PARKING FUNCTIONS AND DESCENT ALGEBRAS

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Abstract. We show that the notion of parkization of a word, a variant of the classical standardization, allows to introduce an internal product on the Hopf algebra of parking functions. Its Catalan subalgebra is stable under this operation and contains the descent algebra as a left ideal.

1. Introduction

Solomon [13] constructed for each finite Coxeter group a remarkable subalgebra of its group algebra, now called its descent algebra.

For the infinite series of Weyl groups, the direct sums of descent algebras can be endowed with some interesting extra structure. This is most particularly the case for symmetric groups (type $A$), where the direct sum $\Sigma = \bigoplus_{n \geq 0} \Sigma_n$ ($\Sigma_n$ being the descent algebra of $S_n$) builds up a Hopf algebra, isomorphic to $\text{Sym}$ (noncommutative symmetric functions) and dual to $Q\text{Sym}$ (quasi-symmetric functions).

It has been understood by Reutenauer [11] and Patras [9] that $\Sigma$ could be interpreted as a subalgebra of the direct sum $\Theta = \sum_{n \geq 0} \mathbb{Z} \Theta_n$ for the convolution product of permutations, which arises when permutations are regarded as graded endomorphisms of a free associative algebra. Indeed, $\Sigma$ is then just the convolution subalgebra generated by the homogeneous components of the identity map. Further understanding of the situation has been provided by Malvenuto and Reutenauer [7], who gave a complete description of the Hopf algebra structure of $\Theta$, and by Poirier-Reutenauer [10], who discovered an interesting subalgebra based on standard Young tableaux.

Finally, the introduction of the Hopf algebra of free quasi-symmetric functions $FQ\text{Sym}$ [1] clarified the picture and brought up a great deal of simplification. Indeed, $FQ\text{Sym}$ is an algebra of noncommutative polynomials over some auxiliary set of variables $a_i$, which is isomorphic to $\Theta$, and is mapped onto ordinary quasi-symmetric function $Q\text{Sym}$ when the $a_i$ are specialized to commuting variables $x_i$, the natural basis $F_\sigma$ of $FQ\text{Sym}$ going to Gessel’s fundamental basis $F_I$. At the level of $FQ\text{Sym}$, the coproduct has a transparent definition (ordered sum of alphabets), and most of its properties become obvious.

There is at least one point, however, on which this construction does not shed much light. It is the original product of the descent algebras $\Sigma_n$, which gives rise on $\text{Sym}$ to a noncommutative analogue of the internal product of symmetric functions (see [6] for the classical case). The introduction of the Hopf structure of $\Sigma = \text{Sym}$ was extremely useful, thanks to the so-called splitting formula [2, 4], a compatibility property between all operations (internal and external product, coproduct). But the embedding of $\text{Sym}$ in $FQ\text{Sym}$ does not seem to bring new information. In particular,
the coproduct dual to the composition of permutations has no nice definition in terms of product of alphabets, and the splitting formula is no more valid in general. Hopf subalgebras in which it remains valid have been studied by Schocker (Lie idempotent algebra, \cite{12}) and by Patras-Reutenauer \cite{10}, this last one being maximal with respect to this property.

There are many combinatorial objects which can be regarded, in one way or another, as generalizations of permutations. Among them are parking functions, on which a Hopf algebra structure \textit{PQSym}, very similar to that of \textit{FQSym}, can be defined \cite{8}. Actually, \textit{FQSym} is a Hopf subalgebra of \textit{PQSym}.

The aim of this note is to show that it is possible to define on \textit{PQSym} an internal product, dual to a natural coproduct corresponding to the Cartesian product of ordered alphabets, exactly as in Gessel’s construction of the descent algebra \cite{3}. This product is very different from the composition permutations or endofunctions, and looks actually rather strange. It can be characterized in terms of the fundamental notion of \textit{parkization} of words defined over a totally ordered alphabet in which each element has a successor.

