THE LEVEL-CROSSING INTENSITY FOR THE DENSITY OF THE IMAGE OF THE LEBESGUE MEASURE UNDER THE ACTION OF A BROWNIAN STOCHASTIC FLOW

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Abstract. In this paper we compute the level-crossing intensity for the density of the image of the Lebesgue measure under the action of a Brownian stochastic flow which is a smooth approximation of the Arratia flow and determine its asymptotic behaviour as the height of the level tends to infinity.

1. Introduction

Let us consider the following stochastic integral equation:

\[ x(u, t) = u + \int_0^t \int_\mathbb{R} \varphi(x(u, s) - q) W(dq, ds), \quad t \geq 0, \]  

where \( u \in \mathbb{R} \) is a fixed parameter, \( W \) is a Wiener sheet on \( \mathbb{R} \times \mathbb{R}_+ \), and the function \( \varphi: \mathbb{R} \to \mathbb{R}_+ \) satisfies the conditions:

(i) \( \varphi \in C^\infty_K(\mathbb{R}, \mathbb{R}_+) \), i.e. \( \varphi \) is non-negative, infinitely differentiable and has compact support;
(ii) \( \varphi(q) = \varphi(-q), \ q \in \mathbb{R} \);
(iii) \( \int_\mathbb{R} \varphi^2(q) \, dq = 1 \).

Under such conditions on the function \( \varphi \) equation (1) has a unique strong solution for every \( u \in \mathbb{R} \). Moreover, for some set \( \Omega \) of full probability (without loss of generality we will assume that it is the whole set of outcomes) the mappings

\[ x(\omega, \cdot, t): \mathbb{R} \to \mathbb{R}, \quad t \geq 0, \quad \omega \in \Omega, \]  

are \( C^\infty \)-diffeomorphisms and the stochastic flow \( \{ \varphi_{s,t}(u), \ u \in \mathbb{R}, \ 0 \leq s \leq t < +\infty \} \) given by

\[ \varphi_{s,t}(\omega, u) := x(\omega, x^{-1}(\omega, u, s), t), \quad u \in \mathbb{R}, \quad 0 \leq s \leq t < +\infty, \quad \omega \in \Omega, \]  

where \( x^{-1}(\omega, \cdot, s) \) is the inverse of the mapping \( x(\omega, \cdot, s) \) (as a rule, we will omit the variable \( \omega \)), is a Brownian stochastic flow of \( C^\infty \)-diffeomorphisms (see [5], [6], and also [9]).

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The covariance function of this Brownian stochastic flow is given by
\[ \Phi(z) := \int_{\mathbb{R}} \varphi(z + q) \varphi(q) \, dq, \quad z \in \mathbb{R}. \]

In other words, for any \( u, v \in \mathbb{R} \) the joint quadratic variation of the Wiener processes \( \{x(u, t), \ t \geq 0\} \) and \( \{x(v, t), \ t \geq 0\} \) has the form
\[ \langle x(u, \cdot), x(v, \cdot) \rangle_t = \int_0^t \Phi(x(u, s) - x(v, s)) \, ds, \quad t \geq 0. \]

Note that the function \( \Phi \) takes value 1 at point 0 and has compact support, whose diameter is not greater than \( 2 d(\varphi) \), where \( d(\varphi) \) is the diameter of the support of the function \( \varphi \). Therefore, as \( d(\varphi) \) tends to zero, the function \( \Phi \) converges pointwise to the function
\[ \mathbb{I}_{\{0\}}(z) = \begin{cases} 1, & \text{if } z = 0, \\ 0, & \text{if } z \neq 0. \end{cases} \]

In [5] it was shown that for any \( n \in \mathbb{N} \) and for any \( u_1, \ldots, u_n \in \mathbb{R} \) in the space \( C([0; 1], \mathbb{R}^n) \) the following weak convergence takes place:
\[ (x(u_1, \cdot), \ldots, x(u_n, \cdot)) \xrightarrow{w} (x_0(u_1, \cdot), \ldots, x_0(u_n, \cdot)), \quad d(\varphi) \to 0^+, \]
where \( \{x_0(u, t), \ u \in \mathbb{R}, \ t \geq 0\} \) is the Arratia flow, i.e. a Brownian stochastic flow with covariance function \( \mathbb{I}_{\{0\}} \).

Recall that the Arratia flow was constructed in [2] as the weak limit of families of coalescing simple random walks and that informally it can be described as a flow of Brownian particles in which any two particles move independently until they meet and after that coalesce and move together. Moreover, it is known (see [2], [7]) that for any \( t > 0 \) and for any interval \( I \subset \mathbb{R} \) the set \( x_0(\mathbb{R}, t) \cap I \) is finite almost surely.

Consider the random measures
\[ \lambda_t := \lambda \circ x^{-1}(\cdot, t), \quad t \geq 0, \]
where \( \lambda \) is the one-dimensional Lebesgue measure. Due to the diffeomorphic property of the mappings (2), these measures are absolutely continuous. Furthermore, it is easy to check that for the corresponding densities the following representation takes place (strict positivity of the derivative \( \frac{\partial}{\partial u}(u, t) \) is implied by Lemma 3.1 stated below)
\[ \frac{d\lambda_t}{d\lambda}(u) \equiv p_t(u) = \frac{1}{\frac{\partial}{\partial u}(x^{-1}(u, t), t)}, \quad u \in \mathbb{R}, \quad t \geq 0. \]

Now if we set
\[ \lambda^0_t := \lambda \circ x^{-1}_0(\cdot, t), \quad t \geq 0, \]
it is natural to expect that, as \(d(\varphi) \to 0^+\), the random measures \(\lambda_t\) converge in some sense to the random measures \(\lambda^0_t\). Therefore, the domains of concentration of the measures \(\lambda_t\) or, in other words, the domains in which the densities \(p_t\) take great values correspond to the atoms of the measures \(\lambda^0_t\) (which obviously coincide with the clusters of the Arratia flow).

It is reasonable to start the study of such high level exceedances of the densities \(p_t\) by investigating their level-crossing intensities. Let us recall the corresponding definition.

Let \(\{\xi(u), \ u \in \mathbb{R}\}\) be a stochastic process, whose trajectories with probability one are not identically equal to a given number \(c \in \mathbb{R}\) on any interval. Then the number \(N([0; 1]; c)\) of crossings by the random process \(\xi\) of the level \(c\) on the interval \([0; 1]\) is well-defined and the number
\[
\mu(c) := \mathbb{E}N([0; 1]; c)
\]
is called the intensity of crossings of the level \(c\) by the stochastic process \(\xi\).

For some classes of stochastic processes \(\{\xi(u), \ u \in \mathbb{R}\}\) the value of \(\mu(c)\) can be computed with the help of the well-known Rice formula (see [11], where upcrossings are considered, and the references therein). As an example we formulate the following theorem, noting on the way that in this theorem and further \(\pi[\eta](\cdot)\) and \(\pi[\eta, \zeta](\cdot, \cdot)\) stand for the densities of the distribution of a random variable \(\eta\) and of the joint distribution of random variables \(\eta\) and \(\zeta\) respectively.

