Homothetic perfect fluid space-times

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Abstract
A brief summary of results on homotheties in General Relativity is given, including general information about space-times admitting an r-parameter group of homothetic transformations for \( r > 2 \), as well as some specific results on perfect fluids. Attention is then focussed on inhomogeneous models, in particular on those with a homothetic group \( H_4 \) (acting multiply transitively) and \( H_3 \). A classification of all possible Lie algebra structures along with (local) coordinate expressions for the metric and homothetic vectors is then provided (irrespectively of the matter content), and some new perfect fluid solutions are given and briefly discussed.

1 Introduction
This paper is devoted to the study of space-times admitting an intransitive group of homotheties, with a view towards those which can be interpreted as perfect fluid solutions of Einstein’s field equations [1].

A collection of important results regarding generic properties of space-times admitting homothetic transformations can be found in [2]-[7] (and references cited therein), and in [8] where the case of multiply transitive action is thoroughly studied by Hall and Steele.

The study of this subject began with the pioneering paper by Cahill and Taub [9], followed by the works of Eardley [10, 11]. From then on, homotheties have been studied in connection with a wealth of situations of physical interest in classical general relativity as well as in cosmology, see [12, 13, 14] for interesting reviews on homothetic solutions.

The paper is organized as follows: section 2 contains a brief summary of results on groups of homotheties and space-times admitting them, the implications that they have on perfect fluids and the physical quantities characterizing them (density, pressure, velocity,...), and we also summarize all the general information about space-times admitting an \( r \)-parameter group of homothetic transformations for \( r > 2 \). This

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includes: dimension of the homothetic and isometric algebras as well as that of the orbits they act on respectively, together with its nature (spacelike, timelike or null), allowed Petrov and Segre types of the Weyl and Ricci tensors, and whether perfect fluid solutions exist or not. Most of the contents of this section is a review of dispersed results in the literature, but we considered useful to gather them all in a single table. Some of the results are, as far as we are aware of, new (see especially those cases where null orbits occur).

In sections 3 and 4 a classification of all possible Lie algebra structures along with (local) coordinate expressions for the metric and homothetic vectors, are given for the cases $H_4$ (acting multiply transitively on three-dimensional orbits, in keeping with our assumption of intransitive action) and $H_3$ respectively. These classes of space-times can be understood as generalizations to the Kantowski-Sachs and Bianchi models respectively. The characterizations provided are independent of the field equations, and therefore they may have applications other than those considered here (perfect fluids).

In particular, section 3 contains some review material on the case $r = 4$, together with some new results, all of them presented in a unified manner, extending the work of Wu [15], Cahill and Taub [9] and Shikin [16]. The general perfect fluid solution is then given in certain, well-defined and invariantly characterized subcases. Whenever this is not possible, a few selected examples are presented. It is assumed that the matter satisfies the weak and dominant energy conditions, and expressions for the kinematical quantities (acceleration, expansion, deceleration parameter, shear and vorticity) are provided for each case.

Finally, section 4 contains some new solutions for the case $r = 3$ and appropriate references to related work on this issue. We summarize the results concerning the topology of the Killing orbits and the Bianchi classification of the homothetic algebras. We distinguish the cases where the Killing subalgebra is Abelian from that where it is non-Abelian. Attention is then restricted to the orthogonally transitive case, giving for each possible Lie algebra structure the coordinate forms of the proper homothetic vector field and the metric. In the Abelian case we distinguish three different classes of such models, depending on the orientation of the fluid flow relative to the homothetic orbits. The case in which we are more interested is the so-called “tilted” where new solutions are found. Finally, we provide explicit forms for the homothetic vector field and the metric in the case of a (maximal) non-Abelian $G_2$, and, although no perfect fluid solutions have been found, we briefly discuss some properties.

2 Basic facts about homotheties

2.1 Definition and properties

Throughout this paper $(M, g)$ will denote a space-time: $M$ then being a Hausdorff, simply connected, four-dimensional manifold, and $g$ a Lorentz metric of signature $(-,+,+,+)$. All the structures will be assumed smooth.

A global vector field $X$ on $M$ is called homothetic if either one of the following
equivalent conditions holds on a local chart

\[ L_X g_{ab} = 2n g_{ab} , \quad X_{;ab} = n g_{ab} + F_{ab} , \]  

where \( n \) is a constant on \( M \), \( L \) stands for the Lie derivative operator, a semi-colon denotes a covariant derivative with respect to the metric connection, and \( F_{ab} = - F_{ba} \) is the so-called homothetic bivector. If \( n \neq 0 \), \( X \) is called proper homothetic and if \( n = 0 \), \( X \) is a Killing vector (KV) on \( M \). For a geometrical interpretation of (1) we refer the reader to [2, 6].

A necessary condition that \( X \) be homothetic is

\[ X^a ;_{bc} = R^a _{bcd} X^d , \]  

where \( R^a _{bcd} \) are the components of the Riemann tensor in the above chart; thus, a homothetic vector field (HVF) is a particular case of affine collineation [7] and therefore it will satisfy

\[ L_X R^a _{bcd} = L_X R_{ab} = L_X C^a _{bcd} = 0 , \]  

where \( R_{ab} (\equiv R^c _{acb}) \) and \( C^a _{bcd} \) stand, respectively for the components of the Ricci and the conformal Weyl tensor.

The set of all HVFs on \( M \) forms a finite dimensional Lie algebra under the usual bracket operation and will be referred to as the homothetic algebra, \( H_r \), \( r \) being its dimension. The set of all KVs on \( M \) also forms a finite dimensional Lie algebra, the Lie algebra of isometries, which will be denoted as \( G_s \) (\( s \) being its dimension), and one has that \( G_s \subseteq H_r \) (i.e., \( G_s \) is a subalgebra of \( H_r \)). Furthermore, it is immediate to see by direct computation that the Lie bracket of an HVF with a KV is always a KV and that, given any two proper HVFs, there always exists a linear combination of them which is a KV. From these considerations it immediately follows that the highest possible dimension of \( H_r \) in a four-dimensional manifold is \( r = 11 \).

If \( r \neq s \) then \( s = r - 1 \) necessarily, and one may choose a basis \( \{ X_1, \cdots, X_{r-1}, X \} \equiv \{ X_A \} _{A=1-r} \) for \( H_r \), in such a way that \( X \) is proper homothetic and \( X_1, \cdots, X_{r-1} \) are Killing vector fields spanning \( G_{r-1} \). If these vector fields in the basis of \( H_r \) are all complete vector fields, then each member of \( H_r \) is complete and Palais’ theorem [4, 17, 18] guarantees the existence of an \( r \)-dimensional Lie group of homothetic transformations of \( M \) (\( H_r \)) in a well-known way; otherwise, it gives rise to a local group of local homothetic transformations of \( M \) and, although the usual concepts of isotropy and orbits still hold, a little more care is required [8].

The following result [8, 19] will be useful:

The orbits associated with \( H_r \) and \( G_{r-1} \) can only coincide if either they are four-dimensional or three-dimensional and null. (This result still holds if \( H_r \) is replaced by the conformal Lie algebra \( C_r \) and does not depend on the maximality of \( H_r \) or \( C_r \)).

The set of zeroes of a proper HVF, i.e., \( \{ p \in M : X(p) = 0 \} \) (fixed points of the homothety), either consists of topologically isolated points, or else is part of a null geodesic. The latter case corresponds to the well-known (conformally flat or Petrov type N) plane waves [2, 20].
At any zero of a proper HVF on $M$ all Ricci and Weyl eigenvalues must necessarily vanish and thus the Ricci tensor is either zero or has Segre type $\{(2, 11)\}$ or $\{(3, 1)\}$ (both with zero eigenvalue), whereas the Weyl tensor is of the Petrov type $O$, $N$ or $III$ [2] (see also [21] for vacuum space-times).

2.2 Perfect fluids

The energy-momentum tensor of a perfect fluid is given by

$$T_{ab} = (\mu + p)u_a u_b + pg_{ab} ,$$

where $\mu$ and $p$ are, respectively, the energy density and the pressure as measured by an observer comoving with the fluid, and $u^a$ ($u^a u_a = -1$) is the four-velocity of the fluid. If $X$ is an HVF then, from Einstein’s Field Equations (EFE) it follows that

$$\mathcal{L}_X T_{ab} = 0 ,$$

and this implies in turn [10]

$$\mathcal{L}_X u_a = nu_a , \quad \mathcal{L}_X p = -2np , \quad \mathcal{L}_X \mu = -2n\mu .$$

Thus, the Lie derivatives of $u_a$, $p$ and $\mu$ with respect to a KV vanish identically.

If a barotropic equation of state exists, $p = p(\mu)$, and the space-time admits a proper HVF $X$ then [22]

$$p = (\gamma - 1)\mu ,$$

where $\gamma$ is a constant ($0 \leq \gamma \leq 2$ in order to comply with the weak and dominant energy conditions). Of particular interest are the values $\gamma = 1$ (pressure-free matter, “dust”) and $\gamma = 4/3$ (radiation fluid). In addition, the value $\gamma = 2$ (stiff-matter) has been considered in connection with the early Universe. Furthermore, values of $\gamma$ satisfying $0 \leq \gamma < 2/3$, while physically unrealistic as regards a classical fluid, are of interest in connection with inflationary models of the Universe. In particular, the value $\gamma = 0$, for which the fluid can be interpreted as a positive cosmological constant, corresponds to exponential inflation, while the values $0 < \gamma < 2/3$ correspond to power law inflation in FRW models [23], but it is customary to restrict $\gamma$ to the range $1 \leq \gamma \leq 2$.

If the proper HVF $X$ and the four-velocity $u$ are mutually orthogonal (i.e., $u^a X_a = 0$) and a barotropic equation of state is assumed, it follows that $\gamma = 2$, i.e., $p = \mu$ stiff-matter [11], on the other hand, if $X^a = \alpha u^a$ the fluid is then shear-free. Further information on this topic can be found in [24, 25, 26].

