Ferromagnetic frustrated spin systems on the square lattice: a Schwinger boson study

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We study a ferromagnetic Heisenberg spin system on the square lattice, with nearest neighbors interaction $J_1$ frustrated by second $J_2$ and third $J_3$ neighbors antiferromagnetic interactions, using a mean field theory for the Schwinger boson representation of spins. For $J_3 = 0$ we find that the boundary between the ferromagnetic and the collinear classical phases shifts to smaller values of $J_2$ when quantum fluctuations are included. Along the line $J_2/|J_1| = 1$ the boundaries between the collinear and incommensurate regions are strongly shifted to larger values with respect to the classical case. We do not find clear evidence for spin gapped phases within the present approximation.

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I. INTRODUCTION

Most current works on frustrated magnetic systems generally deal with competing antiferromagnetic interactions. Recently, some frustrated systems have been discovered where the basic interaction is ferromagnetic. In particular some vanadate and cuprate crystals as $(	ext{CuBr})\text{LaNb}_2\text{O}_7$, $\text{Pb}_2\text{VO}(\text{PO}_4)_2$ $\text{SrZnVO}(\text{PO}_4)_2$, $\text{BaCdVO}(\text{PO}_4)_2$ and $\text{PbZnVO}(\text{PO}_4)$ can be described by a two dimensional Heisenberg model of spin $S = 1/2$ with a ferromagnetic first neighbors interaction and antiferromagnetic further neighbors interactions. Other possible relevant materials are $(\text{CuBr})\text{LaNb}_2\text{O}_7$ which shows collinear order, and $(\text{CuBr})\text{Sr}_2\text{Nb}_3\text{O}_{12}$ which shows a plateau at $M = 1/3$ in the magnetization curve. As one dimensional counterparts, materials like $\text{LiCuVO}$ and $\text{Li}_2\text{ZrCuO}_{4\frac{1}{2}}$ can be modeled by ferromagnetic frustrated spin $S = 1/2$ Heisenberg chains.

In the present work we consider such a two dimensional Heisenberg model on the square lattice (see Fig. 1) with ferromagnetic nearest neighbors interactions $J_1 < 0$, frustrated by next to nearest neighbors antiferromagnetic interactions $J_2 > 0$ and also third neighbors antiferromagnetic interactions $J_3 > 0$, given by the Hamiltonian

$$
H = J_1 \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j + J_2 \sum_{\langle i,j \rangle_2} \vec{S}_i \cdot \vec{S}_j + J_3 \sum_{\langle i,j \rangle_3} \vec{S}_i \cdot \vec{S}_j
$$

(1.1)

where $\vec{S}_i$ is the spin $S$ operator at site $i$ and the range of the interacting neighbor sites $i, j$ is indicated by brackets $\langle i, j \rangle_r$ with $r = 1, 2, 3$. Closely related work has been done for antiferromagnetic $J_1, J_2, J_3$

The classical $S \to \infty$ counterpart of these interactions is described by the scalar product of commuting vectors, where the lowest energy configuration is elementarily obtained. At any value of the exchange constants, it is described by a planar arrangement of vectors rotated by relative angles $\vartheta_x$ in $x$ direction and $\vartheta_y$ in $y$ direction, giving rise to the classical phase diagram in Fig. 2 composed of four different ordered phases:

- F: a ferromagnetic phase, $(\vartheta_x, \vartheta_y) = (0, 0)$.
- CAF: a collinear antiferromagnetic phase showing antiferromagnetic order in one direction of the lattice and ferromagnetic order in the other one, $(\vartheta_x, \vartheta_y) = (\pi, 0)$ or $(0, \pi)$.
- CH: a collinear helicoidal phase showing helicoidal order in one direction of the lattice and ferromagnetic order in the other one, $(\vartheta_x, \vartheta_y) = (0, q)$ or $(q, 0)$ with $\cos(q) = -\frac{J_2 - 2J_3}{4J_3}$.
- H: a helicoidal phase composed by helicoidal order in both directions of the lattice, $(\vartheta_x, \vartheta_y) = (Q, Q)$ with $\cos(Q) = -\frac{J_2 - 2J_3}{4J_3}$.

