mass of $1^{-+}$ fourquark-hybrid mixed states

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We calculate the mass of $J^{PC} = 1^{-+}$ light exotic mesons by QCD sum rules, the mass are extracted from fourquark-hybrid correlation functions. We construct several $1^{-+}$ fourquark molecule currents and hybrid currents, and get one mass around 1.2-1.4GeV and another one around 1.45-1.67GeV. They can be identified as $\eta_1(1400)$ and $\pi_1(1600)$. We also find fourquark current commonly mix with hybrid-like currents under renormalization. The two $1^{-+}$ states are mixed states of fourquark and hybrid.

I. INTRODUCTION

Researches about exotic meson states have a long history, glueball, hybrid meson (quark-antiquark with excited gluon), and multiquark are predicted after the establishment of quantum chromodynamics. The light exotic mesons with $J^{PC} = 1^{-+}$ are somehow strange.

Lattice QCD and most of the phenomenological methods show the mass of $1^{-+}$ hybrid is around 1.7-2.1GeV; they prefer to decay into S- and P-wave mesons. However, experimentally, there are two well-established $1^{-+}$ mesons $\pi_1(1400)$ and $\pi_1(1600)$; both are lighter than predictions. The $\pi_1(1400)$, only decaying into $\eta\pi$, is more strange. Ref. [7] indicate that in the limit of SU(3) flavor symmetry, the $1^{-+}$ hybrid decay to $p$-wave $\eta\pi$ is forbidden, which implies $\pi_1(1400)$ cannot be a glueball.

Some authors consider the $1^{-+}$ exotic mesons as tetraquark (diquark-antidiquark state) or fourquark molecule (mesons bound state). Ref. [7] exhaust all $1^{-+}$ tetraquark configurations, get mass around 1.6GeV and 2.0GeV for two types of quark contents. Ref. [8] gives $1^{-+}$ fourquark molecule mass around 1.4-1.5GeV, but the result receives large uncertainty from the instanton density. Ref. [9] re-examine the researches about $1^{-+}$ exotic mesons, give mass predictions 1.7GeV for treatquark and 1.3GeV for fourquark molecule.

However, if $\pi_1(1400)$ is fourquark, then there exist mesons with strangeness 2 and mesons with 2 units of charge but not observed. Besides, there is a subtle problem for QCD sum rules research about fourquark. The fourquark current, usually can be identified as two meson currents, may easily couple to two mesons. It causes difficulty to extract the correct information about the resonance if the non-bounded mesons give large contribution to the correlation function. Ref. [8] analyzed the diagrams about fourquark correlation function, indicated that the diagrams have no singularity at $s = (\sum_{i=1} m_i)^2$ are not relevant to fourquark state but relevant to two mesons, here the $m_i$ is quark mass. But the validity of this criterion is not clear up to now.

We think that since gluon can couple to quark-antiquark pair, the fourquark state may tend to mix with hybrid state, as long as the symmetry permits. Especially for light fourquark and hybrid, since the involved QCD coupling constant $g \gtrsim 1$ due to the low energy scale. And we find fourquark current commonly mix with hybrid and hybrid-like currents; the $1^{-+}$ fourquark mix with $1^{-+}$ hybrid. Ref. [3] conjecture the strong decay model of $\pi_1(1400)$ can be caused by mixing of fourquark molecule with tetraquark and/or hybrid. It is important that whether we can get two $1^{-+}$ states agree with experiments, if we think there are mixed states.

In this paper, we try to evaluate the mass of $1^{-+}$ mesons from fourquark-hybrid correlation function. Since different mesons have different decay models, the non-bounded mesons states contribution will be suppressed to some extent in fourquark-hybrid correlation function, leaving the states can couple to both currents. Thus for state probably mixed by fourquark and hybrid, we think the mass evaluation based on fourquark-hybrid correlation function is convincible. And we get results agree with experiments.

II. QCD SUM RULES FOR FOURQUARK-HYBRID

A. Currents and Renormalization

The QCD sum rules 10 12 is an effective way to research hadron properties. To evaluate the mass of hadron, the main task is to calculate the correlation function of two currents. For fourquark molecule currents, we choose $\eta\pi$, $\eta'\pi$, $\rho\pi$ and $b_1\pi$ configurations by decay models of $\pi_1(1400)$ and $\pi_1(1600)$. The currents for $\eta\pi$ and

