THE GAUSS MAP AND TOTAL CURVATURE OF COMPLETE
MINIMAL LAGRANGIAN SURFACES IN THE COMPLEX
TWO-SPACE

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Abstract. We give an Osserman-type inequality and the precise maximal number of
exceptional values of the Gauss map for a complete minimal Lagrangian surface with
finite total curvature in the complex two-space. Moreover, we prove that if the Gauss
map of a complete minimal Lagrangian surface which is not a Lagrangian plane omits
three values, then it takes all other values infinitely many times.

1. Introduction

There are many similarities between surfaces in Euclidean 3-space $\mathbb{R}^3$ and Lagrangian
surfaces in the complex 2-space $\mathbb{C}^2$, in particular, the case of minimal surfaces. In fact,
there exists a representation for a minimal Lagrangian surface $M(\subset \mathbb{C}^2)$ in terms of ho-
lorphic data, similar to Weierstrass representation for a minimal surface in $\mathbb{R}^3$ (cf. [25]).
Moreover, the Gauss map $g$ of $M$ is a holomorphic map to the unit 2-sphere $S^2$. On the
representation for $M$, Chen-Morvan [5] first proved that there exists an explicit correspon-
dence in $\mathbb{C}^2$ between minimal Lagrangian surfaces and holomorphic curves (with nonde-
generate condition). Indeed, this correspondence is given by exchanging the orthogonal
complex structure $J$ in $\mathbb{C}^2$ to another one on $\mathbb{R}^4 = \mathbb{C}^2$. More generally, Héléin-Romon
[11, 12] and the first author [1, 2] proved that every Lagrangian surface $S$ in $\mathbb{C}^2$, not
necessarily minimal, is represented in terms of a plus spinor (or a minus spinor) of the
spin$^C$ bundle $(\mathbb{C}_S \oplus \mathbb{C}_S) \oplus (K_S^{-1} \oplus K_S)$ satisfying the Dirac equation with potential (see
[2, Section 1] for details). Here, $\mathbb{C}_S$ and $K_S$ denote respectively the trivial complex line
bundle and the canonical complex line bundle of $S$. Remark that the representation in
terms of plus spinors in $\Gamma(\mathbb{C}_S \oplus \mathbb{C}_S) = \Gamma(S \times \mathbb{C}^2)$ given by the first author is a natural
generalization of the one given by Chen-Morvan. Combining these results, we get the
following:

THEOREM 1.1. ([5], [12]) Let $M$ be a Riemann surface with an isothermal coordinate
$z = u + iv$ around each point. Let $F = (F_1, F_2) : M \to \mathbb{C}^2$ be a holomorphic map satisfying

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|S_1|^2 + |S_2|^2 \neq 0\ everywhere\ on\ M,\ where\ S_1 := (F_2)'_z = dF_2/dz\ and\ S_2 := -(F_1)'_z = -dF_1/dz.\ Then

(1)\ \ f = \frac{1}{\sqrt{2}} e^{i\beta/2} (F_1 - i F_2, F_2 + i F_1)

is a minimal Lagrangian conformal immersion from M to C^2 with constant Lagrangian angle \( \beta \in R/2\pi Z \). The induced metric \( ds^2 \) on M by f and its Gaussian curvature \( K_{ds^2} \) are respectively given by

(2)\ \ ds^2 = (|S_1|^2 + |S_2|^2)|dz|^2, \quad K_{ds^2} = -2\frac{|S_1(S_2)_z - S_2(S_1)_z|}{(|S_1|^2 + |S_2|^2)^{3/2}}.

The Gauss map \( g \) of M is also given by

(3)\ \ g = [-S_2 : S_1] = (-S_2/S_1) : M \to CP^1 = C \cup \{\infty\} \simeq S^2.

Here, by the identification of \( S^2 \) with \( S^2(1) \times \{e^{i\beta/2}, 0\} \subset R^3 \times R^3 \), g can be regarded as the generalized Gauss map of \( F(M) \) in \( R^4 = C^2 \) (cf. [13, 14]). Conversely, every minimal Lagrangian immersion \( f : M \to C^2 \) with constant Lagrangian angle \( \beta \) is congruent with the one constructed as above.