Opposite to the classical limit, the quantum case $S = 1/2$ has been recently analyzed using exact diagonalization (ED) techniques to explore the complete phase diagram in Ref. [21], while more detailed features of the

![FIG. 1. (color online) Square lattice with J1 < 0 ferromagnetic first neighbors couplings, J2 > 0 antiferromagnetic second neighbors couplings and J3 > 0 antiferromagnetic third neighbors couplings.](attachment:fig1.png)
The paper is organized as follows: in section II we present the SBMFT methods, in section III we analyze the model without $J_3$ interactions, and in section IV we analyze the $J_2 = |J_1|$ line with $J_3$ interactions. Section V is devoted to the conclusions.

**II. SCHWINGER BOSON MEAN FIELD THEORY**

The Schwinger boson approach allows to incorporate quantum fluctuations while keeping the rotational invariance of the Heisenberg model (see for instance Ref. [27]). In this method the spin operators are written in terms of two species of bosons $b_i$ and $\bar{b}_i$ via the relations

$$S_i^x = \frac{1}{2} (b_{i,\uparrow}^{\dagger} b_{i,\uparrow} + b_{i,\downarrow}^{\dagger} b_{i,\downarrow})$$

$$S_i^y = \frac{i}{2} (b_{i,\uparrow}^{\dagger} b_{i,\uparrow} - b_{i,\downarrow}^{\dagger} b_{i,\downarrow})$$

$$S_i^z = \frac{1}{2} (b_{i,\uparrow}^{\dagger} b_{i,\downarrow} - b_{i,\downarrow}^{\dagger} b_{i,\uparrow}).$$

(2.1)

In order to represent spin $S$ properly, one must locally fix the bosonic occupation to $2S+1$ states by the constraints

$$b_{i,\uparrow}^{\dagger} b_{i,\uparrow} + b_{i,\downarrow}^{\dagger} b_{i,\downarrow} = 2S$$

(2.2)

at each site $i$.

The Heisenberg Hamiltonian is then a quartic form in bosons, but can be conveniently written as quadratic in bond operators, namely quadratic bosonic operators including one boson from each of the interacting bond sites. Such a factorization is not unique, and different schemes are adopted in case of ferromagnetic or antiferromagnetic interactions. A mixed scheme has been used here.

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**FIG. 2.** Classical phase diagram. F: ferromagnetic phase, CAF: collinear antiferromagnetic phase, CH: collinear helicoidal phase, H: helicoidal phase. A vertical axis at $J_2 = |J_1|$ is drawn for comparison with Fig. 3.

**FIG. 3.** Corrections to the classical phase diagram computed from Schwinger boson mean field fluctuations. On the line $J_3 = 0$ we find the CAF phase for $J_2 > J_2^{SCAF} = 0.41|J_1|$; for $J_2 > J_2^{SCAF} = 0.58|J_1|$ on the classical antiferromagnetic phase was reported. In contrast with linear spin wave theory and, including one boson from each of the interacting bond sites. Such a factorization is not unique, and different schemes are adopted in case of ferromagnetic or antiferromagnetic interactions. A mixed scheme has been used here.
been shown better adapted to include both antiferromagnetic and ferromagnetic short range correlations. It deals with more mean field parameters, but provides quantitatively better results and is our choice to study the present ferromagnetic frustrated system.

Bond operators $A$ and $B$ are defined as

$$A_{i,j} = \frac{1}{2}(b_{i\uparrow}b_{j\downarrow} - b_{i\downarrow}b_{j\uparrow}),$$
$$B_{i,j} = \frac{1}{2}(b_{i\uparrow}b_{j\uparrow} + b_{i\downarrow}b_{j\downarrow}).$$  (2.3)

Notice that

$$A_{i,j}^\dagger A_{i,j} = \frac{1}{4}(|S_i - S_j|^2 - \frac{S}{2}),$$
$$B_{i,j}^\dagger B_{i,j} := \frac{1}{4}(|S_i + S_j|^2 - \frac{S}{2}),$$  (2.4)

where $\mathcal{O}$ : means the bosonic normal order of an operator $\mathcal{O}$, relate non-vanishing $A$ to antiferromagnetic structures and non-vanishing $B$ to ferromagnetic structures. Moreover, expanding the squares yields representations for the SU(2) invariant terms $\vec{S}_i \cdot \vec{S}_j$. The Hamiltonian (1.1) can then be written as

$$H = \sum_{r=1,2,3} J_r \sum_{\langle i,j \rangle_r} (B_{i,j}^\dagger B_{i,j} - A_{i,j}^\dagger A_{i,j}) + H_\lambda$$  (2.5)

where the term

$$H_\lambda = \sum_i \lambda_i (b_{i\uparrow}^\dagger b_{i\downarrow} + b_{i\downarrow}^\dagger b_{i\uparrow} - 2S)$$  (2.6)

forces the local constraints, $\lambda_i$ being the Lagrange multipliers.

