Coherent Permutations with Descent Statistic and the Boundary Problem for the Graph of Zigzag Diagrams

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Abstract

The graph of zigzag diagrams is a close relative of Young’s lattice. The boundary problem for this graph amounts to describing coherent random permutations with descent–set statistic, and is also related to certain positive characters on the algebra of quasisymmetric functions. We establish connections to some further relatives of Young’s lattice and solve the boundary problem by reducing it to the classification of spreadable total orders on integers, as recently obtained by Jacka and Warren.

1 Introduction

Young’s lattice $\mathcal{Y} = \bigcup_{n \geq 0} \mathcal{Y}_n$ is the graded graph whose vertices are Young diagrams, with the grading determined by the number of boxes in diagram, and with the neighbourship relation induced by the inclusion of diagrams [12 p. 288]. We write $\mu \rightarrow \lambda$ when $\lambda$ is an immediate successor of $\mu$ in $\mathcal{Y}$, that is, $\lambda$ is obtained from $\mu$ by adding a box. A path $\lambda_0 \rightarrow \ldots \rightarrow \lambda_n$ in $\mathcal{Y}$ starting with empty diagram $\lambda_0 = \emptyset$ encodes a standard Young tableau of the shape $\lambda_n \in \mathcal{Y}_n$; such a path will be called a standard path in $\mathcal{Y}$.

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dimension function $d$ on $\mathcal{Y}$ which counts the number of standard paths with a given end is uniquely determined by the recursion

$$d(\lambda) = \sum_{\mu : \mu \triangleright \lambda} d(\mu), \quad d(\emptyset) = 1, \quad (1)$$

whose explicit solution is given by Frobenius’ formula or the famous hook formula. Many questions of asymptotic combinatorics and representation theory [25, 27] involve the recursion

$$p(\mu) = \sum_{\lambda : \mu \triangleright \lambda} p(\lambda), \quad p(\emptyset) = 1, \quad (2)$$

which is dual to (1). The boundary problem for $\mathcal{Y}$ asks to describe the convex set of nonnegative solutions to (2), especially the set $\partial \mathcal{Y}$ of extreme solutions which we call the boundary.

The boundary problem has many facets. One interpretation is that there is a bijection $p \leftrightarrow \psi$ between nonnegative solutions $p$ to (2) and linear functionals $\psi : \text{Sym} \rightarrow \mathbb{R}$ on the algebra Sym of symmetric functions, such that $\psi$ vanishes on a particular ideal and assumes nonnegative values on the cone spanned by the basis of Schur functions. By this bijection, the extreme solutions $p$ correspond exactly to multiplicative functionals $\psi$ (such $\psi$ will be called characters.) By another interpretation, a nonnegative solution to (2) is a probability function, which determines the distribution of a Markov chain $X = (X_n)$ with $X_n \in \mathcal{Y}_n$, $n = 0, 1, 2, \ldots$, by virtue of the formula

$$\mathbb{P}(X_0 = \lambda_0, \ldots, X_n = \lambda_n) = p(\lambda_n), \quad \text{for } \emptyset = \lambda_0 \triangleright \cdots \triangleright \lambda_n \in \mathcal{Y}_n. \quad (3)$$

A central result on the boundary problem for Young’s lattice is the identification of $\partial \mathcal{Y}$ with the infinite ‘bi–simplex’

$$\Delta^{(2)} = \{ (\alpha, \beta) : \alpha = (\alpha_j), \beta = (\beta_j), \quad \alpha_1 \geq \alpha_2 \geq \cdots \geq 0, \beta_1 \geq \beta_2 \geq \cdots \geq 0, \quad \sum (\alpha_j + \beta_j) \leq 1 \}, \quad (4)$$

as emerged in the work of Edrei on total positivity [10] and Thoma on characters of the infinite symmetric group [45], where the parameters $(\alpha, \beta)$ appeared as the collections of poles and zeroes of the generating function.
$\Sigma p(n)z^n$, with $p$ evaluated at one–row diagrams $(n)$. See \cite{35} for a related interpretation in terms of a problem of moments. A probabilistic meaning of the parameters was discovered by Vershik and Kerov \cite{46,47}, who recognised in $\alpha_j$ and $\beta_j$ the asymptotic frequencies of the $j$th largest row and the $j$th largest column, respectively, for a diagram $X_n$ that follows the distribution corresponding to $(\alpha, \beta)$. In Section 4 we exhibit yet another construction of $\partial \mathcal{Y}$ based on the comultiplication in $\text{Sym}$, as was communicated to the second author by Sergei Kerov many years ago.

The boundary problems for relatives of Young’s lattice have been intensively studied. Among them is the graph $\mathcal{P}$ of partitions of integers, which has the same vertex set as $\mathcal{Y}$ but multiple edges, as dictated by the branching of partitions of finite sets $[n] := \{1, \ldots, n\}$; so that the analogue of (2) involves some integer coefficients in the right–hand side. From Kingman’s work on partition structures \cite{29,30} we know that the boundary $\partial \mathcal{P}$ is the simplex

$$\Delta = \{ \alpha = (\alpha_j) : \alpha_1 \geq \alpha_2 \geq \ldots \geq 0, \sum \alpha_j \leq 1 \}.$$  

For $X = (X_n)$ a Markov chain on $\mathcal{P}$ corresponding to $\alpha \in \Delta$, the parameter $\alpha_j$ has the same meaning of a row frequency as in the case of Young’s lattice, but the symmetry between rows and columns, which holds for $\mathcal{Y}$, breaks down for $\mathcal{P}$ and only the longest column may have a positive frequency $1 - \Sigma \alpha_j$. The boundary problem for $\mathcal{P}$ may be also stated in terms of characters $\psi : \text{Sym} \to \mathbb{R}$ but this time $\psi$ must be nonnegative on a larger cone in $\text{Sym}$ spanned by the monomial symmetric functions (whence the embedding $\partial \mathcal{P} \hookrightarrow \partial \mathcal{Y}$ which sends $\alpha \in \Delta$ to $(\alpha, 0) \in \Delta(2)$). See \cite{27} for a parametric family of graded graphs which bridge between $\mathcal{Y}$ and $\mathcal{P}$.

The graph of compositions $\mathcal{C}$ has compositions of integers as vertices, and its standard paths of length $n$ correspond to ordered partitions of $[n]$. This graph extends $\mathcal{P}$ in the sense that there is a projection $\mathcal{C} \to \mathcal{P}$ which amounts to discarding the order of parts in composition. The boundary problem for $\mathcal{C}$ is related to the characters $\psi : \text{QSym} \to \mathbb{R}$ of the algebra $\text{QSym}$ of quasisymmetric functions, with the property that $\psi$ must be nonnegative on the cone spanned by the basis of monomial quasisymmetric functions. In \cite{16} the boundary $\partial \mathcal{C}$ was identified with the space $\mathcal{U}$ of open subsets $U \subset [0,1]$ of the unit interval. The connection between the graphs entails a projection $\partial \mathcal{C} = \mathcal{U} \to \partial \mathcal{P} = \Delta$ which assigns to a generic open set $U$ the decreasing sequence $\alpha$ of sizes of its interval components; thus $U$ can be viewed as an arrangement of Kingman’s frequencies $\alpha_j$ in some order. There is an explicit
construction of Markov chains on $P$ and $C$ by means of a simple sampling scheme \cite{29,30,16,20,39}.

Thus, Kingman’s boundary $\partial P$ is a part of the larger Edrei–Thoma–Vershik–Kerov boundary $\partial Y$, and it is also a shadow of the boundary $\partial C$ of the graph of compositions. This suggests to seek for a combinatorial structure to complete the diagram

$$
\partial ? \leftarrow \partial C \\
\downarrow \quad \downarrow \\
\partial Y \leftarrow \partial P
$$

In this paper we argue that a good candidate to fill the gap is the graph of zigzag diagrams $Z$, because $Z$ links to $Y$ in a way very similar to the relation between $C$ and $P$. The vertices in $Z$ are again compositions, which we represent as zigzag diagrams, and the configuration of edges is selected so that each standard path of length $n$ encodes a permutation of $[n]$ with a given set of descents. Alternatively, the vertices of $Z$ can be described as the words in a two–letter alphabet with a natural interpretation of the relation $\mu \nearrow \lambda$, as explained in the next section. In this interpretation, the graph $Z$ is known under the name of subword order and it was studied long ago, see \cite{18,14}. Together with the infinite binary tree, the graph $Z$ forms a dual pair of graphs in the sense of \cite{12,13} (we thank Sergey Fomin for this remark). Recently, the graph $Z$ was considered in \cite{5,40}. However, all the papers listed above are concerned with other problems, different from the boundary problem studied here. Notice also that $Z$ differs from other graphs of compositions \cite{38,26,3} by the configuration of edges.

Although $Z$ cannot be literally projected on $Y$, like $C$ on $P$, their relation fits in the diagram of algebra homomorphisms which respect the cones spanned by the distinguished bases of symmetric functions (s.f.) and quasisymmetric functions (q.s.f):

$$(\text{QSym}, \text{fundamental q.s.f.}) \longrightarrow (\text{QSym}, \text{monomial q.s.f.})$$

$$
\uparrow \quad \uparrow
$$

$$(\text{Sym}, \text{Schur s.f.}) \longrightarrow (\text{Sym}, \text{monomial s.f.})$$

We show that the comultiplication in QSym can be exploited to construct the boundary $\partial Z$, which turns out to be homeomorphic to the space $U^{(2)}$ of
pairs \((U_\uparrow, U_\downarrow)\), where \(U_\uparrow\) and \(U_\downarrow\) are disjoint open subsets of \([0, 1]\). The projection \(\partial Z \to \partial Y\) amounts to the separate record \((U_\uparrow, U_\downarrow) \mapsto (\alpha, \beta)\) of sizes of interval components of the open sets in decreasing order, while the embedding \(\partial C \hookrightarrow \partial Z\) is just \(U \mapsto (U, \emptyset)\). An extension of the paintbox scheme allows to describe the corresponding Markov chains on \(Z\). In loose terms, the oriented paintbox generates a random permutation by grouping \(n\) integers in some number of clusters and arranging the integers within each cluster in either increasing or decreasing order. To show that our construction yields a complete description of the boundary we reduce the boundary problem to a recent characterisation, due to Jacka and Warren [22], of spreadable total orders on \(Z\) (a random total order is spreadable if its probability distribution is invariant under increasing mappings \(Z \to Z\)).

The rest of the paper is organised as follows. In Section 2 we introduce the graph of zigzag diagrams \(Z\). In Section 3 we explain how \(Z\) is related to the algebra QSym. In Section 4 we describe Kerov’s construction of the boundary \(\partial Y\) of the Young graph, and then in Section 5 we apply Kerov’s method to building a part of the boundary \(\partial Z\), which we call the finitary skeleton of \(\partial Z\). In Section 6, using a natural continuity argument, we extend the finitary skeleton to a larger set of boundary points, indexed by oriented paintboxes \((U_\uparrow, U_\downarrow)\). In Section 7 we show that our construction actually yields the whole boundary \(\partial Z\); this is the main result of the paper (Theorem 44). The proof of the completeness claim amounts to a reduction to the remarkable result by Jacka and Warren [22] on spreadable orders. In Section 8 we formulate some complementary results and open problems.

2 The graph of zigzag diagrams

Recall that a composition of \(n\) is a finite sequence \(\lambda = (\lambda_1, \ldots, \lambda_\ell)\) of strictly positive integers whose sum \(|\lambda| = \lambda_1 + \cdots + \lambda_\ell\) equals \(n\). Compositions have a useful graphical representation, as in the following definition.

**Definition 1.** By a zigzag diagram (or simply zigzag) we understand a finite collection of unit boxes such that the \(j\)th box is appended either to the right or below the \((j - 1)\)th box. (Thus, each zigzag comes with a canonical enumeration of its boxes, from left to right and from top to bottom.) We will identify compositions of \(n = 1, 2 \ldots\) and zigzag diagrams with \(n\) boxes: given a composition \(\lambda = (\lambda_1, \ldots, \lambda_\ell)\), the corresponding zigzag shape is drawn so that its \(j\)th row (counted from top to bottom) consists of \(\lambda_j\) boxes, \(j =\)
Given a zigzag diagram $\lambda$, the \textit{conjugate} zigzag $\lambda'$ is obtained by reflecting the zigzag $\lambda$ about the bisectrix of the first quadrant. See Figure 1 for an example.

![Figure 1: The zigzag shape $\lambda = (3, 1, 4)$ and its conjugate $\lambda' = (1^3, 3, 1^2)$](image)

The identification of compositions with zigzag diagrams is similar to the identification of partitions with Young diagrams: both graphical representations unveil a symmetry that appears by transposition of Young diagrams or conjugation of zigzags. Note also that the shape obtained by reflecting a zigzag $\lambda$ about the horizontal axis is a ribbon Young diagram, as defined in [42, p. 345].

There exists an alternative encoding of compositions as \textit{binary words}. Take an alphabet with two symbols, say $\{+, -\}$, and assign to zigzag $\lambda$ a word $w(\lambda) = w_1 \ldots w_{n-1}$ in this alphabet, as follows: for $j = 1, \ldots, n - 1$ let $w_j = +$ if the $j$th box in $\lambda$ is separated from the $(j + 1)$th box by a vertical edge, and let $w_j = -$ otherwise. To illustrate, for $\lambda = (3, 1, 4)$, as on Figure 1, the word is $w(\lambda) = + + - - + + +$. The correspondence between zigzags with $n$ boxes and binary words of length $n - 1$ is a bijection (for $n = 1$, the zigzag with one box is represented by the empty word). As an obvious corollary we obtain that the total number of zigzags with $n$ boxes equals $2^{n-1}$. In terms of binary words, the conjugation $\lambda \rightarrow \lambda'$ amounts to switching $+ \leftrightarrow -$ and reading the resulting word in the reverse order.

