Spatial-Temporal Subset-based Digital Image Correlation: A General Framework

Yuxi Chi, Bing Pan*
Institute of Solid Mechanics, Beihang University, Beijing 100191, China
Email: panb@buaa.edu.cn

Abstract

A comprehensive and systematic framework for easily extending and implementing the spatial-temporal subset-based digital image correlation (DIC) algorithm is presented. The framework decouples the three main factors (shape function, correlation criterion, and optimization algorithm) in DIC, and represents different algorithms in a uniform form. One can freely choose and combine the three factors to meet his own need, or freely add more parameters to extract analytic results. Subpixel translation and a simulated image series with different velocity characters are analyzed using different algorithms based on the proposed framework. And an application of mitigating air disturbance due to heat haze using spatial-temporal DIC (ST-DIC) is demonstrated, proving the applicability of the framework.

Keywords: Digital Image Correlation, Spatial-Temporal, Least Square

1 Introduction

As a non-contact method for shape, displacement and strain measurement, digital image correlation (DIC) has been rapidly developed to a powerful technique in experimental mechanic field. The basic idea of DIC is to match the corresponding points in the reference and deformed images, then the displacement and deformation fields can be obtained. For each calculation point, a subset centered at the point is chosen as representation, and a shape function modeled the continuous deformation is used. Also, a correlation criterion is needed to value the difference between the reference and deformed subsets. Various optimization algorithms can be chosen to optimize the criterion. In the development progress of DIC, researchers have proposed many shape functions[1-2], correlation criteria[3-5], and optimization algorithms[6-9] for accurate measurement.

However, with the increasing demand, more and more challenging scenarios are to be measured, including dynamic testing[10-12], high temperature situation[13-17], etc. In these situations, the noise due to the high sampling frequency or heat haze fluctuates with time, and the noise level is much higher than that in normal condition, thus increasing the measurement error of DIC. To suppress the noise, Pan et.al.[18] proposed to apply Gaussian pre-filter before calculation. This method powerfully reduces the bias error but increases the standard deviation (SD) since the SSSIG[19-20] is also lessen by the filter. In another way, Pan et.al.[21] uses the average of images sequence for tracking deflections to suppress noises. The average operation implies the assumption of constant location and serves as the zero-order temporal shape function (TSF). However, the assumption is too strong and will introduce unmatched errors to the measured results, just like in spatial DIC analysis[2]. To suppress the noise while introducing as little errors as possible, the temporal components therefore should be considered in modeling the deformation. That is, the closer the TSF is to the real movement, the smaller the error will be.

Many researchers have discussed the temporal component in the literal. The idea of introducing temporal components flourished in global-DIC, Besnard et.al.[22] discussed the different velocity fields
(constant, parabolic and discontinuous) and integrated the temporal component into global-DIC[23]. These two works present that the temporal regularity can enhance the performances of DIC. Further, Neggers et.al.[24] integrated time-resolved DIC with finite element method, making the method robust and noise insensitive. Efforts are also spent on subset-based DIC. Broggiato et.al. firstly added the temporal component into the shape function and measured the strain rate. In this work, the author fitted the displacements along time to a parabola to get the strain rate component. Wang et.al.[25] proposed to average the time series with the assumption of the constant velocity. This method showed its merit on random noise suppression over Pan’s method. Tang et.al.[26] later extended it to stereo-DIC for shape reconstruction. The author uses the speckle pattern that changes along time, which correlates image pair with the same timestamp, not the average images. This method is equivalent to measuring the same object using various speckle pattern, thus making it robust against the noise level.

In the development of subset-based DIC considering temporal components, efforts are put into the three factors mentioned above. To take the time continuation into consideration, researchers used the TSF to model the deformation and movement, following the idea of spatial SF. Pan et.al. [21] used both 0-order SF and TSF, zero-mean normalized sum of squared difference (ZNSSD) as the correlation criterion and inverse compositional Gauss-Newton (IC-GN) as the optimization algorithm. Wang et.al. [25] used both 1-order SF and TSF, Newton-Raphson (NR) and sum of squared difference (SSD), and Tang et.al. used 2-order SF, NR and zero-mean cross correlation (ZNCC), respectively. Since Tang’s method used a pattern changing over time, the time continuation did not exist, and TSF is not suitable for this situation. As we can see, the three factors, SF & TSF, correlation criterion, and optimization algorithm, can be freely chosen. However, in these works, formulas are deduced in a coupled form, making it hard for researchers to understand and to improve the existing methods. When more parameters to be considered, or new optimization algorithms to be used, a decoupled form and a uniform framework for spatial-temporal DIC (ST-DIC) will be a great help.

In this paper, we proposed a unified framework for spatial-temporal subset-based DIC based on the least square principle. In this framework, temporal and spatial parameters are treated as equal status, which enables the fast IC-GN algorithm. The influences of three factors mentioned above are decoupled and detailed in this work and it can serve as a guideline for researchers to explore new algorithms. We will start with the basic least squares principles and show how to build a simplest DIC algorithm from scratch. The three factors are then analyzed and added to the framework individually. Thus, new methods can be easily implemented based on this framework. The rest parts are organized as follows. Section 2 details the proposed framework and shows how to use the framework to build a customized DIC algorithm. Two experiments including a simulated experiment and a real-world one are introduced in section 3. Finally, section 4 gives a conclusion.

2 Principles

2.1 Least Squares (LSQ)

Mathematically, least squares is used to find the best approximation that minimizes the sum of squared differences between the true value and the parametric modeled value. Without loss of generality, let \( \mathbf{p} \in \mathbb{R}^n \) be the n-dimensional parameter vector, and \( \mathbf{r}(\mathbf{p}) = [r_1(\mathbf{p}), r_2(\mathbf{p}), \ldots, r_m(\mathbf{p})]^T \) be the m-dimensional residual vector. The least squares problem is expressed as:

\[
\arg\min_{\mathbf{p} \in \mathbb{R}^n} f(\mathbf{p}) = \frac{1}{2} \|\mathbf{r}(\mathbf{p})\|^2 = \frac{1}{2} \mathbf{r}(\mathbf{p})^T \mathbf{r}(\mathbf{p})
\]
From the expression, it can be seen that the LSQ problem is only related to the form of the residual vector \( \mathbf{r}(\mathbf{p}) \). The detailed solution of LSQ problem can be found in Appendix A.

