1 Beginning Concepts

1.1 Introduction

In these lectures, we present geometric approach to the two-dimensional quantum gravity. It became popular since Polyakov’s discovery that first-quantized bosonic string propagating in $\mathbb{R}^d$ can be described as theory of $d$ free bosons coupled with the two-dimensional quantum gravity [1]. In critical dimension $d = 26$, the gravity decouples and Polyakov’s approach reproduces results obtained earlier by different methods (see, e.g., [2] for detailed discussion and references).

Classically, the two-dimensional gravity is a theory formulated on a smooth oriented two-dimensional surface $X$, endowed with a Riemannian metric $ds^2$, whose dynamical variables are metrics in the conformal class of $ds^2$. Classical equation of motion is the two-dimensional Einstein equation with a cosmological term and it describes the metric with constant Gaussian curvature. Since in two dimensions, conformal structure uniquely determines a complex structure, a surface $X$ with the conformal class of $ds^2$ has a structure of a one-dimensional complex manifold—a Riemann surface. We will consider the case when $X$ is either compact (i.e. an algebraic curve), or it is non-compact, having finitely many branch points of infinite order. Except for few cases, the virtual Euler characteristic $\chi(X)$ of the Riemann
surface $X$ is negative so that, according to the Gauss-Bonnet theorem, it admits metrics with constant negative curvature only. Specifically, if $\chi(X) < 0$, then there exists on $X$ a unique complete conformal metric of constant negative curvature $-1$, called the Poincaré, or hyperbolic, metric. In terms of local complex coordinate $z$ on $X$, conformal metric has the form $ds^2 = e^{\phi}|dz|^2$ and its Gaussian curvature is given by $R_{ds^2} = -2e^{-\phi}\phi_{\bar{z}z}$, where subscripts indicate partial derivatives. The condition of constant negative curvature $-1$ is equivalent to the following nonlinear PDE,

$$\frac{\partial^2 \phi}{\partial z \partial \bar{z}} = \frac{1}{2}e^{\phi},$$

(1.1)
called Liouville equation.

Quantization of the two-dimensional gravity (in the conformal gauge) amounts to the quantization of conformal metrics with the classical action given by the Liouville theory. The definition of the latter is a non-trivial problem. Namely, since $\phi(z, \bar{z})$ is not a globally defined function on $X$, but rather a logarithm of the conformal factor of the metric, “kinetic term” $|\phi_z|^2dz \wedge d\bar{z}$ does not yield a $(1,1)$-form on $X$ and, therefore, can not be integrated over $X$. This means that “naive” Dirichlet type functional is not well defined and can not serve as an action for the Liouville theory (it also diverges at the branch points). There are two possible ways to deal with this situation. The first one uses the choice of the background metric on $X$, whereas in the second, a regularization at the branch points is used. In addition, when topological genus of $X$ is zero, one also takes advantage of the existence of a single global coordinate on $X$ [3]; for the case of the non-zero genus one uses a global coordinate, provided by the Schottky uniformization [4]. In the first approach the rich interplay between semi-classical approximation, conformal symmetry and the uniformization of Riemann surfaces seems to be lost, or at least hidden. In the second approach, developed in [3, 4] (see also [5] for a review), and which we will use here, this interplay plays a fundamental role.

Once the action functional is defined, one can quantize the two-dimensional gravity using a method of functional integration, performing the “summation” over all conformal metrics on the Riemann surface $X$, with a hyperbolic metric being a “critical point” of the “integral”. One may refer to it as to the “quantization” of the hyperbolic geometry of Riemann surfaces, as fluctuations around the Poincaré metric “probe” the hyperbolic geometry. In other words, the two-dimensional quantum gravity can be considered as a special topic of
the “quantum geometry” of Riemann surfaces. Among other things, it provides unified
treatment of conformal symmetry, uniformization and complex geometry of moduli spaces.

The following argument illustrates our approach. Let \( X \) be two-dimensional sphere \( S^2 \),
realized as Riemann sphere \( \mathbb{P}^1 \) with complex coordinate \( z \) in standard chart \( \mathcal{O} \). A smooth
conformal metric on \( X \) has the form \( ds^2 = e^\phi |dz|^2 \) with condition \( \phi(z, \bar{z}) \simeq -4 \log |z| \) as
\( z \to \infty \), ensuring the regularity at \( \infty \). Regularized action functional is defined as
\[
S_0(\phi) = \lim_{R \to \infty} \left( \int_{|z| \leq R} (|\phi_z|^2 + e^\phi) d^2z - 8\pi \log R \right), \quad d^2z = \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}, \tag{1.2}
\]
and corresponding Euler-Lagrange equation \( \delta S_0 = 0 \) yields the Liouville equation. However,
it has no global smooth solution, since sphere \( S^2 \), according to Gauss-Bonnet, admits only
metrics of constant positive curvature. At the quantum level one should consider functional
integral
\[
<S^2> = \int \mathcal{D}\phi \, e^{-(1/2\pi h)S_0(\phi)},
\]
where “integration” goes over all smooth conformal metrics on \( S^2 \), \( \mathcal{D}\phi \) symbolizes certain
“integration measure”
\[
\mathcal{D}\phi \doteq \prod_{x \in S^2} d\phi(x),
\]
and positive \( h \) plays the role of a coupling constant. According to the latter remark, this
functional integral does not have critical points, so that the perturbation theory, based on
the “saddle point method”, is not applicable. However, quantity \( <S^2> \) has a meaning of a
partition function/ground state energy of the theory, and as such plays a normalization role
only. Objects of fundamental importance are given by the correlation functions of Liouville
vertex operators \( V_\alpha(\phi)(z) = \exp\{\alpha \phi(z, \bar{z})\} \) with different “charges” \( \alpha \). Namely, according
to Polyakov [1], these correlation functions, which should be calculated in order to find the
scattering amplitudes of non-critical strings, are given by the following functional integral
\[
<V_{\alpha_1}(z_1) \cdots V_{\alpha_n}(z_n)> = \int \mathcal{D}\phi \, V_{\alpha_1}(\phi)(z_1) \cdots V_{\alpha_n}(\phi)(z_n) e^{-(1/2\pi h)S_0(\phi)}. \tag{1.3}
\]
Introducing sources \( \delta(z - z_i) \), localized at insertion points \( z_i \), one can include the product of
the vertex operators into the exponential of the action, so that it acquires a critical point,
given by a singular metric on \( S^2 \). The main proposal of Polyakov [1] is that the “summation”
over smooth metrics with the insertion of vertex operators in (1.3) should be equivalent to
the “summation” over metrics with singularities at the insertion points, without the insertion of the vertex operators! For special $\alpha$’s these singular metrics become complete metrics on Riemann surface $X$—Riemann sphere $\mathbb{P}^1$ with branch points $z_i$. If $\chi(X)$ is negative, a complete hyperbolic metric on $X$ exists and perturbation theory is applicable (see also [7] for similar arguments).

Specifically, consider branch points $z_i$ of orders $2 \leq l_i \leq \infty$, $i = 1, \ldots, n$. According to Poincaré [8] (see also [3, pp. 72-78]), admissible singularities of metric $ds^2 = e^{\phi(z, \bar{z})}|dz|^2$ are of the following form

$$e^{\phi} \simeq l_i^{-2} \frac{r_i^{2/l_i-2}}{(1 - r_i^{1/l_i})^2}, \quad (1.4)$$

when $l_i < \infty$, and

$$e^{\phi} \simeq \frac{1}{r_i^2 \log^2 r_i}, \quad (1.5)$$

when $l_i = \infty$ and $r_i \simeq |z - z_i| \to 0$. Note that the metric

$$\frac{|dz|^2}{|z|^2 \log^2 |z|},$$

used in (1.5), is the Poincaré metric on the punctured unit disc $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$. Assuming for a moment that $\infty$ is a regular point, one also has $\phi \simeq -4 \log |z|$, as $z \to \infty$.

Using the formula

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log |z|^2 = \pi \delta(z),$$

where $\delta(z)$ is the Dirac delta-function, equation (1.1) with asymptotics (1.4)—(1.5) can be rewritten in the following form

$$\frac{\partial^2 \phi}{\partial z \partial \bar{z}} = \frac{1}{2} e^{\phi} - \pi \sum_{i=1}^{n} (1 - 1/l_i) \delta(z - z_i) + 2\pi \delta(1/z), \quad (1.6)$$

where insertion points are present explicitly (cf. [3]). Equation (1.6) is uniquely solvable if

$$\chi(X) = 2 - \sum_{i=1}^{n} (1 - 1/l_i) < 0, \quad (1.7)$$

i.e. if one can “localize curvature” at branch points. As we shall see later, corresponding charges are

$$\alpha_i = \frac{1}{2\hbar} (1 - 1/l_i^2), \quad (1.8)$$
and play a special role in the Liouville theory. Our main interest will be concentrated on the case when branch points $z_i$ are of infinite order, i.e. $l_i = \infty$. In this case Riemann surface $X$ is non-compact, being a Riemann sphere $\mathbb{P}^1$ with $n$ removed distinct points, called punctures. According to (1.5), punctures are “points at infinity” in the intrinsic geometry on $X$, defined by metric $e^{\phi}|dz|^2$, i.e. for any $z_0 \in X$ the geodesic distance $d(z_0, z) \to \infty$ as $z \to z_i$.

This approach does not look like a standard one in the quantum field theory; however, the Liouville theory is not a standard model either! It provides a manifestly geometrical treatment of the theory, as one should expect from the theory of gravity. On the other hand, the standard approach to the two-dimensional quantum gravity [10, 11, 12] (see also [13] for a review), which is based on the free-field representation, does not recover the underlying hyperbolic geometry at a classical limit. These two approaches, perhaps, might reveal the two phases of the two-dimensional quantum gravity; relation between them is yet to be discovered.

Consistent perturbative treatment of the geometrical approach, given in [14, 15], exhibits the richness of the theory. Mathematically, it provides a unified view on the uniformization of Riemann surfaces and complex geometry of moduli spaces, and uses methods of the Teichmüller theory, spectral geometry and theory of automorphic forms. Physically, it realizes conformal bootstrap program of Belavin-Polyakov-Zamolodchikov [16] in the geometrical setting of Friedan-Shenker [17]. In particular, conformal Ward identities can be proved, and the central charge of the Virasoro algebra can be calculated. Detailed exposition of these results will be presented in the course, which will be organized as follows.

In §1.2 we formally define the main objects of the theory: the expectation value of Riemann surfaces and normalized connected forms of multi-point correlation functions with the stress-energy tensor components. In §§2.1—2.3 we recall basic facts from the conformal field theory, adapted to our case: operator product expansion of Belavin-Polyakov-Zamolodchikov, conformal Ward identities and Friedan-Shenker geometric interpretation. In §§3.1—3.2 we give a perturbative definition of the expectation value and correlation functions; in §3.3 we compute one- and two-point correlation functions at the tree level and in the one-loop approximation. In §4.1 we show how conformal Ward identities yield new mathematical results on Kähler geometry of moduli spaces; in §4.2 we discuss corresponding results for the one-loop approximation and compute the central charge of the Virasoro
algebra. In §4.3 we recall basic facts from the Teichmüller theory, and in §4.4 we show how to prove the results, obtained in §§4.1—4.2. In particular, validity of the conformal Ward identities for the quantum Liouville theory follows. Finally, in §4.5 we briefly discuss how to adapt our approach to the case of Riemann surfaces of non-zero genus. All sections are accompanied by exercises, intended to develop the familiarity with various aspects of the mathematical formalism used in this course.

1.2 Quantization of the Hyperbolic Geometry

Let \( X \) be an \( n \)-punctured sphere, i.e. Riemann sphere \( \mathbb{P}^1 \) with \( n \) removed distinct points \( z_1, \ldots, z_n \), called punctures. Without loss of generality we assume \( z_n = \infty \), so that \( X = \mathbb{C} \setminus \{z_1, \ldots, z_{n-1}\} \). Using single global complex coordinate \( z \) on \( \mathbb{C} \), any conformal metric on \( X \) can be represented as

\[
d s^2 = e^{\phi(z, \bar{z})} |dz|^2.
\]

Denote by \( \mathcal{C}(X) \) a class of all smooth conformal metrics on \( X \) having asymptotics (1.3), where for \( i = n \) (\( z_n = \infty \) is now a puncture!) one has \( r_n = |z| \rightarrow r \rightarrow \infty \). These asymptotics ensure that \( \mathcal{C}(X) \) consists of complete metrics on \( X \), which are asymptotically hyperbolic at the punctures. Asymptotics (1.3) imply that the standard expression (1.2) for the Liouville action diverges when \( \phi \in \mathcal{C}(X) \). Properly regularized Liouville action was presented in [3] and has the form

\[
S(\phi) = \lim_{\epsilon \rightarrow 0} \left\{ \int_{X_\epsilon} (|\phi_z|^2 + e^{\phi}) d^2 z + 2\pi n \log \epsilon + 4\pi(n - 2) \log |\log \epsilon| \right\},
\]

where \( X_\epsilon = X \setminus \bigcup_{i=1}^{n-1} \{ r_i < \epsilon \} \cup \{ r > 1/\epsilon \} \). Euler-Lagrange equation \( \delta S = 0 \) yields Liouville equation —the equation for complete conformal metric on \( X \) of constant negative curvature \(-1\). When virtual Euler characteristic \( \chi(X) \) of Riemann surface \( X \) is negative, i.e. when, according to (1.7), \( n \geq 3 \), Liouville equation has a unique solution, denoted by \( \phi_{cl} \).

