Second-Second Moments and Two Observers Testing Quantum Nonlocality

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It is shown that rejection of local realism in quantum mechanics can be tested by Bell-type inequalities for two observers and low-order moments of continuous and unbounded observables. It is proved that one requires three observables for each observer for a maximally entangled state and two observables for a non-maximally entangled state and write down appropriate inequalities and show violation by quantum examples. Finding an example for quadratures or position and momentum is left as an open problem.

1. Introduction

Local realism means that outcomes of measurements by remote observers exist separately for each observer before the measurement is chosen. It has been initially discussed by Einstein, Podolsky, and Rosen (EPR)\(^1\) in the context of measuring position and momentum of an entangled state. However, later Bell,\(^2\) Clauser, Horne, Shimony, and Holt (CHSH)\(^3\) found a simple violation of local realism in a simple entangled state of two spins while measuring spin along different axes, with dichotomic outcomes. Despite the simplicity of the Bell model, it took over 50 years to confirm violation\(^4\)–\(^7\) although the assumptions of the experiments require further research.\(^8\) On the theoretical side, many examples how to reject local realism have been proposed,\(^9\) including many observers\(^11\) or outcomes.\(^12\) The outcome can be a real number from continuous range, a result of position/momentum measurement like in the EPR case.\(^13\)–\(^17\)

In this paper, we focus on a special class of tests local realism, involving moments \((A^k B^l)\) for two separated observers \(A\) and \(B\), with a given maximal degree \(k + l\) and unbounded continuous variables. Note that commonly used dichotomy \(A = \pm 1\) is equivalent to the fourth-moment constraint \((A^2 - 1)^2) = 0\). The moment-based tests have been proposed first by Vallancanti et al.,\(^18\) involving ten observers, later reduced to three observers.\(^19\) The original CHSH inequality can be rewritten in terms of up to fourth moments.\(^20\) Rejection of local realism needs always at least fourth moments\(^21\) for unbounded variables.

Tests of local realism with moments of continuous variables are useful when strong, projective measurements are hard or infeasible. Then weak measurements are more appropriate but at the cost of large noise added to the statistics.\(^22\)–\(^24\) The sharp, discrete clicks are replaced by slightly shifted Gaussian distributions. One can reveal the underlying quantum statistics by subtracting the dominating Gaussian detection noise, or adding more detectors, which is more efficient for low order moments and correlations. This is the case of optical\(^25\) and condensed matter attempts to test local realism,\(^26\)–\(^29\) measuring the flow of charges in mesoscopic junctions. The low-order moments in tests of local realism can be useful also in relativistic quantum field theories where sharp measurement cause problems with renormalization,\(^30\) while moments and correlations can be regularized to avoid infinities.

The aim of the paper is to show how local realism can be rejected in an experiment involving two observers and measuring moments of the type \((A^k B^l)\) with \(k, l \leq 2\), that is, second-second order. It is known that a natural class of inequalities involving such moments is satisfied both in quantum and classical mechanics.\(^31\) We explored a general class of inequalities constructing a positive polynomial being a sum of low order monomials of jointly measurable observables. The violation of the positivity of the average of the polynomial implies the rejection of local realism. We show that such polynomial is not necessarily a sum of squares. Surprisingly, a maximally entangled state requires at least three observables for each observer. However, there exists a class of examples involving non-maximally entangled states and only two observables at each side. Unfortunately, we have not found an example involving only position and momentum (quadratures).

2. Motivation—Weak Measurements

Unlike in quantum optics, where most detections are click-based, measurements in solid state devices, such as tunnel junctions, quantum point contacts, dots, with semiconductors, superconductors, or in quantum Hall regime, are current-based.\(^26\)–\(^29\) It means that the flow of electrons is measured not by click but amplifying tiny voltage measured across the probe. Strong, projective measurements are infeasible because they would be too disturbing for the system. The outcome is then not 0 or 1 but a continuous value of voltage/current. Its statistics is dominated by Gaussian distribution, due to large amplification. The quantum signal is a small shift of the distribution. It can be described in terms of weak measurements, where the detector interacts with
the system instantly but also weakly.\(^{[22–24]}\) The simplest model
of weak measurements uses Gaussian positive operator valued
measure with Kraus operators.\(^{[12]}\)

