Two-dimensional wave propagation in layered periodic media

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Abstract

We study two-dimensional wave propagation in materials whose properties vary periodically in one direction only. High order homogenization is carried out to derive a dispersive effective medium approximation. Whereas one-dimensional materials with constant impedance exhibit no effective dispersion, we show that a new kind of effective dispersion may arise in two dimensions, even in materials with constant impedance. This dispersion is a macroscopic effect of microscopic diffraction caused by spatial variation in the sound speed. This dispersive effect is confirmed by homogenization and direct numerical simulations.

1 Introduction

We consider the propagation of waves with characteristic wavelength $\lambda$ over a distance $L$ in a periodic medium with period $\Omega$ where

$$\Omega < \lambda \ll L.$$ 

Because the wavelength $\lambda$ is larger than the material period $\Omega$, the waves “see” the medium as nearly homogeneous and travel at an effective velocity related to averages of the material properties \cite{9, 2}.

An example of the materials considered in this work is shown in Figure 1. Specifically, we are interested in two-dimensional media whose properties vary along one axis only. Acoustic waves in such media are described by the second-order PDE

$$p_{tt} = K(y) \nabla \cdot \left( \frac{1}{\rho(y)} \nabla p \right).$$ \hfill (1)

Here $p = p(x, y, t)$ is the pressure, $K(y)$ is the bulk modulus, and $\rho(y)$ is the material density. Throughout, we assume that $K$ and $\rho$ are $\Omega$-periodic:

$$K(y + \Omega) = K(y) \quad \rho(y + \Omega) = \rho(y)$$

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Figure 1: Wave propagation in a layered periodic medium. A piecewise-constant medium is shown for simplicity, but arbitrary periodic variation (in $y$) is considered. The terms *normal* and *transverse* are used throughout this work to denote propagation in the directions indicated.

Thus the sound speed $c(y) = \sqrt{K/\rho}$ and the impedance $Z(y) = \sqrt{K\rho}$ are also $\Omega$-periodic. We refer to these as *layered* materials, though the coefficients need not be piecewise-constant. In subsequent sections, we frequently use the terms *normal propagation* and *transverse propagation* to refer to propagation normal to or parallel to the axis of homogeneity, respectively (see Figure 1).

The propagation of a plane wave along the $y$-axis (normal propagation) reduces to a one-dimensional problem that is well-studied. Over long distances, periodic variation in the material impedance leads to a dispersive effect – higher frequencies travel more slowly, due to reflection [9]. We refer to this as *reflective dispersion*. On the other hand, when the impedance does not vary, there is no reflection and – in one dimension – no effective dispersion [9, 7]. Instead, all wavelengths travel at the harmonic average of the sound speed.

The propagation of a plane wave perturbation along any other direction is genuinely two-dimensional, and in general such waves undergo diffraction as well as reflection. A novel observation in the present work is that diffraction can play a role similar to that of reflection in periodic media, leading again to a dispersive effect in which higher frequencies travel more slowly. Thus an effective dispersion arises even in materials with constant impedance. This *diffractive dispersion* is an inherently multidimensional effect, with no one-dimensional analog. Whereas reflective dispersion depends on variation in the material impedance, diffractive dispersion depends on variation in the material sound speed.

It has been observed that reflective dispersion can, in combination with nonlinear effects, lead to the formation of solitary waves [7]. Diffractive dispersion can also lead to formation of nonlinear solitary waves; this is the subject of current research [5].

All code used for computations in this work, along with Mathematica worksheets used to derive the homogenized equations, are available at [http://github.com/ketch/effective Dispersion RR].

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In Section 2 we derive a high order homogenized approximation to the variable-coefficient wave equation (1). In Section 3 we derive and study the effective dispersion relation, showing that the medium is effectively anisotropic and dispersive. In Section 4, we examine the complementary roles played by variation in the impedance and the sound speed. We also validate our homogenized model against direct simulations of the variable-coefficient wave equation (1).

2 Homogenization

Homogenization theory can be used to derive an effective PDE for waves in a periodic medium when the wavelength \( \lambda \) is larger than the period of the medium \( \Omega \). The effective PDE is derived through a perturbation expansion, using \( \delta = \Omega / \lambda \) as a small parameter. We will see that the homogenized PDE depends only on \( \Omega \) and not on \( \lambda \). Since the effective PDE has constant coefficients, it can be used to determine an effective dispersion relation for plane waves in the periodic medium.

The lowest-order homogenized equation for (1) (containing only terms of \( O(\delta^0) \)) is well understood already. In this section we derive homogenized equations including terms up to \( O(\delta^4) \). Additional terms up to \( O(\delta^6) \) are derived for a plane wave propagating in the \( x \)-direction. Our approach is based on the technique used in [1] for one-dimensional wave propagation. We will see that some care is required to extend the technique used there to the case of two-dimensional media of the type shown in Figure 1. We remark that further difficulties arise when considering periodic media in which the coefficients depend on both \( x \) and \( y \); we do not pursue high-order homogenization for such materials here.

It will be convenient to deal with the wave equation (1) in first order form:

\[
\begin{align*}
  p_t + K(y)(u_x + v_y) &= 0, \\
  \rho(y)u_t + p_x &= 0, \\
  \rho(y)v_t + p_y &= 0.
\end{align*}
\]

Here \( u, v \) are the velocities in the \( x \)- and \( y \)-coordinate directions respectively.

We start by introducing a fast scale \( \hat{y} = \delta^{-1}y \). This is the scale on which the material properties vary, so we formally replace the \( \Omega \)-periodic functions \( K(y), \rho(y) \) with \( \lambda \)-periodic functions \( \hat{K}(\hat{y}) \) and \( \hat{\rho}(\hat{y}) \), which are independent of the slow scale \( y \). The dependent variables are assumed to vary on both the fast and slow scales; e.g. \( p = p(x, y, \hat{y}, t) \) and likewise for \( u \) and \( v \). Using the chain rule we find that \( \partial_y \mapsto \partial_{\hat{y}} + \delta^{-1}\partial_{\hat{y}} \). Therefore, system (2) becomes:

\[
\begin{align*}
  \hat{K}^{-1}(y)p_t + u_x + v_y + \delta^{-1}v_{\hat{y}} &= 0, \\
  \hat{\rho}(y)u_t + p_x &= 0, \\
  \hat{\rho}(y)v_t + p_y + \delta^{-1}p_{\hat{y}} &= 0.
\end{align*}
\]