**Remark 1.2.** Set a holomorphic 1-form \( hdz := S_1dz \) on M. In terms of the Weierstrass data \((hdz, g)\) of M, the induced metric \( ds^2 \) and its Gaussian curvature \( K_{ds^2} \) can be rewritten respectively by

(4)\ \ ds^2 = |h|^2(1 + |g|^2)|dz|^2, \quad K_{ds^2} = -\frac{2|g_z|^2}{|h|^2(1 + |g|^2)^{3/2}}.

Moreover, the minimal Lagrangian immersion \( f : M \to C^2 \) is also given by

(5)\ \ f = \frac{1}{\sqrt{2}} e^{i\beta/2} \left( \int ghdz - i \int hdz, \int hdz + i \int ghdz \right)

\quad = \frac{1}{\sqrt{2}} e^{i\beta/2} \left( -\int S_2dz - i \int S_1dz, \int S_1dz + i \int S_2dz \right)

for \((S_1dz, S_2dz) \in \Gamma(K_S^{-1} \oplus K_S)\), which is essentially same as the one proved by Hélein-Romon [11]. On the other hand, the induced metric \( ds^2 \) and its Gaussian curvature \( K_{ds^2} \) of a minimal surface in \( R^3 \) associated with the Weierstrass data \((hdz, g)\) are given respectively by (cf. [25])

(6)\ \ ds^2 = |h|^2(1 + |g|^2)|dz|^2, \quad K_{ds^2} = -\frac{4|g_z|^2}{|h|^2(1 + |g|^2)^4}.

With these understanding, for each \( m \in N \) and a Weierstrass data \((hdz, g)\) on an open Riemann surface M, one can consider the conformal metric \( ds_m^2 := |h|^2(1 + |g|^2)^m|dz|^2 \) on M. In fact, the third author [16] has studied the precise maximal number of exceptional values of g, provided that g is not constant and that \( ds_m^2 \) is complete. Here, we call a value that a function or map never takes an exceptional value of the function or map.
The purpose of this paper is to reveal the relationship between the total curvature and the global behavior of the Gauss map of a complete minimal Lagrangian surface in \( \mathbb{C}^2 \). Our main theorems are Theorems 2.4, 2.7 and 2.9 in Section 2. The paper is organized as follows: In Section 2 we first give a curvature bound for a minimal Lagrangian surface in \( \mathbb{C}^2 \) (Theorem 2.1) and show that the precise maximal number of exceptional values of the Gauss map of a complete minimal Lagrangian surface in \( \mathbb{C}^2 \) is “3” (Corollary 2.2). These results follow from directly Theorem 2.1 and Proposition 2.4 in [16]. Next, we give an Osserman-type inequality (Theorem 2.4) for a complete minimal Lagrangian surface with finite total curvature in \( \mathbb{C}^2 \). Moreover, we show that the equality holds if and only if all the ends of the surface are embedded. The proof is given in Section 3.1. We also provide two applications of the inequality. The first one is to prove that the surface given in Corollary 2.6 is the unique complete minimal Lagrangian surface of the total curvature \(-2\pi\) in \( \mathbb{C}^2 \). The second one is to show that the precise maximal number of exceptional values of the Gauss map of a complete minimal Lagrangian surface with finite total curvature in \( \mathbb{C}^2 \) is “2” (Theorem 2.7). Furthermore, by refining the Mo-Osserman argument in [22], we prove that if the Gauss map of a complete minimal Lagrangian surface in \( \mathbb{C}^2 \) takes on “4” distinct values only a finite number of times, then the surface has finite total curvature (Theorem 2.9). The proof is given in Section 3.2. As a corollary of the result, we obtain that if the Gauss map of a complete minimal Lagrangian surface in \( \mathbb{C}^2 \) which is not a Lagrangian plane omits “3” values, then it takes all other values infinitely many times (Corollary 2.10). This result is a sharpening of Corollary 2.2.

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2. Main results

We first give a curvature bound for a minimal Lagrangian surface \( M \) in \( \mathbb{C}^2 \). Set \( \omega := hdz \). Then the induced metric \( ds^2 \) on \( M \) by \( f \) is rewritten by

\[
(7) \quad ds^2 = |h|^2(1+|g|^2)|dz|^2 = (1+|g|^2)|\omega|^2.
\]

Applying Theorem 2.1 in [16] to the metric \( ds^2 \), we can get the following theorem.

