Kaluza-Klein and Gauss-Bonnet
cosmic strings

Mustapha Azreg-Aïnou and Gérard Clément

1 5 rue du Soleil, 06100 Nice, France
2 Laboratoire de Gravitation et Cosmologie Relativistes,
  Université Pierre et Marie Curie, CNRS/URA769
  Tour 22-12, Boîte 142, 4 place Jussieu, 75252 Paris cedex 05, France

March 28, 1996

Abstract

We make a systematic investigation of stationary cylindrically symmetric solutions to the five-dimensional Einstein and Einstein-Gauss-Bonnet equations. Apart from the five-dimensional neutral cosmic string metric, we find two new exact solutions which qualify as cosmic strings, one corresponding to an electrically charged cosmic string, the other to an extended superconducting cosmic string surrounding a charged core. In both cases, test particles are deflected away from the singular line source. We extend both kinds of solutions to exact multi-cosmic string solutions.

*E-mail: GECL@CCR.JUSSIEU.FR.
1 Introduction

In four spacetime dimensions, the Einstein-Hilbert action augmented by a cosmological constant term is the unique geometrical action (depending only on the metric and the curvature tensor) leading to field equations which involve at most second order derivatives of the metric. This is no longer true if the dimensionality of spacetime is greater than four. As first shown by Lovelock [1], the most general second-order geometrical action is a sum of terms proportional to the successive Euler forms, the number of terms depending on the dimension. From a purely geometrical perspective, there is no compelling reason to truncate this sum by keeping only the Einstein-Hilbert action. The implications of these Lovelock higher-dimensional theories of gravity have been explored in a number of pioneering papers [2-8].

The simplest application of the Lovelock construction is to the case of five dimensions. Kaluza and Klein [9] first suggested the five-dimensional geometrical unification of gravity and electromagnetism, both at the kinematical (equations of motion of charged particles) and dynamical (field equations) levels. The dynamical equations of the theory [10] commonly known as Kaluza-Klein theory are simply the five-dimensional Einstein equations. The dynamical equations of the more general five-dimensional Lovelock theory are the Einstein-Gauss-Bonnet equations (the single, quadratic extra term in the Lagrangian is the Gauss-Bonnet [3] or Lanczos [4] invariant). As far as we know, the only investigation of exact solutions to these five-dimensional equations concerns a cosmological model [7] (the spherically symmetric solutions of [8] are spherical in the four space dimensions, and so do not have the Kaluza-Klein $V_4 \times S_1$ topology). One of the motivations for the present paper was thus the search for exact non-trivial stationary solutions to the Einstein-Gauss-Bonnet equations.

We shall focus in this paper on stationary cylindrically symmetric spacetimes, with four commuting Killing vectors — Kaluza-Klein cosmic string solutions. Such solutions to the five-dimensional Einstein equations were investigated by Ferrari [11] in the static case. Actually, Ferrari’s assumption that the gravitational potential $g_{44}$ goes to a constant at spatial infinity restricted him to ultrastatic solutions with $g_{44}$ constant; the corresponding Kasner-like four-space metrics are singular, except for the special case of the “solenoid” solution (see section 4 below). We shall relax these assumptions, and enlarge our search to the Einstein-Gauss-Bonnet equations, with
the hope that singularities might be regularized by the Gauss-Bonnet term; this will turn out to be true in the case of a particularly interesting solution (section 6 below).

As shown by Witten [14], some spontaneously broken gauge theories lead to the appearance of superconducting cosmic strings carrying a longitudinal electric current. The long-range behaviour of the gravitational field of a straight superconducting cosmic string has been derived by several authors [13] from the solution of the Einstein-Maxwell equations with suitable extended or line sources. One would expect to also be able to describe superconducting cosmic strings in the five-dimensional unified theory, with possible differences with the Einstein-Maxwell description due to the extra Kaluza-Klein scalar field and to the Gauss-Bonnet non linear term. We shall show here that a certain solution to the Einstein-Gauss-Bonnet equations with a line source may be reinterpreted as a superconducting cosmic string solution to the five-dimensional Einstein equations with an effective, Gauss-Bonnet generated longitudinal electric current as extended source.

In the next section we write down in matrix form the stationary cylindrically symmetric Einstein-Gauss-Bonnet equations. The third section is devoted to an exhaustive classification of local solutions to the five-dimensional Einstein equations (no Gauss-Bonnet coupling); the generation of inequivalent global solutions from given local solutions is briefly discussed in section 4. We next go over to the case of the full Einstein-Gauss-Bonnet equations, and first investigate in section 5 a special class of solutions, those for which both the Einstein and Gauss-Bonnet contributions vanish independently; this class includes the five-dimensional extension of the well-known neutral cosmic string spacetime, as well as an electrically charged cosmic string solution. A less trivial exact solution to the full Einstein-Gauss-Bonnet equations is constructed in section 6, and interpreted as a superconducting cosmic string. We then extend in section 7 both the charged and the superconducting cosmic string solutions to exact multi-cosmic string solutions. Our results are briefly summarized in the last section.
2 The Einstein-Gauss-Bonnet equations in the cylindrically symmetric case

The most general geometrical action leading to field equations containing at most second order partial derivatives is, in the case of five spacetime dimensions [4],

\[ S = -\frac{1}{8\pi G_5} \int d^5x \sqrt{g} \left[ \alpha L_0 + \frac{\beta}{2} L_1 + \gamma \frac{L_2}{4} \right], \quad (2.1) \]

where \( L_0, L_1 \) and \( L_2 \) are the cosmological, Einstein-Hilbert and Gauss-Bonnet contributions,

\[ L_0 \equiv 1, \quad L_1 \equiv R, \quad L_2 \equiv R^{ABCD} R_{ABCD} - 4 R^{AB} R_{AB} + R^2 \quad (2.2) \]

(upper-case Roman indices take the values 1 to 5). In this paper we are mainly interested in classical solutions which approach cosmic string metrics at large distances. This is possible only if we choose the cosmological term to be absent, \( \alpha = 0 \). We also normalize the five-dimensional gravitational constant \( G_5 \) by the choice \( \beta = 1 \). The field equations deriving from the geometrical action (2.1) are then

\[ R_{AB} - \frac{1}{2} R g_{AB} + \gamma L_{AB} = 0, \quad (2.3) \]

where \( L_{AB} \) is the covariantly conserved Lanczos tensor [4] defined by

\[ L_{AB} \equiv R_A^{\ CDE} R_{BCDE} - 2 R^{CD} R_{ACBD} - 2 R_{AC} R_B^\ C + R R_{AB} - \frac{1}{4} g_{AB} L_2. \quad (2.4) \]

