Robust Scale-Free Synthesis for Frequency Control in Power Systems

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Abstract—This paper develops a framework for power system stability analysis, that allows for the decentralised design of frequency controllers. The method builds on a novel decentralised stability criterion, expressed as a positive real requirement, that depends only on the dynamics of the components at each individual bus, and the aggregate susceptance of the transmission lines connected to it. The criterion is both robust to network uncertainties as well as heterogeneous network components, and it can be verified using several standard frequency response, state space, and circuit theory analysis tools. Moreover, it allows to formulate a scale free synthesis problem, that depends on individual bus dynamics and leverages tools from $\mathcal{H}_\infty$ optimal control. Notably, unlike similar passivity methods, our framework certifies the stability of several existing (non-passive) power system control schemes and allows to study robustness with respect to delays.

Index Terms—Power systems, frequency control, robust stability, decentralised control synthesis.

I. INTRODUCTION

The composition of the electric grid is in state of flux [2]. Motivated by the need of reducing carbon emissions, conventional synchronous combustion generators, with relatively large inertia, are being replaced with renewable energy sources with little (wind) or no inertia (solar) at all [3]. Alongside, the steady increase of power electronics on the demand side is gradually diminishing the load sensitivity to frequency variations [4]. As a result, rapid frequency fluctuations are becoming a major source of concern for several grid operators [5], [6]. Besides increasing the risk of frequency instabilities, this dynamic degradation also places limits on the total amount of renewable generation that can be sustained by the grid.

One solution that has been proposed to mitigate this degradation is to use inverter-based generation to mimic synchronous generator behavior, i.e. implement virtual inertia [7], [8]. However, while virtual inertia can indeed mitigate this degradation, it is unclear whether this particular choice of control is the most suitable for this task. On the one hand, unlike generator dynamics that set the grid frequency, virtual inertia controllers estimate the grid frequency and its derivative using noisy and delayed measurements, which can lead to noise amplification and instabilities [9], [10]. On the other hand, inverter-based control can be significantly faster than conventional generators. Thus using inverters to mimic generators behavior does not take advantage of their full potential.

However, as more sophisticated controllers are deployed, the dynamics of the power grid become more complex and uncertain, which makes the application of direct stability methods harder. The challenge is therefore to design a control architecture that takes advantage of the added flexibility provided by new resources, while providing stability guarantees. Such an architecture must take into account the effect of delays and measurement noise in the design. And must provide a “plug-and-play” functionality by yielding decentralised, yet not overly conservative, stability certificates. It is the purpose of this paper to provide tools that can be used as a rigorous basis for the robust design of controllers in power systems.

To obtain suitable synthesis tools, the authors argue that it is necessary to take a deliberate step away from the philosophy of standard control approaches. To justify this, suppose that we are able to construct a linear model of a power system about a given operating point $x_0$. If we call such a model $N(x_0)$, to verify that the system is stable about a range of operating points $x_0 \in \mathcal{O}$ would then require checking stability of a set of models $\mathcal{N} = \{N(x_0) : x_0 \in \mathcal{O}\}$. This approach introduces several challenges for power systems. Firstly, these linear models would be high-dimensional (of the order millions of states). Secondly, different operational configurations in $\mathcal{O}$ can render linear models $N(x_0)$ of different sizes. Finally, the set $\mathcal{O}$, and as a result the $\mathcal{N}$, can have uncountably many elements. Thus, trying to apply any method that is based on analysing an individual model, as is the standard practice in model based control, is therefore completely unfeasible. To even consider the corresponding synthesis problems, in which the criteria for elements in this set become coupled through the control variables, would constitute a special form of madness that belongs only in the realm of science fiction.

So how can it be that a problem seemingly so far beyond off the shelf control techniques, can be ‘solved’ so staggeringly well in practice? This is highly suggestive of some underlying structure in power system models that can be exploited to make sweeping simplifications to the corresponding analysis and synthesis problems. We identify a suitable structure by essentially turning the above problem on its head. Instead of trying to test whether a given set of models is stable, we instead look to identify a set of models $\mathcal{N}$ that we can a-priori guarantee to be stable. The key feature of our set is that it is defined entirely by the local network structure. We identify a class that is seemingly well suited to power system problems, and this constitutes the main theoretical contribution of the paper, presented as Theorem 1 in Section III.

Clearly then if we could somehow prove that $\mathcal{N} \subseteq \mathcal{N}$, we would have a method for tackling the original problem.
A. Linearised Power Network Model

We consider a set of $n$ generator buses, indexed by $i \in \{1, \ldots, n\}$, dynamically coupled through an AC network. Assuming operation around an equilibrium, the linearised dynamics are represented by the block diagram in Fig. 1

$$G(s) = \text{diag}(g_i(s))$$

is the diagonal transfer function of generators–with controllers in closed loop–at each bus. Each $g_i(s)$ has as output the angular velocity $\dot{\theta}_i$, and as input the net power at its generator axis, relative to its equilibrium value. This includes an outside disturbance $d_{P,i}$, reflecting variations in mechanical power or local load, minus the variation $P_{N,i}$ in electrical power drawn from the network.

Fig. 1. Block Diagram of Linearised Power Network

Bus Dynamics

\[ G = \text{diag}(g_i := \mathcal{F}_l(G_i, c_i)) \]

\[ d_P \]

\[ \frac{1}{s} L_B \]

\[ P_N \]

\[ \dot{\theta} \]

\[ g_i \]

\[ g_c \]

\[ g_m \]

\[ \mathcal{F}_l \]

\[ \mathcal{F}_g \]

\[ \mathcal{F}_c \]

\[ \mathcal{F}_m \]

\[ \mathcal{F}_l \]

\[ \mathcal{F}_g \]

\[ \mathcal{F}_c \]

\[ \mathcal{F}_m \]

guaranteeing the stability of $\mathcal{N}$. However this is not what we propose. We instead propose to use the set $\mathcal{N}$ as a basis on which to design the network. Doing so has two major advantages:

1) Such a network comes with a-priori stability guarantees about all its operating points and configurations.

2) Since $\mathcal{N}$ is locally defined, the corresponding synthesis questions are decentralised, small, and scale free.

Despite these advantages, the following question remains: in order to meet such a design requirement, would the power system have to be completely redesigned? This is after all where other related techniques such as passivity based control often struggle. We hope to convince the reader through our examples in Section IV that the answer to this question is no. To do so, we show that stability of a range of standard models can all be verified using our techniques.

