Weyl’s Character Formula for $SU(3)$ - A Generating Function Approach

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(March 1996)

Abstract

Using a generating function for the Wigner’s $D$-matrix elements of $SU(3)$

Weyl’s character formula for $SU(3)$ is derived using Schwinger’s technique.

PACS number(s):

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1. INTRODUCTION

In a previous paper we had set up a calculus to do computations on the group $SU(3)$. This is a calculus which is similar to the one developed for $SU(2)$ by Schwinger [2], Bargmann [3]. With the help of this calculus we were able to solve some important problems in group theory related to $SU(3)$. The basic aim of this paper is to apply this calculus to obtain Weyl's character formula for the group $SU(3)$ and thus show that, using our calculus, doing computations on $SU(3)$ is as easy as dealing with $SU(2)$.

A formula for the characters of unitary groups was derived by Weyl [4] using integration on group manifolds. It was also obtained by Freudenthal [5] using a purely Lie algebraic method. Boerner [6] derives the same result by examining Young Frames. (See also the books of Littlewood and Hamermesh [7,8]). This formula is the starting point for computing the dimensions of the Irreducible Representations (IRs) and also for arriving at the branching theorems for the IRs( [10,11]). In this paper Weyl’s formula, for the group $SU(3)$, is derived, for the first time, from a generating function for the matrix elements of $SU(3)$. The generating function method was successfully applied by Schwinger [2] for the computation of Weyl’s character formula for the group $SU(2)$ long back. In essence we are extending his method to $SU(3)$.

The plan of the paper is as follows. In section 1 we collect some results from our previous work on the IRs of $SU(3)$. The next section, that is section 2, is devoted to the computation of the normalizations of the basis states. We then derive Weyl’s character formula for $SU(3)$ using Schwinger’s method in section 3. Section 4 is devoted to a discussion of the results of this paper. We reproduce Schwinger’s original derivation of Weyl’s formula for the characters of $SU(2)$ in the appendix.
2. OVERVIEW OF OUR PREVIOUS RESULTS

In this section we briefly review the results that we need on the group $SU(3)$. Some of these results were obtained by us in a previous paper \[1\].

$SU(3)$ is the group of $3 \times 3$ unitary unimodular matrices $A$ with complex coefficients. It is a group of 8 real parameters. The matrix elements satisfy the following conditions

\[ \begin{align*}
A &= (a_{ij}), \\
A^\dagger A &= I, \\
AA^\dagger &= I, \quad \text{where $I$ is the identity matrix} \\
\det(A) &= 1.
\end{align*} \tag{2.1} \]

A. Parametrization

One well known parametrization of $SU(3)$ is due to Murnaghan \[12\], see also \[13,15,16,18\].

In this we write a typical element of $SU(3)$ as:

\[ D(\delta_1, \delta_2, \phi_3)U_{23}(\phi_2, \sigma_3)U_{12}(\theta_1, \sigma_2)U_{13}(\phi_1, \sigma_1), \tag{2.2} \]

with the condition $\phi_3 = -(\delta_1 + \delta_2)$. Here $D$ is a diagonal matrix whose elements are $\exp(i\delta_1)$, $\exp(i\delta_2)$, $\exp(i\phi_3)$, and $U_{pq}(\phi, \sigma)$ is a $3 \times 3$ unitary unimodular matrix which for instance in the case $p = 1, q = 2$ has the form

\[ \begin{pmatrix}
\cos\phi & -\sin\phi \exp(-i\sigma) & 0 \\
\sin\phi \exp(i\sigma) & \cos\phi & 0 \\
0 & 0 & 1
\end{pmatrix} \tag{2.3} \]

The 3 parameters $\phi_1, \phi_2, \phi_3$ are longitudinal angles whose range is $-\pi \leq \phi_i \leq \pi$, and the remaining 6 parameters are latitude angles whose range is $\frac{1}{2}\pi \leq \sigma_i \leq \frac{3}{2}\pi$.

Now the transformations $U_{23}$ and $U_{13}$ can be changed into transformations of the type $U_{12}$ whose matrix elements are known, by the following device

\[ \begin{align*}
U_{13}(\phi_1, \sigma_1) &= (2, 3)U_{12}(\phi_1, \sigma_1)(2, 3), \\
U_{23}(\phi_2, \sigma_3) &= (1, 2)(2, 3)U_{12}(\phi_2, \sigma_3)(2, 3)(1, 2), \tag{2.4}
\end{align*} \]
where \((1, 2)\) and \((2, 3)\) are the transposition matrices

\[
(1, 2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2, 3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

(2.5)

In this way the expression for an element of the \(SU(3)\) group becomes

\[
D(\delta_1, \delta_2, \phi_3)(1, 2)(2, 3) U_{12}(\phi_2, \sigma_3)(2, 3)(1, 2) U_{12}(\theta_1, \sigma_2)(2, 3) U_{12}(\phi_1, \sigma_1)(2, 3) .
\]

(2.6)

**B. Lie algebra Generators**

Later in order to calculate the normalizations of our un-normalized basis states, we have to first to obtain the representation of the generators of the Lie algebra of \(SU(3)\) as differential operators. For this purpose and also for parametrizing the equivalence classes of \(SU(3)\) we take our \(SU(3)\) generators to be

\[
\pi^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \eta^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \pi^- = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \pi^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
K^- = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K^0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{K}^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]

(2.7)

**C. Irreducible Representations.**

Our parametrization provides us with a defining irreducible representation \(3\) of \(SU(3)\) acting on a 3 dimensional complex vector space spanned by the triplet \(z_1, z_2, z_3\) of complex variables. The hermitian adjoint of the above matrix gives us another defining but inequivalent irreducible representaion \(3^*\) of \(SU(3)\) acting on the triplet \(w_1, w_2, w_3\) of complex variables spanning another 3 dimensional complex vector space. Tensors constructed
out of these two 3 dimensional representations span an infinite dimensional complex vector space.

D. The Constraint

If we impose the constraint

\[ z_1 w_1 + z_2 w_2 + z_3 w_3 = 0, \]  

(2.8)
on this space we obtain an infinite dimensional complex vector space in which each irreducible representation of SU(3) occurs once and only once. Such a space is called a model space for SU(3). Further if we solve the constraint \( z_1 w_1 + z_2 w_2 + z_3 w_3 = 0 \) and eliminate one of the variables, say \( w_3 \), in terms of the other five variables \( z_1, z_2, z_3, w_1, w_2 \) we can write a generating function to generate all the basis states of all the IRs of SU(3). This generating function is computationally a very convenient realization of the basis of the model space of SU(3). Moreover we can define a scalar product on this space by choosing one of the variables, say \( z_3 \), to be a planar rotor \( \exp(i\theta) \). Thus the model space for SU(3) is now a Hilbert space with this ('auxiliary') scalar product between the basis states. The above construction was carried out in detail in a previous paper by us \[I\]. For easy accessibility we give a self-contained summary of some results which are relevant for us here.

