Patterns in transitional shear flows.
Part 2: Nucleation and optimal spacing

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Low Reynolds number turbulence in wall-bounded shear flows en route to laminar flow takes the form of oblique localised turbulent structures. These emerge from uniform turbulence via a spatiotemporal intermittent process in which localised quasi-laminar gaps randomly nucleate and disappear. For slightly lower Reynolds numbers, periodic and approximately stationary laminar-turbulent patterns predominate. The statistics of quasi-laminar regions are analysed in several respects, including the distributions of space and time scales and their Reynolds number dependence. A smooth, but marked transition is observed between uniform turbulence and flow with intermittent quasi-laminar gaps, while the transition from gaps to patterns is more gradual. Wavelength selection in these patterns is analysed via numerical simulations in oblique domains of various sizes. Lifetime measurements in a minimal domain demonstrate the existence of a preferred wavelength. Wavelet transforms are performed on turbulent-laminar patterns, measuring areas and times over which a given wavelength dominates in a large domain. This leads to the quantification of the stability of a pattern as a function of wavelength and Reynolds number. We report that the preferred wavelength maximises the energy and dissipation of the large-scale flow along laminar-turbulent interfaces. This optimal behaviour is primarily due to the advective nature of this large-scale flow, while the role of turbulent fluctuations is secondary in the wavelength selection.

1. Introduction

The transition to turbulence in wall-bounded shear flows is characterised by coexisting turbulent and laminar regions. This phenomenon was first described by Coles & van Atta (1966) and by Andereck et al. (1986) in Taylor-Couette flow. Later, by constructing Taylor-Couette and plane Couette experiments with very large aspect ratios, Prigent et al. (2002, 2003) showed that these coexisting turbulent and laminar regions spontaneously formed regular patterns with a selected wavelength and orientation that depend systematically on Re. These patterns have been simulated numerically and studied intensively in plane Couette flow (Barkley & Tuckerman 2005, 2007; Duguet et al. 2010; Rolland & Manneville 2011; Tuckerman & Barkley 2011), plane Poiseuille flow (Tsukahara et al. 2005; Tuckerman et al. 2014; Shimizu & Manneville 2019; Kashyap 2021), and Taylor-Couette flow (Meseguer et al. 2009; Dong 2009; Wang et al. 2022).

In pipe flow, the other canonical wall-bounded shear flow, only the streamwise direction is long, and transitional turbulence takes the form of flashes (Reynolds 1883) or puffs (Wygnanski & Champagne 1973), which are the one-dimensional analog of bands. In contrast to bands in planar shear flows, experiments and direct numerical simulations show

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that puffs never form regular spatially periodic patterns \cite{Moxey2010,Avila2013}. Instead, the spacing between them is dictated by short-range interactions \cite{Hof2010,Samanta2011}. Puffs have been extensively studied, especially in the context of the model derived by Barkley \cite{Barkley2011a,Barkley2016} from the viewpoint of \textit{excitable media}. In this framework, fluctuations from uniform turbulence trigger laminar gaps \cite{Frishman2022}, who called them \textit{anti-puffs}. Interestingly, spatially periodic solutions like those observed in planar shear flows are produced in a centro-symmetric version of the Barkley model \cite{Barkley2011b}.

In this paper, we will show that in plane Couette flow, as in pipe flow, short-lived localised gaps emerge randomly from uniform turbulence at the highest Reynolds numbers in the transitional range, which we will see is $Re \approx 470$ in the domain which we will study. The first purpose of this paper is to investigate these gaps. The emblematic regular oblique large-scale bands appear at slightly lower Reynolds numbers, which we will see is $Re \approx 430$. If the localised gaps are disregarded, it is natural to associate the bands with a \textit{pattern-forming instability} of the uniform turbulent flow. This was first suggested by Prigent \textit{et al.} \cite{Prigent2003} and later investigated by Rolland \& Manneville \cite{Rolland2011}. Manneville \cite{Manneville2012} and Kashyap \cite{Kashyap2021} proposed a Turing mechanism to account for the appearance of patterns by constructing a reaction-diffusion model based on an extension of the Waleffe \cite{Waleffe1997} model of the streak-roll self-sustaining process. Reetz \textit{et al.} \cite{Reetz2019} discovered a sequence of bifurcations leading to a large-scale steady state that resembles a skeleton for the banded pattern, arising from tiled copies of the exact Nagata \cite{Nagata1990} solutions. The relationship between these pattern-forming frameworks and local nucleation of gaps is unclear.

The adaptation of classic stability concepts to turbulent flows is a major current research topic \cite{Markeviciute2022}. At the simplest level, it is always formally possible to carry out linear stability analysis of a mean flow as in Barkley \cite{Barkley2006}; Bengana \textit{et al.} \cite{Bengana2019}. The mean flow of uniformly turbulent plane Couette flow has been found to be linearly stable \cite{Tuckerman2010}. However, this procedure makes the drastic simplification of neglecting the Reynolds stress entirely in the stability problem and hence its interpretation is uncertain. The next level of complexity and accuracy is to represent the Reynolds stress via a closure model. However, closure models are designed for high-Reynolds-number fully developed turbulence rather than the weak turbulence of transitional wall-bounded shear flows. Indeed, a study using the $(K, \Omega)$ model yielded predictions that are completely incompatible with results from full numerical simulation or experiment \cite{Tuckerman2010}. Another turbulent configuration in which robust large scales emerge are zonal jets, characteristic of geophysical turbulence. For zonal jets, a closure model provided by a cumulant expansion \cite{Srinivasan2012} has led to a plausible stability analysis \cite{Parker2013}. Other strategies are possible for turbulent flows in general; Kashyap \textit{et al.} \cite{Kashyap2022} examined the averaged time-dependent response of uniform turbulence to large-wavelength perturbations and provided evidence for a linear instability in plane channel flow. They computed a dispersion relation which is in good agreement with the natural spacing and angle of patterns.

Classic analyses for non-turbulent pattern-forming flows, such as Rayleigh-Bénard convection or Taylor-Couette flow, yield not only a threshold but also a preferred wavelength, as well as existence and stability ranges for other wavelengths through the
Eckhaus instability (Busse 1981; Ahlers et al. 1986; Riecke & Paap 1986; Tuckerman & Barkley 1990; Cross & Greenside 2009). As the control parameter is varied, this instability causes spatially periodic states to make transitions to other periodic states whose wavelength is preferred. Eckhaus instability is also invoked in turbulent zonal jets (Parker & Krommes 2013).

The second goal of this paper is to study the regular patterns of transitional plane Couette flow and to determine the wavelengths at which they can exist and thrive. At low enough Reynolds numbers, patterns will be shown to destabilise, acquiring a different wavelength. Using an energy analysis formulated in Gomé et al. (2022), we associate the selected wavelength to a maximal dissipation observed for the large-scale flow along the bands.

2. Numerical setup

Plane Couette flow consists of two parallel rigid plates moving at different velocities, here equal and opposite velocities \( \pm U_{\text{wall}} \). Lengths are nondimensionalised by the half-gap \( h \) between the plates and velocities by \( U_{\text{wall}} \). The Reynolds number is defined to be \( R \equiv U_{\text{wall}} h/\nu \). We will require one further dimensional quantity that appears in the friction coefficient – the horizontal mean shear at the walls, which we denote by \( U'_{\text{wall}} \). We will use non-dimensional variables throughout, except when specified. We simulate the incompressible Navier-Stokes equations

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u}, \tag{2.1a}
\]
\[
\nabla \cdot \mathbf{u} = 0, \tag{2.1b}
\]

using the pseudo-spectral parallel code ChannelFlow (Gibson et al. 2019). Since the bands are found to be oriented obliquely with respect to the streamwise direction, we use a doubly periodic numerical domain which is tilted with respect to the streamwise direction of the flow, shown as the oblique rectangle in figure 1. This choice was introduced by Barkley & Tuckerman (2005) and has become common in studying turbulent bands.
The $x$ direction is chosen to be aligned with a typical turbulent band and the $z$ coordinate to be orthogonal to the band. The relationship between streamwise-spanwise coordinates and tilted band-oriented coordinates is:

$$e_{strm} = \cos \theta e_x + \sin \theta e_z \quad (2.2a)$$

$$e_{span} = -\sin \theta e_x + \cos \theta e_z \quad (2.2b)$$

The usual wall-normal coordinate is denoted by $y$ and the corresponding velocity by $v$. Thus the boundary conditions are $u(y = \pm 1) = \pm e_{strm}$ in $y$ and periodic in $x$ and $z$, together with a zero-flux constraint on the flow in the $x$ and $z$ directions. The field visualised in figure 1 comes from an additional simulation we carried out in a domain of size $(L_{strm}, L_y, L_{span}) = (200, 2, 100)$ aligned with the streamwise-spanwise directions. Exploiting the periodic boundary conditions of the simulation, the visualisation shows four copies of the flow instantaneous field.

