Streaming Submodular Maximization with Matroid and Matching Constraints

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Abstract

Recent progress in (semi-)streaming algorithms for monotone submodular function maximization has led to tight results for a simple cardinality constraint. However, current techniques fail to give a similar understanding for natural generalizations such as matroid and matching constraints. This paper aims at closing this gap. For a single matroid of rank $k$ (i.e., any solution has cardinality at most $k$), our main results are:

- A single-pass streaming algorithm that uses $\tilde{O}(k)$ memory and achieves an approximation guarantee of $0.3178$.
- A multi-pass streaming algorithm that uses $\tilde{O}(k)$ memory and achieves an approximation guarantee of $(1 - 1/e - \varepsilon)$ by taking constant number of passes over the stream.

This improves on the previously best approximation guarantees of $1/4$ and $1/2$ for single-pass and multi-pass streaming algorithms, respectively. In fact, our multi-pass streaming algorithm is tight in that any algorithm with a better guarantee than $1/2$ must make several passes through the stream and any algorithm that beats our guarantee $1 - 1/e$ must make linearly many passes.

For the problem of maximizing a monotone submodular function subject to a bipartite matching constraint (which is a special case of matroid intersection), we show that it is not possible to obtain better than $0.3715$-approximation in a single pass, which improves over a recent inapproximability of $0.522$ for this problem. Furthermore, given a plausible assumption, our inapproximability result improves to $1/3 \approx 0.333$.

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1 Introduction

Submodular function optimization is a classic topic in combinatorial optimization (see, e.g., the book [Sch03]). Already in 1978, Nemhauser, Wolsey, and Fisher [NWF78] analyzed a simple greedy algorithm for selecting the most valuable set $S \subseteq V$ of cardinality at most $k$. This algorithm starts with the empty-set $S$, and then, for $k$ steps, adds the element $e$ to $S$ with largest marginal value. Assuming that $f$ is non-negative and monotone, they showed that the greedy algorithm returns a $(1 - 1/e)$-approximate solution. Moreover, the approximation guarantee of $1 - 1/e$ is known to be tight [NW78; Fei98].

A natural generalization of a cardinality constraint is that of a matroid constraint. While a matroid constraint is much more expressive than a cardinality constraint, it has often been the case that further algorithmic developments have led to the same or similar guarantees for both types of constraints. Indeed, for the problem of maximizing a monotone submodular function subject to a matroid constraint, Călinescu, Chekuri, Pál, and Vondrák [Căl+11] developed the more advanced continuous greedy method, and showed that it recovered the guarantee $1 - 1/e$ in this more general setting. Since then, other methods, such as local search [FW14], have been developed to recover the same optimal approximation guarantee.

More recently, applications in data science and machine learning [Kra], with huge problem instances, have motivated the need for space-efficient algorithms, i.e., (semi-)streaming algorithms for (monotone) submodular function maximization. This is now a very active research area, and recent progress has resulted in a tight understanding of streaming algorithms for maximizing monotone submodular functions with a single cardinality constraint: the optimal approximation guarantee is 1/2 for single-pass streaming algorithms, and it is possible to recover the guarantee $1 - 1/e - \varepsilon$ in $O_\varepsilon(1)$ passes. That is it impossible to improve upon 1/2 in a single pass is due to [Fel+20], and the first single-pass streaming algorithm to achieve this guarantee is a simple “threshold” based algorithm [Bad+14] that intuitively selects elements with marginal value at least $\text{OPT}(2k)$. The $(1 - 1/e - \varepsilon)$ guarantee in $O_\varepsilon(1)$ passes can be obtained using smart implementations of the greedy approach [BV14; MBK16; Nor+18; MV19; HK18].

It is interesting to note that simple greedy and threshold-based algorithms have led to tight results for maximizing a monotone submodular function subject to a cardinality constraint in both the “offline” RAM and data stream models. However, in contrast to the RAM model, where more advanced algorithmic techniques have generalized these guarantees to much more general constraint families, current techniques fail to give a similar understanding in the data stream model, both for single-pass and multi-pass streaming algorithms. Closing this gap is the motivation for our work. In particular, current results leave open the intriguing possibility to obtain the same guarantees for a matroid constraint as for a cardinality constraint. Our main algorithmic results make significant progress on this question for single-pass streaming algorithms and completely close the gap for multi-pass streaming algorithms.

**Theorem 1.1.** There is a single-pass semi-streaming algorithm for maximizing a non-negative monotone submodular function subject to a matroid constraint of rank $k$ (any solution has cardinality at most $k$) that stores $O(k)$ elements, requires $O(k)$ additional memory, and achieves an approximation guarantee of 0.3178.

This improves upon the previous best approximation guarantee of $1/4 = 0.25$ [CK15]. Moreover, the techniques are versatile and also yields a single-pass streaming algorithm with an improved approximation guarantee for non-monotone functions (improving from 0.1715 [FKK18] to 0.1921).

Our next result obtains a tight multi-pass guarantee of $1 - 1/e - \varepsilon$, improving upon the previously best guarantee of $1/2 - \varepsilon$ [HTW20].

Theorem 1.2. For every constant $\varepsilon > 0$, there is a multi-pass semi-streaming algorithm for maximizing a non-negative monotone submodular function subject to a matroid constraint of rank $k$ (any solution has cardinality at most $k$) that stores $O(k/\varepsilon^3)$ elements, makes $O(1/\varepsilon^3)$ many passes, and achieves an approximation guarantee of $1 - 1/e - \varepsilon$.

The result is tight (up to the exact dependency on $\varepsilon$) in the following strong sense: any streaming algorithm with a better approximation guarantee than $1/2$ must make more than one pass [Fel+20], and any algorithm with a better guarantee than $(1 - 1/e)$ must make linearly (in the length of the stream) many passes [MV19].

Theorem 1.1 gives an improved approximation guarantee for single-pass streaming algorithms, but does not reach all the way to recover the guarantee $1/2$ known for the cardinality case. It is thus natural to wonder whether we can obtain stronger impossibility results for the problem with a matroid constraint. The most common way for proving such lower bounds is via one-way communication complexity. The input stream is partitioned among players, usually two named Alice and Bob, and each player can send a message to the following player of limited size (corresponding to the memory footprint of the streaming algorithm). We address this question by introducing a new protocol$^1$ that achieves 0.505-approximation guarantee for the two player setting. This rules out the existence of a 1/2 (or better) impossibility result that is based on the two player communication complexity approach.

Theorem 1.3. There is a two player protocol for maximizing a non-negative monotone submodular function subject to a matroid constraint that sends $O(k)$ elements (from Alice to Bob) and has an approximation guarantee of 0.505.

We complement the above results by developing a hard instance for the bipartite matching constraint (which is a special case of the intersection of two matroids). Using this instance, we are able to upper bound the approximation ratio that can be obtained using a streaming algorithm for the problem of maximizing a monotone submodular function subject to a bipartite matching constraint. Our upper bound improves over the previously best known impossibility of 0.52 due to Levin and Wajc [LW21] assuming that the graph can contain parallel edges (which are distinct elements from the point of view of the submodular objective function); the hardness of [LW21] applies even when this is not the case.

Theorem 1.4. No single-pass semi-streaming algorithm can obtain, with probability at least $2/3$, an approximation ratio of 0.3715 for the problem of maximizing a non-negative monotone submodular function subject to a bipartite matching constraint.

Moreover, if no single-pass data stream algorithm with a memory complexity $O(n^{2-\varepsilon})$ can obtain, with probability at least 1/2, an approximation ratio of 1/2+\varepsilon for (unweighted) maximum matching (for any constant $\varepsilon > 0$), then no single pass data stream algorithm with memory complexity $O(n^{2-\varepsilon}/\log n)$ can obtain, with probability at least 2/3, an approximation ratio of 1/3 + $O(\varepsilon)$(\approx 0.333) for the problem of maximizing a non-negative monotone submodular function subject to a bipartite matching constraint.

We note that the assumption in the second part of Theorem 1.4 is plausible given that no single-pass sub-quadratic space data stream algorithm improving over the trivial 1/2-approximation has been found for the maximum matching problem despite all the attention that this problem has received (see, for example, [KMM12; KT17; Kon18]).

$^1$A protocol is a set of algorithms, one for each player.
Additional related work. As mentioned above, C˘ alinescu et al. [C˘ al+11] proposed a \((1 - 1/e)\)-approximation algorithm for maximizing a monotone submodular function subject to a matroid constraint in the offline (RAM) setting, which is known to be tight [NW78; Fei98]. The corresponding problem with a non-monotone objective is not as well understood. A long line of work [Lee+09; FNS11; EN16] on this problem cumulated in 0.385-approximation due to Buchbinder and Feldman [BF19] and an upper bound by [GV11] of 0.478 on the best approximation ratio that can be obtained.

The first semi-streaming algorithm for maximizing a monotone submodular function subject to a matroid constraint was described by Chakrabarti and Kale [CK15], who obtained an approximation ratio of 1/4 for the problem. Prior to this work, this approximation ratio was not improved. However, Chan, Huang, Jiang, Kang and Tang [Cha+17] managed to get an improved approximation ratio of 0.3178 for the special case of a partition matroid in the related preemptive online model. We note that the last approximation ratio is identical to the approximation ratio stated in Theorem 1.1, which points to some similarity that exists between the algorithms. However, the algorithm of [Cha+17] is not a semi-streaming algorithm. The first semi-streaming algorithm for the non-monotone version of the above problem was obtained by Chekuri, Gupta and Quanrud [CGQ15], and achieved a \((1/(4 + e) - \varepsilon) \approx 0.1488\)-approximation. This was later improved to 0.1715-approximation by Feldman, Karbasi and Kazemi [FKK18].

Outline. In Section 2, we introduce notations and definitions used throughout this paper. Afterwards, in Sections 3 and 4, we present and analyze our single-pass and multi-pass algorithms for maximizing submodular functions subject to a matroid constraint, respectively. In Section 5, we explain our hardness result for the bipartite matching constraint, and in Appendix B we describe our two player protocol. It is worth noting that Sections 3 to 5 and Appendix B are independent of each other, and therefore, can be read in any order.

2 Preliminaries

Recall that we are interested in the problems of maximizing a submodular function subject to either a matroid or bipartite matching constraint. In Section 2.1 we give the definitions necessary for formally stating these problems. Then, in Section 2.2 we define the data stream model in which we study the above problems. Finally, in Section 2.3 we present some additional notation and definitions that we use.

2.1 Problems Statement

Submodular Functions. Given a ground set \(\mathcal{N}\), a set function \(f : 2^{\mathcal{N}} \to \mathbb{R}\) is a function that assigns a numerical value to every subset of \(\mathcal{N}\). Given a set \(S \subseteq \mathcal{N}\) and an element \(u \in \mathcal{N}\), it is useful to denote by \(f(u \mid S)\) the marginal contribution of \(u\) to \(S\) with respect to \(f\), i.e., \(f(u \mid S) \triangleq f(S \cup \{u\}) - f(S)\). Similarly, we denote the marginal contribution of a set \(T \subseteq \mathcal{N}\) to \(S\) with respect to \(f\) by \(f(T \mid S) \triangleq f(S \cup T) - f(S)\).

A set function \(f : 2^{\mathcal{N}} \to \mathbb{R}\) is called submodular if for any two sets \(S\) and \(T\) such that \(S \subseteq T \subseteq \mathcal{N}\) and any element \(u \in \mathcal{N} \setminus T\) we have

\[
f(u \mid S) \geq f(u \mid T).
\]

Mirzasoleiman et al. [MJK18] claimed another approximation ratio for the problem (weaker then the one given later by [FKK18]), but some problems were found in their analysis (see Hab+20 for details).
Moreover, we say that $f$ is monotone if for any set $S \subseteq \mathcal{N}$ and any element $u \in \mathcal{N}$ we have $f(u \mid S) \geq 0$, and we say that $f$ is non-negative if $f(S) \geq 0$ for every such set $S$.

**Matroids.** A set system is a pair $M = (\mathcal{N}, \mathcal{I})$, where $\mathcal{N}$ is called the ground set and $\mathcal{I} \subseteq 2^{\mathcal{N}}$ is a collection of subsets of the ground set. We say that a set $S \subseteq \mathcal{N}$ is independent in $M$ if it belongs to $\mathcal{I}$ (otherwise, we say that it is dependent); and the rank of the set system $M$ is defined as the maximum size of an independent set in it. A set system is a matroid if it has three properties: i) The empty set is independent, i.e., $\emptyset \subseteq \mathcal{I}$. ii) Every subset of an independent set is independent, i.e., for any $S \subseteq T \subseteq \mathcal{N}$, if $T \in \mathcal{I}$ then $S \in \mathcal{I}$. iii) If $S \in \mathcal{I}$, $T \in \mathcal{I}$ and $|S| < |T|$, then there exists an element $u \in T \setminus S$ such that $S \cup \{u\} \in \mathcal{I}$.\(^3\)

A matroid constraint is simply a constraint that allows only sets that are independent in a given matroid. Matroids constraints are of interest because they have rich combinatorial structure, and yet are able to capture many constraints of interest such as cardinality, independence of vectors in a vector space and being a non-cyclic sub-graph.

**Problems.** In the Submodular Maximization subject to a Matroid problem ($\text{SMMatroid}$), we are given a non-negative\(^4\) submodular function $f: 2^{\mathcal{N}} \to \mathbb{R}_{\geq 0}$ and a matroid $M = (\mathcal{N}, \mathcal{I})$ over the same ground set. The objective is to find an independent set $S \in \mathcal{I}$ that maximizes $f$. An important special case of $\text{SMMatroid}$ is the Monotone Submodular Maximization subject to a Matroid problem ($\text{MSMMatroid}$) in which we are guaranteed that the objective function $f$ is monotone (in addition to being non-negative and submodular). Naturally, one can obtain better results for $\text{MSMMatroid}$ compared to $\text{SMMatroid}$.

In the Monotone Submodular Maximization subject to a Bipartite Matching problem ($\text{MSMBipartiteMatching}$), we are given a bipartite graph $G = (V, E)$ and a non-negative monotone submodular function $f: 2^E \to \mathbb{R}_{\geq 0}$ whose ground set is the set of edges of the graph. The objective is to find a legal matching in $G$ that maximizes $f$.

### 2.2 Data Stream Model

In the data stream model, the input appears in a sequential form known as the *input stream*, and the algorithm is allowed to read it only sequentially. In the context of our problems, the input stream consists of the elements of the ground set (or edges of the graph) sorted in an adversarially chosen order, and the algorithm is allowed to read the elements/edges from the stream only in this order. Often the algorithm is allowed to read the input stream only once (such algorithms are called *single-pass* algorithms), but in other cases it makes sense to allow the algorithm to read the input stream multiple times, each such reading is called a *pass*.

A trivial way to deal with the restrictions of the data stream model is to store the entire input stream in the memory of the algorithm. Therefore, the goal in this model is to find a high quality solution while using less memory than what is necessary for storing the input stream. People are particularly interested in algorithms that use memory that is nearly linear in the maximum possible size of an output; such algorithms are called *semi-streaming* algorithms.\(^5\) For $\text{SMMatroid}$ and $\text{MSMMatroid}$, this implies that a semi-streaming algorithm is a data stream algorithm that

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\(^3\)The last property is often referred to as the *exchange axiom* of matroids.

\(^4\)The assumption of non-negativity is necessary because we are interested in multiplicative approximation guarantees.

\(^5\)The similar term *streaming* algorithms often refer to algorithms whose space complexity is poly-logarithmic in the parameters of their input. Such algorithms are irrelevant for the problems we consider because they do not have enough space even for storing the output of the algorithm.
uses $O(r \log^{O(1)} |N|)$ space, where $r$ is the rank of the matroid constraint and it is assumed that any single element can be stored in $O(1)$ space. Similarly, for $MSMBipartiteMatching$, a semi-streaming algorithm is a data stream algorithm that uses $O(n \log^{O(1)} n)$ space, where $n$ is the number of vertices in the graph.\footnote{Technically, $O(n \log^{O(1)} n)$ might not be nearly-linear in the size of the maximum possible solution if there are no large matchings in the graph. However, this is the usual definition of semi-streaming algorithms for graph problems.}

The description of submodular functions and matroids can be exponential in the size of their ground sets, and therefore, it is important to define the way in which an algorithm may access them. For our algorithmic results, we make the standard assumption that the algorithm has two oracles: a value oracle and an independence oracle which, given a set $S \subseteq N$ of elements that are stored in the memory of the algorithm, return the value of $f(S)$ and an indicator whether $S \in \mathcal{I}$, respectively. For simplicity, our inapproximability result is proved for the same access model. However, one can verify that our proof applies also to algorithms with a more powerful way to access the objective function $f$, such as the $p$-players model described in [Fel+20].

2.3 Additional Notation and Definitions

Multilinear Extension. A set function $f : 2^N \rightarrow \mathbb{R}$ assigns values only to subsets of $N$. If we think of a set $S$ as equivalent to its characteristic vector $1_S$ (a vector in $\{0, 1\}^N$ that has a value of 1 in every coordinate $u \in S$ and a value of 0 in the other coordinates), then we can view $f$ as a function over the integral vectors in $[0, 1]^N$. It is often useful to extend $f$ to general vectors in $[0, 1]^N$. There are multiple natural ways to do that, however, in this paper we only need the multilinear extension $F$. Given a vector $x \in [0, 1]^N$, let $R(x)$ denote a random subset of $N$ that includes each element $u \in N$ with probability $x_u$ independently. Then,

$$F(x) = \mathbb{E}[f(R(x))] = \sum_{S \subseteq N} f(S) \prod_{u \in S} x_u \prod_{u \not\in S} (1 - x_u).$$

One can observe that, as is implied by its name, the multilinear extension is a multilinear function. This implies that, for every vector $x \in [0, 1]^N$, the partial derivative $\frac{\partial F}{\partial x_u}(x)$ is equal to $F(x + (1 - x_u) \cdot 1_u) - F(x - x_u \cdot 1_u)$. Note that in the last expression we have used $1_u$ as a shorthand for $1_{\{u\}}$. We often also use $\partial_u F(x)$ as a shorthand for $\frac{\partial F}{\partial x_u}(x)$. When $f$ is submodular, its multilinear extension $F$ is known to be concave along non-negative directions [C˘ al+11].

General Notation. Given a set $S \subseteq N$ and an element $u \in N$, we denote by $S + u$ and $S - u$ the expressions $S \cup \{u\}$ and $S \setminus \{u\}$, respectively. Additionally, given two vectors $x, y \in [0, 1]^N$, we denote by $x \vee y$ and $x \wedge y$ the coordinate-wise maximum and minimum operations, respectively.

Additional Definitions from Matroid Theory. Matroid theory is very extensive, and we refer the reader to [Sch03] for a more complete coverage of it. Here, we give only a few basic definitions from this theory that we employ below. Given a matroid $M = (N, \mathcal{I})$, a set $S \subseteq N$ is called cycle if it is a dependent set that is minimal with respect to inclusion (i.e., every subset of $S$ is independent). An element $u \in N$ is called a self-loop if $\{u\}$ is a cycle. Notice that such elements cannot appear in any feasible solution for either $SMMatroid$ or $MSMMatroid$, and therefore, one can assume without loss of generality that there are no self-loops in the ground set.

The rank of a set $S \subseteq N$, denoted by rank$_M(S)$, is the maximum size of an independent set $T \in \mathcal{I}$ which is a subset of $S$. The subscript $M$ is omitted when it is clear from the context.
We also note that \( \text{rank}_M(N) \) is exactly the rank of the matroid \( M \) (i.e., the maximum size of an independent set in \( M \)), and therefore, it is customary to define \( \text{rank}(M) = \text{rank}_M(N) \). We say that a set \( S \subseteq N \) spans an element \( u \in N \) if adding \( u \) to \( S \) does not increase the rank of the set \( S \), i.e., \( \text{rank}(S) = \text{rank}(S + u) \)—observe the analogy between this definition and being spanned in a vector space. Furthermore, we denote by \( \text{span}_M(S) \triangleq \{ u \in N \mid \text{rank}(S) = \text{rank}(S + u) \} \) the set of elements that are spanned by \( S \). Again, the subscript \( M \) is dropped when it is clear from the context.

3 Single-Pass Algorithm

In this section, we present a single-pass semi-streaming algorithm for the Monotone Submodular Maximization subject to a Matroid problem (MSMMatroid). Recall that in this problem we are given a non-negative monotone submodular function \( f : 2^N \to \mathbb{R}_{\geq 0} \), and the objective is to maximize \( f \) subject to a single matroid constraint \( M = (N, I) \). The properties of the algorithm we present are given by the following theorem.

**Theorem 1.1.** There is a single-pass semi-streaming algorithm for maximizing a non-negative monotone submodular function subject to a matroid constraint of rank \( k \) (any solution has cardinality at most \( k \)) that stores \( O(k) \) elements, requires \( \tilde{O}(k) \) additional memory, and achieves an approximation guarantee of 0.3178.

Our algorithm can be extended to the case in which the objective function is non-monotone (i.e., the SMMatroid problem) at the cost of obtaining a lower approximation factor, yielding the following theorem. However, for the sake of concentrating on our main new ideas, we devote this section to the proof of Theorem 1.1, and defer the proof of Theorem 3.1 to Appendix A.

**Theorem 3.1.** There is a single-pass semi-streaming algorithm for maximizing a non-negative (not necessarily monotone) submodular function subject to a matroid constraint of rank \( k \) (any solution has cardinality at most \( k \)) that stores \( O(k) \) elements, requires \( \tilde{O}(k) \) additional memory, and achieves an approximation guarantee of 0.1921.

Throughout this section, we denote by \( P_M := \{ x \in \mathbb{R}_{\geq 0}^N : x(S) \leq \text{rank}(S) \forall S \subseteq N \} \) the matroid polytope of \( M \).

3.1 The algorithm

The algorithm we use to prove Theorem 1.1 appears as Algorithm 1. This algorithm gets a parameter \( \varepsilon > 0 \) and starts by initializing a constant \( \alpha \) to be approximately the single positive value obeying \( \alpha + 2 = e^\alpha \). We later prove that the approximation ratio guaranteed by the algorithm is at least \( \frac{1}{\alpha+2} - \varepsilon \), which is better than the approximation ratio stated in Theorem 1.1 for a small enough \( \varepsilon \). After setting the value of \( \alpha \), Algorithm 1 defines some additional constants \( m, c, \) and \( L \) using \( \varepsilon \) and \( \alpha \). We leave these variables representing different constants as such in the procedure and analysis, which allows for obtaining a better understanding later on of why these values are optimal for our analysis. We also note that, as stated, Algorithm 1 is efficient (i.e., runs in polynomial time) only if the multilinear extension and its partial derivatives can be efficiently evaluated. If that cannot be done, then one has to approximate \( F \) and its derivatives using Monte-Carlo simulation, which is standard practice (see, for example, [C˘ al+11]). We omit the details to keep the presentation simple, but we note that, as in other applications of this standard technique, the incurred error can easily be kept negligible, and therefore, does not affect the guarantee stated in Theorem 1.1.
Algorithm 1 uses sets $A_i$ and vectors $a_i \in [0,1]^N$ for certain indices $i \in \mathbb{Z}$. Throughout the algorithm, we only consider finitely many indices $i \in \mathbb{Z}$. However, we do not know upfront which indices within $\mathbb{Z}$ we will use. To simplify the presentation, we therefore use the convention that whenever the algorithm uses for the first time a set $A_i$ or vector $a_i$, then $A_i$ is initialized to be $\emptyset$ and $a_i$ is initialized to be the zero vector. The largest index ever used in the algorithm is $q$, which is computed toward the end of the algorithm at Line 13.

For each $i \in \mathbb{Z}$, the set $A_i$ is an independent set consisting of elements $u$ that already arrived and for which the marginal increase with respect to a reference vector $a$ (at the moment when $u$ arrives) is at least $c^i$. More precisely, whenever a new element $u \in \mathcal{N}$ arrives and its marginal return $\partial_u F(a)$ exceeds $c^i$ for an index $i \in \mathbb{Z}$ in a relevant range, then we add $u$ to $A_i$ if $A_i + u$ remains independent. When adding $u$ to $A_i$, we also increase the $u$-entry of the vector $a$ by $\frac{c^i}{\partial_u F(a)}$. The vector $a$ built up during the algorithm has two key properties. First, its multilinear value approximates $F$ up to a constant factor. Second, one can derive from a $a$ vector $s$ (see Algorithm 1) such that $F(s)$ is close to $F(a)$ and $s$ is contained in the matroid polytope $P_M$.

Whenever an element $u \in \mathcal{N}$ arrives, the algorithm first computes the largest index $i(u) \in \mathbb{Z}$ fulfilling $c^{i(u)} \leq \partial_u F(a)$. It then updates sets $A_i$ and vectors $a_i$ for indices $i \leq i(u)$. Purely conceptually, the output of the algorithm would have the desired guarantees even if all infinitely many indices below $i(u)$ where updated. However, to obtain an algorithm running in finite (actually even polynomial) time and linear memory, we do not consider indices below $\max\{b, i(u) - \text{rank}(M) - L\}$ in the update step. Capping the considered indices like this has only a very minor impact in the analysis because the contribution of the vectors $a_i$ to the multilinear extension value of the vector $a$ is geometrically decreasing with decreasing index $i$.

In the algorithm, and also the analysis that follows, we sometimes use sums over indices that go up to $\infty$. However, whenever this happens, beyond some finite index, all terms are zero. Hence, such sums are well defined.