Notation: $\mathcal{H}_\infty$ denotes the space of transfer functions of stable linear, time-invariant systems. This is the Hardy space of functions that are analytic on the open right half plane $\mathbb{C}_+$ with bounded norm $\|g(s)\| := \sup_{s \in \mathbb{C}_+} |g(s)|$. $\mathbb{D}$ denotes the subset of $\mathcal{H}_\infty$ that is continuous on the extended imaginary axis $\{j \omega \mid \omega > 0\}$. $\mathcal{R}$ denotes the set of (not necessarily proper) real rational functions, and $\mathcal{R}\mathcal{H}_\infty := \mathcal{R} \cap \mathcal{H}_\infty$. Finally, we follow standard control terminology and denote $\mathcal{F}_l(G, c)$ to the lower linear fractional transformation.

II. Problem Description

In this section we describe the frequency domain power network model used in this paper and formulate the proposed scale free design problem.

A. Linearised Power Network Model

The network power fluctuations $P_N$ are given by a linearised DC model of the power flow equations. More precisely,

$$P_N(s) = \frac{1}{s} L_B$$

(1)

where $L_B$ is a undirected weighted Laplacian matrix of the network with elements given by

$$[L_B]_{ij} = \frac{\partial}{\partial \theta_j} \sum_{j=1}^{n} k_{ij} \sin (\theta_i - \theta_j)|_{\theta = \theta_0}$$

where $\theta_0$ are the angles at steady state and the line parameters $k_{ij} := |V_i||V_j|b_{ij}$, with $|V_i|$ and $b_{ij}$ being the (constant) voltage magnitude at bus $i$ and the line susceptance, respectively. Without loss of generality, we assume that the steady state angle difference $(\theta_{0,i} - \theta_{0,j})$ across each line is smaller than $\frac{\pi}{2}$. A fact to be used later is that since the angle appears only within the sin function,

$$|L_B|_{ii} \leq \frac{\gamma_2}{2} := \sum_{j=1}^{n} k_{ij}.$$  \hspace{1cm} (2)

Finally, to explicitly account in our model for the controllers, we further open the loop of each $g_i(s)$ and define a generalized plant model $G_i(s)$ for each bus given by

$$\begin{bmatrix} \dot{\theta}_i(s) \\ z_i(s) \end{bmatrix} = \begin{bmatrix} G_{i,11}(s) & G_{i,12}(s) \\ G_{i,21}(s) & G_{i,22}(s) \end{bmatrix} \begin{bmatrix} d_{P,i}(s) - P_{N,i}(s) \\ P_{c,i}(s) \end{bmatrix}$$  \hspace{1cm} (3)

where the signal $z_i(s)$ specifies the measurements available for designing the local controller and

$$P_{c,i}(s) = c_i(s)z_i(s)$$  \hspace{1cm} (4)

denotes the controller’s power injection, with $c_i(s)$ being the dynamics of the controller to be designed. Note that in general these transfer functions and the controller $c_i(s)$ need not be scalar. The ‘generalised plant’ captures both the inertial dynamics at the bus, and specifies the measurements available for control system design.

Combining (1), (3), (4) leads to the following generic linear model for power systems frequency control:

$$\begin{bmatrix} \dot{\theta}_i(s) \\ y_i(s) \end{bmatrix} = G_i(s) \begin{bmatrix} d_{P,i}(s) - P_{N,i}(s) \\ P_{c,i}(s) \end{bmatrix},$$

$$P_{c,i}(s) = c_i(s)y_i(s),$$

$$P_N(s) = \frac{1}{s} L_B \dot{\theta}(s).$$  \hspace{1cm} (5)

Remark 1 (Model Assumptions): The linearised network model (5) implicitly makes the following assumptions which
are standard and well-justified for frequency control on transmission networks (2): (i) bus voltage magnitudes $|V_i|$ are constant for all $i$, (ii) lines $ij$ are lossless, and (iii) reactive power flows do not affect bus voltage phase angles and frequencies. See, e.g., [13], [14], [15] for applications of similar models for frequency control within the control literature.

Example: Although (5) is rather generic and can account for many generator models, for the purpose of illustrating our analysis and design framework in this paper we will mostly use the classical swing equations as our generator model. That is, we will consider the generator dynamics described by

$$\dot{m_i} + d_i \dot{\theta}_i = P_G, i + d_{P,i} - P_{N,i},$$

where $m_i$ and $d_i$ are the generator’s inertia and damping respectively. This leads to a generalised plant transfer function

$$G_i(s) = \left[ \frac{1}{m_i + d_i} \frac{1}{m_j + d_j} \right],$$

where the particular choice of $G_{i,21}(s)$ and $G_{i,22}(s)$ would depend on the measured signal $y_i(s)$. For example, if angular velocity measurements were available, then both $G_{i,21}(s) = G_{i,22}(s) = \frac{1}{m_i + d_i}$.

B. Stability Definition

We use the following internal stability definition to analyse eq. (5).

**Definition 1 (Stability):** Equation (5) is stable if and only if

$$\left[ \begin{array}{c} G \\ I \end{array} \right] (I + NG)^{-1} \left[ \begin{array}{c} N \\ I \end{array} \right] \in \mathbb{H}_{\infty}^{2n \times 2n},$$

where $G = \text{diag}(F_i(G_1, c_1), \ldots, F_i(G_n, c_n))$ and $N = \frac{1}{2L_B}$. This definition means that if the external signals (the disturbances from the loads) are bounded and tend to zero, then the internal signals $(P_N, \dot{\theta})$ will tend to zero. This does not necessarily mean that the ‘state variables’ $\theta$ will tend to $\theta_0$, since they $P_N$ do not appear explicitly in the internal signals. However, since

$$P_N = L_B (\theta - \theta_0),$$

it is clear that if $\lim_{t \to \infty} P_N(t) = 0$, then $\lim_{t \to \infty} \dot{\theta}(t) - \dot{\theta}_0 \in \text{Ker}(L_B)$. Therefore satisfying the stability requirement in Definition 1 ensures that the phases differences across transmission lines will return to their equilibrium values.

C. The Scale Free Design Problem

In this section we discuss the problem of designing the controllers $c_i(s)$ in eq. (5). The purpose is to justify the rationale behind the results presented in Section III. From the perspective of synthesis with provable guarantees, there are numerous challenges here, the three largest being:

1) The matrix $L_B$ is very large, and will change dimension if components are add to or removed from power system.
2) The controllers enter into eq. (5) in a decentralised manner.
3) The entries of $L_B$ depends on the operating point.