In \cite{8}, various Hopf subalgebras of \textit{PQSym} have been introduced. We shall show that the Catalan subalgebra \textit{CQSym} (based on the Catalan family of nondecreasing parking functions, or equivalently, non-crossing partitions) is stable under this new internal product, and contains the descent algebra as a left ideal. Moreover, the splitting formula remains valid for it.

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2. Parking functions and parkization

A parking function on \([n] = \{1, 2, \ldots, n\}\) is a word \(a = a_1a_2\cdots a_n\) of length \(n\) on \([n]\) whose non-decreasing rearrangement \(a^\uparrow = a'_1a'_2\cdots a'_n\) satisfies \(a'_i \leq i\) for all \(i\). Let \(PF_n\) be the set of such words.

One says that \(a\) has a \textit{breakpoint} at \(b\) if \(|\{a_i \leq b\}| = b\). Then, \(a \in PF_n\) is said to be \textit{prime} if its only breakpoint is \(b = n\). Let \(PPF_n \subset PF_n\) be the set of prime parking functions on \([n]\).

For a word \(w\) on the alphabet \(1, 2, \ldots\), denote by \(w[k]\) the word obtained by replacing each letter \(i\) by \(i + k\). If \(u\) and \(v\) are two words, with \(u\) of length \(k\), one defines the \textit{shifted concatenation}

\begin{equation}
    u \bullet v = u \cdot (v[k])
\end{equation}

and the \textit{shifted shuffle}

\begin{equation}
    u \ll v = u \ll (v[k])
\end{equation}

The set of permutations is closed under both operations, and the subalgebra spanned by this set is isomorphic to \(S\) \cite{7} or to \textit{FQSym} \cite{11}. 

Clearly, the set of all parking functions is also closed under these operations. The prime parking functions exactly are those which do not occur in any nontrivial shifted shuffle of parking functions. These properties allowed us to define a Hopf algebra of parking functions in [8].

This algebra, denoted by $\text{PQSym}$, for Parking Quasi-Symmetric functions, is spanned as a vector space by elements $F_a$ ($a \in \text{PF}$), the product being defined by
\begin{equation}
F_a F_{a''} := \sum_{a \in a' \uplus a''} F_a.
\end{equation}

For example,
\begin{equation}
F_{12} F_{11} = F_{1233} + F_{1323} + F_{1332} + F_{3123} + F_{3132} + F_{3312}.
\end{equation}

The coproduct on $\text{PQSym}$ is a natural extension of that of $\text{FQSym}$. Recall (see [7, 1]) that if $\sigma$ is a permutation,
\begin{equation}
\Delta F_{\sigma} = \sum_{u \cdot v = \sigma} F_{\text{Std}(u)} \otimes F_{\text{Std}(v)},
\end{equation}
where $\text{Std}$ denotes the usual notion of standardization of a word.

For a word $w$ over a totally ordered alphabet in which each element has a successor, we defined in [8] a notion of parkized word $\text{Park}(w)$, a parking function which reduces to $\text{Std}(w)$ when $w$ is a word without repetition.

For $w = w_1 w_2 \cdots w_n$ on $\{1, 2, \ldots\}$, we set
\begin{equation}
d(w) := \min\{i|\#\{w_j \leq i\} < i\}.
\end{equation}
If $d(w) = n + 1$, then $w$ is a parking function and the algorithm terminates, returning $w$. Otherwise, let $w'$ be the word obtained by decrementing all the elements of $w$ greater than $d(w)$. Then $\text{Park}(w) := \text{Park}(w')$. Since $w'$ is smaller than $w$ in the lexicographic order, the algorithm terminates and always returns a parking function.

For example, let $w = (3, 5, 1, 1, 11, 8, 8, 2)$. Then $d(w) = 6$ and the word $w' = (3, 5, 1, 1, 10, 7, 7, 2)$. Then $d(w') = 6$ and $w'' = (3, 5, 1, 1, 9, 6, 6, 2)$. Finally, $d(w'') = 8$ and $w''' = (3, 5, 1, 1, 8, 6, 6, 2)$, that is a parking function. Thus, $\text{Park}(w) = (3, 5, 1, 1, 8, 6, 6, 2)$.

