Asymptotic Expansions of Feynman Diagrams on the Mass Shell in Momenta and Masses

V.A. Smirnov

*Nuclear Physics Institute of Moscow State University
Moscow 119899, Russia*

Abstract

Explicit formulae for asymptotic expansions of Feynman diagrams in typical limits of momenta and masses with external legs on the mass shell are presented.

\[\text{E-mail: smirnov@theory.npi.msu.su}\]
1 Introduction

Explicit formulae for asymptotic expansion of Feynman diagrams in various limits of momenta and masses have been obtained in the simplest form (with coefficients homogeneous in large momenta and masses) in [1, 2, 3] (see also [4] for a brief review). They hold at least when external momenta are off the mass shell. In the large mass limit, one can also apply the same ‘off-shell’ formulae for any values of the external momenta. If however some of the large external momenta are on the mass shell these formulae are generally non-valid. It is the purpose of this paper to present explicit formulae for asymptotic expansions of Feynman diagrams in two typical limits of momenta and masses with external legs on the mass shell: the limit of the large momenta on the mass shell with the large mass and the Sudakov limit, with the large momenta on the massless mass shell.

To derive these formulae we shall follow a method applied in ref. [5] for operator product expansions within momentum subtractions and later for diagrammatic and operator expansions within dimensional regularization and renormalization [3]. This method starts with constructing a remainder of the expansion and using then diagrammatic Zimmermann identities.

In the next section we shall remind explicit formulae of asymptotic expansions for the off-shell limit of large momenta and masses and illustrate how the remainder of the expansion is constructed. In Section 3 we shall generalize these formulae for the on-shell limit of large momenta and masses and, in Section 5, for the Sudakov limit. These formulae will be illustrated through one-loop examples.

2 Off-shell limit of large momenta and masses

It was an idea of Zimmermann [3] to derive operator product expansion using subtractions of leading asymptotics. Anikin and Zavialov have systematically developed this idea [4] and constructed a remainder of the operator product expansions within BPHZ scheme in such a way that it has the same combinatorial structure as the $R$-operation (i.e. renormalization at the diagrammatical level) itself. Let us consider asymptotic expansion of a Feynman diagram $F_I(Q, q, M, m)$ corresponding to a graph $F_I$ in the limit when the momenta $Q \equiv \{Q_1, \ldots, Q_i, \ldots\}$ and the masses $M \equiv \{M_1, \ldots, M_i, \ldots\}$ are larger than $q \equiv \{q_1, \ldots, q_i, \ldots\}$ and $m \equiv \{m_1, \ldots, m_i, \ldots\}$. Suppose for simplicity that the external momenta are non-exceptional. To obtain explicitly the asymptotic expansion with homogeneity of coefficients in the large momenta and masses, one uses the same strategy and constructs the remainder in the form $\mathcal{R} F_I(Q, q, M, m)$, with an operation $\mathcal{R}$ which can be represented, e.g., by the forest formula [4]

$$\mathcal{R} = \sum_{F} \prod_{\gamma \in F} M_\gamma.$$

(1)
Here the sum runs over forests (sets of non-overlapping subgraphs) composed of one-particle-irreducible (1PI) or asymptotically irreducible (AI) subgraphs and \( \mathcal{M}_\gamma \) is a (pre-)subtraction operator. Note that for the usual \( R \)-operation that removes ultraviolet divergences similar sum is over UV-divergent 1PI subgraphs. Subtractions in 1PI subgraphs remove UV divergences while (pre-)subtractions in AI subgraphs remove first terms of the asymptotic behaviour and provide desirable asymptotic behaviour of the remainder in the considered limit. Thus, if an initial diagram has divergences then the corresponding operation \( \mathcal{R} \) includes usual UV counterterms. One can consider asymptotic expansion of regularized diagrams: in this case \( \mathcal{R} \) involves only pre-subtractions. Let us suppose, for simplicity, that the initial diagram is UV and IR finite.

