LIOUVILLE THEOREM
AND A PRIORI ESTIMATES OF RADIAL SOLUTIONS
FOR A NON-COOPERATIVE ELLIPTIC SYSTEM

PAVOL QUITTNER

Department of Applied Mathematics and Statistics, Comenius University
Mlynská dolina, 84248 Bratislava, Slovakia
email: quittner@fmph.uniba.sk

Abstract. Liouville theorems for scaling invariant nonlinear elliptic systems (saying that the system does not possess nontrivial entire solutions) guarantee a priori estimates of solutions of related, more general systems. Assume that $p = 2q + 3 > 1$ is Sobolev subcritical, $n \leq 3$ and $\beta \in \mathbb{R}$. We first prove a Liouville theorem for the system

$$
-\Delta u = |u|^{2q+2}u + \beta |v|^{q+2}|u|^q u
-\Delta v = |v|^{2q+2}v + \beta |u|^{q+2}|v|^q v,
$$

in the class of radial functions $(u, v)$ such that the number of nodal domains of $u, v, u - v, u + v$ is finite. Then we use this theorem to obtain a priori estimates of solutions to related elliptic systems. In the cubic case $q = 0$, those solutions correspond to the solitary waves of a system of Schrödinger equations, and their existence and multiplicity have been intensively studied by various methods. One of those methods is based on a priori estimates of suitable global solutions of corresponding parabolic systems. Unlike the previous studies, our Liouville theorem yields those estimates for all $q \geq 0$ which are Sobolev subcritical.

Keywords. Liouville theorem, a priori estimate, elliptic system, Schrödinger equation

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1. Introduction and main results

We are mainly interested in a priori estimates of radial solutions of the problem
\begin{align*}
-\Delta u + \lambda u + \gamma v &= |u|^{2q+2}u + \beta|v|^{q+2}|u|^q u, \\
-\Delta v + \lambda v + \gamma u &= |v|^{2q+2}v + \beta|u|^{q+2}|v|^q v,
\end{align*}
\begin{equation}
\tag{1}
\end{equation}
\begin{align*}
u &= v = 0 \text{ on } \partial \Omega \text{ if } \partial \Omega \neq \emptyset,
\end{align*}
where either \( \Omega = B_R := \{ x \in \mathbb{R}^n : |x| < R \} \) or \( \Omega = \mathbb{R}^n, n \leq 3, \lambda, \gamma, \beta \in \mathbb{R}, p := 2q + 3 \in (1, p_S) \), and \( p_S \) denotes the critical Sobolev exponent:
\begin{equation*}
p_S := \begin{cases}
\frac{n+2}{n-2}, & \text{if } n \geq 3, \\
\infty, & \text{if } n \in \{1, 2\}.
\end{cases}
\end{equation*}
In the cubic case \( p = 3 \), solutions of (1) correspond to the solitary waves of a system of Schrödinger equations and their existence and multiplicity have been intensively studied by various (mainly variational) methods; see the references in [16] or [9] if \( \gamma = 0 \) or \( \gamma \neq 0 \), respectively. The case \( p \neq 3 \) has also been studied, see [8, 7] and the references therein.

Topological and global bifurcation arguments often require a priori estimates of solutions and such estimates have been obtained for \( n \leq 3, p = 3 \) and positive solutions in [3, 10, 9], for example, by proving and/or using suitable Liouville theorems for the related scaling invariant problem
\begin{align*}
-\Delta u &= |u|^{2q+2}u + \beta|v|^{q+2}|u|^q u, \\
-\Delta v &= |v|^{2q+2}v + \beta|u|^{q+2}|v|^q v,
\end{align*}
\begin{equation}
\tag{2}
\end{equation}
and the corresponding Dirichlet problem in a halfspace.

Another method of proving existence and multiplicity results for (1) is to consider the corresponding parabolic problem
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u + \lambda u + \gamma v &= |u|^{2q+2}u + \beta|v|^{q+2}|u|^q u, \\
\frac{\partial v}{\partial t} - \Delta v + \lambda v + \gamma u &= |v|^{2q+2}v + \beta|u|^{q+2}|v|^q v,
\end{align*}
\begin{equation}
\tag{3}
\end{equation}
\begin{align*}
u &= v = 0 \text{ on } \partial \Omega \times (0, \infty) \text{ if } \partial \Omega \neq \emptyset,
\end{align*}
and use the fact that (if \( \gamma = 0 \), then) the number of zeroes and intersections of radial solutions of (3) is nonincreasing in time. Such arguments have been used in [30, 16] if \( n \leq 3, p = 3, \lambda > 0 = \gamma \), and they again require a priori estimates of suitable global solutions of (3).

The arguments in the proofs of a priori estimates in [3, 10, 9] or [30, 16] do not allow one to cover the full subcritical range \( p < p_S \) if \( n = 3 \) or \( 2 \leq n \leq 3 \), respectively (see Remark 6 for more details). The main result of this paper is a Liouville theorem for radial solutions of (2), (possibly nonradial) solutions of (2) with \( n = 1 \), and solutions of the problem
\begin{align*}
-u_{xx} &= |u|^{2q+2}u + \beta|v|^{q+2}|u|^q u, \\
v_{xx} &= |v|^{2q+2}v + \beta|u|^{q+2}|v|^q v,
\end{align*}
\begin{equation}
\tag{4}
\end{equation}
\begin{align*}
u(0) &= v(0) = 0
\end{align*}
(see Theorem 1). Using that theorem we obtain the required a priori estimates (for both (1) and (3)) in the full subcritical range. In the case of (3) we will also assume \( p \geq 3 \).
(i.e. \( q \geq 0 \)) in order to avoid some technical problems with local existence and uniqueness of solutions (if \( q < 0 \), then the nonlinearity in (3) is not Lipschitz continuous).

To formulate our results more precisely, let us introduce some notation first. By a nontrivial solution we understand a solution \((u, v)\) such that \((u, v) \neq (0, 0)\).

