COVERINGS BY OPEN CELLS

MÁRIO J. EDMUNDO, PANTELIS E. ELEFTHERIOU, AND LUCA PRELLI

Abstract. We prove that in a semi-bounded o-minimal expansion of an ordered group every non-empty open definable set is a finite union of open cells.

1. Introduction

We fix an arbitrary o-minimal expansion \( \mathcal{R} = (\mathcal{R}, <, +, 0, \ldots) \) of an ordered group. Recall that by [3] \( \mathcal{R} \) is semi-bounded if it has no poles; that is, in \( \mathcal{R} \) there is no definable bijection between a bounded and an unbounded interval. See [3] for other characterizations of semi-boundedness. In this note we prove the following theorem.

**Theorem 1.1.** If \( \mathcal{R} \) is semi-bounded, then every non-empty open definable set is a finite union of open cells.

As explained in [13, Subsection 2.1], there are three possibilities for an arbitrary o-minimal expansion \( \mathcal{R} = (\mathcal{R}, <, +, 0, \ldots) \) of an ordered group:

(A) \( \mathcal{R} \) is linear (that is, its first-order theory \( \text{Th}(\mathcal{R}) \) is linear ([10])). In this case by [10], there exists \( \mathcal{S} \equiv \mathcal{R} \) with \( \mathcal{S} \) a reduct of an ordered vector space \( \mathcal{V} = (V, <, +, 0, \{d\}_{d \in D}) \) over an ordered division ring \( D \) (with the same addition and linear ordering the underlying group of \( \mathcal{S} \)).

(B) \( \mathcal{R} \) is not linear. In this case, the theory of every interval in \( \mathcal{R} \) with the induced structure is not linear and so no interval in \( \mathcal{R} \) is elementarily equivalent to a reduct of an interval in an ordered vector space ([10])). Therefore, by the Trichotomy theorem ([14, Theorem 1.2]), a real closed field whose ordering agrees with that of \( \mathcal{R} \) is definable on some interval \((-e, e)\). There are now two sub-cases to consider:

(B1) \( \mathcal{R} \) is semi-bounded.

(B2) \( \mathcal{R} \) is not semi-bounded. In this case, one can endow the whole structure \( \mathcal{R} \) with a definable real closed field. Indeed, let \( \sigma : (a, b) \to (c, +\infty) \) be a pole in \( \mathcal{R} \); that is, a definable bijection (with say, \( \lim_{t \to b} \sigma(t) = +\infty \)). Without loss of generality, and using translations, we may assume that \( a = c = 0 \) and \( b < e \). But then, being inside a real closed field, the intervals \((0, e)\) and \((0, b)\) are in definable bijection and so \((0, e)\) and \((0, +\infty)\) are in definable bijection. Now it is easy to get a real closed field on the whole of \( \mathcal{R} \).

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A version of Theorem 1.1 in the field case (B2) was proved by Wilkie in [16], for bounded open definable subsets. There are simple examples that show that in this case the boundedness assumption is required. On the other hand, a version of Theorem 1.1 in the linear case (A) was proved by Andrews in [1]. Here we generalize these two results to the semi-bounded non-linear case. Moreover, we also prove a stronger result in the linear case, which we state next. For the notion of 'linear decomposition' and 'star', see Section 2 below. For the notion of 'stratification', see [2, Chapter 4, (1.11)]. By Lemma 2.6, Corollary 2.14 and Proposition 2.16 below, we have:

**Theorem 1.2.** Assume that $R = (\mathbb{R}, <, 0, +, \{\lambda\}_{\lambda \in D})$ is an ordered vector space over an ordered division ring $D$. Let $D$ be a linear decomposition of $\mathbb{R}^n$. Then there is decomposition $C$ of $\mathbb{R}^n$ that refines $D$, such that for every $C \in C$, the star of $C$ is an open (usual) cell. Moreover, $C$ is a stratification of $\mathbb{R}^n$.

An important example of a semi-bounded, non-linear o-minimal structure is the expansion $B$ of the real ordered vector space $\mathbb{R}_{\text{vect}} = (\mathbb{R}, <, +, \{d\}_{d \in \mathbb{R}})$ by all bounded semi-algebraic sets. Every bounded interval in $B$ admits the structure of a definable real closed field. For example, the field structure on $(-1, 1)$ induced from $\mathbb{R}$ via the semi-algebraic bijection $x \mapsto \sqrt{1 + x^2}$ is definable in $B$. By [15, 11, 12], $B$ is the unique structure that lies strictly between $\mathbb{R}_{\text{vect}}$ and the real field. The situation becomes significantly more subtle when $R$ is non-archimedean, and the study of definable sets and groups in the general semi-bounded setting has recently regained a lot of interest ([4, 6, 7, 8, 13]).

We expect that our main theorem on coverings by open cells (Theorem 1.1) will find numerous applications in the theory of locally definable manifolds in o-minimal structures. Some of those are exhibited in [5]. As stated in that reference, a strengthened result of coverings would yield further applications. We state the desired result here as a Conjecture:

**Conjecture.** Every definable set is a finite union of relatively open definable subsets which are definably simply connected.

**Structure of the paper.** Section 2 contains the stratification result (Theorem 1.2) for the linear case. Section 3 contains the covering by open cells (Theorem 1.1) for the semi-bounded non-linear case.

**Notation.** We recall the standard notation for graphs and “generalized cylinders” of definable maps.

- If $f : X \to R$ is a definable map, we denote by $\Gamma(f)$ the graph of $f$.
- If $f, g : X \to R$ are definable maps or the constant maps $-\infty$ and $+\infty$ on $X$ with $f(x) < g(x)$ for all $x \in X$, we write $f < g$ and set:
  - $(f, g)_X = \{(x, y) \in X \times R : f(x) < y < g(x)\};$
  - $[f, g]_X = \{(x, y) \in X \times R : f(x) \leq y < g(x)\};$
  - $(f, g]_X = \{(x, y) \in X \times R : f(x) < y \leq g(x)\};$
  - $[f, g]_X = \{(x, y) \in X \times R : f(x) \leq y \leq g(x)\}.$

We also use the same notation for functions $f, g : Y \to R$ whose domain $Y$ contains $X$ and whose restrictions on $X$ are as above.

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2. The linear case

We assume in this section that \( \mathcal{R} = \langle R, <, 0, +, \{\lambda\}_{\lambda \in D} \rangle \) is an ordered vector space over an ordered division ring \( D \). For basic properties on such o-minimal structures we refer the reader to [2, Chapter 1, Section 7].

A function \( f : R^n \to R \) of the form \( f(x_1, \ldots, x_n) = \lambda_1 x_1 + \ldots + \lambda_n x_n + a \), where \( \lambda_i \in D \) and \( a \in R \), is called linear (or affine). For a definable set \( X \subseteq R^n \), we denote by \( L(X) \) the set of restrictions on \( X \) of linear functions and by \( L_\infty(X) \) the set \( L(X) \cup \{ \pm \infty \} \), where we regard \( -\infty \) and \( +\infty \) as constant functions on \( X \). The functions from \( L(X) \) are called linear functions on \( X \). Clearly, if two linear functions have the same restrictions on \( X \) then their restrictions on \( \text{cl}(X) \) are equal as well.

We define linear cells in \( R^n \) inductively as follows:

- a linear cell in \( R \) is either a singleton subset of \( R \), or an open interval with endpoints in \( R \cup \{ \pm \infty \} \),
- a linear cell in \( R^{n+1} \) is a set of the form \( \Gamma(f) \), for some \( f \in L(X) \), or \( (f, g)_X \), for some \( f, g \in L_\infty(X) \), \( f < g \), where \( X \) is a linear cell in \( R^n \).

In either case, \( X \) is called the domain of the defined cell.

We refer the reader to [2, Chapter 3, (2.10)] for the definition of a decomposition of \( R^n \). A linear decomposition of \( R^n \) is then a decomposition \( \mathcal{C} \) of \( R^n \) such that each \( B \in \mathcal{C} \) is a linear cell. The following can be proved similarly to [2, Chapter 3, (2.11)].

**Theorem 2.1** (Linear CDT).

1. Given any definable sets \( A_1, \ldots, A_k \subseteq R^n \), there is a linear decomposition \( \mathcal{C} \) of \( R^n \) that partitions each \( A_i \).
2. Given a definable function \( f : A \to R \), there is a linear decomposition \( \mathcal{C} \) of \( R^n \) that partitions \( A \) such that the restriction \( f_{|B} \) to each \( B \in \mathcal{C} \) with \( B \subseteq A \) is linear.

