STRUCTURE AND PROPERTIES OF NEUTRON STARS 
IN THE RELATIVISTIC MEAN-FIELD THEORY 

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Properties of rotating neutron stars with the use of relativistic mean-field theory are considered. The performed analysis of neutron star matter is based on the nonlinear Lagrangian density. The presence of nonlinear interaction of vector mesons modifies the density dependence of the $\rho$ field and influences bulk parameters of neutron stars. The observed quasi-periodic X-ray oscillations of low mass X-ray binaries can be used in order to constrain the equation of state of neutron star matter. Having assumed that the maximum frequency of the quasi periodic oscillations originates at the circular orbit it is possible to estimate masses and radii of neutron stars.

1. Introduction 

Computation of the equation of state is one of the key problems in the construction of a reliable model of a neutron star. Therefore the aim of this paper is to study such neutron star parameters, namely the masses and radii, which are the most sensitive ones to the form of the equation of state. The influence of rotation on these parameters is also estimated. The properties of neutron star matter in high-density and neutron-rich regime are considered with the use of the relativistic mean-field approximation. This approach implies the interaction of nucleons through the exchange of meson fields, so the model considered here comprises: nucleons, electrons and scalar, vector-isoscalar and vector-isovector mesons. Consequently the contributions coming from these components characterize the pressure and energy density. In this paper the nonlinear vector-isoscalar self-interaction is dealt. Modification of this type was proposed by Bodmer in order to achieve good agreement with the Dirac-Brückner calculations at high densities. The first version of the $\rho$ meson field introduction is of a minimal type without any nonlinearity and consists only of the coupling of this field to nucleons. This case is enlarged by the nonlinear vector-isoscalar and vector-isovector interaction which modifies the density dependence of the $\rho$ mean field and the energy symmetry. Such an extension of the neutron star model was inspired by the paper in which the authors indicate that there exists

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a relationship between the neutron-rich skin of a heavy nucleus and the properties of a neutron star crust. For a more realistic neutron star description the composite equation of state has been employed. The presented forms of the EOS for neutron rich matter have been combined with those of Negele and Vautherin \(^3\) the Machleidt-Holinde-Elster Bonn potential \(^4\) and Haensel-Pichon \(^5\). It has been found that neutron stars of minimal configurations strongly depend on the proton fraction which in turn is connected with the presence of the nonlinear \(\omega - \rho\) interaction. The observed X-ray radiation can be used to investigate phenomena occurring in the strong gravitational fields near neutron stars. The Rossi X-Ray Timing Explorer discovered high-frequency X-ray brightness oscillations from several low-mass X-ray binaries. Observations of kilohertz QPO sources can show the existence of circular orbits around neutron stars. Using the relation between the orbit radius \(r\) and the mass of the neutron star for given value of the observed frequency \(\nu_{\text{QPO}}\) one can see that the orbit radius is determined by the mass of a neutron star which in turn depends on the particular choice of the equation of state.

The outline of this paper is as follows. In Sect.1 the employed equations of state are introduced with and without the additional \(\omega - \rho\) meson interaction. Then the impact of the introduction of this nonlinear interaction on the neutron star parameters is studied. In Sect.2 general properties of nonrotating and rotating neutron stars are calculated. The results are employed to construct the solutions of the equations of motion obtained for a particle confined to the equatorial plane. The obtained results for the chosen form of the EOS followed by the discussion on their implications are presented in Sect3.

2. The equation of state

It is the equation of state (EOS) of the matter which determines the characteristic of a neutron star. Thus the reliable form of the equation of state is the basic input to the structure equations. However, the knowledge of the EOS at densities around and beyond nuclear density \(\rho_0 \simeq 2.5 \times 10^{14} g/cm^3\) is incomplete. In general the properties of nuclear matter at extreme conditions remain one of the most important problem. There have been several attempts to determine the proper form of the equation of state for dense nuclear matter. The first one is the nonrelativistic Breückner-Bethe theory based on the use of the free nucleon-nucleon interaction with a variational method for the many body correlations. An alternative approach is the relativistic field -theoretical method. The effective relativistic field theory of interacting hadrons makes an optional approach to nuclear many body problem and makes the description of the bulk properties of finite nuclei and binding energy of nuclear matter reliable and reproduces the experimental data \(^6\) \(^7\). These theories are effective ones, as coupling constants are determined by the bulk properties of nuclear matter such as saturation density, binding energy, compressibility and the symmetry energy see also \(^8\). The bulk properties put certain limits on these theories and enable extrapolation to higher densities. The goal is to extend the
RMF theory to the system with density reaching the level of several values of the nuclear density. However, the complete and more realistic description of a neutron star requires taking into consideration not only the interior region of a neutron star but also the remaining layers, namely the inner and outer crust and the surface. The adequate density regions are presented below:

