ALGEBRAICITY OF THE ZETA FUNCTION
ASSOCIATED TO A MATRIX OVER
A FREE GROUP ALGEBRA

CHRISTIAN KASSEL AND CHRISTOPHE REUTENAUER

Abstract. Following and generalizing a construction by Kontsevich, we associate a zeta function to any matrix with entries in a ring of noncommutative Laurent polynomials with integer coefficients. We show that such a zeta function is an algebraic function.

1. Introduction

Fix a commutative ring $K$. Let $F$ be a free group on a finite number of generators $X_1, \ldots, X_n$ and

$$KF = K\langle X_1, X_1^{-1}, \ldots, X_n, X_n^{-1} \rangle$$

be the corresponding group algebra: equivalently, it is the algebra of non-commutative Laurent polynomials with coefficients in $K$. Any element $a \in KF$ can be uniquely written as a finite sum of the form

$$a = \sum_{g \in F} (a, g) g ,$$

where $(a, g) \in K$.

Let $M$ be a $d \times d$-matrix with coefficients in $KF$. For any $n \geq 1$ we may consider the $n$-th power $M^n$ of $M$ and its trace $\text{Tr}(M^n)$, which is an element of $KF$. We define the integer $a_n(M)$ as the coefficient of 1 in the trace of $M^n$:

$$(1.1) \quad a_n(M) = \langle \text{Tr}(M^n), 1 \rangle .$$

Let $g_M$ and $P_M$ be the formal power series

$$(1.2) \quad g_M = \sum_{n \geq 1} a_n(M) t^n \quad \text{and} \quad P_M = \exp \left( \sum_{n \geq 1} a_n(M) \frac{t^n}{n} \right) .$$

They are related by

$$g_M = t \frac{d \log(P_M)}{dt} .$$

We call $P_M$ the zeta function of the matrix $M$ by analogy with the zeta function of a noncommutative formal power series (see Section 2.1); the two concepts will be related in Proposition 4.1.

2010 Mathematics Subject Classification. (Primary) 05A15, 14G10, 68Q70; (Secondary) 05E15, 68R15, 20M35, 14H05

Key words and phrases. Noncommutative formal power series, language, zeta function, algebraic function.
The motivation for the definition of $P_M$ comes from the well-known identity expressing the inverse of the reciprocal polynomial of the characteristic polynomial of a matrix $M$ with entries in a commutative ring

$$\frac{1}{\det(1 - tM)} = \exp\left(\sum_{n \geq 1} \text{Tr}(M^n) \frac{t^n}{n}\right).$$

Note that for any scalar $\lambda \in K$, the corresponding series for the matrix $\lambda M$ become

$$g_{\lambda M}(t) = g_M(\lambda t) \quad \text{and} \quad P_{\lambda M}(t) = P_M(\lambda t).$$

Our main result is the following; it was inspired by Theorem 1 of [15].

**Theorem 1.1.** For each matrix $M \in M_d(KF)$ where $K = \mathbb{Q}$ is the ring of rational numbers, the formal power series $P_M$ is algebraic.

The special case $d = 1$ is due to Kontsevich [15]. A combinatorial proof in the case $d = 1$ and $F$ is a free group on one generator appears in [17].

Observe that by the rescaling equalities (1.3) it suffices to prove the theorem when $K = \mathbb{Z}$ is the ring of integers.

It is crucial for the veracity of Theorem 1.1 that the variables do not commute: for instance, if $a = x + y + x^{-1} + y^{-1} \in \mathbb{Z}[x, x^{-1}, y, y^{-1}]$, where $x$ and $y$ are commuting variables, then $\exp(\sum_{n \geq 1} (a^n, 1) t^n/n)$ is a formal power series with integer coefficients, but not an algebraic function (this follows from Example 3 in [7, Sect. 1]).

The paper is organized as follows. In Section 2 we define the zeta function $\zeta_S$ of a noncommutative formal power series $S$ and show that it can be expanded as an infinite product under a cyclicity condition that is satisfied by the characteristic series of cyclic languages.

In Section 3 we recall the notion of an algebraic noncommutative formal power series and some of their properties.

In Section 4 we reformulate the zeta function of a matrix as the zeta function of a noncommutative formal power series before giving the proof of Theorem 1.1; the latter follows the steps sketched in [15] and relies on the results of the previous sections as well as on an algebraicity result by Andrè [2] elaborating on an idea of D. and G. Chudnovsky.

We concentrate on two specific matrices in Section 5. We give a closed formula for the zeta function of the first matrix; its nonzero coefficients count the planar rooted bicubic maps as well as Chapoton’s “new intervals” in a Tamari lattice (see [9, 22]).

### 2. Cyclic formal power series

#### 2.1. General definitions

As usual, if $A$ is a set, we denote by $A^*$ the free monoid on $A$: it consists of all words on the alphabet $A$, including the empty word $1$. Let $A^+ = A - \{1\}$.

Recall that $w \in A^+$ is primitive if it cannot be written as $w'$ for any integer $r \geq 2$ and any $u \in A^+$. Two elements $w, w' \in A^+$ are conjugate if $w = uw$ and $w' = vu$ for some $u, v \in A^+$.

