Planckian relaxation delusion in metals

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Abstract. We present a critical review of recent attempts to introduce a new quantum (‘Planckian’) limit for the temperature dependence of the inelastic scattering rate of electrons in metals. We briefly discuss the main experimental facts and some simple theoretical models explaining the linear-in-temperature growth of resistivity (starting from very low temperatures) in superconducting cuprates and some similar systems. There is no commonly accepted theoretical explanation for such behavior up to now. We also discuss the known quantum limits for electrical conductivity (resistance). It is shown that the universal Planckian limit for the inelastic relaxation rate proposed in some papers is a kind of delusion related to a certain procedure to represent the experimental data.

Keywords: resistivity of metals, relaxation time, effective mass, quantum fluctuations, high-temperature superconductors

1. Introduction

Linear in temperature ($T$-linear) growth of electrical resistivity of cuprates and some other correlated systems in the wide temperature region from pretty low to rather high temperatures has remained among the most challenging problems of the physics of high-temperature superconductors (HTSCs) for many years. Observed in cuprates since the early experiments [1, 2], it became one of the marked properties of these compounds in the normal phase in optimally doped systems, starting with the temperature of the superconducting transition $T_c$ and continuing up to the highest temperatures achievable before the samples are destroyed. Later, a similar behavior of resistivity was also observed in some other systems.

By itself, the $T$-linear growth of resistivity in metals is not at all surprising: it is practically always observed, though at fairly high temperatures $T > \Theta_D/5$, where $\Theta_D$ is the Debye temperature, which is usually several hundred degrees. What is surprising is the fact that the $T$-linear growth in cuprates takes place starting at significantly lower temperatures. The commonly accepted explanation for such behavior is still lacking, though over the years a number of theoretical models with a purported explanation were proposed. But this is not the aim of the current paper.

The fact of the matter is that recently a number of interesting papers have appeared [3, 4] in which, after rather detailed analyses of a wide range of experimental material on many compounds, it was shown that, in the $T$-linear region of resistivity, the scattering rate of electrons (inverse relaxation time) is pretty accurately described by the dependence $\Gamma(T) = 1/\tau = \kappa_B T/\hbar$, where $\kappa$ ~ 1, and $\kappa_B$ is the Boltzmann constant, which weakly varies from one material to another. In particular, for systems in the vicinity of quantum critical points (on the phase diagram of cuprates and some similar systems), the value of $\kappa$ is in the interval of 0.7–1.1 and is seemingly universal and independent of the strength of interaction, leading to relaxation of the electronic current. Furthermore, it was discovered that a similar dependence gives a rather good description of the data for a number of usual metals (Cu, Au, Al, Ag, Pb, Nb, etc.) in the $T$-linear region of their resistivity. In this case, the values of $\kappa$ belong to a noticeably wider interval from 0.7 to 2.8 [3, 4]. In connection with these (and similar) results, the concept of the universal (independent of interaction strength) ‘Planckian’ upper limit of the scattering rate $1/\tau_p = \Gamma_p \sim \kappa_B T/\hbar$ was introduced in Ref. [5].

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Below, we shall present a short review of the relevant experimental data and discuss some theoretical models, imposing quantum limitations on the resistivity of metals, with the aim of understanding the degree of validity of the concept of 'Planckian' relaxation in metals.

2. Experiments

The temperature dependence of resistivity at quite low temperatures ($T < 0.2\Theta_D \ll E_F$, $\Theta_D$ — Debye temperature, $E_F$ — Fermi energy) is described by the following expression:

$$\rho(T) = \rho_0 + \rho_1(T) = \rho_0 + a_{ee} \left( \frac{k_B T}{E_F} \right)^2 + a_{ep} \left( \frac{T}{\Theta_D} \right)^5,$$  

(1)

where $\rho_0$ is residual resistivity at zero temperature due to impurity scattering, the second term is the contribution of electron–electron scattering, while the third term is the low temperature contribution from electron–phonon scattering, described by the Bloch–Grüneisen theory [6]. For fairly high temperatures $T > 0.2\Theta_D$, this theory gives

$$\rho(T) - \rho_0 = AT.$$

This behavior of resistivity is clearly seen in experiments, as shown in Fig. 1, where we show the data for a number of simple metals [7]. These results show that the temperature dependence of resistivity (conductivity) is almost totally related to the processes of inelastic scattering by phonons. In the case of a significant contribution of scattering by other collective excitations, e.g., spin fluctuations, we can write down in fact very similar expressions.

From these data, it is seen that for temperatures $T > \Theta_D/5$ the resistivity of a metal grows linearly with temperature. In most metals, $\Theta_D \sim 200–600$ K. At the same time, the critical temperature of the superconducting transition in cuprates is commonly several dozen degrees, so that $T_c \sim \Theta_D/5$. Thus, for a rather long time there was a hope [8] that $T$-linear growth of resistivity in cuprates could be explained by the usual electron–phonon scattering, taking into account that it takes place outside the region of low temperature power—like growth, which is masked by a superconducting transition. There were some cases of a few exceptional samples with low values of $T_c$, where $T$-linear growth of resistivity was observed starting from anomalously low temperatures $T < 10$ K [9], but these were rather rare. There were no experiments in the normal phase at low enough temperatures simply because the destruction of the superconductive state in typical cuprates required extremely strong magnetic fields.

This situation changed in recent years after such experiments were successfully performed [3, 4]. These studies presented data on resistivity in a number of high-temperature cuprate superconductors as well as in some analogous systems in very high magnetic fields, suppressing superconductivity. A detailed analysis of these experiments and data obtained by other authors has shown that $T$-linear behavior of resistivity is conserved in the normal phase, in many cases, up to the lowest temperatures.

Typical examples of experimental data from Refs [3, 4] are shown in Figs 2–4. In these papers, characterized by the very detailed analysis of the experimental situation, many additional data can be found. These results further complicated the question of the nature of $T$-linear resistivity in the systems under consideration. It should be stressed that this problem remains unsolved to date, but its analysis is not the aim of the present work.

For us, the main focus now is the analysis of experimental data presented in Refs [3, 4], which allowed the authors to determine the temperature dependence of relaxation time $\tau$
from the values of resistivity and to attain rather unexpected results and reach unanticipated conclusions.

The main idea of the analysis performed in Refs [3, 4] was as follows. Let us write the Drude expression for conductivity as

\[ \sigma = \frac{n e^2 \tau}{m^*} = \frac{n e^2 v_F \tau}{\hbar k_F}, \]

where \( v_F \) is the electron velocity on the Fermi surface, \( m^* \) is the effective mass, and \( p_F = m^* v_F = \hbar k_F \) is the Fermi momentum. Correspondingly, for resistivity we have

\[ \rho = \frac{m^*}{n e^2 \tau} = \frac{\hbar k_F}{n e^2 v_F \tau}. \]

The effective mass \( m^* \) or the value of \( \hbar k_F / v_F \) in Refs [3, 4] were determined from the electron contribution to the specific heat, which can be measured at low temperatures, or from measurements of the de Haas–van Alphen effect, which are

1 For shortness, we write here all expressions for the one-band model. In the multiple band case, we have to take into account contributions from all pockets of the Fermi surface. Appropriate expressions can be found in Refs [3, 4].
also made at fairly low temperatures. These measurements in fact determine the value of the effective mass \( m^* \) and Fermi wave vector \( k_F \) as some average values for each of the pockets of the Fermi surface.

Electron density \( n \), entering expressions (3) and (4), can be calculated (for systems of different dimensionality) as

\[
\begin{align*}
n(a) & = \frac{2k_F}{\pi d_a^2} \text{ (quasi-one-dimensional case)}, \\
n(b) & = \frac{k_F^2}{2\pi d} \text{ (quasi-two-dimensional case)}, \\
n(c) & = \frac{k_F^2}{\pi}, \quad \text{(three-dimensional case)},
\end{align*}
\]

where \( d_a \) and \( d \) are the distances between adjacent conducting chains (oriented along the \( a \) axis) in the directions of the \( b \) and \( c \) axes in a quasi-one-dimensional system, while \( d \) is the distance between adjacent conducting planes in the quasi-two-dimensional case.

If we express the temperature-dependent part of resistivity as in (2) and introduce the \( T \)-linear (Planckian) relaxation rate as

\[
\Gamma = \frac{1}{\tau} = \frac{k_B T}{\hbar}, \quad \tau \sim \text{const},
\]

we immediately obtain

\[
A = \frac{d\rho}{dT} = \frac{k_B}{\hbar} m^* \frac{T}{n e^2} \quad \text{or}
\]

\[
ne^2 \frac{d\rho}{nk_B T} = \frac{m^*}{k_B T} = \frac{\tau}{\hbar},
\]

which gives working formulae to represent experimental data in the \( T \)-linear region of resistivity [3, 4].

In Refs [3, 4], a very detailed analysis of experimental data was performed for a rather wide set of systems (compounds) with very different electronic structures, from high-temperature superconducting copper oxides and iron-based superconductors to organic metals like \( \text{TMTSF}_2\text{PF}_6 \), to compounds like \( \text{Sr}_2\text{Ru}_2\text{O}_7 \), \( \text{CeCoIn}_5 \), \( \text{UPt}_3 \), \( \text{CeRu}_2\text{Si}_2 \), where a \( T \)-linear growth of resistivity is observed.

