Dominant Mixed Feedback Design for Stable Oscillations

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Abstract—We present a design framework that combines positive and negative feedback for robust stable oscillations in closed loop. The design is initially based on graphical methods, to guide the selection of the overall strength of the feedback (gain) and of the relative proportion of positive and negative feedback (balance). The design is then generalized via linear matrix inequalities. The goal is to guarantee robust oscillations to bounded dynamic uncertainties and to extend the approach to passive interconnections. The results of this article provide a system-theoretic justification to several observations from system biology and neuroscience pointing at mixed feedback as a fundamental enabler for robust oscillations.

Index Terms—Closed-loop systems, linear matrix inequalities (LMIs), mixed-feedback, nonlinear control systems.

I. INTRODUCTION

The combination of positive and negative feedback, or mixed feedback, is a recurring structure in biological oscillators across scales. Negative feedback reduces the feedback error between desired and actual outputs while positive feedback amplifies it, leading to instabilities. These often manifest as hysteresis and oscillations [1]. Examples include cell cycles at the molecular level [2], neural spiking and bursting at the cellular level, and control of rhythmic movements at the network level [3]. The role of mixed feedback is not only to enable oscillations but also to make these oscillators robust and adaptive, see, e.g., [2], [4] for biochemical oscillators and [5], [6] for neural oscillators. In engineering, mixed feedback loops can be traced in the biochemical oscillators of synthetic biology [7], in neuromorphic circuits [8], and in robotic locomotion [9] (see also [10] for an introduction to the role of endogenous oscillators in robotic locomotion). The use of mixed feedback in these domains often suffers from a lack of design tools, among which we have harmonic balance methods [11], and specific methods for relaxation oscillations [12].

In writing this article, we are strongly influenced by the system-theoretic characterization of neuronal excitability in [5] and [13]. These papers point at mixed feedback as a fundamental enabler for nonlinear behaviors. We build upon this view. Our goal is to develop a systematic design based on mixed feedback for stable oscillations in closed loop. As in the preliminary results of [14], we investigate the simplest realization of a mixed feedback controller, given by the parallel interconnection of two stable first-order linear networks whose action is combined into a sigmoidal nonlinearity (see Section II). The problem of mixed feedback design is indeed the problem of finding suitable values for the control parameters to guarantee stable oscillations in closed loop.

The analysis and design approaches proposed in this article are based on dominance theory and differential dissipativity [15], [16]. We use dominance theory to determine whether a closed-loop high-dimensional mixed-feedback system has a low dimensional “dominant” behavior (see Section III). We take advantage of the fact that the attractors of a 2-dominant system correspond to the attractors of a planar system (2-dimensional). This means that the Poincaré–Bendixson theorem can be used on a 2-dominant system to certify oscillations, even if the system has a large dimension. Later in this article, we use differential dissipativity to extend our results to open nonlinear systems.

The goal is to develop a robust design framework for mixed-feedback oscillators, which mimics classical linear robust theory, and to propose a passivity-based approach for interconnections.

The novel contributions of this article are summarized as follows.

1) Via root locus analysis (Section IV), we show why fast positive feedback and slow negative feedback are needed for 2-dominance of the mixed feedback closed loop, thus, to support oscillations. Our findings, based on necessary conditions for 2-dominance, agree with and support several observations from biology, where positive feedback is typically fast.

2) In addition to extending the early results of [14] and [17], the discussion in Section V shows how classical Nyquist arguments can be used to provide a graphical quantification of robustness for mixed-feedback oscillators. To the best of authors’ knowledge, the analysis of robustness in Section V-B is new.

3) From Section VI, we generalize our approach by using linear matrix inequalities (LMIs) for oscillator design, extending the approach in [15] and [16] to control synthesis.

4) From Section VIII, we develop state-feedback design for robustness and passivity of oscillators.

Our design shows strong similarities with classical approaches for stabilization of equilibria. The fundamental difference is that the mixed-feedback controller “destabilizes” the closed-loop system while enforcing a specific degree of dominance.

We believe that our paper contributes to clarify the role of mixed feedback in enabling robust oscillations, as observed by other scientific communities (e.g., system biology, neuroscience). Our paper extends this literature through quantitative tools for robust design of mixed-feedback oscillators that apply to large dimensional systems.

II. MIXED FEEDBACK CONTROLLER

We consider the mixed feedback closed loop with minimal nonlinearities, as illustrated in Fig. 1. It consists of linear plant dynamics, \( P(s) \), in feedback with a mixed feedback controller, given by positive and negative passive linear networks, \( \mathcal{C}_p(s) \) and \( \mathcal{C}_n(s) \), and by a static saturation nonlinearity \( \varphi \). We assume that \( P(s) \) is an asymptotically stable and strictly proper single input single output transfer function. \( \mathcal{C}_p(s) = \frac{1}{\tau_p s + 1} \) and \( \mathcal{C}_n(s) = \frac{1}{\tau_n s + 1} \) are first order lags with time constants \( \tau_p \) and \( \tau_n \). We set \( \tau_p \neq \tau_n \), and both time constants are slower than any of the plant time constant. The motivation for these assumptions is to show that the generation of oscillations via mixed feedback works...
characterized by the prolonged system
\[
\begin{align*}
\dot{x} &= f(x) \\
\dot{\delta x} &= \partial f(x) \delta x \\
(x, \delta x) &\in \mathbb{R}^n \times \mathbb{R}^n.
\end{align*}
\]  

Here, \(\partial f(x)\) represents the Jacobian of \(f\) computed at \(x\). \(\delta x \in \mathbb{R}^n\) represents a generic tangent vector at \(x \in \mathbb{R}^n\). Along any generic trajectory \(x(\cdot)\) of (3), the related subtrajectory \(\delta x(\cdot)\) of (4) can be interpreted as a small perturbation/infinitesimal variation, as clarified in [21].