Given a set $A$ and a commutative ring $K$, let $K\langle A \rangle$ be the algebra of noncommutative formal power series on the alphabet $A$. For any element
$S \in K\langle \langle A \rangle \rangle$ and any $w \in A^*$, we define the coefficient $(S, w) \in K$ by

$$S = \sum_{w \in A^*} \langle S, w \rangle w .$$

As an example of such noncommutative formal power series, take the characteristic series $\sum_{w \in L} w$ of a language $L \subseteq A^*$. In the sequel we shall identify a language with its characteristic series.

The generating series $g_S$ of an element $S \in K\langle \langle A \rangle \rangle$ is the image of $S$ under the algebra map $\varepsilon : K\langle \langle A \rangle \rangle \to K[[t]]$ sending each $a \in A$ to the variable $t$.

We have

$$\begin{equation}
(2.1) \quad g_S - (S, 1) = \sum_{w \in A^+} \langle S, w \rangle t^{|w|} = \sum_{n \geq 1} \left( \sum_{|w| = n} \langle S, w \rangle \right) t^n ,
\end{equation}$$

where $|w|$ is the length of $w$.

The zeta function $\zeta_S$ of $S \in K\langle \langle A \rangle \rangle$ is defined by

$$\begin{equation}
(2.2) \quad \zeta_S = \exp \left( \sum_{w \in A^+} \langle S, w \rangle t^{|w|} \right) = \exp \left( \sum_{n \geq 1} \left( \sum_{|w| = n} \langle S, w \rangle \right) t^n \right) .
\end{equation}$$

The formal power series $g_S$ and $\zeta_S$ are related by

$$\begin{equation}
(2.3) \quad t \frac{d \log(\zeta_S)}{dt} = t \frac{\zeta_S'}{\zeta_S} = g_S - (S, 1) ,
\end{equation}$$

where $\zeta_S'$ is the derivative of $\zeta_S$ with respect to the variable $t$.

### 2.2. Cyclicity

**Definition 2.1.** An element $S \in K\langle \langle A \rangle \rangle$ is cyclic if

(i) $\forall u, v \in A^*$, $(S, uv) = (S, vu)$ and

(ii) $\forall w \in A^+, \forall r \geq 2$, $(S, w^r) = (S, w)^r$.

Cyclic languages provide examples of cyclic formal power series. Recall from [4, Sect. 2] that a language $L \subseteq A^*$ is cyclic if

1. $\forall u, v \in A^*$, $uv \in L \iff vu \in L$,

2. $\forall w \in A^+, \forall r \geq 2$, $w^r \in L \iff w \in L$.

The characteristic series of a cyclic language is a cyclic formal power series in the above sense.

Let $L$ be any set of representatives of conjugacy classes of primitive elements of $A^+$.

**Proposition 2.2.** If $S \in K\langle \langle A \rangle \rangle$ is a cyclic formal power series, then

$$\zeta_S = \prod_{\ell \in L} \frac{1}{1 - \langle S, \ell \rangle t^{|\ell|}} .$$

**Proof.** Since both sides of the equation have the same constant term 1, it suffices to prove that they have the same logarithmic derivative. The logarithmic derivative of the RHS multiplied by $t$ is equal to

$$\sum_{\ell \in L} \frac{|\ell| \langle S, \ell \rangle t^{|\ell|}}{1 - \langle S, \ell \rangle t^{|\ell|}} ,$$
which in turn is equal to

\[ \sum_{\ell \in L, k \geq 1} |\ell| (S, \ell)^k t^{k|\ell|}. \]

In view of (2.1) and (2.3) it is enough to check that for all \( n \geq 1 \),

\( (2.4) \)

\[ \sum_{|w|=n} (S, w) = \sum_{\ell \in L, k \geq 1, k|\ell|=n} |\ell| (S, \ell)^k. \]

Now any word \( w = u^k \) is the \( k \)-th power of a unique primitive word \( u \), which is the conjugate of a unique element \( \ell \in L \). Moreover, \( w \) has exactly \( |\ell| \) conjugates and since \( S \) is cyclic, we have

\[ (S, w) = (S, u^k) = (S, u)^k = (S, \ell)^k. \]

From this Equation (2.4) follows immediately. \( \Box \)

**Corollary 2.3.** If a cyclic formal power series \( S \) has integer coefficients, i.e., \( (S, w) \in \mathbb{Z} \) for all \( w \in A^* \), then so has \( \zeta_S \).

### 3. Algebraic noncommutative series

This section is essentially a compilation of well-known results on algebraic noncommutative series.

Recall that a system of proper algebraic noncommutative equations is a finite set of equations

\[ \xi_i = p_i \quad i = 1, \ldots, n, \]

where \( \xi_1, \ldots, \xi_n \) are noncommutative variables and \( p_1, \ldots, p_n \) are elements of \( K(\xi_1, \ldots, \xi_n, A) \), where \( A \) is some alphabet. We assume that each \( p_i \) has no constant term and contains no monomial \( \xi_j \). One can show that such a system has a unique solution \( (S_1, \ldots, S_n) \), i.e., there exists a unique \( n \)-tuple \( (S_1, \ldots, S_n) \) such that \( S_i = p_i(S_1, \ldots, S_n, A) \) for all \( i = 1, \ldots, n \) and each \( S_i \) has no constant term (see [20], [18, Th. IV.1.1], or [21, Prop. 6.6.3]).

