Separating Bichromatic Polygons by Fixed-angle Minimal Triangles

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Abstract
Separation of desired objects from undesired ones is one of the most important problems in computational geometry. It is tended to cover the desired objects by one or a couple of geometric shapes in a way that all of the desired objects are included by the covering shapes, while the undesired objects are excluded. We study separation of poly-lines by minimal triangles with a given fixed angle and present $O(N \log N)$-time algorithm, where $N$ is the number of all the desired and undesired polygons. By a minimal triangle, we mean a triangle in which all of its edges are tangential to the convex hull of the desired polygon. The motivation for studying this separation problem stems from that we need to separate bichromatic objects that are modeled by polylines not points in real life scenarios.

Keywords: Computational geometry, bichromatic separation, minimal triangle, polyline separation, fixed-angle triangle.

1. Introduction
Separation of points is one of the essential and practical problems in computational geometry. Two sets, $P$ and $Q$, of (resp.) blue and red points and a specific geometric shape in the plane are considered, and the blue points of $P$ are aimed to be covered by the specific geometric shape in a way that none of the red points of $Q$ is covered. Various versions of the separation of objects have been defined in which the optimization of parameters such as perimeter, area and the number of edges of separating shape was followed. The polygon with the minimum number of edges is triangle, and we consider triangles as separating shapes in this paper. The separation of points has different applications in facility location [1], VLSI layout design [2], image processing [3], data mining [4], computer graphics [5], statistics and classification [6].

In this paper, we consider a new version of the problem and we present an $O(N \log N)$-time algorithm to separate two bichromatic object sets $P$ and $Q$ by fixed-angle minimal triangles which (a) contain all objects of $P$, (b) avoid all objects of $Q$ and (c) are minimal, i.e. their three sides are tangential to the convex hull of $P$. The objects we consider are polylines. The motivation for studying this separation problem stems from that we need to separate bichromatic objects that are modeled by polylines not points in real life scenarios. We first, give the algorithm for lines, then extend it to polylines in section 8.

The four main steps of the algorithm are presented in sections 4, 5, 6 and 7. Each section outlines one step, proves lemmas involved and gives its time complexity. In section 4, we use Seara’s [1] idea to compute a star-shaped structure for computing separating triangles. We consider trajectory of the...
\(\theta\)-vertex of \(\theta\)-triangles in section 5. In the third main step of the algorithm, in section 6, we compute the event points and finally, we handle those event points and the pseudo-code of our algorithm, as a whole, is given regarding the related sections in section 7.

2. Related Works

There has been a fair amount of work on different kinds of separators. In 1983 a linear-time algorithm for the separation of colored points by a line was given [7]. In 1986 the separation of points by a circle was posed and solved by O’Rourke et al. [8].

Edelsbrunner and Preparata [9] solved the problem of separation of points by a convex polygon with the minimum number of edges and Fekete [10] showed that the separation of points by a general polygon with the minimum number of edges is an \(NP - complete\) problem, so an approximation algorithm was posed using a polynomial time by [11]. The separation of points has also been studied for separators of the form of strips and wedges [12]. Sarkar and Stojmenovic [13] considered polygon separation problem and presented a parallel algorithm to construct a separating convex \(k - gon\) with smallest \(k\). In 2002 Eckstein et al. [14] worked on the problem of finding an axis-aligned rectangle \(B\) such that \(P \cap B = \emptyset\) and the cardinality of \(Q \cap B\) is maximized. Finding an axis-aligned rectangle \(B\) that maximizes the difference between the numbers of points of \(Q\) and \(P\) inside \(B\), i.e., \(|B \cap Q| - |B \cap P|\), has been studied by Liu and Nediak [15].

Demaine et al. [16] studied the separability of two point sets inside a polygon by means of chords or geodesic lines and provided necessary and sufficient conditions for the existence of a chord or geodesic path separates the two sets. Separability of two point sets for the case where separator is an L-shape with orientation of \(\vartheta\) was studied by Sheikh et al. [17]. The problem of deciding whether the two point sets can be separated by two disjoint isothetic rectangles was solved by Moslehi and Bagheri [18]. Separability of imprecise bichromatic points was investigated by Sheikh et al. [19]. Their results include algorithms for finding certain separators (which separate bichromatic points with probability 1), possible separators (which separate with non-zero probability), most likely separators (which separate with maximum probability), and maximal separators (which maximize the expected number of correctly classified points). Bandyapadhyay and Banik [20] considered a collection of geometric problems involving points colored by two colors, referred to as bichromatic problems and presented low polynomial time exact algorithms for those problems. Xue et al. [21] studied the linear separability problem for stochastic geometric objects under the well-known unipoint and multipoint uncertainty models. Let \(S = S_R \cup S_B\) be a given set of stochastic bichromatic points, and define \(n = \min \{|S_R|, |S_B|\}\) and \(N = \max \{|S_R|, |S_B|\}\).

