SYSTOLIC GROWTH OF LINEAR GROUPS

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Abstract. We prove that the residual girth of any finitely generated linear group is at most exponential. This means that the smallest finite quotient in which the $n$-ball injects has at most exponential size. If the group is also not virtually nilpotent, it follows that the residual girth is precisely exponential.

1. Introduction

Let $\Gamma$ be a group with a finite generating subset $S$, and $|\cdot|_S$ the corresponding word length. We assume for convenience that $S$ is symmetric and contains the unit, so that $S^n$ is equal to the $n$-ball. The following three functions are attached to $(\Gamma, S)$:

- the growth: the cardinal $b_{\Gamma,S}(n)$ of $S^n$;
- the systolic growth: the function $\sigma_{\Gamma,S}$ mapping $n$ to the smallest $k$ such that some subgroup $H$ of index $k$ contains no nontrivial element of the $n$-ball; if no such $k$ exists, we define it as $+\infty$;
- the residual girth, or normal systolic growth $\sigma'_{\Gamma,S}$: same definition, with the additional requirement that $H$ is normal.

The growth is always defined and is at most exponential, while the systolic growth and residual girth take finite values if and only if $\Gamma$ is residually finite, and in this case they can be larger than exponential, as the example in [BSe] show. Furthermore, we have the obvious inequalities

$$b_{\Gamma,S}(n) \leq \sigma_{\Gamma,S}(2n + 1) \leq \sigma'_{\Gamma,S}(2n + 1).$$

The asymptotic behavior of these functions, for finitely generated groups, does not depend on the finite generating subset.

A simple example for the residual girth grows strictly faster than the systolic growth is the case of the integral Heisenberg group, for which the growth and systolic growth behaves as $n^4$ while the residual girth grows as $n^6$ (see [BSt] [C]). Also the systolic growth may grow faster than the growth and actually can grow arbitrarily fast. We show here that in linear groups, this is not the case.

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**Theorem 1.1.** Assume that $\Gamma$ admits a faithful finite-dimensional representation over a field (or a product of fields). Then the residual girth (and hence the systolic growth) of $\Gamma$ are at most exponential. In particular, if $\Gamma$ is not virtually nilpotent, then its residual girth and its systolic growth are exponential.

Such a result was asserted by Gromov [G, p.334] for subgroups of $\text{SL}_d(\mathbb{Z})$, under some technical superfluous additional assumption (non-existence of nontrivial unipotent elements).

The proof of Theorem 1.1 consists in finding small enough quotient fields of the ring of entries, while ensuring that the $n$-ball is mapped injectively. The argument can be simplified in case $\Gamma \subset \text{GL}_d(\mathbb{Q})$, since then reduction modulo $p$ for all $p$ large enough work with no further effort; in this case the finite quotients are explicit, while in the general case we only find a suitable quotient field using a counting argument.

**Example 1.2.** The group $\mathbb{Z} \rtimes \mathbb{Z}$ has an exponential residual girth. Another example is $(\mathbb{Z}/6\mathbb{Z}) \rtimes \mathbb{Z}$, which is linear over a product of 2 fields, but not over a single field.

**Remark 1.3.** Closely related functions are the residual finiteness growth, which maps $n$ to the smallest number $s_{\Gamma,S}(n)$ such that for every $g \in S^n \setminus \{1\}$, there is a finite index subgroup of $\Gamma$ avoiding $g$, and $s_{\Gamma,S}^\alpha(n)$ defined in the same way with only normal finite index subgroups. For finitely generated group that are linear over a field, a polynomial upper bound for these functions is established in [BM], and in the case of higher rank arithmetic groups, the precise behavior is obtained in [BK]: for instance, for $\text{SL}_d(\mathbb{Z})$ for $d \geq 3$, the normal residual finiteness growth grows as $n^{d^2-1}$.

2. Preliminaries on polynomials over finite fields

**Lemma 2.1.** Let $F$ be a finite field with $q$ elements. Given an integer $n \geq 1$, the number of irreducible monic polynomials of degree $n$ in $F[t]$ is $\leq q^n/n$ and $\geq (q^n - q^{n-1})/n$.

**Proof.** The case $n = 1$ being trivial, we can assume $n \geq 1$. By Gauss’ formula this number $N_q(n)$ is equal to $\left(1/n\right) \sum_{d|n} \mu(n/d)q^d$, where $\mu$ is Möbius’ function. Let $p > 1$ be the smallest prime divisor of $n$. Then

$$\sum_{d|n} \mu(n/d)q^d = q^n - q^{n/p} + \sum_{d|n, d > p} \mu(n/d)q^d \leq q^n - q^{n/p} + \sum_{d|n, d > p} q^d \leq q^n - q^{n/p} + \sum_{k=0}^{n/p-1} q^k \leq q^n$$

A similar argument shows that $nN_q(n) \geq q^n - q^{1+n/p}$, which is $\geq q^n - q^{n-1}$ if $n \geq 3$; the cases $n \leq 2$ being trivial. \qed
Lemma 2.2. Let $F$ be a field with $q$ elements. Let $P \in F[t]$ be a nonzero polynomial of degree $\leq n$. Then $P$ survives in a quotient field of $F[t]$ of cardinal $\leq 2nq$.