For simplicity, from here on, we omit the hats over the coefficients, with the understanding that all material coefficients are \( \lambda \)-periodic functions of \( \hat{y} \). Next we assume that \( p, u, \) and \( v \) can be written as
power series in $\delta$; e.g. $p(x,y,\hat{y},t) = \sum_{i=0}^{\infty} \delta^i p_i(x,y,\hat{y},t)$. Plugging these expansions into (3) yields

$$K^{-1} \sum_{i=0}^{\infty} \delta^i p_{i,t} + \sum_{i=0}^{\infty} \delta^i u_{i,x} + \sum_{i=0}^{\infty} \delta^i v_{i,y} + \delta^{-1} \sum_{i=0}^{\infty} \delta^i v_{i,\hat{y}} = 0,$$  \hspace{2cm} (4a)

$$\rho \sum_{i=0}^{\infty} \delta^i u_{i,t} + \sum_{i=0}^{\infty} \delta^i p_{i,x} = 0,$$  \hspace{2cm} (4b)

$$\rho \sum_{i=0}^{\infty} \delta^i v_{i,t} + \sum_{i=0}^{\infty} \delta^i p_{i,y} + \delta^{-1} \sum_{i=0}^{\infty} \delta^i p_{i,\hat{y}} = 0,$$  \hspace{2cm} (4c)

where $(\cdot)_{i,x}$ denotes differentiation of $(\cdot)_i$ with respect to $x$. Next we equate terms of the same order in $\delta$; at each order we apply the averaging operator

$$\langle \cdot \rangle := \frac{1}{\lambda} \int_{0}^{\lambda} (\cdot) d\hat{y}$$

(5)

to obtain the homogenized leading order system and corrections to it. Thus the homogenized equations don’t depend on the fast scale $\hat{y}$. Note that the averaging operator averages in one period in $y$; i.e., from 0 to $\Omega$; therefore, in $\hat{y}$ it averages from 0 to $\lambda$.

We present the derivation of the first two corrections in detail. Since the derivation of higher-order terms is similar (but increasingly tedious), we give the higher-order results without detailed derivations. Most of the process is mechanical, but for each system we must make an intelligent ansatz to obtain an expression for the non-homogenized solution of the corresponding system.

2.1 Homogenized $\mathcal{O}(1)$ system

Taking only $\mathcal{O}(\delta^{-1})$ terms in (4) gives

$$v_0,\hat{y} = 0,$$  \hspace{2cm} (6a)

$$p_0,\hat{y} = 0,$$  \hspace{2cm} (6b)

which implies $v_0 = v(x,y,t) =: \hat{v}_0(x,y,t)$ and $p_0 = p(x,y,t) =: \hat{p}_0(x,y,t)$ where we use a bar to denote variables that are independent of the fast scale $\hat{y}$. Note that we can’t assume that $u_0$ is independent of $\hat{y}$; we will see in the following sections that it is not. This is in contrast to the homogenization of similar systems in 1D where, to leading order, all dependent variables are independent of the fast scale

[1].

Taking only the $\mathcal{O}(1)$ terms in (4) gives

$$K^{-1} \hat{p}_{0,t} + u_{0,x} + \hat{v}_{0,y} + v_{1,\hat{y}} = 0,$$  \hspace{2cm} (7a)

$$\rho u_{0,t} + \hat{p}_{0,x} = 0,$$  \hspace{2cm} (7b)

$$\rho \hat{v}_{0,t} + \hat{p}_{0,y} + p_{1,\hat{y}} = 0.$$

(7c)
Next we apply the averaging operator \( \langle \cdot \rangle \) to (7). This eliminates the terms \( v_{1,\hat{y}} \) and \( p_{1,\hat{y}} \), which are periodic with mean zero. We have no way to determine the average of \( \rho u_{0,t} \) because both \( \rho \) and \( u_0 \) depend on \( \hat{y} \). We therefore divide (7b) by \( \rho \) and then apply \( \langle \cdot \rangle \), yielding

\[
K_h^{-1}\bar{p}_{0,t} + \bar{u}_{0,x} + \bar{v}_{0,y} = 0, \\
\rho_h\bar{u}_{0,t} + \bar{p}_{0,x} = 0, \\
\rho_m\bar{v}_{0,t} + \bar{p}_{0,y} = 0.
\]

Here and elsewhere the subscripts \( m \) and \( h \) denote the arithmetic and harmonic average, respectively:

\[
\rho_m := \langle \rho \rangle, \quad \rho_h := \langle \rho^{-1} \rangle^{-1}.
\]

We see already that the effective medium is anisotropic, as indicated by the appearance of these different averages of \( \rho \) in (8b) and (8c). In particular, we see that plane waves propagating parallel to the \( y \)-axis travel with speed \( \sqrt{K_h / \rho_m} \) while plane waves propagating parallel to the \( x \)-axis travel with speed \( \sqrt{K_h / \rho_h} \). We discuss this in more detail in section 3.1.

Combining (7b) and (8b) yields

\[
\bar{u}_0 = \frac{\rho_h}{\rho(\hat{y})}\bar{u}_0,
\]

confirming that \( u_0 \) varies on the fast scale \( \hat{y} \). Importantly, this shows that propagation in \( x \) is affected by the heterogeneity in \( y \) even at the macroscopic scale.

### 2.2 Derivation of \( O(\delta) \) system

Next we obtain expressions for \( u_1, v_1 \) and \( p_1 \) in (7). To do so, we use the following ansatz:

\[
v_1 = \bar{v}_1 + A(\hat{y})\bar{u}_{0,x} + B(\hat{y})\bar{v}_{0,y}, \\
p_1 = \bar{p}_1 + C(\hat{y})\bar{p}_{0,y}.
\]

This ansatz is chosen in order to reduce system (7) to a system of ODEs. Substituting the ansatz (10), the relation (9), and the homogenized leading order system (8) into the the \( O(1) \) system (7) we get:

\[
A_{\hat{y}} = K^{-1}K_h - \rho^{-1} \rho_h, \\
B_{\hat{y}} = K^{-1}K_h - 1, \\
C_{\hat{y}} = \rho\rho_m^{-1} - 1.
\]

Equations (11) represent boundary value ODEs with the normalization conditions that \( \langle A \rangle = \langle B \rangle = \langle C \rangle = 0 \). Note that \( \langle A_{\hat{y}} \rangle = \langle B_{\hat{y}} \rangle = \langle C_{\hat{y}} \rangle = 0 \), which implies that \( A, B \) and \( C \) are periodic. To solve
these boundary value problems we must specify the material functions \( \rho, K \). In Appendix A we work out the resulting values for a layered medium and a sinusoidally-varying medium.

