The Thom Class and Localization of SUSY QM Generating Functional

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Abstract

We demonstrate the usage of explicit form of the Thom class found by Mathai and Quillen for the definition of generating functional of a simple supersymmetric quantum mechanical model.

1 Backgrounds on superformalism

Definition 1.1 The Grassmann algebra $\Lambda_n(\mathbb{C})$ is an $\mathbb{C}$ algebra freely generated by the set of $n$ anticommuting generators $\{\eta_1, \ldots, \eta_n\}$, i.e.

$$\{\eta_i, \eta_j\} = \eta_i \eta_j + \eta_j \eta_i = 0 \ \forall i, j \in \hat{n}$$

Similarly we define $\Lambda_n(\mathbb{R})$.

The straightforward generalization of this is the notion of super($\mathbb{Z}_2$) graded algebra.

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Definition 1.2 Let $M$ be a $\mathbb{Z}_2$ graded module over a $\mathbb{Z}_2$ graded ring $R$. Moreover, let $M$ be endowed with a grading compatible multiplication

$$m : M \otimes_R M \to M$$

Then $M$ is called the superalgebra over $R$.

Clearly, the Grassmann algebra $\Lambda_n(\mathbb{C})$ is an example of superalgebra over $R = \mathbb{C}$. It’s the linear span of 1 and the monomials

$$\{\eta_{i_1} \ldots \eta_{i_k} \mid 1 \leq i_1 < \ldots < i_k \leq n, \ k \in \mathbb{n}\}$$

$R$ is considered with trivial grading here and an element in $\Lambda_n(\mathbb{C})$ is even (odd) if it has in the expansion with respect to (3) the even (odd) monomials only.

In a superalgebra one can define so called generalized Lie bracket putting

$$[a, b] := a \cdot b - (-1)^{\deg(a)\deg(b)} b \cdot a$$

for homogeneous elements and expanding it by linearity. A superalgebra is called commutative if the bracket (4) vanishes. Given two superalgebras $A$ and $B$ one can form the tensor product $A \otimes B$ of them in the usual way, i.e. one makes the tensor product of underlying $\mathbb{Z}_2$ graded modules and endows it with the multiplication defined by $(a \otimes b) \cdot (a' \otimes b') := (-1)^{\deg(b')\deg(a')} aa' \otimes bb'$.

Suppose we have an arbitrary superalgebra. Then one can define the Pfaffian by the following

Definition 1.3 Suppose $n = 2m$ and let $\omega$ be a skew-symmetric $n \times n$ matrix of even elements in an arbitrary superalgebra. The Pfaffian of $\omega$ with respect to the vector of odd elements $\{\eta_i\}_{i=1}^n$ is defined as

$$\frac{1}{m!}\left(\frac{1}{2} \eta^t \omega \eta\right)^m = Pf(\omega)\eta_1 \ldots \eta_n$$

where $(\cdot)^t$ denotes the transpose.

Another useful tool in calculations with superquantities is Berezin or fermionic integral.
Definition 1.4 The Berezin integral of an element $x$ in $\Lambda_n(\mathbb{C})$, resp. $\Lambda_n(\mathbb{R})$ generated by $\{\eta_1, \ldots, \eta_n\}$ is defined as the coefficient of $x$ staying before the $\eta_1 \ldots \eta_n$ monomial, i.e. when

$$x = \sum_{k=0}^{n} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \alpha_{i_1 \ldots i_k}^{(k)} \eta_{i_1} \ldots \eta_{i_k}$$

(6)

then $\int D\eta x := \alpha_{1 \ldots n}^{(n)}$. One can extend this definition to the elements of $A \otimes \Lambda_n(\mathbb{C})$ where $A$ is an arbitrary superalgebra as $\int D\eta a \otimes x := a \int D\eta x$.

With the notion of Berezin integral one can express the Pfaffians as

$$Pf(\omega) = \int D\eta e^{\frac{1}{2} \eta^t \omega \eta}$$

(7)

and the following holds

**Proposition 1.1** Let $A$ be any commutative superalgebra, $n = 2m$, $\omega$ a skew-symmetric $n \times n$ matrix of even elements, $\{\eta_i\}_{i=1}^n$ be a vector of odd elements. Then

$$e^{\frac{1}{2} \eta^t \omega \eta} = \sum_I Pf(\omega_I) \eta^I$$

(8)

where $I$ runs over all subsets of $\hat{n}$ with even cardinality and for $I = \{i_1, \ldots, i_k\}$, $i_1 < \ldots < i_k$, $\eta^I$ denotes $\eta_{i_1} \ldots \eta_{i_k}$ while $(\omega_I)_{i'j'}$ denotes the submatrix of $(\omega)_{ij}$ with $i'$, $j'$ in $I$.

