Relativistic second order dissipative hydrodynamics from effective fugacity quasi-particle model

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In this work a second order relativistic viscous hydrodynamic model has been presented based on the effective fugacity quasi-particle model (EQPM). The hydro model has been derived from the effective relativistic second-order transport equation under EQPM for a multi-particle (two component) system and solving it in Grad's 14 moment method. The EQPM model describes the strongly interacting thermal system of QCD interactions through its fugacity parameters extracted from an updated lattice equations of state. The proper time evolution of temperature and pressure anisotropy is observed to be affected significantly due to the inclusion of EQPM model compared to an ideal system.

Keywords: Effective fugacity quasi-particle model, second order relativistic hydrodynamics

I. INTRODUCTION

In last few decades, during the flourishing era of investigating deconfined quark-gluon plasma (QGP) through heavy-ion collision experiments, relativistic hydrodynamics has been proved to be the most trusted tool to describe the system properties [1–4]. Hydrodynamics provides the collective behavior of the system through evolution equations of the macroscopic state variables. However, since hydrodynamics only deals with bulk properties of the systems and do not care about the single particle distributions, the microscopic dynamics of the system is not included in this theory. But a microscopic theory is essential in its own merit since the input parameters of these evolution equations crucially depend upon particle interactions and underlying dynamics. Many particle kinetic theory has been so far served well for this purpose by describing the macroscopic thermodynamic quantities and transport coefficients in terms of single particle distribution functions and system interactions [5–12]. Techniques that consistently bridge between these two theories have been quite successful in describing the matter created in heavy ion collision experiments.

In the due course, a few earlier attempts were made with ideal hydrodynamics considering the system as a weakly interacting gas which seems to offer first handedly a sensible description of the data. However, a closer inspection to some of the bulk observables such as multiplicity, radial or elliptic flow reveals the need to include dissipative hydrodynamics in order to describe the space-time evolution of the system [13]. However, the first order or the Navier-Stokes theory results in parabolic equations of motion of the thermodynamic state variables leading to severe causality violation problem. This crisis requires the introduction of a second order theory which provides hyperbolic equations of motion resulting in finite time scale for the thermodynamic flows to dissipate. This second order hydrodynamic theory which is famously known as Muller-Israel-Stewart theory, despite of having some issues with numerical stability, has been served the study of dynamical evolution of the system quite authentically [14–22].

In this work, the second order relativistic hydro equations have been derived and solved for a 1+1 boost invariant system including the effective fugacity quasiparticle model (EQPM) [23]. The hydro equations have been derived from relativistic transport equation of covariant kinetic theory by solving them applying Grad’s 14 moment method within a viscous medium. The crux of the work lies into appropriate modeling of the equilibrium, isotropic momentum distribution of gluons and quarks in the hot QCD medium under EQPM scheme. The EQPM scheme, with a recent (2+1)-flavor lattice QCD equation of state (LEOS) [39] at physical quark masses, have been exploited in the present manuscript.

The basic idea of quasiparticle models are to describe the hot QCD equations of state (EOSs) in terms of non-interacting or weakly interacting effective gluons and quarks. Previously, second order relativistic hydro have been studied using different quasiparticle theories, especially within the scope of effective mass models [24–26]. The novelty of EQPM model resides with the fact that the hot QCD medium effects present in the QCD EOSs (either computed within improved perturbative QCD (pQCD) or lattice QCD simulations) can be implemented with a single temperature dependent fugacity parameter for each parton. The resulting particle distribution can describe all the thermodynamic quantities consistently without introducing a temperature dependent mass and the dispersion relation of the underlying theory remains linear. The mean field force term induced by the system’s collective behavior is simply implemented by the fugacity parameters and conserves the particle current and energy-momentum tensor perfectly. Further, the model has been recently extended to be used

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with finite quark chemical potential as well \[27\]. So, the extracted hydrodynamic equations with this model and their solutions are expected to implement a realistic equation of state of the medium within the formalism of a consistently developed covariant kinetic theory.

The manuscript is organized as follows. Section II deals with the covariant formalism of the EQPM model itself, first order theory using Chapman-Enskog technique to extract first order transport coefficients and the Grad’s moment method to derive the second order hydrodynamic equations as well as its solution for a 1+1 boost invariant system. Section III displays the results involving proper time evolution of the concerned thermodynamic quantities and section-IV ends the manuscript with a conclusion and possible outlook.

\section{II. FORMALISM}

\subsection{A. Covariant theory for EQPM model}

As mentioned earlier, the basic idea of EQPM model is to map the strongly interacting hot QCD effects into a medium consisting of practically non-interacting quasi-quarks and quasi-gluons through a parameter called effective fugacity \( z_{q/g} \). Including this parameter, the equilibrium single particle distribution function for this quasi-partons belonging to \( k^{th} \) species \(( k = q, g) \) takes the following form \[24\] \[28\],

\[
    f_k^0 = \frac{1}{z_k^{-1} \exp \left\{ \frac{\omega_{p_k}}{T} - \frac{\mu_k}{T} \right\} + 1} , \tag{1}
\]

with \( E_{p_k} \) as the energy of each bare particle and \( \mu_Bk \) as baryon chemical potential for \( k^{th} \) species. \( T \) is the temperature of the system at local thermal equilibrium. We define the four momentum of the quasi-particles as \( p_k^\mu = (\omega_{p_k}, \vec{p}_k) \), where the three momenta \( |\vec{p}_k| \) is the same as of the bare particles but the energy of each quasi-particle differs from that of bare particles by a dispersion relation,

\[
    \omega_{p_k} = E_{p_k} + \Delta_k, \quad \Delta_k = T^2 \partial_T ln z_k . \tag{2}
\]

For the current purpose we are considering a massless QGP for which bare particle energy equals with its momentum, \( E_{p_k} = |\vec{p}_k| \). The equilibrium distribution function of quasi-particles from Eq. \(1\) can be written alternatively as the following,

\[
    f_k^0 = \frac{1}{\exp \left\{ \frac{\omega_{p_k} - \mu_k}{T} \right\} + 1} , \tag{3}
\]

with \( u^\mu \) as the hydrodynamic four-velocity and \( \mu_k = \mu_{Bk} + \Delta_k + T ln z_k \) denoting \( exp(\mu_k/T) \) as the total effective fugacity.

