PRESENTING GENERALIZED SCHUR ALGEBRAS IN TYPES $B$, $C$, $D$

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Abstract. We give explicit presentations by generators and relations of certain generalized Schur algebras (associated with tensor powers of the natural representation) in types $B$, $C$, $D$. This extends previous results in type $A$ obtained by two of the authors. The presentation is compatible with the Serre presentation of the corresponding universal enveloping algebra. In types $C$, $D$ this gives a presentation of the corresponding classical Schur algebra (the image of the representation on a tensor power) since the classical Schur algebra coincides with the generalized Schur algebra in those types. This coincidence between the generalized and classical Schur algebra fails in type $B$, in general.

Introduction

Throughout the paper we work over $\mathbb{Q}$. Vector spaces are over $\mathbb{Q}$ unless we say otherwise. In fact, all the results are valid over any field of characteristic zero.

In [D] a new presentation of a generalized Schur algebra was given. This presentation is compatible with Lusztig’s modified form $\mathfrak{U}$ of the universal enveloping algebra $\mathfrak{U}$, in which the Cartan generators are replaced by a system of orthogonal idempotents corresponding to weight space projectors. We are interested in the generalized Schur algebras $S(\pi)$ associated to the set $\pi = \pi(r)$ of dominant weights in a given tensor power $\mathfrak{E} \otimes r$ of the natural representation $\mathfrak{E}$ of a simple Lie algebra of type $B$, $C$, or $D$. Our main result is a presentation by generators and relations of $S(\pi)$ which is directly compatible with Serre’s presentation of $\mathfrak{U}$; i.e., the generators of the zero part are Cartan generators instead of idempotents. These results are formulated in Section 2 and proved
in Section 3; they extend results of [DG2, DG1] for type A to the other classical types.

We are also interested in the classical Schur algebras $S(r)$, which we define to be the image of the representation $\mathfrak{U} \to \text{End}(E^{\otimes r})$. These algebras are closely linked to classical invariant theory, and interest in them goes back to Schur and Weyl. They are in types $B$–$D$ the commuting algebras for the action on tensors of an appropriate Brauer algebra; see [B]. (In type $A$ the classical Schur algebras are the commuting algebras for the natural action of symmetric groups, acting by place permutation.) As explained in [D, §7.4], the classical Schur algebra coincides with the corresponding generalized Schur algebra $S(\pi)$ if and only if the set $\pi_0$ of highest weights labeling composition factors of $E^{\otimes r}$ coincides with the set $\pi$ of dominant weights occurring in $E^{\otimes r}$. The inclusion $\pi_0 \subset \pi$ is obvious. In types $A$, $C$, and $D$ it turns out that this inclusion is equality; in type $B$ this is not so — see 1.3.3 and 1.3.4. Thus the Schur algebra $S(r)$ coincides with the generalized Schur algebra $S(\pi)$ in types $A$, $C$, and $D$, but in type $B$ the Schur algebra $S(r)$ is in general a proper quotient of $S(\pi)$.

We rely on Weyl for the computation of the set $\pi_0$ of highest weights labeling composition factors of tensor powers of $E$. In order to make this exposition somewhat self-contained, we collect the relevant results from Weyl’s book in an appendix.

Although we work in characteristic zero throughout the paper, it should be noted that the Schur algebras and generalized Schur algebras considered herein are defined over $\mathbb{Z}$, so their study can be undertaken in any characteristic.

Another interesting problem is to find a basis of $S(\pi)$ which is in some sense compatible with a Poincare-Birkhoff-Witt basis of $\mathfrak{U}$. As an application of Littelmann’s path model, we describe such a basis in Section 4.

### 1. Generalized Schur algebras

Let $\mathfrak{g}$ be a reductive Lie algebra. The theory of generalized Schur algebras was introduced by Donkin in [Do1, Do2, Do4]. A generalized Schur algebra is a certain quotient of $\mathfrak{U} = \mathfrak{U}(\mathfrak{g})$ obtained by throwing away all but finitely many simple modules. More precisely, let $\pi$ be a saturated set of dominant weights, meaning that if $\lambda \in \pi$ and if $\mu$ is a dominant weight such that $\mu \preceq \lambda$ (in the usual dominance order), then $\mu \in \pi$. The generalized Schur algebra determined by $\pi$ is the algebra $S(\pi) := \mathfrak{U}(\mathfrak{g})/\mathcal{I}$, where $\mathcal{I}$ is the ideal of $\mathfrak{U}(\mathfrak{g})$ consisting of all elements
of \( \mathfrak{U}(\mathfrak{g}) \) annihilating every simple \( \mathfrak{U} \)-module of highest weight belonging to \( \pi \).

In this paper, we take \( \mathfrak{g} \) to be a simple Lie algebra of classical type \( B_n \), \( C_n \), or \( D_n \) and we always take \( \pi \) equal to the set \( \Pi^+(\mathbb{E}^{\mathfrak{r}}) \) of dominant weights of \( \mathbb{E}^{\mathfrak{r}} \), where \( \mathbb{E} \) is the natural representation of \( \mathfrak{g} \). Donkin [Do2, Do3] showed that, in types \( A \) and \( C \), the generalized Schur algebra \( S(\pi) \) determined by this choice of \( \pi \) coincides with the Schur algebra \( S(\mathbb{r}) \). We extend this to type \( D \); see Proposition 1.3.3 ahead. The corresponding result is not generally true in type \( B \).

1.1. Basic notation. In types \( B_n, C_n, D_n \) let \( \mathfrak{g} \) be defined by the form given by

\[
\begin{pmatrix}
0 & I & 0 \\
I & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}; \quad \begin{pmatrix}
0 & I \\
-I & 0 \\
I & 0
\end{pmatrix}; \quad \begin{pmatrix}
0 & I \\
0 & I
\end{pmatrix},
\]

where \( I \) is the \( n \times n \) identity matrix.

Let \( m = \dim \mathbb{E} \). Set \( e_{i,j} = (\delta_{ik}\delta_{lj})_{1 \leq k, l \leq m} \). The set \( \{e_{i,j} \mid 1 \leq i, j \leq m\} \) is a basis of \( \mathfrak{gl}_m \). In type \( A_{n-1} \) we have \( m = n \). The \( H_i := e_{i,i} \) \((1 \leq i \leq n)\) form a basis for the diagonal Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{gl}_n \). In types \( B_n, C_n, D_n \) we have respectively \( m = 2n + 1, 2n, 2n \) and we set \( H_i = e_{i,i} - e_{n+i,n+i} \) \((1 \leq i \leq n)\); the \( H_i \) form a basis for the diagonal Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{so}_{2n} \) respectively.

In all types \( A-D \), let \( \{\varepsilon_j\} \) be the basis of \( \mathfrak{h}^* \) dual to the basis \( \{H_i\} \) of \( \mathfrak{h} \); so that \( \varepsilon_j(H_i) = \delta_{ij} \). Define a bilinear form \((\ , \)\) on \( \mathfrak{h}^* \) such that \((\varepsilon_i, \varepsilon_j) = \delta_{ij} \).

Denote by \( \alpha_1, \ldots, \alpha_n \) a fixed choice of simple roots in types \( B_n, C_n, D_n \). Let \( (a_{ij}) \) be the Cartan matrix, defined by \( a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i) \).

1.2. Weights. We regard weights as \( n \)-tuples of rational numbers, determining linear functionals on \( \mathfrak{h} \) by recording their values on the basis \( H_1, \ldots, H_n \). In other words, we identify the linear combination \( \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \) with the tuple \( \lambda = (\lambda_1, \ldots, \lambda_n) \).

Fix simple root vectors \( e_i \in \mathfrak{g}_{\alpha_i}, f_i \in \mathfrak{g}_{-\alpha_i} \). The fundamental weights \( \varpi_j \) \((1 \leq j \leq n)\) are the elements of \( \mathfrak{h}^* \) defined by \( \varpi_j(h_i) = \delta_{ij} \), where \( h_i := [e_i, f_i] \). The lattice of integral weights is the free abelian group \( X \) generated by the \( \varpi_j \), and the set \( X^+ \) of dominant weights is the cone \( \sum \mathbb{N}\varpi_i \). One checks that \( X \) is generated by \( \varepsilon_1, \ldots, \varepsilon_n \) in type \( C_n \) and by \( \varepsilon_1, \ldots, \varepsilon_n \) together with the element \( (\varepsilon_1 + \cdots + \varepsilon_n)/2 \) in types \( B_n, D_n \). As usual, the dominance (partial) order on \( X \) is defined by declaring that \( \lambda \leq \mu \) (for \( \lambda, \mu \in X \)) if \( \mu - \lambda \in \mathbb{N}\alpha_1 + \cdots + \mathbb{N}\alpha_n \).
We identify $\mathfrak{h}^*$ with $\mathbb{Q}^n$ by regarding $\varepsilon_1, \ldots, \varepsilon_n$ as the standard basis of $\mathbb{Q}^n$; this identifies $X$ with a subgroup of $\mathbb{Q}^n$. In fact, under this identification, we have

\begin{equation}
X = \begin{cases} 
\mathbb{Z}^n & \text{(type } C_n); \\
\mathbb{Z}^n \cup ((\frac{1}{2}, \ldots, \frac{1}{2}) + \mathbb{Z}^n) & \text{(types } B_n, D_n). 
\end{cases}
\end{equation}

