Field space entanglement entropy, zero modes and Lifshitz models

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ABSTRACT

The field space entanglement entropy of a quantum field theory is obtained by integrating out a subset of its fields. We study an interacting quantum field theory consisting of massless scalar fields on a closed compact manifold \( M \). To this model we associate its Lifshitz dual model. The ground states of both models are invariant under constant shifts. We interpret this invariance as gauge symmetry and subject the models to proper gauge fixing. By applying the heat kernel regularization one can show that the field space entanglement entropies of the massless scalar field model and of its Lifshitz dual are agreeing.

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1. Introduction

In quantum physics the entanglement entropy is a powerful and intriguing observable (for reviews see [1,2]) and has become the subject of intensive investigation during the last decade. Entanglement entropy provides valuable insight in condensed matter physics (for reviews see [3–6]), quantum field theory (for reviews see [7,8]) and black hole theory (for reviews on holographic entanglement see [9–11]).

Geometric entanglement entropy measures the entropy of a spatial subsystem after tracing out the environment. Only rather few examples of field theories are known, however, for which the entanglement entropy can be computed exactly. In 1 + 1 dimensional conformal field theories the replica technique pioneered by Cardy and Calabrese [12,8] provides an indispensable computational tool. These methods have also been extended to free field theories in higher dimensions [13] and to interacting theories using the AdS/CFT correspondence [9,11]. For a variational approximation to the entanglement entropy see [14].

The 2 + 1 dimensional quantum Lifshitz model is a continuum limit of the dimer model [15–17] at the RK point (for a review see [18]) via a height function [19,20]. One of the remarkable properties of the model is that its ground state wave functional is exactly known and that it has two-dimensional conformal invariance. Therefore techniques of conformal field theory are applicable and analytic calculations become possible; [21–27] demonstrated that the entanglement entropy for various geometries could be expressed and calculated in terms of the free energy of the associated conformal field theory.

Correlation functions of a \( n \) dimensional interacting quantum field theory are coinciding with the large equal-time correlation functions of an associated “dual” Lifshitz model in \( (n + 1) \) dimensions. This specific relationship has been one of the central issues in stochastic quantization ([36], for reviews see [37,38]) and can be summarized as follows: One starts from a Euclidean field theory model in \( n \) dimensions and introduces an ergodic stochastic process possessing the \( n \) dimensional Euclidean path integral measure as its unique equilibrium measure. The stochastic process evolves in an additional time and is defined through a Fokker–Planck equation. Its drift force is taken to be the negative variation of the \( n \) dimensional Euclidean action with respect to the fields. Employing the Feynman–Kac formula for the positive semidefinite Fokker–Planck Hamiltonian one is lead to a functional integral of a field theory model – called dual Lifshitz model – in \( (n + 1) \) dimensions. The partition function of the original \( n \) dimensional model is expressed as the norm square of the ground state of the new \( (n + 1) \) dimensional model.

The prototypical example of the duality of field theory models due to the stochastic quantization scheme is the duality of a free scalar field and the Lifshitz model. In recent years novel interpretations, applications and generalizations of this duality were given (for reviews see [39,40]), Lifshitz-type models of gauge and gravity theories [41,42] attracted great interest.

In this work we are interested in the field space entanglement entropy of quantum field theories, which is obtained by integrating out a subset of its fields. Various measures for entanglement as well as the formulation of holographic entanglement entropy have been investigated in this context [28–30]. We address the question
to which extent the field space entanglement entropies of quantum field theories and their Lifshitz duals are agreeing.

Specifically we compare the field space entanglement entropy of the dual Lifshitz model associated to a model of two interacting massless scalar fields on a compact and closed manifold $M$ with the field space entanglement entropy of the same model, yet defined on $\mathbb{R} \times M$. The ground states of these two models are Gaussian wave functionals invariant under constant shifts and thus are not normalizable. This is related to the existence of zero modes of the Laplacian on compact and closed manifolds. In fact, zero modes represent gauge degrees of freedom, see [31] for a gauge theory interpretation of zero modes in the discussion of the quantization of a massless scalar field on $S^4$. The – non-gauge theory - role of zero modes for the geometrical entanglement entropy is discussed in [32,27,33,34] and for the field space entanglement entropy in [28,35].

