THE STATIONARY HORIZON AND SEMI-INFINITE GEODESICS IN THE
DIRECTED LANDSCAPE

OFER BUSANI, TIMO SEPPÄLÄINEN, AND EVAN SORENSEN

Abstract. The stationary horizon is a stochastic process consisting of coupled Brownian motions, indexed by their real-valued drifts. It was recently constructed by the first author as the scaling limit of the Busemann process for exponential last-passage percolation. We show that the stationary horizon is invariant under the KPZ fixed point. We use this result to show that the stationary horizon describes the Busemann process of the directed landscape, across all directions of growth. We show the existence of semi-infinite geodesics in the directed landscape, simultaneously across all initial points and all directions. We show that, as a function of the direction, the set of discontinuities in the Busemann process is a countable dense set, denoted \( \Xi \), and this is exactly the set of directions in which not all geodesics coalesce. For directions \( \xi \in \Xi \), from every initial point \( p \in \mathbb{R}^2 \), there exist at least two distinct geodesics from \( p \) that eventually split. Across all initial points, this creates two distinct families of geodesics, each of which has a coalescing structure. This is analogous to the result for the exponential corner growth model proved by Janjigian, Rassoul-Agha, and the second author, as well as the result for Brownian last-passage percolation proved by the last two authors. We use tools developed in the latter work to deal with the continuum of initial points in the plane.

1. Introduction

1.1. The KPZ fixed point and directed landscape. The study of the Kardar-Parisi-Zhang (KPZ) class of 1+1 dimensional stochastic models of growth and interacting particles has advanced to the point where the conjectured universal scaling limits of this class have been rigorously constructed. These two interrelated objects are the \textit{KPZ fixed point}, initially derived as the limit of the totally asymmetric simple exclusion process (TASEP) [MQR21], and the \textit{directed landscape} (DL), initially derived as the limit of Brownian last-passage percolation (BLPP) [DOV18]. The KPZ fixed point describes the height of a growing interface, while the directed landscape describes the random environment through which growth propagates. The two objects are related by a variational formula, recorded as Equation (2.2) below. Evidence for the universality claim comes from rigorous scaling limits of exactly solvable models [NQR20, Vir20, QS20, DV21].

Date: Friday 25th March, 2022.
2020 Mathematics Subject Classification. 60K35, 60K37.
Key words and phrases. Busemann function, coalescence, directed landscape, KPZ fixed point, semi-infinite geodesic, stationary horizon.
Our paper studies the global geometry of the directed landscape, through the analytic and probabilistic properties of its Busemann process. Our construction of the Busemann process begins with the recent construction of individual Busemann functions by Rahman and Virág [RV21]. The remainder of this introduction describes the context of our work and gives brief previews of some results. The organization of the paper is explained in Section 1.5.

1.2. Semi-infinite geodesics and Busemann functions. In growth models of first- and last-passage type, semi-infinite geodesics trace the paths of infection all the way to infinity and hence are central to understanding the large-scale structure of the evolution. Their study was initiated by Licea and Newman in first-passage percolation in the 1990s [LN96, New95] with the first results on existence, uniqueness and coalescence. Since the work of Hoffman [Hof08], Busemann functions have been a key tool for studying semi-infinite geodesics.

Closer to the present work, the study of semi-infinite geodesics began in directed last-passage percolation with the application of the Licea-Newman techniques to the exactly solvable exponential model by Ferrari and Pimentel [FP05]. Georgiou, Rassoul-Agha, and the second author [GRAS17a, GRAS17b] showed the existence of semi-infinite geodesics in directed last-passage percolation with general weights under mild moment conditions. Using this, Janjigian, Rassoul-Agha, and the second author [JRS19] showed that geometric properties of the semi-infinite geodesics can be found by studying analytic properties of the Busemann process. In the special case of exponential weights, they used the distribution of the Busemann process from [FS20] and the work of Coupier [Cou11] to show that all geodesics in a given direction coalesce if and only if that direction is not a discontinuity of the Busemann process. On the other hand, if $\xi$ is a direction of discontinuity of the Busemann process, then from every initial point there are exactly two semi-infinite geodesics in direction $\xi$, and these two geodesics belong in two distinct coalescing families.

In [SS21b] the second and third author extended this work to the semi-discrete setting, by deriving the distribution of the Busemann process and analogous results for semi-infinite geodesics in BLPP. Again all semi-infinite geodesics in a given direction coalesce if and only if that direction is not a discontinuity of the Busemann process. In each direction of discontinuity there are two coalescing families of semi-infinite geodesics and from each initial point at least two semi-infinite geodesics. Compared to LPP on the discrete lattice, the semi-discrete setting of BLPP gives rise to additional nonuniqueness. In particular, [SS21b] developed a new coalescence proof to handle the non-discrete setting.

In the directed landscape, Rahman and Virág [RV21] showed the existence of semi-infinite geodesics, almost surely in a fixed direction across all initial points, as well as almost surely from a fixed initial point across all directions. Furthermore, all semi-infinite geodesics in a fixed direction coalesce almost surely. This allowed [RV21] to construct a Busemann function for a fixed direction. Starting from this definition, we construct the full Busemann process across all directions. Through the properties of this process, we establish a classification of uniqueness and coalescence of semi-infinite geodesics in the directed landscape.

1.3. The stationary horizon as the Busemann process of the directed landscape. The stationary horizon (SH) is a cadlag process indexed by the real line whose states are Brownian motions with drift (Definition D.1 in Appendix D). SH was constructed by the first author [Bus21] as the diffusive scaling limit of the Busemann process of exponential last-passage percolation from [FS20]. Remarkably, the last two authors also discovered in [SS21b] that the stationary horizon, restricted to nonnegative drifts, is the Busemann process of Brownian last-passage percolation.

The present paper establishes that the stationary horizon is the Busemann process of the directed landscape. This is the central result that gives access to the properties of the Busemann process. It verifies the universality of SH conjectured by [Bus21]. Furthermore, it provides us with computational tools for studying the geometric features of DL, in particular, its semi-infinite geodesics.

The characterization of the Busemann process of DL comes from a combination of two results. (i) The Busemann process evolves as a KPZ fixed point from a family of coupled initial conditions. (ii) The stationary horizon is the unique invariant distribution under the KPZ fixed point evolution, subject to conditions satisfied by the Busemann process. That is, if we start the KPZ fixed point
from the family of initial data given by the stationary horizon and evolve each component under the same realization of DL, then, after a global height shift, the distribution of this initial data is preserved under the evolution. Furthermore, the stationary horizon is the unique invariant distribution subject to an asymptotic slope condition (Theorem 2.1). Our invariance result is an infinite-dimensional extension of the previously proved fact that Brownian motion with diffusion coefficient \( \sqrt{2} \) and arbitrary drift is invariant for the KPZ fixed point (for this, see [MQR21, Pim21a, Pim21b]). For the invariance of a single Brownian motion our result gives a strengthened uniqueness statement (Remark 2.2).

1.4. Non-uniqueness of geodesics and random fractals. Among the key questions of interest is the uniqueness of semi-infinite geodesics in the directed landscape. We show the existence of a countably infinite random set of directions \( \xi \) such that, from each initial point in \( \mathbb{R}^2 \), two semi-infinite geodesics in direction \( \xi \) emanate, separate immediately or after some time, and never return back together. It is interesting to relate this result and its proof to earlier work on disjoint finite geodesics.

The set of exceptional pairs of points between which there is a non-unique geodesic in the directed landscape \( L \) was studied in [BGH22]. Their approach relied on [BGH21] which studied the random non-decreasing function \( z \mapsto L(y, s; z, t) - L(x, s; z, t) \) for fixed \( x < y \) and \( s < t \). This process is locally constant except on an exceptional set of Hausdorff dimension \( \frac{1}{2} \). From here [BGH22] showed that for fixed \( s < t \) and \( x < y \), the set of \( z \in \mathbb{R} \) such that there exist disjoint geodesics from \( (x, s) \) to \( (z, t) \) and from \( (y, s) \) to \( (z, t) \) is exactly the set of local variation of the function \( z \mapsto L(x, s; z, t) - L(y, s; z, t) \), and therefore has Hausdorff dimension \( \frac{1}{2} \). Going further, they showed that for fixed \( s < t \), the set of pairs \( (x, y) \in \mathbb{R}^2 \) such that there exist two disjoint geodesics from \( (x, s) \) to \( (y, t) \) also has Hausdorff dimension \( \frac{1}{2} \), almost surely.

Our focus is on the limit of the measure studied in [BGH21], namely, the non-decreasing function \( \xi \mapsto W_\xi(y, s; x, s) = \lim_{t \rightarrow \infty} |L(y, s; \xi, t) - L(x, s; \xi, t)| \) which is exactly the Busemann function in direction \( \xi \). The Lebesgue-Stieltjes measure defined by this function is called the shock measure in [RV21]. Analogously to [BGH22], the support of the shock measure corresponds to the existence of disjoint geodesics (Theorem 3.18), but in contrast to [BGH22], the shock measure is supported on a countable discrete set instead of on a set of Hausdorff dimension \( \frac{1}{2} \) (Theorem 3.4(iii) and Remark 3.5). We encounter a Hausdorff dimension \( \frac{1}{2} \) set if we look along a fixed time level \( s \) for those space-time points \( (x, s) \) out of which there are disjoint semi-infinite geodesics in some direction (Theorem 2.3(iv)). An analogous result was established for BLPP in [SS21b] from properties of Brownian motion. In the present paper we utilize the results of [BGH22]. In particular, we show that if there exist disjoint geodesics from \( (x, s) \) to some point \( (y, t) \), then there also exist disjoint semi-infinite geodesics from \( (x, s) \) in some direction (Lemma 6.18).

1.5. Organization of the paper. In Section 2, we define notation and the model of the directed landscape. We also state our two main results. Theorem 2.1 states the invariance and uniqueness of the stationary horizon under the KPZ fixed point, while Theorem 2.3 describes the global structure of semi-infinite geodesics for all directions in the directed landscape. In Section 3, we state more technical theorems regarding the Busemann process and semi-infinite geodesics. Section 4 states several open problems. Section 5 summarizes the results of [RV21], which we use as the starting point for constructing the Busemann process. The proofs of the theorems are contained in Section 6. The proof of Theorem 2.1 is at the beginning of the section, while the proof of Theorem 2.3 is at the very end. The appendices contain some technical lemmas and background from the literature.

1.6. Acknowledgements. Erik Bates showed us that the Hausdorff dimension of the set \( \pi_1(D_{s,u}) \) in Theorem 6.16(ii) is one-half. Duncan Dauvergne explained the mixing of the directed landscape under spatial shifts, recorded as Lemma B.4. E.S. thanks also Jeremy Quastel and Daniel Remenik for helpful discussions.

The work of O. Busani was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy–GZ 2047/1, projekt-id 390685813, and partly performed at University of Bristol. T. Seppäläinen was partially supported by National Science Foundation grant DMS-1854619 and by the Wisconsin Alumni Research Foundation.
E. Sorensen was partially supported by T. Seppäläinen under National Science Foundation grant DMS-1854619.

2. The model and main theorems

2.1. Notation. We use the following notations in this paper.

(i) \( \mathbb{Z}, \mathbb{Q} \) and \( \mathbb{R} \) are restricted by subscripts, as in for example \( \mathbb{Z}_{>0} = \{1, 2, 3, \ldots \} \).
(ii) \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \) denote the standard basis vectors in \( \mathbb{R}^2 \).
(iii) Equality in distribution is denoted by \( \overset{d}{=} \).
(iv) We say that \( X \sim \text{Exp}(\rho) \) if \( X \) is exponentially distributed with rate \( \rho > 0 \), or equivalently, with mean \( \rho^{-1} \).
(v) The increments of a function \( f : \mathbb{R} \to \mathbb{R} \) are denoted by \( f(x, y) = f(y) - f(x) \).
(vi) Increment ordering of \( f, g : \mathbb{R} \to \mathbb{R} : f \leq_{\text{inc}} g \) means that \( f(x, y) \leq g(x, y) \) for all \( x < y \).
(vii) For \( s \in \mathbb{R} \), we let \( H_s = \{(x, s) : x \in \mathbb{R} \} \) be the spatial points at time \( s \).
(viii) A two-sided standard Brownian motion is a continuous random process \( \{B(x) : x \in \mathbb{R} \} \)
such that \( B(0) = 0 \) almost surely and such that \( \{B(x) : x \geq 0 \} \) and \( \{B(-x) : x \geq 0 \} \) are two independent standard Brownian motions on \([0, \infty)\).
(ix) If \( B \) is a two-sided standard Brownian motion, then \( \{cB(x) + \mu x : x \in \mathbb{R} \} \) is a two-sided Brownian motion with diffusivity \( c > 0 \) and drift \( \mu \in \mathbb{R} \).

(x) The parameter domain of the directed landscape is \( \mathbb{R}^4_+ = \{(x, s; y, t) \in \mathbb{R}^4 : s < t \} \).

2.2. Geodesics in the directed landscape. The directed landscape is a random continuous function \( \mathcal{L} : \mathbb{R}^2_+ \to \mathbb{R} \) that has been shown to be the scaling limit of a large class of models in the KPZ universality class, and is expected to be a universal limit of such models. As an example, we cite the theorem for convergence of exponential last-passage percolation in Theorem C.6 in Appendix C. The directed landscape satisfies the metric composition law: for \( (x, s; y, u) \in \mathbb{R}^4_+ \) and \( t \in (s, u) \),

\[
\mathcal{L}(x, s; y, u) = \sup_{z \in \mathbb{R}} \{\mathcal{L}(x, s; z, t) + \mathcal{L}(z, t; y, u)\}.
\]

This implies the reverse triangle inequality: for \( s < t < u \) and \( (x, y, z) \in \mathbb{R}^3 \), \( \mathcal{L}(x, s; z, t) + \mathcal{L}(z, t; y, u) \leq \mathcal{L}(x, s; y, u) \). Furthermore, over disjoint time intervals \((s_i, t_i), 1 \leq i \leq n \), the processes \((x, y) \mapsto \mathcal{L}(x, s_i; y, t_i)\) are independent.

One can define geodesics in the directed landscape as follows. Define the length of a continuous path \( g : [s, t] \to \mathbb{R} \) as

\[
\mathcal{L}(g) = \inf_{k \in \mathbb{N}} \inf_{s = t_0 < t_1 < \cdots < t_k < t} \sum_{i=1}^{k} \mathcal{L}(g(t_{i-1}), t_{i-1}; g(t_i), t_i),
\]

where the second infimum is over all partitions \( s = t_0 < t_1 < \cdots < t_k < t \). By the reverse triangle inequality, \( \mathcal{L}(g) \leq \mathcal{L}(g(s), s; g(t), t) \). We call \( g \) a geodesic if equality holds. When this occurs, every partition \( s = t_0 < t_1 < \cdots < t_k < t \) satisfies

\[
\mathcal{L}(g(s), s; g(t), t) = \sum_{i=1}^{k} \mathcal{L}(g(t_{i-1}), t_{i-1}; g(t_i), t_i).
\]

By the developments in Sections 12 and 13 of [DOV18], for a fixed \((x, s; y, t) \in \mathbb{R}^4_+ \), there exists a unique geodesic between \((x, s)\) and \((y, t)\). In general, across all points, there exist leftmost and rightmost geodesics. The leftmost geodesic \( g^L \) is such that for each \( u \in (t, s) \), \( g^L(u) \) is the leftmost maximizer of \( \mathcal{L}(x, s; z, u) + \mathcal{L}(z, u; y, t) \) over \( z \in \mathbb{R} \), and the analogous fact holds for the rightmost geodesic. In [DOV18], it was shown that geodesics in the directed landscape almost surely have a Hölder regularity of \( \frac{3}{4} - \varepsilon \) for any \( \varepsilon > 0 \).

A semi-infinite geodesic starting from \((x, s) \in \mathbb{R}^2 \) is a continuous path \( g : [s, \infty] \to \mathbb{R} \) such that \( g(s) = x \), and for all \( t > s \), the restriction of \( g \) to the domain \([s, t]\) is a geodesic between \((x, s)\) and \((g(t), t)\). Such an infinite path \( g \) has direction \( \xi \in \mathbb{R} \) if \( \lim_{t \to \infty} g(t)/t \) exists and equals \( \xi \). Two
semi-infinite geodesics $g_1$ and $g_2$ coalesce if there exists $t$ such that $g_1(u) = g_2(u)$ for all $u \geq t$. If $t$ is the minimal such time, then $(g_1(t), t)$ is the coalescence point. Two semi-infinite geodesics $g_1, g_2 : [s, \infty) \to \mathbb{R}$ are disjoint if $g_1(t) \neq g_2(t)$ for all $t > s$.

2.3. The KPZ fixed point. The KPZ fixed point $h_t(\cdot; h)$ started from initial data $h$ is a Markov process taking values in the space of upper-semicontinuous functions. The present paper restricts attention to continuous initial data $h$ that satisfy $|h(x)| \leq a + b|x|$ for some constants $a, b > 0$. This space is preserved under the KPZ fixed point [MQR21]. If $h$ is the initial data for the KPZ fixed point, then we may represent the process $\{h_t(\cdot; h)\}_{t \geq 0}$ as

$$h_t(y; h) = \sup_{x \in \mathbb{R}} \{h(x) + L(x, 0; y, t)\},$$

where $L$ is the directed landscape (see Section 2.2). This formula had been expected to hold for several years before rigorously proved in [NQR20]. If $h$ is a two-sided Brownian motion with diffusion coefficient $\sqrt{2}$ and arbitrary drift, then for each $t > 0$, $h_t(\cdot; h) - h_t(0; h) \overset{d}{=} h$ (see [MQR21, Pim21a, Pim21b]).

2.4. The stationary horizon. The stationary horizon is a process $G = \{G_\xi\}_{\xi \in \mathbb{R}}$ whose values $G_\xi$ lie in the space $C(\mathbb{R})$ of continuous $\mathbb{R} \to \mathbb{R}$ functions, endowed with the Polish topology of uniform convergence on compact sets. The paths $\xi \mapsto G_\xi$ lie in the Skorokhod space $D(\mathbb{R}, C(\mathbb{R}))$ of right-continuous $\mathbb{R} \to C(\mathbb{R})$ functions with left limits. For each $\xi \in \mathbb{R}$, $G_\xi$ is a two-sided Brownian motion with diffusivity $\sqrt{2}$ and drift $2\xi$. The distribution of a $k$-tuple $(G_{\xi_1}, \ldots, G_{\xi_k})$ can be realized as an image of $k$ independent Brownian motions with drift. Appendix D contains more information on this process, including the precise Definition D.1.

For any compact set $K \subseteq \mathbb{R}$, the process $\xi \mapsto G_\xi|_K$ of functions restricted to $K$ is a jump process. Figure 2.1 shows a simulation of the stationary horizon $G_\xi$ for various values of $\xi$. The jump process behavior is manifested by each pair of trajectories remaining together in a neighborhood of the origin before separating for good, both forward and backward on $\mathbb{R}$.

Our first result is the invariance of the stationary horizon under the KPZ fixed point. This generalizes the invariance of a single Brownian motion with drift.
Theorem 2.1. Let \( G = \{ G_\xi \}_{\xi \in \mathbb{R}} \) be the stationary horizon independent of the directed landscape \( \{ L(x,0;y,t) : x, y \in \mathbb{R}, t > 0 \} \) restricted to positive times. For each \( \xi \in \mathbb{R} \), let \( G_\xi \) evolve under the KPZ fixed point with the same environment \( \mathcal{L} \), i.e., for each \( \xi \in \mathbb{R} \),
\[
h_t(y;G_\xi) = \sup_{x \in \mathbb{R}} \{ G_\xi(x) + L(x,0;y,t) \}, \quad \text{for all } y \in \mathbb{R} \text{ and } t > 0.
\]

(Invariance) For each \( t > 0 \) we have the equality in distribution \( \{ h_t(\cdot;G_\xi) - h_t(0;G_\xi) \}_{\xi \in \mathbb{R}} \overset{d}{=} G \) between random elements of \( D(\mathbb{R},C(\mathbb{R})) \).

(Uniqueness) For each \( k \in \mathbb{Z}_{>0} \) and \((\xi_1, \ldots, \xi_k) \in \mathbb{R}^k \), \((G_{\xi_1}, \ldots, G_{\xi_k})\) is the unique invariant distribution on \( C(\mathbb{R})^k \) such that, for each \( i = 1, \ldots, k \),
\[
\lim_{|x| \to \infty} \frac{G_{\xi_i}(x)}{x} = 2 \xi_i, \quad \text{almost surely.}
\]

In particular, \( G \) is the unique invariant distribution on \( D(\mathbb{R},C(\mathbb{R})) \) such that, on a single event of probability one, for each \( \xi \in \mathbb{R} \), the limit (2.3) holds.

Remark 2.2. In the above generality, the uniqueness result was previously unknown even in the case \( k = 1 \), where we consider the KPZ fixed point from a single initial condition. A uniqueness result had previously been obtained by Pimentel for \( \xi = 0 \) in [Pim21a, Pim21b] under the following condition on the initial data \( h_0 \): there exist \( \gamma_0 > 0 \) and \( \psi(r) \) such that for all \( \gamma > \gamma_0 \) and \( r \geq 1 \),
\[
P(\gamma^{-1} h(\gamma^2 x) \leq r | x, \forall x \geq 1) \geq 1 - \psi(r), \quad \text{where } \lim_{r \to \infty} \psi(r) = 0.
\]

2.5. Semi-infinite geodesics. A significant consequence of Theorem 2.1 is that the stationary horizon characterizes the distribution of the Busemann process of the directed landscape (Theorem 3.2). The Busemann process in turn is used to construct semi-infinite geodesics, simultaneously from all initial points and in all directions, that are called Busemann geodesics (Theorem 3.7). A detailed study of the Busemann process and Busemann geodesics comes in Section 3. In this section we state in the next theorem our conclusions for general semi-infinite geodesics. The random countably infinite dense set \( \Xi \) of directions is later characterized in (3.4) below as the discontinuity set of the Busemann process, and its properties stated in Theorem 3.4.

Theorem 2.3. Statements (i)–(iv) below all hold on a single event of full probability, except for the last sentence of Item (iv). There exists a random countably infinite dense subset \( \Xi \) of \( \mathbb{R} \) such that parts (ii)–(iv) below hold.

(i) Every semi-infinite geodesic has a direction \( \xi \in \mathbb{R} \). From each initial point \( p \in \mathbb{R}^2 \) and for each direction \( \xi \in \mathbb{R} \), there exists at least one semi-infinite geodesic from \( p \) in direction \( \xi \).

(ii) When \( \xi \notin \Xi \), all semi-infinite geodesics in direction \( \xi \) coalesce. There exists a random set of initial points, having Lebesgue measure 0, outside of which the semi-infinite geodesic in each direction \( \xi \notin \Xi \) is unique.

(iii) When \( \xi \in \Xi \), there exist at least two families of semi-infinite geodesics in direction \( \xi \), called the \( \xi^- \) and \( \xi^+ \) geodesics. From every initial point \( p \in \mathbb{R}^2 \) there exists both a \( \xi^- \) geodesic and a \( \xi^+ \) geodesic which eventually separate and never come back together. All \( \xi^- \) geodesics coalesce, and all \( \xi^+ \) geodesics coalesce.

(iv) The random set
\[
\mathcal{S} := \{ p \in \mathbb{R}^2 : \text{For some } \xi \in \Xi, \text{ there exist disjoint directed geodesics from } p \text{ in direction } \xi \}
\]
is dense in \( \mathbb{R}^2 \), and for each fixed \( p \in \mathbb{R}^2 \), \( P(p \in \mathcal{S}) = 0 \). For each \( s \in \mathbb{R} \), the set \( \{ x \in \mathbb{R} : (x,s) \in \mathcal{S} \} \) has Hausdorff dimension \( \frac{1}{2} \) on an \( s \)-dependent full probability event.

Remark 2.4 (Busemann geodesics and general geodesics). The proof of Theorem 2.3 is accomplished by controlling all semi-infinite geodesics with Busemann geodesics. Namely, from each initial point \( p \) and in each direction \( \xi \), all semi-infinite geodesics lie between the leftmost and rightmost Busemann geodesics (Theorem 3.13(i)). Furthermore, for all initial points \( p \) outside a random set of Lebesgue measure zero and all directions \( \xi \notin \Xi \), the two extreme Busemann geodesics coincide and thereby
Figure 2.2. On the left is a depiction of the non-uniqueness described in Item (ii): The geodesics from the initial point split, but come back together to coalesce, forming a bubble. On the right, we see two coalescing families of geodesics. The blue/thin paths depict the $\xi^-$ geodesics, while the red/thick paths describe the $\xi^+$ geodesics. From each point, the $\xi^-$ and $\xi^+$ geodesics split eventually at points in $S$, and the $\xi^-$ and $\xi^+$ families each have a coalescing structure.

imply the uniqueness of the semi-infinite geodesic from $p$ in direction $\xi$ (Theorem 2.3(ii)). Even more generally, whenever $\xi \notin \Xi$, all semi-infinite geodesics in direction $\xi$ are Busemann geodesics (Theorem 3.20(viii)). This is presently unknown for $\xi \in \Xi$, but can be expected to hold by virtue of what is known about exponential last-passage percolation [JRS19].

Our work therefore gives a nearly complete, detailed description of the global behavior of semi-infinite geodesics across all directions and initial points, in the directed landscape. The conjecture that all semi-infinite geodesics are Busemann geodesics is equivalent to the following statement: In Item (iii), for $\xi \in \Xi$, there are exactly two families of coalescing semi-infinite geodesics in direction $\xi$. That is, each $\xi$-directed semi-infinite geodesic coalesces either with the $\xi^-$ or the $\xi^+$ geodesics.

Remark 2.5 (Non-uniqueness of geodesics). The nonuniqueness of geodesics from initial points in a set of Lebesgue measure 0 described in Theorem 2.3(ii) is temporary in the sense that the geodesics must coalesce. This forms a temporary “bubble.” Theorem 3.15(ii) states that the first point of intersection after the geodesics split is the coalescence point. Hence, there cannot be more than one bubble formed by these geodesics. This is in contrast to the non-uniqueness of Item (iii) of Theorem 2.3, where geodesics do not come back together. See Figure 2.2 for a comparison. We discuss this non-uniqueness in more detail in Section 3.3.

Remark 2.6. We also note that in the article [RV21], the authors allude to non-uniqueness of geodesics. They showed that for a fixed initial point, with probability one, there are at most countably many directions with a non-unique geodesic. On page 18 of [RV21], they note that the set of directions with a non-unique geodesic “should be dense over the real line.” Our methods show that this set is in fact dense, and furthermore, it is the set $\Xi$ of discontinuities of the Busemann process.

3. Detailed description of the Busemann process and semi-infinite geodesics

3.1. The Busemann process. The Busemann process $\{W_{\xi\Box}(p; q)\}$ is indexed by points $p, q \in \mathbb{R}^2$, a direction $\xi \in \mathbb{R}$ and a sign $\Box \in \{-, +\}$. The following theorems describe this global process. The parameter $\Box \in \{-, +\}$ denotes the left- and right-continuous versions of this process as a function of $\xi$. 


Theorem 3.1. On $(\Omega,\mathcal{F},\mathbb{P})$, there exists a process
\[ \{W_{\xi}(p; q) : \xi \in \mathbb{R}, \square \in \{-, +\}, p, q \in \mathbb{R}^2\} \]
satisfying the following properties. All the properties below hold on a single event of probability one simultaneously for all directions $\xi \in \mathbb{R}$ and signs $\square \in \{-, +\}$, unless otherwise specified.

(i) (Continuity) As an $\mathbb{R}^4 \to \mathbb{R}$ function, $(x, y, s, t) \mapsto W_{\xi}(x, y, s, t)$ is continuous.

(ii) (Additivity) For all $p, q, r \in \mathbb{R}^2$, $W_{\xi}(p; q) + W_{\xi}(q; r) = W_{\xi}(p; r)$. In particular, $W_{\xi}(p; q) = -W_{\xi}(q; p)$ and $W_{\xi}(p; p) = 0$.

(iii) (Monotonicity along a horizontal line) Whenever $\xi_1 < \xi_2$, $x < y$, and $t \in \mathbb{R}$,
\[ W_{\xi_1}(y, t; x, t) \leq W_{\xi_2}(y, t; x, t) \leq W_{\xi_2}(y, t; x, t) \leq W_{\xi_1}(y, t; x, t). \]

(iv) (Backwards evolution as the KPZ fixed point) For all $x, y \in \mathbb{R}$ and $s < t$,
\[ W_{\xi}(x, s; y, t) = \sup_{z \in \mathbb{R}} \{\mathcal{L}(x, z : s) + W_{\xi}(z, t; y, t)\}. \]

(v) (Regularity in the direction parameter) The process $\xi \mapsto W_{\xi}$ is right-continuous in the sense of uniform convergence on compact sets of functions $\mathbb{R}^4 \to \mathbb{R}$, and $\xi \mapsto W_{\xi}$ is left-continuous in the same sense. Furthermore, the restrictions to compact sets are locally constant in the parameter $\xi$; for each $\xi \in \mathbb{R}$ and compact set $K \subseteq \mathbb{R}^4$ there exists a random $\varepsilon = \varepsilon(\xi, K) > 0$ such that, whenever $\xi - \varepsilon < \alpha < \xi < \beta < \xi + \varepsilon$ and $\square \in \{-, +\}$, we have the equalities
\[ W_{\alpha}(x, s; y, t) = W_{\xi}(x, s; y, t) \quad \text{and} \quad W_{\beta}(x, s; y, t) = W_{\xi}(x, s; y, t) \]
for all $(x, s; y, t) \in K$.

(vi) (Busemann limits) For all $\xi \in \mathbb{R}$, $t \in \mathbb{R}$, and $x < y \in \mathbb{R}$, and any sequence $(z_n, u_n)$ with $u_n \to \infty$ and $z_n/u_n \to \xi$,
\[ W_{\xi}(y, t; x, t) \leq \lim_{n \to \infty} W_{\xi}(y, t; z_n, u_n) - W_{\xi}(x, t; z_n, u_n) \leq \lim_{n \to \infty} W_{\xi}(y, t; x, t). \]

In particular, by the additivity of Item (ii), if $W_{\xi}(p; q) := W_{\xi}(p; q) = W_{\xi}(p; q)$ for all $p, q \in \mathbb{R}^2$, then for all such $p, q$ and sequences $r_n = (z_n, u_n)$ as above,
\[ W_{\xi}(p; q) = \lim_{n \to \infty} W_{\xi}(p; r_n) - W_{\xi}(q; r_n). \]

Furthermore,

(vii) (Agreement for fixed directions) For each $\xi \in \mathbb{R}$, there exists a $\xi$-dependent full probability event on which, for all $p, q \in \mathbb{R}^2$ and any sequence $r_n = (z_n, u_n)$ with $u_n \to \infty$ and $z_n/u_n \to \xi$,
\[ W_{\xi}(p; q) = W_{\xi}(p; q) = \lim_{n \to \infty} W_{\xi}(p; r_n) - W_{\xi}(q; r_n). \]

The following describes the distribution of the Busemann process. The key to the proof of Item (iii) is Theorem 2.1.

Theorem 3.2. The following hold.

(i) (Independence) For each $T \in \mathbb{R}$, the processes
\[ \{W_{\xi}(x, s; y, t) : \xi \in \mathbb{R}, \square \in \{-, +\}, x, y \in \mathbb{R}, s, t \geq T\} \]
and $\{\mathcal{L}(x, s; y, t) : x, y \in \mathbb{R} : s < t \leq T\}$
are independent.

(ii) (Stationarity) The process
\[ \{\mathcal{L}(v), W_{\xi}(p; q) : v \in \mathbb{R}^4, p, q \in \mathbb{R}^2, \xi \in \mathbb{R}, \square \in \{-, +\}\} \]
is stationary under space-time shifts. That is, for $r = (z, u) \in \mathbb{R}^2$,
\[ \{\mathcal{L}(x + z, s + u; y + z, t + u), W_{\xi}(p + r; q + r) : (x, s; y, t) \in \mathbb{R}^4, p, q \in \mathbb{R}^2, \xi \in \mathbb{R}, \square \in \{-, +\}\} \]
\[ d = \{ \mathcal{L}(x, s; y, t), W_{\xi \oplus}(p; q) : (x, s; y, t) \in \mathbb{R}^4, p, q \in \mathbb{R}^2, \xi \in \mathbb{R}, \square \in \{-, +\} \}. \]

(iii) (Distribution of the Busemann process along a horizontal line) For each \( t \in \mathbb{R} \), as random elements of the Skorokhod space \( D(\mathbb{R}, C(\mathbb{R})) \), the following equality in distribution holds:

\[ \{ W_{\xi +}(\cdot, t; 0, t) \}_{\xi \in \mathbb{R}} \overset{d}{=} \{ G_\xi(\cdot) \}_{\xi \in \mathbb{R}}, \]

where \( G \) is the stationary horizon.

