FEEDBACK STABILIZATION OF LINEAR AND BILINEAR
UNBOUNDED SYSTEMS IN BANACH SPACE

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Abstract. We consider linear control systems of the form
\[ \dot{y}(t) = Ay(t) - \mu BCy(t) \]
where \( \mu \) is a positive real parameter, \( A \) is the state operator and generates a linear \( C_0 \)-semigroup of contractions \( S(t) \) on a Banach space \( X \), \( B \) and \( C \) are respectively the operators of control and observability, which are defined in appropriate spaces in which they are unbounded in some sense. We aim to show the exponential stability of the above system under sufficient conditions which are expressed in term of admissibility and observability properties. The uniform exponential stabilization using bilinear control is considered as well. Applications to transport and heat equations are also provided.

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1. Introduction

Let us consider the following linear control system
\[ \dot{y}(t) = Ay(t) + Bu(t), \quad t \geq 0, \quad y(0) = y_0 \in X \]
augmented with the output \( z(t) = Cy(t), \quad t \geq 0 \), where \( X \) and \( U \) are two Banach spaces representing respectively, the state and observation/control space, \( A : D(A) \subset X \rightarrow X \) is the system operator, which generates a \( C_0 \)-semigroup of contractions \( S(t) \) on \( X \), the space \( X \) is endowed with norm \( \| \cdot \|_X \), and let \( X_{-1} \) denote the completion of \( X \) w.r.t to the norm \( \|x\|_{-1} := \|(A-\eta I)^{-1}x\|_X, \quad x \in X \) for some (or equivalently all) \( \eta \) in resolvent

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set $\rho(A)$ of $A, B \in \mathcal{L}(U, X_{-1})$ is the control operator and $C \in \mathcal{L}(W, U)$ is the observation operator, where $W$ is a Banach space such that the injections $X_1 \hookrightarrow W \hookrightarrow X$ are continuous ($X_1$ being the space $D(A)$ equipped with the graph norm). Then, closing the system (1) with the control $u(t) = -\mu Cy(t)$, ($\mu > 0$ is the gain control) one obtains the following Cauchy problem

$$\dot{y}(t) = Ay(t) - \mu BCy(t), \quad t > 0, \quad y(0) = y_0 \in X,$$

which is well-posed in $X$ whenever $A - \mu BC$ is a generator of a $C_0$-semigroup on $X$ (cf. [20], Section II.6).

We further consider the bilinear system

$$\dot{y}(t) = Ay(t) + v(t)By(t), \quad y(0) = y_0 \in X.$$  

The well-posedness of the systems like (1) and (3) has been studied in many works using different approaches (see e.g. [1, 13, 14, 23, 27, 35, 38]).

In several practical situation, the modeling gives rise to unbounded control systems of form (1) or (3), where the closed loop operator is of type Weiss-Staffans, Miyadera-Voigt or Desch-Schappacher (see e.g. [5, 21, 26, 32, 34, 36, 37]). This fact often occurs when the control is exercised through the boundary or a point for systems governed by partial differential equations.

The problem of feedback stabilization of some classes of linear and nonlinear systems has been investigated in case of bounded and unbounded control operators in [2, 3, 4, 5, 6, 7, 8, 9]. Feedback stabilization of the bilinear system (3) has been investigated in the case of a bounded control operator by numerous authors using various control approaches, such as quadratic control laws, sliding mode control, piecewise constant feedback and optimal control laws (see [10, 31] and the references therein). Recently, the question of stabilization of bilinear systems with unbounded control operator has been treated in [12, 18, 19, 30]. In [12], the author considered the case where $A$ is self-adjoint and $B$ is positive self-adjoint and bounded from some subspace $V$ of $H$ to its dual space $V^\prime$, then he established the weak and strong stabilizability of the system (3) for all $y_0 \in D(A)$ using nonlinear control. Moreover, in [18, 19] it has been supposed that the linear operator $B$ is relatively bounded w.r.t $A$ from $H$ to an extension $X$ of $H$ with a continuous embedding $H \hookrightarrow X$. Then, under an exact observability condition, it has been shown that (3) is strongly stabilizable, and a polynomial decay estimate of the stabilized state has been provided in the case of positive self-adjoint control operator. In [30], the exponential stabilizability of bilinear systems has been considered for Miyadera’s control operator, and the stabilizing control is a switching one which leads to a closed-loop system like (2) evolving in a reflexive state space. More recently, the case of nonreflexive state space was considered in the context of bounded control operator [31]. In this paper, we deal with a wide class of linear/bilinear systems evolving on a nonreflexive state space with unbounded control operators, including control operators of type Weiss-Staffans, Miyadera-Voigt or Desch-Schappacher. Then we will give sufficient conditions for exponential
stabilizability of infinite dimensional systems that can be described by the systems (1) or (3).

The paper is organized as follows: In the second section, we provide some tools that will be required for the stabilization problem, then state and show the main result in which we present sufficient conditions for exponential stabilization of the linear system (1) with a feedback control involving the output, which leads to closed-loop operator of Weiss-Staffans’s type. Next, we provide applications to bilinear system (3) with control operator of Miyadera-Voigt or Desch-Schappacher type. Applications to transport and heat equations are presented as well.

2. The main results

2.1. Preliminary on linear semigroups. Let us recall some notions and properties related to linear $C_0-$semigroups.

- The duality pairing between the space $X$ and its dual $X^*$ is denoted by $\langle \cdot , \cdot \rangle$, where $X^*$ is the set of all bounded linear forms on $X$ and the pairing between $y \in X$ and $\phi \in X^*$ is denoted by $\langle y, \phi \rangle$. The duality map $J$ from $X$ to $X^*$ is in general a multi-valued operator; i.e. for each $y \in X$, $J(y)$ is by definition the (nonempty) set of all $\phi \in X^*$ such that $\langle y, \phi \rangle = \|y\|_X^2 = \|\phi\|^2_X$, where $\| \cdot \|$ denotes the norm of $X^*$ associated to $\| \cdot \|_X$.