**Definition 2.** Let $\pi_n$ be a permutation of the set $[n] := \{1, \ldots, n\}$, written $\pi_n = \pi_n(1) \ldots \pi_n(n)$ in the usual one–row notation. The \textit{zigzag shape} of $\pi_n$, denoted $zs(\pi_n)$, is a zigzag $\lambda$ with $n$ boxes such that in the corresponding binary word $w(\lambda)$, $w_j(\lambda) = +$ if $\pi_n(j) < \pi_n(j + 1)$ and $w_j(\lambda) = -$ if $\pi_n(j) > \pi_n(j + 1)$ ($j = 1, \ldots, n - 1$). Thus, $zs(\pi_n) = \lambda$ means that the numbers $\pi_n(1), \ldots, \pi_n(n)$ inscribed consecutively in the boxes of $\lambda$ increase from left to right along the rows and decrease down the columns. For example, $zs(\pi_8) = (3, 1, 4)$ for $\pi_8 = 13842567$. 

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This definition can be rephrased as follows. Each permutation $\pi_n$ can be seen as a succession of increasing runs. The last position of each increasing run, except the last run, is called a descent. For example, the permutation 13842567 breaks in three increasing runs 138, 4, 2567 and has two descents 3 and 4 corresponding to the entries $\pi_8(3) = 8$ and $\pi_8(4) = 4$. The sizes of consecutive runs comprise the composition $\lambda = zs(\pi_n)$. Each of the descent positions in $\pi_n$ corresponds to the rightmost box in some row of $\lambda$ (different from the last row), or can be associated with a horizontal edge separating two boxes. Clearly, the zigzag shape of $\pi_n$ is uniquely determined by the descent positions. This encoding of descents by zigzags goes back to MacMahon [33, Ch. IV, Section 156].

The zigzag shape of $\pi_n$ can also be determined from the inverse permutation $\pi^{-1}$ by recording the sizes of increasing subsequences of consecutive integers. For instance, for 13842567 the inverse permutation $\pi^{-1} = 15246783$ breaks into subsequences 123, 4, 5678 of sizes (3, 1, 4).

We need a few more definitions. For $\pi_n$ a permutation of $[n]$ and $\pi_m$ a permutation of $[m]$ with $m < n$, we say that the permutations are coherent if the integers 1, $\ldots$, $m$ appear in $\pi_n$ in the same relative order as in $\pi_m$; we may also say that $\pi_n$ extends $\pi_m$, and that $\pi_m$ is a restriction of $\pi_n$. In particular, $\pi_n$ extends $\pi_{n-1}$ if $\pi_n$ can be obtained by inserting element $n$ in the beginning, in the end or between any two elements of the row $\pi_{n-1}(1) \ldots \pi_{n-1}(n-1)$.

An infinite sequence of coherent permutations $\pi = (\pi_n)$ defines a total order on $\mathbb{N} := \{1, 2, \ldots\}$ and will be also called an arrangement, each $\pi_n$ is then the restriction of the arrangement $\pi$ to $[n]$. We will use symbol $<$ for arrangement as a relation on $\mathbb{N}$. Thus $i < j$ means that $\pi^{-1}_n(i) < \pi^{-1}_n(j)$ for every $n \geq \max(i, j)$. It is useful to note that $\pi = (\pi_n)$ is uniquely determined by a single integer sequence $(\pi^{-1}_n(n), n = 1, 2, \ldots)$ assuming values in $[1] \times [2] \times \ldots$. We call $r_n := \pi^{-1}_n(n)$ the initial rank of $n$, meaning that for $\pi^{-1}_n(n) = r$ the integer $n$ is the $r$th $<$-smallest in $[n]$. Representing $\pi$ via initial ranks is rather convenient because passing from $\pi_n$ to $\pi_{n+1}$ just amounts to appending $r_{n+1}$ to the row $r_1, \ldots, r_n$.

Let $Z_0 = \{\emptyset\}$ be the one–element set which contains the empty zigzag $\emptyset$, and for $n \geq 1$ let $Z_n$ be the set of all $2^{n-1}$ zigzags with $n$ boxes.

**Definition 3.** The graph of zigzag diagrams has the set of vertices $Z := \cup_{n \geq 0} Z_n$, and the set of edges defined by the following neighbourhood relation, further denoted $\nearrow$. Let $\emptyset \nearrow (1)$, and for $n \geq 2$ and two zigzags $\mu \in Z_{n-1}$, $\lambda \in Z_n$ let $\mu \nearrow \lambda$ if $\mu = zs(\pi_{n-1})$ and $\lambda = zs(\pi_n)$ for some coherent
permutations $\pi_{n-1}$ and $\pi_n$. We use the same symbol $Z$ to denote both the
graph and its set of vertices.

Thus, $Z$ is an infinite graded graph which also can be viewed as a graded
(or ranked) poset. Spelled out in detail, in terms of compositions, the imme-
diate followers of $(\mu_1, \ldots, \mu_\ell)$ are compositions $(\mu_1+1, \ldots, \mu_\ell), \ldots, (\mu_1, \ldots, \mu_\ell+1)$, the composition $(1, \mu_1, \ldots, \mu_\ell)$, and all compositions
$$(\mu_1, \ldots, \nu_j+1, \nu_{j+1}, \ldots, \mu_\ell)$$
obtained by splitting a part $\mu_j$ in two positive parts $\nu_j + \nu_{j+1} = \mu_j$ and
incrementing the first of these two parts by 1. Equivalently, in terms of
zigzags, we split a zigzag $\mu$ into two zigzag subdiagrams, $\mu(1)$ and $\mu(2)$, then
according as the edge between them is horizontal or vertical, we shift $\mu(2)$ by
one unit to the right or down and finally add a new box in between. In the
two extreme cases when either $\mu(1)$ or $\mu(2)$ is empty, this means appending
a new box either above the first box of the zigzag or to the right of its last
box. Note that there are $n$ immediate followers $\lambda$ for any $\mu \in Z_{n-1}$. A few
levels of the graph are shown on Figure 2.

![Figure 2: The beginning of the zigzag graph](image)

**Remark 4.** In terms of binary words, the relation $\mu \nearrow \lambda$ (for $\mu \neq \emptyset$) holds if
and only if $w(\mu)$ is a subword of $w(\lambda)$, that is, the word $w(\mu)$ can be obtained
from the word $w(\lambda)$ by deleting one of the symbols. It follows that for any
zigzag $\lambda$ with $|\lambda| \geq 2$, the number of vertices $\mu \nearrow \lambda$ equals the total number
of plus– and minus–clusters in $w(\lambda)$. In the ‘word’ realisation, the graph $Z$
is known under the name *subword order*, see [48, 4].
We define a standard path of length \( n \) in the graph \( \mathcal{Z} \) as a sequence \( \lambda_0 \rightarrow \ldots \rightarrow \lambda_n \) starting at the root \( \lambda_0 = \emptyset \). Likewise, a standard infinite path is an infinite sequence of neighbouring zigzags \( \lambda_0 \rightarrow \lambda_1 \rightarrow \lambda_2 \ldots \).

**Proposition 5.** For each \( n \) and \( \lambda_n \in \mathcal{Z}_n \) there is a canonical bijection between the permutations of \([n]\) with the zigzag shape \( \lambda_n \) and the standard paths in \( \mathcal{Z} \) of length \( n \) leading to \( \lambda_n \). These bijections are consistent for various values of \( n \), hence define a bijection between the set of arrangements and the set of infinite standard paths in \( \mathcal{Z} \).

**Proof.** Note that the position of \( n \) in a permutation of \([n]\) is always a descent, unless this is the last position. From this, inspecting all permutations \( \pi_{n-1} \) extending a given permutation \( \pi_{n-1} \) with zs(\( \pi_{n-1} \)) = \( \lambda_{n-1} \), we see that zs(\( \pi_n \)) is determined by \( \lambda_{n-1} \) and the position of \( n \) in \( \pi_n \), and that the correspondence between the extensions of \( \pi_{n-1} \) and the successors of \( \lambda_{n-1} \) in \( \mathcal{Z} \) is bijective. The claim follows by induction in \( n \). \( \square \)

Proposition 5 becomes less evident when \( \mathcal{Z} \) is introduced as the subword order. In this form the statement was proved in [48].

The dimension function \( d \) on \( \mathcal{Z} \) counts the number of permutations with a given descent set (equivalently, with a given zigzag shape). It is easily seen that for a zigzag \( \lambda \in \mathcal{Z}_n \), the dimension \( d(\lambda) \) equals the number of standard Young tableaux whose shape is the ribbon Young diagram \( \bar{\lambda} \), the image of \( \lambda \) under the reflection about a horizontal axis. Passing to the conjugate zigzag \( \lambda \mapsto \lambda' \) yields an involutive automorphism of \( \mathcal{Z} \) hence preserves the dimension.

With the graph of zigzag diagrams we associate recursions of the same form (1), (2) as for Young’s lattice, but with understanding that \( \lambda, \mu \in \mathcal{Z} \). Obviously, we may equally well write the normalisation condition for (2) as \( p((1)) = 1 \).

**Definition 6.** Let \( \mathcal{V} \) denote the convex set of nonnegative solutions to recursion (2) where \( \lambda \) and \( \mu \) range over \( \mathcal{Z} \). We call the subset of extreme solutions the boundary of \( \mathcal{Z} \) and denote it \( \partial \mathcal{Z} \).

Note that \( p(\lambda) \leq 1 \) for any \( \lambda \in \mathcal{Z} \). Using this observation it is readily seen that \( \mathcal{V} \) is compact and metrisable in the product topology (which corresponds to the pointwise convergence of functions on \( \mathcal{Z} \)). By a well-known general theorem, it follows that \( \partial \mathcal{Z} \) is a \( G_\delta \) subset of \( \mathcal{V} \), hence a Borel space.
Moreover, by the very definition $V$ is a projective limit of finite-dimensional simplices. By [21, p. 164, Proposition 10.21] this implies that the set $V$ is a Choquet simplex: that is, any its point is uniquely representable as a convex mixture of extremes. Thus, there is a canonical bijection between $V$ and the set of all probability measures (‘mixing measures’) on the space $\partial \mathbb{Z}$.

There is an equivalent probabilistic description of the set $V$ which is useful for our purposes. Given a probability measure on the set of arrangements we shall speak of a random arrangement $\Pi = (\Pi_n)$ which is a sequence of coherent random permutations. By an easy application of Kolmogorov’s measure extension theorem, the law of $\Pi$ (which is the joint distribution of the $\Pi_1, \Pi_2, \ldots$) is uniquely determined by the marginal distributions of the $\Pi_n$’s. The following proposition determines a family of random arrangements $\Pi = (\Pi_n)$ for which the descent set of $\Pi_n$ is a sufficient statistic, for $n = 1, 2, \ldots$

**Proposition 7.** The formula

$$P(\Pi_n = \pi_n) = p(zs(\pi_n)), \quad n = 1, 2, \ldots \tag{5}$$

establishes an affine homeomorphism $p \leftrightarrow \Pi$ between $V$ and the convex set of random arrangements $\Pi = (\Pi_n)$ with the property that for every $n$ all values of $\Pi_n$ with the same zigzag shape are equally likely.

**Proof.** This is a direct consequence of the definitions. Indeed, given $p \in V$, we define, for each $n = 1, 2, \ldots$ a random permutation $\Pi_n$ of $[n]$ with the probability law $P_n(\Pi_n = \pi_n) = p(zs(\pi_n))$. The relation (2) just means that, for every $n$, the measures $P_n$ and $P_{n-1}$ are consistent with the projection $\pi_n \mapsto \pi_{n-1}$ (removing $n$ from $\pi_n$), hence, by Kolmogorov’s theorem, $(P_n)$ uniquely determines a probability measure $P$ on the set of arrangements, that is, a random arrangement $\Pi$. Clearly, such $\Pi$ satisfies (3). Conversely, if a random arrangement $\Pi = (\Pi_n)$ with distribution $P$ satisfies (3) then (2) also holds by the rule of addition for probabilities. \(\square\)

In slightly different terms, not uncommon in this area of combinatorial probability [39], we understand each $p \in V$ as a probability function, which by the formula

$$P(X_n = \lambda) = p(\lambda)d(\lambda), \quad \lambda \in \mathbb{Z}$$

and an obvious analogue of (3) determines the probability law of a transient Markov chain $X = (X_n)$ whose state at time $n = 0, 1, \ldots$ is an element of $\mathbb{Z}_n$. The process $X$ viewed in the inverse time $n = \ldots, 2, 1, 0$ is again a Markov
chain with transition probabilities not depending on a particular choice of $p$, namely
\[ \mathbb{P}(X_{n-1} = \mu \mid X_n = \lambda) = 1(\mu \prec \lambda) \frac{d(\mu)}{d(\lambda)}, \]
where $1(\cdots)$ equals 1 when $\cdots$ is true and equals 0 otherwise. By the Markov property, the probability law of $(X_n, \ldots, X_0)$ is completely determined by the law of $X_n$, and the latter may be viewed as a mixture of Dirac distributions on $\mathbb{Z}_n$.