**Definition 2.2.** Let \( \mathcal{C} \) be a linear decomposition of \( R^n \) and \( X \) a subset of \( R^n \). Denote

\[
\text{Star}_\mathcal{C}(X) = \{ D \in \mathcal{C} : X \cap \text{cl}(D) \neq \emptyset \}.
\]

The star of \( X \) with respect to \( \mathcal{C} \), denoted by \( \text{st}_\mathcal{C}(X) \), is then

\[
\text{st}_\mathcal{C}(X) = \bigcup \text{Star}_\mathcal{C}(X).
\]

We just write \( \text{Star}(X) \) and \( \text{st}(X) \) if \( \mathcal{C} \) is clear from the context.

In what follows, if \( k > 0 \), then \( \pi : R^{k+1} \to R^k \) denotes the usual projection map onto the first \( k \)-coordinates, and if \( \mathcal{C} \) is a linear decomposition of \( R^{k+1} \), then \( \pi(\mathcal{C}) \) denotes the linear decomposition \( \{ \pi(C) : C \in \mathcal{C} \} \) of \( R^k \).

**Lemma 2.3.** Let \( \mathcal{C} \) be a linear decomposition of \( R^n \) and \( X \) a subset of \( R^n \). Then:

(i) If \( n > 1 \), then \( \text{Star}_{\pi(\mathcal{C})}(\pi(X)) = \pi(\text{Star}_\mathcal{C}(X)) \).

(ii) If \( X \) is an open union of cells in \( \mathcal{C} \), and \( C \in \mathcal{C} \) with \( C \subseteq X \), then \( \text{st}(C) \subseteq X \).

**Proof.** (i) \( \subseteq \). Let \( D \in \text{Star}(\pi(X)) \). Since \( \pi \) is open, for any open set \( U \) containing \( X \), \( \pi(U) \) is an open set containing \( \pi(X) \). Thus \( D \cap \pi(U) \neq \emptyset \), which
partitions each member of \( G \). Clearly, \( G \) is a finite collection \( C \) and the inductive hypothesis, there is a special linear decomposition of \( R \) of Theorem 1.1 for the linear case (see Corollary 2.19 below).

\[ \pi^{-1}(D) \cap U \neq \emptyset. \] Hence, by the definition of linear decomposition, there is some \( D' \in \text{Star}(X) \) such that \( \pi(D') = D \).

\[ \ni. \] Let \( D \in \text{Star}(X) \). For any open set \( U \) containing \( \pi(X) \), \( \pi^{-1}(U) \) is an open neighborhood of \( X \). Therefore \( \pi^{-1}(U) \cap D \neq \emptyset \), and \( U \cap \pi(D) \neq \emptyset \). Hence \( \pi(D) \) belongs to \( \text{Star}(\pi(X)) \).

(ii) Since \( X \) is open, for every \( B \in \text{Star}(C) \), \( B \cap X \neq \emptyset \), and hence \( B \subseteq X \). \( \square \)

One would expect that \( \text{st}_C(X) \) is an open set. However, the following example shows that this is not the case.

**Example 2.4.** Consider points \( a_{-1} < a_0 < a_1 < a_2 < a_3 \) in \( R \) and let \( C \) be a linear decomposition of \( R^2 \) that contains the following cells: \( (a_{-1}, a_0) \times (a_0, a_2), (a_0, a_1) \times (a_0, a_2), \{a_0\} \times (a_0, a_1), \{a_0\} \times (a_1, a_3) \) and the point \( (a_0, a_1) \). Then the star of the point \( (a_0, a_1) \) is the union of the above cells, which is not open.

Below we define a special kind of a linear decomposition \( C \) of \( R^n \) that remedies the above problem. In fact, such a \( C \) will give us that every \( \text{st}_C(X) \) with \( X \in C \) is an open (usual) cell (see Proposition 2.16 below). From this we obtain the version of Theorem 1.1 for the linear case (see Corollary 2.19 below).

**Definition 2.5.** A special linear decomposition of \( R^n \) is a linear decomposition of \( R^n \) defined by induction on \( n \) as follows. Any linear decomposition of \( R \) is special. A linear decomposition \( C \) of \( R^{k+1} \), \( k > 0 \), is special if:

- \( \pi(C) \) is a special linear decomposition of \( R^k \);
- for any cells \( \Gamma(h; A) \) and \( (f, g)_B \) in \( C \), there is no \( c \in \text{cl}(A) \cap \text{cl}(B) \) such that \( f(c) < h(c) < g(c) \).

Before providing the nice consequences of special linear decompositions, we prove that they always exist.

**Lemma 2.6.** For any linear decomposition \( D \) of \( R^n \), there is a special linear decomposition \( C \) of \( R^n \) that refines \( D \) (that is, every linear cell in \( D \) is a union of linear cells in \( C \)).

**Proof.** By induction on \( n \). For \( n = 1 \), take \( C = D \). Now assume that \( n = k + 1 \) and the lemma holds for \( k > 0 \). Let \( D \) be a linear decomposition of \( R^{k+1} \). Choose a finite collection \( F \) of linear maps \( f : R^k \to R \) such that any linear map that appears in the definition of any linear cell from \( D \) is a restriction of a map from \( F \).

Now set

\[ \mathcal{G} = \{ \Gamma(f) \cap \Gamma(g) : f, g \in F \} \text{ and } \mathcal{G}' = \{ \pi(A) : A \in \mathcal{G} \} \cup \pi(D). \]

Clearly, \( \mathcal{G}' \) is a finite collection of definable subsets of \( R^k \). By the linear CDT and the inductive hypothesis, there is a special linear decomposition \( C' \) of \( R^k \) that partitions each member of \( \mathcal{G}' \).

**Claim 2.7.** For any \( f, g \in F \), either \( f < g \) or \( f = g \) or \( f > g \) on any \( B \in C' \).

Let \( B \in C' \) and let \( A = \Gamma(f) \cap \Gamma(g) \). Since \( \pi(A) \) is a union of members of \( C' \), we have either \( B \subseteq \pi(A) \) or \( B \cap \pi(A) = \emptyset \). In the first case \( f = g \) on \( B \). In the second case, \( B \) is a disjoint union of the open definable subsets \( \{ b \in B : f(b) < g(b) \} \) and \( \{ b \in B : g(b) < f(b) \} \). Since \( B \) is definably connected, one of the two sets is equal to \( B \). \( \square \)
Let $C$ be the linear cell decomposition of $R^{k+1}$ with $\pi(C) = C'$ such that for any $B \in C'$ the set of cells in $C$ with domain $B$ is defined by all functions from $F$. Since $C'$ refines $\pi(D)$, the choice of $F$ and Claim 2.7 imply that $C$ refines $D$.

To conclude we need to show that $C$ is special. Let $(f, g)_B \in C$. Then $f, g \in F$ and for any $h \in F$ we have on $B$ either $h < f$, or $h = f$, or $h = g$ or $h > g$, and so either $h(c) \leq f(c)$ or $g(c) \leq h(c)$, for any $c \in \cl(B)$. In particular, for any $\Gamma(h_{|A}) \in C$ there is no $c \in \cl(A) \cap \cl(B)$ such that $f(c) < h(c) < g(c)$.

\[ \square \]

We now aim towards Proposition 2.16 below. But before we will require several preliminary lemmas.

**Lemma 2.8.** Let $(R, <)$ be a dense linear order, $X \subseteq R^n$, and $\overline{X} = \cl(X)$. Let $f, g : \overline{X} \to R$ be continuous functions, and $C = (f, g)_X$. Then

1. $\cl(\Gamma(f|_X)) = \Gamma(f)$;
2. $\cl(C) = [f, g]_{\overline{X}}$.

In particular, $\pi(\cl(C)) = \cl(\pi(C))$.

