Thermal form factor approach to the ground-state correlation functions of the XXZ chain in the antiferromagnetic massive regime

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Abstract
We use the form factors of the quantum transfer matrix in the zero-temperature limit in order to study the two-point ground-state correlation functions of the XXZ chain in the antiferromagnetic massive regime. We obtain novel form factor series representations of the correlation functions which differ from those derived either from the $q$-vertex-operator approach or from the algebraic Bethe Ansatz approach to the usual transfer matrix. We advocate that our novel representations are numerically more efficient and allow for a straightforward calculation of the large-distance asymptotic behaviour of the two-point functions. Keeping control over the temperature corrections to the two-point functions we see that these are of order $T^\infty$ in the whole antiferromagnetic massive regime. The isotropic limit of our result yields a novel form factor series representation for the two-point correlation functions of the XXX chain at zero magnetic field.

Keywords: form factor expansion, correlation functions, integrable quantum spin chains

\textsuperscript{*} Dedicated to the memory of Petr Petrovich Kulish.
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1. Introduction

Two-point correlation functions of the Heisenberg XXZ chain can be studied by means of form factor expansions. These have turned out to be particularly useful for extracting the large-distance asymptotics \[13, 15, 17, 26, 32\] and are currently the only efficient means to study time dependent correlation functions analytically \[9, 10, 18, 28\]. We distinguish usual form factors from thermal form factors. Usually form factors are understood as matrix elements of local operators between the ground state and excited states of a given Hamiltonian or transfer matrix. The usual form factors of the XXZ chain were studied in \[17, 21, 22, 25, 27, 29\]. Finite-temperature static correlation functions can also be expanded in terms of matrix elements of certain non-local operators between the dominant state and excited states of the quantum transfer matrix \[13, 15\]. In order to distinguish them from the usual form factors we have introduced the term ‘thermal form factors’ in \[13\]. Thermal form factor expansions are particularly convenient for studying the large-distance asymptotics of thermal correlation functions, since the latter is determined by a few terms of the series as long as the temperature remains strictly finite. In the zero-temperature limit infinitely many terms have to be taken into account. Then thermal form factor expansions generate different but equivalent expansions of the zero-temperature static two-point functions. These are the subject of this work.

Form factor densities of the XXZ chain were first obtained within the $q$-vertex operator approach \[22\]. This elegant method is designed to work directly for the infinite chain and applies only to the antiferromagnetic massive ground state regime. Still, unlike other methods, it also works for the fully anisotropic XYZ chain \[33\]. An alternative approach to the calculation of form factors, at least of the XXZ chain, is the algebraic Bethe Ansatz approach. Combining what is called ‘the solution of the quantum inverse problem’ with a remarkable scalar product formula \[40\] due to Slavnov, the form factors of the finite-length XXZ chain were expressed by certain determinants \[29\]. Although the determinants cannot be computed explicitly, they can be evaluated numerically by solving the underlying Bethe Ansatz equations on a computer. This approach turned out to be efficient for the calculation of experimentally relevant correlation functions \[7, 11, 38, 39\].

The form factor formulae for finite spin chains of length $L$ obtained within the algebraic Bethe Ansatz approach hold for arbitrary values of the parameters of the model, which are the strength of the external magnetic field $h$ and the anisotropy parameter $\Delta$. In the thermodynamic limit, $L \to \infty$, the $\Delta$–$h$ parameter plane is separated into three different regimes (or ground state phases) depicted in figure 1. Like all other quantities which can be calculated by means of the algebraic Bethe Ansatz, when $L$ is sent to infinity, the form factors become functionals of a few basic functions such as the dressed energy, dressed momentum and dressed phase. It is not easy to actually perform this limit, and the situation is different in the different ground state regimes sketched in figure 1. In the antiferromagnetic critical regime for $h > 0$ the ground state has finite magnetisation and the low-lying excitations are of particle–hole type, much like in the case of free Fermions. The corresponding Bethe roots are real. For this case the thermodynamic limit of the particle–hole form factors was analysed in \[25, 27\]. A formula for the summation of all particle–hole form factors was obtained in \[26\]. This formula made it possible to determine the large-distance asymptotics of the two-point functions including the non-universal amplitudes.
At zero magnetic field and also in the whole antiferromagnetic massive regime the ground state magnetisation of the finite-size system vanishes. The lowest-lying excitations above the massive ground state involve non-real Bethe roots. This makes the analysis of the thermodynamic limit of certain determinants that are part of the Bethe Ansatz form factors more involved as these determinants are rather singular in this limit. In the antiferromagnetic massive regime there is one state which is almost degenerate with the ground state (a ‘pseudo ground state’). It is the only other state with only real Bethe roots. The matrix element of $\sigma^z$ between this state and the ground state is, in a sense, the simplest non-vanishing form factor. Its thermodynamic limit was obtained in [5, 6, 21]. The thermodynamic limit of all other form factors of $\sigma^z$ in the antiferromagnetic massive regime was obtained in [17]. The corresponding form factor series represents the longitudinal two-point functions not only asymptotically but at all distances.

The critical regime at $h = 0$ is, so far, the least well understood from a Bethe Ansatz perspective. In particular, the calculation of the thermodynamic limit of Bethe Ansatz form factors is still open in this regime. Note, however, that in this regime rather explicit results for the correlation amplitudes, that determine the leading large-distance asymptotic decay of the two-point functions, were obtained [35, 36] from a clever combination of perturbation theory applied to the Gaussian conformal field theory [34] with results from the $q$-vertex operator approach to the form factors of the XYZ model [33].

The amplitude densities in the form factor series for the longitudinal correlation functions obtained in [17] involve higher dimensional residues originating from the fact that one has to sum up the contributions of the non-real Bethe roots whose loci are constrained by the higher-level Bethe equations. These higher dimensional residues make a numerical analysis beyond the so-called two-spinon contribution hard. As we shall see below, the form factor series obtained from the zero temperature limit of a thermal form factor expansion is free of this difficulty. The reason for this stems from the different role played by the magnetic field for the usual transfer matrix and for the quantum transfer matrix. As opposed to the Bethe Ansatz equations of the usual transfer matrix, the Bethe Ansatz equations and hence the eigenvalues and eigenvectors of the quantum transfer matrix depend parametrically on the magnetic field, even in the antiferromagnetic massive regime. As we shall see below, in the zero-temperature limit, although the individual terms in the form factors expansion of the two-point functions...
do depend on the magnetic field, their sum does not. In such a way, one does recover that the zero-temperature correlation functions are field independent. In the presence of a magnetic field the Bethe root patterns for all low-temperature ‘excitations’ of the quantum transfer matrix are of particle–hole type [16]. In the zero-temperature limit the particle and hole roots become unconstrained and densely fill two curve segments in the complex plane. No higher-level Bethe Ansatz equations have to be satisfied. Using this fact we shall obtain rather explicit expressions for form factor densities in the limit which are different from those obtained for the usual transfer matrix [17] and also from those obtained within the vertex operator approach [22, 33].

The paper is organised as follows. In the remainder of this introduction we recall the definition of the model, the form factor series expansions of the two-point functions and our recent results [16] for the spectrum and Bethe root patterns of the quantum transfer matrix in the antiferromagnetic massive regime at finite magnetic field \( h \). Then in section 2 we present our results for the amplitudes in the form factor series in the low-temperature limit. The amplitudes can be decomposed in three factors, a universal part, a determinant part and a factoring part, which will be treated separately. In section 3 we show how the form factor series can be written as series of multiple integrals corresponding to integration over particle and hole parameters. We compare with previous results which were interpreted in terms of multi-spinon contributions and we perform numerical tests against known exact results and purely numerical calculations in order to assess the efficiency of our novel series representations. In section 4 we perform and discuss the isotropic limit. We conclude in section 5 with a summary and the discussion of open questions. Almost all technical details of the calculations are deferred to a series of appendices.

1.1. Hamiltonian and correlation functions

The Hamiltonian defining the spin-\( \frac{1}{2} \) XXZ chain in a magnetic field of strength \( h \) along the magnetic anisotropy direction is

\[
H = J \sum_{j=-L+1}^{L} \left( \sigma_j^+ \sigma_{j-1}^- + \sigma_j^- \sigma_{j-1}^+ + \Delta(\sigma_j^z \sigma_{j-1}^z - 1) \right) - \frac{h}{2} \sum_{j=-L+1}^{L} \sigma_j^z,
\]

where the \( \sigma_j^a \) are Pauli matrices \( \sigma^a \) acting on the \( j \)th factor of the tensor-product space of states \( \mathcal{H} = \mathbb{C}^{2^{2L}} \) of \( 2L \) spins \( \frac{1}{2} \). The intrinsic parameters of the model are the strength \( J > 0 \) of the exchange interaction and the real anisotropy parameter \( \Delta \). The exchange interaction merely fixes the energy scale, \( \Delta \) and \( h/J \) are the two physical parameters which determine the ground state phase diagram [50]. We shall use the standard reparameterisation \( \Delta = (q + q^{-1})/2 \) with \( q = e^{-\gamma} \). In the following we will consider easy-axis anisotropy corresponding to \( \Delta > 1 \). Hence, we assume that \( \gamma > 0 \). We shall also assume that the magnetic field is positive and below the lower critical field \( h_\ell \). The latter as well as several other functions we shall encounter below are sometimes conveniently expressed in terms of the elliptic modulus \( k = k(q) = \delta_2^0(0, q)/\delta_2^0(0, 0) \), where the \( \delta_j(x, q) \) are Jacobian Theta functions (see [48]). Denoting the complementary modulus by \( k' = \sqrt{1 - k^2} \) and the complete elliptic integral of the first kind by \( K = K(k) \) we have

\[
h_\ell/J = 2(q^{-1} - q) \delta_2^0(0, q) = 8Kk'sh(\pi K/K')/\pi,
\]

where \( K' = K(k') \).

Our goal is to derive efficient series representations for the two-point correlation functions of the Hamiltonian (1). Correlation functions of two operators \( X, Y \) acting on \( \mathcal{H} \) are
defined as

\[ \langle XY \rangle = \frac{\text{Tr} e^{-H/T}XY}{\text{Tr} e^{-H/T}}, \]  

where \( T \) is the temperature. In this work we shall focus on the longitudinal and transversal two-point functions \( \langle \sigma^z \sigma^z_{m+1} \rangle \) and \( \langle \sigma^+ \sigma^-_{m+1} \rangle \) in the thermodynamic limit \( L \to \infty \).

Static temperature dependent correlation functions can be treated most efficiently within the quantum transfer matrix formalism [20] which was originally developed to calculate numerically the free energy per lattice site of quantum spin systems in the thermodynamic limit [42, 43] and turned out to be compatible with the integrable structure of vertex models connected with the Yang–Baxter equation [30, 31, 41]. The quantum transfer matrix associated with a spin model like (1) can be introduced as the column-to-column transfer matrix of a vertex model on a \( 2L \times N \) rectangular lattice, where \( 2L \) is the number of lattice sites along the spin chain and \( N \) is the number of auxiliary lattice sites in perpendicular direction. The perpendicular direction may be interpreted as the imaginary time direction in a path-integral realisation of the partition function of the spin chain. \( N \) is called the Trotter number. It was shown in [42] that the ‘Trotter limit’ \( N \to \infty \) of a single dominant eigenvalue of the quantum transfer matrix determines the free energy per lattice site of the spin chain in the thermodynamic limit. In [20] it was realised that, likewise, the corresponding dominant eigenvector determines all temperature dependent correlation functions. They can be written as expectation values of products of ‘monodromy matrix elements’ and quantum transfer matrices with respect to the dominant state (see [20] and appendix A).

Expanding those expressions in a basis of eigenvectors of the quantum transfer matrix we obtained ‘thermal form factor expansions’ in [13]. These are series of the form

\[ \langle \sigma^z \sigma^z_{m+1} \rangle - \langle \sigma^z \rangle \langle \sigma^z_{m+1} \rangle = \sum_{n=1}^{\infty} A_n^{zz} \rho_n^m, \quad \langle \sigma^+ \sigma^-_{m+1} \rangle = \sum_{n=1}^{\infty} A_n^{-+} \rho_n^m, \]  

where the \( \rho_n \) are ratios of eigenvalues of ‘excited states’ of the quantum transfer matrix by the dominant eigenvalue in the Trotter limit and where the amplitudes \( A_n^{zz} \) and \( A_n^{-+} \) are products of two thermal form factors, each being a normalised matrix element of an entry of the monodromy matrix taken between the dominant state and an excited state of the quantum transfer matrix (for more details see [13] and appendix A). The sums run over all relevant excited states, i.e. over all excited states compatible with the conservation of the \( z \)-component of the total spin.

If \( T > 0 \) the absolute values \( |\rho_n| \) form a decreasing sequence, \( 1 \geq |\rho_1| \geq |\rho_2| \geq |\rho_3| \ldots \), with only finitely many of the \( |\rho_i| \) equal to \( |\rho_1| \). These determine the large-\( m \) asymptotic behaviour and, together with the corresponding amplitudes \( A_n^{zz} \) or \( A_n^{-+} \), can be obtained numerically from the expressions derived in [13]. The situation changes in the limit \( T \to 0^+ \). In this limit infinitely many eigenstates of the quantum transfer matrix degenerate and have to be summed up in order to obtain the asymptotic behaviour of the two-point functions. Nevertheless, the situation remains comfortable, since the eigenvalue ratios and amplitudes simplify in this limit. An analysis of the \( T \to 0^+ \) limit of the two-point functions of the XXZ chain in the critical regime was carried out in [13, 15]. Here we perform a similar analysis for the antiferromagnetic massive regime. We are going to build on our recent paper [16]. Based on a set of mild assumptions we have classified in that paper all excitations of the quantum transfer matrix in the low-temperature limit for the model in the antiferromagnetic massive regime, and we have calculated the corresponding eigenvalue ratios \( \rho_n \). This is equivalent to having determined all correlation lengths \( \xi_n = -1/\ln \rho_n \).
1.2. Low-temperature spectrum of correlation lengths

For the description of the low-temperature spectrum of correlation length we have to introduce a number of functions that determine the physical properties of the XXZ chain at $T = 0+$. These are the dressed momentum $p$, the dressed energy $\varepsilon$ and the dressed phase $\varphi$. In the antiferromagnetic massive regime we can express these functions explicitly in terms of known special functions. We define the dressed momentum as

$$ p(x) = \frac{1}{4} + \frac{x}{2\pi} + \frac{1}{2\pi} \ln \left( \frac{\partial^2_k(x + i\gamma/2, q^2)}{\partial^2_k(x - i\gamma/2, q^2)} \right) $$

(5)

and the dressed energy as

$$ \varepsilon(x) = \frac{h}{2} - \frac{4J\text{sh}(\gamma)}{\pi} \text{dn} \left( \frac{2Kx}{\pi} \bigg| k \right), $$

(6)

where $\text{dn}$ denotes the Jacobian elliptic $\text{dn}$-function. Note that the dressed energy depends explicitly on the magnetic field $h$.

The Jacobian Theta functions and the Jacobian elliptic functions are special cases of functions that can be expressed in terms of $q$-multi factorials which, for $|q| < 1$ and $a \in \mathbb{C}$, are defined as

$$(a; q_1, \ldots, q_p) = \prod_{n_1, \ldots, n_p = 0}^{\infty} (1 - aq_1^{n_1} \cdots q_p^{n_p}). $$

(7)

We shall make extensive use of $q$-multi factorials below, when we describe the amplitudes in the form factor expansions of two-point functions. Here we need them to define the dressed phase

$$ \varphi(x_1, x_2) = i \left( \frac{\pi}{2} + x_{12} \right) + \ln \left\{ \frac{\Gamma_q \left( 1 + \frac{ix_{12}}{2\gamma} \right) \Gamma_q \left( \frac{1}{2} - \frac{ix_{12}}{2\gamma} \right)}{\Gamma_q \left( 1 - \frac{ix_{12}}{2\gamma} \right) \Gamma_q \left( \frac{1}{2} + \frac{ix_{12}}{2\gamma} \right)} \right\}. $$

(8)

where $x_{12} = x_1 - x_2$, $|\text{Im} x_2| < \gamma$ and where the $q$-Gamma function $\Gamma_q$ is given in terms of $q$ factorials

$$ \Gamma_q(x) = \frac{(1 - q)^{[q]}(x; q)}{(q^{x}; q)}, $$

(9)

In [16] we have conjectured that at temperatures low enough all excitations of the quantum transfer matrix can be parameterised by an even number of complex parameters located inside the strip $|\text{Im} x| < \gamma/2$. Referring to [16] we call the parameters in the upper half plane particles and denote them by $y_i$, $i = 1, \ldots, n_p$. The parameters in the lower half plane will be called holes and will be denoted $x_j$, $j = 1, \ldots, n_h$. Here $n_p$ and $n_h$ are non-negative integers. Their difference is

$$ n_h - n_p = 2s, $$

(10)

where $s \in \mathbb{Z}$ is the conserved pseudo spin of the quantum transfer matrix (for a definition see appendix A). In our form factor expansions (4) $s$ is fixed and equal to the spin of the operator\footnote{In analogy with conformal field theory the spin of an operator is defined by the value it changes the spin of a state it is acting on.} that stands to the right in the two-point functions, i.e. $s = 0$ for the longitudinal correlation.
function \( \langle \sigma_i^z \sigma_{m+1}^z \rangle \) and \( s = 1 \) for \( \langle \sigma_i^z \sigma_{m+1}^z \rangle \) leaves the \( z \)-component of the total spin unchanged, while \( \sigma_{m+1}^z \) changes it by \( +1 \).

Then, up to corrections of the order \( T^\infty \), the particles and holes are determined by the higher-level Bethe Ansatz equations

\[
\begin{align*}
\varepsilon(y_j) &= 2\pi i T (\ell_j + F(y_j)), & j = 1, \ldots, n_p, \quad (11a) \\
\varepsilon(x_j) &= 2\pi i T (m_j + F(x_j)), & j = 1, \ldots, n_h, \quad (11b)
\end{align*}
\]

where \( \ell_j, m_j \in \mathbb{Z} \) and where we assume the \( \ell_j \) and \( m_j \) to be mutually distinct. \( F \) is the ‘shift function’ defined by

\[
2\pi i F(x) = i\alpha + \alpha \gamma + \sum_{\ell=1}^{n_p} \varphi(x, y_\ell) - \sum_{\ell=1}^{n_h} \varphi(x, x_\ell).
\]

Here \( \alpha \) is an auxiliary twist related to the magnetic field, which serves as a regularisation parameter and will be set equal to zero at the end of the calculation. The parameter \( k \in \{0, 1\} \) distinguishes between two sectors of excitations of the quantum transfer matrix corresponding to staggered and non-staggered contributions to the form factor series below.

In the limit \( T \to 0^+ \) at finite \( n_p \) and \( n_h \) the higher-level Bethe Ansatz equations (11) decouple, \( i\pi T \ell_j \) and \( i\pi m_j T \) turn into independent continuous variables, and the particles and holes become free parameters on the curves

\[
B_{\pm} = \{ x \in \mathbb{C} | \text{Re} \varepsilon(x) = 0, -\pi/2 \leq \text{Re} x \leq \pi/2, 0 < \pm \text{Im} x < \gamma \}. \quad (13)
\]

These curves are shown in figure 2. As we can see, the massive regime is distinguished from the massless regime by the opening of a ‘band gap’ at the critical field \( h_c \).