Loosely speaking, a mixing measure on $\partial \mathbb{Z}$ may be seen as a kind of initial distribution for the time-reversed process $\ldots, X_2, X_1, X_0$ which ‘starts at the infinite level of $\mathbb{Z}$’. By the general theory of Markov chains [24], $\partial \mathbb{Z}$ is a subset of the larger entrance Martin boundary for the inverse chain, and in Section 8 we conjecture that the boundaries coincide. Our purpose is to describe, as explicitly as possible, the boundary $\partial \mathbb{Z}$ of the graph of zigzag diagrams and the associated random processes on $\mathbb{Z}$.

3 The algebra $\text{QSym}$

The boundary problem for $\mathbb{Z}$ is intrinsically related to the algebra $\text{QSym}$ of quasisymmetric functions. Usually $\text{QSym}$ is introduced as a graded algebra of formal power series in infinitely many indeterminates $x_1, x_2, \ldots$. Specifically, elements of $\text{QSym}$ are finite linear combinations of the quasisymmetric monomial functions
\[ M_\lambda(x) = \sum_{1 \leq i_1 < \cdots < \lambda \ell < \infty} x_{i_1}^{\lambda_1} \cdots x_{i_\ell}^{\lambda_\ell}, \]
where $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is an arbitrary composition (possibly, the empty one). The grading is defined by setting $\deg M_\lambda = |\lambda|$. Clearly, $\text{QSym}$ contains the algebra $\text{Sym}$ of symmetric functions as a graded subalgebra.

In $\text{QSym}$ there is another homogeneous basis formed by the fundamental quasisymmetric functions $F_\lambda$, see [13, 42, 44]:
\[ F_\lambda = M_\lambda + \sum_{\mu \text{ is thinner than } \lambda} M_\mu, \]
where ‘$\mu$ is thinner than $\lambda$’ means that the composition $\mu$ can be obtained from $\lambda$ by splitting some parts of the composition $\lambda$ into smaller parts, so...
that at least one part \( \lambda_j \) is replaced by a nontrivial composition of \( \lambda_j \). For example,

\[
F_{(2,2)} = M_{(2,2)} + M_{(1,1,2)} + M_{(2,1,1)} + M_{(1,1,1,1)}.
\]

Note that \( \deg F_\lambda = |\lambda| \). The role of functions \( F_\lambda \) is similar to that of Schur functions in the algebra \( \text{Sym} \).

For most of our purposes it suffices to simply regard \( \text{QSym} \) as a commutative unital \( \mathbb{R} \)-algebra with distinguished basis \( \{ F_\lambda, \lambda \in \mathcal{Z} \} \) indexed by compositions, including the unity \( F_\emptyset = 1 \). The multiplication rule in this basis is the following. For \( \pi_k \) a permutation of \([k]\) and \( \pi_l \) a permutation of \([l]\) let \( \text{shuffle}(\pi_k, \pi_l) \) be the set of permutations \( \pi_{k+l} \) of \([k+l]\) such that \( \pi_k \) and \( \pi_{k+l} \) are coherent, and the relative order of integers \( k + 1, \ldots, k + l \) in \( \pi_{k+l} \) is the same as in the row \( k + \pi_l(1), \ldots, k + \pi_l(l) \). Then for \( \mu = zs(\pi_k) \) and \( \nu = zs(\pi_l) \) the product is defined as

\[
F_\mu F_\nu = \sum_{\pi_{k+l} \in \text{shuffle}(\pi_k, \pi_l)} F_{zs(\pi_{k+l})}.
\]

(6)

By [42, p. 483, Exercise 7.93] this operation is commutative, associative and indeed well defined. That is to say, the definition is not sensitive to the choice of permutations \( \pi_k, \pi_l \) provided they have given zigzag shapes. It is worth noting that the multiplication rule of the fundamental quasisymmetric functions is much simpler than that of Schur functions.

Of special importance for us is the instance of (6) in which \( \pi_k = \pi_{n-1} \) is a permutation of \([n-1]\) and \( \pi_l = \pi_1 \) is the sole permutation of \([1]\). Since \( \pi_n \in \text{shuffle}(\pi_{n-1}, \pi_1) \) just means that \( \pi_n \) extends \( \pi_{n-1} \), we have

\[
F_\mu F_{(1)} = \sum_{\lambda: \mu \triangleright \lambda} F_\lambda.
\]

(7)

The graph \( \mathcal{Z} \) is a multiplicative graph associated to \( \text{QSym} \) in the sense of Vershik–Kerov [24], meaning that (7) mimics the branching in \( \mathcal{Z} \), and that the cone spanned by \( \{ F_\lambda, \lambda \in \mathcal{Z} \} \) is stable under multiplication. A consequence of the multiplicativity is that the formula

\[
\psi(F_\lambda) = p(\lambda)
\]

establishes a bijection between probability functions and linear functionals \( \psi : \text{QSym} \rightarrow \mathbb{R} \) with the properties
Let \( QSym^* \) be the algebraic dual space to the vector space \( QSym \). Denoting \( K \) the cone generated by \( \{ F_\lambda, \lambda \in \mathbb{Z} \} \), condition (ii) says that \( \psi \) belongs to the dual cone \( K^* \subset QSym^* \).

Proposition 8. Under the correspondence (8), \( p \in \partial \mathcal{Z} \) if and only if \( \psi \) is multiplicative, in the sense that

\[
(ii) \quad \psi(F_\lambda) \geq 0 \quad \text{for} \quad \lambda \in \mathbb{Z} ,
\]

\[
(iii) \quad \psi(F_\lambda F_\lambda) = \psi(F_\lambda) \quad \text{for} \quad \lambda \in \mathbb{Z} .
\]

Proof. By virtue of the multiplicativity of \( \mathcal{Z} \), this follows from the general ‘ring theorem’ [28, 25]. See also Proposition 48 in the Appendix below. We apply this result to the quotient of \( QSym \) by the ideal generated by \( F_{(1)} - 1 \), with the cone being the image of the cone \( K \) by the canonical projection. \( \square \)

A functional \( \psi \) which fulfills (i)–(iv) will be called a character of \( QSym \). The set of characters is a subset of the space \( QSym^* \) endowed with the weak topology.

Corollary 9. The boundary \( \partial \mathcal{Z} \) and the set of characters of \( QSym \) are compact topological spaces homeomorphic to each other.

Proof. Since the weak topology on \( QSym^* \) coincides with the topology of simple convergence on the basis elements \( F_\lambda \), the correspondence (8) is a homeomorphism. On the other hand, by Proposition (8) the boundary is closed in \( QSym^* \), hence in \( \mathcal{V} \). Since \( \mathcal{V} \) is compact, so is the boundary. \( \square \)

Besides multiplication, there are two other important operations in \( QSym \) of involution and comultiplication, which we introduce next.

Definition 10. Let \( \text{inv} : QSym \to QSym \) be the involutive linear map defined by \( \text{inv}(F_\lambda) = F_{\lambda'} \), where \( \lambda' \) is the conjugate of \( \lambda \in \mathcal{Z} \).

Using (6) it is readily checked that \( \text{inv} \) is an algebra automorphism. Moreover, \( \text{inv} \) extends the canonical involution \( \omega \) of the algebra \( \text{Sym} \). See Subsection 8.1 for more detail.
Definition 11. (i) The comultiplication is a graded algebra homomorphism $\delta : \text{QSym} \to \text{QSym} \otimes \text{QSym}$. In the basis of monomial functions $\delta$ can be defined by splitting the family of variables $x$ into two disjoint infinite subsets $x'$ and $x''$. A generic quasisymmetric function $F$ becomes then a function $F(x', x'')$ of two families of variables $x'$ and $x''$. Because $F$ is separately quasisymmetric in both $x'$ and $x''$, the function $F$ is an element of $\text{QSym} \otimes \text{QSym}$.

(ii) Equivalently, $\delta$ can be defined on the basis of fundamental quasisymmetric functions as

$$\delta(F_\lambda) = \sum_{\mu, \nu : \lambda = \mu \sqcup \nu} F_\mu \otimes F_\nu,$$ (9)

where, for $\lambda \in \mathbb{Z}_n$, the sum is over all $n+1$ splittings $\lambda = \mu \sqcup \nu$ in two zigzags $\mu$ and $\nu$ one of which can be empty.

The fact that (9) is indeed an algebra homomorphism can be checked directly using (6). Note also that (9) is dual to the rule [44, Equation (67)]. From (i) it follows that $\delta$ extends the comultiplication in Sym, which can be defined in exactly the same way. However, in contrast to the algebra Sym, the comultiplication in QSym is not cocommutative, since switching the factors in $\text{QSym} \otimes \text{QSym}$ does not preserve $\delta$.

Proposition 12. The canonical embedding $\text{Sym} \hookrightarrow \text{QSym}$ induces a projection of the boundaries $\partial \mathcal{Z} \to \partial \mathcal{Y}$.

Proof. Suppose $\psi : \text{QSym} \to \mathbb{R}$ satisfies (i)–(iv). The Schur functions $S_\lambda$, $\lambda \in \mathcal{Y}$, expand with nonnegative coefficients in the basis of fundamental quasisymmetric functions [42, Theorem 7.19.7], hence the cone generated by $\{S_\lambda, \lambda \in \mathcal{Y}\}$ is thinner than the cone $K = \text{conv}\{F_\lambda, \lambda \in \mathcal{Z}\}$, hence $\psi$ is nonnegative on the Schur functions. That $\psi$ is also a character of Sym follows from Proposition 8 since $F_{(1)} = S_{(1)}$ and the embedding is a homomorphism. □

Thus, we can expect a priori that the parameters $(\alpha, \beta)$ of $\partial \mathcal{Y}$ will enter, in some way, in the description of the boundary of the graph of zigzag diagrams.

4 Kerov’s construction of $\partial \mathcal{Y}$

The algebra Sym of symmetric functions with the basis $\{S_\lambda, \lambda \in \mathcal{Y}\}$ of Schur functions underlies Young’s graph $\mathcal{Y}$, which is multiplicative and has
the branching read from the relation

\[ S_\mu S_{(1)} = \sum_{\lambda: \mu \triangleright \lambda} S_\lambda. \]

Replacing the functions \( F_\lambda \) by the functions \( S_\lambda \) in the formula (8) and the subsequent discussion, we identify the elements of the boundary \( \partial \mathcal{Y} \) with the characters \( \psi \) of the algebra \( \text{Sym} \). There exists a simple construction of the characters which is based on the comultiplication in the algebra \( \text{Sym} \). This construction, due to Kerov, did not appear in the literature before. We present it in detail because in the next sections it will be used as a prompt for a more involved construction of the characters of \( \text{QSym} \).

As mentioned above, the comultiplication \( \delta : \text{Sym} \to \text{Sym} \otimes \text{Sym} \) is consistent with the comultiplication in \( \text{QSym} \) under the embedding \( \text{Sym} \to \text{QSym} \). We have

\[ \delta(S_\lambda) = \sum_{|\mu|+|\nu|=|\lambda|} c(\lambda; \mu, \nu)S_\mu \otimes S_\nu, \tag{10} \]

where the coefficients \( c(\lambda; \mu, \nu) \) coincide with the structure constants of the multiplication, the so-called Littlewood–Richardson coefficients. This coincidence is a manifestation of the fact that \( \text{Sym} \) is a self–dual Hopf algebra, see [32, Ch. I, section 5, Example 25]. The fact that the coefficients \( c(\lambda; \mu, \nu) \) are nonnegative plays a crucial role in the construction.

**Proposition 13.** Let \( \psi_1 \) and \( \psi_2 \) be two characters of \( \text{Sym} \) and let \( \omega_1 \) and \( \omega_2 \) be two nonnegative numbers whose sum equals 1. Define a linear functional \( \psi : \text{Sym} \to \mathbb{R} \) by

\[ \psi(S_\lambda) = \sum_{|\mu|+|\nu|=|\lambda|} c(\lambda; \mu, \nu)\omega_1^{|\mu|} \omega_2^{|\nu|} \psi_1(S_\mu)\psi_2(S_\nu). \]

Then \( \psi \) is a character too.

**Proof.** Let us check properties (i)–(iv) above (with \( F \)-functions replaced by \( S \)-functions). Property (i) is trivial because \( \delta(1) = 1 \otimes 1 \). Property (ii) follows from the nonnegativity of coefficients \( c(\lambda; \mu, \nu) \). Regarding (iii), observe that

\[ \delta(S_{(1)}) = S_{(1)} \otimes 1 + 1 \otimes S_{(1)}. \]

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Since $\psi_1(S^{(1)}) = 1$, $\psi_2(S^{(1)}) = 1$, and $\omega_1 + \omega_2 = 1$, this implies $\psi(S^{(1)}) = 1$. Hence (iii) follows from (iv). To prove (iv), introduce a one–parameter family of algebra endomorphisms $r_\omega : \text{Sym} \to \text{Sym}$, where $\omega \in \mathbb{R}$, by setting

$$r_\omega(S_\lambda) = \omega^{1|\lambda}S_\lambda$$

(11)

($r_\omega$ is an automorphism for $\omega \neq 0$). By the very definition of $\psi$, it coincides with the superposition of $\psi_1 \otimes \psi_2$, $r_{\omega_1} \otimes r_{\omega_2}$, and $\delta$. Therefore, $\psi$ is multiplicative. $\square$

Set $\delta^{(2)} = \delta$ and define $\delta^{(3)} : \text{Sym} \to \text{Sym}^{\otimes 3}$ as the superposition map

$$\delta^{(3)} = (\delta \otimes \text{id}) \circ \delta^{(2)} = (\text{id} \otimes \delta) \circ \delta^{(2)}.$$  

Here $\text{id} : \text{Sym} \to \text{Sym}$ is the identity map and the second equality above follows from the associativity of the comultiplication. Continuing this procedure we recursively define the iterated map $\delta^{(k)} : \text{Sym} \to \text{Sym}^{\otimes k}$ for any $k = 2, 3, \ldots$. Clearly, this is a graded algebra morphism. Just as in the case $k = 2$, $\delta^{(k)}$ can be defined by splitting the family of variables into $k$ disjoint subsets.

