Finding Determinant Forms of Certain Hybrid Sheffer Sequences

Monairah Alansari 1, Mumtaz Riyasat 2,*®, Subuhi Khan 2 and Kaleem Raza Kazmi 2,3

1 Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia; malansari@kau.edu.sa
2 Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India; subuhi2006@gmail.com or subuhi.mm@amu.ac.in
3 Department of Mathematics, Faculty of Science & Arts-Rabigh, King Abdulaziz University, Jeddah 21589, Saudi Arabia; kkazmi@kau.edu.sa
* Correspondence: mumtazrst@gmail.com or mumtazriyasat.rs@amu.ac.in

Received: 12 October 2019; Accepted: 11 November 2019; Published: 14 November 2019

Abstract: In this article, the integral transform is used to introduce a new family of extended hybrid Sheffer sequences via generating functions and operational rules. The determinant forms and other properties of these sequences are established using a matrix approach. The corresponding results for the extended hybrid Appell sequences are also obtained. Certain examples in terms of the members of the extended hybrid Sheffer and Appell sequences are framed. By employing operational rules, the identities involving the Lah, Stirling and Pascal matrices are derived for the aforementioned sequences.

Keywords: Sheffer sequences; extended hybrid Sheffer sequences; fractional operators; operational rules; Riordan matrix

1. Introduction

Fractional calculus is a branch of mathematics that deals with the real or complex number powers of the differential operator. It is shown in [1] that the exploitation of integral transforms with special polynomials is an effective way to accord with fractional derivatives. Riemann and Liouville [2,3] were the first to use the integral transforms to deal with fractional derivatives. Since differentiation and integration are usually regarded as discrete operations, therefore it is useful to evaluate a fractional derivative. We recall the following definitions:

Definition 1. The Euler Γ-function [4] is given by

\[ \Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad \Re(x) > 0. \]  

Definition 2. The Euler’s integral ([5], p. 218) is given by (see also [1])

\[ a^{-v} = \frac{1}{\Gamma(v)} \int_0^\infty e^{-at}t^{v-1}dt, \quad \min\{\Re(v),\Re(a)\} > 0. \]  

Let \( \mathbb{K} \) be a field of characteristic zero and \( \mathcal{F} \) be the set of all formal power series in \( t \) over \( \mathbb{K} \). Let

\[ f(t) = \sum_{k=0}^\infty a_k t^k, \]
where \( a_k \in \mathbb{K} \) for all \( k \in \mathbb{N} := \{0, 1, 2, \ldots\} \). The order \( o(f(t)) \) of a power series \( f(t) \) is the smallest integer \( k \) for which the coefficient of \( t^k \) does not vanish. The series \( f(t) \) has a multiplicative inverse, denoted by \( f(t)^{-1} \) or \( \frac{1}{f(t)} \), if and only if \( o(f(t)) = 0 \), then \( f(t) \) is called an invertible series. The series \( f(t) \) has a compositional inverse \( \overline{f}(t) \) such that

\[
f(\overline{f}(t)) = \overline{f}(f(t)) = t,
\]

if and only if \( o(f(t)) = 1 \), then \( f(t) \) with \( o(f(t)) = 1 \) is called a delta series.

**Definition 3.** An invertible series \( g(t) \) and delta series \( f(t) \) with

\[
f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!}, \quad f_0 = 0; f_1 \neq 0,
\]

\[
g(t) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!}, \quad g_0 \neq 0,
\]

for all \( n \geq 0 \) and \( y \in \mathbb{K} \), where \( n \) being the degree of polynomial and \( \mathbb{K} \) a field of characteristic zero.

Let \( (s_n(x))_{n \in \mathbb{N}} \) be a Sheffer sequence for \( (g(t), f(t)) \) and suppose

\[
x^n = \sum_{k=0}^{n} b_{n,k} s_k(x),
\]

then the Sheffer sequence \( s_n(x) \) can be expressed by the following determinant form:

\[
s_0(x) = \frac{1}{b_{0,0}},
\]

\[
s_n(x) = \frac{(-1)^n}{b_{0,0} b_{1,1} \ldots b_{n,n}} \begin{vmatrix}
1 & x & x^2 & \cdots & x^{n-1} & x^n \\
b_{0,0} & b_{1,0} & b_{2,0} & \cdots & b_{n-1,0} & b_{n,0} \\
0 & b_{1,1} & b_{2,1} & \cdots & b_{n-1,1} & b_{n,1} \\
0 & 0 & b_{2,2} & \cdots & b_{n-1,2} & b_{n,2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & b_{n-1,n-1} & b_{n,n-1}
\end{vmatrix},
\]

where \( b_{n,k} \) is the \( (n,k) \) entry of the Riordan matrix \( (g(t), f(t)) \) which defines an infinite, lower triangular array \( (b_{n,k})_{0 \leq k \leq n < \sigma_0} \) according to the following rule:

\[
b_{n,k} = \left[ \frac{t^n}{c_n} \right] g(t) \frac{(f(t))^k}{c_k},
\]

where the functions \( \frac{g(t)(f(t))^k}{c_k} \) are called the column generating functions of the Riordan matrix.
A vast literature associated with the matrix and other approaches to several special polynomials and corresponding hybrid forms can be found, see [8–17]. These matrix forms helps in solving various algorithms and in finding the solution of numerical and a general linear interpolation problems.

