Exactness of the Born Approximation and Broadband Unidirectional Invisibility in Two Dimensions

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Achieving exact unidirectional invisibility in a finite frequency band has been an outstanding problem for many years. We offer a simple solution to this problem in two dimensions that is based on our solution to another more basic open problem of scattering theory, namely finding potentials \( v(x, y) \) whose scattering problem is exactly solvable via the first Born approximation. Specifically, we find a simple condition under which the first Born approximation gives the exact expression for the scattering amplitude whenever the wavenumber for the incident wave is not greater than a given critical value \( \alpha \). Because this condition only restricts the \( y \)-dependence of \( v(x, y) \), we can use it to determine classes of such potentials that have certain desirable scattering features. This leads to a partial inverse scattering scheme that we employ to achieve perfect (non-approximate) broadband unidirectional invisibility in two dimensions. We discuss an optical realization of the latter by identifying a class of two-dimensional isotropic active media that do not scatter incident TE waves with wavenumber in the range \((\alpha/\sqrt{2}, \alpha]\) and source located at \( x = \infty \), while scattering the same waves if their source is relocated to \( x = -\infty \).

I. INTRODUCTION

In 1926, the year Born published his monumental work on the probabilistic interpretation of quantum mechanics \cite{Born1926}, he also laid the foundations of quantum scattering theory and introduced the celebrated Born approximation \cite{Born1954}. The latter proved to be an extremely powerful tool for performing scattering calculations in different areas of physics \cite{Banerjee2017,Bennewitz2018}. The Born approximation of order \( N \) corresponds to the approximation scheme in which one neglects all but the first \( N + 1 \) terms in the standard series solution of the Lippmann-Schwinger equation \cite{Siegfried1983}. For a scattering potential that is proportional to a coupling constant \( \lambda \), this leads to an approximate expression for the scattering amplitude \( f \) which is a polynomial \( f_N \) of degree \( N \) in \( \lambda \); \( N \)-th order Born approximation corresponds to \( f \approx f_N \).

Because of its central importance, Born approximation is discussed in standard textbooks on quantum mechanics \cite{Banerjee2017,Bennewitz2018}, optics \cite{BornWolf1999}, and scattering theory \cite{Siegfried1983}. But, surprisingly, none of these address the natural problem of inquiring into potentials for which the \( N \)-th order Born approximation gives the exact expression for the scattering amplitude, i.e., \( f = f_N \). In this article, we offer a solution of this problem for the case \( N = 1 \) in two dimensions, i.e., identify potentials in two dimensions for which the first Born approximation is exact. This may seem as a purely academic problem, but its solution proves to have far-reaching consequences; it paves the way for devising a method of engineering scattering potentials which we employ to address another outstanding problem of basic importance, namely, achieving perfect unidirectional invisibility in a tunable finite frequency band.

The study of invisible potentials has been a subject of research for many decades. In one dimension, a potential is invisible from the left (respectively right), if it does not reflect a left- (respectively right-) incident plane wave and transmits it without changing the amplitude or phase of the wave, i.e., it is transparent. Reciprocity theorem implies that the transparency cannot be unidirectional \cite{Mostafazadeh2018}. A nonreal scattering potential can, however, support unidirectional reflectionlessness and invisibility \cite{Loran2018,Mostafazadeh2018,Mostafazadeh2018}. The principal example is

\[
v(x) = \begin{cases} 
3e^{2i\beta x} & \text{for } x \in [0, L], \\
0 & \text{for } x \notin [0, L],
\end{cases}
\]

where \( \lambda \) is a coupling constant, \( \beta \) is a nonzero real parameter, and \( L := \pi/|\beta| \). This potential is unidirectionally invisible from the right (respectively left) for an incident wave with wavenumber \( k = |\beta| \), if \( \beta > 0 \) (respectively \( \beta < 0 \)) and \( \beta^2|\lambda| \ll 1 \). The latter condition is to ensure the validity of the first Born approximation. For sufficiently large values of \( \beta^2|\lambda| \), the first Born approximation is unreliable and the unidirectional invisibility of the potential \cite{Loran2018} breaks down \cite{Mostafazadeh2018}.

