GEOMETRIC PROPERTIES OF NONLINEAR INTEGRAL TRANSFORMS ON THE NOSHIRO-WARSCHAWSKI CRITERION

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Abstract. In this paper we discuss univalence and quasiconformal extensibility of two nonlinear integral transforms \( \int_0^z (f'(u))^\alpha du \) and \( \int_0^z (f(u)/u)^\alpha du \) for holomorphic functions which satisfy the Noshiro-Warschawski criterion. Various approaches using pre-Schwarzian derivatives, subordination properties, holomorphic motions and Loewner theory are taken to this problem.

1. Introduction

1.1. Integral Transforms. Let \( \mathcal{A} \) be the family of analytic functions defined in \( \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\} \) with \( f(0) = 0 \) and \( f'(0) = 1 \). Let \( \mathcal{LU} \) and \( \mathcal{ZF} \) be the subclasses of \( \mathcal{A} \) which are defined by

\[ \mathcal{LU} := \{f \in \mathcal{A} : f'(z) \neq 0, \forall z \in \mathbb{D}\} \]

and

\[ \mathcal{ZF} := \{f \in \mathcal{A} : f(z)/z \neq 0, \forall z \in \mathbb{D}\} \]

In 1915, Alexander [Ale15] first observed the integral transform defined by \( J[f](z) = \int_0^z f(u)/u \, du \) on the class \( \mathcal{ZF} \) maps the class of starlike functions onto the class of convex functions. Thus one might be expected that \( J[f] \) always produces a univalent function for all \( f \in \mathcal{S} \), where \( \mathcal{S} \) is the subclass of \( \mathcal{A} \) consisting of univalent functions on \( \mathbb{D} \). However in 1963, Krzyż and Lewandowski [KL63] gave the counterexample \( f(z) = z/(1 - iz)^{1-i} \) which is \( \pi/4 \)-spirallike but transformed to a non-univalent function. In 1972, Kim and Merkes [KM72] extended this type of transform by introducing a complex parameter \( \alpha \in \mathbb{C} \) as

\[ J_{\alpha}[f](z) := \int_0^z \left( \frac{f(u)}{u} \right)^{\alpha} du \]

for \( f \in \mathcal{ZF} \), where the branch is chosen so that \( J_{\alpha}[f](0) = 0 \). In their investigation it was shown \( J_{\alpha}[\mathcal{S}] \subset \mathcal{S} \) when \( |\alpha| \leq 1/4 \) while \( J_{\alpha}[\mathcal{S}] \not\subset \mathcal{S} \) if \( |\alpha| > 1/2 \) and \( \alpha \neq 1 \) (consider \( J_{\alpha}[K](z) \) and Royster’s example [Roy65], where \( K(z) := z/(1 - z)^2 \) is the Koebe function).

Another object of investigation in the studies of integral transforms is \( I_{\alpha}[f] \), defined by

\[ I_{\alpha}[f](z) := \int_0^z (f'(u))^\alpha du. \tag{1} \]

on \( \mathcal{LU} \). Then \( J_{\alpha}[f] \) is represented by \( J_{\alpha}[f] = I_{\alpha}[J[f]] \). In 1975, Pfaltzgraf [Pfa75] proved that \( I_{\alpha}[\mathcal{S}] \subset \mathcal{S} \) if \( |\alpha| \leq 1/4 \). On the other hand, Royster’s example again shows that there exists a function \( f \in \mathcal{S} \) such that \( I_{\alpha}[f] \not\in \mathcal{S} \) if \( |\alpha| > 1/3 \) and \( \alpha \neq 1 \).

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Up to now, nothing better estimates of the range of $|\alpha|$ have been obtained in the problems of univalence of $I_\alpha[f]$ and $J_\alpha[f]$. The reader may be referred to [Dur83] for basic terminology in the theory of univalent functions and [Goo83, Chapter 15] for the basic information of the integral transforms on $S$.

1.2. The Noshiro-Warschawski criterion. For a function $f \in A$, the condition on which $f'(D)$ lies in the right half-plane ensures univalence of $f$ on $D$. This is referred to as the Noshiro-Warschawski criterion due to Noshiro [Nos35] and Warschawski [War35] independently. The original form of the theorem is the following (see also [AA75, Theorem 8]).

**Theorem 1.A** (The Noshiro-Warschawski criterion). A non constant function $f$ that is analytic in a convex domain $D$ is univalent in $D$ if

$$\text{Re} \left\{ e^{-ic} f'(z) \right\} \geq 0$$

for all $z \in D$, where $c$ is a fixed real number.

As special cases, Alexander [Ale15] showed the case when $D$ is the unit disk and $f'(D)$ is contained in a half-plane bounded by a straight line through the origin, and Wolff [Wol34] showed when $D$ is the right half-plane. On the other hand, Tims [Tim51] and Herzog and Piranian [HP51] showed that the assumption of convexity of $D$ is essential, that is, Theorem 1.A does not work in any non-convex domains. For example, a counterexample of Theorem 1.A is easily given by $f(z) = z^{1+\frac{\pi}{2}}$ defined in $\{ z : |\arg z| < \beta \}$, where $\beta$ is a constant satisfying $\pi/2 < \beta < \pi$. In fact, $f$ is defined on a non-convex domain and $\text{Re} f' > 0$ there, but is not univalent.

In what follows we will treat functions which satisfies the theorem in which $D$ and $c$ are specified, i.e., $D = \mathbb{D}$ and $c = 1$. The family of such functions in $f \in A$ is denoted by $\mathcal{R}$. Then Theorem 1.A claims that $\mathcal{R} \subset S$. However, comparing to the other typical subclasses of $S$, a geometric characterization of $\mathcal{R}$ is not known.

