Spiraling elliptic solitons in generic nonlocal nonlinear media

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We have introduced a class of spiraling elliptic solitons in generic nonlocal nonlinear media. The spiraling elliptic solitons carry the orbital angular momentum. This class solitons are stable for any degree of nonlocality except for the local case when the response function of the material is Gaussian function. © 2022 Optical Society of America

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Optical spatial solitons in nonlocal nonlinear media are attracting increasing attention during recent years in both theoretical [1–10] and experimental [11–14] aspects of research. The nonlocality plays an important role in the nonlinear evolution of waves. It may drastically modify the properties of solitons. The solitons in bulk Kerr media may undergo catastrophic collapse [15, 16]. The nonlocality of an arbitrary shape can eliminate collapse in all physical dimensions [6]. Nonlocality can support vortex solitons [17, 18] and multipole solitons [19] which are unstable in local nonlinear media.

In theoretical aspect, ellipse-shaped solitons have been reported in saturable nonlinear media, such as elliptic incoherent solitons [20], elliptic dark solitons [21], and spiraling elliptic solitons [22]. In experimental aspect, coherent elliptic solitons [23] in lead glass which is nonlocal nonlinear media and elliptic incoherent spatial solitons [24] in photorefractive screening nonlinear media are observed.

In this Letter, we use the variational approach to derive the analytical spiraling elliptic solitons solution in generic nonlocal nonlinear media. We analyze the potential function to study the stability properties of the class of solitons.

The propagation of the optical beams in the nonlocal cubic nonlinear media can be modeled by the following generic dimensionless nonlocal nonlinear Schrödinger equation (NNLSE) [3, 7],

\[ i \frac{\partial \psi}{\partial z} + \frac{1}{2} \nabla_\perp^2 \psi + \Delta n \psi = 0, \tag{1} \]

where \( \psi = \psi(x, y, z) \) is a paraxial beam, \( z \) is the longitudinal coordinate, \( \nabla_\perp^2 = \partial_x^2 + \partial_y^2 \), \( x \) and \( y \) are the transverse coordinates, \( \Delta n = \int \int R(x - x', y - y') |\psi(x', y', z)|^2 dxdy' \) is the normalized nonlinear perturbation of refraction index, and \( R \) is the nonlinear response of the medium which is normalized, real and symmetric such that \( \int \int R(x, y) dxdy = 1 \). We suppose the material response to be Gaussian function [7, 25], i.e. \( R(x, y) = 1/(2\pi w_n^2) \exp[-(x^2 + y^2)/2w_n^2] \), where \( w_n \) is the normalized characteristic length of the material response function.

By the variational approach [26], Eq.(1) can be interpreted as an Euler-Lagrange equation corresponding to a vanishing variation

\[ \delta \int \int \int l(\psi, \psi^*, \psi_z, \psi_x, \psi_y, \psi_y^*) dxdydz = 0, \tag{2} \]

where the Lagrangian density \( l \) is given by [27, 28]

\[ l = \frac{i}{2} (\psi \frac{\partial \psi}{\partial z} - \overline{\psi} \frac{\partial \psi^*}{\partial z}) + \frac{1}{2} (|\frac{\partial \psi}{\partial x}|^2 + \frac{\partial \psi}{\partial y})^2 \]

\[ + \frac{1}{2} |\psi|^2 \int \int R(x - \xi, y - \eta) |\psi(\xi, \eta)|^2 d\xi d\eta. \tag{3} \]

We introduce a trial function,

\[ \psi(x, y, z) = A(z)G[X/b(z)]G[Y/c(z)] \exp(i\phi), \tag{4} \]

where the Gaussian envelope is \( G(t) = \exp(-t^2/2) \) the phase is \( \phi = B(z)X^2 + \Theta(z)XY + Q(z)Y^2 + \varphi(z), \) and \( X = x \cos \beta(z) + y \sin \beta(z), \) \( Y = -x \sin \beta(z) + y \cos \beta(z). \) Corresponding to the trial function we can obtain its power, \( P = \frac{1}{2} \int \int |\psi(x, y)|^2 dxdy = \pi A^2 b c \) and orbital angular momentum (OAM), \( M = Im \int \int \psi^* (\mathbf{r} \times \nabla \psi) d\mathbf{r} = 1/2P(b^2 - c^2) \) where \( \mathbf{r} = xe_x + ye_y. \) Substituting the trial function above to the variational principle Eq.(2), we obtain the reduced variational equation

