On an Elasto-Acoustic Transmission Problem in Anisotropic, Inhomogeneous Media.

R. Picard

August 13, 2018

Department of Mathematics, TU Dresden, 01062 Dresden, Germany

Abstract. We consider a coupled system describing the interaction between acoustic and elastic regions, where the coupling occurs not via material properties but through an interaction on an interface separating the two regimes. Evolutionary well-posedness in the sense of Hadamard well-posedness supplemented by causal dependence is shown for a natural choice of generalized interface conditions. The results are obtained in a real Hilbert space setting incurring no regularity constraints on the boundary and almost none on the interface of the underlying regions.

1 Introduction

Similarities between various initial boundary value problems of mathematical physics have been noted as general observations throughout the literature. Indeed, the work by K. O. Friedrichs, [2], already showed that the classical linear phenomena of mathematical physics belong – in the static case – to his class of symmetric positive hyperbolic partial differential equations, later referred to as Friedrichs systems, which are of the abstract form

\((M_1 + A) u = f,\) \hspace{1cm} (1)

with \(A\) at least formally, i.e. on \(C^\infty\)-vector fields with compact support in the underlying region \(\Omega\), a skew-symmetric differential operator and the \(L^\infty\)-matrix-valued multiplication-operator \(M_1\) satisfying the condition

\[ \text{sym} (M_1) := \frac{1}{2} (M_1 + M_1^*) \geq c > 0 \]
for some real number $c$. Indeed, a typical choice of boundary condition is, when $A$ is skew-selfadjoint\footnote{To assume $A$ to be skew-selfadjoint is less restrictive than one might think. For this we note that for example typical dissipative boundary conditions actually give rise to natural skew-selfadjoint spatial operators $A$. [20]. That skew-selfadjoint $A$ is a quite common assumption but may not be recognized. As a typical example we consider the popular transcription of the wave equation $\partial_0^2 - \Delta_D$, where $\Delta_D$ denotes the Laplacian with a homogeneous Dirichlet boundary condition in a bounded domain $\Omega$, into a first order system of the form $\partial_0 + A$, where $A = \begin{pmatrix} 0 & \Delta_D \\ 1 & 0 \end{pmatrix}$ is indeed skew-selfadjoint due to the standard choice of Hilbert space setting.} ($A$ m-accretive would be sufficient). Problem (1) can be considered as the static problem associated with the dynamic problem ($\partial_0$ denotes the time-derivative)

$$\partial_0 M_0 + M_1 + A$$

(2)

with $M_0$ selfadjoint $L^\infty$-multiplication-operator and $M_0 \geq 0$, which were also addressed in [3]. It is noteworthy, that even the temporal exponential weight factor, which plays a central role in the approach of [15], is introduced as an ad-hoc formal trick to produce a suitable $M_1$ for a well-posed static problem. For the so-called time-harmonic case, where $\partial_0$ is replaced by $i\omega$, $\omega \in \mathbb{R}$, we replace $A$ simply by $i\omega M_0 + A$ to arrive at a system of the form (1).

Operators of the Friedrichs type (2), can be generalized to obtain a fully time-dependent theory allowing for operator-valued coefficients, indeed, in the time-shift invariant case, for systems of the general form

$$\left(\partial_0 M \left(\partial_0^{-1}\right) + A\right) U = F$$

(Evo-Sys)

where $A$ is – for simplicity – skew-selfadjoint and $M$ an operator-valued – say – rational function (regular at 0) as an abstract coefficient. The meaning of the so-called material law operator $M \left(\partial_0^{-1}\right)$ is in terms of a suitable function calculus associated with the (normal) operator $\partial_0$, [19] Chapter 6]. This spacious class of operators allows for a large class of material laws including – the recently of great interest – meta-materials.

We shall refer to such systems as evo-systems (or evolutionary equations) to distinguish them from the special subclass of classical (explicit) evolution equations.