Following the discussion in the previous section, the correlation functions of the puncture operators (correlation functions of the punctures), “symbolized” in (1.3), should be depicted by the following functional integral

\[
< X > = \int_{\mathcal{C}(X)} \mathcal{D}\phi \ e^{-\left(1/2\pi\hbar\right)S(\phi)}.
\]
We call \( \langle X \rangle \) the “expectation value” of Riemann surface \( X \) and consider it as the main object of the two-dimensional quantum gravity, which encodes all the information about the theory. Being unable to give a nonperturbative definition, we define \( \langle X \rangle \) using the perturbation expansion around classical solution \( \phi = \phi_{cl} \). (Similar approach was proposed in \([18]\) for the case when \( \phi_{cl} \) corresponds to the Fubini-Study metric on \( \mathbb{P}^1 \)). We describe this procedure in \( \S 3.1 \).

Classical Liouville theory is conformally invariant: this is another way of saying that dynamical variables are conformal metrics \( ds^2 \) on Riemann surface \( X \). This invariance implies, in particular, that the stress-energy tensor of the Liouville theory, which “measures” the respond of the theory to the local deformations of the metric, is traceless. Its \((2,0)\)-component \( T(\phi)(z) \) is given by

\[
T(\phi) = \frac{1}{h}(\phi_{zz} - \frac{1}{2}\phi_z^2),
\]

is conserved on classical equations of motion

\[
\partial_z T_{cl} = 0,
\]

where \( T_{cl}(z) = T(\phi_{cl})(z) \), and has the transformation law of a projective connection (times \( 1/h \)) under holomorphic change of coordinates, i.e.

\[
\tilde{T}(w) = T(f(w)) f'(w)^2 + \frac{1}{h} S(f)(w),
\]

where \( z = f(w) \). Here \( S \) stands for the Schwarzian derivative of (locally) holomorphic function \( f \),

\[
S(f) = \frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2,
\]

which satisfies transformation law

\[
S(f \circ g) = S(f) \circ g(g')^2 + S(g),
\]

called the Cayley identity.

From \([13]\) it follows that \( T(\phi) \) has a second order poles at punctures

\[
T(\phi)(z) = \frac{1}{2h(z - z_i)^2} + o(|z - z_i|^{-2}), \text{ as } z \to z_i, \ i = 1, \ldots, n - 1,
\]

and

\[
T(\phi)(z) = \frac{1}{2hz^2} + o(|z|^{-2}), \text{ as } z \to \infty.
\]
Since $T_{cl}$ is holomorphic on $X$, we also have

$$T_{cl}(z) = \sum_{i=1}^{n-1} \left( \frac{1}{2h(z-z_i)^2} + \frac{c_i}{z-z_i} \right), \quad (1.17)$$

where, in virtue of (1.16),

$$\sum_{i=1}^{n-1} c_i = 0, \quad \sum_{i=1}^{n-1} z_i c_i = 1 - \frac{n}{2}. \quad (1.18)$$

As we shall see in §4.1, expansion (1.17) is ultimately related with the Fuchsian uniformization of Riemann surface $X$.

The $(0,2)$-component $\bar{T}(\phi)(z)$ of the stress-energy tensor is given by

$$\bar{T}(\phi) = \frac{1}{h} (\phi_{zz} - \frac{1}{2} \phi_z^2)$$

and has similar properties.

We emphasize that the “modification” of the stress-energy tensor, i.e. addition of total derivative $\phi_{zz}/h$ to “free-field” term $-\phi_z^2/2h$ in (1.11), is a crucial feature of the theory. It is an artifact of the property that $\phi$ is not a scalar field, but transforms like a logarithm of the conformal factor of metric under the holomorphic change of coordinates

$$\tilde{\phi}(w, \bar{w}) = \phi(f(w), \overline{f(w)}) + \log |f'(w)|^2, \quad z = f(w). \quad (1.19)$$

Therefore the naive “free-field” expression for the stress-energy tensor has a wrong transformation law and only the addition of a total derivative term yields the correct transformation law (1.13). This modification of the stress-energy tensor goes back to Poincaré [8].

Multi-point correlation functions of the holomorphic and anti-holomorphic components of the stress-energy tensor in the presence of punctures are defined by the following functional integral

$$< \prod_{i=1}^{k} T(z_i) \prod_{j=1}^{l} \bar{T}(\bar{w}_j) X > \equiv \int_{\mathcal{C}(X)} D\phi \ \prod_{i=1}^{k} T(\phi)(z_i) \prod_{j=1}^{l} \bar{T}(\phi)(\bar{w}_j) \ e^{-\left(1/2\pi h\right) S(\phi)}. \quad (1.20)$$

We present detailed prescription for calculating these correlation functions in Section 3.2. Here, following the standards of quantum field theory, we introduce a normalized connected (reduced) form of these correlation functions. Denote by $I$ the set $\{z_1, \ldots, z_k, \bar{w}_1, \ldots, \bar{w}_l\}$, so
that correlation function (1.20) can be simply depicted as \(< I >\). The normalized connected form of (1.20), which we denote by
\[
<< \prod_{i=1}^{k} T(z_i) \prod_{j=1}^{l} \bar{T}(\bar{w}_j) X >> = << I >>, \tag{1.21}
\]
is defined by the following inductive formula
\[
<< I >> = \frac{< I >}{< X >} - \sum_{r=2}^{k+l} \sum_{I_1 \cup \cdots \cup I_r} << I_1 >> \cdots << I_r >>, \tag{1.22}
\]
where summation goes over all representations of set \( I \) as a disjoint union of subsets \( I_1, \ldots, I_r \).

Thus, for example,
\[
<< T(z)X >> = \frac{< T(z)X >}{< X >}, \quad << \bar{T}(\bar{z})X >> = \frac{< \bar{T}(\bar{z})X >}{< \bar{X} >},
\]
\[
<< T(z)T(w)X >> = \frac{< T(z)T(w)X >}{< X >} - << T(z)X >> << T(w)X >>,
\]
\[
<< T(z)\bar{T}(\bar{w})X >> = \frac{< T(z)\bar{T}(\bar{w})X >}{< X >} - << T(z)X >> << \bar{T}(\bar{w})X >>.
\]

Next, following Schwinger [19] (see e.g. also [20, Ch. 7.3]) we present a generating functional for the correlation functions with stress-energy tensor components. In our geometrical setting, Schwinger’s external sources are represented by Beltrami differentials on Riemann surface \( X \). Recall that Beltrami differential \( \mu \) is a tensor of the type \((-1, 1)\) on \( X \), i.e. it has a transformation law
\[
\tilde{\mu}(w, \bar{w}) = \mu(f(w), \bar{f}(w)) \frac{f'(w)}{\bar{f}'(\bar{w})}, \tag{1.23}
\]
under the holomorphic change of coordinates \( z = f(w) \). In addition, Beltrami differentials are bounded in the \( L^\infty \) norm:
\[
||\mu||_\infty = \sup_{z \in X} |\mu(z, \bar{z})| < \infty.
\]

The generating functional for the normalized multi-point correlation functions of the stress-energy tensor is introduced by the following expression
\[
Z(\mu, \tilde{\mu}; X) = \frac{Z(\mu, \tilde{\mu}; X)}{< X >}, \tag{1.24}
\]
where
\[
Z(\mu, \tilde{\mu}; X) = \int_{\mathcal{C}(X)} \mathcal{D}\phi \exp\left\{-\frac{1}{2\pi\hbar} S(\phi) + \text{v.p.} \int_X (T(\phi)\mu + T(\phi)\tilde{\mu}) d^2 z\right\}. \tag{1.25}
\]
Here, since $X$ admits a single global coordinate, the $(2,0)$-component of the stress-energy tensor ($(0,2)$-component) can be considered as quadratic differential $Tdz^2$ ($\bar{T}d\bar{z}^2$), so that $T\mu dz \wedge d\bar{z}$ ($\bar{T}\mu d\bar{z} \wedge dz$) is a $(1,1)$-form on $X$ and can be integrated over $X$. The integral in (1.23) is understood in the principal value sense in virtue of singularities (1.15)—(1.16).

Generating functional for the normalized connected multi-point correlation functions is defined in a standard fashion

$$\frac{1}{\hbar} \mathcal{W}(\mu, \bar{\mu}; X) = \log \mathcal{Z}(\mu, \bar{\mu}; X),$$

(1.26)

so that

$$h \ll \prod_{i=1}^{k} T(z_i) \prod_{j=1}^{l} \bar{T}(\bar{w}_j)X >> = \frac{\delta^{k+l} \mathcal{W}(\mu, \bar{\mu}; X)}{\delta \mu(z_1) \cdots \delta \mu(z_k) \delta \bar{\mu}(\bar{w}_1) \cdots \delta \bar{\mu}(\bar{w}_l)} |_{\mu=\bar{\mu}=0}. $$

(1.27)

In §3.2 we present a formalism for calculating the generating functional $\mathcal{W}$ in all orders of the perturbation theory.

**Problems**

1. Evaluate explicitly the integral of the Gaussian curvature of the conformal metric on $S^2$ and show that $S^2$ does not admit a smooth constant negative curvature metric.

2. Prove all properties of the stress-energy tensor, mentioned in the lecture: the transformation law (1.13), conservation on classical solution (1.12), and representation (1.17).

3. Consider as an example, formal power series $A$ and $B$ in infinitely many variables $x_i, i \in \mathbb{N}$,

$$A = \sum_{I} a_I x_I, \quad B = \log A = \sum_{I} b_I x_I,$$

where $I = \{i_1, \ldots, i_l\}$ runs through all finite subsets of the set $\mathbb{N}$ of natural numbers, and $x_I = x_{i_1} \cdots x_{i_l}$, and prove that normalized connected multi-point correlation functions, defined by formula (1.27), satisfy relation (1.22).

### 2 Conformal Symmetry

#### 2.1 Operator Product Expansion

The basic fact that $ds^2 = e^{\phi}|dz|^2$ is a conformal metric on Riemann surface $X$, can be reformulated by saying that positive quantity $e^{\phi(z, \bar{z})}$ has transformation properties of a $(1,1)$-tensor on $X$. This implies that at the classical level, vertex operators $V_\alpha(\phi)(z, \bar{z})$ are tensors
of type \((\alpha,\alpha)\) on \(X\), i.e. they satisfy transformation law

\[
\tilde{V}_\alpha(\tilde{\phi})(w,\bar{w}) = V_\alpha(\phi)(f(w),\bar{f(w)})|f'(w)|^{2\alpha},
\]

under holomorphic change of coordinates \(z = f(w)\). Therefore, they can be considered as primary fields with classical conformal dimensions (weights) \(\Delta = \bar{\Delta} = \alpha\).

One should expect that at the quantum level conformal symmetry of the theory is preserved, so that vertex operators and components of the stress-energy tensor satisfy operator product expansions (OPE) of Belavin-Polyakov-Zamolodchikov

\[
T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{1}{z-w} \frac{\partial T}{\partial w}(w) + \text{regular terms}, \tag{2.1}
\]

\[
\bar{T}(\bar{z})\bar{T}(\bar{w}) = \frac{c/2}{(\bar{z}-\bar{w})^4} + \frac{2\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{1}{\bar{z}-\bar{w}} \frac{\partial \bar{T}}{\partial \bar{w}}(\bar{w}) + \text{regular terms}, \tag{2.2}
\]

\[
T(z)\bar{T}(\bar{w}) = \text{regular terms}, \tag{2.3}
\]

and

\[
T(z)V_\alpha(w,\bar{w}) = \frac{\Delta_\alpha}{(z-w)^2}V_\alpha(w,\bar{w}) + \frac{1}{z-w} \frac{\partial V_\alpha}{\partial w}(w,\bar{w}) + \text{regular terms}, \tag{2.4}
\]

\[
\bar{T}(\bar{z})V_\alpha(w,\bar{w}) = \frac{\bar{\Delta}_\alpha}{(\bar{z}-\bar{w})^2}V_\alpha(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}} \frac{\partial V_\alpha}{\partial \bar{w}}(w,\bar{w}) + \text{regular terms}, \tag{2.5}
\]

where \(c = c(h)\) is the central charge of the theory, \(\Delta_\alpha = \bar{\Delta}_\alpha = \Delta_\alpha(h)\) are conformal dimensions of Liouville vertex operators, and “regular terms” denote expressions, which are regular as \(z \to w\). These OPE are understood as exact expansions of the correlation functions.

