Non-classical Berry-Esseen inequality and accuracy of the weighted bootstrap

Mayya Zhilova

School of Mathematics
Georgia Institute of Technology
Atlanta, GA  30332-0160 USA
e-mail: mzhilova@math.gatech.edu

Abstract: We study accuracy of a weighted bootstrap procedure for estimation of quantiles of Euclidean norm of a sum of independent random vectors with zero mean and bounded fourth moment. We establish higher-order approximation bounds with error terms depending explicitly on a sample size and a dimension. These results lead to improvements of accuracy of a weighted bootstrap procedure for general log-likelihood ratio statistics. The key element of our proofs of the bootstrap accuracy is a multivariate Berry-Esseen inequality in a non-classical form. We consider a problem of approximation of distributions of two sums of zero mean independent random vectors, such that summands with the same indices have equal moments up to at least the second order. The derived approximation bound is uniform on the set of all Euclidean balls in $\mathbb{R}^p$. This approximation is an extension of a Gaussian one. The theoretical results are illustrated with numerical experiments.

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1. Introduction

In this paper we study accuracy of a weighted (or a multiplier) bootstrap procedure for estimation of quantiles of statistics of the form $\|S_n\|$, where $\|\cdot\|$ denotes $\ell_2$-norm in $\mathbb{R}^p$, and

$$S_n \overset{\text{def}}{=} \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$$

for independent random vectors $X_1, \ldots, X_n \in \mathbb{R}^p$ such that $\forall i = 1, \ldots, n$

$$E X_i = 0, \quad E (\|X_i\|^4) < \infty,$$

and $\text{Var}(X_i)$ is positive definite.

We consider the setting when the sample size $n$ is bounded, and approximation errors depend on $n$ and dimension $p$ explicitly. This allows to assess accuracy and limitations of bootstrap approximation in terms of dimension and sample size. Estimation of distribution of statistics of the type $\|S_n\|$ is necessary for
construction of confidence sets and hypothesis testing in some important statistical models and problems, such as linear regression model with unknown distribution of errors, general log-likelihood ratio statistic, construction of elliptical confidence sets for multivariate sample mean.

The weighted bootstrap procedure for \( \|S_n\| \) is defined as follows. Introduce the random variables
\[
e_1, \ldots, e_n \in \mathbb{R}, \text{ i.i.d., independent of } \{X_i\},
\[
Ee_i = 0, \ E(e_i^2) = 1, \ E(e_i^3) = 1, \ E(e_i^4) < \infty.
\]
Define for the sum \( S_n \) its bootstrap version:
\[
S_n^\circ \overset{\text{def}}{=} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i e_i.
\]
The unknown quantiles of the initial statistic \( \|S_n\| \) are approximated with the quantiles of \( \|S_n^\circ\| \) conditioned on the sample \( X_1, \ldots, X_n \). Denote the upper quantile function of \( \|S_n^\circ\| \) as
\[
Q^\circ(\alpha) \overset{\text{def}}{=} \inf \{ t \in \mathbb{R} : P^\circ (\|S_n^\circ\| > t) \leq \alpha \},
\]
where \( \alpha \in (0, 1) \), and \( P^\circ(\cdot) \overset{\text{def}}{=} P(\cdot | X_1, \ldots, X_n) \). One of the main results of the paper is the following approximation bound between \( P(\|S_n\| > Q^\circ(\alpha)) \) and \( \alpha \); if \( X_1, \ldots, X_n \) are i.i.d., then
\[
\left| P(\|S_n\| > Q^\circ(\alpha)) - \alpha \right| \leq C_\Sigma \frac{\sqrt{E(\|X_1\|^4)E(e_i^4)}}{\sqrt{n}},
\]
where the constant \( C_\Sigma \) depends on the largest eigenvalue of the matrix \((\Var X_1)^{-1}\).
This bound implies, in particular, that if the random vector \( X_1 \) is sub-gaussian, then the ratio \( p/\sqrt{n} \) has to be small in order to keep the bootstrap approximation accurate. This complies with the results by [7] for residual bootstrap, the authors showed that the bootstrap least squares estimate in high dimensional linear regression model converges in Mallow’s distance to the original estimator if \( p^2/n \to 0 \). [28] extended these results for M-estimators.

Conditions (1.1) on the bootstrap weights play an important role for obtaining the accuracy in approximating bound (1.4). [26] used the condition \( E(e_i^3) = 1 \) in order to obtain the second order accuracy of the wild bootstrap (or Wu’s bootstrap, first proposed by [50]) approximation to the least squares estimate in a linear regression model. [30] studied validity and higher-order accuracy of the wild bootstrap under the condition \( E(e_i^3) = 1 \) on the weights, in context of linear contrasts in high dimensional linear models and for bootstrapping F-tests. Here we impose condition \( E(e_i^3) = 1 \) in order to obtain a higher-order accuracy as well. Consider the first two moments of the bootstrap sum (1.2) w.r.t. the joint distribution of \( \{X_i\} \) and \( \{e_i\} \). By \( Ee_i = 0, \ E(e_i^4) = 1 \) it holds
\[
E S_n = E S_n^\circ, \quad E(S_n S_n^\top) = E(S_n^\circ S_n^\circ^\top).
\]
Using (1.5) and normal approximation between distributions of \( \|S_n\| \) and \( \|S_n^a\| \)
(e.g. the results of [4]), one can obtain an approximation bound similar to (1.4),
with an error term \( C_3^{3/2} \sqrt{\text{E}} \left( \|X_1\|^3 \right) \text{E}(\varepsilon_1^3) / \sqrt{n} \), which is less sharp than (1.4)
in the ratio between \( p \) and \( n \). Using also the condition \( \text{E}(\varepsilon_1^3) = 1 \), we obtain
\[
\forall \alpha \in \mathbb{R}^p \, \text{E} \{ (\alpha^\top S_n)^3 \} = \text{E} \{ (\alpha^\top S_n^a)^3 \},
\]  
(1.6)
and this property leads to the improved error term in (1.4). In order to employ
the information about the third moments, as in (1.6), one needs to use an ap-
proximation, which is more general than the normal one. For this purpose we
consider a multivariate Berry-Esseen type bound. Before introducing the latter
result, let us mention that approximation (1.4) leads also to an improvement
of accuracy of a weighted bootstrap procedure for general log-likelihood ratio
statistics. [47] considered weighted bootstrap for estimation of quantiles of log-
likelihood ratio, they showed that if a parametric model is not severely misspeci-
fied, then the accuracy of bootstrap log-likelihood ratio quantiles corresponds to
the accuracy of normal approximation between statistics of the type \( \|S_n\| \) and
\( \|S_n^a\| \). Using inequality (1.4), we infer that the accuracy of weighted bootstrap
for log-likelihood ratio depends rather on accuracy of Wilks-type bounds, then
on normal approximation.

The key element in the proofs of our theoretical results about accuracy of
the bootstrap is a multivariate Berry-Esseen inequality in a non-classical form,
which might be interesting by itself. We consider a problem of approximation of probability distribution of the sum \( S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \), where \( X_i \in \mathbb{R}^p \) are independent random vectors such that \( \text{E} X_i = 0 \) and \( \text{E} (\|X_i\|^K) < \infty \) for some \( K \geq 3 \). The approximating distribution corresponds to the following sum
\[
\tilde{S}_n \overset{\text{def}}{=} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i,
\]  
(1.7)
where \( Y_1, \ldots, Y_n \in \mathbb{R}^p \) are independent random vectors, independent of \( \{X_i\}_{i=1}^{n} \),
such that \( \text{E} (\|Y_i\|^K) < \infty \),
\[
\text{E}(X^k_i) = \text{E}(Y^k_i) \, \forall k = 1, \ldots, K - 1,
\]  
(1.8)
and \( Y_i = Z_i + U_i \) for some independent random vectors \( Z_i, U_i \in \mathbb{R}^p \) such that
\( Z_i \) are normally distributed with \( \text{E} Z_i = 0 \). Throughout the paper the condition
\( \text{E}(X^k) = \text{E}(Y^k) \, \forall k = 1, \ldots, K \) on the higher-order moments of random vectors
\( X = (x_1, \ldots, x_p)^\top \in \mathbb{R}^p \) and \( Y = (y_1, \ldots, y_p)^\top \in \mathbb{R}^p \) denotes that for all degrees
\( k = 1, \ldots, K \) and for all indices \( 1 \leq i_1, \ldots, i_k \leq p \)
\[
\text{E}(x_{i_1} \cdots x_{i_k}) = \text{E}(y_{i_1} \cdots y_{i_k}).
\]  
(1.8)
In Lemma 2.1 (in Section 2) we show that if \( X_i \) has a continuous probability dis-
tribution, then the corresponding random vectors \( Z_i, U_i \) always exist. However,
in general, the continuity condition is not necessary for existence of \( Z_i \) and \( U_i \).
The probability distribution of such constructed random vector \( S_n \) turns out to
be a rather good approximation of the distribution of the initial sum $S_n$. One of the main results in the paper is the following uniform Berry-Esseen type bound: for the set $\mathcal{B}$ of all Euclidean balls in $\mathbb{R}^p$ and for i.i.d. $X_1, \ldots, X_n$

$$\sup_{B \in \mathcal{B}} |\mathbf{P}(S_n \in B) - \mathbf{P}(\tilde{S}_n \in B)| \leq C_{K, \Sigma} \left\{ \frac{\mathbf{E}(\|X_1\|^K + \|Y_1\|^K)}{n^{1/2}} \right\}^{1/(K-2)},$$

(1.9)

where constant $C_{K, \Sigma}$ depends on $K$ and on the largest eigenvalue of the matrix $(\text{Var } Z_1)^{-1}$. We study also the case of independent but not necessarily identically distributed summands $X_i$. Bound (1.9) includes the classical Berry-Esseen inequality, when the approximating distribution is purely Gaussian, i.e. $Y_i \sim \mathcal{N}(0, \text{Var } X_i)$ and $K = 3$. If $K > 3$, this bound exploits more information about coinciding moments, than Gaussian approximation does, which leads to a better accuracy.

Our proof of bound (1.9) is based on the work of [4], where the author obtained the multivariate Berry-Esseen inequality in purely standard Gaussian case, uniformly on the set of all Euclidean balls, and also on the set of all convex sets in $\mathbb{R}^p$. The results by [4] have the best known dependence of the approximation error on dimension. In this paper we extend the proof of [4] to the “quasi-Gaussian” case, i.e. for the approximation with the sum $\tilde{S}_n$ of the convolutions $Y_i = Z_i + U_i$, where $Z_i$ are normally distributed. This approach allows us to use both the properties of Gaussian distribution and the higher moments condition (1.7).

Below we give an overview of the existing literature and discuss contribution of this paper to it. Weighted bootstrap is a general version of the classical Efron’s bootstrap (proposed by [13]). According to the latter method, bootstrap approximation of an empirical measure of a random sample is constructed by weighing the empirical measure with multinomial random weights, conditioned on the sample. Later [31] extended this scheme for general exchangeable random weights. Let us mention the papers [38, 34, 35] as some of the first works about general weighted bootstrap. Let us also refer to the book by [3] and to the paper by [22] for exhaustive literature reviews about this topic. One of the basic ways of studying the properties of bootstrap procedures is to consider asymptotic approximations of distributions of an initial statistic and its bootstrap estimate, e.g. using central limit theorems or their refinements with Edgeworth expansions (see the books [17, 29, 43]). Accuracy of bootstrap procedures is usually studied using Edgeworth expansions or Berry-Esseen-type inequalities (the latter technique had been first used by [44] and [26] in the framework of bootstrap).

In the asymptotic high-dimensional setting when both the parameter dimension $p$ and the sample size $n$ are large, [7, 28, 30] studied accuracy of Efron’s and wild bootstrap for linear regression model; [8] studied generalized bootstrap for estimating equations also in high-dimensional asymptotic framework. Most of the non-asymptotic results (i.e. without using asymptotic arguments w.r.t. $n$) about bootstrap are quite recent. [1] studied generalized weighted bootstrap for construction of non-asymptotic confidence bounds in $\ell_r$ -norm ($r \in [1, \infty]$)
for the mean value of high dimensional random vectors with a symmetric and bounded (or with Gaussian) distribution. [10] studied Gaussian approximation and multiplier bootstrap for maxima of sums of high-dimensional vectors in a very general set-up. [11] extended the results from maxima to general hyperrectangles and sparsely convex sets.

In the approximation bound (1.4) obtained in this work we do not use any asymptotic arguments. Moreover, this bound has a better accuracy than a Gaussian approximation does. This justifies that weighted bootstrap can outperform Gaussian approximation if the bootstrap weights are properly chosen (for example, as in (1.1)). We apply this result for construction of confidence sets for least squares estimate in a linear regression model, and for likelihood-based confidence sets.

The problem of approximation of a probability distribution of the sum $S_n$ belongs to the class of Central Limit Problems, which has a long history of studies. In particular, convergence of a distribution of $S_n$ in context of convergence of its moments had been considered by P. Chebyshev, A. Markov, P. Lévy (see the paper by [27] for detailed overview). [21] studied convergence of distribution of $S_n$ in case of i.i.d. scalar summands, to standard normal law, under higher moments condition; the author obtained a higher-order accuracy using Edgeworth expansion. [51] introduced pseudomoments, which characterize closeness of moments of two distributions, for estimation of convergence rates in limit theorems. These characteristics turned out to be very useful for refining the classical limit theorems, without imposing condition of uniform asymptotic negligibility of the summands. Such limit theorems are called non-classical. In the multivariate finite-dimensional case some of the first non-classical results about normal approximation on closed convex sets had been obtained by [33, 37, 48]. [32, 40, 41, 14, 9, 15, 4, 5, 10, 11] studied normal approximation in finite-dimensional space. Let us refer to the books [6, 52, 42] for comprehensive overview of earlier results on these topics. The results of [4] have the best known dependence on dimension among the Berry-Esseen type bounds for $\|S_n\|$.

To the best of our knowledge, the problem of approximation of probability distribution of $S_n$ under the higher moments condition (1.7) and with explicit dependence on $p$, had not been studied before.

**Structure of the paper**

The results about accuracy of bootstrap rely on Berry-Esseen type inequalities, for this reason we first present the latter results in Section 2. Section 3 contains theoretical results regarding bootstrap accuracy. In Sections 3.2 and 3.3 we consider the weighted bootstrap for the linear regression model and for log-likelihood ratio statistics correspondingly. Sections A and B contain proofs of results from Sections 2 and 3 respectively. Section 4 presents results of numerical experiments.
Notation

\[ \| \cdot \| \] denotes Euclidean norm for vectors and spectral norm for matrices; \( S^+_p \) denotes space of symmetric positive definite real-valued matrices of size \( p \times p \); \( \mathcal{B} \) is the set of all Euclidean balls in \( \mathbb{R}^p \); \( I_p \) is the identity matrix of size \( p \times p \); if \( X \) is a vector in \( \mathbb{R}^p \), \( X^k \) stands for the tensor power \( X \otimes k \); \( C \) indicates positive generic constant unless specified otherwise.

2. Non-classical Berry-Esseen inequality

Consider independent random vectors \( X_1, \ldots, X_n \in \mathbb{R}^p \) such that \( \forall i = 1, \ldots, n \) \( \mathbf{E} X_i = 0 \), \( \mathbf{Var}(X_i) \in S^+_p \), \( \mathbf{E}(\|X_i\|^K) < \infty \) for some integer \( K \geq 3 \). Let \( Y_1, \ldots, Y_n \in \mathbb{R}^p \) be independent random vectors, and such that \( \forall i = 1, \ldots, n \) \( Y_i \) is independent of \( X_1, \ldots, X_n \),

\[
\mathbf{E} (\|Y_i\|^K) < \infty, \\
\mathbf{E}(X_i^k) = \mathbf{E}(Y_i^k) \ \forall k = 1, \ldots, K - 1,
\]

(2.1)

A formal definition of the equality of the higher-order moments of vector-valued random variables (as in (2.1)) is given in (1.8). We assume also that \( \forall i = 1, \ldots, n \) \( \exists \) independent r.v. \( Z_i, U_i \in \mathbb{R}^p \), such that

\[
Y_i \overset{d}{=} Z_i + U_i, \quad \mathbf{E} Z_i = \mathbf{E} U_i = 0, \\
Z_i \sim \mathcal{N}(0, \Sigma_{z,i}) \text{ for some } \Sigma_{z,i} \in S^+_p.
\]

(2.2)

In Lemma 2.1 below it is shown that continuity of a probability distribution of \( X_i \) is sufficient for existence of the r.v. \( Y_i \) described above. Consider the following sums of mutually independent zero mean random vectors:

\[
S_n \overset{\text{def}}{=} \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i, \quad \tilde{S}_n \overset{\text{def}}{=} \frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i.
\]

(2.3)

We establish uniform approximation bounds between probability distributions of \( S_n \) and \( \tilde{S}_n \) on the set \( \mathcal{B} \) of all Euclidean balls in \( \mathbb{R}^p \). Theorem 2.1 treats the case when \( \{X_i\}_{i=1}^n \) are i.i.d.; the case of independent but not necessarily identically distributed vectors \( \{X_i\}_{i=1}^n \) is considered in Theorem 2.2.

