On the fourth moment theorem for the complex multiple Wiener-Itô integrals

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Abstract
In this paper, a product formula of Hermite polynomials is given and then the relation between the real Wiener-Itô chaos and the complex Wiener-Itô chaos (or: multiple integrals) is shown. By this relation and the known multivariate extension of the fourth moment theorem for the real multiple integrals, the fourth moment theorem (or say: the Nualart-Peccati criterion) for the complex Wiener-Itô multiple integrals is obtained.

Keywords: Central Limit Theorem; Complex Gaussian Isonormal Process; Complex Hermite Polynomials; Fourth Moment Theorem; Wiener-Itô Chaos Decomposition.

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Contents

1 Introduction 2

2 Preliminaries 5
2.1 Some properties of complex Hermite polynomials 5
2.2 Complex Gaussian isonormal process and complex Wiener-Itô chaos 6

3 Relation between real and complex Wiener-Itô chaos 10
3.1 A product formula and relation between real multiple integrals and complex multiple integrals 10
3.2 Ito’s complex multiple integrals revisited 12
4 Proof of theorems
4.1 Proof of Proposition 2.9 and Theorem 3.1-3.3 ........................................ 16
4.2 Proof of Theorem 1.1-1.2 ................................................................. 22

5 Some generalized results ................................................................. 24

6 Appendix: multiple integrals by means of the divergence operator .................. 26

Notations

\( \mathcal{H} \) : a real separable Hilbert space

\( \mathcal{H}^{\otimes m} \) : the \( m \) times symmetric tensor product of \( \mathcal{H} \)

\( \mathcal{H} \oplus \mathcal{H} \) : the Hilbert space direct sum

\( \mathcal{H}_C \) : the complexification of \( \mathcal{H} \)

\( \mathcal{H}_C^{\otimes m} \) : the \( m \) times symmetric tensor product of \( \mathcal{H}_C \)

\( X, Y \) : the real Gaussian isonormal process over \( \mathcal{H} \)

\( W \) : the real Gaussian isonormal process over \( \mathcal{H} \oplus \mathcal{H} \)

\( X_C, Y_C \) : the complexification of \( X, Y \)

\( Z \) : the complex Gaussian isonormal process over \( \mathcal{H}_C \)

\( \mathcal{H}_n(X), \mathcal{H}_n(Y), \mathcal{H}_n(W) \) : the \( n \)-th Wiener-Itô chaos of \( X, Y, W \)

\( \mathcal{H}_n^C(X), \mathcal{H}_n^C(W) \) : the complexification of \( \mathcal{H}_n(X), \mathcal{H}_n(W) \)

\( \mathcal{H}_{m,n}(Z) \) : the \( (m,n) \)-th complex Wiener-Itô chaos of \( Z \)

\( \text{symm}(f \otimes g) \) : symmetrizing tensor product of \( f \) and \( g \)

1 Introduction

In a seminal paper [22], Nualart and Peccati showed that the convergence in distribution of a normalized sequence of real multiple Wiener-Itô integrals towards a standard Gaussian law is equivalent to convergence of just the fourth moment to 3, which is called the Nualart-Peccati criterion or the fourth moment theorem. Shortly afterwards, Peccati and Tudor [23] gave a multivariate extension of this characterization. After the publication of the two beautiful papers, there are already several proofs of the criterion such as [1, 9, 10, 12, 14, 17, 21]. Especially, Nualart and Ortiz-Latorre [21] presented a crucial methodological breakthrough, linking the criterion to Malliavin operators, Nourdin and Peccati [14] established the combination of Stein’s method and Malliavin calculus, and the recent papers [10] by Ledoux, Azoodeh, Campese and Poly were from the point of view of spectral theory of general Markov diffusion generators. For details, please refer to the monograph [16] written by Nourdin and Peccati. In addition, Nourdin and Peccati [15] showed that the convergence in distribution of a sequence of real multiple Wiener-Itô integrals towards a centered \( \chi^2 \) law is equivalent to convergence of just the fourth moment and the third moment, and the multivariate extension of this theorem was shown by Nourdin and Rosiński recently [19]. Hu, Lu, Nourdin, Nualart and Poly [4, 13, 18]
strengthened the convergence in law to the uniform convergence of the densities and the total variation convergence (which is equivalent to the $L^1(\mathbb{R}^d)$ convergence of the densities) respectively.

Since both the real multiple Wiener-Itô integrals and the complex multiple Wiener-Itô integrals were established by K. Itô almost at the same time in 1950s [3, 5], the question naturally arises if the Nualart-Peccati criterion is still valid for the complex multiple Wiener-Itô integrals. The principle aim of this paper is to give a positive answer to the above-presented question. Our main results are the following two Nualart-Peccati criterions in abstract complex Wiener-Itô chaos (see Definition 2.7).

For the rest of the paper, we shall denote by $\zeta \sim \mathcal{CN}(0, \sigma^2)$ a symmetric complex Gaussian variable, i.e., $\zeta = \xi_1 + i\xi_2$ with $\xi_i \sim \mathcal{N}(0, \frac{1}{2}\sigma^2)$ and independent.

**Theorem 1.1.** Consider a sequence of random variable $F_k$ being the fixed $(m, n)$-th complex Wiener-Itô multiple integrals, $m + n \geq 2$ and suppose that $E[|F_k|^2] \to \sigma^2$ as $k \to \infty$, where $|\cdot|$ is the absolute value (or modulus) of a complex number.

1) If $m \neq n$, as $k \to \infty$, the following two assertions are equivalent:
   - (i) The sequence $(F_k)$ converges in distribution to $\zeta \sim \mathcal{CN}(0, \sigma^2)$;
   - (ii) $E[|F_k|^4] \to 2\sigma^4$.

2) If $m = n$ and $E[F_k^2] \to \sigma^2(a + ib)$ where $a, b \in \mathbb{R}$ such that $a^2 + b^2 < 1$, that is to say the matrix $C = \begin{bmatrix} 1 + a & b \\ b & 1 - a \end{bmatrix}$ is positive definite, the following two assertions are equivalent:
   - (i) The sequence $(\text{Re}F_k, \text{Im}F_k)$ converges in distribution to a jointly normal law with the covariance $\frac{\sigma^2}{2} C$;
   - (ii) $E[|F_k|^4] \to (a^2 + b^2 + 2)\sigma^4$.

3) If $m = n$ and $E[F_k^2] \to \sigma^2(a + ib)$ where $a, b \in \mathbb{R}$ such that $a^2 + b^2 = 1$, i.e., the matrix $C$ is degenerated, the following three assertions are equivalent:
   - (i) The sequence $(\text{Re}F_k, \text{Im}F_k)$ converges in distribution to a jointly normal law with the covariance $\frac{\sigma^2}{2} C$;
   - (ii) $E[|F_k|^4] \to 3\sigma^4$;
   - (iii) $E[F_k^4] \to 3(a + ib)^2\sigma^4$.

**Remark 1.** Especially, in 3), if $a = \pm 1$, i.e. the sequence $(F_k)$ (or the sequence $(iF_k)$) is real, then as $k \to \infty$, it converges in distribution to $\mathcal{N}(0, \sigma^2)$ if and only if $E[F_k^4] \to 3\sigma^4$, which is just the original Nualart-Peccati criterion.

**Theorem 1.2.** Let $\xi(\alpha_1, \alpha_2) = G_1(\alpha_1) + iG_2(\alpha_2)$ be a complex random variable such that $G_i(\alpha_i)$, $i = 1, 2$, being independent variables having centered $\chi^2$ distributions with $\alpha_i$ degree of freedom respectively. Consider a sequence of random variable $F_k$ belonging to the $(m, n)$-th complex Wiener-Itô chaos, $m + n \geq 2$ being an even number and suppose that $E[|F_k|^2] \to \sigma^2$ as $k \to \infty$. 
1) If \( m \neq n \), as \( k \to \infty \), the following two assertions are equivalent:

(i) The sequence \((F_k)\) converges in distribution to \( \xi(\sigma^2/2, \sigma^2/2) \);

(ii) \( E[F_k^3 + 3|F_k|^2F_k] \to 8(1 - i)\sigma^2 \) and \( E[|F_k|^4] \to 2\sigma^4 + 24\sigma^2 \).

2) If \( m = n \) and \( E[F_k^2] \to \sigma^2(\alpha + \beta) \) where \( \alpha, \beta \in \mathbb{R} \) such that \( \alpha^2 + \beta^2 < 1 \), as \( k \to \infty \), the following two assertions are equivalent:

(i) The sequence \((F_k)\) converges in distribution to \( \xi(1 + a^2\sigma^2, 1 - a^2\sigma^2) \);

(ii) \( E[F_k^3 + 3|F_k|^2F_k] \to 8[1 + a - i(1 - a)]\sigma^2 \) and \( E[|F_k|^4] \to (2 + a^2)\sigma^4 + 24\sigma^2 \).

Remark 2. In the above theorem, when \( m + n \) is an odd integer, there does not exist any \((F_k)\) with bounded variances converging in distribution to \( \xi(\alpha_1, \alpha_2) \) as \( k \to \infty \).

Remark 3. It follows from Theorem 5.2 and Corollary 5.5 in [13] by Nourdin, Nualart and Poly that we can strengthen the convergence in law of Theorem 1.1 (when \( C \) is positive definite) and Theorem 1.2 to the convergence in total variation. Denote by \( \Gamma(F_k) \) the Malliavin matrix of \( F_k = (\text{Re}F_k, \text{Im}F_k) \). As a consequence of Theorem 5.2 and Corollary 5.5 in [13], we deduce that \( E[\det \Gamma(F_k)] \) is bounded away from zero. Then each \( F_k \) admits a density and the above convergence in total variation is equivalent to the convergence of the densities in \( L^1(\mathbb{R}^2) \).
2 Preliminaries

2.1 Some properties of complex Hermite polynomials

Consider a 1-dimensional complex-valued Ornstein-Uhlenbeck process \[3\]
\[
dC_t = -e^{i\theta}C_t dt + \sqrt{\rho \cos \theta} dB_t
\] (2.1)
where \(C_t = C_1(t) + iC_2(t), \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \rho > 0, \) and \(\zeta_t\) is a complex Brownian motion. It is clear that this complex-valued process can be represented by the 2-dimensional nonsymmetric (when \(\theta \neq 0\)) Ornstein-Uhlenbeck process
\[
\begin{bmatrix} dC_1(t) \\ dC_2(t) \end{bmatrix} = \begin{bmatrix} -\cos \theta & \sin \theta \\ -\sin \theta & -\cos \theta \end{bmatrix} \begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix} dt + \sqrt{\rho \cos \theta} \begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix}
\]
Its generator is
\[
A_\theta = \frac{\rho \cos \theta}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + (-x \cos \theta + y \sin \theta) \frac{\partial}{\partial x} - (x \sin \theta + y \cos \theta) \frac{\partial}{\partial y}
\]
which is nonsymmetric (when \(\theta \neq 0\)) but normal, where \(\frac{\partial f}{\partial z} = \frac{1}{2} (\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y}), \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} (\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y})\) are the formal derivative of \(f\) at point \(z = x + iy\) with \(x, y \in \mathbb{R}\). We call \(\partial := \frac{\partial}{\partial z}\) and \(\bar{\partial} := \frac{\partial}{\partial \bar{z}}\) the complex annihilation operators. In \[3\] Theorem 2.7, the authors show that for any \(\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\) and \(\rho > 0, A_\theta\) satisfies that
\[
A_\theta J_{m,n}(z, \rho) = -[(m + n) \cos \theta + i(m - n) \sin \theta] J_{m,n}(z, \rho),
\] (2.3)
where \(J_{m,n}(z, \rho)\) is the so-called complex Hermite polynomials (or say: Hermite-Laguerre-Itô polynomials) given by
\[
\begin{align*}
J_{0,0}(z, \rho) &= 1, \\
J_{m,n}(z, \rho) &= \rho^{m+n} (\partial^*)^m (\bar{\partial}^*)^n 1, \quad m, n \in \mathbb{N},
\end{align*}
\] (2.4)
where \((\partial^*)\phi(z) = -\frac{\partial}{\partial z}\phi(z) + \frac{\partial}{\partial \bar{z}}\phi(z), (\bar{\partial}^*)\phi(z) = -\frac{\partial}{\partial z}\phi(z) + \frac{\partial}{\partial \bar{z}}\phi(z)\) for \(\phi \in \mathcal{C}_0^t(\mathbb{R}^2)\) are the adjoint of the operators \(\partial, \bar{\partial}\) respectively (the complex creation operator).

\[
\left\{ (m! n! \rho^{m+n})^{-\frac{1}{2}} J_{m,n}(z, \rho) : m, n \in \mathbb{N} \right\}
\]
is a complete orthonormal system \[3\] of \(L^2_\mathbb{C}(\mathbb{C}, \nu)\) with \(d\nu = \frac{1}{\pi \rho} e^{-\frac{\rho^2 + \bar{\rho}^2}{\rho}} d\rho d\bar{\rho}\). If \(\rho = 2\), we will often write \(J_{m,n}(z)\) instead of \(J_{m,n}(z, \rho)\).