2.2 DIC: A special LSQ

2.2.1 A simple DIC algorithm from scratch

As mentioned before, DIC uses shape function to model the deformation between the reference and the deformation subsets and aims to optimize the parameters in shape function to minimize the difference between the modeled deformation and the true counterpart. Therefore, DIC can be seen as a special LSQ. Here, we rewrite the DIC problem in the form of LSQ and build a straightforward DIC algorithm from scratch.

Consider a subset with center located at \([x_c, y_c]^T\). The first-order shape function with parameters \( \mathbf{p} \) map the pixel at \( \mathbf{X} = [x, y]^T \) to the deformation location as:

\[
\mathbf{\tilde{X}}(\mathbf{p}) = \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \mathbf{X} + \begin{bmatrix} u & u_x & u_y \\ v & v_x & v_y \end{bmatrix} \begin{bmatrix} 1 \\ \Delta x \\ \Delta y \end{bmatrix}
\] (2)

where \( \Delta x = x - x_c, \Delta y = y - y_c, \mathbf{p} = [u, u_x, u_y, v, v_x, v_y]^T \).

Noting the reference subset as \( \mathbf{F} \) and the deformation subset as \( \mathbf{G} \), the residual component at \( \mathbf{X} \) can be expressed as:

\[
\mathbf{r}(\mathbf{p})_\mathbf{X} = \mathbf{F}(\mathbf{X}) - \mathbf{G}(\mathbf{\tilde{X}}(\mathbf{p}))
\] (3)

As we can see in (A.8), the key to solve LSQ problem is the Jacobi matrix. Here we write the Jacobi vector at \( \mathbf{X} \) as:

\[
\mathbf{J}(\mathbf{p})_\mathbf{X} = \frac{\partial \mathbf{r}(\mathbf{p})_\mathbf{X}}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{p}} \left( \mathbf{F}(\mathbf{X}) - \mathbf{G}(\mathbf{\tilde{X}}(\mathbf{p})) \right)
\]

\[
= - \frac{\partial \mathbf{G}}{\partial \mathbf{X}} \frac{\partial \mathbf{\tilde{X}}}{\partial \mathbf{p}}
\] (4)

\[
= - \begin{bmatrix} G_{\tilde{x}} \\ G_{\tilde{y}} \end{bmatrix}^T \begin{bmatrix} 1 & \Delta \tilde{x} & \Delta \tilde{y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \Delta \tilde{x} & \Delta \tilde{y} \end{bmatrix}
\]

where \( G_{\tilde{x}} \) and \( G_{\tilde{y}} \) are the derivatives of the deformation subset \( \mathbf{G} \) with respect to \( x \) and \( y \), respectively.

For every pixel location \( \mathbf{X} \) in the subset, we can write the corresponding Jacobi vector \( \mathbf{J}(\mathbf{p})_\mathbf{X} \). This is a row vector of 6 elements, and we can vertically stack the Jacobi vectors to assemble the Jacobi matrix \( \mathbf{J}(\mathbf{p}) \). If there are \( n \times n \) pixels in the subset, the shape of the Jacobi matrix will be \( n^2 \)-by-6.

Using (A.9), we can establish the same iterative form as:

\[
\mathbf{p}^{(k+1)} = \mathbf{p}^{(k)} + \Delta \mathbf{p}
\]

\[
= \mathbf{p}^{(k)} - \mathbf{H}(\mathbf{p}^{(k)})^{-1} \mathbf{J}(\mathbf{p}^{(k)})^T \mathbf{r}(\mathbf{p}^{(k)})
\] (5)
In this example, a simple first-order shape function and a simple sum of squared difference (SSD) criterion are used to describe the DIC problem, and Gauss-Newton method to solve the parameters. This is the classical algorithm named FA-NR. Note that the accurate name should be FA-GN, because the approximate Hessian matrix, not the exact Hessian matrix, is used. The latter is required in the Newton-Raphson method.

2.3 Optimization algorithm

Mathematically, the DIC problem is a nonlinear LSQ problem that minimizes the difference between the reference image and the deformation counterpart with the deformation parameters. In DIC, considered its special characteristics that two images have equal status in the matching problem, these methods can be divided into forward and inverse methods according to the role of reference during iterations. Also, methods can also be divided into additive and compositional methods according to the way of parameters updating. Hence, there are 4 basic methods: forward additive, forward compositional, inverse additive and inverse compositional methods. Further, to determine the increment of parameters, there are countless mathematical methods such as Newton-Raphson, Gauss-Newton, Levenberg-Marquardt, trust region, dogleg and many other methods. These traits can be freely combined into an optimization algorithm.

The mostly used forward-additive Newton-Raphson (FA-NR) method\cite{6} and inverse-compositional Gauss-Newton (IC-GN) method\cite{9} can be easily built on the proposed framework. Here, we will briefly introduce the two basic classic methods. The simplest and most intuitive FA method has been detailed in 2.2.1 and the compositional methods are described below.

2.3.1 Compositional methods

The compositional method is rather different with the additive ones. In compositional methods, the first order shape function is written in homogeneous form:

\[
\begin{bmatrix}
1 \\
\tilde{x} - x_C \\
\tilde{y} - y_C
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
u & 1 + u_x & u_y \\
v & v_x & 1 + v_y
\end{bmatrix}
\begin{bmatrix}
1 \\
x - x_C \\
y - y_C
\end{bmatrix}
\]

and in the local coordinate as:

\[
\tilde{X}_H =
\begin{bmatrix}
\tilde{x} \\
\tilde{y}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
u & 1 + u_x & u_y \\
v & v_x & 1 + v_y
\end{bmatrix}
\begin{bmatrix}
1 \\
x \\
y
\end{bmatrix}
\]

where \(X_H\) is the local coordinates in the homogeneous form.