At mean field level, we perform a Hartree-Fock decoupling introducing a uniform Lagrange multiplier $\lambda$ and translationally invariant parameters $\alpha, \beta$ for the expectation values of each type of bond operator present in the Hamiltonian (as mentioned above, this decoupling is not unique). As it is known, the most severe approximation here is the violation of the local boson number constraint in Eq. (2.2), which is only respected on average. We are thus dealing with the Lagrange multiplier $\lambda$, six $\alpha$’s and six $\beta$’s as independent variational parameters, the latter set as expectation values of bond operators

$$\alpha_1 = \langle A_{r,r+\hat{x}} \rangle, \quad \tilde{\alpha}_1 = \langle A_{r,r+\hat{y}} \rangle,$$
$$\alpha_2 = \langle A_{r,r+\hat{x}+\hat{y}} \rangle, \quad \tilde{\alpha}_2 = \langle A_{r,r+\hat{x}-\hat{y}} \rangle,$$
$$\alpha_3 = \langle A_{r,r+\hat{y}} \rangle, \quad \tilde{\alpha}_3 = \langle A_{r,r+\hat{x}} \rangle,$$  (2.7)

and similar expressions relating the $\beta$’s to $\langle B_{i,j} \rangle$ expectation values. For compact notation we write $\alpha_{i,j} = \langle A_{i,j} \rangle$, $\beta_{i,j} = \langle B_{i,j} \rangle$, using site indices to indicate the range of the bond $\langle i,j \rangle_r$ ($r = 1, 2, 3$) as well as the possible orientations along the lattice described in Eq. (2.7). The mean field Hamiltonian then reads

$$H_{MF} = \sum_{r=1,2,3} J_r \sum_{\langle i,j \rangle_r} (\beta_{i,j}^\dagger \beta_{i,j} + B_{i,j}^\dagger B_{i,j} - \alpha_{i,j}^\dagger \alpha_{i,j} - \alpha_{i,j}^\dagger \omega_{\vec{k}}(\vec{k}) + \text{const},$$  (2.9)

where

$$\eta_{\vec{k}} = \left( \begin{array}{c} d_{\vec{k}\uparrow} \\ d_{\vec{k}\downarrow} \end{array} \right)$$  (2.10)

contains the Bogoliubov bosonic operators, with dispersion relation $\omega(\vec{k})$, and $\text{const}$ stands for non operator terms.

Finally, we compute self consistently the mean field parameters by minimizing the ground state (Bogoliubov vacuum) energy with respect to $\lambda$ and equating $\alpha_{i,j}$ and $\beta_{i,j}$ with the ground state expectation values of the corresponding operators. Such computation is done numerically on finite lattices of $N$ sites with periodic boundary conditions, allowing the study of large system sizes, up to $10^4$ sites in the present work.

We must stress that our procedure is not suited for the ferromagnetic phase, where parameters $\alpha_{i,j}$ vanish and the Hamiltonian in Eq. (2.8) is already particle conserving: the Schwinger bosons vacuum simply violates the constraint in Eq. (2.2), even on average. We use then the exact energy of a fully polarized (ferromagnetic) state, $E_F = 2NS^2(J_1 + J_2 + J_3)$, for comparison with SBMFT energies, or extrapolations thereof, to determine the ferromagnetic phase boundaries.

Once the self consistent equations are solved, the tools above allow to compute any kind of observable on the ground state. In the present work we have set $S = 1/2$ and studied four quantities: the dispersion relation, its gap, the modulated magnetization $M_n$ (defined below) and the spin correlation function.

When the dispersion relation shows a zero mode, Bose condensation indicates an ordered phase, in the sense that the spin structure factor shows a maximum at a pitch angle $\tilde{\theta} = (\theta_x, \theta_y)$ commensurate with the finite lattice, related to the position of the zero mode of the dispersion relation $\tilde{k}_{min}$ by $\tilde{\theta} = 2\tilde{k}_{min}$. Notice that Bose condensation depends on boson density, related in the SB...
approach to the spin $S$ representation by the constraint in Eq. (2.2). As we study numerically the lowest density case, $S = 1/2$, such ordered phases will also be present for larger $S$. For illustration purpose, in Fig. 4 we show the dispersion relation at coupling values $J_2 = 0.75|J_1|$ and $J_3 = 0$ (well inside the CAF classical phase).

