Photon and polariton fluctuations in arrays of QED-cavities

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Abstract – We propose to detect the Mott insulator-superfluid quantum phase transition in an array of coupled cavities by studying the polariton and photon fluctuations in a block of linear dimension \(M\) (in units of the lattice constant of the array). We explicitly show this for a one-dimensional array; the analysis can be however extended to higher dimensions. In the Mott phase polariton fluctuations are independent of the block size. In the superfluid phase they grow logarithmically with \(M\), the prefactor being related to the compressibility of the system. In the case of photon fluctuations, the critical behaviour is encoded in the subleading scaling with the block dimension, while the leading behaviour is linear in \(M\) and non-critical. Our results have been obtained by means of the density matrix renormalization group numerical algorithm.

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The recent suggestion [1–3] to realize Mott and superfluid phases in arrays of coupled QED-cavities has stimulated a flurry of activity on strongly interacting photonic systems [4–12]. In the presence of randomness a glassy phase of polaritons was shown to appear in the phase diagram [5]. Coupled cavities can be engineered to behave as quantum simulators for a variety of interacting spin models [7,9,10]; they can support soliton excitations [6,13]. A different realization of the strongly interacting regime with photons in an optical guide was proposed in [14], while polariton blockade effects were studied in a resonantly excited photonic quantum dot [15]. As candidates for simulating strongly interacting models, coupled cavities present new characteristics as compared to other systems, like optical lattices [16] or Josephson junction arrays [17]. Most notably, it is possible to access their local properties.

In this paper we would like to exploit the local addressability of cavity arrays to propose a method in order to probe the critical behaviour of the Mott insulator-superfluid transition. Our suggestion is based on the study of the fluctuations in the number of photons and polaritons. To our knowledge, this issue was not addressed so far in the literature. While it is well known how to distinguish a regime of photon blockade, how to detect the transition itself is, to a large extent, an unexplored problem. We study in details a one-dimensional array. Our method of detection, however, can be easily extended to higher dimensions.

The model. – We suppose that inside each cavity a single two-level atom interacts with photons via a Jaynes-Cummings Hamiltonian [18]. While this situation is simpler to simulate, we do not expect, for the purposes of our work, any qualitative change if other atomic level structures [1] are considered. Meanwhile, by varying the number of atoms per site we only expect a change of the location of the phase boundary between the Mott insulator and the superfluid, as is discussed in ref. [5]. The Hamiltonian for an array of \(L\) coupled cavities is then given by

\[
\mathcal{H} = \sum_{i=1}^{L} \left[ \epsilon \sigma_i^+ \sigma_i^- + \omega a_i^\dagger a_i + \beta (\sigma_i^+ a_i + h.c.) \right] - t \sum_{(i,j)} a_i^\dagger a_j, \tag{1}
\]

where \(\epsilon\) denotes the transition energy between the two atomic levels, \(\omega\) is the resonance frequency of the cavity, \(\beta\) is the atom-field coupling constant (\(\epsilon, \omega, \beta > 0\)), and \(t\) is the inter-cavity photon hopping, which is assumed to be constant for all nearest neighbours and zero otherwise. The atomic and photonic raising/lowering operators are denoted as \(\sigma_i^\pm\), and \(\{a_i^\dagger, a_i\}\), respectively, the subscript

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i indicating the lattice site. The total number of atomic plus photonic excitations (i.e., the number operator for polaritons) on the i-th site is given by \( n_{i,\text{pol}} = n_{i,\text{ph}} + \sigma_{i}^{+} \sigma_{i}^{-} \), where \( n_{i,\text{ph}} = a_{i}^{\dagger} a_{i} \) is the photon number operator. Our analysis is restricted to the case of zero relative detuning \( \Delta = \omega - \epsilon \); we also work in the canonical ensemble with a fixed polariton density \( \rho \equiv N/L = 1 \), \( N \) being the total number of polaritons in \( L \) cavities.

The equilibrium phase diagram associated to the model in eq. (1) is characterized by two distinct phases, with polariton Mott insulating (MI) regions surrounded by the superfluid (SF) phase. In the MI phase polaritons are localized on each site due to the photon blockade [19], and there is a gap in the spectrum. A finite hopping renormalizes this gap, which eventually vanishes at a critical value of \( t \). For large hoppings excitations are delocalized and the system enters the SF phase.

