Leibniz-Yang-Mills Gauge Theories and the 2-Higgs Mechanism

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A quadratic Leibniz algebra \((\mathcal{V}, [~,~], \kappa)\) gives rise to a canonical Yang-Mills type functional \(S\) over every space-time manifold. The gauge fields consist of 1-forms \(A\) taking values in \(\mathcal{V}\) and 2-forms \(B\) with values in the subspace \(\mathcal{W} \subset \mathcal{V}\) generated by the symmetric part of the bracket. If the Leibniz bracket is anti-symmetric, the quadratic Leibniz algebra reduces to a quadratic Lie algebra, \(B \equiv 0\), and \(S\) becomes identical to the usual Yang-Mills action functional. We describe this gauge theory for a general quadratic Leibniz algebra. We then prove its (classical and quantum) equivalence to a Yang-Mills theory for the Lie algebra \(\mathfrak{g} = \mathcal{V}/\mathcal{W}\) to which one couples massive 2-form fields living in a \(\mathfrak{g}\)-representation. Since in the original formulation the \(B\)-fields have their own gauge symmetry, this equivalence can be used as an elegant mass-generating mechanism for 2-form gauge fields, thus providing a "higher Higgs mechanism" for those fields.

INTRODUCTION

Leibniz algebras are a simple generalization of Lie algebras, even though they have been much less studied than their antisymmetric specialization. Recently [1–4] it became clear that the gauge sector within gauged supergravity theories constructed by means of the "method of the embedding tensor" [5–11] is based on precisely non-Lie Leibniz algebras—although, in such applications, such theories are generally based on many more, mathematically only partially clarified additional data than the one of a Leibniz algebra [12–16].

It is thus remarkable, that already a Leibniz algebra alone can be used to define a consistent generalization of a Yang-Mills functional. Implicitly this is already a consequence of [15]—the functional [9] and its gauge transformations (10), (11) below result from a specialization and simplification of the formulas found there for the case of ordinary Leibniz algebras—but this particular case was neither clearly highlighted nor worked out in any detail.

We want to fill this gap in the present note: First we present the gauge fields, their field strengths, and, last but not least, the gauge symmetries, using nothing else but the one given Leibniz relation, equation (1) below. Therefore, the corresponding calculations and framework can be useful also in the context of other, more general considerations, not relying on the particular functional we study here, namely the Leibniz-YM functional \(S\). Its definition requires an additional non-degenerate bilinear form \(\kappa\), for which gauge invariance induces a suitable compatibility condition, namely equation (2) below. This additional structure will be shown to restrict the original Leibniz algebra considerably, requiring a Loday cohomology class, characterizing the given Leibniz algebra otherwise, to be zero. As we will see, this in turn makes it possible to reformulate the theory in purely Lie algebraic terms. What we find is that in this reformulation, one obtains an ordinary YM-gauge theory for a Lie algebra \(\mathfrak{g}\), to which, however, the originally present 2-form gauge fields are coupled in a \(\mathfrak{g}\)-representation. Somewhat surprisingly, we find that they now have an induced mass, all of them. This is physically interesting for several reasons, one of which being that if massless, we should have observed the 2-form fields already in experiments.

Although the gauge theory presented here may be viewed upon also as a particular higher gauge theory [17], which therefore can be related to \(L_\infty\)-algebras [18, 19] and for the description of which super-geometrical methods are most suitable [20, 21], we explicitly avoid this language here so as to keep the presentation self-contained and more widely accessible. We remark at this point, however, that while the presence of an \(L_\infty\)-algebra is necessary for the construction of a higher gauge theory [20], it is not sufficient, at least if one wants non-trivial interactions between different form-degree gauge fields. Correspondingly, although every Leibniz algebra is shown to give rise to an \(L_\infty\)-algebra in a canonical way in [1], this does not yet guarantee the existence of a consistent and non-trivial gauge invariant functional for gauge fields of arbitrarily high form degree. The present construction is only the first step into the direction of the conjecturally present hierarchy of higher gauge theories given for every Leibniz algebra with adequate bilinear forms.

