GEODESICS, BIGEODESICS, AND COALESCENCE IN FIRST PASSAGE PERCOLATION IN GENERAL DIMENSION

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Abstract. We consider geodesics for first passage percolation (FPP) on $\mathbb{Z}^d$ with iid passage times. As has been common in the literature, we assume that the FPP system satisfies certain basic properties conjectured to be true, and derive consequences from these properties. The assumptions are roughly as follows: (i) the standard deviation of the passage time on scale $r$ is of some order $\sigma_r$, with $\{\sigma_r, r > 0\}$ growing approximately as a power of $r$; (ii) the tails of the passage time distributions for distance $r$ satisfy an exponential bound on scale $\sigma_r$, uniformly over $r$; and (iii) the limit shape boundary has curvature uniformly bounded away from 0 and $\infty$ (a requirement we can sometimes limit to a neighborhood of some fixed direction.) The main a.s. consequences derived are the following: (a) for one-ended geodesic rays with a given asymptotic direction $\theta$, starting in a natural halfspace $H$, for the hyperplane at distance $R$ from $H$, the density of “entry points” where some geodesic ray first crosses the hyperplane is at most $c(\log R)^K/(R\sigma_R)^{(d-1)/2}$ for some $c, K$, (b) the system has no bigeodesics, i.e. two-ended infinite geodesics, (c) given two sites $x, y$, and a third site $z$ at distance at least $\ell$ from $x$ and $y$, the probability that the geodesic from $x$ to $y$ passes through $z$ is at most $c(\log \ell)^K/(\ell\sigma_{\ell})^{(d-1)/2}$ for some $c, K$, and (d) in $d = 2$, the probability that the geodesic rays in a given direction from two sites have not coalesced after distance $r$ “decays like $r^{-\xi}$,” where $\xi$ is roughly the order of transverse geodesic wandering. Our entry-point density bound compares to a natural conjecture of $c/(R\sigma_R)^{(d-1)/2}$, corresponding to a spacing of order $(R\sigma_R)^{1/2}$ between entry points, which is the conjectured scale of the transverse wandering.

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1. Introduction.

We consider coalescence of geodesic rays, and other related properties of geodesics, in a family of models of first passage percolation (FPP) on $\mathbb{Z}^d$ with $d \geq 2$; geodesic rays are semi-infinite paths.
for which every finite subpath is a geodesic. A doubly infinite path with the same property is called a bigeodesic.

In the spirit of various past works on FPP ([11], [24], [25], [30]), we take as assumptions a few basic properties believed to hold generally, but unproven for any specific FPP model. One such property states that fluctuations of passage times have an exponential bound on the scale of their standard deviation; a second says that the boundary of the limit shape has uniform curvature in a certain sense. We also assume for technical purposes that this standard deviation behaves mildly regularly as a function of distance and direction. See A2 below for details. Our purpose is to show how certain relatively strong conclusions about geodesic behavior, including aspects of coalescence, flow from essentially just the basic properties.

We will use percolation “site/bond” terminology in the lattice \( \mathbb{Z}^d \), rather than “vertex/edge.” For \( d = 2 \) it is known for lattice FPP that under (arguably) mild hypotheses,

(i) every geodesic ray has an asymptotic direction \([29]\);
(ii) for a fixed direction \( \theta \), with probability 1, there is exactly one geodesic ray with asymptotic direction \( \theta \) starting from a given site \([1], [24], [29]\);
(iii) for a fixed direction \( \theta \), with probability 1, for all sites \( x, y \), the geodesic rays with asymptotic direction \( \theta \) from \( x \) and \( y \) eventually coalesce \([13], [24]\);
(iv) for a fixed direction \( \theta \), with probability 1 there is no bigeodesic for which either asymptotic direction is \( \theta \) \([1], [14]\>; a weaker form is in \([24]\).

Here by asymptotic direction we mean the value \( \lim_{n} v_n / |v_n| \in S^{d-1} \), where \( v_0, v_1, \ldots \) are the sites, in order, of the geodesic ray and \( S^{d-1} \) is the \((d-1)\)-sphere, and \( |\cdot| \) denotes Euclidean length. Note that (iv) does not rule out the existence of all bigeodesics, as it allows a random null set of \( \theta \) values for which such bigeodesics exist. We will call a geodesic ray with asymptotic direction \( \theta \) a \( \theta \)-ray. Given a halfspace \( H \) we call a \( \theta \)-ray a halfspace \( \theta \)-ray from \( H \) if its first site, and no other, is contained in \( H \); here \( r \in \mathbb{R} \) and \( \alpha \in S^{d-1} \). We may omit the “from \( H \)” if the appropriate halfspace is either apparent or not relevant.

For \( d \geq 3 \), under our hypotheses we will (among other things) prove (i), and prove (ii) with “at least one” in place of “exactly one,” but it’s not clear whether (iii) and the “exactly one” in (ii) should be true. Assuming continuous distributions of passage times to prevent ties between paths, \( a \text{ priori} \), for any two geodesic rays \( \Gamma, \tilde{\Gamma} \), any of 3 things may happen:

1. \( \Gamma, \tilde{\Gamma} \) are disjoint, i.e. they have no bonds in common;
2. \( \Gamma, \tilde{\Gamma} \) coalesce, that is, there is a site \( v \in \Gamma \cap \tilde{\Gamma} \) such that the segments of \( \Gamma, \tilde{\Gamma} \) up to \( v \) are disjoint, and the two rays from \( v \) onwards coincide;
3. \( \Gamma \cap \tilde{\Gamma} \) is a single segment consisting of finitely many bonds.

We refer to the phenomenon in (3) as temporary touching; when it occurs, the last site in the segment is called a branching site; see Figure 1. Despite the complications temporary touching and branching create, we can still quantify some aspects of the coalescence of \( \theta \)-rays in the following way. Below we will associate to each direction \( \theta \) a vector \( z_\theta \), chosen so the hyperplane \( \{ x \in \mathbb{R}^d : x \cdot z_\theta = 1 \} \) is tangent to the unit ball of the norm associated to the FPP process at the boundary point of the ball in direction \( \theta \). We define the hyperplanes and halfspaces

\[
H_{\theta,s} = \{ x \in \mathbb{R}^d : x \cdot z_\theta = s \}, \quad H_{\theta,s}^+ = \{ x \in \mathbb{R}^d : x \cdot z_\theta \geq s \}, \quad H_{\theta,s}^- = \{ x \in \mathbb{R}^d : x \cdot z_\theta \leq s \}.
\]
For $\theta, \theta_0 \in S^{d-1}$ (close together), consider the $H_{\theta_0,s}$-entry points of $\theta$-rays, meaning those lattice sites in $H_{\theta_0,s}^+$ where some $\theta$-ray from $H_{\theta_0,0}^-$ first intersects $H_{\theta_0,s}^+$. We may ask, what is the density of such $H_{\theta_0,s}$-entry points per unit volume near $H_{\theta_0,s}$, and how does it decrease as $s \to \infty$? Does it approach 0, and if so how fast? We call this density the $H_{\theta_0,s}$-crossing density, postponing a precise definition for later.

Note that $\theta$-rays passing through different $H_{\theta_0,s}$-entry points cannot be assumed disjoint up to those entry points, due to the possibilities of temporary touching and branching.

If all halfspace $\theta$-rays from $H_{\theta_0,0}^-$ coalesce a.s., their $H_{\theta_0,s}$-crossing density must approach 0 as $s \to \infty$. The converse is true for $d = 2$, but for $d \geq 3$ the $H_{\theta_0,s}$-crossing density approaching 0 does not in itself guarantee coalescence of all the $\theta$-rays. We can make equivalence classes of halfspace $\theta$-rays from $H_{\theta_0,0}^-$ by writing $\Gamma \sim \tilde{\Gamma}$ if $\Gamma, \tilde{\Gamma}$ eventually coalesce; the $H_{\theta_0,s}$-crossing density will approach 0 if the number of equivalence classes is finite, but it is not clear that the converse holds. One could also ask about the existence of finite equivalence classes; if they exist with positive probability then their starting points must have a positive density near $H_{\theta_0,0}$, but again the possibility of temporary touching means such a positive density is not immediately ruled out by the $H_{\theta_0,s}$-crossing density approaching 0.

For $d = 2$, we can predict the $H_{\theta_0,s}$-crossing density heuristically from the transverse wandering exponent of geodesics, that is, the value $\xi$ such that for a geodesic of length $s$, the maximum distance of the geodesic from the straight line connecting its endpoints is of order $s^\xi$. There must exist halfspace $\theta$-rays, one passing through each $H_{\theta_0,s}$-entry point, and any such rays must remain disjoint at least until they cross $H_{\theta_0,s}$, since there is a.s. no branching or temporary touching for a fixed $\theta$ in two dimensions. Heuristically, to remain disjoint until $H_{\theta_0,s}^+$ the $\theta$-rays should be spaced apart by order $s^\xi$, so the $H_{\theta_0,s}$-crossing density should be $s^{-\xi}$. For $d \geq 3$ this predicts an $H_{\theta_0,s}$-crossing density of at least $s^{-(d-1)\xi}$, but it is not clear a priori that the crossing density shouldn’t
be greater, since the $\theta$–rays can weave around one another without meeting, and we cannot rule out the branching of some $\theta$–rays each into multiple $\theta$–rays.

We will show that in fact the $H_{\theta_0,s}$–crossing density approaches 0 faster than $s^{-(d-1)\xi+\epsilon}$ for all $\epsilon > 0$; one can in fact replace the factor $s^\epsilon$ here with a large power of $\log s$. Along the way we will obtain results about the regularity of geodesics, and their transverse fluctuations. We will follow a heuristic of Newman (presented at the AIM 2015 workshop “First-passage percolation and related models”) in using the convergence to 0 of the crossing density to show nonexistence of bigeodesics. Reformulated to our context, it goes roughly as follows: suppose the crossing-point density bound holds not just for each single geodesic ray direction, say $\theta_0$–rays, but for the union of all $\theta$–rays over $|\theta - \theta_0| \leq \epsilon$ for some $\epsilon$, for each fixed $\theta_0$; we’ll call these near-$\theta_0$–rays. It is known under (again arguably) mild hypotheses that every geodesic ray has an asymptotic direction, so any bigeodesic must have an asymptotic direction “each way”; the two directions should always be $\pm \theta$ for some $\theta$. Let $\mathcal{G}_{\theta_0,\epsilon}$ be the set of all bigeodesics which have (by some directional labeling) forward asymptotic direction $\theta$ and backward direction $-\theta$, with $|\theta - \theta_0| \leq \epsilon$. Then a geodesic in $\mathcal{G}_{\theta_0,\epsilon}$ has a well-defined $H_{\theta_0,s}^+$–entry point for all $s \in \mathbb{R}$. For $R > 0$ consider the set

$$\mathcal{P}_{\theta_0,R} = \{ x \in H_{\theta_0,R}^+ : x \text{ is the } H_{\theta_0,R}^+ \text{–entry point of some bigeodesic in } \mathcal{G}_{\theta_0,\epsilon} \}.$$  

The density of $\mathcal{P}_{\theta_0,R}$ is bounded above by the density near $H_{\theta_0,R}$ of entry points of halfspace near-$\theta_0$–rays, so by our assumption it approaches 0 as $R \to \infty$. But by translation invariance, $\mathcal{P}_{\theta_0,R}$ has the same density as $\mathcal{P}_{\theta_0,0}$ for all $R$, so the density of $\mathcal{P}_{\theta_0,0}$ must be 0. By stationarity this means $\mathcal{G}_{\theta_0,\epsilon} = \emptyset$ a.s. If this holds for all $\theta_0$, then by compactness, there are a.s. no bigeodesics.

The preceding heuristic assumes we know that the $H_{\theta_0,s}$–crossing density approaches 0 as $s \to \infty$; a second heuristic for why that should be true is as follows. Suppose the transverse fluctuations of a typical geodesic of length $s$ are of some order $\Delta_s$, behaving roughly like $s^\xi$. We can divide $H_{\theta_0,0}$ and $H_{\theta_0,s}$ each into blocks of size $\Delta_s$. If the $H_{\theta_0,s}$–crossing density of near-$\theta_0$ rays is much more than $\Delta_s^{(d-1)}$ (i.e. “one entry point per block”) then a typical block in $H_{\theta_0,s}$ has many $H_{\theta_0,s}^+$–entry points near it. This should mean we can find many pairs of blocks, say $B_0$ in $H_{\theta_0,0}$ and $B_s$ in $H_{\theta_0,s}$, for which there are a large number of near-$\theta_0$ rays from $B_0$ passing through $B_s$ with distinct $H_{\theta_0,s}^+$–entry points. The distinct entry points means none of these near-$\theta_0$ rays have coalesced when reaching $H_{\theta_0,s}$ despite starting and ending their paths from $H_{\theta_0,0}$ to $H_{\theta_0,s}$ within $\Delta_s$ of each other. (Note such coalescence is “temporary” if two near-$\theta_0$ rays have different asymptotic directions.) Thus a primary ingredient is to show there is a low probability of such overly–densely–packed nearly–parallel geodesics.

In addition to $\xi$, the other exponent of central interest is the $\chi$ for which the standard deviation of the passage time over distance $r$ “grows like $r^\chi$.” Our precise assumptions related to this standard deviation are given in A2 below. It is known [11] that under “reasonable” hypotheses and definitions, $\chi, \xi$ are related by $\chi = 2\xi - 1$.

As mentioned above, our assumptions of basic model properties cannot be verified for any specific model of FPP; the best known exponential bounds are on scale $r^{1/2}$ ([20], [31], [15]) whereas for $d = 2$ the conjectured value of $\chi$ is $1/3$. An exponential bound on scale $r^{1/3}$ is known for certain integrable models of last passage percolation (LPP) for $d = 2$, however— in [8] (extracted from [4]) and [23] for LPP on $\mathbb{Z}^2$ with exponential passage times, in [26], [27] for LPP based on a Poisson process in the unit square, and in [12] for LPP on $\mathbb{Z}^2$ with geometric passage times.
For $d \geq 3$ there is no generally-agreed-upon value of $\chi$ in the physics literature. Heuristics and simulations suggest that $\chi$ should decrease with dimension; simulations in [32] for a model believed to be in the same (KPZ) universality class as FPP show a decrease from $\chi = .33$ to $\chi = .054$ as $d$ increases from 2 to 7. Some have predicted the existence of a finite upper critical dimension, possibly as low as 3.5, above which $\chi = 0$ ([16], [22]); others predict that $\chi$ is positive for all $d$ ([3], [28]), with simulations in [21] showing $\chi > 0$ all the way to $d = 12$, decaying approximately as $1/(d + 1)$. Our results here require $\chi > 0$ so they only have content below the upper critical dimension, should it be finite.

In [6], Basu, Hoffman, and Sly show that there are no bigeodesics for last-passage percolation (LPP) on $\mathbb{Z}^2$ when passage times are exponential, essentially by following Newman’s heuristic of bounding the density of entry points, which in turn involves bounding the probability of overly-densely-packed parallel geodesics. (See also the earlier papers [31], [19], and see [5] for a proof avoiding results from integrable probability.) The paper [6] exploits key ingredients not available in our general context—the restriction to $d = 2$, the exponential bound on the scale of the standard deviation in [8] (extracted from [4]), and the fact that the rescaled passage time distributions converge to a limit (Tracy-Widom) which has negative mean. We need here a completely different heuristic and proof to control overly-densely-packed geodesics, and this is the core of our main proof; see Remark 4.1.

In two dimensions one can use bounds on the probability of overly-densely-packed geodesics to bound the probability of non-coalescence before traveling distance $\epsilon r^c$, for $\theta$-rays which start at separation $r^c$. For the integrable case of LPP in $d = 2$ with exponential passage times, such a bound on non-coalescence probabilities was proved in [7]. But again, strong use is made of $d = 2$ and bounds obtained through integrable probability, so the methods do not apply in our context. The results are at the optimal rate, analogous to removing the powers of log in our Theorems 1.5 and 1.7. The reliance on integrable probability was removed in [33], but the results are still restricted to LPP in $d = 2$ with exponential passage times.

Let us now formalize our definitions. Let $E^d$ denote the set of all bonds (i.e. nearest-neighbor pairs) of $\mathbb{Z}^d$. The passage times of bonds are a collection of nonnegative iid variables $\tau = \{\tau_e : e \in E^d\}$. For $x, y \in V$, a (self-avoiding) path $\Gamma$ from $x$ to $y$ is a finite sequence of alternating sites and bonds of $G$, of the form $\Gamma = (x = x_0, \langle x_0, x_1 \rangle, x_1, \ldots, \langle x_{n-1}, x_n \rangle, x_n = y)$, with all $x_i$ distinct; we may identify a path by just the sequence of sites, or view it as a string of line segments, as convenient, when clear from the context. The passage time of $\Gamma$ is

$$T(\Gamma) := \sum_{e \in \Gamma} \tau_e,$$

and the passage time from $x$ to $y$ is

$$T(x, y) := \inf\{T(\Gamma) : \Gamma \text{ is a path from } x \text{ to } y \text{ in } \mathbb{Z}^d\}.\tag{1.1}$$

A path $\Gamma$ achieving the infimum in (1.1) is called a geodesic from $x$ to $y$. For technical reasons we extend (1.1) to $x, y \in \mathbb{R}^d$ as follows. Define $Z : \mathbb{R}^d \to \mathbb{Z}^d$ by $Z(x) = z$ for all $z \in \mathbb{Z}^d$ and $x \in z + [-1/2, 1/2]^d$, where $+$ denotes translation, and set

$$T(x, y) = T(Z(x), Z(y)), \quad x, y \in \mathbb{R}^d.$$

We assume the following.
A1. $\tau_e$ properties.

(i) $\tau_e$ is a continuous random variable.
(ii) There exists $\lambda > 0$ such that $E e^{\lambda \tau_e} < \infty$.

A1 guarantees that there is exactly one geodesic from $x$ to $y$ a.s., for each $x, y$; we denote it $\Gamma_{xy}$. As is standard, since passage times $T(x, y)$ are subadditive, assumption (ii) guarantees the a.s. existence (positive and finite for $x \neq 0$) of the limit

$$g(x) = \lim_{n \to \infty} \frac{T(0, nx)}{n} = \lim_{n \to \infty} \frac{ET(0, nx)}{n} = \inf_n \frac{ET(0, nx)}{n}$$

for $x \in \mathbb{Z}^d$; $g$ extends to $x$ with rational coordinates by considering only $n$ with $nx \in \mathbb{Z}^d$, and then to a norm on $\mathbb{R}^d$ by uniform continuity. We let $\mathfrak{B}_g$ denote the unit ball of $g$, and write $y_0$ for the positive multiple of $\theta$ which lies in $\partial \mathfrak{B}_g$ (so $g(y_0) = 1$.) The tangent hyperplane to $\partial \mathfrak{B}_g$ at $y_0$ will be unique under our hypotheses, and there is a vector $z_0$ such that this tangent hyperplane is $\{x \in \mathbb{R}^d : x \cdot z_0 = 1\} = H_{\theta,1}$.

An infinite self-avoiding path $\Gamma = (x = x_0, (x_0, x_1), x_1, (x_1, x_2), \ldots )$ is a geodesic ray if every finite segment of $\Gamma$ is a geodesic. Given $\theta$ in the sphere $S^{d-1}$, we say $\Gamma$ is a $\theta$-ray if $\lim_n x_n/|x_n| = \theta$.

Throughout the paper, $c_1, c_2, \ldots$ and $C_1, C_2, \ldots$ and $\epsilon_0, \epsilon_1, \ldots$ represent unspecified constants which depend only on $d$ and the distribution of the passage times $\tau_e$. We use $C_i$ for constants which occur outside of proofs and may be referenced later; any given $C_i$ has the same value at all occurrences. We use $c_i$ for those which do not recur and are only needed inside one proof. For the $c_i$’s we restart the numbering with $c_0$ in each proof, and the values are different in different proofs.

To avoid technical clutter, at various times we will assume (sometimes tacitly) that certain points of $\mathbb{R}^d$ are lattice sites, and certain (large) real numbers are integers; the modifications to be made when this fails are trivial.

As mentioned above, we cannot formally establish simple hypotheses on the distribution of $\tau_e$ under which our conclusions hold, due to the lack of any results establishing an exponential bound on the scale of the standard deviation. Instead we will assume certain “macroscopic” properties which one expects to be consequences of such hypotheses, as follows. To that end, we call a nonnegative function $\{\sigma(r) = \sigma_r, r > 0\}$ powerlike (with exponent $\chi$) if there exist $0 < \gamma_1 < \chi < \gamma_2$ and constants $C_i$ such that

$$\lim_{r \to \infty} \frac{\log \sigma_r}{\log r} = \chi$$

and for all $s \geq r \geq C_1, \quad C_2 \left( \frac{s}{r} \right)^{\gamma_1} \leq \frac{\sigma_s}{\sigma_r} \leq C_3 \left( \frac{s}{r} \right)^{\gamma_2}.$$

If (1.2) holds with $\gamma_2 < 1$ we say $\sigma(\cdot)$ is sublinearly powerlike. Note that (1.2) implies that for all $r, s \geq C_1$,

$$\frac{\sigma_s}{\sigma_r} \leq \frac{1}{C_2} + C_3 \frac{s}{r}.$$

A2. System properties.

(i) Unique scale: there exist a sublinearly powerlike function $\sigma_r = \sigma(r), n \geq 1$ with exponent $\chi \in (0, 1)$, and constants $\eta < 1/2, C_i > 0$ such that for all $x, y \in \mathbb{R}^d$ with $|x - y| \geq C_4$,

$$P\left( |T(x, y) - ET(x, y)| \geq t\sigma(|x - y|) \right) \leq C_5 e^{-C_6 t} \quad \text{for all } t > 0,$$

$$\text{var}(T(x, y)) \geq C_7 \sigma^2(|x - y|)$$
(ii) Local curvature near $\theta_0$: for some specified $\theta_0 \in S^{d-1}$, for some $\epsilon_0 > 0$ and constants $C_i > 0$, for all $\theta \in S^{d-1}$ with $|\theta - \theta_0| < \epsilon_0$, and all $y \in H_{\theta,1}$ with $|y - y_\theta| < \epsilon_0$, we have
\[ C_S |y - y_\theta|^2 \leq d(y, \mathfrak{B}_g) \leq C_g |y - y_\theta|^2. \]

Sometimes in place of A2(ii) we assume the following:

(ii') Globally uniform curvature: for some $\epsilon_0 > 0$, for all $\theta \in S^{d-1}$ and all $y \in H_{\theta,1}$ with $|y - y_\theta| < \epsilon_0$, (1.6) holds.

Note that A2(ii) guarantees that $H_{\theta,1}$ is unique for $\theta$ near $\theta_0$, and A2(ii') guarantees the same thing for all $\theta \in S^{d-1}$. A2(i) is very close to the assumptions (A1)-(A4) in Theorem 1.1 of [11].

When A2(ii) holds, for $y \in H_{\theta,1}$ with $|y - y_\theta| \geq \epsilon_0$ we have by (1.6) and convexity that
\[ d(y, \mathfrak{B}_g) \geq C_S \epsilon_0 |y - y_\theta|, \]
so for all $y \in H_{\theta,1}$,
\[ d(y, \mathfrak{B}_g) \geq C_S \epsilon_0^2 \left( \left| \frac{y - y_\theta}{\epsilon_0} \right| \wedge \left( \left| \frac{y - y_\theta}{\epsilon_0} \right| \right)^2 \right). \]

Remark 1.1. If $\sigma(\cdot)$ is powerlike, then so is the nondecreasing function $\hat{\sigma}(r) = \sup_{s \leq r} \sigma(s)$, and if A2(ii) holds for $\sigma(\cdot)$ then it also holds for $\hat{\sigma}(\cdot)$. By (1.2) we have $C_2 \leq \sigma(r)/\sigma(r) \leq 1$ for all $r$. By further increasing $\hat{\sigma}$ (though by at most a constant factor) we may make it strictly increasing and continuous while still preserving these properties. Therefore without loss of generality we always assume $\sigma(\cdot)$ is strictly increasing and continuous.

In addition, under (1.4), (1.5) is equivalent to the assumption that there exist constants $\eta < 1/2, C_{10} > 0$ such that for all $x, y \in \mathbb{R}^d$ with $|x - y| \geq C_4$ the following both hold:
\[ P \left( T(x, y) - ET(x, y) > \eta \sigma(|x - y|) \right) \geq C_10 > 0, \]
\[ P \left( T(x, y) - ET(x, y) < -\eta \sigma(|x - y|) \right) \geq C_10 > 0. \]

Remark 1.2. An equivalent way to state the local curvature condition A2(ii) is as follows. Let $B_r(x)$ denote the closed Euclidean ball of radius $r$ centered at $x$, and recall $z_\theta$ is perpendicular to the tangent plane $H_{\theta,1}$ to $\mathfrak{B}_g$ at $y_\theta$. Define the cones
\[ J(\theta, \epsilon) := \left\{ u \in \mathbb{R}^d : u \neq 0, \left| \frac{u}{|u|} - \theta \right| < \epsilon \right\}. \]

There exist constants $C_{11} < C_{12}$ as follows. For all $\theta \in J(\theta_0, \epsilon_0) \cap S^{d-1}$ we have
\[ B_{C_{11}|z_\theta|}(y_\theta - C_{11}z_\theta) \cap J(\theta, \epsilon) \subset \mathfrak{B}_g \cap J(\theta, \epsilon) \subset B_{C_{12}|z_\theta|}(y_\theta - C_{12}z_\theta) \cap J(\theta, \epsilon). \]

Equation (1.11) says that near $y_\theta$, $\mathfrak{B}_g$ is sandwiched between two balls which also have tangent plane $H_{\theta,1}$ at $y_\theta$. From this we see that for some $\epsilon_1$ determined by $\epsilon_0$, the angle $\psi_{z_\theta, z_\alpha}$ between $z_\theta$ and $z_\alpha$ (equivalently between hyperplanes $H_{\theta,0}$ and $H_{\alpha,0}$) grows at most linearly in $|\theta - \alpha|$:
\[ |\theta - \theta_0| < \epsilon_1, |\alpha - \theta| < \epsilon_1 \Rightarrow \psi_{z_\theta, z_\alpha} \leq C_{13}|\alpha - \theta|, \]
and also
\[ |\theta - \theta_0| < \epsilon_1, |\alpha - \theta| < \epsilon_1 \Rightarrow |y_\alpha - y_\theta| \leq C_{14}|\alpha - \theta|^2. \]
It follows from A2(i) that \( \text{var}(T(x,y)) \) is of order \( \sigma^2(|x-y|) \). When A2(i) holds for some \( \sigma(\cdot) \), the corresponding transverse wandering function is given by

\[
\Delta(r) = \Delta_r = (r\sigma_r)^{1/2}.
\]

Since \( \sigma(\cdot) \) is continuous and increasing, the inverse function \( \Delta^{-1} \) is well-defined. For \( \xi = (1+\chi)/2 \in (\frac{1}{2}, 1) \) we have \( \Delta(r) \asymp r^\xi \) and \( \Delta^{-1}(a) \asymp a^{1/\xi} \) in the sense that

\[
\lim_{r \to \infty} \frac{\log \Delta(r)}{\log r} = \xi, \quad \lim_{a \to \infty} \frac{\log \Delta^{-1}(a)}{\log a} = \frac{1}{\xi}.
\]

To motivate the definition of \( \Delta(r) \), consider two sites \( x, y \) separated by distance \( r \), and a third site \( z \) at some distance \( \Delta \ll r \) from the line segment connecting \( x \) to \( y \), somewhere near the middle of this line. The Euclidean distance via \( z \), that is, \( |y-z| + |z-x| \), exceeds the “straight” distance \( |y-x| \) by order \( \Delta^2/r \), and under the local curvature assumption A2(ii), the same will be true for the distance in the norm \( g \). Heuristically, the geodesic may nonetheless pass through \( z \), meaning \( T(x,z) + T(z,y) = T(x,y) \), if the fluctuation scale \( \sigma_r \) of the random “distance” \( T(\cdot, \cdot) \) is larger than the deterministic excess distance, that is, \( \sigma_r \geq \Delta^2/r \) or equivalently \( \Delta \leq \Delta(r) \).

**Remark 1.3.** Let us call a hyperplane \( H \subset \mathbb{R}^d \) rationally oriented if \( H \cap \mathbb{Z}^d \) spans \( H \); then \( H \cap \mathbb{Z}^d \) is an infinite lattice. When \( H_{\theta,0} \) is rationally oriented, we can apply the multidimensional ergodic theorem (see [18], Appendix 14.A) to obtain the existence of a (nonrandom) crossing density for \( \theta \)-rays. For other \( \theta \) an additional argument would be required for this; since we are primarily interested in upper bounds, we avoid the matter by defining the crossing density as a lim sup.

Let \( L_\theta(u) \) denote the line though a point \( u \) in direction \( \theta \), and write \( L_\theta \) for \( L_\theta(0) \). The \( \theta \)-projection of a point \( x \) into a hyperplane \( H_{\theta,s} \) is the projection along \( L_\theta \).

For \( A \subset H_{\theta,s} \) let \( C_{\theta,s}(A) \) be the set of all \( H_{\theta,s}^+ \)-entry points \( x \) of halfspace \( \theta \)-rays from \( H_{\theta,0} \), for which the \( \theta \)-projection of \( x \) into \( H_{\theta,s} \) lies in \( A \). Similarly let \( C_{J(\theta,\epsilon),s}(A) \) be the set of all sites \( x \) which are \( H_{\theta,s}^+ \)-entry points of halfspace \( \alpha \)-rays from \( H_{\theta,0} \) for some \( \alpha \in J(\theta,\epsilon) \), for which the \( \theta \)-projection of \( x \) into \( H_{\theta,s} \) lies in \( A \). Formally, the mean \( H_{\theta,s} \)-crossing density \( \bar{\rho}_\theta(s) \) for \( \theta \)-rays, and the mean combined \( H_{\theta,s} \)-crossing density \( \bar{\rho}_{J(\theta, \epsilon)}(s) \), are given by

\[
\bar{\rho}_\theta(s) = \limsup_{r \to \infty} \frac{E(|C_{\theta,s}(B_r(s\theta) \cap H_{\theta,s})|)}{\text{Vol}_{d-1}(B_r(s\theta) \cap H_{\theta,s})},
\]

\[
\bar{\rho}_{J(\theta, \epsilon)}(s) = \limsup_{r \to \infty} \frac{E(|C_{J(\theta,\epsilon),s}(B_r(s\theta) \cap H_{\theta,s})|)}{\text{Vol}_{d-1}(B_r(s\theta) \cap H_{\theta,s})},
\]

where \( \text{Vol}_{d-1}(\cdot) \) denotes \((d-1)\)-dimensional volume. The corresponding almost-sure values are the \( H_{\theta,s} \)-crossing density \( \rho_\theta(s) \) and the combined \( H_{\theta,s} \)-crossing density \( \rho_{J(\theta, \epsilon)}(s) \) given by

\[
\limsup_{r \to \infty} \frac{|C_{\theta,s}(B_r(s\theta) \cap H_{\theta,s})|}{\text{Vol}_{d-1}(B_r(s\theta) \cap H_{\theta,s})} = \rho_\theta(s) \text{ a.s.,}
\]

\[
\limsup_{r \to \infty} \frac{|C_{J(\theta,\epsilon),s}(B_r(s\theta) \cap H_{\theta,s})|}{\text{Vol}_{d-1}(B_r(s\theta) \cap H_{\theta,s})} = \rho_{J(\theta, \epsilon)}(s) \text{ a.s.;}
\]

such nonrandom constants exist because the lim sup is a tail random variable.
Remark 1.4. Continuing from Remark 1.3 for rationally oriented $\theta$ we may replace lim sup in (1.15)–(1.18) with limit, and we have

\begin{equation}
\rho_{\theta}(s) = \overline{\rho}_{\theta}(s), \quad \rho_{J(\theta,\epsilon)}(s) = \overline{\rho}_{J(\theta,\epsilon)}(s).
\end{equation}

Existence of a limit in (1.15) and (1.16) follows here from periodicity in $x$ of $P(x \in C_{\theta,s}(\mathbb{R}^d))$ and $P(x \in C_{J(\theta,\epsilon),s}(\mathbb{R}^d))$, and the equality with almost sure limits follows from the multidimensional ergodic theorem (see [18], Appendix 14.A.)

The following is our main result.

Theorem 1.5. Suppose for some FPP process on $\mathbb{Z}^d$, A1 and A2(i),(ii) hold for some $\theta_0, \epsilon_0$. There exists $\epsilon_2$ as follows.

(i) With probability 1, for all $\theta \in J(\theta_0, \epsilon_2)$ and $x \in \mathbb{Z}^d$, there is at least one $\theta$–ray from $x$.

(ii) There exist constants $C_i$ such that

\begin{equation}
\overline{\rho}_{J(\theta_0, \epsilon_2)}(s) \leq C_{15} \frac{C_{16}}{(\log s)^{C_{16}}} \sigma_{d-1} for all $s \geq C_{17},$
\end{equation}

and for all $\theta \in J(\theta_0, \epsilon_2)$,

\begin{equation}
\rho_{\theta}(s) \leq C_{15} \frac{C_{16}}{\Delta_{d-1}} \frac{\sigma_{d-1}}{s} for all $s \geq C_{17}.$
\end{equation}

(iii) With probability 1, there exists no bigeodesic containing a subsequential $\theta$–ray with $\theta \in J(\theta_0, \epsilon_2)$.

(iv) Suppose also $A2(ii')$ holds. Then with probability 1, (a) every geodesic ray has an asymptotic direction, (b) for every $\theta \in S_{d-1}^d$ and every $x \in \mathbb{Z}^d$ there is at least one $\theta$–ray from $x$, and (c) there are no bigeodesics.

We will prove (i), (iv)(a), and (iv)(b) in Section 3, (ii) in Section 5, and (iii) and (iv)(c) in Section 6.

Theorem 1.5(iv) improves on existing results even for $d = 2$ (though under somewhat stronger hypotheses), as it rules out bigeodesics in all directions simultaneously, instead of almost surely for a fixed direction as in [1], [14], [24].

As we have noted, one expects the spacing of entry points at distance $R$ to be of the same order as the transverse fluctuation of geodesics at the same distance $R$; in other words, two geodesics which are close enough that their transverse fluctuations allow them to coalesce should generally do so. This means the bound (1.20) should be sharp up to the power of log in the numerator. The bound (1.21) is likely not sharp, though, as we expect the combining of a small sector of directions should not significantly increase the number of entry points; a bound like (1.21) should apply to the mean combined crossing density as well. But for the purpose of banning bigeodesics, the fact that the bound in (1.20) approaches 0 as $s \to \infty$ is sufficient.

Remark 1.6. Suppose we fix $\theta$ and consider the collection of all halfspace $\theta$–rays from $H_{\theta,0}^-$. By the time these reach $H_{\theta,R}^+$, based on (1.21) enough coalescence or temporary touching must occur so that on average at least $\Delta_{d-1}/(\log R)^{C_{16}}$ $\theta$–rays pass through any given $H_{\theta,R}^+$–entry point $x$; this number is roughly of order $R^{(d-1)\xi}$. To the extent this is due to temporary touching rather than coalescence, these $\theta$–rays will later separate again. But this separating can occur at most only slowly: it can be shown using Proposition 4.10 that the number of $H_{\theta,2R}^+$–entry points of $\theta$–rays
passing through such an \( x \) is with high probability at most of order \( (\log R)^K \) for some \( K \). All the while there should be significant additional coalescence and/or temporary touching initiated between \( H_{\theta, R} \) and \( H_{\theta, 2R} \), since the crossing point density is lower for \( H_{\theta, 2R} \). All this brings up the question, do the conclusions of Theorem 1.5 already force coalescence of all \( \theta \)-rays? We do not have an answer.

When \( \theta \) is fixed it will be convenient to express a general \( u \in \mathbb{R}^d \) in terms of a basis \( B_\theta \) in which the first vector is \( y_\theta \), and the other \( d - 1 \) form an orthonormal basis for \( H_{\theta, 0} \). (The particular choice of orthonormal basis does not matter.) In a mild abuse of notation we will simply write \( u = (u_1^\theta, u_2^\theta)_\theta \) for the representation in this basis, with \( u_1^\theta = u \cdot z_\theta \in \mathbb{R} \) and \( u_2^\theta \in \mathbb{R}^{d-1} \); we call these \( \theta \)-coordinates. The corresponding decomposition of \( u \) is

\[
 u = (u_1^\theta, 0)_\theta + (0, u_2^\theta)_\theta = u_1^\theta y_\theta + (u - u_1^\theta y_\theta)
\]

(see Figure 2), and we refer to \( u_1^\theta y_\theta \) and \( u - u_1^\theta y_\theta \) as the first and second \( \theta \)-components of \( u \).

For \( d = 2 \) it is known ([13], [21]) that, under hypotheses weaker than the combination of A1 and A2(i), (ii), for each fixed \( \theta \in J(\theta_0, e_0) \), with probability one there is a unique \( \theta \)-ray, which we denote \( \Gamma_{x, \theta}^0 \), starting from each \( x \in \mathbb{Z}^2 \), and any two such \( \theta \)-rays eventually coalesce; there is no temporary touching or branching. (We note again, however, that there must be a random countable set of directions \( \theta \) for which branching does occur, producing multiple \( \theta \)-rays from a single site.) So we may ask, how far do two \( \theta \)-rays go before they coalesce? To formulate the question more precisely we first make some definitions. Fix \( \theta, \tilde{\theta} \in J(\theta_0, e_0) \). A \( \tilde{\theta} \)-start site is a site in \( H_{\theta, 0}^- \) which is adjacent to a site in the interior of \( H_{\tilde{\theta}, 0}^+ \). A \( \tilde{\theta} \)-start site \( x \) is a \( \theta \)-source (in a configuration \( \tau \) if \( \Gamma_{x, \theta}^0 \) is a halfspace \( \theta \)-ray from \( H_{\theta, 0}^- \). With probability one, for any two \( \tilde{\theta} \)-start sites \( x, y \), there exists a unique coalescence site \( U_{xy}^\theta \) such that \( \Gamma_{x, \theta}^0 \) and \( \Gamma_{y, \theta}^0 \) are disjoint up to \( U_{xy}^\theta \) and coincide from \( U_{xy}^\theta \) onward. The \( \tilde{\theta} \)-coalescence time of \( \Gamma_{x, \theta}^0 \) and \( \Gamma_{y, \theta}^0 \) is the \( \tilde{\theta} \)-coordinate \( (U_{xy}^\theta)_{\tilde{\theta}} \). Throughout the preceding, it would be convenient to take \( \theta = \tilde{\theta} \), but we will need \( \tilde{\theta} \) to be rationally oriented.

We can bound the tail of the coalescence time as a consequence of Theorem 1.5(ii), as follows.

