A host–parasitoid system with predation-driven component Allee effects in host population

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Allee effects and parasitism are common biological phenomena observed in nature, which are believed to have significant impacts in ecological conservation programmes. In this article, we investigate population dynamics of a discrete-time host–parasitoid system with component Allee effects induced by predation satiation in host to study the synergy effects of Allee effects and parasitism. Our model assumes that parasitism attacks the host after the density dependence of the host. The interactions of component Allee effects and parasitism can lead to extremely rich dynamics that include but are not limited to extinction of both species due to \textit{Allee effects} at their low population density, multiple attractors, strange interior attractors and even crisis of strange attractor due to high parasitism. We perform local and global analysis to study the number of equilibria and their stability; and study the extinction and permanence of our host–parasitoid system. One of the most interesting results shows that the combination of strong Allee effects and parasitism may promote the coexistence of both host and parasite at their high population density. In addition, component Allee effects may destroy interior equilibrium under different values of parameters’ ranges.

\textbf{Keywords:} component Allee effects; strong Allee effects; discrete-time host–parasitoid system; extinction; permanence

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1. Introduction

An \textit{Allee effect} is a positive correlation between the per capita growth rate and population abundance or density [2,40]. Empirical evidence of Allee effects has been reported in many natural populations including plants [19], insects [32], marine invertebrates [47], birds and mammals [6]. Various mechanisms at low population sizes, such as the need of a minimal group size necessary to successfully raise offspring, produce seeds, forage, and/or sustain predator attacks, or enhanced genetic inbreeding have been proposed as potential sources of the Allee effect [5,24,33,44,46]. There is a clear distinction between the \textit{component Allee effect} and the \textit{strong Allee effect} [46]. The \textit{component Allee effect} is refereed to a positive relationship between any
measurable component of individual fitness (such as the per capita birth rate, survival) with population size or density, which can be induced by ecological processes such as mating limitation, cooperative breeding and cooperative feeding [9]. The synergistic interaction between component Allee effects and negative density-dependent factors such as predation, competition can generate strong Allee effects where there exists a critical population threshold. This critical population level is called an Allee threshold below which population goes extinct while beyond which population may sustain. Weak Allee effects are the cases when there is no such critical threshold but still there is a positive correlation between the per capita growth and population density or size at low population densities/sizes. Excellent reviews on Allee effects and their ecological implications can be found in [7] and some related recent work can be found in [1,3,14,15,25,29,34,35,41,45,50].

A large body of literature is available on modelling of the interactions between two populations using discrete time difference equations when the population has non-overlapping generations (e.g. when the reproduction and the life cycle is seasonal). Nicholson and Bailey [39] introduced a host–parasitoid interaction model by using two discrete time difference equations which have become a standard model for modelling many insect–parasitoid or plant–herbivore interactions. Due to increasing number of evidence of endangered species extinction, the impacts of Allee effects have received significant recognition among ecologists and conservations biologists. In recent years, there are fair amount of mathematical modelling work in studying the dynamics of host–parasitoid systems subject to Allee effects based on the modified discrete-time Nicholson-Bailey model (see, e.g. [3,25,26,30,31,34]) in addition to the approaches of the differential equations (e.g. see the work of Hilker et al. [21], Morozov et al. [38], Petrovskii et al. [42]).

The study of Jang and Diamond [26] considered host–parasitoid system with Allee effects on host where host’s density-dependent growth regulation is modelled by overcompensatory Ricker model. They observed that, the Allee effect has a negative impact on the coexistence of both host and parasitoid populations. While the host’s growth is regulated by strong Allee effect and parasitism is aggregative, Liu et al. [34] observed that the dynamical complexities (chaos, sensitivity to initial conditions) of the model is reduced due to the presence of Allee effect. The stability of the system including Allee effect is completely dependent on the choice of the underlying model framework, choice of density-dependent growth and how the Allee effect is incorporated in the model system. The types of the density dependence and parasitism in host life cycle as well as the effects of the density dependence in parasite have considerable impacts on the dynamics of the host–parasitoid interaction [51]. In particular, if the parasitism acts before the density-dependent effect takes place, the host can undergo a stable dynamics above its parasite-free carrying capacity [36]. This type of host–parasitism interactions has been approximately modelled by modifying the classical Nicholson–Bailey model (see the work of Jang [25], Jang and Diamond [26], Kon [31]). In many plant–herbivore (host–parasite) interactions, the leaves are attacked by parasites (or herbivore) when they form good canopies. Thus, in such system, parasitism occurs after the density dependence of the host. This type of host–parasitism (or plant–herbivore) interactions has been modelled and studied by Kang et al. [30]. In this article, we focus on the modified model from Kang et al. [30] by incorporating the component Allee effects on the host.

In this article, we develop and study the dynamics of a host–parasitoid discrete-time model incorporating: (1) a component Allee effect in host population where the Allee effect is induced by predator saturation; (2) parasitism occurring after the growth phase of host, thus, its parasitism population has strong nonlinear correlation with host population. The purpose of this article has twofold:

(i) To understand the effects of component Allee effects on the interactions of host–parasitism.
(ii) To understand the synergetic interactions of component Allee effects and parasitism on the host populations.
The organization of the remaining paper is as follows: In Section 2, we consider a single-species population subject to a component Allee effect where the growth of host is regulated by Ricker dynamics. Then, we incorporate this dynamics in the host population in the presence of parasitoid, where the attacks of parasitoid to host are after host’s density regulation taking effects. In Section 3, we investigate the local and global dynamics of the host–parasitoid system. In Section 4, we focus on the effect of interactions between Allee effects and parasitism. Numerical simulation is performed to support the conclusion. In Section 5, we conclude our results and discuss their biological implications. The detailed proofs of our analytical results are provided in the last section.

2. Formulation of models

In this section, we first introduce the dynamics of single species with component Allee effects, then we formulate a host–parasite model where the attacks of parasitoid to host are after host’s density regulation taking effects.

2.1. A single-species population model with Allee effects

Many ecologists suggest that one of the most possible common Allee effects occurs in species subject to predation by a generalist predator with a saturating functional response [10,45,46]. We consider the following single-species model (1) with component Allee effects due to predator saturation after scaling. This model follows the same approach of Schreiber [45] and Kang [27,29].

\[ N_{t+1} = F(N_t) = N_t e^{r(1-N_t)} I(N_t) \]

where \( r \) denotes the maximum birth-rate of species; \( m \) represents the predation intensity; \( b \) is the proportional to the handling time and all the three parameters \( r, b, m \) are positive. Here, \( I(N) = e^{-rm/(1+bn)} \) is a component Allee effect that describes the probability of escaping predation by a predator with a saturating functional response [20,45]. The values of parameters \( r, b, m \) can lead to no Allee effects, weak Allee effects and strong Allee effects. The dynamics of single-species model (1) can be extremely complicated and rich. Schreiber [45] summarized the dynamics of a similar discrete-time single species.

