HILBERT-POINCARÉ SERIES AND GORENSTEIN PROPERTY FOR
CLOSED PATH POLYOMINOES

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Abstract. In this paper we compute the reduced Poincaré-Hilbert series of the coordinate ring attached to a closed path \( P \) having no zig-zag walks, as a combination of the Poincaré-Hilbert series of convenient simple thin polyominoes. As a consequence we find the Krull dimension and the regularity of \( K[P] \) and we prove that the \( h \)-polynomial is exactly the rook polynomial of \( P \). Finally we characterize the Gorenstein prime closed paths using the \( S \)-property.

1. Introduction

A polyomino is a finite collection of unitary squares joined edge by edge. In 2012 A.A. Qureshi defined a polyomino ideal attached to a polyomino \( P \) as the ideal generated by all inner 2-minors of \( P \) in the polynomial ring over \( K \) in the variables \( x_v \) where \( v \) is a vertex of \( P \). Such an ideal is called the polyomino ideal of \( P \) and it is denoted by \( I_P \). For more details see [17].

The study of the main algebraic properties of the quotient ring \( K[P] = S/I_P \) depending on the shape of \( P \) has become an exciting line of research. For instance, several mathematicians have studied the primality of \( I_P \) and when \( K[P] \) is a normal Cohen-Macaulay domain. For some references to these results we mention [6], [4], [5], [11], [12], [13], [15], [16], [19], [21]. We mention also other references, particularly inspiring for this work, for which we provide now some details. In [1] the author classifies all convex polyominoes whose coordinate rings are Gorenstein and compute the regularity of the coordinate ring of any stack polyomino in terms of the smallest interval which contains its vertices. In [8] the authors give a new combinatorial interpretation of the regularity of the coordinate ring attached to an \( L \)-convex polyomino, as the rook number of \( P \), that is the maximum number of rooks which can be arranged in \( P \) in non-attacking positions. In [20] it is showed that if \( P \) is a simple thin polyomino, which is a polyomino not containing the square tetromino, then the \( h \)-polynomial \( h(t) \) of \( K[P] \) is the rook-polynomial \( r_P(t) = \sum_{i=0}^{n} r_i x^i \) of \( P \), whose coefficient \( r_i \) represents the number of distinct possibilities of arranging \( i \) rooks on cells of \( P \) in non attacking positions (with the convention \( r_0 = 1 \)). Gorenstein simple thin polyominoes are also characterized using the \( S \)-property and finally it is conjectured that a polyomino is thin if and only if \( h(t) = r_P(t) \). In this paper we give also a partial support to this conjecture, since we provide an affirmative answer for a particular class of non-simple thin polyominoes, namely closed paths. In [14] it is also discussed this conjecture for a certain class of polyominoes. In a recent paper [18] the authors introduce a particular equivalence relation on the rook complex of a simple polyomino and they conjecture that the number of equivalence classes of \( i \) non-attacking rooks arrangements is exactly the \( i \)-th coefficient of the \( h \)-polynomial in the reduced Poincaré-Hilbert series. Moreover they prove it for the class of parallelogram polyominoes and by a computational method also for all simple polyominoes with rank at most eleven.

The aim of this paper is to study the Poincaré-Hilbert series of certain non-simple polyominoes \( P \), having only one hole, relating them to the Poincaré-Hilbert series of simple thin polyominoes included in \( P \). As a consequence we compute their Krull dimension and finally the regularity.

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and the Gorenstein property of closed path polyominoes. Roughly speaking, a closed path is a sequence of cells, similar to a pearl necklace on a table, in which each pearl corresponds to a cell, producing only one hole. This class of polyominoes is defined in [6], where the authors characterize their primality using the non-existence of zig-zag walks. In Section 2 we introduce the principal definitions and notations, which are fundamental along the paper. In Section 3, considering a particular polyomino \(\mathcal{L}\) defined in the same section and a generic polyomino \(\mathcal{C}\), we define the class of \((\mathcal{L}, \mathcal{C})\)-polyominoes and we provide an explicit formula for the Poincaré-Hilbert series of the related coordinate rings, depending on the Poincaré-Hilbert series of some polyominoes obtained eliminating specific cells. In a particular case we compute also the Krull dimension of the coordinate ring of a polyomino belonging to this class. In Section 4 we assume that \(\mathcal{P}\) is a closed path polyomino and we deal the case in which \(\mathcal{P}\) has no \(L\)-configuration but it contains a ladder of at least three steps. In fact the case in which \(\mathcal{P}\) has an \(L\)-configuration is a particular case of \((\mathcal{L}, \mathcal{C})\)-polyomino, where \(\mathcal{C}\) is a simple path, so by a known property we fulfill the class of closed paths without zig-zag walks for what concern the study of the Poincaré-Hilbert series. In order to reach our aim we use, in a particular case, some considerations on the initial ideal of \(I_{\mathcal{P}}\) without zig-zag walks for what concern the study of the Poincaré-Hilbert series. In order to reach our aim we use, in a particular case, some considerations on the initial ideal of \(I_{\mathcal{P}}\) with respect to some monomial orders, defined in [6]. In Section 5 we prove that the \(h\)-polynomial of \(K[\mathcal{P}]\), where \(\mathcal{P}\) is a prime closed path polyomino, is the rook polynomial of \(\mathcal{P}\), obtaining as a consequence the regularity and the Krull dimension of \(K[\mathcal{P}]\). Finally we characterize all Gorenstein prime closed paths using the \(S\)-property. We conclude with some considerations about the Cohen-Macaulayness and Krull dimension of weakly closed paths (see [5]) and with some open questions.

2. Polyominoes and polyomino ideals

Let \((i, j), (k, l) \in \mathbb{Z}^2\). We say that \((i, j) \leq (k, l)\) if \(i \leq k\) and \(j \leq l\). Consider \(a = (i, j)\) and \(b = (k, l)\) in \(\mathbb{Z}^2\) with \(a \leq b\). The set \([a, b] = \{(m, n) \in \mathbb{Z}^2 : i \leq m \leq k, j \leq n \leq l\}\) is called an interval of \(\mathbb{Z}^2\). In addition, if \(i < k\) and \(j < l\) then \([a, b]\) is a proper interval. In such a case we say \(a, b\) the diagonal corners of \([a, b]\) and \(c = (i, l), d = (k, j)\) the anti-diagonal corners of \([a, b]\). If \(j = l\) (or \(i = k\)) then \(a\) and \(b\) are in horizontal (or vertical) position. We denote by \([a, b]\) the set \(\{(m, n) \in \mathbb{Z}^2 : i < m < k, j < n < l\}\). A proper interval \(C = [a, b]\) with \(b = a + (1, 1)\) is called a cell of \(\mathbb{Z}^2\); moreover, the elements \(a, b, c, d\) are called respectively the lower left, upper right, upper left and lower right corners of \(C\). The sets \([a, c]\), \([c, b]\), \([b, d]\) and \([a, d]\) are the edges of \(C\). We put \(V(C) = \{a, b, c, d\}\) and \(E(C) = \{\{a, c\}, \{c, b\}, \{b, d\}, \{a, d\}\}\). Let \(\mathcal{S}\) be a non-empty collection of cells in \(\mathbb{Z}^2\). The set of the vertices and of the edges of \(\mathcal{S}\) are respectively \(V(\mathcal{S}) = \bigcup_{C \in \mathcal{S}} V(C)\) and \(E(\mathcal{S}) = \bigcup_{C \in \mathcal{S}} E(C)\), while rank \(\mathcal{S}\) is the number of cells belonging to \(\mathcal{S}\). If \(C\) and \(D\) are two distinct cells of \(\mathcal{S}\), then a walk from \(C\) to \(D\) in \(\mathcal{S}\) is a sequence \(C : C = C_1, \ldots, C_m = D\) of cells of \(\mathbb{Z}^2\) such that \(C_i \cap C_{i+1}\) is an edge of \(C_i\) and \(C_{i+1}\) for \(i = 1, \ldots, m - 1\). In addition, if \(C_i \neq C_j\) for all \(i \neq j\), then \(\mathcal{C}\) is called a path from \(C\) to \(D\). We say that \(C\) and \(D\) are connected in \(\mathcal{S}\) if there exists a path of cells in \(\mathcal{S}\) from \(C\) to \(D\). A polyomino \(\mathcal{P}\) is a non-empty, finite collection of cells in \(\mathbb{Z}^2\) where any two cells of \(\mathcal{P}\) are connected in \(\mathcal{P}\). For instance, see Figure 1.

![Figure 1. A polyomino.](image)

We say that a polyomino \(\mathcal{P}\) is simple if for any two cells \(C\) and \(D\) not in \(\mathcal{P}\) there exists a path of cells not in \(\mathcal{P}\) from \(C\) to \(D\). A finite collection of cells \(\mathcal{H}\) not in \(\mathcal{P}\) is a hole of \(\mathcal{P}\) if any two cells of \(\mathcal{H}\) are connected in \(\mathcal{H}\) and \(\mathcal{H}\) is maximal with respect to set inclusion. For example, the
polyomino in Figure 1 is not simple with an hole. Obviously, each hole of \( P \) is a simple polyomino and \( P \) is simple if and only if it has no any hole.

Consider two cells \( A \) and \( B \) of \( \mathbb{Z}^2 \) with \( a = (i, j) \) and \( b = (k, l) \) as the lower left corners of \( A \) and \( B \) and \( a \leq b \). A cell interval \([A, B]\) is the set of the cells of \( \mathbb{Z}^2 \) with lower left corner \((r, s)\) such that \( i \leq r \leq k \) and \( j \leq s \leq l \). If \((i,j)\) and \((k,l)\) are in horizontal (or vertical) position, we say that the cells \( A \) and \( B \) are in horizontal (or vertical) position.

Let \( P \) be a polyomino. Consider two cells \( A \) and \( B \) of \( P \) in vertical or horizontal position. The cell interval \([A, B]\), containing \( n > 1 \) cells, is called a block of \( P \) of rank \( n \) if all cells of \([A, B]\) belong to \( P \). The cells \( A \) and \( B \) are called extremal cells of \([A, B]\). Moreover, a block \( B \) of \( P \) is maximal if there does not exist any block of \( P \) which contains properly \( B \). It is clear that an interval of \( \mathbb{Z}^2 \) identifies a cell interval of \( \mathbb{Z}^2 \) and vice versa, hence we can associated to an interval \( I \) of \( \mathbb{Z}^2 \) the corresponding cell interval denoted by \( P_I \). A proper interval \([a, b]\) is called an inner interval of \( P \) if all cells of \( P_{[a,b]} \) belong to \( P \). We denote by \( I(P) \) the set of all inner intervals of \( P \). An interval \([a, b]\) with \( a = (i, j) \), \( b = (k, l) \) and \( i < k \) is called a horizontal edge interval of \( P \) if the sets \( \{(\ell, j), (\ell + 1, j)\} \) are edges of cells of \( P \) for all \( \ell = i, \ldots, k - 1 \). In addition, if \( \{(i - 1, j), (i, j)\} \) and \( \{(k, j), (k + 1, j)\} \) do not belong to \( E(P) \), then \([a, b]\) is called a maximal horizontal edge interval of \( P \). We define similarly a vertical edge interval and a maximal vertical edge interval.

We follow [15] and we call a zig-zag walk of \( P \) a sequence \( \mathcal{W} : I_1, \ldots, I_\ell \) of distinct inner intervals of \( P \) where, for all \( i = 1, \ldots, \ell \), the interval \( I_i \) has either diagonal corners \( v_i, z_i \) and anti-diagonal corners \( u_i, v_{i+1} \) or anti-diagonal corners \( v_i, z_i \) and diagonal corners \( u_i, v_{i+1} \), such that:

1. \( I_1 \cap I_\ell = \{v_1 = v_{\ell+1}\} \) and \( I_i \cap I_{i+1} = \{v_{i+1}\} \), for all \( i = 1, \ldots, \ell - 1 \);
2. \( v_i \) and \( v_{i+1} \) are on the same edge interval of \( P \), for all \( i = 1, \ldots, \ell \);
3. for all \( i, j \in \{1, \ldots, \ell\} \) with \( i \neq j \), there exists no inner interval \( J \) of \( P \) such that \( z_i, z_j \) belong to \( J \).