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1 We call both tetraquark and fourquark molecule as fourquark when the distinction between them are not important for discussion.
\[ J_{qq}^\mu = (\bar{u}\gamma^5\gamma^\mu u + \bar{d}\gamma^5\gamma^\mu d + \theta\bar{s}\gamma^5\gamma^\mu s)(\bar{u}\gamma^5u - \bar{d}\gamma^5d), \]
\[ J_{qq}^{\mu\nu} = (\bar{u}\gamma^5\gamma^\mu u + \bar{d}\gamma^5\gamma^\mu d + \theta\bar{s}\gamma^5\gamma^\mu s)(\bar{u}\gamma^5\gamma^\nu u - \bar{d}\gamma^5\gamma^\nu d) + \{\mu \leftrightarrow \nu\}, \] (1)

with \( \theta = -2 \) for \( \eta \pi \) and 1 for \( \eta'\pi \). The currents couple to \( 1^+ \) \( \rho \pi \) and \( b_1 \pi \) are:
\[ J_{b_1 q}^\mu = e^{\mu\nu\rho\beta} (\bar{u}\sigma_{\alpha\beta}d \bar{d}\gamma_5\gamma^\mu u - \bar{d}\sigma_{\alpha\beta}u \bar{u}\gamma^5\gamma^\mu d), \]
\[ J_{b_1 q}^{\mu\nu} = e^{\mu\nu\rho\beta} (\bar{u}\sigma_{\alpha\beta}d \bar{d}\gamma_5\gamma^\mu u - \bar{d}\sigma_{\alpha\beta}u \bar{u}\gamma^5\gamma^\mu d). \]

Here \( \sigma^{0123} = +1 \), \( \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \). There also has a \( \rho \pi \) current couple to \( 1^- \) \( \rho \pi \) and \( b_1 \pi \):
\[ J_{b_2 q}^{\mu\nu} = e^{\mu\nu\rho\beta} (\bar{u}\gamma_\alpha d \bar{d}\gamma_5\gamma^\mu u - \bar{d}\gamma_\alpha u \bar{u}\gamma^5\gamma^\mu d). \]

(2)

The hybrid currents are:
\[ J_H^\mu = i(\bar{u}G^{\mu\nu}\gamma_\nu u - \bar{d}G^{\mu\nu}\gamma_\nu d), \]
\[ J_H^{\mu\nu} = \bar{u}(G^{\mu\nu}\sigma_\alpha - G^{\rho\sigma}\sigma_\rho \sigma_\nu \sigma_\mu)u + \{u \leftrightarrow d\}. \]

(4)

Here we always write \( gT^aG^{\mu\nu} \) as \( G^{\mu\nu} \) for simplicity. To evaluate the fourquark-hybrid correlators, the fourquark currents need to be renormalized; renormalization about composite operator can see, e.g. Ref. 12. The general fourquark current can be written as:
\[ J_{qq} = \bar{\Psi}_a \Gamma_A \Psi_b \bar{\Psi}_c \Gamma_B \Psi_d. \]

(5)

Here \( a, b, c, d \) are flavor indices; \( \Gamma_A, \Gamma_B \) are general \( \gamma \) and colour matrices. The 1/\( \epsilon \)-pole at \( O(\epsilon) \) exist when \( b = c \) and/or \( a = d \). Assume \( b = c \) and \( a \neq d \), for zero-momentum insertion Green function:
\[ \langle 0 | J_{qq} | m_{a,i} A^{\mu} \bar{\Psi}_{d,j}^{\dagger} | 0 \rangle, \]

(6)

here \( i, j \) are spin indices; \( m, r, n \) are colour indices. The 1/\( \epsilon \)-pole is canceled by (see Fig. 1):
\[ \langle 0 | (J_1 + J_2) \bar{\Psi}_a^{m_{a,i}} A^{\mu} \bar{\Psi}_{d,j}^{\dagger} | 0 \rangle, \]

(7)

with:
\[ J_1 = -\frac{1}{\epsilon} \frac{m}{32\pi^2} \bar{\Psi}_a \Gamma_A G^{a\beta} \sigma_{\alpha\beta} \Gamma_B \Psi_d, \]
\[ J_2 = \frac{1}{\epsilon} \frac{1}{48\pi^2} \bar{\Psi}_a \Gamma_A D_{\alpha} G^{a\alpha\beta} \gamma_\beta \Gamma_B \Psi_d. \]

(8)

Here \( m \) is the mass of the quark in the loop; we keep the mass of quark in \( Eq. \ 7 \) \( D_{\alpha} G^{a\alpha\beta} = gT^a D_{\alpha} G^{a\alpha\beta}, \)
\[ D_{\alpha}^b = \partial_\alpha \delta^{ab} + g f^{abc} A_5^c \] is covariant derivative at adjoint representation. \( Eq. \ 8 \) is evaluated at dimension \( D = 4 - 2\epsilon \). The \( J_1 \) vanish at massless limit, we use it only when evaluating \( m(\bar{q}q) \) contributions. Note the chiral suppression of \( J_1 \) does not mean the mixing of fourquark and hybrid is must small, since the mixing is not simply determined by the numerical factor in \( Eq. \ 9 \).