Using perturbation theory, we will prove the validity of (2.1)—(2.5) and calculate the value of \(c\), which turns out to be \(1 + 12/h\). We will also show that conformal dimensions of geometric vertex operators coincide with their classical charges. Actually, the latter result can be verified directly, using the identification between correlation function of vertex operators and expectation value. Namely, consider the thrice-punctured sphere with punctures at \(z_1, z_2\) and \(z_3 = \infty\). It is isomorphic (global conformal symmetry) to the normalized sphere with punctures \(0, 1, \infty\). This isomorphism is given by the fractional-linear transformation \(w = z_1 - z/z_1 - z_2\); using it is easy to see that

\[
\tilde{e}^{\tilde{\phi}(w)} = |z_1 - z_2|^2 e^{\phi(z)},
\]
and (cf. [21])

\[ S(\phi) = S(\tilde{\phi}) + 2\pi \log |z_1 - z_2|. \]

Performing the change of variables \( \phi \mapsto \tilde{\phi} \) in functional integral (1.10), we get

\[ <X>_{z_1,z_2,\infty} = \frac{<X>_{0,1,\infty}}{|z_1 - z_2|^{1/h}}, \quad \text{(2.6)} \]

which shows that scaling dimension \( d = \Delta + \bar{\Delta} \) of the puncture operator is \( 1/h \), so that \( \Delta = 1/2h \). Indeed, \( <X>_{z_1,z_2,\infty} \) should be considered as a three-point correlation function of the puncture operators \( V_\alpha(\phi)(z) \), inserted at points \( z_1, z_2 \) and \( \infty \). According to [16], the three-point correlation function has the form

\[ <V_\alpha(z_1)V_\alpha(z_2)V_\alpha(z_3)> = \frac{C}{|z_1 - z_2|^d|z_1 - z_3|^d|z_2 - z_3|^d}. \]

Since \( V_\alpha(\infty) = \lim_{z\to\infty} |z|^{-2d}V_\alpha(1/z) \), we get from (2.6) that \( d = 1/h \). (Note that if one interpretes \( <X>_{z_1,z_2,\infty} \) as a two-point correlation function, one gets a wrong value \( 1/2h \) for the scaling dimension of the puncture operator!).

It is also instructive to compare the Liouville theory with other solvable models of the conformal field theory, like minimal models of BPZ and WZW model. It is well-known that the latter are examples of the so-called rational conformal field theories, which have the property that “physical” correlation functions of primary fields are finite sums of the products of the holomorphic and anti-holomorphic conformal blocks. Quantum groups naturally enter into this picture, with \( R \)-matrices describing the monodromy of the conformal blocks. In the case of the quantum Liouville theory, one can also expect the chiral splitting of expectation value \( <X> \) into a “sum” (or rather an integral)

\[ <X> = \sum_{\beta} F_\beta(X)\bar{F}_\beta(X) \]

of the products of the holomorphic and anti-holomorphic conformal blocks. However, such decomposition is not yet known. Possible role of quantum groups in the quantum geometry, as well as the role of Poisson-Lie groups in the hyperbolic geometry, is yet another problem to be investigated.
2.2 Conformal Ward Identities

According to BPZ [16], conformal symmetry of the theory, expressed through OPE, can be equivalently formulated in terms of the infinite sequence of conformal Ward identities (CWI), that relate correlation functions involving stress-energy tensor components with correlation functions without them. Specifically, OPE (2.1) in the presence of punctures can be rewritten as

\[
<T(z)X> = \sum_{i=1}^{n-1} \left( \frac{\Delta}{(z-z_i)^2} + \frac{1}{z-z_i} \frac{\partial}{\partial z_i} \right) <X>,
\]

(2.7)

where \( \Delta \) stands for the conformal dimension of the puncture operator. Similarly, OPE (2.1) yields the following CWI

\[
<T(z)T(w)X> = \frac{c/2}{(z-w)^4} <X> + \{ \frac{2}{(z-w)^2} + \frac{1}{z-w} \frac{\partial}{\partial w} \}
+ \sum_{i=1}^{n-1} \left( \frac{\Delta}{(z-z_i)^2} + \frac{1}{z-z_i} \frac{\partial}{\partial z_i} \right) <T(w)X>.
\]

(2.8)

Correlation functions, involving (0, 2)-component \( \bar{T} \) of the stress-energy tensor, satisfy similar CWI (see, e.g., [22]).

Global PSL(2, C)-symmetry of the theory also imposes constraints on correlation functions [16]. Fixing \( z_n = \infty \) reduces this symmetry to the invariance under translations and dilations, and yields the following equations

\[
\sum_{i=1}^{n-1} \frac{\partial}{\partial z_i} <X> = 0, \quad \sum_{i=1}^{n-1} (z_i \frac{\partial}{\partial z_i} + \Delta) <X> = \Delta <X>,
\]

(2.9)

(cf. relations (1.18)), so that

\[
<T(z)X> = \frac{\Delta}{z^2} <X> + O(|z|^{-3}), \quad z \to \infty
\]

(2.10)

(cf. asymptotics (1.16)).

Using (2.9), one can fix the global conformal symmetry by setting \( z_{n-2} = 0, z_{n-1} = 1, z_n = \infty \) and obtain normalized form of CWI. Namely, denoting by \( \mathcal{L}(z) \) and \( \mathcal{L}(\bar{z}) \) the following first order differential operators

\[
\mathcal{L}(z) = \sum_{i=1}^{n-3} R(z, z_i) \frac{\partial}{\partial z_i}, \quad \mathcal{L}(\bar{z}) = \sum_{i=1}^{n-3} R(\bar{z}, \bar{z}_i) \frac{\partial}{\partial \bar{z}_i},
\]

(2.11)
where
\[ R(z, z_i) = \frac{1}{z - z_i} + \frac{z_i - 1}{z} - \frac{z_i}{z - 1} = \frac{z_i(z_i - 1)}{z(z - 1)(z - z_i)}, \quad (2.12) \]
we can rewrite CWI (2.4), and its complex conjugate, in the following succinct form
\[ \langle\langle T(z)X \rangle\rangle_0 = \langle\langle T(z)X \rangle\rangle - T_s(z) = L(z) \log <X>, \quad (2.13) \]
\[ \langle\langle \bar{T}(\bar{z})X \rangle\rangle_0 = \langle\langle \bar{T}(\bar{z})X \rangle\rangle - \bar{T}_s(\bar{z}) = \bar{L}(\bar{z}) \log <X>, \quad (2.14) \]
where
\[ T_s(z) = \sum_{i=1}^{n-1} \frac{\Delta}{(z - z_i)^2} + \frac{(2 - n)\Delta}{z(z - 1)}. \quad (2.15) \]

CWI (2.8), and its complex conjugate, can be written as follows
\[ \langle\langle T(z)T(w) \rangle\rangle = \frac{c/2}{(z - w)^4} \]
\[ + \{2R_w(z, w) + R(z, w)\frac{\partial}{\partial w} + L(z)\} \langle\langle T(w)X \rangle\rangle, \quad (2.16) \]
\[ \langle\langle \bar{T}(\bar{z})\bar{T}(\bar{w}) \rangle\rangle = \frac{c/2}{(\bar{z} - \bar{w})^4} \]
\[ + \{2R_{\bar{w}}(\bar{z}, \bar{w}) + R(\bar{z}, \bar{w})\frac{\partial}{\partial \bar{w}} + \bar{L}(\bar{z})\} \langle\langle \bar{T}(\bar{w})X \rangle\rangle. \quad (2.17) \]
Finally,
\[ \langle\langle T(z)\bar{T}(\bar{w})X \rangle\rangle = L(z) \langle\langle T(w)X \rangle\rangle = \bar{L}(\bar{z})L(\bar{w}) \log <X>. \quad (2.18) \]

According to BPZ [16], relations (2.13)–(2.18) state, at the level of correlation functions, that puncture operators are primary fields and \( T(z) \), \( \bar{T}(\bar{z}) \) are generating functions of the holomorphic and anti-holomorphic Virasoro algebras, that mutually commute in virtue of (2.18).

Similar formulas can be obtained for multi-point correlation functions. Namely, introducing the following first-order differential operators
\[ D(z, w) = R(z, w)\frac{\partial}{\partial w} + 2R_w(z, w), \quad \bar{D}(\bar{z}, \bar{w}) = R(\bar{z}, \bar{w})\frac{\partial}{\partial \bar{w}} + 2R_{\bar{w}}(\bar{z}, \bar{w}), \]
and using notation (cf. §1.2)
\[ \langle\langle 1, \ldots, k; 1, \ldots, \bar{l} \rangle\rangle = \langle\langle \prod_{i=1}^{k} T(z_i) \prod_{j=1}^{l} \bar{T}(\bar{w}_j)X \rangle\rangle, \]

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we get for $k + l > 2$

$$< < 1, \ldots, k; 1, \ldots, l > > = \left( \sum_{i=2}^{k} D(z_i, z_i) + \mathcal{L}(z_1) \right) < < 2, \ldots, k; 1, \ldots, l > > ,$$  \hspace{1cm} (2.19)

$$< < 1, \ldots, k; 1, \ldots, l > > = \left( \sum_{j=2}^{l} D(\bar{w}_j, \bar{w}_j) + \bar{\mathcal{L}}(\bar{w}_1) \right) < < 1, \ldots, k; 2, \ldots, l > > .$$  \hspace{1cm} (2.20)

These formulas, combined with (2.13)—(2.18), express multi-point correlations functions with stress-energy tensor components through expectation value $< X >$.

According to BPZ, constraints imposed by CWI (together with constraints from possible additional symmetries) allow to solve the theory completely. In most interesting examples (minimal models of BPZ, WZW model) this is indeed the case and complete solution can be obtained by representation theory (of the Virasoro algebra, Kac-Moody algebra). The simplest case of the minimal models of BPZ corresponds to the discrete series representations of the Virasoro algebra (see [16, 23] and [24] for a review).

In our geometrical approach, correlation functions are defined through a functional integral, and we need to affirm the validity of (2.13)—(2.18). This will be done in §4, thus providing dynamical proof of the conformal symmetry. Note that in our formulation, we do not use representation theory of the Virasoro algebra and calculate the central charge and conformal dimensions through CWI. In doing so, we tacitly assume that certain analog of the “reconstruction theorem” exists, so that one may indeed talk about Virasoro algebra representations involved. It is interesting to understand their realization, as well as the structure of corresponding conformal blocks. Contrary to the case of minimal models, where algebraic constructions based on the Verma modules have been used, in case of the two-dimensional gravity constructions should be geometrical. Since in our case $c > 1$, we should have “principal series” representations, as opposed to the discrete series for minimal models with $c < 1$.

Conformal Ward identities (2.13)—(2.18) (more precisely, OPE (2.1)—(2.5)) can be written as a single universal Ward identity for generating functional $\mathcal{W}$, and formally coincides with the Ward identity in the light-cone gauge, derived by Polyakov [32] (see also [17, 33]). Namely, we have identities

$$\left( \frac{\partial}{\partial \bar{z}} + \pi \mu \frac{\partial}{\partial z} + 2\pi \mu \frac{\partial \mathcal{W}}{\partial \mu(z)} \right)(\mu, \bar{\mu}; X)$$
\[
\pi h c \mu_{zzz} + \frac{\partial}{\partial \bar{z}}(hT_s(z) + L(z))\left\{W(\mu, \bar{\mu}; X) + h \log <X>\right\}, \tag{2.21}
\]
and
\[
\left(\frac{\partial}{\partial \bar{z}} + \pi \mu \frac{\partial}{\partial \bar{z}} + 2\pi \bar{\mu} \bar{z}\right)\frac{\delta W}{\delta \mu(z)}(\mu, \bar{\mu}; X)
\]
\[
= -\frac{\pi h c}{12} \mu_{zzz} + \frac{\partial}{\partial \bar{z}}(h\bar{T}_s(\bar{z}) + \bar{L}(\bar{z}))\left\{W(\mu, \bar{\mu}; X) + h \log <X>\right\}, \tag{2.22}
\]
which should be understood at the level of generating functions. Using formula
\[
\frac{\partial}{\partial \bar{z}} R(z, w) = \pi \delta(z - w),
\]
where \(z, w \neq 0, 1\), one easily gets (2.13)–(2.18) from (2.21)–(2.22).

