A weakly convergent fully inexact Douglas-Rachford method with relative error tolerance

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Abstract

Douglas-Rachford method is a splitting algorithm for finding a zero of the sum of two maximal monotone operators. Each of its iterations requires the sequential solution of two proximal subproblems. The aim of this work is to present a fully inexact version of Douglas-Rachford method wherein both proximal subproblems are solved approximately within a relative error tolerance. We also present a semi-inexact variant in which the first subproblem is solved exactly and the second one inexactly. We prove that both methods generate sequences weakly convergent to the solution of the underlying inclusion problem, if any.

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1 Introduction

Douglas-Rachford method [5], originally proposed for solving discretized heat equations, was extended by Lions and Mercier [16] for finding a zero of the sum of two maximal monotone operators. This method is, presently, the subject of intense research due to its efficiency, its use for solving PDEs, large-scale optimization problems (even some non-convex ones), imaging problems, and its connection with the alternating direction method (see [10,3,12,13,15,11,2,14] and the references therein).

Eckstein and Bertsekas [6] proved that Douglas-Rachford method can be regarded as an instance of the proximal point method applied to an implicitly defined operator. Recently, Eckstein and Yao [8] cleverly used this result to propose an inexact version of Douglas-Rachford method with relative error tolerance based on Solodov and Svaiter hybrid proximal-extrageradient method [20,19,23,22]. Each iteration of Douglas-Rachford method requires the sequential solution of two proximal subproblems. Eckstein-Yao algorithm is a semi-inexact version of this method in the sense that it allows for inexactness on the first proximal subproblem and requires the second one to be solved exactly.

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Complexity of Eckstein-Yao inexact Douglas-Rachford method was derived by Marques Alves and Geremia [17].

The main contribution of this work is the introduction and analysis of a fully inexact version of Douglas-Rachford method wherein both proximal subproblems are solved approximately within a relative error tolerance. To the best of our knowledge, this is the first fully inexact version of Douglas-Rachford method with a relative error tolerance. We also propose a semi-inexact version for the cases where one of the proximal subproblems can be solved exactly. In the semi-inexact case of Douglas-Rachford, our proposal is to solve the first subproblem exactly and the second one inexactly, as of now, open questions. Weak convergence, in infinite dimensional spaces, of their version allows for errors in the second subproblem. Weak convergence of our version. Their version allows for errors in the second one. Weak convergence, in infinite dimensional spaces, of their version does not require the introduction of a relaxation factor, as ours do, so that their inexact version is formally closer than ours to the exact method. Their version allows for errors in the second subproblem, while our semi-inexact version allows for errors in the second one. Weak convergence, in infinite dimensional spaces, of their version is an interesting open question, while we prove here weak convergence of our version. Their version has a very good practical performance [8] Section 7] and nice complexity properties [17], while the practical performance and complexity properties of our version are, as of now, open questions. Weak convergence on Douglas-Rachford method was established by the author in [25] for the inexact version with the summand error tolerance. Here we use the techniques and ideas of that work to prove weak convergence under a relative error tolerance.

This work is organized as follows. In Section 2 we establish the notation and prove some basic results. In Section 3 we present a fully inexact Douglas-Rachford method and analyze its convergence properties. In Section 4 we present a semi-inexact Douglas-Rachford method and analyze its convergence properties. In Appendices A and B we prove some technical results.
2 Basic definitions and results

From now on $H$ is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|x\| = \sqrt{\langle x, x \rangle}$.

In $H \times H$ we consider the canonical inner product and associated norm of Hilbert spaces products.

We are concerned with the inclusion problem

\[ 0 \in A(x) + B(x) \]  

(2.1)

where $A : H \rightrightarrows H$ and $B : H \rightrightarrows H$ are maximal monotone operators. The extended solution set \([7]\) of this problem is

\[ S_e(A, B) = \{(z, w) : w \in B(z), -w \in A(x)\}. \]  

(2.2)

It is trivial to verify that

\[ 0 \in A(z) + B(z) \iff (z, w) \in S_e(A, B) \text{ for some } w \in H. \]

The $\varepsilon$-enlargement \([4]\) of a maximal monotone operator $T : H \rightrightarrows H$ is defined as

\[ T^{[\varepsilon]}(z) = \{v \in H : \langle z - z', v - v' \rangle \geq -\varepsilon \ \forall z' \in H, v' \in T(z')\} \quad (\varepsilon \geq 0, \ z \in H). \]  

(2.3)

The following elementary properties of the $\varepsilon$-enlargement will be used in the sequel.

**Proposition 2.1.** If $T : H \rightrightarrows H$ is maximal monotone, then

1. $T^{[\varepsilon=0]} = T$;
2. $\lambda(T^{[\varepsilon]}) = (\lambda T)^{[\lambda \varepsilon]}$ for any $\varepsilon \geq 0$ and $\lambda > 0$.

**Proof.** Item 1 follows trivially from (2.3) and the maximal monotonicity of $T$ while item 2 follows directly from (2.3). $\square$

In each iteration of Douglas-Rachford method, one shall compute $(I + \lambda T)^{-1} \zeta$, first with $T = A$ and $\zeta = x_{k-1} - \lambda b_{k-1}$ and then with $T = B$ and $\zeta = y_{k} + \lambda b_{k-1}$. Let $T$ be a maximal monotone operator in $H$. Computation of $z = (I + \lambda T)^{-1} \zeta$ can be decoupled in an inclusion and an equation, which we call the proximal inclusion-equation system:

\[ v \in T(z), \quad \lambda v + z - \zeta = 0. \]  

(2.4)

If we allow errors in the inclusion, by means of the $\varepsilon$-enlargement of $T$, and error in the equation we get

\[ v \in T^{[\varepsilon]}(z), \quad \lambda v + z - \zeta = r \]

where $\varepsilon$ is the error in the inclusion and $r$ is the residual in the equation at (2.4). In some sense, $\|r\|^2 + 2\lambda \varepsilon$ quantifies the overall error in the inexact solution of (2.4), due to the next result proved in [21].