In according to [4], we recall the definition of a closed path polyomino, and the configuration of cells characterizing its primality. We say that a polyomino \( P \) is a closed path if it is a sequence of cells \( A_1, \ldots, A_n, A_{n+1}, n > 5 \), such that:

1. \( A_1 = A_{n+1} \);
2. \( A_i \cap A_{i+1} \) is a common edge, for all \( i = 1, \ldots, n \);
3. \( A_i \neq A_j \), for all \( i \neq j \) and \( i, j \in \{1, \ldots, n\} \);
4. For all \( i \in \{1, \ldots, n\} \) and for all \( j \notin \{i - 2, i - 1, i, i + 1, i + 2\} \) then \( V(A_i) \cap V(A_j) = \emptyset \), where \( A_{-1} = A_{n-1} \), \( A_0 = A_n \), \( A_{n+1} = A_1 \) and \( A_{n+2} = A_2 \).

A path of five cells \( C_1, C_2, C_3, C_4, C_5 \) of \( P \) is called an L-configuration if the two sequences \( C_1, C_2, C_3 \) and \( C_3, C_4, C_5 \) go in two orthogonal directions. A set \( B = \{B_i\}_{i=1, \ldots, n} \) of maximal horizontal (or vertical) blocks of rank at least two, with \( V(B_i) \cap V(B_{i+1}) = \{a_i, b_i\} \) and \( a_i \neq b_i \) for all \( i = 1, \ldots, n - 1 \), is called a ladder of \( n \) steps if \( a_i, b_i \) is not on the same edge interval of \( [a_{i+1}, b_{i+1}] \) for all \( i = 1, \ldots, n - 2 \). For instance, in Figure 2(A) there is a closed path having an L-configuration and a ladder of three steps. We recall that a closed path has no zig-zag walks if and only if it contains an L-configuration or a ladder of at least three steps (see [4, Section 6]).

With reference to [5], a finite non-empty collection of cells \( P \) is called a weakly closed path if it is a path of cells \( A_1, \ldots, A_{n-1}, A_n = A_0 \) with \( n > 6 \) such that:

1. \( |V(A_0) \cap V(A_1)| = 1 \);
2. \( V(A_2) \cap V(A_0) = V(A_{n-1}) \cap V(A_1) = \emptyset \);
3. \( V(A_i) \cap V(A_j) = \emptyset \) for all \( i \in \{1, \ldots, n\} \) and for all \( j \notin \{i - 2, i - 1, i, i + 1, i + 2\} \), where the indices are reduced modulo \( n \).

A finite collection of cells of \( P \), made up of a maximal horizontal (resp. vertical) block \([A, B]\) of \( P \) of length at least two and two distinct cells \( C \) and \( D \) of \( P \), not belonging to \([A, B]\), with
$V(C) \cap V([A, B]) = \{a_1\}$ and $V(D) \cap V([A, B]) = \{a_2, b_2\}$ where $a_2 \neq b_2$, is called a weak ladder if $[a_2, b_2]$ is not on the same maximal horizontal (resp. vertical) edge interval of $\mathcal{P}$ containing $a_1$ (see Figure 2(B)).

**Figure 2.** Examples of a closed path and a weakly closed path.

Let $\mathcal{P}$ be a polyomino. We set $S_\mathcal{P} = K[x_v | v \in V(\mathcal{P})]$, where $K$ is a field. If $[a, b]$ is an inner interval of $\mathcal{P}$, with $a, b$ and $c, d$ respectively diagonal and anti-diagonal corners, then the binomial $x_ax_b - x_cx_d$ is called an inner 2-minor of $\mathcal{P}$. We define $I_\mathcal{P}$ as the ideal in $S_\mathcal{P}$ generated by all the inner 2-minors of $\mathcal{P}$ and we call it the polyomino ideal of $\mathcal{P}$. We set also $K[\mathcal{P}] = S_\mathcal{P}/I_\mathcal{P}$, which is the coordinate ring of $\mathcal{P}$.

We recall some notions on the Hilbert function and the Poincaré-Hilbert series of a graded $K$-algebra $R/I$. Let $R$ be a graded $K$-algebra and $I$ be an homogeneous ideal of $R$. Then $R/I$ has a natural structure of graded $K$-algebra as $\bigoplus_{k \in \mathbb{N}} (R/I)_k$. The numerical function $H_{R/I} : \mathbb{N} \to \mathbb{N}$ with $H_{R/I}(k) = \dim_K (R/I)_k$ is called the Hilbert function of $R/I$. The formal series $HP_{R/I}(t) = \sum_{k \in \mathbb{N}} H_{R/I}(k)t^k$ is called the Poincaré-Hilbert series of $R/I$. It is known by Hilbert-Serre theorem that there exists a polynomial $h(t) \in \mathbb{Z}[t]$ with $h(1) \neq 0$ such that $HP_{R/I}(t) = \frac{h(t)}{(1-t)^d}$, where $d$ is the Krull dimension of $R/I$. Moreover, if $R/I$ is Cohen-Macaualy then $\text{reg}(R/I) = \deg h(t)$. Recall also that if $S = K[x_1, \ldots, x_n]$ then $HP_S(t) = \frac{1}{(1-t)^n}$.

We will use frequently the following well known results (see for instance [24, Chapter 5] for a reference)

**Proposition 2.1.** Let $R$ be a graded $K$-algebra and $I$ be a graded ideal of $R$. Let $q$ be an homogeneous element of $R$ of degree $m$ and let

$$0 \longrightarrow R/(I : q) \longrightarrow R/I \longrightarrow R/(I, q) \longrightarrow 0$$

be a short exact sequence. Then $HP_{R/I}(t) = HP_{R/(I, q)}(t) + t^m HP_{R/(I,q)}(t)$.

**Proposition 2.2.** Let $A$ and $B$ be standard graded $K$-algebras over a field $K$. Then $HP_{A \otimes_K B}(t) = HP_A(t) \cdot HP_B(t)$.

**Remark 2.3.** We will often use also the following elementary fact: if $X$ is a set of indeterminates, $X_1, X_2 \subset X$ form a partition of $X$ into disjoint non empty subsets and $I$ is an ideal of $K[X]$ with $K$ a field, then $K[X]/I \cong K[X_1]/I_1 \otimes_K K[X_2]/I_2$, where $I_j = I \cap K[X_j]$ for $j \in \{1, 2\}$ (mentioned for instance in [2, Section 2.2]).

If $\mathcal{P}$ is a polyomino then we have that $HP_K[\mathcal{P}](t) = \frac{h(t)}{(1-t)^2}$ for some $h(t) \in \mathbb{Z}[t]$ with $h(1) \neq 0$, so we denote $h(t) = h_{K[\mathcal{P}]}(t)$ along the paper. Finally, if $n \in \mathbb{N}$ as usual we denote by $[n]$ the set $\{1, 2, \ldots, n\}$. 
3. **Poincaré-Hilbert series of certain non-simple polyominoes**

In this section we consider a particular class of non simple polyominoes that we introduce in the following definition.

**Definition 3.1.** Let $\mathcal{L}$ be the union of the two cell intervals $[A,A_r]$, consisting of the cells $A, A_1, \ldots, A_r$, and $[A,B_s]$, consisting of the cells $A, B_1, \ldots, B_s$, where $A, A_r$ and $A, B_s$ are respectively in horizontal and vertical position with $r, s \geq 2$. We denote by $a, b$ and $c, d$ respectively the diagonal and anti-diagonal corners of $A$, by $d_i$ and $a_i$ respectively the upper left and upper right corners of $B_i$ for $i \in [s]$ and by $b_j$ and $c_j$ respectively the upper and lower right corners of $A_j$ for $j \in [r]$. Let $\mathcal{C}$ be a polyomino. We say that a polyomino $\mathcal{P}$ is an $(\mathcal{L}, \mathcal{C})$-polyomino if $\mathcal{P} = \mathcal{L} \sqcup \mathcal{C}$ and it is satisfied one and only one of the following four conditions (see also Figure 3):

1. $V(\mathcal{L}) \cap V(\mathcal{C}) = \{a_{s-1}, a_s, b_{r-1}, b_r\}$
2. $V(\mathcal{L}) \cap V(\mathcal{C}) = \{a_{s-1}, a_s, c_{r-1}, c_r\}$
3. $V(\mathcal{L}) \cap V(\mathcal{C}) = \{d_{s-1}, d_s, b_{r-1}, b_r\}$
4. $V(\mathcal{L}) \cap V(\mathcal{C}) = \{d_{s-1}, d_s, c_{r-1}, c_r\}$

![Figure 3](image.jpg)

**Figure 3.** Examples of the different cases of $(\mathcal{L}, \mathcal{C})$-polyominoes

If $\mathcal{P}$ is an $(\mathcal{L}, \mathcal{C})$-polyomino, we define the following related polyominoes that we will use along the paper:

- $\mathcal{P}_1 = \mathcal{P}\setminus[A, A_r]$;
- $\mathcal{P}_2 = \mathcal{P}\setminus[A, B_s]$;
- $\mathcal{P}_3 = \mathcal{P}\setminus([A, A_r] \cup [A, B_s]) = \mathcal{C}$;
- $\mathcal{P}_4 = \mathcal{P}\setminus\{A, A_1, B_1\}$;
- $\mathcal{P}'_1 = \mathcal{P}\setminus[A_1, A_r]$;
- $\mathcal{P}'_2 = \mathcal{P}\setminus[B_1, B_s]$.

**Lemma 3.2.** Let $\mathcal{P}$ be an $(\mathcal{L}, \mathcal{C})$-polyomino. Then $S_\mathcal{P}/(I_\mathcal{P}, x_a, x_d, x_c, x_b) \cong K[\mathcal{P}_4]$.

**Proof.** Observe that $I_\mathcal{P}$ can be written in the following way

$$I_\mathcal{P} = I_{\mathcal{P}_4} + (x_a x_b - x_b x_c) + \sum_{i=1}^r (x_a x_{a_i} - x_c x_{d_i}) + \sum_{i=1}^r (x_d x_{a_i} - x_b x_{d_i}) + \sum_{i=1}^s (x_a x_{b_i} - x_d x_{c_i}) + \sum_{i=1}^r (x_c x_{b_i} - x_b x_{c_i}).$$

It follows that $(I_\mathcal{P}, x_a, x_d, x_c, x_b) = (I_{\mathcal{P}_4}, x_a, x_d, x_c, x_b)$, in particular $S_\mathcal{P}/(I_\mathcal{P}, x_a, x_d, x_c, x_b) = S_{\mathcal{P}_4}/(I_{\mathcal{P}_4}, x_a, x_d, x_c, x_b) \cong S_{\mathcal{P}_4}/I_{\mathcal{P}_4} = K[\mathcal{P}_4]$.

**Proposition 3.3.** Let $\mathcal{P}$ be an $(\mathcal{L}, \mathcal{C})$-polyomino. If $I_\mathcal{P}$ is prime, then $K[\mathcal{P}_i]$ and $K[\mathcal{P}_j]$ are domains for $i \in [4]$ and $j \in \{1, 2\}$. 


Proof. We may assume that \( \mathcal{P} \) is an \((\mathcal{L}, \mathcal{C})\)-polyomino such that \( \{b_{r-1}, b_r\} \subset V(\mathcal{L}) \cap V(\mathcal{C}) \), since similar arguments can be used in the other cases. We prove that \( K[\mathcal{P}_1] \) is a domain. Observe that \( I_{\mathcal{P}} \) is a toric ideal since \( I_{\mathcal{P}} \) is a prime binomial ideal. Then there exists a map \( \phi : S_{\mathcal{P}} \to K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}] \) with \( x_{ij} \mapsto t_i^a t_j^b \) for all \((i, j) \in \mathcal{P}\) such that \( I_{\mathcal{P}} \) is \( \ker \phi \). We define \( \phi_1 : S_{\mathcal{P}} \to K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}] \), by \( \phi_1(x_v) = 0 \) if \( v \in \{a, c_1, \ldots, c_r, b_1, \ldots, b_{r-2}\} \) and \( \phi_1(x_v) = \phi(x_v) \) otherwise.

