Symmetric Monopoles

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Abstract We discuss the spectral curves and rational maps associated with $SU(2)$ Bogomolny monopoles of arbitrary charge $k$. We describe the effect on the rational maps of inverting monopoles in the plane with respect to which the rational maps are defined, and discuss the monopoles invariant under such inversion. We define the strongly centred monopoles, and show they form a geodesic submanifold of the $k$-monopole moduli space. The space of strongly centred $k$-monopoles invariant under the cyclic group of rotations about a fixed axis, $C_k$, is shown to consist of several surfaces of revolution, generalizing the two surfaces obtained by Atiyah and Hitchin in the 2-monopole case. Geodesics on these surfaces give a novel type of $k$-monopole scattering.

We present a number of curves in $TP_1$ which we conjecture are the spectral curves of monopoles with the symmetries of a regular solid. These conjectures are based on analogies with Skyrmions.

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0. Introduction

In recent years, there has been considerable interest in Bogomolny monopoles, which are particle-like solitons in a Yang-Mills-Higgs theory in three spatial dimensions. In this paper, we shall only consider $SU(2)$ Bogomolny monopoles in (flat) $\mathbb{R}^3$, which are the finite energy solutions of the Bogomolny equations (1.2) [1]. Solutions are labelled by their magnetic charge, a non-negative integer $k$, and are physically interpreted as static, non-linear superpositions of $k$ unit charge magnetic monopoles. There is a $4k$-dimensional manifold of (gauge inequivalent) $k$-monopole solutions, known as the $k$-monopole moduli space $M_k$, and on this there is a naturally defined Riemannian metric, which is hyperkähler [2].

For monopoles moving at modest speeds compared with the speed of light, it is a good approximation to model $k$-monopole dynamics by geodesic motion on the moduli space $M_k$. This was conjectured some time ago [3], and the consequences explored in some detail [2, 4]. Very recently, the validity of the geodesic approximation has been proved analytically by Stuart [5].

Most studies of Bogomolny monopoles have been concerned either with the general structure of the $k$-monopole moduli space $M_k$, and its metric, or with a detailed study of $M_2$ and the geodesics on it, which describe 2-monopole scattering and bound orbits. Little work has been done on $k$-monopole dynamics for $k > 2$. (The case $k = 0$ is trivial, and if $k = 1$, one has a single monopole which moves along a line at constant speed.) In this paper, we investigate classes of $k$-monopole solutions which are invariant under various symmetry groups. We consider monopoles invariant under inversion in a fixed plane, monopoles invariant under a cyclic group of rotations about a fixed axis, and monopoles invariant under the symmetry groups of the regular solids, that is, the tetrahedral, octahedral and icosahedral groups. The existence of $k$-monopoles with cyclic symmetry was previously shown in [6]. A submanifold of the moduli space $M_k$, consisting of all $k$-monopoles invariant under a fixed symmetry group, is a geodesic submanifold. We can therefore describe various examples of monopole scattering with symmetry, by finding geodesics on such submanifolds.

In Sections 1 and 2 we summarize the various ways of characterizing monopoles, and recall the spectral curves and rational maps associated with monopoles. In Section 3 we show how the rational map changes when a monopole is inverted in the plane with respect to which the rational map is defined, and we investigate the monopoles which are invariant under this inversion. In Section 4 we consider in detail the holomorphic geometry associated with the centre of a monopole. We state (for the first time precisely), both in terms of the spectral curve and the rational map, the condition for a monopole to be centred. We also define the total phase of a monopole, and introduce the notion of
a strongly centred monopole – one whose centre is at the origin and whose total phase is 1. In Section 5 we show that the space of strongly centred \( k \)-monopoles is a hyperkähler submanifold of \( M_k \), of dimension \( 4k - 4 \), which is totally geodesic in \( M_k \).

Spectral curves of \( k \)-monopoles are curves in \( TP_1 \), the tangent bundle to the complex projective line, satisfying a number of constraints. In Section 6 we consider the action of symmetry groups on general curves in \( TP_1 \), and present various classes of curves with cyclic symmetry, and with the symmetries of regular solids. In Section 7 we consider the rational maps associated with \( k \)-monopoles which are symmetric under the cyclic group \( C_k \), and this gives some information on which curves in \( TP_1 \) with cyclic symmetry are, in fact, spectral curves. The strongly centred monopoles with \( C_k \) symmetry are parametrized by a number of geodesic surfaces of revolution in the moduli space \( M_k \). We deduce, using the geodesic approximation, a class of novel \( k \)-monopole scattering processes, symmetric under the cyclic group \( C_k \). In these, there is a simultaneous collision of \( k \) unit charge monopoles in a plane, with an \( l \)-monopole and a \((k - l)\)-monopole emerging back-to-back along the line through the \( k \)-monopole centre, perpendicular to the initial plane. Here \( l \) can be any integer in the range \( 0 < l < k \). The outgoing monopole clusters both become axisymmetric about the line separating them, in the limit of infinite separation. A purely planar \( k \)-monopole scattering process, with \( C_k \) symmetry, is also possible.

The investigation of rational maps with cyclic symmetry suggests that some monopoles and their spectral curves have the symmetry of a regular solid. We make some precise conjectures about this, and in the last Section we briefly summarize some results on Skyrmions, another type of static soliton in a three-dimensional field theory, which tend to support these conjectures.

Finally a warning is necessary for the reader who wishes to delve into the literature on this subject. There are a number of places in the theory of monopoles where one has to make choices and establish conventions. Most of these are to do with the orientation of \( \mathbb{R}^3 \) and the induced complex structure on the twistor space \( TP_1 \) of all oriented lines in \( \mathbb{R}^3 \). Different authors have made different conventions, and minor sign inconsistencies can appear to result if the literature is only read in a cursory manner.

1. Monopoles and the five-fold way

As outlined in the Introduction, we wish to present some results on monopoles and their scattering, with various symmetries. Before doing this we need to review some of the theory behind monopoles and the different points of view from which they can be studied. Further details of this material can be found in the book by Atiyah and Hitchin [2] and in the references contained therein.

To define a monopole we start with a pair \((A, \phi)\) consisting of a connection 1-form \( A \)
on \( \mathbb{R}^3 \) with values in \( LSU(2) \), the Lie algebra of \( SU(2) \), and a function \( \phi \) (the Higgs field) from \( \mathbb{R}^3 \) into \( LSU(2) \). The value of the Yang-Mills-Higgs energy on this pair is defined to be

\[
E(A, \phi) = \int_{\mathbb{R}^3} (|F_A|^2 + |\nabla_A \phi|^2) d^3 x
\]

(1.1)

where \( F_A = dA + A \wedge A \) is the curvature of \( A \), \( \nabla_A \phi = d\phi + [A, \phi] \) is the covariant derivative of the Higgs field, and the norms are taken using the usual norms on 1-forms and 2-forms and an invariant, positive definite inner product on \( LSU(2) \). We call the integrand in \( E \) the energy density of the monopole. A standard trick, due to Bogomolny, can be used to show that the Yang-Mills-Higgs energy is minimised by the solutions of the Bogomolny equations

\[
\star F_A = \nabla_A \phi
\]

(1.2)

where \( \star \) is the usual Hodge star on forms on \( \mathbb{R}^3 \). These equations and indeed the energy are invariant under gauge transformations, where the gauge group \( G \) of all maps \( g \) from \( \mathbb{R}^3 \) to \( SU(2) \) acts by

\[
(A, \phi) \mapsto (gAg^{-1} - dgg^{-1}, g\phi g^{-1}).
\]

(1.3)

Finiteness of the Yang-Mills-Higgs energy, and the Bogomolny equations, imply certain asymptotic boundary conditions at infinity in \( \mathbb{R}^3 \) on the pair \((A, \phi)\) which are spelt out in detail in [2]. In particular, \(|\phi| \to c\) for some constant \( c \) which cannot change with time. Following [2], we fix \( c = 1 \).

A monopole, then, is a gauge equivalence class of solutions to the Bogomolny equations subject to these boundary conditions. In some suitable gauge there is a well-defined Higgs field at infinity

\[
\phi^\infty: S^2_\infty \to S^2 \subset LSU(2)
\]

(1.4)

going from the two sphere of all oriented lines through the origin in \( \mathbb{R}^3 \) to the unit two-sphere in \( LSU(2) \). This Higgs field at infinity has a degree, or winding number, which, for a solution of the Bogomolny equations, is a positive integer \( k \) called the magnetic charge of the monopole.

Before discussing the moduli space of all solutions of the Bogomolny equations we need to be a little more precise and talk about framed monopoles. We say a pair \((A, \phi)\) is framed if

\[
\lim_{x_3 \to \infty} \phi(0, 0, x_3) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]

(1.5)

The gauge transformations fixing such pairs are those \( g \) with \( \lim_{x_3 \to \infty} g(0, 0, x_3) \) diagonal. Notice that every monopole can be gauge transformed until it is framed. So the space of monopoles modulo gauge transformations is the same as the space of framed monopoles.
modulo those gauge transformations that fix them. We define a framed gauge transformation to be one such that \( \lim_{x_3 \to \infty} g(0,0,x_3) = 1 \). The quotient of the set of all framed monopoles of charge \( k \) by the group of framed gauge transformations is a manifold called the moduli space of (framed) monopoles of charge \( k \) and denoted \( M_k \). The constant diagonal gauge transformations (a copy of \( U(1) \)) act on \( M_k \) and the quotient is called the reduced moduli space \( N_k \). This action is not quite free, the element \(-1\) acts trivially so the group \( U(1)/\{\pm 1\} \) acts freely on \( M_k \).

The dimension of \( M_k \) is \( 4k \) and these parameters can be understood as follows. In the case that \( k = 1 \) there is a spherically symmetric monopole called the Bogomolny-Prasad-Sommerfield (BPS) monopole, or unit charge monopole. Its Higgs field has a single zero at the origin, and its energy density is peaked there so it is reasonable to think of the origin as the centre or location of the monopole. The Bogomolny equations are translation invariant so this monopole can be translated about \( \mathbb{R}^3 \) and also rotated by the circle of constant diagonal gauge transformations. This in fact generates all of \( M_1 \) which is therefore diffeomorphic to \( \mathbb{R}^3 \times S^1 \). The coordinates on \( M_1 \) specify the location of the monopole and what can be thought of as an internal phase.

If one was optimistic one would be tempted to think that \( M_k \) consists of unit charge monopoles located at \( k \) points with \( k \) internal phases. Even more optimistically one might hope that, as the Higgs field of the unit charge monopole vanishes at its point of location, these \( k \) points are where the Higgs field vanishes. This is not true in general but it is asymptotically correct. There is an asymptotic region of the moduli space consisting of approximate superpositions of \( k \) unit charge monopoles located at \( k \) widely separated points and with \( k \) arbitrary phases. Although it is not possible, in general, to assign to a charge \( k \) monopole \( k \) points or locations in \( \mathbb{R}^3 \) it is possible to assign to the monopole a centre which can be thought of as the average of the locations of the \( k \) particles making up the monopole. The important property of this centre is that if we act on the monopole by an isometry of \( \mathbb{R}^3 \) the centre moves by the same isometry. It is also possible to assign to a \( k \)-monopole a total phase; this is essentially the product of the phases of the \( k \) unit charge monopoles. Whereas in the case of the centre we are essentially adding up all the individual locations and dividing by \( k \), to get a phase for the monopole we would want to multiply the individual phases and take a \( k \)th root. Taking a \( k \)th root of a complex number is, of course, ambiguous and we have to content ourselves instead with being able to define the product of all the phases – the total phase. If we act on the monopole by a constant gauge transformation corresponding to an element \( \mu \) of \( U(1) \) then the total phase changes by \( \mu^{2k} \). The power of two here is because it is \( U(1)/\{\pm 1\} \) which acts freely on the monopole, and the power of \( k \) is because this is the total phase.

The natural metric on the moduli space \( M_k \) is obtained from the \( L_2 \) metric on the
fields \((A, \phi)\), taking due account of gauge invariance. Since a large part of the moduli space \(M_k\) describes \(k\) well-separated unit charge monopoles, many geodesics on \(M_k\) correspond to the scattering of \(k\) unit charge monopoles, and we shall discuss below some particularly symmetric cases of such scattering.