Remark 3.3. While Item (iii) only describes the Busemann process along a horizontal line, combining this with Item (i) and Theorem 3.1(iv) gives a description of the global Busemann process.

From the distribution of the Busemann process, we can describe the random set of its discontinuities. For \( p, q \in \mathbb{R}^2 \), define the sets

\[ \Xi(p; q) = \{ \xi \in \mathbb{R} : W_{\xi -}(p; q) \neq W_{\xi +}(p; q) \}, \quad \text{and} \quad \Xi = \bigcup_{p, q \in \mathbb{R}^2} \Xi(p; q). \]

**Theorem 3.4.** The following hold on a single event of probability one.

(i) For each \( t \in \mathbb{R} \), the set \( \Xi(x, t; -x, t) \) is nondecreasing as a function of \( x \in \mathbb{R}_{\geq 0} \). For any \( t \in \mathbb{R} \) and any sequence \( x_k \to \infty \),

\[ \Xi = \bigcup_k \Xi(x_k, t; -x_k, t). \]

(ii) The set \( \Xi \) is countably infinite and dense in \( \mathbb{R} \), while for each fixed \( \xi \in \mathbb{R} \), \( P(\xi \in \Xi) = 0 \).

(iii) For each \( p \neq q \in \mathbb{R}^2 \), the set \( \Xi(p; q) \) is discrete, that is, it has no limit points in \( \mathbb{R} \). For each open interval \( I \subseteq \mathbb{R} \setminus \Xi(p; q) \), the function \( \xi \mapsto W_{\xi -}(p; q) = W_{\xi +}(p; q) \) is constant on \( I \).

Remark 3.5. Item (i) states that all discontinuities of the Busemann process are present on each horizontal level. Item (iii) states that the processes \( \xi \mapsto W_{\xi \pm}(p; q) \) are the left- and right-continuous versions of a jump process. This function defines a random signed measure whose support is a discrete set. As mentioned in the introduction, this process is called the “shock measure” in [RV21]. When \( p \) and \( q \) lie along the same horizontal line, this function is monotone, so we obtain a nonnegative measure (See Theorem 3.1(iii)). In this case, the support of this measure is exactly the set of directions for which the coalescence point of the semi-infinite geodesics in direction \( \xi \) changes, although there are some technical considerations regarding which geodesic we are considering (See Definition 3.16 and Theorems 3.17-3.18 for precise statements).

3.2. Busemann geodesics. With the Busemann process in place, we make the following definition.

**Definition 3.6.** For \( (x, s) \in \mathbb{R}^2 \) and \( t \geq s \), let \( g_{\xi \oplus, L}^{(x, s)}(t) \) and \( g_{\xi \oplus, R}^{(x, s)}(t) \) be, respectively, the leftmost and rightmost maximizers of \( \mathcal{L}(x, s; y, t) + W_{\xi \oplus}(y, t; 0, t) \) over \( y \in \mathbb{R} \).

As noted in the introduction, Rahman and Virág showed the existence of semi-infinite geodesics, almost surely for fixed initial points across all directions and almost surely for a fixed direction across all initial points. We show how to construct semi-infinite geodesics simultaneously across both all initial points and all directions. The next theorem shows that \( g_{\xi \oplus, L}^{(x, s)} \) and \( g_{\xi \oplus, R}^{(x, s)} \) both define semi-infinite geodesics from \( (x, s) \) in direction \( \xi \). Theorem 5.4(iv), quoted from [RV21], states that for a fixed direction \( \xi \), with probability one at times \( t > s \), the maximizers \( z \) of the function \( \mathcal{L}(x, s; z, t) + W_{\xi}(z, t; 0, t) \) are exactly the points on semi-infinite \( \xi \)-directed geodesics from \( (x, s) \). Theorem 3.7 clarifies this on a global scale — that is, across all directions, initial points and signs, one can construct semi-infinite geodesics from the Busemann process. Furthermore, this theorem shows the new result that choosing leftmost (or rightmost) maximizers at each time \( t > s \) gives a well-defined continuous function that is the leftmost (or rightmost) geodesic between any two of its points. This fact is used heavily in the present paper. See the discussion in Section 3.3 for further clarification on leftmost and rightmost geodesics.
Theorem 3.7. The following hold on a single event of probability one across all initial points \((x, s) \in \mathbb{R}^2\), directions \(\xi \in \mathbb{R}\) and signs \(\Box \in \{-, +\}\).

(i) Let \(s = t_0 < t_1 < t_2 < \cdots\) be an arbitrary increasing sequence with \(t_n \to \infty\). For each \(i \geq 1\), let \(g(t_i)\) be any maximizer of \(\mathcal{L}(g(t_{i-1}), t_{i-1}; z, t_i) + W_{\xi \Box}(z; 0, t_i)\) over \(z \in \mathbb{R}\). Then, pick any geodesic of \(\mathcal{L}\) from \((g(t_{i-1}), t_{i-1})\) to \((g(t_i), t_i)\), and for \(t_{i-1} < t < t_i\), let \(g(t)\) be the location of this geodesic at time \(t\). Then, regardless of the choices made at each step, the following hold.

(a) The path \(g : [s, \infty) \to \mathbb{R}\) is a semi-infinite geodesic.

(b) For any \(s \leq t < u\),

\[
\mathcal{L}(g(t), t; g(u), u) = W_{\xi \Box}(g(t), t; g(u), u).
\]

(c) For every \(u > t \geq s\), \(g(u)\) maximizes \(\mathcal{L}(g(t), t; z, u) + W_{\xi \Box}(z; u, 0, u)\) over \(z \in \mathbb{R}\).

(d) The geodesic \(g\) has direction \(\xi\), i.e., \(g(t)/t \to \xi\) as \(t \to \infty\).

(ii) For \(S \in \{L, R\}\), \(g_{(x,s)}^{\xi \Box; S} : [s, \infty) \to \mathbb{R}\) is a semi-infinite geodesic from \((x, s)\) in direction \(\xi\). Moreover, for any \(s \leq t < u\), we have that

\[
\mathcal{L}(g^{\xi \Box; S}_{(x,s)}(t), t; g^{\xi \Box; S}_{(x,s)}(u), u) = W_{\xi \Box}(g^{\xi \Box; S}_{(x,s)}(t), t; g^{\xi \Box; S}_{(x,s)}(u), u),
\]

and \(g^{\xi \Box; S}_{(x,s)}(u)\) is the leftmost/rightmost (depending on \(S\)) maximizer of \(\mathcal{L}(g^{\xi \Box; S}_{(x,s)}(t), t; z, u) + W_{\xi \Box}(z; u, 0, u)\) over \(z \in \mathbb{R}\).

(iii) The path \(g^{\xi \Box; L}_{(x,s)}\) is the leftmost geodesic between any two of its points and \(g^{\xi \Box; R}_{(x,s)}\) is the rightmost geodesic between any two of its points.

Definition 3.8. We refer to the geodesics constructed in Theorem 3.7(i) as \(\xi \Box\) Busemann geodesics, or simply \(\xi \Box\) geodesics.

Remark 3.9. The geodesics \(g^{\xi \Box; L}_{(x,s)}\) and \(g^{\xi \Box; R}_{(x,s)}\) are special types of Busemann geodesics. By Items (ii) and (iii) of Theorem 3.7, for any sequence \(t_0 < t_1 < t_2 < \cdots\) with \(t_n \to \infty\), the path \(g = g^{\xi \Box; L}_{(x,s)}\) can be constructed by choosing \(g(t_i)\) to be the leftmost maximizer of \(\mathcal{L}(g(t_{i-1}), t_{i-1}; z, t_i) + W_{\xi \Box}(z; t_i, 0, t_i)\) over \(z \in \mathbb{R}\), and then, for \(t \in (t_{i-1}, t_i)\), \(g(t)\) is the location at time \(t\) of the leftmost geodesic from \((g(t_{i-1}), t_{i-1})\) to \((g(t_i), t_i)\). The analogous statement holds for \(L\) replaced with \(R\) and ‘leftmost’ replaced with ‘rightmost.’

3.3. Non-uniqueness of geodesics. Theorem 3.7 establishes global existence of semi-infinite geodesics from each initial point and into each direction. Questions about uniqueness naturally arise. We know from Theorem 4 of [RV21], recorded in the present paper as Theorem 5.1(iii), that for a fixed initial point and a fixed direction, there almost surely is a unique semi-infinite geodesic. However, this uniqueness does not extend globally to all initial points and directions simultaneously. In Theorem 5.1, originally proved as Theorem 5 in [RV21], \(g^{\xi \ell}_{p}\) denotes the leftmost semi-infinite geodesic from the point \(p\) in direction \(\xi\), and \(g^{\xi \ell, r}_{p}\) denotes the rightmost such semi-infinite geodesic, alluding to the potential existence of exceptional points and directions with a non-unique geodesic. In the present paper, we rigorously establish the existence of such points and directions. Theorem 3.13(i) below establishes that in general, across all points and directions,

\[
g^{\xi \ell, -L}_{p} = g^{\xi \ell}_{p}, \quad \text{while} \quad g^{\xi \ell, +R}_{p} = g^{\xi \ell, r}_{p}.
\]

The \(L/R\) distinction and the \(\pm\) distinction hint at two types of non-uniqueness of the semi-infinite geodesic from the point \(p\) in direction \(\xi\). These two types of non-uniqueness were both present already in the semi-discrete Brownian last-passage percolation [SS21a, SS21b].

The first type of nonuniqueness is a consequence of the non-discrete nature of the directed landscape, and does not appear in the discrete corner growth model with exponential weights. To describe this type of non-uniqueness, we introduce some random sets. For \(\xi \in \mathbb{R}\) and \(\Box \in \{-, +\}\), let \(\text{NU}_{\xi \Box}^0\) be the set of points \(p \in \mathbb{R}^2\) such that the \(\xi \Box\) geodesic from \(p\) is not unique. Let \(\text{NU}_{\xi \Box}^1\) be the subset of \(\text{NU}_{\xi \Box}^0\) such that the two \(\xi \Box\) geodesics split immediately from the initial point. We may represent these sets as

\[
\text{NU}_{\xi \Box}^0 = \{(x, s) \in \mathbb{R}^2 : g^{\xi \Box; L}_{(x,s)}(t) < g^{\xi \Box; R}_{(x,s)}(t) \text{ for some } t > s\}, \quad \text{and}
\]
Figure 3.1. In this figure, \((x, s) \in NU_0\), and \((y, t) \in NU_1 \subseteq NU_0\).

\[
NU_1^\square = \{(x, s) \in NU_0^\square : \text{there exists } \varepsilon > 0 \text{ such that } g_{(x, s)}^{\xi_L}(t) < g_{(x, s)}^{\xi_R}(t) \text{ for all } t \in (s, s + \varepsilon)\}.
\]

For \(i = 0, 1\), let
\[
NU_i = \bigcup_{\xi \in \mathbb{R}, \square \in \{-, +\}} NU_{i \xi}^\square.
\]

Figure 3.1 illustrates the distinction between points in \(NU_0\) and \(NU_1\). Theorem 3.10(ii) establishes that, along each fixed horizontal level \(s\), \(NU_0\) is nonempty with probability one. We note that \(NU_0\) is nonempty iff \(NU_1\) is nonempty, as follows. We have \(NU_1 \subseteq NU_0\), so one direction is immediate. If \((x, s) \in NU_0^\xi\), then there exists \(t \geq s\) and a point \((y, t) \in NU_1^\xi\) where the geodesics split. See Figure 3.1.

By Theorem 3.1(vii), for a fixed \(\xi \in \mathbb{R}\) and \(\square \in \{-, +\}\), we can drop the \(\pm\) distinction and simply write \(g_p^{\xi_L} = g_p^{\xi_R}\). Hence, we can drop the \(\pm\) distinction for the sets \(NU_{i \xi}^\square\). Then, by Theorem 3.13(i) below, on a \(\xi\)-dependent probability-one event, \(NU_0^\xi\) is exactly the set of points \(p \in \mathbb{R}^2\) such that the semi-infinite geodesic from \(p\) in direction \(\xi\) is not unique. By Theorem 3.15(i), on a single event of probability one, for each direction \(\xi\) and sign \(\square\) \(\in \{-, +\}\), all \(\xi \square\) geodesics coalesce. Therefore, for all \(p \in NU_0\), two \(\xi \square\) geodesics split, but they eventually must come back together. Hence, the set of points \(p \in \mathbb{R}^2\) such that \(g_{(p, s)}^{\xi_L}(t) < g_{(p, s)}^{\xi_R}(t)\) for all \(t > s\) is empty. In particular, the existence of \(\varepsilon > 0\) in the definition of \(NU_{1 \xi}^\square\) (3.8) is essential.

For \(\xi \in \mathbb{R}\) and \(\square \in \{-, +\}\), we do not presently know whether \(NU_{1 \xi}^\square\) is strictly contained in \(NU_{0 \xi}^\square\). However, in the case of Brownian last-passage percolation, the set \(NU_1\) plays a significant role as the set of points from which the leftmost and rightmost competition interfaces have different directions (see specifically Theorem 4.32(ii) in \cite{SS21} for the precise statement). We do not currently know whether an analogous characterization is true in the directed landscape. We make the explicit distinction between the two sets to allow for exploration of the sets in future work.

Looking at the global collection of semi-infinite geodesics, Theorem 3.4(ii) states that the set of \(\xi \in \mathbb{R}\) such that \(W_{\xi^-}(p; q) \neq W_{\xi^+}(p; q)\) for some \(p, q \in \mathbb{R}^2\) is countably infinite. Furthermore, Theorem 2.3(iii) states that whenever \(\xi \in \Xi\), there are at least two semi-infinite geodesics with direction \(\xi\) from every initial point. By Theorem 3.10(ii) below, along a fixed horizontal line, \(NU_0\) is at most countable, and therefore \(NU_0\) is a strict subset of \(\mathbb{R}^2\). Hence, in general, \(NU_{0 \xi}^- \cup NU_{0 \xi}^+\) is not the set of initial points such that the semi-infinite geodesic in direction \(\xi\) is not unique. Instead,
NU₀ is simply the set of points \( p \in \mathbb{R}^2 \) such that the \( \xi \square \) Busemann geodesic from \( p \) is not unique. In other words, the set \( NU_0 \) only captures the non-uniqueness resulting from the \( L/R \) distinction.

We now describe the sets \( NU_1 \) and \( NU_0 \). For \( s \in \mathbb{R} \), recall that \( \mathcal{H}_s = \{(x,s) : x \in \mathbb{R}\} \) are the space-time points at time level \( s \). Also recall that when \( \xi \) is fixed and \( i \in \{0,1\} \), we can drop the \( \pm \) distinction and write \( NU_i = NU_i^+ = NU_i^- \).

**Theorem 3.10.** On a single event of probability one, for \( i = 0,1 \), the set \( NU_i \) satisfies

(3.10) \[
NU_i = \bigcup_{\xi \in \mathcal{Q}} NU_i^\xi.
\]

In particular, the following hold.

(i) For each \( p \in \mathbb{R}^2 \), \( \mathbb{P}(p \in NU_0) = 0 \), and the full probability event of the theorem can be chosen so that \( NU_0 \) contains no points of \( \mathbb{Q}^2 \).

(ii) For each \( s \in \mathbb{R} \), there exists an \( s \)-dependent full probability event on which \( NU_0 \cap \mathcal{H}_s \) is nonempty and at most countably infinite.

We collect facts about the global collection of Busemann geodesics, keeping track of the \( L/R \) and \( \pm \) distinctions.

**Theorem 3.11.** The following hold on a single event of full probability.

(i) For \( s < t, x \in \mathbb{R} \), \( \xi_1 < \xi_2 \), and \( S \in \{L,R\} \),

\[
g_{1}^{\xi_1,-S}(t) \leq g_{1}^{\xi_1+S}(t) \leq g_{1}^{\xi_2,-S}(t) \leq g_{1}^{\xi_2+S}(t).
\]

(ii) Let \( \xi \in \mathbb{R} \), let \( K \subseteq \mathbb{R} \) be a compact set, and let \( T = \max K \). Then, there exists a random \( \varepsilon = \varepsilon(\xi, T, K) > 0 \) such that, whenever \( \xi - \varepsilon < \alpha < \beta < \xi + \varepsilon \), \( \square \in \{-,+\} \), \( S \in \{L,R\} \), and \( x, s \in K \),

\[
g_{\square}^{\xi,S}(t) = g_{\square}^{\xi,-S}(t), \quad \text{and} \quad g_{\square}^{\xi+S}(t) = g_{\square}^{\xi,-S}(t), \quad \text{for all } t \in [s,T].
\]

(iii) For each initial point \( (x,s) \in \mathbb{R}^2 \), \( t > s \), \( \square \in \{-,+\} \), and \( S \in \{L,R\} \),

\[
\lim_{\xi \to \pm \infty} g_{\square}^{\xi,S}(t) = \pm \infty.
\]

(iv) For all \( \xi \in \mathbb{R} \), \( \square \in \{-,+\} \), \( s < t \) and \( x < y \), \( g_{\square}^{\xi,L}(t) \leq g_{\square}^{\xi,R}(t) \). More generally, if \( x < y \), \( s \in \mathbb{R} \), and \( g_1 \) is a \( \xi \square \) geodesic from \( (x,s) \) and \( g_2 \) is a \( \xi \square \) geodesic from \( (y,s) \) such that \( g_1(t) = g_2(t) \) for some \( t > s \), then \( g_1(u) = g_2(u) \) for all \( u > t \). In other words, if the paths defined by \( g_1 \) and \( g_2 \) intersect, they coalesce, and the coalescence point is the first point of intersection.

(v) For all \( \xi \in \mathbb{R} \), \( \square \in \{-,+\} \), \( S \in \{L,R\} \), \( x \in \mathbb{R} \), and \( s < t \),

(3.11) \[
\lim_{(w,s) \to (x,s)} g_{\square}^{\xi,S}(t) = g_{\square}^{\xi,L}(t), \quad \text{and} \quad \lim_{(y,s) \to (x,s)} g_{\square}^{\xi,S}(t) = g_{\square}^{\xi,R}(t),
\]

and if \( g_{\square}^{\xi,L}(t) = g_{\square}^{\xi,R}(t) = g_{\square}^{\xi,S}(t) \), then for \( S \in \{L,R\} \),

(3.12) \[
\lim_{(w,u) \to (x,s)} g_{\square}^{\xi,S}(t) = g_{\square}^{\xi}(t).
\]

Furthermore,

(3.13) \[
\lim_{x \to \pm \infty} g_{\square}^{\xi,S}(t) = \pm \infty.
\]

**Remark 3.12.** In general, Item (i) cannot be extended to mix \( L \) with \( R \). In particular, pick a point \( (x,s) \in NU_0 \), where \( NU_0 \) is defined as in (3.9). Then, there exist directions \( \xi_1 < \xi_2 \) such that for some \( t > s \),

\[
g_{\xi_2,-L}(t) = g_{\xi_2,+L}(t) < g_{\xi_1,-R}(t) = g_{\xi_1,+R}(t).
\]

This follows by the same reasoning as Remark 4.6 in [SS21b].

Item (iv) is an extension of Theorem 5.1 (a collection of results from [RV21] presented below) to all directions and all pairs of initial points on the same horizontal level (see also the discussion
in Section 3.3). It is not true that for all \( \xi \in \mathbb{R} \), \( s < t \), and \( x < y \), \( g_{(x,s)}^{\xi,+R}(t) \leq g_{(y,s)}^{\xi,-L}(t) \). This is discussed further in Remark 3.21 below.

The proof of the next theorem uses Busemann geodesics to give control over the entire collection of semi-infinite geodesics, whether constructed from the Busemann process as in Theorem 3.7 or not.

**Theorem 3.13.** The following hold on a single event of probability one. Let \( (x, s) \in \mathbb{R}^2 \) and \( \xi \in \mathbb{R} \). Let \((x_n, t_n)\) be any sequence such that \( t_n \to \infty \) and \( x_n/t_n \to \xi \). For each \( n \) let \( g_n : [s, t_n] \to \mathbb{R} \) be a geodesic from \((x, s)\) to \((x_n, t_n)\).

(i) For each \( t \geq s \),

\[
g_{(x,s)}^{\xi,-L}(t) \leq \liminf_{n \to \infty} g_n(t) \leq \limsup_{n \to \infty} g_n(t) \leq g_{(x,s)}^{\xi,+R}(t).
\]

In particular, \( g_{(x,s)}^{\xi,-L} \) is the leftmost semi-infinite geodesic from \((x, s)\) in direction \( \xi \), while \( g_{(x,s)}^{\xi,+R} \) is the rightmost (among all semi-infinite geodesics in direction \( \xi \), not just Busemann geodesics).

(ii) Suppose there is a unique semi-infinite geodesic from \((x, s)\) in direction \( \xi \), denoted by \( g_{(x,s)}^{\xi} \).

Let \( T > s \). Then for all sufficiently large \( n \),

\[
g_{(x,s)}^{\xi}(t) = g_n(t) \quad \text{for all } t \in [s, T].
\]

**Remark 3.14.** We note that Item (ii) is the same as Corollary 3.1 in [RV21]. We provide a new method to prove this result that uses the regularity of the Busemann process.

3.4. **Coalescence and the global geometry of geodesics.** We now describe the global structure of the semi-infinite geodesics via the Busemann process, beginning with coalescence.

**Theorem 3.15.** On a single event of full probability, the following hold across all directions \( \xi \in \mathbb{R} \) and signs \( \Box \in \{ -, + \} \).

(i) For all \( p, q \in \mathbb{R}^2 \), if \( g_1 \) and \( g_2 \) are \( \xi \Box \) Busemann geodesics from \( p \) and \( q \), respectively, then \( g_1 \) and \( g_2 \) coalesce. If the first point of intersection of the two geodesics is not \( p \) or \( q \), then the first point of intersection is the coalescence point of the two geodesics.

(ii) Let \( g_1 \) and \( g_2 \) be two distinct \( \xi \Box \) Busemann geodesics from an initial point \((x, s)\) in \( \text{NU}_0^{\xi \Box} \). Then there exist \( t_1 < t_2 \) in \([s, \infty)\) such that \( g_1(t) \neq g_2(t) \) if and only if \( t \in (t_1, t_2) \). That is, the geodesics split at some time \( t_1 \geq s \) and they coalesce exactly when they first meet again at time \( t_2 \).

(iii) For each compact set \( K \subseteq \mathbb{R}^2 \), there exists a random \( T = T(K, \xi, \Box) < \infty \) such that for any two \( \xi \Box \) geodesics \( g_1 \) and \( g_2 \) whose starting points lie in \( K \), \( g_1(t) = g_2(t) \) for all \( t \geq T \). That is, there is a time level \( T \) after which all semi-infinite geodesics started from points in \( K \) have coalesced into a single path.

For two fixed initial points on the same horizontal level, as the direction parameter varies, a constant Busemann process corresponds to a constant coalescence point of the geodesics. However, to make this statement precise, the general non-uniqueness of geodesics across all points requires us to carefully specify the choice of left and right geodesic.

**Definition 3.16.** For \( s \in \mathbb{R} \) and \( x < y \), let \( z_{\Box}^{\xi}(y, s; x, s) \) be the coalescence point of \( g_{(y,s)}^{\xi,-L} \) and \( g_{(x,s)}^{\xi,+R} \).

**Theorem 3.17.** On a single event of probability one, for all reals \( \alpha < \beta \), \( s \), and \( x < y \), the following are equivalent.

(i) \( W_{\alpha,+}(y, s; x, s) = W_{\beta,-}(y, s; x, s) \).

(ii) \( z_{\Box}^{\alpha,+}(y, s; x, s) = z_{\Box}^{\beta,-}(y, s; x, s) \).
There exist \( t > s \) and \( z \in \mathbb{R} \) such that there are paths \( g_1 : [s, t] \to \mathbb{R} \) (connecting \((x, s)\) and \((z, t)\)) and \( g_2 : [s, t] \to \mathbb{R} \) (connecting \((y, s)\) to \((z, t)\)) such that for all \( \xi \in (\alpha, \beta), \square \in \{-, +\}, \) and \( u \in [s, t], \)
\[
g_1(u) = g_{\xi, R}^{\Delta}(u) = g_{(x,s)}^{\alpha+R}(u) = g_{(y,s)}^{\beta-R}(u) < g_2(u) = g_{(x,s)}^{\alpha+L}(u) = g_{(y,s)}^{\beta-L}(u).
\]

**Theorem 3.18.** On a single event of probability one, for all reals \( \xi, s, t, \) and \( x < y \), the following are equivalent.

(i) \( W_{\xi-}(y, s; x, s) = W_{\xi+}(y, s; x, s). \)

(ii) \( \bar{z}_{\xi-}(y, s; x, s) = \bar{z}_{\xi+}(y, s; x, s). \)

(iii) \( g_{\xi, R}^{\xi-}(t) = g_{\xi, R}^{\xi+}(t) \) for some \( t > s. \) In other words, the paths \( g_{\xi, R}^{\xi-} \) and \( g_{\xi, R}^{\xi+} \) intersect.

**Remark 3.19.** In Item (iii), if \( \xi \in \Xi \), then despite intersecting, the geodesics \( g_{\xi, R}^{\xi-} \) and \( g_{\xi, R}^{\xi+} \) cannot coalesce. This follows by the next theorem.

The next theorem of this section is a complete classification of the directions in which all semi-infinite geodesics coalesce.

**Theorem 3.20.** On a single event of probability one, the following are equivalent.

(i) \( \xi \notin \Xi. \)

(ii) \( g_p^{\xi-} = g_p^{\xi+} \) for all \( p \in \mathbb{R}^2 \) and \( S \in \{L, R\}. \)

(iii) All semi-infinite geodesics in direction \( \xi \) coalesce (whether they are Busemann geodesics on not).

(iv) For all \( p \in \mathbb{R}^2 \setminus NU_0 \), there is a unique geodesic starting from \( p \) with direction \( \xi. \)

(v) There exists \( p \in \mathbb{R}^2 \) such that there is a unique semi-infinite geodesic from \( p \) with direction \( \xi. \)

(vi) There exists \( p \in \mathbb{R}^2 \) such that \( g_p^{\xi-} = g_p^{\xi+}. \)

(vii) There exists \( p \in \mathbb{R}^2 \) such that \( g_p^{\xi-} = g_p^{\xi+}. \)

Under these equivalent conditions, the following also holds.

(viii) From any \( p \in \mathbb{R}^2 \), all semi-infinite geodesics in direction \( \xi \) are Busemann geodesics.

**Remark 3.21.** The equivalence (i)\( \Leftrightarrow \) (vi) implies that for all \( \xi \in \Xi \), from every initial point \( p \in \mathbb{R}^2 \), the semi-infinite geodesics \( g_p^{\xi-} \) and \( g_p^{\xi+} \) are distinct. Using the equivalence (i)\( \Leftrightarrow \) (vii), the same is true if \( L \) is replaced with \( R. \) Since \( g_p^{\xi-} \) and \( g_p^{\xi+} \) are both leftmost geodesics between any two of their points (Theorem 3.7(iii)) then if \( \xi \in \Xi \), these two geodesics must separate at some time \( t \geq s \), and they cannot ever come back together. Furthermore, for all \( \xi \in \Xi \), there are two coalescing families of geodesics, namely the \( \xi- \) and \( \xi+ \) geodesics. See Figure 3.2.

Using this fact and the coalescence of Theorem 3.15(i), whenever \( \xi \in \Xi \), for \( s \in \mathbb{R} \), and \( x < y \), \( g_{\xi, R}^{\xi+}(t) > g_{\xi, R}^{\xi-}(t) \) for sufficiently large \( t \), as alluded to in Remark 3.12.

### 4. Open problems

Before proving the main theorems, we state several open problems.

(i) Can one describe the set \( \mathcal{G} \) more globally instead of just on a fixed horizontal line, as in Theorem 2.3(iv)?

(ii) Can one show that all semi-infinite geodesics are Busemann geodesics? Recall by Theorem 3.20(viii) that when \( \xi \notin \Xi \), the answer is yes. The question can be equivalently posed as follows: in the exceptional directions \( \xi \) where there are two distinct coalescing families of geodesics, can one show that every semi-infinite geodesic in direction \( \xi \) coalesces with one of these families.

(iii) For \( \xi \in \mathbb{R} \) and \( \square \in \{-, +\} \) is the set \( NU_{0}^{\square} \) a strict subset of \( NU_{0}^{\Delta} \)? See Equations (3.7), (3.8) for the definitions. That is, are there \( \xi \square \) geodesics that stick together for some time, separate, then come back together, or must they separate immediately? See Figure 3.1.
Figure 3.2. When $\xi \in \Xi$, from every point, there are two distinct semi-infinite geodesics with direction $\xi$. Further, there are two coalescing families of geodesics in this direction. In this figure, the $\xi-$ and $\xi+$ geodesics agree in along the purple (medium thickness) region, then they split into the $\xi-$ geodesics (blue/thin) and $\xi+$ geodesics (red/thick).

(iv) We know the set $NU_0$ almost surely has Lebesgue measure 0 in $\mathbb{R}^2$ by Theorem 3.10(i). Further, we know that it is almost surely countable along a fixed horizontal line by Item (ii) of the same theorem. On the global scale, is the set countable? If not, can one compute its Hausdorff dimension?

(v) In Brownian last-passage percolation (BLPP), it was shown in [SS21b] that the analogue of the inclusion $NU_0 \subseteq \mathcal{S}$ holds. This is because, in BLPP, the analogue of the set $\mathcal{S}$ is exactly the set of initial points for which for some point to the northeast, the (finite) geodesic takes a vertical step before it takes a horizontal step. In the directed landscape, we do not have such a description, so it is not clear whether the same inclusion should hold.

(vi) Are the sets $\mathcal{S}_L$ and $\mathcal{S}_R$ defined in (6.74) equal, as is the case for the analogous sets in BLPP? See Remark 6.20.

5. Summary of the Rahman-Virág results

The paper [RV21] shows existence of the Busemann function for a fixed direction and all initial points simultaneously. They first proved existence and coalescence of semi-infinite geodesics and then used those to construct the Busemann functions. Below is a summary of their results.

**Theorem 5.1** ([RV21], Theorems 3–5 and Corollary 3.2). The following hold.

(i) For fixed initial point $p$, there exist almost surely leftmost and rightmost semi-infinite geodesics $g^\xi_\ell$ and $g^\xi_r$ from $p$ in every direction $\xi$ simultaneously. There are at most countably many directions $\xi$ such that $g^\xi_\ell \neq g^\xi_r$.

(ii) For fixed direction $\xi$, there exist almost surely leftmost and rightmost geodesics $g^\xi_\ell$ and $g^\xi_r$ in direction $\xi$ from every initial point $p$.

(iii) For fixed $p = (x,s) \in \mathbb{R}^2$ and $\xi \in \mathbb{R}$, $g := g^\xi_\ell = g^\xi_r$ with probability one. In particular, there exist $a > 1$ and a random constant $C > 0$ with $E(a^{C^3}) < \infty$ such that

\[
|g(s + t) - x - \xi t| \leq C t^{2/3}[1 \vee (\log \log t)^{1/3}] \quad \text{for } t > 0.
\]

(iv) Fix a direction $\xi$ and a horizontal level $s \in \mathbb{R}$. Then, with probability one, for all $x < y$, $g^\xi_{(x,s)}$ stays weakly to the left of $g^\xi_{(y,s)}$. 
(v) For a fixed direction \(\xi\), with probability one, there are at most countably many \(x \in \mathbb{R}\) for which \(g_{(x,0)}^{\xi} \neq g_{(x,0)}^{\xi,r}\).

(vi) For fixed direction \(\xi\), with probability one, all semi-infinite geodesics in direction \(\xi\) coalesce.

**Remark 5.2.** We note that the leftmost and rightmost semi-infinite geodesics could be the same, as is the case with probability one for a fixed initial point and direction. We note that this theorem gives probability one statements for either a fixed direction or a fixed initial point. The work of the present paper deals with results across all points and directions. In the notation of [RV21], the \(\ell\) and \(r\) in the above theorem are replaced by \(-\) and \(+\), respectively. However, as we have demonstrated in Section 3.3, non-uniqueness is properly characterized by two parameters \(\square \in \{-, +\}\) and \(S \in \{L, R\}\). We altered above the notation of [RV21] to avoid confusion.

**Remark 5.3.** The upper bound in (5.1) is slightly different than the one appearing in [RV21, Eq. 3.3] but still holds from the proof therein.

