Hadamard functions of inverse $M$-Matrices

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Abstract

We prove that the class of GUM matrices is the largest class of bi-potential matrices stable under Hadamard increasing functions. We also show that any power $\alpha \geq 1$, in the sense of Hadamard functions, of an inverse $M$-matrix is also inverse $M$-matrix showing a conjecture stated in Neumann [15]. We study the class of filtered matrices, which include naturally the GUM matrices, and present some sufficient conditions for a filtered matrix to be a bi-potential.

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1 Introduction and Basic Notations

A nonnegative matrix $U$ is said to be a potential if it is nonsingular and its inverse satisfies

$$U_{ij}^{-1} \leq 0 \text{ for } i \neq j, \quad U_{ii}^{-1} > 0$$

$$\forall i \sum_j U_{ij}^{-1} \geq 0,$$

that is $U^{-1}$ is an $M$-matrix which is row diagonally dominant. We denote this class of matrices by $\mathcal{P}$. In addition we say that $U$ is a bi-potential if $U^{-1}$ is also column

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diagonally dominant. We denote this class by $\text{biP}$. We note that $\mathcal{P}, \text{biP}$ are contained in $\mathcal{M}^{-1}$ the class of inverses of $M$-matrices.

The class of potentials matrices play an important role in probability theory. They represent the potential (from where we have taken the name) of a transient continuous time Markov Chain $(X_t)_{t \geq 0}$, with generator $-U^{-1}$. That is,

$$U_{ij} = \int_0^\infty (e^{-U^{-1}t})_{ij} \, dt = \int_0^\infty \mathbb{P}_i\{X_t = j\}\, dt$$

is the mean expected time expended at site $j$ when starting the chain at site $i$. Clearly $U$ is a bi-potential if both $U$ and $U'$ are potentials.

To get a discrete time interpretation take $K_0 = \max_i \{U^{-1}_{ii}\}$. For any $k \geq K_0$ the matrix $P_k = I - \frac{1}{k}U^{-1}$ is nonnegative, sub-stochastic and

$$U^{-1} = k(I - P_k),$$

If we can take $k = 1$, then $U^{-1} = I - P$ (with $P = P_1$) and $U$ is the mean expected number of visits of a Markov chain $(Y_n)_{n \in \mathbb{N}}$ whose transition probability is given by $P$, in fact

$$U_{ij} = \sum_{n \geq 0} P^n_{ij} = \sum_{n \geq 0} \mathbb{P}_i\{Y_n = j\}.$$

We notice that if $U$ is a potential then it is diagonally dominant on each row, in the sense that for all $i, j$ we have $U_{ii} \geq U_{ji}$. The probabilistic prove of this fact is that

$$U_{ji} = f_{ji}U_{ii},$$

where $f_{ji} \leq 1$ is the probability that the Markov process $(X_t)$, starting from $j$ ever reaches the state $i$. If $U$ is a bi-potential, then it is also diagonally dominant on each column. In this case we just say it is diagonally dominant.

For any nonnegative matrix $U$ we define the following quantity

$$\tau(U) = \inf\{t \geq 0 : I + tU \notin \text{biP}\},$$

which is invariant under permutations that is $\tau(U) = \tau(IUI'')$. We point out that if $U$ is a positive matrix then $\tau(U) > 0$. We shall study some properties of this function $\tau$. In particular we are interested in matrices for which $\tau(U) = \infty$. The next result shows that $\tau(U) = \infty$ is a generalization of the class bi$\mathcal{P}$.

**Proposition 1.1** Assume $U$ is a nonnegative matrix, which is nonsingular and $\tau(U) = \infty$, then $U \notin \text{biP}$. 

**Proof.** It is direct from the observation that

$$t(I + tU)^{-1} \xrightarrow{t \to \infty} U^{-1}.$$

$\square$
Remark 1.1 We shall prove later on that the reciprocal is also true: if $U$ is in the class $bi\mathcal{P}$, then $\tau(U) = \infty$.

The following notion will play an important role in this article.

**Definition 1.1** Given a matrix $B$ we say that a vector $\mu$ is a right equilibrium potential if

\[ B\mu = 1, \]

where $1$ is the constant vector of ones. Similarly it is defined the notion of a left equilibrium potential, which is the right equilibrium potential for $B'$. When $B$ is nonsingular we denote the unique right and left equilibrium potentials by $\mu_B$ and $\nu_B$.

We denote by $\bar{\mu} = 1'\mu$ the total mass of $\mu$. In the nonsingular case, it is not difficult to see that $\bar{\nu} = \bar{\mu}$.

Notice that for a matrix $U \in bi\mathcal{P}$ the right and left equilibrium potentials are nonnegative. This is exactly the same as the fact that the inverse is row and column diagonally dominant.

**Definition 1.2** Constant Block Form (CBF) matrices can be defined recursively in the following way: given two CBF matrices $A, B$ of sizes $p$ and $n-p$ respectively, and numbers $\alpha, \beta$ we produce the new CBF matrix by

\[
U = \begin{pmatrix}
A & \alpha 1_p 1_p' \\
\beta 1_{n-p} 1_p' & B_{n-p}
\end{pmatrix},
\]

where the vector $1_p$ is the vector of ones of size $p$. We also say that $U$ is in increasing CBF if $\min\{A, B\} \geq \min\{\alpha, \beta\}$.

We recall the following two definitions introduced in [11] and [14], that generalize the concept of ultrametric matrices introduced in [10] (see also [13]).

**Definition 1.3** A nonnegative CBF matrix $U$ is in Nested Block Form (NBF) if in (\ref{eq:1.1}) $A, B$ are NBF matrices and

- $0 \leq \alpha \leq \beta$;
- $\min\{A_{ij}, A_{ji}\} \geq \alpha$ and $\min\{B_{kl}, B_{lk}\} \geq \alpha$;
- $\max\{A_{ij}, A_{ji}\} \geq \beta$ and $\max\{B_{kl}, B_{lk}\} \geq \beta$.

**Definition 1.4** A nonnegative matrix $U$ of size $n$, is said to be Generalized Ultrametric Matrix (GUM) if it is diagonally dominant that is for all $i, j$ it holds $U_{ii} \geq \max\{U_{ij}, U_{ji}\}$, and $n \leq 2$, or $n > 2$ and every three distinct elements $i, j, k$ has a preferred element. Assume this element is $i$, which means
• $U_{ij} = U_{ik}$;
• $U_{ji} = U_{ki}$;
• $\min\{U_{jk}, U_{kj}\} \geq \min\{U_{ji}, U_{ij}\}$;
• $\max\{U_{jk}, U_{kj}\} \geq \max\{U_{ji}, U_{ij}\}$.

By definition the transpose of a GUM matrix is also GUM. We note that an ultrametric matrix is a symmetric GUM. The study of the chain associated to an ultrametric matrix was done in [5] and for a GUM in [6].

In the next Theorem we summarize the main results in [11] and [14] concerning GUM matrices.

**Theorem 1.1** Let $U$ be a nonnegative matrix.

• $U$ is a GUM matrix iff it is permutation similar to a NBF.
• If $U$ is a GUM, then it is nonsingular iff it does not contain a row of zeros and no two rows are the same.
• If $U$ is a non-singular GUM then $U \in biP$.

It is clear that if $U$ is a GUM then $I + tU$ is a nonsingular GUM. In particular $\tau(U) = \infty$.

We introduce a main object of this article.

**Definition 1.5** Given a function $f$ and a matrix $U$, the matrix $f(U)$ is defined as $f(U)_{ij} = f(U_{ij})$. We shall say that $f(U)$ is a Hadamard function of $U$.

Given two matrices $A, B$ of the same size we denote by $A \odot B$ the Hadamard product of them. So $(A \odot B)_{ij} = A_{ij}B_{ij}$.

Given a vector $a$ we denote by $D_a$ the diagonal matrix, whose diagonal is $a$. For example we have $D_aD_b = D_a \odot D_b = D_{a \odot b}$.

The class of CBF matrices (and its permutations) is closed under Hadamard functions. Similarly, the class of increasing CBF (and its permutations) is closed under increasing Hadamard functions.

On the other hand the class of NBF, and therefore also the class of GUM matrices, is stable under Hadamard nonnegative increasing functions. We summarize this result in the following proposition.

**Proposition 1.2** Assume $U$ is a GUM and $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an increasing function. Then $f(U)$ is a GUM. In particular $\tau(f(U)) = \infty$, and if $f(U)$ is nonsingular then $f(U) \in biP$. A sufficient condition for $f(U)$ to be nonsingular is that $U$ is nonsingular and $f$ is strictly increasing.
Proof. It is clear that \( f(U) \) is a GUM matrix and therefore \( \tau(f(U)) = \infty \). Then, from Proposition 1.1 we have that \( f(U) \in biP \) as long as it is nonsingular. If \( U \) is nonsingular then it does not contain a row (or column) of zeros and there are not two equal rows (or columns). This condition is stable under strictly non-negative functions, so the result follows. \( \Box \)

One of our main results is a sort of reciprocal of the previous one. We shall prove that if \( \tau(f(U)) = \infty \) for all increasing nonnegative functions \( f \), then \( U \) must be a GUM (see Theorem 2.4).

The last concept we need for our work is the following.

**Definition 1.6** We say that a nonnegative matrix \( U \) is in class \( \mathcal{T} \) if

\[ \tau(U) = \inf\{ t > 0 : (I + tU)^{-1} \not\geq 0 \text{ or } 1'(I + tU)^{-1} \not\geq 0 \}, \]

and \( I + \tau(U)U \) is nonsingular whenever \( \tau(U) < \infty \).

We shall prove that every nonnegative matrix \( U \) which is a permutation of an increasing CBF, is in class \( \mathcal{T} \).

We remark here that our purpose is to study Hadamard functions of matrices and not spectral functions of matrices, which are quite different concepts. For spectral functions of matrices there are deep and beautiful results for the same classes of matrices we consider here. See for example the work of Bouleau [3] for filtered operators. For \( M \) matrices see the works of Varga [16], Micchelli and Willoughby [12], Ando [1], Fiedler and Schneider [8], and the recent work of Bapat, Catral and Neumann [2] for \( M \)-matrices and inverse \( M \)-matrices.

## 2 Main Results

**Theorem 2.1** Assume \( U \in \mathcal{P} \) and \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) is a nonnegative strictly increasing convex function. Then \( f(U) \) is nonsingular and \( \det(f(U)) > 0 \). Also \( f(U) \) has a right nonnegative equilibrium potential. Moreover if \( f(0) = 0 \) we have \( M = U^{-1}f(U) \) is an \( M \)-matrix. If \( U \in bi\mathcal{P} \) then \( f(U) \) also has a left nonnegative equilibrium potential.

Note that \( H = f(U)^{-1} \) is not necessarily a \( Z \)-matrix, that is for some \( i \neq j \) it can happen that \( H_{ij} > 0 \), as the following example will show. Therefore the existence of a nonnegative right equilibrium potential, which is

$$\forall i \quad H_{ii} + \sum_{j \neq i} H_{ij} \geq 0.$$
does not imply that the inverse is row diagonally dominant, that is

$$\forall i \ H_{ii} \geq \sum_{j \neq i} |H_{ij}|.$$  

**Example 2.1** Consider the matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Then \( U = (I - P)^{-1} \in bi\mathcal{P} \). Consider the strictly convex function \( f(x) = x^2 - \cos(x) + 1 \). A numerical computation gives

$$ (f(U))^{-1} \approx \begin{pmatrix} 0.3590 & -0.0975 & 0.0027 \\ -0.0975 & 0.2372 & -0.0975 \\ 0.0027 & -0.0975 & 0.3590 \end{pmatrix},$$

which is not a \( Z \)-matrix.