2.3 The “dimensional count-down”

In this subsection, the maximal Lie algebra of global HVF on $M$ will be denoted as $\mathcal{H}_r$ ($r$ being its dimension), and it will be assumed that at least one member of it is proper homothetic.

The case of multiply transitive action is thoroughly studied in [8], and we shall refer the reader there for details; nevertheless, and for the sake of completeness, we
summarize in the following table the results given there, which follow invariably from considerations on the associated Killing subalgebra and the fixed point structure of the proper HVF. Furthermore, we have added a few other results, also in the literature or following straightforwardly from those, so as to complete the study down to dimension 4.

| $r$ | $O_m$ | $K_n$ | Petrov | Segre | Interpretation | PF, info. |
|-----|-------|-------|--------|-------|----------------|-----------|
| 11  | $M$   | $M$   | $O$    | 0     | Flat           | $\exists$ |
| 10  | $M$   | $M$   | -      | -     | Not Possible   | $\exists$ |
| 9   | $M$   | $M$   | -      | -     | Not Possible   | $\exists$ |
| 8   | $M$   | $M$   | $O$    | $(2,11)$ | Gen. Plane wave | $\exists$ |
| 7   | $M$   | $M$   | $N$    | 0, $(2,11)$ | Gen. Plane wave | $\exists$ |
| 7   | $M$   | $T_3$ | $O$    | $(1,11)1$ | Tachyonic Fluid | $\exists$ |
| 7   | $M$   | $N_3$ | -      | -     | Not Possible   | $\exists$ |
| 7   | $N_3$ | $N_3$ | $O$    | $(2,11)$ | Gen. Plane wave | $\exists$ |
| 7   | $M$   | $S_3$ | $O$    | $(1,(111))$ | Perfect Fluid | FRW |
| 6   | $M$   | $M$   | -      | -     | Not Possible   | $\exists$ |
| 6   | $N_3$ | $N_3$ | $N$    | $(2,11)$ | Gen. Plane wave | $\exists$ |
| 5   | $M$   | $M$   | -      | -     | Not Possible   | $\exists$ |
| 5   | $M$   | $N_3$ | -      | -     |                | $\exists$ |
| 5   | $N_3$ | $N_3$ | -      | -     | Not Possible   | $\exists$ |
| 5   | $M$   | $T_3$ | $D,N,O$| $\{1,1(11)\}, \{2,(11)\}$ | LRS | $\exists$ |
| 5   | $M$   | $S_3$ | $D,O$  | $\{(1,1)11\}, \{(2,1)1\}$ | LRS | $\exists$ |
| 4   | $M$   | $N_3$ | $II,III,D,N,O$ | $\{(1,1(11))\}, \{(2,11)\}$ | Plane waves | $\exists$ |
| 4   | $N_3$ | $N_3$ | -      | -     | Not Possible   | $\exists$ |
| 4   | $M$   | $T_3$ | -      | -     |                | $\exists$ |
| 4   | $M$   | $S_3$ | -      | -     |                | $\exists$ |
| 4   | $O_3$ | $N_2$ | $N,O$  | $\{3,1\}, \{2,11\}, \{(1,1)11\}$ | $\exists$ |
| 4   | $O_3$ | $T_2$ | $D,O$  | $(1,111)$ | $\exists$ |
| 4   | $O_3$ | $S_2$ | $D,O$  | $\{-11\}$ | $\exists$ |

**Table 2.1**

The first entry in the table gives the dimension of the group of homotheties, the second and third entries stand for the nature and dimension of the homothetic and Killing orbits respectively (e.g.: $N_2$, $T_2$ and $S_2$ denote Null, Timelike and Spacelike two-dimensional orbits respectively, $O_3$ stands for three-dimensional orbits of either nature, timelike, spacelike or null), the fourth and fifth entries give the Petrov and Segre type(s) of the associated Weyl and Ricci tensors (in the latter case it is to be understood that all possible degeneracies of the given types, can in principle occur, including vacuum when possible). Finally, the last two entries give respectively the possible interpretation whenever it is in some sense unique, and the existence or non-existence of perfect fluid solutions for that particular case, along with some supplementary information; thus FRW stands for Friedmann-Robertson-Walker, LRS for Locally Rotationally Symmetric, and Bianchi refers to that family of perfect fluid solutions. The cases that cannot arise are labeled as “Not Possible”, and wherever
no information is given on the Petrov and Segre types, it is to be understood that all types are possible in principle. The Segre type of the Ricci tensor of the case described in the last row, is unrestricted except in that it must necessarily have two equal (spacelike) eigenvalues; perfect fluid solutions of these characteristics constitute special cases of spherically, plane or hyperbolically symmetric perfect fluid space-times. For further information on LRS space-times, see [27, 28]; for the case $r = 4$ transitive and null three-dimensional Killing orbits, see [1, 29]. Regarding spatially homogeneous Bianchi models, see [24, 30, 33, 36]; and for the last three cases occurring in the table, see respectively [31, 32], [1, 32], and [1].

The case $r = 3$ has an associated Killing subalgebra $G_2$ and the respective dimensions of their orbits are 3 and 2 (see for instance [33, 34, 39, 42] and references cited therein). When the Killing subalgebra has null orbits, the metric is of Kundt’s class [49] and perfect fluids are excluded. If the Killing orbits are timelike, the solutions can then be interpreted as special cases of axisymmetric stationary space-times (provided that regularity conditions hold on the axis [1, 39]), and if they are spacelike as special cases of inhomogeneous cosmological solutions or cylindrically symmetric space-times. In both cases, perfect fluid solutions have been found.

3 The $H_4$ case

The associated isometric group for perfect fluid space-times acts necessarily on $T_3$, $S_3$ or $S_2$ orbits (see Table 2.1). In the intransitive case, the $G_3$ must act multiply transitively on two-dimensional surfaces of maximal symmetry $S_2$, which are then of constant (positive, zero or negative) curvature and admit orthogonal surfaces [44].

Possibly no problem in this context has been more exhaustively studied than that of spherically symmetric homothetic space-times, beginning with the seminal paper of Cahill and Taub [1] and continuing with recent papers by Ori and Piran [45], Carr and Yahil [46], Henriksen and Patel [47], and Foglizzo and Henriksen [48] among others (see references therein). Homothetic space-times with plane symmetry are also considered by Shikin [16].

What we attempt in this section, rather than presenting a survey of the models existing in the literature, is to study in a unified manner all possible cases, i.e., homothetic orbits of either nature (timelike, spacelike or null at every point $p \in M$ and the more general case in which their nature varies from point to point) as well as the different possibilities for the curvature $k$ of the isometry orbits (spherical and plane symmetry as well as the $k = -1$ case); thus, extending previous works by Wu [15], where only spacelike homothetic orbits are considered (i.e., type $B$ and some type $C$ solutions in our classification below), and by Cahill and Taub [1], and Shikin [16] where only spherical and plane symmetry are considered respectively.

As it is well known, the space-time metric can be written as [1]:

$$ds^2 = -A^2(r, t)dt^2 + B^2(r, t)dr^2 + F^2(r, t)(dy^2 + \Sigma^2(y, k)dz^2), \quad (8)$$
\[ \Sigma(y, k) = \begin{cases} 
\sin y & k = +1 \\
 y & k = 0 \\
\sinh y & k = -1 
\end{cases} \] (9)

The Killing vectors being \( \xi_1 = \sin z \partial_y + \sum \cos z \partial_z \), \( \xi_2 = \cos z \partial_y - \sum \sin z \partial_z \), and \( \xi_3 = \partial_z \), where the dash denotes a derivative with respect to \( y \), and they satisfy the following commutation relations

\[ [\xi_1, \xi_2] = k\xi_3 \quad [\xi_2, \xi_3] = \xi_1 \quad [\xi_3, \xi_1] = \xi_2 \] (10)

Assuming the existence of a proper HVF, \( X \), and since its commutator with a KV must be a KV, the Jacobi identities imply the following structures for \( H^4 \) when \( I \) or \( II \):

\[ [X, \xi_i] = 0 \quad i = 1, 2, 3 \quad k = 0, \pm1 \] (11)

\[ X = X^t(t, r) \partial_t + X^r(t, r) \partial_r \] (12)

\[ [X, \xi_i] = \xi_1 \quad [X, \xi_2] = \xi_2 \quad [X, \xi_3] = 0 \quad k = 0 \] (13)

\[ X = X^t(t, r) \partial_t + X^r(t, r) \partial_r - y \partial_y \] (14)

Requiring the coordinate system to be a comoving one (i.e., \( u_a = -A(t, r) \delta_a^t \)) the HVF takes the form

\[ X = X^t(t) \partial_t + X^r(r) \partial_r + X^y(y) \partial_y \equiv \dot{X} + X^y(y) \partial_y \] (15)

and the following possibilities then arise:

\begin{align*}
(A) \quad \dot{X} &= \partial_t \\
(B) \quad \dot{X} &= \partial_r \\
(C) \quad \dot{X} &= \partial_t + \partial_r
\end{align*}

The form (A) corresponds to \( u \) being tangent to the timelike homothetic orbits, \( T_3 \), (B) corresponds to the case of spacelike homothetic orbits, \( S_3 \), orthogonal to the fluid flow, and (C) is the most general (tilted) case, including also the possibility of having null homothetic orbits, \( N_3 \), when the functions \( A(t, r) \) and \( B(t, r) \) in (8) equal each other.

The homothetic equation (I) specialized to the metric (8) yields then the following possibilities:

**Case (A)**

| \( A \) | \( k \) | \( X \) | \( A^2(t, r) \) | \( B^2(t, r) \) | \( F^2(t, r) \) |
|---|---|---|---|---|---|
| I | -1, 0, 1 | \( \partial_t \) | \( e^{2t} H^2(r) \) | \( e^{2t} H^2(r) \) | \( e^{2t} f^2(r) \) |
| II | 0 | \( \partial_t - y \partial_y \) | \( e^{2nt} H^2(r) \) | \( e^{2nt} H^2(r) \) | \( e^{2(n+1)t} f^2(r) \) |

**Table 3.1**
where \( t \) has been scaled in \( AI \) so as to make \( n = 1 \) in (1). Solving now the field equations for a perfect fluid in each one of the two above cases, we find that the only possible solution for the family \( AI \) has an equation of state \( \mu + 3p = 0 \) and it admits (at least) one further Killing vector, thus being a particular case of \( \mathcal{H}_5 \) and we shall not study it here.