Since the first order shape function is an affine transform and all the invertible affine transformations forms an affine group, the increment \(\Delta p\) can be composited with the current parameter \(p\) to get an updated warp matrix. As described in\cite{7-8}, the increment \(\Delta p\) warp the location \(X_H\) in the warped image to \(W(\Delta p)X_H\), thus the composited warp matrix transform the location \(X_H\) in the original image to \(W(p)W(\Delta p)X_H\). That is, the parameters are updated as:

\[
W(p^{(k+1)}) = W(p^{(k)})W(\Delta p)
\]
In light of (8), the residual component at $X_H$ can be expressed as:

$$ r(p)_{x_n} = F(X_H) - G(W(p)W(\Delta p)X_H) $$

and the Jacobi vector at $X$ is expressed as:

$$ J(p)_{x_n} = \frac{\partial}{\partial \Delta p} (F(X_H) - G(W(p)W(\Delta p)X_H)) $$

$$ = - \frac{\partial G}{\partial W(p)X_H} \frac{\partial W(p)X_H}{\partial p} $$

(10)

where $W(p)X_H$ is the warped location, namely $\tilde{X}_H$ in (4). That is, in forward compositional method, the increment $\Delta p$ is calculated in the same way as (5), but parameters are updated using the formula (8).

In the inverse compositional methods, the increment $\Delta p$ is added to the reference. We write the residual component at $X_H$ in the same way as:

$$ r(p)_{x_n} = F(W(\Delta p)X_H) - G(W(p)X_H) $$

(11)

then using the chain rule gives the Jacobi vector at $X_H$:

$$ J(p)_{x_n} = \frac{\partial}{\partial \Delta p} (F(W(\Delta p)X_H) - G(W(p)X_H)) $$

$$ = \frac{\partial F}{\partial X_H} \frac{\partial W(\Delta p)X_H}{\partial \Delta p} $$

$$ = \begin{bmatrix} F_x^T & 1 & x & y & 0 & 0 & 0 \\ F_y^T & 0 & 0 & 1 & x & y \end{bmatrix} $$

$$ \begin{bmatrix} X_H^T & 0 \\ 0 & X_H^T \end{bmatrix} $$

(12)

Where the increment $\Delta p$ is calculated using (A.8), it needs to be composite with the current parameters $p$. Let $X_H' = W(\Delta p)X_H$ be the warped location, $X_H$ will be $X_H = W(\Delta p)^{-1}X_H'$ and the residual component can be expressed as:

$$ r(p)_{x_n} = F(W(\Delta p)X_H) - G(W(p)X_H) $$

$$ = F(X_H') - G(W(p)W(\Delta p)^{-1}X_H') $$

(13)

It is clear that the parameters are updated by the following formula:

$$ W(p^{(k+1)}) = W(p^{(k)})W(\Delta p)^{-1} $$

(14)

Note that when inverse compositional method is used, we can only update the rows with respect to $x$ and $y$. The two rows contain all the parameters, and the two rows in updated warp matrix depend only on the two rows in the current warp matrix. This trick can further improve the calculation efficiency.

2.3.2 Discussion on the optimization algorithm

As seen above, the DIC is a special case of nonlinear LSQ. The model function in DIC is the shape
function. The residual vector in DIC is the greyscale difference between the reference subset and the modeled deformation subset while in general LSQ, the residual vector is the true value with the modeled one. What makes DIC different is the image that serves as a function mapping the location to greyscale. This function makes the comparison happens on the greyscale rather than the location itself. As we can see in the expression of Jacobi matrix, the derivatives of the parameters are firstly propagated through the shape function to the location, then go through image gradient to the greyscale.

In forward methods, the increment is added to the warped subset. Hence, the Jacobi vector contains the gradient of the warped image, which requires to calculate the gradient of the warped image and the warped local coordinates in each iteration, as showed in (4). Because the compositional method is equivalent with the additive methods in first order approximation, as showed in (4) and (10), both additive and compositional methods have the same form of Jacobi vectors.

In inverse compositional method, the increment is added to the reference subset. Consequently, the Jacobi is independent from the warped subset as showed in (12) and only the gradient of reference image is used. The Jacobi vector is only the function of the reference gradient and the pixel coordinate and has no connection with the parameters $p$. This feature allows us to calculate the Jacobi and the Hessian matrices in advance without repeated calculations in the iterations as in forward methods, thus greatly improving the calculation speed and making IC-GN to be the defacto standard algorithm in DIC.

### 2.4 Correlation criterion

In LSQ problem, the correlation criterion or object function is simply selected as the L2-Norm of the residual vector based on the least square principle. In the development history of DIC, various criteria have been proposed. In early days, simple and intuitive criteria are often selected, including cross-correlation (CC), sum of absolute difference (SAD), i.e., the L1-Norm of the residual vector, sum of squared difference (SSD), namely the L2-Norm, and parametric sum of squared difference (PSSD). Considering the robustness, researchers proposed the normalized criteria, including zero-mean cross-correlation (ZNCC), zero-mean normalized sum of squared difference (ZNSSD), and parametric sum of square difference (PSSD$_{ab}$). Actually, as has been proved by Pan et.al[27], these normalized criteria are equivalent. As the simplified version, those criteria without normalization are also equivalent. Here, we just take SSD and ZNSSD for comparison and show how to use ZNSSD in this framework.

ZNSSD consider the offset and scale changes of the greyscale by subtract the mean value then rescale to length one. Mathematically, ZNSSD between two images $F$ and $G$ is:

$$\text{ZNSSD}(F,G) = \frac{\|F - \bar{F} - G - \bar{G}\|^2}{\|F - \bar{F}\| \|G - \bar{G}\|}$$

$$= \left(\frac{1}{n^2 \Delta F} - \frac{1}{n^2 \Delta G}\right)^2$$

where $\Delta F$ is the standard derivation of the pixels’ greyscale in the subset, $n$ is the subset size. The scaling factor $1/n^4$ has no effect on the optimization. The residual component at $X$ can therefore be expressed as:
With the residual expression using ZNSSD, we can easily port the algorithms using SSD to ZNSSD. Here we take the classic FA-NR (4) and IC-GN (12) as examples. For FA-NR method, calculating the Jacobi vector at $\mathbf{X}$ using the chain rule, we can obtain a similar expression:

$$
\mathbf{r}(p)_{\mathbf{x}} = \frac{\mathbf{F}(\mathbf{X}) - \mathbf{F}}{\Delta \mathbf{F}} - \frac{\mathbf{G}(\hat{\mathbf{X}}(p)) - \mathbf{G}}{\Delta \mathbf{G}}
$$