$$\hat{K}(a) = (2g/\pi)^{1/4} \exp(-g(\hat{A} - a)^2)$$  \hspace{1cm} (1)

where \(g\) is the strength of the measurement of the operator \(\hat{A}\) with the outcome \(a\). In the limit \(g \to 0\) we have \(\hat{K}(\pi 2g)^{1/2} \to 1\) so
there is no measurement at all (no dependence on \(a\)). The actually measured probability \(p'(a) = \text{Tr} [\hat{K}(a) \hat{\rho} \hat{K}^\dagger(a)]\) of the outcome \(a\) at the state \(\hat{\rho}\) has a form of convolution

$$p'(a) = \int D(a - A)p(A)dA,$$

$$p(a) = (\delta(A - \hat{A})) = \text{Tr}[\delta(A - \hat{A})\hat{\rho}]$$ \hspace{1cm} (2)

with the dominating detection noise \(D(a) = \sqrt{2g/\pi}e^{-2g|a|^2}\), with \((a^2)_{\hat{A}} = 1/4g\), diverging at \(g \to 0\). Here \(p(A)\) is the probability of the outcome \(A\) in the case of a strong, projective measurement \(g \to \infty\) \((p(A) = \lim_{g \to \infty} F(A))\), to which the noise \(D\) is added. The advantage of weak measurements is their low invasiveness, that is, the state after the measurement reads

$$\int da \hat{K}(a) \hat{\rho} \hat{K}^\dagger(a) = \exp(-g\hat{X}^2/2)\hat{\rho}$$  \hspace{1cm} (3)

with \(\hat{A}\) is in the eigenbasis of \(\hat{A}\) are decreased and this effect is proportional to \(g\).

To retrieve \(p\) from \(p'\), one has to make deconvolution, which is a terrible task, requiring Fourier transform of \(p'\) and back. A simpler approach involves only moments, that is,

$$\langle a \rangle_{p'} = \langle A \rangle_{p}, \quad \langle a^2 \rangle_{p'} = \langle A^2 \rangle_{p} + 1/4g$$ \hspace{1cm} (4)

If a second separate observer makes measurement of \(\hat{B}\) (compatible with \(\hat{A}\), that is, \(\hat{A}\hat{B} = \hat{B}\hat{A}\)) with the outcome \(b\) then the joint Kraus operator reads

$$\hat{K}(a, b) = (2g/\pi)^{1/2} \exp(-g(\hat{A} - a)^2 - g(\hat{B} - b)^2)$$  \hspace{1cm} (5)

with the outcome probability

$$p'(a, b) = \int D(a - A)D(b - B)p(A, B)dAdB,$$

$$p(A, B) = (\delta(A - \hat{A})\delta(B - \hat{B}),)$$ \hspace{1cm} (6)

where again \(p\) corresponds to strong, projective results. The correlations with respect to \(p'\) and \(p\) are related

$$\langle ab \rangle_{p'} = \langle AB \rangle_{p}, \quad \langle a^2 b \rangle_{p'} = \langle A^2 B \rangle_{p} + (B)^2_{p}/4g,$$

$$\langle a^2 b^2 \rangle_{p'} = \langle A^2 B^2 \rangle_{p} + (A^2)^2_{p}/4g + (B^2)^2_{p}/4g + 1/16g^2$$ \hspace{1cm} (7)

It is clear from the above relations that higher moments/correlations will involve high powers of \(1/g\), which is diverging in the weak limit \(g \to 0\). This is why keeping the order of moments/correlation low is desired from practical point of view.