The fundamental weights in type $C_n$ are given explicitly by the equalities

\begin{equation}
\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \quad (1 \leq i \leq n). \tag{1.2.2}
\end{equation}

In type $B_n$ the fundamental weights are given by

\begin{equation}
\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \quad (1 \leq i \leq n-1), \quad \varpi_n = (\varepsilon_1 + \cdots + \varepsilon_n)/2 \tag{1.2.3}
\end{equation}

and in type $D_n$ by

\begin{equation}
\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \quad (1 \leq i \leq n-2), \quad \varpi_{n-1} = (\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n)/2, \quad \varpi_n = (\varepsilon_1 + \cdots + \varepsilon_n)/2. \tag{1.2.4}
\end{equation}

The set $X^+$ of dominant weights is the set of all $\lambda = (\lambda_1, \ldots, \lambda_n) \in X$ satisfying

\begin{equation}
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \quad \text{(types } B_n, C_n); \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n| \quad \text{(type } D_n). \tag{1.2.5}
\end{equation}

1.3. **Signed compositions.** We shall need the following notations. Write

\begin{align*}
\Lambda(n, r) &= \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n \mid \sum \lambda_i = r\}; \\
\overline{\Lambda}(n, r) &= \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \sum |\lambda_i| = r\}.
\end{align*}

The first set is the set of $n$-part compositions of $r$ and the second is the set of $n$-part **signed** compositions of $r$. Note that we allow 0 to appear in a composition. Set

\begin{align*}
\Lambda^+(n, r) &= \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n \mid \sum \lambda_i = r, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}. \\
\Lambda^-(n, r) &= \{\lambda^- \mid \lambda \in \Lambda^+(n, r)\}, \quad \Lambda^+(n, r) = \Lambda^-(n, r) \cup \Lambda^+(n, r).
\end{align*}

As usual, we identify members of $\Lambda^+(n, r)$ with partitions of not more than $n$ parts. Given $\lambda \in \Lambda^+(n, r)$, set $\lambda^- := (\lambda_1, \ldots, \lambda_{n-1}, -\lambda_n)$ (its associated weight), and let

\begin{align*}
\Lambda^-(n, r) &= \{\lambda^- \mid \lambda \in \Lambda^+(n, r)\}, \quad \Lambda^+(n, r) = \Lambda^-(n, r) \cup \Lambda^+(n, r) \\
\Lambda^+(n, r) \text{ and } \Lambda^-(n, r).
\end{align*}

Label the finite-dimensional simple $g$-modules $L$ by their highest weight $\lambda \in X^+$ (regarded as a vector of $\mathfrak{h}$-eigenvalues on $H_1, \ldots, H_n$).
Proposition 1.3.1. (a) The set of weights $\Pi$ of $E \otimes r$ is the set of all signed $n$-part compositions of $r - 2j$ for $0 \leq j \leq [r/2]$ (i.e. the union of the $\Lambda(n, r - 2j)$ for $0 \leq j \leq [r/2]$) in types $C_n$, $D_n$ and the set of all signed $n$-part compositions of $r - j$ for $0 \leq j \leq r$ (i.e. the union of the $\Lambda(n, r - j)$ for $0 \leq j \leq r$) in type $B_n$.

(b) The set $\Pi^+$ of dominant weights of $E \otimes r$ is the union over $0 \leq j \leq [r/2]$ of the sets $\Lambda^+(n, r - 2j)$ in type $C_n$, the union over $0 \leq j \leq r$ of the sets $\Lambda^+(n, r - j)$ in type $B_n$, and the union over $0 \leq j \leq [r/2]$ of the sets $\Lambda^\pm(n, r - 2j)$ in type $D_n$.

Proof. In types $C_n$, $D_n$ the weights of $E$ are $\{\pm \varepsilon_1, \ldots, \pm \varepsilon_n\}$. In type $B_n$ the weights of $E$ are $\{\pm \varepsilon_1, \ldots, \pm \varepsilon_n\} \cup \{0\}$. The weights of $E^\otimes r$ are given by all expressions of the form

$$w_1 + \cdots + w_r$$

where $w_i$ is a weight of $E$ for each $i$. (The $w_i$ are not necessarily distinct.) Part (a) is now clear.

To prove part (b), combine part (a) with (1.2.5).

Proposition 1.3.2. The set $\pi = \Pi^+(E^\otimes r)$ of dominant weights in $E^\otimes r$ is a saturated subset of $X^+$, for types $B_n$, $C_n$, $D_n$.

Proof. One can decompose $E^\otimes r$ into a direct sum of irreducible modules. The set of dominant weights of an irreducible is necessarily saturated (in characteristic zero), and the union of saturated sets is necessarily saturated. In fact, this argument shows that the set of dominant weights of any $g$-module must be a saturated set.

We remark that one can also give a combinatorial proof of the preceding result.

Proposition 1.3.3. The Schur algebras $S(r)$ in types $C_n$, $D_n$ are generalized Schur algebras determined by the saturated set $\pi = \Pi^+(E^\otimes r)$.

Proof. $S(r)$ is by definition the image of the representation $\mathfrak{U} \rightarrow \text{End}(E^\otimes r)$, so $S(r) \simeq \mathfrak{U}/A$ where $A$ is the annihilator of $E^\otimes r$. By Wedderburn theory, the simple $S(r)$-modules are the direct summands of $E^\otimes r$, and thus must have highest weight belonging to the set $\pi = \Pi^+(E^\otimes r)$.

Let $S(\pi) = \mathfrak{U}/\mathcal{I}$ be Donkin’s generalized Schur algebra determined by the saturated set $\pi$. Here $\mathcal{I}$ is the ideal of $\mathfrak{U}$ consisting of the elements annihilating every simple $\mathfrak{U}$-module of highest weight belonging to $\pi$. Clearly $A \subseteq \mathcal{I}$.

Let $\pi_0$ be the set of highest weights of composition factors appearing as a direct summand in a Wedderburn decomposition of $E^\otimes r$. Weyl
computed the decomposition of tensor space for the special orthogonal and symplectic groups. His results show that, in types $C_n$ and $D_n$, $\pi_0 = \pi$. (See the appendix for a detailed summary of Weyl’s results, with references.) This justifies the equality $A = I$, in types $C$, $D$. The proof is complete. □

Remark 1.3.4. The preceding result often fails for type $B_n$. Indeed, the natural module $E$ has two dominant weights but only one highest weight, so for any $n$ the needed equality $\pi_0 = \pi$ fails already for $r = 1$. Moreover, in type $B_2 = \mathfrak{so}_5$ we have $\pi_0 = \{(2,0), (1,1), (0,0)\}$ but the set $\pi$ is $\{(2,0), (1,1), (1,0), (0,0)\}$. Here we relied on Weyl’s Theorem A.3 in the appendix for a description of $\pi_0$ in type $B$, and Proposition 1.3.1 above for the set $\pi$.

Interestingly, the desired equality $\pi_0 = \pi$ holds when $n = 1$ and $r \geq 2$. It would be useful to classify those pairs $n$ and $r$ for which $\pi_0 = \pi$ for type $B_n$. ◊

1.4. The idempotent presentation. Let $\pi = \Pi^+(E^{\otimes r})$, in types $B$, $C$, and $D$. In [D, 6.13], it was shown that the generalized Schur algebra $S(\pi)$ is isomorphic with the associative algebra (with 1) on generators $e_i, f_i$ ($1 \leq i \leq n$), $1_\lambda$ ($\lambda \in W\pi$) with the relations

\begin{align*}
(R1) \quad & 1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda, \quad \sum_{\lambda \in W\pi} 1_\lambda = 1 \\
(R2) \quad & e_i f_j - f_j e_i = \delta_{ij} \sum_{\lambda \in W\pi} (\alpha_i^\vee, \lambda) 1_\lambda \\
(R3) \quad & e_i 1_\lambda = \begin{cases} 1_{\lambda+\alpha_i} e_i & \text{if } \lambda + \alpha_i \in W\pi \\ 0 & \text{otherwise} \end{cases} \\
(R4) \quad & f_i 1_\lambda = \begin{cases} 1_{\lambda-\alpha_i} f_i & \text{if } \lambda - \alpha_i \in W\pi \\ 0 & \text{otherwise} \end{cases} \\
(R5) \quad & 1_\lambda e_i = \begin{cases} e_i 1_{\lambda-\alpha_i} & \text{if } \lambda - \alpha_i \in W\pi \\ 0 & \text{otherwise} \end{cases} \\
(R6) \quad & 1_\lambda f_i = \begin{cases} f_i 1_{\lambda+\alpha_i} & \text{if } \lambda + \alpha_i \in W\pi \\ 0 & \text{otherwise} \end{cases} \\
(R7) \quad & \sum_{s=0}^{1-a_{ij}} (-1)^s \binom{1-a_{ij}}{s} e_i^{1-a_{ij}-s} e_j^s e_i^s = 0 \quad (i \neq j)
\end{align*}
Here $W$ is the Weyl group attached to the Lie algebra $\mathfrak{g}$, the $a_{ij}$ are as before, and $\alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i)$ for $i = 1, \ldots, n$. Note that $W\pi$ is equal to $\Pi(E^{\otimes r})$, the set described explicitly in Proposition 1.3.1.

