Attribute reduction and rule acquisition of formal decision context based on two new kinds of decision rules

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Abstract. This paper mainly studies the rule acquisition and attribute reduction for formal decision context based on two new kinds of decision rules, namely I-decision rules and II-decision rules. The premises of these rules are object-oriented concepts, and the conclusions are formal concept and property-oriented concept respectively. The rule acquisition algorithms for I-decision rules and II-decision rules are presented. Some comparative analysis of these algorithms with the existing algorithms are examined which shows that the algorithms presented in this study behave well. The attribute reduction approaches to preserve I-decision rules and II-decision rules are presented by using discernibility matrix.

Keywords: Formal concept analysis, object-oriented and property-oriented concept lattice, rule acquisition, attribute reduction

1. Introduction

German scholar Wille put forward Formal concept analysis (FCA) in 1982[1]. FCA is a mathematical theory for qualitative analysis of relation data between object and attribute that uses a formal context.
as input to identify a set of formal concepts formed in a concept lattice. A formal context is a binary relation between objects set and attributes set to specify which object possess what attribute. A formal concept consists of two parts (an extent and an intent) by two derivation operators. The extent of a formal concept is an objects subset that are instances of the concept, while the intent is the subset of attributes possessed by the objects. Therefore, formal concepts are the mathematization of philosophical concepts. As a practical tool for knowledge discovery, FCA has been successfully used in several areas, for instance data mining, information retrieval, social network analysis and machine learning [2][3][4][5][6]. In addition, some natural generalizations of derivation operators were proposed which induces some notions, for example, object-oriented concepts, property-oriented concepts, formal fuzzy concepts and three-way concepts [7][8][9][10][11].

Attribute reduction for formal context plays an essential part in FCA. By attribute reduction, more compact knowledge can be discovered and the computational complexity for constructing concept lattices can be reduced. In general, an attribute reduction is a minimal attributes subset which preserves some specific properties of formal context. There are mainly two problems involved in attribute reduction: the criterion of reduction with semantic interpretation and reduction computing method. For formal context, there are two typical criteria of attribute reduction: (1) To preserve the extents set of all formal concepts calculated from the formal context [12][13][14]. In this case, the concept lattice induced from the reduced context and the one derived from the initial context are isomorphic. (2) To preserve the extents set of all object concepts. This kind of attribute reduction is also called granular reduction [15]. In order to compute attribute reductions, CR (clarification and reduction) method [12][13] and DM (discernibility matrix) method [14] were proposed. CR method is established by using meet-irreducible elements in formal concept lattice, whereas DM method is based on discernibility attributes between related formal concepts. These two reduction methods have been extensively studied and applied to attribute reductions for various kinds of concept lattices [16][17][18][19][20][21][22][23][24].

A formal decision context (Fdc) is a formal context in which the attributes are consisted of conditional attributes and decision attributes [25][26]. The knowledge associated with a formal decision context is usually expressed as decision rules to revealing the dependency between conditional and decision attributes. A decision rule is an implication in which the premise and conclusion are concepts of conditional context and decision context respectively. The criteria of attribute reduction for formal decision contexts can be roughly categorized into two groups: to preserve a kind of consistency [13][27][28][29], and to preserve a specific kind of decision rules [30][31][32][33][34] of Fdcs. Qin et al [35] made a comparative research on attribute reduction of formal context and Fdc under the framework of local reduction.

We note that the existing approaches on rule acquisition and attribute reduction pay more attention to decision rules generated by formal concepts and few work has been completed on other types for decision rules. Theoretically speaking, decision rules can be designed by using formal concepts, object-oriented concepts or property-oriented concepts. A specific type of decision rules provide a particular kind of decision knowledge. In this study, we further investigate attribute reduction methods and rule acquisition methods for Fdcs based on two kinds of decision rules, namely I-decision rule and II-decision rule. This paper is structured as follows. In Section 2, the basic definitions of FCA such as formal concept, property-oriented concept and object-oriented concept are concisely recalled. In Section 3, we propose algorithms for I-decision rule acquisition and make some comparative analysis
with the existing algorithms presented in [33]. In addition, we present attribute reduction method for Fdc to preserve I-decision rules. In Section 4, the algorithms for II-decision rule acquisition are presented. We analyze the relationships between I-decision rules and II-decision rules, and accordingly, attribute reduction method to preserve II-decision rules is examined. Section 5 concludes.

2. Preliminaries

In this section, some related notions of FCA are introduced to make this paper self-contained. Please refer to [1,7,8] for details.

2.1. Formal context and concept lattice

The input object-attribute relational data are described by a formal context in FCA.

Definition 2.1. [1] A formal context \((U, M, I)\) constitutes by two sets \(U\) and \(M\), and a binary relation \(I \subseteq U \times M\), where \(U\) (objects set) and \(M\) (attributes set) are both finite nonempty sets. For \(x \in U\) and \(a \in M\), \((x, a) \in I\) indicates that the object \(x\) possesses the attribute \(a\).

In a formal context \(\mathcal{C} = (U, M, I)\), Wille [1] defined two concept forming operators \(\uparrow\) and \(\downarrow\) as follows: for \(O \subseteq U, C \subseteq M\),

\[
O^\uparrow = \{ a \in M | \forall x \in O((x, a) \in I) \} \quad (1)
\]

\[
C^\downarrow = \{ x \in U | \forall a \in C((x, a) \in I) \} \quad (2)
\]

That is to say, \(O^\uparrow\) is the maximal attributes set had by all objects in \(O\), and \(C^\downarrow\) is the maximal set of objects that possess all attributes in \(C\). A formal concept generated by \(\mathcal{C}\) is a pair \((O, C)\) with two sets \(O \subseteq U\) and \(C \subseteq M\) such that \(O^\uparrow = C\) and \(C^\downarrow = O\), where \(O\) and \(C\) are regarded as the extent and intent of \((O, C)\) respectively. Denote the family of all formal concepts of \(\mathcal{C}\) by \(L(\mathcal{C})\). \((L(\mathcal{C}), \leq)\) constitutes a complete lattice [1], referred to as the concept lattice of \(\mathcal{C}\), where the order relation \(\leq\) is given by:

\[
(O_i, C_i) \leq (O_j, C_j) \iff O_i \subseteq O_j \wedge C_j \subseteq C_i
\]

for any \((O_i, C_i), (O_j, C_j) \in L(\mathcal{C})\). In addition, the infimum and supremum of \((L(\mathcal{C}), \leq)\) are defined as follow:

\[
\wedge_{q \in Q}(O_q, C_q) = (\cap_{q \in Q} O_q, (\cup_{q \in Q} C_q)^\uparrow) \quad (3)
\]

\[
\vee_{q \in Q}(O_q, C_q) = ((\cup_{q \in Q} O_q)^\downarrow, \cap_{q \in Q} C_q) \quad (4)
\]

where \(Q\) is an index set and \(\{(O_q, C_q) | q \in Q\} \subseteq L(\mathcal{C})\). For \(\forall x \in U\) and \(\forall a \in M\), we write \({\{x\}}^\uparrow\) and \({\{a\}}^\downarrow\) simply as \(x^\uparrow\) and \(a^\downarrow\) respectively. In addition \(\forall O \subseteq U\) and \(\forall A \subseteq M\), \((O^\uparrow, O^\downarrow)\) and \((C^\downarrow, C^\uparrow)\) are both formal concepts. In what follows, \((O^\uparrow, O^\downarrow)\) and \((C^\downarrow, C^\uparrow)\) are referred to as the formal concepts generated by \(O\) and \(C\) respectively. Customarily, the formal contexts are all assumed to be canonical [14] in the following discussion, i.e., \(\forall x \in U\) and \(\forall a \in M\) there has \(x^\uparrow \neq \emptyset\), \(x^\uparrow \neq M\), \(a^\downarrow \neq \emptyset\) and \(a^\downarrow \neq U\).
2.2. Property (Object) oriented concept lattice

FCA and rough set theory [36] are two efficaciously and closely connected mathematical tools for dealing with data. Over the years, much scholars have been trying to contrast and combine these two theories [7 8 9]. For a formal context \( \mathcal{C} = (U, M, I) \), based on rough approximation operators, Duntsch and Gediga [7] presented a pair of operators \( \Diamond : P(U) \to P(M) \) and \( \Box : P(M) \to P(U) \) as below: for any \( O \subseteq U, C \subseteq M \)

\[
O^\Diamond = \{ a \in M \mid \exists x \in O((x, a) \in I) \}
\]

(5)

\[
C^\Box = \{ x \in U \mid \forall a \in M((x, a) \in I \to a \in C) \}
\]

(6)

These operators are used to construct property-oriented concepts [7]. Similarly, Yao [8 9] considered a pair of operators \( \Box : P(U) \to P(M) \) and \( \Diamond : P(M) \to P(U) \):

\[
O^\Box = \{ a \in M \mid \forall x \in U((x, a) \in I \to x \in O) \}
\]

(7)

\[
C^\Diamond = \{ x \in U; \exists a \in C((x, a) \in I) \}
\]