An alternative approach does not require subtraction of detection noise but measuring twice the same observable by two identical and independent detectors. For a single party the Kraus operator reads

$$\hat{K}(a, a') = (2g/\pi)^{1/2} \exp(-g(\hat{A} - a)^2 - g(\hat{A} - a')^2)$$  \hspace{1cm} (8)

Then the outcome probability reads

$$p'(a, a') = \int D(a - A)D(a' - A)p(A)dA$$ \hspace{1cm} (9)

In this case

$$\langle a \rangle_{p'} = \langle a' \rangle_{p'} = \langle A \rangle_{p}, \quad \langle aa' \rangle_{p'} = \langle A^2 \rangle_{p}$$ \hspace{1cm} (10)

The correlation \(\langle aa' \rangle\) does not contain the noise because the detectors are uncorrelated. This idea generalizes to two parties using four detectors altogether as depicted in Figure 1. The full Kraus operator reads

$$\hat{K}(a, a', b, b') = (2g/\pi) \exp(-g(\hat{A} - a)^2 - g(\hat{A} - a')^2) \times \exp(-g(\hat{B} - b)^2 - g(\hat{B} - b')^2)$$  \hspace{1cm} (11)

with the outcome probability

$$p'(a, a', b, b') = \int D(a - A)D(a' - A)D(b - B)D(b' - B)p(A, B)dAdB$$ \hspace{1cm} (12)

The correlations read

$$\langle ab \rangle_{p'} = \langle a'b' \rangle_{p'} = \langle AB \rangle_{p}, \quad \langle a^2 b \rangle_{p'} = \langle A^2 B \rangle_{p}$$ \hspace{1cm} (13)

Measurements of third moments of electric current current in the mesoscopic junction are very hard experimentally\(^{[33–36]}\) while measurements of fourth moments have not yet succeeded. Of course, subtracting the large noise or splitting \(a\) into \(a\) and \(a'\) opens formally a loophole when testing local realism, but 1) the noise (also applied to \(a - a'\)) is well identified and there is no reason to take it into account to support local realism 2) even condensed matter tests of local realism subtracting this noise are still
The quantum test of such inequality requires identification of $A, B$ and gives up to $\lambda$ equal coefficients at $A^2 B^2$ (independent of $x$, $y$, and 3) no terms $\langle A^2 B_1 \rangle$, $\langle A B^2 \rangle$.

Nevertheless, already (16) generalized to three observers can be violated.[19] Here we stick to two observers, $A$ and $B$. One can rewrite standard CHSH inequality in terms of moments $\langle A^2 B_1 \rangle$ with $k + l \leq 4$. However, it involves pure fourth moments $\langle A^2 B_1 \rangle$.

The proof of positivity and impossibility of decomposition into polynomial squares is given in Appendix A (compare also with Choi example[40]). Unfortunately, we have not found any quantum violation of (20), yet we failed to prove that the inequality holds in the general quantum cases. Nevertheless, in the next sections, we show that the violating cases exist but the polynomials, inequalities, and violating states and observables are complicated.

4. Maximally Entangled State—Three Choices

First note that we can reduce the discussion to pure states that is, $\hat{p} = |\psi\rangle \langle \psi|$. Otherwise

$$\hat{p} = \sum q_i |\psi_i\rangle \langle \psi_i|$$

with $|\psi_i\rangle \langle \psi_i| = \delta_{ij}$ and $q_i \geq 0$, $\sum q_i = 1$ but also

$$\langle A^2 B_1 \rangle = \sum q_i |\psi_i\rangle \langle \psi_i| A^2 B_1 \langle \psi_i|$$

If a positive $p_i$ exists for each pure state $|\psi_i\rangle$ and gives up to second-second moments as predicted by quantum mechanics then $\sum q_i p_i$ will be the final probability.

Focusing on pure states, for two observers we can make Schmidt (singular value) decomposition

$$|\psi\rangle = \sum \phi_i |ij\rangle$$

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in certain tensor basis $|\psi\rangle \equiv |i\rangle_\lambda \otimes |j\rangle_\phi$ with real nonnegative $\phi_j$ satisfying $\sum_\phi \phi_j^2 = 1$. For a maximally entangled state $\phi_j = 1/\sqrt{N}$ for all $j$ where $N$ is the number of basis states in the decomposition. Note that the dimension of $H_\lambda$ and/or $H_\phi$ can be larger than $N$, that is, some basis states may not appear in the decomposition. While maximally entangled states give the largest violation of CHSH or other inequalities, here counterintuitively they are useless if any of the observers, $A$ or $B$, has only two choices. In this case, one can explicitly construct the local probability $\hat{p}$, see Appendix B.