**Theorem 1.7.** Suppose for some FPP process on \( \mathbb{Z}^2 \), A1 and A2(i), (ii) hold for some \( \theta_0, e_0 \). There exist constants \( c_3, c_4 \) as follows. Fix \( \theta \in J(\theta_0, e_0) \), and suppose \( x, y \) are \( \theta \)-start sites with \( C_{18} \leq |x - y| \leq \Delta_r \). Then for all \( r \geq \Delta^{-1}(|x - y|) \),

\[
 C_{19} \frac{|x - y|}{\Delta(r) \log r} \leq P \left( (U_{xy}^\theta)_{\tilde{\theta}} \geq r \right) \leq C_{20} \frac{(\log r)^{C_{21}} |x - y|}{\Delta(r)}.
\]

A consequence of (1.22) is that \( P((U_{xy}^\theta)_{\tilde{\theta}} \geq r) \propto r^{-(1+\chi)/2} \) in the sense that

\[
 \lim_{r \to \infty} \frac{\log P((U_{xy}^\theta)_{\tilde{\theta}} \geq r)}{\log r} = \frac{1 + \chi}{2}.
\]

A natural scaling in Theorem 1.7 is \( r = t \Delta^{-1}(|x - y|) \) with \( t \geq 1 \). If we strengthen the assumption in A2 that \( \sigma(\cdot) \) is powerlike to instead require that \( \sigma(s) \sim C s^\chi \) for some \( C \) (as is known to be true with \( \chi = 1/3 \) for integrable models of LPP [8]), then (1.22) says that for \( t \geq 1 \),

\[
 \frac{C_{7e}}{t^{(1+\chi)/2} (\log(t|x - y|))^{1/2}} \leq P \left( (U_{xy}^\theta)_{\tilde{\theta}} \geq t \Delta^{-1}(|x - y|) \right) \leq \frac{C_{7f} (\log(t|x - y|))^{C_{7e}}}{t^{(1+\chi)/2}}
\]
One expects that the probability in (1.23) is actually of order $t^{-(1+\chi)/2}$, uniformly in $|x-y|$, without any logs involved; such a result is proved for 2d LPP in [7], with related bounds in [31] and [33].

Let $e_j$ denote the $j$th unit coordinate vector. Under assumptions much milder than ours, it is proved in [1] that for all sequences $v_k$ in $\mathbb{Z}^2$ with $|v_k| \to \infty$ we have $P(0 \in \Gamma_{-v_k,v_k}) \to 0$ as $k \to \infty$. In particular, taking $v_k = k e_1$ solves a conjecture made in [10]. Here, under stronger hypotheses, in general dimension we can establish a rate at which this probability converges to 0; as with (1.21) we expect this rate to be optimal up to the power of log in the numerator. The statement is as follows.

**Theorem 1.8.** Suppose for some FPP process on $\mathbb{Z}^d$, (i) and (ii) hold for some $\theta_0, \epsilon_0$. There exist constants $\epsilon_1, C_i$ as follows. Suppose $u, v \in \mathbb{Z}^d$ with $|v|_1^\theta_0 \geq |u|_1^\theta_0 \geq C_22$, and $(v-u)/|v-u| \in J(\theta_0, \epsilon_1)$. Then

$$P(0 \in \Gamma_{uv}) \leq \frac{C_23(\log |u|)^{C_24}}{\Delta(|u|)^{d-1}}.$$  

(1.24)

If we replace assumption (ii) with (ii'), then the same is true without the assumption $(v-u)/|v-u| \in J(\theta_0, \epsilon_1)$.

2. The cost of bad geodesic behavior

Let

$$h(x) = ET(0, x), \quad x \in \mathbb{Z}^d.$$  

Then $h(x) - g(x)$ is nonnegative by subadditivity of $h$. A variant of the following bound was proved in [2] with error term $C_{25}|x|^{1/2} \log |x|$; in [35] this was improved to $C_{25}|x|^{1/2}(\log |x|)^{1/2}$. The proof in [2] adapts readily to the present situation and we don’t need improvement to $(\log |x|)^{1/2}$, so we will work here without that improvement.

**Proposition 2.1.** Assume (i) and (ii). There exists $C_{25}$ such that for all $|x| \geq 2$,

$$g(x) \leq h(x) \leq g(x) + C_{25}\sigma(|x|) \log |x|.$$  

(2.1)

**Proof.** Defining

$$\tilde{\sigma}(r) = r^{\gamma_1} \sup_{s \leq r} \frac{\sigma(s)}{s^{\gamma_1}}$$

we have from (1.2) that $\sigma(r) \leq \tilde{\sigma}(r) \leq \sigma(r)/C_2$, while $\tilde{\sigma}(r)/r^{\gamma_1}$ is nondecreasing. Therefore A2(i) is valid for $\tilde{\sigma}$ in place of $\sigma$, so we may use $\tilde{\sigma}$ throughout.

The proof of (2), Proposition 3.4) only uses the existence of an exponential bound on scale $|x|^{1/2}$ from [20], so under A2(i) it is valid for $\tilde{\sigma}(|x|)$ in place of $|x|^{1/2}$. To obtain (2), Theorem 3.2) from (2), Proposition 3.4), both with the replacement error term $C_{25}\tilde{\sigma}(|x|) \log |x|$, we need only the fact that this error term can be expressed as $C_{25}|x|^{\gamma_1} \varphi(|x|)$ with $\gamma_1 > 0$ and $\varphi(r) = \tilde{\sigma}(r)(\log r)/r^{\gamma_1}$ nondecreasing.

Recall that $\Gamma_{xy}$ denotes the geodesic between $x$ and $y$. In general we view $\Gamma_{xy}$ as an undirected path, but at times we will refer to, for example, the first point of $\Gamma_{xy}$ with some property. Hence when appropriate, and clear from the context, we view $\Gamma_{xy}$ as a path from $x$ to $y$.

Note that $|u|$ always refers to the Euclidean length of $u$, not to the length of the vector of $\theta$–coordinates. As a norm based on $\theta$–coordinates, we use

$$|u|_{\theta, \infty} := \max(|u^\theta_1|, |u^\theta_2|),$$
which satisfies
\[(2.2) \quad (1 + |y|)|u|_{\theta, \infty} \geq |u_1^\theta||y| + |u_2^\theta| \geq |u|.
\]
We define “distance via hyperplane”: noting that \(u \in H_{\theta,u_1^\theta}\), for \(A \subset \mathbb{R}^d\) with \(A \cap H_{\theta,u_1^\theta} \neq \emptyset\) let
\[d_{\theta}(u, A) = d(u, A \cap H_{\theta,u_1^\theta}),\]
and note that \(d_{\theta}(u, L_{\theta}) = |u_2^\theta|\). Finally define the \(\theta\)-ratio of \(0 \neq u \in \mathbb{R}^e\) to be \(|u_2^\theta|/|u_1^\theta| \in [0, \infty]\), and note that if \(H_{\theta,0} \perp \theta\) then this is just the tangent of the angle between \(u\) and \(L_{\theta}\); in general the \(\theta\)-ratio is a surrogate for this tangent.

A number of our arguments involve the following theme: for points \(x, y \in \mathbb{R}^2\), if \(y\) is far enough from the straight line from \(0\) to \(x\), then \(\Gamma_{\alpha x}\) is unlikely to pass through \(y\), because \(g(y) + g(x - y)\) significantly exceeds \(g(x)\). To express this in more than a crude way, since (1.4) involves centering at the expectation, we also need to quantify how increases in \(g(x)\) relate to increases in \(h(x)\), but for the moment we consider just \(g\). For Euclidean distance the following is useful:
\[(2.3) \quad \ell + \frac{m^2}{2\ell} \geq (\ell^2 + m^2)^{1/2} \geq \ell + \min\left(\frac{m}{3}, \frac{m^2}{3\ell}\right) \quad \text{for all } \ell, m > 0.
\]
Under the local curvature assumption A1(ii), the following analog for \(g\)-distance is straightforward, under A1(ii), proved similarly to Lemma 2.3 below (see also Remark 1.2). There exist constants \(\epsilon_1 > 0\) and \(C_i > 0\) such that for \(|\theta - \theta_0| < \epsilon_1\) and \(u = (u_1^\theta, u_2^\theta) \in \mathbb{R}^d\) satisfying \(|\theta - u/|u|| < \epsilon_1\), we have
\[(2.4) \quad u_1^\theta + C_{26}\min\left(|u_2^\theta|, \frac{|u_2^\theta|^2}{u_1^\theta}\right) \leq g(u) \leq u_1^\theta + C_{27}\min\left(|u_2^\theta|, \frac{|u_2^\theta|^2}{u_1^\theta}\right).
\]
This is really the significance of the local curvature assumption for the boundary of \(\mathcal{B}_g\): it says that in dealing with vectors with direction near \(\theta_0\), the norm \(g\) is “Euclidean-like” in that (2.4) holds, and, most importantly, there is consequently a discrepancy as in (2.26) below in the triangle inequality.

In (2.4), we can view the “min” term as a lower bound for the cost, in extra distance, of deviating by \(|u_2^\theta|\) from a target point at \(g\)-distance \(u_1^\theta\) in direction \(\theta\). To obtain a probability cost of such deviation by a geodesic, in view of (1.4) roughly we can divide the extra distance by \(\sigma(u_1^\theta)\); we will incorporate an extra log factor to handle the entropy that arises when handling many scales of \(|u|\) with a single bound. Keeping this in mind, we define first \(\sigma^*(s)\) and \(\Phi(s)\) by
\[\Phi(s) = \frac{s}{C_3\sigma^*(s) \log(2 + s)} = \frac{s^{\gamma_2}}{C_3\sup_{t \leq s} \frac{t^{1-\gamma_2}}{\sigma_t \log(2 + t)}}.
\]
Here factoring out a power of \(s\) ensures that \(\Phi\) is strictly increasing, and \(C_3, \gamma_2\) are from (1.2). Note that by (1.2) we have
\[(2.5) \quad C_3^{-1}\sigma_s \leq \sigma^*(s) \leq \sigma_s,
\]
the first inequality being valid for \(s \geq C_{28}\) for some \(C_{28}\). We then define
\[(2.6) \quad \Xi(s) = (s\sigma(s) \log(2 + s))^{1/2}, \quad D_{\theta}(u) = \begin{cases} \min\left(\frac{|u_2^\theta|^2}{\Xi(u_1^\theta)}, \Phi(|u|_{\theta, \infty})\right) & \text{if } u_1^\theta \geq 0, \\ \Phi(|u|_{\theta, \infty}) & \text{if } u_1^\theta < 0. \end{cases}
\]
Roughly speaking, if we ignore the above-mentioned entropy-controlling log factors in $\Phi$ and $\Xi$, then for a geodesic or geodesic ray with ultimate direction $\theta$, $D_\theta(u)$ represents the cost of that geodesic deviating from direction $\theta$ to pass through $u$. When the direction of $u$ is far from $\theta$, this cost has form $\Phi(|u|_{\theta,\infty})$, and when it is close to $\theta$ the cost has form $|u_2^\theta|^2/\Xi(u_1^\theta)^2$. Note that by (1.2) and (2.5), for some $C_{29}$, for all $s \geq C_{29}$,

$$
\frac{1}{C_3\Phi(s)} \leq \frac{\left(\Xi(s)\right)^2}{s} \leq \frac{1}{\Phi(s)}.
$$

This tells us in part which term in the “min” in (2.6) is smaller: we have for $|u| \geq 2C_{29}$ that

$$
D_\theta(u) = \begin{cases} 
\Phi(|u|_{\theta,\infty}) & \text{if } |u_2^\theta| \geq u_1^\theta, \\
\frac{|u_2^\theta|^2}{\Xi(u_1^\theta)^2} & \text{if } |u_2^\theta| \leq C_3^{-1/2}u_1^\theta.
\end{cases}
$$

Here we used the fact that $0 \leq u_1^\theta \leq |u_2^\theta|$ implies $|u|_{\theta,\infty} = |u_2^\theta|$, and $|u_2^\theta| \leq u_1^\theta$ implies $|u|_{\theta,\infty} = u_1^\theta$.

Let $\Pi_{xy}$ denote the line segment from $x$ to $y$. To deal with paths from 0 to some $ry_\theta$ it is useful to have the following symmetric version of $D_\theta$:

$$
D_{\theta,r}(u) = \begin{cases} 
D_\theta(u) & \text{if } u_1^\theta \leq \frac{r}{2}, \\
D_\theta(ry_\theta - u) & \text{if } u_1^\theta > \frac{r}{2}.
\end{cases}
$$

This makes the right half of the region

$$
E_{\theta,r,c} := \{u : D_{\theta,r}(u) \leq c\}
$$

symmetric with the left half; this region is a “tube” surrounding $\Pi_{0,ry_\theta}$ bounded by the shell \{ $u : |u_2^\theta| = c^{1/2}\Xi(u_1^\theta)$ \}, augmented by a “tilted cylinder” around each endpoint; see Figure 3. (In a mild abuse of terminology, we will simply call it a cylinder.) The cylinder around 0 is

$$
\mathcal{C}_{\theta,c} = \{(u_1^\theta, u_2^\theta) : |u_1^\theta| \leq \Phi^{-1}(c), |u_2^\theta| \leq \Phi^{-1}(c)\}.
$$

We call this the 0–cylinder of $E_{\theta,r,c}$; it has one inside end in the hyperplane $H_{\theta,\Phi^{-1}(c)}$, and an outside end in $H_{\theta,-\Phi^{-1}(c)}$. We thus call $E_{\theta,r,c}$ a tube-and-cylinders region. By monotonicity of $\Phi$ and $\Xi$, \{$u : D_{\theta,r}(u) = c$\} is the boundary of the tube-and-cylinders region.

By (2.8), on \{$u : 0 < u_1^\theta = |u_2^\theta|$\} (which is a cone boundary), $\Phi(|u|_{\theta,\infty})$ is the “min” in (2.6); this uses the fact that $u_1^\theta = |u|_{\theta,\infty}$ on that cone boundary. This means that the boundary of the 0–cylinder meets the shell in the inside end of the 0–cylinder, the intersection being the $(d-2)$-sphere of radius $c^{1/2}\Xi(\Phi^{-1}(c))$ around $\Phi^{-1}(c)y_\theta$ in $H_{\theta,\Phi^{-1}(c)}$; we call this $(d-2)$-sphere $S_{\theta}(c)$.

Let $\psi_{ab}$ denote the angle, taken in $[0, \pi]$, between nonzero vectors $a$ and $b$. The vector $\theta$ (or its multiple $y_\theta$) and the vector $z_\theta \perp H_{\theta,\theta}$ need not be parallel, but we can bound the angle between them as follows. Let $a > 0$ be such that $az_\theta \in H_{\theta,1}$; then $1 = a z_\theta \cdot z_\theta$ so $a = 1/|z_\theta|^2$. Also, $g(az_\theta) \geq 1$, and $az_\theta$ is the orthogonal projection of $y_\theta$ onto the line through 0 and $z_\theta$ so

$$
y_\theta \cdot \frac{z_\theta}{|z_\theta|} = |az_\theta|.
$$

From lattice symmetry, there exists an orthonormal basis for $\mathbb{R}^d$ containing $y_\theta$ and consisting of vectors in $\partial \mathcal{B}_0$ having the same Euclidean length; by inverting basis vectors we may assume $z_\theta$ has all nonnegative coefficients in this basis. Then the convex hull of the basis vectors includes a
multiple \( b \theta \) with \( b > 0 \); the convex hull is contained in \( \mathcal{B}_\theta \) so we must have \( b \leq a \). The minimum Euclidean length of vectors in this convex hull is \( |y_\theta|/\sqrt{d} \), so

\[
(2.9) \quad \frac{y_\theta}{|y_\theta|} \geq |\theta_0| \geq \frac{1}{\sqrt{d}}.
\]

Therefore \( \psi_{y_\theta,z_\theta} = \psi_{\theta,z_\theta} \leq \arccos 1/\sqrt{d} \); alternatively, we can say the angle between \( \theta \) and \( H_{\theta,0} \) is at least \( \arcsin 1/\sqrt{d} \). This has several consequences. First, for all \( u \in \mathbb{R}^d \),

\[
(2.10) \quad \frac{|u_\theta^2|}{\sqrt{d}} = \frac{d_\theta(u, L_\theta)}{\sqrt{d}} \leq d(u, L_\theta) \leq |u_\theta^0|, \quad |u_1^0|y_\theta| \leq \sqrt{d}|u|, \quad |u_2^\theta| \leq \sqrt{d-1}|u|, \quad |u_\theta - u_1^\theta| y_\theta| \leq \frac{d-1}{d} |u_2^\theta|
\]

(see Figure 2). Second, there exists \( \epsilon_5, \epsilon_6 > 0 \) and \( C_{30} \) as follows. Suppose \( \alpha, \theta \in S^{d-1} \) and the angle \( \psi_{\alpha,z_\theta} \) between \( H_{\alpha,0} \) and \( H_{\theta,0} \) is at most \( \epsilon_5 \). Suppose also that for some \( v \), \( L_\theta(v) \) intersects \( H_{\alpha,0} \) and \( H_{\theta,0} \) in points \( x_\alpha \) and \( x_\theta \) respectively. Then

\[
|x_\alpha - x_\theta| \leq C_{30} \psi_{\alpha,z_\theta} |x_\theta|,
\]

which by (1.12) means that

\[
(2.11) \quad \psi_{\alpha \theta} < \epsilon_6 \implies |x_\alpha - x_\theta| \leq C_{31} \psi_{\alpha \theta} |x_\theta|.
\]

In addition, for \( \alpha' \in S^{d-1} \), letting \( w_\alpha = L_\alpha(v) \cap H_{\theta,0}, w_{\alpha'} = L_{\alpha'}(v) \cap H_{\theta,0} \) we have

\[
(2.12) \quad \psi_{\alpha \theta} < \epsilon_6, \psi_{\alpha' \theta} < \epsilon_6 \implies |w_\alpha - w_{\alpha'}| \leq C_{31} \psi_{\alpha \theta}|v - x_\theta|.
\]

Here and in what follows, for a line \( L \), a hyperplane \( H \), and a point \( w \), we write \( w = L \cap H \) as a shorthand for \( \{w\} = L \cap H \).
Note that in both (2.11) and (2.12), we can view the context as starting with a line \( L_\theta(v) \) through \( v \) that intersects a hyperplane \( H_{\theta,0} \) at an angle of at least \( \arcsin \sqrt{d} \). Equation (2.11) bounds the change in the intersection point if we rotate the hyperplane around \( 0 \) from \( H_{\theta,0} \) to \( H_{\alpha,0} \), keeping the line fixed. Equation (2.12) bounds the change in the intersection point if we instead rotate the line through \( v \) from direction \( \theta \) to \( \alpha' \) (both near \( \theta \)), keeping the hyperplane fixed.

Another consequence of (2.11) and (2.12) is the following, relating change in \( \theta \)-coordinate values to change in the angle \( \theta \). When we change from \( \theta \) to \( \alpha \), in comparing \( x_2^\theta \) to \( x_2^\alpha \) it is not appropriate to simply consider \( |x_2^\theta - x_2^\alpha| \), as these are coordinate vectors under different bases, used in different spaces (\( H_\theta \), vs \( H_{\alpha,0} \)). Instead we compare them in \( \mathbb{R}^d \) by considering \( |(0, x_2^\theta) - (0, x_2^\alpha)| \).

**Lemma 2.2.** Suppose A2(ii) holds for some \( \theta_0 \) and \( \epsilon_0 > 0 \), and let \( \epsilon_6 \) be as in (2.11), (2.12). There exist \( C_i \) as follows. Suppose \( \alpha, \theta \in S^{d-1} \) with \( \psi_{\alpha,0} \leq \epsilon_6 \), and \( 0 \neq x \in \mathbb{R}^d \). Then

\[
\max \left( |x_1^\theta - x_1^\alpha|, |(0, x_2^\theta) - (0, x_2^\alpha)\right) \leq C_{32} \psi_{\alpha,0} |x|.
\]

**Proof.** Let \( 0 \neq x \in \mathbb{R}^d \) and \( \alpha, \theta \in S^{d-1} \) with \( \psi_{\alpha,0} \leq \epsilon_6 \). Let \( w_\theta = x_1^\theta y_\theta \) be the first \( \theta \)-component of \( x \), so that

\[
|x - w_\theta| = |x_2^\theta|, \quad |w_\theta| = |y_\theta||x_1^\theta|.
\]

Similarly let \( w_\alpha = x_1^\alpha y_\alpha \), and let \( q = L_\theta \cap H_{\alpha,x_\alpha^1} \) (noting \( x \in H_{\alpha,x_\alpha^1} \)). Then from (2.10), (2.11), and (2.12),

\[
|y_\theta||x_1^\theta - x_1^\alpha| = |w_\theta - x_1^\alpha y_\theta| \\
\leq |w_\theta - q| + |q - w_\alpha| + |x_1^\alpha| |y_\alpha - y_\theta| \\
\leq C_{31} \psi_{\alpha,0}|w_\theta| + C_{31} \psi_{\alpha,0}|w_\alpha| + c_1 \psi_{\alpha,0}|x_1^\alpha| \\
\leq c_2 \psi_{\alpha,0}|x|,
\]

and using the last two inequalities in (2.15),

\[
|(0, x_2^\theta) - (0, x_2^\alpha)\right) \leq |w_\theta - q| + |q - w_\alpha| \leq c_2 \psi_{\alpha,0}|x|.
\]

\( \square \)

We can use (2.10) to relate the \( \theta \)-ratio of \( u \) to the tangent of \( \psi_{u,\theta} \):

\[
u_\theta > 0, \quad \frac{|u_\theta|}{u_1} \leq \frac{|y_\theta|}{2} \implies \tan \psi_{u,\theta} = \frac{d(u, L_\theta)}{u \cdot \theta} \leq \frac{|u_\theta^0|}{u_1|y_\theta| - |u_2^\theta|} \leq \frac{2}{|y_\theta|} \frac{|u_\theta^0|}{u_1^0}.
\]

In the other direction, by (2.9), the tangent of angle between \( \theta \) and \( u - u_1^\theta y_\theta \) has magnitude at least \( 1/\sqrt{d - 1} \) so letting \( w \) be the closest point to \( u \) in \( L_\theta \), satisfying \( |w| = u \cdot \theta \), we have

\[
|u_1^\theta y_\theta - w| \leq \sqrt{d - 1}|u - w| = \sqrt{d - 1}|w| \tan \psi_{u,\theta}
\]

so

\[
u_\theta > 0, \quad \tan \psi_{u,\theta} = \frac{d(u, L_\theta)}{u \cdot \theta} \leq \frac{1}{2\sqrt{d - 1}} \implies \frac{|u_\theta^0|}{|y_\theta|u_1^\theta} \leq \frac{\sqrt{d}|u - w|}{|w| - |u_1^\theta y_\theta - w|} \leq \frac{2\sqrt{d}d(u, L_\theta)}{u \cdot \theta} \implies \frac{|u_\theta^0|}{u_1^\theta} \leq 2\sqrt{d}|y_\theta| \tan \psi_{u,\theta}.
\]
The bound on the angle between $y_0$ and $z_0$ also gives information about the 0-cylinder of $E_{θ,r,c}$. If $u$ lies in either end of $C_{θ,c}$ then $|u| \geq |y_0|\Phi^{-1}(c)/\sqrt{d}$, while if $u$ lies in the side of $C_{θ,c}$ then $|u| \geq \Phi^{-1}(c)/\sqrt{d}$. Thus

$$u \in \partial C_{θ,c} \implies \frac{|y_0| \wedge 1}{\sqrt{d}} \Phi^{-1}(c) \leq |u| \leq (|y_0| + 1)\Phi^{-1}(c).$$

(2.19)

Let $µ = g(e_1)$. Convexity and lattice symmetry yield that $µ\mathcal{B}_g$ contains the $ℓ^1$–unit ball in $\mathbb{R}^d$ and is contained in the $ℓ^∞$–unit ball, so

$$\frac{µ}{\sqrt{d}} |x| \leq g(x) \leq µ\sqrt{d}|x| \text{ for all } x \in \mathbb{R}^d.$$ 

(2.20)

In addition, from the triangle inequality,

$$|x_2| |y_0| - |x_1| |y_0| \leq |x| \leq |x_1| |y_0| + |x_2|.$$ 

(2.21)

Finally, given $λ \geq 1$, by (2.5), for some $C_{33}(λ)$, for all $s \geq C_{33}$,

$$Φ(s) \geq λ^{(1−χ)/2}Φ\left(\frac{s}{λ}\right).$$

(2.22)

Hence suppose $c$ is large and $x$ lies in the tube portion of $E_{θ,r,c}$, that is, $|x_2| \leq c^{1/2}ξ(x_1)$, and suppose $λ\Phi^{-1}(c) \leq x_1 \leq r/2$ for some $λ \geq 1$. Then using (2.7) and (2.22), the $θ$–ratio of $x$ satisfies

$$\frac{|x_2|}{x_1} \leq \frac{c^{1/2}ξ(x_1)}{x_1} \leq \frac{c^{1/2}}{Φ(λ\Phi^{-1}(c))^{1/2}} \leq λ^{−(1−χ)/4}.$$ 

(2.23)

Define the slabs

$$Ω_{θ}(s, t) = \{x \in \mathbb{R}^d : s \leq x_1 \leq t\}.$$ 

Of particular interest are $H_{θ,s}^{fat} := Ω_{θ}(s, s + µ\sqrt{d})$ and $H_{θ,s}^{fat} := Ω_{θ}(s − µ\sqrt{d}, s)$, which we call the fattened and backwards-fattened $H_{θ,s}$, respectively; generically we call any such slab of thickness $µ\sqrt{d}$ a fattened hyperplane. This thickness is chosen so that, by (2.20), any lattice path crossing a fattened hyperplane must have at least one site in it. If $x \in H_{θ,s}^{fat} \cup H_{θ,s}^{fat}$ then by (2.20) the $θ$–projection $\hat{x}$ of $x$ into $H_{θ,s}$ satisfies

$$|x − \hat{x}| \leq \sqrt{d} g(x − \hat{x}) \leq d.$$ 

(2.24)

More generally, for a set $B$ contained in some $H_{θ,s}$ we write $B^{fat}$ and $B^{fat}$ for $[s − µ\sqrt{d}, s] × B$ and $[s, s + µ\sqrt{d}] × B$ (in $θ$–coordinates), respectively. The values $θ, s$ will be uniquely determined by $B$ in all instances here.

Given $δ > 0$, a geodesic $Γ_{xy}$, and a site $u$ preceding a site $v$ in $Γ_{xy}$, we say that $x, u, v$ form a $δ$-fat triangle in $Γ_{xy}$ if $d(u, Π_{xy}) \geq δ|v − x|$. For $0 < δ < 1/2$ it follows straightforwardly from (2.3) that

$$|u − x| + |v − u| − |v − x| \geq (δ^2 ∧ δ)|v − x|,$$

(2.25)

that is, the extra distance associated with this triangle is at least $(δ^2 ∧ δ)|v − x|$. An analog for $g$ is the following variant of (2.4).
Lemma 2.3. Suppose $A1$ and $A2(i),(ii)$ hold for some $\theta_0$ and $\epsilon_0 > 0$. There exists $C_{34}$ as follows. For all $\delta > 0$, all $\theta \in S^{d-1}$ with $\psi_{\theta \theta_0} < \epsilon_0$, all $u,v \in \mathbb{R}^d$ with $v/|v| = \theta$ and $d(u,\Pi_{0v}) \geq \delta|v|$, we have

\begin{equation}
(2.26) \quad g(u) + g(v - u) - g(v) \geq C_{34}(\delta^2 \wedge \delta)|v|.
\end{equation}

Proof. Let $u,v$ be as in the lemma statement. Consider first the case of $g(u) \geq g(v)$; we then have from (2.20) that

\begin{equation}
(2.27) \quad g(u) + g(v - u) \geq g(v) + \frac{\mu}{\sqrt{d}}|v - u| \geq g(v) + \frac{\delta \mu}{\sqrt{d}}|v|.
\end{equation}

Consider next $u \notin \Omega_{\theta}(0,g(v))$, noting that this slab has $0,v$ in its boundary. From symmetry we may assume $u \in H^+_{\theta,g(v)}$. But then $g(u) \geq g(v)$ so (2.27) applies.

Finally consider $g(u) < g(v)$ with $u \in \Omega_{\theta}(0,g(v))$, so $u^0 \in [0,g(v)]$. We let $\hat{w} = u^0 y_\theta$ be the first $\theta$–component of $u$. Then $\hat{w} \in \Pi_{0v}$ so from symmetry we may assume $|\hat{w}| \geq |v|/2$, so that using (2.20),

\begin{equation}
(2.28) \quad \frac{u}{g(\hat{w})} - y_\theta = \frac{|u - \hat{w}|}{g(\hat{w})} \geq \frac{d(u,\Pi_{0v})}{g(v)} \geq \frac{\delta}{\mu \sqrt{d}}.
\end{equation}

Note that $H_{\theta,g(\hat{w})}$ contains $u,\hat{w}$ and is tangent to $\partial(g(\hat{w})\mathcal{B}_g)$ at $\hat{w}$. It therefore follows from (2.28) and (1.8) that

\begin{equation}
(2.29) \quad g(u) \geq g(\hat{w}) \left(1 + d_g \left(\frac{u}{g(\hat{w})},\mathcal{B}_g\right)\right) \geq g(\hat{w}) \left(1 + \frac{c_2 \mu}{\sqrt{d}} (\delta \wedge \delta^2)\right) \geq g(\hat{w}) + c_3 (\delta \wedge \delta^2)|v|.
\end{equation}

Now $H_{\theta,g(\hat{w})}$ contains $u$ and is tangent to the boundary of the translate $v + g(\hat{w} - v)\mathcal{B}_g$ at $\hat{w}$. It follows that $g(u - v) \geq g(\hat{w} - v)$. Therefore using (2.29),

\begin{equation}
\begin{split}
(2.30) \quad g(u) + g(v - u) & \geq g(\hat{w}) + g(\hat{w} - v) + c_3 (\delta \wedge \delta^2)|v| \geq g(v) + c_3 (\delta \wedge \delta^2)|v|,
\end{split}
\end{equation}

as desired. \hfill \Box

The proof of the next proposition is based on the fact that if a path $\gamma$ from 0 to some site $r y_\theta$ contains a site $u$ with $D_{\theta,r}(u) \geq t$, then there are necessarily 3 sites in $\gamma$ (one of which is an endpoint, 0 or $r y_\theta$) which form a $\delta$–fat triangle for some appropriate $\delta$, and Lemma 2.3 can be used to help show that the probability of this is small. Analogous results based on the same general principle appear in [1] for an integrable last passage percolation model in $d = 2$, and in [17] for FPP in $d = 2$ under hypotheses similar to ours here.

Proposition 2.4. Suppose $A1$ and $A2(i),(ii)$ hold for some $\theta_0$ and $\epsilon_0 > 0$. There exist constants $C_i$ as follows. For all $r,\theta$ with $\psi_{\theta \theta_0} < \epsilon_0$ and $0 \neq r y_\theta \in \mathbb{Z}^d$, and all $t > 0$,

\begin{equation}
(2.30) \quad P \left( \max_{u \in \Gamma_{0,r y_\theta}} D_{\theta,r}(u) \geq t \right) \leq C_{35} e^{-C_{36} t \log t}.
\end{equation}
Proof. It is enough to consider \( t \) sufficiently large (not depending on \( r, \theta \)). Let \( U = (U_1^\theta, U_2^\theta) \) be the site which maximizes \( D_{\theta,r}(\cdot) \) over \( \Gamma_{0x} \), with ties broken arbitrarily, and let \( C = D_{\theta,r}(U) \geq t \) be the corresponding maximum value. By monotonicity of \( \Phi \), \( U \) must lie on the boundary of \( \{ u : D_{\theta,r}(u) \leq C \} \). From symmetry we may assume \( U_1^\theta \leq r/2 \). We show there exists \( W \in \Gamma_{0x} \) such that \( U, W \) form a \( \delta \)-fat triangle for appropriate \( \delta \). Fix \( \kappa \) large enough so

\[
\kappa \geq \frac{2\sqrt{d}(1 + |y_\theta|)}{|y_\theta| \wedge 1}, \quad \frac{C_3}{\kappa^{(1-\gamma)/2}} \frac{\log(2 + \kappa s)}{\log(2 + s)} \leq 1 \text{ for all } s \geq 0,
\]

and \( \kappa^{-(1-\chi)/4} \leq \left( \frac{\epsilon_0}{4} \wedge \frac{1}{4\sqrt{d}(1 + |y_\theta|)C_3^{1/2}} \right) |y_\theta|. \) \quad \tag{2.31}

Note that from (2.5), (1.2), and (2.31), given \( \delta > 0 \), provided we take \( \kappa \) sufficiently large,

\[
\left( \frac{\Xi(\kappa s)}{\kappa} \right)^2 = \frac{s}{\kappa} \sigma_s \log(2 + \kappa s) \leq \frac{C_3}{\kappa^{1-\gamma}} s \sigma_s \log(2 + \kappa s) \leq \delta \Xi(s)^2.
\]

There are four cases; see Figure 3.

Case 1: Suppose that \( U_1^\theta \leq \Phi^{-1}(C) \leq r/2 \kappa \). Then \( U \) lies on the boundary of the 0-cylinder \( \{ u : \Phi(|u|_{\theta,\infty}) \leq C \} \). Let \( W \) be the first point of \( \Gamma_{0x} \) after \( U \) with \( W_1^\theta = \kappa \Phi^{-1}(C) \), and let \( W \) be the first site in \( \Gamma_{0x} \) after \( W \). Then \( W, W \) must lie in the “tube” portion of \( E_{\theta,r,C} \), meaning \( |W_2^\theta| \leq C^{1/2} \Xi(W_1^\theta) \). Provided \( t \) (and hence \( C \) and \( W_1^\theta \)) are large, from (2.7), (2.22), and (2.31), \( \theta \)-ratio of \( W \) satisfies

\[
\frac{|W_2^\theta|}{|W_1^\theta|} \leq C^{1/2} \Xi(W_1^\theta) \leq \left( \frac{C}{\Phi(W_1^\theta)} \right)^{1/2} \kappa^{-(1-\chi)/4} \leq \frac{\epsilon_0 |y_\theta|}{4},
\]

and hence by (2.17),

\[
\tan \psi_{\theta W} \leq \frac{2}{|y_\theta|} \kappa^{-(1-\chi)/4} \leq \frac{\epsilon_0}{2},
\]

so \( \psi_{\theta W} < \epsilon_0 \). Now \( \min \{ d_\theta(u, L_\theta) : u_1^\theta \leq \Phi^{-1}(C), D_{\theta,r}(u) = C \} \) is achieved on \( S_\theta(C) \) so is equal to \( C^{1/2} \Xi(\Phi^{-1}(C)) \). Hence by (2.10),

\[
d(U, L_\theta) \geq \frac{1}{d^{1/2}} C^{1/2} \Xi(\Phi^{-1}(C)).
\]

On the other hand, letting \( V \) denote the closest point to \( U \) in \( \Pi_{0W} \), we have \( |V| \leq |U| \leq (1 + |y_\theta|) \Phi^{-1}(C) \), so from (2.7), (2.31), and (2.34),

\[
d(V, L_\theta) \leq |V| \tan \psi_{\theta W} \leq \frac{2(1 + |y_\theta|)}{|y_\theta|} \kappa^{-(1-\chi)/4} \Phi^{-1}(C) \leq \frac{2(1 + |y_\theta|)C^{1/2}}{|y_\theta|} \kappa^{-(1-\chi)/4} C^{1/2} \Xi(\Phi^{-1}(C)) \leq \frac{1}{2d^{1/2}} C^{1/2} \Xi(\Phi^{-1}(C)) \leq \frac{1}{2d^{1/2}} C^{1/2} \Xi(\Phi^{-1}(C)),
\]

so using (2.7) again, we must have

\[
d(U, \Pi_{0W}) = d(U, V) \geq \frac{1}{2d^{1/2}} C^{1/2} \Xi(\Phi^{-1}(C)) \geq \frac{1}{2d^{1/2}} \Phi^{-1}(C).
\]

\[
(2.35)
\]
Now in $\theta$–coordinates $\overline{W}/\kappa = (\Phi^{-1}(C), \kappa^{-1}W^\theta_2)_\varrho$, and from (2.32) and (2.7),

\[
\frac{|W^\theta_2|}{\kappa} \leq \frac{C^{1/2}\Xi(\kappa\Phi^{-1}(C))}{\kappa} \leq C^{1/2}\Xi(\Phi^{-1}(C)) \leq \Phi^{-1}(C),
\]

so $\overline{W}/\kappa$ lies in the boundary of the 0-cylinder of $E_{\theta,r,C}$, and hence by (2.19),

\[
\frac{|y_\theta| \wedge 1}{\sqrt{d}} \Phi^{-1}(C) \leq \frac{|\overline{W}|}{\kappa} \leq (1 + |y_\theta|)\Phi^{-1}(C).
\]

From (2.35) and the second inequality in (2.37) we obtain

\[
d(U, \Pi_0) \geq \frac{1}{2\kappa d^{1/2}(1 + |y_\theta|)} |\overline{W}|.
\]

Since $|W - \overline{W}| \leq 1$ it is then straightforward that

\[
d(U, \Pi_{0,W}) \geq \frac{1}{4\kappa d^{1/2}(1 + |y_\theta|)} |W|,
\]

meaning $0, U, W$ form a $\delta$–fat triangle for $\delta = (4\kappa d^{1/2}(1 + |y_\theta|))^{-1}$. Then from Proposition 2.1 and Lemma 2.3,

\[
h(U) + h(W - U) - h(W) \geq g(U) + g(W - U) - g(W) - C_{25}\sigma_{|w|} \log |w| \geq c_1 |W|.
\]

Let $K_{t,1}$ satisfy

\[
2^{K_{t,1}} \leq \frac{\kappa(|y_\theta| \wedge 1) \Phi^{-1}(t)}{\sqrt{d}} \leq \frac{3 \kappa(|y_\theta| \wedge 1) \Phi^{-1}(t)}{2} \leq 2^{K_{t,1}}.
\]
so from \((2.37)\), since \(|W - \overline{W}| \leq 1\),

\[
2^{K_{t,1} - 3} \leq \frac{\kappa(|y_0| \wedge 1)\Phi^{-1}(t)}{2\sqrt{d}} \leq \frac{\kappa(|y_0| \wedge 1)|\Phi^{-1}(C)|}{2\sqrt{d}} \leq \frac{|W|}{2} \leq |W|.
\]

Since \(U\) lies in the boundary of the 0-cylinder of \(E_{\theta,r,C}\), as in \((2.37)\) we have using \((2.31)\)

\[
|U| \leq (1 + |y_0|)\Phi^{-1}(C) \leq \frac{2\sqrt{d}(1 + |y_0|)}{\kappa(|y_0| \wedge 1)|W|} \leq |W|.
\]

Consider now the events

\[A_k : \text{there exist } u, w \in \mathbb{Z}^d \text{ with } 2^{k-3} < |w| \leq 2^k, |u| \leq |w|, \quad h(u) + h(w - u) - h(w) \geq c_1|w|, \quad T(0, u) + T(u, w) = T(0, w).\]

We have by \((2.38)\) that

\[
(2.40) \quad P \left( \sup_{u \in \Gamma_0, r y_0} D_{\theta,r}(u) \geq t \text{ and Case 1 holds} \right) \leq 2 \sum_{k = K_{t,1}}^{\infty} P(A_k).
\]

Here the factor of 2 accounts for the fact we assumed \(U_1^\theta \leq r/2\). Let \(A_k(u, w)\) denote the event that one of the following holds:

\[
(2.41) \quad h(u) - T(0, u) \geq \frac{c_1}{3}|w|, \quad h(w - u) - T(u, w) \geq \frac{c_1}{3}|w|, \quad T(0, w) - h(w) \geq \frac{c_1}{3}|w|.
\]

For \(u, w\) as in the event \(A_k\), one of these inequalities must hold, so \(A_k \subset \cup_{u,w} A_k(u, w)\), where the union is over \(u, w\) as in the definition of \(A_k\). For each such \(u, w\) we have from \((1.4)\) that

\[
P(A_k(u, w)) \leq 3C_5 \exp \left(-C_0 c_1 2^{k-3}/6\sigma(2^{k+1})\right).
\]

Summing the last bound over \(u, w, k\) and using \((2.39)\), \((2.40)\) yields

\[
P \left( \sup_{u \in \Gamma_0, r y_0} D_{\theta,r}(u) \geq t \text{ and Case 1 holds} \right) \leq 2 \sum_{k = K_{t,1}}^{\infty} P(A_k) \leq \sum_{k = K_{t,1}}^{\infty} c_2 2^{2k} \exp \left(-c_3 2^k/\sigma(2^k)\right) \leq c_4 \exp \left(-c_5 \Phi^{-1}(t)/\sigma(\Phi^{-1}(t))\right) \leq c_4 e^{-c_6 t \log t}.
\]

\[
(2.42)
\]

**Case 2:** Suppose \(U_1^\theta \leq \Phi^{-1}(C)\) and \(\Phi^{-1}(C) > r/2\kappa\). (The latter means “the cylinder is not small relative to the tube.”) Then \(U\) again lies on the boundary of the 0-cylinder in \(E_{\theta,r,C}\), but this time we take \(W = r y_0\). We consider two subcases.