Algorithm 1: Single-Pass Semi-Streaming Algorithm for MMMatroid

1: Set $\alpha = 1.1462$, $m = \left\lceil \frac{3\alpha}{\epsilon} \right\rceil$, $c = \frac{m}{m-\alpha}$, and $L = \left\lceil \log_c \left( \frac{2c}{\epsilon(c-1)} \right) \right\rceil$.
2: Set $a = 0 \in [0,1]^N$ to be the zero vector, and let $b = -\infty$.
3: for every element arriving $u \in \mathcal{N}$, if $\partial_u F(a) > 0$ do
4:  Let $i(u) = \left\lfloor \log_c (\partial_u F(a)) \right\rfloor$. ▷ Thus, $i(u)$ is largest index $i \in \mathbb{Z}$ with $c^i \leq \partial_u F(a)$.
5:  for $i = \max\{b, i(u) - \text{rank}(M) - L\}$ to $i(u)$ do
6:    if $A_i + u \in \mathcal{I}$ then
7:      $A_i \leftarrow A_i + u$.
8:      $a_i \leftarrow a_i + \frac{c^i}{\partial_u F(a)}1_u$.
9:  Set $b \leftarrow h - L$, where $h$ is largest index $i \in \mathbb{Z}$ satisfying $\sum_{j=i}^{\infty} |A_j| \geq \text{rank}(M)$.
10: $a \leftarrow \sum_{i=b}^{\infty} a_i$.
11: Delete from memory all sets $A_i$ and vectors $a_i$ with $i \in \mathbb{Z} < b$.
12: Set $S_k \leftarrow \emptyset$ for $k \in \{0,\ldots,m-1\}$.
13: Let $q$ be largest index $i \in \mathbb{Z}$ with $A_i \neq \emptyset$.
14: for $i = q$ to $b$ (stepping down by 1 at each iteration) do
15:    while $\exists u \in A_i \setminus S_{(i \mod m)}$ with $S_{(i \mod m)} + u \in \mathcal{I}$ do
16:      $S_{(i \mod m)} \leftarrow S_{(i \mod m)} + u$.
17: return a rounding $R \in \mathcal{I}$ of the fractional solution $s := \frac{1}{m} \sum_{k=0}^{m-1} 1_{S_k}$ with $f(R) \geq F(s)$.

Finally, we provide details on the return statement in Line 17 of the algorithm. This statement
is based on the fact stated in [C˘ a˘l+11], namely that a point in the matroid polytope can be rounded losslessly to an independent set. More formally, given any point \( y \in P_M \) in the matroid polytope, there is an independent set \( I \in \mathcal{I} \) with \( f(I) \geq F(y) \). Moreover, assuming that the multilinear extension \( F \) can be evaluated efficiently, then such an independent set \( I \) can be computed efficiently. As before, if one is only given a value oracle for \( f \), then the exact evaluation of \( F \) can be replaced by a strong estimate obtained through Monte-Carlo sampling, leading to a randomized algorithm to round \( y \) to an independent set \( I \) with \( f(I) \geq (1 - \delta)F(y) \) for an arbitrarily small constant \( \delta > 0 \).

We highlight that we can assume in what follows that there is at least one element \( u \in \mathcal{N} \) which gets considered in the for loop on Line 3, i.e., it fulfills \( \partial_u F(a) > 0 \) when appearing in the for loop. Note that if this does not happen, then we are in a trivial special case where \( a \) remains the zero vector and \( \partial_u F(a) = 0 \) for all \( u \in \mathcal{N} \), which corresponds to \( f(\mathcal{N}) = f(\varnothing) \). In this case, all sets \( S_k \) for \( k \in \{0, \ldots, m - 1\} \) are empty, which implies that \( s \) is the zero vector, and one can simply return \( R = \varnothing \), which fulfills \( f(R) \geq F(s) \), and is even a global maximizer of \( f(S) \) over all sets \( S \subseteq \mathcal{N} \).

### 3.2 Analysis of Algorithm 1

We now show that Algorithm 1 implies Theorem 1.1. Let \( \varepsilon \in (0, 1] \) in what follows. As mentioned, we sometimes restrict the considered index range for \( i \) in the algorithm to make sure that the algorithm has a finite running time and only uses limited memory. This happens in particular in Line 10 when updating \( a \), where we only consider indices starting from \( b \). However, for the analysis, it is convenient to look at the vector \( a_{\text{all}} = \sum_{i=-\infty}^{\infty} \pi_i \) obtained without this lower bound, where \( \pi_i \) is the vector \( a_i \) when the algorithm terminates. In the definition of \( a_{\text{all}} \), we also consider indices \( i \in \mathbb{Z} \) together with corresponding vectors \( \pi_i \) that have been removed from memory in Line 11. Here, the vector \( \pi_i \) is simply the last vector \( a_i \) before it got removed from memory in Line 11. Similarly, we let \( A_i \subseteq \mathcal{N} \) be the set \( A_i \) at the end of the algorithm or, in case \( A_i \) got removed from memory at some point, \( A_i \) is the set \( A_i \) right before it got removed from memory.

Note that because the coordinates of the vectors \( a_i \) never decrease throughout the algorithm, every vector \( a \) encountered throughout Algorithm 1 is upper bounded, coordinate-wise, by \( a_{\text{all}} \). In the following, we compare both the value of an optimal solution and the value \( f(R) \geq F(s) \) of the returned set to \( F(a_{\text{all}}) \). We start by making sure that the different steps of the algorithm are well defined. For this, we first show that \( a_{\text{all}} \), and therefore also any vector \( a \) encountered through Algorithm 1, is contained in the box \([0, 1]^N\), which implies that the computations of partial derivatives \( \partial_u F(a) \) are well defined.

**Observation 3.2.** \( a_{\text{all}} \in [0, 1]^N \). Consequently, throughout the algorithm, the vector \( a \) is also contained in \([0, 1]^N\).

**Proof.** Consider an element \( u \in \mathcal{N} \) and the moment when \( u \) was considered in the for loop at Line 3 of Algorithm 1. Let \( i(u) \) be the index computed at Line 4 of the algorithm. Hence, for the vector \( a \) at that moment we have \( c^{i(u)} \leq \partial_u F(a) \). Thus,

\[
a_{\text{all}}(u) \leq \sum_{j=-\infty}^{i(u)} \frac{c^j}{m \cdot \partial_u F(a)} \leq \frac{1}{m} \sum_{j=-\infty}^{0} c^j = \frac{1}{m} \frac{c}{c - 1} = \frac{1}{\alpha} \leq 1,
\]

where the second inequality follows from \( c^{i(u)} \leq \partial_u F(a) \), and the second equality holds by the definition of \( c \), i.e., \( c = \frac{m}{m - \alpha} \).

Moreover, we highlight that the fractional point \( s \) rounded at the end of Algorithm 1 at Line 17 is indeed in the matroid polytope \( P_M \). This holds because it is a convex combination of the sets \( S_k \).
for \( k \in \{0, \ldots m-1\} \), each of which is an independent set by construction. Hence, the rounding performed in Line 17 is indeed possible, as discussed.

We now bound the memory used by the algorithm. Note that, for any constant \( \varepsilon \) (which implies that \( c \) is also a constant), the guarantee in the next lemma becomes \( O(\text{rank}(M)) \), which is the guarantee we need in order to prove Theorem 1.1.

**Lemma 3.3.** At any point in time, the sum of the cardinalities of all sets \( A_i \) that Algorithm 1 has in memory is \( O(L \cdot \text{rank}(M)) = O\left( \frac{\log\left(\frac{2c}{\varepsilon(c-1)}\right) \text{rank}(M)}{\log c} \right) \).

**Proof.** It suffices to bound the number of elements \( \sum_{i=b}^{\infty} |A_i| \) after Line 11. Indeed, we never have more than that many elements in memory plus the number of elements added in a single iteration of the for loop at Line 3, which is at most \( \text{rank}(M) + L = O(L \cdot \text{rank}(M)) \). Hence, consider the state of the algorithm at any moment right after the executing of Line 11. We have

\[
\sum_{i=b}^{\infty} |A_i| = \sum_{i=b}^{b+L} |A_i| + \sum_{i=b+L+1}^{\infty} |A_i| < (L + 1) \text{rank}(M) + \text{rank}(M) = O(L \cdot \text{rank}(M)) ,
\]

where the inequality follows from the fact that the first sum has \( L + 1 \) terms, each is the cardinality of an independent set, which is upper bounded by \( \text{rank}(M) \); moreover, the second term in the sum is strictly less than \( \text{rank}(M) \) by the definition of \( h \) in Algorithm 1 (note that \( h = b + L \)). \hfill \Box

We now start to relate the different relevant quantities to \( F(a_{\text{all}}) \). We start by upper bounding the value of \( F(a_{\text{all}}) \) as a function of the sets \( \overline{A}_i \).

**Lemma 3.4.**

\[
F(a_{\text{all}}) \leq f(\varnothing) + \frac{1}{m} \sum_{i \in \mathbb{Z}} |\overline{A}_i| \cdot c^i .
\]

**Proof.** One can think of the vector \( a_{\text{all}} \) as being constructed iteratively starting with the zero vector \( w = 0 \) as follows. Whenever Algorithm 1 is at Line 8, we update \( w \) by \( w \leftarrow w + \frac{c^i}{m \partial F(a)} 1_u \), where \( a \in [0, 1]^N \) is the current vector \( a \) of the algorithm at that moment in the execution. Note that we have \( w \geq a \) because \( a = \sum_{j=b}^{\infty} a_j \), for the current value of \( b \) and the current vectors \( a_j \), whereas \( w = \sum_{j \in \mathbb{Z}} a_j \). Hence, by submodularity of \( f \), we have that the increase of \( F(w) \) in this iteration is upper bounded by

\[
F\left( q + \frac{c^i}{m \cdot \partial F(a)} 1_u \right) - F(q) \leq F\left( a + \frac{c^i}{m \cdot \partial F(a)} 1_u \right) - F(a) = \frac{c^i}{m} .
\]

Hence, the total change in \( F(w) \) starting from \( F(0) = f(\varnothing) \) to \( F(a_{\text{all}}) \) is therefore obtained by summing the above left-hand side over all occurrences when algorithm is at Line 8, which leads to

\[
F(a_{\text{all}}) - F(0) \leq \frac{1}{m} \sum_{i \in \mathbb{Z}} |\overline{A}_i| \cdot c^i ,
\]

which completes the proof. \hfill \Box

A key difference between the fractional point \( s \), which is constructed during the algorithm, and the point \( a_{\text{all}} \), is that sets \( \overline{A}_i \) for indices below \( \overline{b} \) have an impact on the value of \( F(a_{\text{all}}) \) (but not on \( F(s) \)), as reflected in the upper bound on \( F(a_{\text{all}}) \) in Lemma 3.4. The following lemma shows that this difference in index range is essentially negligible because the impact of the sets \( \overline{A}_i \) in these bounds decreases exponentially fast with decreasing index \( i \).
Lemma 3.5.  
\[ \frac{1}{m} \sum_{i=-\infty}^{\bar{b}-1} c^i \cdot |\overline{A}_i| \leq \frac{c^\bar{b}}{m(c-1)} \text{rank}(M) \leq \frac{\varepsilon}{2c} \cdot \frac{1}{m} \sum_{i=\bar{b}}^{q} c^i \cdot |\overline{A}_i| . \]

Proof. The first inequality of the statement follows from $|\overline{A}_i| \leq \text{rank}(M)$ for $i \in \mathbb{Z}$, which holds because $\overline{A}_i \in \mathcal{I}$. The second one follows from

\[ \frac{1}{m} \sum_{i=\bar{b}}^{q} c^i \cdot |\overline{A}_j| \geq \frac{1}{m} \sum_{i=\bar{b}+L}^{q} c^i \cdot |\overline{A}_j| \geq \frac{1}{m} \text{rank}(M) \cdot c^{\bar{b}+L} \geq \frac{1}{m} \text{rank}(M) \cdot c^\bar{b} \cdot \frac{2c}{\varepsilon(c-1)} , \tag{1} \]

where the second inequality follows by the fact that $\bar{b} + L$ is the value of $h$ at the end of the algorithm, which fulfills by definition $\sum_{i=h}^{\infty} |A_j| \geq \text{rank}(M)$, and the third inequality follows by our definition of $L$. 

Combining Lemma 3.5 with Lemma 3.4, now leads to the following lower bound on $F(a_{all})$, described only in terms of sets $|\overline{A}_i|$ that have not been deleted from memory when the algorithm terminates.

Corollary 3.6.  
\[ F(a_{all}) - f(\emptyset) \leq \left(1 + \frac{\varepsilon}{2c}\right) \cdot \frac{1}{m} \sum_{i=\bar{b}}^{q} c^i \cdot |\overline{A}_i| . \]

Proof. The statement follows from

\[ \left(1 + \frac{\varepsilon}{2c}\right) \cdot \frac{1}{m} \sum_{i=\bar{b}}^{q} c^i \cdot |\overline{A}_i| \geq \frac{1}{m} \sum_{i=-\infty}^{q} c^i \cdot |\overline{A}_i| \geq F(a_{all}) - f(\emptyset) , \]

where the first inequality is due to Lemma 3.5, and the second one follows from Lemma 3.4. 

Before relating $F(s)$ to $F(a_{all})$, we need the following structural property on the sets $\overline{A}_i$, which will be exploited to show that the sets $S_k$, chosen at the end of the algorithm, lead to a point $s$ of high multilinear value.

Lemma 3.7.  
\[ \overline{A}_i \subseteq \text{span}(\overline{A}_{i-1}) \quad \forall i \in \{\bar{b} + 1, \bar{b} + 2, \ldots, q\} . \]

Proof. Let $u \in \overline{A}_i$, and we show the statement by proving that $u \in \text{span}(\overline{A}_{i-1})$. Consider the state of Algorithm 1 when it performs the for loop at Line 4 when the outer for loop is considering the element $u$ (this is the for loop that adds the element $u$ to sets $A_j$). Note that we have

\[ \bar{b} \geq \max\{b, i(u) - \text{rank}(M) - L\} , \]

due to the following. We clearly have $\bar{b} \geq b$ because the value of $b$ is non-decreasing throughout the algorithm. Moreover, if $i(u) - \text{rank}(M) - L \geq b$, then after the execution of the for loop at Line 4, we have $A_j \neq \emptyset$ for each $j \in \{i(u) - \text{rank}(M) - L, \ldots, i(u)\}$. Hence, right after this execution of the for loop, the value of $b$ will be increased to at least $i(u) - \text{rank}(M) - L$.

Thus, because $u$ got added to $A_i$ for some $i \in \mathbb{Z}_{\geq \bar{b}}$, the algorithm will also add $u$ to $A_{i-1}$ if $A_{i-1} + u \in \mathcal{I}$. Hence, after the execution of this for loop, we have $u \in \text{span}(A_{i-1})$. Finally, because $\overline{A}_{i-1} \supseteq A_{i-1}$, we also have $u \in \text{span}(\overline{A}_{i-1})$. 

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We are now ready to lower bound the value of $F(s)$ in terms of $F(a_{all})$.

**Lemma 3.8.**

$$F(s) \geq f(\emptyset) + \frac{1}{m} \left(1 - c^{-m}\right) \sum_{i=\overline{b}}^{q} c^i \cdot |A_i| \geq \left(1 - c^{-m} - \frac{\varepsilon}{2c}\right) \cdot F(a_{all}) .$$

**Proof.** For $k \in \{0, \ldots, m - 1\}$, we partition $S_k$ into

$$S_k = S^\overline{b}_k \cup S^\overline{b+1}_k \cup \ldots \cup S^b_k ,$$

where $S^i_k$ are the elements that got added to $S_k$ in iteration $i$ of Line 14. Note that because $S_k$ only gets updated in every $m$-th iteration, we have $S^i_k = \emptyset$ for any $i \not\equiv k \pmod{m}$. Moreover, we have

$$|S^i_k| \geq |\overline{A}_i| - |\overline{A}_{i+m}| \quad \forall i \in \mathbb{Z} \text{ with } \overline{b} \leq i \leq q \text{ and } i \equiv k \pmod{m} \quad (2)$$

because of the following. We recall that the sets $S_k$ are constructed by adding elements from sets $\overline{A}_j$ from higher indices $j$ to lower ones. Thus, when elements of $\overline{A}_i$ are considered to be added to $S_k$, the current set $S_k$ only contains elements from sets $A_j$ with $j \geq i + m$ (recall that only elements from every $m$-th set $A_j$ can be added to $S_k$). However, by Lemma 3.7, we have that all those elements are spanned by $A_{i+m}$. Hence, when elements of $\overline{A}_i$ are considered to be added to $S_k$, the set $S_k$ has at most $\text{rank}(\overline{A}_{i+m}) = |\overline{A}_{i+m}|$ many elements since $S_k \in \mathcal{T}$ by construction. Moreover, when elements of the set $\overline{A}_i$ are added to $S_k$, this is done in a greedy way, which implies that the size of $S_k$ after adding elements from $\overline{A}_i$ will be equal to $\text{rank}(\overline{A}_i) = |\overline{A}_i|$. This implies Eq. (2).

The desired relation now follow from

$$F(s) \geq f(\emptyset) + \frac{1}{m} \sum_{i=\overline{b}}^{q} c^i \sum_{k=0}^{m-1} |S^i_k|$$

$$\geq f(\emptyset) + \frac{1}{m} \sum_{i=\overline{b}}^{q} c^i \cdot (|\overline{A}_i| - |\overline{A}_{i+m}|)$$

$$\geq f(\emptyset) + \frac{1}{m} (1 - c^{-m}) \sum_{i=\overline{b}}^{q} c^i \cdot |\overline{A}_i|$$

$$\geq f(\emptyset) + \frac{1}{m} (1 - c^{-m}) \left(1 + \frac{\varepsilon}{2c}\right)^{-1} (F(a_{all}) - f(\emptyset))$$

$$\geq \frac{1}{m} (1 - c^{-m}) \left(1 - \frac{\varepsilon}{2c}\right) F(a_{all}) ,$$

$$\geq \frac{1}{m} \left(1 - c^{-m} - \frac{\varepsilon}{2c}\right) F(a_{all}) ,$$

where the first inequality follows from a reasoning analogous to the one used in the proof of Lemma 3.4, the second one is due to Eq. (2), and the fourth one uses Corollary 3.6.

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*More precisely, we can think of $s$ as being constructed iteratively starting from $w = 0$. Whenever the algorithm adds an element $u \in \mathcal{N}$ to some set $A_i$, with $i \in \{\overline{b}, \ldots, q\}$, then, if it is not also part of the set $S^i_k$ for $k \in \{0, \ldots, m - 1\}$ with $i \equiv k \pmod{m}$, we update $w$ by setting it to $w + \frac{c^i_u}{\partial_u F(a)} 1_u$. The increase $F(w + \frac{c^i_u}{\partial_u F(a)} 1_u) - F(w)$ is at least as big as $F(a + \frac{c^i_u}{\partial_u F(a)} 1_u) - F(a)$, which is $\frac{c^i_u}{m}$.*
Let $OPT$ be an arbitrary (but fixed) optimal solution for our problem. To relate $f(OPT)$ to $F(a_{all})$, we analyze by how much $f(OPT)$ can be bigger than $F(a_{all})$. This difference can be bounded through the derivatives $\partial_u F(a_{all})$, which we analyze first. To this end, for any $u \in \mathcal{N}$, we denote by $\ell(u)$ the largest index $i \in \mathbb{Z}$ such that $u \in \text{span}(\overline{A}_i)$. If no such index exists, we set $\ell(u) = -\infty$.

**Observation 3.9.**

$\partial_u F(a_{all}) \leq c^{\ell(u)+1} \quad \forall u \in \mathcal{N}$.

**Proof.** Because $u \not\in \text{span}(\overline{A}_{\ell(u)+1})$, this implies that $u$ did not get added to the set $A_{\ell(u)+1}$ in Algorithm 1, even though $A_{\ell(u)+1} + u \in \mathcal{I}$, which holds because $A_{\ell(u)+1} + u \subseteq \overline{A}_{\ell(u)+1} + u \in \mathcal{I}$. Hence, when $u$ got considered in Line 3 of Algorithm 1, we had $\partial_u F(a) < c^{\ell(u)+1}$. Finally, by submodularity of $f$ and because $a \leq a_{all}$ (coordinate-wise), we have $\partial_u F(a_{all}) \leq \partial_u F(a) \leq c^{\ell(u)+1}$.

We are now ready to bound the difference between $f(OPT)$ and $F(a_{all})$. **Lemma 3.10** is the first statement in our analysis that exploits monotonicity of $f$.

**Lemma 3.10.**

$$f(OPT) - F(a_{all}) \leq \sum_{u \in OPT} c^{\ell(u)+1} .$$

**Proof.** The result follows from

$$f(OPT) - F(a_{all}) \leq F(a_{all} \lor 1_{OPT}) - F(a_{all})$$

$$\leq \nabla F(a_{all})^T ((a_{all} \lor 1_{OPT}) - a_{all})$$

$$\leq \nabla F(a_{all})^T 1_{OPT}$$

$$= \sum_{u \in OPT} \partial_u F(a_{all})$$

$$\leq \sum_{u \in OPT} c^{\ell(u)+1} ,$$

where the first inequality follows from monotonicity of $F$, the second one because $F$ is concave along non-negative directions, the third one uses again monotonicity of $F$ which implies $\nabla F(a_{all}) \geq 0$, and the last one follows from Observation 3.9.

The following lemma allows us to express the bound on the difference between $f(OPT)$ and $F(a_{all})$ in terms of $F(a_{all})$, which, combined with the previously derived results, will later allow us to compare $F(s)$ to $f(OPT)$ via the quantity $F(a_{all})$.

**Lemma 3.11.**

$$\sum_{u \in OPT} c^{\ell(u)+1} \leq (c - 1) \left(1 + \frac{\varepsilon}{2c}\right) \frac{m}{1 - e^{-m}} F(s) .$$
Proof. We start by expanding the left-hand side of the inequality to be shown:

\[
\sum_{u \in \text{OPT}} c^{\ell(u)+1} = c \cdot \sum_{u \in \text{OPT}} c^{\ell(u)} \\
= c \cdot \sum_{i \in \mathbb{Z}} c^i \cdot |\{u \in \text{OPT}: \ell(u) = i\}| \\
= (c - 1) \sum_{i \in \mathbb{Z}} |\{u \in \text{OPT}: \ell(u) \geq i\}| \\
= (c - 1) \left[ \sum_{i=-\infty}^{\bar{b}-1} c^i \cdot |\{u \in \text{OPT}: \ell(u) \geq i\}| + \sum_{i=\bar{b}}^{q} c^i \cdot |\{u \in \text{OPT}: \ell(u) \geq i\}| \right].
\]

(3)

To upper bound the terms in the first sum, we use

\[
|\{u \in \text{OPT}: \ell(u) \geq i\}| \leq \text{rank}(M) \quad \forall i \in \mathbb{Z},
\]

(4)

which holds because \(\{u \in \text{OPT}: \ell(u) \geq i\} \subseteq \text{OPT}\) and \(\text{OPT} \in \mathcal{I}\). Moreover, for the second sum, we use

\[
|\{u \in \text{OPT}: \ell(u) \geq i\}| \leq |\overline{A}_i| \quad \forall i \in \mathbb{Z}_{\geq \bar{b}},
\]

(5)

which holds due to the following. By the definition of \(\ell(u)\) and Lemma 3.7, we have \(\{u \in \text{OPT}: \ell(u) \geq i\} \subseteq \text{span}(\overline{A}_i)\). Eq. (5) now follows from

\[
|\{u \in \text{OPT}: \ell(u) \geq i\}| = \text{rank}(\{u \in \text{OPT}: \ell(u) \geq i\}) = \text{rank}(\text{span}(\overline{A}_i)) = \text{rank}(\overline{A}_i) = |\overline{A}_i|,
\]

where the first equality holds because \(\{u \in \text{OPT}: \ell(u) \geq i\} \subseteq \text{OPT} \in \mathcal{I}\), the inequality holds because \(\{u \in \text{OPT}: \ell(u) \geq i\} \subseteq \text{span}(\overline{A}_i)\), and the last equation follows from \(\overline{A}_i \in \mathcal{I}\).

We now combine the above proved inequalities to obtain the desired result:

\[
\sum_{u \in \text{OPT}} c^{\ell(u)+1} \leq (c - 1) \left[ \sum_{i=-\infty}^{\bar{b}-1} c^i \cdot \text{rank}(M) + \sum_{i=\bar{b}}^{q} c^i \cdot |\overline{A}_i| \right] \\
= (c - 1) \left[ \text{rank}(M) \cdot \frac{c^\bar{b}}{c - 1} + \sum_{i=\bar{b}}^{q} c^i \cdot |\overline{A}_i| \right] \\
\leq (c - 1) \left( 1 + \frac{\varepsilon}{2c} \right) \sum_{i=\bar{b}}^{q} c^i \cdot |\overline{A}_i| \\
\leq (c - 1) \left( 1 + \frac{\varepsilon}{2c} \right) \frac{m}{1 - c^{-m}} F(s),
\]

where the first inequality follows by Eq. (3), Eq. (4), and Eq. (5), the second inequality follows by Lemma 3.5, and the last one is a consequence of Lemma 3.8.

Combining Lemmas 3.10 and 3.11 we obtain the following lower bound on \(F(s)\) in terms of \(f(\text{OPT})\).

