ON SELF-SIMILAR BERNSTEIN FUNCTIONS AND CORRESPONDING GENERALIZED FRACTIONAL DERIVATIVES

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Abstract. We use the theory of Bernstein functions to analyze power law tail behavior with log-periodic perturbations which corresponds to self-similarity of the Bernstein functions. Such tail behavior appears in the context of semistable Lévy processes. The Bernstein approach enables us to solve some open questions concerning semi-fractional derivatives recently introduced in [12] by means of the generator of certain semistable Lévy processes. In particular it is shown that semi-fractional derivatives can be seen as generalized fractional derivatives in the sense of [16].

1. Introduction

A non-negative function \( \tilde{\psi} : (0, \infty) \to (0, \infty) \) is called a Bernstein function if it is of class \( C^\infty(0, \infty) \) and

\[
-1)^{n-1} \tilde{\psi}^{(n)}(x) \geq 0 \quad \text{for all } n \in \mathbb{N} \text{ and } x > 0.
\]

(1.1)

Its first derivative \( f = \tilde{\psi}' \) is a completely monotone function, i.e.

\[
-1)^n f^{(n)}(x) \geq 0 \quad \text{for all } n \in \mathbb{N}_0 \text{ and } x > 0.
\]

(1.2)

Due to a celebrated result of Bernstein, the completely monotone function \( f \) is the Laplace transform of a unique Borel measure \( \mu \) on \( [0, \infty) \)

\[
f(x) = \tilde{\mu}(x) := \int_0^\infty e^{-xt} d\mu(t).
\]

(1.3)

As a consequence, the Bernstein function admits the representation

\[
\tilde{\psi}(x) = a + bx + \int_0^\infty (1 - e^{-xt}) \, d\phi(t)
\]

(1.4)

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for a unique triplet \([a, b, \phi]\), where \(a, b \geq 0\) and \(\phi\) is a Borel measure on \((0, \infty)\) satisfying \(\int_0^\infty \min\{1, t\} \, d\phi(t) < \infty\), also called the Lévy measure. For details on Bernstein functions, completely monotone functions and their connection to stochastic processes we refer to the monograph [30]. It is well known that in case \(a = b = 0\) the Bernstein function \(\tilde{\psi}(x) = \int_0^\infty (1 - e^{-xt}) \, d\phi(t)\) is the Laplace exponent of a Lévy subordinator \((X_t)_{t \geq 0}\), i.e. \(\mathbb{E}[\exp(-sX_t)] = \exp(-t \cdot \tilde{\psi}(s))\) for all \(t \geq 0, s > 0\), where \((X_t)_{t \geq 0}\) is a Lévy process with almost surely non-decreasing sample paths.

In Section 2 we will introduce a self-similarity property for Bernstein functions which is intimately connected to the following class of functions. We call a function \(\theta : \mathbb{R} \to \mathbb{R}\) admissible with respect to the parameters \(\alpha \in (0, 2) \setminus \{1\}\) and \(c > 1\) if the following three conditions are fulfilled:

\[
\begin{align*}
\text{(1.5)} & \quad \theta(x) > 0 \text{ for all } x \in \mathbb{R}, \\
\text{(1.6)} & \quad \text{the mapping } t \mapsto t^{-\alpha}\theta(\log t) \text{ is non-increasing for } t > 0, \\
\text{(1.7)} & \quad \theta \text{ is } \log(c^{1/\alpha})\text{-periodic.}
\end{align*}
\]

In case \(\alpha \in (0, 1)\) we use an admissible function \(\theta\) to define

\[
\phi(t, \infty) := t^{-\alpha}\theta(\log t) \quad \text{for all } t > 0
\]

as the positive tail of a Lévy measure \(\phi\) concentrated on \((0, \infty)\) which belongs to a semistable distribution \(\nu\) with log-characteristic function

\[
\psi(x) = \int_0^\infty (e^{ixy} - 1) \, d\phi(y) \quad \text{for all } x \in \mathbb{R}
\]

given uniquely by the Fourier transform \(\hat{\nu}(x) = \int_\mathbb{R} e^{ixy} \, d\nu(y) = \exp(\psi(x))\). The corresponding Lévy process \((X_t)_{t \geq 0}\) given by \(\mathbb{E}[\exp(ix \cdot X_t)] = \exp(t \cdot \psi(x))\) for all \(t \geq 0\) and \(x \in \mathbb{R}\) is called a semistable subordinator. For details on semistable distributions and Lévy processes we refer to the monographs [20, 29]. The power law tail behavior with log-periodic perturbations of an admissible function \(\theta\) naturally appears in various applications of natural sciences and other areas; see [33] and section 5.4 in [34]. In recent years the asymptotic fine structure of the corresponding measure \(\phi\) and the function \(\psi\) has drawn some attention; cf. [14] and [13]. Given an admissible function \(\theta\) with respect to \(\alpha \in (0, 1)\) and \(c > 1\) with corresponding semistable Lévy measure \(\phi\) given by (1.8), by Definition 2.2 in [12] the semi-fractional derivative \(\frac{\partial^\alpha}{\partial t^\alpha} x^c\).
of order $\alpha \in (0,1)$ is given by the non-local operator

$$\frac{\partial^{\alpha}}{\partial c, \theta x^{\alpha}} f(x) := -Lf(x) = \int_{0}^{\infty} (f(x) - f(x - y)) \, d\phi(y),$$

where $L$ is the generator of the corresponding semistable Lévy process and at least functions $f$ in the Sobolev space $W^{2,1}(\mathbb{R})$ belong to the domain of the semi-fractional derivative. In terms of the Fourier transform we can equivalently rewrite (1.10) as

$$\widehat{\frac{\partial^{\alpha}}{\partial c, \theta x^{\alpha}}} f(x) := -\widehat{L} f(x) = -\psi(x) \hat{f}(x) \quad \text{for all } x \in \mathbb{R},$$

where the Fourier transform of $f$ is given by $\hat{f}(x) = \int_{\mathbb{R}} e^{ixy} f(y) \, dy$; see [12] for details.

In Section 2, we will start with the elementary observation that for $\alpha \in (0,1)$ there is a one-to-one correspondence between self-similar Bernstein functions given as the Laplace exponent $\tilde{\psi}(x) = -\psi(ix)$ for $x > 0$ and semistable Lévy measures $\phi$ of the form (1.8). In particular, this enables us to show that semi-fractional derivatives of order $\alpha \in (0,1)$ can be seen as a special case of generalized fractional derivatives in the sense of [16]. A constant function $\theta$ corresponds to the complete Bernstein function $\tilde{\psi}(x) = x^\alpha$ and an ordinary fractional derivative of order $\alpha \in (0,1)$. For details on classical fractional derivatives we refer to the monographs [15, 25, 27].

In Section 3 we will prove a discrete approximation formula of the generator in (1.10) involving a generalized Sibuya distribution given in terms of the self-similar Bernstein function.

In case $\alpha \in (1,2)$ we use an admissible function $\theta$ to define

$$\phi(-\infty, -t) := t^{-\alpha} \theta(\log t) \quad \text{for all } t > 0$$

as the negative tail of a Lévy measure $\phi$ concentrated on $(-\infty, 0)$ which belongs to a different semistable distribution $\nu$ with log-characteristic function

$$\psi(x) = \int_{-\infty}^{0} \left( e^{ixy} - 1 - ixy \right) \, d\phi(y) \quad \text{for all } x \in \mathbb{R}. \tag{1.13}$$

Given an admissible function $\theta$ with respect to $\alpha \in (1,2)$ and $c > 1$ with corresponding semistable Lévy measure $\phi$ given by (1.12), by Definition 2.5 in [12] the negative semi-fractional derivative $\frac{\partial^{\alpha}}{\partial c, \theta (-x)^{\alpha}}$ of order $\alpha \in (1,2)$ is given by the non-local operator

$$\frac{\partial^{\alpha}}{\partial c, \theta (-x)^{\alpha}} f(x) := Lf(x) = \int_{0}^{\infty} (f(x + y) - f(x) - y f'(x)) \, d\phi(-y), \tag{1.14}$$
where $L$ is again the generator of the corresponding semistable Lévy process and at least functions $f$ in the Sobolev space $W^{2,1}(\mathbb{R})$ belong to the domain of the semi-fractional derivative. For $\alpha \in (1, 2)$ the function $x \mapsto \psi(-ix)$ cannot be a Bernstein function, but we will show in Section 4 that it has an inverse which is a self-similar Bernstein function. This will enable us to solve an open question from [11] concerning space-time duality for semi-fractional differential equations.