In this paper we intend to study a particular transmission problem between two physical regimes, acoustics and elasto-dynamics, within this general framework to establish its well-posedness, which for evo-systems entails not only Hadamard well-posedness, i.e. uniqueness, existence and continuous dependence, but also the crucial property of causality. For this we will only have to establish the skew-selfadjointness of a suitably constructed operator $A$. Then it is known that the requirement

$$\varrho M \left(0\right) + \text{sym} \left(M' \left(0\right)\right) \geq c_0 > 0$$

(3)

for some number $c_0$ all sufficiently large $\varrho \in ]0, \infty[$, yields the desired well-posedness, see the survey [22]. For the simple Friedrichs type case where we additionally assume

$$M_0 = M \left(0\right) \geq c_0 > 0$$

(4)
for some number $c_0$, which clearly implies (3), we may even use the commonly invoked semi-group theory to establish the desired well-posedness (note that in this case $M_1 = M'(0)$ and all higher derivatives of $M$ vanish). Indeed, under these strong restrictions (2) is congruent to

$$
\partial_0 + \sqrt{M_0^{-1}} M_1 \sqrt{M_0^{-1}} + \sqrt{M_0^{-1}} A \sqrt{M_0^{-1}},
$$

which amounts to having $M_0 = 1$ ($M_1$ replaced by the congruent $\sqrt{M_0^{-1}} M_1 \sqrt{M_0^{-1}}$) and using $\sqrt{M_0} U$ as the new unknown in the corresponding problem of the form (Evo-Sys). With $\sqrt{M_0^{-1}} A \sqrt{M_0^{-1}}$ inheriting its skew-selfadjointness from $A$ we obtain indeed a one-parameter group $\left( \exp \left( t \sqrt{M_0^{-1}} A \sqrt{M_0^{-1}} \right) \right)_{t \in \mathbb{R}}$, which by a simple perturbation argument yields a group $(U(t))_{t \in \mathbb{R}}$ such that $(\chi_{[0,\infty]}(t) U(t))_{t \in \mathbb{R}}$, with $\chi_{[0,\infty]}$ denoting the characteristic function of the interval $[0, \infty]$, is the fundamental solution associated with (5). Thus a fairly general solution can be obtained by convolution with this fundamental solution. Restricting this fundamental solution to its support yields a continuous, one-parameter semigroup $(U(t))_{t \in [0,\infty]}$. In any case we are justified to focus on the underlying skew-selfadjointness of the operator $A$ as a central feature to obtain well-posedness for a large class of general material laws, since we shall be concerned with the interaction between the elastic and the acoustic regimes solely via the interface, not via material interactions through the material law, as for example in piezo-electrics, compare e.g. [10] for a typical effect of the latter type. This specific focus also allows us in the interest of brevity to by-pass the intricacies of the time-dependent theory of [15].

Skew-selfadjointness of an operator $A$, i.e.

$$
A = -A^*,
$$

in a real Hilbert space $H$ results in

$$
\langle u | Au \rangle_H = 0
$$

for all $u \in D(A)$. Moreover, in typical cases skew-selfadjointness of $A$ is a simple consequence of $A$ being congruent to a block matrix of the form

$$
\begin{pmatrix}
0 & -C^* \\
C & 0
\end{pmatrix},
$$

where $C : D(C) \subseteq H_0 \rightarrow H_1$ is a closed, densely defined, linear operator between real Hilbert spaces $H_0$, $H_1$, which is clearly skew-selfadjoint in the direct sum Hilbert space $H = H_0 \oplus H_1$.

The interest of studying the coupling between acoustic and elasticity wave phenomena has a relatively long history in the engineering community, with [7], [8], being earlier references. Originally motivated by submarine noise propagation, this coupling is also of
interest in connection with loudspeaker and hearing aid design, as well as non-destructive testing. Near the close of the last century there has been a rekindled interest in these specific issues, [24], [9]. More recent publications are the numerical investigations [1], [25], [23], and the more mathematically oriented [5], [11], [6], [4], just to mention a few. Here we want to transcend the predominant constant coefficient and – with the notable exception of [4] – largely time-harmonic analysis and consider the time-dependent case in anisotropic, inhomogeneous media. Since we shall consider operator coefficients, this also includes media with non-local behavior. For sake of accessibility we restrict our attention to the autonomous case with classical block-diagonal material laws and no memory effects. We use a functional-analytical setting in real Hilbert space to obtain a well-posedness for this elasto-acoustic transmission problem.