E. Labels for the basis states.

(i). Gelfand-Zetlein labels

Normalized basis vectors are denoted by, \(|M, N; P, Q, R, S, U, V >\). All labels are non-negative integers. All Irreducible Representations(IRs) are uniquely labeled by \((M, N)\). For a given IR \((M, N)\), labels \((P, Q, R, S, U, V)\) take all non-negative integral values subject to the constraints:

\[ R + U = M, \quad S + V = N, \quad P + Q = R + S. \]  

(2.9)
The allowed values can be prescribed easily: $R$ takes all values from 0 to $M$, and $S$ from 0 to $N$. For a given $R$ and $S$, $Q$ takes all values from 0 to $R + S$.

Gelfand-Zetlein basis states are represented by the triangular patterns or arrays

\[
\begin{pmatrix}
  m_{13} & m_{23} & 0 \\
  m_{12} & m_{22} \\
  m_{11}
\end{pmatrix}
\]  \hspace{1cm} (2.10)

The entries are non-negative integers satisfying the betweenness constraints

\[
\begin{align*}
(i) & \quad m_{13} \geq m_{12} \\
(ii) & \quad m_{12} \geq m_{23} \\
(iii) & \quad m_{23} \geq m_{22} \\
(iv) & \quad m_{22} \geq 0 \\
(v) & \quad m_{12} \geq m_{11} \\
(vi) & \quad m_{11} \geq m_{22}
\end{align*}
\]  \hspace{1cm} (2.11)

In terms of these labels our labels $P, Q, R, S, U, V$ can be expressed as

\[
\begin{align*}
M &= m_{13} - m_{23} , \\
N &= m_{23} , \\
P &= \frac{1}{2}(m_{11} + m_{23}) - \frac{1}{2}|m_{11} - m_{23}| - m_{22} + |m_{11} - m_{23}|\epsilon(m_{11} - m_{23}) , \\
Q &= m_{12} - \frac{1}{2}(m_{11} + m_{23}) - \frac{1}{2}|m_{11} - m_{23}| + |m_{11} - m_{23}|\epsilon(-(m_{11} - m_{23})) , \\
P + Q &= m_{12} - m_{22} , \\
R &= M - m_{22} , \\
S &= m_{12} - M , \\
U &= m_{22} , \\
V &= m_{13} - m_{12} ,
\end{align*}
\]  \hspace{1cm} (2.12)

(ii). Young - Weyl Tableau labels.

The Gelfand-Zetlein basis may also be represented by the Young - Weyl tableau with two rows of boxes labelled by 1’s, 2’s, and 3’s as follows
(iii). Quark model labels.

The relation between the above Gelfand-Zetlein labels and the Quark Model labels is as given below.

\[ 2I = P + Q = R + S , \]
\[ 2I_3 = P - Q , \]
\[ Y = \frac{1}{3} (M - N) + V - U , \]
\[ = \frac{2}{3} (N - M) - (S - R) . \]  

(2.14)

F. Explicit realization of the basis states

(i). Generating function for the basis states of \( SU(3) \)

The generating function for the basis states of the IR’s of \( SU(3) \) can be written as

\[ g(p, q, r, s, u, v) = \exp(r(pz_1 + qz_2) + s(pw_2 - qw_1) + uz_3 + vw_3) . \]  

(2.15)

The coefficient of the monomial \( p^P q^Q r^R s^S u^U v^V \) in the Taylor expansion of Eq.(2.13), after eliminating \( w_3 \) using Eq.(2.8), in terms of these monomials gives the basis state of \( SU(3) \) labelled by the quantum numbers \( P, Q, R, S, U, V \).

(ii). Formal generating function for the basis states of \( SU(3) \)

The generating function Eq.(2.13) can be written formally as

\[ g = \sum_{P,Q,R,S,U,V} p^P q^Q r^R s^S u^U v^V |PQRSUV⟩ , \]  

(2.16)

where \( |PQRSUV⟩ \) is an unnormalized basis state of \( SU(3) \) labelled by the quantum numbers \( P, Q, R, S, T, U, V \).
Note that the constraint \( P + Q = R + S \) is automatically satisfied in the formal as well as explicit Taylor expansion of the generating function.

(iii). **Generalized generating function for the basis states of \( SU(3) \)**

It is useful, while computing the normalizations (see below) of the basis states, to write the above generating function in the following form

\[
\mathcal{G}(p, q, r, s, u, v) = \exp(rp z_1 + rq z_2 + sp w_2 + sq w_1 + uz_3 + vw_3).
\] (2.17)

In the above generalized generating function (2.17) the following notation holds.

\[
r_p = rp, \quad r_q = rq, \quad s_p = sp, \quad s_q = -sq.
\] (2.18)

**G. Notation**

Hereafter, for simplicity in notation we assume all variables other than the \( z^i_j \) and \( w^i_j \) where \( i, j = 1, 2, 3 \) are real eventhough at some places we have treated them as complex variables. Our results are valid even without this restriction as we are interested only in the coefficients of the monomials in these real variables rather than in the monomials themselves.

**H. ’Auxiliary’ scalar product for the basis states.**

The scalar product to be defined in this section is ’auxiliary’ in the sense that it does not give us the ’true’ normalizations of the basis astes of \( SU(3) \). However it is computationally very convenient for us as all computations with this scalar product get reduced to simple Gaussian integrations and the ’true’ normalizations themselves can then be got quite easily.

(i). **Scalar product between generating functions of basis states of \( SU(3) \)**

We define the scalar product between any two basis states in terms of the scalar product between the corresponding generating functions as follows:

\[
(g', g) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \int \frac{d^2 z_1}{\pi^2} \frac{d^2 z_2}{\pi^2} \frac{d^2 w_1}{\pi^2} \frac{d^2 w_2}{\pi^2} \exp(-z_1 z_2 - w_1 w_2)
\]
\[ \times \exp((r'(p'z_1 + q'z_2) + s'(p'w_2 - q'w_1) - \frac{-v'}{z_3}(z_1w_1 + z_2w_2) + u'z_3) \]

\[ \times \exp((r(pz_1 + qz_2) + s(pw_2 - qw_1) - \frac{-v}{z_3}(z_1w_1 + z_2w_2) + uz_3), \]

\[ = (1 - v'v)^{-2} \left( \sum_{n=0}^{\infty} \frac{(u'u)^n}{(n!)^2} \right) \exp \left[ (1 - v'v)^{-1}(p'q + q'r + s's) \right]. \tag{2.19} \]

(ii). Modified scalar product

In order to facilitate comparison with Schwinger’s computation, of the Weyl’s character formula for \( SU(2) \), and to be able to see, a little more clearly, the possibilities of generalizations to higher groups we redefine the \( \theta \) part of the scalar product and write the above Eq.(2.19) as

\[ (g', g) = (1 - v'v)^{-2} \exp \left[ \frac{(p'q + q'r + s's) + u'u}{(1 - v'v)} \right]. \tag{2.20} \]

(iii). Choice of the variable \( z_3 \)

To obtain the Eqs.(2.19, 2.20) we have made the choice

\[ z_3 = \exp(i\theta). \tag{2.21} \]

The choice, Eq.(2.21), makes our basis states for \( SU(3) \) depened on the variables \( z_1, z_2, w_1, w_2 \) and \( \theta \).