In this article, we first provide completed analysis and results on the dynamics of Model (1) by adopting some results from Schreiber [45]. Let

\[ N_1^* = \frac{b - 1 - \sqrt{(b+1)^2 - 4bm}}{2b} \quad \text{and} \quad N_2^* = \frac{b - 1 + \sqrt{(b+1)^2 - 4bm}}{2b}. \]

The detailed dynamics of Equation (1) can be summarized as the following lemma:

**Lemma 2.1** Single-species model (1) is positively invariant and bounded in \( \mathbb{R}_+ \). Model (1) has Allee effects if \( mb > 1 \) and has no Allee effects if \( 0 < mb < 1 \). In addition, we have the following scenarios regarding the dynamics of Equation (1):

(i) No Allee effects:
(a) If $0 \leq m < \min\{1, 1/b\}$, then system (1) is persistent with two steady states 0 and $N_2^*$, i.e. there exists positive number $\epsilon$ and $B$, such that for any $N_0 > 0$, we have

$$\epsilon \leq \liminf_{t \to \infty} N_t \leq \limsup_{t \to \infty} N_t \leq B.$$ 

(b) If $(b + 1)^2/4b < m < 1/b$, then system (1) has global stability at 0.

(ii) Weak Allee effects:

(a) If $1/b < m < 1$, then system (1) is persistent with two steady states 0 and $N_2^*$.

(b) If $m > \max\{(b + 1)^2/4b, 1/b\}$, then system (1) has global stability at 0.

(iii) If $1 < m < (b + 1)^2/4b$ and $b > 1$, then (1) has strong Allee effects with three steady states $0, N_i^*, i = 1, 2$ such that $0 < N_i^* < N_2^*$. Let

$$N_a = \max\{F^{-1}(N_i^*)\} \quad \text{and} \quad M = \max_{N_1^* \leq N \leq N_2^*}\{F(N)\},$$

then we have the following two cases:

(a) $F(M) \geq N_1^*$: System (1) converges to 0 if $N_0 \in [0, N_1^*) \cup (A, \infty)$ while it is persistent in $[N_1^*, N_0]$.

(b) $F(M) < N_1^*$, then System (1) has essential extinction, i.e. for a randomly chosen $N_0 > 0$, System (1) converges to 0 with probability 1.

Notes: The per capita growth rate of System (1) is $f(N) = e^{r(1-N-m/(1+bN))}$ which gives

$$f'(N) = r \left[-1 + \frac{bm}{(1 + bN)^2}\right]f(N).$$

This implies that the value of $mb$ measuring the total handling time of predator determines whether System (1) has Allee effects (when $mb > 1$) or not (when $mb < 1$). Lemma 2.1 suggests that System (1) has strong Allee effects when $m$ has intermediate values while it has no or weak Allee effects when the values of $m$ are too large or too small. In addition, the system can have weak Allee effects even if population goes extinct. The results of Lemma 2.1 are important for our study on the dynamics of the host–parasite interaction models. The following theorem provides a sufficient condition on the local and global stability of single-species model (1):

**Theorem 2.2** Assume that $0 \leq m < 1$. Single-species model (1) has local stability at $N_2^*$ if $0 < r < 2/N_2^*(1 - bm/(1 + bN_2^*))$ which also implies that $N_2^*$ is locally asymptotically stable if $0 < r < 2$. If, in addition, the following inequality holds:

$$\min_{N > 0} \left\{1 + rN \left[-1 + \frac{bm}{(1 + bN)^2}\right] e^{r(1-N-m/(1+bN))}\right\} > -1 \quad (2)$$

then single-species model (1) has global stability at $N_2^*$.

Moreover, if the inequalities $m < \min\{1/b, 1\}$ and $r < 2/N_2^*(1 - bm/(1 + bN_2^*))$ hold, then single-species model (1) has global stability at $N_2^*$.

Notes: In the case that $m = 0$, System (1) becomes the well-known Ricker’s map $N_{i+1} = N_i e^{r(1-N_i)}$ which has the global stability at $N_2^* = 1$ if $\min_{N > 0} \{[1 - rN] e^{r(1-N)}\} > -1$ which holds when $0 < r \leq 2$ [8]. In addition, Ricker’s map has period doubling two when $2 < r < 2.5$ [8]. In the case that $0 < m < 1$, Theorem 2.2 suggests that the additional term $I(N)$ may stabilize the system under certain conditions. For example, if $m < \min\{1/b, 1\}$ holds, system (1) can still have global and local stability even when $r > 2$. In addition, the degree of stabilized effects can be varied depending on the value of $mb$, i.e. whether System (1) exhibits weak Allee effects or no
Allee effects. Here, we provide numerical simulations to illustrate the effects of the component Allee effect \( I(N) \) (Figure 1): (a) By comparing Figure 1(b) (the Ricker map) to Figure 1(b) (no Allee effects), Figure 1(c) (weak Allee effects), and Figure 1(d) (strong Allee effects), it seems that the component Allee effect \( I(N) \) can stabilize the system when \( r \) is not large enough, i.e. the system has local stability on \( N^\ast \) even when \( r > 2 \); (b) By comparing Figure 1(b)–(c), it seems that the larger values of \( b \) may destabilize the system; and (c) From Figure 1(d), it suggests that large values of \( r \) can generate essential extinction when the system has strong Allee effects.

### 2.2. Allee effect in host population due to predator satiation

Let \( N_t \) and \( P_t \) be the population density of the host (or plant/prey) and parasite (or herbivore/predator) in successive generations \( t \) respectively. We consider a host–parasitoid interaction with component Allee effects induced by predator saturation in the following form:

\[
N_{t+1} = N_t e^{r(1-N_t-m/(1+bN_t)) - aP_t},
\]

\[
P_{t+1} = N_t e^{r(1-N_t-m/(1+bN_t))} (1 - e^{-aP_t}) = N_{t+1} (e^{aP_t} - 1)
\]

with the following ecological assumptions:

H1: In the absence of parasitoids, the dynamics of host population is density dependent and is governed by the overcompensatory Ricker dynamics described by Equation (1).

