RANDOMNESS AND NON-RANDOMNESS PROPERTIES OF PIATETSKI-SHAPIRO SEQUENCES MODULO M

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Abstract. We study Piatetski-Shapiro sequences \( \lfloor nc \rfloor \) modulo \( m \), for non-integer \( c > 1 \) and positive \( m \), and we are particularly interested in subword occurrences in those sequences. We prove that each block \( \in \{0,1\}^k \) of length \( k < c + 1 \) occurs as a subword with the frequency \( 2^{-k} \), while there are always blocks that do not occur. In particular, those sequences are not normal. For \( 1 < c < 2 \), we estimate the number of subwords from above and below, yielding the fact that our sequences are deterministic and not morphic. Finally, using the Daboussi-Kátai criterion, we prove that the sequence \( \lfloor nc \rfloor \) modulo \( m \) is asymptotically orthogonal to multiplicative functions bounded by 1 and with mean value 0.

1. Introduction

The purpose of this paper is to study properties of Piatetski-Shapiro sequences \( \lfloor nc \rfloor \) modulo \( m \) for positive and non-integer \( c > 1 \).

We will show that the sequence \( (x_n) \) where \( x_n = \lfloor nc \rfloor \mod m \) has some quasi-random properties as well as properties similar to those of a deterministic sequence. For example \( (x_n) \) is \( k \)-normal for \( k \leq c \) but not \( k \)-normal for all \( k \). On the other hand the sequence \( (x_n) \) is asymptotically orthogonal to the Möbius function as it is expected for deterministic sequences. We will be more precise on these statements in Section 2.

Piatetski-Shapiro sequences \( \lfloor nc \rfloor \) are very well studied sequences and are an active area of research. They are named after I. Piatetski-Shapiro, who proved the following prime number theorem \[17\]: if \( 1 < c < \frac{12}{11} \), then

\[
|\{n \leq x : \lfloor nc \rfloor \text{ is prime}\}| \sim \frac{x}{c \log x}.
\]

This asymptotic formula is now known for \( 1 < c < \frac{2817}{2426} \) (see Rivat and Sargos \[20\]), moreover, it is true for almost all \( c \in (1,2) \) (see Leitmann and Wolke \[12\]). We also refer to the paper \[3\] by Baker et al., giving a collection of arithmetic results on Piatetski-Shapiro sequences.

A different line of research is given by \( q \)-multiplicative functions \( \varphi \) along Piatetski-Shapiro sequences. These functions satisfy \( \varphi(q^k a + b) = \varphi(q^k a) \varphi(b) \) for all \( a, k \geq 0 \) and \( 0 \leq b < q^k \). Mauduit and Rivat \[13\] proved an asymptotic formula concerning \( q \)-multiplicative functions \( \varphi : \mathbb{N} \rightarrow \{z : |z| = 1\} \) along \( \lfloor nc \rfloor \), where \( 1 < c < 7/5 \). This contains in particular the result that the Thue–Morse sequence on \( \{-1,+1\} \) (which is \( 2 \)-multiplicative) along \( \lfloor n^c \rfloor \) attains each of its two values with

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Date: December 10, 2018.

2010 Mathematics Subject Classification. Primary: 11B50, 11B83; Secondary: 11K16, 11L07, 11N37.

Key words and phrases. exponential sum, Piatetski-Shapiro sequence, normal number, subword complexity, Möbius orthogonality.

This work was supported by the Austrian Science Foundation FWF, SFB F5502-N26 “Subsequences of Automatic Sequences and Uniform Distribution”, which is a part of the Special Research Program “Quasi Monte Carlo Methods: Theory and Applications”, by the joint ANR-FWF project ANR-14-CE34-0009, I-1751 MuDeRa, Ciência sem Fronteiras (project PVE 407308/2013-0) and by the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme under the Grant Agreement No 648132.
asymptotic density $1/2$, as long as $c < 1.4$. Müllner and Spiegelhofer [16] improved this bound to $1 < c < 1.5$, and very recently, Spiegelhofer [22] obtained the range $1 < c < 2$. Moreover, Mauduit and Rivat’s result was transferred to automatic sequences by Deshouillers, Drmota, and Morgenbesser [5].

A more basic question concerns Piatetski-Shapiro sequences modulo $m$. Rieger [19] proved an asymptotic expression for the number of $\lfloor nc \rfloor$ that lie in a residue class modulo $m$, a result that was sharpened by Deshouillers [4].

Mauduit, Rivat and Sárközy [14] studied pseudorandomness properties of $(\lfloor n^c \rfloor \mod 2)_n$ (more precisely, of the sequence $(\lfloor 2n^c \rfloor \mod 2)_n$). They proved that the well distribution measure and the correlation measure of order $k$ of this sequence are both small; these properties are to be expected from a “good” pseudorandom sequence.

In the present paper, we continue the study of $(\lfloor n^c \rfloor \mod m)_n$ and establish further randomness- and non-randomness properties of this sequence.

2. Results

Let $m$ and $k$ be positive integers; a sequence of integers $(u_n)_n$ is said to be $k$-uniformly distributed modulo $m$ if for every block $B \in \{0, 1, \ldots, m - 1\}^k$

$$\lim_{N \to \infty} \frac{1}{N} \text{Card} \{n < N : (u_n, u_{n+1}, \ldots, u_{n+k-1}) \equiv B(\text{mod } m)\} = \frac{1}{m^k};$$

we equivalently say that the sequence $(u_n \mod m)_n$ is $k$-normal. We further say that $(u_n)_n$ is completely uniformly distributed modulo $m$ if it is $k$-uniformly distributed modulo $m$ for any $k$ or, equivalently, that $(u_n \mod m)_n$ is normal if it is $k$-normal for any $k$.

Our first result says that the Piatetski-Shapiro sequence modulo $m$ is $k$-uniformly distributed modulo $m$ up to some level in $k$.

**Theorem 1.** Suppose that $c > 1$ is not an integer and let $m$ be a positive integer. Then the sequence $(\lfloor n^c \rfloor)_n$ is $k$-uniformly distributed modulo $m$ for $1 \leq k \leq c + 1$.

However, this is no longer true for all $k$, even in a weaker sense.

**Theorem 2.** Let $m \geq 2$ be an integer and let $c > 1$ a real number which is not an integer. Then the sequence $x = (\lfloor n^c \rfloor \mod m)_n$ is not normal. More precisely, there exist some $k$ and a block $B \in \{0, 1, \ldots, m - 1\}^k$ which does not appear in $x$.

Note that this behaviour is different from that of $(s_2(\lfloor n^c \rfloor) \mod 2)_n$ (the Thue-Morse sequence along $(\lfloor n^c \rfloor)_n$) since in this case we have normality for $1 < c < 3/2$ [10].

Next, we discuss the case $1 < c < 2$ in more detail. We recall that the subword complexity $L_k$, $k \geq 1$, of a sequence $u$ with values in $\{0, 1, \ldots, m - 1\}$ is the number of different blocks $B \in \{0, 1, \ldots, m - 1\}^k$ that appear as a contiguous subsequence of $u$. A sequence $u$ is said to be deterministic if its topological entropy $h$ of the corresponding dynamical system is zero, or in other terms

$$h = \lim_{k \to \infty} \frac{1}{k} \log L_k = 0.$$

Among the deterministic sequences, a simple class is that of morphic sequences which are the coding of a fixed point of a substitution, see Allouche and Shallit [2]; they satisfy

$$L_k \ll k^2$$

The following result implies that for $1 < c < 2$, the sequence $(\lfloor n^c \rfloor \mod m)_n$ is deterministic but not morphic.
Theorem 3. Assume that $1 < c < 2$ and let $m \geq 2$ be an integer. There exists a constant $C_1$ such that the subword complexity $L_k$ of the sequence $([n^c] \mod m)_n$ is bounded above by $C_1k^r$, for all $r > \max\{4/(2-c), 6\}$.

Moreover, there is a constant $C_2$ such that $L_k \geq C_2k^3$.

It is a famous conjecture by Sarnak [21] that every bounded deterministic sequence $(u_n)_n$ is asymptotically orthogonal to the Möbius function $\mu$, which means that one has

$$\sum_{n<N} \mu(n) u_n = o(N), \quad (N \to \infty).$$

This is true in the case when $u_n = [n^c] \mod m$. We even have the following result

Theorem 4. Suppose that $c > 1$ is not an integer and that $m \geq 2$ is an integer. Let $G$ be a complex valued function defined on $\{0, 1, \ldots, m-1\}$. Then, for every multiplicative function $f(n)$ with $|f(n)| \leq 1$ and the property

$$\sum_{n<N} f(n) = o(N), \quad (N \to \infty),$$

we have

$$\sum_{n<N} f(n) G([n^c] \mod m) = o(N), \quad (N \to \infty).$$

In Section 3, we define some more notation and study the trigonometric sums and discrepancies relevant for our questions. Theorems 1, 2, 3 and 4 are respectively proved in the four subsequent sections.