Definition 1 (see [15, Definition 2]): The nonlinear system (3) is \(p\)-dominant with rate \(\lambda \geq 0\) if and only if there exist a symmetric matrix \(P\) with inertia \((p, 0, n-p)^1\) and \(\varepsilon \geq 0\) such that the prolonged system (4) satisfies the conic constraint
\[
\begin{bmatrix}
\dot{\delta x} \\
\delta x
\end{bmatrix}^T
\begin{bmatrix}
0 & P \\
P & 2\lambda P + \varepsilon I
\end{bmatrix}
\begin{bmatrix}
\dot{\delta x} \\
\delta x
\end{bmatrix} \leq 0
\]  
for all \((x, \delta x) \in \mathbb{R}^{2n}\). The property is strict if \(\varepsilon > 0\).

The conic constraint (5) can be equivalently formulated as
\[
(\partial f(x)^T + \lambda I)P + P(\partial f(x) + \lambda I) \preceq -\varepsilon I \quad \forall x \in \mathbb{R}^n.
\]

\(p\)-dominance guarantees that \(n-p\) eigenvalues of the Jacobian matrix \(\partial f(x)\) lie to the left of \(-\lambda\) while the remaining \(p\) eigenvalues lie to the right of \(-\lambda\), for each \(x\) [15, Th. 3]. This splitting, uniform with respect to \(x\), is a necessary condition for dominance.

Dominance can be also characterized in the frequency domain, for systems that have a Lure representation.

Theorem 1 (see [16, Corollary 4.5]): Consider the Lure feedback system in Fig. 2(a) given by the negative feedback interconnection of the linear system \(G(s)\) and the static nonlinearity \(\varphi\), satisfying the sector condition \(0 \leq \varphi \leq K\). The closed system is strictly \(p\)-dominant with rate \(\lambda\) if
1) the real part of all the poles of \(G(s)\) is not \(-\lambda\);
2) the shifted transfer function \(G(s - \lambda)\) has \(p\) unstable poles;
3) the Nyquist plot of \(G(s - \lambda)\) lies to the right of the vertical line passing through the point \(-1/K\) on the Nyquist plane [as in Fig. 2(b)].

We are particularly interested in \(p\)-dominant systems with a small degree \(p \leq 2\). A small degree guarantees that the nonlinear system possesses a simple attractor.

Theorem 2 (see [15, Corollary 1]): Consider a \(p\)-dominant system \(\dot{x} = f(x), x \in \mathbb{R}^n\), with dominant rate \(\lambda \geq 0\). Every bounded trajectory of the system asymptotically converges to
1) a unique fixed point if \(p = 0\);
2) a fixed point if \(p = 1\);
3) a simple attractor if \(p = 2\), that is, a fixed point, a set of fixed points and connecting arcs, or a limit cycle.

Theorem 2 shows how dominance theory can be used to shape the behavior of the mixed-feedback closed loop. We will enforce oscillations by deriving the set of the control parameters that guarantees 2-dominance and instability of closed-loop equilibria. For these parameters, Theorem 2 guarantees that a system with bounded trajectories must have a limit cycle. In what follows, we will use the terminology of stable oscillations to denote the existence of a stable limit cycle in the state space of the nonlinear system (3).

The action of the mixed feedback controller is regulated by the parameters \(k \geq 0\) and \(0 \leq \beta \leq 1\), where \(k\) regulates the overall feedback gain while \(\beta\) regulates the balance between positive and negative feedback. The linear part of the mixed feedback controller has transfer function
\[
C(s, k, \beta) = \frac{k((\beta \tau_p + \tau_n) - \tau_p)s + 2\beta - 1}{(\tau_p s + 1)(\tau_n s + 1)}.
\]  

The static nonlinearity \(\varphi\) represents a monotone, slope restricted, bounded actuation stage. In this article, \(\varphi\) is a differentiable, sigmoidal function with slope \(0 \leq \varphi' \leq 1\) (here and in what follows \(\varphi'\) denotes \(\partial \varphi/\partial \varphi\)). We also assume that \(|\varphi| \leq M\), for some finite number \(M\). This guarantees the boundedness of the closed-loop trajectories for any selection of the feedback parameters \(k\) and \(\beta\). For simplicity, each simulation in this article will adopt \(\varphi = \tanh\).

The closed loop can be represented as a Lure system as shown in Fig. 2(a), where
\[
G(s, k, \beta) = -C(s, k, \beta) P(s).
\]  

III. DOMINANCE THEORY

Our analysis (and design) of the mixed feedback controller uses tools from dominance theory and differential dissipativity, which show strong contact points with the theory of monotone systems with respect to high rank cones [18], [19] and with contributions that extend the Poincaré–Bendixon theorem to large dimensional systems [20]. We refer the reader to [15] and [16] for a detailed introduction of the framework and a comparison with the literature. In what follows, we summarize the main results of the theory used in this article.