The renormalized fourquark current at \( O(\epsilon) \) then can be written as:
\[ (J_{qq})_R = J_{qq} + J_1 + J_2. \]

(9)

By \( Eq. \ 8 \) and \( 9 \), the renormalized currents for \( \eta'(\pi), \)
\( b_1 \pi \) and \( \rho \pi \) can be easily obtained.

B. Fourquark-Hybrid correlation functions

By operator product expansion (OPE), QCD sum rules express the correlation function as a sort of condensates. Before evaluating the correlation functions, we first discuss the dispersion relation, which is slightly modified when two currents are different. Consider the correlation function of two currents \( J_a \) and \( J_b \), for simplicity, assume the involved states have same quantum number \( J^{PC} \) and suppress the Lorentz indices, it has:
\[ \Pi(q) = i \int d^4x \ e^{iqx} \langle 0 | T \{ J_a(x) J_b^\dagger(0) \} | 0 \rangle, \]
\[ = \int_0^\infty ds \ \rho(s) \frac{\mathcal{P}(q)}{s - q^2 - i\epsilon}, \]
\[ = \mathcal{P}(q) \left[ PP \int_0^\infty ds \ \rho(s) + i\pi \rho(q^2) \right]. \]

(10)

Here \( PP \) means principal part, and we write:
\[ \langle 0 | J_a(0) | n \rangle \langle n | J_b^\dagger(0) | 0 \rangle = \mathcal{P}(q) f_a(q^2) f_b^* (q^2), \]

(11)

here \( | n \rangle \) is on-shell state with momentum \( q \); \( \mathcal{P}(q) \) corresponding to the tensor structure of \( \Pi(q) \); \( f_a(q^2) \) and \( f_b(q^2) \) are coupling constants about the currents and \( | n \rangle \). Define a factor:
\[ P = (-1)^{N+M}, \]

(12)

here \( N = 0, 1, 2 \) is the number of antihermitian currents; \( M = 0 \) \( (\neq 1) \) if \( \Pi(q) \) is even (odd) under exchange \( q \leftrightarrow -q \). Then we can write:
\[ \rho(s) = \sum_n \delta(s - m_n^2) \text{Re}[f_a(s)f_b^*(s)], \]

(13a)

for \( P = 1 \), and:
\[ \rho(s) = \sum_n \delta(s - m_n^2) i \text{Im}[f_a(s)f_b^*(s)], \]

(13b)
for $P = -1$.

Here the integration is implicitly for continue state $|n\rangle$. $\rho(s)$ become familiar when $J_a = J_b$; in this case, it is given by Eq. (13), and the real part reduces to $|f_a(s)|^2$. We give the derivation of Eq. (10 - 13) in Appendix A which based on Ref [12]. Note for $J_a \neq J_b$, $\rho(s)$ or $\text{Im}(s)$ may not always positive and even change the sign when $s$ varaint.

To extract the $1^{-+}$ vector state contribution, just note for currents given by Eq. (11-13), when couple to vector state $|V\rangle$ with momentum $q$, we can generally write:

$$
(0)J^{\mu}_a(0)|V\rangle = e^{\mu}f_a(q^2),
$$

$$
(0)J^{\mu,\nu}_a(0)|V\rangle = (q^\mu e^\nu \pm q^\nu e^\mu)f_b(q^2).
$$

Here $e^\mu$ is polarization vector; the $+ (-)$ refers to symmetry (anti-symmetry) tensor currents. Then we have:

$$
\Pi_{a,b}^{\mu,\nu}(q) = i \int d^4x e^{iqx} \langle 0|T\{J^a_{\mu}(x)J^b_{\nu}(0)\}|0\rangle,
$$

$$
= (q^{\mu}P^{\nu} \pm q^{\nu}P^{\mu})\Pi_{a,b}^{T,V}(q^2) + \ldots .
$$

Here $P^{\mu,\nu} = g^{\mu,\nu} - \frac{q^{\mu}q^{\nu}}{q^2}$, the ellipsis refers to terms irrelevant with vector states. The superscript $T$ and $V$ in $\Pi_{a,b}^{T,V}(q^2)$ is added to indicate the involved currents are tensor and vector currents, we use this notation later. Note at massless limit, for currents given by Eq. (11-13), only the correlation function with one vector current and one tensor current have nonvanishing perturbative diagrams. We only consider this type of correlation function, each fourquark current gives one. By Eq. (16) and (17), isolate the lowest resonance pole, we have:

$$
\frac{1}{\pi}\text{Im}\Pi_{a,b}(s) \simeq \delta(s-m^2)\text{Re}(f_a f_b^*) + \theta(s-s_0)\rho(s).
$$

Here $s_0$ is continue threshold, $P = 1$ in this case.