For \( AII \), two families arise which depend on the value of \( n \):

| \( n \in (-\infty, -3) \cup (-2, -1) \cup (0, +\infty) \) | \( n \in (-3, -2) \) |
|---|---|
| \( a(cosh(\alpha r))^{-\frac{1}{2}} \) | \( a(sinh(\alpha r))^{-\frac{1}{2}} \) |
| \( \frac{\beta}{a^{2n}} e^{-2nt H^{2(a-1)}} \) | \( \frac{\beta}{a^{2n}} e^{-2nt H^{2(a-1)}} \) |
| \( 2n \) | \( 2n \) |
| \( 2 + 3n \) | \( 2 + 3n \) |

Table 3.2

where

\[
\alpha = \frac{2(n+1)}{n+2}, \quad \beta = \frac{(n+1)(n+3)(3n+2)}{n^2(n+2)},
\]

and \( a \) is a constant. The vorticity is zero and the volume expansion \( \theta \), deceleration parameter \( q \equiv -1 - 3\dot{\theta}/\theta^2 \), acceleration \( \dot{u} \), and non-vanishing shear tensor components \( \sigma_{ab} \) can be given as:

\[
\begin{align*}
\theta &= 2 + 3n e^{-nt}, & q &= -\frac{2}{2 + 3n}, & \dot{u} &= \frac{H'}{H} \partial_r, \\
\sigma_{rr} &= -\frac{2e^{nt} H}{3}, & \sigma_{yy} &= \frac{e^{(2+n)t} f^2}{3H}, & \sigma_{zz} &= \frac{e^{(2+n)t} y^2 f^2}{3H}.
\end{align*}
\]

(20)

(21)

Notice that, depending on the value of \( n \), the solution contracts and decelerates or it expands and inflates.

With regard to the dimensionless scalars, the density parameter \( \Omega \equiv 3\mu/\theta^2 \), the dimensionless acceleration \( W \equiv \dot{u}/\theta \) and the shear parameter \( \Sigma \equiv 3\sigma^2/\theta^2 \) (see [1] for further details), one has for the first family

\[
\Omega = \frac{3(n+1)(n+3)}{(2+3n)n^2(n+2)} \frac{1}{\cosh^2(\alpha r)}, \quad W = \frac{1}{2 + 3n} \tanh(\alpha r),
\]

(22)

the models then being accelerated dominated at large distances; whereas for the second family

\[
\Omega = -\frac{3(n+1)(n+3)}{(2+3n)n^2(n+2)} \frac{1}{\sinh^2(\alpha r)}, \quad W = \frac{1}{2 + 3n} \coth(\alpha r),
\]

(23)

thus being asymptotically spatially homogeneous. In both cases \( \lim_{r \to \infty} \Omega = 0 \), (vacuum dominated models) and \( \Sigma = 1/(2 + 3n)^2 \), as this quantity is non-vanishing for all possible values of \( n \), the models have no isotropic limit.

Case (B)
where \( r \) in case \( BI \) has been re-scaled so as to have \( n = 1 \).

Solving now the field equations for a perfect fluid source, one has:

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Domain} & \text{\( H(t) \)} & \text{\( f^2(t) \)} & \text{\( \mu \)} & \text{\( \partial_r - y \partial_y \)} e^{2nr} H^2(t) \\
\hline
k = \pm 1 & 1 & e^{2t} + \frac{k}{2} & \frac{k^2}{4e^{2t}f^4} & t : e^{2t} + \frac{k}{2} > 0 \\
\hline
k = 0, \pm 1 & 1 & \alpha \sinh 2t + \frac{k}{2} & \frac{k^2 + 4\alpha^2}{4e^{2t}f^4} & t : \alpha \sinh 2t + \frac{k}{2} > 0 \\
\hline
k = \pm 1 & 1 & \alpha \cosh 2t + \frac{k}{2} & \frac{k^2 - 4\alpha^2}{4e^{2t}f^4} & \alpha \in (-\frac{1}{2}, \frac{1}{2}) \\
\hline
\end{array}
\]

where \( \alpha \) is an arbitrary constant, which must be different from zero to prevent the occurrence of further Killing vectors. \( k \neq 0 \) in both the first and last cases since otherwise one would have a vacuum solution in the former case, and negative energy density in the latter. In all three cases, the fluid is irrotational and has a stiff equation of state, i.e., \( \gamma = 2 \). For these solutions one has

\[
\begin{align*}
\dot{\theta} &= \frac{2\dot{f}}{e^r f}, \quad q = \frac{1}{2} - \frac{3}{2} \frac{f\dot{f}}{(f)^2}, \quad \dot{u} = -\partial_r, \quad W = \frac{f}{2f}, \\
\sigma_{rr} &= -\frac{2e^r \dot{f}}{3f}, \quad \sigma_{yy} = \frac{e^r \dot{f}}{3}, \quad \sigma_{zz} = \frac{e^r \Sigma^2 f \dot{f}}{3},
\end{align*}
\]

a dot indicating a derivative with respect to \( t \). In all cases the shear parameter is \( \Sigma = 1/4 \), thus there is no isotropic limit. For all solutions in \( BI \) \( \lim_{t \to \infty} W = 1/2 \), thus corresponding to accelerated dominated models, hence with no FRW limit.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Domain} & \text{\( H(t) \)} & \text{\( f^2(t) \)} & \text{\( \mu \)} \\
\hline
n < -3 & a[S]^{\frac{1}{2(n+1)}} & S & \frac{(n+1)(n+3)}{a^2 n^2} e^{-2n} [S]^{-\frac{2n+3}{n+1}} \\
\hline
n > -1 & a[S]^{\frac{1}{2(n+1)}} & C & \frac{(n+1)(n+3)}{a^2 n^2} e^{-2n} [C]^{-\frac{2n+3}{n+1}} \\
\hline
-3 < n < -1 & a[S]^{\frac{1}{2(n+1)}} & C & \frac{(n+1)(n+3)}{a^2 n^2} e^{-2n} [C]^{-\frac{2n+3}{n+1}} \\
\hline
\end{array}
\]

where

\[
a = \text{constant}, \quad S = \sinh(2(n+1)t), \quad C = \cosh(2(n+1)t).
\]

In both cases the fluid is irrotational, its equation of state being \( p = \mu \), and one has:

\[
\dot{u} = n\partial_r, \quad \theta = e^{-nr} H^{-1} \left( \frac{\dot{H}}{H} + 2\frac{\dot{f}}{f} \right), \quad \Sigma = \frac{n^2}{(2n+3)^2}.
\]
As for the respective dimensionless scalars

\[ \Omega = \frac{3(n+1)(n+3)}{n^2(2n+3)^2} \frac{1}{C^2}, \quad W = \frac{n}{2n+3} \frac{S}{C}, \]  

(28)

\[ q = \frac{-(4n^2+6n+1)}{(2n+3)^2} \frac{C^2}{S^2}, \]  

(29)

for the first case, and

\[ \Omega = -\frac{3(n+1)(n+3)}{n^2(2n+3)^2} \frac{1}{S^2}, \quad W = \frac{n}{2n+3} \frac{C}{S}, \]  

(30)

\[ q = -\frac{-(4n^2+6n+1)}{(2n+3)^2} \frac{S^2}{S^2}, \]  

(31)

for the second case. So, the solutions behave in different ways depending on the value of the parameter \( n \) and the range of values of \( t \) considered.

**Case \((C)\)**

| \( C \) | \( k \) | \( X \) | \( A^2(t, r) \) | \( B^2(t, r) \) | \( F^2(t, r) \) |
|---|---|---|---|---|---|
| Ⅰ | \(-1, 0, 1\) | \( \partial_t + \partial_r \) | \( e^{t+r}H^2(t-r) \) | \( e^{t+r}L^2(t-r) \) | \( e^{t+r}f^2(t-r) \) |
| Ⅱ | 0 | \( \partial_t + \partial_r - y\partial_y \) | \( e^{n(t+r)}H^2(t-r) \) | \( e^{n(t+r)}L^2(t-r) \) | \( e^{(n+1)(t+r)}f^2(t-r) \) |

**Table 3.6**

where \( t \) and \( r \) in \( CI \) have been re-scaled so as to make \( n = 1 \).

Unfortunately, no general solutions to the field equations for a perfect fluid can be given in these cases, and as it turns out when trying to solve them in particular cases, most of the solutions thus found hold only on some open domains of the manifold of the form \( t - r > \text{constant} \), nevertheless we next give two solutions that are valid over the whole space-time manifold, both of them corresponding to the case \( CII \):

\((CII, n = -3)\)

\[ f = 1, \quad H^2 = \alpha e^{(\beta-4)(t-r)}, \quad L^2 = \alpha e^{\beta(t-r)}. \]  

(32)

Then one has

\[ \mu = p = e^{3(t+r)} \alpha^{-1}[(4 - \beta)e^{4-\beta(t-r)} + \beta e^{-\beta(t-r)}], \]  

(33)

where \( \alpha \) and \( \beta \) are constants which can easily be chosen so as to make \( \mu > 0 \) all over \( M \). The dimensionless scalars are

\[ \Omega = \frac{12}{(\beta - 7)^2} \left[ 4 - \beta + \beta e^{-4(t-r)} \right], \quad \theta = \frac{\beta - 7}{2\sqrt{H}e^{-\frac{1}{2}(t+r)}}, \]  

(34)

\[ W = \frac{1 - \beta}{\beta - 7} e^{-2(t-r)}, \quad \Sigma = \frac{(1 - \beta)^2}{(\beta - 7)^2}. \]  