The real Hermite polynomials \(H_n\) are defined by the formula\[3\] p1571
\[
H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}, \quad n = 1, 2, \ldots
\]
\[1\]Note that \(H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}\) in \[20\] \[22\] \[27\] and \(H_n(x) = \frac{(-1)^n}{\sqrt{n!}} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}\) in \[28\], here we use the definition in \[3\] \[8\] \[13\].
The following property gives the fundamental relation between the real and the complex Hermite polynomials [3 Corollary 2.8] by the authors, which plays the important role of the proofs of the main theorems of the present paper (see Theorem 3.1-3.2 below).

**Proposition 2.1.** Let \( z = x + iy \) with \( x, y \in \mathbb{R} \). Then the real and the complex Hermite polynomials satisfy that

\[
J_{m,l-m}(z) = \sum_{k=0}^{l} i^{-k} \sum_{r+s=k} \binom{m}{r} \binom{l-m}{s} (-1)^{l-m-s} H_k(x) H_{l-k}(y),
\]

\[
H_k(x) H_{l-k}(y) = \frac{i^k}{2^l} \sum_{m=0}^{l} \sum_{r+s=m} \binom{k}{r} \binom{l-k}{s} (-1)^s J_{m,l}(z). \tag{2.5}
\]

Thus, both the class \( \{J_{k,l}(z) : k+l = n\} \) and the class \( \{H_k(x)H_l(y) : k+l = n\} \) generate the same linear subspace of \( L_2^2(\mathbb{C}, \nu) \).

Equality (2.3) means that \( J_{m,n}(z, \rho) \) is the eigenfunctions of \( A_\theta \) for all \( \theta \in (-\pi/2, \pi/2) \), and the only difference is the eigenvalue. Especially when \( \theta = 0 \), the normal Ornstein-Uhlenbeck operator \( A_\theta \) degenerates to a symmetric operator \( A_0 = \frac{\rho^2}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \) and it follows from (2.3) that both the real part and the imaginary part of \( J_{m,n}(z, \rho) \) are the eigenfunctions of \( A_0 \) with respect to the same eigenvalue \( -(m+n) \). Moreover, since [3 Proposition A.6]

\[
J_{m,n}(z, \rho) = J_{n,m}(z, \rho), \tag{2.6}
\]

we have that when \( m \neq n \), \( E_\nu[J_{m,n}(z, \rho)^2] = E_\nu[J_{m,n}(z, \rho)J_{n,m}(z, \rho)] = 0 \) which implies that the real part and the imaginary part of \( J_{m,n}(z, \rho) \) are orthogonal and having the same norm in \( L_2^2(\mathbb{C}, \nu) \). We conclude it as a proposition.

**Proposition 2.2.** Let \( J_{m,n}(z, \rho) = f + ig \), then

\[
A_0 f = -(m+n) f, \quad A_0 g = -(m+n) g.
\]

If \( m \neq n \) then \( f, g \) satisfy

\[
\|f\|_{L_2^2(\nu)} = \|g\|_{L_2^2(\nu)}, \quad E_\nu[fg] = 0.
\]

The above basic properties give a heuristic answer to the problem of the relation between real multiple integrals and complex multiple integrals. In fact, an infinite dimensional version of Proposition 2.2 is given by Theorem 3.3 below.

### 2.2 Complex Gaussian isonormal process and complex Wiener-Itô chaos

Before we describe the formulation of Wiener-Itô chaos decomposition theorem for complex Gaussian isonormal process, let us recall the corresponding theory of real Gaussian isonormal process.
The standard approach to define the Wiener-Itô chaos is using Hermite polynomials (please refer to [5], [20, Definition 1.1.1] and [16, Definition 2.2.3]), or see Definition 2.3 below), and an alternative standard (but equivalent by Proposition 2.5) way is using general polynomial vector spaces (please refer to [7, Definition 2.1]). Here we adopt the former.

**Definition 2.3.** For a fixed real separable Hilbert space \( \mathfrak{H} \), an isonormal Gaussian process over \( \mathfrak{H} \), \( X = \{X(h) : h \in \mathfrak{H}\} \) means that \( X \) is a centered Gaussian family defined on some probability space \( (\Omega, \mathcal{F}, P) \) and such that \( E[X(g)X(h)] = \langle g, h \rangle_{\mathfrak{H}} \) for every \( g, h \in \mathfrak{H} \). If \( \{e_i : i \geq 1\} \) is a countable orthonormal basis of \( \mathfrak{H} \) and \( \{\xi_i\} \) is a sequence of i.i.d. standard normal random variables, then \( X \) is uniquely determined in the sense of law by

\[
X(h) = \sum_{i=1}^{\infty} \langle h, e_i \rangle_{\mathfrak{H}} \xi_i. \tag{2.7}
\]

The \( n \)-th Wiener-Itô chaos \( \mathcal{H}_n(X) \) of \( X \) is the closed linear subspace of the real \( L^2(\Omega) \) generated by the random variable of the type \( \{H_n(X(h)) : h \in \mathfrak{H}, ||h|| = 1\} \) where \( H_n \) is the \( n \)-th Hermite polynomial.

For a sequence \( m = \{m_k\}_{k=1}^{\infty} \) of nonnegative integrals with finite sum, we set \( |m| = \sum_{k=1}^{\infty} m_k \) and \( m! = \prod_{k=1}^{\infty} m_k! \) and define a Fourier-Hermite polynomial [20, 27]

\[
H_m := \frac{1}{\sqrt{m!}} \prod_{k=1}^{\infty} H_{m_k}(X(e_k)). \tag{2.8}
\]

Set \( \xi_k = X(e_k) \) and \( \xi = \{\xi_k : k = 1, 2, \ldots\} \) and use the notation \( \xi^m := \prod_{k=1}^{\infty} \xi_k^{m_k} \), then the right hand side of (2.8) is exactly the wick product : \( \xi^m \) : (please refer to [7, Theorem 3.15]). The next two results are well-known for the real isonormal Gaussian process [16, 20].

**Proposition 2.4.** (Wiener-Itô chaos decomposition)

i) The linear space generated by the class \( \{H_n(X(h)) : n \geq 0, ||h|| = 1\} \) is dense in \( L^q(\Omega) \) for every \( q \geq 1 \).

ii) The space \( L^2(\Omega, \sigma(X), P) \) can be decomposed into the infinite orthogonal sum of the subspace \( \mathcal{H}_n(X) \), i.e., \( L^2(\Omega) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n(X) \).

**Proposition 2.5.** (Real multiple Wiener-Itô integral)

i) For any \( m \geq 1 \) the random variables \( \{H_m : |m| = m\} \) form a complete orthonormal system in \( \mathcal{H}_m(X) \).

ii) The linear mapping \( I_m(\text{symm} (\bigotimes_{k=0}^{\infty} e_k^{\otimes m_k})) = \sqrt{m!} H_m \) provides an isometry from the tensor product \( \mathfrak{H}^{\otimes m} \), equipped with the norm \( \sqrt{m!} \cdot ||\cdot||_{\mathfrak{H}^{\otimes m}} \), onto \( \mathcal{H}_m(X) \). For any \( f \in \mathfrak{H}^{\otimes m} \), \( I_m(f) \) is called the real multiple Wiener-Itô integral of \( f \) with respect to \( X \) (please refer to [15] or [22]).
Now, we turn to the definitions of complex isonormal process, which stem from Ito’s work on complex multiple integrals essentially [6]. But we narrate them in the terminology appeared in [7] and [20].

Let \( \{ \eta_i : i \geq 1 \} \) be an independent copy of \( \{ \xi_i \} \) on some probability space \((\Omega, \mathcal{F}, P)\), then \( Y = \{ Y(h) : h \in \mathcal{H} \} \) satisfying

\[
Y(h) = \sum_{i=1}^{\infty} \langle h, e_i \rangle_{\mathcal{B}_\mathcal{H}} \eta_i
\]  

is an independent copy of the isonormal Gaussian process \( X \) over \( \mathcal{H} \). Then we complexify \( \mathcal{H} \) and \( L^2(\Omega) \) in the usual way and denote by \( \mathcal{H}_C \) and \( L^2_C(\Omega) \) respectively.

Suppose that \( \mathcal{H}_C \ni h = f + ig \) with \( f, g \in \mathcal{H} \), we write

\[
X_C(h) := X(f) + iX(g) = \sum_{i=1}^{\infty} \langle h, e_i \rangle_{\mathcal{B}_\mathcal{H}} \xi_i,
\]

which satisfies \( E[X_C(h) \overline{X_C(h_1)}] = \langle h, h_1 \rangle_{\mathcal{B}_\mathcal{H}} \), where \( h_1 \in \mathcal{H}_C \). The complexification of \( \mathcal{H}_n(X) \) is given by

\[
\mathcal{H}^C_n(X) := \mathcal{H}_n(X) + i\mathcal{H}_n(X) = \{ F + iG : F, G \in \mathcal{H}_n(X) \},
\]

which is the closed linear subspace of \( L^2(\Omega) \) generated by the random variable of the type \( \{ H_n(X(h)), h \in \mathcal{H}, \|h\| = 1 \} \). Clearly, \( \{ e_i : i \geq 1 \} \) is still the basis of the complex Hilbert space \( \mathcal{H}_C \).

**Definition 2.6.** Let \( \{ \zeta_i = \xi_i + i\eta_i : i \geq 1 \} \) be a sequence of i.i.d. symmetric complex normal random variables with variance 2 on the probability space \((\Omega, \mathcal{F}, P)\). Then

\[
Z(h) = \frac{X_C(h) + iY_C(h)}{\sqrt{2}} = \frac{1}{\sqrt{2}} \sum_{i=1}^{\infty} \langle h, e_i \rangle_{\mathcal{B}_\mathcal{H}} \xi_i, \quad h \in \mathcal{H}_C
\]

is called a complex isonormal Gaussian process over \( \mathcal{H}_C \), which is a centered symmetric complex Gaussian family and such that

\[
E[Z(h)^2] = 0, \quad E[Z(g)\overline{Z(h)}] = \langle g, h \rangle_{\mathcal{B}_C}, \quad \forall g, h \in \mathcal{H}_C.
\]

**Remark 4.** Eq. (2.12) is exactly Janson’s idea of the isometric complex Gaussian Hilbert space (see Example 1.9, Theorem 1.23 of [7] and [7, p15]).