**Case 2a:** Suppose Case 2 holds with \(\Phi^{-1}(C) \leq r/2\). This means the tube is not completely contained inside the two cylinders; this can only occur when \(\Phi^{-1}(t) \leq r/2\). Using \((2.19)\) we then have

\[
\frac{1 \wedge |y_0|}{2\kappa \sqrt{d}} r < \frac{1 \wedge |y_0|}{\sqrt{d}} \Phi^{-1}(C) \leq |U| \leq (1 + |y_0|)\Phi^{-1}(C) \leq \frac{1 + |y_0|}{2} r,
\]

\[
\frac{1 \wedge |y_0|}{2\kappa \sqrt{d}} r < \frac{1 \wedge |y_0|}{\sqrt{d}} \Phi^{-1}(C) \leq |U| \leq (1 + |y_0|)\Phi^{-1}(C) \leq \frac{1 + |y_0|}{2} r,
\]

\[
\frac{1 \wedge |y_0|}{2\kappa \sqrt{d}} r < \frac{1 \wedge |y_0|}{\sqrt{d}} \Phi^{-1}(C) \leq |U| \leq (1 + |y_0|)\Phi^{-1}(C) \leq \frac{1 + |y_0|}{2} r.
\]
and as in Case 1, using (2.7),
\[ d(U, \Pi_{0W}) \geq d(U, L_\theta) \geq \frac{1}{\sqrt{d}} C^{1/2} \Xi(\Phi^{-1}(C)) \geq \frac{1}{(C_3d)^{1/2}} \Phi^{-1}(C) \geq c_7 |y_0| r = c_7 |W|. \]

Thus 0, U, W form a \( c_7 \)-fat triangle. Define the event
\[ B : \text{there exists } u \in \mathbb{Z}^d \text{ with } \frac{1}{2} |y_0| r < |u| \leq \frac{1 + |y_0|}{2} r, \]
\[ h(u) + h(r y_0 - u) - h(r y_0) \geq c_8 r, \text{ and } T(0, u) + T(u, r y_0) = T(0, r y_0). \]

Similarly to Case 1 we have using \( \Phi^{-1}(t) \leq r/2 \) that
\[
P \left( \sup_{u \in \Gamma_{0, ry_0}} D_{\theta, r}(u) \geq t \text{ and Case 2a holds} \right) \leq 2P(B) \leq c_9 r^d e^{-c_{10r}/\sigma r} \leq c_9 e^{-c_{11 t \log t}}. \tag{2.43} \]

Case 2b: Suppose Case 2 holds with \( \Phi^{-1}(C) > r/2 \). This means the two cylinders contain the entire tube, and here we need only consider \( \Phi^{-1}(t) \geq r/2 \). This is generally similar to Case 1, except that we do not know \( |U| \leq |W| = r|y_0| \).

Analogously to (2.19) we do have
\[ d(U, \Pi_{0, ry_0}) \geq (|y_0| \wedge 1) \Phi^{-1}(t) / \sqrt{d} \geq c_{12} r, \text{ so as in (2.38),} \]
\[ h(u) + h(r y_0 - u) - h(r y_0) \geq c_{13} |U|. \]

We also know from (2.19) that \( |U| \geq c_{14} \Phi^{-1}(t) \). Instead of the events \( A_k \) we use
\[ B_k : \text{there exists } u \in \mathbb{Z}^d \text{ with } 2^{k-1} < |u| \leq 2^k, \]
\[ h(u) + h(r y_0 - u) - h(r y_0) \geq c_{13} |u|, \text{ and } T(0, u) + T(u, r y_0) = T(0, r y_0), \]

and we define \( K_{1,2} \) by
\[ 2^{K_{1,2} - 1} < c_{14} \Phi^{-1}(t) \leq 2^{K_{1,2}}, \]

so that similarly to (2.42),
\[
P \left( \sup_{u \in \Gamma_{0, r_{\epsilon_1}}} D_r(u) \geq t \text{ and Case 2b holds} \right) \leq 2 \sum_{k = K_{1,2}}^\infty P(B_k) \leq 2 \sum_{k = K_{1,2}}^\infty c_{15} 2^{dk} \exp \left( -c_{16} 2^k / \sigma(2^k) \right) \leq c_{17} e^{-c_{18 t \log t}}. \tag{2.44} \]

Case 3: Suppose \( \Phi^{-1}(C) < U_1^\theta \leq r/2 \kappa \), meaning that \( U \) lies on the tube boundary \( \{ u : |u_2^\theta| = C^{1/2} \Xi(u_1^\theta) \} \) near the 0 end (but outside the 0-cylinder.) Similarly to Case 1, let \( \overline{W} \) be the first point of \( \Gamma_{0, ry_0} \) after \( U \) with \( \overline{W}_1^\theta = \kappa U_1^\theta \), and let \( W \) be the first site in \( \Gamma_{0, ry_0} \) after \( \overline{W} \). Then using (2.7) and (2.32),
\[
|\overline{W}| \geq |\overline{W}_{1, y_0}| - |\overline{W}_2^\theta| \geq \kappa |y_0| U_1^\theta - C^{1/2} \Xi(\kappa U_1^\theta) \geq \kappa |y_0| U_1^\theta - \frac{\kappa |y_0|}{2} C^{1/2} \Xi(U_1^\theta) \geq \frac{\kappa |y_0|}{2} \Phi^{-1}(C), \tag{2.45} \]

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and the $\theta$-ratio of $U$ satisfies
\begin{equation}
|U_2^\theta| = \frac{C^{1/2} \Xi(U_1^\theta)}{U_1^\theta} \leq \frac{C^{1/2}}{\Phi(U_1^\theta)^{1/2}} < 1
\end{equation}
so using the next-to-last inequality in (2.45), provided $\kappa$ is large,
\begin{equation}
|U| \leq (1 + |y_\theta|)U_1^\theta \leq \frac{|W|}{2} \leq |W|.
\end{equation}
As in Case 1 let $V$ be the closest point to $U$ in $\Pi_{\theta W}$, so $|V| \leq |U|$. From (2.31) and (2.46), the $\theta$-ratio of $W$ satisfies
\begin{equation}
|W_2^\theta| \leq \frac{C^{1/2} \Xi(\kappa U_1^\theta)}{\kappa U_1^\theta} \leq \kappa^{-(1-\chi)/4} \frac{C^{1/2} \Xi(U_1^\theta)}{U_1^\theta} \leq \kappa^{-(1-\chi)/4} < \frac{|y_\theta|}{2},
\end{equation}
so by (2.17), (2.32), and (2.47),
\begin{equation}
d(V, L_\theta) \leq |V| \tan \psi_{\theta W} \leq |U| \tan \psi_{\theta W} \leq \frac{2}{|y_\theta|} \frac{C^{1/2} \Xi(\kappa U_1^\theta)}{\kappa U_1^\theta} \leq \frac{1}{2\sqrt{d}} C^{1/2} \Xi(U_1^\theta).
\end{equation}
On the other hand, we have from (2.10) that
\begin{equation}
d_{\theta}(U, L_\theta) = C^{1/2} \Xi(U_1^\theta) \quad \text{so} \quad d(U, L_\theta) \geq \frac{1}{\sqrt{d}} C^{1/2} \Xi(U_1^\theta),
\end{equation}
which with (2.49) shows
\begin{equation}
d(U, \Pi_{\theta W}) = |U - V| \geq \frac{1}{2\sqrt{d}} C^{1/2} \Xi(U_1^\theta).
\end{equation}
Thus $0, U, \bar{W}$ form a $\delta$-fat triangle with
\begin{equation}
\delta = \frac{C^{1/2} \Xi(U_1^\theta)}{2\sqrt{d}|W|}.
\end{equation}
From (2.48) and (2.21) we have
\begin{equation}
\frac{1}{2} |y_\theta| \kappa U_1^\theta \leq \frac{1}{2} |y_\theta| W_1^\theta \leq |\bar{W}| \leq 2|y_\theta| W_1^\theta,
\end{equation}
so from (2.7), provided $\kappa$ is large this $\delta$ satisfies
\begin{equation}
\delta \leq \frac{\Phi(U_1^\theta)^{1/2} \Xi(U_1^\theta)}{\kappa |y_\theta| \sqrt{d} U_1^\theta} \leq \frac{1}{\kappa |y_\theta| \sqrt{d}} < 1.
\end{equation}
Hence from Lemma 2.3, (2.48), and (2.7) we have
\begin{equation}
g(U) + g(\bar{W} - U) - g(\bar{W}) \geq C_{34} \frac{C \Xi(W_1^\theta/\kappa)^2}{4d|W|}
\end{equation}
\begin{equation}
\geq \frac{C_{34} C W_1^\theta}{4d\kappa} \sigma \left( \frac{W_1^\theta}{\kappa} \right) \log \left( 2 + \frac{W_1^\theta}{\kappa} \right)
\end{equation}
\begin{equation}
\geq c_{19} C \sigma(|W|) \log(2 + |W|),
\end{equation}
and since $|W - \overline{W}| \leq 1$, the same holds for $W$ in place of $\overline{W}$, possibly with a smaller $c_{19}$. Defining the events

$F_k$: there exist $u, w \in \mathbb{Z}^d$ with $2^{k-1} < |w| \leq 2^k, |u| \leq |w|,$

$$g(u) + g(w - u) - g(w) \geq c_{19} t \sigma(|w|) \log(2 + |w|),$$

and $T(0, u) + T(u, w) = T(0, w)$,

and defining $K_{t, 3}$ by

$$2^{K_{t, 3} - 1} < \frac{k|y|}{2} \Phi^{-1}(t) \leq 2^{K_{t, 3}},$$

in view of (2.45) and (2.47) we have similarly to (2.42) and (2.44), provided $t$ is large:

$$P\left(\sup_{u \in \Gamma_{0, r, r_1}} D_r(u) \geq t \text{ and Case 3 holds}\right) \leq 2 \sum_{k = K_{t, 3}}^{\infty} P(F_k)$$

$$\leq 2 \sum_{k = K_{t, 3}}^{\infty} c_{20} 2^{2dk} \exp(-c_{21} tk)$$

$$\leq c_{22} e^{-c_{23} t \log t}.$$  

(2.51)

Case 4: Suppose $\max(\Phi^{-1}(C), r/2 \kappa) < U_1^\theta \leq r / 2$, meaning that $U$ lies on the tube boundary but not near an end. This can only occur when $\Phi^{-1}(t) \leq r / 2$, and this time we take $W = ry_\theta$. We have using (2.10) that

$$d(U, \Pi_{0W}) \geq d(U, L_\theta) \geq \frac{d_\theta(U, L_\theta)}{\sqrt{d}} = \frac{|U_2^\theta|}{\sqrt{d}} = C^{1/2} \Xi(U_1^\theta) \geq C^{1/2} \Xi(r/2 \kappa) \frac{\sqrt{d}}{\sqrt{d}}$$

so $0, U, ry_\theta$ form a $\delta$-fat triangle with

$$\delta = C^{1/2} \Xi(r/2 \kappa) \frac{1}{\sqrt{d}|\theta|}.$$  

From the definition (2.6) we have

$$\delta^2 r \geq \frac{C \Xi(r/2 \kappa)^2}{d|\theta|^2 r} = \frac{C}{2 \kappa d|y_\theta|^2 \sigma} \left(\frac{r}{2 \kappa}\right) \log \left(2 + \frac{r}{2 \kappa}\right).$$

Similarly to Case 3 we have $\delta < 1$. From (2.21) and (2.7) we obtain

$$|U| \leq |y_\theta| U_1^\theta + C^{1/2} \Xi(U_1^\theta) \leq |y_\theta| U_1^\theta + \Phi(U_1^\theta)^{1/2} \Xi(U_1^\theta) \leq (1 + |y_\theta|) U_1^\theta \leq (1 + |y_\theta|) r.$$

Defining the event

$F$: there exists $u \in \mathbb{Z}^d$ with $|u| \leq (1 + |y_\theta|) r$, $T(0, u) + T(u, ry_\theta) = T(0, ry_\theta)$,

and $g(u) + g(ry_\theta - u) - g(ry_\theta) \geq C_{34} \frac{r}{2 \kappa d|y_\theta|^2} \sigma \left(\frac{r}{2 \kappa}\right) \log \left(2 + \frac{r}{2 \kappa}\right),$

the rest is similar to Case 2a, and we obtain

$$P\left(\sup_{u \in \Gamma_{0, ry_\theta}} D_{\theta, r}(u) \geq t \text{ and Case 4 holds}\right) \leq 2P(F) \leq c_{24} e^{-c_{25} t \log t}.$$  

(2.52)

Putting the 4 cases together, (2.42), (2.43), (2.44), (2.51), and (2.52) complete the proof. \qed
The following is a purely deterministic result about norms on $\mathbb{Z}^d$ when the local curvature assumption is satisfied.

**Lemma 2.5.** Suppose the norm $g$ satisfies the local curvature assumption $A2(ii)$ for some $\theta_0, e_0$. There exist constants $\epsilon_7$ and $C_i$ as follows. Suppose $\ell > C_{37}$, $\psi_{\theta_0} < \epsilon_7$, and $u, v \in H_{\theta, \ell}$ with

$$|u - \ell y_0| < \ell \epsilon_7, \quad \frac{|v - u|}{\ell} < \ell \epsilon_7. \tag{2.53}$$

Then

$$|g(v) - g(u)| \leq C_{38} \left( \frac{|v - u| |u - \ell y_0|}{\ell} + \frac{|v - u|^2}{\ell} \right). \tag{2.54}$$

**Proof.** We bound $g(v) - g(u)$ as the opposite bound is nearly symmetric. Let $\alpha = u/|u|$. By $A2(ii)$ and (1.12), provided we take $\epsilon_7$ small, the first inequality in (2.53) guarantees that the angle between $H_{\theta, 0}$ and $H_{\alpha, 0}$ is at most $c_1 |u - \ell y_0|/\ell$. Since $v - u \in H_{\theta, 0}$, it follows that the orthogonal projection $a$ of $v - u$ into $H_{\alpha, 0}$ satisfies $|a| \leq |v - u|$ and

$$|(v - u) - a| \leq |v - u| \sin \left( \frac{c_1 |u - \ell y_0|}{\ell} \right) \leq c_1 |v - u| |u - \ell y_0| \tag{2.55}$$

(see Figure 4.) We have

$$\sin \psi_{\theta, u} \leq \frac{|u - \ell y_0|}{\ell |y_0|} < \frac{\epsilon_7}{\ell |y_0|},$$

so provided $\epsilon_7$ is small enough,

$$\psi_{\alpha, \theta} \leq \psi_{\alpha, \theta} + \frac{\psi_{\theta, 0}}{\ell} \leq \left( \frac{2}{|y_0|} + 1 \right) \frac{\epsilon_7}{\ell} < e_0. \tag{2.56}$$

Now $u + a$ lies in the tangent plane $H_{\alpha, u^2}$ to $g(u) \mathbb{B}_g$ at $u$, with

$$\frac{|a|}{g(u)} \leq \frac{|v - u|}{\ell} < \ell \epsilon_7,$$

so by $A2(ii)$, (2.20), and (2.56),

$$g(u + a) - g(u) = d_g(u + a, g(u) \mathbb{B}_g) \leq \mu \sqrt{d} d(u + a, g(u) \mathbb{B}_g) \leq c_2 \frac{|a|^2}{g(u)} \leq c_2 \frac{|v - u|^2}{\ell}. \tag{2.57}$$

Combining this with (2.20) and (2.55) yields

$$g(v) - g(u) \leq g(u + a) + g((v - u) - a) - g(u) \leq c_2 \frac{|v - u|^2}{\ell} + c_3 \frac{|v - u| |u - \ell y_0|}{\ell}. \tag{2.58}$$

$\square$

We next consider the transverse increments of $T(0, u)$, that is, we bound $|T(0, u) - T(0, v)|$ when $|g(u) - g(v)|$ is small. (We can’t require it be exactly 0 since $u, v$ are lattice points.) Heuristically, assuming $|u - v| \ll \Delta(|u|)$, $\Delta^{-1}(|u - v|)$ may be viewed as the typical distance traveled by $\Gamma_{u_0}$ and $\Gamma_{v_0}$ before they can get close enough to coalesce. Then $\sigma(\Delta^{-1}(|u - v|))$ becomes the scale of “fluctuations before coalescing,” and we show that $|T(0, u) - T(0, v)|$ is unlikely to be much larger than this scale. A variant of the following proposition, for $d = 2$, appears in [17].
Proposition 2.6. Suppose A1 and A2(i),(ii) hold for some $\theta_0, \epsilon_0$, and let $\epsilon_7$ be as in Lemma 2.5. There exist constants $C_i$ as follows. For all $u,v \in \mathbb{Z}^d$ with

\begin{equation}
|u| \geq C_{39}, \quad \left| \theta_0 - \frac{u}{|u|} \right| < \epsilon_7, \quad |g(u) - g(v)| \leq 4\mu d, \quad \text{and} \quad |u - v| \leq C_{40} \Delta |u|,
\end{equation}

and all $\lambda \geq C_{41}$, we have

\begin{equation}
P(\Gamma(0) \geq \Gamma(0) + \lambda \sigma \Delta^{-1} |u - v| \log |u - v|) \leq C_{42} e^{-C_{43} \lambda |u - v|}.
\end{equation}

Proof. Let

\begin{align*}
\theta &= \frac{u}{|u|}, \quad \ell = c_1 \Delta^{-1} |u - v|, \quad \text{and} \quad t = c_2 \lambda |u - v|,
\end{align*}

with $c_1, c_2$ to be specified; note log $\ell$ and log $|u - v|$ are of the same order. Provided $C_{40}$ is taken small (depending on $c_1$), we have $\ell \leq |u|/2|y_0| = u_0/2$. We consider first the case of “moderately large $\lambda$”:

\begin{equation}
\lambda \sigma \Delta^{-1} |u - v| \log |u - v| \leq 2h(u - v).
\end{equation}

We intersect a tube-and-cylinders region with a fattened hyperplane to get

\begin{align*}
\Upsilon_\ell &= H_{\ell, u_0, \theta} \cap E_{\ell, u_0, \theta},
\end{align*}

so that every path crossing $H_{\ell, u_0, \theta}$ inside $E_{\ell, u_0, \theta}$ must include a site in $\Upsilon_\ell$. Therefore either $\Gamma_0 \not\subset E_{\ell, u_0, \theta}$ or there exists a first site $X$ of $\Gamma_0$ in $\Upsilon_\ell$, in which case

\begin{align*}
T(v, 0) &= T(u, X) + T(X, 0) \quad \text{so} \quad T(v, 0) - T(u, 0) \leq T(v, X) - T(u, X).
\end{align*}

It follows using Proposition 2.4 that

\begin{align*}
P(T(v, 0) - T(u, 0) \geq \lambda \sigma \Delta^{-1} |u - v| |u - v|)
\leq P \left( \sup_{x \in \Gamma_0} D_{\ell, u_0, \theta} > t \right) + \sum_{x \in \Upsilon_\ell} P(T(v, x) - T(u, x) \geq \lambda \sigma \Delta^{-1} |u - v| |u - v|)
\leq C_{35} e^{-C_{36} t} + C_3 \ell^{d-1} \max_{x \in \Upsilon_\ell} P(T(v, x) - T(u, x) \geq \lambda \sigma \Delta^{-1} |u - v| |u - v|).
\end{align*}
Figure 5. Illustration for the proof of Proposition 2.6. $T(0, u)$ and $T(0, v)$ are likely to be close because $\Gamma_{0u}$ has the option to pass through the same site $x \in \mathcal{Y}_\ell$ that $\Gamma_{0v}$ passes through.

Let $x \in \mathcal{Y}_\ell \cap \mathbb{Z}^d$. We claim that
\begin{equation}
(2.61) \quad g(v - x) - g(u - x) \leq c_4(t \log |u - v|)^{1/2} \sigma(\Delta^{-1}(|u - v|)).
\end{equation}
To prove this, let $w = (u_1^\theta - \ell)y_\theta \in \Pi_{0u}$. We first approximate $x$ by its $\theta$–projection $\hat{x} = (w_1^\theta, x_2^\theta)$ into $H_{\theta, u_1^\theta - \ell}$. Then from the definition of $\mathcal{Y}_\ell$,
\begin{equation}
(2.62) \quad g(x - \hat{x}) = |x_1^\theta - w_1^\theta| \leq \mu \sqrt{d}.
\end{equation}
Let $\hat{v}$ be the closest point to $v$ in $H_{\theta, u_1^\theta}$. Since $|g(v) - g(u)| \leq 4\mu d$, provided $|u|$ is large (so by (2.57) $\psi_{uv}$ is small) there exists a point $\overline{v}$ on the line through $v$ and $\hat{v}$ satisfying $g(\overline{v}) = g(u)$ and $g(v - \overline{v}) \leq 5\mu d$. In order to bound $g(v - \hat{v})$, we first observe that
\begin{equation}
(2.63) \quad |u - \hat{v}| \leq |u - v| = \Delta(\ell/c_1) \quad \text{and hence} \quad \frac{|u - \hat{v}|}{g(u)} \leq \frac{\Delta(\ell/c_1)}{\ell},
\end{equation}
and the last fraction can be made small by taking $c_1$ large in the definition of $\ell$ (so $\ell$ itself is large), and it then follows from A2(ii) and (2.20) that
\begin{align}
(2.64) \quad g(v - \hat{v}) &\leq \mu \sqrt{d}|\overline{v} - \hat{v}| + g(v - \overline{v}) = \mu \sqrt{d}d(\overline{v}, H_{\theta, u_1^\theta}) + 5\mu d \leq c_5 \left|\frac{|u - \hat{v}|}{u_1^\theta}\right| + 5\mu d \\
&\leq c_5 \frac{\Delta(\ell/c_1)^2}{\ell} + 5\mu d \leq c_6 \sigma(\Delta^{-1}(|u - v|)).
\end{align}
In view of (2.62) and (2.64), to prove (2.61) it remains to bound $g(\hat{v} - \hat{x}) - g(u - \hat{x})$, for which we use Lemma 2.5 with the origin shifted to $\hat{x}$. First observe that provided $c_1$ (and hence $\ell$) is large, using (2.59) we have
\begin{equation}
(2.65) \quad t = c_2 \lambda \log |u - v| \leq c_7 \frac{\Delta(\ell)}{\sigma_\ell} = c_7 \left(\frac{\ell}{\sigma_\ell}\right)^{1/2} < \Phi(\ell).
\end{equation}
Therefore $\ell > \Phi^{-1}(t)$, meaning that $w, x$ lie in the tube part of the tube-and-cylinders region $E_{\theta, u_1^\ell, \ell}$. Therefore

$$\frac{|w - \hat{x}|}{\ell} = \frac{|x_2^0|}{\ell} \leq \frac{t^{1/2} \Xi((x - u)^0)}{\ell} = \frac{t^{1/2} \Xi(\ell)}{\ell},$$

and hence from the second inequality in (2.65), provided $c_1$ (and therefore $\ell$) is large,

$$\frac{|w - \hat{x}|}{\ell} \leq \frac{t^{1/2} \Xi(\ell)}{\ell} \leq \left(\frac{\Delta \log \ell}{\ell}\right)^{1/2} < \epsilon_7,$$

for $\epsilon_7$ from Lemma 2.5. From (2.63) we also have

$$\frac{|u - \hat{v}|}{\ell} \leq \frac{\Delta(\ell/c_1)}{\ell} < \epsilon_7,$$

so Lemma 2.5 applies, giving

$$g(\hat{v} - \hat{x}) - g(u - \hat{x}) \leq C_{38} \left(\frac{|\hat{v} - u|}{\ell} \frac{|w - \hat{x}|}{\ell} + \frac{|\hat{v} - u|^2}{\ell}\right).$$

Using again $|\hat{v} - u| \leq \Delta(\ell/c_1)$ along with (2.66) and (2.67) lets us conclude

$$g(\hat{v} - \hat{x}) - g(u - \hat{x}) \leq c_8(t \log \ell)^{1/2} \sigma \ell.$$

With (2.62) and (2.64) this proves (2.61).

From (2.61) and Proposition 2.1 provided $\lambda$ is large,

$$\frac{\lambda}{4} \sigma(\Delta^{-1}(\ell)) \log |u - v|$$

and hence from the second inequality in (2.65), provided $c_1$ (and therefore $\ell$) is large,

$$|w - \hat{x}| = |x_2^0| \leq t^{1/2} \Xi((x - u)^0) = t^{1/2} \Xi(\ell),$$

and hence from the second inequality in (2.65), provided $c_1$ (and therefore $\ell$) is large,

$$\frac{|w - \hat{x}|}{\ell} \leq \frac{t^{1/2} \Xi(\ell)}{\ell} \leq \left(\frac{\Delta \log \ell}{\ell}\right)^{1/2} < \epsilon_7,$$

for $\epsilon_7$ from Lemma 2.5. From (2.63) we also have

$$\frac{|u - \hat{v}|}{\ell} \leq \frac{\Delta(\ell/c_1)}{\ell} < \epsilon_7,$$

so Lemma 2.5 applies, giving

$$g(\hat{v} - \hat{x}) - g(u - \hat{x}) \leq C_{38} \left(\frac{|\hat{v} - u|}{\ell} \frac{|w - \hat{x}|}{\ell} + \frac{|\hat{v} - u|^2}{\ell}\right).$$

Using again $|\hat{v} - u| \leq \Delta(\ell/c_1)$ along with (2.66) and (2.67) lets us conclude

$$g(\hat{v} - \hat{x}) - g(u - \hat{x}) \leq c_8(t \log \ell)^{1/2} \sigma \ell.$$

With (2.62) and (2.64) this proves (2.61).

From (2.61) and Proposition 2.1 provided $\lambda$ is large,

$$\frac{\lambda}{4} \sigma(\Delta^{-1}(\ell)) \log |u - v|$$

and hence from the second inequality in (2.65), provided $c_1$ (and therefore $\ell$) is large,

$$|w - \hat{x}| = |x_2^0| \leq t^{1/2} \Xi((x - u)^0) = t^{1/2} \Xi(\ell),$$

and hence from the second inequality in (2.65), provided $c_1$ (and therefore $\ell$) is large,

$$\frac{|w - \hat{x}|}{\ell} \leq \frac{t^{1/2} \Xi(\ell)}{\ell} \leq \left(\frac{\Delta \log \ell}{\ell}\right)^{1/2} < \epsilon_7,$$

for $\epsilon_7$ from Lemma 2.5. From (2.63) we also have

$$\frac{|u - \hat{v}|}{\ell} \leq \frac{\Delta(\ell/c_1)}{\ell} < \epsilon_7,$$

so Lemma 2.5 applies, giving

$$g(\hat{v} - \hat{x}) - g(u - \hat{x}) \leq C_{38} \left(\frac{|\hat{v} - u|}{\ell} \frac{|w - \hat{x}|}{\ell} + \frac{|\hat{v} - u|^2}{\ell}\right).$$

Using again $|\hat{v} - u| \leq \Delta(\ell/c_1)$ along with (2.66) and (2.67) lets us conclude

$$g(\hat{v} - \hat{x}) - g(u - \hat{x}) \leq c_8(t \log \ell)^{1/2} \sigma \ell.$$
and similarly for \( |u - x| \). Hence the right side of (2.69) is bounded above by \( 2C_5 e^{-c_{13}\lambda \log |u-v|} \), which with (2.60) yields that for all \( \lambda \geq c_{14}/2 \) satisfying (2.59),

\[
P( T(v,0) - T(u,0) \geq \lambda \sigma(\Delta^{-1}(|u-v|)) \log |u-v| ) \\
\leq C_{35} e^{-C_{36}t \log t} + 2C_5 e^{-c_{13}\lambda \log |u-v|} \\
\leq c_{15} e^{-c_{16}\lambda \log |u-v|}. (2.71)
\]

It remains to consider “large \( \lambda \),” meaning (2.59) does not hold:

\[
\lambda \sigma(\Delta^{-1}(|u-v|)) \log |u-v| > 2h(u-v).
\]

Here we have using (1.4) that

\[
P( T(v,0) - T(u,0) \geq \lambda \sigma(\Delta^{-1}(|u-v|)) \log |u-v| ) \\
\leq P( T(u,v) \geq h(u-v) + \frac{\lambda}{2} \sigma(\Delta^{-1}(|u-v|)) \log |u-v| ) \\
\leq C_5 \exp \left( -\frac{C_6 \lambda}{2} \frac{\sigma(\Delta^{-1}(|u-v|))}{\sigma(|u-v|)} \log |u-v| \right) \\
\leq C_5 e^{-c_{17}\lambda \log |u-v|}. (2.73)
\]

With (2.60) and (2.71) this proves (2.59). \( \square \)

3. Existence and transverse fluctuations of \( \theta \)-rays

For a geodesic ray \( \Gamma = (v_0, v_1, \ldots) \) (as a sequence of sites), we say \( \Gamma \) is a subsequential \( \theta \)-ray if there exists a subsequence \( \{v_{n_k}\} \) for which \( v_{n_k}/|v_{n_k}| \to \theta \). We say a sequence \( \{\Gamma_n\} \) of geodesics or geodesic rays from \( v_0 \) converges to \( \Gamma \) if for each \( j \geq 1 \), for all sufficiently large \( n \), \( \Gamma[v_0, v_j] \) is an initial segment of \( \Gamma_n \). If \( \{\Gamma_n\} \) is a sequence of geodesics or geodesic rays from a fixed \( v_0 \) with length \( |\Gamma_n| \to \infty \), then \( \{\Gamma_n\} \) has a converging subsequence.

In Proposition 2.4 the bound on the probability is uniform in \( r \). This enables us to turn that lemma (or more precisely, its proof) into a result about \( \theta \)-rays, which is part (ii) of the next proposition. The proposition also includes parts (i), (iv)(a), and (iv)(b) of Theorem 1.5.

**Proposition 3.1.** Suppose A1 and A2(i),(ii) hold for some \( \theta_0, \epsilon_0 \).

(i)

\[
P( \text{for all } v \in \mathbb{Z}^d \text{ and } \theta \in S^{d-1} \text{ with } \psi_{\theta_0} < \epsilon_0, \text{ a } \theta \text{-ray from } v \text{ exists} ) = 1.
\]

If also A2(ii') holds then this is true without the condition \( \psi_{\theta_0} < \epsilon_0 \).

(ii) There exist constants \( C_i \) as follows. For \( t > 1 \),

\[
P( \text{for some } \theta \in S^{d-1} \text{ with } \psi_{\theta_0} < \epsilon_0, \text{ there exists a } \theta \text{-ray } \Gamma \text{ from } v \text{ with} \\
\sup_{u \in \Gamma} D_\theta(u) > t ) \leq C_{44} e^{-C_{45}t \log t}.
\]

If also A2(ii') holds then this is true without the condition \( \psi_{\theta_0} < \epsilon_0 \).
(iii) \[ P \left( \text{for every } v \in \mathbb{Z}^d \text{ and } \theta \in S^{d-1} \text{ with } \psi_{\theta 0} < \epsilon_0, \text{ every subsequential } \theta \text{-ray} \right. \]
\[ \text{from } v \text{ is a } \theta \text{-ray} ) = 1. \]

(3.3)

If also A2(ii') holds then

(3.4) \[ P \big( \text{there exists a geodesic ray with no asymptotic direction} \big) = 0. \]

Proof. Observe that event in (3.2) is contained in the event

\[ A_t : \text{there exist } \theta \in S^{d-1} \text{ with } \psi_{\theta 0} < \epsilon_0 \text{ and } z_n \in \mathbb{Z}^d, \text{ with} \]
\[ |z_n| \to \infty, \quad \theta_n = z_n/|z_n| \to \theta, \text{ and } \sup_{u \in \Gamma_{0,z_n}} D_{\theta_n}(u) > t. \]

This can be seen by fixing \( \Gamma = (v_0, v_1, \ldots) \) (as a sequence of sites) and \( \theta \) as in (3.2), and a site \( u = v_m \in \Gamma \) with \( D_\theta(u) > t \), taking \( z_n = v_{m+n} \) for all \( n \), and noting that \( D_{\theta_n}(u) \to D_\theta(u) \) as \( n \to \infty \).

Further, if \( \Gamma = (v_0, v_1, \ldots) \) is a subsequential \( \theta \)-ray which is not a \( \theta \)-ray, then there are subsequences \( v_n(k)/|v_n(k)| \to \theta \) and \( \zeta(j) = v_n'(j)/|v_n'(j)| \to \zeta \) for some \( \theta \neq \zeta \). But this means that given \( t > 0 \), fixing \( k \) sufficiently large we have \( D_{\zeta(j)}(v_n(k)) > t \) for all sufficiently large \( j \) (depending on \( k \)), which in turn means that \( A_t \) occurs with \( z_j = v_n'(j) \). It follows that the complement of the event in (3.3) is contained in \( \bigcap_{t > 0} A_t \).

Therefore to prove both (3.2) and (3.3) it is enough to show

(3.5) \[ P(A_t) \leq C_{A4} e^{-C_{A4} t \log t}. \]

Note that for fixed \( z_n \) we can use Proposition 2.4, but we cannot sum over possible \( z_n \) as the entropy is too large. However, in the proof of that lemma, all that we use is (after converting the notation for our present context) the existence of a \( \delta \)-fat triangle of sites \( u, w \) in \( \Gamma_{x,z_n} \) for sufficiently large \( \delta \) (depending on \( |u|, |w| \)). The bounds in that proof do not involve \( z_n \), so in the 4 cases there is only necessary to sum over ranges of possible values of \( |u| \) or \( |w| \). Thus the proof of Proposition 2.4 also proves (3.5).

The last sentence in (ii) follows from (3.2) and the compactness of \( S^{d-1} \), and similarly in (iii).

Turning to (i), it is enough to consider \( v = 0 \). Given \( \theta \in S^{d-1} \) with \( \psi_{\theta 0} < \epsilon_0 \), let \( z_n \in \mathbb{Z}^d \) with \( |z_n| \to \infty \) and \( \theta_n = z_n/|z_n| \to \theta \). Then some subsequence of \( \{\Gamma_{0,z_n}\} \) converges to a geodesic ray \( \Gamma_{\infty} = (0 = w_0, w_1, \ldots) \) from 0. If \( \Gamma_{\infty} \) is not a \( \theta \)-ray, it is a subsequential \( \zeta \)-ray for some \( \zeta \in S^{d-1}, \zeta \neq \theta \). It follows readily that

\[ \lim_{j \to \infty} \limsup_n D_{\theta_n}(w_j) = \infty, \]

which means \( A_t \) occurs for all \( t > 0 \). Thus \( \Gamma_{\infty} \) is a \( \theta \)-ray a.s. Equation (3.1) then follows from (3.5), and the last sentence of (i) again follows from compactness of \( S^{d-1} \).

4. Crowded geodesics

We call a path or geodesic from a set \( A \subset \mathbb{R}^d \) to \( B \subset \mathbb{R}^d \) nonreturning if only the first bond \( \langle x_0, x_1 \rangle \) (viewed as a line segment) intersects \( A \) and only the last bond \( \langle x_{m-1}, x_m \rangle \) intersects \( B \). A nonreturning path from \( H^-_{\theta,s_1} \) to \( H^+_{\theta,s_2} \), for some \( s_1 < s_2 \) is called a \( \theta \)-slab path, and similarly for a \( \theta \)-slab geodesic.
In Theorem 1.5(ii), one can view the density of $H^{+}_{0,R}$-entry points as bounding the maximum possible longitudinal density of any set of halfspace $\theta$-rays with distinct $H^{+}_{0,R}$-entry points. Given an upper bound for this density larger than its heuristically-suggested order of $\Delta^{d-1}_R$, we may express the bound as $n^{1+(d-1)\beta_0}/\Delta^{d-1}_R$ for some $\beta_0 > 0$ and $n \geq 1$, which we may read as $n$ $\theta$-rays crossing per volume $(n^{-\beta_0}\Delta_R)^{d-1}$ in $H_{\theta,0}$. Thus to bound the mean $H^{+}_{0,R}$-crossing density our main task is roughly to bound

$$P\left(\text{there exist } n \text{ halfspace } \theta_0 \text{-rays with distinct } H^{+}_{0,R} \text{-entry points originating from } \{0\} \times [-n^{-\beta_0}\Delta_R, n^{-\beta_0}\Delta_R]^{d-1}\right).$$

Obtaining such a bound with $n = (\log R)^K$ for some (large) $K$ corresponds—modulo a few technicalities—to bounding the mean $H^{+}_{0,R}$-crossing density by $(\log R)^{K'}/\Delta^{d-1}_R$, with $K' = (1+(d-1)\beta_0)K$. Similarly, we can bound the mean combined $H^{+}_{\theta_0,R}$-crossing density of $\theta$-rays over $\theta \in J(\theta_0, \epsilon)$ for some $\epsilon$, mainly by bounding

$$P\left(\text{there exist } n \theta_0 \text{-slab geodesics from } H^{+}_{\theta_0,0} \text{ to } H^{+}_{\theta_0,2R} \text{ with distinct } H^{+}_{0,R} \text{-entry points originating from } \{0\} \times [-n^{-\beta_0}\Delta_R, n^{-\beta_0}\Delta_R]^{d-1} \text{ with initial orientation in } J(\theta_0, 2\epsilon)\right),$$

where by the initial orientation of a geodesic we mean its direction from its starting point to its $H^{+}_{0,R}$-entry point.