It is not easy to study charge \(k\) monopoles directly in terms of their fields \((A, \phi)\). However, there are various ways of transforming monopoles to other types of mathematical objects. The five approaches to monopoles can be summarised by the diagram:

\[
\begin{array}{c}
\text{Monopoles} \\
\downarrow \quad \text{Rational maps} \\
\text{Nahm data} \\
\downarrow
\end{array} \quad \leftrightarrow \quad \begin{array}{c}
\text{Holomorphic bundles} \\
\downarrow \quad \text{Rational maps} \\
\text{Spectral curves}
\end{array}
\]

In the top left hand corner of the diagram we have monopoles in \(\mathbb{R}^3\) as we have just defined them. There is a twistor theory for monopoles and the result of applying this shows that monopoles are equivalent to a certain class of holomorphic bundles on the so-called mini-twistor space \(TP_1\), the tangent bundle to the complex projective line \(P_1\). This is indicated by the top horizontal arrow. A careful analysis of the boundary conditions of the monopole shows that the holomorphic bundle is determined by an algebraic curve, called the spectral curve. Monopoles that differ only by a constant diagonal gauge transformation have the same spectral curve. These results are due to Hitchin \([7, 8]\). The vertical arrow on the left is a transformation due to Nahm \([9]\) which turns the monopole into a solution of an ordinary differential equation, called Nahm’s equation, on an interval in \(\mathbb{R}\). It can be shown that Nahm’s equation is equivalent to a Lax pair equation and hence one expects to find associated to it an algebraic curve and indeed that curve is the spectral curve \([10]\). The relationship between solutions of Nahm’s equations and spectral curves was explained by Hitchin in \([8]\). Common to all these approaches to monopoles is a rational map, that is a map

\[
R(z) = \frac{p(z)}{q(z)} \quad (1.6)
\]

from \(\mathbb{C}\) to \(\mathbb{C} \cup \infty\), where \(p\) and \(q\) are polynomials.

The holomorphic bundles, spectral curves or solutions of Nahm’s equations that charge \(k\) monopoles give rise to all satisfy some usually rather nasty constraints. However the rational maps have the enormous advantage that they are easy to describe. One just writes down a polynomial \(p\) of degree less than \(k\) divided by a monic (leading coefficient = 1) polynomial \(q\) of degree \(k\) which has no factor in common with \(p\). We shall denote by \(R_k\) the space of all these based rational maps. Donaldson’s theorem then assures us that any
such rational map arises from some unique charge $k$ monopole [11]. The disadvantage of this approach is that there is no explicit way of describing the monopole, given its rational map. Moreover defining the rational map, as we shall see in the next Section, requires choosing a line and an orthogonal plane in $\mathbb{R}^3$, to define an isomorphism $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, and this breaks the symmetries of the problem. Whereas the Bogomolny equations are invariant under all the isometries of $\mathbb{R}^3$, the transformation to a rational map commutes only with those isometries that preserve the direction of the line. The action of these isometries is given as follows. Let $\lambda \in U(1)$ and $w \in \mathbb{C}$ define a rotation and translation, respectively, in the plane $\mathbb{C}$. Let $t \in \mathbb{R}$ define a translation perpendicular to the plane and let $\mu \in U(1)$ define a constant diagonal gauge transformation. A rational map $R(z)$ then transforms under the composition of all these transformations to

$$\tilde{R}(z) = \mu^2 \exp(2t)\lambda^{-2k}R(\lambda^{-1}(z - w)).$$

Note that this is slightly different to the action described in [2, eq. (2.11)]. This is one of those places discussed in the Introduction where different conventions give rise to different signs.

It would be nice to have a description of the action of the full isometry group on the space of all rational maps. In particular, this would settle easily the conjectures we make below about the existence of monopoles with the symmetries of regular solids. We cannot give this. However, we can do two useful things. Firstly we can describe what happens to the rational map when we invert a monopole in the chosen plane. Secondly we can determine the centre and the total phase of a monopole from its rational map.

Our results will be proved below, after we have introduced the necessary twistor machinery, but they are so simple to describe we will do it here. Let $p(z)/q(z)$ be the unique representation of the rational map with $q$ monic. Then the rational map of the inverted monopole is $I(p)(z)/q(z)$, where $I(p)$ is the unique polynomial of degree less than $k$ such that $I(p)(z)p(z) = 1 \mod q(z)$. If the roots $\beta_1, \ldots, \beta_k$ of $q$ are distinct then $I(p)$ is uniquely determined by the fact that $I(p)(\beta_i)p(\beta_i) = 1$ for all $i = 1, \ldots, k$. For the centre and total phase denote by $q_0$ the average of all the roots of $q$ and by $\Delta(p, q)$ the resultant of $p$ and $q$. The centre of the monopole is $(q_0, (1/2k)\log|\Delta(p, q)|)$ and the total phase is $\Delta(p, q)|\Delta(p, q)|^{-1}$.

It is worth noting that these results appear to be inconsistent with Proposition 3.12 of [2]. There it is argued that if

$$R(z) = \frac{p(z)}{q(z)} = \sum_i \frac{\alpha_i}{z - \beta_i}$$

is the rational map of a charge $k$ monopole, which consists of $k$ well-separated unit charge monopoles, then (using our conventions) the individual monopoles are approximately located at the points $(\beta_i, (1/2)\log|\alpha_i|)$ and have phases $\alpha_i|\alpha_i|^{-1}$. This description implies
that inversion fixes the $\beta_i$ and inverts the $\alpha_i$, whereas we show below in Section 3 that
inversion fixes the $\beta_i$ and inverts the $p(\beta_i)$ which are related to the $\alpha_i$ by
\[
\alpha_i = \frac{p(\beta_i)}{\prod_{j \neq i}(\beta_i - \beta_j)}.
\] (1.9)

From our results it appears more likely that the individual monopoles are located at the
points $(\beta_i, (1/2) \log |p(\beta_i)|)$ and have phases $p(\beta_i)|p(\beta_i)|^{-1}$ but we have no proof of this.

2. Monopoles and rational maps

The rational map of a monopole was originally described by Donaldson in terms of
solutions to Nahm’s equations [11]. Hurtubise then showed how it relates to scattering in
$\mathbb{R}^3$ and to the spectral curve of the monopole [12]. It will be convenient for our purposes
to use the description in terms of spectral curves.

The holomorphic bundle of a $k$-monopole is defined by Hitchin as follows [7]. Let $\gamma$
be an oriented line in $\mathbb{R}^3$ and let $\nabla_\gamma$ denote covariant differentiation using the connection $A$ along $\gamma$. Hitchin considers the ordinary differential equation
\[
(\nabla_\gamma - i\phi)v = 0
\] (2.1)
where $v : \gamma \to \mathbb{C}^2$. The vector space $E_\gamma$ of all solutions to equation (2.1) is two-dimensional
and the union of all these spaces forms a rank two smooth complex vector bundle $E$ over
the space of all oriented lines in $\mathbb{R}^3$. It is shown by Hitchin that this space of all oriented
lines is a complex manifold, in fact isomorphic to $TP_1$. Hitchin then shows that $E$ has a
holomorphic structure if the monopole satisfies the Bogomolny equations. The bundle $E$
has two holomorphic sub-bundles $E^\pm_\gamma$ defined by defining their fibres $(E^\pm_\gamma)_\gamma$ at $\gamma$ to be the
space of solutions that decay as $\pm \infty$ is approached along the line $\gamma$. The set of $\gamma$ where
$(E^+_\gamma)_\gamma = (E^-_\gamma)_\gamma$, so there is a solution decaying at both ends, forms a curve $S$ in $TP_1$ called
the spectral curve of the monopole. It is possible to show that a decaying solution decays
exponentially so the spectral curve is also the set of all lines along which there is an $L_2$
solution. Intuitively one should think of the spectral lines as being the lines going through
the locations of the monopoles. In the case of charge 1, the spectral lines are precisely
those going through the centre of the monopole.

If we describe a typical point in $P_1$ by homogeneous coordinates $[\zeta_0, \zeta_1]$ then we can
cover $P_1$, in the usual way, by two open sets $U_0$ and $U_1$ where $\zeta_0$ and $\zeta_1$ are non-zero,
respectively. On the set $U_0$ we introduce the coordinate $\zeta = \zeta_1/\zeta_0$. Let us also denote by
$U_0$ and $U_1$ the pre-images of these sets under the projection map from $TP_1$ to $P_1$. Then a
tangent vector $\eta \partial/\partial \zeta$ at $\zeta$ in $U_0$ can be given coordinates $(\eta, \zeta)$. These coordinates allow
us to describe an important holomorphic line bundle $L$ on $TP_1$, introduced by Hitchin [7,
which has transition function $\exp(\eta/\zeta)$ on the overlap of $U_0$ and $U_1$. Similarly for any complex number $\lambda$ we define the bundle $L^\lambda$ by the transition function $\exp(\lambda \eta/\zeta)$. Finally, if $n$ is any integer we define the line bundle $L^\lambda(n)$ to be the tensor product of $L^\lambda$ with the $n$-th power of the pull-back under projection $TP_1 \to P_1$ of the dual of the tautological bundle on $P_1$. This has transition function $\zeta^{-n} \exp(\lambda \eta/\zeta)$. The line bundle $L^0$ is clearly trivial so we denote it by $O$, and $L^0(n)$ is denoted by $O(n)$.

To avoid the potential ambiguity in what we mean by ‘transition function’ let us be more explicit. The line bundle $L^\lambda(n)$ has non-vanishing holomorphic sections $\chi_0$ and $\chi_1$ over $U_0$ and $U_1$ respectively and for points in $U_0 \cap U_1$ these satisfy

$$\chi_0 = \zeta^{-n} \exp(\frac{\lambda \eta}{\zeta}) \chi_1. \quad (2.2)$$

If we consider an arbitrary holomorphic section $f$ of this line bundle its restriction to $U_0$ and $U_1$ can be written as $f = f_0 \chi_0$ and $f = f_1 \chi_1$ respectively where $f_0$ and $f_1$ are holomorphic functions on $U_0$ and $U_1$. As a consequence of equation (2.2) these functions must satisfy

$$f_0 = \zeta^n \exp(-\frac{\lambda \eta}{\zeta}) f_1 \quad (2.3)$$

at points in the intersection $U_0 \cap U_1$. In fact it follows immediately that a holomorphic section of $L^\lambda(n)$ is exactly equivalent to a pair of such holomorphic functions $f_0$ and $f_1$ defined on $U_0$ and $U_1$ and satisfying (2.3) on $U_0 \cap U_1$.

With these definitions we can present the results of Hitchin that we need. The sub-bundles $E_{1}^\pm$ satisfy $E_{1}^\pm \simeq L^\pm 1(-k)$ and the quotients satisfy $E/E_{1}^\pm \simeq L_{1}^\mp 1(k)$. For a framed monopole there are explicit isomorphisms so we shall write $=$ instead of $\simeq$. The curve $S$ is defined by the vanishing of the map $E_{1}^+ \to E/E_{1}^-$ and hence by a section of $(E_{1}^+)^* \otimes E/E_{1}^- = O(2k)$. In terms of the coordinates $(\eta, \zeta)$, $S$ is defined by an equation of the form

$$P(\eta, \zeta) \equiv \eta^k + \eta^{k-1} a_1(\zeta) + \ldots + \eta a_{k-1}(\zeta) + a_k(\zeta) = 0, \quad (2.4)$$

where, for $1 \leq r \leq k$, $a_r(\zeta)$ is a polynomial in $\zeta$ of degree at most $2r$.

The space $TP_1$ has a real structure $\tau$, namely, the anti-holomorphic involution defined by reversing the orientation of the lines in $\mathbb{R}^3$. In coordinates it takes the form $\tau(\eta, \zeta) = (-\bar{\eta}/\zeta^2, -1/\zeta)$. The curve $S$ is fixed by this involution, so we say that it is real. The reality of $S$ implies that for $1 \leq r \leq k$,

$$a_r(\zeta) = (-1)^r \zeta^{2r} a_r(\frac{-1}{\zeta}). \quad (2.5)$$

If $k = 1$ the spectral curve has the form

$$\eta = (x_1 + ix_2) - 2x_3 \zeta - (x_1 - ix_2)\zeta^2 \quad (2.6)$$
where \( x = (x_1, x_2, x_3) \) is any point in \( \mathbb{R}^3 \), [7, eq. (3.2)]. Such a curve is called a real section as it defines a section of the bundle \( TP_1 \to P_1 \), and is real in the sense given above. In terms of the geometry of \( \mathbb{R}^3 \) this curve is the set of all oriented lines through the point \( x \), so it is the spectral curve of a BPS monopole located at \( x \). We refer to this curve as the “star” at \( x \).