A stationary cylindrically symmetric five-dimensional metric has four commuting Killing vectors, one of which \( (\partial_4) \) is timelike, and two of which \( (\partial_2 \) and \( \partial_5) \) have closed orbits. Choosing adapted coordinates \( x^1 = \rho, \ x^2 = \varphi, \ x^3 = z, \ x^4 = t, \ x^5 \), such a metric can be parametrized in terms of a \( 4 \times 4 \) symmetrical matrix \( \lambda \) by

\[ ds^2 = -d\rho^2 + \lambda_{ab}(\rho) \, dx^a \, dx^b \quad (2.5) \]
(with \(a, b = 2, \ldots, 5\)). A straightforward computation then gives the Riemann tensor components

\[
R_{1ab1} = \frac{1}{2} (\lambda B)_{ab},
\]

\[
R_{abcd} = -\frac{1}{4} [\lambda (\lambda)_{bc} (\lambda)_{ad} - (\lambda)_{ac} (\lambda)_{bd}],
\]

in terms of the matrices

\[
\chi \equiv \lambda^{-1} \lambda, \quad B \equiv \chi, \rho + \frac{1}{2} \chi^2.
\]

From (2.6), (2.7) we compute the Ricci tensor components

\[
R_{11} = -\frac{1}{2} \text{Tr} B, \quad R^a_{\ b} = \frac{1}{2} [B - \frac{1}{2} \chi^2 + \frac{1}{2} (\text{Tr} \chi) \chi]_{\ ab},
\]

as well as the Lanczos tensor components. We only give here as an example one of the terms entering the computation of \(L^a_{\ b}\),

\[
R^{acde} R_{bced} = \frac{1}{8} [(\text{Tr} \chi^2) \chi^2 - \chi^4]_{\ ab}.
\]

We then arrive at the following form of the field equations (2.3), split into a scalar equation (the (11) components of (2.3)) and a matrix equation,

\[
3 \text{Tr} B + \frac{1}{2} [(\text{Tr} \chi)^2 - \text{Tr} \chi^2] + \gamma \left\{ \frac{1}{2} (\text{Tr} (B \chi^2) - \text{Tr} (B \chi \chi) \text{Tr} \chi) \right. \\
\left. + \frac{1}{4} \text{Tr} B [(\text{Tr} \chi)^2 - \text{Tr} \chi^2] \right\} = 0,
\]

\[
\chi_{,\rho} + 2 \chi,\rho + \frac{1}{2} [(\text{Tr} \chi) \chi + \chi \text{Tr} \chi^2 + (\text{Tr} \chi)^2] + \gamma \left\{ \frac{1}{2} (\chi^3)_{,\rho} - \frac{1}{2} (\text{Tr} \chi) (\chi^2)_{,\rho} \\
+ \frac{1}{2} [(\text{Tr} \chi)^2 - \text{Tr} \chi^2] \chi,\rho - \frac{1}{2} (\text{Tr} \chi,\rho) (\chi^2 - (\text{Tr} \chi) \chi) - \frac{1}{4} (\text{Tr} \chi^2)_{,\rho} \chi \\
+ \frac{1}{4} [(\text{Tr} \chi) \chi^3 - (\text{Tr} \chi^2) \chi^2 - (\text{Tr} \chi)(\text{Tr} \chi^2) \chi + (\text{Tr} \chi)^3] \chi \right\} = 0.
\]
3 Exact solutions of the Einstein equations

In this section we give the complete solution of equations (2.11), (2.12) with \( \gamma = 0 \), i.e. of the stationary cylindrically symmetric Kaluza-Klein equations. Let us define the four invariants associated with the matrix \( \chi \):

\[
f \equiv \text{Tr} \chi, \quad g \equiv \text{Tr} \chi^2, \quad h \equiv \text{Tr} \chi^3, \quad k \equiv \det \chi. \quad (3.1)
\]

Traces of higher powers of \( \chi \) may be computed from these by using the characteristic equation for the \( 4 \times 4 \) matrix \( \chi \),

\[
\chi^4 - f \chi^3 + \frac{1}{2} (f^2 - g) \chi^2 + \left( -\frac{1}{3} h + \frac{1}{2} f g - \frac{1}{6} f^3 \right) \chi + k \equiv 0. \quad (3.2)
\]

For \( \gamma = 0 \) the field equations (2.11), (2.12) may be written, using (3.1), as

\[
3 f,\rho + \frac{1}{2} f^2 + g = 0, \quad (3.3)
\]

\[
\chi,\rho + \frac{1}{2} f \chi + 2 f,\rho + \frac{1}{2} f^2 + \frac{1}{2} g = 0. \quad (3.4)
\]

Tracing the matrix equation (3.4), and eliminating \( g \) between the resulting equation and (3.3), we obtain the differential equation for \( f \)

\[
f,\rho + \frac{1}{2} f^2 = 0. \quad (3.5)
\]

The invariant \( g \) is determined from the solution of this equation by

\[
g = f^2. \quad (3.6)
\]

The matrix \( \chi \) may then be obtained by solving the linear equation

\[
\chi,\rho + \frac{1}{2} f \chi = 0. \quad (3.7)
\]

The differential equation (3.5) is solved either by

\[
f = \frac{2}{\rho}, \quad (3.8)
\]

or by

\[
f = 0. \quad (3.9)
\]
We first consider the case $f = 2/\rho$. In this case, equation (3.7) integrates to

$$\chi = \frac{2}{\rho} A,$$

where $A$ is a constant real matrix constrained by

$$\text{Tr}A^2 = \text{Tr}A = 1.$$

Then, the integration of the first equation (2.8) yields

$$\lambda = Ce^{A \ln \rho^2},$$

where $C$ is a constant real matrix of signature $(- - + -)$. The symmetry of the matrix $\lambda(\rho)$ is ensured by the symmetry conditions on $C$:

$$C = C^T, \quad CA = (CA)^T.$$

A matrix $A$ subject to the constraints (3.11) has four eigenvalues $p_i$ solving the characteristic equation

$$p^4 - p^3 - \frac{1}{3}(c - 1)p + d = 0,$$

where $c \equiv \text{Tr}A^3$, $d \equiv \det A$. These eigenvalues are related by the constraints

$$\sum_{i=1}^{4} p_i = 1, \quad \sum_{i=1}^{4} p_i^2 = 1.$$

The resulting metric is obviously of the Kasner type [16] if the four eigenvalues $p_i$ are real. However, this by no means exhausts the various solutions of the form (3.12). By considering systematically the various matrix types for the matrix $A$, we arrive at the following classification:

1) The four eigenvalues are complex,

$$p_1 = x + iy, \quad p_2 = x - iy, \quad p_3 = \frac{1}{2} - x + iz, \quad p_4 = \frac{1}{2} - x - iz.$$

The Lagrange interpolation formula

$$f(A) = \sum_i f(p_i) \prod_{j \neq i} \frac{A - p_j}{p_i - p_j}$$

is
then gives
\[
\lambda = \rho^{2x} CM \left[ (A - x) \sin(2y \ln \rho + \delta_1) + y \cos(2y \ln \rho + \delta_1) \right] + \rho^{1-2x} CN \left[ (A + x - \frac{1}{2}) \sin(2z \ln \rho + \delta_2) + z \cos(2z \ln \rho + \delta_2) \right]
\] (3.18)

where the matrices \(M\) and \(N\) are quadratic in \(A\).

2) Two eigenvalues are complex and two real. The solution is obtained from (3.18) by the replacement \(y \to iy\).