We denote by \( U^{(\alpha)} \) the Hadamard transformation of \( U \) under \( f(x) = x^\alpha \). In particular \( U^{(2)} = U \odot U \). One of our main results is the stability of \( \mathcal{M}^{-1} \) under powers. This solves a conjecture stated on [15].

**Theorem 2.2** Assume \( U \in \mathcal{M}^{-1} \) and \( \alpha \geq 1 \). Then \( U^{(\alpha)} \in \mathcal{M}^{-1} \). If \( U^{-1} \in \mathcal{P} \) then \( (U^{(\alpha)})^{-1} \in \mathcal{P} \). If \( U \in bi\mathcal{P} \) then \( U^{(\alpha)} \in bi\mathcal{P} \).

The previous result has the following probabilistic interpretation. If \( U \) is the potential of a transient continuous time Markov process then \( U^{(\alpha)} \) is also the potential of a transient continuous time Markov process. In the next result we show the same is true for a potential of a Markov chain. An interesting open question is what is the relation between the Markov chain associated to \( U \) and the one associated to \( U^{(\alpha)} \).

**Theorem 2.3** Assume that \( U^{-1} = I - P \) where \( P \) is a submarkov kernel, that is \( P \geq 0, P1 \leq 1 \). Then for all \( \alpha \geq 1 \) there is a submarkov kernel \( Q(\alpha) \) such that \( (U^{(\alpha)})^{-1} = I - Q(\alpha) \). Moreover if \( P'1 \leq 1 \) then \( Q(\alpha)'1 \leq 1 \).

The next result establishes that the class of GUM matrices is the largest class of potentials stable under increasing Hadamard functions.

**Theorem 2.4** Let \( U \) be a nonnegative matrix such that \( \tau(f(U)) = \infty \), for all increasing nonnegative functions \( f \). Then, \( U \) must be a GUM.
Example 2.2 Given \( a, b, c, d \in \mathbb{R}^+ \) consider the non-singular matrix
\[
U = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
a & b & 1 & 0 \\
c & d & 0 & 1 \\
\end{pmatrix}.
\]

For all increasing non-negative functions \( f \) and all \( t > 0 \): \((\mathbb{I} + tf(U))^{-1}\) is an M-matrix, while \( U \) is not a GUM. Moreover, \( U \) is not a permutation of an increasing CBF. This shows that the last Theorem does not hold if we replace the class \( \text{biP} \) by the class \( \mathcal{M}^{-1} \).

Theorem 2.5 Let \( U \in \text{biP} \) and \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) be a strictly increasing convex function. \( f(U) \) is in \( \text{biP} \) if and only if \( f(U) \) belongs to class \( T \).

Theorem 2.6 If \( U \) is a nonnegative increasing CBF matrix then \( U \) is in class \( T \).

As a corollary of the two previous theorems we obtain the following important result.

Theorem 2.7 Assume that \( U \in \text{biP} \) is an increasing CBF matrix, and \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) is a nonnegative strictly increasing convex function. Then \( f(U) \in \text{biP} \).

3 Proof of Theorems 2.1, 2.2, 2.3 and 2.5

Let us start with a useful lemma.

Lemma 3.1 Assume \( U \in \mathcal{M}^{-1} \). Then for all \( t \geq 0 \) we have \((\mathbb{I} + tU) \in \mathcal{M}^{-1} \). Also if \( U \in \mathcal{P} \) so is \((\mathbb{I} + tU) \) and its right equilibrium potential is strictly positive. In particular if \( U \in \text{biP} \) then so is \( \mathbb{I} + tU \) and its equilibrium potentials are strictly positive. Similarly, let \( 0 \leq s < t \) and assume \( \mathbb{I} + tU \in \text{biP} \), then \( \mathbb{I} + sU \in \text{biP} \) and its equilibrium potentials are strictly positive.

Proof. For some \( k > 0 \) large enough, \( U^{-1} = k(\mathbb{I} - N) \) where \( N \geq 0 \) (and \( N1 \leq 1 \) for the row diagonally dominant case). In what follows we can assume that \( k = 1 \) (it is enough to consider the matrix \( kU \) instead of \( U \)).

From the equality \((\mathbb{I} - N)(\mathbb{I} + N + N^2 + \cdots N^p) = \mathbb{I} - N^{p+1} \) we get that
\[
\mathbb{I} + N + N^2 + \cdots N^p = U(\mathbb{I} - N^{p+1}) \leq U,
\]
and we deduce that the series \( \sum_{i=1}^{\infty} N^t \) is convergent and its limit is \( U \).

Consider now the matrix
\[
N_t = t \left( \left( \frac{1}{1 + t} \right)^{-1} - \mathbb{I} \right) = t \sum_{i=1}^{\infty} \left( \frac{1}{1 + t} \right)^i N^t.
\]
We have that \( N_t \geq 0 \) (and \( N_t \mathbf{1} \leq \mathbf{1} \) whenever \( N \mathbf{1} \leq \mathbf{1} \)). Therefore the matrix \( \mathbb{I} - N_t \) is an \( M \)-matrix (which is row diagonally dominant when \( M \) is so). On the other hand we have

\[
\mathbb{I} + tU = \mathbb{I} + t(\mathbb{I} - N)^{-1} = (t\mathbb{I} + \mathbb{I} - N)(\mathbb{I} - N)^{-1} = (1 + t) \left( \mathbb{I} - \frac{1}{1 + t} N \right)(\mathbb{I} - N)^{-1},
\]
from where we deduce that \( \mathbb{I} + tU \) is nonsingular and its inverse is

\[
(\mathbb{I} + tU)^{-1} = \frac{1}{1 + t} \left( \mathbb{I} - N \right) \left( \mathbb{I} - \frac{1}{1 + t} N \right)^{-1}
\]

\[
= \frac{1}{1 + t} \left( (\mathbb{I} - \frac{1}{1 + t} N)^{-1} - N \left( \mathbb{I} - \frac{1}{1 + t} N \right)^{-1} \right)
\]

\[
= \frac{1}{1 + t} \left( \sum_{l=0}^{\infty} (1 + t)^{-l} N^l - \sum_{l=0}^{\infty} (1 + t)^{-l} N^{l+1} \right)
\]

\[
= \frac{1}{1 + t} (\mathbb{I} - N_t).
\]

The only thing left to prove is that \( N_t \mathbf{1} < \mathbf{1} \) in the row diagonally dominant case, that is when \( N \mathbf{1} \leq \mathbf{1} \). For that it is enough to prove that \( N^l \mathbf{1} < \mathbf{1} \) for large \( l \). From the equality \( U = \sum_{l=1}^{\infty} N^l \), we deduce that \( \sum_{l=1}^{\infty} N^l \mathbf{1} < \infty \), and therefore \( N^l \mathbf{1} \) tend to zero as \( l \to \infty \). This proves the claim.

When \( k \) is not 1 we have the following equality

\[
(\mathbb{I} + tU)^{-1} = \frac{k}{t+k} (\mathbb{I} - \frac{k}{t+k} \sum_{l=1}^{\infty} (\frac{k}{t+k})^l N^l),
\]

where \( N = \mathbb{I} - \frac{1}{k} U^{-1} \).

Finally, assume that \( \mathbb{I} + tU \in bi\mathbb{P} \). Hence \( \mathbb{I} + \beta(\mathbb{I} + tU) \in bi\mathbb{P} \) for all \( \beta \geq 0 \). This implies that

\[
\mathbb{I} + \frac{\beta}{1 + \beta} tU \in bi\mathbb{P}.
\]

Now, it is enough to take \( \beta \geq 0 \) such that \( s = \frac{\beta}{1 + \beta} t \).

This Lemma has two immediate important consequences.

**Corollary 3.1** If \( U \in bi\mathbb{P} \) then \( \tau(U) = \infty \).

**Corollary 3.2** Let \( U \) a nonnegative matrix, then

\[
\tau(U) = \sup \{ t \geq 0 : \mathbb{I} + tU \in bi\mathbb{P} \}
\]

**Proof.** It is clear that \( \tau(U) \leq \sup \{ t \geq 0 : \mathbb{I} + tU \in bi\mathbb{P} \} \). On the other hand if \( \mathbb{I} + tU \in bi\mathbb{P} \) then we get \( \mathbb{I} + sU \in bi\mathbb{P} \) for all \( 0 \leq s \leq t \). This fact and the definition of \( \tau(U) \), implies the result. \( \square \)

**Proof. (Theorem 2.1)** We first assume that \( f(0) = 0 \). We have that \( U^{-1} = k(\mathbb{I} - P) \), for some \( k > 0 \) and \( P \) a sub-stochastic matrix. Without loss of generality we can assume \( k = 1 \), because it is enough to consider \( kU \) instead of \( U \) and \( \tilde{f}(x) = f(x/k) \) instead of \( f \).
Consider $M = (U^{-1}f(U))$. Take $i \neq j$ and compute
\[ M_{ij} = (U^{-1}f(U))_{ij} = (1 - p_{ii})f(U_{ij}) - \sum_{k \neq i} p_{ik}f(U_{kj}). \]
Since $1 - p_{ii} - \sum_{k \neq i} p_{ik} \geq 0$, which is equivalent to $\sum_{k} p_{ik} \leq 1$, and $f$ is convex we obtain
\[ \left(1 - \sum_{k} p_{ik}\right) f(0) + \sum_{k} p_{ik}f(U_{kj}) \geq f\left(\sum_{k} p_{ik}U_{kj}\right) = f(U_{ij}). \]
The last equality follows from the fact that $U^{-1} = I - P$. This shows that $M_{ij} \leq 0$. Consider now a positive vector $x$ such that $y' = x'U^{-1} > 0$ (see [9], Theorem 2.5.3). Then
\[ x'M = x'U^{-1}f(U) = y'f(U) > 0, \]
which implies, by the same cited theorem, that $M$ is a $M$-matrix. In particular $M$ is nonsingular and $\det(M) > 0$. So, $f(U)$ is nonsingular and $\det(f(U)) > 0$. Consider now $\rho$ the right equilibrium potential of $f(U)$. We have
\[ M\rho = U^{-1}f(U)\rho = U^{-1}1 = \mu_U \geq 0, \]
then $\rho = M^{-1}\mu_U \geq 0$, because $M^{-1}$ is a nonnegative matrix. This means that $f(U)$ possesses a nonnegative right equilibrium potential. Since $f(U)$ is nonsingular we also have a left equilibrium potential, which we do not know if it is nonnegative. Then the first part is proven under the extra hypothesis that $f(0) = 0$.

Assume now $a = f(0) > 0$, and consider $g(x) = f(x) - a$, which is a strictly increasing convex function. Obviously $f(U) = g(U) + a11'$, so
\[ \mu_{f(U)} = \frac{1}{1 + a\mu_g(U)}\mu_g(U) \geq 0, \quad \nu_{f(U)} = \frac{1}{1 + a\nu_g(U)}\nu_g(U), \]
where $\mu_g(U) = 1'\mu_g(U)$ and $\nu_g(U) = 1'\nu_g(U)$. Thus $f(U)$ has a nonnegative right equilibrium potential, and a left equilibrium potential. We need to prove that $f(U)$ is nonsingular, and $\det(f(U)) > 0$. This follows immediately from the equality
\[ f(U) = g(U)(I + a1\mu_g(U))1'. \]
Indeed we have
\[ f(U)^{-1} = g(U)^{-1} - \frac{a}{1 + a\mu_g(U)}\mu_g(U)1' \quad \text{and} \quad \det(f(U)) = \det(g(U))(1 + a\mu_g(U)), \]
from which the result is proven.