- A one parameter family $S(t), t \geq 0$, of bounded linear operators from a Banach space $X$ into $X$ is a semigroup on $X$ if (i) $S(0) = I$, (the identity operator on $X$) and (ii) $S(t + s) = S(t)S(s)$ for every $t, s \geq 0$. A semigroup $S(t)$ of bounded linear operators on $X$ is a $C_0-$ semigroup if in addition $\lim_{t \to 0^+} S(t)x = x$ for every $x \in X$. This property guarantees the continuity of the semigroup on $\mathbb{R}^+$. Moreover, one can show (see [33], p. 4) that for every $C_0-$ semigroup $S(t)$, there exist constants $\omega \geq 0$ and $M \geq 1$ such that

$$\|S(t)\| \leq Me^{\omega t}, \forall t \geq 0.$$  (4)

If $\omega = 0$ and $M = 1$, $S(t)$ is called a $C_0-$semigroup of contractions.

The linear operator $A$ defined by $Ax = \lim_{t \to 0^+} \frac{S(t)x - x}{t}$ for $x \in X$ such that $\lim_{t \to 0^+} \frac{S(t)x - x}{t}$ exists in $X$, is the infinitesimal generator of the $C_0-$semigroup $S(t)$. The linear space $D(A) := \{ x \in X : \lim_{t \to 0^+} \frac{S(t)x - x}{t} \in X \}$ is the domain of $A$.

The infinitesimal generator of a contraction $C_0-$semigroup is dissipative, i.e., for every $y \in D(A)$ and for all $y^* \in J(y)$ we have $Re\langle Ay, y^* \rangle \leq 0$ (see [33], pp. 14-15).

- For $x \in D(A)$, we have $Ax = \frac{d}{dt}S(t)x|_{t=0}$ and $y(t) := S(t)y_0$ is differentiable and lies in $D(A)$ for all $t > 0$, and is the unique solution
of the Cauchy problem: \( \dot{y}(t) = Ay(t), t > 0, \ y(0) = y_0 \). Moreover,
for every \( y_0 \in X \); \( y(t) = S(t)y_0 \) is called mild solution of this Cauchy
problem.
• If \( A \) is the infinitesimal generator of a \( C_0 \)-semigroup \( S(t) \), then
\( D(A) \) (the domain of \( A \)) is dense in \( X \) and \( A \) is a closed linear
operator. Moreover, according to Hille-Yosida’s Theorem (see for instance [33], p. 20), a linear operator \( A \) is the infinitesimal generator of a \( C_0 \)-semigroup \( S(t) \) satisfying (1) if and only if (i) \( A \) is closed
and \( D(A) \) is dense in \( X \), and (ii) the resolvent set \( \rho(A) \) of \( A \) contains
the ray \((\omega, +\infty)\) and \( \|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)} \) for \( \lambda > \omega, n = 1, 2, ... \)
In particular, a closed operator \( A \) with densely domain \( D(A) \) in \( X \) is the infinitesimal generator of a \( C_0 \)-semigroup of contractions on
\( X \) if and only if the resolvent set \( \rho(A) \) of \( A \) contains \( \mathbb{R}^+ \) and for all
\( \lambda > 0; \|\lambda R(\lambda, A)\| \leq 1 \) (see [33], p. 8).
• The \( C_0 \)-semigroup \( S(t) \) may be extended to a \( C_0 \)-semigroup \( S_{-1}(t) \)
on \( X_{-1} \), whose generator is the extension \( A_{-1} : D(A_{-1}) := X \subset
X_{-1} \rightarrow X_{-1} \) of \( A : D(A) \subset X \rightarrow X \) to a \( m \)-dissipative operator
from \( X \) to \( X_{-1} \). In particular, we have \( A_{-1}y = Ay \) for all \( y \in D(A) \)
(see [20], p. 126). Using the integral representation of the resolvent one obtains
\( R(A_{-1})y = R(\lambda, A)y, \forall y \in X \), for all \( \lambda \in \rho(A_{-1}) \).
Recall also that \( R(\lambda, A_{-1})y \in X, \forall y \in X_{-1} \) and \( R(\lambda, A_{-1})y = \)
\( R(\lambda, A)y, \forall y \in X, \forall \lambda \in \rho(A_{-1}) \). Moreover, If \( A \) is dissipative, then
so is \( A_{-1} \): For \( z \in X \), we have \( \|S_{-1}(t)z\|_{-1} = \|S(t)z\|_{-1} = \|R(\eta : \)
\( A)S(t)z\|_X \leq \|R(\eta : A)z\|_X = \|z\|_{-1} \). Then by density of \( X \) in \( X_{-1} \),
we conclude that \( S_{-1}(t) \) is a contraction on \( X_{-1} \).
• We have \( \sup_{\lambda \in \rho(A_{-1})} \|\lambda R(\lambda, A_{-1})B\|_{\mathcal{L}(U, X_{-1})} < \infty \),
where \( \rho(A_{-1}) \) is the resolvent set of \( A_{-1} \). Moreover, by the closed
graph theorem we have that \( \lambda R(\lambda, A_{-1})B \in \mathcal{L}(U, X) \). For instance for
\( W = U = X_1 \) and \( Range(B) \subset X \), i.e., \( B \in \mathcal{L}(X_1, X) \) (which is the case of Miyadera’s
operators), we have for all (real) \( \lambda \in \rho(A_{-1}) \) large enough,
\[ \|\lambda R(\lambda, A_{-1})B\|_{\mathcal{L}(X_1, X)} = \|\lambda R(\lambda, A)B\|_{\mathcal{L}(X_1, X)} \leq \]
\[ \|B\|_{\mathcal{L}(X_1, X)}, \forall \lambda > 0. \]
In fact this is also true for any admissible operator \( B \) in the sense of
\((h_2)\) below (see [35], p. 219), that is there exists \( K > 0 \) such that
\[ \|\lambda R(\lambda, A_{-1})B\|_{\mathcal{L}(U, X)} \leq K, \text{ for all } \lambda \text{ large enough}. \quad (5) \]
In general this property does not imply the admissibility of \( B \). (see
[14] [15] [16] [17] [22] [25] [27] [28] [39] for some discussions and partial
results about this implication).
• Let \( \mathfrak{X} \oplus \mathfrak{X}_{-1} \) be a direct (algebraic) decomposition in \( X_{-1} \) between
two subspaces \( \mathfrak{X} \) and \( \mathfrak{X}_{-1} \) such that \( \mathfrak{X} \subset X \) and \( \mathfrak{X} \cap \mathfrak{X}_{-1} = \{0\} \), and
let \( P_X \) denote the projection on \( \mathfrak{X} \) according to the above decomposition.
Moreover, if \( K \) is a linear operator such that \( Range(K) \subset \mathfrak{X} \),
then for every \( \lambda \) large enough, \( \lambda R(\lambda, A_{-1})B \in \mathcal{L}(U, X) \) and
\( \|\lambda R(\lambda, A_{-1})B\|_{\mathcal{L}(U, X)} \leq K \), for all \( \lambda \) large enough.
alent to the boundedness of $B$ implies that the operator defined by is bounded (see [1]).