(iv). Scalar product between the generalized generating functions of the basis states of \( SU(3) \)

For the generalized generating function the scalar product becomes

\[ (G', G) = (1 - v'v)^{-2} \exp \left[ (1 - v'v)^{-1}(r_p' r_p + r_q' r_q + s_p' s_p + s_q' s_q) \right] \]

\[ \times \left[ \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left( u' - v \frac{(r_p's_p' + r_q's_q')}{(1 - v'v)} \right)^n \cdot \left( u - v \frac{r_p s_q + r_q s_p}{(1 - v'v)} \right)^n \right], \tag{2.22} \]

and as in Eq.(2.18)
\( r_p = r_p, \quad r_q = r_q, \quad s_p = s_p, \quad s_q = -s_q, \)
\( r'_p = r'_p, \quad r'_q = r'_q, \quad s'_p = s'_p, \quad s'_q = -s'_q. \)  

(v). Modified scalar product between the generalized generating functions of the basis states of \( SU(3) \)

With our modified scalar product Eq.(2.22) reads as
\[
(G', G'') = (1 - v'v'')^{-2} \exp \left[ \frac{(r_p'r_p'' + r_q'r_q'' + s_p's_p'' + s_q's_q'')}{(1 - v'v'')} \right]
+ \left( u' - v'' \frac{(r_p's_q' + r_q's_p')}{(1 - v'v'')} \right) \left( u'' - v' \frac{(r_p''s_q' + r_q''s_p')}{(1 - v'v'')} \right).
\]  

The notation Eq.(2.23) holds good in this equation also but only for the singly primed and the unprimed variables (hidden in doubly primed ones).

3. COMPUTAION OF THE NORMALIZATIONS

The normalizations using the scalar product Eq.(2.19) were computed by us in our paper [1]. But since we have now changed, the \( \theta \) part of the, scalar product we have to compute these normalizations once again. This is done, as before, by the requirement that the representation matrix be unitary in each irreducible representation.

Now let
\( E = \) The set of quantum numbers used in the basis,
\( |E\rangle = \) Unnormalized basis state,
\( |E\rangle = \) Normalized basis state,
\( (E'|E) = M(E)\delta_{E'E}, \)
\( = \) Scalar product between unnormalized basis states with respect to the auxiliary scalar product,
\( T = \) Generator of \( SU(3) \),
\( (E'|T|E) = \) Matrix element of \( T \) between unnormalized basis states \( E', E \) with respect to the 'auxiliary' scalar product. \hfill (3.25)

The symbol \( \parallel \) stands for the scalar product with respect to the 'auxiliary' measure.

Then assuming that
\[
|E\rangle = N\frac{1}{2}(E)|E\rangle ,
\hfill (3.26)
\]

it can be shown that,
\[
\frac{|N(E)|}{N(E')} = \frac{(E'|T|E)M(E)}{(E\parallel T^*|E')^*M(E')}, \hfill (3.27)
\]

where \( T^* \) is the adjoint of \( T \) and \( (\parallel)^* \) is the complex conjugate of \( (\parallel) \).

Thus we can fix normalizations using an 'auxiliary' scalar product which allows explicit computations even though it is not the 'true' scalar product.

(i). Generators of \( SU(3) \).

The 'auxiliary' matrix elements of \( SU(3) \) are to be computed by the action of the generators of \( SU(3) \) on the generating function of the unnormalized basis states. For this purpose the generators of \( SU(3) \) are needed. The generators of \( SU(3) \) take a particularly convenient form when realized in terms of differential operators in the variables \( r_p, r_q, u, s_p, s_q, v \). These operators act on the unnormalized basis states generated by the generalized generating function \( G \) in Eq.(2.17). These generators are listed below.

\[
\begin{align*}
\hat{\pi}^0 &= r_p \frac{\partial}{\partial r_p} - r_q \frac{\partial}{\partial r_q} - s_q \frac{\partial}{\partial s_q} + s_p \frac{\partial}{\partial s_p}, \\
\hat{\pi}^- &= r_p \frac{\partial}{\partial r_q} - s_p \frac{\partial}{\partial s_q}, \\
\hat{\pi}^+ &= r_q \frac{\partial}{\partial r_p} - s_q \frac{\partial}{\partial s_p}, \\
\hat{K}^- &= r_p \frac{\partial}{\partial u} - v \frac{\partial}{\partial s_q}, \\
\hat{K}^+ &= u \frac{\partial}{\partial r_p} - s_q \frac{\partial}{\partial u}, \\
\hat{K}^0 &= r_q \frac{\partial}{\partial u} - v \frac{\partial}{\partial s_p}.
\end{align*}
\]
\[
\hat{K}^0 = u \frac{\partial}{\partial r_q} - s_p \frac{\partial}{\partial v}, \\
\hat{\eta} = r_p \frac{\partial}{\partial r_p} + r_q \frac{\partial}{\partial r_q} + s_q \frac{\partial}{\partial s_q} + s_p \frac{\partial}{\partial s_p} - 2u \frac{\partial}{\partial u} + 2v \frac{\partial}{\partial v}.
\]

(3.28)

(i). 'Auxiliary’ Matrix Elements of Generators of \(SU(3)\).

Consider \(\hat{\pi}^-\) as given in Eq. (3.28). We have

\[
\left(g', \hat{\pi}^-=g\right) = pq \frac{(\bar{r}'r + \bar{s}'s)}{(1-\bar{v}'v)} \left(g', g\right).
\]

This gives us

\[
(P, Q + 1, R, S, U, V|\hat{\pi}^-|P + 1, Q, , R, S, U, V) = M_3(P, Q, R, S, U, V)
\]

Similarly

\[
(P + 1, Q, R, S, U, V|\hat{\pi}^+|P + 1, R, S, U, V) = M_3(P, Q, R, S, U, V)
\]

(3.30)

We now compute the relative normalizations implied by \(\hat{K}^\pm\). As we said above, to calculate, \((g', \hat{K}^-g)\) we use the generalized generating function

\[
(g', \hat{K}^-g) = \left(r_p \frac{\partial}{\partial u} - v \frac{\partial}{\partial s_q}\right) \left(G', G\right),
\]

where the vertical line at the end of this equation means that after applying differential operator on \((G', G)\), we need to set the values Eq. (2.23). For instance,

\[
(r_ps_q + r_q s_p)| = 0,
\]

\[
(\bar{r}'s'_q + \bar{s}'r'_p)| = 0.
\]

(3.32)

We get

\[
(g', \hat{K}^-g) = \frac{1}{(1-\bar{v}'v)^2} \exp \left( \frac{(pp' + qq')(rr' + ss')}{1-\bar{v}'v} + \bar{u}'u \right) \times \left( rpu' + rpu'v\bar{v}' + v \frac{s'd'}{(1-\bar{v}'v)} \right).
\]

(3.33)
Let
\[ M_1(P, Q, R, S, U, V) = \text{coefficient of } p^P q^q r^r s^s u^u v^v \times p^P q^q r^r s^s u^u v^v \]
in the expansion of \( \frac{(g', g)}{(1 - v'v)} \).

