Extrema of Mass, First Law of Black Hole Mechanics and Staticity Theorem in
Einstein-Maxwell-axion-dilaton Gravity

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Using the ADM formulation of the Einstein-Maxwell axion-dilaton gravity we derived the formulas for the variation of mass and other asymptotic conserved quantities in the theory under consideration. Generalizing this kind of reasoning to the initial data for the manifold with an interior boundary we got the generalized first law of black hole mechanics.

We consider an asymptotically flat solution to the Einstein-Maxwell axion-dilaton gravity describing a black hole with a Killing vector field timelike at infinity, the horizon of which comprises a bifurcate Killing horizon with a bifurcate surface. Supposing that the Killing vector field is asymptotically orthogonal to the static hypersurface with boundary S and a compact interior, we find that the solution is static in the exterior world, when the timelike vector field is normal to the horizon and has vanishing electric and axion-electric fields on static slices.

I. INTRODUCTION

During the last years the discovery of new black hole solutions in theories with nonlinear matter fields (see only a selected list of references [1]) prompted us to study topics related to the stationary problem of nonrotating black holes as well as the subject of stationarity of these objects. Nowadays, the problems of black hole physics of the late 1960s and 1970s are reassessed, taking into consideration nonlinear matter models or general sigma models.

From the historical point of view the idea of a staticity theorem was proposed by Lichnerowicz [2] for the simple case in which there was no black hole. He proved that case for a stationary perfect fluid that was everywhere locally static in the sense that its flow vector was aligned with the Killing vector. The Killing vector itself would have the staticity property of being hypersurface orthogonal.

The next extension of the research was attributed to Hawking [3]. His proof of staticity applied to the vacuum case. He considered black holes that were nonrotating in the sense that the null generator of the horizon was aligned with the Killing vector. Carter considered an extension of this problem to the case of electromagnetic fields and obtained the desired result subject to a certain inequality between the norm of the Killing field and the electric potential [4,5].

By means of the Arnowitt-Deser-Misner (ADM) formalism, Sudarsky and Wald [6] considered an asymptotically flat solution to Einstein-Yang-Mills (EYM) equations with a Killing vector field which was timelike at infinity. By means of the notion of an asymptotically flat maximal slice with compact interior, they established that the solution is static when it had a vanishing Yang-Mills electric field on the static hypersurfaces. If an asymptotically flat solution possesses a black hole, then it is static when it has a vanishing electric field on the static hypersurface. They also presented a new derivation of the mass formula and proved that every stationary solution is an extremum of the ADM mass at fixed Yang-Mills electric charge. On the other hand, every stationary black hole solution is an extremum of the ADM mass at fixed electric charge, canonical angular momentum, and horizon area. One should also mention
the work of Sudarsky and Wald [7], in which they derived new integral mass formulas for stationary black holes in EYM theory. Using the notion of maximal hypersurfaces [8] and combining the mass formulae, they obtained the proof that nonrotating Einstein-Maxwell (EM) black holes must be static and have a vanishing magnetic field on the static slices.

The strong rigidity theorem [9], derived by Hawking, emphasizes that the event horizon of a stationary black hole had to be a Killing horizon; i.e., there had to exist a Killing field $\chi_\mu$ in the spacetime which was normal to the horizon. If this field did not coincide with the stationary Killing field $t_\mu$, then it was shown that the spacetime had to be axisymmetric as well as stationary. It follows that the black hole will be rotating; i.e., its angular velocity of the horizon $\Omega$ will be nonzero ($\Omega$ is defined by the relation $\chi_\mu = t_\mu + \Omega \phi_\mu$, where $\phi_\mu$ is an axial Killing vector field) and the Killing vector field $k_\mu$ will be spacelike in the vicinity of the horizon. The black hole will be enclosed by an ergoregion. On the other hand, if $t_\mu$ coincides with $\chi_\mu$ (so that the black hole is nonrotating) and $t_\mu$ is globally timelike outside the black hole, then one can show that the spacetime is static. The standard black hole uniqueness theorem leaves an open question of the problem of the potential existence of additional stationary black hole solutions of EM equations with a bifurcate horizon which are neither static nor axisymmetric. The situation was recuperated by Wald [10]. He showed that any nonrotating black hole in EM theory, the ergoregion of which was disjoint from the horizon had, to be static, even if the $t_\mu$ was not initially presupposed to be globally timelike outside the black hole.

Chruściel [11] reconsidered the problem of the strong rigidity theorem and gave the corrected version of the theorem in which he excluded the previous assumption about maximal analytic extensions which were not unique.

The uniqueness theorems for black holes are closely related to the problem of staticity. However, the uniqueness theorems are based on stronger assumptions than the strong rigidity theorem. Namely, in the nonrotating case one requires staticity whereas in the rotating case the uniqueness theorem is established for circular spacetimes. The foundations of the uniqueness theorems were laid by Israel [12,13] who established the uniqueness of the Schwarzschild metric and its Reissner-Nordström generalization as static asymptotically flat solutions of the Einstein and EM vacuum field equations. Then, Müller zum Hagen et al. [14] in their works were able to weaken Israel’s assumptions concerning the topology and regularity of the two-surface $V = -t^\mu t_\mu = \text{const}$. Robinson [15] generalized the theorem of Israel concerning the uniqueness of the Schwarzschild black hole [12]. Finally, Bunting and Masood-ul-Alam [16] excluded multiple black hole solutions, using the conformal transformation and the positive mass theorem [17]. Lately, a generalization of the results to electrovacuum spacetimes was achieved [18].

The uniqueness results for rotating configurations, i.e., for stationary, axisymmetric black hole spacetimes were obtained by Carter [19], completed by Hawking and Ellis [1] and the next works of Carter [4,5] and Robinson [20]. They were related to the vacuum case. Robinson also gained [21] a complicated identity which enabled him to expand Carter’s results to electrovac spacetimes.

A quite different approach to the problem under consideration was presented by Bunting [22] and Mazur [23]. Bunting’s approach was based on applying a general class of harmonic mappings between Riemannian manifolds while Mazur’s was based on the observation that the Ernst equations describe nonlinear $\sigma$ model on symmetric space. A review of these new methods presented by Bunting and Mazur was given in Ref. [24].

A recent review which covers in detail various aspects of the uniqueness theorems for nonrotating and rotating
black holes was provided by Heusler \cite{Heusler25}.

Heusler and Straumann in Ref. \cite{Ref27} considered the stationary EYM and Einstein dilaton theories. They showed that the mass variation formula involves only global quantities and surface terms; their results hold for arbitrary gauge groups and any structure of the Higgs field multiplets. In Ref. \cite{Ref26} the same authors studied the staticity conjecture and circularity conditions for rotating black holes in EYM theories. It turned out that contrary to the Abelian case staticity conjecture might not hold for non-Abelian gauge fields like the circularity theorem for these fields. Recently, it has been shown \cite{Ref28} that in the non-Abelian case stationary balck hole spacetimes with vanishing angular momentum need not to be static unless they have vanishing electric Yang-Mills charge. Heusler \cite{Heusler23} demonstrated that any selfcoupled, stationary scalar mapping ($\sigma$ model) from a domain of strictly outer communication, with nonrotating horizon, has to be static. He also proved no-hair conjecture for this model.