The stabilization results. In this part we consider the stability of the system (2). The first task is to guarantee the existence and uniqueness of the solution. Note that if the operator $\mu BC \in \mathcal{L}(W, X_{-1})$ is a Weiss-Staffans perturbation for $A$, then (see e.g. [1-3]) the closed loop system (2) is well-posed. More precisely, for small gain control $\mu > 0$, the operator $(A - \mu BC)|_X$ (i.e. the part of $A_{-1} - \mu BC$ on $X$) with domain $D_\mu := \{ y \in W : (A_{-1} - \mu BC)y \in X \}$ generates a $C_0$-semigroup $T(t)$ on $X$ satisfying the following variation of constants formula (V.C.F)

\begin{equation}
T(t)y_0 = S(t)y_0 - \mu \int_0^t S_{-1}(t-s)BCT(s)y_0, \forall y_0 \in D_\mu.
\end{equation}

Note that in general, we have $D(A) \cap D((BC)|_X) \subset D_\mu$. Moreover, if $W \subset X_1 \cup D((BC)|_X)$, then we have $(A_{-1} - \mu BC)|_X = D(A) \cap D((BC)|_X)$.

This motivates the consideration of the assumptions $(h_1) - (h_4)$ below.

$(h_1)$ the well-posedness assumption: there exists $\alpha_1 > 0$ such that for every $\mu \in (0, \alpha_1)$, the operator $(A_{-1} - \mu BC)|_X$ with domain $D := D((A_{-1} \cap BC)|_X)$ generates a $C_0$-semigroup $T(t)$ on $X$.

Now let us consider the following assumptions for some $T, M > 0$.

$(h_2)$ The admissibility assumption of $B \in \mathcal{L}(U, X_{-1})$:

\[ \int_0^T S_{-1}(T-s)Bu(s)ds \in X, \forall u \in L^1(0, T; U), \]

which implies that

\[ \left\| \int_0^T S_{-1}(T-s)Bu(s)ds \right\|_X \leq M \|u\|_1, \]

for all $u \in L^1(0, T; U)$ (or equivalently for all $u \in W^{1,1}(0, t_1; U)$). This also implies that the operator defined by

\[ B_T : u(\cdot) \in L^1(0, +\infty; U) \mapsto \int_0^T S_{-1}(T-s)Bu(s)ds \]

is bounded (see [37]).

Note that if $X$ is reflexive, then the admissibility assumption $(h_2)$ is equivalent to the boundedness of $B$ (see [37]).
(h₃) The admissibility assumption of $C \in \mathcal{L}(W,U)$:

$$\int_0^T \|CS(t)y\|_U dt \leq M\|y\|_X, \forall y \in D(A).$$

(h₄) Joint-admissibility of $B$ and $C$:

$$\int_0^T \left\| C \int_0^t S_{-1}(r-s)Bu(s)ds \right\|_U \, dr \leq M\|u\|_1, \forall u \in L^1(0,T;U)$$

with $\|u\|_1 = \int_0^T \|u(\tau)\|_X \, d\tau$.

In the sequel, if there is no confusion, we use $\langle z, J(y) \rangle$ for any $y^* \in J(y)$ instead of $\langle z, y^* \rangle$. Also, for any functions $t \mapsto \zeta(t)$, we will write $\phi(\cdot) \in J(\zeta(\cdot))$ if $\phi(t) \in J(\zeta(t)), \forall t \geq 0$.

Now, for the stabilization results we further consider the following observation assumption.

(h₅) The observability condition: for some $\delta > 0$ we have

$$\int_0^T \text{Re} \langle \chi_{(BC)} S(t)y, J(S(t)y) \rangle dt \geq \delta\|S(T)y\|_X^2, \forall y \in D(A).$$

The estimate (7) may be seen as a null-exact controllability inequality in the sense of linear systems.

(h₆) The function $F_J : y \mapsto \{(\chi_{(BC)}y,y^*); y^* \in J(y)\}$ is such that for all $y, z \in X$, there exists $(y^*, z^*) \in J(y) \times J(z)$ such that

$$|\langle \chi_{(BC)}y, y^* \rangle - \langle \chi_{(BC)}z, z^* \rangle| \leq k_1 (\|y\|_{D(C)} + \|z\|_{D(C)}) \|y - z\|_X + k_2 (\|y\|_X + \|z\|_X) \|C(y - z)\|_U, \forall y, z \in D(A) \subset W$$

for some constants $k_i \geq 0, i = 1,2$, where $\|y\|_{D(C)} := \|y\|_X + \|Cy\|_U$.

