Geometry of stable ruled surface over an elliptic curve

Arame Diaw†

March 4, 2019

Abstract

We consider the stable ruled surface $S_1$ over an elliptic curve. There is a unique foliation on $S_1$ transverse to the fibration. The minimal self-intersection sections also define a 2-web. We prove that the 4-web defined by the fibration, the foliation and the 2-web is locally parallelizable.

Keywords: elliptic curve, ruled surface, Riccati foliation and singular web.

Contents

1 Introduction 2

2 Preliminaries 3
   2.1 Some properties on an elliptic curve . . . . . . . . . . . . . . 3
   2.2 Ruled surface over an elliptic curve . . . . . . . . . . . . . . 4

3 Geometry of the ruled surface $S_1$ 9
   3.1 Study of special leaves of the Riccati foliation Ric . . . . . . . 12
   3.2 The geometry of the 4-web $W$ . . . . . . . . . . . . . . . . . 13

*supported by ANR-16-CE40-0008 project "Foliage"
†Univ Rennes, CNRS, IRMAR – UMR 6625, 35000 Rennes, France
1 Introduction

Let $C$ be an elliptic curve on $\mathbb{C}$. In 1955, Atiyah proved in [2] that, up to isomorphism, there are only two indecomposable ruled surfaces over $C$: the semi-stable ruled surface $S_0 \mapsto C$ and the stable ruled surface $S_1 \mapsto C$. In this article, we study the geometry of the stable ruled surface. In fact, the surface $S_1$ can be seen as the suspension over $\mathbb{C}$ of the unique representation onto the dihedral group $< -z, \frac{1}{z} >$ (see [6], page 23). Thus, we have a Riccati foliation $Ric$ on $S_1$ such that the generic leaf is a cover of degree $4$ over $C$ and it is the unique foliation transverse to the fibration. On the other hand, the holomorphic section $\sigma : C \mapsto S_1$ have self-intersection $\sigma . \sigma \geq 1$ and those having exactly $\sigma . \sigma = 1$ form a singular holomorphic 2-web $W$. Finally, taking into account the fibration, we have a singular holomorphic 4-web on $S_1$. The aim of this article is to study the geometry of this 4-web composed by the Riccati foliation, the 2-web $W$ and the $\mathbb{P}^1$-fibration $\pi : S_1 \mapsto C$.

Our first result is the following:

**Proposition 1.1.** The discriminant $\Delta$ of the 2-web $W$ defined by the $+1$ self-intersection sections on $S_1$ is a leaf of the foliation $Ric$.

Using the isomorphism between the curve $C$ and its jacobian, we have the main result:

**Theorem 1.2.** There exists a double cover $\varphi : C \times C \mapsto S_1$ ramified on $\Delta$ on which the lifted 4-web $W$ is parallelizable.

This 4-web is locally comprised of pairwise parallel straight lines: its curvature is zero.

Firstly, we show these results using only the properties of an elliptic curve and its jacobian and after, we use the theory of birational geometry to illustrate our results with computations on a trivialization $S_1 \longrightarrow C \times \mathbb{P}^1$.

This paper is part of my thesis work under the direction of Frank Loray and Frédéric Touzet.
2 Preliminaries

2.1 Some properties on an elliptic curve

Let $C = \{(x, y) \in \mathbb{C}^2, y^2 = x(x - 1)(x - t)\} \cup \{p_\infty\}$, where $t \in \mathbb{C} \setminus \{0, 1\}$ be an elliptic curve. Throughout this article, we use the following background of an elliptic curve.

**Proposition 2.1.** The set of points of $C$ forms an abelian group, with $p_\infty$ as the 0 element and with addition characterized for any couple of points $p = (x_1, y_1), q = (x_2, y_2)$ in $C$ by:

1. $-p = (x_1, -y_1)$;
2. if $p \neq q, -q$, then $p + q = (x_3, y_3)$ where $x_3 = \lambda^2 + (1 + t) - x_1 - x_2, y_3 = \lambda(x_1 - x_3) - y_1$ and $\lambda = \frac{y_2 - y_1}{x_2 - x_1}$;
3. if $p = q$ and $y_1 \neq 0$, then $2p = (\tilde{x}, \tilde{y})$ where $\tilde{x} = \lambda^2 - (1 + t) - 2x_1, \tilde{y} = \lambda x_1 - \tilde{x} - y_1$ and $\lambda = \frac{3x_1^2 - 2(1 + t)x_1 + t}{2y_1}$;
4. if $p = q$ and $y_1 = 0$ then $2p = p_\infty$.

**Remark 2.2.** The points $p_i = (i, 0)$, where $i = 0, 1, t$ are the points of order 2 on $C$ and the map

\[
\begin{align*}
I: & \quad C \rightarrow C \\
(x, y) & \mapsto (x, -y)
\end{align*}
\]

is an automorphism of $C$ which fixes the points of order 2: it is the standard involution of the curve $C$.

If we denote $Jac(C)$, the jacobian of $C$, we have:

**Lemma 2.3.** There exists a bijection between $C$ and its jacobian defined by this following map:

\[
\begin{align*}
C & \mapsto Jac(C) \\
p & \mapsto [p] - [p_\infty]
\end{align*}
\]

From now on, we will use the isomorphism between the additive group structure $(C, p_\infty)$ and the group structure on $C$ induced by its jacobian.
2.2 Ruled surface over an elliptic curve

Let $C$ be a smooth curve on $\mathbb{C}$.

**Definition 2.4.** A ruled surface over $C$ is a holomorphic map of two dimensional complex variety $S$ onto $C$ $\pi : S \mapsto C$ which makes $S$ a $\mathbb{P}^1$-fibration over $C$.

**Exemple 2.5.** The fiber bundle associated to a vector bundle of rank 2 over $C$ is a ruled surface. We denote it, $\mathbb{P}(E)$.

Conversely, we have the following theorem proved by Tsen in [5] :

**Theorem 2.6.** Let $\pi : S \mapsto C$ be a ruled surface over $C$:

1. there exists a vector bundle $E$ of rank 2 over $C$ such that $S = \mathbb{P}(E)$;
2. there exists a section, i.e a map $\sigma : C \mapsto S$ such that $\pi \circ \sigma = id$;
3. $\mathbb{P}(E) \cong \mathbb{P}(E')$ if and only if there is a holomorphic line bundle $L$ over $C$ such that $E \cong E' \otimes L$.

**Definition 2.7.** A ruled surface $\mathbb{P}(E)$ is decomposable if it has two disjoint sections.

The following lemma whose proof is in ([7], page 16) shows the relationship between the ruled surface $S = \mathbb{P}(E)$ and the vector bundle $E$.

**Lemma 2.8.** There exists a one-to-one correspondance between the line subbundles of $E$ and the sections of $S$. Futhermore, if $\sigma_L$ is the section related to the line subbundle $L$ then:

$$\sigma_L \cdot \sigma_L = \deg E - 2 \deg L$$

where $\deg(E)$ is the degree of the determinant bundle of $E$.