The mathematical rigor of the uniqueness theorems and related topics were subject to the review articles by Chruściel \cite{Chru20,Chru21}.

Presently it seems that the most promising candidates for a theory of quantum gravity are strings theories. Their implications on the theory of gravity and on black holes are widely elucidated. In what follows, we will concentrate on the Einstein-Maxwell axion-dilaton (EMAD) model, which is relevant to the bosonic sector of heterotic string theory.

Much work has been devoted to the so-called axion-dilaton gravity which is the truncation of $N = 4$, $d = 4$ supergravity with only one vector field Refs. \cite{36} - \cite{44}.

Gal’tsov studied \cite{Gal’tsov45} axion-dilaton gravity interacting with $p$-$U(1)$ vectors in four-dimensional spacetime admitting a non-null Killing vector. A new set of supersymmetric stationary solutions of pure $N = 4, d = 4$ supergravity, generalizing the Israel-Wilson-Perjés solution of EM theory was presented by Bergshoeff, Kallosh and Ortin \cite{34}. The authors argued that one vector field is insufficient to generate all the interesting metrics.

Rogatko \cite{37} derived the general Smarr formula and general variation formula for stationary axisymmetric black holes in EMAD gravity. Heusler \cite{46} pointed out that various self-gravitating field theories with massless scalars and vector fields reduce to the $\sigma$ models, effectively coupled to three-dimensional gravity. Using the coset structure to construct conserved currents and closed two-forms, integrating the latter over a spacelike hypersurface, he provides a generalized Smarr formula for stationary black holes with nonrotating Killing horizon, for both EM and EMAD systems.

Using the canonical formalism for the theories with the matter content arising in string theory, Larsen and Wilczek in Ref. \cite{Larsen47} studied the problem of classical hair in string theory. They derived an effective theory for the hair in terms of the horizon variables. It has turned out that the solution of the constraints expresses these variables in terms of hair seeing by an observer at infinity.

Gibbons et al. \cite{48} derived the first law of black hole thermodynamics by means of the variation of moduli fields. They have shown that the ADM mass is extremized at fixed area, angular momentum, and electric and magnetic charges when the moduli fields took the fixed values depending on electric and magnetic charges. It follows from their research that at least mass of any black hole with fixed conserved electric and magnetic charges is given by the mass of the double-extreme black hole with these charges.

One should also mention the work of Creighton and Mann \cite{49} in which they considered gravity coupled to various
types of Abelian and non-Abelian gauge fields with three and four-form field strengths. Using the quasilocal formalism they found the entropy for stationary black holes and derived the first law of black hole thermodynamics for black holes with the gauge fields under consideration.

The problem of staticity in the theory under consideration was studied by Rogatko in Ref. 51. Using the modified Carter arguments it was proved that the condition of vanishing the asymptotic value of the quantity assembled by means of $SL(2, R)$ duals to the $U(1)$ gauge fields and satisfying the inequality for the sum of potentials enabled to satisfy the staticity conditions for fields and metric. This inequality should hold everywhere in the domain of outer communication. However, this assumption has no physical justification. The big challenge will be to eliminate the aforementioned assumption from the considerations.

Our paper is organized as follows. In Sec.II, we present the canonical formalism for the EMAD gravity. Then, in Sec.III we introduce the exact form of the canonical energy and angular momentum in the theory under consideration. We will show that every stationary solution in EMAD gravity is an extremum of the ADM mass at fixed dilaton-electric charge. We also derived the first law of the black holes dynamics and drew a conclusion that any stationary black hole with a bifurcate Killing horizon is an extremum of the ADM mass at fixed dilaton-electric, canonical angular momentum, and horizon area. In Sec.IV, we deal with the staticity problem for nonrotating black holes, finding the conditions on which the solutions are static.

In our paper Greek indices will range from 0 to 3 and denote tensors on four-dimensional manifold. On the other hand Latin indices range from 1 to 3 and denote tensors on a spatial hypersurface $\Sigma$. $g_{\alpha \beta}$ will be the metric of the spacetime, while $h_{ab}$ will live on a spatial hypersurface $\Sigma$. The corresponding covariant derivatives are denoted by $\nabla_\alpha$ and $\nabla_\alpha$.

**II. CANONICAL FORMALISM FOR EINSTEIN-MAXWELL-AXION-DILATON GRAVITY**

The heterotic strings provides an interesting generalization of the EM theory in the so-called low energy limit. The bosonic sector of the dimensionally reduced effective action for the string theory is obtained when we consider six of ten dimensions which have been compactified 32–34. A simplified model of this kind is an EMAD coupled system. It contains a metric $g_{\mu \nu}$, $U(1)$ vector fields $A_\mu$, a dilaton $\phi$, and a three index antisymmetric tensor field $H_{\alpha \beta \gamma}$. In our work we will consider the action of the form 35

$$I = \int d^4x \sqrt{-g} \left[ R - 2(\nabla \phi)^2 - \frac{1}{3} e^{-4\phi} H_{\alpha \beta \gamma} H^{\alpha \beta \gamma} - e^{-2\phi} F_{\alpha \beta} F^{\alpha \beta} \right],$$

where $F_{\mu \nu} = 2\nabla_{[\mu} A_{\nu]}$ and $H_{\alpha \beta \gamma}$ stands for the three-index antisymmetric tensor field defined by

$$H_{\alpha \beta \gamma} = \nabla_\alpha B_{\beta \gamma} - A_\alpha F_{\beta \gamma} + \text{cyclic.}$$

The equations of motion corresponding to the action (1) derived from the variational principle are given by

$$\nabla_\mu \left( e^{-2\phi} F^{\mu \alpha} \right) + \frac{1}{2} e^{-4\phi} H^{\alpha \beta \gamma} F_{\beta \gamma} = 0,$$

$$\nabla_\mu \left( e^{-4\phi} H^{\mu \alpha \beta} \right) = 0,$$
\[
\n\nabla_\mu \nabla^\mu \phi + \frac{1}{3} e^{-4\phi} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} + \frac{1}{2} e^{-2\phi} F_{\alpha\beta} F^{\alpha\beta} = 0. \tag{5}
\]

In this section, we consider the canonical formalism for the theory under consideration. The ADM formalism considered four-geometry to consist of a foliation of three-geometries. Its main idea is that the geometry of the manifold is described in terms of the intrinsic metric and the extrinsic curvature of a three-dimensional hypersurface, along with the lapse function and the shift vector. The lapse and shift relate the intrinsic coordinate on one hypersurface to the intrinsic coordinates on a nearby hypersurface. Spacetime is sliced into spacelike hypersurfaces with each hypersurface labeled by a global time parameter (see, e.g., [52,53]).