For $\langle GG\rangle$ contributions, some diagrams involve renormalization at two-loop level, which is far beyond the scope of this paper, so we just add counterterm (last diagram in Fig. 2) to cancel the non-local pole $log/\epsilon$. This diagram originates from the current:

$$
\bar{\Psi}(\nabla G + G\nabla)\Psi = i\eta(\nabla GP\Psi) + i\nabla DG\Psi,
$$

which involved by renormalization of fourquark current at two-loop level (at $O(g^3)$); here $\gamma$-matrices and Lorentz indices are suppressed; $\nabla^\mu$ is covariant derivative. The $q(\nabla G\Psi)$ gives the last diagram in Fig. 2. We use $\bar{\Psi}G\Psi$ as the configuration of counterterm. For $J^{\mu}_a$ and $J^{\mu}_b$, the current can be used as counterterm are not unique, but different choices may not cause any visible difference in mass prediction.

Note the overall sign of the tensor structure in Eq. (15) can be adjusted. We fix it by requiring the $\text{Im}\Pi_{a,b}(q^2)$ received from perturbative diagram is positive, so that $\rho(s) > 0$ when $s \to \infty$. We perform the OPE calculation to dimension-10 condensate. There have a large number of diagrams involved, the typical diagrams are shown in Fig 2. The OPE results, counterterms, and all diagrams are given in Appendix B. The calculation is quite cumbersome, we write a Mathematica package [15] to evaluate these diagrams. The results are obtained at $\overline{\text{MS}}$ scheme, the $\gamma^5$ is treated by BMHVscheme [17, 18].

C. Numerical analysis

To evaluate the mass of the lowest resonance, QCD sum rules commonly use the Borel (Laplace) transformation [11, 18, 19]. By Eq. (16) we have:

$$
\frac{1}{\pi}\int_0^{\infty} ds \ e^{-s\tau}\text{Im}\Pi_{a,b}(s) \simeq \text{Re}(f_a f_b^*)e^{-m^2\tau}
$$

$$
+ \int_0^{\infty} ds \ e^{-s\tau}\rho(s).
$$

And one can get the mass from the ratio of moments:

$$
R_n = \frac{M_{n+1}(\tau, s_0)}{M_n(\tau, s_0)} \simeq m^2,
$$

with:

$$
M_n(\tau, s_0) = \int_0^{s_0} ds \ s^n e^{-s\tau}\text{Im}\Pi_{a,b}(s).
$$

There have two free parameters $\tau$ and $s_0$ involved. The typical $s_0$ is around the mass square of the next resonance. The typical choice of $\tau$ is by requiring $M_0(\tau, s_0)/M_0(\tau, \infty) > 0.7$, to make the integral dominated by resonance pole; and the contribution of the highest dimensional condensate less than 10%, to make the OPE convergence. This gives Borel window $11 < \tau < 12$. It also commonly use stability criteria [20] that the mass prediction should be stable for $\tau$, and choose $s_0$ from one have $\tau$ stability to one achieve $s_0$ stability.

To evaluate the mass, we use the value of condensates given by Ref. [20], and $\Lambda = 0.353 GeV$ by Ref. [3]. For dimension-6, -8 and -10 condensates, the factorization deviation factors must be included:

$$
\langle \bar{q}q \rangle \rightarrow k_6\langle \bar{q}q \rangle^2; \quad \langle \bar{q}q \rangle\langle qGq \rangle \rightarrow k_8\langle \bar{q}q \rangle\langle qGq \rangle,
$$

$$
\langle \bar{q}Gq \rangle \rightarrow k_{10}\langle \bar{q}Gq \rangle^2.
$$
FIG. 3: Mass predictions and moments for different correlation functions by set $k_8 = k_{10} = 3.5$. The colors stand for different $\tau$ in Appendix. We then get the range of mass spanned by hybrid currents given by Eq. (22). We choose the average value in the Borel window; if Borel window not exist, we choose the value by stability criteria. We then get the figures for $s$ around $7-10 \text{GeV}^2$.

Here $k_8 \approx 3$ by Ref. [21]. The $k_8$ and $k_{10}$ are not clear, Ref. [22] indicate they around 2-5. We first choose $k_8 = k_{10} = 3.5$ to fix Borel window and $s_0$, then consider two extremes $k_8 = k_{10} = 2$ and $k_8 = k_{10} = 5$ to get conservative results. Note the $s$-quark can not couple to the hybrid currents given by Eq. (4) directly, so the $s$-quark contributions is samll. The results are insufficient to distinguish whether the fourquark content in resonance is $\eta \pi$ or $\eta' \pi$. We show the results for $k_8 = k_{10} = 3.5$ in Fig. 3 and give the figures for $k_8 = k_{10} = 2$ and $k_8 = k_{10} = 5$ in Appendix. The conservative range of mass are given by Table 4.