The coproduct on $\text{PQSym}$ is defined by
\begin{equation}
\Delta F_a := \sum_{u \cdot v = a} F_{\text{Park}(u)} \otimes F_{\text{Park}(v)},
\end{equation}
For example,
\begin{equation}
\Delta F_{3132} = 1 \otimes F_{3132} + F_1 \otimes F_{132} + F_{21} \otimes F_{21} + F_{212} \otimes F_1 + F_{3132} \otimes 1.
\end{equation}
The product and the coproduct of $\text{PQSym}$ are compatible, so that $\text{PQSym}$ is a graded bialgebra, connected, hence a Hopf algebra. Let $G_a = F_a^* \in \text{PQSym}^*$ be the dual basis of $(F_a)$. If $\langle \ , \ \rangle$ denotes the duality bracket, the product on $\text{PQSym}^*$ is given by
\begin{equation}
G_a G_{a''} = \sum_{a} \langle G_a \otimes G_{a''}, \Delta F_a \rangle G_a = \sum_{a \in a' \uplus a''} G_a,
\end{equation}
where the convolution $a' \ast a''$ of two parking functions is defined as

\begin{equation}
(a' \ast a'') = \sum_{u,v; a = u \cdot v, \text{Park}(u) = a', \text{Park}(v) = a''} a.
\end{equation}

For example,

\begin{equation}
G_{12} G_{11} = G_{1211} + G_{1222} + G_{1233} + G_{1311} + G_{1322} + G_{1411} + G_{1422} + G_{2311} + G_{2411} + G_{3411}.
\end{equation}

When restricted to permutations, it coincides with the convolution of $[11, 7]$.

The coproduct of a $G_a$ is

\begin{equation}
\Delta G_a := \sum_{u,v; a \in u \sqcup v} G_{\text{Park}(u)} \otimes G_{\text{Park}(v)}.
\end{equation}

For example,

\begin{equation}
\Delta G_{41252} = 1 \otimes G_{41252} + G_1 \otimes G_{3141} + G_{122} \otimes G_{12} + G_{4122} \otimes G_1 + G_{41252} \otimes 1.
\end{equation}

3. POLYNOMIAL REALIZATION OF $\text{PQSym}^*$

In the sequel, we need the following definitions: given a totally ordered alphabet $A$, the evaluation vector $Ev(w)$ of a word $w$ is the sequence of number of occurrences of all the elements of $A$ in $w$. The packed evaluation vector $c(w)$ of $w$ is obtained from $Ev(w)$ by removing all its zeros. The fully unpacked evaluation vector $d(w)$ of $w$ is obtained from $c(w)$ by inserting $i - 1$ zeros after each entry $i$ of $c(w)$ except the last one. For example, if $w = 3117291781329$, $Ev(w) = (4, 2, 2, 0, 0, 0, 2, 1, 2)$, $c(w) = (4, 2, 2, 1, 2)$, and $d(w) = (4, 0, 0, 0, 2, 0, 2, 0, 0, 1, 2)$.

The algebra $\text{PQSym}^*$ admits a simple realization in terms of noncommutative polynomials, which is reminiscent of the construction of $\text{FQSym}$. If $A$ is a totally ordered infinite alphabet, one can set

\begin{equation}
G_a(A) = \sum_{w \in A^*, \text{Park}(w) = a} w.
\end{equation}

These polynomials satisfy the relations (9) and allow to write the coproduct as $\Delta G_a = G_a(A' \ast A'')$ where $A' \ast A''$ denotes the ordered sum of two mutually commuting alphabets isomorphic to $A$ as ordered sets.

Recall from [8] that the sums

\begin{equation}
P^\pi := \sum_{a,a'^\pi = \pi} F_a
\end{equation}

where $a'^\pi$ means the non-decreasing reordering and $\pi$ runs over non-decreasing parking functions, span a cocommutative Hopf subalgebra $\text{CQSym}$ of $\text{PQSym}$.

