LEAST ENERGY SOLUTIONS FOR FRACTIONAL KIRCHHOFF TYPE EQUATIONS INVOLVING CRITICAL GROWTH

YINBIN DENG*
School of Mathematics and information, Guangxi University
Nanning 530004, China
& Department of Mathematics, Central China Normal University
Wuhan 430079, China

WENTAO HUANG
School of Science, East China JiaoTong University
Nanchang 330013, China

Abstract. We study the following fractional Kirchhoff type equation:
\[
\begin{cases}
(a + b \int_{\mathbb{R}^3} |(-\Delta)^s u|^2 dx)(-\Delta)^s u + V(x)u = f(u) + |u|^{2^*_s - 2}u, \ x \in \mathbb{R}^3, \\
u \in H^s(\mathbb{R}^3),
\end{cases}
\]
where \(a, b > 0\) are constants, \(2^*_s = \frac{6}{3 - 2s}\) with \(s \in (0, 1)\) is the critical Sobolev exponent in \(\mathbb{R}^3\), \(V\) is a potential function on \(\mathbb{R}^3\). Under some more general assumptions on \(f\) and \(V\), we prove that the given problem admits a least energy solution by using a constrained minimization on Nehari-Pohozaev manifold and monotone method.

1. Introduction and main results. In this paper, we are concerned with the following fractional Kirchhoff type equation with critical growth:
\[
\begin{cases}
(a + b \int_{\mathbb{R}^3} |(-\Delta)^s u|^2 dx)(-\Delta)^s u + V(x)u = f(u) + |u|^{2^*_s - 2}u, \ x \in \mathbb{R}^3, \\
u \in H^s(\mathbb{R}^3),
\end{cases}
\]
where \(a, b > 0\) are constants, \(2^*_s = \frac{6}{3 - 2s}\) with \(s \in (0, 1)\) is the critical Sobolev exponent in \(\mathbb{R}^3\), the potential \(V \in C^1(\mathbb{R}^3, \mathbb{R})\) and \(f \in C(\mathbb{R}, \mathbb{R})\) is a subcritical perturbation. The fractional Laplacian \((-\Delta)^s\) is a nonlocal operator defined in the Schwartz class \(\mathcal{S}(\mathbb{R}^3)\) as
\[
(-\Delta)^s u(x) = C(s) P.V. \int_{\mathbb{R}^3} \frac{u(x) - u(y)}{|x - y|^{3+2s}} dy,
\]
where \(C(s)\) is a suitable normalized constant and \(P.V.\) means the Cauchy principle value on the integral.

When \(a = 1\) and \(b = 0\), equation (1) can be reduced to the usual fractional Schrödinger equation:
\[
(-\Delta)^s u + V(x)u = f(u) + |u|^{2^*_s - 2}u, \ x \in \mathbb{R}^3.
\]
The fractional Schrödinger equation was introduced by Laskin [31, 32] in the context of fractional quantum mechanics, as a result of expanding the Feynman path integral, from the Brownian like to the Lévy like quantum mechanical paths. In particular, the fractional Laplacian can be understood as the infinitesimal generator of a stable Lévy diffusion processes [2]. It also has various applications in different subjects, such as, the thin obstacle problem [35, 42], optimization [19], finance [12], conservation laws [4], minimal surfaces [6, 8] and see [5] for further details. The non-locality of the fractional Laplacian makes it difficult to study. To overcome this difficulty, Caffarelli and Silvestre [7] introduced the extension method that reduced this nonlocal problem into a local one in higher dimensions. This extension method has been applied successfully and a series of fruitful results have been obtained. By using the Mountain Pass and Linking Theorems, the authors in [39] proved the existence of weak solutions for the nonlocal problem involving fractional Laplacian operators, see also the works of [18, 20] in the respect of variational methods. For related investigations on problem (2) involving critical growth, we refer the readers to [10, 17, 41] and the references therein.

When $s = 1$, equation (1) can be reduced to the following Kirchhoff type problem

$$\begin{cases} \quad - \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right) \Delta u + V(x)u = f(x, u), \quad x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \quad (3)$$

which is related to the stationary analogue of the equation

$$\begin{cases} \quad u_{tt} - \left( a + b \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u = f(x, u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial \Omega, \end{cases} \quad (4)$$

where $f(x, u)$ is a given nonlinear function in $\mathbb{R}^3 \times \mathbb{R}^1$. Equation (4) was proposed by Kirchhoff in 1883 as a generalization of the classical D’Alembert’s wave equations for free vibration of elastic strings, see [30]. Kirchhoff’s model takes into account the changes in length of the string produced by transverse vibrations. In [1], the authors pointed out that the equation (4) models several physical systems, where $u$ describes a process which depends on the average of itself. Nonlocal effect also finds its application in biological systems. For more mathematical and physical background on equation (4), we refer the readers to [3, 9, 14] and the references therein.

Recent studies have been focused on problem (3). In [26], He and Zou studied problem (3) in the case that $f(x, t) = h(t) \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfies the variant Ambrosetti-Rabinowitz type condition ((AR) in short): i.e., for some $\theta > 4$

$$0 < H(t) := \int_0^t h(s) \, ds \leq \frac{1}{\theta} h(t) t \quad \text{for all } t > 0. \quad (5)$$

By using the Mountain Pass Theorem and the Nehari manifold, they obtained the existence and concentration behavior of ground state solutions. For the case when $f(x, t) = |t|^{p-2}t$ and $3 < p \leq 4$, which does not satisfy the variant (AR) condition (5), by constructing a constrained minimization on a new manifold based on the Nehari manifold and the Pohozaev identity, Li and Ye in [33] showed that the existence of a positive ground state solution for the corresponding limiting problem of (3). Then by using a monotone method and a global compactness lemma, they proved that (3) has a positive ground state solution. After that, Guo
[23] generalized the result in [33] to problem (3) with \( f(x, t) = h(t) \in C^1(\mathbb{R}^+, \mathbb{R}) \) satisfies the following general assumptions:

\[
\begin{align*}
(h_1) & \quad h(t) = o(t) \text{ as } t \to 0^+; \\
(h_2) & \quad \lim_{t \to +\infty} \frac{h(t)}{t^4} = 0; \\
(h_3) & \quad \lim_{t \to +\infty} \frac{h(t)}{t} = +\infty; \\
(h_4) & \quad \frac{h(t)}{t} \text{ is strictly increasing in } (0, +\infty).
\end{align*}
\]

By establishing a new Nehari-Pohozaev manifold and using a constrained minimization on the intersection of the manifold and \( H^1(\mathbb{R}^3) \), Guo proved the limiting problem of (3) has a positive ground state solution. Combining this result and a technical condition on \( V \):

\[
(V'_1) \quad V \in C^1(\mathbb{R}^3, \mathbb{R}) \text{ and there exists a positive constant } A < a \text{ such that } |(\nabla V(x), x)| \leq \frac{A}{2|x|^2}, \quad \text{for all } x \in \mathbb{R}^3 \setminus \{0\},
\]

Guo obtained a positive ground state solution for (3).

Recently, Tang and Chen [44] generalized and improved the results in [23] and [33] to problem (3) with \( f(x, t) = h(t) \in C(\mathbb{R}, \mathbb{R}) \) satisfies the assumptions \((h_1) - (h_3)\) and replacing \((h_4)\) and \((V'_1)\) by the conditions

\[
(h_5) \quad \frac{h(t)t + 6H(t)}{|t|^2} \text{ is nondecreasing on } (-\infty, 0) \cup (0, \infty)
\]

and

\[
(V'_2) \quad V \in C^1(\mathbb{R}^3, \mathbb{R}) \text{ and there exists } \theta \in [0, 1) \text{ such that } 4t^4(V(x) - V(tx)) - (1 - t^4)(\nabla V(x), x) \geq -\frac{\theta a(1 - t^2)^2}{2|x|^2}, \quad \forall t \geq 0, \ x \in \mathbb{R}^3 \setminus \{0\},
\]

respectively. It should be mentioned that the assumption \((h_5)\) is weaker than \((h_4)\), but the assumption \((V'_2)\) is stronger than \((V'_1)\). To obtain a ground state solution of Nehari-Pohozaev type for (3), they took the minimum on the manifold \( \mathcal{M} := \{ u \in H^1(\mathbb{R}^3) \setminus \{0\} \mid J(u) = 0 \} \), where

\[
J(u) := a \int_{\mathbb{R}^3} |\nabla u|^2dx + \frac{1}{2} \int_{\mathbb{R}^3} \left( 4V(x) + (\nabla V(x), x) \right) u^2dx + b \left( \int_{\mathbb{R}^3} |\nabla u|^2dx \right)^2 - \frac{1}{2} \int_{\mathbb{R}^3} \left( h(u)u + 6H(u) \right)dx.
\]

However, different from [23], Tang and Chen [44] did not assume that \( h \in C^1 \). Then \( \mathcal{M} \) may not be a \( C^1 \)-manifold of \( H^1(\mathbb{R}^3) \), which means that the method used in [23] and [33] is no longer applicable. To prove that the minimum of the functional on \( \mathcal{M} \) is a critical point, a new idea was developed in [44].

The existence results for (3) has been well developed in recent years when \( f(x, t) \in C^1(\mathbb{R}^3, \mathbb{R}) \), we refer the readers to [11, 15, 27, 28, 24, 25, 46, 48] and the references therein.

As we have mentioned above, there are many papers dealt with problems (2) and (3), which have only one nonlocal term. The case that the variational problem involve double or more nonlocal terms is much more complicated and has been less studied. Very recently, Fiscella and Valdinoci in [22] proposed a stationary Kirchhoff variational model in bounded domains of \( \mathbb{R}^N \), which took into account the nonlocal
where critical growth.

The following conditions:

\[ - 0 \leq V(x) \leq V_\infty := \lim_{|x| \to \infty} V(x) \text{ for all } x \in \mathbb{R}^3; \]

\( V_2 \) there exists a positive constant \( A < a \) such that \( \| \nabla V(x) \| \leq 2sA^S_u \), where \( S_u \) is given by (10) below.

On the nonlinearity \( f \), we also need the following assumptions:

\( f_1 \) \( f \in C(\mathbb{R}, \mathbb{R}) \) and \( f(t) \equiv 0 \) for all \( t \in (-\infty, 0) \);

\( f_2 \) \( \lim_{t \to 0^+} \frac{f(t)}{t} = 0 \) and \( \lim_{t \to +\infty} \frac{f(t)}{t^{2s-1}} = 0 \);

\( f_3 \) \( f(t) = f(0) f(t) + 6F(t) \) is increasing on \((0, +\infty)\), where \( F(t) = \int_0^t f(\tau)d\tau \);

\( f_4 \) there exist \( \mu > 0 \) and \( 2 < p < 2^*_s \) such that \( f(t) \geq \mu t^{p-1} \) for all \( t \geq 0 \).