**Theorem 1.1.** [3] Let a stochastic process \(\{\xi(u), \ u \in \mathbb{R}\}\) have continuously differentiable paths and satisfy the following three conditions:

(i) the mapping \((u, z) \mapsto \pi[\xi(u)](z)\) is continuous in \(u \in [0; 1]\) and \(z \in U_c\), where \(U_c\) is some neighbourhood of \(c \in \mathbb{R}\);
(ii) the mapping \((u, z_1, z_2) \mapsto \pi[\xi(u), \xi'(u)](z_1, z_2)\) is continuous in \(u \in [0; 1]\), \(z_1 \in U_c\) and \(z_2 \in \mathbb{R}\);
(iii) \(\mathbb{E}w(\xi' ; \delta) \to 0\) as \(\delta \to 0^+\), where \(w(\xi' ; \delta)\) is the modulus of continuity of the stochastic process \(\xi'\) on the interval \([0; 1]\).

Then for any \(c \in \mathbb{R}\)
\[
\mu(c) = \int_0^1 \int_0^{+\infty} |z| \cdot \pi[\xi(u), \xi'(u)](c, z) \, dz \, du.
\] (4)

However, the conditions on the stochastic process ensuring the validity of Formula (4) for every level \(c \in \mathbb{R}\) are often difficult to verify (for instance, in the above theorem such is condition (iii)), and they were checked only for stochastic processes similar to Gaussian. It is easier to verify the conditions ensuring the validity of similar formulae for almost every level \(c \in \mathbb{R}\).

In this paper we use the following version of Rice’s formula (see [3] Exercise 3.8; cf. [11] Formula (3)) also called Banach’s formula.
Theorem 1.2. Let a stochastic process \( \{\xi(u), 0 \leq u \leq 1\} \) be such that almost all its trajectories are absolutely continuous, for any \( u \in [0; 1] \) the distribution of the random variable \( \xi(u) \) has a density \( \pi[\xi(u)](\cdot) \), and the conditional expectation \( E(|\xi'(u)| \mid \xi(u) = c) \) is well-defined. Then for almost every \( c \in \mathbb{R} \)

\[
\mu(c) = \int_0^1 E(|\xi'(u)| \mid \xi(u) = c) \cdot \pi[\xi(u)](c) \, du.
\]

The main aim of this paper is to verify that the level-crossing intensity \( \mu_t(c) \) (with \( c > 0 \)) of the stochastic process \( \{p_t(u), u \in \mathbb{R}\} \) is well-defined for any \( t > 0 \), and to prove that

\[
\mu_t(c) = \pi_t(c) \quad \text{for a.e. } c > 0,
\]

where

\[
\pi_t(c) = \frac{\sqrt{2L''} \cdot e^{\frac{c^2}{8t} \cdot L' t}}{\pi L' \sqrt{\pi t}} \cdot \frac{1}{\sqrt{c}} \cdot \int_0^{+\infty} \frac{e^{-\frac{v^2}{2t} \cdot \sinh v \cdot \sin \frac{\pi v}{L'}}}{\sqrt{1 + \frac{2 \cosh v}{c} + \frac{1}{c^2}}} \, dv
\]

with constants

\[
L' := \int_{\mathbb{R}} \varphi'^2(q) \, dq > 0
\]

and

\[
L'' := \int_{\mathbb{R}} \varphi''^2(q) \, dq > 0,
\]

and also to establish the asymptotic equality

\[
\pi_t(c) = \frac{e^{-\frac{L'}{8}}}{{\pi}^{1/2}} \cdot \sqrt{\frac{c}{\ln c}} \cdot \exp \left[ -\frac{(\ln c)^2}{2L' t} \right] \cdot (1 + o(1)), \quad c \to +\infty.
\]

The structure of the main part of this paper is as follows. In Section 2 we prove the stationarity (for fixed \( t \)) of the stochastic process \( \{x(u, t) - u, u \in \mathbb{R}\} \) and \( \theta \)-homogeneity (see Definition 2.5 below) of the stochastic process \( \{(p_t(u), p'_t(u)), u \in \mathbb{R}\} \) (here the usual derivative is considered), in Section 3 we find the density of the joint distribution of the random variables \( p_t(u) \) and \( p'_t(u) \), and in Section 4 we establish equalities (5) and (6).

2. STATIONARITY AND \( \theta \)-HOMOGENEITY

To prove the stationarity (for fixed \( t \)) of the stochastic process \( \{x(u, t) - u, u \in \mathbb{R}\} \) we will use the version of the Euler–Maruyama discrete approximation scheme
presented in [14]. For \( t > 0 \) and \( n \geq 1 \) set

\[
x^n_0(u, t) \equiv u, \quad u \in \mathbb{R},
\]

\[
x^n_{k+1}(u, t) := x^n_k(u, t) + \xi^n_{k+1}(x^n_k(u, t), t), \quad u \in \mathbb{R}, \quad 0 \leq k \leq n-1,
\]

where

\[
\xi^n_{k+1}(u, t) := \int_{\frac{k+1}{n}}^{\frac{k+2}{n}} \int_{\mathbb{R}} \varphi(u - q) W(dq, ds), \quad u \in \mathbb{R}, \quad 0 \leq k \leq n-1.
\]

By Theorem 4 of paper [14] the stochastic process \( \{x^n(u, 1), \ u \in \mathbb{R}\} \) defined recurrently in this way approximates the stochastic process \( \{x(u, 1), \ u \in \mathbb{R}\} \) in the uniform metric on the interval \([0; 1]\). However, the presented proof of this theorem can be easily extended mutatis mutandis to the case of arbitrary time \( t > 0 \) and interval \([a; b]\), and so its corresponding generalization can be formulated in the following (slightly simplified, but sufficient for our purposes) form.

**Theorem 2.1.** For any time \( t > 0 \) and interval \([a; b]\) \( \subset \mathbb{R} \) there exists a constant \( C > 0 \) depending on the time \( t \), the length of the interval \([a; b]\) and the function \( \varphi \), such that for all \( n \geq 1 \)

\[
E \|x^n(\cdot, t) - x(\cdot, t)\|_{[a; b]} \leq \frac{C}{\sqrt{n}},
\]

where \( \|\cdot\|_{[a; b]} \) is the supremum-norm on the interval \([a; b]\).

**Theorem 2.2.** For any \( t \geq 0 \) the stochastic process \( \{x(u, t) - u, \ u \in \mathbb{R}\} \) is (strictly) stationary.

**Proof.** Fix arbitrary \( t > 0 \) (in the case of \( t = 0 \) there is nothing to prove).

With the help of the principle of mathematical induction, using characteristic functions and taking into account that \( \xi^n_{k+1} \) is independent of \( \mathcal{F}_{k+1} \) for every \( k \in \{0, \ldots, n-1\} \), where \( \{\mathcal{F}_s, \ s \geq 0\} \) is the filtration generated by the Wiener sheet \( W \) (and completed with the sets of zero probability), one can show that all stochastic processes \( \{x^n_k(u, t) - u, \ u \in \mathbb{R}\}, \ 0 \leq k \leq n, \) are stationary.