Taking only the \( \mathcal{O}(\delta) \) terms in (4) gives

\[
\begin{align*}
K^{-1}p_{1,t} + u_{1,x} + v_{1,y} + v_{2,y} &= 0, \\
\rho u_{1,t} + p_{1,x} &= 0, \\
\rho v_{1,t} + p_{1,y} + p_{2,y} &= 0.
\end{align*}
\]

Substituting the ansatz for \( v_1 \) and \( p_1 \) from (10) into (12) and averaging gives

\[
\begin{align*}
K^{-1}h \bar{p}_{1,t} + \bar{u}_{1,x} + \bar{v}_{1,y} &= -\langle K^{-1}C \rangle \bar{p}_{0,yt}, \\
\rho_h \bar{u}_{1,t} + \bar{p}_{1,x} &= -\rho_h \langle \rho^{-1}C \rangle \bar{p}_{0,xy}, \\
\rho_m \bar{v}_{1,t} + \bar{p}_{1,y} &= -\langle \rho A \rangle \bar{u}_{0,xt} - \langle \rho B \rangle \bar{v}_{0,yt}.
\end{align*}
\]

For many materials, including the piecewise-constant and sinusoidal materials that we consider in Section 4, it turns out that (11) leads to \( \langle K^{-1}C \rangle = \langle \rho^{-1}C \rangle = \langle \rho A \rangle = \langle \rho B \rangle = 0 \). We will proceed under this assumption; then we obtain:

\[
\begin{align*}
K^{-1}h \bar{p}_{1,t} + \bar{u}_{1,x} + \bar{v}_{1,y} &= 0, \\
\rho_h \bar{u}_{1,t} + \bar{p}_{1,x} &= 0, \\
\rho_m \bar{v}_{1,t} + \bar{p}_{1,y} &= 0.
\end{align*}
\]

Since the boundary conditions are imposed in the leading order homogenized system (8), system (14) should be solved with homogeneous Dirichlet boundary conditions; therefore, its solution vanishes:

\[
\bar{u}_1 = \bar{v}_1 = \bar{p}_1 = 0.
\]

2.3 Derivation of \( \mathcal{O}(\delta^2) \) system

Taking \( \bar{u}_1 = \bar{v}_1 = \bar{p}_1 = 0 \), we make the following ansatz for the \( \mathcal{O}(\delta) \) solutions \( v_2 \) and \( p_2 \):

\[
\begin{align*}
v_2 &= \bar{v}_2 + D(\hat{y})\bar{u}_{0,xy} + E(\hat{y})\bar{v}_{0,yy}, \\
p_2 &= \bar{p}_2 + F(\hat{y})\bar{p}_{0,yy} + H(\hat{y})\bar{p}_{0,xx},
\end{align*}
\]

From (12b) we get \( u_{1,t} = -\rho^{-1}p_{1,x} \). The ansatz for \( p_1 \) from (10b) gives \( u_{1,t} = -\rho^{-1}C(\bar{p}_{0,x})_y \) and using the homogenized leading order equation (8b) we get \( u_{1,t} = \rho^{-1}p_h C(\bar{u}_{0,y})_t \). Finally we get an expression for the non-homogenized solution \( u_1 \):

\[
u_1 = \rho^{-1}p_h C \bar{u}_{0,y}.
\]
homogenized system (8) and the ODEs for \( A \), \( B \) and \( C \) from (11). Finally, we use the fact that the fast variable coefficients must vanish to obtain

\[
\begin{align*}
D\dot{y} &= K^{-1}K_hC - \rho^{-1}\rho h C - A, \\
E\dot{y} &= K^{-1}K_hC - B, \\
F\dot{y} &= \rho h^{-1}B - C, \\
H\dot{y} &= \rho\rho h^{-1}A,
\end{align*}
\]

with the conditions \( \langle D \rangle = \langle E \rangle = \langle F \rangle = \langle H \rangle = 0 \).

From (4) we take \( \mathcal{O}(\delta^2) \) terms to get:

\[
\begin{align*}
K^{-1}p_{2,t} + u_{2,x} + v_{2,y} + v_{3,y} &= 0, \\
\rho u_{2,t} + p_{2,x} &= 0, \\
\rho v_{2,t} + p_{2,y} + p_{2,gy} &= 0.
\end{align*}
\]

Substituting the ansatz for \( v_2 \) and \( p_2 \) from (16) into (19) and averaging yields

\[
\begin{align*}
K_h^{-1}\bar{p}_2,t + \bar{u}_{2,x} + \bar{v}_{2,y} &= \alpha_1(\bar{u}_{0,xyy} + \bar{v}_{0,yyy}) + \alpha_2(\bar{u}_{0,xxx} + \bar{v}_{0,xyy}), \\
\rho_h \bar{u}_{2,t} + \bar{p}_{2,x} &= \beta_1 \bar{p}_{0,xyy} + \beta_2 \bar{p}_{0,xxx}, \\
\rho_m \bar{v}_{2,t} + \bar{p}_{2,y} &= \gamma_1 \bar{p}_{0,yyy} + \gamma_2 \bar{p}_{0,xyy}.
\end{align*}
\]

Expressions for the coefficients \( \alpha, \beta, \gamma \) appear in appendix A.

### 2.4 Higher order corrections

Following similar, but more involved steps, we find the \( \mathcal{O}(\delta^3) \) and \( \mathcal{O}(\delta^4) \) corrections.

#### 2.4.1 \( \mathcal{O}(\delta^3) \) homogenized correction

The third homogenized correction is:

\[
\begin{align*}
K_h^{-1}\bar{p}_{3,t} + \bar{u}_{3,x} + \bar{v}_{3,y} &= -\langle K^{-1}C \rangle \bar{p}_{2,yt} - \langle K^{-1}N \rangle \bar{p}_{0,gyyt} - \langle K^{-1}P \rangle \bar{p}_{0,xyyt}, \\
\rho_h \bar{u}_{3,t} + \bar{p}_{3,x} &= -\rho_h \langle \rho^{-1}C \rangle \bar{p}_{2,xy} - \rho_h \langle \rho^{-1}N \rangle \bar{p}_{0,xyyy} - \rho_h \langle \rho^{-1}P \rangle \bar{p}_{0,xyxy}, \\
\rho_m \bar{v}_{3,t} + \bar{p}_{3,y} &= -\langle \rho A \rangle \bar{u}_{2,xt} - \langle \rho B \rangle \bar{v}_{2,yt} \\
&\quad - \langle \rho I \rangle \bar{u}_{0,xyyt} - \langle \rho J \rangle \bar{u}_{0,xxyt} - \langle \rho L \rangle \bar{v}_{0,xyyt} - \langle \rho M \rangle \bar{v}_{0,gyyt},
\end{align*}
\]
where the fast-variable functions $I(\hat{y}), J(\hat{y}), L(\hat{y}), M(\hat{y}), N(\hat{y}), P(\hat{y})$ are solutions of the BVPs