**Remark 4.1.** The idea of the bound on (4.2) is as follows, for some values $\beta_i \in (0, 1)$, with some details and definitions altered to reduce technicalities. Consider first the probability there exist $n$ slab geodesics as in (4.2) which are disjoint. Divide the initial portions (i.e. up to $H^{+}_{0,R}$) of these slab geodesics into $n^{\beta_2}$ segments of equal length $\ell = n^{-\beta_2}R$, cutting at the $H^{+}_{0,i\ell}$-entry points for $i \leq n^{\beta_2}$. The “natural” transverse spacing of these segments is $\Delta(\ell) = \Delta(n^{-\beta_2}R)$; for given $i$ let us call the $i$th segments of two slab geodesics $\Gamma^{(1)}, \Gamma^{(2)}$ **neighbors** if the respective $H^{+}_{0,i\ell}$-entry points $u^{(1)}_i, u^{(2)}_i$ satisfy

$$|u^{(1)}_i - u^{(2)}_i| \leq n^{-\beta_0}\Delta_R \ll \Delta \ell, \quad |u^{(1)}_{i+1} - u^{(2)}_{i+1}| \leq n^{-\beta_0}\Delta_R \ll \Delta \ell.$$ 

We show that with very high probability, no pair of neighbors (for any $i$) have passage times that differ by even a small multiple of $\sigma_\ell$. The neighbors have close passage times because, except near their endpoints, they are “geodesics chosen from the same set of possible paths.” The close passage times mean that, modulo a small-probability event, if we fix any one slab geodesic $\Gamma$ and the passage times of its segments, those segment passage times in $\Gamma$ effectively nearly determine the passage times of all neighbor segments of other slab geodesics. In particular, in view of (1.9) and (1.10), we say a segment is **fast** if its passage time is at most $ET(0, \ell y_{\theta_0}) + \frac{\gamma}{8}\sigma_\ell$; up to a small error, for every fast segment of our fixed $\theta$-ray, all neighbor segments in other $\theta$-rays must be fast. We then show that a fixed one of our $n$ slab geodesics (let’s call it “special”) likely has at least of order $n^{\gamma+\beta_3}$ fast segments before crossing $H^{+}_{0,R}$, and every one of those fast segments has one or more neighbors, all of which are also fast. However, we can use the FKG inequality to say that the presence of the special slab geodesic (because it’s a geodesic) stochastically increases the passage times of bonds not in the geodesic; hence the probability is small that all the neighbors will be fast. This provides the desired bound on (4.2), in the disjoint case.
In summary, the fast segments of length \( \ell \) in the special geodesic force all their neighbor segments to be fast (with high probability), while simultaneously reducing the neighbor segments’ probability to be fast, in the FKG sense. These contradictory aspects can only coexist if the probability of crowded geodesics as in (4.2) is very small.

For the general (non-disjoint) case, for segments to qualify as neighbors, we add the requirement that length of their intersection, if any, is not more than \( n^{-\beta_4} \ell \). We consider two possibilities:

(i) **popular site case**: there exists a site in \( H_{\theta_0,R} \) in the intersection of at least \( n^{\beta_3-\beta_4} \) of the \( n \) slab geodesics.

(ii) **no popular site**: there is no site as in (i).

In the no-popular-site case, the arguments from the disjoint case still apply with some modifications. In the popular-site case, we take a subset of \( n^{\beta_5} \) out of the \( n^{\beta_3-\beta_4} \) slab geodesics sharing a common point, with all in the subset having \( H_{\theta_0,R} \)-entry points within distance of order \( n^{-\beta_0} \). Because these slab geodesics share a common point before \( H_{\theta_0,R} \) and have distinct \( H_{\theta_0,R} \)-entry points, they must be disjoint after \( H_{\theta_0,R} \). Hence the disjoint case can be applied to these \( n^{\beta_5} \) geodesics from \( H_{\theta_0,R} \) to \( H_{\theta_0,2R} \), instead of \( n \) geodesics from \( H_{\theta_0,0} \) to \( H_{\theta_0,R} \).

For sites \( u, v \) in a path \( \gamma \), let \( \gamma[u,v] \) denote the segment of \( \gamma \) between \( u \) and \( v \). Given a geodesic \( \Gamma \) from \( H_{\theta_0,0} \) to \( H_{\theta_0,R} \) for some \( R \) and given \( 0 \leq s \leq R \), let \( x_{\theta,s}''(\Gamma) \) denote the \( H_{\theta,s} \)-entry point of \( \Gamma \), and let \( x_{\theta,s}(\Gamma) \) be the last site in \( \Gamma \) before \( x_{\theta,s}''(\Gamma) \). An \( \ell \)-segment of \( \Gamma \) is the segment \( S_{\theta,i}(\Gamma) := \Gamma[x_{\theta,(i-1)\ell}(\Gamma), x_{\theta,i\ell}(\Gamma)] \) for some \( i \geq 1 \).

To start formalizing the argument outlined in Remark 4.1, we show that with high probability, a geodesic contains at least a certain minimum number of fast \( \ell \)-segments.

**Lemma 4.2.** Suppose A1 and A2(i),(ii) hold for some \( \theta_0, \epsilon_0 \), and let \( \eta \) be as in (1.9) and (1.10). There exist constants \( C_i \) as follows. Let

\[
R \geq C_{46}, \quad 0 < \lambda \leq 1 - \gamma_2, \quad C_{47}(\log R)^{1/\lambda} \leq k \leq C_{48}R^{1/2}, \quad \ell = R/k,
\]

with \( \gamma_2 \) from (1.2) and \( k \) an integer. For \( \psi_{\theta_0} < \epsilon_0 \) and \( \Gamma \) a geodesic from \( H_{\theta_0,0} \) to \( H_{\theta_0,R} \), let

\[
N_{\theta}(\Gamma) = \left\{ 1 \leq i \leq k : T(x_{\theta,(i-1)\ell}(\Gamma), x_{\theta,i\ell}(\Gamma)) \leq \frac{1}{k}ET(0,R_{\theta_0}) + \frac{\eta \sigma_{\ell}}{8} \right\}.
\]

Then for all \( v \in H_{\theta,0} \) and all \( w \in H_{\theta,R} \) with

\[
\max(|v|, |w - R_{\theta_0}|) \leq C_{49}\eta^{1/2}k^{(1-\gamma_2)/2} \Delta R,
\]

we have

\[
P\left(N_{\theta}(\Gamma_{vw}) \leq \frac{\eta}{32}k^{1-\lambda} \right) \leq C_{50}e^{-C_{51}k^{3}\eta}.
\]

Note that (4.4) allows the angle \( \psi_{\theta,w,v} \) to be nonzero but not too large. Such a bound is necessary for (4.10) in the proof.

**Proof of Lemma 4.2.** We need a slightly modified definition to take into account that \( \Gamma_{vw} \) might not be a slab geodesic, i.e. we might not have \( x_{\theta,0}(\Gamma_{vw}) = v \) and \( x_{\theta,R}(\Gamma_{vw}) = w \):

\[
\hat{x}_{\theta,i\ell}(\Gamma) = \begin{cases} 
  v & \text{if } i = 0, \\
  x_{\theta,i\ell}(\Gamma) & \text{if } 1 \leq i \leq k - 1 \\
  w & \text{if } i = k,
\end{cases}
\]

and

\[
\hat{x}_{\theta,i\ell}(\Gamma) = \begin{cases} 
  v & \text{if } i = 0, \\
  x_{\theta,i\ell}(\Gamma) & \text{if } 1 \leq i \leq k - 1 \\
  w & \text{if } i = k,
\end{cases}
\]
and define

\[ Y_i(\Gamma) = \frac{T(\hat{x}_{\theta,(i-1)\ell}''(\Gamma), \hat{x}_{\theta,i\ell}''(\Gamma)) - k^{-1}ET(0, R_y\theta)}{\sigma_\ell}, \]

\[ S_k(\Gamma) = \sum_{i=1}^{k} Y_i(\Gamma) = \frac{T(v, w) - ET(0, R_y\theta)}{\sigma_\ell}, \]

\[ \hat{Y}_i(\Gamma) = \frac{T(\hat{x}_{\theta,(i-1)\ell}''(\Gamma), \hat{x}_{\theta,i\ell}''(\Gamma)) - h(\hat{x}_{\theta,i\ell}''(\Gamma) - \hat{x}_{\theta,(i-1)\ell}''(\Gamma))}{\sigma_\ell}. \]

For technical convenience we will assume \( \ell y_\theta \) is a lattice point; the adjustments when this is false are minor.

It follows from (1.2) and (4.3) that provided \( C_{47} \) is large enough in (4.3), we have

\[ \sigma_{R} \leq \frac{C_3}{k^{1-\gamma_2}} \leq \frac{\eta}{16c_1 \log R}, \]

with \( c_1 \) chosen so that \( h(Ry\theta) \leq g(Ry\theta) + c_1 \sigma_{R} \log R \), from Proposition 2.1. With this we obtain that provided \( C_{48} \) is small (so \( \ell \) is large), for some \( c_1 \), for all \( 1 \leq i \leq k \),

\[ h(\hat{x}_{\theta,i\ell}(\Gamma_{vw}) - \hat{x}_{\theta,(i-1)\ell}(\Gamma_{vw})) \geq g\left(k(\hat{x}_{\theta,i\ell}(\Gamma_{vw}) - \hat{x}_{\theta,(i-1)\ell}(\Gamma_{vw}))\right) - c_2 \]

\[ = \frac{1}{k} g\left(k(\hat{x}_{\theta,i\ell}(\Gamma_{vw}) - \hat{x}_{\theta,(i-1)\ell}(\Gamma_{vw}))\right) - c_2 \]

\[ \geq \frac{1}{k} g(Ry\theta) - c_2 \]

\[ \geq \frac{1}{k} \left[h(Ry\theta) - c_2 \sigma_{R} \log R\right] - c_2 \]

\[ \geq \frac{1}{k} ET(0, R_y\theta) - \frac{\eta}{8} \sigma_\ell, \]

where the second inequality follows from \( k(\hat{x}_{\theta,i\ell}(\Gamma_{vw}) - \hat{x}_{\theta,(i-1)\ell}(\Gamma_{vw})) \in H_{\theta,R} \). Therefore

\[ Y_i(\Gamma_{vw}) \geq \hat{Y}_i(\Gamma_{vw}) - \frac{\eta}{8}. \]

The key point is that

\[ S_k(\Gamma_{vw}) \geq (k - N_\theta(\Gamma_{vw})) \frac{\eta}{8} + N_\theta(\Gamma_{vw}) \min_{i \leq k} Y_i(\Gamma_{vw}) = k \frac{\eta}{8} + N_\theta(\Gamma_{vw}) \left( \min_{i \leq k} Y_i(\Gamma_{vw}) - \frac{\eta}{8} \right). \]

Equation (1.2) gives \( \sigma_{R}/\sigma_{r} \geq C^{-1} k^{-\gamma_2} \), so using also (1.4) and (4.8), it follows that

\[ P\left(N_\theta(\Gamma_{vw}) \leq \frac{\eta}{32} k^{1-\lambda}\right) \]

\[ \leq P\left(S_k(\Gamma_{vw}) \geq \frac{\eta}{16} k\right) + P\left(\min_{i \leq k} Y_i(\Gamma_{vw}) \leq \frac{\eta}{8} - 2k^\lambda\right) \]

\[ \leq P\left(T(v, w) - ET(0, R_y\theta) \geq \frac{k \eta \sigma_{R}}{16}\right) + P\left(\min_{i \leq k} Y_i(\Gamma_{vw}) \leq \frac{\eta}{8} - 2k^\lambda\right). \]
To control the next-to-last probability we need to bound $h(w - v) - h(Ry)$. Let $\hat{w}, \hat{v}$ be the $\theta$-projections of $w, v$ into $H_{\theta, R}$ and $H_{\theta, 0}$, respectively, so by (2.24) we have $|w - \hat{w}| \leq d$ and $|v - \hat{v}| \leq d$. From (1.2) and (4.4) we have

$$
\max \left( \frac{|v|^2}{R}, \frac{|w - Ry|^2}{R} \right) \leq C_{49}^2 \eta k^{1-\gamma_2} \sigma_R \leq C_3 C_{49}^2 \eta k \sigma \ell.
$$

and hence provided $C_{48}$ is small (so $\ell$ is large),

$$
\left( \frac{|(\hat{w} - \hat{v}) - Ry|}{R} \right)^2 \leq 4 \left( \frac{|w - Ry|^2}{R} \right)^2 + 4 \left( \frac{|v|^2}{R} \right)^2 + 4 \left( \frac{|w - \hat{w}|}{R} \right)^2 + 4 \left( \frac{|v - \hat{v}|}{R} \right)^2 \leq 8 C_{49}^2 \eta \frac{\sigma \ell}{\ell} + \frac{8 d^2 R^2}{R^2} \leq 9 C_{49}^2 \eta \frac{\sigma \ell}{\ell} < c_0^2.
$$

Therefore A2(ii) applies and, provided we take $C_{49}$ small in (4.4), we get

$$
h(\hat{w} - \hat{v}) \leq g(Ry) + c_3 \frac{|(\hat{w} - \hat{v}) - Ry|^2}{R} \leq g(Ry) + 9 c_3 C_{49}^2 \eta k \sigma \ell \leq g(Ry) + \frac{\eta k \sigma \ell}{64}.
$$

From (4.10),

$$
|w - v| \leq |w - Ry| + R|y| + |v| \leq 2 \left( \frac{C_{49}^2 \eta \sigma \ell}{\ell} \right)^{1/2} R + R|y| \leq 2|y| R.
$$

With Proposition 2.1, (4.6), and (4.11) this gives the desired bound

$$
h(w - v) \leq g(w - v) + C_{25} \sigma (2|y| R) \log (2|y| R) \leq g(\hat{w} - \hat{v}) + 2 \mu \sqrt{d} + c_4 \sigma R \log R \leq g(Ry) + \frac{\eta k \sigma \ell}{64} + 2 \mu \sqrt{d} + \frac{\eta k \sigma \ell}{64} \leq h(Ry) + \frac{3 \eta k \sigma \ell}{64}.
$$

Therefore in (4.9), using (1.2), (1.4), and (4.12) we have

$$
P \left( T(v, w) - ET(0, Ry) \geq \frac{k \eta \sigma \ell}{16} \right) \leq P \left( T(v, w) - ET(v, w) \geq \frac{\eta k \sigma \ell}{64} \right) \leq C_5 \exp \left( -C_6 \frac{\eta k \sigma \ell}{64 \sigma (|w - v|)} \right) \leq C_5 \exp \left( -C_6 \eta k^{1-\gamma_2} \right).
$$

(4.13)
It remains to bound the last probability in (4.9). Write $\Lambda_i$ for $H_{\theta,i}^{fat}$, which must contain $x''_{\theta,i}(\Gamma_{vw})$. For $t = k^{1-\gamma_2}$ we have using Proposition 2.4 and (4.8) that
\[
P \left( \min_{i \leq k} Y_i(\Gamma_{vw}) \leq \frac{\eta}{8} - 2k^\lambda \right)
\leq P(\Gamma_{vw} \notin E_{\alpha,w_1^\alpha},t)
\quad + \quad \sum_{i=1}^k P \left( \Gamma_{vw} \subset E_{\alpha,w_1^\alpha},t, g \left( x''_{\theta,i}(\Gamma_{vw}) - x''_{\theta,(i-1)\ell}(\Gamma_{vw}) \right) > 3\ell, Y_i(\Gamma_{vw}) \leq \frac{\eta}{8} - 2k^\lambda \right)
\quad + \quad \sum_{i=1}^k P \left( \Gamma_{vw} \subset E_{\alpha,w_1^\alpha},t, g \left( x''_{\theta,i}(\Gamma_{vw}) - x''_{\theta,(i-1)\ell}(\Gamma_{vw}) \right) \leq 3\ell, \bar{Y}_i(\Gamma_{vw}) \leq \frac{\eta}{4} - 2k^\lambda \right)
\leq C_{35}e^{-C_{36}t} + \sum_{i=1}^k \sum_{a \in \Lambda_i \cap E_{\alpha,w_1^\alpha}} \sum_{b \in \Lambda_i \cap E_{\alpha,w_1^\alpha}} \sum_{g(b-a) \geq 3\ell} P \left( T(a,b) - ET(0,Ry_\theta) \leq -k^\lambda \sigma_\ell \right)
\quad + \quad \sum_{i=1}^k \sum_{a \in \Lambda_i \cap E_{\alpha,w_1^\alpha}} \sum_{b \in \Lambda_i \cap E_{\alpha,w_1^\alpha}} \sum_{g(b-a) \geq 3\ell} P \left( T(a,b) - ET(a,b) \leq -k^\lambda \sigma_\ell \right).
\tag{4.14}
\]

Note that the lower bound of $3\ell$ in the third line here means the $i$th $\ell$–segment of $\Gamma_{vw}$ has direction quite far from $\theta$. For the last probability we have from (1.4)
\[
P \left( T(a,b) - ET(0,Ry_\theta) \leq -k^\lambda \sigma_\ell \right) \leq C_5 \exp \left( -C_6 k^\lambda \frac{\sigma_\ell}{\sigma(\|b-a\|)} \right) \leq C_5 e^{-c_6 k^\lambda}.
\]

For the next-to-last probability in (4.14) we have from subadditivity and Proposition 2.1 that for all $a, b$ in the double sum,
\[
\frac{1}{k} ET(0,Ry_\theta) \leq ET(0,\ell y_\theta) \leq \ell + c_7 \sigma_\ell \log \ell \leq h(b-a) - \frac{1}{2} g(b-a)
\]
so from (1.4) again,
\[
P \left( T(a,b) - \frac{1}{k} ET(0,Ry_\theta) \leq -k^\lambda \sigma_\ell \right) \leq P \left( T(a,b) - ET(a,b) \leq -\frac{1}{2} g(b-a) \right)
\leq C_5 e^{-c_6 (b-a)/2\sigma(|b-a|)}
\leq C_5 e^{-c_6 \ell/\sigma_\ell}
\leq C_5 e^{-c_6 k^\lambda}.
\tag{4.15}
\]

Here the last inequality follows from the fact that for large $\ell$, by (1.2) and (4.3),
\[
\frac{\ell}{\sigma_\ell} \geq \left( \frac{R}{k} \right)^{1-\gamma_2} \geq c_{10} k^{1-\gamma_2}.
\]

The number of $a$ or $b$ in the sums in (4.14) is at most of order $R^d$, so provided $C_{47}$ is large enough in (4.3), the right side of (4.14) is bounded above by
\[
C_{35}e^{-C_{36} k^{1-\gamma_2}} + c_{11} k R^d e^{-c_6 k^\lambda} + c_{12} k R^d e^{-c_6 k^{1-\gamma_2}} \leq c_{13} e^{-c_{14} k^\lambda}.
\]
With (4.9) and (4.13) this proves (1.5). \qed

**Remark 4.3.** The results to follow involve several different length scales, and other quantities, expressed using $R$ and small (less than 1) powers of the number $n$ of geodesics we are dealing with. We summarize them here for ready reference, with precise definitions to follow:

(i) a length scale $n^{-\beta_0}\Delta_R$ for cubic blocks in each hyperplane $H_{\theta,s}$, with geodesics considered close when they start and end with separation $n^{-\beta_0}\Delta_R$ or less;

(ii) a width $n^{\beta_1}\Delta_R$ for the “target box” to which length–$R$ geodesics are confined, with high probability;

(iii) a length scale $\ell = n^{-\beta_2}R$ for segments of length-$R$ geodesics;

(iv) for a transition (i.e. choice of starting and ending blocks) made by a length–$\ell$ segment of a geodesic, a maximum number $n^{\beta_3}$ of other geodesics which can make the same transition, for that transition to be considered “sparse”;  

(v) a maximum overlap length $n^{-\beta_4}\ell$ for two length–$\ell$ geodesic segments to be considered “low overlap”; this length also serves as the minimum significant backtrack in a geodesic. 

Other quantities are defined in terms of these powers, such as a minimum number $n^{\beta_3-\beta_4}$ of geodesics passing through a site for it to qualify as a popular site in the sense of Remark 4.1.

We start with some formal definitions related to the items in Remark 4.3.

The following definitions are for fixed $R, n, \theta$, which don’t necessarily appear in the notation. For $0 \leq s \leq R$, the home $\theta$–block in a hyperplane $H_{\theta,s}$ is $\{s\} \times [-n^{-\beta_0}\Delta_R, n^{-\beta_0}\Delta_R]^{d-1}$ (in $\theta$–coordinates.) The home block in $H_{\theta,0}$ is denoted $B_{\theta,\text{home}}$. A $\theta$–block in $H_{\theta,s}$ is a translate of the home $\theta$–block in each $\mathbb{B}_\theta$–coordinate direction $i \ (2 \leq i \leq d)$ by an integer multiple of $2n^{-\beta_0}\Delta_R$ (so such $\theta$–blocks tile $H_{\theta,s}$.) The center point of any $\theta$–block is called a $\theta$–block center. An enlarged $\theta$–block has the same center as a block but larger linear dimensions by a factor $2\sqrt{d}$; more precisely it has the form (in $\theta$–coordinates)

$$y + \left( \{s\} \times [-2\sqrt{d}n^{-\beta_0}\Delta_R, 2\sqrt{d}n^{-\beta_0}\Delta_R]^{d-1} \right),$$

with $y$ a $\theta$–block center in $H_{\theta,s}$, and the $+$ denoting translation. The factor $2\sqrt{d}$ ensures that if $u$ lies in a $\theta$–block with center $a$, and $v$ lies outside an enlarged $\theta$–block with center $b$, then $|v-b| \geq 2|u-a|$.

For $\Gamma$ a geodesic which starts in $H_{\theta,0}$ and crosses $H_{\theta,R}$, the pre- $H_{\theta,R}$ segment of $\Gamma$ is the segment of $\Gamma$ from its starting point to $x''_R(\Gamma)$. In what is to follow, as in Lemma 4.2, we will divide the pre-$H_{\theta,R}$ segment of a $\theta$–slab geodesic into sub-segments of some length $\ell$. In this context, a geodesic $\Phi$ is an $(\ell, \theta)$–interval geodesic if for some $i$, the initial site of $\Phi$ is in $H_{\theta,(i-1)\ell}$, and the last bond of $\Phi$ is the first bond of $\Phi$ to cross $H_{\theta,i\ell}$ (so the last site of $\Phi$ must lie in $H_{\theta,i\ell}^{\text{fat}}$.) An $\ell$-segment of a $\theta$–ray (defined before Lemma 4.2) is thus one example of an $(\ell, \theta)$–interval geodesic.

The target $\theta$–box is a tube around $L_\theta$:

$$Q_{R,n,\theta} = \mathbb{R} \times [-2n^{\beta_1}\Delta_R, 2n^{\beta_1}\Delta_R]^{d-1},$$

in $\theta$–coordinates. A geodesic is $\theta$–target-directed if it is contained in the target $\theta$–box. A geodesic from $H_{\theta,0}$ to $H_{\theta,s}$ for some $s > 0$ is $\theta$–target-directed up to $s$ if its pre-$H_{\theta,s}$ segment is target-directed. A target $\theta$–backtrack is a $\theta$–block which intercepts $Q_{R,n,\theta}$.

For $\theta \in S^{d-1}$, $\Gamma$ a geodesic with a designated direction, and sites $x$ preceding $y$ in $\Gamma$, we say $\Gamma$ has a $\theta$–backtrack of size $r$ from $x$ to $y$ if $x_1^{\theta} - y_1^{\theta} \geq r$. 

We may omit the parameters in the preceding terminology when it is clearly understood, e.g. referring to a block rather than a \( \theta \)-block.

Before proceeding to the proof of Theorem [1.5 in the next few lemmas we consider various forms of “bad geodesic behavior” specialized to our context, and apply our previous results to show that these have small probability. These lemmas involve the quantities \( n^{\pm \beta_j} \) as in Remark [4.3] so we now specify the relations that we assume to hold among these exponents.

### A3. Exponent relations, parameters, and definitions.

\( \beta_i \in (0, 1) \) satisfy

\[
\begin{align*}
(4.16) & \quad \frac{2}{1 + \chi} \beta_0 + \frac{3 + \gamma_2}{1 + \chi} \beta_1 < \beta_1 + \beta_2 + \beta_4 < \frac{1 - \chi}{(1 + \chi)(d - 1)}, \quad \gamma_1 \beta_4 > \min(\gamma_2, 1 - \gamma_2) \beta_2 > 2 \beta_1, \\
(4.17) & \quad (1 - \gamma_2) \beta_4 > (2 + \gamma_2) \beta_2, \quad \frac{2(1 + \gamma_2)}{\gamma_1} \beta_1 + \frac{1 + \gamma_2}{2} \beta_2 < \beta_0 < \frac{\chi}{(1 + \chi)(d - 1)}, \\
(4.18) & \quad 3 \beta_3 + 3 \beta_2 + 7(d - 1)(\beta_0 + \beta_1) < 1,
\end{align*}
\]

where \( \gamma_1, \gamma_2 \) are from [1.2]. From the first inequality in (4.16) and the last in (4.17), there exist \( \chi_1, \chi_2 \) with \( \chi_1 < \chi_2 < \chi < \chi_2 < 1 \) for which

\[
(4.19) \quad \beta_1 + \beta_2 + \beta_4 < \frac{1 - \chi_2}{(1 + \chi_2)(d - 1)}, \quad 2 \beta_0 < \frac{2 \chi_1}{(1 + \chi_2)(d - 1)}.
\]

Let \( r, n, \ell \) satisfy

\[
(4.20) \quad R \geq C_{52}, \quad C_{53}(\log R)^{C_{54}} \leq n \leq C_{55} \Delta^{-1} R, \quad \text{and} \quad \ell = n^{-\beta_2} R.
\]

Let \( \epsilon_{\min} = \min\{\epsilon_0, \epsilon_1, \epsilon_5, \epsilon_6, \epsilon_7\} \), with \( \epsilon_7 \) from Lemma [2.5] and the other \( \epsilon_j \) from Section [1], \( B_{\theta_0, \text{cross}} \) is a \( \theta_0 \)-block in \( H_{\theta_0, R} \) with center \( \bar{y} \) for which \( \theta = \bar{y}/|\bar{y}| \) satisfies \( \psi_{\theta_0} < \epsilon_{\min}/2 \). [s, r] with \( 0 \leq s \leq r \) is the largest interval for which

\[
(4.21) \quad Q_{R, n, \theta} \cap \Omega_\theta(s, r) \subset Q_{R, n, \theta} \cap \Omega_{\theta_0}(0, R);
\]

see Figure [10] \( B_{r, \theta, \text{home, +}} \) and \( B_{s, \theta, \text{home, +}} \) are the enlarged home \( \theta \)-blocks in \( H_{\theta, r} \) and \( H_{\theta, s} \), respectively.

The conditions (4.16)–(4.18) can be satisfied by choosing \( \beta_3 < 1 \), then temporarily setting \( \beta_0 = \beta_1 = \beta_2 = 0 \) and choosing \( \beta_4 \) to satisfy the inequalities where \( \beta_4 \) appears, and finally choosing \( \beta_0 \) then \( \beta_2 \) then \( \beta_1 \) similarly, each small enough so that the inequalities still hold when the remaining \( \beta_j \)'s are set to 0.

**Remark 4.4.** The conditions (4.20) yield certain relative scales which we summarize here. Assuming \( R \) is large we have using (4.19) and (4.20)

\[
(4.22) \quad R \geq n^{2/(1+\chi_2)(d-1)}, \quad \frac{R}{\Delta R} = \left( \frac{R}{\sigma_R} \right)^{1/2} \geq R^{(1-\chi_2)/2} \geq n^{(1-\chi_2)/(1+\chi_2)(d-1)}, \quad \sigma_R \geq R^{\chi_1} \geq n^{2\beta_0}.
\]

Putting together (4.22) and the first inequalities in (4.16), (4.19) we obtain

\[
(4.23) \quad R^{(1-\chi_2)/2} \geq n^{\beta_1 + \beta_2 + \beta_4} \geq n^{2\beta_1}.
\]
From (4.22) and the first inequality in (4.19) we also get

\[(4.24) \quad n^{\beta_1 + \beta_2 + \beta_4} \leq \frac{R}{\Delta_R} \quad \text{so} \quad n^{\beta_1} \frac{\Delta R}{\ell} \leq n^{-\beta_4}.\]

By (1.2),

\[(4.25) \quad \frac{\Delta R}{\Delta \ell} \leq C_{56} n^{\beta_2 (1 + \gamma_2) / 2}.\]

From (4.16) and (4.19) we have

\[(4.24) \quad n^{\beta_1 + \beta_2 + \beta_4} \leq \frac{R}{\Delta_R} \quad \text{so} \quad n^{\beta_1} \frac{\Delta R}{\ell} \leq n^{-\beta_4}.\]

From (4.22) and the first inequality in (4.19) we also get

\[(4.26) \quad \ell \chi(1 - \chi_2) = R \chi(1 - \chi_2) n^{-\chi(1 - \chi_2) \beta_2} \geq n^{2 \chi(1 - \chi_2)/(1 + \chi_2) (d - 1)} n^{-\chi(1 - \chi_2) \beta_2} \geq n^{4 \beta_1}.\]

Also from (1.2), (4.22), and (4.19),

\[(4.27) \quad \frac{\ell}{\sigma} \geq C_{2n} n^{-\beta_2 (1 - \gamma_1)} \frac{R}{\sigma_R} = C_{2n} n^{-\beta_2 (1 - \gamma_1)} \frac{R^2}{\Delta_R} \geq C_{2n} n^{-\beta_2 (1 - \gamma_1)} n^{2 (1 - \chi_2)/(1 + \chi_2) (d - 1)} \geq n^{2 (1 + \beta_4)}.

Considering now \(r, s, \text{ and } \epsilon_{\min}\) from A3, observe that by (2.12), \(|\gamma - R y_{\theta_0}| \leq C_{31} \epsilon_{\min} R |y_{\theta_0}|\). Further, by (1.12), the angle between \(H_{\theta_0,0}\) and \(H_{\theta,s}\) is at most \(C_{13} \epsilon_{\min}\); it follows that \(s \leq C_{57} \epsilon_{\min} n^{\beta_1} \Delta_R\), and, using (1.13),

\[(4.28) \quad |r - s| |y_\theta| \geq |\gamma| - C_{58} \epsilon_{\min} n^{\beta_1} \Delta_R \geq (1 - C_{31} \epsilon_{\min}) R |y_{\theta_0}| - C_{58} \epsilon_{\min} n^{\beta_1} \Delta_R
\]

and hence \(r - s \geq (1 - C_{59} \epsilon_{\min}) R\).

Write \(\Pi_{\theta_0}^\infty\) for the infinite line through \(a\) and \(b\).

**Lemma 4.5.** There exist constants \(C_i \geq 0\) as follows. Suppose A1 and A2(i),(ii) hold for some \(\theta_0, \epsilon_0, \text{ and } A3\) holds for some \(R, n, \theta, \ell, \epsilon_{\min}\). For the event

\(G_1:\) (large transverse fluctuation in a geodesic) there exists a \(\theta_0\)-slab geodesic \(\Gamma\) from \(B^\text{fat}_{\theta_0,\text{home}}\)

to \(H^+_{\theta_0,2R}\) with \(H^+_{\theta_0,R}\)-entry point in \(B^\text{fat}_{\theta_0,\text{cross}}\) and with \(\Gamma \not\subset Q_{R,n,\theta}\),

we have

\[(4.29) \quad P(G_1) \leq \exp\left(-C_{60} n^{2 \beta_1 \log R}\right).\]

**Proof.** We must deal with the inconvenience that we may have \(\theta \not= \theta_0\). Suppose \(\tau \in G_1\), and let \(\Gamma\) be a \(\theta_0\)-slab geodesic from \(x \in B^\text{fat}_{\theta_0,\text{home}}\) to \(z \in H^\text{fat}_{\theta_0,2R}\) passing through \(y = x'_R(\Gamma) \in B^\text{fat}_{\theta_0,\text{cross}}\), and let \(u\) be the first site of \(\Gamma\) with \(u \not\in Q_{R,n,\theta}\). Then \(\Gamma_R := \Gamma[x,y]\) is the pre-\(H^+_{\theta_0,R}\) segment of \(\Gamma\), and we recall that \(\gamma\) is the center of \(B^\text{cross}_{\theta_0}\). Using \(\psi_{\theta_0} \leq \epsilon_{\min}\) we obtain straightforwardly that

\[(4.30) \quad |\gamma| \leq 2 |y_{\theta_0}| R, \quad |u| \leq 3 |y_{\theta_0}| R.\]

We now consider four cases; see Figure 6

Case 1. \(u \in \Gamma[x,y]\), so that \(|(u - x)\psi_{\theta_0}| \leq R + 2 \sqrt{\ell} \mu\). Here we find a lower bound for \(d(u, \Pi_{xy})\).

Let \(\varphi = (y - x) \parallel [y - x]\). Define intersection points

\(v = \Pi_{xy} \cap H_{\theta_0, u_1}, \quad \hat{v} = L_\varphi \cap H_{\theta_0, u_1}, \quad w = \Pi_{xy} \cap H_{\theta_0, u_1}, \quad \hat{w} = L_\varphi \cap H_{\theta_0, u_1},\)


and note that all 3 hyperplanes referenced here pass through \( u \). By (2.10),
\[
\sqrt{d}d(u, \Pi_{xy}^\infty) \geq d_\varphi(u, \Pi_{xy}^\infty) = |u - q|,
\]
and from the definition of \( u \),
\[
2n^{\beta_1} \Delta_R \leq d_\theta(u, L_\theta) = |u - \tilde{w}| \leq 3\sqrt{d - 1}n^{\beta_1} \Delta_R.
\]
Further, \( x, y \) being in fattened \( \theta_0 \)-blocks centered on \( L_\theta \) tells us that, first, provided \( R \) is large the \( \theta \)-ratio of \( y - x \) is at most \( 2\sqrt{d - 1}n^{\beta_1} \Delta_R/|y_\theta| R < \epsilon_{\min}|y_\theta|/4 \), so by (2.17) we have \( \psi_\varphi \theta_0 < \epsilon_{\min}/2 \) and hence also \( \psi_\varphi \theta_0 < \epsilon_{\min} \). Second, since \( v \in \Pi_{xy} \),
\[
|v - \hat{v}| = d_\theta_0(v, L_\theta) \leq \max(d_\theta_0(x, L_\theta), d_\theta_0(y, L_\theta)) \leq \sqrt{d - 1}n^{-\beta_0} \Delta_R.
\]
Now by (2.11),
\[
|u - q| \geq |u - v| - |v - q| \geq |u - v| - C_{31}\psi_\varphi \theta_0|u - q|
\]
so
\[
|u - q| \geq \frac{|u - v|}{1 + C_{31}\psi_\varphi \theta_0}.
\]
Combining (2.11), (4.31), (4.32), (4.33), (4.34) we get
\[
(1 + C_{31}\psi_\varphi \theta_0)\sqrt{d}d(u, \Pi_{xy}^\infty) \geq (1 + C_{31}\psi_\varphi \theta_0)|u - q| \geq |u - v| \geq |u - \hat{w}| - |\hat{w} - \hat{v}| - |\hat{v} - v|
\geq (1 - C_{31}\psi_\varphi \theta_0)|u - \hat{w}| - |\hat{v} - v|
\geq 2(1 - C_{31}\psi_\varphi \theta_0)n^{\beta_1} \Delta_R - \sqrt{d - 1}n^{-\beta_0} \Delta_R \geq \frac{3}{2}n^{\beta_1} \Delta_R.
\]
By shrinking some \( \epsilon_i \) we may assume \( C_{31}\epsilon_{\min} < 1/2 \), so since \( \psi_\varphi \theta_0 \leq \epsilon_{\min} \), (4.35) yields
\[
d(u, \Pi_{xy}) \geq d(u, \Pi_{xy}^\infty) \geq \frac{1}{\sqrt{d}}n^{\beta_1} \Delta_R,
\]
while from (4.30),
\[
|y - x| \leq |\bar{y}| + |x| + |y - \bar{y}| \leq 2|y_\theta| R + 2\sqrt{d - 1}n^{-\beta_0} \Delta_R \leq 3|y_\theta| R.
\]
By Lemma 2.3 with \( \delta = n^{\beta_1} \Delta_R/\sqrt{d}|y - x| \) we then have
\[
g(u - x) + g(y - u) - g(y - x) \geq \frac{C_{34}n^{2\beta_1} \Delta_R^2}{|y - x|} \geq c_1n^{2\beta_1} \sigma_R.
\]

Case 2. \( u \in \Gamma[y, z] \cap H_{\theta_0,R}^\infty \). This time we want a lower bound for \( d(y, \Pi_{xy}^\infty) \). Let \( \hat{w} = \Pi_{0u}^\infty \cap H_{\theta_0,R} \) and \( w = \Pi_{xy}^\infty \cap H_{\theta_0,R} \). Then \( u_1^{\theta_0} \leq 2R + \sqrt{d}\mu \leq 3R \) so from (2.11)
\[
|\hat{w} - \bar{y}| = \frac{R}{u_1^{\theta_0}} d_\theta(u, L_\theta) \geq \frac{1}{3}d_\theta(u, L_\theta) \geq \frac{1}{3}(1 - C_{31}\psi_\varphi \theta_0) d_\theta(u, L_\theta) \geq \frac{1}{2}n^{\beta_1} \Delta_R.
\]
Therefore, letting \( \hat{y} \) denote the \( \theta_0 \)-projection of \( y \) into \( H_{\theta_0,R} \), we get
\[
|\hat{w} - \hat{y}| \geq |\hat{w} - \bar{y}| - |\bar{y} - \hat{y}| \geq \frac{1}{2}n^{\beta_1} \Delta_R - \sqrt{d - 1}n^{-\beta_0} \Delta_R \geq \frac{1}{4}n^{\beta_1} \Delta_R,
\]
\[
q = \Pi_{xy}^\infty \cap H_{\varphi, u_1^\varphi}, \quad \hat{q} = L_\theta \cap H_{\varphi, u_1^\varphi}.
\]
where the second inequality uses \( \hat{y} \in B_{\theta_0, \text{cross}} \). Further, we have \( |\tilde{w} - w| \leq |x| \leq \sqrt{d - 1} n^{-\beta_0} \Delta_R \) so using (4.37),

\[
\sqrt{d} d(\hat{y}, \Pi_{xy}) \geq d_{\theta_0}(\hat{y}, \Pi_{xy}^\infty) = |w - \hat{y}| \geq |\tilde{w} - \hat{y}| - |\tilde{w} - w| \geq \frac{1}{4} n^{-\beta_1} \Delta_R.
\]

Thanks to (4.30) we can apply Lemma 2.3 with \( \delta = n^{-\beta_1} \Delta_R/4\sqrt{d}|u| \) and obtain

\[
g(y) + g(u - y) - g(u) \geq g(\hat{y}) + g(u - \hat{y}) - g(u) - c_2 \geq \frac{C_{34} n^{2\beta_1} \Delta_R^2}{16 d |u|} \geq c_3 n^{2\beta_1} \sigma_R,
\]

where the last inequality uses the readily-checked fact that \( |u| \leq c_4 R \).