**Notation 2.9.** We recall that the notation $\sigma_L \cdot \sigma_L$ means the self-intersection of the section $\sigma_L$.

**Remark 2.10.** By the lemma 2.8, $\mathbb{P}(E)$ is decomposable if and only if $E$ is decomposable, i.e $E = L_1 \oplus L_2$ for line subbundles $L_i \subset E$.

Consider $\kappa = \min \{ \sigma, \sigma : C \mapsto S / \pi \circ \sigma = id \}$. This number only depends on the ruled surface $S = \mathbb{P}(E)$. Indeed, it does not change when we replace $E$ by $E \otimes L$ for a line bundle $L$ on $C$.

**Definition 2.11.** The ruled surface $\mathbb{P}(E)$ is stable if $\kappa > 0$. 

4
Definition 2.12. A minimal section of $S$ is a section $\sigma : C \to S$ such that, the self-intersection is minimal. That is to say, $\sigma . \sigma = \kappa$.

Using lemma 2.8, we notice that a minimal section corresponds to a line subbundle of $E$ with maximal degree. Thus, the invariant $\kappa$ can be written as:

$$\kappa = \max \{ \deg(E) - 2 \deg(L), L \hookrightarrow E_1 \}$$

Now, we are interested in indecomposable ruled surfaces over an elliptic curve. Let $O_C(p_\infty)$ be the line bundle related to the divisor $[p_\infty]$. There are unique nontrivial extensions of invertible sheaves:

$$0 \longrightarrow O_C \longrightarrow E_0 \longrightarrow O_C \longrightarrow 0$$

and

$$0 \longrightarrow O_C \longrightarrow E_1 \longrightarrow O_C(p_\infty) \longrightarrow 0$$

Recall the following Atiyah’s theorem as proved in ([1], Th. 6.1):

**Theorem 2.13.** Up to isomorphism, the unique indecomposable ruled surfaces over $C$ are $S_0 = \mathbb{P}(E_0)$ and $S_1 = \mathbb{P}(E_1)$.

**Remark 2.14.** Equivalency, any indecomposable vector bundle $E$ of rank 2 on $C$ takes the form $E = E_i \otimes L$, for $i = 0, 1$ and $L$ a line bundle.

As our aim in this paper is the study of the ruled surface $S_1$, we will show firstly some important properties of $E_1$.

**Lemma 2.15.** The degree of the maximal line subbundles of $E_1$ is zero.

**Proof.** Let $L$ be a subbundle of $E_1$ and consider the quotient $M := E_1 / L$. By the following exact sequence $0 \longrightarrow L \longrightarrow E_1 \longrightarrow M \longrightarrow 0$ and the fact that $E_1$ is indecomposable, we have $H^1(M^{-1} \otimes L) \neq 0$. Thus, due to Serre’s duality, we have $2 \deg(L) \leq \deg(E_1)$ and then $\deg(L) \leq 0$ because $\deg(E_1) = 1$. Since the trivial line bundle $O_C$ is a line subbundle over $E_1$, we have the result.

**Remark 2.16.** By this lemma, we can deduce that the ruled surface $S_1$ is stable. More precisely, up to isomorphism, it is the unique stable ruled surface over an elliptic curve.

If we consider $\max_{E_1} = \{ L \hookrightarrow E_1, \deg L = 0 \}$ the set of line subbundles of $E_1$ having a maximal degree, we have:
Lemma 2.17. The jacobian of $C$ sets the parameters of the set $\max_{E_1}$. More precisely, the map

$$
\begin{align*}
M: \max_{E_1} \rightarrow \text{Jac}(C) \\
L \mapsto [L]
\end{align*}
$$

is a bijection.

To prove this lemma, we have to use a key lemma of Maruyama in ([7], page 8):

Lemma 2.18. Let $E$ be a vector bundle of rank 2 over a curve. If $L_1$ and $L_2$ are distinct maximal line subbundles of $E$ such that $L_1$ and $L_2$ are isomorphic, then $E = L_1 \oplus L_1$.

Now, we can prove lemma 2.17.

Proof.  
1. Let $L_1$ and $L_2$ be two elements in $\max_{E_1}$ such that $L_1 \cong L_2$.
   We have two possibilities, either $L_1 = L_2$ or they are both distinct.
   According to the lemma 2.18, the last case cannot occur because $E_1$ is not decomposable. Thus, the map $M$ is injective.

2. Let $L \in \text{Jac}(C)$ be distinct from the trivial line bundle. If we apply the functor $\text{Hom}(L, -)$ to the exact sequence
   $$
   0 \rightarrow \mathcal{O}_C \overset{f}{\rightarrow} E_1 \overset{g}{\rightarrow} \mathcal{O}_C(p_\infty) \rightarrow 0
   $$
   and we use Riemann Roch’s theorem, we obtain $\dim \text{Hom}(L, E_1) = 1$. There exists a non zero morphism $\tau: L \mapsto E_1$. Thus, if we denote $D$ the effective divisor of zeros of $\tau$, then $L \otimes \mathcal{O}_C(D)$ is a line subbundle of $E_1$. Since $\deg(L) = 0$, $D$ is an effective divisor of zero degree, that is to say $\mathcal{O}_C(D) = \mathcal{O}_C$. Hence, $L$ is a line subbundle of $E_1$.

Remark 2.19. The minimal sections of $S_1$ have self-intersection equal to 1 and they are parametrised by the jacobian which is isomorphic to $C$. Hence for every point $\epsilon \in C$, we denote $\sigma_\epsilon$ the minimal section corresponding via lemma 2.17 to the subbundle isomorphic to $\mathcal{O}_C([\epsilon] - [p_\infty])$.

Lemma 2.20. Let $\sigma_\epsilon$ and $\sigma_{\epsilon'}$ be two minimal sections of $S_1$. If we consider $D$ their intersection divisor, we have:

$$
\pi(D) = [-\epsilon - \epsilon']
$$

where $\pi: S_1 \mapsto C$ is the projection map.
Proof. Intuitively, the divisor $D$ is defined by the points above at which the line bundles related to $\sigma_\epsilon$ and $\sigma_{\epsilon'}$ coincide. More precisely, $\pi(D)$ is a effective divisor equivalent to divisor $\text{det}(E_1) \otimes \mathcal{O}_C([p_\infty] - [\epsilon]) \otimes \mathcal{O}_C([p_\infty] - [\epsilon'])$ which itself is equivalent to divisor $\mathcal{O}_C([-\epsilon - \epsilon'])$. Since the degree of $D$ is equal to 1, we obtain the result.

**Remark 2.21.** Let $Q$ be a point of $S_1$ belonging to the fiber $\pi^{-1}(p)$. If the minimal section $\sigma_\epsilon$ passes through the point $Q$, then the unique other minimal section passing through the same point $Q$ is the section $\sigma_{-p-\epsilon}$. They might be the same for some $Q$.

![Figure 1: intersection of two minimal sections on $S_1$](image)

We also have the following theorem proved by André Weil in [11]:

**Theorem 2.22.** A holomorphic vector bundle on a compact Riemann surface is flat if and only if it is the direct sum of indecomposable vector bundles of degree 0.