\begin{align}
I_{\hat{y}} &= K^{-1}K_h(F - K_h(\langle K^{-1}F \rangle) - \rho^{-1}\rho_h(F - \rho_h(\langle \rho^{-1}F \rangle)) - D, \quad (22a) \\
J_{\hat{y}} &= K^{-1}K_h(H - K_h(\langle K^{-1}H \rangle) - \rho^{-1}\rho_h(H - \rho_h(\langle \rho^{-1}H \rangle)), \quad (22b) \\
L_{\hat{y}} &= K^{-1}K_h(H - K_h(\langle K^{-1}H \rangle), \quad (22c) \\
M_{\hat{y}} &= K^{-1}K_h(F - K_h(\langle K^{-1}F \rangle)) - E, \quad (22d) \\
N_{\hat{y}} &= \rho\rho_m^{-1}(E - \rho_m^{-1}\langle \rho E \rangle) - F, \quad (22e) \\
P_{\hat{y}} &= \rho\rho_h^{-1}(D - \rho_h^{-1}\langle \rho D \rangle) - H, \quad (22f)
\end{align}

with the conditions $\langle I \rangle = \langle J \rangle = \langle L \rangle = \langle M \rangle = \langle N \rangle = \langle P \rangle = 0$ and, for the discontinuous media, $I, J, L, M \in C^0(\hat{y} \in [0, 1])$. For the two types of media considered in this work all coefficients on the right hand side of (21) vanish. Since the boundary conditions are fulfilled by the leading order homogenized system (8), the third homogenized correction vanishes; i.e.,

\begin{equation}
\bar{u}_3 = \bar{v}_3 = \bar{p}_3 = 0. \quad (23)
\end{equation}

### 2.4.2 $O(\delta^4)$ homogenized correction

The fourth correction is given by:

\begin{align}
K_h^{-1}p_{4,t} + \bar{u}_{4,x} + \bar{v}_{4,y} &= \alpha_1(\bar{u}_{2,xyy} + \bar{v}_{2,yyy}) + \alpha_2(\bar{u}_{2,xxx} + \bar{v}_{2,xyy}) \\
&\quad + \alpha_3(\bar{u}_{0,xyyyyy} + \bar{v}_{0,yyyyyy}) + \alpha_4(\bar{u}_{0,xxxxxx} + \bar{v}_{0,xxxyy}) \\
&\quad + \alpha_5(\bar{u}_{0,xxxxxy} + \bar{v}_{0,xxxyyy}), \quad (24a) \\
\rho_h\bar{u}_{4,t} + \bar{p}_{4,x} &= \beta_1\bar{p}_{2,xyy} + \beta_2\bar{p}_{2,xxx} \\
&\quad + \beta_3\bar{p}_{0,xyyyyy} + \beta_4\bar{p}_{0,xxxxxx} + \beta_5\bar{p}_{0,xxxyy}, \quad (24b) \\
\rho_m\bar{v}_{4,t} + \bar{p}_{4,y} &= \gamma_1\bar{p}_{2,xyy} + \gamma_2\bar{p}_{2,xxxx} \\
&\quad + \gamma_3\bar{p}_{0,xyyyyy} + \gamma_4\bar{p}_{0,xxxxxx} + \gamma_5\bar{p}_{0,xxxyy}. \quad (24c)
\end{align}

Expressions for the coefficients $\alpha, \beta, \gamma$ are given in appendix A

### 2.5 Combined homogenized equations

Once we have the homogenized leading order system and the homogenized corrections we can combine them into a single system. This is done by taking $p := \langle p_0 + \delta p_1 + \ldots \rangle = \bar{p}_0 + \delta\bar{p}_1 + \ldots$ and similarly for $u$ and $v$. Combining homogenized systems (8), (14), (20), (21) and (24) we obtain:

\begin{align}
K_h^{-1}p_t + u_x + v_y &= \delta^2[\alpha_1(u_{xyy} + v_{yyy}) + \alpha_2(u_{xxx} + v_{xyy})] \\
&\quad + \delta^4[\alpha_3(u_{xyyyyy} + v_{yyyyyy}) + \alpha_4(u_{xxxxxx} + v_{xxxyy})] \quad (25a) \\
&\quad + \delta^4[\alpha_5(u_{xxxxxy} + v_{xxxyyy})], \\
\rho_h u_t + p_x &= \delta^2[\beta_1p_{xyy} + \beta_2p_{xxx}] + \delta^4[\beta_3p_{xyyyyy} + \beta_4p_{xxxxxx} + \beta_5p_{xxxyy}], \quad (25b) \\
\rho_m v_t + p_y &= \delta^2[\gamma_1p_{xyy} + \gamma_2p_{xxx}] + \delta^4[\gamma_3p_{xyyyyy} + \gamma_4p_{xxxxxx} + \gamma_5p_{xxxyy}], \quad (25c)
\end{align}
where expressions for the coefficients are given in appendix A. Unlike the lowest-order homogenized equation (7), this system is dispersive. The dispersion depends on the direction of propagation.

In general, each coefficient of a $\delta^n$ term in (25) contains a matching factor $\lambda^n$. As a result, each $\delta^n$ term on the right hand side of (25) is proportional to $\Omega^n$ (and independent of $\lambda$). This explains the observation in [7] that the homogenized equations are valid for any choice of the material period $\Omega$. Nevertheless, (25) is only valid for small $\delta$; i.e., for relatively long wavelengths $\lambda > \Omega$.

### 3 Effective dispersion relations

System (25) is a homogeneous, linear hyperbolic system with constant coefficients. Hence its solutions can be completely described by considering the dispersion relation, which relates frequency and wavenumber for a plane wave:

$$p(x, y, t) = p_0 e^{i(k \cdot x - \omega t)}.$$  

(26)

Here $p_0$ is the amplitude, $\omega$ is the angular frequency and $k$ is the wave vector. Let $k = k \langle k_x, k_y \rangle$ with $k_x = \cos \theta, k_y = \sin \theta$, where $\theta$ is the direction of propagation. Because (25) describes a medium that is anisotropic and dispersive, the speed of propagation of a plane wave depends on both the angle $\theta$ and the wavenumber magnitude $k$.