The first two points preclude the possibility of using almost any of the formal synthesis methods from the control literature. This is because the overwhelming majority of these methods need some fixed model of the system, struggle to handle decentralised controller architectures, and struggle to handle problems with lots of variables. The first point is particularly troublesome since it means that it is even hard to define a nominal model within standard frameworks since there are few uncertainty measures that can account for changing the input-output dimension. The final point compounds the problem further by ruling out a scenario based approach in which we try to design controllers valid for all operating points simultaneously. Since $L_B$ is large, there will be too many operating points to even analyse a given design at each one, let alone consider the synthesis problem.

Motivated by the above, we argue that for a synthesis method to have practical value for the power system application, it should meet the following scale free requirements:

(i) Allow each $c_i$ to be designed based only on local component models.

(ii) Guarantee robustness to the entries and size of $L_B$.

At first one might think such requirements make the problem impossible. However this is not necessarily the case; in fact the above points make precise the rationale behind passivity based design. If one took a passivity based approach, one could design the controllers $c_i$ to make $F_i(G_i, c_i)$ passive (as a local synthesis problem), the upshot of which would be that stability could be guaranteed for all possible networks.

The above makes a strong case for using passivity tools for power system problems. However there is considerable extra structure in our problem, since the network model $L_B$ is highly structured. Two natural questions are then, "Is the passivity based approach inherently conservative for this application?", and "do there exist better methods for addressing (i)-(ii) above?". Section III is devoted to these questions.

III. RESULTS

In this section we present results that allow the controllers $c_i$ to be designed based only on local component models, and independently of the operating point.

A. A Scale Free Stability Criterion

In this section we present a stability criterion that allows us to verify stability of eq. (5) independently of the entries of $L_B$. To do so, we leverage the upperbound (2), and perform an equivalent loop transformation, as described in Figure 3 that allows us to substitute $L_B$ with a normalized Laplacian $L := \frac{1}{2B} \frac{1}{2} L_B B^{-\frac{1}{2}}$ such that $0 \leq L \leq I$. Here, $B = \text{diag}(|L_B|_{ii})$, $\bar{P} = \text{diag}(p_i) := \text{diag}(\gamma_i g_i)$ where $\gamma_i$ satisfies (2).

![Fig. 3. Loop Transformation](image)
Thus, we focus our study on the feedback interconnection
\[ y_i(s) = p_i(s) u_i(s) \]
\[ u(s) = -\frac{1}{s} Ly(s) + e(s). \]  
(6)

In the above \( y, u, e \) are signals, \( p(s) \in \mathcal{H}_\infty \) a transfer function, and \( L \) a matrix from the following set
\[ \mathcal{L} := \{ A : A = A^T, 0 \preceq A \preceq I \} . \]

To formally address the scale free requirements, we pose the following problem motivated by [16].

**Problem 1:** Find a set \( \mathcal{P} \) such that if \( \forall i \)
\[ p_i \in \mathcal{P}, \]
then eq. (6) is internally stable for all \( L \in \mathcal{L} \). \( \diamond \)

To illustrate the meaning of Problem 1, let us consider the Extended Strictly Positive Real (ESPR) functions. A set \( \mathcal{L} \) is called Positive Real (PR) if for all \( L \in \mathcal{L} \), and it is well known that the negative feedback interconnection of a PR function is internally stable (e.g. [17]). However, this does not exploit the fact that \( \mathcal{L} \) is strictly larger than ESPR functions that solve Problem 1.

Crucially any such set could be used in exactly the same way as the ESPR property to conduct scale free design, with the advantage of being less restrictive.

Although our solution is distinct to passivity criteria, our solution to Problem 1 is written in terms of PR and ESPR functions. We present the criterion this way because it establishes strong connections to many powerful control methods, including:

1. Multiplier methods and absolute stability criteria [18].
2. State space techniques, e.g., \( \mathcal{H}_\infty \) optimal control [19].
3. The Nyquist criterion.
4. Classical circuit theory.

We will highlight these connections throughout the subsequent sections. We now formally define these function classes.

**Definition 2:** A (not necessarily proper or rational) transfer functions \( g(s) \) is **PR** if:

(i) \( g(s) \) is analytic in \( \text{Re} \{ s \} > 0 \);
(ii) \( g(s) \) is real for all positive real \( s \);
(iii) \( \text{Re} \{ g(s) \} \geq 0 \) for all \( \text{Re} \{ s \} > 0 \).

If in addition \( g \in \mathcal{H}_\infty \) and there exists an \( \epsilon > 0 \) such that \( g(s) - \epsilon \) is **PR**, then \( g(s) \) is **ESPR**.

**Remark 2:** The class of functions **ESPR** is typically only defined for the rational functions. In this case, \( g \in \mathcal{H}_\infty \) is said to be **ESPR** if it is stable, and there exists an \( \epsilon > 0 \) such that \( g(s) - \epsilon \) is **PR**. There does not seem to be a standard notion of an ESPR function in the non-real rational case [17]. We made the choice above because it can be checked using frequency domain methods, and in the real rational setting it coincides with the standard definition.

The following theorem shows that provided \( L \in \mathcal{L} \), and that the elements in the diagonal transfer function are drawn from a parametrised class
\[ \mathcal{P}_h := \{ p \in \mathcal{H}_\infty : p(0) \neq 0, h(s) \left( 1 + \frac{p_i(s)}{s} \right) \in \text{ESPR} \}, \]
then the feedback interconnection is stable. It therefore gives a solution to Problem 1.

**Theorem 1:** If \( h \in \text{PR} \), then for any \( p_1, \ldots, p_m \in \mathcal{P}_h \) and any \( L \in \mathcal{L} \), the feedback interconnection in eq. (6) is stable.