Thus, the canonical formalism divide the metric into spatial and temporal parts, namely
\[
ds^2 = -(N dt)^2 + h_{ab}(dx^a + N^a dt)(dx^b + N^b dt), \tag{6}
\]
where general covariance implies the great arbitrariness in the choice of lapse and shift functions \(N^\mu (N, N^a)\).

In the canonical formulation of the EMAD theory the point in the phase space corresponds to the specification of the fields \((h_{ab}, \pi^{ab}, A_a, E^a, B_{ij}, E^{ij}, \phi, E)\) on a three-dimensional \(\Sigma\) manifold. Here \(h_{ab}\) is a Riemannian metric on \(\Sigma\), \(A_a\) is the \(U(1)\) gauge field on the three-dimensional manifold, \(B_{ij}\) is a Kalb-Ramond antisymmetric tensor field and \(\phi\) is the dilaton field on \(\Sigma\). The field momenta can be found by varying the Lagrangian with respect to \(\nabla_0 h_{ab}, \nabla_0 \phi, \nabla_0 A_a\) and \(\nabla_0 B_{ij}\), where \(\nabla_0\) denotes the derivative with respect to \(t\)-coordinate.

Performing the variations, one has that the momentum \(\pi^{ab}\) canonically conjugate to a Riemannian metric can be expressed by means of the extrinsic curvature \(K_{ab}\) of \(\Sigma\) hypersurface
\[
\pi^{ab} = \sqrt{h} \left( K^{ab} - h^{ab} K \right). \tag{7}
\]

The momenta canonically conjugate to the Kalb-Ramond tensor fields \(B_{ij}\) and to the dilaton \(\phi\) fields are given, respectively, as
\[
\pi^{(H)}_{ij} = 2 e^{-4\phi} E_{ij}, \tag{8}
\]
\[
\pi^{(\phi)} = 4 E, \tag{9}
\]
where
\[
E_{ij} = \sqrt{h} H_{\mu ij} n^\mu, \tag{10}
\]
\[
E = \sqrt{h} \nabla_\mu \phi n^\mu. \tag{11}
\]

\(n^\mu\) is the unit normal timelike vector to the hypersurface \(\Sigma\) in the spacetime. In what follows, for the brevity of considerations, the quantity \(E_{ij}\) will be called as axion-electric field and \(E\) as dilaton-electric field.

The momentum \(\pi^{(A)}_i\) canonically conjugate to the \(U(1)\) gauge fields \(A_a\) is equal to
\[
\pi^{(A)}_i = 4 e^{-2\phi} E_i + 4 e^{-4\phi} E_{ij} A^j, \tag{12}
\]
where the electric field \(E_i\) has the form
\[ E_i = \sqrt{h} F_{\mu i} n^\mu. \]  

(13)

The Hamiltonian, defined by the Legendre transform, yields

\[ H = \pi^{ij} \nabla_0 h_{ij} + \pi^{(i)} \nabla_0 \phi + \pi^{(A)} \nabla_0 A_i + \pi^{ij(H)} \nabla_0 B_{ij} - L_{EMAD} \]

(14)

\[ = N^\mu C_\mu + A_0 \tilde{A} + B_{j0} \tilde{B}^j + \mathcal{H}_{\text{div}}. \]

where \( \mathcal{H}_{\text{div}} \) is the total derivative and has the form

\[ \mathcal{H}_{\text{div}} = \nabla_i (\pi^{i(A)} A_0) - 4 \nabla_i (e^{-4\phi} B_{j0}) + 2 \nabla_a \left( \frac{N_b \pi^{ab}}{\sqrt{h}} \right). \]

(15)

The gauge field \( A_0 \) and \( B_{j0} \) have no associated kinetic terms, so one can regard them as Lagrange multipliers. They correspond to the generalized Gauss law: namely

\[ 0 = \tilde{A} = 2 e^{-4\phi} F_{ij} E^{ij} - \nabla_i \pi^{i(A)}, \]

(16)

\[ 0 = \tilde{B}^j = 4 \nabla_i \left( e^{-4\phi} E^{ij} \right). \]

(17)

As in Refs. [6, 54], in our considerations we will take into account the asymptotically flat initial data (sometimes called the regular initial data); i.e., we assume that there is a region \( C \subset \Sigma \) diffeomorphic to \( R^3 - B \) with \( B \) compact and \( C \) such that the following conditions are satisfied at infinity

\[ h_{ab} \approx \delta_{ab} + \mathcal{O} \left( \frac{1}{r} \right), \]

(18)

\[ \nabla_c h_{ab} \approx \mathcal{O} \left( \frac{1}{r^2} \right), \]

(19)

\[ \pi^{ab} \approx \mathcal{O} \left( \frac{1}{r^2} \right), \]

(20)

\[ E \approx \mathcal{O} \left( \frac{1}{r^2} \right), \]

(21)

\[ \phi \approx \mathcal{O} \left( \frac{1}{r} \right), \]

(22)

\[ E_a \approx \mathcal{O} \left( \frac{1}{r^2} \right), \]

(23)

\[ A_b \approx \mathcal{O} \left( \frac{1}{r} \right), \]

(24)

\[ E_{ij} \approx \mathcal{O} \left( \frac{1}{r^3} \right), \]

(25)

\[ B_{ij} \approx \mathcal{O} \left( \frac{1}{r^3} \right), \]

(26)

where \( \delta_{ab} \) is a spatial metric on the hypersurface for which \( t = \text{const} \). We also assume [54] the following conditions for the lapse and shift functions at infinity:

\[ N \approx 1 + \mathcal{O} \left( \frac{1}{r} \right), \]

(27)

\[ N^a \approx \mathcal{O} \left( \frac{1}{r} \right). \]

(28)
The other requirements will be that $k-th$ order derivatives of the above quantities fall off $k$ power of $r$ faster than described by the above equations.

The EMAD gravity is a theory with constraints. On a hypersurface $\Sigma$, initial data are restricted to the case that at each point $x \in \Sigma$, one has

$$0 = C_0 = \sqrt{h} \left[ -R^{(3)} + \left( \pi_{ab} \pi^{ab} - \frac{1}{2} \pi^2 \right) \left( \frac{1}{h} \right) \right] +$$

$$+ \frac{1}{\sqrt{h}} (2E^2 + 2e^{-2\phi}E_iE^i + e^{-4\phi}E_{ij}E^{ij} + 8e^{-4\phi}E_iE^{ij}A_j) +$$

$$+ 2\sqrt{h} \nabla_a \phi \nabla^a \phi + \sqrt{h} e^{-2\phi} F_{ab} F^{ab} + \frac{1}{3} e^{-4\phi} H_{ijk} H^{ijk},$$

$$0 = C_a = -2 \sqrt{h} \nabla_b \left( \frac{\pi_a b}{\sqrt{h}} \right) + 4E \nabla_a \phi + 2e^{-4\phi}H_{aij} E^{ij} + 4e^{-2\phi}E_i F_{ai} + 8E^{-4\phi} F_{ai} E^{ij} A_j,$$ (30)

$$0 = \tilde{A} = 2e^{-4\phi} F_{ij} E^{ij} - \nabla_i \pi^{i(A)},$$ (31)

$$0 = \tilde{B} = 4 \nabla_i (e^{-4\phi} E^{ij}).$$ (32)

