On the chromatic number
of a simplicial complex

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May 7, 2014

Abstract

In [Ho] A. Hoffman proved a lower bound on the chromatic number of a graph in the terms of the largest and the smallest eigenvalues of its adjacency matrix. In this paper, we define chromatic number of a pure $d$-dimensional simplicial complex and give a lower bound on it in the terms of the spectra of the Laplacian operators, generalizing Hoffman’s result.

1 Introduction

Let $X$ be a finite $k$-regular graph on a set $V$ of $n$ vertices.

Definition 1.1. (a) A subset $A \subseteq V$ of vertices of a graph $X$ is called independent, if there is no edge of $X$ with both ends in $A$. The independence number $i(X)$ of a graph $X$ is the size of the largest independent set of vertices.

(b) The chromatic number $\chi(X)$ of a graph $X$ is the least integer $\chi$ such that the vertex set of $X$ can be partitioned in $\chi$ disjoint independent sets.

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In other words, the chromatic number of a graph is the least number of colors needed to color vertices in such a way that there is no edge having both endpoints of the same color.

Note that $\chi(X) \cdot i(X) \geq |V|$.

Denote the space of real-valued functions on the vertex set $V$ by $C^0$. The Laplacian operator $\Delta$ on $C^0$ is

$$\Delta f(v) = \deg v \cdot f(v) - \sum_{u \sim v} f(u) = \sum_{u \sim v} (f(v) - f(u)),$$

where $f \in C^0$ and $v \in V$.

Denote the spectrum of $\Delta$ by $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_{n-1} = \lambda_{\text{max}}$.

In 1970, A. Hoffman gave a lower bound on the chromatic number of a graph in the terms of the spectrum of the adjacency matrix of graph. For a $k$-regular graph his result may be re-formulated in the terms of the spectrum of the Laplacian operator in the following way.

**Theorem 1.2.** (A. Hoffman, 1970, [Ho]) For a $k$-regular graph $X = (V, E)$ on $n$ vertices

$$i(X) \leq \frac{\lambda_{\text{max}} - k}{\lambda_{\text{max}}} \cdot n,$$

and, therefore,

$$\chi(X) \geq \frac{\lambda_{\text{max}}}{\lambda_{\text{max}} - k}.$$

**Proof.** Let $I \subset V$ be the largest independent subset of $V$, and $C = V \setminus I$ be its complement. Denote $i = |I|$ and $c = |C| = n - i$. Note that both $i$ and $c$ are positive, since the set of edges of $X$ is not empty. Consider the following function $f \in C^0$ on the vertex set $V$:

$$f(v) = \begin{cases} -c & \text{if } v \in I; \\ i & \text{if } v \in C. \end{cases}$$

Then

$$\lambda_{\text{max}} \geq \frac{<\Delta f, f>}{<f, f>} = \frac{i \cdot k \cdot (i + c)^2}{i \cdot c^2 + c \cdot i^2} = \frac{i \cdot k \cdot n^2}{(n - i) \cdot n} = \frac{k \cdot n}{n - i},$$

and hence

$$i = i(X) \leq \frac{\lambda_{\text{max}} - k}{\lambda_{\text{max}}} \cdot n.$$

Since $\chi(X) \geq \frac{n}{i(X)}$, the second bound follows. \qed
Remark 1.3. Hoffman’s Theorem does not require graph to be connected. Moreover, if the graph $X$ is a disjoint union of connected components $X = X_1 \sqcup \cdots \sqcup X_m$, then the spectrum of $X$ is the union of spectra of $X_1, \ldots, X_m$, and consequently Hoffman’s theorem provides
\[
\chi(X) = \max_{i=1, \ldots, m} \chi(X_i) \geq \min_{i=1, \ldots, m} \frac{\lambda_{\max}(X_i)}{\lambda_{\max}(X_i) - k}.
\]

Nevertheless, note that requirement to be $k$-regular imply absence of isolated vertices in $X$.

An important application of Hoffman bound is given in [LPS], where non-bi-partite Ramanujan graphs were constructed. These graphs are shown there to have chromatic number of order $\sqrt{k}$, where $k$ is the degree of regularity, and girth greater or equal to $\frac{4}{3} \log_{k-1} n$, where $n$ is the number of vertices.

In this paper we present a generalization of Hoffman’s result to regular simplicial complexes. As a definition of the chromatic number of a simplicial complex we take the following.

Definition 1.4. (a) A subset $A \subseteq V$ of vertices of a $d$-dimensional simplicial complex $X$ is called independent, if there is no $d$-face in $X$ with all vertices in $A$. The independence number $i(X)$ of a complex $X$ is the size of the largest independent set of vertices.

(b) The chromatic number $\chi(X)$ of a complex $X$ is the least integer $\chi$ such that vertices of $X$ can be partitioned in $\chi$ disjoint independent sets.

In other words, the chromatic number of a $d$-dimensional simplicial complex $X$ is the least number of colors needed to color the vertices of $X$ is such a way that no $d$-face of $X$ is monochromatic, i.e. no $d$-face has all its vertices colored in one color.

Note that this definition allows monochromatic faces of dimension $(d-1)$ and lower. Requiring faces of all dimensions not to be monochromatic would lead back to the chromatic number of the 1-skeleton of the complex.

As in the case of graphs, $\chi(X) \cdot i(X) \geq |V|$.