Now we can generalise Proposition 13.

**Proposition 14.** For any $k = 2, 3, \ldots$ let $\psi_1, \ldots, \psi_k$ be characters of $\text{Sym}$ and let $\omega_1, \ldots, \omega_k$ be real nonnegative numbers whose sum equals 1. Define a linear functional $\psi : \text{Sym} \to \mathbb{R}$ as the superposition

$$(\psi_1 \otimes \cdots \otimes \psi_k) \circ (r_{\omega_1} \otimes \cdots \otimes r_{\omega_k}) \circ \delta^{(k)},$$

(12)

where the endomorphisms $r_\omega$ were introduced in (11). Then $\psi$ is a character too.

**Proof.** By the definition of $\delta^{(k)}$, all the coefficients in the expansion of $\delta^{(k)}(S_\lambda)$ in the natural basis of $\text{Sym}^{\otimes k}$ (the tensor products of $S$–functions) are nonnegative. We also observe that

$$\delta^{(k)}(S^{(1)}) = S^{(1)} \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes S^{(1)} \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes S^{(1)}.$$  

Then we argue as in the proof of Proposition 13. $\square$

**Definition 15.** The above operation will be called $M$–mixing ($M$ for ‘multiplicative’). We will say that the character (12) is obtained by $M$–mixing the characters $\psi_1, \ldots, \psi_k$ in proportions $\omega_1, \ldots, \omega_k$.  

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Observe that the characters of Sym are uniquely specified by their values on the complete homogeneous symmetric functions $h_n = S_{(n)}$ which are generators of the algebra. Using the generating series $H(t) = 1 + \sum_n h_n t^n$ with formal parameter $t$ we set for a character $\psi$

$$\psi(H(t)) = 1 + \sum_{n=1}^{\infty} \psi(h_n) t^n.$$ 

**Proposition 16.** If $\psi$ results from M–mixing the characters $\psi_1, \ldots, \psi_k$ in proportions $\omega_1, \ldots, \omega_k$ then

$$\psi(H(t)) = \psi_1(H(\omega_1 t)) \cdots \psi_k(H(\omega_k t)).$$

**Proof.** Indeed, this follows from the definition of M–mixing and the well–known formula

$$\delta^{(k)}(H(t)) = H(t) \otimes \cdots \otimes H(t) \quad (k \text{ times}).$$

which follows, e.g., from [32 Ch. I, Section 5, Example 25 (a)]. \hfill \Box

The next proposition introduces two elementary characters which will be used to generate further characters by M-mixing.

**Proposition 17.** The linear functionals $\psi_+: \text{Sym} \to \mathbb{R}$ and $\psi_-: \text{Sym} \to \mathbb{R}$ defined by

$$\psi_+(S_{\lambda}) = \begin{cases} 1 & \text{if } \lambda = (n) \\ 0 & \text{if } \lambda \neq (n) \end{cases}, \quad \psi_-(S_{\lambda}) = \begin{cases} 1 & \text{if } \lambda = (1^n) \\ 0 & \text{if } \lambda \neq (1^n) \end{cases}$$

are characters. Here $(n)$ is the one–row diagram and $(1^n) = (n)'$ is the conjugate one–column diagram, $n = 0, 1, 2, \ldots$.

**Proof.** The function $p$ on $Y$ corresponding to $\psi_+$ is the characteristic function of the set of one–row diagrams. Clearly, $p$ is an extreme element in the convex set of solutions to (2). Therefore, by the ring theorem (see Proposition 48), $\psi_+$ is multiplicative. For $\psi_-$ the argument is similar.

An alternative proof consists in checking directly that $\psi_+$ and $\psi_-$ are multiplicative (other properties being obvious). For $\psi_+$ this reduces to the following two claims. Firstly, $h_m h_n = h_{m+n} + \ldots$, where the remaining term does not involve one–row $S$–functions (indeed, it is well known that the
remaining term is a sum of two–row functions). Secondly, if at least one of
the two diagrams $\mu$ and $\nu$ has 2 or more rows then the expansion of
$S_\mu S_\nu$ does not involve one–row $S$–functions (indeed, as is well known, if $S_\lambda$ enters
the expansion of $S_\mu S_\nu$ then $\lambda$ must contain both $\mu$ and $\nu$). For $\psi_-$ one can
apply a similar argument or use the canonical involution of the algebra $\text{Sym}$
which sends $\psi_+$ to $\psi_-$. □

Now we proceed to the explicit construction of characters. Given an
arbitrary collection of real nonnegative numbers $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_l$
whose total sum equals 1, we take the M–mixing of $k$ copies of $\psi_+$ and $l$
copies of $\psi_-$ in proportions $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_l$, respectively. Because
the comultiplication in $\text{Sym}$ is cocommutative, the order in which the couples
$(\alpha_i, \psi_+)$, $(\beta_j, \psi_-)$ are taken plays no role. This is reflected in the symmetry
of formula (13) below with respect to separate permutations of $\alpha_i$’s and $\beta_j$’s.

**Proposition 18.** For the character $\psi$ obtained by the above construction we
have

$$\psi(H(t)) = \frac{\prod_{j=1}^l (1 + \beta_j t)}{\prod_{i=1}^k (1 - \alpha_i t)}.$$  \hspace{1cm} (13)

**Proof.** By the definition of $\psi_+$,

$$\psi_+(H(t)) = \frac{1}{1 - t}.$$

Likewise, denoting by $E(t)$ the generating series for the elementary symmetric
functions $e_n = S_{(1^n)}$ we have

$$\psi_-(E(t)) = \frac{1}{1 - t}.$$  

Together with the classical relation $H(t)E(-t) = 1$ this implies

$$\psi_-(H(t)) = 1 + t.$$

Now the result follows from Proposition 16. □

Arranging the parameters in decreasing order we may identify $(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_l)$ with the element

$$(\alpha_1, \ldots, \alpha_k, 0, 0, \ldots; \beta_1, \ldots, \beta_l, 0, 0, \ldots) \in \Delta^{(2)}.$$
Let $\Delta^{(2)}_{\text{fin}} \subset \Delta^{(2)}$ stand for the subset of all elements of such a kind, that is, $\Delta^{(2)}_{\text{fin}}$ consists of all couples $(\alpha, \beta) \in \Delta^{(2)}$ with finitely many nonzero coordinates whose total sum equals 1. We have shown that to any $(\alpha, \beta) \in \Delta^{(2)}_{\text{fin}}$ there is a character $\psi_{\alpha,\beta}$ such that

$$\psi_{\alpha,\beta}(H(t)) = \frac{\prod_{j=1}^{\infty} (1 + \beta_j t)}{\prod_{i=1}^{\infty} (1 - \alpha_i t)},$$

where the products are actually finite. We call $\{\psi_{\alpha,\beta} : (\alpha, \beta) \in \Delta^{(2)}_{\text{fin}}\}$ the finitary skeleton of the boundary $\partial \mathcal{Y}$.

Let $\text{Sym}^*$ be the algebraic dual space to the vector space $\text{Sym}$. We equip $\text{Sym}^*$ with the weak topology. It is readily seen that the characters form a closed subset of $\text{Sym}^*$. Moreover, the induced topology on the set of characters coincides with the product topology by the embedding $\psi \mapsto (\psi(h_1), \psi(h_2), \ldots)$.

**Proposition 19.** The closure of the finitary skeleton in $\text{Sym}^*$ consists of the characters which are indexed by arbitrary elements $(\alpha, \beta) \in \Delta^{(2)}$ and are specified by

$$\psi_{\alpha,\beta}(H(t)) = e^{\gamma t} \frac{\prod_{j=1}^{\infty} (1 + \beta_j t)}{\prod_{i=1}^{\infty} (1 - \alpha_i t)}, \quad \text{where} \quad \gamma := 1 - \sum_{i=1}^{\infty} \alpha_i - \sum_{j=1}^{\infty} \beta_j. \quad (14)$$

**Proof.** Let $\mathbb{R}^{\infty}$ be the product of countably many copies of $\mathbb{R}$ taken with the product topology. The set $\Delta^{(2)}$ may be viewed as a subset of $\mathbb{R}^\infty \times \mathbb{R}^\infty$. To any $(\alpha, \beta) \in \Delta^{(2)}$ we assign a multiplicative functional $\psi_{\alpha,\beta} : \text{Sym} \to \mathbb{R}$ by making use of (14). Observe that $\Delta^{(2)}$ is compact in the induced topology and $\Delta^{(2)}_{\text{fin}}$ is dense in $\Delta^{(2)}$. Therefore, it suffices to prove that the quantities $\psi_{\alpha,\beta}(h_n)$ are continuous functions in $(\alpha, \beta) \in \Delta^{(2)}$. It is more convenient to consider the values on another set of generators, the Newton power sums $p_1, p_2, \ldots \in \text{Sym}$, which are related to $h_1, h_2, \ldots$ by the well-known formula

$$\log H(t) = \sum_{n=1}^{\infty} \frac{p_n t^n}{n}.$$
From this formula we get
\[ \psi_{\alpha,\beta}(p_1) = 1, \quad \psi_{\alpha,\beta}(p_n) = \sum \alpha_i^n + (-1)^{n-1} \sum \beta_j^n \quad (n \geq 2), \]
and the continuity of \( \psi_{\alpha,\beta}(p_n) \) is readily verified (see, e.g., [27]).  \[ \square \]

Thus, every character \( \psi_{(\alpha,\beta)} \) is, loosely speaking, a \( M \)-mixture of countably many copies of \( \psi_+ \), countably many copies of \( \psi_- \) and a single copy of \( \psi_{0,0} \) in proportions \( \alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots \), and \( \gamma \), respectively. Here \( \psi_{0,0} \) has all parameters equal to 0.

This completes Kerov’s construction. It actually provides all the characters, although the proof of this completeness claim requires further techniques.

**Remark 20.** Note that Kingman’s boundary \( \partial \mathcal{P} \) can be constructed in the same way. In this case we consider Sym with the basis of (augmented) monomial symmetric functions. This basis is not preserved by the canonical involution, and the operation of \( M \)-mixing involves only one elementary character: the corresponding probability function is supported by the partitions \( (n) \in \mathcal{P} \), so that this character may be viewed as a counterpart of \( \psi_+ \). As a consequence, the boundary is just \( \Delta \). A character with the probability function supported by \( \{(1^n)\} \) still exists but it plays the role of \( \psi_{0,0} \), and not the role of \( \psi_{-} \).

# 5 Finitary skeleton of the boundary \( \partial \mathcal{Z} \)

Now we aim to adapt Kerov’s construction to the graph \( \mathcal{Z} \) by dealing with the algebra QSym and its basis \( \{F_\lambda : \lambda \in \mathcal{Z}\} \) of fundamental quasisymmetric functions. As before, we have two elementary characters but the construction becomes more sophisticated because the comultiplication is no longer cocommutative. Another complication is the lack of a nice system of generators like \( \{h_n\} \) or \( \{p_n\} \). As a consequence, we cannot write an analog of formula (14). On the other hand, the action of the comultiplication on the \( F \)-functions has a relatively simple form (compare (9) and (10)), which enables us to control the behaviour of characters on the elements of the basis rather than on a system of generators.
Extending (9) we immediately obtain a simple expression for the iterated comultiplication map \( \delta^{(k)} : \text{QSym} \to \text{QSym}^{\otimes k} \): 
\[
\delta^{(k)}(F_\lambda) = \sum F_{\lambda(1)} \otimes \cdots \otimes F_{\lambda(k)}
\]  
where the summation is over all splittings of zigzag diagram \( \lambda \) at arbitrary positions in \( k \) consecutive zigzag subdiagrams \( \lambda(1) \sqcup \cdots \sqcup \lambda(k) \) some of which may be empty.

The definition of the endomorphisms \( r_\omega \) and of the M–mixing (Definition 15) naturally extends to the algebra \( \text{QSym} \). However, now the resulting character \( \psi \) is sensitive to the ordering of the couples \( (\psi_i, \omega_i) \). Here is an explicit expression for \( \psi \):
\[
\psi(F_\lambda) = \sum \omega_1^{\ell_1} \cdots \omega_k^{\ell_k} \psi_1(F_{\lambda(1)}) \cdots \psi_k(F_{\lambda(k)}) , \quad \lambda \in \mathcal{Z} ,
\]  
where the summation is as above and \( \ell_j = |\lambda(j)| \), so that \( \lambda(j) \in \mathcal{Z}_{\ell_j} \).

We define the elementary characters \( \psi_+, \psi_- : \text{QSym} \to \text{QSym} \) exactly as in Proposition 17, the only change is that \( \lambda \) ranges over \( \mathcal{Z} \) and the \( S \)–functions are replaced by the \( F \)–functions. Both proofs of Proposition 17 remain valid: in the second proof we use the multiplication rule (6). Note that the involution map (Definition 10) interchanges \( \psi_+ \) and \( \psi_- \).