For the completeness of the constraints equations we write here one more time the generalized *Gauss law*, Eqs. (31) and (32). The equations of motion, for the system under consideration, can be formally derived from a Hamiltonian of the form

$$H_V = \int_{\Sigma} d\Sigma \left( N^\mu C_{\mu} + N^\mu A_{\mu} \tilde{A} + N^\mu B_{\mu} \tilde{B} \right).$$ (33)

Here we have used the form of the Hamiltonian proposed in [10] in the case of EYM system and EM gravity. A general variation of the initial data ($\delta h_{ab}, \delta \pi^{ab}, \delta \phi, \delta E, \delta A_a, \delta E^a, \delta B_{ij}, \delta E^{ij}$) of a compact support, will cause a variation in Hamiltonian. After performing integration, we reach to the formula

$$\delta H_V = \int_{\Sigma} d\Sigma \left( P^{ab} \delta h_{ab} + Q_{ab} \delta \pi^{ab} + R^a \delta A_a + S_a \delta E^a + T^{ij} \delta B_{ij} + U_{ij} \delta E^{ij} + Y \delta \phi + Z \delta E \right),$$ (34)

where $P^{ab}, Q_{ab}, R^a, S_a, T^{ab}, U_{ab}, Y, Z$ are given in the forms as follows

$$P^{ab} = \sqrt{h} N a^{ab} + \sqrt{h} \left( h^{ab} \nabla_i N - \nabla^a \nabla^b N \right) - L_{N^a} \pi^{ab},$$ (35)

$$Q_{ab} = (2\pi_{ab} - \pi_{ab}) \frac{N}{\sqrt{h}} + L_{N^a} h_{ab},$$ (36)

$$R^a = -4 \sqrt{h} \nabla_i \left( e^{-2\phi} N F^{ia} \right) - 2N \sqrt{h} e^{-4\phi} H^{aij} F_{ij} + 4 \nabla_i \left( \sqrt{h} Ne^{-4\phi} H^{mia} A_m \right) + 8 \frac{N}{\sqrt{h}} e^{-4\phi} E_i E^{ia} +$$

$$-4L_{N^i} (e^{-2\phi} E^i) - 4L_{N^i} (e^{-4\phi} E^{ai} A_i) - 8(L_{N^i} A_j) e^{-4\phi} E^{aj},$$ (37)

$$S_a = 4e^{-2\phi} \left[ \frac{N E_a}{\sqrt{h}} + 2 \frac{N e^{-2\phi} E_{aj} A_j}{\sqrt{h}} + \nabla_a (NA_0) + L_{N^a} A_a \right],$$ (38)

$$T^{ij} = -2 \nabla_a (N \sqrt{h} e^{-4\phi} H^{aij}) - 2L_{N^a} (e^{-4\phi} E^{ij}),$$ (39)
\[ U_{ij} = \frac{2Ne^{-4\phi}}{\sqrt{\hbar}}(E_{ij} + 4E_{i}A_{j}) + 2e^{-4\phi}[2(\mathcal{L}_{N^i}A_i)A_j + \mathcal{L}_{N^i}B_{ij}] - 4e^{-4\phi}\nabla_i(NB_{j0}), \]  

(40)

\[ \mathcal{Y} = -4\nabla_a(N^aE) - \frac{4e^{-5\phi}Ne_{ij}E_{ij}}{\sqrt{\hbar}} - 2N\sqrt{\hbar}e^{-3\phi}F_{ab}F^{ab} - \frac{4e^{-3\phi}Ne_{i}E_{i}}{\sqrt{\hbar}} + \]

\[ - \frac{4}{3}e^{-5\phi}N\sqrt{\hbar}H_{ijk}H^{ijk} - 32e^{-5\phi}Ne_{i}E_{ij}A_j\sqrt{\hbar} + \]

\[ - 8\left(e^{-3\phi}E^i\mathcal{L}_{N^i}A_i + e^{-5\phi}E^i\mathcal{L}_{N^i}B_{ij} + e^{-3\phi}E^i\nabla_i(NA_0)\right) + \]

\[ + 16\left(e^{-5\phi}E^i\nabla_i(NB_{j0}) + e^{-5\phi}E^iA_j\mathcal{L}_{N^i}A_i\right), \]

\[ Z = 4\left(\frac{NE}{\sqrt{\hbar}} + \mathcal{L}_{N^i}\phi\right). \]  

(42)

In Eq.(33) the form of \( a^{ab} \) is given as follows

\[ a^{ab} = \frac{1}{\hbar}\left[(2a^a_j\pi^{bij} - \pi^{a^{ab}}) - \frac{1}{2}h^{ab}\left(\pi^{ij}\pi^{ij} - \frac{1}{2}\pi^2\right)\right] + \left(R^{ab} - \frac{1}{2}h^{ab}R\right) - \frac{E^2h^{ab}}{\hbar} \]

\[ + \frac{2e^{-4\phi}}{\hbar}\left(E^{ij}E^b_j - \frac{1}{4}h^{ab}E_{ij}E^i_j\right) + 2\frac{e^{-2\phi}}{\hbar}\left(E^aE^b - \frac{1}{2}h^{ab}E^i_i\right) + \]

\[ + \frac{4e^{-4\phi}}{\hbar}\left(2E^aE^b_jA_j - 2E_iE^i_jA^b - h^{ab}E_{ij}A_j\right) + \]

\[ + \left(\nabla_i\phi\nabla^i\phi h^{ab} - 2\nabla^a\phi\nabla^b\phi\right) + 2e^{-2\phi}\left(F_{ij}F^b_j + \frac{1}{4}h^{ab}F_{ij}F^{ij}\right) + e^{-4\phi}\left(-H^{ab}H_{ij}^b + \frac{1}{6}h^{ab}H_{ijk}H^{ijk}\right). \]  

(43)

In Eqs.(35)-\( (42) \) the symbol \( \mathcal{L}_{N^i} \) stands for the Lie derivative taken on the hypersurface \( \Sigma \), with respect to the vector field \( N^i \). The Lie derivative of \( h_{ab}, A_i, B_{ij} \) and \( \phi \) are the ordinary Lie derivatives, while the Lie derivative of \( E_i, E^i, E^i A_j, \pi^{ab} \) are understood as the Lie derivatives of the adequate tensor densities.