We give an upper bound on the independence number of a complex in the terms of spectra of its Laplacian operators, which implies lower bound on the chromatic number of a complex.

Theorem 1.5. Let $X$ be a $d$-dimensional $(k_0, \ldots, k_{d-1})$-regular simplicial complex on a vertex set $V$. Then
\[
i(X) \leq \frac{\lambda_0^{d-1} - (k_0 + 1)(k_1 + 2) \cdots (k_{d-2} + d - 1)k_{d-1}}{\lambda_0 \cdots \lambda_{d-1}} \cdot n,
\]
and, therefore,

\[ \chi(X) \geq \frac{\lambda_0 \cdots \lambda_{d-1}}{\lambda_{\text{max}} \cdots \lambda_{\text{max}} - (k_0 + 1)(k_1 + 2) \cdots (k_{d-2} + d - 1)k_{d-1}}. \]

where \( \lambda_{\text{max}} \) is the largest eigenvalue of the \( j \)-th upper Laplacian operator \( \Delta_j^+ \) of \( X \).

Remark 1.6. Note that we do not require the simplicial complex to be connected in any sense. Nevertheless, by requiring the simplicial complex to be regular we force it to be pure.

The reader is referred to the section 2 where we recall the definition of the Laplacian operators of a simplicial complex. Here, we note that in the one-dimensional case, when a simplicial complex is just a graph, the upper Laplacian operator \( \Delta_0^+ \) coincides with the Laplacian operator \( \Delta \) of a graph defined earlier, and that the Hoffman bound is a special case of Theorem 1.5.

In Section 3, we prove the main result. In Section 4, we give a different formulation of Theorem 1.5 in the terms of the spectra of the stars of the faces of a simplicial complex. In Section 5, we formulate Theorem 1.5 for the case of a \( d \)-dimensional simplicial complex with complete \((d-1)\)-skeleton and give a different, shorter, proof.

This work is part of the Ph.D. thesis being written at the Hebrew University of Jerusalem.

Acknowledgement. The author is grateful to Alex Lubotzky and Ori Parzanchevski for fruitful discussions on the topic, and to ERC for the support.

2 Notations and definitions

An abstract simplicial complex \( X \) on a vertex set \( V \) of \( n \) vertices is a collection of subsets of \( V \) closed under taking subsets. Elements of \( X \) are called faces of the complex.

The dimension of a face \( A \in X \) is its cardinality minus one, \( \dim(A) = |A| - 1 \). A face of dimension \( j \) is called \( j \)-face. The set of all \( j \)-faces of \( X \) is denoted \( X^j \), in particular, \( X^0 = V \). The dimension of a complex, \( \dim(X) \), is the largest dimension of one of its faces. A \( d \)-dimensional complex is called pure, if every face of it is contained in a face of maximal dimension \( d \).

The degree of a \( j \)-face \( A \) is the number of \((j+1)\)-faces containing it, i.e. \( \deg(A) = |\{B \in X^{j+1} \mid A \subseteq B\}| \). A \( d \)-dimensional simplicial complex \( X \) is called \((k_0, \ldots, k_{d-1})\)-regular, if for every \( 0 \leq j \leq d - 1 \), every \( j \)-face is of positive degree \( k_j \). Since we do not allow multiple faces of any dimension, it
is necessary that $k_{d-1} < k_{d-2} < \cdots < k_0$. The property being regular forces simplicial complex to be pure.

For a subset $A \subseteq V$, denote the set of $j$-faces with all vertices in $A$ by

$$X^j(A) = \{(v_0, \ldots, v_j) \in X^j \mid v_l \in A, \text{ for all } l = 0, \ldots, j\}.$$ 

For a $(j-1)$-face $F$ and a subset $A \subseteq V$ denote the set of all $j$-faces containing $F$ and having the remaining vertex in $A$ by

$$X^j(F, A) = \{F \cup \{v\} \in X^j \mid v \in A\}.$$ 

The star of a $j$-face $F$ in $X$ is the subcomplex of $X$ of all faces $G$ for which there exists a face $G'$ containing both $G$ and $F$, i.e.

$$\text{St}_X(F) = \text{St}(F) = \{G \in X \mid \exists G' \in X : F \subset G' \text{ and } G \subset G'\}.$$ 

In the case of a graph $X$, the star of a vertex $v$ is the subgraph consisting of $v$, vertices neighboring to $v$ and the edges connecting them to $v$.

Remark 2.1. The usual definition of the star of a $j$-face $F$ in $X$ is the collection of faces of $X$, which contain $F$ as a subface. Since this collection does not have to be closed under taking subfaces, its closure, which is called the closed star of a face, is usually considered. That is what we have defined above as the star.

**Chromatic and independence numbers** For $1 \leq j \leq d$, a subset $A \subseteq V$ of vertices of a $d$-dimensional simplicial complex $X$ is called $j$-independent, if there is no $j$-face in $X$ with all vertices in $A$. The $j$-th independence number $i_j(X)$ of a complex $X$ is the size of the largest $j$-independent set of vertices.

The $j$-th chromatic number $\chi_j(X)$ of a complex $X$ is the least integer $\chi_j$ such that vertices of $X$ can be partitioned into $\chi_j$ disjoint $j$-independent sets.

In other words, the $j$-th chromatic number of a simplicial complex $X$ is the least number of colors needed to color vertices of $X$ such that no $j$-face of $X$ is monochromatic, i.e. no $j$-face has all its vertices colored in one color.