We construct a minimal example for a violation requiring at least 3 choices for each observer. The inequality, valid classically, reads

$$\langle A^2_i B^2_j \rangle + \langle A^2_i B^2_j \rangle + 2 \sqrt{\langle A^2_i B^2_j \rangle \langle A^2_i B^2_j \rangle}$$

$$+ 2 \sqrt{\langle A^2_i B^2_j \rangle \langle A^2_i B^2_j \rangle} + 2 \sqrt{\langle A^2_i B^2_j \rangle \langle A^2_i B^2_j \rangle}$$

$$\geq 2(\langle A_i B_j \rangle + \langle A_i B_j \rangle + \langle A_i B_j \rangle) - 1$$

(24)

with all correlations measurable. The inequality does not belong to the class of inequalities (18), because it does not satisfy its properties (1) and (2).

To prove (24), note that the following classical inequality holds

$$\langle A_1 B_2 + A_2 B_1 + A_1 B_1 - 1 \rangle^2 \geq 0$$

(25)

for all real numbers $A_i$, $B_i$. On the other hand opening squares we can reduce it to

$$\langle A^2_i B^2_j \rangle + \langle A^2_i B^2_j \rangle + \langle A^2_i B^2_j \rangle$$

$$+ 2(\langle A_1 B_2 A_2 B_1 \rangle + \langle A_2 B_1 A_1 B_2 \rangle + \langle A_1 B_2 A_2 B_1 \rangle)$$

$$\geq 2(\langle A_i B_j \rangle + \langle A_i B_j \rangle + \langle A_i B_j \rangle) - 1$$

(26)

Using Cauchy–Bunyakovsky–Schwarz (CBS) inequality we get

$$\sqrt{\langle A^2_i B^2_j \rangle \langle A^2_i B^2_j \rangle} \geq \langle A_i B_j A_i B_j \rangle$$

(27)

and two others by cyclic shift of 123, and finally (24).

Let us consider the quantum case. The standard Bell state (maximally entangled)

$$\sqrt{2}|\psi\rangle = |++\rangle - |--\rangle$$

(28)

and operators in $|+\rangle$, $|--\rangle$ bases

$$\hat{A}_x = \frac{1}{2} \left( e^{\sqrt{2}i\pi x/1} + 1 \right)$$

(29)

for $x = 1, 2, 3$ (similarly $\hat{B}_y$) then

$$\langle A_i B_j \rangle = \langle A^2_i B^2_j \rangle = (1 - \cos(2\pi(x - y)/3))/4$$

(30)

The operators are in fact projections along regularly distributed axes on the equator of Bloch sphere, see Figure 2. In our case

$$\langle A_i B_j \rangle = 0$$

while $\langle A_i B_j \rangle = 3/8$ for $x \neq y$ and the inequality is violated with the left hand side equal $9/8$ while the right hand side is $2(9/8) - 1 = 10/8 > 9/8$. The violation can be also quickly understood from the fact that $\langle A^2_i B^2_j \rangle = 0$ implies that $A_i B_j = 0$ so either $A_i = 0$ or $B_j = 0$ for each $x$, giving a simpler inequality

$$\langle A^2_i B^2_j \rangle + \langle A^2_i B^2_j \rangle + \langle A^2_i B^2_j \rangle + 1$$

$$\geq 2(\langle A_i B_j \rangle + \langle A_i B_j \rangle + \langle A_i B_j \rangle)$$

(31)

checked by examining all cases, for example, if $A_1 = A_2 = 0$ then it reduces to $\langle A^2_i B^2_j \rangle + 1 \geq 2\langle A_i B_j \rangle$ obviously satisfied. Note that this inequality is only a restricted version of (24), valid if $A_i B_j = 0$, $z = 1, 2, 3$. Moreover, the fact that $\langle A^2_i B^2_j \rangle = 0$ makes it clear that (24) does not satisfy property (2) of (18).

In experimental practice, tests of local realism often cope with the null outcome, that is, both observers register 0 or null—a special outcome if no detection is registered—at a low rate of production of entangled states. It happens, for example, in Clauser–Horne–Eberhard inequality,[41,42] which helps to take into account the finite efficiency of photon detectors. Note that the event with only one observer registers null cannot be removed. Otherwise one has to assume fair sampling, which opens a loophole for local realism.