Proof. Let $m \geq 1$ be the largest number such that every irreducible polynomial of degree $m - 1$ divides $P$. Let us check that $q^m \leq 2nq$; the case $m = 1$ being trivial, we assume $m \geq 2$. By Lemma 2.1 there are $\geq (q^{m-1} - q^{m-2})/(m - 1)$ monic irreducible polynomials of degree $m - 1$. Hence their product, which has degree $\geq q^m - q^{m-2}$, divides $P$. Thus $q^m - q^{m-2} \leq n$. We have $1 - q^{-1} \geq 1/2$; thus $\frac{1}{2}nq^{-1} \leq n$, that is $q^m \leq 2nq$.

Some irreducible polynomial of degree $m$ does not divide $P$, hence the quotient provides a field quotient of cardinal $q^m \leq 2nq$ in which $P$ survives. \qed

Corollary 2.3. Let $F$ be a field with $q$ elements and $P$ a nonzero polynomial in $F[t_1, \ldots , t_k]$, of degree $\leq n$ with respect to each indeterminate. Then $P$ survives in a quotient field of cardinal $\leq (2n)^kq$.

Proof. Induction on $k$. The result is trivial for $k = 0$. Write

$$P = \sum_{i=0}^{n} P_i(t_1, \ldots , t_{k-1})t_k^i.$$ 

Some $P_i$ is nonzero; fix such $i$. Then there exists, by induction, some quotient field $L$ of $F[t_1, \ldots , t_{k-1}]$ of cardinal $\leq (2n)^{k-1}q$ in which $P_i$ survives. Then the image of $P$ in $L[t_k]$ has degree $\leq n$ and is nonzero; hence by Lemma 2.2 it survives in a quotient field of cardinal $2n((2n)^{k-1}q) = (2n)^kq$. \qed

3. Conclusion of the proof

Proposition 3.1. Every finitely generated group that is linear over a field of characteristic $p$ has at most exponential residual girth.

Proof. Such a group embeds into $GL_d(K)$ where $K$ is an extension of degree $b$ of some field $K' = F_q(t_1, \ldots , t_k)$, and hence embeds into $GL_{bq}(K')$. Hence it is no restriction to assume that the group is contained in $GL_d(F_q[t_1, \ldots , t_k])$. We let $S$ be a finite symmetric generating subset with 1; it is actually contained in $GL_d(F_q[t_1, \ldots , t_k][Q^{-1}])$ for some nonzero polynomial $Q$.

Write $S = Q^{-\lambda}T$ with $\lambda$ a non-negative integer and $T \subset \text{Mat}_d(F_q[t_1, \ldots , t_k])$; write $s = \#(S) = \#(T)$. If $x$ is a matrix, let $b(x)$ be the product of all its nonzero entries (thus $b(0) = 1$). Let $m$ be such that every entry of every element of $T$ has degree $\leq m$ with respect to each variable. Then in $T^{2n}$, every entry of every element has degree $\leq 2nm$ with respect to each variable. Define $x_n = \prod_{y \in T^{2n}} b(y - 1)$. Thus $x_n$ is a product of at most $d^2s^{2n}$ polynomials of degree $\leq 2nm$ with respect to each variable. Define $x_n' = x_nQ$; assume that $Q$ has degree $\leq \delta$ with respect to each variable, so that $x_n'$ has degree $\leq 2d^2ms^{2n} + \delta$ with respect to each variable.
Then, by Corollary 2.3, \( x'_n \) survives in a finite field \( F_n \) of cardinal \( q_1 \leq q(4d^2 mns^{2n} + 2\delta)^k \). Thus \( S^n \) is mapped injectively into \( GL_d(F_n) \), which has cardinal

\[
\leq q_1^{d^2} \leq q^{d^2} (4d^2 mns^{2n} + 2\delta)^{kd^2}.
\]

Since \( m, d, k, s, q \) are fixed, this grows at most exponentially with respect to \( n \). \( \square \)

**Proposition 3.2.** Every finitely generated group that is linear over a field of characteristic 0 has at most exponential residual girth.

**Proof.** Similarly as in the proof of Proposition 3.1, we can suppose that the group is contained in \( GL_d(\mathbb{Q}(t_1, \ldots, t_k)) \). We let \( S \) be a finite symmetric generating subset with 1; it is actually contained in \( GL_d(\mathbb{Z}[t_1, \ldots, t_k][r^{-1}Q^{-1}]) \) for some nonzero integer \( r \geq 1 \) and nonzero polynomial \( Q \) with coprime coefficients.