If a formal power series \( S = K(A) \) differs by a constant from such a formal power series \( S_i \), we say that \( S \) is algebraic.

**Example 3.1.** Consider the proper algebraic noncommutative equation

\[ \xi = a\xi^2 + b. \]

(Here \( A = \{a, b\} \).) Its solution is of the form

\[ S = b + abb + aabbb + ababb + \cdots. \]

One can show (see [3]) that \( S \) is the characteristic series of Łukasiewicz’s language, namely of the set of words \( w \in \{a, b\}^* \) such that \( |w|_b = |w|_a + 1 \) and \( |u|_a \geq |u|_b \) for all proper prefixes \( u \) of \( w \).

Recall also that \( S \in K(A) \) is rational if it belongs to the smallest subalgebra of \( K(A) \) containing \( K(A) \) and closed under inversion. By a theorem of Schützenberger (see [5, Th. I.7.1]), a formal power series \( S \in K(A) \) is rational if and only if it is recognizable, i.e., there exist an integer \( n \geq 1 \), a representation \( \mu \) of the free monoid \( A^* \) by matrices with entries in \( K \), a row-matrix \( \alpha \) and a column-matrix \( \beta \) such that for all \( w \in A^* \),

\[ (S, w) = \alpha \mu(w) \beta. \]
We now record two well-known theorems.

**Theorem 3.2.** (1) If \( S \in K\langle A \rangle \) is algebraic, then its generating series \( g_S \in K[[t]] \) is algebraic in the usual sense.

(2) The set of algebraic power series is a subring of \( K\langle A \rangle \).

(3) A rational power series is algebraic.

(4) The Hadamard product of a rational power series and an algebraic power series is algebraic.

(5) Let \( A = \{a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}\} \) and \( L \) be the language consisting of all words on the alphabet \( A \) whose image in the free group on \( a_1, \ldots, a_n \) is the identity element. Then the characteristic series of \( L \) is algebraic.

Items (1)–(4) of the previous theorem are due to Schützenberger [20]. Item (5) to Chomsky and Schützenberger [10] (see [21, Example 6.6.8]).

The second theorem is a criterion due to Jacob [13].

**Theorem 3.3.** A formal power series \( S \in K\langle A \rangle \) is algebraic if and only if there exist a free group \( F \), a representation \( \mu \) of the free monoid \( A^* \) by matrices with entries in \( KF \), indices \( i, j \) and \( \gamma \in F \) such that for all \( w \in A^* \),

\[
(S, w) = ((\mu w)_{i,j}, \gamma).
\]

The following is an immediate consequence of Theorem 3.3.

**Corollary 3.4.** If \( S \in K\langle A \rangle \) is an algebraic power series and \( \varphi : B^* \to A^* \) is a homomorphism of finitely generated free monoids, then the power series

\[
\sum_{w \in B^*} (S, \varphi(w)) \ w \in K\langle B \rangle
\]

is algebraic.

As a consequence of Theorem 3.2(5) and of Corollary 3.4, we obtain the following.

**Corollary 3.5.** Let \( f : A^* \to F \) be a homomorphism from \( A^* \) to a free group \( F \). Then the characteristic series of \( f^{-1}(1) \in K\langle A \rangle \) is algebraic.

4. **Proof of Theorem 1.1**

Let \( M \) be a \( d \times d \)-matrix. As observed in the introduction, it is enough to establish Theorem 1.1 when all the entries of \( M \) belong to \( \mathbb{Z}F \).

We first reformulate the formal power series \( g_M \) and \( P_M \) of (1.2) as the generating series and the zeta function of a noncommutative formal power series, respectively.

Let \( A \) be the alphabet whose elements are triples \([g, i, j]\) where \( i, j \) are integers such that \( 1 \leq i, j \leq d \) and \( g \in F \) appears in the \((i, j)\)-entry \( M_{i,j} \) of \( M \), i.e., \( (M_{i,j}, g) \neq 0 \). We define the noncommutative formal power series \( S_M \in K\langle A \rangle \) as follows: for \( w = [g_1, i_1, j_1] \cdots [g_n, i_n, j_n] \in A^* \), the scalar \((S_M, w)\) vanishes unless we have

(a) \( j_n = i_1 \) and \( j_k = i_{k+1} \) for all \( k = 1, \ldots, n - 1 \),

(b) \( g_1 \cdots g_n = 1 \) in the group \( F \),

in which case \((S_M, w)\) is given by

\[
(S_M, w) = (M_{i_1,j_1}, g_1) \cdots (M_{i_n,j_n}, g_n) \in K.
\]

