Skew Scattering and Side Jump Drive Exciton Valley Hall Effect in Two-Dimensional Crystals

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Exciton Valley Hall effect is the spatial separation of the valley-tagged excitons in the presence of a drag force. Usually, the effect is associated with the anomalous velocity acquired by the particles due to the Berry curvature of the Bloch bands. Here we show that the anomalous velocity plays no role in the exciton valley Hall effect, which is governed by the side-jump and skew scattering mechanisms. We develop microscopic theory of the exciton valley Hall effect in the presence of synthetic electric field and phonon drag and calculate all relevant contributions to the valley Hall current also demonstrating the cancellation of the anomalous velocity. The sensitivity of the effect to the origin of the drag force and to the scattering processes is shown. We extend the drift-diffusion model to account for the valley Hall effect and calculate the exciton density and valley polarization profiles.

Introduction. Atomically-thin transition metal dichalcogenide monolayers (TMDC MLs) form a basis for van der Waals heterostructures, a novel versatile semiconductor platform with fascinating fundamental physics and wide prospects for applications [1, 2]. TMDC MLs demonstrate particularly strong optical response dominated by the robust tightly-bound excitons [3–5] which demonstrate particularly strong light-matter coupling [6, 7]. Transport properties of excitons in two-dimensional (2D) semiconductors attract increasing interest [8–14] due to unusual linear and nonlinear effects like halo formation [15–17] and prospects to observe quantum effects such as weak localization of excitons [18].

Excitons in 2D TMDC possess valley degree of freedom and the chiral optical selection rules allow one to address the states in individual $K_+$ and $K_-$ valleys by the circularly polarized light and control the superpositions of the states by linearly polarized radiation [19–25]. It opens up a possibility to observe and study the transport of valley-tagged excitons [26–28] and, in particular, the valley Hall effect (VHE) [21, 29–32], see also [33].

The VHE results in the generation of opposite fluxes of $K_+$ and $K_-$ excitons in the presence of drag force $F_d$ which produces a net unidirectional flow of the particles, Eq. (1). The exciton valley flux $i_v = (i_+ - i_-)/2$ with $i_\pm$ being the fluxes in the $K_\pm$ valleys reads

$$i_v = \chi N [\mathbf{z} \times \mathbf{F}_d], \quad (1)$$

with the constant $\chi$ and $N$ being the total exciton density. Qualitatively, the chiral structure of states in each valley can be viewed as an effective magnetic field which winds excitons similarly to the cyclotron winding of electrons in a real magnetic field. Due to the time-reversal symmetry of the TMDC MLs [34, 35] this effective magnetic field is opposite in $K_+$ and $K_-$ valley, resulting in the opposite winding directions and giving rise to the valley Hall (VH) flux, Eq. (1) and spatial separation of excitons, Fig. 1. The microscopic mechanisms of the exciton VHE are insufficiently studied and, similarly to the electron VHE, are usually related to the anomalous velocity induced by the force $F_d$ [21, 29, 31, 32, 34, 36, 37].

Here we uncover that exciton VHE in 2D semiconductors is caused by the skew-scattering and side-jump mechanisms, resulting from, respectively, asymmetric exciton scattering by phonons or impurities and exciton displacement in the course of scattering. The anomalous velocity contribution is compensated or absent depending on the origin of the drag force $F_d$. We present the microscopic theory of the exciton VHE and calculate $\chi$ in Eq. (1) for relevant mechanisms of exciton drag and scattering. We demonstrate that the resulting VH current is non-universal and depends both on the origin of the drag force and on the details of the scattering. We also extend the drift-diffusion description of excitonic transport [14, 15, 17] to account for the VH.

Model. The Coulomb-bound electron-hole pair excited by the $\sigma^\pm$-polarized photon is described by the Bloch function [38–40]

$$\Psi_{\text{exc}, \pm} = e^{iKR} \sum_k e^{ik\rho} F(k) |e, ke, \pm\rangle |h, kh, \mp\rangle. \quad (2)$$

Here $R$ and $K$ is the exciton center of mass coordinate and wavevector, $\rho$ and $k$ is the electron-hole rel-
ative motion coordinate and wavevector, $F(k)$ is the Fourier transform of the relative motion envelope $\varphi(\rho) = \sum_k F(k) \exp(i k \rho)$, and $\langle e, k_e, +\rangle$, $\langle h, k_h, +\rangle$ are the electron (hole) Bloch amplitudes and the corresponding wavevectors are given by $k_e = (m_e/M) K + k$, $k_h = (m_h/M) K - k$ with $m_e$, $m_h$ and $M = m_e + m_h$ being the electron, hole and exciton mass, respectively. Equation (2) is valid at the exciton binding energy $E_B$ and its kinetic energy $\xi_k = k^2/2M$ being much smaller than the band gap $E_g$ and the distance to other bands. Correspondingly, the Bloch amplitudes are taken in the lowest order in the $k \cdot p$ band mixing [40, 41]

$$\langle e, k_e, \pm\rangle = \langle c, \pm\rangle \pm \frac{\gamma}{E_g} k_{e,\pm} |v, \pm\rangle + \cdots, \quad (3a)$$

$$\langle h, k_h, +\rangle = |\tilde{v}, +\rangle \mp \frac{\gamma}{E_g} k_{h,\pm} |\tilde{e}, +\rangle + \cdots. \quad (3b)$$

Here $\gamma/h \in \mathbb{R}$ is the interband velocity matrix element, $k_{\pm} = k_e \pm ik_g$, $|c, \pm\rangle$, $|v, \pm\rangle$ are the conduct and valence band Bloch amplitudes at the $K_{\pm}$ points of the Brillouin zone, the tilde denotes the hole Bloch amplitudes, and... denotes the contributions of remote bands; see Supplementary Information [42] for details on the electron-electron and electron-hole representations and the extensions of the model to the multiband case. Equations (2) and (3) allow us to calculate the exciton Berry curvature in the $K_{\pm}$ valley [39, 42, 43]

$$\mathbf{F} = -2 \left[ \xi_e \left( \frac{m_e}{M} \right)^2 + \xi_h \left( \frac{m_h}{M} \right)^2 \right] \mathbf{\tilde{z}}, \quad (4)$$

with $\mathbf{\tilde{z}}$ being the unit vector along the ML normal, $\xi_e$ and $\xi_h$ being corresponding parameters describing the electron and hole Berry curvature (in the two-band model $\xi_e = \xi_h = \gamma^2/E_g^2$, intraband contribution to the exciton position operator)

$$\Omega_K = \frac{1}{2} [\mathbf{F} \times \mathbf{K}], \quad (5)$$

and the matrix element of the exciton scattering [cf. [41]]

$$M_{K'K} = V_c(Q) - V_v(Q) - i\xi \left[ V_c(Q) \frac{m_h}{M} - V_v(Q) \frac{m_e}{M} \right] [K' \times K]_z. \quad (6)$$

Here $Q = K' - K$ is the scattering wavevector, $V_c(Q)$ and $V_v(Q)$ are the Fourier components of the scattering potential acting in the conduction and valence bands, respectively; the hole potential $V_v = -V_c$; Eq. (6) is written disregarding remote bands contribution to the Berry curvature, $\xi = \xi_e = \xi_h$ [42]. In the $K_{\pm}$ valley the signs of $\xi_e$, $\xi_h$ and $\xi$ in Eqs. (4) and (6) should be inverted.

The unidirectional flux of the excitons can be provided by several sources, including (i) synthetic static field acting on the exciton as a whole [39, 43], (ii) phonon drag or wind [17, 44, 45], (iii) exciton density or temperature or-gradient [16, 31]. Here we consider two first options and analyze the exciton Valley Hall effect caused by static field resulting, e.g., from an inhomogeneous strain in the sample, and by the phonon drag. Importantly, in these cases the microscopic origin of the force $F_d$ acting on the excitons, Fig. 1 is qualitatively different. Indeed, the inhomogeneous strain produces the coordinate-dependent energy shifts of the conduction and valence bands which results in the variation of the exciton potential energy,

$$U(R) = (\Xi_e - \Xi_v)[c_{ex}(R) + c_{yy}(R)], \quad (7)$$

with $c_{\alpha\beta}$ being the Cartesian components of the strain tensor and $\Xi_e$, $\Xi_v$ being the conduction and valence band deformation potentials. For the constant strain gradient the synthetic force reads

$$F_d^{(s)} = -\nabla U(R). \quad (8)$$

We stress that this is a potential force because it corresponds to the gradient of the exciton potential energy, Eq. (7). The situation is qualitatively different if the excitons are dragged by the non-equilibrium phonons. The net force due to the phonon drag results from the momentum transfer between the phonons and excitons in the collision, while the exciton motion between the collisions is unaffected by phonons. In this case the drag force [17, 41]

$$F_d^{(ph)} = -\frac{\tau^{ph}}{\hbar} \left( \frac{M}{\hbar} \right)^2 (\Xi_e - \Xi_v)^2 k_B T_{latt}, \quad (9)$$

is not associated with a gradient of any potential energy. In Eq. (9) $\tau^{ph}$ is the phonon momentum relaxation time, $\rho$ is the mass density of the material and $T_{latt}$ is the lattice temperature (strictly speaking, the effective temperature of the acoustic phonons). The difference of the origin of the forces in Eqs. (8) and (9) results, as we show below, in the difference of the VHE mechanisms and in the difference in the constant $\chi$ in Eq. (1).

**Mechanisms of VHE.** Usually, the mechanisms of the VHE are separated into intrinsic (Berry curvature or anomalous velocity related) and scattering-induced [41, 46, 47]. However, these mechanisms are partially interrelated. This is because the scattering is needed to form a steady-state distribution of excitons and provide the $dc$ current in the presence of synthetic fields or drag. Thus, it is convenient to classify the contributions to the excitonic VHE, similarly to those for electrons, as the **anomalous contributions** related to the anomalous velocity and the exciton wavepackets displacements at the scattering being, as we show below, independent of the scatterers density and the skew scattering contribution arising from the asymmetric scattering of excitons by impurities and phonons.