- \(2 \times 10^3 < \rho < 1 \times 10^{11}\) - light metals, electron gas
- \(1 \times 10^{11} < \rho < 2 \times 10^{13}\) - heavy metals, relativistic electron gas
- \(2 \times 10^{13} < \rho < 5 \times 10^{15}\) - nucleons, relativistic leptons

In this paper the composite equation of state for the entire neutron star density span was constructed by joining together the equation of state of the neutron rich matter region given by equations (26)(23), the Negele-Vautherin EOS and Bonn for the relevant density range between \(10^{14}\) and \(5 \times 10^{10}\) g/cm\(^3\) and the Haensel-Pichon EOS for the density region \(9.6 \times 10^{10}\) g/cm\(^3\) to \(3.3 \times 10^{7}\) g/cm\(^3\). Since the density drops steeply near the surface of a neutron star, these layers do not contribute to the total mass of a neutron star in a significant manner. The inner neutron rich region up to density \(\rho \sim 10^{13}\) g/cm\(^3\) influences decisively the neutron star structure and evolution. Different forms of the equations of state are presented in Fig.1. There are original NL1 and TM1 parameter sets describing the neutron rich star interior and the composite EOS constructed by adding Bonn and Negele-Vautherin equations of state to the TM1 one. Of particular interest is the influence of the proton fraction \(Y_p = n_p/n_B\) \((n_p\) and \(n_B\) are the proton and baryon number densities respectively) on the behaviour of the EOS. There are two cases presented in this figure with distinctively different proton fractions, namely the \(Y_p = 0.07\) and \(Y_p = 0.17\). In Fig.2 the form of the equation of state for the original NL3 and NL3 parameter sets and for the NL3 with the additional \(\Lambda_V\) coupling equals 0.025. The simplest but unrealistic neutron star description assumes the presence of neutrons only but it is not possible for neutron star matter to be purely neutron one. The model presented in our work comprises baryons interacting through the exchange of scalar and vector mesons, which gives the Lagrange density function \(\mathcal{L}_0\) consisting of the parts describing the free baryon and meson fields together with the interaction terms. In this model we are dealing with the electrically neutral neutron star matter being in \(\beta\)-equilibrium. Therefore the imposed constrains, namely the charge neutrality and \(\beta\)-equilibrium, imply the presence of leptons. Mathematically it is expressed by adding the Lagrangian of free relativistic leptons \(\mathcal{L}_L = \overline{\psi}_e (i \gamma^\mu \partial_\mu - m_e) \psi_e\) to the Lagrangian function \(\mathcal{L}_{RMF}\). Neutrinos are neglected here since they leak out from the neutron star, whose energy diminishes at the same time. Such a matter possesses a highly asymmetric character caused by the presence of small amounts of protons and electrons.

\[
\mathcal{L}_{RMF} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{4} R^\mu_{\alpha \nu \rho} R^{\alpha \nu \rho \mu} - \frac{1}{4} \Omega^\mu_{\alpha \nu} \Omega^{\alpha \nu \mu} + \frac{1}{2} M_\omega^2 \omega_\mu \omega^\mu + \frac{1}{2} M_\rho^2 \rho_\mu \rho^\mu + g_\rho^2 \rho_\mu \rho^\mu \Lambda_V g_\omega^2 \omega_\mu \omega^\mu - U(\varphi) \tag{1}
\]
The field tensors \( R^a_{\mu\nu} \) and \( \Omega_{\mu\nu} \) and the covariant derivative \( D_\mu \) are given by

\[
R^a_{\mu\nu} = \partial_\mu \rho^a_\nu - \partial_\nu \rho^a_\mu + g_2 \varepsilon^{abc}_\mu \rho^b_\nu
\]

\[
\Omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu
\]

\[
D_\mu = \partial_\mu + i g_\omega \omega_\mu + \frac{1}{2} i g_\rho \rho^a_\mu \sigma^a
\]