By convention, \((S_M, 1) = d\).
Proposition 4.1. The generating series and the zeta function of $S_M$ are related to the formal power series $g_M$ and $P_M$ of (1.2) by

$$g_{SM} - d = g_M \quad \text{and} \quad \zeta_{SM} = P_M.$$  

Proof. For $n \geq 1$ we have

$$\text{Tr}(M^n) = \sum M_{i_1,j_1} \cdots M_{i_n,j_n} = \sum (M_{i_1,j_1}, g_1) \cdots (M_{i_n,j_n}, g_n) g_1 \cdots g_n,$$

where the sum runs over all indices $i_1, j_1, \ldots, i_n, j_n$ satisfying Condition (a) above and over all $g_1, \ldots, g_n \in F$. Then

$$a_n(M) = (\text{Tr}(M^n), 1) = \sum (M_{i_1,j_1}, g_1) \cdots (M_{i_n,j_n}, g_n),$$

where Conditions (a) and (b) are satisfied. Hence,

$$a_n(M) = \sum_{w \in A^* \mid |w|=n} (S, w),$$

which proves the proposition in view of (1.2), (2.1) and (2.2). \hfill \Box

We next establish that $S_M$ is both cyclic in the sense of Section 2 and algebraic in the sense of Section 3.

Proposition 4.2. The noncommutative formal power series $S_M$ is cyclic.

Proof. (i) Conditions (a) and (b) above are clearly preserved under cyclic permutations. Hence we also have

$$(S_M, w) = (M_{i_2,j_2}, g_2) \cdots (M_{i_n,j_n}, g_n) (M_{i_1,j_1}, g_1)$$

when $w = [g_1, i_1, j_1] \cdots [g_n, i_n, j_n]$ such that Conditions (a) and (b) are satisfied. It follows that $(S, uv) = S(vu)$ for all $u, v \in A^*$.

(ii) If $w$ satisfies Conditions (a) and (b), so does $w^r$ for $r \geq 2$. Conversely, if $w^r \ (r \geq 2)$ satisfies Condition (a), then since

$$w^r = [g_1, i_1, j_1] \cdots [g_n, i_n, j_n] [g_1, i_1, j_1] \cdots$$

we must have $j_n = i_1$ and $j_k = i_{k+1}$ for all $k = 1, \ldots, n-1$, and so $w$ satisfies Condition (a).

If $w^r \ (r \geq 2)$ satisfies Condition (b), i.e., $(g_1 \cdots g_n)^r = 1$, then $g_1 \cdots g_n = 1$ since $F$ is torsion-free. Hence $w$ satisfies Condition (b). It follows that $(S, w^r) = (\{(M_{i_1,j_1}, g_1) \cdots (M_{i_n,j_n}, g_n)\})^r = (S, w)^r$. \hfill \Box

Proposition 4.3. The noncommutative formal power series $S_M$ is algebraic.

Proof. We write $S_M$ as the Hadamard product of three noncommutative formal power series $S_1, S_2, S_3$.

The series $S_1 \in K\langle A \rangle$ is defined for $w = [g_1, i_1, j_1] \cdots [g_n, i_n, j_n] \in A^+$ by

$$(S_1, w) = (M_{i_1,j_1}, g_1) \cdots (M_{i_n,j_n}, g_n)$$

and by $(S_1, 1) = 1$. This is a recognizable, hence rational, series with one-dimensional representation $A^* \rightarrow K$ given by $[g, i, j] \mapsto (M_{i,j}, g)$.

Next consider the representation $\mu$ of the free monoid $A^*$ defined by

$$\mu([g, i, j]) = E_{i,j},$$
where $E_{i,j}$ denotes as usual the $d \times d$-matrix with all entries vanishing, except the $(i,j)$-entry which is equal to 1. Set
\[ S_2 = \sum_{w \in A^*} \text{Tr}((\mu w)) \quad w \in K \langle \langle A \rangle \rangle. \]
The power series $S_2$ is recognizable, hence rational. Let us describe $S_2$ more explicitly. For $w = 1$, $\mu(w)$ is the identity $d \times d$-matrix; hence $(S_2, 1) = d$.

For $w = [g_1, i_1, j_1] \cdots [g_n, i_n, j_n] \in A^+$, we have
\[ \text{Tr}((\mu w)) = \text{Tr}(E_{i_1, j_1} \cdots E_{i_n, j_n}). \]
It follows that $\text{Tr}((\mu w)) \neq 0$ if and only if $\text{Tr}(E_{i_1, j_1} \cdots E_{i_n, j_n}) \neq 0$, which is equivalent to $j_n = i_1$ and $j_k = i_{k+1}$ for all $k = 1, \ldots, n-1$, in which case $\text{Tr}((\mu w)) = 1$. Thus,
\[ S_2 = d + \sum_{n \geq 1} \sum [g_1, i_1, i_2] [g_2, i_2, i_3] \cdots [g_n, i_n, i_1], \]
where the second sum runs over all elements $g_1, \ldots, g_n \in F$ and all indices $i_1, \ldots, i_n$.

Finally consider the homomorphism $f : A^* \to F$ sending $[g, i, j]$ to $g$. Then by Corollary 3.5 the characteristic series $S_3 \in K \langle \langle A \rangle \rangle$ of $f^{-1}(1)$ is algebraic.