Proposition 21. The elementary characters of \( \text{QSym} \) defined above are extensions of the elementary characters of \( \text{Sym} \).

Proof. Consider the character \( \psi_+ \) of \( \text{QSym} \) and restrict it to \( \text{Sym} \). Since \( S_{(n)} = F_{(n)} \), we have \( \psi_+(S_{(n)}) = 1 \). If \( \lambda \in \mathcal{Y} \) is not a one–row diagram then the expansion of \( S_\lambda \) in the \( F \)–functions does not involve one–row zigzag diagrams, as follows from [42, Theorem 7.19.7, p. 361]. Therefore, \( \psi_+(S_\lambda) = 0 \). Thus, the restriction of \( \psi_+ \) has the required form. For \( \psi_- \), we use a similar argument or apply the involution. \( \square \)

Now we can select arbitrary weights and apply the M–mixing to some number of copies of \( \psi_+ \) and \( \psi_- \) arranged in a sequence. It is convenient to represent these data in the following form.

Definition 22. Define a finitary oriented paintbox to be a pair \((U_\uparrow, U_\downarrow)\) of disjoint open subsets of the open interval \([0,1]\), each comprised of finitely many interval components and such that their total Lebesgue measure is \( \text{Leb}(U_\uparrow \cup U_\downarrow) = 1 \). Let \( \mathcal{U}^{(2)}_{\text{fin}} \) denote the set of all such pairs.
Think of the interval components of \( U_\uparrow \) as having positive orientation, and think of the interval components of \( U_\downarrow \) as having negative orientation. With this convention, each interval component of the union \( U_\uparrow \cup U_\downarrow \) is characterised by location, length and orientation.

We define M–mixture \( \psi_{(U_\uparrow, U_\downarrow)} \) by assigning, to each interval component \( ]a, b[ \subset U_\uparrow \cup U_\downarrow \), the factor \( \psi_+ \) with weight \( b - a \) in the case \( ]a, b[ \subset U_\uparrow \), or the factor \( \psi_- \) with weight \( b - a \) in the case \( ]a, b[ \subset U_\downarrow \); the order of factors is maintained in accordance with the natural ordering of the intervals, that is, from left to right.

Let
\[
\text{rank} : (U_\uparrow, U_\downarrow) \mapsto (\alpha, \beta)
\]
(16)
be the map assigning to each \( (U_\uparrow, U_\downarrow) \in U_\text{fin}(2) \) two nonincreasing positive sequences \( \alpha = (\alpha_1, \ldots, \alpha_k) \) and \( \beta = (\beta_1, \ldots, \beta_l) \) of lengths of the interval components of \( U_\uparrow \) and \( U_\downarrow \), respectively. Restricting \( \psi_{(U_\uparrow, U_\downarrow)} \) to \( \text{Sym} \) we get the character \( \psi_{\alpha, \beta} \) as defined in Section 4. Conversely, given such \( \alpha \) and \( \beta \) with \( \sum \alpha_i + \sum \beta_j = 1 \), there are as many finitary characters \( \psi_{(U_\uparrow, U_\downarrow)} \) extending \( \psi_{\alpha, \beta} \) as there are distinct arrangements of these \( k + l \) intervals within \( ]0, 1[ \), which is \( (k + l)! \) in the case all lengths are pairwise distinct.

**Proposition 23.** For a finitary oriented paintbox \( (U_\uparrow, U_\downarrow) \) with \( m \) interval components, let \( \omega_1, \ldots, \omega_m \) be the sequence of lengths of these components in the natural order, from left to right. Then for any \( \lambda \in \mathcal{Z} \)
\[
\psi_{(U_\uparrow, U_\downarrow)}(F_\lambda) = \sum \omega_1^{\ell_1} \cdots \omega_m^{\ell_m}
\]
(17)
where the sum expands over all splittings \( \lambda = \lambda(1) \sqcup \cdots \sqcup \lambda(m) \) such that \( \lambda(j) = (\ell_j) \) or \( \lambda(j) = (1^{\ell_j}) \), with \( \ell_j \geq 0 \), according as \( \omega_j \) is the length of a \( U_\uparrow \)–component or a \( U_\downarrow \)–component. In the case \( \ell_j = 0 \) we understand \( \lambda(j) \) as the empty zigzag.

**Proof.** This immediately follows from formula (15) and the definition of \( \psi_+ \) and \( \psi_- \). \( \square \)

In the particular case \( U_\downarrow = \emptyset \) the formula for the probability function corresponding to \( \psi_{U_\uparrow, U_\downarrow} = \psi_{U_\uparrow, \emptyset} \) appeared as \( \Box \) Equation 6.4], this is just the specialisation of \( F_\lambda \) in the variables \( \omega_j \) with no type distinction.

**Definition 24 (Oriented paintbox construction).** We associate with finitary \( (U_\uparrow, U_\downarrow) \) the following construction of a random arrangement \( \Pi = \)
Informally, $\Pi$ breaks $\mathbb{N}$ into blocks and puts them in some order, whereas the order within each of the blocks coincides with either the standard order or with the inverse order. In more detail, let $\xi_1, \xi_2, \ldots$ be an independent sample from the uniform distribution on $[0, 1]$. Keep in mind that the sample values are almost surely distinct and that $\xi_j$ hits the complement to $U_{\uparrow} \cup U_{\downarrow}$ in $[0, 1]$ with probability zero. Define a random permutation $\Pi_n$ of $[n]$ by placing $i$ to the left of $j$ if one of the following conditions holds:

- $\xi_i$ and $\xi_j$ do not hit the same interval component of $U_{\uparrow} \cup U_{\downarrow}$ and $\xi_i < \xi_j$;
- $\xi_i$ and $\xi_j$ hit the same interval component of $U_{\uparrow}$ and $i < j$;
- $\xi_i$ and $\xi_j$ hit the same interval component of $U_{\downarrow}$ and $i > j$.

The number $n$ enters implicitly, in the form of the constraint $n \geq \max(i, j)$, hence the algorithm yields a coherent sequence of permutations, i.e., defines a random arrangement $\Pi = (\Pi_n)$ of $\mathbb{N}$. Let us say that $\Pi$ is directed by $(U_{\uparrow}, U_{\downarrow})$.

In the case $U_{\downarrow} = \emptyset$ the resulting permutation $\Pi_n$ is inverse to a biased riffle shuffle introduced in [8] and further studied in [14]. In [31 Definition 2.1], a more general concept of shuffle is introduced in terms of a measure–preserving mapping $f : [0, 1] \to [0, 1]$. This covers the general finitary $(U_{\uparrow}, U_{\downarrow})$, although that paper is really focussed on mappings $f$ increasing on its intervals of continuity, which in our context still corresponds to $U_{\downarrow} = \emptyset$.

It is known that for Young’s lattice there is also a way to transform an independent sample into a Markov chain on $\mathcal{Y}$, but it is much more complicated as is based on properties of the generalised RSK–correspondence [47].

**Proposition 25.** The distribution of the random arrangement directed by $(U_{\uparrow}, U_{\downarrow})$ is

$$\mathbb{P}(\Pi_n = \pi_n) = \psi_{(U_{\uparrow}, U_{\downarrow})}(F_\lambda), \quad \text{for } \lambda = zs(\pi_n),$$

where $\psi_{(U_{\uparrow}, U_{\downarrow})}(F_\lambda)$ is given by (17).

**Proof.** Let $I_1, \ldots, I_m$ be the interval components of the oriented paintbox and let $\omega_1, \ldots, \omega_m$ be their lengths. For $i = 1, \ldots, m$, let $\ell_i$ be the number of indices $j \in [n]$ such that $\xi_j \in I_i$. If $I_i$ belongs to $U_{\uparrow}$ then $I_i$ contributes

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an increasing segment of length $\ell_i$ to $\Pi_n$, while if $I_i$ belongs to $U_i$, then the contribution is a decreasing segment of length $\ell_i$. By convention, empty and singleton segments may be viewed both as increasing and decreasing. Each splitting of $\pi_n$ in consecutive monotonic segments of lengths $\ell_1, \ldots, \ell_m \geq 0$ contributes to $\mathbb{P}(\Pi_n = \pi_n)$ the probability

$$\mathbb{P}(\xi_{\pi_n(1)}, \ldots, \xi_{\pi_n(\ell_1)} \in I_1; \ldots; \xi_{\pi_n(\ell_1+\ldots+\ell_{m-1}+1)}, \ldots; \xi_{\pi_n(n)} \in I_m) = \omega_1^{\ell_1} \cdots \omega_m^{\ell_m}$$

provided the monotonicity type of each segment $\pi_n(\ell_1+\ldots+\ell_{i-1}+1), \ldots, \pi_n(\ell_1+\ldots+\ell_i)$ agrees with the orientation of $I_i$. Summing over all such splittings yields (17).

**Example 26** ($a$-shuffles [9]). Formula (17) simplifies considerably in the case when $U_i = \emptyset$ and $U_i$ is the equispaced division of $[0, 1]$ in $a$ intervals. In this case the permutation is inverse to a riffle shuffle [9] with $a$ piles and its distribution is given by

$$\mathbb{P}(\Pi_n = \pi_n) = \left(\frac{n+a-k}{n}\right)^{a-n},$$

where $k - 1$ is the number of descents in $\pi_n$. Reciprocally, with the roles of $U_i$ and $U_i$ exchanged, the right-hand side gives the probability $\mathbb{P}(\Pi_n = \pi'_n)$ for the reversed permutation. As $a \to \infty$ both series converge to the distribution of a uniform permutation which has $\mathbb{P}(\Pi_n = \pi_n) = 1/E_{n,k}$, where $E_{n,k}$ is the Eulerian number [11]. In [18] we show that the distributions just described are extreme within the smaller class of coherent permutations which have the number of descents as sufficient statistic. These distributions play for $\mathcal{Z}$ a role comparable with that of Ewens’ distributions for $\mathcal{P}$, in the sense that Ewens’ distributions are extreme among the distributions for exchangeable partitions which have the number of blocks as sufficient statistic [19]. Still, there is a difference: Ewens’ distributions can be decomposed over $\partial \mathcal{P}$ (the mixing measure is known as the Poisson–Dirichlet distribution [39]).

**Example 27** (‘bi–interval’ [17]). Suppose the oriented paintbox is composed of one ↓–interval $]0, \varphi[$ and one ↑–interval $]\varphi, 1[$ with common endpoint $\varphi \in ]0, 1[$. The random zigzag shape $X_n = \text{zs}(\Pi_n)$ has the distribution supported by hook zigzags only, and its probability function is

$$p(1^l, k+1) = \varphi^l (1 - \varphi)^k, \quad k + l = n - 1.$$
The distribution of the resulting arrangement $\prec$ is also easy to describe: with probability $\varphi$ the integer $n$ is $\prec$-ordered below $[n-1]$ and with probability $1-\varphi$ it is $\prec$-ordered above $[n-1]$. Thus the initial ranks $\rho_n := \Pi^{-1}_n(n)$ are independent, assume only extreme values and have a representation

$$\rho_n = 1 \cdot 1(\eta_n = 0) + n \cdot 1(\eta_n = 1),$$

where $\eta_n$'s are independent Bernoulli random variables with $P(\eta_n = 0) = \varphi$.

6 The general oriented paintbox

6.1 The closure

The set $\mathcal{U}_{\text{fin}}^{(2)}$ introduced in Definition 22 is an analog of $\Delta_{\text{fin}}^{(2)}$. We seek now for an analog of $\Delta^{(2)}$ — a topological space which would serve as a set of parameters for the closure of the set of finitary characters. With reference to [16] the logical choice of that space is

$$\mathcal{U}^{(2)} = \{(U^\uparrow, U^\downarrow) : U^\uparrow, U^\downarrow \text{ are open disjoint subsets of } ]0,1[\}.$$  

Let us emphasise that, by definition, $U^\uparrow$ and $U^\downarrow$ are contained in the open interval $]0,1[\] hence include neither 0 nor 1. We endow $\mathcal{U}^{(2)}$ with a kind of Hausdorff distance, defined for $(U^\uparrow, U^\downarrow), (V^\uparrow, V^\downarrow) \in \mathcal{U}^{(2)}$ to be the infimum $\theta$ such that the $\theta$–inflation of the complementary closed set $U^\text{compl} := [0,1] \setminus U^\uparrow$ covers $V^\text{compl}$, the $\theta$–inflation of $V^\text{compl}$ covers $U^\text{compl}$, and the same holds with the roles of $(U^\uparrow, U^\downarrow)$ and $(V^\uparrow, V^\downarrow)$ swapped.

The convergence of a sequence $((V^\uparrow(j), V^\downarrow(j)), j = 1, 2, \ldots)$ to some $(U^\uparrow, U^\downarrow)$ in this metric is described by the following rule. Introduce a notation: for any $\epsilon > 0$, let $(U^\epsilon, U^\epsilon^\uparrow)$ stand for the paintbox obtained from $(U^\uparrow, U^\downarrow)$ by deleting all interval components with length less or equal to $\epsilon$. Then, for arbitrarily small fixed $\epsilon$, different of all interval lengths represented in $U^\uparrow \cup U^\downarrow$, for $j$ large enough, $V^\epsilon(j) \cup V^\epsilon(j)$ must have the same number of intervals as $U^\epsilon \cup U^\epsilon^\uparrow$, with the same orientation pattern. Moreover, the record of the endpoints of all intervals for $(V^\epsilon(j), V^\epsilon(j))$ must approach the similar record for $(U^\uparrow, U^\downarrow)$, as $j \to \infty$.