2. Main results

In the statements to follow, notice that the first, third, fourth, fifth, and sixth relations are identical in all types. In other words, only the second and seventh relations vary by type. (The seventh relation is the same in types $C, D$.)

2.1. Type $B$. The root system for $B_n$ is realized by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i < n$; $\alpha_n = \varepsilon_n$.

**Theorem 2.1.1.** Over $\mathbb{Q}$, the generalized Schur algebra $S(\pi)$ of type $B_n$ is isomorphic with the associative algebra (with 1) on generators $e_i$, $f_i$, $H_i$ ($1 \leq i \leq n$) and with relations

(B1) \[ H_i H_j = H_j H_i \]

(B2) \[ e_i f_j - f_j e_i = \begin{cases} \delta_{ij}(H_i - H_{i+1}) & (i < n) \\ \delta_{ij}(2H_n) & (i = n) \end{cases} \]

(B3) \[ H_i e_j - e_j H_i = (\varepsilon_i, \alpha_j)e_j, \quad H_i f_j - f_j H_i = - (\varepsilon_i, \alpha_j)f_j \]

(B4) \[ \sum_{s=0}^{1-a_{ij}} (-1)^s \left( \sum_{s=0}^{1-a_{ij}} (-1)^s \right) f_i^{1-a_{ij}-s} f_j f_i^s = 0 \quad (i \neq j) \]

(B5) \[ \sum_{s=0}^{1-a_{ij}} (-1)^s \left( \sum_{s=0}^{1-a_{ij}} (-1)^s \right) f_i^{1-a_{ij}-s} f_j f_i^s = 0 \quad (i \neq j) \]

(B6) \[ (H_i + r)(H_i + r - 1)(H_i + r - 2) \cdots (H_i - r) = 0 \]

(B7) \[ (J + r)(J + r - 1)(J + r - 2) \cdots (J - r + 1)(J - r) = 0 \]

where $J = \pm H_1 \pm H_2 \pm \cdots \pm H_n$ varies over all $2^n$ possible sign choices.

Note that the enveloping algebra $\mathfrak{U}(\mathfrak{g}_0^{2n+1})$ is the algebra on the same generators but subject only to the relations (B1)--(B5); moreover, that presentation of $\mathfrak{U}(\mathfrak{g}_0^{2n+1})$ is equivalent to the usual Serre presentation.

The relation (B6) is necessary.
2.2. Type $C$. The root system for $C_n$ is realized by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i < n$; $\alpha_n = 2\varepsilon_n$.

**Theorem 2.2.1.** Over $\mathbb{Q}$, the generalized Schur algebra $S(\pi)$ of type $C_n$ (which coincides with the Schur algebra $S(r)$) is isomorphic with the associative algebra (with 1) on generators $e_i, f_i, H_i$ ($1 \leq i \leq n$) and with relations

\begin{align*}
(C1) & \quad & H_i H_j = H_j H_i \\
(C2) & \quad & e_i f_j - f_j e_i = \begin{cases} 
\delta_{ij}(H_i - H_{i+1}) & (i < n) \\
\delta_{ij} H_n & (i = n)
\end{cases} \\
(C3) & \quad & H_i e_j - e_j H_i = (\varepsilon_i, \alpha_j) e_j, \quad H_i f_j - f_j H_i = -(\varepsilon_i, \alpha_j) f_j \\
(C4) & \quad & \sum_{s=0}^{1-a_{ij}} (-1)^{s} \binom{1-a_{ij}}{s} e_i^{1-a_{ij}-s} e_j e_i^s = 0 \quad (i \neq j) \\
(C5) & \quad & \sum_{s=0}^{1-a_{ij}} (-1)^{s} \binom{1-a_{ij}}{s} f_i^{1-a_{ij}-s} f_j f_i^s = 0 \quad (i \neq j) \\
(C6) & \quad & (H_i + r)(H_i + r - 1)(H_i + r - 2) \cdots (H_i - r) = 0 \\
(C7) & \quad & (J + r)(J + r - 2)(J + r - 4) \cdots (J - r + 2)(J - r) = 0
\end{align*}

where $J = \pm H_1 \pm H_2 \pm \cdots \pm H_n$ varies over all possible sign choices.

Note that the enveloping algebra $\mathfrak{U}(\mathfrak{sp}_{2n})$ is the algebra on the same generators but subject only to the relations (C1)–(C5); that presentation of $\mathfrak{U}(\mathfrak{sp}_{2n})$ is equivalent to the usual Serre presentation.

The relation (C6) is superfluous.

2.3. Type $D$. The root system for $D_n$ is realized by taking $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i < n$; $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$.

**Theorem 2.3.1.** Over $\mathbb{Q}$, the generalized Schur algebra $S(\pi)$ of type $D_n$ (which coincides with the Schur algebra $S(r)$) is isomorphic with the associative algebra (with 1) on generators $e_i, f_i, H_i$ ($1 \leq i \leq n$) and with relations

\begin{align*}
(D1) & \quad & H_i H_j = H_j H_i \\
(D2) & \quad & e_i f_j - f_j e_i = \begin{cases} 
\delta_{ij}(H_i - H_{i+1}) & (i < n) \\
\delta_{ij} (H_{n-1} + H_n) & (i = n)
\end{cases} \\
(D3) & \quad & H_i e_j - e_j H_i = (\varepsilon_i, \alpha_j) e_j, \quad H_i f_j - f_j H_i = -(\varepsilon_i, \alpha_j) f_j
\end{align*}
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\[
\begin{align*}
(D4) & \quad \sum_{s=0}^{1-a_{ij}} (-1)^s \left(1 - a_{ij}\right)^s e_i^{1-a_{ij} - s} e_j^{1} e_i^s = 0 \quad (i \neq j) \\
(D5) & \quad \sum_{s=0}^{1-a_{ij}} (-1)^s \left(1 - a_{ij}\right)^s f_i^{1-a_{ij} - s} f_j^{1} f_i^s = 0 \quad (i \neq j) \\
(D6) & \quad (H_i + r)(H_i + r - 1)(H_i + r - 2) \cdots (H_i - r) = 0 \\
(D7) & \quad (J + r)(J + r - 2)(J + r - 4) \cdots (J - r + 2)(J - r) = 0
\end{align*}
\]

where \( J = \pm H_1 \pm H_2 \pm \cdots \pm H_n \) varies over all possible sign choices.

The enveloping algebra \( \mathfrak{U}(\mathfrak{so}_{2n}) \) is the algebra on the same generators but subject only to the relations (D1)–(D5); that presentation of \( \mathfrak{U}(\mathfrak{so}_{2n}) \) is equivalent to the usual Serre presentation.

The relation (D6) is superfluous.

The proof of all three theorems of this section is given in the next section. Our strategy is to show that the presentation of the theorem is equivalent to the idempotent presentation of section 1.4.

Remark 2.3.8. One can easily show that any \( H_i \), viewed as an operator on \( E \otimes r \), satisfies its minimal polynomial \( P_1(T) \), and similarly that any \( J \), viewed as an operator on \( E \otimes r \), satisfies its minimal polynomial \( P_1(T) \), in type \( B \), or \( P_2(T) \), in types \( C, D \).

3. Proof of the main theorems

We will show that the generalized Schur algebra \( S(\pi) \), with \( \pi = \Pi^+(E \otimes r) \), defined by the presentation in 1.4 is isomorphic with the algebra given by the generators and relations of Theorems 2.1.1, 2.2.1, or 2.3.1, in types \( B_n \)–\( D_n \).