We shall first establish the spatial operator of acoustics and elasticity, respectively, as intimately related skew-selfadjoint operators (mother-descendant mechanism) in a real Hilbert space framework based on the above-mentioned block structure with suitably introduced operators $C$. Then, in Section 3 we apply these observations to a particular interface coupling problem between the two regimes in adjacent regions via a refined mother-descendant mechanism. We emphasize that our setup allows for arbitrary open sets as underlying domains with no additional constraints on boundary regularity and almost no constraints on interface regularity. Indeed we only require the interface to be a Lebesgue null set. The induced homogeneous boundary value constraints and transmission conditions are encoded – as customary – in suitable generalization as containment in the domain of the operator.
2 The Connection of the Spatial Operators of Acoustics and Elasticity

2.1 Basic Ideas

Without loss of generality we may and will assume that all Hilbert spaces used in the following are real.

In many practical cases the desired skew-selfadjointness of the spatial operator \( A \) is evident from its structure as a block operator matrix of the form

\[
A = \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix},
\]

with \( H = H_0 \oplus H_1 \) and \( C : D(C) \subseteq H_0 \to H_1 \) a closed, densely defined, linear operator.

We shall start our exploration by focussing for simplicity and definiteness on the Cartesian situation and on the case of the so-called Dirichlet boundary condition. For this, we initially take \( C \) as the closure \( \tilde{\text{grad}} \) of the classical differential operator

\[
\tilde{\text{C}}_1 (\Omega, \mathbb{R}^3) \subseteq L^2 (\Omega, \mathbb{R}^3) \to L^2 (\Omega, \mathbb{R}^{3 \times 3}),
\]

\[u \mapsto u',\]

where \( u' \) is the derivative (in matrix language the Jacobian) of the vectorfield \( u \). The negative adjoint is the weak extension of the classical divergence operator on matrix fields

\[
\text{div} := -\text{grad}^*.
\]

Thus, the operator of our initial interest is

\[
A = \begin{pmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{pmatrix}.
\]

\(^2\)Note that every complex Hilbert space \( X \) is a real Hilbert space choosing only real numbers as multipliers and

\[
(\phi, \psi) \mapsto \Re \langle \phi | \psi \rangle_X
\]

as new inner product. Note that with this choice \( \phi \) and \( i\phi \) are always orthogonal. Moreover, for any skew-symmetric operator \( A \) we have

\[
x \perp Ax
\]

for all \( x \in D(A) \).

Indeed, since \( \langle x|y \rangle - \langle y|x \rangle = 0 \) (symmetry) we have

\[
\langle x|Ax \rangle - \langle Ax|x \rangle = 0
\]

or by skew-symmetry

\[
0 = \langle x|Ax \rangle - \langle Ax|x \rangle = 2 \langle x|Ax \rangle
\]

for all \( x \in D(A) \).
as a skew-selfadjoint operator in \( H = L^2(\Omega, \mathbb{R}^3) \oplus L^2(\Omega, \mathbb{R}^{3 \times 3}) \). Here \( \mathbb{R}^{3 \times 3} \) is equipped with the standard Frobenius inner product. As an illustration let us consider
\[
\left( \begin{array}{cc}
\varrho & 0 \\
0 & C^{-1}
\end{array} \right) + \left( \begin{array}{cc}
0 & -\text{div} \\
-\text{grad} & 0
\end{array} \right) \left( \begin{array}{c}
v \\
T
\end{array} \right) = \left( \begin{array}{c}
f \\
g
\end{array} \right)
\]
as an associated dynamic problem for finding a solution \( \left( \begin{array}{c}
v \\
T
\end{array} \right) \in L^2(\Omega, \mathbb{R}^3) \oplus L^2(\Omega, \mathbb{R}^{3 \times 3}) \).

Here \( \varrho: L^2(\Omega, \mathbb{R}^3) \rightarrow L^2(\Omega, \mathbb{R}^3) \), and \( C: L^2(\Omega, \mathbb{R}^{3 \times 3}) \rightarrow L^2(\Omega, \mathbb{R}^{3 \times 3}) \) are assumed to be strongly positive definite mappings in order to obtain well-posedness in the sense of our introductory exposition. This type of system can be understood as modeling asymmetric elasticity theory in the sense of [12, 13, 14].