1.3. The aim of the paper. In this paper univalence and quasiconformal extensibility of $J_\alpha[f]$ and $I_\alpha[f]$ on $\mathcal{R}$ are investigated using various approaches, pre-Schwarzian derivatives, subordination properties, holomorphic motions and Loewner theory. In Section 2, We collect some preliminary results on Schwarzian and pre-Schwarzian derivatives and subordinations of analytic functions. Those properties will be used in Section 3 to derive univalence of $J_\alpha[f]$ and $I_\alpha[f]$ on $\mathcal{R}$. Section 4 and 5 will be devoted to quasiconformal extensions of the operator $J_\alpha[f]$. We will try to approach this problem from two sides. One is applying the theory of holomorphic motions and the celebrated $\lambda$-lemma. It enables us to derive quasiconformality of $J_\alpha[f]$ on the full class of $S$. The other is by Loewner theory and a result by Betker. We also consider explicit quasiconformal extension for Betker’s theorem.

2. Preliminaries

2.1. Schwarzian and pre-Schwarzian derivatives. As important quantities to investigate properties of functions $f$ in $LU$, we introduce $T_f$ and $S_f$ defined by

$$T_f := \frac{f''}{f'}, \quad S_f := \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2.$$
$T_f$ and $S_f$ are called the pre-Schwarzian derivative and the Schwarzian derivative respectively. These are considered as elements of the Banach space of functions $f \in \mathcal{LU}$, for which the norm

$$
||T_f|| := \sup_{z \in \mathbb{D}} (1 - |z|^2) |T_f|, \\
||S_f|| := \sup_{z \in \mathbb{D}} (1 - |z|^2)^2 |S_f|,
$$

is finite. Further, in connection with the theory of univalent functions, the following estimates are known. Here, a homeomorphism $f$ is finite. Further, in connection with the theory of univalent functions, the following estimates are known. Here, a homeomorphism $f$ is finite. Further, in connection with the theory of univalent functions, the following estimates are known. Here, a homeomorphism $f$ is finite. Further, in connection with the theory of univalent functions, the following estimates are known. Here, a homeomorphism $f$ is finite. Further, in connection with the theory of univalent functions, the following estimates are known. Here, a homeomorphism $f$ is finite. Further, in connection with the theory of univalent functions, the following estimates are known. Here, a homeomorphism $f$ is finite. Further, in connection with the theory of univalent functions, the following estimates are known. Here, a homeomorphism $f$ is finite. Further, in connection with the theory of univalent functions, the following estimates are known. Here, a homeomorphism $f$ is finite. Further, in connection with the theory of univalent functions, the following estimates are known. Here, a homeomorphism $f$ is finite. Further, in connection with the theory of univalent functions, the following estimates are known. Here, a homeomorphism $f$ is finite. Further, in connection with the theory of univalent functions, the following estimates are known. Here, a homeomorphism $f$ is finite. Further, in connection with the theory of univalent functions, the following estimates are known. Here, a homeomorphism $f$ is finite.

Theorem 2.A. Let $f \in \mathcal{LU}$. Then,

(i) if $||T_f|| \leq 1$, then $f$ is univalent in $\mathbb{D}$,

(ii) if $||T_f|| \leq k < 1$, then $f$ has a quasiconformal extension to $\mathbb{C}$.

(iii) if $f \in \mathcal{S}$, then $||T_f|| \leq 6$,

(iv) if $||S_f|| \leq 2$, then $f$ is univalent in $\mathbb{D}$,

(v) if $f \in \mathcal{S}$, then $||S_f|| \leq 6$.

Becker showed (i) and (ii) in [Bec72, Bec73]. The sharpness of the constant 1 in (i) is due to Becker and Pommerenke [BP84]. (iii) is an easy consequence of the well-known inequality $(1 - |z|^2) f''(z)/f'(z) - 2z \leq 4$ for $f \in \mathcal{S}$. (iv) is firstly shown by Kraus [Kra32] and later reproved by Nehari [Neh49]. Hille [Hil49] showed that the constant 2 is the best possible one with the function $f(z) = ((1 + z)/(1 - z))^{ix}$ $(\varepsilon > 0)$, for it is not univalent for all $\varepsilon > 0$ but $||S_f|| = 2(1 + \varepsilon^2)$ can approach 2. Nehari [Neh49] also verified the assertion (v) and the sharpness is given by the Koebe function as $||S_K|| = 6$.

2.2. Subordination properties. For analytic functions $f$ and $g$, we say that $f$ is subordinate to $g$ if there exists an analytic function $w$ which maps $\mathbb{D}$ into $\mathbb{D}$ such that $w(0) = 0$ and $f(z) = g(w(z))$. This relation is denoted by $f(z) \prec g(z)$. Below we will state two subordination properties which will play central roles in Section 3. The first is a result of differential subordinations due to Hallenbeck and Ruscheweyh.

Theorem 2.B (Hallenbeck and Ruscheweyh [HR75]). Let $p(z)$ be analytic in $\mathbb{D}$ with $p(0) = 1$. Let $q(z)$ be convex univalent in $\mathbb{D}$ with $q(0) = 1$ and suppose $p(z) \prec q(z)$. Then for all $\gamma \neq 0$ with $\Re \gamma > 0$, we have

$$
\gamma z^{-\gamma} \int_0^z u^{\gamma-1} p(u) du < \gamma z^{-\gamma} \int_0^z u^{\gamma-1} q(u) du.
$$

For example, if $f$ satisfies $\Re f'(z)(z/f(z))^{1-\gamma} > 0$, then it implies that $f'(z)(z/f(z))^{1-\gamma} < (1 + z)/(1 - z)$ and Theorem 2.B shows

$$
\left( \frac{f(z)}{z} \right)^\gamma < 1 + \frac{2\gamma}{z^{\gamma}} \int_0^z \frac{u^{\gamma}}{1 - u} du.
$$

In particular, putting $\gamma = 1$ we have

$$
\frac{f(z)}{z} < \frac{-z - 2 \log(1 - z)}{z}
$$

(2)
for all $f \in \mathcal{R}$. This gives the best dominant for $\mathcal{R}$ because if $\phi(z) := -z - 2 \log(1 - z)$ then $\phi'(z) = (1 + z)/(1 - z)$ and therefore $\phi \in \mathcal{R}$.