For instance, $(V^\uparrow(j), V^\downarrow(j)) \to (\varnothing, \varnothing)$ in $\mathcal{U}^{(2)}$ if and only if the length of the longest interval component approaches 0.

**Proposition 28.** $\mathcal{U}_{\text{fin}}^{(2)}$ is dense in $\mathcal{U}^{(2)}$. 

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Proof. Let \((U^\uparrow, U^\downarrow)\) be arbitrary. An approximating sequence 

\[ \{(V^\uparrow(j), V^\downarrow(j))\} \rightarrow (U^\uparrow, U^\downarrow) \]

of finitary paintboxes can be obtained as follows. Choose a sequence \(\epsilon_j \downarrow 0\). Given \(j\), first take \((U^\uparrow_{\epsilon_j}, U^\downarrow_{\epsilon_j})\) and then extend it to a finitary paintbox by breaking the complement \([0,1] \setminus (U^\uparrow_{\epsilon_j} \cup U^\downarrow_{\epsilon_j})\) in finitely many intervals of length not exceeding \(\epsilon_j\). □

**Proposition 29.** \(\mathcal{U}^{(2)}\) is a compact space.

**Proof.** Let us show that any sequence \(\{(V^\uparrow(j), V^\downarrow(j)), j = 1, 2, \ldots\}\) contains a converging subsequence. For any fixed \(\epsilon > 0\), the total number of interval components in \((V^\uparrow(\epsilon_j), V^\downarrow(\epsilon_j))\) does not exceed \(1/\epsilon\). Therefore, passing to a subsequence of indices \(j\), we may assume that the total number of intervals in \((V^\uparrow(\epsilon_j), V^\downarrow(\epsilon_j))\) together with their orientation pattern stabilise as \(j \rightarrow \infty\), and moreover, the record of the endpoints of the intervals is converging. Furthermore, fix \(\epsilon_1 > \epsilon_2 > \ldots\) converging to 0. Then, using Cantor’s diagonal process we can choose a further subsequence of indices \(j\) such that the above convergence property holds for any \(\epsilon_j\). It is readily seen that the resulting subsequence of paintboxes is indeed converging to some element in \(\mathcal{U}^{(2)}\). □

The oriented paintbox construction introduced before for finitary paintboxes (Definition 24) extends literally to arbitrary \((U^\uparrow, U^\downarrow)\) \(\in\) \(\mathcal{U}^{(2)}\). We were careful to formulate the first rule to make it working in the more general situation: when \(\xi_i, \xi_j\) do not hit the same component interval, one of these points may fall in the complement to \(U^\uparrow \cup U^\downarrow\) in which case the order is maintained according as \(\xi_i < \xi_j\) or \(\xi_i > \xi_j\). A somewhat different description of the construction is given in [22].

**Proposition 30.** The correspondence \((U^\uparrow, U^\downarrow) \mapsto \Pi = (\Pi_n)\) between general oriented paintboxes and random arrangements provided by the extension of Definition 24 is continuous.

**Proof.** The assertion means that for any \(n = 1, 2, \ldots\) and any permutation \(\pi_n\) of \([n]\), the probability \(\mathbb{P}(\Pi_n = \pi_n)\) is a continuous function of \((U^\uparrow, U^\downarrow)\). Recall that the random permutation \(\Pi_n\) is uniquely determined by \((U^\uparrow, U^\downarrow)\) and the first \(n\) sample values \(\xi_1, \ldots, \xi_n\). Observe that, for small \(\epsilon > 0\), the probability that \(|\xi_i - \xi_j| \leq \epsilon\) for at least one pair of distinct indices \(i, j \in [n]\)
is $O(\epsilon)$. It follows that the event $\Pi_n = \pi_n$ is determined, up to an event of probability $O(\epsilon)$, by the mutual location of the sample $\xi_1, \ldots, \xi_n$ and $(U_\uparrow^\epsilon, U_\downarrow^\epsilon)$ only. This easily implies the continuity assertion.

**Corollary 31.** Let $\Pi = (\Pi_n)$ be the random arrangement corresponding to an arbitrary $(U_\uparrow, U_\downarrow) \in \mathcal{U}^{(2)}$. Then each probability of the form $\mathbb{P}(\Pi_n = \pi_n)$ depends on the zigzag shape $zs(\pi_n)$ only, so that $\Pi$ determines a probability function $p(\mathcal{U}^{(2)}) \in \mathcal{V}$. Moreover, the corresponding functional $\psi_{(U_\uparrow, U_\downarrow)}(\lambda) = p_{(U_\uparrow, U_\downarrow)}(\lambda)$, is multiplicative and hence is a character. The set of characters obtained in this way is the closure of the set of characters derived from the elements of $\mathcal{U}^{(2)}_{\text{fin}}$.

**Proof.** This immediately follows from Propositions 28 and 30, and for the last claim of the corollary, we also use Proposition 29.

Observe that the projection (16) extends, in a natural way, to a continuous projection \[ \text{rank} : \mathcal{U}^{(2)} \to \Delta^{(2)}, \quad (U_\uparrow, U_\downarrow) \mapsto (\alpha, \beta), \] where the coordinates of $\alpha$ and $\beta$ are the lengths of the interval components in $U_\uparrow$ and $U_\downarrow$, respectively, written in decreasing order. Recall also that there is a natural projection $\partial\mathcal{Z} \to \partial\mathcal{Y}$, see Proposition 12.

**Corollary 32.** Under the projection $\partial\mathcal{Z} \to \partial\mathcal{Y}$, a character $\psi_{(U_\uparrow, U_\downarrow)}$ with $(U_\uparrow, U_\downarrow) \in \mathcal{U}^{(2)}$ is sent to the character $\psi_{\alpha, \beta}$ of $\text{Sym}$ with parameters $(\alpha, \beta) = \text{rank}(U_\uparrow, U_\downarrow)$.

**Proof.** Since the embedding $\text{Sym} \hookrightarrow \text{QSym}$ respects the comultiplication, the claim is true for finitary characters. The general case follows by continuity of the projection.

**Definition 33.** For $n = 2, 3, \ldots$ define an embedding $\iota_n : \mathcal{Z}_n \hookrightarrow \mathcal{U}_{\text{fin}}^{(2)} \subset \partial\mathcal{Z}$ by assigning to each $\lambda \in \mathcal{Z}_n$ a finitary paintbox $\iota_n(\lambda) = (V_\uparrow, V_\downarrow)$ which mimics the zigzag, as follows. Recall that zigzags $\lambda \in \mathcal{Z}$ can be encoded by words $w(\lambda)$ in the binary alphabet $\{+, -\}$, see the discussion following Definition 1. A word $w(\lambda)$ is further transformed into a paintbox $(V_\uparrow, V_\downarrow) \in \mathcal{U}_{\text{fin}}^{(2)}$, with the interval components of $V_\uparrow$ corresponding to the plus–clusters of $w(\lambda)$, and those of $V_\downarrow$ corresponding to the minus–clusters, taking the interval lengths to be proportional to the cluster lengths, in the same order. For example,
\[ w(4, 1^2, 3) = + + + - - - + + , \text{ and its image } \nu_0(4, 1^2, 3) = (V_\uparrow, V_\downarrow) \text{ has } V_\uparrow = ]0, \frac{2}{3} [ \cup \frac{6}{8}, \frac{6}{8} [ \text{ and } V_\downarrow = ]\frac{3}{8}, \frac{6}{8} [. \]

The next result is a law of large numbers for random arrangements derived from oriented paintboxes.

**Proposition 34.** Fix an oriented paintbox \((U_\uparrow, U_\downarrow) \in \mathcal{U}^{(2)}\) and let \(\Pi = (\Pi_n)\) be the corresponding random arrangement as defined above. Write \(\Pi\) as a random standard path \((\lambda_0 \rightarrow \lambda_1 \rightarrow \ldots)\) in \(\mathbb{Z}\). Then almost surely \(\nu_n(\lambda_n)\) converges to \((U_\uparrow, U_\downarrow)\) in the ambient space \(\mathcal{U}^{(2)}\).

**Proof.** Let \((V_\uparrow(n), V_\downarrow(n)) = \nu_n(\lambda_n)\). We have to prove that almost surely \((V_\uparrow(n), V_\downarrow(n)) \rightarrow (U_\uparrow, U_\downarrow)\) as \(n \rightarrow \infty\). We will use the description of the convergence given in the beginning of this section. Fix an arbitrarily small \(\epsilon > 0\) different from the lengths of intervals in the paintbox \((U_\uparrow, U_\downarrow)\). Let \([a, b]\) be an interval component in \(U_\uparrow\) with \(b - a > \epsilon\); we assert that there is a similar interval in \(V_\uparrow(n)\) for all \(n\) large enough. Indeed, let \((\xi_1, \xi_2, \ldots)\) be the i.i.d random variables entering the paintbox construction. Fix an arbitrary index \(j\) such that \(\xi_j\) hits \([a, b]\) \(\subset U_\uparrow\) (it exists almost surely). By the strong law of large numbers applied to \((\xi_m)\), the proportion of natural numbers \(m \in [n]\) placed in \(\Pi_n\) to the left of \(j\) converges to \(a\), and the length of the increasing run containing \(j\) is asymptotic to \((b - a)n\). It follows that the position (on the \(1/(n - 1)\)-scale) of the plus–cluster in \(w(\lambda_n)\) corresponding to this run is close to the interval \([a, b]\).

Conversely, each plus–cluster in \(w(\lambda_n)\), longer than \(\epsilon n\), must correspond to some \(\uparrow\)-interval. This is argued by noting that the probability that an increasing run longer \(\epsilon n\) stems from more than one such interval approaches zero as \(n \rightarrow \infty\), in consequence of the fact that for any interval the maximal index \(j \leq n\) with \(\xi_j\) hitting the interval is likely to satisfy \(n - j = o(n)\), while the minimal \(j\) with \(\xi_j\) hitting the interval is just a finite random variable. A similar argument works for \(\downarrow\)-intervals and minus–clusters. □

**Corollary 35.** The correspondence \((U_\uparrow, U_\downarrow) \mapsto p_{(U_\uparrow, U_\downarrow)}\) defined in Corollary 31 is injective.

Let us summarise the main results of this section:

**Proposition 36.** The correspondence \((U_\uparrow, U_\downarrow) \mapsto p_{(U_\uparrow, U_\downarrow)}\) afforded by the oriented paintbox construction is a homeomorphism of the compact space \(\mathcal{U}^{(2)}\) onto a part of the boundary \(\partial \mathbb{Z}\).
Proof. Indeed, this follows from Corollaries 31 and 35. □

In Section 7 we will show that the image of $U^{(2)}$ actually coincides with the whole boundary $\partial Z$.

### 6.2 Mixing

Here mixing means exploiting the oriented paintbox construction with a random $(U_\uparrow, U_\downarrow)$ chosen from some probability distribution on $U^{(2)}$. Proposition 36 obviously extends to the mixed case, hence yields a continuous injection of the space of probability distributions on $U^{(2)}$ with weak topology into the space $V$ of probability functions on $Z$.

**Example 37 (a random bi–interval).** Suppose as in Example 27 that $]0,1[$ is divided in one $\downarrow$– and one $\uparrow$–interval as $]0,\varphi[ \cup ]\varphi,1[$, but this time $\varphi$ is random with a beta density $P(\varphi \in dx) = x^{\theta_1-1}(1-x)^{\theta_2-1}dx/B(\theta_1, \theta_2)$, where $\theta_1, \theta_2 > 0$. The sequence of initial ranks $(\rho_n)$ is no longer independent. Rather, the number of 1’s in the sequence is a sufficient statistic: the conditional probability of $\rho_n = 1$ given $\rho_1, \ldots, \rho_{n-1}$ is $(k + \theta_1)/(k + \ell + \theta_1 + \theta_2)$ where $k$ is the number of 1’s among $\rho_1, \ldots, \rho_{n-1}$ and $\ell = n - 1 - k$. An astute reader might have noticed that in the representation

$$\rho_n = 1 \cdot 1(\eta_n = 0) + n \cdot 1(\eta_n = 1)$$

the $\eta_n$’s are exchangeable Bernoulli variables that follow Polya’s urn model [11].

### 6.3 Quasi–uniform distributions

For an interval component $]a, \varphi[ \subset U_\uparrow$ we think of the left endpoint $\varphi$ as the initial point, and of $b$ as the terminal point of the interval. For an interval $]a, \varphi[ \subset U_\downarrow$ the convention about the endpoints is reversed. With each $(U_\uparrow, U_\downarrow) \in U^{(2)}$ we associate a probability measure $\nu$ on $[0,1]$ obtained by weeping out the Lebesgue measure of each interval component of $U_\uparrow \sqcup U_\downarrow$ to its initial point:

$$\nu = \sum_{]\varphi, b[ \subset U_\uparrow} \delta_\varphi + \sum_{]a, \varphi[ \subset U_\downarrow} \delta_\varphi + \text{Leb}[0,1\setminus(U_\uparrow \cup U_\downarrow)].$$
Such measures were dubbed quasi–uniform in [22]. In the case $U_\uparrow = \emptyset$ quasi–uniform measures coincide with uniformised measures introduced in [16]. By [22, Lemma 3.3], the quasi–uniform measures can be characterised by the property that the inequalities $\nu[0, x[ \leq x \leq \nu[0, x]$ hold for $\nu$–almost all $x$.