We start with the anomalous contributions illustrated in Fig. 2. The anomalous velocity contribution Fig. 2(a) is readily calculated by virtue of Eq. (4) as [39, 43]

$$v^{(av)} = \frac{N}{2} [\mathbf{F} \times F_d^{(s)}]. \quad (10)$$
Figure 2. Illustration of the anomalous velocity contribution to the VHE (a) and the side-jump contribution (b). We demonstrate the anomalous velocities \( \mathbf{v}_{\sigma} \) for excitons in \( \mathbf{K}_\pm \) valleys (denoted according to the polarization of emission \( \sigma^\pm \)) and exciton displacements \( \delta \mathbf{R}_\pm \). The separation of \( \sigma^+ \) and \( \sigma^- \) polarized excitons due to the anomalous velocity and side-jumps are seen.

\[
\mathbf{F}_{\sigma} = \mathcal{F} \times \mathbf{F} \quad \text{or Berry curvature}
\]

Here \( \mathbf{F}_{\sigma} \) is the potential force related to the synthetic electric field acting on excitons, Eq. (8). Interestingly, the scattering rates do not appear in Eq. (10), thus it is commonly assumed that this contribution is the universal and, thus, dominant one for the electronic and excitonic anomalous or valley Hall effects \cite{31,36,37,39,43}. However, this is not true and, similarly, to the case of free charge carriers this contribution is compensated by the so-called side-jump contribution, Fig. 2(b) related to the shifts of excitonic wavepackets under scattering \cite{41,46,47}. The physical origin of the compensation is clear if one takes into account the anomalous velocity contribution \( \mathcal{F} \times \mathbf{R}_\pm \) at the exciton scattering by an impurity or phonon. Since the net force acting on the exciton in the steady-state conditions is zero, i.e., the force due to external fields or drag is compensated by the friction force due to the impurities and phonons, the total anomalous velocity should vanish.

Microscopically, we calculate the exciton displacement at the scattering as \cite{41,49-51}.

\[
\mathbf{R}_{\mathbf{K}' \mathbf{K}} = -\frac{\xi}{\hbar} V_{\mathbf{e}}(Q) \frac{m_0}{M} - V_{\mathbf{v}}(Q) \frac{m_0}{M} [(\mathbf{K}' - \mathbf{K}) \times \mathbf{z}] + \frac{1}{2} \mathbf{F} \times (\mathbf{K}' - \mathbf{K})]. \tag{11}
\]

There are two contributions due to the side-jump VH flux of excitons in the case of synthetic field acting on excitons. One contribution results from the displacements in the course of the scattering (so-called side-jump accumulation)

\[
i^{(sj,a)}_v = \sum_{\mathbf{K}\mathbf{K}'} \mathbf{R}_{\mathbf{K}' \mathbf{K}} 2\pi \hbar \delta(\mathcal{E}_K - \mathcal{E}_{K'}) |M_{\mathbf{K}' \mathbf{K}}|^2 \delta f_{\mathbf{K}}, \tag{12}
\]

with \( \delta f_{\mathbf{K}} = -\tau_p (\mathbf{v}_K \cdot \mathbf{F}_{\sigma}) f_0(\mathcal{E}_K) \) being the field-induced anisotropic part of the exciton distribution function responsible for the direct flow of excitons. Here \( f_0(\mathcal{E}_K) \) is the exciton equilibrium distribution function, prime denotes the derivative over energy, \( \nu_{K} = \hbar K/M \) is the exciton velocity, and \( \tau_p \) is the exciton momentum relaxation time. The second one results from the work of the synthetic field at the exciton side-jump and can be reduced to the same form as Eq. (12): \( i^{(sj,b)}_v = i^{(sj,a)}_v \) \cite{41}, see SI \cite{42} for details. Calculation shows that the total side-jump current acquires the form

\[
i^{(sj)}_v = -\frac{N}{2} [\mathbf{F} \times \mathbf{F}_{\sigma}] - \frac{\xi}{\hbar} \mathbf{A} \mathbf{N}[\mathbf{z} \times \mathbf{F}_{\sigma}], \tag{13}
\]

where we assumed that the scattering is caused by the short-range impurities or acoustic phonons with \( V_c \) and \( V_r \) independent of the transferred wavevector, \( \mathbf{A} = (\nu m_0 - \nu v)/[M(\nu - 1)] \), \( \nu = \nu_{\text{imp}} \equiv V_{\text{c}}/V_r \) (impurities) or \( \nu = \nu_{\text{ph}} \equiv \Xi_{c}/\Xi_{v} \) (phonons). Combining Eqs. (10) and (13) we observe that the anomalous velocity contribution vanishes, the resulting current depends on the details of the scattering processes and is given by Eq. (1) where

\[
\chi^{(anom,s)} = -\frac{\xi}{\hbar} \mathbf{A} \mathbf{N}. \tag{14}
\]

Note that this compensation does not rely on particular form of the Berry curvature (4), see Refs. \cite{41,42} for detail.

The situation is somewhat different for the anomalous contribution to the current caused by the phonon drag. In this case the only source of the anomalous contribution is the side-jump mechanism, Fig. 2(b) because there is no force acting on excitons between the collisions with phonons. The side-jump current has two contributions, the first one, \( i^{(sj,rel)}_v \) arises due to the relaxation of the anisotropic part of excitonic distribution function and can be recast in the form of Eq. (12). The second contribution arises due to the exciton scattering by anisotropic distribution of phonons \cite{17,41} and can be written as

\[
i^{(sj,anis)}_v = \sum_{\mathbf{K}\mathbf{K}'} \mathbf{R}_{\mathbf{K}' \mathbf{K}} W^{\text{anis}}_{\mathbf{K}' \mathbf{K}} f_0(\mathcal{E}_K), \tag{15}
\]

where \( W^{\text{anis}}_{\mathbf{K}' \mathbf{K}} \propto \nabla T_{\text{latt}} \) is the rate of the exciton-phonon scattering resulting in the phonon drag. The calculation shows that if the exciton scattering is due to the acoustic phonons only, the sum \( i^{(sj,rel)}_v + i^{(sj,anis)}_v = 0 \). The origin of this compensation is clear and related, as before, with the steady-state regime of the exciton transport: The excitonic wavepacket displacements due to collisions with phonons resulting in the phonon drag are compensated by the displacements at the collisions resulting in the momentum relaxation. The net VH current arises provided that in addition to the acoustic phonon scattering, the momentum relaxation takes place at static impurities (or other phonons). The calculations show that \cite{42}

\[
\chi^{(anom,d)} = -\frac{\xi}{\hbar} (A_{\text{imp}} - A_{\text{ph}}) \frac{\tau_p}{\tau_{\text{imp}}^2} \frac{N}{2}. \tag{16}
\]
where $\tau_{\text{imp}}$ is the momentum scattering time at the exciton-impurities collisions.

The comparison of Eqs. (14) and (16) demonstrates that the anomalous contribution to the exciton VHE current strongly depends both on the origin of the drag force and on the details of the scattering processes.

Now let us calculate the skew-scattering contribution to the exciton VHE. This effect arises due to the asymmetric scattering of excitons by impurities or phonons. The direction of asymmetry is opposite in $K_\pm$ valleys and, thus, results in the valley Hall current, as illustrated in Fig. 1. Physical origin of the effect is the spin-orbit interaction which produces an effective magnetic field at exciton scattering by a phonon or an impurity. This effective field has opposite signs in the $K_\pm$ valleys resulting in the VHE. Quantitatively, the effect is related to the asymmetric $\propto |K' \times K|$ contribution in the scattering matrix elements (6). It is noteworthy that in order to calculate the skew-scattering contribution to the VHE one has to go beyond the Born approximation in evaluation of the scattering rate: The squared modulus of the matrix element $|M_{KK'}|^2$ does not contain an asymmetric part. In the $\zeta$-linear regime the asymmetric contributions to the scattering rate are of interference type and appear (i) in the third order in $V_e$, $V_v$, and (ii) in the fourth order as a result of coherent two-impurity or two-phonon scattering [41, 46, 52–54].

Here we focus on the case where the exciton scattering is dominated by the exciton-acoustic phonon interaction. The situation of exciton-impurity scattering is presented in SI [42]. The phonon-induced potential is Gaussian and the third-order contributions $\propto \langle M_{KK'}M_{KK},M_{KK'} \rangle$ with angular brackets denoting the averaging over the phonon density matrix vanish. Thus, we take into account the two-phonon processes described in detail in the VHE. Quantitatively, the effect is related to the asymmetric $\propto |K' \times K|$ contribution in the scattering matrix elements (6). It is noteworthy that in order to calculate the skew-scattering contribution to the VHE one has to go beyond the Born approximation in evaluation of the scattering rate: The squared modulus of the matrix element $|M_{KK'}|^2$ does not contain an asymmetric part. In the $\zeta$-linear regime the asymmetric contributions to the scattering rate are of interference type and appear (i) in the third order in $V_e$, $V_v$, and (ii) in the fourth order as a result of coherent two-impurity or two-phonon scattering [41, 46, 52–54].

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In the presence of the synthetic field (8), the interference of the single- and two-phonon processes results in the in VH current in the form of Eq. (1) with [41, 42]:

$$\chi^{(\text{skew}, 2\text{ph}, s)} = \frac{\xi}{h} N B \frac{k_B T \tilde{\xi}_v}{(\tilde{\xi}_c - \tilde{\xi}_v)^2},$$

where

$$B = 2A_{ph} \left[ \frac{m_e}{M} - \frac{m_h}{M} \tilde{\nu}_{\text{ph}} + 2A_{ph}(1 - \tilde{\nu}_{\text{ph}}) \right],$$

$$\tilde{\nu}_{\text{ph}} = \frac{\tilde{\xi}_c}{\tilde{\xi}_v},$$

and $\tilde{\xi}_c$, $\tilde{\xi}_v$ are the two-phonon deformation potentials. We also take into account the coherent two-phonon processes where the intermediate states lie in the same band [41, 55, 56] with the result [42]

$$\chi^{(\text{skew}, \text{coh}, s)} = \frac{\xi}{h} N A_{ph}.$$  

Under the phonon drag conditions we need to take into account additional contribution to the VH current arising from the skew scattering on the anisotropic phonon distribution. Calculation [42] shows that:

$$\chi^{(\text{skew}, 2\text{ph}, s)} = \frac{\chi^{(\text{skew}, 2\text{ph}, s)}}{2} + \frac{\chi^{(\text{skew}, \text{coh}, s)}}{4}. \tag{19}$$

Results. The microscopic theory demonstrates that the excitonic VHE in two-dimensional semiconductors is driven by the skew scattering and side jump effects. Note that the side-jump contribution is crucial for anomalous transport of excitons and should be taken into account along with the skew scattering for the anomalous exciton Hall effect [57]. Importantly, the coherent two-phonon skew scattering and the side jump contributions have similar order of magnitude and in the case of synthetic field $F_{d}^{(s)}$ exactly compensate each other. Thus, in a synthetic field VHE is driven solely by the skew scattering, and one can expect temperature-dependent VH current since $\chi \propto T$ in Eq. (17).