2. Self-similar Bernstein functions and semi-fractional derivatives

In this section we suppose that $\alpha \in (0, 1)$ in which case the Laplace exponent $\tilde{\psi}$ of the corresponding semistable distribution $\nu$, uniquely given by the Laplace transform $\tilde{\nu}(x) = \int_0^\infty e^{-xy} d\nu(y) = \exp(-\tilde{\psi}(x))$ for $x > 0$, in view of (1.9) can be represented as

\begin{align}
(2.1) \quad \tilde{\psi}(x) = -\psi(ix) = \int_0^\infty (1 - e^{-xy}) \ d\phi(y).
\end{align}

Clearly, $\tilde{\psi}$ is a Bernstein function since $\phi$ integrates $\min\{1, t\}$ on $(0, \infty)$.

**Definition 2.1.** We call a Bernstein function $\tilde{\psi}$ self-similar with respect to $\alpha \in (0, 1)$ and $c > 1$ if it admits the discrete scale invariance

$$
\tilde{\psi}(c^{1/\alpha}x) = c \cdot \tilde{\psi}(x) \quad \text{for all } x > 0.
$$

The following elementary observation is our key result.

**Lemma 2.2.** For fixed $\alpha \in (0, 1)$ and $c > 1$ the following statements are equivalent.

(i) $\tilde{\psi}$ is a self-similar Bernstein function with respect to $\alpha \in (0, 1)$ and $c > 1$.

(ii) $\tilde{\psi}(x) = \int_0^\infty (1 - e^{-xy}) \ d\phi(y)$, where the Lévy measure $\phi$ is given by (1.8) for some admissible function $\theta$ with respect to $\alpha \in (0, 1)$ and $c > 1$.

In either case we have $\tilde{\psi}(x) = x^\alpha \gamma(-\log x)$ for an admissible $C^\infty(\mathbb{R})$-function $\gamma$ with respect to $\alpha \in (0, 1)$ and $c > 1$.

**Proof.** “(i)⇒(ii)”: Since $\tilde{\psi}$ is a Bernstein function, by (1.14) we have

$$
\tilde{\psi}(x) = a + bx + \int_0^\infty (1 - e^{-xt}) \ d\phi(t)
$$

for some $a, b \geq 0$ and a Lévy measure $\phi$ on $(0, \infty)$ integrating $\min\{1, t\}$. Iterating the self-similarity relation shows that $\tilde{\psi}(c^{m/\alpha}x) = c^m \tilde{\psi}(x)$ for all $x > 0$ and $m \in \mathbb{Z}$. As
we see that \( \lim_{x \to 0} \tilde{\psi}(x) = 0 \) and hence \( a = 0 \). By the transformation rule self-similarity now reads as

\[
\tilde{\psi}(e^{1/\alpha}x) = b c^{1/\alpha} x + \int_0^\infty (1 - e^{-xt}) d(c^{1/\alpha} \phi)(t)
\]

\[
= b c x + c \cdot \int_0^\infty (1 - e^{-xt}) d\phi(t) = c \cdot \tilde{\psi}(x),
\]

where \( (c^{1/\alpha} \phi) \) denotes the image measure under scale multiplication with \( c^{1/\alpha} \). Since the Bernstein triplet \([a, b, \phi]\) is unique, we must have \( b = 0 \) and \( (c^{1/\alpha} \phi) = c \cdot \phi \). Hence by Lemma 7.1.6 in [20] we have that \( \phi \) is a \((c^{1/\alpha}, c)\)-semistable Lévy measure on \((0, \infty)\) fulfilling (1.8) for some admissible function \( \theta \) with respect to \( \alpha \in (0, 1) \) and \( c > 1 \). The connection of Corollary 7.4.4 in [20] to admissible functions is made precise by Lemma A.1 in the Appendix.

“\( (ii) \Rightarrow (i) \)”: Clearly, \( \tilde{\psi} \) is a Bernstein function by Theorem 3.2 in [30] and self-similarity follows from (2.2) with \( b = 0 \) which is valid by Lemma 7.1.6 in [20].

Now define \( \gamma(x) := e^{\alpha x} \tilde{\psi}(e^{-x}) \) which is of class \( C^\infty(\mathbb{R}) \) by the product rule and \( \gamma(x) > 0 \) for all \( x \in \mathbb{R} \). Moreover, \( \gamma \) is \( \log(c^{1/\alpha}) \)-periodic since

\[
\gamma(x + \log(c^{1/\alpha})) = e^{\alpha x} c \cdot \tilde{\psi}(c^{-1/\alpha} e^{-x}) = e^{\alpha x} \tilde{\psi}(e^{-x}) = \gamma(x)
\]

and \( t \mapsto t^{-\alpha} \gamma(\log t) = \tilde{\psi}(t^{-1}) \) is non-increasing. \( \square \)

**Remark 2.3.** If we assume that \( \theta \) is smooth in the sense that it is continuous and piecewise continuously differentiable, then it admits a Fourier series representation

\[
\theta(x) = \sum_{k \in \mathbb{Z}} c_k e^{ik\tilde{c}x} \quad \text{with} \quad \tilde{c} = \frac{2\pi \alpha}{\log c}
\]

In this case the function \( \gamma \) appearing in Lemma 2.2 is given by the modified Fourier series

\[
\gamma(x) = \sum_{k \in \mathbb{Z}} c_k \Gamma(ik\tilde{c} - \alpha + 1) e^{ik\tilde{c}x} \quad \text{for} \ x \in \mathbb{R}
\]

which can be seen as follows. By Theorem 3.1 in [12] the coefficients of this series appear in a representation of the log-characteristic function

\[
\psi(x) = -\sum_{k \in \mathbb{Z}} c_k \Gamma(ik\tilde{c} - \alpha + 1) (-ix)^{\alpha - ik\tilde{c}} \quad \text{for} \ x \in \mathbb{R}
\]

and the relation \( \gamma(x) = e^{\alpha x} \tilde{\psi}(e^{-x}) = -e^{\alpha x} \psi(ie^{-x}) \) easily shows (2.4).
A natural question which arises is if for two admissible functions \( \theta_1, \theta_2 \) with respect to the parameters \( \alpha_1 \in (0, 1) \) and \( c_1 > 1 \), respectively \( \alpha_2 \in (0, 1) \) and \( c_2 > 1 \), the composition of the corresponding semi-fractional derivatives given by (1.10) can again be a semi-fractional derivative of order \( \alpha := \alpha_1 + \alpha_2 \). We concentrate on the easiest case when \( \alpha \in (0, 1) \) and \( \theta_1, \theta_2 \) have the same periodicity, i.e. \( c_1^{\alpha_1} = c_2^{\alpha_2} \). In view of (1.11) for the corresponding log-characteristic functions we need to show that

\[
\psi_1(x) \cdot \psi_2(x) = -\psi(x) \quad \text{for all } x \in \mathbb{R},
\]

where \( \psi \) is the log-characteristic function of a semistable distribution corresponding to an admissible function \( \theta \) with respect to the parameters \( \alpha = \alpha_1 + \alpha_2 \in (0, 1) \) and \( c := c_1^{\alpha_1} = c_2^{\alpha_2} > 1 \). Since (2.6) is equivalent to \( \tilde{\psi}_1 \cdot \tilde{\psi}_2 = \tilde{\psi} \) for the corresponding log-Laplace exponents and we have self-similarity

\[
\tilde{\psi}(c^{1/\alpha} x) = \tilde{\psi}_1(c_1^{\alpha_1} x) \cdot \tilde{\psi}_2(c_2^{\alpha_2} x) = c_1 c_2 \cdot \tilde{\psi}_1(x) \tilde{\psi}_2(x) = c^{\alpha_1/\alpha} c^{\alpha_2/\alpha} \tilde{\psi}(x) = c \cdot \tilde{\psi}(x),
\]

in view of Lemma 2.2 this is equivalent to require that \( \tilde{\psi} \) is a Bernstein function.

**Corollary 2.4.** \( \tilde{\psi}_1 \cdot \tilde{\psi}_2 \) is a Bernstein function iff there exists an admissible function \( \theta \) with respect to the parameters \( \alpha = \alpha_1 + \alpha_2 \in (0, 1) \) and \( c = c_1^{\alpha_1} = c_2^{\alpha_2} > 1 \) such that

\[
\frac{\partial^{\alpha_2}}{\partial c_2, \theta_2 x^{\alpha_2}} \frac{\partial^{\alpha_1}}{\partial c_1, \theta_1 x^{\alpha_1}} f = \frac{\partial^\alpha}{\partial c, \theta x^\alpha} f
\]

for suitable functions \( f \in W^{2,1}(\mathbb{R}) \) with \( \frac{\partial^{\alpha_1}}{\partial c_1, \theta_1 x^{\alpha_1}} f \) belonging to the domain of \( \frac{\partial^{\alpha_2}}{\partial c_2, \theta_2 x^{\alpha_2}} f \).

