Magnetoplasmons excitations in Graphene for filling factors $\nu \leq 6$

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(Dated: February 2, 2008)

In the frame of the Hartree-Fock approximation, the dispersion of magnetoplasmons in Graphene is derived for all types of transitions for filling factors $\nu \leq 6$. The optical conductivity components of the magnetoplasmon curves are calculated. It is shown that the electron-electron interactions lead to a strong re-normalization of the apparent Fermi velocity of Graphene which is different for different types of transitions.

PACS numbers: : 71.10.-w, 73.21.-b, 81.05.Uw

I. INTRODUCTION

Graphene is a monolayer of graphite with a band structure composed of two cones located at two inequivalent corners $K$ and $K'$ of the Brillouin zone at which conduction and valence bands merge. This compound has recently received a large attention because of the unusual sequence of quantum Hall states it reveals [1, 2]. In contrast to conventional two-dimensional electron gas (C2DEG) which display a quadratic dispersion law, Graphene exhibits a linear dispersion law $E(k) = \pm v_F |k|$ as a function of the momentum $k$ leading to a Dirac's type Hamiltonian with a Fermi velocity $v_F$ replacing that of the light. Different one-electron band structure models, not including electron-electron interactions, lead to values $v_F \simeq 0.86 \times 10^6 m/s$ with some variance, but this is the value which will adopted in this report. This peculiar dispersion law has two important consequences in contrast to C2DEG (see for instance [3]): (i) the wave functions have a spinor type character and (ii) under a magnetic field $B$ applied perpendicular to the Graphene plane, the Dirac energy spectrum evolves into Landau Levels (LL) with energies given by:

$$E_n = \text{sgn}(n) v_F \sqrt{2e\hbar B|n|} = \text{sgn}(n) E_{10}\sqrt{|n|}$$

where $n$ scans all positive and negative integer values including zero.

Magnetoplasmons (MP), in a two-dimensional electron gas, are excitations between LL, known to be described in terms of excitonic transitions due to electron-electron interactions (EEI): they reveal a specific dispersion as a function of the two-dimensional wave vector $k$ of the exciton. For a C2DEG, the theory, derived in the frame of the Hartree-Fock (HF) approximation, has been first developed [4, 5] for integer values of the filling factor $\nu = N_S \Phi_0 / B$ ($N_S$ and $\Phi_0$ being the two-dimensional carrier concentration and the flux quantum respectively). These studies have been extended to the case of non-integer values of $\nu$ [6] and have also included the calculation of matrix elements for the optical conductivity [7]. The effects of EEI in Graphene have recently been reported on a theoretical basis [8, 9] but with a different model than that of Refs. [4, 5, 6] and restricted to integer values of the filling factor. We have followed here the lines of Ref. [10], which has been shown to reproduce quantitatively experimental results [11] when they are interpreted in terms of MP excitations.

Because of the Kohn’s theorem [10], the EEI effects turn out to be tiny for C2DEG. However this theorem does not apply for a linear dispersion law and therefore EEI are expected to induce significant effects in Graphene. Indeed recent experimental investigations of the magneto-optical transitions in Graphene [11, 12] have been interpreted with an effective velocity $\tilde{c}$, replacing $v_F$ in Eq.1, ranging between 1.03 to 1.18 $10^6 m/s$ and showing a re-normalization of $v_F$ which here, we will show, is mainly due to electron-electron interactions.

On general grounds, the Coulomb energy characteristic of electron-electron interaction in magnetic field is $E_c = e^2/\kappa l_B$ where $\kappa$ is the electronic dielectric constant of the material and $l_B = (eB)^{-1/2}$ the magnetic field length. The magnetoplasmon approach assumes that $E_c$ is smaller than the one-electron energy transitions. In the present case of Graphene $E_c(meV) = 11.2 (\sqrt{B(T)})$ and $E_{10}(meV) = 31 \sqrt{B(T)}$ (see Eq.1) leading to the ratio $E_{10}/E_c = 2.77$ that is a condition better fulfilled for Graphene than for GaAs based C2DEG and furthermore not dependent on the value of the magnetic field. Magneto-excitons should be therefore more stable in Graphene than in C2DEG and the approach derived for these later compounds should be valid. The report is organized as follows: we will first describe the general formalism used to derive the MP dispersion curves (section II). We apply it to the case of filling factors $\nu < 2$ in section III and to the case of $2 < \nu < 6$ in section IV. Results will be discussed and compared to experimental results in section V. The details of calculations are reported in the appendices.

II. GENERAL FORMALISM

In contrast to the GaAs case, Graphene has two valleys which lead to the conclusion that in the absence of spin...
splitting and valley splitting each Landau Level (LL) is, in general, four times degenerate. The fourfold degeneracy of the \( n = 0 \) LL is still due to spin and valley symmetries, but two of these levels have an electron-like character and the two other ones a hole-like character. We will restrict our analysis to the zero temperature case. Because of the peculiar symmetry of the problem the wave functions have a spinor character which can be expressed in the Landau gauge, with the potential vector components of the magnetic field \( A_x = A_z = 0 \) and \( A_y = Bx \), as:

\[
\begin{align*}
F_{np}^{K}(\vec{p}) &= \frac{e}{c_0} \psi_{np}(\vec{p}) \left( -1 \text{sgn}(n) \varphi_{|n|}(x - p) \right), \\
F_{np}^{K'}(\vec{p}) &= \frac{e}{c_0} \psi_{np}(\vec{p}) \left( -1 \text{sgn}(n) \varphi_{|n|}(x - p) \right)
\end{align*}
\]  

(2)

where \( \vec{p} \) is the two-dimensional vector of components \( x \) and \( y \), \( c_0 = 1 \) for \( n = 0 \) and \( 1/\sqrt{2} \) otherwise whereas \( \text{sgn}(n) = 1, 0, -1 \) when \( n > 0, = 0, < 0 \) respectively. \( \varphi_{|n|}(x) \) is the standard normalized Landau wave functions. Note that these wave functions differ from those proposed in Ref. [3] by the phase factor due to the different gauge used. Following the lines of Ref. [3], we call \( A^\pm_{n,n',\sigma,i}(\vec{k}) \), the creation operator of an excitation of energy \( E_{ex} \) corresponding to a transition from LL \( n' \) with spin \( \sigma \) in valley \( i \) to a LL \( n \) of the same spin and same valley. This operator is defined as a function of \( a^+_\sigma \) and \( a\_\sigma \), the standard one particle creation and annihilation operators respectively, as:

\[
A^+_{n,n',\sigma,i}(\vec{k})|0\rangle = \sum_p \exp(ik(x + k_y/2)) \times a^+_n p, \sigma, i |0\rangle
\]

(3)

The total Hamiltonian of the system is written as:

\[
\hat{H}_{total} = \sum_{m,p,\sigma,i} \hbar \omega_{m} a^+_m p, \sigma, i a_{m,p,\sigma,i} + \hat{H}_{int}
\]

(4)

where \( \hbar \omega_{m,i} \) is the corresponding one electron energy of the LL \( m \) with spin \( \sigma \) in valley \( i \). The Coulomb interactions appear in \( \hat{H}_{int} \) as:

\[
\hat{H}_{int} = \frac{1}{2} \int \rho_1 d\rho_2 V(\vec{p}_1 - \vec{p}_2) \left( F^+_{\sigma,i}(\vec{p}_1) \hat{F}_{\sigma,i}(\vec{p}_2) \right) + \text{h.c.}
\]

(5)

where \( \hat{F}^+ \) denotes the scalar product and

\[
\hat{F}_{\sigma,i}(\vec{p}) = \sum_{n,p} F_{n,p}(\vec{p}) a_{n,p,\sigma,i}
\]

\[
V(\vec{p}_1 - \vec{p}_2) = \int \frac{d^2 q}{(2\pi)^2} \tilde{V}(q) e^{i\vec{q}\cdot (\vec{p}_1 - \vec{p}_2)}
\]

(6)

with \( \tilde{V}(q) \) being the 2D Fourier transform of the Coulomb potential \( V(r) = e^2/(qr) \).

After some calculations we obtain an analytic expression for \( \hat{H}_{int} \) which is expressed as:

\[
\hat{H}_{int} = \frac{1}{2} \sum q \tilde{V}(q) \exp(ik_x(p_1 - p_2 - q_y)) \times J_{m,n}(q) J_{n',m'}(-\bar{q}) \times a^+_{n_1,p_1,\sigma_1,i_1} a^+_{n_2,p_2,\sigma_2,i_2} a_{n_3,p_3 + q_x,\sigma_3,i_2} J_{n_4,p_4 - q_y,\sigma_1,i_1}
\]

(7)

In Eq.7, the summation is extended over the ensemble \( n_1, n_2, n_3, n_4 \) of LL, the ensemble \( p_1, p_2 \) of the \( y \)-component of the momentum, the ensemble of spin \( \sigma_1, \sigma_2 \), both valleys \( i_1 \) and \( i_2 \) and the wavevector \( \bar{q} \).

