DIVERGENT SERIES AND SERRE’S INTERSECTION FORMULA FOR GRADED RINGS

DANIEL ERMAN

On a smooth variety, Serre’s intersection formula [Ser00, V.3] computes intersection multiplicities via an alternating sum of the lengths of Tor groups. When the variety is singular, the corresponding sum can be a divergent series. But there are alternate geometric approaches for assigning (often fractional) intersection multiplicities in some singular settings. Our motivating question comes from Fulton, who asks whether an analytic continuation of the divergent series from Serre’s formula can be related to these fractional multiplicities [Ful98, Example 20.4.4].

In this note, we show how ideas from Avramov and Buchweitz [AB93] answer Fulton’s question in the context of graded rings. Namely, [AB93] use a rational function to provide an analytic continuation of the Hilbert series of the total Tor module associated to an intersection (see Theorem 0.1). Evaluating this rational function at \( t = 1 \) provides a notion of intersection multiplicity that extends Serre’s definition (see Corollary 0.2). We then show that the resulting multiplicity agrees with the multiplicities obtained for \( \mathbb{Q} \)-Cartier divisors or on normal surfaces via alternate definitions used in algebraic geometry (see Theorem 0.3).

For an isolated hypersurface singularity, similar techniques are used in [BVS12, Theorem 2.2] to study Hochster’s Theta pairing. Other related results are discussed in Remark 3.4.

We illustrate this with an example. Let \( R := \mathbb{C}[x, y, z]/\langle f \rangle \) be the coordinate ring of a cubic curve. Let \( X = \text{Spec}(R) \subseteq \mathbb{A}^3 \). If \( L, L' \subseteq X \) are lines connecting distinct flex points to the origin, then \( 3L \) and \( 3L' \) are both Cartier divisors, and they intersect only at the origin \( P \in X \). Since \( (3L \cdot 3L')_p = \deg X = 3 \), the local intersection multiplicity \( (L \cdot L')_p \) is \( \frac{1}{3} \).

Now let \( I, J \) be the ideals of these two lines. Directly applying Serre’s formula yields:

\[
\sum_{i \geq 0} (-1)^i \text{length} \left( \text{Tor}^R_{i}(R/I) \right) = 1 - 1 + 1 - 1 + \ldots
\]

To assign a value to this divergent series, we might first consider the series

\[
F(T) := \sum_{i=0}^{\infty} (-1)^i \text{length} \left( \text{Tor}^R_{i}(R/I, R/J) \right) \cdot T^i = 1 - T + T^2 - T^3 + \ldots
\]

as suggested in [Ful98, Example 20.4.4]. But the resulting analytic continuation at \( T = 1 \) (by \( \frac{1}{1-T} \)) would have yielded a multiplicity of \( F(1) = \frac{1}{2} \) instead of \( \frac{1}{3} \).

On the other hand, since everything is graded, there is another natural power series in the picture, namely the alternating sum of the Hilbert series of \( \text{Tor}_i(R/I, R/J) \). This is denoted \( \chi(R/I, R/J)(t) \) in [AB93]. Since \( \text{Tor}_i \) is concentrated entirely in degree \( \lfloor \frac{3i}{2} \rfloor \) we get

\[
\chi(R/I, R/J)(t) := \sum_{i=0}^{\infty} (-1)^i \text{HS}_{\text{Tor}_i(R/I, R/J)}(t) = 1 - t + t^3 - t^4 + t^6 - t^7 + \ldots \in \mathbb{Z}[[t]]
\]

The author was partially supported by NSF grant DMS-1302057.
Using the rational function $\frac{1}{1+t+t^2}$ as our analytic continuation, we obtain the desired intersection multiplicity $\chi(R/I, R/J)(1) = \frac{1}{3}$.

Generalizing this method provides an approach for defining intersection multiplicities in the singular case, at least as long as everything is graded. We fix a field $k$, let $R$ be a graded $k$-algebra (i.e. $R_0 = k$), and let $M, N$ be finitely generated graded $R$-modules. Following [AB93], we define

\[ \chi(M, N)(t) := \sum_{i \geq 0} (-1)^i \text{HS} \text{Tor}_i(M, N)(t), \]

where $\text{HS}_M(t)$ denotes the Hilbert series of a graded module $M$. Note that $\chi(M, N)(t)$ can be thought of as the Hilbert series of the derived tensor product $M \otimes N$.