The tilted box effectively reduces the dimensionality of the system by discarding large-scale variation along the short $x$ direction. The flow in this direction is considered to be statistically homogeneous as it is only dictated by small turbulent scales. In a large non-tilted domain, bands with opposite orientations coexist (Prigent et al. 2003; Duguet et al. 2010; Klotz et al. 2022), but only one orientation is permitted in the tilted box.

We will use two types of numerical domains, with different lengths $L_z$. Both have fixed resolution $\Delta z = L_z/N_z = 0.08$, along with fixed $L_x = 10$ ($N_x = 120$) and $\theta = 24^\circ$. These domains are shown in figure 1.

1. **Minimal Band Units**, an example of which is shown as the dark red box in figure 1. These domains accommodate a single band-gap pair and so are used to study strictly periodic pattern of imposed wavelength $\lambda = L_z$.

2. **Long Slender Boxes**, which have a large $L_z$ direction that can accommodate a large and variable number of gaps and bands in the system. The blue box in figure 1 is an example of such a domain size with $L_z = 240$, but larger sizes ($L_z = 400$ or $L_z = 800$) will be used in our study.

### 3. Nucleation of laminar gaps

We carry out simulations in a Long Slender Box of size $L_z = 800$ for various $Re$ with the uniform turbulent state from a simulation at $Re = 500$ as an initial condition, a protocol called a quench. Figure 2 displays the resulting spatio-temporal dynamics at four Reynolds numbers. Plotted is the ($z,t$) dependence of the cross-flow energy $(v^2 + u_{span}^2)/2$ at $(x = L_x/2, y = 0)$. The cross-flow energy is a useful diagnostic because it is zero for laminar flow. The choice $x = L_x/2$ is arbitrary since there is no large-scale variation of the flow field in the short $x$ direction of the simulation.

Figure 2 demonstrates strong space-time intermittency and encapsulates the main results of this section: the emergence of patterns out of uniform turbulence is a gradual process. At $Re = 500$, barely discernible low-energy regions appear randomly within the turbulent background. At $Re = 460$ the low-energy regions are more pronounced and begin to constitute localised, quasi-laminar gaps within the turbulent flow. These gaps appear sparsely and are not long lived. At $Re = 440$, clearly demarcated, spatially localised quasi-laminar gaps are seen. As $Re$ is further decreased, these quasi-laminar gaps appear more frequently and persist for longer lifetimes. Eventually, the gaps self-organise into persistent, albeit fluctuating, patterns. The remainder of the section will quantify this transition to patterns.
Figure 2: Spatio-temporal visualization of pattern formation with $L_z = 800$, for (a) $Re = 500$, (b) $Re = 460$, (c) 440, (d) 420, (e) 400 and (f) $Re = 380$. Flow at $t = 0$ is initiated from uniform turbulence at $Re = 500$. Color shows local cross-flow energy $(v^2 + u^2_{\text{span}})/2$ at $x = L_x/2$, $y = 0$ (white: 0, red: 0.02). At high $Re$, weak local gaps appear sparsely. When $Re$ is decreased, spatio-temporally intermittent patterns of finite spatial extent emerge. These consist of turbulent cores (dark red) and quasi-laminar gaps (white). For still lower $Re$, quasi-laminar regions live longer, and patterns are more regular and steady.
We consider the $x,y$-averaged cross-flow energy

$$e(z,t) \equiv \frac{1}{L_x L_y} \int_{-1}^{1} \int_{0}^{L_x} \frac{1}{2} (v^2 + u_{\text{span}}^2) (x,y,z,t) \, dx \, dy$$

(3.1)

as a useful diagnostic of quasi-laminar and turbulent zones. The probability density functions (PDFs) of $e(z,t)$ are shown in figure 3a for various values of $Re$. The right tails, corresponding to high-energy events, are broad and exponential for all $Re$. The left, low-energy portions of the PDFs vary qualitatively with $Re$, unsurprisingly since these portions correspond to the weak turbulent events comprising the formation of quasi-laminar gaps. For large $Re$, the PDFs are maximal around $e \simeq 0.007$. As $Re$ is decreased, a low-energy peak emerges at $e \simeq 0.002$, corresponding to the emergence of long-lived, quasi-laminar gaps seen in figure 2. The peak at $e \simeq 0.007$ flattens and gradually disappears. An interesting feature is that the distributions broaden with decreasing $Re$ with both low energy and high energy events becoming more likely. This reflects a spatial redistribution of energy that accompanies the formation of quasi-laminar gaps. This is presumably the effect of turbulent bands extracting energy from the quasi-laminar regions and becoming more intense. (See figure 6 of Gomé et al. (2022).)

An intuitive way to characterise the intermittent creation of gaps is to define turbulent and quasi-laminar regions by thresholding the values of $e(z,t)$. In the following, a region will be called quasi-laminar if $e(z,t) < e_{\text{turb}}$ and turbulent if $e(z,t) \geq e_{\text{turb}}$. As the PDF of $e(z,t)$ evolves with $Re$, we define a $Re$-dependent threshold as a fraction of its average value, $e_{\text{turb}} = 0.75 \overline{e}$. The thresholding is illustrated in figure 3b, which is an enlargement of the flow at $Re = 440$ that shows turbulent and quasi-laminar zones as white and blue areas, respectively. Thresholding within a fluctuating turbulent environment can conflate long-lived quasi-laminar gaps with tiny, short-lived regions where the energy fluctuates below the threshold $e_{\text{turb}}$. These are seen as the numerous small blue spots in figure 3b that differ from the wider and longer-lived gaps. This deficiency is addressed by examining the statistics of the spatial and temporal sizes of quasi-laminar gaps.

We present the length distributions of laminar $L_{\text{lam}}$ and turbulent zones $L_{\text{turb}}$ in figures 3c and 3d at various Reynolds numbers. These distributions have their maxima at very small lengths, reflecting the large number of small-scale, low-energy regions that arise due to thresholding the fluctuating turbulent field. As $Re$ is decreased, the PDF for $L_{\text{lam}}$ begins to develop a peak near $L_{\text{lam}} \simeq 15$, corresponding to the scale of the gaps visible in figure 2. The right tails of the distribution are exponential and shift upwards with decreasing $Re$. The PDF of $L_{\text{turb}}$ also varies with $Re$, but in a somewhat different way. As $Re$ decreases, the likelihood of turbulent length in the range $15 \lesssim L_{\text{turb}} \lesssim 35$ increases above the exponential background, but at least over the range of $Re$ considered, a maximum does not develop. The distributions at large $L_{\text{turb}}$ appear to be independent of $Re$.

It is notable that the laminar-length distributions show the emergence of structure at $Re$ higher than the turbulent-length distributions. This is particularly visible at $Re = 440$, where the distribution of $L_{\text{turb}}$ is indistinguishable from those at higher $Re$, while the distribution of $L_{\text{lam}}$ is substantially altered. This is entirely consistent with impression from the visualisation in figure 2 that quasi-laminar gaps are emerging in a uniform turbulent background. Although the distributions of $L_{\text{lam}}$ and $L_{\text{turb}}$ behave differently, the length scale emerging as $Re$ decreases are of the same order of magnitude for both. This latter aspect is not present in the pipe flow results of Avila & Hof (2013). (See Appendix B for a more detailed comparison.)