**Remark 2.5.** In case \( \theta_1, \theta_2 \) are smooth admissible functions with Fourier series representations as in (2.3)

\[
\theta_1(x) = \sum_{k \in \mathbb{Z}} c_{k,1} e^{i k \tilde{c} x} \quad \text{and} \quad \theta_2(x) = \sum_{\ell \in \mathbb{Z}} c_{\ell,2} e^{i \ell \tilde{c} x},
\]

where \( \tilde{c} = \frac{2 \pi \alpha_1}{\log c_1} = \frac{2 \pi \alpha_2}{\log c_2} = \frac{2 \pi \alpha}{\log c} \), then by (2.5) and the Cauchy product rule we easily get

\[
\tilde{\psi}(x) = \tilde{\psi}_1(x) \cdot \tilde{\psi}_2(x) = \sum_{m \in \mathbb{Z}} d_m \Gamma(i m \tilde{c} - (\alpha_1 + \alpha_2) + 1) x^{\alpha_1 + \alpha_2 - i m \tilde{c}},
\]

where

\[
d_m = \sum_{\ell \in \mathbb{Z}} c_{m-\ell,1} c_{\ell,2} \frac{\Gamma(i (m - \ell) \tilde{c} - \alpha_1 + 1) \Gamma(i \ell \tilde{c} - \alpha_2 + 1)}{\Gamma(i m \tilde{c} - (\alpha_1 + \alpha_2) + 1)}.
\]
Uniqueness of the Fourier coefficients then gives us the representation
\[ \theta(x) = \sum_{m \in \mathbb{Z}} d_m e^{im\hat{c}x} \]
for the admissible function \( \theta \) from Corollary 2.4, provided that (2.8) is a Bernstein function.

Finally, we want to show that the semi-fractional derivative of order \( \alpha \in (0,1) \) can be seen as a special case of a generalized fractional derivative introduced in [16]. Starting with (1.10) for a smooth admissible function \( \theta \), integration by parts shows that
\[ \frac{\partial^\alpha}{\partial x^\alpha} f(x) = \int_0^\infty f'(x-y) y^{-\alpha} \theta(\log y) \, dy \]
as laid out in [12]. Introducing the kernel function \( k(y) := y^{-\alpha} \theta(\log y) \) and restricting considerations to functions with support on the positive real line, this can be interpreted as a semi-fractional derivative of Caputo type
\[ \frac{\partial^\alpha}{\partial x^\alpha} f(x) = \int_0^x f'(x-y) k(y) \, dy = (k \ast f')(x). \]

On the other hand, interchanging the order of integration and differentiation gives a semi-fractional derivative of Riemann-Liouville type
\[ D_{c,\theta}^\alpha f(x) = \frac{d}{dx} \int_0^x f(x-y) k(y) \, dy = (k \ast f)'(x) \]
which is also called of convolution type in [35]. The relationship between these forms is given by the formula
\[ \frac{\partial^\alpha}{\partial x^\alpha} f(x) = D_{c,\theta}^\alpha f(x) - f(0)k(x) =: D_{(k)}^\alpha f(x) \]
which can be derived as (2.33) in [21] and for more general kernel functions \( D_{(k)} f \) is called a generalized fractional derivative in [16]. For further approaches into this direction see [3, 17, 19, 24, 28, 35]. Of particular interest are non-negative locally integrable kernel functions \( k \) such that the operator \( D_{(k)} \) possesses a right inverse \( I_{(k)} \) such that \( D_{(k)} I_{(k)} f = f \). Using the theory of complete Bernstein functions and the relationship to the Stieltjes class, it is shown in [16] that this is possible with
\[ I_{(k)} f(x) = \int_0^x f(y) k^*(x-y) \, dy \]
for locally bounded measurable functions \( f \) if \( (k, k^*) \) forms a Sonine pair of kernels, i.e. \( k \ast k^* \equiv 1 \); cf. also \([32, 26, 36]\). In this case \( I_{(k)} f \) is called a generalized fractional integral of order \( \alpha \) and it also holds that \( I_{(k)} D_{(k)} f(x) = f(x) - f(0) \) for absolutely continuous functions \( f \). As shown in \([8]\), necessarily the kernel function \( k \) must have an integrable singularity at 0. We will now show that a Sonine kernel \( k^* \) may exist for our specific kernel function \( k(y) = y^{-\alpha} \theta(\log y) \) working in the more general framework of Bernstein functions and their relation to completely monotone functions. Hence we aim to extend the list of specific kernel functions given in section 6 of \([19]\) by a new example. Therefore we have to relax the definition of a Sonine pair in the following sense.

**Definition 2.6.** Given a non-negative, locally bounded and measurable function \( k \) on \((0, \infty)\) and a sigma-finite Borel measure \( \rho \) on \((0, \infty)\) we say that \((k, \rho)\) forms a generalized Sonine pair if \( k \ast \rho \equiv 1 \).

We know that \( \tilde{\psi}(x) = \int_0^\infty (1 - e^{-xt}) d\phi(t) \) with \( \phi \) as in \([18]\) is a Bernstein function and thus \( G(x) = \frac{1}{x} \tilde{\psi}(x) \) is completely monotone by Corollary 3.8 (iv) in \([30]\). Integration by parts yields the Laplace transform

\[
G(x) = \int_0^\infty \frac{1}{x} (1 - e^{-xt}) d\phi(t) = \int_0^\infty e^{-xt} k(t) dt = \tilde{k}(x) \quad \text{for} \quad x > 0
\]

of our kernel function \( k(y) = y^{-\alpha} \theta(\log y) \). By Theorem 3.7 in \([30]\) \( G^*(x) = 1/\tilde{\psi}(x) \) is completely monotone since it is the composition of a Bernstein function and the completely monotone function \( x \mapsto \frac{1}{x} \). Hence there exists a Borel measure \( \rho \) on \([0, \infty)\) such that \( G^*(x) = \int_0^\infty e^{-xt} d\rho(t) \) which serves as the generalized Sonine partner of \( k \).

**Lemma 2.7.** For a smooth admissable function \( \theta \) with respect to \( \alpha \in (0, 1) \) and \( c > 1 \) let \( k(y) = y^{-\alpha} \theta(\log y) \) for \( y > 0 \) and \( \rho \) be the Borel measure on \((0, \infty)\) with Laplace transform \( \tilde{\rho}(x) = G^*(x) \) as above. Then \((k, \rho)\) forms a generalized Sonine pair.

**Proof.** We easily calculate the Laplace transform of \((k \ast \rho)(t) = \int_0^t k(t-x) d\rho(x)\) as

\[
\tilde{k}(x) \cdot \tilde{\rho}(x) = G(x) \cdot G^*(x) = \frac{\tilde{\psi}(x)}{x} \cdot \frac{1}{\tilde{\psi}(x)} = \frac{1}{x}
\]

for \( x > 0 \), which is the Laplace transform of \( 1_{(0,\infty)} \). \( \square \)
By virtue of Lemma 2.7 we may interpret
\[ \mathbb{I}^{(k)} f(x) = \int_0^x f(x-y) \, d\rho(y) = (f \ast \rho)(x) \]
as a semi-fractional integral of order \( \alpha \) for locally bounded and measurable functions \( f \). Since \( \mathbb{I}^{(k)} f(0) = 0 \), we get
\[ D^{(k)} \mathbb{I}^{(k)} f(x) = \frac{d}{dx} \int_0^x \mathbb{I}^{(k)} f(x-y) \, dy = \frac{d}{dx} \left( \mathbb{I}^{(k)} f \ast k \right)(x) \]
and for absolutely continuous functions \( f \) with density \( f' \) we get
\[ \mathbb{I}^{(k)} D^{(k)} f(x) = \int_0^x D^{(k)} \mathbb{I}^{(k)} f(x-y) \, dy = \int_0^x \left( k \ast f' \right)(x-y) \, d\rho(y) \]
for all \( x > 0 \).

The semi-fractional integral is also determined by a self-similar Bernstein function.

**Lemma 2.8.** The primitive \( G^*_1(x) := \int_0^x G^*(y) \, dy = \int_0^x \widetilde{\rho}(y) \, dy \) is a self-similar Bernstein function with respect to \( 1 - \alpha \in (0,1) \) and \( d := \frac{c^{\frac{1}{\alpha}}}{c} > 1 \).