(35)
(CII, $n = 1$)

\[ f = 1, \quad H^2 = \alpha L^2, \quad L^2 = \exp \left[ (1 - \alpha)(t - r) - \frac{\beta(1 - \alpha)}{2} e^{-2(t-r)/(1-\alpha)} \right], \quad (36) \]

and then

\[ (\mu - p)e^{t+r}H^2 = 4(1 - \alpha), \quad (37) \]
\[ (\mu + p)e^{t+r}H^2 = 2(1 - \alpha) \left[ (1 - \alpha) + \beta e^{-2(t-r)/(1-\alpha)} \right], \quad (38) \]

where again $\alpha$ and $\beta$ are constants. The energy conditions restrict $\alpha$ to values $0 < \alpha < 1$, and if we demand that the solution be valid over the whole manifold, then $\beta$ must be positive. Notice that in this case, for $\beta \neq 0$, there is no equation of state of the form $p = p(\mu)$. For $\beta = 0$ the solution is an special case of $H_5$. The dimensionless scalars are

\[ \Omega = \frac{3(1 - \alpha) \left[ 3 - \alpha + \beta e^{-\frac{2}{1-\alpha}(t-r)} \right]}{\left[ 3 - \frac{\alpha}{2} + \frac{\beta}{2} e^{-\frac{2}{1-\alpha}(t-r)} \right]^2}, \quad \theta = \frac{3 - \frac{\alpha}{2} + \frac{\beta}{2} e^{-\frac{2}{1-\alpha}(t-r)}}{\sqrt{H e^{\frac{1}{2}}}}, \quad (39) \]

\[ W = \sqrt{\alpha} \left[ -1 + \frac{6}{6 - \alpha + \beta e^{-\frac{2}{1-\alpha}(t-r)}} \right], \quad \Sigma = \left[ -1 + \frac{6}{6 - \alpha + \beta e^{-\frac{2}{1-\alpha}(t-r)}} \right]^2. \quad (40) \]

In both cases there is no FRW limit.

As a final remark to this section, notice that expressions appearing in tables 3.1, 3.3 and 3.6 are completely general, i.e., valid regardless the material content.

### 4 The $H_3$ case

In this section we extend a previous work [33], correct some errors and present new solutions.

The existence of a 3-parameter homothetic group $H_3$, implies that of a $G_2 \subset H_3$ of isometries being the dimensions of their orbits three and two respectively (see section 2.1). When the Killing subalgebra has null orbits, the metric is of Kundt’s class [49] and perfect fluids are excluded [1]. We shall therefore assume in the sequel that the Killing orbits are non-null.

A classification of all such space-times in terms of the Bianchi type of the homothetic algebra can be found in [33]. See also the references cited therein for a (partial) account of papers on this issue. We can summarize the results concerning the topology of the Killing orbits and the Bianchi type of $H_3$ in the following table:

| $G_2$ – type | $G_2$ – orbits | $H_3$ – Bianchi type |
|--------------|----------------|---------------------|
| $G_2I$       | $S^1 \times \mathbb{R}$ | $I, II, III$        |
|              | $\mathbb{R}^2$ | $I, II, III, IV, V, VI, VII$ |
| $G_2II$      | $\mathbb{R}^2$ | $III$               |

Table 4.1
As the above table shows, two different topologies are possible in the Abelian case \((G_2 I)\) \([33]\) namely, \(V_2\) diffeomorphic to \(\mathbb{R}^2\) or to \(S^1 \times \mathbb{R}\); and it follows in the latter case that the only possible Bianchi types for \(\mathcal{H}_3\), irrespectively of the assumed matter content, are \(I, II\) and \(III\) (this holds also if the \(\mathcal{H}_3\) is replaced by a conformal algebra \(\mathcal{C}_3\), see \([33]\)); and for the case \(V_2 \cong \mathbb{R}^2\), the seven soluble Bianchi types can occur. For the non-Abelian case the only possible homothetic algebra is of the Bianchi type \(III\), and its orbits are diffeomorphic to \(\mathbb{R}^2\).

### 4.1 Case \(G_2\) Abelian

In this subsection we shall restrict ourselves to the Abelian case with spacelike isometric orbits diffeomorphic to \(\mathbb{R}^2\), giving appropriate “translation rules” for the other possibilities. Furthermore, we shall assume that the Killing orbits admit orthogonal two-surfaces (i.e., orthogonally transitive \(G_2\) metrics). Cosmological models admitting an Abelian \(G_2\) on spacelike orbits have been studied by Ruiz and Senovilla \([50]\) and Van den Berg and Skea \([52]\) among others. The non-orthogonally transitive case, Wainwright’s classes \(A(i)\) and \(A(ii)\), have been studied in \([53]\) and \([54]\) respectively.

Adapting two coordinates to two commuting KVs, say \(\xi = \partial_x\) and \(\eta = \partial_y\), and choosing two other coordinates, \(t\) and \(z\), on the surfaces orthogonal to the isometry orbits; it follows that the line element can be written in the form (see for instance \([55]\))

\[
ds^2 = -Adt^2 + Bdz^2 + R\left[F(dx + Wdy)^2 + F^{-1}dy^2\right],
\]

where \(A, B, R, F,\) and \(W\) are all functions of \(t\) and \(z\) alone.

All the other cases (i.e., timelike Killing orbits and Killing orbits diffeomorphic to \(S^1 \times \mathbb{R}\) of either nature, spacelike or timelike) can be formally obtained from the above by means of the following substitutions:

\[
\begin{align*}
T_2 \cong \mathbb{R}^2 & \quad V_2 \cong S^1 \times \mathbb{R} \\
\partial_x & \mapsto \partial_t \\
(t, x) & \mapsto i(-x, t) \\
W & \mapsto iW \\
\text{Regularity condition} & \quad \text{on the axis}
\end{align*}
\]

where \(\varphi\) is the angular coordinate (with the standard periodicity \(2\pi\)). Regarding solutions with an Abelian \(G_2\) acting on timelike orbits, (including the astrophysically relevant stationary and axisymmetric models, which have been studied for many years), it is worth mentioning that they have attracted renewed attention, see \([37, 38, 39, 40]\).

It is easy to see from the commutation relations of the proper HVF, \(X\), with \(\xi\) and \(\eta\), and the homothetic equation specialized to the components \(g_{tx}, g_{ty}, g_{zx}\) and \(g_{zy}\) of the metric \((41)\) that, \(X\) must take the form

\[
X = X^t(t, z)\partial_t + X^z(t, z)\partial_z + X^x(x, y)\partial_x + X^y(x, y)\partial_y
\]

\[
\equiv \hat{X} + X^x(x, y)\partial_x + X^y(x, y)\partial_y,
\]

(42)
where $X^x(x, y)$ and $X^y(x, y)$ are linear functions of their arguments that yield for every different Bianchi type the following forms:

| Type | $X^x(x, y)$ | $X^y(x, y)$ |
|------|-------------|-------------|
| I    | 0           | 0           |
| II   | $y$         | 0           |
| III  | $x$         | 0           |
| IV   | $x + y$     | $y$         |
| V    | $x$         | $y$         |
| VI   | $x$         | $qy$        |
| VII  | $-y$        | $x + qy$    |

Table 4.3

As we are interested in perfect fluid solutions for the metric (11), it is always possible to perform a change of coordinates in the $t, z$ plane so as to bring the four-velocity of the fluid to a comoving form, preserving the diagonal form of the metric [55]. As a consequence, the fluid flow velocity $u$ can be written as

$$u = \frac{1}{\sqrt{A}} \frac{\partial}{\partial t}$$

or equivalently

$$u_a = (\sqrt{A}, 0, 0, 0)$$

the Einstein’s field equations taking then a much simpler form.

In this comoving coordinate chart, and taking into account the first equation in (3), it is easy to see that the part of the homothetic vector field orthogonal to the Killing orbits, $\hat{X}$, is

$$\hat{X} = X^t(t)\partial_t + X^z(z)\partial_z .$$

We can now use the remaining coordinate freedom in the $t, z$ plane [$t \rightarrow m(t)$, $z \rightarrow n(z)$] to bring $\hat{X}$ to either of the following three forms

(i) $\hat{X} = \partial_t$

(ii) $\hat{X} = \partial_r$

(iii) $\hat{X} = \partial_t + \partial_r .$

Thus, three classes of perfect fluid solutions arise, depending on the orientation of the fluid flow $u$ relative to the homothetic orbits:

(i) The fluid flow is tangent to the homothetic orbits, and they are then timelike.

(ii) The fluid flow is orthogonal to the homothetic orbits, and therefore they are spacelike.

(iii) “Tilted” fluid flow, i.e., $u$ is neither tangential to nor orthogonal to the homothetic orbits, which are then not constrained -a priori- to being timelike or spacelike, and so that their nature may vary from point to point over the space-time.
4.1.1 Fluid flow tangent to the homothetic orbits

This case, assuming the existence of two hypersurface orthogonal KVs (i.e., diagonal metric) has been thoroughly studied by Wainwright, Hewitt, and collaborators in an interesting series of articles [11, 12, 13], where the properties of these models are analyzed using the qualitative theory of plane autonomous systems, showing that (first-class) self-similar solutions within this family can represent the asymptotic states at later times of more general inhomogeneous $G_2$ models. Uggla [14] found four explicit solutions of this type. We shall not give here any explicit solution belonging to this class (see the above references and those cited therein), but rather provide the general form of the metric functions (including the non-diagonal cases) and briefly discuss the generic behaviour of the kinematical quantities associated with the fluid (acceleration, deceleration parameter, shear,...).