The problem of realizing exact unidirectional invisibility is addressed in Refs. \cite{Loran2018,Mostafazadeh2018}. Similarly to \cite{Loran2018}, the unidirectionally invisible potentials considered in these references display this property for a single or a discrete set of wavenumbers. In one dimension, the problem of constructing potentials that have exact unidirectional invisibility in the entire wavenumber spectrum is solved in \cite{Loran2018,Mostafazadeh2018}. See also \cite{Mostafazadeh2018,Mostafazadeh2018}.

Ref. \cite{Mostafazadeh2018} generalizes the notion of unidirectional invisibility at isolated values of the wavenumber to two and three dimensions and examines its approximate realization for weak potential where the first Born approximation is reliable. A more recent development is the construction of a class of scattering potentials in two dimensions that display exact broadband omnidirectional invisibility \cite{Loran2018}; these do not scatter incident plane waves with an arbitrary incidence angle provided that their
wavenumber $k$ does not exceed a prescribed value. The broadband invisibility achieved in [32] is bidirectional in the sense that the scattering amplitude vanishes regardless of the position of the source of the incident wave. The quest for realizing exact unidirectional invisibility in an extended frequency band, which we address in this article, has been a well-known open problem for many years. This is a much more important problem, because its solution would allow for the design of material with broadband nonreciprocal functionalities.

II. EXACTNESS OF BORN APPROXIMATION

Consider a two-dimensional scattering setup where the scatterer is described by a possibly complex-valued potential $v(x, y)$. The source of the incident wave, which is considered to be a plane wave, can be placed at $x = -\infty$ or $x = +\infty$. We use the terms “left-incident” and “right-incident waves” to refer to these cases, respectively. In general, the wave vector $\mathbf{k}_0$ of the incident wave makes an angle $\theta_0$ with the $x$-axis, i.e., $\mathbf{k}_0 = k(\cos \theta_0 \mathbf{e}_x + \sin \theta_0 \mathbf{e}_y)$, where $\mathbf{e}_j$ is the unit vector along the $j$-axis with $j = x, y$.

We use $\theta_0^{l/r}$ to label $\theta_0$ for the left/right-incident waves. Clearly,

\[ 0 < \cos \theta_0 \leq 1, \quad -1 \leq \cos \theta_0^{l/r} < 0. \quad (2) \]

By definition, if $v(x, y)$ is a scattering potential, the solutions of the Schrödinger equation,

\[ [-\nabla^2 + v(x, y)]\psi(x, y) = k^2 \psi(x, y), \quad (3) \]

tend to plane waves at spatial infinities. In particular, \[ e^{i \mathbf{k}_0 \cdot \mathbf{r}} \] admits the so-called “scattering solutions,” $\psi^{l/r}(\mathbf{r})$, that satisfy $\psi^{l/r}(\mathbf{r}) = e^{i \mathbf{k}_0 \cdot \mathbf{r}} + \sqrt{i/\kappa r} \kappa^{l/r}(\theta) \mathbf{r}$ as $r \to \infty$, where $\mathbf{r} := x \mathbf{e}_x + y \mathbf{e}_y$, $(r, \theta)$ are the polar coordinates of $\mathbf{r}$, and $f^{l/r}(\theta)$ is the scattering amplitude for the left/right-incident waves [33]. The latter stores the scattering properties of the potential. Therefore its determination is the main objective of the scattering theory.

It is not difficult to show that the first Born approximation yields [31]:

\[ f^{l/r}(\theta) = -\frac{\tilde{\nu}\left(k(\cos \theta - \cos \theta_0^{l/r}), k(\sin \theta - \sin \theta_0^{l/r})\right)}{2\sqrt{2\pi}}, \quad (4) \]

where $\tilde{\nu}(\mathbf{r}_x, \mathbf{r}_y) := \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ e^{-i(x\mathbf{r}_x + y\mathbf{r}_y)} v(x, y)$ is the two-dimensional Fourier transform of $v(x, y)$. We wish to find conditions under which [4] gives the exact expression for the scattering amplitudes of the potential. Our main technical tool for achieving this purpose is the transfer-matrix formulation of potential scattering in two dimensions [34]. We summarize its basic ingredients in the sequel.