Temporal measurements of the gaps are depicted in figure 4. Figure 4a shows the procedure by which we define the temporal extents $t_{\text{gap}}$ of laminar gaps. For each laminar
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Figure 3: (a) PDFs of local cross-flow energy $e(z,t)$ defined in (3.1). Maximum at $e \simeq 0.002$ appears for $Re \leq 420$. (b) Illustration of the thresholding $e(z,t) < e_{turb}$, of a laminar-turbulent field at $Re = 440$ with turbulent regions, $e(z,t) > e_{turb}$ in white and quasi-laminar regions in blue. Definitions of laminar and turbulent gaps $L_{gap}$ and $L_{turb}$ are illustrated. (c) PDFs of laminar gap widths $L_{lam}$ showing plateaux near 15 appearing for $Re \leq 440$. (d) PDFs of widths of turbulent regions $L_{turb}$ showing local increase near 20 for $Re \leq 420$.

gap, i.e. a connected zone in $(z,t)$ satisfying $e(z,t) < e_{turb}$, we locate its latest and earliest times and define $t_{gap}$ as the distance between them. Here again, we fix the threshold at $e_{turb} = 0.75 \tau$. Figure 4b shows the temporal distribution of quasi-laminar gaps, via the survival function of their lifetimes. In a similar vein to the spatial gap lengths, two characteristic behaviours are observed: for small times, many points are distributed near zero (as a result of frequent fluctuations near the threshold $e_{turb}$), while for large enough times, an exponential regime is seen:

$$P(t_{gap} > t) \approx e^{-(t-t_0(Re))/\tau_{gap}(Re)} \text{ for } t > t_0(Re)$$

(3.2)

The slope of the exponential tail is extracted at each $Re$ and the resulting characteristic time-scale $\tau_{gap}$ is shown in figure 4c. The evolution of $\tau_{gap}$ with $Re$ displays two regimes, each with nearly exponential dependence on $Re$, but with very different slopes on the semi-log plot. For $Re \geq 470$, the characteristic lifetimes are $\tau_{gap} = O(10^2)$ and vary
weakly with Re. These timescales correspond to the small white events visible in figure 2a and are associated with short-lived, low-energy events on the left tails of the PDFs for e(z,t) in figure 3a. Discounting these events, we refer to such states as uniform turbulence. For Re < 470, \( \tau_{\text{gap}} \) varies rapidly with Re. The abrupt change in slope seen in figure 4c marks the transition between uniform turbulence and the emergence of local gaps as Re is decreased. We denote by \( Re_{\text{gu}} = 470 \) the Reynolds number at which this transition occurs. We stress that as far as we have been able to determine, there are no critical phenomenon associated with this change of behaviour. That is, the transition is smooth and lacks a true critical point. It is nevertheless evident that the dynamics of quasi-laminar gaps change significantly in the region of \( Re = 470 \) and therefore it is useful to define a reference Reynolds number marking this change in behaviour.

Note that typical lifetimes of laminar gaps must become infinite by the threshold \( Re \approx 325 \) below which turbulence is no longer sustained (Lemoult et al. 2016). (We
believe this to be true even for \( Re \lesssim 380 \) when the permanent banded regime is attained, although this is not shown here.) For this reason, we have restricted our study of gap lifetimes to \( Re \gtrsim 380 \) and we have limited our maximal simulation time to \( \sim 10^4 \).

To quantify the distinction between localized gaps and patterns, we introduce a variable \( e_{L/S} \) as follows. Using the Fourier transform in \( z \),

\[
\hat{u}(x, y, k_z, t) = \frac{1}{L_z} \int_0^{L_z} u(x, y, z, t) e^{-ik_z z} \, dz ,
\]

we compute the averaged spectral energy

\[
\hat{E}(y, k_z) \equiv \frac{1}{2} \hat{u} \cdot \hat{u}^* , \quad \hat{E}(k_z) \equiv \langle \hat{E}(y, k_z) \rangle_y
\]

where the overbar designates an average in \( x \) and \( t \). This spectral energy was already described in Gomé et al. (2022, Part 1, figure 3a). We are interested in the ratio of \( \hat{E}(k_z) \) at large scales (pattern scale) to small scales (roll-streak scale), as it evolves with \( Re \).

For this purpose, we define the ratio of large-scale to small-scale maximal energy:

\[
e_{L/S} = \frac{\max_{k_z < 0.5} \hat{E}(k_z)}{\max_{k_z > 0.5} \hat{E}(k_z)}
\]

This quantity is shown as blue squares in figure 4c and is highly correlated to \( \tau_{\text{gap}} \). This correlation is in itself a surprising observation for which we have no explanation. For \( Re \gtrsim 430 \), we have \( e_{L/S} < 1 \), signaling that the dominant peak in the energy spectrum is at the roll-streak scale, while for \( Re \lesssim 430 \), the large-scale pattern begins to dominate the streaks and rolls, as indicated by \( e_{L/S} > 1 \) (light blue area on figure 4c). Note that \( Re = 430 \) is also the demarcation between unimodal and bimodal PDFs of \( e(z, t) \) in figure 3a. The transition from gaps to patterns is smooth. In fact, we do not even observe a qualitative feature sharply distinguishing gaps and patterns. We nevertheless find it useful to define a reference Reynolds number associated to patterns starting to dominate the energy spectrum. This choice has the advantage of yielding a quantitative criterion, which we estimate as \( Re_{pg} \approx 430 \).

In addition to the previous quantitative measures, we also extract the friction coefficient. This is defined as the ratio of the mean wall shear stress \( \mu U_{\text{wall}}' \) to the dynamic pressure \( \rho U_{\text{wall}}^2/2 \), which we write in physical units and then non-dimensional forms as:

\[
C_f \equiv \frac{\mu U_{\text{wall}}'}{2 \rho U_{\text{wall}}^2} = \frac{2 \nu}{U_{\text{wall}}} \frac{U_{\text{wall}}'}{h} \left. \frac{\partial \langle u_{\text{strm}} \rangle_{x,z,t}}{\partial y} \right|_{\text{wall}}
\]

In (3.6), the dimensional quantities \( h, \rho, \mu, \) and \( \nu \) are the half-height, the density, and dynamic and kinematic viscosities, and \( U_{\text{wall}} \) and \( U_{\text{wall}}' \) are the velocity and mean velocity gradient at the wall. We note that the behavior of \( C_f \) in the transitional region has been investigated by Shimizu & Manneville (2019) and Kashyap et al. (2020). Our measurements of \( C_f \) are shown in figure 4d. We distinguish three regimes. In the uniform regime \( Re > Re_{gu} = 470 \), \( C_f \) increases with decreasing \( Re \). In the patterned regime \( Re < Re_{pg} = 430 \), \( C_f \) decreases with decreasing \( Re \). Between the two, in the localised-gap regime \( Re_{pg} < Re < Re_{gu} \), \( C_f \) is approximately constant.

The changes in regimes and the distinction between local gaps and patterns can be further studied by measuring the spatial correlation between quasi-laminar regions within...
(a) Gap-to-gap correlation function $C(\delta z)$ defined by (3.8) for various values of $Re$. (b) plotting $\tanh(10 C(\delta z))$ focuses on the short-range behaviour of $C$. The oscillations at $Re = 420$ are weak at $Re = 460$ and disappear at $Re = 480$. The dots correspond to the first local maximum, indicating the selection of a finite distance between two local gaps.

The flow. We define

$$\Theta(z,t) = \begin{cases} 1 & \text{if } e(z,t) < e_{\text{turb}} \text{ (laminar)} \\ 0 & \text{otherwise (turbulent)} \end{cases}$$

(3.7)

(this is the quantity shown in blue and white in figures 3b and 4a.) We then compute its spatial correlation function:

$$C(\delta z) = \frac{\langle \Theta(z)\Theta(z+\delta z) \rangle_{z,t} - \langle \Theta(z) \rangle_{z,t}^2}{\langle \Theta(z)^2 \rangle_{z,t} - \langle \Theta(z) \rangle_{z,t}^2}. \quad (3.8)$$

Along with $(z,t)$ averaging, $C$ is also averaged over multiple realisations of the quench experiment. As $\Theta$ is a Heaviside function, $C$ can be understood as the average probability of finding quasi-laminar flow at a distance $\delta z$ from other quasi-laminar flow at position $z$. The results are presented in figure 5a. The comparison between different $Re$ values is enhanced by plotting $\tanh(10 C)$, shown in figure 5b. At long range, $C$ approaches zero with some small fluctuations at $Re = 480$, a noisy periodicity at $Re = 460$, and a nearly periodic behaviour for $Re \approx 420$.

In all cases, $C$ initially decreases from one and reaches a first minimum, due to the minimal possible size of a turbulent zone that suppresses the creation of neighbouring laminar gaps in the range $\delta z \lesssim 30$. $C$ has a prominent local maximum $\delta z_{\text{max}}$ right after the initial decrease of $C$, at $\delta z_{\text{max}} \simeq 32$ at $Re = 480$, which increases to $\delta z_{\text{max}} \simeq 41$ at $Re = 420$. These maxima, shown as coloured circles in figure 5b, indicate that gap nucleation is preferred at distance $\delta z_{\text{max}}$ from an existing gap. The increase in $\delta z_{\text{max}}$ and the subsequent extrema as $Re$ is lowered agrees with the trend of increasing wavelength of turbulent bands as $Re$ is decreased in the fully banded regime at lower $Re$ (Prigent et al. 2003; Barkley & Tuckerman 2005).