The main result of our work [16] was an explicit formula for all correlation lengths, or rather all eigenvalue ratios, in the low-temperature regime. At low enough temperatures all excitations are parameterised by solutions of the higher-level Bethe Ansatz equation (11). Thus, instead of \( \rho \), we shall rather write \( \rho = \rho((x_{j=1}^{n_p}, y_{j=1}^{n_h}|k) \). With this change of notation the eigenvalue ratios at finite magnetic field are expressed as follows [16]
\[
\rho(\{x\}|_{j=1}^n, \{y\}|_{j=1}^n|k) = (-1)^k \exp \left\{ 2\pi i \left[ \sum_{j=1}^{n_x} \rho(y_j) - \sum_{j=1}^{n_y} \rho(x_j) \right] \right\} \\
= (-1)^k \prod_{j=1}^{n_x} \frac{\partial_1(y_j - i\gamma/2, q^2)}{\partial_4(y_j - i\gamma/2, q^2)} \prod_{j=1}^{n_y} \frac{\partial_4(x_j - i\gamma/2, q^2)}{\partial_1(x_j - i\gamma/2, q^2)}, \quad (14)
\]

this being valid up to multiplicative corrections of the order \((1 + O(T^\infty))\). Note that the value of the magnetic field \(h\) enters here through the particle and hole parameters.

In the special case when there are neither particles nor holes, \(n_p = n_h = 0\), the eigenvalue ratios reduce to \(\rho(\emptyset, \emptyset|k) = (-1)^k\). As we have discussed in [16] the value \(k = 0\) corresponds to the case when the dominant state eigenvalue is divided by itself, while \(k = 1\) corresponds to an eigenvalue of an excited state which is almost degenerate in absolute value with the dominant state, meaning that up to sign the two eigenvalues differ only by a factor of \((1 + O(T^\infty))\). Using the properties of the dressed momentum function \(p\) in the complex plane, it is not difficult to see [16] that for all other states, for which \(n_p\) or \(n_h\) is non-zero, \(|\rho| < 1\). For small finite temperature the eigenvalue ratios form a sequence of discrete values corresponding to a discrete spectrum of correlation lengths [16]. Their numerical values can be easily obtained from (11) and (14). The largest correlation length for \(s = 0\) corresponds to a single particle–hole excitation with \(n_p = n_h = 1\) and \(\ell_1 = -m_1 = 1\) in (11). The corresponding particle is located at \(x\) and the hole at \(y\). Larger ‘quantum numbers’ \(\ell_1, m_1\) or multiple particle–hole excitations lead to shorter, subdominant correlation lengths.

As \(T \to 0^+\) the eigenvalue ratio corresponding to the largest correlation length converges to

\[
\rho_{\text{max}}(h) = \lim_{T \to 0^+} \max |\rho(\{x\}, \{y\}|0)| \\
\quad \quad = \left[ \frac{1}{\sqrt{k^2}} - \frac{1}{k^2} - 1 \left( \frac{h}{h_f} \right) \right]^2 = \left( \frac{1}{\sqrt{k^2}} - 1 \left( \frac{h}{h_f} \right) \right), \quad (15)
\]

(see [16]) where \(k = k(q)\) is the elliptic modulus. For \(h = 0\) this simplifies to

\[
\rho_{\text{max}}(0) = \frac{1 - \sqrt{1 - k^2}}{1 + \sqrt{1 - k^2}} = k(q^2). \quad (16)
\]

As we know from the analysis of the form factor series of the ordinary transfer matrix [17] and from classical work on the eight-vertex model [23], it is this ratio \(\rho_{\text{max}}(0)\) and not \(\rho_{\text{max}}(h)\) which determines the zero-temperature correlation length of the longitudinal two-point functions in the whole antiferromagnetic massive regime \(0 < h < h_f\). We shall see below that the amplitudes as well depend on the magnetic field through the positions of the particle and hole parameters. The summation (or rather integration) over infinitely many almost degenerate terms in the form factor series causes that the field dependence of the full correlation functions drops out in the end.

2. Zero-temperature limit of the amplitudes in the form-factor expansion

The amplitudes in the thermal form factor expansion of the finite-temperature two-point functions were considered in [13]. They are products of two matrix elements of certain non-local operators between dominant state and excited states of the quantum transfer matrix in the Trotter limit. In [13] it was observed that they can be written as products of three factors.
which were called 'universal part', 'determinant part' and 'factorising part', respectively. It was conjectured that this structure holds for arbitrary form factors involving finite products of local operators at neighbouring sites. For technical reasons dominant state and excited states were considered at different magnetic fields \( h \) and \( h' \) related by the twist parameter \( \alpha \) introduced in (12)

\[
\alpha = \frac{h - h'}{2\gamma T}.
\]

This leads to slightly generalised amplitudes which are more convenient in the intermediate steps of the calculations but in which the limit \( \alpha \to 0 \) has to be performed eventually in order to obtain the physically relevant correlation functions. Using \( \alpha \) has the additional advantage that we may obtain the amplitudes of the longitudinal correlation functions as the second derivative with respect to \( \alpha \) of a properly defined generating function (see appendix A for the definition).

### 2.1. The universal part

The universal part originates from products over ratios of eigenvalues of the quantum transfer matrix evaluated at the Bethe roots of dominant and excited states \([13]\). It depends on the local operators, whose correlation functions are considered, only through the spin \( s \). In order to express it in a compact way and to prepare for rewriting sums over particle and hole parameters as integrals, we introduce two more functions. One corresponds to the leading low-temperature asymptotics of the auxiliary function

\[
a_2(x|\kappa') = a(x|[x_i]_{i=1}^{n_x}, [y_j]_{j=1}^{n_y}) = e^{-\varepsilon(x)/T + 2\pi i F(x)},
\]

which plays a major role in the analysis of the spectrum of the quantum transfer matrix (see \([16]\) and appendix A). \( a_2 \) approximates the exact auxiliary function up to multiplicative \( 1 + \mathcal{O}(T^\infty) \) corrections and depends on a parameter \( \kappa' \) referring to the rescaled magnetic field \( \kappa' = -h'/(2\gamma T) \). The other one is a ratio of \( q \)-gamma and \( q \)-Barnes functions (see appendix F)

\[
\Psi(x) = \prod_{\epsilon = \pm} \frac{1}{\Gamma_{q'}(\epsilon \frac{1}{2} - \epsilon \frac{x}{2\gamma T})} G_{q'}^{\frac{1}{2}} \left( 1 + \epsilon \frac{ix}{2T} \right). \tag{20}
\]

In terms of these functions we can represent the universal part of the amplitudes as

\[
U_{n,s}(\alpha) = U \left( [x_i]_{i=1}^{n_x}, [y_j]_{j=1}^{n_y} \right) = q^{n_x + n_y^2} \left[ 1 - q^{2\pi i F(x)} \right] \prod_{j=1}^{n_y} a'_n(y_j | \kappa') \prod_{j=1}^{n_y} a'_n(y_j | \kappa') \prod_{j=1}^{n_x} \prod_{k=1}^{n_y} e^{\varepsilon(x_j, y_j - \varphi(y_j, x_j))} \\
\times \left[ \prod_{j=1}^{n_x} \prod_{k=1}^{n_y} \Psi(x_j) \right] \left[ \prod_{j=1}^{n_x} \prod_{k=1}^{n_y} \Psi(y_k) \right] \\
\times \prod_{j=1}^{n_x} \prod_{k=1}^{n_y} \Psi(x_j - y_k) \tag{21}
\]

\[
A_n^{xy} = \lim_{\alpha \to 0} U_{n,s}(\alpha) D_n^{xy}(\alpha) F_n^{xy}(\alpha), \quad \text{with } xy = zz, -+, \tag{17}
\]
where equality holds up to multiplicative $1 + \mathcal{O}(T^\infty)$ corrections. Here $x_k = x_j - x_k$, $y_k = y_j - y_k$, and the prime in $a'_n$ denotes the derivative with respect to the first argument. A derivation of the formula is presented in appendix B. The advantages of expressing everything in terms of $q$-gamma and $q$-Barnes functions are first, that the isotropic limit will be rather obvious in this formulation, and second, that we can use the known functional equations among these functions to rewrite the universal part in many useful ways.

### 2.2. The determinant part

The determinant part has its origin in a ratio of two products of determinants \([13]\). We derive its low-temperature limit in the antiferromagnetic massive regime in appendix C. Here we summarise the result.

We introduce a ‘weight function’

$$w(x) = (-1)^k \left[ \prod_{i=1}^{n_0} \vartheta_i(x - x_i, q^2) \right] \left[ \prod_{i=1}^{n_p} \vartheta_i(x - y_i, q^2) \right]$$

(22)

and the ‘deformed kernel’

$$K_n(x) = \frac{1}{2\pi i} (q^{-\alpha} \cot (x - i\gamma) - q^\alpha \cot (x + i\gamma)).$$

(23)

The determinant part is parameterised by two kernel functions $K^\pm$ which are different for the transversal and the longitudinal case. They can be expressed in terms of $K_n$ and will be specified below in (28) and (29). Given the kernel functions we define

$$\frac{v^-(x_j, y)}{2\pi i} = \frac{\text{res} \{w^{-1}\}(x_j)K^-(x_j, y)}{1 - e^{2\pi i F(x_j)}}, \quad V^-(x, y) = \frac{K^-(x, y)}{w(x)},$$

(24a)

$$\frac{v^+(x, y_k)}{2\pi i} = \frac{\text{res} \{w\}(y_k)K^+(x, y_k)}{e^{2\pi i F(y_k)} - 1}, \quad V^+(x, y) = K^+(x, y)w(y),$$

(24b)

\(j = 1, \ldots, n_0; k = 1, \ldots, n_p,\) and the resolvent kernels associated with $K_0$ and $V^\pm$ which are solutions of the linear integral equations

$$R(x - y) = K_0(x - y) - \int_{-\pi/2}^{\pi/2} dz \ K_0(x - z)R(z - y)$$

(25)

and

$$R^-(x, y) = V^-(x, y) - \int_{-\pi/2}^{\pi/2} dz \ R^-(x, z)V^-(z, y),$$

(26a)

$$R^+(x, y) = V^+(x, y) - \int_{-\pi/2}^{\pi/2} dz \ V^+(x, z)R^+(z, y).$$

(26b)
Using these definitions the determinant part can be written as

\[
D_n^{\mathcal{V}}(0) = P_n^{\mathcal{V}} \times \det_{\prod_{-\pi/2 \to \pi/2}} \left(1 + \mathcal{V}^{-}\right) \det_{\prod_{-\pi/2 \to \pi/2}} \left(1 + \mathcal{V}^{+}\right)
\]

\[
\times \det_{m,n=1,\ldots,n_0} \left\{ \delta_{m,n} + v^{-}(x_m, x_n) - \int_{-\pi/2}^{\pi/2} dy \ v^{-}(x_m, y) R^{-}(y, x_n) \right\}
\]

\[
\times \det_{m,n=1,\ldots,n_0} \left\{ \delta_{m,n} + v^{+}(y_m, y_n) - \int_{-\pi/2}^{\pi/2} dy \ R^{+}(y_m, y) v^{+}(y, y_n) \right\}
\]

\[
\times \det_{j,l=1,\ldots,n_p} \det_{k,m=1,\ldots,n_0} \left[ \delta_{jk} + \frac{2\pi i R(y_j - y_k)}{a_0(y_j) a_0(y_k)} \right]^{-1},
\]

(27)

where \(xy = z_{\pm}, -+\). Here the first two determinants on the right-hand side are Fredholm determinants of the integral operators \(\mathcal{V}^{\pm}\) with kernels \(V^{\pm}\), contour \([-\pi/2, \pi/2]\) and measure \(dx\). It is important to note that such Fredholm determinants can be very efficiently calculated numerically [8].

The functions \(K^{\pm}\) in the definition (24a) of the kernel functions have to be specified as

\[
K^{-}(x, y) = K_0(x - y) - K_0(\theta_0 - y),
\]

(28a)

\[
K^{+}(x, y) = K_0(x - y) - K_0(x - \theta_0)
\]

(28b)

in the longitudinal case and as

\[
K^{\pm}(x, y) = K_{\pm 1}(x - y)
\]

(29)

in the transversal case. Note that we have already set \(\alpha = 0\) here and that in the longitudinal case the kernels depend on two parameters \(\theta_\pm\).

The prefactor depends on the same parameters

\[
P_n^{zz} = \frac{4 \sin^2 \left(\frac{\pi k}{2} + \pi \sum_{j=1}^{n_p} (p(y_j) - p(x_j))\right)}{(-q^2; q^2)^4 (1 - e^{2\pi i F(\theta_0)})(1 - e^{-2\pi i F(\theta_0)})} \times \prod_{k=1}^{n_0} \Gamma_q \left(1 + \frac{\theta_0 - y_k}{2i}\right) \Gamma_q \left(1 + \frac{\theta_0 - x_k}{2i}\right) \Gamma_q \left(1 + \frac{\theta_0 - y_k}{2i}\right) \Gamma_q \left(1 + \frac{\theta_0 - x_k}{2i}\right),
\]

(30)

in such a way that \(D_n^{\mathcal{V}}\) is independent of \(\theta_\pm\) (see [24]). Choosing, for instance, \(\theta_+ = \theta_- = \theta\) the prefactor simplifies to

\[
P_n^{zz} = \frac{\sin^2 \left(\frac{k}{2} + \pi \sum_{j=1}^{n_p} (p(y_j) - p(x_j))\right)}{(-q^2; q^2)^4 \sin^2 (\pi F(\theta))},
\]

(31)

but e.g. for numerical calculations other choices may be useful.

In the transversal case the prefactor is simply

\[
P_n^{zz} = \frac{1}{4(-q^2; q^2)^4}.
\]

(32)
2.3. The factorising part

The factorising part is trivial in the longitudinal case, \( F^\pm_n(\alpha) = 1 \), since we have used the generating function approach. In the transversal case the factoring part is of the form

\[
F^\pm_n(\alpha) = \frac{G^\pm_n(\xi)}{(q^{\alpha-1} - q^{1-\alpha})(q^{\alpha} - q^{-\alpha})}. \tag{33}
\]

Using the representation (D.2) derived in appendix D and equations (A.24), (C.6), (C.8), (C.12) (C.16) and (D.1) we obtain the low-temperature limit of the functions in the numerator

\[
G^+\pm_n(\xi) = 1 - q^{1-\alpha}w(\xi)
\]

\[
= (q^{1+\alpha} - q^{-1-\alpha}) \sum_{j=1}^{n_n} \operatorname{res} \{ w \} (y_j) \int_{-\pi/2}^{\pi/2} \frac{dy}{2\pi i} w(y) G_i(y, \xi), \tag{34a}
\]

\[
G^0(\xi) = 1 + q^{1-\alpha}w^{-1}(\xi)
\]

\[
= (q^{-1-\alpha} - q^{1+\alpha}) \sum_{j=1}^{n_n} \operatorname{res} \{ w^{-1} \} (x_j) \int_{-\pi/2}^{\pi/2} \frac{dy}{2\pi i} w(y) G_i(y, \xi). \tag{34b}
\]

Here \( G_i(\xi) \) and \( \overline{G}_i(\xi) \) are the solutions of the linear integral equations

\[
G_i(x, \xi) = -\operatorname{ctg}(x - \xi) + q^{1-\alpha}w(\xi) \operatorname{ctg}(x - \xi + i\gamma)
\]

\[
- \sum_{j=1}^{n_n} V^+(x, y_j) G_i(y_j, \xi) - \int_{-\pi/2}^{\pi/2} dy V^+(x, y) G_i(y, \xi), \tag{35a}
\]

\[
\overline{G}_i(x, \xi) = -\operatorname{ctg}(x - \xi) + q^{1-\alpha}w^{-1}(\xi) \operatorname{ctg}(x - \xi - i\gamma)
\]

\[
- \sum_{j=1}^{n_n} \overline{G}_i(x_j, \xi) V^-(x_j, x) - \int_{-\pi/2}^{\pi/2} dy \overline{G}_i(y, \xi) V^-(y, x). \tag{35b}
\]

With

\[
\frac{1}{2\pi i} \frac{\operatorname{res} \{ w^{-1} \} (x)}{1 - e^{2\pi i f(x)}} = V^-(x, y) = \frac{K_{1-\alpha}(y - x)}{w(y)}, \quad V^+(x, y) = \frac{K_{1+\alpha}(x - y)w(y)}{e^{2\pi i f(y)} - 1}. \quad \tag{36a}
\]

Though we adopt the same symbols \( v^\pm \) and \( V^\pm \) as in (24), we restrict our argument to the transversal case here.

For the physical correlation functions we have to set \( \xi = -i\gamma/2 \) and have to send \( \alpha \to 0 \) in (33). Recall that this limit exists, since \( \overline{G}^+\pm_n(\xi) \) is differentiable in \( \alpha \) and vanishes at \( \alpha = 0 \) [13].

3. Form factor series

For the form factor series (4) we have to sum over all solutions \( x_i \), \( y_j \) of the higher level Bethe Ansatz equations (11) for \( k = 0, 1 \). Proceeding as in our recent work [17] we use
We recall that $B_{\pm}$ defined in (13) are the curves in the upper and lower half planes on which the particles and holes condense in the low-temperature limit. Let us assume that these curves are oriented toward the direction of growing real part. We introduce two simple closed and positively oriented curves going around $B_{\pm}$ and enclosing all particles and holes for small finite temperature and denote them by $\Gamma(B_{\pm})$. Then

$$
\sum_{\{x\}_{j=1}^{n_1}, \{y\}_{j=1}^{n_2} \text{ solutions of HBAEs}} \frac{f(\{x\}_{j=1}^{n_1}, \{y\}_{j=1}^{n_2})}{\det_{j,m=1,\ldots,n} \partial_{y_j} a(y_j|\{u_i\}, \{v_i\}|k) \partial_{u_k} a(u_k|\{v_i\}, \{v_i\}|k)}
= \int_{\Gamma(B_{\pm})} \frac{d^{n_1}v}{n_1!(2\pi i)^{n_1}} \int_{\Gamma(B_{\pm})} \frac{d^{n_2}u}{n_2!(2\pi i)^{n_2}}
\times \prod_{j=1}^{n_1} (1 + a(y_j|\{u_i\}, \{v_i\}|k))\prod_{j=1}^{n_2} (1 + a(u_j|\{v_i\}, \{v_i\}|k))
= (-1)^{n_1} \int_{B_{\pm}} \frac{d^{n_1}v}{n_1!(2\pi i)^{n_1}} \int_{B_{\pm}} \frac{d^{n_2}u}{n_2!(2\pi i)^{n_2}} f(\{u\}_{j=1}^{n_2}, \{v\}_{j=1}^{n_2})(1 + O(T^\infty)),
$$

if $f$ is holomorphic in all its variables on and inside $\Gamma(B_{\pm})$. In the last line we have used that $a(x|\{u_i\}, \{v_i\}|k) = O(T^\infty)$ for $x$ slightly above $B_{\pm}$ or slightly below $B_{\pm}$ and that $a(x|\{u_i\}, \{v_i\}|k) = O(T^{-\infty})$ for $x$ slightly below $B_{\pm}$ or slightly above $B_{\pm}$.

Note that the terms in (37) are the only factors in our expressions for the amplitudes that contain $O(T)$ corrections. It follows from the above consideration that these terms combine in such a way in the form factors series that the remaining corrections to the correlation functions are of order $T^\infty$ in the whole antiferromagnetic massive regime. This is in accordance with our experience with short-range correlation functions [45] which show basically no temperature dependence at small temperatures.