3.1. The algebra \( \Phi \). Write \( \mathfrak{U} = \mathfrak{U}(\mathfrak{g}) \). Given a positive integer \( r \), set

\[
\begin{align*}
P_1(T) & = (T + r)(T + r - 1) \cdots (T - r + 1)(T - r), \\
P_2(T) & = (T + r)(T + r - 2) \cdots (T - r + 2)(T - r)
\end{align*}
\]

polynomials of degree \( 2r + 1 \), \( r + 1 \), respectively. Let \( \Phi \) be the algebra given by the generators and relations of Theorem 2.1.1, 2.2.1, or 2.3.1. Then \( \Phi = \mathfrak{U}/I \). In types \( C_n \) and \( D_n \), \( I \) is the two-sided ideal of \( \mathfrak{U} \) generated by the \( P_1(H_i) \) (\( i = 1, \ldots, n \)) and \( P_2(J) \) for all \( J = \pm H_1 \pm \cdots \pm H_n \). In type \( B_n \), \( I \) is the two-sided ideal of \( \mathfrak{U} \) generated by the \( P_1(H_i) \) (\( i = 1, \ldots, n \)) and \( P_1(J) \) for all \( J \).
From the triangular decomposition \( \mathfrak{U} = \mathfrak{U}^- \Phi \mathfrak{U}^+ \) of \( \mathfrak{U} \) we have a corresponding triangular decomposition \( \Phi = \Phi^- \Phi^0 \Phi^+ \), where each algebra \( \Phi^- \), \( \Phi^0 \), \( \Phi^+ \) is defined to be the image under the appropriate surjective map of the corresponding subalgebra of \( \mathfrak{U} \). Let \( \mathfrak{U}_\mathbb{Z} \) be Kostant’s \( \mathbb{Z} \)-form of \( \mathfrak{U} \) relative to the Chevalley generators \( e_i, f_i \); this is the \( \mathbb{Z} \)-subalgebra of \( \mathfrak{U} \) generated by all \( f_i^{(a)}, e_i^{(c)} \) \((a, c \in \mathbb{N}, 1 \leq i \leq n)\). Then we have equalities \( \mathfrak{U}_\mathbb{Z} = \mathfrak{U}_\mathbb{C} \Phi_\mathbb{C}^0 \mathfrak{U}_\mathbb{C}^+ \), \( \Phi_\mathbb{Z} = \Phi_\mathbb{C} \Phi_\mathbb{C}^0 \Phi_\mathbb{C}^+ \) where the various subalgebras are defined in the obvious manner.

For the moment, regard \( H_1, \ldots, H_n \) as commuting indeterminates. A given set of polynomials in the polynomial ring \( \mathbb{Q}[H_1, \ldots, H_n] \) determines an affine variety in \( \mathbb{Q}^n \).

**Proposition 3.1.2.** Let \( V \subseteq \mathbb{Q}^n \) be the common zero locus of \( P_2(J) \) for all \( J = \pm H_1 \pm H_2 \pm \cdots \pm H_n \), in types \( C_n, D_n \). In type \( B_n \) let \( V \subseteq \mathbb{Q}^n \) be the common zero locus of \( P_2(J) \) for all \( J \), along with \( P_1(H_i) \) for all \( i = 1, \ldots, n \). Then \( V = \Pi \), the set of weights of \( \mathfrak{E}^{\mathfrak{pr}} \).

**Proof.** For fixed \( r \), let \( T = \{-r, -r + 2, \ldots, r - 2, r\} \). This is the set of \( m \in \mathbb{Z} \) satisfying \( |m| \leq r \), \( m \equiv r \) (mod 2). The set \( \Pi \), in types \( C_n \) and \( D_n \), is the set of \( (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \) satisfying \( \sum |\lambda_i| \leq r \).

Consider \( v = (v_1, \ldots, v_n) \in V \), a solution to \( P_2(J) = 0 \) for all choices of \( J = \pm H_1 \pm \cdots \pm H_n \). Then \( \pm v_1 \pm v_2 \pm \cdots \pm v_n \in T \) for all possible sign choices. In particular, for every \( i \) with \( 1 \leq i \leq n \) we have

\[
v_1 + \cdots + v_n \in T \quad \text{and} \quad -v_1 - \cdots - v_{i-1} + v_i - v_{i+1} - \cdots - v_n \in T.
\]

From this we conclude that \( 2v_i \) is an even integer since the sum of any two elements of \( T \) is even. Thus \( v_i \in \mathbb{Z} \) for all \( i \). Since \( \sum |v_i| \leq r \), \( v_i \in \mathbb{Z} \) for all \( i \).

Now take \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \Pi \). We need to prove that \( \pm \lambda_1 + \cdots + \lambda_n \in T \) for all sign choices. By the description of \( \Pi \) given above, we know \( \sum |\lambda_i| \leq r \). For any choice \( \sigma_i \) of signs, we have the congruence \( \sum \sigma_i |\lambda_i| \equiv \sum |\lambda_i| \) (mod 2). Moreover, \( |\sum \sigma_i |\lambda_i|| \leq \sum |\lambda_i| \leq r \). Thus \( \Pi \subseteq V \). This proves that \( V = \Pi \) in types \( C_n \) and \( D_n \).

Now we turn to type \( B_n \). Let \( T' = \{-r, -r + 1, \ldots, r - 1, r\} \) and note that in this case \( \Pi \) is the set of \( (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \) satisfying \( \sum |\lambda_i| \leq r \). By an argument similar to the above, it follows that the set of solutions to the equations \( P_1(J) = 0 \) for all \( J \) coincides with the set of \( (v_1, \ldots, v_n) \in (\frac{1}{2}\mathbb{Z})^n \) such that \( \sum |v_i| \leq r \). If the additional equations \( P_1(H_i) = 0 \) are imposed, then it is clear that the solution set is reduced exactly to \( \Pi = \Pi(\mathfrak{E}^{\mathfrak{pr}}) \). The proof is complete. \( \square \)

**Remark 3.1.3.** The proof shows, in particular, that relations (C6), (D6) are consequences of relations (C7), (D7). \( \diamond \)
We now consider the algebra $\Phi^0 = \mathfrak{U}^0/(\mathfrak{U}^0 \cap I)$. By the PBW theorem, $\mathfrak{U}^0$ is isomorphic with the algebra $\mathbb{Q}[H_1, \ldots, H_n]$ of polynomials in commuting "indeterminates" $H_1, \ldots, H_n$. Define an algebra $\Phi' = \mathfrak{U}^0/I^0$ where $I^0$ is the ideal in $\mathfrak{U}^0$ generated by $P_2(J)$ for all $J$ and $P_1(H_i)$ for all $i$, in types $C_n, D_n$, and is the ideal in $\mathfrak{U}^0$ generated by $P_1(J)$ for all $J$ and by $P_1(H_i)$ for all $i$, in type $B_n$. (In types $C_n, D_n$ the generators $P_1(H_i)$ are not needed.) Given $\lambda \in \Pi$, define an element $1_\lambda \in \Phi^0$ by

$$1_\lambda = \prod_i \frac{P_1^{(\lambda_i)}(H_i)}{P_1^{(\lambda_i)}(\lambda_i)}$$

where $P_1^{(k)}(T)$ equals $P_1(T)$ with factor $(T - k)$ deleted, for a given $k$ satisfying $-r \leq k \leq r$. Since $(H_i - \lambda_i)P_1^{(\lambda_i)}(H_i) = 0$ (by definition of $\Phi$) we see from (3.1.4) that

$$H_i 1_\lambda = \lambda_i 1_\lambda \quad (\lambda \in \Pi, 1 \leq i \leq n).$$

**Proposition 3.1.6.** (a) The algebra $\Phi^0$ is isomorphic with the algebra $\Phi'$.

(b) The set of all $1_\lambda (\lambda \in \Pi)$ is a $\mathbb{Q}$-basis for $\Phi^0$ and a $\mathbb{Z}$-basis for $\Phi^0_{\mathbb{Z}}$; moreover, this set is a set of pairwise orthogonal idempotents in $\Phi^0$ which add up to 1.

**Proof.** View the $H_i$ as coordinate functions on $\mathbb{Q}^n$. The algebra $\Phi'$ is the ring of regular functions on the variety of common zeros of $I^0$. By the previous proposition, this variety is the finite set $\Pi$. The coordinate ring of $\Pi$ is just the product $\prod_{\lambda \in \Pi} \mathbb{Q}_\lambda$ where $\mathbb{Q}_\lambda \cong \mathbb{Q}$ is the function ring of the $\lambda$. (The only functions defined at a single point are the constants.) The explicit isomorphism $\Phi^0 \cong \prod_{\lambda \in \Pi} \mathbb{Q}_\lambda$ is realized by the map (denoted by $\phi$) which sends $f(H_1, \ldots, H_n)$ to $\prod_{\lambda \in \Pi} f(\lambda_1, \ldots, \lambda_n)$. It is easy to check that $1_\lambda(\lambda') = \delta_{\lambda\lambda'}$. Thus, $\phi(1_\lambda)$ is a vector whose entries are all zero except for one entry which equals one. Since $\phi$ is an isomorphism, it follows that the set $1_\lambda (\lambda \in \Pi)$ is a $\mathbb{Q}$-basis for $\Phi'$ and this set is a set of pairwise orthogonal idempotents of $\Phi'$ which add up to 1.