Our observations confirm the absence of large-scale modulation in the uniform regime $Re > 470$ (as defined in figure 4c), but emphasise the presence of (weak) gap interaction at a finite distance in this regime. This preference is stronger as $Re$ decreases and multiple
4. Existence and stability of patterns

In this section, we investigate the existence of a preferred pattern wavelength by using as a control parameter the length $L_z$ of the Minimal Band Unit. In a Minimal Band Unit, the system is constrained and the distinction between local gaps and patterns is lost. $L_z$ is chosen such as to accommodate at most a single turbulent zone and a single quasi-laminar zone, which due to imposed periodicity, can be viewed as one period of a perfectly periodic pattern. By varying $L_z$, we can verify whether a regular pattern of given wavelength $L_z$ can emerge from uniform turbulence, disregarding the effect of scales larger than $L_z$ or of competition with wavelengths close to $L_z$. We refer to these simulations in Minimal Band Units as existence experiments. Indeed, one of the main advantages of the Minimal Band Unit is the ability to create patterns of a given angle and wavelength which may not be stable in a larger domain.

In contrast, in a Long Slender Box, $L_z$ is large enough to accommodate multiple bands and possibly even patterns of different wavelengths. An initial condition consisting of a regular pattern of wavelength $\lambda$ can be constructed by concatenating bands produced from a Minimal Band Unit of size $\lambda$. The stability of such a pattern is studied by allowing this initial state to evolve via the non-linear Navier-Stokes equations. Both existence and stability studies can be understood in the framework of the Eckhaus instability (Kramer & Zimmermann 1985; Ahlers et al. 1986; Tuckerman & Barkley 1990; Cross & Greenside 2009).

In previous studies of transitional regimes, Barkley & Tuckerman (2005) studied the evolution of patterns as $L_z$ was increased. In Section 4.1, we extend this approach to multiple sizes of the Minimal Band Unit by comparing lifetimes of patterns that naturally arise in this constrained geometry. The stability of regular patterns of various wavelengths will be studied in Long Slender Domains ($L_z = 400$) in Section 4.2.

4.1. Temporal intermittency of regular patterns in a short-$L_z$ box

Figure 6a shows the formation of a typical pattern in a Minimal Band Unit of size $L_z = 40$ and at $Re = 440$. While the system cannot exhibit the spatial intermittency seen in figure 2c, temporal intermittency is possible and is seen as alternations between uniform turbulence and patterns. We plot the spanwise velocity at $y = 0$ and $x = L_x/2$.

This is a particularly useful measure of the large-scale flow associated with patterns, seen as red and blue zones surrounding a white quasi-laminar region. The patterned state spontaneously emerges from uniform turbulence and remains from $t \approx 1500$ to $t \approx 3400$. At $t \approx 500$, a short-lived quasi-laminar zone appears at $z = 10$, which can be seen as an attempt to form a pattern.

The pattern is characterised quantitatively by computing the wavenumber that instantaneously maximises the energy of the Fourier mode $k_z$:

$$\hat{\lambda}_{\text{max}}(t) = \frac{2\pi}{\arg\max_{k_z > 0} |\langle \hat{u}(y = 0, k_z, t) \rangle_x|^2},$$

(4.1)

where $\langle \hat{u}(y = 0, k_z, t) \rangle_x$ denotes the $x$ average of the $z$ Fourier transform of the mid-plane velocity. The possible values of $\hat{\lambda}_{\text{max}}$ are integer divisors of $L_z$, i.e. here 40, 20, 10, etc. Figure 6b presents $\hat{\lambda}_{\text{max}}$ and its short-time average $\langle \hat{\lambda}_{\text{max}} \rangle_{t_a}$ with $t_a = 30$ as light and dark blue curves, respectively. When turbulence is uniform, $\hat{\lambda}_{\text{max}}$ varies rapidly between
Figure 6: Pattern lifetimes. (a) Space-time visualization of a metastable pattern in a Minimal Band Unit with $L_z = 40$ at $Re = 440$. Colors show spanwise velocity (blue: $-0.1$, white: 0, red: 0.1). (b) Values of the dominant wavelength $\hat{\lambda}_{\text{max}}$ (light blue curve) and of its short-time average $\langle \hat{\lambda}_{\text{max}} \rangle_t$ (dark blue curve) are shown; see (4.1). A state is defined to be patterned if $\hat{\lambda}_{\text{max}} = L_z$. (c) Survival function of lifetimes of laminar-turbulent patterns in a Minimal Band Unit with $L_z = 40$ for various $Re$. The pattern lifetimes $t_{\text{pat}}$ are the lengths of the time intervals during which $\hat{\lambda}_{\text{max}} = L_z$. (d) Above: characteristic times $\tau_{\text{pat}}$ extracted from survival functions as a function of $L_z$ and $Re$. Below: intermittency factor $\gamma_{\text{pat}}$ for the patterned state: the fraction of time spent in the patterned state.

its discrete allowed values, while $\langle \hat{\lambda}_{\text{max}} \rangle_t$ fluctuates more gently around 10. The flow state is deemed to be patterned when its dominant mode is $\langle \hat{\lambda}_{\text{max}} \rangle_t = L_z$. The long-lived pattern occurring for $1500 \leq t \leq 3400$ in figure 6a is seen as a plateau of $\langle \hat{\lambda}_{\text{max}} \rangle_t$ in figure 6b. There are other shorter-lived plateaus, notably at for $500 \leq t \leq 750$. A similar analysis was carried out by Barkley & Tuckerman (2005); Tuckerman & Barkley (2011) using the Fourier component corresponding to wavelength $L_z$ of the spanwise mid-gap velocity.

Figure 6c shows the survival function $t_{\text{pat}}$ of the pattern lifetimes obtained from $\langle \hat{\lambda}_{\text{max}} \rangle_t$ over long simulation times for various $Re$. This measurement differs from figure 4b which showed lifetimes of gaps in a Long Slender Box and not regular patterns obtained in a Minimal Band Unit. Here, the spatio-temporal intermittency reduces to a
temporal problem, since we consider the flow in the Minimal Band Unit to either contain a pattern or not. Nevertheless the picture is qualitatively similar. As with figure 4b in figure 6c there are many short-lived patterns due to fluctuations. After $t \simeq 200$, the survival functions enter an approximately exponential regime, from which we extract the characteristic times $\tau_{\text{pat}}$ by taking the inverse of the slope.

We then vary $L_z$, staying within the Minimal Box regime $L_z \lesssim 70$ in which only one band can fit. Figure 6d (top) shows that $\tau_{\text{pat}}$ presents a broad maximum in $L_z$ whose strength and position depend on $Re$: $L_z \simeq 42$ at $Re = 440$ and $L_z \simeq 50$ at $Re = 400$. This wavelength corresponds approximately to the natural spacing observed in a Large Slender Box (figure 2). Figure 6d (bottom) presents the fraction of time that is spent in a patterned state, denoted $\gamma_{\text{pat}}$, to reflect that this should be thought of as the intermittency factor for the patterned state. The dependence of $\gamma_{\text{pat}}$ on $L_z$ follows the same trend as $\tau_{\text{pat}}$, but less strongly (the scale of the inset is linear, while that for $\tau_{\text{pat}}$ is logarithmic). For $Re = 480$, the survival function is nearly the same as for 460 and $\tau_{\text{pat}}$ and $\gamma_{\text{pat}}$ are nearly independent of $L_z$; this is the situation for uniform turbulence.

These results complement the Ginzburg-Landau description proposed by Prigent et al. (2003) and Rolland & Manneville (2011). To quantify the bifurcation from featureless to pattern turbulence, they defined an order parameter and showed that it has a quadratic maximum at an optimal wavenumber. This is consistent to the approximate quadratic maximum that we observe in the logarithmic plot of pattern lifetimes, and in the linear plot of $\tau_{\text{pat}}$ with regard to $L_z$.