### 2.3 Friedan-Shenker Modular Geometry

Denote by \(\mathcal{M}_{0,n}\) the moduli space of Riemann surfaces of genus 0 with \(n > 3\) punctures. It can be obtained as a quotient of the space of punctures \(Z_n = \{(z_1, \ldots, z_{n-3}) \in \mathbb{C}^{n-3} | z_i \neq 0, 1 \text{ and } z_i \neq z_j \text{ for } i \neq j\}\) by the action of a symmetric group of \(n\) elements
\[
\mathcal{M}_{0,n} \simeq Z_n/S_n.
\]

Here symmetric group \(S_n\) acts on \(Z_n\) as a permutation of the \(n\)-tuple \((z_1, \ldots, z_{n-3}, 0, 1, \infty)\), followed by the component-wise action of the PSL(2, \(\mathbb{C}\)), normalizing (if necessary) the last three components back to \(0, 1, \infty\). As will be explained in §4.3, moduli space \(\mathcal{M}_{0,n}\) is a complex orbifold of complex dimension \(n - 3\) and admits a natural Kähler structure. Denote by \(d\) the exterior differential on the spaces \(\mathcal{M}_{0,n}\) and \(Z_n\); it has a standard decomposition \(d = \partial + \bar{\partial}\), where
\[
\partial = \sum_{i=1}^{n-3} \frac{\partial}{\partial z_i} dz_i, \quad \bar{\partial} = \sum_{i=1}^{n-3} \frac{\partial}{\partial \bar{z}_i} d\bar{z}_i.
\]

As we shall see in §4.3, vectors \(\partial/\partial z_i\), which form a basis of the (holomorphic) tangent space to \(Z_n\) at point \((z_1, \ldots, z_{n-3})\), corresponding to Riemann surface \(X\), can be represented by harmonic Beltrami differentials on \(X\). Corresponding (holomorphic) cotangent space at point \(X\) can be identified with a linear space of harmonic quadratic differentials on \(X\).
Remarkably, the dual basis to $\partial/\partial z_i$, which consists of $(1,0)$-forms $dz_i$, admits an explicit description on Riemann surface $X$. Namely, introducing

$$P_i(z) = -\frac{1}{\pi} R(z, z_i), \quad i = 1, \ldots, n - 3,$$

we can identify $(1,0)$-forms $dz_i$ on $Z_n$ with quadratic differentials $P_i(z)dz^2$ on $X$, so that under this identification (see [3] and §4.3)

$$\partial = -\frac{1}{\pi} \mathcal{L}(z)dz^2, \quad \bar{\partial} = -\frac{1}{\pi} \bar{\mathcal{L}}(\bar{z})d\bar{z}^2. \quad (2.24)$$

In [17], Friedan-Shenker envisioned a general “philosophy” of the modular geometry, which describes conformal theories in two dimensions in terms of a complex geometry of projective bundles (possibly infinite-dimensional) over moduli spaces of Riemann surfaces. In particular, according to the ideology in [17], expectation value $<X>$ should be interpreted as a Hermitian metric in a certain (holomorphic projective) line bundle over $\mathcal{M}_{0,n}$, and quadratic differential $<< T(z)X >>_0 dz^2$—as a $(1,0)$-component of a canonical metric connection. Using correspondence (2.24), we can read (2.13) as

$$<< T(z)X >>_0 dz^2 = -\frac{1}{\pi} \partial \log <X>, \quad (2.25)$$

in perfect agreement with [17, formula (6)]!

Similarly, the Ward identity (2.18) can be rewritten as

$$<< T(z)\bar{T}(\bar{w})X >> dz^2 d\bar{w}^2 = \frac{1}{\pi^2} \partial \bar{\partial} \log <X>, \quad (2.26)$$

which allows to interpret $<< T(z)\bar{T}(\bar{w})X >>$ as a curvature form of the canonical metric connection (cf. [17, formula (15)]). As we shall see in §§4.1—4.2, these formulas encode remarkable relations between quantum Liouville theory and the Kähler geometry of moduli space $\mathcal{M}_{0,n}$, and provide modular geometry of Friedan-Shenker with a meaningful example.

However, so far our arguments were rather formal, since we did not define rigorously our main objects: the expectation value and correlation functions. This will be done in §§3.1—3.2.

**Problems**

1. (Research problem) Describe the chiral splitting of the correlation functions of puncture operators, conformal blocks, their monodromy, etc. What role do the quantum groups play in this approach?
2 Prove CWI (2.13)—(2.18) for normalized connected correlation functions and (2.19)–(2.20) for multi-point correlation functions.

3 Derive universal Wards identities (2.21)–(2.22) from OPE (2.1)–(2.5).

4 Show that differential operator $\partial_{z}^{3} + \mu \partial_{z} + 2 \mu z$, where $\mu$ is a Beltrami differential and which appears in (2.21)–(2.22), maps projective connections into $(2, 1)$-tensors on $X$.

3 Expectation Value and Correlation Functions

3.1 Expectation Value $<X>$

We define expectation value $<X>$ using the perturbation expansion of functional integral (1.10) around classical solution $\phi = \phi_{cl}$. This expansion will be understood in the sense of formal Laurent series in $\hbar$, thus defining $\log <X>$ as following

$$\log <X> = N - \frac{1}{2\pi \hbar} S_{cl} + \sum_{l=0}^{\infty} \chi_{l} \hbar^{l}.$$ 

Here $N$ is an overall infinite constant, that does not depend on $z_i$ and drops out from all normalized correlation functions, and $S_{cl} = S(\phi_{cl})$ is a classical Liouville action, i.e. the critical value of the action functional. Coefficients $\chi_l$—"higher loop contributions"—are given by the following procedure.

Start with the expansion of the Liouville action around classical solution

$$S(\phi_{cl} + \delta \phi) - S(\phi_{cl}) = \int_{X} \delta \phi (L_0 + 1/2)(\delta \phi) d\rho + \frac{1}{6} \int_{X} (\delta \phi)^{3} d\rho + \cdots ,$$

where $d\rho = e^{\phi_{cl}} d^{2}z$ is a volume form of the Poincaré metric, $\delta \phi$—a variation of $\phi_{cl}$—is a smooth function on $X$, and

$$L_0 = -e^{-\phi_{cl}} \frac{\partial^{2}}{\partial z \partial \bar{z}},$$

is a hyperbolic Laplacian acting on functions, i.e. is Laplace-Beltrami operator of the Poincaré metric on $X$. It is positively definite self-adjoint operator in Hilbert space $H_0(X)$ of square integrable functions on $X$ with respect to the volume form $d\rho$.

Second, "mimic" the saddle point method expansion in the finite-dimensional case (we assume that the reader is familiar with it; see, e.g., [25] for detailed exposition), replacing partial derivatives by variational derivatives and matrices by Schwartz kernels (in the sense
of distributions) of the corresponding operators. There are two fundamental problems which one must resolve along this way.

(a) One needs to define the determinant of differential operator $L_0 + 1/2$. If it was an elliptic operator on a compact manifold, this could be done in a standard fashion by using a heat kernel technique and zeta function of the elliptic operator, or in physical terms, by using the proper time regularization (see, e.g., [26]). In our case Riemann surface $X$ is not compact, operator $L_0$ has a $n$-fold continuous spectrum and a point spectrum (see, e.g., [27]), so that the heat kernel approach is not immediately applicable; additional regularization for the continuous spectrum is needed. However, the determinants of hyperbolic Laplacians can be also defined by means of the Selberg zeta function; for compact Riemann surfaces this definition is equivalent to the standard heat-kernel definition [28, 29]. Thus, following [30], we set

$$\det(2L_0 + 1) \doteq Z_X(2),$$

where $Z_X(s)$ (see, e.g., [27]) is the Selberg zeta function of Riemann surface $X$, defined by

$$Z_X(s) \doteq \prod_{m=0}^{\infty} \prod_{l} (1 - e^{-(s+m)|l|}).$$

Here $l$ runs over all simple closed geodesics on $X$ with respect to the Poincaré metric with $|l|$ being the length of $l$; the infinite product converges absolutely for Re $s > 1$.

(b) Recall the fundamental role played by the inverse matrix of the Hessian of the classical action at the isolated critical point in the standard finite-dimensional formulation of the steepest descent method [25]. In our infinite-dimensional case the analog of the Hessian is operator $2L_0 + 1$. Its inverse $(2L_0 + 1)^{-1}$ is an integral operator, whose kernel—a propagator of the theory—is given by Green’s function $G(z, z')$, which satisfies on $X$ the following PDE

$$-2G_{zz}(z, z') + e^{\phi_{cl}(z)}G(z, z') = \delta(z - z'). \quad (3.2)$$

The Green’s function can be uniquely characterized by the following properties.

1. It is a smooth function on $X \times X \setminus D$, where $D$ is diagonal $z' = z$ in $X \times X$.

2. It is symmetric: $G(z, z') = G(z', z)$, $z, z' \in X$.

3. For fixed $z' \in X$ it satisfies on $X \setminus \{z'\}$ the following PDE

$$(-2\partial^2_{zz} + e^{\phi_{cl}(z)})G(z, z') = 0.$$
For any \( z \in X \) function

\[
G(z, z') + \frac{1}{2\pi} \log |z - z'|^2
\]
is continuous as \( z' \to z \).

For fixed \( z' \in X \) \( G(z, z') \to 0 \) as \( z \to z_i, \ i = 1, \cdots, n \).

Now, the second problem one encounters in the functional case, is the problem of short-distance divergences: Green's function \( G \) and its partial derivatives blow-up at coincident points. These divergences can be regularized in a reparametrization invariant fashion by using the following asymptotic expansion, which goes back to Hadamard [31]

\[
G(z, z') = -\frac{1}{2\pi} \left\{ \log |z - z'|^2 (1 + \frac{1}{2} e^{(\phi_{cl}(z) + \phi_{cl}(z'))/2} |z - z'|^2 + \cdots) \right\}
- \frac{1}{4\pi} (\phi_{cl}(z) + \phi_{cl}(z')) + \cdots,
\]
as \( z' \to z \), where dots indicate higher order terms, which can be evaluated explicitly. Expansion (3.3) admits repeated differentiation with respect to \( z \) and \( z' \). In particular, it enables to regularize the logarithmic divergency of the Green's function on the diagonal in a reparametrization invariant way

\[
G(z, z) = \lim_{z' \to z} (G(z, z') + \frac{1}{2\pi} (\log |z - z'|^2 + \phi_{cl}(z))) \text{.}
\]

This is a outline of the reparametrization invariant regularization scheme for defining expectation value \( <X> \).

It is instructive to present an equivalent description, which is based on the Fuchsian uniformization of Riemann surface \( X \).

First, recall the uniformization theorem (see, e.g. [4]), which states that Riemann surface \( X \) with \( \chi(X) < 0 \) can be represented as a quotient of hyperbolic plane \( H \)—upper half-plane \( H = \{ \zeta \in \mathbb{C} \mid \text{Im} \zeta > 0 \} \)—by the fractional-linear action of a torsion-free finitely generated Fuchsian group \( \Gamma \)

\[
X \simeq H/\Gamma \text{.}
\]

In other words, there exists a holomorphic covering \( J : H \mapsto X \), with \( \Gamma \)—a discrete subgroup of \( \text{PSL}(2, \mathbb{R}) \)—acting as a group of automorphisms (as an abstract group, \( \Gamma \) is isomorphic
to a fundamental group of $X$). In terms of covering map $J$, classical solution $\phi_{cl}$ has the following explicit form

$$e^{\phi_{cl}(z,\bar{z})} = \frac{|(J^{-1})'(z)|^2}{(\text{Im } J^{-1}(z))^2},$$

(3.5)

stating that Poincaré metric $e^{\phi_{cl}(z,\bar{z})}|dz|^2$ is a projection on $X$ of hyperbolic metric $(\text{Im } \zeta)^{-2}|d\zeta|^2$ on $H$. This projection is well-defined, since the hyperbolic metric on $H$ is $\text{PSL}(2, \mathbb{R})$-invariant.

Second, hyperbolic Laplacian $L_0$, lifted to $H$, has the form

$$L_0 = -\text{(Im } \zeta)^2 \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}},$$

and Hilbert space $H_0(X)$ is isomorphic to the space of $\Gamma$-automorphic functions,

$$f(\gamma \zeta) = f(\zeta), \ \gamma \in \Gamma,$$

which are square integrable on $H/\Gamma$ with respect to the volume form of the hyperbolic metric on $H$. Denote by $G_{\Gamma}(\zeta, \zeta')$ the Green’s function of operator $2L_0 + 1$ on $H/\Gamma$ and by $G(\zeta, \zeta')$—the Green’s function of $2L_0 + 1$ on the upper half-plane $H$. It is well-known (see, e.g., [34]), that the Green’s function $G$ admits explicit representation

$$G(\zeta, \zeta') = \frac{1}{2\pi} \int_0^1 \frac{t(1-t)}{(t+u)^2} dt = \frac{1}{2\pi}(2u+1) \log\left(\frac{2u+1}{u}\right) - \frac{1}{\pi},$$

(3.6)

where

$$u = u(\zeta, \zeta') = \frac{|\zeta - \zeta'|^2}{4 \text{Im } \zeta \text{ Im } \zeta'},$$

(3.7)

whereas Green’s function $G_{\Gamma}$ can be obtained by the method of images

$$G_{\Gamma}(\zeta, \zeta') = \sum_{\gamma \in \Gamma} G(\zeta, \gamma \zeta').$$

(3.8)

Series in (3.8) converges uniformly and absolutely on compact subsets of $(H/\Gamma \times H/\Gamma) \setminus D$. It follows from (3.8) that the Green’s function $G_{\Gamma}$ is $\Gamma$-automorphic with respect to both variables

$$G_{\Gamma}(\gamma \zeta, \gamma \zeta') = G_{\Gamma}(\zeta, \zeta') = G_{\Gamma}(\zeta, \zeta'), \ \zeta \neq \zeta' \in H, \ \gamma \in \Gamma.$$  

(3.9)

The Green’s functions on $X$ and on $H/\Gamma$ are related by the following simple formula

$$G(z, z') = G_{\Gamma}(J^{-1}(z), J^{-1}(z')), \ z \neq z' \in X,$$

(3.10)
where according to (3.9), the right hand side does not depend on the choice of a branch of multi-valued function \( J^{-1} : X \to H \). Using representation (3.10), properties of map \( J \) (see, e.g., [4]), and formula (3.3), it is easy to see that regularization scheme described above is essentially equivalent to the subtraction of contribution \( G(\zeta, \zeta') \) of the unit element \( I \in \Gamma \) from series (3.8) as \( \zeta' \to \zeta \). Thus, for instance,

\[
G^\Gamma(\zeta, \zeta) = \sum_{\gamma \in \Gamma, \gamma \neq I} G(\zeta, \gamma \zeta),
\]

(3.11)

and

\[
\partial^2_{\zeta \zeta'} G^\Gamma|_{\zeta' = \zeta} = \sum_{\gamma \in \Gamma, \gamma \neq I} \partial^2_{\zeta \zeta'} G(\zeta, \zeta).
\]

(3.12)

This prescription exhibits “universal” nature of the regularization scheme, which takes full advantage of the knowledge of the underlying hyperbolic geometry of Riemann surfaces—geometry of classical “space-time”.