As with $\text{FQSym}$, one can take the commutative image of the $G_a$, that is, replace the alphabet $A$ by an alphabet $X$ of commuting variables (endowed with an
isomorphic ordering). Then, $G_{\mathbf{a}'}(X) = G_{\mathbf{a}''}(X)$ iff $\mathbf{a}'$ and $\mathbf{a}''$ have the same non-decreasing reordering $\pi$, and both coincide with the generalized quasi-monomial function $M_{\pi} = (P^{\pi})^*$ of $[3]$, that is, the natural basis of the commutative Catalan algebra $CQSym^*$.

Actually, $CQSym^*$ contains $QSym$ as a subalgebra, the quasi-monomial functions being obtained as $M_{I} = \sum_{c(\pi)} = I M_{\pi}$.

As a first application of the polynomial realization, we can quantize $CQSym^*$. Indeed, we can proceed as for the quantization of $QSym$ [14], that is, we map the $a_i$ on $q$-commuting variables $x_i$, that is, $x_j x_i = qx_i x_j$ for $i < j$, $G_{\mathbf{a}'}(X)$ and $G_{\mathbf{a}''}(X)$ are equal only up to a power of $q$ when $\mathbf{a}'$ and $\mathbf{a}''$ have the same non-decreasing reordering $\pi$, and the resulting algebra is not commutative anymore. Deforming the coproduct so as to maintain compatibility with the product, we obtain a self-dual Hopf algebra, which is isomorphic to the Loday-Ronco algebra of plane binary trees [5].

However, our main application will be the definition of an internal product on $PQSym$.

4. THE INTERNAL PRODUCT

Recall that Gessel constructed the descent algebra by extending to $QSym$ the coproduct dual to the internal product of symmetric functions. That is, if $X$ and $Y$ are two totally and isomorphically ordered alphabets of commuting variables, we can identify a tensor product $f \otimes g$ of quasi-symmetric functions with $f(X)g(Y)$. Denoting by $XY$ the Cartesian product $X \times Y$ endowed with the lexicographic order, Gessel defined for $f \in QSym_n$

$$\delta(f) = f(XY) \in QSym_n \otimes QSym_n.$$

The dual operation on $Sym_n$ is the internal product $*$, for which $Sym_n$ is anti-isomorphic to the descent algebra $\Sigma_n$.

This construction can be extended to the commutative Catalan algebra $CQSym^*$, and in fact, even to $PQSym^*$.

Let $A'$ and $A''$ be two totally and isomorphically ordered alphabets of noncommuting variables, but such that $A'$ and $A''$ commute with each other. We denote by $A'A''$ the Cartesian product $A' \times A''$ endowed with the lexicographic order. This is a total order in which each element has a successor, so that $G_{\mathbf{a}}(A'A'')$ is a well defined polynomial. Identifying tensor products $u \otimes v$ of words of the same length with words over $A'A''$, we have

$$G_{\mathbf{a}}(A'A'') = \sum_{\text{Park}(u \otimes v) = \mathbf{a}} u \otimes v.$$

Our main result is the following

**Theorem 4.1.** The formula $\delta(G_{\mathbf{a}}) = G_{\mathbf{a}}(A'A'')$ defines a coassociative coproduct on each homogeneous component $PQSym^*_n$. Actually,

$$\delta(G_{\mathbf{a}}) = \sum_{\text{Park}(\mathbf{a}' \otimes \mathbf{a}'') = \mathbf{a}} G_{\mathbf{a}'} \otimes G_{\mathbf{a}''}.$$
By duality, the formula
\[ F_{a'} * F_{a''} = F_{\text{Park}(a' \otimes a'')} \]
defines an associative product on each \( \text{PQSym}_n \).

**Example 4.2.**

\( \text{(20)} \)
\[ F_{211} * F_{211} = F_{311}; \quad F_{211} * F_{112} = F_{312}; \]
\( \text{(21)} \)
\[ F_{211} * F_{121} = F_{321}; \quad F_{112} * F_{312} = F_{213}; \]
\( \text{(22)} \)
\[ F_{31143231} * F_{23571713} = F_{61385451}. \]

5. **Subalgebras of \( (\text{PQSym}_n, \ast) \)**

The following result is almost immediate.