For fixed direction \(\xi\), [RV21] defines \(\kappa^{\xi}(p,q)\) as the coalescence point of the rightmost geodesics in direction \(\xi\) from \(p\) and \(q\). Then, they define the Busemann function

\[
W_\xi(p; q) = \mathcal{L}(p; \kappa^{\xi}(p,q)) - \mathcal{L}(q; \kappa^{\xi}(p,q)).
\]

**Theorem 5.4** ([RV21], Proposition 3.2, Corollary 3.3, Theorem 6).

(i) For each \(t \in \mathbb{R}\), the process \(x \mapsto W_\xi(x, t; 0, t)\) is a two-sided Brownian motion with diffusivity \(\sqrt{2}\) and drift \(2 \xi\).

Furthermore, the following hold with probability one for a fixed direction \(\xi\).

(ii) Additivity: \(W_\xi(p; q) + W_\xi(q; r) = W_\xi(p; r)\) for all \(p, q, r \in \mathbb{R}^2\).

(iii) Let \(t_n \to \infty\), \(x_n/t_n \to \xi\), and set \(q_n = (x_n, t_n)\). Then, for any given compact interval \(I\) and \(t \in \mathbb{R}\), for all sufficiently large \(n\),

\[
W_\xi(p; 0,0) = \mathcal{L}(p; q_n) - \mathcal{L}(0,0; q_n) \quad \text{for all } p \in I \times \{t\}.
\]

(iv) For all \(s < t\) and \(x, y \in \mathbb{R}\),

\[
W_\xi(x, s; y, t) = \sup_{z \in \mathbb{R}} \{ \mathcal{L}(x, s; z, t) + W_\xi(z, t; y, t) \}.
\]

Moreover, the supremum is attained exactly at those \(z\) such that \((z, t)\) lies on a semi-infinite geodesic from \((x, s)\) in direction \(\xi\).

(v) The function \(W_\xi : \mathbb{R}^4 \to \mathbb{R}\) is continuous.

Moreover,

(vi) For a pair of fixed directions \(\xi_1 < \xi_2\), with probability one, for every \(t \in \mathbb{R}\) and \(x < y\),

\[
W_{\xi_1}(y, t; x, t) \leq W_{\xi_2}(y, t; x, t).
\]

6. **Proofs of the theorems**

6.1. **Invariance of the stationary horizon under the KPZ fixed point.** As in Theorem 2.1, consider a family of initial data for the KPZ fixed point given by \(\{G_\xi\}_{\xi \in \mathbb{R}}\), where \(G\) is the stationary horizon, independent of \(\{\mathcal{L}(x, 0; y, t) : x, y \in \mathbb{R}, t > 0\}\). For \(\xi \in \mathbb{R}\) set

\[
h_t(y; G_\xi) = \sup_{x \in \mathbb{R}} \{G_\xi(x) + \mathcal{L}(x, 0; y, t)\}, \quad \text{for all } y \in \mathbb{R} \text{ and } t > 0.
\]

Define the following state space

\[
\mathcal{Y} := \left\{ \{h_\xi\}_{\xi \in \mathbb{R}} \in D(\mathbb{R}, C(\mathbb{R})) : h_\xi^{\xi_1} \leq_{\text{inc}} h_\xi^{\xi_2} \text{ for } \xi_1 < \xi_2, \right.
\]

\[
\text{and for all } \xi \in \mathbb{R}, \ h_\xi^{\xi}(0) = 0 \ \text{and} \ \lim_{x \to \pm \infty} \frac{h_\xi^{\xi}(x)}{x} = 2\xi.
\]

This is made precise in the following lemma.
Lemma 6.1. The space $\mathcal{Y}$ defined in (6.1) is a Borel subset of $D(\mathbb{R}, C(\mathbb{R}))$. Let $\mathcal{L}$ be the directed landscape, $\{h^\xi\}_{\xi \in \mathcal{Y}}$, $h_0(\cdot; h^\xi) = h^\xi$, and

$$h_t(y; h^\xi) = \sup_{x \in \mathbb{R}} \{h^\xi(x) + \mathcal{L}(x, 0; y; t)\} \quad \text{for } t > 0 \text{ and } y \in \mathbb{R}.$$  

Then $t \mapsto \{h_t(\cdot; h^\xi) - h_t(0; h^\xi)\}_{\xi \in \mathcal{Y}}$ is a Markov process on $\mathcal{Y}$. Specifically, there exists a single event of full probability on which, for each $t > 0$, $\{h_t(\cdot; h^\xi) - h_t(0; h^\xi)\}_{\xi \in \mathcal{Y}} \in \mathcal{Y}$.

Proof. Since we are in the space $D(\mathbb{R}, C(\mathbb{R}))$ of right continuous $\mathbb{R} \to C(\mathbb{R})$ functions with left limits, we may write

$$\mathcal{Y} = \bigcap_{\alpha_1 < \alpha_2 \in \mathbb{Q}} \left\{ \{h^\xi\}_{\xi \in \mathbb{R}} \in D(\mathbb{R}, C(\mathbb{R})) : h_{\alpha_1}(q_1, q_2) \leq h_{\alpha_2}(q_1, q_2) \text{ for } q_1 < q_2 \in \mathbb{Q}, \right. \left. h_{\alpha_1}(0) = 0, \right. \left. \lim_{x \to \pm \infty} \frac{h_{\alpha_1}(x)}{x} = 2\alpha_1 \right\}.$$  

In particular, for the set on the right, $h_{\alpha_1} \leq_{\text{inc}} h_{\alpha_2}$ for $\alpha_1 \leq \alpha_2 \in \mathbb{Q}$ by continuity of each function. Then, $h^\xi_{\alpha_1} \leq_{\text{inc}} h^\xi_{\alpha_2}$ extends to all $\xi_1 < \xi_2 \in \mathbb{R}$ by taking limits of rational numbers $\alpha_1 \searrow \xi_1$ and $\alpha_2 \nearrow \xi_2$. Since $h^\xi(0) = 0$ and $h^\xi_{\alpha_1} \leq h^\xi_{\alpha_2}$ for $\xi_1 \leq \xi_2$, $h^\xi_{\alpha_1}(x) \leq h^\xi_{\alpha_2}(x)$ for $x > 0$, and the inequality flips for $x < 0$. Hence, if the limits above hold for all $\alpha \in \mathbb{Q}$, they extend to all $\xi \in \mathbb{R}$. To finish the proof of measurability of $\mathcal{Y}$, it remains to show that the four quantities $\lim_{x \to \pm \infty} \frac{h^\xi(x)}{x}$ and $\limsup_{x \to \pm \infty} \frac{h^\xi(x)}{x}$ are measurable. For $a \in \mathbb{R}$,

$$\left\{ \lim_{x \to \pm \infty} \frac{h^\xi(x)}{x} > a \right\} = \bigcup_{k \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} \bigcap_{x \geq N} \left\{ \frac{h^\xi(x)}{x} \geq a + \frac{1}{k} \right\},$$

and by continuity of $h^\xi$, the last intersection can be changed to an intersection over all $x \in [N, \infty) \cap \mathbb{Q}$.

Next, we show that $\{h_t(\cdot; h) - h_t(0; h)\}_{\xi \in \mathcal{Y}} \in \mathcal{Y}$ for all $t > 0$. By Lemmas B.8 and B.10, it suffices to show that $\{h_t(\cdot; h)\}_{\xi \in \mathcal{Y}} \in D(\mathbb{R}, C(\mathbb{R}))$ for each $t > 0$. Since $h^\xi_{\alpha_1} \leq_{\text{inc}} h^\xi_{\alpha_2}$, we can apply Lemma A.1 and the global modulus of continuity bounds of Lemma B.2 to conclude that, on a single event of probability one, for each compact $K \subseteq \mathbb{R}$ and $\xi \in \mathbb{R}$, there exists a random $M = M(\xi, t, K) > 0$ such that for all $y \in K$, $\alpha \in (\xi - 1, \xi + 1)$,

$$\sup_{x \in \mathbb{R}} \{h^\alpha(x) + \mathcal{L}(x, 0; y, t)\} = \sup_{x \in [-M, M]} \{h^\alpha(x) + \mathcal{L}(x, 0; y, t)\}.$$  

Then, it follows that $\{h_t(\cdot; h^\xi)\}_{\xi \in \mathcal{Y}}$, as a function $\mathbb{R} \to C(\mathbb{R})$ is right-continuous with left-limits because this is true of $\{h^\xi\}_{\xi \in \mathcal{Y}}$.

The process $t \mapsto \{h_t(\cdot; h^\xi) - h_t(0; h^\xi)\}_{\xi \in \mathcal{Y}}$ is Markovian because, by the metric composition (2.1) of the directed landscape $\mathcal{L}$: for $s < t$,

$$h_t(y; h^\xi) - h_t(0; h^\xi) = \sup_{x \in \mathbb{R}} \{h_s(x; h^\xi) + \mathcal{L}(x, s; y, t)\} - \sup_{x \in \mathbb{R}} \{h_s(x; h^\xi) + \mathcal{L}(x, s; 0, t)\}$$

$$= \sup_{x \in \mathbb{R}} \{h_s(x; h^\xi) - h_s(0; h^\xi) + \mathcal{L}(x, s; y, t)\} - \sup_{x \in \mathbb{R}} \{h_s(x; h^\xi) - h_s(0; h^\xi) + \mathcal{L}(x, s; 0, t)\}.$$  

Since $\mathcal{L}$ has independent temporal increments, the process is Markovian.

$\square$

Lemma 6.2. Fix $\xi \in \mathbb{R}$ and $a > 0$. Consider the KPZ fixed point starting at time 0 from some random function $h$ satisfying

$$(6.2) \quad \frac{h^\xi(x)}{x} \to 2\xi \quad \text{as } x \to \pm \infty \text{ almost surely}.$$  

Let $Z^a_\mathcal{H} \in \mathbb{R}$ denote the set of exit points from the time horizon $\mathcal{H}_0$ of the geodesics associated with $h$ and that terminate in $\{t\} \times [-a, a]$. That is,

$$(6.3) \quad Z^a_\mathcal{H} = \bigcup_{y \in [-a, a]} \arg \max_{x \in \mathbb{R}} \{h^\xi(x) + \mathcal{L}(x, 0; y, t)\}.$$
For $\varepsilon > 0$ small, there exists $t_0(\varepsilon, a) > 0$ such that for any $t > t_0$,
\begin{equation}
P\left(Z^a_j \subset [(\xi - \varepsilon)t, (\xi + \varepsilon)t]\right) \geq 1 - \varepsilon.
\end{equation}

Proof. The idea of the proof is that $h(x) + L(x, 0; y, t)$ is a noisy version of $2\xi x - \frac{x^2}{t}$ and $\arg\max_{x \in \mathbb{R}}(2\xi x - \frac{x^2}{t}) = \xi t$, but the noise cannot change the exit point by much when $t$ is large. Below we prove the result for $\xi > 0$ and the proof for $\xi < 0$ follows by symmetry. The case $\xi = 0$ will be proven separately. Fix $\varepsilon > 0$. Set
\begin{equation}
F(x, t) = C_{DL}t^{1/3}\log^2(2(\sqrt{a^2 + x^2 + t^2} + 2)),
\end{equation}
where $C_{DL}$ is the random positive constant from Lemma B.2. From (6.2), there exists a random $C_\varepsilon > 0$ such that
\begin{align}
\bar{h}(x) + L(x, 0; y, t) &\leq M_U(x, t) := 2\xi x - \frac{x^2}{t} + \varepsilon|x| + C_\varepsilon + F(x, t) \quad \forall y \in [-a, a], x \in \mathbb{R}, \\
\bar{h}(x) + L(x, 0; y, t) &\geq M_L(x, t) := 2\xi x - \frac{x^2}{t} - \varepsilon|x| - C_\varepsilon - F(x, t) \quad \forall y \in [-a, a], x \in \mathbb{R}.
\end{align}
From the lower bound in the display above
\begin{equation}
M_L(\xi t, t) \geq \xi^2 t - 2\xi\varepsilon t - C_\varepsilon - C_{DL}t^{1/3}\log^2(2(\sqrt{a^2 + (\xi t)^2 + t^2} + 2)).
\end{equation}
Next note that from the symmetry of $F$ and $\varepsilon|x|$, $\sup_{x>0} M_U(x, t) = \sup_{x \in \mathbb{R}} M_U(x, t)$. We would now like to find (approximately) where the maximizers of $M_U$ on $x \geq 0$ are. Note that
\begin{equation}
M'_U = 2(\xi + \varepsilon) - \frac{2x}{t} + F'(x, t)
\end{equation}
where
\begin{equation}
F'(x, t) = C_{DL}t^{1/3}2\log(2(\sqrt{a^2 + x^2 + t^2} + 2)) \times (2(\sqrt{a^2 + x^2 + t^2} + 2))^{-1}(a^2 + x^2 + t^2)^{-1/2} \times 2x.
\end{equation}
Note that for $t$ large enough $0 < F'(x, t) < \varepsilon$ uniformly in $x$, which implies that for $t$ large enough
\begin{equation}
\{x : M'_U(x, t) = 0\} \subset (\xi t, (\xi + 2\varepsilon)t).
\end{equation}
and that
\begin{align}
M'_U(x, t) &< 0 \quad \forall x \geq (\xi + 2\varepsilon)t \\
M'_U(x, t) &> 0 \quad \forall x \leq \xi t.
\end{align}
Next we consider the supremum of $x \mapsto M_U$ outside the interval $I_\varepsilon := [(\xi - 3\sqrt{\varepsilon}\varepsilon^{1/2})t, (\xi + 3\sqrt{\varepsilon}\varepsilon^{1/2})t]$. For $\varepsilon$ small enough,
\begin{equation}
[\xi t, (\xi + 2\varepsilon)t] \subset I_\varepsilon.
\end{equation}
From (6.11) and (6.12), we see that to determine the supremum of $M_U$ outside $I_\varepsilon$ it is enough to take the maximum of $M_U$ at the endpoints of $I_\varepsilon$. Plugging the endpoints of the interval on the right-hand side of (6.12) in $M_U$, for small enough $\varepsilon$
\begin{align}
M_U((\xi - 3\sqrt{\varepsilon}\varepsilon^{1/2})t) &= \left[\xi^2 - 7\xi\varepsilon - 6\xi^3\varepsilon^{3/2} + t^{-1}C_\varepsilon + C_{DL}t^{-2/3}\log^2(2(\sqrt{a^2 + ((\xi + 1)^2 + 1)t^2} + 2))\right]t \\
M_U((\xi + 3\sqrt{\varepsilon}\varepsilon^{1/2})t) &= \left[\xi^2 - 7\xi\varepsilon + 6\xi^3\varepsilon^{3/2} + t^{-1}C_\varepsilon + C_{DL}t^{-2/3}\log^2(2(\sqrt{a^2 + ((\xi + 1)^2 + 1)t^2} + 2))\right]t.
\end{align}
It follows that for $\varepsilon$ small enough and $t > t_0(\varepsilon, a, C_{DL}, C_\varepsilon)$
\begin{equation}
\sup_{x \not\in I_\varepsilon} M_U(x, t) \leq \max\{M_U((\xi - 3\sqrt{\varepsilon}\varepsilon^{1/2})t), M_U((\xi + 3\sqrt{\varepsilon}\varepsilon^{1/2})t)\} \leq (\xi^2 - 6\xi\varepsilon)t.
\end{equation}
From (6.6)
\begin{equation}
\{\sup_{x \not\in I_\varepsilon} M_U(x, t) < \sup_{x \in I_\varepsilon} M_U(x, t)\}
\subseteq \{\sup_{x \not\in I_\varepsilon} h(x) + L(x, 0; y, t) < \sup_{x \in I_\varepsilon} h(x) + L(x, 0; y, t) \quad \forall y \in [-a, a]\} \subseteq \{Z^a_j \subset I_\varepsilon\}.
(6.16) \[ \mathbb{P}\left( \sup_{x \notin I_\varepsilon} M_U(x,t) < \sup_{x \in I_\varepsilon} M_L(x,t) \right) \geq 1 - \varepsilon. \]

The display above and (6.15) imply the result for \( \xi \neq 0 \). For \( \xi = 0 \), we follow the same proof, only we set \( I_\varepsilon = [-\sqrt{\varepsilon}t, \sqrt{\varepsilon}t] \).

**Proof of Theorem 2.1.** **Invariance:** By Lemma 6.1, the process \( t \mapsto \{h_t(\cdot; G_{\xi}) - h_t(0; G_{\xi})\}_{\xi \in \mathbb{R}} \) is well-defined as a Markov process on the Borel subset \( \mathcal{Y} \) of \( D(\mathbb{R}, C(\mathbb{R})) \). To show invariance of the stationary horizon \( G \), it is sufficient to prove the invariance for finite-dimensional distributions. That is, for \(-\infty < \xi_1 < \cdots < \xi_k < \infty\), if we consider the KPZ fixed point starting from the \( k \) initial conditions

\[ (G_{\xi_1}, \ldots, G_{\xi_k}) \]

independent of \( \{\mathcal{L}(x,0;y,t) : x, y \in \mathbb{R}, t > 0\} \), and for each \( t > 0 \) and \( 1 \leq i \leq k \),

\[ h_t(y; G_{\xi_i}) = \operatorname{sup}_{x \in \mathbb{R}} \{G_{\xi_i}(x) + \mathcal{L}(x,0;y,t)\}, \]

then for each \( t > 0 \),

\[ (h_t(\cdot; G_{\xi_1}) - h_t(0; G_{\xi_1}), \ldots, h_t(\cdot; G_{\xi_k}) - h_t(0; G_{\xi_k})) \overset{d}{=} (G_{\xi_1}, \ldots, G_{\xi_k}). \]

We prove this via a limiting argument using stability of discrete queues. For \( 1 \leq i \leq k \), define \( \rho_i = \frac{1}{2} - 2^{-4/3} \xi_i N^{-1/3} \), and set \( \rho = (\rho_1, \ldots, \rho_k) \). Let \( \mu^\rho \) be the probability distribution on \((\mathbb{R}^2_{\mathbb{Z}_0})^k \) defined in (C.12) in Appendix C.5. It is identified as the joint distribution of \( k \) horizontal Busemann functions of the exponential corner growth model in Theorem C.8. Let \((I^{N,1}_{i}, \ldots, I^{N,k}_{i})\) be a \( \mu^\rho \)-distributed \( k \)-tuple of random, positive bi-infinite sequences \( \ell = (I^{N,i}_j)_{j \in \mathbb{Z}} \).

We consider discrete LPP with exponential weights, where \( d \) denotes last-passage time, and the weights are independent of \((I^{N,1}_{i}, \ldots, I^{N,k}_{i})\). See Appendix C for the precise definition. For \( 1 \leq i \leq k \), let \( F_{i,N} : \mathbb{R} \rightarrow \mathbb{R} \) be the linear interpolation of the function

\[ m \mapsto \begin{cases} \sum_{j=1}^{m} I^{N,i}_j & m \geq 1 \\ 0 & m = 0 \\ -\sum_{j=m+1}^{0} I^{N,i}_j & m < 0 \end{cases} \]

For \( N \in \mathbb{Z}_{\geq 0} \), define the continuous function

\[ G^{N,i}(x) = 2^{-4/3} N^{-1/3} \left[ F_{i,N}^N(2^{5/3} N^{2/3} x) - 2^{8/3} N^{2/3} x \right] \quad \text{for } x \in \mathbb{R}. \]

Theorems C.8 and D.4 give the distributional limit

\[ (G^N_1, \ldots, G^N_k) \converges \text{distr.} (G_{\xi_1}, \ldots, G_{\xi_k}), \]

on the space \( C(\mathbb{R}, \mathbb{R}^k) \), under the Polish topology of uniform convergence of functions on compact sets.

Next, for \( N \in \mathbb{N} \) sufficiently large, and \( 1 \leq i \leq k \), we consider discrete last-passage percolation with initial data \( F_{i,N}^N \) in an exponential environment, defined in Equation (C.3) of Appendix C. That is, for \( m \in \mathbb{Z} \) and \( n \in \mathbb{Z}_{\geq 0} \), we define

\[ d^N_{i}\mathbb{Z}(m, n) = \sup_{-\infty < \ell \leq m} \{F_{i,N}^N(\ell) + d((\ell,1),(m,n))\}. \]

Next, for \( N \in \mathbb{N} \) and \( 1 \leq i \leq k \), let \( H^i_{t,N} = G^N_{i} \), and for \( t > 0 \), let \( H^i_{t,N} : \mathbb{R} \rightarrow \mathbb{R} \) defined by the linear interpolation of

\[ H^i_{t,N}(y) = 2^{-4/3} N^{-1/3} \left[ d^i(t N + 2^{5/3} N^{2/3} y, t N) - 4 N t - 2^{8/3} N^{2/3} y \right]. \]

Note that \( G^{N,i}_1(0) = 0 \) be construction. By Lemma C.3 and Theorem C.7, for every fixed \( N \in \mathbb{N} \) and all \( t > 0 \) such that \( t N \in \mathbb{Z} \),

\[ (H^i_{t,1,N}(\cdot) - H^i_{t,N}(0), \ldots, H^i_{t,N}(\cdot) - H^i_{t,N}(0)) \overset{d}{=} (G^N_1, \ldots, G^N_k). \]

Then, using (6.19), the proof of (6.17) is complete by the following lemma.
Lemma 6.3. For \( t > 0 \), as \( N \to \infty \), in the topology of uniform convergence on compact sets of functions \( \mathbb{R} \to \mathbb{R}^k \), the process \( (H_{t,N}^1, \ldots, H_{t,N}^k) \) converges in distribution to

\[
(h_t(\cdot; G_{\xi_1}), \ldots, h_t(\cdot; G_{\xi_k})),
\]

where \( (G_{\xi_1}, \ldots, G_{\xi_k}) \) is independent of \( \{\mathcal{L}(x,0;y,t) : x, y, \in \mathbb{R}, t > 0\} \), and for \( 1 \leq i \leq k \),

\[
(6.21) \quad h_t(y; G_{\xi_i}) = \sup_{x \in \mathbb{R}} \{G_{\xi_i}(x) + \mathcal{L}(x,0;y,t)\}.
\]

Proof. Unpacking the definitions,

\[
H_{t,N}^i(y) = \sup_{-\infty < \ell \leq tN + 25/3 N^{2/3} y^3} \left\{ 2^{-4/3} N^{-1/3} \left[ F_i^N(\ell) - 2^{8/3} N^{2/3} x \right] + d((\ell,1), (tN + 25/3 N^{2/3} y^3, tN)) - 4Nt - 2^{8/3} N^{2/3} (y-x) \right\}
\]

\[
(6.22) \quad = \sup_{-\infty < \ell \leq tN + 25/3 N^{2/3} y^3} \left\{ 2^{-4/3} N^{-1/3} \left[ F_i^N(\ell) - 2^{8/3} N^{2/3} x \right] + d((\ell,1), (tN + 25/3 N^{2/3} y^3, tN)) - 4Nt - 2^{8/3} N^{2/3} (y-x) \right\}
\]

\[
(6.23) \quad = \sup_{x \in \mathbb{R}} \{G_i^N(x) + \mathcal{L}_N(x,0;y,t)\},
\]

where \( G_i^N \) is defined as (6.18), and

\[
\mathcal{L}_N(x,0;y,t) = \begin{cases} \frac{d((2/3 N^{2/3} x^3, 1), (tN + 25/3 N^{2/3} x, tN)) - 4Nt - 2^{8/3} N^{2/3} (y-x)}{2^{4/3} N^{1/3}} & \text{if } x \leq y + 2^{-5/3} N^{1/3} t \\ -\infty & \text{otherwise.} \end{cases}
\]

Note that the maximizers in (6.22) are also maximizers of

\[
(6.24) \quad F_i^N(\ell) + d((\ell,1), (tN + 25/3 N^{2/3} y, tN))
\]

over \( \ell \in (-\infty, tN + 25/3 N^{2/3} y) \). When \( tN, tN + 25/3 N^{2/3} y \in \mathbb{Z} \), the maximizers here are precisely the exit points defined in Equation (C.4). For shorthand notation, let \( Z_i^N(y) \) denote this exit point, that is the largest maximizer of (6.24). It immediately follows that \( Z_i^N(x) \leq Z_i^N(y) \) for \( x < y \). If there exists some \( M > 0 \) such that \( |Z_i^N(y)| \leq M2^{5/3} N^{2/3} \), then

\[
\text{line (6.23) = } \sup_{x \in [-M,M]} \{G_i^N(x) + \mathcal{L}_N(x,0;y,t)\}.
\]

By the weak convergence in (6.19) and Theorem C.6 and the independence assumptions, Skorokhod representation ([Dud89, Thm. 11.7.2], [EK86, Thm. 3.18]) gives a coupling of copies of \( \{(G^N_i)_{1 \leq i \leq N}, \mathcal{L}_N\} \) and \( \{(G_{\xi_i})_{1 \leq i \leq n}, \mathcal{L}\} \) such that \( G_i^N \to G_{\xi_i} \) uniformly on compact sets for \( 1 \leq i \leq k \) and \( \mathcal{L}_N \to \mathcal{L} \) uniformly on compact sets. Let \( K \subseteq \mathbb{R} \) be compact. Without loss of generality, take \( K = [a, b] \). Then, for each \( M > 0 \) and \( \varepsilon > 0 \),

\[
\mathbb{P}\left( \max_{1 \leq i \leq k} \sup_{y \in [a, b]} |H_{t,N}^i(y) - h_t(y; G_{\xi_i})| > \varepsilon \right) \\
\leq \mathbb{P}\left( \max_{1 \leq i \leq k} \sup_{y \in [a, b]} \left\{ G_i^N(x) + \mathcal{L}_N(x,0;y,t) \right\} - \sup_{x \in [-M,M]} \left\{ G_{\xi_i}(x) + \mathcal{L}(x,0;y,t) \right\} > \varepsilon \right) \\
+ \mathbb{P}\left( \sup_{x \in [a, b]} \left\{ G_{\xi_i}(x) + \mathcal{L}(x,0;y,t) \right\} > \sup_{x \in [-M,M]} \left\{ G_{\xi_i}(x) + \mathcal{L}(x,0;y,t) \right\} \right) \\
(6.25) \\
+ \sum_{i=1}^k \left[ \mathbb{P}(Z_i^N(a) < -M2^{5/3} N^{2/3}) + \mathbb{P}(Z_i^N(b) > M2^{5/3} N^{2/3}) \right]
\]
Since for each \(i\), \(G_{\xi_i}\) is a Brownian motion with drift, independent of \(\{L(x, 0; y, t) : x, y \in \mathbb{R}, t > 0\}\), the bounds of Lemma B.2 imply that
\[
\lim_{M \to \infty} \mathbb{P}\left( \sup_{x \in \mathbb{R}} \{G_{\xi_i}(x) + L(x, 0; y, t)\} > \sup_{x \in [-M, M]} \{G_{\xi_i}(x) + L(x, 0; y, t)\} \right) = 0.
\]
Further, by the exit point bounds of Lemma C.5,
\[
\lim_{M \to \infty} \limsup_{N \to \infty} \sum_{i=1}^{k} \mathbb{P}(Z_i^N(a) < -M^{2/3}N^{2/3}) + \mathbb{P}(Z_i^N(b) > M^{2/3}N^{2/3}) = 0.
\]
Therefore, by choosing \(M\) sufficiently large in (6.25), the quantity
\[
\limsup_{N \to \infty} \mathbb{P}\left( \sup_{1 \leq i \leq k, y \in [a, b]} |H^i_{t,N}(y) - h_t(y; G_{\xi_i})| > \varepsilon \right)
\]
can be shown to be arbitrarily small and therefore must be 0. □

**Uniqueness:** Let \(k \in \mathbb{N}\) and an increasing vector \(\bar{\xi} \in \mathbb{R}^k\). Let \(\bar{h} = (h^1, ..., h^k)\) be a vector of real functions satisfying
\[
(6.26) \quad \frac{h^i(x)}{x} \to 2\xi_i \quad \text{as} \quad x \to \pm \infty \quad \text{almost surely for all} \quad i \in \{1, ..., k\}.
\]
Suppose further that the increments of \(\bar{h}\) are stationary with respect to the KPZ fixed point i.e.
\[
(6.27) \quad h^i(.) - h^i(0) \sim h_t(\cdot; h^i) - h_t(0; h^i) \quad \forall t \geq 0.
\]
We show that
\[
(6.28) \quad \bar{h}(\cdot) - \bar{h}(0) \sim (G_{\xi_1}, ..., G_{\xi_k})
\]
where \(G\) is the SH. The idea of the proof is to sandwich the geodesics of \(h^i\), going from \(H_0\) and terminating at \([-a, a]\), with those of two Busemann’s with close directions. We then use the Crossing Lemma (recorded as Lemma B.7) to bound the increments of \(h_t(0; h^i)\) with those of the Busemann’s. Finally we use our knowledge of the stationary horizon to show that with high probability the increments of \(h_t(0; h^i)\) agree with those of \(W_{\xi_i}(x, t; 0, t)\) on \([-a, a]\).

For \(\delta > 0\), consider the initial conditions \(\{G_{\xi_i-\delta}\}_{i \in \{1, ..., k\}}\) and \(\{G_{\xi_i+\delta}\}_{i \in \{1, ..., k\}}\) where \(G\) is the stationary horizon. From Theorem 5.1(iii), Theorem 5.4(iv) (using the fact that \(G_{\xi} \sim \text{SH}\). Finally we use our knowledge of the stationary horizon to show that with high probability the increments of \(h_t(0; h^i)\) agree with those of \(W_{\xi_i}(x, t; 0, t)\) on \([-a, a]\).
Since $a$ can be set to an arbitrarily large number, we conclude that (6.34) in fact holds on $\mathbb{R}$. The proof is now complete.

6.2. Construction of the global Busemann process. By Theorem 5.4, there exists an event of full probability, $\Omega_1$, on which the Busemann process along rational directions is well-defined. That is, we define the process

$$\{W_\alpha(p; q) : p, q \in \mathbb{R}^2, \alpha \in \mathbb{Q}\},$$

by (5.2), and on the event $\Omega_1$, this process satisfies all properties listed in Theorem 5.4.

Next, on the event $\Omega_1$, for an arbitrary direction $\xi$, and $t, x, y \in \mathbb{R}$, define

$$W_{\xi^-}(x, t; y, t) = \lim_{\alpha \rightarrow \xi^-} W_\alpha(x, t; y, t), \quad \text{and} \quad W_{\xi^+}(x, t; y, t) = \lim_{\alpha \rightarrow \xi^+} W_\alpha(x, t; y, t).$$

By Theorem 5.4(vi), these limits exist for all $t \in \mathbb{R}$. Complete the definition by setting,

$$W_{\xi\Delta}(x, s; y, t) = \sup_{z \in \mathbb{R}} \{\mathcal{L}(x, s; z, t) + W_{\xi\Delta}(z; t, y, t)\},$$

and finally for $s > t$, $W_{\xi\Delta}(x, s; y, t) = -W_{\xi\Delta}(y, t; x, s)$.

With this construction in place, we prove an intermediate lemma.

**Lemma 6.4.** The following hold on a single event of probability one across all points, directions and signs, unless specified otherwise.

(i) The horizontal Busemann functions are additive. That is, for $x, y, z, t \in \mathbb{R}$, $\xi \in \mathbb{R}$, and $\square \in \{-, +\}$,

$$W_{\xi\square}(x, t; y, t) + W_{\xi\square}(y, t; z, t) = W_{\xi\square}(x, t; z, t).$$

(ii) For every $\xi \in \mathbb{R}$ and $t \in \mathbb{R}$, as $\alpha \uparrow \xi$, as a function of $(x, y) \in \mathbb{R}^2$, $W_{\alpha\square}(y, t; x, t)$ converges uniformly on compact sets of $\mathbb{R}^2$ to $W_{\xi^-}(y, t; x, t)$. The same holds for limits from the right, with $\xi^+$ in place of $\xi^-$.

(iii) Fix $t, \xi \in \mathbb{R}$. Then, on a $(t, \xi)$-dependent full probability event, for every $x, y, z, t \in \mathbb{R}$,

$$W_{\xi^-}(y, t; x, t) = W_{\xi^+}(y, t; x, t).$$

In particular, for each $t$, there exists a $t$-dependent full-probability event on which, for all $\xi \in \mathbb{Q}$ and $x, y \in \mathbb{R}$, $W_{\xi^-}(y, t; x, t) = W_{\xi^+}(y, t; x, t) = W_{\xi}(y, t; x, t)$, where $W_{\xi}$ is the originally defined Busemann function from the construction for all rational directions.