**Theorem 2.1.** Let \( M \) be a minimal Lagrangian surface in \( \mathbb{C}^2 \) whose Gauss map \( g : M \to \mathbb{C} \cup \{\infty\} \) omits more than three distinct values. Then there exists a positive constant \( C \) depending on the set of exceptional values, but not \( M \), such that for all \( p \in M \) we have

\[
|K_{ds^2}(p)|^{1/2} \leq \frac{C}{d(p)},
\]

where \( K_{ds^2}(p) \) stands for the Gaussian curvature of \( M \) at \( p \) and \( d(p) \) stands for the geodesic distance from \( p \) to the boundary of \( M \).
Combining this with Proposition 2.4 in [16], we give the precise maximal number of exceptional values of the Gauss map of the case where \( M \) is complete.

**Corollary 2.2.** The Gauss map of a complete minimal Lagrangian surface in \( \mathbb{C}^2 \) which is not a Lagrangian plane can omit at most three values. Moreover, let \( E \) be an arbitrary set of \( q \) points on \( \mathbb{C} \cup \{ \infty \} \), where \( q \leq 3 \). Then there exists a complete minimal Lagrangian surface in \( \mathbb{C}^2 \) whose image under the Gauss map omits precisely the set \( E \).

Note that an example of the case where \( E = \emptyset \) (i.e., \( q = 0 \)) is given by the following:

\[
F = (F_1, F_2): \mathbb{C} \to \mathbb{C}^2, \quad (F_1, F_2) = \left( \frac{z^3}{3} + z + c_1, \frac{z^2}{2} + c_2 \right) \quad (c_1, c_2 \in \mathbb{C}).
\]

Indeed, the Weierstrass data is \((hdz, g) = (zdz, (z + 1)/z)\) and the resulting surface is a complete minimal Lagrangian surface in \( \mathbb{C}^2 \) whose Gauss map is surjective. See [17] for more details on other function-theoretic properties (e.g., ramification theorem, unicity theorem) of the Gauss map for this class.

Next, we consider the case where \( M \) is complete and has finite total curvature. From (11), the Gaussian curvature \( K_{ds^2} \) is nonpositive, and the total curvature is given by

\[
\int_M K_{ds^2} dA = -\frac{1}{2} \int_M \left( \frac{2|g'|}{1 + |g|^2} \right)^2 du \wedge dv,
\]

where \( z = u + iv \).

**Proposition 2.3.** Let \( M \) be a complete minimal Lagrangian surface with finite total curvature in \( \mathbb{C}^2 \). Then it satisfies

(a) \( M \) is conformally equivalent to \( \overline{M}_\gamma \setminus \{p_1, \ldots, p_k\} \), where \( \overline{M}_\gamma \) is a compact Riemann surface of genus \( \gamma \), and \( p_1, \ldots, p_k \in \overline{M}_\gamma \) ([15]),

(b) The Weierstrass data \((hdz, g)\) is extended meromorphically to \( \overline{M}_\gamma \) ([25]).

We call the points \( \{p_1, \ldots, p_k\} \) ends of \( M \). Thus we easily show that the total curvature of a complete minimal Lagrangian surface in \( \mathbb{C}^2 \) can only take the values \(-2\pi m, m \) a non-negative integer, or \(-\infty \).

It is well-known theorem of Cohn-Vossen that if \( M \) is a complete Riemannian 2-manifold with finite total curvature and finite Euler characteristic \( \chi(M) \), then

\[
\int_M K_{ds^2} dA \leq 2\pi \chi(M).
\]

For complete minimal Lagrangian surfaces in \( \mathbb{C}^2 \), a stronger result can be made.

**Theorem 2.4.** Let \( M \) be a complete minimal Lagrangian surface with finite total curvature in \( \mathbb{C}^2 \). Then the following inequality holds:

\[
\int_M K_{ds^2} dA \leq 2\pi (\chi(M) - k),
\]
where $dA$ is the area element of $M$, $\chi(M)$ is the Euler characteristic of $M$ and $k$ is the number of ends of $M$. The equality holds if and only if all ends are embedded.

Note that for complete minimal Lagrangian surfaces in $\mathbb{C}^2$ the Cohn-Vossen inequality can never be an equality because $k \geq 1$. We call inequality (9) an Osserman-type inequality for complete minimal Lagrangian surfaces in $\mathbb{C}^2$.