**Proposition 38.** The above correspondence $(U_\uparrow, U_\downarrow) \mapsto \nu$ is a homeomorphism between the space $U(2)$ with the Hausdorff–type metric as defined in Section 6.1, and the space of quasi–uniform measures endowed with the weak topology.

**Proof.** It is a straightforward check that the metric agrees with the Skorohod topology on the space of càdlàg distribution functions for quasi–uniform measures. \qed

The role of this class of distributions stems from the following observation. Suppose $\nu$ is a quasi–uniform distribution corresponding to some $(U_\uparrow, U_\downarrow)$. Let $\xi_j$’s be independent uniformly distributed on $[0, 1]$ and $\varphi_j$ be the initial point of the $(U_\uparrow, U_\downarrow)$–interval discovered by $\xi_j$, with the convention $\varphi_j = \xi_j$ in case $\xi_j$ falls in $[0, 1] \setminus (U_\uparrow \cup U_\downarrow)$. Then the probability law of $\varphi_j$ is $\nu$. The value $\varphi_j$ may be defined intrinsically in terms of the induced order $\prec$ as the frequency of integers ordered below $j$,

$$\varphi_j = \lim_{n \to \infty} \frac{\#\{1 \leq i \leq n : i \prec j\}}{n},$$

which suggests to call $\varphi_j$ the **height** of $\prec$ at $j$.

### 6.4 An alternative description

The sequence of heights alone determines only a weak order on $\mathbb{N}$ by the formula

$$i \precsim j \iff \varphi_i \leq \varphi_j. \quad (18)$$

The relation $\precsim$ projects to a total order on $\mathbb{N}/ \sim$ when factored by the equivalence relation

$$i \sim j \iff \varphi_i = \varphi_j. \quad (19)$$

The random partition of $\mathbb{N}$ into some collection of blocks associated with $\sim$ is exchangeable, i.e. its probability law is invariant under all bijections $\mathbb{N} \to \mathbb{N}$, and the realisation of $\sim$ via (19) is an instance of Kingman’s paintbox representation of extreme exchangeable equivalence relations on $\mathbb{N}$, also called
similarly, the realisation of $\preceq$ by (18) is a version of the representation of extreme exchangeable ordered partitions, also called composition structures [16, 20].

The strict order $\prec$ extends the weak order $\preceq$ (off the diagonal $\{(i, i) : i \in \mathbb{N}\}$) by arranging in a certain way the integers within each block of $\sim$. The modus operandi of this extension is the following. Divide the collection of intervals making up $U_\uparrow \cup U_\downarrow$ into bi–intervals $]a, \varphi[ \cup \varphi, b[\text{ for each initial point } \varphi$. If $\varphi$ is the initial point of a single interval, either $]a, \varphi[ \subset U_\downarrow$ or $]\varphi, b[\subset U_\uparrow$, we adopt the convention that another component is empty, with $b = \varphi$ or $a = \varphi$, respectively. The correspondence between bi–intervals and initial points is bijective. It is seen from the oriented paintbox construction that for a generic initial point $\varphi$ the equality $\varphi_n = \varphi$ means that $\xi_n$ hits the bi–interval $]a, \varphi[ \cup \varphi, b[\text{ corresponding to } \varphi$, thus those $n$’s comprise a nonsingleton block of $\sim$, call it $B_\varphi$. By the construction, in the case $\xi_n$ hits $]a, \varphi[\text{ the integer } n$ is $\prec$–ordered below all smaller $j$’s with $j \in B_\varphi$, while in the case $\xi_n \in ]\varphi, b[\text{ the integer } n$ is $\prec$–ordered above all smaller $j$’s with $j \in B_\varphi$. Write $s_n = \downarrow$ in the first case, $s_n = \uparrow$ in the second case, and write $s_n = \cdot$ if $\xi_n$ misses $U_\uparrow \cup U_\downarrow$. All this encodes $\prec$ into a random sequence $((\varphi_n, s_n), n = 1, 2, \ldots)$ by the rule:

\[
(i < j) \& (i \prec j) \iff \varphi_i < \varphi_j \text{ or } (\varphi_i = \varphi_j) \& (s_j = \uparrow). \tag{20}
\]

For $n = 1, 2, \ldots$ the pairs $(\varphi_j, s_j)$ are independent and identically distributed, with $\varphi_n$ having a quasi–uniform distribution $\nu$ and the conditional law of $s_n$ given by

\[
\mathbb{P}(s_n = \uparrow | \varphi_n = \varphi) = \frac{b - \varphi}{b - a}, \quad \mathbb{P}(s_n = \downarrow | \varphi_n = \varphi) = \frac{\varphi - a}{b - a}
\]

for each $\varphi \in [0, 1]$ being the initial point of a bi–interval $]a, \varphi[ \cup \varphi, b[\text{, and } \mathbb{P}(s_n = \cdot | \varphi_n = \varphi) = 1$ for each $\varphi \in [0, 1] \setminus (U_\uparrow \cup U_\downarrow)$.

The representation (20) of $\prec$ involving the heights is canonical; other representations may be obtained by replacing $\varphi_n$’s by $\theta_n = f(\varphi_n)$ with $f : [0, 1] \to \mathbb{R}$ strictly increasing on the support of $\nu$. From a statistical perspective, extending the relation $i \preceq j \iff \theta_i \leq \theta_j$ may be regarded as ‘breaking ties’ in the sequence with repetitions $(\theta_n)$, and this extension is very natural since it agrees with spreadability of the resulting total order (the concept of spreadability is discussed in the next section).
Remark 39. For \( i, j \in B_\phi \) the number of integers \( \triangleleft \)-ordered between \( i \) and \( j \) has a binomial distribution. Moreover, no integer \( n > \max(i, j) \) can be \( \triangleleft \)-ordered between \( i \) and \( j \). Each \( B_\phi \) is therefore a saturated \( \triangleleft \)-chain with the property that there are only finitely many elements between any \( i, j \in B_\phi \).

7 Completeness of the description of the boundary

The random order \( \triangleleft \) induced by an oriented paintbox is not exchangeable, since its distribution may be affected by bijections \( N \rightarrow N \) (with the sole exception \( U_\uparrow \cup U_\downarrow = \emptyset \) when each \( \Pi_n \) is a uniform random permutation). Still, the order \( \triangleleft \) possesses a weaker symmetry property of spreadability. Spreadability is the key property which allows reducing description of all random arrangements \( \Pi \) with descent sets as sufficient statistics to a recent result by Jacka and Warren [22]. See [1] for a survey of earlier results on spreadability and its relation to exchangeability, and see [23] for an updated exposition (in [23] ‘spreadability’ is also called ‘contractability’).

For a subset \( B \subset N \) let us call ranking the standard order isomorphism of \( B \) onto some \([n]\) or onto \( N \), thus the ranking sends \( b \in B \) to \( j \) when \( b \) is the \( j \)th smallest element in \( B \). For \( \pi_n \) a permutation of \([n]\) and \( j \in [n] \), let \( \tau_j(\pi_n) \) be the permutation of \([n-1]\) obtained by removing \( j \) from the sequence \( \pi_n(1), \ldots, \pi_n(n) \) and replacing the elements in the reduced sequence by their ranks.

Proposition 40. Suppose \( \Pi_n \) is a random permutation of \([n]\) which satisfies

\[
\mathbb{P}(\Pi_n = \pi_n) = p(\text{zs}(\pi_n)) \tag{21}
\]

for some function \( p : \mathbb{Z}_n \rightarrow [0, 1] \). Then the distribution of \( \tau_j(\Pi_n) \) is the same for all \( j = 1, \ldots, n \) and satisfies the counterpart of (21) with the function \( p \) extended to \( \mathbb{Z}_{n-1} \) by the virtue of recursion (2) for the graph \( Z \).

Proof. It is sufficient to consider the case when \( \Pi_n \) is uniformly distributed on the set of permutations with some fixed zigzag shape \( \lambda \), because the general case is a mixture. Given \( j = 2, \ldots, n \), divide the set of such permutations in two disjoint subsets \( A \) and \( B \), with \( A \) being the set of permutations in which \( j - 1 \) and \( j \) occupy adjacent positions, and \( B \) being all other permutations which have \( j - 1 \) and \( j \) separated. We have \( \tau_{j-1}(A) = \tau_j(A) \), meaning
the equality of multisets, just because \( \tau_{j-1}(\pi_n) = \tau_j(\pi_n) \) for \( \pi_n \in A \). The latter does not hold for \( B \), but exchanging \( j \) and \( j-1 \) preserves the zigzag shape and, moreover, yields a bijection of \( B \), thus also \( \tau_{j-1}(B) = \tau_j(B) \). Therefore \( \tau_j(\Pi_n) \overset{d}{=} \tau_{j-1}(\Pi_n) \) (equality of distributions), hence by induction all distributions \( \tau_j(\Pi_n) \) are the same for \( 1 \leq j \leq n \). The second assertion is true for \( j = n \) (by the definition of graph \( Z \) and \( [21] \) hence, from the above, it is true for all \( j \leq n \).

For \( \Pi \) a random arrangement of \( \mathbb{N} \) and an infinite set \( B \subset \mathbb{N} \) let \( \Pi|_B \) denote the random arrangement obtained by ranking \( B \), and for a finite set \( B \subset \mathbb{N} \) with \#\( B = n \) let \( \Pi|_B \) be the random permutation of \( [n] \) obtained in the same way. Call \( \Pi \) spreadable if \( \Pi|_B \) has the same probability distribution as \( \Pi \) for every infinite \( B \subset \mathbb{N} \). For \( \Pi = (\Pi_n) \) the identity \( \Pi|_n = \Pi_n \) holds by definition, and in view of the fact that the law of \( \Pi \) is completely determined by the laws of coherent permutations \( \Pi_n \), the spreadability property just says that \( \Pi|_B \overset{d}{=} \Pi_n \) for each \( B \subset \mathbb{N} \) with \#\( B = n \) and each \( n \). This definition of spreadability for \( \subset \) agrees with that for the random array \( Z_{i,j} = 1(i \prec j) \) (see [1]), meaning \( (Z_{i,j}) \overset{d}{=} (Z_{T(i),T(j)}) \) for all increasing \( T : \mathbb{N} \to \mathbb{N} \). The next corollary follows by iterated application of Proposition 40.

**Corollary 41.** If random permutations \( (\Pi_n) \) are coherent and each \( \Pi_n \) is conditionally uniform given \( \text{vs}(\Pi_n) \), then the arrangement \( \Pi = (\Pi_n) \) is spreadable.

The above definition of spreadability works equally well for the set \( \mathbb{Z} \). Note also that the oriented paintbox construction can be trivially modified to produce random arrangements of \( \mathbb{Z} \), we just need to label the random variables \( \xi_j \) by \( \mathbb{Z} \) instead of \( \mathbb{N} \).

**Proposition 42.** (Jacka and Warren [22, Theorem 3.4]) Each spreadable random arrangement on \( \mathbb{Z} \) has the same probability law as the one defined by the oriented paintbox \( (U_1, U_4) \) chosen at random from some probability distribution on \( U(2) \).

To apply Proposition 42 it remains to show that

**Lemma 43.** Each spreadable arrangement on \( \mathbb{N} \) has a distributionally unique extension to a spreadable arrangement on \( \mathbb{Z} \).
Proof. For $\triangleleft$ a spreadable arrangement on $\mathbb{Z}$ the distribution is uniquely determined by the distributions of $\triangleleft|_{[-n,n]}$ for $n = 1, 2, \ldots$. On the other hand, for $\triangleleft$ a spreadable arrangement on $\mathbb{N}$ the restriction $\triangleleft|_{[2n+1]}$ can be uniquely transported to $[-n,n]$. It is easy to check that the pushforwards are consistent for various values of $n$, hence determine an arrangement of $\mathbb{Z}$.

Putting the things together we arrive at our principal conclusion.

**Theorem 44.** The boundary $\partial \mathcal{Z}$ of the graph of zigzag diagrams is homeomorphic to $\mathcal{U}^{(2)}$. Each coherent sequence of permutations $\Pi$ with descent set as a sufficient statistic can be represented by the oriented paintbox construction with some random $(U_\uparrow, U_\downarrow) \in \mathcal{U}^{(2)}$. The distribution of the paintbox $(U_\uparrow, U_\downarrow)$ representing a given $\Pi$ is unique.

**Proof.** The second claim follows from Corollary 41, Lemma 43, and Proposition 42. Then the first claim becomes evident due to Proposition 36. The third claim follows by either recalling that the set $\mathcal{V}$ of probability functions is a Choquet simplex or, more constructively, by appealing to Proposition 34 which allows to identify a random oriented paintbox with the almost sure limit of zigzags $\text{zs}(\Pi_n)$, suitably embedded in $\mathcal{U}^{(2)}$. □

### 8 Complements and open questions

#### 8.1 The involution

The conjugation of zigzags $\lambda \mapsto \lambda'$, defined and interpreted in terms of binary encoding in Section 2, differs from another natural involutive operation on zigzags, see [42, Exercise 7.94]. In terms of descent sets $D \subset [n-1]$, the latter amounts to the operation $D \mapsto [n-1] \setminus D$, which is the same as just switching + ↔ −. In terms of ribbon Young diagrams $\lambda$ (see Section 2), our version is related to the conventional symmetry of (skew) Young diagrams, while the alternative version is related to the reflection about the bisectrix of the first quadrant. An advantage of our definition is that our operation induces a natural involution $\pi_n \mapsto \pi'_n$ on the standard paths in the graph $\mathcal{Z}$ (in terms of infinite paths this means inversion of total order on $\mathbb{N}$), while another version leads to a cumbersome transformation of paths.