(3.34)

Assume similar meanings to the coefficients \( M, M_2, M_4, M_5, \) and \( M_6 \). (See Table below.)

| \( M(\cdots) \) | \( \exp[(1-v')^{-1}(p'p+q'q)(r'r+s's)+u'u] \) | (2I+1+V)! | \( \frac{1}{P!Q!R!S!U!V!(2I+1)} \) |
|-----------------|-----------------------------------|-----------------------------|
| \( M_1(\cdots) \) | \( \exp[(1-v')^{-1}(p'p+q'q)(r'r+s's)+u'u] \) | (2I+1+V)! | \( \frac{1}{P!Q!R!S!U!V!(2I+1)} \) |
| \( M_2(\cdots) \) | \( \exp[(1-v')^{-1}(p'p+q'q)(r'r+s's)+u'u] \) | (2I+2+V)! | \( \frac{1}{P!Q!R!S!U!V!(2I+1)(2I+2)} \) |
| \( M_3(\cdots) \) | \( (r'r+s's) \times \exp[(1-v')^{-1}(p'p+q'q)(r'r+s's)+u'u] \) | (2I+3+V)! | \( \frac{1}{P!Q!R!S!U!V!(2I+1)(2I+2)(2I+3)} \) |
| \( M_4(\cdots) \) | \( (pp' + qq')(r'r' + ss') \times \exp[(1-v')^{-1}(p'p+q'q)(r'r+s's)+u'u] \) | Not needed. |
| \( M_5(\cdots) \) | \( \exp[(1-v')^{-1}(p'p+q'q)(r'r+s's)] \) | (2I+1+V)! | \( \frac{1}{P!Q!R!S!U!V!(2I+1)(2I+2)} \) |
| \( M_6(\cdots) \) | \( \exp[(1-v')^{-1}(p'p+q'q)(r'r+s's)+u'u] \) | (2I+2+V)! | \( \frac{1}{P!Q!R!S!U!V!(2I+1)(2I+2)} \) |

where in the above table in the first column \( \cdots = (P, Q, R, S, U, V) \).

Matching coefficients of like powers we get (see Table above.),

\[
(P, Q + 1, R, S + 1, U, V \| \hat{K}^-|P, Q, R, S, U, V + 1) = M_1(, P, Q, R, S, U, V),
\]

\[
(P, Q, R, S, U + 1, V \| \hat{K}^-|P + 1, Q, R + 1, S, U, V) = M_5(P, Q, R, S, U, V) + M_6(P, Q, R, S, U, V - 1).
\]

(3.35)

Similarly,

\[
(P + 1, Q, R + 1, S, U + 1, V \| \hat{K}^+|P, Q, R, S, U + 1, V) = M_1(P, Q, R, S, U, V),
\]

\[
(P, Q, R, S, U, V + 1 \| \hat{K}^+|P, Q + 1, R, S + 1, U, V) = M_6(, P, Q, R, S, U - 1, V) + 2M_1(P, Q, R, S, U, V) + M_4(P, Q, R, S, U, V).
\]

(3.36)

Likewise
\[ (P, Q, R, S, U + 1, V) \parallel \hat{K}^0 | P, Q + 1, R + 1, S, U, V \rangle = M_5(P, Q, R, S, U, V) \]
\[ + M_6(P, Q, R, S, U, V - 1), \]
\[ (P + 1, Q, R, S + 1, U, V \parallel \hat{K}^0 | P, Q, R, S, U, V + 1 \rangle = -M_1(P, Q, R, S, U, V). \]
\[ \text{(3.37)} \]

and

\[ (P, Q + 1, R + 1, S, U, V) \parallel \hat{K}^0 | P, Q, R, S, U + 1, V \rangle = M_1(P, Q, R, S, U, V), \]
\[ (P, R, S, U, V + 1 \parallel \hat{K}^0 | P + 1, Q, R + 1, S, U, V \rangle = -M_1(P, Q, R, S, U - 1, V) \]
\[ - 2M_1(P, Q, R, S, U, V) \]
\[ - M_4(P, Q, R, S, U, V). \]
\[ \text{(3.38)} \]

A. New Normalizations

In this subsection we calculate the normalizations of unnormnalized basis states by requiring that the matrix elements of the generators of \( SU(3) \) be hermitian adjoints with respect to each other in pairs.

Accordingly, by using Eq. (3.27) together with Eq. (3.35) and Eq. (3.36) we get

\[
\left| \frac{N(P + 1, Q, R + 1, S, U, V)}{N(P, Q, R, S, U + 1, V)} \right| = \frac{(V + 2I + 2)(U + 1)(2I + 1)}{(P + 1)(R + 1)(2I + 2)}, \]
\[ \text{(3.39)} \]

\[
\left| \frac{N(P, Q + 1, R, S + 1, U, V)}{N(P, Q, R, S, U + 1, V)} \right| = \frac{(U + 2I + 2)(V + 1)(2I + 1)}{(Q + 1)(S + 1)(2I + 2)}, \]
\[ \text{(3.40)} \]

and by applying Eq. (3.27), this time, to Eq. (3.37) and Eq. (3.38) we get,

\[
\left| \frac{N(P, Q + 1, R + 1, S, U, V)}{N(P, Q, R, S, U + 1, V)} \right| = \frac{(V + 2I + 2)(U + 1)(2I + 1)}{(Q + 1)(R + 1)(2I + 2)}, \]
\[ \text{(3.41)} \]

\[
\left| \frac{N(P + 1, Q, R, S + 1, U, V)}{N(P, Q, R, S, U + 1, V)} \right| = \frac{(U + 2I + 2)(V + 1)(2I + 1)}{(P + 1)(S + 1)(2I + 2)}. \]
\[ \text{(3.42)} \]
The normalization constant \( N(P,Q,R,S,U,V) \) is uniquely fixed by the above constraints Eqs.(3.39-3.42). The solution is given by

\[
N(P,Q,R,S,U,V) = \frac{(U + 2I + 1)!(V + 2I + 1)!}{P!Q!R!S!U!V!(2I + 1)}.
\] (3.43)

From the above Eq.(3.43) we see that the normalization for the basis states is the same as that obtained in our earlier work [1] though our present inner product is slightly different from the one used in that reference.

Below we briefly summarize the results of this section.

(i). 'Auxiliary' normalizations of unnormalized basis states

The scalar product between two unnormalized basis states, computed using our 'auxiliary scalar product, is given by

\[
M(PQRSUV) \equiv (PQRSUV|PQRSUV),
= \frac{(V + P + Q + 1)!}{P!Q!R!S!U!V!(P + Q + 1)}. \] (3.44)

(ii). Scalar product between the unnormalized and normalized basis states

The scalar product, computed using our 'auxiliary' scalar product, between an unnormalized basis state and a normalized one is denoted by \( (PQRSUV\|PQRSUV > \) and is given below

\[
(PQRSUV\|PQRSUV >= N^{-1/2}(PQRSUV) \times M(PQRSUV). \] (3.45)

(iii). 'True' normalizations of the basis states

We call the ratio of the 'auxiliary' norm of the unnormalized basis state represented by \(|PQRSUV\rangle\) and the scalar product of the normalized Gelfand-Zeitlin state, represented by \(|PQRSUV >\), with the unnormalized basis state as 'true' normalization. It is given by

\[
N^{1/2}(PQRSUV) \equiv \frac{(PQRSUV\|PQRSUV)}{(PQRSUV\|PQRSUV >},
= (\frac{(U + P + Q + 1)!(V + P + Q + 1)!}{P!Q!R!S!U!V!(P + Q + 1)})^{1/2}. \] (3.46)
Before leaving this section we just note that the new normalizations computed by using our slightly different scalar product for the basis states are not much different from the ones computed using our previous scalar product.