**Proof.** (1) This is a special case of a general simple fact about continuous maps in Hausdorff topological spaces.

(2) Clearly, $[f, g]_{\overline{X}} = \cl(C')$, where $C' = (f, g)_{\overline{X}}$. So we need to show that $\cl(C') = \cl(C)$. Since $C \subseteq C'$, it is enough to show that $C' \subseteq \cl(C)$. Let $(x, y) \in C'$ and let $U \times (a, b)$ be an open neighborhood of $(x, y)$ with $U$ an open neighborhood of $x$ and $a < y < b$. We may assume that $f(x) < a < y < b < g(x)$. Since $f$ and $g$ are continuous at $x$, there is an open $V$ with $x \in V \subseteq U$ such that $f(v) < a$ and $b < g(v)$ for all $v \in V$. Since $x \in \overline{X}$, there is $v \in V \cap X$; and so, $(v, y) \in (U \times (a, b)) \cap C$. Thus $(x, y) \in \cl(C)$ as required.

\[ \square \]

For $X \subseteq R^n$ a subset, $x = (x_1, \ldots, x_n) \in X$ and $\epsilon \in R^{>0}$ below we let $V_X(x, \epsilon) = \{ u = (u_1, \ldots, u_n) \in X : |x_i - u_i| < \epsilon \text{ for all } i \}$ (the $\epsilon$-neighborhood of $x$ in $X$).

**Lemma 2.9.** If $X \subseteq R^n$ is a linear cell and $x \in X$ then there is $\epsilon \in R^{>0}$ such that $2x - y \in X$ for all $y \in V_X(x, \epsilon)$.

**Proof.** We prove the result by induction on $n$. Let $n = 1$. If $X$ is a singleton, take any $\epsilon \in R^{>0}$. If $X$ is an open interval, take any $\epsilon \in R^{>0}$ such that $V_X(x, \epsilon) \subseteq X$.

Suppose that the result holds for $n$ and we prove it for $n + 1$.

Let $X = \Gamma(f|Z)$, and $x = (z, f(z))$ where $z \in Z$. By induction, there is $\epsilon \in R^{>0}$ such that $2z - u \in Z$ for all $u \in V_Z(z, \epsilon)$. By linearity of $f$, it follows that, if $y = (u, f(u)) \in V_X(x, \epsilon)$, then $2x - y = (2z - u, 2f(z) - f(u)) = (2z - u, f(2z - u)) \in X$ since $u \in V_Z(z, \epsilon)$.

Let $X = (f, g)_Z$, and $x = (z, v)$, where $z \in Z$ and $f(z) < v < g(z)$. Fix $\delta' \in R^{>0}$ such that $v - \delta', v + \delta' \subseteq (f(z), g(z))$ and by continuity of $f$ and $g$ fix $\delta \in R^{>0}$ such that $v - \delta', v + \delta' \subseteq (f(u), g(u))$ for all $u \in V_Z(z, \delta)$. By induction, there is $\epsilon \in R^{>0}$ such that $2z - u \in Z$ for all $u \in V_Z(z, \epsilon)$. Choose $\epsilon < \delta', \delta$. Let $y = (u, w) \in V_X(x, \epsilon)$. Then $u \in V_Z(z, \epsilon)$ and $2z - u \in Z$; $w \in (v - \epsilon, v + \epsilon) \leq (f(u), g(u))$ and so
such that $h$ and $f$ would be disjoint union of the open definable subsets $\{ \epsilon > g \}$ such that $f|_A = h|_A < k|_A$.

Proof. Since $A \subseteq cl(B)$, for any $h$ and $k$ with $(h, k)_B \in C$ we have $h|_A \leq k|_A$. Therefore, there is a linear cell $(h, k)_B \in C$ which is above $D = \Gamma(f)$ and is such that $f|_A = h|_A \neq k|_A$ and $h|_A \leq k|_A$. We show that $h|_A < k|_A$.

If $k = +\infty$ the claim holds. Assume that $k \neq +\infty$ and let $p = k - h$. Then $p|_A \geq 0$. We have to show that $p|_A > 0$. If not let $a \in A$ be such that $p|_A(a) = 0$. By Lemma 2.9 here is $\epsilon \in R^{> 0}$ such that $2a - b \in A$ for all $b \in V_A(a, \epsilon)$. Since $\{ b \in A : p|_A(b) > 0 \}$ is an open definable subset of $A$ which is non empty (because $h|_A \neq k|_A$) and $A$ is definably connected, $\{ b \in A : p|_A(b) = 0 \}$ is a closed non open definable subset of $A$. So there is a $c \in V_A(a, \epsilon)$ such that $p|_A(c) > 0$. But then since $2a - c \in A$, we have $p|_A(2a - c) = 2p|_A(a) - p|_A(c) = -p|_A(c) < 0$ contradicting the fact $p|_A \geq 0$.

Below we also need the following remark.

Remark 2.11. Let $A \subseteq R^n$ be a subset. We say that $A$ is convex if for all $x, y \in A$ and for all $q \in Q \cap [0, 1]$ we have $qx + (1 - q)y \in A$. See [9, Definition 3.1].

The following hold:

- The intersection of two convex sets is convex.
- Every linear cell is convex.

We are now ready to prove the main lemma for what follows below.

Lemma 2.12. Let $C$ be a special linear decomposition of $R^n$, $n > 1$, $D, E \in C$ two linear cells of the form

$$D = \Gamma(f) \text{ and } E = \Gamma(g),$$

where $f \in L(B)$, $g \in L(A)$, and $A \subseteq cl(B)$. Then:

$$f|_A < g \text{ or } f|_A = g \text{ or } f|_A > g.$$

Proof. Assume not. Then there is $c \in A$ such that $f(c) = g(c)$; otherwise $A$ would the be disjoint union of the open definable subsets $\{ x \in A : f(x) < g(x) \}$ and $\{ x \in A : g(x) < f(x) \}$ contradicting the fact that $A$ is definably connected. Since $f|_A \neq g$, there exists $d \in A$ such that $f(d) \neq g(d)$. We may assume that $f(d) < g(d)$. By Lemma 2.10, there is a linear cell $F \in C$ of the form $F = (h, k)_B$ such that $f|_A = h|_A < k|_A$. We next show that there is a point $e \in A$, such that $h(e) < g(e) < k(e)$ which contradicts the fact that $C$ is special.
If \( g(d) < k(d) \), then let \( e = d \). So assume \( k(d) \leq g(d) \). We will choose \( e \) to be “between” \( c \) and \( d \). We first see that there is \( q_0 \in (0,1] \cap \mathbb{Q} \), such that

\[
g_0g(d) + (1-q_0)g(c) < q_0k(d) + (1-q_0)k(c)
\]

Indeed, if not, then \( k(c) \leq g(c) \). But \( g(c) = f(e) < k(c) \), a contradiction. On the other hand, since \( f(d) < g(d) \) and \( f(c) = g(c) \), we have that for every \( q \in (0,1] \cap \mathbb{Q} \),

\[
qf(d) + (1-q)f(c) < qg(d) + (1-q)g(c).
\]

Hence, if we let \( e = q_0d + (1-q_0)c \), then \( e \in A \) (by Remark 2.11) and we have \( f(e) = h(e) < g(e) < k(e) \), proving our claim.

\[\square\]

**Lemma 2.13.** Let \( C \) be a special linear decomposition of \( \mathbb{R}^n \), \( n > 1 \), and \( D, E \in C \) such that \( D \cap cl(E) \neq \emptyset \). Then:

\[
\pi(D) \subseteq cl(\pi(E)) \Rightarrow D \subseteq cl(E).
\]

**Proof.** Let \( A = \pi(D) \) and \( B = \pi(E) \); so \( A \subseteq cl(B) \). We have the following possibilities for \( E \): (1) \( E = \Gamma(f_B) \) or (2) \( E = (f,g)_B \); and the following possibilities for \( D \): (a) \( D = \Gamma(h_A) \) or (b) \( D = (h,k)_A \).

By Lemma 2.8, if (1) then \( cl(E) = \Gamma(f_{cl(B)}) \), and if (2) then \( cl(E) = [f,g]_{cl(B)}. \)

Suppose (1). If (a), since \( D \cap cl(E) \neq \emptyset \), there is \( a \in A \) with \( f(a) = h(a) \) and so by Lemma 2.12, \( f_{|A} = h_{|A} \) and therefore \( D \subseteq cl(E) \). On the other hand, case (b) under (1) cannot happen: as \( C \) is special, there is no \( a \in A \) such that \( h(a) < f(a) < k(a) \) and so \( D \cap cl(E) = \emptyset \), contradicting the assumption of the lemma.