We can combine (25) into a single second-order equation by differentiating (25a) with respect to $t$, differentiating (25b) and (25c) with respect to $x$ and $y$ respectively, and equating mixed partial derivatives. By substituting (26) into the result, we obtain the effective dispersion relation, up to $O(\delta^4)$:

$$\omega^2 = \frac{K_h}{\rho_h \rho_m} k^2 (k_x^2 \rho_m + k_y^2 \rho_h) + \delta^2 \frac{K_h}{\rho_h \rho_m} k^4 \left[ (\alpha_2 + \beta_2) k_x^2 + (\alpha_1 + \beta_1) k_y^2 \right]$$

$$+ k_y^2 \rho_h \left[ (\alpha_2 + \gamma_2) k_x^2 + (\alpha_1 + \gamma_1) k_y^2 \right] + \delta^4 \frac{K_h}{\rho_h \rho_m} k^6 [k_x^2 \rho_m \left( (\alpha_4 + \alpha_2 \beta_2 - \beta_4) k_x^4 \right. ]$$

$$- (\alpha_5 - \alpha_2 \beta_1 - \alpha_1 \beta_2 + \beta_5) k_x^4 k_y^2 + (\alpha_3 + \alpha_1 \beta_1 - \beta_3) k_y^4)$$

$$+ k_y^2 \rho_h \left[ (\alpha_4 + \alpha_2 \gamma_2 - \gamma_5) k_x^4 - (\alpha_5 - \alpha_2 \gamma_1 - \alpha_1 \gamma_2 + \gamma_4) k_x^2 k_y^2 \right. + (\alpha_3 + \alpha_1 \gamma_1 - \gamma_3) k_y^4) \].$$

(27)

It is important to keep in mind that although (27) is accurate to $O(\delta^4)$, it is obtained by homogenization and is not expected to be valid for wavelengths shorter than $\Omega$, the medium period.

#### 3.1 Effective sound speed

Taking only $O(1)$ terms in (27) we obtain the effective sound speed:

$$c_{\text{eff}} = \omega/k = \sqrt{\frac{K_h}{\rho_h} k_x^2 + \frac{K_h}{\rho_m} k_y^2}.$$  

(28)

The effective sound speed, which indicates the speed of very long wavelength perturbations, depends on the direction of propagation. For normally incident waves ($\theta = \pi/2$), we have $c_{\text{eff}} = \sqrt{K_h/\rho_m}$.
Figure 2: Polar plot of the effective sound speed (28) with $K_h = 1$, $\rho_h = 1$ and $\rho_m = 1$ (blue), $\rho_m = 2$ (red), $\rho_m = 4$ (cyan), $\rho_m = 8$ (black).

which is the effective sound speed in a 1D layered medium [9]. For transverse waves ($\theta = 0$), we have $c_{\text{eff}} = \sqrt{K_h/\rho_h}$. Since the harmonic average is less than or equal to the arithmetic average, long-wavelength normal waves never travel faster than their transverse wave counterparts. This is intuitively reasonable since transverse propagating waves undergo no reflection.

In Figure 2 we plot $c_{\text{eff}}$ as a function of $\theta$ for material parameters $K_h = 1$, $\rho_h = 1$ and different values of $\rho_m$. When $\rho_h = \rho_m$, the sound speed is the same in all directions so we obtain the blue line in Figure 2. As $\rho_m$ increases (corresponding to strong impedance variation and thus more reflection), the effective speed in $y$ decreases and we obtain the red, cyan and black lines in Figure 2. In Figure 3 we take the initial condition

$$p_0(x, y) = 5e^{-\frac{x^2+y^2}{2\sigma^2}}$$

$$u_0 = v_0 = 0$$

with $\sigma^2 = 10$ and a piecewise-constant medium

$$K(y), \rho(y) = \begin{cases} (K_A, \rho_A) & \text{if } (y - \lfloor y \rfloor - \frac{1}{2}) < 0, \\ (K_B, \rho_B) & \text{if } (y - \lfloor y \rfloor - \frac{1}{2}) > 0. \end{cases}$$

with $\rho_A = 8 + \sqrt{56}$, $\rho_B = 8/\rho_A$ and $K_A = K_B = 1$ which gives $K_h = \rho_h = 1$ and $\rho_m = 8$, corresponding to the black line in Figure 2. We show the homogenized leading order pressure (left) and the finite volume pressure (right) at $t = 5$. The predicted anisotropic behavior is observed in both solutions.

3.2 Normally and transversely incident waves

The dispersion relation has some special properties for waves that are aligned with the coordinate axes. Recall the definitions of normal and transverse waves illustrated in Figure 1.
Figure 3: Homogenized leading order solution (left) and finite volume solution (right) for $p$ at $t = 5$
using $\rho_A = 8 + \sqrt{56}$, $\rho_B = 8/\rho_A$ and $K_A = K_B = 1$ which gives $K_h = \rho_h = 1$ and $\rho_m = 8$.
This corresponds to the effective sound speed distribution shown in figure 2 (black line). Note that
the horizontal lines in the right figure are the effect of the medium structure, and not of numerical
resolution (which is much finer).

For initial data that is constant in $x$ (i.e., normal waves), (25) reduces to a one-dimensional
equation:

$$K_h^{-1} p_t + v_y = \delta^2 \alpha_1 v_{yyy} + \delta^4 \alpha_3 v_{yyyyy},$$  \hspace{1cm} (31a)

$$\rho_m u_t + p_y = \delta^2 \gamma_1 p_{yyy} + \delta^4 \gamma_3 p_{yyyyy}. \hspace{1cm} (31b)$$

This system was obtained previously in [9, 1, 7]. The coefficients on the right hand side are all
proportional to the difference of squared impedance; for instance

$$\alpha_1 = - \left( Z_A^2 - Z_B^2 \right) \frac{(K_A - K_B)}{192K_m^2 \rho_m} \lambda^2.$$

(32)

Hence all of the dispersive terms in (31) vanish when the impedance is constant (see Appendix A).