This assumption is motivated by the fact that here, the state space is a general Banach space, i.e. without any smoothness property that evolves the duality mapping. In particular, if the state space $X$ is smooth, so that $J$ is Lipschitz-continuous and $X$ is reflexive (see e.g. [30]), then (h₆) is verified under the admissibility of $B$, as in that case the operator $BC$ will be bounded from $W$ to $X$ (see [37]).

Let us now state our main result.

**Theorem 1.** Let assumptions (h₁)–(h₆) hold. Then there exists $\alpha > 0$ such that for any $\mu \in (0,\alpha)$, the closed-loop system (2) is exponentially stable.

**Proof.** According to assumption (h₁), the operator $A_{BC} := (A_{-1} - \mu BC)|_X$ with domain $D(A_{BC}) = D(A) \cap D((BC)|_X) = D$ generates a $C_0$–semigroup $T(t)$ on $X$ for $\mu > 0$ small enough (says $\mu \in (0,\alpha_1)$). Moreover, $y(t) := T(t)y_0$ is the unique mild solution of (2) and satisfies the following V.C.F

$$y(t) = S(t)y_0 - \mu \int_0^t S_{-1}(t-s)BCy(s)ds, \forall y_0 \in D.$$ (8)
Let $y_0 \in D$ be fixed. Then for all $t \geq 0$, we have $(A_{-1} - \mu BC)|_X y(t) \in X$. Moreover, $y(t)$ has a weak derivative $A_{BC} y(t) = T(t) A_{BC} y_0$, which is weakly continuous and hence bounded

$$
\left| \frac{d}{dt} (y(t), f) \right| \leq L \|A_{BC} y_0\|_X \|f\|, \forall f \in X^* (L > 0)
$$

in any bounded time-interval. Thus $y(t)$ (and so is $\|y(t)\|_X$) is Lipschitz continuous (recall that $\|y\|_X^2 = \sup_{f \in X^* \|f\| \leq 1} |\langle y, f \rangle|$). It follows that $\|y(t)\|_X$ is differentiable almost everywhere and (see [24]) for a.e $t > 0$ we have

$$
(9) \quad \frac{d}{dt} \|y(t)\|_X^2 = 2 \Re \langle A_{BC} y(t), J(y(t)) \rangle.
$$

Here $J$ is the duality mapping of $X$ (recall that $A_{BC} y(t) \in X$), which by integrating and using the dissipativeness of $A$ implies

$$
(10) \quad 2 \mu \int_s^t \Re \langle BC y(\tau), J(y(\tau)) \rangle d\tau \leq \|y(s)\|_X^2 - \|y(t)\|_X^2, \quad t \geq s \geq 0.
$$

According to $(h_6)$, we have

$$
\Re \langle \chi(BC) S(t)y_0, J(S(t)y_0) \rangle \leq
\quad
K \left( \|S(t)y_0\|_{D(C)} + \|y(t)\|_{D(C)} \right) \|S(t)y_0 - y(t)\|_X +
\quad
K \left( \|S(t)y_0\|_X + \|y(t)\|_X \right) \|C(S(t)y_0 - y(t))\|_U +
\quad
\Re \langle BC y(t), J(y(t)) \rangle
$$

where $K = \max(k_1, k_2)$, where we have used that $\chi(BC) y(t) = BC y(t)$, as $y(t) \in X, \forall t \geq 0$.

From the admissibility assumption $(h_3)$, we have

$$
\int_0^T \|CS(t)y_0\|_U dt \leq M \|y_0\|_X
$$

and for all $t \in [0, T]$, we have

$$
\|S(t)y_0 - y(t)\|_X = \mu \left\| \int_0^t S_{-1}(t - s) BC y(s) ds \right\|_X
\quad \leq \mu \|B_T(C y(\cdot))\|_1,
$$

where $B_T$ is the bounded operator defined by

$$
B_T : u(\cdot) \in L^1(0, +\infty; U) \mapsto \int_0^T S_{-1}(T - s) Bu(s) ds.
$$

Hence

$$
\|S(t)y_0 - y(t)\|_X \leq \mu M \|C y(\cdot)\|_1
$$

where $\|C y(\cdot)\|_1 := \int_0^T \|C y(\tau)\|_U d\tau$.

It follows from this and $(h_4)$ that

$$
\int_0^T \|C(S(t)y_0 - y(t))\|_U dt = \mu \int_0^T \left\| C \int_0^t S_{-1}(t - s) BC y(s) ds \right\|_U dt
$$
This together with (10) and (12) implies

We conclude that

Then taking $y(t)$ instead of $y_0$ in the last estimate, it comes

From the variation of constants formula (8) we deduce that for all $t \geq 0$, we have

Then taking $y(kT)$ instead of $y_0$, it comes via (11)