By the definition of $\Phi$ we have an algebra surjection $\mathfrak{U} \to \Phi$. By restriction, this induces an algebra surjection $\mathfrak{U}^0 \to \Phi^0$. The canonical quotient map $\mathfrak{U} \to \mathfrak{U}/I$ induces, upon restriction to $\mathfrak{U}^0$, a map $\mathfrak{U}^0 \to \mathfrak{U}/I = \Phi$. The image of this map is $\Phi^0$ and its kernel is $\mathfrak{U}^0 \cap I$, so $\Phi^0 \cong \mathfrak{U}^0/(\mathfrak{U}^0 \cap I)$. Clearly $I^0 \subseteq \mathfrak{U}^0 \cap I$. Thus we obtain an algebra surjection:

$$\Phi' = \mathfrak{U}^0/I^0 \to \Phi^0.$$
The dimension of $\Phi'$ is the cardinality of $\Pi$, the set of weights appearing in the representation $E^{\otimes r}$.

We consider the quotient of the polynomial ring $\mathbb{Q}[H_1, \ldots, H_n]$ (the variables $H_i$ commute) by the ideal $I^0$. It suffices to show that this quotient is isomorphic with $|\Pi|$ copies of the base field $\mathbb{Q}$. By the Chinese remainder theorem, applied repeatedly to the factors of the polynomial $P_1(H_i)$, for each $i = 1, \ldots, n$, we obtain an isomorphism

$$\mathbb{Q}[H_1, \ldots, H_n]/I^0 \simeq \prod_{\lambda_1, \ldots, \lambda_n} \mathbb{Q}[H_1, \ldots, H_n]/(H_1 - \lambda_1, \ldots, H_n - \lambda_n, P_k(J))$$

where $k = 1$ in types $C_n$, $D_n$ and $k = 2$ in type $B_n$. In the product, each integer $\lambda_i$ belongs to the interval $[-r, r]$. Each factor in the product is either $\mathbb{Q}$ or zero because the relations $H_1 - \lambda_1 = 0, \ldots, H_n - \lambda_n = 0$ make each variable $H_i$ a constant. Consider the factor for a selection of constants $\lambda_1, \ldots, \lambda_n$. If those values satisfy the identity $P_k(J) = 0$ for every choice of $J$, then the selection of constants gives a weight of the $r$th tensor power of $E$, and the factor $\mathbb{Q}[H_1, \ldots, H_n]/(H_1 - \lambda_1, \ldots, H_n - \lambda_n, P_k(J))$ is $\mathbb{Q}$. If the selected values do not satisfy the relation $P_k(J) = 0$ for every choice of $J$, then the evaluation of such a relation at $\lambda_1, \ldots, \lambda_n$ gives a nonzero constant in the ideal, so the quotient $\mathbb{Q}[H_1, \ldots, H_n]/(H_1 - \lambda_1, \ldots, H_n - \lambda_n, P_k(J))$ is 0.

It follows that $\Phi^0$ has the same dimension as $\Phi'$. Thus the surjection (3.1.7) is an isomorphism of algebras. This proves assertions (a) and (b). \qed

**Remark 3.1.8.** (a) The argument shows in particular that $I^0 \cap I = I^0$, an equality which is not obvious from the definitions.

(b) From the proposition and (3.1.5) it follows immediately (by multiplication by $1 = \sum_{\lambda} 1_{\lambda}$) that in $\Phi$ we have the equality $H_i = \sum_{\lambda \in \Pi} \lambda_i 1_{\lambda}$ for any $i$. \qed

**Proposition 3.1.9.** The elements $e_i, f_i$ (1 \(\leq i \leq n\)), $1_{\lambda}$ ($\lambda \in \Pi$) of $\Phi$ satisfy relations (R1)–(R8), with $\Pi = W\pi$.

**Proof.** Relation (R1) was proved in the previous proposition. To prove (R2), first consider the case $i \neq j$. Then relations (B2), (C2), (D2) all assert that $e_if_j - f_je_i = 0$, which is precisely relation (R2) in this case. Now suppose $i = j$ is strictly less than $n$. Then $\alpha_i' = \alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for types $B_n, C_n$, and $D_n$. Thus $(\alpha_i', \lambda) = \lambda_i - \lambda_{i+1}$. Since $i < n$, relations (B2), (C2), (D2) all assert that $e_if_i - f_ie_i = H_i - H_{i+1}$. By Remark 3.1.8(b),

$$H_i - H_{i+1} = \sum_{\lambda \in \Pi}(\lambda_i - \lambda_{i+1}) 1_{\lambda}.$$
This proves (R2) in the case \( i = j < n \). Now consider the final remaining case \( i = j = n \). In type \( B_n \), we have \( \alpha_n^\vee = 2\alpha_n = 2\varepsilon_n \), and \( e_n f_n - f_n e_n = 2H_n \). Relation (R2) for type \( B_n \) now follows since \( 2H_n 1_\lambda = (\alpha_n^\vee, \lambda) 1_\lambda \) for all \( \lambda \). In type \( C_n \), \( \alpha_n^\vee = \alpha_n/2 = \varepsilon_n \) and \( e_n f_n - e_n f_n = H_n \). Relation (R2) for type \( C_n \) now follows since \( H_n 1_\lambda = (\alpha_n^\vee, \lambda) 1_\lambda \) for all \( \lambda \). Finally, in type \( D_n \), \( \alpha_n^\vee = \alpha_n = \varepsilon_{n-1} + \varepsilon_n \) and \( e_n f_n - e_n f_n = H_{n-1} + H_n \). In exactly the same way as for the other cases, it follows at once that (R2) holds for type \( D_n \).

We now prove relation (R3). First, from relations (B3), (C3), and (D3) we see that \( e_j H_i = (H_i - (\varepsilon_i, \alpha_j)) e_j \) and so from (3.1.4) we obtain the equality (in \( \Phi \))

\[
(3.1.10) \quad e_j 1_\lambda = P(H_1, \ldots, H_n) e_j
\]

where \( P(H_1, \ldots, H_n) \) is defined by

\[
(3.1.11) \quad P(H_1, \ldots, H_n) = \prod_i \frac{P_1^{(\lambda_i)}(H_i - (\varepsilon_i, \alpha_j))}{P_1^{(\lambda_i)}(\lambda_i)}.
\]

From Remark 3.1.8(b) and the definition of \( P(H_1, \ldots, H_n) \) we obtain the equality

\[
(3.1.12) \quad P(H_1, \ldots, H_n) = \sum_{\mu \in \Pi} P(\mu_1, \ldots, \mu_n) 1_\mu.
\]

In order to analyze this expression, first note that by its definition \( P_1^{(\lambda_i)}(x) = 0 \) for all integers \( x \) except \( x = \lambda_i \) or \( |x| \geq r + 1 \). Thus for a given \( \mu \in \Pi \), \( P(\mu_1, \ldots, \mu_n) = 0 \) unless, for all \( i \in \{1, \ldots, n\} \), one of the conditions \( \mu_i - (\varepsilon_i, \alpha_j) = \lambda_i \) or \( |\mu_i - (\varepsilon_i, \alpha_j)| \geq r + 1 \) holds.

Now suppose there exists \( \mu \in \Pi \) and \( i \in \{1, \ldots, n\} \) such that \( |\mu_i - (\varepsilon_i, \alpha_j)| \geq r + 1 \). In this case, even though \( P(\mu_1, \ldots, \mu_n) \) need not vanish, the product \( 1_\mu e_j \) is necessarily zero as we now show. We have the equality \( 1_\mu (H_i - \mu_i) = 0 \), so trivially \( 1_\mu (H_i - \mu_i) e_j = 0 \). But, using relation (B3), (C3), or (D3), we may rewrite the last equality in the form

\[
(3.1.13) \quad 1_\mu e_j (H_i + (\varepsilon_i, \alpha_j) - \mu_i) = 0.
\]

The rightmost factor, \( H_i + (\varepsilon_i, \alpha_j) - \mu_i \) can be expressed via Remark 3.1.8(b) as the sum

\[
H_i + (\varepsilon_i, \alpha_j) - \mu_i = \sum_{\nu \in \Pi} (\nu_i + (\varepsilon_i, \alpha_j) - \mu_i) 1_\nu
\]

in which every coefficient differs from zero since we are under the assumption that \( |\mu_i - (\varepsilon_i, \alpha_j)| \geq r + 1 \) and each component of an element of \( \Pi \) lies in the interval \([-r, r]\). It follows that \( H_i + (\varepsilon_i, \alpha_j) - \mu_i \) is an
invertible element of $\Phi$ and so we can multiply equation (3.1.13) by its inverse on the right to obtain the desired result that $1\mu e_j = 0$.

Thus, the only $\mu \in \Pi$ for which $P(\mu_1, \ldots, \mu_n)1\mu e_j \neq 0$ is determined by $\mu_i - (\varepsilon_i, \alpha_j) = \lambda_i$ for all $i \in \{1, \ldots, n\}$. Thus $\mu = \lambda + \alpha_j$. Moreover, one easily sees that $P(\lambda + \alpha_j) = 1$. Relation (R3) now follows. The proofs of relations (R4)–(R6) are similar. Finally, (R7) and (R8) hold since these are among the defining relations for $\mathfrak{U}(g)$. The proof is complete.