**Proof:** Let \( P = \text{diag} \{ p_1, \ldots, p_m \} \). Since \( P \in \mathcal{H}_\infty^{n \times n} \), the interconnection of \( P \) and \( \frac{1}{s} L \) is stable if and only if
\[ \frac{1}{s} L (I + \frac{1}{s} PL)^{-1} \in \mathcal{H}_\infty^{n \times n}. \]

Since \( L \) is symmetric, we can factorize it as
\[ L = QXQ^* , \]
where \( \epsilon \leq X \leq I, Q \in \mathbb{C}^{n \times (n-m)}, m > 0, Q^* Q = I, \epsilon > 0 \). Hence
\[ \frac{1}{s} L (I + \frac{1}{s} PL)^{-1} = QXQ^* (sI + PQQX^*)^{-1} = QX (sI + Q^* PQX^*)^{-1}Q^*. \]

Clearly then it is sufficient to show that
\[ (sI + Q^* PQX^*)^{-1} \in \mathcal{H}_\infty^{(n-m) \times (n-m)}. \]
(7)

The above can be immediately recognized as an eigenvalue condition:
\[ -s \notin \lambda(Q^*P(s)QX), \forall s \in \mathbb{C}_+. \]

By Theorem 1.7.6 of [20], for any \( s \in \mathbb{C} \):
\[ \lambda(Q^*P(s)QX) \subset \text{Co}(p_i(s), i \in N) \times [\epsilon, 1]. \]

In the above \( \times \) denotes the product \( S_1 \times S_2 = \{ ab : a \in S_1, b \in S_2 \} \). Therefore it is sufficient to show that
\[ 0 \notin \text{Co}(s + p_i(s), i \in (1, \ldots, n)) \times [\epsilon, 1], \]
for all \( s = \bar{c} + i \). Observe that since each \( p_i \) is bounded, this condition is trivially satisfied for large \( s \). It is therefore enough to check that this holds for \( s \in \mathbb{C}_+, |s| < R \), for sufficiently large \( R \). This can be done using the separating hyperplane theorem, applied pointwise in \( s \). In particular, [18] holds for any given \( s \) if and only if there exists a nonzero \( h \in \mathbb{C} \) and \( \gamma > 0 \) such that \( \forall i \in \{i, \ldots, n\} \):
\[ \text{Re} \left\{ h(s + kp_i(s)) \right\} > \gamma, \forall \epsilon \leq k \leq 1. \]  
(9)

By a very minor adaptation of the argument in Theorem 2 of [18], we can use a function to define this \( h \) pointwise in \( s \). From the conditions of the theorem and the maximum modulus principle, for any \( R \geq 0 \), there exists a \( \gamma > 0 \) such that \( \forall s \in \mathbb{C}_+, |s| \leq R \):
\[ \text{Re} \left\{ h(s) \left( 1 + \frac{p_i(s)}{s} \right) \right\} \geq \gamma. \]

Since \( h(s) \) is **PR** for all \( k^* \geq 0 \),
\[ \text{Re} \left\{ h(s + p_i(s)) + k^* sh(s) \right\} \geq \gamma. \]

Dividing through by \( (1 + k^*) \) shows that under these conditions
\[ \text{Re} \left\{ h(s) \left( 1 + \frac{p_i(s)}{s(1+k^*)} \right) \right\} \geq \gamma. \]

Therefore (9) is satisfied on the entire right half plane for the required range of \( k \) values. Consequently (7) is satisfied, and the result follows.
B. A Scale Free Analysis Method

Theorem 1 suggests the following simple scale free analysis method for checking stability of eq. (5):
1) Define a function \( h(s) \in \mathcal{P} \).
2) Check that \( \gamma \mathcal{F}_i (G_i, c_i) \in \mathcal{P} \).

Conceptually, specifying the function \( h(s) \) a priori is no different to specifying the passivity property. Crucially, the above can be done on a component by component basis, and guarantees stability independently of the operating point, and independently of the network size.

Rather than simply checking that (2) holds, instead we propose to find the largest \( \gamma \) such that \( \gamma \mathcal{F}_i (G_i, c_i) \in \mathcal{P} \). The following lemma shows that we can use this to deduce whether (2) holds. It is useful to do this because it will give our criteria robustness guarantees, and will also provide a synthesis objective in subsequent sections.

**Lemma 1:** Let \( p \in \mathcal{H}_\infty \) and \( h \in \mathcal{P} \). If \( p \in \mathcal{P}_h \), then for all \( 0 < \gamma \leq 1, \gamma p \in \mathcal{P} \).

**Proof:** Since \( p \in \mathcal{P}_h \), there exists an \( \epsilon \) such that \( h(s) \left( 1 + \frac{p(s)}{s} \right) - \epsilon \in \mathcal{P} \). Therefore
\[
\frac{1 - \gamma h(s) + h(s) \left( 1 + \frac{p(s)}{s} \right) - \epsilon}{\gamma} \in \mathcal{P}.
\]
This implies \( h(s) \left( 1 + \gamma \frac{p(s)}{s} \right) - \gamma \epsilon \in \mathcal{P} \). Consequently \( \gamma p(s) \in \mathcal{P}_h \) for all \( 0 < \gamma \leq 1 \) as required.

Based on the above, we define the following scale free analysis problem.

**Problem 2:** Given \( h, G_i, c_i \), solve
\[
\gamma^*_i = \sup_{\gamma > 0} \gamma \quad \text{s.t.} \quad \gamma \mathcal{F}_i (G_i, c_i) \in \mathcal{P}.
\]

Therefore by Lemma 1 if \( \gamma_i < \gamma^*_i \), then \( \gamma_i \mathcal{F}_i (G_i, c_i) \in \mathcal{P}_h \), and the difference \( \gamma^*_i - \gamma_i \) gives a measure of robustness. We now summarise some techniques for solving Problem 2. These both illustrate how to solve the problem, and also give insight into how the function \( h(s) \) should be selected.

1) **Frequency response methods:** Probably the simplest way to check that a function is ESPR is to plot its frequency response. The required result is the following.

**Lemma 2:** Let \( g \in \mathcal{A}_0 \). Then \( g \) is ESPR if and only if there exists an \( \epsilon > 0 \) such that
\[
\text{Re} \{ g(j\omega) \} > \epsilon, \quad \forall \omega \in \mathbb{R} \cup \{ \infty \}.
\]

**Proof:** Denote \( \phi(s) = \frac{1}{1 + s} \), and define \( z := \phi(s) \) and \( G(z) := g(\phi^{-1}(z)) \). Since \( \phi \) maps the open right half plane to the open unit circle
\[
\sup_{\text{Re}[s] > 0} \text{Re} \{ g(s) \} = \sup_{|z| < 1} \text{Re} \{ G(z) \}.
\]
Since \( g(s) \in \mathcal{A}_0 \), \( G(z) \) is analytic in the open unit disc, and continuous on the unit circle. Therefore by the maximum modulus principle
\[
\sup_{|z| < 1} \text{Re} \{ G(z) \} = \max_{t \in [0,2\pi]} \text{Re} \{ G(e^{jt}) \} = \max_{\omega \in \mathbb{R} \cup \{ \infty \}} \text{Re} \{ g(s) \}.
\]
The result is now immediate from Definition 2.