The second is a fundamental subordination principle in Geometric Function Theory. The original idea is due to Littlewood.

**Theorem 2.C** (Kim and Sugawa [KS02, p.195]). Let $g$ be locally univalent in $\mathbb{D}$. For an analytic function $f$ in $\mathbb{D}$, if $f'(\mathbb{D}) \subset g'(\mathbb{D})$, then we have $||T_f|| \leq ||T_g||$. In particular, $f$ is uniformly locally univalent on $\mathbb{D}$.

Theorem 2.C has a wide range of applications so that we might hope that the inequality $||S_f|| \leq ||S_g||$ also holds for functions $f$ and $g$ with $f'(\mathbb{D}) \subset g'(\mathbb{D})$. However, one can show that the inequality does not always hold under this assumption. Here we note that the Schwarz-Pick lemma shows that all analytic self-mappings $\omega$ of the unit disk satisfy

$$\frac{|\omega'(z)|}{1 - |\omega(z)|^2} \leq \frac{1}{1 - |z|^2}$$

for all $z \in \mathbb{D}$.

**Proposition 2.1.** Let $g$ be locally univalent and $f$ be analytic in $\mathbb{D}$ respectively. If $f'(\mathbb{D}) \subset g'(\mathbb{D})$, then we have

$$||S_f|| \leq ||S_g|| + ||T_\omega|| \cdot ||T_g||,$$

where $\omega = g^{-1} \circ f'$. In particular $f$ is uniformly locally univalent on $\mathbb{D}$.

**Proof.** By assumption we have $T_f = T_g \circ w \cdot w'$ and hence (3) implies that

$$\left(1 - |z|^2\right)^2 \left|\frac{f''}{f'} - \frac{1}{2} \left(\frac{f''}{f'}\right)^2\right| = \left(1 - |z|^2\right)^2 \left|\frac{g''}{g'} \cdot \omega' - \frac{1}{2} \left(\frac{g''}{g'} \cdot \omega'\right)^2\right| \leq \left(1 - |z|^2\right)^2 \left|\frac{g''}{g'} \cdot \omega'\right|^2 + (1 - |z|^2) \frac{|\omega|^2}{|\omega'|^2} \left|\frac{g''}{g'} \cdot \omega'\right|^2 \leq ||S_g|| + ||T_\omega|| \cdot ||T_g|| \quad \square$$

The term $||T_\omega|| \cdot ||T_g||$ in (4) is eliminated in only a few cases. $||T_g|| = 0$ if and only if $g$ is an affine transform and then $||S_g||$ also vanishes. Therefore $||S_f|| = 0$, which implies that $f$ is a Möbius transformation. $||T_\omega|| = 0$ if and only if $\omega$ is an affine transform which is equivalent to the case that one can write $f(z) = a + b$, where $a, b \in \mathbb{C}$ are complex constants.

### 3. Univalence of $J_\alpha[f]$ and $I_\alpha[f]$ for $f \in \mathcal{R}$

#### 3.1. Univalence of $J_\alpha[f]$ when $\alpha \in \mathbb{C}$

Firstly we give a sharp estimation of the norm of $T_{J_\alpha[f]}$ for a function $f \in \mathcal{R}$ and make use of Theorem 2.A to obtain the range of $|\alpha|$ which ensures univalence of $J_\alpha[f]$.

Taking a logarithmic differentiation we have

$$||T_{J_\alpha[f]}|| = |\alpha| \cdot ||T_f||.$$
Then it suffices to estimate $||T_{J[f]}||$. Let us suppose that $f \in \mathcal{R}$. By (2) and Theorem 2.4, we have

$$||T_{J[f]}|| \leq \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \frac{\phi'(z)}{\phi(z)} - \frac{1}{z} \right|$$

for all $f \in \mathcal{R}$, where $\phi(z) = -z - 2 \log(1 - z)$ as defined in Section 2.2. Then a computation shows that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \frac{\phi'(z)}{\phi(z)} - \frac{1}{z} \right| = \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| \frac{2(z + (1 - z) \log(1 - z))}{(1 - z)z(z + 2 \log(1 - z))} \right|$$

$$= \sup_{z \in \mathbb{D}} \frac{1 - |z|^2}{|z|} \left| 1 + \frac{1 + z}{1 - z} \cdot \frac{z}{z + 2 \log(1 - z)} \right|.$$