Corollary 3.12.

\[
F(s) \geq \frac{1 - c^{-m} - \frac{\varepsilon}{2c}}{(c - 1) \left( 1 + \frac{\varepsilon}{2c} \right) m + 1} \cdot f(\text{OPT}).
\]
Proof.

\[ f(\text{OPT}) \leq (c - 1) \left( 1 + \frac{\varepsilon}{2c} \right) \frac{m}{1 - c^{-m}} \cdot F(s) + F(a_{\text{all}}) \]
\[ \leq (c - 1) \left( 1 + \frac{\varepsilon}{2c} \right) \frac{m}{1 - c^{-m}} \cdot F(s) + \left( 1 - c^{-m} - \frac{\varepsilon}{2c} \right)^{-1} F(s) \]
\[ \leq \frac{(c - 1) \left( 1 + \frac{\varepsilon}{2c} \right) m + 1}{1 - c^{-m} - \frac{\varepsilon}{2c}} \cdot F(s) , \]

where the first inequality follows from Lemmas 3.10 and 3.11, and the second one from Lemma 3.8.

Our result for \textit{MSMMatroid}, i.e., Theorem 1.1, now follows from Corollary 3.12 and our choice of parameters \(\alpha, m,\) and \(c\), which have been chosen to optimize the ratio. This leads to a lower bound on \(F(s)\) in terms of \(f(\text{OPT})\), which in turn leads to a lower bound on \(f(R)\) because \(f(R) \geq F(s)\).

Proof of Theorem 1.1. We have

\[ f(R) \geq F(s) \]
\[ \geq \frac{1 - c^{-m} - \frac{\varepsilon}{2c}}{(c - 1) \left( 1 + \frac{\varepsilon}{2c} \right) m + 1} f(\text{OPT}) \]
\[ \geq \frac{1 - e^{-\alpha} - \frac{\varepsilon}{2c}}{(c - 1) \cdot \left( 1 + \frac{\varepsilon}{2c} \right) m + 1} \cdot f(\text{OPT}) \]
\[ = \frac{1 - e^{-\alpha} - \frac{\varepsilon}{2c}}{\alpha \left( c + \frac{\varepsilon}{2} \right) + 1} \cdot f(\text{OPT}) \]
\[ \geq \left( \frac{1 - e^{-\alpha}}{\alpha + 1} \cdot \frac{1}{c + \frac{\varepsilon}{2} - \frac{\varepsilon}{2c}} \right) \cdot f(\text{OPT}) \]
\[ \geq \left( \frac{1}{\alpha + 2} \cdot \frac{1}{c + \frac{\varepsilon}{2} - \frac{\varepsilon}{2c}} \right) \cdot f(\text{OPT}) , \]

where the second inequality is due to Corollary 3.12, the third one follows from \(e^{-m} = (1 - \alpha/m)^m \leq e^{-\alpha}\), the equality uses again \(c = \frac{m}{m - \alpha}\), and the last inequality uses our choice of value for \(\alpha\) (note the inequality would have held as an equality if \(\alpha\) was obeying \(e^\alpha = \alpha + 2\), and we chose a value that is close).

By our choice of \(m = \lceil 3\alpha/\varepsilon \rceil\), we obtain the following bound on \(c\):

\[ c = \frac{m}{m - \alpha} = \frac{1}{1 - \frac{\alpha}{m}} \leq \frac{1}{1 - \frac{\varepsilon}{3}} \leq 1 + \frac{\varepsilon}{2} . \]

Plugging this bound into Eq. (6) and using \(c \geq 1\), we get

\[ F(R) \geq \left( \frac{1}{\alpha + 2} \cdot \frac{1}{1 + \varepsilon} - \frac{\varepsilon}{2} \right) \cdot f(\text{OPT}) \]
\[ \geq \left( \frac{1}{\alpha + 2} \cdot (1 - \varepsilon) - \frac{\varepsilon}{2} \right) \cdot f(\text{OPT}) \]
\[ \geq \left( \frac{1}{\alpha + 2} - \varepsilon \right) \cdot f(\text{OPT}) , \]

as desired.
4 Multi-pass Algorithm

In this section we present multi-pass a semi-streaming algorithm for the Monotone Submodular Maximization subject to a Matroid problem (MSMMatroid). Recall that in this problem we are given a non-negative monotone submodular function \( f : 2^N \rightarrow \mathbb{R}_{\geq 0} \), and the objective is to maximize \( f \) subject to a single matroid constraint \( M = (N, \mathcal{I}) \).

Badanidiyuru and Vondrák [BV14] described an algorithm called “Accelerated Continuous Greedy” that obtains \( 1 - 1/e - O(\varepsilon) \) approximation for MSMMatroid for every \( \varepsilon \in (0, 1) \). Their algorithm is not a data stream algorithm, but it enjoys the following nice properties.

- The algorithm includes a procedure called “Decreasing-Threshold Procedure”. This procedure is the only part of the algorithm that directly accesses the input.
- The Decreasing-Threshold Procedure is called \( O(\epsilon^{-1}) \) times during the execution of the algorithm.
- In addition to the space used by this procedure, Accelerated Continuous Greedy uses only space that is linear in the space necessary to store the outputs of the various executions of the Decreasing-Threshold Procedure.
- The Decreasing-Threshold Procedure returns a base \( D \) of \( M \) after every execution, and this base is guaranteed to obey Eq. (7) stated below. The analysis of the approximation ratio of Accelerated Continuous Greedy treats Decreasing-Threshold Procedure as a black box except for the fact that its output is a base \( D \) of \( M \) obeying Eq. (7), and therefore, this analysis will remain valid even if Decreasing-Threshold Procedure is replaced by any other algorithm with the same guarantee.

Let us now formally state the property that the output base of Decreasing-Threshold Procedure obeys. Let \( P_M \) be the matroid polytope of \( M \), and let \( F \) be the multilinear extension of \( f \). Decreasing-Threshold Procedure gets as input a point \( x \in (1 - \varepsilon) \cdot P_M \), and its output base \( D \) is guaranteed to obey

\[
F(x') - F(x) \geq \varepsilon[(1 - 3\varepsilon) \cdot f(OPT) - F(x')],
\]

where \( x' = x + \varepsilon \cdot 1_D \) and OPT denotes an optimum solution.

Most of this section is devoted to proving the following proposition, which shows there exists a semi-streaming algorithm that can replace the Decreasing-Threshold Procedure is Accelerated Continuous Greedy.

**Proposition 4.1.** There exists a semi-streaming algorithm that given the input of MSMMatroid and a vector \( x \in (1 - \varepsilon) \cdot P_M \) produces a base \( D \) obeying Eq. (7). Furthermore, this algorithm makes only \( O(\varepsilon^{-2}) \) passes over the input stream.

Before getting to the proof of Proposition 4.1, let us explain how it implies the main result of this section (Theorem 1.2). We recall that, due to a result of McGregor and Vu [MV19], this theorem is tight in the sense that any data stream algorithm that obtains a constant approximation ratio \( \alpha > 1 - 1/e \) for MSMMatroid and uses a constant number of passes must have a space complexity that is linear in \( |N| \) (see Section 4.4 for a more detailed discussion of the result of McGregor and Vu [MV19]).

**Theorem 1.2.** For every constant \( \varepsilon > 0 \), there is a multi-pass semi-streaming algorithm for maximizing a non-negative monotone submodular function subject to a matroid constraint of rank \( k \) (any solution has cardinality at most \( k \)) that stores \( O(k/\varepsilon^3) \) elements, makes \( O(1/\varepsilon^3) \) many passes, and achieves an approximation guarantee of \( 1 - 1/e - \varepsilon \).
Proof. Observe that the theorem is trivial when \( \epsilon \geq 1 - 1/e \), and therefore, we assume below that \( \epsilon < 1 - 1/e \). Furthermore, for simplicity, we describe an algorithm with an approximation ratio of \( 1 - 1/e - O(\epsilon) \) rather than a clean ratio of \( 1 - 1/e - \epsilon \). However, one can switch between the two ratios by scaling \( \epsilon \) by a constant.

Let us denote the algorithm whose existence is guaranteed by Proposition 4.1 by ALG, and consider an algorithm called Data Stream Continuous Greedy (DSCG) obtained from the Accelerated Continuous Greedy algorithm of [BV14] when every execution of the Decreasing-Threshold Procedure by the last algorithm is replaced with an execution of ALG. We explain below why DSCG has all the properties guaranteed by the theorem. We begin by recalling that since the approximation ratio analysis of Accelerated Continuous Greedy in [BV14] treats the Decreasing-Threshold Procedure as a black box that has the guarantee stated in Eq. (7), and ALG also this guarantee, this analysis can be applied as is also to DSCG, and therefore, DSCG is a \((1 - 1/e - O(\epsilon))\)-approximation algorithm.

Recall now that Accelerated Continuous Greedy accesses its input only through the Decreasing-Threshold Procedure, which implies that DSCG is a data stream algorithm just like ALG. Furthermore, since Accelerated Continuous Greedy accesses the Decreasing-Threshold Procedure \( O(\epsilon^{-1}) \) times, the number of passes used by DSCG is larger by a factor of \( O(\epsilon^{-1}) \) compared to the number of passes used by ALG (which is \( O(\epsilon^{-2}) \)). Hence, DSCG uses \( O(\epsilon^{-3}) \) passes.

It remains to show that the space complexity of DSCG is low enough to make it a semi-streaming algorithm. Since Accelerated Continuous Greedy uses linear space in the space necessary to keep the \( O(\epsilon^{-1}) \) bases that it receives from the Decreasing-Threshold Procedure, the space complexity of DSCG is larger than the space complexity of the semi-streaming algorithm ALG only by \( O(k/\epsilon) \), which is nearly linear in \( k \) for any constant \( \epsilon \).

We now get to the proof of Proposition 4.1. This proof is split between three subsections. Section 4.1 describes an alternative analysis of a well known single pass semi-streaming algorithm for \textsc{MSMMatroid}. Using this analysis, we are able to show in Section 4.2 an \( O(\epsilon^{-2}) \)-passes semi-streaming algorithm that given the input of \textsc{MSMMatroid} is able to produce a solution that is close (in some sense) to be a local optimum for the problem. Finally, in Section 4.3 we show how the algorithm from Section 4.2 implies Proposition 4.1.

### 4.1 Alternative Analysis for a Known Single Pass Algorithm

The first data stream algorithm for \textsc{MSMMatroid} was described by Chakrabarti and Kale [CK15]. In this section we consider a variant of their algorithm. This variant is a special case of an algorithm that was described by Huang, Thiery and Ward [HTW20] (based on ideas of Chekuri et al. [CGQ15]), and it appears as Algorithm 2. Algorithm 2 gets a parameter \( c > 1 \) and a base \( S_0 \) of \( M \) that it starts from, and intuitively, it inserts every arriving element into its solution (at the expense of an appropriate existing element) whenever such a swap is beneficial enough in some sense. In the pseudocode of Algorithm 2, we denote by \( u_1, u_2, \ldots, u_n \) the elements of \( N \setminus S_0 \) in the order of their arrival. Similarly, we denote by \( S_i \) the solution of the algorithm immediately after it processes element \( u_i \) for every integer \( 1 \leq i \leq n \). Finally, we denote the elements of the base \( S_0 \) by \( u_{1-|S_0|}, u_{2-|S_0|}, \ldots, u_0 \) in an arbitrary order. This notation allows us to define, for every integer \( 1 - |S_0| \leq i \leq n \) and set \( T \subseteq N \),

\[
 f(u_i : T) = f(u_i | \{u_j \in T \mid 1 - |S_0| \leq j < i\}) .
\]

In other words, \( f(u_i : T) \) is the marginal contribution of \( u_i \) with respect to the elements of \( T \) that appear in the input stream of the algorithm before \( u_i \).


Algorithm 2 SINGLE LOCAL SEARCH PASS \((S_0, c)\)

\[
\begin{align*}
1: & \text{ for every element } u_i \in N \setminus S_0 \text{ that arrives do} \\
2: & \quad \text{Let } C_i \text{ be the single cycle in } S_{i-1} + u_i. \\
3: & \quad \text{Let } u'_i \text{ be the element in } C_i - u_i \text{ minimizing } f(u'_i : S_{i-1}). \\
4: & \quad \text{if } f(u_i \mid S_{i-1}) \geq c \cdot f(u'_i : S_{i-1}) \text{ then} \\
5: & \quad \quad \text{Set } S_i \leftarrow S_{i-1} - u'_i + u_i. \\
6: & \quad \text{else} \\
7: & \quad \quad \text{Set } S_i \leftarrow S_{i-1}. \\
8: & \text{ return } S_n.
\end{align*}
\]

One can verify that the solution of Algorithm 2 remains a base of \(M\) throughout the execution of the algorithm. Furthermore, it is known that Algorithm 2 achieves 4-approximation for 

MSMMatroid. However, we need to prove a slightly different property of it. Specifically, we show below that when \(S_0\) is far from being a local optimum (in some sense), then the value of the final solution \(S_n\) of Algorithm 2 is much larger than the value of the initial solution \(S_0\). In Section 4.2 we show how this property of Algorithm 2 can be used to find an approximate local optimum in \(O(\varepsilon^{-2})\) passes.

Let \(B\) be an arbitrary base of \(M\) (intuitively, one can think of \(B\) as the optimal solution, although this will not always be the case). We begin the analysis of Algorithm 2 by showing a lower bound on the sum of the marginal contributions of the elements of \(B \setminus S_0\) with respect to the solutions held by Algorithm 2 when these elements arrive. Let \(A\) be the set of all elements that belong to the solution of Algorithm 2 at some point (formally, \(A = \bigcup_{i=0}^{n} S_i\)).

**Lemma 4.2.** \(\sum_{u_i \in B \setminus S_0} f(u_i \mid S_{i-1}) \geq f(S_0 \cup B) + \frac{1}{\varepsilon-1} \cdot f(S_0) - \frac{c}{\varepsilon-1} \cdot f(S_n).\)

**Proof.** By the submodularity of \(f\),

\[
\sum_{u_i \in B \setminus S_0} f(u_i \mid S_{i-1}) \geq \sum_{u_i \in B \setminus S_0} f(u_i \mid A) \geq f(B \mid A) = f(B \cup (A \setminus S_0) \mid S_0) - f(A \setminus S_0 \mid S_0) \]

\[
\geq f(B \mid S_0) - \sum_{u_i \in A \setminus S_0} f(u_i \mid S_0 \cup \{j \in A \mid 1 \leq j < i\}) \geq f(B \mid S_0) - \sum_{u_i \in A \setminus S_0} f(u_i \mid S_{i-1}),
\]

where the third inequality holds by the monotonicity of \(f\).

Let us now upper bound the second term in the rightmost side. Since all the elements of \(A \setminus S_0\) were accepted by Algorithm 2 into its solution,

\[
\sum_{u_i \in A \setminus S_0} f(u_i \mid S_{i-1}) \leq \frac{c}{\varepsilon-1} \cdot \sum_{u_i \in A \setminus S_0} [f(u_i \mid S_{i-1}) - f(u'_i : S_{i-1})] = \frac{c}{\varepsilon-1} \cdot \sum_{u_i \in A \setminus S_0} [f(u_i \mid S_{i-1}) - f(u'_i \mid S_{i-1} + u_i - u'_i)] = \frac{c}{\varepsilon-1} \cdot \sum_{u_i \in A \setminus S_0} [f(S_i - f(S_{i-1})) = \frac{c}{\varepsilon-1} \cdot [f(S_n) - f(S_0)],
\]

where the second inequality holds by the submodularity of \(f\), and the last equality holds since \(S_i = S_{i-1}\) for every integer \(1 \leq i \leq n\) for which \(u_i \notin A\). The lemma now follows by combining the two above inequalities. \(\square\)
To complement the last lemma, we need to upper bound the marginal contributions of the elements $u'_i$ corresponding to the elements $u_i \in A$ with respect to the solutions of Algorithm 2 when the last elements arrive. We prove such an upper bound in Corollary 4.7 below. However, proving this upper bound requires us to present a few additional definitions as well as properties of the objects defined. We begin by constructing an auxiliary directed graph $G$ whose vertices are the elements of $N$. Furthermore, for every element $u_i \in N \setminus S_0$, we create edges for the graph $G$ in the following way. Note that there is a single element $c_i \in C_i$ that does not belong to $S_i$. The graph $G$ includes edges from $c_i$ to every other element of $C_i$. Let us now prove some properties of the graph $G$.

**Observation 4.3.** For every element $u \in N$, let us define

$$Val(u) = \begin{cases} f(u : S_n) & \text{if } u \in S_n, \\ f(u : S_i) & \text{if } u \in A \setminus S_n, \text{ and } u \text{ was removed from} \\ \text{the solution of Algorithm 2 when } u_i \text{ arrived}, \\ f(u | S_i) & \text{if } u \notin A \text{ and } u = u_i. \end{cases}$$

Then, for every edge $uv$ of $G$ such that $u \in A$, $Val(u) \leq Val(v)$.

**Proof.** Since there is an edge from $u$ to $v$, $u$ must have been removed from $A$ when some element $u_i$ arrived, and $v$ was another element of the cycle $C_i$. If $v \neq u_i$, then the fact that $u$ was removed (rather than $v$) implies

$$Val(u) = f(u : S_i) \leq f(v : S_i) \leq Val(v),$$

where the second inequality holds since $Val(v)$ is equal to $f(v : S_j)$ for some $j \geq i$. Otherwise, if $v = u_i$, then the fact that $u$ was removed following the arrival of $v$ implies

$$Val(u) = f(u : S_i) \leq \frac{f(v \mid S_i)}{c} \leq \frac{Val(v)}{c} \leq Val(v),$$

where the last inequality holds since the monotonicity of $f$ guarantees that $Val(v)$ is non-negative.

**Corollary 4.4.** If $u$ and $v$ are two elements of $A$ such that $v$ is reachable from $u$ in $G$, then $Val(u) \leq Val(v)$.

**Proof.** The corollary follows from Observation 4.3 because the construction of $G$ guarantees that the vertices of $N \setminus A$ are all sources of $G$ (i.e., vertices that do not have any edge entering them).

**Observation 4.5.** $G$ is acyclic; and every element $u \in N$ that is not a sink of $G$ is spanned by the elements of $\delta^+(u)$, where $\delta^+(u) = \{v \mid uv \text{ is an edge of } G\}$.

**Proof.** Every edge $e$ of $G$ was created due to some cycle $C_i$. Furthermore, the edge $e$ goes from a vertex that does not appear in $S_i$ or any solution that Algorithm 2 has at a later time point to a vertex that does belong to $S_i$. Therefore, if we sort the vertices of $G$ by the largest index $i$ for which they belong to $S_i$ (a vertex that does not belong to $S_i$ for any $i$ is placed before all the vertices that do belong to $S_i$ for some $i$), then we obtain a topological order of $G$, which implies that $G$ is acyclic.

To prove the second part of the observation, we note that whenever the construction of $G$ includes edges leaving a node $u$, this implies that these edges go to all the vertices of $C - u$ for some cycle $C$ that includes $u$. Therefore, $\delta^+(u) \supseteq C - u$ spans $u$. 

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To use the last observation, we need the following known lemma (a similar lemma appeared earlier in [Var11] in an implicit form, and was made explicit in [CGQ15]).

**Lemma 4.6 (Lemma 13 of [FKK18]).** Consider an arbitrary directed acyclic graph $G = (V, E)$ whose vertices are elements of some matroid $M'$. If every non-sink vertex $u$ of $G$ is spanned by $\delta^+(u)$ in $M'$, then for every set $S$ of vertices of $G$ which is independent in $M'$ there must exist an injective function $\psi_S$ such that, for every vertex $u \in S$, $\psi_S(u)$ is a sink of $G$ which is reachable from $u$.

**Corollary 4.7.** $\sum_{u_i \in B \setminus A} f(u'_i : S_{i-1}) + c \cdot \sum_{u_i \in B \cap (A \setminus S_0)} f(u'_i : S_{i-1}) \leq f(S_n | \emptyset) - f(S_0 | S_0 \setminus B)$.

**Proof.** Let $\psi_B$ be the function whose existence is guaranteed by Lemma 4.6 (recall that $B$ is a base of $M$, and therefore, is independent in $M$). Consider now an element $u_i \in B \setminus A$, and let $P_i$ be the path in $G$ from $u_i$ to $\psi_B(u_i)$ whose existence is guaranteed by Lemma 4.6. If we denote by $u''_i$ the element that appears in this path immediately after $u_i$ (there must be such an element because $u_i \notin A \supseteq S_n$, and therefore, is not a sink of $G$), then $\text{Val}(u''_i) \leq \text{Val}(\psi_B(u_i))$ according to Corollary 4.4. Additionally, since $u_i$ was rejected by Algorithm 2 immediately upon arrival, both $u'_i$ and $u''_i$ are elements of $C_i - u_i$, and thus, due to the way in which Algorithm 2 selects $u'_i$,

$$f(u'_i : S_{i-1}) \leq f(u''_i : S_{i-1}) \leq \text{Val}(u''_i) \leq \text{Val}(\psi_B(u_i)) = f(\psi_B(u_i) : S_n),$$

where the last equality holds since $\psi_B(u_i)$ is a sink of $G$, and therefore, belongs to $S_n$.

Consider now an element $u_i \in B \cap (A \setminus S_0)$. Since $\psi_B(u_i)$ is reachable in $G$ from $u_i$, $\text{Val}(u_i) \leq \text{Val}(\psi_B(u_i))$. Therefore, the fact that $u_i$ was added upon arrival to the solution of Algorithm 2 implies

$$f(u'_i : S_{i-1}) \leq \frac{f(u_i | S_{i-1})}{c} \leq \frac{\text{Val}(u_i)}{c} \leq \frac{\text{Val}(\psi_B(u_i))}{c} = f(\psi_B(u_i) : S_n).$$

Combining both the above inequalities, we get

$$\sum_{u_i \in B \setminus A} f(u'_i : S_{i-1}) + c \cdot \sum_{u_i \in B \cap (A \setminus S_0)} f(u'_i : S_{i-1}) \leq \sum_{u_i \in B \setminus S_0} f(\psi_B(u_i) : S_n)$$

$$= \sum_{u_i \in S_n} f(\psi_B(u_i) : S_n) - \sum_{u_i \in B \cap S_0} f(\psi_B(u_i) : S_n) \leq f(S_n | \emptyset) - f(S_0 | S_0 \setminus B) = f(S_n | \emptyset) - f(S_0 | S_0 \setminus B),$$

where the first equality holds because $\psi_B$ is a bijection from $B$ to $S_n$ by Lemma 4.6, and the third inequality holds since $\sum_{u_i \in B \cap S_0} \text{Val}(u_i) \geq \sum_{u_i \in B \cap S_0} f(u_i : S_0) \geq \sum_{u_i \in B \cap S_0} f(u_i | (S_0 \setminus B) \cup (S_0 \cap \{u_1-k, u_2-k, \ldots, u_{i-1}\})) = f(S_0 \cap B | S_0 \setminus B)$.

We are now ready to state and prove our main result regarding Algorithm 2. In the beginning of the section, we claimed that this result shows that the difference $f(S_n) - f(S_0)$ is large whenever $S_0$ is not close to being a local optimum. Formally, the closeness of $S_0$ to being a local optimum is represented by the expression $f(B | S_0) - f(S_0 | \emptyset)$ that appears in Proposition 4.8. One can observe that the output of the standard local search algorithm for MSMMatroid (presented in [NWF78]) is guaranteed to make this expression non-positive, which is the reason we think of solutions that make this expression small as close to being local optima (although, it is quite possible that there exist swaps of the standard local search algorithm that can improve such solutions significantly).

**Proposition 4.8.** Algorithm 2 is a semi-streaming algorithm, and it outputs a base $S_n$ of $M$ that obeys $(c - 1) \cdot f(S_n | \emptyset) + \frac{3c-2}{c-1}[f(S_n) - f(S_0)] \geq f(B | S_0 \setminus B) - f(S_0 | \emptyset).$
Proof. The first part of the proposition holds because implementing Algorithm 2 requires us to maintain only two bases of the matroid \( M \), the input base \( S_0 \) and the current solution of the algorithm. The rest of this proof is devoted to proving the second part of the proposition.

Observe that whenever Algorithm 2 changes its solution while processing element \( u_i \), the value of this solution changes by

\[
f(u_i \mid S_{i-1}) - f(u'_i \mid S_{i-1} + u_i - u'_i) \geq c \cdot f(u'_i \mid S_{i-1}) - f(u'_i \mid S_{i-1} + u_i - u'_i) \geq (c - 1) \cdot f(u'_i \mid S_{i-1} + u_i - u'_i) \geq 0,
\]

where the second inequality holds by the submodularity of \( f \) and the last inequality follows from the monotonicity of \( f \). This implies that \( f(S_i) \) is a non-decreasing function of \( i \), and therefore,

\[
f(S_n) - f(S_0) \geq \sum_{u_i \in B \setminus (A \setminus S_0)} [f(u_i \mid S_{i-1}) - f(u'_i \mid S_{i-1} + u_i - u'_i)] \\
\geq \sum_{u_i \in B \setminus (A \setminus S_0)} [f(u_i \mid S_{i-1}) - f(u'_i \mid S_{i-1})] \geq \sum_{u_i \in B \setminus (A \setminus S_0)} [f(u_i \mid S_{i-1}) - f(u'_i \mid S_{i-1}) - c \cdot \sum_{u_i \in B} f(u'_i \mid S_{i-1})] \\
\geq \sum_{u_i \in B \setminus S_0} f(u_i \mid S_{i-1}) - c^2 \cdot \sum_{u_i \in B \setminus (A \setminus S_0)} f(u'_i \mid S_{i-1}) - c \cdot \sum_{u_i \in B} f(u'_i \mid S_{i-1}),
\]

where the second and third inequalities hold by the submodularity of \( f \), the penultimate inequality holds since the elements of \( B \setminus A \) where not added by Algorithm 2 to its solution, and the last inequality holds by the monotonicity of \( f \).