If \( J \subset \mathbb{R} \) is an interval and \( v \in C(J, \mathbb{R}) \), then we define
\[
z(v) = z_f(v) := \sup \{ j : \exists x_1, \ldots, x_j+1 \in J, x_1 < x_2 < \cdots < x_{j+1}, \\]
\[
v(x_i) - v(x_{i+1}) < 0 \text{ for } i = 1, 2, \ldots, j,\]
where \( \sup(\emptyset) := 0. \) We usually refer to \( z_f(v) \) as the zero number of \( v \) in \( J \). Note that \( z_f(v) \) is actually the number of sign changes of \( v \); it coincides with the number of zeros of \( v \) if \( J \) is open, \( v \in C^1(J) \) and all its zeros are simple. If \( v : \mathbb{R}^n \rightarrow \mathbb{R} \) is a continuous, radially symmetric function, i.e. \( v(x) = \tilde{v}(|x|) \) for some \( \tilde{v} \in C([0, \infty), \mathbb{R}) \), then we define \( z(v) := z(\tilde{v}) \).

Given \( C_1, C_2, C_3, C_4 \geq 0 \), set
\[
\mathcal{K} = \mathcal{K}(C_1, C_2, C_3, C_4) := \{ (u, v) : z(u) \leq C_1, z(v) \leq C_2, z(u - v) \leq C_3, z(u + v) \leq C_4 \},
\]
\[
\mathcal{K}^+ = \mathcal{K}^+(C_4) := \{ (u, v) : u, v \geq 0, z(u - v) \leq C_3 \},
\]
\[
\mathcal{K}^* := \{ (u, v) \in \mathcal{K} : u \neq \pm v \},
\]
and notice that \( \mathcal{K}^+ \subset \mathcal{K}(0,0,C_3,0) \).

The following Liouville theorem has already been proved in [3] in the case of nonnegative solutions, \( \beta < 1 \) and \( p = 3 \). Notice also that if \( \beta \in (-1, \infty) \) or \( \beta > 0 \) and one considers nonnegative solutions, then the nonexistence of nontrivial (radial and nonradial) solutions to problems occurring in the following theorem has been studied in [26, 11] or [27], respectively.

**Theorem 1.** Assume \( n \leq 3 \) and \( p = 2q + 3 \in (1, p_S) \). Let \( C_1, C_2, C_3, C_4 \geq 0 \) be fixed. If \( \beta \neq -1 \), then system (2) does not possess nontrivial classical radial solutions satisfying \((u, v) \in \mathcal{K} \) and system (2) with \( n = 1 \) does not possess nontrivial classical solutions satisfying \((u, v) \in \mathcal{K} \). If \( \beta = -1 \), then all classical radial solutions of (2) satisfying \((u, v) \in \mathcal{K} \) and all classical solutions of system (2) with \( n = 1 \) satisfying \((u, v) \in \mathcal{K} \) are of the form \((c, \pm c)\), where \( c \in \mathbb{R} \). Problem (1) does not possess nontrivial classical solutions satisfying \((u, v) \in \mathcal{K} \) for any \( \beta \in \mathbb{R} \).

Theorem 1 combined with scaling and doubling arguments from [19], and an argument due to [2] (based on the Sturm comparison theorem) yield the following result:

**Theorem 2.** Assume \( \Omega = \mathbb{R}^n \) or \( \Omega = B_R, n \leq 3, \lambda, \gamma \in \mathbb{R} \) and \( p = 2q + 3 \in (1, p_S) \). Let \( C_1, C_2, C_3, C_4 \geq 0 \) be fixed. Let \( B \) be a compact set in \( \mathbb{R} \setminus \{-1\} \) and \( B^* \) be a compact set in \( \mathbb{R} \). Then there exists \( C \) such that any classical radial solution \((u, v) \in \mathcal{K} \) of (1) with \( \beta \in B \), and any classical radial solution \((u, v) \in \mathcal{K}^* \) of (1) with \( \beta \in B^* \) satisfies \( \|(u, v)\|_{\infty} \leq C \).

The proof of Theorem 2 shows that this theorem remains true for solutions of large classes of systems which are perturbations of the scaling invariant system (2). In particular, the estimate \( \|(u, v)\|_{\infty} \leq C \) in Theorem 2 is locally uniform with respect to \( \lambda \) and \( \gamma \).

A straightforward modification of the proof of Theorem 2 (cf. [19]) also guarantees universal singularity estimates. More precisely, if \( \Omega := B_R \setminus \{0\}, R > 2, \) and \( p, \lambda, \gamma, B, B^* \) are as in Theorem 2, then there exists \( C > 0 \) such that any classical radial solution \((u, v) \in \mathcal{K} \) of the system of PDEs in (1) with \( \beta \in B \), and any classical radial solution
(u, v) ∈ K* of the system of PDES in (1) with β ∈ B* satisfies the estimate

\[ |u(x)| + |v(x)| ≤ C|x|^{-2/(p−1)}, \quad 0 < |x| < 1. \]

(The solution (u, v) need not satisfy the boundary condition in (1).)

Theorem 1 and 24 guarantee that the related scaling invariant parabolic problem

\[
\begin{align*}
  u_t - \Delta u &= |u|^{2q+2} u + \beta |v|^{q+2} |u|^q u, \\
 v_t - \Delta v &= |v|^{2q+2} v + \beta |u|^{q+2} |v|^q v,
\end{align*}
\]

in \( \mathbb{R}^n \times \mathbb{R} \),

(5)
do not possess nontrivial radial solutions satisfying \((u, v)(\cdot, t) ∈ K \) for all \( t ∈ \mathbb{R} \), and problems (5) with \( n = 1 \) and

\[
\begin{align*}
  u_t - u_{xx} &= |u|^{2q+2} u + \beta |v|^{q+2} |u|^q u, \\
 v_t - v_{xx} &= |v|^{2q+2} v + \beta |u|^{q+2} |v|^q v,
\end{align*}
\]

