SATOH THEOR Y ON THE $q$-TODA HIERARCHY AND ITS EXTENSION

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Abstract. In this paper, we construct the Sato theory including the Hirota bilinear equations and tau function of a new $q$-deformed Toda hierarchy (QTH). Meanwhile the Block type additional symmetry and bi-Hamiltonian structure of this hierarchy are given. From Hamiltonian tau symmetry, we give another definition of tau function of this hierarchy. Afterwards, we extend the $q$-Toda hierarchy to an extended $q$-Toda hierarchy (EQTH) which satisfy a generalized Hirota quadratic equation in terms of generalized vertex operators. The Hirota quadratic equation might have further application in Gromov-Witten theory. The corresponding Sato theory including multi-fold Darboux transformations of this extended hierarchy is also constructed. At last, we construct the multicomponent extension of the $q$-Toda hierarchy and show the integrability including its bi-Hamiltonian structure, tau symmetry and conserved densities.

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1. Introduction

The Toda lattice and KP hierarchy are completely integrable systems which have many important applications in mathematics and physics including the theory of Lie algebra representation, orthogonal polynomials and random matrix model [1-5]. KP and Toda systems have many kinds of reduction or extension, for example BKP, CKP hierarchy, extended Toda hierarchy (ETH) [6,7], bigraded Toda hierarchy (BTH) [8-14] and so on.

The $q$-calculus (also called quantum calculus) traces back to the early 20th century. Many mathematicians have important works in the area of $q$-calculus and $q$-hypergeometric series [15,16]. The $q$-deformation of classical nonlinear integrable system started in 1990’s by means of $q$-derivative $\partial_q$ instead of usual derivative with respect to $x$ in the classical system. As we know, the $q$-deformed integrable system reduces to a classical integrable system when $q$ goes to 1.

Several $q$-deformed integrable systems have been presented, for example the $q$-deformed Kadomtsev-Petviashvili ($q$-KP) hierarchy is a subject of intensive study in the literature [17]-[24]. Basing on a similar $q$-operator as $q$-KP hierarchy in [20,21], the $q$-Toda equation was studied in [25,26] but not for a whole hierarchy. This paper will be devoted to the further studies on the whole $q$-Toda hierarchy (QTH) and its extended hierarchy with logarithmic flows.

Adding additional logarithmic flows to the Toda lattice hierarchy, it becomes the extended Toda hierarchy [6] which governs the Gromov-Witten invariant of $CP^1$. Therefore what is the application in Gromov-Witten theory of the $q$-deformed extended Toda hierarchy becomes a natural question which is one motivation for us to do this work. The extended bigraded Toda hierarchy (EBTH) [6] is the extension of the bigraded Toda hierarchy (BTH) which includes additional logarithmic flows [8,14]. The Hirota bilinear equation of the EBTH was equivalently constructed in our early paper. One can also consider the bigraded extension of the extended QTH which might be included in our future work.

The multicomponent 2D Toda hierarchy was considered from the point of view of the Gauss-Borel factorization problem, non-intersecting Brownian motions and matrix Riemann-Hilbert problem [31]-[34]. In fact the multicomponent 2D Toda hierarchy in [32] is a periodic reduction of the bi-infinite matrix-formed two dimensional Toda hierarchy. The coefficients of the multicomponent 2D Toda hierarchy take values in complex finite-sized matrices. In this paper, we also construct the multicomponent extension of the $q$-Toda hierarchy and show the integrability including its bi-Hamiltonian structure, tau symmetry.

This paper is arranged as follows. In the next section we recall a factorization problem and construct the Lax equations of the $q$-Toda hierarchy. In Section 3-7, we will give the Sato theory of the $q$-Toda hierarchy (QTH) including Hirota bilinear equations, the tau function, vertex operators and Hirota quadratic equations. Basing on the double dressing structure of this hierarchy, the Block type Lie symmetry [14,35] of the QTH was given in Section 8. In Section 9-11, we generalize the Sato theory of the $q$-Toda hierarchy to the extended $q$-Toda hierarchy (EQTH). To prove the integrability of this new extended hierarchy, the bi-Hamiltonian structure and tau symmetry of the EQTH are constructed. In Section 12, the multi-fold Darboux transformation of the EQTH was given which can produce new solutions from seed solutions as used in [27-30]. In Section 13-15, we construct the multicomponent extension of the $q$-Toda hierarchy and show the integrability including the bi-Hamiltonian structure, tau symmetry and conserved densities of this matrix hierarchy.
2. Factorization and dressing operators

Now we will consider the shift operator $\Lambda_q$ acting on these functions as $(\Lambda_q g)(x) := g(qx)$, i.e. $\Lambda_q := e^{\epsilon x \partial_x}$, $q = e^\epsilon$. A Left multiplication by $X$ is as $X\Lambda_q^j$, $(X\Lambda_q^j)(g)(x) := X(x) \circ g(q^j x)$ with defining the product $(X(x)\Lambda_q^j) \circ (Y(x)\Lambda_q^j) := X(x)Y(q^j x)\Lambda_q^{i+j}$.

The Lie algebra
\[ g = \left\{ \sum_j X_j(x)\Lambda_q^j \right\}, \]
has the following important splitting
\[ g = g_+ \circ g_-, \quad \text{(2.1)} \]
where
\[ g_+ = \left\{ \sum_{j \geq 0} X_j(x)\Lambda_q^j \right\}, \quad g_- = \left\{ \sum_{j < 0} X_j(x)\Lambda_q^j \right\}. \]

For the corresponding Lie group $G$ whose Lie algebra is $g$, the splitting (2.1) leads us to consider the following factorization of $g \in G$
\[ g = g_-^1 \circ g_+, \quad g_+ \in G_+ \quad \text{(2.2)} \]
where $G_\pm$ have $g_\pm$ as their Lie algebras. $G_+$ is the set of invertible linear operators of the form $\sum_{j \geq 0} g_j(x)\Lambda_q^j$, while $G_-$ is the set of invertible linear operators of the form $1 + \sum_{j < 0} g_j(x)\Lambda_q^j$. Then the set $g$ of Laurent series in $\Lambda_q$ as an associative algebra is a Lie algebra under the standard commutator. Similar as [36], the factorization (2.2) belong to the big cell [4] and the factorization is defined only locally to avoid the generation of additional problems connected with these local aspects.

Now we introduce the following free operators $W_0, \bar{W}_0 \in G$
\[ W_0 := e^{\sum_{j=0}^{\infty} t_j^j \Lambda_q^j}, \quad \text{(2.3)} \]
\[ \bar{W}_0 := e^{\sum_{j=0}^{\infty} \bar{t}_j^{-j} \Lambda_q^{-j}}, \quad \text{(2.4)} \]
where $t_j \in \mathbb{C}$ will play the role of continuous times.

We define the dressing operators $W, \bar{W}$ as follows
\[ W := S \circ W_0, \quad \bar{W} := \bar{S} \circ \bar{W}_0, \quad S \in G_-, \quad \bar{S} \in G_+. \quad \text{(2.5)} \]
Given an element $g \in G$ and denote $t = (t_j), j \in \mathbb{N}$, one can consider the factorization problem in $G$
\[ W \circ g = \bar{W}, \quad \text{(2.6)} \]
i.e. the factorization problem
\[ S(t) \circ W_0 \circ g = \bar{S}(t) \circ \bar{W}_0. \quad \text{(2.7)} \]
Observe that $S, \bar{S}$ have expansions of the form
\[ S = 1 + \omega_1(x)\Lambda_q^{-1} + \omega_2(x)\Lambda_q^{-2} + \cdots \in G_-, \]
\[ \bar{S} = \bar{\omega}_0(x) + \bar{\omega}_1(x)\Lambda_q + \bar{\omega}_2(x)\Lambda_q^2 + \cdots \in G_. \quad \text{(2.8)} \]
Also we define the symbols of $S, \bar{S}$ as $S, \bar{S}$
\begin{align}
S & = 1 + \omega_1(x)\lambda^{-1} + \omega_2(x)\lambda^{-2} + \cdots , \\
\bar{S} & = \bar{\omega}_0(x) + \bar{\omega}_1(x)\lambda + \bar{\omega}_2(x)\lambda^2 + \cdots .
\end{align}
(2.9)

The inverse operators $S^{-1}, \bar{S}^{-1}$ of operators $S, \bar{S}$ have expansions of the form
\begin{align}
S^{-1} & = 1 + \omega'_1(x)\lambda^{-1} + \omega'_2(x)\lambda^{-2} + \cdots \in G_-, \\
\bar{S}^{-1} & = \bar{\omega}'_0(x) + \bar{\omega}'_1(x)\lambda + \bar{\omega}'_2(x)\lambda^2 + \cdots \in G_+.
\end{align}
(2.10)

Also we define the symbols of $S^{-1}, \bar{S}^{-1}$ as $S^{-1}, \bar{S}^{-1}$ as following
\begin{align}
S^{-1} & = 1 + \omega'_1(x)\lambda^{-1} + \omega'_2(x)\lambda^{-2} + \cdots , \\
\bar{S}^{-1} & = \bar{\omega}'_0(x) + \bar{\omega}'_1(x)\lambda + \bar{\omega}'_2(x)\lambda^2 + \cdots .
\end{align}
(2.11)

The Lax operators $L \in G$ of the $q$-deformed Toda hierarchy are defined by
\begin{align}
L & := W \circ \Lambda_q \circ W^{-1} = \bar{W} \circ \Lambda_q^{-1} \circ \bar{W}^{-1},
\end{align}
(2.12)

and have the following expansions
\begin{align}
L & = \Lambda_q + U(x) + V(x)\Lambda_q^{-1}.
\end{align}
(2.13)

In fact the Lax operators $L \in G$ are also be equivalently defined by
\begin{align}
L & := S \circ \Lambda_q \circ S^{-1} = \bar{S} \circ \Lambda_q^{-1} \circ \bar{S}^{-1}.
\end{align}
(2.14)

3. Lax equations of QTH

In this section we will use the factorization problem (2.6) to derive Lax equations. Let us first introduce some convenient notation on the operators $B_j$ defined as follows
\begin{align}
B_j := \frac{L^{j+1}}{(j+1)!}.
\end{align}
(3.1)

Now we give the definition of the $q$-Toda hierarchy(QTH).

**Definition 1.** The $q$-Toda hierarchy is a hierarchy in which the dressing operators $S, \bar{S}$ satisfy following Sato equations
\begin{align}
\epsilon \partial_t S & = -(B_j)_- S, \\
\epsilon \partial_t \bar{S} & = (B_j)_+ \bar{S}.
\end{align}
(3.2)

Then one can easily get the following proposition about $W, \bar{W}$.

**Proposition 1.** The dressing operators $W, \bar{W}$ are subject to following Sato equations
\begin{align}
\epsilon \partial_t W & = (B_j)_+ W, \\
\epsilon \partial_t \bar{W} & = (B_j)_+ \bar{W}.
\end{align}
(3.3)

From the previous proposition we derive the following Lax equations for the Lax operators.

**Proposition 2.** The Lax equations of the QTH are as follows
\begin{align}
\epsilon \partial_t L & = [(B_j)_+, L].
\end{align}
(3.4)

To show the relation of the QTH and the $q$-KP type hierarchy [20] [21], we will do the following remark.
Remark 1. The q-Toda hierarchy can be treated as a generalization of the q-KdV hierarchy [20] in terms of the same multiplication shift operator $\Lambda_q$. The q-KP hierarchy in [27] is in fact a more general generalization of the q-KdV hierarchy in [20] after rewriting the operator $\Delta_q$ as $\Delta_q=1$. Therefore the q-Toda hierarchy can be treated as a special reduction of the q-KP hierarchy in [27] in terms of an operator $\Delta_q=\Lambda_q-1$ after a certain transformation. The operator $\Lambda_q$ in this paper is different from the q-derivative operator in [22, 24] in which $D_q f(x) = \frac{f(qx)-f(x)}{(q-1)x}$ which leads to a different hierarchy.

To see this kind of hierarchy more clearly, the q-Toda equations as the $t_0$ flow equations will be given in the next subsection.