Consider a nonlinear system of the form
\[
\dot{x} = f(x) \quad x \in \mathbb{R}^n
\]  
where \(f\) is a continuously differentiable vector field. Dominance theory makes use of the system linearization along arbitrary trajectories, also for nonresonant plants, whose dynamics typically have a fast decay rate.
IV. FAST POSITIVE/SLOW NEGATIVE MIXED FEEDBACK

We first show that fast positive feedback and slow negative feedback, \( \tau_p < \tau_n \), are necessary for the design of oscillators based on 2-dominance. This agrees with several observations from biology and neuroscience, which identify in the interplay between fast-positive and slow-negative feedback a source of robust oscillations.

For reasons of space, we keep the discussion short. More details can be found in the arXiv version [22]. We focus on the eigenvalues of the Jacobian of the closed-loop system, computed at a generic point. The goal is to identify the set of parameters that guarantees a uniform splitting of these eigenvalues, for all \( x \) (necessary condition for dominance).

Recall that \( \varphi \) is the only nonlinearity of the system and its derivative satisfies \( 0 < \partial \varphi \leq 1 \). As a consequence, for fixed time constants \( \tau_p \) and \( \tau_n \), the eigenvalues of the Jacobian of the linearized mixed feedback closed loop are contained within the root loci of \( G(s, 1, \beta) \), for any \( \beta \in [0, 1] \). We can, thus, take advantage of root-locus analysis to derive conditions on \( \tau_p \) and \( \tau_n \).

The mixed feedback controller (1) introduces two poles and a zero in open loop

\[
\begin{align*}
    p_p &= -\frac{1}{\tau_p} \\
p_n &= -\frac{1}{\tau_n} \\
z_0 &= -\frac{1}{\beta(\tau_p + \tau_n)}.
\end{align*}
\]

The position of the zero \( z_0 \) is a function of the balance parameter \( \beta \). It approaches \( +\infty \) as \( \beta \rightarrow 0 \) approaches the critical value \( \beta^* = \frac{\tau_p}{\tau_p + \tau_n} \) and crosses 0 for \( \beta = 0.5 \). As \( \beta \) increases within \( (0, \beta^*) \) (weak positive feedback), \( z_0 \) remains to the left of \( p_n \) and moves towards \( -\infty \) on the real axis. While \( z_0 \) approaches \( p_n \) from right as \( \beta \) increases in \( (\beta^*, 1) \) (strong positive feedback).

For \( \tau_p < \tau_n \) and \( \beta^* < \beta < 1 \), the closed-loop system admits a root locus of positive feedback convention, as shown in Fig. 3. For small \( k \geq 0 \), all the poles of the linearized system are stable. In this case, the system is compatible with 0-dominance for \( \lambda = 0 \). The equilibrium at 0 remains stable and no oscillations occur. Note also that the poles of the mixed feedback controller lie to the right of the poles of the plant for a sizable interval of gains \( 0 \leq k < k_c \) (\( k_c \) could be \( \infty \) for plants with small relative degree), as guaranteed by the position of the zero \( z_0 \). Thus, the system is also compatible with 2-dominance with rate \( \lambda > 1/\tau_p \), but such that \( -\lambda \) remains to the right of the poles of \( P(s) \). Furthermore, when \( z_0 \) has positive real part, \( k \) large enough guarantees that the poles of the linearized system cross the imaginary axis. The origin of the closed loop becomes unstable and nonlinear behaviors like multistability and oscillations may appear.

Asymptotes belongs to right-half plane. This guarantees that the origin becomes unstable for large \( k \). We can draw the following conclusions.

1) There is an unstable region for \( k \) sufficiently small.
2) For \( \beta^* < \beta < 1 \) and for a suitable subset of \( 0 < \beta < \beta^* \), the mixed feedback system can be 2-dominant. Nonlinear behaviors like multistability and oscillations may emerge when \( k \) is sufficiently large.

V. 2-DOMINANT MIXED FEEDBACK FOR OSCILLATIONS

A. Control Design

In this section, we outline a design procedure to find the parameter ranges of gain and balance, \( k \) and \( \beta \), for stable oscillations. For reasons of space, we omit proofs and analysis. The results are a straightforward extension of the preliminary results in [14] to general plants. We refer the reader to the arXiv version of this article [22] for details.

Following the discussions in Section III, the design procedure has two steps. The first makes use of Theorem 1 to identify gain and balance ranges that guarantee 2-dominance.

**Theorem 3:** For any constant reference \( r \) and any \( \beta \in [0, 1] \), the mixed feedback system in Fig. 1 is 0-dominant with rate \( \lambda = 0 \) for any gain \( 0 \leq k < k_0 \), where

\[
k_0 = \begin{cases} 
    \infty & \text{if } \min_\omega \Re(G(j\omega, 1, \beta)) \geq 0 \\
    \frac{1}{\min_\omega \Re(G(j\omega, 1, \beta))} & \text{otherwise.}
\end{cases}
\]

**Theorem 4:** Consider a rate \( \lambda \) for which the transfer function \( G(s - \lambda, 1, \beta) \) has two unstable poles. Then, for any constant reference \( r \) and any \( \beta \in [0, 1] \), the mixed feedback system in Fig. 1 is 2-dominant with rate \( \lambda \) for any gain \( 0 \leq k < k_2 \), where

\[
k_2 = \begin{cases} 
    \infty & \text{if } \min_\omega \Re(G(j\omega - \lambda, 1, \beta)) \geq 0 \\
    \frac{1}{\min_\omega \Re(G(j\omega - \lambda, 1, \beta))} & \text{otherwise.}
\end{cases}
\]

Theorems 3 and 4 state conditions on the gain \( k \) for 0-dominance and 2-dominance, respectively. These agree with the discussion in Section IV. The root locus analysis suggests that \( k_2 \) is greater than \( k_0 \), since 2-dominance is compatible with unstable equilibria. To explore the oscillatory regime, we will look at the range of gains \( k_0 < k < k_2 \).