It is now clear that $S_M$ is the Hadamard product of $S_1$, $S_2$, and $S_3$:
\[ S_M = S_1 \odot S_2 \odot S_3. \]
Since by [5, Th. I.5.5] the Hadamard product of two rational series is rational, $S_1 \odot S_2$ is rational as well. It then follows from Theorem 3.2(4) and the algebraicity of $S_3$ that $S_M = S_1 \odot S_2 \odot S_3$ is algebraic. □

Since $M$ has entries in $\mathbb{Z}_F$, the power series $g_{S_M} = g_M + d$ belongs to $\mathbb{Z}[[t]]$. It follows by Corollary 2.3 and Proposition 4.2 that the power series $P_M = \zeta_{S_M}$ has integer coefficients as well. Moreover, by Theorem 3.2(1) and Proposition 4.3
\[ t \frac{d \log(P_M)}{dt} = g_M \]
is algebraic.

To complete the proof of Theorem 1.1, it suffices to apply the following algebraicity theorem.

**Theorem 4.4.** If $f \in \mathbb{Z}[[t]]$ is a formal power series with integer coefficients such that $t \frac{d \log f}{dt}$ is algebraic, then $f$ is algebraic.

Note that the integrality condition for $f$ is essential: for the transcendental formal power series $f = \exp(t)$, we have $t \frac{d \log f}{dt} = t$, which is even rational.

**Proof.** This result follows from cases of the Grothendieck–Katz conjecture proved in [2] and in [6]. The conjecture states that if $Y' = AY$ is a linear system of differential equations with $A \in M_d(\mathbb{Q}(t))$, then far from the poles of $A$ it has a basis of solutions that are algebraic over $\mathbb{Q}(t)$ if and only if for almost all prime numbers $p$ the reduction mod $p$ of the system has a basis of solutions that are algebraic over $\mathbb{F}_p(t)$. 

Let us now sketch a proof of the theorem (see also Exercise 5 of [1, p. 160]). Set \( g = t'f/f \) and consider the system \( y' = (g/t)y; \) it defines a differential form \( \omega \) on an open set \( S \) of the smooth projective complete curve \( \mathcal{S} \) associated to \( g \). We now follow [2] § 6.3, which is inspired from [11]. First, extend \( \omega \) to a section (still denoted \( \omega \)) of \( \Omega_{\mathcal{S}/\mathbb{F}_p}(-D) \), where \( D \) is the divisor of poles of \( \omega \). For any \( n \geq 2 \), we have a differential form \( \sum_{i=1}^n p_i^*(\omega) \) on \( S^n \), where \( p_i : S^n \rightarrow S \) is the \( i \)-th canonical projection; this form goes down to the symmetric power \( S^{(n)} \). Now let \( J \) be the generalized Jacobian of \( S \) which parametrizes the invertible fibre bundles over \( \mathcal{S} \) that are rigidified over \( D \). There is a morphism \( \varphi : S \rightarrow J \) and a unique invariant differential form \( \omega_J \) on \( J \) such that \( \omega = \varphi^*(\omega_J) \). For any \( n \geq 2 \), \( \varphi \) induces a morphism \( \varphi^{(n)} : S^{(n)} \rightarrow J \) such that \( (\varphi^{(n)})^*(\omega_J) = \sum_{i=1}^n p_i^*(\omega) \). For \( n \) large enough, \( \varphi^{(n)} \) is dominant and if \( \omega_J \) is exact, then so is \( \omega \). To prove that \( \omega_J \) is exact, we note that \( J \), being a scheme of commutative groups, is uniformized by \( \mathbb{C}^n \). We can now apply Theorem 5.4.3 of loc. cit., whose hypotheses are satisfied because the solution \( f \) of the system has integer coefficients.

Alternatively, one can use a special case of a generalized conjecture of Grothendieck-Katz proved by Bost, namely Corollary 2.8 in [6, Sect. 2.4]: the vanishing of the \( p \)-curvatures in Condition (i) follows by a theorem of Cartier from the fact that the system has a solution in \( \mathbb{F}_p(t) \), namely the reduction mod \( p \) of \( f \) for all prime numbers \( p \) for which such a reduction of the system exists (see Exercise 3 of [1, p. 84] or Theorem 5.1 of [14]): Condition (ii) is satisfied since \( \mathbb{C}^n \) satisfies the Liouville property.

A nice overview of such algebraicity results is given in Chambert-Loir’s Bourbaki report [8]; see especially Theorem 2.6 and the following lines.

5. Examples

Kontsevich [15] computed \( P_\omega \) when \( \omega = X_1 + X_1^{-1} + \cdots + X_n + X_n^{-1} \) considered as a \( 1 \times 1 \)-matrix, obtaining

\[
(5.1) \quad P_\omega = \frac{2^n}{(2n-1)^{n-1}} \frac{(n-1 + n(1-4(2n-1)t^2)^{1/2})^{n-1}}{(1+4(2n-1)t^2)^{n}}
\]

which shows that \( P_\omega \) belongs to a quadratic extension of \( \mathbb{Q}(t) \).

We now present similar results for the zeta functions of two matrices, the first one of order 2, the second one of order \( d \geq 3 \).