Let us introduce some additional notation before stating the first result. \( M > 0 \) is a generic constant, in the proof in Section A.2 we show that one can take \( M \geq 72.5 \). \( \bar{C}_\mathcal{B} > 0 \) is an isoperimetric constant of the set \( \mathcal{B} \) corresponding to the standard Gaussian measure in \( \mathbb{R}^p \), namely, for \( Z_0 \sim \mathcal{N}(0, I_p) \) and \( \forall r, \varepsilon > 0 \)

\[
\mathbf{P} (r \leq \|Z_0\| \leq r + \varepsilon) \leq \bar{C}_\mathcal{B} \varepsilon;
\]

(2.4)

due to results of [2], the constant \( \bar{C}_\mathcal{B} \) is dimension-free (see Section A.1 for more detail). In the statement of Theorem 2.1 we use \( C_\mathcal{B} \overset{\text{def}}{=} \max\{1, \bar{C}_\mathcal{B}\} \). In
the proof of Theorem 2.1 we use function \( \varphi(x) : \mathbb{R}^p \mapsto [0, 1] \), which is at least \( K - 1 \) times continuously differentiable approximation of the indicator function \( \mathbb{I}\{x \in B_{r+1} \setminus B_r\} \), where \( B_r \in \mathcal{B} \) is some Euclidean ball of radius \( r \); constants \( C_\varphi, \tilde{C}_\varphi, \tilde{C}_{\varphi,1} \) enter upper bounds on supremum norms of the higher-order derivatives \( \varphi^{(K-2)}(x) \) and \( \varphi^{(K-1)}(x) \) (see Lemma A.3 in Section A.2). In the statement of Theorem 2.1 we use \( C_\varphi \defeq \max\{1, \tilde{C}_\varphi, \tilde{C}_{\varphi,1}\} \). For the case of i.i.d. summands \( X_i \) (and hence i.i.d. \( Z_i \)) denote \( \Sigma_z \defeq \Sigma_{z,i} = \text{Var}(Z_i) \). Since \( \Sigma_z \in \mathcal{S}_p^+ \), we can define \( C_z \defeq \|\Sigma_z^{-1/2}\| \).

**Theorem 2.1.** Consider the random vectors \( \{X_i\}_{i=1}^n \) introduced above, suppose that they are i.i.d., and that there exist i.i.d. approximating random vectors \( \{Y_i\}_{i=1}^n \) meeting conditions (2.1) and (2.2). It holds for the sums \( S_n \) and \( \bar{S}_n \) defined in (2.3)

\[
\sup_{B \in \mathcal{B}} \left| \mathbb{P}(S_n \in B) - \mathbb{P}(\bar{S}_n \in B) \right| \leq MC_\varphi C_\varphi \frac{\left\{ C_z^K \mathbb{E} \left( \|X_1\|^K + \|Y_1\|^K \right) \right\}^{1/(K-2)}}{n^{1/2}}.
\]

**Remark 2.1** (The case of Gaussian approximation). If the approximating random vectors \( Y_i \) have purely Gaussian distribution, then \( U_i \equiv 0 \), \( Y_i \sim \mathcal{N}(0, \text{Var}(X_i)) \), \( \Sigma_z = \text{Var}(X_i) \), and \( C_z = \|\text{Var}(X_i)\|^{-1/2} \). Furthermore, if \( K = 3 \) and \( Y_i \) are Gaussian, then the bound in Theorem 2.1 is similar to the classical multivariate Berry-Esseen inequality by [4]. If \( K > 3 \) and \( Y_i \) are Gaussian, the term \( \|X_1\| \) enters the bound above with a better power, than in the classical case where \( K = 3 \).

**Remark 2.2** (Accuracy of the approximation). As it is mentioned in Remark 2.1, in case when \( K = 3 \) and \( Y_i \sim \mathcal{N}(0, \mathbf{I}_p) \), Theorem 2.1 is almost identical to the results by [4], which are the best known in dependence on \( p \). Moreover, [32] proved the lower bound \( \Delta_n \geq C \mathbb{E}(\|X_1\|^3)n^{-1/2} \) for the class of all convex sets in \( \mathbb{R}^p \). We conjecture that the optimal error term in Theorem 2.1 is \( \leq MC_\varphi C_z C_z^K \mathbb{E}(\|X_1\|^K + \|Y_1\|^K)n^{-(K-2)/2} \); this complies with the results of [21]. We leave this improvement of the accuracy for the future work.

**Remark 2.3** (Dependence on \( C_z \)). The approximation bound in Theorem 2.1 depends on \( C_z = \|\Sigma_z^{-1/2}\| \), where \( \Sigma_z \) is a covariance matrix of the Gaussian part \( Z_i \) of the approximating distribution \( Y_i \). In Lemma 2.1 below we show that if \( X_i \) are continuously distributed, then there exist random vectors \( U_i \) such that \( \Sigma_z \) is positive definite. Therefore, it holds \( 0 < \Sigma_z \leq \text{Var}(X_i) \) and \( \|\{\text{Var}(X_i)\}\|^{-1/2} \leq C_z < \infty \). Bounding the value \( C_z \) from above is important for better understanding of the considered approximation, we leave this problem for the future work as well.

Now let us consider the case when the random summands \( X_i \) are not necessarily identically distributed. Denote \( \Sigma_z \defeq n^{-1} \sum_{i=1}^n \Sigma_{z,i} \) and \( \mathcal{C}_z \defeq \|\Sigma_z^{-1/2}\| \). Constant \( C_{\varphi,2} \) is similar to \( \tilde{C}_\varphi, \tilde{C}_{\varphi,1} \) in the previous theorem. \( C_{\varphi,2} \) enters an upper bound on supremum norm of \( \tilde{\varphi}^{(K)}(x) \), where \( \tilde{\varphi} \) is \( K \) times continuously differentiable approximation of the indicator function \( \mathbb{I}\{x \in B_{r+1} \setminus B_r\} \) for \( B_{r+1}, B_r \in \mathcal{B} \) (see Section A.3).
Theorem 2.2. Consider random vectors \( \{X_i\}_{i=1}^{\infty} \) introduced above, suppose that they are independent but not necessarily identically distributed, and that there exist independent approximating vectors \( \{Y_i\}_{i=1}^{\infty} \) meeting conditions (2.1) and (2.2). It holds for the sums \( S_n \) and \( \tilde{S}_n \) defined in (2.3)

\[
\sup_{B \in \mathcal{B}} \left| P(S_n \in B) - P(\tilde{S}_n \in B) \right| \\
\leq M_K \left\{ C_{\phi,2}^{K}\sum_{i=1}^{n} E\left(\|X_i\|^K + \|Y_i\|^K\right) \right\}^{1/(K+1)} \frac{C_{z}^{K} \sum_{i=1}^{n} E\left(\|X_i\|^K + \|Y_i\|^K\right)}{n^{K/2}},
\]

where \( M_K \) is defined as \( 2 \{(K - 1)\}^{-1/(K+1)} < 1.7 \) for \( K \geq 3 \), and \( C_{\phi} \) is the isoperimetric constant introduced in (2.4).

Corollary 2.1 below follows directly from the previous theorems and triangle inequality.

Corollary 2.1. Consider random vectors \( \{X_i\}_{i=1}^{n} \) introduced above, and suppose that there exist independent approximating vectors \( \{Y_i\}_{i=1}^{n} \) meeting conditions (2.1) and (2.2). Consider also independent random vectors \( X'_1, \ldots, X'_n \in \mathbb{R}^p \), that are independent of \( \{X_i\}_{i=1}^{n} \), \( \{Y_i\}_{i=1}^{n} \), and such that \( \forall i = 1, \ldots, n \)

\[
E\left(\|X'_i\|^K\right) < \infty, \\
E(X'_i) = E(X'_i) \quad \forall k = 1, \ldots, K - 1.
\]

Let also

\[
S'_n \overset{\text{def}}{=} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X'_i, \quad \Delta'_n \overset{\text{def}}{=} \sup_{B \in \mathcal{B}} \left| P(S_n \in B) - P(S'_n \in B) \right|.
\]

1. If conditions of Theorem 2.1 are fulfilled and \( \{X'_i\}_{i=1}^{n} \) are i.i.d., then

\[
\Delta'_n \leq M_{C_{\phi}} C_{\phi} \left\{ C_{z}^{K} E\left(\|X_1\|^K + \|Y_1\|^K\right) \right\}^{1/(K-2)} \frac{1}{n^{K/2}}
\]

\[
+ M_{C_{\phi}} C_{\phi} \left\{ C_{z}^{K} E\left(\|X_1\|^K + \|Y_1\|^K\right) \right\}^{1/(K-2)}.
\]

2. If conditions of Theorem 2.2 are fulfilled and \( \{X'_i\}_{i=1}^{n} \) are not necessarily identically distributed, then

\[
\Delta'_n \leq M_K \left\{ C_{\phi,2}^{K} C_{\phi} \right\}^{1/(K+1)} \left\{ C_{z}^{K} \sum_{i=1}^{n} E\left(\|X_i\|^K + \|Y_i\|^K\right) \right\}^{1/(K+1)} \frac{1}{n^{K/2}}
\]

\[
+ M_K \left\{ C_{\phi,2}^{K} C_{\phi} \right\}^{1/(K+1)} \left\{ C_{z}^{K} \sum_{i=1}^{n} E\left(\|X_i\|^K + \|Y_i\|^K\right) \right\}^{1/(K+1)}.
\]
In the following theorem we consider values of a function \( f : \mathbb{R}^p \to \mathbb{R} \), which is at least \( K \) times continuously differentiable on \( \mathbb{R}^p \). The statement shows how well the value \( \mathbb{E} f(S_n) \) can be approximated with \( \mathbb{E} f(S'_n) \) upon condition (2.5).

**Theorem 2.3.** Consider independent random vectors \( X_1, X'_1, \ldots, X_n, X'_n \in \mathbb{R}^p \) such that \( \forall i = 1, \ldots, n \) \( \mathbb{E} X_i = \mathbb{E} X'_i = 0 \), \( \mathbb{V} a r(X_i), \mathbb{V} a r(X'_i) \in S_p^+ \), \( \mathbb{E}(\|X_i\|^K), \mathbb{E}(\|X'_i\|^K) < \infty \), and

\[
\mathbb{E}(X^k_i) = \mathbb{E}(X'^k_i) \quad \forall k = 1, \ldots, K - 1. \tag{2.6}
\]

Let a function \( f : \mathbb{R}^p \to \mathbb{R} \) be at least \( K \) times continuously differentiable; constant \( C_f > 0 \) is such that \( \sup_x |f^{(K)}(x)h^K| \leq C_f \|h\|^K \) for all \( h \in \mathbb{R}^p \). Then it holds for \( S'_n \overset{\text{def}}{=} \frac{1}{\sqrt{n}} \sum_{i=1}^n X'_i \)

\[
\left| \mathbb{E} f(S_n) - \mathbb{E} f(S'_n) \right| \leq \sum_{i=1}^n C_f \frac{\mathbb{E} (\|X_i\|^K + \|X'_i\|^K)}{K!n^{K/2}}.
\]

If \( X_1, \ldots, X_n \) are identically distributed, and so are \( X'_1, \ldots, X'_n \), then

\[
\left| \mathbb{E} f(S_n) - \mathbb{E} f(S'_n) \right| \leq C_f \frac{\mathbb{E} (\|X_1\|^K + \|X'_1\|^K)}{K!n^{(K-2)/2}}.
\]

**Lemma 2.1** (Existence of the approximating distribution). Let a random vector \( X \in \mathbb{R}^p \) have continuous probability distribution, such that \( \mathbb{E} X = 0 \), \( \mathbb{V} a r X \in S_p^+ \), and \( \mathbb{E}(\|X\|^K) < \infty \) for some integer \( K \geq 2 \). Then there exists a random vector \( Y \overset{\text{def}}{=} Z + U \), where \( Z, U \in \mathbb{R}^p \) are independent, \( Z \sim \mathcal{N}(0, \Sigma_z) \) for some \( \Sigma_z \in S_p^+ \), and \( \mathbb{E}(X^k) = \mathbb{E}(Y^k) \) for all \( k = 0, \ldots, K \).

**Proof of Lemma 2.1.** Denote

\[
m_k \overset{\text{def}}{=} \mathbb{E}(X^k), \quad u_k \overset{\text{def}}{=} \mathbb{E}(U^k) \quad \text{for } k = 0, 1, 2, \ldots \tag{2.7}
\]

Conditioning on \( U \) leads to \( \mathcal{L}(Y | U) = \mathcal{N}(U, \Sigma_z) \) and to the following system of linear equations:

\[
m_0 = \mathbb{E}(Z + U)^0 = u_0, \\
m_1 = \mathbb{E}(Z + U) = u_1, \\
m_2 = \mathbb{E}(Z + U)^2 = u_2 + \Sigma_z, \\
m_3 = \mathbb{E}(Z + U)^3 = u_3, \\
\vdots
\]

\[
m_K = \mathbb{E}(Z + U)^K = K! \sum_{l=0}^{[K/2]} S_{p1_k} \frac{u_{K-2l} \otimes \text{vec}(\Sigma_z)^l}{l!(K-2l)!},
\]

where \( S_{p1_k} \) is the symmetrizer operator acting on the \( K \)-th tensor power of \( \mathbb{R}^p \); this formula for the raw moments of the multivariate normal distribution is...
In order to prove the lemma’s statement, it is sufficient to show that there exists a truncated Hamburger moment problem:

\[
\Sigma_k \text{ given in the work of [20]. The solution } \{u_k(\Sigma_k)\}_{k=0}^K \text{ of this system depends on } \Sigma_k \text{ continuously. Moreover, }
\]

\[
\text{if } \Sigma_k = 0, \text{ then } u_k(\Sigma_k) = m_k \ \forall \ k = 0, \ldots, K. \quad (2.8)
\]

In order to prove the lemma’s statement, it is sufficient to show that there exists a truncated moment problem:

\[
\Sigma_k \text{ with real coefficients. A polynomial } p \text{ s.t. the solution } \{u_k(\Sigma_k)\}_{k=0}^K \text{ also solves the following multivariate truncated Hamburger moment problem:}
\]

\[
\text{given a } p \text{-dimensional real multisequence } \{u_k(\Sigma_k)\}_{k=0}^K, \text{ does there exist a positive Borel measure } \mu \text{ on } \mathbb{R}^p \text{ s.t. } \int_{\mathbb{R}^p} x_k d\mu(x) = u_k(\Sigma_k) \ \forall k = 0, \ldots, K?
\]

The work of [12] provides necessary and sufficient conditions for solvability of multivariate truncated moment problems. Before stating these conditions we introduce some notation. Let \( \mathcal{P}_K \) denote space of polynomials of degree \( \leq K \) with real coefficients. A polynomial \( p = p(x) = \sum_{|i| \leq K} a_i x^i \in \mathcal{P}_K \) is non-negative, if \( p \geq 0 \) for all \( x \in \mathbb{R}^p \).

\[
\text{Here } i \overset{\text{def}}{=} (i_1, \ldots, i_p) \in \mathbb{N}_0^p \text{ denotes multi-index, } |i| = \sum_{j=1}^p i_j, \text{ and } x^i = x_1^{i_1} \cdots x_p^{i_p}. \text{ For a multisequence } \{u_i\}_{|i| \leq K} \text{ the Riesz functional } L : \mathcal{P}_K \mapsto \mathbb{R} \text{ is defined as } L(\sum_{|i| \leq K} a_i x^i) = \sum_{|i| \leq K} a_i u_i. \text{ If the Hamburger moment problem is soluble, we can write}
\]

\[
L(p) = \sum_{|i| \leq K} a_i u_i = \int_{\mathbb{R}^p} p(x) d\mu(x). \quad (2.9)
\]

[12] showed that a multisequence \( \{u_i\}_{|i| \leq K} \) solves the multivariate Hamburger truncated moment problem if there exists an extension \( \{\tilde{u}_i\}_{|i| \leq K+2} \) of \( \{u_i\}_{|i| \leq K} \) (i.e. \( \tilde{u}_i = u_i \text{ for all } |i| \leq K \)), such that for the corresponding Riesz functional

\[
\tilde{L}(\sum a_i x^i) = \sum_{|i| \leq K+2} a_i \tilde{u}_i \text{ it holds:}
\]

\[
\text{if } p \in \mathcal{P}_{K+2} \text{ and } p \text{ is non-negative, then } \tilde{L}(p) \geq 0. \quad (2.10)
\]

Let us consider the moment sequence \( \{m_k\}_{k=0}^K \) defined in (2.7). By the theorem of [12] there exists an extension \( \{\tilde{m}_k\}_{k=0}^{K+2} \) s.t. its corresponding Riesz functional

\[
\tilde{L}_m(\sum_{|i| \leq K+2} a_i x^i) = \sum_a m_a \tilde{m}_a \text{ satisfies (2.10). The extension } \{\tilde{m}_k\}_{k=0}^{K+2} \text{ leads to the extended sequence } \{\tilde{u}_k(\Sigma_k)\}_{k=0}^{K+2}. \text{ Property (2.8), continuity of the solutions } \{\tilde{u}_k(\Sigma_k)\}_{k=0}^{K+2} \text{ w.r.t. } \Sigma_k, \text{ and (2.9) imply that there exists some } \Sigma_x \in \mathcal{S}_x^+ \text{ s.t. the corresponding Riesz functional } \tilde{L}_u(p) = \sum_{|i| \leq K+2} a_i \tilde{u}_i(\Sigma_x) > 0 \text{ for all } p = \sum_{|i| \leq K+2} a_i x^i \in \mathcal{P}_{K+2} \text{ such that } p > 0. \text{ Due to continuity of } \mathcal{L}(X), \text{ } P(X = x_0) = 0 \text{ for any } x_0 \in \mathbb{R}^p \text{ such that } p(x_0) = \sum_{|i| \leq K+2} a_i x_0^i = 0., \text{ which finishes the proof. } \blacksquare
3. Validity and accuracy of the weighted bootstrap

3.1. Weighted bootstrap for \( \|S_n\| \)

Consider independent random vectors \( X_1, \ldots, X_n \in \mathbb{R}^p \) with \( E X_i = 0, \text{Var}(X_i) \in S_p^+ \), and \( E(\|X_i\|^4) < \infty \). The bootstrap random weights \( \varepsilon_1, \ldots, \varepsilon_n \), are taken as in (1.1). Below are some examples of such random weights (here \( z_i \sim \mathcal{N}(0,1), \) independent of \( e_i, c_i, b_i \) given below):

- \( \sqrt{1 - 2^{-2/3} z_i + 2^{-1/3} (e_i - 1)}, \) for \( e_i \sim \exp(1) \);
- \( 1/2 z_i + 1/2 (c_i - 1), \) for \( c_i \sim \chi_1^2 \);
- \( \sqrt{1 - 3^{-2/3} z_i + 2^{3/4} (b_i - 0.5)}, \) for \( b_i \sim \text{Bernoulli}(0.5) \).