4.2. Pattern stability in a large domain

To study the stability of a pattern of wavelength $\lambda$, we prepare an initial condition for a Long Slender Box concatenating repetitions of a single band produced in a Minimal Band Unit. We add small-amplitude noise to this initial pattern so that the repeated bands do not all evolve identically. Figures 7a and 7b show two examples of such simulations. Depending on the value of $Re$ and of the initial wavelength $\lambda$, the pattern destabilises to either another periodic pattern (figure 7a for $Re = 400$) or to localised patterns surrounded by patches of featureless turbulence (figure 7b for $Re = 430$).

It can be seen that patterns often occupy only part of the domain. For this reason, we turn to the wavelet decomposition (Meneveau 1991; Farge et al. 1992) to quantify patterns locally. In contrast to a Fourier decomposition, the wavelet decomposition quantifies the signal as a function of space and scale. From this, we are able to define a local dominant wavelength, $\tilde{\lambda}_{\text{max}}(z,t)$, similar in spirit to $\hat{\lambda}_{\text{max}}(t)$ in (4.1), but now at each space-time point. (See Appendix A for details.) Figures 7c and 7d show $\tilde{\lambda}_{\text{max}}(z,t)$ obtained from wavelet analysis of the simulations visualised in figures 7a and 7b.

We now use the local wavelength $\tilde{\lambda}_{\text{max}}(z,t)$ to quantify the stability of an initial wavelength. We use a domain of length $L_z = 400$ and we concatenate $n = 7$ to 13 repetitions of a single band to produce a pattern with initial wavelength $\lambda(n) \equiv 400/n \approx 57, 50, 44 \ldots 31$. (We have rounded $\lambda$ to the nearest integer value here and in what follows.) After adding low-amplitude noise, we run a simulation lasting 5000 time units, compute the wavelet transform and calculate from it the local wavelengths $\tilde{\lambda}_{\text{max}}(z,t)$. We then compute

$$H_{\lambda}(t) = \left\langle \frac{1}{L_z} \int_0^{L_z} \Theta \left( \epsilon_{\lambda} - |\lambda - \tilde{\lambda}_{\text{max}}(z,t)| \right) \ dz \right\rangle_{t_a}.$$  \hspace{1cm} (4.2)

The short-time average $\langle \cdot \rangle_{t_a}$ is taken over time $t_a = 30$ as before. $\Theta$ is the Heaviside function and $\epsilon_{\lambda}$ is a threshold which selects $z$-values such that $\tilde{\lambda}_{\text{max}}$ is closer to $\lambda(n)$.
Figure 7: Simulation in a Long Slender Box from a noise-perturbed periodic pattern with (a) initial $\lambda = 57$ at $Re = 400$ and (b) initial $\lambda = 40$ at $Re = 430$. Colors show spanwise velocity (red: 0.1, white: 0, blue: $-0.1$). (c) and (d) show the local dominant wavelength $\tilde{\lambda}_{\text{max}}(z,t)$ determined by wavelet analysis (see Appendix A) corresponding to the simulations shown in (a) and (b). Color at $t = 0$ shows the wavelength $\lambda$ of the initial condition. (e) shows the wavelet-defined $H_\lambda(t)$ defined in (4.2), which quantifies the proportion of the domain which retains initial wavelength $\lambda$ as a function of time for cases (a) and (b). Circles indicate the times for (a) and (b) after which $H_\lambda$ is below the threshold value $H_{\text{stab}}$ for a sufficiently long time. (f) Ensemble-averaged $\bar{t}_{\text{stab}}$ of the decay time of an imposed pattern of wavelength $\lambda$ for various values of $Re$. The relative stability of a wavelength, whether localised or not, is measured by $t_{\text{stab}}$ via the wavelet analysis.
than to its two neighboring values $\lambda(n + 1)$ and $\lambda(n - 1)$. Thus, $H_N$ represents the proportion of $L_z$ in which we consider the dominant mode $\tilde{\lambda}_{\text{max}}$ to be $\lambda$. In practice, because patterns in a Long Slender Box still fluctuate in width, a steady pattern may have $H_N$ somewhat less than 1. If $H_N \ll 1$, a pattern of wavelength $\lambda$ is present in only a very small part of the flow.

Figure 7e shows how wavelet analysis via the Heaviside-like function $H_N(t)$ quantifies the relative stability of the pattern in the examples shown in figures 7a and 7b. The flow in figure 7a at $Re = 400$ begins with $\lambda = 57$, i.e. 7 bands. The red curve in figure 7e shows $H_N$ decaying quickly and roughly monotonically. One additional gap appears at around $t = 2300$ and starting from then, $H_N$ remains low. This corresponds to the initial wavelength $\lambda = 57$ losing its dominance to $\lambda = 40, 44$ and 50 in the visualisation of $\tilde{\lambda}_{\text{max}}(z, t)$ in figure 7f. By $t = 5000$, the flow shows 9 bands with a local wavenumber $\lambda$ between 40 and 44.

The flow in figure 7b at $Re = 430$ begins with $\lambda = 40$, i.e. 10 bands. The blue curve in figure 7e representing $H_N$ initially decreases and drops fairly suddenly around $t \approx 1000$ as several gaps disappear in figure 7b. $H_N$ then fluctuates around a finite value, which is correlated to the presence of gaps whose local wavelength is the same as the initial $\lambda$, visible as zones where $\tilde{\lambda}_{\text{max}} = 40$ in figure 7d. The rest of the flow can be mostly seen as locally featureless turbulence, where the dominant wavelength is small ($\tilde{\lambda}_{\text{max}} \approx 10$). The local patterns fluctuate in width and strength, and $H_N$ evolves correspondingly after $t = 1000$. The final state reached in figure 7b at $Re = 430$ is characterised by the presence of intermittent local gaps.

The lifetime of an initially imposed pattern wavelength $\lambda$ is denoted $t_{\text{stab}}$, and is defined as follows: We first define a threshold $H_{\text{stab}} = 0.2$ (marked by a horizontal dashed line on figure 7e). If $H_N(t)$ is statistically below $H_{\text{stab}}$, the imposed pattern will be considered as unstable. Following this principle, $t_{\text{stab}}$ is defined as the first time $H_N$ is below $H_{\text{stab}}$, with two further conditions to dampen the effect of short-term fluctuations. First, $H_N(t)$ must be below $H_{\text{stab}}$ for a period of $\Delta t_1 = 100$ after $t_{\text{stab}}$. This avoids selecting a local minimum of little importance. Second, $t_{\text{stab}}$ must obey $\langle H_N(t) \rangle_{t \in [t_{\text{stab}}, t_{\text{stab}} + \Delta T]} < H_{\text{stab}}$, with $\Delta T = 2000$, so as to ensure that the final state is on average below $H_{\text{stab}}$. The corresponding times in case (a) and (b) are marked respectively by a red and a blue circle in figure 7e.

Repeating this experiment over multiple realisations of the initial pattern (i.e. different noise realizations) yields an ensemble-averaged $\bar{t}_{\text{stab}}$. This procedure estimates the time for an initially regular and dominant wavelength to disappear from the flow domain, regardless of the way in which it does so and of the final state approached. Figure 7f presents the dependence of $\bar{t}_{\text{stab}}$ on $\lambda$ for different values of $Re$. We note that a most-stable wavelength emerges from the uniform state, at around $\lambda \approx 40$ at $Re = 440$, similarly to the results from the existence study on figure 6d which showed a preferred wavelength of around 42 at $Re = 440$. Consistently with what was observed in Minimal Band Units of different sizes, the most stable wavelength grows with decreasing $Re$.

4.3. Discussion

Our study of the existence and stability of large-scale modulations of the turbulent flow is summarised in figure 8. This figure resembles the existence and stability diagrams presented for usual (non-turbulent) hydrodynamic instabilities such as Rayleigh-Bénard convection and Taylor-Couette flow (Busse 1981; Ahlers et al. 1986; Cross & Greenside 2009). In classic systems, instabilities appear with increasing control parameter, while
here gaps and bands emerge from uniform turbulent flow as Re is lowered. Therefore, we plot the vertical axis in figure 8 with decreasing upwards Reynolds.

We recall that the existence study of §4.1 culminated in the measurement of $\gamma_{\text{pat}}(\lambda, \text{Re})$, the fraction of simulation time that is spent in a patterned state, plotted in figure 6d. The parameter values at which $\gamma_{\text{pat}}(\lambda, \text{Re}) = 0.45$ (an arbitrary threshold that covers most of our data range) are shown as black circles in figure 8. The dashed curve is an interpolation of the iso-$\gamma_{\text{pat}}$ points and separates two regions, with patterns more likely to exist above the curve than below. The minimum of this curve is estimated to be $\lambda \simeq 42$. This is a preferred wavelength at which patterns first statistically emerge as $\text{Re}$ is decreased from large values.