(16)

Here we take the classic FA-NR (4) and IC-GN (12) as examples. For FA-NR method, calculating the Jacobi vector at $\mathbf{X}$ using the chain rule, we can obtain a similar expression:

$$
\mathbf{J}(p)_{\mathbf{x}} = -\frac{1}{\Delta \mathbf{G}} \frac{\partial \mathbf{G}}{\partial \hat{\mathbf{X}}} \frac{\partial \hat{\mathbf{X}}}{\partial p}
$$

(17)

$$
= -\frac{1}{\Delta \mathbf{G}} \begin{bmatrix} G_x^T \\ G_y^T \end{bmatrix}^T \begin{bmatrix} 1 & \Delta \tilde{x} & \Delta \tilde{y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \Delta \tilde{x} & \Delta \tilde{y} \end{bmatrix}
$$

Compare (17) with (4), we can find that the Jacobi vector is just proportional to that in SSD criterion:

$$
\mathbf{J}_{\text{ZNSSD}}(p)_{\mathbf{x}} = \frac{1}{\Delta \mathbf{G}} \mathbf{J}_{\text{SSD}}(p)_{\mathbf{x}}
$$

(18)

Also, the Hessian matrix is also proportional to that in SSD criterion:

$$
\mathbf{H}_{\text{ZNSSD}}(p) = \frac{1}{\Delta \mathbf{G}^2} \mathbf{H}_{\text{SSD}}(p)
$$

(19)

Then, the iterative updating parameter vector can be express as:

$$
\Delta \mathbf{p} = -\mathbf{H}(p)^{-1} \mathbf{J}(p)^T \mathbf{r}(p)
$$

$$
= -\Delta \mathbf{G} \mathbf{H}_{\text{SSD}}(p)^{-1} \mathbf{J}_{\text{SSD}}(p)^T \mathbf{r}(p)
$$

(20)

Similarly, in IC-GN method, the Jacobi vector at $\mathbf{X}_H$ is expressed as:

$$
\mathbf{J}(p)_{\mathbf{x}_H} = \frac{1}{\Delta \mathbf{F}} \frac{\partial \mathbf{F}}{\partial \mathbf{X}_H} \frac{\partial \mathbf{W}(\Delta \mathbf{p}) \mathbf{X}_H}{\partial \Delta \mathbf{p}}
$$

$$
= \frac{1}{\Delta \mathbf{F}} \begin{bmatrix} F_x^T \\ F_y^T \end{bmatrix}^T \begin{bmatrix} \mathbf{X}_H^T & 0 \\ 0 & \mathbf{X}_H^T \end{bmatrix}
$$

(21)

The similar connections between SSD and ZNSSD in IC-GN method is showed below:

$$
\mathbf{J}_{\text{ZNSSD}}(p)_{\mathbf{x}} = \frac{1}{\Delta \mathbf{F}} \mathbf{J}_{\text{SSD}}(p)_{\mathbf{x}}
$$

$$
\mathbf{H}_{\text{ZNSSD}}(p) = \frac{1}{\Delta \mathbf{F}^2} \mathbf{H}_{\text{SSD}}(p)
$$

(22)

$$
\Delta \mathbf{p} = -\mathbf{H}(p)^{-1} \mathbf{J}(p)^T \mathbf{r}(p)
$$

$$
= -\Delta \mathbf{F} \mathbf{H}_{\text{SSD}}(p)^{-1} \mathbf{J}_{\text{SSD}}(p)^T \mathbf{r}(p)
$$
We have seen how to replace the criterion using ZNSSD while keeping the rest part (shape function and optimization algorithm). When we change the correlation criterion from SSD to ZNSSD, the Hessian and Jacobi matrices can be reused. We only need to calculate the standard deviation and the residual vector in the subset. Note that the residual vector is different from that in SSD criterion while the Hessian matrix and Jacobi matrix are the same.

2.5 Parameters and shape function

In traditional DIC that does not consider the temporal components, only spatial parameters are optimized. Different shape functions (SF) are used to describe the deformations. The zero-order SF considering the translation have only two parameters, i.e. \( u \) and \( v \). The first-order SF consider the translation and affine transformation, which add the first-order spatial derivatives \( (u_x, u_y, v_x, v_y) \). The second-order SF consider higher order of deformation, and second-order spatial derivatives \( (u_{xx}, u_{xy}, u_{yy}, v_{xx}, v_{xy}, v_{yy}) \) are added. Also, any other parameters that can describe deformation can be added into the SF, that is the main idea of the integrate DIC (i-DIC)\[28\]. When the temporal components are considered, there also comes the time shape function (TSF). Similarly, zero-order TSF (no additional parameters) implies the constant location between the image series while first-order TSF (add \( u_t, v_t \)) describe the linear change of location, that is, the constant velocity. Also, second-order TFS (add \( u_{tt}, v_{tt} \)) can fit the constant acceleration condition.

In the deduction of the aforementioned methods, only first-order SF is used for simplicity. But in fact, these formulas do not depend on the selection of shape functions. In the Jacobi matrices showed above, the Jacobi of shape function propagates the derivatives from parameters to the location. They share the same form as \( \frac{\partial \mathbf{X}}{\partial \mathbf{p}} \), depending on the optimization algorithm used. That is, the shape function can be arbitrarily chosen according to the form of deformation.