3) The four eigenvalues are real. The matrix \(A\) may be diagonalized, leading to the five-dimensional Kasner metric
\[
ds^2 = -d\rho^2 - \rho^{2p_1} d\varphi^2 - \rho^{2p_2} dz^2 + \rho^{2p_3} dt^2 - \rho^{2p_4} (dx^5)^2,
\] (3.19)

where the \(p_i\) are constrained by (3.15)

4) Two of the real eigenvalues coincide. The Jordan normal form \([17]\) of the matrix \(A\) is
\[
A = \begin{pmatrix}
p_1 & \epsilon_1 & 0 & 0 \\
0 & p_1 & 0 & 0 \\
0 & 0 & p_3 & 0 \\
0 & 0 & 0 & p_4
\end{pmatrix},
\] (3.20)

with \(\epsilon_1 = 0\) or 1. The resulting matrix \(\lambda\)
\[
\lambda = C \text{diag}(\rho^{2p_1}, \rho^{2p_1}, \rho^{2p_3}, \rho^{2p_4}) + \epsilon_1 \rho^{2p_1} \ln \rho^2 CH
\] (with \((CH)_{ab} = 0\) except for \(a = b = 2\)) differs by a logarithm from the Kasner metric.

5) The four eigenvalues are equal pairwise, \(p_1 = p_2 = (1 - \sqrt{3})/4, p_3 = p_4 = (1 + \sqrt{3})/4\). The Jordan normal form of \(A\) and the most general associated matrix \(C\) satisfying the symmetry conditions (3.13) are
\[
A = \begin{pmatrix}
p_1 & \epsilon_1 & 0 & 0 \\
0 & p_1 & 0 & 0 \\
0 & 0 & p_3 & \epsilon_2 \\
0 & 0 & 0 & p_3
\end{pmatrix}, \quad C = \begin{pmatrix}
(1 - \epsilon_1)a_1 & b_1 & 0 & 0 \\
b_1 & c_1 & 0 & 0 \\
0 & 0 & (1 - \epsilon_2)a_2 & b_2 \\
0 & 0 & b_2 & c_2
\end{pmatrix}.
\] (3.22)

We then find that for \(\epsilon_1\epsilon_2 = 1\), det \(C \geq 0\) so that the metric cannot have the Lorentz signature; for \(\epsilon_1\epsilon_2 = 0\), the solution is of the form (3.21).
6) Three of the eigenvalues coincide and are nonzero, \( p_1 = p_2 = p_3 = -p_4 = 1/2 \). Putting \( B \equiv A - 1/2 \), with \( B^4 = -B^3 \), we obtain the solution
\[
\lambda = \rho C(1 + B^3) + 2 \rho \ln \rho C(B - B^3) + 2(\ln \rho)^2 C(B^2 + B^3) - \frac{1}{\rho} CB^3, \quad (3.23)
\]
with \( B \) of the Jordan normal form
\[
B = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & \epsilon_1 & 0 \\
0 & 0 & 0 & \epsilon_2 \\
0 & 0 & 0 & 0
\end{pmatrix}. \quad (3.24)
\]

7) Three of the eigenvalues vanish. Then the characteristic equation for \( A \) reduces to
\[
A^4 = A^3, \quad (3.25)
\]
leading to the solution
\[
\lambda = C(1 - A^3) + 2 \ln \rho C (A - A^3) + 2(\ln \rho)^2 C(A^2 - A^3) + \rho^2 CA^3. \quad (3.26)
\]
with \( A \) of the Jordan normal form
\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & \epsilon_1 & 0 \\
0 & 0 & 0 & \epsilon_2 \\
0 & 0 & 0 & 0
\end{pmatrix}. \quad (3.27)
\]
The solution (3.26) with \( \epsilon_1\epsilon_2 = 1 \) \( (A^3 \neq A^2) \) is studied in Section 6. For \( \epsilon_1 = 1, \epsilon_2 = 0 \) \( (A^3 = A^2 \neq A) \) we recover a subcase of case 4 (see Section 5, eq. (5.6)). For \( \epsilon_1 = \epsilon_2 = 0 \) \( (A^2 = A) \), the solution (a particular Kasner metric) is the product of a naked cosmic string metric by the Klein circle,
\[
ds^2 = -d\rho^2 - \alpha^2 \rho^2 d\varphi^2 - dz^2 + dt^2 - (dx^5)^2. \quad (3.28)
\]

We now consider the second, trivial solution of the scalar equations (3.5) and (3.6), \( f = g = 0 \). Equation (3.7) is then solved by
\[
\chi = A \quad \text{(3.29)}
\]
where the constant matrix \( A \) is now constrained by
\[
\text{Tr}A = \text{Tr}A^2 = 0. \quad (3.30)
\]
The resulting metrical matrix $\lambda(\rho)$ is

$$\lambda = Ce^{A\rho},$$  \hspace{1cm} (3.31)

where the constant matrix $C$ again obeys the symmetry conditions (3.13). From the characteristic equation

$$A^4 - \frac{h}{3} A + k = 0,$$ \hspace{1cm} (3.32)

it follows that if either $h$ or $k$ is nonzero, $A$ has either four complex eigenvalues (the solution $\lambda(\rho)$ is then analogous to (3.18) with $\ln(\rho)$ replaced by $\rho$) or two real and two complex eigenvalues. We consider in more detail the special case $h = k = 0$, for which the characteristic equation (3.32) reduces to

$$A^4 = 0.$$ \hspace{1cm} (3.33)

The solutions of this equation may be classified according to the rank of the matrix $A$:

a) $r(A) = 3$. The Jordan normal form of $A$ and the most general associated matrix $C$ are

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & a & b \\ 0 & a & b & c \\ a & b & c & d \end{pmatrix}.$$ \hspace{1cm} (3.34)

It results from (3.34) that $\det C \geq 0$, so that this solution does not lead to a Lorentzian metric.

b) $r(A) = 2$. There are two possible Jordan normal forms for $A$. The first form and a typical associated matrix $C$,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$ \hspace{1cm} (3.35)

lead to the five-dimensional metric

$$ds^2 = -d\rho^2 - 2 d\varphi dt - \frac{1}{2} (\rho dt + 2 dz)^2 + dz^2 - (dx^5)^2,$$ \hspace{1cm} (3.36)
which is the product of a Petrov type N metric [19] by the Klein circle. In the case of the second normal form of $A$ with the most general associated matrix $C$,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & a & 0 & b \\ a & c & b & d \\ 0 & b & 0 & e \\ b & d & e & m \end{pmatrix}, \quad (3.37)$$

we find again $\det C \geq 0$ (non-Lorentzian metric).

c) $r(A) = 1$. Then the Jordan normal matrix $A$ with a typical matrix $C$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (3.38)$$

lead to the metric

$$ds^2 = -d\rho^2 - d\varphi^2 + 2dz\,dt + \rho\,dt^2 - (dx^5)^2, \quad (3.39)$$

which is the product of a flat three-dimensional spacetime [20] [19] by two tori.

d) $r(A) = 0$, i.e. $A = 0$. This obvious solution,

$$ds^2 = -d\rho^2 - d\varphi^2 - dz^2 + dt^2 - (dx^5)^2, \quad (3.40)$$

is the five-dimensional Minkowski metric with two dimensions ($\varphi, x^5$) compactified.