The field equations derived from the Euler-Lagrange equations for meson fields \( \varphi, \omega_\mu \) and \( \rho^a_\mu \) are the Klein-Gordon equations with source terms coming from the baryon fields. They are coupled differential equations

\[
\partial_\mu \partial^\mu \varphi + m_s^2 \varphi + g_2 \varphi^2 + g_3 \varphi^3 = g_s \bar{\psi} \psi
\]

\[
\partial_\mu \Omega^{\mu\nu} + M_\omega^2 \omega_\nu + \frac{\xi}{6} g_\omega^4 (\omega_\mu \omega^\mu) \omega_\nu + 2 g_\rho^2 g_\omega^2 \Lambda_\omega^\nu \omega^\rho_\mu \rho^\mu = g_\omega \bar{\psi} \gamma^\nu \psi
\]
Table 1. Parameter sets employed in this paper.

| Parameter | NL1 | NL3 | NL3s | TM1 |
|-----------|-----|-----|------|-----|
| M         | 938 MeV | 938 MeV | 938 MeV | 938 MeV |
| $M_\infty$ | 795.4 MeV | 782.5 MeV | 795.359 MeV | 783.5 MeV |
| $M_p$     | 763 MeV | 763 MeV | 763 MeV | 763 MeV |
| $m_s$     | 492 MeV | 492 MeV | 492 MeV | 492 MeV |
| $g_\omega$ | 12.17 fm$^{-1}$ | 2.03 fm$^{-1}$ | 12.17 fm$^{-1}$ | 7.23 fm$^{-1}$ |
| $g_3$     | 0 | 1.666 | −36.259 | 0.618 |
| $g_\omega$ | 10.077 | 9.696 | 10.138 | 10.029 |
| $g_\rho$  | 13.866 | 12.889 | 13.285 | 12.614 |
| $g_\omega \xi/6$ | 0 | 0 | 0 | 71.308 |
| $\Lambda_V$ | 0 | 0 | 0.025 | 0 |

$$D_\mu R^{\mu\nu} + M_p^2 \rho^{\nu} + 2g_\omega^2 g_\omega^2 \Lambda_V \omega_\mu \omega_\nu \rho_\mu \rho_\nu = g_\rho \bar{\psi} \gamma^\nu \sigma^a \psi.$$ \hspace{1cm} (10)

Sources that appear in the equations of motion are the baryon current

$$J_B^\nu = (\rho_B, J_B^3) = \bar{\psi} \gamma^\nu \psi$$

and the isospin current which exists only in the asymmetric matter

$$J^\nu = (\rho, J^3) = \frac{1}{2} \bar{\psi} \gamma^\nu \sigma^a \psi.$$ \hspace{1cm}

In this model we are dealing with static, homogenous, infinite matter, which sets certain simplifications on the Euler-Lagrange equations, namely the derivative terms in the equations for meson fields vanish due to translational invariance of infinite matter, the rotational symmetry causes the disappearance of spatial components of vector meson fields. In the mean field approach the meson fields are treated as classical ones after replacing them by their mean value ($\bar{\varphi}_0 = < \varphi >, \omega_0 = < \omega >, \rho_{03} = < \rho_{03} >$), baryon currents appearing in the Euler-Lagrange equations are replaced by their ground state expectation value which is the quantum selfconsistent fermion system. In the case of the $\rho$ meson only the neutral state is kept. Having inserted the stated above simplifications the field equations are reduced to

$$m_s^2 \varphi_0 + g_2 \varphi_0 + g_3 \varphi_0^3 = g_s \rho_s$$ \hspace{1cm} (11)

$$M_\omega^2 \omega_0 + \xi g_\omega^4 \omega_0^3 + (g_\omega g_\rho)^2 \Lambda_V \omega_\mu \omega_\nu \rho_{03}^2 = g_\omega J_B^0$$ \hspace{1cm} (12)

$$M_\rho^2 \rho_{03} + (g_\omega g_\rho)^2 \Lambda_V \omega_\mu \omega_\nu \rho_{03} = \frac{1}{2} g_\rho J_3^0$$ \hspace{1cm} (13)

The densities occurring in these equations are the scalar density $\rho_s$ and the conserved baryon density $\rho_B$ and the isospin density $\rho_3$. They are defined as

$$\rho_s = < \bar{\psi} \gamma^0 \psi > = \frac{4}{(2\pi)^4} \int_0^{k_F} \frac{d^3kM^*}{\sqrt{k^2 + M^*}}$$ \hspace{1cm} (14)

$$\rho_B = < \bar{\psi} \gamma^0 \psi > = \rho_p + \rho_n$$ \hspace{1cm} (15)
\[ \rho_3 = \langle \bar{\psi} \gamma^0 \sigma^3 \psi \rangle = \frac{1}{2} (\rho_p - \rho_n) \]  

where \( \rho_p \) and \( \rho_n \) are the proton and neutron densities. The expression describing the scalar density \( \rho_s \) relates the value of the Fermi momentum \( k_F \) and the baryon density \( \rho_B \).