More examples of the bootstrap weights satisfying (1.1) can be found in the works of [26] and [30].

The bootstrap approximation of the sum \( S_n \) is its weighted version \( S^n_{ab} \) defined in (1.2). The probability distribution of \( S^n_{ab} \) is taken conditioned on \( \{X_i\}_{i=1}^n \).

Theorems 3.1 and 3.2 show that the bootstrap quantile function \( Q^n_{ab}(\alpha) \) of \( \|S^n_{ab}\| \) defined in (1.3) is a rather good approximation of the true unknown quantile function \( Q(\alpha) \):

\[
Q(\alpha) \overset{\text{def}}{=} \inf \{ t \in \mathbb{R} : P(\|S_n\| > t) \leq \alpha \} \text{ for } \alpha \in (0,1).
\]

In the statements in Section 3, including the theorems presented below, we use notation from the previous Section 2, in particular, values \( M, \tilde{C}_B, C_\phi, C_z \) were introduced before Theorem 2.1.

**Theorem 3.1.** Consider the random vectors \( \{X_i\}_{i=1}^n \) introduced above. Suppose that they are i.i.d., and that there exist i.i.d. approximating random vectors \( \{Y_i\}_{i=1}^n \) meeting conditions (2.1) and (2.2) for \( K = 4 \). It holds for \( \alpha \in (0,1) \)

\[
|P(\|S_n\| > Q^n_{ab}(\alpha)) - \alpha| \leq \Delta_1 + \Delta_{1,c},
\]

where

\[
\Delta_1 \overset{\text{def}}{=} 2MC_\phi C_z^2 \left( E(\|X_1\|^4) E(\varepsilon_1^4) + E(\|Y_1\|^4) \right)^{1/2}/n^{1/2},
\]

\[
\Delta_{1,c} = \begin{cases} 0, & \text{if } \|S_n\| \text{ and } \|S^n_{ab}\| \text{ are continuously distributed}; \\ \Delta_1(1 + 2\tilde{C}_B C_z), & \text{otherwise}. \end{cases}
\]

**Theorem 3.2.** Let the conditions of Theorem 3.1 be fulfilled, except that \( \{X_i\}_{i=1}^n \) and \( \{Y_i\}_{i=1}^n \) are not assumed to be identically distributed. It holds for \( \alpha \in (0,1) \)

\[
|P(\|S_n\| > Q^n_{ab}(\alpha)) - \alpha| \leq \Delta_2 + \Delta_{2,c},
\]

where

\[
\Delta_2 \overset{\text{def}}{=} 2MC_\phi C_z^2 \left( E(\|X_1\|^4) E(\varepsilon_1^4) + E(\|Y_1\|^4) \right)^{1/2}/n^{1/2},
\]

\[
\Delta_{2,c} = \Delta_2(1 + 2\tilde{C}_B C_z),
\]
where
\[
\Delta_2 \overset{\text{def}}{=} 2.8 \left\{ C_{\phi,2} \overline{C}_\phi \right\}^{1/5} \left\{ \frac{C_4}{n} \sum_{i=1}^{n} \left[ E \left( \|X_{i}\|^4 \right) E \left( \|Y_{i}\|^4 \right) \right] \right\}^{1/5},
\]
\[
\Delta_{2,c} \overset{\text{def}}{=} \begin{cases} 0, & \text{if } \|S_n\| \text{ and } \|S_a\| \text{ are continuously distributed;} \\ \Delta_2 (1 + 2 \overline{C}_\phi C_4 z), & \text{otherwise.} \end{cases}
\]

### 3.2. Weighted bootstrap for linear regression model

Let \( y \overset{\text{def}}{=} (y_1, \ldots, y_n) \top \in \mathbb{R}^n \) be a data sample with
\[
y_i = \Psi_i \top \theta^* + \epsilon_i,
\]
(3.2)

\( \theta^* \in \mathbb{R}^p \) is unknown, \( \Psi_i \in \mathbb{R}^p \) are deterministic regressors such that the matrix \( \Psi \Psi^\top \) is invertible, where \( \Psi \overset{\text{def}}{=} (\Psi_1, \ldots, \Psi_n) \in \mathbb{R}^{p \times n} \). The random errors \( \epsilon_1, \ldots, \epsilon_n \) are independent, \( E \epsilon_i = 0 \), \( E (\epsilon_i^4) < \infty \), for all \( i = 1, \ldots, n \), and \( \text{Var}(\epsilon_i) > 0 \) are unknown. Let \( \epsilon \overset{\text{def}}{=} (\epsilon_1, \ldots, \epsilon_n) \top \). The least squares estimate of the parameter \( \theta^* \) reads as
\[
\tilde{\theta} = (\Psi \Psi^\top)^{-1} \Psi y = \theta^* + (\Psi \Psi^\top)^{-1} \Psi \epsilon.
\]

Consider the normalized quadratic loss
\[
T^2 \overset{\text{def}}{=} \left\| (\Psi \Psi^\top)^{1/2} (\tilde{\theta} - \theta^*) \right\|^2 = \left\| (\Psi \Psi^\top)^{-1/2} \sum_{i=1}^{n} \Psi_i \epsilon_i \right\|^2.
\]

The weighted bootstrap estimate of \( T \) can be written as
\[
T^\sigma = \left\| (\Psi \Psi^\top)^{-1/2} \sum_{i=1}^{n} \Psi_i \epsilon_i \right\|,
\]
(3.3)

where the random i.i.d. bootstrap weights \( \epsilon_1, \ldots, \epsilon_n \) are independent of \( \{ \epsilon_i \}_{i=1}^{n} \) and meet conditions (1.1). For the linear regression model the weighted bootstrap estimate given in (3.3) coincides with the wild bootstrap estimate introduced by [50] (see also [26, 18, 30]).

Denote similarly to \( Q_{T}^\sigma(\alpha) \) in (1.3)
\[
Q_{T}^\sigma(\alpha) \overset{\text{def}}{=} \inf \{ t \in \mathbb{R} : P^\sigma (T > t) \leq \alpha \},
\]
where \( \alpha \in (0, 1) \), and \( P^\sigma(\cdot) \overset{\text{def}}{=} P(\cdot | \epsilon_1, \ldots, \epsilon_n) \). Let \( Y_1, \ldots, Y_n \) be independent approximating vectors meeting conditions (2.1) and (2.2) for \( K = 4 \) and for \( X_i := (\Psi \Psi^\top)^{-1/2} \Psi_i \epsilon_i \). Applying Theorem 3.2 to \( T \) and \( Q_{T}^\sigma(\alpha) \), we obtain the following confidence statement

**Theorem 3.3.** Consider the linear regression model (3.2) with random errors \( \{ \epsilon_i \} \) and approximating vectors \( \{ Y_i \}_{i=1}^{n} \) described above. It holds
\[
\left| P\left( \| (\Psi \Psi^\top)^{1/2} (\tilde{\theta} - \theta^*) \| > Q_{T}^\sigma(\alpha) \right) - \alpha \right| \leq \Delta_{2,T} + \Delta_{2,T,c},
\]

where
\[
\Delta_{2,T} \overset{\text{def}}{=} 2.8 \left\{ C_{\phi,2} \overline{C}_\phi \right\}^{1/5} \left\{ \frac{C_4}{n} \sum_{i=1}^{n} \left[ E \left( \|X_{i}\|^4 \right) E \left( \|Y_{i}\|^4 \right) \right] \right\}^{1/5},
\]
\[
\Delta_{2,T,c} \overset{\text{def}}{=} \begin{cases} 0, & \text{if } \|S_n\| \text{ and } \|S_a\| \text{ are continuously distributed;} \\ \Delta_{2,T} (1 + 2 \overline{C}_\phi C_4 z), & \text{otherwise.} \end{cases}
\]
where

\[ \Delta_{2,T} \overset{\text{def}}{=} 2.8 \left( C_{\varphi,2} \bar{C}_{\varphi} \right)^{1/5} \left\{ \frac{\mathbb{C}_2 \sum^n_{i=1} \mathbb{E} \left( \| (\Psi \Psi^T)^{-1/2} \psi_i \|_4^4 \right) \mathbb{E}(\varepsilon_i^4) \mathbb{E}(\varepsilon_i^4)}{n^2} \right\}^{1/5}, \]

\[ \Delta_{2,T,c} \overset{\text{def}}{=} \begin{cases} 0, & \text{if } T \text{ and } T^* \text{ are continuously distributed;} \\ \Delta_{2,T}(1 + 2\bar{C}_{\varphi}\bar{C}_z), & \text{otherwise}. \end{cases} \]

### 3.3. Weighted bootstrap for log-likelihood ratio statistics

Here we consider a weighed bootstrap procedure for estimation of quantiles of log-likelihood ratio statistics. Before describing the procedure and formulating the theoretical result, we give some necessary definitions.

Let \( y = (y_1, \ldots, y_n) \) denote the data sample, \( y_1, \ldots, y_n \) are i.i.d. random observations from a probability space \((\Omega, \mathcal{F}, P)\). Introduce some known parametric family \( \{P_{\theta}\} \overset{\text{def}}{=} \{P_{\theta} \ll \mu_0, \theta \in \Theta \subset \mathbb{R}^p\} \), here \( \mu_0 \) is a \( \sigma \)-finite measure on \((\Omega, \mathcal{F})\), which dominates all \( P_{\theta} \) for \( \theta \in \Theta \). The true data distribution \( P \) is not assumed to belong to the family \( \{P_{\theta}\} \), thus our analysis includes the case when the parametric family \( \{P_{\theta}\} \) is misspecified. \( \{P_{\theta}\} \) induces the following (quasi)log-likelihood function for the sample \( y \):

\[ L(\theta) = L(\theta, y) \overset{\text{def}}{=} \log \left( \frac{dP_{\theta}}{d\mu_0}(y) \right). \]

The target parameter \( \theta^* \) is defined by projecting the true probability distribution \( P \) on the parametric family \( \{P_{\theta}\} \), using Kullback-Leibler divergence:

\[ \theta^* \overset{\text{def}}{=} \arg\min_{\theta \in \Theta} \text{KL}(P, P_{\theta}) = \arg\max_{\theta \in \Theta} \mathbb{E}L(\theta). \]

The (quasi) maximum likelihood estimate (MLE) is defined as

\[ \tilde{\theta} \overset{\text{def}}{=} \arg\max_{\theta \in \Theta} L(\theta). \]

Let \( Q_L(\alpha) \) denote the upper quantile function of square root of the two times log-likelihood ratio statistic:

\[ Q_L(\alpha) \overset{\text{def}}{=} \inf \left\{ t \geq 0 : P \left( L(\tilde{\theta}) - L(\theta) > t^2/2 \right) \leq \alpha \right\}. \]

\( Q_L(\alpha) \) is a critical value of the likelihood-based confidence set \( \mathcal{E}(\alpha) \):

\[ \mathcal{E}(t) \overset{\text{def}}{=} \left\{ \theta : L(\tilde{\theta}) - L(\theta) \leq t^2/2 \right\}, \quad \mathcal{P} \{ \theta^* \in \mathcal{E}(Q_L(\alpha)) \} > 1 - \alpha. \]
Distribution of $L(\tilde{\theta}) - L(\theta^*)$ depends on the unknown parameter $\theta^*$ and $P$. Hence, in general, quantiles of $L(\theta) - L(\theta^*)$ are also unknown.

Consider the weighted (or the multiplier) bootstrap procedure, which allows to estimate the distribution of $L(\tilde{\theta}) - L(\theta^*)$. Let $u_1, \ldots, u_n$ be i.i.d. random variables:

$$u_i \overset{\text{def}}{=} \varepsilon_i + 1, \text{ for } \varepsilon_i \text{ defined in (1.1), independent of } y.$$  

The bootstrap log-likelihood function $L^a(\theta)$ equals to the initial one $L(\theta)$ weighted with the random bootstrap weights $u_i$:

$$L^a(\theta) \overset{\text{def}}{=} \sum_{i=1}^n \log \left( \frac{dP_{\theta}}{d\mu_0}(y_i) \right) u_i.$$  

Let $P^a(\cdot) \overset{\text{def}}{=} P(\cdot | y)$ and $E^a(\cdot) \overset{\text{def}}{=} E(\cdot | y)$. It holds $E^a L^a(\theta) = L(\theta)$, therefore,

$$\tilde{\theta} = \arg\max_{\theta \in \Theta} L(\theta) = \arg\max_{\theta \in \Theta} E^a L^a(\theta),$$  

and the MLE $\tilde{\theta}$ can be considered as a bootstrap analogue of the unknown target parameter $\theta^*$. The bootstrap likelihood ratio statistic is defined as

$$L^a(\tilde{\theta}^a) - L^a(\tilde{\theta}) \overset{\text{def}}{=} \sup_{\theta \in \Theta} L^a(\theta) - L^a(\tilde{\theta}).$$  

$L^a(\tilde{\theta}^a) - L^a(\tilde{\theta})$ can be computed for each i.i.d. sample of the bootstrap weights $u_1, \ldots, u_n$, thus we can calculate empirical probability distribution function of $L^a(\tilde{\theta}^a) - L^a(\tilde{\theta})$ and estimate its quantiles. Denote

$$Q_L^a(\alpha) \overset{\text{def}}{=} \inf \left\{ t \geq 0 : P^a \left( L^a(\tilde{\theta}^a) - L^a(\tilde{\theta}) > t^2 / 2 \right) \leq \alpha \right\}. \quad (3.5)$$

Theorem 3.4 below provides a two-sided bound on the coverage error of the likelihood confidence set $Q_L^a(\alpha)$ based on the bootstrap quantile $Q_L^a(\alpha)$. Let us introduce some additional notation before stating the theorem. Denote $\ell_i(\theta) \overset{\text{def}}{=} \log \left( \frac{dP_{\theta}}{d\mu_0}(y_i) \right)$, $d_0^a \overset{\text{def}}{=} -E\ell_i'(\theta^*)$, here $\ell_i'(\theta) \overset{\text{def}}{=} \nabla_\theta \ell_i(\theta)$. Take $X_i := d_0^{-1} \ell_i'(\theta^*)$.