The Appell sequence \( A_n(x) \) for \((g(t), t)\) are defined by

\[
\frac{1}{g(t)} e^{xt} = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!},
\]

which also satisfies the functional equation

\[
A_n(x + w) = \sum_{k=0}^{n} \binom{n}{k} A_k(x) w^{n-k}.
\]

The multi-variable forms of special polynomials are studied in a different way via operational techniques. These may also help in solving problems in classical and quantum mechanics associated with special functions. We recall the following definitions:

**Definition 4.** The 2-variable truncated exponential polynomials (2VTEP) (of order \( r \)) \( e^{(r)}_n(x, y) \) are defined by the following generating function, series expansion and operational rule ([18], p. 174 (30)):

\[
e^{yt} J_0(2t \sqrt{-x}) = \sum_{n=0}^{\infty} S_n(x, y) t^n n!, \tag{10}
\]

where \( J_0(x) \) are the regular cylindrical Bessel function, of zero-th order

\[
e^{(r)}_n(x, y) = n! \sum_{k=0}^{\left[ \frac{n}{r} \right]} y^k x^{n-rk} \frac{k!}{(n-rk)!}, \tag{11}
\]

\[
e^{(r)}_n(x, y) = \exp \left( yD_y yD_x^n \right) \{ x^n \}, \quad \left\{ D_y := \frac{\partial}{\partial y} \right\}. \tag{12}
\]

Using operational techniques and by convoluting the 2VTEP \( e^{(r)}_n(x, y) \) with Sheffer sequences [19], a class of hybrid Sheffer sequences namely the 2-variable truncated exponential-Sheffer sequences (2VTES) \( e^{(r)}_n(x, y) \) are introduced.

**Definition 5.** The exponential generating function and operational rule for the 2VTES \( e^{(r)}_n(x, y) \) are given by

\[
\frac{1}{g(t)} e^{xt} = \sum_{n=0}^{\infty} e^{(r)}_n(x, y) \frac{t^n}{n!},
\]

where

\[
e^{(r)}_n(x, y) = \exp \left( yD_y yD_x^n \right) \{ s_n(x) \}. \tag{13}
\]

**Remark 1.** Taking

\[
f(t) = \overline{f(t)} = t
\]

in 2VTES \( e^{(r)}_n(x, y) \), we find the 2-variable truncated exponential-Appell sequences (2VTEAS) \( eA^{(r)}_n(x, y) \) [20], which are defined by the following generating function and operational rule:

\[
\frac{1}{g(t)} e^{xt} = \sum_{n=0}^{\infty} eA^{(r)}_n(x, y) \frac{t^n}{n!}, \tag{15}
\]
Theorem 1. For the extended hybrid Sheffer sequences, the following integral representation holds true:

\[ e_{\nu}^{A_n^{(r)}(x, y)} = \exp \left( yD_y D_y^r \right) \{ A_n(x) \}. \]  

(16)

The Equation (12) gives the operational rule to introduce the 2VTEP \( e_n^{(r)}(x, y) \) while (14) and (16) define the operational connections between the 2VTES \( e_s^{(r)}(x, y) \) and the Sheffer sequences and 2VTEAS \( e_{\nu}^{A_n^{(r)}(x, y)} \) and the Sheffer sequences, obtained by utilizing Equation (12).

The Euler’s integral forms the basis of new generalizations of special polynomials. Additionally, the combination of the properties of exponential operators with suitable integral representations yields an efficient way of treating fractional operators. Dattolli et al. \([1,21,22]\) used the Euler’s integral to find the operational definitions and the generating relations for the generalized and new forms of special polynomials.

In this article, the exponential operational rule and generating function of the truncated exponential Sheffer are applied on an integral transform to introduce the extended forms of the hybrid Sheffer sequences. The determinant forms and other properties for these sequences are studied via fractional operators and Riordan matrices.

2. Extended Hybrid Sheffer Sequences

We show that the combination of exponential operators with the integral transform for the 2VTES \( e_s^{(r)}(x, y) \) will give rise to a new class of extended hybrid Sheffer sequences, namely the extended truncated exponential-Sheffer sequences (ETESS). Here, we define the extended hybrid Sheffer sequences by the following operational rule:

**Definition 6.** The extended hybrid Sheffer sequences are defined by the following operational rule:

\[ \left( \alpha - \frac{\partial}{\partial y} y \frac{\partial^r}{\partial x^r} \right)^{-\nu} \{ s_n(x) \} = e_{\nu}^{(r)} s_n(x, y; \alpha). \]  

(17)

**Theorem 1.** For the extended hybrid Sheffer sequences, the following integral representation holds true:

\[ e_{\nu}^{(r)} s_n(x, y; \alpha) = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-nt} t^{\nu-1} e_{\nu}^{(r)} (x, yt) dt. \]  

(18)

**Proof.** Replacing \( \alpha \) by \( \left( \alpha - \left( \frac{\partial}{\partial y} y \frac{\partial^r}{\partial x^r} \right) \right) \) in integral (2) and then operating the resultant expression on \( s_n(x) \), we find

\[ \left( \alpha - \frac{\partial}{\partial y} y \frac{\partial^r}{\partial x^r} \right)^{-\nu} s_n(x) = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-nt} t^{\nu-1} \exp \left( ty \frac{\partial}{\partial y} y \frac{\partial^r}{\partial x^r} \right) s_n(x) dt, \]  

(19)

which in view of Equation (14) gives

\[ \left( \alpha - \frac{\partial}{\partial y} y \frac{\partial^r}{\partial x^r} \right)^{-\nu} s_n(x) = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-nt} t^{\nu-1} e_{\nu}^{(r)} (x, yt) dt. \]  

(20)

Denoting the right hand side of Equation (20) by a new class of extended hybrid Sheffer sequences, i.e., \( e_{\nu}^{(r)} s_n(x, y; \alpha) \) yields assertion (18).