### 4 From local to global solutions

The local solutions of the Kaluza-Klein equations given in the preceding section are actually equivalence classes—from each solution other solutions may be obtained by linear coordinate transformations $x^a = L^a_{\ b}\ x^b$ mixing the four commuting Killing vectors together (the corresponding transformations on the matrices $A$ are similarity transformations $A' = L^{-1}AL$). However, because two of our Killing vectors ($\partial_2$ and $\partial_5$) have closed orbits, some of these transformations (those with $L^a_m \neq \delta^a_m$ for $m = 2, 5$) lead to new
solutions which are not globally equivalent to the old solutions. Here we only give two examples of magnetic spacetimes obtained from Minkowski spacetime by mixing $\partial_2$ and $\partial_5$ together.

In the first example (the Kaluza-Klein equivalent of the Melvin [21] magnetic universe), the only non-zero non-diagonal element of the transformation matrix $L$ is $L_{25} = \beta$, leading to the metric [22]

$$ds^2 = -d\rho^2 - \rho^2 (d\varphi + \beta dx^5)^2 - dz^2 + dt^2 - (dx^5)^2,$$  \hspace{1cm} (4.1)

where $\varphi$ is periodic with period $2\pi$ and $x^5$ is periodic with period $2\pi a$. Using the standard Kaluza-Klein decomposition of the five-dimensional metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + g_{55} (dx^5 + k A_\mu dx^\mu)^2$$  \hspace{1cm} (4.2)

($\mu, \nu = 1, \cdots, 4$, and $k$ is the Kaluza constant) into four-scalar, four-vector and four-tensor fields $g_{55}, A_\mu$ (electromagnetic potential), $g_{\mu\nu}$ (four-dimensional metric), the metric (4.1) may be rewritten in the form

$$ds^2 = -d\rho^2 - \frac{\rho^2}{1 + \beta^2 \rho^2} d\varphi^2 - dz^2 + dt^2 - (1 + \beta^2 \rho^2) [dx^5 + \frac{\beta \rho^2}{1 + \beta^2 \rho^2} d\varphi]^2$$  \hspace{1cm} (4.3)

first given by Gibbons and Maeda [23], and by Ferrari [11]. The resulting four-dimensional metric interpolates between the Minkowski metric for $\rho = 0$ and the cylindrical Minkowski metric (the four-dimensional part of (3.40)) for $\rho \to \infty$, while the azimuthal vector potential corresponds to a longitudinal magnetic field $B_z = 2k^{-1} \beta (1 + \beta^2 \rho^2)^{-3/2}$, and the scalar field is parabolic. The magnetic flux through a plane $z = \text{const.}$ is

$$\Phi = \oint_{\rho = \infty} A_\mu dx^\mu = \frac{2\pi}{\beta k}. \hspace{1cm} (4.4)$$

Our second example (flux string) is obtained from Minkowski spacetime by a transformation matrix $L$ which is diagonal except for $L_{52} = \nu$, leading to the metric with a dislocation in the fifth dimension,

$$ds^2 = -d\rho^2 - \rho^2 d\varphi^2 - dz^2 + dt^2 - (dx^5 + \nu d\varphi)^2.$$  \hspace{1cm} (4.5)

The azimuthal vector potential $A_\varphi = k^{-1} \nu$ is pure gauge, except for a singularity on the axis $\rho = 0$. The corresponding longitudinal magnetic field
$B^z = 2\pi k^{-1} \nu \delta^2(x)$ is that of a flux tube concentrated on this axis. Other five-dimensional generalizations of four-dimensional metrics singular on the axis $\rho = 0$ are generated by non-zero $L^2_2$ (the static cosmic string metric (3.28)), $L^3_2$ (longitudinal dislocations [24]), and $L^4_2$ (spinning cosmic strings [25]).

5 Solutions with vanishing Lanczos tensor

Now we consider the full five-dimensional Einstein-Gauss-Bonnet equations (2.11), (2.12) with $\gamma \neq 0$. This system of coupled first-order differential equations seems very difficult to disentangle, so we first look for special solutions of the form

$$\chi = \alpha(\rho) A. \quad (5.1)$$

The $\gamma = 0$ solutions (3.10) and (3.29) are of this form. As shown in detail in [18], it turns out that the ansatz (5.1) selects the metrics for which the Einstein tensor vanishes ($\gamma = 0$ solutions) and the Lanczos tensor in (2.3) also vanishes.

As we have seen in section 3, a first class of solutions of the five-dimensional Einstein equations is given by

$$\chi = \frac{2}{\rho} A, \quad (5.2)$$

with $\text{Tr} A = \text{Tr} A^2 = 1$. Inserting (5.2) into the left-hand side of the full matrix equation (2.12), and expanding in powers of $A$, we find that the term linear in $A$ vanishes identically, while the combination of the terms in $A^2$ and $A^3$ reduces to $-(8/\rho^4) (A^3 - A^2)$. So our ansatz solves equation (2.12) for $\gamma \neq 0$ only for matrices $A$ such that

$$A^3 = A^2. \quad (5.3)$$

Computing the resulting matrix $B$ from (2.8),

$$B = \frac{2}{\rho} (A^2 - A), \quad (5.4)$$

we then find $\text{Tr} B = \text{Tr} (B\chi) = \text{Tr} (B\chi^2) = 0$, so that the scalar equation (2.11) is also satisfied. Matrices $A$ obeying (5.3) are of the Jordan normal
form (3.27) with $\epsilon_1 \epsilon_2 = 0$. For $\epsilon_1 = \epsilon_2 = 0$, the five-dimensional metric (3.28) corresponds to a neutral cosmic string (or to a global, magnetic generalization such as (4.1)). For $\epsilon_1 = \epsilon_2 = 0$, we choose

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -\alpha^2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

(5.5)

to obtain the five-dimensional metric

$$ds^2 = -d\rho^2 - \alpha^2 \rho^2 d\phi^2 - dz^2 - 2 dt dx^5 - 2 \ln \rho (dx^5)^2. \quad (5.6)$$

Following the standard Kaluza-Klein dimensional reduction (4.2), we interpret the solution (5.6) as describing a straight charged cosmic string with gravitational and electric potentials

$$\bar{g}_{44} = \frac{1}{2 \ln \rho}, \quad A_4 = \frac{1}{2k \ln \rho}. \quad (5.7)$$

There is however a problem with this interpretation, due to the fact that the scalar field $g_{55} = -2 \ln \rho$ does not go to a constant at spatial infinity. We recall that the sourceless five-dimensional Einstein equations lead to the effective four-dimensional Einstein equations with source,

$$\bar{R}_{\mu \nu} - \frac{1}{2} \bar{R} \delta_{\mu \nu} = \frac{k^2}{2} g_{55} \left( F_{\mu \rho} F^{\nu \rho} - \frac{1}{4} \delta_{\mu \nu} F_{\rho \sigma} F^{\rho \sigma} \right) + |g_{55}|^{-1/2} D_{\mu} D^{\nu} |g_{55}|^{1/2}. \quad (5.8)$$

