Optimality certificates for convex minimization
and Helly numbers

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Abstract
We consider the problem of minimizing a convex function over a subset of \( \mathbb{R}^n \) that is not necessarily convex (minimization of a convex function over the integer points in a polytope is a special case). We define a family of duals for this problem and show that, under some natural conditions, strong duality holds for a dual problem in this family that is more restrictive than previously considered duals.

1 Introduction
Insights obtained through duality theory have spawned efficient optimization algorithms (combinatorial and numerical) which simultaneously work on a pair of primal and dual problems. Striking examples are Edmonds’ seminal work in combinatorial optimization, and interior-point algorithms for numerical/continuous optimization.

Compared to duality theory for continuous optimization, duality theory for mixed-integer optimization is still underdeveloped. Although the linear case has been extensively studied, see, e.g., [4, 5, 11, 12], nonlinear integer optimization duality was essentially unexplored until recently. An important step was taken by Morán et al. for conic mixed-integer problems [10], followed up by Baes et al. [2] who presented a duality theory for general convex mixed-integer problems. The approach taken by Moran et al. was essentially algebraic, drawing on the theory of subadditive functions. Baes et al. took a more geometric viewpoint and developed a duality theory based on lattice-free polyhedra. We follow the latter approach.

Given \( S \subseteq \mathbb{R}^n \) and a convex function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), we consider the problem

\[
\inf_{s \in S} f(s).
\]

We describe a geometric dual object that can be used to certify optimality of a solution to (1). For simplicity, let us consider the situation when the infimum of \( f \) over \( \mathbb{R}^n \) and over \( S \) is attained, and let \( x_0 \in \text{arg inf}_{x \in \mathbb{R}^n} f(x) \). We say that a closed set \( C \) is an \( S \)-free neighborhood of \( x_0 \) if \( x_0 \in \text{int}(C) \) and \( \text{int}(C) \cap S = \emptyset \). Using the convexity of \( f \), it follows that for any \( \bar{s} \in S \) and any \( C \) that is an \( S \)-free neighborhood of \( x_0 \), the following holds:

\[
f(\bar{s}) \geq \inf_{z \in \text{bd}(C)} f(z) =: L(C),
\]

where \( \text{bd}(C) \) denotes the boundary of \( C \) (to see this, consider the line segment connecting \( \bar{s} \) and \( x_0 \) and a point at which this line segment intersects \( \text{bd}(C) \)). Thus, an \( S \)-free neighborhood of \( x_0 \) can be interpreted as a “dual object” that provides a lower bound of the type (2). As a consequence, the following is true.
Proposition 1 (Strong duality). If there exist \( \bar{s} \in S \) and \( C \subseteq \mathbb{R}^n \) that is an S-free neighborhood of \( x_0 \), such that equality holds in (2), then \( \bar{s} \) is an optimal solution to (1).

This motivates the definition of a dual optimization problem to (1). For any family \( \mathcal{F} \) of S-free neighborhoods of \( x_0 \), define the \( \mathcal{F} \)-dual of (1) as

\[
\sup_{C \in \mathcal{F}} L(C). \tag{3}
\]

Assuming very mild conditions on \( S \) and \( f \) (e.g., when \( S \) is a closed subset of \( \mathbb{R}^n \) disjoint from \( \text{arg inf}_{x \in \mathbb{R}^n} f(x) \)), it is straightforward to show that if \( \mathcal{F} \) is the family of all S-free neighborhoods of \( x_0 \), then strong duality holds, i.e., there exists \( \bar{s} \in S \) and \( C \in \mathcal{F} \) such that the condition in Proposition 1 holds. However, the entire family of S-free neighborhoods is too unstructured to be useful as a dual problem. Moreover, the inner optimization problem (2) of minimizing on the boundary of \( C \) can be very hard if \( C \) has no structure other than being S-free. Thus, we would like to identify subfamilies \( \mathcal{F} \) of S-free neighborhoods that still maintain strong duality, while at the same time, are much easier to work with inside a primal-dual framework. We list below three subclasses that we expect to be useful in this line of research. First, we need the concept of a gradient polyhedron:

Definition 2. Given a set of points \( z_1, \ldots, z_k \in \mathbb{R}^n \),

\[
Q := \{ x \in \mathbb{R}^n : \langle a_i, x - z_i \rangle \leq 0, \ i = 1, \ldots, k \}
\]

is said to be a gradient polyhedron of \( z_1, \ldots, z_k \) if for every \( i = 1, \ldots, k \), \( a_i \in \partial f(z_i) \), i.e., \( a_i \) is a subgradient of \( f \) at \( z_i \).