From \[24\] it can be recalled that the particle four flow and energy-momentum tensor under EQPM have the following form,

\[
    N^\mu(x) = \sum_{k=1}^{N} \nu_k \int \frac{d^3|\vec{p}_k|}{(2\pi)^3} \omega_{p_k} p_k^\mu f_k(x, p_k) \tag{4}
\]

\[
    T^\mu\nu(x) = \sum_{k=1}^{N} \nu_k \int \frac{d^3|\vec{p}_k|}{(2\pi)^3} p_k^\mu p_k^\nu f_k(x, p_k) \tag{5}
\]

Here \( \langle p^\mu \rangle = \Delta^\mu p_\mu \) and \( \langle (p^\mu)^2 \rangle = \frac{1}{2} \{ \Delta^{\mu\nu} \Delta^{\alpha\beta} + \Delta^{\mu\alpha} \Delta^{\nu\beta} \} p_\mu p_\beta \) are the irreducible tensors of rank one and two respectively, with \( \Delta^\mu\nu = g^{\mu\nu} - u^\mu u^\nu \) as the projection operator. Throughout the analysis the metric \( g^{\mu\nu} \) has taken to be \( g^{\mu\nu} = (1, -1, -1, -1) \).

From above equations the expressions of particle number density, energy density and pressure take the following forms respectively,

\[
    n(x) = \sum_{k=1}^{N} \nu_k \int \frac{d^3|\vec{p}_k|}{(2\pi)^3} f_k(x, p_k) , \tag{6}
\]

\[
    \epsilon(x) = \sum_{k=1}^{N} \nu_k \int \frac{d^3|\vec{p}_k|}{(2\pi)^3} \omega_{p_k} f_k(x, p_k) , \tag{7}
\]

\[
    P(x) = \frac{1}{3} \sum_{k=1}^{N} \nu_k \int \frac{d^3|\vec{p}_k|}{(2\pi)^3} |\vec{p}_k| f_k(x, p_k) , \tag{8}
\]

with \( f_k(x, p_k) \) is the single particle momentum distribution belonging to \( k^{th} \) species, that is a function of spacetime coordinate \( x \) and particle momenta \( p_k^\mu \). \( \nu_k \) is the corresponding degeneracy factor.

The relativistic transport equation of a single quasi-particle distribution function for a process \( p_k + p_l \rightarrow p_k' + p_l' \), including the mean field force term resulting from collective excitation of quasi-quarks and quasi-gluons is given by,

\[
    \frac{1}{\omega_{p_k}} p_k'^\mu \partial_\mu f_k(x, p_k) + \vec{F}_k \cdot \vec{\nabla}_{p_k} f_k = \sum_{k'=1}^{N} C_{kl}[f_k, f_l] , \quad [k = 1, ..., N] \tag{9}
\]

with \( \vec{F}_k \) as the mean field force on \( k^{th} \) particle and \( C_{kl} \) as the collision integral respectively defined by,

\[
    F_k^\mu = -\partial_\mu \{ \Delta_k u^\mu u^\mu \} , \quad \tag{10}
\]
Here, $d\Gamma_{p_i} = \frac{d^3p_i}{(2\pi)^3}$ denotes the phase space factor and $W = \frac{1}{4}(\pi)^4 \delta^4(p_k + p_l - p_k^0 - p_l^0)$ is the interaction cross section for the corresponding dynamical processes with $M$ as the scattering amplitudes. The conservation of particle current $\partial_\mu \Pi^{\mu\nu} = 0$ and energy-momentum tensor $\partial_\mu T^{\mu\nu}$ can be trivially realized from Eq. (14) and (15) with the help of Eq. (16).

Before concluding this section the expressions for shear and bulk viscous pressure tensor under EQPM can be derived respectively as

$$\Pi = - T^2 \sum_{k=1}^{N} \nu_k \frac{1}{|p_k|} \frac{1}{p_k^0} (1 + \sum_{l=1}^{N} \nu_l \Delta_k) \sum_{l}^{N} \nu_l \frac{1}{|p_l|} \frac{1}{p_l^0} (1 + \sum_{l=1}^{N} \nu_l \Delta_l) \phi_k (\nu_k)$$

with

$$\phi_k = \frac{\gamma}{T} \sum_{k=1}^{N} \nu_k \Delta_k \phi_k$$

where the primed quantities are the distribution function and deviation function with final state momenta.

Operating the derivatives of the left hand side of Eq. (16) on the equilibrium distribution function from (23), we obtain a number of thermodynamic forces as follows.