(8)

where \( O \subseteq U \) and \( C \subseteq M \). Modal-style approximate operators and \( \uparrow, \downarrow \) are closely related. Obviously, we know \( O^\Diamond = \{ a \in M \mid a^\downarrow \cap O \neq \emptyset \} \), \( O^\Box = \{ a \in M \mid a^\uparrow \subseteq O \} \), \( C^\Diamond = \{ x \in U \mid x^\uparrow \cap C \neq \emptyset \} \) and \( C^\Box = \{ x \in U \mid x^\downarrow \subseteq C \} \). In addition, for any \( O_i, O_j, O_k \subseteq U \) and \( C_i, C_j, C_k \subseteq M \), the following properties hold:

1. \( O_j \subseteq O_k \Rightarrow O_j^\Diamond \subseteq O_k^\Diamond, O_j^\Box \subseteq O_k^\Box \);
2. \( C_j \subseteq C_k \Rightarrow C_j^\Diamond \subseteq C_k^\Diamond, C_j^\Box \subseteq C_k^\Box \);
3. \( O_i^\Diamond \subseteq O_i \subseteq O_i^\Box, C_i^\Diamond \subseteq C_i \subseteq C_i^\Box \);
4. \( O_i^\Diamond \cap O_i^\Box = O_i^\Diamond, O_i \uparrow \cap C_i = C_i \uparrow \cap O_i^\Box = C_i^\Box \);
5. \( (O_j \cup O_k)^\Diamond = O_j^\Diamond \cup O_k^\Diamond, (O_j \cap O_k)^\Box = O_j^\Box \cap O_k^\Box \);
6. \( (C_j \cup C_k)^\Diamond = C_j^\Diamond \cup C_k^\Diamond, (C_j \cap C_k)^\Box = C_j^\Box \cap C_k^\Box \).

We call a pair \((O, C)\) with \( O \subseteq U \) and \( C \subseteq M \) a property-oriented concept [7] of \( \mathcal{C} \) if \( O^\Diamond = C \) and \( C^\Box = O \). Let \( L_P(\mathcal{C}) = \{(O, C) \mid O \subseteq U, C \subseteq M, O^\Diamond = C, C^\Box = O \} \) be the family of all property-oriented concepts of \( \mathcal{C} \). \( (L_P(\mathcal{C}), \leq) \) is a complete lattice [7], denoted as the property-oriented concept lattice of \( \mathcal{C} \) with the order relation \( \leq \) is given by:

\[
(O_i, C_i) \leq (O_j, C_j) \iff O_i \subseteq O_j (\iff C_i \subseteq C_j)
\]

for any \((O_i, C_i), (O_j, C_j) \in L_P(\mathcal{C})\). The infimum and supremum of \((L_P(\mathcal{C}), \leq)\) are defined as follows:

\[
\land_{q \in Q}(O_q, C_q) = (\bigcap_{q \in Q} O_q, (\cap_{q \in Q} C_q)^\Diamond)
\]

(9)

\[
\lor_{q \in Q}(O_q, C_q) = (\bigcup_{q \in Q} O_q, \cup_{q \in Q} C_q)^\Box
\]

(10)
∀O ⊆ U and ∀C ⊆ M, \((O^{\Diamond}, O^{\Diamond})\) and \((C^{\square}, C^{\square})\) are called the property-oriented concepts derived from \(O\) and \(C\) separately.

Analogously, we call a pair \((O, C)\) with an objects subset \(O\) of and a attributes subset \(C\) an object-oriented concept [8] of \(\mathcal{C}\) if \(O^{\square} = C\) and \(C^{\Diamond} = O\). \(L_O(\mathcal{C})\) is referred as the family of all object-oriented concepts. \((L_O(\mathcal{C}), \leq)\) is a complete lattice, where the order relation is given by \((Y_i, D_i) \leq (Y_j, D_j) \iff Y_i \subseteq Y_j (\leftrightarrow D_i \subseteq D_j)\) and is called the object-oriented concept lattice of \(\mathcal{C}\). The meet and join of \((L_O(\mathcal{C}), \leq)\) are given by [8]:

\[
\begin{align*}
\bigwedge_{q \in Q}(O_q, C_q) &= (\bigcap_{q \in Q}O_q)^{\Diamond}, (\bigcap_{q \in Q}C_q) \\
\bigvee_{q \in Q}(O_q, C_q) &= (\bigcup_{q \in Q}O_q, (\bigcup_{q \in Q}C_q)^{\Diamond})
\end{align*}
\]

Obviously, \(\forall O \subseteq U\) and \(\forall C \subseteq M\), \((O^{\Diamond}, O^{\Diamond})\) and \((C^{\square}, C^{\square})\) are object-oriented concepts. They are said to be the object-oriented concepts derived from \(O\) and \(C\) separately.

### 3. I-decision rules acquisition and related attribute reduction

A formal decision context (Fdc) is a formal context in which the attributes are consisted of conditional attributes and decision attributes.

**Definition 3.1.** [25][26] A Fdc is a quintuple \(\mathcal{C} = (U, M, I, N, J)\) with two formal contexts \((U, M, I)\) and \((U, N, J)\), \(M\) and \(N\) are regarded as the sets of conditional attributes and decision attributes respectively with \(M \cap N = \emptyset\).

In addition for a Fdc \(\mathcal{C} = (U, M, I, N, J)\), \((U, M, I)\) and \((U, N, J)\) are called conditional context and decision context of \(\mathcal{C}\) and denoted by \(\mathcal{C}_M = (U, M, I)\) and \(\mathcal{C}_N = (U, N, J)\) respectively. In order to distinguish, these operators given by (1), (2), (5), (6), (7) and (8) for \(\mathcal{C}_M\) will be rewrited as \(\uparrow_M, \downarrow_M, \Diamond_M\) and \(\Box_M\), whereas these operators for \(\mathcal{C}_N\) will be denoted by \(\uparrow_N, \downarrow_N, \Diamond_N\) and \(\Box_N\) respectively.

For Fdc \(\mathcal{C}\), we are interested in revealing the dependency relationships between conditional and decision attributes. It is usually expressed as an implication with the form \((O, C) \rightarrow (Z, D)\) and called decision rule, where \((O, C)\) and \((Z, D)\) are concepts from \(\mathcal{C}_M\) and \(\mathcal{C}_N\) respectively. The rule acquisition and attribute reduction methods with respect to several kinds of decision rules have been extensively investigated, for example:

1. \((O, C) \rightarrow (Z, D): (O, C) \in L(\mathcal{C}_M), (Z, D) \in L(\mathcal{C}_N), O \subseteq Z\) and \(O, C, Z, D\) are non-empty [29][30][31][32];
2. \((O, C) \rightarrow (Z, D): (O, C) \in L_O(\mathcal{C}_M), (Z, D) \in L_O(\mathcal{C}_N), O \subseteq Z\) and \(O \neq \emptyset\) and \(Z \neq U\) [34];
3. \((O, C) \rightarrow (Z, D): (O, C) \in L_P(\mathcal{C}_M), (Z, D) \in L_P(\mathcal{C}_N), O \subseteq Z\) and \(O \neq \emptyset\) and \(Z \neq U\) [34];
4. \((O, C) \rightarrow (Z, D): (O, C) \in L_P(\mathcal{C}_M), (Z, D) \in L(\mathcal{C}_N), O \subseteq Z\) and \(O \neq \emptyset\) and \(Z \neq U\) [33];
5. \((O, C) \rightarrow (Z, D): (O, C) \in L_O(\mathcal{C}_M), (Z, D) \in L(\mathcal{C}_N), O \subseteq Z\) and \(O \neq \emptyset\) and \(Z \neq U\) [33].

These decision rules are mutually different and present various kinds of decision information. Ren et al. [33] proposed some rule acquisition algorithms for the decision rule (5). In this part, we further research the rule acquisition and attribute reduction methods for this kind of decision rules. We propose new rule acquisition methods and make some comparative study on the rule acquisition algorithms presented in [33] and the rule acquisition algorithms presented in this paper. Furthermore, we present related attribute reduction methods.
3.1. Rule acquisition methods for I-decision rules

In this subsection, we assume that $\mathcal{C} = (U, M, I, N, J)$ is a Fdc, $\mathcal{C}_M = (U, M, I)$ and $\mathcal{C}_N = (U, N, J)$ are the conditional context and decision context of $\mathcal{C}$ respectively. The notion of I-decision rules is proposed by Ren et al. [33]. Here we make some modifications on technical terms to fit for this study.

**Definition 3.2.** [33] Assume that $(O, C) \in L_O(\mathcal{C}_M), (Y, D) \in L(\mathcal{C}_N)$. If $O \subseteq Y$, $O \neq \emptyset$ and $Y \neq U$, then $(O, C) \rightarrow (Y, D)$ is said to be a I-decision rule of $\mathcal{C}$, $(O, C)$ and $(Y, D)$ are the premise and conclusion of $(O, C) \rightarrow (Y, D)$ respectively.

The semantics of I-decision rule $(O, C) \rightarrow (Y, D)$ can be interpreted as follows. By $C^{O,M} = O \subseteq Y = D^{\downarrow M}$, we know that if an object $x \in U$ has at least one conditional attribute of $C$, then $x \in C^{O,M} \subseteq D^{\downarrow M}$ and hence $x$ possess all decision attributes in $D$. In the following, we define $\mathcal{RI}(\mathcal{C})$ as the set of all I-decision rules of $\mathcal{C}$.