The class of AI subgraphs is determined by the consider limit. For example, in the off-shell limit of momenta and masses a subgraph \( \gamma \) is called AI if

(a) in \( \gamma \) there is a path between any pair of external vertices associated with the large external momenta \( Q_i \);

(b) \( \gamma \) contains all the lines with the large masses;

(c) every connectivity component \( \gamma_j \) of the graph \( \hat{\gamma} \) obtained from \( \gamma \) by collapsing all the external vertices with the large external momenta to a point is 1PI with respect to the lines with the small masses.

Note that, generally, \( \gamma \) can be disconnected. One can distinguish the connectivity component \( \gamma_0 \) that contains external vertices with the large momenta.

The pre-subtraction operator is nothing but the Taylor expansion operator in the small momenta and masses:

\[
\mathcal{M}_\gamma = \mathcal{T} a_{\gamma, q^\gamma, m^\gamma}.
\] (2)

The operator \( \mathcal{T} \) performs Taylor expansion in the corresponding set of variables; \( q^\gamma \) are the small external momenta of the subgraph \( \gamma \) (i.e. all its external momenta apart from the large external momenta of \( \Gamma \)); \( m^\gamma \) is a set of small masses of \( \gamma \). The degrees of subtraction \( a_\gamma \) are chosen as \( a_\gamma = \omega_\gamma + \pi \) where \( \omega_\gamma \) is the UV degree of divergence and \( \pi \) is the number of oversubtractions. The asymptotic behaviour of the remainder is governed by the number \( \pi \). Note that these operators are by definition applied to integrands of Feynman integrals in loop momenta.

To derive the asymptotic expansion it is sufficient then to write down the identity

\[
F_\Gamma = (1 - \mathcal{R})F_\Gamma + \mathcal{RF}_\Gamma
\]

and then the diagrammatic Zimmermann identity [4] for the difference \( 1 - \mathcal{R} \), i.e. the difference of two \( R \)-operations. (Remember that if the initial diagram was renormalized we would obtain the difference \( R - \mathcal{R} \).) In our case, this identity takes the form

\[
1 - \mathcal{R}_\Gamma = \sum_{\gamma} \mathcal{R}_{\Gamma/\gamma} \mathcal{M}_\gamma^{a_\gamma}.
\] (3)

where \( \Gamma/\gamma \) are reduced graphs and the sum is over AI subgraphs.
Taking then the number of oversubtractions $\pi \to \infty$ and using the fact that product of two operators corresponding to two different AI subgraphs gives zero (so that $R_{\Gamma/\gamma}$ in (1) can be replaced by unity) one arrives at an explicit formula for the asymptotic expansion:

$$F_{\Gamma}(Q, q, M, m; \varepsilon) \xrightarrow{M \to \infty} \sum_{\gamma} F_{\Gamma/\gamma}(q, m; \varepsilon) \circ T_{q^\gamma m^\gamma} F_{\gamma}(Q, q^\gamma, M, m^\gamma; \varepsilon),$$

(4)

where the sum is again in AI subgraphs and the symbol $\circ$ denotes the insertion of the polynomial that stands to the right of it into the reduced vertex of the reduced diagram $\Gamma/\gamma$.

Note that individual terms in (4) possess UV and IR divergences which are mutually cancelled (see [2, 3, 4] for details).

3 Large momentum expansion on the mass shell

Let us again consider the large momentum and mass expansion of a Feynman integral $F_{\Gamma}$ corresponding to a graph $\Gamma$. We suppose that the external momenta are non-exceptional, as above. Moreover, we imply that $(\sum_{i \in I} Q_i)^2 \neq M_j^2$, for any subset of indices $I$. Let now the large external momenta be on the mass shell, $Q_j^2 = M_j^2$. To obtain explicit formulae of the corresponding asymptotic expansion let us again start from the remainder $RF_{\Gamma}$ where $R$ is given by (1), with the same class of AI subgraphs but a different subtraction operator $M$.