3.1. The q-Toda equations. As a consequence of the factorization problem (2.6) and Sato equations, after taking into account that $S \in G_-$ and $\bar{S} \in G_+$, the $t_0$ flow of $\mathcal{L}$ in the form of $\mathcal{L} = \Lambda_q + U + V\Lambda_q^{-1}$ is as

$$\epsilon \partial_{t_0}\mathcal{L} = [\Lambda_q + U, V\Lambda_q^{-1}],$$
(3.5)

which lead to q-Toda equation

$$\epsilon \partial_{t_0} U = V(qx) - V(x),$$
(3.6)

$$\epsilon \partial_{t_0} V = U(x)V(x) - V(x)U(q^{-1}x).$$
(3.7)

From Sato equation we deduce the following set of nonlinear partial differential-difference equations

$$\begin{cases}
\omega_1(x) - \omega_1(qx) = \epsilon \partial_{t_1}(e^{\phi(x)} \cdot e^{-\phi(x)}), \\
\epsilon \partial_{t_1} \omega_1(x) = -e^{\phi(x)} e^{-\phi(q^{-1}x)}.
\end{cases}$$
(3.8)

Observe that if we cross the first two equations, then we get

$$\epsilon^2 \partial_{t_1}^2 \phi(x) = e^{\phi(qx)} e^{-\phi(x)} - e^{\phi(x)} e^{-\phi(q^{-1}x)}$$

which is the q-Toda equation. To give a linear description of the QTH, we introduce wave functions $\psi, \bar{\psi}$ defined by

$$\psi = W \cdot \chi, \quad \bar{\psi} = \bar{W} \cdot \bar{\chi},$$
(3.9)

where

$$\chi(z) := z^{\frac{\log x}{x}}, \quad \bar{\chi}(z) := z^{-\frac{\log x}{x}},$$
(3.10)

and the “·” means the action of an operator on a function. Note that $\Lambda_q \cdot \chi = z\chi$ and the following asymptotic expansions can be defined

$$\psi = z^{\frac{\log x}{x}}(1 + \omega_1(x)z^{-1} + \cdots) \psi_0(z), \quad \psi_0 := e^{\sum_{j=1}^{\infty} \frac{\omega_j}{j^{\frac{1}{j}}}}, \quad z \to \infty,$$
$$\bar{\psi} = z^{-\frac{\log x}{x}}(\bar{\omega}_0(x) + \bar{\omega}_1(x)z + \cdots) \bar{\psi}_0(z), \quad \bar{\psi}_0 := e^{\sum_{j=0}^{\infty} \frac{\bar{\omega}_j}{j^{\frac{1}{j}}}}, \quad z \to 0.$$
(3.11)

We can further get linear equations in the following proposition.

Proposition 3. The wave functions $\psi, \bar{\psi}$ are subject to following Sato equations

$$\mathcal{L} \cdot \psi = z\psi, \quad \mathcal{L} \cdot \bar{\psi} = z\bar{\psi},$$
(3.12)

$$\epsilon \partial_{t_j} \psi = (B_j)_+ \cdot \psi, \quad \epsilon \partial_{t_j} \bar{\psi} = (B_j)_+ \cdot \bar{\psi}.$$
4. Hirota bilinear equations of the QTH

Basing on above, Hirota bilinear equations which are equivalent to Lax equations of the QTH can be derived in following proposition.

**Proposition 4.** $W$ and $\bar{W}$ are wave operators of the $q$-Toda hierarchy if and only the following Hirota bilinear equations hold

$$W\Lambda^r_qW^{-1} = \bar{W}\Lambda^{-r}_q\bar{W}^{-1}, \ r \in \mathbb{N}. \quad (4.1)$$

**Proof.** The proof is complicated but quite standard. One can refer the similar proofs in [7,9]. □

To give a description in terms of wave functions, following symbolic definitions are needed. If the series have forms

$$W(x, t, \Lambda_q) = \sum_{i \in \mathbb{Z}} a_i(x, t)\Lambda^i_q \quad \text{and} \quad \bar{W}(x, t, \Lambda_q) = \sum_{i \in \mathbb{Z}} b_i(x, t)\Lambda^i_q,$$

then we denote their corresponding left symbols $\mathcal{W}$, $\bar{\mathcal{W}}$ and right symbols $\mathcal{W}^{-1}$, $\bar{\mathcal{W}}^{-1}$ as following

$$\mathcal{W}(x, t, \lambda) = \sum_{i \in \mathbb{Z}} a_i(x, t)\lambda^i, \quad \mathcal{W}^{-1}(x, t, \lambda) = \sum_{i \in \mathbb{Z}} a'_i(x, t)\lambda^i,$$

$$\bar{\mathcal{W}}(x, t, \lambda) = \sum_{i \in \mathbb{Z}} b_i(x, t)\lambda^i, \quad \bar{\mathcal{W}}^{-1}(x, t, \lambda) = \sum_{j \in \mathbb{Z}} b'_j(x, t)\lambda^j.$$

With above preparation, it is time to give another form of Hirota bilinear equation (see following proposition) after defining residue as $\text{Res}_\lambda \sum_{n \in \mathbb{Z}} \alpha_n\lambda^n = \alpha_{-1}$ using the similar proof as [3,7,9].

**Proposition 5.** Let $S$ and $\bar{S}$ are wave operators of the $q$-Toda hierarchy if and only if for all $m \in \mathbb{Z}$, $r \in \mathbb{N}$, the following Hirota bilinear identity hold

$$\text{Res}_\lambda \left\{ \lambda^{r+m-1} \mathcal{W}(x, t, \lambda)\mathcal{W}^{-1}(q^{-m}x, t', \lambda) \right\} = \text{Res}_\lambda \left\{ \lambda^{-r+m-1}\bar{\mathcal{W}}(x, t, \lambda)\bar{\mathcal{W}}^{-1}(q^{-m}x, t', \lambda) \right\}. \quad (4.2)$$

To give Hirota quadratic function in terms of tau functions, we need to define and prove the existence of the tau function of the QTH firstly in the next section.

5. Tau-functions of QTH

We firstly introduce the following sequences:

$$t - [\lambda] := (t_j - \epsilon(j - 1)!\lambda^j, 0 \leq j \leq \infty). \quad (5.1)$$

A function $\tau \in \mathbb{C}$ depending on the dynamical variables $t$ and $\epsilon$ is called the tau-function of the QTH if it provides symbols related to wave operators as following,
\[ S := \frac{\tau(e^{-\frac{x}{2}}x, t_j - \frac{(j-1)!}{\lambda_j}e, \epsilon)}{\tau(e^{-\frac{x}{2}}x, t, \epsilon)}, \tag{5.2} \]

\[ S^{-1} := \frac{\tau(e^{\frac{x}{2}}x, t_j + \frac{(j-1)!}{\lambda_j}e, \epsilon)}{\tau(e^{\frac{x}{2}}x, t, \epsilon)}, \tag{5.3} \]

\[ \bar{S} := \frac{\tau(e^{\frac{x}{2}}x, t_j + \epsilon(j-1)!\lambda_j, \epsilon)}{\tau(e^{-\frac{x}{2}}x, t, \epsilon)}, \tag{5.4} \]

\[ \bar{S}^{-1} := \frac{\tau(e^{-\frac{x}{2}}x, t_j - \epsilon(j-1)!\lambda_j, \epsilon)}{\tau(e^{\frac{x}{2}}x, t, \epsilon)}. \tag{5.5} \]

**Proposition 6.** Given a pair of wave operators \( S \) and \( \bar{S} \) of the QTH, there exists corresponding invertible tau-functions.

**Proof.** Here, we shall note that the tau function \( \tau(x, t) \) corresponding to the wave operators \( S \) and \( \bar{S} \) is in fact \( \tau(q^{-\frac{1}{2}}x, t) \).

The system is equivalent to:

\[
\log S = \left( \exp \left( -\epsilon \sum_{j=0}^{\infty} j!\lambda^{-j+1}\partial_{t_j} \right) - 1 \right) \log \tau,
\]

\[
\log \bar{S} = \left( \exp \left( \epsilon x \partial_x + \epsilon \sum_{j=0}^{\infty} j!\lambda^{j+1}\partial_{t_j} \right) - 1 \right) \log \tau.
\]

Then using the standard method in [7, 9] will help us to derive the existence of tau function of this hierarchy.

\[ \square \]

After giving tau functions of the QTH, what is the Hirota bilinear equation in terms of the tau function becomes a natural question which will be answered in the next section in terms of vertex operators.

**6. Vertex operators and Hirota quadratic equations**

In this section we continue to discuss on the fundamental properties of the tau function of the QTH, i.e., the Hirota quadratic equations of the QTH. So we introduce the following vertex operators

\[
\Gamma^{\pm_a} := \exp \left( \pm \frac{1}{\epsilon} \sum_{j=0}^{\infty} t_j \frac{\lambda_{j+1}}{(j+1)!} \right) \times \exp \left( \mp \frac{\epsilon}{2} x \partial_x \mp [\lambda^{-1}]_{\partial} \right),
\]

\[
\Gamma^{\pm_b} := \exp \left( \pm \frac{1}{\epsilon} \sum_{j=0}^{\infty} t_j \frac{\lambda_{j-1}}{(j+1)!} \right) \times \exp \left( \mp \frac{\epsilon}{2} x \partial_x \mp [\lambda]_{\partial} \right),
\]

where

\[
[\lambda]_{\partial} := \epsilon \sum_{j=0}^{\infty} j!\lambda^{j+1}\partial_{t_j}.
\]
Theorem 1. The invertible \( \tau(t, \epsilon) \) is a tau-function of the QTH if and only if it satisfies the following Hirota quadratic equations of the QTH.

\[
\text{Res}_\lambda \lambda^{r-1} \left( \Gamma^a \otimes \Gamma^{-a} \right) (\tau \otimes \tau) = \text{Res}_\lambda \lambda^{r-1} \left( \Gamma^{-b} \otimes \Gamma^b \right) (\tau \otimes \tau)
\]  

(6.1)

computed at \( x = q^l x' \) for each \( l \in \mathbb{Z}, r \in \mathbb{N} \).

Proof. We just need to prove that the Hirota quadratic equations are equivalent to the right side in Proposition [5]. By a straightforward computation we can get the following four identities

\[
\begin{align*}
\Gamma^a \tau &= \tau(q^{-\frac{1}{2}} x, t) \mathcal{W}(x, t, \lambda) \lambda^{\log x/\epsilon}, \\
\Gamma^{-a} \tau &= \lambda^{-\log x/\epsilon} \mathcal{W}^{-1}(x, t, \lambda) \tau(q^{\frac{1}{2}} x, t), \\
\Gamma^{-b} \tau &= \tau(q^{-\frac{1}{2}} x, t) \mathcal{W}(x, t, \lambda) \lambda^{x/\epsilon}, \\
\Gamma^b \bar{\tau} &= \lambda^{-\log x/\epsilon} \bar{\mathcal{W}}^{-1}(x, t, \lambda) \tau(q^{\frac{1}{2}} x, t).
\end{align*}
\]  

(6.2) - (6.5)

The proof of four equations eq.(6.2)-eq.(6.5) can be derived by similar QTH as in \[7, 9\]. By substituting four equations eq.(6.2)-eq.(6.5) into the Hirota quadratic equations (6.1), eq.(4.2) is derived. □

Doing a transformation on the eq.(6.1) by \( \lambda \rightarrow \lambda^{-1} \), then the eq.(6.1) becomes

\[
\text{Res}_\lambda \lambda^{r-1} \left( \Gamma^a \otimes \Gamma^{-a} - \Gamma^{-a} \otimes \Gamma^a \right) (\tau \otimes \tau) = 0
\]  

(6.6)

computed at \( x = q^l x' \) for each \( l \in \mathbb{Z}, r \in \mathbb{N} \). That means

\[
\frac{d\lambda}{\lambda} \left( \Gamma^a \otimes \Gamma^{-a} - \Gamma^{-a} \otimes \Gamma^a \right) (\tau \otimes \tau)
\]  

(6.7)

is regular in \( \lambda \) computed at \( x = q^l x' \) for each \( l \in \mathbb{Z} \). The eq.(6.7) i is exactly the \( q \)-version of the Hirota quadratic equation of the Toda hierarchy as a corollary in [7].