H2: The parasitoid attacks hosts after the hosts’s density-dependent growth regulation takes effect.

H3: The encounter between host and parasitoid follows the Poisson distribution, where the parasitoids are distributed randomly in space. Consequently, the probability that an individual host will escape from being parasitized when the parasitoid population is of size \( P \) is \( e^{-aP} \), the zeroth term of Poisson distribution and \( a \) is the search area by an individual parasitoid.
If the host is not subject to component Allee effects, i.e., \( m = 0 \), then a host–parasitoid interaction with parasite attacking after density dependence in host can be described as follows:

\[
\begin{align*}
N_{t+1} &= N_t e^{r(1-N_t) - aP_t}, \\
N_{t+1} &= N_t e^{r(1-N_t)(1 - e^{-aP_t})} = N_{t+1}(e^{aP_t} - 1),
\end{align*}
\] (4)

Model (4) has been extensively studied by Jang and Diamond [26] and Kang et al. [30]. Based on their study, we summarize some basic dynamics of system (4) in the following lemma:

**Lemma 2.3** System (4) is bounded and invariant in \( \mathbb{R}^2_+ \). Its basic dynamics can be summarized below:

(i) If \( 0 < r < 2 \) and \( a > 1 \), then system (4) is permanent in \( \mathbb{R}^2_+ \), i.e. there exists \( 0 < \epsilon < B \) such that for any \( N_0 > 0, P_0 > 0 \), we have

\[
\epsilon \leq \liminf_{t \to \infty} \min\{N_t, P_t\} \leq \limsup_{t \to \infty} \max\{N_t, P_t\} \leq B.
\]

(ii) If \( a > 1 \), system (4) has unique interior equilibrium while if \( a < 1 \), system (4) can have two interior equilibria under certain range of \( r \).

(iii) System (4) has bi-stability for certain parameter ranges when \( r > 2 \), i.e. the system can have boundary attractors on \((N,0)\) and an interior attractor.

In addition to the summary above, the numerical simulations suggest that Equation (4) has global stability at \((1,0)\) if \( 0 < r < 2 \) and \( a < 1 \). The main purpose of this article is to investigate how Allee effects induced by predator satiation affect host–parasite interactions by comparing dynamics of Equation (4) to Equation (3). We also would like to explore how predation may possibly promote the coexistence when the host species suffers from strong Allee effects at its high density. In order to proceed our study, we first give the following lemma regarding the basic dynamical properties of Equation (3):

**Lemma 2.4** System (3) is positively invariant in \( \mathbb{R}^2_+ \) and is attracted to the compact set \([0, e^{r-1}/r] \times [0, e^{r-1}/r]\).

### 3. Mathematical analysis

In this section, we first study the stability of boundary equilibria of system (3), then we investigate its the interior equilibria. In addition, we provide sufficient conditions for the extinction and permanence of the system.

#### 3.1. Boundary equilibria and their stability

System (3) has a trivial equilibrium \( E_0 = (0,0) \) for all parameter values. Depending on the values of \( r, m, b, \) system (3) may have \( E_1 = (N^*_1, 0) \) and \( E_2 = (N^*_2, 0) \) as their boundary equilibria where sufficient conditions of their existence and stability are summarized in the following theorem:

**Theorem 3.1 (Boundary equilibria)** Sufficient conditions for the existence and stability of boundary equilibria of system (3) is depicted in Table 1.

Theorem 3.1 provides sufficient condition for the existence of boundary equilibria as well as their local stability of system (3). In the following subsection, we focus on the interior equilibrium and their stability.
Table 1. Sufficient conditions for the existence and local stability of boundary equilibria for system (3) where \( E_0 = (0,0), E_1 = (N_1^*, 0) \) and \( E_2 = (N_2^*, 0) \).

| Boundary equilibria | Sufficient conditions for existence | Stability |
|---------------------|-------------------------------------|-----------|
| Only \( E_0 \)     | \( 4mb > (b + 1)^2 \)               | Global stable; \( E_0 \) is a saddle; \( E_2 \) is locally stable if \( 0 < r < 2 \) and \( aN_2^* < 1 \) |
| \( E_0, E_2 \)     | \( 0 \leq m < 1 \)                 | \( E_0 \) is locally stable; \( E_1 \) is always unstable where it is a saddle if \( aN_1^* < 1 \), while it is a source if \( aN_1^* > 1 \); and \( E_2 \) is locally asymptotically stable if \( aN_2^* < 1 \) and \( -2 < rN_2^* \left( \frac{4mb}{b + 1 + \sqrt{(b + 1)^2 - 4mb}} - 1 \right) < 0 \) |
| \( E_0, E_1, E_2 \) | \( 1 < m < \frac{(b + 1)^2}{4b} \) and \( b > 1 \) | \( E_0 \) is locally stable; \( E_1 \) is always unstable and \( E_2 \) is locally asymptotically stable if \( aN_2^* < 1 \) and \( -2 < rN_2^* \left( \frac{4mb}{b + 1 + \sqrt{(b + 1)^2 - 4mb}} - 1 \right) < 0 \) |

3.2. Interior equilibria and their stability

The interior equilibrium of Equation (3) should satisfy the following equations:

\[
P = \frac{r(1 - N - m/(1 + bN))}{a} \quad (5)
\]

\[
N = \frac{P}{e^{ap} - 1} \quad (6)
\]

Let \( f_1(N) = r(1 - N - m/(1 + bN))/a \) and \( f_2 = P/e^{ap} - 1 \), then we can easily see that \( f_2 \) is a decreasing function with respect to \( P \geq 0 \), thus, we can denote the inverse of \( f_2 \) as \( f_2^{-1} \). In addition, we have \( \lim_{P \to 0} f_2(P) = 1/a \). Let

\[
f_1'(N) = \frac{r}{a} \left( -1 + \frac{bm}{(1 + bN)^2} \right) \quad \text{and} \quad f_1''(N) = -\frac{rmb^2}{a(1 + bN)^3},
\]

then we can see that \( f_1 \) is a convex function that increases when \( N \in [0, N_c] \) and decreases when \( N \in [N_c, \infty) \) where \( N_c = \sqrt{bm} - 1/b \) subject to \( bm > 1 \). Then, we have the following theorem regarding the number of interior equilibria and their stability:

Table 2. Sufficient conditions for the existence and local stability for system (3).