### 3.1. The longitudinal two-point functions

In order to apply the above to the form factor series for the longitudinal correlation functions we define the form factor density
\[ A^Z(\{ x_i \}_{i=1}^{n_x}, \{ y_j \}_{j=1}^{n_y}) \]

\[ = \left[ \frac{2}{(1 - q^4)} \Gamma_q \left( \frac{1}{2} \right) G_q \left( \frac{1}{2} \right) \right]^{2n_y} \prod_{j=1}^{n_y} \left( 1 - e^{-2\pi i F(x_j)}(1 - e^{-2\pi i F(y_j)}) \right) \times \left[ \prod_{j,k=1}^{n_y} \frac{\psi(x_j)\psi(y_j)}{\psi(x_j - y_k)} \right] \times \frac{\sin^2 \left( \frac{\pi}{2} + \pi \sum_{j=1}^{n_y} (p(y_j) - p(x_j)) \right)}{(-q^2; q^2)^4 \sin^2 (\pi F(\theta))} \times \frac{\det_{d, \theta} (1 + \hat{V}^-)}{\det_{d, \theta} (1 + \hat{V}^+)} \times \frac{\det_{m,n=1, \ldots, n_y} \{ \delta_{m,n} + (\chi_m - \chi_n) - \int_{-\pi/2}^{\pi/2} dy \frac{\chi_m}{\chi_n} R^+(y, \chi_n) \}}{\det_{m,n=1, \ldots, n_y} \{ \delta_{m,n} + (\chi_m - \chi_n) - \int_{-\pi/2}^{\pi/2} dy \frac{\chi_m}{\chi_n} R^+(y, \chi_n) \}}. \]

(39)

Here we have combined (21) and (27) for \( n_b = n_p \) and \( \alpha = 0 \). Except for the factors in (37) we have supplied a factor of \((-1)^{\phi_b}\) which will be absorbed by the integrals. For the longitudinal case the integral operators are fixed by the kernels (24) and (28). For simplicity we have set \( \theta_0 = \theta = \theta \), but since the expression is anyway independent of \( \theta_0 \) we could also use (30) instead of (31) here.

There is a single excited state with \( n_p = n_b = 0 \). For this state \( k = 1 \), the products and finite determinants in (39) are equal to one, and the remaining Fredholm determinants can be calculated [21]. This term describes the staggered order in the antiferromagnetic massive regime at zero temperature and is equal to the square of the staggered magnetisation first obtained by Baxter [5, 6]

\[ A^Z(\emptyset, \emptyset|1) = \frac{(q^2; q^2)^4}{(-q^2; q^2)^4}. \]

(40)

Using the latter result as well as our previous result (14) for the eigenvalue ratios and the summation formula (38) we obtain the form factor series

\[ \langle \sigma^z m+1 \rangle = (-1)^m \frac{(q^2; q^2)^4}{(-q^2; q^2)^4} + \sum_{n=1}^{\infty} \frac{\sin \left( \frac{\pi}{2} + \pi \sum_{j=1}^{n_y} (p(u_j) - p(v_j)) \right)}{(-q^2; q^2)^4} \times \left[ A^Z(\{ u_i \}_{i=1}^{n}, \{ v_j \}_{j=1}^{n}|0) + (-1)^m A^Z(\{ u_i \}_{i=1}^{n}, \{ v_j \}_{j=1}^{n}|1) \right] \]

(41)

for the longitudinal two-point functions, holding, for every fixed \( m \), up to multiplicative corrections of the form \((1 + O(T^\infty))\). Here we have also taken into account that the magnetisation per lattice site \( \langle \sigma^z \rangle \) vanishes in the antiferromagnetic massive regime.

Note that the form factor densities satisfy the identity

\[ A^Z(\{ u_i \}_{i=1}^{n}, \{ v_j \}_{j=1}^{n}|0) = A^Z(\{ u_i + \pi \delta_{i,k} \}_{i=1}^{n}, \{ v_j \}_{j=1}^{n}|1) = A^Z(\{ u_i \}_{i=1}^{n}, \{ v_j + \pi \delta_{i,k} \}_{j=1}^{n}|1) \]

(42)

for \( k = 1, \ldots, n \). Taking into account the quasi periodicity of the momentum function \( p \) it follows that the integrands in (41) are \( \pi \)-periodic in all variables \( u_i, v_j, i, j = 1, \ldots, n \). It further
follows from the definition of $A^z$ that the integrand is holomorphic in every $u_j$, $j = 1, \ldots, n$, inside the strip $-\gamma \leq \text{Im } u_j < 0$ and in every $v_j$, $j = 1, \ldots, n$, inside the strip $0 < \text{Im } v_j < \gamma$. This means that the integration contours $B_\pm$ in (41) can be deformed and shifted inside their respective strips. Since the dependence on the magnetic field entered only through these contours, it follows that, in the low-temperature limit, the form factor series (41) is independent of the magnetic field in the full antiferromagnetic massive regime, $0 < h \ll h_f$, or, in other words, that the magnetic field dependence is contained in the temperature corrections of the form $(1 + O(T^{-\infty}))$ and thus is ‘exponentially small’. Hence, as claimed above, it has turned out that the dependence on the magnetic field of the individual form factors cancel each other out once the summation is performed. One should keep in mind, however, that this behaviour is not uniform in $m$. If we keep $T$ small but fixed, the magnetic field dependence comes back for $m$ large enough.

We choose to deform the contours $B_\pm$ into straight line segments $[-\pi/2, \pi/2] \pm i\gamma/2$. This choice corresponds to the limit of $B_\pm$ for $h \to 0$. It seems to be particularly useful for numerical calculations. With this choice the form factor series for the longitudinal two-point functions finally becomes

$$
\langle \sigma^z_{m+1} \sigma^z_{m+1} \rangle = (-1)^m \frac{(q^2; q^2)_{d}}{(-q^2; q^2)_{d}} \times A(z) \left( \{ u \}_{j=1}^n, \{ v \}_{j=1}^n \right).
$$

This series is valid up to multiplicative temperature corrections of the form $(1 + O(T^{-\infty}))$. Together with the analogous result (47) for the transversal correlation functions below it is the main result of this work. We would like to emphasize that it is different from the previously know form factor series which were obtained by means of the $q$-vertex operator approach [22] or by applying the algebraic Bethe Ansatz approach to the usual transfer matrix [17]. We claim that our new series representation, based on form factors of the quantum transfer matrix, provides a more efficient exact description of the longitudinal two-point functions at $T = 0$, since it does neither involve multiple-contour integrals (as the representation in [22]) nor multiple-residue integrals (as the representation in [17]). Instead we have to deal with Fredholm determinants which, as we believe, are more efficient in numerical calculations.

**Remark.** In the limit $T \to 0^+$ the series representation (43) holds in the whole antiferromagnetic massive regime $\Delta > 1$, $|h| < h_f$ and, in particular, also if the phase boundary $h = h_f$ is approached from below. Hence, when approached from below the leading asymptotic behaviour of the longitudinal two-point function on the phase boundary is

$$
\langle \sigma^z_{m+1} \sigma^z_{m+1} \rangle \sim (-1)^m \frac{(q^2; q^2)_{d}}{(-q^2; q^2)_{d}} g(m, h).
$$

Remarkably this can be reproduced if we approach the phase boundary from above and introduce an appropriate scaling function. Using the the techniques developed in [13] it can be shown [12] that, asymptotically for large $m$ and small positive $h - h_f$,

$$
\langle \sigma^z_{m+1} \sigma^z_{m+1} \rangle \sim (-1)^m \frac{(q^2; q^2)_{d}}{(-q^2; q^2)_{d}} g(m, h).
$$
where
\[ g(m, h) = \sqrt{c} \frac{2^{3/6} (2k)}{A^6} \left( \frac{2k}{1 - k^2} \right)^{1/4} \left( \frac{h}{h_\ell} - 1 \right)^{-1/4} \frac{1}{\sqrt{m}} \]  
(46)

and \( A \) is the Glaisher–Kinkelin constant. Approaching the phase boundary from above in such a way that \( g(m, h) = 1 \) equation (46) reproduces (44).

3.2. The transversal two-point functions

The form factor series for the transversal case follows as well from our results in the previous subsections. We have to combine (21) and (27) for \( n_h = n_p + 2 \) with (33) and have to send \( \alpha \) to zero. Using the summation formula (38) we obtain a form factor series of the form

\[ \langle \sigma_1 a_{m+1}^+ \rangle \]
\[ = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)!} \int_{-\pi}^{\pi} \frac{d^n u}{(2\pi)^n} \int_{-\pi}^{\pi} \frac{d^n v}{(2\pi)^n} e^{-2\pi i m \sum_{j=1}^{n} p(u_j) + 2\pi i m \sum_{j=1}^{n} p(v_j)} \times A^{e-2}(\{u_{j=1}^{n+2}, \{v_{j=1}^{n+2}\}|k), \]
(47)

where the amplitude densities are defined as

\[ A^{e-2}(\{x_{j=1}^{n+2}, \{y_{j=1}^{n+2}\}|k) \]
\[ = \lim_{\alpha \to 0} \frac{q G_\alpha(0,0) G_\alpha(0,0)}{2 \gamma (q^{-1} - q)} \]
\[ \times \left[ \frac{2}{1 - q^4} \frac{1}{\gamma q^{i} \frac{4 \gamma q^{i}}{2}} \right] \times \left[ \prod_{j=1}^{n+2} (1 - e^{-2\pi i F(x_j)} \prod_{j=1}^{n+2} (1 - e^{-2\pi i F(y_j)} \right] \]
\[ \times \left[ \prod_{j=1}^{n+2} \prod_{k=1}^{n} \frac{e^{i(x_j, x_k) - \rho(x_j, x_k)}}{\prod_{j=1}^{n+2} \prod_{k=1}^{n} \Psi(x_j - x_k)} \right] \]
\[ \times \frac{1}{4(-q^2 - q^{-2})^4} \frac{\det_{x \in [-\pi/2, \pi/2]} (1 + \tilde{V}^{-}) \det_{x \in [-\pi/2, \pi/2]} (1 + \tilde{V}^{+}) \}
\[ \times \det_{m,n=1, \ldots, n_p = 2} \left\{ \delta_{m,n} + v^{-}(x_m, x_n) - \int_{-\pi/2}^{\pi/2} dy \, v^{-}(x_m, y)R^{-}(y, x_n) \right\} \]
\[ \times \det_{m,n=1, \ldots, n_p = 2} \left\{ \delta_{m,n} + v^{+}(y_m, y_n) - \int_{-\pi/2}^{\pi/2} dy \, R^{+}(y_m, y)v^{+}(y, y_n) \right\} \]
(48)

Here we adopt the usual conventions for \( n_p = 0 \): the set of hole-rapidities is the empty set, products and integrals over empty sets of holes are replaced by 1. In the transversal case the integral operators are fixed by the kernel functions (24) and (29).

3.3. Discussion and numerical test cases: the longitudinal case

It is an interesting question how our new form factor series are related with those known previously. Both, the \( q \)-vertex operator approach and the algebraic Bethe Ansatz approach applied to the usual transfer matrix, employ different pictures of elementary excitations. For the Hamiltonian and for the usual transfer matrix in the antiferromagnetic massive regime
these are pairs of spinons, parameterised by pairs of real spinon rapidities. The form factor series of the longitudinal two-point functions in this ‘spinon basis’ as obtained, for instance, in [17] is of the form

\[
\langle \sigma_1^\uparrow \sigma_{m+1}^\downarrow \rangle = (-1)^m \frac{(q^2; q^2)^4}{(-q^2; q^2)^4} + \sum_{k=0,1} \frac{(-1)^{2k}}{(2\pi)^2} \int \frac{d^2u}{(2\pi)^2} e^{2\pi in \sum_{j=1}^n p(u_j)} \mathcal{F}^{zz}(\{u_{i}\}_{i=1}^n | k).
\]

(49)

For the form factor density \(\mathcal{F}^{zz}(\{u_{i}\}_{i=1}^n | k)\) in the general case see [17, 22]. We do not want to reproduce it here. Since we are unable so far to relate this \(n\)-spinon form factor density to the \(n\)-particle-\(n\)-hole form factor density \(\mathcal{A}^{zz}(\{u_{i}\}_{i=1}^m, \{v_{j}\}_{j=1}^n | k)\) in the general case, we restrict ourselves to \(n = 1\). In this case we have an explicit result [18] for \(\mathcal{F}^{zz}\) obtained by numerical comparison with a two-spinon form factor formula due to Lashkevich [33]. Namely

\[
\mathcal{F}^{zz}(\{u_1, u_2\} | k) = \frac{32q(q^2; q^2)^2 \sin^2 \left( \pi \left( p(u_1) + p(u_2) + \frac{k}{2} \right) \right) \sin^2(u_2) \sin^2 \left( \frac{|u_2|}{2} \right)}{(q^4; q^4)^2 (q^4e^{2imn}; q^4, q^4)^2 (q^4e^{2imn}; q^4)}
\]

\[
\times \prod_{\sigma = \pm} \sin((u_1 + i\gamma/2) \sin((u_1 - i\gamma/2)/2)
\]

(50)

where \(u_{12} = u_1 - u_2\).

When comparing (43) and (49) a rather natural guess about the relation of the integrals on the right-hand side of both equations is that the term corresponding to the \(n\)-ph contributions in (43) is equal to the \(2n\)-spinon term in (49). Since the integrals to be compared look like Fourier integrals and since we expect that they are pairwise equal for all \(m \in \mathbb{R}\), we expect a simple relation between the integrands. So far we are unable to prove any relation, but we can provide at least a conjecture supported by strong numerical evidence. Let us consider the case \(n = 1\). In this case we can calculate \(\mathcal{A}^{zz}(\{u_1\}, \{u_2\} | 1)\) numerically with high accuracy and compare with (50).

A naïve guess would be that \(\mathcal{F}^{zz}(\{u_1, u_2\} | 1)\) would equal \(2\mathcal{A}^{zz}(\{u_1 - i\gamma\}, \{u_2\} | 1)\). But this cannot be true, because of the different properties of the two functions. While \(\mathcal{F}^{zz}(\{u_1, u_2\} | 1)\) is symmetric in \(u_1, u_2\) and has a double zero at \(u_1 = u_2\), neither of the two properties does hold for \(2\mathcal{A}^{zz}(\{u_1 - i\gamma\}, \{u_2\} | 1)\). This observation gives us a hint which might be the true relationship between the two functions.

**Conjecture 1.** Inside the strip \(0 < \text{Im} u_1, \text{Im} u_2 < \gamma\) we have

\[
\mathcal{F}^{zz}(\{u_1, u_2\} | 1) = \mathcal{A}^{zz}(\{u_1 - i\gamma\}, \{u_2\} | 1) + \mathcal{A}^{zz}(\{u_2 - i\gamma\}, \{u_1\} | 1).
\]

(51)

Here both sides of the equation can be computed with several digits accuracy, which leaves little doubt about the correctness of the conjecture. Comparing the combinatorial factors in (43) and (49) and noting that

\[\text{Henceforth denoted } n\text{-ph amplitude.}\]

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\[ \frac{1}{(n!)^2} = \frac{1}{(2n)!} \binom{2n}{n} \tag{52} \]

It is tempting to speculate that

\[ \mathcal{F}^{zz}(\{0, \ldots, 0\}) = \sum_{(S_1, S_2) \in \mathcal{P}_2(|\{0, \ldots, 0\}|)} \mathcal{A}^{zz}(S_1 - i\gamma, S_2), \tag{53} \]

where \( \mathcal{P}_2(M) \) is the set of all ordered pairs of disjoint subsets of \( M \), and \(|M|\) denotes the number of elements in \( M \). The difficulty in testing (53) even numerically comes from the fact that no efficient expressions for the higher-spinon amplitudes \( \mathcal{F}^{zz} \) beyond (50) are known. So far the four-spinon amplitudes were computed only in the isotropic limit [10]. Note, however, that explicit expressions for the two-spinon amplitudes \( \mathcal{F}^+ \) of the transversal correlation functions are available (see equation (60) below). We have compared these numerically with the corresponding two-hole amplitudes of our approach. Since there are no particles involved in this case, no symmetrisation is necessary, just a proper identification of rapidity variables.

As in the longitudinal case the numerical agreement of both types of amplitudes was perfect.

Unlike in the case of the spinon-based approach it seems not too hard to evaluate the first few higher-ph contributions to the representation (43) of the longitudinal correlation function. We denote the term in the sum on the right-hand side of (43) that involves the \( 2j \)-fold integrals for \( k = 0 \) and \( k = 1 \) by \( I_j(m) \) and set

\[ \langle \sigma_1^z \sigma_{m+1}^z \rangle_{2n} = \sum_{j=0}^{n} I_j(m), \quad I_0(m) := (-1)^m \left( \frac{q^2; q^4}{(q^2; q^4)^4} \right), \tag{54} \]

which includes all contributions up to \( n \) particles and \( n \) holes. The 2-ph contribution to the nearest-neighbour correlator, \( I_1(1) \), for example, is a four-fold integral. For its numerical calculation it is crucial that the integration contours are chosen as \([-\pi/2, \pi/2] \pm i\gamma/2\), since the factors \( e^{\pm i\gamma} \) are real on these contours. Then the properties of the corresponding amplitudes under complex conjugation guarantee that the integral is real. We can use these properties as well as the fact that the amplitudes are symmetric in the particle variables and in the hole variables separately to reduce the computational cost.

We have computed the \( n \)-fold integrals by means of the Gauß–Legendre quadrature rule with \( N \) sampling points. \( N \) was increased until the relative change of the result when incrementing \( N \) to \( N + 2 \) became sufficiently small\(^7\). An improvement of the numerical accuracy may be possible, but we content ourselves to a naive approach here. Gfortran was used to compile the programs with openmp. The computations were mainly performed on an 8-core workstation (Xenon E5-2620, 2GHz). Typical runs consumed \( \mathcal{O}(10^2) \) seconds respectively minutes or hours (cpu time) for \( n = 2 \) respectively 3 or 4.

For illustrational purposes the values of \( I_j(1), j = 1, 2, 3 \), for various values of \( \gamma \) are listed in table 1.

The short-distance correlation functions \( \langle \sigma_1^z \sigma_{m+1}^z \rangle, m = 1, 2, 3 \), are known exactly [44]. For example, the neighbour-correlator has the representation

\[ \text{if } m = 1, \text{ less than } 0.01\% \text{ for } n = 2 \text{ and } n = 4 \text{ for } \gamma \geq 0.3. \text{ For } \gamma = 0.2 \text{ it is } 0.02\% \text{ for } n = 2 \text{ and } 0.2\% \text{ for } n = 4. \]

A similar accuracy could not be achieved for \( n = 6 \), as the maximum \( N \sim 30 \) due to cpu time limitations. As a rule of thumb, we expect an error of the order of 10% in this case.
This gives us the opportunity to test the $n$-ph approximations $\langle \sigma_i^z \sigma_{m+1}^z \rangle_{2n}$ obtained from our form factor expansion. We set

$$r_n(m) = \frac{\langle \sigma_i^z \sigma_{m+1}^z \rangle_{2n}}{\langle \sigma_i^z \sigma_{m+1}^z \rangle_{1}}. \tag{56}$$

The data for these ratios in table 2 clearly demonstrate that the form factor expansion converges quickly towards (55). See also figure 3.

The higher-ph contributions become important as the system approaches the isotropic point. On the contrary, excitations higher than 3-ph seem almost negligible for $\gamma > 0.5$. For
m = 2, 3 the ratios \( r_3(m) \) stay closer to 1 (table 3). This suggests that the contribution from higher particle–hole excitations becomes less important.