FIG. 4. (color online) Dispersion relation for $J_2 = 0.75|J_1|$ and $J_3 = 0$ and for a system of size $N = 100 \times 100$ (in arbitrary scale, with darker zones indicating lower energy). The bosons condense at points $k_x = \pm \pi/2$, $k_y = 0$, which correspond to a CAF phase ($\pm \pi, 0$).

In the case of long range order, the vanishing of the gap in the thermodynamical limit is usually recovered after a finite size scaling analysis. Another issue arising at finite sizes is that related to commensurability. As the gap is obtained through the value of the minima of the dispersion relation on the reciprocal lattice, in the case that a thermodynamical minimum does not fit with the available momenta values at finite sizes, numerical difficulties may show up (see section IV).

FIG. 5. Evolution of the lowest couplings $J_2(N)$ tractable within SBMFT along the line $J_3 = 0$ with the inverse of the system size. Tentative linear extrapolation realized for sizes $N \times N$ with $N = 52, 56, 60, 68, 80, 100$.

The classical phases on this line, shown in Fig. 2, are F and CAF, separated by a critical value $J_2^{\text{class}} = 0.5|J_1|$. The quantum case was studied for $S = 1/2$: based on ED of the model and coupled cluster methods, Richter et. al predict a simple shift of the critical coupling to lower $J_2 = 0.39|J_1|$, while Shannon et al estimate by ED a CAF phase only for larger $J_2 \gtrsim 0.6|J_1|$ and predict the presence of a quadrupolar (bond-nematic) phase in the critical area, $0.4 \lesssim J_2/|J_1| \lesssim 0.6$.

We have studied with SBMFT systems of sizes ranging from $N = 4 \times 8$ (finding excellent agreement with ED, for ground state energies) up to $N = 100 \times 100$.

As mentioned in Sect. III our procedure does not provide a self-consistent solution for the F phase. We first analyze the values of $J_2$ above which SBMFT solutions are obtained. These values turn out to be sensitive dependent on the system size. Above $J_2 \approx 0.56|J_1|$ we reach solutions for all explored systems, up to $100 \times 100$ sites; but approaching the F phase we get oscillatory behaviour with the system trapped in metastable configurations, and the tractable sizes reduce as down to $20 \times 20$ at $J_2 \approx 0.4|J_1|$. The size dependence of the lowest couplings $J_2(N)$ tractable within SBMFT is roughly linear in $1/N$, as shown in Fig. 5 suggesting an infinite size extrapolation to $J_2^\infty = 0.58|J_1|$. Thus we estimate that, investing enough CPU time, one can treat systems of arbitrary size only when $J_2 > J_2^\infty$.

For $J_2$ above $J_2^\infty$ the observables computed from the SBMFT self-consistent solutions correspond to a CAF phase, showing staggered magnetization along one of the lattice axes. An example of the dispersion relation, for $J_2 = 0.75|J_1|$ and $J_3 = 0$ in a large system of size $N = 100 \times 100$, is the one shown in Fig. 4. The boson modes become gapless at momentum points $\mathbf{k} = (\pm \pi/2, 0)$, showing that the bosons do condense. The condensation momenta correspond to ordering angles $\theta = (\pi, 0)$. The same pattern (alternatively with $\theta = (0, \pm \pi)$) is found for $J_2 > J_2(N)$, $N = 20, 40, 60, 80, 100$.

The dispersion relation gap goes clearly to zero for

III. NEAREST NEIGHBORS FRUSTRATION $J_2$

In this section we analyze the case $J_3 = 0$, that is a system with ferromagnetic first neighbors couplings and only second neighbors antiferromagnetic interactions.

We measure modulated magnetizations by considering the ground state from the bosonic representation in Eq. (2.1) and following average over the lattice:

$$M_i^2(\mathbf{q}) = \frac{1}{N(N+2)} \sum_{i,j} \langle S_i \cdot S_j \rangle e^{i\mathbf{q} \cdot (\mathbf{R}_i - \mathbf{R}_j)},$$

where $\mathbf{q} = (\theta_x, \theta_y)$ and $\mathbf{R}_i$ is the position of spin $S_i$ with respect to some reference site. It amounts to an improved normalization of the spin structure factor, that fits better small systems and tends to moderate the weight of strong on-site terms. It is straightforwardly computed from the spin correlations.