**Definition 3.3.** Let $(O_1, C_1) \rightarrow (Y_1, D_1) \in \mathcal{RI}(\mathcal{C}), (O_2, C_2) \rightarrow (Y_2, D_2) \in \mathcal{RI}(\mathcal{C})$. If $O_2 \subseteq O_1 \subseteq Y_1 \subseteq Y_2$, then we say that $(O_2, C_2) \rightarrow (Y_2, D_2)$ can be implied by $(O_1, C_1) \rightarrow (Y_1, D_1)$ and denoted by $(O_1, C_1) \rightarrow (Y_1, D_1) \Rightarrow (O_2, C_2) \rightarrow (Y_2, D_2)$.

Assume that $(O_1, C_1) \rightarrow (Y_1, D_1) \Rightarrow (O_2, C_2) \rightarrow (Y_2, D_2)$. By $O_2 \subseteq O_1 \subseteq Y_1 \subseteq Y_2$, it follows that $C_2 = O_2^{\downarrow M} \subseteq C_1$ and $D_2 = Y_2^{\uparrow N} \subseteq Y_1^{\uparrow N} = D_1$. If an object $x$ possesses at least one conditional attribute of $C_2$, then $x$ possesses at least one conditional attribute of $C_1$ and hence it has all decision attributes in $D_1$ by $(O_1, C_1) \rightarrow (Y_1, D_1) \in \mathcal{RI}(\mathcal{C})$. Consequently, $x$ possess all decision attributes in $D_2$ by $D_2 \subseteq D_1$. We conclude that the decision information associated with $(O_2, C_2) \rightarrow (Y_2, D_2)$ can be inferred from that associated with $(O_1, C_1) \rightarrow (Y_1, D_1)$.

For $(O, C) \rightarrow (Y, D) \in \mathcal{RI}(\mathcal{C})$, if there exists $(O_1, C_1) \rightarrow (Y_1, D_1) \in \mathcal{RI}(\mathcal{C})$ such that $(O_1, C_1) \rightarrow (Y_1, D_1) \Rightarrow (O, C) \rightarrow (Y, D)$ and $(O_1, C_1) \rightarrow (Y_1, D_1) \neq (O, C) \rightarrow (Y, D)$ (i.e., $(O_1, C_1) \neq (O, C)$ or $(Y_1, D_1) \neq (Y, D)$), then we call $(O, C) \rightarrow (Y, D)$ is redundant in $\mathcal{RI}(\mathcal{C})$. Otherwise, $(O, C) \rightarrow (Y, D)$ is called a necessary I-decision rule of $\mathcal{C}$. Clearly, necessary rules are more significant than redundant rules. We regard $\overline{\mathcal{RI}(\mathcal{C})}$ as the set of all necessary I-decision rules.

**Theorem 3.4.** (1) $(\mathcal{RI}(\mathcal{C}), \Rightarrow)$ is a partially ordered set, i.e., rule implication relation $\Rightarrow$ satisfies:

a) Reflexivity: $r \Rightarrow r$ for each $r \in \mathcal{RI}(\mathcal{C})$;

b) Anti-symmetry: $r_1 \Rightarrow r_2$ and $r_2 \Rightarrow r_1$ imply $r_1 = r_2$ for any $r_1, r_2 \in \mathcal{RI}(\mathcal{C})$;

c) Transitivity: $r_1 \Rightarrow r_2$ and $r_2 \Rightarrow r_3$ imply $r_1 \Rightarrow r_3$ for any $r_1, r_2, r_3 \in \mathcal{RI}(\mathcal{C})$.

(2) $r \in \overline{\mathcal{RI}(\mathcal{C})}$ iff $r' \Rightarrow r$ implies $r'' \Rightarrow r$ for any $r' \in \mathcal{RI}(\mathcal{C})$ in the sense that $r$ is a minimal element of $(\mathcal{RI}(\mathcal{C}), \Rightarrow)$.

(3) If $O \in Ext LO(\mathcal{C}_M) \cap Ext LN(\mathcal{C}_N), O \neq \emptyset$ and $O \neq U$, then $(O, O^{\downarrow M}) \in L_O(\mathcal{C}_M)$, $(O, O^{\uparrow N}) \in L(\mathcal{C}_N)$ and $(O, O^{\downarrow M}) \rightarrow (O, O^{\uparrow N})$ is a necessary I-decision rule.

The proof of this Theorem is simple and obvious. We now study the method of necessary I-decision rule acquisition. Intuitively speaking, a decision rule $(O, C) \rightarrow (Y, D) \in \mathcal{RI}(\mathcal{C})$ is necessary if, under the condition of $O \subseteq Y$, $O$ is as large as possible and $Y$ is as small as possible. If $(O, C)$ is given, since $O \subseteq O^{\downarrow N \downarrow N} \subseteq Y^{\downarrow N \downarrow N} = Y$, then we have $O^{\downarrow N \downarrow N}$ is the smallest $Y$ such that $O \subseteq Y$.
and \( Y \in ExtL(\mathcal{C}_N) \). We note that different extents in \( ExtL_O(\mathcal{C}_M) \) may generate same formal concepts in \( L(\mathcal{C}_N) \). Therefore, the object-oriented concepts in \( L_O(\mathcal{C}_M) \) need to be classified. Let \( R_1 \) be a binary relation on \( ExtL_O(\mathcal{C}_M) = \{O \subseteq U \mid \exists C \subseteq M((O, C) \in L_O(\mathcal{C}_M))\} \) given by:

\[
R_1 = \{(O, Y) \in ExtL_O(\mathcal{C}_M) \times ExtL_O(\mathcal{C}_M) | O^{\uparrow N} = Y^{\uparrow N}\}
\]

(13)

In other words, \((O, Y) \in R_1\) is equivalent to \( O \) and \( Y \) are extents in \( ExtL_O(\mathcal{C}_M) \) and they generate same formal concepts in \( L(\mathcal{C}_N) \). Clearly, \( R_1 \) is an equivalence relation on \( ExtL_O(\mathcal{C}_M) \). In the following, we denote by \([O]_{R_1}\) the equivalence class based on \( R_1 \) for \( O \in ExtL_O(\mathcal{C}_M) \). The following theorem presents an approach to derive necessary I-decision rules.

**Theorem 3.5.** For a Fdc \( \mathcal{C} = (U, M, I, N, J) \), we have

\[
\mathcal{R}_I(\mathcal{C}) = \{([O]_{R_1}, ([O]_{R_1})_{\square M}) \to (O^{\uparrow N \downarrow N}, O^{\uparrow N}) | O \in ExtL_O(\mathcal{C}_M), O \neq \emptyset, O^{\uparrow N \downarrow N} \neq U\}
\]

(14)

**Proof:**

(1) Let \( H = \{([O]_{R_1}, ([O]_{R_1})_{\square M}) \to (O^{\uparrow N \downarrow N}, O^{\uparrow N}) | O \in ExtL_O(\mathcal{C}_M), O \neq \emptyset, O^{\uparrow N \downarrow N} \neq U\} \). We firstly prove that \( H \subseteq \mathcal{R}_I(\mathcal{C}) \), i.e., \([([O]_{R_1}, ([O]_{R_1})_{\square M}) \to (O^{\uparrow N \downarrow N}, O^{\uparrow N}) \) is a necessary I-decision rule for any \( O \in ExtL_O(\mathcal{C}_M) \) with \( O \neq \emptyset \) and \( O^{\uparrow N \downarrow N} \neq U \). In fact, For any \( Z \in [O]_{R_1} \), it follows that \((Z, Z_{\square M}) \in L_O(\mathcal{C}_M) \). According to formula (12), the supremum of \{\((Z, Z_{\square M})|Z \in [O]_{R_1}\}\} in \( L_O(\mathcal{C}_M) \) is given by:

\[
\bigvee_{Z \in [O]_{R_1}} (Z, Z_{\square M}) = ([O]_{R_1}, ([O]_{R_1})_{\square M}) \Rightarrow (O^{\uparrow N \downarrow N}, O^{\uparrow N})
\]

Consequently, \([O]_{R_1} \in ExtL_O(\mathcal{C}_M) \). Additionally, \([([O]_{R_1})^{\uparrow N} = \bigcap Z \in [O]_{R_1} Z^{\uparrow N} = O^{\uparrow N} \) and therefore \([O]_{R_1} \in [O]_{R_1} \) is the maximum element in \([O]_{R_1}\). We have \([O]_{R_1} \neq \emptyset \) from \( O \neq \emptyset \). By combining that facts \([([O]_{R_1})^{\uparrow N} = O^{\uparrow N \downarrow N} \), \( O^{\uparrow N \downarrow N} \neq U \) and \([O]_{R_1} \neq \emptyset \), we conclude \((([O]_{R_1}, ([O]_{R_1})_{\square M}) \to (O^{\uparrow N \downarrow N}, O^{\uparrow N}) \) is indeed a I-decision rule.