Case 3. \( u \in \Gamma(y, z) \cap \Omega_{\theta_0}(R - n^{-\beta_1} \Delta_R, R) \), meaning there is a backtrack from \( y \) to \( u \) but not a large one. Again let \( \tilde{w} = \Pi_{\theta_0}^\infty \cap H_{\theta_0, R} \) and \( w = \Pi_{\theta_0}^\infty \cap H_{\theta_0, R} \). (These intersections exist provided \( \epsilon_{\text{min}} \) is small.) Then \( \tilde{w}, \hat{w} \notin Q_{R,n,\theta} \) so using (2.11),

\[
|\tilde{w} - \hat{w}| = d_{\theta_0}(\tilde{w}, L_\theta) \geq (1 - C_{31} \psi_{\theta_0}) d_{\theta}(\tilde{w}, L_\theta) \geq n^{-\beta_1} \Delta_R.
\]

Then as in (4.37) we have \( |\tilde{w} - \hat{y}| \geq \frac{1}{4} n^{-\beta_1} \Delta_R \). Using \( u \in \Omega_{\theta_0}(R - n^{-\beta_1} \Delta_R, R) \) and \( \psi_{\theta_0} \leq \epsilon_{\text{min}} \) we obtain readily that \( |u - \tilde{w}| \leq 2 |y_0| n^{-\beta_1} \Delta_R \). Since \( u \in H_{\theta_0, R - n^{-\beta_1} \Delta_R}^+ \) we have \( g(u) \geq R - n^{-\beta_1} \Delta_R \geq R/2 \) so \( |u| \geq \mu R/2\sqrt{d} \). We can therefore conclude, using also similarity of the triangles \( \Delta 0xu \) and \( \Delta \tilde{w}wu \), that

\[
|w - \tilde{w}| = \frac{|u - \tilde{w}|}{|u|} |x| \leq \frac{4 \sqrt{d} |y_0| n^{-\beta_1} \Delta_R}{\mu R} \sqrt{d - 1} n^{-\beta_0} \Delta_R \leq c_5 n^{\beta_1 - \beta_0} \sigma_R.
\]
Hence similarly to (4.38) we get \( \sqrt{d}(y, \Pi_{xu}) \geq \sqrt{d}(y, \Pi_{xu}^{\infty}) \geq \frac{1}{2} n^{\beta_1} \Delta_R \). As with (4.39) we then get from (4.30) and Lemma 2.3 with \( \delta = n^{\beta_1} \Delta_R / 2\sqrt{d}|u| \) that

\[
(4.40) \quad g(y) + g(u - y) - g(u) \geq C_{34} n^{2\beta_1} \Delta_R^2 / 4d|u| \geq c_6 n^{2\beta_1} \sigma_R.
\]

Case 4. \( u \in \Gamma[y, z] \cap H_{\theta_0, R-n^{\beta_1} \Delta_R}^{-} \), meaning there is a large backtrack from \( y \) to \( u \). Here we let \( \tilde{u} \) be the first site of \( \Gamma \) after \( y \) which lies in \( H_{\theta_0, R-n^{\beta_1} \Delta_R}^{-} \). Then similarly to (4.30) we have \(| \tilde{u} | \leq 2R|y_{0}| \), while using (2.20) gives

\[
\mu \sqrt{d}(y, \Pi_{x\tilde{u}}) \geq \mu \sqrt{d}(y, H_{\theta_0, R-n^{\beta_1} \Delta_R}^{-}) \geq d_{g}(y, H_{\theta_0, R-n^{\beta_1} \Delta_R}^{-}) \geq n^{\beta_1} \Delta_R,
\]

so similarly to (4.39) and (4.40) we get

\[
(4.41) \quad g(y) + g(\tilde{u} - y) - g(\tilde{u}) \geq c_7 n^{2\beta_1} \sigma_R.
\]

Thus in all of Cases 1–4 there are points \( a \) preceding \( b \) in \( \Gamma[x, z] \), with \((a, b) = (u, y), (y, u)\), or \((y, \tilde{u})\), satisfying \( \max|a - x, |b - x| \leq 3R|y_{0}| \) and

\[
g(a - x) + g(b - a) - g(b - x) \geq 2c_8 n^{2\beta_1} \sigma_R.
\]

It follows from Proposition 2.1 and (4.20) that

\[
h(a - x) + h(b - a) - h(b - x) \geq c_8 n^{2\beta_1} \sigma_R.
\]

With this we can proceed as in Case 1 of Proposition 2.4 and sum over all \( O(R^d) \) choices for each of \( a, b \) to obtain

\[
(4.42) \quad P(G_1) \leq c_9 R^{2d} \exp\left(-C_{10} n^{2\beta_1}\right) \leq \exp\left(-\frac{1}{2} C_{10} n^{2\beta_1}\right),
\]

which proves (4.29). \( \square \)

**Lemma 4.6.** There exist constants \( C_i > 0 \) as follows. Suppose A1 and A2(i),(ii) hold for some \( \theta_0, \epsilon_0 \), and A3 holds for some \( R, n, \theta, \ell, \epsilon_{\min} \). For the event

\[
G_2 : \text{(geodesic evading enlarged } \theta \text{-block)} \text{ There exists a geodesic } \Gamma \text{ from } B_{\theta_0, \text{home}}^{\text{fat}} \text{ to } B_{\theta_0, \text{cross}}^{\text{fat}} \text{ which contains a site in } (H_{\theta, s, \text{home}, +}^{\text{fat}} \backslash B_{s, \theta, \text{home}, +}^{\text{fat}}) \cup (H_{\theta, r, \text{home}, +}^{\text{fat}} \backslash B_{s, \theta, \text{home}, +}^{\text{fat}}),
\]

we have

\[
(4.43) \quad P(G_2) \leq \exp\left(-C_{60} n^{2\beta_1} / \log R\right).
\]

For the event \( G_2 \), the idea is that \( B_{s, \theta, \text{home}, +}^{\text{fat}} \) lies in front of, and nearly parallel to, the much smaller block \( B_{\theta_0, \text{home}}^{\text{fat}} \), so any geodesic from \( B_{\theta_0, \text{home}}^{\text{fat}} \) to \( B_{\theta_0, \text{cross}}^{\text{fat}} \) will very likely cross \( H_{\theta, s}^{\text{fat}} \) by passing through \( B_{s, \theta, \text{home}, +}^{\text{fat}} \), whereas \( G_2 \) says to the contrary. The picture with the larger \( B_{r, \theta, \text{home}, +}^{\text{fat}} \) just behind the smaller \( B_{\theta_0, \text{cross}}^{\text{fat}} \) is analogous.

**Proof of Lemma 4.6.** Suppose \( \tau \in G_2 \) and \( \Gamma = \Gamma[x, y] \) is a geodesic as described in \( G_2 \), from some \( x \in B_{\theta_0, \text{home}}^{\text{fat}} \) to \( y \in B_{\theta_0, \text{cross}}^{\text{fat}} \), containing a site \( u \in H_{\theta, s}^{\text{fat}} \backslash B_{s, \theta, \text{home}, +}^{\text{fat}} \). Let \( \hat{u} = (s, u_{\theta_0}^{\theta}) \) be the \( \theta \)-projection of \( u \) into \( H_{\theta, s} \) and \( \hat{x} = (0, x_{\theta_0}^{\theta}) \) the \( \theta_0 \)-projection of \( x \) into \( H_{\theta_0, 0} \), so \(|u - \hat{u}| \leq d\) and
$|x - \hat{x}| \leq d$, by (2.24). Let $\alpha = (y - x)/|y - x|$; we want a lower bound for $D_{\alpha,|(y-x)|_1}(u - x)$, so Proposition 2.4 can be applied. To do this we first consider $\theta$ in place of $\alpha$. Let

$$w = (s, \hat{x} \hat{s})_\theta = L_\theta(\hat{x}) \cap H_{\theta,s}, \quad z = (s, 0)_\theta = L_\theta(0) \cap H_{\theta,s}, \quad q = (x^0, 0)_\theta = L_\theta(\hat{x}) \cap H_{\theta,s},$$

so

$$|w - \hat{x}| = (\hat{u} - \hat{x})^0 |y_\theta|, \quad |\hat{u} - w| = |(\hat{u} - \hat{x})^0_2|, \quad |\hat{x} - q| = w - z.$$

See Figure 7. Since $\hat{u} \notin B_{s,\theta,home,+}$ we have $|\hat{u} - z| \geq 2\sqrt{dn}^{\beta_0} \Delta_R$. Since $\hat{x} \in B_{\theta,home}$ we have $|\hat{x}| \leq \sqrt{dn}^{-\beta_0} \Delta_R$, and therefore from (2.11) we obtain

$$|w - z| = |\hat{x} - q| \leq |\hat{x}| + |q| \leq |\hat{x}| + C_{31} \psi_{\theta,0} |\hat{x} - q| \quad \text{so} \quad |w - z| \leq (1 - C_{31} \epsilon_{\min})^{-1} \sqrt{dn}^{-\beta_0} \Delta_R$$

and then

$$|\hat{u} - \hat{x}|^0_2 = |\hat{w} - w| \geq |\hat{u} - z| - |w - z| \geq (2 - (1 - C_{31} \epsilon_{\min})^{-1}) \sqrt{dn}^{-\beta_0} \Delta_R \geq \frac{1}{2} \sqrt{dn}^{-\beta_0} \Delta_R.$$

Hence using (2.10) and (2.2),

$$|\hat{u} - \hat{x}|_{\alpha, \infty} \geq \frac{|(\hat{u} - \hat{x})^0_2|}{\sqrt{d - 1(1 + |y_\alpha|)}} \geq c_1 n^{-\beta_0} \Delta_R,$$

so using (4.23) and the last inequality in (4.22),

$$\Phi(|\hat{u} - \hat{x}|_{\alpha, \infty}) \geq c_2 \Phi(n^{-\beta_0} \Delta_R) \geq \Phi(R^{1/2}) \geq R^{(1 - \chi_2)/2} \geq n^{2\beta_1}.$$

Having (4.45), by (2.8), to obtain a lower bound for $D_{\alpha,|(y-x)|_1}(u - x)$ we need only obtain a lower bound for $|(\hat{u} - \hat{x})^0_2|^2/\Xi((\hat{u} - \hat{x})^0_1)^2$, under the added condition

$$|\hat{u} - \hat{x}|^0_2 \leq (\hat{u} - \hat{x})^0_1.$$

From the definition of $s$, there must exist a point $v \in H_{\theta,0,0} \cap H_{\theta,s} \cap Q_{R,n,\theta}$, so using (2.11) we have

$$|w - \hat{x}| \leq C_{31} \psi_{\theta,0} |w - v| \leq 4C_{31} \psi_{\theta,0} \sqrt{dn}^{\beta_1} \Delta_R.$$

Now in view of (4.28) the $\theta$–ratio of $y - x$ satisfies

$$\frac{|y^\theta_0 - x^\theta_0|}{|y^\theta_1 - x^\theta_1|} \leq \frac{|(y - \hat{y})^\theta_2| + |x^\theta_2|}{r - s} \leq \frac{5dn^{\beta_1} \Delta_R}{R}.$$
so from \((2.17)\),
\[
\psi_{\alpha \theta} \leq \frac{10dn^{\beta_1} \Delta R}{|y_\theta|^R}.
\]

From Lemma 2.2, (2.21), and (4.46) we obtain that, after reducing \(\epsilon_{\text{min}}\) if necessary,
\[
|\langle \hat{u} - \hat{x} \rangle_2^\alpha| \leq \langle |\hat{u} - \hat{x} \rangle_1^\alpha + C_{32} \psi_{\alpha \theta} |\hat{u} - \hat{x}|
\]
\[
\leq \langle |\hat{u} - \hat{x} \rangle_1^\alpha + \frac{1}{2} \langle |\hat{u} - \hat{x} \rangle_1^\alpha \rangle
\]
so from (4.47),
\[
|y_\theta| \langle |\hat{u} - \hat{x} \rangle_1^\alpha \rangle \leq 2|y_\theta| \langle |\hat{u} - \hat{x} \rangle_1^\alpha \rangle = 2|w - \hat{x}| \leq c_3 \epsilon_1 n^{\beta_1} \Delta R.
\]

Using (4.22) and the first inequalities in (4.16) and (4.19), we obtain \(R/\Delta R \geq 2c_4n^{2\beta_1 + \beta_0}\). From this along with Lemma 2.2 \((4.44)\), (4.46), (4.48) and (4.49), we get
\[
\langle |\hat{u} - \hat{x} \rangle_2^\alpha \rangle \geq \langle |\hat{u} - \hat{x} \rangle_2^\alpha \rangle - C_{32} \psi_{\alpha \theta} |\hat{u} - \hat{x}|
\]
\[
\geq \langle |\hat{u} - \hat{x} \rangle_2^\alpha \rangle - C_{32} \psi_{\alpha \theta}(1 + |y_\theta|)(|\hat{u} - \hat{x} \rangle_1^\alpha \rangle
\]
\[
\geq \frac{\sqrt{d}}{2} n^{-\beta_0} \Delta R \left( 1 - c_4 \frac{n^{2\beta_1 + \beta_0} \Delta R}{R} \right)
\]
\[
\geq \frac{\sqrt{d}}{4} n^{-\beta_0} \Delta R.
\]

From (4.20) and the first inequalities in (4.16) and (4.19) we have
\[
\Delta_R^{1-\chi_2} \geq C_5 n^{-(d-1)/n(1-\chi_2)/(d-1)} \geq n^{(3+\gamma_2)\beta_1 + 2\beta_0}
\]
and hence using (1.2) and (4.49),
\[
\mathbb{E}(\langle |\hat{u} - \hat{x} \rangle_1^\alpha \rangle^2 \rangle \leq c_4 \frac{n^{-2\beta_0} \Delta_R^2 R}{n^{3\gamma_1 \Delta_R} \log R} \geq \frac{\Delta_R^{1-\chi_2}}{n^{\beta_1(1+\gamma_2) + 2\beta_0}} \geq n^{2\beta_1}.
\]

With (4.45) this shows that
\[
\mathcal{D}_\alpha(x) \langle |y - x \rangle_1^\alpha \rangle \geq n^{2\beta_1},
\]
and then summing (2.30) over \(x \in B_{\theta_0,\text{home}}^\text{flat}\) and \(y \in B_{\theta_0,\text{cross}}^\text{flat}\) shows
\[
P(G_2) \leq c_3 R^{2(d-1)} \exp \left( -C_{36} n^{2\beta_1} \right) \leq \exp \left( -\frac{1}{2} C_{36} n^{2\beta_1} \right).
\]

\[\square\]

**Lemma 4.7.** There exist constants \(C_i > 0\) as follows. Suppose \(A1\) and \(A2(i),(ii)\) hold for some \(\theta_0, \epsilon_0\), and \(A3\) holds with \(\theta = \theta_0\) for some \(R, n, \ell, \epsilon_{\text{min}}\). For the events

\(G_3\): (backtrack) some \(\theta_0\)–target-directed \((\ell, \theta_0)\)–interval geodesic contains a \(\theta_0\)–backtrack of \(\frac{1}{2} n^{-\beta_1} \ell\),

\(G_4\): (quick sidestep in a direction–\(\theta_0\) geodesic) There exists an \((\ell, \theta_0)\)–interval geodesic \(\Gamma \subset Q_{R,n,\theta}\)

and sites \(u, v \in \Gamma\) with \(\langle |u - v \rangle_1^\alpha \rangle \leq n^{-\beta_1} \ell\) and \(h(u - v) - h(|u - v \rangle_1^\alpha |y_\theta) \geq \eta \sigma_\ell\),

and \(h(u - v) \geq \eta \sigma_\ell\) for some \(\eta > 0\).
we have
\[ P(G_3 \cup G_4) \leq \exp \left( -C_{60} \frac{n^{2\beta_1}}{\log R} \right). \]

Proof. Since \( \theta = \theta_0 \) we'll simply call it \( \theta \). We can handle \( G_3 \cup G_4 \) as one, as follows. Suppose \( \tau \in G_3 \). This means there exists an \((\ell, \theta)\)-interval geodesic \( \Gamma_{xy} \subset Q_{R,n,\theta} \) containing sites \( u \) preceding \( v \) with \( u_1^q - v_1^q \geq n^{-\beta_4} \ell/2 \). By choosing a different \( v \in \Gamma \) we may assume also \( n^{-\beta_4} \ell/2 \leq u_1^q - v_1^q \leq n^{-\beta_4} \ell \).

We have using (4.27)
\[ h(v - u) \geq g(v - u) \geq g((v_1^q - u_1^q)y_\theta) \geq \frac{n^{-\beta_4} \ell}{2} \geq \frac{\eta}{8} \sigma_\ell. \]

This shows that for the event

\[ G_5 : \text{(bad-direction segment in a geodesic)} \]

There exists an \((\ell, \theta)\)-interval geodesic \( \Gamma \subset Q_{R,n,\theta} \)
and sites \( u \) preceding \( v \) in \( \Gamma \) with \(|(v - u)|_1^q \leq n^{-\beta_4} \ell \) and
\[ h(v - u) - h(|(v - u)|_1^q y_\theta)_{1_{\{v - u\}} \geq 0} \geq \frac{\eta}{8} \sigma_\ell, \]

(4.53) \[ |(u - v)|_1^q \leq n^{-\beta_4} \ell, \quad h(u - v) - h(|(u - v)|_1^q y_\theta)_{1_{\{v - u\}} \geq 0} \geq \frac{\eta}{8} \sigma_\ell. \]

Let \( \alpha = (y - u)/|y - u| \); we want a lower bound for \( D_{\alpha,|(y-u)|_1^q}(v - u) \) so we can use Proposition 2.4. We may assume \( u_1^q \leq (i - 1/2) \ell \), as the other case is symmetric. Then \( u, y \in Q_{R,n,\theta} \) with \(|y - u|_1^q \geq \ell/2 \), so from (4.24), provided \( R \) is large the \( \theta \)-ratio of \( y - u \) is at most \( 8 \sqrt{d - 1} n^{-\beta_4} \Delta_R / \ell \leq 8 \sqrt{d - 1} n^{-\beta_4} \phi < |y_\theta| \epsilon_{\min}/2 \), so by (2.17).

(4.54) \[ \psi_{\alpha \theta} \leq \tan \psi_{\alpha \theta} \leq \frac{16 n^{\beta_4} \Delta_R}{|y_\theta| \ell} < \epsilon_{\min}. \]

Let
\[ \lambda = \frac{1}{2|y_\theta|(1 + (\inf_{\varphi} |y_\varphi|)^{-1})}, \quad t_0 = \frac{\lambda \mu \eta \sigma_\ell}{32 \sqrt{d}}, \quad t = \max \left( v_1^q - u_1^q, t_0 \right), \quad q = u + t y_\theta. \]

Define the intersection points
\[ p = L_\alpha(u) \cap H_{\theta, q}, \quad w = L_\alpha(u) \cap H_{\alpha, v_1^q}, \quad x = L_\alpha(u) \cap H_{\theta, v_1^q}, \]

so that \( v - u = (w - u) + (v - w) \) is the decomposition of \( v - u \) into \( \alpha \)-components. From (2.20) and (4.53), provided \( R \) (and hence \( \ell \) and \(|u - v|\)) is large we have
\[ |v - u| \geq \frac{\mu}{\sqrt{d}} g(v - u) \geq \frac{\mu}{2 \sqrt{d}} h(v - u) \geq \frac{\mu \eta \sigma_\ell}{16 \sqrt{d}} = \frac{2t_0}{\lambda}. \]

From (2.11) and (4.54) we have
\[ |w - x| \leq C_{31} \psi_{\alpha \theta} |v - w|, \]

(4.56) \[ |w - x| \leq C_{31} \psi_{\alpha \theta} |v - w|, \]

while from (2.12),
\[ |p - q| \leq C_{31} \psi_{\alpha \theta} |q - u| = C_{31} \psi_{\alpha \theta} |y_\theta| \ell. \]
Using (2.2), (4.26), and (4.55) we then get
\begin{equation}
\Phi(|v - u|_{\alpha,\infty}) \geq c_1 \Phi(\sigma_t) \geq t^{\lambda_0(1 - \chi_0)} \geq n^{4\beta_1}.
\end{equation}

As a shorthand we say a point \( y \) is strictly behind a hyperplane \( H_{\theta,t} \) if \( y \) lies in the interior of \( H_{\theta,t}^c \), and \( y \) is \( \varphi \)-behind a point \( z \) if \( z \) is strictly behind \( y \). We now consider 3 cases; see Figure 8.

**Case 1.** \( w \in H_{\alpha,u^\alpha_1}^{-} \), meaning \( v \) (hence also \( w \)) is \( \alpha \)-behind \( u \). Here using (2.8) and (4.58) we get

\begin{equation}
D_{\alpha,\mu}(y - u_1^\alpha)(v - u) = \Phi(|v - u|_{\alpha,\infty}) \geq n^{4\beta_1},
\end{equation}

so with (4.59) and (4.54), from Proposition 2.4

\begin{equation}
P(G_5 \text{ and Case 1 holds}) \leq C_{35}e^{-C_{36}n^{2\beta_1}}.
\end{equation}

**Case 2.** \( w \in H_{\alpha,u^\alpha_1}^{+}, v_1^\alpha - u^\alpha_1 < t_0 \). This means \( t = t_0, v_1^\alpha = u_1^\alpha \geq u^\alpha_1 \), and \( v \) is strictly behind \( H_{\theta,u^\alpha_1,t}^c \). (See Figure 8.) Now \( p, u, w, x \) lie on the line \( L_\alpha(u) \), with \( u, x \) strictly behind \( H_{\theta,u^\alpha_1,t}^c \) and \( \theta \)-behind \( \lambda \in H_{\theta,u^\alpha_1,t}^c \) and \( u \) \( \alpha \)-behind \( w \). If \( |v_2^\alpha - u_2^\alpha| \geq |v_1^\alpha - u_1^\alpha| \) then by (2.8), (4.59) applies, so we assume

\begin{equation}
|w - u| = |y_\alpha||(v_1^\alpha - u_1^\alpha)| > |y_\alpha||v_2^\alpha - u_2^\alpha| = |y_\alpha||v - w|.
\end{equation}

If \( w \) is \( \alpha \)-behind \( p \) then \( w \in H_{\theta,t}^{-} \) and \( |w - u| \leq |p - u| \); if \( w \) is not \( \alpha \)-behind \( p \) then \( w \in H_{\theta,u^\alpha_1,t}^c \)

\[ |w - p| \leq |w - x| \]. Either way we have using (4.54), (4.55), (4.56), (4.57), (4.61)

\[ |w - u| \leq |p - u| + |w - p|1_{\{w \in H_{\theta,t}^{-}\}} \leq |q - u| + |p - q| + |w - x| \]

\[ \leq 2|y_\alpha|t + \frac{1}{2}|w - u| \leq \lambda|y_\alpha||v - u| + \frac{1}{2}|w - u| \]

so \( |w - u| \leq 2\lambda|y_\alpha||v - u| \), which with (4.61) yields

\[ |v - u| \leq |v - w| + |w - u| \leq (1 + |y_\alpha|^{-1})|w - u| \leq 2\lambda|y_\alpha|(|1 + |y_\alpha|^{-1})|w - u| < |v - u|, \]

a contradiction. This means (4.61) cannot hold, so (4.59) always applies in Case 2, and therefore as with (4.60).

\begin{equation}
P(G_5 \text{ and Case 2 holds}) \leq C_{35}e^{-C_{36}n^{2\beta_1}}.
\end{equation}

**Case 3.** \( w \in H_{\alpha,u^\alpha_1}^{+}, v_1^\alpha - u_1^\alpha \geq t_0 \). From the definition of \( G_5 \), this means

\begin{equation}
t_0 \leq t = v_1^\theta - u_1^\theta \leq n^{-\beta_1} \ell, \quad v \in H_{\theta,u^\alpha_1,t}^c, \quad x = p,
\end{equation}

and the indicator function in (4.53) is 1. As in Case 2 we may assume (4.61). From (4.56), (4.57), (4.54), and (4.61),

\[ |w - u| \leq |q - u| + |p - q| + |w - x| \leq 2|y_\alpha|t + \frac{|y_\alpha|}{2}|v - w| \leq 2|y_\alpha|t + \frac{1}{2}|w - u|, \]

so

\begin{equation}
|y_\alpha||v_1^\alpha - u_1^\alpha| = |w - u| \leq 4|y_\alpha|t.
\end{equation}

We claim that

\begin{equation}
|v - q| \geq n^{\beta_1} \Delta_t.
\end{equation}
Figure 8. Illustrations for Lemma 4.7. Top: Case 2, in which $v$ lies strictly behind $H_{\alpha,u_1^q}$ but ahead of $H_{\alpha,u_1^q}$, and $t = t_0$. Bottom: Case 3, in which $v$ lies ahead of both $H_{\alpha,u_1^q} + t_0$ and $H_{\alpha,u_1^q}$, and $t \geq t_0$. In both cases $w$ may lie on either side of $x$ or $p$. In Case 1 (not shown) there is an $\alpha$–backtrack: $v$ lies behind $H_{\alpha,u_1^q}$.

Suppose not; we will show that the second inequality in (4.53) is contradicted. From (4.26) and the fact that $t \geq t_0$, the $\theta$–ratio of $v - u$ satisfies

$$\frac{|v_2^\theta - u_2^\theta|}{|v_1^\theta - u_1^\theta|} = \frac{|v - q|}{t} < \frac{n^{\beta_1} \Delta_t}{t} \leq c_2 n^{\beta_1} \ell^{-\chi_1(1-\chi_2)/2} < n^{-\beta_1}.$$  \hspace{1cm} (4.66)

Since $v$ lies in the tangent plane $H_{\alpha,u_1^q} + t$ to $u + t\mathcal{B}_g$ at $u + ty_\theta = q$, it follows from (1.6), using the first inequality in (4.66), that

$$d(v, u + t\mathcal{B}_g) \leq C_9 n^{2\beta_1} \sigma_t.$$  \hspace{1cm} (4.67)

Hence by (2.20),

$$g(v - u) \leq t + C_9 \mu \sqrt{n^{2\beta_1}} \sigma_t = g((v_1^\theta - u_1^\theta)y_\theta) + C_9 \mu \sqrt{n^{2\beta_1}} \sigma_t.$$
From (4.61) and (4.64) we have
\[
|v - u| \leq (1 + |y_0|^{-1})|w - u| \leq 4|y_0|(1 + |y_0|^{-1})t.
\]
With (1.2), the last inequality in (4.16), (4.63), (4.67), and Proposition 2.1 this shows that
\[
h(v - u) - h((v_1^\theta - u_1^\theta)y_\theta) \leq g(v - u) - g((v_1^\theta - u_1^\theta)y_\theta) + C_{25}\sigma(v - u)\log|v - u|
\leq \left(C_9\mu\sqrt{dn^2\beta_1} + c_3\log t\right)\sigma_t
\leq c_4n^{2\beta_1}\sigma_t
\leq c_5n^{2\beta_1-\gamma_1}\sigma_t
\leq \frac{\eta}{8}\sigma_t,
\]
(4.69)
which contradicts (4.53); this proves our claim (4.65). Note that (4.68) is not affected by the contradiction established.

We then have using Lemma 2.2, (4.54), (4.65), and (4.68) that
\[
|(v - u)_{\alpha}| \geq |v - q| - C_{32}\psi_\alpha v - u| \geq n^{\beta_1}\Delta_t - c_6n^{\beta_1}\Delta_t R t \geq n^{\beta_1}\left(\Delta_t - c_7\frac{t\Delta R}{t}\right).
\]
(4.70)
It follows from (1.2), the first inequality in (4.17), and (4.63) that
\[
\left(\frac{\ell\Delta_t}{t}\right)^2 = \frac{\ell^2\sigma_t}{t} \geq C_3^{-1}n^{(1-\gamma_2)\beta_4}\ell\sigma_t \geq C_3^{-1}n(1-\gamma_2)\beta_4\Delta R \geq n^{\beta_2}\Delta R,
\]
which with (4.70) shows that
\[
|(v - u)_{\alpha}| \geq \frac{1}{2}n^{\beta_1}\Delta_t.
\]
From this and (4.64),
\[
\frac{|v_{\alpha} - u_{\alpha}|^2}{\Xi(|v_1^\theta - u_1^\theta|)} \geq \frac{n^{2\beta_1}\Delta_t^2}{4\Xi(4t)^2} \geq c_8n^{2\beta_1}\log t \geq c_8n^{2\beta_1}\log R,
\]
where the first inequality uses $|y_\theta| \leq 2|y_0|$, from (1.13). Then in view of (4.58) we conclude
\[
D_{\alpha,|(y - u)_{\alpha}|}(v - u) \geq c_8n^{2\beta_1}\log R.
\]
(4.71)
Therefore as with (4.60),
\[
P(G_5 \text{ and Case 3 holds}) \leq C_{35}\exp\left(-c_9\frac{n^{2\beta_1}}{\log R}\right),
\]
(4.72)
which with (4.60) and (4.62) yields
\[
P(G_3 \cup G_4) \leq P(G_5) \leq 3C_{35}\exp\left(-c_9\frac{n^{2\beta_1}}{\log R}\right).
\]
(4.73)
\[\square\]
Lemma 4.8. There exist constants $C_i > 0$ as follows. Suppose $A1$ and $A2(i),(ii)$ hold for some $\theta_0, \epsilon_0$, and $A3$ holds with $\theta = \theta_0$ for some $R, n, \ell, \epsilon_{\min}$. For the event

$G_0$: (there are close $(\ell, \theta)$-interval geodesics with dissimilar passage times) There exist $i \leq n/\ell$

and $u, w \in H_{\theta,(i-1)\ell}^{\text{fat}} \cap Q_{R,n,\theta} \cap \mathbb{Z}^d$ and $v, x \in H_{\theta,\ell}^{\text{fat}} \cap Q_{R,n,\theta} \cap \mathbb{Z}^d$ with

$$|u - w| \leq 2\sqrt{dn^{-\beta_0} \Delta_R}, \quad |v - x| \leq 2\sqrt{dn^{-\beta_0} \Delta_R}, \quad |T(u, v) - T(w, x)| \geq \frac{\eta}{8} \sigma_\ell,$$

we have

$$(4.74) \quad P(G_0) \leq \exp \left( -C_{60} \frac{n^{2\beta_1}}{\log R} \right).$$

Proof. Let $i \leq n/\ell$ and suppose $u, w \in H_{\theta,(i-1)\ell}^{\text{fat}} \cap Q_{R,n,\theta} \cap \mathbb{Z}^d$ and $v, x \in H_{\theta,\ell}^{\text{fat}} \cap Q_{R,n,\theta} \cap \mathbb{Z}^d$ with

$$(4.75) \quad |u - w| \leq 2\sqrt{dn^{-\beta_0} \Delta_R}, \quad |v - x| \leq 2\sqrt{dn^{-\beta_0} \Delta_R}.$$ We have

$$(4.76) \quad |T(u, v) - T(w, x)| \leq |T(u, v) - T(u, x)| + |T(u, x) - T(w, x)|,$$

and we want to use Proposition 2.6 to bound the probability that either difference on the right exceeds $\eta \sigma_\ell/8$. Let $\hat{u}, \hat{v}, \hat{w}, \hat{x}$ be the $\theta$-projections of $u, w, v, x$, with $\hat{u}, \hat{w}$ into $H_{\theta,(i-1)\ell}$ and $\hat{v}, \hat{x}$ into $H_{\theta,\ell}$. Since $\hat{u}, \hat{v}, \hat{x} \in Q_{R,n,\theta}$ we have using (4.24)

$$|\hat{v} - \hat{u} - \ell y_\theta| \leq |\hat{v} - i\ell y_\theta| + |\hat{u} - (i - 1)\ell y_\theta| \leq 2n^{\beta_1} \Delta_R \leq \epsilon_{\min} \ell,$$

and using (2.4)

$$(4.77) \quad |\hat{v} - \hat{x}| \leq |v - x| + 2d \leq 3\sqrt{dn^{-\beta_0} \Delta_R} < \epsilon_{\min} \ell,$$

so Lemma 2.5 gives

$$|g(\hat{v} - \hat{u}) - g(\hat{x} - \hat{u})| \leq C_{38}(6\sqrt{dn^{\beta_1} - \beta_0} + 9dn^{-2\beta_0}) \Delta_R^2 \leq c_1 n^{\beta_1 + \beta_2 - \beta_0} \sigma_R.$$ We may assume the points are labeled so that $g(\hat{x} - \hat{u}) \leq g(\hat{v} - \hat{u})$; there then exists a point $\hat{z} \in \Pi_{\hat{u},\hat{v}}$ (close to $\hat{v}$) with $g(\hat{z} - \hat{u}) = g(\hat{x} - \hat{u})$, and $z \in \mathbb{Z}^d$ with $|z - \hat{z}| \leq \sqrt{d}$. We then have

$$(4.78) \quad g(\hat{v} - \hat{z}) = g(\hat{v} - \hat{u}) - g(\hat{z} - \hat{u}) \leq c_1 n^{\beta_1 + \beta_2 - \beta_0} \sigma_R$$

and

$$|T(u, v) - T(u, x)| \leq |T(u, z) - T(u, x)| + T(z, v),$$

and the latter implies

$$(4.79) \quad P \left( |T(u, v) - T(u, x)| \geq \frac{\eta}{8} \sigma_\ell \right) \leq P \left( |T(u, z) - T(u, x)| \geq \frac{\eta}{16} \sigma_\ell \right) + P \left( T(z, v) \geq \frac{\eta}{16} \sigma_\ell \right).$$

Checking the conditions of Proposition 2.6 for the first probability on the right of (4.79), we note first that

$$|g(z - u) - g(x - u)| \leq g(z - \hat{z}) + 2g(u - \hat{u}) + g(x - \hat{x}) \leq 4\mu \sqrt{d}.$$ Further,

$$|v - u| \geq |\hat{v} - \hat{u}| - |\hat{v} - \hat{x}| - |u - \hat{u}| \geq \frac{1}{\mu \sqrt{d}} g(\hat{v} - \hat{u}) - 2d \geq \frac{\ell}{2\mu \sqrt{d}}.$$
so using (4.25), (4.75), and the second inequality in (4.17), for $C_{40}$ from Proposition 2.6

$$\Delta(|v - u|) \geq c_2\Delta \geq c_3n^{-(1+\gamma_2)/2}\Delta_R \geq 2C_{40}^{-1}\sqrt{d}n^{-\beta_0}\Delta_R \geq C_{40}^{-1}|v - x|.$$ 

In applying Proposition 2.6 to the first probability on the right of (4.79) we take $\lambda = n^{2\beta_1}$, so we need to verify that for this $\lambda$,

$$\frac{\eta}{16}\sigma \geq \lambda\sigma\Delta^{-1}(|v - x|)\log|v - x|.$$

Let

$$A = \left(\frac{16\lambda}{C_2\eta}\log|v - x|\right)^{1/\gamma_1}.$$ 

If we can show that

$$\Delta \geq C_3 A^{(1+\gamma_2)/2}|v - x|,$$

then using (1.2),

$$\Delta\left(\frac{\ell}{A}\right) \geq C_3^{-1/2}A^{-1+\gamma_2/2}\Delta \geq |v - x| \quad \text{and hence} \quad C_2\Delta \leq C_2\left(\frac{\ell}{\Delta^{-1}(|v - x|)}\right)^{1/\gamma_1} \leq \frac{\sigma\Delta}{\sigma\Delta^{-1}(|v - x|)}$$

which is equivalent to (4.80). In fact we have using the second inequality in (4.17), along with (4.25), (4.75), and $\log|v - x| \leq \log R \leq n^{2\beta_1}$, that

$$C_3 A^{(1+\gamma_2)/2}|v - x| \leq c_4n^{-\beta_0}\Delta_R\left(n^{4\beta_1}\right)^{(1+\gamma_2)/2}\gamma_1 \leq c_5n^{-\beta_0+2\beta_1(1+\gamma_2)/\gamma_1}\Delta_R \leq c_6n^{-\beta_0+2\beta_1(1+\gamma_2)/\gamma_1+\beta_2(1+\gamma_2)/2}\Delta \leq \Delta,$$
proving (4.81) and hence also (4.80). Now (4.80) and Proposition 2.6 give

(4.82) \[ P \left( |T(u, z) - T(u, x)| \geq \frac{\eta}{16} \sigma \right) \leq C_{42} \exp \left( -C_{43} n^{2\beta_1} \right). \]

Turning now to the last probability in (4.79), we have using (1.2), the second inequality in (4.17), and (4.78) that

\[ h(v - z) \leq c_7 + 2g(v - z) \leq c_8 + 2g(\hat{v} - \hat{z}) \leq 2c_1 n^{\beta_1 + \beta_2 - \beta_0} \sigma_R \leq n^{\beta_1 + \beta_2 (1 + \gamma_2) - \beta_0} \sigma_\ell \leq \frac{\eta}{32} \sigma_\ell \]

and similarly

\[ |v - z| \leq n^{\beta_1 + \beta_2 (1 + \gamma_2) - \beta_0} \sigma_\ell, \]

so from (1.2), (1.4), and (4.26),

\[ P \left( T(z, v) \geq \frac{\eta}{16} \sigma \right) \leq P \left( T(z, v) - ET(z, v) \geq \frac{\eta}{32} \sigma \right) \leq C_5 \exp \left( -\frac{C_6}{32} \frac{\sigma_\ell}{\sigma(\sigma_\ell)} \right) \]

\[ \leq \exp \left( -\ell^\chi_1 (1 - \chi_2) \right) \leq \exp \left( -n^{2\beta_1} \right). \]

Combining this with (4.76), (4.79), and (4.82), along with a similar computation for the second term on the right in (4.76) in place of the first term, we get

\[ P \left( |T(u, v) - T(w, x)| \geq \frac{\eta}{4} \sigma \right) \leq c_9 \exp \left( -c_{10} n^{2\beta_1} \right). \]

Summing this over the \( O(R^{4(d-1)}) \) possible values of \((u, v, w, x)\) we get as in (4.42) that

(4.83) \[ P(G_6) \leq c_9 \exp \left( -\frac{1}{2} c_{10} n^{2\beta_1} \right). \]

\[ \square \]

**Lemma 4.9.** There exist constants \( C_i > 0 \) as follows. Suppose A1 and A2(i),(ii) hold for some \( \theta_0, \epsilon_0, \) and A3 holds with \( \theta = \theta_0 \) for some \( R, n, \ell, \epsilon_{\min} \). For the event

\( G_7 \) : (unusual-speed short segment) There exist \( u, v \in Q_{R, n, \theta} \cap \mathbb{Z}^d \) with

\[ |u - v| \leq 3n^{-\beta_1} \ell, \quad |T(u, v) - h(v - u)| \geq \frac{\eta}{8} \sigma(\ell |y_\theta|/2) \ell, \]

we have

(4.84) \[ P(G_7) \leq \exp \left( -C_{60} \frac{n^{2\beta_1}}{\log R} \right). \]

**Proof.** When \(|u - v| \leq 3n^{-\beta_1} \ell\) we have using (1.2) and the last inequality in (4.16) that

\[ \sigma(|u - v|) \leq 3^\gamma C_2^{-1} n^{-\beta_1} \sigma_\ell \leq \frac{\eta}{8} n^{-2\beta_1} \sigma_\ell, \]

so (1.4) says

\[ P \left( |T(u, v) - h(|u - v|)| \geq \frac{\eta}{8} \sigma_\ell \right) \leq C_5 \exp \left( -C_6 n^{2\beta_1} \right). \]

As in (4.42), summing over the \( O(R^{2d}) \) possible values of \((u, v)\) gives

(4.85) \[ P(G_7) \leq c_1 \exp \left( -\frac{1}{2} c_2 n^{2\beta_1} \right). \]
We are now ready for the core of our main proof, given by the next proposition.