Note that $i_1(X)$ and $\chi_1(X)$ are the independence number and the chromatic number, respectively, of the 1-skeleton of $X$. Also, note that $n \leq i_l(X) \cdot \chi_l(X)$, for all $1 \leq l \leq d$.

The $j$-th chromatic number of a complex does not depend on the structure of the complex in dimensions higher than $j$. Therefore, by the chromatic number $\chi(X)$ of a $d$-dimensional simplicial complex, we will mean its $d$-th chromatic number $\chi_d(X)$.
Lemma 2.2. The following inequality holds

\[ \chi_d(X) \leq \left\lceil \frac{\chi_1(X)}{d} \right\rceil, \]

where \([x]\) denotes the ceiling of a number \(x\).

Proof. Let \(\chi_1 = \chi_1(X)\), \(\chi_d = \chi_d(X)\) and \(k = \left\lceil \frac{\chi_1(X)}{d} \right\rceil\). Let \(V = A_1 \sqcup \cdots \sqcup A_{\chi_1}\) be a 1-coloring of \(X\) in \(\chi_1\) colors, i.e. there is no edge with both endpoints in \(A_j\) for all \(1 \leq j \leq \chi_1\).

Let \(B_1 = A_1 \sqcup \cdots \sqcup A_d, B_2 = A_{d+1} \sqcup \cdots \sqcup A_{2d}, \ldots, B_k = A_{(k-1)d+1} \sqcup \cdots \sqcup A_{\chi_1}\).

We now show that the sets \(B_1, \ldots, B_k\) are \(d\)-independent. Assume the opposite, i.e. there exists a \(d\)-face \(F\) with all vertices in \(B_j\) for some \(1 \leq j \leq k\). Since \(B_j\) is a union of at most \(d\) sets, one of them contains an edge of the face \(F\). This contradicts the assumption on \(A_1, \ldots, A_{\chi_1}\) to be 1-independent.

Since \(B_1, \ldots, B_k\) are \(d\)-independent, \(\chi_d \leq k\).

This lemma shows that a lower bound on \(\chi_d(X)\) provides a proportional lower bound on \(\chi_1(X)\), which is the standard chromatic number of the 1-skeleton of \(X\).

Laplacian operators A \(j\)-face \(A \in X^j\) with an ordering of its vertices is called directed, or oriented, and denoted \([A]\). For \(0 \leq j \leq d\), denote by \(C^j(X, \mathbb{R})\), or just \(C^j\), when does not lead to ambiguity, the vector space of all real-valued antisymmetric functions on directed \(j\)-faces of the complex \(X\). That is, a function \(f \in C^j\), if for a \(j\)-face \(F = \{v_0, \ldots, v_j\} \in X^j\) and a permutation \(\pi \in S_j\) the following condition holds

\[ f([v_{\pi(0)}, \ldots, v_{\pi(j)}]) = sgn(\pi)f([v_0, \ldots, v_j]). \]

An inner product on \(C^j\) is defined as

\[ \langle f, g \rangle = \sum_{A \in X^j} f([A])g([A]). \]

Note, that the sum runs over non-oriented \(j\)-faces of \(X\), but the functions are evaluated on some orientation of a face. Since the functions \(f\) and \(g\) are anti-symmetric, an orientation of each face may be chosen arbitrarily.

For \(0 \leq j \leq d-1\), the \(j\)-th co-boundary map \(\delta_j : C^j \rightarrow C^{j+1}\) is defined as

\[ (\delta_j f)([v_0, \ldots, v_{j+1}]) = \sum_{i=0}^{j+1} (-1)^i f([v_0, \ldots, \widehat{v}_i, \ldots, v_{j+1}]). \]
Note that this defines a cochain complex, i.e. $\delta_{j+1} \circ \delta_j = 0$.

For $0 \leq j \leq d - 1$, the $j$-th boundary map $\partial_j : C^{j+1} \to C^j$ is defined as

$$(\partial_j f)([v_0, \ldots, v_j]) = \sum_{u \in V_j} f([u, v_0, \ldots, \hat{v}_i, \ldots, v_j]).$$

The $j$-th boundary map is adjoint of the $j$-th co-boundary map.

The following operators on $C^j$ are called lower and upper $j$-th Laplacians of the complexes

$$\Delta_j^- = \delta_{j-1} \circ \partial_{j-1} \text{ and } \Delta_j^+ = \partial_j \circ \delta_j.$$ 

The sum of the $j$-th lower and upper Laplacians is called the $j$-th Laplacian operator, and denoted

$$\Delta_j = \Delta_j^- + \Delta_j^+ : C^j \to C^j.$$ 

Both lower and upper Laplacians are positive semi-definite. The spectrum of $(j + 1)$-th lower Laplacian differs from the spectrum of $j$-th upper Laplacian only by the multiplicity of the zero eigenvalue. The $j$-th lower Laplacian vanishes on boundaries, i.e. on the image of the $j$-th boundary operator $\partial_j$. Analogously, the $j$-th upper Laplacian vanishes on co-boundaries, that is on the image of the $(j - 1)$-st co-boundary operator $\delta_{j-1}$.

Note that in the case of graphs, the upper Laplacian operator on $C^0$ of a graph coincides with the Laplacian operator defined earlier.