**Theorem 2.** For the extended hybrid Sheffer sequences \( e_{\nu}^{(r)} s_n(x, y; \alpha) \), the following generating function holds true:

\[ \frac{e^{\nu \mathcal{T}(w)}}{g(\mathcal{T}(w))[\alpha - yD_y D_y^r \mathcal{T}(w)]^\nu} = \sum_{n=0}^{\infty} e_{\nu}^{(r)} s_n(x, y; \alpha) \frac{w^n}{n!}. \]  

(21)
which on using Equation (13) in the right hand side gives
\[
\sum_{n=0}^{\infty} e_{n}^{(r)}(x,y;\alpha) \frac{w^{n}}{n!} = \sum_{n=0}^{\infty} \frac{1}{\Gamma(v)} \int_{0}^{\infty} e^{-at} t^{v-1} e_{n}^{(r)}(x,y)t \frac{w^{n}}{n!} \, dt, \tag{22}
\]

which on using Equation (13) in the right hand side gives
\[
\sum_{n=0}^{\infty} e_{n}^{(r)}(x,y;\alpha) \frac{w^{n}}{n!} = \frac{e^{T(w)}}{\Gamma(v) g(f(w))} \int_{0}^{\infty} e^{-\left(x-y\frac{\partial y}{\partial w}(T(w))\right)} t^{v-1} \, dt. \tag{23}
\]

Making use of Equation (2) in the right hand side of the above equation assertion (21) is obtained.

\[\square\]

A recurrence relation is an equation that recursively defines a sequence or multidimensional array of values, once one or more initial terms are given; each further term of the sequence or array is defined as a function of the preceding terms. Differentiating generating function (21), with respect to \( x, y \) and \( \alpha \), we find the following differential recurrence relations for the extended hybrid Sheffer sequences:
\[
f(D_{x}) \{ e_{n}^{(r)}(x,y;\alpha) \} = n e_{n-1}^{(r)}(x,y;\alpha), \tag{24}
\]
\[
f(D_{y}) \{ e_{n}^{(r)}(x,y;\alpha) \} = v n (n-1) \ldots (n-r) e_{n-r}^{(r)}(x,y;\alpha), \tag{25}
\]
\[
D_{x} \{ e_{n}^{(r)}(x,y;\alpha) \} = -v e_{n}^{(r)}(x,y;\alpha), \tag{26}
\]

The combination of monomiality principle \([21]\) with operational methods opens new possibilities to deal with the theoretical foundations of special polynomials and also to introduce new families of special polynomials. The Sheffer sequences are quasi-monomial. To frame the extended hybrid Sheffer sequences within the context of monomiality principle, the following result is proved:

**Theorem 3.** The extended hybrid Sheffer sequences \( e_{n}^{(r)}(x,y;\alpha) \) are quasi-monomial with respect to the following multiplicative and derivative operators:
\[
\hat{M}_{e_{n}^{(r)}} = \left( x - vyD_{y} yD_{x}^{r-1} - \frac{g'(D_{x})}{g(D_{x})} \right) \frac{1}{f'(D_{x})}, \tag{27}
\]
and
\[
\hat{P}_{e_{n}^{(r)}} = f(D_{x}), \tag{28}
\]
respectively.

\[\square\]

**Proof.** We recall the following recurrence relations for the truncated-Sheffer polynomials \( e_{n}^{(r)}(x,y) \) from \([19]\):
\[
\left( x + ryD_{y} D_{x}^{r-1} - \frac{g'(D_{x})}{g(D_{x})} \right) \frac{1}{f'(D_{x})} \{ e_{n}^{(r)}(x,y) \} = e_{n}^{(r)}(x,y), \tag{29}
\]
and
\[
f(D_{x}) \{ e_{n}^{(r)}(x,y) \} = n e_{n-1}^{(r)}(x,y), \quad n \geq 1. \tag{30}
\]

Consider the operation: (\( \Theta \)): Replacement of \( y \) by \( yt \), multiplication by \( \frac{1}{\Gamma(v)} e^{-at} t^{v-1} \) and then integration with respect to \( t \) from \( t = 0 \) to \( t = \infty \).

Now, operating (\( \Theta \)) on Equation (29) and then using relation (18) with (26) and further in view of recurrence relation \( \hat{M} \{ p_{n}(x) \} = p_{n+1}(x) \) assertion (27) follows.

Again, operating (\( \Theta \)) on Equation (30) and then using relation (18) and further in view of recurrence relation \( \hat{P} \{ p_{n}(x) \} = np_{n-1}(x) \) assertion (28) follows.

\[\square\]
Remark 2. Using expressions (27) and (28) of the operators in monomiality principle equation $\hat{M}\{p_n(x)\} = n p_n(x)$, we deduce the following consequence of Theorem 3:

Corollary 1. The extended hybrid Sheffer sequences $v e^{(r)} s_n(x, y; \alpha)$ satisfy the following differential equation:

$$\left( x - ryD_y y r\frac{\partial}{\partial x} x r - n \right) v e^{(r)} s_n(x, y; \alpha) = 0. \tag{31}$$

Remark 3. Taking

$$f(w) = f(w) = w, \quad g(w) = 1, \quad \frac{g'(D_x)}{g(D_x)} = 0 \quad \text{and} \quad s_n(x) = x^n \tag{32}$$

so that $v e^{(r)} s_n(x, y; \alpha) = v e^{(r)} s_n(x, y; \alpha)$,

in extended hybrid Sheffer sequences $v e^{(r)} s_n(x, y; \alpha)$, we obtain as a special case the extended truncated exponential polynomials $v e^{(r)} s_n(x, y; \alpha)$. The corresponding results are given in Table 1.

| S. No. | Results | Expressions |
|--------|---------|-------------|
| I.     | Operational rule | $\left( \alpha - y \frac{\partial}{\partial x} x \right)^n x^n = v e^{(r)} s_n(x, y; \alpha)$ |
| II.    | Generating function | $\hat{M}_{e^{(r)}} = x - ryD_y y r\frac{\partial}{\partial x} x r; \quad \hat{P}_{e^{(r)}} = D_x$ |
| III.   | Multiplicative and derivative operators | $\hat{M}_{e^{(r)}} = x - ryD_y y r\frac{\partial}{\partial x} x r; \quad \hat{P}_{e^{(r)}} = D_x$ |
| IV.    | Differential equation | $\left( xD_x = ryD_y y r\frac{\partial}{\partial x} x r - n \right) v e^{(r)} s_n(x, y; \alpha) = 0$ |

Note. It should be noted that for $\alpha = \nu = 1$ and $y = D_y^{-1}$, the extended truncated-exponential-Sheffer polynomials $v e^{(r)} s_n(x, y; \alpha)$ reduce to 2VTEP $s_n^{(r)}(x, y)$. For the same choice of parameters $\alpha, \nu$ and variable $y$ the extended truncated-exponential polynomials $v e^{(r)} s_n(x, y; \alpha)$ reduce to the 2VTEP $s_n^{(r)}(x, y)$.