In the bi-potential case use $U'$ instead of $U$ to obtain the existence of a nonnegative left equilibrium potential for $f(U)$.
\[ \square \]
Proof. (Theorem 2.5) Using the same ideas as above we can assume that \( f(0) = 0 \).
Also we have \( U^{-1}(I + tf(U)) = M_t \) is a \( M \)-matrix, for all \( t \geq 0 \). Therefore \( I + tf(U) \) is nonsingular for all \( t \), and we denote by \( \mu_t \) and \( \nu_t \) the equilibrium potentials for \( I + tf(U) \).

Assume first that \( f(U) \) is in class \( T \) which means that
\[
\tau(f(U)) = \min\{ t > 0 : \mu_t \not\geq 0 \text{ or } \nu_t \not\geq 0 \}.
\]
We prove that for all \( t \geq 0 \), \( \mu_t, \nu_t \) are nonnegative. Since
\[
M_t \mu_t = U^{-1} 1 = \mu_U,
\]
we obtain that \( \mu_t = M_t^{-1} \mu_U \geq 0 \), because \( M_t^{-1} \) is a nonnegative matrix. Thus, \( \tau(f(U)) = \infty \) and \( f(U) \) is nonsingular. From Proposition 1.1 we get \( f(U) \in biP \).

Reciprocally if \( f(U) \in biP \) then \( \tau(f(U)) = \infty \) and the result follows. \( \square \)

Lemma 3.2 Assume that \( U \in P \). Then any principal square submatrix \( A \) of \( U \) is also in class \( P \). The same is true if we replace \( P \) by \( biP \).

Proof. By induction and a suitable permutation is enough to prove the result for \( A \) the restriction of \( U \) to \( \{1, \ldots, n-1\} \times \{1, \ldots, n-1\} \), where \( n \) is the size of \( U \). Assume that
\[
U = \begin{pmatrix} A & b \\ c' & d \end{pmatrix} \quad \text{and} \quad U^{-1} = \begin{pmatrix} \Lambda & -\zeta \\ -\theta' & \theta \end{pmatrix}.
\]
Since \( A^{-1} = \Lambda - \frac{1}{\theta} \zeta \theta' \) we obtain that the off diagonal elements of \( A^{-1} \) are non-positive. It is quite easy to see that the result will follow as soon as \( A^{-1} 1 \geq 0 \).

Since \( U \in P \) we have that \( \Lambda 1 - \zeta \geq 0 \) and \( \theta \geq \theta' 1 \). Therefore,
\[
A^{-1} 1 = \Lambda 1 - \frac{1}{\theta} \zeta \theta' 1 = \Lambda 1 - \frac{\theta' 1}{\theta} \zeta \geq \Lambda 1 - \zeta \geq 0.
\]

\( \square \)

In what follows given a vector \( a \) we denote by \( D_a \) the diagonal matrix, whose diagonal is \( a \).

Lemma 3.3 Assume \( U \in biP \) and \( \alpha \geq 1 \). If
\[
U = \begin{pmatrix} A & b \\ c' & d \end{pmatrix},
\]
then there exists a nonnegative vector \( \eta \) such that
\[
A^{(\alpha)} \eta = b^{(\alpha)}.
\]
Proof. We first perturb the matrix $U$ to have a positive matrix. Consider $\epsilon > 0$ and the positive matrix $U_\epsilon = U + \epsilon 1' \cdot 1$. It is direct to prove that

$$U_\epsilon^{-1} = U^{-1} - \frac{\epsilon}{1 + \epsilon \mu_U} \mu_U (\nu_U)',$$

where $\mu_U = 1' \mu_U$ is the total mass of $\mu_U$. Then $U_\epsilon \in bi\mathcal{P}$ and its equilibrium potentials are given by

$$\mu_{U_\epsilon} = \frac{1}{1 + \epsilon \mu_U} \mu_U, \quad \nu_{U_\epsilon} = \frac{1}{1 + \epsilon \nu_U} \nu_U.$$

We decompose the inverse of $U_\epsilon$ as

$$U_\epsilon^{-1} = \begin{pmatrix} \Lambda_\epsilon & \zeta_\epsilon \\ \zeta_\epsilon' & \theta_\epsilon \end{pmatrix},$$

and we notice that $U_\epsilon \zeta_\epsilon + \theta_\epsilon b_\epsilon = 0$ which implies that

$$b_\epsilon = U_\epsilon \lambda_\epsilon,$$

with $\lambda_\epsilon = -\frac{1}{\theta_\epsilon} \zeta_\epsilon \geq 0$. Also we mention here that $\lambda_\epsilon$ is a sub-probability vector, that is $1' \lambda_\epsilon \leq 1$. This follows from the fact that $U_\epsilon^{-1}$ is column diagonally dominant.

Take now the matrix $V_\epsilon = D_{b_\epsilon}^{-1} U_\epsilon$. It is direct to check that $V_\epsilon \in \mathcal{M}^{-1}$ and its equilibrium potentials are

$$\mu_{V_\epsilon} = \lambda_\epsilon, \quad \nu_{V_\epsilon} = D_{b_\epsilon} \nu_{U_\epsilon}.$$

Thus $V_\epsilon \in bi\mathcal{P}$ and we can apply Theorem 2.1 to get that the matrix $V_\epsilon^{(a)}$ posses a right equilibrium potential $\eta_\epsilon \geq 0$, that is, for all $i$

$$\sum_j (V_\epsilon^{(a)})_{ij} (\eta_\epsilon)_j = 1,$$

which is equivalent to

$$\sum_j \frac{(U_\epsilon)_{ij}^{(a)}}{(b_\epsilon)_{ij}^{(a)}} (\eta_\epsilon)_j = 1.$$

Hence

$$U_\epsilon^{(a)} \eta_\epsilon = b_\epsilon^{(a)}.$$

Recall that the matrix $U^{(a)}$ is nonsingular. Since obviously $U_\epsilon^{(a)} \to U^{(a)}$ as $\epsilon \to 0$, we get

$$\eta_\epsilon \to \eta = (U^{(a)})^{-1} b^{(a)},$$

and the result follows. \(\Box\)

Proof. (Theorem 2.2) Consider first the case where $U \in bi\mathcal{P}$. We already know that $U^{(a)}$ is nonsingular and that it has left and right nonnegative equilibrium potentials. Therefore, in order to prove that $U^{(a)} \in bi\mathcal{P}$ is enough to prove that $(U^{(a)})^{-1}$ is a $Z$-matrix.
that is for i ≠ j we have ((U(α)⁻¹)_{ij}) is non positive. An argument based on permutations shows that it is enough to prove the claim for i = 1, j = n, where n is the size of U.

Decompose U(α) and its inverse as follows

\[
U(α) = \begin{pmatrix} A(α) & b(α) \\ c(α)' & d(α) \end{pmatrix} \quad \text{and} \quad (U(α))^{-1} = \begin{pmatrix} Ω & -β \\ -α' & δ \end{pmatrix}
\]

We need to show that β ≥ 0. We notice that

\[
det(U(α)) = det\left(\begin{pmatrix} A(α) & b(α) \\ c(α)' & d(α) \end{pmatrix}\right) > 0
\]

which implies that

\[
b(α) = A(α) \begin{pmatrix} β \\ δ \end{pmatrix}.
\]

Therefore, \( \frac{β}{δ} = η ≥ 0 \), where η is the vector given in Lemma 3.3. Thus β ≥ 0 and the result is proven for the case U ∈ biP.

Now consider U the inverse of the M-matrix M. Using Theorem 2.5.3 in [9], we get the existence of two positive diagonal matrices D, E such that DME is a strictly row and column diagonally dominant M-matrix. Thus V = E⁻¹UD⁻¹ is in biP from where it follows that V(α) ∈ biP. Hence, U(α) = E(α)V(α)D(α) is the inverse of an M-matrix. The rest of the result is proven in a similar way.

Proof. (Theorem 2.3) By hypothesis we have U = I - P, where P ≥ 0 and P1 ≤ 1. We notice that U is diagonally dominant on each row, that is for all i, j

\[U_{ii} ≥ U_{ji}.\]

Also we notice that U = I + PU and therefore U_{ii} ≥ 1.

According to Theorem 2.2 we know that H = (U(α))⁻¹ is a row diagonally dominant M-matrix. The only thing left to prove is that the diagonal elements of H are dominated by one, that is H_{ii} ≤ 1 for all i. It is enough to prove this for i = n, where n is the size of U.

Consider the following decompositions

\[
U = \begin{pmatrix} A & b \\ c' & d \end{pmatrix} \quad \text{and} \quad (U(α))^{-1} = \begin{pmatrix} Ω & -β \\ -α' & δ \end{pmatrix}
\]

\[
U^{-1}U(α) = \begin{pmatrix} Ξ & -ζ \\ -χ' & ρ \end{pmatrix}
\]

A direct computation gives that

\[γ = ρδ + χ'β ≥ ρδ.\]
Since by hypothesis $\gamma \leq 1$, to conclude that $\delta \leq 1$ it is enough to prove that $\rho \geq 1$. We have
\[
\rho = (1 - p_{nn})U_{nn}^{\alpha} - \sum_{j \neq n} p_{nj}U_{jn}^{\alpha} = U_{nn}^{\alpha} - \sum_{j} p_{nj}U_{jn}^{\alpha} = U_{nn}^{\alpha} - \sum_{j} p_{nj}U_{jn}U_{jn}^{\alpha-1}.
\]
On the other hand we have $U_{jn}^{\alpha-1} \leq U_{nn}^{\alpha-1}$ and $\sum_{j} p_{nj}U_{jn} = U_{nn} - 1$ from where we deduce
\[
\rho \geq U_{nn}^{\alpha-1} \geq 1.
\]
This finishes the case when $P1 \leq 1$. The rest of the result is proven by using $U'$ instead of $U$. $\square$

4 Proof of Theorem 2.4

Let $n$ be the dimension of $U$. Notice that $U$ is a GUM if and only if $n \leq 2$ or every principal sub-matrix of size 3 is a GUM.

Since by hypothesis the matrix $I + tU$ is a bi-potential, it is diagonally dominant
\[
1 + tU_{ii} \geq tU_{ij},
\]
and we deduce that $U_{ii} \geq U_{ij}$ (by taking $t \to \infty$). This proves the result when $n \leq 2$. So, in the sequel we assume $n \geq 3$.

Consider $A$ any principal sub-matrix of $U$, of size $3 \times 3$. Since $I + tf(A)$ is a principal sub-matrix of $I + tf(U)$, we deduce that $I + tf(A) \in \mathcal{B}\mathcal{P}$ (as long as $I + tf(U) \in \mathcal{B}\mathcal{P}$). If the result holds for the $3 \times 3$ matrices, we deduce that $A$ is GUM implying that $U$ is also a GUM.

Thus, in what follows we consider that $U$ is a $3 \times 3$ matrix that verifies the hypothesis of the Theorem. After a suitable permutation we can further assume that
\[
U = \begin{pmatrix}
a & b_1 & b_2 \\
c_1 & d & \alpha \\
c_2 & \beta & c
\end{pmatrix},
\]
where $\alpha = \min\{U_{ij} : i \neq j\} = \min\{U\}$ and $\beta = \min\{U_{ji} : U_{ij} = \alpha, i \neq j\}$.