This suggests a simple frequency gridding approach for solving Problem 2. In particular it shows that Problem 2 is equivalent to
\[
\gamma^*_i = \sup_{\gamma > 0, \epsilon > 0, \omega > 0} \gamma \quad \text{s.t.} \quad \text{Re} \left\{ h(j\omega) \left( 1 + \frac{\gamma \mathcal{F}_i (G_i, c_i)}{j\omega} \right) \right\} \geq \epsilon,
\]
which is easily tackled with a host of numerical methods. Perhaps more importantly the frequency domain characterisation also connects Problem 2 to a host of classical criteria. To see this consider the following corollary of Lemma 2 which highlights a simple geometrical feature of \( \mathcal{P}_h \).

**Corollary 1:** Let \( h \in \mathcal{P}_r \), \( p \in \mathcal{H}_\infty \), and assume that \( p(0) \neq 0 \). The following are equivalent:

(i) \( p \in \mathcal{P}_h \).
(ii) For every \( \omega \in \mathbb{R} \cup \{ \infty \} \), \( p(j\omega) / j\omega \) lies in the interior of the halfplane
\[
\left\{ z \in \mathbb{C} : \text{Re} \{ e^{j\omega} (1 + z) \} \geq 0 \right\}.
\]

**Proof:** \( p \in \mathcal{P}_h \) if and only if \( h(s) \left( 1 + \frac{z}{s} \right) \in \mathcal{H}_r \). The red curve shows the ray \( (-\infty, -1] \).

Fig. 4. The black curve shows the frequency response of a transfer function and the shaded region the halfplane obtained if \( \angle h(j\omega) = \frac{\pi}{2} \).
\( P_h \). It is a classical result of Brockett and Willems [18] that the converse of this statement is also true. Observe also that the transfer function sketched in Figure 2 lies in a single half-plane for all positive frequencies. This is not required by Corollary 1 however it is a consequence of the off-axis circle criterion that if \( p(j\omega)/j\omega \) lies in a single half plane for all \( \omega \geq 0 \), then there exists an \( \mathcal{H} \in P_h \). We will show how to exploits this fact in Section IV-B. This also connects Theorem 1 to the Nyquist based results in [21], [22].

2) State space methods: If we restrict ourselves to the space of real rational transfer functions, state space techniques can also be employed. The following simple extension of the Kalman-Yakobovich-Popov (KYP) lemma is the required result. It shows that if we have a state space realisation of the component model and \( h \), we can solve Problem 2 by solving an Linear Matrix Inequality (LMI).

**Lemma 3:** Let \( p, h \in \mathcal{H}, \gamma \geq 0 \), and suppose that \( p(s), h(s)/s \) have minimal realisations

\[
p(s) = \begin{bmatrix} A_1 & B_1 & C_1 & D_1 \end{bmatrix}, \quad \frac{h(s)}{s} = \begin{bmatrix} A_2 & B_2 & C_2 & D_2 \end{bmatrix}.
\]

The following are equivalent:

(i) \( \gamma p \in \mathcal{H} \),

(ii) There exists an \( X > 0 \) such that

\[
\begin{bmatrix} A^TX + XA & CT - XB \\ C - B^TX & - (D + D^T) \end{bmatrix} < 0,
\]

where

\[
A = \begin{bmatrix} A_1 & B_1 & C_1 & A_2 ^T \end{bmatrix}, \quad B = \begin{bmatrix} 0 & B_2 \end{bmatrix},
\]

and

\[
C = \begin{bmatrix} \gamma C_1 & \gamma D_1 & C_2 & A_2 \end{bmatrix}, \quad D = C_2 B_2.
\]

Before proving the above, observe in particular that the LMI in Lemma 3 is affine in \( \gamma \). This means that we may solve Problem 1 by computing

\[
\gamma^* := \sup_{\gamma > 0} \gamma \quad \text{s.t.} \quad \begin{bmatrix} A^TX + XA & CT - XB \\ C - B^TX & - (D + D^T) \end{bmatrix} < 0,
\]

where \( A, B, C, D, \gamma \) are as in Lemma 3(ii).

**Proof:** We are required to show that the condition

\[
\gamma^* := \sup_{\gamma > 0} \gamma \quad \text{s.t.} \quad \begin{bmatrix} A^TX + XA & CT - XB \\ C - B^TX & - (D + D^T) \end{bmatrix} < 0
\]

is equivalent to the LMI in (ii). By the extension of the KYP lemma from [19] Lemma 2.3, if

\[
G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix},
\]

then the condition \( G \in \mathcal{H} \) is equivalent to the existence of an \( X > 0 \) such that

\[
\begin{bmatrix} A^TX + XA & CT - XB \\ C - B^TX & - (D + D^T) \end{bmatrix} < 0.
\]

Therefore we need only show that

\[
\gamma^* := \sup_{\gamma > 0} \gamma \quad \text{s.t.} \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]

where \( A, B, C, D \) are given as in (ii). Applying standard formulae for multiplying state space realisations shows that

\[
\frac{h(s)}{s} = \begin{bmatrix} A_1 & B_1 \gamma C_1 & B_2 \\ 0 & A_2 & B_2 \gamma D_1 C_2 & 0 \\ C_2 A_2 & C_2 B_2 & 0 & 0 \end{bmatrix}^T,
\]

and assume that \( (A, B_2) \) is stabilizable and that \( (C_2, A) \) is detectable. Then there exists a strictly proper controller \( c(s) \) such that \( F_i(M, c) \in \mathcal{H} \) if and only if there exist matrices \( X_1, X_2, Y_1, Y_2 \) such that

\[
\begin{bmatrix} AX_1 + B_2 X_2 & 0 \\ C_1 X_1 + D_1 X_2 - B_1^T & 0 \\ Y_1 A + Y_2 C_2 & Y_1 B_1 + Y_2 D_2 - C_1^T \\ -D_1 & 0 \end{bmatrix} + (*)^T \prec 0,
\]

\[
\begin{bmatrix} -X_1 & I \\ -Y_1 & -I \end{bmatrix} \prec 0.
\]
In [19] they also give an explicit realisation of a controller that renders $\mathcal{F}_1 (M, c) \in \text{ESPR}$, though due to space limitations we omit this.

Theorem 2 allows Problem 3 to be solved as follows. By computing a minimal realisation $M_r$ of the transfer function
\[
\begin{bmatrix}
\gamma(s) \\
0 \\
\end{bmatrix}
\begin{bmatrix}
G(s) + \begin{bmatrix}
\bar{h}(s) & 0 \\
0 & 0 \\
\end{bmatrix}
\end{bmatrix}
\]
and checking the LMI in Theorem 2, $\gamma \in$ can be computed to arbitrary precision using a bisection over $\gamma$.