The structure of the paper is as follows: We first present the gauge fields ("connections"), their field strengthes ("curvatures"), and the Leibniz-YM functional. The gauge symmetries and a proof of gauge invariance of \(S\) is subject of the subsequent section. The techniques and most formulas presented there hold true for every Leibniz algebra; only in the very last step, the invariance of \(S\), the existence of a compatible \(\kappa\) is used. In the ensuing section, however, we make use precisely of the restrictions posed on a Leibniz algebra by \(\kappa\) and turn to the main finding of this paper, namely the above-mentioned equivalence of the Leibniz-YM functional with a Yang-Mills theory coupled to massive 2-form fields that one obtains by an appropriate field redefinition. We conclude with a short summary and outlook.
LEIBNIZ ALGEBRAS, THE GAUGE FIELDS, AND THE FUNCTIONAL

Let \((\mathcal{V}, [\cdot, \cdot])\) be a Leibniz algebra, i.e., a vector space \(\mathcal{V}\) equipped with a bilinear product, denoted by a bracket, which satisfies

\[
[x, [y, z]] = [[x, y], z] + [y, [x, z]]
\]

for all \(x, y, z \in \mathcal{V}\). We call it quadratic, if it carries a non-degenerate symmetric bilinear form \(\kappa\) satisfying the obvious invariance condition

\[
\kappa([x, y], z) + \kappa(y, [x, z]) = 0.
\]

Evidently, if the bracket is antisymmetric, \([x, y] = -[y, x]\), a (quadratic) Leibniz algebra becomes a (quadratic) Lie algebra.

The Yang-Mills (YM) functional defined for every quadratic Lie algebra plays an essential role in the high energy physics of elementary particles. Here we show that this functional can be defined already for a quadratic Leibniz algebra, without the need to specify any additional data. And its definition reduces to the usual YM-theory precisely for those Leibniz algebras which are also Lie algebras.

For its description we first split the bracket into its symmetric and anti-symmetric parts,

\[
[x, y]_\pm := \frac{1}{2} ([x, y] \pm [y, x]).
\]

Let us denote the vector space generated by the image of the symmetric part by \(\mathcal{W}\),

\[
\mathcal{W} := \{ x \in \mathcal{V} \exists y, z \in \mathcal{V}, x = [y, z]_+ \}.
\]

The gauge fields of the Leibniz-YM theory are

\[
A \in \Omega^1(\mathcal{M}, \mathcal{V}) \quad \text{and} \quad B \in \Omega^2(\mathcal{M}, \mathcal{W}),
\]

i.e., 1-forms and 2-forms on spacetime \(\mathcal{M}\) with values in \(\mathcal{V}\) and \(\mathcal{W}\), respectively. Evidently, if the symmetric part of the bracket vanishes, and only then, the Leibniz algebra becomes a Lie algebra, \(\mathcal{W}\) becomes the 0-vector inside \(\mathcal{V}\), and these gauge fields reduce to the Lie algebra valued connection 1-forms we are used to from ordinary YM gauge theories. Next we define the generalization of the curvature 2-forms \(F\):

\[
F = dA + \frac{1}{2}[A, A] - B,
\]

\[
G = dB + [A, B] - 2 + \frac{1}{6}[A, A]_+.
\]

To avoid any misunderstanding, some remark on the notation: If there is a product of differential forms, a wedge product is understood. Thus, using a basis \(e_a\) of \(\mathcal{V}\), we have, e.g., \(A = A^a \otimes e_a\) for (ordinary) 1-forms \(A^a = A_\mu(x)dx^\mu\), and a term like \([A, A]\) stands for \(A^a \wedge A^b \otimes [e_a, e_b]\). Since \(\mathcal{W} \subset \mathcal{V}\), subtracting \(B\) from the other terms in \((6)\) is a meaningful operation. We observe that for \(\mathcal{V} = \mathfrak{g}\), a Lie algebra, not only \(B = 0\), but also \(G = 0\), since the symmetric part of the bracket vanishes identically in this case. The above definitions imply in particular

\[
F \in \Omega^2(\mathcal{M}, \mathcal{V}) \quad \text{and} \quad G \in \Omega^3(\mathcal{M}, \mathcal{W}).
\]

