On intrinsic structure of wave function of fermion triplet in external monopole field

V.M. Red’kov
Institute of Physics, Belarus Academy of Sciences
F Skoryna Avenue 68, Minsk 72, Republic of Belarus
e-mail: redkov@dragon.bas-net.by

Abstract
Using the Weyl-Tetrode-Fock tetrad formalism, the problem of the fermion triplet in the external monopole field (‘t Hooft-Polyakov’s) is examined all over again. Spherical solutions corresponding to the total conserved momentum \( J_i = (l_i + S_i + T_i) \), are constructed. The \((\theta, \phi)\)-dependence is expressed in terms of the Wigner’s functions \(D_{j-m, \pm 1/2}(\phi, \theta, 0)\) and \(D_{j-m, \pm 3/2}(\phi, \theta, 0)\). The radial system of 12 equations decomposes into two sub-systems (each of the \(m\) has 6 equations) by diagonalizing some complicated inversion operator (acting on Lorentzian and isotopic coordinates: \(\hat{N}_S = (\hat{\pi} \otimes \hat{\Pi}_{\text{bisp.}}) \otimes \hat{P}\)). The case of minimal \(j = 1/2\) is considered separately. Further and more detailed analysis has been accomplished for the case of simplest external monopole-like field, namely, the one produced by putting the Dirac monopole potential into the non-Abelian scheme. Now, a discrete operation being able to be diagonalized, contains an additional complex parameter \(A\): \(\hat{N}_S^A = (\hat{\pi}_A \otimes \hat{\Pi}_{\text{bisp.}}) \otimes \hat{P}\). The same quantity enters (as a parameter) basic wave functions. Evidently, this quantity can manifest itself at calculating matrix elements of some physical observables \(\langle \Psi_{jm}^A | \hat{G} | \Psi_{jm'}^{A'} \rangle\). In particular, there have been analyzed the \(N_A\)-parity selection rules arisen, those depending on this \(A\)-parameter explicitly. As shown in the paper, such an \(A\)-freedom pointed is a specific consequence of that there exists additional matrix operations (symmetry of the relevant Hamiltonian). In the Cartesian isotopic gauge, the discrete \(N_A\)-operation depends on space coordinates explicitly:

\[
\hat{N}_A^\text{Cart.} = \left( \exp(iA \vec{n}_\theta \cdot \vec{n}_\phi) \hat{\Pi}_{\text{vect.}} \right) \otimes \hat{\Pi}_{\text{bisp.}} \otimes \hat{P}.
\]

The wave functions considered exhibit else one kind of freedom (named as \(B\)-freedom and associated in turn with its own symmetry transformations acting on the Hamiltonian). The entering of those \(A\) and \(B\) parameters looks most neatly in the Schwinger unitary gauge: \((\delta = \pm 1)\):

\[
\Psi_{jm}^{A,B}(x) = \left[ T_{+1} \otimes \Phi_{jm}^+(x) + e^{iB} T_0 \otimes \Phi_{jm}^0(x) + e^{iA} \delta T_{-1} \otimes \Phi_{jm}^-(x) \right]
\]

There has been examined the form of the transformation introduced: \(U(A' \rightarrow A')\) and \(U(B'' \rightarrow B')\), relating all possible basic wave functions associated with different values of \(A\) and \(B\); their explicit forms are calculated in the unitary and Cartesian gauge.
1. Introduction

The puzzle of monopole seems to be one of still yet unsolved problems of particle physics. Apparently, together with the search of new decided experiments and solutions of some nonlinear systems of equations, the task of analyzing already established results is worth attention too. So, the basic frame of the present investigation will be analysis of particle isotopic triplet with Lorentz spin \( S = \frac{1}{2} \) (earlier a similar approach was developed for the doublet case \([1]\); the triplet case looks in some aspect like the doublet one except there exist quite noticeable distinctions resulting from other isotopic structure of the triplet case which provides some novel physical features.

Some remarks on techniques used below might be of help; let us give them. The innovations of the present treatment consists in utilizing, instead of the so-called monopole harmonics formalism [2-11], the most conventional Wigner’s \( D \)-function techniques [12-19], and what is more, in applying the generally relativistic tetrad formalism of Tetrode-Weyl-Fock-Ivanenko (TWFI) [20-29]. Unification through the use of the generalized Schrödinger’s basis [29] will apparently simplify the real calculations carried out. In addition, the latter makes possible to reveal connection between the monopole topics and the Pauli’s investigation [30] concerning the problem of allowable spherically symmetric wave functions in quantum mechanics; his results bear on the Dirac’s \( eg \)-quantization condition (\( e \) and \( g \) are respectively an electric and magnetic charge).

After those opening and general statements, some more particular remarks referring to the present study and delineating its contains are to be given. In Sec. 1, a starting form of the relevant wave equation is specified by choosing particular bases simultaneously in the tetrad and isotopic frames; those are respectively the spherical tetrad basis and isotopic Schwinger’s (about terminology used see in Supplement A). The radial equations found by separation of variables are rather complicated. We simplify them by searching a suitable operator which could be diagonalized simultaneously with the \( \vec{j}_2 \) and \( j_3 \). As well known, the usual space reflection (\( P \)-reflection) operator for a bispinor field has to be followed by certain transformation in the isotopic space so that a required quantity could be constructed. However, the solution of this problem known to date is not general as much as possible: some possibilities have remained unused. For this reason the question of reflection symmetry \([31-59]\) in the particle triplet-monopole system is examined here in full detail. As a result we find that there exist two different possibilities depending on what type of external monopole potential is analyzed. So, in case of the simplest one \([60]\) that can be regarded as result of embedding the Abelian potential into the non-Abelian scheme the composite reflection operator \( \hat{N}_A \) is determined ambiguously: it depends on an arbitrary complex numerical parameter \( A \), \( e^{iA} \neq 0 \). In turn, if we treat the non-trivial monopole case \([61-63]\), this \( e^{iA} \) must be equated to 1.

Else one problem deserving to be mentioned concerns distinctions between the Abelian and non-Abelian monopoles. We draw attention to the fact that, in the non-Abelian case, two systems: a free triplet and a triple affected by monopole potential (wether a trivial or not-trivial one), they have their spherical symmetry operators \( \vec{j}_2 \), \( j_3 \) identically alike. Correspondingly, in both these cases, isotriplet wave functions do not vary at all in their dependence on angular variables \( \theta, \phi \). This non-Abelian wave functions’ property sharply contrasts with the Abelian one where both electronic wave functions and corresponding symmetry operators undergo significant transformations in the external monopole field \( (D^i_{m\mu \nu} \) designates the Wigner’s \( D \)-functions; the value \( eg = 0 \) relates to the free electronic
\[ J_{1}^{eg} = l_{1} + \left( i \sigma^{12} - e g \right) \cos \phi \sin \theta , \quad J_{2}^{eg} = l_{2} + \left( i \sigma^{12} - e g \right) \sin \phi \sin \theta , \quad J_{3}^{eg} = l_{3} \quad (1.1) \]

\[ \Phi_{j \mu}^{eg}(t, r, \theta, \phi) = \frac{e^{-i \epsilon t}}{r} \begin{pmatrix} f_{1}(r) & D_{j, -m,eg-1/2}(\phi, \theta, 0) \\ f_{2}(r) & D_{j, -m,eg+1/2}(\phi, \theta, 0) \\ f_{3}(r) & D_{j, -m,eg-1/2}(\phi, \theta, 0) \\ f_{4}(r) & D_{j, -m,eg+1/2}(\phi, \theta, 0) \end{pmatrix} \quad (1.2) \]

In other words, we may state that one of the fundamental features underlying the theory of the non-Abelian monopole is that such a field does not destroy (or does not touch) the angular dependence which is dictated solely by isotopic structure arguments. In that sense, it represents an analogue of a spherically symmetric Abelian potential \( A_{\mu}(x) = (A_{0}(r), 0, 0, 0) \) rather than the Abelian monopole potential \( A_{\mu}(x) = (0, 0, 0, A_{\phi} = g \cos \theta) \).

In that connection we might draw attention to the fact that the designation itself monopole anticipates interpretation of the vector triplet \( A_{\mu}(x) \) as carrying, in a new situation, the old Abelian monopole quality and essence, although a real degree of their similarity would be much less than one might expect.

In Sec. 3, we proceed further with the reflection symmetry and give some attention to the question of explicit form of the above operator \( \hat{N}_{S}^{\alpha} \) (here the sign \( S \) stands for the Schwinger’s isotopic gauge) in some other isotopic gauges. There is no reason \textit{a priori} to choose a unique gauge as more preferred that all others; the particular choice above was dictated by convenience only. That restriction can easily be relaxed, for example, by demonstration of several more gauges. We are chiefly interested in the Dirac unitary and Cartesian gauges (as the most frequently used in the literature).

In Sec. 4, because the matter of discrete (in particular, \( P \)-inversion) always gets much attention on many physical reasons (in non-Abelian as well as Abelian theories) and because the freedom in choosing a discrete operator \( \hat{N}_{\alpha} \) seems at first glance rather unusual and even puzzled, we proceed with further studying monopole-affected manifestations of that symmetry. In the Abelian case, because of the well known monopole \( P \)-violation, electronic wave function in monopole field do not obey a fundamental structural condition

\[ \Psi_{l, m, \delta}^{eg}(t, \vec{r}) = \text{(matrix)} \Psi_{l, m, \delta}^{eg}(t, \vec{r}) \quad (1.3a) \]

which does guarantee the existence of the parity selection rules. Instead, only the following

\[ \Psi_{l, m, \delta}^{eg}(t, \vec{r}) = \text{(matrix)} \Psi_{l, m, \delta}^{eg}(t, \vec{r}) \quad (1.3b) \]

holds. It should be noted that in the literature there have been several suggestions as to how obtain a certain (formal) covariance of the monopole situation with respect to the \( P \)-symmetry. A single actual outcome of such attempts is that all those really imply the pseudoscalar character (under \( P \)-inversion) of a magnetic charge. To avoid misunderstanding, one thing should be noticed: the use of various gauges at description of the monopole (Dirac, Schwinger, Wu-Yang’s and so on) brings some peculiarities to this. Indeed, in Schwinger representation, a pseudo-scalar nature of a magnetic charge has usual sense, whereas the same situation in two other gauges seems like as if the common \( P \)-operation needs to be additionally improved through slight alterations (those latter
can be found from the involved gauge transformations). But, admittedly, pseudoscalar-based suggestions do not permit to overcome the non-existence of the discrete symmetry selection rules for matrix elements at considering one particle problem in a fixed monopole potential. In contrast to the Abelian monopole case, the non-Abelian situation is quite different: a relation with the required structure there exists (at first, just the case $N_A=0$-symmetry is analyzed)

$$\Psi_{\epsilon jm\delta}^{\text{triplet}} (t, \mathbf{-r}) = (\text{matrix}) \Psi_{\epsilon jm\delta}^{\text{triplet}} (t, \mathbf{+r})$$  \hspace{1cm} (1.4)

correspondingly $N_{A=0}$-parity selection rules may be produced; we give some details on this matter. We consider the question of how the complex parameter $A$ being involved into $\hat{N}_A$-operator and corresponding triplet wave functions can manifest itself in matrix elements. To this end, an explicit expression for possible matrix elements is looked into. As a natural and simple illustration, the above problem of parity selection rules is investigated again, but now depending on that $A$-background. Taking in mind the class of composite physical observables, that in general may have some inclusive constituent structure, a new definition of composite scalars and pseudoscalars with respect to the above $\hat{N}_A$-operation naturally occurs. At this, different values of $A$ will lead to different concepts of scalars and pseudoscalars. Correspondingly, $N_A$-parity selection rules arising in sequel for matrix elements (certainly if an observable belongs to the class of those $N_A$-scalars or pseudoscalars) differ basically from each other.

