REPRESENTABLE TOLERANCES IN VARIETIES

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Abstract. We discuss two possible ways of representing tolerances: first, as a homomorphic image of some congruence; second, as the relational composition of some compatible relation with its converse. The second way is independent from the variety under consideration, while the first way is variety-dependent. The relationships between these two kinds of representations are clarified.

As an application, we show that any tolerance on some lattice \( \mathcal{L} \) is the image of some congruence on a subalgebra of \( \mathcal{L} \times \mathcal{L} \). This is related to recent results by G. Czédli and E. W. Kiss [CK].

1. Introduction

Recall that a tolerance on some algebra is a binary, compatible, symmetric and reflexive relations. Thus, a congruence is just a transitive tolerance.

It is quite surprising that the study of tolerances (apart from intrinsic interest) has revealed to be essential in the study of congruences. Indeed, present-day research shows that tolerances are becoming increasingly important even in many other at first look seemingly unrelated contexts [C, CCH, CCHL, CG, CHL, CK, G, HM, HMK, KK, L, S, W].

2. A first example

Definition 2.1. Suppose that \( \mathfrak{A}, \mathfrak{B} \) are algebras, and \( \varphi : \mathfrak{B} \to \mathfrak{A} \) is a surjective homomorphism. It is folklore, and easy to see, that if \( \beta \) is a congruence on \( \mathfrak{B} \), then \( \varphi(\beta) = \{ (\varphi(a), \varphi(b)) \mid a \beta b \in B \} \) is a tolerance \( \Theta \) on \( \mathfrak{A} \). In the above situation, we say that \( \Theta \) is an image of \( \beta \). In case we need to specify \( \mathfrak{B} \) explicitly, we shall say that \( \Theta \) is the image of a congruence on \( \mathfrak{B} \).

We now exemplify our methods in the particular case of lattices.
Theorem 2.2. If \( L \) is a lattice and \( \Theta \) is a tolerance on \( L \), then \( \Theta \) is the image of some congruence on some subalgebra of \( L \times L \).

Proof. The partial order \( \leq \) induced by the lattice operations is a compatible relation on \( L \), thus also \( \leq \cap \Theta \) is compatible. Hence the binary relation \( \leq \cap \Theta \) can be considered as a subalgebra \( B \) of \( L \times L \). Let \( \varphi : B \to A \) be the first projection, and \( \beta \) on \( B \) be the kernel of the second projection.

We shall show that \( \varphi(\beta) = \Theta \). Indeed, if \( a \Theta b \), then \( a = a \lor a \Theta a \lor b \), thus \( (a, a \lor b) \in B \), since \( a \leq a \lor b \). Similarly, \( (b, a \lor b) \in B \). Trivially, \( (a, a \lor b) \beta (b, a \lor b) \), \( \varphi(a, a \lor b) = a \), \( \varphi(b, a \lor b) = b \), thus \( \Theta \subseteq \varphi(\beta) \).

Conversely, if \( (a, b) \in \varphi(\beta) \), then there is \( c \in L \) such that \( (a, c) \in B \), \( (b, c) \in B \), hence \( a \Theta c \), \( a \leq c \), \( c \Theta b \), \( b \leq c \), thus \( a = a \land c \Theta c \land b = b \). Hence \( \varphi(\beta) \subseteq \theta \), and the theorem is proved.

Notice that we have not used all the properties of a lattice, thus Theorem 2.2 allows some strengthening, see Proposition 5.1 below. Also, the proof of Theorem 2.2 applies not only to lattices, but also to lattices with additional operations, provided the additional operations respect the lattice order, that is, the order remains a compatible relation with respect to the additional operations.

Notice also that if \( L \) belongs to some variety of lattices \( \mathcal{V} \), then every subalgebra of \( L \times L \) belongs to \( \mathcal{V} \). In fact, we do not need the full assumption that \( \mathcal{V} \) is a variety: we get that if \( \mathcal{V} \) is a class of lattices, and \( \mathcal{V} \) is closed under subalgebras and finite products then every tolerances on some lattice in \( \mathcal{V} \) is the image of some congruence of some lattice in \( \mathcal{V} \).

Following \[\text{CCH}, \text{CK}\], we say that the tolerances of \( \mathcal{V} \) are images of its congruences if every tolerance in \( \mathcal{V} \) is the image of a congruence in \( \mathcal{V} \). By G. Czédli and G. Grätzer \[\text{CG}\] the variety of all lattices, and by G. Czédli and E. W. Kiss \[\text{CK}\], every variety of lattices have the above property. Theorem 2.2 thus furnishes another proof of Czédli and E. W. Kiss result.