The metric (in comoving coordinates) for the case of four-velocity tangent to the $H_3$ orbits (i.e., form (i) of the proper homothetic vector field) takes the form:

$$ds^2 = e^{2nt} \{ -A(z)dt^2 + B(z)dz^2 + e^{at}R(z)[F(t, z)(dx + W(t, z)dy)^2 + F^{-1}(t, z)dy^2] \}$$

where $n$ is the homothetic constant. By rescaling the coordinate $z$ one can set

$$A(z) = B(z) .$$

Then, the functional form of the metric functions can be worked out for each Bianchi type. Thus, one has:

$$\begin{align*}
(1) & \quad \alpha = 0 , \quad F = f(z) , \quad W = w(z) , \\
(II) & \quad \alpha = 0 , \quad F = f(z) , \quad W = w(z) - t , \\
(III) & \quad \alpha = -1 , \quad F = e^{-t}f(z) , \quad W = e^t w(z) , \\
(IV) & \quad \alpha = -2 , \quad F = f(z) , \quad W = w(z) - t , \\
(V) & \quad \alpha = -2 , \quad F = f(z) , \quad W = w(z) , \\
(VI) & \quad \alpha = -(1+q) , \quad F = e^{-(1-q)t}f(z) , \quad W = e^{(1-q)t}w(z) , \\
(VII) & \quad \alpha = -q ,
\end{align*}$$

$$F = \frac{2}{\sqrt{4-q^2}} \left[ \sqrt{1 + c(z)^2 + g(z)^2 + c(z) \cos(\sqrt{4-q^2}t) + g(z) \sin(\sqrt{4-q^2}t)} \right],$$

$$W = \frac{q}{2} + \frac{\sqrt{4-q^2} [ -g(z) \cos(\sqrt{4-q^2}t) + c(z) \sin(\sqrt{4-q^2}t) ]}{\sqrt{1 + c(z)^2 + g(z)^2 + c(z) \cos(\sqrt{4-q^2}t) + g(z) \sin(\sqrt{4-q^2}t)}} .$$

Notice that the form of these functions again holds for any energy-momentum tensor, since no use has been made of the field equations in deducing them.

The cases studied by Wainwright and collaborators correspond to types $I$, $III$, $V$, and $VI$, since these are the only ones in which the function $W$ can be set equal to zero. For type $VII$, $W = \frac{q}{2}$ implies the existence of a further Killing vector tangent to the Killing orbits and the metric would then admit a multiply transitive group $H_i$ of homotheties. Notice also, that the diagonal cases are separable in the variables $t$, $z$,...
z, thus being special cases of the solutions studied by Ruiz and Senovilla [50] or those of Agnew and Goode [51] for \( \gamma = 2 \).

Specializing equation (43) to the matter variables \( \mu \) and \( p \), we obtain
\[
\mu = e^{-2nt}\hat{\mu}(z), \quad p = e^{-2nt}\hat{p}(z),
\] (57)
and by computing the kinematical quantities associated to the fluid velocity vector (43) for the metric (48)
\[
\theta = \frac{3n + \alpha}{e^{nt}\sqrt{A}}, \quad \mathbf{q} = -\frac{\alpha}{3n + \alpha}, \quad \dot{u} = \frac{1}{2} A' \frac{dz}{dt},
\] (58)
where a dash denotes a derivative with respect to \( z \) and \( \mathbf{q} \) here denotes the deceleration parameter. The non-vanishing components of the shear tensor are
\[
\sigma_{zz} = -e^{nt}\sqrt{A}\frac{\alpha}{3}, \quad \sigma_{xx} = e^{nt}\frac{RF}{6\sqrt{A}}\left(3\frac{\dot{F}}{F} + \alpha\right),
\] (59)
\[
\sigma_{xy} = e^{nt}\frac{RF}{6\sqrt{A}}\left(3W\frac{\dot{F}}{F} + W\alpha + 3\dot{W}\right),
\] (60)
\[
\sigma_{yy} = e^{nt}\frac{R}{6\sqrt{A}F}\left(-3\frac{\dot{F}}{F} + \alpha + 3W^2F\dot{F} + W^2F^2\alpha + 6F^2W\dot{W}\right),
\] (61)
thus, the shear scalar is
\[
\sigma^2 = \frac{3\left(\frac{\dot{F}}{F}\right)^2 + \alpha^2 + 3F^2\dot{W}^2}{12Ae^{2nt}}.
\] (62)

If one assumes that the fluid has an equation of state of the form (6), from the contracted Bianchi identities it follows
\[
\gamma(\mu + \mu\theta)u^a + \gamma\mu\dot{u}^a + (\gamma - 1)\mu,_{ba}g^{ba} = 0,
\] (63)
where \( \mu \equiv \mu,_{,t}u^t \). Contracting the above expression with \( u^a \), one gets
\[
\mu = -\gamma\mu\theta.
\] (64)
Assuming \( \theta \neq 0 \) (the case \( \theta = 0 \), although mathematically possible, is not physically interesting since it would correspond to a non-expanding universe, thus contradicting observations) and substituting (57) and (58) in equation (64), it follows
\[
\gamma = \frac{2n}{3n + \alpha}.
\] (65)
Specializing the quantities \( \gamma \) and \( \mathbf{q} \) to each Bianchi type, one gets

| Type | I | II | III | IV | V | VI | VII |
|------|---|----|-----|----|---|----|-----|
| \( \gamma \) | \( \frac{2}{3} \) | \( \frac{2}{3} \) | \( \frac{2n}{3n - 1} \) | \( \frac{2n}{3n - 2} \) | \( \frac{2n}{3n - 2} \) | \( \frac{2n}{3n - (1 + q)} \) | \( \frac{2n}{3n - q} \) |
| \( \mathbf{q} \) | 0 | 0 | \( \frac{2}{3n - 1} \) | \( \frac{2}{3n - 2} \) | \( \frac{2}{3n - 2} \) | \( \frac{1 + q}{3n - (1 + q)} \) | \( \frac{q}{3n - q} \) |
Notice that the only shear-free solution is the type I one. For types II to VI there is no limit where $\Sigma \equiv 3\sigma^2/\theta^2$ becomes zero; and for type VII, $\Sigma \to 0$ for some spatial limit if and only if $c(z) \to 0$, $g(z) \to 0$ and $q = 0$ (i.e., $F \to 1$ and $W \to 0$). Thus, only types I and VII can have solutions with a FRW limit, but in both cases $\mu + 3p = 0$ and $q = 0$, so they are not very relevant from a physical point of view.

### 4.1.2 Fluid flow orthogonal to the homothetic orbits

The case of spatially homothetic orbits was thoroughly studied by Eardley [10] where a classification scheme of these models was given and their dynamical properties were studied. Luminet [56] constructed a convenient basis of 1-forms and gave its explicit form in terms of a standard coordinate basis $\{dx^a\}$ as well as the expression of the homothetic vector in the dual basis $\{\partial/\partial x^a\}$. He also proved a theorem showing that perfect fluid models of a certain class were incomplete in the sense of Hawking and Ellis [57].

For the sake of completeness, we just mention that the form of the metric, corresponding to a homothetic vector field of the form (ii), can be formally obtained from (48) to (56) by simply reversing the roles of the coordinates $t$ and $z$.

The expressions of the acceleration, expansion, deceleration parameter and shear scalar are given by

\[
\dot{u} = ndz, \quad \theta = \frac{1}{2e^{nz}\sqrt{A}} \left\{ \frac{\dot{A}}{A} + 2\frac{\dot{R}}{R} \right\},
\]

\[
q = \frac{2 \left[ 4 \left( \frac{\dot{A}}{A} \right)^2 + \frac{\dot{A}^2}{A^2} + 4 \left( \frac{\dot{R}}{R} \right)^2 - 3 \frac{\dot{A}}{A} - 6 \frac{\dot{R}}{R} \right]}{\left( \frac{\dot{A}}{A} + 2\frac{\dot{R}}{R} \right)^2},
\]

\[
\sigma^2 = \frac{3 \left( \frac{\dot{A}}{A} - \frac{\dot{R}}{R} \right)^2 + 3 \left( \frac{\dot{F}}{F} \right)^2 + 3F^2(W)^2}{12Ae^{2nz}}.
\]

In this case, the homothetic vector field, $X$, and the four-velocity $u$ are mutually orthogonal, thus if a barotropic equation of state is assumed, then necessarily $p = \mu$, i.e., stiff-matter and by simple inspection of the field equations this is seen to be equivalent to:

\[
\frac{\ddot{R}}{R} = (2n + \alpha)^2.
\]

### 4.1.3 “Tilted” fluid flow

Finally, the form (iii) for the proper homothetic vector field is precisely the case we are currently interested in, namely, $u$ not tangent nor orthogonal to the homothetic orbits.
Specializing now the homothetic equation to the metric (41) we obtain
\[
 ds^2 = e^{n(t+z)} \left\{ -A(t-z) dt^2 + B(t-z) dz^2 \\
 + e^{\alpha t} R(t-z) \left[ F(dx + W dy)^2 + \frac{F^{-1}}{2} dy^2 \right] \right\}, 
\]
(70)
where again \( n \) is the homothetic constant. The parameter \( \alpha \) and the functional form of the metric functions for each Bianchi type, are those given by (50)-(56) after effecting the following substitutions
\[
t \mapsto \frac{t + z}{2}, \quad z \mapsto t - z.
\]
(71)
The kinematical quantities being
\[
 \dot{u} = \frac{1}{2} \left( n - \frac{A'}{A} \right) dz, \quad \theta = \frac{3n + \alpha + B' + 2 R'}{2 e^{n(t+z)/2} A},
\]
(72)
\[
 \sigma^2 = \left( \frac{\alpha}{2} - \frac{B'}{B} + \frac{R'}{R} \right)^2 + 3 \left( \frac{F}{F'} \right)^2 + 3 F^2 (W_t)^2 \frac{12 e^{n(t+z)} A}{e^{n(t+z)} A},
\]
(73)
where a dash denotes here a derivative with respect to \( t - z \). A careful study shows that there are no shear-free solutions in this case admitting a maximal group \( H_3 \) of homotheties: shear-free solutions are not possible for types \( II \) and \( IV \); for the Bianchi types \( I \), \( V \) and \( VII \), the shear-free condition implies that the functions \( F \) and \( W \) must be constants and therefore a further Killing vector tangent to the Killing orbits occurs; for types \( III \) and \( VI \), \( W \) must vanish, the type \( III \) solution then being a homogeneous Bianchi \( VI \) model, and the type \( VI \) one such that \( \mu + p = 0 \), thus not corresponding to a perfect fluid.