We consider the following families.

– The family \( \mathcal{F}_{\max} \) of maximal convex S-free neighborhoods of \( x_0 \), i.e., those S-free neighborhoods that are convex, and are not strictly contained in a larger convex S-free neighborhood.

– The family \( \mathcal{F}_\partial \) of convex S-free neighborhoods that are also gradient polyhedra for some finite set of points in \( \mathbb{R}^n \).

– The family \( \mathcal{F}_{\partial, S} \) of convex S-free neighborhoods that are also gradient polyhedra for some finite set of points in \( S \).

We propose the above families so as to leverage a recent surge of activity analyzing their structure; the surveys [3] and Chapter 6 of [6] provide good overviews and references for this whole line of work. This well-developed theory provides powerful mathematical tools to work with these families. As an example, this prior work shows that for most sets \( S \) that occur in practice (which includes the integer and mixed-integer cases), the family \( \mathcal{F}_{\max} \) only contains polyhedra. This is good from two perspectives:

– polyhedra are easier to represent and compute with than general S-free neighborhoods,

– the inner optimization problem (2) of computing \( L(C) \) becomes the problem of solving finitely many continuous convex optimization problems, corresponding to the facets of \( C \).

Of course, the first question to settle is whether these three families actually enjoy strong duality, i.e., do we have strong duality between (1) and the \( \mathcal{F}_{\max} \)-dual, \( \mathcal{F}_\partial \)-dual and \( \mathcal{F}_{\partial, S} \)-dual? It turns out that the main result in [2] shows that for the mixed-integer case, i.e., \( S = C \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \) for some convex set \( C \), the \( \mathcal{F}_\partial \)-dual enjoys strong duality under conditions of the Slater type from continuous optimization. It is not hard to strengthen their result to also show that the \( \mathcal{F}_{\max} \cap \mathcal{F}_\partial \)-dual is a strong dual, under some additional assumptions.

In this paper, we give conditions on \( S \) and \( f \) such that strong duality holds for the dual problem (3) associated with \( \mathcal{F}_{\max} \cap \mathcal{F}_\partial \cap \mathcal{F}_{\partial, S} \). Below we give an explanation as to why this family is very desirable. If these conditions on \( S \) and \( f \) are met, our result is stronger than Baes et al. [2]. For example, when \( S \) is the set of integer points in a compact convex set and \( f \) is any convex function, our
certificate is a stronger one. However, our conditions on $S$ and $f$ do not cover certain mixed-integer problems; whereas, the certificate from Baes et al. still exists in these settings. Nevertheless, it can be shown that in such situations, a strong certificate like ours does not necessarily exist.

**Definition 3.** A strong optimality certificate of size $k$ for (1) is a set of points $z_1, \ldots, z_k \in S$ together with subgradients $a_i \in \partial f(z_i)$ such that

$$Q := \{ x \in \mathbb{R}^n : \langle a_i, x - z_i \rangle \leq 0, \ i = 1, \ldots, k \} \text{ is } S\text{-free},$$

$$\langle a_i, z_j - z_i \rangle < 0 \text{ for all } i \neq j .$$

Recall that $a \in \partial f(z)$ if $f(x) \geq f(z) + \langle a, x - z \rangle$ holds for all $x \in \mathbb{R}^n$. Since $Q$ is $S$-free, for every $s \in S$ there is some $i$ such that $\langle a_i, s - z_i \rangle \geq 0$ and hence $f(s) \geq f(z_i)$. Thus, Property 4 implies that $\min_{s \in S} f(s) = \min_{i \in [k]} f(z_i)$ holds. In other words, given a strong optimality certificate, we can compute (1) by simply evaluating $f(z_1), \ldots, f(z_k)$. This implies that if a strong certificate exists, then the infimum of $f$ over $S$ is attained.

In order to verify that $z_1, \ldots, z_k$ together with $a_1, \ldots, a_k$ form a strong optimality certificate, one has to check whether the polyhedron $Q$ is $S$-free. Deciding whether a general polyhedron is $S$-free might be a difficult task. However, Property 4 ensures that $Q$ is maximal $S$-free, i.e., $Q$ is not properly contained in any other $S$-free closed convex set: Indeed, Property 5 implies that $Q$ is a full-dimensional polyhedron and that $\{ x \in Q : \langle a, x \rangle = 0 \}$ is a facet of $Q$ containing $z_i \in S$ in its relative interior for every $i \in [k]$. Since every closed convex set $C$ that properly contains $Q$ contains the relative interior of at least one facet of $Q$ in its interior, $C$ cannot be $S$-free.

