EQUIVARIANT COMPRESSION OF CERTAIN DIRECT LIMIT GROUPS AND AMALGAMATED FREE PRODUCTS

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Abstract. We give a means of estimating the equivariant compression of a group $G$ in terms of properties of open subgroups $G_i \subset G$ whose direct limit is $G$. Quantifying a result by Gal, we also study the behaviour of the equivariant compression under amalgamated free products $G_1 *_H G_2$ where $H$ is of finite index in both $G_1$ and $G_2$.

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1. Introduction. The Haagerup property, which is a strong converse of Kazhdan’s property (T), has translations and applications in various fields of mathematics such as representation theory, harmonic analysis, operator K-theory and so on. It implies the Baum–Connes conjecture and related Novikov conjecture [7]. We use the following definition of the Haagerup property.

Definition 1.1. A locally compact second countable group $G$ is said to satisfy the Haagerup property if it admits a continuous proper affine isometric action $\alpha$ on some Hilbert space $\mathcal{H}$. Here, proper means that for every $M > 0$, there exists a compact set $K \subset G$ such that $||\alpha(g)(0)|| \geq M$ whenever $g \in G \setminus K$. We say that the action is continuous if the associated map $G \times \mathcal{H} \to \mathcal{H}$, $(g, v) \mapsto \alpha(g)(v)$ is jointly continuous.

Convention 1.2. Throughout this paper, all actions are assumed continuous and all groups will be second countable and locally compact.

Recall that any affine isometric action $\alpha$ can be written as $\pi + b$ where $\pi$ is a unitary representation of $G$ and where $b : G \to \mathcal{H}$, $g \mapsto \alpha(g)(0)$ satisfies

$$\forall g, h \in G : b(gh) = \pi(g)b(h) + b(g).$$

(1)

In other words, $b$ is a 1-cocycle associated to $\pi$.

In [13], the authors define compression as a means to quantify how strongly a finitely generated group satisfies the Haagerup property. More generally, assume that $G$ is a compactly generated group. Denote by $S$ some compact generating subset and equip $G$ with the word length metric relative to $S$. Using the triangle inequality, one checks easily that any 1-cocycle $b$ associated to a unitary action of $G$ on a Hilbert space is Lipschitz. On the other hand, one can look for the supremum of $r \in [0, 1]$ such that there exists $C, D > 0$ with

$$\forall g \in G : \frac{1}{C}|g|^r - D \leq \|b(g)\| \leq C|g| + D.$$
**Definition 1.3.** The above supremum, denoted \( R(b) \), is called the compression of \( b \) and taking the supremum over all proper affine isometric actions of \( G \) on all Hilbert spaces leads to the **equivariant Hilbert space compression** of \( G \), denoted \( \alpha_2^H(G) \). Suppose now that \( G \) is no longer compactly generated but still has a proper length function. Then, define \( \alpha_2^H(G) \) to be the supremum of \( R(b) \) but over all large-scale Lipschitz 1-cocycles.

The equivariant Hilbert space compression contains information on the group. First of all, if \( \alpha_2^H(G) > 0 \), then \( G \) is Haagerup. The converse was disproved by T. Austin in [4], where the author proves the existence of finitely generated amenable groups with equivariant compression 0. Further, it was shown in [13] that if for a finitely generated group \( \alpha_2^H(G) > 1/2 \), then \( G \) is amenable. This result was generalized to compactly generated groups in [9] and it provides some sort of converse for the well-known fact that amenability implies the Haagerup property. Much effort has been done to calculate the explicit equivariant compression value of several groups and classes of groups, see e.g. [2, 5, 12, 19, 20].

Given two finitely generated group \( G \) and \( H \) the group \( \bigoplus_H G \) is no longer finitely generated. However, we can view \( \bigoplus_H G \) as a subspace of \( G \wr H \) and so equip \( \bigoplus_H G \) with a natural proper metric. In this article, we are motivated by comparing the compression of \( \bigoplus_H G \) with \( G \wr H \). We assume that a given group \( G \), equipped with a proper length function \( l \), can be viewed as a direct limit of open (hence closed) subgroups \( G_1 \subset G_2 \subset G_3 \subset \ldots \subset G \). We equip each \( G_i \) with the subspace metric from \( G \). Our main objective will be to find bounds on \( \alpha_2^H(G) \) in terms of properties of the \( G_i \). Note that, as each \( G_i \) is a metric subspace of \( G \), we have \( \alpha_2^H(G) \leq \inf_{i \in \mathbb{N}} \alpha_2^H(G_i) \). The main challenge is to find a sensible lower bound on \( \alpha_2^H(G) \).

The key property that we introduce is the \((\alpha, l, q)\) polynomial property, which we shorten to \((\alpha, l, q)\)-PP (see Definition 2.5 below). Precisely, we obtain the following result.

**Theorem 1.4.** Let \( G \) be a locally compact, second countable group equipped with a proper length function \( l \). Suppose there exists a sequence of open subgroups \( (G_i)_{i \in \mathbb{N}} \), each equipped with the restriction of \( l \) to \( G_i \), such that \( \lim G_i = G \) and \( \alpha = \inf \{ \alpha_2^H(G_i) \} > 0 \). If \( (G_i)_{i \in \mathbb{N}} \) has \((\alpha, l, q)\)-PP, then there are the following two cases:

\[
l \geq q \Rightarrow \alpha_2^H(G) \geq \frac{\alpha}{2l + 1},
\]

or,

\[
l \leq q \Rightarrow \alpha_2^H(G) \geq \frac{\alpha}{l + q + 1}.
\]

We use this result to obtain a lower bound of the compression of the following examples. Let \( F : [0, 1] \times \mathbb{R}^{\geq 0} \rightarrow \mathbb{R} \) be the function

\[
F(\alpha, d) = \begin{cases} 
    d(2\alpha - 1) & \text{if } 2\alpha \geq 1 \\
    0 & \text{otherwise.}
\end{cases}
\]
Theorem 1.5. Let $G$ and $H$ be finitely generated groups where $H$ has polynomial growth of degree $d \geq 1$. Then,

$$\alpha_2^H\left(\bigoplus_H G\right) \geq \frac{\alpha_2^H(G)}{1 + F(\alpha_2^H(G), d) + 2\alpha_2^H(G)(1 + d)},$$

where $\bigoplus_H G$ is equipped with the subspace metric from $G \rtimes H$.