$$N_k = (p_k^0 - p_k^0) \langle \Pi^{\mu\nu} \{ (p_k, u) - h_k \} X_{\mu\nu} + (p_k^0) \sum_{a=1}^{N-1} \nu_a \Delta \phi = T \sum_{k=1}^{N} \nu_k \Delta_k \sum_{l=1}^{N} \nu_l \frac{1}{|p_l|} \frac{1}{p_l^0} (1 + \sum_{l=1}^{N} \nu_l \Delta_l) \phi_k$$

with

$$X = \frac{\partial \cdot u}{\partial t}$$

$$X_{\mu\nu} = \frac{\partial_{\mu} T - \partial_{\nu} P}{T}$$

$$X_{\mu\nu} = \frac{(\partial_{\mu} u)_{\nu} - \frac{h_k}{nh} \partial_{\nu} T}{T}$$

$$\langle \Pi^{\mu\nu} \rangle = \langle \Pi^{\mu\nu} \rangle$$

The next to leading order particle distribution function $f_k = f_k^0 + f_k^1 (1 \pm f_k^0) \phi_k$ is expressed in terms of deviation in particle’s momentum distribution $\phi_k$. The linearized collision term becomes

$$\mathcal{L}_{kl}[\phi_k] = \nu_l \int d\Gamma_{pk} d\Gamma_{p_k^0} d\Gamma_{p_k^1} \frac{1}{|p_k|} \frac{1}{p_k^0} (1 + \sum_{l=1}^{N} \nu_l \Delta_l) \phi_k$$

and

$$C_{kl}[f_k, f_l] = \frac{\nu_l}{\omega_k} \int d\Gamma_{pk} d\Gamma_{p_k^0} d\Gamma_{p_k^1} W \times [f_k(p_k) f_l(p_l)(1 \pm f_k^0)(1 \pm f_l^0)]$$

$$- f_k(p_k) f_l(p_l)(1 \pm f_k^0)(1 \pm f_l^0)$$

(11)

in particle’s momentum distribution $\phi_k$. The linearized collision term becomes,

$$\mathcal{L}_{kl}[\phi_k] = \nu_l \int d\Gamma_{pk} d\Gamma_{p_k^0} d\Gamma_{p_k^1} f_k^0 (1 \pm f_k^0) (1 \pm f_l^0) \phi_k$$

$$- f_k^0 (1 \pm f_k^0) W(p_k, p_l)$$

(15)

where the primed quantities are the distribution function and deviation function with final state momenta.

B. First order theory

In this section the first order transport coefficients, particularly shear viscos coefficient will be estimated from covariant multi-component kinetic theory using a technique called Chapman-Enskog method. The method has been detailedly discussed in (27, 30, 32). This is basically an iterative technique, where from the known lower order distribution function the unknown next order can be determined by successive approximation. Using this method to first order, Eq. (16) turns out to be,

$$p_k^0 \partial_{\mu} f_k + \omega_k \frac{1}{|p_k|} \frac{1}{p_k^0} \partial_{\nu} f_k = - \sum_{l=1}^{N} \mathcal{L}_{kl}[\phi_l], \quad [k = 1, \ldots, N].$$

(14)

The next to leading order particle distribution function $f_k = f_k^0 + f_k^1 (1 \pm f_k^0) \phi_k$ is expressed in terms of deviation

$$p_k^0 \partial_{\mu} f_k^0 + \omega_k \frac{1}{|p_k|} \frac{1}{p_k^0} \partial_{\nu} f_k^0 = - \sum_{l=1}^{N} \mathcal{L}_{kl}[\phi_l], \quad [k = 1, \ldots, N].$$

(14)

The unknown coefficient $C_{kl}$ is further decomposed as $C_{kl} = C_k(p_k, x) \langle p_k^0 \rangle \langle p_k^0 \rangle$. Now putting (21) into Eq. (13), contracting terms having compatible tensorial
ranks and finally comparing with macroscopic definition of shear viscous pressure tensor as following,

\[
\langle \Pi^{\mu\nu} \rangle = 2n\langle \nabla^{\mu} u^{\nu} \rangle ,
\]

we reach the expression for shear viscosity as follows,

\[
\eta = \frac{1}{10} \sum_{k=1}^{N} \nu_k \int d^4p_k f_{k}^{0}(1 \pm \vec{f}_{k}^{0})(p^\mu p^\nu)\langle p_{\mu\nu}p_{\rho\sigma} \rangle C_k ,
\]

with \(d^4p_k = \frac{d^3p}{(2\pi)^3} p^0\). The next task is to extract \(C_k^{\mu\nu}\) from Eq. (22). For this purpose, the variational approximation method has been employed where \(C_k\) is expressed by a polynomial of degree \(p\) as follows,

\[
C_k = \sum_{s=0}^{p} C_k^{(p)s}\tau_k^s .
\]

The superscript \(s\) on \(C_k^{(p)s}\) in Eq. (23) is the coefficient index belonging to each power of the polynomial expansion whereas the same on \(\tau_k\) indicates the \(s^{th}\) power of scaled energy itself. The coefficients \(C_k^{(p)s}\) are functions independent of particle momenta, depending only upon particle mass and thermodynamic macroscopic quantities. Multiplying both side of Eq. (22) with \(\nu_k \tau_k^s(p_{\mu
u},p_{\rho\sigma})\) and integrating over \(d^4p_k\), we find the following recurrence relation,

\[
\sum_{l,s} C_l^{\mu\nu} C_l^{(p)s}\tau_l^s = \nu_k \gamma_k^r ,
\]

with

\[
\gamma_k^r = -\frac{1}{T} \int d^4p_k f_{k}^{0}(1 \pm \vec{f}_{k}^{0})(p^\mu p^\nu)\langle p_{\mu\nu}p_{\rho\sigma} \rangle \tau_k^r .
\]

and

\[
C_l^{\mu\nu} = \nu_k \nu_l \left[ \tau_k^r(p_l^\mu p_l^\nu), \tau_k^s(p_k\rho\sigma) \right] + \nu_k \delta_{kl} \sum_{m} \nu_m \left[ \tau_k^{r\mu}(p_{l\rho\sigma}^\mu p_{k\nu}^\nu), \tau_k^{s\nu}(p_{k\rho\sigma}^\nu p_{l\nu\mu}) \right] - \nu_k \tau_k^r \left[ \tau_k^r(p_{l\rho\sigma}^\mu p_{k\nu}^\nu), \tau_k^{s\nu}(p_{k\rho\sigma}^\nu p_{l\nu\mu}) \right] - \nu_k \delta_{kl} \sum_{m} \nu_m \left[ \tau_k^{r\mu}(p_{l\rho\sigma}^\mu p_{k\nu}^\nu), \tau_k^{s\nu}(p_{k\rho\sigma}^\nu p_{l\nu\mu}) \right] .
\]

The bracket quantity denotes,

\[
\left[A, B\right] = \int d^4p_k d^4p_{\nu} d^4p_{\rho} d^4p_{\sigma} f_{k}^{0}\left(1 \pm \vec{f}_{k}^{0}\right)(1 \pm \vec{f}_{l}^{0})WAB .
\]

From the principle of detailed balance, it can be shown that \(C_l^{\mu\nu} = C_l^{\mu\nu\bar{s}}\).

