Conformal vectors in general space-times

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Abstract
In an earlier paper [14] the author wrote the homothetic equations for vacuum solutions in a first order formalism allowing for arbitrary alignment of the dyad. This paper generalises that method to conformal vectors in non-vacuum spaces. The method is applied to metrics admitting a three parameter motion group on non-null orbits.

PACS number 0420J
Mathematics Subject Classification: 83C15, 83C20

1 Introduction and Notation

In [14] I gave the homothetic Killing equations in vacuum, written out in a first order form without the assumption that the spinor dyad used was aligned to either the symmetry or the curvature in any way, and indeed allowing for the dyad to be non-normalised. In this paper I will generalise to conformal vectors in non-vacuum spaces-times. Equations for this case have been given in [9,11] in a particularly specialised notation, although the former did not use a first order formalism.

A conformal vector $\xi^a$ by definition satisfies the equation
\[ \xi_{a;b} = F_{ab} + \psi g_{ab}. \] (1)

Here $\psi$, the divergence, is an arbitrary scalar, and $F_{ab}$ will be called the conformal bivector. If $\psi$ is constant we have a homothetic vector, if $\psi_a$ is covariantly constant we have a special conformal vector [16].

Let $\{o^A, \ell^A\}$ be a spinor dyad, with $o_A \ell^A = \chi$. A complex null tetrad is related to this dyad in the standard way:
\[ \ell^a = o^A o^A'; \quad n^a = \ell^A o^{A'}; \quad m^a = o^A \ell^{A'}; \quad \overline{m}^a = \ell^A o^{A'}, \] ([15], (4.5.19)), and $\ell_a n^a = -m_a \overline{m}^a = \chi \chi$. As in [14], we define components of the conformal:
\[ \xi_a = \xi_n \ell_a + \xi_{\ell n} a - \xi_{\ell m} m_a - \xi_{m} \overline{m} a, \] (2)

with $\{\ell^a, n^a, m^a, \overline{m}^a\}$ a Newman-Penrose tetrad. Thus, for example, $\chi \chi \xi_\ell = \chi_a \ell^a$.

For the conformal bivector $F_{ab}$ we define its anti-self dual by
\[ -F_{ab} = \frac{1}{2} (F_{ab} + i F^{*}_{ab}) \] (3)
and then

\[-F_{ab} = (\chi\chi)^{-1} \left( 2\phi_{00} \ell_{|a}m_{b|} + 2\phi_{01} (\ell_{|[a}m_{b]} - m_{|[a}m_{b]} - 2\phi_{11} n_{|[a}m_{b]} \right), \tag{4}\]

where

\[\phi_{11} = (\chi\chi)^{-1} F_{ab}^{\ell^a}m^b \tag{5}\]

\[\phi_{01} = \frac{1}{2}(\chi\chi)^{-1} (F_{ab}n^a\ell^b - F_{ab}m^a\ell^b) \tag{6}\]

\[\phi_{00} = (\chi\chi)^{-1} F_{ab}m^a n^b \tag{7}\]

Most of the equations in this paper will be given using the compacted GHP-formalism, see [5,15,16]. In this formalism, we simplify notation by concentrating on those spin coefficients of good weight, that is, those that transform homogeneously under a spin-boost transformation of the dyad: if

\[o^A \mapsto \lambda o^A \quad i^A \mapsto \mu i^A\]

a weighted quantity \(\eta\) of type \(\{r', r; t', t\}\) undergoes a transformation

\[\eta \mapsto \lambda^{r'}\bar{\chi}^{t'}\mu^{r}\bar{\chi}^{t}\eta.\]

These weights will be referred to as the Penrose-Rindler (PR) weights, to distinguish them from the more familiar GHP-weights \((p, q)\) in e.g. [5,16], where \(p = r' - r\) and \(q = t' - t\). A second advantage of this notation is that it is indifferent to the scaling of the tetrad.