Since $U$ is diagonally dominant we have $\min\{a, d, e\} \geq \alpha$. Take $f$ increasing such that $f(\alpha) = 0$ and $f(x) > 0$ for $x > \alpha$. Then,
\[
I + f(U) = \begin{pmatrix}
1 + f(a) & f(b_1) & f(b_2) \\
f(c_1) & 1 + f(d) & 0 \\
f(c_2) & f(\beta) & 1 + f(e)
\end{pmatrix},
\]

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is a bi\( P \)-matrix whose inverse we denote by

\[
\begin{pmatrix}
\delta & -\rho_1 & -\rho_2 \\
-\theta_1 & \gamma_1 & -\gamma_2 \\
-\theta_2 & -\gamma_3 & \gamma_4
\end{pmatrix}.
\]

In particular we obtain

\[
\begin{pmatrix}
1 + f(d) & 0 \\
f(\beta) & 1 + f(e)
\end{pmatrix}^{-1} = \begin{pmatrix}
\gamma_1 & -\gamma_2 \\
-\gamma_3 & \gamma_4
\end{pmatrix} - \frac{1}{\delta} \begin{pmatrix}
\theta_1 \\
\theta_2
\end{pmatrix} \begin{pmatrix}
\rho_1 \\
\rho_2
\end{pmatrix},
\]

and we deduce that

\[0 = \gamma_2 = \theta_1 \rho_2. \quad (4.1)\]

• **Case** \( \rho_2 = 0 \). We deduce that \( f(b_2) = 0 \), and then

\[b_2 = \alpha, \text{ and } c_2 \geq \beta, \quad (4.2)\]

where the last conclusion follows from the definition of \( \beta \). Therefore we have that

\[U = \begin{pmatrix}
a & b_1 & \alpha \\c_1 & d & \alpha \\c_2 & \beta & e
\end{pmatrix}, \quad (4.3)\]

and we should prove that \( U \) is GUM.

Now consider another increasing function \( g \) such that \( g(\beta) = 0 \) and \( g(x) > 0 \) for \( x > \beta \). Then,

\[\mathbb{I} + g(U) = \begin{pmatrix}
1 + g(a) & g(b_1) & 0 \\
g(c_1) & 1 + g(d) & 0 \\
g(c_2) & 0 & 1 + g(e)
\end{pmatrix}.
\]

Its inverse is of the form

\[
\begin{pmatrix}
\tilde{\delta} & -\tilde{\rho}_1 & 0 \\
-\tilde{\theta}_1 & \tilde{\gamma}_1 & 0 \\
-\tilde{\theta}_2 & -\tilde{\gamma}_3 & \tilde{\gamma}_4
\end{pmatrix}.
\]

As before we deduce that \( 0 = \tilde{\gamma}_3 = \tilde{\theta}_2 \tilde{\rho}_1 \).

– **Subcase** \( \tilde{\theta}_2 = 0 \). We have \( g(c_2) = 0 \) which implies \( c_2 = \beta \). Thus, in this situation we have that \( U \) is

\[U = \begin{pmatrix}
a & b_1 & \alpha \\c_1 & d & \alpha \\\beta & \beta & e
\end{pmatrix}.
\]

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By permuting rows and columns 1, 2, if necessary, we can assume that $b_1 \leq c_1$. Consider the situation where $c_1 < \beta$, of course implicitly we should have $\alpha < \beta$. Under a suitable increasing transformation $h$ we have

$$
I + h(U) = \begin{pmatrix}
1 + h(a) & 0 & 0 \\
0 & 1 + h(d) & 0 \\
h(\beta) & h(\beta) & 1 + h(e)
\end{pmatrix},
$$

and its inverse

$$
\begin{pmatrix}
\frac{1}{1+h(a)} & 0 & 0 \\
0 & \frac{1}{1+h(d)} & 0 \\
\frac{1}{h(\beta)} & \frac{1}{1+h(e)} & 1
\end{pmatrix}.
$$

The sum of the third row is then

$$
\frac{1}{1+h(e)} \left( 1 - h(\beta) \left( \frac{1}{1+h(a)} + \frac{1}{1+h(d)} \right) \right),
$$

and this quantity can be made negative by choosing an appropriate function $h$. The idea is to make $h(\beta) \to \infty$ and

$$
\frac{h(\beta)}{\max\{h(a), h(d)\}} \to 1.
$$

Therefore, $c_1 \geq \beta$ and $U$ is a GUM.

**Subcase** $\bar{\rho}_1 = 0$. We have $g(b_1) = 0$ and then $b_1 \leq \beta$. Take again an increasing function, denoted by $\ell$, such that

$$
I + \ell(U) = \begin{pmatrix}
1 + \ell(a) & 0 & 0 \\
\ell(c_1) & 1 + \ell(d) & 0 \\
\ell(c_2) & 0 & 1 + \ell(e)
\end{pmatrix},
$$

and its inverse

$$
\begin{pmatrix}
\frac{1}{1+\ell(a)} & 0 & 0 \\
\frac{\ell(c_1)}{(1+\ell(a))(1+\ell(d))} & \frac{1}{1+\ell(d)} & 0 \\
\frac{\ell(c_2)}{(1+\ell(a))(1+\ell(e))} & 0 & \frac{1}{1+\ell(e)}
\end{pmatrix}.
$$

The sum of the first column is

$$
\frac{1}{1+\ell(a)} \left( 1 - \frac{\ell(c_1)}{(1+\ell(d))} - \frac{\ell(c_2)}{(1+\ell(e))} \right),
$$

which can be made negative by repeating a similar argument as before, if both $c_1 > \beta$ and $c_2 > \beta$.  

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So, if \( c_1 > \beta \) we conclude that \( c_2 \leq \beta \), but we know that \( c_2 \geq \beta \) (see \( 1.2 \)) and we deduce that \( c_2 = \beta \). Thus, \( \alpha \leq b_1 \leq \beta < c_1 \) and

\[
U = \begin{pmatrix}
a & b_1 & \alpha \\
c_1 & d & \alpha \\
\beta & \beta & e
\end{pmatrix},
\]

which is a GUM.

Therefore we can continue under the hypothesis \( c_1 \leq \beta \leq c_2 \).

* **Subsubcase** \( b_1 < \beta \). Again we must have \( \alpha < \beta \). Under this conditions we have that \( c_2 > \alpha \). Using an increasing function \( k \) we get

\[
I + k(U) = \begin{pmatrix}
1 + k(a) & 0 & 0 \\
k(c_1) & 1 + k(d) & 0 \\
k(c_2) & k(\beta) & 1 + k(e)
\end{pmatrix},
\]

and its inverse is

\[
\begin{pmatrix}
\frac{1}{1+k(a)} & 0 & 0 \\
-\frac{k(c_1)}{(1+k(a))(1+k(d))} & \frac{1}{1+k(d)} & 0 \\
-\frac{k(c_2)(1+k(d))-k(\beta)k(c_1)}{(1+k(a))(1+k(d))(1+k(e))} & -\frac{k(\beta)}{(1+k(d))(1+k(e))} & \frac{1}{1+k(e)}
\end{pmatrix}.
\]

The sum of the third row is

\[
\frac{1}{(1+k(e))} \left( 1 - \frac{k(c_2)}{1+k(a)} + \frac{k(\beta)k(c_1)}{(1+k(a))(1+k(d))} - \frac{k(\beta)}{1+k(d)} \right).
\]

If \( c_1 < \beta \) we can assume that \( k(c_1) = 0 \), and we can make this sum to be negative by choosing large \( k \). Thus we must have \( c_1 = \beta \), in which case the sum under study is proportional to

\[
1 - \frac{k(c_2)}{1+k(a)} + \frac{k(\beta)^2}{(1+k(a))(1+k(d))} - \frac{k(\beta)}{1+k(d)} \tag{4.4}
\]

If \( c_2 = \beta \) then

\[
U = \begin{pmatrix}
a & b_1 & \alpha \\
\beta & d & \alpha \\
\beta & \beta & e
\end{pmatrix},
\]

is a GUM. So we must analyze the case where \( c_2 > \beta \) in \( (4.3) \). We will arrive to a contradiction by taking an asymptotic as before. Consider a fixed number \( \lambda \in (0, 1) \). Choose a family \( (k_r)_{r \in \mathbb{N}} \) such that as \( r \to \infty \)

\[
k_r(\beta) \to \infty, \quad \frac{k_r(\beta)}{k_r(c_2)} \to \lambda, \quad \frac{k_r(c_2)}{k_r(a)} \to 1, \quad \frac{k_r(d)}{k_r(a)} \to \phi,
\]

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where $\phi = 1$ if $d > \beta$, and $\phi = \lambda$ if $d = \beta$. The asymptotic of (4.4) is then

$$1 - 1 + \frac{\lambda^2}{\phi} - \frac{\lambda}{\phi}.$$

This quantity is strictly negative for the two possible values of $\phi$, which is a contradiction, and therefore $c_2 = \beta$.

To finish with the **Subcase** $\tilde{\rho}_1 = 0$, which will in turn finish with **Case** $\rho_2 = 0$, we consider

* **Subcase** $b_1 = \beta$. We recall that we are under the restrictions $c_1 \leq \beta \leq c_2$ and

$$U = \begin{pmatrix} a & \beta & \alpha \\ c_1 & d & \alpha \\ c_2 & \beta & e \end{pmatrix}.$$

Notice that if $c_2 = \beta$ then $U$ is GUM. So for the rest of this subcase we assume $c_2 > \beta$. Also if $c_1 = \alpha$ we can permute 1 and 2 to get

$$\Pi U \Pi' = \begin{pmatrix} d & \alpha & \alpha \\ \beta & a & \alpha \\ \beta & c_2 & e \end{pmatrix},$$

which is also in NBF, and $U$ is a GUM. Thus we can assume that $c_1 > \alpha$, and again of course we have $\alpha < \beta$.

Take an increasing function $m$ such that

$$\mathbb{I} + m(U) = \begin{pmatrix} 1 + m(a) & m(\beta) & 0 \\ m(c_1) & 1 + m(d) & 0 \\ m(c_2) & m(\beta) & 1 + m(e) \end{pmatrix},$$

We take the asymptotic under the following restrictions:

$$\frac{m(\beta)}{m(a)} \to \lambda \in (0, 1), \frac{m(c_1)}{m(a)} \to \lambda, \frac{m(e)}{m(a)} \to 1, \frac{m(c_2)}{m(a)} \to 1, \frac{m(d)}{m(a)} \to \phi,$$

where $\phi = 1$ if $d > \beta$, and it is $\lambda$ if $d = \beta$. The limiting matrix for $\frac{1}{m(a)}(\mathbb{I} + m(U))$ is

$$V = \begin{pmatrix} 1 & \lambda & 0 \\ \lambda & \phi & 0 \\ 1 & \lambda & 1 \end{pmatrix},$$

whose determinant is $\Delta = \phi - \lambda^2 > 0$. Therefore $V$ must be in $bi\mathcal{P}$. On the other hand the inverse of $V$ is given by

$$V^{-1} = \frac{1}{\Delta} \begin{pmatrix} \phi & -\lambda & 0 \\ -\lambda & 1 & 0 \\ -(\phi - \lambda^2) & 0 & \phi - \lambda^2 \end{pmatrix},$$

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and the sum of the first column is
\[
\frac{\lambda^2 - \lambda}{\Delta} < 0,
\]
which is a contradiction.