By this theorem, the Atiyah’s bundle $E_1$ is not flat because $\deg E_1 = 1$. However what can we say about its associated ruled surface? The answer of this question is given by Frank Loray and David Marin in [6]. Consider $C$ as a torus $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ and let $\varrho: \mathbb{Z} + \tau\mathbb{Z} \to \mathbb{PSL}_2(\mathbb{C})$ be the representation of the fundamental group of $C$ defined by $\varrho(1) = -z$ and $\varrho(\tau) = \frac{1}{z}$.

Up to conjugacy in $\mathbb{PSL}_2(\mathbb{C})$, it is the unique representation onto the 4-order group $\Gamma = \langle -z, \frac{1}{z} \rangle$.

The orbits of the elements of $\mathbb{C} \times \mathbb{P}^1$ modulo the action given by the representation and the universal cover of $C$ form a ruled surface over $C$, denoted by $\tilde{E}$. It is obvious to see, the horizontal foliation of $\mathbb{C} \times \mathbb{P}^1$ lifts to a regular foliation $\mathcal{F}$ transverse to the fibration of $\tilde{E}$. The foliated surface $(\tilde{E}, \mathcal{F})$ is
called the suspension of $C$. Let $\mathcal{R}_{[x_0, z_0]}$ be a leaf passing through a the point $[x_0, z_0]$ of $\overline{E}$. Then, by definition, we have a isomorphism between $\mathcal{R}_{[x_0, z_0]}$ and the quotient $C/G$ where $G = \{ \alpha \in \mathbb{Z} + \tau \mathbb{Z} / \varrho(\alpha)(z_0) = z_0 \}$. Thus, we can deduce that every leaf of foliation $\mathcal{R}$ is a cover of $C$. It is not difficult to show that the intersection of any fiber of $\overline{E}$ and the leaf $\mathcal{R}_{[x_0, z_0]}$ is given by the set $\{ \varrho(\alpha)(z_0), \alpha \in \mathbb{Z} + \tau \mathbb{Z} \}$. Using the finitude of the representation, we have every leaf of the foliation is a cover of finite degree. Moreover, the monodromy of the foliation $\mathcal{R}$ on any fiber is the representation $\varrho$. 

Note that, the action of $< -z, \frac{1}{z} >$ in $\mathbb{P}^1$ gives two kind of orbits: orbits of order 4 and three special orbits of order 2 given by $(-1, 1), (-i, i)$ and $(0, \infty)$. Therefore, the foliation $\mathcal{R}$ has a generic leaf which is a cover of order 4 of $C$ and three special leaves, which is a cover of order 2.

**Proposition 2.23.** The ruled surface $\overline{E}$ over $C$ is indecomposable such that its invariant $\kappa = 1$. It is the ruled surface $S_1$.

**Proof.** If $\overline{E}$ is a decomposable ruled surface then its invariant $\kappa = 0$. Indeed, let $F$ be a leaf of $\mathcal{R}$ and $\sigma_0$ a minimal section of $\overline{E}$, then we have: $F \equiv 4\sigma_0 + bf$ or $F \equiv 2\sigma_0 + bf'$, where $f$ represents a fiber. Using the fact that $\overline{E}$ is decomposable, we can find a section $\sigma$ such that $\sigma.\sigma_0 = 0$. Since $F.\sigma \geq 0$ and $F^2 = 0$, we obtain that $\sigma_0.\sigma_0 = 0$.

Eventually, if we assume that $\overline{E}$ is a decomposable ruled surface, we have $F \equiv 4\sigma_0$ or $F \equiv 2\sigma_0$ and then $F.\sigma = 0$. The section $\sigma$ does not meet any leaf of $\mathcal{R}$, which does not make a sense because the foliation is regular.

The ruled surface $\overline{E}$ is then indecomposable. Hence, it is either isomorphic to $S_0$ or $S_1$.

By the same arguments above, $\overline{E}$ is not isomorphic to $S_0$, if not there would be a section which not intersects any leaf of $\mathcal{R}$. Thus, the ruled surface $\overline{E}$ is isomorphic to $S_1$ by Atiyah’s theorem.

As up to conjugacy the representation $\varrho$ is the unique representation onto the 4-order group $\Gamma = < -z, \frac{1}{z} >$, we deduce that the isomorphism between $\overline{E}$ and $S_1$ is the identity. 

In summary, we have:

**Theorem 2.24.** The ruled surface $S_1$ has a Riccati foliation Ric with irreducible monodromy group $< -z, \frac{1}{z} >$.

**Remark 2.25.** The foliation $\text{Ric}$ has a generic leaf which is cover of degree 4 over $C$ and three special leaves which are covers of degree 2 over $C$. 

8
3 Geometry of the ruled surface $S_1$

Let $\pi: S_1 \mapsto C$ be the stable ruled surface over $C$.

**Proposition 3.1.** The automorphism group of $S_1$ is a group of order 4 which is isomorphic to the 2-torsion group in $C$.

**Proof.** Let $\psi: S_1 \mapsto S_1$ be a non trivial automorphism of $S_1$. Since the self-intersection is invariant by automorphism $\psi$ preserves the set of +1 self-intersection sections on $S_1$. More precisely, for any $\epsilon \in C$, there exists a unique point $r_\epsilon \in C$ such that $\psi(\sigma_\epsilon) = \sigma_{r_\epsilon}$. The automorphism $\psi$ induces an automorphism $\bar{\psi}$ of $C$ such that for any point $\epsilon \in C$ we have $\bar{\psi}(\epsilon) = r_\epsilon$.

If we define $C$ as the complex torus $C/\mathbb{Z} + \tau \mathbb{Z}$, we can write for any $z \in C$, $\bar{\psi}(z) = az + b$, where $a, b \in C$ and $a (\mathbb{Z} + \tau \mathbb{Z}) = \mathbb{Z} + \tau \mathbb{Z}$.

If we assume this automorphism has a fix point $\epsilon_0$, then by definition we have $\psi(\sigma_{\epsilon_0}) = \sigma_{\epsilon_0}$. Hence, using the lemma 2.20 we obtain that for any $p \in C$, $\psi(\sigma_{-p-\epsilon_0}) = \sigma_{-p-\epsilon_0}$. For any fiber, the automorphism $\psi$ is Mobius map which fixes at least three points: it is the trivial automorphism, which does not make sense by hypothesis.

Therefore, the automorphism $\bar{\psi}$ has no fixed points, it is a translation like $\bar{\psi}(\tau) = \tau + b$. As by definition we have: $\bar{\psi}(-p-\tau) = -p - \psi(\tau)$, the point $b$ is a point of order 2 of $C$.