The function \( J_{m,n}(\bar{q}) \) is defined as:

\[
J_{m,n}(\bar{q}) = c^\_m c^\_n \{ \text{sgn}(m) \text{sgn}(n) J_{m-1,n-1}(\bar{q}) + J_{m,n}(\bar{q}) \}
\]

(8)

with the usual definition of the integral \( J_{m,n}(\bar{q}) \) valid for \( m > n \):

\[
J_{m,n}(\bar{q}) = \int dx e^{ikx - \frac{q_y}{2}} \varphi_m(x + \frac{q_y}{2}) \varphi_n(x - \frac{q_y}{2}) = \left( \frac{n!}{m!} \right) e^{-\frac{q_y^2}{2}} (q_y + i\bar{x})^{m-n} \mathcal{L}_{m-n}(q_y^2/2)
\]

(9)

where \( \mathcal{L}_n(x) \) are the Laguerre polynomials. For \( m < n \) the relation \( J_{m,n}(\bar{q}) = J_{n,m}(-\bar{q}) \) holds.

Using the random phase approximation (RPA) to treat the combination of creation and annihilation operators we arrive to the following expression for the Exciton energies (the notation \[0\]) representing the ground state of the system:

\[
E_{ex}(\bar{k}) A^+_{n,n',\sigma,i}(\bar{k})|0\rangle = \hbar \omega_{n',\sigma,i} A^+_{n,n',\sigma,i}(\bar{k})|0\rangle + \sum_{n_2} [\bar{E}_{n',n_2,n_2}(0) - \bar{E}_{n,n_2,n_2}(0)] f_{n_2,i}^\sigma \times A^+_{n,n',\sigma,i}(\bar{k})|0\rangle + \sum_{n_2,n_4} \tilde{V}_{n',n_2,n_4}(k_x, k_y) f_{n_2,i}^\sigma A^+_{n,n',\sigma,i}(\bar{k})|0\rangle
\]

(10)

where \( f_{n,i}^\sigma \) is the filling factor of LL \( n \) with spin \( \sigma \) in valley \( i \). The matrix elements \( \tilde{E} \) and \( \tilde{V} \) are given by:

\[
\tilde{V}_{n_1,n_2,n_3,n_4}(\bar{q}) = \tilde{V}(q) J_{n_1,n_2}(\bar{q}) J_{n_3,n_4}(\bar{q})
\]

\[
\tilde{E}_{n_1,n_2,n_3,n_4}(\bar{k}) = \int dq \tilde{V}_{n_1,n_2,n_3,n_4}(q) e^{i\bar{k}q}
\]

(11)
We note, at that level, that Eq. 10 is formally equivalent to that obtained for C2DEG [6] except for the definition of the different matrix elements which here takes into account the spinor character of the wave functions. In this equation, the second term (second line) is a measure of the difference of exchange energies of the LL $n'$ and $n$. The third line is related to the direct electron-hole Coulomb interaction (Exciton binding energy). Both terms are involving excitons of same spin and same valley. The last term of Eq. 10 describes the simultaneous annihilation and creation of excitons at different points of the Brillouin zone (RPA contribution): it includes all possible transitions without restriction to spin or valley indices. The exchange terms deserve a special attention in the present case. The corresponding expression for the exchange in Eq. 10 reads as:

$$E_{n,m,n,m}(0) = |c_n|^2 |c_m|^2 \sqrt{2} \int dx e^{-x^2} \left\{ L_{|m|} L_{|n|} + \frac{sgn(n) sgn(m)}{2} L_{|m|-1} L_{|n|-1} \right. \\
+ \frac{sgn(n) sgn(m)}{\sqrt{|m||n|}} L_{|m|-1} L_{|n|-1} \right\}$$ (12)

where all Laguerre polynomials have arguments $x^2$. In Eq.10 (second line), the summation over $n_2$, for these terms, has to include all LL from $-\infty$ to $+\infty$. The evaluation of the exchange contributions to the different situations is given in appendix A.

Solving Eq.10 results in diagonalizing an Hamiltonian, the size of which depends on the number of transitions which are assumed to be coupled by electron-electron interactions. In reality this number is very large for Graphene due to the existence of interband transitions but since $E_c$ is smaller than the energy of transitions, we can reasonably assume, in the spirit of the HF approximation, that EEI will not couple, at first order, transitions with different one-electron energies. In that case, the problem reduces to solve the Hamiltonian for each type of optical transitions which depend on the value of the filling factor. However, even if one solves the whole problem in successive steps, one has to keep in mind that a common energy scale should be adopted for all transitions in order to compare such results with experimental ones.

When writing the Hamiltonian, using Eq. 10, for a given set of transitions in the basis $\phi$ corresponding to these transitions such that $\hat{H}_{tot} \phi = E_{ex} \langle \vec{k} \rangle \phi$, we end up with a matrix which is not symmetric, as in C2DEG [6], but in addition, here, many matrix elements are complex as this will be seen in appendix B. To make the treatment easier to follow, we adopt the same technique as used in [6] which consists in writing the hamiltonian in a new basis $\Psi = \hat{M} \phi$ where $\hat{M}$ is a diagonal unitary matrix. The new Hamiltonian is then expressed as:

$$\hat{H} = \hat{M} \hat{H}_{tot} \hat{M}^{-1}$$ (13)

which is now symmetric and has only real matrix elements.

In the calculations we will neglect the spin splitting $\Delta_S$ which is small in the case of Graphene [13] but will not change anyway the conclusions since the optical transitions conserve the spin.

We will furthermore assume in the following that there exists some valley splitting $\Delta_V$ first suggested by Gusynin et al. [14]. The existence of such a valley splitting, has been recently supported by different models. Some of these models include different types of electron-phonon interactions [17, 10]: they all predict a linear dependence of $\Delta_V$ with the magnetic field. Another one [17] invokes EEI with strain induced gauge field yielding to a valley splitting which varies like $B$. We assume here, for convenience, that $\Delta_V$ is larger than $\Delta_S$ in such a way the electrons remain in the same valley (here the valley $K$ for instance) for filling factor $\nu < 1$. This is not necessary true since experimental results [18] tend to favor a situation, where for $\nu \ll 1$, the system becomes spin polarized. The splitting $\Delta_V$ will not be included in the present calculations but its consequence will be discussed in each case where its contribution could be relevant. All the following results for energies are given in units of $E_c$ and as a function of $K = |\vec{k}| t_B$.

III. MAGNETOPLASMON ENERGIES FOR $\nu < 2$

One electron optical transitions in Graphene

![Graphene Optical Transitions](image)

FIG. 1: (Color on line) Schematic diagram of one-electron transitions used in the magneto-plasmon model for $1 < \nu < 2$. On the left part of the figure (a) are shown the transitions implying the $n = 0$ LL. On the right part of the figure (b) are shown the first interband (electron-hole) transitions from the $n = -2, -1$ LL to the $n = 1, 2$ LL respectively. For $\nu < 1$, the transition $\alpha^+$ in (a) disappears and the new transition $\gamma^-$ appears. The splitting of LL $n$ mimics the spin splitting.

For $\nu < 2$, the typical transitions to be analyzed are displayed in Fig.1. There are two kinds of one electron transitions, in each valley, those which imply the $n = 0$ LL (Fig.1a) and those which correspond to interband (electron-hole) transitions (Fig.1b) are only rep-
represented those from LL \( n = -2, -1 \) to \( n = 1, 2 \). As already mentioned in the preceding section one can treat independently the one-electron transitions implying the \( n = 0 \) LL and those involving the interband transitions.

### A. Magnetoplasmon energies for transitions implying the \( n = 0 \) LL

The different one-electron energy transitions which are considered in this case are displayed in Fig. 1a for \( 1 < \nu < 2 \). In this figure the splitting of LL \( n \) mimics the spin-splitting \( \Delta_S \) for clarity but, as already said, this splitting is not taken into account in the present calculations. The Hamiltonian to be solved is therefore a matrix of rank 5 written, first, in the basis \( \phi = (\alpha^+, \alpha^-, \beta^+, \beta^-, \gamma^+ , \gamma^-) \) (see Fig. 1a for notations) and then transformed according to Eq. 13. The corresponding diagonal matrix \( \hat{M} \) is denoted here \( \hat{M}_{01<\nu<2} \) and has the following diagonal elements:

\[
\hat{M}_{01<\nu<2} = \begin{pmatrix}
    h_{11} & \sqrt{f_0^+}V_{0101} & \sqrt{f_0^+}V_{0101} & \sqrt{f_0^+}V_{0101} \\
    \sqrt{f_0^+}V_{0101} & h_{22} & \sqrt{f_0^+}V_{0101} & \sqrt{f_0^+}V_{0101} \\
    \sqrt{f_0^+}V_{0101} & \sqrt{f_0^+}V_{0101} & h_{33} & \sqrt{f_0^+(1-f_0^+)}(V_{0011} - E_{0011}) \\
    \sqrt{f_0^+(1-f_0^+)}(V_{0011} - E_{0011}) & \sqrt{f_0^+(1-f_0^+)}V_{0011} & \sqrt{f_0^+(1-f_0^+)}V_{0011} & h_{55}
\end{pmatrix}
\]  
(15)

The different matrix elements are given in appendices A (Eq. A1) and B (Eqs. B1, B2, B3).

