Mixed finite element formulation based on the discontinuous stress approximation

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Abstract. The paper is devoted to the development of the mixed form of finite element method for the calculation of constructions and structures. We consider the approach based on the approximation of discontinuous stress fields and the use of the penalty function method to satisfy the equilibrium equations. It is shown that the continuity of both normal and tangential stresses only on the adjacent sides of the finite elements contributes to the expansion of the class of statically admissible stress fields. At the same time, the consistent approximation is provided, both of the main part of the functional of additional energy, and its penalty part. Using the given approach, solutions of some problems were obtained and analyzed, in particular, the calculation of the cantilever beam-wall. The results obtained were compared with the decisions on the FEM in displacements and stresses, as well as with the exact solutions. It is shown that solutions based on the approximation of discontinuous stress fields in mixed form FEM converge to exact values from below and are more flexible in comparison with the solutions obtained by FEM in displacements. In addition, the accuracy of determining stresses is higher here.

1. Introduction

Nowadays, the calculation of structure is carried out, as a rule, using the finite element method (FEM). Many fundamental works have been devoted to the development of the theoretical principles of the FEM and questions of its application in structural mechanics [1-3]. They consider the basic variational principles and the corresponding FEM formulations, on the basis of which the finite element models can be constructed to solve different problems.

There are three main forms of FEM, each form is an analogue of one of the three classical methods of structural mechanics of rod systems – the displacement method, the force method, and the mixed method. The most widely used engineering calculations method is the FEM in displacement method form. The main unknowns in this case here are the displacements, the stresses are secondary and can be calculated by numerically differentiating the displacements. The finite element method in displacements is widely used to solve geometrically and physically nonlinear problems [4-6], problems with unilateral constraints [7-10], problems of stability and dynamics of structures [11-14].

The advantages and disadvantages of the displacement method are well known. The huge advantage of this FEM form is its complete formalization, as well as good stability of the solution with the guaranteed convergence to the lower boundary. However, the accuracy of stresses determination is much lower than the displacements, although it is the stress values which are more important in structural strength analysis. In addition, since the approximate solution in displacements corresponds to the lower boundary, the stress values are also underestimated relative to the exact values.
Castiliano’s minimum principle of additional energy and the related FEM schemes in the form of the force method, as well as the Reissner variational principle (mixed method), have not received such a wide application. However, in some cases, these approaches can be more effective, especially with the regard to calculating stresses. In addition, performing of dual calculations based on the alternative forms of FEM allows, as a rule, to obtain a two-side estimate of the exact solution of the problem [1].

The variational formulation of the mixed method is based on the principle of the stationarity of various forms of the Reissner functional. Some variants of the FEM in a mixed formulation were considered in [2, 3, 15-20]. With this approach, displacements and stresses within each FE are approximated simultaneously, therefore, there is no need to overestimate the requirements for the continuity of the desired functions and their derivatives. On the contrary, it is possible to set the necessary approximations, and since the mixed variational principles lead to a mixed form of relations between stresses and displacements for the finite element, more accurate solution can be obtained.

On the other hand, the Reissner functional is not convex; its surface at the point of stationary has a prominent saddle-like character. The system of resolving equations corresponding to the formulation of a mixed type is not symmetric and positive definite. Therefore, the approximate solution obtained by the FEM in the mixed form is characterized by some imbalance in the fulfillment of the equilibrium conditions and the conditions for the compatibility of deformations and, when crushing the grid, it can approach the exact solution both from below and above. These circumstances to some extent make the direct use of the Reissner functional in the finite element method more difficult [3].

This paper is devoted to the development of the mixed form FEM. It is proposed to use the approximation of discontinuous stress fields to construct a solution to the plane problem of the elasticity theory. Such an approach may have advantages in terms of more accurately fulfilling the equilibrium conditions, as well as in solving contact problems and some problems related to the stress concentration.

2. Statement and solution of the problem
Let’s consider a mixed finite element model that allows you to solve the problems of the elasticity theory directly both in displacements and stresses effectively. It is based on the approximation of discontinuous stress fields and using the penalty method to satisfy equilibrium equations. The variational formulation of the mixed problem corresponds to the principle of stationarity of various forms of the Reissner functional, which directly includes the components of displacements as stresses as well.