**Proposition 2.2** ([21, Corollary 1]). Suppose $T : H \rightrightarrows H$ is maximal monotone, $\lambda < 0$, and $\zeta \in H$. If

\[ v^* \in T(z^*), \quad \lambda v^* + z^* - \zeta = 0, \quad v \in T^{[\varepsilon]}(z), \quad \lambda v + z - \zeta = r \]

then $\|\lambda(v^* - v)||^2 + \|z^* - z||^2 \leq \|r\|^2 + 2\lambda \varepsilon$. 

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The next lemma will be instrumental in the convergence analysis of the inexact methods we propose in this work.

**Lemma 2.3.** Let $z, w \in X$ and $\lambda > 0$. Suppose

$$\lambda a + y = z - \lambda w + r, \quad a \in A^e(x),$$

$$\lambda b + x = y + \lambda w + s, \quad b \in B^u(y),$$

and define $z' = z - \lambda(a + b)$, $w' = w - \lambda^{-1}(x - y)$. For any $(z^*, w^*) \in S_c(A, B)$,

$$\|(z^*, \lambda w^*) - (z, \lambda w)\|^2 \geq \|(z^*, \lambda w^*) - (z', \lambda w')\|^2 + \|\lambda(a + w)\|^2$$

$$- [\|s\|^2 + 2\lambda(\langle r, a + b \rangle + \varepsilon + \mu)].$$

**Proof.** In view of item [22] on Proposition [2.1], it suffices to prove the lemma for the case $\lambda = 1$, which we assume now. In this case

$$a + y = z - w + r, \quad b + x = y + w + s, \quad z' = z - (a + b), \quad w' = w - (x - y).$$

Fix $(z^*, w^*) \in S_c(A, B)$ and let

$$\pi = \langle z^* - z', z' - z \rangle + \langle w^* - w', w' - w \rangle.$$  

In view of definition [2.2], $w^* \in B(z^*)$ and $-w^* \in A(z^*)$. It follows from these inclusions, the inclusions $a \in A^e(y)$ and $b \in B^u(x)$, and definition [2.3], that

$$\langle z^* - z', z' - z \rangle = \langle z^* - z', w^* - b \rangle + \langle z^* - z', -w^* - a \rangle$$

$$= \langle z^* - x, w^* - b \rangle + \langle x - z', w^* - b \rangle + \langle z^* - y, -w^* - a \rangle + \langle y - z', -w^* - a \rangle$$

$$\geq \langle x - z', w^* - b \rangle + \langle y - z', -w^* - a \rangle - \varepsilon - \mu.$$  

Direct combination of this inequality with the definitions of $\pi$ and $w'$ yields

$$\pi \geq \langle x - z', w^* - b \rangle + \langle y - z', -w^* - a \rangle - \varepsilon - \mu + \langle w^* - w', y - x \rangle.$$  

Since the expression at the right hand-side of the above inequality does not depend on $w^*$, we can substitute $b$ for $w^*$ is this expression and use the definitions of $z'$ and $w'$ to conclude that

$$\pi \geq \langle y - z', -b - a \rangle + \langle b - w', y - x \rangle - \varepsilon - \mu$$

$$= \langle y - x + r + s, -a - b \rangle + \langle s, y - x \rangle - \varepsilon - \mu$$

$$= \langle x - y, a + b \rangle - [\langle r, a + b \rangle + \langle s, a + b + x - y \rangle + \varepsilon + \mu].$$  

Therefore,

$$\|(z^*, w^*) - (z, w)\|^2 = \|(z^*, w^*) - (z', w')\|^2 + \|(z', w') - (z, w)\|^2 + 2\pi$$

$$\geq \|(z^*, w^*) - (z', w')\|^2 + \|a + b\|^2 + \|x - y\|^2$$

$$+ 2\langle x - y, a + b \rangle - 2[\langle r, a + b \rangle + \langle s, a + b + y - x \rangle + \varepsilon + \mu]$$

$$= \|(z^*, w^*) - (z', w')\|^2 + \|a + b + x - y\|^2$$

$$- 2[\langle r, a + b \rangle + \langle s, a + b + y - x \rangle + \varepsilon + \mu].$$

To end the proof, observe that

$$\|a + b + x - y\|^2 - 2\langle s, a + b + x - y \rangle = \|a + b + x - y - s\|^2 - \|s\|^2 = \|a + w\|^2 - \|s\|^2$$

and combine the two above equations. \(\square\)
3 A fully inexact Douglas-Rachford method with relative error tolerance

In this section we present an Inexact Douglas-Rachford method wherein both proximal subproblems are to be solved within a relative error tolerance. We prove that the sequences generated by this method converge weakly to a point in $S_e(A, B)$, whenever the solution set of \((2.1)\) is non-empty.