Suppose (2). If (a), since \( D \cap cl(E) \neq \emptyset \), there is \( a \in A \) with \( f(a) \leq h(a) \leq g(a) \). Since \( C \) is special, \( f(a) = h(a) \) or \( h(a) = g(a) \) and so by Lemma 2.12, \( f_{|A} = h_{|A} \) or \( h_{|A} = g_{|A} \), and therefore \( D \subseteq cl(E) \). If (b), there is \( a \in A \) such that \([f(a),g(a)] \) intersects \((h(a),k(a)) \). Since \( C \) is special, we have \( f(a) = h(a) \) and \( g(a) = k(a) \) and so by Lemma 2.12, \( f_{|A} = h_{|A} \) or \( g_{|A} = k_{|A} \) and therefore \( D \subseteq cl(E) \).

\[\square\]

**Corollary 2.14.** Let \( C \) be a special linear decomposition of \( \mathbb{R}^n \), \( n > 0 \), and \( D, E \in C \) such that \( D \cap cl(E) \neq \emptyset \). Then \( D \subseteq cl(E) \).

In particular, \( C \) is a stratification of \( \mathbb{R}^n \).

**Proof.** The statement trivially holds if \( D = E \); hence assume \( D \neq E \). We work by induction on \( n \). For \( n = 1 \), the assumption \( D \cap cl(E) \neq \emptyset \) implies that \( E \) is an open interval and \( D \) is one of its endpoints. So now assume \( n > 1 \).
Clearly, \( \pi(D) \cap cl(\pi(E)) \neq \emptyset \) (using Lemma 2.8), and hence by inductive hypothesis, \( \pi(D) \subseteq cl(\pi(E)) \). By Lemma 2.13, \( D \subseteq cl(E) \).

**Lemma 2.15.** Let \( X \subseteq \mathbb{R}^n \). Then, for any subset \( X \subseteq \mathbb{R}^n \), \( st(X) \) is open.

**Proof.** It suffices to show that \( st(X) \cap cl(E) = \emptyset \) for any \( E \in \mathbb{C} \) with \( st(X) \cap E = \emptyset \). Suppose this is not the case. Then some \( D \in \text{Star}(X) \) meets \( cl(E) \). Then by Corollary 2.14, \( cl(E) \) contains \( D \) and so \( cl(D) \). As \( X \) meets \( cl(D) \), it meets \( cl(E) \), and hence \( E \subseteq st(X) \), which is a contradiction.

**Proposition 2.16.** Let \( \mathcal{C} \) be a special linear decomposition of \( \mathbb{R}^n \), \( n > 0 \), and \( C \in \mathcal{C} \). Then \( U = st(C) \) is an open (usual) cell.

**Proof.** By Lemma 2.15, \( U \) is open. So it remains to prove that \( U \) is a cell. Before that we need a few preliminaries.

Since \( \mathcal{C} \) is a linear decomposition, for every \( B \in \text{Star}(\pi(C)) \), \( \pi^{-1}(B) \cap U \) is a union of linear cells in \( \mathcal{C} \) which are either graphs of linear maps, or cylinders between linear maps, with domain \( B \). By Lemma 2.3(i), \( U \subseteq \bigcup \{ \pi^{-1}(B) : B \in \text{Star}(\pi(C)) \} \), and hence

\[
U = \bigcup \{ \pi^{-1}(B) \cap U : B \in \text{Star}(\pi(C)) \}.
\]

We claim that for every \( B \in \text{Star}(\pi(C)) \),

\[
\pi^{-1}(B) \cap U = (f_B, g_B)_B,
\]

for some \( f_B, g_B \in L_\infty(B) \) with \( f_B < g_B \).

Fix \( B \in \text{Star}(\pi(C)) \). Let \( f_B \) be the bottom function with domain \( B \) defining the bottom cell of \( \pi^{-1}(B) \cap U \) and let \( g_B \) be the top function with domain \( B \) defining the top cell of \( \pi^{-1}(B) \cap U \). (Recall that this latter set is a union of linear cells in \( \mathcal{C} \) which are either graphs of linear maps, or cylinders between linear maps, with domain \( B \).

**Claim 2.17.** If \( C = (l, k)_P \) then \( E = (f_B, g_B)_B \) is the unique cell in \( \text{Star}(C) \) such that \( \pi(E) = B \) and \( f_B|_P = l \) and \( g_B|_P = k \). In particular, \( \pi^{-1}(B) \cap U = (f_B, g_B)_B \).

Suppose that \( E \) is not a cell in \( \text{Star}(C) \). Then there are cells \( \Gamma(h_B^{i_1}), \ldots, \Gamma(h_B^{i_m}) \) in \( \text{Star}(C) \) such that \( f_B < h_B^{i_1} < \cdots < h_B^{i_m} < g_B \). Since \( B \in \text{Star}(\pi(C)) \) we have \( P = \pi(C) \subseteq cl(B) \), and since \( \Gamma(h_B^{i_1}) \in \text{Star}(C) \) we have \( C \subseteq cl(\Gamma(h_B^{i_1})) = \Gamma(h_B^{i_1}_{cl(B)}) \) (by Lemma 2.8). Hence \( cl(C) = [l, k]_{cl(P)} \subseteq \Gamma(h_B^{i_1}_{cl(B)}) \) (by Lemma 2.8) and therefore, \( l = (h_B^{i_{cl(B)}})|_P = k \) which is absurd.

By the choice of \( f_B \) and \( g_B \), \( E = (f_B, g_B)_B \) is then the unique cell in \( \text{Star}(C) \) such that \( \pi(E) = B \). Since \( C \subseteq cl(E) = [f_B, g_B]_{cl(B)} \) we have \( cl(C) = [l, k]_{cl(P)} \subseteq [f_B, g_B]_{cl(B)} \). Since \( C \) is special we must have \( f_B|_P = l \) and \( g_B|_P = k \).

**Claim 2.18.** If \( C = \Gamma(l_P) \) then there are cells \( \Gamma(h_B^{i_1}), \ldots, \Gamma(h_B^{i_m}) \) in \( \text{Star}(C) \) such that \( f_B < h_B^{i_1} < \cdots < h_B^{i_m} < g_B \) and \( h_B^{i_1}|_P = l \). Moreover, \( (h_B^{i_1}, h_B^{i_1+1})_B, (f_B, h_B^{i_1+1})_B \) and \( (h_B^{i_m}, g_B) \) are all cells in \( \text{Star}(C) \). In particular, \( \pi^{-1}(B) \cap U = (f_B, g_B)_B \).

Let \( h_B^{i_1} < \cdots < h_B^{i_m} \) be all the linear functions that appear in the definition of a linear cell of \( \pi^{-1}(B) \cap U \). (Recall that \( U = st(C) \) and \( \pi^{-1}(B) \cap U \) is a union
of linear cells in \( \mathcal{C} \) which are either graphs of linear maps, or cylinders between linear maps, with domain \( B \). Then \( \Gamma(h_{1|B}^1), \ldots, \Gamma(h_{m|B}^m) \) are cells in \( \text{Star}(C) \) and 
\[
(h_{1|B}^1, h_{1|B}^{i+1})_B, (f_B, h_{1|B}^1)_B \text{ and } (h_{m|B}^m, g_B) \text{ are all cells in } \text{Star}(C).
\]

Since \( P \subseteq B \) and \( \Gamma(h_{1|B}^i) \in \text{Star}(C) \), by Lemma 2.12 we must have \( h_{1|B}^i|_P = l \).

We conclude the proof by the proposition by induction on \( n \). If \( n = 1 \), then \( C \) is a point and \( U \) is an open interval or \( C \) is an open interval and \( U = C \). Now assume that \( n = k + 1 \) and the result holds for \( k > 0 \).

Let \( D = \text{st}(\pi(C)) \), \( f = \bigcup_{B \in \text{Star}(\pi(C))} f_B \) and \( g = \bigcup_{B \in \text{Star}(\pi(C))} g_B \). Then

\[
U = (f, g)_D.
\]

By inductive hypothesis, \( D \) is a usual cell. To show that \( f, g \) are continuous, we need to show that for every \( A, B \in \text{Star}(\pi(C)) \), and \( A \subseteq cl(B) \),

\[
(f_B|_A = f_A \text{ and } g_B|_A = g_A)
\]

Indeed, for any \( B, B' \in \text{Star}(\pi(C)) \), if \( cl(B) \cap cl(B') \neq \emptyset \), then the intersection of \( cl(B) \cap cl(B') \) with the domain of \( f \) (resp. \( g \)) is a union of cells \( A \in \mathcal{C} \) such that \( A \subseteq cl(B) \cap cl(B') \) (by Corollary 2.14) and \( A \in \text{Star}(\pi(C)) \).