**Proof.** From Lemma 2.2 we get the scaling relation
\[ G^*(c^{1/\alpha} x) = \frac{1}{\psi(c^{1/\alpha} x)} = \frac{1}{c \cdot \psi(x)} = c^{-1} G^*(x) \]
for all \( x > 0 \), which implies \( (c^{1/\alpha} \rho) = c^{-1} \cdot \rho \) for the measure \( \rho \) in Lemma 2.7. In particular it follows that \( \rho(\{0\}) = 0 \) and by the Fubini-Tonelli theorem we get
\[ G^*_1(x) = \int_0^x G^*(y) \, dy = \int_0^x \int_0^\infty e^{-yt} \, d\rho(t) \, dy \]
where the Borel measure \( \mu \) on \( (0, \infty) \) is given by \( d\mu(t) := \frac{1}{t} d\rho(t) \) and integrates \( \min\{1, t\} \) as shown in the proof of Theorem 3.2 in [30]; cf. Proposition 3.5 in [30].
Thus the primitive $G^*_t(x)$ is a Bernstein function and the scaling relation gives us

$$G^*_t(c^{1/\alpha}x) := \int_0^\infty \left(1 - e^{-xc^{1/\alpha}t}\right) d\mu(t) = c^{1/\alpha} \int_0^\infty \frac{1 - e^{-xc^{1/\alpha}t}}{c^{1/\alpha} t} d\rho(t)$$

$$= c^{1/\alpha} \int_{0+}^\infty \frac{1 - e^{-xt}}{t} d(c^{1/\alpha} \rho)(t) = c^{1/\alpha} \int_0^\infty (1 - e^{-xt}) d\mu(t) = c^{1/\alpha} G^*_t(x).$$

Now let $d = c^{1/\alpha} > 1$ then $d^{1/\alpha} = c^{1/\alpha}$ and we have $G^*_t(d^{1/\alpha}x) = d \cdot G^*_t(x)$ for all $x > 0$. Hence $G^*_t$ is a self-similar Bernstein function with respect to $1 - \alpha \in (0, 1)$ and $d > 1$.

Remark 2.9. By Lemma 2.2 the Lévy measure $\mu$ corresponding to $G^*_t$ is given by $\mu(t, \infty) = t^{\alpha-1} \sigma(\log t)$ for an admissible function $\sigma$ with respect to the parameters $1 - \alpha \in (0, 1)$ and $d > 1$. If this admissible function $\sigma$ is smooth, we get a Sonine pair $(k, k^*)$ in the original sense as follows. Integration by parts yields

$$G^*_t(x) = x \int_0^\infty \frac{1}{x} (1 - e^{-xt}) d\mu(t) = x \int_0^\infty e^{-xt} t^{\alpha-1} \sigma(\log t) dt$$

and hence we get as the derivative

$$G^*(x) = \int_0^\infty e^{-xt} t^{\alpha-1} \sigma(\log t) dt - x \int_0^\infty e^{-xt} t^{\alpha} \sigma(\log t) dt$$

$$= \int_0^\infty e^{-xt} \left(t^{\alpha-1} \sigma(\log t) - \frac{d}{dt} (t^{\alpha} \sigma(\log t))\right) dt$$

$$= \int_0^\infty e^{-xt} t^{\alpha-1} ((1 - \alpha) \sigma(\log t) - \sigma'(\log t)) dt$$

$$=: \int_0^\infty e^{-xt} k^*(t) dt,$$

where the kernel $k^*$ is non-negative by Lemma A.1. To show that $(k, k^*)$ is a Sonine pair simply calculate the Laplace transform as in the proof of Lemma 2.7. Note that in general we cannot expect $k^*$ to be completely monotone as in the approach of [16] with complete Bernstein functions. Nor can we expect that the admissible function $\sigma$ is smooth in general.

3. Discrete approximation of the generator

Recall that by (1.10) the semi-fractional derivative operator of order $\alpha \in (0, 1)$ is given by the negative generator of the continuous convolution semigroup $(\nu^s)_s \geq 0$, where $\nu$ is the semistable distribution with log-characteristic function (1.9). If the tail
of the Lévy measure is given by \( \phi(t, \infty) = \Gamma(1 - \alpha) t^{-\alpha} \), i.e. the admissible function \( \theta \equiv \Gamma(1 - \alpha) \) is constant, it is well known that the semi-fractional derivative coincides with the ordinary Riemann-Liouville fractional derivative of order \( \alpha \in (0, 1) \) and can be approximated by means of the Grünwald-Letnikov formula

\[
\frac{\partial^\alpha}{\partial x^\alpha} f(x) = \lim_{h \downarrow 0} h^{-\alpha} \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j f(x - jh)
\]

for functions \( f \) in the Sobolev space \( W^{2,1}(\mathbb{R}) \); see section 2.1 in [21] for details. The coefficients in the Grünwald-Letnikov approximation formula appear in a discrete distribution on the positive integers called Sibuya distribution which first appeared in [31]. A discrete random variable \( X_\alpha \) on \( \mathbb{N} \) is Sibuya distributed with parameter \( \alpha \in (0, 1) \) if

\[
P(X_\alpha = j) = (-1)^{j-1} \binom{\alpha}{j} = (-1)^{j-1} \frac{\alpha(\alpha - 1) \cdots (\alpha - j + 1)}{j!} \quad \text{for } j \in \mathbb{N}.
\]

For further details and extensions of the Sibuya distribution we refer to [5, 6, 18] and the literature mentioned therein. Using (3.2) we may rewrite (3.1) as

\[
\frac{\partial^\alpha}{\partial x^\alpha} f(x) = \lim_{h \downarrow 0} h^{-\alpha} \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j f(x - jh)
\]

\[
= \lim_{h \downarrow 0} h^{-\alpha} \left( f(x) - \sum_{j=1}^{\infty} P(X_\alpha = j) f(x - jh) \right)
\]

\[
= \lim_{h \downarrow 0} h^{-\alpha} \left( f(x) - \int_{\mathbb{R}} f(x - hy) dP_{X_\alpha}(y) \right)
\]

\[
= \lim_{h \downarrow 0} h^{-\alpha} (f \ast (\varepsilon_0 - \mathbb{P}_{hX_\alpha}))(x)
\]

which shows that the Grünwald-Letnikov formula is in fact a discrete approximation of the generator. For further relations of the Sibuya distribution to fractional diffusion equations see [22, 23].

Our aim is to generalize this formula for semi-fractional derivatives by means of the corresponding self-similar Bernstein functions. Note that for the Bernstein function \( \tilde{\psi}(x) = x^\alpha \) the nominator on the right-hand side of (3.2) is given by \( \tilde{\psi}^{(j)}(1) \) and hence we may define a semi-fractional Sibuya distribution in the following way.
Definition 3.1. Given an admissible function \( \theta \) with respect to \( \alpha \in (0, 1) \) and \( c > 1 \), a semi-fractional Sibuya distributed random variable \( X_\theta \) on \( \mathbb{N} \) is given by

\[
P(X_\theta = j) = \frac{(-1)^{j-1}}{j!} \frac{\widetilde{\psi}(j)(1)}{\widetilde{\psi}(1)} \quad \text{for } j \in \mathbb{N},
\]

where \( \widetilde{\psi} \) is the corresponding self-similar Bernstein function.

Clearly, the expression in (3.3) is non-negative by (1.1). Moreover, by monotone convergence and a Taylor series approach justified by Proposition 3.6 in [30] we get

\[
\frac{(-1)^{j-1}}{j!} \frac{\widetilde{\psi}(j)(1)}{\widetilde{\psi}(1)} = \frac{1}{\widetilde{\psi}(1)} \sum_{j=1}^{\infty} \frac{\widetilde{\psi}(j)(1)}{j!} \lim_{\varepsilon \downarrow 0} \left( (\varepsilon - 1)^j - \widetilde{\psi}(1) \right)
\]

\[
= \frac{1}{\widetilde{\psi}(1)} \lim_{\varepsilon \downarrow 0} \left( \sum_{j=0}^{\infty} \frac{\widetilde{\psi}(j)(1)}{j!} (\varepsilon - 1)^j - \widetilde{\psi}(1) \right)
\]

\[
= \frac{1}{\widetilde{\psi}(1)} \lim_{\varepsilon \downarrow 0} \left( \frac{1}{\varepsilon - 1} \left( \widetilde{\psi}(\varepsilon) - \widetilde{\psi}(1) \right) = \frac{\widetilde{\psi}(1) - \widetilde{\psi}(0)}{\widetilde{\psi}(1)} = 1.
\]

This shows that indeed (3.3) defines a proper distribution on \( \mathbb{N} \) with pgf

\[
G(z) = \sum_{j=1}^{\infty} P(X_\theta = j) z^j = \frac{1}{\widetilde{\psi}(1)} \sum_{j=1}^{\infty} \frac{\widetilde{\psi}(j)(1)}{j!} (-z)^j
\]

\[
= 1 - \frac{1}{\widetilde{\psi}(1)} \sum_{j=0}^{\infty} \frac{\widetilde{\psi}(j)(1)}{j!} ((1 - z) - 1)^j = 1 - \frac{\widetilde{\psi}(1 - z)}{\widetilde{\psi}(1)}
\]

for \( |z| \leq 1 \). Note that in the above arguments self-similarity of the Bernstein function is not needed. Hence (3.3) defines a proper distribution on \( \mathbb{N} \) for every Bernstein function \( \widetilde{\psi} \) with \( \widetilde{\psi}(0) = 0 \).