The second step focuses on the stability of the closed-loop equilibria. This allows us to use Theorem 2 to certify oscillations. Denote by \( u_1 \), \( y_1 \) the input–output pair of \( G(s, k, \beta) \) and by \( u_2 \), \( y_2 \) the input–output pair of \( \varphi' \), as shown in Fig. 2(a). At each equilibrium, the closed-loop system satisfies

\[
\begin{align*}
    y_1 &= G(0, k, \beta)u_1 = -kP(0)(2\beta - 1)u_1 \\
    y_2 &= \varphi(u_2) \\
    y_1 &= u_1 - r - y_2 \\
    u_2 &= y_1 
\end{align*}
\]
that is
\[ \frac{y_1}{kP(0)(2\beta - 1)} + r = \varphi(y_1). \]  

(9)

The slope \( \frac{1}{kP(0)(2\beta - 1)} \) determines the number and the position of the closed-loop equilibria, whose stability can be checked by local analysis. Full details are provided in [22]. For \( k_0 < k < k_2 \), oscillations occur when the equilibria are unstable.

By combining these two steps, we can locate the parametric ranges of \( k \) and \( \beta \) that guarantee a stable oscillation in the mixed feedback closed loop.

B. Robust Oscillations to Plant Uncertainties

The combination of circle criterion for dominance (Theorem 1) and of Nyquist criterion for instability of closed-loop equilibria opens the way to robust analysis (and design). Mimicking classical stability theory, the robustness of the closed-loop oscillations to plant uncertainties is captured by the maximal perturbation that the Nyquist locus can undertake before (i) entering the shaded region in Fig. 2(b) (robustness measure for 2-dominance) and (ii) changing the number of turns around the \(-1\) point of the complex plane, for each equilibrium (robustness of the instability of each equilibrium). This leads to quantifiable bounds on the multiplicative/additive plant uncertainties that the closed loop can sustain while preserving oscillations.

Consider a bounded, fast (poles lie to the left of the dominance rate \(-\lambda\)) additive uncertainty, \( \mathcal{P}_\Delta(s) = \mathcal{P}(s) + \Delta(s) \). Then, the perturbed transfer function reads
\[ G_\Delta(s, k, \beta) = G(s, k, \beta) + \Delta(s)C(s, k, \beta). \]  

(10)

To apply the circle criterion for dominance, we consider the shifted transfer function \( G_\Delta(s - \lambda, k, \beta) \). The shifted perturbation on the nominal plant reads \( \Delta(s - \lambda)C(s - \lambda, k, \beta) \), which corresponds to a graphical perturbation on the nominal Nyquist locus bounded by
\[ \delta := \sup_{\omega \in (0, \infty)} |\Delta(j\omega - \lambda)C(j\omega - \lambda, k, \beta)|. \]

Thus, by continuity, for \( \delta \) sufficiently small (i.e., for \( |\Delta(j\omega - \lambda)| \) sufficiently small at each frequency \( \omega \)), the Nyquist plot of \( G_\Delta(s - \lambda, k, \beta) \) must remain to the right of the vertical axis passing through \(-1\). That is, the perturbed closed-loop system remains 2-dominant.

A similar argument applies to the stability/instability of each equilibrium. The equilibria are now related to \( \mathcal{P}_\Delta \) in (9)
\[ \frac{y_1}{kP(0)(2\beta - 1)} + r = \varphi(y_1). \]

This shows that the perturbation on the positions of the equilibria is a function of the dc gain \( \Delta(0) \). Their stability/instability can be studied via local analysis, using the Nyquist criterion on the system linearized at each equilibrium. This amounts to classical robustness analysis (for stability/instability) and enforces additional constraints on the perturbation bound \( \delta \).

The simplicity of the analysis above leaves open questions about design. Finding gain \( k \) and balance \( \beta \) to achieve a prescribed level of robustness remains a challenging problem. Optimizing such parameters using Nyquist analysis is hard and typically requires several iterations. This motivates the development of a more systematic and general framework to support design, based on \( p \)-dissipativity and LMIs.

Remark 1: We have focused on fast dynamic uncertainties for reasons of simplicity. These guarantee that the poles of the perturbed plant remain to the left of the \(-\lambda\) axis. In the context of classical robust stability, this assumption would correspond to the restriction to stable uncertainties. The analysis of robustness to a wider class of uncertainties requires a more general version of the circle criterion for dominance [16], and an extended use of the Nyquist criterion for the robustness analysis of stable/unstable equilibria.

VI. \( p \)-Dissipativity and LMIs

A. \( p \)-Dissipativity

As in Section III, we summarize here a minimal set of results on differential dissipativity [15], [16]. These are needed to develop LMI-based design of mixed-feedback oscillators.