As for the off-shell limit the operator $M_\gamma$ happens to be a product $\prod_i M_{\gamma_i}$ of operators of Taylor expansion in certain momenta and masses. For connectivity components $\gamma_i$ other than $\gamma_0$ (this is the component with the large external momenta), the corresponding operator performs Taylor expansion of the Feynman integral $F_{\gamma_i}$ in its small masses and external momenta. (Note that its small external momenta are generally not only the small external momenta of the original Feynman integral but as well the loop momenta of $\Gamma$.) Consider now $M_{\gamma_0}$. The component $\gamma_0$ can be naturally represented as a union of its 1PI components and cut lines (after a cut line is removed the subgraph becomes disconnected; here they are of course lines with the large masses). By definition $M_{\gamma_0}$ is again factorized and the Taylor expansion of the 1PI components of $\gamma_0$ is performed as in the case of c-components $\gamma_i$, $i = 1, 2, \ldots$.

It suffices now to describe the action of the operator $M$ on the cut lines. Let $l$ be such a line, with a large mass $M_j$, and let its momentum be $P_l + k_l$ where $P_l$ is a linear combination of the large external momenta and $k_l$ is a linear combination of the loop momenta and small external momenta. If $P_l = Q_i$ then the operator $M$ for this component of $\gamma$ is

$$T_{k_l^2} \frac{1}{\kappa k_l^2 + 2Q_i k_l + i0} \bigg|_{\kappa=1}.$$
(In what follows we shall omit $i0$, for brevity.) In all other cases, e.g. when $P_l = 0$, or it is a sum of two or more external momenta, the Taylor operator $\mathcal{M}$ reduces to ordinary Taylor expansion in small (with respect to this line considered as a subgraph) external momenta, i.e.

$$T_k \frac{1}{(k_l + P_l)^2 - M_l^2} \equiv T_k \frac{1}{(kk_l + P_l)^2 - M_l^2} \bigg|_{k=1}. \quad (6)$$

Note that in all cases apart from the cut lines with $P_l^2 = M_j^2$ the action of the corresponding operator $\mathcal{M}$ is graphically described (as for the off-shell limit) by contraction of the corresponding subgraph to a point and insertion of the resulting polynomial into the reduced vertex of the reduced graph.

Using the same Zimmermann identity (3) (note however that the symbol $\Gamma/\gamma$ in $\mathcal{R}_{\Gamma/\gamma}$ does not now have its proper meaning) with the new pre-subtraction operator we now obtain the following explicit form of the asymptotic expansion in the on-shell limit

$$F_{\Gamma}(Q, M, q, m; \epsilon) \sim \sum_{\gamma} \mathcal{M}_\gamma F_{\Gamma}(Q, M, q, m; \epsilon). \quad (7)$$

Let us apply this general formula to a one-loop propagator-type diagram consisting of two lines, with masses $M$ and $m$, and an external momentum $p$ with $p^2 = M^2$:

$$\int d^4k \frac{1}{(k^2 - 2pk)(k^2 - m^2)}. \quad (8)$$

In the limit $M \to \infty$, general formula (7) gives contributions corresponding to two subgraphs: the subgraph $\gamma$ associated with the heavy mass $M$ and the graph $\Gamma$ itself. The contribution of $\Gamma$ is obtained as formal Taylor expansion of the propagator with the mass $m$ at $m = 0$:

$$\sum_{j=0}^{\infty} (m^2)^j \int d^4k \frac{1}{(k^2 - 2pk)^{j+1}}. \quad (9)$$

The integral involved is easily calculated by the $\alpha$-representation. We have

$$i \pi^{d/2}(M^2)^{-\epsilon} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(j + \epsilon)\Gamma(-2j - 2\epsilon + 1)}{\Gamma(-j - 2\epsilon + 2)} (m^2/M^2)^j. \quad (10)$$

According to prescription (5) the contribution of $\gamma$ comes from the formal Taylor expansion of another factor in the integrand $1/(k^2 - 2pk)$ with respect to $k^2$:

$$\sum_{j=0}^{\infty} (-1)^j \int d^4k \frac{(k^2)^j}{(-2pk)^{j+1}(k^2 - m^2)}. \quad (11)$$