Our result also allows to consider spaces $\bigoplus_H G_h$ where $G_h$ actually depends on the parameter $h \in H$. For example, we take a collection of finite groups $F_i$ with $F_0 = \{0\}$ and look at $G = \bigoplus_{i \in \mathbb{N}} F_i$. This is the first available lower bound for the equivariant compression of groups of this type.

Theorem 1.6. Let $\{F_i\}_{i \in \mathbb{N}}$ be a collection of finite groups. Equip $G = \bigoplus_{i \in \mathbb{N}} F_i$ with the length function $l(g) = \min \{n \in \mathbb{N} : g \in \bigoplus_{i=0}^n F_i\}$. Then, $\alpha_2^H(G, l) > 1/3$.

We give a proof of Theorem 1.4 in Section 2.2 and apply to these concrete examples in Section 2.3. Note that our result can also be viewed as a study of the behaviour of the Haagerup property and the equivariant compression under group constructions has been studied extensively (see e.g. [11, 18], Chapter 6 of [1, 7, 8]).

In Section 3, we quantify part of [12] to study the behaviour of the equivariant compression under certain amalgamated free products $G_1 \ast_H G_2$ where $H$ is of finite index in both $G_1$ and $G_2$. Suppose $H$ is a closed finite index subgroup inside groups compactly generated groups $G_1$ and $G_2$ and there exists proper affine isometric actions $\beta_i : G_i \to \text{Aff}(V_i)$ on Hilbert spaces $V_i$. In [12], the author shows that if there exists a non-trivial closed subspace $W \subset V_1 \cap V_2$ that is fixed by the restricted actions $\beta_i|_H$ then the product $G_1 \ast_H G_2$ also admits a proper affine isometric action on a Hilbert space. We quantify this result.

Theorem 1.7. With the above assumptions $\alpha_2^H(G_1 \ast_H G_2) \geq \frac{\alpha_2^H(H)}{2}$.

2. The equivariant compression of direct limits of groups

2.1. Preliminaries and formulation of the main result. Suppose $G$ is a locally compact second countable group equipped with a proper length function $l$, i.e. closed $l$-balls are compact. Assume that there exists a sequence of open subgroups $G_i \subset G$ such that $\lim G_i = G$, i.e. $G$ is the direct limit of the $G_i$. We equip each $G_i$ with the restriction of $l$ to $G_i$. It will be our goal to find bounds on $\alpha_2^H(G)$ in terms of the $\alpha_2^H(G_i)$. Clearly, as the $G_i$ are subgroups then an upper bound of the equivariant compression is the infimum of the equivariant compressions of the $G_i$. The challenge is to find a sensible lower bound. The next example will show that it is not enough to only consider the $\alpha_2^H(G_i)$.

Example 2.1. Consider the wreath product $\mathbb{Z} \ltimes \mathbb{Z}$ equipped with the standard word metric relative to $\{(\delta_1, 0), (0, 1)\}$, where $\delta_1$ is the characteristic function of $\{0\}$. Let $\mathbb{Z}^{(\mathbb{Z})} = \{f : \mathbb{Z} \to \mathbb{Z} : f \text{ is has finite support} \}$ be equipped with the subspace metric from $\mathbb{Z} \ltimes \mathbb{Z}$. Consider the direct limit of groups

$$\mathbb{Z} \hookrightarrow \mathbb{Z}^3 \hookrightarrow \mathbb{Z}^5 \hookrightarrow \cdots \hookrightarrow \mathbb{Z}^{(\mathbb{Z})}$$
where $\mathbb{Z}^{2n+1}$ has the subspace metric from $\mathbb{Z}^2$. This metric is quasi-isometric to the standard word metric on $\mathbb{Z}^{2n+1}$ and so each term has equivariant compression 1. So $\mathbb{Z}^2$ is a direct limit of groups with equivariant compression 1 but by [2] has equivariant compression less than $3/4$. On the other hand the sequence

$$Z \to Z \to \ldots \to Z,$$

is a sequence of groups with equivariant compression 1 and the equivariant compression of the direct limit is 1. □

Given a sequence of 1-cocycles $b_i$ of $G_i$, then in order to predict the equivariant compression of the direct limit, it will be necessary to incorporate more information on the growth behaviour of the $b_i$ than merely the compression exponent $R(b_i)$. The growth behaviour of 1-cocycles can be completely caught by so called conditionally negative definite functions on the group (See Proposition 2.3 and Theorem 2.4 below).

**Definition 2.2.** A continuous map $\psi : G \to \mathbb{R}^+$ is called conditionally negative definite if $\psi(g) = \psi(g^{-1})$ for every $g \in G$ and if for all $n \in \mathbb{N}$, $\forall g_1, g_2, \ldots, g_n \in G$ and all $a_1, a_2, \ldots, a_n \in \mathbb{R}$ with $\sum_{i=1}^n a_i = 0$, we have

$$\sum_{i,j} a_i a_j \psi(g_i^{-1}g_j) \leq 0.$$

**Proposition 2.3** (Example 13, page 62 of [10]). Let $\mathcal{H}$ be a Hilbert space and $b : G \to \mathcal{H}$ a 1-cocycle associated to a unitary representation. Then, the map $\psi : G \to \mathbb{R}$, $g \mapsto \|b(g)\|^2$ is a conditionally negative definite function on $G$.

**Theorem 2.4** (Proposition 14, page 63 of [10]). Let $\psi : G \to \mathbb{R}$ be a conditionally negative definite function on a group $G$. Then, there exists an affine isometric action $\alpha$ on a Hilbert space $\mathcal{H}$ such that the associated 1-cocycle satisfies $\psi(g) = \|b(g)\|^2$.

These two results imply that we can pass between conditionally negative definite functions and 1-cocycles associated to unitary actions.