The Dirac equations for baryons that are obtained from the Lagrangian function have the following form

\[ i\gamma^\mu D_{\mu} \psi - (M - g_s \varphi_0) \psi = 0 \]  

with \( M^* = M - g_s \varphi_0 \) being the effective nucleon mass which is generated by the nucleon and scalar field interaction. The isovector meson field \( \rho \) is considered two-fold in this paper. The first is that of a meson field introduced in a minimal version and we are dealing with the simplest possible coupling of the \( \rho \) meson to nucleons. There is no mutual interaction between the \( \rho \) and \( \omega \) mesons, which meets the requirements \( \Lambda_V = 0 \) and the equation (13) gives simplified density dependence for the field \( \rho_{03} \)

\[ \rho_{03} = \frac{g_\rho}{2M^2_\rho} (n_p - n_n). \]  

Taking into account the additional \( \omega - \rho \) interaction (the coupling constant \( \Lambda_V \neq 0 \)) the self-consistency equations for vector mesons were obtained. The mutual interactions of vector meson fields result in the effective meson masses which can be obtained with the help of the following substitution in the Euler-Lagrange equations

\[ M_{\text{eff},\omega}^2 = M_\omega^2 + 2g_\omega^2 g_\rho^2 \Lambda_V \rho_{03}^2 \]  

and

\[ M_{\text{eff},\rho}^2 = M_\rho^2 + 2g_\rho^2 g_\omega^2 \Lambda_V \omega^2. \]  

The forms of the effective masses of mesons versus the baryon number density \( n_B \) are presented in Fig.3. The symmetry energy \( E_s \) dependence on the isovector density takes the following form

\[ E_s = \frac{g_\rho^2}{2M_\rho^2} (n_p - n_n)^2 + 3\Lambda_V (g_\rho g_\omega)^2 \rho_{03}^2 \omega_0^2. \]  

Fig.4 shows the dependence of \( \rho \) field expectation value on the baryon number density \( \rho_B \) for the following parameter sets: the original NL3 (\( \Lambda_V = 0 \)) and for the extreme values of the parameter \( \Lambda_V \). Due to the growth of the meson effective mass the value of the \( \rho \) field is significantly smaller in the presence of the \( \omega - \rho \) coupling. The proton fraction \( Y_p \) versus the baryon number density \( n_B \) is presented in Fig.5. The pressure and the energy density are related to the trace of the energy momentum tensor \( T_{\mu\nu} \):

\[ P = \frac{1}{3} \langle T_{ii} \rangle, \quad \epsilon = \langle T_{00} \rangle \]

\[ T_{\mu\nu} = 2 \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - g_{\mu\nu} \mathcal{L} \]
where the Lagrangian function \( \mathcal{L} = \mathcal{L}_{RMF} + \mathcal{L}_L \). The total pressure \( P \) of the neutron star matter consists of pressures coming from the fermion and meson fields

\[
P = \frac{1}{2} M_p^2 \rho_{03} + \frac{1}{2} M_\omega^2 \rho_{03}^2 + \frac{1}{2} 4 \xi \omega_0^4 + g_\omega^2 \rho_{03}^2 \Lambda V (\omega_0^2 + \rho_{03}^2) - U(\varphi_0) + P_F \tag{23}
\]

with the fermion pressure \( P_F \) being the sum of the lepton and nucleon contributions

\[
P_F = \sum_{p,n} \frac{1}{3 \pi^2} \int_0^{k_F} \frac{k^2 dk}{\sqrt{k^2 + (M - g_s \varphi_0)^2}} + \frac{1}{3 \pi^2} \int_0^{k_F} \frac{k^2 dk}{\sqrt{k^2 + m_e^2}} \tag{24}
\]

Analogously, the total energy density \( \varepsilon \) includes terms coming from meson and fermion fields

\[
\varepsilon = \frac{1}{2} g_\omega \rho \omega - \frac{1}{2} M_\omega^2 \rho_{03}^2 + g_\omega \rho_B - g_\omega^2 \rho_{03}^2 \Lambda V (\omega_0^2 + \rho_{03}^2) - \frac{1}{2} M_\omega^2 \omega_0^4 - \frac{1}{24} g_\omega^4 \xi \omega_0^4 + U(\varphi_0) + \varepsilon_F \tag{25}
\]

with \( \varepsilon_F \)

\[
\varepsilon_F = \sum_{p,n} \frac{1}{3 \pi^2} \int_0^{k_F} k^2 dk \sqrt{k^2 + (M - g_s \varphi_0)^2} + \frac{1}{3 \pi^2} \int_0^{k_F} k^2 dk \sqrt{k^2 + m_e^2}. \tag{26}
\]