Let \( J := (I_{\mathcal{P}}, \{x_a, x_c, x_{c_1}, x_{b_j} \mid i \in [r], j \in [r - 2]\}) \), we prove that \( J = \ker \phi_1 \). Let \( f \in \ker \phi_1 \), we can write \( f = \tilde{f} + \beta g \) where \( \beta \in S_{\mathcal{P}}, g \in \{\{x_a, x_c, x_{c_1}, x_{b_j} \mid i \in [r], j \in [r - 2]\} \) and \( \tilde{f} \) not containing variables in the set \( \{x_a, x_c, x_{c_1}, x_{b_j} \mid i \in [r], j \in [r - 2]\} \). Since \( \phi_1(f) = 0 \), we have \( \phi(\tilde{f}) = 0 \), so \( \tilde{f} \in \ker \phi = I_{\mathcal{P}} \). For the other inclusion it suffices to prove that \( I_{\mathcal{P}} \subseteq \ker \phi_1 \). In such a case observe that, for this configuration, if \( f = x_{i_1}x_{i_2} - x_{j_1}x_{j_2} \) is a generator of \( I_{\mathcal{P}} \) then \( \{x_{i_1}, x_{i_2}\} \cap \{x_{a, x_c, x_{c_1}, x_{b_j} \mid i \in [r], j \in [r - 2]\} \neq \emptyset \) if and only if \( \{x_{j_1}, x_{j_2}\} \cap \{x_{a, x_c, x_{c_1}, x_{b_j} \mid i \in [r], j \in [r - 2]\} \neq \emptyset \), so in all possible cases we have \( \phi_1(f) = 0 \). Therefore \( J = \ker \phi_1 \) and \( J \) is a prime ideal. As in Lemma 3.2 we have also that \( J = (I_{\mathcal{P}}, x_a, x_c, x_{c_1}, x_{b_j} : i \in [r], j \in [r - 2]\) and \( K[\mathcal{P}_1] \cong S_{\mathcal{P}}/J \) is a domain. The proof for this case is done. All other cases, considering the other polyominoes, can be proved in a similar way.

If \( \mathcal{P} \) is an \((\mathcal{L}, \mathcal{C})\)-polyomino, our aim is to provide a formula for the Poincaré-Hibert series of \( K[\mathcal{P}] \), involving the Poincaré-Hibert series of \( K[\mathcal{P}_1], K[\mathcal{P}_2], K[\mathcal{P}_3] \) and \( K[\mathcal{P}_1] \) in the hypotheses that \( K[\mathcal{P}] \) is an integral domain. In particular, let \( (i_1, i_2, i_3, i_4) \) be a permutation of the set \( \{a, b, c, d\} \), our strategy consists in considering four short exact sequences:

\[
0 \to S_{\mathcal{P}}/(I_{\mathcal{P}}, x_{i_1}) \to S_{\mathcal{P}}/I_{\mathcal{P}} \to S_{\mathcal{P}}/(I_{\mathcal{P}}, x_{i_1}) \to 0
\]

\[
0 \to S_{\mathcal{P}}/(I_{\mathcal{P}}, x_{i_1}, x_{i_2}) \to S_{\mathcal{P}}/(I_{\mathcal{P}}, x_{i_1}) \to S_{\mathcal{P}}/(I_{\mathcal{P}}, x_{i_1}, x_{i_2}) \to 0
\]

\[
0 \to S_{\mathcal{P}}/(I_{\mathcal{P}}, x_{i_1}, x_{i_2}, x_{i_3}) \to S_{\mathcal{P}}/(I_{\mathcal{P}}, x_{i_1}, x_{i_2}) \to S_{\mathcal{P}}/(I_{\mathcal{P}}, x_{i_1}, x_{i_2}, x_{i_3}) \to 0
\]

\[
0 \to S_{\mathcal{P}}/(I_{\mathcal{P}}, x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}) \to S_{\mathcal{P}}/(I_{\mathcal{P}}, x_{i_1}, x_{i_2}, x_{i_3}) \to S_{\mathcal{P}}/(I_{\mathcal{P}}, x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}) \to 0
\]

From the exact sequence above, we will obtain the Poincaré-Hibert series of \( S_{\mathcal{P}}/I_{\mathcal{P}} \) by a repeated application of Proposition 2.1 and considering in each case an opportune permutation \( (i_1, i_2, i_3, i_4) \) of the set \( \{a, b, c, d\} \) in order to compute the Poincaré-Hibert series of the rings in the intermediate steps. To reach our aim we provide several preliminary lemmas, distinguishing the different possibilities for the set \( V(\mathcal{L}) \cap V(\mathcal{C}) \).

Lemma 3.4. Let \( \mathcal{P} \) be an \((\mathcal{L}, \mathcal{C})\)-polyomino such that \( V(\mathcal{L}) \cap V(\mathcal{C}) = \{a_{s-1}, a_s, b_{r-1}, b_r\} \). Suppose that \( I_{\mathcal{P}} \) is prime. Then:

1. \( S_{\mathcal{P}}/((I_{\mathcal{P}}, x_a) : x_d) \cong K[\mathcal{P}_1] \otimes_K K[x_{b_1}, \ldots, x_{b_{r-2}}] \);
2. \( S_{\mathcal{P}}/((I_{\mathcal{P}}, x_a, x_d) : x_c) \cong K[\mathcal{P}_2] \otimes_K K[x_{a_1}, \ldots, x_{a_{s-2}}] \);
3. \( S_{\mathcal{P}}/((I_{\mathcal{P}}, x_a, x_d, x_c) : x_b) \cong K[\mathcal{P}_3] \otimes_K K[x_{b_1}, x_{a_1}, \ldots, x_{a_{s-2}}, x_{b_1}, \ldots, x_{b_{r-2}}] \).

Proof. (1) Firstly observe that \( I_{\mathcal{P}} \) can be written in the following way:

\[
I_{\mathcal{P}} = I_{\mathcal{P}_1} + (x_a x_b - x_c x_d) + \sum_{i=1}^{s} (x_a x_{a_i} - x_c x_{d_i}) + \sum_{j=1}^{r} (x_a x_{b_j} - x_d x_{c_j}) + \sum_{j=1}^{r} (x_c x_{b_j} - x_b x_{c_j}) + \sum_{k,l \in [r]} (x_{c_k} x_{b_l} - x_{c_l} x_{b_k}) + \{(x_{c_{r-1}} x_v - x_c x_{a_1} [c_{r-1}, v] \in I(\mathcal{P}), u = v - (1, 0)\}),
\]

If \( t = (t_1, \ldots, t_d) \) and \( a = (a_1, \ldots, a_d) \in \mathbb{Z}^d \), then \( t^a = t_1^{a_1} \cdots t_d^{a_d} \).
Consider the ideal $I_P$ plus the ideal generated by $x_a$ in $S$, so
\[
(I_P, x_a) = (I_{P_1}, x_a) + (x_c x_d) + \sum_{i=1}^s (x_c x_{d_i}) + \sum_{j=1}^r (x_d x_{c_j}) + \sum_{j=1}^r (x_c x_{b_j} - x_b x_c) + \sum_{k,l \in [r]} \sum_{k < l} (x_{c_k} x_{b_l} - x_{c_l} x_{b_k}) + \{x_{c_{r-1}} x_v - x_{c_r} x_u | [c_{r-1}, v] \in \mathcal{I}(P), u = v - (1,0)\}.
\]
We prove that $(I_P, x_a) : x_d = (I_{P_1}, x_a) + (x_c) + \sum_{i=1}^r (x_{c_i})$. It follows trivially from the previous equality that $(I_P, x_a) : x_d \supseteq (I_{P_1}, x_a) + (x_c) + \sum_{i=1}^r (x_{c_i})$. Let $f \in S$ such that $x_d f \in (I_P, x_a)$. Then
\[
x_d f = g + \alpha x_a + \beta x_c x_d + \sum_{i=1}^s \gamma_i x_{d_i} x_c + \sum_{j=1}^r \delta_j x_d x_{c_j} + \sum_{i,j} \omega_{ij} x_{c_i} x_{b_j} - x_{b_j} x_{c_i} + \sum_{k,l \in [r]} \sum_{k < l} \nu_{kl} (x_{c_k} x_{b_l} - x_{c_l} x_{b_k}) + \sum_{[c_{r-1}, v] \in \mathcal{I}(P)} \lambda_v (x_{c_{r-1}} x_v - x_{c_r} x_u),
\]
where $g \in I_{P_1}$, $\alpha, \beta, \gamma_i, \delta_j, \omega_{ij}, \nu_{kl} \lambda_v \in S_P$ for all $i, k \in [s], j, l \in [r]$ and for all $v \in V(P)$ such that $[c_{r-1}, v] \in \mathcal{I}(P)$. As a consequence:
\[
x_d \left(f - \beta x_c - \sum_{j=1}^r \delta_j x_{c_j}\right) = g + \alpha x_a + \gamma_i x_{d_i} x_c + \delta_j x_d x_{c_j} - \omega_{ij} x_{c_i} x_{b_j} - x_{b_j} x_{c_i} + \sum_{k,l \in [r]} \sum_{k < l} \nu_{kl} (x_{c_k} x_{b_l} - x_{c_l} x_{b_k}) + \sum_{[c_{r-1}, v] \in \mathcal{I}(P)} \lambda_v (x_{c_{r-1}} x_v - x_{c_r} x_u) x_{c_r}.
\]
Hence we obtain that $x_d (f - \beta x_c - \sum_{j=1}^r \delta_j x_{c_j}) \in I_{P_1} + (x_a) + (x_c) + \sum_{i=1}^r (x_{c_i})$. Since $K[P_1]$ is a domain for Proposition 3.3 and $a, c, \in \not \in V(P_1)$ for all $i \in [r]$ then $I_{P_1} + (x_a) + (x_c) + \sum_{i=1}^r (x_{c_i})$ is a prime ideal in $S_P$. Since $x_d \not \in I_{P_1}$, we have $f - \beta x_c - \sum_{j=1}^r \delta_j x_{c_j} \in I_{P_1} + (x_a) + (x_c) + \sum_{i=1}^r (x_{c_i})$, so $f \in I_{P_1} + (x_a) + (x_c) + \sum_{i=1}^r (x_{c_i})$, that is $(I_P, x_a) : x_d \subseteq (I_{P_1}, x_a) + (x_c) + \sum_{i=1}^r (x_{c_i})$. In conclusion we have $(I_P, x_a) : x_d = (I_{P_1}, x_a) + (x_c) + \sum_{i=1}^r (x_{c_i})$ and as a consequence $S_P/(I_P, x_a) : x_d = S_P/(I_{P_1} + (x_a, x_c, x_{c_1}, \ldots, x_{c_r}) \approx S_{P_1}/I_{P_1} \otimes_K K[x_{c}] | v \in V(P \setminus P_1)]/(x_a, x_c, x_{c_1}, \ldots, x_{c_r}) = K[P_1] \otimes_K K[x_{b_1}, \ldots, x_{b_{r-2}}]$. By similar computations as in the first part of (1) we can prove that $(I_P, x_a, x_d) : x_c = (I_{P_2}, x_a, x_d) + \sum_{i=1}^s (x_{d_i})$ and $(I_P, x_a, x_d, x_c) : x_b = (I_{P_3}, x_a, x_d, x_c) + \sum_{i=1}^s (x_{d_i}) + \sum_{j=1}^r (x_{c_j})$ and the claims (2) and (3) follow by using similar arguments as in the last part in (1).

In the previous result we examine, for a $(L, C)$-polyomino, the case $V(L) \cap V(C) = \{a_{s-1}, a_s, b_{r-1}, b_r\}$. In order to examine the other cases we need other preliminary results involving the polyominoes $P_1$ and $P_2$.

**Lemma 3.5.** Let $P$ be an $(L, C)$-polyomino. Then $S_{P_2}/(I_{P_2}, x_b, x_c) \cong K[P_2]$. Moreover, if $I_{P_2}$ is a prime ideal then $(I_{P_2}, x_b, x_c)$ is a prime ideal of $S_P$.