To establish the determinant form for the extended hybrid Sheffer sequences, the following result is proved:

**Theorem 4.** For the extended hybrid Sheffer sequences $v e^{(r)} s_n(x, y; \alpha)$ of degree $n$, the following holds:

$$\begin{align*}
\begin{vmatrix}
\frac{1}{n!} v e^{(r)} s_n(x, y; \alpha) & v e^{(r)} s_1(x, y; \alpha) & \cdots & v e^{(r)} s_{n-1}(x, y; \alpha) & v e^{(r)} s_n(x, y; \alpha)
\end{vmatrix}
\end{align*} \quad \text{and} \quad
\begin{align*}
\begin{vmatrix}
0 & b_{0,1} & b_{1,1} & \cdots & b_{n-1,1} & b_{n,1} \\
1 & b_{0,0} & b_{1,0} & \cdots & b_{n-1,0} & b_{n,0} \\
0 & b_{0,2} & b_{1,2} & \cdots & b_{n-1,2} & b_{n,2} \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 0 & b_{n,n-1} \\
\end{vmatrix}
\end{align*} \quad \text{where } b_{n,k} \text{ is the } (n, k) \text{ entry of the Riordan matrix } (g(t), f(t)).
$$


Proof. Operating \( O : \exp \left( \alpha - y \frac{d}{dy} - x \frac{d}{dx} \right)^{-\nu} \) on both sides of Equation (6) and then using operational rule (17) and operational rule given in Table 1(I), for \( n = 0, 1, 2, \ldots \) in the LHS and RHS, respectively of the resulting equation assertion (33) follows.

\[ \square \]

Remark 4. Taking

\[ f(w) = f(w) = w \quad \text{and} \quad g(f(w)) = g(w); \quad e^{\nu(s)}_{\nu(s)}A_{\nu(s)}(x, y; \alpha), \]

in extended hybrid Sheffer sequences \( e^{\nu(s)}_{\nu(s)}A_{\nu(s)}(x, y; \alpha) \), we get the extended hybrid Appell sequences \( e^{\nu(s)}_{\nu(s)}A_{\nu(s)}(x, y; \alpha) \). The corresponding results are given in Table 2.

| S. No. | Results | Expressions |
|--------|---------|-------------|
| I.     | Operational rule | \( \left( \alpha - y \frac{d}{dy} - x \frac{d}{dx} \right)^{-\nu} \{ A_N(x) \} = e^{\nu(s)}_{\nu(s)}A_{\nu(s)}(x, y; \alpha) \) |
| II.    | Generating function | \( \frac{g(w)(\alpha - yD_y - xD_x)}{g(w)} = \sum_{n=0}^\infty \frac{e^{\nu(s)}_{\nu(s)}A_{\nu(s)}(x, y; \alpha)}{n!} \) |
| III.   | Multiplicative and derivative operators | \( \mathcal{M}_{\nu(s)}A = \left( x - ryD_y \frac{\partial}{\partial x} - \frac{e^{\nu(s)}_{\nu(s)}(D_x)}{g(D_x)} \right), \quad \mathcal{P}_{\nu(s)}A = D_x \) |
| IV.    | Differential equation | \( (xD_x - ryD_y \frac{\partial^{r+1}}{\partial x^{r+1}} - \frac{e^{\nu(s)}_{\nu(s)}(D_x)}{g(D_x)}D_x - n) e^{\nu(s)}_{\nu(s)}A_{\nu(s)}(x, y; \alpha) = 0 \) |

For the extended hybrid Appell sequences \( e^{\nu(s)}_{\nu(s)}A_{\nu(s)}(x, y; \alpha) \) of degree \( n \), the following holds:

\[ e^{\nu(s)}_{\nu(s)}A_{\nu(s)}(x, y; \alpha) = \frac{1}{g_0}, \]

\[ e^{\nu(s)}_{\nu(s)}A_{\nu(s)}(x, y; \alpha) = \left( \frac{-1}{g_0} \right)^n \] (34)

Remark 5. Taking \( g(f(w)) = 1, \quad \frac{e^{\nu(s)}_{\nu(s)}(D_x)}{g(D_x)} = 0; \quad a_{n, 0} = \delta_{n, 0} \) and \( s_n(x) = p_n(x) \) in the results for extended hybrid Sheffer sequences \( e^{\nu(s)}_{\nu(s)}s_n(x, y; \alpha) \), we get the corresponding results for the extended hybrid associated Sheffer sequences \( e^{\nu(s)}_{\nu(s)}p_n(x, y; \alpha) \).

In the next section, examples of some members belonging to the extended hybrid Sheffer and Appell families are considered.
3. Examples

By making suitable selections for the pair of function \((g(t), f(t))\) in the results derived for extended hybrid Sheffer sequences, we obtain results for the particulars members of the extended hybrid Sheffer family. The following examples illustrate this process:

**Example 1.** Taking

\[
g(w) = \frac{1}{(1-w)^{\beta+1}}, \quad f(w) = \frac{w}{w-1} \quad \text{and} \quad g(f(w)) = (1-w)^{\beta+1}
\]

in extended hybrid Sheffer sequences \(v^{\nu(\nu)}s_n(x, y; \alpha)\), we get the extended hybrid associated Laguerre sequences \(v^{\nu(\nu)}L_n^{(\nu)}(x, y; \alpha)\). The corresponding results are given in Table 3.