| Model (3) | No interior | One interior | Two interior |
|-----------|-------------|--------------|--------------|
| Existence | 0 < r < 2, \( 2(d) \) and \( 2(e); aN_1^* > 1 \) (Figure 2) | \( aN_2^* > 1 \) (Figure 2(a) and \( 2(b) \) | \( f_1(N_c) > f_2^{-1}(N_c) \) and \( aN_2^* < 1 \) (Figure 2(f) and 2(g)) |
| Local Stable | DNE | 1 > \( 1 - rN^* + \frac{bmN^*}{(1 + bN^*)^2} (a - r)N^* + \frac{rm}{1 + bN^*} \) \( r - \frac{1 + bN^*}{bmr} > 1 + N^* \left( a - \frac{r + \frac{1 + bN^*}{bmr}}{(1 + bN^*)^2} \right) - 1 \) | One is always unstable, the other one has the same local stable condition as the case with one interior |
Figure 2. The schematic nullclines for system (3): the solid line is $P = r(1 - N - m/(1 + bN))/a$; the dashed line is $N = P/e^aP - 1$ and $N_c = \sqrt{bm - 1/b}$. Figure (d) and (h) have no Allee effects; figure (b) and (f) have weak Allee effects while the remaining figures have strong Allee effects. In the case of figure (c), system (3) has no interior equilibrium due to strong Allee effects.

**Theorem 3.2** Let $(N^*, P^*)$ be an interior equilibrium of Equation (3), i.e. a positive solution of Equations (5) and (6). Sufficient conditions on the existence of the number of interior equilibria and their stability are listed in Table 2. In addition to the cases listed in the tables, other cases on the existence of interior equilibrium are illustrated in Figure 2.

Theorem 3.2 indicates that component Allee effects on the host species can destroy interior equilibria in the following sense:

Without component Allee effects, the host–parasite model (4) can have one interior equilibrium when $a > 1$ (Lemma 2.3) while in the presence of component Allee effects, the host–parasite model (3) does not have interior equilibrium if $a > \frac{1}{N_1} > 1$ (Figure 2(c)).
3.3. Global dynamics: extinction and persistence

Theorem 3.3 (Sufficient conditions for permanence) If \( 0 \leq m < 1 \), then the host is persistent in system (3). If in addition, \( aN_2^* > 1 \) and conditions in Theorem 2.2 hold, then system (3) is permanent in \( \mathbb{R}^2_+ \), i.e. there exists positive numbers \( \epsilon \) and \( B \) such that for any initial condition with \( N_0 > 0, P_0 > 0 \), we have

\[
\epsilon \leq \liminf_{t \to \infty} \min\{N_t, P_t\} \leq \limsup_{t \to \infty} \max\{N_t, P_t\} \leq B.
\]

System (4) is permanent if \( 0 < r < 2 \) and \( a > 1 \), while in the presence of component Allee effects, one necessary condition for the permanence of systems (3) is that \( a > 1/N_2^* > 1 \) from Theorem 3.3. Thus, component Allee effects on the host species requires higher attacking rate \( a \) for the permanence of host and parasitoid.

Theorem 3.4 (Sufficient conditions for extinction) If \( 4mb > (b + 1)^2 \), then system (3) has global stability at \((0,0)\); while if \( 1 < m < (b + 1)^2/4b \), \( b > 1 \) and \( N_0 < N_2^* \), then it converge to \((0,0)\). If \( ae^{-r}/r < 1 \), then the population of parasitoid of system (3) converges to 0.

Theorem 3.4 indicates that when the host species suffers from strong Allee effects, i.e. \( b > 1 \), \( 1 < m < (b + 1)^2/4b \), both host and parasitoid cannot survive at the low population density of host, i.e. the population of host is below the Allee threshold \( N_2^* \). This also suggests that strong Allee effects of host make both host and parasitoid prone to extinct at the low density of host.

4. Interaction of Allee effects and parasitism

In this section, we use numerical simulations to investigate the effects of parasitism when single host population has essential extinction (i.e. the host population goes extinct with probability one for any randomly chosen initial population \( N_0 > 0 \)) due to strong Allee effects, i.e. we focus on how parasitism may promote the coexistence of the host–parasitoid model (3) when host population has essential extinction at its single state and initial conditions of both host and parasite are large enough (e.g. above its Allee threshold). From the study by Schreiber [45], host population model (1) has essential extinction when \( r = 4.5 \), \( rm = 5 \), \( b = b^* > 9 \) where \( b^* \) is some threshold such Equation (1) suffers from essential extinction when \( r = 4.5 \), \( rm = 5 \). For example, if \( r = 4.5 \), \( rm = 5 \), \( b = 10 > b^* \), then the host population (1) has essential extinctions due to strong Allee effects. However, the host–parasitoid model (3) with the same values of \( r, m, b \) has an interior attractor when the attacking coefficient is \( a = 2.5 \) (Figure 3). This finding is supported by several experimental evidences that parasitoids enhance plant and animal fitness (see, for example, [4,11,16,18,43,49]).

From Figure 3, we can conclude that parasitism combined with strong Allee effects can save host from extinction: Without parasitism, the host population suffers from essential extinction due to strong Allee effects, while in the presence of parasitism, both host and parasite can survive at their high density. The possible biological explanation for this is that host species has scramble intra-specific competition (also called overcompensatory, i.e. a high population density this season leads to a low population density in the next season), a high population density of parasite can bring down the current high population density such that it can have a population density that is above the Allee threshold in the next generation. In addition, we summarize the follows from simulations

(i) The host–parasitoid model (3) with component Allee effects in host can have multiple attractors. If host species is subject to essential extinction, then Equation (3) can have
Figure 3. Figure 3(a) is the nullclines, blue regions in Figure 3(c) are basins of attractions, and Figure 3(b) represents the dynamics: blue is host and red is parasitism for system (3) when $r = 4.5, m = 1.1, b = 10, a = 1.05$. The blue regions of Figure 3(c) are basins of attractions of a locally asymptotically stable interior equilibrium of system (3) while the white regions are basins of attractions of $E_0$: (a) Nullclines for system (3) when $r = 4.5, m = 1.1, b = 10, a = 1.05$, (b) stable interior equilibrium for system (3) when $r = 4.5, m = 1.1, b = 10, a = 1.05$, (c) basins of attractions for system (3) when $r = 4.5, m = 1.1, b = 10, a = 1.05$.