One can note that the eigenvalues of \( \hat{M}_{01<\nu<2} \) are identical for \( f_0^+ = 0 \) or 1 (that is \( \nu = 1 \) or 2) whereas those for non integer values of \( \nu \) are symmetric with respect to \( \nu = 1.5 \). For \( \nu < 1 \) the corresponding Hamiltonian \( \hat{M}_{01<\nu<1} \) has to be written in the basis \( \phi = (\alpha^-, \beta^+, \beta^-, \gamma^+ , \gamma^-) \) (see Fig.1a), replacing \( f_0^+ \) by \( f_0^- \) with a new diagonal matrix \( \hat{M} \) denoted now \( \hat{M}_{01<\nu<1} \) which has the following elements:

\[
\hat{M}_{01<\nu<1} = \begin{pmatrix}
    e^{i\nu \varphi} & e^{-i\nu \varphi} & e^{-i\nu \varphi} & e^{i\nu \varphi} \\
    e^{-i\nu \varphi} & e^{i\nu \varphi} & e^{i\nu \varphi} & e^{-i\nu \varphi} \\
    e^{-i\nu \varphi} & e^{i\nu \varphi} & e^{i\nu \varphi} & e^{-i\nu \varphi} \\
    e^{i\nu \varphi} & e^{-i\nu \varphi} & e^{-i\nu \varphi} & e^{i\nu \varphi} \sqrt{1-f_0^-}
\end{pmatrix}
\]  
(16)

where \( f_0^- \) is the partial filling factor of the spin-down \( n = 0 \) LL.

The corresponding expressions for the matrix elements are given in appendices A (Eq. A1) and B (Eqs. B1, B2, B3). It turns out that the eigenvalues of \( \hat{M}_{01<\nu<1} \) are symmetric of those obtained for \( \hat{M}_{01<\nu<2} \) with respect to \( \nu = 1 \). If we adopt a model where \( \Delta_V \) is smaller than \( \Delta_S \) for \( \nu < 1 \), we obtain an Hamiltonian with the same eigenvalues which shows that the MP results do not depend on this assumption.

Results for the MP dispersion curves are displayed in Fig.2. For \( \nu = 1 \) or 2 one obtains, for the dispersion curves, a solution \( Ed(K) \) three times degenerate and one solution \( Eu(K) \) which have the following analytical expressions:

\[
Ed(K) = E_{10} + C_1 + \frac{3}{4} \alpha_0 - E_{0110}(K)
\]

\[
Eu(K) = Ed(K) + 4V_{0101}(K)
\]  
(17)

where \( E_{10} = E_1 - E_0 = 2.77 \times e^2/(\kappa l_B) \) is the one-electron energy for these transitions and \( C_1 \) defined in appendix A (Eq. A2) is a quantity a priori divergent which will be discussed in section V.

For non integer values of \( \nu \), the solutions \( Ed(K) \) remain twice degenerate, the high energy solution remains close to \( Eu(K) \) and two new solutions appear. The linear dispersion near \( K \approx 0 \) for \( Eu(K) \) is due to the RPA contribution entering Eq. 10. As compared to the solutions found in C2DEG for \( \nu = 2 \) \( [7] \), this contribution is the same whereas that of the exciton binding energy is different. The solutions for \( K \approx 0 \) will be further discussed in section V below.

Following the lines of Ref. [7] we have also calculated, in the frame of the MP picture, the optical conductivity (see appendix C, Eq. C4) which predicts that \( Eu(K) \) should be optically active in both polarizations of the light (note that the optical vectors are proportional to \( v_F^2 \)).

The MP model has been derived without including the valley splitting \( \Delta_V \): if such a splitting is introduced, for
the $n = 0$ LL, we expect a corresponding splitting of the optical transition independent on the relative magnitude of $\Delta V$ and $\Delta S$.

B. Magnetoplasmon energies for transitions from the $n = -2,-1$ to $n = 1,2$ LL

We discuss now the case of interband transitions displayed in Fig. 1b. There are, in this case, eight possible one-electron transitions and the Hamiltonian is written first in the basis $\phi = (I_{K}, J_{K}, I_{K}^+, J_{K}^+, I_{K}s, J_{K}s, I_{K}^+, J_{K}^+)$ and then transformed according to Eq. (13). For $0 < \nu < 2$, the corresponding diagonal matrix $\tilde{M}$ denoted here as $\tilde{M}_{0 < \nu < 2}$ has the following diagonal elements:

$$\tilde{M}_{0 < \nu < 2} = \{e^{-\nu \phi}, e^{\nu \phi}, e^{-\nu \phi}, e^{\nu \phi}, e^{-\nu \phi}, e^{\nu \phi}, e^{-\nu \phi}, e^{\nu \phi}\} \tag{18}$$

For these transitions the symmetrized excitonic Hamiltonians $H_{0 < \nu < 1}$ have matrix elements which are given in appendices A (Eqs. A3, A4) and B (Eqs. B4, B5, B6).

The dispersion of MP energies, in units of $E_c$, are displayed in Fig. 3 as a function of $K$. The corresponding one electron energy for these transitions is $E_{I_{12}} = E_{1} - E_{-2} = (\sqrt{2} + 1)E_{10} = 6.69 \times e^2/(\kappa l_B)$.

As for the preceding case, the solutions are identical for $\nu = 1$ or 2 and symmetric with respect to $\nu = 1$. For integer values of $\nu$, the eigen-values of the Hamiltonian can be expressed analytically and arranged in two groups: (i) two single solutions $E_{I_{12}}^{+/-}(K)$ displayed in the left part of the Fig. 3 and (ii) two other sets of solutions $E_{I_{12}}^{+/-}(K)$, three times degenerate (see dotted circles in Fig. 3), displayed in the right part of the figure. They are expressed as:

$$E_{I_{12}}^{+/-}(K) = (\sqrt{2} + 1)(E_{10} + C_1) + \Delta C_2$$

$$+ 4V_{-12} - E_{-12} - 1$$

$$\pm \sqrt{(\alpha_0^2)^2 + (4V_{-12} - E_{-12} - 2)^2} \tag{19}$$

and
\[ EI_{\nu}^{+/-}(K) = (\sqrt{2} + 1)(E_{10} + C_1) + \Delta C_2 \]
\[ -E_{-122-1} = \pm \sqrt{(\alpha_0/16)^2 + (E_{-122-2})^2} \]  

(20)

where all matrix elements entering Eqs. 19 and 20 are function of \( K \) and given in appendices A (Eq. A4) and B (Eqs. B5, B6). Only the solutions \( EI_{\nu}^{+/-}(K) \) are symmetric with respect to \( \nu \) (see appendix C, Eq. C5). For non-integer values of the filling factor the results are very close to those presented in Fig. 3 except for two solutions of the two groups of degenerate transitions which are no longer degenerate for \( K \approx 0 \).

In contrast to the case of transitions implying the \( n = 0 \) LL, a splitting of the transitions equal to \( \alpha_0/8 \) and due to electron-electron interactions is expected for \( K \approx 0 \).

Note however, here, that the introduction of a valley splitting \( \Delta \nu \) should only provide an additional component either linear in \( B \) or in \( \sqrt{B} \) depending on the origin of this valley splitting.

C. Magnetoplasmon energies for transitions from the \( n = -3, -2 \) to \( n = 2, 3 \) LL

In this case the corresponding Hamiltonian \( \tilde{H}_{12}^{3 \nu_{\leq 2}} \) has the same structure that \( H_{12}^{3 \nu_{\leq 2}} \) and therefore only the values of matrix elements are different. They are given in appendices A (Eqs. A5, A6) and B (Eqs. B7, B8). The dispersion of MP energies, in units of \( E_c \), is displayed in Fig. 4 as a function of \( kl_B \). The corresponding one electron energy for these transitions is \( EI_{32} = E_2 - E_{-32} = (\sqrt{3} + \sqrt{2})E_{10} = 8.72 \times e^2/(kl_B) \). The solutions are formally identical to those given in Eqs. 19 and 20 with the appropriate changes for the matrix elements given in the appendices A (Eq. A5) and B (Eqs. B7, B8). The splitting of the transitions for \( K \approx 0 \) is here equal to \( \alpha_0/16 \).

We, now, evaluate the exciton energies for \( \nu > 2 \).

IV. MAGNETOPLASMON ENERGIES FOR \( 2 < \nu < 6 \)

We will concentrate the report for filling factors \( 2 < \nu < 6 \). The contributions of exchange are given in appendix A (Eq. A7). It turns out that the problem to solve is symmetric with respect to \( \nu = 4 \) and therefore we will detail the treatment for \( 2 < \nu < 4 \) and will note only the main changes for \( 4 < \nu < 6 \).

A. Magnetoplasmon energies for \( 2 < \nu < 4 \)

In this case we have to treat the problem depicted in Fig. 5 for the one electron energy transitions. Note that we have here two types of transitions those implying the \( n = 0 \) LL and those between LL \( n = 1 \) and \( n = 2 \). Because the corresponding one electron energies are different they are treated independently.