In Sec. 5 we are especially interested in the question: where does the above ambiguity come from? It is quite easily understandable that this possibility is closely connected with the fact of decoupling of three isotopic components in the wave equation itself, so that one may change independently three isotopic amplitudes and in the same time not destroy already given $j, m$-structure, only touching the $N_A$-structure. In more physically oriented studies, that possibility may be thought of in terms of electric charge’s characteristics of different isotopic components; namely, together with $j^2$ and $j_3$, the third component of isotopic spin $t_3$ (electric charge operator) may be diagonalized upon the wave functions $\Psi_{\epsilon jm}$. In addition, the situation can be thought of in terms of a hidden symmetry: there are two operators, $t_3$ and $\hat{N}_A$, commuting with the Hamiltonian but not commuting with each other. For the present paper’s purposes just it is an additional symmetry-based formal approach that seems more appropriate and really being made of use. So, just on those aspects of the problem we are going to concentrate further analysis. Thus, it is noted that every particular value $A$ merely governs basis states $\Psi_{\epsilon jm\delta}^{A}(x)$ with no change in the whole functional space; connection between different bases

$$\Psi_{\epsilon jm\delta}^{A'}(x) = V(A', A)\Psi_{\epsilon jm\delta}^{A}(x)$$

can be factorized as follows (for more detail on the used designations see in Sec. 5)

$$V(A', A) = e^{+i\frac{A' - A}{2}} D(A' - A) \Delta(A' - A)$$  \hspace{1cm} (1.5a)

Two of these operations vary in their acting on particular components of the triplet wave functions and also vary in their affecting the $\hat{N}_A$-operator

$$\Delta \hat{N}_A \Delta^{-1} = \hat{N}_{A'} \text{ but } D \hat{N}_A D^{-1} = \hat{N}_A$$  \hspace{1cm} (1.5b)

but both of two do not change the composite momentum components. The second relation in (1.5b) implies that, though such a $D$-symmetry can potentially prove itself in explicit
expressions for matrix elements (analogously the $A$-symmetry), this cannot happen to the $N$-parity selection rules because the $D$-symmetry does not affect the $\hat{N}_A$-operator.

The acting of $D$- and $\Delta$-transformations may be spelled out in a straightforward way. Let a general wave function for the triplet be

$$\Psi_{e_jm\delta}(x) = \left[T_{+1} \otimes \Phi^{(+)}(x) + e^{iB} T_0 \otimes \Phi^{(0)}(x) + \delta e^{iA} T_{-1} \otimes \Phi^{(-)}(x) \right] \quad (1.6a)$$

then the operations $\Delta(\Gamma)$ and $D(\Gamma)$ will act on those functions (1.6a) according to

$$\Psi^{A'(B)}_{e_jm\delta}(x) = \Delta(A' - A) \psi^{A(B)}_{e_jm\delta}(x), \quad \Psi^{A'(B')}_{e_jm\delta}(x) = D(B' - B) \psi^{A(B)}_{e_jm\delta}(x) \quad (1.6b)$$

with no connections between $(B' - B)$ and $(A' - A)$: those perimeters are completely independent.

The operation $\Delta(A' - A)$ in itself represents one-parametric rotation belonging to the complex 3-dimensional group $SO(3.C)$; the $D(B' - B)$ in turn is a certain ‘conformal’ transformation in isotopic space. As may be thought, useful and somewhat intriguing are their respective forms in Cartesian isotopic gauges, which are calculated. It is explicit dependence on angular coordinate $\theta, \phi$ that makes them so exciting and not trivial in appearance:

$$\Delta^{\text{Cart.}}(A' - A) = \exp \left[ i \frac{A' - A}{2} \bar{t} \bar{n}_{\theta,\phi} \right], \quad D^{\text{Cart.}}(\Gamma) = \left[ I + (e^{-it} - I) \bar{t}_0 \right] \quad (1.6c)$$

where $\bar{t}_0$ is a $\theta, \phi$-dependent matrix.

It may be repeated else one time and stressed that both these symmetry operations occur only if the case of special monopole potential is at hand; instead, for the ‘t Hooft-Polyakov potential as well as for the free isotopic triplet case no such additional symmetries (tied with the operators $j^2, j_3$) will arise. This latter seems quite natural in the light of the genuinely non-Abelian monopole presence and less natural or even somewhat strange as being concerned to the free triplet system. Indeed, this mean that we have come to be able to diagonalize an electric charge operator in the special monopole field and cannot do so in the other (monopole free) situation. In any case, just this shows that the case of triplet in the simplest monopole potential exhibits quite definite features which set it apart from the other systems.

2. Separation of variables and discrete operator

The basic tetrad-based equation (see in Supplement A)

$$\left[ i \gamma^\alpha(x) \left( \partial_{\alpha} + \Gamma_{\alpha}(x) - ie t^a W^a_{\alpha}(x) \right) - (m + \kappa \Phi^a(x) t^a) \right] \Psi(x) = 0 \quad (2.1)$$

at the use of Schwinger unitary gauge in isotopic space and the spherical tetrad basis, takes the form

$$\left[ \gamma^0 \left( i \partial_t + erF(r) t^3 \right) + i \gamma^3 \left( \partial_r + \frac{1}{r} \right) + \frac{1}{r} \Sigma_{\theta,\phi} + \right.$$

$$+ \frac{1}{r} \left( er^2K(r) + 1 \right) \left( \gamma^1 \otimes t^2 - \gamma^2 \otimes t^1 \right) - (m + \kappa r \Phi(r) t^3) \right] \Psi^S = 0 \quad , \quad (2.2a)$$
where

$$\Sigma_{\theta, \phi} = \left[ i\gamma^1 \partial_\theta + \gamma^2 \frac{i\partial_\phi + (i\sigma^1 + t^3)\cos \theta}{\sin \theta} \right]. \quad (2.2b)$$

The Dirac matrices $\gamma^a$ are chosen in the Weyl spinor representation; the isotopic ones $t^a$ are specified in the so-called cyclic basis

$$t^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad t^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$  

Having diagonalized the operators $\hat{j}^2$ and $j_3$, the wave functions with quantum numbers $j, m$ are constructed as

$$\Psi^S_{jm}(t, r, \theta, \phi) = \frac{1}{r} e^{-i\epsilon t} \times \left[ T_{+1} \otimes \begin{pmatrix} f_1D_{-3/2} \\ f_2D_{-1/2} \\ f_3D_{-3/2} \\ f_4D_{-1/2} \end{pmatrix} + T_0 \otimes \begin{pmatrix} h_1D_{-1/2} \\ h_2D_{+1/2} \\ h_3D_{-1/2} \\ h_4D_{+1/2} \end{pmatrix} + T_{-1} \otimes \begin{pmatrix} g_1D_{+1/2} \\ g_2D_{+3/2} \\ g_3D_{+1/2} \\ g_4D_{+3/2} \end{pmatrix} \right]$$

where $D_\sigma$ denotes the Wigner functions $D^{j_m, \sigma}_{jm}(\phi, \theta, 0)$ ($j = 1/2, 3/2, \ldots$); $f_i, g_i, h_i$ represent radial functions; and $T_{-1}, T_{+1}, T_0$ designate the basic vector of isotopic space ($t^3 T_s = s T_s$):

$$T_{+1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad T_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad T_{-1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$  

The first step is to separate the variables and work out a radial system\footnote{At this certainly an old result will be reproduced again, though the working method itself is a new one.}. To this end, having used the known relations for Wigner’s $D$-functions [18]

$$\partial_\theta D_{+1/2} = \frac{1}{2}(aD_{-1/2} - bD_{+3/2}) ;$$

$$[\sin^{-1} \theta (i\partial_\phi - (1/2)\cos \theta)]D_{+1/2} = \frac{1}{2}(-aD_{-1/2} - bD_{+3/2}) ;$$

$$\partial_\theta D_{-1/2} = \frac{1}{2}(bD_{-1/2} - aD_{+1/2}) ;$$

$$[\sin^{-1} \theta (i\partial_\phi + (1/2)\cos \theta)]D_{-1/2} = \frac{1}{2}(-bD_{-1/2} - aD_{+1/2}) ;$$

$$\partial_\theta D_{+3/2} = \frac{1}{2}(bD_{+1/2} - cD_{+5/2}) ;$$

$$[\sin^{-1} \theta (i\partial_\phi - (3/2)\cos \theta)]D_{+3/2} = \frac{1}{2}(-bD_{+1/2} - bD_{+5/2}) ;$$

$$\partial_\theta D_{-3/2} = \frac{1}{2}(cD_{-5/2} - bD_{-1/2}) ,$$

$$[\sin^{-1} \theta (i\partial_\phi + (3/2)\cos \theta)]D_{-3/2} = \frac{1}{2}(-cD_{-5/2} - bD_{-1/2}) \quad (2.4a)$$
where
\[ a = (j + 1/2), \quad b = \sqrt{(j - 1/2)(j + 3/2)}, \quad c = \sqrt{(j - 3/2)(j + 5/2)} \]
we find the action of \(\Sigma_{\theta, \phi}\) on \(\Psi_{\epsilon jm}^S(x)\):
\[
\Sigma_{\theta, \phi} \Psi_{\epsilon jm}^S(x) = \frac{1}{r} e^{-i\epsilon t} \left[ i b \ T_{+1} \otimes \begin{pmatrix} -f_4 D_{-3/2} \\ f_3 D_{-1/2} \\ f_2 D_{-3/2} \\ -f_1 D_{-1/2} \end{pmatrix} + i a T_0 \otimes \begin{pmatrix} -h_4 D_{-1/2} \\ h_3 D_{+1/2} \\ h_2 D_{-1/2} \\ -h_1 D_{+1/2} \end{pmatrix} + i b T_{-1} \otimes \begin{pmatrix} -g_4 D_{+1/2} \\ g_3 D_{+3/2} \\ g_2 D_{+1/2} \\ -g_1 D_{+3/2} \end{pmatrix} \right]. \tag{2.4b}
\]
In addition, as a simple matter of calculation we get the action of a mixing term \((W \equiv (er^2K(r) + 1)):\)
\[
\frac{W}{r} (\gamma^1 \otimes t^2 - \gamma^2 \otimes t^1) \Psi_{\epsilon jm}^S(x) = \frac{e^{-i\epsilon t}}{r} \sqrt{2} \frac{W}{r} \times \left[ T_{+1} \otimes \begin{pmatrix} 0 \\ i h_3 D_{-1/2} \\ 0 \\ -i h_1 D_{-1/2} \end{pmatrix} + T_0 \otimes \begin{pmatrix} -if_4 D_{-1/2} \\ +i g_3 D_{+1/2} \\ +if_2 D_{-1/2} \\ -i g_1 D_{+1/2} \end{pmatrix} + T_{-1} \otimes \begin{pmatrix} -ih_4 D_{+1/2} \\ 0 \\ +ih_2 D_{+1/2} \\ 0 \end{pmatrix} \right]. \tag{2.4c}
\]
on one can note that three distinct isotopic components mingle just through the terms which are proportional to the Wigner’s functions \(D_{\pm1/2}\). Finally, after simple calculating we shall find the following set of radial equations (for saving space, the notation \(\tilde{F} = erF(r), \tilde{\Phi} = \kappa r \Phi(r)\) is used):
\[
(\epsilon + \tilde{F}) f_3 - i \frac{d}{dr} f_3 - \frac{ib}{r} f_4 - (m + \tilde{\Phi}) f_1 = 0,
\]
\[
(\epsilon + \tilde{F}) f_4 + i \frac{d}{dr} f_4 + \frac{ib}{r} f_3 + i \sqrt{2} \frac{W}{r} h_3 - (m + \tilde{\Phi}) f_2 = 0,
\]
\[
(\epsilon + \tilde{F}) f_1 + i \frac{d}{dr} f_1 + \frac{ib}{r} f_2 - (m + \tilde{\Phi}) f_3 = 0,
\]
\[
(\epsilon + \tilde{F}) f_2 - i \frac{d}{dr} f_2 - \frac{ib}{r} f_1 - i \sqrt{2} \frac{W}{r} h_1 - (m + \tilde{\Phi}) f_4 = 0; \tag{2.5a}
\]
\[
(\epsilon - \tilde{F}) g_3 - i \frac{d}{dr} g_3 - \frac{ib}{r} g_4 - i \sqrt{2} \frac{W}{r} h_4 - (m - \tilde{\Phi}) g_1 = 0,
\]
\[
(\epsilon - \tilde{F}) g_4 + i \frac{d}{dr} g_4 + \frac{ib}{r} g_3 - (m - \tilde{\Phi}) g_2 = 0,
\]
\[
(\epsilon - \tilde{F}) g_1 + i \frac{d}{dr} g_1 + \frac{ib}{r} g_2 + i \sqrt{2} \frac{W}{r} h_2 - (m - \tilde{\Phi}) g_3 = 0,
\]
\[
(\epsilon - \tilde{F}) g_2 - i \frac{d}{dr} g_2 - \frac{ib}{r} g_1 - (m - \tilde{\Phi}) g_4 = 0; \tag{2.5b}
\]
\]
\[ \epsilon h_3 - i \frac{d}{dr} h_3 - \frac{ia}{r} h_4 - i \frac{\sqrt{2W}}{r} f_4 - mh_1 = 0, \]
\[ \epsilon h_4 + i \frac{d}{dr} h_4 + \frac{ia}{r} h_3 + i \frac{\sqrt{2W}}{r} g_3 - mh_2 = 0, \]
\[ \epsilon h_1 + i \frac{d}{dr} h_1 + \frac{ia}{r} h_2 + i \frac{\sqrt{2W}}{r} f_2 - mh_3 = 0, \]
\[ \epsilon h_2 - i \frac{d}{dr} h_2 - \frac{ia}{r} h_1 - i \frac{\sqrt{2W}}{r} g_1 - mh_4 = 0. \quad (2.5c) \]