By Lemma 2.12, there are 3 possibilities: (i) \( f_B|_A > f_A \), (ii) \( f_B|_A = f_A \), (iii) \( f_B|_A < f_A \).

If we assume (i) we get a contradiction since in that case \( U \) is not open. Let us assume (iii).

If \( C = \Gamma(l|_P) \), then since \( C \) is special, by Lemma 2.12, and using the notation of Claim 2.18, we have \( (h_{1|B}^1)|_A \leq f_A \). Since \( P \subseteq cl(A) \subseteq cl(B) \), we have \( l = (h_{1|B}^1)|_P > (f_A)|_P \geq (h_{1|B}^1)|_P = l \), which is absurd.

If \( C = (l, k)_P \), then by Claim 2.17, \( F = (f_A, g_A)_A \) and \( E = (f_B, g_B) \) are in \( \text{Star}(C) \) and \( l = (f_A)|_P = (f_B)|_P \) and \( k = (g_A)|_P = (g_B)|_P \). Since \( C \) is special, by Lemma 2.12, we have \( (g_B)|_A \leq f_A \). Since \( P \subseteq cl(A) \), we have \( (g_B)|_P \leq (f_A)|_P \).

Hence, if \( (x, y) \in C \), then \( l(x) < y < k(x) = (g_B)|_P(x) \leq (f_A)|_P(x) \) and so \( (x, y) \notin [f_A, g_A]_{cl(A)} = cl(F) \). So \( C \subseteq cl(F) \) which is absurd.

**Corollary 2.19.** If \( \mathcal{R} = (\mathcal{R}, <, 0, +, \{\lambda\}_{\lambda \in D}) \) is an ordered vector space over an ordered division ring \( D \), then every non-empty open definable set is a finite union of open cells.

**Proof.** Let \( X \subseteq \mathcal{R}^n \) be an open definable subset and take \( \mathcal{C} \) a special linear decomposition of \( \mathcal{R}^n \) that partitions \( X \). By Lemma 2.3(ii),

\[
X = \bigcup_{C \in \mathcal{C}, C \subseteq X} \text{st}(C).
\]

Then apply Proposition 2.16. \( \square \)

3. THE SEMI-BOUNDED NON-LINEAR CASE

We assume in this section that \( \mathcal{R} \) is semi-bounded and non-linear. So, as we saw in the Introduction, there exists a definable real closed field \( \langle I, 0_I, 1_I, +, \cdot, <_I \rangle \) on some interval \( I \subset \mathcal{R} \) which, without loss of generality, can be assumed to be of the form \( I = (-\epsilon, \epsilon), 0_I = 0 \) and \( <_I \) is the restriction of \( < \) to \( I \). Here we will use the existence of this "short" definable real closed field to adapt Wilkie’s proof ([16]) in
o-minimal expansions of real closed fields.

In the next lemmas the semi-boundedness assumption of $R$ is not required.

**Lemma 3.1** ([16], Lemma 1). Let $C$ be a cell in $R^n$. Then there exists an open cell $D$ in $R^n$ with $C \subseteq D$ and a definable retraction $H : D \to C$ (that is, a continuous map such that $H|_C = \text{id}_C$).

**Lemma 3.2.** Let $C$ be a cell in $R^n$. Suppose that $h : C \to R$ is a continuous definable map and let $U$ be an open definable subset of $R^{n+1}$. Suppose further that $\Gamma(h) \subseteq U$. Then there exist definable maps $f, g : C \to R$ and cells $C_1, \ldots, C_m \subseteq C$ such that:

1. $f < h < g$;
2. $C = C_1 \cup \cdots \cup C_m$;
3. for each $i$, $f|_{C_i}$ and $g|_{C_i}$ are continuous;
4. for each $i$, $\Gamma(h|_{C_i}) \subseteq [f|_{C_i}, g|_{C_i}] \subseteq U$.

**Proof.** Since $U$ is open and $\Gamma(h) \subseteq U$, by definable choice ([2, Chapter 6, (1.2)] there exists definable maps $f, g : C \to R$ such that $f < h < g$ and $[f, g]C \subseteq U$. By cell decomposition, there are cells $C_1, \ldots, C_m \subseteq C$ covering $C$ such that for each $i$, $f|_{C_i}$ and $g|_{C_i}$ are continuous. Now the rest is clear.

The following is also needed:

**Lemma 3.3.** Let $C$ be a cell in $R^n$. Suppose that $f, g : C \to R$ are continuous definable maps such that $f < g$ and let $V, W \subseteq U$ be open definable subsets of $R^{n+1}$. Suppose further that $(f, g)C \subseteq U$, $\Gamma(f) \subseteq V$ and $\Gamma(g) \subseteq W$. Then there exist definable maps $f', g' : C \to R$ and cells $C_1, \ldots, C_m \subseteq C$ such that:

1. $C = C_1 \cup \cdots \cup C_m$;
2. for each $i$, $f'|_{C_i}$ and $g'|_{C_i}$ are continuous;
3. $f < f' < g'$;
4. for each $i$, $\Gamma(f'|_{C_i}) \subseteq V$ and $\Gamma(g'|_{C_i}) \subseteq W$;
5. for each $i$, $(f'|_{C_i}, g'|_{C_i})C_i \subseteq U$, $(f|_{C_i}, g|_{C_i})C_i \subseteq U$ and $[f|_{C_i}, g|_{C_i}]C_i \subseteq U$.

**Proof.** Since $(f, g)C \subseteq U$, $\Gamma(f) \subseteq V$ and $\Gamma(g) \subseteq W$ and $V, W \subseteq U$ be open definable subsets of $R^{n+1}$, by definable choice ([2, Chapter 6, (1.2)] there exists definable maps $f', g' : C \to R$ such that

1. $f < f' < g'$;
2. $\Gamma(f') \subseteq V$ and $\Gamma(g') \subseteq W$;
3. $(f', g')C \subseteq U$, $(f, g)C \subseteq U$ and $[f', g']C \subseteq U$.

By cell decomposition, there are cells $C_1, \ldots, C_m \subseteq C$ covering $C$ such that for each $i$, $f|_{C_i}$ and $g|_{C_i}$ are continuous. Now the rest is clear.

Below we let

$$d^{(n)}(\overline{x}, \overline{y}) = \max\{|x_i - y_i|_1 \leq n\}$$

denote the standard distance in $R^n$ (where we denote by $\overline{z} = (z_1, \ldots, z_n)$ the elements of $R^n$). This distance is a continuous definable function (by [2, Chapter 6 (1.4)]). Moreover, if $B \subseteq R^n$ is a nonempty definable subset and $\overline{a} \in R^n$, then

$$d^n(\overline{a}, B) = \inf\{d^n(\overline{a}, \overline{z}) : \overline{z} \in B\}$$
is well defined (by ([2, Chapter 1 (3.3)]) and \( d^n(\pi,B) = 0 \) if and only if \( \pi \in cl(B) \) (the if part of this equivalence is immediate and for the only if part one can use the curve selection ([2, Chapter 6 (1.5)])).

Let \( \pi : R^{n+1} \rightarrow R^n \) be the projection onto the first \( n \) coordinates. We say that an open definable subset \( U \) of \( R^{n+1} \) has \( I \)-short height if for every \( \overline{r} \in \pi(U) \) we have

\[
\sup \{|t - s| : t, s \in U_{\overline{r}}\} \in I
\]

where \( U_{\overline{r}} = \{y \in R : (\overline{r}, y) \in U\} \).

We now prove the analogue of [16, Lemma 2] for open definable subsets with \( I \)-short height. The argument of the proof is similar, one just has to observe that the field operations are used in Wilkie’s proof in a uniform way and only along fibers. Since in our case our fibers are \( I \)-short, such field operations, in the field \( I \), can also be used in exactly the same way.