The final result of the stability study in section §4.2, shown in figure 7f, was $\bar{t}_{\text{stab}}(\text{Re}, \lambda)$, a typical duration over which a pattern initialised with wavelength $\lambda$ would persist. The colours in figure 8 show $\bar{t}_{\text{stab}}$. This region also has its minimum at $\lambda \approx 42$. The pattern existence and stability zones are similar in shape and in their lack of symmetry with respect to line $\lambda = 42$. The transition seen in figures 7a and 7f, from $\lambda = 57$ to $\lambda = 44$ at $\text{Re} = 400$ corresponds to motion from a light blue to a dark blue area in the top row of figure 8. This change in pattern wavelength resembles the Eckhaus instability which, in classic hydrodynamics, leads to transitions from unstable wavelengths outside a stability band, to stable wavelengths inside.

An important result of this section is that wavelength 40–44 is preferred, however weakly, up to $\text{Re} = 460$, a regime in which no steady patterns are found (see Section 3). The presence of a most-probable wavelength confirms the initial results of Prigent et al. (2003) and those of Rolland & Manneville (2011). This is also consistent with the instability study of Kashyap et al. (2022) in plane Poiseuille flow. However, contrary to classic pattern-forming instabilities, the turbulent-laminar pattern does not emerge from an exactly uniform state, but instead from a state in which local gaps are intermittent, as established in Section 3. In Section 5, we will emphasise the importance of the mean flow in the wavelength selection that we just described.

5. Optimisation of the mean flow

This section is devoted to the dependence of various energetic features of the patterned flow on the domain length $L_z$ of a Minimal Band Unit. We fix the Reynolds number at
Figure 9: Energy analysis for the patterned state at $Re = 400$ as a function of $L_z$. (a) Spatially-averaged total energy $\langle E \rangle$, mean TKE $\langle K \rangle$ ($\times 5$), mean total dissipation $\langle D \rangle$, mean kinetic dissipation $\langle \epsilon \rangle$ ($\times 3$), for the patterned state at $Re = 400$ as a function of $L_z$. (b) Energy in each of the $z$-Fourier components of the mean flow (equations (5.1) and (5.2)).

$Re = 400$. In the existence study of §4, the wavelength $\lambda \simeq 44$ was found to be selected by patterns. (Recall the upper-most curves corresponding to $Re = 400$ in figure 6d.) We will show that this wavelength also extremises quantities in the energy balances of the flow.

5.1. Average energies in the patterned state

We first decompose the flow into a mean and fluctuations, $\mathbf{u} = \mathbf{u} + \mathbf{u}'$, where the mean is taken over the homogeneous directions $x$ and $t$. We compute energies of the total flow $\langle E \rangle \equiv \langle \mathbf{u} \cdot \mathbf{u} \rangle / 2$ and of the fluctuations (turbulent kinetic energy) $\langle K \rangle \equiv \langle \mathbf{u}' \cdot \mathbf{u}' \rangle / 2$, where $\langle \cdot \rangle$ is the $(x, y, z)$ average. Figure 9a shows these quantities as a function of $L_z$ for the patterned state at $Re = 400$. At $L_z = 44$, $\langle E \rangle$ is maximal and $\langle K \rangle$ is minimal. As a consequence, the mean-flow energy $\frac{1}{2} \langle \mathbf{u} \cdot \mathbf{u} \rangle = \langle E \rangle - \langle K \rangle$ is also maximal at $L_z = 44$. Figure 9a additionally shows average dissipation of the total flow $\langle D \rangle \equiv \langle |\nabla \times \mathbf{u}|^2 \rangle / Re$ and average dissipation of turbulent kinetic energy $\langle \epsilon \rangle \equiv \langle |\nabla \times \mathbf{u}'|^2 \rangle / Re$, both of which are minimal at $L_z = 44$.

The mean flow is further analysed by computing the energy of each spectral component of the mean flow. For this, the $x$ and $t$ averaged flow $\overline{\mathbf{u}}$ is decomposed into Fourier modes in $z$:

$$\overline{\mathbf{u}}(y, z) = \overline{\mathbf{u}}_0(y) + 2\mathcal{R} \left( \overline{\mathbf{u}}_1(y)e^{i2\pi z/L_z} \right) + \overline{\mathbf{u}}_{>1}(y, z)$$

(5.1)

where $\overline{\mathbf{u}}_0$ is the uniform component of the mean flow, $\overline{\mathbf{u}}_1$ is the trigonometric Fourier coefficient corresponding to $k_z = 2\pi/L_z$ and $\overline{\mathbf{u}}_{>1}$ is the remainder of the decomposition, for $k_z > 2\pi/L_z$. (We have omitted the hats on the $z$ Fourier components of $\overline{\mathbf{u}}$.) The energies of the spectral components relative to the total mean energy are

$$e_0 = \frac{\langle \overline{\mathbf{u}}_0 \cdot \overline{\mathbf{u}}_0 \rangle}{\langle \overline{\mathbf{u}} \cdot \overline{\mathbf{u}} \rangle}, \quad e_1 = \frac{\langle \overline{\mathbf{u}}_1 \cdot \overline{\mathbf{u}}_1 \rangle}{\langle \overline{\mathbf{u}} \cdot \overline{\mathbf{u}} \rangle}, \quad e_{>1} = \frac{\langle \overline{\mathbf{u}}_{>1} \cdot \overline{\mathbf{u}}_{>1} \rangle}{\langle \overline{\mathbf{u}} \cdot \overline{\mathbf{u}} \rangle}$$

(5.2)

These are presented in figure 9b. It can be seen that $e_0 \gg e_1 > e_{>1}$ and also that all have an extremum at $L_z = 44$. In particular, $L_z = 44$ minimizes $e_0$ ($e_0 = 0.95$)
while maximizing the trigonometric component \((e_1 = 0.025)\) along with the remaining components \((e_{>1} \approx 0.011)\). Note that for a banded state at \(Re = 350, L_z = 40\), Barkley & Tuckerman (2007) found that \(e_0 \approx 0.70, e_1 \approx 0.30\) and \(e_{>1} \approx 0.004\), consistent with a strengthening of the bands as \(Re\) is decreased.

5.2. Mean flow spectral balance

We now investigate the spectral contributions to the budget of the mean flow \(\bar{u}\), dominated by the mean flow’s two main spectral components \(\bar{u}_0\) and \(\bar{u}_1\). The balances can be expressed as (Gomé et al. [2022] Part 1):

\[
\hat{A}_0 - \hat{\Pi}_0 - \hat{D}_0 + I = 0 \quad \text{for} \quad \bar{u}_0 \quad \text{and} \quad \hat{A}_1 - \hat{\Pi}_1 - \hat{D}_1 = 0 \quad \text{for} \quad \bar{u}_1
\]

where \(I\) is the rate of energy injection by the viscous shear, and \(\hat{\Pi}_0, \hat{D}_0\) and \(\hat{A}_0\) stand for, respectively, production, dissipation and advection (i.e. non-linear interaction) contributions to the energy balance of mode \(\bar{u}_0\) and similarly for \(\bar{u}_1\). These are defined by

\[
I = \frac{2}{Re} R \left\{ \int_{-1}^{1} \frac{\partial}{\partial y} \left( \bar{u}_j^*(k_z = 0) \hat{\bar{u}}_{yj}(k_z = 0) \right) dy \right\} = \frac{1}{Re} \left( \frac{\partial \bar{u}_{\text{strm}}}{\partial y} \right)_{1} + \frac{\partial \bar{u}_{\text{strm}}}{\partial y} \right|_{1}
\]

\[
\hat{\Pi}_0 = R \left\{ \int_{-1}^{1} \frac{\partial \hat{\bar{u}}_{xy}}{\partial x_i}(k_z = 0) \bar{u}_i(k_z = 0) dy \right\}
\]

\[
\hat{D}_0 = \frac{2}{Re} R \left\{ \int_{-1}^{1} \hat{\bar{u}}_{ij}(k_z = 0) \hat{\bar{u}}_{ij}(k_z = 0) dy \right\}
\]

\[
\hat{A}_0 = -R \left\{ \int_{-1}^{1} \hat{\bar{u}}_{xy}(k_z = 0) \frac{\partial \bar{u}_{xy}}{\partial x_i}(k_z = 0) dy \right\}
\]

where \(R\) denotes the real part. We define \(\hat{\Pi}_1, \hat{D}_1\) and \(\hat{A}_1\) similarly by replacing \(k_z = 0\) by \(k_z = 2\pi/L_z\) in (5.4a)-(5.4d).