This finishes with the subcase \( \rho_2 = 0 \) and we return to (4.1) to consider now the following case

- **Case** \( \theta_1 = 0 \). Under this condition we get \( c_1 = \alpha \) and

\[
U = \begin{pmatrix}
a & b_1 & b_2 \\
\alpha & d & \alpha \\
c_2 & \beta & e
\end{pmatrix}.
\]

Consider the transpose of \( U \) and permute on it 2 and 3, to obtain the matrix

\[
\tilde{U} = \begin{pmatrix}
a & c_2 & \alpha \\
b_2 & e & \alpha \\
b_1 & \beta & d
\end{pmatrix},
\]

where now \( b_1 \geq \beta \). Clearly the matrix \( \tilde{U} \) verifies the hypothesis of the Theorem and has the shape of (4.3), that is we are in the "case \( \rho_2 = 0 \)" which we already know implies that \( \tilde{U} \) is GUM. Therefore \( U \) itself is GUM.

\[\square\]

5 Filtered Matrices, sufficient conditions for classes \( biP \) and \( \mathcal{T} \)

A class of matrices of our interest is the class of filtered matrices, which turn to be a generalization of GUM matrices. They were introduced as operators in [7] to generalize the class of self adjoint operators whose spectral decomposition is written in terms of conditional expectations (see for instance [3], [4] and [10]).

The basic tool to construct these matrices are partitions of \( \mathcal{J}_n = \{1, \cdots, n\} \). The components of a partition \( \mathcal{R} \) are called atoms. We denote by \( \sim \) the equivalence relation induced by \( \mathcal{R} \). Then \( i, j \) are in the same atom of \( \mathcal{R} \) if and only if \( i \sim j \).

A partition \( \mathcal{R} \) is coarser or equal than \( \mathcal{Q} \) if the atoms of \( \mathcal{Q} \) are contained in the atoms of \( \mathcal{R} \). We denote this (partial) order relation by \( \mathcal{R} \preceq \mathcal{Q} \). For example in \( \mathcal{J}_4 \) we have \( \mathcal{R} = \{\{1, 2\}, \{3, 4\}\} \preceq \mathcal{Q} = \{\{1\}, \{2\}, \{3, 4\}\} \). The coarsest partition is the trivial one \( \mathcal{N} = \{\mathcal{J}_n\} \) and the finer one is the discrete partition \( \mathcal{F} = \{\{1\}, \{2\}, \cdots, \{n\}\} \).
**Definition 5.1** A filtration is an strictly increasing sequence of comparable partitions $\mathcal{F} = \{\mathcal{R}_0 \prec \mathcal{R}_1 \prec \cdots \prec \mathcal{R}_k\}$. A filtration in wide sense is an increasing sequence of comparable partitions $\mathcal{G} = \{\mathcal{R}_0 \preceq \mathcal{R}_1 \preceq \cdots \preceq \mathcal{R}_k\}$.

The difference between these two concepts is that in the latter repetition of partitions is allowed.

**Definition 5.2** A filtration $\mathcal{F} = \{\mathcal{R}_0 \prec \cdots \prec \mathcal{R}_k\}$ is called dyadic if each non-trivial atom of $\mathcal{R}_s$ is divided into two atoms of $\mathcal{R}_{s+1}$.

The following example is the simplest dyadic filtration $\mathcal{F} = \{\mathcal{N} \prec \{\{1\}, \{2, \cdots, n\}\} \prec \cdots \prec \{\{1\}, \{2\}, \cdots, \{i\}, \{i+1, \cdots, n\}\} \prec \cdots \prec \mathcal{F}\}$.

Each partition $\mathcal{R}$ induces an incidence matrix $F := F(\mathcal{R})$ given by

$$F_{ij} = \begin{cases} 1 & \text{if } i \overset{\mathcal{R}}{\sim} j \\ 0 & \text{otherwise} \end{cases}$$

A vector $v \in \mathbb{R}^n$ is said to be $\mathcal{R}$-measurable if $v$ is constant on the atoms of $\mathcal{R}$, that is $i \overset{\mathcal{R}}{\sim} j \Rightarrow v_i = v_j$.

This can be expressed in terms of standard matrix operations as

$$F(\mathcal{R})v = D_{w_\mathcal{R}}v,$$

where $w_\mathcal{R} = F(\mathcal{R})1$ is the vector of sizes of the atoms. Recall that $D_z$ is the diagonal matrix associated to the vector $z$. The set of $\mathcal{R}$-measurable vectors is a linear subspace of $\mathbb{R}^n$.

**Definition 5.3** A matrix $U$ is said to be filtered if there exists a filtration in wide sense $\mathcal{G} = \{\mathcal{Q}_0 \preceq \mathcal{Q}_1 \preceq \cdots \preceq \mathcal{Q}_l\}$, vectors $a_1, \cdots, a_l, b_1, \cdots, b_l$ with the restriction that $a_s, b_s$ are $\mathcal{Q}_{s+1}$-measurable, such that

$$U = \sum_{s=0}^l D_{a_s}F(\mathcal{Q}_s)D_{b_s}, \quad (5.1)$$

There is no loss of generality if we assume that $\mathcal{Q}_0 = \mathcal{N}$ and $\mathcal{Q}_l = \mathcal{F}$, that is $F(\mathcal{Q}_0) = 11'$ and $F(\mathcal{Q}_l) = I$, the identity matrix. Let us see that (5.1) can be simply written in terms of a filtration. Indeed, notice that if $a_s$ and $b_s$ are $\mathcal{Q}_s$-measurable then

$$D_{a_s}F(\mathcal{Q}_s)D_{b_s} = D_{a_s}D_{b_s}F(\mathcal{Q}_s) = D_{a_s \odot b_s}F(\mathcal{Q}_s),$$
where the vector \( \mathbf{a}_s \odot \mathbf{b}_s \) is the Hadamard product of \( \mathbf{a}_s \) and \( \mathbf{b}_s \) which is also \( \mathcal{Q}_s \)-measurable. Hence a sum of terms of the form

\[
D_{a_s} F(\mathcal{Q}_s) D_{b_s} + D_{a_{s+1}} F(\mathcal{Q}_{s+1}) D_{b_{s+1}} + \cdots + D_{a_{s+r}} F(\mathcal{Q}_{s+r}) D_{b_{s+r}},
\]

with \( \mathcal{R} = \mathcal{Q}_s = \cdots = \mathcal{Q}_{s+r} \), can be reduced to the sum of two terms as

\[
D_C F(\mathcal{R}) + D_{a_{s+r}} F(\mathcal{R}) D_{b_{s+r}},
\]

where \( C = \sum_{k=0}^{r-1} a_{s+k} \odot b_{s+k} \), which is \( \mathcal{R} \)-measurable. In this way representation \( \text{5.1} \) can be written as

\[
U = \sum_{s=0}^{k} D_{C_s} F(\mathcal{R}_s) + D_{m_s} F(\mathcal{R}_s) D_{n_s}, \tag{5.2}
\]

where \( \mathcal{F} = \{ \mathcal{R}_0 \prec \mathcal{R}_1 \prec \cdots \prec \mathcal{R}_k \} \) is a filtration, \( \mathcal{N} = \mathcal{R}_0 \), \( \mathcal{F} = \mathcal{R}_k \), \( C_s \) is \( \mathcal{R}_s \)-measurable, \( m_s, n_s \) are \( \mathcal{R}_{s+1} \)-measurable and \( m_k = 0 \). We shall always consider this reduced representation of \( \text{5.1} \), and we shall say that \( U \) is \textit{filtered} with respect to the filtration \( \mathcal{F} \).

If all \( m_s, n_s \) are \( \mathcal{R}_s \)-measurable then \( \text{5.1} \) reduces to the form

\[
U = \sum_{s=0}^{k} D_{C_s+m_s \odot n_s} F(\mathcal{R}_s), \tag{5.3}
\]

and \( U \) is a symmetric matrix.

We are mainly interested in a decomposition like \( \text{5.2} \) with the vectors \( m_s, n_s \) having the following special structure:

\[
m_s = \Gamma_s \odot \mathbf{p}_s, \hspace{1em} n_s = \mathbf{q}_s, \tag{5.4}
\]

where \( \Gamma_s \) is \( \mathcal{R}_s \)-measurable and \( \{ \mathbf{p}_s, \mathbf{q}_s \} \) is a \( \mathcal{R}_{s+1} \)-measurable partition, that is, they are \( \mathcal{R}_{s+1} \)-measurable \( \{0, 1\} \)-valued vectors with disjoint support: \( \mathbf{p}_s \odot \mathbf{q}_s = 0 \) and \( \mathbf{p}_s + \mathbf{q}_s = 1 \). If this is the case we say that \( U \) is a Special Filtered Matrix (SFM)

\[
U = \sum_{s=0}^{k} D_{C_s} F(\mathcal{R}_s) + D_{\Gamma_s} D_{\mathbf{p}_s} F(\mathcal{R}_s) D_{\mathbf{q}_s}. \tag{5.5}
\]

Notice that \( \Gamma_k = 0 \).

It is not difficult to see that every CBF matrix is filtered. This is done by induction. Assume that

\[
U = \begin{pmatrix} A & \alpha \mathbf{1}_{1-p} \mathbf{1}'_{1-p} \\ \beta \mathbf{1}_{1-p} \mathbf{1}'_{1-p} & B \end{pmatrix}.
\]

20
Define $\mathcal{R}_0 = \mathcal{N}$ and $\mathcal{R}_1 = \{\{1, \ldots, p\}, \{p + 1, \ldots, n\}\}$. Take $C_0 = \alpha 1_n, \Gamma_0 = (\beta - \alpha) 1_n, p_0 = (0_p, 1_{n-p})', q_0 = (1_p, 0_{n-p})'$. Then

$$D_{C_0}F(\mathcal{R}_0) + D_{\Gamma_0}D_{p_0}F(\mathcal{R}_0)D_{q_0} = \begin{pmatrix} \alpha 1_p 1_p' & \alpha 1_p 1_{n-p}' \\ \beta 1_{n-p} 1_p' & \alpha 1_{n-p} 1_{n-p}' \end{pmatrix}.$$  

The key step is that $A - \alpha, B - \alpha$ are also in CBF. We have that $C_0, \Gamma_0$ are $\mathcal{R}_0$-measurable and $p_0, q_0$ is a $\mathcal{R}_1$-measurable partition. We also notice that if $0 \leq \alpha \leq \beta$ then $C_0 \geq 0, \Gamma_0 \geq 0$.

The induction also shows that $U$ can be decomposed as in (5.4) $U = \sum_{s=0}^{k} D_{C_s}F(\mathcal{R}_s) + D_{\Gamma_s}D_{p_s}F(\mathcal{R}_s)D_{q_s}$, where $\mathcal{F} = \{\mathcal{R}_0 \prec \cdots \prec \mathcal{R}_k\}$ is a dyadic filtration, $C_s, \Gamma_s$ are $\mathcal{R}_s$-measurable and $\{p_s, q_s\}$ is a $\mathcal{R}_{s+1}$-measurable partition.

We summarize now the representation form for the class of CBF, NBF and GUM matrices.

**Proposition 5.1** $V$ is a permutation of a CBF if and only if there exists a dyadic filtration $\mathcal{F} = \{\mathcal{R}_0 \prec \cdots \prec \mathcal{R}_k\}$, a sequence of vectors $C_0, \ldots, C_k, \Gamma_0, \ldots, \Gamma_k$ verifying that for all $i$: $C_s, \Gamma_s$ are $\mathcal{R}_s$-measurable and a sequence $\{p_s, q_s\}$ of $\mathcal{R}_{s+1}$-measurable partitions, such that

$$V = \sum_{s=0}^{k} D_{C_s}F(\mathcal{R}_s) + D_{\Gamma_s}D_{p_s}F(\mathcal{R}_s)D_{q_s},$$

that is $V$ is a SFM. Also $V$ is a permutation of an increasing CBF matrix if and only if there is a decomposition where $\Gamma_0, C_s, \Gamma_s : s = 1 \ldots, k$, are nonnegative. On the top of this $V$ is a nonnegative matrix if and only if $C_0$ is nonnegative.