D. Better Scale Free Design Criteria

Since we propose to use Theorem 1 as a basis for design, a natural question is “does there exist a more general criterion that can be used to conduct scale free design?”. Once again restricting ourselves to the rational setting, in the context of our notion of scale free design, this question can be formulated mathematically as follows:

Problem 4: Does there exist a class of transfer functions $Q \subset \mathcal{R}$ such that:

(i) $Q \supset R \cap \mathcal{P}_h$;

(ii) Theorem 1 holds with $\mathcal{P}_h$ replaced by $Q$.

Clearly if such a $Q$ existed, it could be used to impose a less restrictive requirement on the subsystems. Consider now the following relaxation of the sets $\mathcal{P}_h$:

\[
\overline{\mathcal{P}}_h := \left\{ p \in \mathcal{R} : p(0) \neq 0, h(s) \left( 1 + \frac{2p(s)}{s} \right) \in \text{PR} \right\}.
\]

The only difference between $\mathcal{P}_h$ and $\overline{\mathcal{P}}_h$ is that the ESPR requirement has been replaced by a PR one; a very small difference (c.f. definition 2). The following theorem shows that if a set $Q$ exists, it must lie in this gap $\mathcal{P}_h \subset Q \subset \overline{\mathcal{P}}_h$. This result is quite striking from the context of decentralised control. It shows that if we want to do analysis and design based on a subsystem requirement alone (i.e. no communication) in such a way as to be robust against all possible interconnections, to all intents and purposes we can do no better than Theorem 1.

Theorem 3: Let $h \in \mathcal{R} \cap \text{PR}$. If $\mathcal{P}_h \neq \emptyset$, then for any $n \geq 2$ there exist $p_1, \ldots, p_n \in \mathcal{P}$ and a normalised Laplacian matrix $L \in \mathbb{R}^{n \times n}$ such that the feedback interconnection of $P(s) = \text{diag}(p_1(s), \ldots, p_n(s))$ and $\frac{1}{2} L$ is unstable.

Proof: First observe that since $\mathcal{P}_h \neq \emptyset$, there exists a $p \in \mathcal{R} \cap \mathcal{P}_h$ and an $\epsilon > 0$ such that
\[
h(s) \left( 1 + \frac{2p(s)}{s} \right) - \epsilon \in \text{PR} \setminus \text{ESPR},
\]
and consequently (by lemma 2), for some $\omega_0 \geq 0
\[
h(j\omega_0) \left( 1 + \frac{2p(j\omega_0)}{j\omega_0} \right) - \epsilon = 0.
\]
It then follows that for any strictly proper $q \in \mathcal{R} \mathcal{H}_\infty$ satisfying $\|q\|_\infty = |q(j\omega_0)| = \epsilon$ (such a $q$ must exist by [24](Lemma 1.14)),
\[
h(s) \left( 1 + \frac{2p(s)}{s} \right) - q(s) \in \text{PR} \setminus \text{ESPR}
\]
\[
h(j\omega_0) \left( 1 + \frac{2p(j\omega_0)}{j\omega_0} \right) - q(j\omega_0) = 0.
\]
Rearranging the above shows that
\[
h(s) \left( 1 + \frac{2p(s) - q(s)}{h(s)} \right) \in \text{PR},
\]
and also that
\[
\left( 1 + \frac{2p(j\omega_0) - j\omega_0 q(j\omega_0)}{j\omega_0} \right) = 0.
\]
This means that $\bar{p}(s) := p(s) - \frac{q(s)}{h(s)}$ satisfies $\bar{p} \in \mathcal{P}_h$ and
\[
\left( 1 + \frac{2\bar{p}(j\omega_0)}{j\omega_0} \right) = 0.
\]
Since $1 + \frac{2\bar{p}(j\omega_0)}{j\omega_0} = 0$, this shows that $(I + \frac{1}{s}PL)^{-1} \notin \mathcal{R} \mathcal{H}_\infty$.

Remark 3: It turns out that classes $Q$ that solve Problem 4 do exist (this is the realm of function classes such as $\text{PR}$ or $\text{WPR}$, as discussed extensively in [17]), and we could have used such classes (and indeed have in previous research [11]) to broaden the applicability of our results. This comes at the price of more cumbersome solutions to the analysis and synthesis problems, which is why we have elected for using the ESPR class in this paper.

IV. Examples

In this section we give three examples illustrating the scale free results from the previous section. The main goal of this section is three fold. Firstly, we illustrate how the stability of existing power system models can be analyzed using the proposed methodology. Secondly, apply our analysis framework provide delay robustness guarantees for the swing dynamics with delayed droop control. Finally, we analyze the robust stability of automatic generation control (AGC) and design novel AGC controllers –using our synthesis tool– that can significantly increase the robustness margins.

A. Stability of the Swing Equations

In this example we will show that our criteria can be used to verify stability of the swing equations when there is no control. It is of course no great surprise that this model is stable; in fact passivity tools are perfectly adequate for this case, and show stability for any $L_B$. It is nevertheless reassuring that our conditions can easily cover this case, and will set the stage for future examples where passivity tools are no longer viable.

If we have no control, then for all $i$, $c_i = 0$, and consequently
\[
\mathcal{F}_1 (G_i, c_i) = \frac{1}{m_is + d_i},
\]
where \( m_i \geq 0 \) and \( d_i > 0 \). To solve Problem 2 we are therefore required to find the largest \( \gamma_i \) such that

\[
\frac{\gamma_i}{ms + d_i} \in \mathcal{P}_h.
\]

The following shows that there exists an \( h \) such that the above is satisfied for all finite values of the parameters. Therefore, provided \( (m_i, d_i, \gamma_i) \) are finite, the swing equation model is stable by Theorem 1.