(iv) For every $\xi \in \mathbb{R}$, $\square \in \{-, +\}$ and $t \in \mathbb{R}$, the function $x \mapsto W_{\xi\square}(x, t; 0, t)$ is continuous and satisfies

$$\lim_{x \rightarrow \pm \infty} \frac{W_{\xi\square}(x, t; 0, t)}{x} = 2\xi.$$

(v) For each $(x, s; y, t) \in \mathbb{R}^4$, $\xi \in \mathbb{R}$, and $\square \in \{-, +\}$, $W_{\xi\square}(x, s; y, t)$ is finite, and in the case $s < t$, the following limits hold

$$\lim_{z \rightarrow \pm \infty} \mathcal{L}(x, s; z, t) + W_{\xi\square}(z, t; y, t) = -\infty.$$

**Proof.** Item (i) follows from the respective property along the rational directions (Theorem 5.4(ii)) after taking limits.

**Item** (ii): The monotonicity of the horizontal Busemann process from Theorem 5.4(vi) extends to all directions by limits. That is, whenever $\xi_1 < \xi_2$, $x < y$, and $t \in \mathbb{R}$,

$$W_{\xi_1^-}(y, t; x, t) \leq W_{\xi_1^+}(y, t; x, t) \leq W_{\xi_2^-}(y, t; x, t) \leq W_{\xi_2^+}(y, t; x, t).$$

Hence, the limits in Item (ii) exist and agree with the limits taken across the rational directions. Without loss of generality, we take the compact set to be $[a, b]^2$. Then, by (6.39) and Lemma A.2, for $\alpha < \xi$, $\square \in \{-, +\}$, and $a \leq x \leq y \leq b$,

$$0 \leq W_{\xi^-}(y, t; x, t) - W_{\alpha\square}(y, t; x, t) \leq W_{\xi^-}(b, t; a, t) - W_{\alpha\square}(b, t; a, t),$$

and for general $(x, y) \in [a, b]^2$,

$$|W_{\xi^-}(y, t; x, t) - W_{\alpha\square}(y, t; x, t)| \leq |W_{\xi^-}(b, t; a, t) - W_{\alpha\square}(b, t; a, t)|,$$

so the convergence is uniform on compact sets.
Item (iii): We know that for $\alpha \in \mathbb{Q}$, $x \mapsto W_\alpha(x, t; 0, t)$ is a two-sided Brownian motion with diffusivity $\sqrt{2}$ and drift $2\alpha$. Hence, for $x, y, t \in \mathbb{R}$, $W_\alpha(y, t; x, t) \sim \mathcal{N}(2\alpha(y - x), 2|y - x|)$. By (6.39), for $x < y$, $W_{\xi -}(y, t; x, t) \leq W_{\xi +}(y, t; x, t)$, and both sides have the same distribution. Hence, for fixed $\xi, t \in \mathbb{R}$, on a $(\xi, t)$-dependent event of probability one, $W_{\xi -}(y, t; x, t) = W_{\xi +}(y, t; x, t)$ for all $x, y \in \mathbb{R}$. However, since the convergence of the Busemann functions from each side is uniform on compacts, (Item (ii)), both the functions $(x, y) \mapsto W_{\xi -}(y, t; x, t)$ and $(x, y) \mapsto W_{\xi +}(y, t; x, t)$ inherit continuity from the continuity for each rational direction (Theorem 5.4(v)), and equality extends to all $x, y \in \mathbb{R}$. The proof of the particular statement for rational directions $\xi$ is nearly identical: we add the extra note that for $x < y$,

$$W_{\xi -}(y, t; x, t) \leq W_{\xi}(y, t; x, t) \leq W_{\xi +}(y, t; x, t),$$

and all three terms in the string of inequalities have the same distribution.

Item (iv) The continuity was shown in the proof of the previous item. The limits (6.37) hold on an event of probability one for all rational $\xi$ and integers $t$ by Item (iii) and the fact that each function $x \mapsto W_\xi(x, t; 0, t)$ is a Brownian motion with diffusion coefficient $\sqrt{2}$ and drift $2\xi$ (Theorem 5.4(i)). By Theorem 5.4(iv), for all such rational directions, $t \mapsto W_\xi(-t; 0, 0)$ evolves as a KPZ fixed point backwards in time, and by Theorem 5.4(ii), $W_\xi(x, t; 0, t) = W_\xi(x; 0, 0) + W_t(0, 0; 0, t)$. Then, by Lemma B.10 and the temporal reflection invariance of Theorem B.1(v), the limits (6.37) extend to all $t \in \mathbb{R}$ and rational $\xi$. The monotonicity of (6.39) extends (6.37) to all $\xi, t \in \mathbb{R}$ and $\square \in \{-, +\}$.

Item (v) The finiteness of the horizontal Busemann functions $W_{\xi \|}(x, t; y, t)$ follows by taking limits and the monotonicity of (6.39). By the limit in Item (iv) and the bounds in Lemma B.2, the limits (6.38) hold for all $s < t$ on a single event of probability one. By (6.36), $W_{\xi \|}(x, s; y, t)$ is well-defined and finite. 

We prove one more lemma before the proof of Theorem 3.1. As in Definition 3.6, for $(x, s) \in \mathbb{R}^2$ and $t \geq s$, let $g_{\xi \|, L}^{L}(t)$ and $g_{\xi \|, R}^{L}(t)$ be, respectively, the leftmost and rightmost maximizers of $L(x, s; z, t) + W_{\xi \|}(z, t; 0, t)$ over $z \in \mathbb{R}$. By Lemma 6.4(iv)–(v), these maximizers are well-defined.

**Lemma 6.5.** The following holds on a single event of full probability, across all compact sets $K$ and directions $\xi \in \mathbb{R}$. Let $K \subseteq \mathbb{R}$ be a compact set, and let $\xi > 0$. Let $T > \max K$. Then, there exists a random $Z = Z(\xi, T, K) \in (0, \infty)$ such that for all $x, s \in K, \square \in \{-, +\}$, and $S \in \{L, R\}$, $|g_{\xi \|, S}^{L}(T)| \leq Z$.

**Proof.** Assume by way of contradiction that the statement is false. Then, there exists a sequence $z_n$ of maximizers of $L(x_n, s_n; z, T) + W_{\xi \|}(z, T; 0, T)$ over $z \in \mathbb{R}$ where $x_n, s_n \in K$ for all $n$, but $|z_n| \rightarrow \infty$. Furthermore, since $K$ is compact and $T > \max K$, there exist constants $0 < c_1, c_2 < \infty$ such that $c_1 < T - s_n < c_2$ for all $n$. By Lemma 6.4 (iv) and continuity of the KPZ fixed point (using the temporal reflection invariance of Theorem B.1(v)),

$$\inf_{x,s \in K} \sup_{z \in \mathbb{R}} \{L(x, s; z, T) + W_{\xi \|}(z, T; 0, T)\} > -\infty.$$  

Then, by (6.41),

$$\lim_{n \rightarrow \infty} \inf \{L(x_n, s_n; z_n, T) + W_{\xi \|}(z_n, T; 0, T)\} > -\infty.$$  

However, by Lemma 6.4(iv), there exists a constant $a > 0$ such that $|W_{\xi \|}(z, T; 0, T)| \leq a|z|$ for all $z \in \mathbb{R}$. This fact and the bounds of Lemma B.2, combined with the facts that $x_n, s_n$ lie in a compact set, $T - s_n$ is bounded away from 0 and $\infty$, and $|z_n| \rightarrow \infty$ gives a contradiction to (6.42).

**Proof of Theorem 3.1.** **Item (i):** The continuity of the Busemann functions for points along the same horizontal line is Lemma 6.4(iv). The general continuity follows from (6.36) and the continuity of the KPZ fixed point given initial data satisfying the conditions of Lemma 6.4(iv).

**Item (ii):** Recall Definition 3.6. First, we show that for $s < t$, $x \in \mathbb{R}$, $\xi_1 < \xi_2$, and $S \in \{L, R\}$,

$$-\infty < g_{\xi_1 \| - S}^{L}(t) \leq g_{\xi_1 \| + S}^{L}(t) \leq g_{\xi_2 \| - S}^{L}(t) \leq g_{\xi_2 \| + S}^{L}(t) < \infty.$$  


The finiteness of the maximizers comes from Lemma 6.4(v). The rest follows from the monotonicity of (6.39) and Lemma A.1. Next, we show that for \((x, s, y, t) \in \mathbb{R}^4\) and \(\xi \in \mathbb{R}\), \(W_\alpha(x, s; y, t)\) converges pointwise to \(W_{\xi^{-}}(x, s; y, t)\) as \(Q \ni \alpha \nearrow \xi\). The same holds for limits from the right, with \(\xi^{-}\) replaced by \(\xi^{+}\). (In fact, the convergence is uniform by Theorem 3.1(v), but this will be proven later in the paper.) We know the limits of the Busemann functions hold for \(s = t\) by definition. Since we define \(W_{\xi_{\square}}(x, s; y, t) = -W_{\xi_{\square}}(y, t; x, s)\), it suffices to assume \(s < t\). By (6.43) and the additivity of Lemma 6.4(i), for all \(\alpha \in [\xi - 1, \xi + 1] \cap Q\) and \(\square \in \{-, +\},\)

\[
W_\alpha(x, s; y, t) = \sup_{z \in \mathbb{R}} \{\mathcal{L}(x, s; z, t) + W_\alpha(z, t; y, t)\}
= \sup_{z \in \mathbb{R}} \{\mathcal{L}(x, s; z, t) + W_\alpha(z, 0; t)\} + W_\alpha(0, t; y, t)
= \sup_{z \in [g^{(\xi^{+}, L, g^{(\xi^{-})})}]} \{\mathcal{L}(x, s; z, t) + W_\alpha(z, 0; t)\} + W_\alpha(0, t; y, t).
\]

By Lemma 6.4(ii), \(W_\alpha(z, t; y, t)\) converges uniformly on compact sets to \(W_{\xi^{-}}(x, t; y, t)\) for \(Q \ni \alpha \nearrow \xi\) and to \(W_{\xi^{+}}(x, t; y, t)\) for \(Q \ni \alpha \searrow \xi\). This implies the desired pointwise convergence of the Busemann functions. The additivity now follows from the additivity of Busemann functions for each rational direction (Theorem 5.4(ii)).

Item (iii) was previously proven as Equation (6.39).

Item (iv): This follows directly from the construction (6.36). We postpone the proofs of Items (v)–(vii) until after the proof of Theorem 3.2. \(\square\)

Proof of Theorem 3.2. Items(i)–(ii): These follow from the construction of the Busemann functions from semi-infinite geodesics in rational directions, and then their extension via limits. Here, we are using the stationarity under space-time shifts of Theorem B.1(i)–(ii) and the fact that \(\{\mathcal{L}(x, s; y, t) : s, y \in \mathbb{R}, s < t \leq T\}\) is independent of \(\{\mathcal{L}(x, s; y, t) : s, y \in \mathbb{R}, T \leq s < t\}\) for \(T \in \mathbb{R}\).

Item (iii): By the additivity of Theorem 3.1(ii) and the variational definition (6.36), for \(x \in \mathbb{R}, s < t,\) and \(\square \in \{-, +\},\)

\[
W_{\xi_{\square}}(x, s; 0, s) = W_{\xi_{\square}}(x, s; 0, t) - W_{\xi_{\square}}(0, s; 0, t)
= \sup_{y \in \mathbb{R}} \{\mathcal{L}(x, s; y, t) + W_{\xi_{\square}}(y, t; 0, t)\} - \sup_{y \in \mathbb{R}} \{\mathcal{L}(0, s; y, t) + W_{\xi_{\square}}(y, t; 0, t)\}.
\]

By Item (i), Theorem 3.1(iii), and Items (ii) and (iv) of Lemma 6.4, \(\{W_{\xi^{+}}(\cdot, 0; 0, t) : \xi \in \mathbb{R}\}_{t \in \mathbb{R}}\) is a reverse-time Markov process with state space \(\mathcal{Y}\), defined in (6.1). By the stationarity of Item (ii), the law of \(\{W_{\xi^{+}}(\cdot, 0; 0, t) : \xi \in \mathbb{R}\}\) must be invariant for this process. By the temporal reflection invariance of the directed landscape (Theorem B.1(v)), \(\{W_{\xi^{+}}(\cdot, 0; 0, t) : \xi \in \mathbb{R}\}_{t \in \mathbb{R}}\) is also invariant for the KPZ fixed point, forward in time. The uniqueness of Theorem 2.1 completes the proof. \(\square\)

Proof of Theorem 3.1(v)–(vii). We first prove Item (v), then Item (vii), and finish with Item (vi).

Item (v): By Theorem 3.2(iii) and Theorem D.5(v), there exists an event of probability one, on which, for each \(\xi > 0\), each integer \(T\) and each compact set \(K \subseteq \mathbb{R}^2\), there exists a random \(\varepsilon = \varepsilon(\xi, T, K) > 0\) such that, for all \(\xi - \varepsilon < \alpha < \xi < \beta < \xi + \varepsilon\) and \((x, y) \in K,\)

\[
(6.44)\quad W_{\alpha_{\square}}(y, T; x, T) = W_{\xi^{-}}(y, T; x, T), \quad \text{and} \quad W_{\beta_{\square}}(y, T; x, T) = W_{\xi^{+}}(y, T; x, T).
\]

Let \(K \subseteq \mathbb{R}^4\) be compact, and let \(T\) be the smallest integer greater than \(\sup\{t \vee s : (x, s; y, t) \in K\}\). Let

\[
A := \inf\{g^{(\alpha^{-1})}_{(x,s)} \wedge g^{(\alpha^{-1})}_{(y,t)} : (x, s; y, t) \in K\}, \quad \text{and} \quad B := \sup\{g^{(\alpha^{+1})}_{(x,s)} \vee g^{(\alpha^{+1})}_{(y,t)} : (x, s; y, t) \in K\}.
\]

By Lemma 6.5 and (6.43), \(-\infty < A < B < \infty\). By the additivity of Theorem 3.1(ii), for all \((x, s; y, t) \in K\) and \(\xi \in (\alpha - 1, \alpha + 1),\)

\[
W_{\xi_{\square}}(x, s; y, t) = W_{\xi_{\square}}(x, s; 0, T) - W_{\xi_{\square}}(y, t; 0, T)
\]
By (6.44), the conclusion follows.

**Item (vii):** Fix $\xi \in \mathbb{R}$. By Lemma 6.4(iii), there exists an event of probability one, on which, for every $T \in \mathbb{Z}$, and $x, y \in \mathbb{R}$, $W_{\xi^-}(x; T; y, T) = W_{\xi^+}(x; T; y, T)$. Now, let $x, s, y, t \in \mathbb{R}$ be arbitrary and assume without loss of generality that $s \leq t$. Choose an integer $u \geq t$. Then, by the additivity of Theorem 3.1(ii) and the variational definition of the Busemann functions (6.36), for $\xi \in \mathbb{R}$ and $\Box \in \{-, +\}$,

$$W_{\xi^\Box}(x, s; y, t) = W_{\xi^\Box}(x, s; 0, 0) - W_{\xi^\Box}(y, s; 0, 0)$$

$$= \sup_{z \in \mathbb{R}} \{\mathcal{L}(x, s; z, u) + W_{\xi^\Box}(z, 0, u)\} - \sup_{z \in \mathbb{R}} \{\mathcal{L}(y, s; z, u) + W_{\xi^\Box}(z, 0, u)\}$$

$$= \sup_{z \in \mathbb{R}} \{\mathcal{L}(x, s; z, u) + W_{\xi^\Box}(z, u; 0, u)\} - \sup_{z \in \mathbb{R}} \{\mathcal{L}(y, s; z, u) + W_{\xi^\Box}(z, u; 0, u)\}.$$ 

The right-hand side is the same for both choices of $\Box$ by choice of $t$ and the full-probability event. In the case that $\xi$ is rational, Lemma 6.4(iii) also implies that this quantity is the same when the $\Box$ is removed, and the Busemann function is the same as originally defined for rational directions. Hence, for each $\xi \in \mathbb{R}$, on a $\xi$-dependent full-probability event, $W_{\xi} := W_{\xi^-} = W_{\xi^+}$, and this agrees with the construction of the Busemann functions from Section 5. In particular, additivity, Theorem 5.1(iii) and a union bound imply that, on a $\xi$-dependent event of full probability, for each $p, q \in \mathbb{Q}^2$ and $r_n = (z_n, u_n)$ with $u_n \to \infty$ and $z_n/u_n \to \xi$,

$$\lim_{n \to \infty} \mathcal{L}(p; r_n) - \mathcal{L}(q; r_n) = W_{\xi}(p, q).$$

Now, on this event, let $p, q \in \mathbb{R}^2$ be arbitrary. Choose $p', q' \in \mathbb{Q}^2$ such that the time coordinate of $p'$ is less than the time coordinate of $p$ and the time coordinate of $q'$ is greater than the time coordinate of $q$. Then, by metric composition of the directed landscape (Equation 2.1),

$$\mathcal{L}(p; r_n) - \mathcal{L}(q; r_n) \leq \mathcal{L}(p'; r_n) - \mathcal{L}(p'; p) - (\mathcal{L}(q'; r_n) + \mathcal{L}(q'; q')),$$

and so

$$\limsup_{n \to \infty} \mathcal{L}(p; r_n) - \mathcal{L}(q; r_n) \leq W_{\xi}(p'; q') - \mathcal{L}(p'; p) - \mathcal{L}(q'; q').$$

Sending $p' \to p$ and $q' \to q$ and using continuity of the directed landscape and the Busemann function (Item (i)),

$$\limsup_{n \to \infty} \mathcal{L}(p; r_n) - \mathcal{L}(q; r_n) \leq W_{\xi}(p; q).$$

An analogous argument gives the appropriate lower bound.

**Item (vi):** Considering the global event of full probability on which the previous Items of Theorem 3.1 hold, take the intersection of this event with the event on which, for all rational directions $\xi$, The limits (3.3) hold. By Lemma B.6, for arbitrary $\xi \in \mathbb{R}$, $t \in \mathbb{R}$, $s < y \in \mathbb{R}$, $\alpha, \beta \in \mathbb{Q}$ with $\alpha < \xi < \beta$, and a sequence $(z_n, u_n)$ with $u_n \to \infty$ and $z_n/u_n \to \xi$,

$$W_{\alpha}(y, t; x, t) \leq \liminf_{n \to \infty} \mathcal{L}(y, t; z_n, u_n) - \mathcal{L}(x, t; z_n, u_n)$$

$$\leq \limsup_{n \to \infty} \mathcal{L}(y, t; z_n, u_n) - \mathcal{L}(x, t; z_n, u_n) \leq W_{\beta}(y, t; x, t).$$

Sending $\mathbb{Q} \ni \alpha \nearrow \xi$ and $\mathbb{Q} \ni \beta \searrow \xi$ and using Item (v) completes the proof. $\Box$

Before proving Theorem 3.4, we prove a lemma that states that all discontinuities of the Busemann process are present simultaneously along each horizontal line.

**Lemma 6.6.** On a single event of probability one, for each $t \in \mathbb{R}$, $\xi \notin \Xi$ if and only if $W_{\xi^-}(y, t; x, t) = W_{\xi^+}(y, t; x, t)$ for all $x < y$. 


Proof. By the additivity of Theorem 3.1(ii), it suffices to show that $\xi \notin \Xi$ if and only if $W_{\xi^-}(x; t; 0, t) = W_{\xi^+}(x; t; 0, t)$ for all $x \in \mathbb{R}$. By the general construction of the Busemann process from (6.36), $\xi \in \Xi$ if and only if $W_{\xi^-}(x; t; 0, t) \neq W_{\xi^+}(x; t; 0, t)$ for some $x \in \mathbb{R}$. Thus, it suffices to show that, for $s < t$, $W_{\xi^-}(x; s; 0, s) = W_{\xi^+}(x; s; 0, s)$ for all $x \in \mathbb{R}$ if and only if $W_{\xi^-}(x; t; 0, t) = W_{\xi^+}(x; t; 0, t)$ for all $x \in \mathbb{R}$.

By the monotonicity of Theorem 3.1(iii) and Lemma A.2, for $0 < z < Z$,

$$0 \leq W_{\xi^+}(z; t; 0, t) - W_{\xi^-}(z; t; 0, t) \leq W_{\xi^+}(Z; t; 0, t) - W_{\xi^-}(Z; t; 0, t),$$

for $0 < z < 0$,

$$0 \geq W_{\xi^+}(z; t; 0, t) - W_{\xi^-}(z; t; 0, t) \geq W_{\xi^+}(Z; t; 0, t) - W_{\xi^-}(Z; t; 0, t).$$

By the additivity of Theorem 3.1(i), we know that for $\xi \in \mathbb{R}$ and $\Box \in \{-, +\}$,

$$W_{\xi^\Box}(x; s; 0, s) = \sup_{z \in \mathbb{R}} \{L(x, s; z, t) + W_{\xi^\Box}(z; t; 0, t)\} - \sup_{z \in \mathbb{R}} \{L(0, s; z, t) + W_{\xi^\Box}(z; t; 0, t)\}.$$  

Therefore, if $W_{\xi^-}(z; t; 0, t) = W_{\xi^+}(z; t; 0, t)$ for all $z \in \mathbb{R}$, then $W_{\xi^-}(x; s; 0, s) = W_{\xi^+}(x; s; 0, s)$ for all $z \in \mathbb{R}$. Conversely, assume that $W_{\xi^-}(w; t; 0, t) \neq W_{\xi^+}(w; t; 0, t)$ for some $w \in \mathbb{R}$. Assume that $w > 0$, and the case $w < 0$ follows by symmetry. By way of contradiction, assume that for all $x \in \mathbb{R}$, $W_{\xi^-}(x; s; 0, s) = W_{\xi^+}(x; s; 0, s)$. Then, by (6.47), the function

$$x \mapsto f(x) := \sup_{z \in \mathbb{R}} \{L(x, s; z, t) + W_{\xi^+}(z; t; 0, t)\} - \sup_{z \in \mathbb{R}} \{L(x, s; z, t) + W_{\xi^-}(z; t; 0, t)\}$$

is constant in $x$. By (6.45), for all $Z \geq w$,

$$0 < W_{\xi^+}(w; t; 0, t) - W_{\xi^-}(w; t; 0, t) \leq W_{\xi^+}(Z; t; 0, t) - W_{\xi^-}(Z; t; 0, t).$$

By Lemma B.2, for sufficiently large $x$ and $\Box \in \{-, +\}$,

$$\sup_{z \in \mathbb{R}} \{L(x, s; z, t) + W_{\xi^\Box}(z; t; 0, t)\} = \sup_{z \geq w} \{L(x, s; z, t) + W_{\xi^\Box}(z; t; 0, t)\}.$$ 

Then, for such large $x$, $f(x) > 0$. Conversely, by another application of Lemma B.2, if $x < 0$ with $|x|$ sufficiently large,

$$\sup_{z \in \mathbb{R}} \{L(x, s; z, t) + W_{\xi^\Box}(z; t; 0, t)\} = \sup_{z \leq 0} \{L(x, s; z, t) + W_{\xi^\Box}(z; t; 0, t)\},$$

and so by (6.46), $f(x) \leq 0$ for such $x$, a contradiction to the fact that $f$ must be constant. \hfill \Box

**Proof of Theorem 3.4.** **Item (i):** By the monotonicity of Theorem 3.1(vi) and Lemma A.2, for $a \leq x \leq y \leq b$,

$$0 \leq W_{\xi^+}(y; t; x, t) - W_{\xi^-}(y; t; x, t) \leq W_{\xi^+}(b; t; a, t) - W_{\xi^-}(b; t; a, t),$$

so the set $\Xi(x; t; -x, t)$ is nondecreasing in $x$. By Lemma 6.6, on a single event of probability one, for any $t \in \mathbb{R}$, $\xi \in \Xi$ if and only if $W_{\xi^-}(y; t; x, t) \neq W_{\xi^+}(y; t; x, t)$ for some $x < y$, so (3.5) follows.

**Item (ii):** By Theorems 3.2(iii) and D.5(v), there exists an event of probability one on which, for each integer $N \in \mathbb{Z}$, $W_{\xi^+}(N; 0; -N, 0) = W_{\xi^-}(N; 0; -N, 0)$ for all but countably many directions $\xi$. The countability of $\Xi$ then follows from (6.48) and Lemma 6.6. The density of the set $\Xi$ follows directly from Theorem D.5(viii). The fact that $P(\xi \in \Xi)$ for each $\xi \in \mathbb{R}$ is a direct consequence of Theorem 3.1(vii).

**Item (iii):** Let $p \neq q \in \mathbb{R}^2$ and $\xi \in \mathbb{R}$. We show that $\xi$ is not a limit point of $\Xi(p; q)$. By Theorem 3.1(v), there exists $\varepsilon > 0$ such that for all $0 < \varepsilon < \alpha < \xi < \beta < \xi + \varepsilon$ and $\Box \in \{-, +\}$, $W_{\alpha^\Box}(p; q) = W_{\xi^-}(p; q)$ and $W_{\beta^\Box}(p; q) = W_{\xi^+}(p; q)$. Hence, $\Xi(p; q) \cap \left( (\xi - \varepsilon, \xi + \varepsilon) \setminus \{\xi\} \right) = \emptyset$, and $\Xi(p; q)$ is a discrete set, as desired. Additionally, for any two successive points $\xi_1 < \xi_2$ of $\Xi(p; q)$, the function $\xi \mapsto W_{\xi^\pm}(p; q)$ is everywhere locally constant, so it must be constant on $I = (\xi_1, \xi_2)$. \hfill \Box
Consider the equality (3.6) and Lemma 6.7, for geodesic because the weight of the path in between any two points is optimal by Lemma 6.7. From additivity of the Busemann functions (Theorem 3.1(ii)),

Further, for any $t_{i-1} \leq t < u \leq t_i$, it must hold that $\mathcal{L}(g(t); t; g(u), u) = \mathcal{L}(g(t), t; g(u), u)$, for otherwise, by additivity of the Busemann functions (Theorem 3.1(ii)),

Thus, additivity extends (3.6) to all $s \leq t < u$. Therefore, the path is a semi-infinite geodesic because the weight of the path in between any two points is optimal by Lemma 6.7. From the equality (3.6) and Lemma 6.7, for every $t \geq s$, $g(t)$ maximizes $\mathcal{L}(x, s; z, t) + \mathcal{W}(z, t; 0, t)$ over $z \in \mathbb{R}$.

Before we show directedness of all the geodesics, we show that for $S \in \{L, R\}$, $g_{t,x}^{\xi_S}$ are semi-infinite geodesics and that they are the leftmost/rightmost geodesics between any two of their points. Take $S = R$, and the result for $S = L$ follows similarly. Knowing that these quantities
Lemma 6.8. Let \( g \) be as defined above. For \( s < t < u \) let \( z_u \) be the rightmost maximizer of 
\[ L(g(t), t; z, u) + W_{\xi}(z, u; 0, t) \]
over \( z \in \mathbb{R} \), and let \( w_t \) be the rightmost maximizer of 
\[ L(x, s; z, t) + W_{\xi}(z, u; 0, t) \]
on \( z \in \mathbb{R} \). By definition of \( g(u) \) and \( g(t) \) as the rightmost maximizers, we have \( w_t < g(t) \) and \( z_u \leq g(u) \) in general. Assume, to the contrary, that \( g(t) \neq w_t \) and \( u \neq z_u \). We first prove a contradiction in the case \( w_t < g(t) \). For the proof, refer to Figure 6.1 for clarity. Let \( \gamma_1 : [s, u] \rightarrow \mathbb{R} \) be the rightmost geodesic from \((x, s)\) to \((u, t)\) (which passes through \((w_t, t)\)), and let \( \gamma_2 \) be the concatenation of the rightmost geodesics from \((x, s)\) to \((g(t), t)\) followed by the rightmost geodesic from \((g(t), t)\) to \((z_u, u)\). By Item (i)(b) for \( i = 1, 2 \), the weight of the portion of any point of \( \gamma_i \) is equal to the Busemann function between the points. Since \( w_t < g(t) \) and \( z_u \) is the rightmost such maximizer, this gives the contradiction.

Now, we consider the case \( z_u < g(u) \). Define \( \gamma_1 \) and \( \gamma_2 \) as in the previous case. Since \( z_u < g(u) \), there is some point \((y, v)\) with \( t \leq v < u \) such that \( \gamma_1 \) splits from or crosses \( \gamma_2 \) at \((y, v)\). Then, define \( \hat{\gamma} \) as in the previous case. Again, the weight of the portion of the path \( \hat{\gamma} \) is equal to the Busemann function between the two points. Specifically, \( L(g(t), t; g(u), u) = L(g(t), t; z, u) + W_{\xi}(z, u; 0, t) \) over \( z \in \mathbb{R} \). This contradicts the definition of \( z_u \) as the rightmost such maximizer. □

Finally, we show the global directedness of all Busemann geodesics constructed in the manner described in Item (i). By (6.43), for \( t \geq s \) and \( \alpha < \xi < \beta \) with \( \alpha, \beta \in \mathbb{Q} \) (dropping the \( \pm \) superscript by Theorem 3.1(vii)),
\[ g_{(x,s)}^{\alpha,L}(t) \leq g_{(x,s)}^{\xi,L}(t) \leq g(t) \leq g_{(x,s)}^{\xi,R}(t) \leq g_{(x,s)}^{\beta,R}(t). \]
By Theorem 5.4(iv), on an event of probability one, for all rational \( \alpha \), the maximizers of \( L(x, s; z, t) + W_{\xi}(z, t; 0, t) \) over \( z \in \mathbb{R} \) are exactly the locations \( z \) where an \( \alpha \)-directed geodesic goes through \((z, t)\). Therefore, \( g_{(x,s)}^{\alpha,L}(t)/t \rightarrow \alpha \) and \( g_{(x,s)}^{\beta,R}(t)/t \rightarrow \beta \) when \( \alpha \) and \( \beta \) are rational. Sending \( \mathbb{Q} \ni \alpha \nearrow \xi \) and \( \mathbb{Q} \ni \beta \searrow \xi \), using the monotonicity of (6.43) completes the proof of directedness. □

We postpone the proof of Theorem 3.10 until after we prove parts of Theorem 3.11 and Theorem 3.13.

Proof of Theorem 3.11, Items (i)–(iii). Item (i) was already proven as Equation (6.43).

Item (ii): This follows a similar proof as the proof of Theorem 3.1(v), except that \( K \) is now a compact subset of \( \mathbb{R} \), and \( T \) is now chosen to be any integer greater than \( k \), so \( \varepsilon \) depends on \( T \) as well. Set
\[ A = \inf \{ g_{(x,s)}^{(\alpha-1)-L}(T) : x, s \in K \}, \quad \text{and} \quad B = \sup \{ g_{(x,s)}^{(\alpha+1)+R}(T) : x, s \in K \}, \]
and similarly, \( -\infty < A < B < \infty \). Then, for all \( 0 < \varepsilon < 1 \) sufficiently small, all \( \xi - \varepsilon < \alpha < \xi \) and all \( x, s \in K \), the functions \( z \mapsto L(x, s; z, T) + W_{\xi}(z, T; 0, T) \) and \( z \mapsto L(x, s; z, t) + W_{\xi}(z, t; 0, T) \) agree on the set \( [A, B] \), which contains all maximizers. Hence, for such \( \alpha, \varnothing \in \{ -, + \} \), and \( S \in \{ L, R \} \), \( g_{(x,s)}^{\alpha,S}(T) = g_{(x,s)}^{\xi-S}(T) \). Since \( g_{(x,s)}^{\alpha,L} : [s, \infty) \rightarrow \mathbb{R} \) and \( g_{(x,s)}^{\alpha,R} : [s, \infty) \rightarrow \mathbb{R} \) define semi-infinite geodesics that are, respectively, the leftmost and rightmost geodesics between any of their points (Theorem 3.7(ii)–(iii)), it must also hold that for \( S \in \{ L, R \} \) and \( t \in [T, T] \), \( g_{(x,s)}^{\alpha,S}(t) = g_{(x,s)}^{\xi-S}(t) \).

Otherwise, taking \( S = L \) without loss of generality, there would exist two distinct leftmost geodesics depending on \( x, s, \xi \), and \( \varnothing \), for shorthand notation, let \( g(t) = g_{(x,s)}^{\xi,R}(t) \) and the same for \( t \) replaced with \( u \). By what was just proved, it is sufficient to prove the following lemma.
from \((x, s)\) to \((g_{(x, s)}^{\xi_L}(T), T)\), a contradiction. The proof for the \(\xi^+\) geodesics where \(\beta\) is sufficiently close to \(\xi\) from the right is analogous.