| S. No. | Results | Expressions |
|-------|---------|-------------|
| I.    | Operational rule | \((\alpha - y \frac{\beta}{\nu} \frac{\partial}{\partial y})^{-\nu} L_n^{(\nu)}(x) = v^{\nu(\nu)}L_n^{(\nu)}(x, y; \alpha)\) |
| II.   | Generating function | \[
\exp\left[ x \nu(\nu) y \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right] = \sum_{n=0}^{\infty} v^{\nu(\nu)}L_n^{(\nu)}(x, y; \alpha) \frac{w^n}{n!}
\]
| III.  | Multiplicative and derivative operators | \[
\mathcal{M} v^{\nu(\nu)}L = \left( x \nu(\nu) y \frac{\partial}{\partial x} \frac{\partial}{\partial y} - \frac{\beta+1}{(1-D_y)^{\nu}} \right) \frac{\partial}{\partial y} \frac{\partial}{\partial y} = D_y^{D_y-1} v^{\nu(\nu)}L (x, y; \alpha)
\]
| IV.   | Differential equation | \[
\left( x \nu(\nu) y \frac{\partial}{\partial x} \frac{\partial}{\partial y} - \frac{\beta+1}{(1-D_y)^{\nu}} \right) \frac{\partial}{\partial y} \frac{\partial}{\partial y} = 0
\]

For the extended hybrid associated Laguerre sequences \(v^{\nu(\nu)}L_n^{(\nu)}(x, y; \alpha)\) of degree \(n\), the following holds:

\[
v^{\nu(\nu)}L_0^{(\nu)}(x, y; \alpha) = 1,
\]

\[
v^{\nu(\nu)}L_n^{(\nu)}(x, y; \alpha) = (-1)^{\frac{n(n+1)}{2}}.
\]

**Example 2.** Taking

\[
g(w) = e^{\beta \frac{w^2}{2}}, \quad f(w) = f(w) = w \quad \text{and} \quad g(f(w)) = e^{\beta \frac{w^2}{2}}
\]

in extended hybrid Sheffer sequences \(v^{\nu(\nu)}s_n(x, y; \alpha)\), we get the extended hybrid Hermite sequences (EhHS) of variance \(\beta, v^{\nu(\nu)}H_n^{(\nu)}(x, y; \alpha)\). The corresponding results are given in Table 4.
For the extended hybrid Hermite sequences $e_{\nu}(\hat{e})H_n^\beta(x, y; \alpha)$ of degree $n$, the following holds:

\[ e_{\nu,0}H_n^\beta(x, y; \alpha) = 1, \]

\[ e_{\nu,j}H_n^\beta(x, y; \alpha) = (-1)^n, \]

in extended hybrid Sheffer sequences $e_{\nu}(\hat{e})S_n(x, y; \alpha)$, we get the extended hybrid Genocchi sequences (EhGS) $e_{\nu}(\hat{e})G_n(x, y; \alpha)$. The corresponding results are given in Table 5.

### Table 5. Results for $e_{\nu}(\hat{e})G_n(x, y; \alpha)$.

| S. No. | Results | Expressions |
|--------|---------|-------------|
| I.     | Operational rule | $\left( x - y \frac{\partial}{\partial x} y \frac{\partial}{\partial y} \right)^{-\nu} \{ G_n(x) \} = e_{\nu}(\hat{e})G_n(x, y; \alpha) $ |
| II.    | Generating function | $\left( \frac{\hat{e}}{x} \right) e^{\hat{w}}(x - yD_y y \hat{w})^{-\nu} = \sum_{n=0}^{\infty} e_{\nu}(\hat{e})G_n(x, y; \alpha) \frac{\hat{w}^n}{n!} $ |
| III.   | Multiplicative and derivative operators | $M_{e_{\nu}(\hat{e})C} = \left( x - ryD_y y \frac{\partial}{\partial y} \right)^{-\nu} \left( \frac{\partial^{2}(D_y-1)}{(e^{\hat{w}}+1)} \right) C, \quad \hat{p}_{e_{\nu}(\hat{e})C} = D_x $ |
| IV.    | Differential equation | $\left( xD_x - ryD_y y \frac{\partial}{\partial y} - \frac{\partial^{2}(D_y-1)-2}{(e^{\hat{w}}+1)} - n \right) e_{\nu}(\hat{e})G_n(x, y; \alpha) = 0 $ |
Example 4. Taking
\[ g(w) = \left( \frac{2}{1 + \sqrt{1 - w^2}} \right)^\lambda, \quad f(w) = \frac{-w}{1 + \sqrt{1 - w^2}}; \]
\[ f(w) = \frac{-2w}{1 + w^2} \quad \text{and} \quad g(f(w)) = (1 + w^2)^\lambda \]
in the extended hybrid Sheffer sequences \( e^{(\gamma)}_{\nu}(x; y; \alpha) \), we get the extended hybrid Gegenbauer sequences \( (\text{EhGeS}) e^{(\gamma)}_{\nu}C^{(\lambda)}_n(x; y; \alpha) \). The corresponding results are given in Table 6.

| S.No. | Results                       | Expressions                                      |
|-------|-------------------------------|--------------------------------------------------|
| I.    | Operational rule              | \( (x - y)^\mu \frac{\partial}{\partial x}C^{(\lambda)}_n(x) = e^{(\gamma)}_{\nu}C^{(\lambda)}_n(x; y; \alpha) \) |
| II.   | Generating function           | \( \frac{\exp\left(\frac{w}{\sqrt{1-w^2}}\right)}{(1-w^2)\Gamma\left(\frac{n+\lambda}{2}\right)} = \sum_{n=0}^{\infty} e^{(\gamma)}_{\nu}C^{(\lambda)}_n(x; y; \alpha)\frac{w^n}{n!} \) |
| III.  | Multiplicative and derivative operators | \( \nabla_{e^{(\gamma)}_{\nu}C^{(\lambda)}_n(x; y; \alpha)} = \left( x - ryD_yy \frac{\partial}{\partial x} \right)C^{(\lambda)}_n(x; y; \alpha) \) |
| IV.   | Differential equation         | \( \left( x - ryD_yy \frac{\partial}{\partial x} \right)C^{(\lambda)}_n(x; y; \alpha) = -D_y\left(1 + \frac{1}{\sqrt{1-D_y}} \right) - n \) |

For the extended hybrid Gegenbauer sequences \( e^{(\gamma)}_{\nu}C^{(\lambda)}_n(x; y; \alpha) \) of degree \( n \), the following holds:
\[ e^{(\gamma)}_{\nu}C^{(\lambda)}_0(x; y; \alpha) = 1, \]
\[ e^{(\gamma)}_{\nu}C^{(\lambda)}_n(x; y; \alpha) = (-1)^{n(n-1)}\frac{n!(\lambda+n-1)!}{(\lambda+1)!}\frac{\mu}{2^{n-1}(\lambda+2)!} \]
In the next section, the applications of operational rules are considered.