Notice that in [33], the homothetic constant \( n \) was set equal to 1 from the beginning, before choosing the coordinates in the surfaces orthogonal to the Killing orbits. By doing so, some solutions were left out, since, although one can always re-scale the proper HVF \( X \) with a factor \( 1/n \), such an scaling can not always be reabsorbed by a redefinition of the coordinates. We correct here that error.

As was pointed out before, all diagonal \( (W = 0) \), perfect fluid solutions (admitting an orthogonally transitive Abelian \( G_2 \) with flat spacelike orbits) and such that the metric functions \( A, B, R \) and \( F \) are separable in the variables \( t \) and \( z \) are already known [50, 51].

We shall next present some new exact solutions not included in [33], which have been obtained assuming \( W = 0 \) (diagonal), but which are not of separable variables in the above sense.

**Type III:**
\[
 R = 1 , \quad f = e^{\lambda(t-z)} , \quad \frac{A}{B} = \frac{1 - 2\lambda}{1 + 2\lambda} , \quad A = a \exp \left( n(t-z) + \frac{\alpha}{2 - n} e^{(\frac{1}{2} - n)(t-z)} \right) .
\]
(74)
where \(0 < \lambda < 1/2\) and \(\lambda^2 + n^2 = 1/2\) and

\[
(\mu - p)Ae^{n(t+z)} = \frac{4\lambda}{1 + 2\lambda} \left(\frac{1}{2} - n\right)^2,
\]

\[
(\mu + p)Ae^{n(t+z)} = \frac{4\lambda}{1 + 2\lambda} \left(n - \frac{1}{2}\right) \frac{A'}{A},
\]

\(\lambda, \alpha\) and \(a\) are constants. In order to satisfy the energy-conditions, if \(n > 0\), \(\alpha\) must also be positive and the value of \(n\) is then restricted to \((1/2, \sqrt{2}/2)\); for negative values of \(n\), \(\alpha\) is also negative and \(n \in (-\sqrt{2}/2, -1/2)\), in these cases a barotropic equation of state does not exist. When \(\alpha = 0\) the solution is a particular case of a homogeneous Bianchi VI model.

**Type III:**

\[
R = 1, \quad f = e^{\lambda(t-z)}, \quad (75)
\]

\[
\frac{A}{B} = e^{(2n-1)(t-z)}, \quad \lambda^2 = (n-1)^2, \quad n \neq \frac{1}{2},
\]

\[B = a \exp \left(\frac{-\lambda + 2(n-1)^2}{2n-1}(t-z)\right) \left[\exp \left(-2n(t-z)\right) - 1\right]^c,
\]

where \(a\) and \(c\) are constants

\[
\mu = p = \frac{1}{2A} \left[2n - \frac{3 + c}{2}\right] e^{-n(t+z)}.
\]

**Type III:**

\[
R = e^{\lambda(t-z)}, \quad f = e^{-\frac{\lambda}{2}(t-z)}, \quad A = a^2 e^{(2\lambda + 2n-1)(t-r)}, \quad B = b^2, \quad (76)
\]

where \(a\) and \(b\) are constants, \(\lambda = -(2n-1) + \sqrt{3n^2 - 2n + 1/4}\), and

\[
(\mu - p)e^{n(t+z)} = \frac{1}{B} \lambda(2n - 1 - 2\lambda),
\]

\[
(\mu + p)e^{n(t+z)} = \frac{1}{A} \left(n^2 + \frac{\lambda}{2} - \frac{3}{4}\right) + \frac{1}{B} \lambda(\lambda + 2n - \frac{3}{2}).
\]

From where it follows that, in order that the solution satisfies the weak and dominant energy conditions all over the manifold, one must have

\[n \in (0.8210368162407501..., \frac{3}{2}),\]

and, again, a barotropic equation of state \(p = p(\mu)\) does not exist, except in the case \(n = 3/2\), when \(\mu = p\).

**Type V:**

\[
ds^2 = \frac{e^{t+z}}{f^2} |\varphi|^{2c^2 + 2c} \left\{\frac{-M^4 \varphi^2 dt^2 + dz^2}{M^4 \varphi^2 - 1}\right\} + |\varphi|^{-2c} dx^2 + |\varphi|^{2c+2} dy^2, \quad (77)
\]
\[ \mu = p = \frac{f_o^2}{e^{t+z}} \frac{M^4 \varphi^2 - 1}{2M^2 |\varphi|^{2\gamma+2\varepsilon+2}}, \quad n = 1, \]

where \( c, M \) and \( f_o \) are constants, and \( \varphi \) is a function of \( t - z \) given implicitly by

\[ M^2(t - z) = \ln |\varphi| - \frac{M^4}{2} \varphi^2. \]

Type V:

\[ R = 1, \quad f = \epsilon^{\lambda(t-z)}, \quad A = \epsilon^{2(n-1)(t-z)}, \quad \lambda^2 = (n-1)(n-3), \]

\[ A = a \epsilon^{(n+1)(t-z)} \left[ 1 - \epsilon^{-2(n-1)(t-z)} \right]^{\frac{\epsilon}{2(n-1)}}, \]

\[ p = \mu = \frac{(n-1)(4-c)}{2A} e^{-n(t+z)}, \]

with \( a \) and \( c \) constants, and \( n \) restricted to being \( n < 1 \) or \( n > 3 \) in order to satisfy the energy conditions.

For type VI two solutions have been obtained. For the sake of simplicity, we will give them in non-comoving coordinates. Thus

\[ ds^2 = \frac{e^{2t}}{F^2} \left\{ -dt^2 + dz^2 + e^{-2t} B^2 dx^2 + e^{-2qt} S^2 dy^2 \right\}, \quad (79) \]

\[ u_t = -\frac{e^t}{F} \cosh a, \quad u_z = \frac{e^t}{F} \sinh a. \quad (80) \]

The first solution is

\[ F = f_o B, \quad S = C^{1 - \frac{\alpha \sin c}{1 - q}} E, \quad B = C^{-\frac{\alpha \sin c}{1 - q}} E, \quad (81) \]

\[ \mu = p = e^{-2t} 2q(1 - q) f_o^2 \alpha C^{-2 - \frac{2 \sin c}{1 - q}} (\alpha \cos c + 2 \alpha \sin c) E^2, \]

where

\[ A = \alpha^2 e^{(1-q)z} - e^{-(1-q)z}, \quad C = \alpha^2 e^{(1-q)z} + e^{-(1-q)z}, \]

\[ E = \exp \left[ \frac{2q \cos c}{1 - q} \tan^{-1} \left( \alpha e^{(1-q)z} \right) \right], \]

and

\[ \cosh a = \frac{\sqrt{1 + \sin c} C}{\sqrt{4\alpha A \cos c + 8\alpha^2 \sin c}}, \quad \sinh a = \frac{-(1 + \sin c) A + 2\alpha \cos c}{\sqrt{1 + \sin c} \sqrt{4\alpha A \cos c + 8\alpha^2 \sin c}}, \]

where \( c, \alpha \) and \( f_o \) are constants. In order to have positive energy density, the parameter \( q \) is restricted to \( q \in (0, 1) \). A particular, simpler case can be obtained by choosing \( \cos c = 0 \) and \( \sin c = 1 \).
The other solution is

\[ F = f_o B, \quad S = R^{1+\frac{q^2}{c^2} T^{1+\frac{c}{q}}} , \quad B = R^{\frac{q^2}{c^2} T^{1+\frac{c}{q}}} , \quad (82) \]

\[ \mu = p = e^{-2t} \frac{q - 1}{c} \alpha^2 f_o \left[ c^2 R^2 - q^2 T^2 \right] R^{1+\frac{q^2}{c^2} T^{1+\frac{c}{q}}} , \]

where

\[ R = e^{\frac{1-q}{2} z} - \alpha^2 e^{-\frac{1-q}{2} z} , \quad T = e^{\frac{1-q}{2} z} + \alpha^2 e^{-\frac{1-q}{2} z} , \]

and

\[ \cosh a = \frac{(c - q) T R}{2 \alpha \sqrt{c^2 R^2 - q^2 T^2}} , \quad \sinh a = \frac{q T^2 - c R^2}{2 \alpha \sqrt{c^2 R^2 - q^2 T^2}} \]

again \( c, \alpha \) and \( f_o \) are constants. Notice that the solution is only valid for

\[ c^2 R^2 - q^2 T^2 > 0 . \]

Notice that the solutions that have a stiff matter equation of state \((p = \mu)\) can be derived from vacuum solutions (also admitting an Abelian \( G_2 \)) using a method proposed by Wainwright et al. [58]

### 4.2 Non-Abelian \( G_2 \)

For a non-Abelian \( G_2 \), a local system of coordinates can be chosen in which the Killing vectors, say \( \xi \) and \( \eta \), are

\[ \xi = \partial_1 , \quad \eta = x^1 \partial_1 + \partial_2 . \quad (83) \]

Now, supposing the existence of a proper homothetic vector field, say \( X \), one can see that the only allowed Bianchi type for \( H_3 \) is \( III \), i.e.:

\[ [\xi, \eta] = \xi , \quad [\xi, X] = 0 , \quad [\eta, X] = 0 . \quad (84) \]

Taking into account (83) and (84) one easily comes to the following form of \( X \):

\[ X^a = (e^{x_2} X^1(x^3, x^4), X^2(x^3, x^4), X^3(x^3, x^4), X^4(x^3, x^4)) . \quad (85) \]

Note that, since the orbits associated with \( H_3 \) and \( G_2 \) can not coincide [8], the components \( X^3 \) and \( X^4 \) of the homothetic vector field cannot both vanish. Also, notice that the homothetic constant can in this case be set equal to one without altering our choice of coordinates.

#### 4.2.1 The orthogonally transitive case

As in the previous Abelian case, we will restrict our attention just to the orthogonally transitive \( G_2 \) metrics. Regarding non-orthogonally transitive \( G_2 \), a discussion can be found in [59] where a study of perfect fluid solutions with four-velocity orthogonal to the isometric orbits is given. In that reference it is also assumed that the Killing vector \( \xi \) is hypersurface orthogonal and the homothetic vector field, \( X \), is orthogonal to the fluid velocity. These assumptions imply that the fluid is to be “stiff” \((p = \mu)\)
without any a priori assumption of an equation of state. No explicit solutions are known so far, but it is shown that solutions with pressure and matter positive on an open set can in principle exist by suitably specifying the initial conditions.