To prove the stationarity of the stochastic process \( \{x(u, t) - u, \ u \in \mathbb{R}\} \) note that from Theorem 2.1 it follows that for any \( u_1, \ldots, u_m \in \mathbb{R} \)

\[
E \max_{1 \leq k \leq m} |x^n_k(u_k, t) - x(u_k, t)| \to 0, \quad n \to \infty,
\]

and so

\[
(x^n(u_1, t) - u_1, \ldots, x^n(u_m, t) - u_m) \xrightarrow{w} (x(u_1, t) - u_1, \ldots, x(u_m, t) - u_m), \quad n \to \infty.
\]
Therefore, for arbitrary $h > 0$ and any Borel sets $\Delta_1, \ldots, \Delta_m \in \mathcal{B}(\mathbb{R})$ we have

$$
P \{ [x(u_1 + h, t) - (u_1 + h)] \in \Delta_1, \ldots, [x(u_m + h, t) - (u_m + h)] \in \Delta_m \} =
$$

$$
= \lim_{n \to \infty} P \{ [x_n^m(u_1 + h, t) - (u_1 + h)] \in \Delta_1, \ldots, [x_n(u_m + h, t) - (u_m + h)] \in \Delta_m \} =
$$

$$
= \lim_{n \to \infty} P \{ [x_n^m(u_1, t) - u_1] \in \Delta_1, \ldots, [x_n(u_m, t) - u_m] \in \Delta_m \} =
$$

$$
P \{ [x(u_1, t) - u_1] \in \Delta_1, \ldots, [x(u_m, t) - u_m] \in \Delta_m \}.
$$

Thus, the distributions of the random vectors

$$(x(u_1 + h, t) - (u_1 + h), \ldots, x(u_m + h, t) - (u_m + h))$$

and

$$(x(u_1, t) - u_1, \ldots, x(u_m, t) - u_m)$$

coincide. The theorem is proved. \[\square\]

**Corollary 2.3.** For any $t \geq 0$ the three-dimensional stochastic process

$$\left\{ \left( x(u, t) - u, \frac{\partial x}{\partial u}(u, t), \frac{\partial^2 x}{\partial u^2}(u, t) \right), \; u \in \mathbb{R} \right\}$$

is stationary.

Now we will need several definitions (see [15, Chapter 5]).

**Definition 2.4.** A family of mappings $\{\theta_z : \Omega \to \Omega, \; z \in \mathbb{R}\}$ is called a family of (spatial) shifts if

(i) the mapping $\mathbb{R} \times \Omega \ni (z, \omega) \mapsto \theta_z(\omega) \in \Omega$ is measurable;

(ii) $\theta_y \circ \theta_z = \theta_{y+z}$ for any $y, z \in \mathbb{R}$;

(iii) $\theta_0$ is the identity mapping;

(iv) $P \circ \theta_z^{-1} = P$ for any $z \in \mathbb{R}$.

**Definition 2.5.** A stochastic process $\{\xi(u), \; u \in \mathbb{R}\}$ is called a $\theta$-homogeneous stochastic process if

$$\forall \omega \in \Omega \; \forall u, z \in \mathbb{R} : \; \xi(\theta_z \omega, u) = \xi(\omega, u + z).$$

**Definition 2.6.** A mapping $G : \Omega \times \mathbb{R} \to \mathbb{R}$ is called a $\theta$-homogeneous random transformation if

$$\forall \omega \in \Omega \; \forall u, z \in \mathbb{R} : \; G(\theta_z \omega, u) = G(\omega, u + z) - z.$$

**Definition 2.7.** A mapping $A : \Omega \times \mathbb{R} \times \mathbb{R}_+ \to E$ taking values in some measurable space $(E, \mathcal{E})$ is called a $\theta$-homogeneous random field if it is measurable and

$$\forall \omega \in \Omega \; \forall u, z \in \mathbb{R} \; \forall t \geq 0 : \; A(\theta_z \omega, u, t) = A(\omega, u + z, t).$$

**Definition 2.8.** A stochastic flow $\{\varphi_{s,t}(u), \; u \in \mathbb{R}, \; 0 \leq s \leq t < \infty\}$ is called a $\theta$-homogeneous stochastic flow if

$$\forall \omega \in \Omega \; \forall u \in \mathbb{R} \; \forall s, t \geq 0, \; s \leq t : \; \varphi_{s,t}(\theta_z \omega, u) = \varphi_{s,t}(\omega, u + z) - z.$$
Lemma 2.9. For any $t \geq 0$ the stochastic processes
\[
\left\{ \left( x(u, t) - u, \frac{\partial x}{\partial u}(u, t), \frac{\partial^2 x}{\partial u^2}(u, t) \right), \ u \in \mathbb{R} \right\}
\]
and
\[
\left\{ (p_t(u), p'_t(u)), \ u \in \mathbb{R} \right\}
\]
are $\theta$-homogeneous.

Proof. For the proof it is enough to note that the canonical representation of the stochastic process
\[
\left\{ \left( x(u, t) - u, \frac{\partial x}{\partial u}(u, t), \frac{\partial^2 x}{\partial u^2}(u, t) \right), \ u \in \mathbb{R} \right\}
\]
on the space
\[
\Omega = \{ \omega = (\omega_1, \omega_2, \omega_3) \mid \omega_1 \in C^2(\mathbb{R}), \ \omega'_1 > -1, \ \omega_2 = \omega'_1 + 1, \ \omega_3 = \omega''_1 \}
\]
with the Borel $\sigma$-algebra generated by the uniform metric is $\theta$-homogeneous with respect to the standard spatial shifts (for these shifts conditions (ii) and (iii) of Definition 2.4 are obviously satisfied, condition (i) follows from their continuity, and condition (iv) follows from Corollary 2.3). After this one can directly verify the $\theta$-homogeneity of the stochastic process \( \{ x^{-1}(u, t) - u, \ u \in \mathbb{R} \} \) and then that of the stochastic process \( \{ (p_t(u), p'_t(u)), \ u \in \mathbb{R} \} \).

The main results of the third section of this paper are based on the following theorem.

Theorem 2.10. [15, Chapter 5, Theorem 4.11] Let \( \{ \varphi_{s,t} : \mathbb{R} \to \mathbb{R}, \ 0 \leq s \leq t < +\infty \} \) be a $\theta$-homogeneous stochastic flow of $C^1$-diffeomorphisms and \( A : \Omega \times \mathbb{R} \times \mathbb{R}_+ \to E \) be a $\theta$-homogeneous random field taking values in some measurable space \( (E, \mathcal{E}) \). Then for any \( s, t \geq 0, \ s \leq t, \) the following propositions hold true:

(i) the random transformation \( \varphi_{s,t}^{-1}(\cdot) \) is a $\theta$-homogeneous diffeomorphism;
(ii) the stochastic process \( \frac{\partial \varphi_{s,t}}{\partial u}(\cdot) \) is $\theta$-homogeneous;
(iii) the stochastic process \( \frac{\partial^2 \varphi_{s,t}}{\partial u^2}(\varphi_{s,t}^{-1}(\cdot)) \) is $\theta$-homogeneous;
(iv) for any \( E \)-measurable function \( f : E \to \mathbb{R}_+ \) and any \( u \in \mathbb{R} \)
\[
E f(A(\varphi_{s,t}(u), t)) = E \left[ f(A(0, t)) \cdot \frac{1}{\frac{\partial \varphi_{s,t}}{\partial u}(\varphi_{s,t}^{-1}(0))} \right],
\]
provided the expectation on the right-hand side is well-defined and finite.
3. The joint distribution of $p_t(u)$ and $p'_t(u)$

In this section we show that the joint distribution of the random variables $p_t(u)$ and $p'_t(u)$ has a density and find its form.

To begin with, we prove the following lemma.

**Lemma 3.1.** The following representation takes place:

$$\frac{\partial x}{\partial u}(u, t) = \exp \left[ -\frac{1}{2} L't + \int_0^t \int_\mathbb{R} \varphi'(x(u, s) - q) W(dq, ds) \right], \quad t \geq 0, \quad u \in \mathbb{R}. \quad (7)$$

Moreover, for any $u \in \mathbb{R}$ the stochastic process

$$w_u(t) := \frac{1}{\sqrt{L'}} \int_0^t \int_\mathbb{R} \varphi'(x(u, s) - q) W(dq, ds), \quad t \geq 0, \quad (8)$$

is a standard Wiener process.