We note the following corollary for later reference.

**Corollary 3.1.14.** With $\pi = \Pi^+(E^\otimes r)$, there is a surjective algebra homomorphism $S(\pi) \rightarrow \Phi$ given by $e_i \rightarrow e_i$, $f_i \rightarrow f_i$, $1\lambda \rightarrow 1\lambda$.

3.2. The algebra $S$. Let $S = S(\pi)$ be the generalized Schur algebra, given by the generators and relations (R1)–(R8) of 1.4, for types $B_n$–$D_n$. Let $\Pi = \Pi(E^\otimes r) = W\pi$, the set of weights of $E^\otimes r$. Define elements $H_i \in S$ by

$$H_i = \sum_{\lambda \in \Pi} (\varepsilon_i, \lambda)1\lambda = \sum_{\lambda \in \Pi} \lambda_i 1\lambda.$$  

(3.2.1)

**Proposition 3.2.2.** With $H_i$ as above, the elements $H_i$, $e_i$, $f_i$ in $S$ satisfy the relations (B1)–(B7), (C1)–(C7), (D1)–(D7) in types $B_n$, $C_n$, $D_n$ respectively.

**Proof.** The first, third, fourth, fifth, and sixth relations are the same for all types, so the argument differs only for the second and seventh relations. Moreover, the fourth and fifth relations are the same as (R7) and (R8), so we only need to establish the first, second, third, sixth, and seventh relations.

It follows from the definition (3.2.1) of the elements $H_i$ and the commutativity of the $1\lambda$ that the elements $H_i$, $H_j$ commute in $S$. This proves the first relation (B1), (C1), (D1).

We consider the third relation. We will show that $H_i e_j - e_j H_i = (\varepsilon_i, \alpha_j)e_j$. At this point, it is convenient to set $1\lambda = 0$ for all $\lambda \in X - \Pi$. Then the sums in (R1), (R2), and (3.2.1) can be taken over $X$. The relations (R3), (R5) may be expressed by the single equality

$$e_i 1\lambda = 1_{\lambda + \alpha_i} e_i \quad (\text{all } i, \lambda \in X)$$

(3.2.3)

and (R4), (R6) may be expressed as

$$f_i 1\lambda = 1_{\lambda - \alpha_i} f_i \quad (\text{all } i, \lambda \in X).$$

(3.2.4)
From (3.2.1), (3.2.3), by reindexing the first sum we obtain
\[ H_i e_j - e_j H_i = \sum_{\lambda \in \Pi} (\varepsilon_i, \lambda) 1_\lambda e_j - \sum_{\lambda \in \Pi} (\varepsilon_i, \lambda) e_j 1_\lambda \]
\[ = \sum_{\lambda \in \Pi} (\varepsilon_i, \lambda) e_j 1_\lambda - \sum_{\lambda \in \Pi} (\varepsilon_i, \lambda) e_j 1_\lambda \]
\[ = \sum_{\lambda \in \Pi} (\varepsilon_i, \lambda) e_j 1_\lambda - \sum_{\lambda \in \Pi} (\varepsilon_i, \lambda) e_j 1_\lambda \]
\[ = \sum_{\lambda \in \Pi} (\varepsilon_i, \lambda) e_j 1_\lambda = (\varepsilon_i, \alpha_j) e_j, \]

by the second part of (R1), where the sums are taken over \( X \). This proves the first part of (B3), (C3), and (D3). The proof of the second part is entirely similar, using (3.2.4) instead of (3.2.3).

We consider the sixth relation. We have equalities
\[ (H_i + r) \cdots H_i \cdots (H_i - r) \]
\[ = \left( \sum_{\lambda \in \Pi} (\varepsilon_i, \lambda) 1_\lambda + r \right) \cdots \left( \sum_{\lambda \in \Pi} (\varepsilon_i, \lambda) 1_\lambda \right) \cdots \left( \sum_{\lambda \in \Pi} (\varepsilon_i, \lambda) 1_\lambda - r \right) \]
\[ = \left( \sum_{\lambda \in \Pi} ((\varepsilon_i, \lambda) + r) 1_\lambda \right) \cdots \left( \sum_{\lambda \in \Pi} (\varepsilon_i, \lambda) 1_\lambda \right) \cdots \left( \sum_{\lambda \in \Pi} ((\varepsilon_i, \lambda) - r) 1_\lambda \right) \]
\[ = \sum_{\lambda \in \Pi} \left[ ((\varepsilon_i, \lambda) + r) \cdots (\varepsilon_i, \lambda) \cdots ((\varepsilon_i, \lambda) - r) \right] 1_\lambda \]

(since the 1_\lambda are orthogonal idempotents). The last expression above equals 0, i.e., all its coefficients equal 0, because for \( \lambda \in \Pi \), \( (\varepsilon_i, \lambda) = \lambda_i \) is an integer between \(-r\) and \( r \). This proves (B6), (C6), and (D6).

We consider (B2). We must show that \( e_i f_i - f_i e_i = \delta_{ij}(H_i - H_{i+1}) \), when \( i < n \) and \( e_i f_i - f_i e_i = \delta_{ij}(2H_n) \), when \( i = n \). When \( i \neq j \), \( e_i f_i - f_i e_i = 0 \) by (R2), so we are reduced to the case \( i = j \). By (3.2.1), for \( i < n \) we have
\[ H_i - H_{i+1} = \sum_{\lambda \in \Pi} (\varepsilon_i - \varepsilon_{i+1}, \lambda) 1_\lambda \]
\[ = \sum_{\lambda \in \Pi} (\alpha_i, \lambda) 1_\lambda = \sum_{\lambda \in \Pi} (\alpha_i^\vee, \lambda) 1_\lambda = e_i f_i - f_i e_i, \]

by relation (R2). (We used the equality \( \alpha_i = \alpha_i^\vee \) for \( i < n \).) For \( i = n \), we have \( 2H_n = \sum_{\lambda \in \Pi} (2\varepsilon_n, \lambda) 1_\lambda = \sum_{\lambda \in \Pi} (\alpha_n^\vee, \lambda) 1_\lambda = e_n f_n - f_n e_n \), by (R2). This proves (B2). The proof of (C2) and (D2) is similar.

We consider (B7). Consider \( J = \sum_{i=1}^n \sigma_i H_i \) for a given choice of signs \( (\sigma_1, \ldots, \sigma_n) \) in \( \{1, -1\}^n \). Then we have
\[ J = \sum_{i=1}^n \sigma_i \sum_{\lambda \in \Pi} (\varepsilon_i, \lambda) 1_\lambda = \sum_{i=1}^n \sum_{\lambda \in \Pi} \sigma_i \lambda_i 1_\lambda. \]

For any integer \( s \) we have equalities
\[ (J + s) = \left( \sum_i \sum_{\lambda} \sigma_i \lambda_i 1_\lambda \right) + \left( \sum_{\lambda} s 1_\lambda \right) \]
\[ = \sum_{\lambda} \left( \left( \sum_i \sigma_i \lambda_i \right) + s \right) 1_\lambda \]
where \( i \) varies from 1 to \( n \) and \( \lambda \) varies over \( \Pi \). Hence we obtain

\[
(J + r) \cdots J \cdots (J - r) = \prod_{s=-r}^{r} \sum_{\lambda \in \Pi} \left( \sum_{i=1}^{n} \sigma_i \lambda_i + s \right) \lambda
\]

The last expression vanishes since, for each \( \lambda \in \Pi \), \( \pm \lambda_1 \pm \cdots \pm \lambda_n \) is an integer between \(-r\) and \(r\). This proves (B7). The proof of (C7), (D7) is similar. \(\square\)

**Corollary 3.2.5.** There is a surjective algebra homomorphism \( \Phi \to S \) mapping \( e_i \to e_i \), \( f_i \to f_i \), and \( H_i \to H_i = \sum_{\lambda \in \Pi} (\varepsilon_i, \lambda) \lambda \).

### 3.3. Conclusion of the proof.

Corollary 3.1.14 shows that \( S \) is a quotient of \( \Phi \). Corollary 3.2.5 shows that \( \Phi \) is a quotient of \( S \). It follows that \( \Phi \) is isomorphic with \( S = S(\pi) \), and hence Theorems 2.1.1, 2.2.1, and 2.3.1 are proved.

### 4. A basis for \( S(\pi) \)

Let \( g \) be a Lie algebra of classical type defined over \( \mathbb{Q} \). As before, let \( \tau = \Pi^{\pm}(E^{\otimes r}) \). We would like to give a canonical basis for the generalized Schur algebra \( S(\pi) \) in terms of the elements of the universal enveloping algebra.