**Algorithm I: inexact DR method with relative error tolerance.**

(0) Let $z_0, w_0 \in H$, $\lambda > 0$, $0 < \sigma < \nu < 1$. For $k = 1, 2, 3, \ldots$

(1) find $a_k, y_k, \varepsilon_k, b_k, x_k, \mu_k$ such that

$$a_k \in A^{[\varepsilon_k]}(y_k), \quad b_k \in B^{[\mu_k]}(x_k), \quad \|\lambda a_k + y_k - (z_{k-1} - \lambda w_{k-1})\|^2 + \|\lambda b_k + x_k - (y_k + \lambda w_{k-1})\|^2 + 2\lambda(\varepsilon_k + \mu_k) \leq \frac{\sigma^2}{4}(\|a_k + b_k\|^2 + \|x_k - y_k\|^2);$$

(3.1)

(2) set

$$\delta_k = \|\lambda a_k + y_k - (z_{k-1} - \lambda w_{k-1})\|^2 + \|\lambda b_k + x_k - (y_k + \lambda w_{k-1})\|^2 + 2\lambda(\varepsilon_k + \mu_k),$$

$$\rho_k = \|\lambda(a_k + b_k)\|^2 + \|x_k - y_k\|^2,$$

$$t_k = \begin{cases} 0 \quad \text{if } a_k + b_k = x_k - y_k = 0 \\ \nu \max \left\{ 0, \sqrt{\frac{4\delta_k}{\sigma^2 \rho_k}} - \frac{\|\lambda(a_k + w_{k-1})\|^2}{\rho_k} \right\} \quad \text{otherwise}, \end{cases}$$

$$z_k = z_{k-1} - (1 - t_k)\lambda(a_k + b_k), \quad w_k = w_{k-1} - (1 - t_k)\lambda^{-1}(x_k - y_k);$$

We did not specify how to compute $a_k, y_k, \varepsilon_k, b_k, x_k, \mu_k$, which adds generality to the method. In Appendix \[A\] we show that under some mild conditions (on $A$ and $B$) step (2) is computable. Whenever $t_k = 0$, $z_k = z_{k-1} - \lambda(a_k + b_k)$ and $w_k = w_{k-1} - \lambda^{-1}(x_k - y_k)$, so that formally we retrieve a Douglas-Rachford iteration. In those iterations where $t_k \neq 0$, we will have an under-relaxed Douglas-Rachford iteration.

To simplify the convergence analysis, let $r_k$ and $s_k$ denote the residuals in the equations of the proximal inclusion-equations systems to be solved for $A$ and $B$ at the $k$-th iteration, that is,

$$r_k = \lambda a_k + y_k - (z_{k-1} - \lambda w_{k-1}), \quad s_k = \lambda b_k + x_k - (y_k + \lambda w_{k-1}), \quad (k = 1, 2, \ldots).$$

(3.3)

With this notation, \((3.1)\) and the first line of \((3.2)\) writes

$$a_k \in A^{[\varepsilon_k]}(y_k), \quad \lambda a_k + y_k = z_{k-1} - \lambda w_{k-1} + r_k,$$

$$b_k \in B^{[\mu_k]}(x_k), \quad \lambda b_k + x_k = y_k + \lambda w_{k-1} + s_k,$$

$$\delta_k = \|r_k\|^2 + \|s_k\|^2 + 2\lambda(\varepsilon_k + \mu_k), \quad \delta_k \leq \frac{\sigma^2}{4}\rho_k.$$
It follows from the definition of $t_k$ at (3.2) that for all $k$,

$$0 \leq t_k \leq \nu < 1, \quad \frac{\nu}{\sigma} \sqrt{4\delta_k \rho_k} - \nu \|\lambda(a_k + w_{k-1})\|^2 \leq t_k \rho_k, \quad t_k^2 \rho_k \leq \frac{4\nu^2}{\sigma^2} \delta_k. \quad (3.5)$$

From now on in this section,

$$p_k = (z_k, \lambda w_k), \quad (k = 0, 1, 2, \ldots). \quad (3.6)$$

First we will prove that the sequence $\{p_k\}$ converges Féjer to the set of points $(z, \lambda w)$ where $(z, w) \in S_e(A, B)$.

**Lemma 3.1.** For any $(z, w) \in S_e(A, B)$ and all $k$

$$\|(z, \lambda w) - p_{k-1}\|^2 \geq \|(z, \lambda w) - p_k\|^2 + (1 - t_k) \left\{ \frac{\nu - \sigma}{\sigma} \sqrt{4\delta_k \rho_k} + (1 - \nu) \|\lambda(a_k + w_{k-1})\|^2 \right\},$$

$$\|(z, \lambda w) - p_0\|^2 \geq \|(z, \lambda w) - p_k\|^2 + \sum_{i=1}^{k} (1 - t_i) \left\{ \frac{\nu - \sigma}{\sigma} \sqrt{4\delta_i \rho_i} + (1 - \nu) \|\lambda(a_i + w_{i-1})\|^2 \right\}. \quad (3.7)$$

*Proof.* Fix $(z, w) \in S_e(A, B)$ and let $p^* = (z, \lambda w)$. Define, for $k = 1, 2, \ldots,$

$$z_k' = z_{k-1} - \lambda(a_k + b_k), \quad w_k' = w_{k-1} - \lambda^{-1}(x_k - y_k), \quad p_k' = (z_k', \lambda w_k'). \quad (3.7)$$

It follows from (3.6), (3.4), and Lemma 2.3 that

$$\|p^* - p_{k-1}\|^2 \geq \|p^* - p_k'\|^2 + \|\lambda(a_k + w_{k-1})\|^2 - [\|s_k\|^2 + 2\lambda(\{r_k, a_k + b_k\} + \varepsilon_k + \mu_k)].$$