Fig.6 shows the pressure \( P \) as a function of the energy density \( \varepsilon \) for different parameter sets.
Figure 1: The form of equation of state for the NL1, TM1 parameter sets together with the composite EOS. Variations of the EOS with the proton fraction $Y_p$ are also presented.

Figure 2: The equation of state for the original NL3 parameter set and for the NL3 supplemented with the $\Lambda_V$ coupling.
Figure 3: The effective $\rho$ and $\omega$ meson masses.

Figure 4: The mean value of the $\rho$ meson field versus the baryon number density $n_B$. 
3. The geometry of the spacetime around the neutron star

The number of observed rotating neutron stars is increasing making it worthwhile to study the influence of rotation on the structure and parameters of neutron stars. The spacetime outside a rotating neutron star is much more complicated than the metric outside a non-rotating one. It seems to be interesting to investigate not only the properties of rotating neutron stars but the phenomena occurring in the vicinity of the neutron stars as well. In general the metric of a stationary, axisymmetric, asymptotically flat spacetime has the form

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]  

(27)

with \( g_{\mu\nu} \) being the metric tensor. The metric potentials \( \gamma, \rho \) and \( \alpha \) and the angular velocity \( \omega \) of the stellar fluid are functions of the radial coordinate \( r \) and the polar angle \( \theta \) only. The solution of Einstein equations for stationary rotating black holes, namely the Kerr metric in Boyer-Lindquist coordinates and geometrized units \((G = c = 1)\) is given by

\[ ds^2 = -(1 - \frac{2Mr}{\Sigma}) dt^2 - (4Mar \frac{sin^2 \theta}{\Sigma}) dt d\phi \]  

\[ + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + (r^2 + a^2 + 2Ma^2 r \frac{sin^2 \theta}{\Sigma}) sin \theta d\phi^2 \]  

(28)

Here \( \Delta, \Sigma \) and \( a \) are of the following form

\[ \Delta = r^2 - 2Mr + a^2 \]  

(29)
\[ \Sigma = r^2 + a \cos \theta \]
\[ a = \frac{J}{M} \]
where \( J \) is the angular momentum. Setting \( a = 0 \) the above metric reduces to the Schwarzschild one
\[ ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{30} \]
In this case the mass \( M \) and the radius \( R \) of the star are obtained by numerical solution of the relativistic equations for hydrostatic equilibrium, the Tolman-Oppenheimer-Volkoff equations (the TOV equations). The RMF Lagrangian density function \( L \) now is supplemented by the standard gravitational part \( L_G = \frac{R}{2\kappa} \). In order to compute the structure of rapidly rotating fluid body the numerical method developed by Butterworth and Ipser is used. Apart from this exact numerical treatment there is a perturbative Hartle’s method which is based on the assumption that rotating massive body is no longer spherically symmetric. Both spherical and quadrupole deformations of the structure of the star are the effects of the rotation. The star is distorted, and expanding the metric functions through second order in the stars rotational velocity \( \Omega \) one can obtain the following form of the perturbed metric
\[ ds^2 = -e^{\gamma + \rho}dt^2 + e^{2\alpha}(r^2d\theta^2 + dr^2) + e^{\gamma - \rho}r^2\sin^2\theta(d\phi - \omega dt)^2 + O(\Omega^3) \tag{31} \]
where metric functions in this perturbed line element can be calculated from Einstein’s field equations and given as solutions of Hartle’s stellar structure equations, \( \omega \) is the same as in the nonperturbative line element \( \frac{\Theta}{1 - \frac{2M}{r}} \). The metric functions are determined with the use of Einstein equations
\[ G_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{32} \]
The matter source is assumed to be a perfect fluid with the stress-energy tensor \( T_{\mu\nu} \) given by
\[ T_{\mu\nu} = (\varepsilon + P)u_\mu u_\nu + g_{\mu\nu}P \tag{33} \]
where \( \varepsilon \) is the total energy density, \( P \) the pressure. The assumption of uniform rotation of a star means that the value of rotational velocity \( \Omega \) is constant throughout the star. For uniformly rotating bodies there is a relation among components of the four-velocity vector \( u^\varphi = \Omega u^t \). The nonzero components of the four-velocity vector \( u^\mu \) of the matter satisfying the normalization condition \( u^\mu u_\mu = -1 \) are of the form
\[ u^t = e^{-(\gamma + \rho)/2}(1 - v^2)^{-1/2}, \quad u^\varphi = \Omega u^t \tag{34} \]
v is the physical velocity of the fluid, relative to the local zero angular momentum observer and can be expressed in terms of the angular velocity \( \Omega \) as
\[ v = (\Omega - \omega)\sin \theta e^{-\rho} \quad \Omega = \frac{d\varphi}{dt} \tag{35} \]
The absolute limit on stable neutron star rotation is determined by the Kepler frequency $\nu_K$. It determines the frequency at which the mass shedding at the stellar equator sets in. The result of the work of Haensel and Zdunik shows that the value of the Kepler frequency can be estimated knowing the value of the mass and radius of the corresponding nonrotating star and the empirical relation was given