To summarize, we have presented the set of rules for the perturbative definition of expectation value \( \langle X \rangle \). In particular, at the tree level,

\[
\log \langle X \rangle_{\text{tree}} = -\frac{1}{2\pi \hbar} S_{cl},
\]

(3.13)

and one-loop contribution is given by

\[
A = \log \langle X \rangle_{\text{loop}} = -\frac{1}{2} \log \det(2L_0 + 1) = -\frac{1}{2} \log Z_X(2).
\]

(3.14)

Higher multi-loop contributions can be also written down explicitly. However, in their definition additional regularization of integrals

\[
\int_X G^k(z, z) d\rho
\]

at the punctures is required. This regularization is similar to the one used in the derivation of the Selberg trace formula (see, e.g., [27]).

### 3.2 Generating Functional \( Z(\mu, \bar{\mu}; X) \)

Here we derive perturbation expansion for the generating functional for correlation functions involving stress-energy tensor components. To illustrate our approach, we consider first the example of free bosons on \( \mathcal{C} \), where one can perform all calculations explicitly.
Namely, let \( \psi \) be a real-valued scalar field on \( \mathbb{C} \) with the classical action given by Dirichlet functional
\[
S_{\text{free}}(\psi) = \int |\psi_z|^2 d^2z,
\]
which defines a conformal theory of free bosons with stress-energy tensor
\[
T_{\text{free}}(\psi) = -\psi_z^2/2\hbar, \quad \bar{T}_{\text{free}}(\psi) = -\psi_{\bar{z}}^2/2\hbar.
\]
Generating functional for the correlation functions of these stress-energy tensor components is given by the following functional integral
\[
Z_{\text{free}}(J, \bar{J}) \equiv \int D\psi \exp\left\{ -\frac{1}{2\pi\hbar}S_{\text{free}}(\psi) + \int (T_{\text{free}}(\psi)J + \bar{T}_{\text{free}}(\psi)\bar{J}) d^2z \right\}, \tag{3.15}
\]
where \( J \) is an external source. Functional integral (3.15) is Gaussian and its integrand is an exponential of a quadratic form of the following differential operator
\[
-\frac{1}{2\pi\hbar}(L_{\text{free}} - \pi\partial_z J \partial_z - \pi\partial_{\bar{z}} \bar{J} \partial_{\bar{z}}),
\]
where \( L_{\text{free}} = -\partial_z^2 \) is a Laplacian of Euclidean metric \( |dz|^2 \) on \( \mathbb{C} \). Denoting \( G_{\text{free}} = (L_{\text{free}})^{-1} \), we get as a result of Gaussian integration
\[
Z_{\text{free}}(J, \bar{J}) \equiv Z_{\text{free}}(J, \bar{J})/Z_{\text{free}}(J, \bar{J})|_{J=\bar{J}=0} = \det(1 - \pi G_{\text{free}}(\partial J \partial + \bar{\partial} \bar{J} \bar{\partial}))^{-1/2}. \tag{3.16}
\]
Therefore
\[
\frac{1}{\hbar}W_{\text{free}}(J, \bar{J}) = \frac{1}{2} \log \det(1 - \pi G_{\text{free}}(\partial J \partial + \bar{\partial} \bar{J} \bar{\partial})) = \sum_{k=1}^{\infty} \frac{\pi^k}{2k} \text{tr}\{G_{\text{free}}(\partial J \partial + \bar{\partial} \bar{J} \bar{\partial})\}^k, \tag{3.17}
\]
where the sum contains only the terms with even \( k \). In particular, one gets from (3.17) that
\[
<T_{\text{free}}(z)T_{\text{free}}(w)> = \frac{1}{\hbar} \frac{\delta^2 W_{\text{free}}}{\delta J(z) \delta J(w)}|_{J=\bar{J}=0} = \frac{1/2}{(z-w)^4},
\]
so that \( c = 1 \), as it should be for the theory of free bosons.

Next, consider the case of the Liouville theory. We will describe the perturbation expansion of generating functional (1.25).
First, write $\phi = \phi_{cl} + \chi$ and expand the Liouville action and the stress-energy tensor around classical solution $\phi_{cl}$. Using (3.1) with $\delta \phi$ replaced by $\chi$, we get

$$S(\phi_{cl} + \chi) = S_{cl} + \frac{1}{2} \int_X \chi (2L_0 + 1) \rho + S_{int}(\chi), \quad (3.18)$$

where

$$S_{int}(\chi) = \sum_{k=3}^{\infty} \frac{1}{k!} \int_X \chi^k d\rho, \quad (3.19)$$

and

$$T(\phi_{cl} + \chi) = T_{cl} + \frac{1}{h} (\chi_{zz} - \chi_z(\phi_{cl})_z - \frac{1}{2} \chi_z^2). \quad (3.20)$$

Second, using Stokes’ formula, we obtain

$$\int_X T(\phi) \mu d^2z = \int_X T_{cl} \mu d^2z + \frac{1}{h} \int_X (\frac{1}{2} \chi_z^2 + \chi \omega) d^2z,$$

where

$$\omega = \mu_{zz} + (\phi_{cl})_z \mu_z + (\phi_{cl})_{zz} \mu, \quad (3.21)$$

as it follows from from transformation laws (1.19) and (1.23), is well-defined $(1,1)$-form on $X$.

Third, using the variational analog of identity

$$F(x)e^{\lambda x} = F(\frac{d}{d\lambda})e^{\lambda x},$$

and expansions (3.18)–(3.20), generating functional (1.25) can be represented as classical term

$$\exp\{\text{v.p.} \int_X (T_{cl} \mu + \bar{T}_{cl} \bar{\mu}) d^2z\}$$

times the result of an application of the following “pseudo-variational” operator

$$S_{int}(h \frac{\delta}{\delta \xi}) \approx \exp\{-\frac{1}{2\pi h} S_{int}(h \frac{\delta}{\delta \xi})\}$$

to Gaussian functional integral

$$\int_{\mathcal{C}(\mathcal{X})} \mathcal{D}X \exp\{-\frac{1}{2\pi h} \int_X \{\chi (L_0 + 1/2)(\chi) + \pi(\chi_z^2 \mu + \chi_z^2 \bar{\mu} - \xi \chi)\} d\rho\},$$

where

$$\xi = f + \bar{f}, \quad f = e^{-\phi_{cl}} \omega, \quad (3.22)$$
and $f$ is a globally defined function on $X$. Gaussian integration with respect to $\chi$ can be performed explicitly. The final result reads

$$Z(\mu, \bar{\mu}; X) = \exp\{\text{v.p.} \int_X (T_d \mu + \bar{T}_d \bar{\mu}) d^2 z \} \det\left\{ \frac{G}{(1 - 2\pi Ge^{-\phi_{cl}(\partial \mu \bar{\partial} + \partial \bar{\mu} \bar{\partial}))} \right\}^{1/2}$$

where $G = (2L_0 + 1)^{-1}$. It is understood that one should first apply $S_{\text{int}}(h \delta / \delta \xi)$ with subsequent substitution $\xi = e^{-\phi_{cl}(\omega + \bar{\omega})}$.

Note that determinant contribution to this formula looks similar to free theory case (3.16) after replacement $\phi_{cl} \rightarrow 0$ and $G \rightarrow G_{\text{free}}$. Specific features of the Liouville theory are reflected in the second line of (3.23).

From (3.23) it is straightforward to get perturbation expansion for generating functional $\mathcal{W}$. As in the free case, one should apply standard rule

$$\log \det(1 - A) = \sum_{k=1}^{\infty} \frac{1}{k} \text{tr} A^k,$$

expand the exponentials in (3.23) and use the regularization scheme, indicated in §3.1.

### 3.3 Examples

Here we illustrate our formalism by considering perturbation expansion of one- and two-point correlation functions.

(i) **One-point correlation function $\langle\langle T(z) X \rangle\rangle$**

According to definition (1.27),

$$\langle\langle T(z) X \rangle\rangle = \frac{1}{h} \delta \mathcal{W} |_{\mu=0}, \quad \text{(3.24)}$$

and it is clear that

$$\langle\langle T(z) X \rangle\rangle_{\text{tree}} = T_{cl}(z). \quad \text{(3.25)}$$

In order to obtain the expression for $\langle\langle T(z) X \rangle\rangle_{\text{loop}}$, we need to collect all terms of order $h^0$ in formula (3.24). One can get these terms in two different ways.

First, consider such terms as coming from the “universal” expression log det. It always has order $h^0$ and contributes to the one-loop expansion of $\langle\langle T(z) X \rangle\rangle$ through

$$\frac{\delta \text{tr}(G \partial \mu \bar{\partial})}{\delta \mu(z)} |_{\mu=0} = G_{zz'} |_{z'=z}.$$
Second, terms of order $h^0$ come from the interaction. Namely, expand $\log S_{\text{int}}(h\delta/\delta\xi)$ into power series in $h$ and apply its first term (which is of order $h^2$) to the first term of the corresponding expansion of

$$\exp\left\{\frac{\pi}{h} \int_X \xi(G(1 - 2\pi Ge^{-\phi_{cl}}(\partial\mu\partial + \bar{\partial}\bar{\mu}\bar{\partial}))^{-1}(\xi)d\rho}\right\},$$

keeping only terms quadratic in $\xi$; according to (3.21), they also are quadratic in $\mu$. The result is proportional to

$$\frac{\delta}{\delta\mu(z)} \{ \int_X G(z', z')G(\xi)(z')d\rho' \} = \int_X G(z', z')\mathcal{D}_z(G)(z, z')d\rho'.$$

(3.26)

Here, $\mathcal{D}_z \doteq \partial^2_{zz} - (\phi_{cl})_z\partial_z$, and we have used (3.21)–(3.22) and the Stokes’ formula.

Thus we obtain

$$<< T(z)X >>_\text{loop} = -\pi G_{zz}|_{z' = z} - v.p.\pi \int_X G(z', z')\mathcal{D}_z(G)(z, z')d\rho'.$$

(3.27)

Here $G(z, z)$ is given by (3.4) and in accordance with (3.3), the regularized value of $G_{zz'}$ at diagonal $z' = z$ is defined as

$$G_{zz'}|_{z' = z} = \lim_{z' \to z} (G_{zz'}(z, z') + \frac{1}{2\pi} e^{\phi_{cl}(z)}\frac{z - z'}{z - z'}).$$

(3.28)

Note that the integral in (3.27) is a principal value integral, since $\mathcal{D}_z(G)(z, z')$ has a second order pole at $z' = z$.

(ii) Two-point correlation functions $<< T(z)T(w)X >>$ and $<< T(z)\bar{T}(\bar{w})X >>$

According to definition (1.27),

$$<< T(z)T(w)X >> = \frac{1}{h} \frac{\delta^2\mathcal{W}}{\delta\mu(z)\delta\mu(w)}|_{\mu = 0}.$$  

(3.29)

At the tree level, this correlation function has order $h^{-1}$ and the only term of this order comes from integral (3.26) and is quadratic in $\xi$. We get the following expression

$$<< T(z)T(w)X >>_{\text{tree}} = \frac{2\pi}{h} \mathcal{D}_z\mathcal{D}_w G(z, w).$$

(3.30)

Similarly,

$$<< T(z)\bar{T}(\bar{w})X >>_{\text{tree}} = \frac{2\pi}{h} \mathcal{D}_z\mathcal{D}_w G(z, w).$$

(3.31)
One-loop contributions to $<< T(z)T(w)X >>>$ and $<< T(z)\bar{T}(\bar{w})X >>>$ are more complicated and consist of several terms. In particular, $<< T(z)T(w)X >>>_{\text{loop}}$ contains term

$$2\pi^2 G_{zw}(z, w),$$

which comes from expansion of the universal expression $\log \det$. As we shall see in §4.2, it is this term that contributes to the quantum correction to the central charge.