**Item (iii):** By the monotonicity of Item (ii), the limits \(\lim_{\xi \to \infty} g_{(x, s)}^{\xi, S}(t)\) and \(\lim_{\xi \to -\infty} g_{(x, s)}^{\xi, S}(t)\) exist in \(\mathbb{R} \cup \{-\infty, \infty\}\). Furthermore, by this monotonicity and the spatial reflection invariance of the directed landscape (Theorem B.1(v)), it is sufficient to show that

\[
\lim_{\xi \to \infty} g_{(x, s)}^{\xi, -L}(t) = \infty.
\]

First, we show that (6.50) holds with probability one for fixed initial point \((x, s)\) and fixed \(t > s\). It is sufficient to take \((x, s) = (0, 0)\) and then \(t > 0\). We know that \(W_{\xi_L}(z; t; 0, t)\) is a two-sided Brownian motion with drift \(2\xi\) and diffusivity \(\sqrt{2}\), independent of the random function \((x, y) \mapsto \mathcal{L}(x, 0; y, t)\) (Theorem 3.2(i)). Using the skew stationarity of Theorem B.1(iv) with \(c = -\xi\), we obtain

\[
\begin{align*}
\arg \sup_{z \in \mathbb{R}} \{\mathcal{L}(0, 0; z, t) + W_{\xi_L}(z; t; 0, t)\} \\
&= \arg \sup_{z \in \mathbb{R}} \{\mathcal{L}(0, 0; z, t) + \sqrt{2} B(z) + 2\xi z\} \\
&= \arg \sup_{z \in \mathbb{R}} \{\mathcal{L}(0, 0; z - \xi t, t) - 2\xi z + \xi^2 t + \sqrt{2} B(z) + 2\xi z\} \\
&= \arg \sup_{z \in \mathbb{R}} \{\mathcal{L}(0, 0; z - \xi t, t) + \sqrt{2} (B(z) - B(\xi t))\} \\
&= \arg \sup_{z \in \mathbb{R}} \{\mathcal{L}(0, 0; z - \xi t, t) + \sqrt{2} B(z - \xi t)\} \\
&= \arg \sup_{z \in \mathbb{R}} \{\mathcal{L}(0, 0; z, t) + \sqrt{2} B(z)\} + \xi t.
\end{align*}
\]

Since the process \(\{\mathcal{L}(0, 0; z, t) : z \in \mathbb{R}\}\) is continuous and independent of \(B\), Theorem 2 in [Pim14], combined with the parabolic decay of \(\mathcal{L}\) (Lemma B.2) implies that \(\mathcal{L}(0, 0; z, t) + \sqrt{2} B(z)\) almost surely has a unique maximizer. Therefore, for each \(\xi \in \mathbb{R}\), the distribution of \(g_{(x, s)}^{\xi, -L}(t)\) is that of a fixed, almost surely finite, random variable plus \(\xi t\). Since we know \(\lim_{\xi \to \infty} g_{(x, s)}^{\xi, -L}(t)\) exists, the limit must be infinite.

Now, consider the event of probability one, on which for each triple \((w, q_1, q_2)\) \(\in \mathbb{Q}^3\) with \(q_1 < q_2\),

\[
\lim_{\xi \to \infty} g_{(w; q_1)}^{\xi, -L}(q_2) = \infty.
\]

On this event, let \((x, s, t) \in \mathbb{R}^3\) with \(s < t\) be arbitrary. Assume, by way of contradiction, that

\[
(6.52) \quad z := \sup_{\xi \in \mathbb{R}} g_{(x, s)}^{\xi, -L}(t) < \infty,
\]

and let \(g : [s, t]\) denote the leftmost geodesic from \((x, s)\) to \((z, t)\). For this proof, refer to Figure 6.2 for clarity. By the assumption (6.52) and the fact that \(g_{(x, s)}^{\xi, -L}\) is the leftmost geodesic between any two of its points (Theorem 3.7(iii)), \(g_{(x, s)}^{\xi, -L}(t) \leq g(t)\) for all \(\xi \in \mathbb{R}\) and \(t > s\). Let \(q_1 \in (s, t)\) be rational. Choose \(w \in \mathbb{Q}\) such that \(w < g(q_1)\). By continuity of geodesics, we may choose \(q_2 \in (q_1, t) \cap \mathbb{Q}\) to be sufficiently close to \(t\) so that \(|g(q_2) - z| < 1\). Next, by (6.51), we may choose \(\xi\) sufficiently large so that

\[
(6.53) \quad g_{(w; q_1)}^{\xi, -L}(q_2) > z + 1 > g(q_2) \geq g_{(x, s)}^{\xi, -L}(q_2).
\]

Since \(w < g(q_1)\), the paths \(g_{(w; q_1)}^{\xi, -L}\) and \(g_{(x, s)}^{\xi, -L}\) must cross at some point \((\hat{z}, \hat{t})\) with \(\hat{t} \in (q_1, q_2)\). By Theorem 3.7(ii), both \(g_{(w; q_1)}^{\xi, -L}(q_2)\) and \(g_{(x, s)}^{\xi, -L}(q_2)\) are the leftmost maximizer of \(\mathcal{L}(\hat{z}, \hat{t}; y, q_2) + W_{\xi, -L}(y; q_2; 0, q_2)\) over \(y \in \mathbb{R}\). This contradicts (6.53).

We postpone the proof of Items (iv) and (v) until after the proof of Theorems 3.13 and 3.10. \(\Box\)
Proof of Theorem 3.13. Item (i): Let $\alpha < \xi < \beta$. By directedness of geodesics, for all sufficiently large $n$,
\[
g_{(x,s)}^{\alpha,-L}(t_n) < x_n < g_{(x,s)}^{\beta,+R}(t_n).
\] Since $g_{(x,s)}^{\alpha,-L}$ is the leftmost geodesic between any of its points and $g_{(x,s)}^{\beta,+R}$ is the rightmost, the entire path $g_n : [s, t_n] \to \mathbb{R}$ must stay weakly between the portions of $g_{(x,s)}^{\alpha,-L}$ and $g_{(x,s)}^{\beta,+R}$ from time $s$ to time $t_n$. Hence, for all $t \geq s$,
\[
g_{(x,s)}^{\alpha,-L}(t) \leq \liminf_{n \to \infty} g_n(t) \leq \limsup_{n \to \infty} g_n(t) \leq g_{(x,s)}^{\beta,+R}(t).
\] By Theorem 3.11(ii), taking limits as $\alpha \nearrow \xi$ and $\beta \searrow \xi$ completes the proof.

Item (ii): If there is a unique semi-infinite geodesic from $(x, s)$ in direction $\xi$, then $g_{(x,s)}^{\xi,-L} = g_{(x,s)}^{\xi,+R}$. The result follows by the same reasoning as in Item (i), but also applying Theorem 3.11(ii).

Proof of Theorem 3.10. We prove (3.10). If $(x, s) \in \text{NU}^0_0$, then there exists $\xi \in \mathbb{R}$ and $\Box \in \{-, +\}$ such that $g_{(x,s)}^{\xi\Box,L}(t) < g_{(x,s)}^{\xi\Box,R}(t)$ for some $t > s$. By Theorem 3.7(ii), there exists a rational direction $\alpha$ (greater than $\xi$ if $\Box = +$ and less than $\xi$ if $\Box = -$) such that
\[
g_{(x,s)}^{\alpha,L}(t) = g_{(x,s)}^{\xi\Box,L}(t) < g_{(x,s)}^{\xi\Box,R}(t) = g_{(x,s)}^{\alpha,R}(t).
\] Hence, $(x, s) \in \text{NU}^\alpha_0$. An analogous proof shows that $\text{NU} = \bigcup_{\xi \in \mathbb{Q}} \text{NU}^\xi_0$.

Item (i): By Theorem 5.1(iii), for fixed direction $\xi$ and fixed initial point $p$, there is a unique semi-infinite geodesic from $p$ in direction $\xi$, implying $(x, s) \notin \text{NU}^\xi_0$. The result now follows directly from (3.10) and a union bound.

Item (ii) By Theorem 5.1(v) and a union bound, for fixed time $s$, there is a $s$-dependent full probability event on which, for each rational direction $\xi$, $\text{NU}^\xi_0 \cap \mathcal{H}_s$ is at most countably infinite. The fact that $\text{NU}_0 \cap \mathcal{H}_s$ is at most countable then follows from (3.10). We postpone the proof that $\text{NU}_0 \cap \mathcal{H}_s$ is nonempty until the end of this subsection.
**Remaining proofs of Theorem 3.11.** Item (iv): We first prove a weaker result, namely that for \( s \in \mathbb{R}, x < y, \xi \in \mathbb{R}, \square \in \{-, +\}, \) and \( S \in \{L, R\}, \)

\[
(6.54) \quad g_{(x,s)}^{\xi, S}(t) \leq g_{(y,s)}^{\xi, S}(t) \quad \text{for all} \ t \geq s.
\]

Assume that for some \( t > s, z := g_{(x,s)}^{\xi, L}(t) = g_{(y,s)}^{\xi, L}(t). \) By continuity of geodesics, it is sufficient to show that \( g_{(x,s)}^{\xi, L}(u) = g_{(y,s)}^{\xi, L}(u) \) for all \( u > t. \) By Theorem 3.7(ii), for \( u > t, \) both \( g_{(x,s)}^{\xi, L}(u) \) and \( g_{(y,s)}^{\xi, L}(u) \) are the leftmost maximizer of \( w \mapsto L(z, t; w, t) + W_{\xi}(w, t; 0, t), \) so they are equal.

Now, to prove the stated result, we follow a similar argument as Item 2 of Theorem 5 in [RV21] (recorded as Theorem 5.1(iv) in the present paper), adapted to give a global result across all direction, signs, and pairs of points along the same horizontal line. Let \( g_1 \) be a \( \xi \square \) geodesic from \((x, s)\), and let \( g_2 \) be a \( \xi \square \) geodesic from \((y, s)\), and assume that \( g_1(t) = g_2(t) =: g(t) \) for some \( t > s. \) By continuity of geodesics, we may take \( t \) to be the minimal such time. Send choose \( q \in (g_1(r), g_2(r)) \cap \mathbb{Q}. \) See Figure 6.3. By Theorem 3.10(i), there is a unique \( \xi \square \) Busemann geodesic from \((q, r)\), which we shall call \( g = g_{(x,s)}^{\xi, L} = g_{(y,s)}^{\xi, R}. \) Then, by (6.54), for \( u \geq r, \)

\[
g_{(x,s)}^{\xi, R}(u) \leq g(u) \leq g_{(y,s)}^{\xi, L}(u).
\]

Specifically, for \( u = t \), all the inequalities above are equalities, and by applying (6.54) twice, equality holds for all \( u \geq t. \)

**Item (v):** We prove the first limit in (3.11), and the second follows analogously. By Item (iv), \( z := \lim_{w \searrow x} g_{(w,s)}^{\xi, L}(t) \) exists and is less than or equal to \( g_{(x,s)}^{\xi, L}(t). \) Further, by the same monotonicity, for all \( w \in [x - 1, x], \) all maximizers of \( L(x, s; y, t) + W_{\xi}(y, t; 0, t) \) over \( y \in \mathbb{R} \) lie in the common compact set \([g_{(x-1,s)}^{\xi, L}(t), g_{(x,s)}^{\xi, R}(t)]. \) By continuity of the directed landscape, as \( w \nearrow s, \) the function \( y \mapsto L(w, s; y, t) + W_{\xi}(y, t; 0, t) \) converges uniformly on compact sets to the function \( y \mapsto L(x, s; y, t) + W_{\xi}(y, t; 0, t). \) Hence, Lemma A.3 implies that \( z \) is a maximizer of \( L(x, s; y, t) + W_{\xi}(y, t; 0, t) \) over \( y \in \mathbb{R}. \) Since \( z \leq g_{(x,s)}^{\xi, L}(t), \) and \( g_{(x,s)}^{\xi, L}(t) \) is the leftmost such maximizer, equality holds.

The proof of (3.12) is similar: in this case, Lemma 6.5 implies that for all \((w, u)\) sufficiently close to \((x, s), \) the maximizers of \( L(w, u; y, t) + W_{\xi}(y, t; 0, t) \) lie in a common compact set. Then, by Lemma A.3, every subsequential limit of \( g_{(w,u)}^{\xi, S}(t) \) as \((w, u) \to \) \((x, s)\) is a maximizer of \( L(x, s; y, t) + W_{\xi}(y, t; 0, t) \) over \( y \in \mathbb{R}. \) By assumption, there is only one such maximizer, so the desired convergence holds.
Lastly, (3.13) follows from applying Lemmas 6.4(iv) and B.9, as well as the temporal reflection invariance of Theorem B.1(v).

It remains to show that $\text{NU}_0 \cap \mathcal{H}_s$ is nonempty on a $s$-dependent full probability event. We prove some technical lemmas first. For $s \in \mathbb{R}$, recall that $\mathcal{H}_s = \{(x, s) : x \in \mathbb{R}\}$.

**Lemma 6.9.** On a single event of full probability, the following holds. Let $\xi \in \mathbb{R}$, $\square \in \{-, +\}$, $s < t$, $\xi \in \mathbb{R}$, and assume that there is a nonempty interval $I = (a, b) \subseteq \mathbb{R}$ such that no $\xi \square$ geodesic starting on level $s$ crosses $I \times \{t\}$. That is, for all $x \in \mathbb{R}$, $g_{(x,s)}^{\xi \square, R}(t) \leq a$ or $g_{(x,s)}^{\xi \square, L}(t) \geq b$. Then, the set $\text{NU}_0^{\square} \cap \mathcal{H}_s$ is nonempty.

**Proof.** Choose some $y \in (a, b)$, and let $\varepsilon > 0$ be such that $(y - \varepsilon, y + \varepsilon) \subseteq (a, b)$. Let

$$\hat{x} = \sup\{x \in \mathbb{R} : g_{(x,s)}^{\xi \square, R}(t) < y\}.$$

By assumption that no $\xi \square$ geodesic passes through $I \times \{t\}$ and the monotonicity of Theorem 3.11(iv), for all $x < \hat{x}$ and $S \in \{L, R\}$, $g_{(x,s)}^{\xi \square, S}(t) < y - \varepsilon$, while for all $x > \hat{x}$ and $S \in \{L, R\}$, $g_{(x,s)}^{\xi \square, S}(t) > y + \varepsilon$. Then, by taking limits via Equation (3.11) of Theorem 3.11(v),

$$g_{(\hat{x}, s)}^{\xi \square, L}(t) < y - \varepsilon < y + \varepsilon < g_{(\hat{x}, s)}^{\xi \square, R}(t),$$

so $(\hat{x}, s) \in \text{NU}_0^{\xi \square}$. \hfill $\square$

**Lemma 6.10.** Let $I \subseteq \mathbb{R}$ be an interval of length $|I| = l > 0$. There exists $p(l) > 0$ such that

(6.55) $\mathbb{P}(g_{(z,0)}^{0,S}(1) \notin I, \text{ for all } z \in \mathbb{R}, S \in \{L, R\}) > p.$

**Remark 6.11.** In the lemma above, we wrote $g_{(z,0)}^{0,S}$ without reference to the sign $\square \in \{-, +\}$ or $S \in \{L, R\}$. By the monotonicity of Theorem 3.11(iv) and since all 0-directed geodesics are Busemann geodesics (Theorem 3.20(viii)), if the event in (6.55) holds, then there are no geodesics from $(z, 0)$ in direction 0 that pass through $I \times \{1\}$.

**Proof.** Since the direction 0 is fixed, we will drop the superscript from $g_p^0$ in the proof. Let $M_1, M_2, M_3 > 0$ be large numbers to be determined later. We define

$$a_1 = -M_3 \quad b_1 = -M_3 + 1 \quad p_1 = (a_1, 0)$$
$$a_2 = M_3 - 1 \quad b_2 = M_3 \quad p_2 = (b_2, 0).$$

Let $W(\cdot) := W_0(\cdot, 1; 0, 1)$ be the Busemann function at level 1 in direction 0, in particular, $W$ is a Brownian motion conditioned to vanish at the origin. Define the following events

$$A_W := \left\{ \max_{y \in (b_1, a_2)} W(y) \leq 1, \max_{y \in [a_1, b_1]} W(y) \in [M_2, M_2 + 1], \max_{y \in [a_2, b_2]} W(y) \in [M_2, M_2 + 1], \sup_{y \geq b_2} W(y) - (y - a_1)^2 + M_1 \log^{8/3}(2\sqrt{a_1^2 + y^2 + 1}) < -1, \sup_{y \leq a_1} W(y) - (y - b_2)^2 + M_1 \log^{8/3}(2\sqrt{b_2^2 + y^2 + 1}) < -1 \right\}$$

and

$$B_\mathcal{L} := \left\{ -(y - x)^2 - M_1 \log^{8/3}\left(2\sqrt{x^2 + y^2 + 1 + 2}\right) \leq \mathcal{L}(x, 0; y, 1) \leq -(y - x)^2 + M_1 \log^{8/3}\left(2\sqrt{x^2 + y^2 + 1 + 2}\right) \text{ for every } x, y \in \mathbb{R} \right\}.$$

To show (6.55), it is enough to show the following

(6.56) $A_W \cap B_\mathcal{L} \subseteq \{g_{p_1}(1) \notin (-\infty, b_1]\}$
(6.57) $A_W \cap B_\mathcal{L} \subseteq \{g_{p_2}(1) \notin [a_2, \infty]\}$
(6.58) $A_W \cap B_\mathcal{L} \subseteq \{g_q(1) \notin [b_1, a_2] : q = (z, 0), z \in (a_1, b_2)\}.$

Indeed, (6.56)–(6.57) will imply that on $A_W \cap B_\mathcal{L}$, $g_q(1) \notin [b_1, a_2]$, for any $p = (z, 0)$ where $z \in (-\infty, a_1] \cup [b_2, \infty)$. Then, (6.58) will show that no infinite geodesic (in direction 0) starting
We see that (6.59) indeed holds, which concludes the proof of (6.56)–(6.57). We now turn to prove

On $A_W \cap B_L$

\[
\text{arg max}_{y \geq a_1} L(a_1, 0; y, 1) + W(y) \in [a_1, b_1],
\]

which is equivalent to showing that

\[
(6.59) \quad \sup_{y \in [a_1, b_1]} L(a_1, 0; y, 1) + W(y) > \sup_{y \geq b_1} L(a_1, 0; y, 1) + W(y).
\]

On $A_W \cap B_L$

\[
(6.60) \quad \sup_{y \in [a_1, b_1]} L(a_1, 0; y, 1) + W(y) \geq -(b_1 - a_1)^2 - M_1 \log^{8/3} (2(\sqrt{a_1^2 + b_1^2 + 1} + 2)) + M_2.
\]

On the other hand,

\[
\sup_{y \geq b_1} L(a_1, 0; y, 1) + W(y) = \max \left\{ \sup_{y \in [b_1, a_2]} L(a_1, 0; y, 1) + W(y), \sup_{y \in [a_2, b_2]} L(a_1, 0; y, 1) + W(y), \sup_{y \geq b_2} L(a_1, 0; y, 1) + W(y) \right\}.
\]

On $A_W \cap B_L$

\[
\sup_{y \in [b_1, a_2]} L(a_1, 0; y, 1) + W(y) \leq M_1 \log^{8/3} (2(\sqrt{b_1^2 + a_2^2 + 1} + 2)) + 1
\]

\[
\sup_{y \in [a_2, b_2]} L(a_1, 0; y, 1) + W(y) \leq -(a_2 - a_1)^2 + M_1 \log^{8/3} (2(\sqrt{a_1^2 + b_2^2 + 1} + 2)) + M_2
\]

\[
\sup_{y \geq b_2} L(a_1, 0; y, 1) + W(y) \leq -1.
\]

It follows that

\[
(6.61) \quad \sup_{y \geq b_1} L(a_1, 0; y, 1) + W(y) \leq [M_2 - (a_2 - a_1)^2]^+ + M_1 \log^{8/3} (2(\sqrt{a_1^2 + b_2^2 + 1} + 2)) + 1.
\]

For a given $M_1 > 1$, we set $M_2 = M_1^2$ and $M_3 = \frac{1}{2} M_1$. For large enough $M_1$, using (6.61) and

\[
(6.60) \quad \sup_{y \in [a_1, b_1]} L(a_1, 0; y, 1) + W(y) \geq \frac{15}{16} M_1^2 > \frac{14}{16} M_1^2 \geq \sup_{y \geq b_1} L(a_1, 0; y, 1) + W(y).
\]

We see that (6.59) indeed holds, which concludes the proof of (6.56)–(6.57). We now turn to prove

(6.58). It is enough to show that

\[
(6.62) \quad \sup_{y \in [b_1, a_2]} L(z, 0; y, 1) + W(y) < \sup_{y \in [a_1, b_1]} L(z, 0; y, 1) + W(y) \quad \forall z \in [a_1, b_2].
\]

On $A_W \cap B_L$ we have

\[
\sup_{y \in [b_1, a_2]} L(z, 0; y, 1) + W(y) \leq M_1 \log^{8/3} (2(\sqrt{a_1^2 + b_2^2 + 1} + 2)) + 1 \quad \forall z \in [a_1, b_2]
\]

\[
-(b_2 - a_1)^2 - M_1 \log^{8/3} (2(\sqrt{a_1^2 + b_2^2 + 1} + 2)) + M_2 \leq \sup_{y \in [a_1, b_1]} L(z, 0; y, 1) + W(y).
\]

For large enough $M_1$

\[
\sup_{y \in [b_1, a_2]} L(z, 0; y, 1) + W(y) \leq M_1 \log^{8/3} (2(\sqrt{2M_1^2 + 1} + 2)) + 1 < \frac{14}{16} M_1^2 \leq \sup_{y \in [a_1, b_1]} L(z, 0; y, 1) + W(y),
\]

which implies (6.62), which in turns implies (6.58). From Lemma B.2, there exists a random constant $C > 0$ such that

\[
\mathbb{P}(C > m) \leq ce^{-dm^{3/2}}
\]
for some universal constants $c,d > 0$, and
\[
|\xi(x, 0; y, 1) + (x - y)^2| \leq C \log^{4/3} \left(2(\sqrt{x^2 + y^2 + 1} + 2)\right) \log^{2/3}(\sqrt{x^2 + y^2 + 1} + 2)
\]
In particular, we can take $M_1$ large enough so that
\[
\mathbb{P}(B_L) \geq 1/2.
\]
On the other hand, there exists (typically small) $1 > c_W > 0$ such that
\[
(6.63) \quad \mathbb{P}(A_W) > c_W.
\]
As $B_L$ and $A_W$ are independent we see that
\[
\mathbb{P}(g_z(1) \not\in [-M + 1, M - 1], \text{ for all } z \in \mathbb{R}) \geq \mathbb{P}(B_L \cap A_W) \geq \frac{c_W}{2},
\]
which implies the result. \hfill \square

**Proof that the set $NU_0 \cap H_s$ is nonempty on an $s$-dependent full probability event.** It suffices to take $s = 0$. By Lemma 6.9, we have the following containment of events.

\[
(6.64) \quad \{NU_0 \cap H_0 \text{ is nonempty}\} \supseteq \{NU_0^0 \cap H_0 \text{ is nonempty}\} \supseteq \bigcup_{a < b, x \in \mathbb{R}} \{g_{(x)}(1) \leq a \} \cup \{g_{(x)}(1) \geq b\}.
\]

By choosing an interval with rational endpoints contained in $(a, b)$, the union on the right-hand side of (6.64) can be taken over all $a < b \in \mathbb{Q}$. Furthermore, by the limits of Theorem 3.11(v), the intersection can be taken over all $x \in \mathbb{R}$. Hence, the event on the right in (6.64) is measurable. Further, this event is invariant under the following shift

\[
(6.65) \quad \{\xi(x, 0, y, 1), W_0(z, 1; 0, 1) : x, y, z \in \mathbb{R}\} \rightarrow \{\xi(x + w, 0, y + w, 1), W_0(z + w, 1; w, 1) : x, y, z \in \mathbb{R}\},
\]

because it moves the interval $(a, b)$ on the right-hand side of (6.64) to $(a - w, b - w)$. By Lemma B.4, the process $\{\xi(x, 0, y, 1) : x, y \in \mathbb{R}\}$ is mixing under the shift. The process $\{W_{\xi}(z, 1; 0, 1) : z \in \mathbb{R}\}$ is mixing under the shift because it is a Brownian motion with drift and hence has independent increments. By independence of Theorem 3.2(i) and Lemma B.5, $\{\xi(x, 0, y, 1), W_{\xi}(z, 1; 0, 1) : x, y, z \in \mathbb{R}\}$ is mixing under the shift and therefore ergodic. Hence,

\[
\mathbb{P}\left(\bigcup_{a < b, x \in \mathbb{R}} \{g_{(x)}(1) \leq a \} \cup \{g_{(x)}(1) \geq b\}\right) \in \{0, 1\}.
\]

By Lemma 6.10, this event has positive probability and therefore has probability one. The desired conclusion now follows from (6.64). \hfill \square

### 6.4. Proofs of coalescence and global structure of the geodesics

In this section, we prove the results of Section 3.4. We first prove some lemmas. This approach for proving global coalescence was developed in the BLPP setting in [SS21b] (See Section 8 in that paper).

**Lemma 6.12.** The following holds globally on a single event of probability one. Let $s \in \mathbb{R}$ and $x < y$. Assume that, for some $\alpha \leq \xi$ and $\square_1, \square_2 \in \{-, +\}$, we have that $W_{\alpha \square_1}(y, s; x, s) = W_{\xi \square_2}(y, s; x, s)$. If $t > s$ and $g_{(x)}^{\square_1}(t) \leq g_{(y)}^{\alpha \square_2}(t)$, then for all $u \in [s, t]$,

\[
(6.66) \quad g_{(x)}^{\alpha \square_1}(u) = g_{(x)}^{\xi \square_2}(u), \quad \text{and} \quad g_{(y)}^{\alpha \square_1}(u) = g_{(y)}^{\xi \square_2}(u).
\]

**Proof.** If $\alpha = \xi$, we assume without loss of generality that $\square_1 = -$ and $\square_2 = +$. That way, whenever $w < z$ and $t \in \mathbb{R}$, Theorem 3.11(iii) implies the monotonicity

\[
(6.67) \quad W_{\alpha \square_1}(z, t; w, t) \leq W_{\xi \square_2}(z, t; w, t).
\]

For the rest of the proof, we will suppress the $\square_1, \square_2$ notation. By additivity and the definition of the Busemann functions, we may write

\[
W_{\xi}(y, s; x, s) = W_{\xi}(y, s; 0, t) - W_{\xi}(x, s; 0, t)
\]

\[
= \sup_{z \in \mathbb{R}} \{\mathcal{L}(y, s; z, t) + W_{\xi}(z, t; 0, t)\} - \sup_{z \in \mathbb{R}} \{\mathcal{L}(x, s; z, t) + W_{\xi}(z, t; 0, t)\},
\]

where $\mathcal{L}(y, s; z, t)$ denotes the local time at $z$ of the Brownian motion starting at $y$ at $s$.
and the same with $\xi$ replaced by $\alpha$. Recall that \( g_{(x,s)}^{\xi,l}(t) \) and \( g_{(x,s)}^{\xi,d}(t) \) are, respectively, the leftmost and rightmost maximizers of \( \mathcal{L}(x, s; z, t) + W_\xi(z, t; 0, 0) \) over \( z \in \mathbb{R} \). Understanding that these quantities depend on \( s \) and \( t \), we use the shorthand notation \( g_x^{\xi,R} = g_{(x,s)}^{\xi,d}(t) \), and similarly with the other quantities. Then, we have

\[
\mathcal{L}(x, s; g_x^{\xi,R}(t), W_\xi; t; 0, t) = \sup_{z \in \mathbb{R}} \{ \mathcal{L}(x, s; z, t) + W_\xi(z, t; 0, 0) \} - \sup_{z \in \mathbb{R}} \{ \mathcal{L}(x, s; z, t) + W_\alpha(z, t; 0, t) \}
\]

(6.69) Rearranging the first and last lines of this inequality yields

\[
\mathcal{L}(x, s; g_x^{\xi,R}(t), t; 0, t) = \mathcal{L}(x, s; g_x^{\xi,R}(t), t; 0, t)
\]

(6.70) where the middle equality came from the assumption \( W_\xi(z, t; 0, 0) = W_\alpha(z, t; 0, t) \). Rearranging the first and last lines of this inequality yields

\[
W_\xi(g_y^{\alpha,L}(t); g_x^{\xi,R}(t)) \leq W_\alpha(g_y^{\alpha,L}(t); g_x^{\xi,R}(t)).
\]

However, the assumption that \( g_x^{\xi,R} \leq g_y^{\alpha,L} \) combined with (6.67) implies that this inequality is an equality. Hence, inequalities (6.69) and (6.70) are also equalities. From the equality (6.69), we must have that

\[
\mathcal{L}(x, s; g_x^{\xi,R}(t), t; 0, t) = \sup_{z \in \mathbb{R}} \{ \mathcal{L}(x, s; z, t) + W_\alpha(z, t; 0, t) \},
\]

so \( g_x^{\xi,R} \) is a maximizer of \( \mathcal{L}(x, s; z, t) + W_\alpha(z, t; 0, t) \) over \( z \in \mathbb{R} \). By geodesic ordering (Theorem 3.11(i)), \( g_x^{\xi,R} \geq g_y^{\alpha,R} \), and \( g_y^{\alpha,R} \) is by definition the rightmost maximizer. Hence, \( g_x^{\xi,R} = g_x^{\alpha,R} \).

An analogous argument applied to (6.70) implies that \( g_y^{\alpha,L} = g_y^{\xi,L} \). Going back to the full notation, we have shown that

\[
g_{(x,s)}^{\alpha_1,R}(t) = g_{(x,s)}^{\xi_1,R}(t), \quad \text{and} \quad g_{(y,s)}^{\alpha_1,L}(t) = g_{(y,s)}^{\xi_1,L}(t).
\]

Since \( g_{(x,s)}^{\alpha_1,R} \) and \( g_{(x,s)}^{\xi_2,R} \) are both the rightmost geodesics between any two of their points and similarly with the leftmost geodesics from \( y, s \) (Theorem 3.7(iii)), Equation (6.66) holds for all \( u \in [s, t] \), as desired.

**Lemma 6.13.** The following holds globally on a single event of probability one. Let \( s \in \mathbb{R} \) and \( x < y \). If, for some \( \alpha < \xi \) and \( \square_1, \square_2 \in \{ -, + \} \) we have that \( W_{\square_1}(y, s; x, s) = W_{\square_2}(y, s; x, s) \), then \( g_{(x,s)}^{\alpha_1,R} \) coalesces with \( g_{(y,s)}^{\alpha_2,L} \), \( g_{(x,s)}^{\xi_1,R} \) coalesces with \( g_{(y,s)}^{\xi_2,L} \), and the coalescence point for the two pairs of geodesics is the same.

**Proof.** By Theorem 3.7(ii), \( g_{(x,s)}^{\alpha_1,R}(t)/t \to \alpha \) while \( g_{(y,s)}^{\alpha_1,L}(t)/t \to \alpha \) as \( t \to \infty \). By this and continuity of geodesics, there exists a minimal time \( t > s \) such that \( z := g_{(x,s)}^{\xi_1,R}(t) = g_{(y,s)}^{\alpha_1,L}(t) \). By Lemma 6.12,

\[
g_{(x,s)}^{\alpha_1,R}(u) = g_{(x,s)}^{\xi_2,R}(u), \quad \text{and} \quad g_{(y,s)}^{\alpha_1,L}(u) = g_{(y,s)}^{\xi_2,L}(u) \quad \text{for all} \quad u \in [s, t].
\]

Since \( t \) was chosen to be minimal, Theorem 3.11(iv) implies that the pair \( g_{(x,s)}^{\alpha_1,R}, g_{(y,s)}^{\alpha_1,L} \) and the pair \( g_{(x,s)}^{\xi_1,R}, g_{(y,s)}^{\xi_1,L} \) both coalesce at \( (z, t) \).

**Proof of Theorem 3.15. Item (i):** Let \( g_1 \) and \( g_2 \) be \( \xi \) Busemann geodesics from \( (x, s) \) and \( (y, t) \), respectively, and take \( s \leq t \) without loss of generality. Let \( a = (g_1(t) \wedge y) - 1 \) and \( b = (g_1(t) \vee y) + 1 \). By Theorem 3.11(iv), for all \( u \geq t \),

\[
g_{(x,t)}^{\xi,R}(u) \leq g_1(u) \wedge g_2(u) \leq g_1(u) \vee g_2(u) \leq g_{(b,t)}^{\xi,L}(u).
\]

By Theorem 3.1(v), there exists \( \alpha \), sufficiently close to \( \xi \), (from the left for \( \square = - \) and from the right for \( \square = + \)) such that \( W_{\square}(b, t; a, t) = W_{\alpha}(b, t; a, t) \). Then, by Lemma 6.13, \( g_{(a,t)}^{\xi,R} \) coalesces with \( g_{(b,t)}^{\xi,L} \). Then, for \( u \) large enough, all inequalities in (6.71) are equalities, and \( g_1 \) and \( g_2 \) coalesce.

