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ON KATZ’S BOUND FOR NUMBER OF ELEMENTS WITH GIVEN TRACE AND NORM

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Abstract. In this note an improvement of the Katz’s bound on the number of elements in a finite field with given trace and norm is given. The improvement is obtained by reducing the problem to estimating the number of rational points on certain toric Calabi-Yau hypersurface, and then to use detailed cohomological calculations by Rojas-Leon and the second author for such toric hypersurfaces.

1. Introduction

Let \( p \) be a prime and \( \mathbb{F}_q \) be the finite field of \( q \) elements of characteristic \( p \). Given \( a, b \in \mathbb{F}_q^* \), and positive integer \( m \geq 2 \), let

\[
N_m(a, b) = \#\{\alpha \in \mathbb{F}_{q^m} | \text{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\alpha) = a, \text{Norm}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\alpha) = b\}.
\]

Motivated by various applications, it is of interest to give a sharp estimate for the number \( N_m(a, b) \). The case \( b = 0 \) is trivial.

Katz [2] proved the following bound:

Theorem 1.1. Let \( a, b \in \mathbb{F}_q^* \) and \( n \geq 1 \). Then

\[
|N_{n+1}(a, b) - \frac{q^{n+1} - 1}{q(q - 1)}| \leq (n + 1)q^{\frac{n-1}{2}}.
\]

This bound was used by Moisio [3] to improve some cases of the explicit bound in Wan [5] on the number of irreducible polynomials in an arithmetic progression of \( \mathbb{F}_q[x] \). In the case \( n + 1 = 3 \), the Katz bound also plays a significant role in Cohen and Huczynska [1] for their proof of the existence of a cubic primitive normal polynomial with given norm and trace.

If \( a = 0 \), Katz’s bound can be improved in an elementary way using character sums [3]:

\[
|N_{n+1}(0, b) - \frac{q^n - 1}{q - 1}| \leq (d - 1)q^{\frac{n-1}{2}},
\]

where \( d = \gcd(n + 1, q - 1) \).

In this note, we give a uniform improvement of Katz’s bound in the case \( a \neq 0 \).

Theorem 1.2. Let \( a, b \in \mathbb{F}_q^* \) and \( n \geq 1 \). Then

\[
|N_{n+1}(a, b) - \frac{q^n - 1}{q - 1}| \leq nq^{\frac{n-1}{2}}.
\]
In the case that $n + 1$ is a power of $p$, this improvement was first proved by Moisio [3] using Deligne’s estimate for hyper-Kloosterman sums. Moreover, in the case $n + 1 = 3$ also the bounds
\[ 3 \left\lceil \frac{q + 1 - 2\sqrt{q}}{3} \right\rceil \leq N_3(a, b) \leq 3 \left\lfloor \frac{q + 1 + 2\sqrt{q}}{3} \right\rfloor. \]
were obtained in [3] by using the Hasse’s bound for elliptic curves together with a divisibility result. In corollary 2.4, we extend such divisibility bounds to $N_\ell(a, b)$, where $\ell \geq 3$ is any prime.

In the general case, our proof of Theorem 1.2 consists of two steps. The first step is to reduce it to estimating the number of $\mathbb{F}_q$-rational points on certain toric Calabi-Yau hypersurface over $\mathbb{F}_q$. The second step is to use the detailed cohomological calculations in Rojas-Leon and Wan [4] for such toric hypersurfaces. In the case $n + 1 = 3$, the above improved bounds should significantly reduce the amount of calculations in [1].

2. Proof of Theorem 1.2

Let $u = b/a^{n+1} \in \mathbb{F}_q^*$. Let $N(u)$ denote the number of $\mathbb{F}_q$-rational points on the toric hypersurface
\[ Y_u : X_1 + \cdots + X_n + \frac{u}{X_1 \cdots X_n} = 1 = 0. \]

Lemma 2.1.
\[ N_{n+1}(a, b) = \frac{q^n - 1}{q - 1} + (-1)^n \left( N(u) - \frac{(q-1)^n - (-1)^n}{q} \right). \]

Proof. Write the equation of $Y_u$ in the form
\[ X_1 + \cdots + X_{n+1} = 1 \]
\[ X_1 \cdots X_{n+1} = u. \]

Let $\psi$ be the canonical additive character of $\mathbb{F}_q$. Now
\[ q(q-1)N(u) = \sum_{x_1, \ldots, x_{n+1}} \sum_v \psi(v(x_1 + \cdots + x_{n+1} - 1)) \sum_\chi \chi(u^{-1}x_1 \cdots x_{n+1}), \]
where $x_1, \ldots, x_{n+1}$ run over $\mathbb{F}_q^*$, $v$ runs over $\mathbb{F}_q$, and $\chi$ runs over the multiplicative character group of $\mathbb{F}_q$.

