WEAK SOLUTIONS FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH ADDITIVE FRACTIONAL NOISE

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ABSTRACT. We give a new approach to prove the existence of a weak solution of
\[ dx_t = f(t, x_t)dt + g(t)dB^H_t \]
where \( B^H_t \) is a fractional Brownian motion with values in a separable Hilbert space for suitable functions \( f \) and \( g \). Our idea is to use the implicit function theorem and the scaling property of the fractional Brownian motion in order to obtain a weak solution for this equation.

1. INTRODUCTION

Recently there has been a great interest in the study of stochastic partial differential equations (SPDE) driven by coloured noise. The main motivation comes from finance and physics. In finance, it is well known that the market evolves according to a geometric fractional \( H \)-Brownian motion with (see [2] [10] and [7]). In physics, fractional mechanics and its generalization to stochastic mechanics have been developed in recent years (see [14] and [15]).

We are interested to deal with the following kind of SPDE
\[
(1) \quad dx_t = f(t, x_t) \, dt + g(t) \, dB^H_t,
\]
where \( B^H_t \) is a fractional Brownian motion (fBm) with Hurst parameter \( H \in (0, 1) \), \( f : [0, T] \times \mathbb{H} \to \mathbb{H} \) and \( g : [0, T] \to L(\mathbb{H}, \mathbb{H}) \) are continuous maps and a Hilbert space \( \mathbb{H} \) which will be defined precisely in the section section 3 below.

From a mathematical point of view, the problem is to guarantee the existence of solutions for the equation (1). There are, in the literature, different notions of solution for SPDE of this kind (for example [5], [9] and [20]). In this article we show the existence of a weak solution for the equation (1).

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There are several works in this direction. Here, we highlight the contributions of D. Nualart and Y. Oukine in [18] that was improved by B. Boufoussi and Y. Ouknine in [4], J. Snupárková in [21] and, more recently by Z. Li, W. Zhan and L. Xu in [13]. In the above works the proofs are based on the Girsanov Theorem and the Picard iteration argument. Here we present an alternative idea to prove the existence of a weak solution using the scaling property of the Fractional Brownian motion. We adapt a result about deterministic dynamical systems of J. Robbin [19] to prove the existence of a weak solution to of equation (1) under certain hypotheses.

The work is organized as follow: in section 2 we give the basic definitions about fractional Brownian motion on Hilbert spaces and its integration theory following the references [3], [6], [10], [16] and [17]. Finally, in section 3 we establish our main result and give a simple example.

2. FRACTIONAL BROWNIAN MOTION

We start reviewing the basic concepts on fractional Brownian motion (fBm) and stochastic integral with respect to fBm.

Fix a standard filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t, t \in [0, T]\}, \mathbb{P})\), see for instance [3], [16] and [17].

A fractional brownian motion with Hurst parameter \(H \in (0, 1)\) is a continuous and centered Gaussian process \((B^H_t)_{t \geq 0}\) with covariance function

\[
\mathbb{E} [B^H_t B^H_s] = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}).
\]

A very important result on fBm theory is the following integral representation

\[
B^H_t = \int_0^t K_H(t, s) \, dB_s,
\]

where

- \(B_s\) is the standard Brownian motion.

- \(K_H(t, s) := \begin{cases} b_H \left[ (\frac{t-s}{s})^{H-1/2} - (H - 1/2)s^{1/2-H} \int_s^t (u-s)^{H-1/2} u^{H-3/2} \, du \right] & H < 1/2 \\ c_H s^{1/2-H} \int_s^t |u-s|^{H-3/2} u^{H-1/2} \, du & H > 1/2 \end{cases}\)

- \(b_H = \sqrt{\frac{2H}{(1-2H)\beta(1-2H, H+1/2)}}\)

- \(c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-1/2)}}\).
\[\beta(a, b) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)}\]

where \(\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx\) is the Gamma function. (see [3], [16] and [17]).