It was discovered that, for all these quite different systems, the experimental data for the thus defined relaxation rate are well described by a dependence like (8), and the value of \( \tau \) for the majority of these systems falls in the interval 0.7–1.1 and seems to be universal (independent of the peculiarities of the electronic spectrum or the strength of interaction, leading to the scattering of electrons). Moreover, it was also shown that a similar dependence appropriately describes the data for a number of usual metals as well (Cu, Au, Al, Ag, Pb, Nb), though the values of \( \tau \) for them are in a wider interval from 0.7–2.8 [3, 4].

**Table 1.** Experimental values of some parameters for a number of quasi-two-dimensional systems.

| Compound                  | Doping          | \( n, 10^{23} \) m\(^{-3} \) | \( m^*/m_0 \) | \( A/\alpha \Omega \) K\(^{-1} \) | \( \tau \) |
|---------------------------|-----------------|-----------------------------|---------------|-------------------------------|--------|
| \( \text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8.5} \) (Bi2212) | \( p = 0.23 \) | 6.8                         | 8.4 ± 1.6     | 8.0 ± 0.9                     | 1.1 ± 0.3 |
| \( \text{Bi}_2\text{Sr}_2\text{CuO}_{1-x} \) (Bi2201)      | \( p = 0.40 \) | 3.5                         | 7 ± 1.5       | 8 ± 2                         | 1.0 ± 0.4 |
| \( \text{La}_2-x\text{Sr}_x\text{CuO}_4 \) (LSCO)         | \( p = 0.26 \) | 7.8                         | 9.8 ± 1.7     | 8.2 ± 1.0                     | 0.9 ± 0.3 |
| \( \text{La}_{1-x}\text{Nd}_x\text{Sr}_x\text{CuO}_4 \) (Nd-LSCO) | \( p = 0.24 \) | 7.9                         | 12 ± 4        | 7.4 ± 0.8                     | 0.7 ± 0.4 |
| \( \text{Pr}_2-x\text{Ce}_x\text{CuO}_2 \) (PCCO)         | \( x = 0.17 \) | 8.8                         | 2.4 ± 0.1     | 1.7 ± 0.3                     | 0.8 ± 0.2 |
| \( \text{La}_{1-x}\text{Ce}_{x}\text{Cu}_2 \) (LCCO)       | \( x = 0.15 \) | 9.0                         | 3.0 ± 0.3     | 3.0 ± 0.4                     | 1.2 ± 0.3 |
| \( \text{TMTSF}_2\text{PF}_6 \) (TMSTSF)                | \( p = 11 \) kbar | 1.4                         | 1.15 ± 0.2    | 2.8 ± 0.4                     | 1.0 ± 0.3 |

In Table 1, we show, as an example, the experimental values of some parameters under discussion, determined in Ref. [4], for a number of quasi-two-dimensional systems (hole doped and electron doped cuprates, organics).

In Refs [3, 4], further details can be found on similar data for all the systems mentioned above.

All this is nicely illustrated by the consolidated graph shown in Fig. 5, where data are shown on a logarithmic scale representing Eqn (10). These results seem to be quite nontrivial and apparently confirm the concept of the universal ‘Planckian’ mechanism of electronic relaxation in metals, which was introduced earlier [5] and applied to cuprate physics. Furthermore, the value of \( \tau \approx 1 \), which is observed for such a wide set of materials with very different electronic spectra and quite different Fermi surfaces, suggests the idea that (8) is actually the universal quantum upper limit for the inelastic (temperature-dependent) relaxation rate for electrons in metals.

To explain such temperature behavior of resistivity for such different systems from the lowest temperatures, a number of complicated theoretical models were recently proposed [10–13], including some very exotic ones taken from the physics of black holes, cosmology, and superstring theory (see, e.g., [14–17]). Below, we shall limit ourselves to a simple analysis based on the traditional approaches of the quantum theory of solids.
3. Quantum estimates for resistivity of metals

Let us recall some elementary theoretical estimates with respect to the conductivity (resistivity) of metals. Drude expressions for elastic and inelastic scattering are written as
\[
\sigma_0 = \frac{ne^2}{m} \tau_0, \quad \rho_0 = \frac{m}{ne^2} \frac{1}{\tau_0},
\]
(11)

\[
\sigma(T) = \frac{ne^2}{m} \tau(T), \quad \rho(T) = \frac{m}{ne^2} \frac{1}{\tau(T)},
\]
(12)

where \(\tau_0\) is the mean free time due to elastic impurity scattering, determining the residual resistivity, and \(\tau(T)\) is the relaxation time due to inelastic scattering by phonons (or some other collective excitations), or due to electron–electron scatterings. Mass \(m\) here is always understood as free electron mass (band structure mass in a solid!), which does not include corrections due to electron-electron or electron-phonon interactions (see more details below). Then, assuming the additive contributions of different scattering mechanisms (Matissen rule), the total resistivity is written as
\[
\rho(T) = \frac{m}{ne^2} \left( \frac{1}{\tau_0} + \frac{1}{\tau(T)} \right) = \frac{m}{ne^2} (\Gamma_0 + \Gamma(T)),
\]
(13)

where we have introduced the appropriate scattering rates:
\[
\Gamma_0 = \frac{1}{\tau_0},
\]
(14)

\[
\Gamma(T) = \frac{1}{\tau(T)}.
\]
(15)

In the general theory of interacting fermions (electrons), by order of magnitude we have \(\Gamma \sim \text{Im} \Sigma\), where \(\Sigma\) is the electron self-energy, taking into account all the relevant interactions. A consistent approach to calculating conductivity (resistivity) requires, of course, the treatment of a full two-particle Green’s function [18].

3.1 Ioffe–Regel limit

The most prominent quantum limitation for the conductivity (resistivity) of metals is the Ioffe–Regel limit [19], which is relevant to strongly disordered systems. For \(T = 0\), we have
\[
\sigma_0 = \frac{ne^2}{m} \tau_0 = \frac{ne^2}{m} l,
\]
(16)

where \(l = \tau_0 / \tau_0\) is the mean free path. The usual kinetic theory is valid for \(p_F l / h > 1\) or \(E_F \tau_0 / h > 1\). Taking into account \(n = \rho_0 / (3\pi^2 h^3)\), we get the estimate for conductivity in the Ioffe–Regel limit as \(\rho_0 l / \rho_0 \sim 1\) or \(E_F \tau_0 / l \sim 1\):

\[
\sigma_{\text{IR}} = \frac{e^2}{\hbar^2} \frac{p_F}{3\pi^2} \sim \frac{e^2}{\hbar a},
\]
(17)

where \(a \sim n^{-1/3}\) is the average distance between electrons. For typical metallic densities, \(a\) is of the order of the interatomic distance (lattice constant). In this case (for \(n \sim 10^{23} \text{ cm}^{-3}\)), corresponding resistivity \(\rho_{\text{IR}} = \sigma_{\text{IR}}^{-1} \sim 150–300 \mu\Omega\text{ cm}\). For the majority of usual (‘good’) metals, e.g., for Cu, \(\rho(T = 300 \text{ K}) \sim 1 \mu\Omega\text{ cm}\), so that this limit remains unachievable even at the highest temperatures, not exceeding the melting temperature. However, this is not so in strongly disordered (highly-resistive) alloys, where the resistivity can approach the Ioffe–Regel limit.

An estimate quite close to (17) was also proposed by Mott for the so-called ‘minimal metallic conductivity’ \(\sigma_c\), at which a discontinuous metal–insulator transition takes place at \(T = 0\) (Anderson localization) (achieved as disorder grows, e.g., the mean free path is reduced) [20]. In fact, as shown in the scaling theory of localization [21, 22], this transition is continuous, and \(\sigma_c\) plays the role of the characteristic conductivity scale in the vicinity of the Anderson transition:
\[
\sigma \sim \sigma_c \left( \frac{1 - l}{l_c} \right)^{(d-2)/2},
\]
(18)

where \(\sigma_c \sim e^2/(\hbar a^{d-2}), 2 < d < 4\) is a dimension of space, and the critical value of the mean free path can be estimated from \(p_F l_c / h \sim 1\), i.e., \(l_c \sim a\). Here, we introduced the critical exponent of localization length, which in the self-consistent theory of localization [18, 23] is given by
\[
v = \frac{1}{d-2},
\]
(19)

so that \(v = 1\) for \(d = 3\). Current numerical calculations of the Anderson transition give values of \(v\) in the interval 1.5–1.8 [24]. For us, it is not these details that are important, but the actuality of the continuous nature of this transition and the nonexistence of any critical level of conductivity (resistivity).

Strictly speaking, these estimates are valid only for \(T = 0\), when the Anderson transition is a well-defined quantum phase transition in a system of noninteracting electrons. At finite temperatures and taking into account electron–electron interactions, the situation is much more complicated, and we shall not discuss it here.

In most cases, the growth of resistivity with temperature slows down and saturates as resistivity approaches the Ioffe–Regel limit [25]. In such strongly disordered metals (highly-resistive alloys), an empirical Mooij rule is at work: after reaching a resistivity level (at low temperatures) of the order of \(\rho_{\text{IR}}\), the temperature dependence of resistivity becomes very weak, and in a wide temperature interval from low to room temperatures, or even higher, it is often observed that the temperature coefficient of resistivity becomes negative [25, 26]. This fact does not to date have a commonly accepted explanation.