Moreover, $V$ is a GUM if and only if $C_s, \Gamma_s : i = 0, \cdots, k$ are non-negative, and for $i = 0, \ldots, k - 1$ it holds

$$\Gamma_s \leq C_{s+1} + \Gamma_{s+1}.$$  

(5.6)

Finally, $V$ is an ultrametric matrix if and only if there is a decomposition with $\Gamma_s = 0$ for all $s$. □

**Remark 5.1** We can assume without loss of generality that each $p_s, q_s$ is obtained as follows. The nontrivial atoms $A_1, \cdots A_r$ of $\mathcal{R}_s$ are divided into the new atoms

$$A_{1,1}, A_{1,2}, \cdots, A_{r,1}, A_{r,2}$$

of $\mathcal{R}_{s+1}$. Consider also $B_1, \cdots B_r$ the set of trivial atoms in $\mathcal{R}_s$ (that is the singleton atoms). Take $q_s$ be the indicator of $A_{1,1} \cup \cdots \cup A_{r,1}$, $p_s$ be the indicator of $A_{1,2} \cup \cdots \cup A_{r,2} \cup B_1 \cup \cdots \cup B_r$ and $\Gamma_s = 0$ on $B = B_1 \cup \cdots \cup B_r$, which is $\mathcal{R}_s$-measurable. We point out that the partition $\mathcal{R}_{s+1}$ is obtained from $\mathcal{R}_s$ refined by $p_s$. Also the important relation holds

$$D_{p_s}F(\mathcal{R}_s)p_s = D_{p_s}F(\mathcal{R}_{s+1})1$$  

(5.7)
Example 5.1 Consider the CBF matrix

\[
U = \begin{pmatrix}
    a & \alpha_2 & \alpha_1 & \alpha_1 \\
    \beta_2 & b & \alpha_1 & \alpha_1 \\
    \beta_1 & \beta_1 & c & \hat{\alpha}_2 \\
    \beta_1 & \beta_1 & \beta_2 & d
\end{pmatrix}.
\]

U is a NBF if the following constraints are verified: \(\alpha_1 \leq \beta_1\), \(\alpha_1 \leq \min\{\alpha_2, \hat{\alpha}_2\}\), \(\beta_1 \leq \min\{\beta_2, \beta_2\}\), \(\alpha_2 \leq \beta_2\), \(\hat{\alpha}_2 \leq \hat{\beta}_2\) and finally the diagonal dominates over each row and column, that is \(\beta_2 \leq \min\{a, b\}, \beta_2 \leq \min\{c, d\}\).

U is filtered with respect to the dyadic filtration \(\mathcal{R}_0 = \{1, 2, 3, 4\} \prec \mathcal{R}_1 = \{\{1, 2\}, \{3, 4\}\} \prec \mathcal{R}_2 = \{\{1\}, \{2\}, \{3\}, \{4\}\}\) and can be written as

\[
U = D_{C_0} F(\mathcal{R}_0) + D_{\Gamma_0} D_{p_0} F(\mathcal{R}_0) D_{q_0} + D_{C_1} F(\mathcal{R}_1) + D_{\Gamma_1} D_{p_1} F(\mathcal{R}_1) D_{q_1} + D_{C_2} F(\mathcal{R}_2),
\]

where

\[
C_0 = \begin{pmatrix}
    \alpha_1 \\
    \alpha_1 \\
    \alpha_1 \\
    \alpha_1
\end{pmatrix}, \quad \Gamma_0 = \begin{pmatrix}
    \beta_1 - \alpha_1 \\
    \beta_1 - \alpha_1 \\
    \beta_1 - \alpha_1 \\
    \beta_1 - \alpha_1
\end{pmatrix}, \quad p_0 = \begin{pmatrix}
    0 \\
    0 \\
    1 \\
    0
\end{pmatrix}, \quad q_0 = \begin{pmatrix}
    1 \\
    1 \\
    0 \\
    0
\end{pmatrix},
\]

\[
C_1 = \begin{pmatrix}
    \alpha_2 - \alpha_1 \\
    \alpha_2 - \alpha_1 \\
    \hat{\alpha}_2 - \alpha_1 \\
    \hat{\alpha}_2 - \alpha_1
\end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix}
    \beta_2 - \alpha_2 \\
    \beta_2 - \alpha_2 \\
    \hat{\beta}_2 - \hat{\alpha}_2 \\
    \hat{\beta}_2 - \hat{\alpha}_2
\end{pmatrix}, \quad p_1 = \begin{pmatrix}
    0 \\
    1 \\
    0 \\
    0
\end{pmatrix}, \quad q_1 = \begin{pmatrix}
    1 \\
    1 \\
    0 \\
    0
\end{pmatrix},
\]

and

\[
C_2 = \begin{pmatrix}
    a - \alpha_2 \\
    b - \alpha_2 \\
    c - \hat{\alpha}_2 \\
    d - \hat{\alpha}_2
\end{pmatrix}.
\]

The constrains are translated into: the positivity of these vectors and the ones induced by (5.7). We point out that we can also choose, for example, \(\Gamma_1 = (0, \beta_2 - \alpha_2, 0, \hat{\beta}_2 - \hat{\alpha}_2)'\), but in this case \(\Gamma_1\) is not \(\mathcal{R}_1\)-measurable. As we will see in subsection (5.1) this measurability condition will play an important role.

Example 5.2 Consider the nonnegative CBF matrix

\[
U = \begin{pmatrix}
    2 & 2 & 2 \\
    2 & 2 & 1 \\
    2 & 1 & 2
\end{pmatrix}.
\]

This matrix is a SFM and can be decomposed as in (5.7). Nevertheless no such decomposition can have all terms nonnegative. In particular no permutation of \(U\) is an increasing CBF.
Remark 5.2 Notice that the class of CBF is stable under Hadamard functions. Nevertheless there are examples of filtered matrices for which \( f(U) \) is not filtered. Consider the matrix

\[
U = D_\alpha F_1 + D_a F_1 D_b + D_\beta F_2,
\]

where \( F_1 = F(N) = 11' \) and \( F_2 = I \). We have \( \alpha \) is a constant vector, and we confound it with the constant \( \alpha \in \mathbb{R} \). The vectors \( a, b, \beta \) are all \( F \)-measurable. Then \( U \) is filtered and moreover

\[
U = \alpha + ab' + D_\beta \tag{5.8}
\]

Take \( \alpha = \beta = 0 \) and \( a = (2, 3, 5, 7)' \) and \( b = (11, 13, 17, 19)' \). Then all the entries of \( U \) are different. As \( f \) runs all possible functions \( f(U) \) runs over all \( 4 \times 4 \) matrices. This implies that some of them cannot be written as in (5.8), because in this representation we have at most 13 free variables. Still is possible that each \( f(U) \) is decomposable as in (5.7) using maybe a different filtration. A more detailed analysis shows that this is not the case. For example if we choose the filtration \( N \prec \{(1, 2), (3, 4)\} \prec F \) then every matrix \( V \) filtered with respect to this filtration verifies that

\[
V_{13} = V_{23} = V_{14} = V_{24}.
\]

Matrices of the type \( F(R) \) are related to conditional expectations (in probability theory). Indeed, let \( R = \{A_1, A_2, \cdots, A_r\} \) and \( n_\ell = \#(A_\ell) \) be the size of each atom. It is direct that \( w = w_R = F(R)1 \) is a \( R \)-measurable vector that verifies \( w_i = n_\ell \) for \( i \in A_\ell \). Then

\[
E_R = D_w^{-1} F(R) D_w^{-1}
\]

is the matrix of conditional expectation with respect to the \( \sigma \)-algebra generated by \( R \). This matrix \( E = E_R \) satisfies:

\[
EE = E; \quad E' = E; \quad E1 = 1;
\]

\[
\forall v \in \mathbb{R}, Ev \text{ is } R \text{ - measurable;}
\]

if \( v \) is \( R \)-measurable, then \( Ev = v \).

Therefore, \( E \) is the orthogonal projection over the subspace of all \( R \)-measurable vectors. In the case of the trivial partition \( N \) one gets \( E_N = \frac{1}{n} 11' \) the mean operator.

Remark 5.3 The \( L^2 \) space associated to \( \{1, \cdots, n\} \) endowed with the counting measure, is identified with \( \mathbb{R}^n \) with the standard euclidian scalar product. In this way each vector of \( \mathbb{R}^n \) can be seen as a function in \( L^2 \), and \( E \) is an orthogonal projection. The product \( D_v E \) (as matrices) is the product of the operators \( D_v \) and \( E \), where \( D_v \) is the multiplication by the function \( v \). Notice that \( ED_v \) and \( E(v) \) are quite different. The former is an operator (a matrix) and the latter is a function (vector). They are related by \( E(v) = ED_v(1) \), where \( 1 \) is the constant function.
Let \( R, Q \) be two partitions, then \( R \preceq Q \) is equivalent to \( \mathbb{E}_R \mathbb{E}_Q = \mathbb{E}_Q \mathbb{E}_R = \mathbb{E}_R \). This commutation relation can be written as a commutation relation for \( F(R) \) and \( F(Q) \). In fact,
\[
F(R)F(Q) = \mathbb{E}_R D_{w_R} \mathbb{E}_Q D_{w_Q} = \mathbb{E}_R \mathbb{E}_Q D_{w_R} D_{w_Q} = \mathbb{E}_R D_{w_R} D_{w_Q} = F(R)D_{w_Q};
\]
\[
F(Q)F(R) = (F(R)F(Q))' = D_{w_Q} F(R).
\]

### 5.1 An algorithm for filtered matrices: conditions to be in \( biP \)

In this section we explain a backward algorithm to determine when a filtered matrix is in class \( biP \). Assume that \( U \) has a representation as in (5.1)
\[
U = \sum_{s=0}^\ell D_{a_s} F(Q_s) D_{b_s},
\]
where we assume further that \( a_s, b_s \) are all non-negative. In particular \( U \) is a non-negative matrix.

We introduce the conditional expectations \( \mathbb{E}_s = \mathbb{E}_Q = D_{F(Q_s)}^{-1} F(Q_s) \) and the normalized factors: \( a_s = a_s \odot F(Q_s) \mathbf{1}, b_s = b_s \). Then \( U \) can be written as
\[
U = \sum_{s=0}^\ell D_{a_s} \mathbb{E}_s D_{b_s} = \sum_{s=0}^\ell a_s \mathbb{E}_s b_s,
\]
where we have identified vectors (functions) and the operator of multiplication they induce. We shall use this notation throughout this section. Finally, we remind that \( \mathbb{E}_\ell = \mathbb{I} \).