For completeness we include the details of the proof but at the end we follow a more constructive argument suggested to us by Oleg Belegradek. For that we need the following observations which are true in arbitrary o-minimal expansions of ordered groups:

**Remark 3.4.** If \( \theta : [a, b] \rightarrow [c, d] \) is a continuous, definable, strictly decreasing function, \( \theta(a) = d \) and \( \theta(b) = c \), then \( \theta \) is bijective.

Indeed, as \( \theta \) is definable and continuous, \( \theta([a, b]) \) is definable, closed, and bounded by [2, Chapter 6 (1.10)], and hence it a finite union of closed intervals and singletons, by \( o \)-minimality. Since \( \theta \) is strictly decreasing, \( \theta([a, b]) \) is densely ordered, and so is a closed interval, which must be \([c, d]\).

**Remark 3.5.** Let \( V \subseteq R^n \) be an open definable subset, and let \( \{\theta_{\overline{r}} : [a_{\overline{r}}, b_{\overline{r}}] \rightarrow [c_{\overline{r}}, d_{\overline{r}}]\}_{\overline{r} \in V} \) be a uniformly definable family of strictly decreasing functions with \( \theta_{\overline{r}}(a_{\overline{r}}) = d_{\overline{r}} \) and \( \theta_{\overline{r}}(b_{\overline{r}}) = c_{\overline{r}} \). (So by the previous remark all \( \theta_{\overline{r}} \)'s are bijective).

Suppose that all \( a_{\overline{r}}, b_{\overline{r}}, c_{\overline{r}}, d_{\overline{r}} \) are continuous functions in \( \overline{r} \), and moreover, the map

\[
\{\langle \overline{r}, y \rangle : \overline{r} \in V \text{ and } y \in [a_{\overline{r}}, b_{\overline{r}}] \} \rightarrow R : (\overline{r}, y) \mapsto \theta_{\overline{r}}(y)
\]

is continuous. Then the map

\[
\gamma : \{\langle \overline{r}, z \rangle : \overline{r} \in V \text{ and } z \in [a_{\overline{r}}, d_{\overline{r}}] \} \rightarrow R : (\overline{r}, y) \mapsto \theta_{\overline{r}}^{-1}(z)
\]

is continuous. Indeed, for each \( \overline{r} \), let \( \overline{\theta}_{\overline{r}} : R \rightarrow R \) be given by

\[
\overline{\theta}_{\overline{r}}(y) = \begin{cases} 
\alpha_{\overline{r}} + \beta_{\overline{r}} - y & \text{for } y < \alpha_{\overline{r}} \\
\beta_{\overline{r}} & \text{for } \alpha_{\overline{r}} \leq y \leq \beta_{\overline{r}} \\
\beta_{\overline{r}} + \gamma_{\overline{r}} - y & \text{for } y > \beta_{\overline{r}}
\end{cases}
\]

Then \( \{\overline{\theta}_{\overline{r}} : R \rightarrow R\}_{\overline{r} \in V} \) is a uniformly definable family of strictly decreasing functions such that \( \overline{\theta}_{\overline{r}} \) extends \( \theta_{\overline{r}} \) for all \( \overline{r} \in V \), and \( V \times R \rightarrow R : (\overline{r}, y) \mapsto \overline{\theta}_{\overline{r}}(y) \) is a continuous function.

Now \( \overline{\pi} : V \times R \rightarrow R : (\overline{r}, y) \mapsto \overline{\theta}_{\overline{r}}^{-1}(z) \) is also a continuous function since, for any \((a,b) \subseteq R\),

\[
\overline{\pi}^{-1}((a,b)) = \{(\overline{r}, z) \in V \times R : a < \overline{\theta}_{\overline{r}}^{-1}(z) < b\} = \{(\overline{r}, z) \in V \times R : \overline{\theta}_{\overline{r}}(a) < z < \overline{\theta}_{\overline{r}}(b)\},
\]
which is open. Therefore, since $\gamma$ extends $\gamma$, we have that $\gamma$ is also continuous as required.

**Lemma 3.6.** Let $C$ be a cell in $R^n$. Suppose that $f, g : C \to R$ are continuous definable maps such that $f < g$ and let $U$ be an open definable subset of $R^{n+1}$ with $I$-short height. Suppose further that $[f,g]_C \subseteq U$ (respectively $(f,g)_C \subseteq U$). Then there exists an open definable subset $V$ of $R^n$ and continuous definable maps $F, G : V \to R$ such that:

1. $C \subseteq V$;
2. $F|_C = f$ and $\Gamma(F) \subseteq U$ (respectively $\Gamma(G) \subseteq U$);
3. $G|_C = g$;
4. $F < G$;
5. for all $\bar{x} \in V$ and all $y \in R$ with $F(\bar{x}) \leq y < G(\bar{x})$, (respectively $F(\bar{x}) < y \leq G(\bar{x})$), $(\bar{x}, y) \in U$.

**Proof.** We prove the unparenthesized statement, the parenthetical one being similar.

Applying Lemma 3.1 we obtain an open cell $D$ in $R^n$, with $C \subseteq D$, and a continuous definable retraction $H : D \to C$.

Let

$$V = \{ \bar{x} \in D : d^{(n)}(\bar{x}, H(\bar{x})) < d^{(n+1)}((\bar{x}, f \circ H(\bar{x})), U^c) \},$$

where $U^c = R^{n+1}\setminus U$. Clearly $V$ is open in $R^n$ and (1) holds since $\Gamma(f) \subseteq U$. Putting $F = f \circ H|_V$ we see that (2) holds. Also note that for all $\bar{x} \in V$, $F(\bar{x}) < g \circ H(\bar{x})$ and

$$J_{\bar{x}} := [0, g \circ H(\bar{x}) - F(\bar{x})] \subseteq \{ t \in R_{\geq 0} : F(\bar{x}) + t \in U_{\bar{x}} \} \subseteq I$$

since $U$ has $I$-short height.

By o-minimality and the fact that $\Gamma(F) \subseteq U$, there are well defined definable maps $z_0 : V \to I$ and $y_0 : V \to R$ given by

$$z_0(\bar{x}) = \sup \{ t \in J_{\bar{x}} : [F(\bar{x}), F(\bar{x}) + t] \subseteq U_{\bar{x}} \}$$

and

$$y_0(\bar{x}) = F(\bar{x}) + z_0(\bar{x}).$$

Now observe that $y_0 : V \to R$ satisfies the conditions (3), (4) and (5) for $G$ ((3) is satisfied because $(f,g)_C \subseteq U$, by hypothesis, and $f = F|_C$), but maybe $y_0$ is not continuous. Thus we need to find a continuous definable map $G : V \to R$ such that $F < G \leq y_0$ and $G|_C = y_0$.

Consider the definable set

$$S = \{ (\bar{x}, y) \in R^{n+1} : \bar{x} \in V \text{ and } F(\bar{x}) \leq y \leq g \circ H(\bar{x}) \}$$

and the definable continuous maps $\theta_1, \theta_2 : S \to I$ given by

$$\theta_1(\bar{x}, y) = 1_I - 1_I(y - F(\bar{x})) - 1_I(g \circ H(\bar{x}) - F(\bar{x}))^{-1}_I,$$

where $1_I$ is the neutral element for the multiplication $\cdot$, $-1_I$ is the difference and $-1_I$ is inversion in the field $I$, and

$$\theta_2(\bar{x}, y) = \inf\{ d^{(n+1)}((\bar{x}, t), U^c) : F(\bar{x}) \leq t \leq y \}.$$ 

Note that since $U$ has I-short height we do have $\theta_1(S) \subseteq I$ and $\theta_2(S) \subseteq I$.

Fix $\bar{x} \in V$. Then the continuous definable map $(\theta_1 \cdot \theta_2)(\bar{x}, -)$ decreases monotonically and strictly from $d^{(n+1)}((\bar{x}, F(\bar{x})), U^c)$ to $0_I = 0$ on $[F(\bar{x}), y_0(\bar{x})]$ and is identically $0_I = 0$ on $[y_0(\bar{x}), g \circ H(\bar{x})]$. 