Suppose the probability is dominated by the null event $A = B = 0$ so that $\hat{p} \rightarrow r \hat{p}$ with $r$ being the (small) entanglement rate and $1 - r$ being the probability of the null event. Then the example (24) scales down everything except $-1$ on the right hand side at small entanglement production rate and violation disappears. We can get rid of the null event by redefining $A'_i = 1 - A_i$, $B'_j = 1 - B_j$ when the inequality reads

$$\langle (1 - A'_1)(1 - B'_1) \rangle + \langle A'_2 B'_1 \rangle + \langle A'_2 B'_1 \rangle + 2 \sqrt{\langle A'_2 B'_1 \rangle \langle A'_2 (1 - B'_1) \rangle}$$

$$+ 2 \sqrt{\langle A'_2 B'_1 \rangle \langle A'_2 B'_1 \rangle} + 2 \sqrt{\langle A'_2 B'_1 \rangle \langle (1 - A'_1) B'_1 \rangle}$$

$$\geq 2(\langle (1 - A'_1)(1 - B'_1) \rangle + \langle A'_2 B'_1 \rangle + \langle A'_2 B'_1 \rangle) - 1$$

(32)
where the free terms (numbers) cancel at both sides. Thanks to the cancellation the inequality keeps being violated when non-null probability is scaled by \( r \). Operationally the change of variables corresponds to taking complementary projection.

5. Non-Maximally Entangled State—Two Choices

To get violation of a classical inequality with two choices for each observer, we will need a non-maximally entangled state. The inequality reads in this case

\[
\langle A_1^2 B_2^2 \rangle + \langle B_2^2 A_1^2 \rangle + \langle (A_1 + B_1)^2 \rangle/4 \\
+ \sqrt{\langle (A_1 - B_1)^2 \rangle} \left( \sqrt{\langle A_1^2 B_2^2 \rangle} + \sqrt{\langle A_2^2 B_1^2 \rangle} \right) \\
+ 2 \sqrt{\langle A_1^2 B_2^2 \rangle \langle A_2^2 B_1^2 \rangle} \\
\geq 2 \langle (A_1^2 B_2^2) + (B_2^2 A_1^2) \rangle
\]

The inequality does not belong to the class (18) because it does not satisfy its properties (1), (2), and (3).

We prove it starting from

\[
(A_1 B_2 + B_1 A_2 - (A_1 + B_1)/2)^2 \geq 0
\]

expanded into

\[
\langle A_1^2 B_2^2 \rangle + \langle B_2^2 A_1^2 \rangle + 2 \langle A_1 B_1 A_2 B_2 \rangle + \langle (A_1 + B_1)^2 \rangle/4 \\
+ \langle (A_1 - B_1)(A_1 - B_1) \rangle \geq 2 \langle (A_1^2 B_2^2) + (B_2^2 A_1^2) \rangle
\]

Using CBS inequality

\[
\sqrt{\langle A_1^2 B_2^2 \rangle \langle A_2^2 B_1^2 \rangle} \geq \langle A_1 B_2 A_2 B_1 \rangle
\]

and

\[
\sqrt{\langle (A_1 - B_1)^2 \rangle} \left( \sqrt{\langle A_1^2 B_2^2 \rangle} + \sqrt{\langle A_2^2 B_1^2 \rangle} \right) \\
\geq \langle (A_1 - B_1)(A_1 - B_1) \rangle
\]

we get (33).

Let us take \( \hat{A}_1 = \hat{B}_1 = |+\rangle \langle +| \) and \( \hat{A}_2 = |n_+\rangle \langle n_+|, \hat{B}_2 = |n_-\rangle \langle n_-| \) with \( |n_+\rangle = \cos \phi |+\rangle + \sin \phi |-\rangle \), Figure 3, and the state

\[
|\psi\rangle = a |++\rangle + b |-\rangle
\]

\[
\alpha = \frac{\sin^2 \phi}{\sin^2 \phi + \cos^4 \phi}, \quad \beta = \frac{\cos^2 \phi}{\sin^2 \phi + \cos^4 \phi}
\]

We have

\[
\langle A_1^2 B_2^2 \rangle = \alpha^2 \text{ for } j \geq 1, \\
\langle B_2^2 A_1^2 \rangle = 0 \text{ for } j \geq 1, \\
\langle A_2^2 B_1^2 \rangle = \beta^2 \sin^2 \phi, \\
\langle B_1^2 A_2^2 \rangle = \alpha^2 \cos^2 \phi \text{ for } j \geq 1
\]

Then the inequality reads \( a^2 (2 \cos^2 \phi + 1) \geq 4 a^2 \cos^2 \phi \) which is violated whenever \( \cos^2 \phi > 1/2 \), that is, \( \phi < \pi/4 \), although the violation is quite weak, see Figure 4. Note also that the violation disappears when the state becomes either maximally entangled or a simple product.