5.1. Computing \( P_M \) for a \( 2 \times 2 \)-matrix. Consider the following matrix with entries in the ring \( \mathbb{Z}(a,a^{-1},b,b^{-1},d,d^{-1}) \), where \( a, b, d \) are noncommuting variables:

\[
(5.2) \quad M = \begin{pmatrix} a + a^{-1} & b \\ b^{-1} & d + d^{-1} \end{pmatrix}
\]

Proposition 5.1. We have

\[
(5.3) \quad g_M = 3 \frac{(1 - 8t^2)^{1/2} - 1 + 6t^2}{1 - 9t^2}
\]
\[ P_M = \frac{(1 - 8t^2)^{3/2} - 1 + 12t^2 - 24t^4}{32t^6}. \]

Expanding \( P_M \) as a formal power series, we obtain
\[
P_M = 1 + \sum_{n \geq 1} \frac{3 \cdot 2^n}{(n + 2)(n + 3)} \left( \frac{2n + 2}{n + 1} \right) t^{2n}.
\]

**Proof.** View the matrix \( M \) under the form of the graph of Figure 1, with two vertices 1, 2 and six labeled oriented edges. We identify paths in this graph and words on the alphabet \( A = \{a, a^{-1}, b, b^{-1}, d, d^{-1}\} \). Let \( B \) denote the set of nonempty words on \( A \) which become trivial in the corresponding free group on \( a, b, d \) and whose corresponding path is a closed path. Then the integer \( a_n(M) \) is the number of words in \( B \) of length \( n \).

We define \( B_i \) \((i = 1, 2)\) as the set of paths in \( B \) starting from and ending at the vertex \( i \); we have \( B = B_1 + B_2 \). Each set \( B_i \) is a free subsemigroup of \( A^* \), freely generated by the set \( C_i \) of closed paths not passing through \( i \) (except at their ends). The sets \( C_i \) do not contain the empty word. We have
\[
B_i = C_i^+ = \sum_{n \geq 1} C_i^n \quad (i = 1, 2)
\]

Given a letter \( x \), we denote by \( C_i(x) \) the set of closed paths in \( C_i \) starting with \( x \). Any word of \( C_i(x) \) is of the form \( xwx^{-1} \), where \( w \in B_j \) when \( i \xrightarrow{x} j \); such \( w \) does not start with \( x^{-1} \). Identifying a language with its characteristic series and using the standard notation \( L^* = 1 + \sum_{n \geq 1} L^n \) for any language \( L \), we obtain the following two equations:
\[
C_1(a) = a(C_1(a) + C_1(b))^* a^{-1} \tag{5.5}
\]

and
\[
C_1(b) = b(C_2(d) + C_2(d^{-1}))^* b^{-1} \tag{5.6}
\]

Applying the algebra map \( \varepsilon \) and taking into account the symmetries of the graph, we see that the four noncommutative formal power series \( C_1(a), C_1(a^{-1}), C_2(d), C_2(d^{-1}) \) are sent to the same formal power series \( u \in \mathbb{Z}[[t]] \), while \( C_1(b), C_2(b^{-1}) \) are sent to the same formal power series \( v \). It follows from (5.5) and (5.6) that \( u \) and \( v \) satisfy the equations
\[
u = t^2(u + v)^* = \frac{t^2}{1 - u - v} \quad \text{and} \quad v = t^2(2u)^* = \frac{t^2}{1 - 2u}.
\]

**Figure 1.** A graph representing \( M \)

We define \( B_i \) \((i = 1, 2)\) as the set of paths in \( B \) starting from and ending at the vertex \( i \); we have \( B = B_1 + B_2 \). Each set \( B_i \) is a free subsemigroup of \( A^* \), freely generated by the set \( C_i \) of closed paths not passing through \( i \) (except at their ends). The sets \( C_i \) do not contain the empty word. We have
\[
B_i = C_i^+ = \sum_{n \geq 1} C_i^n \quad (i = 1, 2)
\]

Given a letter \( x \), we denote by \( C_i(x) \) the set of closed paths in \( C_i \) starting with \( x \). Any word of \( C_i(x) \) is of the form \( xwx^{-1} \), where \( w \in B_j \) when \( i \xrightarrow{x} j \); such \( w \) does not start with \( x^{-1} \). Identifying a language with its characteristic series and using the standard notation \( L^* = 1 + \sum_{n \geq 1} L^n \) for any language \( L \), we obtain the following two equations:
\[
C_1(a) = a(C_1(a) + C_1(b))^* a^{-1} \tag{5.5}
\]

and
\[
C_1(b) = b(C_2(d) + C_2(d^{-1}))^* b^{-1} \tag{5.6}
\]

Applying the algebra map \( \varepsilon \) and taking into account the symmetries of the graph, we see that the four noncommutative formal power series \( C_1(a), C_1(a^{-1}), C_2(d), C_2(d^{-1}) \) are sent to the same formal power series \( u \in \mathbb{Z}[[t]] \), while \( C_1(b), C_2(b^{-1}) \) are sent to the same formal power series \( v \). It follows from (5.5) and (5.6) that \( u \) and \( v \) satisfy the equations
\[
u = t^2(u + v)^* = \frac{t^2}{1 - u - v} \quad \text{and} \quad v = t^2(2u)^* = \frac{t^2}{1 - 2u}.
\]
from which we deduce
\[ t^2 = u(1 - u - v) = v(1 - 2u). \]
The second equality is equivalent to \((u - v)(u - 1) = 0\). Since \(C_1(a)\) does not contain the empty word, the constant term of \(u\) vanishes, hence \(u - 1 \neq 0\). Therefore, \(u = v\).