Let $g(z) := 1 + \frac{1 + z}{1 - z} \cdot \frac{z}{z + 2 \log(1 - z)}$. It is obvious that $g$ is symmetric with respect to the real axis. Next, we will show that all the coefficients of $g$ are negative. $g$ is written as

$$g(z) = 1 + \frac{1 + z}{1 - z} \cdot \frac{z}{z + 2 \log(1 - z)}$$

$$= 1 - \frac{1 + z}{1 - z} \cdot \frac{1}{-1 - 2 \sum_{n=1}^{\infty} \frac{z^n}{n+1}}$$

$$= 1 - \frac{1 + z}{1 - z + 2 \sum_{n=1}^{\infty} \frac{z^{n+1}}{n+1} - 2 \sum_{n=1}^{\infty} \frac{z^{n+1}}{n+1}}$$

$$= 1 - \frac{1 + z}{1 - 2 \sum_{n=1}^{\infty} \frac{z^{n+1}}{(n+1)(n+2)}}.$$

Thus $g$ has negative coefficients. This fact implies that $\sup_{z \in \mathbb{D}} |g(z)| = \sup_{r \in (0,1)} -g(r)$. Therefore,

$$||T_{J[f]}|| \leq \sup_{z \in \mathbb{D}} \frac{1 - |z|^2}{|z|} \left| 1 + \frac{1 + z}{1 - z} \cdot \frac{z}{z + 2 \log(1 - z)} \right|$$

$$= \sup_{r \in (0,1)} \left[ \frac{-(1 + r)^2}{r + 2 \log(1 - r)} - \frac{1 - r^2}{r} \right].$$

Let us set

$$h(r) := \frac{-(1 + r)^2}{r + 2 \log(1 - r)} - \frac{1 - r^2}{r}.$$ (5)

Simple calculation shows that $h'(r)$ has only one critical point $r_0$ in $r \in (0,1)$ which is the root of the equation

$$2(r^2 + 1)(r - 1)[\log(1 - r)]^2 - 2r(r - 1)^2 \log(1 - r) + r^3(r + 3) = 0.$$ (6)

By numerical experiments, we have $r_0 \approx 0.329423$ and $h(r_0) \approx 1.055681$.

In consequence, the following is obtained.

**Theorem 3.1.** Let $f \in \mathcal{R}$. Then we have the sharp estimate

$$||T_{J[f]}|| \leq |\alpha| \cdot h(r_0)$$

where $h(r_0) \approx 1.055681$, where $h$ is the function defined in (5) and $r_0 \approx 0.329423$ is the unique root of the equation (6) in $t \in (0,1)$. 

Applying Theorem 2.2 to the above estimate, we can deduce the range of \(|\alpha|\) of which \(J_\alpha[f]\) is univalent in \(D\) and has a quasiconformal extension to \(C\).

**Corollary 3.2.** Let \(f \in \mathcal{R}\) and \(k \in [0, 1)\). Then,
1. If \(|\alpha| \leq 1/h(r_0) \approx 0.947255\), then \(J_\alpha[f] \in S\),
2. If \(|\alpha| < k/h(r_0)\), then \(J_\alpha[f]\) can be extended to a \(k\)-quasiconformal mapping of \(C\).

### 3.2. Univalence of \(J_\alpha[f]\) when \(\alpha \in \mathbb{R}\).

In the previous subsection we dealt with \(J_\alpha[f]\) in the case that \(\alpha\) is a complex number. On the other hand, some geometric property of \(J_\alpha[f]\) on typical subclasses of \(S\) under the restriction of \(\alpha \in \mathbb{R}\) have been also investigated. The following is a list of some fundamental results. Here, we denote by \(K, S^*, C\) the well-known classes of convex, starlike and close-to-convex functions in \(A\), respectively.

**Theorem 3.2A (Merkes and Wright [MW71]).** Let \(\alpha \in \mathbb{R}\). Then the followings are true:

1. Let \(f \in K\). If \(\alpha \in [-1, 3]\) then \(J_\alpha[f] \in C\), otherwise there exists a function \(g \in K\) such that \(J_\alpha[g] \notin S\).
2. Let \(f \in S^*\). If \(\alpha \in [-\frac{1}{2}, \frac{2}{3}]\) then \(J_\alpha[f] \in C\), otherwise there exists a function \(g \in S^*\) such that \(J_\alpha[g] \notin S\).
3. Let \(f \in C\). If \(\alpha \in [-\frac{1}{2}, 1]\) then \(J_\alpha[f] \in C\), otherwise there exists a function \(g \in C\) such that \(J_\alpha[g] \notin C\).
4. Let \(f \in K\). If \(\alpha \in [-\frac{1}{3}, \frac{2}{3}]\) then \(I_\alpha[f] \in C\), otherwise there exists a function \(g \in K\) such that \(J_\alpha[g] \notin S\).
5. Let \(f \in C\). If \(\alpha \in [-\frac{1}{3}, 1]\) then \(I_\alpha[f] \in C\), otherwise there exists a function \(g \in C\) such that \(J_\alpha[g] \notin C\).

In this section we derive the similar result as above for the class \(\mathcal{R}\). Suppose that \(f \in \mathcal{R}\). Again, we will make use of the relation \([2]\), namely,

\[
J[f]'(z) < q(z) = \frac{-z - 2\log(1 - z)}{z} < \frac{1 + z}{1 - z}
\]

for all \(f \in \mathcal{R}\). Here \(q\) is a convex function because \(\text{Re}[1 + zq''(z)/q'(z)] > 1 + (-1)^\alpha q''(-1)/q'(-1) = -1 + 1/(2(2\log 2 - 1)) > 0\) for all \(z \in D\) (see Fig. 1). Since

![Fig. 1, the shape of the boundary of q(D)](image)
Let \( f < g \) follows \( f^\alpha < g^\alpha \) for an \( \alpha \in \mathbb{R} \) and \((J[f])^\alpha = J_\alpha [f]^\alpha\), one problem is raised to find the largest \( \alpha_0 \in \mathbb{R} \) such that \( \text{Re} [q(z)^\alpha] > 0 \) for all \( z \in \mathbb{D} \). It is equivalent to find the smallest \( \beta_0 \in \mathbb{R} \) such that the sector domain \( \Delta_{\beta_0} := \{w : |\arg w| < \pi \beta_0/2\} \) contains \( q(\mathbb{D}) \). Then \( \alpha_0 = 1/\beta_0 \) (remark that \( z \in \Delta_{\beta_0} \) then \( 1/z \in \Delta_{\beta_0} \)).