The evolution equations for the EMAD system can be obtained from Eq.(34), by means of the Hamiltonian’s principle, taking variations of compact support of the hypersurface \( \Sigma \). Then, one establishes

\[ \pi^{ab} = -\mathcal{P}^{ab}, \]  

(44)

\[ \dot{h}_{ab} = \mathcal{Q}_{ab}, \]  

(45)

\[ \dot{\pi}^{(A)}a = -\mathcal{R}^a, \]  

(46)

\[ \dot{A}_a = \mathcal{S}_a, \]  

(47)

\[ \dot{\pi}^{(H)ij} = -\mathcal{T}^{ij}, \]  

(48)

\[ \dot{B}_{ij} = \mathcal{U}_{ij}, \]  

(49)

\[ \dot{\pi}^{(\phi)} = -\mathcal{Y}, \]  

(50)
As was pointed out by Regge and Teitelboim \cite{Regge:1973td, Teitelboim:1973db}, Eq. (34) depicts rather the volume contribution to the Hamiltonian and when one considers the perturbation to the Hamiltonian \( H \) satisfying asymptotic boundary conditions at infinity, the nonvanishing surface terms arise due to integration by parts, in order to put \( \delta H \) in the form of Eq. (33). These surface terms can be excluded by adding the additional surface terms \cite{Regge:1973td}. The result may be written as

\[
H = H_V + \int_{S^\infty} dS \left[ N \left( \nabla^a h^a - \nabla_i h_{im}^m \right) + \frac{2N^b \pi^b_i}{\sqrt{h}} + 4e^{-4\phi} N^a A_a E^{ij} A_j + 4e^{-2\phi} (N A_0 + N^a A_a) E^i - 4e^{-4\phi} (N B_{j0} + N^n B_{j0}) E^{ij} \right].
\]  

A direct calculation can visualize that for all asymptotically flat perturbations and for \( N^\mu, A_0, B_{j0} \) satisfying adequate asymptotic conditions at infinity, one arrives at

\[
\delta H = \int_{\Sigma} d\Sigma \left( P^{ab} \delta h_{ab} + Q_{ab} \delta \pi^{ab} + R^a \delta A_a + S_a \delta E^a + T^{ij} \delta B_{ij} + U_{ij} \delta E^{ij} + Y \delta \phi + Z \delta E \right).
\]  

### III. Extrema of Mass and First Law of Black Holes Mechanics

One can define \cite{VanishingPhases} the canonical energy on the constraint submanifold of the phase space as the Hamiltonian function corresponding to the case when \( N^\mu \) is an asymptotic translation at infinity (i.e., \( N \to 1, N^a \to 0 \)). From Eq. (52), one obtains

\[
\mathcal{E} = m + \mathcal{E}^{(\phi-F)} + \mathcal{E}^{(\phi-B)},
\]  

where \( m \) is the ADM mass, defined as

\[
m = \frac{1}{16\pi} \int_{S^\infty} dS_a \left( \nabla_b h^{ab} - \nabla^a h_b^{\ a} \right),
\]  

The quantities \( \mathcal{E}^{(\phi-F)} \) and \( \mathcal{E}^{(\phi-B)} \) are equal to

\[
\mathcal{E}^{(\phi-B)} = \frac{1}{4\pi} \int_{S^\infty} dS_i e^{-4\phi} B_{0j} E^{ij},
\]  

\[
\mathcal{E}^{(\phi-F)} = \frac{1}{4\pi} \int_{S^\infty} dS_i e^{-2\phi} A_0 E^i.
\]  

In the stationary case, \( A_0 \) is uniquely determined up to a time-independent gauge transformation, by the condition \( \dot{A}_a = 0 \) and \( \dot{\pi}^{(A)a} = 0 \) for all time when \( N^\mu \) is taken to be stationary Killing field. One can show \cite{VanishingPhases}, that these conditions lead to the relation

\[
\mathcal{E}^{(\phi-F)} = V_F Q^{(\phi-F)},
\]  

where \( V_F = (A_0 A_0)^\frac{1}{2} \) and

\[
Q^{(\phi-F)} = \pm \frac{1}{4\pi} \int_{S^\infty} e^{-2\phi} \left| E^a r_a \right| dS,
\]  

\[
\dot{\phi} = Z.
\]  

(51)
$r_a$ is the unit radial vector in the metric $\delta_{ab}$. By analogy with the notion of the Yang-Mills charge \cite{56}, we will call expression (59) as a *dilaton-electric* charge.

In the case of the quantity $\mathcal{E}(\phi - B)$ the situation is similar. A time-independent gauge transformation yields that $\dot{B}_{ij} = 0$, $\dot{\pi}^{(H)ij} = 0$. Taking into account Eq.(48) one can show that contracting Eq.(48) with $B_{j0}$, we reach to the condition of the asymptotical constancy of the magnitude of $B_{j0}$

$$V_{(H)} = \lim_{r \to \infty} (B_{j0}B_{j0})^{\frac{1}{2}}. \tag{60}$$

In addition considering the condition $\dot{\pi}^{(H)ij} = 0$, having in mind the asymptotical behavior of $A_i, B_{ij}, E_i, E_{ij}$, one concludes that

$$\mathcal{E}(\phi - B) = V_{(B)}Q^{(\phi - B)}, \tag{61}$$

where

$$Q^{(\phi - B)} = \pm \frac{1}{4\pi} \int_{S^\infty} e^{-\phi} |E_{ij}r_ir_j| \, dS. \tag{62}$$

Using the definition of $E_{ij}$, Eq.(10), we see that because of the skew symmetricity of $E_{ij}$, expression (62) is equal to zero.

We choose $N^\mu$ to be the stationary Killing vector field and select $A_0$ and $B_{j0}$ in a manner that, $A_i, E_i, B_{ij}, E_{ij}, \phi, E$ are time independent. This choice makes the right-hand side of Eq.(53) vanish. Having in mind expressions (54), (56) and (57) we reach to the results which in some aspects generalize the theorem revealed by Sudarsky and Wald \cite{6} in the case of EYM gravity, to the case of EMAD gravity.

**Theorem**

Consider $(h_{ab}, \pi^{ab}, A_a, E^a, B_{ij}, E_{ij}, \phi, E)$ to be smooth data for a stationary asymptotical flat solution of the EMAD gravity. The initial data hypersurface $\Sigma$ has only one asymptotic region and has no interior boundary. Moreover, consider $(\delta h_{ab}, \delta \pi^{ab}, \delta A_a, \delta E^a, \delta B_{ij}, \delta E_{ij}, \delta \phi, \delta E)$ to be an arbitrary smooth asymptotically flat solution of the linearized constraint equations. Then, the following is satisfied:

$$0 = \delta \mathcal{E} = \delta m + V_{(F)} \delta Q^{(\phi - F)}. \tag{63}$$

From Eq.(58) we see that every stationary solution to EMAD gravity is an extremum of the ADM mass at fixed *dilaton-electric* charge described by Eq.(59).