Suppose that \((O_1, C_1) \to (Y_1, D_1) \Rightarrow ([O]_{R_1}, ([O]_{R_1})_{\square M}) \to (O^{\uparrow N \downarrow N}, O^{\uparrow N}) \) where \((O_1, C_1) \to (Y_1, D_1) \in \mathcal{R}_I(\mathcal{C}) \). It follows that \((O_1, C_1) \in L_O(\mathcal{C}_M), (Y_1, D_1) \in L(\mathcal{C}_N) \) and \([O]_{R_1} \subseteq O_1 \subseteq Y_1 \subseteq O^{\uparrow N \downarrow N} \). By \( O \subseteq \bigcup [O]_{R_1} \subseteq Y_1 \subseteq O^{\uparrow N \downarrow N} \), it can be known \( O^{\uparrow N \downarrow N} \subseteq Y^{\uparrow N \downarrow N} = Y_1 \subseteq O^{\uparrow N \downarrow N} \). Consequently \( Y_1 = O^{\uparrow N \downarrow N} \) and hence \((Y_1, D_1) = (O^{\uparrow N \downarrow N}, O^{\uparrow N}) \). In addition, by \( O \subseteq \bigcup [O]_{R_1} \subseteq O_1 \subseteq O^{\uparrow N \downarrow N} \), we have \( O^{\uparrow N \downarrow N} \subseteq O^{\uparrow N \downarrow N} \subseteq O^{\uparrow N \downarrow N} = O^{\uparrow N \downarrow N} \), it follows \( O^{\uparrow N \downarrow N} = O^{\uparrow N \downarrow N} \). Consequently, we know \( O_1 \in [O]_{R_1} \) and \( O_1 \subseteq \bigcup [O]_{R_1} \). Thus \( O_1 = \bigcup [O]_{R_1} \) and \((O_1, C_1) = ([O]_{R_1}, ([O]_{R_1})_{\square M})\). We can conclude that \((O_1, C_1) \to (Y_1, D_1) \Rightarrow ([O]_{R_1}, ([O]_{R_1})_{\square M}) \to (O^{\uparrow N \downarrow N}, O^{\uparrow N}) \) is a necessary I-decision rule.

(2) Secondly, we prove that \( \mathcal{R}_I(\mathcal{C}) \subseteq H \). Suppose that \((O, C) \to (Y, D) \) is a necessary I-decision rule. By (1) we have \([O]_{R_1} \subseteq [O]_{R_1} \subseteq (O^{\uparrow N \downarrow N}, O^{\uparrow N}) \) is a I-decision rule. By \( O \subseteq Y \) we obtain \( O^{\uparrow N \downarrow N} \subseteq Y^{\uparrow N \downarrow N} = Y \) and therefore \([([O]_{R_1}, ([O]_{R_1})_{\square M}) \to (O^{\uparrow N \downarrow N}, O^{\uparrow N}) \Rightarrow (O, C) \to (Y, D) \). From \((O, C) \to (Y, D) \) is necessary, we can conclude \((O, C) \to (Y, D) = ([O]_{R_1}, ([O]_{R_1})_{\square M}) \to (O^{\uparrow N \downarrow N}, O^{\uparrow N}) \) consequently \( \mathcal{R}_I(\mathcal{C}) \subseteq H \) as required. \( \square \)
Algorithm 1: Acquisition of necessary I-decision rules

**Input:** A Fdc $\mathcal{C} = (U, M, I, N, J)$.

**Output:** $\mathcal{R}_I(\mathcal{C})$ // the set of necessary I-decision rules.

1) Construct the object-oriented concept lattices $L_O(\mathcal{C}_M)$
2) Compute $R_1$ by using formula (13)
3) Compute equivalence class $[O]_{R_1}$ for each $O \in ExtL_O(\mathcal{C}_M)$
4) Compute $\mathcal{R}_I(\mathcal{C})$ via applying Theorem 3.4.
5) Output $\mathcal{R}_I(\mathcal{C})$

In what follows, $(\cup [O]_{R_1}, (\cup [O]_{R_1})_{\square M}) \rightarrow (O^{\uparrow N \downarrow N}, O^{\uparrow N})$ is called the necessary I-decision rule generated by $O$. By using Theorem 2 we propose Algorithm 1 to compute necessary I-decision rules.

Theorem 3.5 ensures the validity of Algorithm 1. Then we analyze its time complexity. If $L_O(\mathcal{C}_M)$ is constructed via the algorithms proposed by Outrata and Vychodil [37], then the running time of Step 1 to construct $L_O(\mathcal{C}_M)$ is $O(|U||M|^2|L_O(\mathcal{C}_M)|)$. Running Steps 2-5 takes $O(|L(\mathcal{C}_N)||U||M| + |L_O(\mathcal{C}_M)|)$ in a worst-case. To summary, the global running time is at most $O(|U||M||L(\mathcal{C}_N)| + |L_O(\mathcal{C}_M)|(|U||M|^2 + |L(\mathcal{C}_N)|))$.  

**Example 3.6.** Let us consider a Fdc $\mathcal{C} = (U, M, I, N, J)$ presented by Table 1, where $U = \{1, 2, 3, 4, 5\}$ is objects set, $M = \{a, b, c, d, e, f\}$ is conditional attributes set, and $N = \{d_1, d_2, d_3\}$ is decision attributes set. The value in Table 1 is $\times$ represents the homologous object possesses the homologous attribute, while not have otherwise.

![Table 1](image)

By direct computation, we have

$$L_O(\mathcal{C}_M) = \{(\emptyset, \emptyset), (3, e), (4, f), (24, df), (34, ef), (35, ce), (135, ace), (234, def), (245, bdf), (345, cef), (1345, aef), (2345, bcdef), (U, M)\}.$$  

$$L(\mathcal{C}_N) = \{(\emptyset, N), (4, d_2d_3), (235, d_1d_2), (1235, d_1), (2345, d_2), (U, \emptyset)\}.$$  

Here, for simplicity, set notion is separator-free, e.g., $245$ substitutes for set $\{2, 4, 5\}$ and $bdf$ stands for set $\{b, d, f\}$. 


The Hasse diagrams of $L_O(C_M)$ and $L(C_N)$ are depicted in Fig. 1 and Fig. 2 respectively. Additionally, $\emptyset \uparrow^{R_1} = \{\emptyset\}$, $3 \uparrow^{R_1} = \{3, 35\}$, $4 \uparrow^{R_1} = \{4\}$, $24 \uparrow^{R_1} = \{24, 34, 234, 245, 345, 2345\}$, $135 \uparrow^{R_1} = \{135\}$, $[1345]_{R_1} = \{1345, U\}$. Since $\emptyset \uparrow^N = N$, $35 \uparrow^N = d_1d_2$, $4 \uparrow^N = d_2d_3$, $2345 \uparrow^N = d_2$, $135 \uparrow^N = d_1$ and $U \uparrow^N = \emptyset$, we have four necessary I-decision rules:

\[
\begin{align*}
(r_1) : (4, f) & \rightarrow (4, d_2d_3) \\
(r_2) : (35, ce) & \rightarrow (235, d_1d_2) \\
(r_3) : (135, ace) & \rightarrow (1235, d_1) \\
(r_4) : (2345, bcdef) & \rightarrow (2345, d_2)
\end{align*}
\]

We observe that there are fifteen I-decision rules in $R_I(C)$:

\[
\begin{align*}
(3, e) & \rightarrow (235, d_1d_2), (3, e) \rightarrow (1235, d_1), (3, e) \rightarrow (2345, d_1), (4, f) \rightarrow (4, d_2d_3), \\
(4, f) & \rightarrow (2345, d_2), (34, ef) \rightarrow (2345, d_2), (24, df) \rightarrow (2345, d_2), (235, ce) \rightarrow (235, d_1d_2), \\
(35, ce) & \rightarrow (1235, d_1), (35, ce) \rightarrow (2345, d_2), (135, ace) \rightarrow (1235, d_1), (345, cef) \rightarrow (2345, d_2), \\
(234, def) & \rightarrow (2345, d_2), (245, bdf) \rightarrow (2345, d_2), (2345, bcdef) \rightarrow (2345, d_2).
\end{align*}
\]

Theorem 3.5 presents an approach to compute necessary I-decision rules via an equivalence relation on $\text{Ext}L_O(C_M)$. Actually, decision rules can also be derived based on a classification of formal
concepts in \( L(\mathcal{C}_N) \). Let \( R_2 \) be a binary relation on \( \text{ExtL}(\mathcal{C}_N) \) given by:

\[
R_2 = \{(O, Y) \in \text{ExtL}(\mathcal{C}_N) \times \text{ExtL}(\mathcal{C}_N) | O^M \cap Y^M = \emptyset\}
\] (15)

In other words, \((O, Y) \in R_2\) is equivalent to \(O \) and \(Y\) are extents in \(\text{ExtL}(\mathcal{C}_N)\) and they generate same object-oriented concepts in \(L_O(\mathcal{C}_M)\). \(R_2\) is clearly an equivalence relation on \(\text{ExtL}(\mathcal{C}_N)\). For each \(O \in \text{ExtL}(\mathcal{C}_N)\) we denote \([O]_{R_2}\) as the equivalence class based on \(R_2\).