### 2.2 Symmetric Elasticity as a Descendant of Asymmetric Elasticity.

To illustrate the mother-descendant mechanism, as introduced in [18, 10], see also [21], we first perform the transition to classical (symmetric) elasticity using this concept.

We recall from [17] the following simple but crucial lemma.

**Lemma 2.1.** Let \( C: D(C) \subseteq H \rightarrow Y \) be a closed densely-defined linear operator between Hilbert spaces \( H, Y \). Moreover, let \( B: Y \rightarrow X \) be a continuous linear operator into another Hilbert space \( X \). If \( C^*B^* \) is densely defined, then
\[
(BC)^* = (C^*B^*)^*.
\]

**Proof.** It is
\[
C^*B^* \subseteq (BC)^*.
\]

If \( \phi \in D((BC)^*) \) then
\[
\langle BCu|\phi \rangle_X = \langle u|(BC)^*\phi \rangle_H
\]
for all \( u \in D(C) \). Thus, we have
\[
\langle Cu|B^*\phi \rangle_Y = \langle BCu|\phi \rangle_X = \langle u|(BC)^*\phi \rangle_H
\]
for all \( u \in D(C) \) and we read off that \( B^*\phi \in D(C^*) \) and
\[
C^*B^*\phi = (BC)^*\phi.
\]

Thus we have
\[
(BC)^* = C^*B^*.
\]

If now \( C^*B^* \) is densely defined, we have for its adjoint operator
\[
(C^*B^*)^* = BC.
\]

\( \square \)
As a consequence we have that the descendant
\[
\begin{pmatrix}
1 & 0 \\
0 & B
\end{pmatrix}
\begin{pmatrix}
0 & -C^* \\
C & 0
\end{pmatrix}
\begin{pmatrix}
0 & B^* \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
0 & -C^* B^* \\
BC & 0
\end{pmatrix}
\]
indeed inherits its skew-selfadjointness from its mother \(\begin{pmatrix}
0 & -C^* \\
C & 0
\end{pmatrix}\) (with \(C\) replaced by \(BC\)).

Remark 2.2. Clearly, the role of the components can be interchanged so that
\[
\begin{pmatrix}
D & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & -C^* \\
C & 0
\end{pmatrix}
\begin{pmatrix}
D^* & 0 \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
0 & -DC^* \\
CD^* & 0
\end{pmatrix}
\]
with \(D : H \rightarrow Y\) such that \(CD^*\) is densely defined, is also a valid descendant construction.

These construction can be combined. In general, a repeated application of the mother-descendant mechanism may, however, depend on the order in which they are carried out. This fact has been overlooked in [16]. An illuminating example is choosing \(C\) as the weak \(L^2(\mathbb{R})\) derivative \(\partial\) and \(B = D\) as the cut-off by the characteristic function \(\chi_{[-1/2,1/2]}\) of the symmetric unit interval \([-1/2,1/2]\) yielding
\[
\begin{pmatrix}
0 & \chi_{[-1/2,1/2]} \\
\chi_{[-1/2,1/2]} (\partial \chi_{[-1/2,1/2]}) & 0
\end{pmatrix}
\]
if first the construction with \(B\) and then with \(D\) is carried out. In reverse order we obtain
\[
\begin{pmatrix}
0 & \chi_{[-1/2,1/2]} \\
\chi_{[-1/2,1/2]} (\partial \chi_{[-1/2,1/2]}) & 0
\end{pmatrix}
\]
In comparison (7) models vanishing at \(\pm 1/2\) for the second component, whereas (8) leads to vanishing at \(\pm 1/2\) of the first component.