Using Cauchy-Schwarz inequality, the definition of $\rho_k$ in (3.2), and the third line on (3.4) we conclude that

$$\|s_k\|^2 + 2(\{r_k, a_k + b_k\} + \varepsilon_k + \mu_k) \leq \|s_k\|^2 + 2\lambda(\varepsilon_k + \mu_k) + 2\|r_k\|\sqrt{\rho_k} \leq \delta_k - \|r_k\|^2 + 2\|r_k\|\sqrt{\rho_k}$$

and $\|r_k\| \leq \sqrt{\delta_k} \leq \sigma\sqrt{\rho_k}/2$. Since the expression on the right hand-side of the above equality is increasing for $\|r_k\| \leq \sqrt{\rho_k}$, its maximum value on $[0, \sqrt{\delta_k}]$ is attained at $\|r_k\| = \sqrt{\delta_k}$. Combining these observations with the two above inequalities we conclude that

$$\|p^* - p_{k-1}\|^2 \geq \|p^* - p_k'\|^2 + \|\lambda(a_k + w_{k-1})\|^2 - 2\sqrt{\delta_k \rho_k}.$$ 

Since $p_k = t_k p_{k-1} + (1 - t_k)p_k'$ and $\|p_k - p_k'\|^2 = \rho_k$,

$$\|p^* - p_k\|^2 = t_k\|p^* - p_{k-1}\| + (1 - t_k)\|p^* - p_k'\|^2 \geq t_k\|p^* - p_{k-1}\| + (1 - t_k) \left[ \|p^* - p_k'\|^2 + \|\lambda(a_k + w_{k-1})\|^2 - 2\sqrt{\delta_k \rho_k} \right] \quad (3.5)$$

$$= \|t_k(p^* - p_{k-1}) + (1 - t_k)(p^* - p_k')\|^2 + (1 - t_k)t_k\|p_{k-1} - p_k'\|^2$$

$$+ (1 - t_k) \left[ \|\lambda(a_k + w_{k-1})\|^2 - 2\sqrt{\delta_k \rho_k} \right]$$

$$= \|p^* - p_k\|^2 + (1 - t_k) \left[ t_k \rho_k + \|\lambda(a_k + w_{k-1})\|^2 - 2\sqrt{\delta_k \rho_k} \right].$$

To end the proof of the first inequality, use the above inequality and the next to last inequality in (3.5). The second inequality of the lemma follows trivially from the first one. 

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Lemma 3.2. If $S_{e}(A, B) \neq \emptyset$, then

1. $\{z_k\}$ and $\{w_k\}$ are bounded;
2. $r_k \to 0$, $s_k \to 0$, $\varepsilon_k \to 0$ and $\mu_k \to 0$ as $k \to \infty$;
3. $a_k + w_{k-1} \to 0$ and $y_k - z_{k-1} \to 0$ as $k \to \infty$;
4. $b_k - w_k \to 0$ and $x_k - z_k \to 0$ as $k \to \infty$.

**Proof.** Take $(z, w) \in S_{e}(A, B)$. It follows from Lemma 3.1 that $\{(z_k, \lambda w_k)\}$ is bounded, which proves item 1 and that

$$
\sum_{k=1}^{\infty} \sqrt{\delta_k \rho_k} < \infty, \quad \sum_{k=1}^{\infty} \|a_k + w_{i-1}\|^2 < \infty.
$$

Since $\delta_k \leq \sigma^2 \rho_k / 4$, $\sqrt{\delta_k \rho_k} \geq 2\delta_k / \sigma$ and it follows from the first above inequality that $\delta_k \to 0$ as $k \to \infty$. This result, together with the third line of (3.4) proves item 2. The first limit in item 3 follows trivially from the second above inequality while the second limit follows from the first one, the first limit in item 2 and the definition of $r_k$ in (3.3).

It follows from the update formula for $z_k$ and $w_k$, and from (3.4) that

$$
z_k - x_k + r_k + s_k = t_k \lambda(a_k + b_k), \quad w_k - b_k + s_k = t_k(x_k - y_k).
$$

Squaring both sides of each one of the above equations and adding them we conclude that

$$
\|z_k - x_k + r_k + s_k\|^2 + \|w_k - b_k + s_k\|^2 = t_k^2 \rho_k.
$$

Since $\delta_k \to 0$ as $k \to \infty$, it follows from the above equations and the last inequality in (3.5) that

$$
t_k^2 \rho_k \to 0, \quad z_k - x_k + r_k + s_k \to 0, \quad w_k - b_k + s_k \to 0 \quad \text{as} \quad k \to \infty.
$$

Item 4 follows from the above result and from item 2.

**Theorem 3.3.** If $S_{e}(A, B) \neq \emptyset$, then $\{(x_k, b_k)\}$, $\{(y_k, -a_k)\}$ and $\{(z_k, w_k)\}$ converge weakly to a point in this set.

**Proof.** Suppose a subsequence $\{(z_{k_n}, w_{k_n})\}$ converges weakly to some $(z, w)$. It follows from Lemma 3.2 that

$$
x_{k_n} \rightarrow z, \quad y_{k_n} \rightarrow z, \quad x_{k_n} - y_{k_n} \rightarrow 0, \quad \mu_{k_n} \rightarrow 0, \quad \varepsilon_{k_n} \rightarrow 0,
$$

$$
b_{k_n} \rightarrow w, \quad a_{k_n} \rightarrow w, \quad a_{k_n} - b_{k_n} \rightarrow 0,
$$

as $k \to \infty$.

Since $b_{k_n} \in B[\mu_{k_n}](x_{k_n})$, $a_{k_n-1} \in A[\varepsilon_{k_n-1}](y_{k_n-1})$ for $n = 1, 2, \ldots$, it follows from Lemma 3.2 that $w \in B(z)$ and $-w \in A(w)$. Therefore, $(z, w) \in S_{e}(A, B)$.