Let $G(\chi)$ denote the Gauss sum
\[ G(\chi) = \sum_{x \in \mathbb{F}_q^*} \psi(x)\chi(x). \]
It follows that
\[
q(q-1)N(u) = (q-1)^{n+1} + \sum_{v\neq 0} \psi(-v) \sum_{\chi} \tilde{\chi}(u) \prod_{i=1}^{n+1} \sum_{x_i} \psi(v x_i) \chi(x_i)
\]
\[
\times_{i \rightarrow x_i / v} (q-1)^{n+1} + \sum_{v\neq 0} \psi(-v) \sum_{\chi} \tilde{\chi}(uv^{n+1})G(\chi)^{n+1}
\]
\[
= (q-1)^{n+1} + \sum_{\chi} G(\chi)^{n+1} \tilde{\chi}(u) \sum_{v\neq 0} \psi(-v) \chi^{n+1}(v)
\]
\[
= (q-1)^{n+1} + \sum_{\chi} G(\chi)^{n+1} \tilde{\chi}(u)(-1)^n v^{n+1}.
\]
(2.0.1)

Next we express $N_{n+1}(a,b)$ in terms of Gauss sums. We use the abbreviated notations $\text{Tr}$ and $\text{Norm}$ in place of $\text{Tr}_{\mathbb{F}_{q^{n+1}}/\mathbb{F}_q}$ and $\text{Norm}_{\mathbb{F}_{q^{n+1}}/\mathbb{F}_q}$. Let $\psi_{n+1} = \psi \circ \text{Tr}$ be the canonical additive character of $\mathbb{F}_{q^{n+1}}$ and let $\alpha \in \mathbb{F}_{q^{n+1}}$ with $\text{Tr}(\alpha) = 1$.

Now,
\[
q(q-1)N_{n+1}(a,b) = \sum_{x \in \mathbb{F}_{q^{n+1}}} \sum_{v} \psi(v(\text{Tr}(x - \alpha a))) \sum_{\chi} \chi(b^{-1} \text{Norm}(x))
\]
\[
= \sum_{v} \psi(-av) \sum_{\chi} \tilde{\chi}(b) \sum_{x} \psi_{n+1}(vx) \chi(\text{Norm}(x))
\]
\[
= q^{n+1} - 1 + \sum_{v \neq 0} \psi(-av) \sum_{\chi} \tilde{\chi}(b) \sum_{x} \psi_{n+1}(vx) \chi(\text{Norm}(x))
\]
\[
\times_{i \rightarrow x / v} q^{n+1} - 1 + \sum_{v \neq 0} \psi(-av) \sum_{\chi} \tilde{\chi}(b v^{n+1}) \sum_{x} \psi_{n+1}(x) \chi(\text{Norm}(x)),
\]

since $\text{Norm}(v) = v^{n+1}$.

By the Davenport-Hasse identity the inner sum
\[
\sum_{x} \psi_{n+1}(x) \chi(\text{Norm}(x)) = (-1)^n G(\chi)^{n+1},
\]
and therefore
\[
q(q-1)N_{n+1}(a,b) = q^{n+1} - 1 + (-1)^n \sum_{\chi} G(\chi)^{n+1} \tilde{\chi}(b) \sum_{v \neq 0} \psi(-av) \chi^{n+1}(v)
\]
\[
= q^{n+1} - 1 + (-1)^n \sum_{\chi} G(\chi)^{n+1} \tilde{\chi}((-1)^n a^{n+1} b/A^{n+1}).
\]

Comparing this expression with (2.0.1), one finds that
\[
N_{n+1}(a,b) = \frac{q^{n+1} - 1}{q(q-1)} + (-1)^n \left( N(u) - \frac{(q-1)^{n+1}}{q(q-1)} \right).
\]