1) For a given wavenumber $k$, let $\mathcal{F}_k$ denote the space of complex-valued functions $\xi$ such that $\xi(p) = 0$ for $|p| \geq k$:

\[ \mathcal{F}_k := \{ \xi : \mathbb{R} \to \mathbb{C} | \xi(p) = 0 \text{ for } |p| \geq k \}. \]

The transfer matrix of a scattering potential $v(x, y)$ is a $2 \times 2$ matrix $M$ with operator entries $M_{ij}$ acting in $\mathcal{F}_k$.

2) We can express $M$ as the time-ordered exponential of a non-Hermitian effective $2 \times 2$ matrix Hamiltonian $H(x)$ with operator entries $H_{ij}(x)$ acting in $\mathcal{F}_k$:

\[ M = \mathcal{T} \exp \left\{ -i \int_{-\infty}^{\infty} dx \ H(x) \right\} := I - i \int_{-\infty}^{\infty} dx \ H(x) + \\
(-i)^2 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{x_2} dx_1 \ H(x_2) H(x_1) \cdots, \quad (5) \]

where $\mathcal{T}$ is the time-ordering operation with $x$ playing the role of time, and $I$ is the identity operator for the space $\mathcal{F}_k$ of two-component state vectors with components belonging to $\mathcal{F}_k$. The entries of $H(x)$ are defined by

\[ H_{ij}(x) \xi(p) := \frac{\epsilon_i e^{-i \varpi(p)} x}{2 \varpi(p)} v(x, i \partial_x) \left[ e^{i \varpi(p)} \xi(p) \right], \quad (6) \]

where $\epsilon_i := (-1)^{i-1}$, $\varpi(p) := \sqrt{k^2 - p^2}$, $v(x, i \partial_x)$ is the linear operator acting in $\mathcal{F}_k$ according to

\[ v(x, i \partial_x) \xi(p) := \frac{1}{2\pi} \int_{-k}^{k} dq \ \tilde{\nu}(x, p - q) \xi(q), \quad (7) \]

and $\tilde{\nu}(x, \mathbf{r}_y) := \int_{-\infty}^{\infty} dy \ e^{-i x y} v(x, y)$ is the Fourier transform of $v(x, y)$ with respect to $y$.

3) The scattering amplitudes $f^{l/r}$ are given by

\[ f^{l/r}(\theta) = \frac{-ik|\cos \theta|}{\sqrt{2\pi}} \times \left\{ \begin{array}{ll} T^{l/r}_±(k \sin \theta) & \text{for } \cos \theta < 0, \\ T^{l/r}_±(k \sin \theta) & \text{for } \cos \theta > 0, \end{array} \right. \quad (8) \]

where $T^{l/r}_±$ are the elements of $\mathcal{F}_k$ fulfilling

\[ M_{22} T^{l}_±(p) = -2\pi M_{21} \delta(p - p_0^l), \quad (9) \]
\[ T^{l}_±(p) = M_{12} T^{l}_±(p) + 2\pi (M_{11} - I) \delta(p - p_0^l), \quad (10) \]
\[ M_{22} T^{r}_±(p) = -2\pi (M_{22} - I) \delta(p - p_0^r), \quad (11) \]
\[ T^{r}_±(p) = M_{12} [T^{r}_±(p) + 2\pi \delta(p - p_0^r)]. \quad (12) \]

$\delta(\cdot)$ stands for the Dirac delta function, $p_0^{l/r} := k \sin \theta_0^{l/r}$, and $I$ is the identity operator acting in $\mathcal{F}_k$.

Because $M_{22}$ is in general an integral operator, [19] and [14] are linear integral equations. According to [5], we can solve the scattering problem for the potential $v(x, y)$ provided that we determine the transfer matrix $M$ and solve [19] and [14]. Refs. [24] and [26] offer details of the application of this scheme for solving concrete scattering problems.