For \( m \geq 4 \) explicit formulae are not available so far. We therefore compare our results against standard numerical methods, the DMRG and brute force diagonalisation (the Lanczos method). We utilised the software library ALPS ver. 2 [4]. For the observable we chose

\[
g_{zz}(m) = (-1)^m (\langle \sigma^z_1 \sigma^z_{m+1} \rangle - I_0(m)),
\]

which vanishes asymptotically and is expected to be positive for any \( m \). The above observable, measured by the DMRG and the Lanczos method, will be compared with the expansion

\[
g_{zz}(m) = (-1)^m \sum_{j=1}^{m} I_j(m).
\]

For this purpose we have applied the Lanczos method to chains of various lengths under periodic boundary conditions. For our DMRG calculations we employed open boundary conditions\(^8\) and chose parameters \( \text{MAXSTATES} = 50 \sim 150 \). The anisotropy parameter \( \Delta \) was varied between 1.1 and 2. The data were extrapolated to the thermodynamic limit assuming the form

\[
g_{ll}^{zz}(m) \sim g_{zz}(m) + C(m) e^{-\ell},
\]

where \( g_{ll}^{zz}(m) \) is a finite-\( L \) datum for \( g_{zz}(m) \).

\(^8\) In order to reduce the boundary effect for DMRG, we took the average

\[
g_{zz}(m) = -\frac{1}{L} \sum_{n=0}^{L-1} \langle \sigma^z_n \sigma^z_{n+1} \rangle - I_0(m).
\]

Typically we chose \( \ell = 10 \).
First, we considered system sizes $12 \leq L \leq 24$ within the Lanczos method and $48 \leq L \leq 64$ within the DMRG. Figure 4 (left) shows the resultant values of $g^{zz}(5)$ and $g^{zz}(5)_{ph}$ for $1.1 \leq \Delta \leq 1.5$. One immediately recognizes differences. The discrepancy is partly due to the large correlation lengths $\xi$ in the selected range of $\Delta$ (see table 4). We thus increased the system size up to $L = 38$ within the Lanczos method and up to $L = 112$ within DMRG. The assumption (58) then works well for the DMRG for $\Delta \approx 1.4, 1.5$ with $\tau \propto 1/\xi$.

On the other hand, the Lanczos data do not necessarily obey (58) for the whole range $12 \leq L \leq 38$. We nevertheless fitted the data according to (58) and the result is plotted in figure 4 (right). The coincidence of the DMRG with the form factor expansion data is improved remarkably, while it becomes slightly better for the Lanczos method, as expected. Probably, the agreement with the Lanczos data could be further improved if we would consider the ground state together with the first excited state and take the arithmetic average. Such kind of analysis is justified as ground state and first excited state degenerate in the

![Figure 5](image5.png)

**Figure 5.** Comparison of $g^{zz}(m)$ estimated by the Lanczos method (squares) and by DMRG (triangles) against $g^{zz}(m)_{ph}$. The spin distance $m$ is 3 (left panel) or 8 (right panel). The red circles in the left panel denote the exact values.

![Figure 6](image6.png)

**Figure 6.** Plots of $g^{zz}(m)$ versus $m$ for $\Delta = 2$ obtained by three different methods. The curves are almost indistinguishable.
thermodynamic limit, and it is this average which corresponds to the zero-temperature limit of
the static correlation functions. Supplementary, in figure 5, we show $g_{mzz}(\xi)$ for some more
values of $m$ corresponding to larger system size data. The better agreement for larger $L$ suggests
that the three independent results eventually coincide in the limit $\xi \to L_0$, namely larger $L$ or larger
$\Delta$ (where $\xi$ is small). This is consistent with the observation that $g_{zz}(m_{\text{ph}})$ for $\Delta = 2$
(see figure 6) exhibits already a good agreement with the asymptotic form as $\xi \sim 5.29$.

Summarising, we have confirmed the efficiency of the form factor expansion for arbitrary
distance and its consistency with standard numerical methods. The numerical accuracy
reaches a satisfactory level, except for the vicinity of the isotropic point. We shall discuss this
problem separately in section 4.

3.4. Numerical test cases: the transversal case

The numerical analysis of the previous subsection can be performed for the transversal case in
a parallel manner. We thus only briefly summarise our results.

The form factor series of the transverse correlation function in the spinon basis, an analogous
formula to (49), reads

$$
\langle \sigma_{l} \sigma_{m+1} \rangle = \sum_{u \in \mathbb{N}} \frac{(-1)^{u}}{(2u)!} \int_{-\pi/2}^{\pi/2} d^{2i} u \sum_{j=1}^{2n+1} \sum_{P(u)} e^{-2\pi i m u} \mathcal{F}^{-\tau}([u]k).
$$

(59)

Table 4. Correlation lengths for various values of $\Delta$.

| $\Delta$ | 1.1  | 1.2  | 1.3  | 1.4  | 1.5  | 2.0  |
|---------|------|------|------|------|------|------|
| $\xi$   | 8482.8 | 347.131 | 85.1433 | 37.0497 | 21.0729 | 5.29593 |

Figure 7. Comparison of $g_{zz}(m)$ with its asymptotic form derived in [17]. For $\Delta = 1.5$
(left), due to large $\xi$, $g_{zz}(m)$ still deviates considerably from its asymptotic form. For
$\Delta = 2$ (right) $g_{zz}(m)$ exhibits already a good agreement with the asymptotic form
as $\xi \sim 5.29$.
The explicit integrand for the 2 spinon case was obtained in [22]

\[ \mathcal{F}^{-+}((u_1, u_2) | k) = 4 \vartheta^2 \left( \frac{u_1 + u_2 + k\pi}{2}, q \right) (q^2; q^4)^2(q^4; q^4)^4 \]

\[ \times \prod_{j=1,2} \frac{\sin^2 u_{12}}{\vartheta_2(u_j - i\gamma/2, q^2) \vartheta_2(u_j + i\gamma/2, q^2)} \]

\[ \times \prod_{\sigma = \pm} \frac{(q^4 e^{2i\sigma\pi}; q^4, q^4)^2(q^4 e^{2i\sigma\pi}; q^4)^4}{(q^2 e^{2i\sigma\pi}; q^4, q^4)^2(q^4 e^{2i\sigma\pi}; q^4)^4}. \]  

(60)

On the other hand, we have our novel form factor series in (47). Comparing the two leads us to

**Conjecture 2.** Inside the strip \( 0 < \text{Im } u_1, \text{Im } u_2 < \gamma \) we have

\[ \mathcal{F}^{-+}((u_1, u_2) | 0) = - \mathcal{A}^{-+}((u_1 - i\gamma, u_2 - i\gamma) | 0). \]  

(61)

We have tested this conjecture numerically. The numerical evidence is rather convincing. Since the higher spinon contributions are not known explicitly, we refrain from further discussion here.

The formula (47) is numerically efficient as in the longitudinal case. Set

\[ g^{-+}(m) = (-1)^m (\sigma_1^- \sigma_{m+1}^+). \]

Figure 8 shows the convergence of \( g^{-+}(1) \) to its exact values near the isotropic point with increase in \( n_h \). The curves are indexed by \((n_h, n_p)\): \((2, 0) + (3, 1)\) means the sum of contributions from the sectors \((n_h, n_p) = (2, 0)\) and \((n_h, n_p) = (3, 1)\), for example. The data obtained by the Lanczos method \((L \leq 24)\) deviate from these two results for small anisotropy. When \( \Delta = 2 \), however, the correlation length is sufficiently small and all three results coincide with reasonable accuracy (table 5). The nice agreement supports the validity of the form factor series (47).
4. The isotropic limit

Within the vertex operator approach the isotropic limit was considered, for instance, in [22]. The isotropic point $\Delta = 1, h = 0$ in the ground state phase diagram of the XXZ chain is located at the boundary of the antiferromagnetic massive regime (see figure 1). In our formulae for the ground state correlation functions, which are independent of the magnetic field, it can be reached by sending $\gamma \to 0$ and hence $q \to 1$. As is well known this limit requires also a rescaling of the rapidities $x, y \to \gamma u, \gamma v$ before sending $\gamma \to 0$. Here we are going to perform the isotropic limit for our form factors densities and the form factor series for the longitudinal two-point functions, leaving the transversal case for future study.

We remark that $G = G_{\gamma \to 0}$ and $G' = G'_{\gamma \to 0}$ (see appendix F). This is enough to perform the isotropic limit for the momentum $p$, the shift function $F$, the weight functions $w$ and the function $\Psi$ occurring in the universal part of the amplitudes. We shall denote the limiting functions by hats, $\hat{f}(u) = \lim_{\gamma \to 0} f(\gamma u)$. Then we obtain the momentum

$$\hat{p}(u) = \frac{1}{4} + \frac{1}{2\pi i} \ln \left( \frac{\text{ch} \left( \frac{\pi}{2} u + \frac{i}{2} \right)}{\text{ch} \left( \frac{\pi}{2} u - \frac{i}{2} \right)} \right)$$

(62)
in the isotropic limit. The closely related weight function turns into
\[
\hat{w}(u) = (-1)^k \prod_{j=1}^{n_u} \frac{\text{th} \left( \frac{u}{2} (u - v_j) \right)}{\text{th} \left( \frac{u}{2} (u - u_j) \right)}.
\] (63)

For the limit of the shift function we first recall the expression of the two-spinon scattering phase [19],
\[
\theta_f(u) = \frac{1}{2\pi i} \ln \left\{ \frac{\Gamma(1 - iu)}{\Gamma(1 + iu)} \right\}.
\] (64)

In terms of this scattering phase the rescaled dressed phase \( \varphi \) and the rescaled shift function \( F \),
\[
\varphi(u, v) = \lim_{\gamma \to 0^+} \varphi(\gamma u, \gamma v) \quad \text{and} \quad \hat{F}(u) = \lim_{\gamma \to 0^+} F(\gamma u),
\] (65)
turn into
\[
\hat{\varphi}(u, v) = \frac{i\pi}{2} - 2\pi i \theta_f(u - v),
\] (66)
\[
\hat{F}(u) = \frac{k}{2} + \sum_{j=1}^{n_u} (\theta_f(u - u_j) - \theta_f(u - v_j)).
\] (67)

The isotropic limit of the function \( \Psi \) is simply
\[
\hat{\Psi}(x) = \prod_{\epsilon = \pm} \frac{1}{\Gamma(\frac{1}{2} - \epsilon \frac{iu}{2})} \Gamma(\epsilon \frac{iu}{2}) G^2 \left( \frac{1 + \epsilon \frac{iu}{2}}{2} \right)
\] (68)

With this we have gathered all what is needed to deal with the universal part of the amplitudes.

For the determinant part we note that for our basic kernel function \( K_0 \)
\[
\lim_{\gamma \to 0^+} \frac{d}{d \gamma} \varphi(\gamma u) = \frac{du}{\pi} \frac{1}{1 + u^2} = \frac{du}{\pi} \hat{K}_0(u).
\] (69)

In the integrals in the determinant part the rescaling connected with the isotropic limit leads to the replacement of the integration interval \([-\pi/2, \pi/2]\) by \([-\pi/(2\gamma), \pi/(2\gamma)]\) which in the limit \( \gamma \to 0^+ \) goes to \((-\infty, \infty)\). We set
\[
\hat{\tilde{K}}^-(u, v) = \hat{K}_0(u - v) - \hat{K}_0(u + v),
\] (70a)
\[
\hat{\tilde{K}}^+(u, v) = \hat{K}_0(u - v) - \hat{K}_0(u + v)
\] (70b)

and
\[
\hat{\tilde{\nu}}^-(u_j, v) = \frac{2\pi i \text{res} \{ \hat{w}^{-1} \} (u_j) \hat{K}^-(u_j, v)}{1 - e^{2\pi i \hat{F}(u_j)}}, \quad \hat{\tilde{\nu}}^{-1}(u, v) = \hat{w}^{-1}(u) \hat{K}^- (u, v),
\] (71a)
\[
\hat{\tilde{\nu}}^+(u_k) = \frac{2\pi i \text{res} \{ \hat{w} \} (u_k) \hat{K}^+(u_k, v_k)}{e^{2\pi i \hat{F}(v_k)} - 1}, \quad \hat{\tilde{\nu}}^+(u, v) = \hat{K}^+(u, v) \hat{w}(v).
\] (71b)
We further define the corresponding resolvent kernels in the isotropic limit as solutions of linear integral equations,

$$\dot{R}(u, v) = \dot{V}(u, v) - \int_{-\infty}^{\infty} dz \dot{R}(u, z)\dot{V}(z, v),$$  \hspace{1cm} (72a)$$

$$\dot{R}^+(u, v) = \dot{V}^+(u, v) - \int_{-\infty}^{\infty} dz \dot{V}^+(u, z)\dot{R}^+(z, v),$$  \hspace{1cm} (72b)$$

which completes the definitions needed in the description of the isotropic limit of the finite determinants in (39).

In order to perform the isotropic limit of the Fredholm determinants it is useful to distinguish the cases $k = 0$ and $k = 1$. We show in appendix G that

$$\det_{\text{det}[\tau/2\pi/2]} (1 + \hat{V}^\pm) = ((-1)^{k+1} q^2; q^2)^2 \det_{\text{det}[\pi/2\pi/2]} (1 + \hat{W}_{q,k}^\pm),$$  \hspace{1cm} (73)$$

where $\hat{W}_{q,k}^\pm$ are integral operators with kernels

$$W_{q,k}^+(u, v) = (w^{-1}(u) - (-1)^k(R_k(u - v) - R_k(\theta_- - v)),$$  \hspace{1cm} (74a)$$

$$W_{q,k}^-(u, v) = (R_k(u - v) - R_k(u - \theta_+))w(v) - (-1)^k$$  \hspace{1cm} (74b)$$

defined in terms of two functions

$$R_0(u) = \frac{1}{2\pi i} \partial_u \ln \begin{bmatrix} \Gamma_{q^2}^1 \left(1 + \frac{iu}{2\pi} \right) \Gamma_{q^2}^1 \left(\frac{1}{2} - \frac{iu}{2\pi} \right) \\ \Gamma_{q^2}^1 \left(1 - \frac{iu}{2\pi} \right) \Gamma_{q^2}^1 \left(\frac{1}{2} + \frac{iu}{2\pi} \right) \end{bmatrix},$$  \hspace{1cm} (75a)$$

$$R_1(u) = \frac{1}{2\pi i} \partial_u \ln \begin{bmatrix} \Gamma_{q^2}^1 \left(1 - \frac{iu}{2\pi} \right) \\ \Gamma_{q^2}^1 \left(1 + \frac{iu}{2\pi} \right) \end{bmatrix}.$$  \hspace{1cm} (75b)$$

The Fredholm determinants on the right-hand side of (73) provide us with an alternative representation of the determinant part of the longitudinal correlation functions from which we can easily obtain the isotropic limit.

By virtue of the results of appendix F the limits

$$\hat{W}_{q,k}^\pm(u, v) = \lim_{\gamma \to 0^+} \frac{W_{q,k}^\pm(\gamma u, \gamma v)}{\gamma}$$  \hspace{1cm} (76)$$

of the kernel functions exist and define integral operators acting on the real line. The corresponding Fredholm determinants are finite. Because of the prefactor $((-1)^{k+1} q^2; q^2)^2$ in (73), however, the Fredholm determinants $\det_{\text{det}[\pi/2\pi/2]} (1 + \hat{V}^\pm)$ vanish for $k = 1$ and with them the corresponding amplitudes

$$\hat{A}^{\pm}((u_{k=1}^n, |v_j|_{j=1}^n) | 1) = \lim_{\gamma \to 0^+} (-iy)^{2n} A^{\pm}((\gamma u_{k=1}^n, |\gamma v_j|_{j=1}^n) | 1) = 0.$$  \hspace{1cm} (77)$$

For $k = 0$, on the other hand, the prefactor $((-1)^{k+1} q^2; q^2)^2$ in (73) diverges, but when the Fredholm determinant is inserted into the formula for the amplitudes is canceled by the denominator in such a way that
\( \hat{A}^{zz}(\{u\}_j^{n_1}, \{v\}_j^{n_2}) = \lim_{\gamma \to 0} (-i\gamma)^{2n}A^{zz}(\{\gamma u\}_j^{n_1}, \{\gamma v\}_j^{n_2})|0\rangle \) (78)

stays finite in the isotropic limit. Setting \( \hat{W}^+(u, v) = \hat{W}_0^+(u, v) \) we obtain the explicit expressions

\[
\hat{W}^-(u, v) = (1 - \hat{\nu}^{-1}(u)) (\theta^+_F(u - v) - \theta^-_F(\theta_+ - v)), \quad (79a)
\]

\[
\hat{W}^+(u, v) = (\theta^+_F(u - v) - \theta^-_F(\theta_+ - v))(1 - \hat{\nu}(v)) \quad (79b)
\]

for the remaining kernel functions in the isotropic limit.

Using all the above, the final result for the non-vanishing amplitudes is

\[
\hat{A}^{zz}(\{u\}_j^{n_1}, \{v\}_j^{n_2}) = \left[ \frac{1}{2(1/2)G^4(1/2)} \right]^{2n} \prod_{j=1}^{n} \left( 1 - e^{-2\pi i F(v_j)}(1 - e^{-2\pi i F(y_j)}) \right) \times \prod_{j,k=1}^{n} e^{i\varphi(v_j,v_k) - \varphi(y_j,y_k)} \prod_{k=1}^{n} \hat{\psi}(x_k) \hat{\psi}(y_k) \times \hat{p}^{zz} \det (1 + \hat{W}) \det (1 + \hat{W}^+) \times \det_{u,v} \left\{ \delta_{j,k} + \hat{\nu}^{-1}(u_j, u_k) - \int_{-\infty}^{\infty} dv \hat{\nu}^{-1}(u_j, v) \hat{R}^- (v, u_k) \right\} \times \det_{j,k=1,\ldots,n} \left\{ \delta_{j,k} + \hat{\nu}^{-1}(v_j, v_k) - \int_{-\infty}^{\infty} dv \hat{R}^+ (v_j, v) \hat{\nu}^{-1}(v_j, v_k) \right\}, \quad (80)
\]

where

\[
\hat{p}^{zz} = \frac{4 \sin^2 \left( \pi \sum_{j=1}^{n_2} (\hat{p}(v_j) - \hat{p}(u_j)) \right)}{(1 - e^{2i\pi F(\theta_i)})(1 - e^{-2i\pi F(\theta_i)})} \times \prod_{k=1}^{n} \Gamma \left( \frac{1}{2} + \frac{\theta_i - v_k}{2i} \right) \Gamma \left( \frac{1}{2} + \frac{\theta_i - u_k}{2i} \right) \Gamma \left( \frac{1}{2} + \frac{\theta_i - v_k}{2i} \right) \Gamma \left( \frac{1}{2} + \frac{\theta_i - u_k}{2i} \right), \quad (81)
\]

if we choose to keep \( \theta_i \) and \( \theta_- \) independent. This simplifies to

\[
\hat{p}^{zz} = \frac{\sin^2 \left( \pi \sum_{j=1}^{n_2} (\hat{p}(v_j) - \hat{p}(u_j)) \right)}{\sin^2 (F(\theta))}, \quad (82)
\]

for \( \theta_+ = \theta_- = \theta \).

Finally, we end up with the following form factor series for the longitudinal two-point functions in the isotropic limit

\[
\langle \sigma^z \sigma^z_{m+1} \rangle = \sum_{n=1}^{\infty} \frac{1}{\Omega(n)^2} \int_{\mathbb{R}_+} \frac{du}{(2\pi)^n} \int_{\mathbb{R}_+} \frac{dv}{(2\pi)^n} e^{-2\pi i m \sum_{j=1}^{n_2} (\hat{p}(u_j) - \hat{p}(v_j))} \times \hat{A}^{zz}(\{u\}_j^{n_1}, \{v\}_j^{n_2}), \quad (83)
\]

where \( \hat{A}^{zz}(\{u\}_j^{n_1}, \{v\}_j^{n_2}) \) is defined in (80).