Conversely, for any point $p_i$ of order 2 on $C$, we can define an automorphism $\Phi_i$, on $S_1$ such that for any point $p \in C$, $\Phi_i$ restricts to the fiber $\pi^{-1}(p)$ is the unique Mobius map which associates the points of the sections $(\sigma_{p_0}, \sigma_{p_1}, \sigma_{p_2}, \sigma_{p_3})$ to the points of the sections $(\sigma_{p_0+p_1}, \sigma_{p_0+p_1}, \sigma_{p_1+p_2}, \sigma_{p_2+p_3})$ respectively. It is defined by:

\[\begin{align*}
\Phi_i: & \quad S_1 \quad \longrightarrow \quad S_1 \\
\quad z = \sigma_\omega(p) & \quad \longmapsto \quad z' = \sigma_{\omega+p_i}(p)
\end{align*}\]

There exists a one-to-one correspondance between the automorphisms of the fiber bundle $S_1$ and the points of order 2 in $C$ which preserves the group structure. Hence we have :

\[Aut_C(S_1) = \{\Phi_0, \Phi_1, \Phi_t, \Phi_\infty = Id\}\]

**Proposition 3.2.** The automorphism group of $S_1$ preserves the foliation Ric.

**Proof.** Using the fact that the fundamental group of $C$ is abelian, we can extend the monodromy map over every fiber and regard it as automorphism on $S_1$ which fixes the basis $C$. Thus, we obtain that the monodromy group of the foliation Ric is a subgroup of order 4 of $Aut_C(S_1)$ : they are isomorphic. The group $Aut_C(S_1)$ preserves the Riccati foliation on $S_1$. \qed
Corollary 3.3. The Riccati foliation $\text{Ric}$ is the unique Riccati foliation on the ruled surface $S_1$.

Proof. Let $\mathcal{F}_1$ be a Riccati foliation on $S_1$. As its monodromy group is an abelian subgroup of $\text{PGL}(\mathbb{C}, 2)$, we have three possibilities for its monodromy representation:

- If the conjugacy class of the monodromy is the linear class defined by the group $\langle az, bz \rangle$, there exists two disjoint invariant sections of $S_1$. Hence $S_1$ is a ruled surface related to the direct sum of two line bundles over $C$. It does not make sense because $S_1$ is indecomposable.

- If the conjugacy class of the monodromy is the euclidian class defined by the group $\langle z + 1, z + s \rangle$, there exists an invariant section on $S_1$ with zero self-intersection. In fact by the Camacho Sad’s theorem (in [3]), any invariant curve of regular foliation has a zero self-intersection. This monodromy representation does not make sense in $S_1$ because we have $\min \{\sigma.\sigma, \sigma : C \mapsto S_1 / \pi \circ \sigma = \text{id}\} = 1$.

The only remaining possibility is that the monodromy has image the group $\langle -z, z \rangle$. Thus, the foliation $\mathcal{F}_1$ is conjugated to $\text{Ric}$ by an element in $\text{Aut}_C(S_1)$. As this automorphism group of the fibration $S_1$ preserves the foliation $\text{Ric}$, we have $\text{Ric} = \mathcal{F}_1$.

Lemma 3.4. There exists a ramified double cover of the ruled surface $S_1$ defined by the map:

$$\begin{align*}
\varphi : \quad C \times \text{Jac}(C) &\rightarrow S_1 \\
(p, \epsilon) &\mapsto z = \sigma_\epsilon (p)
\end{align*}$$

such that its involution is defined by:

$$\begin{align*}
i : \quad C \times \text{Jac}(C) &\rightarrow C \times \text{Jac}(C) \\
(p, \epsilon) &\mapsto (p, -p - \epsilon)
\end{align*}$$

Proof. According to the lemma [2.20] three minimal sections cannot meet at the same point, then we deduce for any $p \in C$, the morphism

$$\begin{align*}
\varphi_p : \quad \text{Jac}(C) &\rightarrow \pi^{-1}(p) \\
\epsilon &\mapsto \sigma_\epsilon (p)
\end{align*}$$

is not constant: it is a ramified cover between Riemann surfaces. Furthermore, by the remark [2.21] we know that at most two minimal sections can pass through a given point, then the map $\varphi$ is a ramified double cover. \qed
The immediate consequence of this lemma is the following:

**Theorem 3.5.** There exists a singular holomorphic 2-web \( \mathcal{W} \) on \( S_1 \) defined by the minimal sections whose discriminant \( \Delta \) is a leaf of the foliation \( \text{Ric} \).

**Proof.** By lemma 3.4, for any point \( P \in \pi^{-1}(p) \) there exists a minimal section \( \sigma_r \) passing through this point. Likewise, by lemma 2.20 the minimal section \( \sigma_{-p-r} \) intersects transversally \( \sigma_r \) at the point \( P \). As the sections \( \sigma_r \) and \( \sigma_{-p-r} \) are distinct if and only if \( 2r \neq -p \), we deduce that there exists a singular holomorphic 2-web on \( S_1 \) such that its discriminant is defined by:

\[
\Delta = \bigcup_{p \in C} \{ P \in \pi^{-1}(p) : P \in \sigma_r, 2r = -p \}
\]

In order to prove that \( \Delta \) is a leaf of the Riccati, we need the following:

**Lemma 3.6.** There exists a linear foliation \( \mathcal{F} \) on \( C \times \text{Jac}(C) \) such that \( \varphi_* \mathcal{F} = \text{Ric} \).

**Proof.** Assume that \( C \times \text{Jac}(C) \simeq (\mathbb{C}/\mathbb{Z} + \tau \mathbb{Z}) \times (\mathbb{C}/\mathbb{Z} + \tau \mathbb{Z}) \), and let be \((x, y)\) its local coordinates. If we consider the linear foliation \( \tilde{\mathcal{F}} := dx + 2dy \) on \( \mathbb{C}_x \times \mathbb{C}_y \), then \( \tilde{\mathcal{F}} \) is invariant by the action of the lattice \( \mathbb{Z} + \tau \mathbb{Z} \). Thus we can lift the foliation \( \tilde{\mathcal{F}} \) to a foliation, \( \mathcal{F} \) on \( C \times \text{Jac}(C) \) such that the monodromy is defined by:

\[
\left\{ \begin{array}{ll}
\xi : & \Lambda \rightarrow \text{Aut}(C) \\
& \lambda \rightarrow z \mapsto z - \frac{1}{2} \lambda
\end{array} \right.
\]

where \( \Lambda \) is the lattice \( \mathbb{C}/\mathbb{Z} + \tau \mathbb{Z} \).