**Remark 2.5.** When $M$ is conformally equivalent to $\overline{M}_\gamma \setminus \{p_1, \ldots, p_k\}$, inequality (9) can be rewritten by

$$d \geq 2(\gamma - 1 + k),$$

where $d$ is the degree of $g$ considered as a map of $\overline{M}_\gamma$. Note that, for minimal surfaces in $\mathbb{R}^3$, inequality (9) is rewritten by

$$d \geq \gamma - 1 + k.$$

Theorem 2.4 is useful in showing the global properties of complete minimal Lagrangian surfaces with finite total curvature in $\mathbb{C}^2$. In regard to the Osserman inequality for other classes of surfaces in space forms, for example, see [3], [7], [19], [21], [24], [27], [28] and [26]. Here we give two applications. The first one is to characterize the following example as the unique complete minimal Lagrangian surface of the total curvature $-2\pi$ in $\mathbb{C}^2$. The result corresponds to the fact that the Enneper surface and the catenoid are the unique complete minimal surfaces in $\mathbb{R}^3$ of the total curvature $-4\pi$ [24, Theorem 3.3].

**Corollary 2.6.** A complete minimal Lagrangian surface in $\mathbb{C}^2$ whose total curvature is $-2\pi$ must have the following data:

$$F = (F_1, F_2) : \mathbb{C} \to \mathbb{C}^2, \quad (F_1, F_2) = (az^2 + b, 2az + c) \quad (a, b, c \in \mathbb{C}).$$

**Proof.** Since $d = 1$, $g : M \to S^2(= \mathbb{C} \cup \{\infty\})$ is a conformal diffeomorphism, and $M$ is conformally equivalent to $S^2 \setminus \{p_1, \ldots, p_k\}$. However, by (10), $2(k - 1) \leq 1$. Thus $k = 1$. If $k = 1$, then $M$ is conformally equivalent to the complex plane $\mathbb{C}$. In fact, we may identify $M$ with $g(M) \subset \mathbb{C} \cup \{\infty\}$ and assume $(\mathbb{C} \cup \{\infty\}) \setminus g(M) = \{\infty\}$ after a suitable Möbius transformation. Then $g(z) = z$ and the holomorphic 1-form $h \, dz$ has no zeros on $\mathbb{C}$ because the metric $ds^2$ is nondegenerate. Since $h \, dz$ is extended meromorphically to $S^2 = \mathbb{C} \cup \{\infty\}$, $h(z)$ is a polynomial in $z$ and therefore constant. Set that $h = 2a$ ($a \in \mathbb{C}$). Then we have

$$F_1 = \int gh \, dz = az^2 + b, \quad F_2 = \int h \, dz = 2az + c.$$  \hfill $\Box$

The second one is to give the precise maximal number of exceptional values of the Gauss map of a complete minimal Lagrangian surface with finite total curvature in $\mathbb{C}^2$. 


THEOREM 2.7. Let $M = \overline{M}_\gamma \setminus \{p_1, \ldots, p_k\}$ be a complete minimal Lagrangian surface with finite total curvature in $\mathbb{C}^2$ and $g : M \to \mathbb{C} \cup \{\infty\}$ the Gauss map. Let $d$ be the degree of $g$ considered as a map of $\overline{M}_\gamma$. Assume that $M$ is not a Lagrangian plane. Then we have

$$D_g := \sharp((\mathbb{C} \cup \{\infty\})\setminus g(M)) \leq 2 + \frac{2}{R}, \quad \frac{1}{R} = \frac{\gamma - 1 + k/2}{d} < \frac{1}{2}. \quad (12)$$

In particular, the Gauss map can omit at most two values.

REMARK 2.8. In the case of a complete minimal surface with finite total curvature in $\mathbb{R}^3$, Osserman [24] proved that the Gauss map can omit at most three values. However, at present, there exists no example whose Gauss map omits precise three values (see [6], [18] and [29] for details).