**Proof.** The properties of the integral with respect to a Wiener sheet imply that the stochastic process $\{w_u(t), \ t \geq 0\}$ defined by (8) is a continuous $(\mathcal{F}_t)$-adapted martingale with the quadratic variation

$$\langle w_u \rangle_t = \frac{1}{L'} \int_0^t \int_\mathbb{R} \varphi'^2(x(u, s) - q) dq \, ds = t, \quad t \geq 0,$$

and so, by Lévy’s characterizing theorem it is a standard Brownian motion.

Furthermore, differentiating both sides of the equation for $x(u, t)$ (see [9, Theorem 3.3.3]), we obtain that

$$\frac{\partial x}{\partial u}(u, t) = 1 + \int_0^t \int_\mathbb{R} \varphi'(x(u, s) - q) \frac{\partial x}{\partial u}(u, s) W(dq, ds), \quad t \geq 0. \quad (9)$$

Using Itô’s formula, it is easy to show that the stochastic process defined by the right-hand side of (7) satisfies the stochastic integral equation (9) (with respect to the unknown stochastic process $\{\frac{\partial x}{\partial u}(u, t), \ t \geq 0\}$). Therefore, equation (7) is now implied by the uniqueness of the strong solution of this equation. \qed

**Corollary 3.2.** For any $t > 0$ and $u \in \mathbb{R}$ the distribution of the random variable $\frac{\partial x}{\partial u}(u, t)$ has a density of the form

$$\pi \left[ \frac{\partial x}{\partial u}(u, t) \right] (z) = \frac{1}{\sqrt{2\pi L't}} \cdot \exp \left[ -\left( \frac{\ln z + \frac{3}{2} L't}{2L't} \right)^2 + L't \right], \quad z > 0.$$
Theorem 3.3. For any $t > 0$ and $u \in \mathbb{R}$ the distribution of the random variable $p_t(u)$ has a density of the form

$$
\pi[p_t(u)](z) = \frac{1}{\sqrt{2\pi L't}} \cdot \exp \left[ -\frac{(\ln z + \frac{3}{2}L't)^2}{2L't} + L't \right], \quad z > 0.
$$

(10)

Proof. Set

$$
E := (0; +\infty),
$$

$$
\mathcal{E} := \mathcal{B}((0; +\infty)),
$$

and, for fixed $t_0 > 0$,

$$
A(u, t) := p_{t_0}(u), \quad u \in \mathbb{R}, \quad t \geq 0.
$$

Note that the mapping $(\omega, u, t) \mapsto A(\omega, u, t)$ is measurable, since such is the mapping $(\omega, u, t) \mapsto x(\omega, u, t)$.

Therefore, for the stochastic flow (3) and the function

$$
f(z) := \frac{1}{z} \cdot \mathbb{1}\{z \in \Delta\}, \quad z \in E,
$$

where the set $\Delta \in \mathcal{E}$ is arbitrary, by Theorem 2.10 with $s = 0$ and $t = t_0$ we have (for convenience, until the end of the proof we write $t$ instead of $t_0$)

$$
E \left[ \frac{1}{p_t(x(u, t))} \cdot \mathbb{1}\{p_t(x(u, t)) \in \Delta\} \right] = E \left[ \frac{1}{p_t(0)} \cdot \mathbb{1}\{p_t(0) \in \Delta\} \cdot p_t(0) \right],
$$

so that

$$
P\{p_t(0) \in \Delta\} = E \left[ \frac{\partial x}{\partial u}(u, t) \cdot \mathbb{1}\left\{ \frac{1}{\partial x(\omega, u, t)} \in \Delta \right\} \right].
$$

(11)

However, from Corollary 3.2 it follows that

$$
E \left[ \frac{\partial x}{\partial u}(u, t) \cdot \mathbb{1}\left\{ \frac{1}{\partial x(\omega, u, t)} \in \Delta \right\} \right] =
$$

$$
= \int_{\Delta} \frac{1}{\sqrt{2\pi L't}} \cdot \exp \left[ -\frac{(\ln z + \frac{3}{2}L't)^2}{2L't} + L't \right] dz.
$$

(12)

Since the set $\Delta$ is arbitrary, equalities (11) and (12) imply the required assertion. \hfill \Box

Remark 3.4. Thus, for any $t > 0$ and $u \in \mathbb{R}$ the following equality in distribution holds:

$$
p_t(u) \overset{d}{=} \frac{\partial x}{\partial u}(u, t).
$$
Corollary 3.5. For any \( t > 0 \) and \( u \in \mathbb{R} \) the following equality holds:

\[
P \{ p_t(u) > c \} = \frac{e^{-\frac{L}{2}\sqrt{Lt}}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{c \cdot \ln c}} \cdot \exp \left[ -\frac{(\ln c)^2}{2Lt} \right] \cdot (1 + \sigma(1)), \quad c \to +\infty.
\]

Proof. To prove this equality one should re-write its left-hand side in the form of an integral of the density of the distribution of the random variable \( p_t(u) \) from Formula (10), to reduce it by the change-of-variable formula to an integral of the density \( p \) of the standard Gaussian distribution, and finally to use the well-known (e. g., see [12, Chapter 1]) relation

\[
\int_c^{+\infty} p(u) \, du = \frac{1}{c} \cdot p(c) \cdot (1 + \sigma(1)), \quad c \to +\infty.
\]

To prove the next result we will need an auxiliary lemma concerning the conditional expectation of the Itô integral with respect to a Wiener process.

Lemma 3.6. Let a continuous stochastic process \( \{ \xi_t, \ t \geq 0 \} \) be independent of a Wiener process \( \{ \beta_t, \ t \geq 0 \} \) and the Itô integral

\[
\int_0^t \xi_s \, d\beta_s, \quad t \geq 0,
\]

be well-defined. Then for any Borel set \( \Delta \subset \mathbb{R} \) the following equality holds:

\[
\mathbb{E} \left( \mathbb{I} \left\{ \int_0^t \xi_s \, d\beta_s \in \Delta \right\} \bigg| \xi \right) = \int_{\Delta} \frac{1}{\sqrt{2\pi} \sigma_t} e^{-\frac{v^2}{2\sigma_t^2}} \, dv \cdot \mathbb{I} \{ \sigma_t > 0 \} + \mathbb{I} \{ 0 \in \Delta \} \cdot \mathbb{I} \{ \sigma_t = 0 \},
\]

where

\[
\sigma_t := \left( \int_0^t \xi_s^2 \, ds \right)^{1/2}.
\]

Proof. The proof of this lemma can be obtained in a standard way by approximating indicator functions with smooth bounded functions and is therefore omitted. \( \square \)

Now fix arbitrary \( u \in \mathbb{R} \) and define the stochastic processes

\[
X_1(t) := \frac{\partial x}{\partial u}(u, t), \quad t \geq 0,
\]

\[
X_2(t) := \frac{\partial^2 x}{\partial u^2}(u, t), \quad t \geq 0.
\]
Lemma 3.7. For any $t > 0$ the joint distribution of the random variables $X_1(t)$ and $X_2(t)$ has a density of the form

$$
\pi[X_1(t), X_2(t)](z_1, z_2) = \frac{e^{\frac{-z_1^2}{2t}}}{\pi \sqrt{2} \pi L''t} \cdot \frac{1}{\sqrt{z_1}} \int_0^{+\infty} \frac{e^{-\frac{v}{2t} \sinh v} \sin \frac{\pi v}{t} \pi^2}{\left( (1 + 2z_1 \cosh v + z_1^2) + \frac{(L'z_2)^2}{\pi^2} \right)^{3/2}} dv, \quad z_1 > 0, \quad z_2 \in \mathbb{R}.
$$