The source is the electromagnetic stress-energy tensor if $g_{55}(x)$ converges rapidly enough to a limiting value $g_{55}(\infty) = -16\pi G/c^2k^2$. In the present case, $g_{55}(x)$ has no limit so that the Kaluza-Klein constant $k$ is undetermined. This problem with infrared logarithmic divergences is a familiar one in the case of infinite straight string sources, and should disappear in the case of a closed charged cosmic string. Geodesic motion in the five-dimensional metric (5.6) is obtained by combining geodesic motion in the four-dimensional metric (4.27) of [19] (with $\omega^i = 0$, $\alpha = -1$) with a uniform motion in the $z$ direction. As discussed in [19], the string singularity at $\rho = 0$ is “harmless”, in the sense that it deflects away all timelike and lightlike geodesics. Another interesting
property of the metric (5.6) is that, for $\alpha = 1$, spatial sections are flat, which suggests that multi-charged cosmic string solutions are also possible. These shall be investigated in section 7.

We now turn to the second class of solutions of the five-dimensional Einstein equations,

$$\chi = A, \quad (5.9)$$

with $\text{Tr}A = \text{Tr}A^2 = 0$. Then the full equation (2.12) with $\gamma = 0$ is satisfied, while the full equation (2.11) gives $\text{Tr}A^4 = 0$. This implies, from the trace of equation (3.2), $\det A = 0$, so that the characteristic equation reduces to a special case of equation (3.32),

$$A^4 = p^3 A, \quad (5.10)$$

showing that the matrix $A$ has the four eigenvalues $0, p, jp, j^2 p$ with $j = e^{2\pi i / 3}$. A typical solution is

$$ds^2 = -d\rho^2 + 2 \cos \left( \frac{\sqrt{3} p \rho}{2} - \frac{\pi}{3} \right) e^{-p\rho/2} d\varphi^2 - 4 \cos \frac{\sqrt{3} p \rho}{2} e^{-p\rho/2} d\varphi dt + 2 \cos \left( \frac{\sqrt{3} p \rho}{2} + \frac{\pi}{3} \right) e^{-p\rho/2} dt^2 - e^{p\rho} dz^2 - (dx^5)^2. \quad (5.11)$$

Such a metric, which does not approach, up to logarithms, a cosmic string metric at large distances, does not admit a satisfactory physical interpretation. Note that both solutions (5.6) and (5.11) are the product of the Klein circle or of the real line by a cylindrically symmetric solution of the four-dimensional Einstein equations [19], which explains why their Lanczos tensor vanishes identically [4].

### 6 A Gauss-Bonnet superconducting cosmic string

We now inquire whether there is a non-trivial (i.e. with non-zero Lanczos tensor) solution of the five-dimensional Einstein-Gauss-Bonnet equations, which at the same time is asymptotic, up to logarithms, to the cosmic string metric (3.28). Our strategy is to expand the matrix $\lambda(\rho)$ in powers of $\gamma$,

$$\lambda(\rho) = \lambda_0(\rho) + \gamma \lambda_1(\rho) + \cdots, \quad (6.1)$$
where \( \lambda_0(\rho) \) is an exact solution of the Einstein equations (\( \gamma = 0 \)) with the desired asymptotic behaviour, and to solve perturbatively the full \( \gamma \neq 0 \) equations to obtain the \( n^{\text{th}} \) order term \( \lambda_n(\rho) \) from the lower order terms. Because the Einstein-Gauss-Bonnet equations (2.11), (2.12) are non-linear equations in \( \chi = \lambda^{-1}\lambda_{\rho}\rho \), with the zeroth order \( \chi_0(\rho) \) going as \( 1/\rho \) for the cosmic string metric (3.28), the asymptotic behaviour of the full solution (6.1) will automatically be governed by that of \( \lambda_0(\rho) \). As an added bonus, we will find that, owing to the special algebraic properties of \( \chi_0 \), the linearized solution \( \chi = \chi_0 + \gamma \chi_1 \) is actually an exact solution of the full non-linear equations!

A solution of the five-dimensional Einstein equations which is asymptotic, up to logarithms, to the cosmic string metric (3.28) must be of the form (3.12),

\[
\lambda_0 = C e^{A \ln \rho^2},
\]

where the matrix \( A \) has the eigenvalues \( (1, 0, 0, 0) \), \text{i.e.} must obey the algebraic relation

\[
A^4 = A^3
\]

without necessarily being diagonalizable. If \( A^2 = A \), or \( A^3 = A^2 \) with \( A^2 \neq A \), we recover the neutral cosmic string metric (3.28) or the charged cosmic string metric (5.6) respectively, which as we have seen solve trivially the Einstein-Gauss-Bonnet equations. So we assume \( A^3 \neq A^2 \), corresponding to the Jordan normal form (3.27) with \( \epsilon_1 = \epsilon_2 = 1 \),

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Now we linearize the Einstein-Gauss-Bonnet equations (2.11), (2.12), using the perturbative expansion for the matrix \( \chi = \lambda^{-1}\lambda_{\rho}\rho \),

\[
\chi(\rho) = \frac{2}{\rho} A + \gamma \chi_1(\rho) + \cdots.
\]

The linearized equations are

\[
3 \text{Tr} \chi_{1,\rho} + \frac{1}{\rho} [4 \text{Tr}(A \chi_1) + 2 \text{Tr} \chi_1] = 0,
\]

16
\[ \chi_{1,\rho} + 2\text{Tr}\chi_{1,\rho} + \frac{1}{\rho} \left[ \chi_1 + (\text{Tr}\chi_1)A + 2\text{Tr}(A\chi_1) + 2\text{Tr}\chi_1 \right] - \frac{8}{\rho^4} (A^3 - A^2) = 0. \] (6.6)

Combining the first equation with the trace of the second equation, we obtain the two scalar equations

\[ \text{Tr}(A\chi_1) = \text{Tr}\chi_1, \quad \text{Tr}\chi_{1,\rho} + \frac{2}{\rho} \text{Tr}\chi_1 = 0. \] (6.7)

The second of these equations is solved by

\[ \text{Tr}\chi_1 = \frac{2c}{\rho^2} \] (6.8)

\((c\text{ constant})\). The second equation (6.6) now reduces to

\[ \chi_{1,\rho} + \frac{1}{\rho} \chi_1 + \frac{2c}{\rho^3} A - \frac{8}{\rho^4} (A^3 - A^2) = 0, \] (6.9)

which is solved by

\[ \chi_1 = \frac{2}{\rho} D + \frac{2c}{\rho^3} A - \frac{4}{\rho^4} (A^3 - A^2), \] (6.10)

with \(D\) a constant matrix such that \(\text{Tr}(AD) = \text{Tr}D = 0\). Actually the first two terms result from the expansion to first order of \(\chi = 2(A + \gamma D)/(\rho - \gamma c)\), and may be gauged away by suitable coordinate transformations. So the genuine first-order perturbation is

\[ \chi_1 = -\frac{4}{\rho^3} (A^3 - A^2). \] (6.11)