To obtain the mass, for specific $s_0$, if the Borel window exist, we choose the average value in the Borel window; if Borel window not exist, we choose the value by stability criteria. We then get the range of mass spanned by $s_0$. Note when fixing the Borel window, we choose the absolute values of the contributions of each condensate.

For $\Pi_{\eta(\eta')\pi,H}^V(q^2)$ and $\Pi_{\eta(\eta')\pi,H}^T(q^2)$, the Borel window and $s_0$ stability not exist. The $\tau$-stability is achieved when $s_0 \approx 5.5$ for $\Pi_{\eta(\eta')\pi,H}^V(q^2)$ and $\approx 4.5$ for $\Pi_{\eta(\eta')\pi,H}^T(q^2)$ (the curve for different $\tau$ intersect at these $s_0$). We choose $4 \leq s_0 \leq 6$ for these configurations. Note the $s_0$ stability at $s_0 \approx 2$ in Fig. 3(b) and $s_0 \approx 1$ in Fig. 3(c) are artificial because $s_0$ lower than the corresponding mass square.

When $s_0 = 4$, the $\Pi_{\eta(\eta')\pi,H}^T(q^2)$ have Borel window around 0.33-0.36; the $\Pi_{\eta(\eta')\pi,H}^V(q^2)$ has $\tau_1 \approx \tau_2 \approx 0.38$, we treat is as no Borel window. The $\tau$ stability achieved when $s_0 \approx 3$ (Fig. 3(e)). The $s_0$ stability exists, but starting with $s_0 \approx 5$, the $\tau$ stability becomes worse (Fig. 3(d)). We choose $3 \leq s_0 \leq 5$ for these configurations.

The $\Pi_{\eta(\eta')\pi,H}^T(q^2)$ have no dimension-8 and -10 condensate contributions at leading order, and dominated by dimension-6 condensate. The result is then sensitive with factorization deviation and hard to get a convincible result; we will not discuss it. It is interesting to compare with flux-tube model [23] that, the $1^-\text{hybrid}$ decay into $f_1\pi$ and $b_1\pi$, but not to $\rho\pi$. These facts imply the $1^-\text{hybrid}$ may be easier to couple to $b_1\pi$ than $\rho\pi$.

When $k_8 = k_{10} = 2$, the lower bound of mass prediction decrease to roughly 1.4GeV for $\pi_1(1600)$ and $\approx 1.2\text{GeV}$ for $\pi_1(1400)$. It still has a roughly 0.2GeV gap between them. So we conclude there have two $1^-\text{states}$. This result also implies the factorization deviation for dimension-8 and -10 condensates should be large.

Recall for each correlation function, we fix the sign of $\text{Im}\Pi_{\eta(\eta')\pi,h}$ by requiring the perturbative diagram give positive imaginary part. But for $\Pi_{\eta(\eta')\pi,H}^T(q^2)$ and $\Pi_{\eta(\eta')\pi,H}^V(q^2)$, the $\rho(s)$ is negative for $s \lesssim 8$. A close look find for $\tau$ around or large than Borel window, the moment $\mathcal{M}_0(\tau, s_0)$ is always negative, which indicates the contribution of resonance is negative. Such a situation is not strange, since for $\rho(s)$ given by Eq. (13) with $J_a \neq J_h$, the sign of contributions given by different states need not be same. Alternately, one can consider two currents mixing scenario. For example, $J_a \sim i\partial (J_A + \epsilon J_B)$ and $J_h \sim J_A - \epsilon J_B$, then:

$$\Pi_{a,b}(s) \sim \Pi_{A,A}(s) - \epsilon^2 \Pi_{B,B}(s).$$

(22)

If $\text{Im}\Pi_{A,B}(s) \gg \text{Im}\Pi_{A,A}(s)$ at large $s$, the $\text{Im}\Pi_{a,b}(s)$

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2 The $\rho_{\eta(\eta')}(\bar{\psi}\psi)^2$ in Ref. [23] already contain the deviation factor. But we find use fixed $\alpha_s$ make the results less stable, so we will not use this value.
TABLE I: Mass predictions by each correlation function in the unit of GeV, use notation in Eq. (15). The lower bound is obtained by set $s_0$ equal to the lower bound of $s_0$ and set $k_8 = k_{10} = 2$; the upper bound is obtained by set $s_0$ equal to the upper bound of $s_0$ and set $k_8 = k_{10} = 5$.