The transfer matrix $M$, the functions $T^{l/r}_±$, and the scattering amplitude $f^{l/r}$ depend on the wavenumber $k$. A critical observation underlying the present study is that under a fairly simple condition on the potential,
the Dyson series for the transfer matrix [5] truncates for wavenumbers not exceeding a critical value \( \alpha \);

\[
M = I - i \int_{-\infty}^{\infty} dx \mathbf{H}(x) \quad \text{for} \quad k \leq \alpha.
\]

Furthermore, the same condition allows for an explicit solution of the integral equations yielding \( T_{l/r}^{1/r} \). As we show in Appendix A, this provides explicit formulas for \( T_{l/r}^{1/r} \) and \( f_{l/r}(\theta) \) for \( k \leq \alpha \), and proves:

**Theorem 1.** Let \( v(x, y) \) be a scattering potential satisfying

\[
\tilde{v}(x, \tilde{R}_y) = 0 \quad \text{for} \quad \tilde{R}_y \leq \alpha,
\]

where \( \alpha \) is a given positive real parameter. Then the first Born approximation provides the exact solution of the scattering problem for \( v(x, y) \) whenever the wavenumber \( k \) of the incident wave does not exceed \( \alpha \).

This result is reminiscent of the notion of a quasi-exactly solvable potential [37]. The time-independent Schrödinger equation for such a potential can be solved to determine finitely many low-lying bound state energies and the corresponding eigenfunctions. The potentials fulfilling (14) may also be viewed as quasi-exactly solvable, because their scattering problem is exactly solvable for energies \( k^2 \leq \alpha^2 \). Notice also that according to a result of Ref. [32], condition (14) implies omnidirectional invisibility of the potential for \( k \leq \alpha/2 \) for both left- and right-incident waves, i.e., for these values of \( k \), \( f_{l/r}(\theta) = 0 \). Theorem 1 is a much stronger result, because it provides an explicit formula for the scattering amplitudes \( f_{l/r}(\theta) \) also for \( k \in (\alpha/2, \alpha] \) where they need not vanish.

Let us also note that Theorem 1 does not imply that the first Born approximation gives the exact solution of the Schrödinger equation \( \mathbf{H} \) for \( k \leq \alpha \). It is indeed not difficult to show that the second and higher order terms in the Born series expansion of the wave function are generally nonzero, but for \( k \leq \alpha \) they yield evanescent waves which do not contribute to the scattering amplitude of the potential.

### III. Perfect Broadband Unidirectional Invisibility

Consider the scattering of left- and right-incident waves by a potential \( v(x, y) \) satisfying (14), and suppose that \( k \leq \alpha \). Then (1) holds, and (2) implies

\[
-2\alpha \leq -2k \leq k(\cos \theta - \cos \theta_0^l) \leq k \leq \alpha, \quad (15)
\]

\[
-\alpha \leq -k \leq k(\cos \theta - \cos \theta_0^r) \leq 2k \leq 2\alpha. \quad (16)
\]

According to (1) and (15), \( f^l(\theta) = 0 \) for all possible values of \( \theta_0^l \) and \( \theta \) provided that \( \tilde{v}(\tilde{R}_x, \tilde{R}_y) = 0 \) for \( \tilde{R}_y \in [-2\alpha, \alpha] \). In other words this is a sufficient condition for the left-invisibility of \( v(x, y) \) whenever \( k \leq \alpha \).

For scattering potentials of physical interest, this condition is equivalent to

\[
\tilde{v}(\tilde{R}_x, y) = 0 \quad \text{for} \quad \tilde{R}_x \in [-2\alpha, \alpha), \quad (17)
\]

where \( \tilde{v}(\tilde{R}_x, y) := \int_{-\infty}^{\infty} dx e^{-i\tilde{R}_x x} v(x, y) \) is the Fourier transform of \( v(x, y) \) with respect to \( x \). Similarly, we can use (16) to obtain the following sufficient condition for the right-invisibility of the potential for \( k \leq \alpha \).

\[
\tilde{v}(\tilde{R}_x, y) = 0 \quad \text{for} \quad \tilde{R}_x \in (-\alpha, 2\alpha]. \quad (18)
\]

Because (17) and (18) do not coincide, there is a range of values of \( k \) for which only one of these conditions holds.