In [7, 8] Hitchin lists all the properties that a curve in \( TP_1 \) has to satisfy to be a spectral curve. We are interested in one of these here. From the definition of the spectral curve we see that over the spectral curve the line bundles \( E_1^+ \) and \( E_1^- \) coincide as sub-bundles of \( E_1 \); in particular they must be isomorphic. This is equivalent to saying that the line bundle \( E_1^+ \otimes (E_1^-)^* = L^2 \) is trivial over the curve or that it admits a non-vanishing holomorphic section \( s \). The real structure \( \tau \) can be lifted to an anti-holomorphic, conjugate linear map between the line bundles \( L^2 \) and \( L^{-2} \) and hence the section \( s \) can be conjugated to define a new (holomorphic) section \( \tau(s) = \tau \circ s \circ \tau \) of \( L^{-2} \) over \( S \). Tensoring these defines a section \( \tau(s)s \) of \( L^{-2} \otimes L^2 = \mathcal{O} \) and because \( S \) is compact and connected this is a constant. Because of the framing this constant will be 1. Notice that given only \( S \) and the fact that \( L^2 \) is trivial over \( S \), if we can choose a section \( s \) such that \( \tau(s)s = 1 \) then it is unique up to multiplication by a scalar of modulus one. This circle ambiguity in the choice of \( s \) corresponds to the framing of the monopole. In fact, let \( \mu \) be a complex number of modulus one corresponding to a constant diagonal gauge transformation with diagonal entries \( \mu \) and \( \mu^{-1} \). Then it is possible to follow through the proof in Hitchin [7, pp. 593-4] and show that if we phase rotate a framed monopole by \( \mu \), the isomorphism \( E_1^+ \to L(-k) \) is multiplied by \( \mu \) and the isomorphism \( E_1^- \to L^*(-k) \) is multiplied by \( \mu^{-1} \). The section \( s \) of \( E_1^+ \otimes (E_1^-)^* = L^2 \) is therefore multiplied by \( \mu^2 \). Notice that this is consistent with the fact that the group \( U(1)/\{\pm 1\} \) acts freely on the moduli space \( M_k \) of framed monopoles.

To define the rational map we fix the fibre \( F \) of \( TP_1 \to P_1 \) where \( \zeta = 0 \) and identify it with \( \mathbb{C} \). This corresponds to picking an orthogonal splitting of \( \mathbb{R}^3 \) as \( \mathbb{C} \times \mathbb{R} \). Each point \( z \) in \( \mathbb{C} \) is identified with a point in \( F \) by setting \( z = \eta \), and hence with an oriented line, the line \( \{(x_1, x_2, x_3) \mid x_3 \in \mathbb{R}\} \) with \( z = x_1 + ix_2 \). The intersection of \( F \) with \( S \) defines \( k \) points counted with multiplicity and \( q(z) \) is defined to be the unique monic polynomial of degree \( k \) which has these \( k \) points as its roots. Thus \( q(z) = P(z, 0) \), where \( P \) is given by eq.(2.4). Recall from eq.(2.3) that a holomorphic section \( s \) of the bundle \( L^2 \) is determined locally by functions \( s_0 \) and \( s_1 \), on \( U_0 \cap S \) and \( U_1 \cap S \) respectively, such that

\[
s_0(\eta, \zeta) = \exp\left(\frac{-2\eta}{\zeta}\right)s_1(\eta, \zeta).
\]  

Let \( p(z) \) be the unique polynomial of degree \( k - 1 \) such that \( p(z) = s_0(z, 0) \mod q(z) \). The rational map of the monopole is then \( R(z) = p(z)/q(z) \). If the roots of \( q(z) \) are
distinct complex numbers $\beta_1, \ldots, \beta_k$ then the polynomial $p(z)$ is determined by the fact that $p(\beta_i) = s_0(\beta_i, 0)$ for all $i = 1, \ldots, k$.

Notice that we have departed at this point from the convention of Hurtubise [12]. There the rational map is defined using $\tau(s)$. We will see in the next Section precisely what this means but it may be helpful here to make a brief remark about the construction of the rational map as scattering data in $\mathbb{R}^3$. More details are given in [12] and [2]. The points where $S$ intersects $F$ correspond, of course, to the lines in the $x_3$-direction admitting a solution of eq.(2.1) decaying at both ends. Assume these lines are distinct and label them by the corresponding complex numbers $\beta_i$. Pick for each line a solution $v(\beta_i, x_3)$ decaying at both ends. In the regions where $x_3$ is large positive and large negative there are choices of asymptotically flat gauge such that

$$\lim_{x_3 \to \infty} (x_3)^{-k/2}e^{x_3}v(\beta_i, x_3) = v_i^+(1) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(2.8)

and

$$\lim_{x_3 \to -\infty} (x_3)^{-k/2}e^{-x_3}v(\beta_i, x_3) = v_i^-(0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

(2.9)

The rational map with our conventions is determined by

$$p(\beta_i) = \frac{v_i^+}{v_i^-}. \quad (2.10)$$

This agrees with the results stated in Chapter 16 of ref.[2], although Hurtubise’s conventions give $p(\beta_i) = v_i^- / v_i^+$.

3. Inverting rational maps

Consider the inversion map $I: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $I(x_1, x_2, x_3) = (x_1, x_2, -x_3)$. This inverts $\mathbb{R}^3$ in the $(x_1, x_2)$ plane. The inversion map induces an anti-holomorphic map on the twistor space $TP_1$ which we shall denote by the same symbol and which in the standard coordinates on $TP_1$ is

$$I(\eta, \zeta) = (\frac{-\bar{\eta}}{\zeta^2}, \frac{1}{\zeta}). \quad (3.1)$$

To see this note that the real section defined by the point $I(x_1, x_2, x_3)$ has equation

$$\eta = (x_1 + ix_2) + 2x_3\zeta - (x_1 - ix_2)\zeta^2. \quad (3.2)$$

So a point $I(\eta, \zeta)$ is on this curve if and only if

$$\frac{-\bar{\eta}}{\zeta^2} = (x_1 + ix_2) + 2x_3\frac{1}{\zeta} - (x_1 - ix_2)\frac{1}{\zeta^2}. \quad (3.3)$$
Conjugating this equation and clearing the denominators we recover

\[ \eta = (x_1 + ix_2) - 2x_3\zeta - (x_1 - ix_2)\zeta^2, \quad (3.4) \]

the equation of the real section defined by the point \((x_1, x_2, x_3)\). This confirms the formula for \(I\). Notice that \(I\) is very similar to the real structure \(\tau\); in fact \(I \circ \tau(\eta, \zeta) = (\eta, -\zeta)\).

If we invert the monopole defined by the spectral curve \(S\) and section \(s\) we obtain a new curve \(I(S)\) and a new section \(I(s)\). The definition of \(I(S)\) is straightforward; it is just the image of \(S\) under the map \(I\). We shall consider \(I(s)\) in a moment. Because \(\tau(S) = S\) it follows that \((\eta, \zeta)\) is on \(I(S)\) precisely when \((\eta, -\zeta)\) is on \(S\). In particular, the intersection of \(I(S)\) and the fibre \(F\) is just the intersection of \(S\) and \(F\), since \(\zeta = 0\). So if we denote by \(I(p)\) and \(I(q)\) the numerator and denominator of the rational map for the inverted monopole, we see that \(I(q) = q\).

Now consider the section \(s\). Notice that both \(\tau\) and \(I\) interchange the two coordinate patches \(U_0\) and \(U_1\). The section \(\tau(s)\) is defined locally by

\[ \tau(s)_0(\eta, \zeta) = \bar{s}_1(\tau(\eta, \zeta)) \quad , \quad \tau(s)_1(\eta, \zeta) = \bar{s}_0(\tau(\eta, \zeta)) \quad (3.5) \]

and the section \(I(s)\) by

\[ I(s)_0(\eta, \zeta) = \bar{s}_1(I(\eta, \zeta)) \quad , \quad I(s)_1(\eta, \zeta) = \bar{s}_0(I(\eta, \zeta)) \quad (3.6) \]

Hence \(I(p)\) is defined by

\[ I(p) = I(s)_0(\eta, 0) \mod q = \bar{s}_1 \circ \tau(\eta, 0) \mod q \quad (3.7) \]

using the fact that \(\tau(\eta, 0) = I(\eta, 0)\). From the relation \(\tau(s)s = 1\) and equation (3.5) it follows that \((\bar{s}_1 \circ \tau)s_0 = 1\) and hence

\[ I(p)p = (\bar{s}_1 \circ \tau(s, 0))s_0(\eta, 0) \mod q = 1 \mod q. \quad (3.8) \]

Eq.(3.8), and the requirement that the degree of \(I(p)\) is less than \(k\), determine \(I(p)\) uniquely. If the roots of \(q\) are the distinct complex numbers \(\beta_1, \ldots, \beta_k\), a useful alternative way of obtaining \(I(p)\) is to notice that it is the unique polynomial of degree less than \(k\) such that \(I(p)(\beta_i)p(\beta_i) = 1\) for all \(i = 1, \ldots, k\).

It is interesting to consider the subset of monopoles that are invariant under inversion. Their spectral curves are given by polynomials \(P(\eta, \zeta)\) which are even in \(\zeta\). Their rational maps satisfy \(p^2 = 1 \mod q\), so that \(I(p) = p\). Let us calculate how many such rational maps there are, for a given \(q\). If the roots \(\beta_i\) are distinct, then \(p(\beta_i) = \pm 1\) for all \(i\). To
understand what happens when the roots are not necessarily all distinct, let the distinct roots of \( q \) be denoted by \( \beta_1, \ldots, \beta_d \) and assume that \( \beta_i \) has multiplicity \( n_i \). Of course, \( n_1 + \cdots + n_d = k \). Now, given a polynomial \( p \) of degree less than \( k \) we can associate to it a list of its values and the values of its derivatives up to order \( n_i - 1 \) at the points \( \beta_i \). This defines a linear map

\[
P_k \rightarrow \mathbb{C}^k
\]  

(3.9)

from \( P_k \), the space of polynomials of degree less than \( k \), to \( \mathbb{C}^k \). We claim that this map is a linear isomorphism. Notice that both these spaces have dimension \( k \) so it is enough to check that the map has no kernel. However if a polynomial \( p \) is killed by this map then it must contain a factor \( (z - \beta_i)^{n_i} \) for each \( i \). Hence it is possible to divide \( p \) by a polynomial of degree \( k \), but this means \( p \equiv 0 \). Notice that in the case of distinct roots the construction of the inverse to the map in equation (3.9) is Lagrange interpolation. Assume now that \( p^2 = 1 \mod q \). Then by repeated differentiation we deduce that \( p(\beta_i) = \pm 1 \) and that \( p^{(1)}(\beta_i), \ldots, p^{(n_i-1)}(\beta_i) \) are all zero. Because the map in equation (3.9) is an isomorphism there is, for each choice of signs of the \( p(\beta_i) \), a unique \( p \) satisfying these conditions. Conversely, given such a \( p \) it follows that \( p^2 - 1 \) has a zero of degree at least \( n_i \) at \( \beta_i \) so that it is divisible by each of the factors of \( q \) and hence \( p^2 = 1 \mod q \). So given \( q \) there are \( 2^d \) possible choices of \( p \) making \( p / q \) invariant under inversion, where \( d \) is the number of distinct roots of \( q \).

Let us denote by \( IM_k \) the set of monopoles invariant under inversion. In general this has several components which we denote by \( \{ IM_k^m : m = 0, 1, \ldots, k \} \). \( IM_k^m \) is the component of \( IM_k \) for which (while the roots of \( q \) are distinct) \( m \) values of \( p(\beta_i) \) are \( +1 \), and \( k - m \) values are \( -1 \). Note that \( IM_k^m \) and \( IM_k^{k-m} \) are isomorphic; one is obtained from the other by multiplying \( p \) by \(-1 \). The simplest of the components is \( IM_k^0 \). Here \( p(z) \equiv 1 \), so the rational maps are of the form

\[
R(z) = \frac{1}{q(z)}.
\]  

(3.10)

Clearly, \( IM_k^0 \) is a submanifold of the moduli space \( M_k \). We now prove that \( IM_k \) and hence all of the \( IM_k^m \) are submanifolds. We have seen that the rational maps of the inversion symmetric monopoles satisfy the equation

\[
p^2 = 1 \mod q.
\]  

(3.11)

We would like to formulate this equation as the zero set of a smooth map of maximal rank so that we can apply the implicit function theorem to show that \( IM_k \) is a submanifold. Note that this equation holds precisely when \( p^2 - 1 \) is zero in the \( k \)-dimensional vector space

\[
V_q = \mathbb{C}[z] / < q >
\]  

(3.12)
where \( \langle q \rangle \) is the ideal generated by \( q \). This space depends on the point \( q \) so we think of it as the fibre of a vector bundle \( V \to R_k \), where the fibres in fact depend on \( q \) but not on \( p \). We define a section of \( V \to R_k \) by

\[
h(p, q) = p^2 - 1 + \langle q \rangle \in V_q.
\] (3.13)

Now the points of interest, \( IM_k \), are where the image of the section \( h \) intersects the zero submanifold \( V^0 \subset V \), in other words where the section vanishes. For a function the condition for its zero set to be a submanifold is, of course, that the derivative of the function at every point on the zero set should be onto. The condition in the case of a section of a vector bundle is similar but we are interested only in the vertical component of the derivative. To define the vertical component of the derivative of \( h \), note that at a point on the zero submanifold we can write the tangent space to \( V \) as a direct sum of the tangent space to the fibre, which is naturally identified with the fibre, and the tangent space to \( V^0 \), which is naturally identified with the tangent space to \( R_k \) by the projection \( V \to R_k \). That is

\[
T_{(0, p, q)} V \simeq V_q \oplus T_{(p, q)} R_k.
\] (3.14)

Then the tangent map to \( h \) at a point \( (p, q) \) where \( h(p, q) = 0 \) can be projected onto \( V_q \). The condition for the zero set of \( h \) to be a submanifold is that this projected map is always onto.