### 2 General Equations

The conformal equations themselves, \([1]\), are unaffected by the curvature or the fact that \(\psi\) is not constant and so are the same as in [14]:

\[\mathfrak{b}\xi_\ell = -\pi\xi_m - \kappa\xi_{\bar{m}};\tag{8a}\]
\[\mathfrak{b}'\xi_\ell = -\tau\xi_m - \tau\xi_{\bar{m}} - (\phi_{01} + \bar{\phi}_{01}) + \psi;\tag{8b}\]
\[\mathfrak{d}\xi_\ell = -\rho\xi_m - \sigma\xi_{\bar{m}} + \phi_{11};\tag{8c}\]
\[\mathfrak{b}\xi_n = -\tau'\xi_m - \tau'\xi_{\bar{m}} + (\phi_{01} + \bar{\phi}_{01}) + \psi;\tag{8d}\]
\[\mathfrak{b}'\xi_n = -\kappa'\xi_m - \bar{\pi}'\xi_{\bar{m}};\tag{8e}\]
\[\mathfrak{d}\xi_n = -\rho'\xi_m - \sigma'\xi_{\bar{m}} - \bar{\phi}_{00};\tag{8f}\]
\[\mathfrak{b}\xi_m = -\bar{\tau}'\xi_{\bar{m}} - \kappa\xi_n - \phi_{11};\tag{8g}\]
\[\mathfrak{b}'\xi_m = -\bar{\pi}'\xi_{\bar{m}} - \tau\xi_n + \bar{\phi}_{00};\tag{8h}\]
\[\mathfrak{d}\xi_m = -\bar{\sigma}'\xi_{\bar{m}} - \sigma\xi_n;\tag{8i}\]
\[\mathfrak{d}'\xi_m = -\bar{\rho}'\xi_{\bar{m}} - \rho\xi_n + (\phi_{01} - \bar{\phi}_{01}) - \psi;\tag{8j}\]
Here \( Ψ(\text{correcting a minor typo in } [14]) \):

the Ricci scalar (see [15]).

where equation takes the spinor form

Bianchi identities lead to equations for the derivatives of the \( \phi_{ij} \). The anti-self-dual of this equation takes the spinor form

\[
\nabla_{CC'}\phi_{AB} = (\Psi_{ABCD}ε_{D'C'} + Φ_{ABD'C'}ε_{DC'}) \xi^{DD'} - Λ(ε_{BC}\xi_{AC'} + \epsilon_{AC}\xi_{BC'}) +
\]

\[
\frac{1}{2}(ε_{AC}\psi_{BC'} + \epsilon_{BC}\psi_{AC'})
\]

(9)

Here \( Ψ_{ABCD} \) is the (totally symmetric) Weyl spinor, \( Φ_{ABA'B'} \) the Ricci spinor and \( 24\Lambda = R \), the Ricci scalar (see [15]).

The components of the Weyl and Ricci spinors are given in [15] (4.11.6) and (4.11.8) respectively, and then resolving equation (9) we get the (first) integrability conditions

\[
\]

\[
(10a)
\]

\[
(10b)
\]

\[
(10c)
\]

\[
(10d)
\]

\[
(10e)
\]

\[
(10f)
\]

\[
(10g)
\]

\[
(10h)
\]

\[
(10i)
\]

\[
(10j)
\]

\[
(10k)
\]

\[
(10l)
\]

where \( Π = \chi Λ \). These equations, which can also be obtained from applying the commutators to equations (8), are equivalent to equations (20)–(22) in [11].

| \( r' \) | \( t' \) | \( t \) | \( r \) |
|---|---|---|---|
| 0 | 0 | 1 | 0 |
| −1 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | −1 | 0 | −1 |

Table 1: weights of components
Note that there are four pairs of equations with the same Weyl curvature terms ($c/i; d/j; e/k$ and $f/l$). We can eliminate the Weyl curvature terms between these pairs to give equations equivalent to (23) in [11]:

\[ \begin{align*}
\mathcal{P} \phi_{01} + \mathcal{P}^\prime \phi_{11} - \kappa \phi_{00} - 2 \rho \phi_{01} - \tau \phi_{11} - \frac{3}{2} \mathcal{B} \psi &= (\Phi_{11} - 3 \Pi) \xi_\ell + \Phi_{00} \xi_n - \Phi_{10} \xi_m - \Phi_{01} \xi_{\overline{m}} \\
\mathcal{D} \phi_{01} + \mathcal{B} \phi_{11} - \sigma \phi_{00} - 2 \tau \phi_{01} - \rho \phi_{11} - \frac{3}{2} \mathcal{D} \psi &= \Phi_{12} \xi_\ell + \Phi_{01} \xi_n - (\Phi_{11} + 3 \Pi) \xi_m - \Phi_{02} \xi_{\overline{m}} \\
\mathcal{D}^\prime \phi_{01} + \mathcal{P} \phi_{00} - \rho \phi_{00} - 2 \tau \phi_{01} - \sigma \phi_{11} + \frac{3}{2} \mathcal{D}^\prime \psi &= -\Phi_{21} \xi_\ell - \Phi_{10} \xi_n + \Phi_{20} \xi_m + (\Phi_{11} + 3 \Pi) \xi_{\overline{m}} \\
\mathcal{B}^\prime \phi_{01} + \mathcal{D} \phi_{00} - \tau \phi_{00} - 2 \rho \phi_{01} - \kappa \phi_{11} + \frac{3}{2} \mathcal{B}^\prime \psi &= -\Phi_{22} \xi_\ell - (\Phi_{11} - 3 \Pi) \xi_n + \Phi_{21} \xi_m + \Phi_{12} \xi_{\overline{m}}
\end{align*} \] (11a, 11b, 11c, 11d)

All these equations are easily checked to be consistent as far as spin and boost weight are concerned, and reduce to the equations of [14] for a homothety in vacuum.

### 3 Second integrability conditions

Since a conformal transformation preserves the conformal structure, \( \mathcal{L}_\xi C^a_{bcd} = 0 \), and resolving the spinor version of this equation leads to second integrability conditions involving the \( \Psi_i \). The Ricci tensor is not preserved under a conformal transformation, but instead we have, cf. [6], for the trace-free Ricci tensor and Ricci scalar

\[ \mathcal{L}_\xi S_{ab} = 2 \psi_{;ab} - \frac{1}{2} (\psi_{;cd} g^{cd}) g_{ab} \quad \mathcal{L}_\xi R = -2 \psi R + 6 (\psi_{;cd} g^{cd}). \] (12)

Resolving these gives second integrability conditions involving the \( \Phi_{ij} \) and \( \Pi \). The same integrability conditions arises from applying the commutators to the components of the conformal bivector of course. Using the Bianchi identities and the GHP notation these equations can be reduced to a compact form. Firstly, define the zero weight derivative operator

\[ \mathcal{L}_\xi = \xi_\ell \mathcal{P} + \xi_n \mathcal{P}^\prime - \xi_m \mathcal{D} - \xi_{\overline{m}} \mathcal{D}^\prime, \]

and let

\[ X_{00} = \phi_{00} - \kappa \xi_\ell - \tau \xi_n + \sigma \xi_m + \rho \xi_{\overline{m}}, \quad X_{11} = \phi_{11} + \kappa \xi_n + \tau \xi_\ell - \sigma \xi_{\overline{m}} - \rho \xi_m. \]

(Note that under the Sachs * operation, \( X_{11} \) and \( X_{00} \) are unchanged but \( \overline{X}_{11} = X_{00} \) and \( \overline{X}_{00} = X_{11} \)). Then we find that for the Weyl tensor components

\[ \mathcal{L}_\xi \Psi_i + 2 \psi \Psi_i = i X_{00} \Psi_{i-1} - 2 (2 - i) \phi_{01} \Psi_i + (i - 4) X_{11} \Psi_{i+1}, \] (13)
The Ricci tensor components are more involved because of the presence of the second derivatives of the divergence. I will write them as

\[ L_\xi \Phi_{ab} + 2\psi \Phi_{ab} + \Upsilon_{ab} \psi = aX_{00} \Phi_{(a-1)b} + bX_{00} \Phi_{a(b-1)} + (a - 2)X_{11} \Phi_{(a+1)b} \]
\[ + (b - 2)X_{11} \Phi_{a(b+1)} - 2((1 - a) \phi_{01} + (1 - b) \bar{\phi}_{01}) \Phi_{ab} \]

where \( \Upsilon_{ab} \) are differential operators given below. Equations (14) are equivalent to Collinson and French’s equations (2.2) [3] and Kolassis and Ludwig’s equations (43)–(45) [9]; equations (14) are equivalent to [9] equations (47)–(49).