As a convenient mother operator to start from we take the above-mentioned theory of asymmetric elasticity of Nowacki, [12, 14]. Indeed, classical (symmetric) elasticity theory can be considered as a descendant in the above sense of the form
\[
\begin{pmatrix}
0 & -\text{Div} \\
-\text{Grad} & 0
\end{pmatrix}
\]
where
\[
\text{Grad} := \iota_{\text{sym}}^* \text{grad}
\]
and
\[
\text{Div} := \text{div} \iota_{\text{sym}}
\]
with
\[ \iota_{\text{sym}} : L^2(\Omega, \text{sym}[\mathbb{R}^{3 \times 3}]) \to L^2(\Omega, \mathbb{R}^{3 \times 3}), \]
\[ T \mapsto T, \]
where \text{sym}[\mathbb{R}^{3 \times 3}] denotes the image of \( \mathbb{R}^{3 \times 3} \) under the mapping \text{sym}, i.e. we have in the descendant construction \( B = \iota_{\text{sym}}^* \). Note that
\[ \iota_{\text{sym}}^* T = \text{sym}(T) \]
for all \( T \in L^2(\Omega, \mathbb{R}^{3 \times 3}) \).

\subsection*{2.3 Acoustics as a Descendant of Asymmetric Elasticity.}

The spatial operator used in the acoustics model can also be introduced as a descendant of asymmetric elasticity. It is actually the scalar version corresponding to the asymmetric elasticity case.

Indeed, classical acoustics can be considered as a descendant of the form
\[
\begin{pmatrix} 0 & \text{grad} \\ \text{div} & 0 \end{pmatrix},
\]
where we re-use the classical notations by letting
\[ \overline{\text{div}} := \text{trace grad} \]
and
\[ \text{grad} := \text{div trace}^* \]
with
\[ \text{trace} : L^2(\Omega, \mathbb{R}^{3 \times 3}) \to L^2(\Omega, \mathbb{R}), \]
\[ T = (T_{ij})_{i,j} \mapsto \text{trace} T := \sum_i T_{ii}, \]
i.e. \( B = \text{trace} \). Note that
\[ \text{trace}^* p = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} \]
for all \( p \in L^2(\Omega, \mathbb{R}) \).

Remark 2.3. The acoustic system can also be constructed by applying \( B = \text{trace} \) to the symmetric elasticity operator \( \text{sym} \). Note that the pressure distribution \( p \) is in both cases obtained from the stress as
\[ p := -\text{trace} T. \]
3 An Interface Coupling Between Acoustics and Elasticity

We will now combine the two descendant constructions above to obtain an interface coupling set-up for the skew-selfadjoint operator $A$. We assume $\Omega_0 \cup \Omega_1 \subseteq \Omega$, such that the orthogonal decomposition

$$L^2(\Omega, \mathbb{R}) = L^2(\Omega_0, \mathbb{R}) \oplus L^2(\Omega_1, \mathbb{R})$$

holds.

Then, with the respective canonical embeddings into $L^2(\Omega, \mathbb{R}^{3 \times 3})$ we obtain

$$B : L^2(\Omega, \mathbb{R}^{3 \times 3}) \rightarrow L^2(\Omega_0, \text{sym}[\mathbb{R}^{3 \times 3}]) \oplus L^2(\Omega_1, \mathbb{R})$$,

$$T \mapsto \left( t^*_{L^2(\Omega_0, \text{sym}[\mathbb{R}^{3 \times 3}])} t^*_{\text{sym}} T - t^*_{L^2(\Omega_1, \mathbb{R})} \text{trace } T \right),$$

and so

$$B = \left( t^*_{L^2(\Omega_0, \text{sym}[\mathbb{R}^{3 \times 3}])} t^*_{\text{sym}} - t^*_{L^2(\Omega_1, \mathbb{R})} \text{trace } \right).$$

With this we get as a descendant construction

$$A = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 & -\text{div} \\ -\text{grad} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix}$$

$$\subseteq \begin{pmatrix} 0 \\ -\text{Grad}_{\Omega_0} \\ \text{div}_{\Omega_1} \end{pmatrix} \begin{pmatrix} -\text{Div}_{\Omega_0} & -\text{grad}_{\Omega_1} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & C_{\Omega_1}^{-1} \end{pmatrix}.$$
where the first component is to be understood in the sense of (10). From the inclusion (11), (12), we read off that the resulting evo-system

