Philosophers are concerned with the reality of things. As investigators of perception we are more concerned with the reality of appearances. If we claim an experimental result such as an aftereffect of motion, or a brightness or colour contrast or whatever, we are concerned that the reported appearance is a genuine phenomenon. Although ‘veridical’ observations are beloved of philosophers seeking certainty, and are sought by the natural sciences as empirical data for suggesting and testing hypotheses, departures from accepted object reality can be very interesting for us as illusory though genuine phenomena calling for perceptual explanation. Ultimately though, it may be suggested, what is only ‘appearance’ and what is object ‘reality’ depends on the accounts of the physical sciences. For as science changes its descriptions, accepted reality changes; so the divide between ‘appearance’ and ‘reality’ is labile, with subtle interactions between perceptions and conceptions. But which aspects or levels of description and explanation of the sciences are accepted as reference reality—from which deviating perceptions are seen as veridical (even though surprising) or as illusory? If the accepted reference is the deeper levels of physics then virtually all perception—all appearance—must be rejected, for we do not see matter or objects, or indeed anything, quite as science describes them. Yet we can hardly reject all appearances, all perceptions, as illusory—for it is by perception rather than accounts of physics by which we live to survive another day. And to call everything illusory simply makes the word ‘illusion’ pointless, meaningless, for then there is nothing to contrast it. We need some concept of reality to make ‘illusion’ and ‘appearance’ meaningful.

Representational theories of perception assume there is something that is ‘represented’ in nonillusory perceptions. But our criteria for, say, visual perceptions being veridical or being illusory are weak: there is general lack of obvious conflict with other sensory perception, such as touch. We may also see internal conflict (necessarily illusions?) in pictures of ‘impossible objects’.

Traditionally philosophers have been described as being either ‘Materialists’, with Empiricism as the only route to reality; or as ‘Idealists’, accepting innate a priori knowledge available without experience or experiment, and holding that reality is or at least depends upon mind. All philosophers will admit, though, that just what
materials are (or matter is) remains mysterious, and the status of ideas is even more uncertain. In any case why 'Idealism'? Surely it would be better to call the notion that ideas or mind are necessary for reality—'Idea-ism'.

The local dialect of Bristol, in England where this is written, disconcertingly introduces 'l's on the ends of words. Bristol itself appears as 'Bristo' on old maps. In the local dialect an idea is always called an 'ideal'—"Oh what a good idea!"—which is confusing and (for a merchant city) usually untrue. Was some influential early Idealist philosopher a Bristolian, to confuse 'Idealism' with what should be 'Idea-ism'?

What Idealist—Idea-ist—philosophers hold is that all reality is, or in some way depends on, mind. Bishop Berkeley, of course, thought that existence of matter depends on it being perceived; but he did think that matter is different from mind. Other Idealists identify matter and mind. As mind is traditionally associated with consciousness, awareness or sensation, so for them all reality is and depends on appearances. The snag is that this leads by a slippery slope to solipsism; for why should I accept other people's appearances (or their reality as objects) as real? Shared appearances can be surprising, suggesting they have a common origin beyond our individual minds. Philosophers in Oxford (which has more professional philosophers than did ancient Athens) have grappled with such questions for many years, including the Idealist F H Bradley whose book Appearance and Reality (1893) argued for a monistic unity of things, such that a truth must be the whole truth, and in a sense all things are one thing in a universal mind. Bradley defended a priori knowledge, and throughout his life resented and attacked Empiricism as though it were evil. He served to inspire Bertrand Russell to revolt against Idealism, ultimately to accept experimental science and observations as the sole bringers of new knowledge, which was also Wittgenstein's view. Yet science, too, seeks for unifying accounts—though it also delights in analogies, which require differences! The question here, which Bradley discusses in depth, is whether similarities below appearances (allowing useful analogies) are identities in reality.

Moving on from Bradley, and rejecting Idealism, Oxford philosophy sought to capture reality in intuitions expressed in normal language. The assumption was that common intuitions, or appearances, are true. But common sense had to be rejected, with the startling nonintuitive discoveries of modern science. The bizarre accounts of relativity and quantum mechanics had a devastating effect on Oxford and indeed all philosophy. For, like perception, philosophy has little to offer when appearances are totally rejected. The counter-intuitive claims of science cast doubt on Materialism—as matter was described so differently from appearances. They knocked Idealism—as such unexpected 'reality' through experiment upset the notion of a priori knowledge explained as reality being (or depending on) mind or ideas. Interestingly, though, the observer's mind is supposed to have a part to play in the physics of relativity and quantum mechanics; though mind does not seem to have a solo part, as cosmology persuades us that the physical universe had a dominating role long before perception entered the scene of the Universe.