This provides the basic motivation for achieving broadband unidirectional invisibility. It leads to the following theorem whose proof we give in Appendix B.

**Theorem 2.** A scattering potential \( v(x, y) \) is unidirectionally right- (respectively left-) invisible for wavenumbers \( k \in (\alpha/\sqrt{2}, \alpha] \), if it satisfies (14), (15), and

\[
\tilde{v}(\tilde{R}_x, y) \neq 0 \quad \text{for} \quad \tilde{R}_x \in [-2\alpha, -\alpha), \quad (19)
\]

(respectively (14), (17), and \( \tilde{v}(\tilde{R}_x, y) \neq 0 \) for \( \tilde{R}_x \in [\alpha, 2\alpha] \)).

The conditions appearing in the hypothesis of Theorems 1 and 2 are not difficult to satisfy. In Appendix C we construct an infinite class of scattering potentials fulfilling these conditions. A simple example is

\[
v(x, y) := \frac{3 \epsilon \alpha(y-x)}{[(x/a - i)(y/b + i)]^2}, \quad (20)
\]

where \( \epsilon \) is a real or complex coupling constant, and \( a \) and \( b \) are real parameters. For all values of \( \epsilon, a, \) and \( b \), this potential displays unidirectional right-invisibility whenever the wavenumber of the incident wave is in the range \((\alpha/\sqrt{2}, \alpha] \). To see this, we evaluate the Fourier transform of the right-hand side of (20) with respect to \( y \) and use the result to check that it fulfills (14). According to Theorem 1, this shows that we can employ (4) to calculate the scattering amplitudes of (20) for \( k \leq \alpha \). The result of this calculation is

\[
f_{l/r}(\theta) = -\pi \sqrt{2} \epsilon \alpha \beta k^2 \mathcal{X}(ak, c^{l/r}) \mathcal{X}(bk, c^{l/r}), \quad (21)
\]

where

\[
\mathcal{X}(\xi, \zeta) := \left\{ \begin{array}{ll}
\zeta e^{-\xi \zeta} & \text{for} \quad \zeta > 0, \\
0 & \text{for} \quad \zeta \leq 0,
\end{array} \right.
\]

\[
c^{l/r} := \sin \theta - \sin \theta_0^{l/r} - \alpha/k,
\]

\[
c^{l/r} := \cos \theta_0^{l/r} - \cos \theta - \alpha/k.
\]

Because, \( \cos \theta_0^l < 0 \), for \( k \leq \alpha \) we have \( c^r < 0 \). This implies that \( \mathcal{X}(ak, c^r) = 0 \). Therefore, \( f^r(\theta) = 0 \) for all \( \theta \in [0, 2\pi] \), i.e., the potential (20) is right-visible. It is easy to check that for \( \theta_0^l \in (-\pi/2, 0) \) there are ranges of values of \( \theta \) within the interval \((\pi/2, \pi) \) for which \( f^l(\theta) \neq 0 \). Therefore, the right-invisibility of the potential is unidirectional.
IV. OPTICAL REALIZATION

Complex scattering potentials in two dimensions may be used to model the scattering of transverse electric (TE) waves by the inhomogeneities of an effectively two-dimensional nonmagnetic isotropic medium $M$. Let $\varepsilon(x,y)$ label the permittivity profile of $M$ and suppose that as $r \to \infty$, $\varepsilon(x,y)$ tends to a constant value $\varepsilon_\infty$. For a TE wave propagating in $M$, we can express its electric field in the form $\hat{\epsilon}_0 e^{-i\omega t} \psi(x,y) e_z$, where $\hat{\epsilon}_0$ is a constant amplitude, $t$ labels time, $e_z$ is the unit vector along the $z$-axis, $\omega := \sqrt{\varepsilon_0 / \varepsilon_\infty} \nu$, $\hat{\epsilon}_0$ and $c$ are respectively the permittivity and the speed of light in vacuum, and $k$ is the wavenumber. We can use Maxwell’s equations to identify $\psi(x,y)$ with a solution of the Schrödinger equation (23) for the potential

$$v(x,y) = k^2 [1 - \hat{\epsilon}(x,y)],$$

(22)

where $\hat{\epsilon}(x,y) := \varepsilon(x,y)/\varepsilon_\infty$ is the relative permittivity of $M$.