3. Notation, trigonometric sums, discrepancy

3.1. Notation. For a real number $u$, we let $e(u) = \exp(2\pi i u)$.

For a real number $c$ we use Knuth’s notation for the falling factorials defined recursively by $c^0 = 1$ and $c^k = c^{k-1}(c-k+1)$.

For $k$ a positive integer and $h = (h_1, h_2, \ldots, h_k) \in \mathbb{R}^k$, we let $\|h\|_\infty = \max\{|h_1|, \ldots, |h_k|\}$.

For a real number $x$ we let $x = \lfloor x \rfloor + \{x\}$ be its only decomposition as a sum of an integer and an element in $[0, 1)$. We notice that the map $x \mapsto \{x\}$ permits to identify $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and the interval $[0, 1)$. We let $\|x\| = \min\{\|x\|, 1-\|x\|\}$ be the so-called distance of $x$ to the nearest integer. For a positive integer $k$ we identify $\mathbb{T}^k$ and $[0, 1)^k$. An interval $I$ in $[0, 1)^k$ is a cartesian product $\prod_{1 \leq i \leq k}[a_i, b_i)$, with $0 \leq a_i \leq b_i \leq 1$ for $1 \leq i \leq k$; its Lebesgue measure $\prod_{1 \leq i \leq k}(b_i-a_i)$ is denoted by $\lambda(I)$. For an interval $I \subset [0, 1)^k$, we denote by $\chi_I$ its indicator (also called characteristic) function.

The discrepancy of a finite set $X = \{x_1, x_2, \ldots, x_N\}$ of elements of $\mathbb{R}^k$ is defined by

$$D_N(X) = D_N(x_1, x_2, \ldots, x_N) = \sup_{I \subset [0, 1)^k} \left| \frac{1}{N} \sum_{n=1}^{N} \chi_I(\{x_n\}) - \lambda(I) \right|.$$ 

3.2. Trigonometric sums over polynomials.

Proposition 1. Let $k \geq 1$ and $P(n) = \sum_{i=0}^{k} \alpha_i n^i$ be a polynomial of degree $k$ with real coefficients. Let $q, R, h$ be positive integers and $p$ an integer such that

$$\gcd(p, q) = 1, \quad \left| \alpha_k \frac{p}{q} \right| \leq \frac{1}{q^2} \text{ and } 2hk!R^{2-1/2k-2} \leq q.$$
For $N \geq R$ we have

$$\frac{1}{N} \sum_{1 \leq n \leq N} e(hP(n)) \ll \left( \frac{1}{R} + \frac{q}{N} \right)^{1/2^{k-1}}.$$  

Proof. Our first step is to use, for $k \geq 2$, the Weyl-van der Corput method to reduce the evaluation of the left hand side of (3) to the evaluation of geometric sums. We apply Lemma 2.7 of [8] with

$$q = k - 1, \quad Q = 2^{k-1}, \quad I = (0, N], \quad H_j = R_j = R^{1/2^{k-1-j}} \text{ for } 1 \leq j \leq k - 1,$$

notice that the condition $R \leq N = |I|$ is fulfilled and get

$$\frac{1}{N} \sum_{1 \leq n \leq N} e(hP(n)) \ll \left( \frac{1}{R} + \frac{1}{NR_1 \cdots R_{k-1}} \sum_{1 \leq r_1 \leq R_1} \cdots \sum_{1 \leq r_{k-1} \leq R_{k-1}} \sum_{n=1}^{N-r_1-\cdots-r_{k-1}} e(\alpha_k h! r_1 \cdots r_{k-1} n) \right)^{1/2^{k-1}}.$$  

Let now $\ell = h! r_1 \cdots r_{k-1}$; for any $M$ we have

$$\left| \sum_{n=1}^{M} e(\alpha_k \ell n) \right| \leq \frac{2}{|e(\alpha_k \ell) - 1|} = \frac{1}{|\sin(\pi \alpha_k \ell)|}.$$  

We are thus looking for a lower bound for $|\sin(\pi \alpha_k \ell)|$. We have

$$|\sin(\pi \alpha_k \ell)| = \left| \sin \left( \frac{\ell p}{q} + \pi \ell \left( \frac{\alpha_k}{q} - \frac{p}{q} \right) \right) \right|.$$  

Relation (2) implies $1 \leq \ell \leq q - 1$ (in particular $q \geq 2$) and $\gcd(p, q) = 1$: the number $\ell p$ is never $0$ modulo $q$. On the other hand, we have

$$\left| \ell \left( \frac{\alpha_k}{q} - \frac{p}{q} \right) \right| \leq \frac{\ell}{q^2} \leq \frac{1}{2q}.$$  

The last two relations imply

$$|\sin(\pi \alpha_k \ell)| \geq \sin \left( \frac{\pi}{2q} \right) \geq \frac{1}{q}$$  

and so for any $M$ one has $\left| \sum_{n=1}^{M} e(\alpha_k \ell n) \right| \leq q$. This easily implies the validity of Proposition 1 when $k \geq 2$.

The case when $k = 1$ is now straightforward. We have

$$\frac{1}{N} \sum_{1 \leq n \leq N} e(hP(n)) = \frac{1}{N} \sum_{1 \leq n \leq N} e(\alpha_k h n).$$  

Relation (2) implies $2h \leq q$ and the previous reasoning implies

$$\frac{1}{N} \sum_{1 \leq n \leq N} e(hP(n)) \ll \frac{q}{N},$$  

and (3) is satisfied for any value of $R$. \hfill \Box
3.3. Trigonometric sums involving the function $n^c$ at consecutive arguments.

**Proposition 2.** Let $c > 1$ be a non integral real number, let $L = \lfloor c \rfloor + 1$ and let $m$ be a positive integer. For any $L$-tuple $h = (h_0, h_1, \ldots, h_{L-1})$ of integers which are not all 0 and any positive integer $N$, we have

$$
\sum_{n=N}^{2N-1} e \left( \frac{1}{m} \sum_{\ell=0}^{L-1} h_\ell (n + \ell)^c \right) \ll_{L,c,m} \|h\|_\infty N^{1 - \frac{\|h\|}{2^{c+1}}},
$$

as soon as

$$
\|h\|_\infty = o \left( N^{1 - \{c\}} \right).
$$

**Proof.** In order to apply classical upper bounds for trigonometrical sums, we need to have a lower and an upper bound for the absolute value of some derivative of the function $f$ defined by $f(x) = \frac{1}{m} \sum_{\ell=0}^{L-1} h_\ell (x + \ell)^c$.

If $\frac{1}{m} \sum_{\ell=0}^{L-1} h_\ell \neq 0$, then for $x \in [N, 2N - 1]$ and any integer $k \leq c + 1$ we have

$$
N^{c-k} \ll_{L,c,m} |f^{(k)}(x)| \ll_{L,c,m} \|h\|_\infty N^{c-k},
$$

which is fine for our purpose.

But if $\sum_{\ell=0}^{L-1} h_\ell = 0$, the order of magnitude of $f^{(k)}$ is no longer $N^{c-k}$. In that case, we use the Taylor expansion for $(x + \ell)^c$; the next term is now $\frac{c}{m} \sum_{\ell=0}^{L-1} h_\ell \ell (x + \ell)^{c-1}$. If $\sum_{\ell=0}^{L-1} h_\ell \ell \neq 0$, we have $N^{c-1-k} \ll_{L,c,m} |f^{(k)}(x)| \ll_{L,c,m} \|h\|_\infty N^{c-1-k}$ and we are done; if $\sum_{\ell=0}^{L-1} h_\ell \ell = 0$, we go to the next term in the Taylor expansion, and so on...

However, since the vector $h$ is non zero, there exists $r \in [0, L - 1]$ such that

$$
\sum_{\ell=0}^{L-1} h_\ell \ell^r \neq 0,
$$

if it were not the case, we would have $Ah = 0$, where $A = (j^i)_{0 \leq i, j \leq L-1}$ is the transposed matrix of an invertible Vandermonde matrix and $h$ a non zero vector, which is not possible.