One electron optical transitions in Graphene: \( 2 < \nu < 4 \)

For \( 2 < \nu < 3 \), we have to write, first, the Hamiltonian in the basis \( \phi = \{ \alpha^+, \alpha^-, \beta^+, \beta^-, \gamma^-, \gamma^- \} \) and for \( 3 < \nu < 4 \) in the basis \( \phi = \{ \alpha^+, \alpha^-, \beta^+, \gamma^-, \gamma^+ \} \) and transformed them according to Eq. 13. For \( 2 < \nu < 3 \) and \( 3 < \nu < 4 \), the corresponding diagonal matrices \( \tilde{M} \), denoted here as \( \tilde{M}_{12, \nu_{\leq 3}} \) and \( \tilde{M}_{12, \nu_{\leq 4}} \) respectively, have the following diagonal elements:

\[
\tilde{M}_{12, \nu_{\leq 3}} = \{ 1, 1, 1, 1 \} \begin{pmatrix}
\frac{1}{\sqrt{1 - f_1}}, & \frac{1}{\sqrt{f_1}}
\end{pmatrix}
\]

\[
\tilde{M}_{12, \nu_{\leq 4}} = \{ 1, 1, 1 \} \begin{pmatrix}
\frac{1}{\sqrt{1 - f_1^*}}, & \frac{1}{\sqrt{f_1^*}}
\end{pmatrix}
\]

(21)

where \( f_1^* \) and \( f_1 \) are the partial filling factors of the spin-down and spin-up \( n = 1 \) LL respectively, attached to the valley K’ with our convention. One gets for the corresponding Hamiltonians \( \tilde{H}_{12, \nu_{\leq 3}} \) and \( \tilde{H}_{12, \nu_{\leq 4}} \) the following expressions:
are given by:

\[ H_{2<\nu<3}^{\parallel} = \begin{bmatrix}
    h_{11} & V_{0101} & \sqrt{1-f_1^+V_{0101}} & 0 \\
    V_{0101} & h_{22} & V_{0101} & \sqrt{1-f_1^+V_{0101}} & 0 \\
    \sqrt{1-f_1^-V_{0101}} & 0 & \sqrt{1-f_1^-V_{0101}} & 0 & h_{44} \\
    0 & 0 & 0 & 0 & h_{55}
\end{bmatrix} \]  \quad (22)

and

\[ H_{3<\nu<4}^{\parallel} = \begin{bmatrix}
    h_{11} & V_{0101} & \sqrt{1-f_1^+V_{0101}} & 0 & 0 \\
    V_{0101} & h_{22} & V_{0101} & \sqrt{1-f_1^+V_{0101}} & 0 & 0 \\
    \sqrt{1-f_1^-V_{0101}} & 0 & \sqrt{1-f_1^-V_{0101}} & 0 & h_{44} \\
    0 & 0 & 0 & 0 & \sqrt{f_1^-V_{1212}} & h_{55}
\end{bmatrix} \]  \quad (23)

where the new matrix elements entering these matrices are given in appendices A (Eqs. A7) and B (Eqs. B9, B10). The resulting dispersion curves are displayed in Fig.6. The corresponding one-electron energies for both types of transitions are \( E_{10} = 2.77 \times e^2/(\kappa l_B) \) for the higher ones and \( E_{12} = E_2 - E_1 = 1.15 \times e^2/(\kappa l_B) \) for the lower ones.

\[ E_{02<\nu<3}^{\parallel}(K) = \frac{1}{2}[h_{11} + h_{44} + 2V_{0101}] \pm \sqrt{(h_{44} - h_{11} - 2V_{0101})^2 + 12V_{0101}^2(1 - f_1^-)} \]  \quad (24)

and a third one \( E_{12<\nu<3}(K) = h_{55}(K) \). These transitions are all optically active with a relative weight depending on the filling factor.

For \( 3 < \nu < 4 \), it remains one eigen-value solution identical to \( Ed(K) \) (Eq.17) and two groups of optically active non degenerate solutions denoted as \( E_{03<\nu<4}^{\parallel}(K) \) and \( E_{13<\nu<4}(K) \) with the following analytical expressions:

\[ E_{03<\nu<4}^{\parallel}(K) = \frac{1}{2}[h_{11} + h_{33} + 2V_{0101}] \pm \sqrt{(h_{33} - h_{11} - 2V_{0101})^2 + 8V_{0101}^2(1 - f_1^+)} \]  \quad (25)

and:

\[ E_{13<\nu<4}^{\parallel} = \frac{1}{2}[h_{44} + h_{55}] \pm \sqrt{(h_{44} - h_{55})^2 + 4V_{1212}^2f_1^+} \]  \quad (26)

The corresponding optical vectors for these transitions are given in appendix C (Eqs. C7 and C8).

**B. Magnetoplasmon energies for \( 4 < \nu < 6 \)**

It is easy to see that, in this case, the structures of the corresponding Hamiltonians \( H_{4<\nu<5}^{12} \) and \( H_{5<\nu<6}^{12} \) are

\[ \text{FIG. 6: (Color on line) Variation of the magnetoplasmon energies in units of } e^2/(\kappa l_B) \text{ as a function of } kl_B \text{ for different filling factors } 2 < \nu < 4. \text{ The dotted circles denote the degeneracy of the transitions.} \]
symmetric with respect to those given in Eqs. 22 and 23. At present, this is the $n = 1$ LL attached to the valley $K$ (in our convention) which starts to be filled and the notation $f_1^\pm$ refer to this LL. Of course some of the diagonal matrix elements are changed but results are formally similar and the corresponding exciton dispersion curves are displayed in Fig.6. All the eigen-value solutions of Fig.6 can be expressed analytically:

For $4 < \nu < 5$ one gets one solution $E_d 2(K) = h_{55}(K)$ which is not optically active and two groups of optically active solutions:

$$E_{01}^{\pm < \nu < 5} = \frac{1}{2} [h_{11} + h_{22} \pm \sqrt{(h_{11} - h_{22})^2 + 4V_0^2(1 - f_1^\pm)}]$$

and:

$$E_{14}^{\pm < \nu < 5} = \frac{1}{2} [h_{33} + h_{44} + V_{1212} \pm \sqrt{(h_{44} - h_{33} + V_{1212})^2 + 8V_1^2 f_1^\pm}]$$

for which all matrix elements, dependent on $K$, are given in appendix B (Eq. B11).

For $5 < \nu < 6$ one gets two solutions $E_d 2(K) = h_{44}(K)$ (same expression as for $4 < \nu < 5$) which are not optically active, one optically active solution $E_{05}^{< \nu < 6}(K) = h_{11}(K)$ and two other optically active solutions denoted as $E_{15}^{< \nu < 6}$:

$$E_{15}^{\pm < \nu < 6} = \frac{1}{2} [h_{22} + h_{33} + 2V_{1212} \pm \sqrt{(h_{22} - h_{33} - 2V_{1212})^2 + 12V_1^2 f_1^\pm}]$$

The related matrix elements, dependent on $K$, are given in appendix B (Eq. B12).

We will not discuss the case of interband transitions for this configuration of filling factors but their corresponding Hamiltonians $H_{2 < \nu < 6}^{12}$ and $H_{2 < \nu < 6}^{23}$ are modified in two respects: for both of them the exchange contributions entering the diagonal elements are different and for $H_{2 < \nu < 6}^{12}$ the transitions are now filling factor dependent in such a way the corresponding transitions disappear at $\nu = 6$.

The case of filling factors $6 < \nu < 10$ and following ones will not be discussed as well but the corresponding treatment is formally similar to the case $2 < \nu < 6$ with a different set of one electron energies, exchange contributions and matrix elements.

We are now focussing the discussion on the results obtained for $K \simeq 0$ which could be compared to magneto-optical absorption measurements.

V. DISCUSSION OF THE RESULTS FOR $K \simeq 0$

For $K = |k|l_B| \simeq 0$, all the Hamiltonians are reduced to their diagonal elements which are given in appendix B. The reason is that all off-diagonal elements are proportional to $K$ or $K^2$. We will call the corresponding solutions, at $K \simeq 0$, $E_{nM_P}^{0,n+1}$ and $E_{M_P}^{0,|n\pm|}$ for intra-LL transitions and interband transitions respectively. We restrict the discussion to those solutions which are optically active. All results are function of $\alpha_0 = \frac{1}{2} \sqrt{\frac{3}{2}} = 0.627$ in Coulomb units. We then get the following results:

For the transitions $E_{M_P}^{01}$ which involve the $n = 0$ LL:

$$E_{M_P}^{01} = E_{10} + C_1 - \frac{3}{4} \alpha_0$$

(for $\nu = 1, 2, 3, 4, 5$)

$$E_{M_P}^{01} = E_{10} + C_1 + \frac{\alpha_0}{4} (-9 + \frac{5}{2} (6 - \nu))$$

(for $5 < \nu < 6$) (30)

As clearly apparent in Fig. 6 and 7, this transition is split for non integer values of $\nu > 2$. The high energy component of this split level has an energy which increases with $\nu$ but its oscillator strength decreases with $\nu$ going to zero at integer value of $\nu$.