\[
(\partial_0 M(0) + A) \begin{pmatrix} v_{\Omega_0} + v_{\Omega_1} \\ T_{\Omega_0} \\ p_{\Omega_1} \end{pmatrix} = \begin{pmatrix} f_{\Omega_0} + f_{\Omega_1} \\ F_{\Omega_0} \\ g_{\Omega_1} \end{pmatrix}
\]

indeed yields

\[
\partial_0 \left( g_{*,\Omega_0} + \kappa_{\Omega_1}^{-1} \right) (v_{\Omega_0} + v_{\Omega_1}) + \text{Div}_{\Omega_0} T_{\Omega_0} + \text{grad}_{\Omega_1} p_{\Omega_1} = f_{\Omega_0} + f_{\Omega_1},
\]

which in turn – according to (10) – splits into equations in \(\Omega_0\) and in \(\Omega_1\)

\[
\begin{align*}
\partial_0 g_{*,\Omega_0} v_{\Omega_0} - \text{Div}_{\Omega_0} T_{\Omega_0} &= f_{\Omega_0}, \\
\partial_0 \kappa_{\Omega_1}^{-1} v_{\Omega_1} + \text{grad}_{\Omega_1} p_{\Omega_1} &= f_{\Omega_1}.
\end{align*}
\]

The second block row yields another pair of equations

\[
\begin{align*}
\partial_0 C^{-1} T_{\Omega_0} - \text{Grad}_{\Omega_0} v_{\Omega_0} &= F_{\Omega_0}, \\
\partial_0 C_{\Omega_1} p_{\Omega_1} + \text{div}_{\Omega_1} v_{\Omega_1} &= g_{\Omega_1}.
\end{align*}
\]

The actual system models now generalize natural transmission conditions on the common boundary part \(\dot{\Omega}_0 \cap \dot{\Omega}_1\) and the homogeneous Dirichlet boundary condition on \(\dot{\Omega}_0 \setminus \dot{\Omega}_1\) and the standard homogeneous Neumann boundary condition on \(\dot{\Omega}_1 \setminus \dot{\Omega}_0\) without assuming any smoothness of the boundary via containment of the solution \(U = \begin{pmatrix} v_{\Omega_0} + v_{\Omega_1} \\ T_{\Omega_0} \\ p_{\Omega_1} \end{pmatrix}\) in the operator domain \(D(\partial_0 M(0) + A)\). Since we do not have maximal regularity in this case, this does not mean that \(U \in D(A)\), but we do have

\[
\partial_0^{-1} U \in D(A)
\]

as a form of expressing generalized boundary constraints and and transmission conditions.

If, however, we assume sufficient regularity of the boundary and solution one can easily motivate that the model yields a generalization of classical transmission conditions on \(\dot{\Omega}_0 \cap \dot{\Omega}_1\). Indeed, with

\[
\begin{pmatrix} v_{\Omega_0} + v_{\Omega_1} \\ T_{\Omega_0} \\ p_{\Omega_1} \end{pmatrix} \in D(A)
\]

we have (noting for the smooth exterior unit normal vector fields \(n_{\dot{\Omega}_0}, n_{\dot{\Omega}_1}\) on the boundaries of \(\Omega_0\) and \(\Omega_1\), respectively, that \(n_{\dot{\Omega}_0} = -n_{\dot{\Omega}_1}\) on \(\dot{\Omega}_0 \cap \dot{\Omega}_1\) with

\[
\tilde{A} = \begin{pmatrix} 0 & \text{Grad}_{\Omega_0} & \text{grad}_{\Omega_1} \\ \text{Grad}_{\Omega_0} & 0 & 0 \\ \text{div}_{\Omega_2} & 0 & 0 \end{pmatrix},
\]