We propose to detect the MI-SF transition by analyzing the fluctuations of the occupation number inside a block composed by a subset of \( M \) adjacent cavities. Since the number operator is the canonically conjugated variable with respect to the phase of the whole many-body wave function, we expect that its fluctuations are strongly suppressed in the incoherent MI regime, and, by contrast, greatly enhanced in the coherent SF phase. The dispersion of particle number on a given subsystem with \( M \) sites is quantified by the variance of the corresponding probability distribution:

\[
\delta n_{\alpha}^{2}(M) = \left( \left\langle \sum_{i \in M} n_{i,\alpha}^{2} \right\rangle - \left\langle \sum_{i \in M} n_{i,\alpha} \right\rangle^{2} \right),
\]

where \( \alpha = \text{pol/ph} \) stands for polariton/photon fluctuations, and \( \langle \cdot \rangle \) denotes an average on the system ground state. This can be obtained from the two-point correlation functions of the related number operator: \( C_{\alpha}^{ij} = \langle n_{i,\alpha} n_{j,\alpha} \rangle - \langle n_{i,\alpha} \rangle \langle n_{j,\alpha} \rangle \), such that \( \delta n_{\alpha}^{2}(M) = \sum_{(i,j) \in M} C_{\alpha}^{ij} \).

In the present work we suppose that the system is in its ground state; we neglect decoherence, and assume that spontaneous photon emission and cavity loss characteristic times are much longer than the time scale over which the array can reach the ground state. We are aware that this is a strong assumption that may not be fulfilled in the experiments. We do not try to argue on this very important issue in the present work; rather we are interested in describing a method that is suitable to detect the various (equilibrium and non-equilibrium) many-body phases of an array of cavities. We describe our proposal by using the (equilibrium) MI-SF transition; the essential features of the method can be as well applied to the non-equilibrium case. We will come back to this point in the concluding section.

All the numerical data presented below have been obtained by means of the density matrix renormalization group (DMRG) algorithm with open-boundary conditions.

**Polariton fluctuations.** – We first study fluctuations in the polariton number. The number of polaritons inside a given subsystem can be experimentally measured by instantaneously switching off the effective polaritonic hopping (this can be achieved by changing the detuning \( \Delta \), such to increase the relative strength of the atom-field coupling with respect to the photon hopping). In this way, admitting radiative losses on long time scales, a quantum-jump picture immediately suggests that the polariton number is exactly given by the number of photons emitted from the selected region [20].

In fig. 1 we show numerical data displaying \( \delta n_{\text{pol}}^{2}(M) \) as a function of the photon hopping \( t \) in a system with \( L = 128 \) cavities; the various curves correspond to different sizes \( M \) of the block inside the system. The MI-SF transition point has been located numerically with high accuracy at \( t^{*} / \beta \approx 0.198 \) [5]. On-site fluctuations cannot reveal critical features at the phase transition, because they correspond to a local property; whenever \( t \neq 0 \), the solid curve with full circles corresponding to \( M = 1 \) has always non-zero values. Therefore, an unambiguous characterization of the phase boundary is impossible in this context. This was already pointed out in ref. [21] for the on-site boson number fluctuations in the Bose-Hubbard model, where it has been shown that the number probability distribution evolves from a Poissonian, in the non-interacting gas, to a sharply peaked distribution, in the insulator. We obtained very similar distribution probabilities for the on-site polariton number in our system, that are characterized by sub-Poissonian statistics.
In particular, we found that \( M \) blocksize, while in the SF phase the dependence on MI phase fluctuations tend to be independent of the prefactor at the transition point: \( \langle \bar{\delta} n \rangle \) horizontal dashed line indicates the estimate of the logarithmic plot \( \delta n \) for the variance \( \langle \bar{\delta} n \rangle \) as a function of \( t \): from bottom to top \( t = 0.12, 0.16, 0.2, 0.24, 0.28, 0.32 \); continuous lines are logarithmic fits of the corresponding data. In numerical fits we dropped the first and the latter 10 points (i.e., we kept values for \( M \in [11, L/2 - 10] \)).

From fig. 1 we notice that, by increasing \( M \), in the MI phase fluctuations tend to be independent of the block size, while in the SF phase the dependence on \( M \) is evident. In particular, we found that \( \delta n^2_{pol}(M) \) as a function of \( M \) saturates to a finite constant value in the MI phase, while it diverges logarithmically in the SF phase (see inset of fig. 2, where we plotted the same curves of fig. 1, but fixing \( t \) and varying \( M \)). The logarithmic divergence is directly related to the fact that density correlation functions in the superfluid phase have a power law decaying uniform term [22]:

\[
C^\text{pol}_r = \langle n^\text{pol}_{i+r} n^\text{pol}_i \rangle - \langle n^\text{pol}_i \rangle^2 \sim \frac{2}{K} \frac{1}{(2\pi \rho r)^2} + \cdots .
\]

In the previous expression \( \rho \) is the particle (in this case the polariton) density and \( K \) is the so-called Luttinger parameter [23], that is proportional to the square root of the compressibility of the system. Indeed, from the definition of \( \delta n^2_{pol}(M) \), one immediately concludes that, in the SF phase

\[
\delta n^2_{pol}(M) \sim \frac{1}{K \pi^2 \rho^2} \ln M .
\]  

At the SF-MI transition the coefficient \( K \) at integer densities is known to be equal to \( K_c = 1/2 \) [23]. Therefore, for a polariton density \( \rho = 1 \), the logarithmic prefactor suddenly jumps from a value \( c_0 = 2/\pi^2 \) just inside the SF phase, to zero in the MI, with a characteristic Kosterlitz-Thouless behaviour.