**Proposition 5.1.** The homogeneous components \( \text{CQSym}_n \) of the Catalan algebra are stable under the internal product \( \ast \).

**Example 5.2.**

\( \text{(23)} \)
\[ P_{1123} * P_{1111} = P_{1134}; \quad P_{1111} * P_{1123} = P_{1123}. \]
\( \text{(24)} \)
\[ P_{1123} * P_{1112} = 2P_{1134} + P_{1234}; \quad P_{1122} * P_{1224} = P_{1134} + P_{1233} + 2P_{1234}. \]
\( \text{(25)} \)
\[ P_{1123} * P_{1224} = 2P_{1134} + 5P_{1234}. \]

It is interesting to observe that these algebras are non-unital. Indeed, as one can see on the first example just above

**Proposition 5.3.** The element \( J_n = P^{(1^n)} \) is a left unit for \( \ast \), but not a right unit.

The splitting formula is valid in \( \text{CQSym}_n \). That is,

**Proposition 5.4.** Let \( \mu_r \) denote the \( r \)-fold product map from \( \text{CQSym}^\otimes r \) to \( \text{CQSym} \), \( \Delta^r \) the \( r \)-fold coproduct with values in \( \text{CQSym}^\otimes r \), and \( \ast_r \) the internal product of the \( r \)-fold tensor product of algebras \( \text{CQSym}^\otimes r \). Then, for \( f_1, \ldots, f_r, g \in \text{CQSym} \),
\[ (f_1 \cdots f_r) * g = \mu_r[(f_1 \otimes \cdots \otimes f_r) \ast_r \Delta^r(g)]. \]

This is exactly the same formula as with the internal product of \( \text{Sym} \), actually, an extension of it, since we have

**Theorem 5.5.** The Hopf subalgebra of \( \text{CQSym} \) generated by the elements \( J_n \), which is isomorphic to \( \text{Sym} \) by \( j : S_n \mapsto J_n \), is stable under \( \ast \), and thus also \( \ast \)-isomorphic to \( \text{Sym} \). Moreover, the map \( f \mapsto f * J_n \) is a projector onto \( \text{Sym}_n \), which is therefore a left \( \ast \)-ideal of \( \text{CQSym}_n \).
If \( i < j < \ldots < r \) are the letters occurring in \( \pi \), so that as a word \( \pi = i^{m_i} j^{m_j} \cdots r^{m_r} \), then
\[
P^\pi \star J_n = J_{m_i} J_{m_j} \cdots J_{m_r}.
\] (27)

In the classical case, the non-commutative complete functions split into a sum of ribbon Schur functions, using a simple order on compositions. To get an analogous construction in our case, we have defined a partial order on non-decreasing parking functions.

Let \( \pi \) be a non-decreasing parking function and \( \text{Ev}(\pi) \) be its evaluation vector. The successors of \( \pi \) are the non-decreasing parking functions whose evaluations are given by the following algorithm: given two non-zero elements of \( \text{Ev}(\pi) \) with only zeros between them, replace the left one by the sum of both and the right one by 0. For example, the successors of 113346 are 111146, 113336, and 113344.

By transitive closure, the successor map gives rise to a partial order on non-decreasing parking functions. We will write \( \pi \preceq \pi' \) if \( \pi' \) is obtained from \( \pi \) by successive applications of successor maps.

The Catalan ribbon functions are defined by
\[
P^\pi =: \sum_{\pi' \geq \pi} R_{\pi'}.
\] (28)

The \( R_{\pi} \) are the pre-images of the ordinary ribbons under the projection \( f \mapsto f \star J_n \):

**Proposition 5.6.** Let \( I \) be the composition obtained by discarding the zeros of the evaluation of an non-decreasing parking function \( \pi \). Then
\[
R_{\pi} \star J_n = j(R_I).
\] (29)

More precisely, if \( I = (i_1, \ldots, i_p) \), this last element is equal to \( R_{1^{i_1}2^{i_2} \cdots p^{i_p}} \), that is, the Catalan ribbon indexed by the only non-decreasing word of evaluation \( d(\pi) \).

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