Since \( S(\pi) \) is the direct sum of endomorphism algebras of simple factors of \( E^{\otimes r} \), we consider first the problem of finding a basis for the full matrix algebra \( \text{End}(M) \) of an irreducible module \( M \). To give a basis for \( \text{End}(M) \) in terms of the elements of the universal enveloping algebra, we will exploit Littelmann’s basis for \( M \).

#### 4.1. Littelmann’s basis.

Let \( h \) be a Cartan subalgebra of \( g \). Let \( R \subset h^\star \) be the set of roots of \( h \) in \( g \), i.e., the nonzero eigenvalues for the adjoint representation of \( h \) on \( g \), and let \( S \) be the set of simple roots relative to some hyperplane in \( h^\star \). For each simple root \( \alpha \), let \( s_\alpha \) be the corresponding element of the Weyl group of the root system. Fix a reduced expression \( s_{\alpha_1} \cdots s_{\alpha_m} \) for the longest word \( w_0 \) in the Weyl group of \( g \) in terms of the set \( \{ s_\alpha \mid \alpha \in S \} \).

Let \( X_Q \) be the rational span of the weight lattice of \( g \) within \( h^\star \). Consider the set of paths \( x : [0,1] \to X_Q \) that begin at the origin, and take \( \Pi \) to be the free \( \mathbb{Z} \)-module on that set. In [L1, §1], Littelmann defines certain operators \( \{ \tilde{f}_\alpha \}_{\alpha \in S} \) and \( \{ \tilde{e}_\alpha \}_{\alpha \in S} \) on \( \Pi \). The value of \( \tilde{f}_\alpha \) at a path \( x(t) \) with endpoint \( x(1) \) either is the 0-element of \( \Pi \) or is a
particular path $\tilde{f}_\alpha \cdot x$ with endpoint $x(1) - \alpha$. In an inverse sense, the
value of $\tilde{e}_\alpha$ at a path $x(t)$ with endpoint $x(1)$ either is the 0-element of
$\Pi$ or it is a particular path $\tilde{e}_\alpha \cdot x$ with endpoint $x(1) + \alpha$. Let $A$ be the
algebra generated by those operators.

Consider any path $P_\lambda$ in the dominant chamber that terminates at
an integral weight $\lambda$. $P_\lambda$ generates an irreducible $A$-module $M_\lambda$. In
terms of the chosen reduced expression $w_0 = s_{\alpha_1} \cdots s_{\alpha_m}$, consider the
elements of $A$ of the form $\tilde{f}_{\alpha_1}^{m_1} \cdots \tilde{f}_{\alpha_m}^{m_m}$. Littelmann shows that the
elements $\tilde{f}_{\alpha_1}^{m_1} \cdots \tilde{f}_{\alpha_m}^{m_m} \cdot P_\lambda$ span $M_\lambda$. Moreover, in [L2, §§6–7], he gives a
geometric description of a set $S_\lambda$ of sequences of exponents $n_1, ..., n_m$
such that, as $(n_1, ..., n_m)$ ranges over $S_\lambda$, the elements
$\tilde{f}_{\alpha_1}^{m_1} \cdots \tilde{f}_{\alpha_m}^{m_m} \cdot P_\lambda$
form a basis for $M_\lambda$.

For each simple root $\alpha$, let $f_\alpha$ and $e_\alpha$ be nonzero elements of the
root spaces of $g$ corresponding respectively to the roots $-\alpha$ and $\alpha$, satisfying $[e_\alpha, f_\alpha] = \alpha^\vee \in h^*$. Let $L(\lambda)$ be the irreducible $g$-module of
highest weight $\lambda$, and $v_\lambda$ be a highest weight vector. In [L2, Theorem
10.1], using the theory of crystal bases, Littelmann shows that as the
sequences $(n_1, ..., n_m)$ range over $S_\lambda$, the elements $f_{\alpha_1}^{m_1} \cdots f_{\alpha_m}^{m_m} \cdot v_\lambda$
form a basis for $L(\lambda)$.

Let $S_\lambda^{opp}$ be the set of sequences $(n_m, ..., n_1)$, where $(n_1, ..., n_m)$ ranges
over $S_\lambda$. Let $u_\lambda$ be a lowest weight vector in $L(\lambda)$. By the same theory of
crystal bases, as $(t_1, ..., t_m)$ ranges over $S_\lambda^{opp}$, the elements
$e_{\alpha_1}^{t_1} \cdots e_{\alpha_m}^{t_m} \cdot u_\lambda$
form a basis for $L(\lambda)$.

4.2. Basis for $\text{End}(L(\lambda))$. Consider the dual module $L(\lambda)^*$. Its weights
are the negatives of the weights of $L(\lambda)$, and its highest weight is
$-w_0(\lambda)$, since the lowest weight of $L(\lambda)$ is $w_0(\lambda)$. Furthermore, the
lowest weight of $L(\lambda)^*$ is $-\lambda$, and as a lowest weight vector, we can take the element $\chi_{v_\lambda}$ that is 1 at $v_\lambda$ and 0 at all weight vectors of other
weights. Hence, by the preceding paragraph, as $(t_1, ..., t_m)$ ranges over
$S_{-w_0(\lambda)}^{opp}$, the elements $e_{\alpha_1}^{t_1} \cdots e_{\alpha_m}^{t_m} \cdot \chi_{v_\lambda}$, form a basis for $L(\lambda)^*$.

Let $1_\lambda$ be the element of $\text{End}(L(\lambda))$ that acts as 1 on the highest
weight line of $L(\lambda)$ and acts as 0 on the other weight spaces.

Theorem 4.2.1. Let $\lambda$ be a dominant integral weight. As \(\{n_1, ..., n_m\}\)
range over $S_\lambda$, and as \(\{t_1, ..., t_m\}\) range over $S_{-w_0(\lambda)}^{opp}$, the elements

$$\{f_{\alpha_1}^{n_1} \cdots f_{\alpha_m}^{n_m} 1_\lambda e_{\alpha_1}^{t_1} \cdots e_{\alpha_m}^{t_m}\}$$

form a basis for $\text{End}(L(\lambda))$. 
Proof. Under the natural identification of $L(\lambda) \otimes L(\lambda)^*$ with $\text{End}(L(\lambda))$, the element
\[
f_{\alpha_1}^{n_1} \cdots f_{\alpha_m}^{n_m} \cdot \nu_\lambda \otimes e_{\alpha_1}^{t_1} \cdots e_{\alpha_m}^{t_m} \cdot \chi_\nu_\lambda
\]
of $L(\lambda) \otimes L(\lambda)^*$ corresponds to the element
\[
f_{\alpha_1}^{n_1} \cdots f_{\alpha_m}^{n_m} 1_\lambda \cdot e_{\alpha_1}^{t_1} \cdots e_{\alpha_m}^{t_m}
\]
of $\text{End}(L(\lambda))$. □

Corollary 4.2.2. In Lie algebras of type $B_n$, $C_n$, or $D_n$, as \{n_1, ..., n_m\} range over $S_\lambda$, and as \{t_1, ..., t_m\} range over $S^{\text{opp}}_\lambda$, the elements
\[
\{f_{\alpha_1}^{n_1} \cdots f_{\alpha_m}^{n_m} 1_\lambda \cdot e_{\alpha_1}^{t_1} \cdots e_{\alpha_m}^{t_m}\}
\]
form a basis for $\text{End}(L(\lambda))$.

Proof. In those types, $w_0 = -I$. □

4.3. Basis for $S(\pi)$. Consider the generalized Schur algebra $S(\pi)$ for $\pi = \Pi^+ (E^{\otimes r})$ in types $B_n$, $C_n$, and $D_n$. $S(\pi)$ is isomorphic with the direct sum $\bigoplus_{\lambda \in \pi} \text{End}(L(\lambda))$. In [D, 6.10] it is proved that an idempotent $1_\mu$ ($\mu \in W \pi = \Pi(E^{\otimes r})$) acts on any $S(\pi)$-module $M$ as 1 on the eigenspace of $M$ of value $\mu$, and acts as 0 on the eigenspaces of $M$ of all other values.

Theorem 4.3.1. The algebra $S(\pi)$ has a basis consisting of all elements of the form
\[
f_{\alpha_1}^{n_1} \cdots f_{\alpha_m}^{n_m} 1_{\lambda(i)} \cdot e_{\alpha_1}^{t_1} \cdots e_{\alpha_m}^{t_m}
\]
such that
\[
\{n_1, ..., n_m\} \in S_\lambda; \{t_1, ..., t_m\} \in S^{\text{opp}}_{\lambda}
\]
as $\lambda$ varies over $\pi$.