They showed that the separable-probability of \(S\) can be computed in \(O(nN^{d-1})\) time for \(d \geq 3\) and \(O\left(\min \left\{nN \log N, N^{2/3}\right\}\right)\) time for \(d = 2\), while the expected separation-margin of \(S\) can be computed in \(O(nN^d)\) time for \(d \geq 2\).

In 2018 Har-Peled and Jones [22] studied the separability of \(n\) points in the plane by the minimum number of lines to separate all its pairs from each other. They showed that the minimum number of lines needed to separate \(n\) points, picked randomly (and uniformly), in the unit square, is \(\Theta(n^{2/3})\).

Bonnet et al. [23] worked on Separation of two bichromatic point sets by a minimum-size set of lines that separate them from each other. They showed that, in its full generality, parameterized by the number of lines \(k\) in the solution, the problem is unlikely to be solvable significantly faster than the brute-force \(n^{O(k)}\)-time algorithm, where \(n\) is the total number of points and separation with a minimum-size set of axis-parallel lines can be solved in time \(O(9^\beta)\) (assuming that \(\beta\) is the smallest set).

Abrahamsen et al. [24] showed that separation of groups of objects in the plane by the shortest fences is \(NP - hard\) for the case where the input consists of polygons of two colours with \(n\) corners in total and
they gave a randomized 4/3.2965-approximation algorithm for polygons and any number of colours. Separating cycle problem has investigated Arkin et al. [25]. For a given set of pairs of points in the plane, the goal is to find a simple tour of minimum length that separates the two points of each pair to different sides. They proved hardness of the problem and provided some approximation algorithms under various settings. Misra et al. [26] worked on the special case of the separation problem when the points lie on a circle and demonstrated a polynomial-time algorithm for that case. Acharyya et al. [27] worked on different problems for bichromatic point set on a plane and proposed in-place algorithms for computing (a) an arbitrarily oriented monochromatic rectangle of maximum size in $R^2$ and (b) an axis-parallel monochromatic cuboid of maximum size in $R^3$.

Various versions of the separation of points by other objects such as L-shapes and well-covered rectangle were posed. A thorough study is presented by Seara [1]. In 2017 the separation of two bichromatic point sets by a minimal triangle with a fixed angle was posed by Moslehi and Bagheri [28] and solved by an $O(n \log n)$-time algorithm. We study the separation of polylines by a triangle and present the relevant algorithm of $O(N \log N)$-time, in fact, we generalize their solution for polylines.

3. Preliminaries and algorithm description

We begin with some rudimentary definitions as useful for our discussion. In section 3.1, we show that separation of line segments is different from separation of points. The vertices of separating triangle cannot be placed anywhere in the plane, in section 3.2 we compute the area in which it is feasible to place the vertices of separating triangles, i.e. $\overline{A}$. Since an angle of separating triangles is fixed, so the trajectory of its corresponding vertex is a set of arcs which is called $\theta$-cloud that is computed in section 3.3. The topology of separating triangles is changed in some event points which is defined in section 3.4. Handling of these events and the pseudo-code of our algorithm are given in section 3.5 and finally in section 3.6 we extend the results to polylines.

3.1 Separating Blue and Red Lines by a Triangle

We let $P = \{p_1^1, p_2^1, \ldots, p_n^1\}$ be a set of $n$ blue line segments and $Q = \{q_1^1, q_2^1, \ldots, q_m^1\}$ be a set of $m$ red line segments. The goal is to calculate the minimal triangles with a fixed angle in a way that each of them includes all members of $P$ and excludes all members of $Q$.

First of all, let us check whether this problem can be reduced to bichromatic point set separation problem. Can we consider just the end points of the red and blue lines and separate these two point sets to separate the red and blue lines? We show that the mentioned idea is wrong. In the Fig. 1 the red end points are separated from the blue end points by a triangle but it contains some parts of the red lines.

In the following, we show blue lines by thin gray lines and red lines by thick black lines. As a triangle is convex and the convex hull of a collection of objects is defined as the minimum convex polygon that includes the relevant objects, the separating triangle of relevant objects includes the convex hull of $P$, i.e. $CH(P)$. Also all the three sides of the minimal separating triangle are tangential to $CH(P)$. If there is a triangular separator, $CH(P)$ will be monochromatic and otherwise, two sets of $P$ and $Q$ will be without any triangular separator, resulted from the convexity of a triangle. It is clear that we could calculate $CH(P)$ and its monochromatic state in $O(n \log n)$ time. Then, we suppose $CH(P)$ as monochromatic. Fig. 2 shows the convex hull of the blue lines avoiding the red lines.

3.2 Calculation of feasible area $\overline{A}$
In this section, we have three definitions.

**Definition 1.** Given a convex polygon \( Y \), and a point \( x \) outside \( Y \), by the **supporting lines** of \( x \) to \( Y \), we mean the two lines passing through \( x \) and are tangent to \( Y \). Consider Fig. 3.

The supporting line can be calculated in \( O(n \log n) \) time [29].