To calculate the tangent map to \( h \) note that if \( h(p, q) = 0 \) then

\[
p^2 - 1 - \alpha q = 0
\] (3.15)

for some polynomial \( \alpha \). The equivalent relation on tangent vectors is given by differentiating and substituting to obtain

\[
\dot{h} = 2p\dot{p} - \left( \frac{p^2 - 1}{q} \right) \dot{q} + \langle q \rangle.
\] (3.16)

Hence the tangent map to \( h \) is defined by

\[
T_{(0, p, q)} h: (\dot{p}, \dot{q}) \mapsto (2p\dot{p} - \left( \frac{p^2 - 1}{q} \right) \dot{q} + \langle q \rangle) \oplus (\dot{p}, \dot{q})
\] (3.17)

and the composition of this with the projection onto \( V_q \) is

\[
(\dot{p}, \dot{q}) \mapsto 2p\dot{p} - \left( \frac{p^2 - 1}{q} \right) \dot{q} + \langle q \rangle.
\] (3.18)

To show that this map is onto it is sufficient to show that

\[
\dot{p} \mapsto 2p\dot{p} + \langle q \rangle
\] (3.19)
is onto, which it is unless we can find a polynomial $\dot{p}$ of degree less than $k$ such that $2p\dot{p} = 0 \mod q$. But by unique factorisation, because $q$ has $k$ factors and $\dot{p}$ less than $k$, one of the factors of $q$ would have to occur in $p$. But by assumption $p$ and $q$ have no common factors so the proof is complete.

Each of the spaces $IM^m_k$ has dimension $2k$, since there are $2k$ real parameters in $q$ but none in $p$. Moreover, as they are defined by imposing a symmetry, these spaces are totally geodesic subspaces of $M_k$. $IM^0_k$ is naturally diffeomorphic to the moduli space of $k$ flux vortices in the critically coupled abelian Higgs model, since $k$-vortex solutions are also parametrised by a single monic polynomial of degree $k$ [13]. However, the metrics in the monopole and vortex cases will be different.

We are not sure what kind of monopole configurations lie in the various spaces $IM^m_k$, but we conjecture that for $m = 0$ (or $m = k$), the energy density is always confined to a finite neighbourhood of the plane $x_3 = 0$, whereas for $0 < m < k$ it is possible for there to be monopole clusters arbitrarily far from the plane $x_3 = 0$, arranged symmetrically with respect to inversion in this plane. The examples discussed in Section 7 are consistent with this conjecture. If the roots of $q$ are distinct and well-separated, then the configurations always consist of a set of unit monopoles with their centres in the $x_3 = 0$ plane, provided the monopole positions are as given in the last paragraph of Section 1.

Finally notice that it follows from equations (3.5) and (3.6) and the fact that $\tau(\eta, 0) = I(\eta, 0)$ that using $\tau(s)$ to construct the rational map is the same as using $I(s)$, and hence the $p(\beta_i)$ occurring in the rational map defined using $\tau(s)$ would be the reciprocal of the $p(\beta_i)$ we use, and would give the rational map as defined by Hurtubise.

4. Centred monopoles and rational maps

We remarked earlier that although the positions and internal phases of the $k$ ‘particles’ in a charge $k$ monopole are only asymptotically well-defined, every monopole has a well-defined centre and total phase. This arises naturally in the twistor picture. If $S$ is the spectral curve of a monopole then it intersects every fibre of $TP_1 \to P_1$ in $k$ points counted with multiplicity. If we add these points together we obtain a new curve which is given by an equation $\eta + a_1(\zeta) = 0$. This curve is a real section and hence $a_1$ is of the form

$$a_1(\zeta) = -k((c_1 + ic_2) - 2c_3\zeta - (c_1 - ic_2)\zeta^2).$$  \hspace{1cm} (4.1)

The point $c = (c_1, c_2, c_3)$ is the centre of the monopole. To define the total phase requires a little more work.

Let $(\eta_1, \zeta) \ldots (\eta_k, \zeta)$ be the $k$ points in $S$, which are in the fibre of $TP_1 \to P_1$ over the point $\zeta$. We claim that there is a well-defined linear map

$$L^2_{(\eta_1, \zeta)} \otimes \ldots \otimes L^2_{(\eta_k, \zeta)} \to L^2_{(\eta_1 + \ldots + \eta_k, \zeta)}$$  \hspace{1cm} (4.2)
which, when suitably interpreted, gives rise to a global, holomorphic map. To define this map we recall from equation (2.7) that a section of $L^2$ is determined locally by a pair of functions $s_0$ and $s_1$, on $U_0 \cap S$ and $U_1 \cap S$ respectively, such that

$$s_0(\eta, \zeta) = \exp\left(-\frac{2\eta}{\zeta}\right)s_1(\eta, \zeta). \quad (4.3)$$

and we therefore have

$$s_0(\eta_1, \zeta) \ldots s_0(\eta_k, \zeta) = \exp\left(-\frac{2(\eta_1 + \ldots + \eta_k)}{\zeta}\right)s_1(\eta_1, \zeta) \ldots s_1(\eta_k, \zeta)$$

$$= \exp\left(\frac{2a_1(\zeta)}{\zeta}\right)s_1(\eta_1, \zeta) \ldots s_1(\eta_k, \zeta). \quad (4.4)$$

It follows that the functions $s_0^k$ and $s_1^k$ defined by

$$s_0^k(\eta, \zeta) = s_0(\eta_1, \zeta) \ldots s_0(\eta_k, \zeta) \quad \text{and} \quad s_1^k(\eta, \zeta) = s_1(\eta_1, \zeta) \ldots s_1(\eta_k, \zeta) \quad (4.5)$$

define a global holomorphic section $s^k$ of $L^2$ over the real section $\eta + a_1(\zeta) = 0$. Moreover because $\tau(s)s = 1$ we must have $\tau(s^k)s^k = 1$. The bundle $L^2$ over any real section is trivial and we fix as a choice of trivialisation $f$ over $\eta = k((c_1 + ic_2) - 2c_3\zeta - (c_1 - ic_2)\zeta^2)$

$$f_0(\eta, \zeta) = \exp 2k(c_3 + (c_1 - ic_2)\zeta)$$

$$f_1(\eta, \zeta) = \exp 2k(-c_3 + (c_1 + ic_2)/\zeta). \quad (4.6)$$

It is easy to check that this non-vanishing section $f$ satisfies $\tau(f)f = 1$. If we divide $s^k$ by $f$ we obtain a holomorphic function which must be constant. In fact because $\tau(s^k)s^k = 1$ and $\tau(f)f = 1$ this constant is a complex number of modulus 1. We define $s^k/f$ to be the total phase of the monopole. Notice that if we act on the monopole by a constant gauge transformation $\mu$ then $s$ is replaced by $\mu^2s$ and the total phase is multiplied by $\mu^{2k}$.

Some readers may be concerned that our definition of the total phase depends on the chosen family of trivialisations of $L^2$ over each real section. It would appear that we could arbitrarily scale these over each real section and change the definition of the total phase. However it follows from the construction of $L$ in [7] that the group of translations of $R^3$ acts on the bundle $L^2$, covering its action on $TP_1$. The family of sections we have described is translation invariant and therefore unique up to one overall choice of scale.

Let us now see how to construct the centre and total phase of a monopole from its rational map. Notice first that if we restrict the equation of the spectral curve to the fibre $\zeta = 0$ we obtain an equation of the form

$$\eta^k - k(c_1 + ic_2)\eta^{k-1} + \ldots = 0 \quad (4.7)$$
and hence \( c_1 + ic_2 \) is the average of the points of intersection of the spectral curve with \( \zeta = 0 \) or the average of the zeros of \( q \).

Comparing the construction of the rational map of a monopole we see that

\[
s_0^k(k(c_1 + ic_2), 0) = \prod_i p(\beta_i) = \Delta(p, q)
\]

(4.8)

the resultant of \( p \) and \( q \). It follows that

\[
\frac{s^k}{f} = \Delta(p, q) \exp(-2kc_3).
\]

(4.9)

So, if \( R(z) = p(z)/q(z) \) is the rational map of a monopole with \( q_0 \) the average of the roots of \( q \) and \( \Delta(p, q) \) the resultant of \( p \) and \( q \), then the centre of the monopole is

\[
(q_0, (1/2k) \log |\Delta(p, q)|)
\]

(4.10)

and the total phase is

\[
\Delta(p, q)|\Delta(p, q)|^{-1}
\]

(4.11)

It follows that a monopole is centered if and only if the zeroes of \( q \) sum to zero and \(|\Delta(p, q)| = 1 \). It will be useful to use a stronger notion of centring than this. We call a monopole strongly centred if it is centred and the total phase is 1. From what we have just proven a monopole is strongly centred if and only if its rational map satisfies

\[
q_0 = 0 \quad \text{and} \quad \Delta(p, q) = 1.
\]

(4.12)

We shall denote the space of strongly centred monopoles by \( M_{k,0} \) and show in the next section that it is a (totally) geodesic submanifold of the moduli space \( M_k \).

5. Strongly centred monopoles

Atiyah and Hitchin show that there is a \( k \)-fold covering of the \( k \)-monopole moduli space

\[
\widetilde{M}_k \rightarrow M_k,
\]

(5.1)

and an isometric splitting \( \widetilde{M}_k = X \times \mathbb{R}^3 \times S^1 \) for some hyperkähler manifold \( X \). We shall construct such a covering and splitting explicitly using the twistor space of \( M_k \). Given this, the submanifold \( X \times \{0\} \times \{1\} \) is clearly a totally geodesic submanifold of \( \widetilde{M}_k \) and because (5.1) is a finite covering the image of this submanifold under projection to \( M_k \) is also totally geodesic in \( M_k \). We shall show that this is, in fact, the space \( M_{k,0} \) of strongly centred monopoles.
Recall from [2] the basic facts about the twistor space of a hyperkähler manifold. If $M$ is a hyperkähler manifold then the tangent space at any point of $M$ has complex structures $I, J$ and $K$ defined on it which satisfy the quaternion algebra relations. In fact we can define a family of complex structures on the tangent space by forming combinations $aI + bJ + cK$ as long as $a^2 + b^2 + c^2 = 1$. This family is clearly a two-sphere. The union of all these complex structures for all points defines a two-sphere bundle $Z \to M$ called the twistor space of $M$. It is in fact a complex manifold. The details are given on page 39 of Atiyah and Hitchin’s book [2]. They are

**Theorem 1**  Let $M$ be a hyperkähler manifold of real dimension $4n$ and $Z$ its twistor space. Then

(i) $Z$ is a holomorphic fibre bundle $Z \to P_1$ over the complex projective line,
(ii) the bundle admits a family of holomorphic sections each with normal bundle isomorphic to $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$,
(iii) there exists a holomorphic section $\omega$ of $\bigwedge^2 T_F^* \otimes \mathcal{O}(2)$ defining a symplectic form on each fibre $F$,
(iv) $Z$ has a real structure $\tau$ compatible with (i), (ii) and (iii) and covering the antipodal map on $P_1$.

Conversely, the parameter space of real sections of any complex manifold $Z$ of complex dimension $2n + 1$ satisfying (i) through (iv) is a $4n$-dimensional manifold with a natural hyperkähler structure for which $Z$ is the twistor space.