If either $M$ or $N$ has finite projective dimension, and if $M \otimes N$ has finite length, then $\chi(M, N)(t)$ is a Laurent polynomial in $\mathbb{Z}[t, t^{-1}]$. Thus $\chi(M, N)(1) \in \mathbb{Z}$ and equals the alternating sum of the lengths of the $\text{Tor}_i$, and it was shown by Peskine-Szpiro [PS74, Théorème, 2(iv)] that this extension of Serre’s formula satisfies Vanishing and Positivity.¹

In the case where $M$ and $N$ both have infinite projective dimension, $\text{Tor}_i$ can be nonzero for infinitely many values of $i$, and thus Serre’s formula would yield a divergent sum. However, Avramov and Buchweitz observed that $\chi(M, N)(t)$ admits a simple analytic continuation as a rational function.

**Theorem 0.1** (Avramov-Buchweitz). Let $k$ be a field, $R$ a finitely graded $k$-algebra, and let $M, N$ finitely generated, graded $R$-modules. Then

\[ \chi(M, N)(t) = \frac{1}{(1-t)^{\dim M + \dim N - \dim R}} \cdot e_{M,N}(t) \]

where $e_{M,N}(t) \in \mathbb{Q}(t)$ is a rational function. Moreover, $e_{M,N}(1) \in \mathbb{Q}_{>0}$.

The theorem essentially appears as [AB93, Proposition 8].² Using this rational function as an analytic continuation of $\chi(M, N)(t)$, we can always define $\chi(M, N)(1) \in \mathbb{Q} \cup \{\infty\}$.

**Corollary 0.2.** Let $k$ be a field, $R$ a finitely generated $k$-algebra, and let $M, N$ finitely generated, graded $R$-modules.

1. $\chi(M, N)(1) = \infty \iff \dim M + \dim N > \dim R$.
2. Positivity: $\chi(M, N)(1) \in \mathbb{Q}_{>0} \iff \dim M + \dim N = \dim R$.
3. Nonvanishing: $\chi(M, N)(1) = 0 \iff \dim M + \dim N < \dim R$.

The corollary, which has a precursor in [PS74, Théorème, 2(iv)] in the case where $M$ or $N$ has finite projective dimension, is an immediate consequence of Theorem 0.1.

For a graded singularity $\text{Spec} R$, one thus has an intersection multiplicity $\chi(M, N)(1)$. The contribution of this note is the observation that this definition recovers the multiplicities obtained by two alternate approaches from algebraic geometry: from $\mathbb{Q}$-Cartier divisors on normal varieties, or from a resolution of singularities of a normal surface.

¹The corresponding result fails in the non-graded case. See [DHM85].
²That proposition assumes that $R$ is generated in degree 1, but their proof works equally well for a nonstandard grading.
Theorem 0.3. Let $R$ be a graded $k$-algebra that is a normal domain, and let $X = \text{Spec } R$. Let $C, D \subseteq X$ be an integral curve and an effective divisor that intersect properly at the origin $P \in X$ and that are defined by homogeneous ideals $I_C, I_D \subseteq R$. If either: $D$ is $\mathbb{Q}$-Cartier or $X$ is a surface, then we have

$$(D \cdot C)_P = \chi(R/I_D, R/I_C)(1)$$

and lies in $\mathbb{Q}_{>0}$, where $(D \cdot C)_P$ denotes the local intersection multiplicity at $P$.

See [Ful98, Example 7.1.16] or [Mum61, II(b)] for the definition of $(D \cdot C)_P$ on a normal surface. For a Cartier divisor $E$ defined locally by a function $f$, and intersecting a curve $C$ properly at point $P$, we use the standard definition $(E \cdot C)_P := \text{length } O_{C,P}/\langle f \rangle$. Then for a $\mathbb{Q}$-Cartier divisor $D$ where $e_D$ is Cartier, we take the natural local version of [Kol13, Definition 1.4], setting $(D \cdot C)_P := -\frac{1}{e}(eD \cdot C)_P$.

This note is structured as follows. In §1, we gather some of the background on Hilbert series that we will use in our proof. §2 contains the proof of Theorem 0.3. In §3, we consider examples of computing $\chi(M, N)(1)$ and we discuss the relation with some other results from the literature.

Acknowledgements

We are grateful to many people for helpful conversations and references: Dima Arinkin, Lucho Avramov, Bhargav Bhatt, Ragnar–Olaf Buchweitz, Andrei Căldăraru, William Fulton, Craig Huneke, and Kevin Tucker.