There is a common belief that this is not so in ‘strange’ metals like high-temperature superconducting copper oxides [27, 28] (in the region of optimal doping) and in a number of other systems [29] where the \(T\)-linear growth of resistivity continues even after overpassing values of the order of \(\rho_{\text{IR}} = \sigma_{\text{IR}}^{-1} \sim 100–300 \mu\Omega\text{ cm}\), up to the highest possible temperature \(T \sim 1000 \text{ K}\). However, here, we always face the problem of the correct estimate of resistivity in the Ioffe–Regel limit. From the estimates given above, it becomes clear that it depends significantly on the concentration of current carriers (which in cuprates is much lower than in usual metals), so that, for instance, for \(n \sim 10^{21} \text{ cm}^{-3}\), we obtain \(\rho_{\text{IR}} \sim 1–10 \mu\Omega\text{ cm}\). In recent experiments involving weakly doped cuprates [30], it was clearly shown that their resistivity saturates at \(\rho_{\text{IR}} \sim 3–5 \mu\Omega\text{ cm}\), in complete accordance with the concentration of carriers obtained from measurements of the Hall effect (at \(T \sim 300 \text{ K}\)). Thus, it is quite possible that in the experiments cited above on optimally doped cuprates, the correct value of the Ioffe–Regel limit was simply not achieved up to the highest possible temperatures (before the destruction of samples).
However, our main interest in the following will be related to resistivity of very ‘pure’ metals in the $T$-linear region.

3.2 Planckian relaxation
The idea of a Planckian relaxation mechanism in metal at high enough temperatures seems to be very attractive. Let us give some elementary arguments making it seemingly justified and based upon quite general quantum mechanical estimates using the uncertainty principle [15]. At finite temperatures $T > 0$, different processes of inelastic scattering (electron–phonon, interaction with spin fluctuations, etc.) are at work. Precisely these processes are responsible for establishing thermodynamic equilibrium in an electron system—the Fermi distribution. In a system of interacting particles at finite temperatures, the particle (electron) distribution function is qualitatively of the same form [18]. The conductivity of a metal (degenerate case) is determined by electron distribution in a layer $\sim k_B T$ around the Fermi level (chemical potential). Let us make an elementary estimate using the time–energy uncertainty relation:

$$\Delta E \tau > \hbar,$$

(20)

where $\tau$ is the lifetime of a quantum state, while $\Delta E$ is the uncertainty of its energy. In our case, $\tau = \tau(T)$, and it seems natural to take $\Delta E \sim k_B T$, so that we immediately obtain an estimate:

$$\Gamma(T) = \frac{1}{\tau(T)} < \frac{k_B T}{\hbar} \equiv \Gamma_p = \frac{1}{\tau_p},$$

(21)

where $\alpha = 1$. It is now evident that, according to this elementary estimate, the Planckian relaxation defines an upper limit for resistivity due to inelastic scatterings:

$$\rho(T) - \rho_0 = \frac{m_e}{ne} \Gamma(T) < \frac{m_e}{ne} \frac{k_B T}{\hbar} \equiv \rho_p(T).$$

(22)

Obviously, this estimate is of a rather speculative nature for a system of many, in general, strongly interacting particles, but it correlates well with the results of experiments described above.

If such a universal upper limit exists for the relaxation rate, the qualitative picture of the temperature dependence of resistivity of metals can be suggested as shown in Fig. 6 [31]. The idea here is that in ‘usual’ metals all the temperature dependence of resistivity develops below the Ioffe–Regel limit, while the ‘Planckian’ limit is reached in unusual systems like HTSC cuprates, which are near the quantum critical point and where the Ioffe–Regel limit may be surpassed with the growth of the temperature (however, see Ref. [30]).

From the estimates given above, it is easy to derive the ratio of ‘Planckian’ to Ioffe–Regel resistivity [15],

$$\frac{\rho_p(T)}{\rho_{IR}} \sim \frac{k_B T}{E_F},$$

(23)

so that this limit may be exceeded for $k_B T > E_F$, which can be realized experimentally only in systems with low enough values of $E_F$ (low carrier concentration), such as copper oxides or in multiple band systems (with several pockets of the Fermi surface). In any case, according to this picture, resistivity does not exceed the upper limit determined by the ‘Planckian’ relaxation rate, which is achieved in ‘strange’

 metals. It is quite surprising that the experimental data quoted above seem to confirm the attainment of this limit for many (!) metals, including some quite ‘usual’ ones.

3.3 Electron–phonon interaction
Consider the most important case for the theory of metals—electron–phonon interaction, which will be described within the Eliashberg–McMillan theory, as the modern generalization of the Bloch–Grüneisen theory [32]. Within this theory, the high-temperature ($T > \Theta_{IR}/5$) phonon contribution to resistivity is given by [33, 34]

$$\rho_{ph}(T) = \frac{4\pi m T}{m^2e^2} \int_0^{\infty} \frac{d\omega}{\omega} \chi_p^2(\omega) F(\omega) = \frac{2\pi m T}{me^2} \lambda_{IR},$$

(24)

where $\chi_p^2(\omega) F(\omega$) is the transport Eliashberg–McMillan function [33, 34] ($F(\omega)$ is the phonon density of states), determining the transport electron–phonon coupling constant as

$$\lambda_{IR} = 2 \int_0^{\infty} \frac{d\omega}{\omega} \chi_p^2(\omega) F(\omega).$$

(25)

For most metals, we have $\chi^2 \approx \chi_{IR}^2$, where [34]

$$\lambda = 2 \int_0^{\infty} \frac{d\omega}{\omega} \chi^2(\omega) F(\omega)$$

(26)

is the Eliashberg–McMillan coupling constant [32] determining the temperature of superconducting transition. Then, we get a simple estimate:

$$\rho_{ph}(T) \approx \frac{2\pi m T}{m^2e^2} \lambda = 2\pi \lambda \rho_p(T),$$

(27)

where in the last equality in the definition of $\rho_p(T)$ from (22) we put $\alpha = 1$. In fact, the values of $\lambda > 1$ are not very rare in metals [32], and it becomes clear that even the usual electron–phonon interaction can easily break the Planckian limit, so that $\rho_{ph}(T) > \rho_p(T)$ does not seem exotic. This simple example immediately casts certain doubts on the universality of ‘Planckian’ relaxation, though it seems that our arguments, based on the uncertainty principle, must work for any system and interaction. However, this simple example contradicts it.
4. Elementary model of scattering by quantum fluctuations in metals

4.1 Scattering by quantum fluctuations

Here, following mainly Ref. [35], we shall consider an elementary, though realistic enough, model of electron scattering by quantum fluctuations, which will allow us to analyze our problem in a rather general form. Consider the usual Hamiltonian of electrons in a metal interacting with some Bose–like quantum fluctuations of some arbitrary nature (N is the total number of atoms in a crystal):²

$$\hat{H}_{\text{int}} = \frac{1}{\sqrt{N}} \sum_{\mathbf{p} \mathbf{q}} g_{\mathbf{p} \mathbf{q}} \hat{b}_{\mathbf{p} \mathbf{q}}^+ \hat{\rho}_{\mathbf{p} \mathbf{q}} \hat{b}_{\mathbf{p} \mathbf{q}}.$$

(28)

where we have used the standard notations of creation and annihilation operators of electrons, and $\rho_{\mathbf{p} \mathbf{q}}$ is an operator of quantum fluctuation of ‘any kind’ (e.g., the density of ions in a solid or collective excitations of an electronic subsystem, including spin excitations, though spin indices are dropped for brevity). Let us introduce the appropriate (Matsubara time) Green’s function as [40]

$$F(\mathbf{q}, \tau) = - \langle T_{\tau} \rho_{\mathbf{p} \mathbf{q}}(\tau) \rho_{\mathbf{p} \mathbf{q}}^+(0) \rangle.$$

(29)

We can then write the usual spectral representation for it as [37]

$$F(\mathbf{q}, \omega_m) = \int_{-\infty}^{\infty} d\omega \frac{A(\mathbf{q}, \omega)}{i\omega_m - \omega},$$

(30)

where $\omega_m = 2\pi m T$, and the spectral density is defined as

$$A(\mathbf{q}, \omega) = Z^{-1} \sum_{mn} \exp \left(-\frac{E_n}{T}\right) |\langle \rho_{\mathbf{p} \mathbf{q}} \rangle_{mn}|^2$$

$$\times \left[ 1 - \exp \left(-\frac{\omega_m}{T}\right) \right] \delta(\omega - \omega_m),$$

(31)

where $\omega_m = E_n - E_m$, $\langle \rho_{\mathbf{p} \mathbf{q}} \rangle_{mn} = \langle m | \rho_{\mathbf{p} \mathbf{q}} | n \rangle$, and $m, n$ enumerate the exact many-particle states of the system.

The dynamic structure factor of fluctuations is defined as [36, 38]

$$S(\mathbf{q}, \omega) = Z^{-1} \sum_{mn} \exp \left(-\frac{E_n}{T}\right) |\langle \rho_{\mathbf{p} \mathbf{q}} \rangle_{mn}|^2 \delta(\omega - \omega_m).$$

(32)

Comparing (31) and (32), we obtain

$$A(\mathbf{q}, \omega) = S(\mathbf{q}, \omega) \left[ 1 - \exp \left(-\frac{\omega}{T}\right) \right].$$

(33)

The electron Green’s function in the Matsubara representation is written in a standard form:

$$G(\epsilon_n, \mathbf{p}) = \frac{1}{\text{i}\epsilon_n - \hat{\Sigma} - \hat{\Sigma}(\epsilon_n, \mathbf{p})},$$

(34)

where $\epsilon_n = (2n + 1)\pi T$, $\hat{\Sigma}$ is the free-electron spectrum counted from the Fermi level (chemical potential), and the

² Further on, we shall use the units $\hbar = k_B = 1$. 