(ii) Parasitism can also lead to the extinction of both host and parasite. For example, when host species with strong Allee effects at its single state has persistence under certain conditions (i.e. satisfying conditions in Part 3 (a) of Lemma 2.1), then the large attacking rate of parasite $a$, may lead to the extinction of both species. The dynamical explanation for this is that the basin boundary of the stable manifold of interior attractor collides with the basin boundary of the stable manifold of $E_0$.

5. Conclusion

Predation is a general term in ecological theory and almost universal in different ecological processes, which has been recognized that predators having a Holling type-II functional/numerical response (also called predation satiation) may drive the prey to have Allee mechanism [17]. The role of Allee effects has been largely recognized that it increases the likelihood of local and global extinction, as per capita growth rate becomes low and may become negative as well [9,37]. When a host species suffers from parasitism and Allee effects induced by predation satiation (i.e. Holling type-II functional response), the interactions between host and parasite can become very complicate, thus, lead to extremely rich dynamics that include but are not limited to extinction of both species due to Allee effects at their low population density, multiple attractors, strange interior attractors and even crisis of strange attractor due to high pressure of parasitism.

In this article, motivated by the study on host–parasitoid models [25,26,30,31,34], we study the dynamics of a host–parasitoid model with component Allee effects in host induced by predation satiation and parasitism occurring after the density-dependent effects of the host. We investigate
the local and global dynamics of the system with emphasis on the possible interior equilibria and extinction/permanence. We also provide a counter example that the combinations of component Allee effects and parasitism in fact can promote the coexistence of two species under certain conditions. The summary of our analytical and numerical results are given as follows:

(i) **Effects of component Allee effects:** component Allee effects can generate and destroy interior equilibrium of host–parasitoid models (Theorem 3.2 as well as Figure 2). **Strong Allee effects** can lead to the extinction of both host and parasitism at the low density of host population (Theorem 3.4).

(ii) **Effects of strong Allee effects and parasitism:** When the host species suffers from essential extinction due to strong Allee effects, parasitism can promote the coexistence of both host and parasite at their high density population (Figure 3 and the related discussion in Section 4). This result, in fact, is supported by several experimental evidences (see, e.g. [4,11,16,18,43,49]).

Due to increasing number of rare and endangered species, dynamics of small populations, including Allee effect, has received a significant importance from ecologists and conservation biologists. With alarming theoretical consequences and empirical evidences of Allee effects, it should not be ignored, perhaps should be given utmost priority in the dynamical systems of interacting population like prey–predator system or host–parasitoid models. Hope that our theoretical and numerical results of our host–parasitoid model with component Allee effects on the host can provide some important biological implications in conservations ecology.

6. Proofs

6.1. **Proof of Lemma 2.1**

*Proof* The fixed points of system (1) are given by the solutions of $F(N) = N$, i.e.

$$Ne^{r(1-N-m/(1+bN))} = N \Rightarrow N = 0 \text{ or } 1 - N - \frac{m}{1 + bN} = 0.$$ 

Thus, if $b > 1$ and $1 < m < (b + 1)^2/4b$, then system (1) has the following two positive interior equilibria:

$$N_1^* = \frac{b - 1 - \sqrt{(b + 1)^2 - 4bm}}{2b} \quad \text{and} \quad N_2^* = \frac{b - 1 + \sqrt{(b + 1)^2 - 4bm}}{2b}.$$ 

Since $(b + 1)^2/4b > 1$, thus $b > 1 > m$ indicates that system (1) has only $N_2^*$ as its positive interior equilibrium.

Let $F(N) = Nf(N)$. The per capita growth rate of System (1) is $f(N) = e^{r(1-N-m/(1+bN))}$ which gives

$$f'(N) = r \left[-1 + \frac{bm}{(1+bN)^2}\right]f(N).$$

This implies that System (1) has Allee effects if $mb > 1$ and has not Allee effects if $mb < 1$.

(i) The map $f(N) = e^{r(1-N-m/(1+bN))}$ mapping $[0, \infty)$ to $[0, \infty)$ is differentiable. If $0 \leq m < 1$, then $f(0) > 1$. In addition, we have the following conditions holds:

(a) $f(N) > 0$ for all $N > 0$. This condition also implies that Equation (1) is positively invariant in $\mathbb{R}_+$. 

(b) There exists a $B > 0$ such that for any initial condition $N_0 > 0$ where
\[
B = \max_{N \geq 0} \{Ne^{r(1-N-m/(1+bN))}\} \leq \max_{N \geq 0} \{Ne^{r(1-N)}\} \leq \frac{e^{r-1}}{r},
\]
thus, we have
\[
\limsup_{t \to \infty} N_t \leq B.
\]
Therefore, the permanence of system (1) is followed from Theorem 3.1 in [28], i.e. there exists positive numbers $\epsilon$ and $B$, such that for any $N_0 > 0$, we have
\[
\epsilon \leq \liminf_{t \to \infty} N_t \leq \limsup_{t \to \infty} N_t \leq B.
\]
(ii) In the case that $m > (b + 1)^2/4b$, $N_1^*$ and $N_2^*$ are imaginary. System (1) has only 0 as its equilibrium, and $1 - N - m/(1 + bN) < 0$. So, $\exists h > 0$ such that
\[
1 - N - \frac{m}{1 + bN} < -h < 0,
\]
therefore,
\[
N_{t+1} = N_te^{r(1-N_t-m/(1+bN_t))} < e^{-h}N_t < e^{-ht}N_0 \to 0 \quad \text{as } t \to \infty.
\]
(iii) Notice that $f'(N) = re^{r(1-N-m/(1+bN))}(-1 + bm/(1 + bN)^2)$, thus $f'(N) < 0$ if $bm < 1$, this is the case when system (1) has no Allee effects. If $bm > 1$ and $0 < m < 1$, then system (1) has only two equilibria $0$ and $N_2^*$ with weak Allee effects. While if $1 < m < (b + 1)^2/4b$ and $b > 1$, then (1) has three steady states $0, N_i^*, i = 1, 2$ such that $0 < N_1^* < N_2^*$, i.e. Equation (1) suffers from strong Allee effects with $N_1^*$ as its Allee threshold.

To prove the rest part of item 3 in the theorem, we can verify the following technical criteria by using the results from Schreiber [45]. Since $F : [0, \infty) \to [0, \infty)$ three times continuously differentiable function, then there exists an interval $[a, b]$ with $b > a > 0$ such that,
(a) $F(N) > 0$ for all $N \in [a, b]$;
(b) $\lim_{n \to \infty} F^n(N) = 0$ for all $N \notin [a, b]$;
(c) $F$ has a unique critical point $C (F'(C) = 0)$ and
(d) The Schwartzian derivative of $F$ on $[a, b]$ is negative: $F'''(N)/F'(N) - \frac{3}{2}(F''(N)/F'(N))^2 < 0$.