Let $r$ be the smallest non-negative integer for which (6) holds. For $k \leq c + 1$, we have

$$
N^{c-r-k} \ll_{L,c,m} |f^{(k)}(x)| \ll_{L,c,m} \|h\|_\infty N^{c-r-k}.
$$

Let us first assume that $0 \leq r \leq L - 3$, a case which may occur only when $c > 2$. We let

$$
q = \lfloor c \rfloor - r - 1.
$$

We notice that

$$
q \geq L - 1 - (L - 3) - 1 = 1
$$

and that

$$
c - r - (q + 2) = c - r - [c] + r + 1 - 2 = c - [c] - 1 = \{c\} - 1.
$$

For $x \in [N, 2N - 1]$, we have

$$
N^{\{c\}-1} \ll_{L,c,m} |f^{(q+2)}(x)| \ll_{L,c,m} \|h\|_\infty N^{\{c\}-1}.
$$

We let

$$
\lambda = \min_{x \in [N, 2N - 1]} |f^{(q+2)}(x)| \quad \text{and} \quad \alpha \lambda = \max_{x \in [N, 2N - 1]} |f^{(q+2)}(x)|.
$$

The previous double inequality implies that there are constants $\kappa_1$ and $\kappa_2$ depending at most on $L, c, m$ such that

$$
\lambda = \kappa_1 N^{\{c\}-1} \quad \text{and} \quad 1 \leq \alpha \leq \kappa_2 \|h\|_\infty.
$$
Theorem 2.8 of [8] implies that we have, with $Q = 2^q$,
\[
\sum_{n=N}^{2N-1} e \left( \frac{1}{m} \sum_{\ell=0}^{L-1} h_\ell (n + \ell)^c \right) \ll_{L,c,m} N \left( \|h\|_\infty^{2} N^{(c)-1} \right)^{1/(4Q-2)} + N^{1-1/2Q} \|h\|_\infty^{1/2Q} + N^{1-2Q+1/Q^2} N^{(c)-1/2Q}.
\]

Using the inequalities
\[
1 \leq Q \leq 2Q \leq 4Q - 2 \leq 4 	imes 2^{r-1} - 2 \leq 2^{r+1} - 2 \quad \text{and} \quad \{c\} - 1 \geq -\|c\|,
\]
one obtains Proposition 2.

Let us now assume that $r = L - 2 = \lfloor c \rfloor - 1$. In this case, we have $c - r - 2 = c - (\lfloor c \rfloor - 1) - 2 = \{c\} - 1$, and, thanks to [7], for $x \in [N, 2N - 1]$, we have
\[
N^{(c)-1} \ll_{L,c,m} |f^{(2)} (x)| \ll_{L,c,m} \|h\|_\infty N^{(c)-1}.
\]

We let
\[
\lambda = \min_{x \in [N, 2N-1]} |f^{(2)} (x)| \quad \text{and} \quad \alpha \lambda = \max_{x \in [N, 2N-1]} |f^{(2)} (x)|.
\]

The previous double inequality implies that there are constants $\kappa_1$ and $\kappa_2$ depending at most on $L, c, m$ such that
\[
\lambda = \kappa_1 N^{(c)-1} \quad \text{and} \quad 1 \leq \alpha \leq \kappa_2 \|h\|_\infty.
\]

Theorem 2.2 of [8] implies that we have
\[
\sum_{n=N}^{2N-1} e \left( \frac{1}{m} \sum_{\ell=0}^{L-1} h_\ell (n + \ell)^c \right) \ll_{L,c,m} N \|h\|_\infty N^{(c)-1/2} + N^{(1-c)/2},
\]
in which case, Proposition 2 is satisfied.

We now consider the last case, when $r = L - 1$. In this case, we have $c - r - 1 = c - (\lfloor c \rfloor) - 1 = \{c\} - 1$ and so
\[
N^{(c)-1} \ll_{L,c,m} |f' (x)| \ll_{L,c,m} \|h\|_\infty N^{(c)-1}.
\]

Since, by hypothesis, the last term is $o(1)$, we can apply The Kusmin-Landau lemma (Theorem 2.1 of [8]) and obtain
\[
\sum_{n=N}^{2N-1} e \left( \frac{1}{m} \sum_{\ell=0}^{L-1} h_\ell (n + \ell)^c \right) \ll_{L,c,m} N^{1-c},
\]
in which case, Proposition 2 is again satisfied.

\[\square\]

3.4. Discrepancy of a perturbed sequence. We will make use of the following elementary property

**Proposition 3.** Let $N \geq 1$, $\delta \geq 0$ and $x_1, x_2, \ldots, x_N$ and $y_1, y_2, \ldots, y_N$ be two families of real numbers such that for all $n \leq N$ we have $|y_n - x_n| \leq \delta$. Then we have
\[
D_N(y_1, y_2, \ldots, y_N) \leq 2D_N(x_1, x_2, \ldots, x_N) + 2\delta.
\]

**Proof.** Let $I = [a, b] \subset [0, 1)$. We let $I_\delta^+ = [(a - \delta, b + \delta) + \mathbb{Z}] \cap [0, 1)$ and notice that $I_\delta^+$ is either an interval or the union of two intervals; we let $I_\delta^-$ to be the interval $[a + \delta, b - \delta]$ if $b - a > 2\delta$ or the empty set otherwise. We have
\[
I_\delta \subset I \subset I_\delta^+ \quad \text{and} \quad \lambda(I_\delta^+) \leq \lambda(I) + 2\delta \quad \text{and} \quad \lambda(I_\delta^-) \geq \lambda(I) - 2\delta.
\]
We thus have
\[ \sum_{1 \leq n \leq N} \chi_I(y_n) - \lambda(I) \leq \sum_{1 \leq n \leq N} \chi_{I^+}(x_n) - \lambda(I) \leq \lambda(I^+) + 2D_N(x_1, \ldots, x_N) - \lambda(I) \leq 2D_N(x_1, \ldots, x_N) + 2\delta \]
and
\[ \sum_{1 \leq n \leq N} \chi_I(y_n) - \lambda(I) \geq \sum_{1 \leq n \leq N} \chi_{I^-}(x_n) - \lambda(I) \geq \lambda(I^-) - D_N(x_1, \ldots, x_N) - \lambda(I) \geq -D_N(x_1, \ldots, x_N) - 2\delta, \]
which implies
\[ D_N(y_1, y_2, \ldots, y_N) \leq 2D_N(x_1, \ldots, x_N) + 2\delta. \]

3.5. The multidimensional Erdős-Turán theorem. For the case \( k = 1 \), Erdős and Turán [7] gave an upper bound for the discrepancy in terms of exponential sums. Their result has been generalised by Koksma [10] and Szüsz [23] in the multidimensional case. The version we give is taken from [6] (Theorem 1.21, page 15).

**Proposition 4.** Let \( X = \{x_1, x_2, \ldots, x_N\} \) be a finite set of elements of \( \mathbb{R}^k \) and \( H \) an arbitrary positive integer. We have
\[ D_N(X) \leq \left( \frac{3}{2} \right)^k \left( \frac{2}{H + 1} + \sum_{0 < \|h\|_\infty \leq H} \frac{1}{r(h)} \left| \frac{1}{N} \sum_{n=1}^N e(h \cdot x_n) \right| \right), \]
where, for \( h = (h_1, h_2, \ldots, h_k) \in \mathbb{Z}^k \), we let \( r(h) = \prod_{i=1}^k \max \{1, |h_i|\} \) and \( u \cdot v \) denote the usual scalar product of two elements \( u \) and \( v \) in \( \mathbb{R}^k \).

3.6. Discrepancy of a polynomial sequence. We give here an upper bound for the discrepancy of a polynomial sequence in terms of rational approximations of its coefficients. This will be useful for the proof of Theorem 2.

**Proposition 5.** Let \( P(x) = \sum_{i=0}^d \alpha_i x^i \) be a polynomial of degree \( d \) with real coefficients. For \( i \in [1, d] \), we let \( Q_i, q_i \) and \( p_i \) be rational integers such that
\[ \gcd(p_i, q_i) = 1, \ 1 \leq q_i \leq Q_i \quad \text{and} \quad \left| \alpha_i - \frac{p_i}{q_i} \right| \leq \frac{1}{q_i \cdot Q_i}. \]
Then we have for any \( k \) such that \( 1 \leq k \leq d \)
\[ D_N(P(1), \ldots, P(N)) \ll_d q^k \log(eq_k) \left( \frac{1}{q_k^{1/2}} + \frac{q_k}{N} \right)^{1/2(k-1)} + q_k^{1/k} \sum_{i=k+1}^d \frac{N^i}{Q_i^i}, \]
where \( q := \prod_{i=k+1}^d q_i \).

**Proof.** We want to separate the contribution of the different \( \alpha_k \)'s to the discrepancy. Relation (10) is trivially true when \( q_k = 1 \) and we may assume that \( q_k \geq 2 \); since \( p_k \) and \( q_k \) are coprime, \( p_k \) is different from 0 and so is \( \alpha_k \). We approximate the higher degree coefficients by rational numbers. We define
\[ y_n := \sum_{i=0}^k \alpha_i n^i + \sum_{i=k+1}^d \frac{p_i}{q_i} n^i, \]
and
\[ z_n := \sum_{i=k+1}^d \left( \alpha_i - \frac{p_i}{q_i} \right) n^i, \]
with the usual convention that \( z_n = 0 \) when \( k = d \).