(a) Results can be interpret as $\pi_1(1600)$.

| $\Pi_{\pi^+\pi^0}^{V, T}(q^2)$ | $\Pi_{\eta^\prime\eta}^{V, T}(q^2)$ | $\Pi_{\eta_0\eta_1}^{V, T}(q^2)$ |
|-----------------|-----------------|-----------------|
| 1.45-1.77       | 1.45-1.77       | 1.36-1.67       |

(b) Results can be interpret as $\pi_1(1400)$.

| $\Pi_{\pi^+\eta_1}^{V, T}(q^2)$ | $\Pi_{\eta_0\eta_1}^{V, T}(q^2)$ | $\Pi_{\eta_0\eta_1}^{V, T}(q^2)$ |
|-----------------|-----------------|-----------------|
| 1.18-1.41       | 1.19-1.43       | 1.17-1.46       |

can change the sign. Here we act $\partial$ on $(J_A + \epsilon J_B)$ is because $J_{\mu\nu}^{(\pi^0)}$ and $J_{\mu\pi}$ are anti-hermitian and have mass dimension 6, while $J_H^{(\pi^0)}$ and $J_H^{(\pi^0)}$ are hermitian and have mass dimension 5. The $J_B$ may also give a resonance to $\text{Im}\Pi_{a,b}(s)$, but this state may be overwhelmed by the remaining states given by $J_A$. It is then delicate to evaluate the second resonance, we will not discuss it.

The $\Pi_{\eta_0\eta_1}^{V, T}(q^2)$ gives a result corresponding to $\pi_1(1400)$, but $\pi_1(1400)$ only decay into $\eta\pi^0$. Since $b_1\pi$ is just on the mass threshold of $\pi_1(1400)$, $b_1\pi$ channel may be forbidden or suppressed by little phase space in $\pi_1(1400)$'s decay.

It should be noted that in Eq. (16), we assume only one lowest resonance pole contribute, but there have two $1^{-+}$ states $\pi_1(1400)$ and $\pi_1(1600)$. It becomes important when two states give compatible contributions, while different currents may prefer to couple to a certain one. By results in Table II and Fig. 4, we think each case is dominated by one resonance.

III. CONCLUSION

From fourquark-hybrid correlation functions, we get the mass of $1^{-+}$ states shown in Table II and Fig. 4. There are mixed states of fourquark and hybrid, and can be identified as $\pi_1(1400)$ and $\pi_1(1600)$. Focus on the overlap range of the mass, Table II give mass 1.2-1.4GeV for $\pi_1(1400)$ and 1.45-1.67GeV for $\pi_1(1600)$, which agree with PDG[5] (1.35GeV for $\pi_1(1400)$ and 1.66GeV for $\pi_1(1600)$). But it should be noted that a recent coupled channel analysis of COMPASS data concluded that there has only one $1^{-+}$ state with mass 1.56GeV[24]. However, we can not judge whether there have one or two $1^{-+}$ states merely by the results obtained from QCD sum rules. More works are needed to solve the problems around $1^{-+}$ states.

The renormalization of fourquark current by Eq. (5-9) indicate the fourquark mix with hybrid is quite common, which is striking and make the fourquark and hybrid more subtle. For $1^{-+}$ state in this paper, the fourquark-hybrid correlation functions are dominated by dimension-6 condensates, which implies the mixing is highly nonperturbative. The mixing of $1^{-+}$ fourquark and hybrid explains why researches based on fourquark or hybrid alone are hard to agree with experiments. To better understand the $1^{-+}$ states, a detailed analysis about mixing is needed.

Appendix A: Dispersion relation in generalized situation

For correlation function about two currents, suppress the Lorentz indices, by definition:

$$\Pi(q) = i \int d^4xe^{iqx} \langle 0|T\{J_a(x) J_b^\dagger(0)\}|0\rangle,$$

$$= i \int d^4xe^{iqx} \left[\theta(x^0)\langle 0|J_a(x) J_b^\dagger(0)|0\rangle + \theta(-x^0)\langle 0|J_b^\dagger(0) J_a(x)|0\rangle\right].$$

(A1)

Insert a complete set of on-shell states:

$$\sum_n \int \frac{d^4p}{(2\pi)^4}\theta(p^0)\delta(p^2 - m_n^2)|n\rangle = 1.$$  

(A2)

By translation invariance $\langle 0|J_a(x)|n\rangle = e^{-ipx}\langle 0|J_a(0)|n\rangle$, here $p$ is momentum of state $|n\rangle$, and write $\langle 0|J_a(0)|n\rangle\langle n|J_b^\dagger(0)|0\rangle = P_n(p)A(p^2)$, here $P_n(p)$ only relevant with the tensor structure of $\Pi(q)$, $A(p^2)$ only relevant with $p^2$. Eq. (A1) then can be written as:

$$\Pi(q) = \sum_n \int d^4xe^{iqx} \int \frac{d^4p}{(2\pi)^4} P_n(p) \left[i\theta(x^0)\theta(p^0)\delta(p^2 - m_n^2)e^{-ipx}A(p^2) + i\theta(-x^0)\theta(p^0)\delta(p^2 - m_n^2)e^{ipx}A^1(p^2)\right].$$