Now let \( \theta \) be a smooth admissible function having Fourier series representation

\[
\theta(x) = \sum_{k=-\infty}^{\infty} c_k e^{ik\tilde{c}x} \quad \text{with} \quad \tilde{c} = \frac{2\pi \alpha}{\log c}.
\]

Then by Theorem 3.1 in [12] the corresponding self-similar Bernstein function can be expressed in terms of the log-characteristic function \( \psi \) as

\[
\widetilde{\psi}(x) = -\psi(ix) = \sum_{k=-\infty}^{\infty} \omega_k x^{\alpha - ik\tilde{c}} \quad \text{with} \quad \omega_k = c_k \Gamma(ik\tilde{c} - \alpha + 1)
\]
and hence for $j \in \mathbb{N}_0$ we have
\[
\frac{\tilde{\psi}^{(j)}(1)}{j!} = \sum_{k=-\infty}^{\infty} \omega_k \left( \alpha - ik\tilde{c} \right).
\]

Note that these coefficients appear in a Grünwald-Letnikov type approximation of the semi-fractional derivative given in Theorem 4.1 of [12] which enables us to prove the following approximation formula.

**Theorem 3.2.** Let $\theta$ be a smooth admissible function with respect to $\alpha \in (0, 1)$ and $c > 1$ with corresponding self-similar Bernstein function $\tilde{\psi}$ and semi-fractional Sibuya distributed random variable $X_\theta$ given by (3.3). Then for $f \in W^{2,1}(\mathbb{R})$ the semi-fractional derivative can be approximated along the subsequence $h_m = c^{-m/\alpha}$ by
\[
\frac{\partial^\alpha}{\partial_{c,\theta} x^\alpha} f(x) = \lim_{m \to \infty} \tilde{\psi}(h_m^{-1}) (f * (\varepsilon_0 - \mathbb{P}_{h_m} x_\theta)) (x).
\]

**Proof.** First note that $h_m^{ik\tilde{c}} = c^{-imk/\alpha} = e^{-2\pi mk} = 1$ and by self-similarity we have $\tilde{\psi}(h_m^{-1}) = h_m^{-\alpha} \tilde{\psi}(1)$ for all $m \in \mathbb{N}$. Hence we get
\[
\tilde{\psi}(h_m^{-1}) (f * (\varepsilon_0 - \mathbb{P}_{h_m} x_\theta)) (x) = h_m^{-\alpha} \tilde{\psi}(1) \left( f(x) - \int_{\mathbb{R}} f(x - h_m y) d\mathbb{P}_{x_\theta}(y) \right)
\]
\[
= h_m^{-\alpha} \tilde{\psi}(1) \left( f(x) - \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j!} \tilde{\psi}^{(j)}(1) f(x - jh_m) \right)
\]
\[
= h_m^{-\alpha} \left( \sum_{k=-\infty}^{\infty} \omega_k f(x) + \sum_{j=1}^{\infty} (-1)^j \sum_{k=-\infty}^{\infty} \omega_k \left( \alpha - ik\tilde{c} \right) f(x - jh_m) \right)
\]
\[
= h_m^{-\alpha} \sum_{j=0}^{\infty} (-1)^j \sum_{k=-\infty}^{\infty} \omega_k h_m^{ik\tilde{c}} \left( \alpha - ik\tilde{c} \right) f(x - jh_m) \to \frac{\partial^\alpha}{\partial_{c,\theta} x^\alpha} f(x),
\]
where the last convergence follows from Theorem 4.1 in [12]. \(\Box\)

4. **Space-time duality for semi-fractional diffusions**

We first give a sufficient condition for an inverse function to be a Bernstein function.

**Lemma 4.1.** Let $f : (0, \infty) \to (0, \infty)$ be a $C^\infty(0, \infty)$-function such that $f'$ is a Bernstein function and $f^{(n)}(x) \neq 0$ for all $x > 0$ and $n \in \mathbb{N}$. Then its inverse $f^{-1}$ is a Bernstein function with $(f^{-1})^{(n)}(x) \neq 0$ for all $x > 0$ and $n \in \mathbb{N}$. 


Proof. We know that $f^{-1}(x) > 0$ and $(f^{-1})'(x) = \frac{1}{f(f^{-1}(x))} > 0$ for all $x > 0$. Moreover, as in Remark A.3 of the Appendix $(f^{-1})''(x) = -\frac{f''(f^{-1}(x))}{(f(f^{-1}(x)))^3} < 0$ for all $x > 0$. For $n \geq 3$ we inductively use the formula for $(f^{-1})^{(n)}$ given in Lemma A.2 of the Appendix. By induction we have

$$\text{sign} \left( \prod_{j=1}^{n-1} \left( \frac{(f^{-1})^{(j)}(x)}{j!} \right)^{k_j} \right) = \prod_{j=1}^{n-1} (-1)^{(j-1)k_j} = (-1)^{\sum_{j=1}^{n-1} (j-1)k_j}$$

and $\text{sign}(f^{(k_1+\cdots+k_{n-1})(f^{-1}(x))}) = (-1)^{k_1+\cdots+k_{n-1}}$ since $k_1 + \cdots + k_{n-1} \geq 2$ for $n \geq 3$ as $k_1 + 2k_2 + \cdots + (n-1)k_{n-1} = n$. This shows that

$$\text{sign} \left( f^{(k_1+\cdots+k_{n-1})(f^{-1}(x))} \prod_{j=1}^{n-1} \left( \frac{(f^{-1})^{(j)}(x)}{j!} \right)^{k_j} \right) = (-1)^{\sum_{j=1}^{n-1} jk_j} = (-1)^n$$

for every summand of the formula in Lemma A.2. Thus $\text{sign}((f^{-1})^{(n)}(x)) = (-1)^{n-1}$ for all $x > 0$ and $n \geq 3$ showing that $f^{-1}$ is a Bernstein function. \[\square\]

We want to apply Lemma 4.1 to the function

$$(4.1) \quad \zeta(x) = \psi(-ix) = \int_{-\infty}^{0} (e^{xy} - 1 - xy) \, d\phi(y) > 0 \quad \text{for } x > 0,$$

where $\phi$ is the semistable Lévy measure for $\alpha \in (1, 2)$ from (1.12) concentrated on the negative axis and $\psi$ is the corresponding log-characteristic function from (1.13). Note that $\zeta(x) \to \infty$ as $x \to \infty$ due to the tail behavior in (1.12) and by dominated convergence we have

$$\zeta'(x) = \int_{-\infty}^{0} y (e^{xy} - 1) \, d\phi(y) = \int_{0}^{\infty} (1 - e^{-xy}) \, y \, d\phi(-y).$$

Since $\phi$ integrates $\min\{1, y^2\}$ and fulfills (1.12), the measure $\mu$ with $d\mu(y) = y \, d\phi(-y)$ integrates $\min\{1, y\}$ and thus $\zeta'$ is a Bernstein function. This also follows from the higher order derivatives

$$\zeta^{(n)}(x) = \int_{-\infty}^{0} y^n e^{xy} \, d\phi(y) \begin{cases} > 0 & \text{if } n \geq 2 \text{ is even}, \\ < 0 & \text{if } n \geq 3 \text{ is odd}, \end{cases}$$

which additionally shows that these derivatives do not vanish. Thus from Lemma 4.1 we conclude that the inverse $\zeta^{-1}$ is a Bernstein function. This will be the key to solve an open problem concerning the following result on space-time duality from [11].