We have proved that all weak limit points of $\{(z_k, w_k)\}$ are in $S_{e}(A, B)$. Therefore, all weak limit points of $\{p_k = (z_k, \lambda w_k)\}$ are in $\Omega$,

$$
\Omega = \{(z, \lambda w) : (z, w) \in S_{e}(A, B)\}.
$$

In Lemma 3.1 we proved that $\{p_k\}$ is Fejér convergent to $\Omega$. Since $\Omega \neq \emptyset$, it follows from these results and from Opial’s Lemma [18] that the bounded sequence $\{p_k\}$ has a unique weak limit point and such a point belongs to $\Omega$. Therefore, $\{p_k\}$ converges weakly to a point in $(z, \lambda w)$ where $(z, w) \in S_{e}(A, B)$. To end the proof, use items 3 and 4 of Lemma 3.2.
4 A semi-inexact Douglas-Rachford method

In this section we present an inexact Douglas-Rachford method wherein, in each iteration, the first proximal subproblem is to be exactly solved and the second proximal subproblem is to be solved within a relative error tolerance.

A possible advantage of solving the second subproblem approximately, instead of the first one, is that the error criterion to be satisfied is readily available during the computation of the second step, thereby obviating the necessity of computing more than once the approximate solution per iteration. Since one of the subproblems is to be solved exactly, following [8], we call this method semi-inexact.

Algorithm II: A semi-inexact Douglas-Rachford method with relative error tolerance.

(0) Let $z_0, w_0 \in H$, $\lambda > 0$, $0 < \sigma < \nu < 1$. For $k = 1, 2, 3, \ldots$,

(1) compute $a_k, y_k$ such that

$$a_k \in A(y_k), \quad \lambda a_k + y_k = z_{k-1} - \lambda w_{k-1},$$

compute $b_k, x_k, \mu_k$ such that

$$b_k \in B(\mu_k)(x_k), \quad \|\lambda b_k + x_k - (y_k + \lambda w_{k-1})\|^2 + 2\lambda \mu_k \leq \sigma^2(\|a_k + b_k\|^2 + \|x_k - y_k\|^2);$$

(2) set

$$\delta_k = \|\lambda b_k + x_k - (y_k + \lambda w_{k-1})\|^2 + 2\lambda \mu_k,$$

$$\rho_k = \|\lambda(a_k + b_k)\|^2 + \|x_k - y_k\|^2,$$

$$t_k = \begin{cases} 0 & \text{if } a_k + b_k = x_k - y_k = 0 \\ \nu^2 \max \left\{ 0, \frac{\delta_k}{\sigma^2 \rho_k} - \frac{\|\lambda(a_k + w_{k-1})\|^2}{\rho_k} \right\} & \text{otherwise,} \end{cases}$$

$$z_k = z_{k-1} - (1 - t_k) \lambda(a_k + b_k), \quad w_k = w_{k-1} - (1 - t_k) \lambda^{-1}(x_k - y_k);$$

(4.3)

From now on in this section, \{z_k\}, \{w_k\}, \{a_k\} etc. are sequences generated by Algorithm II. To simplify the convergence analysis, let $s_k$ denote the residuals in the equation of the inclusion-equation system to be solved for $B$ at the $k$-th iteration, that is,

$$s_k = \lambda b_k + x_k - (y_k + \lambda w_{k-1}), \quad (k = 1, 2, \ldots).$$

With this notation, (4.1a), (4.1b) and the first line of (4.2) writes

$$a_k \in A(y_k), \quad \lambda a_k + y_k = z_{k-1} - \lambda w_{k-1},$$

$$b_k \in B(\mu_k)(x_k), \quad \lambda b_k + x_k = y_k + \lambda w_{k-1} + s_k,$$

$$\delta_k = \|s_k\|^2 + 2\lambda \mu_k, \quad \delta_k \leq \sigma^2 \rho_k.$$
It follows from the definition of \( t_k \) at (4.2) that for all \( k \),

\[
0 \leq t_k \leq \nu^2 < 1, \quad \frac{\nu^2}{\sigma^2} \delta_k - \nu^2 \| \lambda(a_k + w_{k-1}) \|^2 \leq t_k \rho_k, \quad t_k^2 \rho_k \leq \frac{\nu^4}{\sigma^2} \delta_k. \tag{4.5}
\]

From now on in this section,

\[
p_k = (z_k, \lambda w_k), \quad (k = 0, 1, 2, \ldots). \tag{4.6}
\]

We will prove that the sequence \( \{p_k\} \) converges to a point \((z, \lambda w)\) where \((z, w) \in S_e(A, B)\).