For $\mathbf{F} \in V$ let
\[ a_{\mathbf{F}} = F(\mathbf{F}), \quad b_{\mathbf{F}} = \gamma_0(\mathbf{F}), \quad c_{\mathbf{F}} = 0, \quad d_{\mathbf{F}} = d^{(n+1)}((\mathbf{F}, F(\mathbf{F})), U^c), \]
and $\partial_{\mathbf{F}}(-) = (\partial_1 \cdot \partial_2)(\mathbf{F}, -)$ : $[a_{\mathbf{F}}, b_{\mathbf{F}}] \rightarrow [a_{\mathbf{F}}, d_{\mathbf{F}}]$. Then by Remark 3.5,
\[ G : V \rightarrow R : \mathbf{F} \mapsto \partial_{\mathbf{F}}^{-1}(d^{(n)}(\mathbf{F}, H(\mathbf{F}))) \]
is a continuous definable function. Moreover, $a_{\mathbf{F}} < G(\mathbf{F}) \leq b_{\mathbf{F}}$ for all $\mathbf{F} \in V$. In fact, if not then $a_{\mathbf{F}} = G(\mathbf{F})$ and we obtain $(\partial_1 \cdot \partial_2)(\mathbf{F}, G(\mathbf{F})) = d^{(n+1)}((\mathbf{F}, F(\mathbf{F})), U^c)$ contradicting the fact that $d^{(n)}(\mathbf{F}, H(\mathbf{F})) < d^{(n+1)}((\mathbf{F}, F(\mathbf{F})), U^c)$. We also have $G(\mathbf{F}) = b_{\mathbf{F}}$ for all $\mathbf{F} \in C$. Therefore, $G$ satisfies (3), (4) and (5) as required.

We need one more lemma:

**Lemma 3.7.** Let $C$ be a cell in $R^n$. Suppose that $f, g : C \rightarrow R$ are continuous definable maps such that $f < g$ and let $U$ be an open definable subset of $R^{n+1}$. Suppose further that $[f, g]_C \subseteq U$. Then there exists an open definable subset $W$ of $R^n$ and continuous definable maps $F, G : W \rightarrow R$ such that:

1. $C \subseteq W$;
2. $F_C = f$ and $\Gamma(F) \subseteq U$;
3. $G_C = g$ and $\Gamma(G) \subseteq U$;
4. $F < G$;
5. for all $\mathbf{F} \in W$ and all $c \in R$ with $F(\mathbf{F}) \leq c \leq G(\mathbf{F})$, $(\mathbf{F}, y) \in U$.

**Proof.** Applying Lemma 3.1 we obtain an open cell $D$ in $R^n$, with $C \subseteq D$, and a continuous definable retraction $H : D \rightarrow C$.

Let $W'$ be the intersection of
\[ \{ \mathbf{F} \in D : d^{(n)}(\mathbf{F}, H(\mathbf{F})) < d^{(n+1)}((\mathbf{F}, f \circ H(\mathbf{F})), U^c) \} \]
and
\[ \{ \mathbf{F} \in D : d^{(n)}(\mathbf{F}, H(\mathbf{F})) < d^{(n+1)}((\mathbf{F}, g \circ H(\mathbf{F})), U^c) \} \]
where $U^c = R^{n+1} \setminus U$. Clearly $W'$ is open in $R^n$ and (1) holds for $W'$ since $\Gamma(f), \Gamma(g) \subseteq U$. Also (2) and (3) hold for $f \circ H_{|W'}$ and $g \circ H_{|W'}$. Also note that for all $\mathbf{F} \in W'$, $f \circ H_{|W'}(\mathbf{F}) < g \circ H_{|W'}(\mathbf{F})$ so (4) holds for $f \circ H_{|W'}$ and $g \circ H_{|W'}$.

Let $B = \{ f \circ H_{|W'}, g \circ H_{|W'} \} \setminus U$ where
\[ f \circ H_{|W'}, g \circ H_{|W'} = \{ (\mathbf{F}, y) \in W' \times R : y \in [f \circ H_{|W'}(\mathbf{F}), g \circ H_{|W'}(\mathbf{F})] \}, \]
and let
\[ W = W' \setminus \overline{\pi(B)}. \]
Clearly $W$ is open. We now show that $C \subseteq W$, verifying in this way (1). Suppose not and let $c = (c_1, \ldots, c_n) \in C$ be such that $c \in \overline{\pi(B)}$. Let $\epsilon > 0$ be such that $E = \Pi_{i=1}^n[c_i - \epsilon, c_i + \epsilon] \subseteq W'$. By definable choice there is a definable map $\alpha : (0, \epsilon) \rightarrow \pi(B) \cap E$ such that $\lim_{\epsilon \rightarrow 0^+} \alpha(t) = c$. By replacing $\epsilon$ we may assume that $\alpha$ is continuous. Again by definable choice, we see that there exists a definable map $\beta : (0, \epsilon) \rightarrow B \cap [f \circ H_{|E}, g \circ H_{|E}]$ such that $\pi \circ \beta = \alpha$. By replacing $\epsilon$ we may assume that $\beta$ is continuous. Since the definable set $B \cap [f \circ H_{|E}, g \circ H_{|E}]$ is closed and, by [3, Proposition 3.1 (3)], $\beta((0, \epsilon))$ is bounded, the limit $\lim_{\epsilon \rightarrow 0^+} \beta(t)$ exists in this set. If $d$ is this limit, then $\pi(d) = c$ since $\pi \circ \beta = \alpha$. So $d = [f \circ H_{|W'}(c), g \circ H_{|W'}(c)] \cap B \neq \emptyset$ contradicting the fact that $[f \circ H_{|W'}(c), g \circ H_{|W'}(c)] = [f(c), g(c)] \subseteq U$. 
If we put $F = f \circ H_W$ and $G = g \circ H_W$ we see that (2), (3) and (4) hold. On the other hand, if $\pi \in W$ and $y \in R$ are such that $F(\pi) \leq y < G(\pi)$ and, by absurd, $(\pi, y) \notin U$, then $(\pi, y) \in B$ and so $\pi \in \pi(B) \subseteq \pi(B)$ contradicting the fact that $\pi \notin \pi(B)$. Thus (5) also holds. \hfill \Box

Combining Lemmas 3.6 and 3.7 we obtain:

**Lemma 3.8.** Let $C$ be a cell in $\mathbb{R}^n$. Suppose that $f, g : C \to R$ are continuous definable maps such that $f < g$ and let $U$ be an open definable subset of $\mathbb{R}^{n+1}$. Suppose further that $[f, g]_C \subseteq U$ (respectively $(f, g)_C \subseteq U$). Then there exists a cell decomposition $C_1, \ldots, C_l$ of $C$ and for each $i = 1, \ldots, l$ there is an open definable subset $V_i$ of $\mathbb{R}^n$ and continuous definable maps $F_i, G_i : V_i \to R$ such that:

1. $C_i \subseteq V_i$;
2. $F_i|_{C_i} = f|_{C_i}$ and $\Gamma(F_i) \subseteq U$ (respectively $\Gamma(G_i) \subseteq U$);
3. $G_i|_{C_i} = g|_{C_i}$;
4. $F_i < G_i$;
5. for all $\pi \in V_i$ and all $y \in R$ with $F_i(\pi) \leq y < G_i(\pi)$, (respectively $F_i(\pi) < y \leq G_i(\pi)$), $(\pi, y) \in U$.

**Proof.** We prove the unparenthesized statement, the parenthetical one being similar.

Let $H : D \to C$ be as in Lemma 3.1. Choose $\epsilon \in I$ such that $2\epsilon \in I$ and put

$$U_f = U \cap ((f \circ H) - \epsilon, (f \circ H) + \epsilon)_D$$

and

$$U_g = U \cap ((g \circ H) - \epsilon, (g \circ H) + \epsilon)_D.$$ 

Then clearly $U_f$ and $U_g$ are open definable subsets of $U$ with $I$-short height. For example, if $(\pi, y) \in U_f$, then $(f \circ H)(\pi) - \epsilon < y < (f \circ H)(\pi) + \epsilon$.

Since $(f, g)_C \subseteq U$, $\Gamma(f) \subseteq U_f$ and $\Gamma(g) \subseteq U_g$, by Lemma 3.3, there exist definable maps $f', g' : C \to R$ and cells $C_1, \ldots, C_m \subseteq C$ such that:

1. $C = C_1 \cup \cdots \cup C_m$;
2. for each $i$, $f'|_{C_i}$ and $g'|_{C_i}$ are continuous;
3. $f < f' < g$;
4. for each $i$, $\Gamma(f'|_{C_i}) \subseteq U_f$ and $\Gamma(g'|_{C_i}) \subseteq U_g$;
5. for each $i$, $(f'|_{C_i}, g'|_{C_i}) \subseteq U$ and $(f'|_{C_i}, g'|_{C_i}, C_i) \subseteq U$.