Again the violation is quickly understood from the fact that \( \langle (A_1 - B_1)^2 \rangle = 0 \) together with \( \langle A_1^2 (1 - B_1)^2 \rangle = 0 \) implies \( A_1 = B_1 = 0,1 \), and \( \langle A_2^2 B_2^2 \rangle \) implies \( A_2 = 0 \) or \( B_2 = 0 \). In the case \( A_1 = B_1 = 1 \), we have a simpler inequality \( \langle B_1^2 \rangle + \langle A_2^2 \rangle + 1 \geq 2 \langle B_1 \rangle + \langle A_2 \rangle \) which is true in both cases (either \( A_2 = 0 \) or \( B_2 = 0 \)). Comparing with the previous section, the presented example is already robust against low entanglement rate (dominating null event) as all terms scale equally with non-null probability.

6. Discussion and Outlook

We have shown that second-second moments suffice to reject local realism for two observers, with inequalities Equations (24), (32), and (33). However, each observer has to use at least three choices for a maximally entangled state. Two choices suffice for...
a non-maximally entangled state but the proposed example is complicated while the violation is very weak. We suggest several further routes of research:

1. Find an example with a larger violation.
2. Find violation by position and momentum or prove the impossibility.
3. Determine the class of inequalities which hold both in classical and quantum mechanics.
4. Apply these or new examples to realistic setup, adjusting if necessary.

Low-order moments can help to combine tests of local realism with relativity, which need a careful treatment of divergences in high-order correlations function. In the case of weak measurement, a larger violation should help to reduce the effect of background noise, which has to be subtracted from the statistics. Due to the very small violation in the presented examples, it is also important to check how much noise added to the outcome distribution spoils the violation in particular cases.

Appendix A: Positive Polynomial Not Being a Sum of Polynomial Squares

We will show that (20) is nonnegative. Changing variables

\[ \sqrt{2} A_z = A_1 \pm A_2, \quad \sqrt{2} B_z = B_1 \pm B_2 \] (A.1)

the polynomial \( W \) reads

\[ A_1^2 + A_2^2 + B_1^2 + B_2^2 + (A_1^2 + A_2^2)(B_1^2 + B_2^2) - 3 \sqrt{2} A_z B_z (A_+ + B_+) \] (A.2)

Denoting

\[ A = \sqrt{A_1^2 + A_2^2} = \sqrt{A_1^2 + A_2^2}, \quad B = \sqrt{B_1^2 + B_2^2} = \sqrt{B_1^2 + B_2^2} \] (A.3)

we have

\[ W = (A^2 + B^2) + A^2 B^2 - 3 \sqrt{2} A_z B_z (A_+ + B_+) \] (A.4)

From Hölder inequality

\[ (A_+ + B_+)^2 = \left( \frac{A}{A_1} + \frac{B}{B_1} \right)^2 \leq (A^2 + B^2) \left( \frac{A_1^2}{A^2} + \frac{B_1^2}{B^2} \right) \]

\[ = (A^2 + B^2) \left( 2 - \frac{A_1^2}{A^2} - \frac{B_1^2}{B^2} \right) \] (A.5)

We have also

\[ 4(A_z B_z)^2 = 4A_1^2 B_1^2 A_2^2 B_2^2 \leq A^2 B^2 \left( \frac{A_1^2}{A^2} + \frac{B_1^2}{B^2} \right)^2 \] (A.6)

so

\[ (A_z B_z (A_+ + B_+))^2 \leq (A^2 + B^2) A^2 B^2 t^2 (2 - t)/4 \]

\[ \leq A^2 B^2 (A^2 + B^2) 8/27 \] (A.7)

where \( t = A_1^2/A^2 + B_1^2/B^2 \geq 0 \) and we used the fact that the maximum of \( t^2 (2 - t) \) for \( t \geq 0 \) is at \( t = 4/3 \) and equals \( 32/27 \). Therefore,

\[ |A_z B_z (A_+ + B_+)| \leq (2/3)^{1/2} AB \sqrt{A^2 + B^2} \] (A.8)

while

\[ A^2 + B^2 + A^2 B^2 \geq 2 \sqrt{A^2 + B^2} AB \] (A.9)

completing the proof.