2.5.1 Spatial parameters

The first-order SF have been showed in (2), and second-order parameters can be easily added using Taylor’s series as:

\[
\tilde{\mathbf{X}}(\mathbf{p}) = \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \mathbf{X} + \begin{bmatrix} u_x & u_y & u_{xx} & u_{xy} & u_{yy} & u \\ v_x & v_y & v_{xx} & v_{xy} & v_{yy} & v \end{bmatrix} \begin{bmatrix} 1 \\ \Delta x \\ \Delta y \\ \Delta x^2/2 \\ \Delta x \Delta y \\ \Delta y^2/2 \end{bmatrix} 
\]

where the parameters \( \mathbf{p} = [\mathbf{p}_u, \mathbf{p}_v]^T \) contains two parts associated with two directions, \( u \) and \( v \). They are expressed as: \( \mathbf{p}_u = [u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}] \) and \( \mathbf{p}_v = [v, v_x, v_y, v_{xx}, v_{xy}, v_{yy}] \).

Since each part have no connection with the other direction and shares the same status, the Jacobi of shape function can be easily written as:

\[
\frac{\partial \tilde{\mathbf{X}}}{\partial \mathbf{p}} = \begin{bmatrix} \mathbf{X}_H^T & 0 \\ 0 & \mathbf{X}_H^T \end{bmatrix}
\]

where \( \mathbf{X}_H^T = [1, \Delta x, \Delta y, \Delta x^2/2, \Delta x \Delta y, \Delta y^2/2] \).
As we have seen, the change of the shape function only affects the specific form of the last item, and the form of Jacobi remains unchanged. Here, the shape of Jacobi vector at $\mathbf{X}$ is 1-by-12, since the second-order shape function requires 6 parameters in every direction. The shape of the assembled Jacobi matrix will be $n^2$-by-12, depending on the number of pixels in the subset. The rest of the steps are exactly the same as before. Also, because the change of shape function has no effect on other items in Jacobi, we can freely choose the optimization algorithm and correlation criterion.

It should be noted that when compositional methods are used, the homogeneous from of shape function, as in (6) and (7), is relatively complex. The warp matrix transform the current location vector $[1, x, y, x^2/2, xy, y^2/2]^T$ to the warped one: $[1, \tilde{x}, \tilde{y}, \tilde{x}^2/2, \tilde{x}\tilde{y}, \tilde{y}^2/2]^T$. The cross terms in the warp matrix can be deduced by expanding the expressions using the shape function. For the specific form of the second-order warp matrix, please refer to[29].

2.5.2 Temporal parameters

In ST-DIC, we use the central frame as the one to be calculated, and its neighbor frames as the temporal information provider. Similar to the spatial dimension, there is also sampling frequency in the time dimension. The spatial frequency is expressed using the pixel size $dx$, while the temporal frequency is expressed using the time interval $dt$. Here we denote one point using its spatiotemporal coordinate $[x, y, t]$ in the local coordinates. If image series are captured with the equal time interval, the temporal coordinate can be simplified to be integers.

Considering the temporal continuity, the temporal parameters can be added into the shape function. Since images are captured individually, the temporal coordinates of every images are known and keep constant. If we use the first order spatial shape function and first order temporal shape function, the shape function can be expressed as:

$$
\tilde{\mathbf{X}}(\mathbf{p}) = 
\begin{bmatrix}
\tilde{x} \\
\tilde{y}
\end{bmatrix}
= \mathbf{X} + \begin{bmatrix}
u & u_x & u_y & u_t \\
v & v_x & v_y & v_t
\end{bmatrix}
\begin{bmatrix}
1 \\
\Delta x \\
\Delta y \\
\Delta t
\end{bmatrix}
$$

(25)

where $\mathbf{X} = [x, y]^T$ and $\mathbf{p} = [u, u_x, u_y, u_t, v, v_x, v_y, v_t]^T$. and in the homogeneous form in the local coordinates as:

$$
\tilde{\mathbf{X}}_h(\mathbf{p}) =
\begin{bmatrix}
1 \\
\tilde{x} \\
\tilde{y} \\
t
\end{bmatrix} = 
\begin{bmatrix}
1 & 0 & 0 & 0 \\
u & 1+u_x & u_y & u_t \\
v & v_x & 1+v_y & v_t \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
x \\
y \\
t
\end{bmatrix}
$$

(26)

The aim of ST-DIC is to minimize difference between the reference and the warped image series. The matching can be seen as to add another dimension to the residual vector. And we can flatten the time dimension to get the residual vector with the shape of $n^2m$-by-1. Here, $m$ is the number of the image series. The residual vector at $\mathbf{X}$ and $t$ is expressed as:

$$
\mathbf{r}(\mathbf{p})_{\mathbf{X},t} = \mathbf{F} (\mathbf{X}) - \mathbf{G}_t (\tilde{\mathbf{X}}(\mathbf{p}))
$$

(27)
where $G_t$ is the deformation image at time $t$.

Similarly, we just need to write the Jacobi vector with respect to one item, and the whole Jacobi matrix can be obtained through stacking the Jacobi vectors. With this in mind, the time dimension is just the same as spatial dimensions, the only difference is that the time component is constant and has no contribution to the Jacobi. Then we can directly write the Jacobi vector at $X$ and $t$ in the additive method using SSD as:

$$
J(p)_{X,t} = - \begin{bmatrix}
G_x \\
G_y
\end{bmatrix}^T \begin{bmatrix}
1 & \Delta \tilde{x} & \Delta \tilde{y} & \Delta t & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \Delta \tilde{x} & \Delta \tilde{y} & \Delta t
\end{bmatrix}
$$

and in the inverse compositional method as:

$$
J(p)_{X,t} = \begin{bmatrix}
F_x \\
F_y
\end{bmatrix}^T \begin{bmatrix}
X_H^T & 0 \\
0 & X_H^T
\end{bmatrix}
$$

where $X_H^T = [1, \tilde{x}, \tilde{y}, t]$ and $X_H^T = [1, x, y, t]$.