Figure 7. Second order diagram for electron self-energy. Dashed line represents the Green’s function of a quantum fluctuation $F$, continuous line represents the electronic Green’s function $G$. 

self-energy part can be taken in the simplest approximation shown by the diagram in Fig. 7:

$$\Sigma(\epsilon_n, \mathbf{p}) = - \frac{T}{N} \sum_{\mathbf{q}} g_{\mathbf{q}}^2 \sum_{m} F(\mathbf{q}, i\omega_m) G(\epsilon_n + \omega_m, \mathbf{p} + \mathbf{q})$$

$$= - \frac{T}{N} \sum_{\mathbf{q}} g_{\mathbf{q}}^2 \int_{-\infty}^{\infty} d\omega \frac{S(\mathbf{q}, \omega)}{i\omega_m - \omega} \left[ 1 - \exp \left(-\frac{\omega}{T}\right) \right]$$

$$\times \frac{1}{\text{i}\omega_n + i\omega_m - \frac{\omega}{\hat{\epsilon}_{\mathbf{p} \mathbf{q}}}},$$

(35)

where $g_{\mathbf{q}}$ is the coupling constant (matrix element of the interaction potential).

Consider now the case where the average frequency of fluctuation ($\Omega$) is much lower than the temperature $T$, i.e., the classical limit for fluctuations. Then, in Eqn (35), it is sufficient to take into consideration only the term with $m = 0$, going actually to the picture of quasi-elastic scattering by fluctuations:

$$\Sigma(\epsilon_n, \mathbf{p}) = \frac{1}{N} \sum_{\mathbf{q}} g_{\mathbf{q}}^2 \int_{-\infty}^{\infty} d\omega \frac{S(\mathbf{q}, \omega)}{i\omega_n - \frac{\omega}{\hat{\epsilon}_{\mathbf{p} \mathbf{q}}}}$$

$$= \sum_{\mathbf{q}} g_{\mathbf{q}}^2 S(\mathbf{q}) \frac{1}{\text{i}\omega_n - \frac{\omega}{\hat{\epsilon}_{\mathbf{p} \mathbf{q}}}},$$

(36)

where we have introduced the structure factor of fluctuations [38]

$$S(\mathbf{q}) = \frac{1}{N} \int_{-\infty}^{\infty} d\omega S(\mathbf{q}, \omega).$$

(37)

In fact, this is in direct analogy with the well-known Ziman–Edwards approximation in the theory of liquid metals [39–41]. The case of $S(\mathbf{q}) = \text{const}$ corresponds to a totally chaotic distribution of static scattering centers [18].

4.2 Phonons

In the case of scattering by phonons, fluctuation operator $\rho_{\mathbf{q}}$ can be expressed via (Bose) creation and annihilation operators of phonons as [38]

$$\rho_{\mathbf{q}} = \frac{1}{\sqrt{2}} \left( b_{\mathbf{q}}^+ + b_{-\mathbf{q}} \right).$$

(38)

Then,

$$S(\mathbf{q}, \omega) = Z^{-1} \sum_{m} \exp \left(-\frac{E_m}{T}\right) \left| \langle m | b_{\mathbf{q}}^+ b_{-\mathbf{q}} | m \rangle \right|^2$$

$$\times \delta(\omega - \omega_{\mathbf{q}}) + \langle m | b_{\mathbf{q}}^+ b_{-\mathbf{q}} | m \rangle \delta(\omega + \omega_{\mathbf{q}}),$$

(39)
where \( \omega_q \) is the phonon spectrum. Introducing the usual Bose distribution,

\[
n_q = \frac{1}{Z} \sum_m \exp \left( -\frac{E_m}{T} \right) \langle m | b_q^+ b_q | m \rangle = \frac{1}{\exp(\omega_q/T) - 1},
\]

(40)
we get [38]

\[
S(q, \omega) = \left[ (n_q + 1)\delta(\omega - \omega_q) + n_q \delta(\omega + \omega_q) \right] = \delta(\omega - \omega_q) + n_q \left[ \delta(\omega - \omega_q) + \delta(\omega + \omega_q) \right].
\]

(41)
Under the conditions of \( T \gg \omega_q \), we go to the classical limit (equipartition theorem):

\[
n_q = \frac{T}{\omega_q},
\]

(42)
and, accordingly,

\[
S(q) = \frac{2T}{\omega_q}.
\]

(43)
Thus, we obtain the structure factor linearly growing with temperature, while its momentum dependence is determined by the phonon spectrum. Then,

\[
\Sigma(\epsilon_a, p) = T \sum_q \frac{2g_q^2}{\omega_q} \frac{1}{\omega_a - \epsilon_p - q}.
\]

(44)
To simplify the model further, let us assume that the phonon spectrum is dispersionless (Einstein phonon or optical phonon with very weak dispersion), taking \( \omega_q = \Omega_0 \). Then, performing all calculations as in the problem of an electron in a system of random impurities [18], we get

\[
\Sigma(\epsilon_a, p) = -i\pi \text{sign} \epsilon_a \frac{2g_0^2}{\Omega_0} N(0) T,
\]

(45)
where \( N(0) \) is the density of states on the Fermi level. Correspondingly, the damping is written as

\[
\frac{\Gamma(T)}{2} = \pi \frac{2g_0^2}{\Omega_0} N(0) T = \pi \lambda_0 T,
\]

(46)
where we have introduced the usual dimensionless coupling constant of electron–phonon interaction:

\[
\lambda_0 = \frac{2g_0^2 N(0)}{\Omega_0}.
\]

(47)
After standard calculations [18], we obtain the resistivity as

\[
\rho(T) = \frac{m}{ne^2} \Gamma(T) = 2\pi \lambda_0 \rho(T).
\]

(48)
which is essentially the high-temperature limit of the Eliashberg–McMillan theory (27). Now, the constant \( z \) used in the definition of ‘Planckian’ relaxation time (21) is expressed via the parameters of the theory as

\[
z = 2\pi \lambda_0.
\]

(49)
Naturally, it is not universal and is just proportional to the coupling constant.

Practically the same result can be easily obtained in the Eliashberg–McMillan approximation [18], where the expression for Matsubara self-energy is written as

\[
\Sigma(\epsilon_a) = \int_0^\infty d\omega \, \omega^2(\omega) F(\omega)
\]

\[
\times \left( \frac{f(\epsilon') + n(\omega)}{\omega_a - \epsilon' + \omega} + \frac{1 - f(\epsilon') + n(\omega)}{\omega_a - \epsilon' - \omega} \right),
\]

(50)
where \( f(\epsilon) \) and \( n(\omega) \) are standard Fermi and Bose distributions. In the high-temperature limit \( T \gg \omega \), Eqn (50) reduces to

\[
\Sigma(\epsilon_a) = \int_0^\infty d\omega \, \omega^2(\omega) F(\omega)
\]

\[
\times \frac{T}{\omega} \left( \frac{1}{\omega_a - \epsilon' + \omega} + \frac{1}{\omega_a - \epsilon' - \omega} \right)
\]

\[
= T \int_{-\infty}^\infty d\epsilon' \left[ \frac{\omega}{\omega_a - \epsilon'} \right]^2 F(\omega) \frac{1}{\omega_a - \epsilon'}
\]

\[
= -i\pi \lambda T \text{sign} \epsilon_a,
\]

(51)
where we have introduced the standard definition of the coupling constant of the Eliashberg–McMillan theory (26), thus reproducing a result like (46). For resistivity, we again get Eqn (48) with an obvious replacement \( \lambda_0 \to \lambda \), where \( \lambda \) is defined in (26).

4.3 More general model

Now, let us try to avoid the explicit introduction of phonons (or any other quasiparticles related to fluctuations). From Eqn (33), for \( \omega \ll T \) we get

\[
A(q, \omega) \approx \frac{\omega}{T} S(q, \omega),
\]

(52)
or

\[
S(q, \omega) \approx \frac{T}{\omega} A(q, \omega).
\]

(53)
Substituting this expression into Eqn (36), we obtain the following expression for self-energy:

\[
\Sigma(\epsilon_a, p) = \frac{T}{N} \sum_q g_q^2 \int_{-\infty}^\infty \frac{d\omega}{\omega} \frac{A(q, \omega)}{\omega_a - \epsilon_p - q}.
\]

(54)
where everything is determined by the spectral density of fluctuations \( A(q, \omega) \), which is not necessarily of a quasiparticle form. Naturally, for the simplest model with \( A(q, \omega) = \delta(\omega - \Omega_0) \) (Einstein model for fluctuations), from (54) immediately follow the Eqs (45)–(47) derived above. In the case of no \( q \) dependence, from (54) we immediately obtain

\[
\Sigma(\epsilon_a, p) = \frac{T}{N} \sum_q g_q^2 \int_{-\infty}^\infty \frac{d\omega}{\omega} \frac{A(\omega)}{\omega_a - \epsilon_p - q}
\]

\[
= -i\pi \text{sign} \epsilon_a N(0) \frac{T}{g_0^2} \int_{-\infty}^\infty \frac{d\omega}{\omega} A(\omega)
\]

\[
= -i\pi \text{sign} \epsilon_a \frac{2g_0^2}{\Omega_0} N(0) T,
\]

(55)

\( ^3 \) Here and in what follows, we ignore for brevity the insignificant (for our aims) difference between \( \lambda \) and \( \lambda_0 \).
where we have introduced an average frequency of fluctuations:
\[
\langle \Omega \rangle^{-1} = 2 \int_{0}^{\infty} \frac{d\omega}{\omega} A(\omega),
\]
(56)

This result is actually equivalent to (45).