7. Bi-Hamiltonian structure and tau symmetry

To describe the integrability of the QTH, we will construct the bi-Hamiltonian structure and tau symmetry of the QTH in this section.

In this section, we will consider the QTH on Lax operator

\[
\mathcal{L} = \Lambda_q + u + e^\nu \Lambda_q^{-1}, \quad \Lambda_q = e^{\epsilon x \partial_x}.
\]  

(7.1)

Then for \( \tilde{f} = \int f dx, \tilde{g} = \int g dx \), we can define the hamiltonian bracket as

\[
\{ \tilde{f}, \tilde{g} \} = \int \sum_{w, w'} \frac{\delta f}{\delta w}(w, w') \frac{\delta g}{\delta w'} dx, \quad w, w' = u \text{ or } v.
\]  

(7.2)

The bi-Hamiltonian structure for the QTH can be given by the following two compatible Poisson brackets similar as [6].
\{v(x), v(y)\}_1 = \{u(x), u(y)\}_1 = 0,
\{u(x), v(y)\}_1 = \frac{1}{\epsilon} \left[ e^{\epsilon x \partial_x} - 1 \right] \delta(x - y),
(7.3)
\{u(x), u(y)\}_2 = \frac{1}{\epsilon} \left[ e^{\epsilon x \partial_x} e^{v(x)} - e^{v(x)} e^{-\epsilon x \partial_x} \right] \delta(x - y),
(7.4)
\{v(x), v(y)\}_2 = \frac{1}{\epsilon} \left[ e^{\epsilon x \partial_x} - e^{-\epsilon x \partial_x} \right] \delta(x - y).

For any difference operator \( A = \sum_k A_k \Lambda_q^k \), we define residue \( \text{Res} A = A_0 \). In the following theorem, we will prove the above Poisson structure can be as the Hamiltonian structure of the QTH.

**Theorem 2.** The flows of the QTH are Hamiltonian systems of the form

\[ \frac{\partial u}{\partial t_j} = \{u, H_j\}_1, \quad j \geq 0, \tag{7.5} \]

They satisfy the following bi-Hamiltonian recursion relation

\[ \{\cdot, H_{n-1}\}_2 = n\{\cdot, H_n\}_1. \]

Here the Hamiltonians have the form

\[ H_j = \int h_j(u, v; u_x, v_x; \ldots; \epsilon) dx, \quad j \geq 0, \tag{7.6} \]

with

\[ h_j = \frac{1}{(j + 1)!} \text{Res} \mathcal{L}^{j+1}. \tag{7.7} \]

**Proof.** The proof is similar as the proof in [6]. Here we will prove that the flows \( \frac{\partial}{\partial t_n} \) are also Hamiltonian systems with respect to the first Poisson bracket.

Suppose

\[ B_n = \sum_k a_{n+1,k} \Lambda_q^k, \tag{7.8} \]

and from

\[ \frac{\partial \mathcal{L}}{\partial t_n} = [(B_n)_+, \mathcal{L}] = [-(B_n)_-, \mathcal{L}], \tag{7.9} \]

we can derive equation

\[ \epsilon \frac{\partial u}{\partial t_n} = a_{n+1;1}(qx) - a_{n+1;1}(x), \tag{7.10} \]
\[ \epsilon \frac{\partial v}{\partial t_n} = a_{n+1;0}(q^{-1}x)e^{v(x)} - a_{n+1;0}(x)e^{v(qx)}. \tag{7.11} \]
By

\[ d\tilde{h}_n = \frac{1}{(n+1)!} d\text{Res} [\mathcal{L}^{n+1}] \]

\[ \sim \frac{1}{n!} \text{Res} [\mathcal{L}^n d\mathcal{L}] \]

\[ = \text{Res} \left[ a_{n;0}(x) du + a_{n;1}(q^{-1}x)e^{v(x)} dv \right], \tag{7.12} \]

it yields the following identities

\[ \frac{\delta H_n}{\delta u} = a_{n;0}(x), \quad \frac{\delta H_n}{\delta v} = a_{n;1}(q^{-1}x)e^{v(x)}. \tag{7.13} \]

This agree with Lax equation

\[ \frac{\partial u}{\partial t_n} = \{u, H_n\}_1 = \frac{1}{\epsilon} \left[ e^{x\partial_x} - 1 \right] \frac{\delta H_n}{\delta v} = \frac{1}{\epsilon} \left(a_{n;1}(q^x) - a_{n;1}(x)\right), \tag{7.14} \]

\[ \frac{\partial v}{\partial t_n} = \{v, H_n\}_1 = \frac{1}{\epsilon} \left[ 1 - e^{x\partial_x} \right] \frac{\delta H_n}{\delta u} = \frac{1}{\epsilon} \left[a_{n;0}(q^{-1}x)e^{v(x)} - a_{n;0}(x)e^{v(qx)}\right]. \tag{7.15} \]

From the above identities we see that the flows \( \frac{\partial}{\partial t} \) are Hamiltonian systems with the first Hamiltonian structure. The recursion relation follows from the following trivial identities

\[ n \frac{1}{n!} \mathcal{L}^n = \mathcal{L} \frac{1}{(n-1)!} \mathcal{L}^{n-1} = \frac{1}{(n-1)!} \mathcal{L}^{n-1} \mathcal{L}. \]

Then we get,

\[ na_{n;1}(x) = a_{n-1;0}(q^x) + u a_{n-1;1}(x) + e^v a_{n-1;2}(q^{-1}x) \]

\[ = a_{n-1;0}(x) + u(qx)a_{n-1;1}(x) + e^{v(q^x)} a_{n-1;2}(x). \]

This further leads to

\[ \{u, H_{n-1}\}_2 = \{[\Lambda_q e^{v(x)} - e^{v(x)} \Lambda_q^{-1}] a_{n-1;0}(x) + u(x) [\Lambda_q - 1] a_{n-1;1}(q^{-1}x)e^{v(x)}\} \]

\[ = n \left[a_{n;1}(x)e^{v(qx)} - a_{n;1}(q^{-1}x)e^{v(x)}\right]. \]

This is exactly the recursion relation on flows for \( u \). The similar recursion flow on \( v \) can be similarly derived. The theorem is proved till now.

\[ \square \]

Similarly as \([9]\), the tau symmetry of the QTH can be proved in the following theorem.

**Theorem 3.** The QTH has the following tau-symmetry property:

\[ \frac{\partial h_m}{\partial t_n} = \frac{\partial h_n}{\partial t_m}, \quad m, n \geq 0. \tag{7.16} \]

**Proof.** Let us prove the theorem in a direct way

\[ \frac{\partial h_m}{\partial t_n} = \frac{1}{m! n!} \text{Res} \left[-(\mathcal{L}^n)_-, \mathcal{L}^m\right] \]

\[ = \frac{1}{m! n!} \text{Res} \left[(\mathcal{L}^m)_+, (\mathcal{L}^n)_-\right] \]

\[ = \frac{1}{m! n!} \text{Res} \left[(\mathcal{L}^m)_+, \mathcal{L}^n\right] = \frac{\partial h_n}{\partial t_m}. \tag{7.17} \]
This theorem is proved.

This property justifies another alternative definition of the tau function for the QTH.

**Definition 2.** The tau function $\tau$ of the QTH can also be defined by the following expressions in terms of the densities of the Hamiltonians:

$$h_n = \epsilon(\Lambda_q - 1)\frac{\partial \log \tau}{\partial t_n}, \quad n \geq 0.$$  \hfill (7.18)

8. Additional symmetry and Block algebra

In this section, we will put constrained condition eq.(2.14) into construction of the flows of additional symmetries which form the well-known Block algebra.

With the dressing operators given in eq.(2.14), we introduce Orlov-Schulman operators as following

$$M = S\Gamma S^{-1}, \quad \bar{M} = \bar{S}\bar{\Gamma}\bar{S}^{-1},$$  \hfill (8.1)

$$\Gamma = \frac{\log x}{\epsilon} \Lambda_q - 1 + \sum_{n \geq 0} (n + 1)\Lambda_q^n t_n, \quad \bar{\Gamma} = -\frac{\log x}{\epsilon} \Lambda_q.$$  \hfill (8.2)

Then one can prove the Lax operator $L$ and Orlov-Schulman operators $M, \bar{M}$ satisfy the following proposition.

**Proposition 7.** The Lax operator $L$ and Orlov-Schulman operators $M, \bar{M}$ of the QTH satisfy the following

$$[L, M] = 1, [L, \bar{M}] = 1,$$  \hfill (8.3)

$$\partial_n M = [(B_n)_+, M], \quad \partial_n \bar{M} = [(B_n)_+, \bar{M}],$$  \hfill (8.4)

$$\frac{\partial M^m L^k}{\partial t_n} = [(B_n)_+, M^m L^k], \quad \frac{\partial \bar{M}^m \bar{L}^k}{\partial \bar{t}_n} = [(B_n)_+, \bar{M}^m \bar{L}^k].$$  \hfill (8.5)

**Proof.** One can prove the proposition by dressing the following several commutative Lie brackets

$$[\partial_n - \frac{\Lambda_q^{n+1}}{(n + 1)!}, \Gamma] = 0,$$

$$[\partial_n - \frac{\Lambda_q^{n+1}}{(n + 1)!}, \frac{\log x}{\epsilon} \Lambda_q^{-1} + \sum_{n \geq 0} \frac{\Lambda_q^n}{n!} t_n] = 0.$$

We are now to define the additional flows, and further to prove that they are symmetries, which are called additional symmetries of the QTH. We introduce additional independent variables $t_{m,l}^*$ and define the actions of the additional flows on the wave operators as

$$\frac{\partial S}{\partial t_{m,l}^*} = -((M - \bar{M})^m L^l)_- S, \quad \frac{\partial \bar{S}}{\partial \bar{t}_{m,l}^*} = ((M - \bar{M})^m L^l)_+ \bar{S},$$  \hfill (8.6)
where $m \geq 0, l \geq 0$. The following theorem shows that the definition (8.6) is compatible with reduction condition (2.14) of the QTH.

**Proposition 8.** The additional flows (8.6) preserve reduction condition (2.14).

**Proof.** By performing the derivative on $\mathcal{L}$ dressed by $S$ and using the additional flow about $S$ in (8.6), we get

\[
(\partial_{t^*_m, l} \mathcal{L}) = (\partial_{t^*_m, l} S) \Lambda S^{-1} + S \Lambda (\partial_{t_m, l} S^{-1}) \\
= -(M - \bar{M})^m L^l \Lambda S^{-1} - S \Lambda S^{-1} (\partial_{t^*_m, l} S) S^{-1} \\
= -(M - \bar{M})^m L^l \Lambda + \mathcal{L} (M - \bar{M})^m L^l. \\
= -[(M - \bar{M})^m L^l, \mathcal{L}].
\]

Similarly, we perform the derivative on $\mathcal{L}$ dressed by $\bar{S}$ and use the additional flow about $\bar{S}$ in (8.6) to get the following

\[
(\partial_{t^*_m, l} \mathcal{L}) = (\partial_{t^*_m, l} \bar{S}) \Lambda \bar{S}^{-1} + \bar{S} \Lambda (\partial_{t_m, l} \bar{S}^{-1}) \\
= ((M - \bar{M})^m L^l) \Lambda \bar{S}^{-1} - \bar{S} \Lambda \bar{S}^{-1} (\partial_{t^*_m, l} \bar{S}) \bar{S}^{-1} \\
= ((M - \bar{M})^m L^l) \Lambda + \mathcal{L} ((M - \bar{M})^m L^l) \\
= -[(M - \bar{M})^m L^l, \mathcal{L}].
\]

Because

\[
[M - \bar{M}, \mathcal{L}] = 0, \quad (8.7)
\]

therefore

\[
\frac{\partial \mathcal{L}}{\partial t^*_m, l} = -((M - \bar{M})^m L^l), \mathcal{L} = [(M - \bar{M})^m L^l] \mathcal{L}, \quad (8.8)
\]

which gives the compatibility of additional flow of QTH with reduction condition (2.14). ☐

Similarly, we can take derivatives on the dressing structure of $M$ and $\bar{M}$ to get the following proposition.