Since \( 1 - e^{i\theta} = -2i \sin(\theta/2) e^{i\theta/2} \), we obtain
\[
\arg q(e^{i\theta}) = -\theta + \arg \left[ -e^{i\theta} - 2 \log \left( \frac{2 \sin \theta}{2} \right) + i \left( \frac{\theta}{2} + \frac{3\pi}{2} \right) \right]
= -\theta + \arctan \frac{\sin \theta - \theta - 3\pi}{\cos \theta + 2 \log \left( \frac{2 \sin \theta}{2} \right)}.
\]

Set
\[
\zeta(\theta) := \frac{\sin \theta - \theta - 3\pi}{\cos \theta + 2 \log \left( \frac{2 \sin \theta}{2} \right)}.
\] (7)

Since \( \partial \arg q(e^{i\theta})/\partial \theta = \zeta'(\theta)/(1 + \zeta(\theta)^2) - 1 \), \( \beta_0 \) is one of the zeros of \( \zeta'(\theta) - \zeta(\theta)^2 - 1 = 0 \) in \( \theta \in (0, \pi/2) \), where \( \zeta \) is defined by in (7).

**Theorem 3.3.** Let \( \alpha \in \mathbb{R} \) and \( f \in \mathcal{R} \). If \( \alpha \in [-\alpha_0, \alpha_0] \) then \( J_\alpha[f] \in \mathcal{R} \), otherwise there exists a function \( g \in \mathcal{R} \) such that \( J_\alpha[g] \notin \mathcal{R} \), where \( \alpha_0 \approx 1.723078 \) is defined by \( \alpha_0 := \pi/2p(\theta_0) \) and \( \theta_0 \) is the unique root of the equation \( \zeta'(\theta) - \zeta(\theta)^2 - 1 = 0 \) in \( \theta \in (0, \pi/2) \), where \( \zeta \) is defined by in (7).

### 3.3. Univalence of \( I_\alpha[f] \) when \( \alpha \in \mathbb{C} \).

Further investigation goes to show univalence of the transform \( I_\alpha[f] \) defined in (I).

**Theorem 3.4.** The followings are true:

1. If \( \alpha \in [-1, 1] \) then \( I_\alpha[f] \in \mathcal{R} \) for all \( f \in \mathcal{R} \),
2. If \( |\alpha| \leq 1/2 \), then \( I_\alpha[f] \in \mathcal{S} \) for all \( f \in \mathcal{R} \),
3. If \( |\alpha| > 1 \), then there exists a function \( g \in \mathcal{R} \) such that \( I_\alpha[g] \notin \mathcal{S} \).

**Proof.** (1) It is clear that \( I_\alpha[f] \in \mathcal{R} \) when \( \alpha \in [-1, 1] \). (2) Let \( f \in \mathcal{R} \). Then \( f'(z) \prec (1 + z)/(1 - z) \), and hence by Theorem 2.C we obtain the sharp bound \( ||T_f|| \leq 2 \) for \( f \in \mathcal{R} \) (see also [Nun68]). Since \( ||T_{I_\alpha[f]}|| = |\alpha| \cdot ||T_f|| \), it follows from Theorem 2.A(i) that \( I_\alpha[f] \in \mathcal{S} \) if \( |\alpha| < 1/2 \). (3) A counterexample is given by the function \( \phi(z) = -z - 2 \log(1 - z) \) which belongs to \( \mathcal{R} \). In fact, it follows from the calculations that \( ||S_{I_\alpha[\phi]}|| = 2|\alpha|(|\alpha| + 2) \). Then Theorem 2.A(v) shows that \( I_\alpha[\phi] \) is not univalent if \( |\alpha| > 1 \).

### 4. Quasiconformal Extension of \( J_\alpha[f] \) with the \( \lambda \)-Lemma

As investigated in the previous studies, \( J_\alpha[f] \) and \( I_\alpha[f] \) transform univalent functions to functions which have various kinds of geometric properties. Further, under certain assumptions \( J_\alpha[f] \) can be quasiconformally extendable. In the next two sections we will derive sufficient conditions under which \( J_\alpha[f] \) has a quasiconformal
extension. The first approach relies on the celebrated \( \lambda \)-lemma. The argument is straightforward, but it will provide us a quite practical and profound result.

4.1. Holomorphic motions. We would like to recall the definition of holomorphic motions and their fundamental properties. The notion of holomorphic motions was first introduced by Mañé, Sad and Sullivan in 1983.

**Definition 4.A** (Mañé, Sad and Sullivan [MSS83]). Let \( A \subset \hat{\mathbb{C}} \) be a non-empty set. If a mapping \( i : D \times A \to \hat{\mathbb{C}} \) satisfies the following three conditions, \( i \) is said to be a holomorphic motion of \( A \).

1. For any fixed \( z_0 \in A \), \( z_0(\lambda) := i(\lambda, z_0) : D \to \hat{\mathbb{C}} \) is a holomorphic mapping.
2. For any fixed \( \lambda_0 \in D \), \( i_{\lambda_0}(z) := i(\lambda_0, z) : A \to \hat{\mathbb{C}} \) is a one-to-one mapping.
3. \( i_0 \) is identity on \( A \).

As for the theory of holomorphic motions, the following theorem is fundamental.