In the same way, one can define the canonical angular momentum $\mathcal{J}$ on the constraint submanifold of the phase space to be the Hamiltonian function corresponding to the case where $N^\mu$ is an asymptotic rotation at infinity (i.e. $N \to 0, N^a \to \phi^a$)

$$\mathcal{J} = -\frac{1}{16\pi} \int_{S^\infty} dS_i \left( 2\phi^b \pi^{i_b} + 4e^{-2\phi} \phi^a A_a E^i + 4e^{-4\phi} \phi^a B_{a0} E_{ij} + 4e^{-4\phi} \phi^a A_a E_{ij} A_j \right). \tag{64}$$

Converting the surface integral in Eq.(64) and using the constraint equation (30), one gets

$$\mathcal{J} = \frac{1}{16\pi} \int_{\Sigma} d\Sigma \left( 4E_{\phi} \phi + \pi^{ab} \mathcal{L}_{\phi} h_{ab} + 4e^{-2\phi} E_{i} \mathcal{L}_{\phi} A_i + 2e^{-4\phi} E_{ij} \mathcal{L}_{\phi} B_{ij} + 4e^{-4\phi} E_{ij} A_j \mathcal{L}_{\phi} A_i \right). \tag{65}$$
The integral over the hypersurface $\Sigma$ disappear because of the fact that the axial Killing field $\phi^\mu$ is equal to its tangential projection. Thus, in the case when $\Sigma$ is a three-dimensional manifold without boundary $J = 0$, for any axisymmetric solution.

Now, let us consider the case when $\Sigma$ has an asymptotic region and a smooth interior boundary $S$. As in Ref. [6], we will be mostly interested when $N^\mu$ asymptotically reaches a linear combination of a time translation and rotation at infinity (i.e. $N \to 1, N^a \to \Omega \phi^a$, $\phi^a$ is an axial Killing field, $\Omega$ is a constant). One readily finds

$$16\pi (\delta \mathcal{E} - \Omega \delta J) = \int_{\Sigma} d\Sigma \left( P_{ab} \delta h_{ab} + Q_{ab} \delta \pi_{ab} + R^a \delta A_a + S_a \delta E^a + T^{ij} \delta B_{ij} + U_{ij} \delta E_{ij} + Y \delta \phi + Z \delta E \right)$$

(66)

We choose the hypersurface $\Sigma$ to be an asymptotically flat one which intersects the bifurcation sphere $S$ of the stationary black hole. Then, we set $N^\mu = \chi^\mu = t^\mu + \Omega \phi^\mu$ and select $A_0$ and $B_{0j}$ so that $\dot{A}_a = \dot{E}_i = \dot{B}_{ij} = \dot{E}_{ij} = \dot{\phi} = \dot{E} = 0$. By means of Eqs. (44)-(51) the first integral on the right-hand side of Eq. (66) vanishes. All but one surface terms will also be equal to zero because of the fact that $N^\mu = 0$ on $S$. The nonzero term \[6\] is given as follows:

$$\int_S dS_n \nabla_b N \delta h_{cd} (h^{ac} h^{bd} - h^{ab} h^{cd}) = 2\kappa \delta A,$$

(67)

where $\kappa$ is the surface gravity, constant over $S$, $A$ is the area of $S$. Having in mind (66) and (63) we reach to the conclusion.

**Theorem**

Let $(h_{ab}, \pi_{ab}, A_a, E^a, B_{ij}, E_{ij}, \phi, E)$ on a hypersurface be smooth asymptotically flat initial data for a stationary black hole with bifurcation sphere lying on $\Sigma$. Moreover, let $(\delta h_{ab}, \delta \pi_{ab}, \delta A_a, \delta E^a, \delta B_{ij}, \delta E_{ij}, \delta \phi, \delta E)$ be an arbitrary smooth asymptotically flat solution of the linearized constraint equations. Then, the following is satisfied:

$$\delta \mathcal{E} = \delta m + V_F (\delta \phi - E) - \Omega \delta J = \frac{1}{8\pi} \kappa \delta A.$$

(68)

Equation (68) constitutes the extension of the first law of black hole mechanics to the case of EMAD theory. For the EYM case this law was derived by Sudarsky and Wald \[6\]. As was stated this derivation is true for arbitrary asymptotical flat perturbations of a stationary black hole not merely for perturbations to other stationary black holes as was done in the original derivation provided by Bardeen, Carter and Hawking \[57\].

In Ref. [48], Gibbons et al. considered the thermodynamical properties of black holes in the string theory. It turned out that, these properties are conditioned on the values of certain massless scalar fields (referred to as moduli fields) at spatial infinity. The authors stated that, for black holes in the string theory the dependence on the scalar charge would not vanish in the general case. However, when moduli fields at spatial infinity are chosen to extremize the ADM mass at fixed entropy, angular momentum and conserved electric and magnetic charges, this dependence vanishes. Despite the extra term in the first law of black holes mechanics, the integrated version of it (the Smarr formula) is lack of the dependence on the scalar charge.

The EMAD gravity theory can be understand as an $SL(2, R)$ sigma model coupled to a vector field and gravity, so one can verify our results invoking attitude presented in Ref. [48]. Namely, from Eq. (68) we can conclude that any stationary black hole solution to the EMAD gravity, with bifurcate Killing horizon is an extremum of the ADM mass at fixed dilaton-electric charge, canonical angular momentum and horizon area.
In order to check this assertion one ought to consider a simple example of a stationary black hole in the theory under consideration. We postpone this problem to consider in a separate publication elsewhere.

Now, we focus our attention on the problem if the converse results to Eqs. (63) and (68) hold. Namely, whether the initial data for a stationary solution which are an extremum of the ADM mass \(m\) at fixed \(Q(\phi - F)\) are necessary ones and if the initial data which are the extremum of \(m\) at fixed \(Q(\phi - F)\), angular momentum and the horizon area are obligatory initial data for a stationary black hole solution. In EYM gravity this problem was widely elaborated by Sudarsky and Wald [6]. We will follow this line of reasoning in the direction of proving the converse theorems.

Of course, one should be aware that the argumentation is not the complete proof of the conversed theorem and generalization to the asymptotically flat EMAD case should be provided, giving necessary and sufficient conditions for solving the adequate equations for perturbations on a compact support.

Consider any EMAD initial data satisfying the EMAD constraint Eqs. (29)-(32). Suppose, that we have smooth perturbed initial data satisfying the linearized EMAD constraints in the neighborhood of infinity satisfying \(\delta Q(\phi - F) = 0, \delta m \neq 0\). One has to solve the following relations:

\[
\delta C_\mu = S_\mu, \tag{69}
\]

\[
\delta C_i = S_i, \tag{70}
\]

\[
\delta C = S, \tag{71}
\]

where \(S_\mu = -\delta C_\mu, S_i = -\delta C_i, S = -\delta C\). Perturbations \((\delta h_{ab}, \delta \pi^{ab}, \delta A_a, \delta E^a, \delta B_{ij}, \delta \phi, \delta E)\) fall off sufficiently rapidly at infinity, that the relations \(\delta m = \delta Q(\phi - F) = 0\) are satisfied.

We will generalize the arguments provided by Fisher and Marsden [8], who studied the linear stability, in the vacuum case, of compact ( without boundary ) Cauchy hypersurface \(\Sigma\). They proved that an equation of the form of Eq. (69) can be solved for a given smooth source \(S_\mu\) iff \(S_\mu\) is orthogonal to the kernel of the \(L^2\) adjoint operator \(A^\dagger\), of the form \(A(\delta h_{ab}, \delta \pi^{ab}) = \delta C_\mu\).