For the transitions $E_{M_P}^{12}$ which involve the transitions between the $n = 1$ and $n = 2$ LL:

$$E_{M_P}^{12} = (\sqrt{2} - 1)(E_{10} + C_1) + \Delta C_2 - \frac{\alpha_0}{16} (1 + 2\sqrt{2})$$

(for $\nu = 3, 4, 5, 6$)

$$E_{M_P}^{12} = (\sqrt{2} - 1)(E_{10} + C_1) + \Delta C_2 + \frac{\alpha_0}{16} (-14 + (13 - 2\sqrt{2})(\nu - 2))$$

(for $2 < \nu < 3$) (31)

![Graphene: transitions n = 0 to n = 1 and n = 1 to n = 2](image)
In this case also this transition is split for non integer values of $\nu$. The high energy component of this split level has an energy which increases with $\nu$ but its oscillator strength decreases with $\nu$ going to zero at integer value of $\nu$.

The optical active interband transitions $E_{IM}^{12}$ which involve the transitions between the $n = -2, -1$ and $n = 1, 2$ Landau levels are split by an amount $\frac{2\alpha}{\hbar}$ but the mean energy $E_{IM}^{12} = (E_{I}^{1}(0) + E_{I}^{1}(0))/2$ (Eq. 19) has the following expression:

$$E_{IM}^{12} = (\sqrt{2} + 1)(E_{10} + C_{1}) + \Delta C_{2} - \frac{33}{32}a_{0}$$ (32)

whereas the mean energy for the optically active transition implying the $n = -3, -2$ and $n = 2, 3$ Landau levels is expressed as:

$$E_{IM}^{23} = (\sqrt{3} + \sqrt{2})(E_{10} + C_{1}) + \Delta C_{3} - \frac{233}{256}a_{0}$$ (33)

For a given value of the Fermi velocity $v_{F}$ (here equal to $0.86 \times 10^{6} m/s$) the energy $E_{10}$ is determined (here $E_{10} = 2.77$) and in Eqs. 30 to 33 the only unknown parameter is $C_{1}$.

As already said this quantity defined in appendix A (Eq. A2) is divergent. The occurrence of such a problem is not specific of the Graphene properties because it is also present in C2DEG though it was not explicitly formulated. The reason why this term appears here is that we wanted to define a common energy scale for intraband and interband transitions. This will corresponds in C2DEG to impose a common energy scale to cyclotron-type transitions and inter-band excitonic transitions. There was an attempt to treat this later transitions in GaAs, in another context, but using the same theoretical model [21] and indeed the same problem of divergence of the exchange interaction among the valence band levels was found without being able to solve it. Therefore this problem is not specific to Graphene but, in that case, one can solve it, at least, in a semi-empirical way.

The divergence of $C_{1}$ is due to the infinite summation over LL (see Eq. A2) which is physically artificial. We could then define, as was done in Ref. [8], a cut-off value on energy or number of LL, but this limit is quite arbitrary. We propose to treat the problem in a semi-empirical way, using $C_{1}$ as a parameter fitted, for one type of transitions, to experimental data and then to deduce all the re-normalized velocities attached to the other transitions. Doing so, we implicitly assume that all the re-normalization of the velocity, for the fitted transition, is only due to electron-electron interactions neglecting any possible contribution from electron-phonon interaction which may be important in carbon based compounds. Among experimental data which could be used for this fitting, those related to magneto-transmission measurements [11, 12] are those which are expected to reflect the magneto-plasmon picture developed is this study. Another set of data, based on photoconductivity measurements on exfoliated Graphene [22], can also be considered to compare results. We will use the data of Ref. [12], obtained on exfoliated Graphene, to fit $C_{1}$ to the $E_{IM}^{12}$ transition at $\nu = 2$. In this reference, the re-normalized Fermi velocity $\tilde{c}_{01} = (1.12 \pm 0.02) \times 10^{6} m/sec$ which, from Eq.30, leads to a value $C_{1} = 1.31 \pm 0.06$. When injecting this value in Eq. 31, we predict a re-normalized Fermi velocity $\tilde{c}_{12} = (1.163 \pm 0.02) \times 10^{6} m/sec$ to be compared with the corresponding experimental value [12] $\tilde{c}_{12} = (1.18 \pm 0.02) \times 10^{6} m/sec$. The agreement is reasonable. As seen on both experimental and theoretical grounds, the re-normalized velocity differs for different transitions. One can then try to evaluate these velocities for other transitions. The results are given in Table 1 for integer values of $\nu$. The value quoted in the table, from Ref. [22], corresponds to the transition $n = 0$ to $n = 1$ whereas the corresponding velocity for the transition $n = -1$ to $n = 0$ is found to be $(1.07 \pm 0.004) \times 10^{6} m/sec$. This corresponds to an asymmetry of the conduction and valence levels not taken into account in our model but also not reported in Ref. [12] for this transition.

In table 1, we have included the results of the re-normalized Fermi velocity for transitions between LL $n = 3$ to $n = 4$, $n = 2$ to $n = 3$, $n = -3$ to $n = 4$ for which we have calculated the diagonal elements of the corresponding Hamiltonians.

As clearly apparent, from Table 1, $\tilde{c}$ varies strongly with the transition though it seems to be relatively constant for all interband transitions. This qualitative feature is also observed in experiments performed on epitaxial multi-layer Graphene [11, 20]. However in these experiments the reported values of $\tilde{c}_{12}$ for all transitions is the same and equal to $(1.03 \pm 0.01) \times 10^{6} m/s$ which is, at present, not understood.

One could, a priori, think that the results obtained are dependent on the value of $v_{F}$ adopted in the calcula-

| Transition | $\nu$ | $\tilde{c}_{12}^{c}(10^{6} m/s)$ | $\tilde{c}_{12}^{b}(10^{6} m/s)$ |
|------------|------|-------------------------------|-------------------------------|
| 3 to 4     | 8.10 | 0.99 ± 0.02                   |                               |
| 2 to 3     | 6.8  | 1.01 ± 0.02                   |                               |
| 1 to 2     | 4.6  | 1.04 ± 0.02                   |                               |
| 0 to 1     | 2.4  | 1.12 ± 0.02 $^{a}$            | 1.12 ± 0.02                   |
| -1 to 2    | 2    | 1.18 ± 0.02 $^{a}$            | 1.16 ± 0.02                   |
| -2 to 3    | 2    | 1.16 ± 0.02                   |                               |
| -3 to 4    | 2    | 1.16 ± 0.02                   |                               |

$^{a}$from Ref. [12]
$^{b}$from Ref. [22]
tions. In fact one can vary this value over a large range, for instance, from \(0.80 \times 10^6 m/s\) to \(0.90 \times 10^6 m/s\) getting values for \(C_1 = 1.50\) to 1.18 respectively but the quantity which enters the Hamiltonians is in fact \(E_{01} + C_1\) which remains constant, independent on \(v_F\) and equal to 4.08. This value has been adopted to calculate the dispersion curves of Figs. 2, 3, 4, 6 and 7. It is therefore not possible from experimental data on energies to determine \(C_1\) but all transitions are now given with a common energy scale. On the other hand the oscillator strengths of the transitions are proportional to \(v_F^2\) and then absolute transmission measurements could in principle give information on \(v_F\) and therefore on \(C_1\).

VI. CONCLUSIONS

In conclusion, we have developed, within the Hartree-Fock approximation, a full treatment of the magnetoplasmon picture in Graphene valid for a very large range of magnetic fields. This model, applied for filling factors up to 6, shows that the electron-electron interactions induce different effects: (i) for some of the transitions these interactions lead to a splitting of the optical transitions and (ii) they are responsible for a strong re-normalization of the Fermi velocity as observed in magneto-optical experiments. This re-normalization is found to be dependent on the type of investigated transitions. The optical conductivity components have been evaluated showing that the oscillator strength of the optical transitions is proportional to \(v_F^2\) and not to the square of the re-normalized velocity. The theory has been derived for all transitions with a common energy scale which should allow a direct comparison of its predictions with future experimental works.

VII. ACKNOWLEDGMENTS

The GHMFL is "Laboratoire conventionné à l’UJF et l’INPG de Grenoble". The work presented here has been supported in part by the European Commission through the Grant RITA-CT-2003-505474.

APPENDIX A: EXCHANGE CONTRIBUTIONS

We report in this appendix the explicit expressions for the contribution of the exchange energies entering the diagonal elements of the different Hamiltonian matrices in units of Coulomb energies. We introduce the notation \(\alpha_0 = \frac{1}{2}\sqrt{\frac{2}{3}}\) which characterize the exchange interaction in C2DEG at \(\nu = 1\). Applying the expression given in Eq. 12 we obtain successively the contribution of exchange for the different Hamiltonians. To simplify the notations we will drop the superscript \(\sim\) from \(\tilde{E}_{n,m,n,m}(0)\) meaning that all these quantities are real.

1. Exchange contributions to \(\tilde{H}_{1<\nu<2}^{0}\) and \(\tilde{H}_{0<\nu<1}^{0}\)

For \(\tilde{H}_{1<\nu<2}^{0}\) we obtain:

\[
\sum_m (E_{0,m,0,m}(0) - E_{1,m,1,m}(0)) f_m^{+} = \frac{3}{4} \alpha_0 (2f_0^+ - 1) + C_1
\]

\[
\sum_m (E_{0,m,0,m}(0) - E_{1,m,1,m}(0)) f_m^{-} = \frac{3}{4} \alpha_0 + C_1
\]

\[
\sum_m (E_{-1,m,-1,m}(0) - E_{0,m,0,m}(0)) f_m^{+} = \frac{3}{4} \alpha_0 (2f_0^+ - 1) + C_1
\]

where:

\[
C_1 = \frac{1}{\sqrt{2}} \sum_m \int_0^{\infty} dx e^{-x^2} \frac{x^2}{\sqrt{m+1}} L_m^1
\]

and \(L_m^1\) are Laguerre polynomials of argument \(x^2\) in this Appendix.