**Proposition 9.** The additional derivatives act on $M, \bar{M}$ as

\[
\frac{\partial M}{\partial t^*_m, l} = -(M - \bar{M})^m L^l, M, \quad (8.9)
\]

\[
\frac{\partial \bar{M}}{\partial t^*_m, l} = [(M - \bar{M})^m L^l], \bar{M}. \quad (8.10)
\]

**Proof.** By performing the derivative on $M$ given in (8.1), there exists a similar derivative as $\partial_{t^*_m, l} \mathcal{L}$, i.e.,

\[
(\partial_{t^*_m, l} M) = (\partial_{t^*_m, l} S) \Gamma S^{-1} + S \Gamma (\partial_{t_m, l} S^{-1}) \\
= -((M - \bar{M})^m L^l) S \Gamma S^{-1} - S \Gamma S^{-1} (\partial_{t^*_m, l} S) S^{-1} \\
= -((M - \bar{M})^m L^l) M + M ((M - \bar{M})^m L^l) \\
= -[((M - \bar{M})^m L^l), M].
\]

Here the fact that $\Gamma$ does not depend on the additional variables $t^*_m, l$ has been used. Other identities can also be obtained in a similar way. ☐
By the two propositions above, the following theorem can be proved.

**Theorem 4.** The additional flows $\partial_{m,l}$ commute with the $q$-Toda hierarchy flows $\partial_{n}$, i.e.,

$$[\partial_{m,l}^{\ast}, \partial_{n}]\Phi = 0,$$  \hspace{1cm} (8.11)

where $\Phi$ can be $S$, $\bar{S}$ or $\mathcal{L}$, and $\partial_{m,l}^{\ast} = \frac{\partial}{\partial m_{l}}$, $\partial_{n} = \frac{\partial}{\partial n}$.

**Proof.** According to the definition,

$$[\partial_{m,l}^{\ast}, \partial_{n}]S = \partial_{m,l}^{\ast}(\partial_{n}S) - \partial_{n}(\partial_{m,l}^{\ast}S),$$

and using the actions of the additional flows and the $q$-Toda flows on $S$, we have

$$[\partial_{m,l}^{\ast}, \partial_{n}]S = -\partial_{m,l}^{\ast}((B_{n})_{-}S) + \partial_{n}(((M - \bar{M})^{m}\mathcal{L}^{l})_{-}S)$$

$$= -((\partial_{m,l}^{\ast}B_{n})_{-}S - (B_{n})_{-}(\partial_{m,l}^{\ast}S)$$

$$+ [\partial_{n}((M - \bar{M})^{m}\mathcal{L}^{l})]_{-}S + ((M - \bar{M})^{m}\mathcal{L}^{l})_{-}(\partial_{n}S).$$

Using (8.6) and Proposition 7, it equals

$$[\partial_{m,l}^{\ast}, \partial_{n}]S = [((M - \bar{M})^{m}\mathcal{L}^{l})_{-}, B_{n}]_{-}S + (B_{n})_{-}(((M - \bar{M})^{m}\mathcal{L}^{l})_{-}S$$

$$+ [(B_{n})_{+}, (M - \bar{M})^{m}\mathcal{L}^{l}]_{-}S - ((M - \bar{M})^{m}\mathcal{L}^{l})_{-}(B_{n})_{-}S$$

$$= [((M - \bar{M})^{m}\mathcal{L}^{l})_{-}, B_{n}]_{-}S - [((M - \bar{M})^{m}\mathcal{L}^{l})_{-}, (B_{n})_{+}]_{-}S$$

$$+ [(B_{n})_{-}, ((M - \bar{M})^{m}\mathcal{L}^{l})_{-}]S$$

$$= 0.$$

In the proof above, $[(B_{n})_{+}, ((M - \bar{M})^{m}\mathcal{L}^{l})_{-}] = [(B_{n})_{+}, ((M - \bar{M})^{m}\mathcal{L}^{l})_{-}]_{-}$ has been used. The action on $\mathcal{L}$ in the theorem can be proved in similar ways. \hfill $\square$

The commutative property in Theorem 4 means that additional flows are symmetries of the QTH. Since they are symmetries, it is natural to consider the algebraic structures among these additional symmetries. So we obtain the following important theorem.

**Theorem 5.** The additional flows $\partial_{m,l}^{\ast}$ form a Block type Lie algebra with the following relation

$$[\partial_{m,l}^{\ast}, \partial_{n,k}^{\ast}] = (km - nl)\partial_{m+n-1,k+l-1}^{\ast},$$  \hspace{1cm} (8.12)

which holds in the sense of acting on $S$, $\bar{S}$ or $\mathcal{L}$ and $m, n, l, k \geq 0$.

**Proof.** By using (8.6), we get

$$[\partial_{m,l}^{\ast}, \partial_{n,k}^{\ast}]S = \partial_{m,l}^{\ast}(\partial_{n,k}^{\ast}S) - \partial_{n,k}^{\ast}(\partial_{m,l}^{\ast}S)$$

$$= -\partial_{m,l}^{\ast}(((M - \bar{M})^{n}\mathcal{L}^{k})_{-}S) + \partial_{n,k}^{\ast}(((M - \bar{M})^{m}\mathcal{L}^{l})_{-}S)$$

$$= -((\partial_{m,l}^{\ast}(M - \bar{M})^{n}\mathcal{L}^{k})_{-}S - ((M - \bar{M})^{n}\mathcal{L}^{k})_{-}(\partial_{m,l}^{\ast}S)$$

$$+ (\partial_{n,k}^{\ast}(M - \bar{M})^{m}\mathcal{L}^{l})_{-}S + ((M - \bar{M})^{m}\mathcal{L}^{l})_{-}(\partial_{n,k}^{\ast}S).$$
We further get
\[
[\partial_{m,l}^*, \partial_{n,k}^*]S = -\left[ \sum_{p=0}^{n-1} (M - \bar{M})^p (\partial_{m,l}^* (M - \bar{M})) (M - \bar{M})^{n-p-1} L^k + (M - \bar{M})^n (\partial_{m,l}^* L^k) \right]_S \\
-((M - \bar{M})^n L^k)_{-(\partial_{m,l}^* S)} \\
+\left[ \sum_{p=0}^{n-1} (M - \bar{M})^p (\partial_{n,k}^* (M - \bar{M})) (M - \bar{M})^{m-p-1} L^l + (M - \bar{M})^m (\partial_{n,k}^* L^l) \right]_S \\
+((M - \bar{M})^m L^l)_{-(\partial_{n,k}^* S)} \\
= [(nl - km)(M - \bar{M})^{m+n-1} L^{k+l-1}]_S \\
= (km - nl)\partial_{m+n-1,k+l-1}^* S.
\]

Similarly the same results on \( \bar{S} \) and \( L \) are as follows
\[
[\partial_{m,l}^*, \partial_{n,k}^*] \bar{S} = ((km - nl)(M - \bar{M})^{m+n-1} L^{k+l-1})_{+} \bar{S} \\
= (km - nl)\partial_{m+n-1,k+l-1}^* \bar{S}, \\
[\partial_{m,l}^*, \partial_{n,k}^*] L = \partial_{m,l}^* (\partial_{n,k}^* L) - \partial_{n,k}^* (\partial_{m,l}^* L) \\
= [((nl - km)(M - \bar{M})^{m+n-1} L^{k+l-1})_{-}, L] \\
= (km - nl)\partial_{m+n-1,k+l-1}^* L.
\]

Denote \( D_{m,l} = \partial_{m,l+1}^* \), and let Block algebra be the span of all \( D_{m,l}, m, l \geq -1 \). Then by \( \text{(8.12)} \), Block algebra is a Lie algebra with relations
\[
[D_{m,l}, D_{n,k}] = ((m+1)(k+1) - (l+1)(n+1))D_{m+n,l+k}, \quad \text{for } m, n, l, k \geq -1. \tag{8.13}
\]
Thus Block algebra is in fact a Block type Lie algebra which is generated by the set
\[
B = \{ D_{-1,0}, D_{0,-1}, D_{0,0}, D_{1,0}, D_{0,1} \} = \{ \partial_{0,1}^*, \partial_{1,0}^*, \partial_{1,1}^*, \partial_{2,1}^*, \partial_{1,2}^* \}. \tag{8.14}
\]

**Theorem 6.** The Block flows of the \( q \)-Toda hierarchy are Hamiltonian systems in the form
\[
\begin{align*}
\frac{\partial u}{\partial t_{m,l}^*} &= \{ u, H_{m,l}^* \}_1, \\
\frac{\partial v}{\partial t_{m,l}^*} &= \{ v, H_{m,l}^* \}_1, \quad m, l \geq 0.
\end{align*}
\tag{8.15}
\]
They satisfy the following bi-Hamiltonian recursion relation
\[
\frac{\partial}{\partial t_{m,l}^*} = \{ \cdot, H_{m,l-1}^* \}_2 = n\{ \cdot, H_{m,l}^* \}_1. \tag{8.16}
\]
Here the Hamiltonians (depending on \( t_n \)) with respect to \( t_{m,l}^* \) have the form
\[
H_{m,l}^* = \int h_{m,l}^*(u, v; u_x, v_x; \ldots; t_n; \epsilon)dx, \quad n \geq 0, \tag{8.17}
\]
with the Hamiltonian densities \( h_{m,l}^*(u, v; u_x, v_x; \ldots; t_n; \epsilon) \) given by
\[
h_{m,l}^* = \text{Res}(M - \bar{M})^m L^l. \tag{8.18}
\]
**Proof.** The proof is similar as the proof for original Toda flows. \( \square \)
9. Extended q-Toda hierarchy

To define the extended flows, we define the following logarithm

\[
\log_+ \mathcal{L} = W \circ \varepsilon x \partial \circ W^{-1} = S \circ \varepsilon x \partial \circ S^{-1},
\]

\[
\log_- \mathcal{L} = -\bar{W} \circ \varepsilon x \partial \circ \bar{W}^{-1} = -\bar{S} \circ \varepsilon x \partial \circ \bar{S}^{-1},
\]

(9.1)

(9.2)

where \( \partial \) is the derivative about the spatial variable \( x \).

Combining these above logarithmic operators together can derive following important logarithm

\[
\log \mathcal{L} := \frac{1}{2}(\log_+ \mathcal{L} + \log_- \mathcal{L}) = \frac{1}{2}(S \circ \varepsilon x \partial \circ S^{-1} - \bar{S} \circ \varepsilon x \partial \circ \bar{S}^{-1}) := \sum_{i=-\infty}^{+\infty} W_i \Lambda_q^i \in G,
\]

(9.3)

which will generate a series of flow equations which contain the spatial flow in later defined Lax equations. Let us first introduce some convenient notations.

**Definition 3.** The operators \( B_j, D_j \) are defined as follows

\[
B_j := \frac{\mathcal{L}^{j+1}}{(j+1)!}, \quad D_j := \frac{2\mathcal{L}^j}{j!}(\log \mathcal{L} - c_j), \quad c_j = \sum_{i=1}^{j} \frac{1}{i}, \quad j \geq 0.
\]

(9.4)

Now we give the definition of the extended q-Toda hierarchy (EQTH).

**Definition 4.** The extended q-Toda hierarchy is a hierarchy in which the dressing operators \( S, \bar{S} \) satisfy following Sato equations

\[
\varepsilon \partial_t j S = -(B_j)_- S, \quad \varepsilon \partial_t j \bar{S} = (B_j)_+ \bar{S},
\]

(9.5)

\[
\varepsilon \partial_s j S = -(D_j)_- S, \quad \varepsilon \partial_s j \bar{S} = (D_j)_+ \bar{S}.
\]

(9.6)

Then one can easily get the following proposition about \( W, \bar{W} \).