Proof. Note that from representation (7) it follows that

$$
\frac{\partial^2 x}{\partial u^2}(u, t) = \frac{\partial x}{\partial u}(u, t) \cdot \int_0^t \int_{\mathbb{R}} \varphi''(x(u, s) - q) \frac{\partial x}{\partial u}(u, s) W(dq, ds), \quad t \geq 0.
$$

(13)

From (9) and (13) we obtain that

$$
X_1(t) = 1 + \int_0^t \int_{\mathbb{R}} \varphi'(x(u, s) - q) X_1(s) W(dq, ds), \quad t \geq 0,
$$

$$
X_2(t) = \int_0^t \int_{\mathbb{R}} \varphi''(x(u, s) - q) X_1(s) W(dq, ds), \quad t \geq 0.
$$

By the properties of the integral with respect to a Wiener sheet, we have

$$
\langle X_1 \rangle_t = \int_0^t \int_{\mathbb{R}} \varphi''(x(u, s) - q) X_1^2(s) dq ds = L' \int_0^t X_1^2(s) ds, \quad t \geq 0,
$$

$$
\langle X_2 \rangle_t = \int_0^t \int_{\mathbb{R}} \varphi''(x(u, s) - q) X_1^2(s) dq ds = L'' \int_0^t X_1^2(s) ds, \quad t \geq 0,
$$

$$
\langle X_1, X_2 \rangle_t = \int_0^t \int_{\mathbb{R}} \varphi'(x(u, s) - q) \varphi''(x(u, s) - q) X_1^2(s) dq ds = 0, \quad t \geq 0.
$$

Since with probability one $X_1(t) > 0$ for all $t \geq 0$, using Doob’s theorem (see [13, Chapter 5, Theorem 5.12]), we obtain that the pair of the stochastic processes $X_1$ and $X_2$ is a solution of the following system of equations:

$$
\begin{cases}
X_1(t) = 1 + \sqrt{L'} \cdot \int_0^t X_1(s) dW_1(s), \quad t \geq 0, \\
X_2(t) = \sqrt{L''} \cdot \int_0^t X_1(s) dW_2(s), \quad t \geq 0,
\end{cases}
$$

(14)
where the Wiener processes \( \{W_1(t), \ t \geq 0\} \) and \( \{W_2(t), \ t \geq 0\} \) can be defined by the equalities

\[
W_1(t) = \frac{1}{\sqrt{L}} \int_0^t \frac{dX_1(s)}{X_1(s)}, \quad t \geq 0,
\]

\[
W_2(t) = \frac{1}{\sqrt{L''}} \int_0^t \frac{dX_2(s)}{X_1(s)}, \quad t \geq 0.
\]

Note that the Wiener processes \( W_1 \) and \( W_2 \) are independent, since

\[
\langle W_1, W_2 \rangle_t = \frac{1}{\sqrt{L''}} \int_0^t \frac{d \langle X_1, X_2 \rangle_s}{X_1^2(s)} = 0, \quad t \geq 0.
\]

Solving system (14), we get the representations

\[
X_1(t) = \exp \left[ -\frac{1}{2} L't + \sqrt{L} W_1(t) \right], \quad t \geq 0,
\]

\[
X_2(t) = \sqrt{L''} \int_0^t \exp \left[ -\frac{1}{2} L's + \sqrt{L} W_1(s) \right] dW_2(s), \quad t \geq 0.
\]

Now to find the density of the joint distribution of the random variables \( X_1(t) \) and \( X_2(t) \) for \( t > 0 \) note that for any Borel sets \( \Delta_1 \subset (0; +\infty) \) and \( \Delta_2 \subset \mathbb{R} \) we have

\[
\mathbf{P} \{ X_1(t) \in \Delta_1, \ X_2(t) \in \Delta_2 \} = \mathbf{E} \left[ \mathbb{1}\{X_1(t) \in \Delta_1\} \cdot \mathbf{E} ( \mathbb{1}\{X_2(t) \in \Delta_2\} \mid X_1) \right].
\]

However, since as is easily seen the filtrations generated by the stochastic processes \( X_1 \) and \( W_1 \) (and completed with the sets of zero probability) coincide and \( W_1 \) and \( W_2 \) are independent, \( X_1 \) and \( W_2 \) are independent too, and so by Lemma 3.6

\[
\mathbf{E} ( \mathbb{1}\{X_2(t) \in \Delta_2\} \mid X_1) =
\]

\[
= \mathbf{E} \left( \mathbb{1}\left\{ \sqrt{L''} \int_0^t X_1(s) dW_2(s) \in \Delta_2 \right\} \mid X_1 \right) =
\]

\[
= \frac{1}{\sqrt{L''}} \int_{\Delta_2} \frac{1}{\sqrt{2\pi \sigma_t}} e^{-\frac{v^2}{2\sigma_t^2}} dv,
\]

where

\[
\sigma_t := \left( \int_0^t X_1^2(s) ds \right)^{1/2} > 0.
\]
Therefore,
\[
\mathbb{P}\{X_1(t) \in \Delta_1, \ X_2(t) \in \Delta_2\} = \\
= \mathbb{E}\left(\mathbb{I}\{X_1(t) \in \Delta_1\} \cdot \frac{1}{\sqrt{2\pi \sigma_t}} \int_{\Delta_2} \frac{1}{\sqrt{2\pi \sigma_t}} e^{-\frac{v^2}{2\sigma_t^2}} dv \right) = \\
= \frac{1}{\sqrt{L''}} \int_{\Delta_2} \mathbb{E}\left(\mathbb{I}\{X_1(t) \in \Delta_1\} \cdot \frac{1}{\sqrt{2\pi \sigma_t}} e^{-\frac{v^2}{2\sigma_t^2}} \right) dv.
\]

The density of the joint distribution of the random variables \(X_1(t)\) and \(\sigma_t^2 \equiv \int_0^t X_1^2(s) \, ds\) has the following form (see [4, p. 265, Formula 1.10.8]):

\[
\pi \left[X_1(t), \sigma_t^2\right] (z_1, z_2) = \frac{\exp \left[ -\frac{L't}{8} - \frac{1+z_1^2}{2L'z_2} \right]}{2z_1^{3/2} \sqrt{\pi} \sqrt{L''} \cdot z_2} \cdot i\frac{L'}{\pi} \left( \frac{z_1}{L'z_2} \right), \quad z_1, z_2 > 0,
\]

where (see [4, p. 644])

\[
i_v(z) := \frac{z e^{\frac{z^2}{4y}}}{\pi \sqrt{\pi y}} \int_0^\infty \exp \left[ -z \cosh v - \frac{v^2}{4y} \right] \sinh v \sin \frac{\pi v}{2y} \, dv, \quad y, z > 0.
\]