The operators \( \Upsilon_{ab} \) have the same symmetry properties under conjugation as \( \Phi_{ab} \) and are

\[
\begin{align*}
\Upsilon_{00} &= b^2 + \pi \delta + \kappa \delta' \\
\Upsilon_{01} &= \delta b + \rho \delta + \sigma \delta' = b \delta + \sigma \delta' + \kappa \delta + \tau \delta' \\
\Upsilon_{02} &= \sigma^2 + \pi' b + \kappa \delta' \\
\Upsilon_{11} &= \frac{1}{2} (b' \delta + \sigma' \delta + \kappa' \delta' + \tau' \delta + \rho' \delta + \tau \delta') \\
\Upsilon_{12} &= \delta b' + \rho \delta + \sigma \delta' = b' \delta + \sigma' \delta + \kappa \delta + \tau \delta' \\
\Upsilon_{22} &= (b')^2 + \kappa \delta + \pi \delta'
\end{align*}
\]

along with the complex conjugates. The alternate forms here arise from the commutators.

As Geroch points out in the appendix to [4], see also [6], a conformal vector is given locally by its values and first two derivatives at a point. So there are no further true integrability conditions.

4 Surface homogenous metrics

Brinkmann’s Theorem (see e.g. [16]), which follows from equation (12), proves the only vacuum metrics with a proper conformals are pp waves, which have been much studied, see e.g. [12]. So for an application we consider the case of a metrics with a \( G_3 \) of motions on a spacelike surface [16], compare [17] and references therein which classified all the spherical symmetric cases. We will begin with the metric in null coordinates:

\[
ds^2 = 2e^{2F(u,v)} du dv - e^{2X(u,v)} (dx^2 + \Sigma^2(x) dy^2).
\]

Here \( \Sigma(x) = \sin x \) for the spherically symmetric case, \( \Sigma(x) = \sinh(x) \) for pseudo-spherical case and \( \Sigma(x) = 1 \) for the plane symmetric case. The (isometric) isotropy implies such metrics are Petrov type D or O and the Ricci tensor has at least two equal eigenvalues which must correspond to spacelike eigenvectors [16]. The Kimura metric considered as an example in [11] is a special case of this metric (see later).
The obvious normalised Newman-Penrose tetrad
\[ \ell_a = e^F dv, \quad n_a = e^F du, \quad m_a = \frac{1}{\sqrt{2}} e^X (dx + i\Sigma(x) dy), \]
is a Petrov canonical tetrad: of the \( \Psi_i \) only
\[ \Psi_2 = -\frac{1}{3} e^{-2F} (X_{uv} - F_{uv}) + \frac{1}{6} e^{-2X} \Sigma_{xx} \Sigma^{-1} \]
is non-zero and is also close to a Ricci canonical tetrad, as only \( \Phi_{00}, \Phi_{11}, \Phi_{22} \) and \( \Pi \) are not identically zero (see appendix) so the Ricci tensor is diagonalisable, but not necessarily diagonalisable over \( \mathbb{R} \). We find that the only non-zero spin coefficients of good weight are
\[ \rho = e^{-F} X_u, \quad \rho' = -e^{-F} X_v, \]
with the other spin coefficients also being real.

As \( \Sigma_{xx}/\Sigma = -\frac{1}{2} K \) where \( K \) is the (constant) Gaussian curvature of the Killing orbits, we can see that these metrics are conformally flat iff \( X - F = \log |u - K\nu| \) for \( K = \pm 1 \), or separable \( (X - F = P(u) + Q(v)) \) for \( K = 0 \). We assume from hence that the metric is type D, since the conformally flat case is known to have 15 independent conformal vectors. See also [2] for an analysis of these cases.

Suppose we have a conformal vector in a type D surface homogenous space-time [16]. Then equations (13) for \( i = 1, 3 \) give us \( \phi_{11} = \rho \xi_m \) and \( \phi_{00} = -\rho' \xi_m \). If we substitute these expressions into equation (10d) and use the conformal equations and the curvature equation \( \psi' \rho = \rho' - \Psi_2 - 2\Pi \), we find that \( \delta \psi = 0 \), and \( \partial \psi = 0 \) follows as \( \psi \) is real and zero-weighted.