4. Applications

In order to give applications of the operational rules derived in previous sections, we use the following operation:

\[ O: \text{Operating } \exp \left( \alpha - y \frac{\partial}{\partial y} \frac{\partial}{\partial x} \right)^{-v} \text{ on both sides of a given result.} \]

We recall that the Sheffer sequences \( (s_n(x))_{n \in \mathbb{N}} \) are defined by following series representation:

\[ s_n(x) = \sum_{k=0}^{n} b_{n,k} x^k, \quad (39) \]

where \( b_{n,k} \) is the \((n,k)\) entry of the Riordan matrix

\[ \left( \frac{1}{g(f(t))}, f(t) \right), \]

which has the following determinant representation:

\[ b_{n,k} = \frac{(-1)^{n-k}}{a_{n,k} \cdots a_{n,n}} \begin{vmatrix} a_{k+1,k} & a_{k+2,k} & \cdots & a_{n-1,k} & a_{n,k} \\ a_{k+1,k+1} & a_{k+2,k+1} & \cdots & a_{n-1,k+1} & a_{n,k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \end{vmatrix}, \quad (40) \]

where \( a_{n,k} \) is the \((n,k)\) entry of the Riordan matrix \((g(t), f(t))\), whose inverse matrix is \( \left( \frac{1}{g(f(t))}, f(t) \right) \).

Performing operation \( O \) on both sides of Equation (39) and then using appropriate operational rules, we find the following series representation for the extended hybrid Sheffer sequences \( \nu^e_r s_n(x; y; \alpha) \):

\[ \nu^e_r s_n(x, y; \alpha) = \sum_{k=0}^{n} b_{n,k} \nu^e_k(x, y; \alpha). \quad (41) \]

We recall that the generalized Riordan arrays for \( c_n = 1 \) reduce to classical Riordan arrays and for \( c_n = n! \) reduce to exponential Riordan arrays.

- The exponential Riordan matrix \( \left( 1, \frac{1}{t^{r+1}} \right) \) with \( \tilde{f}(t) = \frac{1}{t^{r+1}} \) is the Lah matrix \((L(n,k))_{n,k \in \mathbb{N}}\), whose \((n,k)\) entry is

\[ a_{n,k} = (-1)^k \frac{n!}{k!} \binom{n-1}{n-k} = (-1)^k L(n, k), \quad (42) \]

where \( L(n,k) \) are the Lah numbers.
In view of Equations (40)–(42), we find the following series representation for the sequence \( e^{x,y,z}(r) s_n(x, y; a) \) in terms of Lah matrix:

\[
e^{x,y,z}(r) s_n(x, y; a) = \sum_{k=0}^{n} (-1)^n e^{x,y,z} e_k (x, y; a)
\]

\[
\begin{bmatrix}
L(k + 1, k) & L(k + 2, k) & \cdots & L(n - 1, k) & L(n, k) \\
L(k + 1, k + 1) & L(k + 2, k + 1) & \cdots & L(n - 1, k + 1) & L(n, k + 1) \\
0 & L(k + 2, k + 2) & \cdots & L(n - 1, k + 2) & L(n, k + 2) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & L(n - 1, n - 1) & L(n, n - 1)
\end{bmatrix}
\] (43)

- The exponential Riordan matrix \((1, \log(1 + t))\) with \( \bar{f}(t) = e^{t} - 1 \) is the Stirling matrix of the first kind \((s(n, k))_{n, k \in \mathbb{N}}\), whose \((n, k)\) entry is

\[
a_{n,k} = S(n, k),
\] (44)

where \(S(n, k)\) are the Stirling numbers of the second kind.

In view of Equations (40), (41) and (44), we find the following series representation for the sequence \( e^{x,y,z}(r) s_n(x, y; a) \) in terms of Stirling matrix of the second kind:

\[
e^{x,y,z}(r) s_n(x, y; a) = \sum_{k=0}^{n} (-1)^{n-k} e^{x,y,z} e_k (x, y; a)
\]

\[
\begin{bmatrix}
S(k + 1, k) & S(k + 2, k) & \cdots & S(n - 1, k) & S(n, k) \\
S(k + 1, k + 1) & S(k + 2, k + 1) & \cdots & S(n - 1, k + 1) & S(n, k + 1) \\
0 & S(k + 2, k + 2) & \cdots & S(n - 1, k + 2) & S(n, k + 2) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & S(n - 1, n - 1) & S(n, n - 1)
\end{bmatrix}
\] (45)

- The classical Riordan matrix \(\left(\frac{1}{1-t}, \frac{1}{1-t}\right)\) with \( \bar{f}(t) = \frac{1}{1+t} \) is the Pascal matrix \((\binom{n}{k})_{n, k \in \mathbb{N}}\), whose \((n, k)\) entry is

\[
a_{n,k} = (-1)^{n-k} \binom{n}{k}.
\] (46)

In view of Equations (40), (41) and (46), we find the following series representation for the sequence \( e^{x,y,z}(r) s_n(x, y; a) \) in terms of the Pascal matrix:
expressed as:

\[
\nu^{(r)} S_n(x, y; \alpha) = \sum_{k=0}^{n} \nu^{(r)} e_k^{(r)}(x, y; \alpha)
\]