**Definition 2.7.** For each \( m, n \geq 0 \), we write \( \mathcal{H}_{m,n}(Z) \) to indicate the closed linear subspace of \( L^2(\Omega) \) generated by the random variables of the type \( J_{m,n}(Z(h)), h \in \mathcal{H}_C, \|h\|_{\mathcal{B}_C} = \sqrt{2} \) where \( J_{m,n}(z) \) is the complex Hermite polynomials given by (2.4).

The space \( \mathcal{H}_{m,n}(Z) \) is called the Wiener-Itô chaos of degree of \( (m, n) \) of \( Z \) (or say: \( (m, n) \)-th Wiener-Itô chaos of \( Z \)).
Remark 5. Using the polynomial vector spaces, Janson defines complex Wiener-Itô chaos $H_C^n$. Due to Proposition 2.9, it equals to

$$\bigoplus_{k+l=n} \mathcal{H}_{k,l}(Z)$$

in our notation, but he does not decompose $H_C^n$ into the orthogonal direct sum of $\mathcal{H}_{k,l}(Z)$.

Definition 2.8. Take a complete orthonormal system \{\xi_k\} in $\mathcal{H}_C$. For two sequences $m=\{m_k\}_{k=1}^\infty$, $n=\{n_k\}_{k=1}^\infty$ of nonnegative integrals with finite sum, define a complex Fourier-Hermite polynomial

$$J_{m,n} := \prod_k \frac{1}{\sqrt{2^{m_k+n_k}m_k!n_k!}} J_{m_k,n_k}(\sqrt{2}Z(\xi_k)). \quad (2.13)$$

Remark 6. Let $\xi_k = Z(\xi_k)$ and we use the notation $\xi^m \xi^n = \prod_k \xi^{m_k} \xi^{n_k}$, then the right hand side of (2.13) is exactly the Wick product: $\xi^m \xi^n$ (: please refer to Example 3.31 and Example 3.32 of [7, p31]).

In the following key proposition, the basis of $(m,n)$-th Wiener-Itô chaos $\mathcal{H}_{m,n}(Z)$ and an isometry mapping from $\mathcal{H}_C^{\otimes m} \otimes \mathcal{H}_C^{\otimes n}$ onto $\mathcal{H}_{m,n}(Z)$ are given.

Proposition 2.9. Let $m$, $n$ and $\xi_k$ be as in Definition 2.8. Then:

(i) For any $m, n \geq 0$ the random variables

$$\{J_{m,n} : |m| = m, |n| = n\} \quad (2.14)$$

form a complete orthonormal system in $\mathcal{H}_{m,n}(Z)$.

(ii) The linear mapping

$$\mathcal{I}_{m,n}(\text{symm}(\otimes_{k=1}^\infty \xi_k^{m_k}) \otimes \text{symm}(\otimes_{k=1}^\infty \xi_k^{n_k})) = \sqrt{m!n!} J_{m,n} \quad (2.15)$$

provides an isometry from the tensor product $\mathcal{H}_C^{\otimes m} \otimes \mathcal{H}_C^{\otimes n}$, equipped with the norm $\sqrt{m!n!} |||\cdot|||_{\mathcal{H}_C^{\otimes (m+n)}}$, onto the $(m,n)$-th Wiener-Itô chaos $\mathcal{H}_{m,n}(Z)$.

Proof of Proposition 2.9 is presented in Section 4.

Definition 2.10. For any $f \in \mathcal{H}_C^{\otimes m} \otimes \mathcal{H}_C^{\otimes n}$, we call $\mathcal{I}_{m,n}(f)$ the complex multiple Wiener-Itô integral of $f$ with respect to $Z$.

Remark 7. Janson gives another complex multiple integrals $I_n$ from the viewpoint of Gaussian Hilbert space formally [7, Theorem 7.52]. Actually, using the above Definition 2.8 and Proposition 2.9, his definition is only equivalent to a linear isometric mapping from $\bigoplus_{p+q=n} \mathcal{H}_C^{\otimes p} \otimes \mathcal{H}_C^{\otimes q}$ onto $\bigoplus_{p+q=n} \mathcal{H}_{p,q}(Z)$. In our opinion, (2.15) matches the theory of Itô’s complex multiple integrals [6] better (see [3,34] and Remark 8 for the reason to call “integrals”).
Theorem 2.11. (Complex Wiener-Itô chaos decomposition)

(i) The linear space generated by the class
\[
\left\{ J_{m,n}(Z(h)) : m, n \geq 0, h \in \mathcal{H}_C, \|h\|_{\mathcal{H}_C} = \sqrt{2} \right\}
\]  
(2.16)
is dense in \( L^q_C(\Omega, \sigma(X,Y), P) \) for every \( q \in [1, \infty) \).

(ii) One has that \( L^2_C(\Omega, \sigma(X,Y), P) = \bigoplus_{m=0}^{\infty} \bigoplus_{n=0}^{\infty} \mathcal{H}_{m,n} \). This means that every random variable \( F \in L^2_C(\Omega, \sigma(X,Y), P) \) admits a unique expansion of the type
\[
F = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} F_{m,n},
\]
where \( F_{m,n} \in \mathcal{H}_{m,n}, F_{0,0} = E[F] \) and the series converges in \( L^2_C(\Omega) \).

One can give several different proofs of the above theorem along the line of [16, 20] or [7]. In Subsection 4.1, we will give a simple proof using the connection between the real Wiener-Itô chaos and the complex Wiener-Itô chaos based on Theorem 3.2.

3 Relation between real and complex Wiener-Itô chaos

3.1 A product formula and relation between real multiple integrals and complex multiple integrals

To establish the connection between real multiple integrals and complex multiple integrals, we need a third real isonormal Gaussian process \( W \) over the Hilbert space direct sum of the spaces \( \mathcal{H} \) and \( \mathcal{H} \).

Suppose \( h, f \in \mathcal{H} \), then denote \((h, f)\) the Cartesian product (or say, the order pair) of \( \mathcal{H} \) and \( \mathcal{H} \). Denote by \( \mathcal{H} \oplus \mathcal{H} \) the Hilbert space direct sum of the spaces \( \mathcal{H} \) and \( \mathcal{H} \) with the natural inner product (see [26, p48]), i.e., for any \( h_1, h_2, f_1, f_2 \in \mathcal{H} \),
\[
\langle (h_1, f_1), (h_2, f_2) \rangle_{\mathcal{H} \oplus \mathcal{H}} = \langle h_1, h_2 \rangle_{\mathcal{H}} + \langle f_1, f_2 \rangle_{\mathcal{H}}.
\]

With respect to this inner product, \( \mathcal{H} \oplus \mathcal{H} \) is a Hilbert space. We write \( W = \{W(h, f) : h, f \in \mathcal{H} \} \) the isonormal Gaussian process over \( \mathcal{H} \oplus \mathcal{H} \) and denote by \( \mathcal{H}_n(W) \) the \( n \)-th Wiener-Itô chaos of \( W \).

For any \( 0 < \theta_n < \cdots < \theta_0 < \pi \), denote a \((n+1) \times (n+1)\) matrix
\[
M = M(\theta_0, \ldots, \theta_n)
\]
\[
= \begin{bmatrix}
\sin \theta_0^n & \binom{n}{1} \sin \theta_0^{n-1} \cos \theta_0 & \cdots & \binom{n}{n-1} \sin \theta_0 (\cos \theta_0)^{n-1} & \cos \theta_0^n \\
\sin \theta_1^n & \binom{n}{1} \sin \theta_1^{n-1} \cos \theta_1 & \cdots & \binom{n}{n-1} \sin \theta_1 (\cos \theta_1)^{n-1} & \cos \theta_1^n \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\sin \theta_n^n & \binom{n}{1} \sin \theta_n^{n-1} \cos \theta_n & \cdots & \binom{n}{n-1} \sin \theta_n (\cos \theta_n)^{n-1} & \cos \theta_n^n
\end{bmatrix}
\]  
(3.17)
Theorem 3.1. (A product formula of real multiple integrals) Let Definitions 2.3, 2.6, 2.10 prevail. Denote by \( \mathcal{H}_k(X)\mathcal{H}_l(Y) \) the closed linear subspace of \( L^2(\Omega, \mathcal{F}, P) \) generated by the random variables of the type
\[
\{H_k(X(f))H_l(Y(g)) : \|f\|_\beta = \|g\|_\beta = 1\}.
\]
Suppose that \( \|f\|_\beta^2 + \|g\|_\beta^2 = 1 \), then
\[
H_n(X(f) + Y(g)) = \sum_{l=0}^{n} \binom{n}{l} \|f\|_\beta^l \|g\|_\beta^{n-l} H_l \left( \frac{X(f)}{\|f\|_\beta} \right) H_{n-l} \left( \frac{Y(g)}{\|g\|_\beta} \right).
\] (3.18)
Suppose that \( \|f\|_\beta = \|g\|_\beta = 1 \), for any fixed \( 0 < \theta_n < \cdots < \theta_0 < \pi \), then
\[
H_l(X(f))H_{n-l}(Y(g)) = \sum_k M^{-1}_{l,k} H_n (\cos \theta_k X(f) + \sin \theta_k Y(g)),
\] (3.19)
where \( M^{-1}_{l,k} \) is the \((l,k)\)-entry of \( M^{-1} \), the inverse of \( M \) (see (3.17)). That is to say, \( \mathcal{H}_n(W) \), the \( n \)-th Wiener-Itô chaos of \( W \) satisfies
\[
\mathcal{H}_n(W) = \bigoplus_{k+l=n} \mathcal{H}_k(X)\mathcal{H}_l(Y).
\] (3.20)

Theorem 3.2. (The connection between real and complex Wiener-Itô chaos)

Let Definitions 2.3, 2.6, 2.10 prevail. Suppose that \( \|f\|_\beta^2 + \|g\|_\beta^2 = 1 \), \( \|f\|_\beta^2 + \|\tilde{g}\|_\beta^2 = 1 \), then for any fixed \( \theta \in \mathbb{R} \),
\[
H_n(X(f) + Y(g)) + iH_n(X(\tilde{f}) + Y(\tilde{g}))
= \sum_{k=0}^{n} d_k (J_{k,n-k}(Z(h)) + iJ_{k,n-k}(Z(\tilde{h}))),
\] (3.21)
where \( h = \sqrt{2}e^{i\theta}(f - ig), \) \( \tilde{h} = \sqrt{2}e^{i\theta}(\tilde{f} - i\tilde{g}) \), and
\[
d_k = \frac{1}{2^n} \sum_{r+s=k} (-1)^s \binom{n}{l} \binom{l}{r} \binom{n-l}{s} (\cos \theta)^l (i \cdot \sin \theta)^{n-l}.
\] (3.22)
Suppose that \( \mathcal{H}_n \ni h \) with \( \|h\|_{\mathcal{H}_n} = \sqrt{2} \), then
\[
J_{k,n-k}(Z(h)) = \sum_{j=0}^{n} \tilde{c}_j H_n(X(f_i) + Y(g_i)),
\] (3.23)
where \( f_i + ig_i = \frac{1}{\sqrt{2}}e^{i\theta} \tilde{h} \), and
\[
\tilde{c}_j = \sum_{j=0}^{n} M^{-1}_{j,1} \sum_{r+s=j} \binom{k}{r} \binom{n-k}{s} (-1)^{n-k-s}.
\] (3.24)
That is to say, the complexification of \( \mathcal{H}_n(W) \) satisfies
\[
\mathcal{H}_n(W) := \mathcal{H}_n(W) + i\mathcal{H}_n(W) = \bigoplus_{k+l=n} \mathcal{H}_{k,l}(Z).
\] (3.25)
Proofs of Theorem 3.1-3.2 are presented in Section 4.