Now let us try to simplify these equations (2.6) by diagonalizing additionally a suitable discrete operator: a composite reflection in the Lorentzian and isotopic spaces. The usual bispinor \( P \)-inversion has, in the used tetrad basis, the form:

\[
\hat{\Pi}_{\text{sph}} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \otimes \hat{P} = (-\gamma^5 \gamma^1) \otimes \hat{P} \quad (2.6a)
\]

the \( \hat{P} \) is determined by relation \( \hat{P} \Psi(\theta, \phi) = \Psi(\pi - \theta, \phi + \pi) \). Correspondingly, that \( \hat{\Pi}_{\text{sph}} \) acts upon the composite wave function as follows (the relationship \( \hat{P} D^j_{-m,\sigma} = (-1)^j D^j_{-m,\sigma} \) is taken into account)

\[
\hat{\Pi}_{\text{sph}}. \Psi_{\epsilon j m}^S(t, r, \theta, \phi) = \frac{e^{-i\epsilon t}}{r} (-1)^j \times \left[ T_{+1} \otimes \begin{pmatrix} f_4 D_{+1/2} \\ D_{+3/2} f_3 \\ D_{+1/2} h_4 \\ f_1 D_{+3/2} \end{pmatrix} \right] + T_0 \otimes \left[ \begin{pmatrix} h_4 D_{-1/2} \\ h_3 D_{+1/2} \\ h_2 D_{-1/2} \\ h_1 D_{+1/2} \end{pmatrix} \right] + T_{-1} \otimes \left[ \begin{pmatrix} g_4 D_{-3/2} \\ g_3 D_{-1/2} \\ g_2 D_{-3/2} \\ g_1 D_{-1/2} \end{pmatrix} \right]. \quad (2.6b)
\]

From this it follows immediately that the \( \hat{\Pi}_{\text{sph}} \) cannot be diagonalized upon the functions of the sort (2.3); but an operator with required properties can be constructed through extending of the above \( \hat{\Pi}_{\text{sph}} \) by a special transformation \( \hat{\pi} \) upon isotopic coordinates and taking the unit vectors \{ \( T_{+1}, T_0, T_{-1} \) \} into \{ \( T_{-1}, T_0, T_{+1} \) \} respectively (possibly apart from some number factors). Expecting the \( \hat{\pi} \) to satisfy

\[ \hat{\pi} T_{+1} = \alpha T_{-1}, \quad \hat{\pi} T_0 = \beta T_0, \quad \hat{\pi} T_{-1} = \gamma T_{+1}, \quad (2.7a) \]

we will find its matrix representation

\[ \hat{\pi} = \begin{pmatrix} 0 & 0 & \gamma \\ 0 & \beta & 0 \\ \alpha & 0 & 0 \end{pmatrix}. \quad (2.7b) \]

The required composite operator \( \hat{N} \), being determined by

\[ \hat{N} = (\hat{\pi} \otimes \hat{\Pi}_{\text{sph}}) \quad (2.7c) \]

will act on the wave functions as follows

\[ \hat{N} \Psi_{\epsilon j m}^S(t, r, \theta, \phi) = \frac{e^{-i\epsilon t}}{r} (-1)^j \times \]
\[
\begin{bmatrix}
\gamma T_{+1} \otimes \left( \begin{array}{c}
g_1 D_{-3/2} \\
g_3 D_{-1/2} \\
g_2 D_{-3/2} \\
g_1 D_{-1/2}
\end{array} \right)
\right) + \beta T_0 \otimes \left( \begin{array}{c}
h_4 D_{-1/2} \\
h_3 D_{+1/2} \\
h_2 D_{-1/2} \\
h_1 D_{+1/2}
\end{array} \right) + \alpha T_{-1} \otimes \left( \begin{array}{c}
f_4 D_{+1/2} \\
f_3 D_{+3/2} \\
f_2 D_{+1/2} \\
f_1 D_{+3/2}
\end{array} \right) .
\end{bmatrix} \quad (2.8a)
\]

From the proper value equation \( \hat{N} \Psi_{ejm}^S = N \Psi_{ejm}^S \) it follows

\[
(-1)^{j+1} \gamma g_4 = N f_1 , \quad (-1)^{j+1} \alpha f_1 = N g_4 ,
\]
\[
(-1)^{j+1} \gamma g_3 = N f_2 , \quad (-1)^{j+1} \alpha f_2 = N g_3 ,
\]
\[
(-1)^{j+1} \gamma g_2 = N f_3 , \quad (-1)^{j+1} \alpha f_3 = N g_2 ,
\]
\[
(-1)^{j+1} \gamma g_1 = N f_4 , \quad (-1)^{j+1} \alpha f_4 = N g_1 ; \quad (2.8b)
\]
\[
(-1)^{j+1} \beta h_4 = N h_1 , \quad (-1)^{j+1} \beta h_1 = N h_4 ,
\]
\[
(-1)^{j+1} \beta h_2 = N h_3 , \quad (-1)^{j+1} \beta h_3 = N h_2 . \quad (2.8c)
\]

Then, from (2.8c) we get \( N^2 = (-1)^{2j+2} \beta^2 \); from (2.8b) it follows \( N^2 = (-1)^{2j+2} (\alpha \gamma) \); thus \( \beta^2 = (\alpha \gamma) \). In addition, noting that

\[
\hat{\pi}^2 = \begin{pmatrix}
\alpha \gamma & 0 & 0 \\
0 & \beta^2 & 0 \\
0 & 0 & \alpha \gamma
\end{pmatrix} = \beta^2 I
\]

and accepting that the choice of \( \beta \) is not material to diagonalizing that discrete operator, so that we may take \( \beta = 1 \), and eventually get \( \gamma = \alpha^{-1} \) as a single free numerical parameter at an ambiguous operator \( \hat{\pi}_a \). Thus, we have worked out two values for the \( N_\alpha \)-parity and concomitant with them limitations on radial functions:

\[
N = \delta (-1)^{j+1} , \delta = \pm 1
\]
\[
h_3 = \delta h_2 , \quad h_4 = \delta h_1 , \quad g_4 = \delta \alpha f_1 ,
\]
\[
g_3 = \delta \alpha f_2 , \quad g_2 = \delta \alpha f_3 , \quad g_1 = \delta \alpha f_4 ; \quad (2.9a)
\]

so that the functions with quantum numbers \( \epsilon jm\delta \) are built as

\[
\Psi_{ejm\delta}^{S, (a)} (t, r, \theta, \phi) = \frac{e^{-i \epsilon t}}{r}
\]

\[
\begin{bmatrix}
T_{+1} \otimes \left( \begin{array}{c}
f_1 D_{-3/2} \\
f_2 D_{-1/2} \\
f_3 D_{-3/2} \\
f_4 D_{-1/2}
\end{array} \right)
\right) + T_0 \otimes \left( \begin{array}{c}
h_4 D_{-1/2} \\
h_3 D_{+1/2} \\
h_2 D_{-1/2} \\
h_1 D_{+1/2}
\end{array} \right) + \delta \alpha T_{-1} \otimes \left( \begin{array}{c}
f_4 D_{+1/2} \\
f_3 D_{+3/2} \\
f_2 D_{+1/2} \\
f_1 D_{+3/2}
\end{array} \right) . \quad (2.9b)
\end{bmatrix}
\]

At fixed \( (\epsilon, j, m) \) the quantity \( \delta \) takes two values: +1 or -1; the presence of the \( \alpha \) in (2.9b) reflects the ambiguity in choosing the discrete \( \hat{N}_a \).

Now, we are going to substitute (2.9a) into equations (2.6), so that together with the question of their self-consistency we will be able, by the same token, study the question of the possible commuting of the \( \hat{N}_a \) with the relevant triplet-fermion-monopole Hamiltonian. The whole situation here seems completely analogous to that appeared in studying the doublet-fermion-monopole case [1]. It turns out that the system so obtained will not
be self-consistent if the two terms in the equation (2.1) are valid: those are \( e r F(r) r^3 \) and \( \kappa r \Phi(r) r^3 \). So, we have to set \( F(r) = 0 \) as well as \( \kappa = 0 \); that is to restrict ourselves to the case of purely monopole (not dyon) potential and of null coupling constant \( \kappa \). Thus, we arrive at

\[
\begin{align*}
\epsilon f_3 - i \frac{d}{dr} f_3 - \frac{ib}{r} f_4 - m f_1 &= 0 \\
\epsilon f_4 + i \frac{d}{dr} f_4 + \frac{ib}{r} f_3 + i\delta \frac{\sqrt{2W}}{r} h_2 - m f_2 &= 0 \\
\epsilon f_1 + i \frac{d}{dr} f_1 + \frac{ib}{r} f_2 - m f_3 &= 0 \\
\epsilon f_2 - i \frac{d}{dr} f_2 - \frac{ib}{r} f_1 - i\sqrt{2W} \frac{h_1}{r} - m f_4 &= 0 \quad (2.10a)
\end{align*}
\]

\[
\begin{align*}
\epsilon f_2 - i \frac{d}{dr} f_2 - \frac{ib}{r} f_1 - i\alpha^{-1} \frac{\sqrt{2W}}{r} h_1 - m f_4 &= 0 \\
\epsilon f_1 + i \frac{d}{dr} f_1 + \frac{ib}{r} f_2 - m f_3 &= 0 \\
\epsilon f_4 + i \frac{d}{dr} f_4 + \frac{ib}{r} f_3 + i\delta \alpha^{-1} \frac{\sqrt{2W}}{r} h_2 - m f_2 &= 0 \\
\epsilon f_3 - i \frac{d}{dr} f_3 - \frac{ib}{r} f_4 - m f_1 &= 0 \quad (2.10b)
\end{align*}
\]

\[
\begin{align*}
\epsilon h_2 - i \frac{d}{dr} h_2 - \frac{ia}{r} h_1 - i\delta \frac{\sqrt{2W}}{r} f_4 - \delta m h_1 &= 0 \\
\epsilon h_1 + i \frac{d}{dr} h_1 + \frac{ia}{r} h_2 + i\alpha \frac{\sqrt{2W}}{r} f_2 - \delta m h_2 &= 0 \\
\epsilon h_1 + i \frac{d}{dr} h_1 + \frac{ia}{r} h_2 + i\sqrt{2W} \frac{f_2}{r} - \delta m h_2 &= 0 \\
\epsilon h_2 - i \frac{d}{dr} h_2 - \frac{ia}{r} h_1 - i\delta \alpha \frac{\sqrt{2W}}{r} f_4 - \delta m h_1 &= 0 \quad (2.10c)
\end{align*}
\]

It is easily understandable that one has to distinguish between two distinct situations: depending on whether the characteristic function \( W(r) \) is zero or non-zero (we will remember that the case \( W(r) = 0 \) corresponds to the special simplest monopole potential. The fact is that the foregoing operator \( \hat{N}_\alpha \) with an arbitrary complex-valued \( \alpha \) can be diagonalized upon the functions (2.9b) if and only if the function \( W(r) \) is equal to zero; otherwise \( (W(r) \neq 0) \) we have to take \( \alpha = +1 \) (the value \( \alpha = -1 \) could be chosen as well).