As we shall see in the next section, the reason why Theorem 2.2 holds is that tolerances in lattices are representable in the sense of Definition 3.1 to be introduced in the next section.

3. Representable tolerances

Definition 3.1. Suppose that \( A \) is an algebra, and \( R \) is a compatible reflexive relation on \( A \). Let \( R^- \) denote the converse of \( R \). It is immediate to see that \( R \circ R^- \) is a tolerance on \( A \). Tolerances which are representable in the form \( R \circ R^- \) as above have been called representable in \[\text{L}, \text{Definition 2}\], where it has been shown, among other
things, that not every tolerance is representable. See [L, Section 6] for
more on representable tolerances.

**Theorem 3.2.** If $\Theta$ is a representable tolerance on the algebra $\mathfrak{A}$, then
$\Theta$ is the image of a congruence on some subalgebra of $\mathfrak{A} \times \mathfrak{A}$.

**Proof.** Suppose that $\Theta$ is representable as $\Theta = R \circ R^-$, for $R$ a reflexive
compatible relation on $\mathfrak{A}$. Let $\mathfrak{B}$ be $R$ itself, considered as a subalgebra
of $\mathfrak{A} \times \mathfrak{A}$, and let $\varphi$ be the first projection. Since $R$ is reflexive, we have
that $\varphi$ is surjective. Let $\beta$ on $\mathfrak{B}$ be the kernel of the second projection,
that is, $(a, b) \beta (c, d)$ if and only if $b = d$. We shall show that $\varphi(\beta) = \Theta$.
Indeed, for every $a, c \in A$, and since $\Theta = R \circ R^-$, the following is a
chain of equivalent conditions.

1. $a \Theta c$;
2. There is $b \in A$ such that $(a, b) \in R$ and $(b, c) \in R^-$;
3. There is $b \in A$ such that $(a, b), (c, b) \in R$.
4. There is $b \in A$ such that $(a, b), (c, b) \in B$ (thus, $(a, b)\beta (c, b)$).
5. $(a, c) \in \varphi(\beta)$.

We have shown that $\Theta = \varphi(\beta)$, thus the theorem is proved. \(\Box\)

G. Czédli observed that every tolerance on a lattice is representable,
as a consequence of Lemma 2 in [Cz]. Cf. also [CZ]. See [L, Proposition
11] and Proposition 5.1 below, for some slightly more general results.
Hence Theorem 2.2 is actually a particular case of Theorem 3.2. We
have given a direct proof of Theorem 2.2 since it is relatively short and
simple.

It has been shown in [CCHL, CK] that every tolerance on some
algebra $\mathfrak{A}$ is the image of some congruence $\beta$, for appropriate $\mathfrak{B}$ and $\varphi$.
However, in most cases, $\mathfrak{A}$ belongs to some specified variety $\mathcal{V}$, and it
is a natural request to ask that $\mathfrak{B}$, too, belongs to $\mathcal{V}$. This observation
justifies the next definition.

A note on terminology: we shall say that a tolerance $\Theta$ is in a variety
$\mathcal{V}$ to mean that $\Theta$ is a tolerance on some algebra $\mathfrak{A} \in \mathcal{V}$. Technically,
this is justified since a tolerance on $\mathfrak{A}$ can be seen as a subalgebra of
$\mathfrak{A} \times \mathfrak{A}$ (and $\Theta$ and $\mathfrak{A}$ generate the same variety, since, in the above sense,
$\mathfrak{A}$ is isomorphic to a substructure of $\Theta$). A similar remark applies to
congruences in place of tolerance.

**Definition 3.3.** If $\Theta$ is a tolerance on $\mathfrak{A} \in \mathcal{V}$, we say that $\Theta$ is the
image of a congruence in $\mathcal{V}$ if it is possible to chose $\mathfrak{B} \in \mathcal{V}$, $\beta$ a
congruence on $\mathfrak{B}$, and $\varphi : \mathfrak{B} \to \mathfrak{A}$ a surjective homomorphism such
that $\Theta = \varphi(\beta)$.