**Definition 2.** Given a convex polygon \( Y \), and a line segment \( L \) outside \( Y \), by the **supporting lines** of \( L \) to \( Y \), we mean the two common internal tangent lines of \( L \) and \( Y \). Consider Fig. 4.

**Definition 3.** The vertices of a separating triangle cannot lie in some areas defined by the red line segments and the extension of their supporting lines to \( CH(P) \). Let's consider a red line segment \( S_i = q_i q_i^\wedge \). The extension of its supporting lines defines the four areas in the plane. \( CH(P) \) lies in one of these areas, the opposite area is called **forbidden area** \( A_i \) of \( S_i \).

The vertices of a separating triangle fail to lie in forbidden areas. That is because if a vertex of a separating triangle lies in the forbidden area, a part of a red line segment \( S_i \) will also lie in it. If \( A = \bigcup_i A_i \) then an allowed area for vertices of a separating triangle is a complimentary area of \( A \). See Fig. 5.

The area \( \overline{A} \) is star shaped, often unlimited with borders, where \( CH(P) \) is located in its kernel [4]. The border of this forbidden area consists of line segments of the set \( Q \) and the extension of their supporting lines (in [4], a similar idea is used). It could be calculated in \( O(N \log N) \) time, where \( N = n + m \) [1]. We denote it by \( \sigma \).

### 3.3 Computing the Trajectory of \( \theta \)-vertex of \( \theta \)-triangles Around \( CH(P) \)

In this section, we have some definitions.

**Definition 4.** A **\( \theta \)-triangle** is a triangle with an interior angle of \( \theta \).

**Definition 5.** The **\( \theta \)-vertex** of a **\( \theta \)-triangle** is a vertex such that the angle between corresponding edges is \( \theta \).

Considering Fig. 6, imagine two supporting lines to \( CH(P) \) intersecting at a point \( x \) and the internal angle between them is \( \theta \). Fixing the tangency points of the supporting lines, then the locus of point \( x \) is a circular arc. So, the **\( \theta \)-vertex** of a separating **\( \theta \)-triangle** lies on a trajectory \( \tau \). It consists of the mentioned circular arcs and is obtained when the **\( \theta \)-vertex** of a **\( \theta \)-triangle** that is tangent to \( CH(P) \) rotates over a full \( 2\pi \) turn around \( CH(P) \). For more details see [30].

**Definition 6.** **\( \theta \)-cloud** is the trajectory \( \tau \) of the **\( \theta \)-vertex** [31].

**Lemma 7.** The **\( \theta \)-cloud** can be constructed from \( CH(P) \) in \( O(n) \) time and contains \( O(n) \) vertices and circular arcs [8].

### 3.4 Calculation of Event Points

By specifying the **\( \theta \)-cloud**, the separating triangle is examined based on the position of the **\( \theta \)-vertex** in \( \tau \). It is clear that the set of points belonging to \( \tau \) inside the forbidden area are unusable and it is not possible to locate the vertex of separating triangle in them. We divide \( \tau \) into intervals by which we could find out if a separating triangle is possible for all points of the intervals, based on the position of the vertex at points of the intervals. We draw two supporting lines from **\( \theta \)-vertex** \( w \) to hit the first point in the forbidden area and we consider the supporting lines as directed vectors starting from \( w \). The supporting line such that \( CH(P) \) lies on its right side is called \( l(w) \) and the other supporting
line is called \( r(w) \). The first intersecting point of \( l(w) \) with \( \sigma \) is called \( x \) (if it exists), and the first intersecting point of \( r(w) \) with \( \sigma \) is called \( y \) (if it exists). Tangential lines of \( CH(P) \) passing through \( x \) and \( y \), respectively are called \( L \) and \( R \). See Fig. 7. In Lemma 9 it is shown that the existence of the separating triangle depends on the external angle between \( L \) and \( R \). If the external angle between \( L \) and \( R \) is not less than 180° then the two sets \( P \) and \( Q \) of bichromatic line segments can be separated by a minimal triangle. By rotating \( w \) on \( \tau \) we encounter two groups of important points which we call them event points. These event points divide \( \tau \) into some intervals. The points belong to an interval have the same property regarding the existence of separating triangles. The two types of event points are:

**Event points of type 1:**
This event occurs when \( R \) or \( L \) are changed by turning \( w \) on \( \tau \). While \( w \) moves on \( \tau \), \( R \) (respectively \( L \) ) is changed when the intersection point of the extension of \( r(w) \) (respectively \( l(w) \) ) with border \( \sigma \) is changed.

**Event points of type 2**, that are formed by intersecting \( \tau \) and \( \sigma \).
During rotation of \( w \) on \( \tau \) for tracing changes of \( L \) and \( R \) and testing the external angle between \( L \) and \( R \), it is required to consider intersecting points of \( r(w) \) and \( l(w) \) with \( \sigma \). Regarding the two intersection points in the border of the forbidden area which consists of line segments of set \( Q \), the extension of their supporting lines and unlimited areas, six cases are occurred that is equal to \( \binom{3+2-1}{2} \) (which is 2-combinations of three parts of forbidden area with repetition).