We will show that the polynomial cannot be written as \( \sum Q_i^2 \) where \( Q(A_1, A_2, B_1, B_2) \) are polynomials. Equivalently \( Q \) can be polynomials of \( A_+, B_+ \) (change is linear) but it can contain only \( A_+, B_+, A_-, B_- \). Reducing quadratic form by standard methods we can arrange that only \( Q_1 \) contains \( A_+ \),

\[ Q_1 = A_+ - \alpha A_+ B_+ - \beta A_+ B_- \] (A.10)

Note that \( Q_1 \) cannot contain \( A_-, B_- \) or \( A_+, B_+ \) because otherwise \( Q_1^2 \) would produce terms \( A_+, A_-, A_+ B_+ \) and \( A^2 B^2 \), which cannot be cancelled later. Rearranging remaining quadratic terms, only \( Q_2 \) contains \( A_+ \),

\[ Q_2 = A_+ - \gamma A_+ B_+ + \beta A_+ B_- \] (A.11)

As above, it cannot contain \( B_-, A_-, \) while \( -\beta \) term follows from the fact that \( W \) does not contain \( A_+ B_- \) which can appear only in \( Q_2^2 \) and \( Q_1^2 \). Continuing rearranging, only \( Q_3 \) contains \( B_+ \) and only \( Q_4 \) contains \( B_- \) so

\[ Q_3 = B_+ - \delta B_- A_+ - \eta B_- A_- \]

\[ Q_4 = B_- - \xi B_+ A_+ - \eta B_+ A_- \] (A.12)

Moreover \( \alpha + \gamma = (3/2)^{1/2} = \delta + \xi \) while

\[ \sum Q_j^2 = \alpha^2 A_1^2 B_1^2 + \delta^2 B_1^2 A_1^2 + (\gamma^2 + \xi^2) A_1^2 B_1^2 + \cdots \] (A.13)

where the dotted term can only increase the first terms. On the other hand \( W \) puts constraints

\[ \alpha^2 \leq 1, \delta^2 \leq 1, \gamma^2 + \xi^2 \leq 1 \] (A.14)

giving \( \alpha^2 + \gamma^2 + \delta^2 + \xi^2 \leq 3 \) while \( \alpha^2 + \gamma^2 \geq (\alpha + \gamma)^2/2 = (3/2)^{1/2} \) and the same for \( \alpha \to \delta, \gamma \to \xi \). This would lead to \( (3/2)^{1/2} \leq 3 \) which is not true.

Appendix B: Maximally Entangled State and Two Choices

We will show that, counter-intuitively, two choices \( A_\gamma \) are insufficient in the case of maximally entangled states, that is, there exists \( p \) reproducing moments up to second-second order in agreement
with quantum predictions. In Schmidt decomposition (23), a
maximally entangled state is for \( \mathcal{W} = 1/\sqrt{N} \) with \( j = 1..N \)

Both \( \hat{A}_a \) and \( \hat{B} \) (we postpone the generalization to many \( \hat{B}_j \)
to the end of the proof) can have dimension larger than \( N \). Let us
us the block notation

\[
\hat{B} \rightarrow \begin{pmatrix} \hat{B}_a & \hat{B}_a^\dagger \\ \hat{B}_a & \hat{B}_a^\dagger \end{pmatrix}
\]

with \( \hat{B}_a \) restricted to the space of \( 1..N \). First, we make a diagno-
\[ sialization of \( \hat{A}_a = \sum_{a} a |a\rangle \langle a| \). We define a joint proba-

\[
\hat{A}_1 = \sum_{j} |j\rangle \langle j| \] is projection to the space \( 1..N \). Our
aim is to define positive conditional probability

\[
p(b, a ) = \frac{\hat{A}_1 |b\rangle \langle b|}{p(a)}
\]

for the cases \( p(a ) > 0 \) (\( p(b, a ) = 0 \) if \( p(a ) = 0 \) ) giving
correct \( \langle B \rangle_a \) and \( \langle B^2 \rangle_a \) defined as

\[
\langle B^2 \rangle_a = \langle a | \hat{B}^2 \hat{B}^\dagger | a \rangle / N = \sum_{a} b^2 p(\hat{b}, a ) \]