**Proof.** Assume that $g$ omits $D_g$ values. Let $n_0$ be the sum of the branching orders at the image of these exceptional values. Then we have

$$k \geq dD_g - n_0.$$ 

Let $n_g$ be the total branching order of $g$ on $\overline{M}_\gamma$. Then applying the Riemann-Hurwitz formula to the meromorphic function $g$ on $\overline{M}_\gamma$, we have

$$n_g = 2(d + \gamma - 1). \quad (13)$$

Hence we have

$$D_g \leq \frac{n_0 + k}{d} \leq \frac{n_0 + k}{d} = 2 + \frac{2}{R}. \quad (14)$$

On the other hand, the inequality (10) implies

$$d \geq 2\gamma - 2 + 2k > 2\gamma - 2 + k = 2(\gamma - 1 + k/2),$$

and hence

$$\frac{1}{R} = \frac{\gamma - 1 + k/2}{d} < \frac{1}{2}. \quad \square$$

The number two in Theorem 2.7 is optimal. A famous example of a complete minimal Lagrangian surface with finite total curvature in $\mathbb{C}^2$ whose Gauss map omits exactly two values is the Lagrangian catenoid ([4]). Indeed, the surface is defined as

$$F = (F_1, F_2) : \mathbb{C} \setminus \{0\} \to \mathbb{C}^2, \quad (F_1, F_2) = \left(z, \frac{1}{z}\right).$$

The Weierstrass data is given by $(hdz, g) = (-dz/z^2, -z^2)$ and the Gauss map omits exactly two values, 0 and $\infty$. Note that Castro and Urbano in [4] proved that the only Lagrangian catenoid is a minimal Lagrangian surface with circle symmetry in $\mathbb{C}^2$.

Finally, by refining the Mo-Osserman argument in [22], we get the following:
**Theorem 2.9.** Let $M$ be a complete minimal Lagrangian surface in $\mathbb{C}^2$. If the Gauss map takes on four values only a finite number of times, then $M$ has finite total curvature.

On the other hand, by Theorem 2.7, a complete minimal Lagrangian surface with finite total curvature whose Gauss map omits more than two values must be a Lagrangian plane. One consequence of Theorem 2.9 is therefore:

**Corollary 2.10.** Let $M$ be a complete minimal Lagrangian surface in $\mathbb{C}^2$ which is not a Lagrangian plane. If the Gauss map of $M$ omits three values, then it takes all other values infinitely many times.

3. Proof of Main theorems

3.1. **Proof of Theorem 2.4.** We first prove inequality (9) (or (10)). Since the total curvature is finite, we may assume that $M_\gamma \setminus \{p_1, \ldots, p_k\}$, where $M_\gamma$ is a compact Riemann surface of genus $\gamma$, and $p_1, \ldots, p_k \in M_\gamma$. By a rotation of the surface, we may assume that the Gauss map $g$ has neither zero nor pole at $p_j$ and that the zeros and the poles of $g$ are simple. The simple poles of $g$ coincide with the simple zeros of $hdz$ because the metric $ds^2$ is nondegenerate. By the completeness of $M$, $hdz$ has poles at each end $p_j$ ([20], [25] Lemma 9.6]). Moreover, since the surface is well-defined on $M$, $hdz$ has poles of order $\mu_j \geq 2$ at $p_j$. Indeed, suppose that $\mu_j = 1$. Then we can expand the function $h(z)$ about each end $p_j$ as

$$h(z) = \frac{c_j^{1}}{z - p_j} + \sum_{n=0}^{\infty} c_n^j z^n, \quad c_1^j \neq 0.$$ 

We also are able to expand the function $g(z)h(z)$ about each end $p_j$ as

$$g(z)h(z) = \frac{d_j^{1}}{z - p_j} + \sum_{n=0}^{\infty} d_n^j z^n$$

because $g$ is holomorphic around $p_j$. By [5], if the surface is well-defined on $M$, then we have

$$\int_{\gamma} ghdz - i \int_{\gamma} hdz = 0, \quad \int_{\gamma} hdz + i \int_{\gamma} ghdz = 0$$

for any loop $\gamma \in H_1(M, \mathbb{Z})$. Thus we get that

$$2\pi i (d_j^{1} + i c_j^{1}) = 0, \quad 2\pi i (c_j^{1} - i d_j^{1}) = 0,$$

and therefore $c_j^{1} = d_j^{1} = 0$. Hence $\mu_j = 0$, contrary to our assumption. Applying the Riemann-Roch theorem to $hdz$ on $M_\gamma$, we obtain that