We can recast (10c) too: we find that
\[ \partial \psi = \xi_\ell (-\rho \partial' + \Psi_2 + 2\Pi) - \xi_n (\rho^2 + \Phi_{00}) - \rho (\phi_{01} + \phi_{01} - \psi) . \]
But the conformal equations and the curvature equations mean that this is equivalent to
\[ \partial (\psi + \rho' \xi_\ell + \rho \xi_n) = 0 . \]
Similarly, (10f) gives \( \partial' (\psi + \rho' \xi_\ell + \rho \xi_n) = 0 . \) But each term in the real zero-weighted scalar \( C = \psi + \rho' \xi_\ell + \rho \xi_n \) is also annihilated by \( \delta \) and \( \delta' \), and hence \( C \) is a constant.

Using equations (10i) to (10l) we can now find explicit equations for the derivatives of \( \phi_{01} \). If we set \( \phi_{01} = A + iB \) for real weight \((0,0)\) scalars \( A \) and \( B \) we get
\[ \partial B = \partial' B = \delta A = \delta' A = 0 \]
and
\[ \delta B = -i \xi_m (\rho \partial' + \Psi_2 - \Phi_{11} - \Pi) \quad (17) \]
\[ \partial A = \frac{1}{2} \left( \xi_\ell (3\Psi_2 + 2\Phi_{11}) - \xi_n \Phi_{00} - \rho (2A + C) \right) \quad (18) \]
\[ \partial' A = \frac{1}{2} \left( -\xi_n (3\Psi_2 + 2\Phi_{11}) + \xi_\ell \Phi_{22} - \rho' (2A - C) \right) . \quad (19) \]
The conformal equations now take the form

\[ \begin{align*}
\xi_{\ell}' &= -2A - \rho' \xi_{\ell} - \rho \xi_n + C, \\
\xi_n' &= 2A - \rho' \xi_{\ell} - \rho \xi_n + C, \\
\xi_m' &= -\rho \xi_m, \quad \xi_m'' = -\rho' \xi_m \\
\xi_{\ell}' &= 2iB - C, \\
\xi_n' &= \delta \xi_{\ell} = \delta \xi_n = \delta \xi_m = 0,
\end{align*} \tag{20}\]

plus their conjugates.

The only equations left to be considered are the remaining second integrability equations. Most of these turn out to be already satisfied modulo the conformal equations, curvature equations and Bianchi identities. If we also make use of the commutators and the fact that \( \delta \) and \( \delta' \) annihilate all the spin coefficients and curvature components — which is most easily checked from the coordinate form of the metric — we find there is only one second integrability equation left:

\[ 4 \left( \rho \rho' + \Psi_2 - \Pi - \Phi_{11} \right) C = 0. \tag{25} \]

Now the term in brackets on the right hand side of (17) and the left hand side of (25) occurs in the commutators, so can be written in terms of spin coefficients not of good weight: it is

\[ 4 \alpha^2 - 2\delta \alpha = \frac{1}{2} e^{-2\Sigma_{xx}/\Sigma} = -\frac{1}{2} K e^{-2x}, \]

where \( K \) is the (constant) Gaussian curvature of the Killing orbits [15]. So in the plane symmetric case we have \( B \) constant from (17) etc, and in the other cases \( C = 0 \) from (25).

The 6 unknowns fall naturally into disjoint sets \( \{ \xi_{\ell}, \xi_n, A \} \) and \( \{ \xi_m, \xi_m, B \} \), where everything in the first set is annihilated by both \( \delta \) and \( \delta' \), which is easily seen to be equivalent to being independent of \( x \) and \( y \).

In the non-plane cases, or wherever \( C = 0 \), these sets are completely decoupled. But it is a simple matter to solve for the terms \( \xi_m \) and \( \xi_m \) in the plane symmetric case to get (with \( \Sigma(x) = 1 \), i.e. Cartesian coordinates)

\[ -\xi_m \overline{m}^a - \xi_m m^a = 2B(\partial_x y - x \partial_y) + k_1 \partial_x + k_2 \partial_y + C(x \partial_x + y \partial_y), \]