The quantity \(C_1\) diverges due to the simplifying assumption of the infinite linear dispersion of the Graphene bands. The summation has to be truncated at some level or this parameter has to be fitted to experimental data (see section 5).

For \(\tilde{H}_{0<\nu<1}^{0}\) we obtain for the exchange part the same expressions than those given in Eq.11 replacing \(f_0^+\) by \(f_0^+\).

2. Exchange contributions to \(\tilde{H}_{1<\nu<2}^{12}\) and \(\tilde{H}_{0<\nu<1}^{12}\)

For \(\tilde{H}_{1<\nu<2}^{12}\) one gets:

\[
\sum_m (E_{-1,m,-1,m}(0) - E_{2,m,2,m}(0)) f_m^{+} = \frac{\alpha_0}{16} + CI_{12}
\]

\[
\sum_m (E_{-2,m,-2,m}(0) - E_{1,m,1,m}(0)) f_m^{+} = \frac{-\alpha_0}{16} + CI_{12}
\]

\[
\sum_m (E_{-1,m,-1,m}(0) - E_{2,m,2,m}(0)) f_m^{-} = \frac{-\alpha_0}{16} (2f_0^+ - 1) + CI_{12}
\]

\[
\sum_m (E_{-2,m,-2,m}(0) - E_{1,m,1,m}(0)) f_m^{-} = \frac{-\alpha_0}{16} (2f_0^+ - 1) + CI_{12}
\]
where:

\[ CI_{12} = \frac{1}{\sqrt{2}} \sum_{m=0}^{\infty} \int_{0}^{\infty} dx e^{-x^{2}} \frac{x^{2}}{\sqrt{m+1}} L_{m}^{1}(1 + \frac{L_{1}}{\sqrt{2}}) \]

\[ = (\sqrt{2} + 1)C_{1} + \Delta C_{2} \]

\[ \Delta C_{2} = -\frac{1}{2} \sum_{m=0}^{\infty} \int_{0}^{\infty} dx e^{-x^{2}} \frac{x^{4}}{\sqrt{m+1}} \]

(A4)

CI_{12} in this equation also diverges like C_{1} but \( \Delta C_{2} \) converges to a value -0.156.

Similar expressions hold for \( H_{0\nu<1}^{12} \) when replacing \( f_{0}^{+} \) by \( f_{0}^{-} \).

3. Exchange contributions to \( H_{1\nu<2}^{23} \) and \( H_{0\nu<1}^{23} \)

For \( H_{1\nu<2}^{23} \) one gets:

\[ \sum_{m}(E_{-2,m,-2,m}(0) - E_{3,m,3,m}(0))f_{m}^{-} = \frac{\alpha_{0}}{32} + CI_{23} \]

\[ \sum_{m}(E_{-3,m,-3,m}(0) - E_{2,m,2,m}(0))f_{m}^{-} = -\frac{\alpha_{0}}{32} + CI_{23} \]

\[ \sum_{m}(E_{-2,m,-2,m}(0) - E_{3,m,3,m}(0))f_{m}^{+} = \frac{\alpha_{0}}{32}(f_{0}^{+} - 1) + CI_{23} \]

\[ \sum_{m}(E_{-3,m,-3,m}(0) - E_{2,m,2,m}(0))f_{m}^{+} = -\frac{\alpha_{0}}{32}(f_{0}^{+} - 1) + CI_{23} \]

(A5)

where:

\[ CI_{23} = \frac{1}{\sqrt{2}} \sum_{m=0}^{\infty} \int_{0}^{\infty} dx e^{-x^{2}} \frac{x^{2}}{\sqrt{m+1}} L_{m}^{1}\left(\frac{L_{1}}{\sqrt{2}} + \frac{L_{2}}{\sqrt{3}}\right) \]

\[ = (\sqrt{3} + \sqrt{2})C_{1} + \Delta C_{3} \]

\[ \Delta C_{3} = -\frac{1}{2} \sum_{m=0}^{\infty} \int_{0}^{\infty} dx e^{-x^{2}} \frac{x^{4}}{\sqrt{m+1}}(1 + \sqrt{6} - \frac{x^{2}}{\sqrt{6}}) \]

(A6)

CI_{23} in this equation also diverges like C_{1} but \( \Delta C_{3} \) converges to a value -0.467.

Similar expressions hold for \( H_{0\nu<1}^{23} \) when replacing \( f_{0}^{+} \) by \( f_{0}^{-} \).

Comparing Eqs. A4 and A6 one can formally extend the treatment and find that, for any interband transition from LL \(-p\) to LL \(q = p + 1\) the corresponding divergent term \( CI_{pq} \) entering the exchange contributions is given by \( CI_{pq} = (\sqrt{p} + \sqrt{q})C_{1} + F_{p,q} \) where \( F_{p,q} \) is finite.

4. Exchange contributions to \( H_{2\nu<6}^{23} \)

The different contributions to the exchange for the different Hamiltonians are:

\[ \sum_{m}(E_{0,m,0,m}(0) - E_{1,m,1,m}(0))f_{m}^{\pm} = \alpha_{0}\left(\frac{3}{4} - \frac{7}{8}f_{0}^{\pm}\right) + C_{1} \]

\[ \sum_{m}(E_{1,m,1,m}(0) - E_{2,m,2,m}(0))f_{m}^{\pm} = \frac{\alpha_{0}}{32}(5 + (26 - 4\sqrt{2})f_{0}^{\pm}) + C_{12} \]

(A7)

where \( C_{12} = (\sqrt{2} - 1)C_{1} + \Delta C_{2} \).

Here also formally, when extending the treatment, one finds that, for any intra-LL transition from LL \( p \) to LL \( q = p + 1 \) the corresponding divergent term \( C_{pq} \) entering the exchange contributions is given by \( C_{pq} = (\sqrt{q} - \sqrt{q})C_{1} + G_{p,q} \) where \( G_{p,q} \) is finite. Note however that in general \( G_{p,q} \) is different from \( F_{p,q} \).

APPENDIX B: HAMILTONIAN MATRIX ELEMENTS

We report in this appendix the explicit expressions for the matrix elements of the Hamiltonian matrices in units of Coulomb energies. Results are given as a function of \( K = |k L_{B}| \). For simplicity we adopt the same notation \( h_{ij} \) for noting the matrix elements of all matrices but their expression is specific of the case under consideration. All matrix elements \( \tilde{V}_{n_{1},n_{2},n_{3},n_{4}}(\overrightarrow{q}) \) and \( \tilde{E}_{n_{1},n_{2},n_{3},n_{4}}(\overrightarrow{k}) \) are evaluated using Eq. 11.

1. Matrix elements of \( H_{1\nu<2}^{0} \) and \( H_{0\nu<1}^{0} \)

For \( H_{1\nu<2}^{0} \) we obtain:

\[ h_{11} = E_{10} + \frac{3}{4}\alpha_{0}(2f_{0}^{+} - 1) + f_{0}^{+}(V_{0101} - E_{0110}) + C_{1} \]

\[ h_{22} = h_{33} = h_{44} = E_{10} + \frac{3}{4}\alpha_{0} + (V_{0101} - E_{0110}) + C_{1} \]

\[ h_{55} = E_{10} - \frac{3}{4}\alpha_{0}(2f_{0}^{+} - 1) + (1 - f_{0}^{+}) \times (V_{1010} - E_{1001} - 1) + C_{1} \]

(B1)

The matrix elements \( V_{n_{1},n_{2},n_{3},n_{4}} \) entering the Hamiltonian \( H_{1\nu<2}^{0} \) are:

\[ V_{0101}(K) = V_{0011}(K) = \frac{K}{4}e^{-\frac{K^{2}}{4}} \]

(B2)
with \( \tilde{V}_{0,1,0,1} = V_{0,0,1} \) and \( \tilde{V}_{0,0,1,1} = V_{0,0,1,1}e^{2\varphi} \) where \( \varphi \) is the polar angle of the exciton wave vector.

The matrix elements \( E_{n_1n_2n_3n_4} \) entering Eqs. 15 and B1 are:

\[
E_{0110}(K) = \left\{ \frac{\pi}{2} \left[ \Phi(\frac{1}{2}; \frac{1}{2}; \frac{-K^2}{2}) - \frac{1}{4} \Phi(\frac{3}{2}; 1; \frac{-K^2}{2}) \right] \right\} \tag{B3}
\]

\[
E_{0011}(K) = -\frac{3K^2}{32} \sqrt{\frac{\pi}{2}} \Phi(\frac{5}{2}; 3; \frac{-K^2}{2})
\]

where \( \tilde{E}_{0,1,1,0} = E_{0110} \), \( \tilde{E}_{0,0,1,1} = E_{0011}e^{2\varphi} \) and \( \Phi(a, b; z) \) is the confluent hypergeometric function.