With the help of Eq. (24), (25) and (26) and assuming for a massless quark-gluon system the coefficients \(C_k^{(p)s}\) are species independent, the expression for lowest order approximation of shear viscosity is obtained as follows,

\[
\eta = \frac{T}{10} \left\{ \frac{1}{C_{11}^{\prime}} + \frac{1}{C_{12}^{\prime}} + \frac{1}{C_{21}^{\prime}} + \frac{1}{C_{22}^{\prime}} \right\} ,
\]

with \(\gamma_k^{\prime} = -\frac{1}{T} \int d^4p_k f_{k}^{0}(1 \pm \vec{f}_{k}^{0})(p^\mu p^\nu)\langle p_{\mu\nu}p_{\rho\sigma} \rangle \tau_k^{\prime} .
\)

C. Second order theory

This section will provide the second order hydrodynamic equations for the transport quantities mentioned in the previous sections. Before proceeding further let us summarize the thermodynamic identities, i.e., equations of motion for different first order thermodynamic state variables. Here the Eckart’s definition of velocity has been specified for the choice of reference frame. In Eckart’s frame keeping terms up to second order in gradient and ignoring the terms involving thermal and difffusive forces, the thermodynamic identities are given as follows,

\[
Dn = -n \partial_{\mu} u^{\mu} ,
\]

\[
Dc = -hn \partial_{\mu} u^{\mu} + \Pi^{\mu\nu} \partial_{\nu} u_{\mu} ,
\]

\[
hn Du^{\mu} = \nabla^{\mu} P - D^\mu\nu \partial_{\nu} \Pi^{\rho\sigma} + (nh)^{-1} \Pi^{\mu\nu} \nabla_{\nu} P ,
\]

\[
DT = \gamma \partial_{\mu} u^{\mu} + \delta \Pi^{\mu\nu} \partial_{\nu} u_{\mu} ,
\]

\[
D\vec{\mu} = \gamma \partial_{\mu} u^{\mu} + \delta \Pi^{\mu\nu} \partial_{\nu} u_{\mu} .
\]

Here, \(D = u^{\mu} \partial_{\mu}\) is the covariant time derivative and \(\nabla^{\mu} = \Delta^{\mu\nu} \partial_{\nu}\) is the spatial gradient. \(h = \frac{T}{2\pi} c\) is the enthalpy per particle, \(\mu_{c} = \frac{h}{T}\) is the scaled chemical potential and \(\Pi^{\mu\nu}\) is the total viscous pressure tensor (including both shear and bulk part). The coefficients in Eq. (35) and (36) are given in the appendix.

In order to obtain the second order hydrodynamic equations under EQPM model, we need to solve the relativistic transport equation (11), keeping terms up to second order in gradient over thermodynamic quantities. For this purpose, the Grad’s 14 moment method has been opted for the present case. In this method the relativistic transport equation picks up two additional term compared to Eq. (14) containing second order derivative contributions as follows,

\[
\Pi_{l}^{\mu\nu} \partial_{\mu} f_{k}^{0} + f_{k}^{0}(1 \pm \vec{f}_{k}^{0})\Pi_{k}^{\mu\nu} \partial_{\nu} \phi_{k} + \phi_{k} \Pi_{k}^{\mu\nu} \partial_{\nu} f_{k}^{0} = -\frac{1}{T} \sum_{l=1}^{N} \mathcal{C}_{kl} \phi_{l} ,
\]

with \(\Pi_{k}^{\mu\nu} = \frac{\rho_{k}^{4}}{\rho_{k}^{3}}\) as the scaled particle 4-momenta. However, the mean field force term provides zero contribution in a comoving frame.

To proceed further we need to define the deviation function \(\phi_{k}\) for second order theory in a convenient manner. Noticing the distribution function is a scalar depending on the particle momentum \(p_{\mu}^{k}\) and the space-time coordinate \(x_{\mu}\), the deviation function is expressed...
as a sum of scalar products of tensors formed from $\rho_k^\mu$ and tensor functions of $x_\mu$ such as,

$$\phi_k = A_k(x, \tau_k) + B_k^\mu(x, \tau_k)\langle \Pi_{k\mu} \rangle + \sum_{a=1}^{N_v-1} \frac{1}{\mathcal{I}} B_k^{\mu\nu}(x, \tau_k)\langle \Pi_{k\mu} \Pi_{k\nu} \rangle$$

with the coefficients further expanded in a power series of $\tau_k$ as the following,

$$A_k = \sum_{a=0}^2 A_k^a(x)\tau_k^a, \quad C_k^{\mu\nu} = \{C_k^\mu(x)\}^{\mu\nu},$$

$$B_k^{\mu\nu} = \sum_{a=0}^2 \{B_k^{\mu}(x)\}^\mu \tau_k^a, \quad B_k^{\mu\nu} = \sum_{a=0}^2 \{B_k^{\nu}(x)\}^\nu \tau_k^a$$

(39)

Here $a$ is the index of conserved quantum number. The polynomials in Eq. (39) have been retained up to the first non-vanishing contribution to the irreversible flows.