Eventually we obtain the six-equation system (two cases are distinguished)

\[
W(r) = 0 \quad (\text{arbitrary} \ \alpha),
\]

\[
\begin{align*}
\epsilon f_3 - i \frac{d}{dr} f_3 - \frac{ib}{r} f_4 - m f_1 &= 0 , \\
\epsilon f_4 + i \frac{d}{dr} f_4 + \left( \frac{b}{r} f_3 - m f_2 \right) &= 0 ,
\end{align*}
\]
\[ \epsilon f_1 + \frac{id}{dr} f_1 + \frac{ib}{r} f_2 - m f_3 = 0 , \]
\[ \epsilon f_2 - \frac{id}{dr} f_2 - \frac{ib}{r} f_1 - m f_4 = 0 , \]
\[ \epsilon h_2 - \frac{id}{dr} h_2 - \frac{ia}{r} h_1 - \delta m h_1 = 0 , \]
\[ \epsilon h_1 + \frac{id}{dr} h_1 + \frac{ia}{r} h_2 - \delta m h_2 = 0 . \]  

(2.11a)

Correspondingly,
\[ W(r) \neq 0 \quad (\alpha = +1) \]
\[ \epsilon f_3 - \frac{id}{dr} f_3 - \frac{ib}{r} f_4 - m f_1 = 0 , \]
\[ \epsilon f_4 + \frac{id}{dr} f_4 + \frac{ib}{r} f_3 + i\delta \frac{\sqrt{2}W}{r} h_2 - m f_2 = 0 , \]
\[ \epsilon f_1 + \frac{id}{dr} f_1 + \frac{ib}{r} f_2 - m f_3 = 0 , \]
\[ \epsilon f_2 - \frac{id}{dr} f_2 - \frac{ib}{r} f_1 - i\frac{\sqrt{2}W}{r} h_1 - m f_4 = 0 , \]
\[ \epsilon h_2 - \frac{id}{dr} h_2 - \frac{ia}{r} h_1 - i\delta \frac{\sqrt{2}W}{r} f_4 - \delta m h_1 = 0 , \]
\[ \epsilon h_1 + \frac{id}{dr} h_1 + \frac{ia}{r} h_2 + i\frac{\sqrt{2}W}{r} f_2 - \delta m h_2 = 0 . \]  

(2.11b)

It is the point to notice that everything said above is valid as it stated only when \( j \geq 3/2 \); the case of minimal \( j = 1/2 \) is to be considered separately. First of all, it is a special substitution for the starting wave function that should be used:

\[ \Psi_0^S(t, r, \theta, \phi) = \frac{e^{-i\epsilon t}}{r} \times \]

\[ \begin{bmatrix} T_{+1} \otimes \begin{pmatrix} 0 \\ f_2 D_{-1/2} \\ 0 \\ f_4 D_{-1/2} \end{pmatrix} + T_0 \otimes \begin{pmatrix} h_1 D_{-1/2} \\ h_2 D_{+1/2} \\ h_3 D_{+1/2} \\ h_4 D_{+1/2} \end{pmatrix} + T_{-1} \otimes \begin{pmatrix} g_1 D_{+1/2} \\ 0 \\ g_3 D_{+1/2} \\ 0 \end{pmatrix} \end{bmatrix} . \]  

(2.12a)

The angular operator \( \Sigma_{\theta, \phi} \) acts on \( \Psi_0^S(x) \) according to

\[ \Sigma_{\theta, \phi} \Psi_0^S(x) = \frac{e^{-i\epsilon t}}{r} \left[ T_{+1} \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + i T_0 \otimes \begin{pmatrix} -h_1 D_{-1/2} \\ +h_3 D_{+1/2} \\ +h_2 D_{-1/2} \\ -h_1 D_{+1/2} \end{pmatrix} + T_{-1} \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right] \]  

(2.12b)

and the corresponding radial system will be

\[ (\epsilon + \tilde{F}) f_4 + \frac{id}{dr} f_4 + i\frac{\sqrt{2}W}{r} h_3 - (m + \tilde{\Phi}) f_2 = 0 \]
\[(\epsilon + \tilde{F}) f_2 - i \frac{d}{dr} f_2 - \frac{\sqrt{2W}}{r} h_1 - (m + \tilde{\Phi}) f_4 = 0 \]  
\[(2.13a)\]

\[(\epsilon - \tilde{F}) g_3 - i \frac{d}{dr} g_3 - \frac{\sqrt{2W}}{r} h_4 - (m - \tilde{\Phi}) g_1 = 0 \]

\[(\epsilon - \tilde{F}) g_1 + i \frac{d}{dr} g_1 + \frac{\sqrt{2W}}{r} h_2 - (m - \tilde{\Phi}) g_3 = 0 \]  
\[(2.13b)\]

\[(\epsilon - \tilde{F}) g_1 + i \frac{d}{dr} g_1 + \frac{\sqrt{2W}}{r} h_2 - (m - \tilde{\Phi}) g_3 = 0 \]

\[(\epsilon - \tilde{F}) g_1 + i \frac{d}{dr} g_1 + \frac{\sqrt{2W}}{r} h_2 - (m - \tilde{\Phi}) g_3 = 0 \]  
\[(2.13c)\]

Having taken into account the $\hat{N}_\alpha$-based limitations

\[N = \delta(-1)^{3/2}, \quad \delta = \pm 1: \]

\[g_3 = \delta \alpha f_2, \quad g_1 = \delta \alpha f_4, \quad h_4 = \delta h_1 \quad h_3 = \delta h_2 \]  
\[(2.14a)\]

we arrive at (again $\tilde{\Phi} = 0, \tilde{F} = 0$ have been set)

\[\text{W}(r) = 0, \quad j = 1/2: \]

\[\epsilon f_4 + i \frac{d}{dr} f_4 - m f_2 = 0, \]

\[\epsilon f_2 - i \frac{d}{dr} f_2 - m f_4 = 0, \]

\[\epsilon h_2 - i \frac{d}{dr} h_2 - \frac{i}{r} h_1 - \delta m h_1 = 0, \]

\[\epsilon h_1 + i \frac{d}{dr} h_1 + \frac{i}{r} h_2 - \delta m h_2 = 0. \]  
\[(2.14b)\]

\[\text{W}(r) \neq 0, \quad j = 1/2: \]

\[\epsilon f_4 + i \frac{d}{dr} f_4 + i\delta \frac{\sqrt{2W}}{r} h_2 - m f_2 = 0, \]

\[\epsilon f_2 - i \frac{d}{dr} f_2 - i \frac{\sqrt{2W}}{r} h_1 - m f_4 = 0, \]

\[\epsilon h_2 - i \frac{d}{dr} h_2 - \frac{i}{r} h_1 - i\delta \frac{\sqrt{2W}}{r} f_4 - \delta m h_1 = 0, \]

\[\epsilon h_1 + i \frac{d}{dr} h_1 + \frac{i}{r} h_2 + i \frac{\sqrt{2W}}{r} f_2 - \delta m h_2 = 0. \]  
\[(2.14c)\]
3. The operator $\hat{N}_\alpha$ in the Dirac and Cartesian gauges

Now we proceed further with the reflection symmetry and give some attention to the question of explicit form of the above operator $\hat{N}_\alpha^S$ (the additional sign $S$ of the Schwinger’s isotopic gauge is written) in some other isotopic gauges. Obviously, a priori there is no reason to choose a unique gauge as more preferred that all others; the particular choice above was dictated by convenience only. That restriction can easily be relaxed, for example, by demonstration of several more gauges. We are chiefly interested in the Dirac unitary and Cartesian gauges (as the most frequently used in the literature).

The situation, as a whole, concerning the various representations being employed in the present treatment can be delineated as follows (here, the designations Cart. and cycl. are associated with two different bases for a set of isotopic matrices $t_i$; see in Suppl. A)

\[
\begin{align*}
\Psi_{\text{Cart.}}^C & \quad \xrightarrow{e} \quad \Psi_{\text{Cart.}}^D \quad \xrightarrow{e'} \quad \Psi_{\text{Cart.}}^S, \\
\Psi_{\text{cycl.}}^C & \quad \xrightarrow{V} \quad \Psi_{\text{cycl.}}^D \quad \xrightarrow{U} \quad \Psi_{\text{cycl.}}^S,
\end{align*}
\]

those functions are connected by the relations

\[
\begin{align*}
\Psi_{\text{Cart.}}^D &= O(\vec{c}) \; \Psi_{\text{Cart.}}^C, & \Psi_{\text{Cart.}}^S &= O(\vec{c}') \; \Psi_{\text{Cart.}}^D, \\
\Psi_{\text{cycl.}}^C &= S \; \Psi_{\text{Cart.}}^C, & \Psi_{\text{cycl.}}^D &= S \; \Psi_{\text{Cart.}}^D, & \Psi_{\text{cycl.}}^S &= S \; \Psi_{\text{Cart.}}^S.
\end{align*}
\]

(3.1) where $\vec{c}$ is the known Gibbs parameter on the group $SO(3,R)$ [64])

\[
O(\vec{c}) = \begin{pmatrix}
[1 - (1 - \cos \theta) \cos^2 \phi] & -(1 - \cos \theta) \sin \theta \cos \theta & -\sin \theta \cos \phi \\
-(1 - \cos \theta) \sin \theta \cos \phi & [1 - (1 - \cos \theta) \sin^2 \phi] & -\sin \theta \sin \phi \\
\sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta
\end{pmatrix}
\]

\[
O(\vec{c}') = \begin{pmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad S = \begin{pmatrix}
-1/\sqrt{2} & i/\sqrt{2} & 0 \\
0 & 0 & 1 \\
+1/\sqrt{2} & i/\sqrt{2} & 0
\end{pmatrix}
\]

Above in the work, the $S$-gauge (Schwinger’s) and cyclic basis solely were used. The passage to the Dirac gauge can be with no difficulty performed

\[
\Psi_{\text{cycl.}}^D = U \; \Psi_{\text{cycl.}}^S, \quad U = S \; O(-\vec{c}') \; S^{-1}
\]

(3.2a)

for the matrix $U$ we get

\[
U(\phi) = \begin{pmatrix}
e^{-i\phi} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{+i\phi}
\end{pmatrix}.
\]

(3.2b)

Correspondingly, after translating to the $D$-gauge $\hat{N}_{\text{cycl.}}^D = U(\phi) \; \hat{N}_{\text{cycl.}}^S \; U^{-1}(\phi)$, the discrete operator $\hat{N}_\alpha$ takes the form

\[
\hat{N}_{\text{cycl.}}^D = [ \left( U^{-1} \hat{\pi}_\alpha^S \; U \right) \; U_0 \otimes (-\gamma^5\gamma^1) ] \otimes \hat{P}
\]

(3.2c)

where $U_0$ is a matrix arising out of the commutation rule $\hat{P} \; U(\phi) = U(\phi) \; U_0 \; \hat{P}$ and given by

\[
U_0 = \begin{pmatrix}
-1 & 0 & 0 \\
0 & +1 & 0 \\
0 & 0 & -1
\end{pmatrix}.
\]
Ultimately, for \( \hat{N}_{cyc}^D \) we find the following explicit form
\[
\hat{N}_{cyc}^D = (\hat{\pi}_{cyc}^D \otimes (-\gamma^5 \gamma^1)) \otimes \hat{P}
\]
where the matrix \( \hat{\pi}_{cyc}^D \) is given by
\[
\hat{\pi}_{cyc}^D = \begin{pmatrix}
0 & 0 & 0 & (\alpha^{-1}) e^{-2i\phi} \\
0 & 1 & 0 & 0 \\
\alpha e^{2i\phi} & 0 & 0 \\
\end{pmatrix}.
\] (3.2d)

In the same way we find the determining relations for the Cartesian gauge
\[
\Psi_{cyc}^{Cart.} = U(\theta, \phi) \Psi_{cyc}^S .
\] (3.3a)
\[
U(\theta, \phi) = \begin{pmatrix}
\frac{1}{\sqrt{2}} (1 + \cos \theta) e^{-i\phi} & \frac{1}{\sqrt{2}} \sin \theta e^{-i\phi} & \frac{1}{2} (1 - \cos \theta) e^{-i\phi} \\
\frac{1}{\sqrt{2}} \sin \theta & \cos \theta & -\frac{1}{\sqrt{2}} \sin \theta \\
(1 - \cos \theta) e^{i\phi} & \frac{1}{\sqrt{2}} \sin \theta e^{i\phi} & \frac{1}{\sqrt{2}} (1 - \cos \theta) e^{-i\phi} \\
\end{pmatrix}.
\]

Decomposing the operator \( \hat{\pi}_{cyc}^{S,\alpha} \) into a product of two factors (\( \alpha \equiv e^{iA} \), \( A \) is a complex number)
\[
\hat{\pi}_{cyc}^{S,\alpha} = \begin{pmatrix}
0 & 0 & \alpha^{-1} \\
0 & 1 & 0 \\
\alpha & 0 & 0 \\
\end{pmatrix} = \begin{pmatrix}
e^{-iA} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{iA} \\
\end{pmatrix} \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{pmatrix} \equiv \Delta(A) \hat{\pi}_{cyc}^S.
\]

for the \( N \)-operator in Cartesian gauge we get the factorization
\[
\hat{\pi}_{cyc}^{Cart.A} = \left( U(\theta, \phi) \Delta(A) U^{-1}(\theta, \phi) \right) \left( U(\theta, \phi) \hat{\pi}_{cyc}^S. U^{-1}(\pi - \theta, \phi + \pi) \right).
\] (3.3b)