We believe that this series is a good starting point for studying the asymptotics of the longitudinal two-point functions at the isotropic point [1], including higher order logarithmic corrections. As far as its numerical evaluation is concerned, we are still struggling with
technical difficulties involved in the computation of integrals over infinite intervals. Here we provide a numerical estimation of the ph contributions to $\langle \sigma_1 \sigma_2^z \rangle$ at the isotropic limit based on an extrapolation from $D > 1$.

Figure 10 shows the contributions of the 1-, 2- and 3-ph excitations to $\langle \sigma_1 \sigma_2^z \rangle$ as functions of $\Delta$. When $D < 1.02$ we encounter problems with numerical convergence of the 2- and 3-ph approximations. Extrapolating from $D > 1.02$ to the isotropic point we obtain about 95% (up to 2-ph) and 98% (up to 3-ph) of the exact value $\langle \sigma_1 \sigma_2^z \rangle = 1/3 - 4 \ln(2)/3$, which seems consistent with the fact that the 4-spinon contribution to the dynamic structure factor of the isotropic Heisenberg chain saturates a frequency sum rule to 97% [10]. The extrapolation is justified as the $n$-ph approximation ($n \leq 3$) is a continuous functions of $\Delta$ and the limit $D \to 1+$ exists and is finite as we have seen above.

For future record we supplement an estimate of $\langle \sigma_1 \sigma_2^z \rangle$ obtained by extrapolation of the data for $D > 1$ to the isotropic point: 98% (up to $n_h = 3$) and 99% (up to $n_h = 4$) of the exact value. This seems consistent with the above result.

5. Conclusions

We have derived novel form factor series representations for the ground state two-point correlation function of the XXZ chain in the antiferromagnetic massive regime and of the XXX chain at vanishing magnetic field. These were obtained within the algebraic Bethe Ansatz approach applied to the quantum transfer matrix and are based on our previous work [16] where we analysed the spectrum of the quantum transfer matrix in the antiferromagnetic massive regime. Our novel series are manifestly different from the form factor series obtained within the $q$-vertex operator approach [22] or within the algebraic Bethe Ansatz approach applied to the ordinary transfer matrix [17].

The novel series representations come with a different underlying picture of elementary excitations. As we have argued in [16] the spectrum of correlation lengths of the quantum transfer matrix can be entirely classified in terms of particle–hole excitations. By contrast, the excitations of the ordinary transfer matrix of the XXZ chain in the antiferromagnetic massive...
regime are parameterised by pairs of hole-rapidities interpreted in terms of spinons. Within the algebraic Bethe Ansatz approach a complete characterisation of the corresponding Bethe root patterns involve the solution of a set of transcendental equations, the higher-level Bethe Ansatz equations, which, for any given set of spinon rapidities, determines a set of associated non-real Bethe roots \([2, 17, 47, 49]\). In the thermodynamic limit the form factors still depend on these roots, which makes the summation rather involved and is the reason for the appearance of higher dimensional residues in the description of the form factor densities in the thermodynamic limit \([17]\). In this context the form factor series derived above may be interpreted as the result of a resummation of the contributions from the non-real Bethe roots. To further support such interpretation it would be important to prove our conjecture that the spinon amplitudes can be obtained from the particle–hole amplitudes by the symmetrisation procedure suggested in equation (53).

Our preliminary attempts also suggest that the novel form factor series may turn out to be more efficient in the actual numerical calculation of at least the static correlation functions at any distance\(^9\). This seems to be an implication of our computation of the 3-ph contribution to the two-point functions. We further expect from the specific form of the series that they will turn out to be useful for the calculation of the large-distance asymptotics, in particular also in the isotropic limit. We plan to further dwell upon this issue in our future work.

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Appendix A. Quantum transfer matrix and thermal form factors

In order to make this work more self-contained we review some of our previous results on thermal form factors \([13]\) and on the low-temperature spectrum of correlation lengths \([16]\), adapting the notation to the antiferromagnetic massive regime where necessary.

A.1. Quantum transfer matrix approach to correlation functions

A quantum transfer matrix approach for the calculation of temperature dependent correlation functions of Yang–Baxter integrable quantum chains was devised in \([20]\). Its basic input is the \(R\)-matrix of the underlying vertex model. For the XXZ-chain the relevant vertex model is the six-vertex model with \(R\)-matrix

\[ R \]

\(^9\) It might be possible to obtain numerically more efficient expressions for multi-spinon form factors within the \(q\)-vertex operator approach as well (F Smirnov, private communication).
\[ R(x, y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(x, y) & c(x, y) & 0 \\ 0 & c(x, y) & b(x, y) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad b(x, y) = \frac{\sin(y-x)}{\sin(y-x + i\gamma)}, \quad c(x, y) = \frac{\sin(i\gamma)}{\sin(y-x + i\gamma)}. \] (A.1)

The \( R \)-matrix can be used to define the statistical operator in the canonical ensemble, \( e^{-H/T} \), which is needed to calculate thermal expectation values. For this purpose we first associate a staggered monodromy matrix with every site \( j \in \{-L + 1, \ldots, L\} \) of the XXZ chain,

\[ T_j(x|\kappa) = q^{\alpha j^2} R_{\kappa N} \left( x, \frac{i\beta}{N} \right) R_{\kappa}^1 \left( x, \frac{-i\beta}{N} \right) \cdots R_{\kappa}^1 \left( x, \frac{i\beta}{N} \right) R_{\kappa}^1 \left( x, \frac{-i\beta}{N} \right). \] (A.2)

Here \( N \in 2\mathbb{N} \) is called the ‘Trotter number’, the indices \( j = 1, \ldots, N \) refer to \( N \) auxiliary sites in ‘Trotter direction’, and ‘\( t \)’ means transposition with respect to the first space. \( R \) is acting on. The parameters

\[ \beta = -\frac{2Jsh(\gamma)}{T}, \quad \kappa = -\frac{\hbar}{2\gamma T} \] (A.3)

are rescaled inverse temperature and magnetic field. Defining

\[ \rho_{N,L} = \text{Tr}_1 \ldots \pi \{ T_{-L+1}(0|\kappa) \cdots T_L(0|\kappa) \} \] (A.4)

it is easy to see [20] that

\[ e^{-H/T} = \lim_{N \to \infty} \rho_{N,L}. \] (A.5)

We call \( \rho_{N,L} \) a finite Trotter number approximant to the statistical operator. Using \( \rho_{N,L} \) we can calculate approximations to thermal expectation values which become exact in the limit \( N \to \infty \). In particular, the expectation value of any product of local operators \( \mathcal{O}_j^{(1)} \in \text{End} \mathbb{C}^2, j = 1, \ldots, m + 1, m \in \mathbb{N} \), acting on \( m + 1 \) consecutive sites of the infinite chain, is approximated by

\[ \langle \mathcal{O}_1^{(1)} \cdots \mathcal{O}_{m+1}^{(m+1)} \rangle_k = \lim_{L \to \infty} \frac{\text{Tr}_{-L+1 \ldots L} \{ \rho_{N,L} \mathcal{O}_1^{(1)} \cdots \mathcal{O}_{m+1}^{(m+1)} \}}{\text{Tr}_{-L+1 \ldots L} (\rho_{N,L})} = \frac{\langle \kappa | \text{Tr} \{ \mathcal{O}_1^{(1)} T(0|\kappa) \} \cdots \text{Tr} \{ \mathcal{O}_{m+1}^{(m+1)} T(0|\kappa) \} | \kappa \rangle}{\langle \kappa | \kappa \rangle \Lambda^{m+1}(0|\kappa)}, \] (A.6)

where \( \Lambda(0|\kappa) = \Lambda_0(0|\kappa) \) is the unique eigenvalue of largest modulus of the quantum transfer matrix \( t(\lambda|\kappa) = \text{Tr} T(\lambda|\kappa) \) at \( \lambda = 0 \), and where \(| \kappa \rangle = |0; \kappa \rangle \) is the corresponding eigenvector (see [20] for more details). We call \( \Lambda(0|\kappa) \) the dominant eigenvalue and \(| \kappa \rangle \) the dominant eigenstate. All other states will be called ‘excited states’. Below we shall be dealing with sequences of excited states and their eigenvalues which will be denoted somewhat unspecifically \(| \eta; \kappa \rangle \) and \( \Lambda_\eta(\lambda|\kappa) \), respectively.

### A.2. Thermal form factor expansion

An important class of correlation functions are \( \alpha \)-twisted two-point functions for which \( \mathcal{O}_1^{(1)} = X_0, \mathcal{O}_{m+1}^{(m+1)} = Y_{m+1} \) and \( \mathcal{O}_j^{(j)} = q^{\alpha j} \) for \( j = 2, \ldots, m \). Expanding the right-hand side of (A.6) in a basis of eigenstates of the \( \alpha \)-twisted quantum transfer matrix \( t(\lambda|\kappa + \alpha) \) we obtain the ‘form factor expansion’
\[
\langle X_1 q^n \sum_{j=2}^{s} \sigma_j^z Y_{m+1} \rangle_N \\
= \sum_{\kappa} \left( \lambda_n(0) \kappa' \right) \left( \Lambda_n(0) \kappa \right) \left( \frac{\lambda_n(0) \kappa'}{\lambda_n(0) \kappa} \right)^m,
\]

where \( \kappa' = \kappa + \alpha \). Sending \( \alpha \to 0 \) and \( N \to \infty \) we obtain the two-point functions \( \langle X_1 Y_{m+1} \rangle \).

Due to the symmetries of the Hamiltonian (1) there are only two independent proper two-point functions, \( \langle \sigma_1^z \sigma_{m+1}^z \rangle \) and \( \langle \sigma_1^x \sigma_{m+1}^x \rangle \), say. For this reason we may restrict ourselves to the cases \( X = \sigma^-, Y = \sigma^+ \) and \( X = Y = \sigma^z \) in (A.7). Note that

\[
\left[ T_j(\chi[\kappa]), \frac{1}{2} \sigma^z_j + \eta^z \right] = 0,
\]

where \( \eta^z \) is the pseudo spin operator \( \eta^z = \frac{1}{2} \sum_{k=1}^{N} (\sigma^z)_k \). Equation (A.8) implies that the quantum transfer matrix preserves the pseudo spin. Hence, all eigenstates \( |n; \kappa \rangle \) define definite pseudo spin, \( \eta^z |n; \kappa \rangle = s |n; \kappa \rangle \), \( s = -N/2, \ldots, N/2 \). Furthermore, \( \text{Tr} [X T(0) | \kappa \rangle \} \) changes the pseudo spin by \( s \), if \( \frac{1}{2} \sigma^z, X \rangle = sX \). Hence, for the transversal case \( X = \sigma^- \) and \( Y = \sigma^+ \) the non-vanishing part of the sum over \( n \) in (A.7) is over all states with \( s = 1 \), while in the longitudinal case \( X = Y = \sigma^z \) the sum runs over all states with \( s = 0 \).

For finite Trotter number \( N \) the eigenvalues \( \Lambda_n(x|\kappa) \) and eigenstates \( |n; \kappa \rangle \) of the quantum transfer matrix are parameterised by sets \( \{x_j \}_j^{M}, M = N/2 - s \), of so-called Bethe roots. These are defined with the aid of an auxiliary function

\[
a(x) = a(x|\{x_j \}_j^{M}) = q^{2\pi} \left[ \frac{\sin \left( x + i \gamma \right)}{\sin \left( x - i \gamma \right)} \frac{\sin \left( x + i \gamma \right)}{\sin \left( x - i \gamma \right)} \right]^{M} \prod_{k=1}^{N} \sin \left( x - x_k^j - i \gamma \right) \sin \left( x - x_k^j + i \gamma \right) = -1, \quad j = 1, \ldots, M.
\]

Since every solution corresponds to a state label \( \langle n; \kappa \rangle \) we write in the following \( a_n(x|\kappa) \) instead of \( a(x|\{x_j \}_j^{M}) \) if \( \{x_j \}_j^{M} \) satisfies (A.10).

Any auxiliary function \( a_n(x|\kappa) \) associated with a set of Bethe roots satisfies a nonlinear integral equation [16, 31]. This fact allows one to identify auxiliary functions associated with the dominant state and the ‘low-lying excited states’ of the quantum transfer matrix in the Trotter limit. Furthermore, it is known for long [31] how to write the corresponding eigenvalues as integrals involving the auxiliary functions. Using such type of integral representations it is easy to obtain the eigenvalue ratios

\[
\rho_n(x|\alpha) = \frac{\Lambda_n(x + i \gamma / 2 | \kappa')}{\Lambda_n(x + i \gamma / 2 | \kappa)} \quad \text{(A.11)}
\]

in the Trotter limit. For the XXZ chain in the antiferromagnetic massive regime see [16], where also the explicit expressions (14) for the eigenvalue ratios

\[
\rho_n = \rho_n(-i \gamma / 2 | \kappa) \quad \text{(A.12)}
\]

in the low-temperature limit were obtained.
In this work we study the amplitudes
\[ A_n^{xy} (\xi | \alpha) = \frac{\langle \xi | \text{Tr} \{ X T (\xi | \kappa) \} | n; \kappa' \rangle \langle n, \kappa' | \text{Tr} \{ Y T (\xi | \kappa) \} | \kappa \rangle}{\Lambda_n (\xi | \kappa') (\kappa | \kappa)} \]
\[ \Lambda(\xi | \kappa) (n, \kappa' | n; \kappa') \] (A.13)
in the form factor expansion (A.7) of the two-point functions of the XXZ chain in the antiferromagnetic massive regime in the Trotter limit at low temperatures. Here, we adopt the convention that
\[ X = \sigma^x \text{ if } x = z, \quad X = \sigma^\pm \text{ if } x = \pm, \quad X = q^{\alpha \sigma} \text{ if } x = \alpha, \quad X = \text{id} \text{ if } x = 1 \]
and similarly for \( Y \) and \( y \). We derive explicit expressions for
\[ A_n^{zz} = \lim_{\alpha \to 0_N \to \infty} A_n^{C_0} (0,0), \quad A_n^{-+} = \lim_{\alpha \to 0_N \to \infty} A_n^{-+} (0,0). \] (A.14)
In the longitudinal case we utilise the generating function
\[ A_n^{a1} (0,0) = \langle \kappa | n; \kappa' \rangle \langle n, \kappa' | \kappa \rangle \] (A.15)
which seems to be more convenient than working directly with \( A_n^{C_0} (0,0) \). Setting \( X = Y = q^{\alpha \sigma} \) in (A.7) and acting with the operator \( D_m f_m = f_m - f_{m-1} \), it easy to see that
\[ A_n^{zz} = \lim_{N \to \infty} \frac{1}{2} \left[ \frac{1}{n^2} - \rho_n^{-1/2} \partial_n^2 A_n^{a1} (0,0) \right]_{n=0}. \] (A.16)

In [13] we considered \( A_n^{a1} (\xi | \alpha) \) and \( A_n^{-+} (\xi | \alpha) \) for finite Trotter number and in the Trotter limit. We observed that in both cases the amplitudes consist of three factors
\[ A_n^{xy} (\xi | \alpha) = U_{n,s} (\alpha) D_n^{xy} (\alpha) F_n^{xy} (\xi | \alpha), \] (A.17)
the universal part \( U_{n,s} (\alpha) \), the determinant part \( D_n^{xy} (\alpha) \) and the factorising part \( F_n^{xy} (\xi | \alpha) \). The universal part \( U_{n,s} (\alpha) \) does not depend on the details of the operators \( X, Y \) in (A.13), but only on the spin. Its expression in terms of Bethe roots \( \{ x_j \}_{j=1}^{N/2} \) of the dominant state and \( \{ y_j \}_{j=1}^{N/2-s} \) of an excited state of spin \( s \) takes the form
\[ U_{n,s} (\alpha) = \frac{\prod_{j=1}^{N/2} \rho_n \left( x_j \right) \prod_{j=1}^{N/2-s} \rho_n \left( y_j \right) }{\prod_{j=1}^{N/2} \rho_n \left( x_j \right) \prod_{j=1}^{N/2-s} \rho_n \left( y_j \right) }. \] (A.18)

The determinant part consists of four determinants
\[ D_n^{xy} (\alpha) = \frac{\det_{N/2} \left\{ \delta^x_j + \frac{\rho_n \left( x_j \right) \alpha}{a_0 \left( x_j \right) \alpha} U^x (x_j, x_j) \right\} }{\det_{N/2} \left\{ \delta^x_j + \frac{1}{a_0 \left( x_j \right) \alpha} K \left( x_j - x_j \right) \right\} \det_{N/2} \left\{ \delta^x_j + \frac{1}{a_0 \left( y_j \right) \alpha} K \left( y_j - y_j \right) \right\} \det_{N/2} \left\{ \delta^x_j + \frac{1}{a_0 \left( y_j \right) \alpha} K \left( y_j - y_j \right) \right\} \det_{N/2} \left\{ \delta^x_j + \frac{1}{a_0 \left( x_j \right) \alpha} K \left( x_j - x_j \right) \right\}. \] (A.19)
the operators $X$, $Y$ under consideration

$$U^\phi(x, y) = 2\pi i K_{\pm \phi}(x - y),$$

while

$$U^\phi(x, y) = 2\pi i K_{\phi}(x - y) + iq^{-\alpha} - iq^\alpha, \quad (A.21a)$$

$$U^\phi(x, y) = 2\pi i K_{\phi}(x - y) - iq^{-\alpha} + iq^\alpha. \quad (A.21b)$$

In the longitudinal case the factorising part is simply

$$F_n^{\pm}(\xi(\alpha)) = 1. \quad (A.22)$$

In the transversal case the factorising part is of the form

$$F_n^{\pm}(\xi(\alpha)) = \frac{G^+_n(\xi)G^-_n(\xi)}{(q^{\alpha-1} - q^{1-\alpha})(q^{\alpha} - q^{-\alpha})}, \quad (A.23)$$

where the functions in the numerator are determined by linear integral equations [13]. We describe these functions below in appendix D after having introduced some more notation that is useful for taking the zero temperature limit.

### A.4. Low-temperature limit of auxiliary function and eigenvalue ratio

As can be seen from the previous section we need to know the low-temperature behaviour of the auxiliary functions $a_n(\xi|\kappa)$ and of the eigenvalue ratios $\rho_n^{\phi}(\xi|\alpha)$ in order to calculate the amplitudes in the form factor expansion of the two-point function for $T \to 0^+$. This low-temperature behaviour was obtained in [16].

After taking the Trotter limit the auxiliary functions at small temperatures become

$$a_n(x|\kappa) = a(x|\{x\}_{j=1}^{n_1}, \{y\}_{j=1}^{n_2}|k) = (-1)^k e^{-\frac{\pi x}{K} + \sum_{i=1}^{n_2} r^{(1, 2)} - \sum_{i=1}^{n_1} r^{(1, 1)}}. \quad (A.24)$$

Here $\varepsilon$ and $\varphi$ are the dressed energy and the dressed phase defined in (6) and (8) in the main body of the text. The number $k \in \{0, 1\}$ and the two sets of 'particles' $\{y\}_{j=1}^{n_2}$ and 'holes' $\{x\}_{j=1}^{n_1}$ parameterise all excited states. For given $k$ the latter are determined by the 'higher-level Bethe Ansatz equations'

$$a(x_n|\{x\}_{j=1}^{n_1}, \{y\}_{j=1}^{n_2}|k) = -1, \quad a(y_m|\{x\}_{j=1}^{n_1}, \{y\}_{j=1}^{n_2}|k) = -1, \quad (A.25)$$

where $\text{Im } x_n < 0$, $n = 1, \ldots, n_1$, and $\text{Im } y_m > 0$, $m = 1, \ldots, n_2$. Equations (A.25) are equivalent to equations (11) in the main text. They determine the particles and holes up to the order $T$. Corrections are of order $T^\infty$. The auxiliary functions depend on $\kappa$ through $\varepsilon$ and through the particle and hole parameters. Multiplicative temperature corrections to (A.24) are uniformly of the form $1 + O(T^\infty)$ inside the strip $-\gamma < \text{Im } x < \gamma$ away from the line $\text{Re } \varepsilon(x) = h, -\gamma < \text{Im } x \leq -\gamma/2$.