**Case A**
The tangential lines \( l(w) \) and \( r(w) \) hit the border, \( \sigma \), of the forbidden area at the extensions of some line segments of the set \( Q \).
Consider Fig. 7.
\( w \) : The vertex of a separating triangle which is angled \( \theta \).
\( l(w) \) : A tangential line to \( CH(P) \) from point \( w \), in a way that \( CH(P) \) is located on its right.
\( r(w) \) : A tangential line to \( CH(P) \) from point \( w \), in a way that \( CH(P) \) is located on its left.
\( L \) : The first line segment from border which does not belong to \( Q \) and is extended from one of supporting lines of one of red line segments. The extension of \( l(w) \) will intersect it at point \( x \) so that \( CH(P) \) is located on its right and \( L \) is tangential to \( CH(P) \) by extension.
\( R \) : The first line segment from border which does not belong to \( Q \) and is extended from one of supporting lines of one of red line segments. The extension of \( r(w) \) will intersect it at point \( y \) so that \( CH(P) \) is located on its left and \( R \) is tangential to \( CH(P) \) by extension.
\( i \) : The intersection point between \( L \) and \( l(w) \).
\( j \) : The intersection point between \( R \) and \( r(w) \).
\( x \) : The intersection point between \( L \) and \( r(w) \).
\( y \) : The intersection point between \( R \) and \( l(w) \).

We claim that two edges of the hypothetical separating triangle, which is one of its vertices, are located in the extended lines of \( l(w) \) and \( r(w) \). The third side (in front of \( w \)) is located in the line segment \( l(w) \) at one end and the line segment \( r(w) \) at the other end.
Lemma 8. Consider a convex polygon \( Y \) with \( n \) edges, a point \( g \) outside it, two supporting lines of \( g \) and their tangency points \( s \) and \( t \). For each point \( h \) inside the bounded area by \( Y \) and the two supporting lines of \( g \) (the gray area in Fig. 8), the extensions of supporting lines of point \( h \) intersect "both” two line segments \( gs \) and \( gt \).

**Proof by Induction method**

We examine the induction basis that by concerning our polygon as a triangle (for \( n = 3 \)), if from a point such as \( h \) inside the bounded area \( (sgt) \) we draw supporting lines to any selected points, \( s \) and \( t \) will be tangential to the same points and, as \( h \) is closer to \( st \) than \( g \), then the angle \( \angle sht \) will be larger than the angle \( \angle sgt \) and supporting lines will intersect both \( gs \) and \( gt \) by extension. See Fig. 9. In the case that the whole triangle is within the triangle \( hst \), the claim is effective too.

**Induction Hypothesis:**

It is judged as true for \( n = k \).

**Induction Judgment:**

We show that the judgment is true for \( n = k + 1 \).

We consider a convex polygon with \( k + 1 \) edges and draw two tangential lines from a point outside it such as \( g \) (Fig. 8). We show the tangency points as \( s \) and \( t \). If we connect the two points, we will obtain a convex polygon which its edges are less than \( k + 1 \). A convex polygon is divided into two polygons by drawing one of its chords. If the connecting line \( st \) is one of the sides of the polygon, it will be same as our induction basis which has been examined. Then, the presupposition judgment will be effective for the polygon. Therefore, supporting lines of any point inside the area restricted by the main polygon with \( k + 1 \) edges and the tangential lines of \( hu \) and \( hz \) will intersect both line segments of \( gs \) and \( gt \) by extension and the judgment for a polygon with \( k + 1 \) edges will be effective, too. \( \square \)

We consider \( L \) and \( R \) as directed vectors such that \( CH(P) \) lies on the right side of \( R \) and on the left side of \( L \). External and internal angles between \( R \) and \( L \) are defined as standard way. See Fig. 10.

Lemma 9. Two colored sets of \( P \) and \( Q \) are separable by the \( \theta \)-triangle if and only if the external angle, by extension of \( L \) and \( R \), is not less than \( 180^\circ \).

In order to prove the lemma mentioned above, we must notice that when the external angle of \( R \) and \( L \) is less than \( 180^\circ \), any line segment which intersects with \( l(w) \) and \( r(w) \) will overlap \( CH(P) \). So, there will be no separator triangle. Furthermore, we must show that in the restricted area by \( CH(P), l(w), r(w) \) and \( xi \) and \( yj \) line segments, there is no red line.