Now we are ready to present the Leibniz-YM action functional:

\[
S[A, B] = \frac{1}{2} \int_M ||F||^2 + ||G||^2
\]

\[
= \frac{1}{2} \int_M \kappa(F, *F) + \kappa(G, *G).
\]

Here \(*\) denotes the usual Hodge star operation induced by the (pseudo)metric \(g\) on \(\mathcal{M}\); e.g.,

\[
\kappa(F, *F) = \kappa_{ab} F^a \wedge *F^b = F_{\mu\nu}^a F_{\mu\nu}^b \kappa_{ab} d^\mu \cdot x.
\]

Note that we can use the same pairing \(\kappa\) for \(F\) and \(G\), provided only, as we will assume and in which case we will call \((\mathcal{V}, [\cdot, \cdot], \kappa)\) a split-quadratic Leibniz algebra, its restriction to \(\mathcal{W}\) is non-degenerate—as it is the case anyway for the most relevant case of a positive definite \(\kappa\). For physical reasons, one may want to also introduce two coupling constants \(\lambda_F\) and \(\lambda_G\), multiplying the two contributions to \(S\) appropriately.

GAUGE SYMMETRY AND INVARIANCE

The infinitesimal gauge transformations of the fields \(A\) and \(B\) are parametrized by \(\mathcal{V}\)-valued functions \(\epsilon\) on \(\mathcal{M}\) and \(\mathcal{W}\)-valued 1-forms \(\mu\). They are of the form

\[
\delta A = d\epsilon + [A, \epsilon]_+ + \mu, \quad \delta B = d\mu + [A, \mu]_+ + [F - B, \epsilon]_+ - \frac{1}{2}[A, [A, \epsilon]_+]_+.\]

We will now show that they induce the following transformations of \((6)\) and \((7)\):

\[
\delta F = -[\epsilon, F], \quad \delta G = -2[\epsilon, G]_+.
\]

Before we start the calculations, we specify some of the rules that are valid in the concise notation used here: As indicated before, for every two \(\mathcal{V}\)-valued differential forms \(\alpha\) and \(\beta\), an expression like \([\alpha, \beta]_\pm\) stands, more explicitly, for \(\alpha^a \wedge \beta^b \otimes [e_a, e_b]_\pm\). This has, e.g., the following consequence:

\[
[\alpha, \beta]_\pm = \frac{1}{2} \left( [\alpha, \beta] \pm (-1)^{|\alpha||\beta|}[\beta, \alpha] \right),
\]

where \(|\alpha|\) and \(|\beta|\) denote the form-degrees of \(\alpha\) and \(\beta\), respectively. In particular, if both forms are odd: \([\alpha, \beta]_\pm = \frac{1}{2} \left( [\alpha, \beta] \mp \mp (\alpha, \beta) \right).\) In addition, one verifies

\[
[\alpha, \beta]_\pm = \pm (-1)^{|\alpha||\beta|}[\beta, \alpha]_\pm, \quad d[\alpha, \beta] = [d\alpha, \beta] + (-1)^{|\alpha|}[\alpha, d\beta], \quad d[\alpha, \beta] = [d\alpha, \beta]_\pm + (-1)^{|\alpha|}[\alpha, d\beta],
\]

(14) (15) (16)
while the defining Leibniz condition \((1)\) turns into:

\[
[\alpha, [\beta, \gamma]] - (1)^{|\alpha||\beta|}[\beta, [\alpha, \gamma]] = [[\alpha, \beta], \gamma]. \tag{17}
\]

This shows in particular that \((\Omega^*(M, V), d, [\cdot, \cdot])\) is a graded differential Leibniz algebra, with \([\cdot, \cdot]_\gamma\) and \([\cdot, \cdot]_\beta\) the graded-symmetric and antisymmetric parts of the bracket. These two parts of the bracket do not satisfy the relation \((17)\); instead, one has, for example,

\[
[\alpha, [\beta, \gamma]] - (1)^{|\alpha||\beta|}[\beta, [\alpha, \gamma]] = -\text{Alt}[\alpha, [\beta, \gamma]]. \tag{18}
\]

where \text{Alt} denotes the graded anti-symmetrization projector. In fact, with \((1)\) one shows that the 2-term complex \(\Omega^*(M, W) \to \Omega^*(M, V)\) can be endowed with the structure of a differential graded Lie 2-algebra whose non-vanishing 2-bracket is given by \([\cdot, \cdot]_\beta\); but since we want to avoid such terminologies in the present paper for simplicity of the presentation, we will not go further into this perspective here.