\[ \delta \int L dz = 0, \tag{5} \]

where \( L = \int \int l_0 dxdy, \) and \( l_0 \) denotes the result of inserting the Gaussian ansatz (4) into the Lagrangian density (3). It also can be shown that the Hamiltonian corresponding to Eq.(1) is of the following form

\[ H = \int \int \left[ \frac{1}{2} (|\frac{\partial \psi}{\partial x}|^2 + |\frac{\partial \psi}{\partial y}|^2) - \frac{1}{2} |\psi|^2 \right] \]

\[ \int \int R(x - \xi, y - \eta) |\psi(\xi, \eta)|^2 d\xi d\eta \] \] dxdy. \tag{6} \]

After some algebraic calculations, \( L \) and \( H \) can be an-
alytically determined as
\[
L = \frac{A^2 \pi}{4bc} \left[ -b^2 - c^2 - 4b^4B^2c^2 - 4b^2c^4Q^2 - b^4c^2\Theta^2 \\
    -b^2c^4\Theta^2 + \frac{A^2b^3c^3}{(b^2 + w_m^2)(c^2 + w_m^2)} - 2b^4c^2B' \\
    -2b^2c^4Q' + 2b^4c^2\Theta' - 2b^2c^4\Theta' - 4b^2c^4\varphi \right].
\]

(7)

\[
H = \frac{A^2 \pi}{4bc} \left[ b^2 + c^2 + 4b^4B^2c^2 + 4b^2c^4Q^2 + b^4c^2\Theta^2 \\
    + b^2c^4\Theta^2 - \frac{A^2b^3c^3}{(b^2 + w_m^2)(c^2 + w_m^2)} \right].
\]

(8)

Following the standard procedures of the variational approach [26], we have \( b' = 2hB, c' = 2cQ, \beta' = (b^2 + c^2)\Theta/(b^2 - c^2), P' = 0, H' = 0 \) and \( M' = 0 \). Primes indicate derivatives with respect to the evolution variable \( z \). So we can rewrite the Hamiltonian of the system as follows,

\[
H = \frac{P}{4} (b^2 + c^2 + \Pi),
\]

(9)

\[
\Pi = \frac{1}{b^2 + c^2} + \frac{1}{c^2} + \frac{4b^2\sigma^2}{(b^2 - c^2)^2} + \frac{4c^2\sigma^2}{(b^2 - c^2)^2} \\
    - \frac{\pi}{\sqrt{(b^2 + w_m^2)(c^2 + w_m^2)}},
\]

(10)

with \( \sigma = M/P = 1/2(b^2 - c^2)\Theta \).

Solitons can be found as the extrema of the potential \( \Pi(b, c) \). Letting \( \partial\Pi/\partial b = 0 \) and \( \partial\Pi/\partial c = 0 \), we can obtain the critical power and the OAM.

\[
P_c = \frac{2(b^2 + c^2)^3}{b^4c^4 [b^4 + 6b^2c^2 + c^4 + 4(b^2 + c^2)w_m^2]},
\]

(11)

\[
\sigma_c^2 = \frac{(b^2 - c^2)^4}{4b^4c^4 [b^4 + 6b^2c^2 + c^4 + 4(b^2 + c^2)w_m^2]}. 
\]

(12)

We can also obtain the rotation velocity \( \omega \equiv \beta' = 2(b^2 + c^2)\sigma/(b^2 - c^2)^2 \). When the input power and OAM are chosen arbitrarily the spiraling elliptic solitons can be found the semi-axis of which are determined by Eq.11 and Eq.12. One example is shown in Fig.1 with \( P_c = 127272.4 \) and \( \sigma_c = 0.560949 \) (other parameters are \( b = 2.0, c = 1.0, \Theta = 0.373966, w_m = 15.0 \) and \( \omega = 0.623277 \)). The isosurface of intensity of the spiraling soliton is obtained from our variational solution. Comparing two half widths obtained from variational solution, \( w_x = \sqrt{b^2\cos^2\omega z + c^2\sin^2\omega z} \) and \( w_y = \sqrt{c^2\cos^2\omega z + b^2\sin^2\omega z} \), with the numerical results we find an excellent agreement as shown in Fig.2 and Fig.3. We introduce a nonlocal parameter \( \alpha = \max(w_m/b, w_m/c) \) to define the degree of nonlocality for the beam in nonlocal nonlinear media. The larger is the nonlocal parameter, the stronger is the degree of nonlocality. In Fig.1, Fig.2 and Fig.3 \( w_m \) is 15.0, and the degree of nonlocality \( \alpha \) is 7.5.