The second term-multiplier can be brought to the form
\[
U(\theta, \phi) \hat{\pi}_{cyc}^S. U^{-1}(\pi - \theta, \phi + \pi) = -I
\] (3.4)
and the first one may be rewritten as
\[
U(\theta, \phi) \Delta(A) U^{-1}(\theta, \phi) = \left( S \, O(-\vec{c}') \, S^{-1} \right) \Delta(A) \left( S \, O(\vec{c}') \, S^{-1} \right).
\]

Now, noting the identity
\[
S^{-1} \Delta(A) S = \begin{pmatrix}
\cos A & -\sin A & 0 \\
-\sin A & \cos A & 0 \\
0 & 0 & 1 \\
\end{pmatrix} \equiv O(\vec{a}) , \quad \vec{a} = (0, 0, -\tan A/2)
\]
where \( (O(\vec{a}) \) is a transformation of the complex rotation group, and also taking into account the known in the theory of the group \( SO(3.) \) [64] relation
\[
O(-\vec{c}'') \, O(\vec{a}) \, O(\vec{c}'') = O(O_{-\vec{c}'',\vec{a}})
\]
one can get
\[
U(\theta, \phi) \Delta(A) U^{-1}(\theta, \phi) = S \, O(O_{-\vec{c}'',\vec{a}}) \, S^{-1}.
\] (3.5a)
The vector \((O_{\vec{c}''_{\vec{a}}}a)\) has the following explicit form
\[
(0_{\vec{c}''_{\vec{a}}}a) = -\tan A/2 \begin{pmatrix} \sin \theta \cos \phi, & \sin \theta \sin \phi, & \cos \theta \end{pmatrix}
\]
so the expression for \(S \ O(0_{\vec{c}''_{\vec{a}}}a) \ S^{-1}\) is as follows
\[
S \ 0(0_{\vec{c}''_{\vec{a}}}a) \ S^{-1} = S \ O(-\tan A/2 \vec{n}_{\theta,\phi} \ S^{-1} = \\
S \ \exp \left[ i \ A \vec{\tau} \vec{n}_{\theta,\phi} \right] \ S^{-1} = \exp \left[ i \ A \vec{\tau} \vec{n}_{\theta,\phi} \right]
\]
here we employ the well-known exponential representation of orthogonal \(3 \times 3\)-matrix (for more details see [64]). Given these relationships (3.4) (3.5c), one can find \(\hat{N}_A\)-operator in Cartesian gauge
\[
\hat{N}_{\text{Cart.}} = \pi_{\text{Cart.}} \otimes (-\vec{\gamma}_5 \vec{\gamma}_1) \otimes \hat{P} = \exp \left[ i \ A \vec{\tau} \vec{n}_{\theta,\phi} \right] \otimes (-\vec{\gamma}_5 \vec{\gamma}_1) \otimes \hat{P}.
\]
In the case \(\alpha = 1\), from (3.6a) it follows
\[
\alpha = +1, \quad \pi_{\text{Cart.}} = -I.
\]
The accomplished expression for \(\pi_{\text{Cart.}}\) shows that the ordinary \(P\)-reflection operator for a single bispinor field can be diagonalized on the composite (triplet’s) wave functions (when \(A \equiv 0\)). This may also be demonstrated by direct calculation over the explicit wave functions \(\Psi_{\text{cycl.}}\) in this gauge. The explicit expression for \(\Psi_{\text{cycl.}}\) is
\[
\Psi_{\text{cycl.}} = e^{-iet} \times \\
\left\{ \begin{array}{c}
T_+ \otimes e^{-i\phi} \left[ \frac{1 + \cos \theta}{2} F - \frac{\sin \theta}{2} H + \frac{1 - \cos \theta}{2} G \right] + \\
T_0 \otimes \left[ \frac{\sin \theta}{\sqrt{2}} F + \cos \theta H - \frac{\sin \theta}{\sqrt{2}} G \right] + \\
T_- \otimes e^{+i\phi} \left[ \frac{1 - \cos \theta}{2} F + \frac{\sin \theta}{2} H + \frac{1 + \cos \theta}{2} G \right] \end{array} \right\}
\]
here the symbols \((F, H, G)\) denote 4-component column-functions
\[
F = \begin{pmatrix} f_1 \ D_{-3/2} \\
 f_2 \ D_{-1/2} \\
 f_3 \ D_{-3/2} \\
 f_4 \ D_{-1/2} \end{pmatrix} , \quad H = \begin{pmatrix} h_1 \ D_{-1/2} \\
 h_2 \ D_{+1/2} \\
 h_3 \ D_{-1/2} \\
 h_4 \ D_{+1/2} \end{pmatrix} , \quad G = \begin{pmatrix} g_1 \ D_{+1/2} \\
 g_2 \ D_{+3/2} \\
 g_3 \ D_{+1/2} \\
 g_4 \ D_{+3/2} \end{pmatrix}.
\]
After simple calculation for the proper values equation
\[
[ I \otimes (-\vec{\gamma}_5 \vec{\gamma}_1) \otimes \hat{P} ] \Psi_{\text{cycl.}} = P \Psi_{\text{cycl.}}
\]
one can arrive at the same relations (2.9a).
It is important to point out that from the diagonalization of this discrete operator one cannot infer that the whole problem of reflection symmetry of the non-Abelian monopole amounts to the Abelian one. Indeed, the non-Abelian $N$-operator acts on the functions (3.7b), but the function $F(x), H(x), G(x)$ carrying individual Abelian properties are just represented in composites functions as some constructing elements, so that the function-multipliers at $T_{-1}, T_0, T_{+1}$ cannot be induced, via any special $U(1)$-gauge transformation, from Abelian ones. The operation of changing the isotopic gauge $S. \Rightarrow \text{Cart.}$ may be regarded as the calculated modification of isotopic gauge in order to translate a non-null transformation upon the isotopic coordinates in Schwinger gauge into the null transformation upon these coordinates in the Cartesian gauge. But a natural consequence of this is that the all individual Abelian qualities of the $T_{-1}, T_0, T_{+1}$’s function-multipliers have vanished completely.

4. $\hat{N}_A$-parity selection rules.

Now we proceed to analyzing the question of what is the meaning of the parameter $A$. Any value of this $A$ specifies one of the possible operators of the complete set of variables; besides, the corresponding mark appears in the designation and expression of the relevant wave functions $\Psi^\alpha_{j\mu\delta}$ and $\Psi^\alpha_{0\delta i}$ (below the quantum numbers $\epsilon, j, m$ are omitted; for definiteness supposing $j \geq 3/2$)

$$\Psi^{S\alpha}_{\delta\mu}(x) = \left[ T_0 \otimes \Phi^\delta(x) + T_{+1} \otimes \Phi^\mu_{+1}(x) + \delta \mu \, e^{iA} T_{-1} \otimes \Phi^\mu_-(x) \right].$$

The complex parameter $A$ gives a limitation on freedom with which the one triplet component proportional to $T_{-1}$ is built from another proportional to $T_{+1}$. Every $N_A$-operator with a fixed $A$ prescribes exactly how three isotopic components link up with each other, so that a unique whole is produced via a kinematical association of dynamically independent subsystems. This characteristic feature of the triplet wave function at $W(r) = 0$, as evidenced from the previous treatment, is formalized through just the $N_A$-operator. It is noticeable that such a freedom refers solely to the couple of $T_{-1}$ and $T_{+1}$.

The presence of the $A$-parameter in the explicit expressions of the wave function finds its corollary in possible manifestation over matrix elements of physical quantities. Let $\hat{G}$ be a certain composite observable, and the triplet functions are written schematically as

$$\Psi^\alpha_{\delta\mu} = (\Psi^0_\delta + \Psi^+_{\mu} + \delta \mu \, e^{iA} \Psi^-_{\mu})$$

then for the $\hat{G}$’s expectation value we can easily obtain the following expansion (the requirement of Hermiticity is imposed on $\hat{G}$)

$$G = <\Psi^A_{\delta\mu} | \hat{G} | \Psi^A_{\delta\mu}> = <\Psi^+_{\mu} | \hat{G} | \Psi^+_{\mu}> +$$

$$<\Psi^0_\delta | \hat{G} | \Psi^0_\delta> + 2 \, Re \, <\Psi^+_{\mu} | \hat{G} | \Psi^0_\delta> + e^{-(A-A^*)} \, <\Psi^-_{\mu} | \hat{G} | \Psi^-_{\mu}> +$$

$$2 \, \delta \mu \, Re \left( e^{iA} <\Psi^+_{\mu} | \hat{G} | \Psi^-_{\mu}> \right) + 2 \, \delta \mu \, Re \left( e^{iA} <\Psi^0_\delta | \hat{G} | \Psi^-_{\mu}> \right).$$

As a simple illustration let us consider in more detail the problem of $\hat{N}_A$-parity selection rules, particularly giving attention to their $A$-dependence. An initial matrix element may be written in the form

$$\int \Psi^A_{\epsilon j M \delta}(\bar{x}) \, \hat{G}(\bar{x}) \, \Phi^A_{\epsilon j' M' \delta'}(\bar{x}) \, dV \equiv \int r^2 \, dr \, \int f^A(\bar{x}) \, d\Omega.$$
Taking into account the relation
\[
f^A(-\vec{x}) = \delta \delta' (-1)^{J+J'} \Psi^A_{\epsilon J'M',\delta}(\vec{x}) \left[ (\hat{N}_A^+ \otimes) \hat{G}(-\vec{x}) (\hat{N}_A \otimes) \right] \Psi^A_{\epsilon J'M',\delta}(\vec{x}) \tag{4.3a}
\]
and supposing that \(\hat{G}(\vec{x})\) satisfies
\[
\left[ (\hat{N}_A^+ \otimes) \hat{G}(-\vec{x}) (\hat{N}_A \otimes) \right] = \Omega^A \hat{G}(\vec{x}) \tag{4.3b}
\]
where \(\Omega^A = +1\) or \(-1\), we get
\[
f^A(-\vec{x}) = \Omega^A \delta \delta' (-1)^{J+J'} f^A(\vec{x}) , \tag{4.3c}
\]
Therefore, the integral from (4.3a) can be transformed into
\[
\int \Psi^A_{\epsilon J'M',\delta}(\vec{x}) \hat{G}(\vec{x}) \Phi^A_{\epsilon J'M',\delta}(\vec{x}) dV =
\]
\[
= \left[ 1 + \Omega^A \delta \delta' (-1)^{J+J'} \right] \int_{V_{1/2}} \Psi^A_{\epsilon J'M',\delta}(\vec{x}) \hat{G}(\vec{x}) \Phi^A_{\epsilon J'M',\delta}(\vec{x}) dV \equiv \tag{4.3d}
\]
where the symbol \(V_{1/2}\) denotes the half-space of integration. The expansion (4.3c) furnishes, in fact, the required selection rules: if \(\delta, \delta', J, J'\) are chosen such that factor \((1 + \Omega^A \delta \delta' (-1)^{J+J'})\) is equal to zero then the corresponding matrix element is null too.