It is our focus to carefully handle zero modes in field space entanglement entropy. Instead of regulating ground states by ad hoc procedures for the zero modes, we interpret these invariances as gauge symmetries and subject the models to proper gauge fixing [43,44].

The presence of gauge symmetries adds a subtle component to the construction of the dual Lifshitz model and its ground state, which we are going to review in the following lines. Remind the origin of Lifshitz dual models, which is lying in the Fokker–Planck formulation of stochastic quantization. Parisi and Wu [36] proposed stochastic quantization without gauge fixing terms. Since the pure action of the gauge model remains invariant under gauge transformations, the associated drift force acts orthogonally to the gauge orbits. As a consequence unbounded diffusion along the orbits takes place and a Fokker–Planck formulation is not possible. In the approach of Zwanziger [45] the drift force of the stochastic process is modified by the addition of a gauge fixing force tangent to the gauge orbits. This provides damping for the gauge modes’ diffusion along the orbits leaving unchanged all gauge invariant expectation values. Zwanziger’s gauge fixing force, however, intrinsically is non-conservative and generally cannot be accommodated in an action formalism required for the Lifshitz model construction. Eventually, the stochastic quantization approach could be generalized [46,43] by specifying a drift force which not only has tangential components along the gauge orbits but where, subsequently, also the Wiener process itself is modified. Expectation values of gauge invariant observables again remain untouched. With [46,43] a well defined Fokker–Planck formulation with normalizable ground state is obtained and the corresponding Lifshitz model can be derived in a consistent way. As a consequence the issue of comparing field space entanglement entropies of quantum field theories and of their Lifshitz duals can be addressed.

For massless scalar fields the naïve set up of a dual Lifshitz model fails in the first step due to the gauge invariance of the model. A consistent formulation is presented in section 2 after the proper discussion of gauge symmetries and gauge fixing.

In section 3 we construct the regularized ground state associated to the massless scalar field model in the Hilbert space of the entire system, whose inner product is given by integration over all scalar fields. In order to obtain a normalizable ground state, we modify the Lagrangian of the massless scalar field model by adding the same gauge fixing term as in section 2.

By applying the heat kernel regularization we can show in section 4 that the field space entanglement entropies of the massless scalar field model and of its Lifshitz dual are agreeing.

2. Regularized ground state of the dual Lifshitz model

Let $M$ be a compact $n$ dimensional closed manifold and consider the action functional of coupled massless real-valued scalar fields $\phi_1(x), \phi_2(x)$ on $M$

$$S^M(\phi_1, \phi_2) = \frac{1}{2} \int_M d\text{vol}(x) [\phi_1 \Delta^M \phi_1 + \phi_2 \Delta^M \phi_2 + \lambda \phi_1 \Delta^M \phi_2]$$

(1)

where $\lambda$ is a coupling parameter. Let $\mathcal{F}$ denote the configuration space of the fields. To this model in a first step we associate a dual Lifshitz model [36–40], such that its ground state, called Lifshitz ground state, is given by

$$\tilde{\Psi}_0^{\text{Lif}}(\phi_1, \phi_2) = N e^{-\frac{i}{2} S^M(\phi_1, \phi_2)},$$

(2)

where $N$ should be a normalization constant. It is well known, however, that $e^{-\frac{i}{2} S^M(\phi_1, \phi_2)}$ is not normalizable due to the invariance of $S^M(\phi_1, \phi_2)$ under constant shifts $(\phi_1, \phi_2) \mapsto (\phi_1 + c_1, \phi_2 + c_2)$. We interpret this invariance of the Lifshitz ground state as being associated to the gauge transformation

$$\mathcal{F} \times \mathbb{R}^2 \hookrightarrow \mathcal{F} \quad (\phi_1, \phi_2, c_1, c_2) \mapsto (\phi_1 + c_1, \phi_2 + c_2).$$

(3)