If the first point of intersection is not \( (y, t) \), then \( g_1(t) \neq y \), and the coalescence point of \( g_1 \) and \( g_2 \) is the first point of intersection by Theorem 3.11(iv).
**Item (ii):** Let \((x, s) \in \text{NU}_{0}^{\xi}\), and let \(g_1\) and \(g_2\) be two distinct \(\xi\) Busemann geodesics from \((x, s)\). Let \(t_1 = \inf_{t \geq s}\{g_1(t) \neq g_2(t)\}\). By continuity of geodesics, without loss of generality, for all \(t\) greater than, but sufficiently close to \(t_1\), \(g_1(t) < g_2(t)\). By Item (i), \(g_1\) and \(g_2\) must coalesce again, and since they both pass through \((g_1(t), t)\) and \((g_2(t), t)\), respectively, so they cannot coalesce at \((g_1(t), t)\) or \((g_2(t), t)\). Hence, their first point of intersection after time \(t_1\) is the coalescence point.

**Item (iii):** Let \(\xi \in \mathbb{R}\), \(\square \in \{-, +\}\), and let the compact set \(K\) be given. Let \(S\) be the smallest integer greater than or equal to \(\max\{s : (x, s) \in K\}\). Set
\[
A := \inf\{g_{(x, s)}^{\xi, L}(S) : (x, s) \in K\}, \quad \text{and} \quad B := \sup\{g_{(x, s)}^{\xi, R}(S) : (x, s) \in K\}.
\]
By Lemma 6.5, \(-\infty < A < B < \infty\). Then, by Theorem 3.11(iv), whenever \(g\) is a \(\xi\) geodesic starting from \((x, s) \in K\),
\[
g_{(A, S)}^{\xi, L}(t) \leq g(t) \leq g_{(B, S)}^{\xi, R}(t) \quad \text{for all } t \geq S.
\]
To complete the proof, let \(T\) be the time at which \(g_{(A-1, S)}^{\xi, R}\) and \(g_{(B+1, S)}^{\xi, L}\) coalesce, which is guaranteed to be finite by Item (i).

**Proof of Theorem 3.17.** (i)⇒(ii) follows from Lemma 6.13.

(ii)⇒(i): Assume \((z, t) := z^{\alpha+}(y, s; x, s) = z^{\beta-}(y, s; x, s)\). By additivity (Theorem 3.1(ii)) and Theorem 3.7(ii),
\[
W_{\alpha+}(y, s; x, s) = W_{\alpha+}(y, s; z, t) - W_{\alpha+}(x, s; z, t) = \mathcal{L}(y, s; z, t) - \mathcal{L}(x, s; z, t) = W_{\beta-}(y, s; z, t) - W_{\beta-}(x, s; z, t) = W_{\beta-}(y, s; x, s).
\]

(ii)⇒(iii): Let \((z, t)\) be as in the proof of (ii)⇒(i). By Theorem 3.7(iii), the restriction of \(g_{(x, s)}^{\alpha+, R}\) and \(g_{(x, s)}^{\beta-, L}\) to the domain \([s, t]\) are both rightmost geodesics between \((x, s)\) and \((z, t)\), and therefore they agree on this restricted domain. Similarly, \(g_{(y, s)}^{\alpha+, L}\) and \(g_{(y, s)}^{\beta-, R}\) agree on the domain \([s, t]\). By the monotonicity of Theorem 3.11(i), and since \((z, t)\) is the common coalescence point, (3.15) holds for \(u \in [s, t]\), as desired.

(iii)⇒(ii) is immediate.

**Proof of Theorem 3.18.** (i)⇒(ii): If \(W_{\xi-}(y, s; x, s) = W_{\xi+}(y, s; x, s)\), then Theorem 3.1(v) implies that for some \(\alpha < \xi < \beta\), \(W_{\alpha+}(y, s; x, s) = W_{\beta-}(y, s; x, s)\). Then, we apply (i)⇒(iii) of Theorem 3.17 to conclude that for some \(t > s\) and \(z \in \mathbb{R}\),
\[
g_{(x, s)}^{\xi-, R}(u) = g_{(x, s)}^{\xi+, R}(u) < g_{(y, s)}^{\xi-, L}(u) = g_{(y, s)}^{\xi+, L}(u), \quad \text{for } u \in [s, t),
\]
whereas for \(u = t\), all terms above equal \(z\). Therefore, \((z, t) = z^{\xi-}(y, s; x, s) = z^{\xi+}(y, s; x, s)\).

(ii)⇒(i): Similarly as in the proof of (ii)⇒(i) of Theorem 3.17, if \((z, t) = z^{\xi-}(y, s; x, s) = z^{\xi+}(y, s; x, s)\), then
\[
W_{\xi-}(y, s; x, s) = \mathcal{L}(y, s; z, t) - \mathcal{L}(x, s; z, t) = W_{\xi+}(y, s; x, s).
\]

(ii)⇒(iii): Assume that \((z, t) = z^{\xi-}(y, s; x, s) = z^{\xi+}(y, s; x, s)\). Then, \(g_{(x, s)}^{\xi-, R}(t) = z = g_{(y, s)}^{\xi+, L}(t)\).

(iii)⇒(ii): Assume that \(g_{(x, s)}^{\xi-, R}(t) = g_{(y, s)}^{\xi+, L}(t)\) for some \(t > s\). Take \(t\) to be the minimal such time, and let \((z, t)\) be the point where the geodesics first intersect. By Theorem 3.11(i) and Theorem 3.11(i), for \(u > s\),
\[
(6.72) \quad g_{(x, s)}^{\xi-, R}(u) \leq g_{(x, s)}^{\xi+, R}(u) \land g_{(y, s)}^{\xi-, L}(u) \leq g_{(y, s)}^{\xi+, L}(u) \leq g_{(y, s)}^{\xi-, R}(u) \lor g_{(y, s)}^{\xi-, L}(u) \leq g_{(y, s)}^{\xi+, L}(u)\]
Specifically, when \(u = t\), all inequalities in (6.72) are equalities. Further, since \(g_{(x, s)}^{\xi-, R}, g_{(x, s)}^{\xi+, R}\) are rightmost geodesics between \((x, s)\) and \((z, t)\) (Theorem 3.7(iii)), \(g_{(x, s)}^{\xi-, R}(u) = g_{(x, s)}^{\xi+, R}(u)\) for \(u \in [s, t]\).
Similarly, \( g_{(y,s)}(u) = g_{(y,s)}^\xi(L) \) for \( u \in [s,t] \). Since \( t \) was chosen minimally for \( g_{(x,s)}^\xi-R(t) = g_{(y,s)}^\xi+L(t) \), we have \( (z,t) = g^\xi-(y,s;x,x,s) = g^\xi+(y,s;x,x,s) \).

**Proof of Theorem 3.20.** (i)⇒(ii): If \( \xi \not\in \Xi \), then \( W_{\xi-} = W_{\xi+} \), so (ii) follows by the construction of the Busemann geodesics from the Busemann functions.

(ii)⇒(iii): Since a geodesic in direction \( \xi \) from \( (x,s) \) must pass through each horizontal level \( t > s \), it is sufficient to show that, for \( s \in \mathbb{R} \) and \( x < y \), whenever \( g_1 \) is a semi-infinite geodesic from \( (x,s) \) in direction \( \xi \) and \( g_2 \) is a semi-infinite geodesic from \( (y,s) \) in direction \( \xi \), \( g_1 \) and \( g_2 \) coalesce. Assuming (ii) and using Theorem 3.13(i), for all \( t > s \),

\[
g_{(x,s)}^{\xi+L}(t) = g_{(x,s)}^{\xi-L}(t) \leq g_1(t) \land g_2(t) \leq g_1(t) \lor g_2(t) \leq g_{(y,s)}^{\xi+R}(t).
\]

By Theorem 3.15(i), \( g_{(x,s)}^{\xi+L} \) and \( g_{(y,s)}^{\xi+R} \) coalesce, so all inequalities above are equalities for large \( t \), and \( g_1 \) and \( g_2 \) coalesce.

(iii)⇒(i): We prove the contrapositive. If \( \xi \in \Xi \), then by Theorem 3.1(i), \( W_{\xi-}(y,0;x,0) < W_{\xi+}(y,0;x,0) \) for some \( x < y \). By (i)⇔(iii) of Theorem 3.18, \( g_{(x,s)}^{\xi-R}(t) < g_{(y,s)}^{\xi+L}(t) \) for all \( t > s \). In particular, \( g_{(x,s)}^{\xi-R}(t) \) and \( g_{(y,s)}^{\xi+L}(t) \) do not coalesce.

(ii)⇒(iv): By definition of \( NU_0 \), whenever \( p \notin NU_0 \), \( g_p^{\xi+R} = g_p^{\xi-L} \) for \( \xi \in \mathbb{R} \) and \( \square \in \{-,+,\} \). Hence, assuming \( p \notin NU_0 \) and \( g_p^{\xi-R} = g_p^{\xi+R} \), we also have \( g_p^{\xi-L} = g_p^{\xi+R} \), so there is a unique geodesic from \( p \) in direction \( \xi \) by Theorem 3.13(i).

(iv)⇒(v): Using Theorem 3.10(i), the full probability event is chosen so that \( NU_0 \) contains no points of \( Q^2 \) and therefore, \( NU_0 \) is not a strict subset of \( \mathbb{R}^2 \).

(v)⇒(vi) and (v)⇒(vii) are direct consequences of Theorem 3.13(i): If there is a unique semi-infinite geodesic in direction \( \xi \) from a point \( p \in \mathbb{R}^2 \), then \( g_p^{\xi-L} = g_p^{\xi+L} = g_p^{\xi-R} = g_p^{\xi+R} \).

(vi)⇒(ii): Let \( p \) be a point from which \( g_p^{\xi-L} = g_p^{\xi+L} \), and call this common geodesic \( g \). Let \( q \) be an arbitrary point in \( \mathbb{R}^2 \). By Theorem 3.15(i), \( g_q^{\xi-L}, g_q^{\xi+L}, g_q^{\xi-R} \), and \( g_q^{\xi+R} \) each coalesce with \( g \), so \( g_q^{\xi-L} \) and \( g_q^{\xi+L} \) coalesce. Since both geodesics are the leftmost geodesics between their points by Theorem 3.7(iii), they must be the same. Similarly, \( g_q^{\xi-R} = g_q^{\xi+R} \).

(vii)⇒(ii): follows by the same proof.

**Item** (viii): Let \( \xi \in \mathbb{R} \setminus \Xi \), and let \( g \) be a semi-infinite geodesic in direction \( \xi \), starting from a point \( (x,s) \in \mathbb{R}^2 \). By Lemma 6.7 and Theorem 3.7(i), it is sufficient to show that for sufficiently large \( t \),

\[
\mathcal{L}(x,s;g(t),t) = W_\xi(x,s;g(t),t).
\]

(we dropped the \( \pm \) distinction since \( W_{\xi-} = W_{\xi+} \)). By Item (iii), \( g \) coalesces with \( g_{(x,s)}^{\xi,R} \). Then, for sufficiently large \( t \), \( g(t) = g_{(x,s)}^{\xi,R}(t) \) and by Theorem 3.7(ii), (6.73) holds.

### 6.5. The exceptional set of geodesics splitting points

We define the following sets for \( S \in \{L,R\} \).

\[
\mathfrak{S}^S := \{(x,s) \in \mathbb{R}^2 : \text{for some } \xi \in \Xi \text{ and } g_{(x,s)}^{\xi-S}(t) < g_{(x,s)}^{\xi+S}(t) \text{ for all } t > s \}.
\]

**Remark 6.14.** These sets \( \mathfrak{S}^S \) are very similar to the set \( \mathfrak{S} \) defined in (2.5). The distinction is that the sets \( \mathfrak{S}^S \) are only concerned with leftmost and rightmost Busemann geodesics. However, the sets are all very closely related, which is made clear in Lemma 6.19.

**Lemma 6.15.** On a single event of full probability, let \( (x,s; y,u) \in \mathbb{R}^4 \). Let \( g : [s,u] \to \mathbb{R} \) be the leftmost (resp. rightmost) geodesic between \( (x,s) \) and \( (y,u) \). Then, \( (g(t), t) \in \mathcal{L}^L \) (resp. \( \mathfrak{S}^R \)) for some \( t \in [s,u] \). Furthermore, the direction \( \xi \) for which \( g_{(x,s)}^{\xi-L} \) and \( g_{(x,s)}^{\xi+L} \) split at \( (g(t), t) \) can be chosen so that

\[
g_{(x,s)}^{\xi-L}(u) \leq y < g_{(x,s)}^{\xi+L}(u).
\]

for rightmost geodesics, the placement of the strict and weak inequalities can be reversed.
Figure 6.4. The black/thin path is the path \( g \). The red/thick paths are the semi-infinite geodesics \( g_{(x,s)}^-(\xi) \) and \( g_{(x,s)}^+(\xi) \) after they split from \( g \). Once the red paths split, they cannot return, or else there would be two leftmost geodesics from \((g(t), t)\) to the point where they come back together.

**Proof.** We prove the statement for leftmost geodesics, and the statement for rightmost geodesics is analogous. Set
\[
(6.75) \quad \hat{\xi} := \sup \{ \xi \in \mathbb{R} : g_{(x,s)}^{\xi}(u) \leq y \} = \inf \{ \xi \in \mathbb{R} : g_{(x,s)}^{\xi}(u) > y \}.
\]
The monotonicity of Theorem 3.11(i) guarantees that the second equality holds, and that the definition is independent of the choice of \( \Box \in \{-, +\} \). Theorem 3.11(iii) guarantees that \( \hat{\xi} \in \mathbb{R} \). By Theorem 3.11(ii) and the definition of \( \hat{\xi} \),
\[
(6.76) \quad g_{(x,s)}^{\hat{\xi}}(u) \leq y = g(u) < g_{(x,s)}^{\hat{\xi}}(u).
\]
By Theorem 3.7(iii) and the definition of \( g \) as the leftmost geodesic,
\[
(6.77) \quad g_{(x,s)}^{\hat{\xi}}(t) \leq g(t) \leq g_{(x,s)}^{\hat{\xi}}(t) \quad \text{for } t \in [s, u].
\]
Furthermore, once \( g_{(x,s)}^{\hat{\xi}} \) splits from \( g \), the two paths must stay split or else there would be two leftmost geodesics between some pair of points. See Figure 6.4. Set \( \hat{t} = \inf \{ t > s : g_{(x,s)}^{\hat{\xi}}(t) < g_{(x,s)}^{\hat{\xi}}(t) \} \). By (6.76) and continuity of geodesics, \( \hat{t} \in [s, u) \). If \( \hat{t} = s \), then \((x, s) = (g(s), s) \in \mathcal{G} \). If \( \hat{t} > s \), then by (6.77), \( g_{(x,s)}^{\hat{\xi}}(t) = g(t) = g_{(x,s)}^{\hat{\xi}}(t) \) for \( t \in [s, \hat{t}) \), and so \((g(\hat{t}), \hat{t}) \in \mathcal{G} \).

We will make use of some results from [BGH22]. For \((p, q) \in \mathbb{R}_+^2 \), let \( \mathcal{G}_{(p,q)} \) denote the set of geodesics from \( p \) to \( q \). For \( y_1 < y_2 \) and \( s < t \), let
\[
\mathcal{D}_{(y_1, y_2, s, u)} := \{ x \in \mathbb{R} : \exists g_1 \in \mathcal{G}_{(x,s; y_1, u)}, g_2 \in \mathcal{G}_{(x,s; y_2, u)} : g_1(t) < g_2(t) \text{ for all } t \in (s, u) \},
\]
and for \( s < u \), let
\[
\mathcal{D}_{s, u} = \{ (x, y) \in \mathbb{R}^2 : \exists g_1, g_2 \in \mathcal{G}_{(x,s; y, u)} : g_1(t) < g_2(t) \text{ for all } t \in (s, u) \},
\]
and let \( \pi_1(\mathcal{D}_{s, u}) \subseteq \mathbb{R} \) be the projection onto the first coordinate. In other words, \( \mathcal{D}_{(y_1, y_2, s, u)} \) is the set of points \( x \in \mathbb{R} \) such that some geodesic from \((x, s)\) to \((y_1, u)\) is disjoint from some geodesic from \((x, s)\) to \((y_2, u)\). \( \mathcal{D}_{s, u} \) is the set of pairs \((x, y)\) such that there exist two disjoint geodesics from \((x, s)\) to \((y, u)\) (only meeting at the endpoints). We have the following.

**Theorem 6.16** ([BGH22], Theorems 1.9-1.10). The following hold.

(i) For \( y_1 < y_2 \) and \( s < u \), \( \mathcal{D}_{(y_1, y_2, s, u)} \) has Hausdorff dimension \( \frac{1}{2} \) on a \((y_1, y_2, s, u)\)-dependent full-probability event.

(ii) For \( s < u \), \( \mathcal{D}_{(s, u)} \) and \( \pi_1(\mathcal{D}_{(s, u)}) \) have Hausdorff-dimension \( \frac{1}{2} \) on an \( s, u \)-dependent full-probability event.
are both the leftmost geodesic between their points (Theorem 3.7(iii)).

Thus, for all \( y \) points lie on the level \( g \) geodesic from \((\hat{x},s)\) implies that \(\hat{\xi}\). Then, by Lemma 6.7, another \(\xi\) geodesic is formed by following \(g\) from \((x,s)\) to \((\hat{\xi},\hat{t})\), then by following \(g_{\{x,s\}}^{\xi,+}\) onwards. This violates the assumption \((x,s) \notin NU_0\).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.5.png}
\caption{The black/thin paths are the two disjoint geodesics \(g_1\) and \(g_2\) from \((x,s)\) to \((y,u)\), while the two red/thick paths are \(g_{\{x,s\}}^{\xi,-}\) and \(g_{\{x,s\}}^{\xi,+}\). If \(g_{\{x,s\}}^{\xi,\pm}(t) < g_2(t)\) for some \(t \in [s,u]\) as is the case in this figure, then there are two \(\xi+\) geodesics from \((x,s)\)—the other one is given b by following \(g_2\) from \((x,s)\) to \((\hat{\xi},\hat{t})\), then by following \(g_{\{x,s\}}^{\xi,+}\) onwards. This violates the assumption \((x,s) \notin NU_0\).}
\end{figure}

\textbf{Remark 6.17.} Item (i) is slightly different from the result stated in [BGH22]. There, the two initial points lie on the level \(s\), travelling to the exceptional point on level \(t > s\). However, by the temporal reflection invariance of Theorem B.1(v), the distribution of the two sets is the same. The result on \(\pi_1(D_{x,u})\) was not shown in [BGH22], but can be derived from their results in a fairly simple manner by identifying the exceptional set \(D_{x,u}\) as the support of a random measure. We thank Erik Bates for showing this to us.

For \(s \in \mathbb{R}\), recall that \(H_s = \{(x,s) : x \in \mathbb{R}\}\)

\begin{lemma}
On a single event of full probability, for each \(s < u\), the following inclusions hold.
\[\pi_1(D_{s,u}) \times \{s\} \subseteq (\mathcal{G}^L \cap \mathcal{G}^R) \cup NU_0 \cap H_s, \text{ and } (\mathcal{G}^L \cup \mathcal{G}^R) \cap H_s \subseteq \bigcup_{q_1 < q_2 \in \mathbb{Q}} D_{(q_1,q_2,s,u)} \times \{s\}.\]
\end{lemma}

\textbf{Proof.} We start with the first inclusion. Let \(x \in \pi_1(D_{s,u})\), and assume that \((x,s) \notin NU_0\). We show that \((x,s) \in \mathcal{G}^L\), and the proof that \((x,s) \in \mathcal{G}^R\) is analogous. By assumption, there exists \(y \in \mathbb{R}\) such that there are two geodesics \(g_1, g_2\) from \((x,s)\) to \((y,u)\) with \(g_1(t) < g_2(t)\) for all \(t \in (s,u)\). We can assume that these geodesics are maximal in the sense that \(g_1\) is the leftmost geodesic from \((x,s)\) to \((y,u)\), and \(g_2\) is the rightmost. By Lemma 6.15, for some \(\xi \in \{-,+,\}\), \(g_{\{x,s\}}^{\xi,-}(u) \leq y < g_{\{x,s\}}^{\xi,+}(u)\). Since \(g_1\) is the leftmost geodesic between \((x,s)\) and \((y,t)\), Theorem 3.7(iii) implies that \(g_{\{x,s\}}^{\xi,-}(t) \leq g_1(t)\) for all \(t \in [s,u]\). We now claim that \(g_{\{x,s\}}^{\xi,+}(t) \geq g_2(t)\) for all \(t \in [s,u]\).

If not, then since we know that \(g_{\{x,s\}}^{\xi,+}(u) > y = g_2(u)\), then \(g_{\{x,s\}}^{\xi,+}\) and \(g\) must cross again at some point \((\hat{\xi},\hat{t})\). Then, by Lemma 6.7, another \(\xi+\) geodesic is formed by following \(g_2\) from \((x,s)\) to \((\hat{\xi},\hat{t})\), then following \(g_{\{x,s\}}^{\xi,+}\) from \((\hat{\xi},\hat{t})\) onward. Thus, \((x,s) \in NU_0^{\xi+} \subseteq NU_0\), contradicting our assumption. See Figure 6.5. Thus, for all \(t \in (s,u)\),
\[g_{\{x,s\}}^{\xi,-}(t) \leq g_1(t) < g_2(t) \leq g_{\{x,s\}}^{\xi,+}(t).\]

The strict inequality for the first and last term above extends to all \(t > s\) since \(g_{\{x,s\}}^{\xi,-}\) and \(g_{\{x,s\}}^{\xi,+}(t)\) are both the leftmost geodesic between their points (Theorem 3.7(iii)).

Now, we show the second inclusion. If \((x,s) \in \mathcal{G}^L \cup \mathcal{G}^R\), then for some \(\xi \in \mathbb{R}\) and \(S \in \{L,R\}\), \(g_{\{x,s\}}^{\xi,-}(t) < g_{\{x,s\}}^{\xi,S}\). In particular, set \(y_1 = g_{\{x,s\}}^{\xi,-}(u)\) and \(y_2 = g_{\{x,s\}}^{\xi,S}(u)\). Then, there are disjoint geodesics \(g_1, g_2\) from \((x,s)\) to \((y_1,u)\), \((y_2,u)\) respectively such that \(g_1(t) < g_2(t)\) for \(t \in (s,u)\). Hence,
Lemma 6.19. The following hold.

(i) \( \mathcal{S}^L \) and \( \mathcal{S}^R \) are almost surely dense in \( \mathbb{R}^2 \).

(ii) Let \( \mathcal{S} \) be the set defined in (2.5). Then,

\[
\mathcal{S} \setminus \text{NU}_0 = (\mathcal{S}^L \cup \mathcal{S}^R) \setminus \text{NU}_0 = (\mathcal{S}^L \cap \mathcal{S}^R) \setminus \text{NU}_0.
\]

(iii) For each \( s \in \mathbb{R} \), \( \mathcal{S}^L \cap \mathcal{S}^R \cap \mathcal{H}_s \) and \( (\mathcal{S}^L \cup \mathcal{S}^R) \cap \mathcal{H}_s \) have Hausdorff dimension \( \frac{1}{2} \) on an \( s \)-dependent full probability event.

Remark 6.20. In Brownian last-passage percolation BLPP, the analogues of \( \mathcal{S}^L \) and \( \mathcal{S}^R \) are equal and have the characterization as the set of initial points for which some geodesic travels initially vertical (Theorems 2.10 and 4.30 in [SS21b]). Furthermore, in the BLPP case, the analogue of this set contains NU_0. We do not presently know whether either of these things are true in the directed landscape. However, we do note that the Hausdorff dimension \( \frac{1}{2} \) statement of Lemma 6.19 is analogous to Theorem 2.10 in [SS21b].

Proof. Item (i): Let \( (x, s) \in \mathbb{R}^2 \). We show the existence of a sequence \( (y_n, t_n) \in \mathcal{S}^L \) converging to \( (x, s) \). The proof of the density of \( \mathcal{S}^R \) is analogous. Let \( g \) be the leftmost geodesic from \( (x, s) \) to \( (x, s+1) \). Then, for each \( n \geq 1 \), \( g_{[s,s+n-1]} \) is also the leftmost geodesic from \( (x, s) \) to \( (x, s+n^{-1}) \). By Lemma 6.15, for each \( n \), there exists a point \( (x_n, t_n) \in \mathcal{S}^L \) such that \( x_n = g(t_n) \) and \( s \leq t_n \leq s + n^{-1} \). The proof is complete by continuity of geodesics.

Item (ii): When \( p \notin \text{NU}_0 \), the L/R distinction is not present, so \( \mathcal{S}^L \setminus \text{NU}_0 = \mathcal{S}^R \setminus \text{NU}_0 \). This proves the second equality. Further, when \( p \notin \text{NU}_0 \), Theorem 3.13(i) implies that \( g_{(x,s)}^{\xi^-} \) and \( g_{(x,s)}^{\xi^+} \) are the leftmost and rightmost semi-infinite geodesics from \( (x, s) \) in direction \( \xi \). (Dropping the L/R superscript, since they give the same geodesic outside \( \text{NU}_0 \)). Hence, there exists disjoint semi-infinite geodesics from \( (x, s) \) in direction \( \xi \) if and only if \( g_{(x,s)}^{\xi^-} \) and \( g_{(x,s)}^{\xi^+} \) are disjoint. This proves the first equality.

Item (iii): Recall that for a countably infinite collection \( A_i \) of sets, the Hausdorff dimension of their union is the maximum of the Hausdorff dimensions of the \( A_i \). By Theorem 3.10(ii), \( \text{NU}_0 \cap \mathcal{H}_s \) is at most countable and therefore has Hausdorff dimension 0 on an \( s \)-dependent full probability event. Then, the first inclusion of (6.78) and Theorem 6.16(ii) (fixing \( u = s + 1 \) for example), the Hausdorff dimension of \( \mathcal{S}^L \cap \mathcal{S}^R \cap \mathcal{H}_s \) is at least \( \frac{1}{2} \). By the second inclusion of (6.78) and Theorem 6.16(i), the Hausdorff dimension of \( (\mathcal{S}^L \cup \mathcal{S}^R) \cap \mathcal{H}_s \) is at most \( \frac{1}{2} \).

6.6. Proof of Theorem 2.3. We first prove some preliminary lemmas.

Lemma 6.21. Let \( G \) be the Stationary Horizon. There exists \( C > 0 \) such that for any \( \mu > 0 \) and \( \varepsilon < 1/(2000\mu^2) \)

\[
\mathbb{P}(G_{-\mu}(x) \neq G_{\mu}(x) \text{ for some } x \in [-\varepsilon, \varepsilon]) \leq C\varepsilon^{1/2}\mu.
\]

Proof. Consider the sequence \( G^N \) defined in [Bus21, Eq (4.6)]. Let \( \mathbb{P}_{G^N} \) be its distribution on \( D(\mathbb{R}, C(\mathbb{R})) \). From [BF22, Corollary 5.5], it follows that for large enough \( N \)

\[
\mathbb{P}_{G^N}(A) \leq C\varepsilon^{1/2}\mu,
\]

where

\[
A := \{X \in D(\mathbb{R}, C(\mathbb{R})) : \pi_{-\mu}(X) \neq \pi_{\mu}(X) \text{ on the interval } [-\varepsilon, \varepsilon]\},
\]

and where \( \pi : D(\mathbb{R}, C(\mathbb{R})) \to C(\mathbb{R}) \) is the projection map. From [Bus21, Theorem 1]

\[
\mathbb{P}_{G^N} \to \mathbb{P}_G.
\]

Also, the set \( A \subset D(\mathbb{R}, C(\mathbb{R})) \) is open in the Skorokhod topology. It now follows from (6.81) that

\[
\mathbb{P}_G(A) \leq \liminf_{N \to \infty} \mathbb{P}_{G^N}(A) \leq C\varepsilon^{1/2}\mu,
\]

which implies the result. \( \square \)
Define the interval $I^e = \{(x,0) : x \in [-\varepsilon, \varepsilon]\}$, and the set
\[
(6.83) \quad A^{T,\varepsilon} = \{\exists \xi \in [-T, T] : \text{there exist disjoint geodesics in direction } \xi \text{ emanating from } I^e\}
\]

**Lemma 6.22.** There exists a constant $C > 0$ such that for every $\varepsilon > 0$ and $T \in \mathbb{N}$,
\[
\mathbb{P}(A^{T,\varepsilon}) \leq C\varepsilon^{1/2}T.
\]

**Proof.** Let $a = (-\varepsilon,0)$ and $b = (\varepsilon,0)$ be the endpoints of the interval $I^e$. For each $T \in \mathbb{N}$, there exist unique infinite geodesics $g_a^{T}$ and $g_b^{T}$, emanating from $a$ and $b$ respectively, and going in directions $-T$ and $T$ respectively. By Theorem 3.11(i),
\[
g_a^{T}(s) \leq g_\xi^T(s) \leq g_b^{T}(s) \quad z \in I^e, \xi \in [-T, T], s > 0.
\]

It is not hard to see that
\[
(6.84) \quad \mathbb{P}(A^{T,\varepsilon}) \leq \mathbb{P}(g_z^{T,R}(\delta) \neq g_z^{T,L}(\delta) \text{ for some } z \in I^e) \quad \text{for all } \delta > 0.
\]

From the scale invariance of the directed landscape, the construction of the semi-infinite geodesics from maximizers, and Theorem 3.2(i)–(iii),
\[
(6.85) \quad g_\xi^T(\delta) \overset{d}{=} \arg\max_y \delta^{1/3}L(z,0;\delta^{-2/3}y,1) + G_\xi(y) \quad z \in I^e, \xi \in [-T, T],
\]

where $G$ is the stationary horizon, independent of \{$L(x,0;y,1) : x, y \in \mathbb{R}$\}. Define the interval
\[
J^{e,\delta} = \{(x,\delta) : -2\varepsilon \leq x \leq 2\varepsilon\}.
\]

From Theorem 5.1(iii) and Lemma 6.21,
\[
(6.86) \quad \mathbb{P}(G_{-T} = G_T \text{ on } J^{e,\delta}) \geq 1 - C(\varepsilon)^{1/2}T
\]
\[
(6.87) \quad \mathbb{P}(g_a^{T}(\delta) \in J^{e,\delta}), \mathbb{P}(g_b^{T}(\delta) \in J^{e,\delta}) \geq 1 - Ce^{-c\delta^{-2/3}(2\varepsilon-T\delta)^3}.
\]

It now follows from (6.85) that for every $\delta > 0$,
\[
(6.88) \quad \mathbb{P}(g_z^{T,R}(\delta) \neq g_z^{T,L}(\delta) \text{ for some } z \in I^e) \leq \mathbb{P}(g_z^{T}(\delta) \notin J^{e,\delta}) + \mathbb{P}(g_a^{T}(\delta) \notin J^{e,\delta}) + \mathbb{P}(G_{-T} \neq G_T \text{ on } J^{e,\delta})
\]
\[
\leq 8\varepsilon^{1/2}T + 2Ce^{-c\delta^{-2/3}(2\varepsilon-T\delta)^3}.
\]

As $\delta > 0$ in (6.84) is arbitrary, we can choose it so that $2Ce^{-c\delta^{-2/3}(2\varepsilon-T\delta)^3} < \varepsilon^{1/2}T$. Using (6.86)–(6.87) in (6.88)
\[
\mathbb{P}(g_z^{T}(\delta) \neq g_z^{T}(\delta) \text{ for some } z \in I^e) \leq C\varepsilon^{1/2}T.
\]

Using this in (6.84) we obtain the result. \hfill \square

**Proof of Theorem 2.3.** Item (i): First, we show that if $g$ is a semi-infinite geodesic starting from $(x,s)$, then
\[
(6.89) \quad -\infty < \liminf_{t \to \infty} \frac{g(t)}{t} \leq \limsup_{t \to \infty} \frac{g(t)}{t} < \infty.
\]

We show the latter inequality, and the first follows analogously. Assume, by way of contradiction, that $\limsup_{t \to \infty} g(t)/t = \infty$. By the directedness of Theorem 3.7(ii), for each $\xi \in \mathbb{R}$, there exists an infinite sequence $t_i \to \infty$ such that $g(t_i) > g_{(x,s)}^{\xi,R}(t_i)$ for all $i$. Since $g_{(x,s)}^{\xi,R}$ is the rightmost geodesic between any two of its points (Theorem 3.7(iii)), we must have that $g(t) \geq g_{(x,s)}^{\xi,R}(t)$ for all $\xi \in \mathbb{R}$ and all $t \in \mathbb{R}$. By Theorem 3.11(iii), $g(t) = \infty$ for each $t > s$, a contradiction.