In order to apply the original one dimension Erdős-Turán inequality, we have to estimate trigonometrical sums. We have

\[
\sum_{0 < h \leq H} \frac{1}{h} \left| \frac{1}{N} \sum_{n \leq N} e(hP(n)) \right| = \sum_{0 < h \leq H} \frac{1}{h} \left| \frac{1}{N} \sum_{n \leq N} e(h(y_n + z_n)) \right|
\]

\[
= \sum_{0 < h \leq H} \frac{1}{h} \left| \frac{1}{N} \sum_{n \leq N} e(hy_n) + \frac{1}{N} \sum_{n \leq N} e(hy_n) (e(hz_n) - 1) \right|
\]

\[
\leq \sum_{0 < h \leq H} \frac{1}{h} \left( \left| \frac{1}{N} \sum_{n \leq N} e(hy_n) \right| + \frac{1}{N} \sum_{n \leq N} |e(hy_n) (e(hz_n) - 1)| \right)
\]

\[
\leq \sum_{0 < h \leq H} \frac{1}{h} \left| \frac{1}{N} \sum_{n \leq N} e(hy_n) \right| + \sum_{0 < h \leq H} \frac{1}{h} \sum_{n \leq N} 2\pi |hz_n|.
\]

The last sum is easily treated thanks to (9). We have

\[
\sum_{0 < h \leq H} \frac{1}{h} \sum_{n \leq N} 2\pi |hz_n| \leq 2\pi H \sum_{i=k+1}^d \frac{N_i}{q_i} \cdot Q_i \leq 2\pi H \sum_{i=k+1}^d \frac{N_i}{Q_i}.
\]

We thus have

\[
\sum_{0 < h \leq H} \frac{1}{h} \left| \frac{1}{N} \sum_{n \leq N} e(hP(n)) \right| \leq \sum_{0 < h \leq H} \frac{1}{h} \left| \frac{1}{N} \sum_{n \leq N} e(hy_n) \right| + 2\pi H \sum_{i=k+1}^d \frac{N_i}{Q_i}.
\]

Thus, we want to estimate \( \left| \frac{1}{N} \sum_{n \leq N} e(hy_n) \right| \). The following lemma will permit us to reduce the question to the evaluation of trigonometrical sums over polynomials of degree \( k \).

**Lemma 1.** With the above notation, for any integer \( r \), there exists a polynomial \( Q_r \) of degree \( k \) with leading coefficient \( \alpha_k \) such that

\[
y_{q^r - q^k} - q^k Q_r(t) \in \mathbb{Z} \quad \text{for any} \quad t \in \mathbb{Z}.
\]

**Proof.** We recall that \( q = \prod_{i=k+1}^d q_i \). This implies that for any integer \( t \) the sum \( \sum_{i=k+1}^d \frac{p_i}{q_i} (tq + r)^i \) is equal, up to an integer, to \( \sum_{i=k+1}^d \frac{p_i}{q_i} r^i \) which is independent of \( t \). By the binomial expansion, the first part of \( y_{q^r} \), namely \( \sum_{i=0}^k \alpha_i (qt + r)^i \), is easily seen to be a polynomial of degree \( k \) with leading coefficient \( \alpha_k q^k \). \( \square \)

We have

\[
\left| \frac{1}{N} \sum_{n \leq N} e(hy_n) \right| = \left| \frac{1}{N} \sum_{r \leq q} \sum_{n \equiv r \mod q} e(hy_n) \right| \leq \sum_{0 \leq r < q} \frac{1}{N} \sum_{n \equiv r \mod q} e(hy_n) \leq \sum_{0 \leq r < q} \frac{1}{N} \sum_{0 \leq t \leq \left( N - r \right) / q} e(hq^k Q_r(t)).
\]

We define the integers \( R \) and \( H \) by

\[
R = \left\lfloor q_k^{1/2} \right\rfloor \quad \text{and} \quad H = \left\lfloor q_k^{(1/2^k)} / (2k! q^k) \right\rfloor.
\]
We may assume that \( H \geq 1 \), since otherwise Proposition \( \Box \) is trivial. We readily check that the condition \( 2Hq^kR^{2-1/2^{k-2}} \leq q_k \) holds and that as soon as \( N \) is large enough we have \( R \leq (N - q)/q \). We can thus apply Proposition \( \Box \) which implies that for \( 1 \leq h \leq H \) we have

\[
\left| \sum_{0 \leq t \leq (N-r)/q} e(hq^kQ_r(t)) \right| \ll ((N-r)/q) \left( \frac{1}{R} + \frac{q_k}{(N-r)/q} \right)^{1/2^{k-1}} \ll N \left( \frac{1}{R} + \frac{q_k}{N} \right)^{1/2^{k-1}}.
\]

This leads to

\[
\sum_{0 < h \leq H} \frac{1}{h} \left| \frac{1}{N} \sum_{n \leq N} e(hy_n) \right| \ll q \log(eH) \left( \frac{1}{R} + \frac{q_k}{N} \right)^{1/2^{k-1}} \ll q \log(eq_k) \left( \frac{1}{q_k^{1/2}} + \frac{q_k}{N} \right)^{1/2^{k-1}}.
\]

We combine this, Proposition \( \Box \) and \( \Box \), getting Proposition \( \Box \). \( \square \)

We notice that the optimal choice of \( H \) and \( R \) permits to replace the term \( q_k \) in \( \Box \) by \( q^{f(k)} \) where \( f(k) \) tends to \( 1 \) as \( k \) tends to infinity, but this is irrelevant for our application.

4. Proof of Theorem \( \Box \)

To show that the sequence \( (\lfloor n^c \rfloor)_n \) is \( k \)-uniformly distributed modulo \( m \), it is enough to show that for any \( B = (b_0, b_1, \ldots, b_{k-1}) \in \{0, 1, \ldots, m-1\}^k \) we have, as \( N \) tends to infinity

\[
\frac{1}{N} \text{Card}\{n \in [N, 2N) : (\lfloor n^c \rfloor, \lfloor (n+1)^c \rfloor, \ldots, \lfloor (n+k-1)^c \rfloor) = B\} - \frac{1}{m^k} \text{ tends to } 0.
\]

Thanks to the straightforward equivalence

\[
\lfloor n^c \rfloor \equiv b \pmod{m} \iff \left\{ \frac{n^c}{m} \right\} \in \left\lfloor \frac{b}{m}, \frac{b+1}{m} \right\rfloor
\]

and the definition of the discrepancy given above, we have

\[
\left(12\right) \quad \frac{1}{N} \text{Card}\{n \in [N, 2N) : (\lfloor n^c \rfloor, \lfloor (n+1)^c \rfloor, \ldots, \lfloor (n+k-1)^c \rfloor) = B\} - \frac{1}{m^k} \leq D_N \left( \left\{ \left( \frac{n^c}{m}, \frac{(n+1)^c}{m}, \ldots, \frac{(n+k-1)^c}{m} \right) : n \in [N, 2N) \right\} \right).
\]

To evaluate the right hand side of \(12\), we use Proposition \( \Box \) with \( H = N^{\frac{\|c\|}{(k+2)^{2^{k-1}}}} \). Combining it with Proposition \( \Box \) we obtain

\[
\left| \frac{1}{N} \text{Card}\{n \in [N, 2N) : (\lfloor n^c \rfloor, \lfloor (n+1)^c \rfloor, \ldots, \lfloor (n+k-1)^c \rfloor) = B\} - \frac{1}{m^k} \right| \ll_{k,c,m} N^{-\frac{\|c\|}{(k+2)^{2^{k-1}}}}.
\]

Theorem \( \Box \) is thus proved. \( \square \)

5. Proof of Theorem \( \Box \)

5.1. Coefficients of polynomials with large discrepancy. Our first step is to show that if a sequence which is close to a polynomial has a large discrepancy, the non constant coefficients of the underlying polynomial have very good approximations with rationals with bounded denominators.
Theorem 5. Let $\delta$ be a positive number, $d$ a natural integer. There exists a positive integer $M(\delta, d)$ having the following property:

Let $P(x) = \sum_{k=1}^{d} \alpha_k x^k$ be a polynomial of degree $d$ such that for any sufficiently large $N$, for any $\eta = (\eta_1, \ldots, \eta_N)$ with $\|\eta\|_\infty \leq \delta$, we have

\begin{equation}
D_N(P(1) + \eta_1, P(2) + \eta_2, \ldots, P(N) + \eta_N) > 4\delta.
\end{equation}

Then for all sufficiently large $N$, we have

\begin{equation}
\forall k \in [1, d], \exists (p_k, q_k) \text{ with } \gcd(p_k, q_k) = 1, 1 \leq q_k \leq M(\delta, d) \text{ and } \left| \alpha_k - \frac{p_k}{q_k} \right| \leq N^{-\frac{(k+2)!}{2(d+2)}}.
\end{equation}

Proof. The perturbation by $\eta$ will be useful for the application, but we can easily cope with it: by Proposition 3, the bounds $\|\eta\|_\infty \leq \delta$ and (13) imply

\begin{equation}
D_N(P(1), P(2), \ldots, P(N)) > \delta,
\end{equation}

which is the condition we are going to use from now on.