In order for the medium to display perfect broadband unidirectional invisibility for wavenumbers $k \in (\alpha, 2\alpha)$, it suffices to select a reasonable value for $\varepsilon_\infty$, identify the left-hand side of (22) with one of the potentials fulfilling the hypothesis of Theorem 2, and solve this equation for $\hat{\epsilon}(x,y)$.

For example consider the relative permittivity profile corresponding to the potential (20), i.e.,

$$\hat{\epsilon}(x,y) = 1 + \frac{\lambda_0 e_{\alpha(y-x)}}{|(x/a - i) (y/b + i)|^2},$$

(23)

where $\lambda_0 := -\lambda/k^2$ is a free dimensionless coupling constant. By construction, Eq. (21) gives an exact description of the scattering of TE waves by the optical medium possessing the relative permittivity profile (23) whenever their wavenumber does not exceed $\alpha$. As we noted above, this equation establishes the right-invisibility of the medium for TE waves with wavenumber $k \leq \alpha$ (and their superpositions). Fig. 1 provides a graphical demonstration of the unidirectionality of the right-invisibility of this medium for wavenumbers in the range $(\alpha / \sqrt{2}, \alpha)$. Here we have set $\theta_0 = -\pi/4$, $\alpha = 2\pi / 500$ nm, and $a = b = 1$ μm. As expected, the medium described by (23) has a nonzero scattering amplitude for left-incident waves with incidence angle $\theta_0 = -\pi/4$ provided that their wavelength $\lambda := 2\pi/k$ lies between 500.0 nm and 707.1 nm.

V. CONCLUDING REMARKS

In this article we have offered a solution of the old problem of finding potentials whose scattering problem can be solved exactly via the first Born approximation. These provide the first examples of quasi-exactly solvable scattering potentials, because the first Born approximation gives the exact expression for their scattering amplitudes whenever the wavenumber of the incident wave does not exceed a prescribed value $\alpha$.

Condition (14) that ensures the exactness of the first Born approximation for wavenumbers $k \leq \alpha$ does not restrict the x-dependence of the potential $v(x,y)$. Because, according to (21), such a potential may be obtained by performing the inverse Fourier transform of the scattering amplitudes $f^{1/r}(\theta)$, we can fix the x-dependence of the potential by demanding that $f^{1/r}(\theta)$ has a certain desirable behavior for $k \leq \alpha$. This yields an extremely effective partial inverse scattering prescription which we have presently employed for achieving perfect broadband unidirectional invisibility. The latter admits optical realizations involving certain two-dimensional nonmagnetic isotropic media with regions of gain and loss. We expect a two-dimensional analog of the setup employed in Ref. 20 to allow for an experimental detection of this effect.

The generalization of our results to three dimensions does not pose any major difficulty. In particular, one can pursue the above-mentioned idea of partial inverse scattering in three dimensions. The progress in this direction can have interesting applications in acoustics.
Appendix A: Proof of Theorem 1

The proof of Theorem 1 rests on (13) and the fact that we can use this equation to obtain closed-form expressions for \( T_{±}^{1/r} (p) \) and \( f_{1/r} (\theta) \) whenever \( k \leq \alpha \). In the following we outline a derivation of these expressions and provide a proof of Theorem 1.

We begin by noting that for each \( p \in [-k, k], -k + p \leq 0 < \alpha \). Using this inequality in (24), we find

\[
v(x, i\partial_p)\xi(p) = \frac{1}{2\pi} \int_{\alpha}^{k+p} dq \tilde{v}(x, q)\xi(p - q). \tag{24}
\]

To establish (13), we set \( k \leq \alpha \) and use (24) to show that for all \( \xi_1, \xi_2 \in \mathcal{F}_k \),

\[
v(x_2, i\partial_p)[\xi_2(p)v(x_1, i\partial_p)\xi_1(p)] = 0.
\]

This together with (6) imply

\[
H_{ij}(x_2)H_{i'j'}(x_1)\xi(p) = 0 \text{ for } k \leq \alpha. \tag{25}
\]

Hence \( H(x_2)H(x_1) \) vanishes, and (6) implies (13).