Hence the quotient $\mathcal{F}/\mathbb{R}^2$ is the true configuration space of the coupled system giving rise to the trivializable principal $\mathbb{R}^2$-bundle $\mathcal{F} \hookrightarrow \mathcal{F}/\mathbb{R}^2$, with trivialization

$$\omega(\phi_1, \phi_2) = \frac{1}{V_M} \left( \int_M d\text{vol}(x) \phi_1 \int_M d\text{vol}(x) \phi_2 \right) = (\Pi(\phi_1), \Pi(\phi_2)),

(4)

where $V_M = \int_M d\text{vol}(x)$ is the volume of $M$, and $\Pi$ is the projector onto the constant parts of the scalar fields. In fact, the space $\mathcal{F}/\mathbb{R}^2$ of the physical degrees of freedom can be identified with the space of pairs of non-constant scalar fields.

Evidently $\omega$ fulfills $\omega(\phi_1 + c_1, \phi_2 + c_2) = \omega(\phi_1, \phi_2) + (c_1, c_2)$ (see [43,44] for further details, and [31] for a BRST approach).

As outlined in the previous section we interpret the Lifshitz model as being associated to the Fokker–Planck Hamiltonian arising in the stochastic quantization of the gauge model of massless scalar fields (1). Following the general procedure in [46,43,44] the underlying stochastic process has to be adapted judiciously, according to the choice of a gauge fixing function $(w^* S_{\text{gf}})(\phi_1, \phi_2)$. The gauge invariant action $S^M(\phi_1, \phi_2)$ then is replaced by the gauge fixed total action $S^{\text{tot}}_{\text{gf}}(\phi_1, \phi_2)$, which is to be used for the proper definition of the Lifshitz model.

For the gauge fixing function we choose

$$S_{\text{gf}}(c_1, c_2) = \frac{\mu}{2} (c_1^2 + c_2^2 + \lambda c_1 c_2),

(5)

which for $\mu > 0$ and $-2 < \lambda < 2$ is normalizable. Thus we arrive at a gauge fixed total action

$$S^{\text{tot}}_{\text{gf}}(\phi_1, \phi_2) = S^M(\phi_1, \phi_2) + (w^* S_{\text{gf}})(\phi_1, \phi_2) = \frac{1}{2} \int_M d\text{vol}(x) [\phi_1 D^M \phi_1 + \phi_2 D^M \phi_2 + \lambda \phi_1 D^M \phi_2],

(6)

where we introduced $D^M = \Delta^M + \frac{\mu}{V_M} \Pi$. The Laplacian $\Delta^M$ has discrete spectrum

$$0 = \nu_0(\Delta^M) < \nu_1(\Delta^M) \leq \nu_2(\Delta^M) \leq \ldots \to \infty,$n

(7)

where each eigenvalue appears the same number of times as its multiplicity. Let $\lambda^M_i$ denote the orthonormal basis of eigenfunctions of $\Delta^M$ such that
\[ \Delta^M \chi^M_\alpha = \nu_{\alpha} (\Delta^M) \chi^M_\alpha. \]  
(8)

Then \( \chi^M_\alpha \) are also eigenfunctions of \( D^M \) with the eigenvalues
\[ D^M \chi^M_\alpha = \nu_{\alpha} (\Delta^M) \chi^M_\alpha + \frac{H}{V_M} \delta_{\alpha 0} \chi^M_\alpha. \]  
(9)

The strictly positive spectrum of \( D^M \) reads
\[ \text{Spec}(D^M) = \left\{ \nu_{\alpha} (\Delta^M) |_{\alpha \neq 0}, \frac{H}{V_M} \right\}. \]  
(10)

With respect to the orthonormal basis \( \{ \chi^M_\alpha \} \) we rewrite
\[ \phi_1 = \sum_\alpha \phi_{1,\alpha} \chi^M_\alpha, \quad \phi_2 = \sum_\alpha \phi_{2,\alpha} \chi^M_\alpha. \]  
(11)