Since \(C_1 = C_1(a) + C_1(a^{-1}) + C_1(b)\) and \(C_2 = C_2(d) + C_2(d^{-1}) + C_2(b^{-1})\), we have \(\varepsilon(C_1) = \varepsilon(C_2) = 2u + v = 3u\). Therefore, \(\varepsilon(B_1) = \varepsilon(B_2) = 3u/(1 - 3u)\) and
\[
(5.8) \quad g_M = \varepsilon(B) = \frac{6u}{1 - 3u}.
\]
Let us now compute \(u\) using (5.7) and the equality \(u = v\). The formal power series \(u\) satisfies the quadratic equation \(2u^2 - u + t^2 = 0\). Since \(u\) has zero constant term, we obtain
\[
u = 1 - (1 - 8t^2)\frac{1}{2}.
\]
From this and (5.8), we obtain the desired form for \(g_M\).

Let \(P(t)\) be the right-hand side in Equation (5.4). To prove \(P_M = P(t)\), we checked that \(tP'(t)/P(t) = g_M\) and the constant term of \(P(t)\) is 1. □

**Remark 5.2.** We found Formula (5.4) for \(P(t)\) as follows. We first computed the lowest coefficients of \(g_M\) up to degree 10:
\[
g_M = 6 (t^2 + 5t^4 + 29t^6 + 181t^8 + 1181t^{10}) + O(t^{12}).
\]
From this it was not difficult to find that
\[
P_M = 1 + 3t^2 + 12t^4 + 56t^6 + 288t^8 + 1584t^{10} + O(t^{12}).
\]
Up to a shift, the sequence (5.9) of nonzero coefficients of \(P_M\) is the same as the sequence of numbers of “new” intervals in a Tamari lattice computed by Chapoton in [9, Sect. 9]. (We learnt this from [16] where this sequence is listed as A000257.) Chapoton gave an explicit formula for the generating function \(\nu\) of these “new” intervals (see Eq. (73) in loc. cit.). Rescaling \(\nu\), we found that
\[
P(t) = (\nu(t^2) - t^4)/t^6\]
has up to degree 10 the same expansion as (5.9). It then sufficed to check that \(tP'(t)/P(t) = g_M\).

By [10] the integers in the sequence A000257 also count the number of planar rooted bicubic maps with \(2n\) vertices (see [23, p. 269]). Planar maps also come up in the combinatorial interpretation of (5.1) given in [17, Sect. 5] for \(n = 2\).

Note that the sequence of nonzero coefficients of \(g_M/6\) is listed as A194723 in [16].

### 5.2. A similar \(d \times d\)-matrix.
Fix an integer \(d \geq 3\) and let \(M\) be the \(d \times d\)-matrix with entries \(M_{i,j}\) defined by
\[
M_{i,i} = a_i + a_i^{-1} \quad \text{and} \quad M_{i,j} = \begin{cases} b_{ij} & \text{if } i < j, \\ b_{ji}^{-1} & \text{if } j < i, \end{cases}
\]
where \(a_1, \ldots, a_d, b_{ij} (1 \leq i < j \leq d)\) are noncommuting variables. This matrix is a straightforward generalization of (5.2).
Proceeding as above, we obtain two formal power series $u$ and $v$ satisfying the following equations similar to (5.7):

$$u = t^2(u + (d - 1)v) = \frac{t^2}{1 - u - (d - 1)v}$$

and

$$v = t^2(2u + (d - 2)v) = \frac{t^2}{1 - 2u - (d - 2)v},$$

We deduce the equality $u = v$ and the quadratic equation $u(1 - du) = t^2$.

We finally have

$$g_M = \frac{d(d + 1)u}{1 - (d + 1)u},$$

which leads to

$$g_M = \frac{d(d + 1)}{2} \left\{ 1 - (4d^2)^{1/2} - 1 + 2(d + 1)^2 \right\}.$$ 

Its expansion as a formal power series is the following:

$$g_M = d(d + 1) \left\{ t^2 + (2d + 1) t^4 + (5d^2 + 4d + 1) t^6 
+ (14d^3 + 14d^2 + 6d + 1) t^8 
+ (42d^4 + 48d^3 + 27d^2 + 8d + 1) t^{10} \right\} + O(t^{12}).$$

When $d = 2, 3, 4$, the sequence of nonzero coefficients of $g_M/d(d + 1)$ is listed respectively as A194723, A194724, A194725 in [16]. It is also the $d$-th column in Sequence A183134. These sequences count the $d$-ary words either empty or beginning with the first letter of the alphabet, that can be built by inserting $n$ doublets into the initially empty word.