**Theorem 3.7.** For a Fdc \(\mathcal{C} = (U, M, I, N, J)\), we have

\[
\mathcal{R}_I(\mathcal{C}) = \{(O^M \cap Y^M, O^M) \rightarrow (\cap [O]_{R_2}, (\cap [O]_{R_2})^\uparrow_N) | O \in \text{ExtL}(\mathcal{C}_N), O \neq U, O^M \cap Y^M \neq \emptyset\}
\] (16)

**Proof:**

(1) Firstly, we prove that \(\{t_O | O \in \text{ExtL}(\mathcal{C}_N)\} \subseteq \mathcal{R}_I(\mathcal{C})\), i.e., \(t_O\) is a necessary I-decision rule for any \(O \in \text{ExtL}(\mathcal{C}_N)\), where \(t_O = (O^M \cap Y^M, O^M) \rightarrow (\cap [O]_{R_2}, (\cap [O]_{R_2})^\uparrow_N)\). In fact, For any \(Y \in [O]_{R_2}\), it follows that \((Y, Y^\uparrow_N) \in L(\mathcal{C}_N)\). By formula (3), the infimum \(\{(Y, Y^\uparrow_N) \mid Y \in [O]_{R_2}\}\) in \(L(\mathcal{C}_N)\) is given by:

\[
\wedge_{Y \in [O]_{R_2}} (Y, Y^\uparrow_N) = (\cap [O]_{R_2}, (\cup Y \in [O]_{R_2} Y^\uparrow_N)^\downarrow_N^\uparrow_N
\]

Consequently, \( \cap [O]_{R_2} \in \text{ExtL}(\mathcal{C}_N)\). Additionally, by \((\cap [O]_{R_2})^M = \cap Y \in [O]_{R_2} Y^M = O^M\), we have \(\cap [O]_{R_2} \in [O]_{R_2}\) and \(\cap [O]_{R_2}\) is clearly the least element in \([O]_{R_2}\). By \(O^M \cap Y^M = (\cap [O]_{R_2})^M \subseteq \cap [O]_{R_2}\), it follows that \(t_O\) is a I-decision rule.

Suppose that \((O_1, C_1) \rightarrow (Y_1, D_1) \in \mathcal{R}_I(\mathcal{C})\) and \((O_1, C_1) \rightarrow (Y_1, D_1) \Rightarrow t_O\). Then we know that \(O^M \cap Y^M \subseteq O_1 \subseteq Y_1 \subseteq \cap [O]_{R_2}\). By \(O^M \cap Y^M \subseteq O_1 \subseteq \cap [O]_{R_2} \subseteq O\) we have \(O^M \cap Y^M \subseteq O_1 \subseteq O^M \cap Y^M \subseteq O^M \cap Y^M\). Consequently \(O_1 = O^M \cap Y^M\) and hence \((O_1, C_1) = (O^M \cap Y^M, O^M)\). In addition, by \(O^M \cap Y^M \subseteq Y_1 \subseteq \cap [O]_{R_2} \subseteq O\) we have \(O^M \cap Y^M \subseteq Y_1 = Y_1^M \cap O^M \subseteq O^M \cap Y^M\) and thus \(Y_1^M \cap O^M = O^M \cap Y^M\). Therefore we obtain \(Y_1 \in [O]_{R_2}\) and hence \(\cap [O]_{R_2} \subseteq Y_1\). We conclude that \(\cap [O]_{R_2} = Y_1\) and \((O_1, D_1) = (\cap [O]_{R_2}, (\cap [O]_{R_2})^\uparrow_N)\). Consequently, \(t_O = (O_1, C_1) \rightarrow (Y_1, D_1)\) and \(t_O\) is thus a necessary I-decision rule.

(2) Secondly, we prove that \(\mathcal{R}_I(\mathcal{C}) \subseteq \{t_O | O \in \text{ExtL}(\mathcal{C}_N)\}\). Assume \((O, C) \rightarrow (Y, D)\) is a necessary I-decision rule. According to (1), \((Y^M \cap O^M, Y^M) \rightarrow (\cap Y \in [O]_{R_2}, (\cap Y \in [O]_{R_2})^\uparrow_N)\) is a I-decision rule. By \(O \subseteq Y\) and \(O \in \text{ExtL}(\mathcal{C}_M)\) we have \(O = Y^M \cap O^M \subseteq Y^M \cap Y^M \subseteq \cap Y \in [O]_{R_2} \subseteq Y\). Consequently, \((Y^M \cap O^M, Y^M) \rightarrow (\cap Y \in [O]_{R_2}, (\cap Y \in [O]_{R_2})^\uparrow_N) \Rightarrow (O, C) \rightarrow (Y, D)\). By \((O, C) \rightarrow (Y, D)\) is necessary, we obtain \((O, C) \rightarrow (Y, D) = (Y^M \cap O^M, Y^M) \rightarrow (\cap Y \in [O]_{R_2}, (\cap Y \in [O]_{R_2})^\uparrow_N)\). Thus \((O, C) \rightarrow (Y, D) \in \{t_O | O \in \text{ExtL}(\mathcal{C}_N)\}\). Consequently, \(\mathcal{R}_I(\mathcal{C}) \subseteq \{t_O | O \in \text{ExtL}(\mathcal{C}_N)\}\) as required.

By using Theorem 3.7, we put forward Algorithm 2 to acquire necessary I-decision rules.

Then we analyze its time complexity. Assume that \(L(\mathcal{C}_N)\) is computed by using the algorithms presented in [37]. Running Step 1 for generating \(L(\mathcal{C}_N)\) needs \(O(|U||N|^2|L(\mathcal{C}_N)|)\). The running time of Steps 2-5 is at most \(O(|L_O(\mathcal{C}_M)|(|U||N| + |L(\mathcal{C}_N)|))\). Therefore, Algorithm 2 needs at most \(O(|U||N||L_O(\mathcal{C}_M)| + |L(\mathcal{C}_N)|(|U||N|^2 + |L(\mathcal{C}_M)|))\).
3.2. Attribute reduction based on I-decision rules

In this subsection, we present an attribute reduction method for Fdc which preserve I-decision rules.
Let $\mathcal{C} = (U, M, I, N, J)$ be a Fdc, $\mathcal{C}_M = (U, M, I)$, $\mathcal{C}_N = (U, N, J)$ and $E \subseteq M$. A Fdc $\mathcal{C}(E) = (U, E, I_E, N, J)$ generates by $E$ and $I_E = I \cap (U \times E)$, called a subcontext of $\mathcal{C}$. In order to distinguish, the operators given by (7) and (8) for $(U, E, I_E)$ will be expressed as $\Box_E$ and $\Diamond_E$ respectively. In other words, $\forall O \subseteq U, \forall C \subseteq E$, we know $\hat{O}^{C,E} = \{ m \in E | \forall x \in U ((x, m) \in I \rightarrow x \in O) \}$, $C^{\bigtriangleup_E} = \{ g \in U | \exists m \in C ((g, m) \in I) \}$. Obviously it follows $O^{\Box_E} = O^{\bigtriangleup_M} \cap E$ and $C^{\bigtriangleup_E} = C^{\bigtriangleup_M}$.

**Definition 3.9.** Assume that $\mathcal{C} = (U, M, I, N, J)$ is a Fdc, $E \subseteq M$, $\mathcal{C}(E) = (U, E, I_E, N, J)$ is the formal decision subcontext of $\mathcal{C}$, $(O, C) \rightarrow (Y, D) \in \mathcal{R} I(\mathcal{C}(E))$, $(O_1, C_1) \rightarrow (Y_1, D_1) \in \mathcal{R} I(\mathcal{C})$. If $O_1 \subseteq O \subseteq Y \subseteq Y_1$, we call $(O, C) \rightarrow (Y, D)$ imply $(O_1, C_1) \rightarrow (Y_1, D_1)$, denoted by $(O, C) \rightarrow (Y, D) \Rightarrow (O_1, C_1) \rightarrow (Y_1, D_1)$. In this case, the decision information associated with $\mathcal{C}$ can be implied by that of $\mathcal{R} I(\mathcal{C}(E))$. In this case, the decision information associated with $\mathcal{C}$ can be deduced from that of $\mathcal{C}(E)$.

**Definition 3.10.** Let $E \subseteq M$. We call $E$ a I-consistent set of $\mathcal{C}$ if $\mathcal{R} I(\mathcal{C}(E)) \Rightarrow \mathcal{R} I(\mathcal{C})$. In addition, if $E$ is a I-consistent set and $\forall H \subseteq E$ is not a I-consistent set of $\mathcal{C}$, then $E$ is regarded as a I-reduction of $\mathcal{C}$.

From the definition, a I-reduction of $\mathcal{C}$ is a minimal subset $E$ of conditional attributes such that the I-decision rules obtained from $\mathcal{C}$ can be implied by that of $\mathcal{R} I(\mathcal{C}(E))$. In this case, the decision information associated with $\mathcal{C}$ can be deduced from that of $\mathcal{C}(E)$.

**Theorem 3.11.** Let $E \subseteq M$. $E$ is a I-consistent set of $\mathcal{C}$ iff for any $(Y, D) \in L(\mathcal{C}_N)$ and $(O, C) \in L_O(\mathcal{C}_M)$ with $O \subseteq Y$, there exists $(O', C') \in L_O(U, E, I_E)$ such that $O \subseteq O' \subseteq Y$.

**Proof:** Suppose $E$ is a I-consistent set of $\mathcal{C}$, $(Y, D) \in L(\mathcal{C}_N)$, $(O, C) \in L_O(\mathcal{C}_M)$ and $O \subseteq Y$. We have $(O, C) \rightarrow (Y, D)$ is a I-decision rule and hence there exists $(O_1, C_1) \rightarrow (Y_1, D_1) \in \mathcal{R} I(\mathcal{C}(E))$ such that $(O_1, C_1) \rightarrow (Y_1, D_1) \Rightarrow (O, C) \rightarrow (Y, D)$. By Definition 5 we have $O \subseteq O_1 \subseteq Y$ and consequently $O \subseteq O_1 \subseteq Y$ with $(O_1, C_1) \in L_O(U, E, I_E)$.