To have an estimation of the effect and provide a link with electronic VHE it is convenient to consider an effective VH conductivity $\epsilon^2 \chi N$ and take into account that the factors $A \sim 1$. As a result, for anomalous and coherent skew scattering contributions $\epsilon^2 \chi N \sim (\epsilon^2/h)\Xi N$. Using generic estimates for $\xi \sim 10\ldots100$ Å [31] and $N \sim 10^{12}$ cm$^{-2}$ (which is significantly below the Mott density due to the high binding energy of the exciton) the VH conductivity $\epsilon^2 \chi N \sim (10^{-4}\ldots10^{-3})(\epsilon^2/h)$ and contains the temperature dependent factor $\beta(T) \sim \Xi_\nu k_B T/(\Xi_c - \Xi_v)^2$. For the typical values of deformation potentials in the range of units to tens of eV [18, 58, 59] $\beta(T) \ll 1$, but this contribution has a specific temperature dependence.

Figure 3. Exciton density profile at $t = 0$ [panel (a)] and $t = 0.5$ ns [panel (b)] calculated after Eqs. (20) in the presence of $F_{d}^{(s)}$. Color shows exciton valley polarization degree $\chi^{(\text{skew}, s)}$. Parameters: $D = 3$ cm$^{-2}$/s, $\tau_0 = 1$ ns, $\tau_s = 0.3$ ns, drift velocity $F_{d}^{(s)}/(\tau_s/M) = 1$ μm/ns, valley Hall angle $\beta = 2\chi N/(\tau_s = 0.1$, initial spot size 0.33 μm.

Valley separation of diffusing excitons. Unlike free charge carriers, the exciton transport is usually monitored optically via the photoluminescence emission with temporal and spatial resolution [14, 15, 26–28] and the valley polarization of excitons is directly translated into the circular polarization of the radiation. Extending the theory of Refs. [15, 17, 46] we formulate the set of drift-diffusion equations for the exciton density $N(r, t) = \ldots$
$N_+(r,t)+N_-(r,t)$ and pseudospin component $S_z(r,t) = [N_+(r,t) - N_-(r,t)]/2$, with $N_\pm$ being the densities of excitons in the $K_\pm$ valleys. The analysis [42] shows that

$$\frac{\partial N}{\partial t} = D\Delta N - \frac{\tau_p}{M} F_d \cdot \nabla N - 4\chi[F_d \times \nabla] z S_z - \frac{N}{\tau_0}, \quad (20)$$

$$\frac{\partial S_z}{\partial t} = D\Delta S_z - \frac{\tau_p}{M} F_d \cdot \nabla S_z - \chi[F_d \times \nabla N] z - \frac{S_z}{\tau_s}, \quad (21)$$

where $\tau_0$ is the exciton lifetime, $\tau_s$ is the spin polarization lifetime. Note that the value of the parameter $\chi$ depends, generally, on the mechanism behind the drag force $F_d$ as shown above. In addition to the standard diffusive $\propto D\Delta$ and drift $\propto F_d \cdot \nabla$ terms and the contributions responsible for the finite lifetime of particles and valley polarization, Eqs. (20) describe, via the terms $\propto \chi$ the coupling between the exciton and valley polarization propagation due to the valley Hall effect. These contributions result from the valley Hall current, $\partial S_z/\partial t \propto \nabla \cdot \mathbf{i}_s$, and the generation of the direct exciton current due to the anomalous Hall effect, $\partial N/\partial t \propto \nabla \cdot S_z \chi(z \times F_d)$. The product $2\chi M/\tau_p$ can be identified with the valley Hall angle because it describes the rotation of the flux of valley-polarized excitons due to the valley Hall effect. The exciton and exciton valley polarization degree spatial distributions calculated by solving Eqs. (20) are presented in Fig. 3. In calculation we assumed that at $t=0$ a Gaussian cloud of excitons was formed by a short light pulse and that the constant drag force $F_d \parallel x$ is applied. Thus, the exciton cloud spreads due to the diffusion and drifts along the $x$-axis due to the drag force. Importantly, the valley polarization distribution (shown by color) arises due to the VHE. The exciton polarization degree is an odd function of the transversal to the $F_d$ coordinate, $y$ [42].

Conclusion. To conclude, we have developed the microscopic theory of the exciton valley Hall effect in two-dimensional transition metal dichalcogenides. The theory takes into account all relevant contributions to the valley Hall current. We demonstrate the cancellation of the anomalous velocity effect by the part of the side-jump contribution and stress that this cancellation is general and does not depend on the specifics of the model. We show that the resulting valley Hall current is driven by the side-jump and skew scattering mechanisms and calculate microscopically these contributions. The resulting valley Hall current depends crucially both on the details of the scattering and on the origin of the drag force, i.e., whether the excitons drift due to a synthetic field or due to the phonon drag. We include the valley Hall effect in the drift-diffusion description of the exciton transport and illustrate the valley separation of the excitons due to the valley Hall effect.

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[15] Marvin Kulig, Jonas Zipfel, Philipp Nagler, Sofia Blanter, Christian Schüller, Tobias Korn, Nicola Paradiso, Mikhail M. Glazov, and Alexey Chernikov, “Exciton diffusion and halo effects in monolayer semiconductors,” Phys. Rev. Lett. 120, 207401 (2018).

[16] Raúl Perea-Causín, Samuel Brem, Roberto Rosati, Rolando Jago, Marvin Kulig, Jonas D. Ziegler, Jonas Zipfel, Alexey Chernikov, and Ermin Malic, “Exciton propagation and halo formation in two-dimensional materials,” Nano Letters 19, 7317–7323 (2019).

[17] M. M. Glazov, “Phonon wind and drag of excitons in monolayer semiconductors,” Phys. Rev. B 100, 045426 (2019).

[18] M. M. Glazov, “Quantum interference effect on exciton transport in monolayer semiconductors,” Phys. Rev. Lett. 124, 166802 (2020).

[19] G. Sallen, L. Bouet, X. Marie, G. Wang, C. R. Zhu, W. P. Han, Y. Lu, P. H. Tan, T. Amand, B. L. Liu, and B. Urbaszek, “Robust optical emission polarization in monolayers through selective valley excitation,” Phys. Rev. B 86, 081301 (2012).

[20] Aaron M. Jones, Hongyi Yu, Nirmal J. Ghimire, Sanfeng Wu, Grant Aivazian, Jason S. Ross, Bo Zhao, Jiaqiang Yan, David G. Mandrus, Di Xiao, Wang Yao, and Xiaodong Xu, “Optical generation of excitonic valley coherence in monolayer wse2,” Nat Nano 8, 634–638 (2013).

[21] Hongyi Yu, Gui-Bin Liu, Pu Gong, Xiaodong Xu, and Wang Yao, “Dirac cones and Dirac saddle points of bright excitons in monolayer transition metal dichalcogenides,” Nat Commun 5, 3876 (2014).

[22] Hongyi Yu, Xiaodong Cui, Xiaodong Xu, and Wang Yao, “Valley excitons in two-dimensional semiconductors,” National Science Review 2, 57–70 (2015).

[23] M. M. Glazov, E. L. Ivchenko, G. Wang, T. Amand, X. Marie, B. Urbaszek, and B. L. Liu, “Spin and valley dynamics of excitons in transition metal dichalcogenide monolayers,” physica status solidi (b) 252, 2349–2362 (2015).

[24] Mikhail Tokman, Yongrui Wang, and Alexey Belyanin, “Valley entanglement of excitons in monolayers of transition-metal dichalcogenides,” Phys. Rev. B 92, 075409 (2015).

[25] G. Wang, X. Marie, B. L. Liu, T. Amand, C. Robert, F. Cadiz, P. Renucci, and B. Urbaszek, “Control of exciton valley coherence in transition metal dichalcogenide monolayers,” Phys. Rev. Lett. 117, 187401 (2016).

[26] Pasqual Rivera, Kyle L. Seyler, Hongyi Yu, John R. Schajibein, Jiaqiang Yan, David G. Mandrus, Wang Yao, and Xiaodong Xu, “Valley-polarized exciton dynamics in a 2D semiconductor heterostructure,” Science 351, 688–691 (2016).

[27] Nils Lundt, Lukasz Dusanowski, Evgeny Sedov, Petr Stepanov, Mikhail M. Glazov, Sebastian Klembt, Martin Klaas, Johannes Beierlein, Ying Qin, Seaatinn Ton-gay, Maxime Richard, Alexey V. Kavokin, Sven Höfling, and Christian Schneider, “Optical valley Hall effect for highly valley-coherent exciton-polaritons in an atomically thin semiconductor,” Nature Nanotechnology 14, 770–775 (2019).

[28] Dmitrii Umuchek, Alberto Ciarrocchi, Ahmet Avsar, Zhe Sun, Kenji Watanabe, Takashi Taniguchi, and Andras Kis, “Valley-polarized exciton currents in a van der Waals heterostructure,” Nature Nanotechnology 14, 1104 (2019).

[29] Yun-Mei Li, Jian Li, Li-Kun Shi, Dong Zhang, Wen Yang, and Kai Chang, “Light-induced exciton spin Hall effect in van der Waals heterostructures,” Phys. Rev. Lett. 115, 166804 (2015).

[30] T. Yu and M. W. Wu, “Valley depolarization dynamics and valley Hall effect of excitons in monolayer and bilayer MoS2,” Phys. Rev. B 93, 045414 (2016).

[31] Masaru Onga, Yijin Zhang, Toshiya Ideue, and Yoshihiro Iwasa, “Exciton Hall effect in monolayer MoS2,” Nature Materials 16, 1193 (2017).