For particular sets $S$, the properties of $S$-free sets that are maximal have been extensively studied and are much better understood than general $S$-free sets. For instance, if $S = (\mathbb{R}^d \times \mathbb{Z}^n) \cap C$ where $C$ is a closed convex subset of $\mathbb{R}^{n+d}$, maximal $S$-free sets are polyhedra with at most $2^n$ facets [6]. In particular, if $S = \mathbb{Z}^2$ the characterizations in [7, 8] yield a very simple algorithm to detect whether a polyhedron is maximal $\mathbb{Z}^2$-free.

In order to state our main result, we need the notion of the Helly number $h(S)$ of the set $S$, which is the largest number $m$ such that there exist convex sets $C_1, \ldots, C_m \subseteq \mathbb{R}^n$ satisfying

$$\bigcap_{i \in [m]} C_i \cap S = \emptyset \quad \text{and} \quad \bigcap_{i \in [m] \setminus \{j\}} C_i \cap S \neq \emptyset \text{ for every } j \in [m].$$

**Theorem 4.** Let $S \subseteq \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that

(i) $\emptyset \notin \partial f(s)$ for all $s \in S$,

(ii) $h(S)$ is finite, and

(iii) for every polyhedron $P \subseteq \mathbb{R}^n$ with $\text{int}(P) \cap S \neq \emptyset$ there exists an $s^* \in \text{int}(P) \cap S$ with $f(s^*) = \inf_{s \in \text{int}(P) \cap S} f(s)$.

Then there exists a strong optimality certificate of size at most $h(S)$.

Let us first comment on the assumptions in Theorem 4. If $\emptyset \notin \partial f(s^*)$ for some $s^* \in S$, then $s^*$ is an optimal solution to (1), as well as to its continuous relaxation over $\mathbb{R}^n$. An easy certificate of optimality in this case is the subgradient $\emptyset$. A quite general situation in which (ii) is always satisfied is the case $S = (\mathbb{R}^d \times \mathbb{Z}^n) \cap C$ where $C \subseteq \mathbb{R}^{d+n}$ is a closed convex set. In this situation, one has $h(S) \leq 2^n(d + 1)$. The characterization of closed sets $S$ for which $h(S)$ is finite has received a lot of attention, see, e.g., [1]. Finally, note that (iii) implies that the minimum in (1) actually exists. As an example, (iii) is fulfilled whenever $S$ is discrete (every bounded subset of $S$ is finite) and the set $\{ x \in \mathbb{R}^n : f(x) \leq \alpha \}$ is bounded and non-empty for some $\alpha \in \mathbb{R}$ (implying that the set is actually bounded for every $\alpha \in \mathbb{R}$). This latter condition is satisfied, e.g., when $f$ is strictly convex and has a minimizer. Another situation where (iii) is satisfied is when $S$ is a finite set, e.g., $S = C \cap \mathbb{Z}^n$ where $C$ is a compact convex set.

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Also, if conditions (i) and (ii) hold, but (iii) does not hold, a strong optimality certificate may not exist. For example, consider 

$$S = \{ x \in \mathbb{Z}^2 : \sqrt{2}x_1 - x_2 \geq 0, \ x_1 \geq \frac{1}{3}, \ x_2 \geq 0 \}$$

and $f(x) = \sqrt{2}x_1 - x_2$. In this case, no strong optimality certificate can exist, as the infimum of $f$ over $S$ is 0, but it is not attained by any point in $S$.

## 2 Proof of Theorem 4

We make use of the following observation. Let $\text{conv}(\cdot)$ denote the convex hull and $\text{vert}(P)$ denote the set of vertices of a polyhedron $P$.

**Lemma 5.** Let $S \subseteq \mathbb{R}^n$ and $V \subseteq S$ finite such that $V = \text{conv}(V) \cap S = \text{vert}(\text{conv}(V))$. Then we have $|V| \leq h(S)$.

**Proof.** Let $V = \{v_1, \ldots, v_m\}$ and for every $i \in [m]$ let $C_i := \text{conv}(V \setminus \{v_i\})$. Since $V = \text{conv}(V) \cap S = \text{vert}(\text{conv}(V))$, we have $C_i \cap S = V \setminus \{v_i\}$ for every $i \in [m]$. Thus, $C_1, \ldots, C_m$ satisfy (6) and hence $m \leq h(S)$.