To construct the fractional Brownian motion we consider the Schwartz space \(S(\mathbb{R})\) of rapidly decreasing smooth functions on \(\mathbb{R}\) provided with the inner product \(\langle \cdot, \cdot \rangle\) given by

\[
\langle f, g \rangle_H = \begin{cases} 
H(2H - 1) \int_0^T \int_0^T f(s)g(t)|t - s|^{2H - 2}dsdt & \text{for } H > 1/2 \\
\int_0^T K^*_Hf(s)K^*_Hg(s)ds & \text{for } H < 1/2.
\end{cases}
\]

where

\[K^*_Hf(s) := K(T, s)f(s) + \int_s^T (f(t) - f(s))\partial_tK_H(t, s)dt.\]

Let \(S'(\mathbb{R})\) be the dual of \(S(\mathbb{R})\) the space of tempered distributions on \(\mathbb{R}\). We consider the mapping \(f \in S(\mathbb{R}) \rightarrow \exp(1/2||f||^2_H) \in \mathbb{R}\). This map is positive definite in \(S(\mathbb{R})\) thus, by the Bochner-Minlos theorem (see for instance [12, pp. 257]), we guarantee the existence of a probability measure \(P^H\) on the Borel \(\sigma\)-algebra \(B(S'(\mathbb{R}))\) such that

\[
\int_{S'(\mathbb{R})} e^{i\langle \omega, f \rangle}dP^H(\omega) = e^{-1/2||f||^2_H} \quad \text{for all } f \in S(\mathbb{R}).
\]

We will denote by \(I_{[0,t]}\) the usual characteristic function of the interval \([0, t]\).

We define \(B^H : [0, T] \times S'(\mathbb{R}) \rightarrow \mathbb{R}\) by

\[B^H(t, \omega) = B^H_t(\omega) = \langle \omega, I_{[0,t]} \rangle\]

as an element of \(L^2(P^H)\).

It is a simple consequence of the Kolmogorov continuity theorem that \(B^H\) is a fractional Brownian motion with Hurst parameter \(H\).

In order to construct the Wiener integral with respect to the fBm, we follow [3]. Consider the functions \(R_H : [0, T] \times [0, T] \rightarrow [0, \infty)\) given by

\[R_H(t, s) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}),\]

and the closure \(\mathcal{H}\) of the vector space spanned by the functions

\[\{R_H(t, \cdot), t \in [0, T]\}\]

with respect to the scalar product

\[\langle R_H(t, \cdot), R_H(s, \cdot) \rangle := R_H(t, s) \quad \text{for all } t, s \in [0, T].\]
The Wiener integral with respect to the fBm is defined as the linear extension of the isometric map

$$R^H_H(t, \cdot) \rightarrow B^H_t \quad \text{for all } t \in [0, T],$$

from $\mathcal{H}$ in $L^2(\mathbb{P}^H)$.

To define the stochastic integral of a function with respect to the fBm we follow the usual path [3], [16] and [17]. We start defining it for a simple function $\phi = \sum_i a_i \chi_{[t_i, t_{i+1})}$ as

$$\int_0^T \phi(s) \, dB^H_s = \sum_i a_i (B^H_{t_{i+1}} - B^H_{t_i}).$$

Then, we extend it to $L^2(\mathbb{P}^H)$ and we have the isometry

$$\mathbb{E} \left[ \left\langle \int_0^T f(s) \, dB^H_s, \int_0^T g(s) \, dB^H_s \right\rangle \right] = \langle f, g \rangle_H,$$

see [3] and [11].

Now, we review the construction of a fractional Brownian motion in a Hilbert space $\mathbb{H}$.

Let $\mathbb{H}$ be a separable Hilbert space. We will denote by $L(\mathbb{H})$ the space of continuous linear maps from $\mathbb{H}$ to $\mathbb{H}$. Let $Q \in L(\mathbb{H})$ be a symmetric, non-negative trace class operator.

Let $\{\lambda_n\}_{n \in \mathbb{N}}$ be a discrete family of eigenvalues of $Q$, counted with multiplicity, and let $\{e_n\}_{n \in \mathbb{N}}$ be the normalized eigenvectors, this is

$$Q(e_n) = \lambda_n e_n.$$

Let $\{B^n_t\}$ be independent 1-dimensional fBm with parameter $H$. Then the series

$$B^H_t = \sum_{n=1}^{\infty} \sqrt{\lambda_n} B^n_t e_n, \quad t \geq 0$$

converges almost surely and in $L^p$ for $p \geq 1$ (see [3], [10], [16] and [17]). We say that $B^H_t$ is a trace-class fractional Brownian motion in $\mathbb{H}$ with covariance $Q$ and Hurst parameter $H$.