4. GENERATING FUNCTION FOR THE WIGNER’S $D$-MATRIX ELEMENTS OF $SU(3)$.

In this section we first derive, very briefly, a generating function for the Wigner’s $D$-matrix elements of $SU(3)$. This generating function will be our starting point in the next section.

We know from Eq.(2.16) that
\[
g(p, q, r, s, u, v, z_1, z_2, w_1, w_2) = \sum_{P,Q,R,S,U,V} p^P q^Q r^R s^S u^U v^V |PQRSUV\rangle ,
\]
(4.47)
where $|PQRSUV\rangle$ is an unnormalized basis state in the IR labeled by the two positive integers $(M = R + U, N = S + V)$.

We know from Eq.(3.43),
\[
|PQRSUV\rangle = N^{(1/2)} (PQRSUV)|PQRSUV\rangle ,
\]
(4.48)
where $2I = P + Q$ and $|PQRSUV\rangle$ is a normalized basis state.

Therefore
\[
g = \sum_{PQRSUV} \left( \frac{(U + 2I + 1)!(V + 2I + 1)!}{P!Q!R!S!U!V!(2I + 1)!} \right)^{(1/2)} p^P q^Q r^R s^S u^U v^V |PQRSUV\rangle .
\]
(4.49)

Now,
\[
Ag(p, q, ...) = \sum_{PQRSUV} \sum_{P'Q'R'S'U'V'} \left( \frac{(U + 2I + 1)!(V + 2I + 1)!}{P!Q!R!S!U!V!(2I + 1)!} \right)^{(1/2)}
\]
\[
\times D_{PQRSUV, P'Q'R'S'U'V'}^{(M=R+U, N=S+V)} \times p^P q^{P'} r^R s^S u^U v^V \times |PQRSUV\rangle .
\]
(4.50)

To get a generating function for the matrix elements alone we have to take the inner product of this transformed generating function with the generating function for the basis states.
Thus,

\[
\begin{align*}
(g(p', q', r', s', u', v'; z_1, z_2, z_3, w_1, w_2), & \quad Ag(p, q, r, s, u, v; z_1, z_2, z_3, w_1, w_2)) \\
= \sum_{PQRSUV} \sum_{P'R'S'U'V'} \sum_{P''R''S''U''V''} & \frac{(U + 2I + 1)!(V + 2I + 1)!}{P!Q!R!S!U!V!(2I + 1)} \quad (1/2) \\
\times (P'' Q'' R'' S'' U'' V'') & \times D_{PQRSUV, P'R'S'U'V'}^{(M=R+U, N=S+V)}(A) \\
\times p' q' r' s' u' v' & \times p'' q'' r'' s'' u'' v''.
\end{align*}
\]

But we know from Eq. (3.43),

\[
(P'' Q'' R'' S'' U'' V'') \times D_{PQRSUV, P'R'S'U'V'}^{(M=R+U, N=S+V)}(A) >
\]

\[
\begin{align*}
= & \left( \frac{(U' + 2I' + 1)!(V' + 2I' + 1)!}{P'!Q'!R'!S'!U'!V'!(2I' + 1)} \right)^{-1/2} \times \frac{(V' + P' + Q' + 1)!}{P''!Q''!R''!S''!U''!V''!(P' + Q')} \\
\times \delta_{P'P'} \delta_{Q'Q'} \delta_{R'R'} & \delta_{S'S'} \delta_{U'U'} \delta_{V'V'}.
\end{align*}
\]

Substituting this formula and changing the double primed variables to single primed ones, we get

\[
(g(p', q', r', s', u', v'; z_1, z_2, z_3, w_1, w_2), \quad Ag(p, q, r, s, u, v; z_1, z_2, z_3, w_1, w_2))
\]

\[
= \sum_{PQRSUV; P''R'S'U'V'} \frac{(U + 2I + 1)!(V + 2I + 1)!}{P!Q!R!S!U!V!(2I + 1)} \times \left( \frac{P''!Q''!R''!S''!U''!V''!(2I' + 1)!}{(U' + 2I' + 1)!(V' + 2I' + 1)!} \right)^{(1/2)} \\
\times \frac{(V' + P' + Q' + 1)!}{P''!Q''!R''!S''!U''!V''!(P' + Q' + 1)} \times D_{PQRSUV, P'R'S'U'V'}^{(M=R+U, N=S+V)}(A) \\
\times p' q' r' s' u' v' & \times p'' q'' r'' s'' u'' v''.
\]

(4.53)
Next we calculate this inner product using the explicit realization for the generating function. For this purpose it is advantageous, as will be seen in a minute, to use the generalized generating function for the basis states

\[ G = \exp(r_p z_1 + r_q z_2 + s_p w_2 + s_q w_1 + u z_3 + v w_3) \]

\[ = \exp \left( \begin{pmatrix} r_p & r_q & u \\ z_1 & z_2 & s_q \\ z_3 & w_1 & w_2 & w_3 & s_p \end{pmatrix} \right). \]  

(4.54)

When any element \( A \in SU(3) \) acts on this generating function it undergoes the following transformation

\[ AG = \exp \left( \begin{pmatrix} r_p & r_q & u \\ z_1 & z_2 & s_q \\ z_3 & w_1 & w_2 & w_3 & s_p \end{pmatrix} A \right) \left( \begin{pmatrix} r_q & s_q \\ r_p & s_p \\ u & v \end{pmatrix} \right). \]  

(4.55)

As is clear from the above equation we can let the triplets \( r_p, r_q, u \) and \( s_q, s_p, v \) undergo the transformation instead of the triplets \( z_1, z_2, z_3 \) and \( w_1, w_2, w_3 \). Therefore we can write the transformed generating function as

\[ AG = G(r_p'', r_q'', u'', s_q'', s_p'', v''), \]  

(4.56)

where

\[ r_p'' = a_{11} r_p + a_{21} r_q + a_{31} u \]

\[ r_q'' = a_{12} r_p + a_{22} r_q + a_{32} u \]

\[ u'' = a_{13} r_p + a_{23} r_q + a_{33} u, \]

\[ s_q'' = a_{11}^* s_q + a_{21}^* s_p + a_{31}^* v \]

\[ s_p'' = a_{12}^* s_q + a_{22}^* s_p + a_{32}^* v \]

\[ v'' = a_{13}^* s_q + a_{23}^* s_p + a_{33}^* v. \]  

(4.57)
To continue with our computation we have to take the inner product of this transformed generating function with the (untransformed) generating function of the basis states.