Thus for all $k \geq 0$, we have

This together with (11) and (12) implies

$$
\mu \delta \left( \|y((k+1)T)\|_X^2 - 2 \left( \frac{\mu M^2}{1-\mu} \right)^2 \|y(kT)\|_X^2 \right) - 2c\mu^2 \|y(kT)\|_X^2 \leq \|y(kT)\|_X^2 - \|y((k+1)T)\|_X^2.
$$
Hence
\[(1 + \mu \delta) \| y((k + 1)T) \|_X^2 \leq \left( 2\delta \mu \left( \frac{\mu M^2}{1 - M\mu} \right)^2 + 2c\mu^2 + 1 \right) \| y(kT) \|_X^2, \ k \geq 0, \]
from which we derive
\[(14) \quad \| y(kT) \|_X^2 \leq q^k \| y_0 \|_X^2, \ \forall k \geq 0 \]
with \[q := \frac{1 + 2\mu^2}{1 + 2\mu \delta} \left( \delta \mu \left( \frac{M^2}{1 - M\mu} \right)^2 + c \right) \], which lies in \((0, 1)\) for \(\mu \to 0^+\). Moreover, using the following well known property of linear \(C_0\)-semigroups:
\[\| y(t) \|_X \leq Ne^{wt} \| y_0 \|_X, \ t \geq 0, \quad (\text{for some constants } N, w > 0), \]
we deduce, by taking \(k = E(t/T)\), that
\[(15) \quad \| y(t) \|_X \leq M'e^{-\sigma t} \| y_0 \|_X, \ \forall t \geq 0, \]
where \(M', \sigma\) are independent of \(y_0\).
This estimate extends to all initial data in \(X\) by density of \(D\) in \(X\). □

In the previous theorem, we have considered the case where the domain of the generator \(A_{BC}\) is \(\mathcal{D}(A) \cap \mathcal{D}((BC)|_X)\). In the next result, we will state another stabilization result which only requires that the domain of the generator \(A_{BC}\) is independent of the gain control \(\mu\), provided some further conditions are fulfilled. Let us consider the following assumption.

\((h_7)\) Compatibility condition: Range \((B_\lambda C) \subset W\), for some/all \(\lambda \in \rho(A)\) holds with \(B_\lambda := \lambda \Re(\lambda, A_{-1})B\).

Notice that under the compatibility assumption, it comes from the closed graph theorem and the resolvent property that \(\text{Range}(B_\lambda C) \subset W\). Let us define the following scalar valued function
\[f_{BC}(y) = \limsup_{\lambda \to +\infty} \Re(B_\lambda Cy, J(y)).\]
Note that by (5), we have \(f_{BC}(y) \in \mathbb{R}, \ \forall y \in X\). Moreover, if \(\overline{BC}\) is the operator defined by
\[\overline{BC}y := \lim_{\lambda \to +\infty} B_\lambda Cy, \ \forall y \in D(\overline{BC}) := \{ y \in W : \lim_{\lambda \to +\infty} B_\lambda Cy \text{ exists in } X \}, \]
then we have
\[(16) \quad f_{BC}(y) = \Re(\overline{BC}y, J(y)), \ \forall y \in D(\overline{BC}).\]
If in addition \(\text{Range}(B) \subset X\), then the relation (16) holds in \(X\).
Let us consider the following assumptions:

\((h_5)'\) The observability condition: for some \(\delta > 0\) we have
\[(17) \quad \int_0^T f_{BC}(S(t)y)dt \geq \delta \| S(T)y \|_X^2, \ \forall y \in D(A), \]
Accordingly (see [1]), the closed loop operator \( (A, all(t)) \)

\( |f_{BC}(y) - f_{BC}(z)| \leq k_1 (\|y\|_{D(C)} + \|z\|_{D(C)}) \|y - z\|_X + \)

\( k_2(\|y\|_X + \|z\|_X) \|C(y - z)\|_U, \quad \forall y, z \in D(A) \subset W \)

for some constants \( k_i \geq 0, i = 1, 2 \).

From the proof of Theorem [1] we deduce the following result.

**Corollary 1.** Assume that for some \( \mu_1 > 0 \), the domain of \( A_{BC} := (A_{-1} - \mu BC)|_X \) is independent of \( \mu \in (0, \mu_1) \), and let assumptions \((h_2) - (h_4), (h_5)' - (h_6)' \) and \( (h_7) \) hold.

Then there exists \( \alpha > 0 \) such that for any \( \mu \in (0, \alpha) \), the closed-loop system [2] is exponentially stable.

**Proof.** Under the assumptions \((h_2) - (h_4)\) and the compatibility condition \((h_7)\), the operator \( \mu BC \in \mathcal{L}(W, X) \) is a Weiss-Staffans perturbation for \( A \). Accordingly (see [1]), the closed loop operator \( (A_{-1} - \mu BC)|_X \) with domain \( D(A_{BC}) = \{ y \in W : (A_{-1} - \mu BC)y \in X \} \) generates a \( C_0 \)-semigroup \( T(t)y_0 \) on \( X \) and \( y(t) := T(t)y_0 \) is the unique mild solution of [1] and satisfies the formula [1] for all \( y_0 \in D(A_{BC}) \).

Let \( y_0 \in D(A_{BC}) \) be fixed. Then we have \( (A_{-1} - \mu BC)|_X y(t) \in X \) for all \( t \geq 0 \), and \( ||y(t)||_X \) is differentiable almost everywhere and [9] holds for \( y_0 \in D(A_{BC}) \). Moreover, for every \( y \in D(A_{BC}) \), we have

\[ \Re(\lambda R(\lambda, A_{-1})A_{BC}y, J(y)) = \Re(\lambda R(\lambda, A_{-1})A_{-1}y, J(y)) - \mu \Re(B_\lambda Cy, J(y)) \]

Because \( A \) is dissipative, we have for all \( y \in X \)

\[ \Re(\lambda R(\lambda, A_{-1})A_{-1}y, J(y)) = \Re(-y + \lambda R(\lambda, A)y, J(y)) \leq 0. \]

It follows that

\[ \Re(\lambda R(\lambda, A_{-1})A_{BC}y, J(y)) \leq -\mu \Re(B_\lambda Cy, J(y)). \]

Since \( y \in D(A_{BC}) \), it comes that

\[ \lambda R(\lambda, A_{-1})A_{BC}y = \lambda R(\lambda, A)A_{BC}y \to A_{BC}y \text{ in } X, \text{ as } \lambda \to +\infty. \]

Hence

\[ \Re(A_{BC}y, J(y)) \leq -\mu \limsup_{\lambda \to +\infty} \Re(B_\lambda Cy, J(y)). \]

The remainder of the proof is exactly the same as in the proof of Theorem [1], which leads to the estimate [15] with constants \( M', \sigma \) which are independent of \( y_0 \). Then we conclude by density of \( D(A_{BC}) \) in \( X \).