**Proposition 4.10.** Suppose $A1$ and $A2(i),(ii)$ hold for some $\theta_0, \epsilon_0$. There exist constants $\beta_j \in (0, 1)$ and $C_i$ as follows. Let $R, n, \epsilon_{\min}$ be as in $A3$, and let $B_{\theta_0,\mathrm{cross}}$ be a $\theta_0$–block in $H_{\theta_0, R}$ with center point $\bar{y}$. Let $\theta = |\bar{y}|,$ and suppose $\psi_{\theta_0} < \epsilon_{\min}/2$. Then

\[ P\left( \text{there exist } n \text{ } \theta_0 \text{–} \text{slab geodesics from } B_{\theta_0,\text{home}}^{\text{flat}} \text{ to } H_{\theta_0,2R}^{+} \right) \]

with distinct $H_{\theta_0,R}^{+}$–entry points in $D_{\theta_0,\text{cross}}^{\text{flat}} \leq \exp \left( -C_{61}n^{2\beta_1}/\log R \right).$

(4.86)

The upper bound on $n$ in (4.20) can be written

\[ \frac{n}{(n-\beta_0 \Delta R)^{d-1}} \leq C_{55}n^{\beta_0(d-1)}, \]

which is not really a restriction at all, since the density of $H_{\theta_0,R}^{+}$–entry points is bounded. It is only there for technical use in Lemmas 4.5–4.9.

**Proof of Proposition 4.10.** Let $s, r$, and $\beta_j \in (0, 1)$ be as in $A3$.

As a shorthand, a $\theta_0$–slab geodesic from $B_{\theta_0,\text{home}}^{\text{flat}}$ to $H_{\theta_0,2R}^{+}$ will be called a $2R$–geodesic. Given a $2R$–geodesic $\Gamma$, we can decompose the pre-$H_{\theta_0,R}$ segment of $\Gamma$ into an initial bond and $n^{\beta_2}$ $\ell$–segments; every such $\ell$–segment is an ($\ell, \theta_0$)–interval geodesic. An ($\ell, \theta$)-interval geodesic $\Psi$ is good if

1. $\Psi$ is contained in $Q_{R,n,\theta}$,
2. $\Psi$ contains no backtrack of $\frac{1}{2}n^{-\beta_4} \ell$.

Let $G_8$ be the event in (4.86), and define the event

$G_9 : \text{there exist } n \theta \text{–} \text{target-directed } 2R \text{–} \text{geodesics with distinct } H_{\theta_0,R}^{+} \text{–} \text{entry points in } B_{\theta_0,\text{cross}}^{\text{flat}}$, so

(4.87) $G_8 \subset G_9 \cup G_1$.

Note that $\tau \in G_9 \setminus G_3$ says that every $\ell$–segment of every $\theta$–target-directed $2R$–geodesic is a good ($\ell, \theta$)–interval geodesic.

Our main task is to bound $P(G_9)$. There are at most $c_{1R}(\beta_0+\beta_1)(d-1)$ $\theta_0$–blocks intersecting $H_{\theta_0,2R} \cap Q_{R,n,\theta}$, and we denote the $j$th one (in some arbitrary order) as $B_{2R,j}$. When $\tau \in G_9$, there exists a subcollection $\mathcal{G}$ of size at least

\[ g_n = c_2n^{1-(\beta_0+\beta_1)(d-1)} \]

out of the $n$ $2R$–geodesics, which for some $m$ all have $H_{\theta_0,R}^{+}$–entry point in block $B_{2R,m}^{\text{flat}}$. We call such a subcollection $\mathcal{G}$ a crowded set (via $B_{\text{cross}}$ and $B_{2R,m}$), and fix such a $\mathcal{G}$ and $m$. Let $y^*$ be the center of $B_{2R,m}$, let $\theta^* = (y^* - \bar{y})/|y^* - \bar{y}|$, and let

\[ Q_{R,n,\theta^*} = \bar{y} + \left( \mathbb{R} \times [-4\sqrt{d-1}n^{\beta_1} \Delta_R, 4\sqrt{d-1}n^{\beta_1} \Delta_R]^{d-1} \right) \text{, in } \theta^*\text{–coordinates.} \]
Figure 10. Illustration for the proof of Proposition 4.10. The dashed line is a typical geodesic from the crowded set $\mathcal{G}$. The primary $\theta$–slab geodesic crosses the left gray box, which is part of the square tube $Q_{R,n,\theta}$ surrounding $L_\theta$. The secondary $\theta^*$–slab geodesic crosses the right gray box, which is part of the square tube $Q^*_{R,n,\theta^*}$ surrounding $L_{\theta^*}(\overline{y})$. The hash marks bound the named blocks (such as $B_{\theta_0,\text{home}}$) in the hyperplanes as shown.

This is a tube around $L_{\theta^*}(\overline{y}) = \prod_\infty y, y^*$ with cross section larger than that of $Q_{R,n,\theta}$ in each dimension by a factor $2\sqrt{d-1}$. It is straightforward to show that due to this larger cross section we have
\[ Q^*_{R,n,\theta^*} \cap \Omega_{\theta_0}(R,2R) \supset Q_{R,n,\theta} \cap \Omega_{\theta_0}(R,2R). \]

Analogously to $[s,r]$, there is a largest interval $[s^*,r^*]$ for which
\[ Q_{R,n,\theta} \cap \Omega_{\theta^*}(s^*,r^*) \subset Q_{R,n,\theta} \cap \Omega_{\theta_0}(R,2R); \]
see Figure 10. Since $y^* \in Q_{R,n,\theta}$ it is easily checked that the $\theta$–ratio of $y^* - \overline{y}$ is bounded by $c_3 n^{\beta_1} \Delta_R/R$, and hence
\[ (4.88) \quad \psi_{\theta^*} \leq c_4 \frac{n^{\beta_1} \Delta_R}{R} \leq \frac{\epsilon_{\min}}{4} \quad \text{and hence} \quad \psi_{\theta^*} \theta_0 \leq \frac{\epsilon_{\min}}{2}. \]

There is a “nuisance possibility” we must deal with here: by assumption the $g_0$ (or more) $2R$–geodesics in the crowded set $\mathcal{G}$ have distinct $H_{\theta^+_0,R}$–entry points, but this does not guarantee they have distinct entry points for the hyperplanes $H_{\theta^+_r}$ (behind $H_{\theta_0,R}$) and $H_{\theta^*_r,s^*}$ (ahead of $H_{\theta_0,R}$).

However, at least one of the following options must be true:

(I) there is a subset $\mathcal{G}_1 \subset \mathcal{G}$ with $|\mathcal{G}_1| \geq g_0^{1/3}$ which all have the same $H_{\theta^+_0,R}$–entry point,

(II) there is a subset $\mathcal{G}_2 \subset \mathcal{G}$ with $|\mathcal{G}_2| \geq g_0^{1/3}$ which all have the same $H_{\theta^*_r,s^*}$–entry point,

(III) there is a subset $\mathcal{G}_3 \subset \mathcal{G}$ with $|\mathcal{G}_3| \geq g_0^{1/3}$ which all have distinct $H_{\theta^*_r,s^*}$–entry points, and which all have distinct $H_{\theta^+_0,R}$–entry points.

If (I) occurs in some $\tau$, then since the geodesics in $\mathcal{G}_1$ have distinct $H_{\theta^+_0,R}$–entry points, uniqueness of finite geodesics means these geodesics must be “disjoint from $H_{\theta_0,R}$ to $H_{\theta_0,2R}$”, or more precisely,
the geodesics $\Gamma[x'_{R}(\Gamma), x''_{R}(\Gamma)]$, $\Gamma \in \mathfrak{G}_{1}$, are disjoint, except for possibly sharing a starting point $x'_{R}(\Gamma)$. Similarly, if (II) occurs then the geodesics in $\mathfrak{G}_{2}$ must all have disjoint pre-$H_{\theta_{0},R}$ segments, except for possibly sharing the same $x''_{R}(\Gamma)$. In (III) we cannot conclude any such disjointness.

In $H_{\theta, s}$ and $H_{\theta, r}$ we have the enlarged home $\theta$–blocks $B_{s, \theta, \text{home}, +}$ and $B_{r, \theta, \text{home}, +}$, respectively, centered on the line $L_{\theta}$. In $H_{\theta^{*}, s}$ and $H_{\theta^{*}, r}$ we can define analogous shifted enlarged home $\theta^{*}$–blocks by translating an enlarged $\theta^{*}$–block within the hyperplane so that it is centered on $L_{\theta}(\overline{y})$; we denote these shifted enlarged home $\theta^{*}$–blocks by $\tilde{B}_{s, \theta^{*}, \text{home}, +}$ and $\tilde{B}_{r, \theta^{*}, \text{home}, +}$, respectively. The analog of the event $G_{2}$ for the geodesic segments from $H_{\theta_{0},R}$ to $H_{\theta_{0},2R}$ is the following:

$G_{10}$: (geodesic after $H_{\theta_{0},R}$ evading enlarged $\theta$–blocks) There exists a geodesic $\Gamma$ from $B_{\theta_{0}, \text{cross}}^{\text{fat}}$ to $B_{2R,m}^{\text{fat}}$ which contains a site in $(H_{\theta, s}^{\text{fat}} \setminus \tilde{B}_{s, \theta^{*}, \text{home}, +}) \cup (H_{\theta, r}^{\text{fat}} \setminus \tilde{B}_{r, \theta^{*}, \text{home}, +})$.

In view of (4.88) we essentially can apply Lemma 4.6 to bound $P(G_{10})$, the only thing being different for $G_{10}$ is that the tube $Q_{R,n,\theta^{*}}$ is fatter by a constant factor $2\sqrt{d-1}$. This makes no material difference so we have

$$P(G_{10}) \leq \exp \left( -C_{60} \frac{n^{2\beta_{1}}}{\log R} \right).$$

Consider now $\tau \in G_{9} \setminus (G_{3} \cup G_{2} \cup G_{10})$ and suppose option (III) occurs. We have the following situation: each $\Gamma \in \mathfrak{G}_{3}$ contains a unique $\theta$–slab geodesic from $B_{\theta, \text{home}, +}^{\text{fat}}$ to $B_{\theta, \text{home}, +}^{\text{fat}}$, which we call the primary $\theta$–slab geodesic of $\Gamma$ and denote $\Gamma^{\text{pri}}$. $\Gamma$ also contains a unique $\theta^{*}$–slab geodesic from $\tilde{B}_{\theta^{*}, \text{home}, +}^{\text{fat}}$ to $B_{\theta^{*}, \text{home}, +}^{\text{fat}}$, which we call the secondary $\theta^{*}$–slab geodesic of $\Gamma$ and denote $\Gamma^{\text{sec}}$. A site $x \in H_{\theta, r}$ is called popular for $\mathfrak{G}_{3}$ (in $\tau$) if there exist $n^{\beta_{1}-\beta_{4}}/8$ $2R$–geodesics $\Gamma \in \mathfrak{G}_{3}$ for which $x$ lies in the primary $\theta$–slab geodesic of $\Gamma$. A key observation is that if such a popular site $x$ exists, then since $\{\Gamma \in \mathfrak{G}_{3} : x \in \Gamma\}$ have distinct $H_{\theta, r}^{+}$–entry points, the $n^{\beta_{1}-\beta_{4}}/8$ (or more) geodesics $\{\Gamma^{\text{sec}} : \Gamma \in \mathfrak{G}_{3}, x \in \Gamma\}$ must be disjoint. Based on the preceding discussion we can now restate the options as follows, with option (III) split into two suboptions.

(I) there is a subset $\mathfrak{G}_{1} \subset \mathfrak{G}$ with $|\mathfrak{G}_{1}| \geq g_{n}^{1/3}$ for which $\{\Gamma^{\text{sec}} : \Gamma \in \mathfrak{G}_{1}\}$ are disjoint,

(II) there is a subset $\mathfrak{G}_{2} \subset \mathfrak{G}$ with $|\mathfrak{G}_{2}| \geq g_{n}^{1/3}$ for which $\{\Gamma^{\text{pri}} : \Gamma \in \mathfrak{G}_{1}\}$ are disjoint,

(IIIa) there is a subset $\mathfrak{G}_{3a} \subset \mathfrak{G}$ with $|\mathfrak{G}_{3a}| \geq n^{\beta_{3}-\beta_{4}}$ for which $\{\Gamma^{\text{sec}} : \Gamma \in \mathfrak{G}_{3a}\}$ are disjoint,

(IIIb) there is a subset $\mathfrak{G}_{3b} \subset \mathfrak{G}$ with $|\mathfrak{G}_{3b}| \geq n^{1/3}$ for which no popular site exists for $\mathfrak{G}_{3b}$.

In bounding the probabilities for options (I)–(IIIb) we only use the geodesics $\Gamma^{\text{pri}}$ or $\Gamma^{\text{sec}}$. This means the original angle $\theta_{0}$ is no longer involved, except that we have effectively replaced $R$ with $r - s$ (for geodesics $\Gamma^{\text{pri}}$) or $r^{*} - s^{*}$ (for geodesics $\Gamma^{\text{sec}}$). In view of (4.28) this has negligible effect. In the interest of expositional and notational clarity, we can therefore henceforth assume $\theta = \theta_{0}$ and $[r, s] = [0, R]$ for options (II) and (IIIb), where we deal with geodesics $\Gamma^{\text{pri}}$.

The most difficult of the options to control is (IIIb) where we must deal with the lack of disjointness; in fact our proof of a probability bound for that case will essentially subsume the simpler proofs for the other 3 cases. Hence we consider the events

$$G_{\text{bad}} = G_{3} \cup G_{1} \cup G_{4} \cup G_{6} \cup G_{7} \cup G_{2} \cup G_{10}$$

$$G_{11} : \tau \in G_{9} \setminus G_{\text{bad}}$$

and we call $\mathfrak{G}_{3b}$ a crowded subset.
Recall that $S_{\theta,i}(\Gamma)$ denotes the $i$th $\ell$-segment of $\Gamma$. We arbitrarily number the target $\theta$–blocks $1$ through $(2\sqrt{d-1})^{d-1} n^{(\beta_3+\beta_4)(d-1)}$ in each $H_{\theta,i}$. For $\Gamma \in \mathcal{G}_{3b}$ we say $S_{\theta,i}(\Gamma)$ (or just $\Gamma$) makes a \textit{transition from $j$ to $k$} if $x''_{(i-1)\ell}(\Gamma)$ is in fattened target $\theta$–block $j$ in $H^\text{fat}_{(i-1)\ell}$ and $x''_{i\ell}(\Gamma)$ is in fattened target $\theta_0$–block $k$ in $H^\text{fat}_{i\ell}$; we call this transition the $(i,j,k)$ \textit{transition} and write $S_{\theta,i}(\Gamma) \in T_i(j,k)$. For $\Gamma^{(1)}, \Gamma^{(2)} \in \mathcal{G}_{3b}$ and $i \leq n^{\beta_2}$, $S_{\theta,i}(\Gamma^{(1)})$ and $S_{\theta,i}(\Gamma^{(2)})$ are called \textit{neighboring} if they make the same transition. A given transition $(i,j,k)$ is called \textit{sparse} if the number of $2R$–geodesics in $\mathcal{G}_{3b}$ making that transition is at most $n^{\beta_3}$.

The definitions of “transition” and “neighboring,” among others here, also make sense for general $(\ell,\theta)$-interval geodesics that are not part of a $2R$–geodesic.

When the $i$th $\ell$-segments of some $\Gamma, \hat{\Gamma} \in \mathcal{G}_{3b}$ intersect, $S_{\theta,i}(\Gamma) \cap S_{\theta,i}(\hat{\Gamma})$ is necessarily a subsegment of both $S_{\theta,i}(\Gamma)$ and $S_{\theta,i}(\hat{\Gamma})$, with some endpoints $v = (v_1^\theta, v_2^\theta)$ and $w = (w_1^\theta, w_2^\theta)$, labeled so $v$ precedes $w$ in $\Gamma$. We call $\Gamma[v,w] = \hat{\Gamma}[v,w]$ the overlap segment. The projection of the overlap segment onto the first $\theta$–coordinate is an interval in $\mathbb{R}$ containing $v_1^\theta, w_1^\theta$ which we call the \textit{projected overlap interval}; we denote it $\mathcal{O}(S_{\theta,i}(\Gamma), S_{\theta,i}(\hat{\Gamma}))$. The \textit{$\theta$–overlap} of $S_{\theta,i}(\Gamma)$ and $S_{\theta,i}(\hat{\Gamma})$ is $|\mathcal{O}(S_{\theta,i}(\Gamma), S_{\theta,i}(\hat{\Gamma}))|$. If two neighboring $i$th $\ell$-segments have $\theta$–overlap at most $n^{-\beta_4}\ell$, we say they are \textit{low-overlap neighbors}; see Figure 11. We claim the following.

\textbf{Claim 1.} If $\tau \in G_{11}$ with crowded subset $\mathcal{G}_{3b}$, then every $\ell$-segment of every $\Gamma \in \mathcal{G}_{3b}$ making a non-sparse transition has a low-overlap neighbor.

To prove Claim 1, fix $\tau \in G_{11}$. For fixed $i,j,k$ and $\Gamma \in \mathcal{G}$ with $S_{\theta,i}(\Gamma)$ making a non-sparse $(i,j,k)$ transition, suppose $S_{\theta,i}(\Gamma)$ has no low-overlap neighbors. Then

\begin{equation}
(4.90) \sum_{\hat{\Gamma} \in T_i(j,k), \hat{\Gamma} \neq \Gamma} |\mathcal{O}(S_{\theta,i}(\hat{\Gamma}), S_{\theta,i}(\Gamma))| \geq (|T_i(j,k)| - 1)n^{-\beta_4}\ell \geq \frac{1}{2} n^{-\beta_3-\beta_4}\ell.
\end{equation}

We must deal with the technical complication that there may be backtracks, meaning that

(i) not all projected overlap intervals, for $S_{\theta,i}(\Gamma)$ and some $S_{\theta,i}(\hat{\Gamma})$, are necessarily contained in $[(i-1)\ell, i\ell]$, and

(ii) having two projected overlap intervals intersect in an interval of positive length need not mean that the corresponding overlap segments have nonempty intersection.

Issue (i) is readily dealt with: since we are assuming $\tau \in G_3$, every $\mathcal{O}(S_{\theta,i}(\hat{\Gamma}), S_{\theta,i}(\Gamma))$ in \textbf{(4.90)} is contained in $[(i-2)\ell, i\ell]$. It then follows from \textbf{(4.90)} that some point $a \in [(i-2)\ell, i\ell]$ must be in at least $\frac{1}{4} n^{-\beta_3-\beta_4}$ of these intervals. For issue (ii), let $F_a(\Gamma)$ and $F'_a(\Gamma)$ be the first and last points, respectively, of $\Gamma$ in $H_{\theta,a}$. Because we are assuming $\tau \in G_3$ and $\Gamma$ has no low-overlap neighbors, if $a$ lies in some $\mathcal{O}(S_{\theta,i}(\hat{\Gamma}), S_{\theta,i}(\Gamma))$, then the corresponding overlap segment $S_{\theta,i}(\hat{\Gamma}) \cap S_{\theta,i}(\Gamma)$ must contain either $F_a(\Gamma)$ or $F'_a(\Gamma)$, because no segment of $\Gamma$ lying entirely between $F_a(\Gamma)$ and $F'_a(\Gamma)$ can have projected length more than $n^{-\beta_4}\ell/2$. It follows that either $F_a(\Gamma)$ is contained in $S_{\theta,i}(\hat{\Gamma}) \cap S_{\theta,i}(\Gamma)$ for at least $\frac{1}{4} n^{-\beta_3-\beta_4}$ of the neighbors $\hat{\Gamma}$ in \textbf{(4.90)}, or the same is true for $F'_a(\Gamma)$. But this makes $F_a(\Gamma)$ or $F'_a(\Gamma)$ a popular site for $\mathcal{G}_{3b}$, a contradiction to $\tau \in G_{11}$. This proves Claim 1.

\textbf{Claim 2.} If $\tau \in G_{11}$ with crowded subset $\mathcal{G}_{3b}$, then there exists $\Gamma \in \mathcal{G}_{3b}$ such that every $\ell$–segment of $\Gamma$ has a low-overlap neighbor.

To prove Claim 2, note that for each $i$, the number of possible transitions by the $i$th $\ell$-segment of a target-directed $2R$–geodesic is at most the square of the number of target $\theta$–blocks in each $H_{\theta,i\ell}$, so the number of $(i,j,k)$ such that $\ell$–segment $i$ can transition from block $j$ to block $k$ is at
Figure 11. Geodesic $\Gamma$ (dashed curve) for which every $\ell$–segment of the pre-$H_{\theta,R}$ segment has a low-overlap neighbor (gray curves.) The low-overlap neighbors are $\ell$–segments of other geodesics in the crowded subset $\mathcal{E}_{3b}$ of the crowded group $\mathcal{E}$. Later in the proof we allow similar but more general “low-overlap partners,” which are still geodesics but which do not have to be $\ell$–segments of geodesics in the crowded subset. Of primary interest are the fast $\ell$–segments of $\Gamma$, for which the corresponding partners are (modulo a small-probability event) “forced” by Lemma 4.8 to be disjointly semi-fast.

most $n^{\beta_2+2(\beta_0+\beta_1)(d-1)}$. It follows that the total number of sparse transitions made by all $\Gamma \in \mathcal{E}_{3b}$ (summed over $\mathcal{E}_{3b}$) is at most $n^{\beta_3+\beta_2+2(\beta_0+\beta_1)(d-1)}$, which is less than $g_n^{1/3}$ by (4.18). Hence there exists some $\Gamma \in \mathcal{E}_{3b}$ making no sparse transitions, and Claim 2 follows from Claim 1.

Our definition of low-overlap neighbor requires that such a neighbor be an $\ell$–segment of another $2R$–geodesic in our specified $\mathcal{E}_{3b}$. We now loosen this restriction, and say for $\Gamma$ a $2R$–geodesic, any $(\ell, \theta)$–interval geodesic $\Psi$ is a low-overlap partner of the $i$th $\ell$–segment of $\Gamma$ if the following hold:

(a) $\Psi$ is a good $(\ell, \theta)$-interval geodesic,
(b) $\Psi$ and $S_{\theta,i}(\Gamma)$ make the same transition, and
(c) the $\theta$–overlap of $\Psi$ and $S_{\theta,i}(\Gamma)$ is at most $n^{-\beta_4 \ell}$.

It follows that every low-overlap neighbor is a low-overlap partner, for $\tau \in G_{11}$. Define the event

\[ G_{12} : \text{there exist a target-directed } 2R \text{-geodesic from } B^\text{fat}_{\theta,\text{home}} \text{ to } H^+_{\theta,2R} \]

\[ \text{with } H^0_{\theta,R} \text{-entry point in } B^\text{fat}_{\theta,\text{cross}} \text{ for which every } \ell \text{-segment in the pre-} H_{\theta,R} \]

\[ \text{segment has a low-overlap partner; } \tau \notin G_{\text{bad}}. \]

We then conclude from Claim 2 that

\[ G_{11} \subset G_{12} \cup G_{\text{bad}}. \]

We now bound $P(G_{12})$. Define

\[ N_L = |B^\text{fat}_{\theta,\text{home}} \cap \mathbb{Z}^d|, \quad N_R = |B^\text{fat}_{\theta,\text{cross}} \cap \mathbb{Z}^d|, \quad \Omega_i = \Omega_{\theta}((i - 1/6)\ell, (i + 7/6)\ell). \]
Let $\gamma_0$ be a $\theta$–slab path from $B_{\theta,\text{home}}^{\text{fat}}$ to $B_{\theta,\text{cross}}^{\text{fat}}$. Given $x, y \in \Omega_1 \cap V$ we can define the set of low-overlap constrained paths

$$\mathcal{P}_i(x, y, \gamma_0) = \left\{ \gamma : \gamma \text{ is a path from } x \text{ to } y \text{ in } \mathbb{Z}^d \text{ with } \gamma \subset \Omega_i, \text{ and either } \gamma \cap \gamma_0 = \emptyset \text{ or for some } \gamma \right\}$$

(4.92)

sites $u$ preceding $v$ in $\gamma_0$ with $|u^\theta_1 - v^\theta_1| \leq n^{-\beta_1} \ell$ we have $\gamma \cap \gamma_0 = \gamma_0[u, v]$}

and for $\gamma \in \mathcal{P}_i(x, y, \gamma_0)$ with $\gamma \cap \gamma_0 = \gamma_0[u, v]$, define

$$T^{\text{dis}, i}(\gamma, \gamma_0) = T(\gamma) - T(\gamma \cap \gamma_0) + h((v^\theta_1 - u^\theta_1)y_0).$$

Note that part of the passage time $T(\gamma)$ comes from the overlap segment $\gamma[u, v]$; in defining $T^{\text{dis}, i}(\gamma, \gamma_0)$ we replace this part of the passage time with the approximation $h((v^\theta_1 - u^\theta_1)y_0)$, so that it does not depend on the passage times of bonds in the overlap segment. Next define the disjoint passage times

(4.93)

$$T^{\text{dis}, i}(x, y, \gamma_0) := \inf \{ T^{\text{dis}, i}(\gamma, \gamma_0) : \gamma \in \mathcal{P}_i(x, y, \gamma_0) \},$$

so that $T^{\text{dis}, i}(x, y, \gamma_0)$ is not affected by the passage times of the bonds in $\gamma_0$.

The case of interest is the following: given $\tau \notin G_{\text{bad}}$, if $\Gamma$ is a $\theta$–slab geodesic from $B_{\theta,\text{home}}^{\text{fat}}$ to $B_{\theta,\text{cross}}^{\text{fat}}$, and $\Psi = \Psi[x, y]$ is a low-overlap partner of $S_{\theta,i}(\Gamma)$, then, due to the bound on backtracks in low-overlap partners we have $\Psi \subset \Omega_i$, so $\Psi \in \mathcal{P}_i(x, y, \Gamma)$. Suppose $\gamma \in \mathcal{P}_i(x, y, \Gamma)$ with $\gamma \cap \Gamma = \Gamma[u, v]$ for some $u, v$. Since $v$ lies in the hyperplane $H_{\theta,v_i}^{\theta}$ tangent to $u + |(v - u)_1^1|B_{\theta}$ at $u + (v - u)_1^1y_0$, we have using Lemma 2.1, (1.2), and (4.20) (taking $C_{54}$ there sufficiently large) that

$$h(v - u) \geq g(v - u) \geq g((v - u)_1^1y_0) \geq h((v - u)_1^1y_0) - C_{25}\sigma((v - u)_1^1y_0)|\log ((v - u)_1^1y_0)| \geq h((v - u)_1^1y_0) - c_8n^{-\gamma_1\beta_4}\sigma(\ell|y_0|/2) \log \ell \geq h((v - u)_1^1y_0) - \frac{\eta}{8}\sigma(\ell|y_0|/2).$$

Since $\tau \notin G_{\tau}$, this yields that for all $\gamma \in \mathcal{P}_i(x, y, \Gamma)$

(4.94)

$$T(\gamma) - T^{\text{dis}, i}(\gamma, \Gamma) = T(u, v) - h((v^\theta_1 - v_1^\theta)y_0) \geq T(u, v) - h(v - u) - \frac{\eta}{8}\sigma(\ell|y_0|/2) \geq -\frac{\eta}{4}\sigma(\ell|y_0|/2).$$

Since also $\tau \notin G_0$,

(4.95)

$$|T(\Psi) - T(S_{\theta,i}(\Gamma))| < \frac{\eta}{8}\sigma(\ell).$$

For $\Gamma$ a $\theta$–slab geodesic from $B_{\theta,\text{home}}^{\text{fat}}$ to $B_{\theta,\text{cross}}^{\text{fat}}$, we say $\Gamma$ is clean if $\Gamma$ contains no $\theta$–backtrack of $\frac{1}{2}n^{-\beta_4}\ell$, and no pair $u, v$ as in the event $G_{\text{bad}}$; a $2R$–geodesic is called clean if its pre-$H_{\theta,R}$ segment is clean. For $2 \leq i \leq n^{\beta_2} - 1$, as in Lemma 4.2 let us call $S_{\theta,i}(\Gamma)$ a fast segment if

$$T(S_{\theta,i}(\Gamma)) \leq \frac{1}{k}ET(0, R_{\theta}) + \frac{\eta}{8}\sigma(\ell|y_0|/2),$$

so that $N_{\theta}(\Gamma)$ (from Lemma 4.2) is the number of fast $\ell$–segments in $\Gamma$. We say an $\ell, \theta$–interval geodesic $\Psi = \Psi[x, y]$ is semi-fast if its passage time satisfies

$$T(\Psi) < \frac{1}{k}ET(0, R_{\theta}) + \frac{3}{4}\eta\sigma(\ell|y_0|/2).$$
and disjointly semi-fast if

\[ T^{\text{dis},i}(\Psi, \Gamma) < \frac{1}{k} E T(0, R y_\theta) + \frac{\eta}{2} |\ell| y_\theta / 2. \]

Thus for \( \tau \in G_{12} \) we have the following: there exists a clean \( \theta \)-target-directed geodesic \( \Gamma \) from \( B_{\theta,\text{home}} \) to \( B_{\theta,\text{cross}} \) for which every \( S_{\theta,i}(\Gamma) \) has a low-overlap partner \( \Psi_i = \Psi_i[y_\theta, z_\theta] \), and for each fast \( \ell \)-segment \( S_{\theta,i}(\Gamma) \),

\[ T^{\text{dis},i}(\Psi_i, \Gamma) \leq T(\Psi_i) + \frac{\eta}{4} |\ell| y_\theta / 2 \leq T(S_{\theta,i}(\Gamma)) + \frac{3\eta}{8} |\ell| y_\theta / 2 \leq \frac{1}{k} E T(0, R y_\theta) + \frac{\eta}{2} |\ell| y_\theta / 2, \]

meaning \( \Psi_i \) is disjointly semi-fast. Here the first two inequalities use (4.94) and (4.95). Thus for the event

\[ G_{13} : \text{there exists a clean \( \theta \)-target-directed \( \theta \)-slab geodesic \( \Gamma \) from \( B_{\theta,\text{home}} \) to \( B_{\theta,\text{cross}} \) for which every fast \( \ell \)-segment \( S_{\theta,i}(\Gamma) \) has a disjointly semi-fast low-overlap partner,} \]

we have

\[ G_{12} \subset G_{13}, \]

so we now want to bound \( P(G_{13}) \). Let \( a_1, \ldots, a_{N_L} \) and \( b_1, \ldots, b_{N_R} \) be the sites of \( \mathbb{Z}^d \) in \( J_L \) and \( J_R \) respectively, and let

\[ n_0 = \eta \gamma^2 \beta_2 / 8. \]

Defining events

\[ H_{jk} : \Gamma_{a_jb_k} \] is a clean \( \theta \)-target-directed \( \theta \)-slab geodesic, \( N_\theta(\Gamma_{a_jb_k}) \geq n_0 \), and every fast \( \ell \)-segment \( S_{\theta,i}(\Gamma_{a_jb_k}) \) with \( 2 \leq i \leq n_0 - 1 \), has a disjointly semi-fast low-overlap partner, we have

\[ P(G_{13}) \leq \sum_{j=1}^{N_L} \sum_{k=1}^{N_R} \left( P(N_\theta(\Gamma_{a_jb_k}) < n_0) + P(H_{jk}) \right). \]

(We note here that when \( \tau \in H_{jk} \), the corresponding \( \Gamma_{a_jb_k} \) can serve as the special geodesic in the heuristic in Remark 4.1) Using the last inequality in (4.16), Lemma 4.2 yields that

\[ P \left( N_\theta(\Gamma_{a_jb_k}) < n_0 \right) \leq c_6 R^{2(d-1)} \exp \left( -c_7 n^{1-(1-\gamma)\beta_2} \right) \leq c_8 \exp \left( -c_9 n^{2\beta_1} \right). \]

We next bound \( P(H_{jk}) \). Suppose we fix both of the following:

\[ \text{a clean \( \theta \)-target-directed \( \theta \)-slab path \( \gamma \) from \( a_j \) to \( b_k \), and the times \( \tau_\gamma = \{ \tau_e : e \in \gamma \} \).} \]

These determine the set

\[ \mathcal{I}(\gamma, \tau_\gamma) = \{ 2 \leq i \leq n_0^\beta_2 - 1 : \) \( S_{\theta,i}(\gamma) \) is a fast \( \ell \)-segment} \}

For each such \( \gamma \) we have the event

\[ H_\gamma : \text{every fast \( \ell \)-segment \( S_{\theta,i}(\gamma) \), } 2 \leq i \leq n_0^\beta_2 - 1 \), has a disjointly semi-fast low-overlap partner. Conditionally on the fixed objects in (4.99), we may view the events \( H_\gamma \), as well as the event \( \{ \Gamma_{a_j b_k} = \gamma \} \), as determined by the unconditioned passage times \( \{ \tau_e : e \notin \gamma \} \). In this context,
$\{\Gamma_{u_j v_k} = \gamma\}$ is an increasing event (that is, its indicator is an increasing function of $\{\tau_e : e \notin \gamma\}$), whereas $H_\gamma$ is a decreasing event. It follows from the FKG inequality that

$$P(H_{jk} \mid \tau_\gamma, \Gamma_{u_j v_k} = \gamma) = P(H_\gamma \mid \tau_\gamma, \Gamma_{u_j v_k} = \gamma) \leq P(H_\gamma \mid \tau_\gamma) = P(H_\gamma \mid \mathcal{I}(\gamma, \tau_\gamma)).$$

(4.100)

For $I \subset \{1, \ldots, n^{\beta_2} - 2\}$, the events

$$H_{\gamma, I} : \text{ for all } i \in I, S_{\theta, i}(\gamma) \text{ has a disjointly semi-fast low-overlap partner}$$

satisfy

(4.101) $$P(H_\gamma \mid \mathcal{I}(\gamma, \tau_\gamma) = I) = P(H_{\gamma, I} \mid \mathcal{I}(\gamma, \tau_\gamma) = I) = P(H_{\gamma, I}),$$

since the event $H_{\gamma, I}$ is independent of $\{\tau_e, e \in \gamma\}$. For a clean $\theta$–target-directed $\theta$–slab path $\gamma$ from $B_{\theta, \text{home}}$ to $B_{\theta, \text{cross}}$, let

$$\mathcal{M}(\gamma) = \{(i, j, k) : \gamma \text{ makes transition } (i, j, k)\}.$$ and let $\mathfrak{M}$ be the set of all possible values of $\mathcal{M}(\gamma)$ as $\gamma$ varies over all such paths. For $M \in \mathfrak{M}$ and $I \subset \{2, \ldots, n^{\beta_2} - 1\}$ define events

$$F_{M, I} : \text{ for every } (i, j, k) \in M \text{ with } i \in I, \text{ there exists a semi-fast good } (\ell, \theta)\text{–interval geodesic } \Psi \text{ making transition } (i, j, k).$$

We claim that

(4.102) $$H_{\gamma, I} \setminus (G_3 \cup G_6 \cup G_7) \subset F_{\mathcal{M}(\gamma), I} \setminus (G_3 \cup G_6).$$

In fact, suppose $\tau \in H_{\gamma, I} \setminus (G_3 \cup G_6 \cup G_7), i \in I$, and $(i, j, k) \in \mathcal{M}(\gamma)$. Then there are sites $u$ preceding $v$ in $S_{\theta, i}(\gamma)$ and a disjointly semi-fast $(\ell, \theta)$–interval geodesic $\Psi \subset Q_{R, n, \theta}$ for which $\Psi$ makes transition $(i, j, k), \quad \Psi \cap \gamma = \gamma[u, v], \quad |u_1^\theta - v_1^\theta| \leq n^{-\beta_2} \ell.

Since $\tau \notin G_7$ and $\gamma$ is clean,

$$T(\Psi) = T^{\text{dis}, i}(\Psi, \gamma) + T(\Psi[u, v]) - h(u_1^\theta - v_1^\theta y_\theta) \leq T^{\text{dis}, i}(\Psi, \gamma) + T(\Psi[u, v]) - h(u - v) + h(u - v) - h(u_1^\theta - v_1^\theta y_\theta) \leq \frac{1}{k} ET(0, R y_\theta) + \frac{3}{4} \eta \sigma(\ell y_\theta)/2.$$ (4.103)

Thus $\Psi$ is semi-fast. Since $\tau \notin G_3, \Psi$ is also good, so $\tau \in F_{\mathcal{M}(\gamma), I}$, proving the claim. This shows that

(4.104) $$P(H_{\gamma, I}) \leq P(F_{\mathcal{M}(\gamma), I} \setminus (G_3 \cup G_6)) + P(G_3 \cup G_6 \cup G_7).$$

Let $\overline{u}_{i j}$ denote the center point of the $j$th block in $H_{\ell \ell}$, and define the event

$$\mathcal{F}_{M, I} : \text{ for every } (i, j, k) \in M \text{ with } i \in I, \text{ we have } T(\overline{u}_{(i-1) j}, \overline{u}_{i k}) \leq \frac{1}{k} ET(0, R y_\theta) + \frac{7}{8} \eta \sigma(\ell y_\theta)/2.$$ Then $F_{\mathcal{M}(\gamma), I} \setminus G_6 \subset \mathcal{F}_{\mathcal{M}(\gamma), I}$ so by (4.104),

$$P(H_{\gamma, I}) \leq P(\mathcal{F}_{\mathcal{M}(\gamma), I} \setminus (G_3 \cup G_6)) + P(G_3 \cup G_6 \cup G_7).$$
With (4.100) and (4.101) we obtain from this that for the functions
\[ f_0(\gamma, I) = P(H_{\gamma, I}), \quad f_1(M, I) = P(M_{\gamma, I}(G_3 \cup G_0)) + P(G_3 \cup G_0 \cup G_7), \]
we have
\[
P(H_{jk}) = E\left( P\left( H_{jk} \mid \{ \tau_e : e \in \Gamma_{u_jv_k}, \Gamma_{u_jv_k} \} \right) \right)
\leq E\left( f_0(\Gamma_{u_jv_k}, \{ \tau_e : e \in \Gamma_{a_jb_k} \}) 1_{\{ |\Gamma_{u_jv_k}, \{ \tau_e : e \in \Gamma_{a_jb_k} \}| \geq n_0 - 2 \}} \right)
\leq E\left( f_1(\{ \Gamma_{u_jv_k}, \{ \tau_e : e \in \Gamma_{a_jb_k} \}) 1_{\{ |\Gamma_{u_jv_k}, \{ \tau_e : e \in \Gamma_{a_jb_k} \}| \geq n_0 - 2 \}} \right).
\]
(4.105)

As a comment on the last two expressions, we can view \( \gamma, I \) as parameters in the probability \( P(H_{\gamma, I}) \) expressed by the function \( f_0 \), with this probability for a given \( (\gamma, I) \) calculated for a random configuration \( \tau \). When we calculate the second expectation in (4.105), we can view it as choosing the parameters \( \gamma, I \) randomly using a completely separate independent passage time configuration \( \tau' \). Our ability to separate the choice of \( \tau \) from the choice of parameters (functions of \( \tau' \)) is a consequence of the FKG property in (4.100) and of (4.101). When we next replace \( f_0 \) with \( f_1 \) in the last line, the parametrization no longer uses the full path \( \gamma \) but rather only the transitions of \( \gamma \). This formulation means that to bound \( P(H_{jk}) \), instead of summing over \( M \) and \( I \) in \( f_1(M, I) \) (which involves too much entropy), we are averaging over \( \Gamma_{u_jv_k} \).