The foliation \( \mathcal{F} \) is transverse to the first projection on \( C \times \text{Jac}(C) \) with a monodromy group isomorphic to the group of points of order 2 \{\( p_\infty, p_0, p_1, p_t \)\}. Moreover, if \( \mathcal{F}_{(p,\omega)} \) is the leaf passing through the point \( (p, \omega) \), then by definition we have:

\[ i(\mathcal{F}_{(p,\omega)}) = \mathcal{F}_{(p,-\omega)} \]

where \( i \) is the involution of the ramified double cover \( \varphi \). Hence, \( \varphi_* \mathcal{F} \) the direct image of the foliation \( \mathcal{F} \) by \( \varphi \) is a Riccati foliation on \( S_1 \) having the same monodromy group than \( \text{Ric} \). Using the uniqueness of \( \text{Ric} \) by the corollary 3.3, we obtain the result. \( \square \)

As by definition the curve \( G = \{(2p,-p) / p \in C \} \) is a leaf of the foliation \( \mathcal{F} \), using the foregoing lemma we can deduce that \( \varphi(G) = \Delta \) is a leaf of \( \text{Ric} \). Which completes the proof of theorem 3.5. \( \square \)
3.1 Study of special leaves of the Riccati foliation $\text{Ric}$

According to lemma 3.6, if $P = \sigma_\omega(p) \in S_1$ then the Riccati leaf passing through at this point is given by

$$\text{Ric}_P = \{(q, z) / z = \sigma_\omega(q), 2\omega' = 2\omega + p - q\}$$

Thus, if we use this characterisation of the Riccati leaves on $S_1$, we have the following lemma:

**Lemma 3.7.** There exists three special leaves $\text{Ric}_0$, $\text{Ric}_1$ and $\text{Ric}_t$ of the foliation $\text{Ric}$ which are double cover of $C$. More precisely, they are respectively the set of fixed points of the automorphisms $\Phi_0$, $\Phi_1$ and $\Phi_t$.

**Proof.** We just give the proof for the leaf $\text{Ric}_0$ because it is the same process for the other special leaves.

- Let $\text{Ric}_0$ be the Riccati leaf passing through the point $z_0 = \sigma_{p_0}(p_0)$, then by definition, we have:

$$\text{Ric}_0 = \{(p, z) / z = \sigma_\omega(p), 2\omega = p_0 - p\}$$

According to the monodromy of $\text{Ric}$, if the leaf $\text{Ric}_0$ passes through the point $z = \sigma_\omega(p)$ then it passes through the points $\sigma_{\omega+p_i}(p)$, where $p_i$ is a point of $C$ of order 2. Since $2\omega = p_0 - p$, we deduce from lemma 2.20 that: $\sigma_{\omega+p_0}(p) = \sigma_\omega(p)$ and $\sigma_{\omega+p_i}(p) = \sigma_{\omega+p_i}(p)$ thus, $\text{Ric}_0$ meets any fiber of $S_1$ twice. It is a double cover over $C$.

- Let $\Phi_0$ be the automorphism of $S_1$ related to the point $p_0$ defined by:

$$\Phi_0 : S_1 \rightarrow S_1$$

$$z = \sigma_\omega(p) \rightarrow z' = \sigma_{\omega+p_0}(p)$$

and consider the set of its fix points

$$\text{Fix}_0 = \{(p, z) / \Phi_0(p, z) = (p, z)\}$$

If $z = \sigma_\omega(p)$ is the fix point of $\Phi_0$, then by lemma 2.20 we have $2\omega = p_0 - p$, and therefore $z \in \text{Ric}_0$. Conversely, if $z \in \text{Ric}_0$, we have $2\omega = p_0 - p$, and according to lemma 2.20 we have, $\sigma_{\omega+p_0}(p) = \sigma_\omega(p)$.

Thus, we deduce that:

$$\text{Ric}_0 = \{(p, z) / \Phi_0(p, z) = (p, z)\}$$

According to all the foregoing, we have:

**Remark 3.8.** The 2-web given by the +1 self-intersection sections, the Riccati foliation and the $\mathbb{P}^1$-bundle $\pi : S_1 \rightarrow C$ form a singular holomorphic 4-web $W$ on $S_1$. 

12
3.2 The geometry of the 4-web W

Let \((x, y)\) be the local coordinates of \(\mathbb{C}^2\). As the linear foliations \(\mathcal{G}\) and \(\mathcal{H}\) respectively defined by \(dy = 0\) and by \(dy + dx = 0\) are invariant by the action of the lattice \(\mathbb{Z} + \tau\mathbb{Z}\), we can lift them to a decomposable 2-web \(W'\) on \(C \times Jac(C)\).

**Proposition 3.9.** The direct image \(\varphi^*(W')\) of the 2-web \(W'\) by the ramified cover \(\varphi\) is the 2-web \(W\) on \(S_1\) defined by the minimal sections.

**Proof.** As by definition the 2-web \(W\) is invariant by the involution of the ramified double cover \(\varphi\), its direct image is also a singular holomorphic 2-web on \(S_1\). Let \((p', \epsilon') \in C \times Jac(C)\) and consider

- \(A_{\epsilon'} = \{(p, \epsilon) \in C \times Jac(C) / \epsilon = \epsilon'\}\)
- \(B_{\epsilon'} = \{(p, \epsilon) \in C \times Jac(C) / \epsilon = -p + (p' + \epsilon')\}\),

the leaves of the 2-web \(W'\) passing through this point. Since, using lemma \(2.20\) we have: \(\varphi(A_{\epsilon'}) = \sigma_{\epsilon'}\) and \(\varphi(B_{\epsilon'}) = \sigma_{-p-\epsilon}\), then the leaves of \(\varphi^*(W')\) are the minimal sections of \(S_1\) which are the same along the discriminant \(\Delta\). \(\square\)

The local study of the 4-web \(W\) on \(S_1\) is the same as the 4-web on \(C \times Jac(C)\) given by the 2-web \(W'\), the foliation \(\mathcal{F}\) and the \(Jac(C)\)-bundle defined by the first projection on \(C \times Jac(C)\).

**Theorem 3.10.** Outside the discriminant locus \(\Delta\), the 4-web \(W\) is locally parallelizable.

**Proof.** According to the foregoing, the pull-back of 4-web \(W\) by the ramified cover \(\varphi\) is locally the 4-web defined by \(W(x, y, y + x, y + 2x)\) on \(\mathbb{C}^2\). It is a holomorphic parallelizable web. \(\square\)

**Remark 3.11.** An immediate consequence of theorem 3.10 is that the curvature of the 4-web \(W\) is zero.

The second part of this paper aims to use the theory of birational geometry in order to find the theoretic results of the first part by computations on the birational trivialisation \(C \times \mathbb{P}^1\).
4 Geometry of 4-web W after elementary transformations

Let $\pi: S_1 \hookrightarrow C$ be the $\mathbb{P}^1$-bundle and $\{p_0, p_1, p_\infty\}$, the set of points of order 2 in $C$.

**Definition 4.1.** An elementary transformation at the point $P \in \pi^{-1}(p)$ is the birational map given by the composition of the blow-up of the point $P$, followed by the contraction of the proper transform of the fiber $\pi^{-1}(p)$.

**Remark 4.2.** After elementary transformation at the point $P$, we obtain a new ruled surface with a point $\tilde{P}$ which is the contraction of the proper transform of the fiber $\pi^{-1}(p)$.

How many elementary transformations do you need to trivialize the ruled surface $S_1$?

**Lemma 4.3.** The ruled surface $S_1$ is obtained after three elementary transformations at the points $\tilde{P}_0 = (p_0, 0)$, $\tilde{P}_1 = (p_1, 1)$ and $\tilde{P}_\infty = (p_\infty, \infty)$ on the trivial bundle $C \times \mathbb{P}^1$.