If one wants to use the second order temporal shape function, then just write the shape function as:

$$
\tilde{X}(p) = \begin{bmatrix}
\tilde{x} \\
\tilde{y}
\end{bmatrix} = X + \begin{bmatrix}
u & u_x & u_y & u_t & u_{tt} \\
v & v_x & v_y & v_t & v_{tt}
\end{bmatrix} \begin{bmatrix}
1 \\
\Delta x \\
\Delta y \\
\Delta t \\
\Delta t^2 / 2
\end{bmatrix}
$$

and in homogeneous form in the local coordinates as:

$$
\tilde{X}_H(p) = \begin{bmatrix}
1 \\
\tilde{x} \\
\tilde{y} \\
\tilde{t}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
u & 1 + u_x & u_y & u_t & u_{tt} \\
v & v_x & 1 + v_y & v_t & v_{tt} \\
0 & 0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
1 \\
x \\
y \\
\tilde{t} \\
\tilde{t}^2 / 2
\end{bmatrix}
$$

The Jacobi vector at $X$ and $t$ in the additive method are identically expressed as:

$$
J(p)_{X,t} = - \begin{bmatrix}
G_x \\
G_y
\end{bmatrix}^T \begin{bmatrix}
\tilde{X}_H^T & 0 \\
0 & \tilde{X}_H^T
\end{bmatrix}
$$

and in the inverse compositional method as:
Once the form of the Jacobi matrix is obtained, the problem is solved.

As we have seen, all the deductions are totally the same as before. The three factors are decoupled and can be freely combined to form a new algorithm.

2.6 A tutorial of building a customized DIC algorithm

Follow the same procedure, any other parameters (e.g. strain rate), can also be added into the framework. Here, we will completely show how to build a DIC algorithm following the proposed framework.

Step 1: Write the shape function including the strain rate elements with the help of Taylor’s expansion as:

\[
\tilde{X}(\mathbf{p}) = \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \mathbf{X} + \begin{bmatrix} u & u_x & u_y & u_t & u_{xt} & u_{yt} \\ v & v_x & v_y & v_t & v_{xt} & v_{yt} \end{bmatrix} \begin{bmatrix} 1 \\ \Delta x \\ \Delta y \\ \Delta t \\ \Delta x \Delta t \\ \Delta y \Delta t \end{bmatrix}
\]

then expand the cross items in homogeneous form in the local coordinates as:

\[
\begin{align*}
\tilde{x} t &= xt + ut + u_x xt + u_y yt + u_{xt} xt^2 + u_{yt} yt^2 \\
\tilde{y} t &= yt + vt + v_x xt + v_y yt + v_{xt} xt^2 + v_{yt} yt^2
\end{align*}
\]  

(35)

omit the high order terms and rewrite the shape function in the homogeneous form as:

\[
\begin{bmatrix}
1 \\
\tilde{x} \\
\tilde{y} \\
t \\
\tilde{x} t \\
\tilde{y} t
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
u & 1 + u_x & u_y & u_t & u_{xt} & u_{yt} \\
v & v_x & 1 + v_y & v_t & v_{xt} & v_{yt} \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & u & 0 & 0 & 1 + u_x & u_y \\
0 & v & 0 & 0 & v_x & 1 + v_y
\end{bmatrix} \begin{bmatrix}
1 \\
x \\
y \\
t \\
x t \\
y t
\end{bmatrix}
\]

(36)

\[
\tilde{X}_H(\mathbf{p}) = \mathbf{W}(\mathbf{p}) \mathbf{X}_H
\]

With the help of shape function, the deformation warped subset can be interpolated using the deformation image and warped locations. Then, the warped subset will be used to calculation the residual vector.

Step 2: Choose the optimization algorithm. If forward additive method is preferred, then the Jacobian vector at one point is calculated using (32). Or if inverse compositional method is chosen, it follows (33). Then stack the Jacobian vectors into the Jacobian matrix, the Jacobian and Hessian Matrices can be calculated in advance.
Step 3: Choose the correlation criterion. If we use SSD criterion, then the updating parameter vector \( \Delta \mathbf{p} \) can be calculated using (A.8). And if ZNSSD is used, then \( \Delta \mathbf{p} \) should be calculated using (20) or (22) according to the optimization algorithm chosen. Where the residual vector should also be calculated using (3) or (16) according to the correlation criterion.

Step 4: Updating the parameters and perform iteration. According to the optimization algorithm, update the parameters using the calculated updating vector and iterate until converge.

### 3 Experiments

#### 3.1 Simulation Experiment

##### 3.1.1 Sub-pixel translation analysis

A sub-pixel translation experiment is performed to analysis the robustness against noise. The original image was downloaded from the DIC challenge[30]. And a series of translated image were generated using Fourier transform. With a step of 0.05 pixel, 20 deformed images are simulated with the \( u \) displacement following the formula as:

\[
u(t) = \frac{t}{20} \text{ pixels}\]

(37)

Then, Gaussian noises of zero mean and different variance were added to the generated image series. Here, 5 noise level range from 1% to 5% of the maximum greyscale (255 in 8-bit gray image) were added.

After the image generation, the traditional spatial DIC method and the temporal method described in 2.6 were used for comparison, where the former use the first-order spatial shape function while the latter use the first-order spatial and temporal shape function. In both methods, the spatial subset size is 31\( \times \)31 with 10 pixels step in both directions, and the calculation points are the same where the temporal method use 5 frames in the temporal dimension. Both methods use the ZNSSD criterion and IC-GN optimization algorithm. In each frame, we use the mean L1-norm to evaluate the measurement error between calculated displacements and the given one:

\[
\text{error} = \frac{1}{N} ||u_m - u||_1
\]

(38)

where \( N \) is the number of calculation points, \( u_m \) is the measured displacement vector of all calculation points and \( u \) is the given one.

The sub-pixel translation results are showed in Figure 1. The left two subplot shows the results of ST-DIC and the results of spatial DIC is showed in the right. In the zero-noise case, the errors show sinusoidal trend with respect to the sub-pixel displacement in both methods. The errors and standard deviation (SD) are increasing along with the noise level as expected in both methods. In each level of noise, the errors and SD using ST-DIC are lower than that of spatial DIC. And it can be seen intuitively that the result of processing 5% noise image using ST-DIC has the same level of error and SD as the result of processing 3% noise image using spatial DIC, both in errors and SD. It is convincing that the ST-DIC can further improve the robustness against image noise.
In the sub-pixel translation experiment, the displacement and time are in linear relationship. Hence, the first-order temporal shape function, which implies the linear velocity, can perfectly fit this situation.