where constants \( B, k_1 \) and \( k_2 \) give the known Killing vectors in the spacelike surface. These are the only possible conformals vectors lying completely in the Killing orbits, as \( \xi_{\ell} = \xi_n = 0 \) implies \( A = C = 0 \). The term with \( C \) is the standard homothety in the plane, which from the equations is coupled to the components orthogonal to the Killing orbit. That such coupling must be present also follows from the fact that we cannot have a conformal vector with a fixed point at which the divergence in non-zero in Petrov type D [7]: as \( C = \psi + \rho' \xi_{\ell} + \rho \xi_n \), any conformal vector with \( C \) non-zero cannot be tangent to the orbit of the known Killing vector. Since we can subtract Killing vectors from any proper conformal vector, it follows that in looking for other conformal vectors in a plane symmetric metric we can assume we have \( B = 0, \xi_m = Ce^X(x + iy)/\sqrt{2} \) and solve the remaining equations for \( \xi_{\ell}, \xi_n \) and \( A \).
For the non-plane cases the left hand side of equation \((17)\) is \(\frac{1}{2}iKe^{-2X}\xi_m\), where the Gaussian curvature \(K = \pm 1\). So using equation \((23)\) \(\delta\delta B = -Ke^{-2X}B\), or in coordinates \(\nabla^2 B = -2KB\) for \(\nabla^2\) the Laplacian on the Killing orbit. As is well-known this equation has exactly three independent solutions. In our coordinates they are

\[
\Sigma'(x), \quad \Sigma(x) \sin y, \quad \text{and} \quad \Sigma(x) \cos y.
\]

We have now solved the \(\{\xi_m, \xi_m, B\}\) set, since \(\xi_n = \xi_\ell = 0\) are again the three known Killing vectors. Similarly to the plane symmetric case, we can restrict the search for proper conformal vectors in the non-plane cases to those vectors with \(\xi_m = B = 0\).

At this stage it is as well to convert to a coordinate form of the main conformal equations. Firstly, equations \((24)\) implies that \(\xi_\ell = Y^2(v)e^{F(u,v)}\) and \(\xi_n = Y^1(u)e^{F(u,v)}\).

The surface orthogonal part of the conformal vector, \(\xi_\ell a + \xi_n a\), is then \(Y^1(u)\partial_u + Y^2(v)\partial_v\) and \(\psi = C + Y^1X_u + Y^2X_v = C + \xi^a X_a\).

The other derivatives of \(\xi_n\) and \(\xi_\ell\) can be combined to give expressions for \(A\) and \(C\).

Writing \(Z\) for \(F - X\) equations \((20)\) and \((21)\) give

\[
\begin{align*}
4A &= Y^1_{,u} + 2Y^1F_{,u} - Y^2_{,v} - 2Y^2F_{,v} \quad (26) \\
2C &= Y^1_{,u} + 2Y^1Z_{,u} + Y^2_{,v} + 2Y^2Z_{,v} \quad (27)
\end{align*}
\]

These expressions for \(A\) and \(C\) can then be substituted into the integrability conditions \((18)\) and \((19)\) to give

\[
\begin{align*}
Y^1_{,uu} + 2Z_{,u} Y^1_{,u} + 2Z_{,uu} Y^1 &= -2Z_{,uv} Y^2 \\
Y^2_{,vv} + 2Z_{,v} Y^2_{,v} + 2Z_{,vv} Y^2 &= -2Z_{,uv} Y^1.
\end{align*}
\]

However, in coordinate form it is easy to see that both these latter equations are identically satisfied modulo equation \((27)\), a fact that can be proved in the GHP notation with sufficient work. Since we can consider equation \((26)\) as defining \(A\), with a little algebra we have, cf. \([13]\):

**Theorem 1** The surface homogeneous metric

\[
\begin{equation}
\begin{aligned}
ds^2 &= e^{2X(u,v)} \left(2e^{2Z(u,v)} du dv - (dx^2 + \Sigma^2(x)dy^2)\right),
\end{aligned}
\end{equation}
\]

if type D, admits the conformal vector

\[
\xi^a = Y^1(u)\partial_u + Y^2(v)\partial_v + C \left(x\partial_x + y\partial_y\right)
\]

iff the equation

\[
2C = Y^1_{,u} + 2Y^1Z_{,u} + Y^2_{,v} + 2Y^2Z_{,v}
\]

is satisfied for constant \(C\), which must be zero in the non-plane symmetric cases.