We can now use the algorithm developed in [7] to study the inverse of \( I + U \). In what follows, we take the convention \( 0 \cdot \infty = 0/0 = 0 \). This algorithm is defined by the backward recursion starting with the values \( \lambda_\ell = \mu_\ell = \kappa_\ell = 1, \sigma_\ell = (1 + a_\ell b_\ell)^{-1} \) and for \( s = \ell - 1, \ldots, 0 \):
\[
\lambda_s = \lambda_{s+1} [1 - \sigma_{s+1} a_{s+1} \mathbb{E}_{s+1}(\kappa_{s+1} b_{s+1})];
\]
\[
\mu_s = \mu_{s+1} [1 - \sigma_{s+1} b_{s+1} \mathbb{E}_{s+1}(\kappa_{s+1} a_{s+1})];
\]
\[
\kappa_s = \mathbb{E}_{s+1}(\lambda_s) = \mathbb{E}_{s+1}(\mu_s);
\]
\[
\sigma_s = (1 + \mathbb{E}_s(\kappa_s a_s b_s))^{-1};
\]
from where we obtain the recursion
\[
\kappa_{s-1} = \mathbb{E}_s(\kappa_s) - \frac{\mathbb{E}_s(\kappa_s a_s) \mathbb{E}_s(\kappa_s b_s)}{1 + \mathbb{E}_s(\kappa_s a_s b_s)}.
\]

The algorithm continues until some \( \lambda \) or \( \mu \) are negative otherwise we arrive to \( s = 0 \). If this is the case then \( I + U \) is nonsingular and its inverse is of the form \( I - N \) where
\[
N = \sum_{s=0}^\ell \sigma_s \lambda_s a_s \mathbb{E}_s b_s \mu_s.
\]
We also have that
\[ \lambda_{-1} = (I - N)1, \quad \text{and} \quad \mu_{-1} = (I - N)'1, \]
where \( \lambda_{-1}, \mu_{-1} \) are obtained from the first two formulae in (5.9) for \( s = -1 \). Therefore, if they are also non-negative the matrix \( I + U \) is a \( biP \)-matrix.

In this way we have that a sufficient condition for \( I + U \) to be a \( biP \)-matrix, is that the algorithm works for \( s = -1, \ldots, 0 \) and all the \( \lambda, \mu \) are nonnegative, including \( \lambda_{-1}, \mu_{-1} \). In this situation we have that \( \lambda \) (and \( \mu \)) is a decreasing non-negative sequence of vectors.

Sufficient treatable conditions involve the recurrence (5.10). Starting from \( \kappa_\ell = 1 \) we assume this recurrence has a solution such that \( \kappa_s \in [0, 1] \) for all \( s = \ell, \ldots, -1 \). We shall study closely this recursion for the class of SFM, and we shall obtain sufficient conditions to have \( I + U \) in \( biP \).

Before studying this problem, we discuss further the algorithm. We have the following relations:
\[
\left( I + \sum_{k=s}^{\ell} a_k E_k b_k \right)^{-1} = I - \sum_{k=s}^{\ell} \sigma_k \lambda_k a_k E_k b_k \mu_k = I - N_s,
\]
\[
\lambda_{s-1} = (I - N_s)1, \quad \mu_{s-1} = (I - N_s)'1
\]
That is, the algorithm imposes that all the matrices:
\[
I + a_\ell E_\ell b_\ell, \ldots, I + \sum_{k=s}^{\ell} a_k E_k b_k, \ldots, I + \sum_{k=0}^{\ell} a_k E_k b_k = I + U,
\]
are in class \( biP \).

We now assume that \( U \) is a SFM with a decomposition like
\[
U = \sum_{s=0}^{k} D_{C_s} F(\mathcal{R}_s) + D_{\Gamma_s} D_{p_s} F(\mathcal{R}_s) D_{q_s},
\]
where \( \mathbb{F} = \mathcal{R}_0 \prec \cdots \prec \mathcal{R}_k \) is a filtration, \( C_s, \Gamma_s \) are nonnegative \( \mathcal{R}_s \)-measurable and \( \{p_s, q_s\} \) is a \( \mathcal{R}_{s+1} \)-measurable partition. Again we put \( E_s = D_{F(\mathcal{R}_s)1} F(\mathcal{R}_s) \) and the normalized factors:
\[
c_s = C_s \odot F(\mathcal{R}_s)1, \quad \gamma_s = \Gamma_s \odot F(\mathcal{R}_s)1,
\]
which are \( \mathcal{R}_s \)-measurable. Since diagonal matrices commute we get that \( U \) has a representation of the form
\[
U = \sum_{s=0}^{k} C_s E_s + \gamma_s p_s E_s q_s,
\]
with \( \gamma_k = 0 \). In the previous algorithm we can make two steps at each time and consider \( \kappa_s \) in place of \( \kappa_{2s} \), \( \lambda_s \) instead of \( \lambda_{2s+1} \), \( l_s \) instead of \( \lambda_{2s} \). We also introduce \( d_s = 1/\kappa_s \).
to simplify certain formulae (this vector can take the value $\infty$). We get, starting from $\kappa_k = l_k = 1, \sigma_k = (1 + c_k)^{-1}$, that for $s = k - 1, \ldots, 0$

$$\begin{align*}
\lambda_s &= \sigma_{s+1} l_{s+1}; \\
l_s &= \lambda_s [1 - \gamma_s p_s E_s (q_s / (c_{s+1} + d_{s+1}.jpa)); \\
\kappa_s &= E_s (l_s); \\
\sigma_s &= 1 / (1 + \kappa_s c_s) = d_s / (c_s + d_s).
\end{align*}$$

Similar recursions hold for $\mu, m$, which are the analogous of $\lambda, l$. Relation (5.10) takes the form

$$\frac{1}{d_s} = E_s \left( \frac{1}{c_{s+1} + d_{s+1}} \right) - \gamma_s E_s \left( \frac{p_s}{c_{s+1} + d_{s+1}} \right) E_s \left( \frac{q_s}{c_{s+1} + d_{s+1}} \right) \quad (5.11)$$

The inverse of $I + U$ is $I - N$ where

$$N = \sum_{s=0}^{k} c_s \sigma_s l_s E_s m_s + \sum_{s=0}^{k-1} \gamma_s \lambda_s p_s E_s q_s m_s = \sum_{s=0}^{k} c_s \sigma_s l_s E_s m_s + \gamma_s \lambda_s p_s E_s q_s m_s. \quad (5.12)$$

Again $\lambda_{-1} = (I - N) 1 = s_0 l_0$, and similarly $\mu_{-1} = s_0 m_0$.

Let us introduce the following function

$$\rho_s = E_s (p_s) p_s + E_s (q_s) q_s.$$

**Theorem 5.1** Assume that the backward recursion (5.11) has a non-negative solution starting with $d_k = 1$. Assume moreover that this solution verifies for $s = k - 1, \ldots, 0$

$$\rho_s \gamma_s \leq c_{s+1} + d_{s+1}. \quad (5.13)$$

Then $\lambda_s, l_s, \mu_s, m_s, s_s: i = k, \ldots, 0$, as well as $\lambda_{-1}, \mu_{-1}$, are well defined and nonnegative. Therefore, $I + U \in bP$ and its inverse is $I - N$ where $N$ is given by (5.12).

The proof of this result is based on the following lemma.

**Lemma 5.1** Assume $x, y$ are nonnegative vectors and $E$ is a conditional expectation. If $x E(y) \leq 1$ then $E(xy) \leq 1$.

**Proof.** We first assume that $y$ is strictly positive. Since $x \leq 1 / E(y)$ and $E$ is an increasing operator, we have

$$E(xy) \leq E \left( \frac{1}{E(y)} y \right) = \frac{E(y)}{E(y)} = 1.$$

For the general case consider $(y + \epsilon 1) / (1 + \epsilon |x|_\infty)$ instead of $y$ and pass to the limit $\epsilon \to 0$. □

**Proof.** (Theorem 5.1) We notice that condition (5.13) implies that

$$\frac{q_s \gamma_s}{c_{s+1} + d_{s+1}} E_s (q_s) \leq 1.$$
Since \( \gamma_s \) is \( E_s \)-measurable and \( q_s = q_s^2 \) we obtain

\[
\gamma_s E_s \left( \frac{q_s}{c_{s+1} + d_{s+1}} \right) = E_s \left( \frac{\gamma_s q_s^2}{c_{s+1} + d_{s+1}} \right).
\]

This last quantity is bounded by one by Lemma 5.1. Similarly we have

\[
\gamma_s E_s \left( \frac{p_s}{c_{s+1} + d_{s+1}} \right) \leq 1,
\]

which implies that the algorithm is not stopped, all the coefficients are non-negative including \( \lambda_{-1}, \mu_{-1} \).

\[\blacksquare\]

**Corollary 5.1** Assume that for \( s = k - 1, \ldots, 0 \) we have

\[
\rho_s \gamma_s \leq c_{s+1} + \gamma_{s+1}.
\]  

(5.14)

Then the recursion (5.11) has a nonnegative solution that verifies (5.13). In particular, \( \mathbb{I} + tU \) is in class \( biP \) for all \( t \geq 0 \), and \( U \) is in \( biP \) if it is nonsingular.

**Proof.** Let us consider first the case \( t = 1 \). We prove by induction that \( \gamma_s \leq d_s \). For \( i = k \) we have \( 0 = \gamma_k \leq d_k = 1 \). We point out that if we multiply in (5.11) by \( \gamma_s \) we get

\[
\frac{\gamma_s}{d_s} = E_s \left( \frac{\gamma_s}{c_{s+1} + d_{s+1}} \right) - E_s \left( \frac{\gamma_s p_s}{c_{s+1} + d_{s+1}} \right) E_s \left( \frac{\gamma_s q_s}{c_{s+1} + d_{s+1}} \right),
\]

which is of the form \( x + y - xy \), where \( x = E_s \left( \frac{\gamma_s p_s}{c_{s+1} + d_{s+1}} \right) \). The inequality (5.14), the induction hypothesis \( \gamma_{s+1} \leq d_{s+1} \) and Lemma 5.1 imply \( 0 \leq x \leq 1, 0 \leq y \leq 1 \). In particular

\[
0 \leq \frac{\gamma_s}{d_s} \leq 1,
\]

and the induction is completed. Theorem 5.1 shows that \( \mathbb{I} + U \) is in class \( biP \). We notice that \( tU \) also verifies condition (5.12) because this condition is homogeneous, and the result follows. \[\blacksquare\]

**Remark 5.4** We notice that condition (5.14) can be expressed in terms of the original coefficients \( C, \Gamma \) in the dyadic case. In fact (see (5.7))

\[
p_s E_s(p_s) = D_p F_{(R_s)}^{-1} \ F(R_s) p_s = D_p F_{(R_s)}^{-1} \ F(R_{s+1}) 1,
\]

which implies that

\[
\rho_s = (1/F(R_s)) 1 \odot F(R_{s+1}) 1.
\]

Then inequality (5.14) is

\[
\Gamma_s \leq C_{s+1} + \Gamma_{s+1},
\]

which is the condition for having a GUM (see (5.6)). We mention here that condition (5.14) is more general than having a GUM, as the following example shows.
Remark 5.5 Consider the matrix $U_\beta$

$$U_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \beta & \beta & 1 & 0 \\ \beta & \beta & 0 & 1 \end{pmatrix} = D_{\Gamma_0}D_{\rho_0}F(R_0)D_{\rho_0} + I,$$

where $R_0 = N$, $\Gamma_0 = \beta(1,1,1,1)' \leq C_1 = (1,1,1,1)'$. We compute $c_0 = 0$, $\gamma_0 = 4\beta$, $c_1 = C_1$, $\gamma_1 = 0$ and also $\rho_0 = 1/2$.

It is direct to check that $U_\beta^{-1} = U_{-\beta}$. Then for all $\beta \geq 0$ the matrix $U_\beta \in M_{-1}$. Also $U_\beta \in bi\mathcal{P}$ if and only if $0 \leq \beta \leq 1/2$. When $\beta \geq 0$ the condition (5.6), that is

$$\Gamma_0 \leq C_1 + \Gamma_1,$$

is equivalent to $\beta \leq 1$. Then, this condition does not ensure that $U \in bi\mathcal{P}$ (this happens because the filtration is not dyadic). Nevertheless, the analogue condition in terms of the normalized factors (5.14)

$$\rho_0 \gamma_0 \leq c_1 + \gamma_1,$$

which is equivalent to $\beta \leq 1/2$, is the right one.