Next, we note that according to (13) and (24),

\[ (M_{ij} - \delta_{ij})I(M_{i'j'} - \delta_{i'j'})I = 0 \text{ for } k \leq \alpha, \tag{26} \]

where \( \delta_{ij} \) denotes the Kronecker delta symbol, and 0 stands for the zero operator acting in \( \mathcal{F}_k \). A simple consequence of (26) is a straightforward calculation of \( T_{±}^{1/r} \). To see this, we express (9) and (11) in the form

\[
T_{±}^{1} (p) = -(M_{22} - 1)T_{±}^{1} (p) - 2\pi M_{21}\delta(p - p_0'), \tag{27}
\]

\[
T_{±}^{r} (p) = -(M_{22} - 1)T_{±}^{r} (p) + 2\pi \delta(p - p_0'). \tag{28}
\]

Similarly to (10) and (12), the right-hand sides of these equations consist of terms obtained by applying operators of the form \( M_{i'j'} - \delta_{i'j'} I \) to functions belonging to \( \mathcal{F}_k \). In light of (26), this implies \( (M_{ij} - \delta_{ij})I T_{±}^{1/r} (p) = 0 \). Using this relation in (10), (12), (27), and (28), we obtain

\[
T_{±}^{1} (p) = -2\pi M_{21}\delta(p - p_0'), \tag{29}
\]

\[
T_{±}^{r} (p) = 2\pi\delta(p - p_0'). \tag{30}
\]

\[
T_{±}^{r} (p) = 0. \tag{31}
\]

\[
T_{±}^{r} (p) = 2\pi M_{12}\delta(p - p_0'). \tag{32}
\]

To obtain more explicit formulas for \( T_{±}^{1/r} (p) \), we introduce

\[
h_{ij}(p, q) := \tilde{v}(\epsilon_i\omega(p) - \epsilon_j\omega(p - q), q),
\]

and use (10), (24), and (14) to show that

\[
(M_{ij} - \delta_{ij})\xi(p) = \frac{-i\epsilon_i}{4\pi\omega(p)} \int_{\alpha}^{k+p} dq h_{ij}(p, q)\xi(p - q).
\]

Substituting this equation in (29) and (32) and using the result together with the identities, \( p = k\sin \theta \) and \( \omega(p) = k|\cos \theta| \), in (6), we obtain (4). This proves Theorem 1.

Appendix B: Proof of Theorem 2

Suppose that (14) and (18) hold. Then (4) applies, and the potential is right-invisible for all \( \theta_0' < k \leq \alpha \). To ensure that it is not left-invisible, we must determine values of \( k \) within the interval \( (0, \alpha] \) and ranges of values of \( \theta_0' \) and \( \theta \) for which \( f_{1/r}(\theta) \neq 0 \), i.e., \( \tilde{v}(k(\cos \theta - \sin \theta_0')) \neq 0 \). Conditions (14) and (18) violate this inequality unless \( \sin \theta - \sin \theta_0' > \alpha/k \) and \( \cos \theta_0' - \cos \theta \geq \alpha/k \). We have used these relations together with (2) to show that

\[
-\frac{\pi}{2} < \theta_0' < 0, \quad \frac{\pi}{2} < \theta < \pi, \quad \frac{\alpha}{\sqrt{2}} < k \leq \alpha,
\]

and \( \varphi_k < \theta - \theta_0' < 2\pi - \varphi_k \), where \( \varphi_k := 2\arcsin(\alpha/\sqrt{2}k) \in [\pi/2, \pi] \). Similarly, we find the following necessary conditions for unidirectional left-invisibility of \( v(x, y) \):

\[ \varphi_k < \theta_0' < \theta < 2\pi - \varphi_k \text{ and } \pi < \theta_0' < \frac{3\pi}{2}, \quad 0 < \theta < \frac{\pi}{2}, \quad \frac{\alpha}{\sqrt{2}} < k \leq \alpha. \]

This completes the proof of Theorem 2.