\( u = v = 0 \) on \( \{0\} × \mathbb{R} \),

(6)
do not possess nontrivial solutions satisfying \((u, v)(\cdot, t) ∈ K \) for all \( t ∈ \mathbb{R} \). These parabolic Liouville theorems together with scaling and doubling arguments in 20 immediately imply the following universal \( L^∞ \)-estimate for global solutions of (3) (see 20 Corollary 5) for a more general statement:

**Corollary 3.** Assume \( Ω = \mathbb{R}^n \) or \( Ω = B_R, n ≤ 3, \beta \neq -1 \) and \( p = 2q + 3 \in (1, p_S) \). Then there exists \( C > 0 \) such that any global radial classical solution of (3) with \((u, v)(\cdot, t) ∈ K \) for all \( t ∈ (0, ∞) \) satisfies the following estimate:

\[
\|(u, v)(\cdot, t)\|_∞ ≤ C(1 + t^{−1/(p−1)}), \quad t ∈ (0, ∞).
\]

The constant \( C = C(β, λ, γ) \) in Corollary 3 is locally uniform for \( β ∈ \mathbb{R} \setminus \{0\} \) and \( λ, γ ∈ \mathbb{R} \). Notice also that \( K \) or \( K^+ \) is invariant with respect to the semiflow generated by (3) if \( γ = 0 \) or \( γ ≤ 0 \), respectively.

Corollary 3 can be used to prove the following uniform \( H^1 \)-estimate for global radial solutions of (3) with bounded energy and initial data in \( H^1 \cap K \) or \( H^1 \cap K^+ \). By \( H^1 \) we denote the set of radial functions in \( H^1 \) and by \( \| \cdot \| \) the norm in \( H^1(Ω, \mathbb{R}^2) \). We also set

\[
\begin{align*}
  \mathcal{U} &:= (u, v), \\
  \mathcal{F}(\mathcal{U}) &:= (|u|^{2q+2} u + \beta |v|^{q+2} |u|^q u, |v|^{2q+2} v + \beta |u|^{q+2} |v|^q v), \\
  G(\mathcal{U}) &:= \frac{1}{p+1} \mathcal{F}(\mathcal{U}) \cdot \mathcal{U} \quad \text{(hence} \nabla G = \mathcal{F})..
\end{align*}
\]

(7)

**Proposition 4.** Assume \( Ω = \mathbb{R}^n \) or \( Ω = B_R, n ≤ 3, \beta \neq -1, \lambda > 0 ≥ γ, p = 2q + 3 \in [3, p_S) \). If \( Ω = \mathbb{R}^n \), then assume also \( λ + γ > 0 \). Let \( \mathcal{U}_0 ∈ H^1_0(Ω, \mathbb{R}^2) \). If \( γ = 0 \) or \( γ < 0 \), then assume also \( \mathcal{U}_0 ∈ K \) or \( \mathcal{U}_0 ∈ K^+ \), respectively. Assume that the solution of (3) with initial data \( \mathcal{U}(\cdot, 0) = \mathcal{U}_0 \) is global and satisfies \( |E(t)| ≤ C_E \) for \( t > 0 \), where

\[
E(t) := \frac{1}{2} \int_Ω (|\nabla \mathcal{U}(x, t)|^2 + λ|\mathcal{U}(x, t)|^2) \, dx + γ \int_Ω (uv)(x, t) \, dx - \int_Ω G(\mathcal{U}(x, t)) \, dx.
\]

Then

\[
\|\mathcal{U}(\cdot, t)\| ≤ C = C(\|\mathcal{U}_0\|, C_E).
\]

(8)
The $H^1$-estimate in Proposition 4 is based on the universal $L^\infty$-estimates in Corollary 3, but the universality of those estimates is not needed: It would be sufficient to use $L^\infty$-estimates which can depend on $\|U_0\|$ and $C_E$, and such estimates could likely be obtained directly from the elliptic Liouville theorem (Theorem 1) by using the approach in [13] (hence we would not need the parabolic Liouville theorems in [24]). On the other hand, universal $L^\infty$-estimates as in Corollary 3 also enable one to prove the existence of periodic solutions of related problems with time-periodic coefficients, for example (see [4, Section 6]), and such results cannot be obtained by using the weaker estimates depending on $\|U_0\|$ and $C_E$.

As already mentioned, the authors of [16, 30] use the properties of the parabolic semiflow in order to prove the existence and multiplicity of nontrivial radial solutions of (1) with $n \leq 3$, $\lambda > 0$ and $q = \gamma = 0$. More precisely, paper [30] deals with positive radial solutions, $\Omega = B_R$ and $\beta \leq -1$, and paper [16] with nodal radial solutions of various generalizations of (1) and $\beta < 0$ (or $\beta < \beta_0$, where $\beta_0 > 0$ is small enough). In both papers, a priori estimates of suitable global solutions of (3) play an important role. If we consider initial data $U_0 \in A$, where $A$ is the domain of attraction of the zero solution, then the solution of (3) is global and the corresponding energy function $E(t)$ is bounded, hence estimate (8) is true (provided the remaining assumptions in Proposition 4 are satisfied). Estimate (8) then also guarantees that the solutions of (3) with initial data $U_0 \in \partial A$ are global and satisfy (8), and these particular global solutions are used in [16, 30] in order to find solutions of (1) with prescribed number of nodal domains or intersections. The arguments in [30] also require some compactness of those particular global solutions, and such compactness is guaranteed by the next proposition.

**Proposition 5.** Let the assumptions of Proposition 4 be satisfied. If $\Omega = \mathbb{R}^n$, then assume also that $U_0$ is compactly supported and $n \geq 2$. Then the trajectory $t \in [0, \infty) \to H^1_r(\Omega, \mathbb{R}^2) : t \mapsto U(\cdot, t)$ is compact.