The variational functional associated with equation (1) is defined by

\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^3} (a((\Delta)^s u)^2 + V(x)u^2)dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |(\Delta)^s u|^2dx \right)^2 - \int_{\mathbb{R}^3} F(u)dx - \frac{1}{2s} \int_{\mathbb{R}^3} |u|^{2s}dx \]

for \( u \in H^s(\mathbb{R}^3) \). We can prove that \( I \in C^1(H^s(\mathbb{R}^3), \mathbb{R}) \) and a critical point of \( I \) in \( H^s(\mathbb{R}^3) \) corresponds to a weak solution of (1). Moreover, we note that the corresponding limiting equation of (1) is:

\[ \begin{cases} 
(a + b \int_{\mathbb{R}^3} |(-\Delta)^s u|^2dx)(-\Delta)^s u + V_\infty u = f(u) + |u|^{2^*_s - 2}u, & x \in \mathbb{R}^3, \\
u \in H^s(\mathbb{R}^3), 
\end{cases} \]
which is the Euler-Lagrange equation associated with the functional
\[
I^\infty(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left( a|(-\Delta)^{\frac{s}{2}} u|^2 + V_\infty u^2 \right) dx + \frac{b}{4} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^4 dx
- \int_{\mathbb{R}^3} F(u) dx - \frac{1}{2} \int_{\mathbb{R}^3} |u|^2^2 dx. \tag{9}
\]

Before stating our main results, we need introduce the following Nehari-Pohozaev manifold:
\[
\mathcal{M}^\infty := \left\{ u \in H^s(\mathbb{R}^3) \setminus \{0\} \mid G^\infty(u) = 0 \right\},
\]
where
\[
G^\infty(u) := 2sa \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + 4s \int_{\mathbb{R}^3} V_\infty u^2 dx + 2sb \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx
- \int_{\mathbb{R}^3} \left[ (4s-3)f(u)u + 6F(u) \right] dx - 2s \int_{\mathbb{R}^3} |u|^2^2 dx.
\]

For any \( u \in \mathcal{M}^\infty \), we define that
\[
m^\infty := \inf_{u \in \mathcal{M}^\infty} I^\infty(u).
\]

Now, our first result can be stated as follows.

**Theorem 1.1.** Assume that \((f_1) - (f_4)\) hold and \( s \in \left( \frac{3}{4}, 1 \right) \). Then problem (8) exists a nonnegative solution \( \tilde{u} \) such that \( I^\infty(\tilde{u}) = m^\infty > 0 \), if either \( \frac{4s}{3-2s} < p < 2^*_s \) for all \( \mu > 0 \) or \( 2 < p \leq \frac{4s}{3-2s} \) for \( \mu > 0 \) sufficiently large.

**Remark 1.** We mention here that, to ensure the associated functional has a Mountain Pass geometry, the condition \( 2^*_s > 4 \) is necessary. So, the parameter \( s \) must lie in \( \left( \frac{3}{4}, 1 \right) \).

To prove Theorem 1.1, the main difficulties lie in three aspects: Firstly, the double nonlocal terms due to the presence of fractional Laplacian operator \( (-\Delta)^{\frac{s}{2}} \) and integral \( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx \) in equation (8), lead to some additional difficulties and make the study become interesting. Secondly, without the assumption that \( f \in C^1(\mathbb{R}, \mathbb{R}) \), then \( \mathcal{M}^\infty \) is not a \( C^1 \)-manifold. It seems difficult to prove that the minimum of \( I^\infty \) on \( \mathcal{M}^\infty \) is a critical point. The arguments in [23] and [33] can not be applied in this paper and some new tricks will be developed. Thirdly, the unboundedness of the domain \( \mathbb{R}^3 \) and the critical Sobolev exponent lead to the lack of compactness. What’s more, to estimate some precise threshold value for \( m^\infty \), we are in situation to solve a factional order algebra equation. Thus more careful analysis is needed.

Based on Theorem 1.1, under the conditions \((V_1) - (V_2)\), we can prove the existence of a least energy solution to equation (1).

**Theorem 1.2.** Assume that \((V_1) - (V_2)\) and \((f_1) - (f_4)\) hold with \( s \in \left( \frac{3}{4}, 1 \right) \). Then problem (1) exists a nonnegative nontrivial least energy solution \( w \), if either \( \frac{4s}{3-2s} < p < 2^*_s \) for all \( \mu > 0 \) or \( 2 < p \leq \frac{4s}{3-2s} \) for \( \mu > 0 \) sufficiently large.

Since we do not assume that \( f \) satisfies the variant \((AR)\) type condition, it seems difficult to get the boundedness of any \((PS)\) sequence even if a \((PS)\) sequence has been obtained. In order to overcome this difficulty, we make use of the monotone
method due to Jeanjean [29]. First, for \( \lambda \in [\delta, 1] \), we introduce a family of \( C^1 \)-functionals defined as
\[
I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left( a|(-\Delta)^s u|^2 + V(x)u^2 \right) dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |(-\Delta)^s u|^2 dx \right)^2 - \lambda \int_{\mathbb{R}^3} F(u) dx - \frac{\lambda}{2s} \int_{\mathbb{R}^3} |u|^{2^*_s} dx,
\]
where \( \delta \in (0, 1) \). By Proposition 1 below, for a.e. \( \lambda \in [\delta, 1] \), there is a bounded \((PS)_{c_\lambda}\) sequence \( \{u_n\} \subset H^s(\mathbb{R}^3) \), where \( c_\lambda \) is given in Lemma 3.1. In general, we do not know \( \int_{\mathbb{R}^3} |(-\Delta)^s u_n|^2 dx \to \int_{\mathbb{R}^3} |(-\Delta)^s u|^2 dx \) only by \( u_n \rightharpoonup u \) in \( H^s(\mathbb{R}^3) \).

Fortunately, by introducing the functional
\[
J_\lambda(u) = \frac{a + bA^2}{2} \int_{\mathbb{R}^3} |(-\Delta)^s u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} V(x)u^2 dx - \lambda \int_{\mathbb{R}^3} F(u) dx - \frac{\lambda}{2s} \int_{\mathbb{R}^3} |u|^{2^*_s} dx,
\]
where \( A^2 = \lim_{n \to \infty} \int_{\mathbb{R}^3} |(-\Delta)^s u_n|^2 dx \), it is easy to check that \( \{u_n\} \) is a \((PS)_{c_\lambda + \frac{bA^2}{2}}\) sequence for \( J_\lambda \) and the weak limit \( u \in H^s(\mathbb{R}^3) \) of \( \{u_n\} \) is a critical point of \( J_\lambda \). To prove that \((PS)_{c_\lambda}\) condition of \( I_\lambda \) for a.e. \( \lambda \in [\delta, 1] \) (\( \delta \) is given in Lemma 3.3), we need establish a version of global compactness lemma related to the functional \( J_\lambda \) and its limiting functional
\[
J_\lambda^\infty(u) = \frac{a + bA^2}{2} \int_{\mathbb{R}^3} |(-\Delta)^s u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} V_\infty u^2 dx - \lambda \int_{\mathbb{R}^3} F(u) dx - \frac{\lambda}{2s} \int_{\mathbb{R}^3} |u|^{2^*_s} dx.
\]
Finally, choosing a critical point sequence \( \{(\lambda_n, u_{\lambda_n})\} \subset [\delta, 1] \times H^s(\mathbb{R}^3) \) with \( \lambda_n \to 1 \) as \( n \to \infty \), we can prove that \( \{u_{\lambda_n}\} \) is a bounded \((PS)_{c_1}\) sequence for \( I = I_1 \). By using the global compactness lemma again, we complete the proof of Theorem 1.2.

The outline of this paper is as follows. In Section 2, we devote to showing the weak sequential continuity of \( I_\lambda \) in \( H^s(\mathbb{R}^3) \) by direct calculations due to the existence of the nonlocal term \( \int_{\mathbb{R}^3} |(-\Delta)^s u|^2 dx \).

Notations. Throughout this paper, we make use of the standard notations as follows. The Hilbert space \( H^s(\mathbb{R}^3) \) is defined as
\[
H^s(\mathbb{R}^3) := \left\{ u \in L^2(\mathbb{R}^3) \mid (-\Delta)^s u \in L^2(\mathbb{R}^3) \right\}.
\]
We endow the space \( H^s(\mathbb{R}^3) \) with the inner product and norm by
\[
(u, v) := \int_{\mathbb{R}^3} (-\Delta)^s u (-\Delta)^s v dx + \int_{\mathbb{R}^3} uv dx
\]
and
\[
\|u\|_{H^s} := \left( \int_{\mathbb{R}^3} |(-\Delta)^s u|^2 + u^2 |dx \right)^{\frac{1}{2}}
\]
respectively, where
\[
\int_{\mathbb{R}^3} |(-\Delta)^s u|^2 dx = \frac{C(s)}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3 + 2s}} dx dy.
\]
It is well known that \( H^s(\mathbb{R}^3) \) is continuously embedded into \( L^q(\mathbb{R}^3) \) for \( 2 \leq q \leq 2^*_s \) and compactly embedded into \( L^q_{loc}(\mathbb{R}^3) \) for \( 1 \leq q < 2^*_s \). \( L^q(\mathbb{R}^3) \) is the usual Lebesgue
space with the standard norms
\[ \|u\|_q := \left( \int_{\mathbb{R}^3} |u|^q dx \right)^{\frac{1}{q}}, \quad 1 \leq q < \infty. \]

From [13, 16] we know that \( D^{s,2}(\mathbb{R}^3) \) is continuously embedded into \( L^{2^*}(\mathbb{R}^3) \) and there exists a best constant \( S_s > 0 \) such that
\[ S_s = \inf_{u \in D^{s,2}(\mathbb{R}^3)} \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx}{\left( \int_{\mathbb{R}^3} |u|^{2^*_s} dx \right)^{\frac{2}{2^*_s}}}, \tag{10} \]
where \( D^{s,2}(\mathbb{R}^3) \) is defined by
\[ D^{s,2}(\mathbb{R}^3) := \left\{ u \in L^{2^*}(\mathbb{R}^3) \mid (-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R}^3) \right\} \]
endowed with the norm
\[ \|u\|_{D^{s,2}} := \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx \right)^{\frac{1}{2}}. \]

From assumption \((V_1)\) and \( a > 0 \) is fixed, we can define an equivalent norm on the fractional Sobolev space \( H^s(\mathbb{R}^3) \) as
\[ \|u\| := \left( \int_{\mathbb{R}^3} (a (-\Delta)^{\frac{s}{2}} u)^2 + V(x)u^2 dx \right)^{\frac{1}{2}}. \]

2. Ground state solutions of Nehari-Pohozaev type for (8). In this section, we will use a constrained minimization on \( \mathcal{M}^\infty \) to get a nonnegative ground state solution of Nehari-Pohozaev type for the limiting problem (8).

**Lemma 2.1.** Under the assumption \((f_3)\), we have that
\[
\frac{(4s-3)(1-t^{8s})}{8s} f(\tau) \tau - 2(4s-3) + 6t^{8s} F(\tau + t^6 F(t^{4s-3}\tau)) \geq 0, \quad \forall t \geq 0, \quad \tau \in \mathbb{R}.
\]

**Proof.** Without loss of generality, we may assume that \( \tau \neq 0 \) and set
\[ h(t) := \frac{(4s-3)(1-t^{8s})}{8s} f(\tau) \tau - 2(4s-3) + 6t^{8s} F(\tau + t^6 F(t^{4s-3}\tau)). \]
By a direct computation, we have
\[ h'(t) = t^{8s-1} \frac{(4s-3)f(t^{4s-3}\tau)t^{4s-3}\tau + 6F(t^{4s-3}\tau)}{t^{2(4s-3)\tau^2}} - \frac{(4s-3)f(\tau) \tau + 6F(\tau)}{\tau^2}. \]
It follows from \((f_3)\) that \( h(t) \geq h(1) = 0, \quad \forall t \geq 0. \) \(\square\)

**Lemma 2.2.** Assume that \((f_1)-(f_3)\) hold. Then for any \( t > 0, \)
\[ I^\infty(u) \geq I^\infty(u_t) + \frac{1-t^{8s}}{8s} G^\infty(u) + \frac{a(1-t^{4s})^2}{4} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx, \quad \forall u \in H^s(\mathbb{R}^3), \]
where \( u_t(x) := t^{4s-3}u(t^{-2}x). \)