□
Remark 1. From the proof of Theorem 1 (resp. Corollary 1), we can see that the results remain true if we assume that \( (h_5) \) (resp. \( (h_5)' \) ) holds for some element of the duality set \( J(S(t)y) \) provided that \( (h_6) \) (resp. \( (h_6)' \) ) holds for every \( (y^*, z^*) \in J(y) \times J(z) \).

3. Applications

In this section we will apply the result of the previous section to the bilinear system (3). More precisely, we will investigate the exponential stability of (3) under the bang-bang feedback control \( v(t) = -\mu 1_{\{t \geq 0; B y(t) \neq 0\}} \). As special cases, we will consider Miyadera-Voigt’s and Desch-Schappacher’s control operators.

3.1. Stabilization of unbounded bilinear systems. First let us note that if \( B \) is decomposable according to \( B = B C \) with \( B \) and \( C \) satisfying the conditions of Theorem 1 then we can see that (3) is exponentially stabilizable by the control \( v(t) = -\mu 1_{\{t \geq 0; B y(t) \neq 0\}} \).

In the following corollary, we provide a result that extends the one of (30) to the case of non reflexive state space.

Corollary 2. Let \( B \in L(X_1, X) \) be such that

\( (m_1) \) there exists \( M > 0 \) such that

\[
\int_0^T \|B S(t)y\|_X dt \leq M\|y\|_X, \forall y \in D(A),
\]

\( (m_2) \) there exists \( \delta > 0 \) such that

\[
\int_0^T \text{Re}\{(B S(t)y, J(S(t)y))\} dt \geq \delta\|y\|^2_X, \forall y \in D(A),
\]

\( (m_3) \) the function \( F_I : y \mapsto \{(B y, y^*); y^* \in J(y)\} \) is such that for all \( y, z \in X \), there exists \( (y^*, z^*) \in J(y) \times J(z) \) such that

\[
\|\langle B y, y^* \rangle - \langle B z, z^* \rangle \| \leq k_1(\|y\|_{D(B)} + \|z\|_{D(B)}) \|y - z\|_X + k_2(\|y\|_X + \|z\|_X) \|y - z\|_{D(B)}, \forall y, z \in D(A),
\]

for some constants \( k_i \geq 0, i = 1, 2 \).

Then there exists \( \alpha > 0 \) such that for any \( \mu \in (0, \alpha) \), the control \( v(t) = -\mu 1_{\{t \geq 0; B y(t) \neq 0\}} \) guarantees the exponential stabilization of (3).

Proof. Let us first observe that under the assumption of the corollary, the operator \( \mu B \) may be seen as a Miyadera’s perturbation of \( A \), for \( \mu > 0 \) small enough, and we have \( D((A_{-1} - \mu B)|_X) = D(A) \).

Moreover, we have \( x = \mathcal{X} = X \) and \( \mathcal{X}_{-1} = \{0\} \).

Let us take \( B = i : X \hookrightarrow X_{-1} \) (the embedding \( X \hookrightarrow X_{-1} \)) with \( U = X \) and \( W = X_1 \), so that \( B = C \in L(X_1, X) \). Then, since \( \text{Range}(B) = \text{Range}(C) \subset X \), the compatibility condition \( (h_7) \) follows from the fact that \( R(\lambda, A_{-1})C = R(\lambda, A)C \), while the others conditions of Theorem 1 are clearly satisfied. Moreover, the well-posedeness follows from [29] (see...
also [20], p. 199) and ([1], Theorem 18 and its remark). Finally, by observing that \( v(y)B y = -\mu B y \) with \( v(y) = -\mu 1_{\{y \in \ker B\}} \), we can see that the bilinear system [3], closed with the feedback control \( v(t) = -\mu 1_{\{t \geq 0; B y(t) \neq 0\}} \), leads to the system in closed-loop [2]. Hence according to Theorem [1] we have the exponential stability for small gain control \( \mu > 0 \).

We have the following result regarding the case of Desch-Schappacher’s control operator.

**Corollary 3.** Let \( B \in \mathcal{L}(X, X_{-1}) \) be such that for some \( \mu_1, T, \beta > 0 \) we have

\[
\begin{align*}
(ds)_1 & \text{ the domain of } A_{BC} := (A_{-1} - \mu BC)|_X \text{ is independent of } \mu \in (0, \mu_1), \\
(ds)_2 & \text{ for all } u \in L^1(0, T; X), \text{ we have } \int_0^T S_{-1}(T - s)Bu(s)ds \in X, \\
(ds)_3 & \text{ there exists } \delta > 0 \text{ such that } \\
\int_0^T \lim \sup_{\lambda \to +\infty} Re\langle B_\lambda S(t)y, J(S(t)y) \rangle dt \geq \delta ||S(T)y||_X^2, \forall y \in X, \\
(ds)_4 & \text{ for all } y, z \in X, \text{ there exists } (y^*, z^*) \in J(y) \times J(z) \text{ such that } \\
|\lim \sup_{\lambda \to +\infty} \langle B_\lambda y, y^* \rangle - \lim \sup_{\lambda \to +\infty} \langle B_\lambda z, z^* \rangle| \leq k \left( ||y||_X + ||z||_X \right) ||y - z||_X, \forall y, z \in X,
\end{align*}
\]

for some constant \( k \geq 0 \).

Then there is \( \alpha > 0 \) such that for any \( \mu \in (0, \alpha) \), the control \( v(t) = -\mu 1_{\{t \geq 0; B y(t) \neq 0\}} \) guarantees the exponential stabilization of [3].