Problems

1. Calculate two-loop contribution $X_1$ to $\log <X>$.

2. Show the equivalence of regularization schemes on $X$ and on $H/\Gamma$. In particular, prove that

$$G(J(\zeta), J(\zeta)) = G_\Gamma|_{\zeta' = \zeta} - \frac{1 - \log 2}{\pi}$$

and

$$G_{zz'}|_{z' = z = J(\zeta)} J'(\zeta)^2 = \partial^2_{\zeta}\zeta G_\Gamma|_{\zeta' = \zeta} + \frac{1}{12\pi} S(J)(\zeta).$$

3. Complete all details of the calculations of generating functional $Z_{\text{free}}(J, \bar{J})$.

4. Complete details for the Liouville case; in particular, verify formulas (3.27)—(3.31).

5. Compute two-loop contribution to correlation function $<< T(z)X >>>$ and one-loop contribution to $<< T(z)T(w)X >>>$ and $<< T(z)\bar{T}(\bar{w})X >>>$.

4 Ward Identities and Modular Geometry

4.1 Semi-classical Approximation

(i) One-point correlation function $<< T(z)X >>>$

Consider Ward identity (2.13). At the tree level it reads

$$T_{cl}(z) - T_s(z) = -\frac{1}{2\pi \hbar} \mathcal{L}(z) S_{cl}. \tag{4.1}$$

From (2.11)—(2.12) it follows that the right hand side of (4.4) has only simple poles at $z = z_i$. According to (1.17) and (2.13), it is equivalent to the statement that classical dimension of the puncture operator is $1/2\hbar$ (cf. discussion in §1.1 and §2.1). Using constraints (1.18), we then have

$$T_{cl}(z) - T_s(z) = \frac{1}{\hbar} \sum_{i=1}^{n-3} c_i R(z, z_i),$$

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so that (4.1) is equivalent to the following relations

\[ c_i = -\frac{1}{2\pi} \partial_i S_{cl}, \quad i = 1, \ldots, n - 3. \] (4.2)

In other words, coefficients \( c_i \) are “conjugate” to punctures \( z_i \), and classical action \( S_{cl} \) plays a role of generating function.

What is the meaning of coefficients \( c_i \)? Actually, they have been around for more than hundred years, were called “accessory parameters” by Poincaré and are closely related with the Fuchsian uniformization of Riemann surface \( X \). Namely, as it follows from (3.5),

\[ T_{cl}(z) = \frac{1}{\hbar} S(J^{-1})(z). \] (4.3)

Next, according to Klein and Poincaré (see, e.g. [5] for more details), the Schwarzian derivative of the inverse function to uniformization map \( J \) has the property that the monodromy group of associated second order linear differential equation

\[ \frac{d^2 y}{dz^2} + \frac{1}{2} S(J^{-1})(z)y = 0, \quad z \in X, \]

coincides (up to a conjugation) with Fuchsian group \( \Gamma \), uniformizing Riemann surface \( X \)! Therefore, in the representation

\[ S(J^{-1})(z) = \sum_{i=1}^{n-1} \left( \frac{1}{2(z - z_i)^2} + \frac{c_i}{z - z_i} \right), \] (4.4)

which follows from (1.17) and (1.3), coefficients \( c_i \) are uniquely determined by this global condition. It turns out to be an extremely difficult mathematical problem to characterize accessory parameters as functions of the punctures, i.e. as functions on \( \mathbb{Z}_n \). On the other hand, CWI (4.1) implies that coefficients \( c_i \) possess a remarkable property (4.2)! These relations, conjectured by Polyakov [6] on the basis of CWI, were rigorously proved in [3]. The proof is based on fundamental facts from the Teichmüller theory, which we present in §4.3.

\textbf{(ii) Two-point correlation functions} \( \ll T(z)T(w)X \gg \) and \( \ll T(z)\bar{T}(w)X \gg \)

Start with the CWI (2.18). Using (3.31), at the tree level it can be written as

\[ 4D_z D_w G(z, w) = -\frac{1}{\pi^2} \mathcal{L}(z)\bar{\mathcal{L}}(w)S_{cl}. \] (4.5)
Denoting $P(z, w) = 4D_z D_w G(u, v)$, we can rewrite (4.5) as the following relation on $Z_n$

$$P(z, w) dz^2 d\bar{w}^2 = -\partial \bar{\partial} S_{cl}. \hspace{1cm} (4.6)$$

It turns out that kernel $P(z, w)$ can be described explicitly: it is finite-dimensional and is related with the Weil-Petersson Hermitian metric on space $Z_n$. Namely, let $H_2(X)$ be the Hilbert space of quadratic differentials on $X$—tensors of type $(2, 0)$, which are square integrable with respect to the measure $e^{-\phi_{cl}} d^2 z$ on $X$. If $P, Q \in H_2(X)$, then

$$(P, Q) = \int_X P(z) \overline{Q(z)} e^{-\phi_{cl}(z)} d^2 z < \infty. \hspace{1cm} (4.7)$$

Denote by $H^{2,0}(X)$ the space of harmonic quadratic differentials on $X$, i.e. the subspace in $H_2(X)$, consisting of holomorphic quadratic differentials—zero modes of operator $\bar{\partial}$ in the space $H_2(X)$, that is a finite-dimensional vector space of complex dimension $n - 3$. It is easy to show that $P(z, w)$ is the kernel of orthogonal Hodge projection operator

$$P : H_2(X) \mapsto H^{2,0}(X).$$

Indeed, using formulas (3.5)—(3.8), one gets

$$P(z, w) = \mathcal{P}_\Gamma(J^{-1}(z), J^{-1}(w))(J^{-1})'(z)^2(\bar{J}^{-1})'(\bar{w})^2, \hspace{1cm} (4.8)$$

where

$$\mathcal{P}_\Gamma(\zeta, \zeta') = \frac{12}{\pi} \sum_{\gamma \in \Gamma} \frac{\gamma'(\zeta')^2}{(\zeta - \gamma \zeta')^4}. \hspace{1cm} (4.9)$$

Next, recall that Hilbert space $H_2(X)$, lifted to upper half-plane $H$ by map $Q \mapsto q = Q \circ J(J')^2$, is isomorphic to space $H_2(H/\Gamma)$, consisting of automorphic forms of weight 4 with respect to Fuchsian group $\Gamma$,

$$q(\gamma \zeta) \gamma'(\zeta)^2 = q(\zeta), \hspace{0.5cm} \gamma \in \Gamma,$$

which are square integrable on $H/\Gamma$ with respect to measure $(\text{Im} \zeta)^2 d^2 \zeta$. Therefore, for any $q \in H_2(H/\Gamma)$,

$$\mathcal{P}_\Gamma(q) = \int_{H/\Gamma} \mathcal{P}_\Gamma(\zeta, \zeta') q(\zeta') (\text{Im} \zeta')^2 d^2 \zeta' = \frac{12}{\pi} \int_H \frac{(\text{Im} \zeta')^2}{(\zeta - \zeta')^4} q(\zeta') d^2 \zeta'.$$
According to the Ahlfors lemma \[35, \text{Ch. VI.D, Lemma 2}\], the result of integration is either $q$, if $q \in \mathcal{H}^{2,0}(H/\Gamma)$, or zero, if $q$ is orthogonal to $\mathcal{H}^{2,0}(H/\Gamma)$.

In terms of arbitrary basis $P_i$ in $\mathcal{H}^{2,0}(X)$, the kernel $P(z, w)$ can be written as

$$P(z, w) = \sum_{i,j=1}^{n-3} g^{ij} P_i(z) P_j(w), \quad (4.10)$$

where matrix $\{g^{ij}\}$ is inverse to the Gram matrix of the basis $P_i$ with respect to inner product $(4.7)$, i.e. the matrix with elements $g_{ij} = (P_i, P_j), \ i, j = 1, \ldots, n - 3$. In particular, if $P_i$ is a basic, dual to the basic of harmonic Beltrami differentials corresponding to vector fields $\partial/\partial z_i$, then

$$g^{ij} = \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right)_{WP}, \quad (4.11)$$

where $(\ , \ )_{WP}$ stands for the Weil-Petersson Hermitian metric on $Z_n$ (see §4.3). Combining formulas $(4.10)$, $(4.11)$ and $(4.11)$, we finally obtain

$$(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j})_{WP} = -\frac{\partial^2 S_{cl}}{\partial z_i \partial z_j}, \quad (4.12)$$

Relation $(4.12)$, which is equivalent to CWI $(2.18)$ at the tree level, states that the Weil-Petersson metric on space $Z_n$ is Kähler, with a potential given by classical action $S_{cl}$! This potential is a smooth single-valued function on $Z_n$, but not on moduli space $\mathcal{M}_{0,n}$, where it is a Hermitian metric in a certain holomorphic line bundle over $\mathcal{M}_{0,n}$ (see [21]). The Kähler property of the Weil-Petersson metric on the Teichmüller space was proved by Ahlfors and Weil [36, 37]. Relation $(4.12)$, which establishes a connection between Fuchsian uniformization and Weil-Petersson geometry through classical Liouville action $S_{cl}$ as the Kähler potential, was not known to the founders of the Teichmüller theory. It was proved only recently [3], after the importance of the quantum Liouville theory [1, 6] has been appreciated. Comparing $(4.12)$ and $(4.12)$ yields

$$\frac{\partial c_i}{\partial z_j} = \frac{1}{2\pi} \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right)_{WP}, \quad (4.13)$$

so that the Weil-Petersson metric also “measures” a deviation of accessory parameters $c_i$ from being holomorphic functions on $Z_n$.

Finally, consider CWI $(2.16)$. Using $(3.25)$ and $(3.30)$, we get at the tree level

$$\frac{2\pi}{\hbar} D_z D_w G(z, w) = \frac{c_{cl}/2}{(z - w)^2} + (2R_w(z, w) + R(z, w) \frac{\partial}{\partial w} + \mathcal{L}(z))T_{cl}(w). \quad (4.14)$$
Similarly to the previous case, denoting \( \tilde{P}(z, w) = 4D_zD_wG(z, w) \) and setting
\[
\tilde{P}_\Gamma(\zeta, \zeta') = \tilde{P}(J(\zeta), J(\zeta'))J'(\zeta)^2J'(\zeta')^2,
\]
we get
\[
\tilde{P}_\Gamma(\zeta, \zeta') = P(\zeta, \bar{\zeta}') = \frac{12}{\pi} \sum_{\gamma \in \Gamma} \frac{\gamma'(\zeta')^2}{(\zeta - \gamma\zeta')^4}. \tag{4.15}
\]
This symmetric kernel \( \tilde{P}_\Gamma(\zeta, \zeta') = \tilde{P}_\Gamma(\zeta', \zeta) \) is the third derivative of the so-called meromorphic Eichler integral for Fuchsian group \( \Gamma \), which is regular at the punctures and has a simple pole at \( \zeta = \zeta' \) (see [9, Ch.V, Sect. 7]), thus establishing the connection between CWI (4.14) and Eichler integrals. The right hand side of (4.14), in virtue of (4.1) and (4.2), essentially consists of \( \partial c_i/\partial z_j \), partial derivatives of accessory parameters in holomorphic directions. In [3] we proved explicit formulas expressing these derivatives through kernel \( \tilde{P}(z, w) \) [3, §4.3]; it is easy to show that these results can be put together into single relation (4.14)!

Moreover, as it follows from (3.30) and (4.15),
\[
<< T(z)T(w)X >> = \frac{6/h}{(z - w)^4} + O(|z - w|^{-3}) \quad \text{as } z \to w,
\]
so that, as expected, \( c_{cd} = 12/h \).

Thus we have seen that even at the tree level, CWI for the two-dimensional quantum gravity are highly non-trivial and yield new important information about the uniformization of the Riemann surfaces and complex geometry of moduli spaces.