The proof in [30] guaranteeing the existence of positive solutions of (1) with prescribed number of intersections required $\Omega = B_R$, $p = 3$, and the authors of [30] also assume $\gamma = 0$. Propositions 4 and 5 enable one to prove analogous results also for $\Omega = \mathbb{R}^n$ and $p \in (3, p_S)$. In addition, one can also consider the case $\gamma < 0$: If $\Omega = B_R$, then in order to guarantee the stability of the zero solution, one has to assume $\lambda + \gamma > -\lambda_1$, where $\lambda_1$ is the first eigenvalue of the negative Dirichlet Laplacian in $\Omega$.

Similarly, Proposition 4 indicates that many arguments from [16] guaranteeing the existence of solutions of (1) with prescribed number of nodal domains in the cubic case $p = 3$ can also be used if $p \in (3, p_S)$.

**Remark 6.** (i) The proofs of Liouville theorems used in [3, 10, 9] heavily depend on the choice $p = 3$: The arguments in those proofs cannot be used if $n = 3$ and $p > 3$, for example.

(ii) The bounds of global solutions of (3) in [30, 16] are proved by integral estimates (cf. [6]) which require $p := 2q + 3 < p_{CL} := (3n + 8)/(3n - 4)$. Condition $p < p_{CL}$ can likely be improved to $p < p_S$ by a bootstrap argument due to [21] (see also [22] or [14, 15] for applications of this argument to more general or rescaled problems), but only if $\beta > -1$. If $\beta \leq -1$, then a modification of that bootstrap argument could likely improve the condition $p < p_{CL}$ slightly if $n = 3$ (to $p < p_{CL} + 1/5$), but not for $n = 2$, cf. [1]. Our results guarantee that the required a priori estimates remain true for any $p \in [3, p_S)$ if $n \leq 3$ and $\beta \neq -1$. 
On the other hand, if \( p = 3, n \leq 3, k \) is a fixed positive integer, \( \lambda > 0 \geq \gamma \) and \( \lambda + \gamma > 0 \), then the integral estimates in [30] of suitable global positive solutions \((u, v)\) of

\[
\text{Equation (1)}
\]

implies where \( G_\beta > 0 \) if arguments (see [4], for example) imply

\[
\text{Equation (2)}
\]

Assume that the \( \omega \)-limit set of such global solution \((u, v)\) contains a positive stationary solution of the form \((u^*, u^*)\). Since the norms of such positive stationary solutions tend to \( \infty \) as \( \beta \to -1 \), this would yield a contradiction if \( \beta \) is close to \(-1\). Consequently, the topological arguments in the proof of [30] Theorem 1.1] leading to the existence of stationary solutions satisfying \( z(u - v) = k \) can be used whenever \( \beta < -1 + \varepsilon_k \), where \( \varepsilon_k > 0 \) is small enough. Our bounds based on Liouville theorems are locally uniform with respect to \( \beta \) only for \( \beta \in \mathbb{R} \setminus \{-1\} \), hence such arguments cannot be used. The reason is that we are using the universal estimates in Corollary [3] which are true for all solutions in \( \mathcal{K} \) including solutions of the form \((u, u)\), hence they cannot be uniform as \( \beta \) approaches \(-1\).

2. Proofs

Proof of Theorem 1. Due to scaling and doubling arguments (see [19]), we only have to prove the nonexistence for bounded solutions. Assume that \((u, v) \in \mathcal{K}\) is a nontrivial bounded radial solution of (2) and \((u, v) \neq (c, \pm c)\) if \( \beta = -1 \). Consider \( u, v \) as functions of the radial variable \( r = |x|, \Delta u(r) = u''(r) + \frac{\lambda - 1}{r}u'(r) \). System (2) possesses nontrivial radial solutions of the form \( W_0 := (w, \pm w) \) or \( W_1 := (w, 0) \) or \( W_2 := (0, w) \), where \( z(w) < \infty \), only if \( \beta = -1 \), and such solutions are of the form \((c, \pm c)\) with \( c \neq 0 \) (see [18] Theorem 2.2) in the case of \( W_1, W_2 \) or \( W_0 \) and \( \beta > -1 \), and see [23 Proposition 4] in the case of \( W_0 \) and \( \beta < -1 \), hence we have \( u \neq v, u \neq -v, u \neq 0 \) and \( v \neq 0 \). Replacing \( u \) by \(-u\) and/or \( v \) by \(-v\) if necessary, we may assume that there exists \( R_0 \geq 0 \) such that

\[
u(r) > v(r) > 0 \text{ for } r > R_0. \tag{9}\]

Assume first \( n \leq 2 \) or \( n = 3 \) and \( p = 2q + 3 \leq 3 \). Set \( w := u - v \) if \( \beta \leq 0 \), and \( w := u \) otherwise. If \( r > R_0 \), then \( w(r) > 0 \) and \( -\Delta w \geq wp \), which contradicts the corresponding Liouville-type theorem for inequalities in exterior domains, see [7], for example. The same argument applies to (possibly nonradial) solutions of (2) in \( \mathbb{R}^3 \), and to solutions of (1). Consequently, we just have to prove the nonexistence of bounded radial solutions of (2) satisfying (5) in the case of \( n = 3 \) and \( p = 2q + 3 \in (3, 5) \).

Theorem 1 for \( n = 1 \) (which we have just proved) together with scaling and doubling arguments (see [1], for example) imply

\[
|u(r)| + |v(r)| + r(|u'(r)| + |v'(r)|) \leq C^x r^{-2/(p-1)}, \quad r > 0. \tag{10}\]

If \( \beta > -1 \), then

\[
C_1 |U|^{p+1} \leq G(U) \leq C_2 |U|^{p+1} \text{ for any } U = (u, v), \tag{11}\]

where \( G \) is defined in (7). In addition, the Rellich-Pohozaev identity [26, Lemma 3.6] (which is true also for nodal solutions) implies

\[
\int_0^R c_p G(U(r)) r^2 \, dr = R^3 (2G(U(R)) + |U'(R)|^2 + \frac{1}{R} U(R) \cdot U'(R)), \tag{12}\]

where \( c_p \) is a constant depending on \( |U(R)|, |U'(R)|, \) and \( p \).
where \( c_p := 5 - p > 0 \). Now (12), (11) and (10) imply
\[
\int_0^R |U(r)|^{p+1} r^2 \, dr \leq CR^{-\frac{2-p}{p-1}} \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty,
\]
which yields a contradiction.