For an exposition of theory of Laplacian operators of an abstract simplicial complex, we address the reader to [HJ] and references therein.

In this paper we deal with spectra of the upper Laplacian operators

$$\Delta_j^+ = \partial_j \circ \delta_j : C^j \to C^j, \text{ for } 0 \leq j \leq d - 1.$$ 

Denote the spectrum of $\Delta_j^+$ by

$$0 = \lambda_j^0 \leq \cdots \leq \lambda_{|X^j| - 1}^j = \lambda_{\max}^j.$$ 

We denote by $\lambda_{\max}(X)$ the largest eigenvalue of $\Delta_j^+(X)$, if there is a need to emphasize the complex. The following lower bound is given in [HJ].

**Theorem 2.3.** (D. Horak, J. Jost, 2011, [HJ], Theorem 3.4) Let $X$ be a $d$-dimensional $(k_0, \ldots, k_{d-1})$-regular simplicial complex. Then for all $0 \leq j \leq d - 1$

$$\lambda_{\max}^j(X) \geq k_j + (j + 1).$$

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3 Main theorem

We are now ready to prove the main result:

**Theorem 1.5.** Let $X$ be a $d$-dimensional $(k_0, \ldots, k_{d-1})$-regular simplicial complex on a finite vertex set $V$. Then

$$i_d(X) \leq \frac{\lambda_0^0 \cdots \lambda_{max}^{d-1} - (k_0 + 1)(k_1 + 2)\cdots(k_{d-2} + d - 1)k_{d-1}}{\lambda_0^0 \cdots \lambda_{max}^{d-1}} \cdot n,$$

and, therefore,

$$\chi_d(X) \geq \frac{\lambda_0^0 \cdots \lambda_{max}^{d-1} - (k_0 + 1)(k_1 + 2)\cdots(k_{d-2} + d - 1)k_{d-1}}{\lambda_0^0 \cdots \lambda_{max}^{d-1}},$$

where $\lambda_{max}^j$ is the largest eigenvalue of the $j$-th upper Laplacian operator $\Delta^+_j$ of $X$.

**Proof.** Let $I \subset V$ be the largest $d$-independent set of vertices of $X$, i.e. there is no $d$-face with all vertices in $I$. Denote the complement $C = V \setminus I$ and $i = |I|, c = |C|$. Note that both $i$ and $c$ are positive. The desired inequality is obtained by bounding from above and below the number $|X^1(I, C)|$ of edges, i.e. 1-faces, with exactly one vertex in $I$.

A bound from above, we obtain, by considering a function $f_0 \in C^0$ on the vertices of $X$:

$$f_0(v) = \begin{cases} -c & \text{if } v \in I; \\ i & \text{if } v \in C. \end{cases}$$

By the min-max principle,

$$\lambda_{max}^0 \geq \frac{<\Delta^+_0 f_0, f_0>}{< f_0, f_0 >} = \frac{<\delta_0 f_0, \delta_0 f_0 >}{< f_0, f_0 >} = \frac{|X^1(I, C)| \cdot (i + c)^2}{i \cdot c^2 + c \cdot i^2} = \frac{|X^1(I, C)| \cdot n}{(n - i)i},$$

where $\lambda_{max}^j$ is the largest eigenvalue of the $j$-th upper Laplacian operator $\Delta^+_j$ of $X$. 

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and therefore
\[ \lambda_0 \cdot \frac{(n - i) \cdot i}{n} \geq |X^1(I, C)|. \quad (3) \]

A bound from below we obtain by a series of consecutive bounds on the number of \( j \)-faces with exactly one vertex \textit{not} in \( I \), i.e. in \( C \), for \( j = d - 2, \ldots, 0 \).

Fix a \((d - 2)\)-face \( F = \{v_0, \ldots, v_{d-2}\} \in X^{d-2}(I) \) with all vertices in \( I \) and its ordering \([F] = [v_0, \ldots, v_{d-2}]\). Denote by
\[ k_{d-2}^I(F) = |X^{d-1}(F, I)| = |\{F \cup \{v\} \in X^{d-1} \mid v \in I\}| \]
the number of \((d-1)\)-faces containing \( F \) and having the remaining vertex in \( I \), and by
\[ k_{d-2}^C(F) = |X^{d-1}(F, C)| = |\{F \cup \{v\} \in X^{d-1} \mid v \in C\}| \]
the number of \((d-1)\)-faces containing \( F \) and having the remaining vertex in \( C \). The regularity assumption implies
\[ k_{d-2}^I(F) + k_{d-2}^C(F) = k_{d-2}. \quad (4) \]

Since \( X \) is pure, \( k_{d-2}^C \) is not equal to zero. Although the case \( k_{d-2}^C(F) = k_{d-2} \) (and hence, \( k_{d-2}^I(F) = 0 \)) requires separate treatment, which is provided after the general case, it will be shown to fit the bound proved for the general case.

In this notations, remembering that \( I \) is \( d \)-independent, we can give an exact formula and a bound for the number of \( d \)-faces containing \( F \) and having exactly one vertex in \( C \). Namely,
\[ \left| \{H = \{v_0, \ldots, v_{d-2}, v_I, v_C\} \in X^d \mid v_I \in I \text{ and } v_C \in C\} \right| = k_{d-2}^I(F) \cdot k_{d-1} \leq k_{d-2}^I(F) \cdot k_{d-2}^C(F). \quad (5) \]

There also might exist \( d \)-faces containing \( F \) and having both remaining vertices in \( C \). They are not counted in \((5)\).