10
\[ 0 = \left\langle \left( \begin{array}{c} v_{\Omega_0} + v_{\Omega_1} \\ T_{\Omega_0} \\ p_{\Omega_1} \end{array} \right) \right| A \left( \begin{array}{c} v_{\Omega_0} + v_{\Omega_1} \\ T_{\Omega_0} \\ p_{\Omega_1} \end{array} \right) \right\rangle \\
= -\langle v_{\Omega_0} \mid \text{Div } T_{\Omega_0} \rangle - \langle T_{\Omega_0} \mid \text{Grad} \Omega_0 v_{\Omega_0} \rangle + \\
+ \langle p_{\Omega_1} \mid \text{div} \Omega_1 v_{\Omega_1} \rangle + \langle v_{\Omega_1} \mid \text{grad} \Omega_1 p_{\Omega_1} \rangle \\
= -\int_{\Omega_0 \cap \Omega_1} v_{\Omega_0}^T T_{\Omega_0} n_{\Omega_0} \, do + \int_{\Omega_0 \cap \Omega_1} n_{\Omega_1}^T (p_{\Omega_1} v_{\Omega_1}) \, do \\
= -\int_{\Omega_0 \cap \Omega_1} v_{\Omega_0}^T T_{\Omega_0} n_{\Omega_0} \, do - \int_{\Omega_0 \cap \Omega_1} v_{\Omega_1}^T (p_{\Omega_1} n_{\Omega_0}) \, do. \]

Since \((v_{\Omega_0} + v_{\Omega_1}) \in D(\text{Grad})\) is by construction admissible we may assume that \(v_{\Omega_0} = v_{\Omega_1}\) on the interface and conclude that

\[ T_{\Omega_0} n_{\Omega_0} + p_{\Omega_1} n_{\Omega_0} = 0 \tag{14} \]

is a needed transmission condition. In particular, we see that

\[ n_{\Omega_0} \times T_{\Omega_0} n_{\Omega_0} = 0. \]

Inserting the explicit transmission condition \([14]\) now yields

\[ 0 = \int_{\Omega_0 \cap \Omega_1} (v_{\Omega_0} - v_{\Omega_1})^T (p_{\Omega_1} n_{\Omega_0}) \, do \\
= \int_{\Omega_0 \cap \Omega_1} p_{\Omega_1} n_{\Omega_0}^T (v_{\Omega_0} - v_{\Omega_1}) \, do \]

which, with \(p_{\Omega_1}\) being arbitrary, now implies

\[ n_{\Omega_0}^T v_{\Omega_0} = n_{\Omega_0}^T v_{\Omega_1} \]

i.e. the continuity of the normal components

\[ v_{\Omega_0,n} = v_{\Omega_1,n}, \]

as a complementing transmission condition. These more or less heuristic considerations motivate to take the above evo-system as a appropriate generalization to cases, where the boundary does not have a reasonable normal vector field.

All in all, we summarize our findings in the following well-posedness result.

**Theorem 3.1.** If \(\varphi_{\Omega_0}, C_{\Omega_0}\) and \(n_{\Omega_1}, c_{\Omega_1}\) are selfadjoint, strictly positive definite, continuous operators on \(L^2(\Omega_0, \mathbb{R}^3)\), \(L^2(\Omega_0, \text{sym } [\mathbb{R}^3 \times \mathbb{R}^3])\), and on \(L^2(\Omega_1, \mathbb{R}^3)\), \(L^2(\Omega_1, \mathbb{R})\), respectively, the evo-system \([13]\) is Hadamard well-posed. Moreover, the solution depends causally on the data.
Remark 3.2.

1. Since $M(0) \gg 0$, we could construct a fundamental solution of $\partial_0 + \sqrt{M(0)^{-1}} A \sqrt{M(0)^{-1}}$, which in turn is obtained from the unitary group

$$
\left( \exp \left( -t \sqrt{M(0)^{-1}} A \sqrt{M(0)^{-1}} \right) \right)_{t \in \mathbb{R}}
$$

as described above.

2. We note that we may actually allow for completely general – say, for simplicity, rational – material laws as long as condition (3) is warranted. The above simple choice has been used as a more approachable illustrating example, which links up more explicitly with cases considered elsewhere.

References

[1] B. Flemisch, M. Kaltenbacher, and B. I. Wohlmuth. Elasto-acoustic and acoustic-acoustic coupling on non-matching grids". International Journal for Numerical Methods in Engineering, 67(13):1791–1810, 2006.

[2] K. O. Friedrichs. Symmetric hyperbolic linear differential equations. Comm. Pure Appl. Math. 7, 345-392, 1954.

[3] K. O. Friedrichs. Symmetric positive linear differential equations. Communications on Pure and Applied Mathematics, 11(3):333–418, 1958.