An important aspect of any family of soliton solutions is their stability properties. We can study the stability characteristics of our analytical soliton solution by means of the analysis of the potential function \( H(b, c) \). So we search the second derivative of the potential \( \Pi(b, c) \) with respect to \( b \) and \( c \), then substituting Eq.11 and Eq.12 into it we get

\[
\frac{\partial^2\Pi}{\partial b^2} = \frac{2(b^2 + c^2)}{b^4c^4 (b^2 + w_m^2) [b^4 + 6b^2c^2 + c^4 + 4(b^2 + c^2)w_m^2]} \\
    \left[ b^2c^2 (b^4 + 14b^2c^2 + c^4) + (b^6 + 18b^4c^2 + 33b^2c^4 \\
    + 4b^6)w_m^2 + (b^4 + 5b^2c^2 + 16c^4)w_m^2 \right],
\]

(13)

\[
\frac{\partial^2\Pi}{\partial bc} = -\frac{2(b^2 + c^2)}{b^4c^4 [b^4 + 6b^2c^2 + c^4 + 4(b^2 + c^2)w_m^2]} \\
    \left[ b^4 + 14b^2c^2 + c^4 + 12(b^2 + c^2)w_m^2 \right],
\]

(14)

\[
\frac{\partial^2\Pi}{\partial c^2} = \frac{2(b^2 + c^2)}{b^4c^4 (c^2 + w_m^2) [b^4 + 6b^2c^2 + c^4 + 4(b^2 + c^2)w_m^2]} \\
    \left[ b^2c^2 (b^4 + 14b^2c^2 + c^4) + (4b^6 + 33b^4c^2 + 18b^2c^4 \\
    + 4b^6)w_m^2 + (4b^4 + 5b^2c^2 + 4c^4)w_m^2 \right].
\]

(15)

From Eq.13, Eq.14 and Eq.15 we can easily get \( \partial^2\Pi/\partial b^2 > 0 \), \( \partial^2\Pi/\partial bc^2 > 0 \) and \( \Delta \equiv \partial^2\Pi/\partial b^2 - (\partial^2\Pi/\partial bc^2)^2 > 0 \) when \( w_m \neq 0 \) (the degree of nonlocality \( \alpha \) is not zero). So \( b \) and \( c \) of the spiraling elliptic solitons what we have got analytically by the variational approach are corresponding to the minima of the potential \( \Pi(b, c) \). So the soliton solutions are stable for any degree of nonlocality except for the local case. But we should mention that when the degree of nonlocality decreases the low-intensity oscillating tails which indicate the appearance of dispersive waves radiated by the soliton will occur [22, 29]. The radiative tails take a portion of radiated OAM from the soliton, and the reduction of OAM in the main soliton leads to the slow reduction of ellipticity of the transverse rotating profile [22]. This can be verified in Fig.4 and Fig.5. In Fig.4 and Fig.5 \( w_m \) is 8.0, and the degree of nonlocality \( \alpha \) is 4.0.

In conclusion, we have obtained spiraling elliptic solitons in generic nonlocal nonlinear media by use of the variational approach. We show that this class of solitons...
are stable for any degree of nonlocality except for the local case. Because of the appearance of dispersive waves radiated by the soliton the ellipticity of the spiraling elliptic solitons will reduce when the degree of nonlocality becomes lower. Our theoretical results have been confirmed by direct numerical simulations of the NNLSE.

Fig. 1. (color online) Propagation dynamics of the spiraling elliptic soliton in nonlocal nonlinear media. The isointensity plot is at the level $I_m/2$ of the elliptic soliton with $I_m = 20256.03$ where $I_m = \max|\psi|^2$. The normalized characteristic length of the material response function $w_m$ is 15, and the degree of nonlocality $\alpha$ is 7.5.

Fig. 2. (color online) Evolution of the beam width of the spiraling elliptic soliton in the direction of x axis. Numerically obtained half width $w_x$ (black line) is compared to the variational result (red line). The parameters $w_m$ and $\alpha$ are 8.0 and 4.0 respectively.

Fig. 3. (color online) Same as Fig.2 but with plot corresponding to the beam width of the spiraling elliptic soliton in the direction of y axis.

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