We recall two main results from Gomé et al. [2022] Part 1: first, \(\hat{A}_1 \approx -\hat{A}_0\). This term represents the energetic transfer between modes \(\bar{u}_0\) and \(\bar{u}_1\) via the self-advection of the mean flow (the energetic spectral influx from \((\bar{u} \cdot \nabla)\bar{u}\)). Second, \(\hat{\Pi}_1 < 0\), and this term approximately balances the negative part of TKE production. This is a feedback from turbulent fluctuations to the component \(\bar{u}_1\) of the mean flow.

Each term contributing to the balance of \(\bar{u}_0\) and \(\bar{u}_1\) is shown as a function of \(L_z\) in figures 10a and 10b. We do not show \(\hat{A}_0\) because \(\hat{A}_0 \approx -\hat{A}_1\).

We obtain the following results:

1. Production \(\hat{\Pi}_0\), dissipation \(\hat{D}_0\) and energy injection \(I\) are nearly independent of \(L_z\), varying by no more than 6% over the range shown. These \(k_z = 0\) quantities correspond to uniform fields in \(z\) and hence it is unsurprising that they depend very little on \(L_z\).

2. The non-linear term \(\hat{A}_1 \approx -\hat{A}_0\), i.e. the transfer from \(\bar{u}_0\) to \(\bar{u}_1\) which is the principal source of energy of \(\bar{u}_1\), has a maximum at \(L_z \approx 44\). This is the reason for which \(\bar{u}_0\) is minimised by \(L_z \approx 44\) (see figure 9b): more energy is transferred from \(\bar{u}_0\) to \(\bar{u}_1\).

3. Production \(\hat{\Pi}_1\) increases with \(L_z\) and does not show an extremum at \(L_z \approx 44\).
Figure 10: Spectral energy balance of the mean flow components (a) $\overline{u}_0$ and (b) $\overline{u}_1$. See equation (5.3).

(it instead has a weak maximum at $L_z \simeq 50$). In all cases, $\tilde{\Pi}_1 < \tilde{A}_1$: the TKE feedback on the mean flow, although present, is not dominant and not selective.

(4) Dissipation $\tilde{D}_1$ accounts for the remaining budget and its extremum at $L_z \simeq 44$ corresponds to maximal dissipation.

The turbulent kinetic energy balance is also modified with changing $L_z$. This is presented in Appendix C. The impact of TKE is however secondary, because the feedback on the mean flow, via $\tilde{\Pi}_1$, is not the leading term that fuels $\overline{u}_1$, and does not participate in maximising the energy of $\overline{u}_1$ at the preferred wavelength.

6. Conclusion and discussion

We have explored the appearance of patterns from uniform turbulence in plane Couette flow at $Re \leq 500$. We used numerical domains of different sizes to quantify the competition between featureless (or uniform) turbulence and (quasi-) laminar gaps. In Minimal Band Units, intermittency reduces to a random alternation between two states: uniform or patterned. In large slender domains, however, gaps nucleate randomly and locally in space, and the transition to patterns takes place continuously via the regimes presented in Section 3: the uniform regime in which gaps are rare and short-lived (above $Re \simeq 470$), and another regime ($Re < 470$) in which gaps are more numerous and long-lived. Below $Re \simeq 430$, the large-scale spacing of these gaps starts to dominate the energy spectrum, which is a possible demarcation of the patterned regime. In this latter regime, with further decreasing in $Re$, gaps eventually fill the entire flow domain, forming regular patterns. The distinction between these regimes is observed in both gap lifetime and friction factor.

Spatially isolated gaps were already observed by Prigent et al. (2003), Barkley & Tucker (2005) and Rolland & Manneville (2011). (See also Manneville (2015, 2017) and references therein.) Our results confirm that pattern emergence, mediated by randomly-nucleated gaps, is necessarily more complex than the supercritical Ginzburg-Landau framework initially proposed by Prigent et al. (2003) and later developed by Rolland & Manneville (2011). However, this does not preclude linear processes in the appearance
of patterns, such as those reported by Kashyap et al. (2022) from an ensemble-averaged linear response analysis.

The intermittency between uniform turbulence and gaps that we quantify here in the range $380 \lesssim Re \lesssim 500$ is not comparable to that between laminar flow and bands present for $325 \lesssim Re \lesssim 340$. The latter is a continuous phase transition in which the laminar flow is absorbing: laminar regions cannot spontaneously develop into turbulence and can only become turbulent by contamination from neighbouring turbulent flow. This is connected to the existence of a critical point at which the correlation length diverges with a power-law scaling with $Re$, as characterised by important past studies (Shi et al. 2013; Chantry et al. 2017; Lemoult et al. 2016) which showed a connection to directed percolation. The emergence of gaps from uniform turbulence is of a different nature. Neither uniform turbulence nor gaps are absorbing states, since gaps can always appear spontaneously and can also disappear, returning the flow locally to a turbulent state. While the lifetimes of quasi-laminar gaps do exhibit an abrupt change in behaviour at $Re = 470$ (figure 4c), we observe no evidence of critical phenomena associated with the emergence of quasi-laminar gaps from uniform turbulence. Hence, the change in behaviour appears to be in fact smooth. This is also true in pipe flow where quasi-laminar gaps form, but not patterns (Avila & Hof 2013; Frishman & Grafke 2022).

We used the pattern wavelength as a control parameter, via either the domain size or the initial condition, to investigate the existence of a preferred pattern wavelength. We propose that the finite spacing between gaps, visible in both local gaps and patterned regimes, is selected by the preferred size of their associated large-scale flow. Once gaps are sufficiently numerous and patterns are established, their average wavelength increases with decreasing $Re$, with changes in wavelength in a similar vein to the Eckhaus picture. The influence of the large-scale flow in wavelength selection is analysed in Section 5, where we carried out a spectral analysis like that in Gomé et al. (2022, Part 1) for various sizes of the Minimal Band Unit. In particular, we investigated the roles of the turbulent fluctuations and of the mean flow, which is in turn decomposed into its uniform component $\overline{u}_0$ and trigonometric component $\overline{u}_1$, associated to the large-scale flow along the laminar-turbulent interface. Our results demonstrate a maximisation of the energy of $\overline{u}_1$ by the wavelength naturally preferred by the flow, and this is primarily associated to the advective term $(\overline{\mathbf{u}} \cdot \nabla)\mathbf{u}$ in the mean flow equation. This term redistributes energy between modes $\overline{u}_0$ and $\overline{u}_1$ and is mostly responsible for energising the large-scale along-band flow. Turbulent fluctuations are of secondary importance in driving the large-scale flow and do not play a significant role in the wavelength selection.

This result resonates with certain optimality principles underpinning classical pattern formation and for which Rayleigh-Bénard convection are a canonical example: Malkus (1954) and Busse (1981) (and references therein) proposed a principle of maximal heat transport, or equivalently maximal dissipation, obeyed by convective turbulent solutions. The maximal dissipation principle, as formulated by Malkus (1956) in shear flows, occurs in other systems such as von Kármán flow (Ozawa et al. 2001; Mihelich et al. 2017). (This principle has been somewhat controversial. Disputed by Reynolds & Tiederman (1967) within the context of stability theory, it was recently revisited with statistical closures by Markeviciute & Kerswell (2022)). In our case, the flow maximises the transport of momentum and the dissipation of the large-scale flow, analogous to the principles mentioned by Malkus (1956) and Busse (1981). Explaining this mere observation from a guiding principle remains a tremendous challenge.

It is essential to understand the creation of the large-scale flow around a randomly emerging laminar hole. The statistics obtained in our tilted configuration must be extended to large streamwise-spanwise domains, in which short-lived and randomly-
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Nucleated holes might align in the streamwise direction (Manneville 2017, Fig. 5), presumably before the regime of long-lived gaps is attained. Furthermore, a more complete dynamical picture of gap creation is needed. The excitable model of Barkley (2011a) might provide a proper framework, as it accounts for both the emergence of anti-puffs (Frishman & Grafke 2022) and of periodic solutions (Barkley 2011b). Connecting this model to the Navier-Stokes equations is, however, a formidable challenge. Our work emphasises the necessity of including the effects of the advective large-scale flow to adapt this model to the establishment of the laminar-turbulent patterns observed in planar shear flows.