**Proof.** \( \forall u \in H^s(\mathbb{R}^3), \ t > 0, \) we set \( u_t(x) := t^{4s-3}u(t^{-2}x), \) it follows that
\[
I^\infty(u_t) = \frac{a}{2} t^{4s} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_t|^2 dx + \frac{1}{2} t^{8s} \int_{\mathbb{R}^3} V_{\infty} u_t^2 dx + \frac{b}{4} t^{8s} \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_t|^2 dx \right)^2 - t^6 \int_{\mathbb{R}^3} F(t^{4s-3}u_t) dx - \frac{1}{2s} t^{2s+2} \int_{\mathbb{R}^3} |u_t|^2 dx.
\]
From Lemma 2.1 and Sobolev inequality, we have
\begin{align*}
I^\infty(u) - I^\infty(u_t) &= \frac{1-t^{4s}}{8s} \left( 2sa \int_{\mathbb{R}^3} |(-\Delta)^{2s}u|^2 \, dx + 4s \int_{\mathbb{R}^3} V_{\infty} u^2 \, dx + 2sb \left( \int_{\mathbb{R}^3} |(-\Delta)^{2s}u|^2 \, dx \right)^2 \
&\quad - \int_{\mathbb{R}^3} \left( (4s-3) f(u) + 6F(u) \right) \, dx - 2s \int_{\mathbb{R}^3} |u|^{2^*} \, dx \right) \
&\quad + \frac{a(1-t^{4s})^2}{4} \int_{\mathbb{R}^3} |(-\Delta)^{2s}u|^2 \, dx \
&\quad + \int_{\mathbb{R}^3} \left[ \frac{(4s-3)(1-t^{4s})}{8s} f(u) u - \frac{2(4s-3) + 6t^{4s}}{8s} F(u) + t^6 F(t^{4s-3}u) \right] \, dx \
&\quad + \int_{\mathbb{R}^3} \left[ \frac{(4s-3)(1-t^{4s})}{8s} - \frac{2(4s-3) + 6t^{4s}}{8s} + \frac{t^{2s-2^*}}{2^*} \right] |u|^{2^*} \, dx \
&\geq \frac{1-t^{4s}}{8s} G^\infty(u) + \frac{a(1-t^{4s})^2}{4} \int_{\mathbb{R}^3} |(-\Delta)^{2s}u|^2 \, dx,
\end{align*}
where we used the fact that
\[ k(t) := \frac{(4s-3)(1-t^{4s})}{8s} - \frac{2(4s-3) + 6t^{4s}}{8s} + \frac{t^{2s-2^*}}{2^*} \geq k(1) = 0, \quad \forall t \geq 0. \]

So, Lemma 2.2 is proved. \(\square\)

**Lemma 2.3.** Under the assumptions (f1) – (f3), then for any \( u \in H^s(\mathbb{R}^3) \setminus \{0\} \), there is a unique \( t_0 > 0 \) such that \( u_{t_0} \in \mathcal{M}^\infty \). Moreover, \( I^\infty(u_{t_0}) = \max_{t > 0} I^\infty(u_t) \).

**Proof.** For any \( u \in H^s(\mathbb{R}^3) \setminus \{0\} \) and \( t > 0 \), we consider \( y(t) := I^\infty(u_t) \). It is easy to check that \( y(t) > 0 \) for \( t > 0 \) small and \( y(t) \to -\infty \) as \( t \to +\infty \), which gives that \( y(t) \) has a critical point \( t_0 > 0 \) corresponding to its maximum, i.e., \( y(t_0) = \max_{t > 0} y(t) \) and \( y'(t_0) = 0 \). Thus
\begin{align*}
2sa t_0^{4s-1} \int_{\mathbb{R}^3} |(-\Delta)^{2s}u|^2 \, dx + 4s t_0^{8s-1} \int_{\mathbb{R}^3} V_{\infty} u^2 \, dx + 2sb t_0^{8s-1} \left( \int_{\mathbb{R}^3} |(-\Delta)^{2s}u|^2 \, dx \right)^2 \
- t_0^5 \int_{\mathbb{R}^3} \left( (4s-3) f(t_0^{4s-3}u) t_0^{4s-3}u + 6F(t_0^{4s-3}u) \right) \, dx - 2s t_0^{2s-2^*-1} \int_{\mathbb{R}^3} |u|^{2^*} \, dx = 0,
\end{align*}

and hence \( G^\infty(u_{t_0}) = 0, \quad u_{t_0} \in \mathcal{M}^\infty \) and \( I^\infty(u_{t_0}) = \max_{t > 0} I^\infty(u_t) \).

Moreover, we claim that the critical point of \( y(t) \) is unique. Indeed, by contradiction, we suppose that there exist two points \( t_1, \ t_2 > 0 \) such that \( G^\infty(u_{t_i}) = 0 \) for \( i = 1, 2 \). Similarly to the proof of Lemma 2.2, we can deduce that
\begin{align*}
I^\infty(u_{t_1}) &\geq I^\infty(u_{t_2}) + \frac{t_1^{4s} - t_2^{4s}}{8s t_1^{8s}} G^\infty(u_{t_1}) + \frac{a(t_1^{4s} - t_2^{4s})^2}{4t_1^{4s}} \int_{\mathbb{R}^3} |(-\Delta)^{2s}u|^2 \, dx \\
&= I^\infty(u_{t_2}) + \frac{a(t_1^{4s} - t_2^{4s})^2}{4t_1^{4s}} \int_{\mathbb{R}^3} |(-\Delta)^{2s}u|^2 \, dx,
\end{align*}
and

\[ I^∞(u_t) ≥ I^∞(u_t) + \frac{t^{8s}-t^{8s}}{8s^2t^{14s}}G^∞(u_t) + \frac{a(t^{14s}-t^{14s})^2}{4t^{14s}} \int_{\mathbb{R}^3} |(−Δ)^{\frac{s}{2}} u|^2 \ dx \]

\[ = I^∞(u_t) + \frac{a(t^{14s}-t^{14s})^2}{4t^{14s}} \int_{\mathbb{R}^3} |(−Δ)^{\frac{s}{2}} u|^2 \ dx. \]

This implies that \( t_1 = t_2 \). So, \( t_0 > 0 \) is the unique critical point of \( y(t) \).

**Lemma 2.4.** Assume that \((f_1)-(f_3)\) hold. There holds

\[ m^∞ = \inf_{u \in H^s(\mathbb{R}^3) \setminus \{0\}} \max_{t > 0} I^∞(u_t) > 0. \]

**Proof.** It follows from Lemma 2.3 that

\[ m^∞ = \inf_{u \in H^s(\mathbb{R}^3) \setminus \{0\}} \max_{t > 0} I^∞(u_t). \]

Next, we claim that \( m^∞ > 0 \). Indeed, it follows from \((f_1)-(f_2)\) that there exists a constant \( C > 0 \) such that

\[ |F(u)| ≤ \frac{1}{2} V^∞ u^2 + \bar{C}|u|^{2^*_s}, \quad ∀u \in H^s(\mathbb{R}^3). \quad (11) \]

Thus for any \( u \in \mathcal{M}^∞ \), by (11), Lemma 2.2 and Sobolev inequality, we have

\[ I^∞(u) ≥ I^∞(u_t) \]

\[ ≥ \frac{a}{2} t^{4s} \int_{\mathbb{R}^3} |(−Δ)^{\frac{s}{2}} u|^2 \ dx + \frac{1}{2} \int_{\mathbb{R}^3} V^∞ u^2 \ dx \]

\[ − t^{6} \int_{\mathbb{R}^3} F(t^{4s-3}u) \ dx - \frac{1}{2s} t^{2s-2} \int_{\mathbb{R}^3} |u|^{2^*_s} \ dx \]

\[ ≥ \frac{a}{2} t^{4s} \int_{\mathbb{R}^3} |(−Δ)^{\frac{s}{2}} u|^2 \ dx - \left( \bar{C} + \frac{1}{2s} \right) t^{2s-2} \int_{\mathbb{R}^3} |u|^{2^*_s} \ dx \]

\[ ≥ \frac{a}{2} t^{4s} \int_{\mathbb{R}^3} |(−Δ)^{\frac{s}{2}} u|^2 \ dx - \left( \bar{C} + \frac{1}{2s} \right) \mathcal{S}_s^{\frac{2^*_s}{2}} t^{2s-2} \left( \int_{\mathbb{R}^3} |(−Δ)^{\frac{s}{2}} u|^2 \ dx \right)^{\frac{2^*_s}{2}} \]

\[ = \frac{sa}{3} \left[ \frac{a\mathcal{S}_s^{\frac{2^*_s}{2}}}{2^*_sC} \right]^{\frac{3-2s}{2}}, \]

if we take

\[ t = \left[ \frac{a\mathcal{S}_s^{\frac{2^*_s}{2}}}{2^*_sC} \right]^{\frac{1}{2s-1}} \left( \int_{\mathbb{R}^3} |(−Δ)^{\frac{s}{2}} u|^2 \ dx \right)^{\frac{1}{2s}} \quad \text{and} \quad \bar{C} = \bar{C} + \frac{1}{2s}. \]

As a result, we complete the proof.

Throughout this section, the norm on the \( H^s(\mathbb{R}^3) \) is taken as

\[ ||u|| := \left( \int_{\mathbb{R}^3} (a|−Δ)^{\frac{s}{2}} u|^2 + V^∞ u^2) \ dx \right)^{\frac{1}{2}}. \]

Now, we establish a splitting lemma as follows.
Lemma 2.5. Assume that \((f_1) - (f_2)\) hold. If \(u_n \rightharpoonup u\) in \(H^s(\mathbb{R}^3)\), then
\[
I^\infty(u_n) = I^\infty(u) + I^\infty(u_n - u) + \frac{b}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}(u_n - u)|^2 dx \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx + o_n(1),
\]
(12)
\[
\langle (I^\infty)'(u_n), u_n \rangle = \langle (I^\infty)'(u), u \rangle + \langle (I^\infty)'(u_n - u), u_n - u \rangle + 2b \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}(u_n - u)|^2 dx \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx + o_n(1)
\]
and
\[
G^\infty(u_n) = G^\infty(u) + G^\infty(u_n - u) + 4sb \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}(u_n - u)|^2 dx \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx + o_n(1),
\]
(13)
\(\)where \(o_n(1) \to 0\) as \(n \to \infty\).

Proof. Set \(v_n = u_n - u\), we have \(v_n \rightharpoonup 0\) in \(H^s(\mathbb{R}^3)\). It follows from \((f_1) - (f_2)\) and Brezis-Lieb lemma that
\[
\|u_n\|^2 = \|u\|^2 + \|v_n\|^2 + o_n(1),
\]
(15)
\[
\int_{\mathbb{R}^3} F(u_n)dx = \int_{\mathbb{R}^3} F(u)dx + \int_{\mathbb{R}^3} F(v_n)dx + o_n(1)
\]
(16)
and
\[
\|u_n\|_{2^*_s}^2 = \|u\|_{2^*_s}^2 + \|v_n\|_{2^*_s}^2 + o_n(1).
\]
(17)
Moreover,
\[
\left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u_n|^2 dx \right)^2 = \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx \right)^2 + \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}v_n|^2 dx \right)^2 + 2 \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}v_n|^2 dx \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx + o_n(1).
\]
(18)
From (15)-(18), we know that (12) holds.