**Proposition 10.** The dressing operators \( W, \bar{W} \) are subject to following Sato equations

\[
\varepsilon \partial_t j W = (B_j)_+ W, \quad \varepsilon \partial_t j \bar{W} = (B_j)_+ \bar{W},
\]

(9.7)

\[
\varepsilon \partial_s j W = \left(\frac{\mathcal{L}^j}{j!}(\log_+ \mathcal{L} - c_j) - (D_j)_- \right) W, \quad \varepsilon \partial_s j \bar{W} = \left(-\frac{\mathcal{L}^j}{j!}(\log_- \mathcal{L} - c_j) + (D_j)_+ \right) \bar{W}.
\]

(9.8)

From the previous proposition we derive the following Lax equations for the Lax operators.

**Proposition 11.** The Lax equations of the EQTH are as follows

\[
\varepsilon \partial_t j \mathcal{L} = [(B_j)_+, \mathcal{L}], \quad \varepsilon \partial_s j \mathcal{L} = [(D_j)_+, \mathcal{L}], \quad \varepsilon \partial_t j \log \mathcal{L} = [(B_j)_+, \log \mathcal{L}],
\]

\[
\varepsilon (\log \mathcal{L})_{s_j} = -[(D_j)_-, \log_+ \mathcal{L}] + [(D_j)_+, \log_- \mathcal{L}].
\]

(9.9)

(9.10)

To see this kind of hierarchy more clearly, the Hirota quadratic equations of the EQTH will be given in next subsection.
10. Generalized vertex operators and Hirota quadratic equations

Introduce the following sequences:

\[ t - [\lambda] := (t_j - \epsilon(j - 1)!\lambda^j, 0 \leq j \leq \infty). \tag{10.1} \]

A scalar function depending only on the dynamical variables \( t, s \) and \( \epsilon \) is called the tau-function of the EQTH if it provides symbols related to wave operators as following,

\[
S := \frac{\tau(e^{s_0 - \frac{\epsilon}{2} x}, t_j - \epsilon(j - 1)!\lambda^j, s; \epsilon)}{\tau(e^{s_0}, t, s; \epsilon)}, \tag{10.2}
\]

\[
S^{-1} := \frac{\tau(e^{s_0 + \frac{\epsilon}{2} x}, t_j + \epsilon(j - 1)!\lambda^j, s; \epsilon)}{\tau(e^{s_0 + \frac{\epsilon}{2} x}, t, s; \epsilon)}, \tag{10.3}
\]

\[
\bar{S} := \frac{\tau(e^{s_0 + \frac{\epsilon}{2} x}, t_j + \epsilon(j - 1)!\lambda^j, s; \epsilon)}{\tau(e^{s_0}, t, s; \epsilon)}, \tag{10.4}
\]

\[
\bar{S}^{-1} := \frac{\tau(e^{s_0 - \frac{\epsilon}{2} x}, t_j - \epsilon(j - 1)!\lambda^j, s; \epsilon)}{\tau(e^{s_0 - \frac{\epsilon}{2} x}, t, s; \epsilon)}. \tag{10.5}
\]

The proof of the existence of the tau function of the EQTH is a also standard, one can refer the similar proof in \cite{7, 9}.

Remark: We need to note that the tau function of the EQTH is unique up to a multiplication of an arbitrary function depending on extended variables \( s_j, j > 0 \) for a pair of given wave functions.

In this section we continue to discuss on the fundamental properties of the tau function of the EQTH, i.e., the Hirota quadratic equations of the EQTH. So we introduce the following vertex operators

\[
\Gamma^{\pm a} := \exp \left( \pm \frac{1}{\epsilon} \sum_{j=0}^{\infty} t_j \frac{\lambda^{j+1}}{(j + 1)!} + s_j \frac{\lambda^j}{j!} (\log \lambda - c_j) \right) \times \exp \left( \pm \frac{\epsilon}{2} \partial_{s_0} \mp [\lambda^{-1}] \partial_{s_0} \right),
\]

\[
\Gamma^{\pm b} := \exp \left( \pm \frac{1}{\epsilon} \sum_{j=0}^{\infty} t_j \frac{\lambda^{-j-1}}{(j + 1)!} - s_j \frac{\lambda^{-j}}{j!} (\log \lambda - c_j) \right) \times \exp \left( \pm \frac{\epsilon}{2} \partial_{s_0} \mp [\lambda] \partial_{s_0} \right),
\]

where

\[
[\lambda] \partial_{s_0} := \epsilon \sum_{j=0}^{\infty} j! \lambda^{j+1} \partial_{s_j}.
\]

Because of the logarithm \( \log \lambda \), the vertex operators \( \Gamma^{\pm a} \otimes \Gamma^{\mp a} \) and \( \Gamma^{\pm b} \otimes \Gamma^{\mp b} \) are multi-valued. There are monodromy factors \( M^a \) and \( M^b \) respectively as following among different branches around \( \lambda = \infty \)

\[
M^a = \exp \left\{ \pm \frac{2\pi i}{\epsilon} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} (s_j \otimes 1 - 1 \otimes s_j) \right\}, \tag{10.6}
\]

\[
M^b = \exp \left\{ \pm \frac{2\pi i}{\epsilon} \sum_{j=0}^{\infty} \frac{\lambda^{-j}}{j!} (s_j \otimes 1 - 1 \otimes s_j) \right\}. \tag{10.7}
\]
In order to offset the complication, we need to generalize the concept of vertex operators which leads it to be not scalar-valued any more but take values in a differential operator algebra in $\mathbb{C}$. So we introduce the following vertex operators

\begin{align*}
\Gamma_a^\delta &= \exp \left( - \sum_{j>0} \frac{j! \lambda^{j+1}}{\epsilon} (\epsilon x \partial_x) s_j \right) \exp(\log x \partial_{s_0}), \\
\Gamma_b^\delta &= \exp \left( - \sum_{j>0} \frac{j! \lambda^{-(j+1)}}{\epsilon} (\epsilon x \partial_x) s_j \right) \exp(\log x \partial_{s_0}), \\
\Gamma_a^{\#} &= \exp(\log x \partial_{s_0}) \exp \left( \sum_{j>0} \frac{j! \lambda^{j+1}}{\epsilon} (\epsilon x \partial_x) s_j \right), \\
\Gamma_b^{\#} &= \exp(\log x \partial_{s_0}) \exp \left( \sum_{j>0} \frac{j! \lambda^{-(j+1)}}{\epsilon} (\epsilon x \partial_x) s_j \right).
\end{align*}

Then

\begin{align*}
\Gamma_a^{\#} \otimes \Gamma_a^\delta &= \exp(\log x \partial_{s_0}) \exp \left( \sum_{j>0} \frac{j! \lambda^{j+1}}{\epsilon} (\epsilon x \partial_x) (s_j - s'_j) \right) \exp(\log x \partial_{s_0}), \\
\Gamma_b^{\#} \otimes \Gamma_b^\delta &= \exp(\log x \partial_{s_0}) \exp \left( \sum_{j>0} \frac{j! \lambda^{-(j+1)}}{\epsilon} (\epsilon x \partial_x) (s_j - s'_j) \right) \exp(\log x \partial_{s_0}).
\end{align*}

After computation we get

\begin{align*}
\left( \Gamma_a^{\#} \otimes \Gamma_a^\delta \right) M^a &= \exp \left\{ \pm \frac{2\pi i}{\epsilon} \sum_{j>0} \frac{\lambda^j}{j!} (s_j - s'_j) \right\} \\
&\exp \left( \pm \frac{2\pi i}{\epsilon} ((s_0 + \log x) - (s'_0 + \log x + \sum_{j>0} \frac{\lambda^j}{j!} (s_j - s'_j)) \right) \left( \Gamma_a^{\#} \otimes \Gamma_a^\delta \right) \\
&= \exp \left( \pm \frac{2\pi i}{\epsilon} (s_0 - s'_0) \right) \left( \Gamma_a^{\#} \otimes \Gamma_a^\delta \right),
\end{align*}

\begin{align*}
\left( \Gamma_b^{\#} \otimes \Gamma_b^\delta \right) M^b &= \exp \left\{ \pm \frac{2\pi i}{\epsilon} \sum_{j>0} \frac{\lambda^{-j}}{j!} (s_j - s'_j) \right\} \\
&\exp \left( \pm \frac{2\pi i}{\epsilon} ((s_0 + \log x) - (s'_0 + \log x + \sum_{j>0} \frac{\lambda^{-j}}{j!} (s_j - s'_j)) \right) \left( \Gamma_b^{\#} \otimes \Gamma_b^\delta \right) \\
&= \exp \left( \pm \frac{2\pi i}{\epsilon} (s_0 - s'_0) \right) \left( \Gamma_b^{\#} \otimes \Gamma_b^\delta \right).
\end{align*}

Thus when $s_0 - s'_0 \in \mathbb{Z} \epsilon$, $(\Gamma_a^{\#} \otimes \Gamma_a^\delta)$ $(\Gamma_b^{\#} \otimes \Gamma_b^\delta)$ and $(\Gamma_a^{\#} \otimes \Gamma_a^\delta)$ $(\Gamma_b^{\#} \otimes \Gamma_b^\delta)$ are all single-valued near $\lambda = \infty$.

Now we should note that the above vertex operators take value in differential operator algebra $\mathbb{C}[\partial, x, t, s, \epsilon] := \{ f(x, t, \epsilon) | f(x, t, s, \epsilon) = \sum_{i \geq 0} c_i(x, t, s, \epsilon) \partial^i \}$. Then we can get the following important theorem similar as [7,9].
Theorem 7. The invertible \( \tau(t,s,\epsilon) \) is a tau-function of the EQTH if and only if it satisfies the following Hirota quadratic equations of the EQTH.

\[
\text{Res}_\lambda \lambda^{r-1} \left( \Gamma^a \# \otimes \Gamma^a \right) \left( \Gamma^a \otimes \Gamma^{-a} \right) (\tau \otimes \tau) = \text{Res}_\lambda \lambda^{r-1} \left( \Gamma^b \# \otimes \Gamma^b \right) \left( \Gamma^{-b} \otimes \Gamma^b \right) (\tau \otimes \tau)
\]

(10.14)

computed at \( s_0 - s'_0 = l\epsilon \) for each \( l \in \mathbb{Z}, r \in \mathbb{N} \).

11. Bi-Hamiltonian structure of the EQTH

As another important part of the Sato theory, the bi-Hamiltonian structure of the EQTH will be constructed in the next section similar as [6].

Theorem 8. The flows of the EQTH are Hamiltonian systems of the form

\[
\frac{\partial u_i}{\partial t_{k,j}} = \{u_i, H_{k,j}\}_1, \quad \frac{\partial v_i}{\partial t_{k,j}} = \{v_i, H_{k,j}\}_1, \quad k = 0, 1; \ j \geq 0,
\]

(11.1)

with \( t_{0,j} = t_j, t_{1,j} = s_j \). They satisfy the following bi-Hamiltonian recursion relation

\[
\{\cdot, H_{1,n-1}\}_2 = n\{\cdot, H_{1,n}\}_1 + 2\{\cdot, H_{0,n-1}\}_1, \quad \{\cdot, H_{0,n-1}\}_2 = (n+1)\{\cdot, H_{0,n}\}_1.
\]

Here the Hamiltonians have the form

\[
H_{k,j} = \int h_{k,j}(u,v;u_x,v_x;\ldots;\epsilon)dx, \quad k = 0, 1; \ j \geq 0,
\]

(11.2)

with

\[
h_{0,j} = \frac{1}{(j+1)!} \text{Res} \mathcal{L}^{j+1}, \quad h_{1,j} = \frac{2}{j!} \text{Res} \left[ \mathcal{L}^j (\log \mathcal{L} - c_j) \right].
\]

(11.3)

Proof. For the \( q \)-Toda hierarchy, the proof was already given in the Theorem 2.

Here we will prove that the flows \( \frac{\partial}{\partial \tau_{0,n}} \) are also Hamiltonian systems with respect to the first Poisson bracket. Like in [6], the following identity has been proved

\[
\text{Res} \left[ \mathcal{L}^n d(S\epsilon x \partial_x S^{-1}) \right] \sim \text{Res} \mathcal{L}^{n-1} d\mathcal{L},
\]

(11.4)

which show the validity of the following equivalence relation:

\[
\text{Res} \left( \mathcal{L}^n d \log_+ \mathcal{L} \right) \sim \text{Res} \left( \mathcal{L}^{n-1} d\mathcal{L} \right).
\]

(11.5)

Here the equivalent relation \( \sim \) is up to a \( x \)-derivative of another 1-form.