Using the low-temperature formula for the eigenvalues $\Lambda_n(x|\kappa)$ obtained in [16] we see that the eigenvalue ratios behave as

$$\rho_n(\xi|\alpha) = \rho_n^{\phi}(\xi|\alpha) \times \begin{cases} 1 & -\gamma < \text{Im } x < 0, \\ \frac{1 + a_n(x|\kappa)}{1 + a_n(x|\kappa)} & \text{Im } x > 0, \end{cases} \quad (A.26)$$
where
\[
\rho_n^{(0)}(x|\alpha) = (-1)^k \exp \{ (i\pi k - \alpha\gamma) \ln_{x>0} \} 
\]
\[
\times \left( \frac{\cos(x + i\gamma)}{\cos(x)} \right)^{\frac{n_e}{2}} \prod_{j=1}^{n_e} \frac{\sin(x - y_j + i\gamma)}{\sin(x - y_j)} \left| \prod_{j=1}^{n_h} \frac{\sin(x - x_j)}{\sin(x - x_j + i\gamma)} \right| 
\]
\[
\times \exp \left\{ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dy \, K(x - y + i\gamma/2)[\gamma/2] \left[ \sum_{k=1}^{n_e} \varphi(y, y_k) - \sum_{k=1}^{n_h} \varphi(y, x_k) \right] \right\}. 
\]
\[
(A.27)
\]

Here
\[
K(x|\delta) = \frac{1}{2\pi i} (\cotg (x - i\delta) - \cotg (x + i\delta)) 
\]
by definition, and
\[
I_{\text{condition}} = \begin{cases} 
1 & \text{if condition is satisfied} \\
0 & \text{else.} 
\end{cases} 
\]
\[
(A.29)
\]

As before \((A.26)\) and \((A.27)\) hold up to multiplicative corrections of the form \(1 + \mathcal{O}(T^\infty)\) inside the strip \(-\gamma < \text{Im} \, x < \gamma\) away from the line \(\text{Re} \, \varepsilon(x) = h, -\gamma < \text{Im} \, x \leq -\gamma/2\).

### Appendix B. Low-temperature limit of the universal part

In this appendix we use \((A.24)\) and \((A.26)\), \((A.27)\) to calculate the universal part \((A.18)\) of the amplitudes in the Trotter limit at low temperature.

Step 1. Universal part expressed by a contour integral.

Using that the Bethe roots \(\{x_j^\prime\}\) of the dominant state in \((A.18)\) are simple zeros of the function \(1 + a_0(x|\kappa)\) and that the Bethe roots \(\{y_j\}\) of the excited states in \((A.18)\) are simple zeros of \(1 + a_n(x|\kappa)\), we may rewrite \((A.18)\) as
\[
U_{n,\alpha}(x) = \left[ \prod_{j=1}^{n_e} \frac{1}{\rho_n(y_j|\alpha)} \right] \exp \left\{ \int_{C} \frac{dy}{2\pi i} \ln(\rho_n(y|\alpha)) \partial_y \ln \left( \frac{1 + a_0(y|\kappa)}{1 + a_n(y|\kappa)} \right) \right\}. 
\]
\[
(B.1)
\]
Here the contour \(C\), sketched in figure B1, encircles all Bethe roots of the dominant state as well as all Bethe roots with negative imaginary part of the excited state, while the Bethe roots of the excited state which have positive imaginary part and all other singularities of the integrand are outside \(C\). We assume that the temperature is low enough for the general low-temperature picture developed in [16] to hold true. In [17] we found that, for \(T \to 0^+\), all Bethe roots condense to the curves \(B_s\) determined by \(\text{Re} \, \varepsilon(x) = 0, 0 < \pm \text{Im} \, x < \gamma\). These curves are sketched in figure 2 in the main text. The Bethe roots of the dominant state all condense to \(B_d\). In the Trotter limit the excited states have infinitely many Bethe roots located on \(B_d\) and only finitely many on \(B_s\). The latter were called close roots or particles in [16]. We denote them by \(y_j, j = 1, \ldots, n_p\). We define a rectangular contour \(C\) starting at \(-\pi/2\) and joining the points \(-\pi/2, -\pi/2 - i\gamma/2, \pi/2 - i\gamma/2, \text{and } \pi/2\) in a counterclockwise manner\(^{10}\). Then \(B_s\) and hence all Bethe roots of the dominant state and all Bethe roots with negative real part of the excited states are located inside \(C\). The only other singularities of the integrand inside \(C\) are a finite number of \(n_h\) zeros of \(1 + a_n(x|\kappa)\) which are also zeros of

\(^{10}\) This contour is different from the contour \(C\) in [16] as it is only half as wide. This choice turns out to be more suited for the analysis of the \(T \to 0^+\) limit of the form factors.
They were called holes in [16]. We denote them by \( x_j, j = 1, \ldots, n_h \). The holes are excluded from \( \mathcal{C} \) by construction. We can achieve the exclusion by adding contours \( \mathcal{C}_j \) to \( \mathcal{C} \) starting at \( p_{-2} \) going straight to \( x_j \), going around it in a small circle and going straight back to \( p_{-2} \) (see figure B1).

Step 2. ‘Straightening the contour’.

We now perform the integrals over the contours \( \mathcal{C}_j \) (see figure B1) and integrate partially in the integral over \( \mathcal{C} \). Then some care is necessary with the definition of the logarithms. Following [16] we define for any point \( x \) on \( \mathcal{C} \) and \( f = a, 1 + a, 1 + a^{-1}, \) where \( a = a_0(\cdot|\kappa) \) or \( a = a_0(\cdot|\kappa') \).

\[
\ln f(x) = \int_{\mathcal{C}_x} dy \, \partial_x \ln f(y). \tag{B.2}
\]

Here \( \mathcal{C}_x \) is the simple contour which starts at \(-\pi/2 - i0 \) and runs along \( \mathcal{C} \) up to the point \( x \). The function \( \ln f \) is holomorphic along \( \mathcal{C} \) by construction and can be used in partial integration.

For the integral over \( \mathcal{C} \) a partial integration results in

\[
\exp \left\{ \int_{\mathcal{C}_x} \frac{dy}{2\pi i} \ln(\rho(y|\kappa)) \partial_y \ln \left( \frac{1 + a_0(y|\kappa)}{1 + a_0(y|\kappa')} \right) \right\} = \rho_n(-\pi/2|\kappa)^{-s} \exp \left\{ -\int_{\mathcal{C}_x} \frac{dy}{2\pi i} \frac{\rho'_n(y|\kappa)}{\rho_n(y|\kappa)} \ln \left( \frac{1 + a_0(y|\kappa)}{1 + a_0(y|\kappa')} \right) \right\}. \tag{B.3}
\]

The integrals over the \( C_j \) can be calculated as follows

\[
\begin{align*}
\exp \left\{ \int_{\mathcal{C}_j} \frac{dy}{2\pi i} \ln(\rho_j(y|\kappa)) \partial_y \ln \left( \frac{1 + a_0(y|\kappa)}{1 + a_0(y|\kappa')} \right) \right\} &= \exp \left\{ \int_{\mathcal{C}_j} \frac{dy}{2\pi i} \ln \left( \frac{\rho_j(y|\kappa)}{y - x_j} \right) + \ln(y - x_j) \right\} \\
&\times \left[ \partial_y \ln \left( \frac{1 + a_0(y|\kappa)}{1 + a_0(y|\kappa')} \right) - \frac{1}{y - x_j} \right] \\
&= -\frac{1 + a_0(-\pi/2|\kappa') \rho'_n(x_j|\kappa)(1 + a_0(x_j|\kappa))}{1 + a_0(-\pi/2|\kappa')} \frac{a'_0(x_j|\kappa')}{a_0(x_j|\kappa)}. \tag{B.4}
\end{align*}
\]

**Figure B1.** The contour \( \mathcal{C} = \mathcal{C} + \sum_{j=1}^{n_h} \mathcal{C}_j \). Here for \( n_h = 2 \). The branch cuts of \( \ln(\rho_n(y|\kappa)) \) originating from \( x_j, j = 1, \ldots, n_h \), leave the contour \( \mathcal{C} \) at \(-\pi/2 \).
Here the first logarithms in the square brackets under the second integral are holomorphic inside \( \mathcal{C}_j \). The second logarithm in the first square bracket is defined with a branch cut originating from \( x_j \) and going through \(-\pi/2\). For the second equation see appendix E. Inserting (B.3) and (B.4) into (B.1) we obtain

\[
U_{n,\pi}(x) = (-1)^n \sum_{j=1}^n \frac{1}{\rho_h(y_j|\alpha)} \prod_{j=1}^n \frac{\rho_n'(y_j|\alpha) (1 + \alpha_0(y_j|\kappa))}{\alpha_n'(y_j|\kappa')} \rho_h(-\pi/2|\alpha)^{-s} \left( \frac{1 + \alpha_n(-\pi/2|\kappa')}{1 + \alpha_0(-\pi/2|\kappa)} \right)^n \exp \left\{ -\int_{C^-} \frac{dy}{2\pi i} \rho_n(y|\alpha) \ln \left( \frac{1 + \alpha_0(y|\kappa)}{1 + \alpha_n(y|\kappa')} \right) \right\}.
\]  

(B.5)

The next steps now consist of inserting the low-temperature expressions (A.24) and (A.26) into the various terms on the right-hand side of this equation.

Step 3. Low-temperature limit of the integral term and replacing \( \rho_h(\cdot|\alpha) \) by its low-temperature limit.

Following essentially the same reasoning as in equations (31)–(35) of our paper [16] and using that

\[
\rho_n(y|\alpha) = \lim_{\kappa \to y} \rho_n^{(0)}(y|\alpha) \frac{1 + \alpha_n(y|\kappa')}{1 + \alpha_0(y|\kappa)} = \text{res} \{ \rho_n^{(0)}(y|\alpha) \} \frac{\alpha_n'(y|\kappa')}{1 + \alpha_0(y|\kappa)},
\]  

(B.8)

up to multiplicative corrections of the form \( 1 + O(T^\infty) \). Using (A.26) we further see that

\[
\rho_n(y|\alpha) = \lim_{\kappa \to y} \rho_n^{(0)}(y|\alpha) \frac{1 + \alpha_n(y|\kappa')}{1 + \alpha_0(y|\kappa)} = \text{res} \{ \rho_n^{(0)}(y|\alpha) \} \frac{\alpha_n'(y|\kappa')}{1 + \alpha_0(y|\kappa)},
\]  

(B.8)

up to multiplicative corrections of the form \( 1 + O(T^\infty) \). Equation (B.6) also implies the identity

\[
\int_{-\pi/2}^{\pi/2} \frac{dy}{2\pi i} (\partial_{\kappa} \ln (\rho_n^{(0)}(y|\alpha))) 2i y s = s \ln (\rho_n^{(0)}(-\pi/2|\alpha)) - i \pi s (n_p + s) - s \int_{-\pi/2}^{\pi/2} \frac{dy}{\pi} \ln (\rho_n^{(0)}(y|\alpha)).
\]  

(B.9)

Inserting (B.7)–(B.9) and (A.25) into (B.5) we obtain the following low-temperature expression for the universal part of the amplitudes
which is again valid up to multiplicative corrections of the form $1 + O(T^\infty)$.

Step 4. Inserting explicit expressions, evaluating remaining integrals.

If we insert (A.24) and (A.27) into equation (B.10) we obtain an expression containing explicit functions and integrals over explicit functions. The only slightly cumbersome task that remains is to calculate these integrals. This can be done in various ways. One way is to use Fourier series representations and the convolution theorem for Fourier series. We have gathered some formulae needed in that case in appendix E. Before presenting the final formulae we give a few intermediate results.

First of all, using (E.7)

\[ \exp \left\{ -s \int_{-\pi/2}^{\pi/2} \frac{dy}{\pi} \ln (\rho_n^{(0)}(y|\alpha)) \right\} = (-1)^{k+n_\alpha+n_\gamma} q^{-s^2} \exp \left\{ -is \left( \sum_{j=1}^{n_\gamma} y_j + \sum_{j=1}^{n_\alpha} y_j \right) \right\} \quad \text{(B.11)} \]

Next, replacing $\ln (\alpha_0(y|\kappa)) - \ln (\alpha_n(y|\kappa')) - 2i\gamma$ by its low-$T$ limit (A.24), introducing the ‘periodic form of the dressed phase’

\[ \varphi_p(x, z) = \varphi(x, z) - i(\pi/2 + x - z) \quad \text{(B.12)} \]

and using (B.6) we obtain

\[ \exp \left\{ \int_{-\pi/2}^{\pi/2} \frac{dy}{2\pi i} (\partial_y \ln (\rho_n^{(0)}(y|\alpha))) [\ln (\alpha_0(y|\kappa)) - \ln (\alpha_n(y|\kappa')) - 2i\gamma] \right\} = (-1)^{k+n_\alpha+n_\gamma} q^{-s^2} \exp \left\{ i(n_\alpha + s) \left( \sum_{j=1}^{n_\gamma} y_j - \sum_{j=1}^{n_\alpha} y_j \right) \right\} \times \exp \left\{ \int_{-\pi/2}^{\pi/2} \frac{dy}{2\pi i} (\partial_y \ln (\rho_n^{(0)}(y|\alpha))) \left( \sum_{j=1}^{n_\gamma} \varphi_p(y, y_j) - \sum_{j=1}^{n_\gamma} \varphi_p(y_j, y) \right) \right\} \quad \text{(B.13)} \]

Then we insert (A.26) and (A.27) into the third factor on the right-hand side of (B.10), implying that
Moreover

\[
\prod_{j=1}^{n_p} \frac{f_{\alpha}^{(y)}(x_j|x)}{\prod_{j=1}^{n_p} \text{res} \left\{ f_{\alpha}^{(y)}(y_j|x) \right\}} \left( (-1)^{(k+\alpha)q} q^{-m \alpha} \right) \exp \left\{ 2 \pi i \sum_{j=1}^{n_p} x_j - \sum_{j=1}^{n_p} y_j \right\} \left( \frac{1}{\sin(i \gamma)} \right)^{2(q+1)} \\
= \left[ \prod_{j=1}^{n_p} \prod_{k=1}^{n_p} \frac{\sin(x_j - x_k)}{\sin(x_j - x_k + i \gamma)} \right] \left[ \prod_{j=1}^{n_p} \frac{\sin(y_j - y_k)}{\sin(y_j - y_k + i \gamma)} \right] \exp \left\{ \sum_{j=1}^{n_p} K(x_j - y + i \gamma/2|\gamma/2) - \sum_{j=1}^{n_p} K(y_j - y + i \gamma/2|\gamma/2) \right\} \\
\times \left\{ \sum_{j=1}^{n_p} \phi_p(y_j, x_j) - \sum_{j=1}^{n_p} \phi_p(x_j, y_j) \right\}.
\]

Moreover

\[
\frac{1}{2 \pi i} \partial_{\gamma} \ln \left( \rho_{\alpha}^{(y)}(y|x) \right) = -\frac{q}{\pi} + \sum_{j=1}^{n_p} K(y - y_j + i \gamma/2|\gamma/2) - \sum_{j=1}^{n_p} K(y - y_j + i \gamma/2|\gamma/2) \\
+ \int_{-\pi}^{\pi} \frac{dz}{2 \pi i} K(z - y + i \gamma/2|\gamma/2) \partial_{\gamma} \left[ \sum_{j=1}^{n_p} \phi_p(z, y_j) - \sum_{j=1}^{n_p} \phi_p(z, x_j) \right].
\]

All the remaining integrals can now be calculated e.g. by means of equations (E.2)–(E.6) in appendix E. This leads to the following expression for the universal part of the amplitudes expressed in terms of \( q \)-multi factorials.

\[
U_{\alpha,\alpha}(\gamma) = \left[ \prod_{j=1}^{n_p} \frac{1 - e^{-2 \pi i F(x_j)}}{\alpha_a(x_j|x')} \right] \left[ \prod_{j=1}^{n_p} \frac{1 - e^{-2 \pi i F(y_j)}}{\alpha_a(y_j|x')} \right] \\
\times \left( (-1)^{\gamma + \alpha} \right) \exp \left\{ 2 \pi i \sum_{k=1}^{n_p} x_k - 2 \pi i \sum_{k=1}^{n_p} y_k \right\} \\
\times \left[ \prod_{j=1}^{n_p} \sin(x_j) \prod_{j=1}^{n_p} \sin(y_j) \right] \\
\times \left[ \prod_{j=1}^{n_p} \prod_{k=1}^{n_p} \sin(x_j - y_k) \sin(y_j - x_k) \right] \left[ \prod_{j=1}^{n_p} (q^2 e^{2 i \gamma}; q^4) (q^4 e^{2 i \gamma}; q^4) (q^4 e^{2 i \gamma}; q^4) \left( q^{-2 i \gamma}; q^4 \right)^2 \right] \\
\times \left[ \prod_{j=1}^{n_p} (q^2 e^{2 i \gamma}; q^4) (q^4 e^{2 i \gamma}; q^4) (q^4 e^{2 i \gamma}; q^4) \left( q^{-2 i \gamma}; q^4 \right)^2 \right] \\
\times \left[ \prod_{j=1}^{n_p} \prod_{k=1}^{n_p} (q^2 e^{-2 i \gamma}; q^4) (q^4 e^{-2 i \gamma}; q^4) \left( q^{-2 i \gamma}; q^4 \right)^2 \right] \\
\times \left[ \prod_{j=1}^{n_p} \prod_{k=1}^{n_p} \prod_{\sigma = \pm 1} (q^2 e^{2 i \gamma}; q^4) (q^4 e^{2 i \gamma}; q^4) \left( q^{2 i \gamma}; q^4 \right)^2 \right].
\]
From here we arrive at equation (21) in the main body of the text if we replace systematically the $q$-multi factorials by $q$-gamma and $q$-Barnes function and the sine functions by $q$-numbers using the formulae collected in appendix F.

**Appendix C. Low-temperature limit of the determinant part**

In the Trotter limit the determinants in (A.19) turn into Fredholm determinants of linear integral operators defined by their kernels and by certain contours and ‘measures’ (or ‘weight functions’). All of this was described in some detail in [13]. Here we have to adapt the notation to the antiferromagnetic massive regime.