Thus instead of (2) we define the regularized Lifshitz ground state with respect to the gauge fixed total action (6) by
\[ \Psi_0^{\text{Li}}(\phi_1, \phi_2) = \prod_{\alpha \neq 0} \Psi_0^{\text{Li}}(\phi_{1,\alpha}, \phi_{2,\alpha}) \Psi_0^{\text{Li}}(\phi_{1,0}, \phi_{2,0}). \]  
(12)

where
\[ \Psi_0^{\text{Li}}(\phi_{1,0}, \phi_{2,0}) = \text{Ne}^{-\frac{1}{4} \nu_0 (\Delta^M) \left( \phi_{1,0}^2 + \phi_{2,0}^2 + \lambda_1 \phi_{1,0} \phi_{2,0} \right)} \]  
(13)

as well as
\[ \Psi_0^{\text{Li}}(\phi_{1,\alpha}, \phi_{2,\alpha}) = \text{Ne}^{-\frac{1}{4} \frac{\nu_{\alpha}}{\nu_0} \left( \phi_{1,\alpha}^2 + \phi_{2,\alpha}^2 + \lambda_1 \phi_{1,\alpha} \phi_{2,\alpha} \right)}. \]  
(14)

3. Regularized ground state of the massless scalar field model

As a next step we construct the ground state associated to the massless scalar field model \( S^M(\phi_1, \phi_2) \). It will be convenient to define this model on \( \mathbb{R} \times M \) by the action functional
\[ S^{R \times M} = \int_\mathbb{R} d^M L^M(\phi_1, \phi_2, \phi_1, \phi_2) \]  
(15)

with
\[ L^M(\phi_1, \phi_2, \phi_1, \phi_2) = \frac{1}{2} \int d^M \mu(x) \left[ \phi_1^2 + \phi_2^2 + \lambda \phi_1 \phi_2 \right] + S^M(\phi_1, \phi_2). \]  
(16)

This system, like the dual Lifshitz model before, admits the gauge symmetry (3) under constant shifts. A straightforward Hamiltonian analysis following the Dirac constraint quantization approach [47, 48] shows that the corresponding physical states are wave functions \( \Psi(\phi_1, \phi_2) \) which are constant under the field transformations (3). The ground state is
\[ \tilde{\Psi}_0(\phi_1, \phi_2) = \text{Ne}^{-\tilde{S}(\phi_1, \phi_2)}. \]  
(17)

where \( \tilde{S}(\phi_1, \phi_2) = \frac{1}{2} \int d^M \mu(x) \left[ \phi_1 \left( \Delta^M \right)^{1/2} + \phi_2 \left( \Delta^M \right)^{1/2} \right] + \lambda \phi_1 \left( \Delta^M \right)^{1/2} \phi_2. \]  
(18)

Since the ground state is constant along the gauge orbits, it is not normalizable with respect to the Hilbert space of the entire system whose inner product is given by integration over all scalar fields. In order to obtain a normalizable ground state, we modify the Lagrangian (16) by adding the same gauge fixing term \( (w^* S^M)(\phi_1, \phi_2) \) as before. Hence the system remains unchanged along the physical degrees of freedom and we get
\[ L_{\text{reg}}^M(\phi_1, \phi_2, \phi_1, \phi_2) = \frac{1}{2} \int d^M \mu(x) \left[ \phi_1^2 + \phi_2^2 + \lambda \phi_1 \phi_2 \right] + s_{\text{reg}}(\phi_1, \phi_2). \]  
(19)

The mode expansion on \( M \) leads to the regularized ground state
\[ \Psi_{\text{reg}}(\phi_1, \phi_2) = \prod_{\alpha \neq 0} \Psi_{\text{reg}}(\phi_{1,\alpha}, \phi_{2,\alpha}) \Psi_{\text{reg}}(\phi_{1,0}, \phi_{2,0}). \]  
(20)

where
\[ \Psi_{\text{reg}}(\phi_{1,\alpha}, \phi_{2,\alpha}) = \text{Ne}^{-\frac{1}{4} \frac{\nu_{\alpha}}{\nu_0} \left( \phi_{1,\alpha}^2 + \phi_{2,\alpha}^2 + \lambda_1 \phi_{1,\alpha} \phi_{2,\alpha} \right)}, \]  
(21)

as well as
\[ \Psi_{\text{reg}}(\phi_{1,0}, \phi_{2,0}) = \text{Ne}^{-\frac{1}{4} \frac{\nu_0}{\nu_0} \left( \phi_{1,0}^2 + \phi_{2,0}^2 + \lambda_1 \phi_{1,0} \phi_{2,0} \right)}. \]  
(22)

The modified ground state agrees, when restricted to the physical degrees of freedom, with the ground state ((17), (18)) of the original system. Its essential effect is to provide an integrable contribution along the gauge orbits and thus, like in the dual Lifshitz model, gives a well defined state in the full Hilbert space.