Let $g : [0, T] \rightarrow L(\mathbb{H})$ be a continuous map. We define the integral of $g$ with respect to the trace-class fBm $B^H_t$ as

$$\int_0^t g(t) \, dB^H_t := \sum_{n,m=1}^{\infty} \left( \int_0^t \sqrt{\lambda_m} \, g(t)e_m, v_n > dB^m_t \right) v_n,$$

where $\{v_m\}_{m \in \mathbb{N}}$ is any orthonormal basis of $\mathbb{H}$. 
Using the isometry (2) we get that
\[
E \left| \int_0^t g(t) \, dB_t^H \right|^2 = \sum_{n,m=1}^{\infty} E \left| \int_0^t \sqrt{\lambda_m} < g(t)e_m, v_n > \, dB_t^n \right|^2
\]
\[= \sum_{n,m=1}^{\infty} \lambda_m \langle \langle g(t)e_m, v_n \rangle \rangle_H > \text{ if } H > 1/2,
\]
Thus
\[
E \left| \int_0^t g(t) \, dB_t^H \right|^2 = \begin{cases} 
\sum_{j,k=1}^{\infty} \lambda_k H(2H - 1) \int_0^T \int_0^T < g(t)e_k, g(s)e_k > \times |t-s|^{2H-2} \, dsdt & \text{if } H \geq 1/2, \\
\sum_{j,k}^{\infty} \lambda_k \int_0^T < K_H^* g(s)e_k, K_H^* g(s)e_k > \, ds & \text{if } H < 1/2,
\end{cases}
\]
where
\[
<K_H^* g(s)e_k, K_H^* g(s)e_k> = K(T, s)^2 \|g(s)e_k\|^2 + 2K(T, s) \int_s^T < g(s)e_k, (g(t) - g(s))e_k > \partial_t K_H(t, s) \, dt + \int_s^T \int_s^T < (g(r) - g(s))e_k, (g(t) - g(s))e_k > \times \partial_t K_H(t, s) \partial_s K_H(r, s) \, drdt.
\]

The next lemma is a consequence of the above set up and it shows estimates that we will use below.

**Lemma 1.** Let \(g, h : [0, T] \rightarrow L(\mathbb{H}, \mathbb{H})\) be continuous functions such that

\[
\sup_{r \in [0, T]} \|g(r)e_j\|^2_H < \infty \quad \text{and} \quad \sup_{r \in [0, T]} \|r h(r)e_j\|^2_H < \infty.
\]

Then for any positive real numbers \(a\) and \(k\),
\[
\sum_j \lambda_j E \left| \int_0^T g(a^k s)(e_j) \, dB_t^H \right|^2 < \infty
\]
\[
\sum_j \lambda_j E \left| \int_0^T sh(a^k s) dB_t^H \right|^2 < \infty.
\]
Proof. It follows directly from the construction of the fractional Brownian motion $B^H$ and (2) we get
\[
\sum_j \lambda_j a^{2Hk} \mathbb{E} \left| \int_0^T g(a^k s)(e_j) \, d(B^H_t) \right|^2 \leq C(H, T) a^{2Hk} \sup_{r \in [0,T]} ||g(a^k r) e_j||^2_H \sum_j \lambda_j
\]
\[
\sum_j \lambda_j a^{2Hk} \mathbb{E} \left| \int_0^T sh(a^k s)(e_j) \, d(B^H_t) \right|^2 \leq C(H, T) a^{2Hk} \sup_{r \in [0,T]} ||r h(a^k r) e_j||^2_H \sum_j \lambda_j
\]
where for $H > 1/2$,
\[
C(H, T) = H(2H - 1) \int_0^T \int_0^T |t - s|^{2H - 2} \, dsdt
\]
and for $H < 1/2$
\[
C(H, T) = \int_0^T K(T, s)^2 \, ds + 2 \int_0^T \int_s^T |K(T, s) \partial_t K_H(t, s)| \, dt \, ds + \int_0^T \int_s^T \int_s^T |\partial_t K_H(t, s) \partial_t K_H(r, s)| \, dr \, dt \, ds.
\]
Now, the hypothesis (4) guarantee the result. \qed

3. MAIN RESULT

We consider stochastic differential equations over $\mathbb{H}$ of the following type
\[
(5) \quad dx_t = f(t, x_t) \, dt + g(t) \, dB^H_t,
\]
where
- $B^H_t$ is a trace-class fractional Brownian motion in $\mathbb{H}$ with covariance $Q$ and Hurst parameter $H$.
- $f : [0, T] \times \mathbb{H} \to \mathbb{H}$ and $g : [0, T] \to L(\mathbb{H}, \mathbb{H})$ are continuous maps.
We say that a process $x_t$ is a weak solution to equation (5) starting at $x_0 \in \mathbb{H}$ if
\[
x_t - x_0 = \int_0^t f(s, x_s) \, ds + \int_0^t g(s) \, dB^H_s,
\]
for some fractional Brownian motion $B^H_s$ with Hurst parameter $H$.