**C.1. Determinants in the numerator**

We begin our discussion with the determinants in the numerator in (A.19). Let

$$\begin{align*}
\text{det} & = \frac{\prod_{k=1}^{n} (1 + a_0(x|\kappa'))}{1 - \frac{a_0(x|\kappa)}{a_0(x|\kappa')}}.
\end{align*}$$

(C.1)

The function $\rho_n(x|\kappa')(1 + a_0(x|\kappa'))/(1 + a_n(x|\kappa'))$ is meromorphic inside the strip $-\gamma/2 < \text{Im} x < \gamma/2$, where the Bethe roots are located for low enough temperature. Its only poles and zeros inside this strip are simple poles at the Bethe roots of the excited state $|n; \kappa\rangle$ and simple zeros at the Bethe roots of the dominant state $|\kappa\rangle$. This becomes clear when we write this function explicitly in terms of products over Bethe roots. Let us now define a simple closed contour $\Gamma_n$ located inside the strip $-\gamma/2 < \text{Im} x < \gamma/2$ and encircling all Bethe roots of the states $|n; \kappa\rangle$ and $|\kappa\rangle$ but none of the possible zeros of $1 - a_0(x|\kappa)/a_n(x|\kappa')$. Then, for any function $f$ which is holomorphic inside the strip $-\gamma/2 < \text{Im} x < \gamma/2$

$$\int_{\Gamma_n} \frac{dM^\sigma(x)}{2\pi i} f(x) = \begin{cases} 
\sum_{j=1}^{N/2} \frac{f(x_j^+)}{\int_{\Gamma_n} \rho_0(x_j^+|\kappa)} & \text{if } \sigma = -, \\
\sum_{j=1}^{N/2} \frac{f(x_j^-)}{\int_{\Gamma_n} \rho_0(x_j^-|\kappa')} & \text{if } \sigma = +.
\end{cases}$$

(C.2)

With these formulae we can interpret the determinants in the numerator of (A.19) as Fredholm determinants with kernels $\mathcal{U}^{\sigma,j}$, contour $\Gamma_n$ and measures $dM^\pm$ (see [13]), and we can take the Trotter limit.

The temperature dependence of the Fredholm determinants comes from the measures $dM^\pm$. We will consider these measures in the low-temperature limit. The contour $\Gamma_n$ sketched in figure C1 consists of an upper part with $\text{Im} x > 0$, a lower part with $\text{Im} x = -\gamma/2$ and a left and a right part connecting $\pm\pi/2$ with $\mp\pi/2 - i\gamma/2$. The contributions to the Fredholm determinants stemming from the left and right parts of the contour cancel each other due to the $\pi$-periodicity of the integrand. For this reason it suffices to consider the measures on the upper and lower part of the contour.

We start by calculating the explicit form of the function $\rho_n^{(0)}$ for $-\gamma < \text{Im} x < 0$. Inserting $\varphi_n$ equation (B.12), into (A.27) and calculating the integrals by means of the formulae in appendix E, we obtain
The last equation defines the function \( w(x) \) in the entire complex plane, which will be needed below. It can be nicely expressed in terms of Jacobian theta functions or in terms of the dressed momentum

\[
\rho_n^{(0)}(x|\alpha) = (-1)^{k+1} g^{-\alpha} \exp \left\{ -2ixx + i \sum_{k=1}^{n_k} x_k - i \sum_{k=1}^{n_k} y_k, \right. \\
\left. \prod_{k=1}^{n_k} \frac{\sin(x - x_k)}{\sin(x - y_k + i\gamma)} \right\} \\
\exp \left\{ \int_{-\pi/2}^{\pi/2} \frac{dy}{2\pi i} K(x - y + i\gamma/2, \gamma/2) \left[ \sum_{k=1}^{n_k} \varphi_p(y, y_k) - \sum_{k=1}^{n_k} \varphi_p(y, x_k) \right] \right\} \\
= (-1)^k (4q)^i \prod_{k=1}^{n_k} \sin(x - x_k) \prod_{\sigma = \pm 1} \left( \frac{q^{2i\sigma(x-x_k)} q^x}{q^{2i\sigma(x-y_k)} q^y} \right) \\
\left[ \prod_{k=1}^{n_k} \frac{1}{\sin(x - y_k)} \prod_{\sigma = \pm 1} \left( \frac{q^{2i\sigma(x-y_k)} q^x}{q^{2i\sigma(x-y_k)} q^y} \right) \right] = w(x). \tag{C.3}
\]

The function \( \rho_n^{(0)} \) has a jump discontinuity across the real axis which comes from the explicit prefactor in (A.27) and from the pole at \( y = x \) in the integration kernel. It can be calculated e.g. by ‘pulling \( x \) across the real axis’. It follows that

\[
\rho_n^{(0)}(x|\alpha) = (-1)^k \rho_n^{(0)}(x|\alpha) \left[ \sum_{k=1}^{n_k} \varphi_p(x, x_k) - \sum_{k=1}^{n_k} \varphi_p(x, y_k) \right]. \\
\tag{C.4a}
\]

Replacing \( x \) by \( -i\gamma/2 \) this reproduces equation (14) as it should be.
\( \rho_n^{(0)}(x) = \begin{cases} w(x) & \text{for } -\gamma < \text{Im } x < 0, \\ w(x)e^{-2\pi i F(x)} & \text{for } 0 < \text{Im } x < \gamma. \end{cases} \quad (C.5) \)

Combining now (A.26) and (C.5) and using the explicit low-temperature form of the auxiliary functions (A.24) we see that, up to multiplicative corrections of the form \( 1 + \mathcal{O}(T^\infty) \),

\[
dM^{\pm}(x) = \begin{cases} \frac{dx}{w^{\pm 1}(x)e^{2\pi i F(x)}} & \text{for } x \text{ on the upper part of } \Gamma_n, \\ \frac{dx}{1 - e^{2\pi i F(x)}} & \text{for } x \text{ on the lower part of } \Gamma_n. \end{cases} \quad (C.6)\]

With this we can simplify Fredholm determinants of the form \( \det_{\Gamma_n} (1 + \widehat{U}^\pm) \), where \( \widehat{U}^\pm \) are integral operators with regular kernels of the form \( U^\pm(x, y) \). First note that for any function \( f \) holomorphic on and inside \( \Gamma \)

\[
[f(1 + \widehat{U}^-)](y) = f(y) + \int_{\Gamma_n} dM^-(x) f(x)U^-(x, y) = f(y) + \sum_{j=1}^{n_0} f(x_j)v^-(x_j, y) + \int_{-\pi/2}^{\pi/2} dx f(x)V^-(x, y) + \mathcal{O}(T^\infty), \quad (C.7)
\]

where

\[
v^-(x_j, y) = \frac{2\pi i \text{res}_{w^{-1}} \{ w^{-1}(x_j)U^-(x_j, y) \}}{1 - e^{2\pi i F(x_j)}} = V^-(x, y) = w^{-1}(x)U^-(x, y). \quad (C.8)
\]

In (C.7) we have pushed the upper part of \( \Gamma_n \) down and the lower part of \( \Gamma_n \) up to the interval \([-\pi/2, \pi/2]\). Pushing up the lower part produces the sum over holes which are simple poles of \( w^{-1} \) (see (C.4a)).

Equation (C.7) shows that we may interpret \( 1 + \widehat{U}^- \) as an integral operator acting on functions supported on \([-\pi/2, \pi/2] \cup \{x_1, \ldots, x_{n_0}\} \). Its determinant is

\[
det = \begin{vmatrix} \delta(x - y) + V^-(x, x) & V^-(x, x_1) & \ldots & \ldots & V^-(x, x_{n_0}) \\ v^-(x, y) & 1 + v^-(x_1, x_1) & v^-(x_1, x_2) & \ldots & v^-(x_1, x_{n_0}) \\ \vdots & \vdots & \ldots & \ldots & \vdots \\ v^-(x_{n_0}, y) & v^-(x_{n_0}, x_1) & v^-(x_{n_0}, x_2) & \ldots & 1 + v^-(x_{n_0}, x_{n_0}) \end{vmatrix}.
\]

Here we would like to extract the Fredholm determinant corresponding to the upper left block \( 1 + \widehat{V}^- \), using the identity

\[
det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(D - CA^{-1}B) \quad (C.9)
\]

for block matrices.

The kernel of the inverse of \( 1 + \widehat{V}^- \) may be expressed with the aid of the resolvent defined by

\[
R^-(x, y) = V^-(x, y) - \int_{-\pi/2}^{\pi/2} dz R^-(x, z)V^-(z, y). \quad (C.10)
\]
Then
\[
\det_{\{\gamma, \gamma^*\}} (1 + \widetilde{U}) = \det_{\{x, [\pi/2, \pi/2]\}} (1 + \widetilde{V}) \\
\times \det_{m, n=1, \ldots, n} \left\{ \delta_{m, n} + \nu^{-}(x_{m}, x_{n}) - \int_{-\pi/2}^{\pi/2} dy \nu^{-}(x_{m}, y)R^{-}(y, x_{n}) \right\}.
\] (C.11)

up to multiplicative corrections of the form \(1 + \mathcal{O}(T^\infty)\).

A very similar reasoning may be applied to the other Fredholm determinant in the numerator, \(\det_{\{\gamma, \gamma^*\}} (1 + \widetilde{U}^+).\) Setting
\[
\nu^{+}(x, y) = \frac{2\pi i \text{res} \{w\}(y)\mathcal{U}^{+}(x, y)}{\mu \mathcal{U}^{+}(\eta, y) - 1}, \quad \mathcal{V}^{+}(x, y) = \mathcal{U}^{+}(x, y)w(y),
\] (C.12)

and introducing the resolvent kernel \(R^{+}\) as the solution of the linear integral equation
\[
R^{+}(x, y) = \mathcal{V}^{+}(x, y) - \int_{-\pi/2}^{\pi/2} dz \mathcal{V}^{+}(x, z)R^{+}(z, y)
\] (C.13)
we obtain
\[
\det_{\{\gamma, \gamma^*\}} (1 + \widetilde{U}^+) = \det_{\{x, [\pi/2, \pi/2]\}} (1 + \widetilde{V}^+) \\
\times \det_{m, n=1, \ldots, n} \left\{ \delta_{m, n} + \nu^{+}(y_{m}, y_{n}) - \int_{-\pi/2}^{\pi/2} dy R^{+}(y_{m}, y)\nu^{+}(y, y_{n}) \right\}.
\] (C.14)

which is again valid up to multiplicative corrections of the form \(1 + \mathcal{O}(T^\infty)\).

Equations (C.11) and (C.14) provide computable and efficient expressions for the determinants in the numerator of (A.19) in the Trotter limit and for low temperatures. In the longitudinal case we have to substitute
\[
\frac{2\pi i \text{res} \{w\}(y)\mathcal{U}^{+}(x, y)}{\mu \mathcal{U}^{+}(\eta, y) - 1} = \mathcal{V}^{+}(x, y)
\] (C.15)
into (C.8) and (C.12), while, in the transversal case
\[
\mathcal{U}^{\pm}(x, y) = K_{\alpha, \pm 1}(x - y).
\] (C.16)

Since we are working with the generating function in the longitudinal case, we still have to explain how to perform the derivative with respect to \(\alpha\). We can proceed the same way as in our work on the massless regime [13] using an idea going back to [24]. The idea is to extract a factor linear in \(\alpha\) from each of the determinants in the numerator.

### C.2. Extraction of \(\alpha\) in the longitudinal case

We define
\[
U^{\pm}_{\theta}(x, y) = K_{\alpha}(x - y) - K_{\alpha}(x - \theta),
\] (C.17a)
\[
U^{\pm}_{\theta}(x, y) = K_{\alpha}(x - y) - K_{\alpha}(\theta - y).
\] (C.17b)

Then
\[
\lim_{\theta \to i\infty} U^{\pm}_{\theta}(x, y) = \mathcal{U}^{\pm}(x, y).
\] (C.18)

Thus, in the longitudinal case we may substitute \(U^{\pm}_{\theta}(x, y)\) into (C.8) and (C.12) and then send \(\theta \to i\infty\).
We further define a function

\[
f(x) = \exp \left\{ \int_{C_n} \frac{dy}{2\pi i} \cotg (x - y) \ln \frac{1 + a_n(y|\kappa')}{1 + a_0(y|\kappa)} \right\},
\]

where \( C_n \) is a contour enclosing all Bethe roots of the dominant state and of the excited state as well as the point \(-i\gamma/2\), but none of the holes. The logarithm is defined along the contour as in (B.2). Following the reasoning of appendix A.3 of [24] one can show that the ratios

\[
\frac{\det \alpha_{M^+; I^+}(1 + \tilde{U}_0^+)}{q^{\alpha f^{\pm 1} + i\gamma} - q^{-\alpha f^{\pm 1} + i\gamma}}
\]

are independent of \( \theta \).

As is clear from its definition (A.15), the function \( A_n(\alpha) \) has a double zero in \( \alpha \) at \( \alpha = 0 \).

It originates from the determinants in the numerator of the determinant part. Since

\[
\lim_{\theta \to \infty} f(\theta) \text{ exists, we may conclude that}
\]

\[
\frac{1}{2} \partial^2_{\alpha} \left. \det \frac{(1 + \tilde{U}^-)}{dM^+; I^+_n} \left|_{\alpha = 0} \right. \right) = \lim_{\alpha \to 0} \frac{4 \det \alpha_{M^+; I^+}(1 + \tilde{U}^+)}{(q^\alpha - q^{-\alpha})^2} = \lim_{\alpha \to 0} \frac{\det \alpha_{M^+; I^+}(1 + \tilde{U}^+_0)}{q^{\alpha f^{\pm 1}(\theta + i\gamma)} - q^{-\alpha f^{\pm 1}(\theta - i\gamma)}} \frac{\det \alpha_{M^+; I^+}(1 + \tilde{U}^+_0)}{q^{\alpha f^{\pm 1}(\theta + i\gamma)} - q^{-\alpha f^{\pm 1}(\theta - i\gamma)}}
\]

\[
= \frac{4 \det \alpha_{M^+; I^+}(1 + \tilde{U}^-_0) \det \alpha_{M^+; I^+}(1 + \tilde{U}^+_0)}{(1 - f(\theta - i\gamma)) f(\theta - i\gamma)(1 - f(\theta + i\gamma)) f(\theta + i\gamma)} \left. \right|_{\alpha = 0}.
\]

Here we can insert

\[
\frac{f(\theta - i\gamma)}{f(\theta + i\gamma)} = \frac{a_n(\theta|\kappa')}{a_0(\theta|\kappa')},
\]

which follows from the nonlinear integral equations satisfied by the auxiliary functions [16].

We want to perform the low temperature limit in (C.21). This is now easy for the first fraction on the right-hand side. The numerator is of a form such that we can apply the formulae of the previous subsection, and for the denominator we can use (C.22) and (A.24). For the second fraction on the right-hand side we can utilise the fact that \( \theta_+ \) and \( \theta_- \) are free parameters. Choosing \( \theta_+ = \theta_- = \theta \) this fraction equals one, and in the low-temperature limit we end up with

\[
\frac{1}{2} \partial^2_{\alpha} \left. \det \frac{(1 + \tilde{U}^-)}{dM^+; I^+_n} \left|_{\alpha = 0} \right. \right) = -\frac{\det \alpha_{M^+; I^+}(1 + \tilde{U}^-_0) \det \alpha_{M^+; I^+}(1 + \tilde{U}^+_0)}{\sin^2(\pi F(\theta))} \left. \right|_{\alpha = 0},
\]

being valid up to multiplicative corrections of the form \( 1 + \mathcal{O}(T^\infty) \). For the determinants in the numerator we have to substitute (C.8), (C.11) and (C.12), (C.14) with \( U^+(x, y) = U^+_0(x, y) \).
Alternatively it is possible to keep the two free parameters $\theta_+$ and $\theta_-$ in the low-temperature limit. In that case the resulting expression looks more involved, since the factors $f(\theta_+ + i\gamma)$ and $1/f(\theta_+ + i\gamma)$ do not cancel each other anymore and have to be calculated using similar techniques as in appendix B. Here we only give the final result

\[ f(\theta + i\gamma) = \exp \left\{ \frac{1}{2} \left[ \gamma \alpha - i\pi k + i \sum_{j=1}^{n_n} (y_j - x_j) \right] \right\} \prod_{j=1}^{n_n} \frac{q^2 e^{2i(\theta - y_j)}}{(q^2 e^{2i(\theta - y_j)}; q^4)(q^2 e^{2i(\theta - y_j)}; q^4)}, \]  

(C.24)

and leave the details to the reader. Choosing $\theta_+$ and $\theta_-$ independently is sometimes advantageous, e.g. in numerical calculations.

C.3. Determinants in the denominator

With the determinants in the denominator we can proceed in a similar way as with the determinants in the numerator. We introduce a measure $\alpha_n(x|\kappa')$

\[ dm(x) = \frac{dx}{1 + \alpha_n(x|\kappa')}. \]  

(C.25)

In the Trotter limit the second determinant in the denominator of (A.19) becomes a Fredholm determinant $\det_{\text{dm},C_\ell}(1 + \bar{K})$ with measure $dm$ and with respect to a contour $C_\ell$ which includes all Bethe roots but excludes the holes of the state $|n; \kappa'\rangle$ (see [13]).

For any $\pi$-periodic function, holomorphic on and inside $C_\pi$, \begin{align*}
[ (1 + \bar{K}) f ](x) &= f(x) + \int_{C_\pi} dm(y) K_0(x - y) f(y) \\
&= f(x) + \sum_{k=1}^{n_n} \frac{2\pi i K_0(x - y_k)f(y_k)}{\alpha_\ell(y_k|\kappa')} - \sum_{k=1}^{n_n} \frac{2\pi i K_0(x - x_k)f(x_k)}{\alpha_\ell'(x_k|\kappa')} \\
&+ \int_{-\pi/2}^{\pi/2} dy K_0(x - y)f(y) + \mathcal{O}(T^\infty). \tag{C.26}
\end{align*}

This shows that we may interpret $1 + \bar{K}$ as an integral operator acting on functions supported on $[-\pi/2, \pi/2] \cup \{y_1, \ldots, y_n; x_1, \ldots, x_{n_n}\}$. The determinant of this integral operator is

\[
\delta(x - y) + K_0(x - y) = \begin{vmatrix}
K_0(y_1 - y) & 1 + \frac{2\pi i K_0(x - y_1)}{\alpha_\ell(y_1|\kappa')} & \cdots & \cdots & \frac{2\pi i K_0(x - y_{n_n})}{\alpha_\ell(x_{n_n}|\kappa')}

K_0(y_2 - y) & \frac{2\pi i K_0(x - y_2)}{\alpha_\ell(y_2|\kappa')} & \cdots & \cdots & \frac{2\pi i K_0(x - y_{n_n})}{\alpha_\ell(x_{n_n}|\kappa')}

\vdots & \vdots & \ddots & \ddots & \vdots

K_0(x_{n_n} - y) & \frac{2\pi i K_0(x_{n_n} - y_1)}{\alpha_\ell(y_1|\kappa')} & \cdots & \cdots & 1 + \frac{2\pi i K_0(x_{n_n} - y_{n_n})}{\alpha_\ell(x_{n_n}|\kappa')}
\end{vmatrix}.
\]

Now we can proceed as above. We introduce the resolvent kernel $R$ as the solution of the linear integral equation

\[ R(x - y) = K_0(x - y) - \int_{-\pi/2}^{\pi/2} dz K_0(x - z)R(z - y). \tag{C.27} \]
Then, using (C.9), we end up with
\[
\det_{\text{det}\mathcal{C}_0} (1 + \tilde{K})_{\text{det}\mathcal{C}_0} = \det_{dx,[-\pi/2,\pi/2]} (1 + \tilde{K}_0) \det_{dx,[-\pi/2,\pi/2]} (1 + \tilde{K}_0) = \left| \begin{array}{cc} \delta_{ji} + \frac{2\pi i (y_i - y_j)}{a'_0(y_j)} & -\frac{2\pi i (y_i - x_n)}{a'_0(x_n)} \\ \frac{2\pi i (y_i - y_j)}{a'_0(y_j)} & \delta_{ji} - \frac{2\pi i (y_i - x_n)}{a'_0(x_n)} \end{array} \right|, \tag{C.28}
\]
where we have neglected multiplicative corrections of the form $1 + \mathcal{O}(T^{-\infty})$. It follows from equation (A.24) that the finite determinant on the right-hand side goes to one as $T \to 0^+$. But here the multiplicative corrections are of the form $1 + \mathcal{O}(T)$. Therefore we keep the finite determinant. It nicely combines with the $\mathcal{O}(T)$ contributions of the universal part.