The above definition is a local version of the mentioned notion from
[CCH, CK] that the tolerances of $\mathcal{V}$ are images of its congruences.
Though in the above definitions $\mathcal{V}$ is intended to stand for a variety, our results generally hold for an arbitrary class $\mathcal{V}$ which is closed under taking subalgebras and products, in particular, for quasivarieities. Actually, in most cases, it is enough to assume that $\mathcal{V}$ is closed under taking subalgebras and finite products.

As an immediate consequence of Theorem 3.2, we get:

**Corollary 3.4.** If all tolerances in the variety $\mathcal{V}$ are representable, then the tolerances of $\mathcal{V}$ are the images of its congruences (actually, it is enough to suppose that $\mathcal{V}$ is closed under taking subalgebras and finite products).

The converse of Corollary 3.4 is not true. By [CCHL, CK], in every variety defined by the empty set of equations the tolerances of $\mathcal{V}$ are the images of its congruences, but, by [L, Proposition 10] (see also Proposition 5.5 below), there exists a non representable tolerance on some algebra (which trivially belongs to a variety defined by an empty set of equations).

However, it is possible to show that, within a given variety, a tolerance is the image of some congruence if and only if it is the image of some representable tolerance (see Corollary 4.5 below). This can be useful, since if we want to show that, for a variety $\mathcal{V}$, the tolerances of $\mathcal{V}$ are the images of its congruences, it is enough to show that the tolerances of $\mathcal{V}$ are the images of its representable tolerances.

4. **Weakly representable tolerances**

There is a version of Theorem 3.2 dealing with a more general notion of representability.

**Definition 4.1.** Let $\lambda$ be a nonzero cardinal.

We say that a tolerance is $\lambda$-weakly representable if it is the intersection of at most $\lambda$ representable tolerances.

A tolerance is weakly representable if it is $\lambda$-weakly representable, for some nonzero cardinal $\lambda$. Thus, representable is the same as 1-weakly representable. See again [L, Section 6] for more informations about weakly representable tolerances.

In the statement of the next theorem, $+$ denotes cardinal sum, that is, $\lambda + 1 = \lambda$, if $\lambda$ is infinite.

**Theorem 4.2.** If $\Theta$ is a $\lambda$-weakly representable tolerance on the algebra $\mathfrak{A}$, then $\Theta$ is the image of a congruence on some subalgebra of the power $\mathfrak{A}^{\lambda+1}$. 
Hence, if \( V \) is a class closed under subalgebras and products (in particular, if \( V \) is a variety), then every weakly representable tolerance belonging to \( V \) is the image of a congruence in \( V \).

**Proof.** Suppose that \( \Theta = \bigcap_{i \in \lambda} \Theta_i \), where each \( \Theta_i \) has the form \( R_i \circ R_i^{-} \), for certain reflexive compatible relations \( R_i \). Let \( \mathcal{B} \) be the subalgebra of \( A \times A^\lambda \cong A^{\lambda+1} \) whose base set is \( B = \{(a, (a_i)_{i \in \lambda}) \mid a, a_i \in A, \text{ and } a \; R_i \; a_i, \text{ for each } i \in \lambda \} \). The assumption that each \( R_i \) is compatible implies that \( \mathcal{B} \) is indeed a subalgebra of \( A \times A^\lambda \).

Let \( \varphi : \mathcal{B} \to A \) be the first projection, and let \( \beta \) be the kernel of the second projection \( \pi : \mathcal{B} \to A^\lambda \). The same arguments as in the proof of Theorem 3.2 show that \( \varphi(\beta) = \Theta \).

The converse of Theorem 4.2 does not hold, in general; see Proposition 5.5. However, we expect that the converse of Theorem 4.2 is true, under some mild assumptions on \( A \) or \( V \).

Theorem 4.2 gives us the possibility of improving Corollary 3.4.

**Corollary 4.3.** If all tolerances in a variety \( V \) are weakly representable, then the tolerances of \( V \) are the images of its congruences (indeed, it is enough to assume that \( V \) is closed under subalgebras and arbitrary products).

Extending Definition 2.1 in the natural way, if \( A \) and \( C \) are algebras, we say that a tolerance \( \Theta \) on \( A \) is the image of some tolerance \( \Psi \) on \( C \) if there is some surjective homomorphism \( \psi : C \to A \) such that \( \Theta = \{(\psi(a), \psi(b)) \mid a \; \Psi \; b\} \). It is trivial to see that the image of some tolerance, in the above sense, is again a tolerance.