Conversely, assume that $(O, C) \rightarrow (Y, D) \in \mathcal{R} I(\mathcal{C})$. We have $(Y, D) \in L(\mathcal{C}_N)$, $(O, C) \in L_O(\mathcal{C}_M)$ and $O \subseteq Y$. From the assumption, there exists $(O', C') \in L_O(U, E, I_E)$ such that $O \subseteq O' \subseteq Y$. From $O' \subseteq Y$, we know $(O', C') \rightarrow (Y, D) \in \mathcal{R} I(\mathcal{C}(E))$ and $(O', C') \rightarrow (Y, D) \Rightarrow (O, C) \rightarrow (Y, D)$. We can conclude $\mathcal{R} I(\mathcal{C}(E)) \Rightarrow \mathcal{R} I(\mathcal{C})$ and $E$ is a I-consistent set of $\mathcal{C}$.

**Theorem 3.12.** Let $E \subseteq M$. $E$ is a I-consistent set of $\mathcal{C}$ iff $Y^{\Box_M \Diamond_M} = Y^{\Box_E \Diamond_E}$ for any $Y \in Ext L(\mathcal{C}_N)$.

**Proof:** Suppose $E$ is a I-consistent set of $\mathcal{C}$ and $Y \in Ext L(\mathcal{C}_N)$. It follows that $Y^{\Box_M \Diamond_M} \in Ext L_O(\mathcal{C}_M)$ and $Y^{\Box_M \Diamond_M} \subseteq Y$. From Theorem 3.11, there exists $Z \in Ext L_O(U, E, I_E)$ such that $Y^{\Box_M \Diamond_M} \subseteq Z \subseteq Y$. Since $Y^{\Box_E \Diamond_E} = (Y^{\Box_M} \cap E)^{\Diamond_E} = (Y^{\Box_M} \cap E)^{\Diamond_M} \subseteq Y^{\Box_M \Diamond_M}$, we obtain $Y^{\Box_M \Diamond_M} \subseteq Z = Z^{\Box_E \Diamond_E} \subseteq Y^{\Box_E \Diamond_E} \subseteq Y^{\Box_M \Diamond_M}$.
and thus \( Y \square_M \bowtie_M = Y \square_E \bowtie_E \) as required.

Conversely, assume that \((O, C) \in L_O(\mathcal{C}_M), (Y, D) \in L(\mathcal{C}_N) \) and \( O \subseteq Y \). By the assumption and \( Y \in ExtL(\mathcal{C}_N) \) we obtain \( Y \square_M \bowtie_M = Y \square_E \bowtie_E \). Therefore, \( O = O \square_M \bowtie_M \subseteq Y \square_M \bowtie_M = Y \square_E \bowtie_E \subseteq Y \).

Thus \( O \subseteq Y \square_E \bowtie_E \subseteq Y \) with \( Y \square_E \bowtie_E \in ExtL_O(U, E, I_E) \). From Theorem 3.11, \( E \) is a \( I \)-consistent set of \( \mathcal{C} \).

Let \( \mathcal{U}_I(\mathcal{C}) = \{ Y \square_M \bowtie_M | Y \in ExtL(\mathcal{C}_N) \} \). From Theorem 3.7, \( O \in \mathcal{U}_I(\mathcal{C}) \) iff \((O, O \square_M)\) is the premise of a necessary \( I \)-decision rule. For any \((O_1, C_1), (O_2, C_2) \in L_O(\mathcal{C}_M) \), let \( \beta(((O_1, C_1), (O_2, C_2)) \) be the condition \( O_1 \in \mathcal{U}_I(\mathcal{C}) \land (O_2, C_2) \prec (O_1, C_1) \) or \( O_2 \in \mathcal{U}_I(\mathcal{C}) \land (O_1, C_1) \prec (O_2, C_2) \) and

\[
D_I(((O_1, C_1), (O_2, C_2)) = \begin{cases} 
C_1 \cup C_2 - C_1 \cap C_2, & \text{if } \beta(((O_1, C_1), (O_2, C_2)), \\
\emptyset, & \text{otherwise}.
\end{cases}
\]

where \((O_2, C_2) \prec (O_1, C_1)\) means that \((O_2, C_2)\) is a direct sub-concept of \((O_1, C_1)\), i.e., \( O_2, C_2 \leq (O_1, C_1) \) and \((O_2, C_2) \neq (O_1, C_1) \) and \( (O_2, C_2) \leq (O_3, C_3) \leq (O_1, C_1) \) implies \( (O_2, C_2) = (O_3, C_3) \) or \((O_3, C_3) = (O_1, C_1)\).

**Theorem 3.13.** Let \( E \subseteq M \). \( E \) is a \( I \)-consistent set of \( \mathcal{C} \) iff for any \((O_1, C_1), (O_2, C_2) \in L_O(\mathcal{C}_M) \), if \( D_I(((O_1, C_1), (O_2, C_2)) \neq \emptyset \), then \( E \cap D_I(((O_1, C_1), (O_2, C_2)) \neq \emptyset \).

**Proof:**

**Necessity.** Assume that \((O_1, C_1), (O_2, C_2) \in L_O(\mathcal{C}_M) \) and \( E \cap D_I(((O_1, C_1), (O_2, C_2)) \neq \emptyset \). It follows that \( \beta(((O_1, C_1), (O_2, C_2)) \) holds. Without losing generality, we suppose that \( O_i \in \mathcal{U}_I(\mathcal{C}) \land (O_j, C_j) \prec (O_i, C_i) \). By \( O_i \in \mathcal{U}_I(\mathcal{C}) \), there exists \( Y \in ExtL(\mathcal{C}_N) \) such that \( O_i = Y \square_M \bowtie_M \) and hence \( C_i = O_i \square_M = Y \square_M \). Since \( E \) is a \( I \)-consistent set, we obtain \( Y \square_M \bowtie_M = Y \square_E \bowtie_E \) from Theorem 3.12.

Therefore, from \( (E \cap C_i) \square_M = (E \cap C_i) \dot{\bowtie}_E = (E \cap Y) \square_M = Y \square_E \bowtie_E = Y \square_M \bowtie_M = O_i \), we have \( (E \cap C_i) \bowtie_M \subseteq C_i \subseteq O_i \subseteq (E \cap C_i) \square_M \) and thus \( E \cap C_i \neq E \cap C_j \). Therefore \( E \cap (C_i \cap C_j - C_i \cap C_j) \neq \emptyset \). That is \( E \cap D_I(((O_1, C_1), (O_2, C_2)) \neq \emptyset \).

**Sufficiency.** By Theorem 3.12, it suffices to prove that \( O \square_E \bowtie_E = O \square_M \bowtie_M \) for any \( O \in ExtL(\mathcal{C}_N) \).

If there exists \( O \in ExtL(\mathcal{C}_N) \) such that \( O \square_E \bowtie_E \neq O \square_M \bowtie_M \), then \( O \square_E \bowtie_E \subseteq O \square_M \bowtie_M \). By the assumption and \( O \square_E \bowtie_E \subseteq O \square_M \bowtie_M \). By combining the facts \( O \square_E \bowtie_E \subseteq O \square_M \bowtie_M \), \( O \square_E \bowtie_E \subseteq O \square_M \bowtie_M \), and \( O \square_E \bowtie_E \subseteq O \square_M \bowtie_M \), we obtain \( O \square_E \bowtie_E \subseteq O \square_M \bowtie_M \). It follows that there exists \( O_i, C_i \in L_O(\mathcal{C}_M) \) such that \( O_i \subseteq O \subseteq O \square_M \bowtie_M \). Consequently, \( O \square_E \bowtie_E \subseteq C_i \subseteq O \square_M \bowtie_M \). By the assumption, we obtain \( E \cap (O \square_M - C_i) \neq \emptyset \) and therefore \( E \cap (O \square_M - O \square_E \bowtie_E) \neq \emptyset \). Hence there exists \( e \in E \) such that \( e \in O \square_M \) and \( e \notin O \square_E \bowtie_E \). Consequently, \( e \in E \cap O \square_M = O \square_E \). This contradicts the fact that \( O \square_E \bowtie_E \subseteq O \square_M \bowtie_M \). Thus \( O \square_E \bowtie_E = O \square_M \bowtie_M \) for each \( O \in ExtL_O(\mathcal{C}_M) \) and \( E \) is a \( I \)-consistent set.

For any \((O_1, C_1), (O_2, C_2) \in L_O(\mathcal{C}_M), D_I(((O_1, C_1), (O_2, C_2)) \) is conditional attributes set which discerns \((O_1, C_1)\) and \((O_2, C_2)\). In what follows,

\[
f = \bigwedge_{D_I(((O_1, C_1), (O_2, C_2))) \neq \emptyset} \bigvee D_I(((O_1, C_1), (O_2, C_2)))
\]
is called the discernibility function of \( C \). Here each attribute in \( D_I((O_1, C_1), (O_2, C_2)) \) is taken as a Boolean variable and \( f \) is a CNF (conjunctive normal form) formula in classical propositional logic. By the technique of attribute reduction proposed in [38], we can get the theorem for computing reductions as follows.