Using a similar argument as Lemma 2.7 in [43], we can show that there exists a subsequence of \(\{u_n\}\), still denoted by \(\{u_n\}\) such that
\[
\sup_{\varphi \in H^s(\mathbb{R}^3), \|\varphi\| \leq 1} \left| \int_{\mathbb{R}^3} (f(u_n) - f(u) - f(v_n)) \varphi dx \right| = o_n(1).
\]

Thus, one has
\[
\left| \int_{\mathbb{R}^3} (f(u_n) - f(u) - f(v_n)) u_n dx \right|
\]
\[
\leq \|u_n\| \sup_{\varphi \in H^s(\mathbb{R}^3), \|\varphi\| \leq 1} \left| \int_{\mathbb{R}^3} (f(u_n) - f(u) - f(v_n)) \varphi dx \right| = o_n(1).
\]
From this, we have
\[
\int_{\mathbb{R}^3} f(u_n)u_n\,dx = \int_{\mathbb{R}^3} f(u)\,dx + \int_{\mathbb{R}^3} f(v_n)v_n\,dx + \int_{\mathbb{R}^3} f(v_n)\,dx
+ \int_{\mathbb{R}^3} f(u)v_n\,dx + \int_{\mathbb{R}^3} (f(u_n) - f(u) - f(v_n))u_n\,dx
\]
(19)
Combining (15), (17)-(18) and (19), we can obtain (13).
Finally, noting that
\[
G^\infty(u) = 2s\langle(I^\infty)'(u), u\rangle + 2s \int_{\mathbb{R}^3} V_\infty u^2\,dx - \int_{\mathbb{R}^3} ((2s - 3)f(u) + 6F(u))\,dx,
\]
the equality (14) follows from (13), (16) and (19).

In the following, we give an important energy estimate for \(m^\infty\).

**Lemma 2.6.** Assume that (\(f_1\)) – (\(f_4\)) hold. There holds
\[m^\infty < g(T) := \frac{a}{2}S^{\frac{4s}{3-2s}}T^{4s} + \frac{b}{4}S^{\frac{4s}{3-2s}}T^{8s} - \frac{1}{2^s}s^{\frac{4s}{3-2s}}T^{2s-2s};\]
if either \(\frac{4s}{3-2s} < p < 2\) for all \(\mu > 0\) or \(2 < p \leq \frac{4s}{3-2s}\) for \(\mu > 0\) sufficiently large.
Where \(T\) is a unique positive solution of equation \(aS^{\frac{2s}{3-2s}}y^{4s} + bS^{\frac{2s}{3-2s}}y^{8s} - S^{\frac{2s}{3-2s}}y^{2s-2s} = 0\).

**Proof.** Let \(\psi \in C_0^\infty(\mathbb{R}^3)\) be a cut-off function such that \(\psi(x) = 1\) if \(|x| \leq r\) and \(\psi(x) = 0\) if \(|x| \geq 2r\). For \(\varepsilon > 0\), we define
\[
u_\varepsilon(x) = \psi(x)U_\varepsilon(x), \quad x \in \mathbb{R}^3,
\]
where \(U_\varepsilon(x) = \varepsilon^{\frac{3-2s}{2}}u^*(\frac{x}{\varepsilon})\), \(u^*(x) = \frac{u(x)}{\|u\|_{L^2}^2}\) and
\[U(x) = \kappa(\tau^2 + |x - x_0|^2)^{\frac{-3-2s}{2}}\]
with \(\kappa \in \mathbb{R}\setminus\{0\}, \tau > 0\) and \(x_0 \in \mathbb{R}^3\). From [40, 45], we know that
\[
\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2\,dx \leq S^{\frac{2s}{3}} + O(\varepsilon^{3-2s}),
\]
\[
\int_{\mathbb{R}^3} |u_\varepsilon|^2\,dx = S^{\frac{3}{3}} + O(\varepsilon^3)
\]
and
\[
\int_{\mathbb{R}^3} |u_\varepsilon|^q\,dx = \begin{cases} O(\varepsilon^{3-\frac{3-2s}{3-2s}}), & \text{for } q \geq \frac{3}{3-2s}, \\ O(\varepsilon^{\frac{3}{3-2s}}\log \varepsilon), & \text{for } q = \frac{3}{3-2s}, \\ O(\varepsilon^{\frac{3}{3-2s}}), & \text{for } q < \frac{3}{3-2s}. \end{cases}
\]
In particular, since \(s > \frac{3}{4}\), one has
\[
\int_{\mathbb{R}^3} |u_\varepsilon|^2\,dx = O(\varepsilon^{3-2s}).
\]

By Lemma 2.3 and Lemma 2.4, there exists a \(t_\varepsilon > 0\) such that
\[0 < m^\infty \leq \max_{t > 0} I^\infty((u_\varepsilon)_t) = I^\infty((u_\varepsilon)_t).\]
(20)

Next, we claim that there exist two constants \(t_*, t^* > 0\) such that \(t_\varepsilon \leq t_\varepsilon \leq t^*\). Indeed, we first prove that \(t_\varepsilon\) is bounded from below by a positive constant.
Otherwise, we could find a sequence $\varepsilon_n \to 0$ such that $t_{\varepsilon_n} \to 0$. By the above estimations, up to a subsequence, we have $(u_{\varepsilon_n})_{t_{\varepsilon_n}} \to 0$ in $H^s(\mathbb{R}^3)$. Therefore,

$$0 < m^\infty \leq I^\infty((u_{\varepsilon_n})_{t_{\varepsilon_n}}) \to I^\infty(0) = 0,$$

which is a contradiction. On the other hand, it follows from $(f_4)$ that

$$0 \leq I^\infty((u_{\varepsilon})_{t_{\varepsilon}}) \leq \frac{a}{2} t_{\varepsilon}^{4s} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_{\varepsilon}|^2 dx + \frac{1}{2} t_{\varepsilon}^{8s} \int_{\mathbb{R}^3} V_\infty u_{\varepsilon}^2 dx + \frac{b}{4} t_{\varepsilon}^{8s} \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_{\varepsilon}|^2 dx \right)^2 - \frac{\mu}{p} t_{\varepsilon}^{(4s-3)p+6} \int_{\mathbb{R}^3} |u_{\varepsilon}|^p dx \leq Ct_{\varepsilon}^{4s} + Ct_{\varepsilon}^{2s} - Ct_{\varepsilon}^{(4s-3)p+6},$$

which implies that there exists $t^* > 0$ such that $t_{\varepsilon} \leq t^*$. Thus, the claim is proved.

Therefore, we conclude that

$$I^\infty((u_{\varepsilon})_{t_{\varepsilon}}) \leq \frac{a}{2} S_s^{\frac{3}{4}} t_{\varepsilon}^{4s} + \frac{b}{4} S_s^{\frac{3}{2}} t_{\varepsilon}^{8s} - \frac{1}{2s} S_s^{\frac{3}{4}} t_{\varepsilon}^{2s} t_{\varepsilon}^{2s} + O(\varepsilon^{3-2s}) - C\varepsilon \int_{\mathbb{R}^3} |u_{\varepsilon}|^p dx.$$

Let

$$g(t) = \frac{a}{2} S_s^{\frac{3}{4}} t^{4s} + \frac{b}{4} S_s^{\frac{3}{2}} t^{8s} - \frac{1}{2s} S_s^{\frac{3}{4}} t^{2s} t^{2s}, \quad t > 0.$$

By elementary computation, we know that there is a unique $T > 0$ such that $g(T) = \max_{t > 0} g(t)$. Moreover,

$$g'(T) = 2sT^{-1} \left( a S_s^{\frac{3}{4}} T^{4s} + b S_s^{\frac{3}{2}} T^{8s} - S_s^{\frac{3}{4}} T^{2s} \right) = 0.$$

Then

$$m^\infty \leq I^\infty((u_{\varepsilon})_{t_{\varepsilon}}) \leq g(T) + O(\varepsilon^{3-2s}) - C\varepsilon \int_{\mathbb{R}^3} |u_{\varepsilon}|^p dx.$$

Noting that $2s - \frac{3-2s}{2} p < 0$ if $\frac{4s}{3-2s} < p < 2^*_s$, we have

$$\lim_{\varepsilon \to 0} \frac{\mu}{\varepsilon^{3-2s}} \int_{\mathbb{R}^3} |u_{\varepsilon}|^p dx = \begin{cases} +\infty, & \text{for } \mu \varepsilon^{\frac{3-2s}{3-2s}} < p < 2^*_s, \\ \lim_{\varepsilon \to 0} \frac{\mu O(\varepsilon^{\frac{3-2s}{3-2s}} p)}{\varepsilon^{3-2s}}, & \text{for } \frac{3}{3-2s} < p \leq \frac{4s}{3-2s}, \\ \lim_{\varepsilon \to 0} \frac{\mu O(\varepsilon^2 |\log\varepsilon|)}{\varepsilon^{3-2s}}, & \text{for } p = \frac{3}{3-2s}, \\ \lim_{\varepsilon \to 0} \frac{\mu O(\varepsilon^{\frac{3-2s}{3-2s}} p)}{\varepsilon^{3-2s}}, & 2 < p < \frac{3}{3-2s}. \end{cases}$$

When $\frac{4s}{3-2s} < p < 2^*_s$, we can verify that for any $\mu > 0$, $m^\infty < g(T)$ for $\varepsilon > 0$ small. Moreover, if $2 < p \leq \frac{4s}{3-2s}$ and $\varepsilon > 0$ small, we also have $m^\infty < g(T)$ by choosing $\mu = \varepsilon^{-2s}$.

\[ \square \]

**Lemma 2.7.** Under the assumptions $(f_1) - (f_4)$, then $m^\infty > 0$ is achieved at some $\hat{u} \in \mathcal{M}^\infty$. 

Proof. Take $t = 0$ in Lemma 2.1, we have that
\[ f(\tau)\tau - 2F(\tau) \geq 0, \quad \forall \tau \in \mathbb{R}. \quad (21) \]
We set $\Phi^\infty(u) := I^\infty(u) - \frac{1}{8s}G^\infty(u)$, then
\[ \Phi^\infty(u) = \frac{a}{4} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{4s - 3}{8s} \int_{\mathbb{R}^3} (f(u)u - 2F(u)) dx \]
\[ + \frac{4s - 3}{12} \int_{\mathbb{R}^3} |u|^2 dx. \quad (22) \]
Let $\{u_n\} \subset \mathcal{M}^\infty$ be a minimizing sequence for $m^\infty$ such that
\[ I^\infty(u_n) \to m^\infty < g(T), \]
where $g$ is defined in Lemma 2.6. In the following, we divide the proof into three steps.

**Step 1.** The sequence $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$.

From (21)-(22), we know that
\[ m^\infty + o_n(1) = I^\infty(u_n) = \Phi^\infty(u_n) \geq \frac{a}{4} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx. \]
Next, we only need to show the boundedness of $\int_{\mathbb{R}^3} u_n^2 dx$. By $(f_1) - (f_2)$ and $G^\infty(u_n) = 0$, we have that for any $\varepsilon > 0$, there exists a $C_\varepsilon > 0$ such that
\[ 4s \int_{\mathbb{R}^3} V_\infty u_n^2 dx = \int_{\mathbb{R}^3} \left( (4s - 3) f(u_n) u_n + 6F(u_n) \right) dx + 2s \int_{\mathbb{R}^3} |u_n|^2 dx \]
\[ - 2sa \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx - 2sb \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \right)^2 \]
\[ \leq \varepsilon \int_{\mathbb{R}^3} u_n^2 dx + C_\varepsilon \int_{\mathbb{R}^3} |u_n|^2 dx + C. \]
By choosing $\varepsilon > 0$ small enough, we obtain that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$.