Corollary 5.2 Assume that

$$\rho_s \gamma_s \leq \sum_{r=s+1}^k c_r,$$  \hspace{1cm} (5.15)

hold for $s = k - 1, \ldots, 0$. Then the recursion (5.11) has a nonnegative solution that verifies (5.13). In particular, $I + tU$ is in class $bi\mathcal{P}$ for all $t \geq 0$, and $U$ is in $bi\mathcal{P}$ if it is nonsingular.

Proof. Consider the set of inequalities

$$\rho_s \gamma_s \lor \xi_s \leq c_{s+1} + \xi_{s+1},$$

for $i = k-1, \ldots, 0$. A non-negative solution is given by $\xi_s = \sup\{0, \gamma_0 \rho_0 - \sum_{r=1}^s c_r, \ldots, \gamma_k \rho_k - \sum_{r=k+1}^s c_r, \ldots, \gamma_{s-1} \rho_{s-1} - c_s\}$. The hypothesis is that $\xi_k = 0$. Moreover we have that $\xi_s$ is $\mathcal{R}_s$-measurable.

We show, using a backward recursion that $\xi_s \leq d_s$. Indeed, $1/\xi_s = \mathbb{E}_s(1/\xi_s) \geq (c_{s+1} + \xi_{s+1})^{-1}$ by construction while $1/d_s \leq \mathbb{E}_s((c_{s+1} + d_{s+1})^{-1})$. Then the inequality $\rho_s \gamma_s \leq c_{s+1} + \xi_{s+1}$ implies $\rho_s \gamma_s \leq c_{s+1} + d_{s+1}$, from where the result holds (see Theorem 5.1).

5.2 Conditions for class $\mathcal{T}$ and proof of Theorem 2.6

Theorem 5.2 Assume that $U$ has a decomposition

$$U = \sum_{s=0}^\ell a_s \mathbb{E}_s b_s,$$
where \( a_s, b_s \) are nonnegative \( \mathbb{E}_{s+1} \)-measurable. Then \( U \) belongs to the class \( \mathcal{T} \) and moreover
\[
\tau(U) = \inf\{ t > 0 : (I + tU)^{-1} \mathbf{1} \not\succ 0 \text{ or } 1'(I + tU)^{-1} \not\prec 0 \}\]
In particular if \( \tau(U) < \infty \) then \( I + \tau(U)U \in \mathcal{B}\mathcal{P} \).

**Remark 5.6** Since the set of nonsingular matrices is open, then in the previous result when \( \tau(U) < \infty \), we have for \( t > \tau(U) \) sufficiently close to \( \tau(U) \), that the matrix \( I + tU \) is nonsingular.

Theorem 5.2 states that every filtered matrix, with a nonnegative decomposition, is in class \( \mathcal{T} \) which proves Theorem 2.6.

**Proof.** (Theorem 5.2) A warning about the use of vectors and functions. Here we consider vectors or functions on \( \{1, \ldots, n\} \) indistinctively. Thus for two vectors \( a, b \) the product \( ab \) makes sense as the product of two functions, which corresponds to the Hadamard product of the vectors. Also an expression as \( (1 + ab)^{-1} \) is the vector whose components are the reciprocals of the components of \( 1 + ab \). We also recall that \( (a)_i \) is the \( i \)-th component of \( a \).

First, for \( p = 0, \ldots, \ell \) consider the matrices
\[
U(p) = \sum_{s=p}^{\ell} a_s \mathbb{E}_s b_s.
\]
We notice that \( U(0) = U \). We shall prove that \( \tau_p = \tau(U(p)) \) is increasing in \( p \) and \( \tau_\ell = \infty \).

We rewrite the algorithm for \( I + tU \). This takes the form\[\lambda_\ell(t) = \mu_\ell(t) = \kappa_\ell(t) = 1, \sigma_\ell(t) = (1 + ta_\ell b_\ell)^{-1}\]
and for \( p = \ell - 1, \ldots, 0 \):
\[
\begin{align*}
\lambda_p(t) &= \lambda_{p+1}(t)[1 - \sigma_{p+1}(t) t a_{p+1} \mathbb{E}_{p+1}(\kappa_{p+1}(t)b_{p+1})]; \\
\mu_p(t) &= \mu_{p+1}(t)[1 - \sigma_{p+1}(t) t b_{p+1} \mathbb{E}_{p+1}(\kappa_{p+1}(t)a_{p+1})]; \\
\kappa_p(t) &= \mathbb{E}_{p+1}(\lambda_p(t)) = \mathbb{E}_{p+1}(\mu_p(t)); \\
\sigma_p(t) &= (1 + \mathbb{E}_p(\kappa_p(t)ta_p b_p))^{-1};
\end{align*}
\]
(5.16)
Also \( \lambda_{-1}(t), \mu_{-1}(t) \) are defined similarly. If \( \lambda_s(t), \mu_s(t), \sigma_s(t) : s = \ell, \ldots, p \) are well defined then
\[
(I + tU(p))^{-1} = I - N(p, t),
\]
where
\[
N(p, t) = \sum_{s=p}^{\ell} \sigma_s(t)\lambda_s(t) t a_s \mathbb{E}_s b_s \mu_s(t).
\]
(5.17)
If \( \lambda_s(t), \mu_s(t), \sigma_s(t) : s = \ell, \ldots, p \) are nonnegative then \( N(p, t) \geq 0 \), and \( (I + tU(p)) \in \mathcal{M}^{-1} \). Moreover, \( \lambda_{p-1}(t) \) and \( \mu_{p-1}(t) \) are the right and left equilibrium potentials of \( (I + tU(p)) \)
\[
(I + tU(p))\lambda_{p-1}(t) = 1', \text{ and } \mu_{p-1}(t)(I + tU(p)) = 1'.
\]
So, if they are nonnegative, we have \( I + tU(p) \in bi\mathcal{P} \). In particular we have that
\[
(\mathbb{I} + ta_\ell E_\ell b_\ell)^{-1} = (\mathbb{I} + tU(\ell))^{-1} = \mathbb{I} - t(1 + ta_\ell b_\ell)^{-1}a_\ell E_\ell b_\ell.
\]
Since \( E_\ell = \mathbb{I} \) we obtain that \( \lambda_{\ell-1} = \mu_{\ell-1} = (1 + ta_\ell b_\ell)^{-1} \). This means that \( \mathbb{I} + tU(\ell) \in bi\mathcal{P} \) for all \( t \geq 0 \). Therefore \( \tau_\ell = \infty \) and the result is true for \( U(\ell) \). This implies in particular that \( \tau_{\ell-1} \leq \tau_\ell \). Assume the following inductive hypothesis

- \( \tau_{p+1} \leq \cdots \leq \tau_\ell \); and for \( q = p + 1, \ldots, \ell \)
- \( \tau_q = \inf \{ t > 0 : \lambda_{q-1}(t) \not< 0 \text{ or } \mu_{q-1}(t) \not> 0 \} = \inf \{ t > 0 : \lambda_{q-1}(t) \not= 0 \text{ or } \mu_{q-1}(t) \not= 0 \} \);
- \( \lambda_s(t), \mu_s(t) \), for \( s = \ell, \ldots, q - 1 \), are strictly positive for \( t \in [0, \tau_q] \);
- If \( \tau_q < \infty \) we have \( \mathbb{I} + \tau_q U(q) \in bi\mathcal{P} \).

The case \( \tau_{p+1} = \infty \) is clear. Indeed, fix \( t \geq 0 \). From Lemma 3.1, \( \mathbb{I} + tU(p + 1) \in bi\mathcal{P} \) and its equilibrium potential are strictly positive, that is \( \lambda_p(t) > 0, \mu_p(t) > 0 \). Thus, \( \mathbb{I} + tU(p) \) is nonsingular, its inverse is \( \mathbb{I} - N(p, t) \), where \( N(p, t) \geq 0 \) is given by \( 5.17 \). Hence, \( \mathbb{I} + tU(p) \in \mathcal{M}^{-1} \). We conclude that
\[
\tau_p = \inf \{ t > 0 : \mathbb{I} + tU(p) \notin bi\mathcal{P} \} = \inf \{ t > 0 : \lambda_{p-1}(t) \not< 0 \text{ or } \mu_{p-1}(t) \not> 0 \}.
\]
So, if \( \tau_p = \infty \) we have, from Lemma 3.1 that
\[
\lambda_{p-1}(t) > 0, \ \mu_{p-1}(t) > 0,
\]
and the induction step holds in this case.

Now if \( \tau_p < \infty \), by continuity we have \( \mathbb{I} + \tau_p U(p) \in bi\mathcal{P} \), and we shall prove later on that \( \lambda_{p-1}(t), \mu_{p-1}(t) \) are strictly positive in \( [0, \tau_p] \).

We analyze now the case \( \tau_{p+1} < \infty \). We first notice that in the algorithm the only possible problem is with the definition of \( \sigma_p(t) \). Since \( \sigma_p(t_{p+1}) > 0 \) the algorithm is well defined, by continuity, for steps \( \ell, \ldots, p \) on an interval \( [0, \tau_{p+1} + \epsilon] \), for small enough \( \epsilon \). This proves that the matrix \( \mathbb{I} + tU(p) \) is nonsingular in that interval, and that \( \lambda_{p-1}, \mu_{p-1} \) exist in the same interval.

Now, for a sequence \( t_n \downarrow \tau_{p+1} \) either \( \lambda_p(t_n) \) or \( \mu_p(t_n) \) has a negative component. Since there are a finite number of components we can assume with no loss of generality that for a fixed component \( i \) we have \( \lambda_p(t_n)_i < 0 \). Then, by continuity we get that \( \lambda_p(t_{p+1})_i = 0 \), which implies (by the algorithm) that \( \lambda_{p-1}(t_{p+1})_i = 0 \).

Assume now that for some \( t > \tau_{p+1} \) the matrix \( \mathbb{I} + tU(p) \in bi\mathcal{P} \). Again by Lemma 3.1 we will have that \( \mathbb{I} + \tau_{p+1} U(p) \in bi\mathcal{P} \) but its equilibrium potential will satisfy \( \lambda_{p-1}(\tau_{p+1}) > 0 \), which is a contradiction. Therefore we conclude that \( \tau_p \leq \tau_{p+1} \).
The conclusion of this discussion is that the matrix $\mathbb{I} + tU(p)$, for $t \in [0, \tau_{p+1}]$, is nonsingular and its inverse is $\mathbb{I} - N(p, t)$, with $N(p, t) \geq 0$. That is $\mathbb{I} + tU(p) \in \mathcal{M}^{-1}$ and therefore

$$
\tau_p = \inf\{t > 0 : \mathbb{I} + tU(p) \notin \text{biP}\} = \inf\{t > 0 : \lambda_{p-1}(t) \not\geq 0 \text{ or } \mu_{p-1}(t) \not\geq 0\},
$$

and by continuity $\mathbb{I} + \tau_p U(p) \in \text{biP}$.

To finish the proof we need to show that $\tau_p$ coincides with

$$
S = \inf\{t > 0 : \lambda_{p-1}(t) \not\geq 0 \text{ or } \mu_{p-1}(t) \not\geq 0\}.
$$

It is clear that $S \leq \tau_p$. If $S < \tau_p$ then, due to Lemma 3.1 we have that both $\lambda_{p-1}(S) > 0$ and $\mu_{p-1}(S) > 0$, which is a contradiction and then $S = \tau_p$. This shows that $\lambda_{p-1}(t), \mu_{p-1}(t)$ are strictly positive for $t \in [0, \tau_p)$, and the induction is proven. $\square$

**Remark 5.7** It is possible to prove that $\kappa_p(\tau_p) > 0$ when $\tau_p < \infty$, but this is not central to our discussion.

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