**Proof.** Suppose for the moment $A$ is the Grassmann algebra generated by $\{\eta_1, \ldots, \eta_n\}$. Consider the subalgebra generated by $\eta_i$, $i \in I$. Then the homomorphisms $h_I$ which kill all $\eta_k$, $k \notin I$ sends $e^{\frac{1}{2} \eta^t \omega \eta}$ to the corresponding Gaussian expression constructed from $\omega_I$. Hence

$$e^{\frac{1}{2} \eta^t \omega \eta} = \sum_I \int D\eta^I h_I(e^{\frac{1}{2} \eta^t \omega \eta}) = \sum_I \int D\eta^I e^{\frac{1}{2} (\eta^I)^t \omega_I \eta^I} = \sum_I Pf(\omega_I) \eta^I$$

(9)
which verifies (8) in this case. Suppose now $A$ is any commutative superalgebra. Both elements in (8) can be interpreted as elements of free commutative superalgebra $S \otimes \Lambda_n(\mathbb{C})[\eta_1, \ldots, \eta_n]$, where $S$ is the algebra of polynomials in variables $\omega_{ij}$. These elements are equal since polynomials are determined by it values. Therefore we can apply realization homomorphism which takes the elements $\omega_{ij}, \eta_i$ of the former to corresponding element of $A$. Thus (8) holds in $A$.

Using this proposition one can state

**Proposition 1.2** Let $A$ be any commutative superalgebra, $n = 2m$, $\omega$ a skew-symmetric $n \times n$ matrix of even elements, $\{J_i\}_{i=1}^n$ be a vector of odd elements. Then

$$
\int D\eta e^{\frac{1}{2}b^i\omega_{ij} + J^i\eta} = \sum_{I \text{ even}} \epsilon(I, I')(-1)^{|I'|} Pf(\omega_I) J^{I'}
$$

where $I'$ denotes the complement of $I$ in $\hat{n}$ and $\epsilon(I, I')$ is defined as

$$
\eta^I \eta'^{I'} = \epsilon(I, I') \eta_1 \ldots \eta_n.
$$

**Proof.** uses [1]. See [2] for further details.

2 Thom class construction and Euler characteristic

Suppose now $n = 2m$ and we have a compact manifold $M$, $\dim M = n$ with spin structure, i.e. we have an isomorphism of $TM$ with bundle $P \times_{Spin(n)} V$ associated to principal spin frame bundle $P$ over $M$ and standard representation of $Spin(n)$ on $V = \mathbb{R}^n$. Hence $P \times V$ is a principal $Spin(n)$ bundle over $TM$ which is equal to $\pi^* P$, $\pi$ is the $TM$ projection. The forms $\Omega(TM)$ can be identified with basic forms of $\Omega(P \times V)$, i.e. the forms which satisfies

$$
R^*_g \omega = \omega \quad (11)
$$

$$
i_X \omega = 0 \quad (12)
$$
for all $g$ in $Spin(n)$ and all $X$ in its Lie algebra. In particular, the identification isomorphism is given by pull-back with respect to the projection $f : P \times V \to TM$.

Suppose now we have the spin connection $\theta$ in $P$. This connection may be pulled back to $P \times V$. We denote this connection by the same symbol. If one consider its curvature

$$\Omega = d\theta + \theta \wedge \theta \quad (13)$$

then one can define the form on $\Omega(P \times V)_{\text{basic}} \simeq \Omega(TM)$ putting

$$U = \pi^{-m} e^{-x^2} \sum_{I \text{ even}} \epsilon(I,I') Pf(\frac{1}{2} \Omega_I)(dx + \theta x)^I' \quad (14)$$

In fact, one can show that this form is closed (see [2]) and integrates to 1 over the fibres as is easily seen from the identity

$$\pi^{-m} \int_{\mathbb{R}^n} e^{-x^2} d^n x = 1 \quad (15)$$

We recall the Thom class definition. Let $\mathcal{F}$ be an oriented vector bundle with scalar product and let $\mathbb{D}\mathcal{F}$ denotes the unit disk bundle of $\mathcal{F}$. Then one has the Thom isomorphism

$$\pi_* : H^i(\mathcal{F}, \mathcal{F} \setminus \mathbb{D}\mathcal{F}) \to H^{i-n}(M) \quad (16)$$

**Definition 2.1** The Thom class is an element of $H^n(\mathcal{F}, \mathcal{F} \setminus \mathbb{D}\mathcal{F})$ defined as

$$u(\mathcal{F}) := \pi_*^{-1}(1) \quad (17)$$

When passing to the form representation of cohomology classes $\pi_*$ corresponds to integration over the fibres and thus the Thom class is represented by a closed form with support in $\mathbb{D}\mathcal{F}$ which integrates to 1 over the fibres.