One checks that this is the same as the expression in Lemma 2.1. \qed

This lemma reduces Theorem 1.2 to the following

**Theorem 2.2.** Let $u \in \mathbb{F}_q^*$. Then
\[
|N(u) - \frac{(q-1)^n - (-1)^n}{q}| \leq nq^{\frac{n-1}{2}}.
\]
Proof. Over the algebraic closure $\overline{\mathbb{F}}_q$, we can write $u = \lambda^{-(n+1)}$ for some non-zero element $\lambda$. Then $Y_u$ is isomorphic to the toric hypersurface
\[
X_\lambda : X_1 + \cdots + X_n + \frac{1}{X_1 \cdots X_n} - \lambda = 0
\]
whose zeta function over a finite field was studied in detail in [3], see [3] for more elementary description of the results. For a prime $\ell \neq p$, the $\ell$-adic cohomology
\[
H^j_\ell(Y_u \otimes \overline{\mathbb{F}}_q, \mathbb{Q}_\ell) \cong H^j_\ell(X_\lambda \otimes \overline{\mathbb{F}}_q, \mathbb{Q}_\ell)
\]
was calculated in Theorem 2.1 in [4]. In particular, we have
\[
H^j_\ell(Y_u \otimes \overline{\mathbb{F}}_q, \mathbb{Q}_\ell) = 0, \quad j < n - 1 \text{ or } j > 2n - 1,
\]
and there is an exact sequence of Galois modules
\[
0 \rightarrow \mathbb{Q}_\ell^n \rightarrow H^{j-n}_\ell(Y_u \otimes \overline{\mathbb{F}}_q, \mathbb{Q}_\ell) \rightarrow M_u \rightarrow 0,
\]
where $M_u$ is of rank at most $n$ and mixed of weight at most $n - 1$. It follows that
\[
|\text{Tr}(\text{Frob}_u|M_u)| \leq nq^{\frac{n-1}{2}}.
\]
By the $\ell$-adic trace formula,
\[
N(u) = \sum_{j=n}^{2n-2} (-1)^j \binom{n}{j-n+2} q^{j-(n-1)} + (-1)^{n-1} n + (-1)^{n-1} \text{Tr}(\text{Frob}_u|M_u).
\]
Replacing $j$ by $j + n - 2$, one finds
\[
N(u) = \sum_{j=2}^{n} (-1)^{j-n} \binom{n}{j} q^{j-n} + (-1)^{n-1} n + (-1)^{n-1} \text{Tr}(\text{Frob}_u|M_u).
\]
The theorem follows.

Remark. If $u \neq (n+1)^{-(n+1)}$, i.e., $\lambda \notin \{n+1\} \zeta_{n+1} = 1$, then $M_u$ is pure of weight $n - 1$ and of rank $n$. If $u = (n+1)^{-(n+1)}$ (necessarily $p \mid n+1$), then the rank of $M_u$ drops by 1 and thus
\[
|\text{Tr}(\text{Frob}_u|M_u)| \leq (n-1)q^{\frac{n}{2}}.
\]
If $u = (n+1)^{-(n+1)}$ and $n$ is even, then one of the Frobenius eigenvalues has weight $n - 2$ (instead of $n - 1$), and thus
\[
|\text{Tr}(\text{Frob}_u|M_u)| \leq (n-2)q^{\frac{n-1}{2}} + q^{\frac{n-2}{2}}.
\]
All these follow from Proposition 2.6 in [4].

Corollary 2.3. Let $u = (n+1)^{-(n+1)}$. Then
\[
|N(u) - \frac{(q-1)^n - (-1)^n}{q}| \leq (n-1)q^{\frac{n+1}{2}}.
\]
If $n$ is also even, then
\[
|N(u) - \frac{(q-1)^n - (-1)^n}{q}| \leq (n-2)q^{\frac{n-1}{2}} + q^{\frac{n-2}{2}}.
\]
Corollary 2.4. Let $\ell \geq 3$ be a prime number. Let $a, b \in \mathbb{F}_q^*$, Then, we have

$$\ell \left[ \frac{\ell^k-1}{q-1} - (\ell - 1)q^{\left(\ell^2-2\right)/2} \right] \leq N_\ell(a,b) \leq \ell \left[ \frac{\ell^k-1}{q-1} + (\ell - 1)q^{\left(\ell^2-2\right)/2} \right].$$

Proof. Let $R$ be the number of $c \in \mathbb{F}_q$ such that $\ell c = a$ and $c^\ell = b$. It is clear that $R$ is either 0 or 1. Since $\ell$ is a prime, $N_\ell(a,b) - R$ is divisible by $\ell$. If $R = 0$, the corollary is the consequence of Theorem 1.2 and the divisibility of $N_\ell(a,b)$ by $\ell$.

Assume now that $R = 1$. Since $a \neq 0$, $\ell$ cannot be $p$. In this case, we have $a = \ell c$, $b = c^\ell$ and thus $u = b/a^\ell = \ell^{-\ell} \in \mathbb{F}_q^*$. We can apply the stronger estimate in the previous corollary to deduce the desired inequalities for $N_\ell(a,b)$.

References

[1] S. Huczynska and S.D. Cohen, Primitive free cubics with specified norm and trace, Trans. Amer. Math. Soc., 355(2003), 3099-3116.
[2] N. Katz, Estimates for Soto-Andrade sums, J. Reine Angew. Math., 438(1993), 143-161.
[3] M. Moisio, Kloosterman sums, elliptic curves, and irreducible polynomials with prescribed trace and norm, Acta Arith., to appear.
[4] A. Rojas-Leon and D. Wan, Moment zeta functions for toric Calabi-Yau hypersurfaces, Communications in Number Theory and Physics, Vol. 1, No.3 (2007), 539-578.
[5] D. Wan, Generators and irreducible polynomials over finite fields, Math. Comp., 219(1997), 1195-1212.
[6] D. Wan, Lectures on zeta functions over finite fields, Proceedings of 2007 Göttingen summer school on higher dimensional geometry over finite fields, to appear. [arXiv:0711.3651]

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