The divergence \(\psi\) of \(\xi^a\) is then \(C + \xi^a X_a = (X + Z)_{,a} \xi^a + \frac{1}{2} \left(Y^1_{,u} + Y^2_{,v}\right)\).
The form of the metric in this theorem is doubly convenient in that it shows these metrics to be conformally reducible, in the sense of Carot and Tupper [2], by illustrating the conformal scaling which is irrelevant to the conformal vector: we expect a free conformal factor in the metric if all we do is look for conformal vectors. This factor would be fixed by the field equations.

Tupper et al [17] classified the spherically symmetric cases according to the possible extra Killing or homothetic vectors. Their analysis applies with no change to the case where the Killing orbit has negative curvature.

Here I will merely point out that by the Bilyalov–Defrise-Carter Theorem [1,7] and isotropy considerations, the maximal size of the conformal algebra is 6, and in these cases the metric is locally conformal to a metric with a 6-parameter group of motions, and hence locally conformal to a product of two 2-spaces of constant curvature, [16]. Any metric with $Z_{uv} = 0$, for example, is easily shown to have a 6 parameter conformal algebra.

One explicit example with the maximal conformal algebra is the spherically symmetric Kimura case [8] considered in [10,11]. Changing the coordinates in those references to $u = t - r^{-1}$, $v = t + r^{-1}$ we have $F = -\log(u - v) + \log(b_0)$, $Z = -\log b_0$ in [16] and solving (27) with $C = 0$ we have

$$Y^1(u) = k_1 u + k_2 + k_3, \quad Y^2(v) = -k_1 v - k_2 + k_3,$$

which are easily seen to give the two proper conformal vectors ($k_1$ and $k_2$) and fourth Killing vector ($k_3$) given by in [10,11].

A similar calculation can be performed in the case of a metric with a three-parameter Killing group on time-like surfaces. We start with the metric in the form

$$ds^2 = e^{2X(\zeta,\bar{\zeta})} (\Sigma^2(r)dt^2 - dr^2) - 2e^{2F(\zeta,\bar{\zeta})} d\zeta d\bar{\zeta}$$

(31)

see [16], but note I have chosen conformally flat coordinates in the surface orthogonal to the Killing orbit. Since the metrics in the two cases are the same under the complex coordinate transformations

$$(u, v, x, y) \mapsto (\zeta, \bar{\zeta}, i r, i t)$$

plus a switch of signature (essentially the Sachs $*$ operation), similar results arise.

The obvious normalised tetrad

$$\ell_a = \frac{1}{\sqrt{2}} e^X (\Sigma dt + dr), \quad n_a = \frac{1}{\sqrt{2}} e^X (\Sigma dt - dr), \quad m_a = e^F d\zeta$$

is again a Petrov canonical tetrad, with only $\tau = \bar{\tau}'$ non-zero spin coefficients and $\Psi_2$, $\Phi_{11}$, $\Phi_{02}$, $\Phi_{20}$, $\Pi$ non-zero in the curvature. The unknowns split into the same two disjoint sets $\{\xi_\ell, \xi_m, A\}$ and $\{\xi_m, \xi_{\bar{m}}, B\}$ where $\phi_{01} = A + i B$. with everything in the second set annihilated by $\psi$ and $\psi'$. The equations for the first set solve to give the three Killing vectors, and so only the equations for the second set remain:

$$\begin{align*}
\psi \xi_m & = \psi' \xi_m = \delta \xi_m = 0 \\
\delta' \xi_m & = 2i B - \tau \xi_{\bar{m}} - \bar{\tau} \xi_m - C \\
2 \delta B & = -2\tau B - i \phi_{02} \xi_{\bar{m}} - i (3\Psi_2 - 2\Phi_{11}) \xi_m
\end{align*}$$

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plus their conjugates, for constant $C$ which is zero if the Killing orbit is not flat. All other equations are identically satisfied modulo these and the curvature equations etc as before.