According to [29, Example 28.2], the semistable Lévy process $(X_t)_{t \geq 0}$ with $P\{X_t \in A\} = \nu^{st}(A)$ for Borel sets $A \in \mathcal{B}(\mathbb{R})$, where $\nu$ is the semistable distribution with
The log-characteristic function \( \psi \) from (1.13), possesses \( C^\infty(\mathbb{R}) \)-densities \( x \mapsto p(x, t) \) for every \( t > 0 \) with \( P\{X_t \in A\} = \int_A p(x, t) \, dx \) and for \( t = 0 \) we may write \( p(x, 0) = \delta(x) \) corresponding to \( X_0 = 0 \) almost surely. It is shown in [12] that these densities are the point source solution to the semi-fractional diffusion equation

\[
\frac{\partial^\alpha}{\partial c^\alpha} p(x, t) = \frac{\partial}{\partial t} p(x, t)
\]

with the negative semi-fractional derivative of order \( \alpha \) from (1.14) acting on the space variable. Since the semi-fractional derivative is a non-local operator, this equation is hard to interpret from a physical point of view, whereas non-locality in time may correspond to long memory effects [9]. As a generalization of a space-time duality result for fractional diffusions [4] [10] based on a corresponding result for stable densities by Zolotarev [38] [39] it was shown in [11] that space-time duality may also hold for semi-fractional diffusions. Theorem 3.3 in [11] states that for \( x > 0 \) and \( t > 0 \) we have \( p(x, t) = \alpha^{-1} h(x, t) \), where \( h(x, t) \) is the point source solution to the semi-fractional differential equation

\[
\frac{\partial^{1/\alpha}}{\partial d^{1/\alpha}} h(x, t) + \frac{\partial}{\partial x} h(x, t) = t^{-1/\alpha} g(\log t) \delta(x)
\]

with a semi-fractional derivative of order \( 1/\alpha \) acting on the time variable, provided that \( \tau \) and \( g \) are admissible functions with respect to \( 1/\alpha \in (\frac{1}{2}, 1) \) and \( d = c^{1/\alpha} > 1 \) which remains an open problem in [11]. We will now show that indeed \( \tau \) is admissible and \( g(x) = -\alpha \tau'(x) \), provided that \( \tau \) is smooth. Note that in general \( g \) will not be admissible as conjectured in [11] but we will justify the inhomogeneity in (4.3) by different arguments. In [11] the function \( \tau \) appears in the following way. The above inverse \( \zeta^{-1} \) is called \( \xi \) in [11] and its existence is shown in Lemma 4.1 of [11]. It was further shown in Lemma 4.2 of [11] that \( \xi(t) = t^{1/\alpha} g(\log t) \) for a continuously differentiable and \( \log(c) \)-periodic function \( g \). Since we now know that \( \xi = \zeta^{-1} \) is a Bernstein function, in fact \( g \) is a \( C^\infty(\mathbb{R}) \)-function. Since \( g \) is a smooth \( \log(c) \)-periodic function, it is representable by its Fourier series

\[
g(x) = \sum_{n \in \mathbb{Z}} d_n e^{-i dx} \quad \text{with} \quad \tilde{d} = \frac{2\pi}{\log c} = \frac{2\pi \frac{1}{\alpha}}{\log d} \quad \text{for} \quad d = c^{1/\alpha},
\]

where by Lemma 1 in §12 of [2] we have \( |d_n| \leq C \cdot e^{-\tilde{d} |n|} \) for some \( C > 0 \) and all \( n \in \mathbb{Z} \). If we require a little more quality, namely that the Fourier coefficients even
decay as

\[ |d_n| \leq C \cdot e^{-\frac{\varepsilon}{2} \frac{1}{\alpha}} \left| n \right|^{-\frac{3}{2} - \frac{1}{\alpha} - \varepsilon} \] for some \( \varepsilon > 0 \) and all \( n \in \mathbb{Z} \setminus \{0\} \),

then we can define \( \tau \) by the Fourier series

\[ \tau(x) = \sum_{n \in \mathbb{Z}} \frac{d_n}{\Gamma(\text{ind} \hat{d} - \frac{1}{\alpha} + 1)} e^{-\text{ind} x}. \]

**Lemma 4.2.** If (4.5) holds, then the function \( \tau \) in (4.6) is well-defined and a smooth admissible function with respect to \( 1/\alpha \in \left( \frac{1}{2}, 1 \right) \) and \( d = c^{1/\alpha} \). Moreover, \( \xi^{-1} \) is a self-similar Bernstein function with respect to the same parameters.

**Proof.** As shown above \( \xi = \xi^{-1} \) is a Bernstein function and with \( d = c^{1/\alpha} \) and \( \xi(t) = t^{1/\alpha} g(\log t) \) for a \( \log(c) \)-periodic function \( g \) we get

\[ d \cdot \xi(t) = (ct)^{1/\alpha} g(\log(ct)) = \xi(ct) = \xi(d^\alpha t) \]

showing that \( \xi \) is a self-similar Bernstein function with respect to \( 1/\alpha \in \left( \frac{1}{2}, 1 \right) \) and \( d = c^{1/\alpha} \). By Lemma 2.2 we have

\[ \xi(x) = \int_0^{\infty} (1 - e^{-xy}) \, d\mu(y), \]

where \( \mu \) is a semistable Lévy measure with \( \mu(t, \infty) = t^{-1/\alpha \tau(\log t)} \) for an admissible function \( \tau \) with respect to \( 1/\alpha \in \left( \frac{1}{2}, 1 \right) \) and \( d = c^{1/\alpha} \). It remains to show that \( \tau \) is indeed the function we are looking for. Let us assume for a moment that \( \tau \) is smooth and thus admits the Fourier series

\[ \tau(x) = \sum_{n \in \mathbb{Z}} a_n e^{-\text{ind} x} \quad \text{with} \quad \hat{d} = \frac{2\pi}{\log c} = \frac{2\pi^{1/\alpha}}{\log d}. \]

In this case the function \( \gamma \) appearing in Lemma 2.2 fulfills

\[ \gamma(-x) = e^{-\frac{1}{\alpha} x} \xi(x) = g(x) \]

and by (2.4) has the Fourier series representation

\[ g(x) = \gamma(-x) = \sum_{n \in \mathbb{Z}} a_n \Gamma(\text{ind} \hat{d} - \frac{1}{\alpha} + 1) e^{-\text{ind} x}. \]

A comparison with (4.4) and uniqueness of the Fourier coefficients shows that \( a_n \) coincides with \( d_n / \Gamma(\text{ind} \hat{d} - \frac{1}{\alpha} + 1) \) and thus \( \tau \) is indeed the function in (4.6). Finally,
we have to show that the series in (4.6) converges. Using the asymptotic behavior of the gamma function in Corollary 1.4.4 of [1], the Fourier coefficients fulfill
\[ \left| \frac{d_n}{\Gamma(in_d - \frac{1}{\alpha} + 1)} \right| \leq K|d_n| \cdot |n|^{-\frac{1}{2} + \frac{1}{\alpha} + \frac{\varepsilon}{2}} |n|d \]
for a constant \( K > 0 \) and all \( n \in \mathbb{Z} \setminus \{0\} \). According to our assumption (4.5) we obtain
\[ \left| \frac{d_n}{\Gamma(in_d - \frac{1}{\alpha} + 1)} \right| \leq KC|n|^{-2-\varepsilon} \]
for some \( \varepsilon > 0 \) and all \( n \in \mathbb{Z} \setminus \{0\} \) showing that the series in (4.6) converges and the resulting function is continuously differentiable by Theorem 2.6 in [7].

Admissability of \( \tau \) in combination with Theorem 3.3 in [11] finally enables us to completely solve space-time duality for semi-fractional diffusions. The equation (4.3) is derived in [11] by Laplace inversion of the equation
\[ \xi(s) \tilde{h}(x, s) - \frac{1}{s} \xi(s) \delta(x) + \frac{\partial}{\partial x} \tilde{h}(x, s) = -\frac{1}{s} \frac{f(s)}{s + f(s)} \xi(s) \delta(x), \]
where \( f \) is defined in the proof of Theorem 3.1 in [11] as
\[ f(s) = \frac{1}{\alpha} \xi(s)^\alpha m'(\log \xi(s)) \]
and \( m \) is a \( \log(1/\alpha) \)-periodic function given by
\[ \zeta(s) = x^\alpha m(\log x). \]

**Theorem 4.3.** Assume that (4.5) holds, \( \tau \) is given as in Lemma 4.2 and define \( \varphi(t) = -\alpha \tau'(t) \). Then the point source solutions \( p(x, t) \) of the semi-fractional diffusion equation (4.2) of order \( \alpha \in (1, 2) \) in space and \( h(x, t) \) of the semi-fractional equation (4.3) of order \( 1/\alpha \in (1/2, 1) \) in time are equivalent, i.e. \( p(x, t) = \alpha^{-1} h(x, t) \) for all \( x > 0 \) and \( t > 0 \).