Now consider the following function \( f_F \in C^{d-1}(X) \) on directed \((d-1)\)-faces, supported on the star \( St_X(F) \) of \( F \) in \( X \).

Let \( G \) be a \((d-1)\)-face lying in \( St_X(F) \). By the definition of the star, there exist two options for \( G \): either \( F \subset G \); or \( F \not\subset G \), then there exists \( d \)-face \( H \), s.t. \( F \cup G = H \).

If \( F \subset G \), then we define:
\[ \bullet \quad f_F([G]) = f_F([v_0, \ldots, v_{d-2}, v_I]) = (-1)^d \cdot k_{d-2}^C(F), \text{ if } v_I \in I. \]

The number of such faces is \( k_{d-2}^I(F) \).
\[ f_F([G]) = f_F([v_0, \ldots, v_{d-2}, v_I]) = (-1)^{d-1} \cdot k_{d-2}^I(F), \text{ if } v_C \in C. \]

The number of such faces is \( k_{d-2}^C(F) \).

If \( F \) does not lie in \( G \), there exist two options. First is that the \( d \)-face
\[ H = F \cup G \]
if of the form
\[ H = \{v_0, \ldots, v_{d-2}, v_I, v_C\}, \text{ where } v_I \in I \text{ and } v_C \in C, \]
then \( G \) is of the form
\[ G = \{v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{d-2}, v_I, v_2\}, \text{ for some } 0 \leq j \leq d - 2. \]

It follows from (5) that the number of such faces \( G \) in the star of \( F \) is not greater than \( (d - 1) \cdot k_{d-2}^I(F)k_{d-2}^C(F) \). In this case we define
\[ f_F([G]) = f_F([v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{d-2}, v_I, v_C]) = (-1)^j, \text{ where } 0 \leq j \leq d - 2 \text{ and } v_I \in I, v_C \in C. \]

Second option is that the \( d \)-face \( H = F \cup G \) if of the form
\[ H = \{v_0, \ldots, v_{d-2}, v_{C,1}, v_{C,2}\}, \text{ where both } v_{C,1}, v_{C,2} \in C, \]
then \( G \) is of the form
\[ G = \{v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{d-2}, v_{C,1}, v_{C,2}\}, \text{ for some } 0 \leq j \leq d - 2. \]

In this case, and in the case, when \( G \) does not lie in the star of \( F \) we define
\[ f_F([G]) = 0. \]

Now we give a lower bound on the Rayleigh quotient of \( f_F \) with respect to the \( (d - 2) \)-nd upper Laplacian. First, note that
\[ < f_F, f_F > \leq \]
\[ \leq k_{d-2}^I(F) \cdot k_{d-2}^C(F)^2 + k_{d-2}^C(F) \cdot k_{d-2}^I(F)^2 + (d - 1)k_{d-2}^I(F)k_{d-2}^C(F) = \]
\[ = k_{d-2}^I(F)k_{d-2}^C(F) \left(k_{d-2}^I(F) + k_{d-2}^C(F) + d - 1\right) = \]
\[ = k_{d-2}^I(F)k_{d-2}^C(F) \left(k_{d-2}^I + d - 1\right). \tag{6} \]

The co-boundary image \( \delta_{d-1}f_F \) takes value \( (k_{d-2} + d - 1) \) on every \( d \)-face such that it contains \( F \) and has exactly one vertex in \( C \). Number of such faces is exactly \( k_{d-2}^I(F)k_{d-1}, \) hence
\[ < \Delta_{d-1}f_F, f_F > = < \delta_{d-1}f_F, \delta_{d-1}f_F > \geq \]
\[ \geq k_{d-1} \cdot k_{d-2}^I(F) \cdot (k_{d-2} + d - 1)^2. \tag{7} \]
Hence, combining (7) and (8), we get that
\[
\lambda_{\text{max}}^{d-1} \geq \frac{\langle \Delta_{d-1} f_F, f_F \rangle}{\langle f_F, f_F \rangle} \geq \frac{k_{d-1}(k_{d-2} + d - 1)}{k_{d-2}^C(F)}.
\] (8)

Therefore, we get that for every \((d-2)\)-face \(F \in X^{d-2}(I)\)
\[
k_{d-2}^C(F) = |X^{d-1}(F, C)| \geq \frac{(k_{d-2} + d - 1)k_{d-1}}{\lambda_{\text{max}}^{d-1}}.
\] (9)

In the same manner we conduct a similar bound for \((d-3)\)-faces lying in \(I\). Let \(F = [v_0, \ldots, v_{d-3}] \in X^{d-3}(I)\) be a \((d-3)\)-face with all vertices in \(I\).