The next task is to express these unknown coefficients in terms of irreversible flow quantities. Here we make another assumption as before that the coefficient functions $A_k^a, C_k^\mu$ etc. are species independent for a massless QGP. We first recall the traceless part of viscous tensor summarized as follows,

The full expressions of $\tilde{\tau}^{\mu\nu}$ are first expressed for a system with $1+1$ dimensional expansion in the $z$ direction. The concern space-time variables are now the proper time $\tau$ and space-time rapidity $\eta$ which are related to the original variables ($x^\mu = t, z$) as $t = \tau \cosh \eta$ and $z = \tau \sinh \eta$. The hydrodynamic four velocity $u^\mu$ and viscous pressure tensor $\Pi^{\mu\nu}$ are then expressed as follows (23),

$$u^\mu = (\cosh \eta, 0, 0, \sinh \eta), \quad \Pi^{\mu\nu} = \left[ \begin{array}{cccc} \phi \sinh^2 \eta & 0 & 0 & 0 \\ 0 & -\frac{\phi^2}{2} & 0 & 0 \\ 0 & 0 & -\frac{\phi^2}{2} & 0 \\ 0 & 0 & 0 & \phi \cosh^2 \eta \end{array} \right]$$

(51)

(52)

To perform the derivatives in a boost invariant system, we go to the Milne coordinates $(\tau, x, y, \eta)$ in which we find $D = \frac{\partial}{\partial \tau}, \quad \phi = \frac{1}{3} (\nabla^n u^o) = -\frac{2}{3} \tau \frac{\partial}{\partial \tau}$ and $\phi = -\tau^2 \Pi^{\mu\nu}$. Applying these derivatives and ignoring the bulk viscous part for the present case, Eq. (43) and (44) turn out to be respectively,

$$\frac{dt}{d\tau} = -\frac{e + P}{\tau^2} + \frac{\phi}{\tau},$$

$$\frac{d\phi}{d\tau} = -\frac{\phi}{\tau} + \frac{2}{3} \frac{2}{3} \frac{\lambda_\pi}{\tau^2} \frac{\eta}{\tau}$$

(53)

(54)
Now, further from Eq. (7) the energy density can be expressed in terms of temperature and effective chemical potential as follows,

$$\varepsilon = \frac{T^4}{\pi^2} \sum_k \nu_k \left( 3R_k^2 + \Delta_k R_k^2 \right),$$

(55)

with $\Delta_k = \frac{\Delta_{\pi}}{C_{\pi}} = \frac{T}{\eta/s} \frac{\partial^2 P_{\beta k}}{\partial \tau^2}$. The series polynomial is expressed as $R_k^2 = \sum_i (\pm 1)^{l(i-1)} \frac{m_{\pi i}}{T} \pi^i \nu_{\pi i} k$ for bosonic and fermionic particles respectively. $\frac{\partial^2 P_{\pi k}}{\partial \tau^2}$ is the scaled baryon chemical potential for $k^{th}$ species. With the help of Eq. (55), for a system with zero baryon chemical potential, \[ \Delta_k \] reduces to temperature evolution equation. Here, noticing that $\Delta_k$ is a slow varying function of $T$, higher order terms like $(O(\sim T^2))$ have been ignored keeping upto linear terms only. Following this prescription, the final second order hydrodynamic equations for temperature and viscous flow come out to be,

$$\frac{dT}{d\tau} = -a_1 \frac{T}{\tau} + b_1 \frac{\phi}{T} \tau, \quad (56)$$

$$\frac{d\phi}{d\tau} = -\frac{\phi}{\tau} + \frac{2a_2}{3} - b_2 \frac{\phi}{\tau}. \quad (57)$$

The corresponding coefficients are described as follows,

$$a_1 = \frac{(1 + c_2^2) \sum_k \nu_k \left( 3R_k^2 + \Delta_k R_k^2 \right)}{12 \sum_k \nu_k R_k^2 + 8 \sum_k \nu_k \Delta_k R_k^2}, \quad (58)$$

$$b_1 = \frac{12 \sum_k \nu_k R_k^2 + 8 \sum_k \nu_k \Delta_k R_k^2}{\pi^2}, \quad (59)$$

$$a_2 = \frac{\eta}{\tau} = \frac{1}{10C_1}, \quad (60)$$

$$b_2 = \frac{\lambda_\tau}{\tau} = \frac{4}{3} + \frac{1}{3} (1 - 3c_2^2) \frac{C_2}{C_1}. \quad (61)$$

The analytic expressions of $\tau^r$ and $\gamma^r_k$ functions upon which the $C$ functions of Eq. (60) and (61) crucially depend through (48) and (49), are given in the appendix. The values of sound velocity and entropy density are given by the relationships, $c_2^2 = \frac{\sum_k \{ \frac{n_\pi k}{n_p} + \frac{\partial P_{\beta k}}{\partial \tau} \}}{\sum_k \{ \frac{n_\pi k}{n_p} + \frac{\partial P_{\beta k}}{\partial \tau} \}}$ and $s = \frac{\sum_k (n_\pi k + n_p k)}{\sum_k n_{\pi k} + n_p k}$ respectively.

### III. RESULTS

Eqs. (56) and (57) describe the evolution of temperature and viscous pressure tensor within a dissipative QCD medium described by EQPM model. In this section I will present the numerical solution of these two equation for a 1+1 boost invariant system in order to visualize the effect of EQPM model on the temperature and viscous pressure evolution. The fugacity parameters $z_{g,q}$ have been set from the updated lattice EOSs for the current purpose. Before proceeding further, we need to examine the behavior of one important parameter of the hydrodynamic evolution of these coupled equations, which is the relaxation time of shear viscous flow $\tau_\pi$. Fig. 1 shows the temperature dependence of $\tau_\pi$ with and without EQPM model for different values of shear viscosity. As predicted by [37-39], the temperature dependence of $\tau_\pi$ shows an decreasing trend with increasing temperature and being proportional to shear viscosity shows higher value for higher $\eta/s$. For each set of $\eta/s$, $\tau_\pi$ has been plotted with and without considering the EQPM model taken into calculation. Whereas in high temperature domain the plots with and without EQPM model merge with each other indicating that at those temperatures, the quasi-particle properties behave almost like those of the free particles, with $z_k$'s reaching their Stefan-Boltzmann (SB) limit, $z_q/z_g \rightarrow 1$. However, at low temperatures the effect of EQPM model is quite prominent with the enhancement over ideal values (without EQPM model).