**Proof.** Let $R$ be the polyomino obtained from the cells of $P_2$ and renaming the vertices $b$ and $c$ respectively with $d$ and $a$, in particular $S_R = K[x_v | v \in V(P_2) \setminus \{b, c\}]$. Observe that
\[
I_{P_2} = I_R + (x_a x_b - x_c x_d) + \sum_{i=1}^r (x_c x_{b_i} - x_b x_c) + \sum_{k,l \in [r]} \sum_{k < l} (x_{c_k} x_{b_l} - x_{c_l} x_{b_k}) + \{x_c x_{r-1} x_v - x_c x_{r} x_u | [c_{r-1}, v] \in \mathcal{I(P)}, u = v - (1,0)\}
\]
So $(I_{P_2}, x_b, x_c) = (I_R, x_b, x_c)$ and in particular $S_{P_2}/(I_{P_2}, x_b, x_c) = S_{P_2}/(I_R, x_b, x_c) \cong S_R/I_R = K[R] \cong K[P_2]$, since $x_b, x_c$ do not belong to the support of any element of $I_R$ and observing that, a part from the name of the vertices involved, $R = P_2$. Furthermore $S_P/(I_{P_2}, x_b, x_c) \cong$
Hence we obtain that
\[ x \in V(\mathcal{P} \setminus \mathcal{P}_2') \approx K[\mathcal{P}_2] \otimes_K K[x_v \mid v \in V(\mathcal{P} \setminus \mathcal{P}_2')], \]
so also the last claim follows.

**Lemma 3.6.** Let \( \mathcal{P} \) be an \((\mathcal{L}, \mathcal{C})\)-polyomino. Then \( S_{\mathcal{P}} / (I_{\mathcal{P}_1}, x_b, x_d) \cong K[\mathcal{P}_1] \). Moreover, if \( I_{\mathcal{P}_1} \) is a prime ideal then \((I_{\mathcal{P}_1}, x_b, x_d)\) is a prime ideal of \( S_{\mathcal{P}} \).

**Proof.** The result can be obtained reasoning as in the proof of Lemma 3.5. Indeed the arrangements involved in these situations can be considered the same up to one reflection and one rotation. \( \Box \)

**Lemma 3.7.** Let \( \mathcal{P} \) be an \((\mathcal{L}, \mathcal{C})\)-polyomino such that \( V(\mathcal{L}) \cap V(\mathcal{C}) = \{d_{s-1}, d_s, b_{r-1}, b_r\} \). Suppose that \( I_{\mathcal{P}} \) is prime. Then:

1. \( S_{\mathcal{P}} / (I_{\mathcal{P}}, x_b, x_c) : x_b \cong K[\mathcal{P}_1] \otimes_K K[x_b, \ldots, x_{b_{r-2}}] \);
2. \( S_{\mathcal{P}} / (I_{\mathcal{P}}, x_b, x_c) : x_a \cong K[\mathcal{P}_2] \otimes_K K[x_b, \ldots, x_{b_{r-2}}] \);
3. \( S_{\mathcal{P}} / (I_{\mathcal{P}}, x_a, x_b, x_c) : x_d \cong K[\mathcal{P}_3] \otimes_K K[x_b, \ldots, x_{b_{r-2}}, x_{d_1}, \ldots, x_{d_{s-2}}] \).

**Proof.** Arguing as in Lemma 3.4 we obtain the equalities of the following ideals:

1. \( (I_{\mathcal{P}}, x_c) : x_b = (I_{\mathcal{P}_1}, x_c) + (x_a) + \sum_{i=1}^r (x_c_i) \)
2. \( (I_{\mathcal{P}}, x_b, x_c) : x_a = (I_{\mathcal{P}_1}, x_b, x_c) + \sum_{i=1}^r (x_a_i) \)
3. \( (I_{\mathcal{P}}, x_a, x_b, x_c) : x_d = (I_{\mathcal{P}_1}, x_a, x_b, x_c) + \sum_{i=1}^r (x_a_i) + \sum_{i=1}^r (x_c_i) \)

In particular, the second equality above holds having the primality of \((I_{\mathcal{P}_1}, x_b, x_c)\) by Lemma 3.5 and from claim (2) derives. For the sake of completeness we provide its proof.

Observe that \( I_{\mathcal{P}} \) can be written in the following way:

\[
I_{\mathcal{P}} = I_{\mathcal{P}_2} + \sum_{i=1}^s (x_a x_{a_i} - x_c x_{d_i}) + \sum_{i=1}^s (x_d x_{a_i} - x_b x_{d_i}) + \sum_{k, \ell \leq [s]} \sum_{k < \ell} (x_d x_{a_k} - x_d x_{a_\ell}) +
\]
\[ + \left\{ x_{a_s} x_{v} - x_{a_{s-1}} x_{u} \mid [v, a_s] \in \mathcal{I}(\mathcal{P}), u = v + (0, 1) \right\}, \]

It follows:

\[
(I_{\mathcal{P}}, x_b, x_c) = (I_{\mathcal{P}_2}, x_b, x_c) + \sum_{i=1}^s (x_a x_{a_i}) + \sum_{i=1}^s (x_d x_{a_i}) + \sum_{k, \ell \leq [s]} \sum_{k < \ell} (x_d x_{a_k} - x_d x_{a_\ell}) +
\]
\[ + \left\{ x_{a_s} x_{v} - x_{a_{s-1}} x_{u} \mid [v, a_s] \in \mathcal{I}(\mathcal{P}), u = v + (0, 1) \right\}, \]

We prove that \((I_{\mathcal{P}}, x_b, x_c) : x_a = (I_{\mathcal{P}_2}, x_b, x_c) + \sum_{i=1}^s (x_a_i)\). From the previous equality it follows that \((I_{\mathcal{P}}, x_b, x_c) : x_a \supseteq (I_{\mathcal{P}_2}, x_b, x_c) + \sum_{i=1}^s (x_a_i)\). Let \( f \in S \) such that \( x_a f \in (I_{\mathcal{P}}, x_b, x_c) \). Then

\[
x_a f = g + \sum_{i=1}^s \gamma_i x_a x_{a_i} + \sum_{j=1}^s \delta_j x_d x_{a_j} + \sum_{k, \ell \leq [s]} \nu_{k\ell} (x_d x_{a_k} - x_d x_{a_\ell}) + \sum_{[v, a_s] \in \mathcal{I}(\mathcal{P}), u = v + (0, 1)} \lambda_u (x_a x_v - x_{a_{s-1}} x_u),
\]

where \( g \in (I_{\mathcal{P}_2}, x_b, x_c) \), \( \gamma_i, \delta_j, \nu_{k\ell}, \lambda_u \in S_{\mathcal{P}} \) for all \( i, k, j, l \in [s] \) and for all \( v \in V(\mathcal{P}) \) such that \([v, a_s] \in \mathcal{I}(\mathcal{P})\). As a consequence:

\[
x_a (f - \sum_{i=1}^s \gamma_i x_{a_i}) = g + \sum_{j=1}^s (\delta_j x_d) x_{a_j} + \sum_{k, \ell \leq [s]} (\nu_{k\ell} x_d) x_{a_k} - \sum_{k, \ell \leq [s]} (\nu_{k\ell} x_d) x_{a_\ell} +
\]
\[ + \left( \sum_{[v, a_s] \in \mathcal{I}(\mathcal{P}), u = v + (0, 1)} \lambda_u x_{v} \right) x_{a_{s-1}}.
\]

Hence we obtain that \( x_a (f - \sum_{i=1}^s \gamma_i x_{a_i}) \in (I_{\mathcal{P}_2}, x_b, x_c) + \sum_{i=1}^s (x_a_i) \). Since \((I_{\mathcal{P}_2}, x_b, x_c)\) is prime and \( a_i \notin V(\mathcal{P}_2) \) for all \( i \in [s] \) then \((I_{\mathcal{P}_2}, x_b, x_c) + \sum_{i=1}^s (x_a_i)\) is a prime ideal in \( S_{\mathcal{P}} \). By being \( x_a \notin I_{\mathcal{P}_2} \), we
have $f - \sum_{i=1}^{s} \gamma_{i} x_{a_{i}} \in (I_{P_{2}}^{c}, x_{c}) + \sum_{i=1}^{s} (x_{a_{i}})$, so $f \in (I_{P_{2}}^{c}, x_{c}) + \sum_{i=1}^{s} (x_{a_{i}})$, that is $(I_{P}, x_{c}) : x_{d} \subseteq (I_{P_{2}}^{c}, x_{c}) + \sum_{i=1}^{s} (x_{a_{i}})$. In conclusion we have $(I_{P}, x_{c}) : x_{a} = (I_{P_{1}}, x_{c}) + \sum_{i=1}^{s} (x_{a_{i}})$ and as a consequence $S_{P}/((I_{P}, x_{c}) : x_{a}) = S_{P}/(I_{P_{2}}^{c}, x_{c}) + (x_{a}) \cong S_{P}/(I_{P_{2}}^{c}, x_{c}) \otimes_{K} K[x_{c} / v \in V(P \setminus P_{1})](x_{a_{1}}, \ldots, x_{a_{s}}) \cong K[P_{2}] \otimes_{K} K[x_{d_{1}}, \ldots, x_{d_{s-2}}].$ □

We omit to provide the analogous result for the case $V(L) \cap V(C) = \{a_{s-1}, a_{s}, c_{r-1}, c_{r}\}$. In fact, we can reduce it to the case examined in the previous Lemma up to a rotation and a reflection.

**Lemma 3.8.** Let $P$ be an $(L, C)$-polyomino such that $V(L) \cap V(C) = \{d_{s-1}, d_{s}, c_{r-1}, c_{r}\}$. Suppose that $I_P$ is prime. Then:

1. $S_{P}/((I_{P}, x_{c}) : x_{c}) \cong K[P_{1}] \otimes_{K} K[x_{c_{1}}, \ldots, x_{c_{r-2}}]$;
2. $S_{P}/((I_{P}, x_{b}, x_{c}) : x_{d}) \cong K[P_{2}] \otimes_{K} K[x_{d_{1}}, \ldots, x_{d_{s-2}}]$;
3. $S_{P}/((I_{P}, x_{b}, x_{c}, x_{d}) : x_{a}) \cong K[P_{3}] \otimes_{K} K[x_{d_{1}}, x_{c_{1}}, \ldots, x_{c_{r-2}}, x_{d_{1}}, \ldots, x_{d_{s-2}}]$.

**Proof.** The claims follow reasoning as in Lemma 3.4 obtaining the equalities of the following ideals:

1. $(I_{P}, x_{c}) : x_{c} = (I_{P_{1}}, x_{c}) + \sum_{i=1}^{s} (x_{a_{i}})$;
2. $(I_{P}, x_{b}, x_{c}) : x_{d} = (I_{P_{2}}, x_{b}, x_{c}) + \sum_{i=1}^{s} (x_{a_{i}})$.
3. $(I_{P}, x_{b}, x_{c}, x_{d}) : x_{a} = (I_{P_{3}}, x_{b}, x_{c}, x_{d}) + \sum_{i=1}^{s} (x_{a_{i}}) + \sum_{i=1}^{s} (x_{a_{i}})$.

In particular, the first equality follows from the primality of $(I_{P_{2}}, x_{b}, x_{c})$ by Lemma 3.6. The second equality follows from the primality of $(I_{P_{2}}, x_{b}, x_{c})$ by Lemma 3.5. □

**Theorem 3.9.** Let $P$ be an $(L, C)$-polyomino. Suppose that $I_P$ is prime. Then:

$$HP_{K[P]}(t) = \frac{1}{1-t} HP_{K[P_{1}]}(t) + \frac{t}{1-t} \left[ HP_{K[P_{1}]}(t) \right] \left[ HP_{K[P_{2}]}(t) \right] \left[ HP_{K[P_{3}]}(t) \right]$$

**Proof.** Assume that $V(L) \cap V(C) = \{a_{s-1}, a_{s}, b_{r-1}, b_{r}\}$. Consider the following four short exact sequences:

1. $0 \to S_{P}/(I_{P}, x_{a}) \to S_{P}/(I_{P}, x_{a}) \to 0$
2. $0 \to S_{P}/((I_{P}, x_{a}) : x_{d}) \to S_{P}/(I_{P}, x_{a}) \to 0$
3. $0 \to S_{P}/((I_{P}, x_{a}, x_{d}) : x_{c}) \to S_{P}/(I_{P}, x_{a}, x_{d}) \to 0$
4. $0 \to S_{P}/((I_{P}, x_{a}, x_{d}, x_{c}) : x_{b}) \to S_{P}/(I_{P}, x_{a}, x_{d}, x_{c}) \to 0$

Observing that $I_{P} : x_{a} = I_{P}$, because $I_{P}$ is prime, the claim easily follows by repeated applications of Proposition 2.7 and from Proposition 2.2, Lemma 3.2 and Lemma 3.4.