**Proof.** As shown in [11] it remains to carry out Laplace inversion of (4.8). Clearly, \( \frac{\partial}{\partial x} \tilde{h}(x, s) \) is the Laplace transform of \( \frac{\partial}{\partial x} h(x, t) \). Moreover, the first part on the left-hand side of (4.8) is the Laplace transform of the semi-fractional derivative in time \( \frac{\partial^{1/\alpha}}{\partial t^{1/\alpha}} h(x, t) \) as argued in [11]. We may rewrite the right-hand side of (4.8) as follows. From (4.10) we obtain
\[ \zeta'(t) = \alpha x^{\alpha-1} m(\log x) + x^{\alpha-1} m'(\log x) \]
and thus we have by (4.10) and \( \zeta(\xi(s)) = s \)

\[
\zeta'(\xi(s)) = \alpha \left( \frac{1}{\xi(s)} s + \frac{1}{\xi(s)} \xi(s)^{\alpha} m'(\log \xi(s)) \right) = \alpha \frac{1}{\xi(s)} s + \alpha \frac{1}{\xi(s)} f(s),
\]

where the last equality follows from (4.9). This shows that

\[
f(s) = \frac{1}{\alpha} \zeta'(\xi(s)) \xi(s) - s
\]

and the right-hand side of (4.8) can be rewritten as

(4.11) \quad - \frac{1}{s} \frac{f(s)}{s + f(s)} \xi(s) \delta(x) = - \frac{\alpha}{s} \frac{f(s)}{\zeta'(\xi(s))} \delta(x) = - \frac{1}{s} \xi(s) \delta(x) + \alpha \frac{1}{\zeta'(\xi(s))} \delta(x).

Note that the first part \(- \frac{1}{s} \xi(s) \delta(x)\) also appears on the left-hand side of (4.8); cf. Remark 4.4. Moreover, since \( \frac{1}{s} \) is the Laplace transform of \( 1_{(0,\infty)}(t) \), the function \( \frac{1}{s} \xi(s) \) is the Riemann-Liouville semi-fractional derivative of \( 1_{(0,\infty)}(t) \) which is

\[
\frac{d}{dt} \int_0^t 1_{(0,\infty)}(t - s) s^{-1/\alpha} \tau(\log(s)) \, ds = t^{-1/\alpha} \tau(\log(t)).
\]

Now from \( \zeta(\xi(s)) = s \) we get \( \zeta'(\xi(s)) \cdot \xi'(s) = 1 \) and hence it follows from (4.7) and integration by parts

\[
\frac{1}{\zeta'(\xi(s))} = \xi'(s) = \int_0^\infty e^{-st} dt \, d\mu(t)
\]

\[
= \left[ -e^{-st} t^{-1/\alpha} \tau(\log(t)) \right]_0^\infty + \int_0^\infty (-s e^{-st} t + e^{-st}) t^{-1/\alpha} \tau(\log(t)) \, dt
\]

\[
= -s \int_0^\infty e^{-st} t^{-1/\alpha} \tau(\log(t)) \, dt + \int_0^\infty e^{-st} t^{-1/\alpha} \tau(\log(t)) \, dt
\]

which is the Laplace transform of

\[
- \frac{d}{dt} \left( t^{-1/\alpha} \tau(\log(t)) \right) + t^{-1/\alpha} \tau(\log(t))
\]

\[
= -(1 - \frac{1}{\alpha}) t^{-1/\alpha} \tau(\log(t)) - t^{-1/\alpha} \tau'(\log(t)) + t^{-1/\alpha} \tau(\log(t))
\]

\[
= t^{-1/\alpha} \left( \frac{1}{\alpha} \tau(\log(t)) - \tau'(\log(t)) \right).
\]

Putting things together, Laplace inversion of the right-hand side of (4.8) yields

\[
\left( -t^{-1/\alpha} \tau(\log(t)) + \alpha t^{-1/\alpha} \left( \frac{1}{\alpha} \tau(\log(t)) - \tau'(\log(t)) \right) \right) \delta(x) = -\alpha t^{-1/\alpha} \tau'(\log(t)) \delta(x)
\]

concluding the proof. \qed

Remark 4.4. Note that in the stable case we have that \( \tau \) is constant and thus \( \tau' \equiv 0 \). Hence we recover space-time duality for fractional diffusions in [10] as a special case.
Further note that with (4.11) the term \(-\frac{1}{s} \xi(s) \delta(x)\) appears on both sides of (4.8) and can be cancelled. After cancellation the first part on the left-hand side of (4.8) is the Laplace transform of the Riemann-Liouville semi-fractional derivative of \(h(x,t)\) and as in the proof of Theorem 4.3 we may rewrite (4.3) as
\[
\mathbb{D}^{1/\alpha}_{d,t} h(x,t) + \frac{\partial}{\partial x} h(x,t) = t^{-1/\alpha} (\tau(\log t) - \alpha \tau'(\log t)) \delta(x),
\]
where the semi-fractional derivative acts on the time variable and now the inhomogeneity on the right-hand side is non-negative by Lemma A.1.

APPENDIX

We first show an elementary result connecting admissible functions with the representation in Corollary 7.4.4 of [20].

Lemma A.1. Let \(\theta : \mathbb{R} \to (0, \infty)\) be a periodic function and \(\alpha > 0\). Then the following statements are equivalent.

(i) \(\theta(y + \delta) \leq e^{\alpha \delta} \theta(y)\) for all \(y > 0\) and \(\delta > 0\).
(ii) \(\theta(y + \delta) \leq e^{\alpha \delta} \theta(y)\) for all \(y \in \mathbb{R}\) and \(\delta \geq 0\).
(iii) \(\theta(y - \delta) \geq e^{-\alpha \delta} \theta(y)\) for all \(y \in \mathbb{R}\) and \(\delta \geq 0\).
(iv) The mapping \(t \mapsto t^{-\alpha} \theta(\log t)\) is non-increasing for \(t > 0\).

If \(\theta\) is additionally differentiable, then each statement (i)-(iv) is equivalent to

(v) \(\theta'(y) \leq \alpha \theta(y)\) for all \(y \in \mathbb{R}\).

Proof. “(i)\(\Rightarrow\)(ii)”: Note that (ii) is trivially true for \(\delta = 0\). If \(\theta\) has period \(p > 0\) and \(y \leq 0\), choose \(k \in \mathbb{N}\) such that \(y + kp > 0\). Then \(\theta(y + \delta) = \theta(y + kp + \delta) \leq e^{\alpha \delta} \theta(y + kp) = e^{\alpha \delta} \theta(y)\).

“(ii)\(\Rightarrow\)(iii)”: Write \(z = y - \delta\) then \(\theta(y - \delta) = \theta(z) \geq e^{-\alpha \delta} \theta(z + \delta) = e^{-\alpha \delta} \theta(y)\).

“(iii)\(\Rightarrow\)(iv)”: For \(0 < s < t\) write \(t = e^y\) and \(s = e^{y - \delta}\) for some \(y \in \mathbb{R}\) and \(\delta > 0\). Then \(s^{-\alpha} \theta(\log s) = e^{-\alpha(y - \delta)} \theta(y - \delta) \geq e^{-\alpha y} \theta(y) = t^{-\alpha} \theta(\log t)\).

“(iv)\(\Rightarrow\)(i)”: Write \(y = \log t\) and \(\delta = \log \gamma\) for some \(t, \gamma > 1\). Then \(\theta(y + \delta) = \theta(\log(\gamma t)) = (\gamma t)^\alpha (\gamma t)^{-\alpha} \theta(\log(\gamma t)) \leq (\gamma t)^\alpha t^{-\alpha} \theta(\log t) = e^{\alpha \delta} \theta(y)\).

“(ii)\(\Rightarrow\)(v)”: For fixed \(y \in \mathbb{R}\) we have \(\frac{\theta(y + \delta) - \theta(y)}{\delta} \leq e^{\alpha \delta} \theta(y)\) for all \(\delta > 0\) and for differentiable \(\theta\) as \(\delta \downarrow 0\) we get \(\theta'(y) \leq \alpha \theta(y)\).

“(v)\(\Rightarrow\)(iv)”: We have \(\frac{d}{dt}(t^{-\alpha} \theta(\log t)) = t^{-\alpha - 1} (-\alpha \theta(\log t) + \theta'(\log t)) \leq 0\). \(\square\)
Now we derive a formula for higher order derivatives of inverse functions used in Section 3 for which we couldn’t find a suitable reference.