This is known to us from Eq. (2.24) as

\[
(G', G'') = (1 - v'v'')^{-2} \exp \left[ \frac{(r_p' r_p'' + r_q' r_q'' + s_p' s_p'' + s_q' s_q'')}{(1 - v'v'')} \right] \\
+ \left( u' - v' \frac{(r_p' s_q' + r_q' s_p')}{(1 - v'v''')} \right) \left( u'' - v' \frac{(r_p'' s_q'' + r_q'' s_p'')}{(1 - v'v'')} \right) \right].
\]

(4.58)

This expression gets further simplified if we substitute from Eq. (2.18)

\[
r_p' = r_p', \quad r_q' = r_q', \quad s_q' = -s_q', \quad s_p' = s_p.
\]

We, therefore, get

\[
(G', G'') = (1 - v'v'')^{-2} \exp \left[ (1 - v'v'')^{-1} (r_p' r_p'' + r_q' r_q'' + s_p' s_p'' + s_q' s_q'') \right] \\
+ u' (u'' - v' \frac{(r_p'' s_q'' + r_q'' s_p'')}{(1 - v'v'')} \right] \right].
\]

(4.59)

One last simplification can be brought about in the above expression when we recognize that

\[
r_p'' s_q'' + r_q'' s_p'' + u'' v'' = r_p s_q + r_q s_p + v u,
\]

\[
= vu.
\]

(4.60)

This tells us that

\[
r_p'' s_q'' + r_q'' s_p'' = uv - u'' v''.
\]

(4.61)

Substituting this in our expression Eq. (4.59) for the inner product we get,

\[
(G', G'') = (1 - v'v'')^{-2} \exp \left[ (1 - v'v'')^{-1} (r_p' r_p'' + r_q' r_q'' + s_p' s_p'' + s_q' s_q'') \right] \\
+ u' (u'' - v' (uv - u'' v'')) \right] ,
\]

(4.62)
\[ (1 - v'v'')^{-2} \exp \left[ (1 - v'v'')^{-1} (r_p' r_p'' + r_q' r_q'' + s_p' s_p'' + s_q' s_q'') \right. \\
+ \left. \left( u'(u'' - uvv') \right) \left( 1 - v'v'' \right) \right]. \tag{4.62} \]

The expression on the right hand side of Eq.\( (4.62) \) is our generating function for the Wigner's D-matrix elements of \( SU(3) \).

Hereafter we denote the generating function Eq.\( (4.62) \) for the Wigner's D-matrix elements of \( SU(3) \) by the symbol \( G(D(A)) \).

5. Weyl’s Character Formula for \( SU(3) \)

We now apply Schwinger’s method (see appendix), for the case of \( SU(2) \), to obtain Weyl’s formula for the characters of IRs of \( SU(3) \).

Weyl’s character formula for \( (4,10,11) \) \( U(3) \) is written as

\[
\chi_{U(3)}^{(m_{13},m_{23},m_{33})}((D(A))) = \frac{1}{\sqrt{\sum_{m_{12}'} \sum_{m_{22}'} e^{-3i\theta_{33} (m_{13} + m_{23} + m_{33})}}} \sum_{m_{13}}^{m_{13}+1} \sum_{m_{23}}^{m_{23}+1} e^{-3i\theta_{33} \left( \frac{m_{12}'}{2} \frac{m_{22}'}{2} \right)} \times \\
\sin \frac{m_{12}' - m_{22}'}{2} \left( \frac{\theta_{11} - \theta_{22}}{\sin \frac{\theta_{11} - \theta_{22}}{2}} \right). \tag{5.64} \]

where the representations of \( SU(3) \) are given by
\[(p, q, 0) = (m_{12} - m_{23}, m_{23} - m_{33}, 0). \quad (5.65)\]

Below we will try to obtain the form represented by Eq.\((5.64)\) for Weyl’s character formula for \(SU(3)\).

Our starting for applying Schwinger’s method is the generating function for the Wigner’s \(D\)-matrix elements of \(SU(3)\) Eq.\((4.62)\) and its formal equivalent Eq.\((5.67)\). These two forms are reproduced below.

\[
G(D(A)) = \sum_{PQRSUV; P'Q'R'S'U'V'} \left( \frac{(U + 2I + 1)(V + 2I + 1)!}{P!Q!R!S!U!V!(2I + 1)!} \right) \times \left( \frac{P'!Q'!R'!S'!U'!V'!(2I' + 1)}{(U' + 2I' + 1)(V' + 2I' + 1)!} \right) ^{(1/2)}
\]

\[
\times \left( \frac{(V' + P' + Q' + 1)!}{P'!Q'!R'!S'!U'!V'!(P' + Q' + 1)!} \right) \times D^{(M=R+U, N=S+V)}_{PQRSUV; P'Q'R'S'UV'}(A)
\]

\[
x \times p^{P} q^{Q} r^{R} s^{S} u^{U} v^{V} \times p'^{P'} q'^{Q'} r'^{R'} s'^{S'} u'^{U'} v'^{V'},
\]

\[
= (1 - v'v'')^{-2} \exp \left[ \frac{(r_{p} r_{p}'' + r_{q} r_{q}'' + s_{p} s_{p}'' + s_{q} s_{q}'') + u'(u'' - uvv')}{(1 - v'v'')} \right]. \quad (5.66)
\]

We first take up the formal expression, r.h.s. of Eq.\((5.67)\). Since \(P + Q = R + S\) make the following replacements

\[
p' \rightarrow \frac{t\partial}{\partial p}, \quad q' \rightarrow \frac{t\partial}{\partial q}, \quad r' \rightarrow \frac{\partial}{\partial r},
\]

\[
s' \rightarrow \frac{\partial}{\partial s}, \quad u' \rightarrow \frac{t\partial}{\partial u}, \quad v' \rightarrow \frac{t\partial}{\partial v}, \quad (5.68)
\]

in this expression and evaluate the derivatives at \(p = q = r = s = u = v = 0\). This gives us

\[
G(D(A)) = \sum_{PQRSUV; P'Q'R'S'U'V'} \left( \frac{(U + 2I + 1)(V + 2I + 1)!}{P!Q!R!S!U!V!(2I + 1)!} \right) \times \left( \frac{P'!Q'!R'!S'!U'!V'!(2I' + 1)}{(U' + 2I' + 1)(V' + 2I' + 1)!} \right) ^{(1/2)}
\]

\[
\times \left( \frac{(V' + P' + Q' + 1)!}{P'!Q'!R'!S'!U'!V'!(P' + Q' + 1)!} \right) \times D^{(M=R+U, N=S+V)}_{PQRSUV; P'Q'R'S'UV'}(A) \times P!Q!R!S!U!V!
\]
\begin{align}
&\times \delta_{P'} p' \delta_{Q'} Q \delta_{R'} R \delta_{S'} S \delta_{U'} U \delta_{V'} V, \\
&= \sum_{M,N} \frac{(V + P + Q + 1)!}{(P + Q + 1)!} t^{M+N}_{M,N} \chi^{(M,N)}_{SU(3)}(D(A)).
\end{align}

(5.69)

(5.70)

where we have used Eq. \((2.9)\) in writing Eq. \((2.9)\).