Using Lemma 4.2 and Corollary 4.7, the previous inequality implies

\[
f(S_n) - f(S_0) \geq \{ f(S_0 \cup B) + \frac{1}{c-1} \cdot f(S_0) - \frac{c}{c-1} \cdot f(S_0) \} - c \cdot \{ f(S_n \mid \emptyset) - f(S_0 \mid S_0 \setminus B) \} \\
= f(S_0 \cup B) + c \cdot f(\emptyset) + \frac{1}{c-1} \cdot f(S_0) - \frac{c^2}{c-1} \cdot f(S_n) + c \cdot f(S_0 \mid S_0 \setminus B) \\
\geq f(S_0 \cup B) + c \cdot f(\emptyset) + \frac{1}{c-1} \cdot f(S_0) - \frac{c^2}{c-1} \cdot f(S_n) + f(S_0 \mid S_0 \setminus B),
\]

where the second inequality follows from monotonicity of \( f \) and the fact that \( c > 1 \). The proposition now follows by rearranging this inequality (this can be verified by checking term by term). \( \Box \)

### 4.2 Obtaining an Approximate Local Optimum

As promised above, in this section we show that Algorithm 2 can be used to get a solution that is approximately a local optimum (in some sense). The algorithm we use to do that is given as Algorithm 3, and it gets a parameter \( \delta \in (0, 1) \).

Intuitively, Algorithm 3 works by employing the fact that every execution of Algorithm 2 increases the value of its input base \( T_{i-1} \) significantly, unless this input base is close to being a local optimum, and therefore, if the execution produces a base \( T_i \) which is not much better than \( T_{i-1} \), then we know that \( T_{i-1} \) is close to being a local optimum in the sense that we care about. The following lemma proves this formally.

**Lemma 4.9.** If Algorithm 3 does not indicate a failure, then its output set \( T \) obeys \( f(B \mid T \setminus B) - f(T \mid \emptyset) < 5\delta \cdot f(OPT \mid \emptyset) \).
Algorithm 3 Multiple Local Search Passes ($\delta$)

1: Find a base $T_0$ of $M$ using a single pass (by simply initializing $T_0$ to be the empty set, and then adding to it any elements that arrives and can be added to $T_0$ without violating independence in $M$).
2: Let $T_1$ be the output of Algorithm 2 when given $S_0 = T_0$ and $c = 2$.
3: for $i = 1$ to $1 + \lceil 4\delta^{-2} \rceil$ do
4:  let $T_i$ be the output of Algorithm 2 when given $S_0 = T_{i-1}$ and $c = 1 + \delta$.
5:  if $f(T_i) - f(T_{i-1}) \leq \delta^2 \cdot f(T_i \mid \emptyset)$ then
6:     return $T_{i-1}$.
7:  Indicate failure if the execution of the algorithm has arrived to this point.

Proof. Since $T_1$ is a base of $M$, $f(T_1 \mid \emptyset) = f(T_1) - f(\emptyset) \leq f(\text{OPT}) - f(\emptyset) = f(\text{OPT} \mid \emptyset)$. This implies that when Algorithm 3 returns a set $T_{i-1}$, then
\[ f(T_i) - f(T_{i-1}) \leq \delta^2 \cdot f(\text{OPT} \mid \emptyset). \]

Plugging this inequality and the fact that $f(T_i \mid \emptyset) \leq f(\text{OPT} \mid \emptyset)$ (because $T_i$ is a base of $M$) into the guarantee of Proposition 4.8 for the execution of Algorithm 2 that has created $T_i$ yields
\[
5\delta \cdot f(\text{OPT} \mid \emptyset) \geq \delta \cdot f(\text{OPT} \mid \emptyset) + \delta (3\delta + 1) \cdot f(\text{OPT} \mid \emptyset)
\geq \delta \cdot f(T_i \mid \emptyset) + \frac{3\delta + 1}{\delta} [f(T_i) - f(T_{i-1})] \geq f(B \mid T_{i-1} \setminus B) - f(T_{i-1} \mid \emptyset). \]

One could imagine that it is possible for the value of the solution maintained by Algorithm 3 to increase significantly following every iteration of the loop starting on Line 3, which will result in the algorithm indicating failure rather than ever returning a solution. However, it turns out that this cannot happen because the value of the solution of Algorithm 3 cannot exceed $f(\text{OPT})$, which implies a bound on the number of times this value can be increased significantly. This idea is formalized by the next two claims.

Observation 4.10. $f(T_1 \mid \emptyset) \geq \frac{1}{5} f(\text{OPT} \mid \emptyset)$.

Proof. If we set $B = \text{OPT}$, then by applying Proposition 4.8 to the execution of Algorithm 2 on Line 2 of Algorithm 3, we get
\[
f(T_1 \mid \emptyset) + 4[f(T_1) - f(T_0)] \geq f(\text{OPT} \mid T_0 \setminus \text{OPT}) - f(T_0 \mid \emptyset)
\geq f(\text{OPT} \mid T_0) - f(T_0 \mid \emptyset),
\]
where the second inequality follows from the monotonicity of $f$. Since the leftmost side the last inequality is equal to $5f(T_1 \mid \emptyset) - 4f(T_0 \mid \emptyset)$, this inequality implies
\[
5f(T_1 \mid \emptyset) \geq f(\text{OPT} \mid T_0) + 3f(T_0 \mid \emptyset) = f(\text{OPT} \cup T_0) + 2f(T_0) - 3f(\emptyset)
\geq f(\text{OPT}) - f(\emptyset) = f(\text{OPT} \mid \emptyset),
\]
where the second inequality follows again from the monotonicity of $f$. The observation now follows by dividing the last inequality by 5.

Corollary 4.11. Algorithm 3 does not ever indicate failure.
Proof. If \( f(OPT \mid \emptyset) = 0 \), then the value of every base of \( M \) according to \( f \) is \( f(\emptyset) \), which guarantees that Algorithm 3 returns \( T_1 \) during the first iteration of the loop starting on its Line 3. Therefore, we assume below that \( f(OPT \mid \emptyset) > 0 \). Furthermore, assume towards a contradiction that Algorithm 3 indicates failure. By Observation 4.10, this assumption implies that the value of the solution maintained by Algorithm 3 increases by at least \( \delta^2 \cdot f(T_1 \mid \emptyset) \geq \frac{\delta^2}{5} f(OPT \mid \emptyset) \) after every iteration of the loop starting on Line 3. Therefore, after all the \( 1 + \lceil 4\delta^{-2} \rceil \) iterations of this loop, the value of the solution of Algorithm 3 is at least

\[
f(T_1) + (1 + \lceil 4\delta^{-2} \rceil) \cdot \frac{\delta^2}{5} f(OPT \mid \emptyset) > f(\emptyset) + \frac{1}{5} f(OPT \mid \emptyset) + \frac{4}{5} f(OPT \mid \emptyset) = f(OPT),
\]

which is a contradiction since the solution of Algorithm 3 is always kept as a base of \( M \). \( \square \)

We summarize the properties of Algorithm 3 in the following proposition.

Proposition 4.12. Algorithm 3 is an \( O(\delta^{-2}) \)-passes semi-streaming algorithm that outputs a base \( T \) of \( M \) that obeys \( f(B \mid T \setminus B) - f(T \mid \emptyset) < 5\delta \cdot f(OPT) \) and \( f(T \mid \emptyset) \geq \frac{1}{5} f(OPT) \).

Proof. To see why the last part of the proposition also hold, we note that Algorithm 3 outputs the first solution in the list \( T_1, T_2, \ldots, T_{\lceil 4\delta^{-2} \rceil} \) which is not larger by at least \( 4\delta^2 \cdot f(T_1) \geq 0 \) compared to the previous solution. Therefore, the value of this solution is at least \( f(T_1) \), which is at least \( \frac{1}{5} f(OPT) \) by Observation 4.10. \( \square \)

4.3 Proof of Proposition 4.1

In this section we prove Proposition 4.1 using the result we have proved in the previous section. Proposition 4.1 requires us to prove the existence of an algorithm with appropriate properties, and we do so by showing that the algorithm appearing as Algorithm 4 has these properties. Notice that this algorithm gets two parameters: a vector \( x \in (1 - \varepsilon) \cdot P_M \) and a value \( \varepsilon \in (0, 1) \).

Algorithm 4 Procedure for Accelerated Continuous Greedy \((x, \varepsilon)\)

1. Define a new non-negative monotone submodular function \( g : 2^\mathcal{N} \to \mathbb{R}_{\geq 0} \) as \( g(S) = F(x + \varepsilon \cdot 1_S) \) for every set \( S \subseteq \mathcal{N} \).
2. Use Algorithm 3 with \( \delta = 3\varepsilon/25 \) to find a base \( D \) of \( M \) that obeys \( g(D \mid \emptyset) \geq \frac{1}{5} g(OPT_g \mid \emptyset) \) and \( g(OPT \mid D \setminus OPT) - g(D \mid \emptyset) < 5\delta \cdot g(OPT_g \mid \emptyset) \), where \( OPT_g \) is the base of \( M \) maximizing the function \( g \) (notice that we have set here \( B = OPT \)).
3. return \( D \).

The following observation states some immediate properties of Algorithm 4. This observation follows from Proposition 4.12 since Algorithm 4 essentially consists of a single execution of Algorithm 3 with appropriate parameters.

Observation 4.13. Algorithm 4 is a \( O(\delta^{-2}) = O(\varepsilon^{-2}) \)-passes semi-streaming algorithm.

To complete the proof of Proposition 4.1, we still have to show that the output set \( D \) of Algorithm 4 obeys Eq. (7), which we do next.

Lemma 4.14. Let \( D \) be the output set of Algorithm 4. Then, \( F(x') - F(x) \geq \varepsilon [(1 - 3\varepsilon) \cdot f(OPT) - F(x')] \), where \( x' = x + \varepsilon \cdot 1_D \).

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\begin{proof}
We need to consider two cases. The simpler case is when \( g(\text{OPT}_g \mid \emptyset) \geq 5\epsilon \cdot f(\text{OPT}) \). In this case,

\[
F(x') - F(x) = F(x + \epsilon \cdot 1_D) - F(x) = g(D \mid \emptyset) \\
\geq \frac{1}{\epsilon} g(\text{OPT}_g \mid \emptyset) \geq \epsilon \cdot f(\text{OPT}) \geq \epsilon((1 - 3\epsilon) \cdot f(\text{OPT}) - F(x')) .
\]

In the rest of the proof we consider the case of \( g(\text{OPT}_g \mid \emptyset) \leq 5\epsilon \cdot f(\text{OPT}) \). We note that, in this case,

\[
F(x') - F(x) = F(x + \epsilon \cdot 1_D) - F(x) = g(D \mid \emptyset) \\
\geq g(\text{OPT} \mid D \setminus \text{OPT}) - 5\delta \cdot g(\text{OPT}_g \mid \emptyset) \geq g(\text{OPT} \mid D \setminus \text{OPT}) - 25\delta \cdot f(\text{OPT}) .
\]

To further develop the last inequality, we need to use \( \odot \) to denote coordinate-wise multiplication between vectors. Using this notation, we get

\[
F(x') - F(x) \geq g(\text{OPT} \mid D \setminus \text{OPT}) - 25\epsilon \delta \cdot f(\text{OPT}) \\
= [F(x + \epsilon \cdot 1_{D \setminus \text{OPT}} + \epsilon \cdot 1_{\text{OPT}}) - F(x + \epsilon \cdot 1_{D \setminus \text{OPT}})] - 25\epsilon \delta \cdot f(\text{OPT}) \\
\geq [F(x' + \epsilon \cdot 1_{\text{OPT}}) - F(x')] - 25\epsilon \delta \cdot f(\text{OPT}) \\
\geq [F(x' + \epsilon (1_N - x') \odot 1_{\text{OPT}}) - F(x')] - 25\epsilon \delta \cdot f(\text{OPT}) \\
\geq \epsilon [F(x' \lor 1_{\text{OPT}}) - F(x')] - 25\epsilon \delta \cdot f(\text{OPT}) = \epsilon \cdot [(1 - 25\delta)f(\text{OPT}) - F(x')] ,
\]

where the second inequality hold by the monotonicity and submodularity of \( f \), the third and last inequalities hold by the monotonicity of \( f \) and the fourth inequality holds because the submodularity of \( f \) guarantees that \( F \) is concave along non-negative directions (such as \((1_N - x') \odot 1_{\text{OPT}} \)). The lemma now follows by plugging in the value of \( \delta \) into the previous inequality. \qed
\end{proof}

\section*{4.4 A Discussion of a Lower Bound by McGregor and Vu [MV19]}

McGregor and Vu [MV19] showed that any data stream algorithm for the Maximum \( k \)-Coverage Problem (which is a special case of \text{MSMMatroid} in which \( f \) is a coverage function and \( M \) is a uniform matroid of rank \( k \)) that makes a constant number of passes must use \( \Omega(m/k^2) \) memory to achieve \((1 + \epsilon) \cdot (1 - (1 - 1/k)^k)\)-approximation with probability at least 0.99, where \( m \) is the number of sets in the input, and it is assumed that these sets are defined over a ground set of size \( n = \Omega(\epsilon^{-2}k \log m) \). Understanding the implications of this lower bound for \text{MSMMatroid} requires us to handle two questions.

- The first question is how the lower bound changes as a function of the number of passes. It turns out that when the number of passes is not dropped from the asymptotic expressions because it is considered to be a constant, the lower bound of McGregor and Vu [MV19] on the space complexity becomes \( \Omega(m/(pk^2)) \), where \( p \) is the number of passes done by the algorithm.

- The second question is about the modifications that have to be done to the lower bound when it is transferred from the Maximum \( k \)-Coverage Problem to \text{MSMMatroid}. Such modifications might be necessary because of input representation issues. However, as it turns out, the proof of the lower bound given by [MV19] can be applied to \text{MSMMatroid} directly, yielding the same lower bound (except for the need to replace \( m \) with the corresponding value in \text{MSMMatroid}, namely, \( |N| \)). Furthermore, McGregor and Vu [MV19] had to use a very large ground set.
so that random sets will behave as one expects with high probability. When the objective
function is a general submodular function, rather than a coverage function, it can be chosen
to display the above mentioned behavior of random sets, and therefore, $\varepsilon$ can be set to 0.

We summarize the above discussion in the following corollary.

**Corollary 4.15** (Corollary of McGregor and Vu [MV19]). For any $k \geq 1$, any $p$-pass data stream
algorithm for $\text{MSMMatroid}$ that achieves an approximation guarantee of $1 - \left(1 - 1/k\right)^k \leq \left(1 - 1/e - 1/k\right)$ with probability at least 0.99 must use $\Omega\left(|N|/(pk^2)\right)$ memory, and this is the case even when the matroid $M$ is restricted to be a uniform matroid of rank $k$.

5 Impossibility results for bipartite matching constraints

In this section we present our impossibility results for maximizing a monotone submodular function subject to a bipartite matching constraint (i.e., $\text{MSMBipartiteMatching}$). In particular, in the presence of parallel edges, we improve the recent hardness of 0.522 (that relies on complexity theoretic assumptions) to the following unconditional result:

**Theorem 5.1.** Any single-pass data stream algorithm for $\text{MSMBipartiteMatching}$ must use mem-
ory of $\Omega\left(r \log \omega(1) r\right)$ on $r$-vertex graphs if it finds a 0.3715-approximate solution with probability at least $2/3$.

This result is obtained by combining two hardness results: the one-way communication complexity of $\text{CHAIN}_p(n)$ (defined below) and streaming lower bounds for the bipartite maximum matching problem. It is a longstanding open question whether it is possible to devise a data stream algorithm for the maximum matching problem with a better approximation guarantee than 1/2, even if we are allowed to use memory $O(r^{2-\varepsilon})$. The following result basically says that improving over the guarantee 1/3 for maximizing a monotone submodular function subject to a bipartite matching constraint in the data stream model would lead to such a breakthrough.

**Theorem 5.2.** For any constant $0 < \varepsilon < 1$, assuming Theorem 5.3 for $\alpha = 0.5 + \varepsilon$ and $M(r) = O(r^{2-\varepsilon})$, any single-pass data stream algorithm for $\text{MSMBipartiteMatching}$ on $r$-vertex graphs that finds a $(\frac{1}{3} + O(\varepsilon))$-approximate solution with probability at least $2/3$ uses memory $\Omega\left(r^{2-\varepsilon}/\log r\right)$.

Note that Theorems 5.1 and 5.2 imply together Theorem 1.4.

5.1 Preliminaries

Our impossibility results harness the hardness of two problems: bipartite matching in the data
stream model and $\text{CHAIN}_p(n)$.

Bipartite matching in the data stream model. The task of devising optimal data stream
algorithms for finding a matching in (bipartite) graphs remains a notorious open question. In this
problem, the algorithm is provided with a stream of edges of a bipartite graph $G = (V, E)$, and is
allowed to use a limited amount of memory while processing them. We assume that before it starts
to read the stream, the algorithm has unbounded computational power (i.e., unbounded time and
memory) to initialize, and similarly, after reading the last edge from the stream, the algorithm
again has unbounded computational power to produce a matching based on what it has stored in
the memory. Kapralov [Kap21] proved that any (potentially randomized) 0.591-approximate data
stream algorithm with success probability at least 1/2 requires $\Omega\left(|V| \log^{\omega(1)} |V|\right)$ memory. His result
is information theoretic and he shows that there is a hard distribution of $r$-vertex bipartite graphs (for large enough $r$) so that no algorithm of “small” memory can find a “good” solution, even if it has unbounded computational power while processing edges (but limited memory in-between the arrival of edges). In particular, his result implies the following theorem with $M(r) = O(r \log^{O(1)} r)$ and $\alpha = 0.591$.

**Theorem 5.3.** There is an infinite number of positive integers $r$ such that the following holds. Consider a single-pass data stream algorithm $B$ for the bipartite matching problem that uses memory at most $M(r)$. Then, there is an $r$-vertex instance $G$ such that $B$ finds an $\alpha$-approximate matching with probability at most $\frac{1}{2}$ on input $G$.

We have stated the theorem in this general form as we will use it as a template in our general reduction. Indeed, it is a conceivable that the theorem holds with $M(r) = O(r^{2-\varepsilon})$ and $\alpha = 1/2 + \varepsilon$, which leads to our stronger (conditional) lower bound Theorem 5.2. Throughout, we assume that the memory satisfies $x \leq M(r) \leq x^2$ for large enough $x \in \mathbb{R}$, and the approximation guarantee satisfies $0 < \alpha < 1$. Finally, we have the following corollary obtained by running $q$ independent copies of $B$ and outputting the largest found matching among all copies.

**Corollary 5.4.** Let $q \geq 1$ be an integer. There is an infinite number of positive integers $r$ such that the following holds. Consider a single-pass data stream algorithm $B$ for the bipartite matching problem that uses memory at most $M(r)/q$. Then, there is an $r$-vertex instance $G$ such that $B$ finds an $\alpha$-approximate matching with probability at most $1/q$ on input $G$.

**One-way communication complexity of CHAIN$_p(n)$.** The second (hard) problem we use is CHAIN$\np(n)$, introduced by Cormode, Dark and Konrad [CDK19], which is closely related to the Pointer Jumping problem (see [Cha07]). In this problem, there are $p$ players $P_1, \ldots, P_p$. For every $1 \leq i < p$, Player $P_i$ is given a bit string $x^i \in \{0, 1\}^n$ of length $n$, and, for every $2 \leq i \leq p$, Player $P_i$ (also) has as input an index $t^i \in \{1, 2, \ldots, n\}$ (note that the convention in this terminology is that the superscript of a string/index indicates the player receiving it). Furthermore, it is promised that either $x^i_{t^i+1} = 0$ for all $1 \leq i < p$ or $x^i_{t^i+1} = 1$ for all these $i$ values. We refer to these cases as the 0-case and 1-case, respectively. The objective of the players in CHAIN$\np(n)$ is to decide whether the input instance belongs to the 0-case or the 1-case. The first player, based on the input bit string $x^1$ sends a message $M^1$ to the second player. Any player $2 \leq i < p$, based on the message it receives from the previous player (i.e., $M^{i-1}$), the input bit string $x^i$ and index $t^i$ sends message $M^i$ to the next player. The last player based on $M^{p-1}$ and $t^p$ decides if we are in the 0-case or 1-case. Each player has unbounded computational power and can use any (potentially randomized) algorithm. We refer to the collections of the algorithms used by all the players as a protocol. The success probability of a protocol is the probability that its decision is correct, and the communication complexity of a protocol is the size of the maximum message sent (i.e., maximum size of $M^1, \ldots, M^{p-1}$). In [Fel+20], the following lower bound is shown for the CHAIN$\np(n)$ problem which is very similar to the lower bounds proved by [CDK19].

**Theorem 5.5.** ([Theorem 3.3 in [Fel+20]]) For any positive integers $n$ and $p \geq 2$, any (potentially randomized) protocol for CHAIN$\np(n)$ with success probability of at least $2/3$ must have a communication complexity of at least $n/(36p^2)$. Furthermore, this holds when instances are drawn from a known distribution $D(p, n)$.

The distribution $D(p, n)$ is simply the uniform distribution over all 0-case and 1-case instances (see the definition of $D^p$ in Appendix C of [Fel+20]).
5.2 Hardness Reduction for Bipartite Matching Constraint

As mentioned above, we describe a general reduction that harnesses the hardness of the bipartite matching problem in the data stream model. This general reduction formally appears as Theorem 5.6. We note that the general reduction implies Theorem 5.2 and Theorem 5.1 by selecting $\varepsilon$ to be small enough and $p$ to be large enough. Specifically, Theorem 5.2 follows by substituting in the assumptions $M(r) = O(r^{2-\varepsilon})$ and $\alpha = 1/2 + \varepsilon$, and Theorem 5.1 is implied since [Kap21] proved Theorem 5.3 with $M(r) = O(r^{\log^c r})$ for any constant $c > 1$ and $\alpha = 0.591$.

**Theorem 5.6.** Assuming Theorem 5.3, for any $\varepsilon > 0$ and integer $p \geq 2$, any data stream algorithm for $\text{MSM Bipartite Matching}$ that finds a $\frac{1+\varepsilon}{p} \left(1 + \frac{\alpha}{1+\alpha} p\right)$-approximate solution with probability at least $2/3$ must use at least $\frac{M(r)}{5 \cdot 10^p \cdot \log(1+\varepsilon(r))}$ memory, where $r$ denotes the number of vertices of the bipartite graph.

Let $r \gg p$ be a large integer as is guaranteed by Corollary 5.4, i.e., so that for any data stream algorithm $B$ for the bipartite matching problem that uses memory at most $M(r)/q$, there is an $r$-vertex instance $G$ such that $B$ finds an $\alpha$-approximate matching with probability at most $1/q$ on input $G$. Here, we select $q = 10p$. We further let $n = M(r)/(5 \cdot 10p)$. We assume that $r$ and $n$ are selected large enough so that $O(p \log n) < n$ (for the hidden constants appearing in the proofs) and $n/\log_{1+\varepsilon}(r) + p \log(n) < n/(36p^2)$. This allows us to simplify some (unimportant) calculations.

Our approach is to assume the existence of an algorithm $A$ for $\text{MSM Bipartite Matching}$ that finds a $\frac{1+\varepsilon}{p} \left(1 + \frac{\alpha}{1+\alpha} p\right)$-approximate solution with probability at least $2/3$ on any instance. Using this algorithm, we provide a protocol for $\text{CHAIN}_{p+1}(n)$. Our protocol is parameterized by $r$-vertex instances $G_1 = (V_1, E_1), \ldots, G_p = (V_p, E_p)$ to the bipartite matching problem in the streaming model. These instances will later be selected to be “hard” instances using Corollary 5.4 (see Section 5.2.3). Throughout, for $i \in [p]$, we use $m_i$ to denote the smallest power of $(1+\varepsilon)$ that upper bounds the size of a maximum matching in $G_i$.

Each player in our protocol for $\text{CHAIN}_{p+1}(n)$ will simulate $A$ on a monotone submodular function selected from a certain family. We describe this family of submodular functions next. We then, in Section 5.2.2, describe and analyze the protocol assuming a “good” selection of the instances $G_1, \ldots, G_p$. Finally, in Section 5.2.3, we show how to select such instances and explain how it implies Theorem 5.6.

5.2.1 Family of submodular functions

We start by defining an extended ground set based on the edge sets of the graphs $G_1, \ldots, G_p$. For $i \in [p]$, let

$$N_i = \{(e,j) \mid e \in E_i, j \in [n]\}.$$  

In other words, $N_i$ contains $n$ parallel copies of each edge of $G_i$, one for each possible choice of $j \in [n]$. We shall use the notation $N_{\leq i} = N_1 \cup \cdots \cup N_i$, $N_{> i} = N_i \cup N_{i+1} \cup \cdots \cup N_p$, and $N = N_{\leq p}$. Furthermore, for a subset $S \subseteq N$, we let

$$s(i) = \frac{|S \cap N_i|}{m_i} \quad \text{and} \quad s(i, -o_i) = \frac{|\{(e,j) \in S \cap N_i \mid j \neq o_i\}|}{m_i}.$$  

Recall that $m_i$ denotes (an upper bound on) the size of a maximum matching in $G_i$. Hence, assuming the edges in $S \cap N_i$ form a matching in $G_i$, $s(i) \in [0,1]$ denotes the approximation ratio of the considered matching. Similarly, $s(i, -o_i)$ measures the approximation ratio of those edges that do not correspond to some index $o_i$. 