We were not able to find a closed formula for $P_M$ analogous to (5.4). Using Maple, we found that, for instance up to degree 10, the expansion of $P_M$ is

$$1 + \frac{d(d + 1)}{2} t^2 + \frac{d(d + 1)(d^2 + 5d + 2)}{8} t^4 
+ \frac{d(d + 1)(d^3 + 14d^2 + 59d^2 + 38d + 8)}{48} t^6 
+ \frac{d(d + 1)(d^5 + 27d^4 + 271d^3 + 1105d^2 + 904d + 332d + 48)}{384} t^8.$$

Acknowledgement

We are most grateful to Yves André, Jean-Benoît Bost and Carlo Gasbarri for their help in the proof of Theorem 4.4. We are also indebted to François Bergeron and Pierre Guillot for assisting us with computer computations in the process detailed in Remark 5.2 to Frédéric Chapoton for his comments on Section 5.2 and to an anonymous referee for having spotted slight inaccuracies. Thanks also to Stavros Garoufalidis for pointing out References [12, 19], in which the algebraicity of $g_M$ had been proved.

The first-named author was partially funded by LIRCO (Laboratoire International Franco-Québécois de Recherche en Combinatoire) and UQAM (Université du Québec à Montréal). The second-named author is supported by NSERC (Canada).
References

[1] Y. André, *G-functions and geometry*, Aspects of Mathematics, E13, Friedr. Vieweg & Sohn, Braunschweig, 1989.

[2] Y. André, *Sur la conjecture des p-courbures de Grothendieck et Katz*, Geometric aspects of Dwork theory, Vol. I, II, 55–112, Walter de Gruyter GmbH & Co. KG, Berlin, 2004.

[3] J. Berstel, *Transductions and context-free languages*, Leitfäden der Angewandten Mathematik und Mechanik, 38, B. G. Teubner, Stuttgart, 1979.

[4] J. Berstel, C. Reutenauer, *Zeta functions of formal languages*, Trans. Amer. Math. Soc. 321 (1990), no. 2, 533–546.

[5] J. Berstel, C. Reutenauer, *Noncommutative rational series with applications*, Encyclopedia of Mathematics and its Applications, 137, Cambridge University Press, Cambridge, 2011.

[6] J.-B. Bost, *Algebraic leaves of algebraic foliations over number fields*, Publ. Math. Inst. Hautes Études Sci. 93 (2001), 161–221.

[7] M. Bousquet-Mélou, *Algebraic generating functions in enumerative combinatorics, and context-free languages*, V. Diekert and B. Durand (Eds.), STACS 2005, 18–35, Lect. Notes Comput. Sci., 3404, Springer-Verlag, Berlin, Heidelberg, 2005.

[8] A. Chambert-Loir, *Théorèmes d’algébricité en géométrie diophantienne (d’après J.-B. Bost, Y. André, D. & G. Chudnovsky)*, Séminaire Bourbaki, Vol. 2000/2001, Astérisque No. 282 (2002), Exp. No. 866, viii, 175–209.

[9] F. Chapoton, *Sur le nombre d’intervalles dans les treillis de Tamari*, Séminaire Lotharingien de Combinatoire 55 (2006), B55f; arXiv:math/0602368.

[10] N. Chomsky, M. P. Schützenberger, *The algebraic theory of context-free languages*, Computer programming and formal systems, pp. 118–161, North-Holland, Amsterdam, 1963.

[11] D. V. Chudnovsky, G. V. Chudnovsky, *Applications of Padé approximations to the Grothendieck conjecture on linear differential equations*, Number theory (New York, 1983–84), 52–100, Lecture Notes in Math., 1135, Springer, Berlin, 1985.

[12] S. Garoufalidis, J. Bellissard, *Algebraic G-functions associated to matrices over a group-ring*, arXiv:math/0708.4234v4.

[13] G. Jacob, *Sur un théorème de Shamir*, Information and Control 27 (1975), 218–261.

[14] N. Katz, *Nilpotent connections and the monodromy theorem: applications of a result of Turrittin*, Publ. Math. Inst. Hautes Etudes Sci. 39 (1970), 175–232.

[15] M. Kontsevich, *Noncommutative identities*, talk at Mathematische Arbeitstagung 2011, Bonn; arXiv:1109.2469v1.

[16] *The On-Line Encyclopedia of Integer Sequences* (2010), published electronically at [http://oeis.org](http://oeis.org).

[17] C. Reutenauer, M. Robado, *On an algebraicity theorem of Kontsevich*, FPSAC 2012, Nagoya, Japan.

[18] A. Salomaa, M. Soittola, *Automata-theoretic aspects of formal power series*, Texts and Monographs in Computer Science. Springer-Verlag, New York-Heidelberg, 1978.

[19] R. Sauer, *Power series over the group ring of a free group and applications to Novikov-Shubin invariants*, High-dimensional manifold topology, 449–468, World Sci. Publ., River Edge, NJ, 2003.

[20] M. P. Schützenberger, *On a theorem of R. Jungen*, Proc. Amer. Math. Soc. 13 (1962), 885–890.

[21] R. P. Stanley, *Enumerative combinatorics*, vol. 2, with a foreword by Gian-Carlo Rota and Appendix 1 by Sergey Fomin, Cambridge Studies in Advanced Mathematics, 62, Cambridge University Press, Cambridge, 1999.

[22] W. T. Tutte, *A census of planar maps*, Canadian J. Math. 15 (1963), 249–271.