**Theorem 4.B** (\( \lambda \)-Lemma [MSS83]). Let \( A \subset \hat{\mathbb{C}} \) be a non-empty set and \( i \) a holomorphic motion of \( A \).

1. \( i : D \times A \to \hat{\mathbb{C}} \) is a continuous map,
2. for each \( \lambda \), the map \( z \mapsto i_{\lambda}(z) \) is a quasiconformal mapping on \( A \) onto \( \hat{\mathbb{C}} \),
3. \( i \) extends to a holomorphic motion of the closure \( \overline{A} \) of \( A \).

Further, the next striking result is important in our investigations. It was first established by Slodkowski (for another proof of the theorem, see e.g. [AM01, Dou95])

**Theorem 4.C** (Slodkowski [Slo91]). Let \( A \subset \hat{\mathbb{C}} \) be a non-empty set and \( i : D \times A \to \hat{\mathbb{C}} \) a holomorphic motion of \( A \). Then \( i \) has an extension to \( \tilde{i} : D \times \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) such that

1. \( \tilde{i} \) is a holomorphic motion of \( \hat{\mathbb{C}} \),
2. each \( \tilde{i}_{\lambda} = \tilde{i}(\lambda, \cdot) \) is a \( |\lambda| \)-quasiconformal automorphism of \( \hat{\mathbb{C}} \).

4.2. Results. Now, let us define the constant \( c_0 \) by

\[
c_0 := \sup \{ c \in \mathbb{R} : J_\alpha[S] \subset S \text{ for all } \alpha \in \mathbb{C} \text{ with } |\alpha| < c \}
\]

and the function \( i \) by

\[
i(\lambda, z) := J_{c_0\lambda}[f](z) = \int_0^z \left( \frac{f(u)}{u} \right)^{c_0\lambda} \, du \quad (\lambda \in D, \ z \in D).
\]

Then one can easily see that the map \( i \) is a holomorphic motion of \( D \). Therefore by Theorem 4.C, \( i \) is extended to a holomorphic motion \( \tilde{i} \) of \( \hat{\mathbb{C}} \). Furthermore, \( \tilde{i}_{\lambda} \) is a \( |\lambda| \)-quasiconformal automorphism of \( \hat{\mathbb{C}} \) for each \( \lambda \) and it is a very quasiconformal extension of \( J_{c_0\lambda}[f] \). Up to now it is known that \( c_0 \geq 1/4 \). Consequently, we obtain the following theorem.

**Theorem 4.1.** Let \( f \in \mathcal{LU} \) and \( \alpha \) is a complex constant with \( |\alpha| < 1/4 \). Then \( J_\alpha[f] \) can be extended to a \( 4|\alpha| \)-quasiconformal extension to \( \hat{\mathbb{C}} \).

Observing Theorem 4.1, the open problem to find out the maximal range of \( |\alpha| \) of which \( J_\alpha[f] \) belongs to \( S \) for all \( f \in \mathcal{LU} \) is expressed by the following way.

\[1\) The authors would like to thank Professor Toshiyuki Sugawa for pointing this out to us.
Problem 4.2. Find a value $c_1 \in \mathbb{C}$, $|c_1| \in [1/4, 1/2]$ and a function $g \in \mathcal{LU}$ such that $J_{\alpha}[f] \in \mathcal{S}$ for all $f \in \mathcal{LU}$ and all $\alpha$ with $|\alpha| < c_1$, $J_{\alpha}[g]$ is univalent and $J_{\alpha}[g](\mathbb{D})$ is not a Jordan domain.

5. Quasiconformal extension of $J_{\alpha}[f]$ with Loewner chains

In this section we will make use of the theory of Loewner chains and its applications to derive quasiconformal extension conditions for $J_{\alpha}[f]$ under the class $\mathcal{R}$.

5.1. Loewner chains and inverse Loewner chains. Before starting our argument we describe the theory of Loewner chains and results of quasiconformal extensions due to Becker and Betker with some notations and terminology we will use.

Let $f_t(z) = \sum_{n=1}^{\infty} a_n(t)z^n$, $a_1(t) \neq 0$, be a function defined on $\mathbb{D} \times [0, \infty)$, where $a_1(t)$ is a complex-valued, locally absolutely continuous function on $[0, \infty)$. Then $f_t$ is called a Loewner chain if $f_t$ satisfies the following conditions:

1. $f_t$ is univalent in $\mathbb{D}$ for each $t \geq 0$,
2. $|a_1(t)|$ increases strictly monotonically as $t$ increases, and $\lim_{t \to \infty} |a_1(t)| \to \infty$.
3. $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ for $0 \leq s < t < \infty$.

We remark that monotonicity of $a_1(t)$ deduces that $f_s(\mathbb{D}) \neq f_t(\mathbb{D})$ for all $0 \leq s < t < \infty$. The key properties of Loewner chains are that $f_t$ is absolutely continuous on $t \geq 0$ for each $z \in \mathbb{D}$, which implies $\partial_t f_t (\partial_t := \partial/\partial t)$ exists almost everywhere on $[0, \infty)$, and satisfies the partial differential equation

$$\partial_t f_t(z) = z\partial_z f_t(z)p(z,t) \quad (z \in \mathbb{D}, \text{ a.e. } t \geq 0),$$

(8) where $p(z,t)$ is analytic for all $z \in \mathbb{D}$ for each $t \geq 0$, measurable for all $t \geq 0$ for each $z \in \mathbb{D}$ and satisfies $\text{Re} p(z,t) > 0$ for all $z \in \mathbb{D}$ and $t \geq 0$. We call such a function $p$ a Herglotz function. Further, Becker [Bec72, Bec76] showed that if $p$ satisfies

$$\left| \frac{1-p(z,t)}{1+p(z,t)} \right| \leq k \quad (z \in \mathbb{D}, \text{ a.e. } t \geq 0)$$

then $f_0$ has a $k$-quasiconformal extension to $\mathbb{C}$. It enables us to derive various kinds of sufficient conditions under which a function $f \in \mathcal{S}$ has a quasiconformal extension (see e.g. [Hot09, Hot11]).