In the general case, when we can not ignore the momentum dependence of the spectral density of fluctuations, we can use the Eliashberg–McMillan approach, assuming that fluctuations scatter electrons in some narrow (~ T) layer around the Fermi surface. Then, we can introduce the self-energy averaged over the momenta on the Fermi surface:
\[
\Sigma(\epsilon_n) = \frac{1}{N(0)} \sum_{p} \delta(\xi_p) \Sigma(\epsilon_n, p),
\]
(57)

and also an effective (averaged over initial and final momenta on the Fermi surface) interaction:
\[
g_{pp}^{2} A(p - p', \omega) = \frac{1}{N(0)} \sum_{p} \frac{1}{N(0)} \sum_{p'} g_{pp}^{2} \delta(\xi_p) \delta(\xi_{p'})
\]
\[
\equiv \frac{1}{N(0)} \chi^2(\omega) F(\omega),
\]
(58)

where
\[
F(\omega) = \sum_{q} A(q, \omega)
\]
(59)

is the density of states of fluctuations. Then, from (54), we obtain for (57),
\[
\Sigma(\epsilon_n) = \frac{T}{N(0)} \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \chi^2(\omega) F(\omega) N(0) \int_{-\infty}^{\infty} d\xi \frac{1}{i\epsilon_n - \xi}
\]
\[
= -i \text{sign} \epsilon_n T \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \chi(\omega) F(\omega)
\]
\[
= -i \epsilon_n \lambda T \equiv -i \frac{\Gamma(T)}{2} \text{sign} \epsilon_n ,
\]
(60)

where we again introduced the dimensionless coupling constant as in the Eliashberg–McMillan theory defined by Eqn (26), the value of which is in fact determined by the (averaged as in (58)) spectral density of fluctuations \(A(q, \omega)\), which does not necessarily describe any quasiparticles.

Finally, we obtain
\[
\Gamma(T) = 2n \lambda T,
\]
(61)

which is of the same form as Eqn (46) and leads immediately to (48). Strictly speaking, this is not necessarily so if we remember the possible temperature dependence of spectral density \(A(q, \omega)\). Planckian distribution is obtained only in the absence or weakness of this dependence. It is obvious that Eqn (46) immediately follows from (26) and (60) if \(F(\omega) = \delta(\omega - \Omega_0)\), which corresponds to the Einstein spectrum of fluctuations.

### 4.4 Vicinity of the quantum critical point

The simplest model considered above certainly does not explain the \(T\)-linear behavior of resistivity in cuprates and similar systems in the normal state, starting from very low temperatures. It should be noted that this behavior is often related to the closeness of these systems to some quantum critical point.

Let us consider the schematic phase diagram of hole-doped cuprates shown in Fig. 8 [42].

Figure 8. (Color online.) Typical phase diagram of hole-doped cuprates. Superconducting state is observed in the region of temperatures below \(T_c\), shown by dashed line. Antiferromagnetic region (AF) is shown in brown. Pseudogap phase (PG) exists at temperatures below \(T^*\) (yellow) and ends at \(T = 0\) at the critical point \(p^*\) (red dot). Inside the pseudogap phase exists the phase with a charge density wave (CDW) (blue). The region of a strange metal (SM) (white) continuously transforms into the Fermi-liquid region (FL) at higher dopings [42].

![Phase diagram of hole-doped cuprates](image)

In the general case, when we cannot ignore the momentum dependence of the spectral density of fluctuations, we can use the Eliashberg–McMillan approach, assuming that fluctuations scatter electrons in some narrow (~ \(T\)) layer around the Fermi surface. Then, we can introduce the self-energy averaged over the momenta on the Fermi surface:

\[
\Sigma(\epsilon_n) = \frac{1}{N(0)} \sum_{p} \delta(\xi_p) \Sigma(\epsilon_n, p),
\]
(57)

and also an effective (averaged over initial and final momenta on the Fermi surface) interaction:

\[
g_{pp}^{2} A(p - p', \omega) = \frac{1}{N(0)} \sum_{p} \frac{1}{N(0)} \sum_{p'} g_{pp}^{2} \delta(\xi_p) \delta(\xi_{p'})
\]

\[
\equiv \frac{1}{N(0)} \chi^2(\omega) F(\omega),
\]
(58)

where

\[
F(\omega) = \sum_{q} A(q, \omega)
\]
(59)

is the density of states of fluctuations. Then, from (54), we obtain for (57),

\[
\Sigma(\epsilon_n) = \frac{T}{N(0)} \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \chi^2(\omega) F(\omega) N(0) \int_{-\infty}^{\infty} d\xi \frac{1}{i\epsilon_n - \xi}
\]

\[
= -i \text{sign} \epsilon_n T \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \chi(\omega) F(\omega)
\]

\[
= -i \epsilon_n \lambda T \equiv -i \frac{\Gamma(T)}{2} \text{sign} \epsilon_n ,
\]
(60)

where we again introduced the dimensionless coupling constant as in the Eliashberg–McMillan theory defined by Eqn (26), the value of which is in fact determined by the (averaged as in (58)) spectral density of fluctuations \(A(q, \omega)\), which does not necessarily describe any quasiparticles.

Finally, we obtain

\[
\Gamma(T) = 2n \lambda T,
\]
(61)

which is of the same form as Eqn (46) and leads immediately to (48). Strictly speaking, this is not necessarily so if we remember the possible temperature dependence of spectral density \(A(q, \omega)\). Planckian distribution is obtained only in the absence or weakness of this dependence. It is obvious that Eqn (46) immediately follows from (26) and (60) if \(F(\omega) = \delta(\omega - \Omega_0)\), which corresponds to the Einstein spectrum of fluctuations.
just one of the most striking cases — the sharp singularity of the electronic specific heat coefficient \( \gamma = C_{el}/T \) at \( p = p^* \) in the normal state of cuprates, obtained in a strong magnetic field suppressing superconductivity [42].

The nature of the pseudogap state in cuprates has not been completely clarified to date. In particular, it is not clear whether the PG region on the phase diagram of Fig. 8 is some new phase and the line \( T^* \) defines the critical temperature of a true phase transition, or it is the region of some crossover to antiferromagnetic phase AF, with well developed fluctuations of AF short-range order [45–47]. The antiferromagnetic scenario of pseudogap formation has also serious experimental support [42], though some authors believe that the transition to the pseudogap phase is a true phase transition with some still unknown order parameter. A number of specific models (see, e.g., [48]) of such a phase transition have been proposed, allowing us to consider \( p = p^* \) as a true quantum critical point. However, it is still unclear whether we can speak of the quantum critical point in the crossover scenario. In any case, the \( T^* \)-linear behavior of resistivity in cuprates is observed close to optimal doping, which nearly coincides with \( p = p^* \). To the left of this concentration resistivity is demonstrated some dielectric (localization) effects (a strong negative temperature coefficient of resistivity) [49, 50], while to the right we have a crossover to more or less usual Fermi-liquid-like behavior with a quadratic growth of resistivity with temperature.

The closeness of the system to the quantum critical point can, in principle, explain the \( T^* \)-linear growth of resistivity even within the elementary model considered above. To observe such growth at low temperatures, it is sufficient to satisfy the inequality \( \Omega_0 \ll T \), where \( \Omega_0 \) is the characteristic frequency of fluctuations scattering the electrons. In the vicinity of the quantum critical point (of any nature), we can expect a typical ‘softening’ of the appropriate fluctuation mode according to the relation [43, 44]

\[
\Omega_0 \sim |p - p^*|^{\nu},
\]

where \( p \) is the concentration of carriers (holes or electrons). Here, \( \nu \) and \( z \) are the standard critical exponents of the theory of quantum phase transitions, determining the critical behavior of characteristic lengths:

\[
\xi \sim |p - p^*|^{-\nu}, \quad \xi_t \sim |p - p^*|^{-z}. \tag{63}
\]

Here, \( \tau \) denotes the imaginary (Matsubara) time, and above we just put \( \Omega_0 \sim \xi_t^{-1} \).

As shown in Fig. 10, the quantum critical region is defined by the inequality [43, 44]

\[
\xi_t^* > \frac{1}{T}, \quad \Omega_0 < T, \tag{64}
\]

which may guarantee the \( T \)-linear behavior of resistivity in this region in cuprates or similar systems. However, the nature of quantum fluctuations of importance here, as well as the mechanism of their interaction with electrons, remains an open question.