**Step 2.** There exist a sequence $\{y_n\} \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that
\[ \liminf_{n \to \infty} \int_{B_R(y_n)} u_n^2 dx \geq \beta > 0. \quad (23) \]
Suppose by contradiction that for all $R > 0$,
\[ \lim_{n \to \infty} \sup_{y \in \mathbb{R}^3} \int_{B_R(y)} u_n^2 dx = 0. \]
By Lemma 2.4 in [38], we have that
\[ u_n \to 0 \quad \text{in} \quad L^q(\mathbb{R}^3), \quad 2 < q < 2_s^*. \quad (24) \]
Since $G^\infty(u_n) = 0$, we get
\[ 2sa \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx + 4s \int_{\mathbb{R}^3} V_\infty u_n^2 dx + 2s \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \right)^2 \]
\[ - 2s \int_{\mathbb{R}^3} |u_n|^2 dx = o_n(1). \]
From the definition of the constant $\mathcal{S}_s$, we have
\[ a\mathcal{S}_s \left( \int_{\mathbb{R}^3} |u_n|^2 dx \right)^{\frac{2}{s}} + b\mathcal{S}_s^2 \left( \int_{\mathbb{R}^3} |u_n|^2 dx \right)^{\frac{4}{s}} \leq \int_{\mathbb{R}^3} |u_n|^2 dx + o_n(1). \quad (25) \]
Without loss of generality, we may assume that
\[ \int_{\mathbb{R}^3} |u_n|^2^* dx \to S_{s^2}^{T^*} l^{2s^2} \geq 0. \]

It is easy to check that if \( l > 0 \), otherwise \( \|u_n\| \to 0 \) as \( n \to \infty \) which contradicts \( m^\infty > 0 \). Thus, (25) implies that
\[ g'(l) = 2s^{-1} \left( aS_{s^2}^{T^*} l^{4s} + bS_{s^2}^{3T^*} l^{8s} - S_{s^2}^{3T^*} l^{2s^2} \right) \leq 0. \]

Since \( g(t) \) has a unique critical point \( T > 0 \) corresponding to its maximum, it follows from (26) that \( l \geq T \). On the other hand, from (24), one has
\[ m^\infty + o_n(1) = I^\infty(u_n) = I^\infty(u_n) - \frac{1}{8s} G^\infty(u_n) \]
\[ = \frac{a}{4} \int_{\mathbb{R}^3} \frac{(-\Delta)^{\frac{s}{2}} u_n^2 dx + 4s - 3}{12} \int_{\mathbb{R}^3} |u_n|^2^* dx + o_n(1). \]

Thus by the definition of \( S_s \) and the fact that \( aS_{s^2}^{3T^*} T^{4s} + bS_{s^2}^{3T^*} T^{8s} - S_{s^2}^{3T^*} T^{2s^2} = 0 \), we have
\[ m^\infty \geq \frac{a}{4} S_s \left( \int_{\mathbb{R}^3} |u_n|^2^* dx \right)^{\frac{2}{2s}} + \frac{4s - 3}{12} \int_{\mathbb{R}^3} |u_n|^2^* dx + o_n(1) \]
\[ \geq \frac{a}{4} S_s^{\frac{3}{2}} T^{4s} + \frac{4s - 3}{12} S_s^{\frac{3}{2}} T^{2s^2} \]
\[ \geq \frac{a}{2} S_s^{\frac{3}{2}} T^{4s} + \frac{b}{4} S_s^{\frac{3}{2}} T^{8s} - \frac{1}{2s} S_s^{\frac{3}{2}} T^{2s^2}, \]
which contradicts Lemma 2.6.

\textbf{Step 3.} \( m^\infty \) is achieved.

Denote \( \tilde{u}_n(x) = u_n(x + y_n) \), then \( \tilde{u}_n \in \mathcal{M}^\infty \). It is easy to check that \( \{\tilde{u}_n\} \) is still a bounded minimizing sequence for \( m^\infty \). Up to a subsequence, we may assume that there is a \( \tilde{u} \in H^s(\mathbb{R}^3) \) such that
\[
\begin{aligned}
\tilde{u}_n &\rightharpoonup \tilde{u} \quad \text{in} \quad H^s(\mathbb{R}^3), \\
\tilde{u}_n &\to \tilde{u} \quad \text{in} \quad L^q_{loc}(\mathbb{R}^3), \quad 1 \leq q < 2^*, \\
\tilde{u}_n &\to \tilde{u} \quad \text{a.e. in} \quad \mathbb{R}^3.
\end{aligned}
\]

It follows from (23) that there exist \( R, \beta > 0 \) such that
\[ \int_{B_R(0)} \tilde{u}_n^2 dx \geq \beta > 0, \]
which implies that \( \tilde{u} \neq 0 \). Set \( \tilde{v}_n = \tilde{u}_n - \tilde{u} \). By using (27), similarly to Lemma 2.5, we deduce that
\[ \Phi^\infty(\tilde{u}_n) = \Phi^\infty(\tilde{u}) + \Phi^\infty(\tilde{v}_n) + o_n(1) \]
and
\[ G^\infty(\tilde{u}_n) = G^\infty(\tilde{u}) + G^\infty(\tilde{v}_n) + 4sb \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \tilde{v}_n|^2 dx \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \tilde{u}|^2 dx + o_n(1). \]
From this, one has
\[ m^\infty - \Phi^\infty(\tilde{u}) = \Phi^\infty(\tilde{v}_n) + o_n(1) \quad \text{and} \quad G^\infty(\tilde{v}_n) + o_n(1) \leq -G^\infty(\tilde{u}). \quad (30) \]

Without loss of generality, we may assume that \( \tilde{v}_n \neq 0 \). Otherwise, the lemma is trivial. By Lemma 2.3, there exists \( t_n > 0 \) such that \( (\tilde{v}_n)_{t_n} \in \mathcal{M}^\infty \) for any \( n \).

Now we claim that \( G^\infty(\tilde{u}) \leq 0 \). Indeed, suppose by contradiction that \( G^\infty(\tilde{u}) > 0 \), from (30) we know that \( G^\infty(\tilde{v}_n) + o_n(1) < 0 \). It follows from Lemma 2.2, (22) and (30) that
\[
m^\infty - \Phi^\infty(\tilde{u}) = \Phi^\infty(\tilde{v}_n) + o_n(1)
= I^\infty(\tilde{v}_n) - \frac{1}{8s} G^\infty(\tilde{v}_n) + o_n(1)
\geq I^\infty((\tilde{v}_n)_{t_n}) - \frac{\bar{\rho}_s}{8s} G^\infty(\tilde{v}_n) + o_n(1) \geq m^\infty + o_n(1),
\]
which implies that the claim is true since \( \Phi^\infty(\tilde{u}) > 0 \).

Lemma 2.3 also implies that there exists \( \tilde{u}_i \in \mathcal{M}^\infty \). Combining Lemma 2.2, (21)-(22) and Fatou’s lemma, we have that
\[
m^\infty = \liminf_{n \to \infty} \left[ I^\infty(\tilde{u}_n) - \frac{1}{8s} G^\infty(\tilde{u}_n) \right]
= \liminf_{n \to \infty} \left[ \frac{a}{4} \int_{\mathbb{R}^3} (-\Delta)^{\frac{1}{2}} \tilde{u}_n^2 \, dx + \frac{4s - 3}{8s} \int_{\mathbb{R}^3} (f(\tilde{u}_n))\tilde{u}_n - 2F(\tilde{u}_n) \, dx \right.
+ \frac{4s - 3}{12} \int_{\mathbb{R}^3} |\tilde{u}_n|^{\frac{2s}{s-1}} \, dx
\geq \left. \frac{a}{4} \int_{\mathbb{R}^3} (-\Delta)^{\frac{1}{2}} \tilde{u}^2 \, dx + \frac{4s - 3}{8s} \int_{\mathbb{R}^3} (f(\tilde{u}))\tilde{u} - 2F(\tilde{u}) \, dx + \frac{4s - 3}{12} \int_{\mathbb{R}^3} |\tilde{u}|^{\frac{2s}{s-1}} \, dx \right]
= I^\infty(\tilde{u}) - \frac{1}{8s} G^\infty(\tilde{u})
\geq I^\infty(\tilde{u}_i) - \frac{\bar{\rho}_s}{8s} G^\infty(\tilde{u}) \geq m^\infty.
\]

Therefore, we conclude that \( I^\infty(\tilde{u}) = m^\infty \) and \( G^\infty(\tilde{u}) = 0 \). \( \square \)

**Lemma 2.8.** Assume that \((f_1) - (f_3)\) hold. If \( I^\infty(u) = m^\infty \) for \( u \in \mathcal{M}^\infty \), then \( u \) is a critical point of \( I^\infty \).

**Proof.** Suppose by contradiction that \( (I^\infty)'(u) \neq 0 \), there exist \( \rho, \delta > 0 \) such that
\[
\|(I^\infty)'(v)\|_{H^{-\epsilon}(\mathbb{R}^3)} \geq \rho \quad \text{if} \quad \|u - v\| \leq 3\delta, \quad \forall v \in H^s(\mathbb{R}^3).
\]

We first show that
\[
\lim_{t \to 1} \|u_t - u\| = 0.
\]

Otherwise, suppose that there exist \( \varepsilon_0 > 0 \) and a sequence \( \{t_n\} \) such that
\[
\|u_{t_n} - u\|^2 \geq \varepsilon_0 \quad \text{as} \quad t_n \to 1.
\]

Notice that there exist two functions \( U_1, U_2 \in C_0(\mathbb{R}^3) \) such that
\[
\int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}} u - U_1|^2 \, dx < \frac{\varepsilon_0}{20 \max\{a, V_\infty\}} \quad \text{and} \quad \int_{\mathbb{R}^3} |u - U_2|^2 \, dx < \frac{\varepsilon_0}{20 \max\{a, V_\infty\}}.
\]
From this, we get that
\[
\int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}}(u_{t_n} - u)|^2 \, dx \\
\leq 2 \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}}u_{t_n} - U_1|^2 \, dx + 2 \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}}u - U_1|^2 \, dx \\
\leq 6t_n^{4s} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}}u - U_1|^2 \, dx + 6t_n^{2(4s-3)} \int_{\mathbb{R}^3} |U_1(t_n^{-1}x) - U_1(x)|^2 \, dx \\
+ 6(t_n^{2s-3} - 1)^2 \int_{\mathbb{R}^3} U_1^2 \, dx + 2 \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}}u - U_1|^2 \, dx \\
\leq \frac{2\varepsilon_0}{5 \max\{a, V_\infty\}} + o_n(1)
\]
and
\[
\int_{\mathbb{R}^3} |u_{t_n} - u|^2 \, dx \leq 2 \int_{\mathbb{R}^3} |u_{t_n} - U_2|^2 \, dx + 2 \int_{\mathbb{R}^3} |u - U_2|^2 \, dx \\
\leq 6t_n^{4s} \int_{\mathbb{R}^3} |u - U_2|^2 \, dx + 6t_n^{2(4s-3)} \int_{\mathbb{R}^3} |U_2(t_n^{-1}x) - U_2(x)|^2 \, dx \\
+ 6(t_n^{4s-3} - 1)^2 \int_{\mathbb{R}^3} U_2^2 \, dx + 2 \int_{\mathbb{R}^3} |u - U_2|^2 \, dx \\
\leq \frac{2\varepsilon_0}{5 \max\{a, V_\infty\}} + o_n(1).
\]
Thus,
\[
||u_{t_n} - u||^2 = a \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}}(u_{t_n} - u)|^2 \, dx + V_\infty \int_{\mathbb{R}^3} |u_{t_n} - u|^2 \, dx \leq \frac{4}{\varepsilon_0} + o_n(1),
\]
which contradicts (33). It follows from (32) that for \( \delta > 0 \) given above, there exists \( \delta_1 > 0 \) such that
\[
||u_t - u|| \leq \delta \quad \text{if} \quad |t - 1| < \delta_1. \tag{34}
\]
Lemma 2.2 implies that
\[
I^\infty(u_t) \leq I^\infty(u) - \frac{a(1-t^{4s})^2}{4} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}}u|^2 \, dx \\
= m^\infty - \frac{a(1-t^{4s})^2}{4} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}}u|^2 \, dx, \quad \forall t > 0. \tag{35}
\]
Let \( \varepsilon = \min\left\{ \frac{a}{8s} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}}u|^2 \, dx, 1, \frac{a^4}{8s} \right\} \), \( S = \{ v \in H^s(\mathbb{R}^3) \mid ||u - v|| < \delta \} \). It follows from Lemma 2.3 in [47] that there exists a map \( \eta \in C([0,1] \times H^s(\mathbb{R}^3), H^s(\mathbb{R}^3)) \) such that
(i) \( \eta(1, v) = v \) if \( v \notin (I^\infty)^{-1}(|m^\infty - 2\varepsilon, m^\infty + 2\varepsilon|) \cap S_{2\delta} \);
(ii) \( \eta(1, (I^\infty)^{m^\infty + \varepsilon} \cap S) \subset (I^\infty)^{m^\infty - \varepsilon} ; \)
(iii) \( I^\infty(\eta(1, v)) \leq I^\infty(v) \) for all \( v \in H^s(\mathbb{R}^3) \);
(iv) \( \eta(1, v) \) is a homeomorphism of \( H^s(\mathbb{R}^3) \).
Lemma 2.2 implies that \( I^\infty(u_t) \leq I^\infty(u) = m^\infty \) for \( t > 0 \), then by (34) and (ii), we have
\[
I^\infty(\eta(1, u_t)) \leq m^\infty - \varepsilon \quad \text{if} \quad |t - 1| < \delta_1.
\]
On the other hand, from (35) and (iii), we have
\[
I^\infty(\eta(1,u_t)) \leq I^\infty(u_t) \leq m^\infty - \frac{a(1-t^4)^2}{4} \int_{\mathbb{R}^3} |(-\Delta)^s u|^2 dx
\]
\[
\leq m^\infty - \frac{a\delta^2}{4} \int_{\mathbb{R}^3} |(-\Delta)^s u|^2 dx, \quad \text{if } |t-1| \geq \delta_1.
\]
Thus,
\[
\max_{t \in [\frac{1}{2}, \frac{3}{2}]} I^\infty(\eta(1,u_t)) < m^\infty. \tag{36}
\]