Betker introduced the following notion of an inverse Loewner chain. Let $\omega_t(z) = \sum_{n=1}^{\infty} b_n(t)z^n$, $b_1(t) \neq 0$, be a function defined on $\mathbb{D} \times [0, \infty)$, where $b_1(t)$ is a complex-valued, locally absolutely continuous function on $[0, \infty)$. Then $\omega_t$ is said to be an inverse Loewner chain if

1. $\omega_t$ is univalent in $\mathbb{D}$ for each $t \geq 0$,
2. $|b_1(t)|$ decreases strictly monotonically as $t$ increases, and $\lim_{t \to \infty} |b_1(t)| \to 0$.
3. $\omega_s(\mathbb{D}) \supset \omega_t(\mathbb{D})$ for $0 \leq s < t < \infty$,
4. $\omega_s(z) = z$ and $\omega_s(0) = \omega_t(0)$ for $0 \leq s \leq t < \infty$.

$\omega$ also satisfies the partial differential equation:

$$\partial_t \omega_t(z) = -z\partial_z \omega_t(z)q(z,t) \quad (z \in \mathbb{D}, \text{ a.e. } t \geq 0),$$

(9) where $q$ is a Herglotz function. Conversely, we can construct an inverse Loewner chain by means of (9) according to the following lemma:
Lemma 5.A. Let \( q(z, t) \) be a Herglotz function. Suppose that \( q(0, t) \) be locally integrable in \([0, \infty)\) with \( \int_0^\infty \text{Re} \ q(0, t) \mathrm{d}t = \infty \). Then there exists an inverse Loewner chain \( w_t \) with \( q \).

By applying the notion of an inverse Loewner chain, we obtain a generalization of Becker’s result.

**Theorem 5.B** (Betker [Bet92]). Let \( k \in (0, 1] \). Let \( f_t \) be a Loewner chain satisfying \( (8) \) with

\[
\left| \frac{p(z, t) - q(z, t)}{p(z, t) + q(z, t)} \right| \leq k < 1 \quad (z \in \mathbb{D}, \text{a.e.} \ t \geq 0)
\]

where \( q(z, t) \) is a Herglotz function. Let \( \omega_t \) be the inverse Loewner chain which is generated by \( q \) with \( (9) \). Then \( f_t \) and \( \omega_t \) are continuous and injective on \( \overline{\mathbb{D}} \) for each \( t \geq 0 \), and \( f_0 \) has a \( k \)-quasiconformal extension \( \Phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) which is defined by

\[
\Phi \left( \frac{1}{\omega_t(e^{i\theta})} \right) = f_t(e^{i\theta}) \quad (\theta \in [0, 2\pi), t \geq 0).
\]

We obtain Becker’s result for \( q(z, t) = 1 \). In this case \( \omega_t(z) = e^{-t}z \). Further, choosing \( \omega \) as \( p = q \), we have the following corollary:

**Corollary 5.C** (Betker [Bet92]). Let \( \alpha \in (0, 1] \). Suppose that \( f_t \) is a Loewner chain for which \( p \) in \( (8) \) satisfying the condition

\[
\left| \arg p(z, t) \right| \leq \frac{\alpha \pi}{2} \quad (z \in \mathbb{D}, \text{a.e.} \ t \geq 0).
\]

Then \( f_t \) admits a continuous extension to \( \overline{\mathbb{D}} \) for each \( t \geq 0 \) and the map defined by \( (10) \) is a \( \sin(\alpha \pi/2) \)-quasiconformal extension of \( f_0 \) to \( \mathbb{C} \).

In contrast to Becker’s quasiconformal extension theorem mentioned above, the theorem due to Betker does not always give a quasiconformal extension explicitly. The reason is based on the fact that in general it is difficult to express an inverse Loewner chain \( \omega_t \) which has the same Herglotz function as a given Loewner chain \( f_t \) in an explicit form.

More precisely, let \( f_t \) be a given Loewner chain and \( p(z, t) \) be a Herglotz function associated with \( f_t \) by \( (5) \). Fix an arbitrary \( T > 0 \), and define a Herglotz function \( q(z, t) \) by

\[
q(z, t) := \begin{cases} p(z, T - t), & t \in [0, T] \\ 1, & t \in (T, \infty). \end{cases}
\]

It is known to exist a Loewner chain \( h_t \) with the equation \( \partial_t h_t(z) = z \partial_z h_t(z) q(z, t) \). One can see that \( g_t(z) \) defined by

\[
g_t(z) := \begin{cases} (h_t^{-1} \circ h_t)(z), & t \in [0, T] \\ e^{T - t}, & t \in (T, \infty), \end{cases}
\]

is also a Loewner chain whose Herglotz function is \( q \). Such \( g_t \) is uniquely determined by the condition \( g_T(z) = z \). Therefore \( g_t \) is the unique solution of the differential equation

\[
\partial_t g_t(z) = z \partial_z g_t(z)p(z, T - t)
\]

for all \( z \in \mathbb{D} \) and \( t \in [0, T] \). Hence \( \omega_t := g_{T-t} \) is defined on \( z \in \mathbb{D}, \ t \in [0, T] \) and satisfies \( \partial_t \omega_t(z) = -z \partial_z \omega_t(z)p(z, t) \). It is also easily seen that \( \omega_0(z) = z, \ \omega_t(0) = z \).
0, \omega_s(\mathbb{D}) \supset \omega_t(\mathbb{D}) and b_1(t) is monotonically decreasing with \(|b_1| \to 0 \) as \( t \to \infty \).