![Figure 2: Special points of the trivial $\mathbb{P}^1$-bundle over $C$](image-url)
In fact, if we perform the elementary transformations of the three special points \( \tilde{P}_0, \tilde{P}_1 \) and \( \tilde{P}_\infty \) of \( C \times \mathbb{P}^1 \) (see figure 2), we have a ruled surface \( S \) with three special points \( P_0, P_1 \) and \( P_\infty \) (see figure 3).

\[
\sigma.\sigma = \sigma'.\sigma' + r
\]

where \( r = \epsilon_0 + \epsilon_1 + \epsilon_\infty \) such that

\[
\epsilon_i = \begin{cases} 
-1 & \text{if } P_i \in \sigma \\
+1 & \text{if } P_i \notin \sigma
\end{cases}
\]

in particular, \( r \in \{-1, 1, -3, 3\} \). Then, we can deduce that the ruled surface \( S \) has a invariant \( \kappa \leq 1 \).

1. If \( \kappa = 0 \), then there is \( \sigma.\sigma = 0 \) on \( S \), then its strict transform \( \sigma'.\sigma' = \text{odd} \) on \( C \times \mathbb{P}^1 \). This cannot hold because all the sections of the trivial bundle have even self-intersection;

2. If \( \kappa < 0 \), then there exists a +2 self-intersection section of \( C \times \mathbb{P}^1 \) passing through by three points. It is absurd because there exists a
effective divisor equivalent to its normal bundle which contains at least three points.

According to these two cases, after our elementary transformations on the trivial bundle \( C \times \mathbb{P}^1 \), we obtain a ruled surface such that its invariant \( \kappa = 1 \). Therefore, it is a stable ruled surface over an elliptic curve.

4.1 The Riccati foliation on \( S_1 \) after elementary transformations

Proposition 4.4. After elementary transformations of the three special points \( P_0, P_1 \) and \( P_\infty \) on \( S_1 \), the Riccati foliation \( \text{Ric} \) induces a Riccati foliation \( \tilde{\text{Ric}} \) on the trivial bundle \( C \times \mathbb{P}^1 \) such that the points \( \tilde{P}_0 = (p_0, 0) \), \( \tilde{P}_1 = (p_1, 1) \) and \( \tilde{P}_\infty = (p_\infty, \infty) \) are radial singularities.

Proof. As the problem is local, we can prove it on the surface \( C^2 \). In our context, after elementary transformation at the origin, a regular Riccati foliation becomes the pull-back of this holomorphic foliation \( \frac{dz}{dx} = az^2 + bz + c \) where \( a, b, c \in \mathbb{C} \) by the birational map \( C^2 \rightarrow \mathbb{C}^2 ; (x, z) \mapsto (x, xz) \). Thus, we obtain that a Riccati foliation such that the linear part looks like \( xdz - zdx = 0 \). Therefore the origin is a radial singularity. Finally, we can say the foliation \( \text{Ric} \) on \( S_1 \) is after elementary transformations a Riccati foliation on \( C \times \mathbb{P}^1 \) having three radial singularities at the points \( \tilde{P}_0, \tilde{P}_1 \) and \( \tilde{P}_\infty \). \( \square \)

If \( ((x, y), z) \) are coordinates of the trivial bundle \( C \times \mathbb{P}^1 \), then the foliation \( \tilde{\text{Ric}} \) is defined by \( dz = [a(x, y)z^2 + b(x, y)z + c(x, y)] \frac{dx}{2y} \) where \( a, b, c \) are the meromorphic functions with pole of order 1 at the points \( p_0, p_1 \) and \( p_\infty \), i.e, \( a, b, c \in H^0(\mathcal{O}_C(p_0 + p_1 + p_\infty)) \simeq \mathbb{C} < 1, \frac{1}{y}, \frac{1}{y^2} > \). It means that \( a = \frac{a_0 + a_1x + a_2y}{y} \), where \( a_i \) are constant. If we write the same relation for the functions \( b \) and \( c \), we obtain that the foliation \( \tilde{\text{Ric}} \) is locally defined by the following 1-form:

\[
ydz = [(a_0 + a_1x + a_2y)z^2 + (b_0 + b_1x + b_2y)z + (c_0 + c_1x + c_2y)] \frac{dx}{2y}
\]

As the foliation is invariant by the involution \( (x, y) \mapsto (x, -y) \) on \( C \), the coefficients \( a_2, b_2, \) and \( c_2 \) are zero. Furthermore, if we use the relation on an elliptic
curve, \( y^2 = x (x - 1)(x - t) \), and the fact that the points \((0, 0, 0), (1, 0, 1), (p_\infty, \infty)\) are the radial singularities, we have \( \widetilde{Ric} \) is defined by the 1-form:

\[
w := dz + \left[ \frac{-z^2}{4x(x - 1)} - \frac{z}{2x} + \frac{1}{4(x - 1)} \right] dx
\]

**Proposition 4.5.** If we fix a point \((x_0, y_0) \) of \( C \), the monodromy group of the foliation \( \widetilde{Ric} \) along of the fiber \( \pi^{-1}(x_0, y_0) \) is an abelian group isomorphic to a group given by these automorphisms: \( \widetilde{\Phi}_0 : z \mapsto \frac{z - x_0}{z - 1}, \widetilde{\Phi}_1 : z \mapsto \frac{x_0}{z}, \widetilde{\Phi}_t : z \mapsto \frac{x_0(z - 1)}{z - x_0}, \widetilde{\Phi}_\infty : z \mapsto z \).

**Proof.** Let \( \sigma_\infty := \{ z = \infty \}, \sigma_0 := \{ z = 0 \}, \sigma_1 := \{ z = 1 \} \) and \( \sigma_d := \{ z = x \} \) be the four special sections obtained after elementary transformations. By definition of the monodromy group of \( Ric \), we can see that for the point \( p_0 \) of order 2, the automorphism \( \widetilde{\Phi}_0 \) restricted to any fiber is the unique Moebius transformation which relates respectively the points of the sections \( (\sigma_0, \sigma_1, \sigma_\infty, \sigma_d) \) to the points of the sections \( (\widetilde{\sigma}_0, \widetilde{\sigma}_1, \widetilde{\sigma}_\infty, \widetilde{\sigma}_d) \). Using the same process for the other automorphisms, we obtain the result.