1. Background

In this section we summarize some results about Hilbert series. Let $R$ be a finitely generated, graded $k$-algebra with generators in degrees $d_1, \ldots, d_s$. Let $S = k[x_1, \ldots, x_s]$ with generators in the same degrees and with a surjection $S \to R$ of graded rings. We write $HS_M(t)$ for the Hilbert series of a graded module $M$.

Lemma 1.1. For any finitely generated, graded $R$-module $M$, we have

$$HS_M(t) = \frac{1}{(1-t)^{\dim M}} \cdot e_M(t)$$

where $e_M(t) \in \mathbb{Q}(t)$ and $e_M(1) \in \mathbb{Q}_{>0}$.

Proof. See [BH93, Proposition 4.4.1], which is equivalent but in a slightly different form. □

Lemma 1.2. Let $M, N$ be finitely generated $R$-modules. Then

$$\chi(M, N)(t) = \frac{HS_M(t)HS_N(t)}{HS_R(t)}.$$ 

In particular, $\chi(M, N)(t)$ is a rational function.

Proof. This is proven in [AB93, Lemma 7(ii)], where it is cited as mathematical folklore. □

Proof of Theorem 0.1. By Lemma 1.2, we have $\chi(M, N)(t) = \frac{HS_M(t)HS_N(t)}{HS_R(t)}$. We then apply Lemma 1.1 to obtain

$$\chi(M, N)(t) = \frac{\frac{1}{(1-t)^{\dim M}} \cdot e_M(t)}{(1-t)^{\dim M} \cdot e_R(t)} \cdot \frac{\frac{1}{(1-t)^{\dim N}} \cdot e_N(t)}{e_R(t)} = \frac{(1-t)^{\dim M+\dim N}}{(1-t)^{\dim R}} \cdot \frac{e_M(t)e_N(t)}{e_R(t)}.$$
This is exactly the statement of [AB93, Proposition 8(i)], which is only stated in the case
where $R$ is generated in degree 1, but their proof applies just as well in this context.
Since $e_M(1), e_N(1), e_R(1)$ are all strictly positive, it follows that $\frac{e_M(1)e_N(1)}{e_R(1)} \in \mathbb{Q}_{>0}$. Setting
$e_{M,N}(t) := \frac{e_M(t)e_N(t)}{e_R(t)}$ and evaluating at $t = 1$, we obtain all three parts of the theorem. \( \square \)

\section{Proof of Main Result}

\textbf{Proof of Theorem 0.3. Q-Cartier case:} Since $D$ is $\mathbb{Q}$-Cartier, we may assume that $eD$
eq 0 equals a Cartier divisor $E$ defined by the homogeneous principal ideal $I_E = \langle f \rangle$. By definition, we then have

$$(D \cdot C)_P := \frac{1}{e}(E \cdot C)_P = \frac{1}{e} \dim_k \mathcal{O}_{C,P}/\langle f \rangle_C. $$

Since $C$ is not contained in $D$, it is also not contained in $E$ and thus $f|_C \neq 0$. It follows that $\text{Tor}_i^R(R_E, R_C) \neq 0$ if and only if $i = 0$. We can then compute $\chi(R_E, R_C)(1) = \dim_k \text{Tor}_0(R_E, R_C) = \dim_k \mathcal{O}_{C,P}/\langle f \rangle_C$. We thus have

(1) \( (D \cdot C)_P = \frac{1}{e} \chi(R_E, R_C)(1). \)

We next relate the constants $m_D(1)$ and $m_E(1)$. We define $X_0 := \text{Proj} R$, which is a normal projective variety, and we let $D_0 = \text{Proj}(R/I_D)$ and $E_0 = \text{Proj}(R/I_E)$ be the corresponding divisors on $X_0$. Note that $eD_0$ is still equivalent to the Cartier divisor $E_0$. Moreover, if we write $H_0$ for the Weil divisor corresponding to $\mathcal{O}_{X_0}(1)$, then $D_0, E_0, H_0$ are all positive rational multiples of one another, and hence each is ample.