Next, we claim that \(\eta(1,u_t) \cap \mathcal{M}^\infty \neq \emptyset\) for some \(t \in [\frac{1}{2}, \frac{3}{2}]\), which contradicts (36). Indeed, we define
\[
\Phi_1(t) = G^\infty(u_t) \quad \text{and} \quad \Phi_2(t) = G^\infty(\eta(1,u_t)), \quad \forall t > 0.
\]
From Lemma 2.3 and the definition of Brouwer degree, we have
\[
\deg(\Phi_1, \left(\frac{1}{2}, \frac{3}{2}\right), 0) = 1.
\]
In view of (35) and (i), it is clear that \(\eta(1,u_t) = u_t\) for \(t = \frac{1}{2}\) and \(t = \frac{3}{2}\). The homotopy invariance of Brouwer degree gives that
\[
\deg(\Phi_1, \left(\frac{1}{2}, \frac{3}{2}\right), 0) = \deg(\Phi_2, \left(\frac{1}{2}, \frac{3}{2}\right), 0) = 1.
\]
So, there exists \(t_0 \in (\frac{1}{2}, \frac{3}{2})\) such that \(\Phi_2(t_0) = 0\), which means that \(\eta(1,u_{t_0}) \in \mathcal{M}^\infty\). Therefore, the claim is true and we complete the proof. □

**Proof of Theorem 1.1.** Combining Lemma 2.7 and Lemma 2.8, one could directly obtain that \(I^\infty\) has a critical point \(\tilde{u} \in \mathcal{M}^\infty\) such that \(I^\infty(\tilde{u}) = m^\infty > 0\).

Next, we only need to prove that \(\tilde{u}\) is nonnegative. Let us consider the functional
\[
I_+^\infty(u) = \frac{1}{2} \int_{\mathbb{R}^3} (a|(-\Delta)^s \tilde{u}|^2 + V_\infty u^2) dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} (-\Delta)^{2s} u^2 dx \right)^2
\]
\[
- \int_{\mathbb{R}^3} F(u) dx - \frac{1}{2} \int_{\mathbb{R}^3} (u^+)^{2^*_s} dx,
\]
where \(u^+ = \max\{u, 0\}\) and \(u^- = \min\{u, 0\}\). Similarly, we can obtain a nontrivial solution \(u\) of the equation
\[
\left(a + b \int_{\mathbb{R}^3} |(-\Delta)^s u|^2 dx\right)(-\Delta)^s u + V_\infty u = f(u) + (u^+)^{2^*_s-1}. \tag{37}
\]
Multiplying the above equation (37) by \(u^-\) and integrating over \(\mathbb{R}^3\), we find
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x) - u(y))(u^-(x) - u^-(y))}{|x - y|^{3+2s}} dxdy \leq 0.
\]
But we know that
\[ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x) - u(y))(u^-(x) - u^-(y))}{|x - y|^{3+2s}} \, dx \, dy \]
\[ \geq \int_{\{y: u < 0\}} \int_{\{x: u > 0\}} \frac{(u(x) - u(y))(u^-(x) - u^-(y))}{|x - y|^{3+2s}} \, dx \, dy \]
\[ + \int_{\{y: u < 0\}} \int_{\{x: u < 0\}} \frac{|u^-(x) - u^-(y)|^2}{|x - y|^{3+2s}} \, dx \, dy \]
\[ + \int_{\{y: u > 0\}} \int_{\{x: u < 0\}} \frac{(u(x) - u(y))u^-(x)}{|x - y|^{3+2s}} \, dx \, dy \geq 0. \]
Thus, \( u^- = 0 \) and \( u \geq 0 \) is a solution of equation (37). Hence, we can obtain that \( \tilde{u} \geq 0 \).}

3. Proof of Theorem 1.2. In this section, we employ the monotone method developed by Jeanjean in [29] to prove Theorem 1.2. Since we do not assume that \( f \) satisfies (AR) condition, it seems difficult to get a bounded (PS) sequence. In order to overcome this difficulty, we introduce the following abstract result developed by Jeanjean [29].

**Proposition 1.** ([29], Theorem 1.1) Let \( X \) be a Banach space equipped \( \| \cdot \| \), and let \( L \subset \mathbb{R}^+ \) be an interval. We consider a family \( (I_{\lambda})_{\lambda \in L} \) of \( C^1 \)-functionals on \( X \) of the form
\[ I_{\lambda}(u) = A(u) - \lambda B(u), \quad \forall \lambda \in L, \]
where \( B(u) \geq 0 \), \( \forall u \in X \), and such that either \( A(u) \to +\infty \) or \( B(u) \to +\infty \) as \( \|u\| \to \infty \). We assume that there are two points \( (v_1, v_2) \) in \( X \), such that setting
\[ \Gamma = \{ \gamma \in C([0, 1], X) \mid \gamma(0) = v_1, \gamma(1) = v_2 \}, \]
there hold, \( \forall \lambda \in L \),
\[ c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I_{\lambda}(\gamma(t)) > \max\{I_{\lambda}(v_1), I_{\lambda}(v_2)\}. \]
Then, for almost every \( \lambda \in L \), there is a bounded (PS)\(_{c_\lambda} \) sequence in \( X \). Moreover, the map \( \lambda \mapsto c_\lambda \) is continuous from the left.

To use Proposition 1, we set \( L = [\delta, 1] \), where \( \delta \in (0, 1) \), and denote a family of \( C^1 \)-functionals on \( H^s(\mathbb{R}^3) \) as follows
\[ I_{\lambda}(u) = \frac{1}{2} \int_{\mathbb{R}^3} (a|(-\Delta)^{\frac{s}{2}}u|^2 + V(x)u^2) \, dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 \, dx \right)^2 \]
\[ - \frac{\lambda}{2} \int_{\mathbb{R}^3} F(u) \, dx - \frac{\lambda}{2s} \int_{\mathbb{R}^3} |u|^{2s} \, dx, \quad \forall \lambda \in [\delta, 1]. \]
Denoting
\[ A(u) = \frac{1}{2} \int_{\mathbb{R}^3} (a|(-\Delta)^{\frac{s}{2}}u|^2 + V(x)u^2) \, dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 \, dx \right)^2 \to +\infty, \]
as \( \|u\| \to +\infty \), and \( B(u) = \int_{\mathbb{R}^3} F(u) \, dx + \frac{1}{2s} \int_{\mathbb{R}^3} |u|^{2s} \, dx \geq 0 \) for any \( u \in H^s(\mathbb{R}^3) \), we have \( I_{\lambda}(u) = A(u) - \lambda B(u) \).
From Theorem 1.1, we conclude that for any $\lambda \in [\delta, 1]$, the associated limiting equation of $I_\lambda$ as follows:

$$
\begin{cases}
(a + b \int_{\mathbb{R}^3} \frac{|(-\Delta)^{\frac{s}{2}} u|^2}{\lambda} \, dx) (-\Delta)^{\frac{s}{2}} u + V_{\infty} u = \lambda f(u) + \lambda |u|^{2^*_s - 2} u, & x \in \mathbb{R}^3, \\
u \in H^s(\mathbb{R}^3),
\end{cases}
$$

has a nonnegative solution $u_\lambda \in M_\lambda^\infty := \{ u \in H^s(\mathbb{R}^3) \setminus \{0\} \mid G_\lambda^\infty(u) = 0 \}$ satisfying

$$
I_\lambda^\infty(u_\lambda) = m_\lambda^\infty := \inf_{u \in M_\lambda^\infty} I_\lambda^\infty(u),
$$

where

$$
I_\lambda^\infty(u) = \frac{1}{2} \int_{\mathbb{R}^3} (a \frac{|(-\Delta)^{\frac{s}{2}} u|^2}{\lambda} + V_{\infty} u^2) \, dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx \right)^2 - \lambda \int_{\mathbb{R}^3} F(u) \, dx - \frac{\lambda}{2^*_s} \int_{\mathbb{R}^3} |u|^{2^*_s} \, dx
$$

and

$$
G_\lambda^\infty(u) := 2sa \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx + 4s \int_{\mathbb{R}^3} V_{\infty} u^2 \, dx + 2sb \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx \right)^2 - \lambda \int_{\mathbb{R}^3} (4s - 3)f(u) + 6F(u) \, dx - 2s \lambda \int_{\mathbb{R}^3} |u|^{2^*_s} \, dx.
$$

By a standard argument (see e.g. [27, 33, 45]), we can verify that the functional $I_\lambda$ has a Mountain Pass geometry.

**Lemma 3.1.** Assume that $(V_1)$ and $(f_1) - (f_2)$ hold. For all fixed $\lambda \in [\delta, 1]$, $I_\lambda$ satisfies

(i) there exists $\bar{t} > 0$ independent of $\lambda$ such that $I_\lambda((u_1)_{\bar{t}}) \leq 0$, where $u_1$ defined by (39) with $\lambda = 1$;

(ii) $c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(0), I_\lambda((u_1)_{\bar{t}})\}$, where

$$
\Gamma = \{ \gamma \in C([0,1], H^s(\mathbb{R}^3)) \mid \gamma(0) = 0, \gamma(1) = (u_1)_{\bar{t}} \}.
$$

Arguing as in Lemma 2.2, we can establish the following lemma.

**Lemma 3.2.** Assume that $(f_1) - (f_3)$ hold. For all fixed $\lambda \in [\delta, 1]$ and any $t > 0$, there holds

$$
I_\lambda^\infty(u) \geq I_\lambda^\infty(u_t) + \frac{1 - t^{8s}}{8s} G_\lambda^\infty(u) + \frac{a(1 - t^{4s})^2}{4} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx, \quad \forall u \in H^s(\mathbb{R}^3),
$$

where $u_t$ is given in Lemma 2.2.