### 4.2 One-loop Approximation

(i) **One-point correlation function \( << T(z)X >> \)**

Consider CWI (2.13) at the one-loop level: using formulas (3.14) and (3.27) for one-loop contributions to the log \( < X > \) and \( << T(z)X >> \), we get
\[
2\pi G_{zz'}|_{z'=z} + 2\pi \text{v.p.} \int_X G(z', z')D_zG(z, z')d\rho' = \mathcal{L}(z)\log Z_X(2). \tag{4.16}
\]

First, it is not obvious that the left hand side of (4.16) is a holomorphic quadratic differential on \( X \), as it is explicitly stated in the right hand side. However, consider the following formal calculations
\[
\frac{\partial}{\partial z'}(G_{zz'}|_{z'=z}) = ((\partial_z + \partial_{z'})G_{zz'})|_{z'=z}
\]

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\[
G_{zz'}(z, z') + G_{zz'z'}(z, z') |_{z' = z} \\
= \frac{1}{2} (e^{\phi_{ccl}(z)} G_{z'}(z, z') + e^{\phi_{ccl}(z') G_{z}(z, z') - \delta_{z'}(z - z') - \delta_{z}(z - z')) |_{z' = z} \\
= \frac{1}{2} e^{\phi_{ccl}(z)} \frac{d}{dz} G(z, z), \quad (4.17)
\]

and

\[
\partial_{\bar{z}} (G_{zz} - (\phi_{ccl})_{z} G_{z}) = G_{zz} - (\phi_{ccl})_{\bar{z}} G_{z} - (\phi_{ccl})_{z} G_{\bar{z}} \\
= \frac{1}{2} (e^{\phi_{ccl}} G_{z} + (\phi_{ccl})_{z} e^{\phi_{ccl}} G_{z} - e^{\phi_{ccl}} \delta_{z} - (\phi_{ccl})_{z} e^{\phi_{ccl}} \delta - e^{\phi_{ccl}} G_{z} \\
- (\phi_{ccl})_{z} e^{\phi_{ccl}} G_{z} + (\phi_{ccl})_{z} e^{\phi_{ccl}} \delta) = -\frac{1}{2} e^{\phi_{ccl}} \delta_{z}, \quad (4.18)
\]

which use equations (1.1), (3.2) and symmetry property 2 of the Green’s function. Formula (4.18) implies that

\[
\frac{\partial}{\partial \bar{z}} (\int_{X} G(z', z') D_{z} G(z, z') d\rho') = -\frac{1}{2} e^{\phi_{ccl}(z)} \frac{d}{dz} G(z, z), \quad (4.19)
\]

which, together with (4.17), “show” that \(<< T(z)X >>_{loop} \) is holomorphic on \(X\).

One can easily make these arguments rigorous (and valid for the case of compact Riemann surfaces as well).

First, observe that expression (3.27) has a transformation law of a projective connection (times 1/6). Indeed, from (3.4) and classical formula

\[
\lim_{z' \to z} \left\{ \frac{f'(z) f'(z')}{(f(z) - f(z'))^2} - \frac{1}{(z - z')^2} \right\} = \frac{1}{6} S(f)(z),
\]

where \(f\) is a (locally) holomorphic function (see, e.g., [38]), it follows that the first term in (3.27) has transformation law (1.13) (times 1/6). Moreover, as it follows from (1.19), the differential operator \(D\) maps functions on \(X\) into quadratic differentials. Therefore, the second term in (3.27) is a well-defined quadratic differential on \(X\).

Second, using definition (3.28) of \(G_{zz}|_{z'=z}\), property 3 of the Green’s function, asymptotics

\[
G_{z} = -\frac{1}{2\pi} \left\{ \frac{1}{z - z'} + \frac{1}{2} (\phi_{ccl})_{z} \right\} + O(1), \quad z' \to z,
\]

and the Taylor formula, one gets

\[
\frac{\partial}{\partial \bar{z}} (G_{zz}|_{z'=z}) = \frac{1}{2} e^{\phi_{ccl}(z)} \frac{d}{dz} G(z, z).
\]
This justifies (4.17).

Third, the same property 3 and equation (1.1) imply that $\mathcal{D}_z G(z, z')$ is a holomorphic quadratic differential on $X \setminus \{z'\}$. It has the following asymptotics

$$\mathcal{D}_z G(z, z') = \frac{1}{2\pi} \left\{ \frac{1}{(z - z')^2} + \frac{\phi_{cl}}{z - z'} \right\} + O(1), \ z' \to z,$$

so that (4.19) follows from the standard properties of singular integrals [35, Ch. 5.A, Lemma 2].

Next, what about the validity of CWI (4.16) itself? First, it is not difficult to show that

$$<< T(z)X >>_{\text{loop}} = P(-\pi G_{zz'} | z' = z),$$

(4.20)

where $P$ is the Hodge projector, so that $<< T(z)X >>_{\text{loop}} \in \mathcal{H}^{2,0}(X)$. In particular, it has no second order poles at the punctures (see §4.3), so that at one-loop there is no contribution to the conformal dimension, in agreement with the discussion in §2.1. Second, consider explicit formula for the first variation of the Selberg zeta function with respect to moduli, proved in [30, Lemma 3], which states (after setting $s = 2$) that

$$\partial \log Z(2) = -2P(G_{zz'} | z' = z).$$

(4.21)

Combining (4.20) and (4.21), we get (4.16), which “encodes” a variational formula for the Selberg zeta function at a special point $s = 2$!

(ii) Two-point correlation functions

$$<< T(z)T(w)X >> \text{ and } << T(z)\bar{T}(\bar{w})X >>$$

Without presenting explicit (though quite complicated) expressions for correlation function $<< T(z)T(w)X >>_{\text{loop}}$, here we restrict ourselves to a simpler problem of determining its leading singularity as $z \to w$. Direct inspection shows that this singularity is of order $(z - w)^{-4}$ and can be obtained only from term $2\pi^2 G_{zw}^2(z, w)$, described in §3.2. According to property 4 of the Green’s function, it has the form

$$\frac{1}{(z - w)^4},$$

which shows that one-loop correction $c_{cl}$ to the central charge equals 1. Analyzing higher-loop terms, that can only be obtained from the “interaction part” in the expression for generating function $Z(\mu, \bar{\mu}; X)$, one can convince oneself that they do not contribute to the
leading singularity of \( \langle \langle T(z)T(w)X \rangle \rangle \). Thus we have

\[
c = c_{\text{cl}} + c_{\text{loop}} = \frac{12}{\hbar} + 1
\]

for the central charge of the Virasoro algebra. This is in accordance with general “philosophy” of the quantum field theory that only one-loop approximation contributes to the anomaly.

It is also possible to analyze the CWI (2.18) at the one-loop level. Working out explicit expression for \( \langle \langle T(z)\bar{T}(\bar{w})X \rangle \rangle_{\text{loop}} \), one can show that (2.18) in a one-loop approximation is equivalent to the local index theorem for families of \( \bar{\partial} \)-operators on punctured Riemann surfaces (a generalization of the Belavin-Knizhnik theorem \[39\]), proved in \[30\].

### 4.3 Elements of the Teichmüller Theory

Here we present, in a succinct form, basic facts from the Teichmüller theory as it was developed by Ahlfors and Bers \[35, 40, 41\]; our exposition mainly follows \[3\]. Although we restrict ourselves to the case of punctured spheres, compact Riemann surfaces are treated similarly.

Recall that a Riemann surface \( X \) is called marked, if a particular canonical system of generators (up to an inner automorphism) of its fundamental group is chosen. Two marked Riemann surfaces are isomorphic if there exists a complex analytic isomorphism that maps one set of generators into the other (up to a overall conjugation). This equivalence relation is more restrictive than the usual one, so that the corresponding quotient space—Teichmüller space \( \mathcal{T}_{0,n} \) of Riemann surfaces of genus zero with \( n \) punctures—covers the moduli space \( \mathcal{M}_{0,n} \) (and is easier to deal with, since it is isomorphic to an open cell in \( \mathbb{C}^{n-3} \)). All points in \( \mathcal{T}_{0,n} \)—equivalence classes of marked Riemann surfaces—can be obtained from a given point—equivalence class of a marked Riemann surface \( X \)—by deformations of its complex structure.

These deformations can be described by considering the hyperbolic plane \( H \) first. Complex structures on \( H \) correspond to the 1-forms \( d\zeta + \mu d\bar{\zeta} \), where \( \mu \in L^\infty(H) \) is a \((-1,1)\)-form on \( H \) with the property \( \|\|\mu\|\|_\infty < 1 \). The standard complex structure on \( H \) corresponds to the case \( \mu = 0 \). According to the fundamental theorem from the theory of quasi-conformal mappings, all complex structures on \( H \) are isomorphic: there exists diffeomorphism \( f : H \mapsto H \)
satisfying Beltrami equation

\[ \frac{\partial f}{\partial \zeta} = \mu \frac{\partial f}{\partial \bar{\zeta}} \]

on \( H \). However, complex structure \( d\zeta + \mu d\bar{\zeta} \) on \( H \) can be projected on \( X \cong H/\Gamma \) only when \( \mu \) is a Beltrami differential on \( H/\Gamma \), i.e. satisfies the transformation law

\[ \mu(\gamma \zeta) \frac{\gamma'(\zeta)}{\gamma'(\zeta')} = \mu(\zeta), \quad \gamma \in \Gamma. \]

In this case, \( \mu \) can be projected on \( X \), so that \( M = m \circ J^{-1}J'/J' \) is a Beltrami differential on Riemann surface \( X \). Even in this case isomorphism \( f \) will not necessarily be \( \Gamma \)-equivariant: Fuchsian group \( \Gamma^u = f \circ \Gamma \circ f^{-1} \) will not be conjugated to \( \Gamma \) in \( \text{PSL}(2, \mathbb{R}) \), thus yielding Riemann surface \( X^u = H/\Gamma^u \) as a deformation of \( X \).

Specifically, infinitesimal deformations are described by sheaf cohomology group \( H^1(X, TX) \). Using Hodge theory (for the choice of Poincaré metric on \( X \)), one can identify it with the complex linear space \( \mathcal{H}^{-1,1}(X) \) of harmonic Beltrami differentials on \( X \). This introduces a natural complex structure on Teichmüller space \( \mathcal{T}_{0,n} \), with space \( \mathcal{H}^{-1,1}(X) \) being its holomorphic tangent space at point \( X \). The inner product in \( \mathcal{H}^{-1,1}(X) \), induced by the Hodge ∗-operator is given by

\[ (M, N)_{WP} = \int_X M \bar{N} d\rho, \quad (4.22) \]

and defines a Hermitian metric on \( \mathcal{T}_{0,n} \). It is called the Weil-Petersson metric, and it turns out to be Kähler \([36, 37]\). The corresponding holomorphic cotangent space to \( \mathcal{T}_{0,n} \) at point \( X \) is isomorphic to the linear space \( \mathcal{H}^{2,0}(X) \) of harmonic quadratic differentials on \( X \). The pairing between these spaces—Serre duality—is given by integration

\[ \int_X MQ, \quad M \in \mathcal{H}^{-1,1}(X) \quad Q \in \mathcal{H}^{2,0}(X), \quad (4.23) \]

since the product of a Beltrami differential and a quadratic differential is a (1, 1)-form on \( X \). According to \((4.23)\), Hermitian product \((4.22)\) in the tangent space induces Hermitian product \((4.7)\) in the space of harmonic quadratic differentials.

These results generalize verbatim to the case of Riemann surfaces of the type \((g, n)\)—Riemann surfaces of genus \( g \) with \( n \) punctures, satisfying the condition \( \chi(X) = 2 - 2g + n < 0 \). Corresponding Teichmüller space \( \mathcal{T}_{g,n} \) is a complex manifold of dimension \( 3g - 3 + n \) with a natural Kähler structure, given by the Weil-Petersson metric.
In our case the presentation is simplified by considering the space of punctures $Z_{n}$, introduced in §2.2. It plays the role of the “intermediate” moduli space: it is covered by the Teichmüller space $T_{0,n}$ and in turn, covers the moduli space $M_{0,n}$. The complex linear space $H_{2,0}(X)$ can be described explicitly as consisting of meromorphic functions on $\overline{X} = \mathbb{P}^1$ with at most simple poles at the punctures $z_{1}, \ldots, z_{n-1}$, having the order $O(|z|^{-3})$ as $z \to z_{n} = \infty$. It is $n-3$-dimensional and has a special basis $\{P_{i}\}$ associated with punctures,

$$P_{i}(z) = -\frac{1}{\pi} R(z, z_{i}), \ i = 1, \ldots, n-3,$$

where $R(z, z_{i})$ is given by (2.12). As we mentioned, in the cotangent space the Weil-Petersson metric has the form

$$(P, Q) = \int_{X} P(z) \overline{Q(z)} e^{-\phi_{cl}(z, \bar{z})} d^{2}z, \ P, Q \in H^{2,0}(X). \quad (4.24)$$

When lifted to upper half-plane $H$, it coincides with Petersson’s inner product for $\Gamma$-automorphic forms of weight 4.