As before, denote by
\[
k_{d-3}^I(F) = |X^{d-2}(F, I)| = |\{F \cup \{v\} \in X^{d-2} | v \in I\}|
\]
the number of \((d-2)\)-faces containing \(F\) and having the remaining vertex in \(I\), and by
\[
k_{d-3}^C(F) = |X^{d-2}(F, C)| = |\{F \cup \{v\} \in X^{d-2} | v \in C\}|
\]
the number of \((d-2)\)-faces containing \(F\) and having the remaining vertex in \(C\). The regularity condition implies
\[
k_{d-3}^I(F) + k_{d-3}^C(F) = k_{d-3}.
\] (10)

Since the complex is pure, \(k_{d-3}^C(F) \neq 0\). The case \(k_{d-3}^C(F) = 0\) follows the general case. As in (5), we can bound from above the number of \((d-1)\)-faces containing \(F\) and having exactly one vertex in \(C\) by
\[
|\{H = \{v_0, \ldots, v_{d-3}, v_I, v_C\} \in X^{d-1} | v_I \in I \text{ and } v_C \in C\}| \leq k_{d-3}^I(F) \cdot k_{d-3}^C(F).
\] (11)

It follows from (9), that
\[
|\{H = \{v_0, \ldots, v_{d-3}, v_I, v_C\} \in X^{d-1} | v_I \in I \text{ and } v_C \in C\}| \geq k_{d-3}^I(F) \cdot \frac{(k_{d-2} + d - 1)k_{d-1}}{\lambda_{\text{max}}^{d-1}}.
\] (12)

Now consider the following function \(f_F \in C^{d-2}(X)\) on directed \((d-2)\)-faces of \(X\), supported on the star \(St_F(X)\) of \(F\). Let \(G\) be a \((d-2)\)-face lying in \(St_X(F)\). If \(F \subset G\), then we define
\[
\begin{align*}
\bullet \quad f_F(G) &= f_F([v_0, \ldots, v_{d-3}, v_I]) = (-1)^{d-1} \cdot k^C_{d-3}(F), \text{ where } v_I \in I. \text{ The number of such faces is } k^I_{d-3}(F). \\
\bullet \quad f_F(G) &= f_F([v_0, \ldots, v_{d-3}, v_C]) = (-1)^{d-2} \cdot k^I_{d-3}(F), \text{ where } v_C \in C. \text{ The number of such faces is } k^C_{d-3}(F).
\end{align*}
\]

If \( F \) does not lie in \( G \) and \( F \cup G = \{v_0, \ldots, v_{d-3}, v_I, v_C\} \), where \( v_I \in I \) and \( v_C \in C \), we define

\[
\bullet \quad f_F(G) = f_F([v_0, \ldots v_{j-1}, v_{j+1}, \ldots, v_{d-3}, v_I, v_C]) = (-1)^{j}, \text{ where } 0 \leq j \leq d - 3 \text{ and } v_I \in I, v_C \in C.
\]

It follows from (11) that the number of such faces is not greater than \( (d - 2) \cdot k^I_{d-3}(F)k^C_{d-3}(F) \).

In all other cases, we put

\[
\bullet \quad f_F(G) = 0.
\]

As in (10), we give an upper bound

\[
\begin{align*}
< f_F, f_F > &\leq k^I_{d-3}(F) \cdot k^C_{d-3}(F)^2 + k^C_{d-3}(F) \cdot k^I_{d-3}(F)^2 + (d - 2)k^I_{d-3}(F)k^C_{d-3}(F) \\
&= k^I_{d-3}(F)k^C_{d-3}(F)(k^I_{d-3}(F) + k^C_{d-3}(F) + d - 2) \\
&= k^I_{d-3}(F)k^C_{d-3}(F)(k^I_{d-3} + d - 2).
\end{align*}
\]

(13)

The coboundary image \( \delta_{d-2}f_F \) takes value \((k^I_{d-3} + d - 2)\) on every \((d - 1)\)-face, which contains \( F \) and has exactly one vertex in \( C \). Applying bound (12) we obtain

\[
< \Delta_{d-1}f_F, f_F > = < \delta_{d-1}f_F, \delta_{d-1}f_F > \geq k^I_{d-3}(F)(k^I_{d-2} + d - 1)k^I_{d-1} \lambda_{\max}^{d-1} \cdot (k^I_{d-3} + d - 2)^2.
\]

(14)

Hence,

\[
\lambda_{\max}^{d-2} \geq \frac{< \Delta_{d-2}f_F, f_F >}{< f_F, f_F >} \geq \frac{(k^I_{d-3} + d - 2)(k^I_{d-2} + d - 1)k^I_{d-1}}{k^C_{d-3}(F)\lambda_{\max}^{d-1}}.
\]

(15)

Therefore, we get that for every \((d - 3)\)-face \( F \in X^{d-3}(I) \)

\[
k^C_{d-3}(F) = |X^{d-2}(F, C)| \geq \frac{(k^I_{d-3} + d - 2)(k^I_{d-2} + d - 1)k^I_{d-1}}{\lambda_{\max}^{d-1}\lambda_{\max}^{d-2}}.
\]

(16)
Similarly, we obtain a bound for every $j = d - 2, \ldots, 1$. For a $j$-face $F$

$$k_j^C(F) = |X^{j+1}(F, C)| \geq \frac{(k_j + j + 1) \ldots (k_{d-2} + d - 1)k_{d-1}}{\lambda_{\text{max}}^j \ldots \lambda_{\text{max}}^{d-1}}. \quad (17)$$

The bound for $j = 1$ gives that for every edge $F = \{v_0, v_1\} \in X^1(I),$

$$k_1^C(F) = |X^2(F, C)| \geq \frac{(k_1 + 2) \ldots (k_{d-2} + d - 1)k_{d-1}}{\lambda_{\text{max}}^2 \ldots \lambda_{\text{max}}^{d-1}}. \quad (18)$$

Now, almost the last step to take is to give a lower bound on the number of edges outgoing from a vertex $v \in I$ to $C$. Although the approach at this step is the same as above, we produce it for the sake of clearness.