**Theorem 3.14.** For a Fdc \( \mathcal{C} = (U, M, I, N, J) \), if the minimal disjunctive normal form of the discernibility function of \( \mathcal{C} \) is \( f = \bigvee_{i=1}^{t} \bigwedge_{j=1}^{s_i} b_{i,j} \), then \( \{E_i; 1 \leq i \leq t\} \) is the family of all I-reductions of \( \mathcal{C} \), where \( E_i = \{b_{i,j}; 1 \leq j \leq s_i\} \) for any \( 1 \leq i \leq t \).

**Example 3.15.** We reconsider the Fdc \( \mathcal{C} = (U, M, I, N, J) \) given by Table 1. We can conclude \( \mathcal{U}_I(\mathcal{C}) = \{\emptyset, 4, 35, 135, 2345, U\} \). The discernibility matrices are given by Table 2 and Table 3.

|                | (\(\emptyset, \emptyset\)) | (3, e) | (4, f) | (24, df) | (34, ef) | (35, ce) | (135, ace) |
|----------------|-----------------------------|--------|--------|----------|----------|----------|------------|
| (4, f)         |                             |        | f      |          |          |          |            |
| (35, ce)       |                             | c      |        |          |          |          |            |
| (135, ace)     |                             |        |        | a        |          |          |            |
| (2345, bcdef)  |                             |        |        |          |          |          |            |
| (U, M)         |                             |        |        |          |          |          |            |

|                | (234, def)                  | (245, bdf) | (345, cef) | (1345, acef) | (2345, bcdef) | (U, M) |
|----------------|-----------------------------|-------------|-------------|----------------|----------------|---------|
| (4, f)         |                            |             |             |                |                |         |
| (35, ce)       |                            |             |             |                |                |         |
| (135, ace)     |                            |             |             |                |                |         |
| (2345, bcdef)  | bc                          | ce          | bd          |                |                |         |
| (U, M)         | bd                          |             |             |                |                | a       |

The discernibility function of \( \mathcal{C} \) is
\[
f = f \land c \land a \land (b \lor c) \land (c \lor e) \land (b \lor d) = f \land c \land a \land (b \lor d) = (a \land b \land c \land f) \lor (a \land c \land d \land f)
\]
Therefore, there are two I-reductions: \( \{a, b, c, f\} \) and \( \{a, c, d, f\} \).

### 4. II-decision rule acquisition and related attribute reduction

In what follows, we consider another type of decision rules which is generated by object-oriented concept and property-oriented concept.
Definition 4.1. In a Fdc $\mathcal{C} = (U, M, I, N, J)$, for any $(O, C) \in L_O(\mathcal{C}_M)$ and $(Y, D) \in L_P(\mathcal{C}_N)$ with $O \subseteq Y$, $(O, C) \rightarrow (Y, D)$ is said to be a II-decision rule of $\mathcal{C}$.

Assume $(O, C) \rightarrow (Y, D)$ is a II-decision rule. By the definition, $(O, C)$ is an object-oriented concept generated by conditional context $\mathcal{C}_M = (U, M, I)$ while $(Y, D)$ is a property-oriented concept in decision context $\mathcal{C}_N = (U, T, J)$. From $C^O_M = O \subseteq Y = D^\square_N$, we can conclude if an object $x \in U$ has some conditional attributes in $C$, then $x \in C^O_M \subseteq D^\square_N$ and hence the decision attributes had by $x$ are all in $D$. We denote $\mathcal{R}_{II}(\mathcal{C})$ as the set of all II-decision rules of $\mathcal{C}$.

Definition 4.2. For $(O_1, C_1) \rightarrow (Y_1, D_1) \in \mathcal{R}_{II}(\mathcal{C})$, $(O_2, C_2) \rightarrow (Y_2, D_2) \in \mathcal{R}_{II}(\mathcal{C})$, if $O_2 \subseteq O_1 \subseteq Y_1 \subseteq Y_2$, then we say that $(O_2, C_2) \rightarrow (Y_2, D_2)$ can be implied by $(O_1, C_1) \rightarrow (Y_1, D_1)$ and denote this implication relationship by $(O_1, C_1) \rightarrow (Y_1, D_1) \Rightarrow (O_2, C_2) \rightarrow (Y_2, D_2)$.

Assume that $(O_1, C_1) \rightarrow (Y_1, D_1) \Rightarrow (O_2, C_2) \rightarrow (Y_2, D_2)$. By $O_2 \subseteq O_1 \subseteq Y_1 \subseteq Y_2$, it follows that $C_2 = O_2^\square_M \subseteq O_1^\square_M = C_1$ and $D_1 = Y_1^\diamond_N \subseteq Y_2^\diamond_N = D_2$. We conclude that the decision information associated with $(O_2, C_2) \rightarrow (Y_2, D_2)$ can be inferred from $(O_1, C_1) \rightarrow (Y_1, D_1)$.

Let $(O, C) \rightarrow (Y, D) \in \mathcal{R}_{II}(\mathcal{C})$. If there exists $(O_1, C_1) \rightarrow (Y_1, D_1) \in \mathcal{R}_{II}(\mathcal{C}) - \{(O, C) \rightarrow (Y, D)\}$ such that $(O_1, C_1) \rightarrow (Y_1, D_1) \Rightarrow (O, C) \rightarrow (Y, D)$, then $(O, C) \rightarrow (Y, D)$ is called a redundant rule in $\mathcal{R}_{II}(\mathcal{C})$. Otherwise, $(O, C) \rightarrow (Y, D)$ is referred to as a necessary rule in $\mathcal{R}_{II}(\mathcal{C})$. We denote $\overline{\mathcal{R}}_{II}(\mathcal{C})$ as the set of all necessary II-decision rules.

In order to acquire necessary II-decision rules, we define binary relation $S_1$ on $ExtL_O(\mathcal{C}_M)$ and $S_2$ on $ExtL_P(\mathcal{C}_N)$. Let

\[
S_1 = \{(O, Y) \in ExtL_O(\mathcal{C}_M) \times ExtL_O(\mathcal{C}_M) | O^\diamond_N = Y^\diamond_N \} \quad (17)
\]

\[
S_2 = \{(O, Y) \in ExtL_P(\mathcal{C}_N) \times ExtL_P(\mathcal{C}_N) | O^\square_M = Y^\square_M \} \quad (18)
\]

Clearly, $S_1$ and $S_2$ are all equivalence relations. We denote $[O]_{S_1}$ as the equivalence class based on $O$ for each $O \in ExtL_O(\mathcal{C}_M)$ and by $[O]_{S_2}$ the equivalence class based on $O$ for each $O \in ExtL_P(\mathcal{C}_N)$ respectively. The following theorems present approaches to derive necessary II-decision rules.

Theorem 4.3. For a Fdc $\mathcal{C} = (U, M, I, N, J)$, we have

\[
\overline{\mathcal{R}}_{II}(\mathcal{C}) = \{(\cup [O]_{S_1}, (\cup [O]_{S_1})^\square_M) \rightarrow (O^\diamond_N \square_N, O^\diamond_N) | O \in ExtL_O(\mathcal{C}_M) \} \quad (19)
\]

Theorem 4.4. For a Fdc $\mathcal{C} = (U, M, I, N, J)$, we have

\[
\overline{\mathcal{R}}_{II}(\mathcal{C}) = \{(O^\square_M, O^\square_M) \rightarrow (\cap [O]_{S_2}, (\cap [O]_{S_2})^\diamond_N) | O \in ExtL_P(\mathcal{C}_N) \} \quad (20)
\]

We can prove Theorem 4.3 and Theorem 4.4 similarly to Theorem 3.5 and Theorem 3.7 respectively.

Example 4.5. We reconsider the Fdc $\mathcal{C} = (U, M, I, N, J)$ given by Table 1. It can be computed that

\[
L_O(\mathcal{C}_M) = \{((\emptyset, \emptyset), (3, e), (4, f), (24, df), (34, ef), (35, ce), (135, ace), (234, de), (245, bdf), (345, cef), (1345, acef), (2345, bcde), (U, M))\}.
\]
In addition, we have

\[ L_P(\mathcal{C}_N) = \{(\emptyset, \emptyset), (1, d_1), (4, d_2d_3), (1235, d_1d_2), (U, N)\} . \]

We consider \( S_1 \) on \( ExtL_O(\mathcal{C}_M) \). It’s routine to review that \([\emptyset]_{S_1} = \{\emptyset\}, [3]_{S_1} = \{3, 35, 135\}, [4]_{S_1} = \{4\}, [24]_{S_1} = \{24, 34, 234, 345, 245, 1345, 2345, U\} \) and \([\emptyset]_{\ominus N \ominus N} = \emptyset, 3_{\ominus N \ominus N} = 1235, 4_{\ominus N \ominus N} = 4, 24_{\ominus N \ominus N} = U\). Thus, by Theorem 4.3, there are four necessary II-decision rules:

\[ r_1 : (\emptyset, \emptyset) \rightarrow (\emptyset, \emptyset) \]
\[ r_2 : (4, f) \rightarrow (4, d_2d_3) \]
\[ r_3 : (135, ace) \rightarrow (1235, d_1d_2) \]
\[ r_4 : (U, M) \rightarrow (U, N). \]

Now we consider \( S_2 \) on \( ExtL_P(\mathcal{C}_N) \). It is routine to check that \([\emptyset]_{S_2} = \{1, \emptyset\}, [4]_{S_2} = \{4\}, [1235]_{S_2} = \{1235\}, [U]_{S_2} = \{U\} \) and \( \emptyset_{\ominus N \ominus M} = \emptyset, 4_{\ominus M \ominus M} = 4, 1235_{\ominus M \ominus M} = 135, U_{\ominus M \ominus M} = U\). Thus, by Theorem 4.4, we also have four necessary II-decision rules \( r_1, r_2, r_3 \) and \( r_4 \).