**Proof.** This follows from Corollary [1] by taking \( U = W := X, C := I_X \) and \( B := B \in \mathcal{L}(X, X_{-1}) \).

\[\square\]

### 3.2. Examples.

**Example 1.** Let us consider the following system

\[
y(t) = y_u(t) - \mu B y(t), \quad \text{in } (0, 1) \times (0, +\infty), \\
y(1, t) = 0, \quad \text{in } (0, +\infty), \\
y(0, 0) = y_0 \in L^1(0, 1), \quad \text{in } (0, 1).
\]

Here, \( X = L^1(0, 1) \) and the duality map is given for all \( y \in X \) by

\[
J(y) = \{ \xi \in L^\infty(\Omega) : \xi(x) \in \text{sign}(y(x)) \cdot ||y|| \},
\]

where the sign function is defined by

\[
\text{sign}(s) = \begin{cases} 
1, & s > 0, \\
I, & s = 0, \quad \text{with } I = [-1, 1], \\
-1, & s < 0.
\end{cases}
\]

The state operator is defined by \( Ay = y' \) with domain \( D(A) = W^{1,1}_0(0, 1) \) =
\{y \in W^{1,1}(0,1) : y(1) = 0\} and generates the semigroup \(S(t)\) defined for all \(y \in L^1(\Omega)\) by

\[
(S(t)y)(\xi) = \begin{cases}
y(\xi + t), & \text{if } \xi + t \leq 1 \\
0, & \text{else.}
\end{cases}
\]

Let \(\phi = \alpha \delta_1 \in (W^{1,1}(0,1))^1\) where \(\delta_1\) is the Dirac point evaluation in 1 and \(\alpha \in \mathbb{R}\), and let us define the control operator for \(y \in W^{1,1}(0,1)\) by

\(By = y + \phi(y)A_{-1}a\), where \(a(x) = 1\), a.e. \(x \in (0,1)\).

Let us consider the unbounded part of \(B\), which is defined by \(y \mapsto \phi(y)A_{-1}a\), and which may be written in the form \(BC : W^{1,1}(0,1) \to X_{-1}\), where \(B \in \mathcal{L}(\mathbb{C}, X_{-1})\) is defined by \(Bq = qA_{-1}a\), \(q \in U := \mathbb{C}\) and \(C \in \mathcal{L}(W, \mathbb{C})\) is defined by \(Cy = \phi(y) = \alpha y(1), \forall y \in W := W^{1,1}(0,1)\).

The admissibility properties of \(B\) and \(C\) as well as the compatibility condition can be checked for \(T = 1\) (see [1]), which implies the well-posedness of the above system (for \(\mu > 0\) small enough). Let \(\mu \in (0, \mu_1) \subset (0,1)\) with \(0 < \mu_1 < \frac{1}{\alpha}\) for which the well-posedness is guaranteed. Then

\(y \in D((BC)|_X) \iff \phi(y)A_{-1}a \in X \iff \phi(y) = 0 \ (\text{because } a \notin D(A)) \iff By = y\).

Here, one can take \(X = W^{1,1}(0,1)\) and \(X_{-1} = \text{span}(A_{-1}a)\). Then we have \(\chi By = y, \forall y \in W^{1,1}(0,1)\) and so \(\langle \chi By, J(y) \rangle = \|y\|^2\), \(\forall y \in W\). Thus the assumptions \((h_5)\) & \((h_6)\) clearly hold.

Moreover, we have

\(y \in D((A_{-1} - \mu B)|_X) \Rightarrow A_{-1}(y - \mu \phi(y)a) \in X \Rightarrow y - \mu \phi(y)a \in D(A) \Rightarrow y(1) = \mu \phi(y) \Rightarrow By = y \in X \ (\text{recall that } 0 < \alpha \mu < 1) \Rightarrow y \in D(B|_X) \cap D(A)\).

We conclude by Theorem [4] that the system \((\text{IS})\) is exponentially stable for \(\mu > 0\) small enough.

**Example 2.** Let \(\Omega = (0, +\infty)\) and let us consider the following system

\[
y_t(\cdot, t) = -y_x(\cdot, t) + v(t)(1 + k(x))y(\cdot, t), \quad \text{in } (0, +\infty)^2
\]

\[
y(0, t) = 0, \quad \text{in } (0, +\infty)
\]

\[
y(\cdot, 0) = y_0 \in L^1(0, +\infty), \quad \text{in } (0, +\infty)
\]

Here, \(X = L^1(\Omega)\) is the state space, the parameter \(u(t)\) is the bilinear control and the corresponding solution \(z(t) := y(\cdot, t) \in X\) is the state. The function \(k\) is such that \(k \in L^1(0, +\infty)\) and \(\|k\|_X < 1\).

The unbounded operator \(A = -\frac{\partial}{\partial x}\) with domain

\[D(A) = \{y \in W^{1,1}(\Omega); y(0) = 0\}\]
generates a group of isometries \( S(t) \) on \( X \), which is defined for all \( y \in L^1(\Omega) \) by
\[
S(t)y(\xi) = \begin{cases} 
  y(\xi - t), & \text{if } \xi - t \geq 0 \\
  0, & \text{else.}
\end{cases}
\]

Let us define the operators \( B = y + k(x)y \) and \( By = k(x)y \).

Note that if \( k \not\in L^\infty(0, +\infty) \), then \( B \) is not a bounded operator on \( X \).

Let us show that \( B \) is \( A \)-bounded. It comes from
\[
|y(x)| = \left| \int_0^x y'(s)ds \right| \leq \int_0^\infty |y'(s)|ds, \forall y \in D(A)
\]
that \( D(A) \subset D(B) \) and that for all \( y \in D(A) \), we have
\[
\|ky\|_X = \int_0^\infty |k(x)y(x)|dx \leq \|y\|_{D(A)} \|k\|_X.
\]

Hence \( B \in L(X_1, X) \).