We split \( I \) into odd and even values, \( I = I_{\text{odd}} \cup I_{\text{even}} \) and define for \( u, v \in \Omega_i \):
\[ T_{\Omega_i}(u, v) = \inf\{ T(\gamma) : \gamma \text{ is a path from } u \text{ to } v, \gamma \subset \Omega_i \}. \]

Fix \( M \); for each \( i \leq n^{2} \) there exist unique \( j_i, k_i \) for which \( (i, j_i, k_i) \in M \). The events
\[ F_{M,i}^*: T_{\Omega_i}(\pi(i-1), j_i, \pi(k_i)) \leq \frac{1}{k} ET(0, R\theta y) + \frac{7}{8} \eta \sigma(\ell |y|/2) \]
with \( i \) odd are independent, since the regions \( \Omega_i \) are disjoint. Therefore
\[
P(F_{M,i}) \leq P\left( \bigcap_{i \in I_{\text{odd}}} F_{M,i}^* \right)
= \prod_{i \in I_{\text{odd}}} P(F_{M,i}^*)
\leq \prod_{i \in I_{\text{odd}}} P\left( T(\pi(i-1), j_i, \pi(k_i)) \leq \frac{1}{k} ET(0, R\theta y) + \frac{7}{8} \eta \sigma(\ell |y|/2) \right).
\]
(4.106)

With (1.2) and Proposition 2.1 using \( \pi_{i,k_i} - \pi_{i-1,j_i} \in H_{\theta, I} \), we obtain
\[
\frac{1}{k} ET(0, R\theta y) \leq g(\ell y) + \frac{C_{25}}{k} \sigma(\ell |y|) \log(R |y|)
\leq g(\pi_{i,k_i} - \pi_{i-1,j_i}) + \frac{1}{8} \eta \sigma(\ell |y|/2)
\leq h(\pi_{i,k_i} - \pi_{i-1,j_i}) + \frac{1}{8} \eta \sigma(\ell |y|/2).
\]
(4.107)

Since \( \pi_{i,k_i}, \pi_{i-1,j_i} \) lie in \( Q_{R,n,y} \), it follows readily from (4.24) that
\[ |\pi_{i,k_i} - \pi_{i-1,j_i}| \geq |(\pi_{i,k_i} - \pi_{i-1,j_i})| \eta |y| - |(\pi_{i,k_i} - \pi_{i-1,j_i})| \eta |y| \geq \frac{\ell |y|}{2}. \]
Combining this with (1.9) and (4.107) yields
\[
P \left( T(\vec{u}_{(i-1),j_i}, \vec{v}_{i,k_i}) \leq \frac{1}{k} ET(0, R \vec{y}_0) + \frac{7}{8} \eta \sigma (|\vec{y}_0|/2) \right)
\leq P \left( T(\vec{u}_{(i-1),j_i}, \vec{v}_{i,k_i}) < h(\vec{v}_{i,k_i} - \vec{u}_{(i-1),j_i}) + \eta \sigma (|\vec{v}_{i,k_i} - \vec{u}_{(i-1),j_i}|) \right)
\leq 1 - C_{10}.
\]

(4.108)

Since (4.106) also holds for $I_{\text{even}}$, and $\max(I_{\text{odd}},I_{\text{even}}) \geq |I|/2$, this together with the last inequality in (4.16) shows that for $|I| \geq n_0 - 2$,
\[
P(\mathcal{F}_{M,I} \setminus G_3) \leq (1 - C_{10}) |I|/2 \leq (1 - C_{10})^{(n_0 - 2)/2} \leq \exp \left( -C_{10} \eta n^{2 \beta_2}/32 \right) \leq \exp \left( -C_{10} \eta n^{2 \beta_1} \right).
\]

Combining this with (4.105) and Lemmas 4.7, 4.8 and 4.9 we see that for $|I| \geq n_0 - 2$ and all $M$,
\[
f_1(M, I) \leq \exp \left( -C_{10} \eta n^{2 \beta_1} \right) + \exp \left( -C_{60} \frac{n^{2 \beta_1}}{\log R} \right).
\]

With (4.91), (4.96), (4.97), (4.98), and (4.105) this shows
\[
P(\tau \in G_9 \setminus G_{\text{bad}} \text{ and option (IIIb) occurs})
= P(G_{11})
\leq P(G_{13}) + P(G_{\text{bad}})
\leq N_L N_R \left( c_8 \exp \left( -c_9 n^{2 \beta_1} \right) + \exp \left( -C_{10} \eta n^{2 \beta_1} \right) + \exp \left( -C_{60} \frac{n^{2 \beta_1}}{\log R} \right) \right) + P(G_{\text{bad}})
\leq c_{10} R^{2(d-1)} \exp \left( -C_{60} \frac{n^{2 \beta_1}}{\log R} \right) + P(G_{\text{bad}})
\leq \exp \left( -c_{11} \frac{n^{2 \beta_1}}{\log R} \right) + P(G_{\text{bad}}),
\]

(4.109)

where we have used $n^{\beta_1} \geq (\log R)^2$.

The preceding proof of (4.109) applies also to options (I) and (II); the proof could even be substantially simplified for these cases since there is no overlap in the relevant geodesics. To deal with option (IIIia) where the number of geodesics in the crowded subset is smaller, we can simply define $m$ by $n^{\beta_3-\beta_1} = g_m^{1/3}$, so $m = c_{12} n^\lambda$ with $\lambda < 1$, and then repeat the entire proof with $n$ replaced by $m$. (This has the effect of replacing each $\beta_j$ with $\lambda \beta_j$.) We thus conclude using (4.87) and Lemmas 4.5–4.9 that
\[
P(G_8) \leq P(G_9 \setminus G_{\text{bad}}) + P(G_{\text{bad}}) \leq 4 \exp \left( -c_{11} \frac{n^{2 \beta_1}}{\log R} \right) + 5 P(G_{\text{bad}}) \leq \exp \left( -c_{13} \frac{n^{2 \beta_1}}{\log R} \right),
\]

which completes the proof of Proposition 4.10. \hfill $\square$

5. Entry Point Density Bound

In this section we prove Theorem 1.5(ii) and Theorem 1.8.

Proof of Theorem 1.5(ii). The orientation of a finite geodesic from $x$ to $y$ is $(y-x)/|y-x|$. With $\theta_0, R$ fixed, for $\Gamma$ a $\theta_0$–slab geodesic from $H_{\theta_0, -R}$ to $H_{\theta_0, R}$, the initial orientation of $\Gamma$ is the orientation of its pre-$H_{\theta_0, 0}$ segment. The term “initial orientation” also applies to $\theta$–rays from
$H_{\theta_0,-R}$ which cross $H_{\theta_0,0}$. Let $\epsilon_1$ be as in Remark 1.2 and $\epsilon_{\min}$ as in A3, let $\epsilon(R) = (\log R)^{K_1} \Delta_R / R$, with $K_1$ to be specified, and for $\theta \in J(\theta_0, \epsilon_{\min})$ define

$$W_R := W_R(\theta_0, \theta, \tau) := \{ x \in H^s_{\theta_0,0} \cap \mathbb{Z}^d : \text{there exists a } \theta_0 \text{-slab geodesic from } H^-_{\theta_0,-R} \text{ to } H^+_{\theta_0,R} \text{ with } H^+_{\theta_0,0} \text{-entry point } x \text{ and initial orientation in } J(\theta, \epsilon(R)) \},$$

$$Y_R := Y_R(\theta_0, \theta, \tau) := \{ x \in H^s_{\theta_0,0} \cap \mathbb{Z}^d : \text{for some } \theta \in J(\theta_0, \epsilon_{\min}) \text{ there exists a } \theta \text{-ray from } H^-_{\theta_0,-R} \text{ with } H^+_{\theta_0,0} \text{-entry point } x \text{ and initial orientation not in } J(\theta, \epsilon(R)) \},$$

$$Z_R := Z_R(\theta_0, \theta, \tau) := \{ x \in H^s_{\theta_0,0} \cap \mathbb{Z}^d : \text{there exists a } \theta \text{-ray from } H^-_{\theta_0,-R} \text{ with } H^+_{\theta_0,0} \text{-entry point } x \},$$

so

$$Z_R \subset W_R \cup Y_R.$$  \hfill (5.1)

We may think of $W_R$ and $Y_R$ as the sets of entry points of “normal” and “crooked” $\theta$-rays, respectively, though formally $W_R$ is defined in terms of finite geodesics.

Let us first bound the density of $Y_R$. Suppose $x \in Y_R, \theta \in J(\theta_0, \epsilon_{\min})$, and there exists a $\theta$-ray from $H^-_{\theta_0,-R}$ with starting point $a \in H^s_{\theta_0,-R}$ for which the initial orientation $\alpha = (x-a)/|x-a| \notin J(\theta, \epsilon(\epsilon(R))).$ This means $\psi_{\alpha \theta} \geq \epsilon(R)$, and it is easily checked that this implies

$$D_\theta (x-a) \geq c_1 \left( \frac{\epsilon(R)^2 | x-a |^2}{2 (|x-a|)^2} \wedge \Phi(|x-a|) \right) \geq c_2 \epsilon(R)^2 \frac{|x-a|}{\sigma(|x-a|) \log |x-a|},$$

But then, defining events

$$F_{x,j} : \text{for some } a \in \mathbb{Z}^d \text{ with } 2^{j-1} < |x-a| \leq 2^j \text{ and some } \theta \in J(\theta_0, \epsilon_{\min}) \text{ with }$$

$$D_\theta (x-a) \geq c_2 \epsilon(R)^2 \frac{|x-a|}{\sigma(|x-a|) \log |x-a|},$$

there is a $\theta$-ray from $a$ containing $x$, we get from Proposition 3.1 that provided $K_1$ is large,

$$P(x \in Y_R) \leq \sum_{j \geq \log_2 R} \sum_{j \geq \log_2 R} c_3 2^{2j} e^{-c_4 \epsilon(\epsilon(R)^2 j / \sigma(2^j))} \leq c_5 e^{-c_6 \epsilon(R)^2 K_1^{-1}}.$$  \hfill (5.2)

We now turn to our main task, which is bounding the density of $W_R$; see Figure 12. We keep $\theta_0, \theta$ fixed throughout, with $\psi_{\theta \theta_0} < \epsilon_{\min}$. Let $R > 0$ and $n = (\log R)^{K_2}$, with $K_2$ to be specified, satisfying \hfill (4.20). Define

$$\bar{\mathcal{G}}_{R,\theta} = L_\theta \cap H_{\theta_0,-R},$$

let $B_{\theta_0,\text{home}} \subset H_{\theta_0,0}$ be the home $\theta_0$–block as in Section 4 and define the “big block” centered at $\bar{\mathcal{G}}_{R,\theta}$:

$$B_{-R,\theta_0,\theta,\text{start, big}} = \{ -R \} \times \left( \bar{\mathcal{G}}_{R,\theta} + \left[ -8 \sqrt{d} |y_{\theta_0}| \epsilon(R) R, 8 \sqrt{d} |y_{\theta_0}| \epsilon(R) R \right]^{d-1} \right)$$

expressed in $\theta_0$-coordinates. As after Remark 4.3, we can define fattened versions $B^\text{fat}_{\theta_0,\text{home}}$ and $B^\text{fat}_{-R,\theta_0,\theta,\text{start, big}}$. We want to show that the $\theta_0$-slab geodesic in the definition of $W_R$ must have starting point in $B^{\text{fat}}_{-R,\theta_0,\theta,\text{start, big}}$. To that end we take $a \in H_{\theta_0,-R}^\text{fat} \setminus B^\text{fat}_{-R,\theta_0,\theta,\text{start, big}}$ and $x \in B^\text{fat}_{\theta_0,\text{home}}$
and let \( \alpha = (x - a)/|x - a| \). We will show that \( \psi_{\alpha \theta} > \epsilon(R) \), so that indeed \( a \) is not the starting point in question.

Let \( \hat{a}, \hat{x} \) be the \( \theta_0 \)-projections of \( a \) into \( H_{\theta_0, -R} \) and \( H_{\theta_0, 0} \), respectively, and let \( \hat{\alpha} = (\hat{x} - \hat{a})/|\hat{x} - \hat{a}| \), so \( \psi_{\alpha \hat{\alpha}} \leq c_7/R \). Define intersection points

\[
p = L_{\theta}(\hat{x}) \cap H_{\theta_0, -R}, \quad q = L_{\theta}(\hat{x}) \cap H_{\theta_0, \hat{\alpha}^0} \quad \text{so} \quad p - \overline{Y}_{R, \theta} = \hat{x},
\]

and note that both the hyperplanes here contain \( \hat{a} \). By (2.11) we have \(|p - q| \leq C_{31} \psi_{\theta_0 \theta_0} |\hat{a} - p| \leq C_{31} \psi_{\theta_0 \theta_0} (|\hat{a} - \overline{Y}_{R, \theta}| + |\overline{Y}_{R, \theta} - p|) \). Hence the \( \theta \)-ratio of \( \hat{x} - \hat{a} \) satisfies

\[
\frac{|(\hat{x} - \hat{a})_2|}{|y_0||(\hat{x} - \hat{a})_1|^\theta} = \frac{|q - \hat{a}|}{|x - q|} \geq \frac{|\hat{a} - \overline{Y}_{R, \theta}| - |\overline{Y}_{R, \theta} - p| - |p - q|}{|\hat{x} - p| + |p - q|} \geq \left( \frac{1 - C_{31} \psi_{\theta_0 \theta_0}}{|\overline{Y}_{R, \theta}| + C_{31} \psi_{\theta_0 \theta_0} (|\hat{a} - \overline{Y}_{R, \theta}| + |\hat{x}|)} \right) \tag{5.3}
\]

Since \( a \notin B^\text{flat}_{-R, \theta_0, \theta, \text{start, big}} \), we have

\[
|\hat{a} - \overline{Y}_{R, \theta}| \geq 8 \sqrt{d}|y_{\theta_0}| \epsilon(R) R, \quad |\hat{x}| \leq \sqrt{d - 1} n^{-\theta_0} \Delta_R
\]

and from (2.12)

\[
|\overline{Y}_{R, \theta}| \leq | - Ry_{\theta_0} | + | Y_{R, \theta} - (-Ry_{\theta_0}) | \leq R|y_{\theta_0}|(1 + C_{31} \psi_{\theta_0 \theta_0}).
\]

With (5.3) these yield

\[
\frac{|(\hat{x} - \hat{a})_2|}{|y_0||(\hat{x} - \hat{a})_1|^\theta} \geq 7 \sqrt{d \epsilon(R)}.
\]

so from (2.18) (after reducing \( \epsilon_{\min} \) if necessary),

\[
\tan \psi_{\hat{\alpha} \hat{\theta}} \geq \frac{7}{2} \epsilon(R) \quad \text{and hence} \quad \psi_{\alpha \theta} > \psi_{\hat{\alpha} \hat{\theta}} - \psi_{\alpha \hat{\alpha}} > \epsilon(R),
\]
as desired, showing the starting point corresponding to \( x \in B_{\theta_0, \text{home}}^{\text{fat}} \) cannot be a \( \not\in B_{R, \theta_0, \theta, \text{start, big}}^{\text{fat}} \).

In the other direction, if \( y^* \) is the center of one of the blocks \( B_{-R, m} \) intersecting \( B_{-R, \theta, \theta, \text{start, big}}^{\text{fat}} \), then from (2.17), the \( \theta_0 \)-ratio of \( y^* \) satisfies

\[
(5.4) \quad \frac{|(y^*)_{\theta_0}|}{(y^*)_{\theta_0}^d} \leq \frac{8d|y_{\theta_0}|\epsilon(R)R + \sqrt{d - 1}n^{-\beta_0} \Delta_R}{R} \leq 9d|y_{\theta_0}|\epsilon(R) \text{ so } \psi_{\theta_0} y^* \leq 36\epsilon(R) < \frac{\epsilon_{\min}}{2},
\]

which will enable us later to apply Proposition 4.10.

Now the number (which we call \( M_R \)) of \( \theta_0 \)-blocks in \( H_{\theta_0, -R} \) which intersect \( B_{-R, \theta_0, \theta, \text{start, big}}^{\text{fat}} \) satisfies

\[
(5.5) \quad M_R \leq \left( \frac{8\sqrt{d}|y_{\theta_0}|\epsilon(R)R}{n^{-\beta_0} \Delta_R} \right)^{d-1} \leq c_8 n^{-\beta_0(d-1)}(\log R)^{(d-1)K_1}.
\]

We denote these as \( \{B_{-R, m}, m \leq M_R\} \), and define events

\[ A_m : \text{there exist } n \theta_0 \text{-slab geodesics from } B_{-R, m}^{\text{fat}} \text{ to } B_{\theta_0, \text{home}}^{\text{fat}} \text{ with distinct } H_{\theta_0, R}^{\text{fat}} \text{-entry points}. \]

Then from Proposition 4.10 and (5.5), assuming \( K_2 \) is large,

\[
(5.6) \quad P \left( |Z_R(\theta_0, \theta, \epsilon_{\min}) \cap B_{\theta_0, \text{home}}^{\text{fat}}| \geq nM_R \right) \leq \sum_{m=1}^{M_R} P(A_m) \leq M_R \exp \left( -C_{61} \frac{n^{2\beta_1}}{\log R} \right) \leq \exp \left( -\frac{C_{61} n^{2\beta_1}}{2 \log R} \right)
\]

so in view of (5.1) and (5.2),

\[
E \left( |Z_R(\theta_0, \theta, \tau) \cap B_{\theta_0, \text{home}}^{\text{fat}}| \right) \leq nM_R + |Z^d \cap B_{\theta_0, \text{home}}^{\text{fat}}| \left[ \exp \left( -\frac{C_{61} n^{2\beta_1}}{2 \log R} \right) + \sup_{x \in H_{\theta_0, 0}^{\text{fat}} \cap \mathbb{Z}^d} P(x \in Y_R(\theta_0, \theta, \epsilon_1) \right) \leq 2nM_R \leq 2c_8(\log R)^{(1+\beta_0(d-1))K_2+(d-1)K_1}.
\]

Letting \( \hat{\theta}_{0, 0, t} = \{0\} \times [-t, t]^{d-1} \) (in \( \theta_0 \)-coordinates), and noting that (5.7) is also valid for any translate of \( B_{\theta_0, \text{home}}^{\text{fat}} \) in \( H_{\theta_0, 0, t}^{\text{fat}} \), we can sum (5.7) over such translates to obtain

\[
(5.8) \quad \limsup_{t \to \infty} \frac{E \left( |Z_R(\theta_0, \theta, \tau) \cap B_{\theta_0, 0, t}^{\text{fat}}| \right)}{(2t)^{d-1}} \leq \frac{E \left( |Z_R(\theta_0, \theta, \epsilon_{\min}) \cap B_{\theta_0, \text{home}}^{\text{fat}}| \right)}{(2n^{-\beta_0} \Delta_R)^{d-1}} \leq c_8(\log R)^{K_3} \leq \frac{c_8(\log R)^{K_3}}{2^{d-2} \Delta_R^{d-1}},
\]

where \( K_3 = (1 + 2\beta_0(d-1))K_2 + (d-1)K_1 \), proving (1.21).
To prove (1.20), we proceed similarly but in the definition of $B_{-R, \theta_0, \text{start}, \big}$ we replace $\epsilon(R)$ with $\epsilon_{\min}/2d$, and in place of $W_R, Z_R$ we use

$$W_R := W_R(\theta_0, \theta, \tau) := \{ x \in H_{\theta_0,0}^{\text{fat}} \cap \mathbb{Z}^d : \text{for some } \theta \in J(\theta_0, \epsilon_{\min}/2d) \text{ there exists a } \theta_0\text{-slab geodesic from } H_{\theta_0,-R}^- \text{ to } H_{\theta_0,R}^+ \text{ with } H_{\theta_0,0}^{\text{fat}}\text{-entry point } x \text{ and initial orientation in } J(\theta, \epsilon_{\min}/2d) \},$$

$$Z_R := Z_R(\theta_0, \theta, \tau) := \{ x \in H_{\theta_0,0}^{\text{fat}} \cap \mathbb{Z}^d : \text{for some } \theta \in J(\theta_0, \epsilon_{\min}/2d) \text{ there exists a } \theta\text{-ray from } H_{\theta_0,-R}^- \text{ with } H_{\theta_0,0}^{\text{fat}}\text{-entry point } x \},$$

which satisfy $Z_R \subset W_R \cup Y_R$. Then we must also replace $\epsilon(R)$ with $\epsilon_{\min}/2d$ in (5.6). The modified (5.4) still gives $\psi_{\theta_0, R^*} \leq \epsilon_{\min}/2$ so Proposition 4.11 still applies. Hence in place of (5.7) we have

$$E(|Z_R(\theta_0, \theta, \tau) \cap B_{\theta_0, \text{home}}^{\text{fat}}|) \leq 2nM_R \leq c_{419}n^{1+\beta_0(d-1)} \left( \frac{R}{\Delta R} \right)^d \leq c_{10}(\log R)^{(1+\beta_0(d-1))K_2} \left( \frac{R}{\Delta R} \right)^d,$$

(5.9)

and then in place of (5.8),

$$\limsup_{t \to \infty} \frac{E(|Z_R(\theta_0, \theta, \tau) \cap \hat{B}_{\theta_0,0,t}^{\text{fat}}|)}{(2t)^{d-1}} \leq \frac{E(|Z_R(\theta_0, \theta, \tau) \cap B_{\theta_0, \text{home}}^{\text{fat}}|)}{(2n^{-\beta_0} \Delta R)^{d-1}} \leq \frac{c_8(\log R)^{(1+2\beta_0(d-1))K_2}}{2d-2\sigma_R^{d-1}},$$

(5.10)

proving (1.20) and completing the proof of Theorem 1.5(ii). □

**Proof of Theorem 1.8** Suppose A1 and A2(i),(ii) hold for some $\theta_0, \epsilon_0$. By changing $\theta_0$ slightly we may assume $\theta_0$ is rationally oriented. This means there exist vectors $b_j \in H_{\theta_0,0} \cap \mathbb{Z}^d, j \leq d - 1$, which form a basis for $H_{\theta_0,0}$. The sites of form $z_n = \sum_{j=1}^{d-1} n_j b_j$, with $n = (n_1, \ldots, n_{d-1}) \in \mathbb{Z}^{d-1}$ form a lattice in $H_{\theta_0,0}$ which divides $H_{\theta_0,0}$ into parallelepipeds; let $\Lambda_n$ denote the parallelepiped with opposite corners $z_n, z_m$ with $m = (n_1 + 1, \ldots, n_{d-1} + 1).

For $a \neq b$ in $\mathbb{R}^d$ we write $\alpha_{ab}$ for $(b - a)/|b - a|$. Let $u, v$ be as in the theorem statement. If $d_{\theta_0}(0, \Pi_{uv}) \geq c_1\Delta(\|u\|)(\log |u|)^2$ then it is easily checked that for $r = |v - u|/|y_{\alpha_{uv}}|$ we have $D_{\alpha_{uv},r}(-u) \geq c_2\Delta(\|u\|)^2(\log |u|)^4/\Xi(\|u\|)^2 \geq c_3(\log |u|)^3$, from Proposition 2.4

(5.11)

$$P(0 \in \Gamma_{uv}) \leq C_{35}e^{-c_4(\log |u|)^3},$$

and (1.24) follows. Therefore we may assume

(5.12)

d_{\theta_0}(0, \Pi_{uv}) < c_1(\log |u|)^2\Delta(\|u\|) \quad \text{ and hence } \quad \psi_{\alpha_{uv}, \alpha_0} \leq \frac{\epsilon_{\min}}{8},

the last being provided $|u|$ is large. Let $-R = u_{\theta_0}^0 + \mu \sqrt{d}$, so $R, |u|$ are of the same order. Similarly to (5.11) we have for $\chi_2 > \chi$ and $|u|$ large

(5.13)

$$P(y \in \Gamma_{uv} \text{ for some } y \in H_{\theta_0,R}^- \text{ with } |y - u| \geq c_5(\log |u|)^{2/(1-\chi_2)}) \leq C_{35}e^{-c_6(\log |u|)^2}$$
since any such \( y \) satisfies \( D_{\alpha,z}(y-u) \geq c_7 \Phi(\log |u|)^{2/(1-\chi_2)} \geq c_8 (\log |u|)^2 \), with \( \chi_2 \) from A3. Again similarly, we get

\[
(5.14) \quad P\{z \in \Gamma_{u0} \text{ for some } z \in H^+_{\theta_0,0} \text{ with } |z| \geq c_5 (\log |u|)^{2/(1-\chi_2)} \} \leq C_35 e^{-c_6 (\log |u|)^2}.
\]

We will bound \( P(0 \in \Gamma_{ uv}) \) essentially by considering the expected number of sites \( x \) in certain fattened blocks for which the translated event \( x \in \Gamma_{x+u,x+v} \) occurs, and relating this expected number to the expected number of certain entry points, which can be bounded using Proposition 4.10. Note that \( x + u \in H^\text{fat}_{\theta_0,-R} \) for all \( x \in H^\text{fat}_{\theta_0,0} \). For \( x, z \in H^\text{fat}_{\theta_0,0} \cap \mathbb{Z}^d \) and \( y \in H^\text{fat}_{\theta_0,-R} \) define events

\[ C_{x,y,z} : x \in \Gamma_{x+u,x+v}, \text{ } y \text{ is the last site of } \Gamma_{x+u,x+v} \text{ in } H^+_{\theta_0,-R}, \]

\[ z \text{ is the first site of } \Gamma_{y,x+v} \text{ in } H^+_{\theta_0,0}, \]

\[ F_x = \bigcup \left\{ C_{x,y,z} : y \in H^\text{fat}_{\theta_0,-R} \cap \mathbb{Z}^d, |y - (x + u)| \leq c_5 (\log |u|)^{2/(1-\chi_2)}, \right. \]

\[ z \in H^\text{fat}_{\theta_0,0} \cap \mathbb{Z}^d, |z - x| \leq c_5 (\log |u|)^{2/(1-\chi_2)} \}, \]

and let

\[ X_{ uv} = \{ x \in H^\text{fat}_{\theta_0,0} \cap \mathbb{Z}^d : \tau \in F_x \}; \]

see Figure 13. For a given configuration \( \tau \) with \( 0 \in \Gamma_{ uv} \), we either have \( \tau \in F_0 \) (meaning the sites \( y, z \) in the event \( C_{0,y,z} \) are close to \( u, 0 \) respectively) or one of the events in (5.13) or (5.14) occurs. Therefore

\[
(5.15) \quad P(0 \in \Gamma_{ uv}) \leq P(F_0) + 2C_35 e^{-c_6 (\log |u|)^2}.
\]

Define next the random set of entry points

\[ \hat{W}_u := \left\{ z \in H^\text{fat}_{\theta_0,0} \cap \mathbb{Z}^d : \text{ for some } x \in H^\text{fat}_{\theta_0,0} \text{ and } y \in H^\text{fat}_{\theta_0,-R} \text{ with } |z - x| \leq c_5 (\log |u|)^{2/(1-\chi_2)} \]

and \( |y - (x + u)| \leq c_5 (\log |u|)^{2/(1-\chi_2)} \), there is a \( \theta_0 \)-slab geodesic from \( y \) to \( H^+_{\theta_0,R} \)

with \( H^+_0 \)-entry point \( z \} \).

(a variant of \( W_R \) in the previous proof.) If \( \tau \in C_{x,y,z} \) for some \( C_{x,y,z} \subseteq F_x \), then \( z \in \hat{W}_u \). Let \( n = (\log |u|)^K \) with \( K \) to be specified, let \( \beta_j \) be as in A3, and as in section 4, subdivide \( H_{\theta_0,-R} \) and \( H_{\theta_0,0} \) each into blocks of side \( 2n^{-\beta_0} \Delta_R \). Fix a block \( \hat{B} \) of \( H_{\theta_0,0} \) and let \( \overline{z} \) be its center; then \( \overline{z} + u \in H^\text{fat}_{\theta_0,0} \). Let \( \hat{y} \) be the \( \theta_0 \)-projection of \( \overline{z} + u \) into \( H^\text{fat}_{\theta_0,-R} \), let \( Q_{\hat{B}} \) be the block of \( H_{\theta_0,-R} \) containing \( \hat{y} \), and let \( \overline{y} \) be the center of \( Q_{\hat{B}} \). Then \( \overline{z} + u \in Q_{\hat{B}}^\text{fat} \).

From the definition of \( \hat{W}_u \), given \( z \in \hat{W}_u \cap \hat{B}^\text{fat} \) there exist \( x \in H^\text{fat}_{\theta_0,0} \) and \( y \in H^\text{fat}_{\theta_0,-R} \) with \( |z - x| \leq c_5 (\log |u|)^{2/(1-\chi_2)} \) and \( |y - (x + u)| \leq c_5 (\log |u|)^{2/(1-\chi_2)} \). Since by (2.24) \( d_\theta_0(z,H_{\theta_0,0}) \leq d \),
we have $|z - \tilde{z}| \leq d + \sqrt{d - 1}n^{-\beta_0} \Delta_R$. It follows that, provided $|u|$ (and hence $R$) is large,

$$
\begin{align*}
    d(y, Q^\text{fat}_B) &\leq |y - (\tilde{z} + u)| \\
    &\leq |y - (x + u)| + |(x + u) - (z + u)| + |(z + u) - (\tilde{z} + u)| \\
    &\leq 2\zeta_0 (\log |u|)^{2/(1-x_2)} + d + \sqrt{d - 1}n^{-\beta_0} \Delta_R \\
    &\leq 2\sqrt{d - 1}n^{-\beta_0} \Delta_R.
\end{align*}
$$

(5.16)

Let $\{B_{-R,z,j}, j \leq N_d\}$ be those blocks $B$ of $H_{\theta_0,-R}$ satisfying $d(Q^\text{fat}_B, B) \leq 2\sqrt{d - 1}n^{-\beta_0} \Delta_R$; the number $N_d$ of such blocks depends only on $d$, and by (5.16), $y$ must be in one of these blocks (backwards-fattened.) Suppose now that $|\tilde{W}_u \cap \hat{B}^\text{fat}| \geq N_d R$; then some $B^\text{fat}_{-R,z,j}$ with $j \leq N_d$ contains the starting points $y$ of at least $n$ of the corresponding $\theta_0$-slab geodesics from the definition of $\tilde{W}_u$. Let $\overline{\gamma}_j$ be the center of $B^\text{fat}_{-R,z,j}$ and $\zeta_j = (\overline{\gamma} - \overline{\gamma}_j)/|\overline{\gamma} - \overline{\gamma}_j|$. To apply Proposition 4.10 we need to bound $\psi_{\theta_0 \zeta_j}$; for this we will use

$$
\psi_{\theta_0 \zeta_j} \leq \psi_{\theta_0, \alpha_y} + \psi_{\alpha_{uy}, \alpha_{y0}} + \psi_{\alpha_{u0}, \alpha_{zy}} + \psi_{\alpha_{zy}, \alpha_{y0}}.
$$

(5.17)

The first two angles on the right are bounded by assumption and by (5.12), so we will bound the last two. From the bounds on $|z - x|$ and $|y - (x + u)|$ in the definition of $\tilde{W}_u$, provided $|u|$ is large we have $\psi_{\alpha_{u0}, \alpha_{zy}} \leq c_0 (\log |u|)^{2/(1-x_2)}/|u| \leq \epsilon_{\min}/8$. Since the pairs $y, \overline{\gamma}_j$ and $z, \overline{\gamma}$ each lie in the
same (forwards or backwards) fattened block, we have \( \psi_{\alpha_{ij}, \beta_{ij}} \leq c_{440} n^{-\beta_0} \Delta_R / R < \epsilon_{\min}/8 \), again provided \(|u|\) (and hence \(R\)) is large. Provided we take \(\epsilon_4 \leq \epsilon_{\min}/8\) we thus obtain from (5.17) that \(\psi_{\theta_0, \zeta} < \epsilon_4 + 3\epsilon_{\min}/2 < \epsilon_{\min}/2\), and then from Proposition 4.10 that, provided \(K\) is large enough,

\[
P(|\tilde{W}_u \cap \hat{B}^{\text{fat}}| \geq N_d n) \leq N_d \exp \left( -C_{61} n^{2\beta_1} \log R \right) \leq e^{-c_{10}(\log |u|)^2},
\]

and therefore

\[
E\left(|\tilde{W}_u \cap \hat{B}^{\text{fat}}|\right) \leq N_d n + c_{11}(n^{-\beta_0} \Delta_R)^{d-1} e^{-c_{10}(\log |u|)^2} \leq 2N_d n = 2N_d(\log |u|)^K.
\]

Now \(x \in X_{uv}\) implies \(z \in \tilde{W}_u\) for some \(z\) with \(|z - x| \leq c_5(\log |u|)^2/(1-\chi)^2\), so we have

(5.18) \[ E\left(|X_{uv} \cap \hat{B}^{\text{fat}}|\right) \leq c_{12}(\log |u|)^{2(d-1)/(1-\chi^2)} E\left(|\tilde{W}_u \cap \hat{B}^{\text{fat}}|\right) \leq c_{13}(\log |u|)^{K+2(d-1)/(1-\chi^2)}.
\]

Observe that \(P(x \in X_{uv}) = P(F_x)\) has period \(\Lambda_0\), in the sense that it takes the same value at \(x\) and \(x + b_j\) for all \(x\) and all \(j \leq d - 1\). Therefore, letting \(q = \{|n : \Lambda_n \subset \hat{B}\}\), we have from (5.18)

(5.19) \[ \sup_x P(F_x) \leq \sum_{x \in \Lambda_0} P(F_x) = E\left(|X_{uv} \cap \Lambda_0^{\text{fat}}|\right) \leq \frac{1}{q} E\left(|X_{uv} \cap \hat{B}^{\text{fat}}|\right) \leq \frac{c_{14}}{q} (\log |u|)^{K+2(d-1)/(1-\chi^2)}.
\]

Provided \(|u|\) (and hence \(\hat{B}\)) is large, we have \(q|\Lambda_0| \geq |\hat{B}|/2\) so

\[
q \geq \frac{c_{14}}{|\Lambda_0|} n^{-(d-1)/\beta_0} \Delta_R^{d-1} \geq c_{15}(\log |u|)^{-(d-1)\beta_0} K \Delta(|u|)^{d-1}
\]

which with (5.19) shows that, provided \(K\) is large,

(5.20) \[ \sup_x P(F_x) \leq \frac{c_{16}(\log |u|)^{dK}}{\Delta(|u|)^{d-1}}.
\]

This and (5.15) complete the proof. \(\square\)

6. Nonexistence of bigeodesics

In this section we prove parts (iii) and (iv) of Theorem 1.5. We begin with (iii); the main idea is that all bigeodesics as in (iii) are \(\theta\)-rays in one direction and \((-\theta)\)-rays in the other, for some \(\theta \in J(\theta_0, \epsilon_2)\), and the crossing-point density of such bigeodesics is bounded above by \(\overline{\rho}_{J(\theta_0, \epsilon_2)}\) for all \(R\), up to a small error term.

We may assume \(\theta_0\) is rationally oriented. Suppose A2(i),(ii) hold for some \(\theta_0, \epsilon_0\), fix \(\epsilon < \epsilon_0\) to be specified, and suppose

(6.1) \[ P\left(\text{there exists a bigeodesic containing a subsequential } \theta\text{-ray for some } \theta \in J(\theta_0, \epsilon) \right) > 0.
\]

Let \(\Gamma = (x_i, i \in \mathbb{Z})\) be such a bigeodesic; we may assume the subsequential \(\theta\)-ray is \((x_i, i \geq 0)\). By Proposition 3.1(iii), \((x_i, i \geq 0)\) is a \(\theta\)-ray. We write \(\Gamma[x_j, \infty)\) for the \(\theta\)-ray \((x_i, i \geq j)\) and \(\Gamma(-\infty, x_j)\) for the geodesic ray \((x_i, i \leq j)\).

We claim \((x_{-i}, i \geq 0)\) is a \((-\theta)\)-ray for every such \(\theta\) and \(\Gamma\), a.s. If not, some such \(\theta\)-ray is a subsequential \(\varphi\)-ray for some \(\varphi \neq -\theta\), so there exists \(i_k \to \infty\) for which \(x_{-i_k}/|x_{-i_k}| \to \varphi\). Letting
we have from Proposition 3.1 that it is easily checked that we therefore have for all sufficiently large $k$

$$\sup_{u \in \Gamma(x_{-i_k}, \infty)} D_\theta(u - x_{-i_k}) \geq D_\theta(x_0 - x_{-i_k}) \geq c_1 \left( \frac{\psi_{\varphi,-\theta} r_k}{\Xi(r_k)^2} \right)^2 \geq c_2 \psi_{\varphi,-\theta}^2 \frac{r_k}{\sigma(r_k) \log r_k}.$$ But for the events

$$F_{\delta,j} : \text{for some } x \in \mathbb{Z}^d \text{ with } 2^{j-1} < |x| \leq 2^j \text{ and some } \theta \in J(\theta_0, \epsilon)$$

$$D_\theta(-x) \geq c_3 \delta^2 \frac{|x|}{\sigma(|x|) \log |x|},$$

there is a $\theta$–ray from $x$ containing 0, we have from Proposition 3.1 that

$$P(F_{\delta,j}) \leq c_4 \delta^j e^{-c_5 \delta^2 / j \sigma(2^j)} \quad \text{so} \quad P(F_{\delta,j} \text{ i.o. in } j) = 0.$$ The same is true if we translate the events by $x_0$, replacing 0 with $x_0$ and $|x|$ with $|x - x_0|$, so this proves the claim that $(x_{-i}, i \geq 0)$ is a $(-\theta)$–ray. $\Gamma$ is thus a $\theta$–bigeodesic, by which we mean a bigeodesic which is a $\theta$–ray in one direction, and a $(-\theta)$–ray in the opposite direction. We have shown that with probability 1, every bigeodesic $\Gamma$ containing a subsequential $\theta$–ray for some $\theta \in J(\theta_0, \epsilon)$ is actually a $\theta$–bigeodesic, so it has a well-defined entry point $x''_{\theta_0,0}(\Gamma)$ in $H_{\theta_0,0}^+$. Let

$$C^\text{bi}_{J(\theta_0, \epsilon),0}(A) = \left\{ x \in A : x = x''_{\theta_0,0}(\Gamma) \text{ for some } \theta \in J(\theta_0, \epsilon) \text{ and } \theta \text{–bigeodesic } \Gamma \right\}.$$ We may consider the “largest $\theta_0$–backtrack after $x''_{\theta_0,0}(\Gamma)$”, or more precisely, the value

$$R_0(\Gamma) = \min \{ u^\theta_{1,j} : u \in \Gamma[x''_{\theta_0,0}(\Gamma), \infty) \} \wedge 0,$$ which by the preceding is finite for every $\theta$–bigeodesic $\Gamma$ with $\theta \in J(\theta_0, \epsilon)$.