It remains to consider the case \( n = 3, \ p = 2q + 3 \in (3, 5) \) and \( \beta \leq -1 \). Our arguments in this case are inspired by the proof of [18, Theorem 2.5]. In the rest of the proof we denote \( U(r) := r^{2/(p-1)}u(r), \ V(r) := r^{2/(p-1)}v(r) \). Then (10) guarantees
\[
|U(r)| + |V(r)| \leq C^*, \quad r|U'(r)| + r|V'(r)| \leq 2C^*, \quad r > 0. \tag{13}
\]
If \( Z \in \{U, V\} \), then \( Z \) solves the equation
\[
r^2Z'' + arZ' - bZ + F(Z) = 0, \tag{14}
\]
where
\[
a = \frac{2(p - 3)}{p - 1} \in (0, 1), \quad b = \frac{2(p - 3)}{(p - 1)^2} \in (0, \frac{1}{4}),
\]
and
\[
F(Z) = \begin{cases} |U|^{p-1}U + \beta|V|^{q+2}|U|^qU & \text{if} \ Z = U, \\ |V|^{p-1}V + \beta|U|^{q+2}|V|^qV & \text{if} \ Z = V. \end{cases}
\]
Set also
\[
E := -\frac{b}{2}(U^2 + V^2) + \frac{1}{p+1}(|U|^{p+1} + |V|^{p+1}) + \frac{2\beta}{p+1}|UV|^{(p+1)/2},
\]
\[
\varphi := (U')^2 + (V')^2.
\]
Multiplying (14) with \( Z = U \) or \( Z = V \) by \( U' \) or \( V' \), respectively, and adding the resulting equations we obtain
\[
\frac{1}{2}r^2\varphi'(r) + ar\varphi(r) + E'(r) = 0, \tag{15}
\]
and integration by parts yields
\[
\frac{1}{2} \left( \rho^2\varphi(\rho) - r^2\varphi(r) \right) - (1-a) \int_r^\rho s\varphi(s) \, ds + E(\rho) - E(r) = 0, \quad \rho > r. \tag{16}
\]
If \( r > R_0 \), then (9) and \( \beta \leq -1 \) imply \( F(V(r)) \leq 0 \). Assume
\[
V'(r_0) \geq 0 \quad \text{for some} \quad r_0 > R_0. \tag{17}
\]
Then \( V' > 0 \) on \( (r_0, \infty) \), since \( V'' > 0 \) whenever \( V' = 0 \). Fix \( r_1 > r_0 \) and set
\[
\varepsilon := \min(bV(r_1), ar_1V'(r_1)) > 0.
\]
If \( ar_2V'(r_2) < \varepsilon \) for some \( r_2 > r_1 \), then set \( r_3 := \inf \{ r < r_2 : a\rho V'(\rho) < \varepsilon \text{ on } [r, r_2] \} \) and notice that \( r_3 \in [r_1, r_2] \), \( ar_3V'(r_3) = \varepsilon \) and \( bV(r) > bV(r_1) \geq \varepsilon \) for \( r > r_1 \). These estimates, (14) and \( F(V) \leq 0 \) guarantee \( V'' > 0 \) on \( (r_3, r_2) \), hence \( ar_2V'(r_2) > ar_3V'(r_3) = \varepsilon \) which yields a contradiction. Consequently, \( arV'(r) \geq \varepsilon \) for \( r > r_1 \), which contradicts the boundedness of \( V \). Thus (17) fails and we have \( V' < 0 \) on \( (R_0, \infty) \).

If \( V_\infty := \lim_{r \to \infty} V(r) > 0 \), then (14) implies \( r^2V''(r) > bV_\infty/2 =: c_V \) for \( r > r_4 \), hence considering \( R \to \infty \) in the estimate
\[
-V'(r) > V'(R) - V'(r) = \int_r^R V''(\rho) \, d\rho > c_V \int_r^R \frac{1}{\rho^2} \, d\rho = c_V \left( \frac{1}{r} - \frac{1}{R} \right)
\]
we obtain $V'(r) \leq -c_V/r$ for $r > r_4$, which contradicts the boundedness of $V$. Thus $V_\infty = 0$ and $q > 0$ implies $F(V(r)) = o(V(r))$ as $r \to \infty$. Consequently, there exists a positive nonincreasing function $f$ such that $f(r) \to 0$ as $r \to \infty$ and

$$r^2V''(r) + arV'(r) \in (0, f(r)) \quad \text{for } r \text{ large}.$$ 

Assume $(1 - a)rV'(r) < -f(r)$ for some $r$ large. Then $r(rV'(r))' = r^2V''(r) + arV'(r) < r^2V''(r) + arV'(r) - f(r) < 0$, hence $(1 - a)rV'(r) < -f(r)$ for $r > r$. The inequality $|V'(r)| > \frac{f(r)}{1 - a - r}$ contradicts the boundedness of $V$. Hence

$$V(r) + r|V'(r)| = o(1) \quad \text{as } r \to \infty. \quad (18)$$

Fix $M := e^2, \varepsilon_k \searrow 0$ and choose $R_k \nearrow \infty$ such that $R_1 > R_0$ and

$$V(r) + r|V'(r)| < \varepsilon_k \quad \text{for } r \geq R_k. \quad (19)$$

We have two possibilities:

Case A: $(\forall k)(\exists r_k \geq R_k) 0 < U \leq \varepsilon_k$ on $[r_k, Mr_k]$.

Case B: $(\exists k_0)(\forall r \geq R_k_0)(\exists \tilde{r} \in [r, Mr]) U(\tilde{r}) > \varepsilon_{k_0}$.