Other important bit-players are numbers. Are numbers objects (like tables or stars) or are they ideas in mind? Granted we cannot see or in any way sense number, as we can sense tables and stars; yet we are forced into agreements such that numbers do seem to be objective things. For example we all agree that 1 more than 12 is 13, and that 13 is a prime. We all agree that $365/12 = 30.416666666$. Indeed such agreements are more complete than for any sensory objects. So, for many philosophers and scientists, numbers and their properties are more deeply real than matter—and nothing like so inconstant or inconsistent as mind. But then, for the same applies, do we not have to say that negative and imaginary numbers $\sqrt{-1}$ are also 'real'.
Descartes rejected ‘imaginary’ numbers as unreal; but later they were reified as ‘complex’ numbers, and (Jourdain 1959) they may be thought of as operators, to make the machine of mathematics work.

Just as sensed objects can be surprising, so are many properties and structures of numbers. What is so odd is that, although numbers are shared and discussed and surprise us like objects of sense, they do not occupy space. So we cannot see numbers—or can we?

Enter the computer, with graphics.

Computers can now generate wonderful pictures. Computer patterns such as Mandelbrot pictures are, as from another world, disturbingly beautiful. Do they in a way undreamed of by classical philosophy or science bridge appearance and abstract realities of numbers?

There are earlier hints to the powers of computer graphics. Plato would have said that drawings of Euclidian constructions show abstract realities. But are not such drawings guides to how to think rather than giving directly accessed conceptual insights?

There is a hint of visual patterns depicting mathematical properties in graphs. It is very helpful to see an exponential or a logarithmic function, or whatever, plotted as a graph. I cannot think of a probability distribution apart from bell-shaped curves. Oddly, the origins of graphs are missing from histories of mathematics. Apart from maps, and drawings of movements of the planets, and then of trajectories of cannon balls, there seem to be few hints of graphs before Cartesian geometry—which added direction, including negative directions, to Euclidian magnitudes. Newton’s teacher Isaac Barrow did, though, devise a prior graphical form of differential calculus (Boyer 1949), and integration has commonly been done graphically. But graphs do not seem to have been used for showing properties of equations much earlier than a hundred years ago. Graphs are certainly useful for thinking in science and presenting results: it would be interesting to know how useful they are to mathematicians for seeing their realities.

Now that the computer can transcend ordinary graphs by presenting visually, in a few seconds, results of superhuman computations, it is important to ask: Can these pictures allow us actually to see realities of numbers? And can they reveal hidden features of numbers, or of sensory objects? What fractals seem to show is that certain patterns are present in a surprising range of objects, and remain much the same over an enormous size range. It is interesting that a coastline does not have a clear length, for the greater the resolution of measurement the longer it gets. Apart from atomic limits the coastline of a country of finite area is infinite at the highest resolution, and the patterns of indentations look almost the same over a huge range of magnification. This is not at all like Euclidian geometry with its lines and triangles and circles.

Fifty years before computer graphics, around 1915, two French mathematicians, Gaston Julia and Pierre Fatou, investigated rates of growth of equations with complex \[ a + bi, \] where \( i = \sqrt{-1} \) numbers. They found that some sets of such numbers, instead of growing to infinity or shrinking to zero, were drawn towards a particular steady state, or to a cycle of values now called ‘attractors’. When these numbers are plotted on a graph with real numbers on one axis and imaginary numbers on the other (an Argand diagram) they produce weird maps, as of unknown countries with complicated shore lines and patterns of islands in flat lakes. When discovered they were intolerably tedious to produce and investigate; now anyone can generate them on a home computer, in dramatic colours. Benoit Mandelbrot used an Argand diagram to plot what happened when a complex number \( c \) was multiplied by itself (squared), and added to the original number \( c \). The result \( Z \) is squared and added to \( c \), then the
Figure 1. The Mandelbrot set.

Figure 2. Typical Julia sets [detailed mathematical treatment and many more beautiful maps will be found in *The Beauty of Fractals* (Peitgen and Richter 1986)].
second result $Z_2$ is squared and added to $c$, and so on ... Each point was plotted as
black when the number grew, or white if it shrank or remained constant. The resulting
weird shapes of blobs and islands are (as I understand it) envelopes of iterations,
which computers are very good at doing. Newton invented a related method for
finding the roots of an equation $f(x) = 0$ by a dynamic process of iteration where
guessed solutions compete. This can generate visually interesting computer-graphic
patterns. But are they conceptually significant?

In general, these iterative processes represent limited ranges of positive feedback in
nonlinear systems—which can apply to population growth, to electronic control, and
no doubt to some important brain processes. But what has made these computer-
generated pictures take off and become famous is, surely, their immediate visual
appeal. The earlier Julia sets did not catch the public eye or mind before they were
pictured with imaginative variations and careful choice of colours by computer
graphics. But are the more visually interesting patterns the more conceptually or
mathematically interesting? If so, evidently we can read conceptual significance in
visual patterns of which we have no experience. So here, in computer graphics, is a
philosophical tool for exploring claims of mind-full Idealism (or Idea-ism) where a priori
truths are allowed—or for Empiricist Materialism, where, as a priori knowledge is	aboo, we have to learn the hard way to see and understand. These conceptual claims
could perhaps be tested by perceptual experiments with more or less interesting
computer graphics, to answer the deep question of the status of mathematics: Can
abstract realities be seen in appearances?

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