Using determinant representation of binomial coefficient \( \binom{n}{k} \) [23], the above equations can be expressed as:

\[
\nu^{(r)} S_n(x + w, y; \alpha) = \sum_{k=0}^{n} \binom{n}{k} \nu^{(r)} s_k(x, y; \alpha) \nu^{(r)} p_{n-k}(w, y; \alpha),
\]

(48)

\[
\nu^{(r)} A_n(x + w, y; \alpha) = \sum_{k=0}^{n} \binom{n}{k} \nu^{(r)} a_k(x, y; \alpha) \nu^{(r)} e_{n-k}^{(r)}(w, y; \alpha),
\]

(49)

Next, by performing the operation \( (\mathcal{O}) \) on both sides of Equations (5) and (9) and then use of appropriate operational rules in the resultant equations, we find the following functional equations for the extended hybrid sequences \( \nu^{(r)} S_n(x, y; \alpha) \) and \( \nu^{(r)} A_n(x, y; \alpha) \):

\[
\nu^{(r)} S_n(x + w, y; \alpha) = \sum_{k=0}^{n} \left( \binom{n}{k} \right) \nu^{(r)} s_k(x, y; \alpha) \nu^{(r)} p_{n-k}(w, y; \alpha),
\]

(50)

\[
\nu^{(r)} A_n(x + w, y; \alpha) = \sum_{k=0}^{n} \left( \binom{n}{k} \right) \nu^{(r)} a_k(x, y; \alpha) \nu^{(r)} e_{n-k}^{(r)}(w, y; \alpha),
\]

(51)
The other identities for the Sheffer sequences \( s_n(x) \) can be recalled from [10] as:

\[
s_n(x) = \sum_{k=0}^{n} \binom{n}{k} s_{n-k}(0) p_k(x), \quad n = 0, 1, \ldots,
\]

(52)

\[
s_n(x) = \frac{1}{\beta_0} \left( p_n(x) - \sum_{k=0}^{n-1} \binom{n}{k} \beta_{n-k} s_k(x) \right), \quad n = 1, 2, \ldots,
\]

(53)

\[
p_n(x) = \sum_{k=0}^{n} \binom{n}{k} \beta_{n-k} s_k(x), \quad n = 0, 1, \ldots,
\]

(54)

\[
s_n(x) = \sum_{k=0}^{n} \left( \sum_{i=k}^{n} \binom{n}{i} \alpha_{n-i} a_{i,k} \right) x^i, \quad n = 0, 1, \ldots.
\]

(55)

which on performing operation \( \mathcal{O} \) and then using appropriate operational rules in the resultant equations yield the following identities for the hybrid sequences \( e^{(r)} s_n(x; y; \alpha) \):

\[
e^{(r)} s_n(x; y; \alpha) = \sum_{k=0}^{n} \binom{n}{k} s_{n-k}(0) e^{(r)} p_k(x; y; \alpha), \quad n = 0, 1, \ldots,
\]

(56)

\[
e^{(r)} s_n(x; y; \alpha) = \frac{1}{\beta_0} \left( e^{(r)} p_n(x; y; \alpha) - \sum_{k=0}^{n-1} \binom{n}{k} \beta_{n-k} e^{(r)} p_k(x; y; \alpha) \right), \quad n = 1, 2, \ldots,
\]

(57)

\[
e^{(r)} p_n(x; y; \alpha) = \sum_{k=0}^{n} \binom{n}{k} \beta_{n-k} e^{(r)} s_k(x; y; \alpha), \quad n = 0, 1, \ldots,
\]

(58)

\[
e^{(r)} s_n(x; y; \alpha) = \sum_{k=0}^{n} \left( \sum_{i=k}^{n} \binom{n}{i} \alpha_{n-i} a_{i,k} \right) e^{(r)} x^i, \quad n = 0, 1, \ldots.
\]

(59)

Next, we consider the following results for the Appell sequences \( A_n(x) \) ([9], (31–32) p. 1534):

\[
A_n(x) = \frac{1}{\beta_0} \left( x^n - \sum_{k=0}^{n-1} \binom{n}{k} \beta_{n-k} A_k(x) \right), \quad n = 1, 2, \ldots,
\]

(60)

\[
x^n = \sum_{k=0}^{n} \binom{n}{k} \beta_{n-k} A_k(x), \quad n = 0, 1, \ldots.
\]

(61)

which on performing operation \( \mathcal{O} \) and then using the appropriate operational relations in the resultant equations, we obtain the following identities for the sequence \( e^{(r)} A_n(x; y; \alpha) \):

\[
e^{(r)} A_n(x; y; \alpha) = \frac{1}{\beta_0} \left( e^{(r)} A_n(x; y; \alpha) - \sum_{k=0}^{n-1} \binom{n}{k} \beta_{n-k} e^{(r)} A_k(x; y; \alpha) \right), \quad n = 1, 2, \ldots,
\]

(62)

\[
e^{(r)} A_n(x; y; \alpha) = \sum_{k=0}^{n} \binom{n}{k} \beta_{n-k} e^{(r)} A_k(x; y; \alpha), \quad n = 0, 1, \ldots.
\]