Fix a vertex $v$ in $I$, and denote by $k_0^I(v) = |X^1(v, I)|$ the number of edges in $I$ adjacent to $v$ and by $k_0^C(v) = |X^1(v, C)|$ the number of edges adjacent to $v$ having the second vertex in $C$. The regularity condition implies $k_0^I(v) + k_0^C(v) = k_0$. Since $X$ is pure, $k_0^C(v) \neq 0$. As before, let us assume, that also $k_0^I(v) \neq 0$. Consider the following function $f_v \in C^1(X)$ on directed edges of $X$.

$$f_v([v_1, v_2]) = \begin{cases} k_0^C(v) & \text{if } v_1 = v \text{ and } v_2 \in I; \\ -k_0^I(v) & \text{if } v_1 = v \text{ and } v_2 \in C; \\ 1 & \text{if } v \neq v_1 \in I, v_2 \in C \text{ and } \{v_1, v_2\} \in \text{St}_X(v); \\ 0 & \text{otherwise} \end{cases} \quad (19)$$

As in (5) and (11), (12), the number of 2-faces containing $v$ and exactly one vertex in $C$ is bounded via

$$|\{v, v_I, v_C\} \in X^2 \mid v_I \in I, v_C \in C\} | \leq k_0^I(v) \cdot k_0^C(v). \quad (20)$$

and, applying (18),

$$|\{v, v_I, v_C\} \in X^2 \mid v_I \in I, v_C \in C\} | \geq \frac{(k_1 + 2) \ldots (k_{d-2} + d - 1)k_{d-1}}{\lambda_{\text{max}}^2 \ldots \lambda_{\text{max}}^{d-1}} \cdot k_0^I(v). \quad (21)$$

13
The upper bound (20) implies that
\[ < f_v, f_v > \leq k_0^I(v)k_C(v)(k_0 + 1). \] (22)

On each of such 2-face the co-boundary image of \( f_v \) takes value \( (k_0 + 1) \). Hence, applying (21),
\[ < \Delta_1 f_v, f_v > = < \delta_1 f_v, \delta_1 f_v > \geq \frac{(k_1 + 2) \ldots (k_{d-2} + d - 1)k_{d-1}}{\lambda_{max}^2 \ldots \lambda_{max}^{d-1}} \cdot k_C(v) \cdot (k_0 + 1)^2 \] (23)
Therefore, we get that
\[ \lambda_{max} \geq \frac{< \Delta_1 f_v, f_v >}{< f_v, f_v >} \geq \frac{(k_1 + 2) \ldots (k_{d-2} + d - 1)k_{d-1}}{\lambda_{max}^2 \ldots \lambda_{max}^{d-1}} \cdot \frac{1}{k_0^C(v)} \] (24)
and that for every vertex \( v \in I \)
\[ |X^1(v, C)| = k_C^i(v) \geq \frac{(k_0 + 1)(k_1 + 2) \ldots (k_{d-2} + d - 1)k_{d-1}}{\lambda_{max}^1 \lambda_{max}^2 \ldots \lambda_{max}^{d-1}}. \] (25)

Note, that for all \( j = d - 2, \ldots, 0 \) we worked under the assumption \( k_j^C(F) \neq k_j \). In the case of \( k_j^C(F) = k_j \), and hence \( k_j^I(F) = 0 \), the above approach fails, since \( f_F \) takes value zero everywhere. Nevertheless, it follows from Theorem 2.3 and inequality \( k_{d-1}^1 < k_{d-2} < \cdots < k_0 \), that the exactly the same bounds, i.e. (24) and (25) hold in this case.

Summing the latter inequality (25) over all vertices in \( I \), we obtain
\[ \sum_{v \in I} |X^1(v, C)| = |X^1(I, C)| \geq \frac{(k_0 + 1)(k_1 + 2) \ldots (k_{d-2} + d - 1)k_{d-1}}{\lambda_{max}^1 \lambda_{max}^2 \ldots \lambda_{max}^{d-1}} \cdot i. \] (26)
Combining with the upper bound (3) for \( |X^1(I, C)| \), we get that
\[ \lambda_{max}^0 \cdot \frac{(n - i) \cdot i}{n} \geq \frac{(k_0 + 1)(k_1 + 2) \ldots (k_{d-2} + d - 1)k_{d-1}}{\lambda_{max}^1 \lambda_{max}^2 \ldots \lambda_{max}^{d-1}} \cdot i, \] (27)
which implies,
\[ i_{d-1}(X) = i \leq \frac{\lambda_{max}^0 \ldots \lambda_{max}^{d-1} - (k_0 + 1)(k_1 + 2) \ldots (k_{d-2} + d - 1)k_{d-1}}{\lambda_{max}^0 \lambda_{max}^1 \ldots \lambda_{max}^{d-1}} \cdot n. \] (28)
4 Local version of the theorem

As it may be seen, essentially the same proof as given above allows to prove the theorem for non-regular pure simplicial complexes.