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We now recursively define \( p \) families of non-negative monotone submodular functions \( \mathcal{F}_p, \mathcal{F}_{p-1}, \ldots, \mathcal{F}_1 \). Family \( \mathcal{F}_p \) contains a single monotone submodular function \( f_{o_p}: 2^{N_p} \to \mathbb{R}_{\geq 0} \) defined by \( f_{o_p}(S) = \min \{1, s(p)\} \). The use of the subindex \( o_p \) in the last definition is not technically necessary, but it simplifies our notation. For \( i = p-1, \ldots, 1 \), the family \( \mathcal{F}_i = \{ f_{o_{i+1}, \ldots, o_p} \mid o_{i+1}, \ldots, o_p \in [n] \} \) consists of \( n^{p-i} \) monotone submodular functions on the ground set \( N_{\geq i} \) that are defined recursively in terms of the functions in \( \mathcal{F}_{i+1} \) as follows:

\[
f_{o_{i+1}, \ldots, o_p}(S) = \min \left\{ p+1-i, s(i) \left( 1 - \frac{s(i, -o_i)}{p+1-i} \right) f_{o_{i+1}, \ldots, o_p}(S \cap N_{\geq i+1}) \right\}.
\] (8)

To intuitively understand the last definition. One should think of every graph \( G_i \) as having a mass to be covered (in some sense). Every edge of \( G_i \) covers equal amounts of mass from \( G_i, G_{i+1}, \ldots, G_p \), except for the edges of \( G_i \) with the index \( o_i \), which cover only mass of \( G_i \). Furthermore, edges of a single graph \( G_i \) are correlated in the sense that the mass of \( G_{i'} \) that they cover (for any \( i \leq i' \leq p \)) is additive, while edges of different graphs \( G_i, G_{i'} \) that cover the mass of the same graph \( G_{i''} \) do it in an independent way (so if the edges of \( G_i \) cover a \( q_i \) fraction of this mass and the edges of \( G_{i'} \) cover \( q_2 \) fraction, then together they cover only \( 1 - (1-q_i)(1-q_2) \) of the mass of \( G_{i''} \)). Given this intuitive point of view, \( f_{o_{i+1}, \ldots, o_p}(S) \) represents the total mass of the graphs \( G_i, G_{i+1}, \ldots, G_p \) that is covered by the edges of \( S \). Note that this explains why Eq. (8) includes a negative term involving \( s(i, -o_i) \): if there are many edges of \( G_i \) with indexes other than \( o_i \) that appear in \( S \), then a lot of the mass accounted for by \( f_{o_{i+1}, o_{i+2}, \ldots, o_p}(S) \) is counted also by \( s(i) \).

**Observation 5.7.** For every \( i \in [p] \), the functions of \( \mathcal{F}_i \) are non-negative, monotone and submodular.

**Proof.** We prove the observation by downward induction on \( i \). For \( i = p \) it follows because \( f_{o_p} \) is the minimum between a positive constant and the non-negative monotone and submodular function \( s(i) \), and such a minimum is known to also have these properties (see, e.g., Lemma 1.2 of [BF18]).

Assume now that the observation holds for \( i+1 \), and let us prove it for \( i \). The product

\[
\left( 1 - \frac{s(i, -o_i)}{p+1-i} \right) f_{o_{i+1}, \ldots, o_p}(S \cap N_{\geq i+1})
\]

is the product of two non-negative submodular functions, one of which is monotone and the other down-monotone (i.e., \( f(S) \geq f(T) \) for every \( S \subseteq T \)), and such products are known to be non-negative and submodular.\(^8\) Therefore, the sum

\[
s(i) + \left( 1 - \frac{s(i, -o_i)}{p+1-i} \right) f_{o_{i+1}, \ldots, o_p}(S \cap N_{\geq i+1})
\]

is non-negative and submodular since the sum of non-negative and submodular functions also has these properties (see, again, Lemma 1.2 of [BF18]). This sum is also monotone since adding an edge \( e \) to \( S \) either increases the sum by 1, if the index of the edge \( e \) is \( o_1 \), or by \( 1 - f_{o_{i+1}, o_{i+2}, \ldots, o_p}/(p+1-i) \geq 0 \), otherwise. The observation now follows since \( f_{o_{i+1}, \ldots, o_p}(S) \) is the minimum between the above sum and a positive constant. \( \square \)

We let \( \mathcal{F} = \mathcal{F}_1 \). The following lemma proves some useful properties of this functions family.

\(^8\)To see why, note that if \( f \) is a non-negative monotone submodular function and \( g \) is a non-negative down-monotone submodular function, then, with respect to the product \( f \cdot g \), the marginal contribution of an element \( u \) to a set \( S \) that does not include it is given by \( f(S+u) \cdot g(S+u) - f(S) \cdot g(S) = f(u \mid S) \cdot g(S+u) + f(S) \cdot g(u \mid S) \), which is a down-monotone function of \( S \).
Lemma 5.8. The monotone submodular functions in $F$ have the following properties:

(a) For $i \in [p]$, any two functions $f_{o_1, \ldots, o_p}, f'_{o_1', \ldots, o'_p} \in F$ with $o_1 = o_1', \ldots, o_{i-1} = o'_{i-1}$ are identical when restricted to the ground set $N_{\leq i}$.

(b) We can evaluate $f_{o_1, \ldots, o_p}$ on input set $S$, using $n$ memory in addition to the input length.

(c) Let $M_1, \ldots, M_p$ be maximum matchings in $G_1, \ldots, G_p$, respectively. Then

$$f_{o_1, \ldots, o_p}(S) \geq \frac{p}{1 + \varepsilon} \quad \text{for } S = \bigcup_{i=1}^{p} \{(e, o_i) \mid e \in M_i\}.$$  

(d) For a subset $S \subseteq N$ such that $|S \cap N_i| \leq \alpha n_i$ and $\{(e, o_i) \in S \cap N_i \} = \emptyset$ for all $i \in [p]$,

$$f_{o_1, \ldots, o_p}(S) < 1 + \frac{\alpha}{\alpha + 1} p.$$  

Proof. Property (a): This follows by the definition of the submodular functions (8): when $S \subseteq N_{\leq i}$, the value of $f_{o_1, \ldots, o_p}(S)$ only depends on $s(1), s(1, -o_1), \ldots, s(i-1), s(i-1, -o_{i-1})$ and $s(i)$.

Property (b): Given input set $S$, we can evaluate $f_{o_1, \ldots, o_p}(S)$ as follows. First calculate $s(p)$ and $s(p, -o_p)$. This requires us to store two numbers. Furthermore, using these numbers we calculate $f_{o_p}(S \cap N_p)$ and then “free” the memory used for $s(p)$ and $s(p, -o_p)$. Now, suppose we have calculated $f_{o_{i+1}, \ldots, o_p}(S \cap N_{\geq i+1})$. We then calculate $s(i)$ and $s(i, -o_i)$ which allows us to calculate $f_{o_1, \ldots, o_p}(S \cap N_{\geq i})$ from $f_{o_{i+1}, \ldots, o_p}(S \cap N_{\geq i+1})$. Following this calculation, we free the memory used for $f_{o_{i+1}, \ldots, o_p}(S \cap N_{\geq i+1})$ and $s(i+1), s(i+1, -o_{i+1})$. We proceed in this way until we have calculated the desired value $f_{o_1, \ldots, o_p}(S)$. At any point of time we have only stored at most 4 numbers, and each number takes $O(\log n)$ bits to store. Thus the memory that we need is upper bounded by $n$ (since $n$ is selected to be sufficiently large).

Property (c) For $S = \bigcup_{i=1}^{p} \{(e, o_i) \mid e \in M_i\}$, by the selection of $M_i$ and $m_i$, $s(i) \geq 1/(1 + \varepsilon)$ for all $i \in [p]$. Moreover, since we only have items corresponding to the indices $o_1, \ldots, o_p$, we have $s(i, -o_i) = 0$ for $i \in [p]$. Hence, by (8), $f_{o_1, \ldots, o_p}(S) \geq p/(1 + \varepsilon)$.

Property (d): As $S$ does not contain any elements with the indices $o_1, \ldots, o_p$, we have $s(i) = s(i, -o_i)$ for all $i \in [p]$. Furthermore, by assumption, $s(i) \leq \alpha < 1$, which implies $f_{o_p}(S \cap N_p) = s(p)$ and, for $i = p, p-1, \ldots, 1$,

$$f_{o_1, \ldots, o_p}(S \cap N_{\geq i}) = s(i) + \left(1 - \frac{s(i)}{p + 1 - i}\right) f_{o_{i+1}, \ldots, o_p}(S \cap N_{\geq i+1}).$$

In the last equality we only used the fact that $s(i) < 1$ for all $i \in [p]$. Plugging in the stronger inequality $s(i) \leq \alpha$ yields

$$f_{o_1, \ldots, o_p}(S) \leq \alpha \left(1 + \left(1 - \frac{\alpha}{p}\right) \left(1 - \frac{\alpha}{p-1}\right) + \left(1 - \frac{\alpha}{p}\right)^2 \left(1 - \frac{\alpha}{p-2}\right) + \cdots + \left(1 - \frac{\alpha}{p}\right)^{i} \left(1 - \frac{\alpha}{p-1}\right)^{i-1} \left(1 - \frac{\alpha}{1}\right)^{i-1}\right).$$

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Let us find the solution to the terms inside the parenthesis in the last inequality. To that end, we let
\[
h(p) = \left( 1 + \left( 1 - \frac{\alpha}{p} \right) \right) \left( 1 - \frac{\alpha}{p - 1} \right) + \left( 1 - \frac{\alpha}{p} \right) \left( 1 - \frac{\alpha}{p - 2} \right) + \ldots \,.
\]
One can observe that
\[
h(p) = (1 - \frac{\alpha}{p})h(p - 1) + 1 \quad \text{and} \quad h(0) = 1.
\]
Let us now show by induction that \( h(k) \leq \frac{k + \alpha + 1}{\alpha + 1} \), and the inequality is strict for every \( k \geq 1 \). The base case \( h(0) = 1 \) holds as an equality. Now, suppose we have proved \( h(k - 1) \leq \frac{k + \alpha}{\alpha + 1} \), then
\[
h(k) = \left( 1 - \frac{\alpha}{k} \right) h(k - 1) + 1
\leq \left( 1 - \frac{\alpha}{k} \right) \cdot \left( \frac{k + \alpha}{\alpha + 1} \right) + 1
= \left( \frac{k - \alpha}{k} \right) \cdot \left( \frac{k + \alpha}{\alpha + 1} \right) + 1
= \frac{k^2 - \alpha^2}{k(\alpha + 1)} + 1
= \frac{k^2 - \alpha^2 + k(\alpha + 1)}{k(\alpha + 1)}
< \frac{k^2 + k(\alpha + 1)}{k(\alpha + 1)}
= \frac{k + \alpha + 1}{\alpha + 1},
\]
where the first inequality holds since \( \alpha < 1 \leq k \).

Plugging the upper bound we have proved on \( h(p) \) into the upper bound we have on \( f_{o_1,\ldots,o_p}(S) \) produces
\[
f_{o_1,\ldots,o_p}(S) \leq \alpha \cdot h(p) \leq \alpha \cdot \frac{p + \alpha + 1}{\alpha + 1} < 1 + \frac{\alpha}{\alpha + 1}p,
\]
as required. \( \square \)

5.2.2 Description and Analysis of Protocol for CHAIN_{p+1}(n)

We use algorithm \( \mathcal{A} \) for \texttt{MSMBipartiteMatching} to devise Protocol 1 for CHAIN_{p+1}(n). Given CHAIN_{p+1}(n) instance \( x^1, x^2, t^2, \ldots, x^p, t^p, t^{p+1} \), the protocol simulates the execution of \( \mathcal{A} \) on the following stream:

First the elements in \( \{(e, j) \in \mathcal{N}_1 \mid j \in [n] \text{ with } x^j_1 = 1\} \) are given (by the first player).

Then for, \( i = 2, \ldots, p \), the elements in \( \{(e, j) \in \mathcal{N}_i \mid j \in [n] \text{ with } x^j_i = 1\} \) are given (by the \( i \)-th player).

The submodular function to be optimized is \( f_{o_1,\ldots,o_p} \) where \( o_i = t^{i+1} \) for \( i = 1, \ldots, p-1 \). In order for the players to be able to simulate the execution of \( \mathcal{A} \) and any oracle call made to \( f_{o_1,\ldots,o_p} \), Player \( i \) sends to Player \( i+1 \) the state of \( \mathcal{A} \) and the indices \( t^2, \ldots, t^i \). Hence, the communication complexity of the protocol is upper bounded by the memory usage of \( \mathcal{A} \) plus \( p \log(n) \). Note that Player \( i \)
Protocol 1 Reduction from CHAIN$_{p+1}(n)$ to MSMBipartiteMatching

Player $P_i$’s Algorithm for $i = 1, \ldots, p$

1: Initialize $A$ with the received memory state (or initial state if first player).
2: Simulate $A$ on the elements $\{(e, j) \in \mathcal{N}_i \mid j \in [n] \text{ with } x_j = 1\}$.
3: The objective function for $A$ is one of the functions $f_{o_1, \ldots, o_p} \in \mathcal{F}$ with $o_1 = t^2, \ldots, o_{i-1} = t^i$. By Property (a) of Lemma 5.8, these functions are identical when restricted to $\mathcal{N}_{\leq i}$, and so any oracle query from $A$ can be evaluated without ambiguity.
4: Send to $P_{i+1}$ the values $t^2, t^3, \ldots, t^i$ and the memory state of $A$.

Player $P_{p+1}$’s Algorithm

1: The objective function for $A$ can now be determined to be $f_{o_1, \ldots, o_p} \in \mathcal{F}$ with $o_1 = t^2, \ldots, o_{p-1} = t^p$.
2: Initialize $A$ with the received memory state, and ask it to return a solution $S$.
3: If $f_{o_1, \ldots, o_p}(S) \geq 1 + \frac{a}{4+\alpha}p$, output “1-case”; otherwise, output “0-case”.

only needs to know $o_1, \ldots, o_{i-1}$ (and thus indices $t^2, \ldots, t^i$) in order to evaluate the oracle calls by Property (a) of Lemma 5.8 (since at that point only elements of $\mathcal{N}_{\leq i}$ has arrived, and hence, $A$ can only query the oracle for subsets of $\mathcal{N}_{\leq i}$).

We proceed to analyze the success probability of the protocol. The success probability in the 0-case will depend on the selection of $G_1, \ldots, G_p$.

Definition 5.9. We say that the selection of $G_1, \ldots, G_p$ is successful if the following holds: if we select a random 0-case instance of CHAIN$_{p+1}(n)$ from $D(p+1, n)$, then with probability at least $9/10$ the output $S$ of $A$ in Protocol 1 satisfies $|S \cap \mathcal{N}_i| \leq \alpha \cdot m_i$ for all $i \in [p]$.

In other words, the selection of $G_1, \ldots, G_p$ is successful if Protocol 1 is unlikely to find a large matching in any of the graphs. Intuitively, it should be possible to select such graphs since, by Theorem 5.2, any algorithm for finding a large matching requires large memory. The following lemma formalizes this argument.

Lemma 5.10. If $A$ uses memory at most $n/\log_{1+\epsilon}(r)$, there is a successful selection of $G_1, \ldots, G_p$.

The next section is devoted to proving the last lemma. Here we proceed to show how it implies Theorem 5.6.

Lemma 5.11. If the selection of $G_1, \ldots, G_p$ is successful, Protocol 1 succeeds with probability at least $2/3$ on the distribution $D(p+1, n)$.

Below we prove Lemma 5.11. However, before doing so, let us first explain how Lemma 5.11 implies Theorem 5.6. Indeed, suppose toward contradiction that $A$ uses memory less than $n/\log_{1+\epsilon}(r)$ which by selection of $n$ equals $\frac{\mathcal{M}(r)}{5\cdot 10p \cdot \log_{1+\epsilon}(r)}$. Then, Lemma 5.10 says that there is a successful selection of $G_1, \ldots, G_p$, which in turn, by Lemma 5.11, means that Protocol 1 succeeds with probability at least $2/3$. As aforementioned, the communication complexity of Protocol 1 is at most the memory of $A$ plus $p \log(n)$. This contradicts Theorem 5.5 because $n/\log_{1+\epsilon}(r) + p \log(n) < n/(36p^2)$. It follows that $A$ must use memory at least $n/\log_{1+\epsilon}(r)$.

We complete this section with the proof of Lemma 5.11.

Proof of Lemma 5.11. We first analyze the success probability of Protocol 1 in the 1-case. In the 1-case, the elements of $S = \bigcup_{i=1}^p \{(e, o_i) \mid e \in M_i \}$ are elements of the stream, where $M_1, \ldots, M_p$ denote maximum matchings in $G_1, \ldots, G_p$, respectively. Hence, by Property (c) of Lemma 5.8,
there is a solution to \textsc{MSMBipartiteMatching} of value at least \( p/(1 + \epsilon) \). Now, by assumption, \( A \) finds a \( \frac{1 + \epsilon}{p} \left( 1 + \frac{\alpha}{1 + \alpha}p \right) \)-approximate solution with probability at least 2/3. As

\[
\frac{p}{1 + \epsilon} \cdot \frac{1 + \epsilon}{p} \left( 1 + \frac{\alpha}{1 + \alpha}p \right) \geq 1 + \frac{\alpha}{1 + \alpha}p,
\]

Player \( p + 1 \) correctly outputs 1-case, i.e., Protocol 1 succeeds, with probability at least 2/3.

For the 0-case, there is no elements \((e,o_i) \in \mathcal{N}_i\) in the stream for \( i \in [p] \). Moreover, since the selection of \( G_1, \ldots, G_p \) is successful, we have, for a random 0-case instance from \( D(p + 1, n) \), that the solution \( S \) output by \( A \) in Protocol 1 satisfies \(|S \cap \mathcal{N}_i| \leq \alpha \cdot m_i\) for \( i \in [p] \) with probability \( 9/10 \geq 2/3 \). Whenever that happens, Property (d) of Lemma 5.8 says that \( f_{o_1,\ldots,o_p}(S) < 1 + \frac{\alpha}{1 + \alpha}p \).

It follows that Player \( p + 1 \) outputs 0-case, i.e., the protocol succeeds, with probability at least 2/3 for a randomly chosen 0-case instance from \( D(p + 1, n) \). Combining the two cases, we have thus proved that the protocol succeeds with probability at least 2/3 on the distribution \( D(p + 1, n) \).

\[\Box\]

### 5.2.3 The selection of \( G_1, \ldots, G_p \)

We now prove Lemma 5.10, which we restate here for convenience.

**Lemma 5.10.** If \( A \) uses memory at most \( n/\log_{1+\epsilon}(r) \), there is a successful selection of \( G_1, \ldots, G_p \).

**Proof.** We select the graphs \( G_1, \ldots, G_p \) one-by-one, starting from the left. When selecting \( G_i \), we make sure to select a graph such that, on a random 0-case instance from \( D(p + 1, n) \), the probability that \( A \) in Protocol 1 outputs a set \( S \) such that \(|S \cap \mathcal{N}_i| > \alpha \cdot m_i\) is at most \( 1/(10p) \). The lemma then follows by the union bound.

Now, suppose that we have already selected \( G_1, \ldots, G_{i-1} \). We proceed to explain how \( G_i \) is selected. The outline of the argument is as follows. We will simulate the execution of Player \( i \) to obtain a streaming algorithm for the bipartite matching problem that uses memory at most \( 4n \). Hence, by the selection of \( n \) and Corollary 5.4, there must be a graph \( G_i \) for which it is likely to fail.

In order to simulate the execution of the \( i \)th player, we need to be able to evaluate the oracle calls to the submodular function. To this end, observe that the value of a submodular function in \( \mathcal{F} \) on a subset \( S \subseteq \mathcal{N}_{\leq i} \) only depends on the values \( m_1, \ldots, m_i \) and the numbers \(|(e,j) \in S \cap \mathcal{N}_k|\) for \( j \in [n] \) and \( k = 1, 2, \ldots, i \). Since we have selected \( G_1, \ldots, G_{i-1} \), we know the values \( m_1, \ldots, m_{i-1} \). Furthermore, there are \( R \) many possibilities of \( m_i \), where \( R \leq 1 + \log_{1+\epsilon}(r) \), because we consider \( r \)-vertex graphs. Denote these possibilities by \( m_i^1, m_i^2, \ldots, m_i^R \). In the algorithm below that simulates the execution of Player \( i \), we make a copy of \( A \) for each of these possibilities. This allows us to answer any evaluations of the submodular function made by Player \( i \).

More precisely, we simulate the execution of Player \( i \) to obtain the following algorithm \( B \) for the bipartite matching problem in the streaming model:

**Preprocessing** In the unbounded preprocessing phase, we start by sampling a 0-case instance of \( \text{CHAIN}_{p+1}(n) \) \( x^1, x^2, t^2, \ldots, x^p, t^p, t^{p+1} \) from \( D(p + 1, n) \). We then simulate the execution of the first \( i - 1 \) players on this instance, which is possible since we fixed the graphs \( G_1, \ldots, G_{i-1} \). Finally, we use the state of \( A \) received from the previous player (or the initial state if \( i = 1 \)) to initialize \( R \) copies of \( A \), \( A_1 \) for guess \( m_i^1 \), \( A_2 \) for guess \( m_i^2 \), \ldots, \( A_R \) for guess \( m_i^R \).

**Processing stream:** At the arrival of an edge \( e \), we forward the elements \( \{(e,j) \mid x_j^i = 1\} \subseteq \mathcal{N}_i \) to each copy \( A_j \) of \( A \). Note that since each copy has a fixed guess of \( m_i \), we can evaluate any call to the function \( f_{o_1,\ldots,o_p} \), where \( o_i = t^{i+1} \) for \( i = 1, \ldots, p - 1 \).
**Postprocessing** In the unbounded postprocessing phase, we decode the memory state of each copy of $A$ as follows. Let $Y_j$ be the memory state of $A_j$. Consider all possible streams of edges that could lead to this memory state $Y_j$, let $E(Y_j)$ be the edges that appear in all these graphs, and select $M_j$ to be the largest matching in $E(Y_j)$. The output matching is then the largest matching among $M_1, \ldots, M_R$.

We now bound the memory used by $B$ when it processes the stream. It saves the vector $x^i$, which requires $n$ bits, and the indices $o_1, \ldots, o_{p-1}$, which require $p \log(n) < n$ bits. It then runs $R$ parallel copies of $A$, which requires $n/\log_{1+\varepsilon}(r) \cdot R \leq 2n$ bits. Finally, by Property (b) of Lemma 5.8, any submodular function evaluation requires at most $n$ bits. Hence, the total memory usage of the algorithm is at most $5n$. Now by the selection of $n$, namely that $5n \leq \mathcal{M}(r)/q$ for $q = 10p$, we can apply Corollary 5.4 to obtain that there is an $r$-vertex graph such that the described algorithm outputs an $\alpha$-approximate matching with probability at most $1/(10p)$. We select $G_i$ to be this graph, and let $m_i$ be the smallest power of $1 + \varepsilon$ that upper bounds the size of the maximum matching of $G_i$.

Let $j$ be the guess such that $m_i = m_i^j$. Note that (since $m_i^j = m_i$) $A_j$ in $B$ simulates exactly the distribution of Player $i$ when given a random $0$-case instance from $D(p+1, n)$. In particular, Player $i$ will send the state $Y_j$ to Player $i + 1$, and with probability at least $1 - 1/(10p)$ (for a random $0$-case instance from $D(p+1, n)$) the size of the largest matching in $E(Y_j)$ is less than $\alpha \cdot m_i$. Whenever this happens, the output $S$ of $A$ in Protocol 1 must satisfy $|S \cap N_i| \leq \alpha \cdot m_i$ (no matter what the following players do) because the edges of $E(Y_j)$ are the only edges that $A$ can know for sure (given his memory state when Player $i$ stops executing) that they belong to $G_i$. This completes the description of how the graph $G_i$ is selected. Repeating this until all $p$ graphs have been selected yields the lemma.

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A Single-Pass for Non-Monotone Submodular Functions

We now show how our single-pass algorithm for MSMMatroid, i.e., Algorithm 1, can be extended to the non-monotone case, i.e., to SMMatroid, to obtain Theorem 3.1. Only minor modifications are needed to the algorithm. The modified algorithm appears as Algorithm 5. There are two changes compared to Algorithm 1, our algorithm for the monotone case. First, the parameter \( \alpha \) is chosen differently, and is set to be (approximately) the solution to \( e^\alpha = \frac{2^2 + 2a - 1}{a} \); more precisely, we set \( \alpha = 1.9532 \). Second, when updating coordinates of the vectors \( a_i \), we only increase a coordinate, corresponding to some element \( u \in \mathcal{N} \), if the total increase of the coordinate \( u \) so far does not exceed some target value \( p \in (0,1) \), which is set to \( \frac{1}{m(c(1-c))} \).