We next take up the expression on the r.h.s of Eq. \((4.62)\). To exhibit the similarity of our starting point for the computation of the characters of \(SU(3)\) with that of Schwinger’s computation for \(SU(2)\) we write this as

\[ G(D(A)) = (1 - V'A^\dagger V)^{-2} \exp \left[ \frac{(R' AR + S'A^\dagger S - uu'vv')}{(1 - V'A^\dagger V)} \right]. \]

(5.71)

where \(A \in SU(3)\) and

\[ V = (s_q s_p v), \quad R = (r_p r_q u), \quad S = (s_q s_p v) \]

\[ V' = \begin{pmatrix} 0 \\ 0 \\ v' \end{pmatrix}, \quad R' = \begin{pmatrix} r'_p \\ r'_q \\ u' \end{pmatrix}, \quad S' = \begin{pmatrix} s'_q \\ s'_p \\ 0 \end{pmatrix}. \]

(5.72)

We now choose \(A\) to be diagonal. This is possible since every unitary matrix can be diagonalized. So let

\[ A = \begin{pmatrix} e^{-i\theta_{11}} & 0 & 0 \\ 0 & e^{-i\theta_{22}} & 0 \\ 0 & 0 & e^{-i\theta_{33}} \end{pmatrix}, \]

(5.73)

(5.74)

where

\[ \theta_{11} + \theta_{22} + \theta_{33} = 0. \]

(5.75)
With this choice for the matrix $A$ the Eq.(5.71) becomes

$$G(D(A)) = \frac{1}{(1 - e^{i\theta_{33}v'})^2} \exp \left[ \frac{e^{-i\theta_{11}r_p' r_p} + e^{-i\theta_{22}r_q' r_q} + e^{i\theta_{22}r_q' r_p} + e^{i\theta_{11}r_p' r_q} + u'(e^{-i\theta_{33}u - uu'})}{(1 - e^{i\theta_{33}v'})} \right],$$

$$= \left( \frac{\exp e^{-i\theta_{33}u'} u}{(1 - e^{i\theta_{33}v'})^2} \right) \times \left( \frac{\exp e^{-i\theta_{33}u'} u}{(1 - e^{i\theta_{33}v'})^2} \right) \times \left[ \frac{(rr' + ss'e^{-i\theta_{33}})(e^{-i\theta_{11}p} + e^{-i\theta_{22}q})}{(1 - e^{i\theta_{33}v'})} \right].$$

(5.76)

We now replace $u'$ with $t(\frac{\partial}{\partial u})$ and evaluate the derivatives at $u = 0$. Similarly we replace $p'$ with $t(\frac{\partial}{\partial p})$ and evaluate at $p = 0$. With $\lambda$ standing for the fraction $\frac{r(1 + e^{-i\theta_{33}v'})}{(1 - e^{i\theta_{33}v'})}$ these manipulations give us

$$G(D(A)) = \left( \sum_{U=0}^{\infty} t^U e^{-i\theta_{33}U} \right) \times \left( \frac{1}{1 - t\lambda e^{-i\theta_{11}}} \right) \times \left( \frac{1}{1 - t\lambda e^{-i\theta_{22}}} \right),$$

$$= \left( \sum_{U=0}^{\infty} t^U e^{-i\theta_{33}U} \right) \times \frac{1}{t\lambda(e^{-i\theta_{11}} - e^{-i\theta_{22}})} \times \left\{ \frac{1}{1 - t\lambda e^{-i\theta_{11}}} - \frac{1}{1 - t\lambda e^{-i\theta_{22}}} \right\},$$

$$= \left( \sum_{U=0}^{\infty} t^U e^{-i\theta_{33}U} \right) \times \frac{1}{t\lambda(e^{-i\theta_{11}} - e^{-i\theta_{22}})} \times \left\{ \sum_{P+Q=0}^{\infty} t^{P+Q} \lambda^{P+Q} (e^{-i(P+Q)\theta_{11}} - e^{-i(P+Q)\theta_{22}}) \right\},$$

$$= \left( \sum_{U=0}^{\infty} t^U e^{-i\theta_{33}U} \right) \times \frac{1}{t\lambda e^{-\frac{1}{2}(\theta_{11} + \theta_{22})}} \times \left\{ \sum_{2I=0}^{\infty} t^{2I} \lambda^{2I} e^{-\frac{1}{2}2I(\theta_{11} + \theta_{22})} (e^{-\frac{1}{2}(2I)\theta_{11}} - e^{+\frac{1}{2}(2I)\theta_{11}}) \right\},$$

$$\times \left\{ \sum_{2I=0}^{\infty} t^{2I} \lambda^{2I} e^{-\frac{1}{2}2I(\theta_{11} - \theta_{22})} (e^{-\frac{1}{2}(2I)\theta_{11}} - e^{+\frac{1}{2}(2I)\theta_{11}}) \right\}. $$
\[
\left( \sum_{U=0}^{\infty} t^U e^{-i\theta_{3U}} \right) \times \sum_{2I=0}^{\infty} t^{(2I-1)} (2I+1) e^\frac{2I+1}{2} e^{i\theta_{3U}} \times \sin \frac{2I}{2}(\theta_{11} - \theta_{22}) \frac{1}{\sin \frac{1}{2}(\theta_{11} - \theta_{22})},
\]

\[
\left( \sum_{U=0}^{\infty} t^U e^{-i\theta_{3U}} \right) \times \sum_{2I=0}^{\infty} t^{(2I-1)} \left( \frac{r'r' + e^{-i\theta_{3U}SS'}}{(1 - t e^{i\theta_{3U}SS'})^{(2I+1)}} \right) e^\frac{2I+1}{2} e^{i\theta_{3U}} \times \sin \frac{2I}{2}(\theta_{11} - \theta_{22}) \frac{1}{\sin \frac{1}{2}(\theta_{11} - \theta_{22})}.
\]

\[
(5.79)
\]

We will now work on the remaining variables. So replace \( v' \) by \( \frac{\partial}{\partial v} \), \( r' \) by \( \frac{\partial}{\partial r} \) and \( s' \) by \( \frac{\partial}{\partial s} \) and evaluate the derivatives at \( r = s = v = 0 \). Doing this we get

\[
G(D(A)) = \left( \sum_{U=0}^{\infty} \sum_{2I=0}^{\infty} S_{V=0, R+S=2I} \sum_{U=0}^{\infty} \sum_{2I+1=0}^{\infty} \sum_{R+S=2I}^{\infty} \frac{(2I + V)!(2I + U + V - 1)}{(2I + 1)} e^{-i\theta_{3U}(\frac{2I}{2} + S + U - V)} \right) \times \sin \frac{2I}{2}(\theta_{11} - \theta_{22}) \frac{1}{\sin \frac{1}{2}(\theta_{11} - \theta_{22})},
\]

\[
(5.78)
\]

Comparing the above equation Eq.(5.78) with Eq.(5.70) we deduce that

\[
\chi_{SU(3)}^{(M,N)}(D(A)) = \sum_{R+S=2I} e^{-i\theta_{3U}(\frac{2I}{2} + S + U - V)} \times \sin \frac{2I+1}{2}(\theta_{11} - \theta_{22}) \frac{1}{\sin \frac{1}{2}(\theta_{11} - \theta_{22})},
\]

\[
= \sum_{R=0}^{M} \sum_{S=0}^{N} e^{-i\theta_{3U}(\frac{2I}{2} + S + U - V)} \times \sin \frac{2I+1}{2}(\theta_{11} - \theta_{22}) \frac{1}{\sin \frac{1}{2}(\theta_{11} - \theta_{22})},
\]

\[
= e^{i\theta_{3}(2m_{13}-m_{23})} \sum_{m'_{12}=m_{13}-m_{23}}^{m_{13}} \sum_{m'_{22}=0}^{m_{13}-m_{23}} e^{-3i\theta_{3}(\frac{m'_{12}+m'_{22}}{2})} \times \sin \frac{m'_{12}-m'_{22}+1}{2}(\theta_{11} - \theta_{22}) \frac{1}{\sin \frac{1}{2}(\theta_{11} - \theta_{22})}.
\]

\[
(5.79)
\]

where use has been made use of Eq(2.14) and Eq(2.12) in arriving at Eq.(5.79).

