | 3.28902         | −3.28121       | −0.27289(9)     |
| 14  | −0.33956          | 3.15520         | −3.21970       | −0.40406(8)     |
| 16  | −0.40016          | 3.03076         | −3.16707       | −0.53647(8)     |
| 18  | −0.46255          | 2.91457         | −3.12267       | −0.67065(6)     |
| 20  | −0.52689          | 2.80570         | −3.08597       | −0.80716(6)     |
| 25  | −0.69789          | 2.56037         | −3.02522       | −1.16274(6)     |
| 30  | −0.88767          | 2.34549         | −3.00540       | −1.54757(6)     |
TABLE IV. Contributions to $F_{21}^{SE}(Z\alpha)$ and $F_{1}^{SE}(Z\alpha)$.

| Z  | $F_{1}^{SE}(Z\alpha)$, this work | $F_{1}^{SE}(Z\alpha)$, Ref. [15] | $F_{21}^{SE}(Z\alpha)$ |
|----|---------------------------------|---------------------------------|------------------------|
| 4  | $-14.71(100)$                  | $-13.85(40)$                   | 7.9(4.8)               |
| 5  | $-14.92(43)$                   | $-14.46(21)$                   | 6.7(2.1)               |
| 6  | $-15.26(25)$                   | $-14.92(12)$                   | 6.3(1.2)               |
| 7  | $-15.51(13)$                   | $-15.283(75)$                  | 5.98(75)               |
| 8  | $-15.753(75)$                  | $-15.584(50)$                  | 5.75(50)               |
| 9  | $-15.921(49)$                  | $-15.856(35)$                  | 5.60(35)               |
| 10 | $-16.188(31)$                  | $-16.113(26)$                  | 5.43(26)               |
| 12 | $-16.631(16)$                  |                                  | 5.03(13)               |
| 14 | $-17.057(10)$                  |                                  | 4.712(75)              |
| 16 | $-17.476(7)$                   |                                  | 4.438(50)              |
| 18 | $-17.896(5)$                   |                                  | 4.182(26)              |
| 20 | $-18.317(4)$                   | $-18.307(3)$                   | 3.935(19)              |
| 25 | $-19.389(2)$                   | $-19.380(2)$                   | 3.336(8)               |
| 30 | $-20.524(1)$                   |                                  | 2.748(5)               |
FIG. 1. The contour $C$ of the integration over the energy of the virtual photon. The poles and the branch cuts of the integrand are shown.
FIG. 2. $F_{21}^{SE}$ as a function of the nuclear charge number (the upper graph). The same plot with the linear part of $F_{21}^{SE}$ subtracted (the lower graph). As an illustration to the fitting procedure, two examples of fits are shown, the linear and the parabolic one.
FIG. 3. The function $F_{1}^{\text{SE}}(Z)$ in different evaluations (the upper graph). The same plot with the linear part of $F_{1}^{\text{SE}}(Z)$ subtracted (the lower graph). Two examples of different fits are given, the linear and the parabolic one.