**Lemma 4.4.** If \( \Theta, \Psi \) and \( \Phi \) are tolerances, \( \Theta \) is an image of \( \Psi \), and \( \Psi \) is an image of \( \Phi \), then \( \Theta \) is an image of \( \Phi \).

**Proof.** Let \( \Theta \) be on \( A \), \( \Psi \) be on \( C \), and \( \Phi \) be on \( D \), and let the assumption of the lemma be witnessed by surjective homomorphisms \( \psi : C \to A \) and \( \varphi : D \to C \). Then \( \psi \circ \varphi : D \to A \) witnesses that \( \Theta \) is an image of \( \Phi \).

**Corollary 4.5.** Let \( V \) be a class of algebras closed under subalgebras and products (in particular, a variety). For every tolerance \( \Theta \) in \( V \), the following conditions are equivalent.

1. \( \Theta \) is the image of a congruence in \( V \).
2. \( \Theta \) is the image of a representable tolerance in \( V \).
3. \( \Theta \) is the image of a weakly representable tolerance in \( V \).

In particular, for every \( V \) as above, the tolerances of \( V \) are the images of its congruences if and only if the tolerances of \( V \) are the images of its (weakly) representable tolerances.
Proof. (1) ⇒ (2) and (2) ⇒ (3) are trivial, since every congruence \(\beta\) is representable (as \(\beta = \beta \circ \beta\)), and since every representable tolerance is weakly representable.

(3) ⇒ (1) If \(\Theta\) is the image of a weakly representable tolerance \(\Psi\) in \(V\), then, by Theorem 4.2, \(\Psi\) is the image of some congruence \(\beta\) in \(V\), hence, by Lemma 4.4, \(\Theta\) is the image of \(\beta\). □

Corollary 4.6. Suppose that \(\Theta\) is a tolerance on the algebra \(\mathfrak{A}\). Then the following conditions are equivalent.

(1) \(\Theta\) is the image of a congruence on some subalgebra of some power \(\mathfrak{A}^I\), for some set \(I\).
(2) For every variety \(V\) such that \(\mathfrak{A} \in V\), \(\Theta\) is the image of a congruence in \(V\).
(3) \(\Theta\) is the image of a congruence in \(V(\mathfrak{A})\), the variety generated by \(\mathfrak{A}\).

In all the preceding conditions we can equivalently replace the word “congruence” with either “representable tolerance” or “weakly representable tolerance”.

Proof. (1) ⇒ (2) is obvious, since if \(\mathfrak{A} \in V\), then every subalgebra of \(\mathfrak{A}^I\) is in \(V\).

(2) ⇒ (3) is trivial.

(3) ⇒ (1) Let \(\Theta\) be an image of \(\gamma\), a congruence on \(\mathfrak{C} \in V(\mathfrak{A})\). By the HSP characterization of \(V(\mathfrak{A})\), there are a set \(I\), an algebra \(\mathfrak{B} \subseteq \mathfrak{A}^I\), and a surjective homomorphism \(\varphi : \mathfrak{B} \rightarrow \mathfrak{C}\). Then \(\beta = \varphi^{-1}(\gamma) = \{(b, b') \mid b, b' \in B \text{ and } (\varphi(b), \varphi(b')) \in \gamma\}\) is a congruence on \(\mathfrak{B}\), and \(\varphi(\beta) = \gamma\), in the sense of Definition 2.1.

Thus \(\Theta\) is an image of \(\gamma\), which is an image of \(\beta\), hence, by Lemma 4.4, \(\Theta\) is an image of \(\beta\), a congruence on \(\mathfrak{A}^I\), and (1) is proved.

The last statement is immediate from Corollary 4.5. □

5. Additional remarks

We first provide a generalization of Theorem 2.2. Its proof exploits exactly the only properties of lattices which were used in the proof of 2.2.

Proposition 5.1. Suppose that \(\mathfrak{A}\) is an algebra with two binary operations \(\vee\) and \(\wedge\) (among possibly other operations), and with a compatible binary relation \(M\), which satisfy the following conditions:

(1) \(a \vee a = a\), for every \(a \in A\).
(2) \(a M(a \vee b)\), and \(b M(a \vee b)\), for every \(a, b \in A\).
(3) \(a = a \wedge c = c \wedge a\), for every \(a, c \in A\) such that \(a M c\).
Then every tolerance \( \Theta \) of \( \mathfrak{A} \) is representable, and is an image of some congruence on some subalgebra of \( \mathfrak{A} \times \mathfrak{A} \).