Let $N$ be a sufficiently large integer. For $i \in [1, d]$ we define

\begin{equation}
N_i = \lfloor N \frac{(i+2)!}{2(d+2)} \rfloor + 1, \quad Q_i = \lfloor N_i^{1-\epsilon} + 1 \rfloor, \quad \varepsilon = \frac{1}{2(d+2)},
\end{equation}

and we let $(p_i, q_i)$ be such that

\begin{equation}
\gcd(p_i, q_i) = 1, 1 \leq q_i \leq Q_i \text{ and } \left| \alpha_i - \frac{p_i}{q_i} \right| \leq \frac{1}{q_i \cdot Q_i}.
\end{equation}

In order to prove the theorem, we shall show that for any $k$ in $[1, d]$, there exists $M_k(\delta, d)$ such that

\begin{equation}
\forall i \in [k, d], \exists (p_i, q_i) \text{ with } \gcd(p_i, q_i) = 1, 1 \leq q_i \leq M_k(\delta, d) \text{ and } \left| \alpha_i - \frac{p_i}{q_i} \right| \leq N^{-\frac{(i+2)!}{2(d+2)}}.
\end{equation}

We first prove (16) when $k = d$. By (15) and Proposition 5, we have

\begin{equation}
\delta \leq D_{N_d}(P(1), \ldots, P(N_d)) \ll_d \log(e q_d) \left( \frac{1}{q_d} + \frac{q_d}{N_d} \right)^{1/2d-1} \ll_d \log(e q_d) \max \left( \frac{1}{q_d^{1/2d}}, \left( \frac{Q_d}{N_d} \right)^{1/2d-1} \right).
\end{equation}

By definition, we have $Q_d \leq 2N_d^{1-\epsilon}$ and so the quantity $\log(e q_d) (Q_d/N_d)^{1/2d-1}$ which is less than $\log(e Q_d) (Q_d/N_d)^{1/2d-1}$ tends to zero as $N_d$ tends to infinity and thus as $N$ tends to infinity; when $N$ is large enough, the term $\log(e q_d) q_d^{-1/2d}$ has to be bounded from below, i.e. $q_d$ has to be bounded from above. This is the case $k = d$ of the theorem.

Assume now that (16) is proved for some $k \in [2, d]$ and let us show that it is also true for $k - 1$. We are going to use Proposition 5 with $N = N_{k-1}$ and start with some preliminary computation.

\begin{equation}
q = \prod_{i=k}^{d} q_i \leq M_k(\delta, d)^d.
\end{equation}

We also have

\begin{equation}
Q_{k-1}^{1/2k-2} \sum_{i=k}^{d} \frac{N_i^0}{Q_i} \leq \sum_{i=k}^{d} \frac{N_i^{i+1}}{Q_i} = \sum_{i=k}^{d} \frac{N_i^{i+1}(i+1)!}{(d+2)!} = \sum_{i=k}^{d} \left( \frac{N_i^{i+1}(i+1)!}{(d+2)!} \right).
\end{equation}
\[
\leq \sum_{i=k}^{d} \left( \frac{N^{i+1}}{N^{1-\varepsilon(i+2)}} \right)^{\frac{(k+1)i}{(d+2)!}} = \sum_{i=k}^{d} \left( \frac{1}{N^{1-\varepsilon(i+2)}} \right)^{\frac{(k+1)i}{(d+2)!}} \leq (d-k+1)N^{\frac{1-(d+2)\varepsilon}{(d+2)}}.
\]

We recall that \(\varepsilon = 1/(2(d+2))\) and obtain
\[
q_{k-1}^{1/2k-2} \sum_{i=k}^{d} \frac{N^i}{Q_i} \leq Q_{k-1}^{1/2k-2} \sum_{i=k}^{d} \frac{N^i}{Q_i} \leq dN^{-\frac{1}{2(d+2)}}.
\]

From that computation and Proposition 5, we get
\[
\delta \leq D_{N_k-1}(x_1, \ldots, x_{N_k-1}) \ll_d M(k, d)^d \log(eq_{k-1}) \max \left( \frac{1}{q_{k-1}}, \left( \frac{q_{k-1}}{N_{k-1}} \right)^{1/2k-2} \right) + q_{k-1}^{1/2k-2} \sum_{i=k}^{d} \frac{N^i}{Q_i} \leq dN^{-\frac{1}{2(d+2)}}.
\]

Arguing as above, we see that for \(N\) large enough, this relation can hold only if \(q_{k-1}\) is bounded from below by \(M'(\delta, d)\), say. We let \(M_{k-1}(\delta, d) = \max(M'(\delta, d), M_k(\delta, d))\).

Induction implies the validity of Theorem 5 with \(M(\delta, d) = M_1(\delta, d)\). \(\square\)

5.2. Non-uniformity modulo \(m \geq 3\) of perturbed polynomials. The following result shows that the sequence of the integral parts of the values of a slightly perturbed real polynomial cannot be uniform modulo any \(m \geq 3\).

**Theorem 6.** Let \(m\) and \(d\) be positive integers with \(m \geq 3\). There exist a positive integer \(N = N(m, d)\) and a block \(B \in [0, 1, \ldots, m-1]^N\) such that for any real polynomial \(P\) of degree \(d\) and any sequence \(\eta = (\eta_n)_n\) of real numbers bounded by \(1/20\) there exists \(n \in [1, N]\) such that

\[
(P(n) + \eta_n) \not\equiv b_n \pmod{m}.
\]

**Proof.** Let \(M = M(1/(20m), d)\) (where \(M(\cdot, \cdot)\) was defined in Theorem 5) and \(N \geq (40mdM!)^{(d+3)!}\) be integers satisfying Theorem 5 (with \(d = d\) and \(\delta = 1/(20m)\)) and define the block \(B\) to consist of \(M!\) digits 2 followed by \(M!\) digits 1 followed by \(N - 2M!\) digits 0.

We assume that there exists a polynomial \(P\) for which (17) does not hold; in particular, for any \(n \in [2M! + 1, N]\) we have \([P(n) + \eta_n] \equiv 0 \pmod{m}\). We let \(R(x) = P(x)/m\) and \(\beta_n = \eta_n/m\). Since \(N \geq 8M!\) the discrepancy of the sequence \((R(n) + \beta_n)_{1 \leq n \leq N}\) is larger than \(1/2\). We can thus apply Theorem 5 with \(\delta = 1/(20m)\). Let us write \(R(x) = \sum_{k=1}^{d} \alpha_k x^k\); for any \(k\) there exist coprime rational integers \(p_k, q_k\) with \(1 \leq q_k \leq M\) and
\[
|\alpha_k - \frac{p_k}{q_k}| \leq N^{-\frac{(k+1)!}{2(d+2)!}} \leq \frac{1}{20md \cdot (2M!)^d},
\]
where the last inequality comes from the choice of \(N\).

For \(k \in [1, d]\) we let \(\varepsilon_k = \alpha_k - \frac{p_k}{q_k}\) and \(r(x) = \alpha_0 + \sum_{k=1}^{d} \varepsilon_k (M!)^k x^k\). Since \(M!/q_k \in \mathbb{N}\), we have for any integer \(\ell\)
\[R(\ell M! \equiv r(\ell) \pmod{1}).\]

For \(\ell \in \{1, 2\}\) we have
\[|r(0) - r(\ell)| \leq \sum_{k=1}^{d} \varepsilon_k (M!)^k \leq \sum_{k=1}^{d} \frac{1}{20md \cdot (2M!)^d (M!)^d 2^d} \leq \frac{1}{20m}.
\]

For \(\ell \in \{1, 2, 3\}\) we have \(|r(\ell) + \beta_{\ell M!} - r(0)| \leq 1/(10m)\), which implies that the three real numbers \(r(\ell) + \beta_{\ell M!}\) belong to an interval of length \(1/(5m) < 1/(3m)\). This relation is incompatible with the fact that \(\{[P(\ell M! + \eta_{\ell M!}) : 1 \leq \ell \leq 3\}\) takes three different values. \(\square\)
5.3. Non uniformity modulo $m \geq 2$ of smoothly perturbed polynomials. The proof of the previous result makes a crucial use of the fact that we can find at least three different digits in base $m$. Indeed, the observation that for any sequence $(b_n)_n \in \{0, 1\}^\mathbb{N}$, there exists a sequence $(\varepsilon_n)_n$ tending to 0 as quickly as we wish such that for any $n$ we have $[2n + 1 + \varepsilon_n] \equiv b_n \pmod{m}$ shows that Theorem 6 cannot be extended without modification to the case when $m = 2$. Theorem 7 shows that the case $m = 2$ can be treated if we add some regularity condition. The next easy lemma explains which regularity we require.

Lemma 2. Let $m \geq 2$ be an integer and $x_1, x_2, x_3$ be a monotonic sequence of real numbers such that $|x_3 - x_1| < 1$. The triplet of the residues modulo $m$ of $[x_1], [x_2]$ and $[x_3]$ cannot be $(0, 0, 0)$ nor $(1, 1, 1)$.