**Lemma A.2.** Let $A, B \subseteq \mathbb{R}$ be open and let $f : A \to B$ be an invertible $C^n(A)$-function for some $n \in \mathbb{N}$ such that $f'(x) \neq 0$ for all $x \in A$. Then $f^{-1} : B \to A$ is of class $C^n(B)$ and for $n \geq 2$ we have

$$(f^{-1})^{(n)}(x) = -\frac{1}{f'(f^{-1}(x))} \sum_{k_1+2k_2+\cdots+(n-1)k_{n-1}=n} \frac{n!}{k_1! \cdots k_{n-1}!} f^{(k_1+\cdots+k_{n-1})}(f^{-1}(x)) \prod_{j=1}^{n-1} \left( \frac{(f^{-1})^{(j)}(x)}{j!} \right)^{k_j}.$$ 

**Remark A.3.** For $n = 1$ it is well known that $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$. For $n = 2$ we have $k_1 = 2$ in Lemma A.3 so that

$$(f^{-1})''(x) = -\frac{1}{f'(f^{-1}(x))} f''(f^{-1}(x)) \cdot ((f^{-1})'(x))^2 = -\frac{f''(f^{-1}(x))}{(f'(f^{-1}(x)))^3}.$$ 

Iterating this procedure leads to a formula for $(f^{-1})^{(n)}$ given in [37] not involving the derivatives of lower order $(f^{-1})', \ldots, (f^{-1})^{(n-1)}$.

**Proof of Lemma A.2.** We use Faà di Bruno’s formula of higher order chain rule

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{n!}{k_1! \cdots k_n!} f^{(k_1+\cdots+k_n)}(g(x)) \prod_{j=1}^{n} \left( \frac{(g^{(j)}(x))}{j!} \right)^{k_j}.$$ 

For $j = n$ we must have $k_n = 1$ and $k_1 = \cdots = k_{n-1} = 0$, otherwise $k_n = 0$ if $k_j \geq 1$ for some $j \in \{1, \ldots, n-1\}$. Hence we get

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{k_1+2k_2+\cdots+(n-1)k_{n-1}=n} \frac{n!}{k_1! \cdots k_n!} f^{(k_1+\cdots+k_{n-1})}(g(x)) \prod_{j=1}^{n-1} \left( \frac{(g^{(j)}(x))}{j!} \right)^{k_j} + f'(g(x)) g^{(n)}(x).$$ 

If $g = f^{-1}$, we know that $\frac{d^n}{dx^n} f(f^{-1}(x)) = 0$ for $n \geq 2$ which directly leads to the stated formula for $(f^{-1})^{(n)}$. □

**References**

[1] Andrews, G.E.; Askey, R.; and Roy, R. (1999) *Special Functions*. Cambridge University Press, Cambridge.

[2] Arnold, V.I. (1983) *Geometrical Methods in the Theory of Ordinary Differential Equations*, 2nd Edition. Springer, New York.
[3] Ascione, G. (2021) Abstract Cauchy problems for the generalized fractional calculus. *Nonlinear Anal.* **209** 112339.

[4] Baeumer, B.; Meerschaert, M.M.; and Nane, E. (2009) Space-time duality for fractional diffusion. *J. Appl. Probab.* **46**, 110–115.

[5] Bouzner, N. (2008) The semi-Sibuya distribution. *Ann. Inst. Stat. Math.* **60** 459–464.

[6] Christoph, G.; and Schreiber, K. (2000) Scaled Sibuya distribution and discrete self-decomposability. *Statist. Probab. Lett.* **48** 181–187.

[7] Folland, G.B. (1992) *Fourier Analysis and Its Applications*. Wadsworth, Belmont.

[8] Hanyga, A. (2020) A comment on a controversial issue: A generalized fractional derivative cannot have a regular kernel. *Fract. Calc. Appl. Anal.* **23**(1) 211–223.

[9] Hilfer, R. (2008) Threefold introduction to fractional derivatives. In: R. Klages et al. (eds.) *Anomalous Transport: Foundations and Applications*. Wiley-VCH, Weinheim, pp. 17–74.

[10] Kelly, J.F.; and Meerschaert, M.M. (2017) Space-time duality for the fractional advection-dispersion equation. *Water Resour. Res.* **53**, 3464–3475.

[11] Kern, P.; and Lage, S. (2021) Space-time duality for semi-fractional diffusions. In: U. Freiberg et al. (eds.) *Fractal Geometry and Stochastics VI*. Progress in Probability **76**, Birkhäuser, Basel, pp. 255–272.

[12] Kern, P.; Lage, S.; and Meerschaert, M.M. (2019) Semi-fractional diffusion equations. *Fract. Calc. Appl. Anal.* **22**(2) 326–357.

[13] Kern, P.; Meerschaert, M.M.; and Xiao, Y. (2018) Asymptotic behavior of semistable Lévy exponents and applications to fractal path properties. *J. Theoret. Probab.* **31** 598–617.

[14] Kevei P. (2020) Regularly log-periodic functions and some applications. *Probab. Math. Statist.* **40**(1) 159–182.

[15] Kilbas, A.A.; Srivastava, H.M.; and Trujillo, J.J. (2006) *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematical Studies **204**, Elsevier, Amsterdam.

[16] Kochubei, A.N. (2011) General fractional calculus, evolution equations, and renewal processes. *Integr. Equ. Oper. Theory* **71** 583–600.

[17] Kochubei, A.N.; Kondratiev, Y.; and da Silva, J.L. (2020) From random times to fractional kinetics. *Interdisciplinary Studies of Complex Systems* **16** 5–32.

[18] Kozubowski, T.J.; and Podgórski, K. (2018) A generalized Sibuya distribution. *Ann. Inst. Stat. Math.* **70** 855–887.

[19] Liu, W.; Röckner, M.; and da Silva, J.L. (2021) Strong dissipativity of generalized time-fractional derivatives and quasi-linear (stochastic) partial differential equations. *J. Functional Anal.* (to appear) [https://doi.org/10.1016/j.jfa.2021.109135](https://doi.org/10.1016/j.jfa.2021.109135).

[20] Meerschaert, M.M.; and Scheffler, H.P. (2001) *Limit Distributions for Sums of Independent Random Vectors*. Wiley, New York.

[21] Meerschaert, M.M.; and Sikorskii, A. (2012) *Stochastic Models for Fractional Calculus*. De Gruyter, Berlin.

[22] Nichols, J.A.; Henry, B.I.; and Angstmann, C.N. (2018) Subdiffusive discrete time random walks via Monte Carlo and subordination. *J. Comput. Phys.* **372** 373–384.

[23] Pachon, A.; Polito, F.; and Ricciuti, C. (2021) On discrete-time semi-Markov processes. *Discrete Cont. Dyn. Syst. B* **26**(3) 1499–1529.

[24] Patie, P.; and Srapionyan, A. (2021) Self-similar Cauchy problems and generalized Mittag-Leffler functions. *Fract. Calc. Appl. Anal.* **24**(2) 447–482.

[25] Podlubny, I. (1998) *Fractional Differential Equations*. Academic Press, San Diego.

[26] Samko, S.G.; and Cardoso, R.P. (2003) Integral equations of the first kind of Sonine type. *Int. J. Math. Math. Sci.* **57** 3609–3632.
[27] Samko, S.G.; Kilbas, A.A.; and Marichev, O.I. (1993) *Fractional Integrals and Derivatives*. Gordon and Breach, London.

[28] Sandev, T.; Metzler, R.; and Chechkin, A. (2018) From continuous time random walks to the generalized diffusion equation. *Fract. Calc. Appl. Anal.* 21(1) 10–28.

[29] Sato, K. (1999) *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge.

[30] Schilling, R.L.; Song, R.; and Vondraček (2012) *Bernstein Functions*, 2nd Edition. De Gruyter, Berlin.

[31] Sibuya, M. (1979) Generalized hypergeometric, digamma, and trigamma distributions. *Ann. Inst. Stat. Math.* 33 177–190.

[32] Sonine, N. (1884) Sur la généralisation d’une formule d’Abel. *Acta Math.* 4 171–176.

[33] Sornette, D. (1998) Discrete-scale invariance and complex dimensions. *Physics Reports* 297 239–270.

[34] Sornette, D. (2000) *Critical Phenomena in Natural Sciences*. Springer, Berlin.

[35] Toaldo, B. (2015) Convolution-type derivatives, hitting-times of subordinators and time-changed $C_0$-semigroups. *Potential Anal.* 42 115–140.

[36] Wick, J. (1968) Über eine Integralgleichung vom Abelschen Typ. *Z. Angew. Math. Mech.* 48(8) T39–T41.

[37] Zabreiko, P.P.; and Lysenko, Y.V. (2001) Exact formulas for higher-order derivatives of inverse functions in Banach spaces (in Russian). *Dokl. Nats. Akad. Nauk Belarusi* 45(2) 27–30.

[38] Zolotarev, V.M. (1961) Expressions of the density of a stable distribution with exponent $\alpha$ greater than one by means of a frequency with exponent $1/\alpha$. *Selected Translations in Mathematical Statistics and Probability* 1, AMS, Providence, pp. 163–167.

[39] Zolotarev, V.M. (1986) *One-Dimensional Stable Distributions*. Translations of Mathematical Monographs 65, AMS, Providence.

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