Theorem 1. Let $\mathbb{H}$ and $\mathbb{H}$ be two Hilbert spaces such that $\mathbb{H} \subset \mathbb{H}$ densely an there is a family of linear operators $\{S_n\}_{n \in \mathbb{N}}$ such that
- $S_n : \mathbb{H} \to \mathbb{H}$,
- $S_n(x) \in \mathbb{H}$ for all $x \in \mathbb{H}$,
- $||S_n - I_\mathbb{H}|| \to 0$. 

Let $f : [0, T] \times \mathbb{H} \to \mathbb{H}$ and $g : [0, T] \to L(\mathbb{H}, \mathbb{H})$ be continuously differentiable maps such that

- $|f(t, x)| + |\partial_x f(t, x)| \leq \phi(t)(1 + |x|)$,
- $|\partial_x f(t, x)v|^2 \leq \phi(t)^2(1 + |x|^2 + |v|^2)$,
- $|g(t)(v)| + |\partial_x g(t)(v)| \leq \phi(t)|v|$.

for a continuous square integrable function $\phi : [0, T] \to \mathbb{R}$.

Then there exists a fractional Brownian motion $\tilde{B}^H$ in $\mathbb{H}$ with covariance $Q$ and Hurst parameter $H$, which is a rescaling of the original $B^H$, and $X : [0, T] \times \Omega \times \mathbb{H} \to \mathbb{H}$ satisfying the equation

$$X(t) - x_0 = \int_0^t f(s, X(s)) \, ds + \int_0^t g(s) \, dB^H_s,$$

for any $t \in [0, T]$.

Furthermore, the dependence of $X$ in $x_0$ is smooth.

**Proof.** We fix $0 < a < 1$ and $k \in 2\mathbb{N}$ such that $kH > 1$. For any $X \in C([0, T], L^2(\Omega, \mathbb{H}))$ we define

$$F_n(X, a, x_0)(t) = S_nX(t) - \int_0^t a^k f(a^k s, x_0 + S_nX(s)) \, ds - \int_0^t a^{Hk} g(a^k s) \, dB^H_s.$$

By Lemma and Fubini’s theorem

$$\mathbb{E}[||F_n(X, a, x_0)(t)||^2] \leq 8\mathbb{E}[||S_nX(t)||^2] + 8 \int_0^t t(a^{2k}\mathbb{E}[||f(a^k s, x_0 + S_nX(s))||^2]) \, dt
+ 8C(H, T) \int_0^t \sum_j \lambda_j a^{2Hk} ||g(a^k s)e_j||^2_H \, ds.$$

Thus,

$$F_n : C([0, T], L^2(\Omega, \mathbb{H})) \times \mathbb{R} \times \mathbb{H} \to C([0, T], L^2(\Omega, \mathbb{H})),
$$

is well defined and $F_n(0, 0, x_0) = 0$.

Now we claim that $F_n$ belong to $C^1$ and $\partial_X F_n(0, 0, x_0)$ is invertible. In fact, we have that

$$D(F_n)(X, a, x_0)(W, b, y)(t) = S_nW(t) - \int_0^t a^k DF_n(a^k s, x_0 + S_nX(s))(S_nW(s), b, y) \, ds$$

$$- \int_0^t k a^{Hk+k-1} s b \partial_x g(a^k s) \, dB^H_s,$$

where $W \in C([0, T], L^2(\Omega, \mathbb{H}))$, $b \in \mathbb{R}$ and $y \in \mathbb{H}$.