**Proof by reductio ad absurdum**

Our absurd hypothesis is that there is at least one red point in the mentioned area. As \( l(w) \) and \( xi \) line segments are tangential to \( CH(P) \), by lemma 8 if there is a red point in the restricted area by \( CH(P) \) and the line segments mentioned, extensions of the supporting lines of it will intersect \( xi \) and \( l(w) \) so that \( l(w) \) hits the border \( \sigma \) before \( L \) line segment at another point unlike our basic hypothesis. Then, the absurd hypothesis is false and the judgment is effective. This claim is also effective for \( yj \) and \( r(w) \) line segments. It can be claimed in Fig. 7 that there is no red line in the restricted area by \( CH(P), l(w) \) and \( r(w) \) and \( xi \) and \( yj \) line segments. That is because if there is any red line, its supporting line will intersect \( l(w) \) and \( r(w) \) and it is unlike our basic hypothesis based on the fact that \( L \) and \( R \) are the first lines from border \( \sigma \) which the extension of \( l(w) \) and \( r(w) \) will intersect them. Then, the absurd hypothesis is false and the judgment is effective. We showed that any line that intersects both \( wx \) and
wy, can be the third side of a separating triangle if and only if the external angle of L and R is not less than 180°. In such a way the third side does not overlap CH(P) and there is no red line inside the formed separator triangle. □

Case B
In this case, one of the tangential lines, l(w) or r(w), intersects σ at one of the red line segments from the set Q, while the other tangential line intersects a supporting line (of a red line segment).
Consider Fig. 11.
R: One of supporting lines of the first line segment from border σ which belongs to Q and intersects r(w), by extension, in a way that CH(P) is located on its right, and will be located in front of w by extension.
x: The intersection point between l(w) and the red line which belongs to Q.
y: The intersection point between r(w) and a supporting line from border.
L: A line passing through x and tangential to CH(P) in a way that CH(P) is located on its left.
k: The intersection point between r(w) and extension of L.
j: The end point of the red line segment reverse to w.
i: The intersection point between the red line and a line extended by R.
We claim that in this case, the lemma 9 is effective too.
Proof.
Unlike case A, l(w) intersects one of red lines by extension and any points in the red lines have their own specific L and R. Unlike a supporting line extension which all points of it have a unique tangential line to CH(P), a red line has a specific tangential line to each point of it. Then, the condition of the third side of the separator triangle depends on the point which l(w) or r(w) hits in the red line by extension.
In Fig. 11, the external angle between R and the line passing through x (i.e., L) determines whether the third side of a separator triangle is formed. For this case, as r(w) has hit supporting line by extension and all points in it have a fixed R, as in case A for L and R, we can, by extension of R and calculation of its intersection point with the red line, divide the red line into two parts. According to the external angle of R and the relevant L of each point after point i, which is the intersection point between R and red line, then, the third side of the separating triangle can be formed without overlapping CH(P) area.
Therefore, with regards to lemma 9 in case A, if the intersection point of x be located on line ij, the third side of the separating triangle will end in l(w) and r(w) so that it does not overlap CH(P) and there is no red line in the area restricted by CH(P), yi and xk. If there is no intersection point between R and the red line then it lies one side of R. If it lies on the right side of R then for any x on the red line there is a separating triangle and if it lies on the left side of R then there is no separating triangle. □
Case C
In the third case, the tangential lines to CH(P), which cross over w, meet the red lines which belong to σ while hitting the border σ.
See Fig. 12.
In this condition both l(w) and r(w) have intersected red lines by extension and each point in red lines has specific L and R. In order to find out whether a separating triangle can be formed, tangential lines must be drawn from x and y to CH(P). If the external angle is not less than 180°, there will be a
separating triangle, otherwise, no separating triangle is formed. As there are infinitely many points in two red lines and each point must be examined separately, we claim that, by turn of around $\tau$, the red lines are divided into finitely many intervals so that all points in the intervals have a unique manner (Fig. 13).

In Fig. 13, if we consider the intersection point of $I(w)$ with red line $L_1$ as $f_2$, which is fixed, and want to examine the position of the intersection point of $r(w)$ with red line $L_2$ as $f_1$ in order for a separating triangle to be formed, by drawing a tangential line to $CH(P)$ from $f_2$, we can define its intersection point with the red line $L_2$ as $f_3$. For a separating triangle to be formed, the external angle of $L$ and $R$ must not be less than 180$^\circ$, so, the other end of the third side of the triangle must be located in the interval from $f_3$ to point $(a_1,b_1)$.

It can be now claimed that, in general, when $f_3$ reaches $f_1$, a separating triangle is formed. Therefore, by assuming $F_1$ as a distance from $f_1$ to beginning point of $L_2$ and $F_3$ as distance from $f_3$ to point $(a_1,b_1)$, we can decide whether a triangle is formed by $F_1-F_3$ so that, by turn of $w$, when the amount is not positive the separating triangle is formed. By the beginning of a line segment we mean the endpoint with the lower y coordinate.

Keeping the issue intact, we assume an arc from $\tau$, around which $w$ is turned, and is related to a circle with radius one and the center of coordinate origin. Then, by applying polar coordinates of $w$ and intersecting red lines, $I(w)$ and $r(w)$, we gain the Equation 1 based on simple high school calculations.

\[
F_1 - F_3 = \left( \frac{A_1 \sin(\theta) + A_2 \cos(\theta) + A_3}{\sin(\theta) + A_4 \cos(\theta) + A_5} \right) - \left( \frac{A_6 \sin(\theta) + A_7 \cos(\theta) + A_8}{A_9 \sin(\theta) + A_{10} \cos(\theta) + A_{11}} \right)
\]

(1)

For all: $0 \leq \theta \leq 2\pi$ and $A_i$ as constant coefficients.