By previous definitions, such defined $\{X_i\}_{i=1}^n$ are i.i.d with zero mean. Moreover, if conditions from Section B.1 are fulfilled, then $E(\|X_i\|^4) < \infty$. Let $Y_1, \ldots, Y_n$ be i.i.d. vectors meeting conditions (2.1) and (2.2) for $K = 4$, and $C_{zL} \overset{\text{def}}{=} \|\{\text{Var}(Z_i)\}^{-1/2}\|$. Now we are ready to formulate the following

**Theorem 3.4.** If the conditions from Section B.1 are fulfilled, then it holds with probability $\geq 1 - 10e^{-x}$

$$|P \{ \theta^* \notin \delta(\tilde{Q}_L^a(\alpha)) \} - \alpha | \leq \Delta_L + \Delta_{L,c},$$

where

$$\Delta_L \leq 2MC_{zL}C_{\phi}C_{zL}^2 \left\{ E \left( \left\| d_0^{-1} \ell_i'(\theta^*) \right\|^4 \right) E(\varepsilon_i^4) + E \left( \left\| Y_i \right\|^4 \right) \right\}^{1/2} \quad (3.6)$$

$$+ C_{zL}C_{zL}(p + x)n^{-1/2}.$$
here $C$ is a generic constant $\geq 0$; a more detailed definition of the error term $\Delta_L$ is given in (B.8), Section B;

$$\Delta_{L,c} \overset{\text{def}}{=} \begin{cases} 0, & \text{if } L(\tilde{\theta}) - L(\theta^*) \text{ and } L^o(\tilde{\theta}) - L^o(\tilde{\theta}) \\
2\Delta_L, & \text{otherwise.} \end{cases}$$

Remark 3.1. The second term in the bound (3.6) comes from Wilks-type approximations for the likelihood ratios $L(\tilde{\theta}) - L(\theta^*)$ and $L^o(\tilde{\theta}) - L^o(\tilde{\theta})$ (see the proof in Section B.1 for more detail); the first term in (3.6) comes from Berry-Esseen type inequality for i.i.d summands (Theorem 2.1), and is similar to the error term in Theorem 3.1. The imposed conditions in Section B.1 include sub-exponential tail behavior of $\varepsilon_i$ and $d_0^{-1}\ell'_i(\theta^*)$, therefore, the first term (3.6) is bounded from above with $Cpn^{-1/2}$ on a set of exponentially large probability. Thus, in Theorem 3.4 both Wilks-type bound and Berry-Esseen type inequality yield similar impacts of $p$ and $n$ in the error of approximation $\Delta_L$.

4. Numerical experiments

This section presents results of simulation studies, illustrating accuracy of the considered Berry-Esseen bounds and weighted bootstrap procedure.

4.1. Berry-Esseen inequality

Figure 1 shows c.d.f.-s of $S_n$, $\tilde{S}_n$ and $\mathcal{N}(0, 1)$ for sample size $n = 50$, dimension $p = 1$ and number $K - 1 = 3$ of equal moments of $S_n$ and $\tilde{S}_n$. Similarly Figure 2 shows c.d.f.-s of $\|S_n\|^2$, $\|\tilde{S}_n\|^2$ and $\chi^2_p$ for $n = 50$, $p = 7$ and $K - 1 = 3$. Distributions of $X_i$ and $Y_i$ are described in the bottom of each of the Figures 1 and 2. The c.d.f.-s are obtained from $15 \cdot 10^3$ i.i.d. samples. Both figures agree with the theoretical results about the higher order Berry-Esseen bounds: the latter approximation has a better accuracy than the Gaussian one.

4.2. Bootstrap

Here we examine accuracy of the weighted bootstrap for $\|S_n\|$ (described in Section 3) by computing coverage probabilities using bootstrap quantiles $Q^o(\alpha)$. All the results are collected in Table 1. Columns $n$, $p$, $\mathcal{L}(\varepsilon_i)$, $\mathcal{L}(X_{i,j})$ show the sample size, dimension, distribution of the bootstrap weights $\varepsilon_i$, and distribution of $X_{i,j}$, where i.i.d. coordinates $X_{i,j}$ are s.t. $X_i = (X_{i,1}, \ldots, X_{i,p})^\top$. Nominal coverage probabilities $1 - \alpha$ are given in the second row 0.975, 0.95, 0.90, 0.85, ..., 0.50. All the rest numbers represent frequencies of the event $\{\|S_n\| \leq Q^o(\alpha)\}$, computed for different $n$, $p$, $\alpha$, $\mathcal{L}(\varepsilon_i)$, and $\mathcal{L}(X_{i,j})$, from $7 \cdot 10^3$ i.i.d. samples $\{X_i\}_{i=1}^n$ and $\{\varepsilon_i\}_{i=1}^n$. We consider two types of the bootstrap weights: first one $\varepsilon_i = z_i + u_i$, with $u_i \sim (Bernoulli(b) - b)\sigma_u$, $b = 0.276, \sigma_u \approx$
2.235, and \( z_i \sim \mathcal{N}(0, \sigma_z^2) \), \( \sigma_z^2 \approx 0.038 \), for this case \( \mathbf{E}\varepsilon_i = 0 \), \( \mathbf{E}(\varepsilon_i^2) = \mathbf{E}(\varepsilon_i^2) = 1 \), therefore \( \varepsilon_i \) meet conditions (1.1). The second type is \( \varepsilon_i \sim \mathcal{N}(0, 1) \), in this case \( \mathbf{E}(\varepsilon_i^3) \neq 1 \), and an approximation accuracy corresponds to classical normal approximation with a larger error term. In this numerical experiment we check, whether the additional condition \( \mathbf{E}(\varepsilon_i^3) = 1 \) improves numerical performance of the weighted bootstrap for \( \|S_n\| \). Table 1 confirms this property for most of the computed coverage probabilities, which agrees with the theoretical results.

Appendix A: Proof of the Non-Classical Berry-Esseen inequality

In this section we prove Theorems 2.1, 2.2, and 2.3. In Section A.1 we provide some bounds on Gaussian surface area of Euclidean balls and ellipsoids in \( \mathbb{R}^p \), these bounds are used in the proofs of the theorems from Sections 2 and 3.

A.1. Gaussian surface area of ellipsoids in \( \mathbb{R}^p \)

In this section we collect bounds on Gaussian surface area (GSA) of Euclidean balls and ellipsoids in \( \mathbb{R}^p \). These bounds are required for the proofs of Theorems 2.1 and 2.2. The following lemma shows that the GSA of any ball or an ellipsoid in \( \mathbb{R}^p \) is bounded with a constant independent of dimension \( p \). This properties are well known (see the works [39, 2, 24, 23]), we give here the proof for the sake of the text’s completeness.
Fig 2. Distribution functions of $\|S_n\|^2$ and $\|\hat{S}_n\|^2$ for $n = 50$, $p = 7$, $K = 4$.

- c.d.f. of $\chi^2_{n,p}$.
- c.d.f. of $\|S_n\|^2$ for $X_i = (X_{i,1}, \ldots, X_{i,p})^T$, $X_{i,j}$ are i.i.d.,
  $X_{i,j} \sim \ln N(0, 1) - 1.649/2.161$;
- c.d.f. of of $\|\hat{S}_n\|^2$ for $Y_i = (Z_{i,1} + U_{i,1}, \ldots, Z_{i,p} + U_{i,p})^T$, $U_{i,j}$ are i.i.d.,
  $U_{i,j} \sim \text{Pareto}(0.5, 4.1) - 0.661 \cdot 4.334$.

### Table 1

| $n$ | $p$ | $\mathcal{L}(X_{i,j})$ | $\mathcal{L}(z_i)$ | Coverage levels |
|-----|-----|----------------------|-------------------|-----------------|
| 400 | 40  | $\chi^2_1 - 1$       | $\mathcal{L}(z_i + u_i)$ | 0.975 | 0.95 | 0.90 | 0.85 | 0.80 | 0.70 | 0.60 | 0.50 |
|     |   | Pareto*              | $\mathcal{L}(z_i)$ | 0.982 | 0.957 | 0.910 | 0.855 | 0.804 | 0.701 | 0.595 | 0.491 |
|     |   | $\ln N^*(2.5)$      | $\mathcal{L}(z_i + u_i)$ | 0.984 | 0.960 | 0.914 | 0.862 | 0.810 | 0.704 | 0.597 | 0.495 |

| 150 | 15  | $\chi^2_1 - 1$       | $\mathcal{L}(z_i + u_i)$ | 0.986 | 0.958 | 0.907 | 0.855 | 0.807 | 0.703 | 0.596 | 0.492 |
|     |   | Pareto*              | $\mathcal{L}(z_i)$ | 0.986 | 0.965 | 0.915 | 0.863 | 0.811 | 0.706 | 0.595 | 0.485 |
|     |   | $\ln N^*(2.5)$      | $\mathcal{L}(z_i + u_i)$ | 0.985 | 0.967 | 0.926 | 0.869 | 0.823 | 0.723 | 0.584 | 0.449 |

| 50  | 5   | $\chi^2_1 - 1$       | $\mathcal{L}(z_i + u_i)$ | 0.985 | 0.963 | 0.906 | 0.850 | 0.794 | 0.694 | 0.587 | 0.483 |
|     |   | Pareto*              | $\mathcal{L}(z_i)$ | 0.988 | 0.970 | 0.917 | 0.861 | 0.801 | 0.692 | 0.578 | 0.467 |
|     |   | $\ln N^*(1.5)$      | $\mathcal{L}(z_i + u_i)$ | 0.984 | 0.964 | 0.914 | 0.858 | 0.794 | 0.654 | 0.515 | 0.388 |

Here Pareto* and $\ln N^*(\sigma^2)$ denote zero mean distributions Pareto(0.5, 4.1) $- 0.661$
and $\ln N(0, \sigma^2) - e^{\sigma^2/2}$ correspondingly.
Lemma A.1 (GSA of Euclidean balls and ellipsoids). There exists a generic constant \( \tilde{C}_\mathcal{B} > 0 \) such that for all \( a \in \mathbb{R}^p \) and \( r \geq 0 \)

\[
\int_{\|x-a\|=r} \phi(x)dx \leq \tilde{C}_\mathcal{B}, \quad (A.1)
\]

where \( \phi(x) \overset{\text{def}}{=} (2\pi)^{-p/2}e^{-\|x\|^2/2} \) is the standard normal density. Moreover, for any matrix \( \Sigma \in S_+^p \) and \( \phi_\Sigma(x) \overset{\text{def}}{=} \{(2\pi)^p \det(\Sigma)\}^{-1/2} e^{-\|\Sigma^{-1/2}x\|^2/2} \), it holds

\[
\int_{\|x\|=r} \phi_\Sigma(x+a)dx \leq \frac{\tilde{C}_\mathcal{B}}{\sqrt{\lambda_{\min}}},
\]

where \( \lambda_{\min} > 0 \) is the smallest eigenvalue of the covariance matrix \( \Sigma \).

Proof of Lemma A.1. At first let us consider the case \( a = 0 \).

\[
\int_{\|x\|=r} \phi(x)dx = \frac{1}{(2\pi)^{p/2}} \int_{\|x\|=r} dx = \frac{1}{(2\pi)^{p/2}} e^{-r^2/2} \cdot r^{p-1} \pi^{p/2} \int_0^\infty \frac{1}{\Gamma(p/2 + 1)} \leq \text{const.} \quad (A.2)
\]

Expression (A.2) is maximized in \( r = \sqrt{p-1} \), which implies (A.3).

Now consider the balls with an arbitrary center \( a \in \mathbb{R}^p \). Let \( Z \sim \mathcal{N}(0, I_p) \), the following expression corresponds to the density function of the r.v. \( \|Z+a\|^2 \), which follows the noncentral chi-squared distribution:

\[
f(x; p, a) = \sum_{k=0}^{\infty} \frac{e^{-\|a\|^2/2}([\|a\|^2/2]_k)^k}{k!} f(x; p + 2k, 0), \quad (A.4)
\]

where \( f(x; p + 2k, 0) \) is a probability density function of the chi-squared distribution with \( p + 2k \) degrees of freedom. Equation (A.4) together with the bound (A.3) imply inequality (A.1).

For the case of ellipsoids with an arbitrary center \( a \in \mathbb{R}^p \) we assume w.l.o.g. that the covariance matrix \( \Sigma \) is diagonal. Let \( Z \sim \mathcal{N}(0, I_p) \). Due to the results of [36] and [19] the c.d.f. of the random variable \( \|\Sigma^{1/2}Z + a\|^2 \) (which is a weighted sum of independent noncentral chi-squared r.v.) reads as

\[
F(x; p, a, \Sigma) = \sum_{k=1}^{\infty} q_k F(x/\lambda_{\min}; p + 2k, 0, I_p), \quad (A.5)
\]

where \( F(x; p + 2k, 0, I_p) \) is a c.d.f. of a chi-squared distribution with \( p + 2k \) degrees of freedom, \( c \) is a positive number, and \( q_k = q_k(p, a, \Sigma) > 0 \) are coefficients which depend on \( p, a \) and \( \Sigma \), s.t. \( \sum_{k=1}^{\infty} q_k = 1 \). Equation (A.5) together with bound (A.3) complete the proof. \( \square \)
Consider an arbitrary Euclidean ball \( B = B_r(x_0) \) defined as \( \{ x \in \mathbb{R}^p : \|x - x_0\| \leq r \} \).

Denote for some number \( \varepsilon \in \mathbb{R} \)
\[
B^\varepsilon \text{ def}= \begin{cases} B_{r+\varepsilon}(x_0), & \text{if } r + \varepsilon \geq 0, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{A.6}
\]

Lemma A.1 implies the following statement.

**Lemma A.2.** Let \( Z \sim \mathcal{N}(a, \Sigma_z) \) for some \( a \in \mathbb{R}^p \) and \( \Sigma_z \in S_p^+ \). It holds for \( C_z \)\( \text{def}= \|\Sigma_z^{-1/2}\| \) and for arbitrary \( \varepsilon > 0 \)
\[
\sup_{B \in \mathcal{B}} \mathbf{P}(Z \in B^\varepsilon \setminus B) \leq \varepsilon \tilde{C}_z C_z.
\]

### A.2. Auxiliary statements and proof of Theorem 2.1

Here we extend the proof of [4] to our setting of “quasi-Gaussian” higher order approximation. The proof uses smoothing of a characteristic function of a Euclidean ball, and induction w.r.t. \( n \). Let us introduce some necessary statements before proving Theorem 2.1. Lemma A.4 follows from a more general Lemma 2.1 in the work of [4], and we do not give its proof here.

**Taylor’s formula**

Below is the Taylor’s formula, which will be used further in the proof: for a sufficiently smooth function \( f : \mathbb{R}^p \mapsto \mathbb{R} \) and \( x, h \in \mathbb{R}^p \)
\[
f(x + h) = f(x) + f'(x)h + \cdots + \frac{1}{s!} f^{(s)}(x) h^s \\
+ \frac{1}{s!} \mathbb{E}(1 - \tau)^{s} f^{(s+1)}(x + \tau h) h^{s+1}, \tag{A.7}
\]

here we use the notation \( f^{(s)}(x) h^s = (h^\top \nabla)^s f(x) \); \( \tau \) is a random variable uniformly distributed on the interval \([0, 1]\), independent of all other random variables; the above formula for the remainder term follows directly from the remainder’s integral form.