Therefore, the discussions and results above also show that
(i) If $1/b < m < 1$, then system (1) is persistent with two steady states $0$ and $N_2^*$.
(ii) If $m > \max\{(b + 1)^2/4b, 1/b\}$, then system (1) has global stability at 0.

\[\blacksquare\]

6.2. Proof of Theorem 2.2

Proof Since $\frac{(b+1)^2}{4b} \geq 1$, thus according to Lemma 2.1, the inequalities $0 \leq m < 1$ indicate that
\[
0 \leq m < 1 \leq \frac{(b + 1)^2}{4b} \Rightarrow (b - 1)^2 - ((b + 1)^2 - 4mb) = 4b(m - 1) < 0 \Rightarrow N_1^* < 0 < N_2^* < 1
\]
\[ 0 \leq m < 1 \leq \frac{(b+1)^2}{4b} \Rightarrow \frac{4bm}{(b+1)^2} < 1 \iff \frac{bm}{(b+1)^2} < \frac{1}{4} < 1. \]

This implies that single-species model (1) has a unique interior equilibrium \( N^*_2 = b - 1 + \sqrt{(b+1)^2 - 4bm/2b} \) such that

\[ F(N) > N \text{ if } 0 \leq N < N^*_2 \quad \text{and} \quad F(N) < N \quad \text{if } N > N^*_2. \]

Since \( F'(N) = [1 + rN(-1 + bm/(1 + bN)^2)]e^{(1-N-m/(1+bN))} \), then the local stability of \( N^*_2 \) is determined by the values of \( F'(N^*_2) = 1 + rN^*_2(-1 + bm/(1 + bN^*_2)^2) \). Thus, if \( -2 < \frac{-b}{rN^*_2(-1 + bm/(1 + bN^*_2)^2)} < 0 \), then \( N^*_2 \) is locally asymptotically stable. The fact that \( 0 \leq m < 1 \) holds indicates the following inequality holds:

\[ rN^*_2 \left( -1 + \frac{bm}{(1 + bN^*_2)^2} \right) = \frac{r(b - 1 + \sqrt{(b+1)^2 - 4bm})}{2b} \times \left( -1 + \frac{4bm}{(b+1 + \sqrt{(b+1)^2 - 4bm})^2} \right) \leq \frac{r(b - 1 + \sqrt{(b+1)^2 - 4bm})}{2b} \left( -1 + \frac{4bm}{(b+1)^2} \right) < 0. \]

This implies that \( F'(N^*_2) < 1 \) and \( -1 + \frac{bm}{(1 + bN^*_2)^2} < 0 \) if \( 0 \leq m < 1 \). Therefore, \( N^*_2 \) is locally asymptotically stable if

\[ rN^*_2 \left( -1 + \frac{bm}{(1 + bN^*_2)^2} \right) > -2 \iff r < \frac{2}{N^*_2 \left( 1 - \frac{bm}{(1 + bN^*_2)^2} \right)}. \]

Also notice that \( N^*_2 \leq 1 \), thus \( N^*_2 \) is locally asymptotically stable if \( 0 < r < 2 \).

Now we should show that \( F(N) \) has a unique critical point. Note that

\[ F'(N) = \left[ 1 + rN \left( -1 + \frac{bm}{(1 + bN)^2} \right) \right] e^{(1-N-m/(1+bN))} \]

\[ = -\left[ \frac{b^2rN^3 + b(2r - b)N^2 - (bmr + 2b - r)N - 1}{(1 + bN)^2} \right] e^{(1-N-m/(1+bN))}. \]

Let \( g(N) = b^2rN^3 + b(2r - b)N^2 - (bmr + 2b - r)N - 1 \), then we have

\[ g'(N) = 3b^2rN^2 + 2b(2r - b)N - (bmr + 2b - r) \]

\[ = 3b^2r \left[ N - \frac{b - 2r - \sqrt{3bmr^2 + (b + r)^2}}{3rb} \right] \left[ N - \frac{b - 2r + \sqrt{3bmr^2 + (b + r)^2}}{3rb} \right] \]

which implies that \( g(N) \) has at least one critical point \( b - 2r - \sqrt{3bmr^2 + (b + r)^2}/3rb \) being negative. Notice that \( g(0) = -1 < 0 \), this indicates that \( g(N) \) has only one positive root, therefore, \( F(N) \) has a unique critical point.

From discussion above, we can conclude that the map \( F(N) \) is unimodal that satisfies Assumption 9.4 in the book of Thieme [48]. Therefore, according to Corollary 9.9 of Thieme [48],
we can conclude that system (1) has global stability at $N^*_2$ if $0 \leq m < 1$ and the inequality (2)
\[ \min_{N > 0} \{0, e^{-f(N)}(1 - m) e^{(1 - N - m/(1 + bN_2^2))} - 1 \} > 0 \]
and the inequality (2) $\min_{N > 0} \{1 - rN \} e^{(1 - N - m/(1 + bN_2^2))} > -1$ holds.

In the case that $m = 0$, System (1) becomes the well-known Ricker’s map
\[ N_{t+1} = N_t e^{(1 - N_t)} \]
and the inequality (2) $\min_{N > 0} \{1 - rN \} e^{(1 - N - m/(1 + bN_2^2))} > -1$ holds when $0 < r < 2$ [8]. This indicates that the Ricker map has the global stability at $N^*_2 = 1$ when $0 < r < 2$.