By calculating the phrase above and changing the variable $x = \sin \theta$ and using the algebraic identity $\sin^2(\theta) + \cos^2(\theta) = 1$, we gain a polynomial which has 8 roots at most. Therefore, the function intersects the x-axis in 8 points at most. So, the function mentioned above is not positive in 8 points at most and red line segments are divided into 8 intervals at most in a way that every point in such interval has the same $L$ and $R$. Then, the lemma 9 is effective.

**Case D**

In the fourth case, the extension of one of the tangential lines to $CH(P)$ crossing $w$ does not hit border $\sigma$ and the other one hits a side of border $\sigma$ where does not belong to $Q$.

See Fig. 14.

$L$ : The first line segment from border $\sigma$ which does not belong to $Q$, rather is the extension of one of supporting lines of one of red line segments, which extension of $I(w)$ intersects it in $x$ in a way that $CH(P)$ is located on its right.

$x$ : The intersection point between $I(w)$ and $L$.
Lemma 10. For each convex polygon such as $Y$ and both parallel lines such as $R$ and $L$ which are tangential to $Y$ and any point such as $g$ in an area restricted by $R$ and $L$ and outside $Y$, if we draw two tangential lines to $Y$ which crosses over $g$, the lines will intersect both $R$ and $L$ by extension. See Fig. 15.

Proof by induction method

We examine the judgment for $n = 3$ to be true when the convex polygon is a triangle (Fig. 16).

Induction hypothesis

We assume the judgment to be true for $n = k$.

Induction judgment

We show that the judgment is true for $n = k + 1$.

Regarding the polygon as convex, if we connect points $s$ and $t$, we will gain a convex drawing a chord in a convex polygon, it is divided into two convex polygons which its edges a $n + 1$ so that the judgment is effective for this polygon according to our induction judgment.

the extension of supporting lines of each point inside the area restricted by the main polyg parallel tangential lines intersect both parallel tangential lines and the judgment will be eflf polygon with $n + 1$ edges (Fig. 17). □

We claim that in this case, the lemma 9 is effective too and it is proved as follows.

Proof by reductio ad absurdum

As in the previous parts, if the external angle of $R$ and $L$ is less than $180^\circ$, then any line segment which ends in $l(w)$ and $r(w)$, will overlap $CH(P)$ and no triangle will be formed. There is also no red line in the area restricted by $CH(P)$ and $l(w)$ and line segment $xj$ in addition to an area restricted by $CH(P)$ and $r(w)$ and $R$. We can use lemmas 8 and 10 in order to prove our claim.

According to lemma 8, as $l(w)$ and $xj$ are tangential to $CH(P)$, so if there be even one red point inside the restricted area, its supporting lines will intersect both $l(w)$ and $xj$ by extension. This is contrary to our basic hypothesis- $L$ is the first line segment from border which is intersected by $l(w)$. Then, the absurd hypothesis is false and the judgment is effective. □

According to lemma 10, as $R$ and $r(w)$ are parallel and tangential to $CH(P)$, if even one red point be inside their restricted area, its supporting lines will intersect $R$ and $r(w)$ by extension. This is contrary to our basic hypothesis - $r(w)$ does not intersect any border in the forbidden area. Then, the absurd hypothesis is false and the judgment is effective so that no red line is inside the area restricted by $r(w)$ and $CH(P)$ and $R$.

Case E

In the fifth case, two tangential lines to $CH(P)$, which cross over $w$, do not hit border by extension. See Fig. 18.

$L$ : A line parallel to $l(w)$ and tangential to $CH(P)$ in a way that $L$ and $l(w)$ are located at opposite sides of $CH(P)$.
\( R \): A line parallel to \( r(w) \) and tangential to \( CH(P) \) in a way that \( R \) and \( r(w) \) are located at both sides of \( CH(P) \).

\( i \): The intersection point between \( l(w) \) and the extension of \( R \).

\( j \): The intersection point between \( r(w) \) and the extension of \( L \).

We claim that in this case, the lemma 9 is effective too.

Proof. Like the previous part and using the lemma 10, we can prove the lemma 9 as true. Only the definition of \( R \) and \( L \) is different which has no effect on the claim of lemma 9. □

**Case F**

In the sixth case, one of the tangential lines to \( CH(P) \) which cross over \( w \), does not hit border by extension and the other tangential lines to \( CH(P) \) hits a side of border which belongs to \( Q \).

Consider the Fig. 19.

\( x \): The intersection point between \( r(w) \) and the red line mentioned.

\( R \): The tangential line to \( CH(P) \) while crossing \( x \).

\( L \): A line parallel to \( l(w) \) and tangential to \( CH(P) \) in a way that \( l(w) \) and \( L \) are located at opposite sides of \( CH(P) \).

\( y \): The intersection point between \( r(w) \) and the extension of \( L \).