Hence (in the second equality we simply change \(v\) to \(z_2\) and \(z_2\) to \(v\))

\[
\mathbb{P}\{X_1(t) \in \Delta_1, \ X_2(t) \in \Delta_2\} = \\
= \int_{\Delta_2} \int_{\Delta_1}^{+\infty} \int_{0}^{+\infty} \exp \left[ -\frac{z_1^2}{2L'v} - \frac{L't}{8} - \frac{1+z_1^2}{2L'v} \right] \cdot i\frac{L'}{2} \left( \frac{z_1}{L'v} \right) \, dz_1 \, dz_2 \, dv = \\
= \int_{\Delta_2} \int_{\Delta_1}^{+\infty} \int_{0}^{+\infty} \exp \left[ -\frac{z_2^2}{2L'v} - \frac{L't}{8} - \frac{1+z_2^2}{2L'v} \right] \cdot i\frac{L'}{2} \left( \frac{z_2}{L'v} \right) \, dz_1 \, dz_2 \, dv = \\
= \int_{\Delta_1} \int_{\Delta_2} \int_{0}^{+\infty} \exp \left[ -\frac{z_1^2}{2L'v} - \frac{L't}{8} - \frac{1+z_1^2}{2L'v} \right] \cdot i\frac{L'}{2} \left( \frac{z_1}{L'v} \right) \, dz_1 \, dz_2 \, dv.
\]
Thus, for $z_1 > 0$ and $z_2 \in \mathbb{R}$ we have

$$\pi[X_1(t), X_2(t)](z_1, z_2) = \int_0^{+\infty} \exp \left[ -\frac{z_1^2}{2L't} - \frac{L't}{8} - \frac{1+z_2^2}{2L't} \right] \cdot \frac{1}{\sqrt{z_1}} \cdot \int_0^{+\infty} e^{-\frac{z_2^2}{2L't}} \cdot \sinh v \sin \frac{\pi v}{L't} \cdot \frac{1}{\sqrt{z_1}} dv =$$

$$= \frac{e^{\frac{z_1^2}{2L't} - \frac{L't}{8}}}{2\pi L'\sqrt{L'L''t}} \cdot \int_0^{+\infty} e^{-\frac{z_2^2}{2L't}} \cdot \frac{1}{\sqrt{z_1}} \cdot \int_0^{+\infty} \left[ \exp \left[ -\frac{1}{2t} \left( \frac{z_2^2}{L''} + \frac{1+2z_1 \cosh v + z_2^2}{L''} \right) \right] \right] dv \cdot e^{-\frac{z_2^2}{2L't}} \sinh v \sin \frac{\pi v}{L't} dr.$$

Finally, since for $K > 0$

$$\int_0^{+\infty} \frac{1}{v^{5/2}} e^{-\frac{K}{v}} dv = \frac{\sqrt{2\pi}}{K^{3/2}},$$

we obtain

$$\pi[X_1(t), X_2(t)](z_1, z_2) =$$

$$= \frac{e^{\frac{z_1^2}{2L't} - \frac{L't}{8}}}{2\pi L'\sqrt{L'L''t}} \cdot \int_0^{+\infty} \frac{1}{\sqrt{z_1}} \cdot \int_0^{+\infty} e^{-\frac{z_2^2}{2L't}} \sinh v \sin \frac{\pi v}{L't} dv \cdot \sqrt{2\pi} dv =$$

$$= \frac{e^{\frac{z_1^2}{2L't} - \frac{L't}{8}}}{\pi \sqrt{2\pi L''t}} \cdot \int_0^{+\infty} \frac{1}{\sqrt{z_1}} \cdot \int_0^{+\infty} e^{-\frac{z_2^2}{2L't}} \sinh v \sin \frac{\pi v}{L't} \left( (1 + 2z_1 \cosh v + z_1^2) + \frac{L'}{L''} z_2^2 \right)^{3/2} dv. \quad \square$$

**Remark 3.8.** The density of the joint distribution of $X_1(t)$ and $X_2(t)$ is symmetric in the second variable:

$$\pi[X_1(t), X_2(t)](z_1, -z_2) = \pi[X_1(t), X_2(t)](z_1, z_2), \quad z_1 > 0, \quad z_2 \in \mathbb{R}.$$

**Theorem 3.9.** For any $t > 0$ and $u \in \mathbb{R}$ the joint distribution of the random variables $p(t)$ and $p'(u)$ has a density of the form

$$\pi[p(t), p'(u)](z_1, z_2) = \frac{e^{\frac{z_1^2}{2L't} - \frac{L't}{8}}}{\sqrt{2\pi L''t}} \cdot \int_0^{+\infty} \frac{1}{\sqrt{z_1}} \cdot \int_0^{+\infty} e^{-\frac{z_2^2}{2L't}} \sinh v \sin \frac{\pi v}{L't} \left( (z_1^2 + 2z_2^3 \cosh v + z_2^4) + \frac{L'}{L''} z_2^2 \right)^{3/2} dv,$$

$$z_1 \geq 0, \quad z_2 \in \mathbb{R}, \quad z_1^2 + z_2^2 \neq 0.$$

**Proof.** The proof of this theorem is similar to that of Theorem 3.3. Set

$$E := (0; +\infty) \times \mathbb{R},$$

$$E := B((0; +\infty) \times \mathbb{R}),$$

$$E := (0; +\infty) \times \mathbb{R},$$

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and, for fixed $t_0 > 0$,

$$A(u, t) := (p_{t_0}(u), p'_{t_0}(u)), \quad u \in \mathbb{R}, \quad t \geq 0.$$  

Note that the mapping $(\omega, u, t) \mapsto A(\omega, u, t)$ is measurable, since such is the mapping $(\omega, u, t) \mapsto x(\omega, u, t)$.

Therefore, for the stochastic flow (3) and the function

$$f(z_1, z_2) = \frac{1}{z_1} \cdot \mathbb{I}\{z_1 \in \Delta_1, \ z_2 \in \Delta_2\}, \quad (z_1, z_2) \in E,$$

where $\Delta_1 \in \mathcal{B}((0; +\infty))$ and $\Delta_2 \in \mathcal{B}(\mathbb{R})$ are arbitrary, by Theorem 2.10 with $s = 0$ and $t = t_0$ we have (for convenience, until the end of the proof we write $t$ instead of $t_0$)

$$\mathbb{E}\left[ \frac{\partial x}{\partial u}(u, t) \cdot \mathbb{I} \left\{ \frac{1}{v_1}, \frac{v_2}{v_1^3} \in \Delta_1, \ -\frac{\partial^2 x}{\partial u^2}(u, t) \in \Delta_2 \right\} \right] =$$

$$= \mathbb{E}\left[ \frac{1}{p_t(0)} \cdot \mathbb{I} \{p_t(0) \in \Delta_1, \ p'_t(0) \in \Delta_2\} \cdot p_t(0) \right],$$

so that

$$\mathbb{P}\{p_t(0) \in \Delta_1, \ p'_t(0) \in \Delta_2\} = \mathbb{E}\left[ X_1(t) \cdot \mathbb{I} \left\{ \frac{1}{X_1(t)} \in \Delta_1, \ -\frac{X_2(t)}{X_1(t)} \in \Delta_2 \right\} \right] =$$

$$= \int_{0}^{+\infty} v_1 \cdot \mathbb{I} \left\{ \frac{1}{v_1} \in \Delta_1, \ -\frac{v_2}{v_1^3} \in \Delta_2 \right\} \cdot \pi[X_1(t), X_2(t)](v_1, v_2) \, dv_1 \, dv_2 =$$

$$= \left[ \frac{z_1 = \frac{1}{v_1}}{\frac{1}{v_1}} \right] = \int_{\Delta_1} \int_{\Delta_2} \frac{1}{z_1} \cdot \pi[X_1(t), X_2(t)] \left( \frac{1}{z_1} - \frac{z_2}{z_1^2} \right) \, dz_1 \, dz_2.$$

From this and from Remark 3.8 it follows that the joint distribution of the random variables $p_t(0)$ and $p'_t(0)$ has a density for which the following representation takes place:

$$\pi[p_t(0), p'_t(0)](z_1, z_2) = \frac{1}{z_1} \cdot \pi[X_1(t), X_2(t)] \left( \frac{1}{z_1} - \frac{z_2}{z_1^2} \right), \quad z_1 > 0, \quad z_2 \in \mathbb{R}. \quad (15)$$