Remark 8. Substituting (3.20) into (3.25), we get
\[ \bigoplus_{k+l=n} H_{k,l}(Z) = \bigoplus_{i+j=n} \left[ H_i(X)H_j(Y) + iH_i(X)H_j(Y) \right], \]
which can be regarded as the infinite dimensional version of Equality (2.5).

Theorem 3.3. Suppose \( \varphi \in \mathcal{S}_C^{\otimes m} \otimes \mathcal{S}_C^{\otimes n} \) and \( F = \mathcal{I}_{m,n}(\varphi) = U + iV \). Then there exist real \( u, v \in (\mathcal{H} \oplus \mathcal{H})^{\otimes (m+n)} \) such that
\[ U = \mathcal{I}_{m+n}(u), \quad V = \mathcal{I}_{m+n}(v), \quad (3.26) \]
where \( \mathcal{I}_p(g) \) is the \( p \)-th real Wiener-Itô multiple integral of \( g \) with respect to \( W \). And if \( m \neq n \) then
\[ E[U^2] = E[V^2], \quad E[UV] = (m + n)! \langle u, v \rangle_{(\mathcal{H} \oplus \mathcal{H})^{\otimes (m+n)}} = 0. \quad (3.27) \]

This theorem can be seen as the infinite dimensional version of Proposition 2.2, the proof is presented in Section 4.

3.2 Ito’s complex multiple integrals revisited

If \( \mathcal{H} \) is a real separable Hilbert space \( L^2(T, \mathcal{B}, \mu) \) and \( \mu \) is a non-atomic measure, then the definition of \( \mathcal{I}_{m,n} \) (see Definition 2.10) coincides with a multiple Wiener-Itô integrals defined by Itô [6]. In fact, at first, Itô gave a continuous complex normal random measure \( M = \{ M(B) : B \in \mathcal{B}, \mu(B) < \infty \} \) on \( (T, \mathcal{B}) \), such that, for every \( B, C \in \mathcal{B} \) with finite measure,
\[ E[M(B)M(C)] = \mu(B \cap C). \]

Next, for the off-diagonal simple function \( f \in \mathcal{S}_C^{\otimes m} \otimes \mathcal{S}_C^{\otimes n} \) of the form
\[ f(t_1, \ldots, t_m, s_1, \ldots, s_n) = \sum a_{i_1 \ldots i_m,j_1 \ldots j_n} 1_{E_{i_1} \times \cdots \times E_{i_m} \times E_{j_1} \times \cdots \times E_{j_n}}, \quad (3.28) \]
with \( 1_B(\cdot) \) the characteristic function of the set \( B \), he defined the multiple integrals \( I_{m,n}(f) \) by
\[ I_{m,n}(f) = \sum a_{i_1 \ldots i_m,j_1 \ldots j_n} M(E_{i_1}) \cdots M(E_{i_m}) \bar{M}(E_{j_1}) \cdots \bar{M}(E_{j_n}). \quad (3.29) \]

And then by density argument, he extended the multiple integrals to any \( f \in \mathcal{S}_C^{\otimes m} \otimes \mathcal{S}_C^{\otimes n} \),
\[ I_{m,n}(f) = \int \cdots \int f(t_1, \ldots, t_m, s_1, \ldots, s_n) dM(t_1) \cdots dM(t_m) dM(s_1) \cdots dM(s_n). \]
Moreover, Itô established the relation between complex multiple integrals and complex Hermite polynomials: suppose that \( h_1(t), \ldots, h_l(t) \) be any orthonormal system in \( \mathfrak{H} \) and \( \alpha_i, \beta_j = 1, \ldots, l \), then

\[
\int \cdots \int h_{\alpha_1}(t_1) \cdots h_{\alpha_m}(t_m) h_{\beta_1}(s_1) \cdots h_{\beta_n}(s_n) dM(t_1) \cdots dM(t_m)dM(s_1) \cdots dM(s_n) = \prod_{k=1}^{l} \frac{2^{-m_k+n_k}}{2} J_{m_k,n_k}(\sqrt{2}z_k),
\]

where \( z_k = \int \hat{h}_k(t) dM(t), k = 1, \ldots, l \) and \( m_k, n_k \) are the number of \( k \) appearing in \( \alpha_i \) and \( \beta_j \) respectively.

**Remark 9.** Here the notation \( 2^{-\frac{j+i}{2}} J_{i,j}(\sqrt{2}z) \) is exactly the notation \( H_{i,j}(z, \bar{z}) \) of [6] by Itô. It follows from (2.15) and (3.30) that \( J_{m,n} \) coincides with \( I_{m,n} \).

Now we turn to express Eq. (3.18)-(3.19) and Eq. (3.21)-(3.23) in terms of Itô’s theory. Set \( \hat{M} = \frac{1}{\sqrt{2}}[M_1 + i M_2] \). Then \( M_1, M_2 \) are two real independent continuous normal system such that, for every \( B, C \in \mathcal{B} \) with finite measure, \( E[M_1(B)M_1(C)] = E[M_2(B)M_2(C)] = \mu(B \cap C) \). Set \( \hat{T} = \{1, 2\} \times T, \mathcal{B}(\hat{T}) = \mathcal{B}(\{1, 2\} \times T) \). And set

\[
\hat{M}(B) = M_1(B_1) + M_2(B_2), \quad \forall B = (\{1\} \times B_1) \cup (\{2\} \times B_2) \in \mathcal{B}(\hat{T}).
\]

Then \( \hat{M} = \{\hat{M}(B) : B = (\{1\} \times B_1) \cup (\{2\} \times B_2), \mu(B_1) + \mu(B_2) < \infty\} \) is a normal random measure on \((\hat{T}, \mathcal{B}(\hat{T}))\) and \( L^2(\hat{T}) = L^2(T) \oplus L^2(T) \).

For any \( \hat{f} = (f_1, f_2) \) with \( f_i \in \mathfrak{H}, i = 1, 2 \). Suppose \( \|f_1\|_{\mathfrak{H}}^2 + \|f_2\|_{\mathfrak{H}}^2 = 1 \), then we have that

\[
\int \hat{f} d\hat{M} = \int f_1 dM_1 + \int f_2 dM_2,
\]

and Eq. (3.18) means that

\[
\int \cdots \int \hat{f}^{\otimes n} d\hat{M}(t_1) \cdots d\hat{M}(t_n)
= H_n(\int \hat{f} d\hat{M})
= \sum_{l=0}^{n} \binom{n}{l} \|f_1\|_\mathfrak{H}^l \|f_2\|_{\mathfrak{H}}^{n-l} H_l(\int f_1 dM_1_{\|f_1\|_\mathfrak{H}}) H_{n-l}(\int f_2 dM_2_{\|f_2\|_{\mathfrak{H}}})
= \sum_{l=0}^{n} \binom{n}{l} \int \cdots \int f_1^{\otimes l} dM_1(t_1) \cdots dM_1(t_l) \int \cdots \int f_2^{\otimes (n-l)} dM_2(t_{l+1}) \cdots dM_2(t_n).
\]

Suppose that \( \|f_1\|_{\mathfrak{H}}^2 = \|f_2\|_{\mathfrak{H}}^2 = 1 \), then Eq. (3.19) means that

\[
\int \cdots \int f_1^{\otimes n} dM_1(t_1) \cdots dM_1(t_l) \int \cdots \int f_2^{\otimes (n-l)} dM_2(t_{l+1}) \cdots dM_2(t_n)
\]
\[ H_n(\int f_1 dM_1)H_{n-1}(\int f_2 dM_2) \]
\[ = \sum_k M_{i,k}^{-1} H_n(\cos \theta_k \int f_1 dM_1 + \sin \theta_k \int f_2 dM_2) \]
\[ = \sum_k M_{i,k}^{-1} \int \widehat{f}^{(k)} \otimes f \, d\widehat{M}(t_1) \cdots d\widehat{M}(t_n), \]

where \( \widehat{f}^{(k)} = (f_1^{(k)}, f_2^{(k)}) = (\cos \theta_k f_1, \sin \theta_k f_2). \)

Moreover, let \( \hat{g} = (g_1, g_2), \) and suppose that \( \|f_1\|_2^2 + \|f_2\|_2^2 = 1, \|g_1\|_2^2 + \|g_2\|_2^2 = 1, \) then Eq. (3.21) means that
\[ \int \cdots \int (\hat{f}^{\otimes n} + \hat{g}^{\otimes n}) \, d\widehat{M}(t_1) \cdots d\widehat{M}(t_n) \]
\[ = H_n(\int \hat{f} \, d\widehat{M}) + iH_n(\int \hat{g} \, d\widehat{M}) \]
\[ = \sum_{k=0}^n d_k (J_{k,n-k}(\int \hat{h} \, d\widehat{M}) + iJ_{k,n-k}(\int \hat{h} \, d\widehat{M})) \]
\[ = \sum_{k=0}^n d_k \int \cdots \int (\hat{h}^{\otimes k} \otimes \hat{h}^{\otimes (n-k)} + i\hat{h}^{\otimes k} \otimes \hat{h}^{\otimes (n-k)}) \, dM(t_1) \cdots dM(t_m) \overline{dM(s_1)} \cdots \overline{dM(s_n)}, \]

where \( \hat{h} = \sqrt{2e^{i\theta}(f_1 - if_2)}, \hat{h} = \sqrt{2e^{i\theta}(g_1 - ig_2)}. \)

Suppose that \( \hat{h} \in \mathcal{H} \) with \( \|\hat{h}\|_{\mathcal{H}} = \sqrt{2}, \) then Eq. (3.23) means that
\[ \int \cdots \int \hat{h}^{\otimes k} \otimes \hat{h}^{\otimes (n-k)} \, dM(t_1) \cdots dM(t_m) \overline{dM(s_1)} \cdots \overline{dM(s_n)} \]
\[ = J_{k,n-k}(\int \hat{h} \, d\widehat{M}) = \sum_{i=0}^n \hat{c}_i H_n(\int \hat{f} \, d\widehat{M}) \]
\[ = \sum_{i=0}^n \hat{c}_i \int \cdots \int \hat{f}^{(i)} \otimes \hat{h} \, d\widehat{M}(t_1) \cdots d\widehat{M}(t_n), \]

where \( \hat{f}^{(i)} = (f_1, g_i) \) and \( f_1 + ig_i = \frac{1}{\sqrt{2}} e^{i\theta} \hat{h} \).