**Lemma A.3** (Properties of a smoothing function \( \varphi \)). Take an arbitrary Euclidean ball \( B \) in \( \mathbb{R}^p \), then for any \( \varepsilon > 0 \) there exists a function \( \varphi \) (which depends only on \( \varepsilon \) and \( B \)) such that
\[
0 \leq \varphi(x) \leq 1, \quad \varphi(x) = \begin{cases} 1, & x \in B; \\ 0, & x \notin B^\varepsilon, \end{cases} \tag{A.8}
\]
and for all \(x, h \in \mathbb{R}^p\), and some constants \(\tilde{C}_\phi, \tilde{C}_{\phi,1} > 0\)

\[
\begin{align*}
|\varphi^{(K-2)}(x)h^{K-2}| & \leq \frac{\tilde{C}_\phi \|h\|^{K-2}}{(\varepsilon/2)^{K-2}} \mathbb{I}\{x \in B^c \setminus B\}, \text{ and} \\
|\{\varphi^{(K-2)}(x) - \varphi^{(K-2)}(y)\}h^{K-2}| & \leq \|x - y\|\frac{\tilde{C}_{\phi,1} \|h\|^{K-2}}{(\varepsilon/2)^{K-1}} (\mathbb{I}\{x \in B^c \setminus B\} + \mathbb{I}\{y \in B^c \setminus B\}).
\end{align*}
\] (A.9)

Furthermore, we can choose \(\varphi\) to have the form

\[
\varphi(x) = \phi(\tilde{\rho}(x)/\varepsilon),
\] (A.11)

where \(\phi : \mathbb{R} \mapsto [0, 1]\) is \(K - 1\) times continuously differentiable non-negative non-increasing function such that

\[
\int_{\mathbb{R}} |\phi'(t)| \, dt = 1,
\] (A.12)

function \(\tilde{\rho}(x) : \mathbb{R}^p \mapsto \mathbb{R}\) and number \(\varepsilon > 0\) are such that

\[
\tilde{\rho}(x) = 0 \text{ for } x \in B, \quad \varepsilon > 0 \text{ for } x \notin B, \\
\tilde{\rho}(x)/\varepsilon \leq 1 \text{ for } x \in B^c, \quad \varepsilon \in (0, B^c).
\]

**Proof of Lemma A.3.** Let \(\phi(t)\) be a sufficiently smooth approximation of a step function (e.g. based on higher-order polynomials)

\[
0 \leq \phi(x) \leq 1, \quad \phi(x) = \begin{cases} 1, & x \leq 0; \\ 0, & x \geq 1, \end{cases}
\] (A.13)

such that the lemma’s conditions are fulfilled. Let the ball \(B = B_r(x_0)\) have center \(x_0\) and radius \(r\). Function \(\tilde{\rho}(x)\) can be taken as follows

\[
\tilde{\rho}(x) = \begin{cases} \|x - x_0\|^2 - r^2, & x \notin B; \\ 0, & x \in B. \end{cases}
\]

Let also \(\varepsilon = \varepsilon^2 + 2r\varepsilon\), then \(\tilde{\rho}(x)/\varepsilon = 0\) for \(x \in B\), and

\[
0 < \tilde{\rho}(x)/\varepsilon \leq 1 \text{ for } x \in B^c \setminus B, \quad \tilde{\rho}(x)/\varepsilon > 1 \text{ for } x \in \mathbb{R}^p \setminus B^c.
\]

Properties (A.9) and (A.10) follow from the representation (A.11). Indeed, definition (A.13) implies that \(\phi'(x) \neq 0\) iff \(x \in (0, 1)\), therefore, \(\phi'(x) = \phi'(x) \mathbb{I}\{x \in (0, 1)\}\). Moreover

\[
\|\varphi'(x)\| \leq \|\phi'\|_{\infty} \frac{2\|x - x_0\|}{\varepsilon} \mathbb{I}\{\tilde{\rho}(x)/\varepsilon \in (0, 1)\} \\
\leq \frac{2\|\phi'\|_{\infty}}{\varepsilon} \mathbb{I}\{x \in B^c \setminus B\}.
\]

Property (A.9) is derived by further differentiation of \(\varphi'(x)\). Inequality (A.10) is derived similarly, using also Taylor’s formula. \(\square\)
Lemma A.4 (Smoothing lemma). Let \( \varepsilon \) be a positive number and \( B \in \mathcal{B} \). Let function \( \varphi : \mathbb{R}^p \to \mathbb{R} \) satisfy (A.8) from Lemma A.3. It holds for arbitrary random variables \( X, Y \in \mathbb{R}^p \):

\[
\sup_{B \in \mathcal{B}} |P(X \in B) - P(Y \in B)| \leq \sup_{B \in \mathcal{B}} |E\varphi(X) - E\varphi(Y)| + \sup_{B \in \mathcal{B}} P(Y \in B^c \setminus B),
\]

the ball \( B^c \) was introduced in (A.6).

Lemma A.5 (Some bounds used in the proof). Let random vector \( X \) be an i.i.d. copy of \( X_i \) from Theorem 2.1, let also \( \Sigma_z, C_z \) be as in Theorem 2.1, then the following properties hold

\[
\forall k \geq 2 \quad C_k^zE(\|X\|^k) \geq 1; \tag{A.14}
\]

\[
\forall 2 \leq j \leq k \quad C_j^zE(\|X\|^j) \leq C_k^zE(\|X\|^k); \tag{A.15}
\]

\[
\forall j \leq K - 1 \quad E(\|Z_0^T \Sigma_z^{-1/2}X\|j) \leq C_zE(\|X\|^{j+1}); \tag{A.16}
\]

\[
\forall 1 \leq j \leq K \quad E \int_{\mathbb{R}^p} \left| f_0^{(j)}(x) \left\{ \Sigma_z^{-1/2}X \right\}^j \right|^2 \, dx \leq \sqrt{j!}C_k^zE(\|X\|^j). \tag{A.17}
\]

where \( Z_0 \sim \mathcal{N}(0, I_p) \), independent from \( X \), and \( f_0(x) \overset{\text{def}}{=} (2\pi)^{-p/2}e^{-\|x\|^2/2} \) denotes p.d.f. of the multivariate standard normal distribution in \( \mathbb{R}^p \).

Proof of Lemma A.5. Bounds (A.14), (A.16) follow from Hölder’s and Cauchy-Schwarz inequalities. Indeed

\[
C_k^zE(\|X\|^k) = \|\Sigma_z^{-1/2}\|^kE(\|X\|^k)
\]

\[
\geq \|\Sigma_z^{-1/2}\|^kE(\|X\|^k)
\]

\[
\geq \{E(\|\Sigma_z^{-1/2}X\|^2)\}^{k/2} = p^{k/2} \geq 1;
\]

\[
E(\|Z_0^T \Sigma_z^{-1/2}X\|j) = E(\|X\|^jE(\|Z_0^T \Sigma_z^{-1/2}X\|X))
\]

\[
\leq E(\|X\|^jE(\|Z_0^T \Sigma_z^{-1/2}X\|X))
\]

\[
\leq E(\|X\|^{j+1})\|\Sigma_z^{-1/2}\|.
\]

Inequality (A.15) is implied by Hölder’s inequality and by the previous bound (A.14)

\[
C_k^zE(\|X\|^j) \leq \left\{ C_k^zE(\|X\|^k) \right\}^{j/k} \leq C_zE(\|X\|^k).
\]

Now we check the property (A.17). Let \( H_j(x) \) denote a multivariate Hermite polynomial defined by the multivariate analogue of Rodrigues’s formula \( f_0^{(j)}(x) = (-1)^j H_j(x) f_0(x) \). Then by the orthogonal property of Hermite polynomials (see e.g. [16]) and Hölder’s inequality, it holds

\[
E \int_{\mathbb{R}^p} \left| f_0^{(j)}(x)X \right|^2 \, dx \leq E \left\{ \int_{\mathbb{R}^p} \left| H_j(x)X \right|^2 f_0(x) \, dx \right\}^{1/2} \leq E(\|X\|^j) \sqrt{j}.
\]
Proof of Theorem 2.1. Denote

\[ \beta \overset{\text{def}}{=} E(\|X_1\|^K + \|Y_1\|^K). \]

With this notation, the theorem’s statement reads as

\[ \Delta_n \overset{\text{def}}{=} \sup_{B \in \mathcal{B}} \left| P(S_n \in B) - P(\tilde{S}_n \in B) \right| \leq MC_{\mathcal{B}} C_\phi \frac{\{C_z^K \beta\}^{1/(K-2)}}{n^{1/2}}. \]  

(A.18)

Below we use induction w.r.t. \( n \).

**Induction basis.** Bound (A.18) holds for all \( n \) s.t. \( 1 \leq n \leq n' \), where \( n' \geq 1 \) is some natural number.

Indeed, due to property (A.14), and since \( C_{\mathcal{B}}, C_\phi \geq 1 \), \( K \geq 3 \), it holds

\[ 1 \leq C_{\mathcal{B}} C_\phi \{C_z^K \beta/2\}^{1/(K-2)}, \]

therefore, since \( \Delta_n \leq 1 \), one can take \( n \leq n' = \left( MC_{\mathcal{B}} C_\phi \{C_z^K \beta/2\}^{1/(K-2)} \right)^2 \geq M^2 \) for the induction basis. In the end of the proof we show that \( M \geq 72.5 \).

In the next steps of the proof we consider \( n > n' \).

**Induction step**

Assume that the following bound holds for all \( l = 1, \ldots, n - 1 \):

\[ \Delta_l \leq MC_{\mathcal{B}} C_\phi \frac{\{C_z^K \beta/2\}^{1/(K-2)}}{l^{1/2}}. \]  

(A.19)

**Our goal is to show that** (A.19) **is also true for** \( l = n \). The first step is to apply the smoothing lemma A.4 to \( \Delta_n \). Let \( \varepsilon > 0 \) be some fixed number, it holds

\[ \Delta_n \overset{\text{def}}{=} \sup_{B \in \mathcal{B}} \left| P(S_n \in B) - P(\tilde{S}_n \in B) \right| \]

\[ \leq \sup_{B \in \mathcal{B}} \left| E\varphi(S_n) - E\varphi(\tilde{S}_n) \right| + \sup_{B \in \mathcal{B}} P(Y \in B^c \setminus B) \]

\[ \leq \sup_{B \in \mathcal{B}} \left| E\varphi(S_n) - E\varphi(\tilde{S}_n) \right| + \varepsilon C_{\mathcal{B}} C_z, \]  

(A.20)

where the function \( \varphi \) is taken from Lemmas A.3 and A.4, and the last inequality follows from Lemma A.2 and the following property of the approximating sum \( \tilde{S}_n \)

\[ \mathcal{L} \left( \tilde{S}_n - \frac{1}{\sqrt{n}} \sum_{i=1}^n U_i \bigg| U_1, \ldots, U_n \right) = \mathcal{N}(0, \Sigma_z). \]  

(A.21)
Below we represent the difference $\E \phi (S_n) - \E \phi (\tilde{S}_n)$ as a telescopic sum and divide this sum into four parts, then we derive a bound for each of this parts, and collect all the bounds together in the end of the proof. Let random vectors $X, Y, Z, U$ be i.i.d. copies of $X_i, Y_i, Z_i, U_i$ correspondingly. Denote 
\[ \overline{X} \overset{\text{def}}{=} \frac{X}{\sqrt{n}}, \quad \overline{X}_i \overset{\text{def}}{=} \frac{X_i}{\sqrt{n}}, \quad \overline{Y} \overset{\text{def}}{=} \frac{Y}{\sqrt{n}}, \quad \overline{Y}_i \overset{\text{def}}{=} \frac{Y_i}{\sqrt{n}} \] (A.22)

Let also 
\[ W_k \overset{\text{def}}{=} \sum_{i=1}^{k-1} \overline{Y}_i + \sum_{i=k+1}^{n} \overline{X}_i \quad \text{for} \quad k = 2, \ldots, n, \]
\[ W_1 \overset{\text{def}}{=} \sum_{i=1}^{n} \overline{Y}_i \quad \text{and} \quad W_n \overset{\text{def}}{=} \sum_{i=1}^{n} \overline{Y}_i = W_{n+1} + \overline{X}, \]

thus 
\[ S_n = \overline{X}_1 + W_1, \quad \tilde{S}_n = W_n + \overline{Y}_n, \quad W_k + \overline{Y} \overset{\text{w}}{=} W_{k+1} + \overline{X}, \]

here $W_k$ is independent of $\overline{X}_k, \overline{Y}_k$ and $\overline{X}, \overline{Y}$; we will use this property further in the proofs. Denote 
\[ \gamma_k \overset{\text{def}}{=} |\E \phi (W_k + \overline{X}) - \E \phi (W_k + \overline{Y})|, \] (A.23)

therefore 
\[
|\E \phi (S_n) - \E \phi (\tilde{S}_n)| \\
= |\E \phi (W_1 + \overline{X}) - \E \phi (W_n + \overline{Y})| \\
= |\E \phi (W_1 + \overline{X}) - \E \phi (W_1 + Y) + \E \phi (W_1 + \overline{Y}) - \E \phi (W_n + \overline{Y})| \\
= |\E \phi (W_1 + \overline{X}) - \E \phi (W_1 + \overline{Y}) + \E \phi (W_2 + \overline{X}) - \E \phi (W_n + \overline{Y})| \\
\leq \sum_{k=1}^{n} \gamma_k.
\]

Define $n_0 \overset{\text{def}}{=} [n/2]$, let $m$ be some natural number $2 \leq m \leq n_0$. Let us split the sum from the last inequality as follows: 
\[ \sum_{k=1}^{n} \gamma_k = \gamma_1 + \sum_{k=2}^{m} \gamma_k + \sum_{k=m+1}^{n_0} \gamma_k + \sum_{k=n_0+1}^{n} \gamma_k. \] (A.24)

**Induction step, part 1**

*We shall prove that*
\[
\gamma_1 \overset{\text{def}}{=} |\E \phi (W_1 + \overline{X}) - \E \phi (W_1 + \overline{Y})| \\
\leq \frac{1.5 C_\phi^2}{(K-2)!} \left\{ \frac{\beta}{(\varepsilon/2)^{(K-2)n(K-2)/2}} \left\{ 2MC_\phi C_\phi \left\{ \frac{C_\phi^K}{\sqrt{2\pi}} \right\}^{1/(K-2)} + \varepsilon C_\phi C_\phi \right\} \right\}. \] (A.25)
It holds
\[
E\varphi(W_1 + X) - \frac{1}{s!} \sum_{s=0}^{K-3} E \left\{ \varphi^{(s)}(W_1)X^s \right\} = \frac{1}{(K-3)!} E \left\{ (1 - \tau)^{K-3} \varphi^{(K-2)}(W_1 + \tau X)X^{K-2} \right\} \tag{A.26}
\]
\[
\leq \frac{\tilde{C}_\phi}{(K-3)!} \cdot 2^{K-2} E \left( \left\| X \right\|^K (1 - \tau)^{K-3} \mathbb{I} \{ W_1 + \tau X \in B^c \setminus B \} \right) \tag{A.27}
\]
\[
\leq \frac{\tilde{C}_\phi \cdot C_0^2}{(K-2)!} \cdot 2^{K-2} \cdot 2 \Delta_{n-1} + \varepsilon \alpha_1 C_{\theta} C_z \tag{A.28}
\]
\[
\leq \frac{\tilde{C}_\phi \cdot C_0^2}{(K-2)!} \cdot 2^{K-2} \cdot 2 \Delta_{n-1} + \varepsilon \alpha_1 C_{\theta} C_z \tag{A.29}
\]
where \( \alpha_1 \overset{\text{def}}{=} \sqrt{n/(n-1)} \); inequality (A.30) is implied by induction assumption (A.19), property (A.21) and Lemma A.2, indeed
\[
P \left( S_{n-1} \in B_{s\alpha_1} \right) - P \left( S_{n-1} \in B \right) \leq 2 \Delta_{n-1} + P \left( S_{n-1} \in B_{s\alpha_1} \setminus B \right) \tag{A.31}
\]
Similar bounds hold for \( Y \), therefore, using independence of \( W_1 \) of \( X \) and \( Y \), and condition \( E(X^k) = E(Y^k) \) \( \forall k = 0, \ldots, K-1 \), we infer
\[
\left| E\varphi(W_1 + X) - E\varphi(W_1 + Y) \right| \leq \frac{\tilde{C}_\phi \cdot C_0^2 \beta}{(K-2)!} \cdot 2^{K-2} \cdot 2 \Delta_{n-1} + \varepsilon C_{\theta} C_z \sqrt{n/(n-1)} \right\},
\]
which implies (A.25) due to induction assumption (A.19).
Induction step, part 2

Below we show that

$$
\gamma_2 + \cdots + \gamma_m
\leq 4\sqrt{2} \left( \frac{\sqrt{m-1} - 1}{(c/2)^{(K-1)}K!} \right) C_{\phi,1} C_z \frac{\beta}{n^{(K-1)/2}} \left( 2MC_{\phi,1}C_{\phi} \left\{ C_{\phi,1} \right\}^{1/(K-2)} \frac{1}{n^{1/2}} + \frac{M}{\sqrt{n}} \right).
$$

(A.32)

Let us fix some integer \( k \) s.t. \( 2 \leq k \leq n - 1 \). Introduce similarly to (A.22)

$$
Z = \frac{Z}{\sqrt{n}}, \quad U = \frac{U}{\sqrt{n}}, \quad Z_i = \frac{Z_i}{\sqrt{n}}, \quad U_i = \frac{U_i}{\sqrt{n}}.
$$

therefore, \( Y = Z + U \), \( Y_i = Z_i + U_i \), and

$$
\sum_{i=1}^{k-1} Y_i = \sum_{i=1}^{k-1} Z_i + \sum_{i=1}^{k-1} U_i = (k - 1)^{1/2} Z + \sum_{i=1}^{k-1} U_i.
$$

where \( Z_0 \sim \mathcal{N}(0, I_p) \), independent of all \( U_i, X_i, X \). Denote

$$
X_{\text{sum}} = \sum_{i=k+1}^{n} X_i, \quad U_{\text{sum}} = \sum_{i=1}^{k-1} U_i, \quad \frac{1}{\theta_k} \overset{\text{def}}{=} \sqrt{(k - 1)/n}.
$$

(A.33)

Using all the notation introduced above, we have:

$$
W_k + X \overset{\text{def}}{=} \Sigma_x^{1/2} Z_0 / \theta_k + U_{\text{sum}} + X_{\text{sum}} + X,
$$

$$
W_k + Y \overset{\text{def}}{=} \Sigma_x^{1/2} Z_0 / \theta_k + U_{\text{sum}} + X_{\text{sum}} + Y.
$$

(A.34)