The importance of this result for us is that the twistor construction of hyperkähler manifolds behaves nicely under natural geometric constructions such as quotients and products. We shall need a number of instances of this. The first is that if we have a product $M = M_1 \times M_2$ of hyperkähler manifolds then the twistor space $Z$ must be a fibre product $Z = Z_1 \times Z_2$ of the twistor spaces $Z_i \to P_1$ of the $M_i$. Recall that the fibre product is defined by $(Z_1 \times Z_2)_z = (Z_1)_z \times (Z_2)_z$. The converse of this theorem is also true. If $Z$ is the twistor space of $M$ and $Z = Z_1 \times Z_2$ is a fibre product and the structures in Theorem 1 decompose in the natural way then $M$ is a product of two hyperkähler manifolds $M_1 \times M_2$. The second is that if $Z_k$, the finite group of $k$-th roots of unity, acts freely on a twistor space $Z$ commuting with all the structures in Theorem 1 then it will act on the corresponding hyperkähler manifold freely and vice-versa.

The twistor space $Z_k$ of the monopole moduli spaces is constructed on page 46 of Atiyah and Hitchin’s book as follows.

**Theorem 2**  The twistor space $Z_k$ of the moduli space $M_k$ is defined by taking two copies of $\mathbb{C} \times R_k$ parametrised by $(\zeta, R(z))$ and $(\tilde{\zeta}, \tilde{R}(z))$ and identifying them over $\zeta \neq 0$. 

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by
\[ \zeta = \zeta^{-1} \]
\[ \tilde{q}(\frac{z}{\zeta}) = \zeta^{-2k}q(z) \]
\[ \tilde{p}(\frac{z}{\zeta}) = e^{-2z/\zeta}p(z) \mod q(z). \]

The symplectic form on the fibre is defined by
\[ \omega = \pi \sum d\tilde{\beta}_i \wedge d\tilde{p}(\tilde{\beta}_i) = \frac{\pi}{\zeta^2} \sum d\beta_i \wedge dp(\beta_i) \]

where the \( \tilde{\beta}_i \) (resp. \( \beta_i \)) are the roots of \( \tilde{q} \) (resp. \( q \)).

The real structure is defined by
\[ \tau(\zeta) = -\zeta^{-1} \]
\[ \tau(\frac{p(z)}{q(z)}) = (-1)^k\zeta^{-2k}(\frac{p(-z\zeta^{-2})}{q(-z\zeta^{-2})}). \]

Define a k-fold cover \( \tilde{R}_k \) of \( R_k \) by considering all pairs \((p(z)/q(z), p_0)\) where \( p_0 \) is a complex number satisfying \( p_0^k = \Delta(p, q) \). Notice that \( \mathbb{Z}_k \) acts freely on \( \tilde{R}_k \) by multiplying \( p_0 \) and the quotient is \( R_k \).

We can now construct a twistor space \( \tilde{Z}_k \) which is a k-fold cover of \( Z_k \) by identifying two copies of \( C \times \tilde{R}_k \) using the same rules as in Theorem 2 and a rule for identifying \( p_0 \) and \( \tilde{p}_0 \). To see what that should be note that
\[ \Delta(\tilde{p}, \tilde{q}) = e^{-2kq_0/\zeta} \Delta(p, q). \tag{5.5} \]

A good choice then is to identify \( p_0 \) and \( \tilde{p}_0 \) by
\[ \tilde{p}_0 = e^{-2q_0/\zeta}p_0. \tag{5.6} \]

Notice that the action of \( \mathbb{Z}_k \) extends to a (free) action on \( \tilde{Z}_k \) and that the quotient is \( Z_k \). It is straightforward now to lift the definitions of the symplectic form and the real structure to \( \tilde{Z}_k \) in such a way that
\[ \tilde{Z}_k \to Z_k \tag{5.7} \]
is a quotient of twistor spaces. We claim now that there is a corresponding k-fold covering of hyperkähler manifolds
\[ \tilde{M}_k \to M_k. \tag{5.8} \]

This follows immediately as long as \( \tilde{M}_k \) is non-empty; that is, as long as we can lift any holomorphic section \( P_1 \to M_k \) to a holomorphic section \( P_1 \to \tilde{M}_k \). To see that this is
possible note that the fibration $\tilde{Z}_k \to Z_k$ restricted to the image of such a section is a $k$-fold covering of $P_1$ without ramification. It must therefore be trivial and hence the section lifts. It follows that we have constructed a $k$-fold covering $\tilde{M}_k$ of the hyperkähler manifold $M_k$.

If $R_{k,0}$ denotes the strongly centred rational maps we can define an isomorphism $\tilde{R}_k \to R_{k,0} \times \mathbb{C}^\times \times \mathbb{C}$ by

$$\begin{aligned}(R(z), p_0) &\mapsto (p_0^{-1}R(z + q_0), p_0, q_0) \quad (5.9)\end{aligned}$$

where $q_0 = (1/k) \sum \beta_i$ is the average of the roots of $q$. This map just sends a monopole to the corresponding strongly centred monopole and the centre and total phase. The inverse map is

$$\begin{aligned}(R(z), p_0, q_0) &\mapsto (p_0 R(z - q_0), p_0). \quad (5.10)\end{aligned}$$

We can also define a subtwistor space $Z_{k,0} \subset Z_k$ by identifying two copies of $\mathbb{C} \times R_{k,0}$ by the rules of Theorem 2.

Atiyah and Hitchin show that $\mathbb{C}^\times \times \mathbb{C}$ is $R_1$ so we can form a twistor space $Z_1$ with two copies of $\mathbb{C}^\times \times \mathbb{C} \times \mathbb{C}$ and this is the twistor space of $M_1 = \mathbb{R}^3 \times S^1$. The map in equation (5.9) can now be seen to extend to a fibre map

$$\tilde{Z}_k \to Z_{k,0} \ast Z_1. \quad (5.11)$$

It is straightforward to check that this map is well-defined, i.e. two things identified in Theorem 2 are still identified after they are mapped by (5.9). We also need to check that the symplectic form on the fibres of $\tilde{Z}_k$ maps to the product symplectic form on the fibres of $Z_{k,0} \ast Z_1$. To see this note that it is enough to work on each fibre and show that the pull-back of the symplectic form on $R_{k,0} \times \mathbb{C}^\times \times \mathbb{C}$ to $\tilde{R}_k$ under the map

$$\begin{aligned}(R(z), p_0) &\mapsto (p_0^{-1}R(z + q_0), p_0, q_0). \quad (5.12)\end{aligned}$$

is the symplectic form of $\tilde{R}_k$. The pull-back of the symplectic form is

$$\begin{aligned}\frac{\pi}{\zeta^2} \left( \sum \frac{d(\beta_i - q_0) \wedge d(p_0^{-1}p(\beta_i))}{p_0^{-1}p(\beta_i)} + k \frac{dq_0 \wedge dp_0}{p_0} \right). \quad (5.13)\end{aligned}$$

Recall that $p_0^k = \prod p(\beta_i)$ and $q_0 = (1/k) \sum \beta_i$ so that

$$\begin{aligned}\sum (\beta_i - q_0) = 0 \quad \text{and} \quad k \frac{dp_0}{p_0} = \sum \frac{dp(\beta_i)}{p(\beta_i)}. \quad (5.14)\end{aligned}$$

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In the expression (5.13) we need to expand the term \( d(p_0^{-1} p(\beta_i)) \). This yields two terms and in the second one, \( d(p_0^{-1})p(\beta_i) \), the factor \( p(\beta_i) \) cancels with the same factor in the denominator. The sum is then over \( d(\beta_i - q_0) \) which vanishes. So we have

\[
\frac{\pi}{\zeta^2} \left( \sum d(\beta_i - q_0) \wedge d(p_0^{-1} p(\beta_i)) + k \frac{dq_0 \wedge dp_0}{p_0} \right)
\]

\[
= \frac{\pi}{\zeta^2} \left( \sum \frac{d\beta_i \wedge dp(\beta_i)}{p(\beta_i)} - \sum \frac{dq_0 \wedge dp(\beta_i)}{p(\beta_i)} + k \frac{dq_0 \wedge dp_0}{p_0} \right)
\]

\[
= \frac{\pi}{\zeta^2} \left( \sum \frac{d\beta_i \wedge dp(\beta_i)}{p(\beta_i)} \right),
\]

(5.15)
as required.

We now have an isomorphism of twistor spaces \( \tilde{Z}_k = Z_{k,0} \ast Z_1 \). We can define a subset of \( Z_{k,0} \ast Z_1 \) by defining it fibre by fibre to correspond to the subset \( R_{k,0} \times \{1\} \times \{0\} \times \mathbb{C} \). It is clear that this is well-defined. The holomorphic sections which lie inside this space define the totally geodesic subspace \( Z_{k,0} \times \{0\} \times \{1\} \) inside \( M_{k,0} \times \mathbb{C} \times \mathbb{C}^\times \) and hence inside \( \tilde{M}_k \). The image of this under the finite covering \( \tilde{M}_{k,0} \) is also totally geodesic and clearly defines the space of strongly centred monopoles as it corresponds to monopoles whose rational map is strongly centered.

6. Symmetric Curves in \( TP_1 \)

In eq. (2.4) we presented the general form of curves in \( TP_1 \) that occur as spectral curves of charge \( k \) monopoles. The coefficients \( a_r(\zeta) \) must satisfy the reality condition (2.5), and the curve is centred at the origin in \( \mathbb{R}^3 \) if \( a_1(\zeta) = 0 \). Here we shall discuss the form of these curves when they are required to be invariant under a group of rotations about the origin.

Let us recall that in \( TP_1 \), the \( P_1 \) of lines through the origin are parametrized by \( \zeta \) with \( \eta = 0 \). The line in the direction of the Cartesian unit vector \( (x_1, x_2, x_3) \) has \( \zeta = (x_1 + ix_2)/(1 + x_3) \). It will be important to consider the homogeneous coordinates \( [\zeta_0, \zeta_1] \) on \( P_1 \), as well as the inhomogeneous coordinate \( \zeta = \zeta_1/\zeta_0 \).

An \( SU(2) \) Möbius transformation on the homogeneous coordinates, \([\zeta_0, \zeta_1] \rightarrow [\zeta'_0, \zeta'_1] \), of the form

\[
\zeta'_0 = -(b + ia)\zeta_1 + (d - ic)\zeta_0
\]

\[
\zeta'_1 = (d + ic)\zeta_1 + (b - ia)\zeta_0
\]

(6.1)

where \( a^2 + b^2 + c^2 + d^2 = 1 \), corresponds to an \( SO(3) \) rotation in \( \mathbb{R}^3 \). The rotation is by an angle \( \theta \) about the unit vector \( (x_1, x_2, x_3) \), where \( x_1 \sin \frac{\theta}{2} = a, \quad x_2 \sin \frac{\theta}{2} = b, \quad x_3 \sin \frac{\theta}{2} = c, \quad \cos \frac{\theta}{2} = d \). The inhomogeneous coordinate \( \zeta \) transforms to

\[
\zeta' = \frac{(d + ic)\zeta + (b - ia)}{-(b + ia)\zeta + (d - ic)}.
\]

(6.2)
Since $\eta$ is the coordinate in the tangent space to $P_1$ at $\zeta$, it follows that if $\zeta$ transforms to $\zeta'$ as in (6.2) then $\eta$ transforms to $\eta'$ via the derivative of (6.2), that is

$$\eta' = \frac{\eta}{(-(b + ia)\zeta + (d - ic))^2}.$$  \hspace{1cm} (6.3)

A curve $P(\eta, \zeta) = 0$ in $TP_1$ is invariant under the Möbius transformation if $P(\eta', \zeta') = 0$ is the same curve. If the curve is the spectral curve of a monopole, then the monopole is invariant under the associated rotation.

The simplest group of symmetries is the cyclic group of rotations about the $x_3$-axis, $C_n$. The generator is the Möbius transformation

$$\begin{align*}
\zeta' &= e^{2\pi i \frac{n}{m}} \zeta, \\
\eta' &= e^{2\pi i} \eta.
\end{align*} \hspace{1cm} (4.5)$$

A curve $P(\eta, \zeta) = 0$ is invariant if all terms of $P$ have the same degree, mod $n$. A curve of the form (2.4) is $C_n$-invariant if all terms have degree $k$, mod $n$. In particular, it is $C_k$-invariant if all terms have degree zero, mod $k$.  