Calculating the above one-loop integral gives

$$i \pi^{d/2} \frac{1}{2}(m^2)^{-\epsilon} \sum_{j=0}^{\infty} \frac{\Gamma((j + 1)/2)\Gamma((j - 1)/2 + \epsilon)}{j!} (m^2/M^2)^{(j+1)/2}. \quad (12)$$
In the sum of two contributions artificial IR and UV poles are cancelled, with the following result:

\[ i\pi^2 \left\{ \frac{1}{\epsilon} + 2 - \ln M^2 + \frac{1}{2} \frac{m^2}{M^2} \left( \ln \frac{M^2}{m^2} + 2 \right) \right\} + \frac{1}{2} \frac{m}{M} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)\Gamma(n-1/2)}{(2n)!} \left( \frac{m^2}{M^2} \right)^n - \frac{1}{2} \frac{m^4}{M^4} \sum_{n=0}^{\infty} \frac{(n+1)!n!}{(2n+3)!} \left( \frac{m^2}{M^2} \right)^n \right\}. \tag{13} \]

The UV pole which is present from the very beginning can be removed by usual renormalization.

4 Explicit formulae for the Sudakov limit

The large momentum limit of the last section can be considered either in Euclidean or Minkowski space. An example of typically Minkowskian situation which has no analogues in Euclidean space is the Sudakov limit \[8\]. One of its version is formulated as the behaviour of a three-point Feynman diagram \[ F_{\Gamma}(p_1, p_2, m) \] depending on two momenta, \( p_1 \) and \( p_2 \), on the massless mass shell, \( p_1^2 = 0 \), with \( q^2 \equiv (p_1 - p_2)^2 \to -\infty \).

We suppose, for simplicity, that there is one small non-zero mass, \( m \). Let us enumerate three end-points of the diagram according to the following order: \( p_1, p_2, q \).

To treat this limit let us use the previous strategy and construct a remainder determined by an appropriate pre-subtraction operator and then apply Zimmermann identities.

Now, we call a subgraph \( \gamma \) of \( \Gamma \) AI if at least one of the following conditions holds:

(i) In \( \gamma \) there is a path between the end-points 1 and 3. The graph \( \hat{\gamma} \) obtained from \( \gamma \) by identifying the vertices 1 and 3 is 1PI.

(ii) Similar condition with 1 \( \leftrightarrow \) 2.

Note that when \( q^2 \) gets large the components at least of \( p_1 \) or \( p_2 \) are large.

The pre-subtraction operator \( \mathcal{M}_\gamma \) is now naturally defined as a product \( \prod_j \mathcal{M}_{\gamma_j} \) of operators of Taylor expansion acting on 1PI components and cut lines of the subgraph \( \gamma \). Suppose that the above condition (i) holds and (ii) does not hold. Let \( \gamma_j \) be a 1PI component of \( \gamma \) and let \( p_1 + k \) be one of its external momenta, where \( k \) is a linear combination of the loop momenta. (We imply that the loop momenta are chosen in such a way that \( p_1 \) flows through all \( \gamma_j \) and the corresponding cut lines). Let now \( q_j \) be other independent external momenta of \( \gamma_j \). Then the operator \( \mathcal{M} \) for this component is defined as\[4\]

\[ T_{k - ((p_1 k)/(p_1 p_2))p_2, q_j, m_j}, \tag{14} \]

where \( m_j \) are the masses of \( \gamma_j \). In other words, it is the operator of Taylor expansion in \( q_j \) and \( m_j \) at the origin and in \( k \) at the point \( \tilde{k} = \frac{(p_1 k)}{(p_1 p_2)}p_2 \) (which depends on \( k \))\[5\].

The formulated prescription for the pre-subtraction operator works at least up to two-loop level. Its justification and clarification for the general situation will be given in future publications.
itself).

For the cut lines we adopt the same prescription. With \( p_1 + k \) as the momentum of the cut line, we have

\[
T_r \frac{1}{\kappa(k_1^2 - m_1^2) + 2p_1k}\Big|_{\kappa=1},
\]

which is similar to (3). If both conditions (i) and (ii) hold the corresponding operator performs Taylor expansion in the mass and the external momenta of subgraphs (apart from \( p_1 \) and \( p_2 \)). For example, \( \mathcal{M}_\gamma \) for the whole graph \( (\gamma = \Gamma) \) gives nothing but Taylor expansion of the integrand in \( m \).