Consider Case A first. If $r^2\varphi(r) \geq 2\varepsilon_k^2$ on $J_k := [r_k, Mr_k]$, then $(19)$ implies $r|U'(r)| \geq \varepsilon_k$ on $J_k$, hence

$$\varepsilon_k \geq |U(Mr_k) - U(r_k)| = \left| \int_{J_k} U'(r) \, dr \right| \geq \int_{J_k} \frac{\varepsilon_k}{r} \, dr = 2\varepsilon_k,$$

which yields a contradiction. Consequently, there exists $\tilde{R}_k \in J_k$ such that $\tilde{R}_k^2\varphi(\tilde{R}_k) < 2\varepsilon_k^2$. Since $U(\tilde{R}_k), V(\tilde{R}_k) \to 0$, we have

$$E(\tilde{R}_k) \to 0, \quad \tilde{R}_k^2\varphi(\tilde{R}_k) \to 0, \quad \tilde{R}_k \to \infty. \quad (20)$$

Next consider Case B. Set $\varepsilon^* := \varepsilon_{k_0}, R^* := R_{k_0}, I_k := [M^{k-1}R^*, MR^*], k = 1, 2, \ldots$. For each $k$ there exists $\tilde{r}_k \in I_k$ such that $U(\tilde{r}_k) \in [\varepsilon^*, C^*]$. Set

$$u_k(\rho) := \tilde{r}_k^{2/(p-1)}u(\tilde{r}_k\rho), \quad v_k(\rho) := \tilde{r}_k^{2/(p-1)}v(\tilde{r}_k\rho), \quad \rho > R_0/\tilde{r}_k.$$ 

Then $u_k, v_k > 0$ are locally bounded, $v_k \to 0$ locally uniformly, $u_k(1) = U(\tilde{r}_k) \in [\varepsilon^*, C^*]$ and

$$0 = \Delta u_k + u_k^p + \beta v_k^{q+2}u_k^{q+1}.$$ 

Consequently, a subsequence $u_{kj}$ converges in $C_{\text{loc}}$ to a positive solution $\bar{u}$ of $\Delta u + u^p = 0$ in $(0, \infty)$. Fix $m \geq 1$ and set $\rho_{j,m} := \tilde{r}_{kj,m}/\tilde{r}_{kj} \in [M^{m-1}, M^{m+1}]$. Then

$$u_{kj}(\rho_{j,m}) = \tilde{r}_{kj}^{2/(p-1)}u(\tilde{r}_{kj,m}) = \rho_{j,m}^{-2/(p-1)}U(\tilde{r}_{kj,m}) \geq \rho_{j,m}^{-2/(p-1)}\varepsilon^*.$$ 

Since $u_{kj} \to \bar{u}$ on $[M^{m-1}, M^{m+1}]$, there exists $\rho_m \in [M^{m-1}, M^{m+1}]$ such that $\bar{u}(\rho_m) \geq \varepsilon^*\rho_m^{-2/(p-1)}$. Hence $\limsup_{\rho \to \infty} \bar{u}(\rho)\rho^{2/(p-1)} \geq \varepsilon^*$ and [25, Remark 9.5] (see also [12, 28]) shows that $\bar{u}(\rho) = b^{1/(p-1)}\rho^{-2/(p-1)}$. Consequently, $U(\tilde{r}_{kj}, \rho) \to b^{1/(p-1)}$, $V(\tilde{r}_{kj}, \rho) \to 0$ and $E(\tilde{r}_{kj}, \rho) \to \infty := -\frac{p-1}{2(p+1)}b^{1/(p-1)}$, locally uniformly with respect to $\rho > 0$. Fix $\varepsilon \in (0, -E_\infty)$ and $0 < \rho_1 < \rho_2$ such that $\log(\rho_2/\rho_1) > 2C^}*^{-1/2}$, and set $J_j := (\tilde{r}_{kj,\rho_1}, \tilde{r}_{kj,\rho_2})$. Assume that

$$\limsup_{j \to \infty} \inf_{r \in J_j} r^2\varphi(r) \geq \varepsilon. \quad (21)$$

Then $(18)$ implies

$$\limsup_{j \to \infty} \inf_{r \in J_j} r|U'(r)| \geq \sqrt{\varepsilon/2},$$
hence for suitable \( j \) large we obtain \( r|U'(r)| \geq \sqrt{2}/2 \) on \( J_j \) and \( \int_{J_j} U'(r) \, dr \geq \int_{J_j} \frac{\sqrt{2}}{2r} \, dr > C^* \), which contradicts (13). Consequently, (21) fails, hence if \( j \) is large, then there exists \( \tilde{R}_j \in J_j \) such that

\[
\tilde{R}_j^2 \phi(\tilde{R}_j) < \varepsilon < -E_{\infty}, \quad E(\tilde{R}_j) \to E_{\infty}, \quad \tilde{R}_j \to \infty.
\]  

(22)

Notice that \( E(0) = 0 \) and \( \lim_{r \to 0+} r^2 \phi(r) = 0 \). In both Case A and B, due to (20) and (22), respectively, we can pass to the limit in (16) with \( r := 0 \) and \( \rho := \tilde{R}_k \) (or \( \rho := \tilde{R}_j \)) to obtain \( \int_0^\infty s\phi(s) \, ds \leq 0 \), which yields a contradiction.

**Proof of Theorem 2.** If \( \Omega = \mathbb{R}^n \), then set \( R = \infty \). Radial solutions \((u, v)\) will be considered as functions of \( r := |x| \in [0, R) \).