For initial data that is constant in $y$ (i.e. tranverse waves), (25) simplifies to

$$K_h^{-1} p_t + u_x = \delta^2 \alpha_2 u_{xxx} + \delta^4 \alpha_4 u_{xxxx} + \delta^6 \alpha_6 u_{xxxxx}, \hspace{1cm} (33a)$$

$$\rho_m u_t + p_x = \delta^2 \beta_2 p_{xxx} + \delta^4 \beta_4 p_{xxxx} + \delta^6 \beta_6 p_{xxxxx}. \hspace{1cm} (33b)$$

Here we have derived and included additional 6th-order corrections, because this case will be of
particular interest in what follows. In this case, all coefficients on the right hand side are proportional
to the difference of squared sound speeds; for instance

$$\alpha_2 = - \left( c_A^2 - c_B^2 \right) \frac{(K_A - K_B)}{192K_m^2 \rho_m} \lambda^2 \hspace{1cm} (34)$$

Hence all of the dispersive terms in (33) vanish when the sound speed is constant. (see Appendix A).

Thus we see that the role played by impedance for normal waves corresponds to the role played
by sound speed for transverse waves, and vice-versa.
Wave behavior in 2D layered periodic media

In the remainder of the paper, we examine wave propagation in different layered periodic materials, comparing the behavior predicted by (27) with full solutions of the variable-coefficient wave equation (1). Solutions of the homogenized equations are obtained using a Fourier spectral collocation method in space and 4th order Runge-Kutta integration in time [10]. Solutions of the variable-coefficient problem are obtained using PyClaw [4], including numerical methods described in [6, 3, 8].

For most of the experiments, we use the piecewise-constant medium (30). Exact expressions for the homogenization coefficients for this medium are given in Appendix A.1. We consider three classes of material parameters: variable impedance (only), variable sound speed (only), and parameters for which both the impedance and the sound speed vary.

4.1 Variable-impedance media

We first consider a medium (30) with $K_A = \rho_A = 4$ and $K_B = \rho_B = 1$. This medium has constant sound speed $c = 1$ and variable impedance $Z_A = 4, Z_B = 1$. Figure 4 shows the sound speed as a function of wavenumber based on the effective dispersion relation (27) for this medium. In this case, only reflective dispersion (or $Z$-dispersion) is present, affecting primarily waves in the normal direction. Waves traveling in the transverse ($x$) direction are unaffected.

Waves propagating obliquely to the material heterogeneity also experience $Z$-dispersion since they encounter interfaces leading to reflections. We take the initial condition (29) with $\sigma^2 = 2$. In figure 5, we show a surface plot of the finite volume pressure at $t = 95$ and slices along the $x$- (solid blue) and $y$-axis (solid red). Only the upper-right quadrant is shown, since the solution is invariant under vertical or horizontal reflection. We also plot slices of the homogenized solution given by (25) (dashed lines).
Figure 5: Propagation of a perturbation in a $Z$-dispersive medium. Left: surface plot of pressure (computed by finite volumes); Right: pressure slices along the $x$-axis (blue) and $y$-axis (red); the dashed lines correspond to the solution of the homogenized equations.

Figure 6: Speed $c = \omega(k)/k$ as a function of wavenumber $k$ for a medium with variable sound speed and constant impedance.

4.2 Variable-sound speed media

Next we consider a piecewise-constant medium (30) with

$$K_A = \frac{1}{\rho_A} = 5/8, \quad (35a)$$
$$K_B = \frac{1}{\rho_B} = 5/2. \quad (35b)$$

This medium has constant impedance $Z = 1$ and variable sound speed $c_A = 5/8, c_B = 5/2$. Figure 6 shows the sound speed as a function of wavenumber based on the effective dispersion relation (27) for this medium. In this case, only diffractive dispersion (or \textit{c-dispersion}) is present, affecting primarily waves traveling in the transverse direction. Waves traveling in the normal ($y$) direction are unaffected.
4.2.1 Propagation of a plane-wave perturbation

In this section we show the effect of $c$-dispersion on an initial perturbation that varies only in $x$:

$$p_0(x, y) = 10e^{-\frac{x^2}{10}} \quad u_0 = v_0 = 0.$$  

The homogenized equations used are given by (33). To demonstrate that the homogenized equations are valid for more general media, we consider both the piecewise constant medium (30) and a smoothly varying medium

$$K(y) = \frac{K_A + K_B}{2} + \frac{K_A - K_B}{2} \sin(2\pi y),$$  

$$\rho(y) = \frac{1}{K(y)},$$  

again with the material parameters (35). For this medium we numerically solve the BVPs defining the homogenization coefficients. Mathematica and Matlab files to solve for these expressions can be found at http://github.com/ketch/effective_dispersio

Figure 7 shows the results for the two media, after the initial pulse has traveled a distance of more than 1000 material layers. Homogenized solutions of differing orders are shown to demonstrate the increasing accuracy of the high-order homogenized approximations. These results are compared with the finite volume solution of the variable-coefficient wave equation (1), which is averaged in the $y$-direction (since the homogenized solution represents this average). The agreement is very good, and close examination of the dispersive tail shows that the approximations are increasingly accurate.

In Figure 8, we show a quiver plot of the velocity superimposed on a color plot of the strain. The presence of diffraction is evident in the variation in vertical velocity (the $v$ values have been increased ten times to make this more evident).

4.2.2 Propagation of a general perturbation

Next we consider the propagation of a pressure perturbation that is localized in both directions. The initial condition is a Gaussian (29) with $\sigma^2 = 2$. Figure 9 shows the results for the piecewise-constant and sinusoidal media. Again, the agreement between the variable-coefficient and homogenized solutions is very good.

4.3 General media

In general, both the impedance and the sound speed may vary in a periodic medium. In this case, normal waves experience $Z$-dispersion, transverse waves experience $c$-dispersion, and oblique waves experience both effects. In Figure 10, we consider such a medium with $K_A = \rho_A = 1$, $K_B = 1.5$ and $\rho_B = 2.5$. For this set of parameters, $Z$-dispersion is stronger, but in general either effect can be dominant, depending on whether the impedance or the sound speed varies more strongly. Figure 11 shows solutions for an initially Gaussian pressure perturbation. Again, we observe excellent agreement between the variable-coefficient and homogenized solutions.
Figure 7: Homogenized equations with different order corrections vs. $y$-averaged finite volume solution for (a) piecewise-constant and (b) sinusoidal media. On the right we show a close-up of the dispersive tail where the differences in the homogenized corrections are more noticeable.

Figure 8: Velocity field (vectors) superimposed on strain (color) for a pulse traveling in a $c$-dispersive medium. The $y$-component of the velocity has been scaled up 10x to emphasize the diffraction.
Figure 9: A Gaussian perturbation propagating in a (a) piecewise-constant and (b) sinusoidal $c$-dispersive medium. Left: surface plot of pressure (by finite volumes); Right: slices along the $x$-axis (blue) and $y$-axis (red); dashed lines are homogenized solutions.