We observe that

$$DF_n(a^k s, x_0 + S_nX)(S_nW, b, y) = \partial_t f(a^k s, x_0 + S_nX(s))ka^{k-1}b + \partial_x f(a^k s, x_0 + S_nX(s))(y + S_nW),$$
is an integrable function for each \( n \in \mathbb{N} \) big enough. In fact,
\[
\int_0^t |\partial_t f(a^k s, x_0 + S_n X(s))|^2 \, dt \leq \int_0^t \phi(s)^2 |S_n X(s)|^2 \, ds
\]
\[
\leq \int_0^t \phi(s)^2 |X(s)|^2 \, ds + \int_0^t \phi(s)^2 |S_n X(s) - X(s)|^2 \, ds,
\]
and
\[
\int_0^t |\partial_x f(a^k s, x_0 + S_n X(s))(y + S_n W)|^2 \, ds \leq \int_0^t \phi(s)^2 (1 + |x_0 + S_n X(s)|^2) \, ds +
\]
\[
+ \int_0^t \phi(s)^2 |y + S_n W|^2 \, ds.
\]
Thus \( F_n \) belong to \( C^1 \).

We have that \( F_n \) verifies the hypothesis of the the implicit function theorem (see [1, pp. 121]) because
\[
\partial_X F_n(0, 0, x_0)W(t) = S_n W(t).
\]
and \( S_n \) is invertible for \( n \) big enough. This follows from
\[
S_n = I + (S_n - I),
\]
with \( ||S_n - I|| \to 0 \).

By the implicit function theorem, there is a neighborhood \( U \) of \( (0, x_0) \in [0, T] \times \mathbb{H} \) and a differentiable map \( G_n : U \to C([0, T], L^2(\Omega, \mathbb{H})) \) such that
\[
F_n(G_n(a, x, a, x)) = 0 \quad \text{for all } (a, x) \in U.
\]

We set
\[
X(t) := S_n G_n(x_0, \epsilon)(t/\epsilon^k).
\]
Then, considering the self-similarity property \( \tilde{B}_{a^k t}^H := a^k H B_t^H \) (see [16], [17], Prop. 2.2]) we get
\[
X(t) - \int_0^t f(r, x_0 + X(r)) \, dr - \int_0^t g(r) \, dB_r^H = S_n G_n(x_0, \epsilon)(t/\epsilon^k)
\]
\[
- \int_0^{t/\epsilon^k} e^k f(\epsilon^k s, x_0 + S_n G_n(x_0, \epsilon)(s)) \, ds
\]
\[
- \int_0^{t/\epsilon^k} e^k g(\epsilon^k s) \, dB_s^H
\]
\[
= F_n(G_n(x_0, \epsilon), \epsilon, x_0)(t/\epsilon^k) = 0.
\]
Thus
\[
\ddot{X} = x_0 + X,
\]
is a solution of the equation (5) that depends smoothly in \( x_0 \).
Remark 1. To show the result above we just need the estimates
\[ \int_0^t t(a^{2k}) \mathbb{E}[|f(a^k s, x_0 + X(s))|^2] \, dt < \infty, \]
and
\[ \int_0^t [\partial_t f(a^k s, x_0 + X(s))k a^{k-1} b + \partial_x f(a^k s, x_0 + X(s))(y + W)]^2 \, ds < \infty. \]

Example 1. We consider the quasi-linear equation in \( L^2(\mathbb{R}) \) given by
\begin{align*}
  &du(t, x) = [\alpha \Delta u(t, x) + \beta u(t, x) + \gamma \cdot \nabla u(t, x)] \, dt + dB_t^H \\
  &u(0, x) = v(x)
\end{align*}
for \( \alpha, \beta \) non-negative constants, \( \gamma \in \mathbb{R}^n \) and \( v(x) \in \mathcal{H}^2(\mathbb{R}) \).

We first observe that \( \mathcal{H}^2(\mathbb{R}) \subset L^2(\mathbb{R}) \) is densely contained with
\[ f(t, u) = [\alpha \Delta u + \beta u + \gamma \cdot \nabla u] \]
and
\[ \partial_u f(t, u)v = [\alpha \Delta v + \beta v + \gamma \cdot \nabla v] \]

Since the Laplace operator is linear, we get
\[ \|\Delta u\|_{L^2} \leq C(1 + \|u\|_{\mathcal{H}^2}) \]
\[ \|u(t, x)\|_{L^2} \leq C(1 + \|u\|_{\mathcal{H}^2}) \]
\[ \|\nabla u(t, x)\|_{L^2} \leq C(1 + \|u\|_{\mathcal{H}^2}), \]
for a positive constant \( C \). It follows that
\[ |f(t, u)| + |\partial_t f(t, u)| \leq C(1 + |u|) \]
and
\[ |\partial_u f(t, u)|^2 \leq C(1 + |u|^2 + |v|^2). \]

By Theorem 1 there is a weak solution for the equation (6).

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