We now decompose the gauge transformations into two parts,

\[
\delta = \delta_\epsilon + \delta_\mu, \quad \text{and first consider (the degree zero derivation) } \delta_\mu, \text{ where } \delta_\mu A = \mu \text{ and } \delta_\mu B = d\mu + [A, \mu].
\]

This yields

\[
\delta_\mu F = d\mu + \frac{1}{2}[A, \mu] + \frac{1}{2} [\mu, A] - d\mu - [A, \mu] = [A, \mu].
\]

Now it is important that \(A\) takes values in \(W, \mu \in \Omega^1(M, W)\), and that this subspace lies in the left-center of the Leibniz algebra, \(W \subset Z_L(V)\): Adding to \((1)\) the same equation with \(x\) and \(y\) exchanged, we see that for every \(w = [x, y] + [y, x] \in W\) one has \(w, z = 0\) for all \(z \in V\). This then proves \(\delta_\mu F = 0\).

Henceforth we will freely use \([\mu, \cdot] = 0 = [B, \cdot]\), such that, e.g., \(\delta_\mu B = d\mu + \frac{1}{2}[A, \mu]\). This also implies that the second equation in \((12)\), which is adapted to show explicitly that the change of \(G\) lies inside \(W\), can be rewritten as

\[
\delta G = -[\epsilon, G]. \tag{19}
\]

Similarly, in some calculations it is useful to note that \([\alpha, \beta]_+ = 0\), which implies, e.g.,

\[
[\alpha, \beta, 0] = -(1)^{|\alpha||\beta|}[\beta, [\alpha, 0]].
\]

We now turn to showing \(\delta_\epsilon G = 0\). Straightforward and simple calculations yield

\[
\delta_\mu (dF) = \frac{1}{2} \left( [F, \mu] - A, d\mu \right),
\]

\[
\delta_\mu ([A, B] + F) = \frac{1}{2} \left( [A, d\mu] + [A, \mu] - [A, F] \right),
\]

\[
\delta_\mu ([A, \frac{1}{2} [A, \mu]] + [A, [A, \mu]] + [A, [A, \mu]]) = \frac{1}{12} \left( [A, [A, \mu]] + [A, [A, \mu]] + [A, [A, \mu]] \right). \tag{20}
\]

Taking the sum of these three equations we obtain

\[
\delta_\mu G = -\frac{1}{2} [[A, A], \mu] + \frac{1}{2} [A, [A, \mu]] + \frac{1}{12} [[A, [A, \mu]], A].
\]

The first two terms on the r.h.s. of this equation cancel due to \([[A, A], \mu] = 2[A, [A, \mu]]\), which follows from \((17)\). And the last term vanishes on the nose, since \(W\) is an ideal of \(V\) (in particular \([V, W] \subset W\), and thus \([A, \mu] \in \Omega^2(M, W)\)), as one shows by symmetrizing \((1)\) with respect to \(y\) and \(z\):

\[
[x, [y, z]]_+ = [[x, y], z]_+ + [y, [x, z]]_+. \tag{20}
\]

The transformation behavior of \(F\) and \(G\) with respect to \(\delta_\epsilon\) can now be proven in a very similar fashion by these techniques. We restrict ourselves to showing only the first equation in \((12)\) in detail, while leaving \((19)\) as an exercise to the reader. We start as follows

\[
\delta_\epsilon (dA + \frac{1}{2}[A, A]) = [dA, \epsilon]_+ + [A, [A, \epsilon]_+].
\]

Now eliminate on both sides \(dA\) by means of \((6)\), thus, e.g., the l.h.s. becomes \(\delta(F + B)\). We are showing this calculation also since it permits to in fact deduce the c-part of \((11)\), provided the first equation in \((12)\) holds true: Using \([F + B, \epsilon]_+ = -[\epsilon, F] + [\epsilon, F - B]_+\), one finds the necessary change of \(B\) to be of the given form, provided only

\[
-\frac{1}{2} [[A, A], \epsilon]_+ + [A, [A, \epsilon]_+] = -\frac{1}{2} [A, [A, \epsilon]_+], \tag{21}
\]

which one verifies by using \((18)\).