Now, that we have established (6.89), we assume by way of contradiction that
\[
\liminf_{t \to \infty} \frac{g(t)}{t} < \limsup_{t \to \infty} \frac{g(t)}{t}.
\]

Choose some $\xi$ between the two values above. By the directedness of Theorem 3.7(ii), there exists a sequence $t_i \to \infty$ such that $g_{(x,s)}^{\xi,R}(t_i) < g(t_i)$ for $i$ even and $g_{(x,s)}^{\xi,R}(t_i) > g(t_i)$ for $i$ odd. This cannot occur since $g_{(x,s)}^{\xi,R}$ is the rightmost geodesic between any two of its points.
By Theorem 3.7(ii), for each $\xi \in \mathbb{R}$ and $(x, s) \in \mathbb{R}^2$, $\mathbf{g}^{\xi, +R}_{(c, s)}$, for example, is a semi-infinite geodesic from $(x, s)$ in direction $\xi$, justifying the claim that there is at least one semi-infinite geodesic from each point and in every direction.

**Item (ii):** This follows from the equivalences (i)$\iff$(iii)$\iff$(iv) of Theorem 3.20. The set $\text{NU}_0$ almost surely has Lebesgue measure 0 by Theorem 3.10(i) and Lemma A.4 since $\mathbb{P}(p \in \text{NU}_0)$ for each $p \in \mathbb{R}^2$.

**Item (iii):** This follows from Remark 3.21.

**Item (iv):** Recall the definition of $\mathcal{G}^L$ and $\mathcal{G}^R$ from (6.74). Note that $\mathcal{G} \supseteq \mathcal{G}^L \cup \mathcal{G}^R$, so the density of $\mathcal{G}^S$ in $\mathbb{R}^2$ follows from Lemma 6.19(i). Next, for $p \in \mathbb{R}^2$, define

$$A^p = \{ \exists \xi \in \mathbb{R} : \text{there exist disjoint geodesics in direction } \xi \text{ emanating from } p \}.$$  

We must show that for a fixed $p \in \mathbb{R}^2$, $\mathbb{P}(A^p) = 0$. From stationarity, it is enough to show the result for $p = (0, 0)$. Note that

$$A^{(0,0)} = \lim_{T \to \infty} \lim_{\varepsilon \to 0} A^{\varepsilon, T},$$

where $A^{\varepsilon, T}$ is defined in (6.83). The result follows by taking the probability of both sides of (6.86). Lastly, fix $s \in \mathbb{R}$. By Theorem 3.10(ii), on an $s$-dependent full-probability event $\text{NU}_0 \cap \mathcal{H}_s$ is at most countable and therefore has Hausdorff dimension 0. The conclusion then follows from Items (ii) and (iii) of Lemma 6.19.

**Appendix A. Auxiliary Results**

For a function $f : \mathbb{R} \to \mathbb{R}$, we denote its increments by $f(x, y) := f(y) - f(x)$. For two functions $f, g : \mathbb{R} \to \mathbb{R}$, we say that $f \leq_{\text{inc}} g$ if $f(x, y) \leq g(x, y)$ for all $x < y$ in $\mathbb{R}$.

**Lemma A.1.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions satisfying $f(x) \vee g(x) \to -\infty$ as $x \to \pm \infty$ and $f \leq_{\text{inc}} g$. Let $x^L_f$ and $x^R_f$ be the leftmost and rightmost maximizers of $f$ over $\mathbb{R}$, and similarly defined for $g$. Then, $x^L_f \leq x^L_g$ and $x^R_f \leq x^R_g$.

**Proof.** By the definition of $x^R_g$ and by the assumption $f \leq_{\text{inc}} g$, for all $x > x^R_g$

$$f(x^R_g, x) \leq g(x^R_g, x) < 0.$$  

Hence, the rightmost maximizer of $f$ must be less than or equal to $x^R_g$. We get the statement for leftmost maximizers by considering the functions $x \mapsto f(-x)$ and $g \mapsto g(-x)$.

**Lemma A.2.** Assume that $f, g : \mathbb{R} \to \mathbb{R}$ satisfy $f \leq_{\text{inc}} g$. Then, for $a \leq x \leq y \leq b$,

$$0 \leq g(x, y) - f(x, y) \leq g(a, b) - f(a, b).$$

**Proof.** The first inequality follows immediately from the assumption $f \leq_{\text{inc}} g$. The second follows from the inequality

$$f(a, b) - f(x, y) = f(a, x) + f(y, b) \leq g(a, x) + g(y, b) = g(a, b) - g(x, y).$$

**Lemma A.3.** Let $S \subseteq \mathbb{R}^n$, and let $f_n : S \to \mathbb{R}$ be a sequence of continuous functions, converging uniformly to the function $f : S \to \mathbb{R}$. Assume that there exists a sequence $\{c_n\}$, of maximizers of $f_n$, converging to some $c \in S$. Then, $c$ is a maximizer of $f$.

**Proof.** $f_n(c_n) \geq f_n(x)$ for all $x \in S$, so it suffices to show that $f_n(c_n) \to f(c)$. This follows from the uniform convergence of $f_n$ to $f$, the continuity of $f$, and

$$|f_n(c_n) - f(c)| \leq |f_n(c_n) - f(c_n)| + |f(c_n) - f(c)|.$$  

This is a well-known fact, but we include its proof because of its simplicity.

**Lemma A.4.** Let $A \subseteq \mathbb{R}^n$ be a random set satisfying $\mathbb{P}(x \in A) = 0$ for Lebesgue-almost every $x \in \mathbb{R}^n$. Then, $A$ almost surely has Lebesgue measure 0.

**Proof.** Let $|A|$ denote the Lebesgue measure of $A$. Then,

$$\mathbb{E}[|A|] = \mathbb{E} \int_{\mathbb{R}^n} 1(x \in A) \, dx = \int_{\mathbb{R}^n} \mathbb{P}(x \in A) \, dx = 0.$$
APPENDIX B. THE DIRECTED LANDSCAPE AND THE KPZ FIXED POINT

The directed landscape satisfies the following symmetries.

**Theorem B.1** ([DOV18], Lemma 10.2 and [DV21], Proposition 1.23). As a random continuous function of \((x, s; y, t) \in \mathbb{R}^4\), the directed landscape \(L\) satisfies the following distributional symmetries, for all \(r, c \in \mathbb{R}\) and \(q > 0\).

(i) (Time stationarity)
\[ L(x, s; y, t) \overset{d}{=} L(x, s + r; y, t + r). \]

(ii) (Spatial Stationarity)
\[ L(x, s; y, t) \overset{d}{=} L(x + c, s; y + c, t). \]

(iii) (Flip symmetry)
\[ L(x, s; y, t) \overset{d}{=} L(-y, -t; -x, -t). \]

(iv) (Skew stationarity)
\[ L(x, s; y, t) \overset{d}{=} L(x + cs, s; y + ct, t) - 2c(t - s)(x - y) + (t - s)c^2. \]

(v) (Spatial and temporal reflections)
\[ L(x, s; y, t) \overset{d}{=} L(-x, s; -y, t) \overset{d}{=} L(y, -t; x; -s). \]

(vi) (Rescaling)
\[ L(x, s; y, t) \overset{d}{=} qL(q^{-2}x, q^{-3}s; q^{-2}y, q^{-3}t). \]

**Lemma B.2** ([DOV18], Corollary 10.7). There exists a random constant \(C\) such that for all \(v = (x, s; y, t) \in \mathbb{R}^4\), we have
\[ |L(x, s; y, t) + \frac{(x - y)^2}{t - s}| \leq C(t - s)^{1/3} \log^{4/3} \left( \frac{2(\|v\| + 2)}{t - s} \right) \log^{2/3} (\|v\| + 2), \]
where \(\|v\|\) is the Euclidean norm.

**Lemma B.3** ([Dau22], Proposition 2.6). For every \(i = 1, \ldots, k\) and \(\varepsilon > 0\), let
\[ (B.1) \quad K_{i, \varepsilon} = \{ (x, s; y, t) \in \mathbb{R}_+^4 : s, t \in [0, \varepsilon], x, y \in [i - 1/4, i + 1/4] \}. \]
Then, there exists a coupling of \(k + 1\) copies of the directed landscape \(L_0, L_1, \ldots, L_k\) so that \(L_1, \ldots, L_k\) are independent, and almost surely, there exists a random \(\varepsilon > 0\) such that for \(1 \leq i \leq k\), \(L_0|_{K_{i, \varepsilon}} = L_i|_{K_{i, \varepsilon}}\).

On a measure space \((\Omega, \mathcal{F}, \mathbb{P})\), a measure-preserving transformation \(T\) is a transformation such that \(T^{-1}E \in \mathcal{F}\) and \(\mathbb{P}(T^{-1}E) = \mathbb{P}(E)\) for all \(E \in \mathcal{F}\). Such a transformation is said to be ergodic if \(\mathbb{P}(E) \in \{0, 1\}\) whenever \(T^{-1}E = E\). The transformation \(T\) is said to be mixing if for all \(A, B \in \mathcal{F}\), \(\mathbb{P}(A \cap T^{-k}B) \to \mathbb{P}(A)\mathbb{P}(B)\) as \(k \to \infty\). By Setting \(A = B\), one sees that mixing implies ergodicity.

**Corollary B.4.** Consider the shift operator \(T_z\) acting on the directed landscape \(L\) as
\[ T_zL(x, s; y, t) = L(x + z, s; y + z; t), \]
where both sides are understood as a process on \(\mathbb{R}_+^4\). Then, \(L\) is mixing under this transformation. That is, for all Borel subsets \(A, B\) of the space \(C(\mathbb{R}_+^4, \mathbb{R})\),
\[ \mathbb{P}(L \in A, T_zL \in B) \overset{|z| \to \infty}{\longrightarrow} \mathbb{P}(L \in A)\mathbb{P}(L \in B). \]

**Proof.** This key Lemma B.3, and we thank Duncan Dauvergne for pointing this out to us. By Theorem B.1(ii), \(L\) is stationary under the shift \(T_z\). By Dynkin’s \(\pi\)-\(\lambda\) theorem, it suffices to show that for compact sets \(K_1, K_2 \subseteq \mathbb{R}_+^4\) and Borel sets \(A \subseteq C(K_1, \mathbb{R})\) and \(B \subseteq C(K_2, \mathbb{R})\),
\[ \mathbb{P}(L|_{K_1} \in A, T_zL|_{K_2} \in B) \overset{|z| \to \infty}{\longrightarrow} \mathbb{P}(L|_{K_1} \in A)\mathbb{P}(L|_{K_2} \in B). \]
Let $k = 2$, and consider the coupling $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2$ of Lemma B.3. Then, using the rescaling and spatial stationarity of Theorem B.1,
\[
\mathbb{P}(\mathcal{L}_1|K_1 \in A, \mathcal{L}_2|K_2 \in B) = \mathbb{P}(\mathcal{L}_0(x, s; y, t)|K_1 \in A, \{\mathcal{L}_0(x + z, s; y + z, t)|K_2 \in B \\
= \mathbb{P}(z^{1/2} \mathcal{L}_0(z^{-1}x, z^{-3/2}s; z^{-1}y, z^{-3/2}t)|K_1 \in A, \\
\quad z^{1/2} \mathcal{L}_0(z^{-1}x + 1, z^{-3/2}s; z^{-1}y + 1, z^{-3/2}t)|K_2 \in B) \\
= \mathbb{P}(z^{1/2} \mathcal{L}_0(z^{-1}x + 1, z^{-3/2}s; z^{-1}y + 1, z^{-3/2}t)|K_1 \in A, \\
\quad z^{1/2} \mathcal{L}_0(z^{-1}x + 2, z^{-3/2}s; z^{-1}y + 2, z^{-3/2}t)|K_2 \in B). \\
\quad (B.2)
\]

Above, we used the shorthand notation $\mathcal{L}_0(x, s; y, t)|K_1$ to denote $\{\mathcal{L}_0(x, s; y, t) : (x, s; y, t) \in K_1\}$. Since $K_1$ and $K_2$ are compact, almost surely for any $\varepsilon > 0$, when $z$ is sufficiently large,
\[(z^{-1}x + 1, z^{-3/2}s; z^{-1}y + 1, z^{-3/2}t) \in K_{1,\varepsilon}, \quad \text{and} \quad (z^{-1}x + 2, z^{-3/2}s; z^{-1}y + 2, z^{-3/2}t) \in K_{2,\varepsilon}, \]
where $K_{i,\varepsilon}$ are defined in (B.1) By the dominated convergence theorem, another application of the rescaling and spatial stationarity, and the independence of $\mathcal{L}_1$ and $\mathcal{L}_2$ from Lemma B.3, as $z \to \infty$, the probability in (B.2) converges to
\[
\mathbb{P}(\mathcal{L}_1|K_1 \in A, \mathcal{L}_2|K_2 \in B) = \mathbb{P}(\mathcal{L}_1|K_1 \in A)\mathbb{P}(\mathcal{L}_2|K_2 \in B). \tag*{\Box}
\]

The following is a well-known fact that follows immediately from the definition of mixing.

**Lemma B.5.** For $i = 1, 2$, let $T_i$ be a mixing transformation on $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$. Then, $T_1 \times T_2$ is a mixing transformation on the product space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, \mathbb{P}_1 \times \mathbb{P}_2)$.

Recall that the KPZ fixed point $h_t(\cdot; \mathfrak{h})$ with initial data $\mathfrak{h}$ at time 0 can be represented as
\[
h_t(y; \mathfrak{h}) = \sup_{x \in \mathbb{R}}\{h(x) + \mathcal{L}(x, 0; y, t)\} \quad \text{for } t > 0.
\]
The KPZ fixed point satisfies the semi-group property. That is, for $0 < s < t$,
\[
h_t(y; \mathfrak{h}) = \sup_{x \in \mathbb{R}}\{h_s(x; \mathfrak{h}) + \mathcal{L}(x, s; y, t)\}.
\]
In this sense, we may say that $h_t$ has initial data $\mathfrak{h}$ sampled at time $s < t$, in which case, we write
\[
h_t(y; \mathfrak{h}) = \sup_{x \in \mathbb{R}}\{h(x) + \mathcal{L}(x, s; y, t)\}.
\]

**Lemma B.6.** Let $\mathcal{L} : \mathbb{R}^4 \to \mathbb{R}$ be a continuous function satisfying the metric composition law (2.1) and such that maximizers in (2.1) exist. Assume that $s < t$, $x_1 < x_2$ and $y_1 < y_2$. Then
\[
\mathcal{L}(x_2, s; y_1, t) - \mathcal{L}(x_1, s; y_1, t) \leq \mathcal{L}(x_2, s; y_2, t) - \mathcal{L}(x_1, s; y_2, t).
\]

**Proof.** This is a standard paths-crossing argument. By planarity, a geodesic from $(x_1, s)$ to $(y_2, t)$ must cross a geodesic from $(x_2, s)$ to $(y_1, t)$. Set $(z, u)$ to be a crossing point. Then,
\[
\mathcal{L}(x_2, s; y_1, t) - \mathcal{L}(x_1, s; y_1, t) \\
\leq \mathcal{L}(x_2, s; z, u) + \mathcal{L}(z, u; y_1, t) - \mathcal{L}(x_1, s; z, u) + \mathcal{L}(z, u; y_1, t) \\
= \mathcal{L}(x_2, s; z, u) + \mathcal{L}(z, u; y_2, t) - \mathcal{L}(x_1, s; z, u) + \mathcal{L}(z, u; y_2, t) \\
\leq \mathcal{L}(x_2, s; y_2, t) - \mathcal{L}(x_1, s; y_2, t). \quad \Box
\]

**Lemma B.7** ([Pim21b], Proposition 3). Let $\mathfrak{h}^1, \mathfrak{h}^2 : \mathbb{R} \to \mathbb{R}$ be two continuous functions, and let $\mathcal{L} : \mathbb{R}^4 \to \mathbb{R}$ be a continuous function satisfying the metric composition law (2.1) and such that maximizers in (2.1) exist. For $i = 1, 2$ and $t > 0$, set
\[
h_t(y; \mathfrak{h}^i) = \sup_{x \in \mathbb{R}}\{h^i(x) + \mathcal{L}(x, 0; y, t)\}. \tag{B.3}
\]
Assuming that maximizers in (B.3) exist, for $t > 0$ and $i = 1, 2$, set
\[
Z_t(y; \mathfrak{h}^i) = \max_{x \in \mathbb{R}}\max\{h^i(x) + \mathcal{L}(x, 0; y, t)\}.
\]
Then, if \( x < y \) and \( Z_t(y; h^1) \leq Z_t(x; h^2) \),

\[
h_t(y; h^1) - h_t(x; h^1) \leq h_t(y; h^2) - h_t(x; h^2).
\]

**Lemma B.8** ([Pim21b], Proposition 4). Let \( h^1, h^2 : \mathbb{R} \to \mathbb{R} \) be two continuous functions, and let \( \mathcal{L} : \mathbb{R}^4 \to \mathbb{R} \) be a continuous function satisfying the metric composition law (2.1) and such that maximizers in (2.1) exist. For \( i = 1, 2 \) and \( t > 0 \), set

\[
\tag{B.4}
h_t(y; h^i) = \sup_{x \in \mathbb{R}} \{ h'(x) + \mathcal{L}(x, 0; y, t) \}.
\]

Then, assuming that maximizers in (B.4) exist for \( i = 1, 2 \), and all \( y \in \mathbb{R} \), if \( h^1 \leq_{\text{inc}} h^2 \), then \( h_t(\cdot; h^1) \leq_{\text{inc}} h_t(\cdot; h^2) \) for all \( t > 0 \).

We prove the following technical lemmas that are needed in the main proofs. We believe these results are fairly well-known but do not have a formal reference.

**Lemma B.9.** Let \( h \) be continuous initial data for the KPZ fixed point, satisfying \( |h(x)| \leq a + b|x| \) for some constants \( a, b > 0 \). Then, on a single event of probability one, for any \( t > 0 \), \( \delta > 0 \) and sufficiently large \( |y| \) (depending on \( t \) and \( \delta \)), all maximizers of \( h(x) + \mathcal{L}(x, 0; y, t) \) over \( x \in \mathbb{R} \) lie in the interval \((y - y^{1/2+\delta}, y + y^{1/2+\delta})\).

**Proof.** By Lemma B.2 and the growth assumption on \( h \),

\[
\tag{B.5}
\sup_{x \in \mathbb{R}} \{ h(x) + \mathcal{L}(x, 0; y, t) \} \geq h(y) + \mathcal{L}(y, 0; y, t) \geq -a - b|y| - C\log^k |y|,
\]

where \( C = C(t) \) is a constant, and \( k \) is sufficiently large. By Lemma B.2, for each \( x \in \mathbb{R} \),

\[
\tag{B.6}
h(x) + \mathcal{L}(x, 0; y, t) \leq -\frac{(x - y)^2}{t} + C_1 \log^{k_1}(|x| + |y|) + a + b|x|,
\]

where \( C_1 \) is a constant depending on \( t \), and \( k_1 \) is sufficiently large. We first note that \( h(x) + \mathcal{L}(x, 0; y, t) \to -\infty \) as \( x \to \pm \infty \) so that maximizers must exist. By comparing (B.6) to (B.5), when \( y \) is sufficiently large, maximizers cannot lie outside the interval \((y - y^{1/2+\delta}, y + y^{1/2+\delta})\). \( \Box \)

**Lemma B.10.** The following holds simultaneously for all initial data and all \( t > s \) on a single event of probability one. Let \( h \) be continuous initial data for the KPZ fixed point, sampled at time \( s \), and satisfying

\[
\lim_{x \to \infty} \frac{h(x)}{x} = \xi_1 \quad \text{and} \quad \lim_{x \to \infty} \frac{h(x)}{x} = \xi_2
\]

for some \( \xi_1, \xi_2 \in \mathbb{R} \). For all \( t > s \), and \( y \in \mathbb{R} \), set

\[
h_t(y; h) = \sup_{x \in \mathbb{R}} \{ h(x) + \mathcal{L}(x, s; y, t) \}
\]

Then, for all \( t > s \),

\[
\lim_{x \to \infty} \frac{h_t(x; h)}{x} = \xi_1 \quad \text{and} \quad \lim_{x \to \infty} \frac{h_t(x, h)}{x} = \xi_2.
\]

**Proof.** We assume that \( h(x)/x \to \xi \) as \( x \to \infty \) and show that for all \( t > s \), \( h_t(x; h)/x \to \xi \) as \( x \to \infty \) as well. The analogous statement for limits as \( x \to -\infty \) follows from the spatial reflection invariance of Theorem B.1.(v).

Let \( \varepsilon > 0 \), and let \( y \) be sufficiently large and positive so that \( h(y) \geq (\xi - \varepsilon)y \). Then, using Lemma B.2, for such sufficiently large positive \( y \),

\[
\tag{B.7}
\sup_{x \in \mathbb{R}} \{ h(x) + \mathcal{L}(x, s; y, t) \} \geq h(y) + \mathcal{L}(y, s; y, t) \geq (\xi - \varepsilon)y - C\log^k(y),
\]

where \( C \) is a constant depending on \( s \) and \( t \), and \( k \) is sufficiently large. Therefore,

\[
\liminf_{y \to \infty} \frac{h_t(y; h)}{y} \geq \xi - \varepsilon.
\]
Now, we prove the upper bound. By the assumptions on the asymptotics of $h$, Lemma B.9 implies that for $\varepsilon > 0$ and sufficiently large $y$,

$$
\sup_{x \in \mathbb{R}} \{ h(x) + L(x, s; y, t) \} \leq \sup_{x \in (y-\varepsilon/3, y+\varepsilon/3)} \left\{ (\xi + \varepsilon)x - \frac{(x-y)^2}{t-s} + C_1 \log^{k_1}(|x| + |y|) \right\}
$$

$$
\leq \sup_{x \in (y-\varepsilon/3, y+\varepsilon/3)} \left\{ (\xi + \varepsilon)x - \frac{(x-y)^2}{t-s} + \varepsilon(x+y) \right\}
$$

$$
= (\xi + 3\varepsilon)y + C(\varepsilon, s, t, \xi),
$$

and so

$$
\limsup_{y \to \infty} \frac{h_t(y; \mathbf{h})}{y} \leq \xi + 3\varepsilon. \quad \square
$$

### Appendix C. The Busemann process for exponential last-passage percolation

#### C.1. Discrete last-passage percolation

Let $\{Y_x : x \in \mathbb{Z}^d\}$ be a collection of nonnegative i.i.d random variables, each associated to a vertex on the integer lattice. For $x, y \in \mathbb{Z} \times \mathbb{Z}$, define the last-passage time as

$$
d(x, y) = \sup_{x \in \Pi_{x, y}} \sum_{k=0}^{|y-x|} Y_{x_k}, \quad (C.1)
$$

where $\Pi_{x, y}$ is the set of up-right paths $\{x_k\}_{k=0}^n$ that satisfy $x_0 = x, x_n = y$, and $x_k - x_{k-1} \in \{e_1, e_2\}$. A maximizing path is called a geodesic. We call this model discrete last-passage percolation (LPP). The most tractable case of discrete LPP is given when $Y_x \sim \text{Exp}(1)$, and we refer to this model as the exponential corner growth model or CGM. We will consider this model specifically for the remainder of this appendix.

#### C.2. Stationary LPP in the quadrant

Choose $x \in \mathbb{Z}^2$ and consider the quadrant $x + \mathbb{Z}_{\geq 0}^2$. Fix a parameter $\rho \in (0, 1)$. Let $\{Y_z : z \in x + \mathbb{Z}_{\geq 0}^2\}$ be i.i.d. $\text{Exp}(1)$ weights in the bulk of the quadrant, and let $\{I_{x+ke_1}, J_{x+le_2} : k, l \in \mathbb{Z}_{\geq 0}\}$ be mutually independent boundary weights such that $I_{x+ke_1} \sim \text{Exp}(\rho)$ and $J_{x+le_2} \sim \text{Exp}(1-\rho)$. These weights are defined under a probability measure $\mathbb{P}^\rho$. We define the increment-stationary process $d^\rho_x$ as follows. First, on the boundary, $d^\rho_x(x) = 0$, and for $k, \ell \geq 1$, 

$$
d^\rho_x(x + ke_1) = \sum_{i=1}^k I_{x+ie_1} \quad \text{and} \quad d^\rho_x(x + \ell e_2) = \sum_{j=1}^\ell J_{x+je_2}. \quad (C.2)
$$

In this model, we can define also define geodesics from $x$ to $y \in x + \mathbb{Z}_{\geq 0}^2$ that travel for some time along the boundary and then enter the bulk. Because exponential random variables have continuous distribution, the maximizing paths for both bulk LPP and stationary LPP are almost surely unique. For $y \in x + \mathbb{Z}_{\geq 0}^2$, if the unique geodesic for the stationary model enters the bulk from the horizontal boundary, define $\tau^\rho_y(x)$ as the unique value $k$ that maximizes in (C.2). Otherwise, define $\tau^\rho_y(x) = 0$. Similarly, define $\tau^\rho_y(y)$ as the exit location from the vertical boundary, or 0 if the geodesic exits from the horizontal boundary.

This model is increment-stationary in the sense that, for any down-right path $\{y_i\}$ in $x + \mathbb{Z}_{\geq 0}^2$, the increments $d^\rho_{x, y_{i+1}} - d^\rho_{x, y_i}$ are mutually independent, and for $y \in x + \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ and $z \in x + \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$,

$$
I^\rho_y := d^\rho_{x}(y) - d^\rho_{x}(y - e_1) \sim \text{Exp}(\rho) \quad \text{and} \quad J^\rho_{y} = d^\rho_{x}(z) - d^\rho_{x}(z - e_2) \sim \text{Exp}(1-\rho).
$$

See Theorem 3.1 in [Sep18] for a proof.
C.3. LPP in the half-plane with boundary conditions and queues. We also define the last-passage model in the upper half-plane with a horizontal boundary. Let \( h = (h(k))_{k \in \mathbb{Z}} \) be a real sequence. For \( m \in \mathbb{Z} \) let \( d^h(m, 0) = h(m) \), and for \( n > 0 \)
\[
(C.3) \quad d^h(m, n) = \sup_{-\infty < k \leq m} \{ h(k) + d((k, 1), (m, n)) \}.
\]
We assume that \( h \) is such that the supremum is almost surely finite and achieved at a finite \( k \). We define the exit point \( Z^h(m, n) \) as
\[
(C.4) \quad Z^h(m, n) = \max\{ k \in \mathbb{Z} : h(k) + d((k, 1), (m, n)) = d^h(m, n) \}.
\]
Geodesics in this model are defined as follows: the path consists of the backwards-infinite horizontal ray \( \{(k, 0) : k \leq Z := Z^h(m, n)\} \), an upward step from \((Z, 0)\) to \((Z, 1)\), and then the LPP path in the bulk from \((Z, 1)\) to \((m, n)\).

The half-plane model with boundary satisfies superadditivity. That is, for \( r \in \{1, 2\}, x \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} \) and \( x + e_r \leq z \) coordinate-wise,
\[
(C.5) \quad d^p(z) \geq d^p(x) + d(x + e_r, z).
\]
The model with boundary condition can be constructed from queuing mappings, which we now define. Let \( I = (I_k)_{k \in \mathbb{Z}} \) and \( \omega = (\omega_k)_{k \in \mathbb{Z}} \) be sequences of nonnegative numbers such that
\[
\lim_{m \to -\infty} \sum_{i=m}^{0} (\omega_i - I_{i+1}) = -\infty.
\]
Let \( F = (F_k)_{k \in \mathbb{Z}} \) be a function on \( \mathbb{Z} \) satisfying \( I_k = F_k - F_{k-1} \). Define the output sequence \( \tilde{F} = (\tilde{F}_\ell)_{\ell \in \mathbb{Z}} \) by
\[
(C.6) \quad \tilde{F}_\ell = \sup_{-\infty < k \leq \ell} \left\{ F_k + \sum_{i=k}^{\ell} \omega_i \right\}, \quad \ell \in \mathbb{Z}.
\]
Now, define the sequences \( \tilde{I} = (\tilde{I}_\ell)_{\ell \in \mathbb{Z}} \) and \( J = (J_k)_{k \in \mathbb{Z}} \) by
\[
\tilde{I}_\ell = \tilde{F}_\ell - \tilde{F}_{\ell-1}, \quad \text{and} \quad J_k = F_k - F_{k-1}.
\]
In queuing terms, \( I_k \) is the time between the arrivals of customers \( k - 1 \) and \( k \), \( \omega_k \) is the service time of customer \( k \), \( \tilde{I}_\ell \) is the interdeparture time between customers \( \ell - 1 \) and \( \ell \), and \( J_k \) is the sojourn time of customer \( k \). We use the mappings \( D \) and \( S \) to describe this queuing process. That is,
\[
\tilde{I} = D(\omega, I), \quad \text{and} \quad J = S(\omega, I).
\]
Exponentially distributed arrival times are invariant for the queue with exponential service times. This is made precise is the following lemma.

Lemma C.1 ([FS20], Lemma B.2). Let \( 0 < \rho < \tau \). Let \((I_k)_{k \in \mathbb{Z}} \) and \( \{\omega_j\}_{j \in \mathbb{Z}} \) be mutually independent random variables with \( I_k \sim \text{Exp}(\rho) \) and \( \omega_j \sim \text{Exp}(\tau) \). Let \( \tilde{I} = D(\omega, I) \) and \( J = S(\omega, I) \). Then, \( \{\tilde{I}_j\}_{j \in \mathbb{Z}} \) is an i.i.d. sequence of \( \text{Exp}(\rho) \) random variables, and for each \( k \in \mathbb{Z} \), \( \{\tilde{I}_j\}_{j \leq k} \) and \( J_k \) are mutually independent with \( J_k \sim \text{Exp}(\tau - \rho) \).

The following lemma shows how to construct the LPP model in the half-plane with boundary from the queuing mappings.

Lemma C.2. Consider last-passage percolation for the environment \( \{Y_x\}_{x \in \mathbb{Z}_2} \). For \( n \geq 1 \), let \( Y^n = \{Y_{m,n}\}_{m \in \mathbb{Z}} \) be the weights along the horizontal level \( n \). Let \( h \) be a function on \( \mathbb{Z} \) that denotes initial data for the LPP model with boundary. Define the sequence \( I^n = (I^n_i)_{i \in \mathbb{Z}} \) by \( I^n_0 = h(i) - h(i-1) \). Let \( I^1 = D(Y^1, I^0) \), and for \( n > 1 \), let \( I^n = D(Y^n, I^{n-1}) \) and \( J^n = S(Y^n, I^{n-1}) \). Then, for each \( n \geq 1 \) and \( m \in \mathbb{Z} \),
\[
(C.7) \quad I^n_m = d^h(m, n) - d^h(m - 1, n), \quad \text{and} \quad J^n_m = d^h(m, n) - d^h(m, n - 1).
\]
Proof. For $m \in \mathbb{Z}$,
\[
d^h(m, 1) = \sup_{-\infty < k \leq m} \{h(k) + d((k, 1), (m, 1))\} = \sup_{-\infty < k \leq m} \left\{h(k) + \sum_{i=k}^m Y_{i,1}\right\},
\]
and so by (C.6) and the definitions below, Equation (C.6) holds for $n = 1$, follow from the definition. Now, assume that the statements hold for some $n \geq 1$. Then, $(d^h(m, n))_{m \in \mathbb{Z}}$ is a function whose increments are given by $I^n$. Then, by definition of $D$, for $m \in \mathbb{Z}$,
\[
I_{m+1}^n = [D(Y^{n+1}, I^n)]_m = \sup_{-\infty < k \leq m} \left\{d^h(k, n) + \sum_{i=k}^m Y_{i,n+1}\right\} - \sup_{-\infty < k \leq m-1} \left\{d^h(k, n) + \sum_{i=k}^{m-1} Y_{i,n+1}\right\}
\]
\[
= \sup_{-\infty < \ell \leq k \leq m} \left\{h(\ell) + d((\ell, 1), (m, n)) + \sum_{i=\ell}^m Y_{i,n+1}\right\}
\]
\[
- \sup_{-\infty < \ell \leq k \leq m-1} \left\{h(\ell) + d((\ell, 1), (m, n)) + \sum_{i=\ell}^m Y_{i,n+1}\right\}
\]
\[
= d^h(m, n + 1) - d^h(m - 1, n + 1).
\]
The last equality is the dynamic programming principle. Similarly,
\[
J_m^n = [S(Y^{n+1}, I^n)]_m = \sup_{-\infty < k \leq m} \left\{d^h(k, n) + \sum_{i=k}^m Y_{i,n+1}\right\} - d^h(m, n)
\]
\[
= d^h(m, n + 1) - d^h(m, n).
\]

To construct the stationary boundary condition, fix a parameter $\rho \in (0, 1)$, and let $h$ be defined so that $h(0) = 0$ and $\{h(k) - h(k - 1)\}_{k \in \mathbb{Z}}$ is a sequence of i.i.d. Exp(1) random variables, independent of the i.i.d. Exp(1) random variables $\{Y_{x,i}\}_{x \in \mathbb{Z} \times \mathbb{Z}_>0}$. Let $\hat{\mathbb{P}}^\rho$ be the probability measure of these random variables. Abusing notation, we denote LPP in the half-plane with this initial data simply as $d^\rho$.