Let \( f_0(x) \) denote p.d.f. of multivariate standard normal distribution: \( f_0(x) \overset{\text{def}}{=} (2\pi)^{-p/2}e^{-||x||^2/2} \). Due to (A.34) it holds for \( y = x/\theta_k + \Sigma_x^{-1/2} X \)

$$
E\varphi(W_k + X) = E\varphi(\Sigma_x^{1/2} Z_0 / \theta_k + U_{\text{sum}} + X_{\text{sum}} + X).
$$

$$
= E \int_{\mathbb{R}^p} \varphi(\Sigma_x^{1/2} y/\theta_k + U_{\text{sum}} + X_{\text{sum}} + X) f_0(x) dx
$$

$$
= \theta_k^p E \int_{\mathbb{R}^p} \varphi(\Sigma_x^{1/2} y + U_{\text{sum}} + X_{\text{sum}}) f_0(\theta_k y - \theta_k \Sigma_x^{-1/2} X) dy.
$$

Expand the density function \( f_0(\theta_k y - \theta_k \Sigma_x^{-1/2} X) \), conditioned on \( X \), by Taylor formula (A.7):

$$
f_0(\theta_k y - \theta_k \Sigma_x^{-1/2} X)
= f_0(\theta_k y) - \theta_k \Sigma_x^{-1/2} X E f_0'(\theta_k y - \tau \theta_k \Sigma_x^{-1/2} X)
= f_0(\theta_k y) + \theta_k^2 E (y - \tau \Sigma_x^{-1/2} X)^\top \Sigma_x^{-1/2} X f_0(\theta_k y - \tau \theta_k \Sigma_x^{-1/2} X).
$$
Consider

\[ I_0 \overset{\text{def}}{=} \theta_k^2 \mathbb{E} \int_{\mathbb{R}^p} \varphi\left(\Sigma_z^{1/2} y + \bar{U}_{\text{sum}} + \bar{X}_{\text{sum}}\right) f_0(\theta_k y) dy = E \varphi(\Sigma_z^{1/2} Z_0/\theta_k + \bar{U}_{\text{sum}} + \bar{X}_{\text{sum}}), \]  

(A.35)

\[ I_1 \overset{\text{def}}{=} \theta_k^2 \mathbb{E} \int_{\mathbb{R}^p} \varphi\left(\Sigma_z^{1/2} y + \bar{U}_{\text{sum}} + \bar{X}_{\text{sum}}\right) \theta_k^2 (y - \tau \Sigma_z^{-1/2} \bar{X})^\top \Sigma_z^{-1/2} \bar{X} \]

\[ \times f_0(\theta_k y - \tau \theta_k \Sigma_z^{-1/2} \bar{X}) dy = \theta_k E \left\{ \varphi(\Sigma_z^{1/2} Z_0/\theta_k + \bar{U}_{\text{sum}} + \bar{X}_{\text{sum}} + \tau \bar{X}) Z_0^\top \Sigma_z^{-1/2} \bar{X} \right\} \]

\[ = \theta_k E \left\{ (Z_0^\top \Sigma_z^{-1/2} \bar{X}) \sum_{j=1}^{N-1} \frac{1}{j!} \varphi^{(j)}(\Sigma_z^{1/2} Z_0/\theta_k + \bar{U}_{\text{sum}} + \bar{X}_{\text{sum}})(\tau \bar{X})^j \right\} \]

\[ + \frac{(1 - \tau_1)^{N-1}}{(N-1)!} (Z_0^\top \Sigma_z^{-1/2} \bar{X}) \varphi^{(N)}(\Sigma_z^{1/2} Z_0/\theta_k + \bar{U}_{\text{sum}} + \bar{X}_{\text{sum}} + \tau \bar{X})(\tau \bar{X})^N \right\}, \]  

(A.36)

where \( N \overset{\text{def}}{=} K - 2 \); in the last equation we again used Taylor formula (A.7) (\( \tau_1 \) is an i.i.d. copy of \( \tau \)) and also the property \( \mathbb{E} \bar{X} = 0 \) together with independence of \( \bar{X} \) from \( Z_0, \bar{U}_{\text{sum}}, \bar{Z} \). It holds

\[ \mathbb{E} \varphi(W_k + \bar{X}) = I_0 + I_1. \]

Analogous relations hold for the second term in \( \gamma_k \):

\[ \mathbb{E} \varphi(W_k + \bar{Y}) = I_0 + J_1, \]

for

\[ J_1 \overset{\text{def}}{=} \theta_k E \left\{ (Z_0^\top \Sigma_z^{-1/2} \bar{Y}) \sum_{j=1}^{N-1} \frac{1}{j!} \varphi^{(j)}(\Sigma_z^{1/2} Z_0/\theta_k + \bar{U}_{\text{sum}} + \bar{X}_{\text{sum}})(\tau \bar{Y})^j \right\} \]

\[ + \frac{(1 - \tau_1)^{N-1}}{(N-1)!} (Z_0^\top \Sigma_z^{-1/2} \bar{Y}) \varphi^{(N)}(\Sigma_z^{1/2} Z_0/\theta_k + \bar{U}_{\text{sum}} + \bar{X}_{\text{sum}} + \tau \bar{Y})(\tau \bar{Y})^N \right\}, \]

here we substituted \( \bar{X} \) with \( \bar{Y} \) in expressions (A.35), (A.36) for \( I_0, I_1 \), and \( I_0 \) remained the same.
Introduce

\[ I_2 = \theta_k E \left\{ (Z_0^\top \Sigma_z^{-1/2} \mathbf{X}) \sum_{j=1}^{N-1} \frac{1}{j!} \varphi^{(j)}(\Sigma_z^{1/2} Z_0/\theta_k + \mathcal{U}_{\text{sum}} + \mathbf{X}_{\text{sum}})(\tau \mathbf{X})^j \right\} \]

\[ + \left( \frac{1 - \tau_1}{N - 1} \right)^{N-1} \left( Z_0^\top \Sigma_z^{-1/2} \mathbf{X} \phi(\Sigma_z^{1/2} Z_0/\theta_k + \mathcal{U}_{\text{sum}} + \mathbf{X}_{\text{sum}})(\tau \mathbf{X})^N \right) \],

\[ J_2 = \theta_k E \left\{ (Z_0^\top \Sigma_z^{-1/2} \mathbf{Y}) \sum_{j=1}^{N-1} \frac{1}{j!} \varphi^{(j)}(\Sigma_z^{1/2} Z_0/\theta_k + \mathcal{U}_{\text{sum}} + \mathbf{X}_{\text{sum}})(\tau \mathbf{Y})^j \right\} \]

\[ + \left( \frac{1 - \tau_1}{N - 1} \right)^{N-1} \left( Z_0^\top \Sigma_z^{-1/2} \mathbf{Y} \phi(\Sigma_z^{1/2} Z_0/\theta_k + \mathcal{U}_{\text{sum}} + \mathbf{X}_{\text{sum}})(\tau \mathbf{Y})^N \right). \]

\[ I_2 = J_2 \] since \( \mathbf{X}, \mathbf{Y} \) are independent of \( Z_0, \mathcal{U}_{\text{sum}}, \mathbf{X}_{\text{sum}}, \tau \) and \( E(\mathbf{X}^k) = E(\mathbf{Y}^k) \) \( \forall k = 0, \ldots, K - 1 \). Therefore, we can write:

\[ \gamma_k = |I_0 + I_1 - I_0 - J_1| = |I_1 - I_2 + J_2 - J_1|. \]

Due to property (A.10) of function \( \varphi(x) \), it holds (since \( N = K - 2 \)):

\[ |I_1 - I_2| \]

\[ \leq \frac{\tilde{C}_\varphi, \theta_k}{(\varepsilon/2)^{N+1}} E \left[ \left( \frac{1 - \tau_1}{N - 1} \right)^{N-1} |Z_0^\top \Sigma_z^{-1/2} \mathbf{X}| \right] \]

\[ \times \left\{ P \left( \Sigma_z^{1/2} Z_0/\theta_k + \mathcal{U}_{\text{sum}} + \mathbf{X}_{\text{sum}} + \tau \tau_1 \mathbf{X} \in B^c \setminus B \mid Z_0, \mathcal{U}_{\text{sum}}, \mathbf{X}, \tau, \tau_1 \right) \]

\[ + P \left( \Sigma_z^{1/2} Z_0/\theta_k + \mathcal{U}_{\text{sum}} + \mathbf{X}_{\text{sum}} \in B^c \setminus B \mid Z_0, \mathcal{U}_{\text{sum}} \right) \right\] \]

\[ \leq \frac{2 \tilde{C}_\varphi, \theta_k}{(\varepsilon/2)^{N+1}} \left[ \frac{1}{(N - 2)!} \sup_{z \in \mathbb{R}_p} P \left( z + S_{n-k} \in B^{\varepsilon \alpha_k} \setminus B \right) \right] \]

\[ \leq \frac{2 \tilde{C}_\varphi, \theta_k \|\Sigma_z^{-1/2}\|}{(\varepsilon/2)^{N+1}} \cdot \frac{E(\|\mathbf{X}\|^{N+2})}{n/(N+2)!} (2\Delta_{n-k} + \varepsilon \alpha_k C_{\beta} C_z). \] \hspace{1cm} (A.37)

where

\[ \alpha_k \overset{\text{def}}{=} \sqrt{n/(n-k)}. \] \hspace{1cm} (A.38)

Inequality (A.37) follows from bound (A.16) in Lemma A.5, from Lemma A.2 and the following bounds (cf. (A.31)):

\[ P \left( S_{n-k} \in B^{\varepsilon \alpha_k} \right) - P \left( S_{n-k} \in B \right) \]

\[ \leq 2 \Delta_{n-k} + P \left( \tilde{S}_{n-k} \in B^{\varepsilon \alpha_k} \right) - P \left( \tilde{S}_{n-k} \in B \right) \]

\[ \leq 2 \Delta_{n-k} + \varepsilon \alpha_k C_{\beta} C_z. \]
The analogous bound holds for $|J_1 - J_2|$:

$$|J_1 - J_2| \leq \frac{2C_{\phi,1}k}{(\varepsilon/2)^{K-1}K!} \frac{E(||X||^K + ||Y||^K)}{n^{K/2}} (2\Delta_{n-k} + \varepsilon_k C_{\beta} C_z).$$

Using that $N = K - 2$, we have for any $k = 2, \ldots, n - 1$

$$\gamma_k \leq |I_1 - I_2| + |J_2 - J_1| \leq \frac{2C_{\phi,1}k}{(\varepsilon/2)^{K-1}K!} \frac{E(||X||^K + ||Y||^K)}{n^{K/2}} (2\Delta_{n-k} + \varepsilon_k C_{\beta} C_z)$$

$$\leq \frac{2\alpha_k C_{\phi,1}||\Sigma_z||^{-1/2}||\Phi||}{(\varepsilon/2)^{K-1}K!} \frac{\beta}{n^{(K-1)/2}K-1} \left(2M C_{\beta} C_{\phi} \left\{C_{\beta} \right\}^{1/(K-2)} + \varepsilon_k C_{\beta} C_z \right),$$

where the last inequality follows from (A.33), (A.38), and induction assumption (A.19). For $2 \leq m$

$$\sum_{j=2}^{m} \frac{1}{\sqrt{k-1}} \leq 1 + \int_2^{m} \frac{dt}{\sqrt{t-1}} = 2\sqrt{m-1} - 2.$$ Moreover, $\alpha_k = \sqrt{n/(n-k)} \leq \sqrt{2}$ for $k \leq n/2$. These properties and the last bound on $\gamma_k$ imply the resulting inequality (A.32).

**Induction step, part 3**

Below it is shown that

$$\gamma_{m+1} + \cdots + \gamma_m \leq \frac{C_{\beta}^{1/(K-2)} 2\sqrt{2} + 2^{(K-2)/2}}{n^{(K-2)/2} (m-1)^{(K-2)/2}}.$$

Let us fix some integer $k$ s.t. $2 \leq k \leq n$. Due to Lemma A.3

$$\gamma_k = |E\Phi(W_k + \bar{X}) - E\Phi(W_k + \bar{Y})|$$

$$= \left| \int_{\mathbb{R}} \phi(t/\varepsilon)dt \left[ P \left\{ \hat{P}(W_k + \bar{X}) \leq t \right\} - P \left\{ \hat{P}(W_k + \bar{Y}) \leq t \right\} \right] \right|$$

$$\leq \int_{\mathbb{R}} |\phi'(t/\varepsilon)| \left[ P \left\{ \hat{P}(W_k + \bar{X}) \leq t \right\} - P \left\{ \hat{P}(W_k + \bar{Y}) \leq t \right\} \right] \frac{dt}{\varepsilon}$$

$$\leq \sup_{B \in \mathcal{B}} \left| P \left\{ W_k + \bar{X} \in B \right\} - P \left\{ W_k + \bar{Y} \in B \right\} \right|. $$

Representations (A.34) imply

$$P \left\{ W_k + \bar{X} \in B \right\} = \left\{ \sum_{j=0}^{1/2} Z_j/\theta_k + \bar{U}_{sum} + \bar{X}_{sum} + \bar{X} \in B \right\}$$

$$= E \int_{\mathbb{R}^p} \left\{ \sum_{j=0}^{1/2} x/\theta_k + \bar{U}_{sum} + \bar{X}_{sum} + \bar{X} \in B \right\} f_0(x)dx$$

$$\text{=} \theta_k^p E \int_{\mathbb{R}^p} \left\{ \sum_{j=0}^{1/2} y + \bar{U}_{sum} + \bar{X}_{sum} + \bar{X} \in B \right\} f_0(\theta_k y - \theta_k \sum_{j=1/2}^{1/2} \bar{X})dy.$$
Similarly
\[ P \{ W_k + \overline{Y} \in B \} = \theta_k^p \mathbb{E} \int_{\mathbb{R}^p} \mathbb{I} \{ \frac{1}{2} y + \overline{U}_{\text{sum}} + \overline{X}_{\text{sum}} \in B \} f_0(\theta_k y - \theta_k \Sigma^{-1/2} \overline{X}) dy. \]

Applying Taylor formula (A.7) with \( s = K - 1 \) to the density function \( f_0(\theta_k y - \theta_k \Sigma^{-1/2} \overline{X}) \), we have
\[
\begin{align*}
&f_0(\theta_k y - \theta_k \Sigma^{-1/2} \overline{X}) = f_0(\theta_k y) + \sum_{j=1}^{K-1} \frac{(-1)^j}{j!} f_0^{(j)}(\theta_k y) \left\{ \theta_k \Sigma^{-1/2} \overline{X} \right\}^j \\
&+ \frac{(-1)^K}{(K-1)!} \mathbb{E} (1 - \tau)^{K-1} f_0^{(K)}(\theta_k y - \tau \theta_k \Sigma^{-1/2} \overline{X}) \left\{ \theta_k \Sigma^{-1/2} \overline{X} \right\}^K.
\end{align*}
\]

Similar expansion holds for \( f_0(\theta_k y - \theta_k \Sigma^{-1/2} \overline{Y}) \), with \( \overline{X} \) replaced by \( \overline{Y} \). Since \( \overline{X}, \overline{Y} \) are independent of \( \overline{U}_{\text{sum}} \) and \( \overline{X}_{\text{sum}} \), and \( \mathbb{E}(\overline{X}^j) = \mathbb{E}(\overline{Y}^j) \) \( \forall j = 0, \ldots, K-1 \), it holds
\[
\begin{align*}
&\left| P \{ W_k + \overline{X} \in B \} - P \{ W_k + \overline{Y} \in B \} \right| \\
&= \frac{\theta_k^p}{(K-1)!} \left| \mathbb{E} (1 - \tau)^{K-1} \int_{\mathbb{R}^p} \mathbb{I} \left\{ \frac{1}{2} y + \overline{U}_{\text{sum}} + \overline{X}_{\text{sum}} \in B \right\} f_0^{(K)}(x) dx \right| \\
&\quad \times \left[ \frac{(1 - \tau)^K}{(K-1)!} \left\{ \theta_k \Sigma^{-1/2} \overline{X} \right\}^K \right] \\
&\leq I_X + I_Y,
\end{align*}
\] (A.40)

where
\[
I_X \overset{\text{def}}{=} \left| \mathbb{E} \int_{\mathbb{R}^p} \mathbb{I} \left\{ \frac{1}{2} x/\theta_k + \overline{U}_{\text{sum}} + \overline{X}_{\text{sum}} + \tau \overline{X} \in B \right\} f_0^{(K)}(x) dx \right| \\
\times \left[ \frac{(1 - \tau)^K}{(K-1)!} \left\{ \theta_k \Sigma^{-1/2} \overline{X} \right\}^K \right],
\]
\[
I_Y \overset{\text{def}}{=} \left| \mathbb{E} \int_{\mathbb{R}^p} \mathbb{I} \left\{ \frac{1}{2} x/\theta_k + \overline{U}_{\text{sum}} + \overline{X}_{\text{sum}} + \tau \overline{Y} \in B \right\} f_0^{(K)}(x) dx \right| \\
\times \left[ \frac{(1 - \tau)^K}{(K-1)!} \left\{ \theta_k \Sigma^{-1/2} \overline{Y} \right\}^K \right].
\]

Consider the term \( I_X \), it holds
\[
P \left( \frac{1}{2} x/\theta_k + \overline{U}_{\text{sum}} + \overline{X}_{\text{sum}} + \tau \overline{X} \in B \mid \overline{U}_{\text{sum}}, \overline{X}, \tau \right)
\leq \Delta_{n-k} + P \left( \frac{1}{2} x/\theta_k + \overline{U}_{\text{sum}} + \overline{S}_{n-k}/\alpha_k + \tau \overline{X} \in B \mid \overline{U}_{\text{sum}}, \overline{X}, \tau \right).
\]
Let us consider the term $I_{X,2}$ where $Z \sim N(0, I)$.