For $J_2$ above $J_2^\infty$ the observables computed from the SBMFT self-consistent solutions correspond to a CAF phase, showing staggered magnetization along one of the lattice axes. An example of the dispersion relation, for $J_2 = 0.75|J_1|$ and $J_3 = 0$ in a large system of size $N = 100 \times 100$, is the one shown in Fig. 4. The boson modes become gapless at momentum points $\mathbf{k} = (\pm \pi/2, 0)$, showing that the bosons do condense. The condensation momenta correspond to ordering angles $\theta = (\pi, 0)$. The same pattern (alternatively with $\theta = (0, \pm \pi)$) is found for $J_2 > J_2(N)$, $N = 20, 40, 60, 80, 100$.

The dispersion relation gap goes clearly to zero for
$N \to \infty$, as shown in Fig. 6 (size scaling for $J_2 = |J_1|$ is shown in the inset). Correspondingly, the spin correlation function exhibits long range order: when $\vec{\theta} = (\pm \pi, 0)$, we observe antiferromagnetic correlations in the $x$ direction and ferromagnetic correlations in the $y$ direction. For example, in Fig. 7 we show the correlations for $J_2 = 0.75|J_1|$. The corresponding modulated magnetization has a maximum at $(\pm \pi, 0)$. Then $M_2^y(\pi, 0)$, shown in Fig. 8 measures the staggered magnetization along the $x$-direction. In general, $M_2^x(\pi, 0)$ and $M_2^x(0, \pi)$ can be used as order parameters for the CAF phase.

We now turn to discuss the region $0.4 \lesssim J_2/|J_1| \lesssim 0.58$, under controversy in the literature, where the largest sizes studied do not provide a self-consistent solution. Notice in Figs. 6 and 7 that in all converged solutions there is no signal of an exotic phase but clear indications of the CAF phase. It turns out that the finite size available solutions scale in this range to the same CAF phase as in the range $J_2/|J_1| < 0.58$.

The energy per site obtained from SBMFT self-consistent solutions is very stable against system size and shows a neat linear dependence with $J_2$, including the region $0.4 \lesssim J_2/|J_1| \lesssim 0.58$. (see Fig. 9). We conclude that lack of convergence of SBMFT self-consistent equations at large system size is an artifact of our present approach, presumably due to the proximity of a ferromagnetic phase, and that solutions obtained for small systems are good estimates of the CAF phase in the region for $0.4 \lesssim J_2/|J_1| < 0.58$. Then, following the criteria in Ref. [24] we set the F-CAF phase boundary at the intersection point between the extrapolated CAF energy and the exact energy per site of a fully polarized state, $E_F/N = 2S^2(J_1 + J_2)$ (here with $S = 1/2$).

Such a point $J_2^{F-CAF}(N)$ appears, depending on size $N$, at a quite precise value of $J_2/|J_1|$ between 0.402 and 0.4078, as shown in Fig. 10. The roughly linear dependence in $1/N$ suggests an extrapolated transition point at $J_2^{F-CAF} = 0.41 |J_1|$. From the results in this Section we conclude that, on the $J_3 = 0$ line, the present SBMFT approach can confirm the CAF phase for $J_2 > J_2^{F-CAF} = 0.41 |J_1|$.

For $J_2 > J_2^{F-CAF} = 0.58 |J_1|$, this can be tested even in the thermodynamical limit. For lower $J_2$, although systems of limited size can be solved, we find CAF observables until the transition to the F phase. A linear extrapolation of the CAF ground state energies suggests a direct first order transition to the F phase at $J_2^{F-CAF} = 0.41 |J_1|$ in accordance with Ref. [24].

**IV. EFFECTS OF NEXT TO NEAREST NEIGHBOR FRUSTRATION $J_3$**

In this section we analyze the influence of third neighbors antiferromagnetic couplings $J_3$ on top of the CAF
from the region

The classical phase diagram in Fig. 2 shows collinear antiferromagnetic order for $0 < J_3 \leq 0.25 |J_1|$, a continuous transition to collinear helicoidal $(q,0)$ (or $(0,q)$) order for $0.25 |J_1| < J_3 < 0.5 |J_1|$, with $q$ decreasing from $\pi$ to $\frac{5}{4} \pi$, and a discontinuous transition to a helicoidal phase $(Q, Q)$ for $J_3 > 0.5 |J_1|$, with $Q$ increasing in a narrow window, from 0.4195$\pi$ to $\frac{5}{4} \pi$ (reaching $Q = 0.4460\pi$ at $J_3 = |J_1|$, the largest value of $J_3$ in the present analysis).