$$d - \sum_{j=1}^{k} \mu_j = 2\gamma - 2,$$
where $d$ denotes the degree of $g$ considered as a map of $\overline{M}_g$. Thus we have

\begin{equation}
(15) \quad d = 2\gamma - 2 + \sum_{j=1}^{k} \mu_j \geq 2\gamma - 2 + 2k = -\chi(M) + k
\end{equation}

and

$$\int_M K ds dz dA = -2\pi d \leq 2\pi (\chi(M) - k).$$

Next we prove that the equality of (15) holds if and only if all ends are embedded. By Proposition 2.3, $F_1$ and $F_2$ are extended meromorphically to $\overline{M}_g$. Thus there exists a neighborhood $U_j$ of each end $p_j$ such that $F_1$ and $F_2$ can be written as

$$F_1(z) = (z - p_j)^{-m_j} F_1(z), \quad F_2(z) = (z - p_j)^{-n_j} F_2(z),$$

where $F_1(p_j), F_2(p_j) \neq 0$, $m_j, n_j \in \mathbb{Z}$. If we suppose that $m_j \geq n_j$, then we have

$$F_1 - i\overline{F_2} = (z - p_j)^{-m_j} (F_1(z) - \overline{i(z - p_j)^{m_j-n_j} e^{i\theta_j(z)} F_2(z)}),$$

$$F_2 + i\overline{F_1} = (z - p_j)^{-m_j} (\overline{F_1(z)} + (z - p_j)^{m_j-n_j} e^{-i\theta_j(z)} \overline{F_2(z)}),$$

where $\theta_j(z) \in \mathbb{R}$. If we suppose that $m_j \leq n_j$, then we have

$$F_1 - i\overline{F_2} = (z - p_j)^{-n_j} (-i\overline{F_2(z)} + (z - p_j)^{n_j-m_j} e^{-i\theta_j(z)} \overline{F_1(z)}),$$

$$F_2 + i\overline{F_1} = (z - p_j)^{-n_j} (\overline{F_2(z)} + i(z - p_j)^{n_j-m_j} e^{i\theta_j(z)} \overline{F_1(z)}),$$

where $\eta_j(z) \in \mathbb{R}$. Hence the surface $f$ twists $c_j$-times along any loop around $p_j$, where $c_j := \max\{m_j, n_j\} \geq 1$. In particular, we get that an end $p$ of $f$ is embedded if and only if $F_1$ and $F_2$ have at most a single pole of order 2 at each end $p_j$, that is, the holomorphic 1-form $hdz$ has a pole of order 2 at $p_j$, that is, each end is embedded. \qed

3.2. Proof of Theorem 2.9. We first recall the notion of chordal distance between two distinct values in the Riemann sphere $\mathbb{C} \cup \{\infty\}$. For two distinct values $\alpha, \beta \in \mathbb{C} \cup \{\infty\}$, we set

$$|\alpha, \beta| := \frac{|\alpha - \beta|}{\sqrt{1 + |\alpha|^2} \sqrt{1 + |\beta|^2}},$$

if $\alpha \neq \infty$ and $\beta \neq \infty$, and $|\alpha, \infty| = |\infty, \alpha| := 1/\sqrt{1 + |\alpha|^2}$. We note that, if we take $v_1, v_2 \in S^2$ with $\alpha = \varpi(v_1)$ and $\beta = \varpi(v_2)$, we have that $|\alpha, \beta|$ is a half of the chordal distance between $v_1$ and $v_2$, where $\varpi$ denotes the stereographic projection of the 2-sphere $S^2$ onto $\mathbb{C} \cup \{\infty\}$.