Denote by
\[
\Lambda_j = \max_{F \in X^j} \lambda_{\text{max}}(\text{St}_X(F)), \text{ for } 1 \leq j \leq d - 1,
\]
the maximum of the largest eigenvalue of the \( j \)-th upper Laplacian of a star over all \( j \)-faces of \( X \) and by
\[
k_j = \min_{F \in X^j} \text{deg}(F), \text{ for } 0 \leq j \leq d - 1.
\]
the smallest degree among \( j \)-faces of \( X \).

Then the following statement may be proven in a way similar to that of Theorem 1.5.

**Theorem 4.1.** Let \( X \) be a pure \( d \)-dimensional simplicial complex on a finite vertex set \( V \), then
\[
i_{d-1}(X) \leq \lambda_{\text{max}}^d \Lambda_1 \cdots \Lambda_{d-1} - (k_0 + 1)(k_1 + 2)\cdots(k_{d-2} + d - 1)k_{d-1} \cdot n,
\]
and, therefore,
\[
\chi_{d-1}(X) \geq \frac{\lambda_{\text{max}}^d \Lambda_2 \cdots \Lambda_{d-1}}{\lambda_{\text{max}}^d \Lambda_1 \cdots \Lambda_{d-1} - (k_0 + 1)(k_1 + 2)\cdots(k_{d-2} + d - 1)k_{d-1}}.
\]

5 Complexes with complete \((d - 1)\)-skeleton

Let \( X \) be a \( d \)-dimensional simplicial complex \( X \) on a vertex set \( V \), with \(|V| = n\). It is said to have a complete \((d - 1)\)-skeleton, if every subset of \( V \) of cardinality \( d \) forms a \((d - 1)\)-face of \( X \). We will call such a complex \( k_{d-1} \)-regular, if every \((d - 1)\)-face is of degree \( k_d \).

**Proposition 5.1.** Let \( X \) be a pure \( d \)-dimensional \( k_{d-1} \)-regular simplicial complex \( X \) with complete \((d - 1)\)-skeleton, then
\[
i_{d-1}(X) \leq \frac{\lambda_{\text{max}}^{d-1} - k_{d-1}}{\lambda_{\text{max}}^{d-1}} \cdot n,
\]
and
\[
\chi_{d-1}(X) \geq \frac{\lambda_{\text{max}}^{d-1}}{\lambda_{\text{max}}^{d-1} - k_{d-1}}.
\]
Since \( X \) is \((k_0, \ldots, k_{d-1})\)-regular with
\[
k_j = n - (j + 1) \text{ for } 0 \leq j \leq d - 2,
\]
and the largest eigenvalues of \( \Delta_j^+ \)
\[
\lambda^j_{\text{max}} = n \text{ for } 0 \leq j \leq d - 2,
\]
the proposition follows directly from the main theorem. Nevertheless, this special case can be proven directly and differently as follows.

**Proof.** As before, let \( I \subset V \) be the largest \((d-1)\)-independent set of vertices of \( X \), let \( C = V \setminus I \) be its complement and \( i = |I|, c = |C| \). Note that, \( i \geq d - 1 \).

Partition \( I \) into a disjoint union of \((d-1)\) non-empty sets \( I = A_0 \sqcup \cdots \sqcup A_{d-2} \), and denote \(|A_j| = a_j\). Consider the following function \( f \in C^{d-1}(X, \mathbb{R}) \)
\[
f([v_0, \ldots, v_{d-2}]) = \begin{cases} 
c & \text{if } v_j \in A_j \text{ for all } 0 \leq j \leq d - 2; \\
(-1)^i a_i & \text{if } v_1 \in C \text{ and } v_j \in A_j \text{ for all other } 0 \leq j \leq d - 2; \\
0 & \text{otherwise.}
\end{cases}
\]
Then
\[
< f, f > = a_0 \cdots a_{d-2} \cdot c(c + a_0 + \ldots a_{d-2}) = a_0 \cdots a_{d-2} \cdot c \cdot n
\]
and, since \( I \) is \((d-1)\) independent,
\[
< \Delta_{d-1} f, f > = a_0 \cdots a_{d-2} \cdot k_{d-1} \cdot (c + a_0 + \ldots a_{d-2})^2 = a_0 \cdots a_{d-2} \cdot k_{d-1} \cdot n^2
\]
And since
\[
\lambda^{d-1}_{\text{max}} \geq \frac{< \Delta_{d-1} f, f >}{< f, f >} = \frac{k_{d-1} \cdot n}{n - i}
\]
the bound follows.

This proof was inspired by the work [PRT] of O. Parzanchevski, R. Rosenthal and R. Tessler on a generalization of the Cheeger inequality.

**Remark 5.2.** The bound presented in Theorem 3 is sharp for the independence number of the full \( d \)-dimensional complex \( K^d_n \) on \( n \) vertices, that is a complex on \( n \) vertices, such that every subset of vertices of cardinality \((d+1)\) forms a face.

Complex \( K^d_n \) has full \((d-1)\)-skeleton and every \((d-1)\)-face is contained in \( k_{d-1} = n - d \) \( d \)-faces. The largest eigenvalue of \( \Delta^+_{d-1} \) is \( \lambda^{d-1}_{\text{max}} = n \).

The independence number of \( K^d_n \) is equal to \( d \), and it equals to the bound given by Theorem 1.5
\[
i(K^d_n) = d = \frac{\lambda^{d-1}_{\text{max}} - k_{d-1}}{\lambda^{d-1}_{\text{max}}} \cdot n.
\]
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