**Example 1.** Let \( (B_1(t), B_2(t)) \) denote 2-dimensional Brownian motion on \( t \in [0, \infty). \) We put \( \zeta_t := \frac{B_1(t) + iB_2(t)}{\sqrt{2}}. \) \( \zeta_t \) is called complex Brownian motion. Extending Theorem 9.6.9 in the textbook by Kuo to 2-dimensional Brownian motion, we have
\[ H_n(\int \hat{f} \, d\widehat{M}) = H_n(\frac{1}{\sqrt{2}} \int_0^\infty f_1(t) \, dB_1(t) + \frac{1}{\sqrt{2}} \int_0^\infty f_2(t) \, dB_2(t)) \]
\[ = \frac{n!}{2n^{2}} \sum_{i_1, \ldots, i_n=1}^2 \int_0^\infty \int_0^{t_{i_1}} \cdots \int_0^{t_{i_n}} f(i_1(t_1) \cdots f(i_n(t_n)) \, dB_{i_1}(t_1) \cdots dB_{i_n}(t_n). \]
Thus, we can express Eq. (3.18) in Itô’s iterated integrals as follows

\[
\sum_{i_1, \ldots, i_n=1}^{2} \int_{0}^{t_1} \int_{0}^{t_2} \cdots \int_{0}^{t_n} f_{i_1}(t_1) \cdots f_{i_n}(t_n) dB_{i_1}(t_1) \cdots dB_{i_n}(t_n)
\]

\[
= \sum_{l=0}^{n} \int_{0}^{t_1} \int_{0}^{t_2} \cdots \int_{0}^{t_{l+2}} f_1(t_1) \cdots f_{l+2}(t_{l+2}) dB_1(t_1) \cdots dB_2(t_{l+2})
\]

\[
\times \int_{0}^{t_1} \int_{0}^{t_2} \cdots \int_{0}^{t_{l+1}} f_2(t_{l+1}) dB_2(t_{l+1}) \cdots dB_2(t_{l+2})
\]

We express Eq. (3.19) in Itô’s iterated integrals as follows

\[
\int_{0}^{t_1} \int_{0}^{t_2} \cdots \int_{0}^{t_{l+2}} f_1(t_1) \cdots f_l(t_l) dB_1(t_1) \cdots dB_l(t_l)
\]

\[
\times \int_{0}^{t_1} \int_{0}^{t_2} \cdots \int_{0}^{t_{l+1}} f_{l+2}(t_{l+1}) dB_{l+2}(t_{l+1}) \cdots dB_2(t_{l+2})
\]

\[
= \binom{n}{l} \sum_{k} \sum_{i_1, \ldots, i_n=1}^{2} \int_{0}^{t_1} \int_{0}^{t_2} \cdots \int_{0}^{t_n} f_{i_1}(t_1) \cdots f_{i_l}(t_l) dB_{i_1}(t_1) \cdots dB_{i_n}(t_n),
\]

We express Eq. (3.21) in Itô’s iterated integrals as follows

\[
\sum_{i_1, \ldots, i_n=1}^{2} \int_{0}^{t_1} \int_{0}^{t_2} \cdots \int_{0}^{t_n} f_{i_1}(t_1) \cdots f_{i_n}(t_n) dB_{i_1}(t_1) \cdots dB_{i_n}(t_n)
\]

\[
= \frac{2^{n/2}}{n!} \sum_{k=0}^{n} d_k \int \cdots \int (\tilde{\eta}^{\otimes k} \otimes \tilde{\eta}^{\otimes(n-k)} + i \tilde{\eta}^{\otimes k} \otimes \tilde{\eta}^{\otimes(n-k)}) d\zeta(t_1) \cdots d\zeta(t_n) d\zeta(s_1) \cdots d\zeta(s_n),
\]

where \(\|f_1\|_\tilde{\eta}^2 + \|f_2\|_\tilde{\eta}^2 = 1, \|g_1\|_\tilde{\eta}^2 + \|g_2\|_\tilde{\eta}^2 = 1\) and \(\tilde{\eta} = \sqrt{2}e^{ig_2}, \tilde{\eta} = \sqrt{2}e^{ig_1}\).

We express Eq. (3.23) in Itô’s iterated integrals as follows

\[
\int \cdots \int (\tilde{\eta}^{\otimes k} \otimes \tilde{\eta}^{\otimes(n-k)}) d\zeta(t_1) \cdots d\zeta(t_n) d\zeta(s_1) \cdots d\zeta(s_n)
\]

\[
= \frac{n!}{2^{n/2}} \sum_{j=0}^{n} \sum_{i_1, \ldots, i_n=1}^{2} \int_{0}^{t_1} \int_{0}^{t_2} \cdots \int_{0}^{t_n} f_{i_1}(t_1) \cdots f_{i_n}(t_n) dB_{i_1}(t_1) \cdots dB_{i_n}(t_n),
\]

where \(\tilde{f}^{(j)} = (f_1^{(j)}, f_2^{(j)}), f_1^{(j)} + if_2^{(j)} = \frac{1}{\sqrt{2}}e^{ig_2}\tilde{f}\).

**Example 2.** Let \(\{\zeta_j = \xi_j^{(1)} + i\xi_j^{(2)} : j \geq 1\}\) be a sequence of i.i.d. symmetric complex normal random variables with variance 1. Set \(T = \{1, 2, \ldots, n, \ldots\}, \#\) be the counting measure on \(T\) and \(M(E) = \sum_{j \in E \subset T} \zeta_j\), which is the complex normal random measure on \((T, \#)\).
This system is not included in Itô’s framework, because \( (T, \hat{z}) \) is not continuous (or say non-atomic, see [6] and [20]). Specifically, set \( \mathcal{H} = l^2 \), \( f_1, f_2 \in \mathcal{H} \) and \( \hat{f} = f_1 + if_2 \) with \( \|f_1\|^2_0 + \|f_2\|^2_0 = 1 \), by direct computation,

\[
\sum_{t_1...t_n,s_1...s_m} \hat{f}(t_1) \cdots \hat{f}(t_n) \hat{f}(s_1) \cdots \hat{f}(s_m) \zeta_{t_1} \cdots \zeta_{t_n} \zeta_{s_1} \cdots \zeta_{s_m} \\
\neq H_{n,m} \left( \sum_{n} \hat{f}(n) \zeta_n, \sum_{n} \hat{f}(n) \zeta_n \right) \\
= 2^{-\frac{n+m}{2}} J_{n,m} \left( \sqrt{2} \sum_{n} \hat{f}(n) \zeta_n \right). \quad \text{(see Remark 3)}
\]

This means that Itô’s multiple integrals with discrete time do not coincide with the chaos decomposition with respect to real or complex Hermite polynomials (see Definition [2.7]). For an alternative theory of stochastic integrals with discrete time and the corresponding chaos decomposition, please refer to the monograph by Privault [24].

4 Proof of theorems

4.1 Proof of Proposition [2.9] and Theorem [3.1–3.3]

**Proof of Proposition [2.9]** (i): We follow the arguments of the real Wiener-Itô chaos [20, Page 7]. Concretely, denote \( P_{m,n} \) be the closure of the linear space generated by the class

\[
\{ p(Z(h_1), \ldots, Z(h_j)) : j \geq 1, h_1, \ldots, h_j \in \mathcal{H}_C \}, \quad \text{(4.31)}
\]

where \( p \) is a polynomial in complex variables \( z_1, \ldots, z_j \) of degree less than \( m \) and \( \bar{z}_1, \ldots, \bar{z}_j \) of degree less than \( n \) (for short, say the degree of \( p \) less than or equal to \( (m, n) \)). We claim that

\[
P_{m,n} = \bigoplus_{k=0}^m \bigoplus_{l=0}^n \mathcal{H}_{k,l}(Z). \quad \text{(4.32)}
\]

Indeed, since \( J_{k,l}(z) \) is a polynomial in complex variable \( z \) of degree \( k \) and \( \bar{z} \) of degree \( l \) (see [3] Theorem 2.15 or [7] Example 3.31), the inclusion \( \bigoplus_{k=0}^m \bigoplus_{l=0}^n \mathcal{H}_{k,l}(Z) \subset P_{m,n} \) is immediate. To prove the converse inclusion, it is enough to check \( P_{m,n} \) is orthogonal to all \( \mathcal{H}_{k,l}(Z) \) for \( k > m \) or \( l > n \), i.e., to show that for any \( h \in \mathcal{H}_C \)

\[
E[p(Z(h_1), \ldots, Z(h_j)) J_{i,k}(Z(h))] = 0. \quad \text{By the Gram-Schmidt orthogonalization process, we can suppose that} \ h \in \{h_1, \ldots, h_j\} \text{ which is an orthonormal system. The equality [3 Corollary 2.8]}
\]

\[
z^r \bar{z}^s = \sum_{i=0}^{r+s} \binom{r}{i} \binom{s}{i} i!2^i J_{r-i,s-i}(z) \quad \text{(4.33)}
\]
implies $E[Z(h)\overline{Z(h)}] = 0$ when $r < l$, $s < k$. This ends the proof together with the independent property of $Z(h_i)$ for $i = 1, \ldots, j$.

Clearly, the random variables of the class $\{J_{m,n} : |m| = m, |n| = n\}$ belong to $\mathcal{P}_{m,n}$ and are orthonormal system. Since $\{e_i\}$ is an orthonormal basis of $\mathcal{H}_C$, every polynomial random variable $p(Z(h_1), \ldots, Z(h_j))$ as (4.31) can be approximated by polynomials $q(Z(e_1), \ldots, Z(e_r))$ with the degree of $q$ less than or equal to $(m, n)$. Thus Eq. (4.33) implies that the random variables of the class $\{J_{m,n} : |m| \leq m, |n| \leq n\}$ are a basis of $\mathcal{P}_{m,n}$. But $\{J_{m,n} : |m| = m, |n| = n\}$ are orthogonal to $\mathcal{P}_{m-1,n} \cup \mathcal{P}_{m,n-1}$. Thus the class $\{J_{m,n} : |m| = m, |n| = n\}$ are a basis of $\mathcal{H}_{m,n}(Z)$.

(ii): The isometry property is deduced from $\|\text{symm}(\otimes_{k=1}^{\infty} |e_k^{(m)}\rangle\rangle\|_{\mathcal{H}_{m,n}}^2 = \frac{m!}{m!}$ [20] Page 8. Since by (i) the span of $\{J_{m,n} : |m| = m, |n| = n\}$ generates $\mathcal{H}_{m,n}(Z)$ and since linear combinations of vectors of the type $\text{symm}(\otimes_{k=1}^{\infty} |e_k^{(m)}\rangle\rangle) \otimes \text{symm}(\otimes_{k=1}^{\infty} |e_k^{(n)}\rangle\rangle)$ are dense in $\mathcal{H}_{m,n}^{\otimes m} \otimes \mathcal{H}_{m,n}^{\otimes n}$, we have that the mapping between $\mathcal{H}_{m,n}^{\otimes m} \otimes \mathcal{H}_{m,n}^{\otimes n}$ and $\mathcal{H}_{m,n}(Z)$ is onto.

We cite a special determinant which is Problem 342 of [25].

**Lemma 4.1.** Let $\mathbb{F}[x, y]$ be the set of 2-variate polynomials over the field $\mathbb{F}$. Set $f_i(a, b) \in \mathbb{F}[a, b]$ be the homogeneous polynomials of degree $i$ with $i = 1, \ldots, n$. Then the $n + 1$ order determinant

$$
\begin{vmatrix}
  f_n(a_0, b_0) & b_0 f_{n-1}(a_0, b_0) & \ldots & b_0^{n-1} f_1(a_0, b_0) & b_0^n \\
  f_n(a_1, b_1) & b_1 f_{n-1}(a_1, b_1) & \ldots & b_1^{n-1} f_1(a_1, b_1) & b_1^n \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  f_n(a_n, b_n) & b_n f_{n-1}(a_n, b_n) & \ldots & b_n^{n-1} f_1(a_n, b_n) & b_n^n \\
\end{vmatrix} = c_1 c_2 \ldots c_n \prod_{0 \leq i < j \leq n} (a_i b_j - a_j b_i),
$$

where $c_i$ is the coefficient of $a^i$ in the polynomial $f_i(a, b)$.