[32] Zunmeng Huang, Yuanda Liu, Kévin Dini, Qinghai Fan, Jin Liu, Timothy Liew, and Weibo Gao, “Robust room temperature valley Hall effect of interlayer excitons,” Nano Letters 20, 1345–1351 (2020).

[33] Shun-ichi Kuga, Shuichi Murakami, and Naoto Nagaoa, “Spin Hall effect of excitons,” Phys. Rev. B 78, 205201 (2008).

[34] Di Xiao, Gui-Bin Liu, Wanxiang Feng, Xiaodong Xu, and Wang Yao, “Coupled spin and valley physics in monolayers of MoS2 and other group-VI dichalcogenides,” Phys. Rev. Lett. 108, 196802 (2012).

[35] Andor Kormanyos, Guido Burkard, Martin Gmitra, Jaroslav Fabian, Viktor Zolyomi, Neil D Drummond, and Vladimir Felko, “kp theory for two-dimensional transition metal dichalcogenide semiconductors,” 2D Materials 2, 022001 (2015).

[36] A. V. Kalameitsev, V. M. Kovaliev, and I. G. Savenko, “Valley acoustoelectric effect,” Phys. Rev. Lett. 122, 256801 (2019).

[37] A. Gianfrate, O. Bleu, L. Dominici, V. Ardizzone, M. De Giorgi, D. Ballarini, G. Lerario, K. W. West, L. N. Pfeiffer, D. D. Solnyshkov, D. Sanvitto, and G. Malpuech, “Measurement of the quantum geometric tensor and of the anomalous Hall drift,” Nature 578, 381–385 (2020).

[38] E. L. Ivchenko, Optical spectroscopy of semiconductor nanostructures (Alpha Science, Harrow UK, 2005).

[39] Wang Yao and Qian Niu, “Berry phase effect on the exciton transport and on the exciton Bose-Einstein condensate,” Phys. Rev. Lett. 101, 106401 (2008).

[40] M. M. Glazov, L. E. Golub, G. Wang, X. Marie, T. Amand, and B. Urbaszek, “Intrinsic exciton-state mixing and nonlinear optical properties in transition metal dichalcogenide monolayers,” Phys. Rev. B 95, 035311 (2017).

[41] M. M. Glazov and L. E. Golub, “Valley Hall effect caused by the phonon and photon drag,” arXiv:2004.05091 (2020).

[42] See Supplementary Information for details.

[43] V. M. Kovaliev and I. G. Savenko, “Quantum anomalous valley Hall effect for bosons,” Phys. Rev. B 100, 121405 (2019).

[44] L. V. Keldysh, “Phonon wind and dimensions of electron-hole drops in semiconductors,” JETP Lett. 23, 86 (1976).

[45] V. S. Bagaev, L. V. Keldysh, N. N. Sibel’din, and V. A. Tsvetkov, “Dragging of excitons and electron-hole drops by phonon wind,” JETP 43, 362 (1976).

[46] M. I. Dyakonov, ed., Spin physics in semiconductors, 2nd ed., Springer Series in Solid-State Sciences 157 (Springer International Publishing, 2017).

[47] N. A. Sinitsyn, “Semiclassical theories of the anomalous Hall effect,” Journal of Physics: Condensed Matter 20,
[48] Nozières, P. and Lewiner, C., “A simple theory of the anomalous Hall effect in semiconductors,” J. Phys. France 34, 901–915 (1973).

[49] Robert Karplus and J. M. Luttinger, “Hall effect in ferromagnetics,” Phys. Rev. 95, 1154–1160 (1954).

[50] V. I. Belinicher, E. L. Ivchenko, and B. I. Sturman, “Kinetic theory of the displacement photovoltaic effect in piezoelectrics,” JETP 83, 649 (1982).

[51] Boris I. Sturman, “Ballistic and shift currents in the bulk photovoltaic effect theory,” Physics-Uspekhi 63 (2019).

[52] L. E. Gurevich and I. N. Yassievich, “Theory of ferromagnetic Hall effect,” Sov. Phys. Solid. State 4, 2091 (1963).

[53] V.N. Abakumov and I.N. Yassievich, “Anomalous Hall effect for polarized electrons in semiconductors,” JETP 34, 1375 (1972).

[54] V. I. Belinicher and B. I. Sturman, “Phonon mechanism of photogalvanic effect in piezoelectrics,” Sov. Phys. Solid State 20, 476 (1978).

[55] I. A. Ado, I. A. Dmitriev, P. M. Ostrovsky, and M. Titov, “Anomalous Hall effect with massive Dirac fermions,” EPL (Europhysics Letters) 111, 37004 (2015).

[56] I. A. Ado, I. A. Dmitriev, P. M. Ostrovsky, and M. Titov, “Sensitivity of the anomalous Hall effect to disorder correlations,” Phys. Rev. B 96, 235148 (2017).

[57] V. K. Kozin, V. A. Shabashov, A. V. Kavokin, and I. A. Shelykh, “Anomalous exciton Hall effect,” arXiv:2006.08717 (2020).

[58] Kristen Kaasbjerg, Kristian S. Thygesen, and Karsten W. Jacobsen, “Phonon-limited mobility in n-type single-layer MoS2 from first principles,” Phys. Rev. B 85, 115317 (2012).

[59] Zhenghe Jin, Xiaodong Li, Jeffrey T. Mullen, and Ki Wook Kim, “Intrinsic transport properties of electrons and holes in monolayer transition-metal dichalcogenides,” Phys. Rev. B 90, 045422 (2014).
Online supplementary information:
Skew Scattering and Side Jump Drive Exciton Valley Hall Effect in Two-Dimensional Crystals

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SI. EXCITON BLOCH FUNCTIONS AND BERRY CURVATURE

Here we demonstrate the way of construction of excitonic Bloch functions within the three-band \( k \cdot p \)-model. We start with the Hamiltonians \( \mathcal{H}_\pm \) describing the electron states (in the electron representation) in the vicinity of the \( K_\pm \) points of the Brillouin zone:

\[
\mathcal{H}_\pm = \begin{pmatrix}
0 & \pm \gamma k_\mp & \pm \gamma' k_\pm \\
\pm \gamma k_\mp & -E_g & 0 \\
\pm \gamma' k_\pm & 0 & -E_g'
\end{pmatrix}
\] (S1)

Here \( k_\pm = k_x \pm i k_y \), the wavevector is reckoned from the corresponding Brillouin zone edge, real constants \( \gamma \) and \( \gamma' \) are the parameters related to the interband momentum matrix elements, \( E_g \) is the band gap and \( E_g' \) is the energy distance from the conduction band to the remote valence band, \( k \cdot p \) mixing between the valence band and the \( k^2 \) diagonal terms are disregarded for brevity. Note that the states in \( K_+ \) and \( K_- \) valleys are related by the time-reversal operation and the relation between \( \mathcal{H}_+ \) and \( \mathcal{H}_- \) establishes the rules of the transformation at the time reversal (i.e., the phases of the Bloch states):

\[
k_\pm \rightarrow -k_\mp.
\] (S2)

Accordingly, in the lowest non-vanishing order in the \( k \cdot p \)-mixing the electron Bloch function acquires the form [1]

\[
|e, k; \pm \rangle \propto |c, \pm \rangle \pm \frac{\gamma}{E_g} k_\pm |v, \pm \rangle \pm \frac{\gamma'}{E_g'} k_\pm |v', \pm \rangle.
\] (S3)

Here \( |c, \pm \rangle, |v, \pm \rangle \), and \( |v', \pm \rangle \) are the Bloch functions at the \( K_\pm \) points of the Brillouin zone, the normalization constant is omitted. Equation (S3) is in agreement with Eq. (3a) of the main text.

In the course of direct optical transition the electron is promoted from the valence band to the conduction band leaving behind an unoccupied state in the same valley. According to the general theory [2–6], the hole state is
associated with the state, obtained by the time reversal operation of the unoccupied state (in this way the linear and angular momentum conservation laws at optical transitions are naturally fulfilled, see also the theory of positronium [7]). Let us for specificity consider the electron-hole pair excited by the $\sigma^+$ light, i.e., where both the electron and unoccupied state are in the $K_+$ valley, correspondingly, the hole is in the $K_-$ valley. The matrix elements $(n, n')$ are the band indices, $\mathcal{K}$ is the time reversal operator, superscript $(h)$ underlines the hole representation) of the hole Hamiltonian are obtained as [2]:

$$
\mathcal{H}_{nn'}(k) = -\mathcal{H}_{\mathcal{K}n',\mathcal{K}n}(-k),
$$

(Eq. S4)

with the result

$$
\mathcal{H}^{(h)}(k) = \begin{pmatrix}
0 & \pm \gamma k_x & \pm \gamma' k_y \\
\pm \gamma k_x & +E_g & 0 \\
\pm \gamma' k_y & 0 & +E'_g
\end{pmatrix}.
$$

(Eq. S5)

For completeness we note that the Hamiltonian for the hole in the $K_+$ valley (i.e., the hole excited by the $\sigma^-$ light, unoccupied state is in the $K_-$ valley) can be obtained from (S5) by the transformation Eq. (S2). The hole state in the lowest order in the $k \cdot p$-mixing reads in agreement with Eq. (3b) of the main text

$$
|h, k, \mp\rangle \propto |\hat{\mathbf{v}}, \mp\rangle \mp \frac{\gamma}{E_g} k'_\pm |\hat{\mathbf{c}}, \mp\rangle,
$$

(Eq. S6)

where tilde denotes the hole Bloch functions.