$$\Omega_K \approx C_{Hz} \sqrt{(M_s/M_\odot)(R_s/10\text{km})^2} = (0.63 - 0.67) \times \Omega_c$$

where, $C_{Hz} = 7700\text{s}^{-1}$ and $\Omega_c$ is the Newtonian value and is equal

$$\Omega_c = \sqrt{M_s/R_s^3}$$

the index $s$ indicates that these values refer to the spherical configuration. From the condition of stationarity and axisymmetry of the metric it is clear that the energy and angular momentum are constants of motion. One of the problem considered in this paper is connected with the existence and location of the marginally stable orbit in the equatorial plane outside the star. In agreement with the predictions of the general theory of relativity for sufficiently compact stars there is a region around the star, in the equatorial plane in which particles moving along geodesics are unstable to radial perturbations. Using the method presented by Bardeen one can calculate the location of the marginally stable orbit. From the condition $p_\mu p^\mu = -1$ the equations of motion of a particle confined to the equatorial plane have the following form

$$\frac{dt}{ds} = p_0^0 = e^{-(\gamma + \rho)}(E - \omega \Phi)$$

$$\frac{d\Phi}{ds} = p_1^0 = \Omega p_0^0 = \frac{\Phi}{r^2 e^{\gamma - \rho}} + \omega e^{-(\gamma + \rho)}(E - \omega \Phi)$$

$$e^{(2\alpha + \gamma + \rho)} \left( \frac{dr}{ds} \right)^2 = Q(r, E, \Phi) = e^{-(\gamma + \rho)}(E - \Phi \omega)^2 - 1 - \frac{\Phi^2}{r^2} e^{-(\gamma + \rho)}$$

where the constants of motion are $E = -p_0$, the energy per unit mass, and $\Phi = p_1$, the angular momentum per unit mass about the axis of symmetry, $s$ is the proper time along the geodesic. The condition for the circular orbit is given by

$$\frac{\partial Q}{\partial r} = 0$$

This allows us to determine the velocity of rotation of a circular orbit

$$v = \pm \left( \sqrt{e^{-2\rho} r^2 \omega^2 + 2r(\gamma' + \rho')} + r^2(\gamma'' - \rho'') \pm e^{-\rho} r^2 \omega' \right) / (2 + r(\gamma' - \rho'))$$

Solutions of this equation determine the velocity of co-rotate and counter-rotate particles respectively, whereas the condition for the stability of the circular orbit is fulfilled when