By (A.17) in Lemma A.5 and (A.33) we have

$$I_2 \leq \Delta_{n-k} \mathbb{E} \int_{R^p} f_0^{(K)}(x) \left( \frac{1 - \tau}{(K - 1)!} \right) \left\{ \theta_k \Sigma_x^{-1/2} \mathbf{X} \right\}^K dx + I_{X,2}. \quad (A.41)$$

By (A.17) in Lemma A.5 and (A.33)

$$\mathbb{E} \int_{R^p} f_0^{(K)}(x) \left( \frac{1 - \tau}{(K - 1)!} \right) \left\{ \theta_k \Sigma_x^{-1/2} \mathbf{X} \right\}^K dx \leq \frac{\|\Sigma_x^{-1/2}\|^K \mathbb{E}(\|X\|^K)}{\sqrt{K!(k - 1)^{K/2}}}.$$

Let us consider the term $I_{X,2}$. By definition (A.38)

$$\tilde{S}_{n-k}/\alpha_k \sim \Sigma_x^{1/2} Z_0/\vartheta_k + \overline{U}_{\text{sum}},$$

where $Z_0 \sim N(0, I_p)$ is independent of all other r.v., and

$$\frac{1}{\vartheta_k} \overset{\text{def}}{=} \sqrt{(n - k)/n}, \quad \overline{U}_{\text{sum}} \overset{\text{def}}{=} \sum_{i=k+1}^n U_i.$$

In this way

$$I_{X,2} = \frac{\theta_k^K}{(K - 1)!} \mathbb{E} \left( 1 - \tau \right)^{K-1} \int_{R^p} \left\{ \Sigma_x^{1/2} x/\theta_k + U_{\text{sum}} + \overline{U}_{\text{sum}} \right\}^K f_0^{(K)}(x) dx \left\{ \Sigma_x^{-1/2} \mathbf{X} \right\}^K$$

$$= \frac{\theta_k^K}{(K - 1)!} \mathbb{E} \left( 1 - \tau \right)^{K-1} \int_{R^p} \int_{R^p} \left\{ \Sigma_x^{1/2} y + \tau \mathbf{X} \in C \right\} f_0^{(K)}(\theta_k x) f_0(y) dy \left\{ \Sigma_x^{-1/2} \mathbf{X} \right\}^K$$

$$= \frac{\theta_k^K}{(K - 1)!} \mathbb{E} \left( 1 - \tau \right)^{K-1} \int_{R^p} \int_{R^p} \left\{ \Sigma_x^{1/2} y + \tau \mathbf{X} \in C \right\} f_0(\theta_k x) f_0(y) dy \left\{ \Sigma_x^{-1/2} \mathbf{X} \right\}^K$$

$$\leq \frac{\overline{U}_{\text{sum}}}{K!} \mathbb{E} \left. f_0^{(K)}(y) \left\{ \Sigma_x^{-1/2} \mathbf{X} \right\}^K \right| dy$$

$$\leq \frac{\|\Sigma_x^{-1/2}\|^K \mathbb{E}(\|X\|^K)}{\sqrt{K!(n - k)^{K/2}}},$$
here equality (A.42) is obtained by integrating by parts $K$ times, and the last inequality follows from (A.17) in Lemma A.5.

Therefore, bound (A.41) reads as

$$I X \leq \left\| \Sigma^{-1/2} z \right\| K E(\left\| X \right\|^K) \left( \frac{\Delta_{n-k}}{(k-1)^{K/2}} + \frac{1}{(n-k)^{K/2}} \right).$$

The analogous inequality holds for the term $I Y$, which implies

$$\gamma_k \leq \left\| \Sigma^{-1/2} z \right\| K \left( \frac{\Delta_{n-k}}{(k-1)^{K/2}} + \frac{1}{(n-k)^{K/2}} \right) \leq \frac{C^k \beta}{\sqrt{K!}} \left( MC^0 \left\{ C^0 K \beta \right\}^{1/(K-2)} \sqrt{2} + \frac{1}{\{n/2\}^{K/2}} \right),$$

here we used induction assumption (A.19) and $k \leq n/2$. Using the bound

$$\sum_{k=m+1}^{n/2} \frac{1}{(k-1)^{K/2}} \leq \int_{m-1}^{+\infty} \frac{dt}{t^{K/2}} = \frac{1}{(m-1)^{K/2-1}} \frac{1}{K/2 - 1} \leq \frac{2}{(m-1)(K-2)/2},$$

we infer the resulting inequality (A.39):

$$\gamma_{m+1} + \cdots + \gamma_{n_0} \leq \frac{C^k \beta}{\sqrt{K!}} \left( MC^0 \left\{ C^0 K \beta \right\}^{1/(K-2)} \frac{2\sqrt{2}}{n^{K/2} \{m-1\}(K-2)/2} + \frac{1}{\{n/2\}^{K/2}} \right),$$

**Induction step, part 4**

For the last part of the sum (A.24) it holds

$$\gamma_{n_0+1} + \cdots + \gamma_n \leq \frac{2C^k \beta}{n^{K/2}(K-2)/2}. \quad (A.43)$$

Indeed, consider the bound (A.40), due to (A.17) from Lemma A.5 and definition (A.33)

$$I X = \left| \left( 1 - \tau \right)^{K-1} (K-1)! \int_{R^{2p}} \left\{ \frac{1}{\theta_k + U_{\text{sum}} + \overline{X}_{\text{sum}} + \tau \overline{X}} v_{\text{sum}} + 1 \right\} f_0^{(K)}(x) dx \right|$$

$$\times \left\{ \theta_k \Sigma^{-1/2} x \right\} K \leq \frac{\theta^K}{K!} \left| \frac{f_0^{(K)}(x)}{K!} \right| \left\{ \Sigma^{-1/2} x \right\} K dx \leq \frac{\left\| \Sigma^{-1/2} \right\| K E(\left\| X \right\|^K)}{\sqrt{K!} (k-1)^{K/2}}.$$

Similar inequality for $I Y$. Consider also

$$\sum_{k=n_0+1}^{n} \frac{1}{(k-1)^{K/2}} \leq \int_{n_0-1}^{+\infty} \frac{1}{t^{K/2}} dt = \frac{2}{K-2} \frac{1}{(n_0 - 1)(K-2)/2} \leq \frac{2}{K-2} \frac{1}{(n/6)(K-2)/2}.$$
Hence,
\[
\sum_{k=n_0+1}^{n} |P \{W_k + \bar{X} \in B \} - P \{W_k + \bar{Y} \in B \}| \\
\leq \frac{\|\Sigma_z^{-1/2}\|K_\beta}{\sqrt{K!}} \frac{2}{K - 2} \frac{6(K-2)/2}{n(K-2)/2} \\
\leq 2 \frac{\|\Sigma_z^{-1/2}\|K_\beta}{n(K-2)/2} = \frac{2\varepsilon K_\beta}{n(K-2)/2}.
\]

**Induction step, collection of all parts together**

Here we sum the bounds (A.20), (A.25), (A.32), (A.39), (A.43) and finalize the proof. Denote \( \delta = \frac{\beta}{n^{1/2}M}. \)

\[
\Delta_n \\leq \varepsilon C_\delta C_z + \frac{1.5 \tilde{C}_\phi C_z^2}{(K-2)!} \Big\{ \frac{2MC_\delta C_\phi (C_z^K \delta)^{1/(K-2)} + \varepsilon C_\delta C_z}{\varepsilon n^{1/2}K^{1/2-K}} \Big\} \\
+ \frac{4\sqrt{2} (\sqrt{m-1} - 1) \tilde{C}_\phi C_z}{\varepsilon n^{1/2}K^{1/2-K}} \frac{\delta}{(K-2)^2} \Big\{ \frac{2MC_\delta C_\phi (C_z^K \delta)^{1/(K-2)} + \varepsilon C_\delta C_z}{\varepsilon n^{1/2}K^{1/2-K}} \Big\} \\
+ \frac{2\sqrt{2} C_z^K \beta}{\sqrt{K!}(m-1)(K-2)/2} MC_\delta C_\phi (C_z^K \delta)^{1/(K-2)} + \frac{2(K-2)/2}{\sqrt{K!}} C_z^K \delta + 2C_z^K \delta.
\]

Taking \( \varepsilon = (C^2 C_\phi \delta)^{1/(K-2)} a \) and \( \sqrt{m-1} = b \{ C_z^K \beta \}^{1/(K-2)} \) for some \( a, b > 0 \), and using that \( K \geq 3, C_\delta, C_\phi \geq 1, C_\phi \geq \tilde{C}_\phi, \tilde{C}_\phi, \) we obtain

\[
\Delta_n \leq C_\delta C_\phi \frac{(C_z^K \delta)^{1/(K-2)} M}{a/M + 1.5(a/2)^{-K-2} (2M + a)} \\
+ \frac{4\sqrt{2} (\sqrt{m-1} - 1) (a/2)^{-K-1} (2M + a)}{K!} \frac{M}{M} + \frac{2\sqrt{2} C_z^K \beta}{\sqrt{K!}(m-1)(K-2)/2} \\
+ 2.6 \{ C_z^K \delta \}^{(K-3)/(K-2)} \frac{1}{M} \Bigg[ \frac{a/M + 1.5(a/2)^{-K-2} (2M + a)}{(K-2)!} \Bigg] \\
\leq C_\delta C_\phi \frac{(C_z^K \delta)^{1/(K-2)} M}{a/M + 1.5(a/2)^{-K-2} (2M + a)} \\
+ \frac{4\sqrt{2} b (2M + a)}{(a/2)^{-K-1} MK!} + \frac{2\sqrt{2}}{\sqrt{K!}b^{K-2}} + 2(K-3)/(K-2) 2.6 \Bigg[ \frac{2M}{M^{K-2}} \Bigg] \\
\leq C_\delta C_\phi \frac{(C_z^K \delta)^{1/(K-2)} M}{a/M + 1.5(a/2)^{-K-2} (2M + a)}.
\]

here we also used the condition \( n^{1/2} > M \{ C_z^K \beta/2 \}^{1/(K-2)} \) from the induction basis; the last inequality holds for parameters \( M \geq 72.5, a = M/1.95, b = 9.25. \)
Moreover, for these values \( 1 < m \leq n/2, \) as it is assumed in the proof. \( \square \)
A.3. Proof of Theorem 2.2

Let us take a smoothing function \( \tilde{\phi} : \mathbb{R}^p \mapsto \mathbb{R} \) such that it satisfies (cf. (A.8) and (A.9) in Lemma A.3)

\[
0 \leq \tilde{\phi}(x) \leq 1, \quad \tilde{\phi}(x) = \begin{cases} 1, & x \in B; \\ 0, & x \notin B \end{cases}, \quad \left| \tilde{\phi}(K)(x)h^K \right| \leq \frac{C_{\phi,2}\|h\|^K}{\varepsilon^K},
\]

for some positive constant \( C_{\phi,2} \in \mathbb{R} \) and for all \( x, h \in \mathbb{R}^p \). For example, one can take \( \tilde{\phi}(x) = \phi(\tilde{\rho}(x)/\varepsilon) \), as in (A.11) in Lemma A.3 with function \( \phi(x) \) \( K \) times continuously differentiable

\[
0 \leq \phi(x) \leq 1, \quad \phi(x) = 0 \text{ for } x \geq 1, \quad \phi(x) = 1 \text{ for } x \leq 0.
\]

Applying the smoothing Lemma A.4 and Lemma A.2, we have

\[
\Delta_n \overset{\text{def}}{=} \sup_{B \in \mathcal{B}} \left| P(S_n \in B) - P(\tilde{S}_n \in B) \right|
\leq \sup_{B \in \mathcal{B}} \left| E\tilde{\phi}(S_n) - E\tilde{\phi}(\tilde{S}_n) \right| + \sup_{B \in \mathcal{B}} P(\tilde{S}_n \in B^c \setminus B) \tag{A.44}
\]

\[
\leq \sup_{B \in \mathcal{B}} \left| E\tilde{\phi}(S_n) - E\tilde{\phi}(\tilde{S}_n) \right| + \varepsilon C_{\phi,2} C_{\varepsilon,2} \tag{A.45}
\]

Denote

\[
\overline{X}_i \overset{\text{def}}{=} \frac{X_i}{\sqrt{n}}, \quad \overline{Y}_i \overset{\text{def}}{=} \frac{Y_i}{\sqrt{n}}
\]

then it holds

\[
S_n \overset{\text{def}}{=} \overline{X}_1 + \ldots + \overline{X}_n, \quad \tilde{S}_n \overset{\text{def}}{=} \overline{Y}_1 + \ldots + \overline{Y}_n.
\]

Below we employ the telescopic sum approach by [25]:

\[
W_k \overset{\text{def}}{=} \sum_{i=1}^{k-1} Y_i + \sum_{i=k+1}^{n} \overline{X}_i \quad \text{for } k = 2, \ldots, n,
\]

\[
W_1 \overset{\text{def}}{=} \overline{X}_2 + \ldots + \overline{X}_n, \quad W_n \overset{\text{def}}{=} \overline{Y}_1 + \ldots + \overline{Y}_{n-1},
\]

thus

\[
S_n = \overline{X}_1 + W_1, \quad \tilde{S}_n = W_n + \overline{Y}_n.
\]

here \( W_k \) is independent of \( \overline{X}_k \) and \( \overline{Y}_k \). Introduce

\[
\tilde{\gamma}_k \overset{\text{def}}{=} \left| E\tilde{\phi}(W_k + \overline{X}_k) - E\tilde{\phi}(W_k + \overline{Y}_k) \right|,
\]

it holds

\[
\left| E\tilde{\phi}(S_n) - E\tilde{\phi}(\tilde{S}_n) \right| \leq \sum_{k=1}^{n} \tilde{\gamma}_k. \tag{A.46}
\]
By Taylor’s formula (A.7)
\[ \tilde{\varphi}(W_k + \bar{X}_k) = \tilde{\varphi}(W_k) + \sum_{j=1}^{K-1} \frac{\varphi^{(j)}(W_k)}{j!} (W_k) X_k^j + \mathbf{E} \left( \frac{(1 - \tau)^{K-1}}{(K - 1)!} \varphi^{(K)}(W_k + \tau X_k) X_k^K \right). \]

Similar expression for \( \tilde{\varphi}(W_k + \tilde{X}_k) \) together with independence of \( W_k, X_k, Y_k, \tau \) and condition (2.1) imply
\[ \tilde{\gamma}_k \leq C_{\phi,2} \frac{\mathbf{E} \left( \|X_k\|^K + \|Y_k\|^K \right)}{K! n^K n^{K/2}}. \] (A.47)

Collecting (A.45), (A.46) and (A.47) implies
\[ \Delta_n \leq \frac{C_{\phi,2}}{K!} \epsilon n^{K/2} \sum_{k=1}^n \mathbf{E} \left( \|X_k\|^K + \|Y_k\|^K \right) + \epsilon \bar{C}_{\phi} \bar{C}_x \]
\[ \leq 2 \left\{ \frac{C_{\phi,2} (\bar{C}_{\phi} \bar{C}_x)^K}{n^{K/2} (K - 1)!} \sum_{k=1}^n \mathbf{E} \left( \|X_k\|^K + \|Y_k\|^K \right) \right\}^{1/(K+1)}, \]
which finishes the proof of Theorem 2.2.