(A3)
We then can write:

\[
[i\theta(x^0)\theta(p^0)\delta(p^2 - m_n^2) + (-1)^{N+M}i\theta(-x^0)\theta(-p^0)\delta(p^2 - m_n^2)]e^{-ipx} \text{Re}[A(p^2)]
\]  
\[
+ [i\theta(x^0)\theta(p^0)\delta(p^2 - m_n^2) - (-1)^{N+M}i\theta(-x^0)\theta(-p^0)\delta(p^2 - m_n^2)]e^{-ipx} i\text{Im}[A(p^2)].
\]  

(A4)

Here \( N = 0, 1, 2 \) is the number of antihermitian currents; \( \mathcal{P}_n(-p) = (-1)^M\mathcal{P}_n(p) \), or equivalently \( \Pi(-q) = (-1)^M\Pi(q) \). Write Eq. (A3) as \( \Pi(q) = \iint (\text{Re} + \text{Im}) \) briefly, under replacement \( x \rightarrow -x \), \( p \rightarrow -p \) and \( q \rightarrow -q \), it becomes:

\[
(-1)^M\Pi(q) = (-1)^{N+2M}\iint \text{Re} - (-1)^{N+2M}\iint \text{Im}.
\]  

(A5)

Thus for \( N + M = \) even, the \( \text{Im}[A(p^2)] \) term vanishes; for \( N + M = \) odd, the \( \text{Re}[A(p^2)] \) term vanishes. By:

\[
\int \frac{d^4p}{(2\pi)^4} \left( i\theta(x^0)\theta(p^0)\delta(p^2 - m_n^2) + i\theta(-x^0)\theta(-p^0)\delta(p^2 - m_n^2) \right)e^{-ipx} = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ipx}}{m_n^2 - p^2 - i\epsilon}.
\]  

(A6)

We then can write:

\[
\Pi(q) = \sum_n \frac{\mathcal{P}_n(q)}{m_n^2 - q^2 - i\epsilon} \left( \frac{1 + (-1)^{N+M}}{2} \text{Re}[A(q^2)] + \frac{1 - (-1)^{N+M}}{2} i\text{Im}[A(q^2)] \right),
\]

\[
= \int_0^\infty ds \sum_n \frac{\mathcal{P}_n(q)}{m_n^2 - q^2 - i\epsilon} \left( \frac{1 + (-1)^{N+M}}{2} \text{Re}[A(q^2)] + \frac{1 - (-1)^{N+M}}{2} i\text{Im}[A(q^2)] \right).
\]  

(A7)

Which then gives the dispersion relation for correlation function with two different currents. Especially for \( N + M = \) even, ignore the tensor structure, it gives:

\[
\frac{1}{\pi} \text{Im} [\Pi(q)] = \sum_n \delta(q^2 - m_n^2)\text{Re}[A(q^2)].
\]  

(A8)

Appendix B: Additional Table and Figures

![Diagrams for fourquark-hybrid correlation function, up to permutation of background gluon. The first diagram in second row use identity \( \langle \bar{\Psi}(x)G^{\mu\nu}(0)\Psi'(x) \rangle = -g^2\langle \bar{\Psi}\Psi \rangle^2/(36(D - 1)) (x^{\mu'}x^{\nu'} - x^{\nu'}x^{\mu'}) \). The diagrams in last row are relevant with renormalization at two-loop level; the last diagram refers to counterterm.](image-url)
TABLE II: OPE Results and counterterms. Here $m = m_u + m_d$. $\langle GG \rangle = g^2 \langle G^{\mu \nu} G_{\mu \nu} \rangle$, $\langle \bar{q} G q \rangle = g \langle \bar{q} T^{\mu \nu} G_{\mu \nu} \rangle$.

The values in each column give the factors of corresponding terms, e.g. $H_{T^\sigma T^\rho}^{V} (q^2) = -1/12960 \pi^5$.

The last four rows give the currents used as counterterms with $q = (u, d)^T$ and $\bar{q} = (\bar{u}, \bar{d})$.