Next, the proper time evolution of the pressure anisotropy, i.e the ratio between longitudinal and transverse pressure $P_L/P_T = (P - \phi)/(P + \frac{\phi}{2})$ has been shown in Fig. 2 for different set of $\eta/s$ ratio with and without EQPM model. The initial values of proper time and temperature have been taken to be the Relativistic Heavy Ion Collider (RHIC) values \[ \tau_i = 0.25 fm \text{ and } T_i = 0.3 GeV \]. The initial value of shear viscous pressure is taken to be the Navier-Stokes value $\phi_i = 4/3$. For higher value of viscosity the curves are enhanced indicating dissipative processes make the system to take larger times to cool down. The effect of EQPM model for each set of $\eta/s$ is clearly distinct from the ideal ones which is more prominent at larger times i.e. for smaller temperatures.

Next, in Fig. 3 and 4 proper time evolution of the pressure anisotropy, i.e the ratio between longitudinal and transverse pressure $P_L/P_T = (P - \phi)/(P + \frac{\phi}{2})$...
has been depicted using EQPM model. In Fig. 3, the EQPM results have been compared with BAMPS data for different $\eta/s$ values with initial temperature $T_i = 0.5 GeV$ and initial time $\tau_i = 0.4 fm$, which correspond to Large Hadron Collider (LHC) initial condition. The EQPM estimate of pressure anisotropy shows considerable agreement with BAMPS result. In Fig. 4, the same has been plotted with $T_i = 0.6 GeV$, $\tau_i = 0.25 fm$ (which are again LHC conditions) and $\eta/s = 1/4 \pi$. In both the cases the initial pressure configuration has been taken to be isotropic $\phi_i = 0$. In Fig. 4, the EQPM result of pressure anisotropy has been compared with other quasi particle models. vHydro stands for the standard second-order viscous hydrodynamics, where QaHydro is the quasiparticle anisotropic hydrodynamics and QvHydro is quasiparticle second-order viscous hydrodynamics. These two quasi particle models describe the thermodynamic system by considering temperature dependent quasiparticle masses. The prediction of pressure anisotropy from current EQPM set up appears to be consistent with these models as well.

The last plot presents proper time evolution of the inverse Reynolds number $R^{-1} = \sqrt{\frac{3}{2} \phi/P}$ in Fig. 5 with the same initial conditions as Fig. 4 and is compared with other quasiparticle estimates discussed earlier. Here also EQPM is observed to predict consistent result with other models.

**IV. CONCLUSION AND OUTLOOK**

In the present work a second order relativistic viscous hydrodynamic theory has been developed including the effects of a strongly interacting thermal medium through EQPM model. The effects of hot QCD EOSs have been included through the temperature dependent fugacity parameter of the quasi-partons which have been extracted from an updated lattice prediction. The second order hydro equations have been formulated from the covariant kinetic theory for a multi-component system that has been consistently developed within the scheme of EQPM model. The relativistic transport equation from this effective kinetic theory has been solved using Grad’s 14 moment method within a viscous medium. Finally, the hydrodynamic equations corresponding to energy density (as well as temperature) and viscous flow have been solved for a 1+1 boost invariant system. The proper time evolution of temperature and pressure anisotropy have shown significant modification due to the inclusion
of EQPM model with respect to the ideal ones. Hence, it can be concluded that the effects of a strongly interacting thermal QCD medium have been embedded into the evolution equations of the system through the EQPM model which is reflected on the behavior of thermodynamic quantities and cooling laws of the system consequently.

Formulating second order dissipative hydrodynamics in the presence of electromagnetic fields is becoming an interesting and relevant topic now a days. Works like [42-47] are worth mentioning and encouraging in this direction. Keeping this scenario in view, an immediate extension of the current work will be treating the system with EQPM model in the presence of a strong electromagnetic field. A number of recent works regarding the first order transport processes in the presence of a strong magnetic field employing EQPM model have been pursued in [48]. Based on these previous works a complete formalism of dissipative magnetohydrodynamics theory within the scope of EQPM model is a sure aim in the future.

Appendix A: Important identities

The coefficients in Eq. (35) and (36) are given for a two component system as the following,

$$ \gamma = \frac{-nh \left( \frac{\partial n}{\partial x_1} \right) \left( \frac{\partial n}{\partial x_2} \right) + n_1 \left( \frac{\partial n}{\partial x_1} \right) \left( \frac{\partial n}{\partial x_2} \right) + n_2 \left( \frac{\partial n_1}{\partial x_2} \right) \left( \frac{\partial n}{\partial x_1} \right)}{\left( \frac{\partial n}{\partial x_1} \right) \left( \frac{\partial n}{\partial x_2} \right) - \frac{\partial n_1}{\partial x_2} \left( \frac{\partial n}{\partial x_1} \right) - \frac{\partial n_2}{\partial x_1} \left( \frac{\partial n}{\partial x_2} \right) + nh \left( \frac{\partial n}{\partial x_1} \right) \left( \frac{\partial n}{\partial x_2} \right)} \left( \frac{\partial n}{\partial x_1} \right) \left( \frac{\partial n}{\partial x_2} \right), \quad (A1) $$