We claim that in this case, the lemma 9 is effective too.

Proof.

As in the case B, \( r(w) \) has hit one of the red lines by extension, and each point in the red lines has its own specific \( L \) or \( R \) because, unlike a supporting line in which all points have a unique tangential line to \( CH(P) \), any red line has a specific tangential line depending on its various points. Then, based on the point which \( r(w) \) hits, the formation of the third side of the triangle will be defined. The external angle between \( L \) and the tangential line crossing \( R \) determines whether the third side of the separating triangle is formed. In this case, the extension of \( l(w) \) has not hit the forbidden area-unlimited-and all points of it have a fixed \( L \). If \( y \) exists and \( y \neq x \) then the third side of the separating triangle can be formed without overlapping \( CH(P) \). There is no red line in the area restricted by \( CH(P), l(w), r(w), L \) and the tangential line to \( CH(P) \) and crossing \( x \) based on the previous lemmas. □

We can now examine the event points based on all of possible cases related to the extension of \( l(w) \) and \( r(w) \). As the external angle between the lines extended by \( R \) and \( L \) determines whether the set of line segments are separable by \( \theta-triangle \) or not, then at the time \( w \) turns around \( CH(P) \) in the \( \tau \), all points have the same condition- either there is \( \theta-triangle \) for each of them or there is no \( \theta-triangle \) for any of them so long as \( R \) and \( L \) are not changed -the sides of border \( \sigma \) which \( l(w) \) and \( r(w) \) intersect are not changed. Depending on in what case of six cases the extensions of \( r(w) \) and \( l(w) \) lie, \( L \) and \( R \) are updated, and in each case, as mentioned, it is decided that the separating triangle exists or does not exist. Therefore, \( \tau \) is divided into a set of intervals and it seems enough to make us able to examine one point in the interval in order to decide whether a separating \( \theta-triangle \) is formed.

We can define 2 events, based on previous considerations, as follow:

**The event of \( R \), \( L \) change**

This event occurs when \( R \) or \( L \) are changed by turning \( w \) on \( \tau \). While \( w \) moves on \( \tau \), \( R \)
(respectively \( L \)) is changed when the intersection point of the extension of \( r(w) \) (respectively \( l(w) \)) with border \( \sigma \) is changed, which occurs in six cases A to F.

Based on previous considerations, some points in \( \tau \), at which \( R \) and \( L \) are changed, are reported as the event points. However, if the intersection point of the extension of \( l(w) \) and \( r(w) \) with border \( \sigma \) be located in the third case, the red line will be divided into 8 intervals at most so that each interval has the same \( R \) and \( L \). Then, 8 events are reported.

### The event of departure

This event occurs when \( \sigma \) and \( \tau \) intersect each other. For each point in \( \tau \) which is located inside forbidden area \( A \), there is not any \( \theta\text{-triangle} \). Then, the intersection point between \( \tau \) and \( \sigma \) is reported as the point of departure.

**Lemma 11.** Given two sets of blue and red lines \( P \) and \( Q \) of total size \( N \), the total number of events of type 1 and 2 is \( O(N) \) and can be calculated at \( O(N) \) time.

**Proof.**

The total number of events is corresponding to the number of the sides of \( \sigma \). The sides are consisted of red lines and supporting lines and as we draw two supporting lines for each red line, the total number of the sides is at most 3 times the number of red lines. Then, the total number of events is \( O(N) \). When we extend any side of \( \sigma \), it may intersect \( \tau \) at many points. The last intersection point of the extension of each side of \( \sigma \) with \( \tau \) is a point related to event of type 1. The intersection point of each side of \( \sigma \) with \( \tau \) is related to event of departure. We remind that \( \sigma \) is a star shaped which \( CH(P) \) is as its kernel. We can define sort of order for the sides of \( \sigma \). The intersection point of \( \sigma \) and \( \tau \) obey the order, too. As we have already calculated the order of sides of \( \sigma \), then we can report all points at \( O(N) \) time. □

### 3.5 Handling the events

Based on the previous considerations, we need to determine whether a separating \( \theta\text{-triangle} \) is formed for an arbitrary point except the end-points in each interval and generalize the results to all points of the interval. If, for any intervals of \( \tau \), no separating triangle is formed, then the set of lines are not separable by \( \theta\text{-triangle} \). We know that all of event points are saved in event queue depending on their order in \( \tau \). While the event queue is not empty, we can put them out of the queue in order to process each.

**The event of departure**

We know that if the two sides related to a concave vertex from \( \sigma \) hits \( \tau \), it can specify two events of departures. These two consecutive sides specify two consecutive events of departures. There is no \( \theta\text{-triangle} \) between two events. Therefore, none of the events between the two events mentioned are processed.

**The event of \( R,L \)**

When we encounter this event, we update \( R,L \) as to decide whether a separating \( \theta\text{-triangle} \) is formed for each of the points between the present event point and the next one. It must be noted that these two events are processed in an \( O(1) \) time.