Writing the spatial part of the conformal vector as $-\xi_m m^a - \xi_m m^a = Y \partial_\zeta + \overline{Y} \partial_{\overline{\zeta}}$, the $p$ and $p'$ equations show that $Y$ is a function of $(\zeta, \overline{\zeta})$ only, and the remaining equations reduce to the Cauchy-Riemann equations, $Y_{\overline{\zeta}} = 0$ and the equivalent of equation (27)

$$2C = Y_\zeta + \overline{Y}_{\overline{\zeta}} + 2(Y Z_\zeta + 2\overline{Y} Z_{\overline{\zeta}}),$$

where $Z(\zeta, \overline{\zeta}) = X(\zeta, \overline{\zeta}) - \lambda(\zeta, \overline{\zeta})$, or serve to define $B$:

$$4B(\zeta, \overline{\zeta}) = i (Y_\zeta - \overline{Y}_{\overline{\zeta}} + 2Y \lambda_\zeta - 2\overline{Y} \lambda_{\overline{\zeta}}).$$

The appearance of the Cauchy-Riemann equations is to be expected, since we are essentially looking for conformals on a Riemannian 2-space. So

**Theorem 2** The timelike-surface homogeneous metric

$$ds^2 = e^{2X(\zeta, \overline{\zeta})} \left( \Sigma^2(r) dt^2 - dr^2 - 2e^{2Z(\zeta, \overline{\zeta})} d\zeta d\overline{\zeta} \right),$$

admits the conformal vector

$$\xi^a = Y(\zeta) \partial_\zeta + \overline{Y}(\overline{\zeta}) \partial_{\overline{\zeta}} + C(t \partial_t + r \partial_r)$$

iff $Y(\zeta)$ is analytic and the equation

$$2C = Y_\zeta + \overline{Y}_{\overline{\zeta}} + 2(Y Z_\zeta + 2\overline{Y} Z_{\overline{\zeta}})$$

is satisfied for constant $C$, which must be zero in the non-plane symmetric cases.

The divergence $\psi$ of $\xi^a$ is then $C + \xi^a X_a = (X + Z)_a \xi^a + \frac{1}{2} (Y_\zeta + \overline{Y}_{\overline{\zeta}})$.

An analysis similar to that of [17] for the spherically symmetric situation can obviously be carried out in this case too.

As in the spherical case we can find a type D metric of the form (31) admitting the maximal 6-parameter conformal group. One such example is given by setting $X(\zeta, \overline{\zeta}) = \ln(\zeta + \overline{\zeta}) + \lambda(\zeta, \overline{\zeta})$ when the metric admits, in addition to the Killing vectors, the conformal vectors

$$\xi^a = (k_1 i \zeta^2 + k_2 \zeta + k_3 i) \partial_\zeta + \text{conjugate}$$

with $k_i$ real constants.
Appendix: Components of the Curvature

In the surface-homogeneous metric (16) with Newman-Penrose tetrad as given the non-zero Ricci tensor components are

\[ \Phi_{11} = \frac{1}{2} e^{-2F} (X_u X_v - F_{uv}) - \frac{1}{4} e^{-2X} \Sigma_{xx} \Sigma^{-1}, \]
\[ \Phi_{00} = \Phi_{22} = e^{-2F} \left( 2F_{,u} X_{,u} - X_{,u}^2 \right), \]
\[ \Pi = \frac{1}{6} e^{-2F} (F_{,uv} + 2X_{,uv} + 3X_{,u}X_{,v}) - \frac{1}{12} e^{-2X} \Sigma_{xx} \Sigma^{-1}. \]

In the timelike case, metric (31), with Newman-Penrose tetrad as given the non-zero curvature components are

\[ \Psi_2 = \frac{1}{3} e^{-2\lambda} (\lambda - X)_{,\xi} + \frac{1}{6} e^{-2X} \Sigma_{rr} \Sigma^{-1}, \]
\[ \Phi_{11} = \frac{1}{2} e^{-2\lambda} (X_{,\xi} X_{,\xi} - \lambda_{,\xi}) + \frac{1}{4} e^{-2X} \Sigma_{rr} \Sigma^{-1}, \]
\[ \Phi_{20} = \Phi_{02} = e^{-2\lambda} \left( 2X_{,\xi} \lambda_{,\xi} - X_{,\xi}^2 - X_{,\xi} \right), \]
\[ \Pi = -\frac{1}{6} e^{-2\lambda} \left( \lambda_{,\xi} + 2X_{,\xi} + 3X_{,\xi} X_{,\xi} \right) - \frac{1}{12} e^{-2X} \Sigma_{rr} \Sigma^{-1}. \]

Acknowledgements

Calculations were carried out using Maple, and in particular the GHPII package of Vu and Carminati [18]. Maple is a registered trademark of Waterloo Maple Inc.

5 References

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