**Lemma 4.1.** For any \((z^*, w^*) \in S_e(A, B)\) and all \( k \)

\[
\|(z^*, \lambda w^*) - p_{k-1}\|^2 \geq \|(z^*, \lambda w^*) - p_k\|^2 + (1 - t_k) \left\{ \frac{\nu^2}{\sigma^2} \delta_k + (1 - \nu^2) \| \lambda(a_k + w_{k-1}) \|^2 \right\},
\]

\[
\|(z^*, \lambda w^*) - p_0\|^2 \geq \|(z^*, \lambda w^*) - p_k\|^2 + \sum_{i=1}^{k} (1 - t_i) \left\{ \frac{\nu^2}{\sigma^2} \delta_i + (1 - \nu^2) \| \lambda(a_i + w_{i-1}) \|^2 \right\}.
\]

**Proof.** Fix \((z^*, w^*) \in S_e(A, B)\) and let \( p^* = (z^*, \lambda w^*) \). Define, for \( k = 1, 2, \ldots \),

\[
z_k' = z_{k-1} - \lambda(a_k + b_k), \quad w_k' = w_{k-1} - \lambda^{-1}(x_k - y_k), \quad p_k' = (z_k', \lambda w_k'). \tag{4.7}
\]

It follows from (4.6), (4.4) and Lemma 2.3 that

\[
\|p^* - p_{k-1}\|^2 \geq \|p^* - p_k'\|^2 + \|\lambda(a_k + w_{k-1})\|^2 - \|s_k\|^2 + 2 \lambda \mu_k
\]

\[
\geq \|p^* - p_k'\|^2 + \|\lambda(a_k + w_{k-1})\|^2 - \sigma^2 \rho_k.
\]

Since \( p_k = t_k p_{k-1} + (1 - t_k)p_k' \) and \( \|p_k - p_k'\|^2 = \rho_k \),

\[
\|p^* - p_{k-1}\|^2 = t_k \|p^* - p_{k-1}\| + (1 - t_k) \|p^* - p_{k-1}\|^2
\]

\[
\geq t_k \|p^* - p_{k-1}\| + (1 - t_k) \left[ \|p^* - p_k'\|^2 + \|\lambda(a_k + w_{k-1})\|^2 - \delta_k \right]
\]

\[
= \|t_k (p^* - p_{k-1}) + (1 - t_k) (p^* - p_k')\|^2 + (1 - t_k) t_k \|p_k - p_k'\|^2
\]

\[
+ (1 - t_k) \left[ \|\lambda(a_k + w_{k-1})\|^2 - \delta_k \right]
\]

\[
= \|p^* - p_k\| + (1 - t_k) [t_k \rho_k + \|\lambda(a_k + w_{k-1})\|^2 - \delta_k].
\]

To end the proof of the first inequality, use the above inequality and the next to last inequality in (4.5).

The second inequality of the lemma follows trivially from the first one. \( \square \)

**Lemma 4.2.** If \( S_e(A, B) \) is nonempty, then

1. \( \{z_k\} \) and \( \{w_k\} \) are bounded;
2. \( s_k \to 0 \) and \( \mu_k \to 0 \) as \( k \to \infty \);
3. \( a_k + w_{k-1} \to 0 \) and \( y_k - z_{k-1} \to 0 \) as \( k \to \infty \);
4. \( b_k - w_k \to 0 \) and \( x_k - z_k \to 0 \) as \( k \to \infty \).
Proof. Take \((z^*, w^*)\) in \(S_e(A, B)\). It follows from Lemma 4.1 that \(\{(z_k, \lambda w_k)\}\) is bounded, which proves item 1 and that

\[
\sum_{k=1}^{\infty} \delta_k < \infty, \quad \sum_{k=1}^{\infty} \|a_k + w_{i-1}\|^2 < \infty.
\]

It follows from the first above inequality that \(\delta_k \to 0\) as \(k \to \infty\). This result, together with the third line of (4.4) proves item 2. Item 3 follows trivially from the second above inequality and the equality in (4.1a).

It follows from the update formulas for \(z_k\) and \(w_k\), and from (4.4) that

\[
z_k - x_k + s_k = t_k \lambda (a_k + b_k), \quad w_k - b_k + s_k = t_k (x_k - y_k).
\]

Squaring both sides of each one of these equalities and adding them we conclude that

\[
\|z_k - x_k + s_k\|^2 + \|w_k - b_k + s_k\|^2 = t_k^2 \rho_k.
\]

Since \(\delta_k \to 0\) as \(k \to \infty\), it follows from the above equations and the last inequality in (4.5) that

\[
t_k^2 \rho_k \to 0, \quad z_k - x_k + s_k \to 0, \quad w_k - b_k + s_k \to 0 \quad \text{as} \quad k \to \infty.
\]

Item 4 follows from the above result and from item 2.

**Theorem 4.3.** If \(S_e(A, B) \neq \emptyset\), then \(\{(x_k, b_k)\}, \{(y_k, -a_k)\}\) and \(\{(z_k, w_k)\}\) converge weakly to a point in this set.

**Proof.** Suppose a subsequence \(\{(z_{k_n}, w_{k_n})\}\) converges weakly to some \((z, w)\). It follows from Lemma 4.2 that

\[
x_{k_n} \xrightarrow{w} z, \quad y_{k_n-1} \xrightarrow{w} z, \quad x_{k_n} - y_{k_n-1} \to 0,
\]

\[
b_{k_n} \xrightarrow{w} w, \quad a_{k_n-1} \xrightarrow{w} -w, \quad a_{k_n-1} + b_{k_n} \to 0, \quad \text{as} \quad k \to \infty.
\]

Since \(b_{k_n} \in B[y_{k_n-1}, x_{k_n}], a_{k_n-1} \in A(y_{k_n-1})\) for \(n = 1, 2, \ldots\), it follows from Lemma 4.2 that \(w \in B(z)\) and \(-w \in A(w)\). Therefore, \((z, w) \in S_e(A, B)\).