### 3.1.2 Simulated vibration analysis

For real world experiments, ST-DIC is used in image sequence analysis where the time step between adjacent image is a known constant. A typical example is to measure vibration using a high-speed camera. In vibration testing, the displacements are rarely linear. And first-order temporal shape function will introduce the under-matched error as in spatial analysis[2]. Here, we simulated two vibration displacements as $u$ and $v$ respectively and applied them to the reference image to generate a series of deformation image, where:

$$u(t) = 10e^{-2t}\sin10t$$

$$v(t) = 10e^{-3t}\sin5t$$

(39)

The imposed displacements are showed in Figure 2. The time step is 0.01s and there are 200 frames in the 2s testing time span. It is clear that the distribution of sample points near the extreme point is not linear, and the local linearity goes better as time increasing. The displacement mode combines the sharp and gentle change. Similarly, noises of 5 level vary from 1% to 5% were added to discuss the robustness.
against the noise. Here, we used three DIC algorithm to measure the given displacement: the simple first-order spatial DIC, the first-order ST-DIC and the ST-DIC with second order temporal shape function described in (30). Other parameters remained unchanged.

![Figure 2](image1.png)

**Figure 2** The imposed displacements \( u \) and \( v \).

For intuitiveness, results of \( u \) and \( v \) are showed in Figure 3 and Figure 4. For the sharp changed displacement \( u \), the constant velocity assumption may not be satisfied near the extreme points. As showed in Figure 3, in the first-order ST-DIC, the under-matched error dominated the bias error near the extreme points, performing much worse than the spatial method. Whereas, the second-order ST-DIC can fit the local non-linearity better and showed the best performance in the initial sharp changed situation (\( t < 1 \)s). The method successfully overcame the under matched errors and suppressed the noise interference. When the displacement change became gentle (\( t > 1 \)s), the first order method showed its superiority over other two methods, the errors were less than the second-order method and showed less SD in the time axis.

![Figure 3](image2.png)

**Figure 3** The measured \( u \) displacements using three DIC methods.

For the \( v \) displacement, the results supported the same conclusion. The first-order ST-DIC showed worse in the sharp changed area but showed the best in the gentle changed area. The second-order ST-DIC showed equally stability in both cases and have better robustness against noise than the classic spatial
DIC under every noise level. In the flat case, when \( t > 1 \) s, the ratio of mean errors along the time of other methods to the linear method are listed in Table 1. The mean errors of the second-order method are 10\%\% to 15\%\% higher than the linear method, while the errors of traditional spatial method are 30\%\% to 40\%\% higher. It can be concluded that the second-order ST-DIC is suitable for both linear and nonlinear displacements due to its great performance on complex displacements and anti-noisiness.

![Figure 4](image)

**Table 1** The mean error ratio (# / ST-DIC (order 1))

| noise / % | Spatial DIC | ST-DIC (order 2) |
|-----------|-------------|-----------------|
| 0         | 1.033       | 1.031           |
| 1         | 1.252       | 1.092           |
| 2         | 1.295       | 1.112           |
| 3         | 1.308       | 1.116           |
| 4         | 1.403       | 1.157           |
| 5         | 1.348       | 1.132           |

### 3.2 Application of mitigating air disturbance due to heat haze

As described in our previous work[16], the air disturbance due heat haze brings adverse effects in thermomechanical testing. The unsteady flow field between the camera and the specimen will bring huge noises to the deformation measurement. The thermal expansion displacement field of an isotropic material will be no longer concentric under the disturbance. Note that the air disturbance and the true deformation is coupled and cannot be easily separated. But in this experiment, the temperature of the specimen is applied stably but the air disturbance is random, the temporal-spatial method described in 2.6 can be used to apply the temporal constraint and mitigate the disturbance to some extent.

Here, we just apply the proposed method to the recoded images in the experiments described in the previous work[16]. The experiment measures the coefficient of thermal expansion (CTE) using 2D-DIC, and the specimen is freely located and heated by the quartz lamps. More details of the experiment set-up can be found in the reference. The images are analyzed using two method (first order spatial-DIC and first order ST-DIC) based on the proposed framework. In both methods, the spatial subset sizes are 51 × 51.
pixels and the grid steps are 15 pixels. In the spatial-temporal method, the temporal subset size is set to 5 to apply enough temporal constraints.

Figure 5 shows the displacement fields at the temperature of 555°C using the two methods. Note that the rigid body displacement has been removed. In the experiments, the contours should be regular concentric circles in ideal situation. As shown in the Fig, the effect of heat haze is obvious. The contours obtained using traditional spatial DIC is more deformed and farther away from concentric circles while that of ST-DIC shows much better displacement fields. Comparing with the contours obtained using ST-DIC, the traditional one shows a compression disturbance near the center. This kind of disturbance is usually due to the heating of quartz lamp strips. It is apparent that the use of temporal constraints can mitigate the disturbance due to heat haze.

Figure 5 The displacement fields at the temperature of 555 °C using the temporal-spatial DIC (left) and traditional spatial DIC (right).

In the experiments, the strain along two directions should be equal and homogeneous within the ROI. Figure 6 gives the mean strain components of \( u_x \) and \( u_y \) using two methods along with temperature. The mean strain obtained using traditional spatial-DIC is showed as dot while that of ST-DIC is showed in line. It is clear that the line goes through the dots in both \( x \) and \( y \) directions, indicating the results of ST-DIC has less fluctuation and higher linearity. Through linear fit, the CTE and the \( R^2 \) along \( x \) and \( y \) can be obtained and showed in Table 2. The difference of CTEs in two directions using ST-DIC is 0.06 \( \mu \varepsilon / ^\circ \text{C} \) while that of spatial-DIC is 0.53 \( \mu \varepsilon / ^\circ \text{C} \). The \( R^2 \) of the ST-DIC is higher than that of spatial-DIC in both directions. It can be concluded that the CTEs obtained using ST-DIC are of higher credibility. To further demonstrate the improvements of ST-DIC, the SD of strain field along with temperature using two methods are showed in Figure 7. The SD errors using ST-DIC is half of that of spatial-DIC in both directions, which means the strain fields are more homogeneous than that of spatial-DIC, indicating the robustness of ST-DIC against the air disturbance due to heat haze.