As already indicated, this functional is not convex; its surface at the stationary point has a prominent saddle-like character. This circumstance significantly complicates its use in the finite element method (the matrix of coefficients in this case is not symmetric and positive definite).

A convex mixed functional can be obtained by subtracting the Lagrange functional from the first form of the Reissner functional:

\[
\Pi(u, \sigma) = \Pi_{u}(u, \sigma) - \Pi_{l}(u) = \int_{\Omega} (\sigma)^T Au d\Omega - \frac{1}{2} \int_{\Omega} (\sigma)^T (E)^{1/2} \sigma d\Omega - \frac{1}{2} \int_{\partial \Omega} (Au)^T E Au d\Omega - \int_{\partial \Omega} (u - u_s)^T (L_s)^T \sigma dS. \tag{1}
\]

under the additional conditions: \((A)^T \sigma + \rho = 0 \in \Omega, \quad (L_s)^T \sigma = g_s \in S_s\).

The price for the convexity of the obtained functional (1) was to move the equilibrium conditions from natural (for \(\Pi_u(u, \sigma)\)) to additional ones. We use the penalty method to satisfy these conditions:

\[
\Pi(u, \sigma) = \int_{\Omega} (\sigma)^T Au d\Omega - \frac{1}{2} \int_{\Omega} (\sigma)^T (E)^{1/2} \sigma d\Omega - \frac{1}{2} \int_{\partial \Omega} (Au)^T E Au d\Omega + \\
+ \alpha \int_{\partial \Omega} [(A)^T \sigma + \rho]^T [(A)^T \sigma + \rho] d\Omega - \int_{\partial \Omega} (u - u_s)^T (L_s)^T \sigma dS. \tag{2}
\]
with the additional condition: 
\[(L_s)^T \sigma = g_s \in S_g.\]

Here is \(\sigma = \{\sigma_x, \sigma_y, \tau_{xy}\}^T\) – the stress vector; \(E\) – the stiffness coefficients matrix; \(A\) – differentiation operations matrix in the equilibrium equations; \(L_s\) – the direction cosine matrix of the external normal to the boundary \(S = S_g + S_u\) of the plane region \(\Omega\); \(g_s\) – the surface force vector at the boundary \(S_g\); \(u_s\) – the vector of given displacements at the \(S_u\); \(\rho = \{\rho_x, \rho_y\}^T\) – the volumetric force vector.

It is easy to see that the reduced functional is equivalent to the variational statement of the problem in a least squares form. This circumstance provides the convexity condition for the functional (2), which makes it convenient for applying the finite element method. Thus, setting an approximating expression for displacements \(u = N_u u^e\) and, accordingly, for stresses \(\sigma = N_\sigma \sigma^e\), and varying the functional (2) in a discrete form, we obtain the following matrix equilibrium equation for a finite element with a symmetric, positive definite coefficient matrix:

\[
[K^e_u] [u^e] + [K^e_{uu}] [u^e] = [R^e] + [U^e],
\]

where \(K^e_u = \int (AN_u)^T E A N_u \, d\Omega\), \(K^e_{uu} = \int (N_u)^T E^4 N_u \, d\Omega - \alpha \int (A^T N_u)^T A^T N_u \, d\Omega\) – are the stiffness and deformability matrixes; \(K^e_{uu} \neq K_{uu}^e = -\int (AN_u)^T N_\sigma \, d\Omega\) – is the mixed matrix; \(N_u, N_\sigma\) – are the form function matrixes for displacements and stresses, respectively; \(u^e, \sigma^e\) – are the vectors of nodal generalized displacements and stresses; \(R^e, U^e\) – are the vectors of generalized reactions and displacements in the nodes of the finite element.

When solving problems in stresses, the main restrictions on the smoothness of the desired functions are imposed by the equilibrium equations – it is necessary to ensure the existence of piecewise continuous derivatives of the stress components. Therefore, according to them, for the plane problem of the elasticity theory, the differentiability of the stresses \(\sigma_x\) only on \(x\), \(\sigma_y\) – only on \(y\), and \(\tau_{xy}\) – both on \(x\) and on \(y\) is required. Thus, normal stresses can undergo discontinuities at sites perpendicular to the element boundaries. In comparison with the continuous approximation, the use of such an assumption contributes to the expansion of the class of statically allowable stress fields, among which the solution is sought. This allows us to minimize the additional energy functional to a greater extent – as a result, the solution of the problem is, on average, closer to the exact one. The use of discontinuous approximations leads to the need to apply a special class of the finite elements. The location of the nodal points here should ensure that the conditions of discontinuity of normal stresses are satisfied.