Define, again,

\[
\Omega = \{(z, \lambda w) : (z, w) \in S_e(A, B)\}.
\]

We have proved that all weak limit points of \(\{(z_k, w_k)\}\) are in \(S_e(A, B)\). Therefore, all weak limit points of \(\{p_k = (z_k, \lambda w_k)\}\) are in \(\Omega\). In Lemma 4.1 we proved that \(\{p_k\}\) is Fejér convergent to \(\Omega\). Since \(\Omega \neq \emptyset\), it follows from these results and from Opial’s Lemma 18 that the bounded sequence \(\{p_k\}\) has a unique weak limit point and such a point belongs to \(\Omega\). Therefore, \(\{p_k\}\) converges weakly to a point in \((z, \lambda w)\) where \((z, w) \in S_e(A, B)\). To end the proof, use items 3 and 1 of Lemma 4.2.
A Computability of step (2)

Proposition A.1. Suppose that $T = A$ and $T = B$ have the following properties

1. for any $v, z \in H$, on can verify whether $v \in T(z)$ or $v \notin T(z)$;
2. for any $c \in H$, on can generate sequences $v_i, z_i, \eta_i$ such that $v_i \in T[\eta_i](z_i)$ for all $i$ and

$$\eta_i \to 0, \|\lambda v_i + y_i - c\| \to 0 \quad \text{as} \quad i \to \infty.$$  \hspace{1cm} (A.1)

Then, step (1) of Algorithms I and II are computable.

Proof. It suffices to consider one iteration of Algorithm I. Assume that we are at iteration $k$ of Algorithm I. If $-w_{k-1} \in A(z_{k-1})$ and $w_{k-1} \in B(z_{k-1})$ then

$$(a_k, y_k, \varepsilon_k) = (-w_{k-1}, z_{k-1}, 0), \quad (b_k, x_k, \mu_k) = (w_{k-1}, z_{k-1}, 0)$$

trivially satisfies criterion (3.1).

Suppose $-w_{k-1} \notin A(z_{k-1})$ or $w_{k-1} \notin B(z_{k-1})$ and let

$$\hat{y} = (I + \lambda A)^{-1}(z_{k-1} - \lambda w_{k-1}), \quad \hat{a} = \lambda^{-1}(z_{k-1} - \lambda w_{k-1} - \hat{y}),$$

$$\hat{x} = (I + \lambda B)^{-1}(\hat{y} + \lambda w_{k-1}), \quad \hat{b} = \lambda^{-1}(\hat{y} + \lambda w_{k-1} - \hat{x}).$$

It follows from these definitions that $\hat{a} \in A(\hat{y}), \hat{b} \in B(\hat{x})$, and

$$\lambda \hat{a} + \hat{y} = z_{k-1} - \lambda w_{k-1}, \quad \lambda \hat{b} + \hat{x} = \hat{y} + \lambda w_{k-1}.$$ 

If $\hat{a} + \hat{b} = \hat{x} - \hat{y} = 0$, then it follows from the above equalities that $w_{k-1} = \hat{b} = -\hat{a}, z_{k-1} = \hat{x} = \hat{y}$ and, consequently, $-w_{k-1} \in A(z_{k-1})$ and $w_{k-1} \in B(z_{k-1})$, in contradiction with our assumption. Therefore, $\hat{a} + \hat{b} \neq 0$ or $\hat{x} - \hat{y} \neq 0$.

In view of the assumption of the proposition, one can generate sequences $\{(a_{k,j}, y_j, \varepsilon_{k,j})\}_{j \in \mathbb{N}}$ and $\{(b_{k,j}, x_{k,j}, \mu_{k,j})\}$ such that

$$a_{k,j} \in A(\varepsilon_{k,j})(y_{k,j}), \varepsilon_{k,j} \leq \frac{1}{2\lambda j^2}, \|\lambda a_{k,j} + y_{k,j} - (z_{k-1} + \lambda w_{k-1})\| \leq \frac{1}{j}$$

$$b_{k,j} \in B(\mu_{k,j})(x_{k,j}), \mu_{k,j} \leq \frac{1}{2\lambda j^2}, \|\lambda b_{k,j} + x_{k,j} - (y_{k-1} + \lambda w_{k-1})\| \leq \frac{1}{j}$$

It follows from the above relations and from Proposition 2.2 that

$$\|\lambda(a_{k,j} - \hat{a})\|^2 + \|y_{k,j} - \hat{y}\|^2 \leq \frac{2}{j^2}. $$

In particular $\|y_{k,j} - \hat{y}\| \leq 2/j$ and

$$\|\lambda b_{k,j} + x_{k,j} - (\hat{y} + \lambda w_{k-1})\| \leq \frac{3}{j}. $$

Using again Proposition 2.2 we conclude that

$$\|b_{k,j} - \hat{b}\|^2 + \|x_{k,j} - \hat{x}\| \leq \frac{10}{j^2}. $$

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Therefore,
\[ y_{k,j} \to \hat{y}, \quad a_{k,j} \to \hat{a}, \]
\[ x_{k,j} \to \hat{x}, \quad b_{k,j} \to \hat{b} \quad \text{as} \quad j \to \infty. \]

and \[ \| \lambda(a_{k,j} - b_{k,j}) \|^2 + \| x_{k,j} - y_{k,j} \|^2 \to \| \lambda(\hat{a} + \hat{b}) \|^2 + \| \hat{x} - \hat{y} \|^2 > 0 \quad \text{as} \quad j \to \infty. \]

Since \[ \| \lambda a_{k,j} + y_{k,j} - (z_{k-1} + \lambda w_{k-1}) \|^2 + \| \lambda b_{k,j} + x_{k,j} - (y_{k-1} + \lambda w_{k-1}) \|^2 + 2\lambda(\varepsilon_{k,j} + \mu_{k,j}) \leq \frac{4}{j^2}, \]

for \( j \) large enough criterion (3.1) will be satisfied.

\[ \Box \]

\section*{B A technical Lemma}

Let \( X \) be a real Banach space with topological dual \( X^\ast \) and let \( \langle x, x^\ast \rangle \) stands for the duality product \( x^\ast(x) \) for \( x \in X \) and \( x^\ast \in X^\ast \). A point-to-set operator \( T : X \rightrightarrows X^\ast \) is monotone if
\[ \langle x - y, x^\ast - y^\ast \rangle \geq 0 \quad \forall x, y \in X, x^\ast \in T(x), y^\ast \in T(y). \]

A monotone operator \( T \) is maximal monotone if it is monotone and its graph is maximal in the family of the graphs of monotone operators.