Table 2 The CTE and \( R^2 \) of different methods

|                  | Spatial DIC | ST-DIC |
|------------------|------------|--------|
| \( \text{CTE}_x \) (\( \mu \varepsilon / ^\circ \text{C} \)) | 19.66      | 19.04  |
|               | Method 1 | Method 2 |
|---------------|----------|----------|
| $R^2$         | 0.9884   | 0.9982   |
| $\text{CTE}_y \ (\mu \varepsilon/{}^\circ C)$ | 19.13    | 18.98    |
| $R^2$         | 0.9975   | 0.9989   |

Figure 6 The mean strain components $u_x$ (left) and $v_y$ (right) using two methods at different temperatures.

Figure 7 The mean strain components $u_x$ (left) and $v_y$ (right) using two methods at different temperatures.

4 Conclusions

In this paper, a comprehensive and systematic framework that can easily extend and implement the spatial-temporal DIC (ST-DIC) algorithm is proposed. The three main factors (optimization algorithms, correlation criterion and shape functions) in DIC problem are decoupled and discussed in detail. The framework unifies the time and space dimension and treats them as the same status. Hence, it can represent the traditional spatial DIC and ST-DIC in a uniform form. Researchers can freely choose the factors or add more parameters to form a new DIC algorithm. Though ZNSSD and IC-GN have become the first choice, the shape function can be modified in different situation.
Further, the 1-order temporal shape function considering the strain rate and the 2-order temporal shape function based on the proposed framework were presented. The two algorithms are deduced in the uniform form and can be easily adapted from the existing code. Simulated sub-pixel translation experiments proved the robustness against noises of the temporal methods. And the simulated image sequence with different velocity characters showed the difference application scenarios of difference temporal methods. The researcher should select different temporal shape function or parameters according to the motion characteristics of the object to be measured. The real-world experiment measuring the CTE shows the robustness of the ST-DIC against the air disturbance due to heat haze. Through the adverse effect cannot be totally eliminated, the proposed method can mitigate the disturbance to some extent. When researchers consider building a ST-DIC algorithms, the proposed framework is hoped to be a great help.

Acknowledgments

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Appendix A

A.1 Linear LSQ

When $\mathbf{r}(\mathbf{p})$ is a linear combination of the parameter vector $\mathbf{p}$, we call this case linear least squares. That is, $\mathbf{r}(\mathbf{p})$ can be expressed as:

$$\mathbf{r}(\mathbf{p}) = \mathbf{A}\mathbf{p} + \mathbf{b} \quad (A.1)$$

We obtain the quadratic form of the object function:

$$f(\mathbf{p}) = \frac{1}{2} \| \mathbf{A}\mathbf{p} + \mathbf{b} \|^2 = \frac{1}{2} \mathbf{p}^T \mathbf{A}^T \mathbf{A}\mathbf{p} + \mathbf{b}^T \mathbf{A}\mathbf{p} + \frac{1}{2} \mathbf{b}^T \mathbf{b} \quad (A.2)$$

Differentiating both sides of (A.2) gives the optimality condition:

$$\nabla f(\mathbf{p}^*) = \mathbf{A}^T \mathbf{A}\mathbf{p}^* + \mathbf{A}^T \mathbf{b} = 0 \quad (A.3)$$

Solving (A.3) gives the close form solution of parameters in linear least squares problem:

$$\mathbf{p}^* = - (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b} \quad (A.4)$$

A.2 Nonlinear LSQ

In general, however, there is not often the case that the residual vector can be expressed as linear
When \( \mathbf{r}(\mathbf{p}) \) is a nonlinear function, we consider the first-order Taylor expansion of the object function about the current parameter vector \( \mathbf{p} \):

\[
f(\mathbf{p} + \Delta \mathbf{p}) = \frac{1}{2} \| \nabla \mathbf{r}(\mathbf{p}) \Delta \mathbf{p} + \mathbf{r}(\mathbf{p}) \|^2 \tag{A.5}
\]

where \( \mathbf{r}(\mathbf{p}) \) can be substituted by introducing the Jacobi matrix:

\[
\nabla \mathbf{r}(\mathbf{p}) = [\nabla r_1(\mathbf{p}), \nabla r_2(\mathbf{p}), \ldots, \nabla r_m(\mathbf{p})]^T = \mathbf{J}(\mathbf{p}) \tag{A.6}
\]

where \( \mathbf{J}(\mathbf{p}) \) is an \( m \)-by-\( n \) matrix, and the element \( J_{ij} \) the partial derivative of \( r_i \) with respect to \( x_j \). Then the nonlinear least squares problem can be iteratively solved. The iterative sub-problem can be expressed as:

\[
\text{arg min}_{\Delta \mathbf{p} \in \mathbb{R}^n} f(\mathbf{p} + \Delta \mathbf{p}) = \frac{1}{2} \| \mathbf{J}(\mathbf{p}) \Delta \mathbf{p} + \mathbf{r}(\mathbf{p}) \|^2 \tag{A.7}
\]

Using the close form solution (A.4) gives the iterative updating parameter vector:

\[
\Delta \mathbf{p}^* = - (\mathbf{J}(\mathbf{p})^T \mathbf{J}(\mathbf{p}))^{-1} \mathbf{J}(\mathbf{p})^T \mathbf{r}(\mathbf{p}) \tag{A.8}
\]

where \( \mathbf{J}(\mathbf{p})^T \mathbf{J}(\mathbf{p}) \) is a good approximation of the Hessian matrix when the residual vector is small or of low non-linearity. Denoting this expression by \( \mathbf{H}(\mathbf{p}) \), we can establish the iterative form:

\[
\mathbf{p}^{(k+1)} = \mathbf{p}^{(k)} - \mathbf{H}(\mathbf{p}^{(k)})^{-1} \mathbf{J}(\mathbf{p}^{(k)})^T \mathbf{r}(\mathbf{p}^{(k)}) \tag{A.9}
\]

Repeating the iteration until convergence gives the optimized parameter vector \( \mathbf{p}^* \). This is the well-known Gauss-Newton method, a straightforward method for nonlinear LSQ problem.