Proof. Same as the proof of Theorem 2.2, using \( M \) in place of \( \leq \): \( \Theta \) is representable as \( R \circ R^- \), with \( R = M \cap \Theta \). The last statement is immediate from Theorem 3.2. \( \square \)

Remark 5.2. Condition (2) in Proposition 5.1 is satisfied in case \( M \) is defined by

\[
a \ M b \text{ if and only if } a \lor b = b,
\]

and \( \mathfrak{A} \) satisfies \( a \lor (a \lor b) = a \lor b \) and \( b \lor (a \lor b) = a \lor b \), for every \( a, b \in A \).

By Proposition 5.1, and writing explicitly the condition that the \( M \) given by Remark 5.2 is compatible, we get:

**Proposition 5.3.** Suppose that \( \mathfrak{A} \) is an algebra with (exactly) two binary operations \( \lor \) and \( \land \) satisfying the following conditions:

1. For every \( a, a', b, b' \in A \), if \( a \lor b = b \) and \( a' \lor b' = b' \), then \( (a' \lor a') \lor (b \lor b') = b \lor b' \) and \( (a' \lor a') \land (b \lor b') = b \land b' \).
2. \( a \lor (a \lor b) = a \lor b \), and \( b \lor (a \lor b) = a \lor b \), for every \( a, b \in A \).
3. For every \( a, c \in A \), if \( a \lor c = c \), then \( a = a \land c = c \land a \).

Then every tolerance \( \Theta \) of \( \mathfrak{A} \) is representable, and is an image of some congruence on some subalgebra of \( \mathfrak{A} \times \mathfrak{A} \).

We now show that the conditions exploited in the proof of Theorem 3.2 actually characterize representable tolerances.

**Proposition 5.4.** Suppose that \( \Theta \) is a tolerance on the algebra \( \mathfrak{A} \). Then \( \Theta \) is representable if and only if \( \Theta \) can be realized as the image of a congruence on some subalgebra \( \mathfrak{B} \) of \( \mathfrak{A} \times \mathfrak{A} \), such that \( \mathfrak{B} \) contains \( \Delta = \{(a, a) \mid a \in A\} \), and in such a way that \( \varphi \) and \( \beta \) in Definition 2.1 can be chosen to be, respectively, the first projection and the kernel of the second projection.

Proof. The construction used in the proof of Theorem 3.2 shows that if \( \Theta \) is representable, then \( \mathfrak{B} \), \( \varphi \), and \( \beta \) can be chosen to satisfy the desired requirements.

Conversely, suppose that we have \( \mathfrak{B} \subseteq \mathfrak{A} \times \mathfrak{A} \), \( \varphi \) and \( \beta \) satisfying the conditions in the statement of the proposition. Being a subalgebra of \( \mathfrak{A} \times \mathfrak{A} \), \( \mathfrak{B} \) can be thought of as a compatible relation on \( \mathfrak{A} \). We shall take \( R = B \). Since \( B \) contains \( \Delta \), then \( R \) is reflexive. By assumption, \( a \Theta c \) if and only if \( (a, c) \in \varphi(\beta) \). Noticing that the equivalence of items (5) and (2) in the proof of Theorem 3.2 holds also in the present situation,
we get that \( a \Theta c \) if and only if there is \( b \in A \) such that \( (a, b) \in R \) and \( (b, c) \in R^- \). This means exactly that \( \Theta = R \circ R^- \).

We now show that the converse of Theorem 4.2 fails in a large class of algebras.

**Proposition 5.5.** For every set \( A \), and every reflexive and symmetric relation \( \Theta \) on \( A \) which is not transitive, there is an algebra \( A \) with base set \( A \) and such that \( \Theta \) is a tolerance on \( A \) which is not weakly representable, but \( \Theta \) is the image of some congruence on some subalgebra of some power of \( A \).

**Proof.** For every \( a, b \in A \) such that \( a \Theta b \), and for every function \( f : A \to \{a, b\} \), add to \( A \) a unary function symbols representing \( f \). It is easy to see that \( \Theta \) is a tolerance on the algebra thus obtained, and that \( \Theta \) is not weakly representable. Indeed, every nontrivial compatible relation \( R \) on \( A \) contains \( \Theta \), and, since \( \Theta \) is not transitive, then \( \Theta \subseteq \Theta \circ \Theta \subseteq R \circ R^- \) (see [L, Proposition 12] for more details).