It is now very easy to check, using the properties \(A\chi_1 = \chi_1 A = 0, \chi_1^2 = 0, \text{Tr}\chi_1 = 0\), that

\[ \chi = \frac{2}{\rho} A - \frac{4\gamma}{\rho^3} (A^3 - A^2) \] (6.12)

is an exact solution of the full Einstein-Gauss-Bonnet equations (2.11), (2.12).
The corresponding metrical matrix, obtained by solving the differential equation \( \lambda_{,\rho} = \lambda \chi \), is

\[
\lambda = C \exp \left[ A \ln \rho^2 + \frac{2\gamma}{\rho^2}(A^3 - A^2) \right] = C[e^A \ln \rho^2 + \frac{2\gamma}{\rho^2}(A^3 - A^2)].
\] (6.13)

Choosing \( A \) in the Jordan normal form (6.4) and

\[
C = \begin{pmatrix}
-\alpha^2 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & a \\
0 & -1 & a & b \\
\end{pmatrix},
\] (6.14)

we finally obtain the exact solution

\[
\begin{align*}
ds^2 &= -d\rho^2 - \alpha^2 \rho^2 d\varphi^2 - 2 dz dx^5 - dt^2 - 4L dt dx^5 \\
&\quad -2(L^2 - p - \gamma/\rho^2) (dx^5)^2, \\
L &= \ln(\rho/\rho_0),
\end{align*}
\] (6.15)

where we have defined \( \ln \rho_0 = a/2, \ p = (a^2 + 2b)/4 \). The corresponding four-dimensional metric and electromagnetic potentials are, according to the Kaluza-Klein dimensional reduction (4.2),

\[
\begin{align*}
\mathbf{g}_{\mu\nu} \ dx^\mu \ dx^\nu &= -d\rho^2 - \alpha^2 \rho^2 d\varphi^2 - \frac{1}{2(L^2 + p + \gamma/\rho^2)} dz^2 \\
&\quad + \frac{(L^2 + p + \gamma/\rho^2)}{(L^2 - p - \gamma/\rho^2)} \left( dt + \frac{L}{(L^2 + p + \gamma/\rho^2)} dz \right)^2, \\
A_\mu \ dx^\mu &= \frac{1}{2k(L^2 - p - \gamma/\rho^2)} (dz + 2L dt).
\end{align*}
\] (6.16)

As in the case of the charged cosmic string solution (5.6), the scale of the electromagnetic field, proportional to the inverse of the Kaluza-Klein constant \( k \), is undetermined because the scalar field \( g_{55}(x) \) does not go to a constant at spatial infinity. However we expect the cylindrical solution to approximate the behaviour of the gravitational and electromagnetic fields not too far from a closed line source modelled by \( \rho = 0 \).

The electromagnetic field (6.16) is mainly electric, with a small magnetic component, at both large \((\rho \gg \rho_0)\) and small \((\rho \ll \rho_0)\) distances. This suggests that the distributional source for this metric is a line of electric...
charge, as borne out by a careful computation of the left-hand side of the Gauss-Bonnet equations (2.3). However, we shall argue that these equations may be reinterpreted as the five-dimensional equations

$$R^A_B - \frac{1}{2} R \delta^A_B = 8\pi G T^A_{\text{eff} B} \quad (6.17)$$

with an effective source $T^A_{\text{eff} B}$ which is the sum of a distributional contribution and of the continuous Gauss-Bonnet contribution $-(\gamma / 8\pi G) L^A B$. The computation of the left-hand side of (6.17) involves the Ricci tensor components given, from equations (2.9) and (6.12), by

$$R^{a b} = \frac{1}{2 \rho} (\rho \chi)_{, b} = [A \Delta \ln \rho + \gamma (A^3 - A^2) \Delta \rho^{-2}]^{a}_{b}, \quad (6.18)$$

where $\Delta$ is the covariant Laplacian. Going over to two-dimensional conformal coordinates $(x, y)$ with

$$(x + iy)^{\alpha} = \rho e^{i \alpha \varphi}, \quad (6.19)$$

and using $\Delta \ln \rho = 2 \pi \alpha \delta^2(x)$, where $\delta^2(x)$ is the two-dimensional covariant Dirac distribution, we obtain from (6.4) the non-vanishing components of the effective source,

$$T^3_{\text{eff} 3} = T^4_{\text{eff} 4} = T^5_{\text{eff} 5} = \frac{1 - \alpha}{4G} \delta^2(x),$$

$$T^3_{\text{eff} 4} = T^4_{\text{eff} 5} = \frac{\alpha}{4G} \delta^2(x),$$

$$T^3_{\text{eff} 5} = -\frac{\gamma}{2\pi G \rho^4}. \quad (6.20)$$

Recalling that the energy-momentum tensor components $T^\mu_{\text{eff} 5}$ are, in the Kaluza-Klein theory, proportional to the electromagnetic current density $j^\mu$, we are thus led to interpret the metric (6.15) as describing an extended superconducting cosmic string (longitudinal current density $T^3_{\text{eff} 5}$) surrounding a longitudinally boosted electrically charged naked cosmic string.

Another effect of the Gauss-Bonnet coupling is to transform the mild logarithmic singularity of the metric (6.15) for $\rho \to 0$ into a strong $\rho^{-2}$ singularity, which at first sight is rather annoying. However, as we shall see by studying geodesic motion in this metric, while the singularity mildly repels test particles for $\gamma = 0$, it becomes strongly repulsive for $\gamma > 0$, so that in
this sense the $\gamma > 0$ solution is less singular than the $\gamma = 0$ one. The first
integrated five-dimensional geodesic equations in a stationary cylindrically
symmetric metric (2.3) are

$$\left(\frac{d\rho}{d\tau}\right)^2 - \Pi_a \lambda^{ab}(\rho) \Pi_b + \epsilon = 0, \quad (6.21)$$

$$\frac{dx^a}{d\tau} = \lambda^{ab}(\rho) \Pi_b, \quad (6.22)$$

where $\tau$ is an affine parameter, $\epsilon$ is a real constant, and the $\Pi_a$ are the
constants of the motion associated with the four Killing vectors. In the case
of the metric (6.13), equation (6.21) reads (after rescaling lengths so that
$\rho_0 = 1$)

$$\left(\frac{d\rho}{d\tau}\right)^2 + (\alpha^{-2}\Pi_2^2 + 2\gamma \Pi_3^2) \rho^{-2} + 2\Pi_3^2 (\ln \rho)^2 - 4\Pi_3 \Pi_4 \ln \rho$$

$$+ 2\Pi_3(p \Pi_3 + \Pi_5) + \Pi_2^2 + \epsilon = 0. \quad (6.23)$$

For $\gamma < 0$, $\Pi_3^2 > -\Pi_2^2/2\gamma \alpha^2$, the effective potential is strongly attractive,
and all geodesics terminate at the singularity $\rho = 0$. For $\gamma = 0$, the effective
potential is repulsive except if $\Pi_2 = \Pi_3 = 0$, allowing only spacelike geodesics
($\epsilon < 0$) to reach the singularity. For $\gamma > 0$, the centrifugal repulsion is
enhanced by a term proportional to $\Pi_3^2$, so that again only spacelike geodesics
can reach the singularity. We also note that the asymptotic behaviour of the
effective potential is dominated by the term $2\Pi_3^2(\ln \rho)^2$ so that, whatever the
value of the Gauss-Bonnet coupling constant $\gamma$, only geodesics with $\Pi_3 = 0$
and $\epsilon < 0$ extend to infinity; the fact that timelike or lightlike geodesics are
bounded is again a pathology of infinite straight cosmic strings.