Before proceeding to the proof of Theorem 2.9, we recall two function-theoretic lemmas.
Lemma 3.1. \[10\] (8.12) in page 136 Let \( g \) be a nonconstant meromorphic function on \( \Delta_R = \{ z \in \mathbb{C}; |z| < R \} \) \((0 < R \leq +\infty)\) which omits \( q \) values \( \alpha_1, \cdots, \alpha_q \). If \( q > 2 \), then for each positive \( \eta \) with \( \eta < (q-2)/q \), then there exists a positive constant \( C' \), depending on \( q \) and \( L := \min_{i<j} |\alpha_i, \alpha_j| \), such that

\[
\frac{|g'_z|}{(1+|g|^2) \prod_{j=1}^q |g, \alpha_j|^{1-\eta}} \leq C' \frac{R}{R^2 - |z|^2}.
\]

Lemma 3.2. \[9\] Lemma 1.6.7 Let \( \sigma^2 \) be a conformal flat metric on an open Riemann surface \( \Sigma \). Then, for each point \( p \in \Sigma \), there exists a local diffeomorphism \( \Phi \) of a disk \( \Delta_R = \{ z \in \mathbb{C}; |z| < R \} \) \((0 < R \leq +\infty)\) onto an open neighborhood of \( p \) with \( \Phi(0) = p \) such that \( \Phi \) is a local isometry, that is, the pull-back \( \Phi^*(\sigma^2) \) is equal to the standard Euclidean metric \( ds^2_{Euc} \) on \( \Delta_R \) and, for a point \( a_0 \) with \( |a_0| = 1 \), the \( \Phi \)-image \( \Gamma_{a_0} \) of the curve \( L_{a_0} = \{ w := a_0 s; 0 < s < R \} \) is divergent in \( \Sigma \).

Proof of Theorem 2.9. Suppose that the Gauss map \( g \) attains four distinct values \( \alpha_1, \cdots, \alpha_4 \) only a finite number of times. We may assume that \( \alpha_4 = \infty \) after a suitable Möbius transformation. Then the assumption of the theorem implies that outside a compact subset \( D \) in \( M \), \( g \) is holomorphic and omits three values \( \alpha_1, \alpha_2, \alpha_3 \). We choose a positive number \( \eta \) with \( 0 < \eta < 1/4 \) and set \( \lambda := 1/(2-4\eta) \). Now we define a new metric

\[
d\sigma^2 = |h|^{-2/\lambda} \left( \frac{1}{|g'_z|^2} \prod_{j=1}^q \left( \frac{|g - \alpha_j|}{\sqrt{1 + |\alpha_j|^2}} \right)^{1-\eta} \right)^{2/\lambda} |dz|^2
\]

on the set \( M' := \{ p \in M \setminus D; g'_z(p) \neq 0 \} \). Since \( h \) and \( g \) are holomorphic, \( d\sigma^2 \) is flat and can be smoothly extended over \( D \). We thus obtain the pseudo-metric \( d\sigma^2 \) on \( M'' := M' \cup D \) which is flat outside the compact set \( D \).

We prove that \( d\sigma^2 \) is complete on \( M'' \). This will be proved by contradiction. If \( d\sigma^2 \) is not complete on \( M'' \), then there exists a divergent curve \( \gamma(t) \) on \( M'' \) with finite length. By removing an initial segment, if necessary, we may assume that there exists a positive distance \( d \) between the curve \( \gamma \) and the compact set \( D \). Thus \( \gamma: [0, 1) \to M' \) and, since \( \gamma \) is divergent on \( M'' \) with finite length, the curve \( \gamma(t) \) tends to either a point where \( g'_z = 0 \) or else the boundary of \( M \) as \( t \to 1 \). However, the former case cannot occur. The reason is as follows. We assume that \( \gamma(t) \) tends to a point \( p_0 \) where \( g'_z = 0 \) as \( t \to 1 \). Taking a local complex coordinate \( \zeta := g'_z \) in a neighborhood of \( p_0 \) with \( \zeta(p_0) = 0 \), we can write

\[
d\sigma^2 = |\zeta|^{-2\lambda/(1-\lambda)} w |d\zeta|^2
\]

for some positive smooth function \( w \). Since \( \lambda/(1-\lambda) > 1 \), we have

\[
\int_{\gamma} d\sigma \geq \tilde{C} \int_{\gamma} \frac{|d\zeta|}{|\zeta|^{\lambda/(1-\lambda)}} = +\infty,
\]