In what follows we assume that the Coulomb interaction is sufficiently weak such that the exciton binding energy $E_B$ is by far smaller than the band gaps $E_g, E'_g$. Thus, the Coulomb interaction can be considered as a perturbation and the exciton state can be presented as a linear superposition of the two-particle Bloch functions [2, 6]

$$
\Psi_{\text{exc}, \pm} = \sum_{k_e, k_h} e^{i(k_e r_e + i k_h r_h)} \Phi(k_e, k_h) |e, k_e, \pm\rangle |h, k_h, \mp\rangle.
$$

(Eq. S7)

Here subscripts $\pm$ denote the polarization of light which excites corresponding exciton, $\Phi(k_e, k_h)$ is the smooth envelope of the exciton wavefunction in the $k$-space, $k_e, k_h$ are the electron and hole wavevectors. The envelope function can be found from the two-particle Schrödinger equation written in the effective mass approximation. For a free exciton the total wavevector $\mathbf{K} = k_e + k_h$ is a good quantum number and

$$
\Phi(k_e, k_h) = \delta_{k_e+k_h, \mathbf{K}} F(k),
$$

(Eq. S8)

with

$$
F(k) = \int d\rho \varphi(\rho) e^{-i k \cdot \rho} d\rho,
$$

is the relative motion envelope function with $\varphi(\rho)$ being the real space envelope function. For completeness we present also the relations between the coordinates (and wavevectors) of the individual carriers, $r_e$ and $r_h (k_e, k_h)$ and center-of-mass/relative motion:

$$
R = \frac{m_e}{M} r_e + \frac{m_h}{M} r_h, \quad \rho = r_e - r_h,
$$

(Eq. S9a)

$$
\begin{align*}
\mathbf{k}_e &= \frac{m_e}{M} \mathbf{K} + \mathbf{k}, \\
\mathbf{k}_h &= \frac{m_h}{M} \mathbf{K} - \mathbf{k},
\end{align*}
$$

(Eq. S9b)

with $m_e, m_h$ and $M = m_e + m_h$ being the electron, hole and exciton mass, respectively.

Equations (S7) and (S8) are in agreement with Eq. (2) of the main text:

$$
\Psi_{\text{exc}, \pm} = e^{i\mathbf{k} \mathbf{R}} U_{\mathbf{K}, \pm}(\rho),
$$

(Eq. S10)

where the function

$$
U_{\mathbf{K}, \pm}(\rho) = \sum_k e^{i k \cdot \rho} F(k) |e, k_e, \pm\rangle |h, k_h, \mp\rangle,
$$

(Eq. S11)

can be formally associated with the exciton Bloch amplitude. Note that the optically active excitons have the center of mass wavevectors in the vicinity of the Brillouin zone center ($\Gamma$-point). This is because electron and hole occupy
opposite valleys of the \( k \)-space. Making use of definition of the exciton Bloch amplitude (S11) one can express the exciton Berry curvature \( z \)-component as

\[
\mathcal{F}_{\pm,z} = 2 \text{Im} \int \frac{\partial \mathcal{U}_{K_{\pm}}(\rho)}{\partial K_y} \frac{\partial \mathcal{U}_{K_{\pm}}(\rho)}{\partial K_x} d\rho = \mp 2 \left[ \xi_e \left( \frac{m_e}{M} \right)^2 + \xi_h \left( \frac{m_h}{M} \right)^2 \right] \hat{z}.
\] (S12)

Here, within the considered three-band model

\[
\xi_e = \frac{\gamma^2}{E_g^2} + \frac{\gamma'^2}{E_g'^2},
\]
(S13a)

\[
\xi_h = \frac{\gamma^2}{2E_g'},
\]
(S13b)

and Eq. (S12) corresponds to Eq. (4) of the main text.

Making use of Eq. (S11) one can readily calculate the contribution to the exciton position operator related to the Bloch bands:

\[
\Omega_{\pm,K} = -\text{Im} \int \mathcal{U}_{K_{\pm}}(\rho) \nabla K \mathcal{U}_{K_{\pm}}(\rho) d\rho = \frac{1}{2} [\mathcal{F}_{\pm} \times K],
\] (S14)

in agreement with Eq. (5) of the main text. In fact, one can also arrive to Eq. (S14) using simple qualitative arguments evaluating the center-of-mass coordinate of the exciton:

\[
\Omega_{\pm,K} = \frac{m_e}{M} \Omega_{\pm,e}(k_e) + \frac{m_h}{M} \Omega_{\pm,h}(k_h),
\] (S15)

where \( \Omega_{\pm,e}, \Omega_{\pm,h} \) are the electron and hole Bloch coordinates and the overline denotes averaging over the relative motion wavevector \( \vec{k} \).

**SII. EXCITON SCATTERING BY EXTERNAL POTENTIAL**

We introduce the perturbation operator \( \mathcal{V} \) related to the external potential (i.e., due to the defects of the crystal or the deformation potential due to phonons) as

\[
\mathcal{V} = \begin{pmatrix} V_e(r) & 0 & 0 \\ 0 & V_e(r) & 0 \\ 0 & 0 & V'_e(r) \end{pmatrix}.
\] (S16)

For simplicity we assume that the perturbation is diagonal in the band indices and does not mix valleys, which is reasonable assumption for the long-wavelength acoustic phonon scattering. The perturbation \( \mathcal{V} \) is presented in the electron representation. It follows from Eq. (S4) that in the hole representation the perturbation has the same form, but the opposite sign.

Making use of definitions of the electron and hole Bloch functions, Eqs. (S3) and (S6), we arrive at the following expressions for the scattering matrix elements for the electron in the \( K_+ \) valley and the hole in the \( K_- \) valley

\[
V^e_{k_e',k_e} = V_e(k_e' - k_e) + \xi [k_e' \times k_e]_z V_e(k_e' - k_e) - \xi' [k_e' \times k_e]_z V'_e(k_e' - k_e),
\] (S17a)

\[
V^h_{k_e',k_e} = -V_e(k_e' - k_e) - i\xi [k_e' \times k_h]_z V_e(k_e' - k_e),
\] (S17b)

with

\[
\xi = \gamma^2 / E_g ^2, \quad \xi' = \gamma'^2 / E_g'^2.
\] (S18)

Equations (S17) together with (S10) allow us to express the exciton scattering matrix element in the form

\[
M_{K',K} = \sum_{k_e,k_h} \delta_{K,k_e+k_h} \delta_{K',k_e'+k_h'} F^*(k') F(k) \left( V^e_{k_e',k_e} \delta_{k_e,k_h} + V^h_{k_e',k_e} \delta_{k_e,k_h} \right)
\]

\[
= V_e(K' - K) - V_e(K' - K) + i \xi V_e(K' - K) \frac{m_e}{M} [K' \times K]_z
\]

\[
+ i \xi' V'_e(K' - K) \frac{m_e}{M} [K' \times K]_z - i \xi V_e(K' - K) \frac{m_h}{M} [K' \times K].
\] (S19)
Note that for the exciton created by the $\sigma^-$ light where the electron occupies the $K_-$ valley and the hole occupies the $K_+$ valley the signs of $\xi$ and $\xi'$ should be reversed.

It is instructive to disregard the contribution $\xi'$ to the Berry curvature and scattering matrix element. In this case in agreement with the main text [Eq. (6)] we obtain from Eq. (S19)

$$M_{K'K} = V_c(K' - K) - V_v(K' - K) - i\xi \left[ V_c(K' - K) \frac{m_h}{M} - V_v(K' - K) \frac{m_e}{M} \right] [K' \times K].$$

(S20)

Equation (S20) can be further simplified in the relevant case of the short range scattering. We introduce the notations

$$\nu = \frac{V_c}{V_v}, \quad A = \frac{\nu m_h - m_e}{M(\nu - 1)},$$

(S21)

and recast

$$M_{K'K} = (V_c - V_v) (1 - i\xi A[K' \times K]).$$

(S22)

Note that the analogous scattering matrix element for electrons takes the form

$$M_{k'e} = V_c \left( 1 + i\frac{\xi}{\nu} [k' \times k_e] \right),$$

(S23)

i.e., it differs from Eq. (S22) by the replacement

$$\nu^{-1} \rightarrow -A.$$  \hspace{1cm} (S24)

It makes possible to use the results of Ref. [1] for calculation of the exciton valley Hall effect.

### SIII. ANOMALOUS CONTRIBUTIONS TO THE EXCITONIC VHE

It is shown in Ref. [1] that for two-dimensional electrons in the presence of a force $F$ the side-jump contribution due to the change of potential energy at scattering, $e^{(s)}_{v}^{(s,j,b)}$, is given by

$$e^{(s)}_{v}^{(s,j,b)} = e^{(s)}_{v}^{(s,j,a)} = -\frac{1}{2} e N_e [F \times F] + \frac{e}{\hbar} N_e \frac{\xi}{\nu} [\hat{z} \times F].$$

(S25)

Here $N_e$ is the electron density per valley, and $i^{(s,j,a)}_{v}$ is the side-jump accumulation contribution, cf. Eq. (12) of the main text. Since the form of the scattering matrix elements for electrons and excitons is similar, cf. Eqs. (S22) and (S23), the corresponding side-jump current for excitons in the presence of synthetic field can be derived from Eq. (S25) using the replacement (S24) together with obvious substitutions (we recall that $N$ is the total density of excitons):

$$F \rightarrow F^d, \quad N_e \rightarrow \frac{1}{2} N.$$  \hspace{1cm} (S26)

As a result, we obtain for the total side-jump current $i^{(s)}_{v} = i^{(s,j,a)}_{v} + i^{(s,j,b)}_{v}$ Eq. (13) of the main text:

$$i^{(s)}_{v} = -\frac{N}{2} [F \times F^d] - \frac{\xi}{\hbar} AN [\hat{z} \times F^d].$$

(S27)

The first term here exactly compensates the anomalous velocity contribution, and the total anomalous current is given by the linear in $A$ part of the side-jump contribution.