Let $n$ be the dimension of $X_0$. For any $r$ we have the short exact sequence

$$0 \to \mathcal{O}_{X_0}(-D_0 + rH_0) \to \mathcal{O}_{X_0}(rH_0) \to \mathcal{O}_{D_0}(rH_0|_{D_0}) \to 0.$$

Thus, for $r \gg 0$ we compute

$$\dim(R/I_D)_r = h^0(D_0, \mathcal{O}_{D_0}(rH_0|_{D_0})) = h^0(X_0, \mathcal{O}_{X_0}(rH_0)) - h^0(X_0, \mathcal{O}_{X_0}(-D_0 + rH_0)),$$

Since $X_0$ is normal, we may apply Asymptotic Riemann-Roch to compute the above two terms up to $O(r^{n-2})$, and after some simple cancellations we obtain:

$$= \frac{H_0^{n-1} \cdot D_0 \cdot r^{n-2}}{(n-1)!} + O(r^{n-2}).$$

In the above line, the intersection product is computed on $X_0$. Since Lemma 1.1 implies that $\text{HS}_{R/I_D}(t) = \frac{1}{(1-t)^n} m_D(t)$, we can also write $\dim(R/I_D)_r = \frac{m_D(1)}{(n-1)!} r^{n-1} + O(r^{n-2})$. We thus conclude that

$$m_D(1) = H_0^{n-1} \cdot D_0.$$

A similar computation yields $m_E(1) = H_0^{n-1} \cdot E_0$, which then implies that

(2) \( m_E(1) = e \cdot m_D(1). \)

Finally, we turn our attention to comparing the series $\chi(R_E, R_C)(t)$ and $\chi(R_D, R_C)(t)$:

$$\chi(R_E, R_C)(t) - e \cdot \chi(R_D, R_C)(t) = \frac{m_E(t)m_C(t)}{m_R(t)} - \frac{e \cdot m_D(t)m_C(t)}{m_R(t)}$$

$$= (m_E(t) - e \cdot m_D(t)) \frac{m_C(t)}{m_R(t)}.$$
But by (2), \( m_E(1) = e \cdot m_D(1) \), and so plugging in \( t = 1 \) yields

\[
\chi(R_E, R_C)(1) - e \cdot \chi(R_D, R_C)(1) = 0.
\]

Combining this with (1), we conclude that

\[
(D \cdot C)_P = \frac{1}{e} \chi(R_E, R_C)(1) = \chi(R_D, R_C)(1).
\]

**Normal surface case:** Next we take the case where \( X \) is a normal surface. We define \( X_0 = \text{Proj}(R) \), which is a smooth curve since \( R \) is a normal, graded ring of dimension 2. Let \( H_0 \) be the ample Cartier divisor corresponding to \( \mathcal{O}_{X_0}(1) \). We define \( C_0 \) and \( D_0 \) as the divisors corresponding to \( C \) and \( D \) on \( X_0 \). Since \( C, D \) are integral, we have that \( C_0 \) and \( D_0 \) is each a single point on \( X_0 \). For all \( r \gg 0 \), we have

\[
\dim(R/I_C)_r = H^0(C_0, \mathcal{O}_{C_0}(r)) = \deg C_0 = 1.
\]

Thus \( m_C(1) = 1 \). Similarly \( m_D(1) = \deg D_0 = 1 \). Moreover, for \( r \gg 0 \), we can use Riemann-Roch to compute

\[
\dim R_r = H^0(X_0, \mathcal{O}_{X_0}(rH_0)) = (\deg H_0) \cdot r + O(1)
\]

and hence we also have \( m_R(1) = \deg H_0 \).

Blowing up the cone point of \( X \), we get a resolution \( \pi : \widetilde{X} \to X \) with exceptional divisor \( E \) where \( E^2 = -\deg H_0 = -m_R(1) \). We use Mumford’s definition (see [Ful98, Example 7.1.16] or [Mum61, II(b)]) to define

\[
(C \cdot D)_P := \pi^*_{\text{num}} C \cdot \pi^*_{\text{num}} D.
\]

where \( \pi^*_{\text{num}} C \) is the unique \( \mathbb{Q} \)-divisor on \( \widetilde{X} \) that pushes forward to \( C \) and that intersects the exceptional divisor in multiplicity 0, and where \( \pi^*_{\text{num}} D \) is similarly defined. It follows that \( \pi^*_{\text{num}} C = \widetilde{C} + \frac{1}{m_R(1)} E \) and similarly for \( D \). We then compute:

\[
\pi^*_{\text{num}} C \cdot \pi^*_{\text{num}} D = \widetilde{C} \cdot \widetilde{D} + \frac{1}{m_R(1)} \left( \widetilde{C} \cdot E + E \cdot \widetilde{D} \right) + \frac{1}{m_R(1)^2} (E^2) = 0 + \frac{2}{m_R(1)} - \frac{1}{m_R(1)} = \frac{1}{m_R(1)} = \frac{m_C(1)m_D(1)}{m_R(1)}.
\]