**Definition 2.5.** Let $G$ be a group equipped with a proper length function $l$ and suppose that $(G_i)_{i \in \mathbb{N}}$ is a normalized nested sequence of open subgroups such that $\lim G_i = G$. Assume that $\alpha := \inf_{i \in \mathbb{N}} \alpha_i^*(G_i) \in (0, 1]$ and $l, q \geq 0$. The sequence $(G_i)_{i}$ has the $(\alpha, l, q)$-polynomial property $(\alpha, l, q)$-PP if there exists:

1. a sequence $(\eta_i)_{i} \subset \mathbb{R}^+$ converging to 0 such that $\eta_i < \alpha$ for each $i \in \mathbb{N}$,
2. $(A_i, B_i)_{i \in \mathbb{N}} \subset \mathbb{R}_{>0} \times \mathbb{R}_{>0}$,
3. a sequence of 1-cocycles $(b_i : G_i \to \mathcal{H}_i)_{i \in \mathbb{N}}$, where each $b_i$ is associated to a unitary action $\pi_i$ of $G_i$ on a Hilbert space $\mathcal{H}_i$ such that

$$\frac{1}{A_i} |g|^{2\alpha - \eta_i} - B_i \leq \|b_i(g)\|^2 \leq A_i |g|^2 + B_i \quad \forall g \in G_i, \forall i \in \mathbb{N}$$

and there is $C, D > 0$ such that $A_i \leq C l^\eta, B_i \leq D l^\eta$ for all $i \in \mathbb{N}$.

Note that the only real restrictions are the inequalities $A_i \leq C l^\eta, B_i \leq D l^\eta$; we exclude sequences $A_i, B_i$ that grow faster than any polynomial. The intuition is that equivariant compression is a polynomial property (this follows immediately from its
definition), so that sequences $A_i$, $B_i$ growing faster than any polynomial would be too dominant and one would lose all hope of obtaining a lower bound on $\alpha_2^q(G)$. On the other hand, if the $A_i$ and $B_i$ grow polynomially, then one can use compression to somehow compensate for this growth. One then obtains a strictly positive lower bound on $\alpha_2^q(G)$ which may decrease depending on how big $l$ and $q$ are. We have the following useful characterisation of $(\alpha, l, q)$-polynomial property.

**Lemma 2.6.** Let $G$ be a locally compact second countable group and $l$ is a proper length metric. Suppose there exists a sequence of open subgroups $(G_i)_{i \in \mathbb{N}}$ such that $\lim G_i = G$. If each $G_i$ are equipped with the restricted length metric from $G$ then $(G_i)_{i \in \mathbb{N}}$ has the $(\alpha, l, q)$-polynomial property if and only if there exists $C, D > 0$ such that for all $\varepsilon > 0$ there exists

1. a sequence $(A_i, B_i)_{i \in \mathbb{N}} \subset \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ such that $A_i \leq C l^i$ and $B_i \leq D l^q$;
2. a sequence of 1-cocycles $(b_i: G_i \to H_i)_{i \in \mathbb{N}}$

such that

$$\frac{1}{A_i} |g|^{2\alpha - \varepsilon} - B_i \leq \|b_i(g)\|^2 \leq A_i |g|^2 + B_i \quad \forall g \in G_i, \forall i \in \mathbb{N}.$$

**Proof.** The “if” direction is obvious. For the “only if” direction fix $\varepsilon > 0$ and suppose $(G_i)_{i \in \mathbb{N}}$ has the $(\alpha, l, q)$-polynomial property with respect to sequences $(\eta_i)_{i \in \mathbb{N}}$ and $(b_i: G_i \to H_i)_{i \in \mathbb{N}}$. Choose $N \in \mathbb{N}$ large enough so that $\eta_k < \varepsilon$ for all $k \geq N$. Thus, $b_k: G_k \to H_k$ satisfies the above conditions for all $k \geq N$. For $k \leq N$ we take the restriction of $b_N$ to $G_k$ to obtain the sequence satisfying the above conditions for all $k \in \mathbb{N}$. \qed

**Proposition 2.7.** Let $G$ be a locally compact second countable group and suppose there exists a sequence of open subgroups $(G_i)_{i \in \mathbb{N}}$ such that $\lim G_i = G$. If $\alpha := \alpha_2^q(G) > 0$ then $(G_i)_{i \in \mathbb{N}}$ has $(\alpha, 0, 0)$-polynomial property.

**Proof.** For all $0 < \varepsilon < \alpha$ there exists a 1-cocycle $b$ such that

$$\frac{1}{A} |g|^\alpha - \varepsilon - B \leq \|b(g)\| \quad \forall g \in G.$$

The restriction of $b$ to each $G_i$ is a 1-cocycle and gives $(G_i)_{i \in \mathbb{N}}$ the $(\alpha, 0, 0)$-polynomial property. \qed

Combining this with Theorem 1.4 we have the following consequence which confirms our intuition.

**Corollary 2.8.** Let $G$ be a locally compact second countable group with a proper length function $l$. If there exists a sequence of open subgroups $(G_i)_{i \in \mathbb{N}}$ such that $\lim G_i = G$ then $(G_i)_{i \in \mathbb{N}}$ has the $(\alpha, l, q)$-polynomial property for some $\alpha \in (0, 1]$ and $l, q \geq 0$ if and only if $\alpha_2^q(G) > 0$.

### 2.2. The proof of Theorem 1.4

**Proof of Theorem 1.4.** First, we can assume that $l$ is uniformly discrete. That is there exists a $c > 0$ such that $l(x) > c$ for all $x \in G \setminus \{e\}$. This is because given a length function $l$ one can define a new length function $l'$ such that $l'(x) = 1$ whenever
0 < l(x) ≤ 1 and l′(x) = l(x) when l(x) ≥ 1. Hence l′ will be quasi-isometric to l and so will not change the compression of G or Gi.