For $y \in \mathbb{Z} \times \mathbb{Z}_\geq 0$ and $z \in \mathbb{Z} \times \mathbb{Z}_{>0}$, we define
\[
(C.8) \quad I_y = d^\rho(y) - d^\rho(y - e_1), \quad \text{and} \quad J_z = d^\rho(z) - d^\rho(z - e_2).
\]

The stationary model in the quadrant is simply a projection of the stationary model in the half-plane. This is made precise in the following lemma.

**Lemma C.3.** Let $I_y$ and $J_z$ be defined as in (C.8). Fix $x = (k, 0)$, where $k \in \mathbb{Z}$. Then, $\{J_{x+je_2}\}_{j \geq 1}$ is a sequence of i.i.d. Exp$(1)$ random variables, independent of the i.i.d. Exp$(\rho)$ random variables $\{Y_{x,i}\}_{i \geq 1}$. With $I_{x+je_1}$ and $J_{x+je_2}$ defined, let the process $\{d^\rho_y(x) : y \in x + \mathbb{Z}^2_{\geq 0}\}$ be defined as in (C.2). Then, under $\hat{\mathbb{P}}^\rho$, for any $y \in x + \mathbb{Z}_{>0}$, the portion of the almost surely unique geodesic to $y$ for the process $d^\rho$ that lies in $x + \mathbb{Z}^2_{\geq 0}$ coincides with the portion of the geodesic from $x$ to $y$ for the process $d^\rho_x$ that lies in $x + \mathbb{Z}^2_{>0}$.

**Proof.** Let $I^0 = \{I_{ie_1}\}$, and for $n \geq 1$, let $Y^n = \{Y_{m,n}\}_{m \in \mathbb{Z}}$, and $I^n = D(Y^n, I^{n-1})$. By Lemma C.2, $J_{x+e_2} = [S(Y^1, I^0)]_k$. For $k$ fixed, we note that the sequence
\[
F_\ell = \begin{cases} -\sum_{i=\ell+1}^{\ell+1} I_{ie_1}, & \ell \leq k \\ -\sum_{i=\ell+1}^{\ell+1} I_{ie_1}, & \ell > k \end{cases}
\]
satisfies $F_k = 0$ and $F_\ell - F_{\ell-1} = I_{ie_1} = I^0_\ell$ for $\ell \in \mathbb{Z}$. Then, by definition of the mappings $S$ and $D$,
\[
J_{x+e_2} = \sup_{-\infty < j \leq k} \left\{-\sum_{i=\ell+1}^{\ell+1} I_{ie_1} + \sum_{i=j}^{\ell+1} Y^n_i\right\},
\]
while, for $\ell \leq k$,
\[
I^1_\ell = \sup_{-\infty < j \leq \ell} \left\{-\sum_{i=\ell+1}^{\ell+1} I_{ie_1} + \sum_{i=j}^{\ell+1} Y^n_i\right\} - \sup_{-\infty < j \leq \ell-1} \left\{-\sum_{i=\ell+1}^{\ell+1} I_{ie_1} + \sum_{i=j}^{\ell+1} Y^n_i\right\}.
\]
Therefore, \( \{(I^1_t)_{t \leq k}, J_{x+e_2}\} \) is a measurable function of \((I_{e_1})_{i \leq k}\) and \(Y^1\), and is therefore independent of \(\{(I_{x+e_1})_{i \geq 1}, Y^2, Y^3, \ldots\}\). By Lemma C.1, \(I^1\) is an i.i.d. sequence of \(\text{Exp}(\rho)\) random variables, \(J_{x+e_2} \sim \text{Exp}(1 - \rho)\), and \(\{(I^1_t)_{t \leq k}, J_{x+e_2}\) are mutually independent.

Now, assume by way of induction, that for some \(n \geq 1\),
\[
\{(I^n_t)_{t \leq k}, J_{x+e_2}, \ldots, J_{x+n e_2}, (I_{x+ie_1})_{i \geq 1}, Y^{n+1}, Y^{n+2}, \ldots\}.
\]
are mutually independent, and \(I^n\) is an i.i.d. sequence of \(\text{Exp}(\rho)\) random variables. Using the same reasoning as in the base case via Lemmas C.2 and C.3, \(\{(I^n_t)_{t \leq k}, J_{x+(n+1)e_1}\) is a measurable function of \((I^n_t)_{t \leq k}\) and \(Y^{n+1}\). Thus, from (C.9), we have
\[
\{(I^{n+1}_t)_{t \leq k}, J_{x+e_2}, \ldots, J_{x+(n+1)e_2}, (I_{x+ie_1})_{i \geq 1}, Y^{n+2}, Y^{n+3}, \ldots\}
\]
are mutually independent, \(I^{n+1}\) is a sequence of i.i.d. \(\text{Exp}(\rho)\) random variables, and \(J_{x+(n+1)e_2} \sim \text{Exp}(1 - \rho)\).

For the second part of the lemma, this follows the same reasoning as Lemma B.3 in [BBS20] and Lemma A.1 in [Sep18]. Suppose that the geodesic to \(y\) for \(d^\rho\) enters the quadrant \(x + Z_2^\rho\) through the edge \((w, z)\) with \(w = x + e_r\) for \(r \in \{1, 2\}\), and suppose that the geodesic from \(x\) to \(y\) for \(d^\rho\) enters the boundary through \((w, z)\) with \(w = x + p e_s\) for \(s \in \{1, 2\}\). For \(1 \leq i \leq \ell\), set \(\eta_i = I_{x+ie_1} = d^\rho(x + ie_1) - d^\rho(x + (i - 1)e_1)\) if \(r = 1\), and \(\eta_i = J_{x+ie_2} = d^\rho(x + je_2) - d^\rho(x + (j - 1)e_2)\) if \(r = 2\). For \(1 \leq j \leq p\), set \(\eta_j = I_{x+je_1}\) if \(s = 1\) and \(\eta_j = J_{x+je_2}\) if \(s = 2\). Then, using (C.5) in the last inequality below,
\[
d^\rho(y) = d^\rho(w) + d(z, y) = d^\rho(x) + \sum_{i=1}^{\ell} \eta_i + d(z, y)
\]
\[
\leq d^\rho(x) + d^\rho(y) = d^\rho(x) + \sum_{j=1}^{p} \eta_j + d(z, y)
\]
\[
= d^\rho(x) + d(z, y) \leq d^\rho(y).
\]
Thus, all inequalities are equalities. Since geodesics are almost surely unique in both models, the desired conclusion follows. \(\square\)

C.4. KPZ scaling of the exponential CGM. We make use of the following.

**Lemma C.4** ([EJS20], Theorem 2.5. See also [SS20], Corollary 3.6 and Remark 2.5b). Recall the stationary LPP model in the quadrant from Section C.2, and let \(\tau_1^x, \tau_2^x\) be the exit times from the boundary. Let \(K = [a, b] \subseteq (0, 1)\). Then, there exist positive constants \(\bar{N}_0, C\) depending only on \(K\) such that for all \(N \geq N_0, b > 0, \) and \(\rho \in K\),
\[
\mathbb{P}^\rho \left( \tau_1^x \left( x + ([N \rho^2 - bN^{2/3}, [N(1 - \rho^2)]) \right) \leq 1 \right) \leq e^{-CN^3}, \quad \text{and}
\]
\[
\mathbb{P}^\rho \left( \tau_2^x \left( x + ([N \rho^2 + bN^{2/3}, [N(1 - \rho^2)]) \right) \geq 1 \right) \leq e^{-CN^3}.
\]

**Lemma C.5.** For \(N \geq 1\), consider LPP with \(\text{Exp}(1)\) bulk weights and boundary conditions for the stationary model in the half-plane as defined above, where \(\rho_N = \frac{1}{2} + cn^{-1/3}\) and \(c\) is some real-valued constant. Assume these are all coupled together under some probability measure \(\mathbb{P}\). Let \(Z^{\rho_N}\) denote shorthand notation for the exit point defined in (C.4) with initial profile given by sums of i.i.d. \(\text{Exp}(\rho_N)\) random variables. Then, for any \(y \in \mathbb{R}\) and \(t > 0\), there exists a constant \(C = C(c, x, t) > 0\) such that
\[
\limsup_{N \to \infty} \mathbb{P}(Z^{\rho_N}([tN + N^{2/3}y], [tN])) \geq M N^{2/3}) \leq e^{-CM^3}, \quad \text{for all } M > 0.
\]

**Proof.** We first show that
\[
\limsup_{N \to \infty} \mathbb{P}(Z^{\rho_N}([tN + N^{2/3}y], [tN])) \leq -MN^{2/3}) \leq e^{-CM^3}.
\]
Let \( y \in \mathbb{R} \), and let \( N \) be large enough so that \( \lfloor tN + N^{2/3}y \rfloor > \lfloor -MN^{2/3} \rfloor \). Set \( x = (\lfloor -MN^{2/3} \rfloor, 0) \).

With this choice of \( x \), consider the coupling of \( d_{p_N}^x \) and \( d_{p_N}^y \) described in Lemma C.3, where geodesics for the two models coincide in the quadrant \( x + \mathbb{Z}_{\geq 0} \). In particular, under this coupling, \( Z_{p_N}^x (\lfloor tN + N^{2/3}y \rfloor, [tN]) \leq \lfloor -MN^{2/3} \rfloor \) if and only if \( \tau_{\mathbb{R}}^x (\lfloor tN + N^{2/3}y \rfloor, [tN]) \geq 1 \). See Figure C.1.

The proof that \( \sup_{N \to \infty} \mathbb{P}(Z_{p_N}^x (\lfloor tN + N^{2/3}y \rfloor, [tN]) > MN^{2/3}) \leq e^{-CM^3} \)

follows analogously, this time setting \( x = (\lfloor MN^{2/3} \rfloor, 0) \), and replacing the use of (C.11) from Lemma C.4 with (C.10).

It was shown in [DV21] that exponential last-passage percolation converges to the directed landscape. We cite their theorem here

**Theorem C.6** ([DV21], Theorem 1.7). Let \( d \) denote last-passage percolation (C.1) with i.i.d. \( \text{Exp}(1) \) weights. There exists a coupling of the directed landscape \( \mathcal{L} \) and identically distributed copies \( d_N \), of \( d \), such that

\[
d_N \left( (sN + 2^{5/3}xN^{2/3}, sN), (tN + 2^{5/3}yN^{2/3}, tN) \right)
= 4N(t - s) + 2^{8/3}N^{2/3}(y - x) + 2^{4/3}N^{1/3}(\mathcal{L} + o_N)(s, x; y, t).
\]

Above \( d_N \) is interpreted as an appropriately interpolated version of the LPP process, and \( o_N : \mathbb{R}^d_+ \to \mathbb{R} \) is a random continuous function such that for every compact \( K \subset \mathbb{R}^d_+ \), there exists a constant \( c > 0 \) such that

\[
\sup_K |o_N| \to 0 \text{ almost surely, and } \mathbb{E} \left[ c \sup_K (o_N^-)^3 + (o_N^+)^3 \right] \to 1.
\]

**C.5. Existence and distribution of the Busemann functions.** In the case of the exponential CGM, Busemann functions are known to exist and are indexed by direction vectors \( u \). We index the direction in terms of a real parameter \( \rho \in (0, 1) \):

\[
u(\rho) = \left( \frac{\rho^2}{\rho^2 + (1 - \rho)^2}, \frac{(1 - \rho)^2}{\rho^2 + (1 - \rho)^2} \right). 
\]
Then for a fixed $\rho \in (0, 1)$ and $x, y \in \mathbb{Z}^2$, the following limit exists almost surely:

$$B_{\rho}^{x,y} = \lim_{n \to \infty} d(-n\mathbf{u}(\rho), y) - d(-n\mathbf{u}(\rho), x).$$

The Busemann functions can be extended to right- and left-continuous processes defined for all directions, as in [Sep18]. Here, we notice that the geodesics are travelling asymptotically to the southwest, whereas the geodesics we construct in the present paper are travelling to the northeast. By the temporal and spatial reflection invariance of Theorem B.1(v), both formulations are equivalent in the directed landscape. The geodesics to the southwest give rise to queueing relations that are more natural in the discrete model, so we use this formulation here.

Fan and the second author [FS20] derived the joint distribution of Busemann functions in the exponential CGM. The joint distribution for finitely many inputs is described by iterating the queueing mapping $D$ introduced in Section C.1. For $n \in \mathbb{N}$, let $\rho^n = (\rho_1, \ldots, \rho_n)$. We define the state spaces

$$\mathcal{Y}^n = \{(I_1, \ldots, I^n) \in (\mathbb{R}_+^n): \lim_{m \to -\infty} \frac{1}{m} \sum_{i=m}^{0} I_i < \lim_{m \to -\infty} \frac{1}{m} \sum_{i=m}^{0} I_i^{k+1}, \text{ for } 1 \leq k \leq n - 1\},$$

$$\mathcal{Y}_\rho^n = \{(I_1, \ldots, I^n) \in (\mathbb{R}_+^n): \lim_{m \to -\infty} \frac{1}{m} \sum_{i=m}^{0} I_i = (\rho_k)^{-1} \text{ for } 1 \leq k \leq n\}.$$

In the definitions above, the limits are assumed to exist. We extend the mapping $D$ to maps that take more than two sequences as inputs. For $k \geq 1$, define the maps $D^{(k)} : \mathcal{Y}^n \to \mathbb{R}_+^n$ inductively as follows. Define $D^{(1)}(I^1) = I^1$, and for $k > 1$,

$$D^{(k)}(I^1, \ldots, I^n) = D(I^1, D^{(k-1)}(I^2, I^3, \ldots, I^n)).$$

Furthermore, define the map $D^{(n)} : \mathcal{Y}^n \to \mathcal{Y}^n$ as

$$[D^{(n)}(I^1, \ldots, I^n)]_i = D^{(i)}(I^1, \ldots, I^n).$$

On the space $\mathcal{Y}^n$, we define the measure $\nu^{\rho^n}$ as follows: $(I^1, \ldots, I^n) \sim \nu^{\rho^n}$ if $(I^1, \ldots, I^n)$ are mutually independent, and for $1 \leq i \leq n$, $I^i$ is a sequence of i.i.d. exponential random variables with rate $\rho_i$. We define the measure $\mu^{\rho^n}$ as

$$\mu^{\rho^n} = \nu^{\rho^n} \circ (D^{(n)})^{-1}.$$ (C.12)

We now cite two theorems.

**Theorem C.7** ([FS20], Theorem 5.4). Let $\rho^n = (\rho_1, \ldots, \rho_n)$ with $1 > \rho_1 > \cdots > \rho_n > 0$ and assume $(I^1, \ldots, I^n) \sim \mu^{\rho^n}$. Let $I^0$ be a sequence of i.i.d. exponential random variables with rate 1, independent of $(I^1, \ldots, I^n)$. Then,

$$(D(I^0, I^1), \ldots, D(I^0, I^n)) \sim \mu^{\rho^n}.$$**

**Theorem C.8** ([FS20], Theorem 3.2). For $\rho \in (0, 1)$, define the sequence $I^\rho$ as $I^\rho_i = B^\rho_{(i-1)e_1, ie_1}$. Let $\rho^n = (\rho_1, \ldots, \rho_n)$ with $1 > \rho_1 > \cdots > \rho_n > 0$. Then,

$$(I^\rho_1, \ldots, I^\rho_n) \sim \mu^{\rho^n}.$$**

**APPENDIX D. THE STATIONARY HORIZON**

To describe the stationary horizon, we introduce some notation from [Bus21]. The map $\Phi : C(\mathbb{R}) \times C(\mathbb{R}) \to C(\mathbb{R})$ is defined as

$$\Phi(f, g)(x) = \begin{cases} f(x) + \left[ W_0(f - g) + \inf_{0 \leq y \leq x} (f(y) - g(y)) \right] - & \text{if } x \geq 0 \\ f(x) - \left[ W_x(f - g) + \inf_{x < y \leq 0} (f(y) - f(x) - [g(y) - g(x)]) \right] - & \text{if } x < 0, \end{cases}$$

where

$$W_x(f) = \sup_{y \leq x} |f(x) - f(y)|.$$ We note that the map $\Phi$ is well-defined only on the appropriate space of functions where the supremums are all finite. This map extends to maps $\Phi^k : C(\mathbb{R})^k \to C(\mathbb{R})^k$ as follows.
As relations of Theorem D.5(ii), one can see that this formulation of the limit is the same. The following hold for the stationary horizon. (Bus21, Theorem 1.2 and SS21b, Theorems 3.9, 3.11, 3.15, 7.20 and Lemma 3.6) Theorem D.5. The stationary horizon \( \{G_\xi\}_{\xi \in \mathbb{R}} \) is a process with state space \( C(\mathbb{R}) \) and with paths in the Skorokhod space \( D(\mathbb{R}, C(\mathbb{R})) \) of right-continuous functions \( \mathbb{R} \to C(\mathbb{R}) \) with left limits. \( C(\mathbb{R}) \) has the Polish topology of uniform convergence on compact sets. The law of the stationary horizon is characterized as follows: For real numbers \( \xi_1 < \cdots < \xi_k \), the \( k \)-tuple \( (G_{\xi_0}, \ldots, G_{\xi_k}) \) of continuous functions has the same law as \( \Phi^k(f_1, \ldots, f_k) \), where \( f_1, \ldots, f_k \) are independent two-sided Brownian motions with drifts \( 2\xi_1, \ldots, 2\xi_k \), and each with diffusion coefficient \( \sqrt{2} \) (as defined in point (ix) in Section 2.1).

Remark D.2. The transformation \( \Phi^k \) is such that for each \( \xi \in \mathbb{R} \), \( G_\xi \) is also a two-sided Brownian motion with diffusion coefficient \( \sqrt{2} \) and drift \( 2\xi \).

For \( N \in \mathbb{N} \), we define \( F_\xi^N \in D(\mathbb{R}, C(\mathbb{R})) \) to be such that, for each \( \xi \in \mathbb{R} \), \( F_\xi^N \) is the linear interpolation of the function \( Z \ni t \mapsto B_{t,0}^{1/2-2^{-4/3}N^{-1/3}} \). Then, for \( \xi \in \mathbb{R} \), we define \( G_\xi^N(x) = 2^{-4/3}N^{-1/3} \left[ F_\xi^N(2^{5/3}N^{2/3}x) - 2^{8/3}N^{2/3}x \right] \).

Remark D.3. The parameterization here is different from the one used in [Bus21], because in the present paper, \( G_\xi \) is a Brownian motion with diffusivity \( \sqrt{2} \) and drift \( 2\xi \), while in [Bus21], \( G_{\mu} \) has diffusivity \( 2 \) and drift \( \mu \). Specifically, \( G_\xi(x) = \tilde{G}_{4\xi}(x/2) \), where \( \tilde{G} \) denotes the parameterization used in the present paper, and \( G \) denotes the parameterization from [Bus21]. Using the scaling relations of Theorem D.5(ii), one can see that this formulation of the limit is the same.

The main theorem of [Bus21] is the following:

**Theorem D.4.** As \( N \to \infty \), the process \( G_\xi^N \) converges in distribution to \( G \) on the path space \( D(\mathbb{R}, C(\mathbb{R})) \). In particular, for any finite collection \( \xi_1, \ldots, \xi_n \),

\[
(G_{\xi_1}^N, \ldots, G_{\xi_n}^N) \Rightarrow (G_{\xi_1}, \ldots, G_{\xi_n}),
\]

where the convergence holds in distribution in the sense of uniform convergence on compact sets of functions in \( C(\mathbb{R})^n \).

The first author [Bus21] first proved this finite-dimensional convergence and then showed tightness of the process to conclude the existence of a limit taking values in \( D(\mathbb{R}, C(\mathbb{R})) \). The last two authors [SS21b] discovered that the stationary horizon is also the Busemann process of Brownian last-passage percolation, up to an appropriate scaling and reflection (see Theorem 5.3 in [SS21b]).

The following collects several facts about the stationary horizon from these two papers. For notation, let \( G_{\xi_+} = G_\xi \), and let \( G_{\xi_-} \) be the limit of \( G_\alpha \) as \( \alpha \nearrow \xi \).

**Theorem D.5** ([Bus21], Theorem 1.2 and SS21b, Theorems 3.9, 3.11, 3.15, 7.20 and Lemma 3.6). The following hold for the stationary horizon.

(i) For each \( \xi \in \mathbb{R} \), with probability one, \( G_{\xi_-} = G_{\xi_+} \), and \( G_\xi \) is a two-sided Brownian motion with diffusion coefficient \( \sqrt{2} \) and drift \( 2\xi \).

(ii) For \( c > 0 \) and \( \nu \in \mathbb{R} \),

\[
\{ cG_{\xi}(c^{-2}x) - 2\nu x : x \in \mathbb{R} \}_{\xi \in \mathbb{R}} \overset{d}{=} \{ G_\xi(x) : x \in \mathbb{R} \}_{\xi \in \mathbb{R}}.
\]

(iii) Spatial stationarity holds in the sense that, for \( y > 0 \),

\[
\{ G_\xi(x) : x \in \mathbb{R} \}_{\xi \in \mathbb{R}} \overset{d}{=} \{ G_\xi(y, x + y) : x \in \mathbb{R} \}_{\xi \in \mathbb{R}}.
\]

(iv) Fix \( x > 0 \), \( \xi_0 \in \mathbb{R} \), \( \xi > 0 \), and \( z \geq 0 \). Then

\[
\mathbb{P}\left( \sup_{a,b \in [-x,x]} |G_{\xi_0 + \xi}(a, b) - G_{\xi_0}(a, b)| \leq z \right) = \mathbb{P}\left( G_{\xi_0 + \xi}(-x, x) - G_{\xi_0}(-x, x) \leq z \right)
\]

\[
= \Phi\left( \frac{z - 2\xi x}{2\sqrt{2x}} \right) + e^{\frac{2\xi}{2}} \left( 1 + \frac{\xi}{2} z + \xi^2 x \right) \Phi\left( \frac{z + 2\xi x}{2\sqrt{2x}} \right) - \xi \sqrt{x} e^{-\frac{(z+2\xi x)^2}{8x}}.
\]
Furthermore, the following hold on a single event of full probability.

(v) Let \( x_0 > 0 \) and consider the process \( G^{x_0} \in D(\mathbb{R}, C([-x_0, x_0])) \) defined by restricting each function \( G_\xi \) to \([-x_0, x_0]\). Then, \( G^{x_0} \) is a jump process with finitely many jumps in any compact interval \([-\xi_0, \xi_0]\), but infinitely many jumps in \( \mathbb{R} \). The number of jumps in a compact interval has finite expectation. In other words, for each \( \xi \in \mathbb{R} \) and compact set \( K \), there exists a random \( \varepsilon = \varepsilon(\xi, K) \) such that for all \( \xi - \varepsilon < \alpha < \xi < \beta < \xi + \varepsilon \), \( \square \in \{-, +\} \), and all \( x \in K \), \( G_{\xi-}(x) = G_\alpha(x) \) and \( G_{\xi+}(x) = G_\beta(x) \).

(vi) For \( x_1 \leq x_2 \), \( \xi \mapsto G_\xi(x_1, x_2) \) is a non-decreasing jump process.

(vii) For each \( \alpha < \beta \), there exist random times \( S_1 = S_1(\alpha, \beta) \) and \( S_2 = S_2(\alpha, \beta) \) with \( S_1 < 0 < S_2 \) such that \( G_\alpha(x) = G_\beta(x) \) for \( x \in [S_1, S_2] \), and \( G_\alpha(x) \neq G_\beta(x) \) for \( x \notin [S_1, S_2] \).

(viii) For each \( \alpha < \beta \) and \( S_1 = S_1(\alpha, \beta) \), there exists a random \( \xi \in (\alpha, \beta) \) such that \( G_{\xi-}(x) = G_{\xi+}(x) \) for \( x \in [-S_1, 0] \), and \( G_{\xi-}(x) > G_{\xi+}(x) \) for \( x < S_1 \). For each \( \alpha < \beta \) and \( S_2 = S_2(\alpha, \beta) \), there exists a random \( \xi \in (\alpha, \beta) \) such that \( G_{\xi-}(x) = G_{\xi+}(x) \) for \( x \in [0, S_2] \), and \( G_{\xi-}(x) < G_{\xi+}(x) \) for \( x > S_2 \).

Theorem 2.1 gives the following corollary, stating a previously unknown fact about the stationary horizon.

**Corollary D.6.** The stationary horizon, as a distribution on \( D(\mathbb{R}, C(\mathbb{R})) \) satisfies the following reflection property:

\[
\{G_{(-\xi)+}(-\cdot)\}_{\xi \in \mathbb{R}} \overset{d}{=} \{G_\xi\}_{\xi \in \mathbb{R}}.
\]

**Proof.** By the spatial reflection invariance of the directed landscape (Theorem B.1(v)), \( \{G_{(-\xi)+}\}_{\xi \in \mathbb{R}} \) is an invariant distribution for the KPZ fixed point such that each marginal satisfies the limit condition (2.3). The result follows from the uniqueness part of Theorem 2.1. \( \square \)

**References**

[BBS20] Márton Balázs, Ofer Busani, and Timo Seppäläinen. Non-existence of bi-infinite geodesics in the exponential corner growth model. *Forum Math. Sigma*, 8:Paper No. e46, 34, 2020.

[BF22] Ofer Busani and Patrik L. Ferrari. Universality of the geodesic tree in last passage percolation. *Ann. Probab.*, 50(1):90–130, 2022.

[BGH21] Riddhipratim Basu, Shirshendu Ganguly, and Alan Hammond. Fractal geometry of Airy\(_2\) processes coupled via the Airy sheet. *Ann. Probab.*, 49(1):485–505, 2021.

[BGH22] Erik Bates, Shirshendu Ganguly, and Alan Hammond. Hausdorff dimensions for shared endpoints of disjoint geodesics in the directed landscape. *Electron. J. Probab.*, 27:Paper No. 1, 44, 2022.

[Bus21] Ofer Busani. Diffusive scaling limit of the Busemann process in Last Passage Percolation. *Preprint*: arXiv:2110.03808, 2021.

[Cou11] David Coupier. Multiple geodesics with the same direction. *Electron. Commun. Probab.*, 16:517–527, 2011.

[Dau22] Duncan Dauvergne. Non-uniqueness times for the maximizer of the KPZ fixed point. *Preprint*: arXiv:2202.01700, 2022.

[DOV18] Duncan Dauvergne, Janosch Ortmann, and Bálint Virág. The directed landscape. *Preprint*: arXiv:1812.00309, 2018. To appear in Acta. Math.

[Dud89] Richard M. Dudley. *Real analysis and probability*. The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1989.

[DV21] Duncan Dauvergne and Bálint Virág. The scaling limit of the longest increasing subsequence. *Preprint*: arXiv:2104.08210, 2021.

[EJS20] Elnur Emrah, Chris Janjigian, and Timo Seppäläinen. Right-tail moderate deviations in the exponential last-passage percolation. *Preprint*: arXiv:2004.04285, 2020.

[EK86] Stewart N. Ethier and Thomas G. Kurtz. *Markov processes: Characterization and convergence*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986.

[FP05] Pablo A. Ferrari and Leandro P. R. Pimentel. Competition interfaces and second class particles. *Ann. Probab.*, 33(4):1235–1254, 2005.

[FS20] Wai-Tong (Louis) Fan and Timo Seppäläinen. Joint distribution of Busemann functions for the exactly solvable corner growth model. *Probability and Mathematical Physics*, 1(1):55–100, 2020.

[GRAS17a] Nicos Georgiou, Firas Rassoul-Agha, and Timo Seppäläinen. Geodesics and the competition interface for the corner growth model. *Probab. Theory Related Fields*, 169(1-2):223–255, 2017.

[GRAS17b] Nicos Georgiou, Firas Rassoul-Agha, and Timo Seppäläinen. Stationary cocycles and Busemann functions for the corner growth model. *Probab. Theory Related Fields*, 169(1-2):177–222, 2017.
[Hof08] Christopher Hoffman. Geodesics in first passage percolation. *Ann. Appl. Probab.*, 18(5):1944–1969, 2008.

[JRS19] Christopher Janjigian, Firas Rassoul-Agha, and Timo Seppäläinen. Geometry of geodesics through Busemann measures in directed last-passage percolation. *Preprint*: arXiv:1908.09040, 2019. To appear in J. Eur. Math. Soc.

[LN96] Cristina Licea and Charles M. Newman. Geodesics in two-dimensional first-passage percolation. *Ann. Probab.*, 24(1):399–410, 1996.

[MQR21] Konstantin Matetski, Jeremy Quastel, and Daniel Remenik. The KPZ fixed point. *Acta Math.*, 227(1):115–203, 2021.

[New95] Charles M. Newman. A surface view of first-passage percolation. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 1017–1023. Birkhäuser, Basel, 1995.

[NQR20] Mihai Nica, Jeremy Quastel, and Daniel Remenik. One-sided reflected Brownian motions and the KPZ fixed point. *Forum Math. Sigma*, 8:Paper No. e63, 16, 2020.

[Pim14] Leandro P. R. Pimentel. On the location of the maximum of a continuous stochastic process. *J. Appl. Probab.*, 51(1):152–161, 2014.

[Pim21a] Leandro P. R. Pimentel. Ergodicity of the KPZ fixed point. *ALEA Lat. Am. J. Probab. Math. Stat.*, 18(1):963–983, 2021.

[Pim21b] Leandro P. R. Pimentel. Brownian aspects of the KPZ fixed point. In *In and out of equilibrium. 3. Celebrating Vladas Sidoravicius*, volume 77 of *Progr. Probab.*, pages 711–739. Birkhäuser/Springer, Cham, [2021] ©2021.

[QS20] Jeremy Quastel and Sourav Sarkar. Convergence of exclusion processes and KPZ equation to the KPZ fixed point. *Preprint*: arXiv:2008.06584, 2020. To appear in J. Amer. Math. Soc.

[RV21] Mustazee Rahman and Bálint Virág. Infinite geodesics, competition interfaces and the second class particle in the scaling limit. *Preprint*: arXiv:2112.06849, 2021.

[Sep18] Timo Seppäläinen. The corner growth model with exponential weights. In *Random growth models*, volume 75 of *Proc. Sympos. Appl. Math.*, pages 133–201. Amer. Math. Soc., Providence, RI, 2018.

[SS20] Timo Seppäläinen and Xiao Shen. Coalescence estimates for the corner growth model with exponential weights. *Electron. J. Probab.*, 25:Paper No. 85, 31, 2020.

[SS21a] Timo Seppäläinen and Evan Sorensen. Busemann process and semi-infinite geodesics in Brownian last-passage percolation. *Preprint*: arXiv:2103.01172, 2021. To appear in Ann. Inst. Henri Poincaré Probab. Stat.

[SS21b] Timo Seppäläinen and Evan Sorensen. Global structure of semi-infinite geodesics and competition interfaces in Brownian last-passage percolation. *Preprint*: arXiv:2112.10729, 2021.

[Vir20] Bálint Virág. The heat and the landscape I. *Preprint*: arXiv:2008.07241, 2020.

Ofer Busani, Universität Bonn, Endenicher Allee 60, Bonn, Germany

Email address: busani@iam.uni-bonn.de

Timo Seppäläinen, University of Wisconsin-Madison, Mathematics Department, Van Vleck Hall, 480 Lincoln Dr., Madison WI 53706-1 Our work 388, USA.

Email address: seppalai@math.wisc.edu

Evan Sorensen, University of Wisconsin-Madison, Mathematics Department, Van Vleck Hall, 480 Lincoln Dr., Madison WI 53706-1388, USA.

Email address: elsorensen@wisc.edu