We can also describe the special leaves of the foliation \( \widetilde{Ric} \). In fact, if we consider \( \phi_i : C \times \mathbb{P}^1 \rightarrow C \times \mathbb{P}^1; (x, y, z) \mapsto (x, y, \Phi_i(z)) \), then according to lemma \( 3.7 \), the special leaves are defined by:

1. \( \widetilde{Ric}_0 := \{(x, y, z), \phi_0(x, y, z) = (x, y, z)\} = \{(x, z), -z^2 - x + 2z = 0\} \)
2. \( \widetilde{Ric}_1 := \{(x, y, z), \phi_1(x, y, z) = (x, y, z)\} = \{(x, z), -z^2 + x = 0\} \)
3. \( \widetilde{Ric}_i := \{(x, y, z), \phi_i(x, y, z) = (x, y, z)\} = \{(x, z), z^2 - 2xz + x = 0\} \)

Now it is natural to ask if we can find the expression of the leaf of order 4 of \( Ric \). To do this, we use the special leaves to find a first integral. Let

\[
f_0 := -z^2 + 2z - x, \quad f_1 := -z^2 + x, \quad f_t := z^2 - 2xz + x
\]

be the polynomials which define respectively the leaves \( \widetilde{Ric}_0, \widetilde{Ric}_1, \widetilde{Ric}_t \) and consider the function \( \gamma : C_x \times \mathbb{P}^1 \rightarrow C_x \times \mathbb{P}^1; (x, z) \mapsto (x, F(x, z)), \) where

\[
F(x, z) = \frac{x f_0^2}{x f_0^2 - (x - 1)f_1^2}
\]

The pull-back \( \gamma^* dy \) of the 1-form \( dy \) by \( \gamma \) is a foliation on \( C_x \times \mathbb{P}^1 \) having the function \( F(x, z) \) as a first integral and such that the curves \( \widetilde{Ric}_0, \widetilde{Ric}_1 \) and \( \widetilde{Ric}_t \) are invariant. Hence, it is the foliation \( \overline{Ric} \). We deduce that:
Lemma 4.6. The foliation \( \tilde{\text{Ric}} \) on \( C \times \mathbb{P}^1 \) has a rational first integral defined by the following function:

\[
F(x, z) = \frac{x(z^2 - 2z - x)^2}{(-z^2 + 2xz - x)^2}
\]

4.2 The generic 2-web after elementary transformations

After elementary transformations at the three special points on \( S_1 \), the generic +1 self-intersection sections (i.e. not passing through the three special points) become the +4 self-intersection sections of \( C \times \mathbb{P}^1 \) passing through the points \((0, 0, 0), (1, 0, 1)\) and \((p_\infty, \infty)\); see figure 4.

Lemma 4.7. A +4 self-intersection section passing through the points \( \tilde{P}_0 \), \( \tilde{P}_1 \) and \( \tilde{P}_\infty \) is either given by the graph \( z = \frac{(1 - x_0)(y_0x - x_0y)}{y_0(x - x_0)} \), or the graph \( z = x \).

![Figure 4: Generic +4 self-intersection section](image)

Proof. If \( \sigma : C \mapsto \mathbb{P}^1 \) is a +4 self-intersection section on the trivial bundle, then it defines a rational map of degree 2 generated by two sections \( \sigma_1 \) and \( \sigma_2 \) of a line bundle of degree 2 over \( C \); more precisely, for any point \((x, y) \in C\),

\[
\sigma(x, y) = (\sigma_1(x, y) : \sigma_2(x, y)).
\]

Since for any line bundle of degree 2 over \( C \), there exists a point \( p = (x_0, y_0) \in C \) such that \( L = [p] + [p_\infty] \), we have two cases:
• if \( p \neq p_\infty \), according to the Riemann Roch’s theorem, \( H^0(L) = \mathbb{C} \langle y - y_0, x - x_0 \rangle \) and then, \( \sigma \) is a graph given by

\[
z = \frac{a(y - y_0) + b(x - x_0)}{c(y - y_0) + d(x - x_0)}, \quad a, b, c, d \in \mathbb{C}
\]

Using the fact that the section passes through the points \( \tilde{P}_0, \tilde{P}_1 \) and \( \tilde{P}_\infty \) and the puiseux parametrisation of elliptic curve at the infinity point is given by \( t \mapsto (\frac{1}{t^2}, \frac{1}{t^3}) \), we obtain a system of equations which solutions

\[
\begin{align*}
    a &= d \frac{x_0(x_0 - 1)}{y_0}, \\
b &= -d(x_0 - 1), \\
c &= 0, \\
d &= d
\end{align*}
\]

where \( d \neq 0 \);

• if \( p = p_\infty \) then \( H^0(L) = \mathbb{C} \langle 1, \frac{1}{x} \rangle \), likewise using the fact that the section passes through the points \( (0, 0, 0), (1, 0, 1) \) and \( (p_\infty, \infty) \), we obtain that \( \sigma \) is the graph \( z = x \).

From now on, unless otherwise mentionned, we will assume that a +4 self-intersection section is a section which passes through the points \( \tilde{P}_0, \tilde{P}_1 \) and \( \tilde{P}_\infty \).

Now using the birational trivialisation of \( S_1 \), we can give another proof to show that the minimal sections of \( S_1 \) define a singular 2-web and its discriminant is a leaf of the Riccati foliation on \( S_1 \).

**Proposition 4.8.** For any point of \( C \times \mathbb{P}^1 \), there exists a +4 self-intersection section which passes through this point.

**Proof.** Let \( (u, v, z) \in C \times \mathbb{P}^1 \) such that \( v \neq 0 \), we have to find the points \( (x_0, y_0) \neq (u, v) \) of \( C \) such that \( z = \frac{(1 - x_0)(y_0u - x_0v)}{y_0(u - x_0)} \).

Using the fact that \( y_0^2 = x_0(x_0 - 1)(x_0 - t) \) and \( v^2 = u(u - 1)(u - t) \), we have the following equation:

\[
(\ast): (u - z)^2x_0^3 - [(2(uz - u))(-z + u) - (-z + u)^2t - v^2]x_0^2 + [(uz - u)^2 - (2(uz - u))(-z + u)t + v^2]x_0 - (uz - u)^2t = 0
\]

1. if \( u - z = 0 \), then \( (\ast) \) becomes \( (x_0 - u) \left( x_0 - \frac{t(u - 1)}{u - t} \right) = 0 \). As by hypothesis \( v \neq 0 \), we obtain two solutions given by the point \( (x_0, y_0) \) such that \( x_0 = \frac{t(u - 1)}{u - t} \) and the point \( p_\infty \).
2. if \((u-z) \neq 0\), then the solutions verify the following second degree equation:

\[
(\star) : (u-z)^2 x_0^2 + \left[(-t-u)z^2 + 2u(t+1)z - u(t+u)\right] x_0 + tu(z-1)^2 = 0
\]

The \(+4\) self-intersection sections define a singular holomorphic 2-web \(W\) such that the discriminant \(\Delta\) is the discriminant of the equation \(\star\). Thus we have:

\[
\Delta := (t-u)z^4 - 4(1-t)uz^3 + 2u(2tu+t-u-2)z^2 - 4u^2(t-1)z + u^2(t-u) = 0
\]

**Lemma 4.9.** The discriminant of the 2-web \(W\) is a leaf of order 4 of the Riccati foliation \(\widetilde{\text{Ric}}\).

**Proof.** In fact, by the definition of the first integral of the foliation \(\widetilde{\text{Ric}}\), we have:

\[
F(x,z) - t = -\frac{(t-u)z^4 - 4(1-t)uz^3 + 2u(2tu+t-u-2)z^2 - 4u^2(t-1)z + u^2(t-u)}{(2xz - z^2 - x)^2}
\]

Therefore, the first integral is constant along of the discriminant \(\Delta\).

According to the foregoing, on the birational trivialisation of \(S_1\), we have a 4-web \(W_4\) defined by the 2-web \(W\), the Riccati foliation \(\widetilde{\text{Ric}}\) and the trivial fiber bundle.