In a similar way as eq. (11.4), we obtain the following equivalence relation

\[
\text{Res} \left[ \mathcal{L}^n d(S\epsilon x \partial_x S^{-1}) \right] \sim -\text{Res} \mathcal{L}^{n-1} d\mathcal{L},
\]

(11.6)

i.e.,

\[
\text{Res} \left( \mathcal{L}^n d \log_- \mathcal{L} \right) \sim \text{Res} \left( \mathcal{L}^{n-1} d\mathcal{L} \right).
\]

(11.7)

Combining (11.5) with (11.7) together can lead to

\[
\text{Res} \left( \mathcal{L}^n d \log \mathcal{L} \right) \sim \text{Res} \left( \mathcal{L}^{n-1} d\mathcal{L} \right).
\]

(11.8)

Then from

\[
\frac{\partial \mathcal{L}}{\partial \tau_{k,n}} = [(B_{k,n})_+, \mathcal{L}] = -(B_{k,n})_- \mathcal{L}, \quad B_{0,n} = B_n, B_{1,n} = D_n,
\]

(11.9)
and supposing
\[ B_{1,n} = \sum_k a_{1,n+1;k} \Lambda^k, \quad (11.10) \]
we can derive equation
\[ \frac{\partial u}{\partial t_{1,n}} = a_{1,n+1;1}(qx) - a_{1,n+1;1}(x) \in \mathbb{C}, \quad (11.11) \]
\[ \frac{\partial v}{\partial t_{1,n}} = a_{1,n+1;0}(q^{-1}x)e^{v(x)} - a_{1,n+1;0}(x)e^{v(qx)} \in \mathbb{C}. \quad (11.12) \]

The equivalence relation \((11.8)\) now readily follows from the above two equations. By using \((11.5)\) we obtain
\[ dh_{1,n} = 2 \frac{d}{n!} Res \left[ \mathcal{L}^n (\log \mathcal{L} - c_n) \right] \]
\[ \approx \frac{2}{(n-1)!} \text{Res} \left[ \mathcal{L}^{n-1} (\log \mathcal{L} - c_{n-1}) d\mathcal{L} \right] + \frac{2}{n!} \text{Res} \left[ \mathcal{L}^{n-1} d\mathcal{L} \right] \]
\[ = \frac{2}{(n-1)!} \text{Res} \left[ \mathcal{L}^{n-1} (\log \mathcal{L} - c_{n-1}) d\mathcal{L} \right] \]
\[ = \text{Res} \left[ a_{1,n;0}(x)du + a_{1,n;1}(q^{-1}x)e^{v(x)}dv \right]. \quad (11.13) \]
It yields the following identities
\[ \frac{\delta H_{1,n}}{\delta u} = a_{1,n;0}(x), \quad \frac{\delta H_{1,n}}{\delta v} = a_{1,n;1}(q^{-1}x)e^{v(x)}. \quad (11.15) \]

This agree with Lax equation
\[ \frac{\partial u}{\partial t_{1,n}} = \{ u, H_{1,n} \}_1 = \frac{1}{\epsilon} \left[ \epsilon x \partial_x - 1 \right] \frac{\delta H_{1,n}}{\delta v} = \frac{1}{\epsilon} \left( a_{1,n+1;1}(qx) - a_{1,n+1;1}(x) \right), \]
\[ \frac{\partial v}{\partial t_{1,n}} = \{ v, H_{1,n} \}_1 = \frac{1}{\epsilon} \left[ 1 - \epsilon x \partial_x \right] \frac{\delta H_{1,n}}{\delta u} = \frac{1}{\epsilon} \left[ a_{1,n+1;0}(q^{-1}x)e^{v(x)} - a_{1,n+1;0}(x)e^{v(qx)} \right]. \]

From the above identities we see that the flows \( \frac{\partial}{\partial t_{1,n}} \) are Hamiltonian systems of the first bi-Hamiltonian structure. For the case of \( k = 1 \) the recursion relation follows from the following trivial identities
\[ n \frac{2}{n!} \mathcal{L}^n (\log_\pm \mathcal{L} - c_n) = \mathcal{L} \frac{2}{(n-1)!} \mathcal{L}^{n-1} (\log_\pm \mathcal{L} - c_{n-1}) - 2 \frac{1}{n!} \mathcal{L}^n \]
\[ = \frac{2}{(n-1)!} \mathcal{L}^{n-1} (\log_\pm \mathcal{L} - c_{n-1}) \mathcal{L} - 2 \frac{1}{n!} \mathcal{L}^n. \]

Then we get, for \( \beta = 1 \),
\[ na_{1,n+1;1}(x) = a_{1,n;0}(qx) + ua_{1,n;1}(x) + e^v a_{1,n;2}(q^{-1}x) - 2a_{0,n+1;1}(x) \]
\[ = a_{1,n;0}(x) + u(qx)a_{1,n;1}(x) + e^{v(q^2x)}a_{1,n;2}(x) - 2a_{0,n+1;1}(x). \]

This further leads to
\[ \{ u, H_{1,n-1} \}_2 = \left[ \left[ \Lambda e^{v(x)} - e^{v(x)} \Lambda^{-1} \right] a_{1,n;0}(x) + u(x) [\Lambda - 1] a_{1,n;1}(q^{-1}x)e^{v(x)} \right] \]
\[ = n \left[ a_{1,n+1;1}(x)e^{v(qx)} - a_{1,n+1;1}(q^{-1}x)e^{v(x)} \right] + 2 \left[ a_{0,n+1;0}(x)e^{v(qx)} - a_{0,n+1;0}(q^{-1}x)e^{v(x)} \right]. \]
This is exactly the recursion relation on flows for $u$. The similar recursion flow on $v$ can be similarly derived. Theorem is proved till now.

Similarly as [6], the tau symmetry of the EQTH can be proved in the following theorem.

**Theorem 9.** The EQTH has the following tau-symmetry property:

$$\frac{\partial h_{\alpha,m}}{\partial t_{\beta,n}} = \frac{\partial h_{\beta,n}}{\partial t_{\alpha,m}}, \quad \alpha, \beta = 0, 1, \ m, \ n \geq 0. \quad (11.16)$$

**Proof.** Let us prove the theorem for the case when $\alpha = 1, \beta = 0$, other cases are proved in a similar way

$$\frac{\partial h_{1,m}}{\partial t_{0,n}} = \frac{2}{m! (n+1)!} \text{Res}[(-\mathcal{L}^{n+1} - \mathcal{L}^m (\log \mathcal{L} - c_m))]$$

$$= \frac{2}{m! (n+1)!} \text{Res}[(\mathcal{L}^m (\log \mathcal{L} - c_m))_+ (\mathcal{L}^{n+1})_-]$$

$$= \frac{2}{m! (n+1)!} \text{Res}[(\mathcal{L}^m (\log \mathcal{L} - c_m))_+ \mathcal{L}^{n+1}] = \frac{\partial h_{0,n}}{\partial t_{1,m}}. \quad (11.17)$$

The theorem is proved.

This property justifies the following alternative definition of another kind of tau function for the EQTH.

**Definition 5.** The tau function $\bar{\tau}$ of the EQTH can be defined by the following expressions in terms of the densities of the Hamiltonians:

$$h_{\beta,n} = \epsilon(\Lambda - 1) \frac{\partial \log \bar{\tau}}{\partial t_{\beta,n}}, \quad \beta = 0, 1; \ n \geq 0, \quad (11.18)$$

with $t_{0,j} = t_j, \ t_{1,j} = s_j$.

With above two different definitions tau functions of this hierarchy, some mysterious connections between these two kinds of tau functions become an open question. One is from Sato theory without fixing extended variables and another is from the Hamiltonian tau symmetry. While considering the constraint of $\bar{\tau}$ down to a sub-manifold only depending on non-extended coordinates, $\bar{\tau}$ and $\tau$ should be the same.

12. **Darboux transformation of the EQTH**

In this section, we will consider the Darboux transformation of the EQTH on Lax operator

$$\mathcal{L} = \Lambda_q + u + v\Lambda_q^{-1}, \quad (12.1)$$
i.e.

$$\mathcal{L}^{[1]} = \Lambda_q + u^{[1]} + v^{[1]}\Lambda_q^{-1} = W\mathcal{L}W^{-1}, \quad (12.2)$$

where $W$ is the Darboux transformation operator. That means after Darboux transformation, the spectral problem about the wave function $\phi$

$$\mathcal{L}\phi = \Lambda_q \phi + u\phi + v\Lambda_q^{-1}\phi = \lambda\phi, \quad (12.3)$$

will become
\[ \mathcal{L}^{[1]} \phi^{[1]} = \lambda \phi^{[1]}. \] (12.4)

To keep the Lax pair of the EQTH invariant, i.e.
\[ \epsilon \partial_t \mathcal{L}^{[1]} = [(B_j^{[1]})_+, \mathcal{L}^{[1]}], \epsilon \partial_s \mathcal{L}^{[1]} = [(D_j^{[1]})_+, \mathcal{L}^{[1]}], \quad B_j^{[1]} := B_j(\mathcal{L}^{[1]}), \quad D_j^{[1]} := D_j(\mathcal{L}^{[1]}), \] (12.5)
\[ \epsilon \partial_t \log \mathcal{L}^{[1]} = [(B_j^{[1]})_+, \log \mathcal{L}^{[1]}], \quad \epsilon \partial_s \log \mathcal{L}^{[1]} = [-(D_j^{[1]})_-, \log \mathcal{L}^{[1]}] + [(D_j^{[1]})_+, \log \mathcal{L}^{[1]}], \] (12.6)

the dressing operator \( W \) should satisfy the following dressing equation
\[ \epsilon \partial_t j W = -W(B_j) + (WB_j W^{-1})_+ W, \quad j \geq 0 \] (12.7)
\[ \epsilon \partial_s j W = -W(D_j) + (WD_j W^{-1})_+ W, \quad j \geq 0. \] (12.8)

where \( W_{t_j} \) means the derivative of \( W \) by \( t_j \). To give the Darboux transformation, we need the following lemma.

**Lemma 1.** The operator \( B := \sum_{n=0}^{\infty} b_n \Lambda_q^n \) is a non-negative difference operator, \( C := \sum_{n=1}^{\infty} c_n \Lambda_q^{-n} \) is a negative difference operator and \( f, g \) (short for \( f(x), g(x) \)) are two functions of the spatial parameter \( x \), following identities hold
\[ (B f \frac{\Lambda_q^{-1}}{1 - \Lambda_q^{-1}} g)_+ = B(f) \frac{\Lambda_q^{-1}}{1 - \Lambda_q^{-1}} g, \quad (f \frac{\Lambda_q^{-1}}{1 - \Lambda_q^{-1}} g B)_- = f \frac{\Lambda_q^{-1}}{1 - \Lambda_q^{-1}} B^*(g), \] (12.9)
\[ (C f \frac{1}{1 - \Lambda_q} g)_+ = C(f) \frac{1}{1 - \Lambda_q} g, \quad (f \frac{1}{1 - \Lambda_q} g C)_- = f \frac{1}{1 - \Lambda_q} C^*(g). \] (12.10)

**Proof.** Here we only give the proof of the eq.(12.9) by direct calculation
\[ (B f \frac{\Lambda_q^{-1}}{1 - \Lambda_q^{-1}} g)_- = \sum_{m=0}^{\infty} b_m (f(q^m x)) \Lambda_m \frac{\Lambda_q^{-1}}{1 - \Lambda_q^{-1}} g)_- \]
\[ = \sum_{m=0}^{\infty} b_m f(q^m x) \left( \frac{\Lambda_q^{-m-1}}{1 - \Lambda_q^{-1}} \right) g \]
\[ = \sum_{m=0}^{\infty} b_m f(q^m x) \frac{\Lambda_q^{-1}}{1 - \Lambda_q^{-1}} g \]
\[ = B(f) \frac{\Lambda_q^{-1}}{1 - \Lambda_q^{-1}} g, \] (12.11)
\[
(f \frac{\Lambda_q^{-1}}{1 - \Lambda_q^{-1}} gB)_- = \sum_{m=0}^{\infty} (f \frac{\Lambda_q^{-1}}{1 - \Lambda_q^{-1}} gb_m \Lambda^m)_-
\]
\[
= \sum_{m=0}^{\infty} (f \frac{\Lambda_q^{-1}}{1 - \Lambda_q^{-1}} \Lambda^m g(q^{-m}x)b_m(q^{-m}x))_- 
\]
\[
= \sum_{m=0}^{\infty} f \frac{\Lambda_q^{-1}}{1 - \Lambda_q^{-1}} g(q^{-m}x)b_m(q^{-m}x) 
\]
\[
= \sum_{m=0}^{\infty} f \frac{\Lambda_q^{-1}}{1 - \Lambda_q^{-1}} b_m(q^{-m}x)g(q^{-m}x) 
\]
\[
= f \frac{\Lambda_q^{-1}}{1 - \Lambda_q^{-1}} B^*(g). \tag{12.12}
\]

Similar proof for the eq. (12.10) can be got easily.