Now assume that $bm < 1$. Let $F(N) = Nf(N)$ where $f(N) = e^{(1 - N - m/(1 + bN_2^2))}$, then we have
\[ f'(N) = r \left[ -1 + \frac{bm}{(1 + bN)^2} \right] f(N) < r[-1 + bm]f(N) < 0 \text{ if } bm < 1. \]

Define $h(y) = -\ln(f(N_2^* e^{-y})) = -r[1 - N_2^* e^{-y} - m/(1 + bN_2^* e^{-y})]$]. Then the Schwarzian derivative of $h(y)$ is negative everywhere when $bm < 1$ since
\[ (Sh)(y) = \frac{h''(y)}{h'(y)} - \frac{3}{2} \left( \frac{h''(y)}{h'(y)} \right)^2 \]
\[ = -\frac{(N_2^* e^{-y})^2(N_2^* e^{-y})^2 + 4bN_2^* e^{-y} + 10mb + 6) + (bm - 1)^2 + 4bN_2^* e^{-y}(1 - bm)}{(N_2^* e^{-y} + b)^2 - b^2 - mb + 1^2}. \]

This indicates that the map $F(N)$ satisfies Conditions stated in Corollary 2.7 of Eduardo [13], therefore, we can conclude that if $m < \min\{1/b, 1\}$ then $F(N)$ has global stability at $N^*_2$ whenever local stability at $N^*_2$. Therefore, $F(N)$ has global stability at $N^*_2$ if the following inequalities hold:
\[ m < \min \left\{ \frac{1}{b}, 1 \right\} \text{ and } r < \frac{2}{N_2^*(1 - bm/(1 + bN_2^2))}. \]

\subsection{Proof of Lemma 2.4}

Proof The positive invariance follows directly from continuity argument since $N = 0$ and $P = 0$ are invariant manifolds for system (3). We now show the boundedness of system (3) in $\mathbb{R}_+^2$.

For any $N_t \geq 0, P_t \geq 0$, we have
\[ N_{t+1} = N_t e^{(1 - N_t - \frac{m}{1 + bN_2^2}) - aP_t} \leq N_t e^{(1 - N_t - m/(1 + bN_2^2))} \leq \frac{e^{r-1}}{r}, \]
\[ P_{t+1} = N_t e^{(1 - N_t - \frac{m}{1 + bN_2^2})}(1 - e^{-aP_t}) = N_{t+1}(e^{aP_t} - 1) \leq N_t e^{(1 - N_t - m/(1 + bN_2^2))} \leq \frac{e^{r-1}}{r}. \]

Thus, after the first two iterations, system (3) maps $\mathbb{R}_+^2$ to the compact set $[0, e^{r-1}/r] \times [0, e^{r-1}/r]$. \qed
6.4. Proof of Theorem 3.1

Proof When \( P = 0 \), system (3) is reduced to Equation (1), and its Jacobian matrix takes the form of Equation (1) at the boundary equilibrium \((\hat{N}, 0)\):

\[
J = \begin{bmatrix}
1 + r\hat{N} & \frac{mb}{(1 + bN)^2} - 1 & -a\hat{N} \\
0 & a\hat{N} & \end{bmatrix}
\] (7)

with eigenvalues \( \lambda_1 = 1 + r\hat{N}(mb/(1 + b\hat{N})^2 - 1) \) and \( \lambda_2 = a\hat{N} \). The values of \( \hat{N} \) can be 0, \( N_1^* \) or \( N_2^* \) determined by \( r, m, b \).

From Lemma 2.1, we know that system (3) has only the trivial boundary equilibrium \( E_0 = (0, 0) \) which is global stable if \( 4mb > (b + 1)^2 \). Since system (3) is positively invariant and attracted to a compact set according to Lemma 2.4, thus we have the following inequality:

\[
N_{t+1} \leq N_t e^{r(1-N_t-m/2bN_t)},
\]

which converges to 0 with the consequence that population of parasite also converge to 0. Therefore, \( E_0 \) is global stable when the inequality \( 4mb > (b + 1)^2 \) holds. In addition, Lemma 2.1 implies that system (3) has only two boundary equilibria \( E_0 = (0, 0) \) and \( E_2 = (N_2^*, 0) \) if \( 0 \leq m < 1 \); and it has three boundary equilibria \( E_0 = (0, 0), E_1 = (N_1^*, 0), E_2 = (N_2^*, 0) \) if \( b > 1, 1 < m < (b + 1)^2/4b \).

At the boundary equilibrium point \((0, 0)\), the Jacobian matrix (7) takes the form of

\[
J|_{(0,0)} = \begin{bmatrix}
e^{r(1-m)} & 0 \\
0 & 0 \end{bmatrix}
\]

with eigenvalues \( \lambda_1 = e^{r(1-m)} \) and \( \lambda_2 = 0 < 1 \). Hence, \( E_0 \) is locally asymptotically stable if \( m > 1 \) and is a saddle if \( m < 1 \). Notice that \( 4mb > (b + 1)^2 \Rightarrow m > 1 \), thus, \( E_0 \) is locally asymptotically stable if \( 4mb > (b + 1)^2 \).

At the boundary equilibrium point \((N_1^*, 0)\), the Jacobian matrix (7) takes the form of

\[
J|_{(N_1^*,0)} = \begin{bmatrix}
1 + rN_1^* & \frac{mb}{(1 + bN_1^*)^2} - 1 & -aN_1^* \\
0 & aN_1^* \end{bmatrix}
\]

with eigenvalues \( \lambda_1 = 1 + rN_1^*(mb/(1 + bN_1^*)^2 - 1) \) and \( \lambda_2 = aN_1^* \). Notice that \( N_1^* = b - 1 - \sqrt{(b + 1)^2 - 4mb/2b} \) and

\[
-(b + 1) + \sqrt{(b + 1)^2 - 4mb} < 0,
\]

\[
\Rightarrow -2(b + 1)\sqrt{(b + 1)^2 - 4mb} + (\sqrt{(b + 1)^2 - 4mb})^2 + (b + 1)^2 - 4mb < 0,
\]

\[
\Rightarrow [(b + 1) - \sqrt{(b + 1)^2 - 4mb})^2 < 4mb \Rightarrow \frac{4mb}{(b + 1 - \sqrt{(b + 1)^2 - 4mb})^2} > 1.
\]

Therefore,

\[
\lambda_1 = 1 + rN_1^* \left( \frac{mb}{(1 + bN_1^*)^2} - 1 \right) = 1 + rN_1^* \left( \frac{4mb}{(b + 1 - \sqrt{(b + 1)^2 - 4mb})^2} - 1 \right) > 1.
\]

This implies that \((N_1^*, 0)\) is always unstable where it is a saddle if \( \lambda_2 = aN_1^* < 1 \) and it is a source if \( \lambda_2 > 1 \).
At the boundary equilibrium point \((N^*_2, 0)\), the Jacobian matrix \((7)\) takes the form of

\[
J|_{(N^*_2,0)} = \begin{bmatrix}
1 + rN^*_2 \left( \frac{mb}{(1 + bN^*_2)^2} - 1 \right) & -aN^*_2 \\
0 & aN^*_2
\end{bmatrix},
\]

where its two eigenvalues are

\[
\lambda_1 = 1 + rN^*_2 \left( \frac{mb}{(1 + bN^*_2)^2} - 1 \right) \quad \text{and} \quad \lambda_2 = aN^*_2.
\]