In our case, the left-hand sides of Eqs. (69)-(71) defined the operator \(B\) which mapped perturbations of EMAD initial data into covariant vector, scalar fields on the hypersurface \(\Sigma\). In order to find to find the adjoint operator \(B^\dagger\) we multiply Eq. (69) by \(M^\mu\), Eq. (70) by \(a^i\) and Eq. (71) by \(a\), and integrate over hypersurface \(\Sigma\). We remark that the equation

\[
\delta H_V = \int_\Sigma d\Sigma \left( N^\mu \delta C_\mu + N^\mu A_\mu \delta \tilde{A} + N^\mu B_{\mu j} \delta \tilde{B}^j \right), \tag{72}
\]

effectively computes the adjoint operator \(B^\dagger\) and Eqs. (64)-(71) show that \(M^\mu\) lies in the kernel of the adjoint operator if \(M^\mu\) is a Killing vector field for the background initial data and

\[
\dot{A}_a = F_{M^\mu} A_a = 0, \tag{73}
\]

\[
\dot{E}^a = F_{M^\mu} E^a = 0,
\]

\[
\dot{B}_{ij} = F_{M^\mu} B_{ij} = 0,
\]
\[ \dot{E}^{ij} = \mathcal{L}_{M^\mu} E^{ij} = 0, \]
\[ \dot{\phi} = \mathcal{L}_{M^\mu} \phi = 0, \]
\[ \dot{E} = \mathcal{L}_{M^\mu} E = 0, \]
in a gauge where the following is satisfied

\[ M^\mu A_\mu = a, \tag{74} \]
\[ M^\mu B_{j\mu} = a_j. \]

By the analogy with the vacuum case, we conjecture that Eqs. (69)-(71) can be solved if there do not exist \((M^\mu, a^j, a)\) satisfying asymptotical conditions, where \(M^\mu\) is a Killing vector field for the background spacetime and Eqs.(73) and (74) are satisfied. Then, a necessary condition for an extremum of mass at fixed charges is that the background EMAD initial data correspond to a stationary solution.

On the other hand, when we take into consideration the case with the boundary \(S\), one should begin with the perturbations \((\delta h_{ab}, \delta \pi^{ab}, \delta A_a, \delta E^a, \delta B_{ij}, \delta \phi^i, \delta E^{ij}, \delta \phi, \delta E)\) satisfying the constraints near infinity and also satisfying \(\delta Q(\phi - F) = \delta J = 0\) and \(\delta m \neq 0\). Once more, we should solve Eqs. (69)-(71) with the additional conditions that \(\delta Q(\phi - F) = \delta J = \delta A = 0\). As in the previous case the background spacetime must admit a Killing field \(M^\mu\) and \(a^j\), \(a\) satisfying Eqs.(73)-(74). In order to have no surface terms, the requirement revealed from the condition \(\delta J = 0\), is that \(M^a \rightarrow \phi^a\), \(M^0 \rightarrow \text{const}\), \(a^j \rightarrow \text{const}\) and \(a \rightarrow \text{const}\) at infinity. The requirement that \(M^\mu = 0\) at \(S\) and \(\nabla_a M^0\) has a constant magnitude on \(S\) assures us that no surface terms will be generated at \(S\). As was pointed in Ref. [6], the fact that the Killing vector field \(M^\mu\) vanishes at \(S\) caused that the null geodesics orthogonal to the surface \(S\) generate a bifurcate horizon.

The nonexistence of the triple \((M^\mu, a^j, a)\) fulfilling the discussed asymptotic conditions at infinity and boundary conditions at the surface \(S\), is obligatory to solve Eqs. (69)-(71) with arbitrary smooth source terms of compact support for \(\delta J = \delta Q(\phi - F) = \delta m = \delta A = 0\). Then, the converse theorem to the theorem (68) takes place; namely in order for EMAD initial data to be an extremum of mass at fixed dilaton-electric charge, canonical momentum and horizon area, the initial data have to coincide with a stationary black hole with a bifurcation surface and a bifurcate Killing horizon.

**IV. THE MASS FORMULAS AND THE STATICITY PROBLEM FOR NONROTATING BLACK HOLES IN EINSTEIN-MAXWELL AXION-DILATON GRAVITY**

Following the attitude presented by Sudarsky and Wald [7], we will deal in this section with the problem of finding the conditions for nonrotating EMAD black holes to be static.

In EYM gravity Sudarsky and Wald [6] were able to give the proof of staticity theorem for nonrotating black holes which with no additional inequalities as was achieved in the first proof provided by Carter [4,5]. Now, we will try to find the similar way of getting rid of the additional assumption in the staticity theorem.

We will consider the spacetime of a stationary black hole with bifurcate Killing horizon. This kind of black holes possesses a Killing vector field \(t^\mu\), which becomes a time translation in the asymptotic region and a Killing vector
field $\chi^\mu$. The vector field $\chi^\mu$ vanishes on the bifurcation sphere. If $\chi^\mu$ does not coincide with the $t^\mu$ field, then the spacetime has an axial Killing field, $\phi^\mu$ fulfilling the relation $\chi^\mu = t^\mu + \Omega \phi^\mu$, where $\Omega$ is constant known as the angular velocity.

Chrusciel and Wald [8] proved that any stationary black hole with bifurcate Killing horizon admits an asymptotically flat maximal hypersurface which is asymptotically orthogonal to $t^\mu$ and its boundary is the bifurcation surface $S$, of the horizon. In what follows, one will take into account the hypersurface $\Sigma$ to be such a maximal hypersurface.

Considering the initial data which are induced on $\Sigma$ and choosing the lapse and shift function to coincide with a Killing field in the spacetime, the Eqs.(44)-(51) have their right-hand sides equal to zero. Then, contracting Eq.(44) we reach to the expression

$$\nabla_i \nabla^i N = \rho N,$$

where $\rho$ is the non-negative quantity in the form as follows

$$\rho = \frac{1}{h} \pi^{ab} \pi_{ab} + 2 E^2 \frac{h}{h} + \frac{e^{-2\phi} E_i E^i}{h} + \frac{8 e^{-2\phi} E_i E^i A_j}{h} + \frac{2 e^{-2\phi} E_i E^i}{h} + \frac{1}{2} e^{-2\phi} F_{ab} F^{ab} + \frac{1}{3} e^{-4\phi} H_{ijk} H^{ijk},$$

Next we use the lapse function in the form defined by $\lambda = -n_a t^a$, where $t^\mu$ is the stationary Killing vector field. The boundary conditions for the lapse function are $\lambda|_S = 0$, $\lambda|_\infty = 1$. Integrating Eq.(75) over $\Sigma$, taking into account that the surface integral over $S_\infty$ is $4 \pi M$ and the surface integral over $S$ is equal to $\kappa A$, one can reach to the mass formula in EMAD gravity, namely,

$$4 \pi M - \kappa A = \int_\Sigma d\Sigma \lambda \rho.$$  

(77)

To proceed further, we take into account mass formula obtained by Bardeen, Carter and Hawking [7]

$$M - \kappa A - 2 \Omega J_H = 2 \int_\Sigma d\Sigma \left( T_{\mu \nu} - \frac{1}{2} T g_{\mu \nu} \right) t^\mu n^\nu,$$

(78)

where $J_H$ is the angular momentum of the black hole, defined [53] in the standard way by $J_H = \frac{1}{16 \pi} \int_S \epsilon_{\alpha \beta \gamma \delta} \nabla^\gamma \phi^\delta$.