7 Multiple cosmic strings

In this section we show how the non-Kasner solutions (5.6) and (6.15) may
be extended to multi-cosmic string solutions, using the construction of [20],
which we first briefly review. Multi-cosmic string spacetimes having less
symmetry than the one-cosmic string spacetimes considered so far, we only
assume here the existence of two commuting Killing vectors, one of which
is $\partial_5$, and the other is either $\partial_t$ (stationary configurations) or $\partial_z$ (parallel
cosmic strings). In this case the five-dimensional metric can be parametrized by
\[ ds^2 = \tau^{-1} h_{ij} \, dx^i \, dx^j + \mu_{ab}(dx^a + A^a_i \, dx^i)(dx^b + A^b_j \, dx^j), \quad (7.1) \]
where \( i \) takes the three values (1, 2, 3) or (1, 2, 4), \( a \) takes the two values (4, 5) or (3, 5), and \( \tau = -\det \mu_{ab} \). Define the twist two-vector \( \omega_a \) such that
\[ \omega_{a,i} = h^{-1/2} \tau \mu_{ab} h^{-1/2} \epsilon^{jkl} A^b_{j,k} \quad \text{(7.2)} \]
\( (h = -\det h_{ij}, \text{ and } \epsilon^{jkl} \text{ is the antisymmetric symbol}), \) and the 3 × 3 matrix field
\[ M = \begin{pmatrix} \mu_{ab} + \tau^{-1} \omega_a \omega_b & -\tau^{-1} \omega_a \\ -\tau^{-1} \omega_b & \tau^{-1} \end{pmatrix}. \quad (7.3) \]
The five-dimensional Einstein equations then reduce to the three-dimensional sigma-model system \[27\]
\[ (M^{-1} M^i)_{;i} = 0, \quad R_{ij} = \frac{1}{4} \text{Tr}(M^{-1} M_i M^{-1} M_j), \quad (7.4) \]
where all geometric symbols refer to the three-dimensional metric \( h_{ij} \). If this metric is (pseudo-)Euclidean, then a class of solutions to the equations (7.4) is given by
\[ M = C e^{A \sigma(x)} e^{A^2 \phi(x)}, \quad (7.5) \]
where the 3 × 3 matrices \( A \) and \( C \) are constrained by
\[ \text{Tr} A = \text{Tr} A^2 = 0, \quad C = C^T, \quad CA = (CA)^T, \quad (7.6) \]
and \( \sigma(x), \phi(x) \) are arbitrary harmonic functions,
\[ \nabla^2 \sigma = 0, \quad \nabla^2 \phi = 0, \quad (7.7) \]
where \( \nabla^2 \) is the flat-space Laplacian. The linearity of equations (7.4) then leads to the existence of multi-centre \[26\] or, in the present case, of multi-cosmic string solutions to the original field equations.

The solution (5.6) with \( \alpha = 1 \) may be put in the form (7.1) with
\[ \mu = \begin{pmatrix} 0 & -1 \\ -1 & -2 \ln \rho \end{pmatrix} \quad (7.8) \]
(τ = 1), \( A^a_i = 0 \), and \( h_{ij} = -\delta_{ij} \) \((i = 1, 2, 3)\). The matrix \( M \) associated with \( (7.8) \) is of the form \((7.5)\) with

\[
A = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad C = \begin{pmatrix}
0 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\((A^2 = 0)\), and \( \sigma(x) = 2 \ln \rho \), where \( \rho \) is the distance of the point \( x \) to the charged cosmic string. The natural generalization to a system of non-parallel charged cosmic strings is the linear superposition of harmonic functions

\[
\sigma(x) = 2 \sum_\alpha c_\alpha \ln \rho_\alpha ,
\]

where \( \rho_\alpha \) is the distance of the point \( x \) to the \( \alpha \)th cosmic string, and \( c_\alpha \) is an arbitrary weight. The resulting multi-cosmic string metric is

\[
ds^2 = -dx^2 - dy^2 - dz^2 - 2 dt \, dx^5 - \sigma(x) \, (dx^5)^2 .
\]

Owing to the special algebraic character of this metric, the only non-vanishing Riemann tensor component for this spacetime is \( R_{\alpha j}^\alpha \) = \(-(1/2)\sigma_{i,j} \), and the resulting Lanczos tensor vanishes identically, as in the one-cosmic string case (the harmonicity condition on \( \sigma \) is not used to prove this result, which also holds on the charged cosmic string sources).

The construction is more involved in the case of the solution \( (6.15) \) with \( \alpha = 1, \gamma = 0 \), which may be rewritten in the form \( (7.1) \) with \( i = (1, 2, 4), \quad a = (3, 5) \). Explicitly,

\[
ds^2 = -dx^2 - dy^2 - dt^2 + \mu_{ab} (dx^a + A^a_i \, dx^i) (dx^b + A^b_j \, dx^j) ,
\]

with

\[
\mu = \begin{pmatrix}
0 & -1 \\
-1 & -2(L^2 - p)
\end{pmatrix} , \quad A^a_i = 2L \, \delta^a_i \, \delta^a_3
\]

\((\tau = 1)\), and \( L = \ln \rho \) (taking \( \rho_0 \) as the unit of length). The corresponding twist field is, according to \( (7.2) \), \( \omega_3 = 0, \quad \omega_5 = 2 \varphi \). The resulting matrix \( M \) may be put in the form \( (7.3) \) with

\[
A = \begin{pmatrix}
0 & 0 & -1 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix} , \quad C = \begin{pmatrix}
0 & -1 & 0 \\
-1 & 2p & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\(22\)
\((A^3 = 0)\), and

\[
\sigma = -2 \Im \ln \zeta, \quad \phi = -2 \Re (\ln \zeta)^2,
\]

(7.15)

with \(\zeta(x) = \rho e^{i\varphi}\). This may be generalized to a multi-cosmic string solution of the five-dimensional Einstein equations by replacing (7.15) by the linear superpositions

\[
\sigma = -2 \Im \sum \alpha c_\alpha \ln \zeta_\alpha, \quad \phi = -2 \Re \sum \alpha c_\alpha^2 (\ln \zeta_\alpha)^2, \quad \ln \zeta_\alpha = \ln \rho_\alpha + i \varphi_\alpha
\]

(7.16)

of harmonic functions singular on cosmic strings rotated and translated in three-dimensional Euclidean space \((x, y, t)\) with respect to the original cosmic string, \(\rho_\alpha\) being the Euclidean distance of the event \(x = (x, y, t)\) to the \(\alpha^{th}\) cosmic string, and \(\varphi_\alpha\) its azimuthal angle relative to this string. From the corresponding matrix \(M\), we extract \(\mu_{ab}\) and \(\omega_a\), and invert equation (7.2) which reduces in this case to the linear equation

\[
(\nabla \wedge A^3)_{i} = \sigma_{,i},
\]

(7.17)

to obtain the metric generated by a system of parallel cosmic strings, moving relative to each other with uniform velocities. This metric is of the form (7.12) with

\[
\mu = \begin{pmatrix} 0 & -1 \\ -1 & -\sigma^2/2 + \phi + 2\rho \end{pmatrix}, \quad A^3_{i} = 2 \sum \alpha c_\alpha \hat{u}_{\alpha i} \ln \rho_\alpha, \quad A^5_{i} = 0
\]

(7.18)

where \(\hat{u}_{\alpha i}\) is the direction in \((x, y, t)\) space of the \(\alpha^{th}\) cosmic string.