The \( \varepsilon \)-enlargement of a maximal monotone operator \( T \) in \( X \) is defined as
\[ T[\varepsilon](x) = \{ x^\ast \in X^\ast : \langle x - y, x^\ast - y^\ast \rangle \geq -\varepsilon \} \quad \varepsilon \geq 0, \ x \in X. \]

\textit{Fitzpatrick} \cite{9} function \( \varphi \) associated with a maximal monotone operator \( T : X \rightrightarrows X^\ast \) is defined as
\[ \varphi : X \times X^\ast \to \mathbb{R}, \quad \varphi(x, x^\ast) = \sup_{y^\ast \in T(y)} \langle x, y^\ast \rangle + \langle y, x^\ast \rangle - \langle y, y^\ast \rangle. \quad (B.1) \]

\textbf{Lemma B.1.} Suppose \( T \) is a maximal monotone operator in \( X \) ad let \( \varphi \) be its Fitzpatrick function. Then

\begin{enumerate}
  \item \( \varphi \) is convex and lower semicontinuous in the weak×weak-* topology of \( X \times X^\ast \);
  \item \( \langle x, v \rangle \leq \varphi(x, v) \);
  \item \( v \in T(x) \iff \langle x, v \rangle = \varphi(x, v) \);
  \item \( v \in T[\varepsilon](x) \iff \varphi(x, v) \leq \langle x, v \rangle + \varepsilon \);
\end{enumerate}

\textbf{Proof.} Items \cite{1} and \cite{2} where proved in \cite{9}.

Item \cite{3} was proved in \cite{24}. For the sake of completeness we present a proof. If follows from (B.1) that
\[ \varphi(x, x^\ast) - \langle x, x^\ast \rangle = \sup_{y^\ast \in T(y)} \langle x - y, y^\ast - x^\ast \rangle. \]

To end the proof, use (2.3) to write
\[ x^\ast \in T[\varepsilon](x) \iff \varepsilon \geq \langle x - y, y^\ast - x^\ast \rangle \quad \forall y \in X, \ y^\ast \in T(y) \]

and combine the two above equations. \( \Box \)
The next technical lemma will be used in the convergence analysis of the inexact Douglas-Rachford method proposed in this work.

**Lemma B.2.** Let $X$ be a real Banach space. If $T_1, \ldots , T_m : X \rightrightarrows X^*$ are maximal monotone operators, $v_{i,k} \in T^{[\varepsilon_{i,k}]}(x_{i,k})$ for $i = 1, \ldots , m$ and $k = 1, 2, \ldots , m$, and

$$
x_{i,k} - x_{j,k} \to 0 \quad (\text{for } i, j = 1, \ldots , m), \quad \sum_{i=1}^{m} v_{i,k} \to v,
$$

$$
x_{i,k} \rightharpoonup x, \quad v_{i,k} \rightharpoonup^{\ast} v_i, \quad \varepsilon_{i,k} \to 0 \quad (\text{for } i = 1, \ldots , m),
$$

as $k \to \infty$, then $v_i \in T_i(x)$ for $i = 1, \ldots , m$.

**Proof.** Let $\varphi_i$ be Fitzpatrick’s function of $T_i$ for $i = 1, \ldots , m$. Since $v_{i,k} \in T^{[\varepsilon_{i,k}]}(x_{i,k})$,

$$
\varphi_i(x_{i,k}, v_{i,k}) \leq \langle x_{i,k}, v_{i,k} \rangle + \varepsilon_{i,k} \quad i = 1, \ldots , m.
$$

Adding these inequalities for $i = 1, \ldots , m$ we obtain, after trivial algebraic manipulations the inequality

$$
\sum_{i=1}^{m} \varphi_i(x_{i,k}, v_{i,k}) \leq \langle x_{1,k}, v \rangle + \left( \sum_{i=1}^{m} v_{i,k} - v \right) + \sum_{i=1}^{m} \langle x_{i,k} - x_{1,k}, v_{i,k} \rangle + \varepsilon_{i,k}.
$$

Each $\varphi_i$ is lower semicontinuous in the weak×weak-∗ topology of $X \times X^*$. Moreover, it follows trivially from the assumptions of the lemma that the sequences $\{x_{i,k}\}$ and $\{v_{i,k}\}$ are bounded for $i = 1, \ldots , m$. Therefore, taking the lim inf as $k \to \infty$ at both sides of the above inequality and using the lower semicontinuity of $\varphi$ we conclude that

$$
\sum_{i=1}^{m} \varphi_i(x, v_i) \leq \langle x, v \rangle.
$$

It also follows trivially from the assumptions of the lemma that $\sum_{i=1}^{m} v_i = v$. So, we can write this above inequality as

$$
0 \geq \sum_{i=1}^{m} \varphi_i(x, v_i) - \langle x, v_i \rangle.
$$

Since all terms of this sum are non-negative, each one is equal to 0 and the conclusion follows from item 3 of Lemma B.1.

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