Take sequences \((\psi_j : G_i \to \mathbb{R})_{i \in \mathbb{N}}\), \((\eta_j)_i\) and \((A, B) = (A_i, B_i)_{i \in \mathbb{N}} \subset \mathbb{R}^+ \times \mathbb{R}^+_0\) satisfying the conditions of \((\alpha, l, q)\)-PP (see Definition 2.5). We assume here, without loss of generality, that the sequences \((A_j)_i\), \((B_j)_i\) are non-decreasing.

For each \(G_i\), define a sequence of maps \((\varphi^i_k : G_i \to \mathbb{R})_{k \in \mathbb{N}}\) by

\[
\varphi^i_k(g) = \begin{cases} 
\exp\left(-\frac{\psi(g)}{k}\right) & \text{if } g \in G_i \\
0 & \text{otherwise.}
\end{cases}
\]

Note that each \(\varphi^i_k\) is continuous as \(G_i\) is open and also closed, being the complement of \(\cup_{g \notin G_i} gG_i\). By \((\alpha, l, q)\)-PP, for all \(i, k \in \mathbb{N}\), we have

\[
\exp\left(\frac{-A_i|g|^2 - B_i}{k}\right) \leq \varphi^i_k(g) \quad \forall g \in G_i, \text{ and}
\]

\[
\varphi^i_k(g) \leq \exp\left(\frac{-|g|^{2\alpha - m} + A_iB_i}{A_i/k}\right) \quad \forall g \in G.
\]

Fix some \(p > 0\), set \(J(i) = (A_i + B_i)^{1+p}\) and define \(\overline{\psi} : G \to \mathbb{R}\) by

\[
\overline{\psi}(g) = \sum_{i \in \mathbb{N}} 1 - \Phi_i(g),
\]

where \(\Phi_i(g) := \varphi^i_{|g|}(g)\). To check that \(\overline{\psi}\) is well defined, choose any \(g \in G\) and note that for \(i > |g|\), we have \(g \in G_i\) and so \(\varphi^i_{|g|}(g) \geq \exp(-\frac{-A_i|g|^2 - B_i}{k})\). Hence

\[
\sum_{i > |g|} 1 - \Phi_i(g) \leq \sum_{i > |g|} 1 - \exp\left(\frac{-A_i|g|^2 - B_i}{(A_i + B_i)^{1+p}}\right) \\
\leq \sum_{i > |g|} 1 - \exp\left(\frac{-|g|^2}{i^{1+p}}\right) \\
\leq \sum_{i > |g|} \frac{|g|^2}{i^{1+p}} = |g|^2 \sum_{i > |g|} \frac{1}{i^{1+p}}.
\]

As

\[
\overline{\psi}(g) = \sum_{i=1}^{\lfloor \frac{1}{p}\rfloor} 1 - \Phi_i(g) + \sum_{i > |g|} 1 - \Phi_i(g),
\]

we see that \(\overline{\psi}\) is well defined and that it can be written as a limit of continuous functions converging uniformly over compact sets. Consequently, it is itself continuous. By Schoenberg's theorem (see [10, Theorem 5.16]), all of the maps \(\varphi^i_k\) are positive definite on \(G_i\) and hence on \(G\) (see [15, Section 32.43(a)])). In other words,

\[
\forall n \in \mathbb{N}, \forall a_1, a_2, \ldots, a_n \in \mathbb{R}, \forall g_1, g_2, \ldots, g_n \in G : \sum_{i,j=1}^{n} a_i a_j \varphi^i_k(g^{-1}g_j) \geq 0.
\]
Hence, $\overline{\psi}$ is a conditionally negative definite map. Moreover, using that $l$ is uniformly discrete, we can find a constant $E > 0$ such that

$$
\overline{\psi}(g) \leq |g| + |g|^2 \sum_{\{l \mid g\}} \frac{1}{l^{1+p}} \leq E|g|^2,
$$

so the 1-cocycle associated to $\overline{\psi}$ via Theorem 2.4 is large-scale Lipschitz.

Let us now try to find the compression of this 1-cocycle. Set $VI : \mathbb{N} \rightarrow \mathbb{R}$ to be the function

$$
VI(i) = (A_i J(i) \ln(2) + A_i B_i)^{\frac{1}{1+p}}.
$$

One checks easily that

$$
|g| \geq VI(i) \Rightarrow \Phi_i(g) = \varphi_{J(i)}(g) \leq \frac{1}{2}.
$$

To make the function $VI$ more concrete, let us look at the values of $A_i$, $B_i$ and $J(i)$. Recall that by assumption, we have $A_i \leq C_i l^j$, $B_i \leq D_i q^l$. Hence for $i$ sufficiently large, we have $J(i) \leq (C_i l^j + D_i q^l)^{1+p} \leq F i^X$ where $F$ is some constant and $X = 1 + p + \max(l, q)$. We thus obtain that there is a constant $K > 0$ such that for every $i$ sufficiently large (say $i > I$ for some $I \in \mathbb{N}$),

$$
VI(i) \leq Ki^{Y/(2a-\eta)},
$$

where

$$
Y = \max(X + l, l + q),
$$

$$
= \max(1 + p + 2l, 1 + p + l + q).
$$

As the sequence $\eta_i$ converges to 0, we can choose any $\delta > 0$ and take $I > 0$ such that in addition $\eta_i < \delta$ for $i > I$. We then have for all $i > I$ that

$$
VI(i) \leq Ki^{Y/(2a-\delta)}.
$$

Together with equation (3), this implies that for $i > I$,

$$
|g| \geq Ki^{Y/(2a-\delta)} \Rightarrow \Phi_i(g) = \varphi_{J(i)}(g) \leq \frac{1}{2}.
$$

For every $g \in G$, set

$$
c(g)_{p,\delta} = \sup \left\{ i \in \mathbb{N} \mid Ki^{Y/(2a-\delta)} \leq |g| \right\}.
$$