**Proof of Theorem 4.4.** Eq. (4.18) can be directly induced from the invariant property of Hermite polynomials [11, 28]

$$
H_n(x \cos \theta + y \sin \theta) = \sum_{l=0}^{n} \binom{n}{l} (\cos \theta)^l (\sin \theta)^{n-l} H_l(x) H_{n-l}(y).
$$

In fact, if $\|f\|_2^2 + \|g\|_2^2 = 1$ we can choose $\theta \in \mathbb{R}$ such that $\cos \theta = \|f\|_2$, $\sin \theta = \|g\|_2$, and let $x = \frac{X(f)}{\|f\|_2}$, $y = \frac{Y(g)}{\|g\|_2}$ in the above equation.

Now we turn to Eq. (4.19). Denote $a = \sin \theta$, $b = \cos \theta$ and let $f_i(a, b) = \binom{i}{0} a^i$. If we choose $\theta = \theta_k$, $k = 0, \ldots, n$, then Eq. (4.34) can be looked as a system of $n + 1$ linear equations in $n + 1$ unknowns $H_l(x) H_{n-l}(y)$, $l = 0, \ldots, n$ and the matrix of coefficients of the system is

$$
M = (b_k f_{n-l}(a_k, b_k))_{k,l},
$$

(4.35)
where \( k, l = 0, \ldots, n, a_k = \sin \theta_k, b_k = \cos \theta_k \). It follows from Lemma 4.1 that the determinant of the coefficient matrix equals to

\[
\prod_{k=0}^{n} \binom{n}{k} \prod_{0 \leq i < j \leq n} (a_i b_j - a_j b_i) = \prod_{k=0}^{n} \binom{n}{k} \prod_{0 \leq i < j \leq n} \sin(\theta_i - \theta_j) \neq 0.
\]

That is to say, the coefficient matrix \( M \) is invertible. Thus the unknowns \( H_i(x) H_{n-l}(y) \) can be expressed as

\[
H_i(x) H_{n-l}(y) = \sum_{k=0}^{n} M_{i,k}^{-1} H_n(x \cos \theta_k + y \sin \theta_k), \quad l = 0, \ldots, n.
\]

Set \( x = X(f), y = Y(g) \) in the above equation displayed, Eq. (3.19) is induced. Since \( \|f \cos \theta_k\|_0^2 + \|g \sin \theta_k\|_0^2 = 1 \), Eq. (3.20) can be deduced from the following Proposition 4.2 and the definition of \( H_k(X) H_l(Y) \). \( \square \)

We divide the proof of Theorem 3.2 into four propositions.

Proposition 4.2. Suppose that \( Y \) is an independent copy of the real isonormal Gaussian process \( X \) over the Hilbert space \( \mathfrak{H} \). Then a realization of the isonormal Gaussian process \( W \) over the Hilbert space direct sum \( \mathfrak{H} \oplus \mathfrak{H} \) is

\[
W(h, f) = X(h) + Y(f), \quad \forall h, f \in \mathfrak{H}.
\]

That is to say, \( \{X(h) + Y(f) : \forall h, f \in \mathfrak{H}\} \) is a centered Gaussian family such that \( \forall h_1, h_2, f_1, f_2 \in \mathfrak{H}, \)

\[
E[(X(h_1) + Y(f_1))(X(h_2) + Y(f_2))] = \langle h_1, f_1 \rangle_{\mathfrak{H}} + \langle h_2, f_2 \rangle_{\mathfrak{H}}.
\]

Proof. Note that \( Y \) is an independent copy of the isonormal Gaussian process \( X \), we have that \( \sum_j a_j X(h_j) + Y(f_j) = X(\sum_j a_j h_j) + Y(\sum_j a_j f_j) \) is 1-dimensional centered normal random variable, where \( a_j \in \mathbb{R} \). Thus \( \{X(h) + Y(f) : \forall h, f \in \mathfrak{H}\} \) is a centered Gaussian family. The independent property of \( X \) and \( Y \) implies that

\[
E[(X(h_1) + Y(f_1))(X(h_2) + Y(f_2))] = \langle h_1, h_2 \rangle_{\mathfrak{H}} + \langle f_1, f_2 \rangle_{\mathfrak{H}} = \langle h_1, f_1 \rangle_{\mathfrak{H}} + \langle h_2, f_2 \rangle_{\mathfrak{H}}.
\]

\( \square \)

Proposition 4.3. If \( \mathfrak{H}_c \ni h = u + i v \) such that \( \| h \|_{\mathfrak{H}_c} = \sqrt{2}, 0 < \theta_n < \cdots < \theta_0 < \pi \) and \( f_i + i g_i = \frac{1}{\sqrt{2}} e^{i \theta_i} h_i \), then

\[
J_{k,n-k}(Z(h)) = \sum_i c_i H_n(X(f_i) + Y(g_i)) \quad \text{ (4.36)}
\]

holds, where \( c_i \) is a constant depending on \( \theta_k \) given by (3.24).
Proof. (2.10) and (2.12) imply that

$$Z(h) = \frac{X_Z(h) + iY_Z(h)}{\sqrt{2}} = \frac{1}{\sqrt{2}}[X(u) - Y(v)] + \frac{i}{\sqrt{2}}[X(v) + Y(u)].$$

It follows from the fundamental relation Eq.(2.5) and Theorem 3.1 that we have

$$J_{k,n-k}(Z(h)) = \sum_j c_j H_j(\text{Re}(Z(h))) H_{n-j}(\text{Im}(Z(h)))$$

$$= \sum_j c_j \sum_i M_{j,i}^{-1} H_n(\cos \theta_i \text{Re}(Z(h)) + \sin \theta_i \text{Im}(Z(h))),$$

$$= \sum_i \tilde{c}_i H_n(\cos \theta_i \text{Re}(Z(h)) + \sin \theta_i \text{Im}(Z(h)))$$

(4.37)

where $c_j = i^{n-k} \sum_{r+s=j} \binom{k}{r} \binom{n-k}{s} (-1)^{n-k-s}, \tilde{c}_i = \sum_j c_j M_{j,i}^{-1}$ are constant. Since

$$\cos \theta_i \text{Re}(Z(h)) + \sin \theta_i \text{Im}(Z(h)) = \cos \theta \sqrt{2}[X(u) - Y(v)] + \sin \theta \sqrt{2}[X(v) + Y(u)]$$

$$= X(f_i) + Y(g_i),$$

(4.38)

and $\|f_i\|_D^2 + \|g_i\|_D^2 = \frac{1}{2} \|b\|_D^2 = 1$, we have that

$$H_n(\cos \theta_i \text{Re}(Z(h)) + \sin \theta_i \text{Im}(Z(h))) = H_n(X(f_i) + Y(g_i))$$

Inserting the above equation displayed into (4.37), we get Eq.(4.36).

Proposition 4.4. If $f, g \in D$ such that $\|f\|_D^2 + \|g\|_D^2 = 1$ and $h = \sqrt{2}e^{i\theta}(f - ig)$ with $\theta \in \mathbb{R}$, then

$$H_n(X(f) + Y(g)) = \sum_{k=0}^n d_k J_{k,n-k}(Z(h))$$

(4.39)

holds, where $d_k$ is a constant depending on $\theta$ given by (3.22).

Proof. It follows from Eq.(4.38), Eq.(4.34) and the fundamental relation Eq.(2.5) that

$$H_n(X(f) + Y(g)) = H_n(\cos \theta \text{Re}(Z(h)) + \sin \theta \text{Im}(Z(h)))$$

$$= \sum_{l=0}^n \binom{n}{l} (\cos \theta)^l (\sin \theta)^{n-l} H_l(\text{Re}(Z(h))) H_{n-l}(\text{Im}(Z(h)))$$

$$= \sum_{k=0}^n d_k J_{k,n-k}(Z(h)),$$

(4.40)

It follows from Proposition 4.2-4.4 and the definition of $\mathcal{H}^C_n(W)$ that one has the following corollary.
Corollary 4.5. Let $\mathcal{H}_n^C(W)$ be as in Equality (2.11). Then $\mathcal{H}_n^C(W)$ is also the closed linear subspace of $L^2_c(\Omega, \sigma(W), P)$ generated by the random variable of the type $\left\{ J_{k,l}(Z(h)) : k + l = n, h \in \mathcal{H}_C, \|h\|_{\mathcal{H}_C} = \sqrt{2} \right\}$.

Proposition 4.6. Let $\zeta_1, \zeta_2 \sim \mathcal{CN}(0, 2)$ be jointly symmetric complex Gaussian (i.e., $c_1\zeta_1 + c_2\zeta_2$ is a symmetric complex Gaussian variable for any $c_i \in \mathbb{C}$ $\forall i$) Then for all $m_1, n_1, m_2, n_2 \in \mathbb{N}$:

$$E[J_{m_1,n_1}(\zeta_1)J_{m_2,n_2}(\zeta_2)] = \begin{cases} m_1!n_1!(E[\zeta_1\zeta_2])^{m_1}(E[\zeta_1\zeta_2])^{n_1}, & \text{if } m_1 = m_2, n_1 = n_2 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By the Laplace transform of the jointly Gaussian distribution, we have that for all $s, t \in \mathbb{C}$,

$$E\left( \exp \left\{ s\zeta_1 + s\zeta_1 - 2|s|^2 \right\} \exp \left\{ t\zeta_2 + t\zeta_2 - 2|t|^2 \right\} \right)$$

$$= \exp \left\{ E(s\zeta_1 + s\zeta_1)(t\zeta_2 + t\zeta_2) \right\}$$

$$= \exp \left\{ stE[\zeta_1\zeta_2] + stE[\zeta_1\zeta_2] \right\}.$$  

Taking the partial derivative $\frac{\partial^{m_1+n_1}}{\partial^{m_1}\partial^{s_1}} \frac{\partial^{m_2+n_2}}{\partial^{m_2}\partial^{t_2}}$ at $s = t = 0$ in both sides of the above equality yields

$$E[J_{m_1,n_1}(\zeta_1)J_{m_2,n_2}(\zeta_2)] = E(J_{m_1,n_1}(\zeta_1)J_{m_2,n_2}(\zeta_2))$$

$$= \begin{cases} m_1!n_1!(E[\zeta_1\zeta_2])^{m_1}(E[\zeta_1\zeta_2])^{n_1}, & \text{if } (m_1, n_1) = (m_2, n_2) \\ 0, & \text{otherwise.} \end{cases}$$

Proof of Theorem 3.2. Eq. (3.21) and Eq. (3.23) are Proposition 4.4 and Proposition 4.3 respectively. Corollary 4.3 implies that $\mathcal{H}_n^C(W) = \sum_{k+l=n} \mathcal{H}_{k,l}(Z)$. Corollary 4.6 implies that $\mathcal{H}_{k,l}(Z)$ are orthogonal for distinct $(k, l)$. Thus (3.25) is valid.

As a by-product of Theorem 3.2, we can give a simple proof of the Wiener-Itô chaos decomposition for complex Gaussian isonormal processes.

An alternative proof of Theorem 2.11 (i) Corollary 4.3 implies that two types random variables

$$\left\{ J_{k,l}(Z(h)) : k + l = n, h \in \mathcal{H}_C, \|h\|_{\mathcal{H}_C} = \sqrt{2} \right\},$$

and

$$\left\{ H_n(X(f) + Y(g)) : f, g \in \mathcal{H}, \|f\|^2_{\mathcal{H}} + \|g\|^2_{\mathcal{H}} = 1 \right\}$$

generate the same linear subspace of $L^2_c(\Omega)$. Since the linear space generated by the class

$$\left\{ H_n(X(f) + Y(g)) : n \geq 0, f, g \in \mathcal{H}, \|f\|^2_{\mathcal{H}} + \|g\|^2_{\mathcal{H}} = 1 \right\}$$

2It is necessary that $E[\zeta_1\zeta_2] = 0$. 

is dense in $L^q_C(\Omega)$, the linear space generated by the class (4.40) is also dense in $L^q_C(\Omega)$.