5. General relations for the Green’s function and conductivity

5.1 Green’s function for a system of interacting particles

Let us recall some general expressions for an arbitrary system of interacting electrons. We have the following general expression for the Matsubara Green’s function [37]:

\[
G(v_n, p) = \frac{1}{i\omega_n - \xi_p - \Sigma(v_n, p)}. \tag{65}
\]

Keeping in mind the electron–phonon coupling and its analogs (quantum fluctuations of a general nature), dropping the momentum dependence of self-energy, we can write the following expression in the standard notations of the Eliashberg–McMillan theory:

\[
\Sigma(v_n) = [1 - z(v_n)] i\omega_n + Z(v_n). \tag{66}
\]

Then,

\[
G(v_n, p) = \frac{1}{Z(v_n)i\omega_n - \xi_p - Z(v_n)} \tag{67}
\]

where we have defined

\[
Z(v_n) = z^{-1}(v_n). \tag{68}
\]

Let us also define

\[
e(p, v_n) = Z(v_n)\xi_p + \text{Re} Z(v_n), \tag{69}
e(T) \text{ sign } v_n = -2Z(v_n)\text{ Im} Z(v_n). \tag{70}
\]

The Green’s function can then be written in the following very general form:

\[
G(v_n, p) = \frac{Z(v_n)}{i\omega_n - e(p, v_n) + (i/2)E(T) \text{ sign } v_n}. \tag{71}
\]

If there is no dependence on \( v_n \) in (69), we can write the Green’s function in the usual form,

\[
G(v_n, p) = \frac{Z}{i\omega_n - e(p) + (i/2)E(T) \text{ sign } v_n}, \tag{72}
\]

which is sufficient for our aims.
These expressions more or less correspond to the usual picture of electron–phonon interaction, when $\lambda \ll 1$, though we can try to ‘drag’ them to the region of $\lambda > 1$, using the Migdal theorem [18], which allows us to disregard vertex corrections. In particular, in this approach, the renormalization factor is [18]

$$Z = \frac{1}{1 + \lambda}$$

(73)

and does not depend explicitly on $\epsilon_n$. Equation (73) is valid within an energy layer of the order of the double Debye frequency around the Fermi level (in the case of interaction with phonons) or the double average frequency of fluctuations ($Q$). Outside this layer, it is obvious that $Z = 1$ fluctuations are practically irrelevant for electrons of high enough energy.

5.2 General expressions for conductivity

The diagonal element of the conductivity tensor at $q = 0$ can be written as [18, 51]

$$\sigma_{xx}(\omega) = \frac{1}{i \omega} \left[ \Phi_{xx}(\omega + i\delta) - \Phi_{xx}(0 + i\delta) \right],$$

(74)

where

$$\Phi_{xx}(i \omega_m) = -2e^2 T \sum_n \sum_p \frac{P_x}{m} J_x(p, p, \epsilon_n, \epsilon_n + \omega_m)$$

$$\times G(\epsilon_n p) G(\epsilon_n + \omega_m p),$$

(75)

which corresponds to the usual loop diagram shown in Fig. 11, where the ‘current’ vertex can be written as

$$J_x(p, p, \epsilon_n, \epsilon_n + \omega_m) \equiv \frac{e}{m} p_p \Xi(p, \epsilon_n, \epsilon_n + \omega_m),$$

(76)

and the ‘bare’ current vertex is $(e/m) p_p$.

Then, rewriting Eqn (75) as

$$\Phi_{xx}(i \omega_m) = -2e^2 T \sum_n \sum_p \frac{P_x}{m^2} \Xi(p, \epsilon_n, \epsilon_n + \omega_m)$$

$$\times G(\epsilon_n p) G(\epsilon_n + \omega_m p),$$

(77)

and performing the standard summation over $n$ and analytic continuation $i \omega_m \to \omega + i\delta$ [51], we obtain the static conductivity at $(\omega \to 0)$:

$$\sigma_{xx} = \frac{e^2}{2\pi} \sum_p \frac{P_x^2}{m^2} \Xi(p) G^R(p, 0) G^A(p, 0),$$

(78)

where we have introduced the static limit $\Xi(p) = \Xi(p, 0 - i\delta, 0 + i\delta)$.

Obviously, the main difficulty here is related to the explicit calculation of the vertex part for the system of interacting particles. The simplest estimate can be obtained using in Eqn (75) the obvious Ward identity [51], which is valid for $\omega_m = 0$:

$$J_x(p, p, \epsilon_n, \epsilon_n) = -e \frac{\partial}{\partial p_p} G^{-1}(\epsilon_n, p) = -e \left[ -v_p \frac{\partial}{\partial p_p} \right],$$

(79)

where velocity $v_p = \partial \xi_p / \partial p_p$. If we assume here that the self-energy is momentum independent, i.e., $\Xi(\epsilon_n, p) = \Sigma(\epsilon_n)$, as usually takes place for electron–phonon interaction, vertex (79) is reduced to the ‘bare’ one:

$$J_x(p, p, \epsilon_n, \epsilon_n) = e v_p^* = -\frac{e}{m} p_p,$$

(80)

which corresponds to $\Xi(p, \epsilon_n, \epsilon_n) = 1$. Using (79) in (75), we get

$$\Phi_{xx}(i \omega_n) = -2e^2 T \sum_n \sum_p \frac{P_x}{m} \frac{\partial}{\partial p_p} G^{-1}(\epsilon_n, p)$$

$$\times G(\epsilon_n p) G(\epsilon_n + \omega_n p)$$

$$= -2e^2 T \sum_n \sum_p \frac{P_x^2}{m^2} G(\epsilon_n p) G(\epsilon_n + \omega_n p),$$

(81)

and then, making all transformations as in going from (75) to (78), we obtain the following approximate relation for static conductivity:

$$\sigma_{xx} = \frac{e^2}{2\pi} \sum_p \frac{P_x^2}{m^2} G^R(p, 0) G^A(p, 0),$$

(82)

which simply corresponds to the choice of $\Xi(p) = 1$ in (78). In contrast to (78), this is certainly a kind of approximation. It is used, for example, in Ref. [48]. What is lost here will become clear below.

Further calculations will be done for the general case of (78). Keeping in mind the typical metal, where all the physics of conductivity is determined in the vicinity of the Fermi surface (and scattering is rather weak), we can use the usual integration over the energy spectrum linearized close to the Fermi level and write (78) as (Re $\Sigma(0)$ can be included in the renormalization of the chemical potential)

$$\sigma_{xx} = \frac{e^2}{2\pi} N(0) \int_{-\infty}^{\infty} d\xi_p \frac{1}{3m^2} \Xi(p\xi) G^R(p, 0) G^A(p, 0)$$

$$= \frac{e^2}{2\pi} \frac{1}{3} v_F^2 N(0) \int_{-\infty}^{\infty} d\xi_p \frac{1}{\xi_p^2 + 1} \Sigma(0)$$

$$= \frac{e^2}{2\pi} \frac{1}{3} v_F^2 N(0) \Xi(p\xi) \frac{1}{2} \Im \Sigma(0) \equiv ne \tau_e.$$ 

(83)

From this expression, it is clear, by the way, that approximation $\Xi(p\xi) = 1$ corresponds to ignoring the difference between $\tau$ and $\tau_e$, which we used everywhere above. For the case of scattering by point-like impurities or Einstein phonons, this is simply always valid — from (83) we immediately get (48). Precisely the same expression for conductivity is used, for instance, in Ref. [48]. It now is clear that the mass renormalization, appearing, for example, in the expression for electronic specific heat, does not enter the expression for $\sigma_{xx}$, where everything is determined by the ‘bare’ (band) mass. This result has been well known, actually, for a long time [52–55].

---

Figure 11. Loop diagram determining conductivity.
The contribution from the possible momentum dependence of $\Sigma(\varepsilon, \mathbf{p})$ to the static conductivity can also be obtained in a rather simple way [51]. Let us write

$$G^R(\varepsilon, \mathbf{p}) = \left[ Z^{-1} - b^{-1} \varepsilon_p + \text{Im} \, \Sigma(\varepsilon, \mathbf{p}) \right]^{-1},$$

(84)

where, as usual, we introduced

$$Z^{-1} = 1 - \frac{\partial}{\partial \varepsilon_p} \text{Re} \, \Sigma(\varepsilon, \mathbf{p})|_{\varepsilon=0},$$

(85)

$$b^{-1} = 1 + \frac{\partial}{\partial \varepsilon_p} \text{Re} \, \Sigma(0, \mathbf{p})|_{\mathbf{p}=\mathbf{p}_F}.$$  

(86)

Then, using Eqn (84), we obtain the density of states renormalization on the Fermi level and get

$$\tilde{N}(0) = -\frac{1}{\pi} \sum_{\mathbf{p}} \text{Im} \, G^R(0, \mathbf{p})$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\varepsilon_p}{b^{-2} \varepsilon_p^2 + \text{Im} \, \Sigma(0, \mathbf{p})} = bN(0).$$

(87)

Now, we may repeat all calculations for conductivity done above using (84) and obtain the final result for static conductivity as [51]

$$\sigma_{xx} = \frac{ne^2}{m} \frac{\Sigma(\mathbf{p}_F)}{2 \text{Im} \, \Sigma(\mathbf{p}_F)} \tilde{N}(0) = \frac{ne^2}{m} \frac{\text{Im} \, \Sigma(\mathbf{p}_F)}{N(0)}.$$  

(88)

5.3 Self-consistent calculation

Let us return to the simplest model of electron–phonon interaction and make the self-consistent calculation, when the electron line in the diagram of Fig. 7 is ‘dressed’, i.e., taken into account all orders of nonintersecting interaction lines. Again, we assume the validity of Migdal’s theorem [18].