Assume to the contrary that there exist \( \beta_k \in B \) and radial solutions \((u_k, v_k) \in \mathcal{K} \) (or \( \beta_k \in B^* \) and \((u_k, v_k) \in \mathcal{K}^* \)) such that \( \| (u_k, v_k) \|_{\infty} \to \infty \). Then there exist \( r_k \in [0, R) \) such that \( M_k := M(u_k, v_k)(r_k) \to \infty \), where

\[
M(u, v) := |u|^{(p-1)/2} + |v|^{(p-1)/2} + |u'|^{(p-1)/(p+1)} + |v'|^{(p-1)/(p+1)}.
\]

The Doubling Lemma in (19) guarantees that we may assume

\[
M(u_k, v_k) \leq 2M_k \quad \text{on} \quad \{ r \in [0, R) : |r - r_k| \leq \frac{k}{M_k} \}.
\]

Set \( \lambda_k := 1/M_k \). We may assume that \( \beta_k \to \beta \) and also that one of the following three cases occur:

Case A: \( r_k/\lambda_k \to c_0 \geq 0 \).

Case B: \( r_k/\lambda_k \to \infty \) and either \( R = \infty \) or \( (R - r_k)/\lambda_k \to \infty \).

Case C: \( R < \infty \) and \( (R - r_k)/\lambda_k \to c_R \geq 0 \).

We set

\[
\hat{u}_k(\rho) := \begin{cases} \lambda_k^{2/(p-1)}u_k(\lambda_k\rho) & \text{in Case A,} \\ \lambda_k^{2/(p-1)}u_k(r_k + \lambda_k\rho) & \text{in Case B,} \\ \lambda_k^{2/(p-1)}u_k(R - \lambda_k\rho) & \text{in Case C,} \end{cases}
\]

and we define \( \tilde{v}_k \) analogously. We also set

\[
\rho_k := \begin{cases} 0 & \text{in Case A,} \\ r_k/\lambda_k & \text{in Case B,} \\ -(R - r_k)/\lambda_k & \text{in Case C,} \end{cases} \quad \tilde{\rho}_k := \begin{cases} r_k/\lambda_k \to c_0 & \text{in Case A,} \\ 0 & \text{in Case B,} \\ (R - r_k)/\lambda_k \to c_R & \text{in Case C.} \end{cases}
\]

Then

\[
\hat{u}_k' + \frac{n-1}{\rho + \rho_k} \hat{u}_k' - \lambda_k^2(\lambda \hat{u}_k + \gamma \tilde{v}_k) + |\hat{u}_k|^{p-1}\hat{u}_k + \beta_k|\tilde{v}_k|^{q+2}|\tilde{u}_k|^q \tilde{u}_k = 0,
\]

\[
\tilde{v}_k' + \frac{n-1}{\rho + \rho_k} \tilde{v}_k' - \lambda_k^2(\lambda \tilde{v}_k + \gamma \hat{u}_k) + |\tilde{v}_k|^{p-1}\tilde{v}_k + \beta_k|\hat{u}_k|^{q+2}|\hat{u}_k|^q \hat{u}_k = 0,
\]

\[
M(\hat{u}_k, \tilde{v}_k)(\rho_k) = 1, \quad \text{and} \quad M(\hat{u}_k, \tilde{v}_k)(\rho) \leq 2 \quad \text{whenever}
\]

\[
\rho \in [0, R/\lambda_k), \quad |\rho - r_k/\lambda_k| \leq k \quad \text{in Case A,}
\]

\[
\rho \in [-(R - r_k)/\lambda_k, (R - r_k)/\lambda_k), \quad |\rho| \leq k \quad \text{in Case B,}
\]

\[
\rho \in [0, R/\lambda_k), \quad |(R - r_k)/\lambda_k - \rho| \leq k \quad \text{in Case C.}
\]
Consequently, a subsequence of \((\tilde{u}_k, \tilde{v}_k)\) (still denoted \((\tilde{u}_k, \tilde{v}_k)\)) converges locally uniformly to a nontrivial solution \((\tilde{u}, \tilde{v}) \in \mathcal{K}\) of problem (2) or (2) with \(n = 1\) or (3) in Case A or B or C, respectively (notice that \((\tilde{u}, \tilde{v})\) is radial in Case A).

In Case C or if \(\beta \neq -1\), then we obtain a contradiction with Theorem 1.

Assume \(\beta = -1\) and consider Case A or B. Then Theorem 1 and \(M(\tilde{u}_k, \tilde{v}_k)(\tilde{p}_k) = 1\) guarantee \((\tilde{u}, \tilde{v}) = (c, \pm c)\), where \(c = 2^{-2/(p-1)}\). Replacing \(v_k\) by \(-v_k\) (and \(C_3\) by \(C_4\)) if necessary, we may assume \((\tilde{u}, \tilde{v}) = (c, c)\). Since \((u_k, v_k) \in \mathcal{K}^*\), we have \(\tilde{w}_k := \tilde{u}_k - \tilde{v}_k \neq 0\) and we also have

\[
\tilde{w}_k'' + P_k \tilde{w}_k' + Q_k \tilde{w}_k = 0, \quad \text{where}
\]

\[
P_k := \frac{n - 1}{\rho + \rho_k}, \quad Q_k := \lambda_k^2 (\gamma - \lambda) + \frac{|u_k|^{p-1} u_k - |v_k|^{p-1} v_k}{u_k - v_k} - \beta_k |u_k v_k|^q u_k v_k.
\]

Notice also that \(\frac{1}{2} P_k' + \frac{1}{4} P_k^2 = \frac{(n-3)(n-1)}{4(\rho + \rho_k)^2}\). Fix \(R_1 > (p - 1)^{-1/2}\) and consider \(R_2 > R_1\) and \(\rho \in (R_1, R_2)\). Since \(\beta_k \to -1\) and \(\tilde{u}_k, \tilde{v}_k \to c\) locally uniformly, we see that

\[
q_k := Q_k - \frac{1}{2} P_k' - \frac{1}{4} P_k^2 = Q_k - \frac{1}{4 \rho^2}
\]

\[
\to e^{p-1}(p - \beta) - \frac{1}{4 \rho^2} = 1 (p + 1) - \frac{1}{4 \rho^2} > 0,
\]

where the convergence is uniform for \(\rho \in (R_1, R_2)\). Set \(W_k(\rho) = \tilde{w}_k(\rho) \exp(\frac{1}{2} \int_1^\rho P_k)\). Then \(W_k'' + q_k W_k = 0\) and \(q_k > 1/2\) on \((R_1, R_2)\) for \(k\) large enough. Since the solution \(W(r) = \sin(\frac{n}{2} r)\) of the equation \(W'' + \frac{1}{2} W = 0\) has at least \(C_3 + 2\) zeroes in \((R_1, R_2)\) for \(R_2\) large enough, the Sturm comparison theorem guarantees that \(z(\tilde{w}_k) = z(W_k) > C_3\) which contradicts \((u_k, v_k) \in \mathcal{K}^*\) and concludes the proof.