(ii) Clearly, $\sigma(W) = \sigma(X,Y)$. It follows from the Wiener-Itô chaos decomposition of $W$ that

$$L^2_C(\Omega, \sigma(W), P) = \bigoplus_{n=0}^\infty \mathcal{H}^C_n(W) = \bigoplus_{n=0}^\infty \bigoplus_{k+l=n} \mathcal{H}_{k,l} = \bigoplus_{k=0}^\infty \bigoplus_{l=0}^\infty \mathcal{H}_{k,l}.$$  

□

Lemma 4.7. Suppose $\varphi(t_1, \ldots, t_m, s_1, \ldots, s_n) \in \mathcal{H}^\otimes_m \otimes \mathcal{H}^\otimes_n$. Let

$$\psi(t_1, \ldots, t_n, s_1, \ldots, s_m, t_1, \ldots, t_n),$$  

(4.42)

then

$$\mathcal{I}_{m,n}(\varphi) = \mathcal{I}_{n,m}(\psi).$$  

(4.43)

Proof. By the linear property of $\mathcal{I}_{m,n}$, we need only to show that (4.43) is valid for $\varphi = f \otimes g$ such that

$$f = \text{symm}(\otimes_{k=1}^\infty \varepsilon_k^{m_k}), \quad g = \text{symm}(\otimes_{k=1}^\infty \bar{\varepsilon}_k^{n_k}),$$

where $m$, $n$ and $\varepsilon_k$ are as in Definition 2.8. Clearly, $\psi = \bar{g} \otimes \bar{f}$. It follows from (2.6) and (2.13) that

$$\mathcal{I}_{m,n}(f \otimes g) = 2^{-\frac{m+n}{2}} \prod_k J_{m_k,n_k}(\sqrt{2}Z(\varepsilon_k)) = \mathcal{I}_{n,m}(\bar{g} \otimes \bar{f}).$$

Proof of Theorem 3.3. Set $p = m + n$. It follows from Proposition 2.3 (or see [16, Theorem 2.7.7], [20, Proposition 1.1.7]) that the multiple integral $\mathcal{I}_p$ provides an isometry from $\mathcal{H} \oplus \mathcal{H}$ onto $\mathcal{H}^p(W)$, i.e.,

$$\mathcal{I}_p((\mathcal{H} \oplus \mathcal{H})^\otimes_p) := \{ \mathcal{I}_p(u) : u \in (\mathcal{H} \oplus \mathcal{H})^\otimes_p \} = \mathcal{H}^p(W).$$

Proposition 2.4 implies that the complex multiple integral $\mathcal{I}_{m,n}$ provides an isometry from $\mathcal{H}^\otimes_m \otimes \mathcal{H}^\otimes_n$ onto $\mathcal{H}_{m,n}(Z)$, i.e.,

$$\mathcal{I}_{m,n}(\mathcal{H}^\otimes_m \otimes \mathcal{H}^\otimes_n) := \{ \mathcal{I}_{m,n}(\varphi) : \varphi \in \mathcal{H}^\otimes_m \otimes \mathcal{H}^\otimes_n \} = \mathcal{H}_{m,n}(Z).$$

Therefore, (3.26) is deduced from Theorem 3.2.

Let $\psi$ be as in (4.42). When $m \neq n$, it follows from Theorem 2.11 and Lemma 4.7 that

$$0 = \langle \mathcal{I}_{m,n}(\varphi), \mathcal{I}_{n,m}(\psi) \rangle_{L^2_\Omega}$$

$$= \langle F, \bar{F} \rangle_{L^2_\Omega} = E[(U + iV)^2]$$

$$= E[U^2] - E[V^2] + 2iE[UV].$$

Thus (3.27) is valid. □
4.2 Proof of Theorem 1.1-1.2

Lemma 4.8. Suppose that \( u, v \in (\mathfrak{F} \oplus \mathfrak{F})^{\otimes q} \) and \( U = \mathcal{I}_q(u), V = \mathcal{I}_q(v) \). Then we have

\[
E[U^2V^2] = 2(E[UV])^2 + E[U^2]E[V^2] + \sum_{r=1}^{q-1} \binom{q}{r} ((q!)^2 \|u \otimes_r v\|^2 + (r!)^2 \binom{q}{r} (2q - 2r)! \|u \tilde{\otimes}_r v\|^2),
\]

where \( u \otimes_r v \) is the \( r \)-th contraction of \( u \) and \( v \), and \( u \tilde{\otimes}_r v \) is the symmetrization of \( u \otimes_r v \).

The lemma is a minor extension of the case \( u = v \) in [12, Lemma 4.1], the reader can also refer to (3.5-6) of [19] for a similar case \( u \in (\mathfrak{F} \oplus \mathfrak{F})^{\otimes q}, v \in (\mathfrak{F} \oplus \mathfrak{F})^{\otimes p} \) where \( p, q \) can be different. But in the present paper we just need \( p = q \) and for a convenience we show it shortly.

Proof. The product formula of real multiple Wiener-Itô integral [12, 16] implies that

\[
UV = \mathcal{I}_q(u) \mathcal{I}_q(v) = \sum_{r=0}^{q} r! \binom{q}{r}^2 \mathcal{I}_2\gamma - 2r(u \tilde{\otimes}_r v).
\]

Using the orthogonality and isometry properties of the integrals \( \mathcal{I}_q \), we have that

\[
E[U^2V^2] = \sum_{r=0}^{q} (r!)^2 \binom{q}{r}^4 (2q - 2r)! \|u \tilde{\otimes}_r v\|_{\mathfrak{F}^{\otimes (2q - 2r)}}^2.
\]  

(4.44)

Clearly, when \( r = q \) in (4.44),

\[
(q!)^2 \|u \tilde{\otimes}_q v\|^2 = [q!(u, v)_{\mathfrak{F}^{\otimes q}}]^2 = (E[UV])^2.
\]  

(4.45)

Along the same line to get (4.26) of [12], using some combinatorics, it is readily checked that when \( r = 0 \) in (4.44),

\[
(2q)! \|u \tilde{\otimes} v\|_{\mathfrak{F}^{\otimes 2q}}^2 = (q!)^2 [(u \otimes v)^2 + \|u \otimes v\|^2 + \sum_{r=1}^{q-1} \binom{q}{r}^2 \|u \otimes_r v\|^2_{\mathfrak{F}^{\otimes (2q - 2r)}}]
\]

\[
= (E[UV])^2 + E[U^2]E[V^2] + (q!)^2 \sum_{r=1}^{q-1} \binom{q}{r}^2 \|u \otimes_r v\|^2_{\mathfrak{F}^{\otimes (2q - 2r)}}.
\]  

(4.46)

Substituting (4.45) and (4.46) into (4.44) yields the desired result. \( \square \)

Proof of Theorem 1.1 Let the notation in Theorem 3.3 prevail. It follows directly from Theorem 3.3 that \( F_k = U_k + V_k = \mathcal{I}_{m+n}(u_k) + i\mathcal{I}_{m+n}(v_k) \). Since \( U_k \)'s and \( V_k \)'s admit moments of all order, \( F_k \)'s also admit moments of all order. The implication
Remark 10. 1) The crucial component of the proof is the inequalities \((4.47)\) and \((4.48)\).

2) That the sequence \((U_k, V_k)\) converges in distribution to a jointly normal law is also equivalent to \(u_k \otimes u_k \to 0, v_k \otimes v_k \to 0\) for \(r = 1, \ldots, m + n - 1\), i.e., \(U_k, V_k\) converge in distribution to \(N(0, 1 + \frac{a^2 + b^2}{2} \sigma^2)\) and \(N(0, 1 + \frac{a^2 + b^2}{2} \sigma^2)\) respectively. Moreover, from the proof we have that \(\text{Cov}(U_k, V_k) = E[U_k^2 V_k^2] - E[U_k^2]E[V_k^2] \to 0\).
3) There exists an alternative proof. In fact, Theorem 1.1 can be implied directly from Theorem 4.2 of [19] and Theorem 3.3.

Proof of Theorem 1.2. The proof is similar to that of Theorem 1.1. If \( m \neq n \) then the condition \( E[F_k^n + 3|F_k|^2F_k] \rightarrow 8(1-i)\sigma^2 \) implies that \( E[F_k^n + 3|F_k|^2F_k] \rightarrow 8(1+i)\sigma^2 \). Then we have

\[
E[U_k^3] = \frac{1}{8}E(F_k + \bar{F}_k)^3 \\
= \frac{1}{8}\{E[F_k^3] + 3|F_k|^2\bar{F}_k] + E[\bar{F}_k^3] + 3|F_k|^2F_k]\}
\]

and similarly \( E[V_k^3] = \frac{1}{8}\{E[F_k^3] + 3|F_k|^2\bar{F}_k] - E[\bar{F}_k^3] + 3|F_k|^2F_k\} \rightarrow 2\sigma^2 \). Then the condition \( E[|F_k|^4] \rightarrow 2\sigma^4 + 24\sigma^2 \) yields that

\[
E[U_k^4] - 12E[U_k^2V_k^2] - 12E[V_k^4] + 2E[U_k^2V_k^2] \rightarrow 2\sigma^4 - 24\sigma^2.
\]

Since \( E[U_k^2] = E[V_k^2] \rightarrow \frac{1}{2}\sigma^2 \), we have that

\[
E[U_k^4] - 12E[U_k^2] + 24E[U_k^2] - 3(E[U_k^2])^2 \\
+ E[V_k^4] - 12E[V_k^2] + 24E[V_k^2] - 3(E[V_k^2])^2 \\
+ 2(E[U_k^2V_k^2] - E[U_k^2]E[V_k^2]) \\
\rightarrow 0.
\]

It follows from (3.5-3.7) in [15] that

\[
E[U_k^4] - 12E[U_k^2] + 24E[U_k^2] - 3(E[U_k^2])^2 \geq 0, \\
E[V_k^4] - 12E[V_k^2] + 24E[V_k^2] - 3(E[V_k^2])^2 \geq 0.
\]

Together with (4.48), the above inequalities yield

\[
E[U_k^4] - 12E[U_k^2] + 24E[U_k^2] - 3(E[U_k^2])^2 \rightarrow 0, \\
\text{Cov}(U_k^2, V_k^2) = E[U_k^2V_k^2] - E[U_k^2]E[V_k^2] \rightarrow 0. \\
\text{(4.49)}
\]

Then we have \( E[U_k^4] - 12E[U_k^2] \rightarrow \frac{3}{4}\sigma^4 - 12\sigma^2 \) and similarly \( E[V_k^4] - 12E[V_k^2] \rightarrow \frac{3}{4}\sigma^4 - 12\sigma^2 \). Together with (4.49), it follows from Theorem 4.5 of [19] that \( (U_k, V_k) \) converges in distribution to independent random variable having identical centered \( \chi^2 \) distribution with \( \frac{\sigma^2}{4} \) degree of freedom.