One of the simplest elements of this type is a rectangle, four nodes at the vertices of which correspond to displacements and tangential stresses, two nodes on each of the sides correspond to normal stresses \(\sigma_x\), two nodes on the upper and lower sides correspond to stresses \(\sigma_y\). The approximation of displacements within a finite element can be similar to that usually used in the FEM in displacement method form. The stress distribution is given by the following approximating polynomials:

\[
\sigma_x = a_1 + a_2 x; \quad \sigma_y = a_3 + a_4 y; \quad \tau_{xy} = a_5 + a_6 x + a_7 y + a_8 xy.
\]

Having substituted the approximation data in the equilibrium equations, we obtain:

\[
\alpha_2 + \alpha_3 + \alpha_5 x + \rho_x = 0; \quad \alpha_3 + \alpha_4 + \alpha_7 y + \rho_y = 0.
\]

Obviously, for the constant volumetric element forces \(\rho_x, \rho_y\), the conditions (5) can be satisfied only if the following condition is identically satisfied: \(\alpha_6 x + \alpha_8 y = 0\).
Since this is not provided by the exact integration of the penalty term of functional (2), a method for artificially lowering the accuracy of calculating the integrals was proposed for the reduced finite element, which facilitates the zeroing of the expression in the integral part of the penalty term [8]. Despite the fact that this finite element gives the acceptable accuracy and convergence of the results of solving problems in stresses, there are significant disadvantages. First, to calculate the coefficients of the deformability matrix related to the penalty term, it is necessary to apply the procedure of numerical integration. Secondly, for the main part of functional (2) and its penalty term, different stress approximations are used, at least, on accuracy — for the penalty term, respectively, of a lower order than for the main part of the functional. The model proposed here for approximating discontinuous stress fields for a finite element of a plane problem of elasticity theory is free from these disadvantages.

Consider the above condition \( a_{y} x = a_{y} y = 0 \). For the finite element and approximation (4) described above, its exact execution is possible only when using the single-point integration scheme of the penalty term, i.e. when the integration order is lowered. However, if you locate nodes of the finite element on the \( x, y \) axes (figure 1 a), condition \( a_{y} x = a_{y} y = 0 \) is reduced to the equality \( a_{y} x = a_{y} y \).

Substituting this equality into approximation (4), we get the following polynomial variants for \( \tau_{xy} \):

\[
\tau_{xy} = a_{x} + a_{y} x + a_{x} y + a_{y} x^{2}; \quad \tau_{xy} = a_{x} + a_{y} x + a_{x} y + a_{y} y^{2}.
\]

For each of the polynomials (6), we define the form functions for nodes and average them. Then the nodal stress vector for the finite element \( \sigma^{e} = \{ \sigma_{2y}, \tau_{2xy}, \sigma_{4x}, \tau_{4xy}, \sigma_{6y}, \tau_{6xy}, \sigma_{8y}, \tau_{8xy} \} \) the following form functions will be a respond:

\[
N_{2y} = \frac{1}{2} - \frac{y}{2b}; \quad N_{2xy} = \frac{1}{4} - \frac{y}{2b} - \frac{x^{2}}{4a^{2}} + \frac{y^{2}}{4b^{2}}; \quad N_{4x} = \frac{1}{2} + \frac{x}{2a} - \frac{x^{2}}{4a^{2}} - \frac{y^{2}}{4b^{2}}; \quad N_{4xy} = \frac{1}{4} + \frac{x}{2a} + \frac{x^{2}}{4a^{2}} - \frac{y^{2}}{4b^{2}};
\]

\[
N_{6y} = \frac{1}{2} + \frac{y}{2b}; \quad N_{6xy} = \frac{1}{4} + \frac{y}{2b} - \frac{x^{2}}{4a^{2}} + \frac{y^{2}}{4b^{2}}; \quad N_{8x} = \frac{1}{2} - \frac{x}{2a}; \quad N_{8xy} = \frac{1}{4} - \frac{x}{2a} + \frac{x^{2}}{4a^{2}} - \frac{y^{2}}{4b^{2}}.
\]

The approximation of the tangential stresses within a given finite element, in particular, for the shape function \( N_{8y} \), is shown in figure 1 b.