A quantitative analysis of the crossover between the two different behaviours has been performed by fitting numerical data according to:

\[
\delta n^2_{pol}(M) = c^\text{pol}_0 \ln \left( \frac{L}{\pi} \sin \left( \frac{\pi}{L} M \right) \right) + A^\text{pol} ,
\]

with \( A^\text{pol} \) and \( c^\text{pol}_0 \) as fitting parameters. The constant term \( A^\text{pol} \) is not important for our purposes, therefore we concentrate on the logarithmic term. In the main panel of fig. 2 we plot the logarithmic prefactor \( c^\text{pol}_0 \) as a function of \( t \): the phase transition point in this situation can be quite clearly identified. In 1D, in particular, a precise criterion for values of \( c^\text{pol}_0 \geq 2/\pi^2 \) in the superfluid phase, clearly identifies the transition point.

Photon fluctuations. – We now concentrate on the photon number fluctuations, a quantity which is more directly measured in quantum optical experiments, where the system is typically in a non-equilibrium state between photon/cavity decay and external pumping. The state of the polariton field is usually retrieved by detecting and characterizing the light emission [24,25].

In this case the situation is quite different as compared to the case of polariton fluctuations. Even at zero hopping the on-site photon number is fluctuating. For example, in the case of zero relative detuning (\( \Delta = 0 \)) and deep in the MI regime, we have \( \langle n^\text{ph}_i \rangle \approx 1/2 \) and \( \langle (n^\text{ph}_i)^2 \rangle \approx 1/2 \). Therefore, even for a perfect insulator, the variance \( \delta n^2_{ph}(M) \) of the photon number distribution inside a block of length \( M \) is proportional to the block size: \( \delta n^2_{ph}(M) \approx M/4 \). The onset of the superfluid behaviour can then be sought in the deviations from this linear growth. These deviations are due to the raising of correlations between distant sites on increasing the hopping strength. Therefore, in the scaling ansatz for the photon fluctuations one should also include a term that is linear in the block dimension, i.e.

\[
\delta n^2_{ph}(M) = \alpha M + c^\text{ph}_0 \ln \left( \frac{L}{\pi} \sin \left( \frac{\pi}{L} M \right) \right) + A^\text{ph} .
\]
fluctuations. The ratio between $c^\text{pol}_0$ and $c^\text{ph}_0$, shown in the inset, is non-universal.

$K_c = 1/2$ do not apply. The ratio between the coefficients of polariton and photon fluctuations is shown in the inset of fig. 4. A marked change at the transition can be noticed. In our opinion, however, this is not obviously related to the critical point; rather we expect that the ratio is a non-universal feature depending on the details of the model and on the value of the couplings.

Indication of the critical behaviour from photon correlations may also be obtained without resorting to a scaling analysis as a function of the block size $M$. This can be achieved by studying the second derivative of photon number fluctuations with respect to the block size, evaluated for blocks of half the length of the system size, that is $\partial^2_M[\delta n^2_{\text{ph}}(M)]|_{M=L/2}$. Although this approach might be slightly less accurate, it can give an interesting insight into the critical region.

We first consider $\partial^2_M[\delta n^2_{\text{ph}}(M)]|_{M=L/2}$ as a function of the hopping parameter $t$. Numerical data in fig. 5 show that, like for the polaritonic number fluctuations, also this quantity can be used to characterize the MI-SF transition for finite sizes. In addition, we can identify a behaviour that is very similar to the one of the logarithmic prefactor $c^\text{pol}_0$ of eq. (5) for the polariton number fluctuations. Namely, the correlation between these two quantities is nearly perfect: apart from a proportionality factor, which depends on the system size, the two curves in the main panels of figs. 2 and 5 are exactly the same. This is shown in the inset of fig. 5 where, for a given value of the photon hopping $t$, we plot the corresponding values of $c^\text{pol}_0$ and of $\partial^2_M[\delta n^2_{\text{ph}}(M)]|_{M=L/2}$ in the two axes, thus displaying a perfect linear correlation.