We are ready to prove Theorem 4. Let us consider the following algorithm (in fact, we will see that this is indeed a finite algorithm):

1. $Q_0 \leftarrow \mathbb{R}^n$, $k \leftarrow 1$
2. while $\text{int}(Q_{k-1}) \cap S \neq \emptyset$ :
   1. $t_k \leftarrow \min \{ f(s) : s \in \text{int}(Q_{k-1}) \cap S \}$
   2. $C_k \leftarrow \{ x \in \mathbb{R}^n : f(x) \leq t_k \}$
   3. $z_k \leftarrow \text{any } s \in \text{int}(Q_{k-1}) \cap S \text{ with } f(s) = t_k$ such that $\text{dim}(F_{C_k}(s))$ is largest possible
   4. $a_k \leftarrow \text{any point in } \text{relint}(\partial f(z_k))$
   5. $Q_k \leftarrow \{ x \in Q_{k-1} \setminus \{a_k, x - z_k\} \leq 0 \}$
   6. $k \leftarrow k + 1$

In the above, $\text{relint}(\cdot)$ denotes the relative interior and $\text{dim}(\cdot)$ the affine dimension. For a closed convex set $C \subseteq \mathbb{R}^n$ and a point $p \in C$ we denote by $F_C(p)$ the smallest face of $C$ that contains $p$.

Remark that iteration $k$ of the algorithm can always be executed, as the set $Q_k$ is a polyhedron and hence by the assumption in (iii) the minimum in (7) always exists. Furthermore, since $a_k \in \text{relint}(\partial f(z_k))$ we have

$$F_k := F_{C_k}(z_k) = \{ x \in C_k : \langle a_k, x - z_k \rangle = 0 \}$$

**Claim 1:** For every $k$ we have that $\langle a_i, z_i - z_k \rangle < 0$ holds for all $i, j \leq k - 1, i \neq j$. Let $k \geq 2$ and assume that the claim is satisfied for all $i, j \leq k - 1, i \neq j$. Since $z_k \in \text{int}(Q_{k-1})$ and $a_i \neq \emptyset$ by assumption (i), we have that $\langle a_i, z_k - z_i \rangle < 0$ for every $i < k$.

It remains to show that $\langle a_k, z_i - z_k \rangle < 0$ for every $i < k$. Since $a_k \in \partial f(z_k)$, we have that $\langle a_k, z_i - z_k \rangle \leq f(z_i) - f(z_k)$ and for $i < k$ by (7) we have $f(z_i) \leq f(z_k)$. Therefore $\langle a_k, z_i - z_k \rangle \leq 0$ and if $\langle a_k, z_i - z_k \rangle = 0$, then $f(z_i) = f(z_k)$.

Assume this is the case. Since $\langle a_i, z_k - z_i \rangle < 0$, we have $z_k \notin F_i$ and in particular

$$F_i \neq F_k$$

By (8) this means that $z_i \in F_k$ holds. Since $F_i$ is the smallest face that contains $z_i$, this implies $F_i \subseteq F_k$. By (9), we have that $\text{dim}(F_i) \geq \text{dim}(F_k)$ and thus $F_i = F_k$, a contradiction to (10).

**Claim 2:** For every $k$ we have that $V := \{ z_1, \ldots, z_k \}$ satisfies $V = \text{conv}(V) \cap S = \text{vert}(\text{conv}(V))$.

It is easy to see that Claim 1 implies $V = \text{vert}(\text{conv}(V))$. For the sake of contradiction, assume there exists some $s \in (\text{conv}(V) \setminus V) \cap S$. By Claim 1, we have $s \in \text{int}(Q_k)$. Therefore by (7) we have $f(s) \geq t_k$. Since $f$ is convex and $s \in \text{conv}(V)$, this implies $f(s) = t_k$. Let $a \in \text{relint}(\partial f(s))$ and
consider $F := F_{C_k}(s) = \{x \in C_k : \langle a, z_i - s \rangle = 0\}$. Since $V \subseteq C_k$, we have that $z_i \in F$ for at least one $i \in [k]$. Due to $\langle a, z_i - s \rangle \leq f(z_i) - f(s)$ we must have $f(z_i) = f_k$ and hence $F_i \subseteq F$. By (8), we further have $\dim(F_i) \geq \dim(F)$, which shows $F_i = F$. However, by Claim 1 we have $z_j \notin F_i$ for all $j \neq i$ and hence $s \notin F_i$, a contradiction since $s \in F$.

Claim 3: The algorithm stops after at most $h(S)$ iterations and $Q := Q_k$ is $S$-free.

Note that the set $V := \{z_1, \ldots, z_k\}$ becomes larger in every iteration. By Claim 2 and Lemma 5 we must have $k \leq h(S)$ and hence the algorithm stops after at most $h(S)$ iterations. Since the algorithm stops if and only if $Q_k$ is $S$-free, this proves the claim.  

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