The above condition (4.3b) is, in fact, a definition of composite scalars and pseudoscalars under the operation of \(N_A\)-inversion. The (4.3b) can be easily enlarged on. To this end, let us introduce more detailed notation for \(\hat{G}\):
\[
\hat{G}(\vec{x}) = \left( \begin{array}{ccc} \hat{g}_{11}(\vec{x}) & \hat{g}_{12}(\vec{x}) & \hat{g}_{13}(\vec{x}) \\ \hat{g}_{21}(\vec{x}) & \hat{g}_{22}(\vec{x}) & \hat{g}_{23}(\vec{x}) \\ \hat{g}_{31}(\vec{x}) & \hat{g}_{32}(\vec{x}) & \hat{g}_{33}(\vec{x}) \end{array} \right) \otimes \hat{G}_0(\vec{x}) \tag{4.4a}
\]
then the condition (4.3b) takes the form
\[
\otimes \hat{G}_0(-\vec{x})^* = \Omega^A \left( \begin{array}{ccc} e^{i(A-A')} \hat{g}_{33}(-\vec{x}) & e^{-iA'} \hat{g}_{32}(-\vec{x}) & e^{-i(A+A')} \hat{g}_{31}(-\vec{x}) \\ e^{+iA} \hat{g}_{23}(-\vec{x}) & \hat{g}_{22}(-\vec{x}) & e^{-iA} \hat{g}_{21}(-\vec{x}) \\ e^{i(A+A')} \hat{g}_{13}(-\vec{x}) & e^{+iA'} \hat{g}_{12}(-\vec{x}) & e^{-i(A-A')} \hat{g}_{11}(-\vec{x}) \end{array} \right) \otimes \hat{G}_0(\vec{x}) . \tag{4.4b}
\]

5. On relating the \(\Psi^A_{\epsilon jm,\delta}\) at different \(A\)’s

All different values of \(A\) give the same Hilbert space of quantum states. The transformation from \(A\)- to \(A'\)-basis is described in the Schwinger’s gauge as follows:
\[
\Psi^A_{\epsilon jm,\delta} = \left[ e^{-iA} \left( \begin{array}{ccc} e^{iA} & 0 & 0 \\ 0 & e^{iA} & 0 \\ 0 & 0 & e^{iA'} \end{array} \right) \otimes I \right] \Psi^A_{\epsilon jm,\delta} \tag{5.1a}
\]
(further we follow only its isotopic part). Let us look into a general structure of the relationship
\[ \Psi_{S'}(x) = V_S(A', A) \Psi_S(x) \, . \] (5.1b)
To this end, factorizing the \( V(A', A) \) according to (\( \Gamma \equiv (A' - A)/2 \))
\[ V_S(A', A) \equiv e^{+i\Gamma} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\Gamma} & 0 \\ 0 & 0 & e^{+i\Gamma} \end{pmatrix} \, (5.2a) \]
or
\[ V_S(A', A) \equiv e^{+i\Gamma} D(\Gamma) \Delta(\Gamma) \, . \] (5.2b)
The term \( \Delta(\Gamma) \) already appeared; it represents a rotation of angle \( \Gamma \) about third isotopic axis \((0,0,1)\). The second transformation \( D(\Gamma) \) acts only on the vector \( T_0 \) in isotopic space, and it changes its length through multiplying it by a complex number (it seems to be referred to the so-called conformal ones). The matrix \( D(\Gamma) \) may be rewritten in the exponential form
\[ D(\Gamma) = e^{-i\Gamma t^0}, \quad t^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \, . \] (5.3)
With the use of explicit expressions for the isotopic rotation generators \( t_i \) and their squares
\[ t_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad t_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \]
\[(t_1)^2 = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}, \quad (t_2)^2 = \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix}, \quad (t_3)^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\]
we can produce a formula for \( t^0 \)
\[ t^0 = \frac{1}{2}(t_1^2 + t_2^2 - t_3^2) \, . \] (5.3b)
Now let us find a representation for \( V(A', A) \) in the Cartesian gauge
\[ V_{\text{Cart.}}(A', A) \equiv e^{+i\Gamma} \left[ U(\theta, \phi) D(\Gamma) U^{-1}(\theta, \phi) \right] \left[ U(\theta, \phi) \Delta(\Gamma) U^{-1}(\theta, \phi) \right] \, . \] (5.4a)
The second term in brackets has been already calculated
\[ U(\theta, \phi) \Delta(\Gamma) U^{-1}(\theta, \phi) = \exp\left[ i \Gamma \vec{\tilde{t}} \vec{n}_{\theta,\phi} \right] \, . \] (5.4b)
At analyzing the third term it is convenient to utilize the exponential representation for \( D(\Gamma) \) and also the identity (well-known in the theory of 3-rotation [64])
\[ U(\vec{n}) \, t_i \, U(-\vec{n}) = U_{ij}(\vec{n}) \, t_j \equiv \vec{t}_i(\theta, \phi) \, . \] (5.4c)
Ultimately, the \( V_{\text{Cart.}}(A', A) \) is written as
\[ V_{\text{Cart.}}(A', A) \equiv e^{+i\Gamma} \exp\left[ -i \Gamma \vec{t}^0(\theta, \phi) \right] \exp\left[ i \Gamma \vec{\tilde{t}} \vec{n}_{\theta,\phi} \right] \, . \] (5.4d)
Here, the first term has the trivial $U(1)$ character, the second is a conformal one; the third describes a rotation of complex 3-dimensional group $SO(3,C)$ about the axis $\vec{n}_{t,\phi}$.

Let us detail an explicit expression for this conformal transformation in Cartesian gauge. Taking into account that the relation $(\vec{t}^0)^2 = \vec{t}^0$, we can obtain the following expansion

$$D^{\text{Cart.}}_{\text{cyl.}}(\Gamma) = U(\theta, \phi) \ D(\Gamma) \ U^{-1}(\theta, \phi) = \left[ I + (e^{-i\tau} - 1) \ \vec{t}^0 \right] \quad (5.5)$$

and, on straightforward calculation, for the $\vec{t}^0$ get

$$\vec{t}^0 = \begin{pmatrix}
\frac{1}{2} \sin^2 \theta & -\frac{1}{\sqrt{2}} \sin \theta \cos \theta e^{-i\phi} & -\frac{1}{2} \sin^2 \theta e^{-2i\phi} \\
-\frac{1}{\sqrt{2}} \sin \theta \cos \theta e^{+i\phi} & \cos^2 \theta & \frac{1}{\sqrt{2}} \sin \theta \cos \theta e^{-i\phi} \\
-\frac{1}{2} \sin^2 \theta e^{+2i\phi} & \frac{1}{\sqrt{2}} \sin \theta \cos \theta e^{+i\phi} & \frac{1}{2} \sin^2 \theta
\end{pmatrix}. $$

As a controlling test on $\vec{t}^0$, we can verify the identity $(\vec{t}^0)^2 = \vec{t}^0$.

Now let us turn again to the structure of the transformation $V_S(A', A)$:

$$\Psi_{S'}^A(x) = V_S(A', A) \Psi_{S}^A(x), \quad V_S(A', A) \equiv e^{+i\tau} \ D(\Gamma) \ \Delta(\Gamma) \quad (5.6a)$$

remembering $\Gamma = (A' - A)/2$ . In accordance with the definition, the relationship

$$V_S(A', A) \ \hat{N}_S^A \ V_S^{-1}(A', A) = \hat{N}_{S'}^A. \quad (5.6b)$$

must hold. However, allowing for the above factorization of $V_S(A', A)$, it is readily verified that only the operator $\Delta(\Gamma)$ (a single term from two ones) makes actually this translation: $\hat{N}_S^A \rightarrow \hat{N}_{S'}^A$, whereas the remaining term do not: $D(\Gamma) \ \hat{N}_S^A \ D^{-1}(\Gamma) = \hat{N}_S^A$. One should remember that $D(\Gamma)$ acts only on the isotopic component proportional to $T_0$. Moreover, it is evident that $D(\Gamma)$ does not affect all the operators from a complete set of variables: $i\partial_t, J^2, J_3$ as well as the $\hat{N}_A$.

The existence of an operation with those properties indicates that there exists a possibility to subject the function $\Psi_{ejmb}^A(x)$ to a transformation of this $D(F)$ kind (here $F$ involved may not be correlated with the above $\Gamma$), so that as a result we arrive at a new wave function with the same quantum numbers. However this situation does not provide us with a paradox. To understand this, it suffices to return to the radial system (2.11a) and give attention to the fact that in this system the two sub-sets of functions $f_1, f_2, f_3, f_4$ and $h_1, h_2$ are completely independent of each other. The latter speaks of the following: the set of operators used above and provided the quantum number $(e, j, m, \delta)$ fixes the wave function apart from an arbitrary complex factor at the multiplet component proportional to $T_0$

$$\Psi_{ejmb}^{A(B)}(\vec{x}) = \left[ T_{+1} \otimes \Phi^+(\vec{x}) + e^{iB} \ T_0 \otimes \Phi_0(\vec{x}) + \delta \ e^{iA} \ T_{-1} \otimes \Phi^-(\vec{x}) \right] \quad (5.7a)$$

The transformation of $D$-kind acts on those functions according to

$$\Psi_{ejmb\mu}^{A(B')}(\vec{x}) = D(B - B') \ \Psi_{ejmb\mu}^{A(B)}(\vec{x}). \quad (5.7b)$$

Evidently, such a $B$-symmetry can prove itself in explicit expression for matrix elements; but this cannot happen to the $N$-parity selection rules because this $D$-symmetry does not affect the $N$-operator.
Supplement A. Some technical material: Dirac and Schwinger gauges in isotopic space; spinor approach by Tetrode-Weyl-Fock-Ivanenko; Schrödinger’s basis and Pauli’s analysis.

This section deals with some representation of the non-Abelian monopole potential, which will be the most convenient one to formulate and analyze the problem of isotopic multiplet in this field. Let us begin describing in detail this matter. The well-known form of the monopole solution introduced by t’Hooft and Polyakov ([61,62]; see also Julia-Zee [63]) may be taken as a starting point. The field $W^{(a)}$ represents a covariant vector with the usual transformation law $W^{(a)} = (\partial x^i/\partial x^i) W^{(a)}_i$ and our first step is the change of variables in 3-space. Thus, the given potentials $(\Phi^{(a)}(x), W^{(a)}_r)$ convert into $(\Phi^{(a)}(x), W^{(a)}_t, W^{(a)}_r, W^{(a)}_\theta, W^{(a)}_\phi)$. Our second step is a special gauge transformation in the isotopic space. The required gauge matrix can be determined (only partly) by the condition $(O_{ab} \Phi^b(x)) = (0, 0, r \Phi(r))$. This equation has a set of solutions since the isotopic rotation by every angle about the third axis $(0,0,1)$ will not change the finishing vector $(0,0,r \Phi(r))$. We fix such an ambiguity by deciding in favor of the simplest transformation matrix. It will be convenient to utilize the known group $SO(3,R)$ parameterization through the Gibbs 3-vector:

$$O = O(\bar{c}) = I + 2 \frac{\bar{c} \times + (\bar{c} \times)^2}{1 + \bar{c}^2}, \quad (\bar{c} \times)_{ab} = -\epsilon_{abc} c_b. \quad (A.1)$$

According to [65], the simplest rotation above is

$$\bar{B} = O(\bar{c})\bar{A}, \quad \bar{c} = \frac{[\bar{B}\bar{A}]}{(\bar{A} + \bar{B})\bar{A}}.$$ 

Therefore,

if \( \bar{A} = r \Phi(r) \bar{n}_{\theta,\phi}, \quad \bar{B} = r \Phi(r)(0,0,1), \)

then

$$\bar{c} = \frac{\sin \theta}{1 + \cos \theta} (+ \sin \phi, -\cos \phi, 0). \quad (A.2)$$

Together with varying the scalar field $\Phi^a(x)$, the vector triplet $W^{(a)}_\beta(x)$ is to be transformed from one isotopic gauge to another under the law [66]

$$W^{(a)}_\alpha(x) = O_{ab}(\bar{c}(x)) W^{(b)}_\alpha(x) + \frac{1}{e} f_{ab}(\bar{c}(x)) \frac{\partial c_b}{\partial x^\alpha}, \quad f(\bar{c}) = -2 \frac{1 + \bar{c} \times}{1 + \bar{c}^2}. \quad (A.3)$$

With the use of (A.3), we obtain the new representation

$$\Phi^{D,(a)} = r \Phi(r) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad W^{D,(a)}_\theta = (r^2 K + 1/e) \begin{pmatrix} -\sin \phi \\ + \cos \phi \\ 0 \end{pmatrix}, \quad W^{D,(a)}_r = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad W^{D,(a)}_t = \begin{pmatrix} 0 \\ 0 \\ r F(r) \end{pmatrix}, \quad W^{D,(a)}_\phi = \begin{pmatrix} -(r^2 K + 1/e) \sin \theta \cos \phi \\ -(r^2 K + 1/e) \sin \theta \sin \phi \\ \frac{1}{e}(\cos \theta - 1) \end{pmatrix}. \quad (A.4)$$

The author highly recommends the book [64] for many further details developing the Gibbs approach to groups $SO(3,R), SO(3,C), SO_0(3,1)$, etc.
It should be noticed that the factor \( (r^2 K(r) + 1/e) \) will vanish when \( K = -1/er^2 \). Thus, only the delicate fitting of the single proportional coefficient (it must be taken as \(-1/e\)) results in the actual formal simplification of the non-Abelian monopole potential.