The dependence of the photon fluctuations on the size is however more complicated than that of $\delta n^2_{\text{pol}}$. In fig. 6 we plot $\partial^2_M[\delta n^2_{\text{ph}}(M)]|_{M=L/2}$ (its absolute value, actually, since it is always negative) as a function of the system size $L$, for several fixed photon hoppings $t$. While the logarithmic prefactors $c^\text{pol}_0$ and $c^\text{ph}_0$ are independent of the system size, this is not the case for the second derivative of photon number fluctuations, which asymptotically drops to zero as a power law with $L$. More specifically, in the free-photon limit ($t \to +\infty$) it decays as $L^{-1}$, while for finite values of the photon hopping and a sufficiently large system size, there is a crossover to a $L^{-2}$ behaviour. This follows from two qualitatively different decays of the photon number fluctuations.
correlation function $C^\text{ph} = \langle n_{i+}\rangle - \langle n_i \rangle^2$, apart from open-boundary effects, in the first regime $C^\text{ph} \approx 1/L$, while in the second case $C^\text{ph} \approx 1/r^2$. We carried out a numerical analysis of the photon correlations $C^\text{ph}$ in our QED-cavity array system, and explicitly found these two distinct behaviours.

It has also been possible to derive an analytic estimate of the crossover scale by exploiting a mapping of the cavity array with a Bose-Hubbard (BH) model, and studying its corresponding boson correlators $C^\text{BH}_{ij}$. It should be kept in mind that the formal analogy between $C^\text{ph}_{ij}$ and $C^\text{BH}_{ij}$ has to be considered only at a qualitative level, since the equivalence of the two models becomes exact only in the limit of a large number of atoms per cavity [1].

The BH Hamiltonian is defined by

$$\mathcal{H} = -J \sum_j (b^\dagger_j b_{j+1} + \text{h.c.}) + \frac{U}{2} \sum_{j=1}^L b^\dagger_j b_j b^\dagger_j b_j,$$

(7)

where $\{b_j^\dagger, b_j\}$ are the boson creation/annihilation operators, $n_i = b^\dagger_i b_i$ the boson number operator, and with $C^\text{BH}_{ij}$ we indicate the corresponding correlation functions. When the depletion of the condensate is not too great ($J \gg U$), we can employ the Bogoliubov approximation, which consists in replacing the boson creation and annihilation operators at a given site $j$ by a $c$-number $z_i \in \mathbb{C}$ plus a fluctuation operator $\beta_j$. Only terms at most of the second order in $\beta_j$ are considered when diagonalizing the Hamiltonian [26–28]. Within this framework and for periodic boundary conditions, we locate the crossover scale between the $L^{-1}$ and $L^{-2}$ behaviour by

$$U/J \lesssim \frac{2\pi^2}{n_0 L^2},$$

(8)

with $n_0$ being the density of the boson condensate. Whenever the inequality is valid, the spectrum of the Bogoliubov quasi-particles at small $k$ is quadratic, and correlations behave as for free bosons: $C_{r}^{-1} \approx 1/L$. Conversely, if the inequality does not hold, the spectrum becomes linear at small $k$, and the free-boson limit fails; in this regime we recover the decay $C^\text{BH}_{r} \approx 1/r^2$ for large $r$, typical of the polariton number correlations [22].

A proper quantitative comparison between photon number correlations in the coupled cavity system and boson number correlations in the BH model has been obtained by numerically solving the BH Bogoliubov equations for a system with open boundaries, and it is shown in fig. 7. The hopping is chosen to be equal in the two models ($t = J$); the polariton/boson density is $\rho = 1$. DMRG data are fitted by varying the on-site repulsion $U$ in the BH model and by minimizing the squared differences between the two curves: $\sigma^2(U) = \sum_r |C^\text{ph}_r - C^\text{BH}_r|^2$. Of course, if $t/\beta$ is too small, the Bogoliubov approximation fails, and we do not find an appropriate minimum. We remark that correlations are always negative, thus indicating photon antibunching, as is revealed by on-site number distributions with negative values of the Mandel parameter [18].

**Conclusions.** – In this paper we studied photon and polariton fluctuations in coupled cavities. By a suitable scaling of the fluctuation detected over a region of the
sample, we showed that it is possible to extract the critical properties of the superfluid to Mott insulator phase transition. Our proposal discussed here for one-dimensional systems can be extended to higher-dimensional lattices, where, unfortunately, the numerical DMRG algorithm is practically unfeasible. Of course, one would expect a behaviour of the fluctuations in the superfluid phase that is different from a logarithmic one, since this is typical of 1D systems. We remark that we confined our study to the equilibrium case; an interesting point that remains to be addressed is to consider the effects of photon and cavity losses, which may lead to new non-equilibrium effects in the physics of such systems. We do however expect that the method proposed here will hold also in the non-equilibrium case. The reason is that our approach ultimately distinguishes whether the system is incompressible or not. While the general principle also holds for driven systems, the detailed behaviour leading to the logarithmic scaling may be modified. As a consequence, the most appropriate scaling ansatz for the non-equilibrium cases has to be checked.

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