For there to be axial symmetry about the $x_3$-axis, with symmetry group $C_{\infty}$, the curve must be invariant under $\zeta \to e^{i\theta} \zeta$, $\eta \to e^{i\theta} \eta$, for all $\theta$. This requires that all terms in $P(\eta, \zeta)$ have degree $k$. There is a unique axially symmetric, strongly centred monopole for each charge $k$. Hitchin has shown that its spectral curve is \cite{7}

$$\begin{align*}
\eta \prod_{l=0}^{m} \left( \eta^2 + \left(l + \frac{1}{2}\right)^2 \pi^2 \zeta^2 \right) &= 0 \quad \text{for} \quad k = 2m + 1, \\
\prod_{l=0}^{m} \left( \eta^2 + \left(l + \frac{1}{2}\right)^2 \pi^2 \zeta^2 \right) &= 0 \quad \text{for} \quad k = 2m + 2.
\end{align*} \hspace{1cm} (6.5)$$

Notice that these curves are not determined by symmetry alone, and that the coefficients of $P$ are transcendental numbers. The only curve of the form (2.4) which has full $SO(3)$ symmetry is $\eta^k = 0$. This is the spectral curve of a unit charge monopole at the origin when $k = 1$, but for $k > 1$ it is not the spectral curve of a monopole.

The groups $C_n$ and $C_{\infty}$ are extended to the dihedral groups $D_n$ and $D_{\infty}$ by adding a rotation by $\pi$ about the $x_1$-axis. This rotation corresponds to the transformation on $TP_1$

$$\zeta' = \frac{1}{\zeta}, \quad \eta' = -\frac{\eta}{\zeta^2}. \hspace{1cm} (6.6)$$

Under this transformation, and for any constant $\nu$,

$$\left(\eta^2 + \nu \zeta^2\right)' = \frac{1}{\zeta^4} (\eta^2 + \nu \zeta^2), \hspace{1cm} (6.7)$$
so each of the axially symmetric monopoles has symmetry group $D_{\infty}$.

It is useful to note that by a similar argument to that in Section 3, the reflection $x_2 \rightarrow -x_2$ corresponds to $\zeta \rightarrow \bar{\zeta}, \eta \rightarrow \bar{\eta}$, so a curve $P(\eta, \zeta) = 0$ is invariant under this reflection if all coefficients in $P(\eta, \zeta)$ are real. The axially symmetric monopoles therefore have this reflection symmetry too.

As an example of finite cyclic or dihedral symmetry, let us consider centred $k = 3$ curves with either $C_3$ or $D_3$ symmetry. Before imposing the symmetry, the curves are of the form

\begin{equation}
\eta^3 + \eta(\alpha_4\zeta^4 + \alpha_3\zeta^3 + \alpha_2\zeta^2 + \alpha_1\zeta + \alpha_0)
+ (\beta_6\zeta^6 + \beta_5\zeta^5 + \beta_4\zeta^4 + \beta_3\zeta^3 + \beta_2\zeta^2 + \beta_1\zeta + \beta_0) = 0
\end{equation}

subject to the reality conditions

\begin{equation}
\alpha_4 = \bar{\alpha}_0, \ \alpha_3 = -\bar{\alpha}_1, \ \alpha_2 = \bar{\alpha}_2,
\beta_6 = -\bar{\beta}_0, \ \beta_5 = \bar{\beta}_1, \ \beta_4 = -\bar{\beta}_2, \ \beta_3 = \bar{\beta}_3.
\end{equation}

$C_3$ symmetry implies that (6.8) reduces to

\begin{equation}
\eta^3 + \alpha\eta\zeta^2 + \beta\zeta^6 + \gamma\zeta^3 - \bar{\beta} = 0
\end{equation}

where $\alpha$ and $\gamma$ are real. By a rotation about the $x_3$-axis, we can orient the curve so that $\beta$ is real, too, and then there is reflection symmetry under $x_2 \rightarrow -x_2$. There is $D_3$ symmetry if $\gamma = 0$; then the curve reduces to

\begin{equation}
\eta^3 + \alpha\eta\zeta^2 + \beta(\zeta^6 - 1) = 0
\end{equation}

with $\alpha$ and $\beta$ real.

The axisymmetric charge 3 monopole has a spectral curve of type (6.11) with $\alpha = \pi^2$ and $\beta = 0$. Also, three separated unit charge monopoles at the vertices of an equilateral triangle can have $D_3$ symmetry. The spectral curve is asymptotic to the product of three stars at

\begin{equation}
(x_1, x_2, x_3) = \left\{(a, 0, 0), \ (a \cos \frac{2\pi}{3}, a \sin \frac{2\pi}{3}, 0), \ (a \cos \frac{4\pi}{3}, a \sin \frac{4\pi}{3}, 0)\right\},
\end{equation}

that is,

\begin{equation}
(\eta - a(1 - \zeta^2))(\eta - a\omega(1 - \omega\zeta^2))(\eta - a\omega^2(1 - \omega^2\zeta^2)) = 0,
\end{equation}

where $\omega = e^{2\pi i/3}$. Equation (6.13), when multiplied out, is a curve of the form (6.11) with $\alpha = 3a^2$ and $\beta = a^3$, or equivalently $\alpha^3 = 27\beta^2$. We shall find out more about the spectral curves of charge 3 monopoles with symmetry $C_3$ or $D_3$ when we consider the rational maps associated with the monopoles (see Section 7).
$C_4$ symmetry is rather a weak constraint on curves with $k = 4$. $D_4$ symmetry, however, implies that a $k = 4$ curve is of the form

$$\eta^4 + \alpha \eta^2 \zeta^2 + \beta \zeta^8 + \gamma \zeta^4 + \beta = 0$$

(6.14)

with $\alpha$, $\beta$ and $\gamma$ real. The axisymmetric charge 4 monopole has this form of spectral curve, with $\alpha = (5/2)\pi^2$, $\beta = 0$ and $\gamma = (9/16)\pi^4$. Four separated unit charge monopoles at the vertices of the square $\{(\pm a, 0, 0), (0, \pm a, 0)\}$ can have $D_4$ symmetry. The spectral curve is asymptotic to a product of stars, and is of the form (6.14), with $\alpha = 4a^2$, $\beta = -a^4$ and $\gamma = 2a^4$. After a $\pi/4$ rotation, the monopoles are at $(\pm a/\sqrt{2}, \pm a/\sqrt{2}, 0)$, and $\alpha = 4a^2$, $\beta = a^4$ and $\gamma = 2a^4$.

There is another interesting asymptotic monopole configuration, with a spectral curve of type (6.14). Consider two well-separated axisymmetric charge 2 monopoles, centred at $(0, 0, b)$ and $(0, 0, -b)$, and with the $x_3$-axis the axis of symmetry. The spectral curve is asymptotic to a product of curves associated with the charge 2 monopoles. Recall that the spectral curve of a centred axisymmetric charge 2 monopole is $\eta^2 + \pi^2 \zeta^2 = 0$. This factorizes as $(\eta + i\pi \zeta)(\eta - i\pi \zeta) = 0$, which is a product of stars at the complex conjugate points $(0, 0, \pm i\pi/2)$. Translation by $b$ gives the curve

$$\eta^2 + 4b\eta \zeta + (4b^2 + \pi^2)\zeta^2 = 0$$

(6.15)

which is the product of stars at $(0, 0, b \pm i\pi/2)$. Similarly, translation by $-b$ gives

$$\eta^2 - 4b\eta \zeta + (4b^2 + \pi^2)\zeta^2 = 0$$

(6.16)

and the product of these is the curve

$$\eta^4 + (2\pi^2 - 8b^2)\eta^2 \zeta^2 + (4b^2 + \pi^2)^2 \zeta^4 = 0.$$  

(6.17)

Since all terms have degree 4 this curve is axisymmetric; however, the true spectral curve of the charge 4 monopole has symmetry $D_4$, as we shall see in the next Section, becoming axisymmetric only in the limit of infinite separation.

Let us now investigate the curves in $TP_1$ with the symmetries of a regular solid. Some of these are special cases of the curves we have already discussed. There are three rotational symmetry groups to consider, those of a tetrahedron, an octahedron and an icosahedron. The direct way to construct a symmetric curve is to find Möbius transformations which generate the symmetry group, and calculate the conditions for the curve to be invariant. For example, a curve of type (6.14), with $D_4$ symmetry, has octahedral symmetry if it is invariant under the transformation

$$\zeta' = i\zeta + 1, \quad \eta' = \frac{-2}{(\zeta + i)^2} \eta,$$

(6.18)
which corresponds to a $\pi/2$ rotation about the $x_1$-axis, and this requires that the curve reduces to

$$\eta^4 + \beta(\zeta^8 + 14\zeta^4 + 1) = 0. \quad (6.19)$$

A more powerful and less laborious approach is to use the theory of invariant bilinear forms and polynomials on $P_1$, as expounded in Klein’s famous book [14].

Consider a homogeneous bilinear form $Q_r(\zeta_0, \zeta_1)$ of degree $r$, and its associated inhomogeneous polynomial $q_r(\zeta)$ defined by

$$Q_r(\zeta_0, \zeta_1) = \zeta_0^r q_r(\zeta). \quad (6.20)$$

Generally $q_r$ has degree $r$, but it may have lower degree. Suppose $Q_r(\zeta_0, \zeta_1)$ is invariant under a Möbius transformation of the form (6.1). Then $q_r(\zeta)$ transforms in a simple way under the corresponding transformation (6.2), namely

$$q'_r(\zeta) = \frac{q_r(\zeta)}{(-(b + ia)\zeta + (d - ic))}. \quad (6.21)$$

On the other hand, $\eta$ transforms as in (6.3). Consider a centred curve in $TP_1$,

$$P(\eta, \zeta) \equiv \eta^k + \eta^{k-2}q_4(\zeta) + \eta^{k-3}q_6(\zeta) + \ldots + q_{2k}(\zeta) = 0. \quad (6.22)$$

If, under a Möbius transformation, each polynomial $q_r(\zeta)$ transforms as in (6.21), and $\eta$ as in (6.3), then each term in the polynomial $P(\eta, \zeta)$ is multiplied by the same factor $(-(b + ia)\zeta + (d - ic))^{-2k}$, so the curve is invariant. It follows that curves invariant under the rotational symmetry group of a regular solid can be constructed from the inhomogeneous polynomials $q_r$ derived from the bilinear forms $Q_r$ invariant under the group.

Let $G$ denote the tetrahedral, octahedral or icosahedral group. Klein has described the ring of bilinear forms, $\text{Inv}_G$, which change only by a constant factor under each transformation of $G$ – for each form these factors define an abelian character of $G$. Let $\text{Inv}_G^*$ be the subring of strictly invariant forms. A form $Q$ is in $\text{Inv}_G^*$ if the roots of the associated polynomial $q$ are invariant under $G$, that is, if they are the union of a set of $G$-orbits on $P_1$.

Generic $G$-orbits on $P_1$ consist of $|G|$ points, i.e. 12, 24 and 60 points respectively for the three groups. The associated forms of degree $|G|$ are always strictly invariant under $G$, and they span a vector space of forms, of dimension two. For each group $G$, there are also three forms of degree less than $|G|$ associated with special orbits of $G$, and these generate the ring $\text{Inv}_G$. Let $V$, $E$ and $F$ be the set of vertices, mid-points of edges, and centres of faces of the centred regular solid (tetrahedron, octahedron or icosahedron) invariant under $G$. Centrally project these points onto the unit sphere, identified with $P_1$, denoting the
resulting sets of points again by \( V, E \) and \( F \). \( V \) is a \( G \)-orbit, so there is a form \( Q_V \) in \( \text{Inv}_G \) and an associated polynomial \( q_V \), such that \( Q_V \) has degree \( |V| \) and \( Q_V = 0 \) at all points of \( V \). Similarly, there are forms and polynomials \( Q_E, Q_F \) and \( q_E, q_F \). Table 1 gives the polynomials \( q_V, q_E \) and \( q_F \) for the three groups \( G \), and a star indicates that the associated form \( Q_V \), \( Q_E \) or \( Q_F \) is strictly \( G \)-invariant. [A choice of orientation has been made for the solids: the tetrahedron has its vertices at \((1/\sqrt{3})(\pm1, \pm1, \pm1)\), with either two or no signs negative; the octahedron has its vertices on the Cartesian axes; the icosahedron has two vertices on the \( x_3 \)-axis and is invariant under the dihedral group \( D_5 \).]

| \( G \)     | \( q_V \)                          | \( q_E \)                          | \( q_F \)                          |
|------------|-----------------------------------|-----------------------------------|-----------------------------------|
| Tetrahedral| \( \zeta^4 + 2\sqrt{3}i\zeta^2 + 1 \) | \( \zeta(\zeta^4 - 1)^* \)        | \( \zeta^4 - 2\sqrt{3}i\zeta^2 + 1 \) |
| Octahedral | \( \zeta(\zeta^4 - 1) \)          | \( \zeta^{12} - 33\zeta^8 \)     | \( \zeta^8 + 14\zeta^4 + 1^* \)   |
|            |                                   | \( -33\zeta^4 + 1 \)             |                                   |
| Icosahedral| \( \zeta(\zeta^{10} + 11\zeta^5 - 1)^* \) | \( \zeta^{30} + 522\zeta^{25} \) | \( \zeta^{20} - 228\zeta^{15} + 494\zeta^{10} \) |
|            |                                   | \( -10005\zeta^{20} - 10005\zeta^{10} \) | \( +228\zeta^5 + 1^* \)  |
|            |                                   | \( -522\zeta^5 + 1^* \)           |                                   |

Polynomials associated with the special orbits \( V, E \) and \( F \) of the rotational symmetry groups of the regular solids. A star(*) denotes that the homogeneous bilinear form \( Q \) related to the polynomial \( q \) is strictly invariant.