![Figure 1. Eight-node finite element with discontinuous stress approximation.](image)

With helping the program that implements the FEM schemes discussed above, numerical studies and comparative analysis of finite element solutions based on the formulation of the displacement method, the force method, and the mixed method were performed.

3. Results and discussion

In order to compare various forms of the finite element method, the problem of determining the stress-strain state of the cantilever plate was solved (figure 2 a). Grids of different densities were used, for example, in figure 2, b shows the the 2×3 grid when calculating the plate by the mixed method.
Figure 2. The problem of the cantilever plate.

The values of stresses and displacements obtained using the approaches considered were compared with the results of the FEM calculation in the displacement method form. Table 1 shows the vertical displacements $\nu$ of the middle of the free edge ($x = 12m, y = 0$) and the stress $\sigma_x$ at the lower point of the fixed edge of the plate ($x = 0, y = 3m$), obtained on different grids of the finite elements.

| Grid | Displacement method | Force method | Mixed method |
|------|---------------------|--------------|--------------|
|      | $\nu$               | $\sigma_x$   | $\nu$       |
|      | $p$                 | $p$          | $p$          |
|      | $Eh$                | $h$          | $Eh$         |
|      | Number of unknowns  | Number of unknowns | Number of unknowns |
| 2x2  | 12 152.5 6.67 24 16.63 42 156.2 15.97 | 3x2 18 156.2 6.83 34 16.20 58 183.8 15.74 | 3x3 24 184.4 8.94 48 14.27 80 189.4 14.39 |
| 4x3  | 40 200.8 10.34 62 14.42 102 197.6 14.52 | 5x4 70 213.1 11.70 98 14.56 158 211.7 14.72 | 6x4 96 216.6 12.19 116 14.90 186 215.9 14.81 |
| Exact solution | 225.0 15.10 | 15.10 | 225.0 15.10 |

The calculation results for the given variants of the finite element grids show the following. FEM in the force method form provides a monotonic convergence of the solution in stresses to exact values from above. The solution obtained by the mixed method, when crushing the grid, approaches the exact solution alternately, then from above, then from below. Moreover, the use of discontinuous approximations of stress fields allows us to achieve better convergence and accuracy of the solution with the same order of the system of resolving equations.

As it can be seen, on rough grids, the stresses obtained by the FEM in the force method form are significantly closer to the exact values than those obtained by the FEM in displacements. Under crushing the grid (with the equivalent dimension of the problem), the accuracy of determining the stress by the force method and the mixed method also remains higher than by the displacement method. So, with the number of unknowns of the order of 100, the values of the stresses obtained by the force method differ from the exact values by 3.6%, the mixed method by 3.8%, and the displacement method by 19.3%. In turn, for the identical grids, the mixed FEM allows one to obtain stresses with the same accuracy as FEM in stresses, and displacements – as FEM in displacements.

Figure 3 illustrates the convergence of solutions obtained using various forms of the FEM. Obviously, according to the accuracy of determining the stresses, the approaches considered here are more effective in comparison with the traditional FEM scheme in displacements.
4. Conclusion
Summarizing the results of numerical researches, we can draw the following conclusions:
1. The stress values in finite element solutions built on the stress approximation are much closer to exact values than those obtained in the displacement method formulation.
2. The use of discontinuous stress approximations contributes to the expansion of the class of statically admissible stress fields. In this case, the same approximation is provided for both the main part of the additional energy functional and its penalty part. This allows us to obtain more accurate stress values as compared to the continuous stress approximation.
3. The mixed finite-element models allow us to obtain alternative solutions to the problems under consideration, thereby ensuring their greater validity and reliability. Under identical grids, the mixed finite element method allows us to obtain stresses with the same accuracy as the FEM in stresses, and displacements – as the FEM in the displacement method form.

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