We then have for every $g \in G$ with $|g|$ large enough, that

$$
\overline{\psi}(g) \geq \sum_{i=1}^{c(g)_{p,\delta}} 1 - \varphi_{J(i)}(g),
$$

$$
\geq \sum_{i=I+1}^{c(g)_{p,\delta}} 1/2 = \frac{c(g)_{p,\delta} - I}{2}.
$$
As \( c(g)_{p,\delta} \geq \left( \frac{|g|}{K} \right)^{2(2\alpha-\delta)/Y} \), we conclude that \( R(b) \geq \frac{2\alpha-\delta}{\max(1+p+2l,1+p+l+q)} \). As this is true for any small \( p, \delta > 0 \), we can take the limit for \( p, \delta \to 0 \) to obtain \( \alpha^\#_2(G) \geq \frac{2\alpha-\delta}{\max(1+p+2l,1+p+l+q)} \). Hence, we have the following two cases:

\[
l \geq q \Rightarrow \alpha^\#_2(G) \geq \frac{\alpha}{1+2l},
\]
or,

\[
l \leq q \Rightarrow \alpha^\#_2(G) \geq \frac{\alpha}{l+q+1}.
\]

\[\square\]

2.3. Examples

Let \( F : [0, 1] \times \mathbb{R} \to \mathbb{R} \) be the function

\[
F(\alpha, d) = \begin{cases} d(2\alpha - 1) & \text{if } 2\alpha \geq 1 \\ 0 & \text{otherwise.} \end{cases}
\]

**Theorem 2.9.** Let \( G \) and \( H \) be finitely generated groups where \( H \) has polynomial growth of degree \( d \geq 1 \). Then,

\[
\alpha^\#_2\left( \bigoplus_H G \right) \geq \frac{\alpha^\#_2(G)}{1 + F(\alpha^\#_2(G), d) + 2\alpha^\#_2(G)(1 + d)},
\]

where \( \bigoplus_H G \) is equipped with the subspace metric from \( G \rtimes H \).

**Remark 2.10.** Theorem 1.3. from [17] provides a lower bound to the compression of \( G \rtimes H \). Under the assumptions in Theorem 2.9, Theorem 1.3. in [17] gives a lower bound \( \alpha^\#_2(G \rtimes H) \geq \alpha^\#_1(G)/2 \). As this bound is in terms of \( L^1 \)-compression, this makes comparison between the bound in Theorem 2.9 and [17, Theorem 1.3.] difficult. However, it is known that \( \alpha^\#_2(G) \leq \alpha^\#_1(G) \leq 2\alpha^\#_2(G) \) for all finitely generated groups \( G \), see the proof of Theorem 1.1. and Theorem 1.3. in [17] and [18, Lemma 2.3.].

We use this to show that under some circumstances the above lower bound is larger than the bound provided in [17, Theorem 1.3.]. Suppose that \( \alpha^\#_1(G)/2 < \alpha^\#_2(G) \). Then, there exists a \( c > 0 \) such that \( \frac{2\alpha^\#_2(G)}{\alpha^\#_1(G)} > 1 + c \). If \( \alpha^\#_2(G) \leq \min \left\{ \frac{c}{2(1+d)}, \frac{1}{2} \right\} \) then by Theorem 2.9

\[
\alpha^\#_2(\bigoplus_H G) \geq \frac{\alpha^\#_2(G)}{1+c} > \frac{\alpha^\#_1(G)}{2}.
\]

Unfortunately, the values of \( \alpha^\#_2 \) are not so well understood and at the time of writing the only known values for \( \alpha^\#_2 \) are \( 1, 1/2, 0 \) and \( \frac{1}{2^k} \) for \( k \in \mathbb{N} \) [2, 4, 18]. In the non-equivariant case any value for compression can be achieved [3]. It is likely that there exists groups such that \( \alpha^\#_2 \) takes values strictly between 0 and 1/2 in which case our theorem can be applied to provide larger lower bounds than \( \alpha^\#_1(G)/2 \).

**Proof.** We consider \( \bigoplus_H G \) to be the group of functions \( f : H \to G \) that have finite support. Let \( f \in \bigoplus_H G \) and let \( \text{Supp}(f) = \{ h_1, \ldots, h_n \} \subset H \). Set the length of \( f \) as
follows

\[ |f|_{G \wr H} = \inf_{\sigma \in S_n} \left( d_H(1, h_{\sigma(1)}) + \sum_{i=1}^{n} d_H(h_{\sigma(i)}, h_{\sigma(i+1)}) + d_H(h_{\sigma(n)}, 1) \right) \]
\[ + \sum_{h \in H} |f(h)|_G. \]

This is the induced length metric from \( G \wr H \) and so this is a proper length function on \( \bigoplus_{H} G \). Consider the following group

\[ G_i = \{ f : H \to G : \text{Supp}(f) \subset B(1, i) \} , \]

and set \( n_i = |B(1, i)| \). Each \( G_i \) is finitely generated and the restricted wreath metric to \( G_i \) is proper and left invariant so the wreath metric and the word metric are quasi-isometric. In particular

\[ |f|_{G \wr H} - 2i|B(1, i)| \leq \sum_{h \in B(1, i)} |f(h)|_G \leq |f|_{G \wr H}, \]

for all \( f \in G_i \). By [14, Proposition 4.1. and Corollary 2.13.] it follows that \( \alpha_2^H(G_i) = \alpha_2^H(G) \) for all \( i \in \mathbb{N} \). Set \( 0 < \alpha < \alpha_2^H(G) \) and consider a 1-cocyle \( b : G \to H \) such that

\[ \frac{1}{C} |g|_{G}^{2\alpha} \leq \| b(g) \|^2 \leq C |g|_{G}^{2\alpha}. \]

Enumerate \( B(1, i) \) so that \( \{ h_1, \ldots, h_{n_i} \} = B(1, i) \) and define a 1-cocyle \( b_i : G_i \to H^{n_i} \), where \( b_i(f) = (b(f(h_1)), \ldots, b(f(h_{n_i}))). \) If \( |f|_{G \wr H} > 4i|B(1, i)| \), then

\[ \| b_i(f) \|_{1/\alpha} = \left( \sum_{j=1}^{i} \| b\left(f(h_{j})\right)\|^{1/\alpha} \right)^{\alpha} \geq 1 \frac{1}{C^{1/\alpha}} \left( \sum_{j=1}^{i} |f(h_{j})|_{G} \right)^{\alpha} \]
\[ \geq \frac{1}{C^{1/\alpha}} \left( |f|_{G \wr H} - 2i|B(1, i)| \right)^{\alpha} \geq \frac{1}{2C^{1/\alpha}} |f|_{G \wr H}^{2\alpha}. \]