Since \( T \) is arbitrary, we obtain our desired inverse Loewner chain.

The above argument indicates that in order to obtain the concrete expression of \( \omega_t \) we need to describe \( h_t \) and \( h_t^{-1} \) by a given \( f_t \), and it is not always possible. Loewner chains for spirallike functions are one of the few known cases in which this method works well. Here \( f \in \mathcal{A} \) is said to be \( \lambda \)-spirallike \(( \lambda \in (-\pi/2, \pi/2)) \) if

\[
\Re \left\{ \frac{e^{-i\lambda} z f'(z)}{f(z)} \right\} > 0
\]

for all \( z \in \mathbb{D} \). We know that \( f_t(z) = e^{i\lambda t} f(z) \) describes an expanding flow for \( \lambda \)-spirallike domains. In this case the corresponding inverse Loewner chain \( \omega_t \) can be written explicitly by

\[
\omega_t(z) := f^{-1}(e^{-i\lambda t} f(z)).
\]

Let \( \alpha \in (-\pi/2, \pi/2) \) be given. Suppose \( |\arg(z f'(z)/f(z)) - \lambda| < \pi\alpha/2 \). Then by Corollary 5.C \( f \) has a continuous extension to \( \overline{\mathbb{D}} \), and the function \( \Phi : \mathbb{C} \to \mathbb{C} \),

\[
\Phi(z) = \begin{cases} 
  f(z), & z \in \mathbb{D} \\
  \frac{1}{f^{-1}(e^{-i\lambda t} f(e^{i\theta}))}, & \theta \in [0, 2\pi), t \geq 0.
\end{cases}
\]

defines a \( \sin(\pi\alpha/2) \)-quasiconformal extension of \( f \). If \( z = 1/f^{-1}(e^{-i\lambda t} f(e^{i\theta})) \) we have

\[
f \left( \frac{1}{z} \right) = e^{-i\lambda t} f(e^{i\theta})
\]

and hence [13] is expressed by

\[
\Phi(z) = \begin{cases} 
  f(z), & z \in \mathbb{D} \\
  \frac{(f(e^{i\theta}))^2}{f(1/z)}, & z \in \mathbb{D}^* := \{ z \in \mathbb{C} : |z| > 1 \},
\end{cases}
\]

where \( f(e^{i\theta}) \) is uniquely determined by the equation \( \arg f(1/z) = \arg f(e^{i\theta}) \) which is deduced by [14], where \( \arg \) represents the \( \lambda \)-argument (for details, see [Sug12]). The function [15] is the same as given in [Sug12].

5.2. Results. Several conditions under which \( f \in \mathcal{R} \) has a quasiconformal extension to the complex plane are known. One of the remarkable results is due to Chuaqui and Gevirtz [CG03] who gave the necessary and sufficient condition under which \( f(\mathbb{D}) \) can be a quasidisk by introducing the notion of property \( M \). Comparing to it, our argument will provide a quantitative result for quasiconformal extensions.

A Loewner chain which corresponds to the class \( \mathcal{R} \) is simply given by

\[
f_t(z) := f(z) + t z.
\]

In fact, a straightforward calculation shows that

\[
\frac{1}{p(z,t)} = \frac{\partial_t f_t(z)}{z \partial_z f_t(z)} = f'(z) + t.
\]

If we assume that \( |\arg f'(z)| \leq \alpha \pi/2 \) for a fixed constant \( \alpha \in (0, \pi/2] \), then it follows from Corollary 5.C that \( f \) has a \( \sin(\alpha \pi/2) \)-quasiconformal extension to \( \mathbb{C} \). Consequently we will obtain the following:
Theorem 5.1. Let $f \in \mathcal{A}$ and $\alpha \in [0, 1)$. If $|\arg f'(z)| \leq \alpha \pi/2$ for all $z \in \mathbb{D}$, then $f$ belongs to $\mathcal{R}$ and has a $\sin(\alpha \pi/2)$-quasiconformal extension to $\mathbb{C}$.

As we have seen above, in this case it does not seem to obtain an explicit quasiconformal extension by (10) because there are no effective means to find a Loewner chain $h_t$ whose Herglotz function is given by (11) with $q(z, t) = f'(z) + t$ and its inverse function $h_t^{-1}(z)$ to define $g_t$ by (12).

We finish this section with the following result.

Theorem 5.2. Let $f \in \mathcal{R}$. Let $\beta_0 \approx 0.580356$ and $\alpha_0 \approx 1/\beta_0 \approx 1.723078$ be constants which are given in Subsection 3.2 and $\alpha \in (-\alpha_0, \alpha_0)$ be fixed. Then $J_\alpha[f]$ has a $\sin(\alpha \pi/2)$-quasiconformal extension to $\mathbb{C}$.

Proof. We have shown in Subsection 3.2 that if $f \in \mathcal{R}$ then \{\frac{f(z)}{z} : z \in \mathbb{D}\} lies in the sector domain $\Delta_{\beta_0} = \{w : |\arg w| < \pi \beta_0/2\}$. It implies that $(f(z)/z)\alpha = J_\alpha[f]'(z) \in \Delta_{\alpha \beta_0}$ for all $z \in \mathbb{D}$, and therefore

$$|\arg J_\alpha[f]'(z)| \leq \frac{\alpha |\beta_0 \pi|}{2}.$$ 

Hence Theorem 5.1 yields our assertion. \qed

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