**Lemma 3.3.** Assume that $(V_1)$ and $(f_1) - (f_4)$ hold. Then there exists $\delta \leq \bar{\delta} < 1$ such that

$$
c_\lambda < m_\lambda^\infty < g_\lambda(T_\lambda) := a \frac{S^2_{\frac{2^*_s}{s}}}{2} T^{4s}_{\lambda} + \frac{b}{4} S^{\frac{2}{s}} T^{8s}_{\lambda} - \frac{\lambda}{2^*_s} \frac{S^{\frac{2}{s}}}{2} T^{2s}_{\lambda} \quad \forall \lambda \in [\delta, 1] \quad (40)
$$

if either $\frac{4s}{4s - 2s} < p < 2^*_s$ for all $\mu > 0$ or $2 < p \leq \frac{4s}{4s - 2s}$ for $\mu > 0$ sufficiently large. Where $T_\lambda$ is a unique positive solution of equation $a S^{\frac{2}{s}} y^{4s} + b S^{\frac{2}{s}} y^{8s} - \lambda S^{\frac{2}{s}} y^{2s-2} = 0$. 

Proof. First, we prove that there exists $\delta \leq \bar{\delta} < 1$ such that $c_\lambda < m_\infty^\sim$, $\forall \lambda \in [\delta, 1]$. Indeed, for any $\lambda \in [\delta, 1]$, we take $t_\lambda \in (0, \bar{t})$ such that $I_\lambda((u_1)_{t_\lambda}) = \max_{t \in [0, \bar{t}]} I_\lambda((u_1)_t)$. Since $I_\lambda((u_1)_t) \rightarrow -\infty$ as $t \rightarrow +\infty$, there is a $\bar{t} > 0$ such that
\[
I_\lambda((u_1)_t) \leq I_1(u_1) - 1, \quad \forall t \geq \bar{t}. \tag{41}
\]
However,
\[
I_1(u_1) \leq I_\lambda((u_1)_{t_\lambda}), \quad \forall \lambda \in [\delta, 1],
\]
which combined with (41), implies that $t_\lambda < \bar{t}$, $\forall \lambda \in [\delta, 1]$. Let $\beta_0 = \inf_{\lambda \in [\delta, 1]} t_\lambda$. Next we claim that $\beta_0 > 0$. Otherwise, if $\beta_0 = 0$, there exists a sequence $\{\lambda_n\} \subset [\delta, 1]$ such that $\lambda_n \rightarrow \lambda_0 \in [\delta, 1]$ and $t_{\lambda_n} \rightarrow 0$, as $n \rightarrow \infty$.

By the definition of $t_\lambda$, one has
\[
0 < c_1 \leq c_{\lambda_n} \leq I_{\lambda_n}((u_1)_{t_{\lambda_n}}) \rightarrow 0, \quad \text{as } n \rightarrow \infty,
\]
which is a contradiction. Thus, $0 < \beta_0 \leq t_\lambda < \bar{t}$, $\forall \lambda \in [\delta, 1]$. Set
\[
\bar{\delta} = \max \left\{ \delta, 1 - \frac{1}{\beta_\delta} \{ \frac{1}{\beta_\delta} \inf_{\theta \in [\beta_\delta, \beta_0]} \int_{\mathbb{R}^3} (V_\infty - V(\theta^2 x)) u_1^2 dx \right\}.
\]

It follows from Lemma 3.2 and the definition of $t_\lambda$ that
\[
m_\infty^\sim \geq m_\infty^\sim = I_1^\sim(u_1) \geq I_{\lambda}^\sim((u_1)_{t_\lambda})
\]
\[
= I_\lambda((u_1)_{t_\lambda}) + \frac{1}{2} t_\lambda^{4s} \int_{\mathbb{R}^3} (V_\infty - V(\lambda^2 x)) u_1^2 dx - (1 - \lambda) t_\lambda^6 \int_{\mathbb{R}^3} F(\theta_{4s-3} u_1) dx - \frac{1 - \lambda}{2s} t_\lambda^{2s-2} \int_{\mathbb{R}^3} |u_1|^{2s} dx
\]
\[
> c_\lambda + \frac{1}{2} \beta_\delta \int_{\mathbb{R}^3} (V_\infty - V(\theta^2 x)) u_1^2 dx - (1 - \lambda) t_\lambda^6 \int_{\mathbb{R}^3} F(\theta_{4s-3} u_1) dx - \frac{1 - \lambda}{2s} t_\lambda^{2s-2} \int_{\mathbb{R}^3} |u_1|^{2s} dx
\]
\[
\geq c_\lambda, \quad \forall \lambda \in [\delta, 1].
\]

The second inequality in (40) is obtained due to Lemma 2.6.

Next, we provide the following version of global compactness lemma, which is adopted to prove that the functional $I_\lambda$ satisfies $(PS)_{c_\lambda}$ condition for $a.e$ $\lambda \in [\delta, 1]$. The proof is similar to [33, 34, 45], so we omit details here.

**Lemma 3.4.** Assume that $(V_1) - (V_2)$ and $(f_1) - (f_\lambda)$ hold. Let $\lambda \in [\delta, 1]$ be fixed and $\{u_n\}$ be a bounded $(PS)_{c_\lambda}$ sequence for $I_\lambda$ with $c_\lambda < g_\lambda(T_\lambda)$. Then there exist $a \in H^s(\mathbb{R}^3)$ and $A \in \mathbb{R}$ such that $J_\lambda'(u) = 0$, where
\[
J_\lambda(u) = \frac{a + b A^2}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} V(x) u^2 dx - \lambda \int_{\mathbb{R}^3} F(u) dx - \frac{\lambda}{2s} \int_{\mathbb{R}^3} |u|^{2s} dx \tag{42}
\]
and either
\[
(i) \quad u_n \rightarrow u \text{ in } H^s(\mathbb{R}^3), \text{ or }
\]
Assume that Lemma 3.5.  

Following result.

According to Lemma 3.4, we know that, for each 1 \leq k \leq l, nontrivial solutions w^1, \cdots, w^l of the following equation

\[(a + bA^2)(-\Delta)^s u + V_{\infty} u = \lambda f(u) + \lambda |u|^{2^*_s - 2} u\] (43)

such that

\[c_\lambda + \frac{bA^4}{4} = J_\lambda(u) + \sum_{k=1}^l J_\lambda^\infty(w^k),\]

where

\[J_\lambda^\infty(u) = \frac{a + bA^2}{2} \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} V_{\infty} u^2 dx - \lambda \int_{\mathbb{R}^3} F(u) dx - \frac{\lambda}{2s} \int_{\mathbb{R}^3} |u|^{2^*_s} dx,\] (44)

\[\|u_n - u - \sum_{k=1}^l w^k (\cdot - y^k_n)\| \to 0,\]

and

\[A^2 = \|(-\Delta)^{s/2} u\|_2^2 + \sum_{k=1}^l \|(-\Delta)^{s/2} w^k\|_2^2.\]

On the convergence of the bounded (PS) sequence \(\{u_n\}\) for \(I_\lambda\), we have the following result.

**Lemma 3.5.** Assume that \((V_1) - (V_2)\) and \((f_1) - (f_4)\) hold. Let \(\lambda \in [\bar{\delta}, 1]\) be fixed and \(\{u_n\} \subset H^s(\mathbb{R}^3)\) be a bounded (PS)\(c_\lambda\) sequence for \(I_\lambda\) with \(c_\lambda < g_\lambda(T_\lambda)\). Then there exist a subsequence of \(\{u_n\}\), still denoted by \(\{u_n\}\) and a \(u_\lambda \in H^s(\mathbb{R}^3)\setminus\{0\}\) such that

\[u_n \to u_\lambda \text{ in } H^s(\mathbb{R}^3).\]

**Proof.** According to Lemma 3.4, we know that, for \(\lambda \in [\bar{\delta}, 1]\), there exist a \(u_\lambda \in H^s(\mathbb{R}^3)\) and \(A_\lambda \in \mathbb{R}\) such that

\[u_n \to u_\lambda \text{ in } H^s(\mathbb{R}^3), \lim_{n \to \infty} \|(-\Delta)^{s/2} u_n\|_2^2 = A_\lambda^2 \quad \text{and} \quad J'_\lambda(u_\lambda) = 0,\]

where \(J'_\lambda\) is given by (42) with \(A\) replaced by \(A_\lambda\). Moreover, either (i) or (ii) occurs.

If (ii) occurs, i.e., there exist an \(l \in \mathbb{N}\) and \(\{y^k_n\} \subset \mathbb{R}^3\) with \(|y^k_n| \to +\infty\) as \(n \to \infty\) for each \(1 \leq k \leq l\), nontrivial solutions \(w^1, \cdots, w^l\) of equation (43) such that

\[c_\lambda + \frac{bA^4}{4} = J_\lambda(u_\lambda) + \sum_{k=1}^l J_\lambda^\infty(w^k),\] (45)

\[\|u_n - u_\lambda - \sum_{k=1}^l w^k (\cdot - y^k_n)\| \to 0,\]

and

\[A_\lambda^2 = \|(-\Delta)^{s/2} u_\lambda\|_2^2 + \sum_{k=1}^l \|(-\Delta)^{s/2} w^k\|_2^2,\] (46)

where \(J_\lambda^\infty\) is given by (44) with \(A\) replaced by \(A_\lambda\). Since \(J'_\lambda(u_\lambda) = 0\), by the similar argument as Proposition 4.1 in [10], we can deduce the following Pohozaev identity

\[P_\lambda(u_\lambda) := \frac{3 - 2s}{2} (a + bA^2) \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u_\lambda|^2 dx + \frac{3}{2} \int_{\mathbb{R}^3} V(x) u_\lambda^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} (\nabla V(x), x) u_\lambda^2 dx - 3\lambda \int_{\mathbb{R}^3} F(u_\lambda) dx - \frac{3}{2s} \lambda \int_{\mathbb{R}^3} |u_\lambda|^{2^*_s} dx = 0.\]
From (V2), (21) and Sobolev inequality, we have

\[ J_\lambda(u_\lambda) = J_\lambda(u_\lambda) - \frac{(4s - 3)\langle J'_\lambda(u_\lambda), u_\lambda \rangle + 2P_\lambda(u_\lambda)}{8s} \]

\[ = \frac{a + bA_\lambda^2}{4} \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{t}{2}} u_\lambda \right|^2 dx - \frac{1}{8s} \int_{\mathbb{R}^3} \langle \nabla V(x), x \rangle u_\lambda^2 dx \]

\[ + \frac{4s - 3}{8s} \lambda \int_{\mathbb{R}^3} (f(u_\lambda)u_\lambda - 2F(u_\lambda)) dx + \frac{4s - 3}{12} \lambda \int_{\mathbb{R}^3} |u_\lambda|^2^2 dx \]

\[ \geq \frac{a + bA_\lambda^2}{4} \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{t}{2}} u_\lambda \right|^2 dx \]

In the following we estimate the value of \( J_\lambda^\infty(w^k) \). For each nontrivial solution \( w^k (k = 1, \ldots, l) \) of problem (43), we have the following Pohozaev identity

\[ P_\lambda^\infty(w^k) := \frac{3 - 2s}{2} (a + bA_\lambda^2) \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{t}{2}} w^k \right|^2 dx + \frac{3}{2} \int_{\mathbb{R}^3} V^\infty |w^k|^2 dx \]

\[ - 3\lambda \int_{\mathbb{R}^3} F(w^k) dx - \frac{3}{2s} \lambda \int_{\mathbb{R}^3} |w^k|^2^2 dx = 0, \quad 1 \leq k \leq l. \]