Therefore, by Lemma 3.7 we obtain that for $z_1 > 0$ and $z_2 \in \mathbb{R}$

$$\pi[p_t(0), p'_t(0)](z_1, z_2) = \frac{1}{z_1} \cdot \frac{e^{-\frac{z_2^2}{2}} \cdot \frac{L'}{2}}{\pi \sqrt{2\pi L''t}} \cdot \sqrt{z_1} \cdot \int_{0}^{+\infty} e^{\frac{-z_1^2}{2} \sinh v} \sin \frac{\pi v}{L} \cdot \frac{\left( 1 + 2z_1 \cosh v + z_1^2 \right)}{(z_1^2 + 2z_1^2 \cosh v + z_1^4 + \frac{L'}{L''} z_1^2)^{3/2}} \, dv =$$

$$= \frac{e^{\frac{z_2^2}{2} - \frac{L'}{8}}}{\pi \sqrt{2\pi L''t}} \cdot z_1^{3/2} \cdot \int_{0}^{+\infty} e^{\frac{-z_1^2}{2} \sinh v} \sin \frac{\pi v}{L} \cdot \frac{\left( (z_1^2 + 2z_1^2 \cosh v + z_1^4 + \frac{L'}{L''} z_1^2)^{3/2} \right)}{dv.$
It remains to note that by Lemma 2.9 the density of the joint distribution of $p_t(u)$ and $p'_t(u)$ does not depend on $u \in \mathbb{R}$.

\[ \square \]

Remark 3.10. The density of the joint distribution of $p_t(u)$ and $p'_t(u)$ is symmetric in the second variable:

\[ \pi[p_t(u), p'_t(u)](z_1, -z_2) = \pi[p_t(u), p'_t(u)](z_1, z_2), \quad z_1 > 0, \quad z_2 \in \mathbb{R}. \]

4. LEVEL-CROSSING INTENSITY FOR THE STOCHASTIC PROCESS $p_t$

By Lemma 2.9 the stochastic process \{ $p_t(u)$, $u \in \mathbb{R}$ \} is strictly stationary, and by Theorem 3.3 all its one-dimensional distributions are continuous. Hence (see \[10\] pp. 146–147) for any $c \in \mathbb{R}$ almost surely it is not identically equal to $c$ on any interval, and so the number $N_t(0; 1; c)$ of crossings of the level $c$ by this stochastic process on the interval $[0; 1]$ and the corresponding level-crossing intensity $\mu_t(c)$ are well-defined.

Theorem 4.1. For any $t > 0$ we have

\[ \mu_t(c) = \frac{\sqrt{2} L'' \cdot e^{\frac{c^2}{2L^2}}}{\pi L' \sqrt{\pi t}} \cdot \frac{1}{\sqrt{c}} \cdot \int_0^{+\infty} e^{-\frac{c^2}{2L^2}} \sinh v \sin \frac{\pi v}{2L} \cdot \frac{dv}{\sqrt{\sqrt{1 + \frac{2 \cosh v}{c^2} + \frac{1}{c^2}}}} \]

for a. e. $c > 0$.

Proof. It is easy to see that the stochastic process \{ $p_t(u)$, $u \in [0; 1]$ \} satisfies the conditions of Theorem 1.2 and so

\[ \mu_t(c) = \overline{\mu}_t(c) \quad \text{for a. e. } c > 0, \]

where

\[ \overline{\mu}_t(c) := \int_0^1 E(|p'_t(u)| \mid p_t(u) = c) \cdot \pi[p_t(u)](c) \cdot du. \]

However, from Theorem 3.9 the strict positivity of the density $\pi[p_t(u)](z)$ for $z > 0$ implied by Theorem 3.3, and Remark 3.10 it follows that

\[ \int_0^1 E(|p'_t(u)| \mid p_t(u) = c) \cdot \pi[p_t(u)](c) \cdot du = \]

\[ = \int_0^{+\infty} \int_{-\infty}^\infty |z| \cdot \pi[p_t(u), p'_t(u)](c, z) \cdot dz \cdot du = 2 \int_0^{+\infty} z \cdot \pi[p_t(0), p'_t(0)](c, z) \cdot dz \]

and so

\[ \overline{\mu}_t(c) = 2 \int_0^{+\infty} z \cdot \pi[p_t(0), p'_t(0)](c, z) \cdot dz. \]
Therefore, by the same Theorem 3.9
\[ \mu_t(c) = 2 \int_0^{+\infty} \left[ \frac{e^{L'' t} - L'' t}{\pi \sqrt{2 L'' t}} \cdot z \cdot \int_0^{+\infty} \frac{c^{3/2} e^{-\frac{v^2}{2}} \sinh \frac{\pi v}{2L''} d\pi}{(c^2 + 2c^3 \cosh v + c^4 + L'' t z)^{3/2}} dv \right] dz = \]
\[ = \frac{\sqrt{2} \cdot e^{L'' t} - L'' t}{\pi \sqrt{2 L'' t}} \cdot c^{3/2} \cdot \int_0^{+\infty} \frac{z dz}{(c^2 + 2c^3 \cosh v + c^4 + L'' t z)^{3/2}} e^{-\frac{v^2}{2}} \sinh \frac{\pi v}{L''} dv. \]

Finally, since for any \( A, B > 0 \)
\[ \int_0^{+\infty} \frac{z dz}{(A + Bz^2)^{3/2}} = \frac{1}{B\sqrt{A}}, \]
we obtain
\[ \mu_t(c) = \frac{\sqrt{2} \cdot e^{L'' t} - L'' t}{\pi \sqrt{2 L'' t}} \cdot c^{3/2} \cdot \int_0^{+\infty} \frac{L'' e^{-\frac{v^2}{2}} \sinh \frac{\pi v}{L''} d\pi}{L'' \sqrt{c^2 + 2c^3 \cosh v + c^4}} dv = \]
\[ = \frac{\sqrt{2L''} \cdot e^{L'' t} - L'' t}{\pi L'' \sqrt{2L''}} \cdot \frac{1}{\sqrt{c}} \cdot \int_0^{+\infty} \frac{e^{-\frac{v^2}{2L''}} \sinh \frac{\pi v}{L''} d\pi}{\sqrt{1 + \frac{2c^2 \cosh v}{c^2}}} dv. \]
\[ \square \]

It seems difficult to find the asymptotics of \( \mu_t(c) \) as \( c \to +\infty \) by direct analytic methods. However, one can find it with the help of a probabilistic approach\footnote{The author is grateful to Prof. A. A. Dorogovtsev for the advice to use this approach.}.