Proof. We assume that the sequence $x_1, x_2, x_3$ is non-decreasing. We have

$$[x_1] \leq [x_3] \leq [x_1 + 1] = [x_1] + 1.$$ 

Since $[x_1]$ and $[x_3]$ have the same residue modulo $m$, they are equal. We have $[x_1] \leq [x_2] \leq [x_3]$, which implies $[x_1] = [x_2]$, a contradiction. The case when the sequence $x_1, x_2, x_3$ is non increasing is treated in a similar way.

Theorem 7. Let $m$ and $d$ be positive integers with $m \geq 2$. There exist a positive integer $N = N(m, d)$ and a block $B \in \{0, 1\}^N$ such that for any real polynomial $P$ of degree $d$ and any real function $\eta \in \mathcal{C}^{d+1}([1, N])$ such that

$$\forall t \in [1, N]: |\eta(t)| \leq 1/20 \text{ and } \eta^{(d+1)}(t) \neq 0,$$

there exists $n \in [1, N]$ such that

$$|P(n) + \eta(n)| \equiv b_n \pmod{m}.$$

Proof. We let $M = M(1/(20m), d)$ (where $M(., .)$ was defined in Theorem 5 and $N \geq (40mdM!)^{(d+3)!}$ be integers satisfying Theorem 5 (with $d = d$ and $\delta = 1/(20m)$) and define the block $B$ to consist of $N$ integers almost all of them being equal to 0, with the exception that, for $k = 1, 2, \ldots, 2d$ one has $b_{(2kM)} = 1$.

We assume that there exists a polynomial $P$ for which (19) does not hold; in particular, we have Card$\{n \in [1, N]: |P(n) + \eta(n)| \equiv 0 \pmod{m}\} = N - 2d$. We let $R(x) = P(x)/m$ and $\beta(x) = \eta(x)/m$. The discrepancy of the sequence $(R(n) + \beta_n)_{1 \leq n \leq N}$ is larger than $1/3$. We can thus apply Theorem 5 with $\delta = 1/(20m)$. Let us write $R(x) = \sum_{k=1}^{d} \alpha_k x^k$; for any $k$ there exist coprime rational integers $p_k, q_k$ with $1 \leq q_k \leq M$ and

$$|\alpha_k - \frac{p_k}{q_k}| \leq N^{-\frac{(k+2)!}{2d(k+2)!}} \leq \frac{1}{20md \cdot (4dM!)^d},$$

where the last inequality comes from the choice of $N$.

For $k \in [1, d]$ we let $\varepsilon_k = \alpha_k - p_k/q_k$ and $r(x) = \alpha_0 + \sum_{k=1}^{d} \varepsilon_k (M!)^k x^k$. Since $M!/q_k \in \mathbb{N}$, we have for any integer $\ell$

$$R(\ell M!) \equiv r(\ell) \pmod{1}.$$

We define a function $f$ by the relation

$$\forall t \in [1, 4d]: f(t) = mr(t) + \eta(t M!).$$

For any $t \in [1, N]$, we have

$$|f(t) - f(0)| \leq m \sum_{k=1}^{d} |\varepsilon_k| (M!)^k t^k + 2 \times \frac{1}{20} \leq \sum_{k=1}^{d} \frac{1}{20d \cdot (4dM!)^d (M!)^d} + \frac{1}{10} \leq 3 \frac{20}{20}.$$
and so for any \( t_1 \) and \( t_2 \) one has \(|f(t_1) - f(t_2)| \leq \frac{3}{10} \).

Since \( r \) is a polynomial of degree \( d \), we have \( f^{(d+1)}(t) = (M!)^{d+1} \eta^{(d+1)}(tM!) \), which is different from 0 by \((18)\). By repeated applications of Rolle’s theorem, we find that \( f' \) vanishes at most \( d \) times on \([1, 4d]\): there exists at least an integer \( \ell_0 \in [1, 4d] \) such that the sequence \( f(\ell_0), f(\ell_0+1), f(\ell_0+2) \) is monotonic.

By Lemma \( \Box \) the triplet \( ([f(\ell_0)], [f(\ell_0 + 1)], [f(\ell_0 + 2)]) \) taken modulo \( m \) cannot be \((0, 1, 0)\) nor \((1, 0, 1)\).

We finally notice that, for any integer \( \ell \), the difference between \( f(\ell) \) and \( P(\ell M!) + \eta(\ell m!) \) is a multiple of \( m \); thus the triple \( ([P(\ell_0 M!) + \eta(\ell_0 M!)], [P((\ell_0 + 1) M!) + \eta((\ell_0 + 1) M!)], [P((\ell_0 + 2) M!) + \eta((\ell_0 + 2) M!)]) \) taken modulo \( m \) cannot be \((0, 1, 0)\) nor \((1, 0, 1)\), contrary to our assumption. This contradiction proves Theorem \( \Box \).

5.4. **Proof of Theorem \( \Box \)** Let \( m \geq 2 \) be an integer and \( c > 1 \) a real number which is not an integer. We let \( d = \lfloor c \rfloor \) and consider the number \( N = N(m, d) \) and the block \( B \) given by Theorem \( \Box \). For any positive real numbers \( X \) and \( t \), we consider the Taylor approximation of order \( d \) of \((X + t)^c\), namely

\[
(X + t)^c = P_X(t) + \eta_X(t), \quad \text{where } P_X(t) = \sum_{k=0}^{d} \frac{c(c-1)\ldots(c-k+1)}{k!} t^k \quad \text{and} \quad |\eta_X(t)| \leq ct^{(d+1)} X^{(c-1)} - 1.
\]

For any sufficiently large integer \( X \), say \( X \geq X_0 \), we have \(|\eta_X(t)| \leq 1/20 \) for any \( t \in [1, N] \). Moreover, the \((d+1)\)-th derivative of \( \eta_X \) is the \((d+1)\)-th derivative of \( t \mapsto (X + t)^c \) and thus does not vanish. We can thus apply Theorem \( \Box \) there exists a block \( B \) of length \( N \) which does not occur in the sequence of the residues modulo \( m \) of the sequence \( ([n^c])_{n \geq X_0} \). Let \( U \) be in \( \{0, 1, \ldots, m - 1\}^{X_0} \); the word \( UB \), concatenation of the words \( U \) and \( B \), never occurs in the sequence of the residues modulo \( m \) of the sequence \( ([n^c])_{n \geq 0} \). This ends the proof of Theorem \( \Box \).

We end this section by noticing that for \( m \geq 3 \) the more general Theorem \( \Box \) is sufficient for proving Theorem \( \Box \).

6. **Proof of Theorem \( \Box \)**

6.1. **An upper bound for the complexity.** Assume that \( L \geq 1 \) (a block length) and \( \varepsilon = 1/(4L^2) \). Write \( a_n = \lfloor n^c \rfloor \) mod \( m \). By Taylor’s theorem (consider the second derivative of \( x^c \), which tends to 0 like \( x^{c-2} \)) there exists a constant \( C \), only depending on \( c \), such that for \( a \geq 4/(2 - c) \) and \( N \geq CL^a \) the following is satisfied: There are reals \( \alpha, \beta \) such that

\[
0 < n^c - (n\alpha + \beta) < \varepsilon
\]

for \( N \leq n < N + L \). We also assume that \( \alpha \) is irrational, which is no loss of generality. This technical condition will be used later, when we apply the three gaps theorem. The number of different factors in \( a \) of length \( L \) occurring at positions \( N < CL^a \) is trivially bounded by \( CL^a \), which gives the first term of the maximum in the theorem.

It remains to consider start positions \( N \geq CL^a \), where linear approximation of quality \( \varepsilon \) can be applied. Any block \( (a_N, \ldots, a_{N+L-1}) \) is obtained by starting from a block \( b = ([N\alpha + \beta] \mod m, \ldots, ([N + L - 1]\alpha + \beta] \mod m) \) and possibly modifying this sequence at indices \( n \) such that \( 1 - \varepsilon \leq \{n\alpha + \beta\} < 1 \). This possible modification consists in adding 1 modulo \( m \).

We begin by estimating the number of factors of \( [n\alpha + \beta] \mod m \). Each such block corresponds to a finite Sturmian word by considering the sequence of differences \( ([n + 1]\alpha + \beta] - [n\alpha + \beta]. \)

Note that such a sequence of differences corresponds to at most \( m \) factors of the Beatty sequence.
This follows by taking the first element \( [n_0 \alpha + \beta] \mod m \) of the considered factor into account and considering partial sums. Using Mignosi [15] we can estimate the number of factors \( b \) by \( O(L^3) \), where here and in the following the implied constant may depend on \( m \).