By (46), for each \( 1 \leq k \leq l \), we have that

\[ 0 = (4s - 3)\langle (J_\lambda^\infty)'(w^k), w^k \rangle + 2P_\lambda^\infty(w^k) \]

\[ = 2s(a + bA_\lambda^2) \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{t}{2}} w^k \right|^2 dx + 4s \int_{\mathbb{R}^3} V^\infty |w^k|^2 dx \]

\[ - \lambda \int_{\mathbb{R}^3} \left( (4s - 3)f(w^k)w^k + 6F(w^k) \right) dx - 2s \lambda \int_{\mathbb{R}^3} |w^k|^2^2 dx \geq G_\lambda^\infty(w^k). \]

Similarly to Lemma 2.3, there exists \( t_k > 0 \) such that \( (w^k)_{t_k} \in \mathcal{M}_\lambda^\infty \). It follows from Lemma 3.2 that

\[ J_\lambda^\infty(w^k) = J_\lambda^\infty(w^k) - \frac{(4s - 3)\langle (J_\lambda^\infty)'(w^k), w^k \rangle + 2P_\lambda^\infty(w^k)}{8s} \]

\[ = \frac{a}{4} \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{t}{2}} w^k \right|^2 dx + \frac{4s - 3}{8s} \lambda \int_{\mathbb{R}^3} (f(w^k)w^k - 2F(w^k)) dx \]

\[ + \frac{4s - 3}{12} \lambda \int_{\mathbb{R}^3} |w^k|^2^2 dx + \frac{bA_\lambda^2}{4} \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{t}{2}} w^k \right|^2 dx \]

\[ = I_\lambda^\infty(w^k) - \frac{1}{8s} G_\lambda^\infty(w^k) + \frac{bA_\lambda^2}{4} \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{t}{2}} w^k \right|^2 dx \]

\[ \geq I_\lambda^\infty((w^k)_{t_k}) - \frac{bA_\lambda^2}{8s} \left( k \right) G_\lambda^\infty(w^k) + \frac{bA_\lambda^2}{4} \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{t}{2}} w^k \right|^2 dx \]

\[ \geq m_\lambda^\infty + \frac{bA_\lambda^2}{4} \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{t}{2}} w^k \right|^2 dx, \quad 1 \leq k \leq l. \]
From (45)-(47), we can calculate
\[
c_\lambda + \frac{bA_\lambda^2}{4} = J_\lambda(u_\lambda) + \sum_{k=1}^{l} J_\lambda^k(w^k) \\
\geq \ln \mu^\infty + \frac{bA_\lambda^2}{4} \int_{\mathbb{R}^3} |(-\Delta)\hat{u}_\lambda|^2 dx + \frac{bA_\lambda^2}{4} \sum_{k=1}^{l} \int_{\mathbb{R}^3} |(-\Delta)\hat{w}^k|^2 dx \\
\geq m_\mu^\infty + \frac{bA_\lambda^2}{4}, \quad \forall \lambda \in [\delta, 1].
\]
Thus, we have \(c_\lambda \geq m_\lambda^\infty\), which contradicts Lemma 3.3. So (i) holds, i.e., \(u_n \rightarrow u_\lambda\)
in \(H^s(\mathbb{R}^3)\), and \(u_\lambda\) is a nontrivial critical point for \(I_\lambda\) with \(I_\lambda(u_\lambda) = c_\lambda\).

At this point, combining Proposition 1 and Lemma 3.1, we deduce that for a.e. \(\lambda \in [\delta, 1]\), there exists a bounded \((PS)\) sequence \(\{u_n\} \subset H^s(\mathbb{R}^3)\) such that \(I_\lambda(u_n) \rightarrow c_\lambda < g_\delta(T_\lambda)\). Then by Lemma 3.5, we know that \(I_\lambda\) has a nontrivial critical point \(u_\lambda \in H^s(\mathbb{R}^3)\) with \(I_\lambda(u_\lambda) = c_\lambda\) for a.e. \(\lambda \in [\delta, 1]\). As a special case we obtain the existence of a sequence \(\{(\lambda_n, u_{\lambda_n})\} \subset [\delta, 1] \times H^s(\mathbb{R}^3)\) with \(\lambda_n \rightarrow 1\) as \(n \rightarrow \infty\) and \(u_{\lambda_n} \neq 0\) satisfying
\[
I'_\lambda(u_{\lambda_n}) = 0 \quad \text{and} \quad I_{\lambda_n}(u_{\lambda_n}) = c_{\lambda_n}. \quad (48)
\]
In order to prove Theorem 1.2, we need to show that the critical point sequence \(\{u_{\lambda_n}\}\) obtained in (48) is bounded and that is a \((PS)_{c_\delta}\) sequence for \(I = I_1\), where \(I\) is given by (7). By applying Lemma 3.5 again, we obtain a nontrivial critical point for \(I\) and the proof is completed.

**Proof of Theorem 1.2.** First, we show that the sequence \(\{u_{\lambda_n}\} \subset H^s(\mathbb{R}^3)\) obtained in (48) is bounded. Since \(I_{\lambda_n}(u_{\lambda_n}) = c_{\lambda_n} \leq c_\delta\) and from (V2), (21) and Sobolev inequality, we have
\[
c_\delta \geq I_{\lambda_n}(u_{\lambda_n}) \\
= I_{\lambda_n}(u_{\lambda_n}) - \frac{G_{\lambda_n}(u_{\lambda_n})}{8s} \\
= \frac{a}{4} \int_{\mathbb{R}^3} |(-\Delta)\hat{u}_{\lambda_n}|^2 dx - \frac{1}{8s} \int_{\mathbb{R}^3} (\nabla V(x), x) u_{\lambda_n}^2 dx \\
+ \frac{4s-3}{8s} \lambda_n \int_{\mathbb{R}^3} (f(u_{\lambda_n})u_{\lambda_n} - 2F(u_{\lambda_n})) dx + \frac{4s-3}{12} \lambda_n \int_{\mathbb{R}^3} |u_{\lambda_n}|^{2s} dx \quad (49) \\
\geq \frac{a}{4} \int_{\mathbb{R}^3} |(-\Delta)\hat{u}_{\lambda_n}|^2 dx - \frac{1}{8s}\|\nabla V(x), x\|_{L^\infty}(\int_{\mathbb{R}^3} |u_{\lambda_n}|^{2s} dx)^{\frac{2}{2s}} \\
\geq \frac{a - A}{4} \int_{\mathbb{R}^3} |(-\Delta)\hat{u}_{\lambda_n}|^2 dx,
\]
where
\[
G_\lambda(u) := 2sa \int_{\mathbb{R}^3} |(-\Delta)\hat{u}|^2 dx + \int_{\mathbb{R}^3} \left(4sV(x) + (\nabla V(x), x)\right) u^2 dx \\
+ 2sb \left(\int_{\mathbb{R}^3} |(-\Delta)\hat{u}|^2 dx\right)^2 \\
- \lambda \int_{\mathbb{R}^3} \left((4s-3)f(u)u + 6F(u)\right) dx - 2s \lambda \int_{\mathbb{R}^3} |u|^{2s} dx \quad (50)
\]
with \( G_{\lambda_n}(u_{\lambda_n}) = 0 \) due to Pohozaev identity and \( I'_{\lambda_n}(u_{\lambda_n}) = 0 \). Since \( I'_{\lambda_n}(u_{\lambda_n}) = 0 \), from \((f_1) - (f_2)\), we have that for any \( \varepsilon > 0 \), there exists a \( C_\varepsilon > 0 \) such that
\[
\|u_{\lambda_n}\|^2 \leq a \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}} u_{\lambda_n}|^2 dx + \int_{\mathbb{R}^3} V(x) u_{\lambda_n}^2 dx + b \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}} u_{\lambda_n}|^2 dx \right)^2
\]
\[
= \lambda_n \int_{\mathbb{R}^3} f(u_{\lambda_n}) u_{\lambda_n} dx + \lambda_n \int_{\mathbb{R}^3} |u_{\lambda_n}|^2 dx
\]
\[
\leq \varepsilon \int_{\mathbb{R}^3} u_{\lambda_n}^2 dx + C_\varepsilon \int_{\mathbb{R}^3} |u_{\lambda_n}|^2 dx.
\]
By choosing \( \varepsilon > 0 \) small enough, we obtain that \( \{u_{\lambda_n}\} \) is bounded in \( H^s(\mathbb{R}^3) \). Thus we have for any \( \varphi \in H^s(\mathbb{R}^3) \)
\[
\left| \langle I'_{\lambda_n}(u_{\lambda_n}), \varphi \rangle - \langle I'(u_{\lambda_n}), \varphi \rangle \right| = \left| (\lambda_n - 1) \int_{\mathbb{R}^3} (f(u_{\lambda_n}) u_{\lambda_n} + |u_{\lambda_n}|^{2s} - 2 u_{\lambda_n}) \varphi dx \right| \to 0
\]
as \( n \to \infty \). And notice that
\[
\lim_{n \to \infty} I(u_{\lambda_n}) = \lim_{n \to \infty} \left[ \lambda_n (u_{\lambda_n}) + (\lambda_n - 1) \int_{\mathbb{R}^3} (f(u_{\lambda_n}) + \frac{1}{2s} |u_{\lambda_n}|^{2s}) dx \right]
\]
\[
= \lim_{n \to \infty} c_{\lambda_n} = c_1,
\]
where we used the fact that the map \( \lambda \mapsto c_\lambda \) is continuous from the left. That is \( \{u_{\lambda_n}\} \) is a bounded \((PS)_{c_1}\) sequence for \( I \). By applying Lemma 3.5, we obtain a nontrivial critical point \( u_0 \in H^s(\mathbb{R}^3) \) for \( I \) and \( I(u_0) = c_1 \).

Finally, we end this proof by showing the existence of a least energy solution for problem (1). Let
\[
d := \inf \{ I(u) \mid u \neq 0, I'(u) = 0 \}.
\]
Then \( 0 \leq d \leq I(u_0) < g(T) \). In fact, for any \( u \) satisfying \( I'(u) = 0 \), by standard argument we see \( \|u\| \geq \rho \) for some positive constant \( \rho \). Similarly to the argument of (49)-(50), we infer
\[
I(u) \geq \frac{a - A}{4} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}} u|^2 dx.
\]
(51)
Therefore, \( d \geq 0 \). In the following we rule out \( d = 0 \). Suppose by contradiction that \( \{u_n\} \) be a critical point sequence of \( I \) satisfying \( \lim_{n \to \infty} I(u_n) = 0 \). From (51), we have
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}} u_n|^2 dx = 0.
\]
This conclusion combined with \( \langle I'(u_n), u_n \rangle = 0 \), we can verify that \( \lim_{n \to \infty} \int_{\mathbb{R}^3} u_n^2 dx = 0 \). Therefore, we obtain \( \lim_{n \to \infty} \|u_n\| = 0 \), which is a contradiction with \( \|u_n\| \geq \rho > 0 \) for all \( n \in \mathbb{N} \).

Let \( \{u_n\} \subset H^s(\mathbb{R}^3) \) be a sequence of nontrivial critical point of \( I \) satisfying \( \lim_{n \to \infty} I(u_n) = d < g(T) \). Similarly, we can deduce that \( \{u_n\} \) is bounded in \( H^s(\mathbb{R}^3) \), i.e., \( \{u_n\} \) is a bounded \((PS)_d\) sequence for \( I \). Similarly to the argument in Lemma 3.5, there exists a nontrivial \( w \in H^s(\mathbb{R}^3) \) such that \( I(w) = d \) with \( I'(w) = 0 \). Arguing as in the proof of Theorem 1.1, we show that the least energy solution \( w \) is nonnegative.

\[\square\]

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E-mail address: ybdeng@mail.ccnu.edu.cn
E-mail address: wthuang1014@aliyun.com