Acting in the spirit of Eqns (66)–(72), we can write the expression for the Green’s function as (simplified variant of Eqn (71))

$$G(\varepsilon_a, \mathbf{p}) = \frac{Z}{\varepsilon_a - Z \varepsilon_p + (i/2) \tilde{\Gamma}(T) \text{ sign } \varepsilon_a},$$

(89)

which corresponds to a choice of $Z(\varepsilon_a) = Z$, $\text{Re} \, \chi(\varepsilon_a) = 0$, and $\text{Im} \, \chi(\varepsilon_a) = -\tilde{\Gamma}(T)$ in Eqn (67). Renormalization factor $Z < 1$ is assumed to be a constant for simplicity. Then, it may seem that, taking $\tilde{\Gamma}(T) = 2\pi\lambda T$, we may introduce the renormalized damping as

$$\tilde{\Gamma}(T) = \Gamma(T) = 2\pi\lambda T = 2\pi\lambda \frac{T}{1 + \lambda},$$

(90)

where we have accounted for the known result due to electron–phonon interactions in the low-temperature limit [18]:

$$Z = \frac{1}{1 + \lambda}.$$  

(91)

Accordingly, for $\lambda \ll 1$ we have (46), and for $\lambda \gg 1$ we get

$$\tilde{\Gamma}(T) = 2\pi T,$$

(92)

i.e., the universal ‘Planckian’ behavior of relaxation rate (21) with $\lambda = 2\pi$ independent of the coupling constant of electrons with fluctuations (phonons). In the general case, for arbitrary values of $\lambda$ we have $\lambda > 2\pi$, so that the upper limit for $\lambda$ appears in a natural way and is defined by (92).

However, substituting (89) into (44), which is obtained in the high-temperature limit, in the model with the Einstein spectrum of fluctuations (phonons), we immediately obtain

$$\Sigma(\varepsilon_a, \mathbf{p}) = \Sigma(\varepsilon_a) = -\frac{2\pi^2 \hbar^2}{\Omega_0} N(0) T \text{ sign } \varepsilon_a$$

$$= -\frac{i}{2} \Gamma(T) \text{ sign } \varepsilon_a,$$

(93)

so that the renormalization factor $Z$ in damping is canceled out, and we just reproduce the usual result derived above without any renormalizations. This is not surprising at all — it is clear from the very beginning that $Z \approx 1$ for temperatures (energies) much higher than the characteristic phonon frequencies (or any other fluctuation quanta scattering the electrons). Similar results are valid not only in the Einstein model but also in the general case, described by the Eliashberg–McMillan approximation (51), (60) [35].

It may also seem that the limitations discussed above may be obtained from completely different considerations. Let us write the quite general expression for the Matsubara Green’s function in the high-temperature limit as

$$G(\varepsilon_a, \mathbf{p}) = \frac{1}{\varepsilon_a - \varepsilon_p + (i/2) \Gamma(T) \text{ sign } \varepsilon_a}.$$  

(94)

After substituting the limiting value of $\Gamma(T)$ from (21) to (94), we obtain

$$G(\varepsilon_a, \mathbf{p}) = \frac{1}{\varepsilon_a - \varepsilon_p + (i/2)\pi T \text{ sign } \varepsilon_a}.$$  

(95)

We then immediately see that the constant $\lambda$, seemingly, cannot reach the value of $\lambda = 2\pi$, as Matsubara frequencies in (95) become in this case even, i.e., the system of fermions ‘turns’ into bosons, which is just impossible because of the general spin-statistics theorem — no interaction (leading, for example, to temperature-dependent relaxation) can change the statistics of particles. Then, apparently, we have to introduce the same limitation again:

$$\Gamma(T) = \frac{1}{\pi \Gamma(T)} < 2\pi T,$$

(96)

which means, notably, that in our model, according to Eqn (61), we always have an inequality $\lambda < 1$. Such a limitation on the coupling constant agrees with recent results of quantum Monte Carlo calculations of electron–phonon interactions [56] but contradicts many years of experience in studies of superconductivity within the Eliashberg–McMillan theory [33, 34]. A discussion of these contradictions and possible solutions can be found in the recent paper [57].

In fact, the arguments given above are incorrect, and the value of $\lambda$ can take any values, including integers, with no paradoxes like the change of statistics. These values are not special, and the system may continuously pass through them with the growth of $\lambda$ (i.e., the coupling strength). This can be easily understood by making, for instance, explicit calculations of the distribution function (see Appendix).

It should be noted, of course, that the $T$-linear behavior of damping in the Green’s function in all cases breaks the
standard criteria for Fermi-liquid behavior [38], so that quasiparticles in the system are badly defined. This is also
seen from the explicit form distribution function, which is rather far from the usual Fermi step-like function.

5.4 Planckian relaxation delusion
In Refs [3, 4], experimental data on resistivity were repre-
dented by Drude expression (13), where the effective mass \( m^* \) was determined from measurements of specific heat or the
de Haas–van Alphen effect, which in the model with electron–phonon or a more general model of scattering by quantum
fluctuations of an arbitrary nature, is obtained from the band
structure effective mass by a simple substitution \( m \rightarrow m^* = m(1 + \lambda) \), which takes into account mass renor-
malization by interactions (at low temperatures!). The
inconsistency of this approach was already stressed in
Ref. [48]. Let us show that precisely this representation of
relaxation in metals. In fact, Eqn (48) for the
limit of resistivity can be identically rewritten as

\[
\rho(T) = \frac{m(1+\lambda)}{ne^2} \frac{\Gamma(T)}{1+\lambda} = \frac{m^*}{ne^2} \tilde{\Gamma}(T),
\]

where

\[
\tilde{\Gamma}(T) = \frac{2\pi}{1+z} \frac{\lambda}{T},
\]

which always leads to

\[
\tilde{\Gamma}(T) < 2\pi T
\]

just imitating the universal ‘Planckian’ behavior of relaxation rate (21) with \( z = 2\pi \) as an upper limit independent of the
coupling constant of electrons with fluctuations (phonons).
The substitution of \( m \rightarrow m^* = m(1 + \lambda) \) in (97) by itself is
correct, despite being used in an expression in the high-
temperature limit. Also correct is the treatment of experi-
mental data in Refs [3, 4], where they used the effective mass \( m^* \), obtained from low-temperature measurements. However,
this approach clouds the crux of the matter, creating the
delusion of universal Planckian behavior.

It is easy to estimate that the experimentally observed
values [3, 4] of \( z \sim 1 \) correspond to the rather typical \( \lambda \sim 0.2 \), while \( z \sim 2.8 \) for lead [3] corresponds to \( \lambda \sim 0.53 \). Calcula-
tions within the Eliashberg–McMillan theory for Pb give \( \lambda \sim 1.5 \) [34]. For Nb, in Ref. [3] it was found that \( z \sim 2.3 \), which,
according to the expressions given above, corresponds to \( \lambda \sim 0.43 \), while the calculations in [34] give \( \lambda \sim 1.26 \).
Suppression by a factor of \( \sim 3 \) of our values of \( \lambda \) may be possibly related to the fact that in expressions for resistivity
we must use \( \lambda_{ee} < \lambda \). However, the calculated values of \( \lambda_{ee} [34] \)
equal 1.19 for Pb and 1.17 for Nb, which does not improve
much the agreement with experiments.

Much more important may be the account of mass renormalization due to electron–electron interactions. In the Landau–Silitn Fermi-liquid theory,
\( \lambda_{ee} = F_1^t/3 \), where \( F_1^t \) is the appropriate coefficient in
the expansion of the Landau function [38]. For typical metals, \( \lambda_{ee} \sim 1 \). Then,

\[
\tau = \frac{2\pi}{1+\lambda+\lambda_{ee}},
\]

so that, taking \( \lambda \sim 0.4 \) and \( \lambda_{ee} \sim 1 \) as typical, we immediately obtain \( \tau \sim 1.04 \), while for \( \lambda \sim 0.5 \) we have \( \tau \sim 1.25 \), in good
agreement with the majority of the data of Refs [3, 4]. For Pb,
taking \( \lambda \sim 1.68 \) [34] and \( \lambda_{ee} = 1 \), we get \( \tau \sim 2.86 \), in reason-
able agreement with the ‘experimental’ value of \( \tau_{exp} = 2.8 \) [3].
Similarly, for Nb we have \( \lambda = 1.26 \) [34], so that, again, using
\( \lambda_{ee} = 1 \), we obtain \( \tau \sim 2.42 \), in good agreement with the
‘experimental’ value of \( \tau = 2.3 \) [3]. The interval of the values of \( \tau = 0.7 - 2.8 \) [3, 4] for \( \lambda_{ee} = 1 \) corresponds to the interval of \( \lambda = 0.25 - 1.5 \), which seems quite reasonable.

More detailed results of such estimates (assuming \( \lambda_{ee} = 1 \))
are given in Table 2.

| Metal | \( \lambda \) | \( 2\pi\lambda/1+\lambda \) | \( 2\pi/1+\lambda+\lambda_{ee} \) | \( \tau_{exp} \) |
|-------|-------------|----------------|----------------|---------------|
| Pb    | 1.68        | 3.93           | 2.86           | 2.8           |
| Nb    | 1.26        | 3.50           | 2.42           | 2.3           |
| Cu    | 0.14        | 0.77           | 0.41           | 1.0           |
| Al    | 0.44        | 1.91           | 1.13           | 1.1           |
| Pd    | 0.35        | 1.63           | 0.93           | 1.1           |

We can now see that, for all metals under consideration
here, with the exception of Cu, experimental data on coefficient \( \tau \) are described rather satisfactorily, and full
agreement can be achieved by minor variations in
\( \lambda_{ee} \) around the value of \( \lambda_{ee} = 1 \). This is not so only for Cu, where
agreement can be reached by the introduction of \( \lambda_{ee} < 0 \),
Negative values of \( \lambda_{ee} \) are possible, taking into account that the
general limitation here [38] is \( F_1^t > -3 \), i.e., \( \lambda_{ee} > -1 \).

Thus we come to the main conclusion: the ‘experimentally’
observed universal Planckian relaxation in metals,
independent of the value of the coupling constant, is nothing
more than a delusion, related to the procedure of representa-
tion of experimental data in Refs [3, 4] (determination of the
effective mass in the expression for resistivity from low-
temperature measurements). Similarly, the same applies to
the results of a recent paper [58], where this procedure was
used to represent experimental data on bilayer graphene near
the ‘magic’ angle of (mis)orientation of the layers.