If $V(L) \cap V(C) = \{d_{s-1}, d_{s}, b_{r-1}, b_{r}\}$ the formula is obtained referring to Lemma 3.7 by an opportune permutation of the set $\{a, b, c, d\}$. For symmetry, we obtain the claim also for the case $V(L) \cap V(C) = \{a_{s-1}, a_{s}, c_{r-1}, c_{r}\}$.

Finally, if $V(L) \cap V(C) = \{d_{s-1}, d_{s}, c_{r-1}, c_{r}\}$ we use again the same argument together with Lemma 3.8. □

**Corollary 3.10.** Let $P$ be an $(L, C)$-polyomino and suppose that $I_{P}$ is a prime ideal and $C$ is a simple polyomino. Then:

$$HP_{K[P]}(t) = h_{K[P_{1}]}(t) + t[h_{K[P_{1}]}(t) + h_{K[P_{2}]}(t) + (1-t)h_{K[P_{3}]}(t)] \frac{1}{(1-t)^{V(P) - \text{rank } P}}$$

In particular $K[P]$ has Krull dimension $|V(P)| - \text{rank } P$.

**Proof.** Since $C$ is a simple polyomino, then $P_{1}, P_{2}, P_{3}$ and $P_{4}$ are simple polyominoes, so we have that $K[P_{1}]$ is a normal Cohen-Macaulay domain of dimension $|V(P_{j})| - \text{rank } P_{j}$ for $j \in \{1, 2, 3, 4\}$ from Corollary 3.3 and Theorem 2.1. We put $|V(P)| = n$ and $\text{rank } P = p$. Observe that

- $|V(P_{1})| = n - 2r$ and $\text{rank } P_{1} = p - r - 1$, so $|V(P_{1})| - \text{rank } P_{1} = n - p - r + 1$;
• $|V(\mathcal{P}_2)| = n - 2s$ and rank $\mathcal{P}_2 = p - s - 1$, so $|V(\mathcal{P}_2)| - \text{rank } \mathcal{P}_2 = n - p - s + 1$;
• $|V(\mathcal{P}_3)| = n - 2s - 2r$ and rank $\mathcal{P}_3 = p - r - s - 1$, so $|V(\mathcal{P}_3)| - \text{rank } \mathcal{P}_3 = n - p - s - r + 1$;
• $|V(\mathcal{P}_4)| = n - 4$ and rank $\mathcal{P}_4 = p - 3$, so $|V(\mathcal{P}_4)| - \text{rank } \mathcal{P}_4 = n - p - 1$.

Then $n - p = |V(\mathcal{P}_4)| - \text{rank } \mathcal{P}_4 + 1 = |V(\mathcal{P}_1)| - \text{rank } \mathcal{P}_1 + (r - 2) + 1 = |V(\mathcal{P}_2)| - \text{rank } \mathcal{P}_2 + (s - 2) + 1$ and $|V(\mathcal{P}_3)| - \text{rank } \mathcal{P}_3 + (s + r - 3) + 1 = n - p - 1$. Therefore the formula for $HP_{K[\mathcal{P}]}(t)$ in the statement follows from Theorem 3.9 after an easy computation. Finally, let $h(t)$ be the polynomial in the numerator of the formula. By [3, Corollary 4.1.10], observe that $h(1) = h_K[\mathcal{P}_4](1) + h_K[\mathcal{P}_3](1) + h_K[\mathcal{P}_2](1) > 0$, so $(1 - t)$ do not divide $h(t)$, that implies $K[\mathcal{P}]$ has Krull dimension $|V(\mathcal{P})| - \text{rank } \mathcal{P}$.

4. Poincaré-Hilbert series of prime closed path polyominoes having no $L$-configuration

In this section we suppose that $\mathcal{P}$ is a prime closed path polyomino having no $L$-configurations, so $\mathcal{P}$ contains a ladder of at least three steps from [13, Section 6]. Let $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ be three maximal horizontal blocks of a ladder of $n$ steps in $\mathcal{P}$, with $n \geq 3$. We may assume without loss of generality that there does not exist a maximal block $\mathcal{K} \neq \mathcal{B}_2, \mathcal{B}_3$ of $\mathcal{P}$ such that $\{\mathcal{K}, \mathcal{B}_1, \mathcal{B}_2\}$ is a ladder of three steps. Moreover, considering convenient reflection or rotation of $\mathcal{P}$, we can suppose that the orientation of the ladder is right/up, as in Figure 4.

![Figure 4](image)

Our aim is to study the Poincaré-Hilbert series of the coordinate ring of $\mathcal{P}$. We split our arguments in two cases. In the first case we suppose that at least one block between $\mathcal{B}_1$ or $\mathcal{B}_2$ contains exactly two cells while in the second one assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ contain at least three cells.

4.1. At least one block between $\mathcal{B}_1$ or $\mathcal{B}_2$ contains exactly two cells. We start with some preliminary definitions that we adopt throughout this subsection. Let $\mathcal{W}$ be a collection of cells consisting of an horizontal block $[A_s, A_1]$ of rank at least two, containing the cells $A_s, A_{s-1}, \ldots, A_1$, a vertical block $[B_1, B_r]$ of rank at least two, containing the cells $B_1, B_2, \ldots, B_r$, and a cell $A$ not belonging to $[A_s, A_1] \cup [B_1, B_r]$, such that $V([A_s, A_1]) \cap V([B_1, B_r]) = \{b\}$, where $b$ is the lower right corner of $A$. Moreover we denote by $a$ the left upper corner of $A$, by $d$ the lower right corner of $B_1$, by $c$ the lower right corner of $A_1$. Moreover, for $i \in [s]$ let $b_i$ and $c_i$ be respectively the left upper and lower corners of $A_i$, for $j \in [r]$ let $a_j$ and $d_j$ be respectively the left and the right upper corners of $B_j$ (Figure 5).

Since $\mathcal{P}$ has no $L$-configurations, it is trivial to check that $\mathcal{P}$ contains a collection of cells $\mathcal{W}$ such that $[A_s, A_1]$ and $[B_1, B_r]$ are maximal blocks of $\mathcal{P}$. In particular, if $\mathcal{M}$ is the collection of cells such that $\mathcal{P} = \mathcal{W} \cup \mathcal{M}$, then we call $\mathcal{W}$:

• 1-Configuration, if $V(\mathcal{W}) \cap V(\mathcal{M}) = \{c_{s-1}, c_s, d_{r-1}, d_r\}$;
• 2-Configuration, if $V(\mathcal{W}) \cap V(\mathcal{M}) = \{b_{s-1}, b_s, d_{r-1}, d_r\}$.

Observe that just one of the following cases can occur:
(1) $|\mathcal{B}_1| = |\mathcal{B}_2| = 2$. In such a case $s = 2$ and $r = 2$ with $\mathcal{B}_1 = [A_2, A_1]$ and $\mathcal{B}_2 = [A, B_1]$, so we obtain an $1$-Configuration.

(2) $|\mathcal{B}_1| > 2$ and $|\mathcal{B}_2| = 2$. In such a case $s > 2$ and $r = 2$ with $\mathcal{B}_1 = [A_s, A_1]$ and $\mathcal{B}_2 = [A, B_1]$, so we have an $1$-Configuration or a $2$-Configuration depending on $\mathcal{M} \cap \{A_s\}$.

(3) $|\mathcal{B}_1| = 2$ and $|\mathcal{B}_2| > 2$. In such a case, after an opportune rotation and reflection, consider a new ladder where $\mathcal{B}_1 = [A_1, A]$ and $\mathcal{B}_2 = [B_1, B_s]$ with $s \geq 2$ and $r > 2$. Let $C$ be a cell of $\mathcal{P}$ such that $I := [C, A_1]$ is a maximal block of $\mathcal{P}$. Therefore we obtain an $1$-Configuration or a $2$-Configuration depending on the position of the cell of $\mathcal{P}\setminus I$ adjacent to $C$.

Moreover we define the following related polyominoes:

- $\mathcal{Q} = \mathcal{P} \setminus \{A\}$;
- $\mathcal{Q}_1 = \mathcal{P} \setminus \{A, A_1, B_1\}$;
- $\mathcal{R}_1 = \mathcal{Q} \setminus \{B_1\}$;
- $\mathcal{R}_2 = \mathcal{Q} \setminus \{B_1, \ldots, B_s\}$;
- $\mathcal{F}_1 = \mathcal{Q} \setminus \{A_1, \ldots, A_s\}$;
- $\mathcal{F}_2 = \mathcal{Q} \setminus \{A_1, B_1, \ldots, B_s\}$.

Let $<^1$ be the total order on $V(\mathcal{P})$ defined as $u <^1 v$ if and only if, for $u = (i, j)$ and $v = (k, l)$, $i < k$, or $i = k$ and $j < l$. Let $Y \subset V(\mathcal{P})$ and consider $<^Y_{\text{lex}}$ be the lexicographical order in $S_\mathcal{P}$ induced by the following order on the variables of $S_\mathcal{P}$:

$$x_u <^Y_{\text{lex}} x_v \iff \begin{cases} u \notin Y \text{ and } v \in Y \\ u, v \in Y \text{ and } u <^1 v \\ u, v \in Y \text{ and } u <^1 v \end{cases}$$

Considering Figure 5 from [6, Theorem 4.9] we know that there exists a set $L \subset V(\mathcal{P})$, with $a, d \in L$ and $b, c, a_1, b_1, c_1, d_1 \notin L$, such that the set of generators of $I_\mathcal{P}$ forms the reduced Gröbner basis of $I_\mathcal{P}$ with respect to $<^L_{\text{lex}}$. Furthermore, in the case of $1$-Configuration also $d_2, \ldots, d_r \notin L$.

For convenience, we denote such a monomial order by $<_\mathcal{P}$. Moreover, let $<_{\mathcal{Q}}, <_{\mathcal{Q}_1}, <_{\mathcal{R}_1}, <_{\mathcal{R}_2}$ be the monomial orders induced in natural way from $<_\mathcal{P}$ respectively on the rings $S_\mathcal{Q}, S_{\mathcal{Q}_1}, S_{\mathcal{R}_1}, S_{\mathcal{R}_2}$.

The following proposition will be useful.

**Proposition 4.1.** Let $\mathcal{P}$ be a closed path polyomino containing a collection of cells of type $\mathcal{W}$. Then the set of inner to $2$-minors of $\mathcal{Q}$ is the reduced Gröbner basis of $I_\mathcal{Q}$ with respect to the monomial order $<_{\mathcal{Q}}$. The same holds for the polyominoes $\mathcal{Q}_1, \mathcal{R}_1$ and $\mathcal{R}_2$ considering respectively the monomial orders $<_{\mathcal{Q}_1}, <_{\mathcal{R}_1}$ and $<_{\mathcal{R}_2}$.

**Proof.** Let $f, g$ be two generators of $I_\mathcal{Q} \subset I_\mathcal{P}$. Since every $S$-polynomial $S(f, g)$ reduces to zero in $I_\mathcal{P}$ then the conditions in lemmas in [6, Section 3] are satisfied for the collection of cells $\mathcal{P}$. A part from the occurrences $f = x_b x_{c_1} - x_c x_{b_1}$ and $g = x_b x_{d_1} - x_d x_{a_1}$, in which the leading terms of $f$
and \( g \) have the greatest common divisor equal to 1, the other conditions of the mentioned lemmas do not involve the cell \( A \). So the same conditions hold also for the collection of cells \( Q \), hence \( S(f, g) \) reduces to zero also in \( I_Q \). By the same argument also the second claim in the statement holds.

\[ \square \]

**Remark 4.2.** Observe that \( Q_1, R_1, R_2, F_1 \) and \( F_2 \) are simple polyominoes, so their related coordinate rings are normal Cohen-Macaulay domains whose Krull dimension is given by the difference between the number of vertices and the number of cells of the fixed polyomino (see [12 Corollary 3.3] and [11 Theorem 2.1]). The polyomino \( Q \) is not simple but it is a weakly closed path and it is easy to see that \( Q \) has a weak ladder in these cases which we are studying. Therefore \( I_Q \) is a prime ideal (equivalently \( K[Q] \) is a domain) from [5 Proposition 4.5]. Moreover, from Proposition 4.1 and arguing as in the proof of [6 Theorem 4.10] we obtain also that \( K[Q] \) is a normal Cohen-Macaulay domain. The Krull dimension of \( K[Q] \) is unknown at the moment.