- **Admissibility of \( B \).** Let \( T > 0 \),
\[
\int_0^T \|BS(t)y\|_X dt = \int_0^T \|k(x)S(t)y\|_X dt = \int_0^T \int_0^\infty |k(x)S(t)y|1_{(0 \leq t \leq x)}dxdt = \int_0^T \int_0^\infty |k(x + t)y(x)|dxdt = \int_0^T \int_0^\infty |y(x)| \left( \int_0^T |k(x + t)| dt \right) dx \leq \|k\|_X \|y\|_X.
\]

This implies the admissibility estimate \( (m_1) \).

- **Assumption \( (h_6) \).** For \( T > 0 \), we have
\[
\langle k(x)y, J(y) \rangle = \|y\|_X \int_0^\infty k(x)|y(x)|dx, \ y \in D(A).
\]

Then
\[
|\langle k(x)y, J(y) \rangle - \langle k(x)z, J(z) \rangle| \leq \|y\|_X \int_0^\infty |k(x)(y(x) - z(x))|dx + \|y - z\|_X \int_0^\infty |k(x)y(x)|dx = \|y\|_X \|B(y - z)\|_X + \|y - z\|_X \|By\|_X
\]
which gives \( (h_6) \).

- **Observation.** For \( T > 0 \), we have
\[
\langle BS(t)y, J(S(t)y) \rangle = \|S(t)y\|^2_X + \langle BS(t)y, J(S(t)y) \rangle \geq \|y\|^2_X - \|y\|_X \|BS(t)y\|_X
\]
then
\[
\int_0^T \langle BS(t)y, J(S(t)y) \rangle dt \geq \|y\|^2_X - \|y\|_X \int_0^T \|BS(t)y\|_X dt \geq (1 - \|k\|_X)\|y\|^2_X.
\]

Hence \( (m_2) \) holds for \( \|k\|_X < 1 \).

From Corollary \[3\], we conclude that the control \( v(t) = -\mu 1_{[t \geq 0, y(t) \neq 0]} \) ensures the exponential stability of the system \[\[14\].\]
Example 3. Consider the following system:

\[
\begin{aligned}
y_t(t) &= y_{xx}(t) + v(t)(y_x(t) + y_x(t)), \quad (x, t) \in [0, 1] \times (0, +\infty), \\
y'(0, t) &= y'(1, t) = 0, \quad t \in (0, +\infty), \\
y(\cdot, 0) &= y_0 \in C_0([0, 1]), \\
y(\cdot) &\in \mathcal{C}_0([0, 1]),
\end{aligned}
\]

where \( u(t) \in \mathbb{R} \) is the control and \( y(t) = y(\cdot, t) \) is the state.

The state space \( X = \mathcal{C}_0([0, 1]) \) is equipped with the supremum norm, the operator \( A = \partial_{xx} \) with domain \( \mathcal{D}(A) = \{ y \in C^2([0, 1]) : y'(0) = y'(1) = 0 \} \) generates a contraction \( \mathcal{C}_0 \)-semigroup \( S(t) \) in \( X := \mathcal{C}_0([0, 1]) \) (see [20], pp. 93-94). The control operator is \( \mathcal{B} = I + \partial_x \), then the \( A \)-boundedness of \( \mathcal{B} \) follows from the following inequalities

\[
|y'(x)| \leq \int_0^1 |y''(s)| \, ds \leq \| y \|_{\mathcal{D}(A)}, \quad y \in \mathcal{D}(A).
\]

In other words, \( \mathcal{B} \in \mathcal{L}(X_1, X) \).

We will show that the stabilization assumptions previously considered are not all required. In particular, here we only need some elements of the duality set. For \( f \in \mathcal{C}_0([0, 1]) \) we have (20), p. 93:

\[
\Lambda(f) := \{ \varphi = f(s_0)\delta_{s_0} : s_0 \in [0, 1] \text{ s.t. } |f(s_0)| = \| f \| = \max_{s \in [0, 1]} |f(s)| \} \subset J(f),
\]

where \( \delta_{s_0} \) is any point measure supported by a point \( s_0 \) where \( |f| \) reaches its maximum. For the expression of the full duality map (see e.g. [11], p. 5).

Let \( y \in \mathcal{D}(A) \) and let \( y^* = |y(s_0)|\delta_{s_0} \in J(y) \), i.e. \( |y(s_0)| = \| y \| \). Thus for \( s_0 \in (0, 1) \) we have \( y'(s_0) = 0 \), so taking into account the Neumann boundary conditions, we deduce that \( y'(s_0) = 0 \) for every \( s_0 \in [0, 1] \) such that \( |y(s_0)| = \| y \| \). Thus we have

\[
\langle y', J(y) \rangle = |y'(s_0)||y(s_0)| = 0, \quad \forall y \in \mathcal{D}(A).
\]

It follows that

\[
\langle \mathcal{B} y, J(y) \rangle = \| y \|^2, \quad \forall y \in \mathcal{D}(A).
\]

Hence \((\text{h}_5)\) and \((\text{h}_6)\) are verified for any element of \( \Lambda(f) \).

Now, from (33), p. 82 we deduce that \( A - \mu \mathcal{B} \) with domain \( \mathcal{D}(A - \mu \mathcal{B}) = \mathcal{D}(A) \) is a generator on \( X \) for \( \mu > 0 \) small enough. Hence using again that \( \langle \mathcal{B} y, J(y) \rangle = \| y \|^2 \) for all \( y \in \mathcal{D}(A) \), we derive directly from (9) that the feedback control \( v(t) = -\mu 1_{\{t \geq 0; y(t) + y_0(t) \neq 0\}} \) results in an exponentially stable closed-loop system for a small gain control \( \mu > 0 \).

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