With the first determinant in the denominator of (A.19) we can proceed in a very similar way as above. We define
\[
dm_0(x) = \frac{dx}{1 + a_0(x|\kappa)} \tag{C.29}
\]
In the Trotter limit the first determinant in the denominator of (A.19) then becomes the Fredholm determinant $\det_{\text{det}\mathcal{C}_0} (1 + \tilde{K})$ where the contour $\mathcal{C}_0$ includes all Bethe roots of the dominant state $|\kappa\rangle$. Following the same steps as above we obtain the low-temperature asymptotic value
\[
\det_{\text{det}\mathcal{C}_0} (1 + \tilde{K}) = \det_{dx,[-\pi/2,\pi/2]} (1 + \tilde{K}_0) \tag{C.30}
\]
valid up to multiplicative corrections of the form $1 + \mathcal{O}(T^{-\infty})$. Alternatively, the Fredholm determinant on the right-hand side can be expressed in terms of the integral operator $\tilde{R}$ connected with the resolvent kernel (C.27) or in terms of $q$-factorials \[21\]
\[
\det_{dx,[-\pi/2,\pi/2]} (1 + \tilde{K}_0) = \frac{1}{\det_{dx,[-\pi/2,\pi/2]} (1 - \tilde{R})} = 2(-q^2; q^2)^2. \tag{C.31}
\]

### Appendix D. Low-temperature limit of the factorising part

In \[13\] we introduced two functions $G_+$ and $\tilde{G}_+$. which determine the factorising part of the transversal correlation functions as solutions of linear integral equations. Here we need these equations in a form which respects the notational conventions for the antiferromagnetic massive regime and is at the same time appropriate for taking the low-temperature limit. Such a form can be obtained e.g. from the linear integral equations in \[13\] by first going back to finite Trotter number and sums over Bethe roots instead of integrals, then switching to the conventions of the antiferromagnetic massive regime and finally using (C.2) to obtain integrals more appropriate for the low-temperature limit. This way we obtain the linear integral equations
\[ G_+ (x, \xi) = - \operatorname{ctg} (x - \xi) + \frac{q^{1-\alpha} \rho_\alpha (\xi | \alpha) \operatorname{ctg} (x - \xi - i \gamma)}{1 + a_\alpha (\xi | \kappa')} + \frac{q^{1+\alpha} \rho_\alpha (\xi | \alpha) \operatorname{ctg} (x - \xi + i \gamma)}{1 + a_\alpha^{-1} (\xi | \kappa')} - \int_{\Gamma_n} dM^+(y) K_{1+\alpha} (x - y) G_+ (y, \xi), \]

\[ \mathcal{G}_- (x, \xi) = - \operatorname{ctg} (x - \xi) + \frac{q^{1-\alpha} \rho_\alpha (\xi | \alpha) \operatorname{ctg} (x - \xi - i \gamma)}{\rho_\alpha (\xi | \alpha)(1 + a_0 (\xi | \kappa))} + \frac{q^{1+\alpha} \rho_\alpha (\xi | \alpha) \operatorname{ctg} (x - \xi + i \gamma)}{\rho_\alpha (\xi | \alpha)(1 + a_0^{-1} (\xi | \kappa))} - \int_{\Gamma_n} dM^- (y) K_{\alpha-1} (y - x) \mathcal{G}_- (y, \xi), \]

where, for low enough temperature, the contour \( \Gamma_n \) is the same as in figure \( C1 \) and where \( \xi \) is outside \( \Gamma_n \) with \( \operatorname{Im} \xi < -\gamma / 2 \).

The functions \( G_- (\xi) \) and \( \mathcal{G}_- (\xi) \) in equation (A.23) can be represented by means of integrals involving \( G_+ (\cdot, \xi) \) and \( \mathcal{G}_+ (\cdot, \xi) \). Starting again from the corresponding equations in [13] and proceeding in a similar way as above we obtain

\[ G_- (\xi) = 1 - \frac{q^{1-\alpha} \rho_\alpha (\xi | \alpha)}{1 + a_\alpha (\xi | \kappa')} - \frac{q^{1+\alpha} \rho_\alpha (\xi | \alpha)}{1 + a_\alpha^{-1} (\xi | \kappa')} - \left( q^{1+\alpha} - q^{1-\alpha} \right) \int_{\Gamma_n} \frac{dM^+(y)}{2 \pi i} G_+ (y, \xi), \]

\[ \mathcal{G}_- (\xi) = - 1 + \frac{q^{1-\alpha}}{\rho_\alpha (\xi | \alpha)(1 + a_0 (\xi | \kappa))} + \frac{q^{1+\alpha}}{\rho_\alpha (\xi | \alpha)(1 + a_0^{-1} (\xi | \kappa))} - \left( q^{1-\alpha} - q^{1+\alpha} \right) \int_{\Gamma_n} \frac{dM^- (y)}{2 \pi i} \mathcal{G}_- (y, \xi). \]

Using (C.6) and arguments similar to those in appendix \( C \) in order to perform the low-temperature limit we obtain equations (34a) of the main text.

**Appendix E. Some Fourier series and integrals**

Many of the integrals occurring in appendix \( B \) can be calculated using the convolution theorem for Fourier series combined with resummation.

**Lemma 1.** \textit{Convolution of Fourier series.} Given the Fourier series representations of two functions \( f, g : [-\pi/2, \pi/2] \rightarrow \mathbb{C} \),

\[ f (x) = \sum_{n \in \mathbb{Z}} f_n e^{2i\pi n x}, \quad g (x) = \sum_{n \in \mathbb{Z}} g_n e^{2i\pi n x}, \tag{E.1} \]
their convolution has the Fourier series representation

\[ \int_{-\pi/2}^{\pi/2} dy f(x - y)g(y) = \pi \sum_{n \in \mathbb{Z}} f_n g_n e^{2i\pi nx}. \] (E.2)

In appendix B one may use the Fourier series

\[ \frac{1}{2\pi i} \cotg(x) = \begin{cases} \frac{1}{2\pi} - \frac{1}{\pi} \sum_{n=1}^{\infty} e^{2\pi inx} & \text{if } \text{Im } x > 0, \\ \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=-\infty}^{-1} e^{2\pi inx} & \text{if } \text{Im } x < 0. \end{cases} \] (E.3)

\[ \varphi_n(x, z) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n(x-z)}}{1 + q^{-2n}} + \sum_{n=-\infty}^{-1} \frac{1}{n} \frac{e^{2\pi i n(x-z)}}{1 + q^{2n}} \quad \text{for } |\text{Im } z| < \gamma \] (E.4)

and the ‘resummation formulae’

\[ \sum_{n=1}^{\infty} \frac{1}{n} \frac{x^n}{1 + q^{2n}} = \ln \left( \frac{(x^n; q^n)}{(x; q^n)} \right). \] (E.5a)

\[ \sum_{n=1}^{\infty} \frac{1}{n} \frac{x^n}{(1 + q^{2n})^2} = \ln \left( \frac{(xq^n; q)^2}{(x; q^2)(xq^n; q^2)} \right). \] (E.5b)

where \(|x| < 1\), as well as the reduction formulae

\[ (xq^n; q, q) = \frac{(x; q^n; q^n)}{(x; q^n)}, \quad (xq^n; q) = \frac{(x; q^n)}{1 - x} \] (E.6)

in order to calculate the remaining integrals and to simplify the result.

In appendix B we frequently encountered certain elementary integrals involving logarithms along lines parallel to the real axis. To make it easier to verify the results of appendix B we briefly discuss these integrals here. In all cases involving integrals over \( \ln \sin(x) \) or its derivatives, the following definition is very helpful

\[ \ln \sin(x) = \begin{cases} \frac{ix}{2} - \ln(2) - ix + \ln(1 - e^{2ix}) & \text{if } \text{Im } x > 0, \\ -\frac{ix}{2} - \ln(2) + ix + \ln(1 - e^{-2ix}) & \text{if } \text{Im } x < 0. \end{cases} \] (E.7)

Here \( \ln \) denotes the principal branch of the logarithm. Note that \( \ln(1 - e^{2ix}) \) is holomorphic and \( \pi \)-periodic in the upper half plane, while \( \ln(1 - e^{-2ix}) \) is holomorphic and \( \pi \)-periodic in the lower half plane. Using (E.7) we obtain, for instance, for \(-\gamma < \text{Im } x < 0\) that

\[ \int_{-\pi/2}^{\pi/2} \frac{dy}{2\pi i} \left( \ln \left( \sin(y - x - i\gamma) \right) - \ln \left( \sin(y - x) \right) \right) = -\frac{\pi}{2} - x - i\gamma. \] (E.8)

We also encountered the integral

\[ I_1 = \int_{z_-}^{z_+} \frac{dz}{2\pi i} \ln(z - z_0), \] (E.9)

where \( z_\pm \) denote the boundary values from above and below the cut of a point on the cut and where the integration contour is a simple closed contour from \( z_- \) to \( z_+ \) that goes around \( z_0 \) (see
Then $\ln(z - z_0)$ is holomorphic and single-valued on the contour. Hence
\[
I_2 = \frac{1}{2\pi i} \ln^2(z - z_0) \bigg|_{z_0}^{z_+} = \ln(z_+ - z_0) - \pi i = \ln(z_- - z_0) + \pi i. \tag{E.10}
\]

Lastly, we needed to know integrals over the same type of contour of the form
\[
I_2 = \int_{z_-}^{z_+} \frac{dz}{2\pi i} \ln(z - z_0)f'(z), \tag{E.11}
\]
where $f$ is holomorphic and single valued on the contour with $f(z_+) = f(z_-)$. For these integrals we use partial integration to obtain
\[
I_2 = f(z_+) - f(z_0) = f(z_-) - f(z_0). \tag{E.12}
\]

**Appendix F. The $q$-gamma family**

In this appendix we collect some basic facts about $q$-gamma and $q$-Barnes functions. They belong to the $q$-analogue of the family of multiple-gamma functions. With the definition
\[
[x]_q = \frac{1 - q^x}{1 - q}, \tag{F.1}
\]
of a ‘$q$-number’ we have the following

**Theorem.** \cite{37} Let $q \in \mathbb{C}, |q| < 1$. The sequence of functional equations
\[
g_r(x + 1) = g_{r-1}(x)g_r(x), \quad r \in \mathbb{N} \tag{F.2}
\]
with boundary conditions
\[
g_r(1) = 1, \quad g_0(x) = [x]_q \tag{F.3}
\]
and
\[
\partial_x^{r+1}\ln g_r(x + 1) \geq 0 \quad \text{for } x \geq 0 \tag{F.4}
\]
uniquely determines a sequence of meromorphic functions $g_r$. 

---

![Figure E1. Integration contour for an integral around the cut of a logarithm with branch point at $z_0$.](image)
The function $\Gamma_q = g_1$ is the $q$-gamma function and $G_q = g_2$ is the $q$-Barnes function. For the whole sequence of multiple $q$-gamma functions infinite product representations exist \cite{37}. Alternatively they can be expressed in terms of $q$-multi factorials. Here we give only the $q$-multi factorial representations of $\Gamma_q$ and $G_q$,
\begin{align}
\Gamma_q(x) &= (1 - q)^{1-x} \frac{(q^x; q)_x}{(q^1; q)}, \quad G_q(x) = (1 - q)^{-x(1-x)(2-x)} \frac{(q^x; q)_{x-1}(q^2; q)_x}{(q; q)_x}.
\end{align}
(F.5)

These definitions together with the functional equations (F.2) were used to obtain the expression (21) for the universal part of the amplitudes in the main text.

For the isotropic limit we have used that the $q$-gamma and $q$-Barnes functions turn into their classical counterparts as $q \to 1$. Here we include a short proof of this fact which is in the spirit of our treatment of the nonlinear integral equations in \cite{17}.

For $|q| < 1$ we obtain the following series expansion for $\ln \Gamma_q$ directly from (F.5),
\begin{align}
\ln \Gamma_q(x) &= \sum_{\rho \geq 1} \frac{1}{\rho} \left\{ \frac{q^\rho - q^\rho}{1 - q^\rho} - q^\rho (1 - x) \right\}.
\end{align}
(F.6)

Upon setting $q = e^{-t}$ and introducing
\begin{align}
f_t(s) &= \frac{1}{s} \left\{ \frac{e^{-sx} - e^{-s}}{1 - e^{-s}} - e^{-s}(x - 1) \right\},
\end{align}
(F.7)
we can rewrite (F.6) as
\begin{align}
\ln \Gamma_q(x) &= \sum_{\rho \geq 1} t f_t((\rho)) = \int_{\tilde{C}_t} ds \frac{f_t(s)}{e^{s} - 1},
\end{align}
(F.8)
where the contour $\tilde{C}_t$ consists of three straight line segments
\begin{align}
\tilde{C}_t &= \left\{ f \to +\infty; \frac{1}{2} \right\} \cup \left\{ \frac{1}{2} + [i\delta; -i\delta] \right\} \cup \left\{ \frac{1}{2}; +\infty \right\}[-i\delta].
\end{align}
(F.9)

The contour integral can be decomposed as
\begin{align}
\ln \Gamma_q(x) &= \int_0^{+\infty} ds f_t(s) - \int_0^{\frac{1}{2}} ds f_t(s) - \sum_{\epsilon = \pm} \epsilon \int_{\tilde{C}_t(\epsilon)} ds \frac{f_t(s)}{e^{s} - 1},
\end{align}
(F.10)
where $\tilde{C}_t(\epsilon) = \tilde{C}_t \cap \mathbb{H}^\epsilon$. Since $f_t$ is smooth, decays exponentially fast as $\text{Re} \, s \to +\infty$ and $f_t(s) = O(s)$ for $s \to 0$, the last two terms on the right-hand side produce $O(t^2)$ contributions, while the first term is a known integral representation of $\ln \Gamma(x)$ (see e.g. \cite{3}).

The calculations are similar for the $q$-Barnes function. The logarithms of the $q$-multi factorials in (F.5) may be expanded into the series
\begin{align}
\ln G_q(x) &= \sum_{\rho \geq 1} \frac{1}{\rho} \left\{ \frac{1}{2} (1 - x)(2 - x)q^\rho + \frac{(1 - x)q^\rho}{1 - q^\rho} - \frac{q^\rho - q^\rho}{(1 - q^\rho)^2} \right\}.
\end{align}
(F.11)

The function
\begin{align}
f_C(s) &= \frac{1}{s} \left\{ \frac{1}{2} (1 - x)(2 - x)e^{-s} + \frac{(1 - x)e^{-s}}{1 - e^{-s}} - \frac{e^{-sx} - e^{-s}}{(1 - e^{-s})^2} \right\},
\end{align}
(F.12)
which satisfies \( f_G(s) = O(1) \) for \( s \to 0^+ \), allows one to recast \( G_y(x) \) into the form

\[
\ln G_y(x) = \sum_{p \geq 1} \gamma_G(p) = \int_0^\infty ds \frac{f_G(s)}{e^s - 1} - \int_0^\infty ds f_G(s) - \sum_{\epsilon = \pm} \epsilon \int_{C_\epsilon} ds \frac{f_G(s)}{e^{-\frac{s}{\epsilon}} - 1}.
\]

The last two terms are \( O(t) \) in the limit \( t \to 0^+ \). This establishes that

\[
G_y(x) \to \tilde{G}(x) = \exp \left\{ \int_0^{+\infty} ds f_G(s) \right\},
\]

pointwise for \( \Re x > 0 \). Performing such a pointwise limit in the functional equation (F.2) for \( r = 2 \), we conclude that \( \tilde{G}(x + 1) = \Gamma(x)\tilde{G}(x) \). Since \( G_y(1) = 1 \) and \( \partial^2_x \ln G_y(x) \geq 0 \) for \( x \geq 0 \) it follows that the same properties hold for \( \tilde{G} \). Thus, \( \tilde{G} \) must be equal to the Barnes \( G \) function owing to the uniqueness theorem of Vigneras [46].

**Appendix G. Fredholm determinants in the isotropic limit**

In this appendix we provide some details of the derivation of equation (73) in the main text. The kernels \( K^\pm \), equation (28), define integral operators \( \tilde{K}^\pm \) acting on \( L^2([-\pi/2, \pi/2]) \). Let us further define an operator \( \tilde{w} \) which acts on \( L^2([-\pi/2, \pi/2]) \) by pointwise multiplication with the values of the function \( w \), equation (22).

The integral operators \( \tilde{V}^\pm \) appearing in the Fredholm determinant contributions to the longitudinal correlation functions can then be written as

\[
\tilde{V}^- = \tilde{w}^{-1} \tilde{K}^-, \quad \tilde{V}^+ = \tilde{K}^- \tilde{w}.
\]

It is not difficult to see that, for \( k = 0, 1 \), the resolvents \( \hat{L}_k^\pm \), defined by

\[
(1 + (-1)^k \hat{K}^-)(1 - (-1)^k \hat{L}_k^-) = 1, \quad (G.2a)
\]

\[
(1 - (-1)^k \hat{L}_k^-)(1 + (-1)^k \hat{K}^-) = 1, \quad (G.2b)
\]

exist as operators on \( L^2([-\pi/2, \pi/2]) \) and can represented as integral operators with kernels

\[
L_k^-(x, y) = R_k(x - y) - R_k(x - \theta_-- y), \quad (G.3a)
\]

\[
L_k^+(x, y) = R_k(x - y) - R_k(x - \theta_+ - y), \quad (G.3b)
\]

where the functions \( R_k, k = 0, 1 \), were introduced in (75a). It is further known that

\[
\det_{\text{dir}[\pi/2, \pi/2]} (1 + \hat{K}^-) = \det_{\text{dir}[\pi/2, \pi/2]} (1 + \hat{K}^+) = (-q^2, q^2)^2. \quad (G.4)
\]
The above equations imply that

\[
\begin{align*}
\det_{\mathbb{S}_{\pi/2}} (1 + \mathbb{V}^-) &= \det_{\mathbb{S}_{\pi/2}} (1 + \mathbb{\tilde{W}}^{-1}\mathbb{K}^-) \\
= & \det_{\mathbb{S}_{\pi/2}} (1 + (-1)^k\mathbb{K}^-) \\
\times & \det_{\mathbb{S}_{\pi/2}} (1 + (-1)^k\mathbb{\tilde{K}}^- + (\mathbb{\tilde{W}}^{-1} - (-1)^k\mathbb{\tilde{K}}^-) \det_{\mathbb{S}_{\pi/2}} (1 - (-1)^k\mathbb{\tilde{L}}^-_k) \\
= & ((-1)^{k+1}q^2, q^2)^2 \det_{\mathbb{S}_{\pi/2}} (1 + (\mathbb{\tilde{W}}^{-1} - (-1)^k\mathbb{\tilde{L}}^-_k). \quad (G.5)
\end{align*}
\]

Similarly

\[
\begin{align*}
\det_{\mathbb{S}_{\pi/2}} (1 + \mathbb{V}^+) = & ((-1)^{k+1}q^2, q^2)^2 \det_{\mathbb{S}_{\pi/2}} (1 + \mathbb{\tilde{L}}^+_k(\mathbb{\tilde{W}} - (-1)^k). \quad (G.6)
\end{align*}
\]

The latter two equations are equivalent to (73) in the main text.

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