**Algorithm 5** Single-Pass Semi-Streaming Algorithm for MSMMatroid

1. Set \( \alpha = 1.9532 \), \( m = \left\lceil \frac{3\alpha}{\epsilon} \right\rceil \), \( c = \frac{m}{m-\alpha} \), \( p = \frac{1}{m(c(1-c))} \), and \( L = \left\lceil \log_\epsilon \left( \frac{2c}{c(1-c)} \right) \right\rceil \).
2. Set \( a = 0 \in [0,1]^\mathcal{N} \) to be the zero vector, and let \( b = -\infty \).
3. for every element arriving \( u \in \mathcal{N} \), if \( \partial_u F(a) > 0 \) do
4. Let \( \ell(u) = \left\lfloor \log_\epsilon \left( \partial_u F(a) \right) \right\rfloor \). \( \ell(u) \) is largest index \( i \in \mathbb{Z} \) with \( c_i \leq \partial_u F(a) \).
5. \( \beta := \max\{b, \ell(u) - \text{rank}(M) - \text{L}\} \).
6. for \( i = \beta \) to \( \ell(u) \) do
7. if \( A_i + u \in \mathcal{I} \) and \( \sum_{i=\beta}^{\ell(u)} a_i(u) \leq p \) then
8. \( A_i \leftarrow A_i + u \).
9. \( a_i \leftarrow a_i + \frac{c_i}{m\partial_u F(a)} \mathbf{1}_u \).
10. Set \( b \leftarrow h - L \), where \( h \) is largest index \( i \in \mathbb{Z} \) satisfying \( \sum_{j=i}^{\infty} |A_j| \geq \text{rank}(M) \).
11. \( a \leftarrow \sum_{i=b}^{\infty} a_i \).
12. Delete from memory all sets \( A_i \) and vectors \( a_i \) with \( i \in \mathbb{Z}_{<b} \).
13. Set \( S_k \leftarrow \emptyset \) for \( k \in \{0, \ldots, m-1\} \).
14. Let \( q \) be largest index \( i \in \mathbb{Z} \) with \( A_i \neq \emptyset \).
15. for \( i = q \) to \( b \) (stepping down by 1 at each iteration) do
16. while \( 3u \in A_i \setminus S_{(i \text{ mod } m)} \) with \( S_{(i \text{ mod } m)} + u \in \mathcal{I} \) do
17. \( S_{(i \text{ mod } m)} \leftarrow S_{(i \text{ mod } m)} + u \).
18. return a rounding \( R \in \mathcal{I} \) of the fractional solution \( s := \frac{1}{m} \sum_{k=0}^{m-1} \mathbf{1}_{S_k} \) with \( f(R) \geq F(s) \).

We recall that most results shown in Section 3 did not rely on monotonicity of \( f \). More precisely, the first result needing monotonicity of \( f \) was Lemma 3.10. Moreover, also Observation 3.9 does not hold anymore, because it used the fact that we add \( u \) to sets \( A_i \) as long as the marginal contribution is large enough and \( A_i + u \) is independent. However, this is not always the case in Algorithm 5 because we will stop adding \( u \) to sets \( A_i \) if the condition \( \sum_{i=\beta}^{\ell(u)} a_i(u) \leq p \) on Line 7 of Algorithm 5 fails.

To circumvent this issue, we provide two bounds on the partial derivative \( \partial_u F(a_{\text{all}}) \), one for elements \( u \in \mathcal{N} \) with \( a_{\text{all}}(u) \leq p \) and one for elements \( u \in \mathcal{N} \) with \( a_{\text{all}}(u) \geq p \). In the first case, the condition \( \sum_{i=\beta}^{\ell(u)} a_i(u) \leq p \) was always fulfilled, and we can replicate the same analysis as in the monotone case. In the second case, we exploit that \( a_{\text{all}}(u) \geq p \) is large to provide a bound on \( \partial_u F(a_{\text{all}}) \) that depends on \( p \).

Recall that \( \ell(u) \) was defined, in Section 3, as largest index \( i \in \mathbb{Z} \) such that \( u \in \text{span}(\mathcal{A}_i) \) (or \( -\infty \), if no such index exists).

**Observation A.1.**

\[ \partial_u F(a_{\text{all}}) \leq c^{\ell(u) + 1} \quad \forall u \in \mathcal{N} \text{ with } a_{\text{all}}(u) \leq p . \]
Proof. Since \( u \notin \text{span}(\overline{A}_{\ell(u)+1}) \) and \( a_{\text{all}}(u) \leq p \), we get that \( u \) did not get added to the set \( A_{\ell(u)+1} \) in Algorithm 1, even though it fulfilled both \( A_{\ell(u)+1} + u \subseteq \overline{A}_{\ell(u)+1} + u \subseteq I \) and \( \sum_{i=b}^{\ell(u)} a_i(u) \leq a_{\text{all}}(u) \leq p \) when it got considered. Hence, when \( u \) got considered in Line 3 of Algorithm 5, we had \( \partial_u F(a) < c^{\ell(u)+1} \). Finally, by submodularity of \( f \) and because \( a \leq a_{\text{all}} \) (coordinate-wise), we have \( \partial_u F(a_{\text{all}}) \leq \partial_u F(a) \leq c^{\ell(u)+1} \). \( \square \)

**Lemma A.2.**

\[
\partial_u F(a_{\text{all}}) \leq \frac{1}{1-p} \cdot c^{\ell(u)+1} \quad \forall u \in \mathcal{N} \text{ with } a_{\text{all}}(u) \geq p .
\]

Proof. Let \( u \in \mathcal{N} \) with \( a_{\text{all}}(u) \geq p \), and let \( a \) be the vector at the beginning of the iteration of the for loop in Line 3 of Algorithm 5 when \( u \) got considered. Moreover, let \( \beta := \max\{b, \ell(u) - \text{rank}(M) - L\} \) be the \( \beta \) computed and used at that iteration. Because the multilinear extension \( F \) is linear in each single coordinate, and in particular the one corresponding to \( u \),

\[
F \left( a + \sum_{i=\beta}^{\ell(u)} \frac{c^i}{m \cdot \partial_u F(a)} 1_u \right) - F(a) = \frac{1}{m} \sum_{i=\beta}^{\ell(u)} c^i .
\]  

(9)

Note that because the values of \( a_i(u) \) are only increased during the iteration of the for loop in Line 3 that considers \( u \), we have

\[
a_{\text{all}}(u) = \sum_{i=\beta}^{\ell(u)} \frac{c^i}{m \cdot \partial_u F(a)} .
\]  

(10)

Due to the same reason, we have \( a(u) = 0 \). Hence, \( a_{\text{all}} \land 1_{\mathcal{N} - u} \geq a \) (coordinate-wise). We thus obtain

\[
\frac{1}{m} \sum_{i=\beta}^{\ell(u)} c^i = F(a + a_{\text{all}}(u) \cdot 1_u) - F(a) \\
\geq F((a_{\text{all}} \land 1_{\mathcal{N} - u}) + a_{\text{all}}(u) \cdot 1_u) - F(a_{\text{all}} \land 1_{\mathcal{N} - u}) \\
= a_{\text{all}}(u) \cdot \partial_u F(a_{\text{all}}) \\
\geq p \cdot \partial_u F(a_{\text{all}}) ,
\]  

(11)

where the first equality is due to Eqs. (9) and (10), the first inequality follows from submodularity of \( f \) and \( a_{\text{all}} \land 1_{\mathcal{N} - u} \geq a \), the second equality is due to multilinearity of \( F \), and the last inequality holds because we are considering an element \( u \in \mathcal{N} \) with \( a_{\text{all}}(u) \geq p \). The result now follows due to

\[
\partial_u F(a_{\text{all}}) \leq \frac{1}{mp} \sum_{i=\beta}^{\ell(u)} c^i \leq \frac{1}{mp} \sum_{i=-\infty}^{\ell(u)} c^i = \frac{1}{mp(c-1)} \cdot c^{\ell(u)+1} = \frac{1}{1-p} \cdot c^{\ell(u)+1} ,
\]

where the first inequality is due to Eq. (11), and the last equality follows from our definition of \( p \), i.e., \( p \triangleq \frac{1}{m(c-1)+1} \). \( \square \)

We now combine Observation A.1 and Lemma A.2 to obtain a result analogous to Lemma 3.10 (which we had in the monotone case).

**Lemma A.3.**

\[
F(a_{\text{all}} \lor 1_{\text{OPT}}) - F(a_{\text{all}}) \leq \sum_{u \in \text{OPT}} c^{\ell(u)+1} .
\]

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Proof. Let
\[ \text{OPT}_{\text{big}} := \{ u \in \text{OPT} : a_{\text{all}}(u) \geq p \} . \]

The result now follows from
\[
F(a_{\text{all}} \lor 1_{\text{OPT}}) - F(a_{\text{all}}) \leq \sum_{u \in \text{OPT}} \partial_u F(a_{\text{all}}) \cdot (1 - a_{\text{all}}(u)) \\
\leq \frac{1}{1 - p} \sum_{u \in \text{OPT}_{\text{big}}} c^{\ell(u)+1} \cdot (1 - a_{\text{all}}(u)) + \sum_{u \in \text{OPT} \setminus \text{OPT}_{\text{big}}} c^{\ell(u)+1} \cdot (1 - a_{\text{all}}(u)) \\
\leq \sum_{u \in \text{OPT}} c^{\ell(u)+1},
\]
where the first inequality uses concavity of \( F \) along non-negative directions, the second one is due to Observation A.1 and Lemma A.2, and the last one holds since \( 1 - a_{\text{all}}(u) \leq 1 - p \) in the first sum (because \( u \in \text{OPT}_{\text{big}} \)) and \( 1 - a_{\text{all}}(u) \leq 1 \) in the second sum.

As in the monotone case, we now would like to relate the difference between \( f(\text{OPT}) \) and \( F(a_{\text{all}}) \) to the above-derived bounds on \( \partial_u F(a_{\text{all}}) \). Lemma 3.10, which we used in the monotone case, does not hold for non-monotone functions \( f \). To avoid the need for monotonicity, we bound the difference \( F(a_{\text{all}} \lor 1_{\text{OPT}}) - F(a_{\text{all}}) \) instead. To relate \( F(a_{\text{all}} \lor 1_{\text{OPT}}) \) to \( f(\text{OPT}) \), we exploit that \( a_{\text{all}} \) has small coordinates, through the following known lemma (for completeness, we prove this lemma here, however, we note that it can also be viewed as an immediate corollary of either Lemma 7 of [CJV15] or Lemma 2.2 of [Buc+14]).

**Lemma A.4.** Let \( f : 2^N \to \mathbb{R}_{\geq 0} \) be a non-negative submodular function with multilinear extension \( F \), and let \( p \in [0,1] \), \( x \in [0,p]^N \), and \( S \subseteq N \). Then \( F(x \lor 1_S) \geq (1 - p)f(S) \).

**Proof.** We use the fact that the multilinear extension is lower bounded by the Lovász extension \( f_L : [0,1]^N \to \mathbb{R}_{\geq 0} \), which is given by
\[
f_L(y) := \int_{t=0}^{1} f(\{ u \in N : y(u) > t \}) dt \quad \forall y \in [0,1]^N.
\]
Hence, \( F(y) \geq f_L(y) \) for all \( y \in [0,1] \) (see, e.g., [Von13] for a formal proof of this well-known fact). The result now follows from
\[
F(x \lor 1_S) \geq f_L(x \lor 1_S) \\
= \int_{t=0}^{1} f(S \cup \{ u \in N : x(u) > t \}) dt \\
\geq \int_{t=p}^{1} f(S \cup \{ u \in N : x(u) > t \}) dt \\
= (1 - p)f(S),
\]
where the last equality uses that \( x \in [0,p]^N \).

By applying Lemma A.4 in our context, we get the following lower bound on \( F(a_{\text{all}} \lor 1_{\text{OPT}}) \) in terms of \( f(\text{OPT}) \).

**Corollary A.5.**
\[
(1 - p - \frac{1}{m}) \cdot f(\text{OPT}) \leq F(a_{\text{all}} \lor 1_{\text{OPT}})
\]
Proof. This is an immediate consequence of Lemma A.4 and the fact that $a_{\text{all}} \in [0, p + \frac{1}{m}]$, which holds due to the following. For any element $u \in \mathcal{N}$, Algorithm 5 does not continue to increase coordinates $a_i(u)$ if the sum of the $a_j(u)$ already surpasses $p$. Moreover, every increase of $u$ happens through an update of one of the $a_i$ vectors by increasing $a_i(u)$ by $\frac{e^{c_i}\epsilon}{m \cdot \partial_{u} F(a)} \leq \frac{1}{m}$, because $c^i \leq c(u) \leq \partial_{u} F(a)$ by choice of $i(u)$.

Finally, by combining the above obtained relations, and using our choices of the parameters $\alpha$, $c$, $p$, and $m$, we obtain the desired result. Note that the bound on the memory requirement of Algorithm 1 also holds for Algorithm 5, as it is unrelated to monotonicity of $f$ or to the minor differences between the two algorithms.

Proof of Theorem 3.1. Because $f(R) \geq F(s)$, it suffices to show that $F(s) \geq 0.1921 \cdot f(\text{OPT})$. The value of OPT and $F(s)$ can be related as follows:

$$
\left(1 - p - \frac{1}{m}\right) \cdot f(\text{OPT}) \leq F(a_{\text{all}} \lor 1_{\text{OPT}})
\leq F(a_{\text{all}}) + \sum_{u \in \text{OPT}} c^{\ell(u)+1}
\leq \frac{1}{1 - e^{-m} - \frac{\epsilon}{2c}} \cdot F(s) + (c - 1) \cdot \left(1 + \frac{\epsilon}{2c}\right) \cdot \frac{m}{1 - e^{-m}} \cdot F(s)
\leq \left(m \cdot (c - 1) \cdot \left(1 + \frac{\epsilon}{2c}\right) + 1\right) \cdot \frac{1}{1 - e^{-m} - \frac{\epsilon}{2c}} \cdot F(s),
$$

where the first inequality is due to Corollary A.5, the second one follows from Lemma A.3, and the third one is implied by Lemmas 3.8 and 3.11 (we recall that these results did not need monotonicity of $f$).

Regrouping terms in the above inequality and simplifying, we obtain the following:\footnote{The first steps of the derivation are analogous to the ones performed in the proof of Theorem 1.1. The only difference is the additional term of $(1 - p - \frac{1}{m})$.}

$$
F(s) \geq \frac{1 - e^{-\alpha}}{m \cdot (c - 1) \cdot \left(1 + \frac{\epsilon}{2c}\right) + 1} \cdot \left(1 - p - \frac{1}{m}\right) \cdot f(\text{OPT})
\geq \frac{1 - e^{-\alpha}}{\alpha \cdot \left(1 + \frac{\epsilon}{2c}\right) + \frac{1}{(c - 1) \cdot \left(1 + \frac{\epsilon}{2c}\right)}} \cdot \left(1 - p - \frac{1}{m}\right) \cdot f(\text{OPT})
\geq \left(1 - e^{-\alpha} \cdot \frac{1 - 2c}{\alpha + 1} \cdot \frac{\epsilon}{2c}\right) \cdot \left(1 - p - \frac{1}{m}\right) \cdot f(\text{OPT})
\geq \left(1 - e^{-\alpha} \cdot \frac{\alpha c}{\alpha + 1} \cdot \frac{\epsilon}{(\alpha + 1)^2}\right) \cdot f(\text{OPT})
\geq \left(1 - e^{-\alpha} \cdot \frac{\alpha}{(\alpha + 1)^2} - \epsilon\right) \cdot f(\text{OPT}),
$$

\[ (12) \]
where the different inequalities hold due to the following. The second inequality uses that \( c = \frac{m}{m - \alpha} \), which implies \( c^{-m} = (1 - \alpha/m)^m \leq e^{-\alpha} \) and \( m(c - 1) = ac \). The third inequality holds because \( c + \varepsilon/2 \geq 1 \) and \( \alpha \cdot (c + \varepsilon/2) + 1 \geq 1 \). The forth one follows from \( c = \frac{m}{m - \alpha} = (1 - \alpha/m)^{-1} \leq (1 - \varepsilon/3)^{-1} \leq 1 + \varepsilon/2 \) by using our definitions of \( c \) and \( m \). The fifth inequality uses that \( (1 + \varepsilon)^{-1} \geq 1 - \varepsilon \) and \( \frac{1 - e^{-\alpha n}}{\alpha + 1} \leq \frac{1}{\beta} \). The sixth inequality holds because \( \frac{1 - e^{-\alpha}}{\alpha + 1} \leq \frac{1}{\beta} \). The re-arrangement of \( p = \frac{1}{\alpha + 1} \). Finally, the last inequality follows from \( c \geq 1 \).

The claimed approximation factor of 1.921 is obtained by plugging in our value of \( \alpha = 1.9532 \) (for a small enough \( \varepsilon > 0 \)).

\[ \square \]

**B Better-than-2 approximation for two-player submodular maximization subject to a matroid constraint**

Let \( M = (\mathcal{N}, \mathcal{I}) \) be a matroid with rank function \( r: 2^{\mathcal{N}} \rightarrow \mathbb{Z}_{\geq 0} \), let \( k \triangleq r(\mathcal{N}) \) be the cardinality of a basis of \( M \), and let \( f: 2^{\mathcal{N}} \rightarrow \mathbb{R}_{\geq 0} \) be a non-negative submodular function. We are interested in maximizing \( f \) under the matroid constraint \( M \) in the following two-player setting. The ground set \( \mathcal{N} \) is partitioned into two sets \( \mathcal{N} = \mathcal{N}_A \cup \mathcal{N}_B \). The first player, Alice, obtains \( \mathcal{N}_A \) and can query \( f \) on any subset of \( \mathcal{N}_A \). She then selects a subset of elements \( Q \subseteq \mathcal{N}_A \) to be forwarded to the second player, Bob. After receiving the elements forwarded by Alice, Bob returns an independent set \( R \in \mathcal{I} \) with \( R \subseteq Q \cup \mathcal{N}_B \). The goal of Alice and Bob is to return an independent set \( R \) of submodular value \( f(R) \) as large as possible, while using only a small message size \( |Q| \). In particular, we are interested in the achievable trade-offs between the message size \( |Q| \) and the approximation guarantee \( f(R)/f(\text{OPT}) \) of the returned set \( R \), where \( \text{OPT} \in \text{argmax}\{f(I): I \in \mathcal{I}\} \) is an optimal solution to the (offline) problem.

In one extreme, Alice can send all of her elements \( Q = \mathcal{N}_A \) to Bob, who can then find (in exponential time) and return an independent set \( I \subseteq \mathcal{N} \) with \( f(I) = f(\text{OPT}) \). In the other extreme, one can observe that a message size of \( \Omega(k) \) is required to be able to obtain a constant approximation guarantee. In particular, a trivial \( 1/2 \)-approximation is obtained by Alice sending a maximizer \( Q \) of \( \max\{f(I): I \in \mathcal{I}, I \subseteq \mathcal{N}_A\} \) to Bob. To obtain a \( 1/2 \)-guarantee, it suffices for Bob to return the better of \( Q \) and a maximizer of \( \max\{f(I): I \in \mathcal{I}, I \subseteq \mathcal{N}_B\} \).

The result we show in this section is that one can beat the factor of \( 1/2 \) with a linear message size.

**Theorem 1.3.** There is a two player protocol for maximizing a non-negative monotone submodular function subject to a matroid constraint that sends \( O(k) \) elements (from Alice to Bob) and has an approximation guarantee of 0.505.

**B.1 The protocol**

We now describe the protocol claimed by **Theorem 1.3.** A key step of our protocol is the use of a continuous greedy algorithm for submodular maximization similar to the ones often used to approximately maximize a submodular function under various constraints. It is well-known that such continuous greedy procedures can be discretized to obtain efficient implementations with an arbitrarily small constant loss in the approximation guarantee compared to their continuous counterparts. However, a straightforward discretization would lead to a message size of \( O(k^2) \) in our case. We use a discretized version developed by Badanidiyuru and Vondrák [BV14], whose goal was to create a very fast algorithm to approximately maximize a non-negative monotone submodular function over a matroid constraint. Even though our focus is not on running time, the method developed in [BV14] has a key property that we can exploit in our context, namely that
the procedure returns a fractional point of linear support. The way we employ the algorithm, this linear support condition translates into a linear message size sent from Alice to Bob.

We start by describing the discretized version of the continuous greedy process that Alice will use in our protocol, which is a minor variation of an algorithm presented in [BV14]. The algorithm is shown below as Algorithm 6, where \( F: [0,1]^N \rightarrow \mathbb{R}_{\geq 0} \) is the multilinear extension of \( f \). Apart from the matroid \( M \), given by an independence oracle, the algorithm takes as input a subset \( S \subseteq N \) and a parameter \( h \in \mathbb{Z}_{\geq 1} \). The set \( S \) describes the elements of the ground set to which the algorithm is applied. It will later be chosen as \( S = N_A \) because Alice uses this algorithm in our protocol and can only apply it to the elements she received. The parameter \( h \) captures the number of steps in the discretization. Hence, the larger \( h \), the better the solution quality. In particular, the larger we choose \( h \), the closer the procedure is to being a continuous greedy algorithm. We will later see that \( h = 125 \) is large enough to obtain Theorem 1.3. Nevertheless, we leave \( h \) as a variable to highlight the dependence on \( h \) of the obtained approximation guarantee, which explains our choice of \( h = 125 \).

Algorithm 6, being a (discretized version of a) continuous greedy procedure, constructs a point \( x \) in the matroid polytope \( P_M := \{ y \in \mathbb{R}^N_{\geq 0} : y(S) \leq r(S) \forall S \subseteq N \} \) of \( M \), with support in \( N_A \), i.e., \( \text{supp}(x) := \{ e \in N : x_e > 0 \} \subseteq N_A \). The point \( x \) is constructed successively over a constant number of \( h \) steps, starting from the zero vector. At each step \( i \in [h] \), an independent set \( C_i \in I \cap 2^N_A \) is constructed, and the final point \( x \) is given by \( x = \frac{1}{h} \sum_{i=1}^h 1_{C_i} \). Since the sets \( C_i \) define the point, and correspond to elements that Alice will pass on to Bob in our protocol, we let the output of Algorithm 6 be the sets \( C_1, \ldots, C_h \) instead of returning the point \( x \).

As mentioned above, Algorithm 6 is a slightly modified version of an algorithm due to [BV14]. The most significant difference between the two is that the original algorithm of [BV14] used a thresholding technique on the weights to speed up the step of repeatedly finding an element \( u \) at the cost of an additional small error term. However, because running time is not our focus here, we replaced the thresholding by a conceptually simpler step to find \( u \). In any case, even our description without thresholding runs in polynomial time, and is therefore not the runtime bottleneck because we also perform steps that require exponential time.

Our protocol to obtain Theorem 1.3 is described in Algorithm 7.

### B.2 Analysis of Algorithm 7 (proof of Theorem 1.3)

To analyze Algorithm 7, we start by introducing some key terminology and notation used throughout our analysis in Appendix B.2.1. Then, before going into technical details, we present in Appendix B.2.2 a thorough outline of the analysis and provide intuition for the proof strategy that leads to the claimed performance guarantee. Finally, in Appendix B.2.3, we fill in all technical details to obtain a full formal proof of the performance guarantee of Algorithm 7.

---

10 Instead of constructing the sets \( C_i \) in the while loop of Algorithm 6 by successively adding elements \( u \), one could also choose \( C_i \in \text{argmax}(F(x + \frac{1}{h}1_C) : C \in I) \). Our analysis would still work out with this choice. However, this would only lead to a very minor simplification of the presentation while turning Algorithm 6 into an exponential-time procedure, whereas our version is polynomial-time if we replace the exact computation of \( F \) by a strongly efficiently computable estimate. This difference is irrelevant for showing Theorem 1.3, as we anyway design an exponential-time method. Nevertheless, for potential future work, and to better understand where exponential running time is currently needed, it may be helpful to know that sets \( C_1, \ldots, C_h \) with the desired properties can be computed efficiently.
Algorithm 6 Discretized version of continuous greedy for matroid constraint

**Input:** Matroid $M = (\mathcal{N}, \mathcal{I})$, set $S \subseteq \mathcal{N}$, and $h \in \mathbb{Z}_{\geq 1}$.

**Output:** Independent sets $C_1, \ldots, C_h \in I \cap 2^S$.

1: $x = 0 \in \mathbb{R}^\mathcal{N}$.
2: for $i = 1$ to $h$ do
3: $C_i \leftarrow \emptyset$.
4: while $r(C_i) < r(S)$ do
5: Find $u \in \text{argmax}\{F(x + \frac{1}{h} \cdot 1_{C_i \cup \{e\}}) : e \in S \text{ with } C_i \cup \{e\} \in \mathcal{I}\}$.
6: $C_i \leftarrow C_i \cup \{u\}$.
7: $x \leftarrow x + \frac{1}{h} \cdot 1_{C_i}$.
8: return $C_1, \ldots, C_h$.

Algorithm 7 Protocol leading to Theorem 1.3

**Alice’s protocol**

1: Determine $\text{OPT}(\mathcal{N}_A) \in \text{argmax}\{f(I) : I \in \mathcal{I}, I \subseteq \mathcal{N}_A\}$.
2: Apply Algorithm 6 to $M$ with $S = \mathcal{N}_A$ and $h = 125$ to obtain $C_1, \ldots, C_h \in \mathcal{I} \cap 2^{\mathcal{N}_A}$, and let $x := \frac{1}{h} \sum_{i=1}^h 1_{C_i}$.
3: Compute $W \in \text{argmax}\{F(x \lor 1_H) : H \in \mathcal{I}, H \subseteq \mathcal{N}_A\}$.
4: Send the elements $Q := \text{OPT}(\mathcal{N}_A) \cup W \cup \bigcup_{i=1}^h C_i$ to Bob.

**Bob’s protocol**

1: return $R \in \text{argmax}\{f(S) : S \in \mathcal{I}, S \subseteq Q \cup \mathcal{N}_B\}$.

B.2.1 Terminology and notation

To show that the set $R$ returned by Algorithm 7 is a 0.505-approximation, we define

$$\Delta := \frac{f(R)}{f(\text{OPT})} - \frac{1}{2} \quad (13)$$

and show that $\Delta \geq 0.005$.