### 4.3 Geometry of the 4-web \(W_4\)

We want to find the slopes of the leaves of \(W_4\) in order to represent it by a differential equation. Let \((x_0, y_0, z_0) \in C \times \mathbb{P}^1\) be a generic point. As the leaves of the 2-web \(W\) passing through this point are respectively the graph \(z = \frac{(1-a_0)(b_0x-a_0y)}{b_0(x-a_0)}\) and \(z = \frac{(1-a_0')(b_0'x-a_0'y)}{b_0'(x-a_0')}\) such that the points \((a_0, b_0)\) and \((a_0', b_0')\) verify the equation (\(\star\)), we deduce that their slopes at the point \((x_0, y_0, z_0)\) are respectively given by the following formulas:

1. \[
Z_1 = \frac{1-a_0-z_0}{x_0-a_0} + \left(\frac{z_0 + (a_0-1)x_0}{x_0-a_0}\right) \left(\frac{3x_0^2 - 2(1+t)x_0 + t}{2x_0(x_0-1)(x_0-t)}\right);
\]
2. \[
Z_2 = \frac{1-a_0'-z_0}{x_0-a_0'} + \left(\frac{z_0 + (a_0'-1)x_0}{x_0-a_0'}\right) \left(\frac{3x_0^2 - 2(1+t)x_0 + t}{2x_0(x_0-1)(x_0-t)}\right).
\]
Thus, the 2-web \( W \) is defined by the following differential equation:
\[
\left( \frac{dz}{dx} \right)^2 - \left( \frac{z^2 + 2(x - 1)z - x}{2x(x - 1)} \right) \frac{dz}{dx} + \frac{z(z - 1)((2tx - x^2 - t)z - x^3 + x^2 - tx + 2)}{4x^2(x - 1)^2(t - x)}
\]

Furthermore, if we consider
\[
Z_0 = \frac{1}{4} \left[ \frac{(z_0^2 + 2(x_0 - 1)z_0 - x_0)}{x_0(x_0 - 1)} \right],
\]

the slope of the foliation \( \tilde{Ric} \) at the point \((x_0, y_0, z_0)\), then the 4-web \( W_4 \) is locally equivalent to the 4-web on the complex plane given by \( W(\infty, Z_0, Z_1, Z_2) \).

**Theorem 4.10.** The 4-web \( W(\infty, Z_0, Z_1, Z_2) \) is locally equivalent to a parallelizable 4-web.

**Proof.** The pull-back of the foliation \( \tilde{Ric} \) by the multiplication of order 2 on \( C \) is another Riccati foliation \( \tilde{Ric}_2 \) on \( C \times \mathbb{P}^1 \) with trivial monodromy.

Let \( M_2: C \mapsto C \), be the multiplication of order 2 on \( C \) then, for any point \((x, y) \in C \) the first projection of \( M_2(x, y) \) is given by the following formula:
\[
pr_1 \circ M_2(x, y) = \frac{(3x^2 - 2(t + 1)x + t)^2}{4x(x - 1)(x - t)} + (1 + t) - 2x
\]

Using the pull-back of the special leaves, we can choose three curves by:

1. \( C_0 := \{(x, y, z), z = z_0 = \frac{-x^2 + t}{2(t - x)}\} \)
2. \( C_1 := \{(x, y, z), z = z_1 = \frac{-x^2 + t}{2y}\} \)
3. \( C_2 := \{(x, y, z), z = z_2 = \frac{-x^2 + t}{2y}\} \)

which are the leaves of \( \tilde{Ric}_2 \). Now, if we consider the map \( \psi: C \times \mathbb{P}^1 \mapsto C \times \mathbb{P}^1 \) which for any coordinate \((X, Z)\) relates:
\[
\psi(X, Z) = \left( \frac{(3X^2 - 2(t + 1)X + t)^2}{4X(X - 1)(X - t)} + (1 + t) - 2X, \frac{Z\mu z_1 - z_0}{Z\mu - 1} \right)
\]
where \( \mu = \frac{z_3 - z_0}{z_3 - z_1} \), then the pull-back of the first integral of \( \tilde{Ric} \) by \( \psi \) is the following meromorphic function:
\[
(\psi^*F)(X, Z) = \frac{(Z^2 - 2Z + 2)^2}{Z^2(Z - 2)^2}
\]
Finally, the foliation $\psi_* \tilde{Ric}$ is locally defined by the 1-form $dZ = 0$. Likewise, the pull-back of the slopes $Z_1$ and $Z_2$ by $\psi$ defines a 2-web such that the leaves verify the following differential equation:

$$
(**) : \left( \frac{dZ}{dX} \right)^2 + \frac{(t - 1)Z^4 + (-4t + 4)Z^3 + (4t - 8)Z^2 + 8Z - 4}{4X(X - 1)(X - t)} = 0
$$

In summary, the 4-web $\psi_* W(\infty, Z_0, Z_1, Z_2)$ is locally equivalent to the web $W(\infty, 0, \beta, -\beta)$, where $\beta$ is a solution of $(**).$ As the 4-web $W(\infty, 0, \beta, -\beta)$ has a constant cross-ratios equal to $-1$ and all the 3 subweb are hexagonal, we can deduce that it is locally parallelizable.

References

[1] M. F. Atiyah. Complex fibre bundles and ruled surfaces. *Proc. London Math. Soc. (3)*, 5:407–434, 1955.

[2] M. F. Atiyah. Vector bundles over an elliptic curve. *Proc. London Math. Soc. (3)*, 7:414–452, 1957.

[3] César Camacho and Alcides Lins Neto. *Geometric theory of foliations*. Birkhäuser Boston, Inc., Boston, MA, 1985. Translated from the Portuguese by Sue E. Goodman.

[4] Robert Friedman. *Algebraic surfaces and holomorphic vector bundles*. Universitext. Springer-Verlag, New York, 1998.

[5] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[6] Frank Loray and David Marín Pérez. Projective structures and projective bundles over compact Riemann surfaces. *Astérisque*, (323):223–252, 2009.

[7] Masaki Maruyama. *On classification of ruled surfaces*, volume 3 of Lectures in Mathematics, Department of Mathematics, Kyoto University. Kinokuniya Book-Store Co., Ltd., Tokyo, 1970.

[8] Shigeru Mukai. *An introduction to invariants and moduli*, volume 81 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2003. Translated from the 1998 and 2000 Japanese editions by W. M. Oxbury.
[9] Luc Pirio. Sur les tissus plans de rang maximal et le problème de Chern. 
*C. R. Math. Acad. Sci. Paris*, 339(2):131–136, 2004.

[10] Olivier Ripoll. Détermination du rang des tissus du plan et autres invariants géométriques. 
*C. R. Math. Acad. Sci. Paris*, 341(4):247–252, 2005.

[11] André Weil. Variété de Picard et variétés jacobienes. In *Séminaire Bourbaki, Vol. 2*, pages Exp. No. 72, 219–226. Soc. Math. France, Paris, 1995.