These two operations on zigzags determine two distinct involutive automorphisms of $\text{QSym}$, $\text{inv}$ and $\hat{\omega}$, which, however, coincide on $\text{Sym}$ with the
canonical involution $\omega$ of that algebra. For $\tilde{\omega}$, a simple proof of this fact is given in [42, solution to Exercise 7.94], and for inv the argument is similar.

8.2 The empty paintbox

The paintbox $U_\uparrow = U_\downarrow = \emptyset$ corresponds to the only exchangeable random arrangement $\Pi = (\Pi_n)$ with each $\Pi_n$ being the uniform random permutation of $[n]$. Thus,

$$p(\emptyset, \emptyset)(\lambda_n) = d(\lambda_n)/n! \quad \lambda_n \in \mathcal{Z}_n.$$  

The corresponding character $\psi(\emptyset, \emptyset)$ projects to the Plancherel character $\psi_{0,0}$ of Sym. Thus, $\psi(\emptyset, \emptyset)$ may be viewed as an analog of the Plancherel character. There is a large literature on properties of the descent set for uniform permutations, e.g., it is well known that in this case the distribution of the number of descents is close to normal with mean $(n - 1)/2$, see [2]. It would be interesting to learn which results on the Plancherel measure on the Young diagrams and the corresponding measure on the paths in the Young graph $\mathcal{Y}$ translate to the graph $\mathcal{Z}$, see [37] for some work in this direction.

8.3 The Martin boundary

The concept of Martin boundary for graded graphs [27] may be well adapted to our context; this corresponds to the entrance boundary [24] for the inverse Markov chain. Given two zigzags $\mu, \lambda \in \mathcal{Z}$ let $d(\mu, \lambda)$ denote the number of paths (if any) $\mu \Rightarrow \cdots \Rightarrow \lambda$ in $\mathcal{Z}$ ascending from $\mu$ to $\lambda$ and set

$$K(\mu, \lambda) = \frac{d(\mu, \lambda)}{d(\lambda)}. \quad (22)$$

For $\lambda_n \in \mathcal{Z}_n$ fixed, the function $\mu \mapsto K(\mu, \lambda_n)$ satisfies the recursion (2) in $\mu$ up to level $n - 1$. A sequence $(\lambda_n \in \mathcal{Z}_n)$ is said to be regular if for any fixed $\mu$ the numerical sequence $K(\mu, \lambda_n)$ has a limit. For a regular sequence $(\lambda_n)$, the function

$$p(\mu) := \lim_{n \to \infty} K(\mu, \lambda_n)$$

is a solution to (2), hence an element of $\mathcal{V}$. The Martin boundary of $\mathcal{Z}$ is the collection of all $p \in \mathcal{V}$ which arise as such limits. One can prove that the Martin boundary contains the boundary $\partial \mathcal{Z}$, hence $\partial \mathcal{Z}$ may be also called the minimal boundary. The inclusion relation between the two boundaries
also holds for the general graded graphs, see for instance [25, Chapter 1, §1]. Examples of graphs are known which have the Martin boundary strictly larger than the minimal boundary [19].

**Conjecture 45.** (i) A sequence \((\lambda_n \in \mathcal{Z}_n)\) is regular if and only if the points \(\iota_n(\lambda_n)\) converge in the compactum \(U^{(2)}\).

(ii) \(\iota_n(\lambda_n) \to (U^\uparrow, U^\downarrow)\) is equivalent to \(K(\mu, \lambda_n) \to p_{(U^\uparrow, U^\downarrow)}(\mu)\) for all \(\mu \in \mathcal{Z}\).

(iii) The Martin boundary of the graph \(\mathcal{Z}\) actually coincides with its minimal boundary \(\partial\mathcal{Z} = U^{(2)}\).

The analogues of (i)–(iii) are known to be true for the Young graph \(\mathcal{Y}\) [27], and for the graphs \(\mathcal{P}\) and \(\mathcal{C}\) [29, 16, 25].

### 8.4 The kernel \(K(\mu, \lambda)\)

Consider the Young graph \(\mathcal{Y}\) and the corresponding kernel \(K(\mu, \lambda)\) given by (22), where \(\mu\) and \(\lambda\) are assumed to be Young diagrams. The algebraic approach of [27] to Thoma’s theorem describing the boundary \(\partial\mathcal{Y}\) relies on the following properties of the kernel.

- The linear span (over \(\mu\)) of the set of functions
  \[ f_\mu(\lambda) = n(n-1) \cdots (n-m+1)K(\mu, \lambda), \quad \mu \in \mathcal{Y}_m, \quad \lambda \in \mathcal{Y}_n \]
  is an algebra \(A\) under pointwise multiplication.

- This algebra \(A\) can be identified with the algebra Sym of symmetric functions in such a way that
  \[ f_\mu = S_\mu + \text{lower degree terms} \]
  where \(S_\mu\) is the Schur function.

Moreover, we have a very detailed information about the functions \(f_\mu\), viewed as elements of Sym: these are certain ‘factorial’ analogues of Schur functions, called the *Frobenius–Schur functions*, see [30].

Similar approach applies to the graph \(\mathcal{P}\): then the corresponding functions \(f_\mu\) are identified with natural factorial analogues of monomial symmetric functions. As an application one gets an algebraic derivation of Kingman’s theorem about \(\partial\mathcal{P}\), see [25]. One can also prove similar claims for
the graph $C$: then $A$ is identified with $Q\text{Sym}$ and the functions $f_\mu$ turn into factorial quasisymmetric monomial functions. Again, this yields an algebraic approach to computing the boundary $\partial C$, hence entails an alternative proof of the main result of [16].

**Problem 46.** Is it possible to extend this approach to the kernel (22) on the graph $Z$?

A positive answer would lead to an alternative proof of Theorem 44 and could also be useful in studying Conjecture 45.

### 8.5 Nonextreme solutions $p \in V$

There has been an extensive study of concrete examples of nonextreme solutions $p$ to recursion (2) for the graphs $P$, $C$, and $Y$. These examples arise from various probabilistic models and the problem of harmonic analysis on the infinite symmetric group, see [16, 39, 7].

**Problem 47.** Find natural examples of nonextreme solutions $p$ to recursion (2) on the graph $Z$.

### 8.6 The graph associated with the peak algebra

A graph $Q$ closely related to $Z$ can be viewed as a natural ‘quasisymmetric analogue’ of the so–called *Schur graph* $S$ which appears in the theory of projective characters of the symmetric groups and is associated with the classical $Q$–Schur functions [34, 6]. The standard finite paths in $Q$ are in bijection with permutations, like for $Z$, but the set of vertices is different. While a generic vertex of $Z$ corresponds to the descent set of a permutation, a vertex of $Q$ corresponds to the *peak set* of a permutation. Thus, the analogue of recursion (2) for $Q$ reflects the branching of coherent permutations with the peak–set statistics. Alternatively, $Q$ can be defined as the ‘multiplicative’ graph associated with the *peak algebra* $\text{Peak} \subset Q\text{Sym}$ and its distinguished basis introduced by Stembridge [43].

Our method of constructing the boundary can be extended to the graph $Q$: here we use the fact that $\text{Peak}$ is closed under comultiplication. The algebra $\text{Peak}$ has no natural involution, hence in the construction of characters we may exploit only one elementary character (as in the cases of $P$ and $C$).
The boundary of $Q$ is parametrised by open sets $U \subset ]0,1[$. The completeness of the description of the boundary can be deduced from Theorem 44 using the fact that Peak can also be realised as a quotient of QSym.

The relations between the boundaries of the graphs is represented by the diagram

\[
\partial Q \longrightarrow \partial Z \leftarrow \partial C
\]

\[
\partial S \longrightarrow \partial Y \leftarrow \partial P
\]

Specifically, by the embedding $\partial P \hookrightarrow \partial Y$ the parameter $\alpha \in \Delta$ goes to $(\alpha, 0) \in \Delta^{(2)}$. The boundary $\partial S$ is again $\Delta$ (as is read from [34]), but the embedding $\partial S \hookrightarrow \partial Y$ sends $\alpha \in \Delta$ to $(\alpha', \beta') \in \Delta^{(2)}$ with $\alpha' = \beta' = (\alpha_1/2, \alpha_2/2, \ldots)$. Likewise, the embedding $\partial C \hookrightarrow \partial Z$ sends $U \in U$ to $(U, \emptyset) \in U^{(2)}$, while by the embedding $\partial Q \hookrightarrow \partial Z$ each $U \in U$ is mapped to the pair $(U_1', U_2')$ which is obtained by halving each component interval of $U$ and composing $U_1'$ of right halves and $U_2'$ of left halves.

8.7 The ring theorem

Proposition 8 refers to the Kerov–Vershik ‘ring theorem’ [28, 25]. For reader’s convenience we present here a complete proof which follows the lines in [28], but provides some details omitted in that paper. Assume that

(i) we are given a commutative unital algebra $A$ over $\mathbb{R}$ together with a convex cone $K$ in $A$ (below we write $a \geq b$ if $a - b \in K$),

(ii) the cone is generating ($K - K = A$) and stable under multiplication ($K \cdot K \subseteq K$),

(iii) the unity $1$ in $A$ is also an order unity, meaning $1 \geq 0$, and that for any $a \in K$ there exists a small $\epsilon > 0$ with $\epsilon a \leq 1$,

(iv) the cone $K$ is generated by a countable set of elements.

Denote by $P \subset A^*$ the set of all linear functionals $\psi : A \rightarrow \mathbb{R}$ which are nonnegative on $K$ and normalised by $\psi(1) = 1$, and observe that $P$ is a convex set.

**Proposition 48.** Under these assumptions, a functional $\psi \in P$ is an extreme point of $P$ if and only if $\psi$ is multiplicative, that is, $\psi(ab) = \psi(a)\psi(b)$ for all $a, b \in A$. 

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Proof. First of all, note that if $\psi$ vanishes on an element $a \geq 0$ then $\psi(ab) = 0$ for any $b \in A$. Indeed, by (ii) and (iii), without loss of generality, we may assume that $0 \leq b \leq 1$. Then we have

$$0 \leq \psi(ab) \leq \psi(ab) + \psi(a(1-b)) = \psi(a) = 0,$$

whence $\psi(ab) = 0$. Assume now that $\psi$ is extreme. To show that $\psi$ is multiplicative, it suffices to prove that $\psi(ab) = \psi(a)\psi(b)$ for $0 \leq a \leq 1$ and any $b$. Set $a' = 1 - a$ and suppose first that both $\psi(a)$ and $\psi(a')$ are nonzero. Then the functionals

$$\psi_a(b) := \frac{\psi(ab)}{\psi(a)} \quad \text{and} \quad \psi_{a'}(b) := \frac{\psi(a'b)}{\psi(a')}$$

are well defined and are both elements of $P$. Since $\psi$ is a convex combination of $\psi_a$ and $\psi_{a'}$ (with coefficients $\psi(a)$ and $\psi(a')$, respectively), and $\psi$ is extreme, we have either $\psi = \psi_a$ or $\psi = \psi_{a'}$. Consequently, we have either $\psi(b) = \psi_a(b)$ or $\psi(b) = \psi_{a'}(b)$, and each of these relations implies $\psi(ab) = \psi(a)\psi(b)$.

Furthermore, if $\psi(a) = 0$ then, as mentioned earlier, $\psi(ab) = 0$, which again implies $\psi(ab) = \psi(a)\psi(b)$. Similarly, $\psi(a') = 0$ implies $\psi(a'b) = \psi(a')\psi(b)$, which is equivalent to $\psi(ab) = \psi(a)\psi(b)$.

Let us check the inverse implication. Denote by $P_{ex} \subset P$ the subset of extreme points. The convex set $P$ lies in the dual vector space $A^*$ which we consider with the weak topology. By (iii) and (iv) there exists a sequence $b_1, b_2, \ldots \in K$ spanning the cone and such that $b_1 \leq 1, b_2 \leq 1, \ldots$. By (ii), the family $\{b_1, b_2, \ldots\}$ contains a basis of $A$. It readily follows that the mapping $\psi \mapsto (\psi(b_1), \psi(b_2), \ldots)$ determines a homeomorphism of $P$ onto a closed subset of the product space $[0,1]^\infty$, hence $P$ is a compact, metrisable, and separable space. By Choquet’s theorem, $P_{ex}$ is a $G_\delta$–subset in $P$, and each $\psi \in P$ is representable as a mixture

$$\psi(a) = \int_{P_{ex}} \varphi(a)\sigma(d\varphi) \quad \forall a \in A$$

with some probability measure $\sigma$ on $P_{ex}$.

Using the embedding $P \hookrightarrow [0, 1]^\infty$ we may view $\sigma$ as a probability measure on the product space $[0, 1]^\infty$, so that the coordinate functions in this space become random variables. Now assume $\psi$ multiplicative. Then, in particular, $\psi(b_i^2) = (\psi(b_i))^2$ for every $i = 1, 2, \ldots$. It follows that each of the coordinate
functions has variance 0, hence is constant almost surely. This means that $\sigma$ is supported by a single point, that is, $\psi \in P_{ex}$, as wanted. \hfill $\Box$

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