For \( H_{0<\varphi<1}^0 \) the matrix elements are identical to those given in Eqs. B1, B2 and B3 replacing \( f_0^+ \) by \( f_0^- \) when appropriate.

2. Matrix elements of \( H_{1<\varphi<2}^1 \) and \( H_{0<\varphi<1}^0 \)

The matrix elements \( h_{ij} \) of \( H_{1<\varphi<2}^1 \) are:

\[
h_{11} = EI_{12} + \frac{\alpha_0}{16} + C_2 + V_{-12-12} - E_{-122-1}
\]

\[
h_{22} = EI_{12} - \frac{\alpha_0}{16} + C_2 + V_{-12-12} - E_{-122-1}
\]

\[
h_{33} = EI_{12} + \frac{\alpha_0}{16} + (2f_0^+ - 1) + C_2 + V_{-12-12} - E_{-122-1}
\]

\[
h_{44} = EI_{12} - \frac{\alpha_0}{16} + (2f_0^+ - 1) + C_2 + V_{-12-12} - E_{-122-1}
\]

\[
h_{55} = EI_{12} + \frac{\alpha_0}{16} + C_2 + V_{-12-12} - E_{-122-1}
\]

\[
h_{66} = EI_{12} - \frac{\alpha_0}{16} + C_2 + V_{-12-12} - E_{-122-1}
\]

\[
h_{77} = EI_{12} + \frac{\alpha_0}{16} + C_2 + V_{-12-12} - E_{-122-1}
\]

\[
h_{88} = EI_{12} - \frac{\alpha_0}{16} + C_2 + V_{-12-12} - E_{-122-1}
\]

\[
h_{12} = h_{34} = h_{56} = h_{78} = V_{-11-22} - E_{-122-2}
\]

\[
h_{13} = h_{15} = h_{24} = h_{26} = h_{28} = V_{-12-12}
\]

\[
h_{35} = h_{37} = h_{46} = h_{48} = h_{58} = V_{-12-12}
\]

\[
h_{14} = h_{16} = h_{23} = h_{25} = h_{27} = V_{-11-22}
\]

\[
h_{36} = h_{38} = h_{45} = h_{47} = h_{57} = h_{67} = V_{-11-22}
\]

where \( EI_{12} = (\sqrt{2} + 1)E_{10} \). The matrix elements \( V_{n_1n_2n_3n_4} \) entering Eq. B4 are:

\[
V_{-12-12}(K) = V_{-11-22}(K) = \frac{K}{8} e^{\frac{K^2}{2}} \left[ (3 - 2\sqrt{2}) \right]
\]

\[
+ (\sqrt{2} - 2) \frac{K^2}{2} + \frac{K^4}{8} \right] \tag{B5}
\]

with \( \tilde{V}_{-1,2,-1,2} = V_{-12-12} \) and \( \tilde{V}_{-1,1,-2,2} = V_{-11-22}e^{2\varphi} \).

The matrix elements \( E_{n_1n_2n_3n_4} \) entering B4 are:

\[
E_{-122-1}(K) = \sqrt{\frac{\pi}{2}} \left[ \Phi(\frac{1}{2}; 1; \frac{-K^2}{2}) \right]
\]

\[
- \Phi(\frac{3}{2}; 1; \frac{-K^2}{2}) + \frac{3}{4} \Phi(\frac{5}{2}; 1; \frac{-K^2}{2}) - \frac{15}{64} \Phi(\frac{7}{2}; 1; \frac{-K^2}{2})
\]

\[
E_{-112-2}(K) = -\frac{K^2}{64} \sqrt{\frac{3(1 + \sqrt{2})}{2}} \Phi(\frac{5}{2}; 3; \frac{-K^2}{2})
\]

\[
- 15(2 + \sqrt{2}) \Phi(\frac{7}{2}; 3; \frac{-K^2}{2}) + \frac{105\sqrt{2}}{16} \Phi(\frac{9}{2}; 3; \frac{-K^2}{2}) \right] \tag{B6}
\]

where \( \tilde{E}_{-1,2,2,-1} = E_{-122-1} \) and \( \tilde{E}_{-1,1,2,-2} = E_{-112-2}e^{2\varphi} \).

For \( H_{1<\varphi<2}^1 \) two columns of the matrix \( H_{1<\varphi<2}^1 \) are inverted but the eigenvalues are the same with \( f_0^+ \) replacing \( f_0^- \) in Eq. B6.

3. Matrix elements of \( H_{2<\varphi<3}^2 \) and \( H_{0<\varphi<1}^0 \)

The matrix elements \( h_{ij} \) of \( H_{2<\varphi<3}^2 \) are similar to those given in Eq. B4 when replacing \( V_{-12-12}, V_{-11-22}, E_{-122-1}, E_{-112-2} \) by \( V_{-23-23}, V_{-22-33}, E_{-233-3}, E_{-223-3} \) respectively and \( EI_{23} = (\sqrt{2} + \sqrt{3})E_{10} \).

The new matrix elements \( V_{n_1n_2n_3n_4} \) are here:

\[
E_{-233-2}(K) = \sqrt{\frac{\pi}{2}} \left[ \Phi(\frac{1}{2}; 1; \frac{-K^2}{2}) \right]
\]

\[
+ \frac{15}{4} \Phi(\frac{3}{2}; 1; \frac{-K^2}{2}) - 795 \frac{7}{192} \Phi(\frac{7}{2}; 1; \frac{-K^2}{2})
\]

\[
+ 315 \frac{9}{128} \Phi(\frac{9}{2}; 1; \frac{-K^2}{2}) - \frac{945}{1536} \Phi(\frac{11}{2}; 1; \frac{-K^2}{2}) \right] \tag{B7}
\]

where \( \tilde{E}_{-2,3,3,-2} = E_{-233-2} \) and \( \tilde{E}_{-2,2,3,-3} = E_{-223-3}e^{2\varphi} \) and the corresponding matrix elements \( V_{n_1n_2n_3n_4} \):
\[ V_{23-23}(K) = V_{22-33}(K) = \frac{K}{8} e^{-\frac{k^2}{2}} (3 - 2\sqrt{6}) + \left(\frac{3\sqrt{6}}{2} - 8\right) K^2 + \left(\frac{9}{8} - \frac{\sqrt{6}}{3}\right) K^4 \]
\[ + \left(\frac{\sqrt{6}}{48} K^6 + \frac{K^8}{192}\right) \]

with \( \bar{V}_{2,3,-2,3} = V_{23-23} \) and \( \bar{V}_{2,2,-2,3} = V_{22-33} e^{2i\phi} \).

4. Matrix elements of \( \bar{H}_{1c}^{12} \)

For \( \bar{H}_{1c}^{12} \) we obtain:
\[ h_{11} = h_{22} = h_{33} = E_{10} + \frac{3}{4} \alpha_0 + (V_{0101} - E_{0110}) + C_1 \]
\[ h_{44} = E_{10} + \alpha_0 \left(\frac{3}{4} - \frac{7}{8} f_1^+\right) + (1 - f_1^+) (V_{0101} - E_{0110}) + C_1 \]
\[ h_{55} = E_{12} + \frac{\alpha_0}{16} \left(1 + \frac{57}{6} \sqrt{2}\right) f_1^- \]
\[ + f_1^- (V_{1212} - E_{1221}) + C_2' \]

For \( \bar{H}_{1c}^{12} \) we have:
\[ h_{11} = h_{22} = E_{10} + \frac{3}{4} \alpha_0 + (V_{0101} - E_{0110}) + C_1 \]
\[ h_{33} = E_{10} + \alpha_0 \left(\frac{3}{4} - \frac{7}{8} f_1^+\right) + (1 - f_1^+) (V_{0101} - E_{0110}) + C_1 \]
\[ h_{44} = E_{12} + \frac{\alpha_0}{32} \left(21 - 10 \sqrt{2}\right) \]
\[ + (V_{1212} - E_{1221}) + C_2' \]
\[ h_{55} = E_{12} + \frac{\alpha_0}{16} \left(1 + \frac{19}{2} \sqrt{2}\right) f_1^+ \]
\[ + f_1^+ (V_{1212} - E_{1221}) + C_2' \]

For \( \bar{H}_{1c}^{12} \) we have:
\[ h_{11} = E_{10} + \frac{3}{4} \alpha_0 + (V_{0101} - E_{0110}) + C_1 \]
\[ h_{22} = E_{10} + \alpha_0 \left(\frac{3}{4} - \frac{7}{8} f_1^+\right) + (1 - f_1^+) (V_{0101} - E_{0110}) + C_1 \]
\[ h_{33} = E_{12} + \frac{\alpha_0}{16} \left(1 + \frac{19}{2} \sqrt{2}\right) f_1^- \]
\[ + f_1^- (V_{1212} - E_{1221}) + C_2' \]
\[ h_{44} = E_{12} + \frac{\alpha_0}{32} \left(21 - 10 \sqrt{2}\right) \]
\[ + (V_{1212} - E_{1221}) + C_2' \]

APPENDIX C: OPTICAL CONDUCTIVITY

In this appendix we derive, following the lines of Ref. [7], the corresponding expressions which allow to calculate the optical matrix elements of the MP curves which enter in the optical conductivity \( \bar{\sigma}(h\omega) \) which has two components:
\[ \sigma_{\parallel} = -ie^2 G_B \frac{2E_M}{\omega} \sum j \left(\frac{M_{ij}}{E_{ij}}\right)^2 \]
\[ \sigma_{\perp} = -ie^2 G_B \frac{2\omega M_{ij} \cdot L_j}{\omega (E_{ij}^2 - \omega^2)} \]

with the corresponding new matrix elements entering Eq. B9, B10, B11 and B12:
\[ V_{2121}(K) = \frac{K}{8} e^{-\frac{k^2}{2}} [1 + \sqrt{2} - \frac{K^2}{2\sqrt{2}}] \]
\[ E_{1221}(K) = E_{1221}(K) \]

where \( \bar{V}_{1,2,1,2} = V_{1212} \)
where the function \([F^sVF]\) denotes the scalar product and the velocity operators \(\vec{V}_s^\dagger\) are:

\[
\vec{V}_\parallel^\dagger = v_F \begin{bmatrix} 0 & -ie^{-i\varphi} \\ ie^{i\varphi} & 0 \end{bmatrix}
\]

\[
\vec{V}_\perp^\dagger = v_F \begin{bmatrix} 0 & e^{-i\varphi} \\ e^{i\varphi} & 0 \end{bmatrix}
\]  

(C3)

and \(\vec{V}_s^\dagger = (\vec{V}_s^\dagger)^*\).