Similar to I-decision rules, in practical application, II-decision rules \((O, C) \rightarrow (Y, D)\) will be restricted by \( O \neq \emptyset \) and \( Y \neq U \). In this case, we will obtain necessary II-decision rules \( r_2 \) and \( r_3 \) in Example 4.5.

For a Fdc \( \mathcal{C} = (U, M, I, N, J) \), \( \mathcal{C}^c = (U, M, I, N, \neg J) \) is called the complement Fdc of \( \mathcal{C} \) where \((g, t) \in \neg J \) is determined by \((g, t) \notin J \) for any \( g \in U \) and \( t \in N \).

**Theorem 4.6.** \( \varphi : L(U, N, \neg J) \rightarrow L_P(U, N, J) \) is a lattice isomorphism, where \( \varphi ((O, C)) = (O, M - C) \) for any \((O, C) \in L(U, N, \neg J)\).

From this theorem, \( L(U, N, \neg J) \) and \( L_P(U, N, J) \) are isomorphic. Thus we have the next theorem.

**Theorem 4.7.** For a Fdc \( \mathcal{C} = (U, M, I, N, J) \), we have

1. \( \mathcal{R}_I(\mathcal{C}) = \{(O, C) \rightarrow (Y, D)|((O, C) \rightarrow (Y, \neg D)) \in \mathcal{R}_I^c(\mathcal{C})\} \)
2. \( \mathcal{R}_{II}(\mathcal{C}) = \{(O, C) \rightarrow (Y, D)|((O, C) \rightarrow (Y, \neg D)) \in \mathcal{R}_I^c(\mathcal{C})\} \)

where \( \mathcal{R}_I^c(\mathcal{C}) \) is the family of all I-decision rules and \( \mathcal{R}_I^c(\mathcal{C}) \) is the family of all necessary I-decision rules of the complement Fdc \( \mathcal{C}^c \).

**Definition 4.8.** Let \( E \subseteq M, (O, C) \rightarrow (Y, D) \in \mathcal{R}_{II}(\mathcal{C}(E)) \), \((O_1, C_1) \rightarrow (Y_1, D_1) \in \mathcal{R}_{II}(\mathcal{C})\). If \( O_1 \subseteq O \subseteq Y \subseteq Y_1 \), then we call \((O, C) \rightarrow (Y, D)\) imply \((O_1, C_1) \rightarrow (Y_1, D_1)\) and denote this implication relationship by \((O, C) \rightarrow (Y, D) \Rightarrow (O_1, C_1) \rightarrow (Y_1, D_1)\).

If for any \((O_1, C_1) \rightarrow (Y_1, D_1) \in \mathcal{R}_{II}(\mathcal{C})\), there exists \((O, C) \rightarrow (Y, D) \in \mathcal{R}_{II}(\mathcal{C}(E))\) such that \((O, C) \rightarrow (Y, D) \Rightarrow (O_1, C_1) \rightarrow (Y_1, D_1)\), then we say that \( \mathcal{R}_{II}(\mathcal{C}(E)) \) imply \( \mathcal{R}_{II}(\mathcal{C})\), denoted by \( \mathcal{R}_{II}(\mathcal{C}(E)) \Rightarrow \mathcal{R}_{II}(\mathcal{C})\).

**Definition 4.9.** In a Fdc \( \mathcal{C} = (U, M, I, N, J), E \subseteq M \), we call \( E \) a II-consistent set of \( \mathcal{C} \) if \( \mathcal{R}_{II}(\mathcal{C}(E)) \Rightarrow \mathcal{R}_{II}(\mathcal{C}) \). In addition, if \( E \) is a II-consistent set and \( \forall H \subseteq E \) is not a II-consistent set of \( \mathcal{C} \), then we call \( E \) is a II-reduction of \( \mathcal{C} \).
Theorem 4.10. Let \( \mathcal{C} = (U, M, I, N, J) \) be a Fdc, \( E \subseteq M \).

1. \( E \) is a II-consistent set of \( \mathcal{C} \) iff \( E \) is a I-consistent set of \( \mathcal{C}^c \).
2. \( E \) is a II-reduction of \( \mathcal{C} \) iff \( E \) is a I-reduction of \( \mathcal{C}^c \).

Proof:
(1) Suppose that \( E \) is a II-consistent set of \( \mathcal{C} \). For each \((O, C) \rightarrow (Y, D) \in \mathcal{R}^c_I(\mathcal{C})\), we have \((O, C) \rightarrow (Y, T - D) \in \mathcal{R}^c_{II}(\mathcal{C})\) and therefore there exists \((O_1, C_1) \rightarrow (Y_1, D_1) \in \mathcal{R}^c_{II}(\mathcal{C}(E))\) such that \((O_1, C_1) \rightarrow (Y_1, D_1) \Rightarrow (O, C) \rightarrow (Y, T - D)\) because of \( E \) is a II-consistent set of \( \mathcal{C} \). We have \( O \subseteq O_1 \subseteq Y_1 \subseteq Y \) and \((O, C) \rightarrow (Y_1, T - D_1) \in \mathcal{R}^c_{II}(\mathcal{C}(E))\). It follows that \((O_1, C_1) \rightarrow (Y_1, T - D_1) \Rightarrow (O, C) \rightarrow (Y, D)\) and \( E \) is a I-consistent set of \( \mathcal{C}^c \) as required.

If \( E \) is a I-consistent set of \( \mathcal{C}^c \), then \( E \) is a II-consistent set of \( \mathcal{C} \) can be proved similarly.

(2) follows directly from (1).

Example 4.11. We reconsider the Fdc \( \mathcal{C} \) given by Table 1. The complement Fdc \( \mathcal{C}^c \) is proposed by Table 4.

| \( a \) | \( b \) | \( c \) | \( d \) | \( e \) | \( f \) | \( d_1 \) | \( d_2 \) | \( d_3 \) |
|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | |
| 2 | | | | | | | | |
| 3 | | | | | | | | |
| 4 | | | | | | | | |
| 5 | | | | | | | | |

Table 5. The discernibility matrix

\[
\begin{array}{cccccccc}
(\emptyset, \emptyset) & (3, e) & (4, f) & (24, df) & (34, ef) & (35, ce) & (135, ace) \\
(\emptyset, \emptyset) & & & & & & a \\
(4, f) & f & & & & & & \\
(135, ace) & & & & & & & \\
(U, M) & & & & & & & \\
\end{array}
\]

By Example 3.6, we obtain
\[
L_O(\mathcal{C}_M) = \{(\emptyset, \emptyset), (3, e), (4, f), (24, df), (34, ef), (35, ce), (135, ace),
(234, def), (245, bdf), (345, cef), (1345, acef), (2345, bedef), (U, M)\}.
\]

In addition, we have
\[
L(U, T, \neg J) = \{(\emptyset, T), (1, d_2d_3), (4, d_1), (1235, d_3), (U, \emptyset)\}.
\]
Therefore, $\mathcal{U}_I(\mathcal{C}^c) = \{\emptyset, 4, 135, U\}$. The discernibility matrices are given by Table 5 and Table 6.

The discernibility function of $\mathcal{C}^c$ is $f \land a \land (b \lor d) = (a \land b \land f) \lor (a \land d \land f)$ and $\mathcal{C}^c$ has two I-reduction $\{a, b, f\}$ and $\{a, d, f\}$. Then we can obtain $\{a, b, f\}$ and $\{a, d, f\}$ are II-reduction of $\mathcal{C}$.

### 5. Conclusions

The knowledge of formal decision context is usually expressed as decision rules. We note that the existing works on knowledge discovery of formal decision context focus mainly on decision rules derived from conditional and decision formal concepts. This paper mainly proposes some novel methods to knowledge discovery for formal decision context based on two new kinds of decision rules, namely I-decision rules and II-decision rules, which are generated by formal concepts, object-oriented concepts and property-oriented concepts. For I-decision rules, via equivalence relations on extents set of conditional (decision) concept lattices, we develop two rule acquisition algorithms. Some comparative analysis of these algorithms with the existing algorithms presented in [33] is conducted. It is shown that the algorithms presented in this paper have lower time complexities than the existing ones. The attribute reduction method for formal decision context to preserve I-decision rules is presented. For II-decision rules, by using isomorphism between concept lattice of a formal context and property-oriented concept lattice of its complement context, the algorithms for rule acquisition are proposed and the attribute reduction method to preserve II-decision rules is examined.

In further research, we will study attribute reduction methods for formal decision context with respect to some other types of decision rules. Moreover, the applications of attribute reduction approaches to three-way decision theory deserve further study.

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