To show that there exists an independent set in $Q \cup \mathcal{N}_B$ with large value, we consider different candidate solutions that Bob can return in our analysis. To this end, it is often useful to consider candidate solutions that use part of an optimal solution. As previously mentioned, let $\text{OPT} \in \text{argmax}\{f(I) : I \in \mathcal{I}\}$ be an arbitrary fixed optimal solution, and we define

$$\text{OPT}_A := \text{OPT} \cap \mathcal{N}_A, \text{ and}$$

$$\text{OPT}_B := \text{OPT} \cap \mathcal{N}_B.$$

Note that elements of $\text{OPT}_B$ are among those that Bob can return, whereas the elements of $\text{OPT}_A$ are, in general, not part of the set $Q \cup \mathcal{N}_B$ of elements from which Bob selects a set $R$ to return.

To create candidate solutions by mixing elements from independent sets we encounter in our protocol with elements of $\text{OPT}$—in particular, $\text{OPT}_B$—we heavily rely on the following well-known exchange theorem of matroids.

**Theorem B.1** ([Bry73; Gre73; Woo74]). Let $M = (\mathcal{N}, \mathcal{I})$ be a matroid, $I, J \in \mathcal{I}$, and let $I_1, I_2$ be a partition of $I$. Then there is a partition $J_1, J_2$ of $J$ such that the following holds simultaneously:

(i) $I_1 \cup J_2 \in \mathcal{I}$, and

(ii) $I_1 \cup J_2 \in \mathcal{I}$, and

...
We start by using the above exchange theorem to split the vector $x = \frac{1}{h} \sum_{i=1}^{h} 1_{C_i}$, which we get through the discretized continuous greedy algorithm, into two parts $x_A$ and $x_B$ as follows. Applying Theorem B.1 to $\text{OPT}$ with partition $\text{OPT}_A$, $\text{OPT}_B$, and the set $C_i$, we obtain that for every $i \in [h]$ there is a partition of $C_i$ into $C_{i,A}$, $C_{i,B}$ such that (i) $\text{OPT}_A \cup C_{i,B} \in I$, and (ii) $\text{OPT}_B \cup C_{i,A} \in I$. We split the vector $x$ into $x = x_A + x_B$ as follows

$$x_A := \frac{1}{h} \sum_{i=1}^{h} 1_{C_{i,A}} \text{, and}$$

$$x_B := \frac{1}{h} \sum_{i=1}^{h} 1_{C_{i,B}} \text{.}$$

Moreover, for $i \in [h]$, we denote by

$$x^i_B := \frac{1}{h} \sum_{j=1}^{i} 1_{C_{j,B}} \text{, and}$$

$$x^i_A := \frac{1}{h} \sum_{j=1}^{i} 1_{C_{j,A}} \text{,}$$

a decomposition $x^i = x^i_A + x^i_B$ of the intermediate solution $x^i = \frac{1}{h} \sum_{j=1}^{i} 1_{C_j}$ obtained in the discretized continuous greedy algorithm after $i$ iterations. By convention, we set $x^0_A$, $x^0_B$, and $x^0$ to be the zero vector in $\mathbb{R}^N$, which corresponds to the fact that the discretized continuous greedy algorithm starts with the zero vector. Note that $x^h = x$.

Instead of showing that there are good candidate independent sets within the elements $Q \cup N_B$ among which Bob selects a solution, it is often more convenient to show that there is a fractional point $y$ with $\text{supp}(y) \subseteq Q \cup N_B$ in the matroid polytope $y \in P_M$ of high multilinear extension value $F(y)$. This approach is sound because it is well-known that the problem of maximizing the multilinear extension on a matroid has no integrality gap (see [C˘ al+11]).

**Lemma B.2.** Let $y \in P_M$. Then there is a set $I \subseteq \mathcal{I}$ with $I \subseteq \text{supp}(y)$ such that $f(I) \geq F(y)$.

Consequently, Bob returns a set $R$ with $f(R) \geq F(y)$ for any point $y \in P_M$ with $\text{supp}(y) \subseteq Q \cup N_B$.

**B.2.2 Outline of analysis and intuition**

To build up intuition, we suggest to think of the special case where $f: 2^\mathcal{N} \rightarrow \mathbb{R}_{\geq 0}$ is a coverage function. More precisely, in this case there is a finite universe $U$ and, to every $e \in \mathcal{N}$, a set $U_e \subseteq U$ is associated. The value of the function $f$ evaluated at some set $S \subseteq \mathcal{N}$ is then given by $f(S) = |\cup_{e \in S} U_e|$. For simplicity, let us use the shorthand notation $U_S := \cup_{e \in S} U_e$ for any $S \subseteq \mathcal{N}$. Hence, $f(S) = |U_S|$. Moreover, we suggest to think of the case $f(\mathcal{N}) = f(\text{OPT})$. In words, $U_{\text{OPT}} = U$, i.e., OPT covers the whole universe. These suggested assumptions are only to simplify the understanding of the proof plan and main ideas, and are never used in any of our statements or proof.

We now argue why there is no instance for which Algorithm 7 has an approximation factor arbitrarily close to $1/2$. Hence, equivalently, $\Delta$ must be bounded away from zero. First, to obtain
Lemma B.4. 

Let us use again the special case of a coverage function to better exemplify things. The elements \( A \) covers fractionally by \( x \) normally expect to lose a factor of \( \frac{1}{2} \). \( A \) and \( B \) do not contain an independent set of high value. First, one can show that the three vectors \( x_A \cap 1_{OPT_B}, x_B \cap 1_{OPT_A}, \) and \( x \) are all in the matroid polytope \( P_M \). By Lemma B.2, this allows us to obtain the following. (Full proof of this and further statements in this subsection are provided in Appendix B.2.3.)

Observation B.3.

(i) \( F(x_A \cup 1_{OPT_B}) \leq (\frac{1}{2} + \Delta) \cdot f(OPT) \).
(ii) \( F(x_B \cup 1_{OPT_A}) \leq (\frac{1}{2} + \Delta) \cdot f(OPT) \).
(iii) \( F(x) \leq (\frac{1}{2} + \Delta) \cdot f(OPT) \).

Indeed, Property (i) and Property (iii) are immediate consequences of Lemma B.2 and the fact that the support of both \( x_A \cap 1_{OPT_B} \) and \( x \) are contained in the elements \( Q \cap N_B \) among which Bob chooses the independent set of largest submodular value. Moreover, Property (ii) also follows from Lemma B.2 and the fact that \( \text{supp}(x_B \cup OPT_A) \subseteq N_A \), which implies that \( f(OPT(N_A)) \geq F(x_B \cup OPT_A) \). Again, \( OPT(N_A) \) is a set that can be returned by Bob because \( OPT(N_A) \subseteq Q \).

Hence, interpreting this in our example of a coverage function, we have that, in a bad instance (one with very small \( \Delta \)), the sets \( C_{i,A} \) used to build \( x_A \) almost only cover elements of \( U_{OPT_B} \). Analogously, the sets \( C_{i,B} \) to build \( x_B \) almost only cover elements of \( U_{OPT_A} \). This has several consequences. In particular, what \( x_A \) or \( x_B \) “covers” has almost no overlap. Hence, \( F(x) \) is close to \( F(x_A) + F(x_B) \).

Another, more important consequence is that the point \( x \) we get through the discretized continuous greedy procedure leads to a high value \( F(x) \). More precisely, recall that the discretized continuous greedy algorithm is run by Alice, and is thus only applied to the elements in \( N_A \). It could be that \( f(N_A) \) is very close to \( \frac{1}{2} f(OPT) \). Hence, by using a greedy approach, one would normally expect to lose a factor of \( 1 - \frac{1}{e} \), which would lead to a lower bound on \( F(x) \) of only about \( \frac{1}{2}(1 - \frac{1}{e}) f(OPT) \). However, we can show that \( F(x) \) is at least about \( \frac{1}{2} f(OPT) \) for small \( \Delta \). The reason for this is the following. Consider an arbitrary iteration \( i \in [h] \) of Algorithm 6, and let us use again the special case of a coverage function to better exemplify things. The elements in \( OPT_A \) where candidates to build the sets \( C_{i,A} \) that the discretized continuous greedy chose. As discussed, \( OPT_A \) covers \( U_{OPT_A} \), which is half of the universe. Part of \( U_{OPT_A} \) may already be fractionally covered by \( x_{i-1} = x_{i-1}^A + x_{i-1}^B \). However, as mentioned, we can assume (for very small \( \Delta \)) that only \( x_{i-1}^B \) (fractionally) covers elements of \( U_{OPT_A} \). Hence, the marginal contribution of \( OPT_A \) at step \( i \) of Algorithm 6 is about \( (\frac{1}{2} - F(x_{i-1})) f(OPT) \). Formalizing these arguments leads to the following.

Lemma B.4.

\[
F(x_A) + \frac{1}{h} \sum_{i=1}^{h} F(x_B^i) \geq \left( \frac{1}{2} - \Delta \right) f(OPT) .
\]
Note that Lemma B.4 indeed implies that $F(x)$ has a value that cannot be far below $\frac{1}{2}f(\text{OPT})$ for small $\Delta$ because of the following. By monotonicity of $f$, which implies monotonicity of $F$, we have $F(x_B^i) \leq F(x_B)$ for $i \in [h]$. Together with Lemma B.4 this implies $F(x_A) + F(x_B) \geq (\frac{1}{2} - \Delta) f(\text{OPT})$. Finally, we recall that $F(x)$ is close to $F(x_A) + F(x_B)$ because $x_A$ and $x_B$ cover different parts of the universe.

Lemma B.4 has another interesting consequence. Namely, if we have a bad instance where $F(x_B)$ is bounded away from zero, then $F(x_B)$ must be close to $\frac{1}{h} \sum_{i=1}^{h} F(x_B^i)$. For otherwise, replacing the term $\frac{1}{h} \sum_{i=1}^{h} F(x_B^i)$ by $F(x_B)$ in Lemma B.4 would lead to a significant increase of the left-hand side in Lemma B.4, thus showing that $F(x_A) + F(x_B)$ (which is close to $F(x)$) has value strictly above $\frac{1}{2}$. Formally, this relation is captured by the following statement.

**Lemma B.5.**

\[
\frac{1}{h} \sum_{i=1}^{h} F(x_B^i) \geq F(x_B) - 4\Delta \cdot f(\text{OPT}) .
\]

Lemma B.5 implies that if $F(x_B)$ is large, then the sequence $F(x_B^1), F(x_B^2), \ldots, F(x_B^h) = F(x_B)$ increases quickly early on. In other words, even for a relatively small index $i^* \in [h]$, we have that $F(x_B^{i^*})$ is close to $F(x_B)$. In case $F(x_B)$ is large, we consider such an index $i^*$ and the point $z := x_B^{i^*} + (1 - \frac{i^*}{h})1_{\text{OPT}}$, which lies in $P_M$ and whose support is contained in the elements among which Bob chooses a set to return. Because $x_B$, and thus also $x_B^{i^*}$, almost exclusively covers elements in $U_{\text{OPT}_A}$, and $\text{OPT}_B$ is covered by the elements in $U_{\text{OPT}_B}$, we have that $F(z)$ is very close to $F(x_B^{i^*}) + (1 - \frac{i^*}{h})f(\text{OPT})$. Hence, if $F(x_B)$ is large, then so is $F(x_B^{i^*})$, and $F(z)$ will be bounded away from $\frac{1}{2}f(\text{OPT})$ as desired.

Hence, it remains to deal with the case where $F(x_B)$ is small. This is where we rely on the set $W$ that Alice passes to Bob, which we have not used so far. We again crucially exploit that $x$ was constructed with a (discretized version of a) continuous greedy algorithm. Because $F(x_B)$ is small, we have that $F(x_A)$ must be close to $\frac{1}{2}f(\text{OPT})$ due to Lemma B.4. Hence, in the context of our example based on $f$ being a coverage function, this implies that $x_A$ (fractionally) covers almost all of $U_{\text{OPT}_B}$ whereas $U_{\text{OPT}_A}$ remains nearly uncovered. This motivates our choice of $W$ as a maximizer of

\[
\max \{ F(x \vee 1_H) : H \in \mathcal{I}, H \subseteq \mathcal{N}_A \} .
\]

Indeed, because it remains to cover $U_{\text{OPT}_A}$, the set $W = \text{OPT}_A$ would be a near maximizer of Eq. (14). Of course, $W = \text{OPT}_A$ would be ideal, in which case Bob could even return OPT. In general, $W \subseteq \mathcal{N}_A$ will be some set that covers nearly all of $U_{\text{OPT}_A}$ but may not overlap with OPT$_A$. We use Theorem B.1 to partition $W$ into $W_A$ and $W_B$ such that $W_A \cup \text{OPT}_B \in \mathcal{I}$ and $W_B \cup \text{OPT}_A \in \mathcal{I}$. We want to show that $W_A$ covers a significant fraction of $U_{\text{OPT}_A}$, in which case we are done because Bob can return $W_A \cup \text{OPT}_B$. Hence, the only bad case is if $U_{\text{OPT}_A}$ is almost entirely covered by $W_B$. We argue that, in this case, the discretized version of the continuous greedy algorithm would already have returned a point $x$ with $F(x)$ significantly larger than $\frac{1}{2}f(\text{OPT})$. The reason is that $\text{OPT}_A \cup W_B$ was a candidate set for each of the sets $C_i$ considered in Algorithm 6. This set nearly “double-covers” the almost uncovered part $U_{\text{OPT}_A}$ of the universe. Now we exploit a crucial property of the (discretized) continuous greedy algorithm, namely that its local improvement is given by the sum of the coverages of the elements that get added. In other words, the double coverage of part of the universe of value $\frac{1}{2}f(\text{OPT})$ also leads to a double contribution to the local improvement, i.e., the gradient would be about $f(\text{OPT})$. This would have been true for all steps of the discretized continuous greedy algorithm, leading to $F(x)$ being close to $f(\text{OPT})$ if both $F(x_B)$ and $\Delta$ are small. However, this shows that it is impossible for both $F(x_B)$ and $\Delta$ to be very small when $x$ covers only about half of the universe $U$. Hence, this case is impossible, showing that, if
both $F(x_B)$ and $\Delta$ are small, then $f(W_A \cup \text{OPT}_B)$ will be significantly larger than $\frac{1}{2} f(\text{OPT})$. The statement below quantifies this relationship.

**Lemma B.6.** $f(W_A \cup \text{OPT}_B) \geq (1 - 4\Delta)f(\text{OPT}) - 3F(x_B)$.

Using that $W_A \cup \text{OPT}_B$ is a set that Bob can return, and must therefore have a value of at most $(\frac{1}{2} + \Delta)f(\text{OPT})$, we can quantify the fact that not both $F(x_B)$ and $\Delta$ can be very small simultaneously as follows.

**Lemma B.7.** $F(x_B) \geq (\frac{1}{6} - \frac{5}{3}\Delta) \cdot f(\text{OPT})$.

Hence, we finally obtain that it is impossible for both $F(x_B)$ and $\Delta$ to be very small, which brings us back to the case of large $F(x_B)$, for which we have already explained why our protocol obtains an approximation factor that is by some constant larger than $\frac{1}{2}$. Putting everything together leads to our claimed lower bound on $\Delta$.

**Theorem B.8.** $\Delta \geq 0.005$.

### B.2.3 Formal proofs

We now fill in the details of the previously outlined analysis plan and provide proofs of the mentioned statements. We start by highlighting some basic properties of the multilinear extension $F$.

**Lemma B.9** (see [Cal+11]). Let $f : 2^N \to \mathbb{R}_{\geq 0}$ be a submodular function and $F : [0,1]^N \to \mathbb{R}_{\geq 0}$ its multilinear extension. Then the following holds.

(i) For any $x \in [0,1]^N$, all entries of the Hessian $\nabla^2 F(x)$ are non-positive. This implies in particular that $F$ is concave along non-negative directions; hence

$$F(y + \lambda z) \geq (1 - \lambda)F(y) + \lambda F(y + z) \quad \forall y \in [0,1]^N, z \in \mathbb{R}_{\geq 0}^N, \lambda \in [0,1] \text{ with } y + z \in [0,1]^N.$$  

(ii) If $f$ is monotone then so is $F$, i.e.,

$$F(y) \leq F(z) \quad y, z \in [0,1]^N \text{ with } y \leq z.$$  

The above properties allow for generalizing the diminishing returns property and further properties of submodular functions to the multilinear extensions, as stated in Lemma B.10 below. The properties listed in Lemma B.10 readily follow from prior work. (For example, Property (ii) and Property (i) are special cases of [CVZ14, Lemma 3.11], and Property (iii) is an immediate consequence of Property (ii) and the monotonicity of $F$.) Nevertheless, we provide a proof of Lemma B.10 for completeness. For two vectors $x, y \in \mathbb{R}^N$, we use the notation $x \lor y$ for the component-wise maximum of $x$ and $y$, and $x \land y$ for their component-wise minimum.

**Lemma B.10.** Let $f : 2^N \to \mathbb{R}_{\geq 0}$ be a submodular function and $F : [0,1]^N \to \mathbb{R}_{\geq 0}$ its multilinear extension. Then the following holds.

(i) $F(w + y) - F(y) \geq F(w + z) - F(z) \quad \forall w, y, z \in [0,1]^N \text{ with } y \leq z \text{ and } w + z \in [0,1]^N.$

(ii) $F(w) + F(y) \geq F(w \lor y) + F(w \land y) \quad \forall w, y \in [0,1]^N.$

(iii) Moreover, if $f$ is monotone, then

$$F(w \lor y) - F(y) \geq F(w \lor z) - F(z) \quad \forall w, y, z \in [0,1]^N \text{ with } y \leq z.$$
Proof. We start by proving Property (i), which follows readily from the non-positivity of the Hessian of $F$:

$$F(w + z) - F(z) - (F(w + y) - F(y)) = \int_0^1 w^T (\nabla F(z + tw) - \nabla F(y + tw)) \, dt$$

$$= \int_0^1 \int_0^1 w^T \nabla^2 F(y + tw + \tau(z - y))(z - y) \, d\tau \, dt$$

$$\leq 0,$$

where the inequality follows from $w \geq 0$, $z - y \geq 0$, and $\nabla^2 F(y + tw + \tau(z - y)) \leq 0$.

Property (ii) follows from Property (i) because

$$F(w \lor y) - F(w) = F((w \lor y) - (w \land y)) - F(w)$$

$$\geq F((w \lor y) - ((w \land y) - w)) - F(w \land y)$$

$$= F(y) - F(w \land y),$$

where the second inequality is due to Property (i) and the last one uses $(w \land y) + (w \lor y) = w + y$.

Finally, Property (iii) can be derived from Property (ii) and the monotonicity of $F$ as follows.

$$F(w \lor y) + F(z) \geq F((w \lor y) \lor z) + F((w \lor y) \land z)$$

$$\geq F(w \lor z) + F(y),$$

where the first inequality is due to Property (ii), and the second one follows from $(w \lor y) \lor z = w \lor z$, which holds because $z \geq y$, and $F((w \lor y) \land z) \geq F(y)$, which holds because $(w \lor y) \land z \geq y$ (again, using $z \geq y$) and $F$ is monotone.

Using these properties of the multilinear extension, we are now ready to provide a formal proof of Observation B.3.

Observation B.3.

(i) $F(x_A \lor 1_{\text{OPT}_B}) \leq (\frac{1}{2} + \Delta) \cdot f(\text{OPT}).$

(ii) $F(x_B \lor 1_{\text{OPT}_A}) \leq (\frac{1}{2} + \Delta) \cdot f(\text{OPT}).$

(iii) $F(x) \leq (\frac{1}{2} + \Delta) \cdot f(\text{OPT}).$

Proof. Property (i): We first observe that $x_A \lor 1_{\text{OPT}_B} \in P_M$ because

$$x_A \lor 1_{\text{OPT}_B} = \left(\frac{1}{h} \sum_{i=1}^h 1_{C_i,A}\right) \lor 1_{\text{OPT}_B}$$

$$= \frac{1}{h} \sum_{i=1}^h (1_{C_i,A} \lor 1_{\text{OPT}_B})$$

$$= \frac{1}{h} \sum_{i=1}^h 1_{C_i,A \cup \text{OPT}_B},$$

which shows that $x_A \lor 1_{\text{OPT}_B}$ is a convex combination of the points $1_{C_i,A \cup \text{OPT}_B}$ for $i \in [h]$; moreover, $1_{C_i,A \cup \text{OPT}_B} \in P_M$, because $C_i \cup \text{OPT}_B \in I$. Furthermore, $\text{supp}(x_A \lor 1_{\text{OPT}_B}) \subseteq Q \cup \mathcal{N}_B$, and $Q \cup \mathcal{N}_B$ are the elements among which Bob chooses the largest-valued independent set $R$ to return. The statement now follows by Lemma B.2, which implies $f(R) \geq F(x_A \lor 1_{\text{OPT}_B})$, and the definition of $\Delta$ (see Eq. (13)).
Property (ii) The proof of this statement is identical to the proof of Property (i) with the roles of \(A\) and \(B\) exchanged.

Property (iii): By construction, \(x = \frac{1}{h} \sum_{i=1}^{h} 1_{C_i}\) is a point in the matroid polytope \(P_M\), because \(C_i \in \mathcal{I}\) for \(i \in [h]\), and its support is contained in \(Q\). Hence, as before, the statement follows by Lemma B.2.

As mentioned, Observation B.3 can be interpreted, in the context of \(f\) being a coverage function, as stating that \(x_A\) heavily overlaps with \(\text{OPT}_B\) and, analogously, \(x_B\) heavily overlaps with \(\text{OPT}_A\), in terms of what parts of the universe they cover. Another way to phrase this is that \(x_A\) has little overlap with \(\text{OPT}_A\) and, analogously, \(x_B\) has little overlap with \(\text{OPT}_B\). These relations, stated formally below, are in some circumstances more convenient.

Lemma B.11.

(i) \(F(x_A \lor 1_{\text{OPT}_A}) \geq \left(\frac{1}{2} - \Delta\right) \cdot f(\text{OPT}) + F(x_A)\).

(ii) \(F(x_B \lor 1_{\text{OPT}_B}) \geq \left(\frac{1}{2} - \Delta\right) \cdot f(\text{OPT}) + F(x_B)\).

Proof. We only prove the first statement. The proof of the second one is identical with the roles of \(A\) and \(B\) exchanged.

\[
F(x_A \lor 1_{\text{OPT}_A}) = F(x_A \lor 1_{\text{OPT}_A}) - F(x_A) + F(x_A) \\
\geq F(x_A \lor 1_{\text{OPT}_A \cup \text{OPT}_B}) - F(x_A \lor 1_{\text{OPT}_B}) + F(x_A) \\
\geq f(\text{OPT}) - \left(\frac{1}{2} + \Delta\right) f(\text{OPT}) + F(x_A) \\
= \left(\frac{1}{2} - \Delta\right) f(\text{OPT}) + F(x_A),
\]

where the first inequality follows from Property (iii) of Lemma B.10 and the second one follows from Lemma B.11 and \(F(x_A \lor 1_{\text{OPT}_A \cup \text{OPT}_B}) \geq F(1_{\text{OPT}}) = f(\text{OPT})\), which holds due to monotonicity of \(F\).

As previously discussed, \(x_A\) and \(x_B\) have little overlap, which implies that \(F(x_A) + F(x_B)\) is close to \(F(x)\). In turn, \(F(x)\) is no larger than \(\left(\frac{1}{2} + \Delta\right) f(\text{OPT})\) because of Lemma B.2. Combining and formalizing these statements leads to the following result.

Lemma B.12. \(F(x_A) + F(x_B) \leq \left(\frac{1}{2} + 3\Delta\right) \cdot f(\text{OPT})\).

Proof. We have

\[
\left(\frac{1}{2} + \Delta\right) \cdot f(\text{OPT}) \geq F(x) \\
= F(x_A) + F(x) - F(x_A) \\
\geq F(x_A) + F(x \lor 1_{\text{OPT}_B}) - F(x_A \lor 1_{\text{OPT}_B}) \\
\geq F(x_A) + F(x_B \lor 1_{\text{OPT}_B}) - F(x_A \lor 1_{\text{OPT}_B}) \\
\geq F(x_A) + F(x_B) - 2\Delta \cdot f(\text{OPT}),
\]

where the first inequality follows by Property (iii) of Lemma B.10, the second one by monotonicity of \(F\), and the last one by Property (ii) of Lemma B.11 and by Property (i) of Observation B.3. The claimed result follows by reordering terms.

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We now start to analyze properties of the discretized continuous greedy procedure, i.e., Algorithm 6. We start by deriving guarantees on the improvement happening in a single iteration. More precisely, the statement below shows that the improvement at some iteration \( i \) is at least as good as the best improvement that could have been achieved in the next iteration. This idea and the corresponding proof technique go back to Badanidiyuru and Vondrák [BV14]. We provide a full proof in our context for convenience.

**Lemma B.13.** Let \( i \in [h] \) and let \( X \subseteq C_i, Y \subseteq N_A \) satisfying

(a) \( |X| = |Y| \), and

(b) \( (C_i \setminus X) \cup Y \in I \).