Consequently we arrive at an explicit expansion similar to (6):

\[
F_\Gamma(p_1, p_2, m, \epsilon) \sim \int d^d k \sum_{\gamma} \mathcal{M}_\gamma F_\Gamma(p_1, p_2, m, \epsilon),
\]

with the new operator \( \mathcal{M}_\gamma \) and the new definition of AI subgraphs.

Let us illustrate these prescriptions through an example of a one-loop triangle Feynman integral, with the masses \( m_1 = m_2 = 0, m_3 = m \) and the external momenta \( p_1^2 = p_2^2 = 0, q = p_1 - p_2 \) (see Fig. 1)

\[
F_\Gamma(p_1, p_2, m, \epsilon) = \int \frac{d^d k}{(k^2 - 2p_1k)(k^2 - 2p_2k)(k^2 - m^2)}.
\]

The set of AI subgraphs consists of two single massless lines as well as the graph \( \Gamma \) itself, with the corresponding pre-subtraction operators \( \mathcal{M}_1^a, \mathcal{M}_2^a \) and \( \mathcal{M}_0^a \). The remainder of the corresponding asymptotic expansion is \( \mathcal{R}^a F_\Gamma \) where

\[
\mathcal{R}^a = (1 - \mathcal{M}_0^a)(1 - \mathcal{M}_1^a - \mathcal{M}_2^a)
\]
and the pre-subtraction operators $M_i^a$ are

\begin{align}
M_i^a \frac{1}{k^2 - m^2} &= \sum_{j=0}^a \frac{(m^2)^j}{(k^2)^{j+1}}, \\
M_i^a \frac{1}{k^2 - 2p_i k} &= \sum_{j=0}^a \frac{(k^2)^j}{(-2p_i k)^{j+1}}, \quad i = 1, 2.
\end{align}

It is implied that each of these operators acts only on the corresponding factor of the integrand and does not touch other two factors. The remainder is UV and IR finite, for any $a$. Its asymptotic behaviour is $(m^2)^{a+1}/(k^2)^{a+2}$ modulo logarithms.

Then the terms of the expansion result from the Zimmermann identity:

\begin{equation}
1 = (1 - R) + R = M_0 + M_1 + M_2 + \ldots
\end{equation}

where we have dropped zero products of different operators and turned to the limit $a \to \infty$ (with $M_i = M_i^\infty$).

All the resulting one-loop integrals are easily evaluated for any order of the expansion. The operator $M_0$ gives the following contribution at $\epsilon \neq 0$:

\begin{equation}
M_0 F_\Gamma = -i\pi^{d/2} \frac{1}{(-q^2)^{1+\epsilon}} \sum_{n=0}^\infty \frac{\Gamma(n+1+\epsilon)\Gamma(-n-\epsilon)^2}{\Gamma(1-n-2\epsilon)} \left(\frac{m^2}{q^2}\right)^n.
\end{equation}

The terms $M_1$ and $M_2$ are not individually regulated by dimensional regularization but their sum exists for general $\epsilon \neq 0$:

\begin{equation}
(M_1 + M_2) F_\Gamma = i\pi^{d/2} \frac{1}{q^2(m^2)^\epsilon} \Gamma(\epsilon)\Gamma(1-\epsilon) \sum_{n=0}^\infty \frac{1}{n!\Gamma(1-n-\epsilon)} \times \left[\ln(-q^2/m^2) + \psi(\epsilon) + 2\psi(n+1) - \psi(1) - \psi(1-\epsilon) - \psi(1-n-\epsilon)\right] \left(\frac{m^2}{q^2}\right)^n.
\end{equation}

Here $\psi$ is the logarithmic derivative of the gamma function.