This is precisely equal to \( \chi(R/I_D, R/I_C)(1) \) by the proof of Theorem 0.1 from above. \( \square \)

3. Examples and remarks

**Example 3.1** (Rational normal cone). Let \( X \subseteq \mathbb{A}^{d+1} = \text{Spec}(k[x_0, \ldots, x_d]) \) be the cone over the rational normal curve of degree \( d \). Let \( L \) be the class of a line through the origin. Note that \( L \) is a \( \mathbb{Q} \)-Cartier divisor and \( L^2 = \frac{1}{d} \). Let \( R \) be the coordinate ring of \( X \), so that \( R \) is \( k[x_0, \ldots, x_d] \) modulo the \( 2 \times 2 \) minors of the matrix \( \begin{pmatrix} x_0 & x_1 & \ldots & x_{d-1} \\ x_1 & x_2 & \ldots & x_d \end{pmatrix} \). Let \( I_1 = \langle x_0, \ldots, x_{d-1} \rangle \) and \( I_2 = \langle x_1, \ldots, x_d \rangle \) be ideals of two different lines \( L_1 \) and \( L_2 \) on \( X \).
Figure 1. In Example 3.2, $X$ is the union of the two planes meeting at a point $P$. Two lines $V(I)$ and $V(J)$ intersect at $P$ with multiplicity $\frac{1}{2}$.

A direct computation (using e.g. Lemma 1.2) yields:

\[
\chi(R/I_1, R/I_2)(t) = 1 - (d - 1)t + (d - 1)^2t^2 - (d - 1)^3t^3 + (d - 1)^4t^4 - \ldots
\]

\[
= \frac{1}{1 + (d - 1)t}.
\]

Evaluating at $t = 1$ we get $\chi(R/I_1, R/I_2)(1) = \frac{1}{1 + (d - 1)} = \frac{1}{d}$.

Example 3.2. The intersection formulas from Corollary 0.2 make sense even when $X$ fails to be normal. For instance, let $R = \mathbb{Q}[x, y, z, w]/(xz, xw, yz, yw)$ be the coordinate ring of two planes in $\mathbb{A}^4$ intersecting at the origin. Let $I = (x, y, w)$ and $J = (y, z, w)$ be the ideals of two lines. Then

\[
\chi(R/I, R/J)(t) = (1 - 2t + 5t^2 - 12t^3 + 29t^4 - 70t^5 + 169t^6 - \ldots)
\]

\[
= \left(\frac{1}{1 + 2t - t^2}\right).
\]

Hence $\chi(R/I, R/J)(1) = \frac{1}{2}$.

Example 3.3. Let $X = \text{Spec}(k[x, y, z, w]/(xw - yz))$ be the cone over the quadric surface. Let $D = V(x, y)$ and $E = V(z, w)$, which are both planes lying in $X$. Note that $D \cap E$ is supported on the origin, so we have

\[
3 = \text{codim } D \cap E > \text{codim } D + \text{codim } E = 2.
\]

Write $I_D, I_E$ for the defining ideals of $D$ and $E$. We have

\[
\chi(R/I_D, R/I_E)(t) = 1 + t^2 + t^4 + \ldots
\]

\[
= \frac{1}{1 - t^2}.
\]

This has a pole at $t = 1$, hence $\chi(R/I_D, R/I_E)(1) = \infty$.

Remark 3.4. We conclude with several remarks:

(1) A similar rational function expression for the total Ext-module $\text{Ext}^p(M, N)$ is given in [ABS97] where it was used to recover and greatly expand on a ramification formula for modular invariants from [BCB95].

(2) In [BVS12], the authors use the series $\chi(M, N)(t)$ in their analysis of Hochster’s Theta pairing in the case of a graded hypersurface singularity. The authors go on to analyze the nongraded case as well, relating the Theta pairing to a linking number of associated cycles, thus proving a conjecture of Steenbrink.
(3) In [Gul86], Gulliksen considers another way to obtain a Serre-like intersection formula in the case of non-regular rings (and not just in the graded case). In particular, he proves that if you take a surjective map \( S \to R \), where \( S \) is a regular ring of minimal embedding dimension, then the resulting sum
\[
\chi^S(M, N) := \sum_i (-1)^i \dim \Tor^S_i(M, N)
\]
will not depend on the choice of the ring \( S \). In the graded case, this formula gives different results than the formula \( \chi(M, N)(1) \). For instance, in the example from the introduction, we could choose \( S := \mathbb{C}[x, y, z] \) and then
\[
\chi^S(R/I, R/J) = 3 - 3 = 0.
\]
In fact, this equals zero by [Ser00, IV.3] because \( \dim R/I + \dim R/J = \dim R < \dim S \), and something similar will hold for every non-trivial example where Corollary 0.2 yields a positive, rational multiplicity.