If \( 2\alpha < 1 \) then \( \| b_i(f) \|_{2} \geq \| b_i(f) \|_{1/\alpha} \) for all \( f \in G_i \) and so it follows that

\[ \frac{1}{4C^{2/\alpha}} |f|_{G \wr H}^{2\alpha} - \frac{i^{2\alpha}}{C} |B(1, i)|^{2\alpha} \leq \| b_i(f) \|_{2}^{2} , \]

for all \( f \in G_i \). Hence \( (G_i)_{i \in \mathbb{N}} \) has the \((\alpha, 0, 2\alpha(1 + d))\) polynomial property.
If $2\alpha \geq 1$ then by Hölder’s inequality $\|b_i(f)\|_2 \geq \frac{1}{2^{-\alpha}} \|b_i(f)\|_1/\alpha$ for all $f \in G_i$ and so it follows that

$$\frac{1}{4C^2/\alpha} |B(1, i)|^{2\alpha-1} \|f_i^2\|_C^{2\alpha} - \frac{i^{2\alpha}}{\alpha} |B(1, i)|^{2\alpha} \leq \|b_i(f)\|_2^2,$$

for all $f \in G_i$. Hence $(G_i)_{i \in \mathbb{N}}$ has a common finite index subgroup $H$. For each $i$ the following result.

Hence $(G_i)_{i \in \mathbb{N}}$ has the $(\alpha, d(2\alpha - 1), 2\alpha(1 + d))$ polynomial property. Thus by Theorem 1.4 and that $\alpha, d \geq 0$ it follows that

$$\alpha^2 \# (\bigoplus_H G) \geq \frac{\alpha}{1 + F(\alpha, d) + 2\alpha(1 + d)},$$

for all $\alpha < \alpha^2(G)$ and so the statement of the theorem holds.

**Theorem 2.11.** Let $\{F_i\}_{i \in \mathbb{N}}$ be a collection of finite groups such that $F_0 = \{1\}$. Let $G = \bigoplus_{i \in \mathbb{N}} F_i$ be equip with the proper length function $l(g) = \min \{ n \in \mathbb{N} : g \in \bigoplus_{i=0}^n F_i \}$. Then $\alpha^2(G) \geq 1/3$.

**Proof.** Set $G_i = \bigoplus_{j=0}^i F_j$ and observe that $\alpha^2(G_i) = 1$ as $G_i$ is finite for all $i \in \mathbb{N}$. Define $f_i : G_i \to \mathbb{R}$ to be the 0-map. This is clearly a 1-cocycle and satisfies

$$\forall g \in G_i : l(g)^2 - i^2 \leq |f_i(g)|^2 \leq l(g)^2 + i^2.$$ 

Hence $(G_i)_{i \in \mathbb{N}}$ has the $(1, 0, 2)$-polynomial property. Thus $\alpha^2(G) \geq 1/3$.

**Example 2.12.** We will use [3] to provide an example of a sequence that does not have $(\alpha, l, q)$-polynomial property for any $\alpha \in (0, 1]$ and $l, q > 0$. Let $\Pi_k$, $k \geq 1$ be a sequence of Lafforgue expanders that do not embed into any uniformly convex Banach space [16]. These are finite factor groups $M_k$ of a lattice $\Gamma$ of $SL_3(F)$ for a local field $F$.

For every $\alpha \in (0, 1]$ there exists a finitely generated group $G$ and a sequence of scaling constants $\lambda_k$ such that $\lambda_k \Pi_k$ has compression $\alpha$ and $G$ is quasi-isometric to $\lambda_k \Pi_k$. Furthermore, $G$ contains the free product $*_{k} M_k$ as a subgroup. Let $\alpha = 0$ and let $G$ and the scaling constants $\lambda_k$ be such that $G$ has compression 0. We can equip $*_{k} M_k$ with a proper left invariant metric coming from $G$. Hence we have a sequence

$$M_1 \hookrightarrow M_1 * M_2 \hookrightarrow \cdots \hookrightarrow *_{k=1}^n M_k \hookrightarrow \cdots \hookrightarrow *_{k} M_k.$$

For each $n > 0$, $*_{k=1}^n M_k$ has equivariant compression $1/2$ [11, Theorem 1.4] however the limit group $*_{k} M_k$ contains a quasi-isometric copy of $\lambda_k \Pi_k$ and so has compression 0. Thus, this sequence cannot have the $(\alpha, l, q)$-polynomial property for any $\alpha \in (0, 1]$ and $l, q > 0$.

3. The behaviour of compression under free products amalgamated over finite index subgroups. It is known that the Haagerup property is not preserved under amalgamated free products. Indeed, $(SL_2(\mathbb{Z}) \rtimes \mathbb{Z}, \mathbb{Z})$ has the relative property $(T)$. So $SL_2(\mathbb{Z}) \rtimes \mathbb{Z}^2 = (\mathbb{Z}_6 \rtimes \mathbb{Z}^2) *_{(\mathbb{Z}_2 \times \mathbb{Z}^2)} (\mathbb{Z}_4 \rtimes \mathbb{Z}^2)$ is not Haagerup. In [12], S.R. Gal proves the following result.

**Theorem 3.1.** Let $G_1$ and $G_2$ be finitely generated groups with the Haagerup property that have a common finite index subgroup $H$. For each $i = 1, 2$, let $\beta_i$ be a proper affine isometric action of $G_i$ on a Hilbert space $V_i (= l^2(\mathbb{Z}))$. Assume that $W < V_1 \cap V_2$ is
invariant under the actions $\beta_i)_H$ and moreover that both these (restricted) actions coincide on $W$. Then, $G_1 *_H G_2$ is Haagerup.