Consider the interval \( I = [1 - \varepsilon, 1) \) and the set

\[ A = \{ n : N \leq n < N + L, \{ n \alpha + \beta \} \in I + \mathbb{Z} \} . \]

We make use of the three gaps theorem (see, for example, the survey by Alessandri and Berthe [1], in particular the remark in section 4), which implies that there are at most three differences \( a_2 - a_1 \) between consecutive elements of the set \( B = \{ n \in \mathbb{N} : \{ n \alpha + \beta \} \in I + \mathbb{Z} \} \) and if three differences occur, the largest one is the sum of the smaller ones.

We distinguish between three cases.

1. All gaps are \( \geq L \). In this case, \( |A| \leq 1 \), so that we have to change the block \( b \) at at most one position by adding 1 modulo \( m \), as noted above. This gives a factor of \( L + 1 \), which implies that this case contributes \( O(L^{3+1}) \) many cases.

2. Exactly one gap is \( < L \). In this case \( A \) is an arithmetic progression, consisting of the elements \( n_j = n_0 + jd \) for some \( n_0 = \min A \) and \( d \geq 1 \), and \( 0 \leq j < k \). Set \( x_j = \{ n_j \alpha + \beta \} \) and \( \delta = x_1 - x_0 \).

First, we want to show that \( \{ x_0, \ldots, x_{k-1} \} \) is an arithmetic progression with difference \( \delta \). Note that \( |\delta| < \varepsilon < 1/2 \). Suppose that we have already shown that \( \{ x_0, \ldots, x_{j-1} \} \) is an arithmetic progression. Clearly we have \( x_j = x_{j-1} + \delta + r \) for some \( r \in \mathbb{Z} \). Suppose that \( r \neq 0 \). Then \( |x_j - x_{j-1}| > 1 - \varepsilon \). Since both \( x_j \) and \( x_{j-1} \) are elements of the interval \( [1 - \varepsilon, 1) \), this is a contradiction to \( \varepsilon < 1/2 \).

Next, we prove that the set \( J = \{ i < k : |n_i | > |n_i \alpha + \beta| \} \) is an interval. To this end, note that \( |n_i | > |n_i \alpha + \beta| \) if and only if \( n_i - (n_i \alpha + \beta) \geq 1 - \{ n_i \alpha + \beta \} \), which is the case if and only if \( n_i - n_i \alpha + \beta \geq \{ n_i \alpha + \beta \} + i \delta - 1 \geq 0 \). Note that the left hand side is a convex function of \( i \), which implies the assertion.

In order to obtain the block \( (a_N, \ldots, a_{N+L-1}) \) from the block \( b = ([N \alpha + \beta] \mod m, \ldots, [(N + L - 1) \alpha + \beta] \mod m) \), we modify \( b \) at indices \( n_j \) for \( j \in J \), where \( J \) is the interval obtained above. These indices form an arithmetic progression in \( [N, N + L - 1] \), of which there are \( O(L^3) \) many. (Note that in fact \( L^2 \log L \) is sufficient.) This implies a contribution of \( O(L^{3+3}) \) for this case.

3. There exist two gaps \( g_1 < g_2 < L \). We are going to show that this case cannot occur. We first note that \( g_1 \alpha \notin \mathbb{Z} \). Otherwise, the set \( B \) would contain an arithmetic progression with difference \( g_1 \), therefore the gap \( g_2 \) would not occur, a contradiction. It follows that \( 0 < \| g_1 \alpha \| < \varepsilon \).

Choose \( n_1, n_2 \in \mathbb{N} \) in such a way that \( n_2 - n_1 = g_1 \) and \( n_1, n_2 \in B \). Consider the \( g_1 \) points \( \{ n_1 \alpha + \beta \}, \{ (n_1 + 1) \alpha + \beta \}, \ldots, \{ (n_1 + g_1 - 1) \alpha + \beta \} \). These points dissect the torus into \( g_1 \) many intervals. Therefore there is an interval \( J = [x, y] \) in \( \mathbb{R} \) of length \( \geq 1/g_1 \geq 1/L = 4L\varepsilon \) such that \( n\alpha + \beta \notin J + \mathbb{Z} \) for \( n_1 \leq n < n_2 \). Assume that \( \{ g_1 \alpha \} < \varepsilon \), the case \( \{ g_1 \alpha \} > 1 - \varepsilon \) being analogous. Then

\[
\{ n \alpha + \beta : n_1 \leq n < n_1 + 2Lg_1 \} = \bigcup_{0 \leq k < 2L} \{ n \alpha + \beta : kn_1 \leq n < n_1 + (k + 1)g_1 \} \\
\subseteq \{ n \alpha + \beta : n_1 \leq n < n_1 + g_1 \} + \bigcup_{0 \leq k < 2L} (kg_1 \alpha + \mathbb{Z}) \\
\subseteq \mathbb{R} \setminus (J' + \mathbb{Z}),
\]

where \( J' = [x + 2L\varepsilon, y] \) has length \( \geq 2\varepsilon \). We now use the fact that \( 0 \neq \| g_1 \alpha \| < \varepsilon \) in order to shift the interval \( J' \) over the interval \( [1 - \varepsilon, 1) \) by using a multiple of \( \alpha \). Set \( \delta = (1 - 2\varepsilon) - (x + 2L\varepsilon) \) and assume that \( n_0 \) is such that \( n_0 \alpha \in \delta + [0, \varepsilon) + \mathbb{Z} \). Then for all \( n \) such that \( n_1 + n_0 \leq n < n_1 + n_02Lg_1 \) we have

\[
n \alpha + \beta \in \mathbb{R} \setminus (J' + \mathbb{Z}) + n_0 \alpha
\]
\[ \frac{b}{2} + L_\varepsilon + 1 + \delta + \lfloor 0, \varepsilon \rfloor + \mathbb{Z} \]
\[ \subseteq \lfloor x + 2L_\varepsilon + 1 \rfloor + (1 - 2\varepsilon) - (x + 2L_\varepsilon + 1) + (0, \varepsilon) + \mathbb{Z} \]
\[ \subseteq [0, 1 - \varepsilon] + \mathbb{Z} . \]

It follows that the sequence \((n\alpha + \beta)_{n \geq 0}\) does not visit \(I + \mathbb{Z}\) for at least \(2L\) many steps, which is a contradiction to the three gaps theorem: we proved the existence of a gap \(\geq 2L\), but it is the sum of the smaller ones, therefore is at most \(2L - 1\).

Therefore case (3) does not occur, and the first part of the theorem is proved.

6.2. A lower bound for the complexity. We use Mignosi \cite{13} again, this time we use the fact that there are at least \(Ck^3\) Sturmian words of length \(k\). Let \(a \in \{0, 1\}^k\) be a subword of a Sturmian word. There exist an irrational \(\alpha_0 < 1\) and some \(\beta_0\) and \(n\) such that \(a_\ell = \lfloor (n + \ell + 1)\alpha_0 + \beta_0 \rfloor - \lfloor (n + \ell)\alpha_0 + \beta_0 \rfloor\) for \(0 \leq \ell < k\). Let \(b \in \{0, \ldots, m - 1\}^k\) be the sequence of partial sums modulo \(m\); moreover, let sequences \(b^{(j)}\) be defined by \(b^{(j)}(\ell) = b(\ell + j) \mod m\). Then one out of \(b = b^{(0)}, \ldots, b^{(m-1)}\) appears as a subword of \(L(\beta_0)\), where \(L(\beta) = \lfloor (n\alpha_0 + \beta) \mod m \rfloor_{n \geq 0}\).

By irrationality of \(\alpha\) and the three gap theorem it is not difficult to show that there is some \(B\) such that, for all \(\beta\), every subword of \(L(\beta)\) of length \(B\) contains one subword taken from the set \(\{b^{(0)}, \ldots, b^{(m-1)}\}\). (Sturmian words are uniformly recurrent.) We are going to show that \(\lfloor n^c \rfloor \mod m_{n \geq 0}\) contains a subword of \(L(\beta)\) of length \(B\), which establishes our claim.

It is elementary to show that there exists an \(\varepsilon > 0\) and some open interval \(I\), such that the following holds: for all \(\beta\) and \(n\) such that \(n\alpha_0 + \beta \in I + \mathbb{Z}\), we have \(\|(n + m)\alpha_0 + \beta\| > \varepsilon\) for all \(m < B\).

We use the denseness of \(n\alpha_0 + \beta \mod 1\): let \(A\) be so large that for all \(\beta\) and \(n\) we have \((n + \ell)\alpha_0 + \beta \in I + \mathbb{Z}\) for some \(\ell < A\).

Finally, let \(x_0\) be so large such that \(\frac{1}{2} f^n(x)(A + B + 1)^2 < \varepsilon\) for \(x \geq x_0\), where \(f(x) = x^c\). We approximate \(x^c\) by a linear function \(x\alpha + \beta\) at some point \(x \geq x_0\) satisfying \(\alpha = f'(x) \equiv a_0 \mod m\).