(4) Serre’s definition contains the implicit idea of attaching a multiplicity to an object like \( R/I \otimes_R R/J \). A similar idea appears in [GS09], where the authors use techniques for summing divergent series to show how to recover numerical data, like degree, in terms of homological data contained in what they call a semi-free resolution.

(5) For any flat family of graded modules \( M_t \) over a base \( T \), the Hilbert series \( M_t \otimes N \) does not depend on \( t \in T \). In particular, \( \chi(M, N)(1) \) is finite whenever \( \dim M + \dim N \leq \dim R \), regardless of whether \( M \otimes N \) has finite length. It is thus possible to define degenerate intersection multiplicities like \( \chi(R/I, R/I)(1) \).

(6) The proof of Theorem 0.3 relies on the fact that \( m_{R/I_D}(1) = c \cdot m_{R/I_D^2}(1) \) when \( R \) is normal and \( I_D \) is a cxdimension 1 ideal. This can fail when \( R \) is not normal. For instance, let \( R = \Spec(\mathbb{C}[x, y, z]/\langle y^2z - x^3 \rangle) \) and let \( D \) be the line defined by the ideal \( I_D = \langle x, y \rangle \). Then \( m_{R/I_D}(1) = 1 \) whereas \( m_{R/I_D^2}(1) = 3 \).

References

[AB93] Luchezar L. Avramov and Ragnar-Olaf Buchweitz, *Lower bounds for Betti numbers*, Compositio Math. 86 (1993), no. 2, 147–158. ↑1, 2, 3, 4

[ABS97] Luchezar L. Avramov, Ragnar-Olaf Buchweitz, and Judith D. Sally, *Laurent coefficients and Ext of finite graded modules*, Math. Ann. 307 (1997), no. 3, 401–415. ↑6

[BCB95] D. J. Benson and W. W. Crawley-Boevey, *A ramification formula for Poincaré series, and a hyperplane formula for modular invariants*, Bull. London Math. Soc. 27 (1995), no. 5, 435–440. ↑6

[BH93] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. ↑3

[BVS12] Ragnar-Olaf Buchweitz and Duco Van Straten, *An index theorem for modules on a hypersurface singularity*, Mosc. Math. J. 12 (2012), no. 2, 237–259, 459 (English, with English and Russian summaries). ↑1, 6

[DHM85] Sankar P. Dutta, M. Hochster, and J. E. McLaughlin, *Modules of finite projective dimension with negative intersection multiplicities*, Invent. Math. 79 (1985), no. 2, 253–291. ↑2

[Ful98] William Fulton, *Intersection theory*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998. ↑1, 3, 5

[GS09] Vassily Gorbounov and Vadim Schechtman, *Homological algebra and divergent series*, SIGMA Symmetry Integrability Geom. Methods Appl. 5 (2009), Paper 034, 31. ↑7
[Gul86] Tor H. Gulliksen, "A note on intersection multiplicities," Algebra, algebraic topology and their interactions (Stockholm, 1983), Lecture Notes in Math., vol. 1183, Springer, Berlin, 1986, pp. 192–194. ↑7

[Kol13] János Kollár, Singularities of the minimal model program, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge, 2013. With a collaboration of Sándor Kovács. ↑3

[Mum61] David Mumford, The topology of normal singularities of an algebraic surface and a criterion for simplicity, Inst. Hautes Études Sci. Publ. Math. 9 (1961), 5–22. ↑3, 5

[PS74] Christian Peskine and Lucien Szpiro, Syzygies et multiplicités, C. R. Acad. Sci. Paris Sér. A 278 (1974), 1421–1424 (French). ↑2

[Ser00] Jean-Pierre Serre, Local algebra, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2000. Translated from the French by CheeWhye Chin and revised by the author. ↑1, 7

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WI

E-mail address: derman@math.wisc.edu

URL: http://math.wisc.edu/~derman/