Figure 10: Speed $c = \omega(k)/k$ for a medium with $Z$- and $c$-dispersion. On the left we show a surface plot and on the right slices along the $x$- (blue) and $y$-axis (red).
Figure 11: Propagation of a Gaussian perturbation in a medium with variable $Z$ and $c$. Left: surface plot of pressure (by finite volumes); Right: slices along the $x$-axis (blue) and $y$-axis (red); dashed lines are homogenized solutions.

Figure 12: Speed $c = \omega(k)/k$ for a medium that macroscopically behaves isotropic for long wavelengths. Left: the $O(\delta^2)$ effective dispersion relation; middle and right: the $O(\delta^4)$ effective dispersion relation.

### 4.3.1 Effectively isotropic media

The materials considered in this work have a strongly anisotropic microstructure, and in general this leads to macroscopically anisotropic behavior. However, by careful choice of material parameters, we can design a medium that is strongly heterogeneous yet effectively isotropic in terms of both the effective (long-wavelength) sound speed and the leading order dispersion (i.e., to $O(\delta^2)$). Due to the approximate symmetry between $Z$- and $c$-variation pointed out in Section 3.2, this can be achieved by taking $Z_A = c_A$ and $Z_B = c_B$.

We take $K_A = 16$ and $\rho_A = \rho_B = K_B = 1$, which yields $Z_A = c_A = 1$ and $Z_B = c_B = 4$. Figure 12 shows the dispersion relation for this medium. The plot on the left shows the effective dispersion relation obtained if only terms up to $O(\delta^2)$ are kept in (27). Up to this order, the effective dispersion is isotropic. For long enough wavelengths and short times the $O(\delta^4)$ terms are negligible and the $O(\delta^2)$ approximation is valid. The middle and right plots show the effective dispersion including $O(\delta^4)$ terms, which lead to significant anisotropy for high frequencies.
Figure 13: 2D wave propagation in a medium with variable $Z$ and $c$, for an initial condition with (a) only low frequency components and (b) some high frequency components. We show surface plots (left) of the pressure (computed by finite volumes) at $t = 65$ and slices along the $x$- (solid blue) and $y$-axis (solid red). We also show slices of the homogenized solution (dashed lines).

For low frequencies and short times the $O(\delta^2)$ solution is a valid approximation so the material behaves isotropically on the macroscale. To illustrate this, we use the initial condition (29) with $\sigma^2 = 5$. Figure 13a shows the pressure and slices along each axis for the variable-coefficient and homogenized solutions. For higher-frequency waves, the $O(\delta^2)$ approximation is less accurate and the solution behaves more anisotropically. This is demonstrated in figure 13b where we use the initial condition (29) with $\sigma^2 = 2$.

5 Discussion

As pointed out in Section 3.2, impedance and sound speed variations seem to play almost dual roles with respect to normal and transverse wave propagation. Equations (31) for normal propagation can be combined to yield the dispersive approximation

$$K_h^{-1} \rho_m p_{ttt} - p_{yy} = \delta^2 (\alpha_1 + \gamma_1) p_{xxxx} + O(\delta^4)$$
where for the piecewise-constant medium (30) the leading dispersive term coefficient is

$$\alpha_1 + \gamma_1 = \frac{\left(Z_A^2 - Z_B^2\right)^2}{192K_m^2\rho_m^2} \lambda^2.$$ 

Meanwhile, equations (33) for transverse propagation can be combined to yield

$$K_h^{-1} \rho_h p_{tt} - p_{yy} = \delta^2(\alpha_2 + \beta_2)p_{xxxx} + \mathcal{O}(\delta^4)$$

where for the piecewise-constant medium (30) the leading dispersive term coefficient is

$$\alpha_2 + \beta_2 = \frac{\left(c_A^2 - c_B^2\right)^2}{192K_m^2\rho_m^2} \rho_h \lambda^2.$$ 

The similarity is striking. It is also striking that all dispersive coefficients in the homogenized equations for normal propagation vanish when \(Z\) is constant, while all dispersive coefficients in the homogenized equations for transverse propagation vanish when \(c\) is constant. We have not tried to prove that this holds in general, but it is true for all media that we have investigated.

A Coefficients of homogenized equations

In this appendix we give the coefficients of the homogenized systems (25), (33) and (31). They are:

$$\alpha_1 = K_h \langle K^{-1}F \rangle,$$

$$\alpha_2 = K_h \langle K^{-1}H \rangle,$$

$$\alpha_3 = K_h \langle K^{-1}U \rangle - K_h^2 \langle K^{-1}F \rangle^2,$$

$$\alpha_4 = K_h \langle K^{-1}W \rangle - K_h^2 \langle K^{-1}H \rangle^2,$$

$$\alpha_5 = K_h \langle K^{-1}V \rangle - 2K_h^2 \langle K^{-1}F \rangle \langle K^{-1}H \rangle,$$

$$\alpha_6 = K_h \langle K^{-1}B \rangle + K_h^3 \langle K^{-1}H \rangle^3 - 2K_h^2 \langle K^{-1}H \rangle \langle K^{-1}W \rangle,$$

$$\beta_1 = -\rho_h \langle \rho^{-1}F \rangle,$$

$$\beta_2 = -\rho_h \langle \rho^{-1}H \rangle,$$

$$\beta_3 = -\rho_h \langle \rho^{-1}U \rangle,$$

$$\beta_4 = -\rho_h \langle \rho^{-1}W \rangle,$$

$$\beta_5 = -\rho_h \langle \rho^{-1}V \rangle,$$

$$\beta_6 = -\rho_h \langle \rho^{-1}B \rangle,$$
and

\[
\begin{align*}
\gamma_1 &= \rho_m^{-1} \langle \rho E \rangle, \\
\gamma_2 &= \rho_h^{-1} \langle \rho D \rangle, \\
\gamma_3 &= \rho_m^{-1} \langle \rho T \rangle - \rho_m^{-2} \langle \rho E \rangle^2, \\
\gamma_4 &= \rho_h^{-1} \langle \rho Q \rangle + \langle \rho^{-1}F \rangle \langle \rho D \rangle + \rho_m^{-1} \langle \rho S \rangle - \rho_m^{-1} \rho_h^{-1} \langle \rho E \rangle \langle \rho D \rangle, \\
\gamma_5 &= \langle \rho D \rangle \langle \rho^{-1}H \rangle + \rho_h^{-1} \langle \rho R \rangle,
\end{align*}
\]