If \( m = n \) and \( a^2 + b^2 < 1 \), we can finish the proof by simply mimicking the arguments of the case \( m \neq n \).

\[\square\]

5 Some generalized results

We generalize Theorem 1.1 in this section. From the proofs of Theorem 1.1-1.2, we find that the condition of the degree of the Wiener-Itô chaos being fixed is not necessary. Thus we have the following version of the fourth moment theorem.
Theorem 5.1. For a fixed \( l \geq 2 \), consider a sequence of random variable \( F_k \in \bigoplus_{m+n=l} \mathcal{H}_{m,n}(Z) \), and suppose that \( E[|F_k|^2] \to \sigma^2 \) and \( E[F_k^2] \to \sigma^2(a + ib) \) where \( a, b \in \mathbb{R} \) such that \( a^2 + b^2 \leq 1 \) as \( k \to \infty \), then the following three assertions are equivalent:

(i) The sequence \((\text{Re}F_k, \text{Im}F_k)\) converges in distribution to a centered jointly normal law with the covariance \( \frac{1}{2}C \), where the matrix \( C \) is as in Theorem 1.1.

(ii) \( E[|F_k|^4] \to (a^2 + b^2 + 2)\sigma^4 \).

(iii) \( u_k \otimes_r u_k \to 0 \), \( v_k \otimes_r v_k \to 0 \) for \( r = 1, \ldots, l - 1 \).

where \( F_k = U_k + iV_k = \mathcal{I}_l(u_k) + i\mathcal{I}_l(v_k) \) (see Theorem 3.2) and \( u \otimes_r v \) is the \( r \)-th contraction of \( u \) and \( v \).

Theorem 5.2. Let \( \xi(\alpha_1, \alpha_2) \) be as in Theorem 1.2. For a fixed even number \( l \geq 2 \), consider a sequence of random variable \( F_k \in \bigoplus_{m+n=l} \mathcal{H}_{m,n}(Z) \). Suppose that \( E[|F_k|^2] \to \sigma^2 \) and \( E[F_k^2] \to \sigma^2(a + ib) \) such that \( a^2 + b^2 < 1 \) as \( k \to \infty \). Then as \( k \to \infty \), the following two assertions are equivalent:

(i) The sequence \((F_k)\) converges in distribution to \( \xi(\frac{1+a}{2}\sigma^2, \frac{1-a}{2}\sigma^2) \);

(ii) \( E[F_k^3 + 3|F_k|^2F_k] \to 8[1 + a - i(1 - a)]\sigma^2 \) and \( E[|F_k|^4] \to (2 + a^2)\sigma^4 + 24\sigma^2 \).

Moreover, we can also show the following multivariate version of Theorem 1.1.

Theorem 5.3. Let \( d \geq 2 \), and let \( l_1, \ldots, l_d \) be positive integers such that \( l_i \neq l_j \) for any \( i \neq j \). Consider vectors \( F_k = (F_{1,k}, \ldots, F_{d,k}) \) with \( F_{i,k} \in \bigoplus_{m+n=l_i} \mathcal{H}_{m,n}(Z) \).

Assume that for \( i = 1, \ldots, d \), as \( k \to \infty \), \( E[|F_{i,k}|^2] \to \sigma_i^2 \) and \( E[F_{i,k}^2] \to \sigma_i^2(a_i + ib_i) \) such that \( a_i^2 + b_i^2 \leq 1 \). As \( k \to \infty \), the following two assertions are equivalent:

(i) The sequence \((F_k)\) converges in distribution to \( \zeta = (\zeta_1, \ldots, \zeta_d) \) such that all \( \zeta_i \) are independent and \((\text{Re}\zeta_i, \text{Im}\zeta_i)\) being centered jointly normal with covariance matrix \( C_i = \frac{\sigma_i^2}{2} \begin{bmatrix} 1 + a_i & b_i \\ b_i & 1 - a_i \end{bmatrix} \);

(ii) The sequence \((F_{i,k})\) converges in distribution to \( \zeta_i \) such that \((\text{Re}\zeta_i, \text{Im}\zeta_i)\) being centered jointly normal with covariance matrix \( C_i \) for \( i = 1, \ldots, d \).

Theorem 5.4. Let \( d \geq 2 \), and let \( l_1, \ldots, l_d \) be positive integers such that \( l_i \neq l_j \) for any \( i \neq j \). Consider vectors \( F_k = (F_{1,k}, \ldots, F_{d,k}) \) with \( F_{i,k} \in \bigoplus_{m+n=l_i} \mathcal{H}_{m,n}(Z) \).

Assume that for \( i = 1, \ldots, d \), as \( k \to \infty \),

(i) \( E[|F_{i,k}|^2] \to \sigma_i^2 \) and \( E[F_{i,k}^2] \to \sigma_i^2(a_i + ib_i) \) such that \( a_i^2 + b_i^2 < 1 \),

(ii) \( E[F_{j,k}^2 F_{i,k}] \to 0 \) and \( E[|F_{j,k}|^2 F_{i,k}] \to 0 \) whenever \( l_i = 2l_j \),
(iii) \( E[F_{i,k}^3 + 3|F_i|k^2 F_{i,k}] \to 8[1 + a_i - i(1 - a_i)]\sigma_i^2 \) and \( E[|F_{i,k}|^4] \to (2 + a_i^2)\sigma_i^4 + 24\sigma_i^2 \).

Then the sequence
\[
(F_{1,k}, \ldots, F_{d,k}) \xrightarrow{law} (\xi_1(1 + \frac{1}{2}\sigma_1^2, \frac{1}{2} - \frac{1}{2}\sigma_1^2), \ldots, \xi_d(1 + \frac{1}{2}\sigma_d^2, \frac{1}{2} - \frac{1}{2}\sigma_d^2))
\]
where all \( \xi_i \) are independent and each \( \xi_i \) is a complex centered \( \chi^2 \) distribution with \((\frac{1 + a_i}{2}\sigma_i^2, \frac{1 - a_i}{2}\sigma_i^2)\) degrees of freedom.

6 Appendix: multiple integrals by means of the divergence operator

To be self-contained and for the reader’s convenience, we present an alternative way to define the complex multiple Wiener-Itô integrals by means of the divergence operator along the routine of the real multiple Wiener-Itô integrals [16]. All the proofs are omitted.

Let \( S \) denote the set of all random variables of the form
\[
F = f(Z(\varphi_1), \ldots, Z(\varphi_m)),
\]
where \( m \in \mathbb{N}, \varphi_1, \ldots, \varphi_m \in \mathcal{F}_C, f \in C^\infty(\mathbb{C}^m) \). Here we assume that \( f \) with its partial derivatives has polynomial growth. A random variable belonging to \( S \) is said to be smooth. Clearly, the space \( S \) is dense in \( L^p(\Omega) \) for every \( q \in [1, \infty) \). For \( p \in \mathbb{N} \) and \( j_1, \ldots, j_p = 1, \ldots, m \), denote
\[
\partial_{j_1} \cdots \partial_{j_p} f = \frac{\partial^p f(z_1, \ldots, z_m)}{\partial z_{j_1} \cdots \partial z_{j_p}}, \quad \bar{\partial}_{j_1} \cdots \bar{\partial}_{j_p} f = \frac{\partial^p f(z_1, \ldots, z_m)}{\partial \bar{z}_{j_1} \cdots \partial \bar{z}_{j_p}}.
\]

**Definition 6.1.** Let \( F \in S \) be given by (6.50). The \( p \)-th Malliavin derivative of \( F \) (with respect to \( Z \)) is the element of \( L^2(\Omega, \mathcal{F}_C^{\otimes p}) \) defined by
\[
\begin{align*}
D^p F &= \sum_{j_1, \ldots, j_p = 1}^m \partial_{j_1} \cdots \partial_{j_p} f(Z(\varphi_1), \ldots, Z(\varphi_m)) \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_p}, \\
\bar{D}^p F &= \sum_{j_1, \ldots, j_p = 1}^m \bar{\partial}_{j_1} \cdots \bar{\partial}_{j_p} f(Z(\varphi_1), \ldots, Z(\varphi_m)) \bar{\varphi}_{j_1} \otimes \cdots \otimes \bar{\varphi}_{j_p}.
\end{align*}
\]

It is routine to show that the two operators \( D^p, \bar{D}^p \) are closable and can be consistently extended to the set \( \mathbb{D}^{p,q} \) and \( \bar{\mathbb{D}}^{p,q} \) which are the closure of \( S \) with respect to the Sobolev norm [16]. The adjoint operators of \( D^p, \bar{D}^p \) are written \( \delta^p, \bar{\delta}^p \) and called the multiple divergence operators of order \( p \).

**Remark 11.** By following the same route of [16], we could define the Malliavin derivatives in Hilbert space and the Hilbert space valued divergences.

**Definition 6.2.** Let \( m, n \geq 0 \) and \( f \in \mathcal{F}_C^{\otimes m} \otimes \mathcal{F}_C^{\otimes n} \). The \((m, n)\)-th multiple integral of \( f \) (with respect to \( Z \)) is defined by \( \mathbb{I}_{m,n}(f) = \delta^m \delta^n(f) \).
Proposition 6.3. Let \( m, n \geq 0 \) and \( f \in \mathcal{H}^{\otimes m}_C \otimes \mathcal{H}^{\otimes n}_C \). For all \( q \in [1, \infty) \), \( \mathcal{I}_{m,n}(f) \in D^{\infty,q} \bigcap \bar{D}^{\infty,q} \). Moreover, for all \( a, b \geq 0 \),

\[
D^a \bar{D}^b \mathcal{I}_{m,n}(f) = \begin{cases} 
\frac{m! n!}{(m-a)!(n-b)!} \mathcal{I}_{m-a,n-b}(f), & \text{if } a \leq m, b \leq n \\
0, & \text{otherwise.}
\end{cases}
\]

Proposition 6.4. (Isometry property of integrals) Fix integrals \( m_i, n_i, i = 1, 2 \), as well as \( f \in \mathcal{H}^{\otimes m_1}_C \otimes \mathcal{H}^{\otimes n_1}_C \), \( g \in \mathcal{H}^{\otimes m_2}_C \otimes \mathcal{H}^{\otimes n_2}_C \). We have

\[
E[\mathcal{I}_{m_1,n_1}(f)\overline{\mathcal{I}_{m_2,n_2}(g)}] = \begin{cases} 
m_1! n_1! \langle f, g \rangle_{\mathcal{H}^{\otimes (m_1+n_1)}}, & \text{if } m_1 = m_2, n_1 = n_2 \\
0, & \text{otherwise.}
\end{cases}
\]

Theorem 6.5. Let \( h \in \mathcal{H}_C \) be such that \( \|h\|_{\mathcal{H}_C} = \sqrt{2} \). Then, for any \( m, n \geq 0 \), we have

\[
\mathcal{I}_{m,n}(h^{\otimes m} \otimes \bar{h}^{\otimes n}) = J_{m,n}(Z(h)).
\]

Moreover, let \( m, n, e_k \) and \( J_{m,n} \) be as in Definition 2.8, we have that

\[
\mathcal{I}_{m,n} (\text{symm}( \otimes_{k=1}^{m} e_k^{\otimes n}) \otimes \text{symm}( \otimes_{k=1}^{n} \bar{e}_k^{\otimes m})) = \sqrt{m! n!} J_{m,n} \]

As a consequence, the linear operator \( \mathcal{I}_{m,n} \) provides an isometry from \( \mathcal{H}^{\otimes m}_C \otimes \mathcal{H}^{\otimes n}_C \) onto the \((m,n)\)-th chaos \( \mathcal{H}_{m,n}(Z) \).

The above theorem and Proposition 2.9 imply that the linear operator \( \mathcal{I}_{m,n} \) is exactly \( \mathcal{I}_{m,n} \) given in Definition 2.10.

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