$$\frac{\partial^2 Q}{\partial r^2} < 0$$
Location of the innermost marginally stable orbit is given by the requirement
\[
\frac{\partial^2 Q}{\partial r^2} = 0 \tag{44}
\]
which leads to the following equation
\[
(1 - v^2) \frac{\partial V}{\partial r^2} = -\left(\gamma'' + \rho''\right)(1 + v^2) - v^2(-\gamma' + \rho')(\gamma' + \rho) + 2\frac{v^2 \omega''}{(\Omega - \omega)}
\]
\[
+ \left(\gamma' + \rho'\right)^2 + 4\frac{v^2 \omega'}{\Omega - \omega}(\gamma' + \rho') - \frac{v^2 \omega'}{\Omega - \omega} \tag{45}
\]
\[
- \frac{6v^2}{r^2} + 2\frac{v^2}{r}(\gamma' + \rho') + 2\frac{\omega^2}{(\Omega - \omega)^2} - 2
\]
where prime denotes a first order partial derivative with respect to \(r\). This allows to obtain the equation for the radius of the marginally stable orbit. Considering the case of a nonrotating, spherically symmetric star the mass of the neutron star as a function of the marginally stable orbit radius can be expressed as \(M = \frac{R_{ms}}{6}\) and the orbital frequency of a point particle in a circular orbit is given by
\[
\nu = \frac{1}{2\pi} \sqrt{\frac{M}{r^3}} \tag{46}
\]
\(M\) is the neutron star mass, \(r\) is the orbit radius. The \(M = \frac{R_{ms}}{6}\) relation on the mass-radius plane is a straight line which intersects the function \(M(R_{orb}, \nu)\) at the point
\[
M_{max} = \frac{2200\text{Hz}}{\nu} M_\odot. \tag{47}
\]
\(M_{max}\) settles the upper bound on the mass of the star. In the case of rotating neutron stars the obtained mass-orbit radius relation is altered. The results depend on the rate of stellar spin which can be characterized by the dimensionless parameter \(j = J/M^2\), \(J\) and \(M\) are the angular momentum and the mass of the star. For very small values of \(j\) the approximate expressions valid to first order of the parameter \(j\) can be incorporated in order to achieve the expressions for the frequency of gas in the circular orbit \(\nu\), the Lense-Thirring precession frequency \(\nu_{LT} = \omega/(2\pi)\) and adequate expressions for the marginally stable orbit
\[
\nu = \frac{1}{2\pi} \sqrt{\frac{M}{r^3}}(1 - \frac{M}{r^3} j) \tag{48}
\]
\[
\nu_{LT} = \frac{2}{2\pi} \frac{M^2}{r^3} j \tag{49}
\]
\[
r_{ms} = 6M(1 - \frac{2}{3} \frac{M}{r^3} j) \tag{50}
\]
\[
\nu_{ms} = \frac{6^{-3/2}}{2\pi M}(1 + \frac{11}{6^{3/2}} j) \tag{51}
\]
Figure 6: The mass-radius relations for the employed equations of state. The neutron star mass as a function of the marginally stable orbit radius is shown (the dotted line) together with the relation $M(R_{\text{orb}}, \nu)$

\[ \nu_{LT,ms} = \frac{1}{\frac{1}{6^3 \pi} \frac{j}{M}} \]

where $r_{ms}$ is the radius of the marginally stable orbit for the rotating neutron star.

4. Conclusions

The properties of non-rotating neutron stars for given equations of state are listed in Table 2. They correspond to the maximum stable star configuration. The employed parameter sets are the TM1 and NL1 for the neutron rich matter without crust and $TM1^*$ which marks the composite EOS namely TM1 one joined with those of Negele and Vautherin and Haensel and Pichon. For the latter case the numerical calculations were performed for the density span from $7 \times 10^{14} \text{g/cm}^3$ (the maximum density for the TM1 parameter set) to $3 \times 10^{17} \text{g/cm}^3$. In the obtained equations of state the original NL3 parameter set and the NL3 one supplemented with the additional $\rho - \omega$ mesons coupling have been also analyzed. For comparison the neutron star parameters for TM1 EOS with changed proton fraction have been obtained. The forms of the equations of state are presented in Figs. 1 and 2.

Each particular column of Table 2 contains: the central density $\rho_c$, the baryon rest mass $M_B$, the gravitational mass $M$, the radius $R_s$, the moment of inertia $I$, the value of the Newtonian limit of the stellar spin $\Omega_c$ and the gravitational redshift $z$.

The observed X-ray radiation can be used to investigate phenomena occurring
Figure 7: The mass-radius relation for the TM1 parameter set with the relation \( M(R_{\text{orb}}, \nu) \). Dotted line represents the neutron star mass as a function of the marginally stable orbit radius.

Figure 8: The effective nucleon mass versus the neutron star radius for the neutron-rich neutron star interior for TM1 and NL3 (\( \Lambda_V = 0.005 \)) parameter sets.
Figure 9: The $\rho$ and $\omega$ field as a function of stellar radius for different parameter sets.