**Corollary 2:** Let \( M \geq 0, D > 0 \) and \( \gamma^* > 0 \). There exists an \( h \in PR \) such that for all \( M \geq m \geq 0, D \geq d > 0, \) and \( \gamma^* \geq \gamma > 0 \),

\[
\frac{\gamma}{ms + d} \in \mathcal{P}_h.
\]

**Proof:** Let \( h(s) = \frac{s}{T^2 + 1} \), and consider Problem 2 which in this case is equivalent to:

\[
\sup_{\gamma > 0} \gamma \\
\text{s.t.} \quad \frac{s}{T} + 1 \left( 1 + \frac{\gamma}{s (ms + d)} \right) - \epsilon \in PR.
\]

Multiplying out the constraint shows that it is equivalent to

\[
(1 - T \epsilon) ms^2 + (d - cm - Tde)s + \gamma - de \in T ms^2 + (T + m) + d \in PR. (11)
\]

Applying eq. (10) shows that above is equivalent to

\[
\left( \sqrt{md(1 - T \epsilon)} - \sqrt{mT (\gamma - de)} \right)^2 \leq (d - cm - Tde)(T + m).
\]

Since \( \gamma^* \) is finite, if \( T \) is sufficiently small, then \( \sqrt{md(1 - T \epsilon)} \geq \sqrt{mT (\gamma - de)} \) for all \( \gamma, d, m \) meeting the constraints in the theorem statement. Therefore a simple sufficient condition for eq. (11) is that

\[
md(1 - T \epsilon) \leq (d - cm - Tde)(T + m).
\]

This simplifies to

\[
Td (d(1 - T \epsilon) - cm) - cm^2 \geq 0.
\]

Since \( d > 0 \), there always exists an \( \epsilon > 0 \) such that the above is satisfied, and therefore eq. (11) holds for all parameter values as required.

**B. Stability of Droop Control Subject to Delay**

In this example we will use our criteria to verify stability of the undamped swing equations when there is droop control subject to delays. In this case

\[
G_i = \frac{1}{m_i s} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad c_i = \frac{1}{r_i} e^{-s \tau_i},
\]

\[
\Rightarrow \mathcal{F}_i(G_i, c_i) = \frac{1}{m_i s + \frac{1}{r_i} e^{-s \tau_i}}. (12)
\]

In the above \( r_i > 0 \) is the so called droop constant, and \( \tau_i \geq 0 \) a measurement delay. Note that for now we have neglected governor and turbine dynamics, and assumed that the damping constant equals zero. These can easily be included, and will be in the next example. This simplified model however serves as a good approximation for delayed droop control, and allows for the derivation of clean analytical criteria.

Since \( \mathcal{F}_i(G_i(s), c_i(s)) \notin \mathcal{R} \), we will use the frequency response methods from Section III-B. Motivated by Corollary 1 we plot the Nyquist diagram of \( \mathcal{F}_i(G_i(s), c_i(s)) / s \) for certain parameter values. This is shown in Figure 5.

From this we observe that as we increase the delay, the curves are pushed closer and closer to the -1 point. This not only shows that passivity tools cannot be used, even for arbitrarily small values of the delay, but also that all the Nyquist diagrams lie within the same halfplane. This suggests that suitable sets \( \mathcal{P}_h \) can be found, but they can also be characterised in terms of the system parameters. In fact, given any set of system parameters \( (m_i, r_i) \) and an upper bound on the delay we may robustly solve Problem 2 analytically.

**Theorem 4:** Let \( \frac{\pi}{4} - 10^{-2} \geq \tau^* \geq 0 \). Then there exists an \( h \in PR \) such that for any \( r > 0, m > 0 \) the solution to the optimisation problem

\[
\gamma^* = \sup_{\gamma > 0} \gamma \\
\text{s.t.} \quad \gamma ms + \frac{1}{r} e^{-s \tau} \in \mathcal{P}_h, \forall 0 \leq \tau \leq rm \tau^* (13)
\]

satisfies

\[
\frac{\pi (\frac{\pi}{2} - \tau^*)}{2m (\tau^*)^2} \left( 1 - 10^{-4} \right) \leq \gamma^* \leq \frac{\pi (\frac{\pi}{2} - \tau^*)}{2m (\tau^*)^2} \left( 1 + 10^{-4} \right). (14)
\]

Before providing the proof of Theorem 4 it is worth discussing its implications. As mentioned earlier, (13) constitutes a robust version of Problem 2 where \( \gamma^* \) is such that \( \frac{\gamma^*}{ms + \frac{1}{r} e^{-s \tau}} \in \mathcal{P}_h \) for every \( \tau \) satisfying \( 0 \leq \tau \leq rm \tau^* \). This leads to the following corollary:

**Corollary 3:** For any network topology such that, for every bus \( i, \gamma_i = 2 \sum k_{ij} < \gamma^* \), with \( \gamma^* \) given by (13), the swing dynamics with delayed droop control (12) are stable for all \( \tau_i \leq r_i mr \tau^* \) where \( (\tau^*, \gamma^*) \) satisfy (14).

We now provide the proof of Theorem 4.

**Proof:** First note that by putting

\[
k = \frac{1}{m \gamma r^2}, \quad s = mars \quad \text{and} \quad t = \frac{s}{mr},
\]

we obtain the following canonical form:

\[
\frac{1}{s (ms + \frac{1}{r} e^{-s \tau})} = \frac{1/k}{s^2 + \frac{r}{s} \left( \frac{1}{s} \right)}.
\]
We will now try to find conditions on the parameters \( k, t \) so that there exists a \( \delta > 0 \) such that \( \forall 0 \leq \hat{\omega} < \infty \)
\[
v := \text{Re} \left\{ (x + jy) \left( k + \frac{1}{j\hat{\omega}(j\hat{\omega} + e^{-jt\omega})} \right) \right\} > \delta,
\]
where \( x = \pi \tau^*, y = 2(\pi - \tau^*) \). We will use this to construct an \( h \in \text{PR} \) such that Corollary 1(ii) holds.