(63)
Further, we recall the following identities for the associated Laguerre, Hermite, Gegenbauer and Genocchi sequences [6]:

\[
L_n^{(\beta+\gamma+1)}(u + v) = \sum_{k=0}^{n} \binom{n}{k} L_k^{(\beta)}(u) L_{n-k}^{(\gamma)}(v),
\]

\[
H_n^{(\beta)}(u + v) = \sum_{k=0}^{n} \binom{n}{k} u^{n-k} H_k^{(\beta)}(v),
\]

\[
x^n = \sum_{0 \leq k \leq n, n-k \equiv (mod 2)} \frac{(k + \lambda) n! \Gamma(\lambda)}{2^n \left(\frac{n-k}{2}\right)! \Gamma\left(\frac{n+k+2\lambda+2}{2}\right)} C_n^{(\lambda)}(x),
\]

\[
G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_k x^{n-k},
\]

which on using the appropriate operation rules in both sides yields the following identities for the sequences \(v^{(r)}L_n^{(\beta)}(x, y; \alpha)\), \(v^{(r)}H_n^{(\beta)}(x, y; \alpha)\) and \(v^{(r)}C_n^{(\lambda)}(x, y; \alpha)\):

\[
v^{(r)}L_n^{(\beta+\gamma)}(u + v; \alpha) = \sum_{k=0}^{n} \binom{n}{k} v^{(r)}L_k^{(\beta)}(u; \alpha) v^{(r)}L_{n-k}^{(\gamma)}(v; \alpha),
\]

\[
v^{(r)}H_n^{(\beta)}(u + v; \alpha) = \sum_{k=0}^{n} \binom{n}{k} v^{(r)}H_k^{(\beta)}(v; \alpha),
\]

\[
v^{(r)}C_n^{(\lambda)}(x, y; \alpha) = \sum_{0 \leq k \leq n, n-k \equiv (mod 2)} \frac{(k + \lambda) n! \Gamma(\lambda)}{2^n \left(\frac{n-k}{2}\right)! \Gamma\left(\frac{n+k+2\lambda+2}{2}\right)} v^{(r)}C_k^{(\lambda)}(x, y; \alpha),
\]

\[
v^{(r)}G_n(x, y; \alpha) = \sum_{k=0}^{n} \binom{n}{k} G_k v^{(r)}C_{n-k}^{(\lambda)}(x, y; \alpha).
\]

Author Contributions: Funding acquisition, M.A.; Supervision, S.K.; Writing-original draft, M.R.; Editing: K.K.

Funding: This work was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under the grant no. G-363-247-1440. The authors, therefore, acknowledge with thanks DSR for technical and financial support.

Acknowledgments: The authors are thankful to the Reviewer(s) for several useful comments and suggestions towards the improvement of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References
1. Dattoli, G.; Ricci, P.E.; Cesarano, C.; Vázquez, L. Special polynomials and fractional calculus. *Math. Comput. Model.* 2003, 37, 729–733.
2. Oldham, H.; Spanier, N. *The Fractional Calculus;* Academic Press: San Diego, CA, USA, 1974.
3. Widder, D.V. *An Introduction to Transform Theory;* Academic Press: New York, NY, USA, 1971.
4. Andrews, L.C. *Special Functions for Engineers and Applied Mathematicians;* Macmillan Publishing Company: New York, NY, USA, 1985.
5. Srivastava, H.M.; Manocha, H.L. *A Treatise on Generating Functions;* Halsted Press: New York, NY, USA, 1984.
6. Roman, S. *The Umbral Calculus;* Academic Press: New York, NY, USA, 1984.
7. Roman, S.; Rota, G. The Umbral Calculus. *Adv. Math.* 1978, 27, 95–188.
8. Costabile, F.A.; Dell’Accio, F.; Gualtieri, M.I. A new approach to Bernoulli polynomials. *Rendiconti di Matematica e delle sue Applicazioni* 2006, 26, 1–12.
9. Costabile, F.A.; Longo, E. A determinantal approach to Appell polynomials. *J. Comput. Appl. Math.* 2010, 234, 1528–1542.
10. Costabile, F.A.; Longo, E. An algebraic approach to Sheffer polynomial sequences. *Integral Transform. Spec. Funct.* 2013, 25, 295–311.
11. Costabile, F.A.; Serpe, A. An algebraic approach to Lidstone polynomials. *Appl. Math. Lett.* 2007, 20, 387–390.
12. He, Y.; Araci, S.; Srivastava, H.M.; Abdel-Aty, M. Higher-Order convolutions for Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. *Mathematics* 2018, 6, 329.
13. Khan, S.; Nahid, T. Determinant forms, difference equations and zeros of the $q$-Hermite-Appell polynomials. *Mathematics* 2018, 6, 258.
14. Khan, S.; Riyasat, M. A determinantal approach to Sheffer-Appell polynomials via monomiality principle. *J. Math. Anal. Appl.* 2015, 421, 806–829.
15. Khan, S.; Riyasat, M. Determinantal approach to certain mixed special polynomials related to Gould-Hopper polynomials. *Appl. Math. Comput.* 2015, 251, 599–614.
16. Natalini, P.; Ricci, P.E. Appell-Type functions and Chebyshev polynomials. *Mathematics* 2019, 7, 679.
17. Srivastava, H.M.; Yasmin, G.; Muhyi, A.; Araci, S. Certain results for the twice- Iterated 2D $q$-Appell polynomials. *Symmetry* 2019, 11, 1307.
18. Dattoli, G.; Migliorati, M.; Srivastava, H.M. A class of Bessel summation formulas and associated operational methods. *Frac. Appl. Anal.* 2004, 7, 169–176.
19. Khan, S.; Yasmin, G.; Ahmad, N. On a new family related to truncated exponential and Sheffer polynomials. *J. Math. Anal. Appl.* 2014, 418, 921–937.
20. Khan, S.; Yasmin, G.; Ahmad, N. A note on truncated exponential-based Appell polynomials. *Bull. Malays. Math. Sci. Soc.* 2017, 40, 373–388.
21. Dattoli, G. Hermit-Bessel and Laguerre-Bessel Functions: A By-Product of the Monomiality Principle, Advanced Special Functions and Applications (Melfi, 1999), 147-164. *Proc. Melfi Sch. Adv. Top. Math. Phys.* 1, Aracne, Rome, 2000. Available online: http://xueshu.baidu.com/usercenter/paper/show?paperid=112e530795f16f9927472459a8391b4&site=xueshu_se (accessed on 5 October 2019).
22. Dattoli, G. Generalized polynomials operational identities and their applications. *J. Comput. Appl. Math.* 2000, 118, 111–123.
23. Wang, W. A determinantal approach to Sheffer sequences. *Linear Algebra Appl.* 2014, 463, 228–254.