It remains to show the gauge invariance of \((9)\). The condition \((2)\) yields

\[
\kappa([\alpha, \beta], \gamma) + (1)^{|\alpha||\beta|} \kappa([\beta, \alpha], \gamma) = 0,
\]

from which we deduce \(\kappa([\epsilon, \beta], [\beta, \epsilon]) + \kappa([\beta, \epsilon], [\epsilon, \beta]) = 0\) for every \(\epsilon \in C^\infty(M, V)\), \(\beta \in \Omega^2(M, V)\). Together with \((12)\) and \((19)\), this then indeed proves

\[
\delta S = 0. \tag{22}
\]

EQUIVALENCE TO STANDARD YM WITH MASSIVE 2-FORM FIELDS

It is time to study the notion of a split-quadratic Leibniz algebra into further depth. Recall, that in addition to the condition \((2)\), which is needed for gauge invariance, we require the restriction \(\kappa\) to \(W\) to be non-degenerate, which is needed for the non-trivial propagation of the \(B\)-gauge fields in \((6)\). This implies

\[
V = W^\perp + W \ni (\xi, w),
\]

so that every element \(v \in W\) can be decomposed uniquely into a part in \(W^\perp\) and another one in \(W\). For every \(\xi, \xi' \in W^\perp\) and \(w \in W\), we have

\[
\kappa([\xi, \xi'], w) = -\kappa([\xi', [\xi, w]) = 0,
\]

since \([V, W] \subset W\). Thus \(W^\perp\) is a subalgebra of \(V\). Since in addition \([\xi, \xi']_+\) needs to lie also in \(W\), it has to vanish, and the restriction of the Leibniz algebra to \(W^\perp\) is a Lie
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ential, so that Leibniz algebras, in the picture of the day differential [22, 23]. Changing the splitting, leads where \(\alpha\) of equation (27) one has \(V\) that one obtains as the quotient of the Leibniz algebra elements of \(g\). Since, in addition, \([w, \xi] = 0 = [w, w']\), we have

\[
[(\xi, w), (\xi', w')] = ([\xi, \xi']_g, \xi \cdot w').
\]

(24)

The decomposition (23) also applies to the bilinear form, \(\kappa((\xi, w), (\xi', w')) = \kappa(\xi, \xi') + \kappa(w, w')\), and then the condition (2) shows that \(\kappa\) is in one-to-one correspondence with a non-degenerate ad-invariant symmetric bilinear form on \(g\) and a like-wise \(g\)-invariant one on \(W\). Thus we find, in a slight generalization of a result established in [16], that every split-quadratic Leibniz algebra is the same as an ordinary quadratic Lie algebra together with a quadratic representation:

\[
(V, [\cdot, \cdot], \kappa) = (g, [\cdot, \cdot], \kappa_g) \times (W, \kappa_W).
\]

(25)

At this point, we want to remark that for every Leibniz algebra \(V\) we have an exact sequence of Leibniz algebras,

\[
0 \to W \to V \to g,
\]

(26)

where \(W\) is again defined as in [4] and \(g\) is the Lie algebra that one obtains as the quotient of the Leibniz algebra \(V\) by its squares. Now, choosing an arbitrary splitting \(\sigma: g \to V\) of the above sequence, we again can represent elements of \(V\) as couples \((\xi, w) \subset g \otimes W\), but now instead of equation (27) one has

\[
(\xi, w), (\xi', w')] = ([\xi, \xi']_g, \xi \cdot w' + \alpha(x, x')),
\]

(27)

where \(\alpha: g \otimes g \to W\) is a cocycle with respect to theoday differential [22, 23]. Changing the splitting, leads to a modification of \(\alpha\) by coboundary of this differential, so that Leibniz algebras, in the picture of the above sequence, are characterized by a cohomology class \([\alpha] \in H^2_{\text{Lie}}(g, W)\) in general. As we showed above, the presence of a split-quadratic form enforces this cohomology class to vanish.