Before continuing let us recall that the mean (combined) $H_{\theta_0,R}$–crossing density is defined in (1.15) and (1.16) by counting entry points in $H_{\theta_0,R}^+$ of geodesic rays having only their initial site in $H_{\theta_0,0}^-$. We could equally well consider a “shift by $R$”: entry points in $H_{\theta_0,0}^+$ of geodesic rays having only their initial site in $H_{\theta_0,-R}^-$. It is enough to consider $R$ for which $H_{\theta_0,-R}$ (and therefore also $H_{\theta_0,0}^+$) contains a lattice point; by stationarity, for such $R$ the shift by $R$ does not alter the crossing point density. For $R > 0$ we split $C^\text{bi}_{J(\theta_0, \epsilon),0}(A)$ into a “large–backtrack” set

$$C^\text{bi, } R^+_{J(\theta_0, \epsilon),0}(A) = \left\{ x \in A : x = x''_{\theta_0,0}(\Gamma) \text{ for some } \theta \in J(\theta_0, \epsilon) \text{ and } \theta \text{–bigeodesic } \Gamma \text{ with } R_0(\Gamma) \geq R \right\}$$

and a small–backtrack set $C^\text{bi, } R^-_{J(\theta_0, \epsilon),0}(A) = C^\text{bi}_{J(\theta_0, \epsilon),0}(A) \setminus C^\text{bi, } R^+_{J(\theta_0, \epsilon),0}(A)$. For $x \in C^\text{bi, } R^-_{J(\theta_0, \epsilon),0}(A)$, there exists a $\theta$–bigeodesic $\Gamma$ with $R > |R_0(\Gamma)|$ and with $H_{\theta_0,0}^+$–entry point $x$, and letting $y_{-R}(\Gamma)$ be the last point of $\Gamma$ in $H_{\theta_0,-R}^-$, we have that $\Gamma[y_{-R}(\Gamma), \infty)$ is a halfspace $\theta$–ray from $H_{\theta_0,-R}^-$ with the same $H_{\theta_0,0}^+$–entry point $x$. Now the mean density of bigeodesic entry points, given by

$$\bar{p}^\text{bi}_{J(\theta_0, \epsilon),0} = \limsup_{r \to \infty} \frac{E(|C^\text{bi}_{J(\theta_0, \epsilon),0}(H_{\theta_0,0}^+ \cap B_r(0))|)}{\text{Vol}_{d-1}(H_{\theta_0,0} \cap B_r(0))}$$
satisfies

$$\bar{p}^\text{bi}_{J(\theta_0, \epsilon),0} \leq \limsup_{r \to \infty} \frac{E(|C^\text{bi, } R^-_{J(\theta_0, \epsilon),0}(H_{\theta_0,0}^+ \cap B_r(0))|)}{\text{Vol}_{d-1}(H_{\theta_0,0} \cap B_r(0))} + \limsup_{r \to \infty} \frac{E(|C^\text{bi, } R^+_{J(\theta_0, \epsilon),0}(H_{\theta_0,0}^+ \cap B_r(0))|)}{\text{Vol}_{d-1}(H_{\theta_0,0} \cap B_r(0))}.$$
for all \( R > 0 \), and, by the preceding remark about \( x \in C_{J(\theta_0,\epsilon)}^{b_i}(A) \), the first lim sup on the right is bounded above by \( P_{J(\theta_0,\epsilon),R} \). The second lim sup is bounded by

\[
c_0 P \left( \text{for some } \theta \in J(\theta_0,\epsilon) \text{ there exists a } \theta \text{-ray from } 0 \text{ which intersects } H_{\theta_0,-R} \right),
\]

which (cf. Lemma 4.7) is readily shown to approach 0 as \( R \to \infty \). Since \( R \) is arbitrary, provided \( \epsilon \leq \epsilon_2 \) it then follows from Theorem 1.5(ii) that \( P_{J(\theta_0,\epsilon),0}^{b_i} = 0 \). Since \( \theta_0 \) is rationally oriented, periodicity of \( P(x \in C_{J(\theta_0,\epsilon)}^{b_i}(H_{\theta_0,0}^{\text{fat}})) \) then means that we have \( P(x \in C^{b_i}_{J(\theta_0,\epsilon),0}(H_{\theta_0,0}^{\text{fat}})) = 0 \) for all sites \( x \in H_{\theta_0,0}^{\text{fat}} \), which proves Theorem 1.5(iii). Then (iv)(c) follows from (iii) and compactness of \( S^{d-1} \).

7. Coalescence time bounds

In this section we prove Theorem 1.7. We start with the upper bound on \( P((U_{xy_i}^\theta)^0_1 \geq r) \), as the lower bound is much simpler.

Let \( \epsilon_3 = \min(\epsilon_0/2, \epsilon_2/2, \epsilon_6) \), where \( \epsilon_2 \) is from Theorem 1.5 and \( \epsilon_6 \) from Lemma 2.2. Let \( \theta \in J(\theta_0,\epsilon_3) \) and let \( x, y \) be \( \theta \)-start sites with second coordinates \( x_2 < y_2 \). Provided \( |y-x| \) is large, there exists \( \tilde{\theta} \) with \( \psi_{\theta,\tilde{\theta}} < \epsilon_3 \) (and thus \( \psi_{\tilde{\theta},\tilde{\theta}} < 2\epsilon_3 \)) for which \( y-x \in H_{\tilde{\theta},0} \). We may assume \( y-x \) makes an angle of at least \( \pi/4 \) with the horizontal axis; then there is exactly one \( \tilde{\theta} \)-start site \( z_k \) at each integer height \( k \). We have \( P((U_{xy_i}^\theta)^0_1 \geq r) = P((U_{0y_i-x}^\theta)^0_1 \geq r - x_1^\theta) = P((U_{x-y,0}^\theta)^0_1 \geq r - y_1^\theta) \), and \( x_1^\theta, y_1^\theta \in [-\mu \sqrt{d}, 0] \). Now one of \( y-x, x-y \) lies in \( H_{\tilde{\theta},0}^- \) so is a \( \tilde{\theta} \)-start site; in case it is \( y-x \) then the preceding shows that \( x, y \) can be replaced by \( 0, y-x \), or equivalently, we may assume \( x = 0 \). The proof is the same in the other case, so we will indeed assume \( x = 0 \) and \( y \in H_{\tilde{\theta},0} \).

For points \( u, v \in H_{\tilde{\theta},0} \) we use notation \([u,v]\) for the interval from \( u \) to \( v \) in \( H_{\tilde{\theta},0} \); we call such an interval an \( H \)-interval. From (2.8) and Proposition 3.1(ii), provided \( \epsilon_3 \) is small we have

\[
P \left( \frac{(U_{0y}^\theta)^0_1 \geq r}{(U_{0y}^\theta)^0_1} \right) \leq P \left( \frac{(U_{0y}^\theta)^0_1 \geq r}{(U_{0y}^\theta)^0_1} \right) + P \left( \frac{(U_{0y}^\theta)^0_1 \geq r}{(U_{0y}^\theta)^0_1} \right)
\]

\[
\leq P \left( \frac{(U_{0y}^\theta)^0_1 \geq r}{(U_{0y}^\theta)^0_1} \right) + P \left( \sup_{u \in \Gamma_0^\theta} D_\theta(u) \geq \Phi(r) \right)
\]

\[
\leq P \left( \frac{(U_{0y}^\theta)^0_1 \geq r}{(U_{0y}^\theta)^0_1} \right) + c_1 e^{-c_2 \Phi(r)},
\]

so it is sufficient to prove (1.22) with \((U_{0y}^\theta)^0_1\) replaced by \((U_{0y}^\theta)^0_1\). For each \( \tilde{\theta} \)-start site \( z \) let \( z \) be its projection horizontally into \( H_{\tilde{\theta},0} \). Throughout this section, “projection” will mean horizontal projection into \( H_{\tilde{\theta},0} \), unless stated otherwise.

Let \( Z_{\tilde{\theta}} \) denote the set of all \( \tilde{\theta} \)-start sites, and \( S_{\tilde{\theta},\tilde{\theta}} \subset Z_{\tilde{\theta}} \) the subset which are \( \theta \)-sources. Let \( r > 1 \), and for each \( z \in Z_{\tilde{\theta}} \) let \( V_z \) be the last site of \( \Gamma_z^\theta \) in \( H_{\tilde{\theta},0}^- \) (so necessarily \( V_z \in S_{\tilde{\theta},\tilde{\theta}} \)) and let \( W_z \) be the \( H_{\tilde{\theta},0}^- \)-entry point of \( \Gamma_z^\theta \). A \((\tilde{\theta}, \theta, r)\)-gap is an open \( H \)-interval \( I \) in \( H_{\tilde{\theta},0} \) with the properties that (i) \( I \) contains no projected \( \theta \)-source \( V_z \), and (ii) the endpoints \( \overline{v}, \overline{w} \) of \( I \) are projections of sources \( v, w \in S_{\tilde{\theta},\tilde{\theta}} \) with \( V_v \neq W_v \). A \((\tilde{\theta}, \theta, r)\)-entry interval is the closed interval between two successive \((\tilde{\theta}, \theta, r)\)-gaps. It then follows from planarity of \( Z^2 \) that any two \( \theta \)-sources \( v, w \) satisfy \( W_v = W_w \) if and only if \( \overline{v}, \overline{w} \) lie in the same \((\tilde{\theta}, \theta, r)\)-entry interval; thus the gaps separate those groups of
halflspace $\theta$–rays from $H^{-}_{\tilde{\theta},0}$ which coalesce before crossing $H^{+}_{\tilde{\theta},r}$. (See Figure 14.) Equivalently,

$$v, w \in S_{\tilde{\theta},\theta}, \ (U_{vw}^{\theta})_{1}^{\tilde{\theta}} \geq r \implies \text{there is a } (\tilde{\theta}, \theta, r)\text{–gap between } v \text{ and } w.$$ 

(It should be emphasized that this is only true for $\theta$–sources, not general $\tilde{\theta}$–start sites.)

We note that $\tilde{\theta}$–start sites are periodic in the sense that $u$ is a $\tilde{\theta}$–start site if and only if $u + y$ is one. We now consider translates of the events in (7.1) corresponding to $\tilde{\theta}$–start sites $u, u + y$ in place of 0, $y$. We have for all $u \in \mathcal{Z}_{\tilde{\theta}}$:

$$P\left((U_{u,u+y}^{\theta})_{1}^{\tilde{\theta}} \geq r\right) = P\left((U_{0y}^{\theta})_{1}^{\tilde{\theta}} \geq r - u_{1}^{\tilde{\theta}}\right) \geq P\left((U_{0y}^{\theta})_{1}^{\tilde{\theta}} \geq r + \mu \sqrt{d}\right),$$

and therefore, averaging over a period,

$$P\left((U_{0y}^{\theta})_{1}^{\tilde{\theta}} \geq r + \mu \sqrt{d}\right) \leq \frac{1}{y^{2}} \sum_{k=0}^{y^{2}-1} P\left((U_{z_{k},z_{k}+y}^{\theta})_{1}^{\tilde{\theta}} \geq r\right).$$

**Figure 14.** *Top:* The gray $\theta$–rays share $W_{u+y}$ as their common $H^{+}_{\tilde{\theta},r}$–entry point; the dashed ones similarly share $W_{u}$. The thick segment in $H_{\tilde{\theta},0}$ is the $(\tilde{\theta}, \theta, r)$–gap $G_{u,u+y}$, separating those gray and dashed $\theta$–rays which are halflspace $\theta$–rays. The hash marks on $H_{\tilde{\theta},0}$ show the corresponding enlarged $(\tilde{\theta}, \theta, r)$–gap ($G_{\min}, G_{\max}$); the lowest start point of a gray geodesic is at $G_{\min}$, and the highest start point of a dashed one is at $G_{\max}$. *Bottom:* The event $A_{uw}$. The $H^{+}_{\tilde{\theta},r}$–entry points from $u$ and $w$ are different, but $V_{u} = V_{w}$. 


For \( \tilde{\theta} \)-start sites \( u, w \), it is possible that \( (U_{uw}^0)_{1}^{\tilde{\theta}} \geq r \) and the continuation \( \Gamma_{u}^0 \cap \Gamma_{w}^0 = \Gamma_{u,w}^0 \) after coalescence backtracks to visit \( H_{\tilde{\theta},0}^- \), so that \( V_u = V_w \); let us call this event \( A_{uw} \). See again Figure 14. Given \( \tau \in A_{uw} \) let \( q \) be first point of \( \Gamma_{u,w}^0 \) in \( H_{\tilde{\theta},0}^- \), and \( p \) the last point of \( \Gamma_{u,w}^0 \) before \( q \) with \( p \in H_{\tilde{\theta},r}^+ \). Then \( (p-u)_1^{\tilde{\theta}} \in [r, r + 2\mu \sqrt{d}] \). If \( |(p-u)_2^{\tilde{\theta}}| \geq 2r \) then by Lemma 2.2 we also have \(|(p-u)_2^{\tilde{\theta}}| \geq |(p-u)_1^{\tilde{\theta}}| \geq r/2 \) so \( D_{\theta}(p-u) \geq \Phi(r/2) \). If instead \(|(p-u)_2^{\tilde{\theta}}| < 2r \) then we use the readily-verified fact that for such \( p \) we have \( q \in H_{\tilde{\theta},0}^- \implies D_{\theta}(q-p) \geq \Phi(r/2) \). Let \( K_r = \lfloor \log_2 r \rfloor \) and define corresponding events for these two situations:

\[ A'_u : \text{ there exist } q, p \text{ with } |p-u| \leq (3 + |y|)r, \ q \in \Gamma_{u}^0, \ \text{ and } D_{\theta}(q-p) \geq \Phi(r/2), \]

\[ A''_u : \text{ there exists } p \in \Gamma_{u}^0 \text{ with } D_{\theta}(p-u) \geq \Phi(r/2), \]

so that from Proposition 3.1(ii), taking \( w = u + y \),

\[ P(A_{u,u+y}) \leq P(A'_u) + P(A''_u) \leq c_4 r d e^{-C_{36} \Phi(r/2)} + C_{35} e^{-C_{36} \Phi(r/2)} \leq e^{-C_{36} \Phi(r/2)/2}. \]

It remains to consider the event that \( (U_{u,u+y}^\theta)_{1}^{\tilde{\theta}} \geq r \) but \( \tau \notin A_{u,u+y} \). In this case, for \( \theta \)-sources \( V_u, V_{u+y} \), coalescence of \( \Gamma_{u}^0, \Gamma_{u+y}^0 \) occurs in \( H_{\tilde{\theta},r}^+ \), so by (7.2) there must be a \( (\tilde{\theta}, \theta, r) \)-gap between \( V_u \) and \( V_{u+y} \); let \( G_{u,u+y} \) be longest such gap, breaking ties arbitrarily. There are 2 cases to consider:

1. \( |\pi - V_u| \geq (log r)^c \) or \( |u+y - V_{u+y}| \geq (log r)^c \)
2. \( \max(d(\pi, G_{u,u+y}), d(u+y, G_{u,u+y})) \leq |y| + 2 \max(|\pi - V_u|, |u+y - V_{u+y}|) \leq |y| + 2(log r)^c \),

where \( c_4 \) is chosen so \( \Phi((log r)^c) \geq \log r \). In case (1) if \( |\pi - V_u| \geq (log r)^c \) then we have \( V_u \in \Gamma_{u}^0 \) and \( D_{\theta}(V_u - u) \geq \frac{1}{2} \Phi((log r)^c) \), and similarly for \( u+y \) in place of \( u \). Therefore by Proposition 3.1(ii) we have

\[ P\left((U_{u,u+y}^\theta)_{1}^{\tilde{\theta}} \geq r, \ \tau \notin A_{u,u+y}, \ \text{ and Case (1) holds}\right) \leq C_{35} e^{-C_{36} \Phi((log r)^c)} \leq C_{35} e^{-C_{36}(log r)^2}. \]

Case (2) is more complicated. We have

\[ |G_{u,u+y}| \leq |V_u - V_{u+y}| \leq |y| + 2(log r)^c; \]

we call any \( (\tilde{\theta}, \theta, r) \)-gap \( G \) short if \( |G| \leq |y| + 2(log r)^c \). Then

\[ P\left((U_{u,u+y}^\theta)_{1}^{\tilde{\theta}} \geq r, \ \tau \notin A_{u,u+y}, \ \text{ and Case (2) holds}\right) \leq P\left(d(\pi, G) \leq |y| + 2(log r)^c \text{ for some short } (\tilde{\theta}, \theta, r) \text{-gap } G \right). \]

We now take \( u = z_k \) and consider the average as in (7.3). Define events

\[ Q_k : (U_{z_k,z_k+y}^\theta)_{1}^{\tilde{\theta}} \geq r, \ \tau \notin A_{z_k,z_k+y}, \]

\[ R_k : d(z_k, G) \leq |y| + 2(log r)^c \text{ for some short } (\tilde{\theta}, \theta, r) \text{-gap } G \text{ in } H_{\tilde{\theta},0}. \]

From (7.4), (7.5), and (7.6),

\[ P\left((U_{0,y}^\theta)_{1}^{\tilde{\theta}} \geq r + \mu \sqrt{d}\right) \leq \frac{1}{y_2} \sum_{k=0}^{y_2-1} [P(Q_k) + P(A_{z_k,z_k+y})] \leq \frac{1}{y_2} \sum_{k=0}^{y_2-1} P(R_k) + 2C_{35} e^{-C_{36}(log r)^2}. \]
We need to bound the average on the right in (7.7). Let \( J_t = \{ y \in H_{\tilde{\theta}, \theta} : |y| \leq t \} \) and let \( N_t \) be the number of short \((\tilde{\theta}, \theta, r)\)-gaps \( G \) intersecting \( J_t \). Each corresponding \((\tilde{\theta}, \theta, r)\)-entry interval (between two such gaps) contains the projection of a \( \theta \)-source, and the halfspace \( \theta \)-rays from these sources have different \( H_{\theta}^+ \)-entry points for each \((\tilde{\theta}, \theta, r)\)-entry interval. It then follows using Theorem 1.5(ii) that for \( \rho_\theta(r) \) from (1.17),

\[
(7.8) \quad \limsup_{t \to \infty} \frac{N_t}{2t} \leq \rho_\theta(r) \leq C_{15} \frac{(\log r)^{C_{16}}}{\Delta_r}.
\]

We now use periodicity as in the proof of Theorem 1.8. By the multidimensional ergodic theorem (see [18], Appendix 14.A), we have

\[
(7.9) \quad \frac{1}{y_2} \sum_{k=0}^{y_2-1} P(R_k) = \lim_{m \to \infty} \frac{1}{2m+1} \sum_{k=-m}^{m} 1_{R_k}, \text{ a.s.}
\]

Now

\[
\sum_{k=-m}^{m} 1_{R_k} \leq \left| \{ u \in \mathbb{Z}_\tilde{\theta} : |u_2| \leq m, d(\pi, G) \leq |y| + 2(\log r)^{c_4} \text{ for some short } (\tilde{\theta}, \theta, r)\text{-gap } G \} \right|
\]

\[
\leq N_{m|y|/y_2}(3|y| + 6(\log r)^{c_4}) \leq N_{m|y|/y_2}|y|(\log r)^{c_4}
\]

so

\[
\frac{1}{2m+1} \sum_{k=-m}^{m} 1_{R_k} \leq \frac{|y|}{y_2} N_{m|y|/y_2} \frac{2m}{2m+1} |y|(\log r)^{c_4}.
\]

With (7.8) and (7.9) this yields

\[
\frac{1}{y_2} \sum_{k=0}^{y_2-1} P(R_k) \leq c_5 \frac{(\log r)^{C_{16}+c_4}|y|}{\Delta_r}.
\]

Combining this with (7.7) we obtain

\[
P((U_{\tilde{\theta}y})^\theta \geq r + \mu \sqrt{d}) \leq c_6 \frac{(\log r)^{C_{16}+c_4}|y|}{\Delta_r},
\]

which establishes the upper bound in (1.22).

Turning to the lower bound, we need to consider the fact that \( z \) and \( V_z \) may lie on opposite sides of a gap. Given a \((\tilde{\theta}, \theta, r)\)-gap \( G \), we let \( G_{\max}, G_{\min} \) denote respectively the highest (lowest) projected \( \theta \)-start site \( \overline{\pi} \) for which \( \overline{V}_z \) lies below (above) \( G \). It is not necessarily true that \( G_{\min} \) is below \( G_{\max} \), but \( G_{\max} \) is in or above \( G \), \( G_{\min} \) is in or below \( G \), and \( G_{\max} \) is at most one vertical unit below \( G_{\min} \). It is easily seen that for every \( \tilde{\theta} \)-start site \( z \) we have

\[
(7.10) \quad D_{\tilde{\theta}}(V_z - z) \geq c_7 \Phi(|\overline{\pi} - \overline{V}_z|);
\]

it then follows readily from Proposition 3.1(ii) that \( G_{\min}, G_{\max} \) always exist for all \( G \), a.s. We call \( G^+ := G \cup (G_{\min}, G_{\max}) \) an enlarged \((\tilde{\theta}, \theta, r)\)-gap. We say an enlarged \((\tilde{\theta}, \theta, r)\)-gap \( G^+ \) is semi-short if \( |G^+| \leq |y| + (\log r)^{c_4} \), and very long otherwise. A key observation is that

\[
(7.11) \quad \text{one of } z = G_{\min}, G_{\max} \text{ must satisfy } |\overline{\pi} - \overline{V}_z| \geq \frac{1}{2}|G^+| - 1.
\]
Let $a_r = \Delta_r \log r$ and define overlapping intervals in $H_{\tilde{\theta},0}$:

$$I_j = [j a_r, (j + 2) a_r],$$

and define events

$$Y_j : I_j \text{ intersects a semi-short enlarged } (\tilde{\theta}, \theta, r)\text{-gap } G^+.\$$

Suppose $I_j$ intersects a semi-short enlarged $(\tilde{\theta}, \theta, r)$-gap $G^+$, for some gap $G$. For each $0 \leq k < y_2$, consider the points $\{z_k + iy : i \in \mathbb{Z}\}$; let $w_k$ be the lowest such point for which $V_{z_k+(i-1)y}$ and $V_{z_k+iy}$ are on opposite sides of $G$. Then $w_k \in G^+$, and the points $\{w_k : 0 \leq k < y_2\}$ are all distinct. This shows that

$$\left| \{u \in \mathbb{Z}_{\tilde{\theta}} : \pi \in G^+, V_u \text{ and } V_{u+y} \text{ are on opposite sides of } G \} \right| \geq y_2;$$

see Figure 15. Now, any given semi-short enlarged $(\tilde{\theta}, \theta, r)$-gap $G^+$ intersects at least one and at most three $H$-intervals $I_j$. It follows that for the events

$$F_k : V_{z_k}\text{ and } V_{z_k+y} \text{ are on opposite sides of some } (\tilde{\theta}, \theta, r)\text{-gap } G \text{ for which } G^+ \text{ is semi-short}$$

we have

$$\frac{y_2}{3} \sum_{j=-\ell}^{\ell} 1_{Y_j} \leq \sum_{k=-\ell \Delta_r \log r}^{(\ell+2) \Delta_r \log r} 1_{F_k},$$

Figure 15. The gray curves are $\theta$-rays. The gap $G$ is marked by the hash marks on $H_{\tilde{\theta},0}$; the enlarged gap $G^+$ is the thickened part of $H_{\tilde{\theta},0}$. We fix a height $k \in [0, y_2)$; the row of dots to the left of $H_{\tilde{\theta},0}$ (other than $V_{w_k}$ and $V_{w_k+y}$) are the sites $z_k + iy, i \in \mathbb{Z}$. $w_k$ is the lowest such site for which $V_{w_k}$ and $V_{w_k+y}$ lie on opposite sides of $G$. There is such a site $w_k$ for each $k \in [0, y_2)$, creating $y_2$ occurrences of events $F_j$. Necessarily at least one of $w_k, w_k+y$ lies in $G^+$. 

Let $a_r = \Delta_r \log r$ and define overlapping intervals in $H_{\tilde{\theta},0}$:

$$I_j = [j a_r, (j + 2) a_r],$$

and define events

$$Y_j : I_j \text{ intersects a semi-short enlarged } (\tilde{\theta}, \theta, r)\text{-gap } G^+.\$$

Suppose $I_j$ intersects a semi-short enlarged $(\tilde{\theta}, \theta, r)$-gap $G^+$, for some gap $G$. For each $0 \leq k < y_2$, consider the points $\{z_k + iy : i \in \mathbb{Z}\}$; let $w_k$ be the lowest such point for which $V_{z_k+(i-1)y}$ and $V_{z_k+iy}$ are on opposite sides of $G$. Then $w_k \in G^+$, and the points $\{w_k : 0 \leq k < y_2\}$ are all distinct. This shows that

$$\left| \{u \in \mathbb{Z}_{\tilde{\theta}} : \pi \in G^+, V_u \text{ and } V_{u+y} \text{ are on opposite sides of } G \} \right| \geq y_2;$$

see Figure 15. Now, any given semi-short enlarged $(\tilde{\theta}, \theta, r)$-gap $G^+$ intersects at least one and at most three $H$-intervals $I_j$. It follows that for the events

$$F_k : V_{z_k}\text{ and } V_{z_k+y} \text{ are on opposite sides of some } (\tilde{\theta}, \theta, r)\text{-gap } G \text{ for which } G^+ \text{ is semi-short}$$

we have

$$\frac{y_2}{3} \sum_{j=-\ell}^{\ell} 1_{Y_j} \leq \sum_{k=-\ell \Delta_r \log r}^{(\ell+2) \Delta_r \log r} 1_{F_k},$$
Figure 16. The event that $L_0 = [0, 2a_r]$ intersects no enlarged $(\tilde{\theta}, \theta, r)$–gap. Because $|v - u| \approx |p_v - p_u|$ is large, the common $H^+_{\tilde{\theta}, r}$–entry point $w$ from $u$ and $v$ must be far from either $p_u$ or $p_v$. This remains true when we shift slightly from $\tilde{\theta}$–coordinates (black lines) to $\theta$–coordinates (gray lines), so $\Gamma^{\theta}_u$ or $\Gamma^{\theta}_v$ must make a large transverse fluctuation to pass through $w$.

and hence by the ergodic theorem,

$$
\frac{y_2}{3 \Delta_r \log r} P(Y_0) \leq \frac{1}{y_2} \sum_{k=0}^{y_2-1} P(F_k)
$$

$$
\leq \frac{1}{y_2} \sum_{k=0}^{y_2-1} \left[ P \left( (U^{\theta}_{z_k, z_k+y})_{\tilde{\theta}} \geq \frac{r}{2} \right) + P \left( \Gamma^{\theta}_{W_{z_k}} \cap H^{-}_{\tilde{\theta}, r/2} \neq \emptyset \right) \right]
$$

(7.12)

Here the second inequality reflects that when $\tau \in F_k$, the $H^+_{\tilde{\theta}, r}$–entry points $W_{z_k} \neq W_{z_k+y}$, so either coalescence occurs in $H^+_{\tilde{\theta}, r/2}$ or both geodesics $\Gamma^{\theta}_{z_k}, \Gamma^{\theta}_{z_k+y}$ backtrack to $H^{-}_{\tilde{\theta}, r/2}$ after entering $H^+_{\tilde{\theta}, r}$.

The third inequality reflects that (i) the first probability on the second line is maximized over $k$ when $z_k, z_k+y$ lie in $H_{\tilde{\theta}, 0}$, and (ii) the second probability on the second line can be bounded as in (7.4). (Also in (7.12), for technical convenience in applying the ergodic theorem, we have assumed $\Delta_r \log r$ is an integer multiple of $y_2$, ensuring $P(Y_j)$ is the same for all $j$. The added technicalities without this assumption are tedious but straightforward, using our assumption $|y| \leq \Delta_r$.)

Next we show $P(Y_0)$ is near 1; with (7.12) this will complete the proof of the lower bound in (1.22). We have

$$
P(Y_0^c) \leq P(I_0 \text{ intersects no enlarged } (\tilde{\theta}, \theta, r)\text{–gap})
$$

(7.13)

$$
+ P(I_0 \text{ intersects a very long enlarged } (\tilde{\theta}, \theta, r)\text{–gap}).
$$
Let us consider the first probability on the right of (7.13). Suppose \( I_0 \) intersects no enlarged \((\tilde{\theta}, \theta, r)\)-gap; then \( I_0 \) is contained in some \((\tilde{\theta}, \theta, r)\)-entry interval \([\pi, \nu]\). See Figure 16. This means \( u, v \) are \( \theta \)-sources with \(|\pi - \nu| \geq 2\Delta_r \log r \), and with \( W_u = W_v = w \) for some \( w \). Now the angle between \( \tilde{\theta} \) and \( H_{\tilde{\theta},r} \) is at least \( \pi/4 \) by (2.9), and it follows by straightforward geometry that the points \( p_u = L_{\tilde{\theta}}(u) \cap H_{\tilde{\theta},r} \) and \( p_v = L_{\tilde{\theta}}(v) \cap H_{\tilde{\theta},r} \) satisfy
\[
|p_u - p_v| - |\pi - \nu| \leq \sqrt{2}|u - \pi| + \sqrt{2}|v - \nu| \leq 2\sqrt{2}
\]
Provided \( \psi_{\tilde{\theta}} \) is small, when we change the angle to \( \theta \) and consider \( q_u = L_{\theta}(u) \cap H_{\theta,w_1^q} \) and \( q_v = L_{\theta}(v) \cap H_{\theta,w_1^q} \), we have via straightforward use of (2.11) and (2.12) that
\[
w_1^\theta \geq \frac{r}{2}, \quad |q_u - q_v| \geq \frac{3}{4}|p_u - p_v| \geq \frac{3}{2}|\pi - \nu|.
\]
Since \( w, q_u, q_v \) all lie in \( H_{\theta,w_1^q} \) follows that
\[
\max((w - u)^\theta_2, (w - v)^\theta_2) = \max(|w - q_u|, |w - q_v|) \geq \frac{1}{4}|\pi - \nu|.
\]
We may assume the first entry in the maximum is the larger one. Then using (2.8), for some \( c_{10}, c_{11}, \)
\[
D_\theta(w - u) \geq D_*(|\pi - \nu|) := \frac{c_{10}|\pi - \nu|}{\sigma^*(|\pi - \nu|) \log |\pi - \nu|} \quad \text{if } |\pi - \nu| \geq r,
\]
\[
\frac{c_{11}|\pi - \nu|^2}{r \sigma_r \log r} \quad \text{if } |\pi - \nu| < r.
\]
Let \( \tilde{K}_r = \lfloor \log_2(2\Delta_r \log r) \rfloor \), and define events
\[
(7.14)\quad E_k: \text{for some } u, v \in \mathcal{Z}_\tilde{\theta} \text{ with } I_0 \subset [\pi, \nu] \text{ and } 2^k < |\pi - \nu| \leq 2^{k+1}, \quad \sup_{w \in \Gamma^\theta_u} D_\theta(w - u) \geq D_*(2^k).
\]
The preceding together with Proposition 3.1(ii) then shows that
\[
P(\text{\(I_0\) intersects no enlarged \((\tilde{\theta}, \theta, r)\)-gap}) \leq \sum_{k=K_r}^{\infty} P(\text{\(E_k\)}) \leq \sum_{k=K_r}^{\infty} c_{12} 2^{2k} \exp \left( -C_{45} D_* (2^k) \log D_*(2^k) \right) \leq e^{-c_{13}(\log r)(\log \log r)}.
\]
We now turn to the last probability in (7.13). Suppose \( I_0 \) intersects a very long enlarged \((\tilde{\theta}, \theta, r)\)-gap \((\bar{f}, \bar{g})\). Then by (7.11), one of \( z = f \) or \( g \) satisfies \(|z - \mathcal{V}_z| \geq \frac{1}{3} |\bar{f} - \bar{g}| \geq \frac{1}{3} (|y| + (\log r)^{c_4}) \) and \( D_\theta(V_z - z) \geq c_{14} \Phi(|z - \mathcal{V}_z|) \). Letting \( K(r, y) = \lfloor \log_2(\frac{1}{3} (|y| + (\log r)^{c_4})) \rfloor \) and defining events
\[
M_k: \text{for some } \tilde{\theta} \text{-start site } z \text{ with } d(z, I_0) \leq 2^k \text{ we have } D_\theta(V_z - z) \geq c_{14} \Phi(2^{k-1}/3),
\]

we see that if $2^{k-1} < |f - g| \leq 2^k$ then $\tau \in M_k$. It follows using Proposition 3.1(ii) that

\[
P(I_0 \text{ intersects a very long enlarged } (\theta, \theta, r)\text{–gap}) \leq \sum_{k=K(r,y)}^{\infty} P(M_k)
\]

\[
\leq c_{15} \sum_{k=K(r,y)}^{\infty} (\Delta_r \log r + 2^{k+1}) e^{-c_{16} \Phi(2^k) \log(2^k)}
\]

\[
\leq c_{17} \Delta_r (\log r) e^{-c_{18} \Phi((\log r)^{c_4}) \log \Phi((\log r)^{c_4})}
\]

\[
\leq e^{-c_{18} (\log r)(\log \log r)/2}.
\]

(7.16)

With (7.13) and (7.14) this shows that $P(Y_0) \geq 1/2$; with (7.12) this completes the proof of the lower bound in (1.22).

The general outline of this proof, and in particular the use of gaps, is analogous to ([7], Section 6), with added complications due to the undirected nature of paths here, which means not all start sites are sources, and backtracking may occur after coalescence.

**References**

[1] Ahlberg, D. and Hoffman, C. (2016). Random coalescing geodesics in first-passage percolation. arXiv:1609.02447 [math.PR]

[2] Alexander, K.S. (1997). Approximation of subadditive functions and rates of convergence in limiting shape results. Ann. Probab. 24 30–55.

[3] Alves, S. G., Oliveira, T. J., and Ferreira, S. C. (2018). Universality of fluctuations in the Kardar-Parisi-Zhang class in high dimensions and its upper critical dimension. Phys. Rev. E 90 020103. arXiv:1405.0974 [cond-mat.stat-mech]

[4] Baik, J., Ferrari, P. L., and Péché, S. (2014). Convergence of the two-point function of the stationary TASEP. Singular phenomena and scaling in mathematical models, 91–110, Springer, Cham. arXiv:1209.0116 [math-ph]

[5] Balázs, Busani, O., and Seppäläinen, T. (2019). Nonexistence of bi-infinite geodesics in the exponential corner growth model. arXiv:1909.06883 [math.PR]

[6] Basu, R., Hoffman, C., and Sly, A. (2018). Nonexistence of bigeodesics in integrable models of last passage percolation. arXiv:1811.04908 [math.PR]

[7] Basu, R., Sarkar, S., and Sly, A. (2019). Coalescence of geodesics in exactly solvable models of last passage percolation. J. Math. Phys. 60 093301, 22 pp. arXiv:1704.05219 [math.PR]

[8] Basu, R., Sidoravicius, V., and Sly, A. (2016). Last passage percolation with a defect line and the solution of the slow bond problem. arXiv:1408.3464 [math.PR]

[9] Benaïm, M. and Rossignol, R. (2008). Exponential concentration for first passage percolation through modified Poincaré inequalities. Ann. Inst. Henri Poincaré Probab. Stat. 44 544–573. arXiv:math/0609730 [math.PR]

[10] Benjamin, I., Kalai, G., and Schramm, O. (2003). First passage percolation has sublinear distance variance. Ann. Probab. 31 1970–1978. arXiv:math/0203262 [math.PR]

[11] Chatterjee, S. (2013). The universal relation between scaling exponents in first-passage percolation. Ann. of Math. (2) 127, no. 2, 663–697. arXiv:1105.4566 [math.PR]

[12] Corwin, I., Liu, Z., and Wang, D. (2016). Fluctuations of TASEP and LPP with general initial data. Ann. Appl. Probab. 26, 2030–2082. arXiv:1412.5087 [math.PR]

[13] Damron, M. and Hanson, J. (2014). Busemann functions and infinite geodesics in two-dimensional first-passage percolation. Comm. Math. Phys. 325, no. 3, pp. 917–963. arXiv:1209.3036 [math.PR]

[14] Damron, M. and Hanson, J. (2017). Bigeodesics in first-passage percolation. Comm. Math. Phys. 349, no. 2, pp. 753–756. arXiv:1512.00804 [math.PR]

[15] Damron, M. and Kubota, N. (2014). Gaussian concentration for the lower tail in first-passage percolation under low moments. Stoch. Proc. Appl. 126, 3065–3076. arXiv:1406.3105 [math.PR]
[16] Fogedby, H. C. (2006). Kardar-Parisi-Zhang equation in the weak noise limit: Pattern formation and upper critical dimension. *Phys. Rev. E* **73** 031104. arXiv:cond-mat/0510268 [cond-mat.stat-mech]

[17] Gangopadhyay, U. (2019). In preparation.

[18] Georgii, H. O. (1998). *Gibbs Measures and Phase Transitions*. de Gruyter Studies in Mathematics **9**, de Gruyter, Berlin.

[19] Georgiou, N., Rassoul-Agha, F., and Seppäläinen, T. (2017). Geodesics and the competition interface for the corner growth model. *Prob. Theory Rel. Fields* **169** 223–255. arXiv:1510.00860 [math.PR]

[20] Kesten, H. (1993). On the speed of convergence in first-passage percolation. *Ann. Appl. Probab.* **3** 296–338.

[21] Kim, S.-W. and Kim, J. M. (2014). A restricted solid-on-solid model in higher dimensions. *J. Stat. Mech.* **2014** P07005.

[22] Le Doussal, P. and Wiese, K. J. (2005). Two-loop functional renormalization for elastic manifolds pinned by disorder in $N$ dimensions. *Phys. Rev. E* **72** 035101. arXiv:cond-mat/0501315 [cond-mat.dis-nn]

[23] Ledoux, M. and Rider, B. (2010) Small deviations for beta ensembles. *Electron. J. Probab.* **15**, 1319–1343. arXiv:0912.5040 [math.PR]

[24] Licea, C. and Newman, C. M. (1996). Geodesics in two-dimensional first-passage percolation. *Ann. Probab.* **24**, 399–410.

[25] Licea, C., Newman, C. M. and Piza, M. S. T., Superdiffusivity in first-passage percolation, *Probab. Theory Rel. Fields* **106** (1996), 559–591.

[26] Löwe, M. and Merkl, F. (2001). Moderate deviations for longest increasing subsequences: The upper tail. *Comm. Pure Appl. Math.* **54**, 1488–1519.

[27] Löwe, M., Merkl, F., and Rolles, S. (2002). Moderate deviations for longest increasing subsequences: The lower tail. *J. Theor. Probab.* **15**, 1031–1047.

[28] Marinari, E., Pagnani, A., Parisi, G., Rácz, Z. (2002). Width distributions and the upper critical dimension of Kardar-Parisi-Zhang interfaces. *Phys. Rev. E* **65** 026136. arXiv:cond-mat/0105158 [cond-mat.stat-mech]

[29] Newman, C. M., A surface view of first passage percolation. *Proceedings of the International Congress of Mathematicians*, Vol. 1, 2 (Zürich, 1994), 1047–1023, Birkhäuser, Basel (1995).

[30] Newman, C. M. and Piza, M. S. T. (1995). Divergence of shape fluctuations in two dimensions. *Ann. Probab.* **23**, 977–1005.

[31] Pimentel, L. (2015). Duality between coalescence times and exit points in last-passage percolation models. *Ann. Probab.* **44**, 3187–3206. arXiv:1307.7769 [math.PR]

[32] Rodrigues, E. A., Oliveira, F. A., and Mello, B. A. (2015). On the existence of an upper critical dimension for systems within the KPZ universality class. *Acta. Phys. Pol. B* **46**, 1231–1234. arXiv:cond-mat/1502.06121 [cond-mat.stat-mech]

[33] Seppäläinen, T. and Shen, X. (2019). Coalescence estimates for the corner growth model with exponential weights, arXiv:1911.03792 [math.PR]

[34] Talagrand, M. (1995). Concentration of measure and isoperimetric inequalities in product spaces. *Publications Mathématiques de l’Institut des hautes Études Scientifiques* **81**(1), 73–205.

[35] Tessera, R. (2018). Speed of convergence in first passage percolation and geodesicity of the average distance. *Ann. Inst. Henri Poincaré Probab. Stat.* **54**, 569–586. arXiv:1410.1701 [math.PR]

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