Under the same conditions as above, we want to give estimates on $\alpha_p^H(G_1 *_H G_2)$ in terms of the equivariant Hilbert space compressions of $G_1$, $G_2$ (see Theorem 3.3 below). Note that the following lemma shows that $\alpha_p^H(G_1) = \alpha_p^H(G_2)$ when $H$ is of finite index in both $G_1$ and $G_2$. We are indebted to Alain Valette for this lemma and its proof. The notation $\alpha_p^H$ refers to the equivariant $L_p$-compression for some $p \geq 1$.

It is defined in exactly the same way as $\alpha_p^G$ except that one considers affine isometric actions on $L_p$-spaces instead of $L_2$-spaces.

**Lemma 3.2.** Let $G$ be a compactly generated, locally compact group, and let $H$ be an open, finite-index subgroup of $G$. Then, $\alpha_p^H(H) = \alpha_p^G(G)$.

**Proof.** As $H$ is embedded $H$-equivariantly, quasi-isometrically in $G$, we have $\alpha_p^H(H) \geq \alpha_p^G(G)$. To prove the converse inequality, we may assume that $\alpha_p^G(H) > 0$.

Let $S$ be a compact generating subset of $H$. Let $A(h)v = \pi(h)v + b(h)$ be an affine isometric action of $H$ on $L_p$, such that for some $\alpha < \alpha_p^H(H)$ we have $\|b(h)\|_p \geq C|h|_S^\alpha$, for every $h \in H$. Now, we induce up the action $A$ from $H$ to $G$, as on p. 91 of [6].

The affine space of the induced action is

$$E := \{f : G \to L_p : f(gh) = A(h)^{-1}f(g), \forall h \in H \text{ and almost every } g \in G\},$$

with distance given by $\|f_1 - f_2\|_p^p = \sum_{x \in G/H} \|f_1(x) - f_2(x)\|_p^p$. The induced affine isometric action $\tilde{A}$ of $G$ on $E$ is then given by $(\tilde{A}(g)f)(g') = f(g^{-1}g')$, for $f \in E$, $g,g' \in G$.

A function $\xi_0 \in E$ is then defined as follows. Let $s_1 = e, s_2, \ldots, s_n$ be a set of representatives for the left cosets of $H$ in $G$. Set $\xi_0(s_i, h) = b(h^{-1})$, for $h \in H$, $i = 1, \ldots, n$. Define the 1-cocycle $\tilde{b}$ on $G$ by $\tilde{b}(g) = \tilde{A}(g)\xi_0 - \xi_0$, for $g \in G$. For an $h \in H$, we then have:

$$\|\tilde{b}(h)\|_p^p = \sum_{i=1}^n \|\xi_0(h^{-1}s_i)\|_p^p = \sum_{i=1}^n \|\xi_0(h^{-1}s_i)\|_p^p \geq \|\xi_0(h^{-1})\|_p^p = \|b(h)\|_p^p.$$  

Set $K = \max_{1 \leq i \leq n} \|\tilde{b}(s_i)\|_p$. Take $T = S \cup \{s_1, \ldots, s_n\}$ as a compact generating set of $G$.

For $g \in G$, write $g = s_i h$ for $1 \leq i \leq n$, $h \in H$. Then,

$$\|\tilde{b}(g)\|_p \geq \|\tilde{b}(h)\|_p - K \geq \|b(h)\|_p - K \geq C|h|_S^\alpha - K \geq C|h|_T^\alpha - K \geq C'|g|_T^\alpha - K'.$$

So the compression of the 1-cocycle $\tilde{b}$ is at least $\alpha$, hence $\alpha_p^H(G) \geq \alpha_p^H(H)$. \qed

The following proof uses a construction by S.R. Gal, see page 4 of [12].

**Theorem 3.3.** Let $V_1$ and $V_2$ be closed subspaces of a Hilbert space. Suppose $H$ is a finite index subgroup of $G_1$ and $G_2$ and suppose there are proper affine isometric actions $\beta_i$ (with compression $\alpha_i$) of each $G_i$ on $V_i$. Assume that $W < V_1 \cap V_2$ is invariant under

---

1We seize this opportunity to correct a misprint in the definition of the vector $\xi_0$ in that construction in p. 91 of [6].
the actions \((\beta_i|_H)\) and moreover that both these (restricted) actions coincide on \(W\). Then, 
\[\alpha_2^\#(G_1 \ast_H G_2) \geq \frac{\min(\alpha_1(G_1), \alpha_1(G_2))}{2}.\]
In particular, \(\alpha_2^\#(G_1 \ast_H G_2) \geq \frac{\alpha_1^\#(H)}{2}\).

Proof. Following [12], let us build a Hilbert space \(W_\Gamma\) on which \(\Gamma = G_1 \ast_H G_2\) acts affinely and isometrically. Let \(\omega\) be a finite alternating sequence of 1’s and 2’s and suppose \(\pi\) is a linear action of \(H\) on some Hilbert space denoted \(\mathcal{H}_\omega\). One can induce up the linear action from \(H\) to \(G_i\), obtaining a Hilbert space
\[V := \{ f : G_i \to \mathcal{H}_\omega \mid \forall h \in H, f(gh) = \pi(h^{-1}f(g))\}
\]
and an orthogonal action \(\pi_i : G_i \to \mathcal{O}(V)\) defined by \(\pi_i(g)f(g') = f(g^{-1}g')\). The subspace
\[\{ f : G_i \to \mathcal{H}_\omega \mid \forall h \in H, f(h) = \pi(h^{-1})f(1), f|_{G_i \setminus H} = 0 \},
\]
can be identified with \(\mathcal{H}_\omega\) by letting an element \(f\) correspond to \(f(1)\). It is clear that the action \(\pi_i\) restricted to \(H\) coincides with the original linear action \(\pi\) via this identification.

So, starting from any linear \(H\)-action on a Hilbert space \(\mathcal{H}_\omega\), we can obtain a linear action of say \(G_1\) on a Hilbert space that can be written as \(\mathcal{H}_\omega \oplus \mathcal{H}_{1\omega}\) for some \(\mathcal{H}_{1\omega}\). We can restrict this action to a linear \(H\)-action on \(\mathcal{H}_{1\omega}\) and we can lift this to an action of \(G_2\) on a space \(\mathcal{H}_{1\omega} \oplus \mathcal{H}_{2\omega}\) and so on, repeating the process indefinitely. Here, we will execute this infinite process twice.