We now want to show what consequences this has for the functional (9). Let us for this purpose split the 1-form gauge field into its two parts, \(A = A_g + A_W\). Then also \(F = F_g + F_W\), where \(F_g = dA_g + \frac{1}{2}[A_g, A_g]_g\) is the ordinary curvature 2-form of the \(g\)-connection \(A_g\) and, on the other hand, \(F_W = dA_W + \frac{1}{2}A_g \cdot A_W - B\). Next we rewrite \(G\) as

\[
G = dB + [A, B] - [A, dA]_+ = -\frac{1}{2}[A, [A, A]],
\]

where we made use of \([A, [A, A]]_+ = \frac{3}{2}[A, [A, A]]\) and implement the above decomposition of \(A:\)

\[
G = dB + A_g \cdot B - \frac{1}{2}dA_g \cdot A_W - \frac{1}{2}A_g \cdot dA_W - \frac{1}{2}A_g \cdot (A_g \cdot A_W).
\]

Now introduce a new 2-form field by setting \(\tilde{B} := B - dA_W - \frac{1}{2}A_g \cdot A_W\); since this change of fields has Jacobian one, it is also a valid operation on the quantum level. A simple calculation then shows that in these new coordinates on field space, (9) takes the form

\[
S[A_g, \tilde{B}] = \int_M \kappa_g(F_g + F_g^*) + \kappa_W(D_g \tilde{B} + dA_g \cdot \tilde{B}) + \kappa_W(D_g \tilde{B}, \tilde{B} + \kappa_W(\tilde{B}, \tilde{B}))
\]

(28)

where \(D_g \tilde{B} = d\tilde{B} + A_g \cdot \tilde{B}\) is the standard covariant derivative for a field taking values in a \(g\)-representation \(W\). Most noteworthy, the action functional now does no more depend on \(A_W\), the part of the 1-form gauge field taking values in \(W \subset V\). This extremely simple gauge symmetry can be identified with the \(\delta_{\mu}\)-part of the gauge symmetries (10) and (11). One may still wonder what happened to the \(W\)-part of the \(\delta_{\mu}\)-transformations. In fact, there is a subtlety in the previous parametrization of the gauge symmetries: For every choice of \(\epsilon \in \Omega^0(M, W)\), one has

\[
\delta_{\epsilon} := \tilde{\epsilon} + \delta_{\mu} := d\epsilon + [A, \epsilon]_+ \equiv 0.
\]

(29)

In absence of a splitting (23) which also respects the bracket structure, it is useful to work with the reducible gauge symmetries (10) and (11). Now, however, we can split the parameter \(\epsilon\) into its two parts \(\epsilon_g\) and \(\epsilon_W\) and forget about the latter parameters, as their effect can be captured by means of a particular, field-dependent \(\mu\)-shift. (Note that this is a legitimate operation due to \([A, \epsilon]_+ = \frac{1}{2}A_g \cdot \epsilon \in \Omega^1(M, W)\). One verifies also that for \(\epsilon = \epsilon_g \in C^\infty(M, g)\) and \(\mu = 0\) the transformations (10) and (11) induce the obvious symmetry of (28):

\[
\delta_{\epsilon_g} A_g = d\epsilon_g + [A_g, \epsilon_g], \quad \delta_{\epsilon_g} \tilde{B} = -\epsilon_g \cdot \tilde{B}.
\]

(30)

Thus, on the classical as well as on the quantum level, the Leibniz Yang-Mills gauge theory naturally associated to a split-quadratic Leibniz algebra turns out to be equivalent to an ordinary Yang-Mills gauge theory with gauge field \(A_g\) to which one couples a massive 2-form field \(B\) taking values in the \(g\)-representation \(W\). The originally present \(\delta_{\mu}\)-symmetry for the \(B\) gauge fields disappeared now, together with \(A_W\).