**Table 1**

All the icosahedral forms are strictly invariant because the icosahedral group \( A_5 \) is simple, and has no non-trivial abelian characters. The tetrahedral forms \( Q_V \) and \( Q_F \) are not strictly invariant, but acquire factors of \( e^{\pm2\pi i/3} \) under a \( 2\pi/3 \) rotation about a 3-fold symmetry axis; so \( Q_V Q_F \) is strictly invariant. In fact, the polynomial associated with \( Q_V Q_F \) is \( \zeta^8 + 14\zeta^4 + 1 \), which has octahedral symmetry. Similarly, the octahedral forms \( Q_V \) and \( Q_E \) acquire factors of \(-1 \) under a rotation by \( \pi/2 \) around a 4-fold symmetry axis, and \( Q_V Q_E \) is strictly invariant.
There are remarkable identities satisfied by the forms \(Q_V, Q_E\) and \(Q_F\) (which remain true if the forms \(Q\) are replaced by the associated polynomials \(q\)), namely

\[
\begin{align*}
Q_V^3 - Q_F^3 - 12\sqrt{3}i Q_E^2 &= 0 \quad \text{for the tetrahedral group} \\
108 Q_V^4 - Q_F^3 + Q_E^2 &= 0 \quad \text{for the octahedral group} \\
1728 Q_V^5 - Q_F^3 - Q_E^2 &= 0 \quad \text{for the icosahedral group.}
\end{align*}
\]

These identities occur, because each term is a strictly invariant form of degree \(|G|\), lying in the two-dimensional vector space of forms associated with the generic \(G\)-orbits.

We can now write down some examples of invariant curves in \(TP_1\), also satisfying the reality conditions (2.5). Recall that invariant curves in \(TP_1\) must be constructed from polynomials derived from strictly invariant forms. The simplest curves with tetrahedral symmetry are

\[
\eta^3 + ia\zeta(\zeta^4 - 1) = 0 \tag{6.24}
\]

where \(a\) is real. After a rotation, (6.24) becomes

\[
\eta^3 + a(\zeta^6 + 5\sqrt{2}\zeta^3 - 1) = 0, \tag{6.25}
\]

which is of the form (6.10), exhibiting manifest \(C_3\) symmetry about the \(x_3\)-axis.

The curves in \(TP_1\) with \(k = 4\), and either octahedral or tetrahedral symmetry, are

\[
\eta^4 + ic\eta\zeta(\zeta^4 - 1) + d(\zeta^8 + 14\zeta^4 + 1) = 0 \tag{6.26}
\]

with \(c\) and \(d\) real. All such curves have tetrahedral symmetry, and if \(c = 0\) the symmetry is octahedral. Finally, the simplest curves with icosahedral symmetry are

\[
\eta^6 + a\zeta(\zeta^{10} + 11\zeta^5 - 1) = 0 \tag{6.27}
\]

with \(a\) real.

We shall discuss in the next Section the possibility that some of these curves are spectral curves of monopoles.

7. Rational Maps of Symmetric Monopoles, and Monopole Scattering

The advantage of working with the rational maps associated with monopoles is that there is a 1 − 1 correspondence between the maps and monopoles. Also, cyclic or axial symmetry about the \(x_3\)-axis, if present, is manifest. The information hidden in the rational map is the full three-dimensional structure of the monopole, and we do not know which maps, if any, characterise monopoles with the symmetries of a regular solid. In this Section, we shall investigate monopoles with cyclic symmetry, and make some conjectures about
monopoles with the symmetries of regular solids. We shall also discover some novel types of geodesic monopole scattering.

Recall that the rational map of a charge \(k\) monopole takes the form

\[
R(z) = \frac{p(z)}{q(z)},
\]

(7.1)

with \(q\) monic of degree \(k\) and \(p\) of degree less than \(k\). Let \(\omega = e^{2\pi i/k}\). Consider the cyclic group of rotations about the \(x_3\)-axis, \(C_k\), generated by the transformation \(z \to z'\), where \(z' = \omega z\). The monopole with rational map \(R(z)\) is \(C_k\) symmetric if \(R(z')\) differs from \(R(z)\) only by a constant phase. We get a class of charge \(k\) monopoles with \(C_k\) symmetry for each irreducible character of \(C_k\). Let us denote the \(l\)th such class of monopoles by \(M^l_k\) \((0 \leq l < k)\). These are the monopoles whose rational maps are of the form

\[
R(z) = \frac{\mu z^l}{z^k - \nu},
\]

(7.2)

where \(\mu\) and \(\nu\) are complex parameters. For these monopoles, \(R(z') = \omega^l R(z)\). \(M^l_k\) is a 4-dimensional geodesic submanifold of the moduli space \(M_k\), since it arises by imposing a symmetry on the monopoles. Its metric is also Kähler, because the set of rational maps (7.2) is a complex submanifold of the set of all maps (7.1).

Since the strongly centred monopoles are geodesic in the moduli space, we shall now restrict attention to rational maps of strongly centred, \(C_k\)-symmetric monopoles. There is no essential loss of generality in doing this. For a monopole with a rational map of type (7.2), the criterion (4.12) for it to be strongly centred reduces to

\[
\mu^k \prod_{i=1}^{k} (\beta_i)^l = 1
\]

(7.3)

where \(\{\beta_i : i = 1, \ldots, k\}\) are the \(k\) roots of \(z^k - \nu = 0\). Eq. (7.3) is equivalent to \(\mu^k \nu^l = \pm 1\), with the lower sign if both \(k\) is even and \(l\) odd, and the upper sign otherwise. The magnitude of \(\mu\) is \(|\mu| = |\nu|^{-l/k}\), and there are \(k\) choices for the phase of \(\mu\). The rational maps we obtain are parametrised by several surfaces of revolution. For given \(k\) and \(l\) there may be one or more surfaces. For \(l = 0\), for example, there are \(k\) distinct surfaces, each with \(\nu\) a good coordinate; \(\mu\) is a distinct, and constant, \(k\)th root of unity on each surface. If \(l \neq 0\), and \(k\) and \(l\) have highest common factor \(h\), there are \(h\) distinct surfaces. As \(\arg \nu\) increases by \(2\pi\), \(\arg \mu\) decreases by \(2\pi l/k\), so \(\arg \nu\) must increase by \(2\pi k/h\) for \(\mu\) to return to its initial value. \(\nu\) is therefore a good coordinate on each surface, but the range of \(\arg \nu\) is \(2\pi k/h\).

For given \(k\), and each \(l\) in the range \(0 \leq l < k\), let us choose one of the surfaces just described, say, the one containing the rational map (7.2) with \(\nu = 1\) and \(\mu = e^{\pi i/k}\).
(if $k$ is even and $l$ odd) or $\mu = 1$ (otherwise). Denote this surface by $\Sigma^l_k$. If there is another surface, for a particular value of $l$, then it is isomorphic to $\Sigma^l_k$, as $\mu$ differs on it simply by a constant phase. Let us now consider the geodesics on $\Sigma^l_k$, and the associated $C_k$-symmetric monopole scattering. The simplest geodesic is when $\nu$ moves along the real axis – the monopole then has no angular momentum.

On $\Sigma^0_k$ the rational maps are of the form

$$R(z) = \frac{1}{z^k - \nu},$$

where $\nu$ is an arbitrary complex number. $\Sigma^0_k$ is therefore a submanifold of the space of inversion symmetric monopoles $IM^0_k$. For $\nu = 0$, the rational map is that of a strongly centred axisymmetric charge $k$ monopole. If $|\nu|$ is large, there are $k$ well-separated unit charge monopoles at the vertices of an $k$-gon in $\mathbb{R}^3$, with $x_1 + ix_2$ a $k$th root of $\nu$, and $x_3 = 0$. The geodesic where $\nu$ moves along the entire real axis corresponds to a simultaneous scattering of $k$ unit charge monopoles in the $(x_1, x_2)$ plane, where the incoming and outgoing trajectories are related by a $\pi/k$ rotation. The configuration is instantaneously axially symmetric when $\nu = 0$. This kind of symmetric planar scattering of $k$ solitons has been observed in a number of models, and can be understood in a rather general way [15].

On $\Sigma^l_k$, with $l \neq 0$, $\nu$ is a non-zero complex number. $\nu = 0$ is forbidden, as the numerator and denominator of $R(z)$ would have a common factor $z^l$. A simple geodesic is with $\nu$ moving along the positive real axis, say towards $\nu = 0$. The rational map is

$$R(z) = \frac{\iota \nu^{l/k}}{z^k - \nu},$$

where $\iota = e^{\pi i/k}$ (if $k$ is even and $l$ is odd) or $\iota = 1$ (otherwise). Then the initial motion is again $k$ unit charge monopoles at the vertices of a contracting $k$-gon in the $(x_1, x_2)$ plane. As $\nu \to 0$, the map approaches

$$R(z) = \frac{\iota}{\nu^{l/k}} \frac{1}{z^k - \nu},$$

which is the map of a charge $(k - l)$ axisymmetric monopole, centred at $(0, 0, (l/2k) \log \nu)$. This is a positive distance along the $x_3$-axis as $\nu$ is small. Following an argument of Atiyah and Hitchin [2,pp.25-6], we deduce that the charge $k$ monopole has split up, with one cluster the charge $k - l$ monopole just described, and a further cluster (or clusters) near the $x_3$-axis, but not so far up. In fact, there is just one other cluster, which is an axisymmetric charge $l$ monopole at a negative distance along the $x_3$-axis. This is seen by inverting the original monopole in the $(x_1, x_2)$ plane. The procedure described in Section 3 shows that the rational map (7.5) transforms under inversion to

$$R(z) = \frac{\tilde{\iota}}{\nu^{(k-l)/k}} \frac{z^{k-l}}{z^k - \nu}.$$
where $\iota \bar{\iota} = 1$, because

$$\iota \bar{\iota} \frac{z^l}{\nu^l/k} \frac{z^{k-l}}{\nu^{(k-l)/k}} = \frac{z^k}{\nu} = 1 \mod z^k - \nu.$$  \hfill (7.8)

The inverted monopole therefore has an axisymmetric charge $l$ monopole cluster at $(0, 0, -((k - l)/(2k)) \log \nu)$, as $\nu \to 0$, while the original monopole has this axisymmetric charge $l$ cluster at $(0, 0, (k - l)/(2k)) \log \nu)$. In the geodesic motion, $k$ unit charge monopoles come in, but the outgoing configuration is of two approximately axisymmetric monopole clusters, of charges $k - l$ and $l$, at distances $ld$ and $-(k - l)d$ along the $x_3$-axis, with $d$ increasing uniformly. This geodesic motion can, of course, also be reversed. The centre of mass of these clusters remains at the origin.

If $k$ is even and $l = k/2$ then the rational maps, and the geodesic monopole motion we have described, have an additional inversion symmetry. $R(z) = z^{k/2}/(\nu^{1/2}(z^k - \nu))$ lies in the space of inversion symmetric maps $IM_k^{k/2}$, and the factor $\iota$ makes no essential difference. Consequently, the outgoing clusters have the same charges and equal speeds. Since $\nu$ was assumed to be real, there is reflection symmetry under $x_2 \to -x_2$. Together with the inversion symmetry, $x_3 \to -x_3$, we obtain an additional rotational symmetry, by $\pi$ about the $x_1$-axis. Hence, monopoles with rational maps of the form (7.5) have $D_k$ symmetry if $k$ is even and $l = k/2$. There is also $D_k$ symmetry if $l = 0$, for any $k$.

The surfaces $\Sigma_0^2$ and $\Sigma_1^2$ are the “rounded cone” and “trumpet” described by Atiyah and Hitchin. These surfaces are not isomorphic, but the geodesics with $\nu$ real (on $\Sigma_0^2$) and $\nu$ real and positive (on $\Sigma_1^2$) are isomorphic. Along the first, two unit charge monopoles scatter through $\pi/2$ in the $(x_1, x_2)$ plane, and along the second they scatter through $\pi/2$ in the $(x_1, x_3)$ plane. There are no analogous isomorphisms in the higher charge cases.