There exists close connection between \( W^{D,(a)}_\alpha(x) \) from (A.4) and the Dirac’s expression for the Abelian monopole potential (supposing that \( \vec{n} = (0, 0, -1) \)):

\[
A^D_\alpha = g \left( 0, \frac{[\vec{r} \vec{n}]}{(r + \vec{r} \vec{n}) r} \right), \quad \text{or} \quad A^D_\phi = -g \cos \theta - 1. \quad (A.5)
\]

So, \( W^{triv,(a)}_{(a)\alpha}(x) \) from (A.4) (produced by setting \( K = -1/er^2 \)) can be thought of as the result of embedding the Abelian potential (A.5) in the non-Abelian gauge scheme: \( W^{(a)D}_\alpha(x) \equiv (0, 0, A^D_\alpha(x)) \). The quantity \( W^{(a)D}_\alpha(x) \) labelled with symbol \( D. \) will be named after its Abelian counterpart; in other words, this potential will be treated as relating to the Dirac’s non-Abelian gauge in the isotopic space.

In Abelian case, the Dirac’s potential \( A^D_\alpha(x) \) can be converted into the Schwinger form \( A^S_\alpha(x) \)

\[
A^S_\alpha = \left( 0, g \frac{[\vec{r} \vec{n}]}{(r + \vec{r} \vec{n}) r} \right), \quad \text{or} \quad A^S_\phi = g \cos \theta \quad (A.6)
\]

by means of the following transformation

\[
A^S_\alpha = A^D_\alpha + \frac{hc}{ie} S \frac{\partial}{\partial x^\alpha} S^{-1}, \quad S(x) = \exp(-ieg/\hbar \chi(x)).
\]

It is possible to draw an analogy between the Abelian and non-Abelian models. That is, we may introduce the Schwinger non-Abelian basis in the isotopic space:

\[
(\Phi^{D,(a)}, W^{D,(a)}_\alpha) \overset{\tilde{\vec{c}}'}{\rightarrow} (\Phi^{S,(a)}, W^{S,(a)}_\alpha), \quad \tilde{\vec{c}}' = (0, 0, -\tan \phi/2); \quad (A.7a)
\]

where

\[
O(\tilde{\vec{c}}') = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Now an explicit form of the monopole potential is given by

\[
W^{S,(a)}_\psi = \begin{pmatrix} 0 & (r^2 K + 1/e) \\ 0 & 0 \end{pmatrix}, \quad W^{S,(a)}_\phi = \begin{pmatrix} -(r^2 K + 1/e) & 0 \\ -\frac{1}{e} \cos \theta & 0 \end{pmatrix},
\]

\[
W^{S,(a)}_r = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad W^{S,(a)}_t = \begin{pmatrix} 0 & 0 \\ 0 & rF(r) \end{pmatrix}, \quad \Phi^{S,(a)} = \begin{pmatrix} 0 & 0 \\ 0 & r \Phi(r) \end{pmatrix} \quad (A.7b)
\]

where the symbol \( S. \) stands for the Schwinger gauge. Both \( D. \)- and \( S. \)-gauges (see (A.4) and (A.7b)) are unitary ones in the isotopic space due to the respective scalar fields \( \Phi^{D,(a)}(x) \) and \( \Phi^{S,(a)}(x) \) are \( x_3 \)-unidirectional, but one of them (Schwinger’s) seems simpler than another (Dirac’s).

For the following it will be convenient to determine the matrix \( O(\tilde{\vec{c}}'') \) relating the Cartesian gauge of isotopic space with Schwinger’s:

\[
O(\tilde{\vec{c}}'') = O(\tilde{\vec{c}}') O(\tilde{\vec{c}}) = \begin{pmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \end{pmatrix},
\]

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\[ \vec{c}'' = (+ \tan \theta/2 \tan \phi/2, - \tan \theta/2, - \tan \phi/2). \] (A.8)

This matrix \(O(\vec{c}'')\) is also well-known in other context as a matrix linking Cartesian and spherical tetrads in the space-time of special relativity (as well as in a curved space-time of spherical symmetry)

\[ x^\alpha = (x^0, x^1, x^2, x^3), \quad dS^2 = [(dx_0)^2 - (dx_1)^2 - (dx_2)^2 - (dx_3)^2], \quad e^\alpha_{(a)}(x) = \delta^\alpha_a \] (A.9a)

and

\[ x'^\alpha = (t, r, \theta, \phi), \quad dS^2 = [dt^2 - dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)], \quad e^\alpha_{(0)} = (1, 0, 0, 0), \quad e^\alpha_{(1)} = (0, 0, 1/r, 0), \quad e^\alpha_{(2)} = (0, 0, 1/r \sin \theta), \quad e^\alpha_{(3)} = (0, 1, 0, 0). \] (A.9b)

Below we review briefly some relevant facts about the tetrad formalism. In the presence of an external gravitational field, the starting Dirac equation \((i\gamma^a \partial/\partial x^a - m)\Psi(x) = 0\) is generalized into \([20-29]\)

\[ [i\gamma^a(x) (\partial_a + \Gamma_\alpha(x)) - m] \Psi(x) = 0 \] (A.10)

where \(\gamma^\alpha(x) = \gamma^\alpha e^\alpha_{(a)}(x)\), and \(e^\alpha_{(a)}(x), \Gamma_\alpha(x) = \frac{1}{2} \sigma^{ab} e^\beta_{(a)} \nabla_a (e^\alpha_{(b)\beta})\), \(\nabla_a\) stand for a tetrad, the bispinor connection, and the covariant derivative symbol, respectively. In the spinor basis:

\[ \psi(x) = \left( \begin{array}{c} \xi(x) \\ \eta(x) \end{array} \right), \quad \gamma^a = \left( \begin{array}{cc} 0 & \tilde{\sigma}^a \\ \sigma^a & 0 \end{array} \right), \quad \sigma^a = (I, +\sigma^k), \quad \tilde{\sigma}^a = (I, -\sigma^k) \]

(\(\sigma^k\) are the two-row Pauli spin matrices; \(k = 1, 2, 3\)) we have two equations

\[ i\sigma^a(x) [\partial_a + \Sigma_\alpha(x)] \xi(x) = m \eta(x), \quad i\tilde{\sigma}^a(x) [\partial_a + \tilde{\Sigma}_\alpha(x)] \eta(x) = m \xi(x) \] (A.11)

where the symbols \(\sigma^a(x), \tilde{\sigma}^a(x), \Sigma_\alpha(x), \tilde{\Sigma}_\alpha(x)\) denote respectively

\[ \begin{align*}
\sigma^a(x) &= \sigma^a e^\alpha_{(a)}(x), \\
\tilde{\sigma}^a(x) &= \tilde{\sigma}^a e^\alpha_{(a)}(x), \\
\Sigma_\alpha(x) &= \frac{1}{2} \Sigma^{ab} e^\beta_{(a)} \nabla_a (e^\alpha_{(b)\beta}), \\
\tilde{\Sigma}_\alpha(x) &= \frac{1}{2} \tilde{\Sigma}^{ab} e^\beta_{(a)} \nabla_a (e^\alpha_{(b)\beta}), \\
\Sigma^{ab} &= \frac{1}{4} (\sigma^a \sigma^b - \sigma^b \sigma^a), \\
\tilde{\Sigma}^{ab} &= \frac{1}{4} (\sigma^a \tilde{\sigma}^b - \tilde{\sigma}^b \sigma^a).
\end{align*} \]

The form of equations (A.10), (A.11) implies quite definite their symmetry properties. It is common, considering the Dirac equation in the same space-time, to use some different tetrads \(e^\beta_{(a)}(x)\) and \(e'^\beta_{(a)}(x)\), so that we have the equation (A.10) and an analogous one with a new tetrad mark. In other words, together with (A.10) there exists an equation on \(\Psi'(x)\), where quantities \(\gamma'^\alpha(x)\) and \(\Gamma'_\alpha(x)\), in contrast with \(\gamma^\alpha(x)\) and \(\Gamma_\alpha(x)\), are based on a new tetrad \(e^\beta_{(b)}(x)\) related to \(e^\beta_{(a)}(x)\) through a certain local Lorentz matrix

\[ e'^\beta_{(b)}(x) = L^a_{(b)}(x) e^\beta_{(a)}(x). \] (A.12a)

It may be shown that these two Dirac equations on functions \(\Psi(x)\) and \(\Psi'(x)\) are related to each other by a definite bispinor transformation:

\[ \xi'(x) = B(k(x)) \xi(x), \quad \eta'(x) = B^+(\tilde{k}(x)) \eta(x). \] (A.12b)
Here, \( B(k(x)) = \sigma^a k_a(x) \) is a local matrix from the \( SL(A.C) \) group; 4-vector \( k_a \) is the well-known parameter on this group ([67]; also see in [64]). The matrix \( L_b^a(x) \) from (A.12a) may be expressed as a function of arguments \( k_a(x) \) and \( k_a^*(x) \):

\[
L_b^a(k, k^*) = \delta^c_b \left[ -\delta^c_c k^n k^n_x + k_c k^c_x + k_c^* k^c_x + i\epsilon^{anmm}_c k_n k_m^* \right]
\]  

(A.12c)

where \( \delta^c_b \) is a special Cronecker symbol

\[
\delta^c_b = 0 \text{ if } c \neq b; = +1 \text{ if } c = b = 0; = -1 \text{ if } c = b = 1, 2, 3 .
\]

It is normal practice that some different tetrads are used in examining the Dirac equation on the same Riemannian space-time background. If there is a need to analyze some correlation between solutions in those distinct tetrads, then it is important to know what are the relevant gauge transformations over the spinor wave functions. In particular, the matrix relating spinor wave functions in Cartesian and spherical tetrads (see (A.9)) is as follows

\[
B = \pm \left( \begin{array}{cc}
cos \frac{\theta}{2} e^{i\phi/2} & \sin \frac{\theta}{2} e^{-i\phi/2} \\
-\sin \frac{\theta}{2} e^{i\phi/2} & \cos \frac{\theta}{2} e^{-i\phi/2}
\end{array} \right) \equiv B(\vec{c}''') = \pm \frac{I - i\vec{\sigma}\vec{c}'''}{\sqrt{1 - (\vec{c}''')^2}} .
\]

(A.12d)

The vector matrix \( L_b^a(\theta, \phi) \) referring to the spinor’s \( B(\theta, \phi) \) is the same as \( O(\vec{c}''') \) from (A.8). It is significant that the two gauge transformations, arising in quite different contexts, correspond so closely with each other.

This basis of spherical tetrad played a substantial role in the present work. This Schrödinger frame of spherical tetrad [29] was used with great efficiency by Pauli [30] when investigating the problem of allowed spherically symmetrical wave functions in quantum mechanics. Below, we briefly review some results of this investigation. Let the \( J^\lambda_i \) denote

\[
J_1 = (l_1 + \lambda \frac{\cos \phi}{\sin \theta}), \quad J_2 = (l_2 + \lambda \frac{\sin \phi}{\sin \theta}), \quad J_3 = l_3 .
\]

At an arbitrary \( \lambda \), as readily verified, those \( J_i \) satisfy the commutation rules of the Lie algebra \( SU(2) \) : \( [J_a, J_b] = i \epsilon_{abc} J_c \). As known, all irreducible representations of such an abstract algebra are determined by a set of weights \( j = 0, 1/2, 1, 3/2, \ldots \ (\text{dim } j = 2j + 1) \). Given the explicit expressions of \( J_a \) above, we will find functions \( \Phi^\lambda_{jm}(\theta, \phi) \) on which the representation of weight \( j \) is realized. In agreement with the generally known method, those solutions are to be established by the following relations

\[
J_+ \Phi^\lambda_{jj} = 0 , \quad \Phi^\lambda_{jm} = \sqrt{(j + m)! (j - m)! (2j)!} \Phi^\lambda_{j-j} ,
\]

(A.13)

\[
J_{\pm} = (J_1 \pm i J_2) = e^{\pm i\phi} \left[ \pm \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} + \frac{\lambda}{\sin \theta} \right].
\]

From the equations \( J_+ \Phi^\lambda_{jj} = 0 \) and \( J_3 \Phi^\lambda_{jj} = j \Phi^\lambda_{jj} \), it follows that

\[
\Phi^\lambda_{jj} = N^\lambda_{jj} e^{ij\phi} \sin^j \theta \left( \frac{1 + \cos \theta} {1 - \cos \theta} \right)^{\lambda/2} , \quad N^\lambda_{jj} = \frac{1}{\sqrt{2\pi}} \frac{1}{2^j} \sqrt{\frac{(2j + 1)}{\Gamma(j + m + 1) \Gamma(j - m + 1)}} .
\]