[4] Y. Gao, P. Li, and B. Zhang. Analysis of transient acoustic-elastic interaction in an unbounded structure. SIAM Journal on Mathematical Analysis, 49(5):3951–3972, 2017.

[5] G. C. Hsiao, R. E. Kleinman, and G. F. Roach. Weak solutions of fluid-solid interaction problems. Math. Nachr., 218:139–163, 2000.

[6] F. Kang and X. Jiang. Variational approach to shape derivatives for elasto-acoustic coupled scattering fields and an application with random interfaces. Journal of Mathematical Analysis and Applications, 456(1):686 – 704, 2017.

[7] H. Lamb. On the vibrations of an elastic plate in contact with water. Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character, 98(690):205–216, 1920.

[8] Melvin Lax. The effect of radiation on the vibrations of a circular diaphragm. The Journal of the Acoustical Society of America, 16(1):5–13, 1944.

[9] C. J. Luke and P. A. Martin. Fluid-solid interaction: Acoustic scattering by a smooth elastic obstacle. SIAM J. Appl. Math., 55(4):904–922, August 1995.

[10] A. J. Mulholland, R. Picard, S. Trostorff, and M. Waurick. On well-posedness for some thermo-piezoelectric coupling models. Mathematical Methods in the Applied Sciences, 39(15):4375–4384, 2016. mma.3866.
[11] D. Natroshvili, D. Sadunishvili, I. Sigua, and Z. Tedashvili. Fluid-solid interaction: acoustic scattering by an elastic obstacle with Lipschitz boundary. *Mem. Differ. Equ. Math. Phys.*, 35:91–127, 2005.

[12] W. Nowacki. Some theorems of asymmetric thermoelasticity. *J. Math. Phys. Sci.*, 2:111–122, 1968.

[13] W. Nowacki. Dynamische Probleme der unsymmetrischen Elastizität. *Prikl. Mekh.*, 6, No.4, 31-50), 1970.

[14] W. Nowacki. *Theory of asymmetric elasticity. Transl. from the Polish by H. Zorski*. Oxford etc.: Pergamon Press; Warszawa: PWN-Polish Scientific Publishers. VIII, 383 p., 1986.

[15] R. Picard. A Structural Observation for Linear Material Laws in Classical Mathematical Physics. *Math. Methods Appl. Sci.*, 32(14):1768–1803, 2009.

[16] R. Picard. Mother Operators and their Descendants. *J. Math. Anal. Appl.*, 403(1):54–62, 2013.

[17] R. Picard. Mother operators and their descendants. *J. Math. Anal. Appl.*, 403(1):54–62, 2013.

[18] R. Picard. Mother Operators and their Descendants. Technical report, TU Dresden, arXiv:1203.6762v2, 2012.

[19] R. Picard and D. F. McGhee. *Partial Differential Equations: A unified Hilbert Space Approach*, volume 55 of *De Gruyter Expositions in Mathematics*. De Gruyter. Berlin, New York. 518 p., 2011.

[20] R. Picard, St. Seidler, S. Trostorff, and M. Waurick. On abstract grad-div systems. *Journal of Differential Equations*, 260(6):4888 – 4917, 2016.

[21] R. Picard, S. Trostorff, and M. Waurick. On some models for elastic solids with micro-structure. *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik*, 95(7):664–689, 2015.

[22] R. Picard, S. Trostorff, and M. Waurick. *Well-posedness via Monotonicity – an Overview*, volume 250, pages 397–452. Springer International Publishing, Cham, 2015.

[23] S. Mönkölä. Numerical simulation of fluid-structure interaction between acoustic and elastic waves. *Jyväskylä Studies in Computing*, 133:136 p., 2011.

[24] A. F. Seybert, T. W. Wu, and X. F. Wu. Radiation and scattering of acoustic waves from elastic solids and shells using the boundary element method. *The Journal of the Acoustical Society of America*, 84(5):1906–1912, 1988.

[25] L. C. Wilcox, G. Stadler, C. Burstedde, and O. Ghattas. A high-order discontinuous galerkin method for wave propagation through coupled elastic-acoustic media. *Journal of Computational Physics*, 229(24):9373–9396, 2010.