**Theorem 4.2.** For any \( t > 0 \) we have
\[ \mu_t(c) = e^{L'' t} \frac{\sqrt{L''}}{\pi \sqrt{2 L''}} \cdot \frac{\sqrt{c}}{\ln c} \cdot \exp \left[ -\frac{(\ln c)^2}{2L'' t} \right] \cdot (1 + \mathcal{O}(1)), \quad c \to +\infty. \]

**Proof.** From equalities (16) and (15) we obtain
\[ \mu_t(c) = 2 \int_0^{+\infty} z \cdot \pi [p_t(0), p_t(0)](c, z) dz = \]
\[ = \frac{2}{c^3} \cdot \int_0^{+\infty} z \cdot \pi [X_1(t), X_2(t)] \left( \frac{1}{c}, \frac{z}{c^3} \right) \cdot \frac{1}{z^2} d\pi = \]
\[ = \frac{2}{c} \cdot \int_0^{+\infty} z \cdot \pi [X_1(t), X_2(t)] \left( \frac{1}{c}, z \right) dz. \]
Note that since
\[
\int_0^\infty z \cdot \pi[X_1(t), X_2(t)] \left( \frac{1}{c}, z \right) dz = \pi[X_1(t)] \left( \frac{1}{c} \right) \cdot \mathbb{E} \left[ (X_2(t))^+ \middle| X_1(t) = \frac{1}{c} \right],
\]
where

\[
(z)_+ := \begin{cases} 
  z, & z \geq 0, \\
  0, & z < 0,
\end{cases}
\]
we have

\[
\overline{\pi}_t(c) = \frac{2}{c} \cdot \pi[X_1(t)] \left( \frac{1}{c} \right) \cdot \mathbb{E} \left[ (X_2(t))^+ \middle| X_1(t) = \frac{1}{c} \right] = \frac{2}{c} \cdot \pi[X_1(t)] \left( \frac{1}{c} \right) \cdot \mathbb{E} \left[ \left( \sqrt{L} \int_0^t X_1(s) dW_2(s) \right)^+ \middle| X_1(t) = \frac{1}{c} \right].
\]

Furthermore, in a standard way by using the approximation of the stochastic integral with partial sums, the independence of the stochastic process \(X_1\) from the Wiener process \(W_2\), and the fact that if \(\xi \sim \mathcal{N}(0; \sigma^2)\), then

\[
\mathbb{E}(\xi) = \sigma \sqrt{\frac{2}{\pi}}.
\]

one can show that

\[
\mathbb{E} \left[ \left( \sqrt{L} \int_0^t X_1(s) dW_2(s) \right)^+ \middle| X_1(t) = \frac{1}{c} \right] = \frac{\sqrt{L}}{\sqrt{2\pi}} \cdot \left( \int_0^t X_1^2(s) ds \right)^{1/2}.
\]

Therefore,

\[
\overline{\pi}_t(c) = \frac{2}{c} \cdot \pi[X_1(t)] \left( \frac{1}{c} \right) \cdot \sqrt{\frac{L}{2\pi}} \cdot \mathbb{E} \left[ \left( \int_0^t X_1^2(s) ds \right)^{1/2} \middle| X_1(t) = \frac{1}{c} \right].
\]

However,

\[
\mathbb{E} \left[ \left( \int_0^t X_1^2(s) ds \right)^{1/2} \mid X_1(t) = \frac{1}{c} \right] = \mathbb{E} \left[ \left( \int_0^t \exp \left[ -L's + 2\sqrt{L'} \left( \frac{s}{t} W_1(t) + \tilde{B}(s) \right) \right] dW_1(t) \right)^{1/2} \middle| W_1(t) = \frac{1}{\sqrt{L'}} \left( \frac{1}{2} L't - \ln c \right) \right],
\]
where

\[
\tilde{B}(s) := W_1(s) - \frac{s}{t} W_1(t), \quad 0 \leq s \leq t.
\]
Since the stochastic process \( \{ \tilde{B}(s), \, 0 \leq s \leq t \} \) does not depend on the random variable \( W_1(t) \), we obtain

\[
E \left( \left( \int_0^t X_1^2(s) \, ds \right)^{1/2} \mid X_1(t) = \frac{1}{c} \right) =
\]

\[
= E \left( \int_0^t \exp \left[ -L's + 2\sqrt{L'} \cdot \frac{s}{t} \cdot \frac{1}{\sqrt{L'}} \cdot \left( \frac{1}{2}L't - \ln c \right) + 2\sqrt{L'} \tilde{B}(s) \right] \, ds \right)^{1/2}
\]

\[
= E \left( \int_0^t \exp \left[ -2 \ln c \cdot \frac{s}{t} + 2\sqrt{L'} \tilde{B}(s) \right] \, ds \right)^{1/2}
\]

Set

\[
B(s) := \frac{1}{\sqrt{t}} \tilde{B}(st), \quad 0 \leq s \leq 1.
\]

Then \( \{ B(s), \, 0 \leq s \leq 1 \} \) is a standard Brownian bridge and

\[
E \left( \int_0^t \exp \left[ -2 \ln c \cdot \frac{s}{t} + 2\sqrt{L'} \tilde{B}(s) \right] \, ds \right)^{1/2}
\]

\[
= E \left( \int_0^t \exp \left[ -2 \ln c \cdot \frac{s}{t} + 2\sqrt{L'} \cdot \sqrt{t} B \left( \frac{s}{t} \right) \right] \, ds \right)^{1/2}
\]

\[
= \sqrt{t} \cdot E \left( \int_0^1 \exp \left[ -2 \ln c \cdot s + 2\sqrt{L't} \cdot B(s) \right] \, ds \right)^{1/2}
\]

Now define functions \( h_c : [0; 1] \to \mathbb{R}, \, c > 1 \), by the equality

\[
h_c(s) := \frac{2c^2 \ln c}{c^2 - 1} \cdot e^{-2 \ln c \cdot s}, \quad 0 \leq s \leq 1, \quad c > 1,
\]

and note that they satisfy the following conditions:

1) \( h_c(s) \geq 0, \, s \in [0; 1], \, c > 1; \)

2) \( \int_0^1 h_c(s) \, ds = 1, \, c > 1; \)

3) \( \forall \varepsilon \in (0; 1) : \lim_{c \to +\infty} \int_\varepsilon^1 h_c(s) \, ds = 0. \)
Therefore,

\[
\lim_{c \to +\infty} \int_0^1 h_c(s) e^{2\sqrt{LT}B(s)} \, ds = e^{2\sqrt{LT}B(0)} = 1.
\]

Since for any \( n \geq 1 \) we have

\[
\mathbb{E} \sup_{c>1} \left( \int_0^1 h_c(s) e^{2\sqrt{LT}B(s)} \, ds \right)^n \leq \mathbb{E} \sup_{c>1} \left( \int_0^1 h_c(s) \, ds \cdot e^{2\sqrt{LT} \max_{0 \leq s \leq 1} B(s)} \right)^n = \mathbb{E} \sup_{c>1} e^{2n\sqrt{LT} \max_{0 \leq s \leq 1} B(s)} = 1 + o(1),
\]

the family of random variables

\[
\left( \int_0^1 h_c(s) e^{2\sqrt{LT}B(s)} \, ds \right)^{1/2}, \quad c > 1,
\]

is uniformly integrable, and so

\[
\lim_{c \to +\infty} \mathbb{E} \left( \int_0^1 h_c(s) e^{2\sqrt{LT}B(s)} \, ds \right)^{1/2} = \mathbb{E} \lim_{c \to +\infty} \left( \int_0^1 h_c(s) e^{2\sqrt{LT}B(s)} \, ds \right)^{1/2} = 1.
\]

Therefore,

\[
\mathbb{E} \left( \int_0^1 \exp \left[ -2 \ln c \cdot s + 2\sqrt{LT}B(s) \right] \, ds \right)^{1/2} = \frac{1}{\sqrt{2 \ln c}} : (1 + o(1)), \quad c \to +\infty.
\]

Thus, taking Corollary 3.2 into account, we finally obtain

\[
\overline{\mu}(c) = \frac{2}{c} \cdot \frac{e^{-L/s}}{\sqrt{2\pi L^t}} \cdot \sqrt{\frac{c}{2\ln c}} \cdot \exp \left[ -\frac{(\ln c)^2}{2L^t} \right] = \frac{1}{\sqrt{2 \ln c}} : (1 + o(1)), \quad c \to +\infty.
\]

The theorem is proved. \( \square \)

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