Appendix C: Construction of potentials displaying broadband unidirectional invisibility

It is easy to see that under multiplication of a function \( g(x) \) by \( e^{i\beta x} \) its Fourier transform \( \tilde{g}(\mathcal{R}) \) changes to \( \tilde{g}(\mathcal{R} - \beta) \). This allows for expressing conditions appearing in the statement of Theorems 1 and 2 in terms of functions with vanishing Fourier transform with respect to \( x \) or \( y \) on the negative or positive \( x \)- or \( y \)-axes (38). To be more specific, let \( w_{±}(x, y) \) and \( v_{±}(x, y) \) be scattering potentials satisfying

\[
\tilde{w}_{±}(\mathcal{R}, y) = 0 \text{ respectively for } \pm \mathcal{R}_x \leq 0, \tag{33}
\]

\[
\tilde{w}_{±}(x, \mathcal{R}_y) = 0 \text{ for } \mathcal{R}_y \leq 0, \tag{34}
\]

\[
v_{±}(x, y) = e^{i\gamma y} \left[ \gamma e^{-i\delta x} w_{-}(x, y) + e^{i\delta x} w_{+}(x, y) \right], \tag{35}
\]

\[
v_{±}(x, y) = e^{i\gamma x} \left[ e^{-i\gamma x} w_{-}(x, y) + \gamma e^{i\gamma x} w_{+}(x, y) \right], \tag{36}
\]

where \( \gamma = 0, 1 \). Then, according to Theorem 2, for \( k \in (\alpha/\sqrt{2}, \alpha] \), \( v_{±} \) is unidirectionally left-invisible, if

\[
\tilde{w}_{±}(\mathcal{R}, y) \neq 0 \text{ for } \mathcal{R}_x \in [0, \alpha], \tag{37}
\]

and \( v_{±} \) is unidirectionally right-invisible, if

\[
\tilde{w}_{±}(\mathcal{R}, y) \neq 0 \text{ for } \mathcal{R}_x \in [-\alpha, 0]. \tag{38}
\]

We can construct specific examples for \( w_{±}(x, y) \) using functions of the form

\[
g_{j}(x) := (x/L_j + i)^{\alpha_j - 1},
\]

where \( j = 1, 2, 3, 4 \), and \( L_j \) and \( n_j \) are respectively positive real numbers and positive integers. It is easy to
check that $\tilde{g}_j(\mathcal{R}) = 0$ for $\mathcal{R} \leq 0$. This in turn implies that the function $\tilde{g}_j(x) := g_j(x)^*$ satisfies $\tilde{g}_j(\mathcal{R}) = 0$ for $\mathcal{R} \geq 0$. In view of these properties of $g$ and $\tilde{g}_j$, we can show that the following choices for $w_\pm(x, y)$ fulfill (33), (34), (37), and (38).

$$w_-(x, y) = \bar{g}_1(x)g_2(y), \quad w_+(x, y) = \bar{g}_3(x)g_4(y),$$

where $\bar{g}_j$ are nonzero real or complex coupling constants. Substituting (33) in (35) and (36), we therefore find scattering potentials that are unidirectionally left- and right-invisible for $k \in (\alpha/\sqrt{2}, \alpha]$. It is important to note that this is true for arbitrary choices of $L_j$ and $n_j \geq 1$, and that linear combinations of potentials of this form (with different choices for $L_j$ and $n_j$) posses the same unidirectional invisibility property.

Now, consider the $w_-(x, y)$ of (39) with $g_1(x)$ and $g_2(y)$ given by $n_1 = n_2 = 1$, and let $a := L_1$ and $b := L_2$. Substituting this choice for $w_-(x, y)$ in (36) and setting $\gamma = 0$ yield the potential (40) with $\bar{z} := -k^2 \bar{z}_-.$

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[38] These potentials are respectively employed in [25–27] and [32] to achieve full-band unidirectional invisibility in one dimension and broadband bidirectional invisibility in two dimensions.