A.4. Proof of Theorem 2.3

repeats the main steps of the proof of Theorem 2.2 except for the smoothing (A.44). Using the notation similar to the proof of Theorem 2.2: \( \bar{X}_k \) def \( X_k/\sqrt{n} \),
\[ W'_k \overset{\text{def}}{=} \sum_{i=1}^{k-1} X'_i + \sum_{i=k+1}^n \bar{X}_i \text{ for } k = 2, \ldots, n, \]
\[ W'_1 \overset{\text{def}}{=} \bar{X}_2 + \cdots + \bar{X}_n, \quad W'_n \overset{\text{def}}{=} \bar{X}'_1 + \cdots + \bar{X}'_{n-1}, \]

we can write
\[ \left| \mathbf{E} f(S_n) - \mathbf{E} f(S'_n) \right| \leq \sum_{k=1}^n \left| \mathbf{E} f(W'_k + \bar{X}_k) - \mathbf{E} f(W'_k + \bar{X}'_k) \right|. \]

By Taylor’s formula (A.7)
\[ f(W'_k + \bar{X}_k) = f(W'_k) + \sum_{j=1}^{K-1} \frac{f^{(j)}(W'_k)}{j!} (W'_k) \bar{X}_k^j + \mathbf{E} \left( \frac{(1 - \tau)^{K-1}}{(K - 1)!} f^{(K)}(W'_k + \tau \bar{X}_k) \bar{X}_k^K \right). \]

Similar expression for \( f(W'_k + \bar{X}'_k) \), independence of \( W'_k, X_k, \tilde{X}_k, \tau \) and condition (2.6) imply
\[ \left| \mathbf{E} f(W'_k + \bar{X}_k) - \mathbf{E} f(W'_k + \bar{X}'_k) \right| \leq C_f \frac{\mathbf{E} \left( \|X_k\|^K + \|X'_k\|^K \right)}{K! n^{K/2}}, \]
therefore
\[ \left| \mathbf{E} f(S_n) - \mathbf{E} f(S'_n) \right| \leq \sum_{k=1}^n C_f \frac{\mathbf{E} \left( \|X_k\|^K + \|X'_k\|^K \right)}{K! n^{K/2}}. \]
If \( \{X_i\} \) are i.i.d., and \( \{X'_i\} \) are also identically distributed, we have
\[
\left| EF(S_n) - EF(S'_n) \right| \leq C_f \frac{E(\|X\|^K + \|X'_\|^K)}{K!n^{(K-2)/2}}.
\]
This finishes the proof of Theorem 2.3.

**Appendix B: Proofs of the properties of the weighted bootstrap**

This section contains proofs of the results from Section 3.

**Proof of Theorem 3.1.** Consider the following approximating random vector
\[
\tilde{S}_n \overset{\text{def}}{=} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \varepsilon_i.
\]
It holds
\[
E(X_k) = E((X_i \varepsilon_i)^k) = E(Y_k^i) \quad \forall k = 1, 2, 3,
\]
here the expectation is taken w.r.t. the joint probability distribution of \( X_i \) and \( \varepsilon_i \). Hence, applying Theorem 2.1 and triangle inequality, we obtain the following approximating bound between distributions of \( \|S_n\| \) and \( \|\tilde{S}_n\| \):
\[
\sup_{t \geq 0} \left| P(\|S_n\| \leq t) - P(\|\tilde{S}_n\| \leq t) \right| \\
\leq MC_B C_f C_2 \frac{E(\|X\|^4) + E(\|Y\|^4)}{n^{1/2}} \\
+ MC_B C_f C_2 \frac{E(\|X\|^4) E(\varepsilon^4) + E(\|Y\|^4)}{n^{1/2}} \\
\leq \Delta_1 \overset{\text{def}}{=} 2MC_B C_f C_2 \frac{E(\|X\|^4) E(\varepsilon^4) + E(\|Y\|^4)}{n^{1/2}}.
\] (B.1)

Introduce the upper quantile function for the r.v. \( \|\tilde{S}_n\| \):
\[
\tilde{Q}(\alpha) \overset{\text{def}}{=} \inf \left\{ t \in \mathbb{R} : P(\|\tilde{S}_n\| > t) \leq \alpha \right\}.
\] (B.2)

Bound (B.1) and definitions (3.1), (B.10) imply
\[
Q(\alpha + 2\Delta_1) \leq \tilde{Q}(\alpha + \Delta_1) \leq Q(\alpha).
\] (B.3)

By definitions (1.3), (B.10), and by inequalities (B.3) we infer
\[
Q(\alpha + \Delta_1) \leq Q^o(\alpha) \leq Q(\alpha - \Delta_1) + \Delta Q^o,
\]
where
\[
\Delta Q^o \overset{\text{def}}{=} \begin{cases} 
0, & \text{if } P^o(\|S_n^o\| > t) \text{ is continuous in } Q^o(\alpha); \\
\Delta_1, & \text{otherwise.}
\end{cases}
\]
Let us define similarly
\[ \Delta_2 \overset{\text{def}}{=} \begin{cases} 0, & \text{if } P(\|S_n\| > t) \text{ is continuous in } Q(\alpha - \Delta_1); \\ \Delta_1, & \text{otherwise.} \end{cases} \]

Collecting the derived bounds, we have
\[
\begin{align*}
\alpha + \Delta_1 &\geq P(\|S_n\| > Q(\alpha + \Delta_1)) \geq P(\|S_n\| > Q^o(\alpha)), \\
\alpha - \Delta_1 &\leq P(\|S_n\| > Q(\alpha - \Delta_1) - \Delta_2) \\
&\leq P(\|S_n\| > Q(\alpha - \Delta_1) + \Delta_1 + (\Delta_1 + \Delta Q^o)\tilde{C}_B C_z) \quad (B.4) \\
&\leq P(\|S_n\| > Q^o(\alpha)) + \Delta_1(1 + 2\tilde{C}_B C_z),
\end{align*}
\]

where inequality (B.4) follows from Theorem 2.1, Lemma A.2, and bound (B.1).

Indeed, for arbitrary \( \varepsilon > 0 \):
\[
P(t < \|S_n\| \leq t + \varepsilon) \leq P\left(t < n^{1/2}\sum_{i=1}^n Y_i \right) \leq 2\Delta_1/2 \\
\leq \varepsilon\tilde{C}_B C_z + \Delta_1.
\]

**Proof of Theorem 3.2.** is similar to the proof of Theorem 3.1, except that the bound (B.1) is replaced with the following one:
\[
\sup_{t \geq 0} P(\|S_n\| \leq t) - P(\tilde{S}_n \leq t) \\
\leq \Delta_2 \overset{\text{def}}{=} 2.8 \left\{ \frac{\tilde{C}_2}{C_4} \sum_{i=1}^n \left[ \mathbb{E}(\|X_i\|^4) \mathbb{E}(\varepsilon_1^4) + \mathbb{E}(\|Y_i\|^4) \right] \right\}^{1/5}.
\]

This inequality follows from Theorem 2.2 for \( K = 4 \), and from triangle inequality.

**Proof of Theorem 3.3.** The statement follows from Theorem 3.2 applied to \( X_i \Rightarrow (\Psi^\top)^{-1/2}\eta_i \), and \( \|S_n\| \Rightarrow T, \|S^o_n\| \Rightarrow T^o \).

**B.1. Conditions and proof of Theorem 3.4**

Recall that \( \ell_i(\theta) \overset{\text{def}}{=} \log\left( \frac{dP_{Y_i}}{dP_{\theta}}(y_i) \right) , \ d_0^\ell \overset{\text{def}}{=} -E\ell_i^\ell(\theta^*) \). By definition of \( \theta^* \), vectors \( \ell_i^\ell(\theta^*) \) are i.i.d. with zero mean. Below are the conditions for Theorem 3.4, they are required for the Wilks type bounds for the log-likelihood ratios \( L(\hat{\theta}) - L(\theta^*) \) and \( L^o(\hat{\theta}) - L^o(\theta^*) \), obtained in [45, 46] and [47] correspondingly. We took the set of conditions below from Section B.3.1 of supplement of [47].

1. The covariance matrix \( \text{Var}\{\ell_i^\ell(\theta^*)\} \) is positive definite;
2. There exist a positive-definite symmetric matrix $v_0^2$ and constants $\bar{g} > 0, \nu \geq 1$ such that $\operatorname{Var}\{\ell'_i(\theta^*)\} \leq v_0^2$ and for all $|\lambda| \leq \bar{g}$

$$\sup_{|\gamma| \leq 1} \log \mathbb{E} \exp \left\{ \lambda \gamma^\top v_0^{-1} \ell'_i(\theta^*) \right\} \leq \nu^2 \lambda^2 / 2;$$

3. There exist a constant $\bar{\omega} > 0$ and for each $r > 0$ a constant $\bar{g}(r)$ such that it holds for all $\theta \in \Theta$ such that $\|d_0(\theta - \theta^*)\| \leq r/\sqrt{n}$, for $j = 1, 2$,

and for all $|\lambda| \leq \bar{g}(r)$

$$\sup_{|\gamma| \leq 1} \log \mathbb{E} \exp \left\{ \bar{\omega}^{-1} \lambda \gamma^\top d_0^{-1}(\ell''_i(\theta) - \mathbb{E}\ell''_i(\theta))d_0^{-1} \gamma \right\} \leq \nu^2 \lambda^2 / 2;$$

4. There exists a constant $a > 0$ s.t. $a^2 d_0^2 \geq v_0^2$;

5. There exists a constant $C_m \geq 0$ such that for each $r \geq 0$, and for all $\theta \in \Theta$ such that $\|d_0(\theta - \theta^*)\| \leq r/\sqrt{n}$

$$\sup_{|\gamma| \leq 1} \|d_0^{-1} \gamma^\top \mathbb{E}\ell''_i(\theta)d_0^{-1}\| \leq C_m;$$

6. For each $r \geq C\sqrt{p + x}$ and for all $\theta \in \Theta$ such that $\|d_0(\theta - \theta^*)\| \leq r/\sqrt{n}$,

it holds for some value $C_6(r) > 0$ s.t. $rC_6(r) \to +\infty$ with $r \to +\infty$

$$\|d_0^{-1} \mathbb{E}\ell'_i(\theta)d_0^{-1}\| \geq C_6(r);$$

7. For the bootstrap weights $\{u_i\}_{i=1}^n$ it holds for all $|\lambda| \leq \bar{g}$

$$\log \mathbb{E}\exp(\lambda(u_i - 1)) \leq \nu^2 \lambda^2 / 2.$$

Proof of Theorem 3.4. Take $S_n := n^{-1/2} \sum_{i=1}^n d_0^{-1} \ell'_i(\theta^*)$, and $S_n^\alpha := n^{-1/2} \sum_{i=1}^n d_0^{-1} \ell'_i(\theta^*) \varepsilon_i$.

Let also $S_n^\alpha(\widetilde{\theta}) := n^{-1/2} \sum_{i=1}^n d_0^{-1} \ell'_i(\widetilde{\theta}) \varepsilon_i$.

[45, 46] showed that under the conditions given above the following non-asymptotic Wilks type inequality ([49]) holds with probability $\geq 1 - 5e^{-x}$:

$$\left| \sqrt{2\{L(\theta) - L(\theta^*)\}} - \|S_n\| \right| \leq \Delta_W(x), \hspace{1cm} (B.6)$$

where $\Delta_W(x) \leq 3\sqrt{p + x} n^{-1/2} \left[ C\sqrt{p + x} + 6\sqrt{2} \varepsilon(\mathbb{g}^2 n)^{-1/2} (2\sqrt{p + \sqrt{2} x} + 4p \{x(\mathbb{g}^2 n)^{-1} + 1\}) \right] \leq C(p + x)/\sqrt{n}.$

[47] obtained the bootstrap version of (B.6). If the conditions above are fulfilled, then the following bounds hold with probability $\geq 1 - 5e^{-x}$:

$$\left| \sqrt{2\{L^\alpha(\theta) - L^\alpha(\theta^*)\}} - \|S_n^\alpha(\theta)\| \right| \leq \Delta^\alpha_W(x),$$

$$\|S_n^\alpha(\theta) - S_n^\alpha(\widetilde{\theta})\| \leq \Delta^\alpha(x),$$
Due to (B.9) denote\(\varepsilon > 0\). Indeed, it holds for arbitrary inequality (B.7) follows from Theorem 3.1 and Lemma A.2 (similarly to bounds), we infer
\[
P \left( \sqrt{2\{L(\tilde{\theta}) - L(\theta^*)\}} > t \right) \\
\geq P \left( \|\mathcal{S}_n\| > t + \Delta W(x) \right) \\
\geq P \left( \|\mathcal{S}_n\| > t - \Delta W^o(x) - \Delta^o(x) \right) - \Delta_L \\
\geq P \left( \sqrt{2\{L^o(\tilde{\theta}^o) - L^o(\tilde{\theta})\}} > t \right) - \Delta_L,
\] where
\[
\Delta_L \overset{\text{def}}{=} \Delta_L + \overline{C}_C \varepsilon \{ \Delta W(x) + \Delta W^o(x) + \Delta^o(x) \}, \\
\overline{\Delta}_L \overset{\text{def}}{=} 2MC \overline{C}_C C_z 2^{\frac{1}{4}} \frac{\left\{ \mathbf{E} \left( \|d_0^{-1} \ell_1'(\theta^*)\|_2^4 \right) \mathbf{E}(\varepsilon_1^4) + \mathbf{E} \left( \|Y_i\|_2^4 \right) \right\}^{1/2}}{n^{1/2}}.
\] Inequality (B.7) follows from Theorem 3.1 and Lemma A.2 (similarly to bounds (B.5)). Indeed, it holds for arbitrary \(\varepsilon > 0\)
\[
P \left( \|\mathcal{S}_n\| > t + \varepsilon \right) \\
\geq P \left( \|n^{-1/2} \sum_{i=1}^n Y_i\| > t + \varepsilon \right) - \overline{\Delta}_L / 2 \\
\geq P \left( \|n^{-1/2} \sum_{i=1}^n Y_i\| > t \right) - \overline{\Delta}_L / 2 - \varepsilon \overline{C}_C C_z L \\
\geq P \left( \|\mathcal{S}_n\| > t \right) - \overline{\Delta}_L - \varepsilon \overline{C}_C C_z L.
\] Similar inequalities in the inverse direction imply
\[
\left| P \left( \sqrt{2\{L(\tilde{\theta}) - L(\theta^*)\}} > t \right) - P \left( \sqrt{2\{L^o(\tilde{\theta}^o) - L^o(\tilde{\theta})\}} > t \right) \right| \leq \Delta_L. \tag{B.9}
\] Denote
\[
\widetilde{Q}_L(\alpha) \overset{\text{def}}{=} \inf \left\{ t \geq 0 : P \left( \sqrt{2\{L^o(\tilde{\theta}^o) - L^o(\tilde{\theta})\}} > t \right) \leq \alpha \right\}. \tag{B.10}
\] Due to (B.9)
\[
Q_L(\alpha + 2\Delta_L) \leq \widetilde{Q}_L(\alpha + \Delta_L) \leq Q_L(\alpha). \tag{B.11}
\] Furthermore, by definitions (3.5), (B.10) and by inequality (B.11), it holds
\[
Q_L(\alpha + \Delta_L) \leq Q_L^o(\alpha) \leq Q_L(\alpha - \Delta_L) + \Delta_Q^o,
\] where
\[
\Delta_{Q^o} \overset{\text{def}}{=} \begin{cases} 
0, & \text{if } P^o \left( \sqrt{2\{L^o(\tilde{\theta}^o) - L^o(\tilde{\theta})\}} > t \right) \text{ is continuous in } Q^o_L(\alpha); \\
\Delta_L, & \text{otherwise.}
\end{cases}
\]
Denote similarly
\[
\Delta_3 \overset{\text{def}}{=} \begin{cases} 
0, & \text{if } P \left( \sqrt{2} \{ L(\tilde{\theta}) - L(\theta^*) \} > t \right) \text{ is continuous in } Q_L(\alpha - \Delta_L); \\
\Delta_L, & \text{otherwise}.
\end{cases}
\]

Collecting the derived bounds, we obtain
\[
\alpha + \Delta_L \geq P \left( \sqrt{2} \{ L(\tilde{\theta}) - L(\theta^*) \} > Q_L(\alpha + \Delta_L) \right) \\
\geq P \left( \sqrt{2} \{ L(\tilde{\theta}) - L(\theta^*) \} > Q_L^0(\alpha) \right),
\]
\[
\alpha - \Delta_L \leq P \left( \sqrt{2} \{ L(\tilde{\theta}) - L(\theta^*) \} > Q_L(\alpha - \Delta_L) - \Delta_3 \right) \\
\leq P \left( \sqrt{2} \{ L(\tilde{\theta}) - L(\theta^*) \} > Q_L^0(\alpha) \right) + 2 \Delta_W(x) + 2 \tilde{\Delta}_L (1 + \tilde{C}_{\Theta} C_{z,L}),
\]
the last inequality follows from (B.6), Theorem 2.1 and Lemma A.2, indeed, it holds for any \( \varepsilon > 0 \):
\[
P \left( t < \sqrt{2} \{ L(\tilde{\theta}) - L(\theta^*) \} \leq t + \varepsilon \right) \\
\leq P \left( t < \| S_n \| \leq t + \varepsilon \right) + 2 \Delta_W(x) \\
\leq P \left( t < \| n^{-1/2} \sum_{i=1}^n Y_i \| \leq t + \varepsilon \right) + 2 \Delta_W(x) + \tilde{\Delta}_L \\
\leq \varepsilon \tilde{C}_{\Theta} C_{z,L} + 2 \Delta_W(x) + 2 \tilde{\Delta}_L.
\]

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