For $H_{T^\sigma T^\rho}^{V} (q^2)$ and $H_{T^\sigma T^\rho}^{V} (q^2)$, the numerator in parenthesis are obtained by using counterterm in last row.

|                | $H_{V^\eta, T^\rho}^{T^\sigma} (q^2)$ | $H_{V^\eta, T^\rho}^{T^\sigma} (q^2)$ | $H_{V^\eta, T^\rho}^{T^\sigma} (q^2)$ | $H_{V^\eta, T^\sigma} (q^2)$ | $H_{V^\eta, T^\sigma} (q^2)$ |
|----------------|------------------------------------------|------------------------------------------|------------------------------------------|------------------------------------------|------------------------------------------|
| $\alpha_s q^6 \log (-q^2/\mu^2)$ | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ |
| $\alpha_s q^6 \log (-q^2/\mu^2) m(\bar{q} q)$ | $1/72 \pi$ | $1/72 \pi$ | $1/72 \pi$ | $1/72 \pi$ | $1/36 \pi$ |
| $\alpha_s q^6 \log (-q^2/\mu^2) m(\bar{q} q)$ | $-101/432 \pi^3$ | $-101/432 \pi^3$ | $23/216 \pi^3$ | $23/216 \pi^3$ | $-47/108 \pi^3$ |
| $\alpha_s q^6 \log (-q^2/\mu^2) (GG)$ | $247/995328 \pi^5$ | $281/995328 \pi^5$ | $-1/497664 \pi^5$ | $-1/497664 \pi^5$ | $113/248832 \pi^5$ |
| $\alpha_s q^6 \log (-q^2/\mu^2) (GG)$ | $-1/12960 \pi^5$ | $-1/12960 \pi^5$ | $-1/12960 \pi^5$ | $-1/12960 \pi^5$ | $-1/12960 \pi^5$ |

$\partial^\mu (\bar{q} G^\alpha \gamma_\mu q) + \{ \mu \leftrightarrow \nu \}$

$\frac{-7g^2}{\epsilon 3456 \pi^5}$

$\frac{161g^2}{\epsilon 13824 \pi^4}$

$\partial^\mu (\bar{q} G^\alpha \gamma_\mu q) - \{ \mu \leftrightarrow \nu \}$

$\frac{17g^2}{\epsilon 6912 \pi^4}$

$\frac{-g^2}{\epsilon 1728 \pi^4}$

$\partial^\alpha (\bar{q} G^{\alpha \beta} \sigma_{\alpha \beta} q)$

$\frac{g^2}{\epsilon 864 \pi^4}$

$\frac{29g^2}{\epsilon 1728 \pi^4}$

$\frac{23g^2}{\epsilon 864 \pi^4}$

$\partial^\alpha (\bar{q} G^{\alpha \beta} \sigma_{\alpha \beta} q)$

$\frac{g^2}{\epsilon 864 \pi^4}$

$\frac{29g^2}{\epsilon 1728 \pi^4}$

$\frac{23g^2}{\epsilon 864 \pi^4}$

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FIG. 5: Mass predictions for different correlation functions by set \( k_8 = k_{10} = 2 \). The colors stand fourquark configurations: red for \( \eta \pi \), green for \( \eta' \pi \), blue for \( b_1 \pi \). (a): Mass versus \( \tau \) for \( \Pi^{V, T}_{\eta(\eta') \pi, H} (q^2) \) and \( \Pi^{T, V}_{b_1 \pi, H} (q^2) \); (b): mass versus \( s_0 \) for \( \Pi^{V, T}_{\eta(\eta') \pi, H} (q^2) \) and \( \Pi^{T, V}_{b_1 \pi, H} (q^2) \); (c): mass versus \( \tau \) for \( \Pi^{T, V}_{V, T \eta(\eta')} \pi, H(q^2) \) and \( \Pi^{V, T}_{b_1 \pi, H} (q^2) \); (d): Mass versus \( s_0 \) for \( \Pi^{V, T}_{\eta(\eta') \pi, H} (q^2) \) and \( \Pi^{V, T}_{b_1 \pi, H} (q^2) \).

FIG. 6: Mass predictions for different correlation functions by set \( k_8 = k_{10} = 5 \). The colors stand fourquark configurations: red for \( \eta \pi \), green for \( \eta' \pi \), blue for \( b_1 \pi \). (a): Mass versus \( \tau \) for \( \Pi^{V, T}_{\eta(\eta') \pi, H} (q^2) \) and \( \Pi^{T, V}_{b_1 \pi, H} (q^2) \); (b): mass versus \( s_0 \) for \( \Pi^{V, T}_{\eta(\eta') \pi, H} (q^2) \) and \( \Pi^{T, V}_{b_1 \pi, H} (q^2) \); (c): mass versus \( \tau \) for \( \Pi^{T, V}_{V, T \eta(\eta')} \pi, H(q^2) \) and \( \Pi^{V, T}_{b_1 \pi, H} (q^2) \); (d): Mass versus \( s_0 \) for \( \Pi^{V, T}_{\eta(\eta') \pi, H} (q^2) \) and \( \Pi^{V, T}_{b_1 \pi, H} (q^2) \).

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