In EMAD the explicit form of Eq.(78) is

$$M - \kappa A - 2 \Omega J_H = 2 \int_\Sigma \left[ 4 \lambda \frac{E^2}{h} + 4 t^m \nabla_m \phi E \frac{h}{\sqrt{h}} + e^{-2\phi} \left( \frac{2 \lambda E_a E^a}{h} + 4 t^m F_{md} E^d \frac{h}{\sqrt{h}} + \lambda F_{ab} F^{ab} \right) + \frac{1}{3} e^{-4\phi} \left( t^m H_{mij} E^{ij} \frac{h}{\sqrt{h}} + \frac{2}{3} \lambda E_{ij} E^{ij} \frac{h}{\sqrt{h}} + \frac{1}{3} \lambda H_{ijk} H^{ijk} \right) \right]$$

(79)

As in Ref. [7], one can show that using the definition of $J_\infty$ and changing the surface integral into a volume integral over the hypersurface $\Sigma$, taking into account the constraint, Eq. [78], and the fact that the integral over $\Sigma$ in the definition of $J_\infty$ vanishes because of the fact that the axial Killing field $\phi^\mu$ is equal to its tangential projection $\phi^i$, one gets that

$$J_\infty = J_H.$$  

(80)

Recalling that the first term in equation for $J_H$ is equal to $J_H$ [8] and computing the other terms, having in mind that on $S$ vectors $t^a$ and $\phi^a$ coincide up to the constant $\Omega$ as the result of vanishing $\chi^a$ on $S$, one readily finds
\[ 4\pi(J_H - J_\infty)\Omega = -\int_S \frac{dS_i}{\sqrt{h}} \left( t^m A_m e^{-2\phi} E^i + t^m B_{mj} e^{-4\phi} E^{ij} + t^m A_m e^{-4\phi} E^{ij} A_j \right). \] (81)

Now, we change the surface integral into the volume one and we take into consideration the asymptotic behavior of \( A_i, E^i, B_{ij}, E^{ij} \) at infinity. One can draw a conclusion that there will be no contribution from the boundary at infinity. Thus, from equation (81) we arrive at the following relation:

\[ 4\pi(J_H - J_\infty)\Omega = \int \d\Sigma \left[ -\frac{\lambda e^{-2\phi} E_a E^a}{\sqrt{h}} - 4\frac{\lambda e^{-4\phi} E_i E^{ij} A_j}{\sqrt{h}} - \frac{\lambda e^{-4\phi} E_{ab} E^{ab}}{\sqrt{h}} + t^m \left( F_{im} e^{-2\phi} E^i - \frac{1}{2} e^{-4\phi} H_{mij} E^{ij} \right) \right] + \int_{S=\infty} dS_i \lambda A_i e^{-2\phi} E^i - \int_{S=\infty} dS_i \lambda B_{ij} e^{-4\phi} E^{ij}. \] (82)

We use Eqs.(47) and (49), \( \dot{B}_{ij} = 0 \) and \( \dot{A}_i = 0 \), for the explicit form of \( L_N, B_{ij} \) and \( L_N, A_i \). Of course, there will be no contribution from the boundary integrals over \( S \), because of the boundary conditions for the lapse function \( \lambda \). Then, we get

\[ 4\pi(J_H - J_\infty)\Omega = \int \d\Sigma \left[ -\frac{\lambda e^{-2\phi} E_a E^a}{\sqrt{h}} - 4\frac{\lambda e^{-4\phi} E_i E^{ij} A_j}{\sqrt{h}} - \frac{\lambda e^{-4\phi} E_{ab} E^{ab}}{\sqrt{h}} + t^m \left( F_{im} e^{-2\phi} E^i - \frac{1}{2} e^{-4\phi} H_{mij} E^{ij} \right) \right] + \int_{S=\infty} dS_i \lambda A_i e^{-2\phi} E^i - \int_{S=\infty} dS_i \lambda B_{ij} e^{-4\phi} E^{ij}. \] (83)

Eliminating, from relation (83), \( J_H \) by means of Eq.(79)

\[ 4\pi M - \kappa A + 8\pi V(\phi) Q(\phi - F) + 8\pi \Omega J_\infty = \int \d\Sigma \lambda \left( \frac{1}{2} e^{-2\phi} F_{ab} F^{ab} + \frac{1}{3} e^{-4\phi} F_{ijk} F^{ijk} - e^{-2\phi} E_a E^a - \frac{8 e^{-4\phi} E_i E^{ij} A_j}{h} - 2 e^{-4\phi} E^{ij} E_{ij} \right). \] (84)

Taking into considerations Eqs.(83), (78) and (77), one can conclude

\[ 8\pi \left( \Omega J_\infty - V(\phi) Q(\phi - F) \right) = \int \d\Sigma \lambda \left( \frac{\pi_{ab} \pi^{ab}}{h} + 2 e^{-2\phi} E_a E^a - \frac{4 e^{-4\phi} E_i E^{ij} A_j}{h} + 16 e^{-4\phi} E^{ij} E_{ij} A_i \right). \] (85)

Theorem 4.2 in Ref. 8 states that the exterior region of the black hole can be foliated by maximal hypersurfaces with boundary \( S \) which are asymptotically orthogonal to the timelike Killing vector field \( t^\mu \), if the strong energy condition \( R_{\mu
u} Z^\mu Z^\nu \geq 0 \) for all timelike vectors \( Z^\mu \) is fulfilled. By the direct calculation, one can check that this is the case in EMAD gravity. Taking the above, one can reach to the main result of this section, namely, the following.

**Theorem**

Consider an asymptotically flat solution to EMAD gravity which has a Killing vector field which is timelike at infinity, describing a stationary black hole comprising a bifurcate Killing horizon with a bifurcation surface \( S \). Suppose, moreover, that the following is satisfied:

\[ \Omega J_\infty - V(\phi) Q(\phi - F) = 0. \]

Then, the solution is static and has vanishing \( E_i \) and \( E_{ij} \) on the static hypersurfaces.
We remark that, in the above formulation of the staticity theorem for non-rotating black holes in EMAD gravity, we managed to get rid of the additional assumption used in the previous proof of the staticity theorem in EM gravity [4] and in the prior attitude to the staticity problem in EMAD gravity [5].

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