For orthogonally transitive $G_2$, the metric can be written as

$$g_{ab} = \begin{pmatrix} e^{-2x^2}a_{11} & e^{-x^2}a_{12} & 0 & 0 \\ e^{-x^2}a_{12} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix} \tag{86}$$

where $\epsilon = \pm 1$ and $a_{ij} = a_{ij}(x^3, x^4)$.

For this case, one can see that, assuming non-null homothetic orbits $V_3$, it is always possible to perform a coordinate change in the two-spaces orthogonal to the Killing orbits, such that it brings the homothetic vector field to the form

$$X^a = (e^{x^2}X^1(x^3, x^4), X^2(x^3, x^4), 1, 0) \tag{87}$$

and the line element can be written as

$$ds^2 = A(e^{-x^2}dx^1 + Wdx^2)^2 + B(dx^2)^2 + F((dx^3)^2 + \epsilon(dx^4)^2) \tag{88}$$

where $A$, $B$, $F$, and $W$ are functions of $x^3$ and $x^4$ alone.

Specializing now the homothetic equation to the homothetic vector (87) and the metric (88) yields the following forms for $X$ and the metric functions

$$X^a = (\alpha e^{x^2}, n, 1, 0) \quad \alpha, n = \text{const} \tag{89}$$

$$A = e^{2(1+n)x^3}a(x^4), \quad B = e^{2x^3}b(x^4), \quad F = e^{2x^3}f(x^4) \tag{90}$$

and

$$W = \begin{cases} -\frac{\alpha}{n} e^{-nx^3}w(x^4) & n \neq 0 \\ -\alpha x^3 + w(x^4) & n = 0 \end{cases} \tag{91}$$

In the case $n \neq 0$, one can still perform the coordinate change

$$\hat{x}^1 = x^1 - \frac{\alpha}{n}e^{x^2} \tag{92}$$

so that the homothetic vector field and $W$ take the forms

$$X^a = (0, n, 1, 0) \quad W = e^{-nx^3}w(x^4) \tag{93}$$

It can be easily shown, just by computing the Einstein and Riemann tensors, that the only vacuum solution with a non-Abelian group of isometries acting orthogonally transitively and admitting a proper homothetic vector field, is Minkowski space-time.

For perfect fluid solutions, equation (6) specialized to the Killing vectors (83) implies for the fluid velocity

$$u_a = \begin{pmatrix} e^{-x^2}u_1(x^3, x^4), u_2(x^3, x^4), u_3(x^3, x^4), u_4(x^3, x^4) \end{pmatrix} \tag{94}$$
For orthogonally transitive $G_2$ models, it is possible to perform a change of coordinates in the $x^3$, $x^4$ plane so as to write the four velocity and the metric as

$$u_a = \left(e^{-x^2 \bar{u}_1(x^3,x^4), \bar{u}_2(x^3,x^4), 0, \bar{u}_4(x^3,x^4)}\right),$$  \hfill (95)

$$ds^2 = A(x^3,x^4)(e^{-x^2} dx^1 + W(x^3,x^4) dx^2)^2 + B(x^3,x^4)(dx^3)^2 + G(x^3,x^4)(dx^4)^2,$$  \hfill (96)

and, as a consequence, the field equations take on a much simpler form. In this coordinate chart and taking into account (3) and (1) specified to the components $g_{13}$, $g_{14}$, $g_{23}$ and $g_{24}$, it is easy to see that $X$ must be of the form

$$X^a = (\alpha e^{x^2}, n, X^3(x^3), X^4(x^4)),$$  \hfill (97)

and the following possibilities then arise

$$\begin{align*}
\text{I} & \quad X^a = (\alpha e^{x^2}, n, 0, 1), \\
\text{II} & \quad X^a = (\alpha e^{x^2}, n, 1, 0), \\
\text{III} & \quad X^a = (\alpha e^{x^2}, n, 1, 1),
\end{align*}$$

In the three cases, for $n \neq 0$ one can perform the coordinate change (24) thus enabling one to set $\alpha$ zero.

The homothetic equation specialized to the metric (96) yields then the following possibilities

| type | $X^a$ | $A(x^3,x^4)$ | $W(x^3,x^4)$ |
|------|-------|-------------|-------------|
| I,a  | $(0,n,0,1)$ | $e^{2(1+n)x^3}a(x^3)$ | $e^{-n x^3}w(x^3)$ |
| I,b  | $(\alpha e^{x^2},0,0,1)$ | $e^{2x^3}a(x^3)$ | $-\alpha x^3 + w(x^3)$ |
| II,a | $(0,n,1,0)$ | $e^{2(1+n)x^3}a(x^4)$ | $e^{-n x^4}w(x^4)$ |
| II,b | $(\alpha e^{x^2},0,1,0)$ | $e^{2x^3}a(x^4)$ | $-\alpha x^3 + w(x^4)$ |
| III,a | $(0,n,1,1)$ | $e^{-(1+n)(x^3+x^4)}a(x^3-x^4)$ | $e^{-\frac{\alpha}{2}(x^3+x^4)}w(x^3-x^4)$ |
| III,b | $(\alpha e^{x^2},0,1,1)$ | $e^{x^3+x^4}a(x^3-x^4)$ | $-\frac{\alpha}{2}(x^3 + x^4) + w(x^3 - x^4)$ |

**Table 4.5**

| type | $B(x^3,x^4)$ | $F(x^3,x^4)$ | $G(x^3,x^4)$ |
|------|-------------|-------------|-------------|
| I    | $e^{2x^3}b(x^3)$ | $e^{2x^3}f(x^3)$ | $e^{2x^3}f(x^3)$ |
| II   | $e^{2x^3}b(x^4)$ | $e^{2x^3}f(x^4)$ | $e^{2x^3}f(x^4)$ |
| III  | $e^{x^3+x^4}b(x^3-x^4)$ | $e^{x^3+x^4}f(x^3-x^4)$ | $e^{x^3+x^4}g(x^3-x^4)$ |

**Table 4.6**
and for the velocity field one has

| type  | $\tilde{u}_1(x^3, x^4)$ | $\tilde{u}_2(x^3, x^4)$ | $\tilde{u}_4(x^3, x^4)$ |
|-------|--------------------------|--------------------------|--------------------------|
| I.a   | $e^{(1+n)x^3} \tilde{u}_1(x^3)$ | $e^{x^3} \tilde{u}_2(x^3)$ | $e^{x^4} \tilde{u}_4(x^3)$ |
| I.b   | $e^{-x^2} \tilde{u}_1(x^4)$ | $-\alpha x^3 e^{x^3} \tilde{u}_1(x^3) + e^{x^3} \tilde{u}_2(x^3)$ | $e^{x^4} \tilde{u}_4(x^3)$ |
| II.a  | $e^{(1+n)x^3} \tilde{u}_1(x^4)$ | $e^{x^3} \tilde{u}_2(x^4)$ | $e^{x^4} \tilde{u}_4(x^4)$ |
| II.b  | $e^{-x^2} \tilde{u}_1(x^4)$ | $-\alpha x^3 e^{x^3} \tilde{u}_1(x^3) + e^{x^3} \tilde{u}_2(x^4)$ | $e^{x^4} \tilde{u}_4(x^4)$ |
| III.a | $e^{\frac{1-n}{2} (x^3 + x^4)} \tilde{u}_1(v)$ | $e^{\frac{x^3}{2}} \tilde{u}_2(v)$ | $e^{\frac{x^3 + x^4}{2}} \tilde{u}_4(v)$ |
| III.b | $e^{\frac{x^3 + x^4}{2}} \tilde{u}_1(v)$ | $-\alpha x^3 e^{x^3} \tilde{u}_1(v) + e^{\frac{x^3 + x^4}{2}} \tilde{u}_2(v)$ | $e^{\frac{x^3 + x^4}{2}} \tilde{u}_4(v)$ |

Table 4.7

where the subcases $a$ and $b$ refer to whether $n \neq 0$ or $n = 0$ respectively, and $v$ stands for $x^3 - x^4$.

Since $u_3 = 0$ in this coordinate chart, the components of the Einstein tensor $G_{13}$, $G_{23}$ and $G_{34}$ must vanish identically, hence

$$W_{,3} = 0,$$  \hspace{1cm} (101)

$$\frac{A_3}{A} = \frac{B_3}{B},$$ \hspace{1cm} (102)

$$0 = \frac{B_3}{B} \left( -\frac{1}{4} F_{\frac{A}{4}} + \frac{1}{2} F_{\frac{A}{4}} + \frac{3}{4} B_{\frac{A}{4}} \right) + \frac{1}{4} G_{\frac{A}{4}} \left( \frac{A_4}{A} + \frac{B_4}{B} \right) - \frac{B_{\frac{A}{4}}}{B}. \hspace{1cm} (103)$$

where $\cdot$ means derivative with respect $x^i$. From these equations we found more explicit forms for the metric functions, that are given by:

**Case I.a**

$$ds^2 = e^{2x^4} \left\{ k e^{2nx^4} \left[ f(x^3) \right]^{\frac{2+n}{n}} \left( e^{-x^2} dx^1 + w e^{-nx^4} dx^2 \right)^2 + b \left[ f(x^3) \right]^{\frac{2+n}{n}} (dx^2)^2 \right. $$

$$+ f(x^3) \left( (dx^3)^2 + \epsilon (dx^4)^2 \right) \right\}, \hspace{1cm} (104)$$

where $k = \pm 1$, $w$ and $b$ are arbitrary constants. Note that $n$ must be different from zero or a third Killing vector occurs and the metric becomes then LRS. In such a case the metric functions $a(x^3)$ and $b(x^3)$ are still proportional to each other, but no relation exits, in principle, with $f(x^3)$.