We are going to use all these introductory facts in the proofs of the next results. With abuse of notation we refer to \( \text{in}(I_P) \), \( \text{in}(I_Q) \), \( \text{in}(I_{Q_1}) \), \( \text{in}(I_{R_1}) \), \( \text{in}(I_{R_2}) \) respectively for the initial ideals of \( I_P \) with respect to \( \prec_P \), of \( I_Q \) with respect to \( \prec_Q \), of \( I_{Q_1} \) with respect to \( \prec_{Q_1} \), of \( I_{R_1} \) with respect to \( \prec_{R_1} \), and of \( I_{R_2} \) with respect to \( \prec_{R_2} \).

**Proposition 4.3.** Let \( P \) be a closed path polyomino containing a collection of cells of type \( W \). Then

\[
\text{HP}_{K[P]}(t) = \text{HP}_{K[Q]}(t) + \frac{t}{1-t} \text{HP}_{K[Q_1]}(t)
\]

**Proof.** Observe that:

\[
I_P = I_Q + (x_b x_{a_1} - x_a x_b) + (x_b x_{d_1} - x_a x_d) + (x_c x_{a_1} - x_a x_c),
\]

\[
I_P = I_{Q_1} + (x_b x_{a_1} - x_a x_b) + (x_b x_{d_1} - x_a x_d) + (x_c x_{a_1} - x_a x_c) + 
\]

\[
+ \sum_{i=1}^r (x_b x_{d_i} - x_d x_{a_i}) + \sum_{i=1}^s (x_b x_{c_i} - x_c x_{b_i}).
\]

From Proposition 4.1 we obtain:

\[
\text{in}(I_P) = \text{in}(I_Q) + (x_a x_b) + (x_a x_d) + (x_a x_c),
\]

\[
\text{in}(I_P) = \text{in}(I_{Q_1}) + (x_a x_b) + (x_a x_d) + (x_a x_c) + \left\{ \max_{\prec_P} \{x_b x_{d_i}, x_d x_{a_i} : i \in [r]\} \right\} + 
\]

\[
+ \left\{ \max_{\prec_P} \{x_b x_{c_i}, x_c x_{b_i} : i \in [s]\} \right\}.
\]

From the above equalities it is not difficult to see that:

- \( (\text{in}(I_P), x_a) = (\text{in}(I_Q), x_a) \), in particular \( S_P/(\text{in}(I_P), x_a) = S_P/(\text{in}(I_Q), x_a) \cong S_Q/\text{in}(I_Q) \).
- \( \text{in}(I_P) : x_a = (\text{in}(I_{Q_1}), x_b, x_c, x_d) \) (see for instance [10 Proposition 1.2.2]), in particular \( S_P/(\text{in}(I_P), x_a) = S_P/(\text{in}(I_{Q_1}), x_b, x_c, x_d) \cong S_{Q_1}/\text{in}(I_{Q_1}) \otimes_K K[x_a] \).

Consider the following exact sequence:

\[
0 \rightarrow S_P/(\text{in}(I_P), x_a) \rightarrow S_P/\text{in}(I_P) \rightarrow S_P/(\text{in}(I_P), x_a) \rightarrow 0
\]

Since for every graded ideal \( I \) of a standard graded \( K \)-algebra \( S \) and for every monomial order \( < \) on \( S \) it is verified that \( S/I \) and \( S/\text{in}_<(I) \) have the same Hilbert function (see [10 Corollary 6.1.5]), then from the above computations and from Propositions 2.1 and 2.2 we obtain \( \text{HP}_{K[P]}(t) = \text{HP}_{K[Q]}(t) + \frac{t}{1-t} \text{HP}_{K[Q_1]}(t) \). \( \square \)
We observed that \( Q \) is not a simple polyomino. Our aim is to provide a formula for the Poincaré-Hilbert series of \( K[\mathcal{P}] \) involving the Hilbert-Poincaré series related to the coordinate rings of simple polyominoes. By the previous result, since \( Q_1 \) is a simple polyomino, we have to study the Poincaré-Hilbert series of \( K[Q] \). We examine 1-Configuration and 2-Configuration separately.

**Theorem 4.4.** Let \( \mathcal{P} \) be a closed path polyomino containing a collection of cells of type \( \mathcal{W} \) with the occurrence of 1-Configuration. Then

\[
\text{HP}_{K[\mathcal{P}]}(t) = \frac{h_{K[\mathcal{R}_1]}(t) + t [h_{K[\mathcal{R}_2]}(t) + h_{K[\mathcal{Q}_1]}(t)]}{(1-t)^{|V(\mathcal{P})|-\text{rank} \mathcal{P}}}
\]

In particular, the Krull dimension of \( K[\mathcal{P}] \) is \( |V(\mathcal{P})| - \text{rank} \mathcal{P} \).

**Proof.** Observe that:

\[
I_Q = I_{\mathcal{R}_1} + \sum_{i=1}^r (x_b x_{d_i} - x_d x_{a_i}),
\]

\[
I_Q = I_{\mathcal{R}_2} + \sum_{i=1}^r (x_b x_{d_i} - x_d x_{a_i}) + \sum_{k,l \in \{r\} \text{ and } k < l} (x_{a_k} x_{d_l} - x_{a_l} x_{d_k}) + \{a_{r-1}, v \} \in \mathcal{I}(Q), u = v - (0,1)\}.
\]

From Proposition 4.1 we obtain:

\[
\text{in}(I_Q) = \text{in}(I_{\mathcal{R}_1}) + \sum_{i=1}^r (x_d x_{a_i}),
\]

\[
\text{in}(I_Q) = \text{in}(I_{\mathcal{R}_2}) + \sum_{i=1}^r (x_d x_{a_i}) + \{\max \{x_{a_k} x_{d_l}, x_{a_l} x_{d_k}\} | k, l \in \{r\}, k < l\} + \{\max \{x_{a_{r-1}}, v, x_{a_r} x_u\} | (a_{r-1}, v) \in \mathcal{I}(Q), u = v - (0,1)\}.
\]

From the above equalities it is not difficult to see that:

- \((\text{in}(I_Q), x_d) = (\text{in}(I_{\mathcal{R}_1}), x_d)\), in particular \( S_Q / (\text{in}(I_Q), x_d) \cong S_{\mathcal{R}_1} / \text{in}(I_{\mathcal{R}_1}) \).
- \((\text{in}(I_Q), x_d) = \text{in}(I_{\mathcal{R}_2}) + \sum_{i=1}^r (x_d x_{a_i})\), in particular \( S_Q / (\text{in}(I_Q), x_d) \cong S_{\mathcal{R}_2} / \text{in}(I_{\mathcal{R}_2}) \otimes_K K[x_{d_1}, \ldots, x_{d_{r-2}}] \).

So, arguing as in the proof of Proposition 4.3 we obtain \( \text{HP}_{K[Q]}(t) = \text{HP}_{K[\mathcal{R}_1]}(t) + t \cdot \frac{\text{HP}_{K[\mathcal{R}_2]}(t)}{(1-t)^{r-2}} \).

Combining such an equality with the claim of Proposition 4.3 we have:

\[
\text{HP}_{K[\mathcal{P}]}(t) = \text{HP}_{K[\mathcal{R}_1]}(t) + t \cdot \left( \frac{\text{HP}_{K[\mathcal{R}_2]}(t)}{(1-t)^{r-2}} + \frac{\text{HP}_{K[\mathcal{Q}_1]}(t)}{1-t} \right)
\]

Set \( |V(\mathcal{P})| = n \) and \( \text{rank} \mathcal{P} = p \). Observe that

- \( |V(\mathcal{R}_1)| = n - 2 \) and \( \text{rank} \mathcal{R}_1 = p - 2 \), so \( |V(\mathcal{R}_1)| - \text{rank} \mathcal{R}_1 = n - p \) and this is the Krull dimension of \( K[\mathcal{R}_1] \) since \( \mathcal{R}_1 \) is simple;
- \( |V(\mathcal{R}_2)| = n - 2r + 1 \) and \( \text{rank} \mathcal{R}_2 = p - r - 1 \), so \( |V(\mathcal{R}_2)| - \text{rank} \mathcal{R}_2 = n - p - r + 2 \) and this is the Krull dimension of \( K[\mathcal{R}_2] \);
- \( |V(\mathcal{Q}_1)| = n - 4 \) and \( \text{rank} \mathcal{Q}_1 = p - 3 \), so \( |V(\mathcal{Q}_1)| - \text{rank} \mathcal{Q}_1 = n - p - 1 \) and this is the Krull dimension of \( K[\mathcal{Q}_1] \).

Therefore, by easy computations we obtain the formula for \( \text{HP}_{K[\mathcal{P}]}(t) \) in the statement. Finally, because of the Cohen-Macaulay property of \( K[\mathcal{R}_1], K[\mathcal{R}_2] \) and \( K[\mathcal{Q}_1] \) and by [3 Corollary 4.1.10], we have that \( h_{K[\mathcal{R}_1]}(1) + h_{K[\mathcal{R}_2]}(1) + h_{K[\mathcal{Q}_1]}(1) > 0 \), so \( \text{dim} K[\mathcal{P}] = |V(\mathcal{P})| - \text{rank} \mathcal{P} \).
Now we want to study the 2-Configuration. In such a case we do not need to use the initial ideals.

**Theorem 4.5.** Let $\mathcal{P}$ be a closed path polyomino containing a collection of cells of type $\mathcal{W}$ with the occurrence of 2-Configuration. Then

$$\text{HP}_{K[\mathcal{P}]}(t) = \frac{(1 + t)h_{K[\mathcal{Q}_1]}(t) + t\left[h_{K[\mathcal{I}_1]}(t) + h_{K[\mathcal{I}_2]}(t)\right]}{(1 - t)^{|V(\mathcal{P})| - \text{rank} \mathcal{P}}}.$$ 

In particular, the Krull dimension of $K[\mathcal{P}]$ is $|V(\mathcal{P})| - \text{rank} \mathcal{P}$.

**Proof.** Arguing as in Lemma 3.3, we obtain the following equalities:

1. $I_\mathcal{Q} = I_{\mathcal{Q}_1}$;
2. $(I_{\mathcal{Q}_1}, x_c) = I_{\mathcal{F}_1} + (x_c) + \sum_{i=1}^s(x_i);
3. (I_{\mathcal{Q}_1}, x_b, x_c) = I_{\mathcal{F}_2} + (x_b, x_c) + \sum_{i=1}^s(x_i).
4. $(I_{\mathcal{Q}_1}, x_b, x_c, x_d) = (I_{\mathcal{Q}_1}, x_b, x_c, x_d)$

Again by the same arguments of Lemma 3.4, we obtain the following equalities:

1. $S_{\mathcal{Q}}/(I_{\mathcal{Q}}) = K[\mathcal{Q}]$;
2. $S_{\mathcal{Q}}/(I_{\mathcal{Q}_1}, x_b) \cong K[\mathcal{F}_1] \otimes_K K[x_{b_1}, \ldots, x_{b_{s-2}}];$
3. $S_{\mathcal{Q}}/(I_{\mathcal{Q}_1}, x_b, x_c) \cong K[\mathcal{F}_2] \otimes_K K[x_d, x_{d_1}, \ldots, x_{d_{r-2}}];$
4. $S_{\mathcal{Q}}/(I_{\mathcal{Q}_1}, x_b, x_c, x_d) \cong K[\mathcal{Q}_1]$

Now applying the opportune exact sequences and arguing as in Theorem 3.9, we obtain the following:

$$\text{HP}_{K[\mathcal{Q}]}(t) = \frac{1}{1 - t} \text{HP}_{K[\mathcal{Q}_1]} + \frac{t}{1 - t} \left[\frac{\text{HP}_{K[\mathcal{I}_1]}(t)}{(1 - t)^{s-2}} + \frac{\text{HP}_{K[\mathcal{I}_2]}(t)}{(1 - t)^{r-1}}\right].$$

So, from Theorem 4.3, we have:

$$\text{HP}_{K[\mathcal{P}]}(t) = \frac{1 + t}{1 - t} \text{HP}_{K[\mathcal{Q}_1]} + \frac{t}{1 - t} \left[\frac{\text{HP}_{K[\mathcal{I}_1]}(t)}{(1 - t)^{s-2}} + \frac{\text{HP}_{K[\mathcal{I}_2]}(t)}{(1 - t)^{r-1}}\right].$$

Finally, we obtain our claims arguing as in the last part of the previous result (or also, for instance, as in Corollary 3.10).