Consider the open nonlinear system of the form
\[ \begin{align*}
\dot{x} &= f(x) + Bu \\
y &=Cx + Du
\end{align*} \]

where \( B \in \mathbb{R}^{n \times m} \), \( C \in \mathbb{R}^{m \times n} \), and \( D \in \mathbb{R}^{m \times m} \). The prolonged system, derived through linearization, reads
\[ \begin{align*}
\dot{x} &= f(x) + Bu \\
\delta x &= \delta f(x)\delta x + B\delta u \\
y &= Cx + Du \\
\delta y &= C\delta x + D\delta u.
\end{align*} \]

Definition 2 (see [15, Definition 3]): The nonlinear system (11) is differentially \( p \)-dissipative with rate \( \lambda \geq 0 \) and differential supply rate
\[ \begin{bmatrix}
\delta y \\
\delta u
\end{bmatrix} \preceq \begin{bmatrix}
Q & L \\
L & R
\end{bmatrix} \begin{bmatrix}
\delta y \\
\delta u
\end{bmatrix} \]

(13)

if there exist some symmetric matrix \( P \) with inertia \((p, 0, n - p)\) and some constant \( \varepsilon \geq 0 \), such that the prolonged system (12) satisfies the conic constraint
\[ \begin{bmatrix}
\delta y \\
\delta u
\end{bmatrix} \preceq \begin{bmatrix}
0 & P \\
P & 2\lambda P + \varepsilon I
\end{bmatrix} \begin{bmatrix}
\delta x \\
\delta u
\end{bmatrix} \leq \begin{bmatrix}
\delta y \\
\delta u
\end{bmatrix} \preceq \begin{bmatrix}
Q & L \\
L & R
\end{bmatrix} \begin{bmatrix}
\delta y \\
\delta u
\end{bmatrix} \]

(14)

for all \((x, \delta x) \in \mathbb{R}^{2n}\) and all \((u, \delta u) \in \mathbb{R}^{2m}\). The property is strict if \( \varepsilon > 0 \).

Equation (14) corresponds to a dissipativity inequality of the form
\[ V(\delta x) \leq s(\delta y, \delta u) \]

for \( \delta y = \delta x^T P \delta x \) and for \( s(\delta y, \delta u) \) given by (13), applied to a prolonged system with shifted Jacobian \( \delta f(x) + \lambda I \). \( p \)-dissipativity replaces the usual constraint \( P > 0 \) with a constraint on its inertia.

We will use of two particular supplies: the gain supply
\[ \begin{bmatrix}
\delta y \\
\delta u
\end{bmatrix} \preceq \begin{bmatrix}
-0I & 0 \\
0 & \gamma^2 I
\end{bmatrix} \begin{bmatrix}
\delta y \\
\delta u
\end{bmatrix} \]

(15)

where we call \( \gamma \) the \( p \)-gain, and the passivity supply
\[ \begin{bmatrix}
\delta y \\
\delta u
\end{bmatrix} \preceq \begin{bmatrix}
-\alpha I & I \\
I & \mu I
\end{bmatrix} \begin{bmatrix}
\delta y \\
\delta u
\end{bmatrix} \]

(16)

where \( \alpha > 0 \) (\( \mu < 0 \)) denotes excess of output (input) passivity, and \( \alpha < 0 \) (\( \mu > 0 \)) denotes shortage of output (input) \( p \)-passivity, respectively.

The notion of \( p \)-gain combined with the small gain theorem for dominance [15], [23] provides a framework for robust control of dominant systems.

Theorem 5 (Small gain interconnection): Let \( \mathcal{S}_i \) be a \( p_i \)-dominant system with input \( u_i \), output \( y_i \), and differential \( p \)-gain \( \gamma_i \in \mathbb{R}_+ \) with rate \( \lambda > 0 \), for \( i \in \{1, 2\} \). Then, the closed system \( \mathcal{S}_i \) defined by the feedback interconnection
\[ u_1 = y_1, \quad u_2 = y_2 \]

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of $\Sigma_1$ and $\Sigma_2$ is $(p_1 + p_2)$-dominant with rate $\lambda$ if $\gamma_1 \gamma_2 < 1$.

Like classical passivity, $p$-passivity is preserved by negative feedback, as clarified by the following theorem (see [15, Th. 4]).

**Theorem 6:** Let $\Sigma_1$ be a $p_i$-passive system from input $u_i$ to output $y_i$ with dominant rate $\lambda_i \geq 0$ and supply rate

$$
\frac{\partial f}{\partial y_i} \begin{bmatrix} \alpha_i I & 1 \\ I & \mu_i I \end{bmatrix} \begin{bmatrix} \delta y_i \\ \delta u_i \end{bmatrix} \leq 0
$$

for $i \in \{1, 2\}$. Then, the closed-loop system defined by the negative feedback interconnection

$$u_1 = -y_2, \quad u_2 = y_1$$

is $p_1 + p_2$-dominant if $\alpha_1 - \mu_2 \geq 0$ and $\alpha_2 - \mu_1 \geq 0$.