Figure 10: The energy symmetry versus the stellar radius for TM1 and the original NL3 ($\Lambda_V = 0$) and the extended NL3 ($\Lambda_V = 0.005$) models.
in the strong gravitational fields near neutron stars. With the Rossi X-Ray Timing Explorer high-frequency X-ray brightness oscillations from several low-mass X-ray binaries have been discovered. Observations of kilohertz QPO sources can show the existence of circular orbits around neutron stars. The higher frequency in a kilohertz pair in the sonic point model can be explained as the orbital frequency of gas in a nearly circular orbit around the neutron star which in the case of Schwarzschild metric is given by the relation \( \Omega = \frac{\nu}{2\pi} \). From this equation one can obtain the relation between the orbit radius \( r \) and the mass of the neutron star for given value of the frequency \( \nu_{QPO} \). Thus the orbit radius is determined by the mass of the neutron star which in turn depends on the particular choice of the equation of state, and as a consequence the allowed region on the mass-radius plane for the observed values of \( \nu_{QPO} \) exists.

Fig. 6 shows the mass-radius plane together with the allowed area for the fixed value of the frequency \( \nu_{QPO} \) in the case of non-rotating stars. Having evaluated the EOS the numerical solutions of the structure equations are obtained and then the mass-radius relations are constructed for chosen forms of the equations of state listed in Table 1. Table 3 contains properties of slowly rotating neutron stars for mentioned earlier equations of state. \( R_{s} \) and \( R_{c} \) denote radii of marginally stable orbits for co-rotating and counter-rotating particles respectively. The condition \( R_{s} < R_{ms} \) (the neutron star radius \( R_{s} \) is smaller than the radius of the marginally stable orbit \( R_{ms} \)) is fulfilled for the employed equations of state. For the TM1 parameter set there exist masses in the relevant range from 0.5\( M_{\odot} \) to 1.4\( M_{\odot} \) for which the condition \( R_{s} < R_{orb} \) is fulfilled, for LN1 the same range is from 1.2\( M_{\odot} \) to 1.6\( M_{\odot} \). The composite EOS changes the range from 0.75\( M_{\odot} \) to 1.5\( M_{\odot} \). For the composite form of the equations of state the minimal star configuration appears. The same result is achieved changing the proton fraction \( Y_{p} \). From Fig.6 one can see that the minimum mass configuration is very sensitive to the proton fraction. The symmetric nuclear matter excluded the existence of stable minimum configuration stars. The value of the proton fraction \( Y_{p} \) is directly connected with the energy symmetry \( E_{s} \), hence the influence of the symmetry energy on neutron star parameters, especially on the minimum configuration is straightforward and comes from the change of the chemical composition of the neutron star matter. Fig.10 compares the symmetry energy \( E_{s} \) as a function of stellar radius \( R \) for parameter sets with...
(NL3 $\Lambda_V = 0.005$) and without (TM1 and NL3 $\Lambda_V = 0$) additional meson coupling. In the first case the symmetry energy prevails over the latter ones. The changes in energy symmetry are caused by the appearance of the effective meson masses which are closely connected with the self-consistency equation for meson fields. Using these equations the form of the meson fields expectation values as function of stellar radius $R$ for different parameter sets are presented in Fig.9. In the case of different from zero $\Lambda_V$ the value of $\rho$ decreases and the energy symmetry tends to the value characteristic to the symmetric matter. This is of special significance for the minimum configuration models. Solutions with minimal masses are sensitive to the proton fraction which in turn is determined by the value of parameter $\Lambda_V$. Thus, the minimum mass configurations are sensitive to the existence of the additional meson coupling and one can conclude that the nonzero $\Lambda_V$ modiﬁes the mass radius dependence. The effective nucleon mass $M^*$ is caused by the nucleon-scalar field coupling. The parameter $\delta = \frac{m}{M}$ versus the stellar radius $R$ is presented in Fig.8 for TM1 and NL3 $\Lambda_V = 0.005$. The idea of introducing the additional meson coupling and finding an analogy between the neutron rich nucleus $^{209}_{\text{Fe}}$ and the behaviour of neutron star crust was introduced by Horowitz and Piekarewicz in their work [1]. As it has been stated in the previous chapter the change of proton fraction $Y_p$ and the behaviour of the energy symmetry are connected to the presence of the additional $\rho - \omega$ coupling and the nonzero value of $\Lambda_V$. The next step is the extension of EOS for neutron rich matter to the case of finite temperature [17]. This will be the subject of future investigations. Since the neutron star parameters varies with temperature, which the allowed area on the mass-radius plane changes as well. For temperatures different from zero the mass and the radius of the star have higher values and any configuration with larger mass altered a marginally stable orbit radius. The most sensitive component of neutron star to temperature are electrons (in the case when neutrinos are neglected), thus the influence of temperature acts in the similar way as the introduction of the parameter $\Lambda_V$.

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