**4.2. $B_1$ and $B_2$ contain at least three cells.** Suppose that $B_1 = [B_1, B]$, consisting of the cells $B_1, \ldots, B_r, B$ with $r \geq 2$, and $B_2 = [A, A_s]$, consisting of the cells $A, A_1, \ldots, A_s$ with $s \geq 2$. We denote by $a, c$ respectively the upper and lower left corners of $A$, by $b, d$ respectively the upper and lower right corners of $A$, by $f, g$ respectively the left and right lower corners of $B$, by $a_i, b_i$ respectively the upper and lower right corners of $A_i$ for $i \in [s]$, by $c_i, d_i$ respectively the lower and upper left corners of $B_i$ for $i \in [r]$. Considering our assumption on the ladder at the beginning of Section 3 and the fact that $\mathcal{P}$ has not any $L$-configuration, we have that $c_1, c_2 \notin V(\mathcal{P}) \setminus V(B_1)$. The described arrangement is summarized in Figure 6.

For our purpose we need to define the following related polyominoes:

- $\mathcal{K}_1 = \mathcal{P} \setminus [B_1, B]$;
- $\mathcal{K}_2 = \mathcal{P} \setminus ([A, A_s] \cup \{B, B_r\})$;
- $\mathcal{K}_3 = \mathcal{P} \setminus ([B_1, B] \cup \{A\})$;
- $\mathcal{K}_4 = \mathcal{P} \setminus \{A, B, A_1, B_r\}$.

**Lemma 4.6.** Let $\mathcal{P}$ be a closed path polyomino having a ladder of at least three steps satisfying the previous assumptions. Then the following hold:

1. $S_{\mathcal{P}}/(I_{\mathcal{P}, x_g}) \cong K[\mathcal{P}]$;
2. $S_{\mathcal{P}}/(I_{\mathcal{P}, x_g, x_d}) \cong K[K_1] \otimes_K K[x_{d_1}, \ldots, x_{d_r}]$;
3. $S_{\mathcal{P}}/(I_{\mathcal{P}, x_g, x_d, x_b}) \cong K[K_2] \otimes_K K[x_a, x_b, x_{a_1}, \ldots, x_{a_{s-2}}]$;
4. $S_{\mathcal{P}}/(I_{\mathcal{P}, x_g, x_d, x_b}) \cong S_{\mathcal{P}}/(I_{\mathcal{P}, x_g, x_d, x_b})$;
Remark 4.7. If suppose that $B_2$ has just two cells (so $s = 1$), then $(I_P, x_g, x_d, x_b)$ is not prime. In fact, set $b = a_0$, denote by $C$ the cell adjacent to $A_1$, and let $b, p$ and $q, a_1$ be respectively the diagonal and anti-diagonal corners of $C$. Observe that in such a case $x_q x_{a_1} \in (I_P, x_g, x_d, x_b)$ but $x_q, x_{a_1} \notin (I_P, x_g, x_d, x_b)$.\[\square\]
Theorem 4.8. Let \( \mathcal{P} \) be a closed path polyomino having a ladder of at least three steps satisfying the assumptions at the beginning of Subsection 4.2. Then
\[
\text{HP}_{K[\mathcal{P}]}(t) = \frac{h_{K[\mathcal{K}]}(t) + t[h_{K[\mathcal{K}]}(t) + 2 \cdot h_{K[\mathcal{K}]}(t) + h_{K[\mathcal{K}]}(t)]}{(1-t)^{|V(\mathcal{P})|-\text{rank} \mathcal{P}}}
\]
In particular \( K[\mathcal{P}] \) has Krull dimension \(|V(\mathcal{P})|-\text{rank} \mathcal{P}\).

Proof. It follows considering the opportune exact sequences by Lemma 4.6 and arguing as done in Theorem 3.9 and Corollary 3.10. \(\square\)

5. Rook-polynomial and Gorenstein property

Let \( \mathcal{P} \) be a polyomino. A \( k \)-rook configuration in \( \mathcal{P} \) is a configuration of \( k \) rooks which are arranged in \( \mathcal{P} \) in non-attacking positions.

![Figure 7. An example of a 4-rook configuration in \( \mathcal{P} \).](image)

The rook number \( r(\mathcal{P}) \) is the maximum number of rooks which can be placed in \( \mathcal{P} \) in non attacking positions. We denote by \( R(\mathcal{P}, k) \) the set of all \( k \)-rook configurations in \( \mathcal{P} \) and we set \( r_k = |R(\mathcal{P}, k)| \) for all \( k \in \{0, \ldots, r(\mathcal{P})\} \), conventionally \( r_0 = 1 \). The rook-polynomial of \( \mathcal{P} \) is the polynomial in \( \mathbb{Z}[t] \) defined as \( r_\mathcal{P}(t) = \sum_{k=0}^{r(\mathcal{P})} r_k t^k \).

We recall that a polyomino is thin if it does not contain the square consisting of four cells. In [20] the authors prove that if \( \mathcal{P} \) is a simple thin polyomino then \( h_{K[\mathcal{P}]}(t) = r_\mathcal{P}(t) \) and, in particular, \( \text{reg} K[\mathcal{P}] = r(\mathcal{P}) \) (see [20] Theorem 3.12)). Now we show how the rook polynomial is related to Poincaré-Hilbert series of the polyominoes considered in this work.

Proposition 5.1. Let \( \mathcal{P} \) be a \((\mathcal{L}, \mathcal{C})\)-polyomino. Then:

1. \( r(\mathcal{P}_1) = r(\mathcal{P}_2) = r(\mathcal{P}) - 1; \)
2. \( r(\mathcal{P}_3) = r(\mathcal{P}) - 2; \)
3. \( r(\mathcal{P}) - 2 \leq r(\mathcal{P}_4) \leq r(\mathcal{P}). \)

Proof. 1) Consider \( \mathcal{P} \) and \( \mathcal{P}_1 = \mathcal{P}\setminus[A, A_r] \). Once we fix a rook in a cell of \([A_1, \ldots, A_{r-1}]\), we cannot place another rook in \([A, A_r]\) in non-attacking position in \( \mathcal{P} \), so \( r(\mathcal{P}) = r(\mathcal{P}_1) + 1 \). Hence \( r(\mathcal{P}_1) = r(\mathcal{P}) - 1 \). In a similar way it can be showed that \( r(\mathcal{P}_2) = r(\mathcal{P}) - 1 \).

2) It follows by similar previous arguments on the intervals \([A, A_r]\) and \([A, B_3]\).

3) Since \( \mathcal{P} = \mathcal{P}_4 \cup [A, A_3] \cup [A, B_4] \), it is obvious that \( r(\mathcal{P}_4) \leq r(\mathcal{P}) \). Moreover, \( \mathcal{P}_4 = \mathcal{P}_3 \cup [A_2, A_r] \cup [B_2, B_4] \), so \( r(\mathcal{P}_3) \leq r(\mathcal{P}_4) \), that is \( r(\mathcal{P}) - 2 \leq r(\mathcal{P}_4) \). In particular, observe that if \( r, s > 3 \) then \( r(\mathcal{P}_4) = r(\mathcal{P}) \), if either \( r = 3 \) or \( s = 3 \) then \( r(\mathcal{P}_4) = r(\mathcal{P}) - 1 \), and if \( r, s = 3 \) then \( r(\mathcal{P}_4) = r(\mathcal{P}) - 2 \). \(\square\)

Theorem 5.2. Let \( \mathcal{P} \) be a \((\mathcal{L}, \mathcal{C})\)-polyomino. Suppose that \( \mathcal{C} \) is a simple thin polyomino. Then \( h_{K[\mathcal{P}]}(t) \) is the rook-polynomial of \( \mathcal{P} \). Moreover, we have that \( \text{reg}(K[\mathcal{P}]) = r(\mathcal{P}) \).
Proof. It is known that \( h_{K[\mathcal{P}]}(t) = h_{K[\mathcal{P}_4]}(t) + t [h_{K[\mathcal{P}_4]}(t) + h_{K[\mathcal{P}_3]}(t) + (1-t) h_{K[\mathcal{P}_3]}(t)] \). We denote by \( r_{\mathcal{P}_j}(t) = \sum r_{j}^{(P)} r_{j}^{(P)} t^k \) the rook-polynomial of \( \mathcal{P}_j \). Since \( \mathcal{C} \) is a simple thin polyomino, then \( \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 \) and \( \mathcal{P}_4 \) are simple thin polyominoes, so \( h_{K[\mathcal{P}_j]}(t) = r_{\mathcal{P}_j}(t) \) for \( j \in \{1, 2, 3, 4\} \). From Proposition 5.1 we have \( \deg h_{K[\mathcal{P}]} = r(\mathcal{P}) \). Then

\[
\begin{aligned}
\quad h_{K[\mathcal{P}]}(t) = \sum_{k=0}^{r(\mathcal{P})} [r_k^{(4)} + r_{k-1}^{(2)} + r_{k-1}^{(3)} - r_{k-2}^{(3)}] t^k,
\end{aligned}
\]

where we set \( r_{j}, r_{k-1}, r_{k-2}, r_{\mathcal{P}-1}, r_{k}^{(4)} \) equal to 0, for all \( j \in \{1, 2, 3\} \) and for \( k \geq r(\mathcal{P}_4) \).

We want to prove that \( r_k^{(4)} + r_{k-1}^{(2)} + r_{k-1}^{(3)} - r_{k-2}^{(3)} \) is exactly the number of ways in which \( k \) rooks can be placed in \( \mathcal{P} \) in non attacking positions, for all \( k \in \{0, \ldots, r(\mathcal{P})\} \). Fix \( k \in \{0, \ldots, r(\mathcal{P})\} \). Observe that:

1. \( r_k^{(4)} \) can be viewed as the number of \( k \)-rook configurations in \( \mathcal{P} \) such that no rook is placed on \( A, A_1 \) and \( B_1 \).
2. Assume that a rook \( \mathcal{T} \) is placed in \( A_1 \). Then we cannot place any rook on a cell of \( [A, A_r] \), so \( r_{k-1}^{(1)} \) is the number of all \( (k-1) \)-rook configurations in \( \mathcal{P}_1 \). Hence \( r_{k-1}^{(1)} \) is the number of all \( k \)-rook configurations in \( \mathcal{P} \) such that a rook is on \( A_1 \). Observe that there are some \( k \)-rook configurations in \( \mathcal{P} \) in which a rook \( \mathcal{T}' \neq \mathcal{T} \) is on \( B_1 \). Paraphrasing, note that \( r_{k-1}^{(1)} \) is the number of all \( k \)-rook configurations in \( \mathcal{P} \) such that \( \mathcal{T} \) is on \( A_1 \) and \( \mathcal{T}' \) is not on \( B_1 \) plus that ones where \( \mathcal{T} \) is on \( A_1 \) and \( \mathcal{T}' \) is on \( B_1 \).
3. Assume that a rook \( \mathcal{T} \) is placed in \( B_1 \). Arguing as before, \( r_{k-1}^{(1)} \) is the number of all \( k \)-rook configurations in \( \mathcal{P} \) such that \( \mathcal{T} \) is on \( B_1 \) and \( \mathcal{T}' \) is not on \( A_1 \) plus those ones where \( \mathcal{T} \) is on \( B_1 \) and \( \mathcal{T}' \) is on \( A_1 \).
4. Assume that a rook is placed on \( A \). Then we cannot place any rook on a cell of \( [A, A_r] \cup [A, B] \), so \( r_{k-1}^{(3)} \) is the number of all \( (k-1) \)-rook configurations in \( \mathcal{P}_4 \), that is the number of \( k \)-rook configurations in \( \mathcal{P} \) such that a rook is placed on \( A \).
5. Fix a rook \( \mathcal{T} \) in \( A_1 \) and another one \( \mathcal{T}' \) in \( B_1 \). Then we cannot place any rook on a cell of \( [A, A_r] \cup [A, B] \), so \( r_{k-2}^{(3)} \) is the number of all \( (k-2) \)-rook configurations in \( \mathcal{P}_4 \). Hence \( r_{k-2}^{(3)} \) is the number of all \( k \)-rook configurations in \( \mathcal{P} \) such that a rook is on \( A_1 \) and another is on \( B_1 \).