**B. Convex Relaxation for LMI Design**

Solutions to (5) and (14) can be obtained via convex relaxation, by confining the system Jacobian within the convex hull of a finite set of linear matrices $A := \{A_1, \ldots, A_N\}$. Namely, for all $x$, we need $\frac{\partial f(x)}{\partial x} = \sum_{i=1}^{N} \rho_i(x) A_i$ for some $\rho_i(x)$ satisfying $\sum_{i=1}^{N} \rho_i(x) = 1$ [15, Sec. VI.B].

Recall that condition (5) is equivalent to

$$\frac{\partial f(x)}{\partial x}^T P + P \frac{\partial f(x)}{\partial x} + 2\lambda P + \epsilon I \preceq 0 \quad \forall x \in \mathbb{R}^n. \tag{18}$$

Thus, if $\frac{\partial f(x)}{\partial x} \in \text{ConvexHull}(A)$ for all $x$, a solution to (18) is given by then any (uniform) solution $P$ to

$$A_i^T P + PA_i + 2\lambda P + \epsilon I \preceq 0, \quad i \in \{1, \ldots, N\} \tag{19}$$

is also a solution to LMI (18).

Likewise, for the supply rate (15) and (16), solutions to (14) can be obtained by finding a solution $P$, with inertia $(p_0, n - p)$, to

$$\begin{bmatrix} A_i^T P + PA_i + 2\lambda P + \epsilon I & PB & CT^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} \leq 0 \tag{20}$$

and to

$$\begin{bmatrix} A_i^T P + PA_i + 2\lambda P + \epsilon I & PB - CT^T \\ B^T P - C & -\mu I & D^T \\ C & D & -\frac{1}{\alpha} \end{bmatrix} \leq 0 \tag{21}$$

respectively for $i \in \{1, \ldots, N\}$.

The convex hull of the mixed feedback controller is given by a family of two matrices, corresponding to the system’s Jacobian associated to $\frac{\partial \varphi}{\partial y}$ and $\frac{\partial \varphi}{\partial u}$.

**VII. LMI DESIGN FOR CLOSED-LOOP OSCILLATIONS**

In this section, we develop LMIs to design oscillators. We consider the generalized mixed feedback closed loop shown in Fig. 5, where we have replaced gain and balance parameters of Section IV with a full state feedback from the controller and plant states. The system has the state-space representation

$$\begin{bmatrix} \dot{x} \\ y \\ u \end{bmatrix} = \begin{bmatrix} A & B & C \\ C & D & \mu \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \varphi(Kx) + r \end{bmatrix} \in \mathbb{R}^{n+2}. \tag{22}$$

**Theorem 7:** If there exist a symmetric matrix $Y$ with inertia $(2, 0, n)$, a matrix $Z$, and $\epsilon > 0$ such that

$$\begin{bmatrix} YA^T + AY + 2\lambda Y + \epsilon I \leq 0 \\ YA^T + ZB^T + AY + BZ + 2\lambda Y + \epsilon I \leq 0 \end{bmatrix} \tag{23}$$

then the state feedback matrix $K = YZ^{-1}$ guarantees $2$-dominance in closed loop with rate $\lambda$.

The feasibility of (23) follows from Section V, Theorem 4, since the selection of gain and a balance corresponds to a particular state feedback $K$. The inertia constraint on $Y$ makes the optimization problem nonconvex. However, there is no need to enforce this constraint explicitly. The first inequality in (23) guarantees that $Y$ has inertia $(2, 0, n)$ whenever two eigenvalues of $A$ fall to the right of $-\lambda$. This also implies that the plant dynamics limit the design of the closed loop, by imposing a constraint on the time constants $\gamma_p$ and $\gamma_n$, which must be sufficiently slow. This affects the achievable oscillation frequency. To recover design flexibility, a precompensation feedback could be introduced. We refer the reader to the arXiv version [22, Remark 5] for details.

To induce a stable oscillation in closed loop, we combine (23) with the following constraint:

$$YA^T + ZB^T + AY + BZ + \epsilon I \leq 0. \tag{24}$$

For $r = 0$ ($r \neq 0$ is similar), the constraint on the inertia of $Y$ in Theorem 7 combined with (24) guarantees that the equilibrium point at the origin is unstable.

In agreement with Section V, the dc gain of the linear open loop component is $-KA^T B$, that is, the slope of the line in (9) is now $\frac{-1}{KA^T B}$. The closed loop has a single equilibrium if $-KA^T B < 1$, which is unstable by (24). This guarantees stable oscillations in closed loop (given the boundedness of the closed-loop trajectories). Multiple equilibria will appear for $-KA^T B > 1$, which may lead to a region of coexistence of oscillations and stable fixed points. To reduce the control gains $|K|$ when $-KA^T B > 1$, we could add the constraint

$$\begin{bmatrix} -\nu & Z \\ Z^T & -I \end{bmatrix} \leq 0 \tag{25}$$

where the constant $\nu > 0$ limits the norm square of matrix $Z$, i.e., by Schur complement $ZZ^T \leq \nu$. Since $K = YZ^{-1}$, if $Y$ does not change dramatically, the parameter $\nu$ effectively control the magnitude of $K$.

**Remark 2:** The combination of (23) and (24) leads to the automatic derivation of state-feedback gains for oscillations. The design takes advantage of the particular feedback structure (22), which is limited to linear plants for simplicity. The approach can be extended to nonlinear plants of the form $\dot{x} = f(x) + Bu$ by taking advantage of more general convex hull relaxations, as discussed in Section VI.B.