**Case I.b**

$$ds^2 = e^{2x^4} \left\{ a(x^3) \left( e^{-x^2} dx^1 - \alpha x^3 dx^2 \right)^2 + b a(x^3) (dx^2)^2 + (dx^3)^2 + \epsilon (dx^4)^2 \right\}, \hspace{1cm} (105)$$

where $b$ is an arbitrary constant different from zero to prevent a non-singular metric, and $\alpha \neq 0$ if the group $G_2$ is to be maximal.

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Case II
Only one possibility arises in this case, namely $\alpha = n = 0$; thus the line element becomes
\[ ds^2 = e^{2x^3} \left\{ a(x^4) \left( e^{-x^2} dx^1 + w(x^4) dx^2 \right)^2 + b(x^4) \left( dx^2 \right)^2 + (dx^3)^2 + \epsilon (dx^4)^2 \right\} , \] (106)

Case III.a
\[ ds^2 = e^{x^3 + x^4} \left\{ e^{n(x^3 + x^4)} a(v) \left( e^{-x^2} dx^1 + w e^{-nx^4} dx^2 \right)^2 + be^{n(x^3 - x^4)} a(v) \left( dx^2 \right)^2 + f(v) \left( dx^3 \right)^2 + g(v) \left( dx^4 \right)^2 \right\} , \] (107)

$w$ being an arbitrary constant and $v = x^3 - x^4$.

Case III.b
\[ ds^2 = e^{x^3 + x^4} \left\{ a(v) \left[ (e^{-x^2} dx^1 - \alpha x^4 dx^2)^2 + b \left( dx^2 \right)^2 \right] + f(v) \left( dx^3 \right)^2 + g(v) \left( dx^4 \right)^2 \right\} . \] (108)

Again, $\alpha$ and $n$ are arbitrary non-null constants. The differential equation (103) for case III can be rewritten as
\[ 0 = \frac{g'}{g} \left( 1 - \frac{a'}{a} \right) - \frac{f'}{f} \left( 1 + n + \frac{a'}{a} \right) + 1 + n \frac{a'}{a} + \left( \frac{a'}{a} \right)^2 + 2 \left( \frac{a'}{a} \right)' , \] (109)

where the prime denotes an ordinary derivative with respect to the variable $v = x^3 - x^4$.

4.2.2 Diagonal case
Since the field equations for a perfect fluid are still so complicated we will make a further assumption that will bring the metric into diagonal form; namely: the Killing vector $\xi$ being hypersurface orthogonal.

The possibilities are now restricted just to diagonal subcases (a), since diagonal subcases (b) do always admit a further Killing vector tangent to the Killing orbits $V_2$.

Computing the Einstein tensor for those metrics, one has
\[ G_{14} = 0 \quad \text{and} \quad G_{24} \neq 0 . \] (110)

Consequently, we have chosen a coordinate chart in such a way that the fluid flow velocity always lies in the two plane spanned by $\partial / \partial x^2$ and $\partial / \partial x^4$ at each point. Therefore, we will have in all cases
\[ u = u_2 dx^2 + u_4 dx^4 , \quad \frac{(u_2)^2}{g_{22}} + \frac{(u_4)^2}{g_{44}} = -1 . \] (111)
A careful analysis of all the possibilities reveals that, in most cases, there exist further KVs and, in the instance of null homothetic orbits, the energy conditions cannot be fulfilled. Apart from these cases, it is worth mentioning that whenever $X$ is orthogonal to $u$, the metric and field equations are

$$ds^2 = e^{2x^3} \left\{ a^2(x^4)e^{-2x^2}(dx^1)^2 - eb^2(x^4)(dx^2)^2 + (dx^3)^2 + \epsilon (dx^4)^2 \right\} , \quad (112)$$

$$0 = 2\epsilon - \frac{1}{b^2} + \frac{a'b'}{ab} + \frac{a''}{a} , \quad (113)$$

$$0 = \left[ \frac{a' - b'}{a - b} \right]^2 - \left[ \frac{b''}{b} - \frac{a''}{a} \right] \left[ - \frac{1}{b^2} + \frac{a'b'}{ab} - \frac{b''}{b} \right] , \quad (114)$$

a prime indicating a derivative with respect to $(x^4)$, and then one necessarily has

$$p = \mu = e^{-2x^3} \left\{ 1 + \frac{b''}{b} \right\} , \quad (115)$$

without previously assuming the existence of a barotropic equation of state.

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**References**

[1] D. Kramer, H. Stephani, M.A.H. MacCallum and E. Herlt. *Exact Solutions of Einstein’s Field Equations*. Deutscher Verlag der Wissenschaften, Berlin (1980).

[2] G.S. Hall, Gen. Rel. Grav., 20 (1988) 671.

[3] G.S. Hall, preprint (1988).

[4] G.S. Hall in *Relativity Today*, Proc. 2nd Hungarian Relativity Workshop 1987, Ed. Z. Perjes, Singapore, World Scientific 1988.

[5] G.S. Hall, Class. Quantum Grav., 5 (1988) L77.

[6] G.S. Hall, J. Math. Phys. 31 (1990) 1198.

[7] G.S. Hall and J. da Costa, J. Math. Phys., 29 (1988) 2465.

[8] G.S. Hall and J.D. Steele, Gen. Rel. Grav., 22 (1990) 457.
[9] M.E. Cahill and A.H. Taub, Comm. Math. Phys. 21 (1971) 1.
[10] D.M. Eardley, Commun Math. Phys., 37 (1974) 287.
[11] D.M. Eardley, Phys. Rev. Lett. 33 (1974) 442.
[12] A.A. Coley, preprint (1995).
[13] B.J. Carr and A.A. Coley, preprint (1996).
[14] B.J. Carr, preprint (1992).
[15] Wu Z. C., Gen. Rel. Grav., 13 (1981) 625.
[16] I.S. Shikin, Gen. Rel. Grav., 11 (1979) 433.
[17] R.S. Palais, Mem. Am. Math. Soc. (1957) no. 2.
[18] F. Brickell and R.S. Clark, Differentiable Manifolds, Van Nostrand (1970)
[19] R.F. Bilyalov, Sov. Phys., 8 (1964) 878.
[20] D. Alexeevski, Ann. Global. Anal. Geom., 3 (1985) p.59.
[21] J.K. Beem, Lett. Math. Phys., 2 (1978) 317.
[22] J. Wainwright, Self-similar solutions of Einstein’s equations in Galaxies, Axisymmetric systems and relativity, ed. M.A.H. MacCallum, Cambridge (1985) C.U.P.
[23] J.D. Barrow and D.H. Sonoda, Phys. Rep. 139 (1986) 1.
[24] C.B.G. McIntosh, Gen. Rel. Grav., 7 (1976) 199.
[25] C.B.G. McIntosh, Phys. Lett. A 69 (1978) 1.
[26] C.B.G. McIntosh, Gen. Rel. Grav., 10 (1979) 61.
[27] G.F.R. Ellis, J. Math. Phys., 8 (1967) 1171.
[28] J.M. Stewart and G.F.R. Ellis, J. Math. Phys. 9 (1968) 1072.
[29] K. Rosquist and R.T. Jantzen, Class. Quantum Grav., 2 (1985) L129.
[30] M.P. Ryan and L.C. Shepley, Homogeneous Relativistic Cosmologies, Princeton Univ. Press. (1975)
[31] A. Barnes, J. Phys., A12 (1979) 1493.
[32] H. Goenner and J. Stachel, J. Math. Phys., 11 (1970) 3358.
[33] J. Carot, L. Mas and A.M. Sintes, J. Math. Phys., 35 (1994) 3560.
[34] C. Uggla, Class. Quantum Grav., 209 (1992) 2287.
[35] L. Hsu and J. Wainwright, Class. Quantum Grav., 3 (1986) 1105.
[36] J. Wainwright and L. Hsu, Class. Quantum Grav., 6 (1989) 1409.
[37] D. Kramer, Class. Quantum Grav., 1 (1984) 611.
[38] D. Kramer, Gen. Rel. Grav., 22 (1990) 1157.
[39] M. Mars and J.M.M. Senovilla, Class. Quantum Grav., 10 (1993) 1633.
[40] J.M.M. Senovilla, Class. Quantum Grav., 9 (1992) L167.
[41] C.G. Hewitt, J. Wainwright and S.W. Goode, Class. Quantum Grav., 5 (1988) 1313.
[42] C.G. Hewitt, J. Wainwright and M. Glaum, Class. Quantum Grav., 8 (1991) 1505.
[43] C.G. Hewitt and J. Wainwright, Class. Quantum Grav., 7 (1990) 2295.
[44] B.G. Schmidt, Z. Naturforsch 22a (1967) 1351.
[45] A. Ori and T. Piran, Phys. Rev. D42 (1990) 1068.
[46] B.J. Carr and A. Yahil, Astroph. J. 360 (1990) 330.
[47] R.N. Henriksen and K. Patel, Gen. Rel. Grav. 23 (1991) 527.
[48] T. Foglizzo and R.N. Henriksen, Phys. Rev. D48 (1993) 4645.
[49] W. Kundt, Z. Phys.163 (1961) 77.
[50] E. Ruiz and J.M.M. Senovilla, Phys. Rev. D45 (1992) 1995.
[51] A.F. Agnew and S.W. Goode, Class. Quantum. Grav., 11 (1994) 1725.
[52] N. Van den Bergh and J. Skea, Class. Quantum Grav., 9 (1992) 527.
[53] N. Van den Bergh, P. Wils and M. Castagnino, Class. Quantum. Grav., 8 (1991) 947.
[54] P. Wils, Class. Quantum. Grav., 8 (1991) 361.
[55] J. Wainwright, J. Phys. A, 14 (1981) 1131.
[56] J.P. Luminet, Gen. Rel. Grav., 9 (1978) 673.
[57] S.W. Hawking and G.F.R. Ellis, The Large Scale Structure of Space-time, Cambridge Univ. Press, Cambridge (1973).
[58] J. Wainwright, W.C.W. Ince and B.J. Marshman, Gen. Rel. Grav., 10 (1979) 259.

[59] N. Van den Bergh, Class. Quantum. Grav., 5 (1988) 861.