Taking into account terms with arbitrary $j$ in (19) and (20) and calculating the corresponding one-loop integrals we come, in the limit $\epsilon \to 0$, to the known result:

\begin{equation}
F_\Gamma |_{\epsilon=0} \sim -\infty \quad (M_0 + M_1 + M_2) F_\Gamma |_{\epsilon=0} = -i\pi^2 \left[\text{Li}_2 \left(\frac{1}{t}\right) + \frac{1}{2} \ln^2 t - \ln t \ln(t-1) - \frac{1}{3} \pi^2\right],
\end{equation}

where $\text{Li}_2$ is the dilogarithm and $t = -q^2/m^2$.

## 5 Conclusion

The explicit formulae of the on-shell asymptotic expansions presented above can be successfully applied for calculation of Feynman diagrams — see, e.g., [9] where the
formulae of Section 3 are used. It looks also natural to apply the formulae for the 
Sudakov limit for analyzing asymptotic behaviour of multiloop diagrams.

Observe that any term of asymptotic expansions \( (7,16) \), in particular \( (13,24) \), is 
calculated much easier than the whole diagram itself. For example, an arbitrary term 
of the expansion of a 2-loop on-shell diagram considered in \( [1] \) can be in principle analytically evaluated by computer. Let us note that the formulae for the Sudakov limit 
give not only the leading asymptotic behaviour (“the leading twist”) but any power 
with respect to the expansion parameter and all the logarithms: in the considered 
one-loop examples all the powers and logarithms were obtained at the same footing. 
Thus it looks reasonable to apply the presented technique to extend well-known results on asymptotic behaviour in the Sudakov limit \( [8, 10, 11, 12] \) (see also references in \( [11] \)) and obtain all powers and logarithms, at least at the 2-loop level.

This research has been supported by the Russian Foundation for Basic Research, 
project 96–01–00654.

Acknowledgments. I am very much grateful to K.G. Chetyrkin, A. Czarnecki, 
K. Melnikov and J.B. Tausk for helpful discussions.

References

[1] S.G. Gorishny, preprints JINR E2–86–176, E2–86–177 (Dubna 1986); Nucl. Phys. 
B319 (1989) 633

[2] K.G. Chetyrkin, Teor. Mat. Fiz. 75 (1988) 26; 76 (1988) 207; K.G. Chetyrkin, 
preprint MPI-PAE/PTh 13/91 (Munich, 1991)

[3] V.A. Smirnov, Commun. Math. Phys. 134 (1990) 109; V.A. Smirnov, Renormal-
ization and asymptotic expansions (Birkhäuser, Basel, 1991)

[4] V.A. Smirnov, Mod. Phys. Lett. A 10 (1995) 1485

[5] S.A. Anikin and O.I. Zavialov, Teor. Mat. Fiz. 27 (1976) 425; Ann. Phys. 116 
(1978) 135; O.I. Zavialov, Renormalized Quantum Field Theory (Kluwer Academic Publishers, 1990)

[6] W. Zimmermann, Ann. Phys. 77 (1973) 570

[7] W. Zimmermann, Commun. Math. Phys. 15 (1969) 208

[8] V.V. Sudakov, Zh. Eksp. Teor. Fiz. 30 (1956) 87

[9] A. Czarnecki and V.A. Smirnov, hep-ph/9608407

\(^3\)The present work hence solves the problem, raised by \( [13] \) and discussed in \( [14] \), of how to expand \( (17) \) without knowing the full result.
[10] J.M. Cornwall and G. Tiktopoulos, *Phys. Rev.* D13 (1976) 3370; V.V. Belokurov and N.I. Ussyukina, *Teor. Mat. Fiz.* 47 (1979) 157; 44 (1980) 147

[11] J.C. Collins, in *Perturbative QCD*, ed. A.H. Mueller, 1989, p. 573

[12] G.P. Korchemsky, *Phys. Lett.* B217 (1989) 330; B220 (1989) 629

[13] J.C. Collins and F.V. Tkachov, *Phys. Lett.* B294 (1992) 403

[14] V.A. Smirnov, *Phys. Lett.* B309 (1993) 397