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Declaration of Interests

The authors report no conflict of interest.

Appendix A. Wavelet transform

We introduce the one-dimensional continuous wavelet transform of the velocity $u(z,t)$ taken along the line $(x,y) = (L_x/2,0)$:

$$\tilde{u}(z,r,t) = C^{-1/2}_ψ r^{-1/2} \int_0^{L_z} \psi^* \left( \frac{z' - z}{r} \right) u(z',t) dz'$$  \hspace{1cm} (A 1)

Here $ψ$ is the Morlet basis function, defined in Fourier space as $\hat{ψ}(k) = \pi^{-1/4} e^{-((k-k_ψ)^2)/2}$ for $k > 0$. Its central wavenumber is $k_ψ = 6/\Delta z$, where $\Delta z$ is the grid spacing. The scale factor $r$ is related to wavelength via $λ \simeq 2\pi r/k_ψ$. $C_ψ \equiv \int |k|^{-1} |\hat{ψ}(k)|^2 dk$ is a normalization constant. Tildes are used to designate wavelet transformed quantities. The inverse transform is:

$$u(z,t) = C^{-1/2}_ψ \int_0^\infty \int_{-\infty}^{\infty} r^{-1/2} ψ \left( \frac{z - z'}{r} \right) \tilde{u}(z',r,t) \frac{dz' dr}{r^2}$$  \hspace{1cm} (A 2)

The wavelet transform is related to the Fourier transform in $z$ by:

$$\tilde{u}(z,r,t) = \frac{1}{2\pi} C^{-1/2}_ψ r^{1/2} \int_{-\infty}^{\infty} \hat{ψ}(r k_z) \hat{u}(k_z,t) e^{ik_z z} dk_z$$  \hspace{1cm} (A 3)

We then define the most energetic instantaneous wavelength as:

$$\tilde{λ}_{max}(z,t) = \frac{2\pi}{k_ψ} \arg\max_r |\tilde{u}(z,r,t)|^2$$  \hspace{1cm} (A 4)

The characteristic evolution of $\tilde{λ}_{max}(z,t)$ is illustrated in figure 11b for the flow case corresponding to figure 11a. Regions in which $\tilde{λ}_{max}$ is large ($> 10$) and dominated by a single value correspond to the local patterns observed in figure 11a. In contrast, in regions where $\tilde{λ}_{max}$ is small ($< 10$) and fluctuating, the turbulence is locally uniform.
Figure 11: Space-time visualisation of a quench experiment at $Re = 430$: (a) spanwise velocity (blue: $-0.2$, white: $0$, red: $0.2$), (b) $\tilde{\lambda}_{\text{max}}(z,t)$ defined by (A 4). $\tilde{\lambda}_{\text{max}}(z,t)$ (b) quantifies the presence of local large-scale modulations within the flow. Dark blue zones where $\tilde{\lambda}_{\text{max}}(z,t) < 10$ correspond to locally featureless turbulence in (a). Large-scale modulation of gaps at different wavelengths are visible by the green-to-red spots in (b).

Figure 12: Space-time fraction of large to small wavelengths obtained by wavelet transform. $f_{L/S}$ crosses $0.5$ at $Re \simeq 427 \simeq Re_{\text{pg}}$.

This space-time intermittency of the patterns is quantified by measuring

$$f_{L/S} = \left\langle \Theta(\tilde{\lambda}_{\text{max}}(z,t) - 10) \right\rangle_{z,t}$$

(A 5)

and is shown in figure 12.

Appendix B. Laminar and turbulent distributions in pipe vs Couette flows.

From figures 3c and 3d of the main text, both distributions of laminar or turbulent lengths, $L_{\text{lam}}$ and $L_{\text{turb}}$, are exponential for large enough lengths, similarly to pipe (Avila & Hof 2013). It is however striking that the distributions of $L_{\text{lam}}$ and $L_{\text{turb}}$ have different shapes for $L_{\text{lam}}$ or $L_{\text{turb}} > 10$ in plane Couette flow: $L_{\text{lam}}$ shows a sharper
Figure 13: Cumulative distribution of (a) laminar gaps and (b) turbulent zones, for various \(Re\).

distribution, whereas \(L_{\text{turb}}\) is more broadly distributed. We present on figures 13a and 13b the cumulative distributions of \(L_{\text{lam}}\) and \(L_{\text{turb}}\) for a complementary analysis.

We focus on the characteristic length \(l^*_{\text{turb}}\) or \(l^*_{\text{lam}}\) for which

\[
\begin{align*}
P(L_{\text{lam}} > l^*_{\text{lam}}) &= 10^{-2}; \\
\text{for example, } l^*_{\text{lam}} &= 15.5 \text{ at } Re = 440; \\
\end{align*}
\]

\[
\begin{align*}
l^*_{\text{turb}} &= 26.5 \text{ at } Re = 400. \text{ We see that } l^*_{\text{turb}} \text{ and } l^*_{\text{lam}} \text{ are of the same order of magnitude. This differs from the same measurement in pipe flow, carried out by Avila & Hof (2013, Fig. 2): } \\
l^*_{\text{lam}} &= 6 \text{ and } l^*_{\text{turb}} \simeq 50 \text{ at } Re = 2800; \\
l^*_{\text{lam}} \simeq 17 \text{ and } l^*_{\text{turb}} \simeq 160 \text{ at } Re = 2500 \text{ (as extracted from their figure 2.).}
\end{align*}
\]

Appendix C. Turbulent kinetic energy balance for various \(L_z\)

In this appendix, we address the balance of turbulent kinetic energy \(\hat{K}(k_z)\), written here in a \(y\)-integrated form at a specific mode \(k_z\) (see equation (5.3) of Gomé et al. (2022, Part 1) and the methodology in, e.g., Bolotnov et al. (2010); Lee & Moser (2015); Mizuno (2016); Cho et al. (2018)):

\[
0 = \hat{\Pi} - \hat{D} + \hat{A} + \hat{T}_{nl}
\]  

respectively standing for production, dissipation, triadic interaction and advection terms.

We recall that \((\cdot)\) is an average in \((x,t)\). The \(y\) evolution of the energy balance was analysed in Gomé et al. (2022, Part 1).

Gomé et al. (2022, Part 1) reported robust negative production at large scales, along with inverse non-linear transfers to large scales. If \(k_{\text{rolls}} = 1.41\) denotes the scale of rolls and streaks, this inverse transfer occurs for \(k_z < k_{\text{LS}} = 0.94\), while a downward transfer occurs for \(k_z > k_{\text{SS}} = 3.6\) (We refer the reader to the figure 5 of Gomé et al. (2022, Part
1). This spectral organization of the energy balance will be quantified by the following transfer terms arising from (C 2):

\[ \hat{T}_{LS} = \sum_{k_z=0}^{k_{LS}} \hat{T}_{nl}(k_z), \quad \hat{T}_{SS} = \sum_{k_z=k_{SS}}^{\infty} \hat{T}_{nl}(k_z), \quad \hat{D}_{LS} = \sum_{k_z=0}^{k_{LS}} \hat{D}(k_z), \quad \hat{A}_{LS} = \sum_{k_z=0}^{k_{LS}} \hat{A}(k_z) \]  

(C 3)

\( \hat{T}_{LS} \) quantifies transfer to large scales, \( \hat{T}_{SS} \) the transfer to small scales, \( \hat{D}_{LS} \) the dissipation at large scales, and \( \hat{A}_{LS} \) is a transfer of energy from the mean flow to the large fluctuating scales.

The variables defined in (C 3) are displayed in figure 14 as function of \( L_z \). \( \hat{T}_{LS} \) is minimal at \( L_z \simeq 44 \). \( \hat{D}_{LS} \) is minimal at \( L_z \simeq 40 \). Contrary to \( \hat{T}_{LS} \), \( \hat{T}_{SS} \) is relatively constant with \( L_z \) (green dashed line in figure 14), with a variation of around 6%. This demonstrates that transfers to small scales are unchanged with \( L_z \). Large-scale advection decays with increasing \( L_z \) and does not play a role in the wavelength preference. Our results confirm that the balance at large-scale is minimised around \( L_z \simeq 44 \), and that TKE will play a less important role, compared to that of the mean flow whose energy and balance are maximised at \( L_z \simeq 44 \).

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