Next, denote by $\{Q_{i}\}_{i=1}^{n-3}$ the basis in $H^{2,0}(X)$, orthogonal to $\{P_{i}\}_{i=1}^{n-3}$, i.e.

$$(P_{i}, Q_{j}) = \delta_{ij}. \quad (4.25)$$

Harmonic Beltrami differentials

$$M_{i}(z, \bar{z}) = e^{-\phi_{cl}(z, \bar{z})} \overline{Q_{i}(z)} \quad (4.26)$$

define quasi-conformal mappings $F_{i}^{\epsilon}$, which infinitesimally “move” only the puncture $z_{i}$ and keep other punctures fixed. Indeed, as it follows from Beltrami equation, the infinitesimal deformation

$$\dot{F}_{i} = (\frac{\partial}{\partial \epsilon} F_{i}^{\epsilon})|_{\epsilon=0},$$

satisfies the $\bar{\partial}$-equation

$$\frac{\partial \dot{F}_{i}}{\partial \bar{z}} = M_{i},$$

and, therefore, admits the following integral representation

$$\dot{F}_{i}(z) = -\frac{1}{\pi} \int M(w) R(w, z) d^{2}w. \quad (4.27)$$
Now from (4.27) and (4.24)–(4.26) it immediately follows that
\[ \dot{F}_i(z_j) = \delta_{ij}, \]  
(4.28)
which is a condition of infinitesimally moving only a given puncture \( z_i \). This shows that vector fields \( \partial/\partial z_i \) on \( Z_n \) at a point corresponding to Riemann surface \( X \), are represented by infinitesimal deformations \( \dot{F}_i, \ i = 1, \ldots, n - 3 \). As it also follows from (4.27), formula (4.28) can be specialized further as
\[ \dot{F}_i(z) = \delta_{ij} + (z - z_j)\dot{F}_iz(z_j) + o(|z - z_j|), \ z \to z_j, \ j \neq n, \]
\[ \dot{F}_n(z) = z\dot{F}_iz(\infty) + o(|z|), \ z \to z_n = \infty. \]  
(4.29)

Finally, we describe the dependence of classical solution \( \phi_{cl}(z; z_1, \ldots, z_{n-3}) \) on the punctures. It is given by the following “conservation law”
\[ \frac{\partial}{\partial z_i}(e^{\phi_{cl}}) = \frac{\partial}{\partial z}(e^{\phi_{cl}}\dot{F}_i), \ i = 1, \cdots, n - 3, \]  
(4.30)
which is a reformulation of Ahlfors’ result on vanishing of the first variation of the Poincaré volume element under quasiconformal deformations with harmonic Beltrami differentials [37].

4.4 Proofs

The reader can find the detailed account in [3, §3-4]; here we just highlight the main points.

In order to prove (4.2), one should use definition of Liouville action (1.9), relation (4.3), properties (4.29) of vector fields \( \dot{F}_i \) and formula
\[ \frac{\partial}{\partial z_i}\phi_{cl} = -(\phi_{cl})z\dot{F}_i - \dot{F}_iz, \]
which follows from conservation law (4.30). Application of Stokes’ formula, with subsequent careful analysis of boundary terms as \( \epsilon \to 0 \), results in relations (4.2).

In order to prove (4.13), consider a one-parameter family of Riemann surfaces \( X^\epsilon_j = \mathbb{C} \setminus \{z_1^\epsilon, \cdots, z_{n-3}^\epsilon, 0, 1\} \), \( j = 1, \ldots, n - 3 \), obtained by deformations \( F^\epsilon_j \) for sufficiently small \( \epsilon \). The map \( F^\epsilon_j \) satisfies Beltrami equation on \( X \)
\[ \frac{\partial F^\epsilon_j}{\partial \bar{z}} = \epsilon M_j \frac{\partial F^\epsilon_j}{\partial z}, \]
\[ 37 \]
and depends holomorphically on $\epsilon$; $z_i^{\epsilon_j} = F_j^{\epsilon}(z_i)$ are also holomorphic in $\epsilon$. Denote by $J_j^{\epsilon}$ uniformization map for the Riemann surface $X_j^{\epsilon}$ and consider the following commutative diagram

$$
\begin{array}{ccc}
H & \xrightarrow{f_j^{\epsilon}} & H \\
J & \downarrow & J_j^{\epsilon} \\
X & \xrightarrow{F_j^{\epsilon}} & X_j^{\epsilon}
\end{array}
$$

where map $f_j^{\epsilon}$ is quasiconformal with Beltrami differential $\mu = M_j \circ J J'/J'$. Next, consider equation

$$
S((J_j^{\epsilon})^{-1} \circ F_j^{\epsilon}) = S(f_j^{\epsilon} \circ J^{-1}),
$$

which follows from the commutative diagram, and use Caley identity (1.14) to obtain

$$
S((J_j^{\epsilon})^{-1} \circ F_j^{\epsilon} (F_j^{\epsilon})^2 + S(F_j^{\epsilon}) = S(f_j^{\epsilon} \circ J^{-1}(J^{-1'})^2 + S(J^{-1}).
$$

Finally, apply partial derivative $\partial/\partial \bar{\epsilon}$ to equation (4.31) and evaluate the result at $\epsilon = 0$

$$
\sum_{i=1}^{n-3} \frac{\partial}{\partial \bar{\epsilon}} c_i^{\epsilon_j} |_{\epsilon=0} R(z, z_i) = \left. \frac{\partial}{\partial \bar{\epsilon}} (f_j^{\epsilon} \circ J^{-1}(J^{-1'})^2 \right|_{\epsilon=0}.
$$

Using relation

$$
\frac{\partial}{\partial \bar{\epsilon}} f_j^{\epsilon} |_{\epsilon=0} = -\frac{1}{2} Q_j,
$$

which is equivalent to the Ahlfors lemma in §4.1, we get

$$
\sum_{i=1}^{n-3} \frac{\partial c_i}{\partial \bar{z}_j} P_i = \frac{1}{2\pi} Q_j,
$$

so that

$$
\frac{\partial c_i}{\partial \bar{z}_j} = \frac{1}{2\pi} (Q_j, Q_i) = \frac{1}{2\pi} (\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}) W P.
$$

In order to prove CWI (1.14), apply partial derivative $\partial/\partial \epsilon$ to equation (4.31) and evaluate the result at $\epsilon = 0$

$$
\{ \frac{\partial}{\partial z_i} + \hat{F}_i(z) \frac{\partial}{\partial z} + \hat{F}_i(z) \} T_{\alpha}(z) = \frac{1}{h} (\hat{\delta}_{i\zeta\zeta} \circ J^{-1} - \hat{\delta}_{i\zeta\zeta})(z).
$$

From the integral representation (4.27) it follows that

$$
\hat{\delta}_{i\zeta\zeta}(z) - \hat{\delta}_{i\zeta\zeta}(z) = 2 \int_X \left\{ \frac{3}{\pi(z-w)^4} - D_zD_w G(z, w) \right\} e^{-\phi_{\alpha}(w)Q_i(w)} d^2 w.
$$

Now (4.14) follows from formulas (2.11), (4.32)—(4.33) and orthogonality condition (1.27).

We will not present here the proofs of the one-loop results [14]. The reader can find necessary technical tools in [30].
4.5 Generalizations

First, CWI for multi-point correlation functions also carry meaningful information about Weil-Petersson geometry. Thus, for instance, CWI for four-point correlation function $<\langle T(z_1)T(z_2)\bar{T}(\bar{w}_1)\bar{T}(\bar{w}_2)X \rangle>$ at the tree level reproduces Wolpert’s result on evaluation of Riemann tensor of the Weil-Petersson metric [12]. Moreover, universal CWI (2.21)−(2.22) carry, in a “compressed form”, important information about the modular geometry; the challenging problem is to “decode” it without appealing to perturbation theory.

Second, our approach can be trivially generalized to the case of Riemann surfaces with the branch points of orders $l_i$, $2 \leq l_i \leq \infty$. Corresponding changes in the definition of Liouville action (1.9) are obvious. One has for the Schwarzian derivative of uniformization map (see, e.g., [8])

$$T_{cl}(z) = \frac{1}{\hbar} S(J^{-1})(z) = \sum_{i=1}^{n} \left\{ \frac{\alpha_i}{(z - z_i)^2} + \frac{c_i}{z - z_i} \right\},$$

where $\alpha_i = (1 - l_i^{-2})/2\hbar$ are classical dimensions $\Delta_i$ of geometric vertex operators $V_{\alpha_i}$ (recall (1.8)). The arguments, used in the punctured case, apply to this case as well and show that after the quantization conformal dimensions $\Delta_i$ remain the same.

Third, our approach can be generalized to Riemann surfaces of type $(g, n)$. In this case (see [4] for details when $n = 0$), one can use single global coordinate on the Schottky cover of Riemann surface $X$, provided by Schottky uniformization. In its terms it is possible to define Liouville action, expectation value $<X>$ by functional integral (1.10) and generating functional (1.25). On Riemann surface $X$ this amounts to the choice of Schottky projective connection $T^S$ and considering the difference $T(\phi) - T^S/\hbar$, which is well-defined quadratic differential on $X$ (cf. [13]).

Finally, we make the following speculative remark. In derivation [1] of Liouville action from $D$-dimensional bosonic string, the following correspondence between Liouville’s coupling constant $\hbar$ and dimension $D$ was established

$$\frac{1}{2\pi \hbar} = \frac{25 - D}{24\pi}$$

(note the replacement $26 \mapsto 25$, which should be considered as a quantum correction).
According to our calculations,

\[ c_{\text{Liouv}} = \frac{12}{h} + 1 = 25 - D + 1 = 26 - D. \]

This expression, when added to contribution \( D \) from the string modes and to the contribution \(-26\) coming from Faddeev-Popov ghosts, yields zero and thus cancels the global conformal anomaly for any \( D \).

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**References**

[1] A.M. Polyakov, Phys. Lett. **103B**, 207 (1981).

[2] L. Alvarez-Gaumé, preprint CERN-TH.6736/92, to appear in the proceedings of the Les Houches Summer School on Gravitation and Quantization, 1992.

[3] P.G. Zograf and L.A. Takhtajan, Funct. Anal. Appl. **19**, 219 (1986); Math. USSR Sbornik **60**, 143 (1988).

[4] P.G. Zograf and L.A. Takhtajan, Math. USSR Sbornik **60**, 297 (1988).

[5] L.A. Takhtajan, In: *New Symmetry Principles in Quantum Field Theory*, Eds J. Frölich, G. ’t Hooft, A. Jaffe, G. Mack, P.K. Mitter, and R. Stora, Plenum Press, New York and London 1992.

[6] A.M. Polyakov, *Lecture at Steklov Institute*, Leningrad 1982, unpublished.

[7] N. Seiberg, Prog. Theor. Phys. Suppl. **102**, 319 (1990).

[8] H. Poincaré, J. Math. Pures Appl. 5 se. **4**, 157 (1898).

[9] I. Kra, *Automorphic Forms and Kleinian Groups*, Benjamin, Reading, Mass., 1972.
[10] V. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov, Mod. Phys. Lett. A3, 819 (1988).

[11] F. David, Mod. Phys. Lett. A3, 1651 (1988).

[12] J. Distler and H. Kawai, Nucl. Phys. B321, 509 (1989).

[13] E. D’Hoker, Mod. Phys. A6, 745 (1991).

[14] L.A. Takhtajan, Mod. Phys. Lett. A8 3529 (1993)

[15] L.A. Takhtajan, preprint hep-th/9403013 to appear in Mod. Phys. Lett A.

[16] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B241, 333 (1984).

[17] D. Friedan and S. Shenker, Nucl. Phys. B281, 509 (1987).

[18] A.B. Zamolodchikov, Phys. Lett. B117, 87 (1982).

[19] J. Schwinger, Proc. Nat. Acad. Sci. 37, 452 (1951).

[20] K. Huang, Quarks, Leptons, and Gauge Fields, World Scientific, Singapore, 1982.

[21] P.G. Zograf, Leningrad Math. J. 1, 941 (1990).

[22] Vl.S. Dotsenko, Adv. Stud. Pure Math. 16, 123 (1988).

[23] D. Friedan, Z. Qiu, and S. Shenker, Phys. Rev. Lett. 52 1575 (1984).

[24] P. Ginsparg, In: Fields, Strings and Critical Phenomena, Eds E. Brézin and J. Zinn-Justin, North-Holland 1989.

[25] F.W.J. Olver, Introduction to Asymptotics and Special Functions, Academic Press, New York and London, 1974.

[26] A.S. Shvarts, J. Soviet Math. 21, 551 (1983).

[27] A.B. Venkov, Spectral Theory of Automorphic Functions and Its Applications, Kluwer, 1991.
[28] E. D’Hoker and D. Phong, Commun. Math. Phys. 104, 537 (1986).

[29] P. Sarnak, Commun. Math. Phys. 110, 113 (1987).

[30] L.A. Takhtajan and P.G. Zograf, Commun. Math. Phys. 137, 399 (1991).

[31] J. Hadamard, Lectures on Cauchy’s Problem in Linear Partial Differential Equations, Yale Univ. Press, New Haven, Conn., 1923.

[32] A. Polyakov, Mod. Phys. Lett. A2, 893 (1987).

[33] H. Verlinde, Nucl. Phys. B337, 652 (1990).

[34] S. Lang, SL2(R), Addison-Wesley, Reading, Mass., 1975.

[35] L.V. Ahlfors, Lectures on Quasiconformal Mappings, Wadsworth & Brooks, Belmont, California, 1987.

[36] A. Weil, Séminaire Bourbaki 1957/58, Secrétariat Math., Paris, 1958, Exposé 168.

[37] L.V. Ahlfors, Ann. Math. (2) 74, 171 (1961).

[38] A.N. Tyurin, Russian Math. Surveys 33, 149 (1978).

[39] A.A. Belavin and V.G. Knizhnik, Sov. Phys. JETP 64, 214 (1986).

[40] L.V. Ahlfors and L. Bers, Ann. Math. (2) 72, 385 (1960).

[41] L. Bers, Proc. Internat. Congr. Math. (Edinburgh, 1958), Cambridge Univ. Press, 349, 1960.

[42] S. Wolpert, Invent. Math. 85, 119 (1986).

[43] H. Sonoda, Nucl. Phys. B281, 546 (1987).