**Proof of Proposition 4** By \(C\) we denote various constants which depend only on \(\|U_0\|\) and \(C_E\).

Problem (3) is well posed in \(H^1\), hence there exists \(\delta = \delta(\|U_0\|) \in (0, 1)\) such that

\[
\|U(t, t)\| \leq C \quad \text{for} \quad t \in (0, \delta].
\]

If \(\Omega = B_R\), then this estimate and Corollary 3 implies

\[
\int_\Omega |U|^2(x, t) dx \leq C, \quad t \geq 0.
\]

Multiplying the first and the second equation in (3) by \(u\) and \(v\), respectively, integrating by parts, summing the identities and using \(\gamma \leq 0\) we obtain

\[
\frac{1}{2} \int_\Omega |U(x, t)|^2 dx \geq -(p + 1) E(t) + \frac{p - 1}{2} \int_\Omega (|\nabla U(x, t)|^2 + (\lambda + \gamma)|U(x, t)|^2) dx.
\]

We also have

\[
C \geq E(t_1) - E(t_2) = \int_{t_1}^{t_2} \int_\Omega |U|^2 dx dt, \quad t_2 > t_1.
\]

Set

\[
\tilde{\lambda} := \begin{cases} 
\lambda & \text{if } \Omega = B_R, \\
\lambda + \gamma & \text{if } \Omega = \mathbb{R}^n.
\end{cases}
\]
and notice that $\tilde{\lambda} > 0$. Now (25), (24) and the boundedness of $E$, and then the Cauchy inequality and (26) guarantee
\[
\int_t^{t+1} \int_\Omega (|\nabla U|^2 + \tilde{\lambda}|U|^2) \, dx \, dt \leq C \left(1 + \int_t^{t+1} \int_\Omega |U| \cdot |U_r| \, dx \, dt\right)
\[
\leq C \left(1 + \left(\int_t^{t+1} \int_\Omega |U|^2 \, dx \, dt\right)^{1/2}\right),
\]
which first shows $\int_t^{t+1} \int_\Omega |U|^2 \, dx \, dt \leq C$, and then
\[
\int_t^{t+1} \|U(\cdot, s)\|^2 \, ds \leq C.
\] (27)

Since $U$ solves the linear equation $U_t = \Delta U - \lambda U + HU$, where the matrix $H = H(x, t)$ satisfies $\|H(\cdot, t)\|_\infty \leq C$ for any $t \geq \delta$ due to Corollary 3, we have
\[
\|U(\cdot, t_0 + \tau)\| \leq C(\|U(\cdot, t_0)\|) \quad \text{whenever} \quad t_0 \geq \delta, \quad \tau \in [0, 2].
\] (28)

Choosing $t_0 = \delta$ in (28) and using (29) we obtain $\|U(\cdot, t)\| \leq C$ for $t \in [0, 2]$. Next (27) guarantees that for each $k = 2, 3, \ldots$ we can find $t_k \in [k - 1, k]$ such that $\|U(\cdot, t_k)\| \leq C$ and (28) guarantees $\|U(\cdot, t)\| \leq C$ for $t \in [k, k + 1]$. This concludes the proof.

**Proof of Proposition 6** If $\Omega = B_R$, then the statement follows from the continuity and boundedness of the trajectory, and the smoothing properties of the semiflow generated by $H$. In fact, standard estimates based on the variation of constant formula guarantee that $U(\cdot, t)$ is bounded in $H^2(B_R, \mathbb{R}^2)$ for $t \geq \delta$, hence the compactness follows from the compact embedding of $H^2(B_R, \mathbb{R}^2)$ into $H^1(B_R, \mathbb{R}^2)$.

Next let $\Omega = \mathbb{R}^n$, $U_0$ be compactly supported and $n \geq 2$. It is well known (see [29], [17], for example), that $H^1_{r_0}(\mathbb{R}^n, \mathbb{R}^2)$ is compactly embedded into $L^s$ if $2 < s < p_s$. It is also easily seen that the function $M(r) := \delta e^{-\varepsilon(r-R)}$, $r > R$, is a supersolution to problem (3) for any $R > 0$ if $\varepsilon, \delta > 0$ are small enough (where the smallness depends only on $\lambda$ and $\sup_{|U_t| = 1} |F(U)|$). More precisely, if $|U_0(r)| \leq M(r)$ for $r > R$ and $|U(R, t)| < M(R)$ for all $t \geq 0$, then $|U(r, t)| \leq M(r)$ for all $r \geq R$ and $t \geq 0$. Fix such $\varepsilon, \delta$.

Since [29] Radial Lemma] guarantees $|U(x, t)| \leq C(n)|x|^{(1-n)/2}|U(\cdot, t)|$ and $U_0$ is compactly supported, we can find $R > 0$ such that the support of $U_0$ is contained in $B_R$ and $|U(R, t)| < \delta$ for all $t$. Consequently, we obtain $|U(r, t)| \leq M(r)$ for all $r \geq R$ and $t \geq 0$, hence the trajectory of $U$ is bounded in $L^1$. This fact and the compactness in $L^s$ guarantee the compactness in $L^2$, and smoothing arguments also prove the compactness in $H^1$. In fact, due to Corollary 3 one can easily show that the mapping $L^2 \to H^1 : U(\cdot, t) \to U(\cdot, t + 1)$ is continuous.

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