**Example 1 (Mixed state-feedback of a first order plant):** For illustration, we consider the design task of driving a first order plant into oscillation using LMIs. The linear component has matrices

$$A = \begin{bmatrix} -\frac{1}{\tau_p} & 0 & 0 \\ \frac{1}{\tau_n} & -\frac{1}{\tau_p} & 0 \\ \frac{1}{\tau_n} & 0 & -\frac{1}{\tau_n} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{\tau} \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}. \tag{26}$$

Fig. 5. Block diagram of the generalized mixed feedback closed loop.
where $\tau_1 = 0.01$, $\tau_p = 0.1$, $\tau_n = 1$. Setting $\lambda = 50$ and solving (23) and (24) with CVX [24], we get

$$Y = \begin{bmatrix} 0.3788 & -0.8923 & -0.2650 \\ -0.8923 & -0.5368 & -0.2545 \\ -0.2650 & -0.2545 & -0.2053 \end{bmatrix}. $$

$Y$ has inertia $(2,0,1)$ and the controller gains read

$$K = Z Y^{-1} = \begin{bmatrix} 0.5284 & 0.9623 & -0.6342 \end{bmatrix}. $$

The dc gain $-K A^{-1} B = 0.8565 < 1$ guarantees a unique unstable equilibrium point and, hence, stable oscillations for $r = 0$ (see Fig. 6, left).

LMI{s} can also be leveraged to handle parametric uncertainties, for example, on the time constants of the mixed feedback controller $C(s, k, \beta)$, as it is often the case in the biological setting. A robust solution to the case of 20% perturbation on $\tau_p$ and $\tau_n$ is illustrated in Fig. 6, right. We refer the reader to the arXiv version [22] for more details.

Even if we did not enforce any specific sign pattern on the controller gains, the controllers show a negative gain associated with the slow network $C_n$ and a positive gain associated with the fast network $C_p$. This is an additional confirmation of the role of mixed-feedback in the generation of stable oscillations.

**VIII. LMI DESIGN FOR ROBUSTNESS AND PASSIVITY**

**A. Robust Oscillations Via Robust 2-Dominance**

In this section, we will consider a robust design for dynamic uncertainties. Our approach takes advantage of Theorem 5, the small gain results for dominance theory.

Consider the mixed feedback closed loop with dynamic uncertain dynamics $w = \Delta(z)$ (not necessarily linear)

$$\begin{align*}
\dot{x} &= A x + B_1 u + B_2 w \\
y &= C_1 x, \quad z = C_2 x \\
u &= \varphi(K x)
\end{align*}$$

(27)

where $A$, $B_1$, and $C_1$ are the nominal state space matrices as (22). $B_2$ and $C_2$ characterize how the uncertain dynamics affect the nominal system.

**Theorem 8:** Suppose that the uncertain dynamics $w = \Delta(z)$ has $0$-gain less than $\frac{1}{2}$ with rate $\lambda$. The closed loop given by (27) and $\Delta$ is $2$-dominant with rate $\lambda$ if there exist a symmetric matrix $Y$ with inertia $(2, 0, n)$, a matrix $Z, \varepsilon > 0$ such that

$$\begin{align*}
\begin{bmatrix} Y A^T + A Y + 2 \lambda Y + \varepsilon I & B_2 & Y C_2^T \\
B_2^T & -\gamma I & 0 \\
C_2 Y & 0 & -\gamma I \end{bmatrix} &\leq 0 \quad (28a) \\
\begin{bmatrix} Y A^T + ZB_1^T + A Y + B_1 z + 2 \lambda Y + \varepsilon I & B_2 & Y C_2^T \\
B_2^T & -\gamma I & 0 \\
C_2 Y & 0 & -\gamma I \end{bmatrix} &\leq 0 \quad (28b)
\end{align*}$$

and $K = Z Y^{-1}$.

For large gains $\gamma$, the feasibility of (28) reduces to the feasibility of (23) (discussed in Section VII). The assumption on the dynamic uncertainties guarantees that $\Delta$ belongs to a family of incrementally stable perturbations whose decay rate is faster than $\lambda$. That is, the perturbed plant dynamics remain faster than the controller dynamics, as in the nominal case.

Together with 2-dominance, to guarantee robust oscillations, we need to ensure robust instability of the equilibrium point at the origin. This can be established via robustness criteria for instability (see, e.g., [25]), or we can use again small gain results for dominance.

1) Consider the system linearization at the equilibrium point given by $\bar{A} = A + BK$. We can combine (23) and (24) with (20), the latter for $A_0 = \bar{A}$, $\lambda = 0$, and for $\gamma = \gamma_{\text{ins}}$ to certify that the instability will be preserved by any perturbation $\Delta$ with $0$-gain less than $1/\gamma_{\text{ins}}$ (for $\lambda = 0$). This follows from Theorem 5.

2) We can also design the feedback $K$ to enforce the desired level of robust instability of the equilibrium. To achieve this, we pair (28) to an additional LMI of the form (28b) where we take $\lambda = 0$ and $\gamma = \gamma_{\text{ins}}$.

**Remark 3:** The control action represented by $K$ has no effect on the closed-loop gains when $\partial \varphi(K x) = 0$. This means that the open-loop plant features limit the achievable $2$-gain. Performances can be improved via precompensation, as discussed in the arXiv version [22, Remark 7].