Here \( \hat{B} \) is Hermitian and \( \hat{B}^\dagger \) means either complex conju-
gation or transpose (equivalent). If suffices to define moments

\[
\langle b^2 \rangle_{a, a} = \sum_{a} b^2 p(\hat{b}, a )
\]

for \( k = 1, 2 \) that satisfy

\[
\langle b^2 \rangle_{a, a} \leq \langle b^2 \rangle_{a, a} p(\hat{b}, a ) \quad \langle B^2 \rangle_a = \sum_{a} \langle b^2 \rangle_{a, a}
\]

because then a positive Gaussian model

\[
p(b, a ) = \frac{p(\hat{b}, a )}{\sqrt{2\pi \sigma \langle b^2 \rangle_{a, a}}}
\]

\[
\times \exp \left( -\frac{(b^2 - \langle b^2 \rangle_{a, a})^2}{2\sigma^2} \right)
\]

explains up to second-second moments. The Gaussian distribu-
tion is only one of options, other choices include, for example,
dichotomic distribution centered at the average. In the case of
equality on (B.5) we have \( p(b, a ) = \delta(b - \langle b^2 \rangle_{a, a} / p(a) ) \).
We define

\[
2N(b)_{a, a} = \sum_{a} |a\rangle \langle a| (a | \hat{B}^2 \hat{B}^\dagger | a \rangle
\]

\[
+ \sum_{a} |a\rangle \langle a| (a | \hat{B}^2 \hat{B}^\dagger | a \rangle
\]

which gives correct \( \langle B \rangle_a \) by the fact that \( \sum_{a} |a\rangle \langle a| \) is identity in
the space containing \( 1..N \) (it does not matter if and how larger).
We also define

\[
\langle b^2 \rangle_{a, a} = \langle a | \hat{B}^2 \hat{B}^\dagger | a \rangle / N
\]

which gives correct \( \langle B^2 \rangle_a \) analogously. Moreover

\[
\langle b^2 \rangle_{a, a} \leq \langle b^2 \rangle_{a, a} p(a)
\]

by the fact that

\[
\langle a | \hat{B}^2 \hat{B}^\dagger | a \rangle \leq \langle a | \hat{B}^2 \hat{B}^\dagger | a \rangle
\]

which follows from CBS inequality

\[
\langle v\rangle \langle w\rangle \leq \langle v w \rangle \langle v \rangle \langle w \rangle
\]

and the fact that \( \langle v\rangle \langle w \rangle \leq 1 \) (both \( |a\rangle \) are the normalized base
vectors, while \( 1_N \) projects them into a subspace).

The full second moments contain \( 1_N \) \( \hat{B}^2 \hat{B}^\dagger \) being a semipositive operator. Let us define \( c(a) = \langle a | \hat{C} | a \rangle \) for \( c > 0 \). Note that \( c = \sum_{a} c(a) = \sum_{j} j \langle j | \hat{C} | j \rangle / N \) does
not depend on \( \pm \). Finally

\[
\langle b^2 \rangle_{a, a} = \langle b^2 \rangle_{a, a} + c(a) c(a) / c
\]

assuming \( c > 0 \). If \( c = 0 \) then \( \hat{C} = 0 \) and \( \langle b^2 \rangle_{a, a} = \langle b^2 \rangle_{a, a} \).
One can easily check that it gives the correct full moments, keeping
the desired inequality satisfied so the probability \( p(b, a ) \) \( \geq 0 \) exists.
For many \( \hat{B}_j \) we simply define

\[
p(b, a ) = \prod_{j} p(b, a )
\]

which completes the proof.

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Conflict of Interest

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