Then

\[
F \left( x^{i-1} + \frac{1}{h} 1_X \right) - F \left( x^{i-1} \right) \geq \frac{1}{h} \sum_{y \in Y} \left[ F \left( \left( x^i + \frac{1}{h} 1_y \right) \land 1_{N} \right) - F \left( x^i \right) \right],
\]

\[
\geq \frac{1}{h} \sum_{y \in Y} \left[ F \left( x^i \lor 1_y \right) - F \left( x^i \right) \right].
\]

**Proof.** We start by proving the first inequality. Let \( C_i = \{u_1, \ldots, u_\rho\} \), where the numbering corresponds to the order in which elements where added to \( C_i \) in the construction of \( C_i \) described by Algorithm 6. Let \( X = \{u_{j_1}, u_{j_2}, \ldots, u_{j_k}\} \) with \( j_1, \ldots, j_k \in [\rho] \) and the numbering of the indices is chosen such that \( j_1 < j_2 < \ldots < j_k \). Because \( |X| = |Y| \) and both \( C_i \in I \) and \( (C_i \setminus X) \cup Y \in I \), classical results in matroid theory guarantee that there is a bijection (often called Brualdi bijection) \( \phi: X \rightarrow Y \) such that

\[
(C_i \setminus \{u\}) \cup \{\phi(u)\} \in I \quad \forall u \in X.
\]

Moreover, we define

\[
\lambda_\ell := \min \left\{ \frac{1}{h}, 1 - x^i(\phi(u_{j_\ell})) \right\} \quad \forall \ell \in [k].
\]

The first inequality now follows by the relations below, whose individual steps will be further
explained in the following.

\[
F\left(x^{i-1} + \frac{1}{h} \mathbf{1}_X\right) - F\left(x^{i-1}\right) \\
= \sum_{\ell=1}^{k} \left[ F\left(x^{i-1} + \frac{1}{h} \mathbf{1}_{\{u_{j_1}, u_{j_2}, \ldots, u_{j_\ell}\}}\right) - F\left(x^{i-1} + \frac{1}{h} \mathbf{1}_{\{u_{j_1}, u_{j_2}, \ldots, u_{j_{\ell-1}}\}}\right) \right] \\
\geq \sum_{\ell=1}^{k} \left[ F\left(x^{i-1} + \frac{1}{h} \mathbf{1}_{\{u_{1}, \ldots, u_{j_\ell}\}}\right) - F\left(x^{i-1} + \frac{1}{h} \mathbf{1}_{\{u_{1}, \ldots, u_{j_{\ell-1}}\}}\right) \right] \\
\geq \sum_{\ell=1}^{k} \left[ F\left(x^{i-1} + \frac{1}{h} \mathbf{1}_{\{u_{1}, \ldots, u_{j_{\ell-1}}\}} + \frac{1}{h} \mathbf{1}_{\phi(\ell)}\right) - F\left(x^{i-1} + \frac{1}{h} \mathbf{1}_{\{u_{1}, \ldots, u_{j_{\ell-1}}\}}\right) \right] \\
\geq \sum_{\ell=1}^{k} \left[ F\left(x^{i} + \lambda_{\ell} \cdot \mathbf{1}_{\phi(\ell)}\right) - F(x^i) \right] \\
= \sum_{\ell=1}^{k} \left[ F\left(\left(x^{i} + \frac{1}{h} \cdot \mathbf{1}_{\phi(\ell)}\right) \land \mathbf{1}_{\mathcal{N}}\right) - F(x^i) \right] \\
= \sum_{y \in Y} \left[ F\left(\left(x^{i} + \frac{1}{h} \cdot \mathbf{1}_{\phi(\ell)}\right) \land \mathbf{1}_{\mathcal{N}}\right) - F(x^i) \right].
\]

The first inequality holds due to Property (i) of Lemma B.10. The second one is a consequence of how the set \( C_i = \{u_1, \ldots, u_\rho\} \) was constructed. More precisely, for any \( \ell \in [k] \), the element \( u_{j_\ell} \) was chosen to be the element \( e \) maximizing the expression \( F(x^{i-1} + \frac{1}{h} \mathbf{1}_{\{u_{1}, \ldots, u_{j_{\ell-1}}\}}) - F(x^{i-1} + \frac{1}{h} \mathbf{1}_{\{u_{1}, \ldots, u_{j_{\ell-1}}\}}) \) among all elements \( e \in \mathcal{N}_A \) with \( \{u_1, \ldots, u_{j_{\ell-1}}\} \cup \{e\} \in \mathcal{I} \). Moreover, \( \{u_1, \ldots, u_{j_{\ell-1}}\} \cup \{\phi(\ell)\} \subseteq (C_i \setminus \{u_{j_\ell}\}) \cup \{\phi(\ell)\} \in \mathcal{I} \); hence, \( \phi(\ell) \) was a candidate element considered when \( u_{j_\ell} \) was picked in the construction of \( C_i \). The third inequality follows from \( \lambda_{\ell} \leq \frac{1}{h} \) and the monotonicity of \( F \). The fourth inequality is implied by Property (i) of Lemma B.10. The penultimate equality is due to the definition of \( \lambda_{\ell} \). Finally, the last equality holds because \( \phi \) is a bijection from \( X \) to \( Y \).

We now show the second inequality of the statement, which follows by properties of \( F \) and holds even for any set \( Y \subseteq \mathcal{N} \). For every \( y \in Y \) we have

\[
\frac{1}{h} F\left(x^i \lor \mathbf{1}_y\right) + \left(1 - \frac{1}{h}\right) F\left(x^i\right) \leq F\left(x^i + \frac{1}{h} \cdot ((x^i \lor \mathbf{1}_y) - x^i)\right) \\
= F\left(x^i + \frac{1}{h} \cdot \left(1 - x^i(y)\right) \cdot \mathbf{1}_y\right) \\
\leq F\left(\left(x^i + \frac{1}{h} \cdot \mathbf{1}_y\right) \land \mathbf{1}_{\mathcal{N}}\right).
\]

where the first inequality holds due to concavity of \( F \) along non-negative directions (Lemma B.9), and the second inequality follows by monotonicity of \( F \). The above inequality implies

\[
\frac{1}{h} \left(F\left(x^i \lor \mathbf{1}_y\right) - F\left(x^i\right)\right) \leq F\left(\left(x^i + \frac{1}{h} \cdot \mathbf{1}_y\right) \land \mathbf{1}_{\mathcal{N}}\right) - F\left(x^i\right) \quad \forall y \in Y,
\]

which shows the second inequality of the statement, as desired. \( \square \)
The statement below, which, as we show, is a consequence of the first inequality of Lemma B.13, shows that the marginal gains of \( F(x^i) \) over the iterations of Algorithm 6 are non-increasing.

**Lemma B.14.**

\[
F(x^{i+1}) - F(x^i) \leq F(x^i) - F(x^{i-1}) \quad \forall i \in [h - 1].
\]

**Proof.** We invoke Lemma B.13 with \( i < h \). Note that by construction of the sets \( C_i \) we have \( |C_i| = r(N_A) = |C_{i+1}| \). Moreover, \( (C_i \setminus X) \cup Y = C_{i+1} \in \mathcal{I} \). Hence, the conditions of Lemma B.13 are satisfied and we have

\[
F(x^i) - F(x^{i-1}) = F(x^i - \frac{1}{h} 1_{C_i}) - F(x^{i-1}) \\
\geq \sum_{y \in C_{i+1}} \left[ F\left((x^i + \frac{1}{h} 1_y) \land 1_N\right) - F(x^i)\right] \\
= \sum_{y \in C_{i+1}} \left[F\left(x^i + \frac{1}{h} 1_y\right) - F(x^i)\right],
\]

where the inequality follows from Lemma B.13 and the second equality from the fact that \( x^i \in \frac{i}{h} P_M \) and \( i < h \), which implies that each entry of \( x^i \) is at most \( 1 - 1/h \). Moreover, for an arbitrary numbering \( C_{i+1} = \{y_1, \ldots, y_p\} \) of the elements in \( C_{i+1} \), we have

\[
\sum_{y \in C_{i+1}} \left[F\left(x^i + \frac{1}{h} 1_y\right) - F(x^i)\right] = \sum_{j=1}^p \left[F\left(x^i + \frac{1}{h} 1_{y_j}\right) - F(x^i)\right] \\
\geq \sum_{j=1}^p \left[F\left(x^i + \frac{1}{h} 1_{\{y_1, \ldots, y_j\}}\right) - F\left(x^i + \frac{1}{h} 1_{\{y_1, \ldots, y_{j-1}\}}\right)\right] \\
= F(x^{i+1}) - F(x^i),
\]

where the inequality follows by Property (i) of Lemma B.10. The desired result is obtained by combining the above inequality with Eq. (15). \( \square \)

We are now ready to prove Lemma B.4.

**Lemma B.4.**

\[
F(x_A) + \frac{1}{h} \sum_{i=1}^h F(x^i_B) \geq \left(\frac{1}{2} - \Delta\right) f(OPT).
\]

**Proof.** For each \( i \in [h] \), we have

\[
F\left(x^{i-1} + \frac{1}{h} 1_{C_{i,A}}\right) - F(x^{i-1}) \geq \frac{1}{h} \left(F\left(x^i \lor 1_{\text{OPT}_A}\right) - F(x^i)\right) \\
\geq \frac{1}{h} \left(F\left(x^i \lor 1_{\text{OPT}_A}\right) - F(x^i_A) - F(x^i_B)\right) \\
\geq \frac{1}{h} \left(F\left(x^i_A \lor 1_{\text{OPT}_A}\right) - F(x^i_A) - F(x^i_B)\right) \\
\geq \frac{1}{h} \left(F\left(x^i_A \lor 1_{\text{OPT}_B}\right) - F(x^i_A) - F(x^i_B)\right) \\
\geq \frac{1}{h} \left(\frac{1}{2} - \Delta\right) f(OPT) - F(x^i_B)\right),
\]

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where the first inequality follows from Lemma B.13 with \( X = C_{i,A} \) and \( Y = \text{OPT}_A \), the second one from Property (i) of Lemma B.10 with \( w = x_i^A, \ z = x_i^B \), and \( y = 0 \) is the zero vector, the third one from monotonicity of \( F \), the forth one is a consequence of Property (iii) of Lemma B.10, and the last one follows from Property (i) of Lemma B.11.

By summing up both sides of the above inequality over all \( i \in [h] \) and reordering terms, we obtain the desired result:

\[
\left( \frac{1}{2} - \Delta \right) f(\text{OPT}) \leq \frac{1}{h} \sum_{i=1}^{h} \left( F \left( x_{i}^{A} + \frac{1}{h} 1_{C_{i,A}} \right) - F \left( x_{i}^{A-1} \right) \right) + \frac{1}{h} \sum_{i=1}^{h} F \left( x_{B}^{i} \right)
\]

\[
\leq \frac{1}{h} \sum_{i=1}^{h} \left( F \left( x_{A}^{i-1} + \frac{1}{h} 1_{C_{i,A}} \right) - F \left( x_{A}^{i-1} \right) \right) + \frac{1}{h} \sum_{i=1}^{h} F \left( x_{B}^{i} \right)
\]

\[
= \frac{1}{h} \sum_{i=1}^{h} \left( F \left( x_{A}^{i} \right) - F \left( x_{A}^{i-1} \right) \right) + \frac{1}{h} \sum_{i=1}^{h} F \left( x_{B}^{i} \right)
\]

\[
= F(x_A) - f(\emptyset) + \frac{1}{h} \sum_{i=1}^{h} F \left( x_{B}^{i} \right)
\]

\[
\leq F(x_A) + \frac{1}{h} \sum_{i=1}^{h} F \left( x_{B}^{i} \right)
\]

where the second inequality is due to Property (i) of Lemma B.10, and the last one follows from \( f(\emptyset) \geq 0 \).

\[\square\]

Lemma B.5, which we mentioned in the outline of the analysis, can now readily be derived from Lemma B.4.

**Lemma B.5.**

\[
\frac{1}{h} \sum_{i=1}^{h} F \left( x_{B}^{i} \right) \geq F(x_B) - 4\Delta \cdot f(\text{OPT})
\]

**Proof.** This results follows from

\[
\frac{1}{h} \sum_{i=1}^{h} F \left( x_{B}^{i} \right) \geq \left( \frac{1}{2} - \Delta \right) f(\text{OPT}) - F(x_A)
\]

\[
\geq F(x_B) - 4\Delta f(\text{OPT})
\]

where the first inequality is due to Lemma B.4 and the second one is a consequence of Lemma B.12.

\[\square\]

We now start to analyze properties of the set \( W \) computed by Alice. We recall that we use Theorem B.1 to partition \( W \) into sets \( W_A \) and \( W_B \) such that \( \text{OPT}_A \cup W_B \in \mathcal{I} \) and \( \text{OPT}_B \cup W_A \in \mathcal{I} \). We fix one such partition and use it throughout the analysis that follows. In our analysis of \( W \), we start by showing that if \( F(x_A) \) is close to \( \frac{1}{2} f(\text{OPT}) \), and \( \Delta \) is small, then \( W_B \) largely overlaps with \( x \). Indeed, the statement of Lemma B.15 implies that in this case \( F(x \vee 1_{W_B}) \) is close to \( \frac{1}{2} f(\text{OPT}) \).

**Lemma B.15.**

\[
F(x \vee 1_{W_B}) \leq 3F(x) - F(x_A) - \left( \frac{1}{2} - \Delta \right) \cdot f(\text{OPT})
\]

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Proof. Lemma B.13 allows us to obtain the following lower bound on the marginal improvement of the last iteration, i.e., iteration $h$, of Algorithm 6:

$$
F(x^h) - F(x^{h-1}) = F\left(x^{h-1} + \frac{1}{h}1_{C_h}\right) - F(x^{h-1}) \\
\geq \frac{1}{h} \sum_{e \in \text{OPT}_A \cup W_B} [F(x \lor 1_e) - F(x)] \\
\geq \frac{1}{h} [F(x \lor 1_{\text{OPT}_A}) - F(x) + F(x \lor 1_{W_B}) - F(x)],
$$

(16)

where the second inequality is implies by the fact that for any set $S \subseteq \mathcal{N}$, we have $\sum_{e \in S} [F(x \lor 1_e) - F(x)] \geq F(x \lor 1_{S}) - F(x)$, which holds because of the following. Let $S = \{e_1, \ldots, e_p\}$ be an arbitrary numbering of the elements of $S$. Then

$$
\sum_{e \in S} [F(x \lor 1_e) - F(x)] = \sum_{i=1}^{p} [F(x \lor 1_{e_i}) - F(x)] \\
\geq \sum_{i=1}^{p} [F(x \lor 1_{\{e_1, \ldots, e_i\}}) - F(x \lor 1_{e_1, \ldots, e_{i-1}})] \\
= F(x \lor 1_{S}) - F(x),
$$

where the inequality follows from Property (iii) of Lemma B.10.

Moreover, by Lemma Lemma B.14, we have for any $i \in [h]$: 

$$
F(x) - F(x^{h-1}) \leq F(x^i) - F(x^{i-1}).
$$

Combining the above relation with Eq. (16) and summing up over all $i \in [h]$ leads to 

$$
F(x) - f(\emptyset) \geq F(x \lor 1_{\text{OPT}_A}) - F(x) + F(x \lor 1_{W_B}) - F(x) \\
\geq F(x_A \lor 1_{\text{OPT}_A}) - 2F(x) + F(x \lor 1_{W_B}) ,
$$

(17)

where the second inequality is due to monotonicity of $F$. Finally, we lower bound the term $F(x_A \lor 1_{\text{OPT}_A})$ by using Property (i) of Lemma B.11 i.e.,

$$
F(x_A \lor 1_{\text{OPT}_A}) \geq \left(\frac{1}{2} - \Delta\right) f(\text{OPT}) + F(x_A),
$$

which, after plugging it into Eq. (17), using $f(\emptyset) \geq 0$, and reordering terms, leads to the desired result.

We recall that our goal in this part is to show that $f(W_A \cup \text{OPT}_B)$ has a large value if both $\Delta$ and $F(x_B)$ is small (which implies that $F(x_A)$ is large). Our proof plan is to first observe that $F(x \lor 1_W)$ is large, and then show that $W_A$ covers a substantial part of what $\text{OPT}_B$ covers. The lemma below deals with the first part by showing that $F(x \lor 1_W)$ is large if both $F(x_A)$ and $\Delta$ are small.

**Lemma B.16.** 

$$
F(x \lor 1_W) \geq \left(\frac{1}{2} - \Delta\right) . f(\text{OPT}) + F(x_A).
$$
Proof. The statement follows from
\[ F(x \lor 1_W) \geq F(x_A \lor 1_{OPT_A}) \geq \left( \frac{1}{2} - \Delta \right) \cdot f(OPT) + F(x_A) , \]
where the first inequality follows from monotonicity of \( F \) and the second one from Property (i) of Lemma B.11.

The next lemma show that, if \( F(x_B) \) and \( \Delta \) are small, then \( W_A \) indeed covers what is not yet covered by \( x_A \). Because, if \( F(x_B) \) is small, the vector \( x_A \) mainly covers what \( OPT_B \) covers, this indeed implies that \( W_A \) covers mainly what is covered by \( OPT_A \).

**Lemma B.17.**
\[ F(x_A \lor 1_{W_A}) \geq (1 - 2\Delta) \cdot f(OPT) - 3F(x_B) . \]

**Proof.** By Property (iii) of Lemma B.10, we have
\[ F(x_A \lor 1_{W_A}) - F(x_A) \geq F(x_A \lor 1_W) - F(x \lor 1_{W_B}) . \tag{18} \]
Finally, the relation follows from
\[ F(x_A \lor 1_{W_A}) \geq F(x_A \lor 1_W) - F(x \lor 1_{W_B}) + F(x_A) \]
\[ \geq F(x \lor 1_W) - 3F(x) + 2F(x_A) + \left( \frac{1}{2} - \Delta \right) \cdot f(OPT) \]
\[ \geq (1 - 2\Delta) \cdot f(OPT) - 3F(x) + 3F(x_A) \]
\[ \geq (1 - 2\Delta) \cdot f(OPT) - 3F(x_B) , \]
where the first inequality is due to Eq. (18), the second one follows from Lemma B.15, the third one from Lemma B.16, and the last one uses \( F(x) \leq F(x_A) + F(x_B) \), which holds due to Property (iii) of Lemma B.10 and \( f(\emptyset) \geq 0 \).

Finally, Lemma B.16 now allows for proving Lemma B.6, which we stated in the outline of the analysis.

**Lemma B.6.** \( f(W_A \cup OPT_B) \geq (1 - 4\Delta)f(OPT) - 3F(x_B) \).

**Proof.** The desired relation follows from
\[ f(W_A \cup OPT_B) = f(W_A \cup OPT_B) - f(OPT_B) + f(OPT_B) \]
\[ \geq F(1_{W_A \cup OPT_B} \lor x_A) - F(1_{OPT_B} \lor x_A) + f(OPT_B) \]
\[ \geq F(1_{W_A} \lor x_A) - F(1_{OPT_B} \lor x_A) + \left( \frac{1}{2} - \Delta \right) f(OPT) \]
\[ \geq (1 - 4\Delta)f(OPT) - 3F(x_B) , \tag{19} \]
where the above-used relations hold due to the following. The first inequality holds due to Property (iii) of Lemma B.10. The second one uses monotonicity of \( F \) and \( f(OPT_B) \geq (1/2 - \Delta)f(OPT) \), which holds because
\[ f(OPT_B) \geq f(OPT) - f(OPT_A) \geq f(OPT) - \left( \frac{1}{2} + \Delta \right) f(OPT) = \left( \frac{1}{2} - \Delta \right) f(OPT) , \]
where the first inequality follows by submodularity and non-negativity of \( f \), and the second one by \( f(OPT_A) \leq f(R) \leq (1/2 + \Delta)f(OPT) \), because OPT\(_A\) is a candidate set that Bob could return. Finally, the last inequality of Eq. (20) follows by Lemma B.17 and Property (i) of Observation B.3.
As discussed, as a quite straightforward consequence of Lemma B.6, we obtain that it is impossible for \( F(x_B) \) and \( \Delta \) both to be very small.

**Lemma B.7.** \( F(x_B) \geq (\frac{1}{6} - \frac{5}{3} \Delta) \cdot f(\text{OPT}). \)

*Proof.* The claimed relation is an immediate consequence of

\[
\left( \frac{1}{2} + \Delta \right) f(\text{OPT}) = f(R) \\
\geq f(W_A \cup \text{OPT}_B) \\
\geq (1 - 4\Delta) f(\text{OPT}) - 3F(x_B) ,
\]

(20)

where the first inequality follows from the fact that \( W_A \cup \text{OPT}_B \in \mathcal{I} \) is a candidate set that Bob could return, and the second one is due to Lemma B.6.

By putting everything together, we obtain our main result of this section.

**Theorem B.8.** \( \Delta \geq 0.005. \)

*Proof.* Let \( i^* \in [h] \) be a maximizer of \( F(x_B^i) - \frac{i}{2h} f(\text{OPT}), \) i.e.,

\[
i^* \in \arg\max \left\{ F(x_B^i) - \frac{i}{2h} f(\text{OPT}) : i \in [h] \right\} \text{, and we define} \\
\beta := F(x_B^{i^*}) - \frac{i^*}{2h} f(\text{OPT}) .
\]

Moreover, we define

\[
\gamma := \frac{2h(F(x_B) - \beta)}{f(\text{OPT})} .
\]

We use these quantities to lower bound the term \( \frac{1}{h} \sum_{i=1}^{h} F(x_B^i) \) as follows:

\[
\frac{1}{h} \sum_{i=1}^{h} F(x_B^i) \leq \frac{1}{h} \sum_{i=1}^{h} \min \left\{ \beta + \frac{i}{2h} f(\text{OPT}), F(x_B) \right\} \\
= \frac{1}{h} \sum_{i=1}^{\lfloor \gamma \rfloor} \left( \beta + \frac{i}{2h} f(\text{OPT}) \right) + \frac{1}{h} (h - \lfloor \gamma \rfloor) F(x_B) \\
= \frac{1}{h} \left[ \lfloor \gamma \rfloor \left( \beta + \frac{1 + \lfloor \gamma \rfloor}{4h} f(\text{OPT}) \right) + (h - \lfloor \gamma \rfloor) F(x_B) \right] \\
\leq \frac{1}{h} \gamma \left( \beta + \frac{1 + \gamma}{4h} f(\text{OPT}) \right) + (h - \gamma) F(x_B) \]

(21)

\[
= F(x_B) + \frac{1}{h} 2h(F(x_B) - \beta) \left[ 1 + \frac{2h(F(x_B) - \beta)}{f(\text{OPT})} \right] f(\text{OPT}) - (F(x_B) - \beta) \\
= F(x_B) - \frac{1}{f(\text{OPT})} (F(x_B) - \beta)^2 + \frac{1}{2h} (F(x_B) - \beta) ,
\]

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where the inequality follows from the fact that the function $g(z) := z(\beta + \frac{1 + z}{4h} f(\text{OPT})) + (h - z) F(x_B)$ is increasing for $z \in [\gamma, \gamma]$ because

$$
\frac{d}{dz} g(z) = \beta + \frac{1 + 2z}{4h} f(\text{OPT}) - F(x_B) \\
\geq \beta + \frac{2\gamma}{4h} f(\text{OPT}) - F(x_B) \\
= 0.
$$

Combining Eq. (21) with Lemma B.5, we get

$$
\frac{1}{f(\text{OPT})} (F(x_B) - \beta)^2 - \frac{1}{2h} (F(x_B) - \beta) - 4\Delta f(\text{OPT}) \leq 0 . \tag{22}
$$

By the definition of $\beta$ we have $\beta \geq F(x_B) - \frac{1}{2} f(\text{OPT})$, and hence $F(x_B) - \beta \leq \frac{1}{2} f(\text{OPT})$. We use this to bound the middle term of the above inequality, which leads to

$$
\frac{1}{f(\text{OPT})} (F(x_B) - \beta)^2 - \left( \frac{1}{4h} + 4\Delta \right) f(\text{OPT}) \leq 0 . \tag{23}
$$

To obtain a lower bound on $\Delta$, we now derive an upper bound on $\beta$, which we will then use, together with the lower bound on $F(x_B)$ provided by Lemma B.7, to eliminate both $\beta$ and $F(x_B)$ in Eq. (23).

To upper bound $\beta$, we observe that

$$
\left( \frac{1}{2} + \Delta \right) f(\text{OPT}) = f(R) \\
\geq F \left( x_B + \left( 1 - \frac{i^*}{h} \right) 1_{\text{OPT}_B} \right) \\
\geq F(x_B^*) + \left( 1 - \frac{i^*}{h} \right) (F(x_B \lor 1_{\text{OPT}_B}) - F(x_B)) \\
\geq F(x_B^*) + \left( 1 - \frac{i^*}{h} \right) \left( \frac{1}{2} - \Delta \right) f(\text{OPT}) \\
= \beta + \left( \frac{1}{2} - \Delta + \frac{i^*}{h} \Delta \right) f(\text{OPT}) \\
\geq \beta + \left( \frac{1}{2} - \Delta \right) f(\text{OPT}) ,
$$

where the used inequalities and equalities hold due to the following. The first inequality follows from the fact that the point $y := x_B + (1 - \frac{i^*}{h}) 1_{\text{OPT}_B}$ satisfies $y \in P_M$, and Lemma B.2 then implies that there is an independent set $I \subseteq \text{supp}(y)$ with $f(I) \geq F(y)$; finally, because $\text{supp}(y) \subseteq Q \cup N_B$, this implies that $I$ is a candidate set for Bob to return. The second inequality is a consequence of Property (iii) of Lemma B.10 and $f(\emptyset) \geq 0$. The third follows by Property (ii) of Lemma B.11. The second equality is due to the definition of $\beta$. The last inequality drops the non-negative term $\frac{i^*}{h} \Delta f(\text{OPT})$. By reordering the terms on the leftmost side and rightmost side of the above relation, we obtain

$$
\beta \leq 2\Delta f(\text{OPT}) .
$$
Using this upper bound on $\beta$ and the lower bound on $F(x_B)$ given by Lemma B.7 within Eq. (23), and dividing by $f(\text{OPT})$, we obtain

$$\left(\frac{1}{6} - \frac{11}{3} \Delta\right)^2 - \left(\frac{1}{4h} + 4\Delta\right) \leq 0.$$ 

For $h \geq 125$ this implies $\Delta \geq 0.005$, as desired. \qed