**B. 2-Passivity and Interconnections**

The mixed feedback closed loop can also be adapted to passive interconnections, taking advantage of Theorem 6. The goal of this section is to design the controller gains to achieve 2-dominance for closed-loop interconnections represented. We assume that $P_{\alpha}(z)$ is a generic external dynamics, $0$-passive with excess of output passivity $\alpha$ at rate $\lambda$. This implies that $P_{\alpha}$ has fast transients and its shifted dynamics are incrementally passive.

**Theorem 9:** Consider a $0$-passive system $P_{\alpha}$ with excess of output passivity $\alpha > 0$ at rate $\lambda$. Then, the closed loop given by (27) and $w = -P_{\alpha}(z)$ is 2-dominant if there exist a symmetric matrix $Y$ with inertia $(2, 0, n)$, a matrix $Z, \mu < \alpha$, and $\varepsilon > 0$ such that

$$\begin{align*}
\begin{bmatrix} Y A^T + A Y + 2 \lambda Y + \varepsilon I & B - Y C_2^T \\
B^T - C Y & -\mu I \end{bmatrix} &\leq 0 \quad (29a) \\
\begin{bmatrix} Y A^T + ZB_1^T + A Y + B_1 z + 2 \lambda Y + \varepsilon I & B - Y C_2^T \\
B^T - C Y & -\mu I \end{bmatrix} &\leq 0 \quad (29b)
\end{align*}$$

and $K = Z Y^{-1}$.

When the shortage of input passivity is large, $\mu \gg 0$, the feasibility of (29) reduces to the feasibility of (23). Equation (29) guarantees that the closed-loop system given by the plant $P$ and the mixed feedback
controller $C$ is 2-passive from $w$ to $z$. This property is later used to guarantee that negative feedback interconnections with 0-passive dynamics preserve 2-dominance. As in the previous section, 2-dominance is not enough to guarantee the stability of equilibria. Additional conditions must be enforced to guarantee the instability of equilibria.

**Example 2 (Controlled oscillations in a large circuit):** Consider the large circuit in Fig. 7. $\Sigma_a$ is given by the interconnection of the mixed feedback controller with an RC circuit (plant). $\Sigma_a$ can be considered as a simplified conductance-based model of a neuron, with the mixed feedback controller modeling fast and slow conductances affecting the dynamics of the neuron membrane. $\Sigma_a$ represents a spatially discretized cable dynamics, modeling how current and voltage distribute along neurites. Their interconnection satisfies $v_0^a = v_0^b$ and $i_0^a = -i_0^b$.

The mixed feedback loop $\Sigma_a$ is given by (22). From (26), $A$ is given by $\tau_i = R_0C_0 = 0.01 (R_0 = 100$ and $C_0 = 10^{-4})$, and we keep $\tau_p = 0.1$ and $\tau_n = 1$, as in the other examples. Considering $v_0^a$ as input and $v_0^b$ as output, we have $B = \frac{1}{C_0} [0 \ 0]^T$ and $C = [1 \ 0 \ 0]$. $\Sigma_b$ has input $v_0^b$ and output $v_0^b$. The model is taken from cable theory [26], where $R_1$ represents the resistance along the fiber and the parallel of $R_2$ and $C_m$ represents the impedance of each segment. For $n$ segments, the admittance of $\Sigma_b$ is recursively described by

$$G_n(s) = \frac{1}{R_1 + \frac{1}{C_m s + \frac{1}{1/R_2 + G_{n-1}(s)}}}$$

with base case $G_1(s) = \frac{1}{R_1 + \frac{1}{C_m/R_2 + 1}}$. It is easy to show that $G_1(s)$ is positive real (passive). The same result holds for the shifted transfer function $G_1(s - \lambda)$, if $\lambda < 1/C_mR_2$, which captures the fact that the zero of $G_1(s)$ lies to the left of $-\lambda$.

Under such condition, by induction, $\Sigma_a$ remains 0-passive for rate $\lambda < 1/C_mR_2$, since addition and inversion in (30) preserve passivity, and the elements on the right-hand side of (30) are all positive real. We can further deduce that $|G_n(j\omega - \lambda)| \leq \frac{1}{\lambda \pi^2}$, if $\lambda < 1/C_mR_2$, which indicates that $\Sigma_a$ has an excess of output passivity. As parameters, we take $C_m = C_0 = 10^{-4}$, $R_1 = R_0 = 100$, and $R_2$ varying in [300, 600].

We consider the mixed state feedback design for passive interconnection and set the dominant rate $\lambda = 15$. For all $R_2 \in [300, 600]$, $\Sigma_b$ is 0-passive with an excess of passivity $\alpha > 30$. This is verified using (21) on a minimal state space realization of (30). Thus, following Theorem 9, the state-feedback gains of the mixed feedback loop $\Sigma_a$ are obtained by setting $\mu = 30$ in (29). We also enforce (24) to destabilize the equilibrium at 0. The solution is

$$Y = \begin{bmatrix} 18836.5 & -724.7 & -85.5 \\ -724.7 & -696.9 & -138.6 \\ -85.5 & -138.6 & -81.7 \end{bmatrix}$$

has inertia (2,0,1) and the controller gains read

$$K = ZY^{-1} = \begin{bmatrix} -3.1117 & 7.1900 & -6.5486 \end{bmatrix}.$$
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