5.5 Once again on the uncertainty principle
Where is the problem with estimates of the relaxation time
based on the energy–time uncertainty relation? This is rather
simple — in Eqn (20), \( \Delta \tau \) should be taken not as \( k_B T \), but as a
real ‘level width’ (spectrum damping) in the system of many
interacting (f) particles:

\[
\Gamma(T) = 2\pi\lambda T.
\]

Then, we immediately have

\[
\frac{\hbar}{\tau} \lesssim 2\pi\lambda T.
\]
so that the use of the uncertainty relation becomes a kind of tautology. Then, the ‘upper limit’ of the high-temperature relaxation rate is not universal and naturally proportional to the coupling constant.

At the same time, the result of Eqn (99) obtained above formally defines some universal upper limit for the relaxation rate of quasiparticles, so that the estimates based on the uncertainty relation can probably be applied here.

6. Conclusions
As we stressed above, our aims did not include an explanation of $T$-linear growth of resistivity, starting from the lowest temperatures, which is observed in copper oxide high-temperature superconductors and some similar systems. In the models of scattering by quantum fluctuations of an arbitrary nature (e.g., phonons), such a behavior appears at temperatures of the order of or higher than the characteristic frequency of these fluctuations, i.e., in the classical limit. At present, it is unclear whether such low-frequency fluctuations exist in the vicinity of the quantum critical point on the phase diagram of cuprates, which could have explained their anomalous behavior. But this is the simplest calculable model which clearly shows that the universal Planckian behavior of the relaxation rate is just absent, as is the universal upper limit for this relaxation rate. A dependence on the interaction parameters in the relaxation rate does not disappear, and it can, in principle, easily overcome the limit of $\sigma \sim 1$.

In the literature, quite a number of microscopic models have been proposed to explain the $T$-linear growth of resistivity (inelastic scattering rate) in metals. We have already mentioned Refs [10, 11, 13], where this behavior appeared in the model of ‘fermion’ condensation and was related, in particular, to the formation of ‘flat’ bands near the Fermi level and the appearance of low-frequency zero-sound excitations, so that scattering by these excitations leads to a $T$-linear growth of the scattering rate.

In Ref. [12], an interesting model was proposed with random interactions of electrons, where the ‘flat’ band also forms, and for a wide interval of parameters, the $T$-linear growth of resistivity is realized. It is unclear, however, what relation, if any at all, the model of interactions assumed in Ref. [12] has to real metals.

Note also that in all these approaches the universal ‘Planckian’ dependence on temperature does not appear, and the dependence on the coupling strength (though probably a weak one) is conserved. A similar situation is realized in a popular phenomenological model of a marginal Fermi liquid [48], which successfully explains many properties of cuprates.

An interesting model was proposed recently in Ref. [59] to explain the anomalous temperature dependence of resistivity in Sr$_3$Ru$_2$O$_3$ by electron–electron scattering processes, taking into account the rather complicated real electronic spectrum of this compound, leading to a many-sheet Fermi surface, with some special ‘hot’ pockets, where electrons are non-degenerate (classical limit). It is the scattering of electrons from ‘cold’ (degenerate) parts of the Fermi surface, with a transition of one of these electrons to the ‘hot’ pocket, which leads to the linear growth of resistivity with temperature. The analogy with the model with scattering by nondegenerate fluctuations (classical limit) considered above is pretty obvious. Of course, the dependence on the value of the appropriate coupling constant does not disappear, and the ‘universal’ Planckian behavior appears only for the renormalized (in the sense discussed above) scattering rate of the quasiparticles.

A special place is occupied by Refs [14–17], based on analogies taken from black hole physics, cosmology, and superstring theory, which pretend to explain the universal ‘Planckian’ behavior of the relaxation rate. Being very interesting from the theoretical point of view, these papers, in our opinion, have a weak relation to solid state physics.

In this paper, we have limited ourselves to an elementary analysis based on the standard approaches of the solid state theory and shown that the observed universal ‘Planckian’ behavior of the electron relaxation rate in many metals [3, 4] has a rather simple explanation not related to some special deep physics but simply to the method of representation of experimental data used in these papers. The use in the Drude expression for conductivity (resistivity) of the effective mass, determined from low-temperature measurements, which includes the renormalization due to many-particle effects (interactions), inevitably leads to appropriate renormalization of the relaxation rate (time), which is replaced by an effective rate $\Gamma$ (98) of relaxation for quasiparticles, which is relatively weakly dependent on the coupling strength and, in principle, produces the universal ‘Planckian’ behavior as an upper limit for this relaxation rate. However, this behavior is a pure delusion and does not reflect any kind of special physics. In microscopic theory, it is the unrenormalized (by interactions) band mass which enters the Drude expression for conductivity, and the relaxation rate is naturally proportional to the strength of interaction, as has been well known since the early days of quantum solid state theory.

The size of the effective relaxation rate $\hbar/\tau = k_B T$ (i.e., in fact, the ‘Planckian’ rate discussed above) in the standard theory, as is also well known, defines some characteristic scale which separates regions of different behavior of the resistivity of metals, as shown in Fig. 12, taken from textbook [25]. There, we also show characteristic temperature dependences of resistivity of metals. The ‘Planckian’ scattering rate determines the diagonal on this figure, but does not define any new quantum limit. In principle, all this has been known

Figure 12. Temperature–disorder plane ($T, \hbar/\tau$), where regions of applicability of different models of electrical resistivity are shown. Also shown are the temperature dependences of resistivity. The ‘Planckian’ relaxation rate determines the diagonal on this plane.
for a long time and does not require any ‘exotic’ approaches to explain it.

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Appendix. Momentum distribution in the case of Planckian relaxation

Consider the momentum distribution corresponding to Green’s function (95):

\[ N(p) = T \sum_n G(i \epsilon_n, p) . \] (A.1)

Performing the standard summation over fermion Matsubara frequencies [18], we obtain

\[
N(p) = -\frac{1}{2\pi} \int f(i) G(i, p) \, \mathrm{d}f(i) \frac{\exp(i \epsilon - \epsilon_p) - G^A(i, p)}{\epsilon - \epsilon_p + i0} \cdot \exp(-i \epsilon \tau) + \int_{-\infty}^{\infty} f(i) A(i, p) . \] (A.2)

where \( f(i) = 1/\exp(\epsilon/T) + 1 \) is the Fermi distribution, and the spectral density \( A(i, p) \) is given by the Lorentzian

\[
A(i, p) = \frac{1}{\pi} \frac{(\alpha/2) T - \epsilon}{\epsilon^2 - (\alpha/2)^2} = \frac{1}{\pi} \frac{\exp(-i \epsilon \tau) + \int_{-\infty}^{\infty} f(i) A(i, p) . \] (A.3)

In Fig. 13, we show distribution functions \( N(p) \), obtained numerically directly from (A.2) with spectral density (A.3) for different values of \( \tau \). However, for special \( \tau = 2\pi n \), \( n = 0, 1, \ldots \), distribution functions can be obtained analytically. For \( \tau = 0 \), we obviously get the usual Fermi distribution \( N(p) = f(\epsilon_p) \). For \( \tau = 2\pi \), we obtain

\[
G(i \epsilon_n, p) = \frac{1}{\tau} \frac{\exp(-i \epsilon \tau) + \int_{-\infty}^{\infty} f(i) A(i, p) . \] (A.4)

where \( m \) is an integer, but sign \( \epsilon_n \) excludes the value of \( m = 0 \). Then, the distribution function of particles is

\[
N(p) = T \sum_n \frac{1}{\tau} \frac{\exp(-i \epsilon \tau) + \int_{-\infty}^{\infty} f(i) A(i, p) . \] (A.5)

Note that the sign before summation over seemingly even Matsubara frequencies remained the same (fermion-like), which is related to initial summing over odd frequencies. The second term in the last expression in (A.5) compensated the contribution of \( m = 0 \), which appeared in the sum over \( m \) in the first term. Making the standard summation over even frequencies \( \epsilon_n = 2\pi Tm \), we get

\[
T \sum_n \frac{1}{\tau} \frac{\exp(-i \epsilon \tau) + \int_{-\infty}^{\infty} f(i) A(i, p) . \] (A.6)

where \( n(\epsilon) = 1/\exp(\epsilon/T) - 1 \) is the Bose distribution. Thus, for \( \tau = 2\pi \),

\[
N(p) = f(\epsilon_p) . \] (A.7)

Note once again that the minus sign before the Bose distribution in (A.7) is related to initial summation over fermion frequencies.

For \( \tau = 4\pi \), again we have summation over odd frequencies, but with the two fermion Matsubara frequencies closest to zero excluded:

\[
N(p) = T \sum_n \frac{1}{\tau} \frac{\exp(-i \epsilon \tau) + \int_{-\infty}^{\infty} f(i) A(i, p) . \] (A.8)

In Fig. 13, for \( \tau = 2\pi \) and \( \tau = 4\pi \) we show both distributions obtained numerically from (A.2) with spectral density (A.3) and obtained directly from (A.7) and (A.8) (obviously, these are identical).

In general, it can be seen that the distribution functions in our model are quite different from the usual step-function of the Fermi-liquid theory.

Note added in proof. After this work was accepted for publication we have learned on the paper [60], where the authors came to similar conclusions in the context of electron–phonon interaction.
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