In the one electron picture the selection rules for optical transitions between \(LL m \rightarrow n\) are \(\delta_{|m|,|n|} \pm 1\)

1. **Optical vectors for \(H_{1<\nu<2}^0\) and \(H_{0<\nu<1}^0\)\n
In the case of \(H_{1<\nu<2}^0\) we obtain the following components of \(\mathcal{M}_\parallel\) and \(\mathcal{M}_\perp\):

\[
\mathcal{M}_\parallel^0 = \frac{v_F}{\sqrt{2}} e^{-\frac{k^2}{2}} \left( \sqrt{f_0^+, 1, 1, 1, -1 - f_0^+} \right)
\]

\[
\mathcal{M}_\perp^0 = i \frac{v_F}{\sqrt{2}} e^{-\frac{k^2}{2}} \left( \sqrt{f_0^+, 1, 1, 1, -1 - f_0^+} \right)
\]  

(C4)

whereas for \(0 < \nu < 1\), \(f_0^+\) has to be replaced by \(f_0^-\).

It can be easily verified that for \(K \approx 0\) where all transitions become degenerate, there is a sum rule such that \(\sum_j |\mathcal{M}_\parallel \cdot \mathcal{L}_j|^2 = 2v_F^2\), independent of the filling factor whereas \(\sum_j (\mathcal{M}_\parallel \cdot \mathcal{L}_j)(\mathcal{M}_\perp \cdot \mathcal{L}_j)^* = uv_F^2\). One therefore recover the selection rules obtained for the one-electron model. Note that the Fermi velocity entering in the optical matrix elements is that existing in the absence of electron-electron interactions.

2. **Optical vectors for \(H_{1<\nu<2}^{12}\) and \(H_{0<\nu<1}^{12}\)**

Following the same approach we get for the components of the corresponding optical vectors \(\mathcal{M}_\parallel\) and \(\mathcal{M}_\perp\):

\[
\mathcal{M}_\parallel^{12} = \frac{v_F}{2} e^{-\frac{k^2}{2}} \left( 1 + K^2 \sqrt{2} - 1 \right) \left( \sqrt{1 - f_0^+, 1, 1, 1, -1, 1, -1} \right)
\]

\[
\mathcal{M}_\perp^{12} = i \frac{v_F}{2} e^{-\frac{k^2}{2}} \left( 1 - K^2 \sqrt{2} + 1 \right) \left( \sqrt{1 - f_0^+, 1, 1, 1, -1, 1, 1} \right)
\]  

(C5)

which are no longer dependent of the filling factor for \(\nu < 2\). It can be shown that the only optical active transition are those corresponding to the solutions \(EI_1^{+/−}(K)\) of Eq. 18.

3. **Optical vectors for \(H_{1<\nu<2}^{23}\) and \(H_{0<\nu<1}^{23}\)**

In this case we get for the components of the corresponding optical vectors \(\mathcal{M}_\parallel\) and \(\mathcal{M}_\perp\):

\[
\mathcal{M}_\parallel^{23} = \frac{v_F}{2} e^{-\frac{k^2}{2}} \left( 1 + \frac{3}{\sqrt{6}} \right) \left( K^2 + \frac{1}{4} \frac{1}{\sqrt{6}} \right)
\]

\[
\mathcal{M}_\perp^{23} = i \frac{v_F}{2} e^{-\frac{k^2}{2}} \left( 1 - \frac{3}{\sqrt{6}} \right) \left( K^2 + \frac{1}{4} \frac{1}{\sqrt{6}} \right)
\]

\[
\{1, 1, 1, 1, 1, 1, 1\}
\]

(C6)

4. **Optical vectors for \(H_{2<\nu<5}^{12}\)**

Here one gets for the components of the corresponding optical vectors \(\mathcal{M}_\parallel\) and \(\mathcal{M}_\perp\) the following relations where we have defined the functions \(p_{\parallel}(K) = \frac{1}{2\sqrt{2}} (1 - K^2 (\sqrt{2} - 1))\) and \(p_{\perp}(K) = \frac{1}{2\sqrt{2}} (1 - K^2 (\sqrt{2} - 1))\):

For \(2 < \nu < 3\):

\[
\mathcal{M}_\parallel^{12}_{2<\nu<3} = \frac{v_F e^{-ip_{\parallel}(K)}}{\sqrt{2}} e^{-\frac{k^2}{2}}
\]

\[
\{1, 1, 1, \sqrt{1 - f_0^+, p_{\parallel}(K)} \sqrt{f_0^+} \}
\]

(C7)

\[
\mathcal{M}_\perp^{12}_{2<\nu<3} = i \frac{v_F e^{-ip_{\perp}(K)}}{\sqrt{2}} e^{-\frac{k^2}{2}} \left( \sqrt{1 - f_0^+, p_{\perp}(K)} \sqrt{f_0^+} \right)
\]

For \(3 < \nu < 4\):

\[
\mathcal{M}_\parallel^{12}_{3<\nu<4} = \frac{v_F e^{-ip_{\parallel}(K)}}{\sqrt{2}} e^{-\frac{k^2}{2}} \left( \sqrt{1 - f_0^+, p_{\parallel}(K)} \sqrt{f_0^+} \right)
\]

\[
\{1, 1, \sqrt{1 - f_0^+, p_{\parallel}(K)} \sqrt{f_0^+} \}
\]

(C8)

For \(4 < \nu < 5\):

\[
\mathcal{M}_\parallel^{12}_{4<\nu<5} = \frac{v_F e^{-ip_{\parallel}(K)}}{\sqrt{2}} e^{-\frac{k^2}{2}} \left( \sqrt{1 - f_0^+, p_{\parallel}(K)} \sqrt{f_0^+} \right)
\]

\[
\{1, \sqrt{1 - f_0^+, p_{\parallel}(K)} \sqrt{f_0^+} \}
\]

(C9)
For $5 < \nu < 6$:  
\[ \overline{M}_{\parallel}^{5<\nu<6} = \frac{v_F e^{-i\nu}}{\sqrt{2}} e^{-\nu^2} \left\{ \sqrt{1 - f_0^2} \frac{p_{\parallel}(K)}{f_0}, \frac{p_{\parallel}(K)}{p_{\parallel}(K)} \right\} \]
\[ \overline{M}_{\perp}^{5<\nu<6} = i \frac{v_F e^{-i\nu}}{\sqrt{2}} e^{-\nu^2} \left\{ \sqrt{1 - f_0^2} \frac{p_{\perp}(K)}{f_0}, \frac{p_{\perp}(K)}{p_{\perp}(K)} \right\} \]
\[ (C10) \]

It can be verified that, for $K \simeq 0$, there is a sum rule such that for the transitions $n = 0$ to $n = 1$, \[ \sum_j |\overline{M}_{\parallel} \cdot \overrightarrow{L}_j|^2 = \frac{6-\nu^2}{4} v_F^2 \] whereas for the transitions $n = 1$ to $n = 2$, \[ \sum_j |\overline{M}_{\parallel} \cdot \overrightarrow{L}_j|^2 = \nu^2 - \frac{\nu^2}{4} v_F^2. \]

We also get \[ \sum_j (\overline{M}_{\parallel} \cdot \overrightarrow{L}_j) (\overline{M}_{\perp} \cdot \overrightarrow{L}_j)^* = i \sum_j |\overline{M}_{\parallel} \cdot \overrightarrow{L}_j|^2. \]
One therefore recovers the selection rules obtained for the one-electron model.

To conclude this part, it is worth comparing these results with those obtained in C2DEG where the introduction of electron-electron interactions change the selection rules [7, 9] with respect to those obtained in the one-electron picture. In the case of Graphene, the selection rules are in general, similar to the results obtained in the one-electron picture, except that the oscillator strength is condensed into one or two branches of the MP curves and that the strength remains proportional to $v_F^2$ and not to the square of the re-normalized velocity.

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