**The Algorithm**

In this section the pseudo-code of our algorithm, as a whole, is given regarding the previous sections. See algorithm 1. The algorithm has four main steps. First of all, we check monochromatic state of
If \( CH(P) \) is not monochromatic, we don’t have any separating triangle, otherwise we need to compute the star-shaped structure and the \( \theta - \text{cloud} \) around \( CH(P) \) regarding sections 4 and 5. In the section 6 as the third main step, some points in \( \tau \), at which \( R \) and \( L \) are changed or \( \sigma \) and \( \tau \) intersect each other, are reported as the event points. Finally, we handle those event points regarding section 7.

Algorithm 1 Report_Minimal_Separating_Triangles

**Input:** A finite set \( P \) of blue line segments and a finite set \( Q \) of red line segments in the plane and an angle \( 0 < \theta < \pi \).

**Output:** All combinatorially different minimal triangles containing all line segments of \( P \), avoiding any line segment of \( Q \) and having an angle \( \theta \).

1. Compute the convex hull of \( P \). If \( CH(P) \) is not monochromatic, we don’t have any separating triangle and return empty.
2. Compute the star-shaped structure and denote it by \( \sigma \) (refer to sec. 4).
3. Compute the \( \theta - \text{cloud} \) around \( CH(P) \) and denote it by \( \tau \) (refer to sec. 5).
4. Let \( w \) be an arbitrary point on \( \tau \) and maintain \( r(w), l(w), L, R, x, y, i \) and \( j \) as defined in sec. 5.
5. Move the apex \( W \) clockwise along \( \tau \) and collect all of two types of event points described in section 6.
6. For each event point computed in 5, handle it according to sec. 7 and store the separating \( \theta - \text{triangle} \) between each pair of consecutive points.

**Theorem 12.** The algorithm Report_Minimal_Separating_Triangles takes \( O(N \log N) \) time in order to specify \( \theta - \text{triangle} \) separability of two given bichromatic sets of lines \( P \) and \( Q \) of total size \( N \) and give a report on all of the separating triangles.

**Proof.** We examine all of the algorithm steps to analyze the time complexity of the algorithm. In step1, computing the convex hull of \( P \) and its monochromatic state takes \( O(N \log N) \) time and in steps 2 to 6, we must gain a star shaped polygon in order to examine the separability of lines with \( \theta - \text{triangle} \) in \( O(N \log N) \) time [1] and also calculate \( \tau \) in \( O(N \log N) \) time [8]. We can specify two events of change and departure by \( \sigma \) and \( \tau \). According to lemma 9, all events are calculated at \( O(N) \) time. As the number of events is \( O(N) \), we can enlist the events in order in \( O(N \log N) \) time. We know that each event is processed at a fixed time. As there are \( O(N) \) events, all events need \( O(N) \) time to be processed. Therefore, the algorithm is carried out in \( O(N \log N) \) time.

3.6 Extending the results to polylines

Let \( P \) be a set of blue polylines with \( n \) blue segments and \( Q \) be a set of red polylines with \( m \) red segments. In time \( O(n \log m) \) we can check if \( CH(P) \) is monochromatic so the problem of the
triangle separability for polylines is easily reduced to the problem of the triangle separability for line segments described in section 3. If \( CH(P) \) is monochromatic, consider the set of line segments of the polylines of \( P \) and \( Q \) and run the algorithm described in section 7. This can be done in \( O(N \log N) \) where \( N = n + m \).

4. Conclusion
We have solved the problem of separating two sets of polylines in the plane with minimal triangles. Our algorithm reports all combinatorially different minimal separating \( \theta \)-triangles of two sets \( P \) and \( Q \) of segments and polylines and it runs in \( O(N \log N) \) time. An interesting problem that can be studied is finding the minimal rectangles.

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Captions
Fig. 1. End points are separated by a triangle while line segments are not separated.
Fig. 2. Separation of blue and red lines.
Fig. 3. Supporting lines of a point.
Fig. 4. Supporting lines of a line.
Fig. 5. Forbidden area $A_i$.
Fig. 6. $\theta$–cloud $\tau$ by circular arcs.
Fig. 7. $l(w)$ and $r(w)$ intersect two supporting lines.

Fig. 8. Induction judgment.

Fig. 9. Induction basis.

Fig. 10. Internal and external angles.

Fig. 11. $l(w)$ intersects a red line segment and $r(w)$ intersects a supporting line.

Fig. 12. $l(w)$ and $r(w)$ intersect two red line segments.

Fig. 13. Turn of $w$ around $\theta$–cloud $\tau$.

Fig. 14. $r(w)$ intersects a supporting line and $l(w)$ does not hit border.

Fig. 15. Tangential lines crossing over $g$ intersect both $R$ and $L$.

Fig. 16. Induction basis.

Fig. 17. Induction judgment.

Fig. 18. $l(w)$ and $r(w)$ don’t intersect border.

Fig. 19. $r(w)$ intersects a red line segment and $l(w)$ does not hit border.

Figures

![Figure 1](image1.png)

![Figure 1](image2.png)