We obtain some \(\ell < A\) such that \(\|([x] + \ell + m)\alpha + \beta\| > \varepsilon\) for all \(m < B\). By Taylor’s formula it follows that \(\lfloor ([x] + \ell + m)^c \rfloor = \lfloor (x + \ell + m)\alpha + \beta \rfloor\) for all \(m < B\), which shows that there is a subword of \(L(\beta)\) of length \(B\) contained in \(\lfloor n^c \rfloor \mod m_{n \geq 0}\).

It follows that \(\lfloor n^c \rfloor \mod m\) has complexity at least \(Ck^3\), which shows (using Allouche and Shallit \cite{2} Corollary 10.4.9) that this sequence is not morphic. In particular, it is not an automatic sequence.

7. Proof of Theorem 4

Theorem 4 is an easy corollary of the following result

**Theorem 8.** Suppose that \(c > 1\) is not an integer. Then we have for every multiplicative function \(f(n)\) with \(|f(n)| \leq 1\) and for every \(\alpha \in \mathbb{Q} \setminus \mathbb{Z}\)

\[ \sum_{n \leq N} f(n) e(\alpha \lfloor n^c \rfloor) = o(N), \quad (N \to \infty). \]

By the Daboussi-Kátai criterion \cite{9} it is sufficient to prove

\[ S := \sum_{n < N} e(\alpha \lfloor pn^c \rfloor - \lfloor qn^c \rfloor) = o(N), \quad (N \to \infty) \]

for sufficiently large (and different) prime numbers \(p, q\).

Actually it is important that we do not have to check (20) for all pairs of (different) primes \(p, q\) since we have to exclude those cases where \((q/p)^c\) is rational. Fortunately this can only occur for finitely many cases.
Lemma 3. Suppose that $c > 0$ is not an integer. Then there exists a constant $L > 0$ such that for all pairs of different primes $p, q > L$ we have

$$(p/q)^c \notin \mathbb{Q}.$$ 

Proof. If there is at most one pair of different primes $p, q$ such that $(p/q)^c \in \mathbb{Q}$ then we set $L = 1$ or $L = \max\{p, q\}$.

Suppose next that there are two different pairs $(p_1, q_1), (p_2, q_2)$ of primes with $p_1 > q_1, p_2 > q_2$

$$(p_1/q_1)^c = r_1 \in \mathbb{Q} \quad \text{and} \quad (p_2/q_2)^c = r_2 \in \mathbb{Q}$$

and suppose that $(p_3, q_3)$ is another pair of different primes with $p_3 > q_3 > \max\{p_1, q_1, p_2, q_2\}$ such that

$$(p_3/q_3)^c = r_3 \in \mathbb{Q}.$$ 

Setting $\lambda_{11} = \log(p_1/q_1), \lambda_{12} = \log(p_2/q_2), \lambda_{13} = \log(p_3/q_3)$ and $\lambda_{21} = \log r_1, \lambda_{22} = \log r_2, \lambda_{23} = \log r_3$ it follows that

$$\frac{\lambda_{11}}{\lambda_{21}} = \frac{\lambda_{12}}{\lambda_{22}} = \frac{\lambda_{13}}{\lambda_{23}} = \frac{1}{c}$$

or equivalently that the matrix

$$M = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \end{pmatrix}$$

has rank 1.

By assumption it might be that one of $p_1, q_1$ coincides with one of $p_2, q_2$ but not both. Hence, by unique factorization it follows that

$$\lambda_{11} = \log(p_1/q_1), \quad \lambda_{12} = \log(p_2/q_2), \quad \lambda_{13} = \log(p_3/q_3)$$

are linearly independent over the rationals.

Furthermore we have the property that $c$ is irrational or equivalently that $\lambda_{11}$ and $\lambda_{21}$ are linearly independent over the rationals. Assuming the contrary it would have

$$\left(\frac{p_1}{q_1}\right)^A = r_1^B$$

for coprime integers $A, B$, that is, $c = A/B$. Recall that the primes $p_1$ and $q_1$ are different. Hence, $p_1$ has to appear on the right hand side, and due to the exponent $B$ it has to appear with an integer multiple of $B$ as its multiplicity. However, due unique factorization this multiplicity has to be $A$ which implies that $B = 1$ and consequently that $c = A$ is an integer. But this is excluded by assumption.

Thus, by the Six Exponential Theorem by Lang [11] and Ramachandra [13] implies that the matrix $M$ has rank 2. This leads to a contradiction and proves the lemma by setting $L = \max\{p_1, q_1, p_2, q_2\}$.

The next ingredient that we need is the following estimate for exponential sums.

Lemma 4. Suppose that $c > 1$ is not an integer. Then we have uniformly for all real numbers $U$ with $|U| \geq \eta$, where $\eta > 0$, and $N \geq 1$

$$\sum_{n \leq N} e(Un^c) \ll |U| N^{1-\frac{1}{2c}}.$$ 

Proof. The proof runs along the same lines as that of Proposition 2.
Now suppose that \( p, q \) are different primes such that \((p/q)^c\) is irrational. Let \( H \) be an arbitrary large number and observe first that we can slightly modify \( S \) from (20):

\[
S = \sum_{n<N} e\left( \alpha \left( ((pn)^c] - ((qn)^c] \right) \right)
\]

\[
= \sum_{n<N} e\left( \alpha ((pn)^c] - \alpha ((qn)^c] \right)
\]

\[
= \sum_{n<N} e\left( \alpha ((pn)^c - (qn)^c) - \alpha ((pn)^c} - \{ (qn)^c \} \right)
\]

\[
= \sum_{0 \leq k_1, k_2 < H} \sum_{n < N, ((pn)^c} \in \{ \frac{k_1}{H}, \frac{k_1+1}{H} \}, ((qn)^c} \in \{ \frac{k_2}{H}, \frac{k_2+1}{H} \} e\left( \alpha ((pn)^c - (qn)^c) - \alpha \left( \frac{k_1}{H} - \frac{k_2}{H} \right) \right) + O \left( \frac{N}{H} \right)
\]

It is, thus, sufficient to study the sums

\[
S_{k_1, k_2} := \sum_{n < N, ((pn)^c} \in \{ \frac{k_1}{H}, \frac{k_1+1}{H} \}, ((qn)^c} \in \{ \frac{k_2}{H}, \frac{k_2+1}{H} \} e\left( \alpha ((pn)^c - (qn)^c) \right)
\]

\[
= \sum_{n < N} e\left( \alpha ((pn)^c - (qn)^c) \right) \chi_{1/H} \left( (pn)^c - \frac{k_1}{H} \right) \chi_{1/H} \left( (qn)^c - \frac{k_2}{H} \right)
\]

For this purpose we approximate the indicator function \( \chi_{1/H} \) with the help of a Lemma due to Vaaler (see [8, Theorem A.6]) and obtain

\[
\left| \chi_{1/H}(x) \chi_{1/H}(y) - \sum_{|h_1|, |h_2| \leq H^3} a_{h_1}(H^{-1}, H^3) a_{h_2}(H^{-1}, H^3) e(h_1x + h_2y) \right| \leq \sum_{|h| \leq H^3} b_h(H^{-1}, H^3) (e(hx) + e(hy)),
\]

where

\[
a_0(H^{-1}, H^3) = \frac{1}{H}, \quad |a_h(H^{-1}, H^3)| \leq \min \left\{ \frac{1}{H}, \frac{1}{\pi |h|} \right\}, \quad |b_h(H^{-1}, H^3)| \leq \frac{1}{H^3 + 1}.
\]

Thus, \( S_{k_1, k_2} \) can be estimated by

\[
|S_{k_1, k_2}| \ll \sum_{|h_1|, |h_2| \leq H^3} |a_{h_1}(H^{-1}, H^3) a_{h_2}(H^{-1}, H^3)| \sum_{n < N} e\left( ((\alpha + h_1) p^c + (-\alpha + h_2) q^c) n^c \right)
\]

\[
+ \sum_{|h| \leq H^3} |b_h(H^{-1}, H^3)| \sum_{n < N} e\left( h p^c n^c \right) + \sum_{|h| \leq H^3} |b_h(H^{-1}, H^3)| \sum_{n < N} e\left( h q^c n^c \right)
\]

\[
\ll c_1(H) N^{1 - \frac{\|c\|}{2H}} + \frac{N}{H^3} + c_2(H) N^{1 - \frac{\|c\|}{2H}},
\]

where \( c_1(H) \) and \( c_2(H) \) are constants depending on \( H \) and where we have used Lemma [4] and the property that (by assumption)

\[
\min_{|h_1|, |h_2| \leq H^3} |((\alpha + h_1) p^c + (-\alpha + h_2) q^c)| > 0.
\]

Summing up and letting \( N \to \infty \) it follows

\[
\limsup_{N \to \infty} \frac{|S|}{N} \ll \frac{1}{H} + H^2 \frac{1}{H^3} \ll \frac{1}{H}.
\]
Since $H$ can be chosen arbitrarily large it finally follows that $S = o(N)$ as $N \to \infty$. This completes the proof of Theorem 8 and thus that of Theorem 4.

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