$$ \delta = \frac{\left( \frac{\partial n}{\partial x_1} \right) \left( \frac{\partial n}{\partial x_2} \right) - \frac{\partial n_1}{\partial x_2} \left( \frac{\partial n}{\partial x_1} \right) - \frac{\partial n_2}{\partial x_1} \left( \frac{\partial n}{\partial x_2} \right) + nh \left( \frac{\partial n}{\partial x_1} \right) \left( \frac{\partial n}{\partial x_2} \right)}{\left( \frac{\partial n}{\partial x_1} \right) \left( \frac{\partial n}{\partial x_2} \right) - \frac{\partial n_1}{\partial x_2} \left( \frac{\partial n}{\partial x_1} \right) - \frac{\partial n_2}{\partial x_1} \left( \frac{\partial n}{\partial x_2} \right) + nh \left( \frac{\partial n}{\partial x_1} \right) \left( \frac{\partial n}{\partial x_2} \right)}, \quad (A2) $$

$$ \gamma_1 = \frac{-n_1 \left( \frac{\partial n}{\partial x_1} \right) \left( \frac{\partial n}{\partial x_2} \right) + n_1 \left( \frac{\partial n}{\partial x_1} \right) \left( \frac{\partial n}{\partial x_2} \right) - n_2 \left( \frac{\partial n_1}{\partial x_2} \right) \left( \frac{\partial n}{\partial x_1} \right) + nh \left( \frac{\partial n}{\partial x_1} \right) \left( \frac{\partial n}{\partial x_2} \right)}{\left( \frac{\partial n}{\partial x_1} \right) \left( \frac{\partial n}{\partial x_2} \right) - \frac{\partial n_1}{\partial x_2} \left( \frac{\partial n}{\partial x_1} \right) - \frac{\partial n_2}{\partial x_1} \left( \frac{\partial n}{\partial x_2} \right) + nh \left( \frac{\partial n}{\partial x_1} \right) \left( \frac{\partial n}{\partial x_2} \right)}, \quad (A3) $$

$$ \delta_1 = \frac{\left( \frac{\partial n}{\partial x_1} \right) \left( \frac{\partial n}{\partial x_2} \right) - \frac{\partial n_1}{\partial x_2} \left( \frac{\partial n}{\partial x_1} \right) - \frac{\partial n_2}{\partial x_1} \left( \frac{\partial n}{\partial x_2} \right) + nh \left( \frac{\partial n}{\partial x_1} \right) \left( \frac{\partial n}{\partial x_2} \right)}{\left( \frac{\partial n}{\partial x_1} \right) \left( \frac{\partial n}{\partial x_2} \right) - \frac{\partial n_1}{\partial x_2} \left( \frac{\partial n}{\partial x_1} \right) - \frac{\partial n_2}{\partial x_1} \left( \frac{\partial n}{\partial x_2} \right) + nh \left( \frac{\partial n}{\partial x_1} \right) \left( \frac{\partial n}{\partial x_2} \right)}. \quad (A4) $$

The expression for $\gamma_2$ and $\delta_2$ can be trivially obtained by mutually interchanging particle index 1 and 2 in Eq. (A3) and (A4) respectively.

The $A^2$, $A^1$ and $A^0$ in Eq. (A2) are given by,

$$ A^2 = \frac{1}{\left\{ \sum_k \nu_k a_k^2 \right\}}; \quad (A5) $$

$$ A^1 = \frac{1}{\left\{ \sum_k \nu_k a_k^2 \right\}} \left[ \left( \sum_k \nu_k a_k^1 \right) \left( \sum_k \nu_k a_k^2 \right) - \left( \sum_k \nu_k a_k^1 \right) \left( \sum_k \nu_k a_k^2 \right) \right], \quad (A6) $$

$$ A^0 = \frac{1}{\left\{ \sum_k \nu_k a_k^2 \right\}} \left[ \left( \sum_k \nu_k a_k^1 \right) \left( \sum_k \nu_k a_k^1 \right) - \left( \sum_k \nu_k a_k^2 \right) \right], \quad (A7) $$

Appendix B: Analytic expression for $\gamma^r_k$ functions

with,

$$ \alpha_k^r = -T \int d\Gamma_{p_k} f_0^k (1 \pm f_0^k) \tilde{q}_k T_k, \quad (A8) $$

$$ a_k^n = \int d\Gamma_{p_k} f_0^k (1 \pm f_0^k) T_k \tilde{r}_k, \quad (A9) $$

$$ \gamma_k^0 = -\frac{1}{3 \pi^2 T} \left\{ I_k^0 - \Delta_k I_k^0 \right\}, \quad (B1) $$

$$ \gamma_k^1 = -\frac{1}{3 \pi^2 T} T_k^0, \quad (B2) $$

$$ \gamma_k^2 = -\frac{1}{3 \pi^2 T} \left\{ I_k^2 + \Delta_k I_k^2 \right\}, \quad (B3) $$

$$ \gamma_k^0 = -\frac{1}{3 \pi^2 T} T_k^0, \quad (B4) $$

FIG. 5: Proper time evolution of inverse Reynolds number.
with \( I^n_k = n!T^{n+1}R^n_k \).