The form $(14)$ represents the Thom class of $TM$ in the following sense. This form doesn’t have the support in $\mathbb{D}TM$ but one can define a fibrewise diffeomorphism, namely
\[ f : TM \to \mathbb{D}TM \]  \hspace{1cm} (18)
\[ f(x) := \frac{x}{\sqrt{1 + \|x\|^2}} \]

and define the complex \( \Omega_d(TM) \) as \( f^*(\Omega(TM, TM \setminus \mathbb{D}TM)) \). Then since integration is invariant under orientation preserving diffeomorphisms we can extend \( \pi^* \) as

\[ \tilde{\pi}^* : \Omega_d(TM) \to \Omega(M) \]  \hspace{1cm} (19)

which induces the isomorphism on cohomology. Thus \( U \in \Omega_d(TM) \) represents the Thom class since it’s closed and integrates to 1 over the fibres.

Now we turn back to proposition 1.2 and apply it for the case of \( A := \Omega(P \times V) \) and \( J = dx + \theta x \). Thus the form \( U \) may be expressed as

\[ U = \pi^{-m} \int \mathcal{D} \eta e^{-s^2 + \frac{1}{2} \eta \Omega(n+(dx+\theta x)'} \eta \]  \hspace{1cm} (20)

If we denote by \( s : M \to TM \) section with isolated zeroes then \( s^*U \) represents Euler class of \( M \) and we have the well-known formula

\[ \int_M s^*U = \sum_i \text{Ind}_i(s) \]  \hspace{1cm} (21)

This can be proved in greater generality for an arbitrary oriented vector bundle over a compact manifold, considering the family of sections \( \{s_t\}_{t \in \mathbb{R}^+}, s_t := ts \) for such \( s \). All the forms \( s^*_tU \) represents the same cohomology class and thus for \( t \to 0 \) we obtain the Euler class by the definition. The integral \( \int s^*_tU \) remains the same for all \( t \) and for \( t \to +\infty \) one obtains RHS of (21). However, in the case of \( TM \) famous Hopf theorem identifies RHS of (21) with Euler characteristic \( \chi(M) \).

Finally if we choose such a section \( s \) then we have from (21)

\[ \chi(M) = \pi^{-m} \int_M \int \mathcal{D} \eta e^{-s^2 + \frac{1}{2} \eta^2 (s^*\Omega(n+(\nabla s)')} \eta \]  \hspace{1cm} (22)

which works even for \( s = 0 \) from the reasoning above.
3 A physical interpretation: SUSY quantum mechanics

We are going to show that (22) is in fact the generating functional for SUSY quantum mechanics. In physics, however, one has to start with the loop space in order to obtain the action in integral form. This is defined as follows

**Definition 3.1** Let $M$ be a differentiable manifold. Loop space $LM$ of $M$ is defined to be the set of all smooth mappings from $S^1$ to $M$.

Of course, one has to define the topology and differentiable structure on $LM$ (see [3]), but we can treat it at least formally in physics. The tangent bundle of $LM$ is defined as

$$T_x LM := \{ X \in LT M \mid \pi_{TM} X(t) = x(t) \ \forall t \in S^1 \} \quad (23)$$

If $M$ is equipped with Riemannian metric then the induced metric on $TLM$ is defined as

$$\hat{g}_x (V_1(x), V_2(x)) := \int_{S^1} dt g_{x(t)} (V_1(x)(t), V_2(x)(t)) \quad (24)$$

where $x \in LM$.

Now if we put $dx^\mu = \psi^\mu$ and $s = \dot{x}^\mu$, i.e. we identify fermionic fields with coordinates in $T^*LM$ and choose our section to be the 'time' derivative of bosonic fields, then we can, at least formally, identify the exponent in (22) with the action of SUSY QM which in coordinates looks like

$$S_{QM} = \int_{S^1} dt \left[ -g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{1}{2} R_{\rho\sigma}^{\mu\nu} \bar{\psi}_\mu \psi_\nu \psi_\sigma + \bar{\psi}_\mu \nabla_t \psi^\mu \right] \quad (25)$$

Here $\nabla_t$ is the 'covariant derivative' defined as

$$\nabla_t \psi^\mu (t) := \ddot{\psi}^\mu (t) + \Gamma^\mu_{\rho\sigma} \dot{x}^\rho \psi^\sigma \quad (26)$$

where $\Gamma^\mu_{\rho\sigma}$ are Christoffel symbols of Levi-Civita connection induced by $g$. The term $\nabla s$ in (22) corresponds to
Finally, we consider the generating functional of this physical model

\[ Z_{SQM} = \int_{LM} \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}x \ e^{iS_{QM}[\psi,\bar{\psi},x]} \]  

(28)

This integral over loop space is not correctly defined, however, we can apply formally on it \((21)\) and 'localize' it at zeroes of \(s = \dot{x}^\mu\). But this means to integrate over the space of constant loops, i.e. the manifold \(M\). This means we recover the \((22)\) for \(s = 0\) in this case and thus 'regularized Euler characteristic' can be identified with Euler characteristic of \(M\). This is the argument physicists use proving the fact that SUSY QM is an example of topological field theory.

References

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