where \( \tilde{C} \) is some positive number. It contradicts that \( \gamma \) has finite length. We conclude that \( \gamma(t) \) must tend to the boundary of \( M \) when \( t \to 1 \).
Then we choose a number $t_0 \in [0, 1)$ such that the length of $\gamma([t_0, 1))$ with respect to the metric $d\sigma^2$ is less than $d/3$. Since $d\sigma^2$ is flat, by Lemma 3.2 there exists a local isometry $\Phi$ satisfying $\Phi(0) = \gamma(t_0)$ from a disk $\Delta_R = \{ w \in \mathbb{C} ; |w| < R \}$ ($0 < R \leq +\infty$) with the standard Euclidean metric $ds^2_E$ onto an open neighborhood of the point $\gamma(t_0)$ with the metric $d\sigma^2$, such that, for a point $a_0$ with $|a_0| = 1$, the $\Phi$-image $\Gamma_{a_0}$ of the line segment $L_{a_0} := \{ w := a_0 s : 0 < s < R \}$ is divergent in $M'$. Since the length of $\gamma([t_0, 1))$ is less than $d/3$ and $\gamma$ is divergent curve in $M$, we have $R \leq d/3$. Hence the image $\Phi(\Delta_R)$ must be bounded away from $D$ by a distance of at least $2d/3$. Moreover, since the zeros of $g'_z$ have been shown to be infinitely far away in the metric, $\Gamma_{a_0}$ must actually go to the boundary of $M$.

For brevity, we denote the function $g \circ \Phi$ on $\Delta_R$ by $g$ in the following. Since $\Phi^*d\sigma^2 = |dz|^2$, we get by (17) that

$$|h| = \left( |g'_z|^3 \prod_{j=1}^3 \left( \frac{\sqrt{1 + |\alpha_j|^2}}{|g - \alpha_j|} \right)^{1-\eta} \right)^\lambda. \quad (18)$$

By Lemma 3.1 we have

$$\Phi^*ds = |h|\sqrt{1 + |g|^2}dz = \left( |g'_z|(1 + |g|^2)^{1/2} \prod_{j=1}^3 \left( \frac{\sqrt{1 + |\alpha_j|^2}}{|g - \alpha_j|} \right)^{1-\eta} \right)^\lambda |dz|$$

$$= \left( \frac{|g'_z|}{(1 + |g|^2) \prod_{j=1}^3 |g, \alpha_j|^{1-\eta}} \right)^\lambda |dz| \leq (C')^\lambda \left( \frac{R}{R^2 - |z|^2} \right)^\lambda |dz|.$$ 

Then, for the length $L$ of $\Gamma_{a_0}$ with respect to the metric $ds^2$, we obtain

$$L = \int_{\Gamma_{a_0}} ds = \int_{L_{a_0}} \Phi^*ds \leq (C')^\lambda \int_{L_{a_0}} \left( \frac{R}{R^2 - |z|^2} \right)^\lambda |dz| \leq (C')^\lambda \frac{R^{1-\lambda}}{1-\lambda} < +\infty$$

because $1/2 < \lambda < 1$. In particular, the length of $L$ is finite. We thus show that if $d\sigma^2$ were not complete on $M''$, then we could find a divergent curve on $M$ with finite length in the original metric $ds^2$, that is, $M$ would not be complete. However it contradicts the assumption. Hence we conclude that $d\sigma^2$ is complete on $M''$.

Since $d\sigma^2$ is also flat outside a compact set, the total curvature of $d\sigma^2$ is finite. Then, by the Huber theorem [15], $M''$ is finitely connected. We thus show that $g'_z$ can have only a finite number of zeros and $M$ is finitely connected. Moreover, by [23, Theorem 2.1], each annular end of $M''$, hence of $M$, is conformally equivalent to a punctured disk. Therefore the Riemann surface $M$ must be conformally equivalent to $\overline{M}\setminus\{p_1, \cdots, p_k\}$, where $\overline{M}$ is a closed Riemann surface and $p_j \in \overline{M}$ ($j = 1, \cdots, k$). In a neighborhood of each of $p_j$, $g$ is holomorphic and omits three values. By the Picard great theorem, $g$ cannot have an
essential singularity, but must have at most a pole. Hence $g$ extends to a meromorphic function on $\overline{M}$. Then we have

$$\int_M K ds^2 dA = -\frac{1}{2} \int_M \left( \frac{2|g'_z|}{1 + |g|^2} \right)^2 du \wedge dv = -2\pi d,$$

where $z = u + iv$ and $d$ denotes the degree of $g$ considered as a map on $\overline{M}$. In particular, $M$ must have finite total curvature. □

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