SUMMARY AND OUTLOOK

In this paper we highlighted the fact that a Yang-Mills functional does not only exist for every quadratic Lie algebra, but already for a split-quadratic Leibniz algebra. We presented an index-free formalism that can be used also for more general gauge theories related to Leibniz algebras. Our main result was to show that the gauge theory (9) can be recast into the form of a YM-theory for a Lie algebra to which 2-form fields are coupled. In the reformulation, where the specific form of split-quadratic Leibniz algebras was used, cf. equations (24)
and (27), the 2-form gauge fields turned into massive 2-form fields without an independent gauge invariance.

We thus obtain a higher version of a Higgs mechanism: The gauge invariance (11) excludes the addition of explicit mass terms for the 2-form fields $B$-fields, similarly to what we know for the usual 1-form gauge fields. But such as an appropriate gauge-invariant addition of 0-form fields can be used to effectively generate masses for the 1-form fields, here it is the 1-forms that generate them for the 2-form fields. Interestingly, this does not need to be implemented by an intricate coupling procedure, it is achieved for free by considering the functional (9) based on a Leibniz algebra.

The presented equivalence was established under the assumption that the (not necessarily positive) scalar product $\kappa$ is non-degenerate upon restriction to $\mathcal{W} \subset \mathcal{V}$. One may want to see what happens when this requirement is relaxed. The presence of the $\mu$-shift symmetry seems to suggest, however, that also more generally such a relation holds—on the quantum level upon an appropriate partial gauge fixing forcing the part of $A$ with values in $\mathcal{W}$ to vanish. Another, more drastic alternative might be to drop the non-degeneracy condition of $\kappa$. Consider, e.g., $\kappa_{\mathcal{W}} \equiv 0$ [24]: then $B$ drops out from the action, $S[A, B] = S[A]$. But since then, and only then, also $S[A] = S[A + A_{\mathcal{W}}]$ for every $A_{\mathcal{W}} \in \Omega^1(M, \mathcal{W})$, we see that effectively $A$ takes values in the quotient $\mathcal{V}/\mathcal{W} = g$, cf. the sequence (26). Although in this case, we can no longer split $A$ into two parts in the same way as before, we now end up with the Yang-Mills gauge theory for the (quadratic) Lie algebra $g$, here without 2-form fields.

In the physically most relevant case of a positive definite $\kappa$, on the other hand, our assumptions are compulsory and the above analysis valid. It would be interesting to see, how the situation extends to a Yang-Mills type functional based on an enhanced Leibniz algebra [15] [16].

In a final paragraph we come back to the language of $L_{\infty}$-algebras, which we touched upon in the Introduction, but avoided otherwise. We saw that there is a functional $S$ associated canonically to every quadratic Leibniz algebra. Underlying to it is a Lie 2-algebra, which results upon truncation of a Lie $\infty$-algebra (an $L_{\infty}$-algebra concentrated in non-positive degrees) associated to every Leibniz algebra [1] [4]. It is suggestive that there exists a higher YM-type functional for arbitrarily high form-degrees associated to every quadratic Leibniz algebra, if equipped also with a non-degenerate invariant anti-symmetric scalar product. This functional should moreover permit consistent truncations to arbitrary levels, the functional $S$ presented here corresponding to level 2, while level 1 would be the YM-theory for the Lie algebra $g$ without 2-form fields. (As already evident from these lowest examples, the truncation is not consistent, if one merely puts higher form degree gauge fields to zero—the functional $S[A, 0]$, resulting from (9) by setting the 2-form fields to zero, is not gauge invariant anymore—truncations always involve appropriate quotient constructions at the highest non-vanishing level). One may expect that some of the higher gauge fields, although probably not all of them, receive masses by a generalization of the above mechanism. To be seen.

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1 We propose the name 2-Higgs mechanism for this. First, it is a higher version of an ordinary Higgs mechanism, and in the language of ‘higher structures’ one conventionally labels a hierarchy of structures by an integer $n \in \mathbb{N}$, with $n = 1$ denoting the original notion. Second, and more importantly, it gives masses to 2-form gauge fields and, more generally, an $n$-Higgs mechanism would do so for $n$-form gauge fields.
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