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Further, employing (A.13) we produce the functions $\Phi^\lambda_{jm}$

$$\Phi^\lambda_{jm} = N^\lambda_{jm} e^{im\phi} \frac{1}{\sin^m \theta} \frac{(1 - \cos \theta)^{\lambda/2}}{(1 + \cos \theta)^{\lambda+1/2}} \times$$

$$\left( \frac{d}{d\cos \theta} \right)^{j-m} \left[ (1 + \cos \theta)^{j+\lambda} (1 - \cos \theta)^{j-\lambda} \right]$$

where

$$N^\lambda_{jm} = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{(2j + 1)(j + m)!}}{2(j - m)!\Gamma(j + \lambda + 1)\Gamma(j - \lambda + 1)}.$$

The Pauli criterion tells us that the $(2j + 1)$ functions $\Phi^\lambda_{jm}(\phi), \ m = -j, ..., +j$ so constructed, are guaranteed to be a basis for a finite-dimension representation, providing that the functions $\Phi^\lambda_{j,-j}(\phi)$, found by this procedure, obey the identity

$$J_- \Phi^\lambda_{j,-j} = 0. \quad (A.15a)$$

After substituting the function $\Phi^\lambda_{j,-j}(\phi)$, the relation (A.15a) reads

$$J_- \Phi^\lambda_{j,-j} = N^\lambda_{j,-j} e^{-i(j+1)\phi} (\sin \theta)^{j+1} \frac{(1 - \cos \theta)^{\lambda/2}}{(1 + \cos \theta)^{\lambda+1/2}} \times$$

$$\left( \frac{d}{d\cos \theta} \right)^{2j+1} \left[ (1 + \cos \theta)^{j+\lambda} (1 - \cos \theta)^{j-\lambda} \right] = 0 \quad (A.15b)$$

which in turn gives the following restriction on $j$ and $\lambda$

$$\left( \frac{d}{d\cos \theta} \right)^{2j+1} \left[ (1 + \cos \theta)^{j+\lambda} (1 - \cos \theta)^{j-\lambda} \right] = 0. \quad (A.15c)$$

But the relation (A.15c) can be satisfied only if the factor $P(\phi)$, subjected to the operation of taking derivative $(d/d\cos \theta)^{2j+1}$, is a polynomial of degree $2j$ in $\cos \theta$. So, we have (as a result of the Pauli criterion)

1. the $\lambda$ is allowed to take values $+1/2, -1/2, +1, -1, ...$

Besides, as the latter condition is satisfied, $P(\phi)$ takes different forms depending on the $(j, \lambda)$-correlation:

$$P(\phi) = (1 + \cos \theta)^{j+\lambda} (1 - \cos \theta)^{j-\lambda} = P^{2j}(\cos \theta), \quad if \quad j = |\lambda|, |\lambda| + 1, ...$$

or

$$P(\phi) = \frac{P^{2j+1}(\cos \theta)}{\sin \theta}, \quad if \quad j = |\lambda| + 1/2, |\lambda| + 3/2, ...$$

so that the second necessary condition resulting from the Pauli criterion is

2. given $\lambda$ according to 1., the number $j$ is allowed to take values $j = |\lambda|, |\lambda| + 1, ...$

We draw attention to that the Pauli criterion $J_- \Phi_{j,-j}(t, r, \theta, \phi) = 0$ affords the condition that is invariant relative to possible gauge transformations. The function $\Phi_{j,m}(t, r, \theta, \phi)$
may be subjected to any gauge transformation. But if all the components \( J_i \) vary in a corresponding way too, then the Pauli condition provides the same result on \((j, \lambda)\)-quantization. In contrast to this, the common requirement to be a single-valued function of spatial points, often applied to produce a criterion on selection of allowable wave functions in quantum mechanics, is not invariant under gauge transformations and can easily be destroyed by a suitable gauge one. Also, it should be noted that the angular variable \( \phi \) is not affected (charged) by the Pauli criterion; instead, a variable that works above is the \( \theta \). Significantly, in the contrast to this, the well-known procedure of deriving the electric charge quantization condition from investigating continuity properties of quantum mechanical wave functions, such a working variable is the \( \phi \).

If the first and second Pauli consequences fail, then we face rather unpleasant mathematical and physical problems (Reader is referred to the Pauli article [30] for more detail about those peculiarities). As a simple illustration, we may indicate the familiar case when \( \lambda = 0 \); if the second Pauli condition is violated, then we will have the integer and half-integer values of the orbital angular momentum number \( l = 0, 1/2, 1, 3/2, \ldots \). As regards the Dirac electron with the components of the total angular momentum in the form (1.2), we have to employ the above Pauli criterion in the constituent form owing to \( \lambda \) changed into \( \Sigma_3 \). Ultimately, we obtain the allowable set \( J = 1/2, 3/2, \ldots \).

A fact of primary practical importance to us is that the functions \( \Phi_{jm}(\theta, \phi) \) constructed above relate directly to the known Wigner \( D \)-functions [18]:

\[
\Phi_{jm}(\theta, \phi) = (-1)^{j-m} D_{jm}(\phi, \theta, 0).
\]

**Supplement B. On complete set of operators.**

For our purposes we may take a Dirac’s generalized operator; in spherical tetrad and Schwinger isotopic basis it is

\[
\hat{K} = +i \gamma^0 \gamma^3 \Sigma_{\theta, \phi} = i \gamma^0 \gamma^3 \left[ i\gamma^1 \partial_\theta + \gamma^2 \frac{i\partial_\phi + (i\sigma^{12} + t^3)\cos \theta}{\sin \theta} \right]. \tag{B.1}
\]

Now, taking into account already known action of the \( \Sigma_{\theta, \phi} \) on wave functions with fixed numbers \( j \ m \), for \( K \Psi_{jm\delta}(x) \) we get

\[
\hat{K} \Psi_{jm\delta}^{(A)}(x) = \frac{e^{-i\delta}}{r} \times
\]

\[
\left[ b T_{+1} \otimes \left( f_1 D_{-3/2} \right) + a e^{iB} T_0 \otimes \left( \delta h_1 D_{1/2} \right) + \delta e^{iA} b T_{-1} \otimes \left( f_1 D_{+1/2} \right) \right].
\]

Correspondingly, from the equation \( \hat{K} \Psi_{jm\delta}^{(A)}(x) = \lambda \Psi_{jm\delta}^{(A)}(x) \) it follows

\[
\begin{align*}
& b f_4 = \lambda f_1, \quad b f_1 = \lambda f_4, \quad b f_3 = \lambda f_2, \quad b f_2 = \lambda f_3, \\
& \delta a h_1 = \lambda h_1, \quad a h_1 = \delta \lambda h_1, \quad \delta a h_2 = \lambda h_2, \quad a h_2 = \delta \lambda h_2. \tag{B.2}
\end{align*}
\]

Let us suppose that all radial functions \( f_1, \ldots, f_4 \), and \( h_1, h_2 \) are not null ones; then from (B.2) we arrive at two non-consistent with each other relations: \( \lambda^2 = b^2 \) and \( \lambda^2 = a^2 \), but
\(a\) and \(b\) are already given as basically different: \(a = (j + 1/2)\), \(b = \sqrt{(j - 1/2)(j + 3/2)}\). Therefore, the equations in (B.2) may be satisfied only as follows:

**1:**

\[
f_i = 0; \quad h_1, \ h_2 \neq 0 \quad \implies \quad \lambda^{(1)} = \delta a \quad (\delta = \pm 1),
\]

and proper wave functions of \(\hat{K}\) are

\[
\Psi_{jm\delta(\delta)}^{(B)}(x) = e^{iB}T_0 \otimes \begin{pmatrix} h_1 \ D_{-1/2} \\ h_2 \ D_{+1/2} \\ \delta \ h_2 \ D_{-1/2} \\ \delta \ h_1 \ D_{+1/2} \end{pmatrix}; \quad (B.3a)
\]

the corresponding system of radial equations is

\[
\begin{align*}
\epsilon \ h_2 - i \frac{d}{dr} \ h_2 - \frac{i a}{r} \ h_1 - \delta \ m \ h_1 &= 0, \\
\epsilon \ h_1 + i \frac{d}{dr} \ h_1 + \frac{i a}{r} \ h_2 - \delta \ m \ h_2 &= 0.
\end{align*}
\]

**2:**

\[
f_i \neq 0; \quad h_1, \ h_2 = 0 \quad \implies \quad \lambda^{(2)} = \mu b \quad (\mu = \pm 1),
\]

and proper wave functions of \(\hat{K}\) are \((f_4 = \mu f_1, \ f_3 = \mu f_2)\)

\[
\Psi_{jm\delta\mu}^{A}(x) = \frac{e^{-ict}}{r} \left[ T_{+1} \otimes \begin{pmatrix} f_1 \ D_{-3/2} \\ f_2 \ D_{-1/2} \\ \mu \ f_2 \ D_{-3/2} \\ \mu \ f_1 \ D_{-1/2} \end{pmatrix} + \delta \ \mu \ e^{iA}T_{-1} \otimes \begin{pmatrix} f_1 \ D_{+1/2} \\ f_2 \ D_{+3/2} \\ \mu \ f_2 \ D_{+1/2} \\ \mu \ f_1 \ D_{+3/2} \end{pmatrix} \right]; \quad (B.4a)
\]

the radial system will be as follows

\[
\begin{align*}
\epsilon \ f_2 - i \frac{d}{dr} \ f_2 - \frac{i b}{r} \ f_1 - \mu \ m \ f_1 &= 0, \\
\epsilon \ f_1 + i \frac{d}{dr} \ f_1 + \frac{i b}{r} \ f_2 - \mu \ m \ f_2 &= 0.
\end{align*}
\]

Thus, the complete set of operators \(\hat{H}, \ \hat{J}^2, \ J_3, \ \hat{N}_A, \ \hat{K}\) leads us to basis functions of two different types:

\[
\Psi_{jm\delta(\delta)}^{(B)}(x), \ \lambda^{(1)} = \delta a \quad \text{and} \quad \Psi_{jm\delta\mu}^{(A)}(x), \ \lambda^{(2)} = \mu b; \quad (B.5)
\]

that is every state with fixed \(\epsilon, \ j, \ m, \ \delta\) is yet triply degenerated; the situation may illustrated by the scheme

\[
\Psi_{\epsilon jm\delta}^{A(B)}(x) \quad \implies \quad \left\{ \Psi_{\epsilon jm\delta(\delta)}^{A}(x) \oplus \Psi_{\epsilon jm\delta\mu}^{B}(x) \right\}. \quad (B.6)
\]

Now let us consider the case of minimal \(j\):

\[
\Psi_{\delta \ \min.}^{t, r}(t, r) = \frac{e^{-ict}}{r} \times
\]

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\[
T_{+1} \otimes \begin{pmatrix}
0 \\
\frac{f_2}{D_{-1/2}} \\
0 \\
\frac{f_4}{D_{-1/2}}
\end{pmatrix} + e^{iB} T_0 \otimes \begin{pmatrix}
\frac{h_1}{D_{-1/2}} \\
\frac{h_2}{D_{+1/2}} \\
\frac{\delta h_2}{D_{-1/2}} \\
\frac{\delta h_1}{D_{+1/2}}
\end{pmatrix} + e^{iA} \delta T_{-1} \otimes \begin{pmatrix}
\frac{f_4}{D_{+1/2}} \\
0 \\
\frac{f_2}{D_{+1/2}} \\
0
\end{pmatrix}.
\]

From the equation $\hat{K}_{\delta} \Psi_{\delta}^{\min.} = \lambda \Psi_{\delta}^{\min.}$ we can get two proper values $\lambda$:

\[
\lambda = \delta : \quad \Psi_{\delta(\lambda=\delta)}^{\min.}(t, r) = e^{iB} T_0 \otimes \begin{pmatrix}
\frac{h_1}{D_{-1/2}} \\
\frac{h_2}{D_{+1/2}} \\
\frac{\delta h_2}{D_{-1/2}} \\
\frac{\delta h_1}{D_{+1/2}}
\end{pmatrix}; \quad (B.7a)
\]

\[
\lambda = 0 : \quad \Psi_{\delta(\lambda=0)}^{\min.}(t, r) = T_{+1} \otimes \begin{pmatrix}
\frac{0}{D_{-1/2}} \\
\frac{f_2}{D_{-1/2}} \\
\frac{0}{D_{-1/2}} \\
\frac{f_4}{D_{-1/2}}
\end{pmatrix} + e^{iA} \delta T_{-1} \otimes \begin{pmatrix}
\frac{f_4}{D_{+1/2}} \\
0 \\
\frac{f_2}{D_{+1/2}} \\
0
\end{pmatrix} \quad (B.7b)
\]
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