Observe that the above is convex in \( k \). It is also easy to show that there exists a finite \( \omega_{\text{max}} \) and \( \delta^* > 0 \) such that eq. (15) holds for \( \hat{\omega} \geq \omega_{\text{max}} \). Consider the following constrained optimisation problem:
\[
v^* = \min_{\hat{\omega}, t} v
\text{s.t. } k = \frac{4\tau^2 \alpha}{\pi (\pi - 2\tau^*)}, 0 \leq \hat{\omega} \leq \omega_{\text{max}}, 0 \leq t \leq \tau^*.
\]
If \( v^* > 0 \), then eq. (15) will hold with \( \delta = \min \{ \delta^*, v^* \} \) for any \( k \geq \frac{4\tau^2 \alpha}{\pi (\pi - 2\tau^*)} \) and \( t \leq \tau^* \). Similarly, if we make \( \tau^* \) a variable and add the constraint
\[
0 \leq \tau^* \leq \frac{\pi}{2} - \epsilon,
\]
the same guarantees hold for any \( \tau^* \) in this interval. The above defines a nonconvex optimisation problem over a bounded domain, with variables \( \hat{\omega}, \omega, t, \tau^* \), and constants \( \alpha, \omega_{\text{max}}, \epsilon \). Certifiable upper and lower bounds can therefore be obtained using standard methods, for example branch and bound. This problem was solved for \( \epsilon = 10^{-2} \) and a suitably chosen \( \omega_{\text{max}} \) using the bminb solver in YALMIP for both \( \alpha = \frac{1}{\lambda T} \) and \( \alpha = \frac{1}{1+10^{-3}} \). In the first case this yielded a strictly positive lower bound, and in the second a strictly negative upper bound. Therefore for the larger \( \alpha \), eq. (15) holds for all \( k \) and \( t \) meeting the constraints, and for the lower value it does not. In terms of the original variables, this shows that given any \( \tau^* \), eq. (15) holds for all
\[
0 \leq \tau \leq m \tau^*, 0 \leq \gamma \leq \frac{\pi}{2m(\tau^*)} \left( 1 - 10^{-4} \right),
\]
and the sign in front of the \( 10^{-4} \) cannot be flipped. Therefore, if eq. (15) implied the existence of a suitable \( P_h \), the proof would be complete. We will now show this. Observe that Equation (15) is very Corollary 1 but with the function \( h \) replaced with the constant \( (x + jy) \). We will now show that eq. (15) implies that there exists an \( h \in \text{PR} \) the condition in Corollary 1 also holds. First note that eq. (15) is equivalent to the existence of a \( \delta_2 \) greater than
\[
\left| \arg \left\{ (k + \frac{1}{j\hat{\omega}(j\hat{\omega} + e^{-jt\omega})}) \right\} + \angle (x + jy) \right| \leq \frac{\pi}{2} - \delta_2.
\]
The result now follows from classical theory on RL and RC multipliers, and the off-axis circle criterion. More specifically, it follows directly from Lemma 1 of [25] that there exists an \( h \in \text{PR} \) of the form
\[
h = \frac{s}{s + T} \sum_{k=1}^{N} \frac{s + \alpha_k}{s + \beta_k},
\]
where \( 0 < \beta_1 < \alpha_1 < \beta_2 < \ldots < \beta_T \), such that for any \( \epsilon \) and interval \( \omega < \omega_i < \infty \)
\[
\left| \arg \left\{ h(j\omega_i) \right\} + \angle (x + jy) \right| < \epsilon.
\]
By choosing \( \epsilon < \delta_2 \) and considering the limits \( \omega \to 0 \) and \( \omega \to \infty \), the result soon follows.

\[\text{TABLE I}
\begin{tabular}{cccccccc}
\hline
\text{m} & \text{d} & \text{T}_g & \text{T}_i & \text{r} & \beta & \text{k} \\
\hline
0.16 & 0.02 & 0.08 & 0.40 & 3.00 & 0.33 & 0.30 \\
0.20 & 0.02 & 0.06 & 0.44 & 2.73 & 0.40 & 0.20 \\
0.12 & 0.02 & 0.07 & 0.30 & 2.82 & 0.38 & 0.40 \\
\hline
\end{tabular}\]

C. Stability of Automatic Generation Control

AGC is an extension of droop control. The primary objective of AGC is to regulate system frequency to the specified nominal value (50/60 Hz), while maintaining the flow of power between buses at their scheduled values. This is achieved by adjusting the power production of the generators based on both \( \delta_i \) and \( P_{N_i} \). A typical controller architecture is shown in Figure 6. Observe in particular that the integral action will drive the signal \( (P_{N_i} - \beta_i \delta) \) to zero in steady state, and hence restore the system frequency and power flows to their scheduled values, even in the presence of persistent disturbances \( d_{P,i} \).

From the control perspective, the synthesis task is to design the parameters \( \beta_i, k_i \). The conventional approach is loosely based on time scale separation arguments and simulation studies (see e.g. [12, §11.1.5]). It is common to select
\[
\beta_i \approx \frac{1}{r_i} + d_i,
\]
with \( k_i \) selected based on simulation studies to act on the time scale of 1-10 minutes. It has been observed that when ‘large’ \( \beta_i \)’s are chosen, stability issues can arise.

We can formally address the design of \( \beta_i, k_i \). Within our framework, the generalised plant is
\[
G_i(s) = \frac{1}{m_s + d} \frac{1}{m_s + d + \frac{1}{(m_s + d)(1 + TS_i)(1 + TS_i)}} \frac{1}{1 + TS_i},
\]
and the standard AGC controller is
\[
c_i(s) = \left[ -\frac{1}{r_i} 0 \right] + \frac{k_i}{s} \left[ -\beta_i 1 \right].
\]
To guide the design of \( c_i(s) \), we solved the analysis problem in Problem 2 for a range of values of the control parameters. For the first set of generator parameters from Table 1 this is shown in Figure 7. From this figure we see that the nominal design, which is marked by a cross, is a reasonable choice, though the stability margin could be further improved by reducing \( \beta_i \) or increasing \( k_i \). We also see that increasing \( \beta_i \) will reduce \( \gamma^* \), justifying the observation that ‘large’ \( \beta_i \)’s can cause stability problems.
We can also design\(\mathcal{H}_\infty\) controllers by solving the synthesis problem in Problem 3 using \(\mathcal{H}_\infty\) methods. Given the need for simple controllers, the value here is more in finding out what values of \(\gamma_{s,1}\) are possible, rather than in the controllers themselves. To this end we fixed the controller parameters \(r_i, \beta_i\) to their values from Table 2. Selecting the best possible \(k_i \in \mathbb{R}\) gives \(\gamma_{s,*} \approx 11\). However, by replacing the constant \(k_i\) with a transfer function \(k_i \in \mathcal{H}\), and solving the synthesis problem using the \(\mathcal{H}_\infty\) method from section III-C yielded
\[
10^4 \leq \gamma_{s,*} \leq 2 \cdot 10^4.
\]
This shows that the use of dynamic control has the potential to greatly increase the margin \(\gamma_{s,1}\). It is interesting to think how this can be exploited in the design of inverters, where the use of more complex controllers is a more realistic prospect.

V. CONCLUSIONS

We present a novel decentralised analysis and design framework for power frequency control in power systems that is robust to bus heterogeneity and network uncertainties. Our framework only uses local information in the analysis and synthesis problems, and as a result is independent of the system size, i.e., scale free. We illustrate the suitability of the framework for power systems by: (i) showing how the robustness of existing schemes can be analyzed and further improved using the newly developed tools; and (ii) providing novel delay robustness margin for the classical swing equations.

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