The first linear \(H\)-action on which we apply the process is obtained as follows. As \(\beta_i(H)(W) = W\) for each \(i = 1, 2\), the restriction to \(H\) of \(\beta_1\), gives naturally a linear \(H\)-action on \(\mathcal{H}_1 := V_1/W\). The second linear \(H\)-action is obtained by similarly noting that the restriction to \(H\) of \(\beta_2\) gives a linear \(H\)-action on \(\mathcal{H}_2 := V_2/W\). We then apply the above process indefinitely.

\[
\begin{align*}
\mathcal{H}_1^* := \mathcal{H}_1 \oplus & \mathcal{H}_1^{21} \oplus \mathcal{H}_1^{121} \oplus \cdots, \quad & \mathcal{H}_2^* := \mathcal{H}_2 \oplus \mathcal{H}_2^{12} \oplus \mathcal{H}_2^{121} \oplus \cdots, \\
\mathcal{H}_1 := & \mathcal{H}_1 \ominus \mathcal{H}_1^{21} \ominus \mathcal{H}_1^{121} \ominus \cdots, \quad & \mathcal{H}_2 := \mathcal{H}_2 \ominus \mathcal{H}_2^{12} \ominus \mathcal{H}_2^{121} \ominus \cdots.
\end{align*}
\]

where for \(\omega\) a sequence of alternating 1’s and 2’s, \(G_i\) acts on \(\mathcal{H}_\omega \oplus \mathcal{H}_{i\omega}\). Note that there are two \(H\)-actions on \(\mathcal{H}_1^*\) as \(H\) acts on the first term \(\mathcal{H}_1\). One can verify that both \(H\)-actions coincide (this fact is also mentioned in [12], page 4). The same is true for \(\mathcal{H}_2^*\).

Denote \(\mathcal{H}_1 = \mathcal{H}_1^* \ominus \mathcal{H}_1\) and similarly, set \(\mathcal{H}_2 = \mathcal{H}_2^* \ominus \mathcal{H}_2\). We denote
\[W_\Gamma = W \oplus \mathcal{H}_1^* \oplus \mathcal{H}_2^* = V_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 = V_2 \oplus \mathcal{H}_2 \oplus \mathcal{H}_1^*.
\]

The above formula, which writes \(W\) as a direct sum in three distinct ways, shows that both \(G_1\) and \(G_2\) act on \(W_\Gamma\). As mentioned before, the actions coincide on \(H\) and so we obtain an affine isometric action of \(\Gamma\) on \(W_\Gamma\). Note that the corresponding 1-cocycle, when restricted to \(G_1\) (or \(G_2\)), coincides with the 1-cocycle of \(\beta_1\) (or \(\beta_2\)).

We inductively define a length function \(\psi_T : \Gamma \to \mathbb{N}\) by \(\psi_T(h) = 0\) for all \(h \in H\) and \(\psi_T(\gamma) = \min\{\psi_T(\eta) + 1 \mid \gamma = \eta g\}\), where \(g \in G_1 \cup G_2\). By applying Proposition 2 in [10] to the Bass–Serre tree of \(G_1 \ast_H G_2\), we see that this map is conditionally negative definite and thus the normed square of a 1-cocycle associated to an affine isometric action of \(\Gamma\) on a Hilbert space.

Let \(\psi_\Gamma\) be the conditionally negative definite function associated to the action of \(\Gamma\) on \(W_\Gamma\). We now find the compression of the conditionally negative definite map
ψ = ψ_Γ + ψ_T. First set

\[ M = \max \left\{ |t_i^j|_{G_i} : i = 1, 2 \text{ and } 1 \leq j \leq [G_i : H] \right\}, \]

where \( t_i^j \) are right coset representatives of \( H \) in \( G_i \) such that \( t_i^1 = 1_{G_i} \) for \( i = 1, 2 \).

Denote \( \alpha = \min(\alpha_1, \alpha_2) \) and fix some \( \varepsilon > 0 \) arbitrarily small. Let \( \gamma \in \Gamma \) and suppose in normal form \( \gamma = g t_i^1 \cdots t_k^j \), where \( g \in G_i \) for some \( i = 1, 2 \). Assume first that \( \psi_T(\gamma) \geq |\gamma|^{\alpha - \varepsilon} M \). In that case, \( \psi(\gamma) \geq |\gamma|^{\alpha - \varepsilon} M \). Else, we have that \( \psi_T(\gamma) < |\gamma|^{\alpha - \varepsilon} M \) and so for all \( \gamma \in \Gamma \) such that \( |\gamma| \) is sufficiently large, we have

\[
\psi(\gamma) \geq \psi_T(\gamma) = \| \gamma \cdot 0 \|^2 \\
\geq (\| g \cdot 0 \| - \psi_T(\gamma) M)^2 \\
\geq (\| \gamma \| - \psi_T(\gamma) M)^{\alpha - \varepsilon / 2} - \psi_T(\gamma) M)^2 \\
\geq (\| \gamma \| - |\gamma|^{\alpha - \varepsilon})^{\alpha - \varepsilon / 2} - |\gamma|^{\alpha - \varepsilon})^2 \\
\geq |\gamma|^{2\alpha - \varepsilon},
\]

where \( \geq \) represents inequality up to a multiplicative constant; we use here that one can always assume, without loss of generality, that the 1-cocycles associated to \( \beta_1 \) and \( \beta_2 \) satisfy \( \| b_i(g_i) \| \geq |g_i|^{-\varepsilon} \) (see Lemma 3.4 in [1]).

So now, by the first case, \( \psi(\gamma) \geq |\gamma|^{\alpha - \varepsilon} \) for all \( \gamma \in \Gamma \) that are sufficiently large. Hence, we obtain the lower bound \( \alpha^{\mathbb{H}}_2(\Gamma) \geq \alpha^{\mathbb{H}}_2(\mathcal{H})/2 \). □

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