Similarly as in [27–30], we can get the \(n\)-fold Darboux transformation in the following theorem which will be used to generate new solutions.

**Theorem 10.** The \(n\)-fold Darboux transformation of EQTH equation is as following

\[
W_n = 1 + t_1^{[n]} \Lambda_q^{-1} + t_2^{[n]} \Lambda_q^{-2} + \cdots + t_n^{[n]} \Lambda_q^{-n} \tag{12.13}
\]

where

\[
W_n \cdot \phi_i |_{i \leq n} = 0. \tag{12.14}
\]

The Darboux transformation leads to new solutions form seed solutions

\[
u^{[n]} = \begin{cases}
u^{[n]} = u + (\Lambda_q - 1)t_1^{[n]}, & \text{if } n \geq 1 \\
u^{[n]} = t_1^{[n]}(x)(\Lambda_q^{-n}v)t^{[n]-1}(q^{-1}x). & \text{if } n = 0
\end{cases} \tag{12.15}
\]

where

\[
W_n = \frac{1}{\Delta_n} \left[ \begin{array}{cccc}
1 & \Lambda_q^{-1} & \Lambda_q^{-2} & \cdots & \Lambda_q^{-n} \\
\phi_1(q^{-1}x) & \phi_1(q^{-2}x) & \phi_1(q^{-n}x) \\
\phi_2(q^{-1}x) & \phi_2(q^{-2}x) & \phi_2(q^{-n}x) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\phi_n(q^{-1}x) & \phi_n(q^{-2}x) & \phi_n(q^{-n}x)
\end{array} \right]
\]

\[
\Delta_n = \left| \begin{array}{cccc}
\phi_1(q^{-1}x) & \phi_1(q^{-2}x) & \cdots & \phi_1(q^{-n}x) \\
\phi_2(q^{-1}x) & \phi_2(q^{-2}x) & \cdots & \phi_2(q^{-n}x) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_n(q^{-1}x) & \phi_n(q^{-2}x) & \cdots & \phi_n(q^{-n}x)
\end{array} \right|
\]

It can be easily checked that \(W_n\phi_i = 0, \ i = 1, 2, \ldots, n\).

Taking seed solution \(u = 0, v = 1\), then using Theorem 10 one can get the \(n\)-th new solution of the EQTH as

\[
u^{[n]} = (1 - \Lambda_q^{-1})\partial_0 \log \tilde{W}_r(\phi_1, \phi_2, \ldots, \phi_n), \tag{12.17}
\]
\[
u^{[n]} = e^{(1-\Lambda_q^{-1})(1-\Lambda_q^{-1})} \log \tilde{W}_r(\phi_1, \phi_2, \ldots, \phi_n), \tag{12.18}
\]
where $\hat{W}_r(\phi_1, \phi_2, \ldots \phi_n)$ is the q-deformed “Wronskian”

$$\hat{W}_r(\phi_1, \phi_2, \ldots \phi_n) = det(\Lambda_q^{j+1}\phi_{n+1-i})_{1\leq i,j\leq n}. \quad (12.19)$$

### 13. Multicomponent $q$-Toda Hierarchy

#### 13.1. Factorization Problem.

In this section, we will denote $G_N$ as a group which contains invertible elements of complex $N \times N$ complex matrices and denote its Lie algebra $g_N$ as the associative algebra of complex $N \times N$ complex matrices $M_N(\mathbb{C})$.

Now we introduce the following free operators $W_{N0}, \tilde{W}_{N0} \in G_N$

$$W_{N0} := \sum_{k=1}^{N} E_{kk} e^{\sum_{j=0}^\infty t_{jk} \Lambda_q^j}, \quad (13.1)$$

$$\tilde{W}_{N0} := \sum_{k=1}^{N} E_{kk} e^{\sum_{j=0}^\infty \bar{t}_{jk} \Lambda_q^{-j}}, \quad (13.2)$$

where $t_{jk}, \bar{t}_{jk} \in \mathbb{C}$ will play the role of continuous times. We define the dressing operators $W_N, \tilde{W}_N$ as follows

$$W_N := S_N \cdot W_{N0}, \quad \tilde{W}_N := \tilde{S}_N \cdot \tilde{W}_{N0}. \quad (13.3)$$

Given an element $g_N \in G_N$ and time series $t = (t_{jk}), \tilde{t} = (\bar{t}_{jk}), s = (s_j); j, k \in \mathbb{N}, 1 \leq k \leq N$, one can consider the factorization problem in $G_N$ \[32\]

$$W_N \cdot g_N = \hat{W}_N, \quad (13.4)$$

i.e. the factorization problem

$$S_N(t, \bar{t}, s) \cdot W_{N0} \cdot g_N = \tilde{S}_N(t, \bar{t}, s) \cdot \tilde{W}_{N0}, \quad S_N \in G_N^{-} \text{ and } \tilde{S}_N \in G_N^{+}. \quad (13.5)$$

Observe that $S_N, \tilde{S}_N$ have expansions of the form

$$S_N = I_N + \beta_1(x)\Lambda_q^{-1} + \beta_2(x)\Lambda_q^{-2} + \cdots \in G_N^{-}, \quad S_N = \beta_0(x) + \beta_1(x)\Lambda_q + \beta_2(x)\Lambda_q^2 + \cdots \in G_N^{+}. \quad (13.6)$$

Also the inverse operators $S_N^{-1}, \tilde{S}_N^{-1}$ of operators $S, \tilde{S}_N$ have expansions of the form

$$S_N^{-1} = I_N + \beta'_1(x)\Lambda_q^{-1} + \beta'_2(x)\Lambda_q^{-2} + \cdots \in G_N^{-}, \quad \tilde{S}_N^{-1} = \beta'_0(x) + \beta'_1(x)\Lambda_q + \beta'_2(x)\Lambda_q^2 + \cdots \in G_N^{+}. \quad (13.7)$$

The Lax operators $L, C_{kk}, \bar{C}_{kk} \in g_N$ are defined by

$$L := W_N \cdot \Lambda_q \cdot W_N^{-1} = \tilde{W}_N \cdot \Lambda_q^{-1} \cdot \tilde{W}_N^{-1}, \quad (13.8)$$

$$C_{kk} := W_N \cdot E_{kk} \cdot W_N^{-1}, \quad \bar{C}_{kk} := \tilde{W}_N \cdot E_{kk} \cdot \tilde{W}_N^{-1}, \quad (13.9)$$

and have the following expansions

$$L = \Lambda_q + u(x) + v(x)\Lambda_q^{-1}, \quad (13.10)$$

$$C_{kk} = E_{kk} + C_{kk,1}(x)\Lambda_q^{-1} + C_{kk,2}(x)\Lambda_q^{-2} + \cdots, \quad \bar{C}_{kk} = \bar{C}_{kk,0}(x) + \bar{C}_{kk,1}(x)\Lambda_q + \bar{C}_{kk,2}(x)\Lambda_q^2 + \cdots.$$
In fact the Lax operators $L, C_{kk}, \bar{C}_{kk} \in g_N$ can also be equivalently defined by
\begin{align}
L := S_N \cdot \Lambda_q \cdot S^{-1}_N &= \bar{S}_N \cdot \Lambda^{-1}_q \cdot \bar{S}^{-1}_N, \\
C_{kk} := S_N \cdot E_{kk} \cdot S^{-1}_N, \quad \bar{C}_{kk} := \bar{S}_N \cdot E_{kk} \cdot \bar{S}^{-1}_N. \quad (13.11)
\end{align}

\section{LAX EQUATIONS OF MQTH}

In this section we will use the factorization problem (13.4) to derive Lax equations. Let us first introduce some convenient notations.

\textbf{Definition 6.} The matrix operators $C_{kk}, \bar{C}_{kk}, B_{jk}, \bar{B}_{jk}$ are defined as follows
\begin{align}
C_{kk} := W_N E_{kk} W^{-1}_N, \quad \bar{C}_{kk} := \bar{W}_N E_{kk} \bar{W}^{-1}_N, \\
B_{jk} := W_N E_{kk} \Lambda^{jk}_q W^{-1}_N, \quad \bar{B}_{jk} := \bar{W}_N E_{kk} \Lambda^{-jk}_q \bar{W}^{-1}_N. \quad (14.1)
\end{align}

Now we give the definition of the multicomponent $q$-Toda hierarchy (MQTH).

\textbf{Definition 7.} The multicomponent $q$-Toda hierarchy is a hierarchy in which the dressing operators $S_N, \bar{S}_N$ satisfy following Sato equations
\begin{align}
\epsilon \partial_{tjk} S_N &= -(B_{jk})_+ \cdot S_N, \quad \epsilon \partial_{tjk} \bar{S}_N = (B_{jk})_+ \cdot \bar{S}_N, \\
\epsilon \partial_{tjk} \bar{S}_N &= -(B_{jk})_- \cdot S_N, \quad \epsilon \partial_{tjk} S_N = (B_{jk})_- \cdot \bar{S}_N. \quad (14.2)
\end{align}

Then one can easily get the following proposition about $W_N, \bar{W}_N$.

\textbf{Proposition 12.} The wave operators $W_N, \bar{W}_N$ satisfy following Sato equations
\begin{align}
\epsilon \partial_{tjk} W_N &= (B_{jk})_+ \cdot W_N, \quad \epsilon \partial_{tjk} \bar{W}_N = (B_{jk})_+ \cdot \bar{W}_N, \\
\epsilon \partial_{tjk} \bar{W}_N &= -(B_{jk})_- \cdot W_N, \quad \epsilon \partial_{tjk} W_N = -(B_{jk})_- \cdot \bar{W}_N. \quad (14.3)
\end{align}

From the previous proposition we can derive the following Lax equations for the Lax operators.

\textbf{Proposition 13.} The Lax equations of the MQTH are as follows
\begin{align}
\epsilon \partial_{tjk} L &= [(B_{jk})_+, L], \quad \epsilon \partial_{tjk} C_{ss} = [(B_{jk})_+, C_{ss}], \quad \epsilon \partial_{tjk} \bar{C}_{ss} = [(B_{jk})_+, \bar{C}_{ss}], \\
\epsilon \partial_{tjk} \bar{L} &= [(B_{jk})_+, \bar{L}], \quad \epsilon \partial_{tjk} \bar{C}_{ss} = [(\bar{B}_{jk})_+, C_{ss}], \quad \epsilon \partial_{tjk} \bar{C}_{ss} = [(\bar{B}_{jk})_+, \bar{C}_{ss}]. \quad (14.4)
\end{align}

To see this kind of hierarchy more clearly, the multicomponent $q$-Toda equations as the $\partial_{t1k}$ flow equations will be given in the next subsection.