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[1] L. D. Landau, Izv. Akad. Nauk Ser. Fiz. 17 (1953) 51.
[2] J. P. Blaizot and J. Y. Ollitrault, Nucl. Phys. A 458 (1986) 745.
[3] J. D. Bjorli, Phys. Rev. D 27 (1983) 140.
[4] S. Jeon and U. Heinz, Int. J. Mod. Phys. E 24 (2015) no.10, 1530010.
[5] P. F. Kolb, J. Sollfrank and U. W. Heinz, Phys. Rev. C 62 (2000) 054909.
[6] H. Song and U. W. Heinz, Phys. Rev. C 77 (2008) 064901.
[7] P. Danielewicz and M. Gyulassy, Phys. Rev. D 31 (1985) 53.
[8] P. B. Arnold, G. D. Moore and L. G. Yaffe, JHEP 0011 (2000) 001. P. B. Arnold, G. D. Moore and L. G. Yaffe, JHEP 0305 (2003) 051.
[9] M. Luzum and P. Romatschke, Phys. Rev. C 78 (2008) 034915.
[10] M. Muller, Z. Phys. A 198 (1967) 329, W. Israel, Annals Phys. 100 (1976) 310, W. Israel and J. M. Stewart, Annals Phys. 118 (1979) 341.
[11] H. Heiselberg and C. J. Pethick, Phys. Rev. D 48 (1993) 2916, H. Heiselberg, Phys. Rev. D 49 (1994) 4739.
[12] J. W. Chen, H. Dong, K. Ohnishi and Q. Wang, Phys. Lett. B 685 (2010) 277.
[13] A. Muronga, Phys. Rev. Lett. 88 (2002) 062302.
[14] R. Baier, P. Romatschke and U. A. Wiedemann, Phys. Rev. C 73 (2006) 064903.
[15] E. Calzetta and J. Peralta-Ramos, Phys. Rev. D 82 (2010) 106003.
[16] A. Monnai and T. Hiranou, Nucl. Phys. A 847 (2010) 283.
[17] A. Jaiswal, Phys. Rev. C 87 (2013) no.5, 051901.
[18] G. S. Denicol, H. Niemi, E. Molnar and D. H. Rischke, Phys. Rev. D 85 (2012) 114047. G. S. Denicol and H. Niemi, Nucl. Phys. A 904-905 (2013) 369c.
[19] U. W. Heinz, D. Bazow and M. Strickland, Nucl. Phys. A 931 (2014) 920.
[20] V. Chandra, R. Kumar and V. Ravishankar, Phys. Rev. C 76 (2007) 054909, V. Chandra and V. Ravishankar, Phys. Rev. D 84 (2011) 074013, V. Chandra and V. Ravishankar, Eur. Phys. J. C 64 (2009) 63, V. Chandra, A. Ranjan and V. Ravishankar, Eur. Phys. J. A 40 (2009) 109.
[21] L. Tinti, A. Jaiswal and R. Ryblewski, Phys. Rev. D 95 (2017) no.5, 054907.
[22] M. Alqahtani, M. Nopoush and M. Strickland, Phys. Rev. C 92 (2015) no.5, 054910.
[23] M. Alqahtani, M. Nopoush, R. Ryblewski and M. Strickland, Phys. Rev. Lett. 119 (2017) no.4, 042301.
[24] S. Mitra and V. Chandra, Phys. Rev. D 96 (2017) no.9, 094003.
[25] S. Mitra and V. Chandra, Phys. Rev. D 94 (2016) no.3, 034025.
[26] S. Mitra and V. Chandra, Phys. Rev. D 97 (2018) no.3, 034032.
[27] S. Mitra, S. Ghosh and S. Sarkar, Phys. Rev. C 85 (2012) 064917.
[28] S. Mitra and S. Sarkar, Phys. Rev. D 87 (2013) no.9, 094026.
[29] S. Mitra and S. Sarkar, Phys. Rev. D 89 (2014) no.5, 054013.
[30] S. R. De Groot, W. A. Van Leeuwen and C. G. Van Weert, Relativistic Kinetic Theory, Principles And Applications, Amsterdam, Netherlands: North-holland (1980).
[31] S. Mitra, U. Gangopadhyaya and S. Sarkar, Phys. Rev. D 91 (2015) no.9, 094012.
[32] A. Muronga, Phys. Rev. C 76 (2007) 014910.
[33] A. Bazavov et al. (HotQCD Collaboration), Phys. Rev. D 90 (2014) 094503.
[34] M. Natsuume and T. Okamura, Phys. Rev. D 77 (2008) 066014.
[35] G. S. Denicol, J. Noronha, H. Niemi and D. H. Rischke, J. Phys. G 38 (2011) 124177.
[36] T. Koide, E. Nakano and T. Kodama, Phys. Rev. Lett. 103 (2009) 052301.
[37] A. El, Z. Xu and C. Greiner, Nucl. Phys. A 806 (2008) 287.
[38] Z. Xu and C. Greiner, Phys. Rev. C 71 (2005) 064901, A. El, Z. Xu and C. Greiner, Phys. Rev. C 81 (2010) 041901.
[39] G. S. Denicol, X. G. Huang, E. Molnár, G. M. Monteiro, H. Niemi, J. Noronha, D. H. Rischke and Q. Wang, Phys. Rev. D 98 (2018) no.7, 076009, G. S. Denicol, E. Molnar, H. Niemi and D. H. Rischke, arXiv:1902.01699 [nucl-th].
[40] J. Hernandez and P. Kovtun, JHEP 1705 (2017) 001.
[41] U. Grsoy, D. Kharzeev and K. Rajagopal, Nucl. Phys. A 931 (2014) 986, U. Gursoy, D. Kharzeev and K. Rajagopal, Phys. Rev. C 89 (2014) no.5, 054905, U. Grsoy, D. Kharzeev, E. Marcus, K. Rajagopal and C. Shen, Phys. Rev. C 98 (2018) no.5, 055201.
[42] D. E. Kharzeev, J. Phys. G 38 (2011) 124061.
[43] V. Roy, S. Pu, L. Rezzolla and D. H. Rischke, Phys. Rev. C 96 (2017) no.5, 054909.
[44] G. Inghirami, L. Del Zanna, A. Beraudo, M. H. Moghadam, F. Becattini and M. Bleicher, Eur. Phys. J. C 76 (2016) no.12, 659.
[45] M. Kurian and V. Chandra, Phys. Rev. D 96 (2017) no.11, 114026, M. Kurian and V. Chandra, Phys. Rev. D 97 (2018) no.11, 116008, M. Kurian, S. Mitra, S. Ghosh and V. Chandra, arXiv:1805.07313 [nucl-th], accepted in