Fix $i = 1, \ldots, m$. Then we can apply Lemma 3.6 to the data $(U_f, f|_{C_i}, f'|_{C_i})$ and obtain the data $(V_f, F_1, F'_1)$ satisfying (1) to (5) of that lemma. Similarly, we can apply Lemma 3.6 to the data $(U_g, g|_{C_i}, g'|_{C_i})$ and obtain the data $(V_g, G_1, G'_1)$ satisfying (1) to (5) of that lemma. On the other hand, we can apply Lemma 3.7 to the data $(U, f'|_{C_i}, g'|_{C_i})$ and obtain the data $(W, F', G')$ satisfying (1) to (5) of that lemma.

Take $V_i = V_f \cap V_g \cap W$ and set $F = F_1|_{V_i}$, $G = G_1|_{C_i}$. Then clearly (1) to (5) hold. \hfill \Box

The following is also required:

**Lemma 3.9.** Let $C$ be a cell in $\mathbb{R}^n$. Suppose that $k : C \to R$ is a continuous definable map and let $U$ be an open definable subset of $\mathbb{R}^{n+1}$. Suppose further that
Then there exists an open definable subset $W$ of $R^n$ and a continuous definable map $K : W \to R$ such that:

1. $C \subseteq W$;
2. $K|_{C} = k$ and $\Gamma(K) \subseteq U$;
3. for all $\mathfrak{r} \in W$ and all $y \in R$ with $K(\mathfrak{r}) \leq y$ (respectively $y \leq K(\mathfrak{r})$), $(\mathfrak{r}, y) \in U$.

Proof. We prove the unparenthesized statement, the parenthetical one being similar.

Applying Lemma 3.1 we obtain an open cell $D$ in $R^n$, with $C \subseteq D$, and a continuous definable retraction $H : D \to C$.

Let

$$W' = \{ (\mathfrak{r} \in D : d^{(n)}(\mathfrak{r}, H(\mathfrak{r})) < d^{(n+1)}((\mathfrak{r}, k \circ H(\mathfrak{r})), U^c) \}$$

where $U^c = R^{n+1} \setminus U$. Clearly $W'$ is open in $R^n$ and (1) holds for $W'$ since $\Gamma(k) \subseteq U$.

Also (2) holds for $k \circ H_{|W'}$.

Let $B = [k \circ H_{|W'}, +\infty]_{W'} \setminus U$ where

$$[k \circ H_{|W'}, +\infty]_{W'} = \{ (\mathfrak{r}, y) \in W' \times R : k \circ H_{|W'}(\mathfrak{r}) \leq y \}$$

and let

$$W = W' \setminus \pi(B).$$

Clearly $W$ is open. We now show that $C \subseteq W$, verifying in this way (1). Suppose not and let $c = (c_1, \ldots, c_n) \in C$ be such that $c \in \pi(B)$. Let $\epsilon > 0$ be such that $E = \Pi^n_{i=1} [c_i - \epsilon, c_i + \epsilon] \subseteq W'$. By definable choice there is a definable map $\alpha : (0, \epsilon) \to \pi(B) \cap E$ such that $\lim_{t \to 0^+} \alpha(t) = c$. By replacing $\epsilon$ we may assume that $\alpha$ is continuous. Again by definable choice, we see that there exists a definable map $\beta : (0, \epsilon) \to B \cap [k \circ H_{|E}, +\infty]_E$ such that $\pi \circ \beta = \alpha$. By replacing $\epsilon$ we may assume that $\beta$ is continuous. Since the definable set $B \cap [k \circ H_{|E}, +\infty]_E$ is closed and, by [3, Proposition 3.1 (3)], $\beta((0, \epsilon))$ is bounded, the limit $\lim_{t \to 0^+} \beta(t)$ exists in this set. If $d$ is this limit, then $\pi(d) = c$ since $\pi \circ \beta = \alpha$. So $d \in [k \circ H_{|E}(c), +\infty] \cap B \neq \emptyset$ contradicting the fact that $[k \circ H_{|E}(c), +\infty] = [k(c), +\infty] \subseteq U$.

If we put $K = k \circ H_{|W}$ we see that (2) holds. On the other hand, if $\mathfrak{r} \in W$ and $y \in R$ are such that $K(\mathfrak{r}) \leq y$ and, by absurd, $(\mathfrak{r}, y) \notin U$, then $(\mathfrak{r}, y) \in B$ and so $\mathfrak{r} \in \pi(B) \subseteq \overline{\pi(B)}$ contradicting the fact that $\mathfrak{r} \notin \overline{\pi(B)}$. Thus (3) also holds. □

Corollary 3.10. If $R$ is a semi-bounded non-linear o-minimal expansion of an ordered group, then every non-empty open definable set is a finite union of open cells.

Proof. This is done by induction on the dimension of the open definable set. For dimension one this is clear. Let $U$ be an open definable subset of $R^{n+1}$. Let $D$ be a cell decomposition of $R^{n+1}$ partitioning $U$. Clearly it is enough to show that each cell $D \in D$ with $D \subseteq U$ can be covered by finitely many open cells (in $R^{n+1}$) each of which is contained in $U$.

Case A: $D = (f, g)_C$ for some cell $C$ in $R^n$ and continuous definable maps $f, g : C \to R$ such that $f < g$.

Let $f' = \frac{f + g}{2}$ and $g' = \frac{f + 2g}{3}$. Then $f', g' : C \to R$ are continuous definable maps such that

- $f < f' < g' < g$;
- $\Gamma(f') \subseteq U$ and $\Gamma(g') \subseteq U$;
• \((f',g)_C \subseteq U\) and \((f,g')_C \subseteq U\).

Now apply Lemma 3.8 to the data \((C, U, f, g')\) and obtain the data \((C_i, V_i, F_i, G_i')\) with \(i = 1, \ldots, l\) satisfying (1) to (5) of that lemma. By the inductive hypothesis there exists a finite collection \(A_i\) of open cell in \(R^n\) contained in \(V_i\) which cover \(V_i\). By (4) and (5) of Lemma 3.8, for each \(A \in A_i\), \((F_{i|A}, G_{i|A})_A\) is an open cell in \(R^{n+1}\) contained in \(U\), and by (1), (2) and (3) of that lemma, \((f_{|C_i}, g_{|C_i})_C \subseteq \bigcup\{(F_{i|A}, G_{i|A})_A : A \in A_i\}\). Thus \((f, g')_C \subseteq \bigcup\{(F_{i|A}, G_{i|A})_A : A \in A_i\} \cup (f_i, g_i')_{\mathbb{R}}\). Hence the same is true for \((f, g)_C = (f, g')_C \cup (f_i, g_i')_{\mathbb{R}}\).

Case B: \(D = \Gamma(h)\) for some continuous definable map \(h : C \to R\) where \(C\) is a cell in \(R^n\). This case reduces to Case A above by Lemma 3.2.

Case C: \(D = (k, +\infty)_C\) (respectively \(D = (-\infty, k)_C\)) for some cell \(C\) in \(R^n\) and continuous definable map \(k : C \to R\).

Then we can apply Lemma 3.9 to the data \((C, U, f, g')\) and obtain the data \((C, W, K)\) satisfying (1) to (3) of that lemma. By the inductive hypothesis there exists a finite collection \(A\) of open cell in \(R^n\) contained in \(W\) which cover \(W\). By (3) of Lemma 3.9, for each \(A \in A\), \((K_{|A}, +\infty)_A\) is an open cell in \(R^{n+1}\) contained in \(U\), and by (1) and (2) of that lemma, \((k_{|C}, +\infty)_C \subseteq \bigcup\{(K_{|A}, +\infty)_A : A \in A\}\).

Similarly for the case \(D = (-\infty, k)_C\).

\[\square\]

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Universidade Aberta and CMAF Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal
E-mail address: edmundo@cii.fc.ul.pt

Department of Pure Mathematics, University of Waterloo, 200 University Ave West, N2L 3G1 Waterloo, Ontario, Canada
E-mail address: pelefthe@uwaterloo.ca

CMAF Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal
E-mail address: lprelli@math.unipd.it