Proof. (By induction on the partial order on $\pi$.) Let $\lambda$ be a maximal weight of $\pi$. The element
\[
f_{\alpha_1}^{n_1} \cdots f_{\alpha_m}^{n_m} 1_\lambda \cdot e_{\alpha_1}^{t_1} \cdots e_{\alpha_m}^{t_m}
\]
is zero in $\text{End}(L(\lambda'))$ for each $\lambda' \in \pi$ different from $\lambda$. Such elements give a basis for the endomorphism algebra of $L(\lambda)$. By induction we may assume the basis has been established as stated for the generalized Schur algebra $S(\pi \setminus \{\lambda\})$. Note that $\pi \setminus \{\lambda\}$ is saturated. □
Appendix A. Irreducible factors in the $r$th tensor power of the natural module

Weyl [W] describes the irreducible factors in the $r$th tensor power of the natural module for a classical group. For convenience, we summarize here the results from [W] needed in the paper.

**Type** $B_n = \text{SO}_{2n+1}$. Let $T$ be a diagram with row lengths $f_1 \geq f_2 \geq \cdots \geq f_t > 0$, with $t \leq n$. The diagram $T'$ associated to $T$ has row lengths $f_1 \geq f_2 \geq \cdots \geq f_t \geq f_{t+1} \geq \cdots \geq f_{2n+1-t}$, where $f_{t+1} = f_{t+2} = \cdots = f_{2n+1-t} = 1$, i.e., $T'$ is obtained by adding $2n-2t+1$ boxes to the first column of $T$. The irreducible $\text{SO}_{2n+1}$-modules associated to $T$ and $T'$ are isomorphic with highest weight of exponents $f_1 \geq f_2 \geq \cdots \geq f_t$.

(Weyl, Chapter 7, equations 9.10 and 9.11.) The pairs $\{T, T'\}$ partition Weyl’s set of permissible diagrams (Weyl p. 155).

The following is Weyl’s Theorem 5.7F for the group $\text{SO}_{2n+1}$.

**Theorem A.1.** For each pair $\{T, T'\}$ of permissible diagrams, the irreducible module corresponding to $T$ is a factor of the $r$th tensor power of the natural module iff $\sum_{j=1}^{t} f_j = m - 2k$, for some $k \geq 0$, and the irreducible module corresponding to $T'$ is a factor of the $r$th tensor power of the natural module iff $\sum_{j=1}^{2n+1-t} f_j = m - 2k$ for some $k \geq 0$.

**Remark.** Weyl’s theorem is stated for the full orthogonal group, but in section 9 of Chapter 6, he shows that irreducible modules for the full orthogonal group remain irreducible for the proper orthogonal group when the dimension of the natural module is odd.

Because $f_j = 1$ for $j = t+1, \ldots, 2n+1-t$, we can write the condition on $T'$ in the theorem as $\sum_{j=1}^{t} f_j + (2n-2t+1) = m - 2k$ for some $k \geq 0$, or equally, as $\sum_{j=1}^{t} f_j + (2n-2t) = m - 2k - 1$ for some $k \geq 0$.

We can give the irreducible factors of the $r$th tensor power in the following theorem:

**Theorem A.2.** The irreducible factors of the $r$th tensor power of the natural module for $\text{SO}_{2n+1}$ are those whose highest characters have exponents $f_1 \geq f_2 \geq \cdots \geq f_t > 0$ with $t \leq n$, where either

(i) $\sum_{j=1}^{t} f_j = m - 2k$, for some $k \geq 0$, or

(ii) $\sum_{j=1}^{t} f_j + (2n-2t) = m - 2k - 1$, for some $k \geq 0$.

The theorem can be restated in a slightly different form:
Theorem A.3. The irreducible factors of the $rt$th tensor power of the natural module for $SO_{2n+1}$ are those whose highest characters have exponents $f_1 \geq f_2 \geq \cdots \geq f_n \geq 0$, where either

(i) $\sum_{j=1}^{n} f_j = m - 2k$, for some $k \geq 0$, or

(ii) $\sum_{j=1}^{n} f_j = m - 2k' - 1$, for some $k' \geq 0$, for which $f_{n-k'} \neq 0$.

Proof. Set $k' = n + k - t$ to interchange parts (ii) of the theorems A.2 and A.3. Note that the condition $k \geq 0$ translates into the condition $t \geq n - k'$. □

Type $C_n = Sp_{2n}$. Consider the diagrams $T$ with row lengths $f_1 \geq f_2 \geq \cdots \geq f_n \geq 0$. For the symplectic group $Sp_{2n}$, the irreducible modules correspond to the diagrams $T$. The irreducible $Sp_{2n}$-module corresponding to the diagram $T$ is a factor of the $rt$th tensor power of the natural module if and only if $f_1 + f_2 + \cdots + f_n = m - 2k$ for some $k \geq 0$.

In terms of highest weights, we have the following (Weyl, Chapter 6, p. 175 and Theorem 7.8D):

Theorem A.4. Consider sequences of integers $f_1 \geq f_2 \geq \cdots \geq f_n \geq 0$. The irreducible factors of the $rt$th tensor power of the natural module for $Sp_{2n}$ are those whose highest characters have exponents $(f_1, f_2, \ldots, f_n)$, where $f_1 + f_2 + \cdots + f_n = m - 2k$, for some $k \geq 0$.

Type $D_n = SO_{2n}$. The permissible diagrams for the full orthogonal group $O_{2n}$ are those whose first two columns have combined length no more than $2n$. The irreducible modules for the full orthogonal group $O_{2n}$ correspond to the permissible diagrams (Weyl Theorem 5.7F).

The set of permissible diagrams can be partitioned into pairs of associated diagrams as follows. Let $T$ be a diagram with row lengths $f_1 \geq f_2 \geq \cdots \geq f_t > 0$, with $t \leq n$. The diagram $T'$ associated to $T$ has row lengths $f_1 \geq f_2 \geq \cdots \geq f_t \geq f_{t+1} \geq \cdots \geq f_{2n-t}$, where $f_{t+1} = f_{t+2} = \cdots = f_{2n-t} = 1$, i.e., $T'$ is a permissible diagram obtained from the permissible diagram $T$ by adding $2n - 2t$ boxes to the first column of $T$. By Weyl’s theorem 5.7F for $O_{2n}$, the irreducible module corresponding to $T$ is a factor of the $rt$th tensor power of the natural module exactly when $\sum_{j=1}^{t} f_j = m - 2k$, for some $k \geq 0$, and the irreducible module corresponding to $T'$ is a factor of the $rt$th tensor power of the natural module exactly when $\sum_{j=1}^{2n-t} f_j = m - 2k'$, for some $k' \geq 0$. Because $f_{t+1} = f_{t+2} = \cdots = f_{2n-t} = 1$, that condition can be written as $\sum_{j=1}^{t} f_j = m - 2(n - t) - 2k'$, for some $k' \geq 0$. 
Consider the pair \( \{T, T'\} \). If \( T' \) satisfies the condition 
\[ \sum_{j=1}^{t} f_j = m - 2(n - t) - 2k', \]
for some \( k' \geq 0 \), then \( T \) satisfies the condition 
\[ \sum_{j=1}^{t} f_j = m - 2k, \]
for the value \( k = (n - t) + k' \geq 0 \). Hence, if the diagram \( T' \) corresponds to a factor of the \( r \)th tensor power, so does the diagram \( T \).

When we restrict the irreducible \( O_{2n} \)-modules corresponding to \( T \) and \( T' \) to the proper orthogonal group \( SO_{2n} \), there are two cases to consider (Weyl, Theorems 5.9A and 7.9). In the first case, \( t < n \). There, \( T \) and \( T' \) are distinct diagrams that correspond to nonisomorphic \( O_{2n} \)-modules. Upon restriction to \( SO_{2n} \), those modules become isomorphic irreducible \( SO_{2n} \)-modules. The highest character of that irreducible \( SO_{2n} \)-module has nonzero exponents \((f_1, f_2, ..., f_t)\), with \( t < n \). In the second case, \( t = n \). There, \( T = T' \), and the corresponding irreducible \( O_{2n} \)-module, upon restriction to \( SO_{2n} \), splits into two nonisomorphic irreducible \( SO_{2n} \)-modules, one whose highest character has exponents \((f_1, f_2, ..., f_{n-1}, f_n)\) and the other whose highest character has exponents \((f_1, f_2, ..., f_{n-1}, -f_n)\).

The following theorem gives the isomorphism classes of the irreducible \( SO_{2n} \)-modules in the \( r \)th tensor power of the natural module, summing up the conclusions of the preceding paragraphs.

**Theorem A.5.** Consider sequences of integers \( f_1 \geq f_2 \geq \cdots \geq f_n \geq 0 \) such that 
\[ \sum_{j=1}^{n} f_j = m - 2k, \]
for some \( k \geq 0 \). The irreducible modules in the \( r \)th tensor power of the natural module for \( SO_{2n} \) are those whose highest characters have exponents \((f_1, f_2, ..., f_{n-1}, f_n)\) with \( f_n = 0 \), and those whose highest characters have exponents \((f_1, f_2, ..., f_{n-1}, -f_n)\) or \((f_1, f_2, ..., f_{n-1}, -f_n)\), with \( f_n > 0 \).

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