The quantum case was studied by Sindzingre et al. by ED in systems up to $N = 36$ sites, for positive $J_2$ and $J_3$, both up to $|J_1|$. For $J_2 > 0.75 |J_1|$ and around the classical boundary between collinear helicoidal and helicoidal phases, the authors find signals of an exotic gapped phase, stating that it is difficult to conclude its precise nature because of large and irregular finite size effects. In particular, on the line $J_2 = |J_1|$, they find a CAF phase for $J_3 \leq 0.35 |J_1|$ and a gapped phase for $J_3 > 0.35 |J_1|$.

We have applied the SBMFT to systems of size $N = 20 \times 20$, $40 \times 40$, $60 \times 60$, $80 \times 80$, $100 \times 100$. For $0 < J_3 \lesssim 0.4 |J_1|$, we find persistence of the CAF phase: the dispersion relation remains gapless at commensurate momenta $\vec{k} = (\pi/2,0)$ (or $(0, \pi/2)$) and the $\theta$-dependent susceptibility has a maximum at $(\pi,0)$ (or $(0,\pi)$). This phase shows a boundary that barely depends on system sizes and can be estimated as $J_3^{CAF-CCH} \approx 0.41 |J_1|$. This amounts to a shift of $0.16 |J_1|$ with respect to the classical value.

For larger $J_3$ the minima of the dispersion relation move to incommensurate values of $\vec{k}$. It gets numerically difficult on a finite lattice to determine the existence of gapless minima. However, though with less precision than in the CAF phase, we find for all studied sizes that, immediately above $J_3^{CAF-CCH}(N)$, a gapless collinear helicoidal phase with $\vec{k} = (q,0)$ (or $(0,q)$) develops. The available values for $q$ are discrete but, as shown in Fig. 11, the dispersion minima position evolves in the same range as the classical ones, simply shifted in $J_3 \rightarrow J_3 + 0.16 |J_1|$. For each point in the figure, the $\theta$-dependent susceptibility shows a maximum at $\theta = (q,0)$ (or $(0,q)$), characterizing collinear helicoidal magnetization order.

The collinear helicoidal phase extends up to $J_3^{CCH-H} \approx 0.56 |J_1|$. Again, such boundary is almost independent of the system sizes. For even larger $J_3$ further numerical difficulties show up. Indeed, the dispersion seems to get gapless at momenta $(Q,Q)$ with $Q$ in the same narrow window found in the classical phase diagram for the helicoidal phase. As the thermodynamical values of minima position in such a narrow range may mismatch the available momenta for a given finite lattice, it is difficult to select the minimum of the dispersion relation amongst neighboring points. For this same reason, a gap may seem to open but could be just a finite size artifact. A refinement was done by choosing different system sizes in order to allow for different distributions of lattice momenta. In general, when a clear minimum is found, it corresponds to a gapless situation with ordering angle $\vec{\theta} = (Q,Q)$ where $0.421 \pi < Q < 0.483 \pi$. In such cases, the order parameter $M_2^2(Q,Q)$ remains finite signaling a helicoidal phase. Thus, the present method shows no
clear indications of the appearance of a gap in the excitation spectrum.

FIG. 11. (color online) Evolution of the incommensurate $q$ along the line $J_2=|J_1|$, in the collinear helicoidal phase for different system sizes. For comparison, we show the classical pitch angle $\vartheta(J_2 = 0.16|J_1|)$ (i.e. plotted with respect to the SBMFT phase boundary $J_3^{CAF-CH}$).

V. CONCLUSIONS

To summarize, along the line $J_3=0$ and within SBMFT we can confirm the collinear antiferromagnetic phase for $J_2 > J_2^{F-CAP} = 0.41|J_1|$. For $J_2^{F-CAP} < J_2 < J_2^\infty = 0.58|J_1|$ the convergence becomes harder, presumably because of dominance of ferromagnetic correlations. However, at tractable system sizes we find CAF observables until the transition to the ferromagnetic phase. A linear extrapolation of the CAF phase ground state energies suggests a direct first order transition to the ferromagnetic phase at $J_2^{F-CAP}$ in good agreement with Ref. [24].

Along the line $J_2=|J_1|$ we have found that the boundaries between the collinear and incommensurate phases are strongly shifted, with respect to the classical case, to larger values of $J_3$: $J_3^{CAF-CH} \approx 0.41|J_1|$ and $J_3^{CH-H} \approx 0.56|J_1|$. We do not find clear evidence of spin-gapped phases within the present approximation.

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