At the phonon drag, the anomalous contribution to the electron VHE has the form [1]

$$\chi^{\text{anom},d}_{el} = \frac{\xi}{\hbar} \left( \frac{1}{\nu_{\text{imp}}} - \frac{1}{\nu_{\text{ph}}} \right) \frac{\tau_p}{\tau_{\text{imp}}} N_e.$$  \hspace{1cm} (S28)

Making the substitutions (S24) and (S26) we get the expression for excitons:

$$\chi^{\text{anom},d} = -\frac{\xi}{2\hbar} (A_{\text{imp}} - A_{\text{ph}}) \frac{\tau_p}{\tau_{\text{imp}}} N,$$  \hspace{1cm} (S29)

in agreement with Eq. (16) of the main text.
SIV. SKEW SCATTERING CONTRIBUTIONS

A. Exciton-impurity scattering

For non-degenerate electrons scattered by short-range impurities, the skew scattering contribution to VHE in the electric field is given by [1]

$$\chi_{\text{el}}^{(\text{skew, imp})} = 4\pi \xi N e \hbar g_U v_k B T \tau_p = 2 \xi N e \frac{k_B T}{n_i U_c}. \quad (S30)$$

Here $U_{c,v}$ are the Fourier components of the scattering potential acting in the conduction and valence bands, cf. Eq. (S16), and $n_i$ is the impurity density. Making the substitutions (S24) and (S26) we obtain for excitons

$$\chi_{\text{el}}^{(\text{skew, imp})} = -A \xi N e \frac{k_B T}{n_i U_c}. \quad (S31)$$

B. Two-phonon scattering

Let us firstly analyze the contributions due to the two-phonon processes described by the Hamiltonian (17) of the main text. Analogous calculation for non-degenerate electrons yields [1]:

$$\chi_{\text{el}}^{(\text{skew, 2ph, s})} = -4 \xi N e \frac{k_B T}{\Xi_v + 2 \Xi_c \Xi_v} \Xi_c^2, \quad (S32)$$

where $\Xi_{c,v}$ and $\tilde{\Xi}_{c,v}$ are the one- and two-phonon constants for the conduction and valence bands. For excitons, we make the following substitutions in accordance with Eq. (S20)

$$\Xi_c \rightarrow \Xi_c - \Xi_v, \quad \Xi_v \rightarrow m_e \Xi_v - m_h \Xi_c / M,$$

$$\tilde{\Xi}_c \rightarrow \Xi_c - \Xi_v, \quad \tilde{\Xi}_v \rightarrow m_e \Xi_v - m_h \Xi_c / M. \quad (S33)$$

Therefore we obtain Eq. (18) of the main text:

$$\chi_{\text{el}}^{(\text{skew, 2ph, s})} = \frac{\xi}{\hbar} N B \frac{k_B T \tilde{\Xi}_v}{(\Xi_c - \Xi_v)^2}, \quad (S34)$$

where

$$B = 2A_{\text{ph}} \left[ \frac{m_e}{M} - \frac{m_h}{M} \tilde{\nu}_{\text{ph}} + 2A_{\text{ph}} (1 - \tilde{\nu}_{\text{ph}}) \right], \quad \tilde{\nu}_{\text{ph}} = \tilde{\Xi}_c / \Xi_v.$$

In a similar fashion Eqs. (19) and (20) of the main text can be derived from the results of Ref. [1].

C. Coherent skew scattering at the phonon drag

Let us now turn to the coherent two-phonon skew scattering processes in the case where the scattering takes place on the anisotropic distribution of the phonons. For clarity and brevity of presentation we consider the case electrons scattered by the anisotropic distribution of phonons and calculate the asymmetric scattering in $K+$ valley as follows:

$$W_{k'k}^{(as)} = \frac{2\pi}{h} \sum_{pp',\nu,\mu} \delta(\varepsilon_{k'} - \varepsilon_k - \nu \hbar s |k - p| + \mu \hbar s |k' - p|) |\delta_{k + k', p + p'}|^2$$

$$\times \frac{1}{2} \left| \frac{M^{(2, \mu)}_{k' p} M^{(1, -\nu)}_{p k}}{\varepsilon_k - \varepsilon_p - \nu \hbar s |k - p| + i0} + \frac{M^{(1, -\nu)}_{k' p'} M^{(2, \mu)}_{p' k}}{\varepsilon_{k'} - \varepsilon_p - \mu \hbar s |k' - p'| + i0} \right|^2. \quad (S35)$$
Here the factor 1/2 allows avoiding double account for the same phonons. Taking the interference term we obtain

\[
\text{Re} \left\{ \frac{M^{(2,\mu)}_{p k} M^{(1,\nu)}_{p k} \left[ M^{(1,\nu)}_{p k} M^{(2,\mu)}_{p k} \right]^*}{(\varepsilon_k - \varepsilon_p + \nu h s |k - p| + i0)(\varepsilon_k - \varepsilon_{p'} - \mu h s |k - p'| - i0)} \right\} \to \pi \text{Im} \left\{ \frac{M^{(2,\mu)}_{p k} M^{(1,\nu)}_{p k} \left[ M^{(1,\nu)}_{p k} M^{(2,\mu)}_{p k} \right]^*}{\varepsilon_k - \varepsilon_{p'} - \mu h s |k - p'|} \right\} \left[ \varepsilon_k - \varepsilon_p + \nu h s |k - p| \right] \left[ \varepsilon_k - \varepsilon_{p'} - \mu h s |k - p'| \right] \delta(\varepsilon_k - \varepsilon_p - \mu h s |k - p|) - \delta(\varepsilon_k - \varepsilon_{p'} - \mu h s |k - p'|)
\]

where we used that at simultaneous change \( p \leftrightarrow p' \), \( \mu \leftrightarrow -\nu \) the product of matrix elements is changed to the complex-conjugated.

Using the property \( \left[ M^{(\nu)}_k \right]^* = M^{(-\nu)}_k \), we obtain

\[
W^{(as)}_{k k'} = (\frac{2\pi}{\hbar})^2 \sum_{p p', \nu, \mu = \pm} \delta(\varepsilon_k - \varepsilon_p - \nu h s |k - p| + \mu h s |k' - p'| \delta_{k + k', p + p'}) \times \varepsilon_k - \varepsilon_{p'} - \mu h s |k - p'| \right\} \left\{ \left( M^{(\mu)}_{k p} M^{(-\nu)}_{k' p'} \right)_p \left( M^{(-\nu)}_{p k} M^{(\nu)}_{p' k'} \right)_p \right\} . \tag{S37}
\]

One can check that the above expression indeed changes sign at substitution \( k \leftrightarrow k' \):

\[
W^{(as)}_{k k} = \frac{(2\pi)^2}{\hbar} \sum_{p p', \nu, \mu = \pm} \delta(\varepsilon_k - \varepsilon_p - \nu h s |k' - p| + \mu h s |k - p|) \delta_{k + k', p + p'} \times \varepsilon_k - \varepsilon_{p'} - \mu h s |k - p'| \right\} \left\{ \left( M^{(\mu)}_{p k} M^{(-\nu)}_{k' p'} \right)_p \left( M^{(-\nu)}_{k' p} M^{(\nu)}_{p k} \right)_p \right\} . \tag{S38}
\]

If we interchange two dummy indices \( \mu \leftrightarrow \nu \) then we see that this expression differs from \( W^{(as)}_{k k} \) just by the complex conjugations which changes the sign of the imaginary part.

In the lowest order in the phonon energy we have

\[
\delta(\varepsilon_k - \varepsilon_{k'} - \nu h s |k - p| + \mu h s |k' - p'|) \delta(\varepsilon_k - \varepsilon_p + \mu h s |k - p|) = \frac{\delta(\varepsilon_k - \varepsilon_p + \mu h s |k' - p|)}{\varepsilon_k - \varepsilon_{p'} - \mu h s |k - p'|} \delta(\varepsilon_k - \varepsilon_p + \nu h s |k - p|)
\]

\[
\approx \mu h s |k' - p| \left[ \frac{\delta(\varepsilon_k - \varepsilon_p)}{\varepsilon_k - \varepsilon_{p'}} + \frac{\delta(\varepsilon_k - \varepsilon_{p'})}{\varepsilon_k - \varepsilon_{p'}} \right] \delta(\varepsilon_k - \varepsilon_p + \nu h s |k - p|) \delta(\varepsilon_k - \varepsilon_{p'}) \delta(\varepsilon_k - \varepsilon_p). \tag{S39}
\]

The electron-phonon matrix element is given by

\[
M^{(\nu)}_{k' k} = i\nu \sqrt{\frac{\hbar q}{2 \rho s}} (\xi_c + i\xi_v \langle k' \times k \rangle) \sqrt{m_q} \delta_{k' - k, \nu q}. \tag{S40}
\]

Then we obtain from Eq. (S37)

\[
W^{(as)}_{k k} = \frac{(2\pi)^2}{\hbar} \sum_{p p', \nu, \mu = \pm} \delta_{k + k', p + p'} \left[ \frac{[k' \times k + k \times p + p \times k + k \times k']}{\varepsilon_k - \varepsilon_{p'}} \right] \times m_{\mu(p - k')} m_{\nu(p - k')}
\]

\[
\times m_{\mu(p - k')} m_{\nu(p - k)} \left\{ \mu_{k'} - p \left[ \frac{\delta'(\varepsilon_k - \varepsilon_p)}{\varepsilon_k - \varepsilon_{p'}} + \frac{\delta(\varepsilon_k - \varepsilon_{p'})}{\varepsilon_k - \varepsilon_{p'}} \right] \delta(\varepsilon_k - \varepsilon_p + \nu |k - p|) \delta(\varepsilon_k - \varepsilon_{p'}) \delta(\varepsilon_k - \varepsilon_p) \right\} . \tag{S41}
\]

Here \( m_q = q n_q \).
Hereafter we assume the following nonequilibrium phonon distribution

\[ n_q = \bar{n}_q (1 + c_q), \quad c_{-q} = -c_q, \]  

(S42)

where \( \bar{n}_q = k_B T / (\hbar s q) \) is the Planck function and \( c_q \) describes an anisotropic correction responsible for the nonequilibrium phonon flux in the system. Then we have from Eq. (S41)

\[
W_{k'k}^{(as)} = \frac{(2\pi)^2}{\hbar} \Xi^2 \epsilon \Xi \hbar s \left( \frac{k_B T}{\hbar s^2} \right)^2 \sum_{\nu,\mu=\pm1, p, p'} \delta_{k+k', p+p'} \frac{[k' \times p + k \times p' + p \times k + p' \times k']_z}{\epsilon_k - \epsilon_{p'}} \times \left\{ \mu |k' - p| \left[ \frac{\delta'(\epsilon_{k'} - \epsilon_p)}{\epsilon_k - \epsilon_{p'}} + \frac{\delta(\epsilon_{k'} - \epsilon_p)}{2} \right] \delta(\epsilon_k - \epsilon_p) + \nu |k - p| \delta(\epsilon_k - \epsilon_p) \delta'(\epsilon_k - \epsilon_p) \right\}. 
\]  

(S43)

In the lowest order in inelasticity we have

\[(1 + c_\mu(p-k'))(1 + c_{\nu(p-k')}) \approx 1 + c_\mu(p-k') + c_{\nu(p-k')}, \]  