Finally we extend our multi-cosmic string solution to the case of the Einstein-Gauss-Bonnet equations with \(\gamma \neq 0\). We again make the ansatz (7.18) with the same harmonic function \(\sigma\) as in (7.16), but a modified function \(\phi\). Noting that the only non-vanishing Riemann tensor components for this metric are

\[
R_{ijkl} = \frac{1}{2} \epsilon_{ijkl} \sigma, \quad R_{i5j5} = \frac{1}{2}(\sigma \sigma, i, j - \phi, i, j) + \frac{1}{4} (3 \sigma, i \sigma, j - \delta_{ij}(\sigma, k)^2),
\]

(7.19)

we find that the non-vanishing Lanczos tensor component \(L_{55} = -(1/2)(\sigma, k, l)^2\) is balanced by the Ricci tensor contribution \(R_{55} = (1/2)\nabla^2 \phi\) for the choice

\[
\phi = \phi_0 + \frac{\gamma}{2} (\nabla \sigma)^2
\]

(7.20)
(where \( \phi_0 \) is the harmonic contribution in (7.16)), corresponding to an exact multi-superconducting cosmic string solution of the full Einstein-Gauss-Bonnet equations.

8 Conclusion

Apart from the trivial five-dimensional extension of the four-dimensional neutral cosmic string solution of the vacuum Einstein equations and its global extensions, we have obtained essentially two new exact cosmic string solutions to the five-dimensional Einstein-Gauss-Bonnet equations.

The electrically charged cosmic string metric (5.6) is a solution independently of the value of the Gauss-Bonnet coupling constant \( \gamma \). Its line source is not “seen” by massive or massless test particles, which are always deflected away without encountering the singularity. On the other hand, the form of the solution (6.15) depends explicitly on the value of \( \gamma \). By reinterpreting the Gauss-Bonnet term as an effective source term, we have identified this five-dimensional metric as that of an extended superconducting cosmic string surrounding a charged core. Again, for \( \gamma \geq 0 \) test particles are deflected away from the singular line source. We have also shown that both solutions may be extended to exact multi-cosmic string solutions, describing static systems of non-parallel charged cosmic strings in the first case, systems of parallel superconducting cosmic strings in uniform relative motion in the second case.

We have not investigated the stability of these classical solutions. The behaviour of test particles leads us to conjecture that the charged cosmic string solution, as well as the superconducting cosmic string solution for \( \gamma \geq 0 \), which both bind test particles, are stable, while the superconducting cosmic string solution for \( \gamma < 0 \) is unstable through collapse onto the line singularity.

Acknowledgments

One of us (M. A.-A.) wishes to thank M. Le Bellac for the kind hospitality afforded in his group at the Institut Non Linéaire de Nice, where part of this work was done. We acknowledge stimulating discussions with B. Linet and J. Madore.
References

[1] Lovelock 1971 J. Math. Phys. 12 498
[2] Buchdal H A 1979 J. Phys. A: Math. Gen. 12 1037
[3] Zwiebach B 1985 Phys. Lett. 156B 315
   Boulware D G and Deser S 1985 Phys. Rev. Lett. 55 2656
   Zumino B 1986 Phys. Reports 137 109
[4] Madore J 1985 Phys. Lett. 110A 289
[5] Müller-Hoissen F 1985 Phys. Lett. 163B 106
[6] Madore J 1986 Class. Quantum Grav. 3 361
   Henriques A B 1986 Nucl. Phys. B277 621
   Ishihara H 1986 Phys. Lett. 179B 217
[7] Deruelle N and Madore J 1986 Mod. Phys. Lett. A1 237
[8] Wheeler J T 1986 Nucl. Phys. B273 732
   Wiltshire D L 1986 Phys. Lett. B169 36
[9] Kaluza T 1921 Sitz. Preuss. Akad. Wiss. K1 966
   Klein O 1926 Z. Phys. 37 895
[10] Jordan P 1947 Ann. Phys., Lpz 18 219
    Thiry Y 1948 C. R. Acad. Sci., Paris 226 216
[11] Ferrari J A 1990 Gen. Rel. Grav. 22 19
[12] Vilenkin A 1981 Phys. Rev. D23 852
[13] Gott J A 1985 Astrophys. J. 288 422
    Hiscock W A 1985 Phys. Rev. D31 3288
    Linet B 1985 Gen. Rel. Grav. 17 1109
[14] Witten E 1985 Nucl. Phys. B249 557
[15] Moss I and Poletti S P 1987 Phys. Lett. B199 34
    Linet B 1989 Class. Quant. Grav. 6 435
    Hellwiwel T M and Konkowski D A 1990 Phys. Lett. A143 438
[16] Kasner E 1921 *Amer. J. Math.* **43** 217

[17] Gantmacher F R 1966 *Théorie des matrices* **1** (Paris: Dunod)

[18] Azreg-Ainou M 1995 *Solutions stationnaires en théorie de Kaluza-Klein*,
Doctoral Thesis (Nice University) gr-qc/9511021

[19] Clément G and Zouzou I 1994 *Phys. Rev.* D **50** 7271

[20] Clément G 1985 *Int. J. Theor. Phys.* **24** 267

[21] Melvin M A 1964 *Phys. Lett.* **8** 65

[22] Dowker F, Gauntlett J P, Kastor D A and Traschen J 1994 *Phys. Rev.* D **49** 2909

Dowker F, Gauntlett J P, Gibbons G W and Horowitz G T 1995 *Phys. Rev.* D **52** 6929

[23] Gibbons G W and Maeda K 1988 *Nucl. Phys.* B **298** 741

[24] Gal’tsov D V and Letelier P S 1993 *Phys. Rev.* D **47** 4273

Tod K P 1994 *Class. Quant. Grav.* **11** 1331

[25] Deser S, Jackiw J and ’t Hooft G 1984 *Ann. Phys.*, N.Y. **152** 220

[26] Clément G 1986 *Gen. Rel. Grav.* **18** 861; *Phys. Lett.* A **118** 11

[27] Maison D 1979 *Gen. Rel. Grav.* **10** 717