The Eq.(5.79) can be recast in the form given by Eq.(5.64) by the replacement
Thus

$$\chi_{SU(3)}^{(m_{13}-m_{23},m_{23},0)}(D(A)) = e^{i\theta_{33}(m_{13}+m_{23})} \sum_{m_{12}=m_{23}}^{m_{13}} \sum_{m_{22}=0}^{m_{23}} e^{-3i\theta_{33}\left(\frac{m'_{12}+m'_{22}}{2}\right)} \times \frac{\sin \frac{m'_{12}-m'_{22}+1}{2}(\theta_{11}-\theta_{22})}{\sin \frac{1}{2}(\theta_{11}-\theta_{22})}.$$  

(5.81)

where the representations of $SU(3)$ are given by

$$(p,q) = (m_{12} - m_{23}, m_{23}).$$  

(5.82)

With this we complete our derivation of Weyl’s character formula for the group $SU(3)$.

6. DISCUSSION

Our aim in this paper has been to compute the Weyl’s formula for the characters of $SU(3)$ by using a method similar to the one Schwinger used in deriving the character formula of $SU(2)$. As was noted by us in the introduction, Weyl’s formula has been known for a quite long time now and it has been derived many times over by many people by using a variety of methods. But to our knowledge ours is the first derivation to have obtained Weyl’s formula from a generating function for the matrix elements of $SU(3)$. For this purpose we made use of our modified scalar product Eq.(2.24) rather than the one (Eq.(2.22)) we had used in our earlier computations [1]. But we need to stress the point that we would get the same results even if we had done computations using our earlier (unmodified) scalar product.

Our generating function itself is part of a calculus for doing computations on $SU(3)$. Since this calculus can be extended to higher groups also we expect the generating function method described in this paper also to carry over to higher groups.

We can try a slightly different approach in generalizing these computations to higher groups. We can start with Weyl’s formula for the general case and then by working back we may be able to recover the two tools of our calculus for the general case. These two tools are (i). a generating function for the basis states and (ii). a scalar product for the basis.
states. Since Weyl computed his formula by using group-invariant integrations and since his character formula for $SU(n)$ seems to hide in itself, at least in the special cases of $SU(2)$ and $SU(3)$, a generating function for the basis as well as a prescription for a scalar product we may be able to connect up our constraint Eq.(2.8) with his construction on the group manifold and our ‘auxiliary’ measure with the group-invariant measure itself. This way one may be able to tieup the concept of a model space of a group with a suitable realization of the group manifold.

With regards to applications we just mention one possibility which may be worth trying. We know that in computing some partition functions one has to make use of group characters and measures. In the normal way of doing things the integrations over the group variables cannot be done exactly. But in our case the characters as well as the (‘auxiliary’) measure can be written in the form of exponential functions and since these can be combined with the partition function exponential one may be able to evaluate the partition function more exactly for all the IRs of the group and then extract the information regarding the IR we are actually interested in as we had done in the case of computing Clebsch-Gordan coefficients \[1\].
Appendix: Weyl’s Character Formula for SU(2) - Schwinger’s Derivation

Below we reproduce Schwinger’s derivation of the Weyl’s character formula of SU(2) almost verbatim.

Let \( u \in SU(2) \) then, for Schwinger the generating function for Wigner’s \( D \)-matrix elements of \( SU(2) \) is given by

\[
\exp(x^*uy).
\]  

(Formula A.1)

Formally this can be written as

\[
\exp(x^*uy) = \sum_{jm} \phi_{jm}(x^*)U_{mm'}^{(j)}\phi_{jm'}(y),
\]

(Formula A.2)

where \( \phi_{jm}(x) \) and \( \phi_{jm'} \) are basis states of the IR labelled by the quantum number \( j \) and \( U_{mm'}^{(j)} \) is the matrix element of \( u \) in the IR labelled by the quantum number \( j \). As is well known for a fixed value of \( j \) the range of \( m \) is, \(-j \leq m \leq +j\).

For simplicity we shall assume the reference system to be so chosen that \( u \) is a diagonal matrix, with eigenvalues \( e^{\pm(i/2)\gamma} \). We replace \( x^* \) with \( t(\frac{\partial}{\partial y}) \) and evaluate the derivatives at \( y_\zeta = 0 \). According to

\[
\phi_{jm}(\frac{\partial}{\partial y}\phi_{jm'}(y))_{y_\zeta=0} = \delta_{m,m'}.
\]  

(Formula A.3)

Substituting this in the above Eq. (A.2) we get

\[
\sum_j t^{2j} \chi^{(j)} = \exp(te^{-i2\gamma} \frac{\partial}{\partial y_1} ; y_1) \cdot \exp(te^{i2\gamma} \frac{\partial}{\partial y_2} ; y_1) \Big|_{y_\zeta=0}.
\]

(Formula A.4)

in which the notation reflects the necessity of placing the derivatives to the left of the powers of \( y_\zeta \). Now

\[
\exp \left( \lambda \frac{\partial}{\partial y} ; y \right) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left( \frac{\partial}{\partial y} \right)^n y^n = \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1 - \lambda}.
\]

(Formula A.5)

and therefore
\[
\sum_j t^{2j} \chi^{(j)} = \frac{1}{1 - t \cdot \exp\left(\frac{-i}{2} \gamma\right)} \times \frac{1}{1 - t \cdot \exp\left(\frac{i}{2} \gamma\right)}
\]
\[
= \frac{1}{1 - 2t \cdot \cos \frac{1}{2} \gamma + t^2}
\]
which is a generating function for the \(\chi^{(j)}\). On writing
\[
\frac{1}{1 - t \cdot \exp\left(\frac{-i}{2} \gamma\right)} \times \frac{1}{1 - t \cdot \exp\left(\frac{i}{2} \gamma\right)} = \frac{1}{2it \cdot \sin \frac{1}{2} \gamma} \left[ \frac{1}{1 - t \cdot \exp\left(\frac{1}{2} \gamma\right)} - \frac{1}{1 - t \cdot \exp\left(\frac{-i}{2} \gamma\right)} \right].
\]
and expanding in powers of \(t\), one obtains
\[
\chi^{(j)}(\gamma) = \frac{\sin(j + \frac{1}{2}) \gamma}{\sin \frac{1}{2} \gamma}
\]
(A.8)
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