(S44)

and since \( \sum_{\nu=\pm} \nu c_{\nu q} = 2c_q \), we obtain

\[
W_{k'k}^{(as)} = 2 \frac{(2\pi)^2}{\hbar} \Xi^2 \epsilon \Xi \hbar s \left( \frac{k_B T}{\hbar s^2} \right)^2 \sum_{p, p'} \delta_{k+k', p+p'} \frac{[k' \times p + k \times p' + p \times k + p' \times k']_z}{\epsilon_k - \epsilon_{p'}} \times \left\{ c_{p-k'} |k' - p| \left[ \frac{\delta'(\epsilon_{k'} - \epsilon_p)}{\epsilon_k - \epsilon_{p'}} + \frac{\delta(\epsilon_{k'} - \epsilon_p)}{2} \right] \delta(\epsilon_k - \epsilon_p) + c_{p-k} |k - p| \delta(\epsilon_k - \epsilon_p) \delta'(\epsilon_k - \epsilon_p) \right\}. 
\]  

(S45)

Hereafter we assume the following asymmetry of the phonon distribution typical for the phonon drag [1, 8]:

\[ c_q = \frac{q}{|q|} \cdot e \quad \text{i.e.} \quad n_q = \frac{k_B T}{\hbar s q} \left( 1 + \frac{q}{|q|} \cdot e \right), \]  

(S46)

where \( e \) is the in-plane vector related to the phonon drag force as:

\[
m \frac{s}{\tau_{ph}} e = F_{\text{drag}} \]  

(S47)

with \( m \) being the electron effective mass. Then we have

\[
W_{k'k}^{(as)} = 2 \frac{(2\pi)^2}{\hbar} \Xi^2 \epsilon \Xi \hbar s \left( \frac{k_B T}{\hbar s^2} \right)^2 \sum_{p, p'} \delta_{k+k', p+p'} \frac{[k' \times p + k \times p' + p \times k + p' \times k']_z}{\epsilon_k - \epsilon_{p'}} \times \left\{ (p - k') \left[ \frac{\delta'(\epsilon_{k'} - \epsilon_p)}{\epsilon_k - \epsilon_{p'}} + \frac{\delta(\epsilon_{k'} - \epsilon_p)}{2} \right] \delta(\epsilon_k - \epsilon_p) + (p - k) \frac{\delta(\epsilon_{k'} - \epsilon_p) \delta'(\epsilon_k - \epsilon_p)}{\epsilon_k - \epsilon_{p'}} \right\} \cdot e.
\]  

(S48)

Here we directed the \( x \) axis along \( e \), \( g \) is the density of states per valley,

\[
\Phi_0 = \left\langle (p_x - k_x') \frac{[k' \times p + k \times p' + p \times k + p' \times k']_z}{\epsilon_{p'} - \epsilon_k} \right\rangle \varphi_{p', p = k = k'}, \quad \Phi_1 = \left\langle (p_x - k_x') \frac{[k' \times p + k \times p' + p \times k + p' \times k']_z}{\epsilon_{p'} - \epsilon_k} \right\rangle \varphi_{p, p = k = k'}, \quad \Phi_2 = \left\langle (p_x - k_x) \frac{[k' \times p + k \times p' + p \times k + p' \times k']_z}{\epsilon_{p'} - \epsilon_k} \right\rangle \varphi_{p, p = k'}. 
\]  

(S49)
Introducing the scattering time
\[ \frac{1}{\tau_{ph}} = \sum_{k'} W_{0k'k} = \sum_{k'} \frac{2\pi}{\hbar^2} \delta(\varepsilon_k - \varepsilon_{k'}) 2 \Xi^2 k_B T \rho_s^2, \]  
where
\[ \frac{1}{\tau_{ph}} = \sum_{k'} \frac{2\pi}{\hbar^2} \delta(\varepsilon_k - \varepsilon_{k'}) 2 \Xi^2 c \frac{k_B T}{\rho_s^2}, \]
we have
\[ W_{0k'k}^{(as)} = -\xi \frac{1}{\tau_{ph}} \frac{h^2}{2mg} \left[-\delta(\varepsilon_{k'} - \varepsilon_k) \Phi_0 + \delta'(\varepsilon_{k'} - \varepsilon_k)(\Phi_1 - \Phi_2)\right] F_{\text{drag},x}. \]
We took here into account Eq. (S47).

Having calculated the asymmetric scattering probability we can now calculate the VHE from the kinetic equation. The anisotropic correction to the distribution function is found from
\[ \frac{\delta f_k}{\tau_p} = \sum_{k'} W_{kk'}^{(as)} f_0(\varepsilon_{k'}). \]
The VHE current is given by
\[ j_y = e \sum_k \delta f_k v_{k,y} = e \tau_p \sum_{k,k'} v_{k,y} W_{kk'}^{(as)} f_0(\varepsilon_{k'}) \equiv e \tau_p \sum_{k,k'} v_{k',y} W_{kk'}^{(as)} f_0(\varepsilon_k). \]
Therefore we need the averages \( \langle \Phi_0,1,2v_{k',y} \rangle_{\varphi_{p',\varphi_k,\varphi_{k'}}} \):
\[ j_y = -e F_{\text{drag},x} \xi \frac{\tau_p}{\tau_{ph}} \frac{h^2}{2mg} \sum_{k,k'} v_{k',y} \left[-\delta(\varepsilon_{k'} - \varepsilon_k) \Phi_0 + \delta'(\varepsilon_{k'} - \varepsilon_k)(\Phi_1 - \Phi_2)\right] f_0(\varepsilon_k). \]
Using the relations
\[ [k' \times p + k \times p' + p \times k + k \times k']_{z} = 2[k' \times p + p \times k + k \times k']_{z}, \]
\[ \varepsilon_{k+k'-p} - \varepsilon_k = \frac{h^2}{2m} (k'^2 + p^2 + 2k \cdot k' - 2k \cdot p - 2k' \cdot p), \]
we obtain
\[ \langle \Phi_0 v_{k',y} \rangle_{\varphi_{p',\varphi_k,\varphi_{k'}}} = m \left( \frac{h^3}{\tau_{ph}} \sum_{k,k'} \left( \frac{\sin \varphi_1 [-\sin \varphi_1 + \sin \varphi_2 + \sin (\varphi_1 - \varphi_2)]}{[1 + \cos (\varphi_1 - \varphi_2) - \cos \varphi_2 - \cos \varphi_1]^2} \right)_{\varphi_1,\varphi_2} \right). \]
Here we introduced \( \varphi_1 = \varphi_{k'} - \varphi_p, \varphi_2 = \varphi_k - \varphi_p \). Similarly,
\[ \langle \Phi_1 v_{k',y} \rangle_{\varphi_{p',\varphi_k,\varphi_{k'}}} = \frac{1}{\hbar k'^2} \left( \frac{\sin \varphi_1 [-\sin \varphi_1 + \sin \varphi_2 + \sin (\varphi_1 - \varphi_2)]}{(1 + x^2)/2 + x \cos (\varphi_1 - \varphi_2)} \right)_{\varphi_1,\varphi_2}, \]
where we introduced \( x = k/k' \). Finally,
\[ \langle \Phi_2 v_{k',y} \rangle_{\varphi_{p',\varphi_k,\varphi_{k'}}} = \frac{1}{\hbar k'^2} \left( \frac{\sin \varphi_1 [-\sin \varphi_1 + \sin \varphi_2 + \sin (\varphi_1 - \varphi_2)]}{1 + x \cos (\varphi_1 - \varphi_2)} \right)_{\varphi_1,\varphi_2}. \]
Therefore we have
\[ j_y = -e F_{\text{drag},x} \xi \frac{\tau_p}{\tau_{ph}} \frac{h^2}{2mg} \sum_{k,k'} k'^2 \left[-\delta(\varepsilon_k - \varepsilon_{k'}) G_0/2\varepsilon_{k'} + \delta'(\varepsilon_k - \varepsilon_{k'}) G(k/k')\right] f_0(\varepsilon_k), \]
where
\[ G_0 = \left( \frac{\sin \varphi_1 [-\sin \varphi_1 + \sin \varphi_2 + \sin (\varphi_1 - \varphi_2)]}{[1 + \cos (\varphi_1 - \varphi_2) - \cos \varphi_2 - \cos \varphi_1]^2} \right)_{\varphi_1,\varphi_2}. \]
\[ G(x) = \left( \frac{x^2 \sin \varphi + x \sin \varphi_2 + \sin (\varphi_1 - \varphi_2)}{(1 + x^2)/2 + x \cos (\varphi_1 - \varphi_2) - x^2 \cos \varphi_2 - x \cos \varphi_1} \right) \left( 1 + \frac{x \cos (\varphi_1 - \varphi_2) - \sin \varphi_1 \cos \varphi_2 + \sin \varphi_1 - \sin (\varphi_1 - \varphi_2)}{1 + x \cos (\varphi_1 - \varphi_2) - \cos \varphi_2 - \cos \varphi_1} \right)_{\varphi_1, \varphi_2}. \] (S63)

Summation over \( k, k' \) is performed as follows:
\[ j = -\frac{e}{\hbar} [\hat{z} \times \mathbf{F}_{\text{drag}}] \xi \frac{\tau_p}{\tau_{\text{ph}}} \Xi \sum_{k, k'} [-\delta(\varepsilon_k - \varepsilon_{k'})G_{0}/2 + \varepsilon_{k'} \delta(\varepsilon_k - \varepsilon_{k'})G(k/k')] f_0(\varepsilon_k) \]
\[ = -\frac{e}{\hbar} [\hat{z} \times \mathbf{F}_{\text{drag}}] \xi \frac{\tau_p}{\tau_{\text{ph}}} \Xi N \left\{ G(1) - \frac{1}{2}[G_0 + G'(1)] \right\}. \] (S64)

Calculations show that \( G(1) = 1 \) and \( G_0 + G'(1) = 1 \). This yields
\[ j = -\frac{e}{2\hbar} \xi N \frac{\tau_p}{\tau_{\text{ph}}} \Xi \left[ \hat{z} \times \mathbf{F}_{\text{drag}} \right]. \] (S65)

We see that the coherent contribution for electrons at phonon drag is a quarter of that in electric field. The same relation holds for excitons. Therefore we finally have
\[ \chi^{(\text{skew,d})} = \frac{\chi^{(\text{skew,2ph,s})}}{2} + \frac{\chi^{(\text{skew,coh,s})}}{4}. \] (S66)

It proves Eq. (20) of the main text.