\subsection{The multicomponent $q$-Toda equations}

As a consequence of the factorization problem (13.4) and Sato equations, after taking into account that $S_N \in G_{N-}$ and $\bar{S}_N \in G_{N+}$ and using the notation $e^{\phi_N} := \beta_0$ in $\bar{S}_N$, $B_{1k}$ has following form
\begin{align}
B_{1k} = E_{kk} \Lambda_q + U_k + \bar{V}_k \Lambda^{-1}_q, \quad 1 \leq k \leq N, \quad (14.8)
\end{align}
and we have the alternative expressions
\begin{align}
U_k := \beta_1(x) E_{kk} - E_{kk} \beta_1(qx) = \epsilon \partial_{t1k} (e^{\phi_N(x)}) \cdot e^{-\phi_N(x)}, \\
V_k = e^{\phi_N(x)} E_{kk} e^{-\phi_N(q^{-1}x)} = -\epsilon \partial_{t1k} \beta_1(x). \quad (14.9)
\end{align}
From Sato equations we deduce the following set of nonlinear partial differential-difference equations
\[
\begin{align*}
  \beta_1(x) E_{kk} - E_{kk} \beta_1(qx) &= \epsilon^2 \partial_{t_1k} (e^{\phi_N(x)} \cdot e^{-\phi_N(x)}), \\
  \partial_{t_1k} \beta_1(x) &= -e^{\phi_N(x)} E_{kk} e^{-\phi_N(q^{-1}x)}. 
\end{align*}
\] (14.10)

These equations constitute what we call the multicomponent \( q \)-Toda equations. Observe that if we cross the two equations in (14.10), then we get
\[
\epsilon^2 \partial_{t_1k} \left( \partial_{t_1k} (e^{\phi_N(x)} \cdot e^{-\phi_N(x)}) \right) = E_{kk} e^{\phi_N(qx)} E_{kk} e^{-\phi_N(x)} - e^{\phi_N(x)} E_{kk} e^{-\phi_N(q^{-1}x)} E_{kk},
\]
which is the matrix extension of the following Toda equation (the case when \( N = 1 \))
\[
\epsilon^2 \partial_{t_11} \partial_{t_11} (\phi_N(x)) = e^{\phi_N(qx)} - \phi_N(x) - e^{\phi_N(x)} - \phi_N(q^{-1}x).
\]

Besides above multicomponent \( q \)-Toda equations, the logarithmic flows the MQTH also contains some extended flow equations in the next subsection.

15. BI-HAMILTONIAN STRUCTURE AND TAU SYMMETRY

To describe the integrability of the MQTH with the matrix-valued Lax operator
\[
L = \Lambda_q + u + v \Lambda_q^{-1}, \quad i.e. \quad L_{ij} = \delta_{ij} \Lambda_q + u_{ij} + v_{ij} \Lambda_q^{-1},
\] (15.1)
we will construct the bi-Hamiltonian structure and tau symmetry of the MQTH in this section. For a matrix \( A = (a_{ij}) \), the vector field \( \partial_A \) over MQTH is defined by
\[
\partial_A = \sum_{i,j=1}^{N} \sum_{k \geq 0} a_{ij}^{(k)} \left( \frac{\partial}{\partial u_{ij}^{(k)}} + \frac{\partial}{\partial v_{ij}^{(k)}} \right) = Tr \sum_{k \geq 0} A^{(k)} \left( \frac{\partial}{\partial u^{(k)}} + \frac{\partial}{\partial v^{(k)}} \right),
\] (15.2)
where
\[
\left( \frac{\partial}{\partial u^{(k)}} \right)_{ji} = \frac{\partial}{\partial u_{ij}^{(k)}}, \quad \left( \frac{\partial}{\partial v^{(k)}} \right)_{ji} = \frac{\partial}{\partial v_{ij}^{(k)}}.
\] (15.3)

For two functionals \( \bar{f} = \int f \, dx, \bar{g} = \int g \, dx \), we have
\[
\partial_A \bar{f} = \int \sum_{i,j=1}^{N} \sum_{k \geq 0} a_{ij}^{(k)} \left( \frac{\partial f}{\partial u_{ij}^{(k)}} + \frac{\partial f}{\partial v_{ij}^{(k)}} \right) dx = \int Tr \sum_{k \geq 0} A^{(k)} \left( \frac{\partial f}{\partial u^{(k)}} + \frac{\partial f}{\partial v^{(k)}} \right) dx.
\] (15.4)

Then we can define the hamiltonian bracket as
\[
\{ \bar{f}, \bar{g} \} = \int \sum \frac{\delta \bar{f}}{\delta w} \{w, w'\} \frac{\delta \bar{g}}{\delta w'} dx, \quad w, w' = u_{ij} \text{ or } v_{ij}, \quad 1 \leq i, j \leq N.
\] (15.5)

The bi-Hamiltonian structure for the MQTH can be given by the following two compatible Poisson brackets which is a generalization in matrix forms of the extended Toda hierarchy in [6]
\[
\begin{align*}
\{ u(x)_{ij}, u(y)_{pq} \} & = \frac{1}{\epsilon} [\delta_{iq} u_{pj}(x) - \delta_{jp} u_{iq}(x)] \delta(x - y), \\
\{ u(x)_{ij}, v(y)_{pq} \} & = \frac{1}{\epsilon} [\delta_{iq} \Lambda_q v_{pj}(x) - \delta_{jp} v_{iq}(x)] \delta(x - y), \\
\{ v(x)_{ij}, v(y)_{pq} \} & = 0.
\end{align*}
\] (15.6) (15.7) (15.8)
\{u(x)_{ij}, u(y)_{pq}\}_2 = \frac{1}{\epsilon} \left[ \delta_{iq}\Lambda_q v_{pj}(x) - \delta_{jp} v_{iq}(x)\Lambda_q^{-1} + \delta_{iq} \sum_{s=1}^{N} u_{sj} \Lambda_q^{-1} u_{ps} - u_{pj} \Lambda_q^{-1} u_{iq} \right.
- u_{iq}(\Lambda_q - 1)^{-1} u_{pj} + \delta_{jp} \sum_{s=1}^{N} u_{is}(\Lambda_q - 1)^{-1} u_{sq} \right] \delta(x - y), \quad (15.9)

\{u(x)_{ij}, v(y)_{pq}\}_2 = \frac{1}{\epsilon} \left[ \delta_{iq} \sum_{s=1}^{N} u_{sj} \Lambda_q^2(\Lambda_q - 1)^{-1} v_{ps}(x) - u_{pj} \Lambda_q(\Lambda_q - 1)^{-1} v_{iq} \right.
- u_{iq}(\Lambda_q - 1)^{-1} \Lambda_q v_{pj}(x) + \delta_{jp} \sum_{s=1}^{N} u_{is}(\Lambda_q - 1)^{-1} v_{sq} \right] \delta(x - y), \quad (15.10)

\{v(x)_{ij}, v(y)_{pq}\}_2 = \frac{1}{\epsilon} \left[ \delta_{iq} \sum_{s=1}^{N} v_{sj} \Lambda_q^2(\Lambda_q - 1)^{-1} v_{ps}(x) - v_{pj} \Lambda_q(\Lambda_q - 1)^{-1} v_{iq} \right.
- v_{iq}(\Lambda_q - 1)^{-1} v_{pj}(x) + \delta_{jp} \sum_{s=1}^{N} v_{is} \Lambda_q^{-1}(\Lambda_q - 1)^{-1} v_{sq} \right] \delta(x - y). \quad (15.11)

In the following theorem, we will prove the above poisson structures can be considered as the bi-Hamiltonian structure of the MQTH.

**Theorem 11.** The flows of the MQTH are Hamiltonian systems of the form

\[
\frac{\partial u_{pq}}{\partial t_{j,k}} = \{u_{pq}, H_{j,k}\}_1, \quad \frac{\partial v_{pq}}{\partial t_{j,k}} = \{v_{pq}, H_{j,k}\}_1, \quad (15.12)
\]

\[
\frac{\partial u_{pq}}{\partial t_{j,k}} = \{u_{pq}, \tilde{H}_{j,k}\}_1, \quad \frac{\partial v_{pq}}{\partial t_{j,k}} = \{v_{pq}, \tilde{H}_{j,k}\}_1, \quad k = 0, 1, \ldots N; \quad j \geq 0. \quad (15.13)
\]

They satisfy the following bi-Hamiltonian recursion relation

\[
\{\cdot, H_{n-1,k}\}_2 = \{\cdot, H_{n,k}\}_1, \quad \{\cdot, \tilde{H}_{n-1,k}\}_2 = \{\cdot, \tilde{H}_{n,k}\}_1. \quad (15.14)
\]

Here the Hamiltonians have the form

\[
F_{j,k} = \int f_{j,k}(u, v; u_x, v_x; \ldots; \epsilon) dx, \quad (15.15)
\]

with the Hamiltonian \(F_{j,k} = H_{j,k}, \tilde{H}_{j,k}\) and the Hamiltonian densities \(f_{j,k} = h_{j,k}, \tilde{h}_{j,k}\) given by

\[
h_{j,k} = TrRes C_{kk} L^j, \quad \tilde{h}_{j,k} = TrRes \bar{C}_{kk} L^j. \quad (15.16)
\]

For readers' convenience, now we will write down the first several Hamiltonian densities explicitly as follows

\[
h_{0,k} = TrRes C_{kk} = Tr E_{kk} = 1, \quad (15.17)
\]

\[
h_{1,k} = TrRes C_{kk} L = Tr[(1 - \Lambda_q)^{-1} u E_{kk} - E_{kk} \frac{\Lambda_q}{1 - \Lambda_q} u] = u_{kk}, \quad (15.18)
\]

\[
\tilde{h}_{0,k} = Tr Res \bar{C}_{kk} = Tr \tilde{\beta}_0 E_{kk} \tilde{\beta}_0^{-1}, \quad (15.19)
\]

\[
\tilde{h}_{1,k} = Tr Res \bar{C}_{kk} L = Tr (\tilde{\beta}_1 E_{kk} \tilde{\beta}_0^{-1} - \tilde{\beta}_0 E_{kk} \tilde{\beta}_0^{-1}(q^{-1} x) \tilde{\beta}_1 (q^{-1} x) \tilde{\beta}_0^{-1}), \quad (15.20)
\]

with

\[
\tilde{\beta}_0 = v \tilde{\beta}_0 (q^{-1} x), \quad \tilde{\beta}_1 = u \tilde{\beta}_0 + v \tilde{\beta}_1 (q^{-1} x). \quad (15.21)
\]
When $N = 1$, the above conserved densities will be the ones of the Toda hierarchy in [6]. Similarly as [6], the tau symmetry of the MQTH can be proved in the following theorem.

**Theorem 12.** The Hamiltonian densities of the MQTH have the following tau-symmetry property:

\[
\frac{\partial h_{\alpha,m}}{\partial t_{j,k}} = \frac{\partial h_{j,k}}{\partial t_{\alpha,m}}, \quad \frac{\partial \bar{h}_{\alpha,m}}{\partial t_{j,k}} = \frac{\partial \bar{h}_{j,k}}{\partial t_{\alpha,m}},
\]

\[
\frac{\partial h_{\alpha,m}}{\partial \bar{t}_{j,k}} = \frac{\partial \bar{h}_{j,k}}{\partial \bar{t}_{\alpha,m}}, \quad \frac{\partial \bar{h}_{\alpha,m}}{\partial \bar{t}_{j,k}} = \frac{\partial h_{j,k}}{\partial \bar{t}_{\alpha,m}},
\]

(15.22)

(15.23)

Proof. Let us prove the theorem for the first equation, other cases can be proved in a similar way

\[
\frac{\partial h_{m,s}}{\partial t_{n,k}} = Tr Res[-(C_{kk}L_n), C_{ss}L^m] = Tr Res[(C_{ss}L^m)_+, (C_{ss}L^n)_-] = Tr Res[(C_{ss}L^m)_+, C_{kk}L^n] = \frac{\partial h_{n,k}}{\partial t_{m,s}}.
\]

(15.24)

This property justifies the following definition of the tau function for the MQTH:

**Definition 8.** The tau function $\tau_N$ of the MQTH can be defined by the following expressions in terms of the densities of the Hamiltonians:

\[
h_{j,n} = \epsilon(\Lambda_q - 1)\frac{\partial \log \tau_N}{\partial t_{j,n}},
\]

(15.25)

\[
\bar{h}_{j,n} = \epsilon(\Lambda_q - 1)\frac{\partial \log \tau_N}{\partial \bar{t}_{j,n}}.
\]

(15.26)

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