From 1), 2), 3), 4) and 5) it follows that \( r_k^{(4)} + r_{k-1}^{(1)} + r_{k-1}^{(2)} + r_{k-1}^{(3)} - r_{k-2}^{(3)} \) is the number of \( k \)-rook configurations in \( \mathcal{P} \).

\( \square \)

**Proposition 5.3.** Let \( \mathcal{P} \) be a closed path satisfying the conditions in Subsection 4.2. Then \( h_{K[\mathcal{P}]}(t) \) is the rook-polynomial of \( \mathcal{P} \) and \( \text{reg}(K[\mathcal{P}]) = r(\mathcal{P}) \).

**Proof.** It can be proved by similar arguments as in Proposition 5.1 that the rook-numbers of \( \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3 \) and \( \mathcal{K}_4 \) satisfy the following:

1. \( r(\mathcal{K}_1) = r(\mathcal{P}) - 1; \)
2. \( r(\mathcal{P}) - 2 \leq r(\mathcal{K}_2) \leq r(\mathcal{P}) - 1; \)
3. \( r(\mathcal{K}_3) = r(\mathcal{P}) - 1; \)
4. \( r(\mathcal{P}) - 2 \leq r(\mathcal{K}_4) \leq r(\mathcal{P}). \)

We denote by \( r_{\mathcal{K}_j}(t) = \sum_{k=0}^{r(\mathcal{K}_j)} r_{k}^{(j)} t^k \) the rook-polynomial of \( \mathcal{K}_j \), for \( j = 1, 2, 3, 4 \). Observe that \( \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3 \) and \( \mathcal{K}_4 \) are simple thin polyominoes, so \( h_{K[\mathcal{K}_j]}(t) = r_{\mathcal{K}_j}(t) \) for \( j \in \{1, 2, 3, 4\} \), and from the above formulas and Theorem 4.8 we have \( \deg h_{K[\mathcal{P}]} = r(\mathcal{P}) \). Moreover

\[
\begin{aligned}
\quad h_{K[\mathcal{P}]}(t) = \sum_{k=0}^{r(\mathcal{P})} [r_k^{(4)} + r_{k-1}^{(2)} + 2r_{k-1}^{(3)}] t^k,
\end{aligned}
\]
where we set \( r^{(j)}_{-1}, r^{(4)}_k, r^{(2)}_l \) equal to 0, for all \( j \in \{1, 2, 3\} \), for \( k \geq r(K_4) \) and \( l \geq r(K_2) \).

Similarly as done in Theorem 5.2, we have that \( r^{(4)}_k + r^{(1)}_{k-1} + 2r^{(2)}_{k-1} + r^{(3)}_{k-1} \) is the number of \( k \)-rook configurations in \( P \), for all \( k \in \{0, \ldots, r(P)\} \). In fact, let \( k \in \{0, \ldots, r(P)\} \). Observe that:

1. \( r^{(4)}_k \) is the number of \( k \)-rook configurations in \( P \) such that no rook is placed on \( A, A_1, B \) and \( B_r \).
2. Fix a rook \( T \) on \( B_r \). Then \( r^{(1)}_{k-1} \) is the number of all \( k \)-rook configurations in \( P \) such that \( T \) is on \( B_r \). Observe that, among these configurations, there are some \( k \)-rook configurations in which \( T' \neq T \) is placed either in \( A \) or in \( A_1 \).
3. Fix a rook \( T \) in \( B \). Then \( r^{(3)}_{k-1} \) is the number of all \( k \)-rook configurations in \( P \) such that \( T \) is on \( B \). As before, among these configurations there are some \( k \)-rook configurations in which \( T' \neq T \) is placed in \( A_1 \).
4. Assume that a rook is placed in \( A \) (resp. \( A_1 \)). Then \( r^{(2)}_{k-1} \) is the number of all \( k \)-rook configurations in \( P \) such that \( T \) is on \( A \) (resp. \( A_1 \)), and no rook is on a cell of \([A, A_s] \cup \{B, B_r\}\).

From 1), 2), 3) and 4) we have the desired conclusion.

In order to complete the cases of closed path polyominoes having no \( L \)-configuration, it remains to consider \( 1 \)-Configuration and \( 2 \)-Configuration introduced in Subsection 4.1. For such cases we mention only the analogous result, omitting the proof since the arguments are the same.

**Proposition 5.4.** Let \( P \) be a closed path satisfying the conditions of \( 1 \)-Configuration or \( 2 \)-Configurations in Subsection 4.1. Then \( h_{K[P]}(t) \) is the rook-polynomial of \( P \) and \( \text{reg}(K[P]) = r(P) \).

Observing that a closed path having an \( L \)-configuration is an \((L, C)\)-polyomino with \( C \) a path of cells and gathering all the results above, we obtain the following general result.

**Theorem 5.5.** Let \( P \) be a closed path having no zig-zag walks, equivalently having an \( L \)-configuration or a ladder of three steps. Then:

- \( K[P] \) is a normal Cohen-Macaulay domain of Krull dimension \( |V(P)| - \text{rank} \, P \);
- \( h_{K[P]}(t) \) is the rook-polynomial of \( P \) and \( \text{reg}(K[P]) = r(P) \).

At this point we are ready to provide the condition for the Gorenstein property of a closed path polyomino with no zig-zag walks. To reach this aim we recall the definition of \( S \)-property given in [20] for a thin polyomino.

**Definition 5.6.** Let \( P \) be a thin polyomino. A cell \( C \) is called single if there exists a unique maximal interval of \( P \) containing \( C \). We say that \( P \) has the \( S \)-property if every maximal interval of \( P \) has only one single cell.

Observe that if \( P \) is a closed path polyomino, then \( P \) has the \( S \)-property if and only if every maximal block of \( P \) contains exactly three cells.

**Theorem 5.7.** Let \( P \) be a closed path having no zig-zag walks. The following are equivalent:

1. \( P \) has the \( S \)-property;
2. \( K[P] \) is Gorenstein.

**Proof.** If \( P \) has no zig-zag walks, then \( K[P] \) is a normal Cohen-Macaulay domain of Krull dimension \( |V(P)| - \text{rank} \, P \) and \( h_{K[P]}(t) = r_P(t) = \sum_{k=0}^s r_k t^k \), where \( s = r(P) \). In such a case it is known (23) that \( K[P] \) is Gorenstein if and only if \( r_i = r_{s-i} \) for all \( i = 0, \ldots, s \).

1) \( \Rightarrow \) 2). Suppose that \( P \) has the \( S \)-property. Fix \( i \in \{0, 1, \ldots, r(P)\} \) and prove that \( r_i = r_{s-i} \). Since \( P \) has the \( S \)-property, \( P \) consists of maximal cell intervals of rank three. If \( i = 0 \) then it is
Define $\mathcal{P}_1 = \mathcal{P}\backslash\{A, A_1, A_2\}$, $\mathcal{P}_2 = \mathcal{P}\backslash\{A, A_1, A_2, C_1, C_2\}$ and $\mathcal{P}_3 = \mathcal{P}\backslash\{A, A_1, A_2, B_1, B_2\}$. We denote by $r_{\mathcal{P}_j}(t) = \sum_{k=0}^{r_{\mathcal{P}_j}} r_k(t)^k$ the rook-polynomial of $\mathcal{P}_j$. Observe that $r(\mathcal{P}_1) = r(\mathcal{P}) - 1 = s - 1$ and $r(\mathcal{P}_2) = r(\mathcal{P}_3) = r(\mathcal{P}) = 2 = s - 2$. By similar arguments as in Theorem 5.2 it is easy to prove that $r_k = r_k^{(1)} + r_k^{(1)} + r_k^{(2)}$ for all $k \in \{1, \ldots, s\}$. Then

$$r_{s-i} = r_{s-i}^{(1)} + r_{a_i}^{(1)} + r_{a-i-1}^{(2)} = r_{s-i}^{(1)}(s-1) - (i-1) + r_{(s-1)-i}^{(1)} + r_{(s-1)-i}^{(1)} + r_{(s-2)-(i-1)}^{(3)}.$$

Since $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ are simple thin polyominoes having the $S$-property, then from Theorem 4.2 of [20] we have: $r_{a_i}^{(1)} = r_{i-1}^{(1)}$, $r_{a_i}^{(2)} = r_{i}^{(1)}$, $r_{a-i-1}^{(1)} = r_{i-1}^{(2)}$ and $r_{a-i-1}^{(2)} = r_{i}^{(3)}$. Hence

$$r_{s-i} = r_{a-i-1}^{(3)} + r_{i}^{(2)} + r_{a-i-1}^{(3)} = r_{i}^{(1)} + r_{i}^{(1)} + r_{i}^{(2)} + r_{i}^{(3)} = r_{s-i}.$$ 

(2) ⇒ 1). Assume that $K[\mathcal{P}]$ is Gorenstein, that is $r_i = r_{s-i}$ for all $i = 0, \ldots, s$. We prove that $\mathcal{P}$ has the $S$-property. First of all, we observe that all the ranks of the maximal intervals of $\mathcal{P}$ cannot be greater or equal to four. In fact, if there exists a maximal interval $I = [A, B]$ with rank $I \geq 4$, then we can consider two distinct cells $C, D \in I\backslash\{A, B\}$. Hence we can obtain an $s$-rook configuration in $\mathcal{P}$ with a rook in $C$ and another one with a rook in $D$, so $r_s \geq 2 > r_0 = 1$, that is a contradiction.

Let $\{A, A_1, A_2, B_1, B_2\}$ be an $L$-configuration of $\mathcal{P}$, as in Figure 8. Consider $\mathcal{P}' = \mathcal{P}\backslash\{A, A_1, A_2\}$, which is a simple thin polyomino. Let $r_{\mathcal{P}'}(t) = \sum_{k=0}^{r_{\mathcal{P}'}} r_k(t)^k$ the rook-polynomial of $\mathcal{P}'$, where $s' = r(\mathcal{P}) - 1$. We prove that $\mathcal{P}'$ has the $S$-property. Suppose that $\mathcal{P}'$ has not the $S$-property so from the case b) ⇒ c) of [20] Theorem 4.2] it follows that either $r_{s'} > 1$ or $r_{s'-1} > r(\mathcal{P}')$. Both cases lead to a contradiction with $r_s = 1$ or $r_{s-1} = r(\mathcal{P})$. By similar arguments we can prove that $\mathcal{P}'' = \mathcal{P}\backslash\{A, B_1, B_2\}$ is a simple thin polyomino having the $S$-property. Since $\mathcal{P}'$ and $\mathcal{P}''$ have the $S$-property, it follows trivially that also $\mathcal{P}$ has the $S$-property. \[\square\]

**Remark 5.8.** With reference to Subsection 4.1 and to the particular weakly closed path $Q$, we mentioned in Remark 4.2 that $K[Q]$ is a normal Cohen-Macaulay domain but its Krull dimension was unknown. From Theorem 4.3 we obtain that $HP_{K[Q]}(t) = h(t) = h(1) - h_{K[Q]}(t)$ with $h(t) = h_{K[Q]}(t)$, where $h(t) = h_{K[Q]}(t)$ and $h_{K[Q]}(t)$ are the rook-polynomials respectively of $\mathcal{P}$ and $Q$. Since $Q$ is contained in $\mathcal{P}$, then $h(1) > 0$. So the Krull dimension of $K[Q]$ is $|V(\mathcal{P})| - r(\mathcal{P}) = |V(Q)| - r(\mathcal{Q})$. Moreover, by the same arguments adopted in this work, we obtain that $h(t)$ is the rook-polynomial of $Q$ and $K[Q]$ is Gorenstein if and only if $Q$ has the $S$-property.

**Concluding remarks.**

In the existing literature the Poincaré-Hilbert series and the Gorenstein property of a polyomino ideal have been provided only for some classes of simple polyominoes. In this work we provide some results in this line of research for a class of non simple polyominoes, that is closed paths having no zig-zag walks, since it is known the Cohen-Macaulayness of their coordinate rings. Considering our results, some questions arise:
(1) The $(L,C)$-polyominoes are non simple polyominoes. We ask if it is possible to characterize the Gorenstein property for these polyominoes with different choices of $C$.

(2) For the Poincaré-Hilbert series of closed paths we consider only those without zig-zag walks. Is it possible to provide similar results for the Poincaré-Hilbert series of a closed path having zig-zag walks?

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