4. Entanglement entropy

So far we provided the mode decompositions of the regularized ground states for the two considered scalar field models. Each mode contributed in a similar way
\[ \Psi_{\text{reg}}(\phi_{1,\alpha}, \phi_{2,\alpha}) = \text{Ne}^{-c_\alpha \frac{1}{4} \nu_{\alpha} \left( \phi_{1,\alpha}^2 + \phi_{2,\alpha}^2 + \lambda_1 \phi_{1,\alpha} \phi_{2,\alpha} \right)}, \]  
(23)

where the positive constants \( c_\alpha \) can be found in the equations of the previous sections (their values are irrelevant for the calculation of the entanglement entropy, however). The ultimate calculation of the field space entanglement entropy is quickly done: A simple analysis along the lines of [49–51,28] tells us that each mode's contribution is governed by the eigenvalues \( \{ \lambda_1, \lambda_2 \} \) of the quadratic form \( c_\alpha \left( \phi_{1,\alpha}^2 + \phi_{2,\alpha}^2 + \lambda_1 \phi_{1,\alpha} \phi_{2,\alpha} \right) \). These eigenvalues are simply \( \{ \lambda_{1,\alpha}, \lambda_{2,\alpha} \} = \left\{ \xi_{\alpha} \frac{\lambda_1}{\xi_{\alpha}}, \xi_{\alpha} \frac{\lambda_2}{\xi_{\alpha}} \right\} \) and one calculates the auxiliary quantity
\[ \xi_{\alpha} = \frac{\sqrt{\lambda_1 \alpha} - \sqrt{\lambda_2 \alpha}}{\sqrt{\lambda_1 \alpha} + \sqrt{\lambda_2 \alpha}} = -\frac{8 + \lambda^2 + 4 \sqrt{4 - \lambda^2}}{\lambda^2}, \]  
(24)

which is independent of \( \alpha \), independent of \( c_\alpha \), independent of the gauge fixing parameter \( \mu \) and is equal for the scalar field model and its Lifshitz dual. Finally one obtains for each mode – also the zero mode – an equal contribution \( s \) to the field space entanglement entropy
\[ s_{\alpha} = -\log(1 - \xi_{\alpha}) - \frac{\xi_{\alpha}}{1 - \xi_{\alpha}} \log(\xi_{\alpha}) = s. \]  
(25)

We perform the heat kernel regularized sum over all the contributions of the modes
\[ S_{\text{reg}}(t) = \sum_{\alpha = 0}^{\infty} s_{\alpha} e^{-t \nu_0 (\Delta^M)} = s \sum_{\alpha = 0}^{\infty} e^{-t \nu_0 (\Delta^M)}. \]  
(26)

Using the asymptotic expansion for the heat kernel of the Laplacian, we finally obtain for the regularized entanglement entropy
\[ S_{\text{reg}}(t) \simeq \sum_{k=0}^{\infty} q_k (\Delta^M) t^{k-n/2}, \quad t \ll 0 \]  
(27)
where \( a_0(\Delta M) \) are the Seeley coefficients [52] of the Laplacian on \( M \). Since
\[
a_0(\Delta M) = \left(4\pi^2\right)^{-\frac{d}{2}} V_M
\]  
\[ (28) \]
one thus arrives at the field space entanglement being proportional to the volume of \( M \) [28–30].

Concluding, we have shown that the field space entanglement entropies of the massless scalar field model and of its Lifshitz dual are agreeing.

5. Outlook

It seems interesting to adapt our present calculation for the case of self interacting scalar fields along the lines of [14].

In a forthcoming publication, we will present calculations of the field space entanglement entropy in the case of a gauge model with Gribov ambiguities [53].

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