where the functions \(D(\hat{y}), E(\hat{y}), F(\hat{y})\) and \(H(\hat{y})\) are solutions of the BVP (18), \(Q(\hat{y}), R(\hat{y}), S(\hat{y}), T(\hat{y}), U(\hat{y}), V(\hat{y})\) and \(W(\hat{y})\) are solutions of

\[
\begin{align*}
Q_{\hat{y}} &= K^{-1}K_h \left( N - CK_h \langle K^{-1}F \rangle \right) - \rho^{-1}\rho_h \left( N - C\rho_h \langle \rho^{-1}F \rangle \right) - I, \\
R_{\hat{y}} &= K^{-1}K_h \left( P - CK_h \langle K^{-1}H \rangle \right) - \rho^{-1}\rho_h \left( P - C\rho_h \langle \rho^{-1}H \rangle \right) - J, \\
S_{\hat{y}} &= K^{-1}K_h \left( P - CK_h \langle K^{-1}H \rangle \right) - L, \\
T_{\hat{y}} &= K^{-1}K_h \left( N - CK_h \langle K^{-1}F \rangle \right) - M, \\
U_{\hat{y}} &= \rho\rho_m^{-1} \left( M - B\rho_m^{-1} \langle \rho E \rangle \right) - N, \\
V_{\hat{y}} &= \rho A \langle \rho^{-1}F \rangle + \rho\rho_h^{-1}I + \rho\rho_m^{-1} \left( L - B\rho_h^{-1} \langle \rho D \rangle \right) - P, \\
W_{\hat{y}} &= \rho A \langle \rho^{-1}H \rangle + \rho\rho_h^{-1}J,
\end{align*}
\]

with the normalization conditions that \(\langle Q \rangle = \langle R \rangle = \langle S \rangle = \langle T \rangle = \langle U \rangle = \langle V \rangle = \langle W \rangle = 0\). For the piecewise constant medium we also require \(Q, R, S, T, U, V, W \in C^0(\hat{y} \in [0, 1])\). The functions \(A(\hat{y}), B(\hat{y})\) and \(C(\hat{y})\) are solutions of the BVP (11). Finally, \(\tilde{B}\) is solved by:

\[
\begin{align*}
\tilde{A}_{\hat{y}} &= K^{-1}K_h \left( \tilde{W} - K_h \langle K^{-1}W \rangle - K_hH \langle K^{-1}H \rangle + K_h^2 \langle K^{-1}H \rangle^2 \right) \\
&\quad - \rho^{-1}\rho_h \left( \tilde{W} - \rho_h \langle \rho^{-1}W \rangle - \rho_hH \langle \rho^{-1}H \rangle + \rho_h^2 \langle \rho^{-1}H \rangle^2 \right), \\
\tilde{B}_{\hat{y}} &= \rho A \langle \rho^{-1}W \rangle + \rho J \langle \rho^{-1}H \rangle + \rho\rho_h^{-1}\tilde{A},
\end{align*}
\]

with the normalization conditions \(\langle \tilde{A} \rangle = \langle \tilde{B} \rangle = 0\). For the piecewise constant medium we also impose \(\tilde{A}, \tilde{B} \in C^0(\hat{y} \in [0, 1])\).

These fast-variable functions and the coefficients themselves depend on the details of the medium. Below we give further details for the piecewise constant medium (30) and the sinusoidal medium (36).

**A.1 Piecewise constant medium**

For the piecewise constant medium it is easy to obtain the fast-variable functions and the coefficients in closed form; however, most of them are too cumbersome to present here. Therefore, we just show
the coefficients of the first non-zero correction and refer to \url{http://github.com/ketch/effective dispersion_RR} for Mathematica files where the rest of the coefficients can be found. The coefficients of the first non-zero correction for the piecewise medium are:

\[
\begin{align*}
\alpha_1 &= -\frac{(K_A - K_B)}{192 K_m^2} \cdot \frac{(Z_A^2 - Z_B^2)}{\rho_m} \lambda^2, \\
\alpha_2 &= -\frac{(K_A - K_B)}{192 K_m^2} \cdot \frac{(c_A^2 - c_B^2)}{\rho_m^2} \lambda^2, \\
\beta_1 &= \frac{(\rho_A - \rho_B)}{192 K_m} \cdot \frac{(Z_A^2 - Z_B^2)}{\lambda^2}, \\
\beta_2 &= \frac{(\rho_A - \rho_B)}{192 K_m} \cdot \frac{(c_A^2 - c_B^2)}{\lambda^2}, \\
\gamma_1 &= -\frac{(\rho_A - \rho_B)}{192 K_m} \cdot \frac{(Z_A^2 - Z_B^2)}{\rho_m^2} \lambda^2, \\
\gamma_2 &= -\frac{(\rho_A - \rho_B)}{192 K_m} \cdot \frac{(c_A^2 - c_B^2)}{\lambda^2}.
\end{align*}
\]

### A.2 Sinusoidal medium

For the sinusoidal medium and for more general $y$-periodic media it is difficult to find closed expressions for the fast-variable functions and for the coefficients. Therefore, we solve the boundary value problems and compute the coefficients numerically. Details can be found at \url{http://github.com/ketch/effective dispersion_RR}. The files available there can easily be modified to produce coefficients for other media.

The numerically computed coefficients for the sinusoidal medium are (taking $\lambda = 1$):

\[
\begin{align*}
\alpha_1 &= 2.2656 \times 10^{-10}, \\
\alpha_2 &= -1.3208 \times 10^{-2}, \\
\alpha_3 &= -1.8927 \times 10^{-11}, \\
\alpha_4 &= -1.8172 \times 10^{-4}, \\
\alpha_5 &= 1.3398 \times 10^{-3}, \\
\alpha_6 &= 6.0711 \times 10^{-6}, \\
\beta_1 &= 2.9249 \times 10^{-4}, \\
\beta_2 &= -1.1033 \times 10^{-2}, \\
\beta_3 &= -5.6345 \times 10^{-7}, \\
\beta_4 &= -2.3474 \times 10^{-5}, \\
\beta_5 &= 1.1465 \times 10^{-3}, \\
\beta_6 &= 6.9060 \times 10^{-6}, \\
\gamma_1 &= 2.2656 \times 10^{-10}, \\
\gamma_2 &= 1.2843 \times 10^{-2}, \\
\gamma_3 &= -1.8927 \times 10^{-11}, \\
\gamma_4 &= -1.3391 \times 10^{-3}, \\
\gamma_5 &= -1.6986 \times 10^{-4}.
\end{align*}
\]
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