Consider \( V(A) \), the variety generated by \( A \). Since \( V(A) \) is unary, then, by [CK, Corollary 4.4], tolerances are images of congruences in \( V(A) \). Then Corollary [L,3] \( \Rightarrow \) (1) implies that \( \Theta \) is the image of some congruence on some subalgebra of some power of \( A \).

On the other hand, under certain conditions, the converse of Theorem 4.2 does hold.

**Proposition 5.6.** Suppose that \( A \) is an algebra in a 3-permutable variety \( V \), and \( \Theta \) is a tolerance on \( A \). Then the following conditions are equivalent.

1. \( \Theta \) is representable.
2. \( \Theta \) is weakly representable.
3. \( \Theta \) is the image of some congruence on some subalgebra of \( A \times A \).
4. \( \Theta \) is the image of some congruence in \( V(A) \).
5. \( \Theta \) is a congruence of \( A \).

If we only assume that every subalgebra of \( A \times A \) has 3-permutable congruences, then Conditions (1), (3) and (5) above are still equivalent.

**Proof.** (1) \( \Rightarrow \) (2) and (3) \( \Rightarrow \) (4) are trivial.

(1) \( \Rightarrow \) (3) and (2) \( \Rightarrow \) (4) follow from, respectively, Theorems 3.2 and 4.2.

(4) \( \Rightarrow \) (5) Clearly \( V(A) \), being a subvariety of \( V \), is 3-permutable, too, hence \( \Theta \) is the image of some congruence on some algebra with 3-permuting congruences. But it is well-known that this implies that \( \Theta \) is a congruence, see Theorem 1.10 in [J, Chapter 7].

(5) \( \Rightarrow \) (1) is trivial, since if \( \Theta \) is a congruence, then \( \Theta = \Theta \circ \Theta^- \).
Under the assumption that every subalgebra of $\mathfrak{A} \times \mathfrak{A}$ has 3-permutable congruences, (3) $\Rightarrow$ (5) holds, again by Theorem 1.10 in [J, Chapter 7]. The implications (1) $\Rightarrow$ (3) and (5) $\Rightarrow$ (1) do not use 3-permutability at all.

We expect that parts of Proposition 5.6 hold under assumptions weaker than 3-permutability. However, globally (that is, if we ask that the conditions hold for every tolerance in a 3-permutable variety), Proposition 5.6 is essentially an empty result, in the sense that the conditions hold only in permutable varieties (in which they are trivially true).

**Corollary 5.7.** Suppose that $\mathcal{V}$ is an $n$-permutable variety, for some $n$. Then the following conditions are equivalent.

1. Every tolerance in $\mathcal{V}$ is weakly representable.
2. Every tolerance in $\mathcal{V}$ is representable.
3. Every tolerance in $\mathcal{V}$ is the image of a congruence in $\mathcal{V}$.
4. $\mathcal{V}$ is permutable.
5. Every tolerance in $\mathcal{V}$ is a congruence.

**Proof.** (1) $\Rightarrow$ (2) and (5) $\Rightarrow$ (1) are trivial.

(2) $\Rightarrow$ (3) follows from Theorem 3.4.

(3) $\Rightarrow$ (4) is [CK, Theorem 5.3].

(4) $\Rightarrow$ (5) is immediate from a classical result from [W], parts of which are due independently to G. Hutchinson [H]. Actually, Conditions (4) and (5) are equivalent for every variety, as follows easily from the above papers, and explicitly stated, e. g., in [C].

**Problems 5.8.** Notice that, again by [CZ, CZ], tolerances in lattices satisfy a property stronger than representability. Indeed, if $\Theta$ is a tolerance on a lattice $\mathfrak{L}$, then

1. there is a compatible relation $R$ such that $\Theta = (R \circ R^-) \cap (R^- \circ R)$, or even

2. there is a compatible relation $R$ such that $a \Theta b$ if and only if there are $c$ and $d$ such that $a R_c R^- b$, $a R^- d R b$, and $d R c$.

(just take $R = \Theta \cap \leq$, $c = a \lor b$ and $d = a \land b$) Which parts of the theory of tolerances on lattices follow just from the assumption (1) or (2)?
Notice that we do not need all the axioms for lattices, in order to get \( (1) \) above: the properties listed in Proposition 5.1, together with their duals suffice.

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