Write \( S = (Qr)^{-\lambda}T \) with \( \lambda \) a non-negative integer and \( T \subset \text{Mat}_d(\mathbb{Z}[t_1, \ldots, t_k]) \); write \( s = \#(S) = \#(T) \). Let \( R \) be an upper bound on coefficients of entries of elements of \( T \), and let \( M \) be an upper bound on the number of nonzero coefficients of entries of elements of \( T \). Then any product of \( 2n \) elements of \( T \) is a sum of \( \leq M^{2n} \) monomials, each with a coefficient of absolute value \( \leq R^{2n} \). Since any entry of an element in \( T^{2n} \) is a sum of at most \( d^{2n-1} \) such products, we deduce that the coefficients of entries of elements of \( T^{2n} \) are \( \leq d^{2n-1} R^{2n} M^{2n} \). There exists a prime \( p_n \in [2d^{2n-1}(RM)^2 n, 4d^{2n-1}(RM)^2 n] \). There exists \( n_0 \) such that for every \( n \geq n_0 \), \( 2d^{2n-1}(RM)^2 n \) is greater than any prime divisor of \( r \), and \( 2d^{2n-1}(RM)^2 n \) is greater than the lowest absolute value of a nonzero coefficient of \( Q \). Now we always assume \( n \geq n_0 \). Then \( S^{2n} \) is mapped injectively into \( GL_d((\mathbb{Z}/p_n\mathbb{Z})[t_1, \ldots, t_k][Q^{-1}]) \).

Let \( m \) be such that every entry of any element of \( T \) has degree \( \leq m \) with respect to each variable. The previous proof provides a quotient \( GL_d(F_n) \) of \( GL_d((\mathbb{Z}/p_n\mathbb{Z})[t_1, \ldots, t_k][Q^{-1}]) \) in which \( S^n \) is mapped injectively, such that \( GL_d(F_n) \) has cardinal

\[
\leq p_n^{d^2} (4d^2 mns^{2n} + 2\delta)^{kd^2}
\]

Here \( m, d, s, k \) are independent of \( n \). The latter number is

\[
\leq (4d^{-1}(dRM)^{2n})^{d^2} (4d^2 mns^{2n} + 2\delta)^{kd^2},
\]

which grows at most exponentially with respect to \( n \). \( \square \)

**Proof of Theorem 1.1.** First assume that \( \Gamma \) is linear over some field. By Propositions 3.1 and 3.2, the residual girth, and hence the systolic growth, is at most exponential. If \( \Gamma \) is not virtually nilpotent, then by the Tits-Rosenblatt alternative, it contains a free subsemigroup on 2 generators and hence has exponential growth, and therefore has at least exponential systolic growth and residual girth.

Now assume that \( \Gamma \) is linear over some product of fields. Let \( A \) be the ring generated by entries of \( \Gamma \). This is a finitely generated reduced commutative ring; hence it has finitely many minimal prime ideals, whose intersection equals the set of nilpotent elements and hence is reduced to zero. Therefore \( \Gamma \) embeds into
a finite product of matrix group over various fields. We conclude that \( \Gamma \) has at most exponential residual girth, using the following two general facts:

- suppose that \( \Gamma_1, \ldots, \Gamma_k \) are finitely generated groups and \( \Gamma_i \) has residual girth asymptotically bounded above by some function \( u_i \geq 1 \), then the residual girth of \( \prod_{i=1}^k \Gamma_i \) is asymptotically bounded above by \( \prod u_i \);
- if \( \Lambda_1 \subset \Lambda_2 \) are finitely generated groups then the residual girth of \( \Lambda_1 \) is asymptotically bounded above by that of \( \Lambda_2 \).

\[ \square \]

**References**

[BK] K. Bou-Rabee and T. Kaletha. Quantifying residual finiteness of arithmetic groups. Compos. Math. 148(03) (2012) 907–920.

[BM] K. Bou-Rabee and D. B. McReynolds. Extremal behavior of divisibility functions. Geometriae Dedicata, to appear (2014). [arXiv:1211.4727]

[BSe] K. Bou-Rabee and B. Seward. Arbitrarily large residual finiteness growth. To appear in J. Reine Angew. Math.

[BSt] K. Bou-Rabee and D. Studenmund. Full residual finiteness growths of nilpotent groups. [arXiv:1406.3763] (2014).

[C] Y. Cornulier. Gradings on Lie algebras, systolic growth, and cohopfian properties of nilpotent groups. ArXiv [arXiv:1403.5299] (2014).

[G] M. Gromov. Systoles and intersystolic inequalities. Actes de la table ronde de géométrie différentielle (Luminy, 1992), 291–362, Sém. Congr., 1, Soc. Math. France, Paris, 1996.

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