**SV. DRIFT-DIFFUSION EQUATIONS**

**A. Derivation**

Here we derive drift-diffusion equations describing valley separation of excitons due to the VHE. To that end, we follow Ref. [9] and SI of Ref. [10]. We introduce the following notations: Let \( N_{\pm} \) be the occupancies (densities) of excitonic states active in \( \sigma^+ \) and \( \sigma^- \) polarizations (we denote these excitons as excitons in \( K_{\pm} \) valleys according to the valley degree of freedom of electron in the exciton),
\[ N = N_+ + N_-. \] (S67a)

is the total exciton density. The \( z \)-component of exciton pseudospin \( S_z \) is given by
\[ S_z = \frac{N_+ - N_-}{2}. \] (S67b)

Note that \( N_{\pm}, N \) and \( S_z \) depend on time \( t \) and position \( r \). The partial flux densities of excitons in \( K_{\pm} \) valleys are denoted as \( i_{\pm} \), thus the total exciton flux density reads
\[ i = i_+ + i_-, \] (S68a)

and the exciton valley flux density is defined as
\[ i'_v = \frac{i_+ - i_-}{2}, \] (S68b)

in accordance with the main text.

In the presence of the drag force \( \mathbf{F}_d \) (either due to the synthetic field or due to the phonon drag) the exciton flux in a given valley reads
\[ i_{\pm} = \frac{\tau_p}{M} \mathbf{F}_d N_{\pm} \pm 2\chi N_{\pm} [\hat{z} \times \mathbf{F}_d], \] (S69)

with \( \tau_p \) being the momentum relaxation time and appropriate constant \( \chi \) which depends, as shown above, on the origin of \( \mathbf{F}_d \). Equations (S69) describe the fluxes induced by the force, i.e., the drift of the excitons. The spatial
gradients of \( N_+ \) and \( N_- \) produce diffusive contributions to the valley fluxes \( i^{(diff)} \) = \(-D \nabla N_\pm \) with \( D \) being the diffusion coefficient of excitons. Accordingly, the fluxes in the presence of both the force field and the density gradients read
\[
\begin{align*}
i &= \frac{\tau_p}{M} F_d N + 4\chi S_z [\hat{z} \times F_d] - D \nabla N, \\
i_v &= \frac{\tau_p}{M} F_d S_z + \chi N [\hat{z} \times F_d] - D \nabla S_z.
\end{align*}
\]
(S70a)
(S70b)

Note that, generally, there are additional terms \( \propto \nabla \times (S_z \hat{z}) \) and \( \propto [\hat{z} \times \nabla N] \) in the right hand sides of Eqs. (S70a) and (S70b), respectively [9], but these terms vanish in the drift diffusion equations for \( N \) and \( S_z \), because the latter equations contain the divergences of the fluxes where these additional terms are nullified.

Making use of the continuity equations and taking into account the finite lifetime of the particles we arrive at the set of the drift-diffusion equations [Eqs. (21) of the main text]
\[
\frac{\partial N}{\partial t} = D \Delta N - \frac{\tau_p}{M} F_d \cdot \nabla N - 4\chi [F_d \times \nabla]_z S_z - \frac{N}{\tau_0},
\]
(S71a)
\[
\frac{\partial S_z}{\partial t} = D \Delta S_z - \frac{\tau_p}{M} F_d \cdot \nabla S_z - \chi [F_d \times \nabla N]_z - \frac{S_z}{\tau_s}.
\]
(S71b)

Note that these equations can be recast in somewhat more symmetric form if one, instead of \( S_z \) introduces the valley polarization density \( P_z = 2S_z \) and the parameter \( \beta = 2\chi \) [cf. Ref. [9]]. In these notations Eqs. (S71) read
\[
\frac{\partial N}{\partial t} = D \Delta N - \frac{\tau_p}{M} F_d \cdot \nabla N - \beta [F_d \times \nabla]_z P_z - \frac{N}{\tau_0},
\]
(S72a)
\[
\frac{\partial P_z}{\partial t} = D \Delta P_z - \frac{\tau_p}{M} F_d \cdot \nabla P_z - \beta [F_d \times \nabla N]_z - \frac{P_z}{\tau_s}.
\]
(S72b)

## B. Analytical solution

The set of Eqs. (S71) admits analytical solution. We consider the situation where unpolarized excitons were created at \( t = 0 \), let \( N_0(r) \) be the initial density profile. Performing the Fourier-transform of Eqs. (S71) and taking into account the initial condition we arrive at the set of the algebraic equations
\[
\begin{align*}
\left[-i\omega + D q^2 + i \frac{\tau_p}{M} (F_d \cdot q) + \tau_0^{-1}\right] N_{\omega,q} + 4i\chi [F_d \times q]_z S_{\omega,q} &= N_{0,q}, \\
\left[-i\omega + D q^2 + i \frac{\tau_p}{M} (F_d \cdot q) + \tau_s^{-1}\right] S_{\omega,q} &= -i\chi [F_d \times q]_z N_{\omega,q}.
\end{align*}
\]
(S73a)
(S73b)

Here \( N_{0,q} \) is the Fourier-transform of the \( N_0(r) \) over the coordinate, \( N_{\omega,q} \) and \( S_{\omega,q} \) denote the Fourier transforms of \( N(r,t) \) and \( S_z(r,t) \) over time and coordinate.

Under reasonable assumptions \( \chi N/\tau_p \ll 1, \tau_s \ll \tau_0 \) and for typical times \( t \gtrsim \tau_s \) one can disregard the term \( \propto S_z \) in Eq. (S73a), while in Eq. (S73b) keep, in the left hand side, only \( \tau_s^{-1} \). As a result
\[
S_{\omega,q} = -i\tau_s \chi [F_d \times q]_z N_{\omega,q},
\]
or transforming to the real space
\[
S_z(r,t) = -\chi \tau_s [F_d \times \nabla N(r,t)]_z.
\]
(S74)

Note that this expression can be directly derived from Eq. (S72b) neglecting time and spatial derivatives of \( S_z \). The solution of Eq. (S73a) with \( \chi S_z \) term neglected can be readily written in the closed analytical form. For example, we consider the initial Gaussian distribution of excitons
\[
N_0(r) = \frac{C}{\pi r_0^2} \exp \left(-r^2/r_0^2\right),
\]
(S75)

with the constant \( C \) and \( r_0 \) being the initial spot radius. Introducing the drift velocity \( v_d = F_d \tau_p/M \) we arrive at
\[
N(r,t) = \sum_q \exp \left[-(D q^2 + \tau_0^{-1})t + i q (r - v_d t)\right] N_{0,q} = \frac{C}{\pi r_0^2 + 4Dt} \exp \left(-\frac{(r - v_d t)^2}{r_0^2 + 4Dt} - \frac{t}{\tau_0}\right).
\]
(S76)
The valley polarization degree \( (t \gg r_0^2/D) \)

\[
P_v = \frac{2S_z(r, t)}{N(r, t)} \approx \frac{\chi \tau_s}{D t} [F_d \times r]_z, \tag{S77}
\]

in agreement with numerical calculations presented in the main text and in Figs. S1 and S2.

Figure S1. Exciton density profile at \( t = 0 \) [panel (a)], \( t = 0.5 \) ns [panel (b)], and \( t = 1 \) ns [panel (c)] calculated after Eqs. (S71) in the presence of \( F_d \parallel x \). Panels (d-f) show the exciton pseudospin \( S_z \). Parameters: \( D = 3 \) cm\(^2\)/s, \( \tau_0 = 1 \) ns, \( \tau_s = 0.3 \) ns, drift velocity \( F_d\tau_p/M = 1 \) \( \mu \)m/ns, valley Hall angle \( \beta = 2\chi M/\tau_p = 0.1 \), initial spot radius \( r_0 = 0.33 \) \( \mu \)m. False color scale is used to highlight the density and pseudospin spatial profiles.

Figure S2. (a) Exciton density profile at \( t = 0.5 \) ns calculated after Eqs. (S71) in the presence of \( F_d \parallel x \). Color shows exciton pseudospin \( S_z \). (b) Exciton polarization degree \( P_v \) calculated numerically (points) and using analytical asymptotics, Eq. (S77). Parameters are the same as in Fig. S1.
[1] M. M. Glazov and L. E. Golub, “Valley Hall effect caused by the phonon and photon drag,” arXiv:2004.05091 (2020).
[2] G. L. Bir and G. E. Pikus, Symmetry and Strain-induced Effects in Semiconductors (Wiley/Halsted Press, 1974).
[3] E. L. Ivchenko, Optical spectroscopy of semiconductor nanostructures (Alpha Science, Harrow UK, 2005).
[4] M. M. Glazov, E. L. Ivchenko, G. Wang, T. Amand, X. Marie, B. Urbaszek, and B. L. Liu, “Spin and valley dynamics of excitons in transition metal dichalcogenide monolayers,” physica status solidi (b) 252, 2349–2362 (2015).
[5] M. M. Glazov, L. E. Golub, G. Wang, X. Marie, T. Amand, and B. Urbaszek, “Intrinsic exciton-state mixing and nonlinear optical properties in transition metal dichalcogenide monolayers,” Phys. Rev. B 95, 035311 (2017).
[6] Gang Wang, Alexey Chernikov, Mikhail M. Glazov, Tony F. Heinz, Xavier Marie, Thierry Amand, and Bernhard Urbaszek, “Colloquium: Excitons in atomically thin transition metal dichalcogenides,” Rev. Mod. Phys. 90, 021001 (2018).
[7] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, Quantum Electrodynamics, Second Edition (vol. 4) (Butterworth-Heinemann, Oxford, 1999).
[8] M. M. Glazov, “Phonon wind and drag of excitons in monolayer semiconductors,” Phys. Rev. B 100, 045426 (2019).
[9] M. I. Dyakonov, ed., Spin physics in semiconductors, 2nd ed., Springer Series in Solid-State Sciences 157 (Springer International Publishing, 2017).
[10] Marvin Kulig, Jonas Zipfel, Philipp Nagler, Sofia Blanter, Christian Schüller, Tobias Korn, Nicola Paradiso, Mikhail M. Glazov, and Alexey Chernikov, “Exciton diffusion and halo effects in monolayer semiconductors,” Phys. Rev. Lett. 120, 207401 (2018).