The general geodesics on the surfaces $\Sigma_k^0$ and $\Sigma_k^l$ $(l \neq 0)$ are presumably analogous to those on the cone $\Sigma_0^2$ or trumpet $\Sigma_1^2$. On $\Sigma_k^0$, they correspond to $k$ unit charge monopoles scattering in the $(x_1, x_2)$ plane with net orbital angular momentum. On $\Sigma_k^l$ $(l \neq 0)$, $k$ unit charge monopoles again come in with net orbital angular momentum. If this is small, the geodesic passes through the trumpet-like surface and two monopole clusters with magnetic charges $l$ and $k - l$ emerge back-to-back on the $x_3$-axis. They also have opposite electric charges, which accounts, physically, for angular momentum conservation. If the initial angular momentum is large, then the geodesic does not pass through the trumpet, but is reflected, and there are $k$ outgoing unit charge monopoles in the $(x_1, x_2)$ plane.

What can we learn about the spectral curves of centred $C_k$-symmetric monopoles from this discussion of rational maps? First, recall that monopoles whose rational maps differ only by a phase have the same spectral curves. We need therefore only consider the
chosen surfaces of rational maps, \( \Sigma^l_k \), and their associated monopoles. Let us also restrict attention to monopoles which are oriented to be reflection symmetric under \( x_2 \rightarrow -x_2 \), which requires \( \nu \) to be real, and choose a fixed phase for \( \mu \) as \( \nu \) varies in magnitude. This restricts us to \( 2k - 1 \) disjoint curves in the surfaces \( \Sigma^l_k \), \( \nu \) real in \( \Sigma^0_k \), \( \nu \) positive and \( \nu \) negative in \( \Sigma^l_k \), and these curves are geodesics. It follows that among the centred curves in \( TP_1 \) of the form (2.4) with \( C_k \) symmetry and oriented, there are \( 2k - 1 \) disjoint loci of spectral curves. (We refer to a connected, one-dimensional submanifold of spectral curves as a locus in the space of curves in \( TP_1 \).) All these spectral curves will have real coefficients because of the reflection symmetry. We have been unable to determine, in general, for which parameter values a curve is a spectral curve, but we can make some qualitative assertions, based on knowledge of the asymptotic monopole configurations, and the axisymmetric configurations. We restrict our remarks to the cases \( k = 3 \) and \( k = 4 \).

For \( k = 3 \), and \( l = 0,1 \) or \( 2 \), there are five loci of spectral curves of the form (6.10), with \( \beta \) real. When \( l = 0 \) there is \( D_3 \)-symmetry, so \( \gamma = 0 \). The locus is asymptotic at both ends to \( \alpha^3 = 27\beta^2 \), with \( \beta \) large and positive at one end, and \( \beta \) large and negative at the other. The axisymmetric monopole, half-way along the locus, has \( \beta = 0 \) and \( \alpha = \pi^2 \). Presumably, \( \alpha \) is positive along the whole locus. The four remaining loci, for \( l = 1 \) and \( l = 2 \), are isomorphic. This is because \( \nu \rightarrow -\nu \) corresponds to a reflection \( x_1 \rightarrow -x_1 \), and because the \( l = 2 \) monopoles are obtained from \( l = 1 \) monopoles by inversion \( (x_3 \rightarrow -x_3) \). Under the first symmetry \( \beta \rightarrow -\beta \), and under the second \( \gamma \rightarrow -\gamma \). Each of the four loci is asymptotic at one end to \( \alpha^3 = 27\beta^2, \gamma = 0, \) with \( \beta \) either positive or negative, and at the other to \( \alpha = \pi^2 - 3b^2, \beta = 0, \gamma = -2b(b^2 + \pi^2) \), with \( b \) either positive or negative. These latter parameters result from taking the product of the spectral curve of a unit charge monopole at \((0,0,b)\) with the spectral curve of an axisymmetric charge 2 monopole at \((0,0,-b/2)\), that is

\[
P(\eta, \zeta) = (\eta - 2b\zeta)(\eta^2 + 2b\eta\zeta + (b^2 + \pi^2)\zeta^2) = \eta^3 + (\pi^2 - 3b^2)\eta\zeta^2 - 2b(b^2 + \pi^2)\zeta^3 = 0. \tag{7.9}
\]

We note that along these four loci, \( \alpha \) passes through 0 , and it is possible that the loci pass through four points of the form \( (\alpha, \beta, \gamma) = (0, \pm a, \pm 5\sqrt{2}a) \), for some \( a \). These points correspond to a charge 3 monopole with tetrahedral symmetry (and four distinct orientations of the tetrahedron). From what we know about Skyrmions and their scattering (see the next Section), we conjecture that tetrahedral charge 3 monopoles do exist, and that their spectral curves are on the loci corresponding to rational maps with \( k = 3 \) and \( l = 1 \) or \( 2 \).

In the case \( k = 4 \), we have seven loci of spectral curves with \( C_4 \) symmetry. Only three of these are essentially different. The four corresponding to the rational maps with \( l = 1 \)

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and $l = 3$, and $\nu$ positive or negative, are isomorphic. The $l = 1$ and $l = 3$ maps, and hence the corresponding monopoles and spectral curves, are related by inversion, and the sign of $\nu$ can be reversed by a $\pi/4$ rotation. The spectral curves along these four loci have no higher symmetry than $C_4$ symmetry.

There are two isomorphic loci of spectral curves corresponding to the $l = 2$ maps. Here there is inversion symmetry, and the spectral curves are therefore $D_4$ symmetric and of the form (6.14). Reversing the sign of $\nu$ again corresponds to a $\pi/4$ rotation, and $\beta$ changes sign. The locus with $\nu$ negative interpolates between the asymptotic parameter values $\alpha = 4a^2, \beta = a^4, \gamma = 2a^4$ with $a$ large (corresponding to four stars at $(1/\sqrt{2})(\pm a, \pm a, 0)$ and the asymptotic values $\alpha = 2\pi^2 - 8b^2, \beta = 0, \gamma = (4b^2 + \pi^2)^2$ with $b$ large (corresponding to two axisymmetric charge 2 monopole clusters on the $x_3$-axis). Along the locus, $\alpha$ changes sign, so it is possible that when $\alpha = 0$ the locus passes through a curve with $\gamma = 14\beta$. This spectral curve would correspond to a charge 4 monopole with octahedral symmetry. Again, by analogy with Skyrmions, we conjecture that such a monopole does exist.

Finally, there is a single locus corresponding to the $l = 0$ maps. This interpolates between the asymptotic parameter values $\alpha = 4a^2, \beta = -a^4, \gamma = 2a^4$ and $\alpha = 4a^2, \beta = a^4, \gamma = 2a^4$, with $a$ large, and passes through the values $\alpha = 5\pi^2/2, \beta = 0, \gamma = 9\pi^4/16$, corresponding to the axisymmetric charge 4 monopole. Presumably, $\alpha$ and $\gamma$ are positive along the entire locus.

In summary, our main result is that in the geodesic scattering of charge $k$ monopoles, with $C_k$ symmetry and angular momentum zero, there are two kinds of motion. First, there is the well-known possibility of $k$ unit charge monopoles scattering in the $(x_1, x_2)$ plane through an angle $\pi/k$. Second, there is the novel possibility of $k$ unit charge monopoles coming in as before, but emerging as charge $l$ and charge $k-l$ axisymmetric monopoles moving back-to-back along the $x_3$-axis. $l$ can have any integer value in the range $0 < l < k$. We conjecture that in the special case $k = 3$, $l = 1$ or 2, the geodesic passes through a configuration with tetrahedral symmetry, and in the case $k = 4$, $l = 2$ the geodesic passes through a configuration with octahedral symmetry.

8. Connections with Skyrmions

The Skyrme model is a theory of a scalar field in three spatial dimensions with values in $S^3$. Finite energy fields are characterised by their degree $B$, identified physically with baryon number. There is a standard potential energy functional whose minima are the Skyrmions [16]. An interesting submanifold of Skyrme fields in $\mathbb{R}^3$ is obtained by calculating the holonomy of $SU(2)$ Yang-Mills instantons along lines parallel to the $x_4$-axis in $\mathbb{R}^4$ [17]. Instantons of charge $k$ give rise to an $(8k - 1)$-dimensional manifold of static Skyrme fields with baryon number $B = k$, and these Skyrme fields become dynamical if the instan-
ton moduli are regarded as time-dependent. Minima of the potential energy functional, restricted to this submanifold of fields, give good approximations to the Skyrmions, at least for $B \leq 4$; the fields have the same symmetries and approximately the same energies, and are easier to compute [18].

Braaten et al. have found the Skyrmions with $B \leq 6$ [19]. The $B = 1$ Skyrmion (the basic one) is spherically symmetric, like a unit charge monopole. The $B = 2$ Skyrmion is axially symmetric, with symmetry group $D_\infty$. This is like the axially symmetric charge 2 monopole. Moreover, two suitably orientated $B = 1$ Skyrmions scatter through $\pi/2$ in a head-on collision, like monopoles. This is particularly clear if the Skyrme field is constrained to the submanifold of instanton-generated fields [17].

The $B = 3$ Skyrmion has tetrahedral symmetry. Three $B = 1$ Skyrmions at the vertices of a large equilateral triangle, and orientated so that the attractive force between them is maximised, relax to the tetrahedral configuration as the triangle becomes smaller. Furthermore, we expect that in a head-on collision of three $B = 1$ Skyrmions, with the same initial configuration, the field will pass close to the tetrahedron and then emerge as a $B = 1$ Skyrmion and a $B = 2$ Skyrmion moving back-to-back along the 3-fold axis of symmetry of the initial triangle. (For Skyrmions there is no precise analogue of geodesic scattering at very slow speeds, so the outgoing Skyrmions would oscillate and emit some radiation, as do monopoles scattering at finite speed.) This type of 3-Skyrmion scattering has not been simulated numerically, but the motion we have described seems natural from the point of view of the instanton-generated $B = 3$ Skyrme fields. It is clearly analogous to the geodesic scattering of a charge 3 monopole on the surface $\Sigma_3$, with zero angular momentum. The existence of the tetrahedrally symmetric $B = 3$ Skyrmion strengthens the conjecture that a charge 3 monopole with this symmetry exists.

The $B = 4$ Skyrmion has octahedral symmetry. Moreover, for instanton-generated Skyrme fields, there is a scattering channel for four $B = 1$ Skyrmions with tetrahedral symmetry (and maximal attraction between the Skyrmions) in which they are at the vertices of a contracting tetrahedron, pass through an octahedrally symmetric configuration and emerge at the vertices of an expanding tetrahedron dual to the first [18]. This motion should be a good approximation to the dynamics of Skyrmions in the full theory. The monopole analogue of this motion would be a geodesic scattering of monopoles, where the spectral curve of the monopoles was at all times of the form (6.26), and instantaneously had octahedral symmetry. From the Skyrmyon results, we therefore conjecture that there is a locus of spectral curves of type (6.26), with $c$ taking all real values and $d$ a symmetric function of $c$. When $c = 0$, the spectral curve and associated monopole would have octahedral symmetry.

We may also make a conjecture about Skyrmion scattering, based on our monopole
results. Namely, in a head-on collision of two $B = 2$ Skyrmions, with their axes of symmetry along the collision path (and isospin-orientated so that their net pion dipoles are in opposite directions), we expect the field to pass close to the octahedral $B = 4$ Skyrmion and emerge as four $B = 1$ Skyrmions at the vertices of an expanding square in a plane perpendicular to the collision axis. This motion, or its reverse, would be analogous to the zero angular momentum geodesic monopole scattering on $\Sigma^2_4$.

The $B = 5$ and $B = 6$ Skyrmions have rather low symmetry. In particular, the $B = 6$ Skyrmion does not have the symmetry of the icosahedron. We therefore have no insight from Skyrmions into the possible existence of icosahedrally symmetric monopoles.

Finally, we remark that the relationship between Skyrmion scattering and monopole scattering is not systematically understood. A $B = 1$ Skyrmion has six degrees of freedom, whereas a unit charge monopole has four. The moduli space of charge $k$ monopoles has dimension $4k$. There is a less well-defined moduli space of Skyrme fields of baryon number $B$, of dimension $6B$, and a well-defined space of instanton-generated Skyrme fields of dimension $8B - 1$. It would be interesting if the charge $B$ monopole moduli space could be identified as a submanifold of either of these latter spaces. This is certainly possible for $B = 2$ [17].

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