The equilibrium point \((N^*_2, 0)\) is locally asymptotically stable if both the eigenvalues lies within the unit circle. Notice that

\[
N^*_2 = b - 1 + \sqrt{(b + 1)^2 - 4bm/2b},
\]

thus, we have

\[
\lambda_1 = 1 + rN^*_2 \left( \frac{mb}{(1 + bN^*_2)^2} - 1 \right) = 1 + rN^*_2 \left( \frac{4mb}{(b + 1 + \sqrt{(b + 1)^2 - 4mb})^2} - 1 \right).
\]

Thus, if \(-2 < rN^*_2 (4mb/(b + 1 + \sqrt{(b + 1)^2 - 4mb})^2 - 1) < 0\), then we have \(|\lambda_1| < 1\). Therefore, \((N^*_2, 0)\) is locally asymptotically stable if

\[
-2 < rN^*_2 \left( \frac{4mb}{(b + 1 + \sqrt{(b + 1)^2 - 4mb})^2} - 1 \right) < 0 \quad \text{and} \quad aN^*_2 < 1.
\]

From proof of Theorem 2.2, we know that if \(0 \leq m < 1\), then we have \(N^*_2 \leq 1\) and \(1 > mb/(1 + bN^*_2)^2 > mb/(1 + b)^2\), therefore,

\[
0 < r < 2
\]

\[
\Rightarrow rN^*_2 \left( 1 - \frac{mb}{(1 + bN^*_2)^2} \right) < 2 \left( 1 - \frac{mb}{(1 + bN^*_2)^2} \right)
\]

\[
\Rightarrow -2 < -2 \left( 1 - \frac{mb}{(1 + bN^*_2)^2} \right) < rN^*_2 \left( -1 + \frac{mb}{(1 + bN^*_2)^2} \right)
\]

\[
\Rightarrow |\lambda_1| < 1.
\]

Thus, \((N^*_2, 0)\) is locally asymptotically stable if the following inequities hold:

\[
0 < r < 2, \ 0 \leq m < 1 \quad \text{and} \quad a < \frac{1}{N^*_2} = \frac{2b}{b - 1 + \sqrt{(b + 1)^2 - 4bm}}.
\]

6.5. Proof of Theorem 3.2

**Proof** From the work of Kang et al. [30], we know that system (4) (i.e. \(m = 0\) in Equation (3)) has no interior equilibrium if \(0 < r < 2, \ a < 1\). Since \(r(1 - N) > r(1 - N - m/(1 + bN))\) for all \(N > 0\), thus, system (3) has no interior equilibrium if \(0 < r < 2, \ a < 1\). The sufficient conditions for the existence of the different number of interior equilibria for systems (3) can be easily concluded from the graphic representation of its nullclines (Figure 2).
Assume that \((N^*, P^*)\) is an interior equilibrium of system (3), then for system (3) we have
\[
e^{aP^*} - 1 = \frac{P^*}{N^*} = \frac{f_1(N^*)}{N^*}.
\]

Thus, the Jacobian matrix of system (3) evaluated at \((N^*, P^*)\) can be described as Equation (8)
\[
J_a|_{(N^*, P^*)} = \begin{bmatrix}
1 + aN^*f_1'(N^*) - aN^* & -aN^* \\
\frac{f_1(N^*)(1 + aN^*f_1'(N^*))}{N^*} & aN^*
\end{bmatrix}.
\]

By the Jury test [12, p57], we see that the interior equilibrium \((N^*, P^*)\) is locally asymptotically stable for system (3) if the following inequalities hold:
\[
2 > 1 + \frac{a(N^*)^2 - af_1(N^*) + a^2N^*f_1'(N^*)(N^*)^2 - f_1(N^*)}{N^*}
\]
\[
> |\text{trace}(J_a)| = |1 + aN^*(1 + f_1'(N^*))|.
\]

Notice that
\[
f_1(N) = \frac{r(1 - N - m/(1 + bN))}{a} \quad \text{and} \quad f_1'(N) = \frac{r}{a} \left(-1 + \frac{bm}{(1 + bN)^2}\right),
\]
then we have
\[
\text{det}(J_a) = \left(1 - rN^* + \frac{bmrN^*}{(1 + bN^*)^2}\right) \left((a - r)N^* + r - \frac{rm}{1 + bN^*}\right)
\]
and
\[
\text{trace}(J_a) = 1 + N^* \left(a - r + \frac{bmr}{(1 + bN^*)^2}\right).
\]

Thus, the interior equilibrium \((N^*, P^*)\) of system (3) is locally asymptotically stable if
\[
1 > \left(1 - rN^* + \frac{bmrN^*}{(1 + bN^*)^2}\right) \left((a - r)N^* + r - \frac{rm}{1 + bN^*}\right)
\]
\[
> \left|1 + N^* \left(a - r + \frac{bmr}{(1 + bN^*)^2}\right)\right| - 1.
\]

6.6. Proof of Theorem 3.3

Proof According to Lemma 2.4, we know that the dynamics of system (3) can be restricted on a compact set \(C = [0, B] \times [0, B]\). According to Lemma 2.1, we also know that the host is persistent in its single state (i.e. \(P = 0\)) for system (3). In order to show that the host is persistent
in system (3), we only need to show that
\[
\sup_{t \to \infty} \liminf_{N_0 \to 0} \frac{N_{t+1}}{N_0} > 1 \quad \text{for any } P_0 \in [0, B]
\]
according to Theorem 2.2 in [22] but also see the related results by Hutson and Moran [23]. Since the omega limit set of \( N = 0 \) is \((0,0)\), then we have
\[
\sup_{t \geq 0} \liminf_{N_0 \to 0} \frac{N_{t+1}}{N_0} \geq e^{r(1-m)} > 1 \quad \text{for any } P_0 \in [0, B].
\]
Thus, the host is persistent in system (3) by applying Hutson’s [22] Theorem 2.2 and its Corollary 2.3 but also see the related results by Hutson and Moran [23] to obtain permanence of \( P \) in system (3) by using the average Lyapunov function \( H(N, P) = P \), i.e. we have
\[
\sup_{t \geq 0} \liminf_{N_0 \to 0} \frac{P_{t+1}}{P_0} \geq aN^*_2 > 1 \quad \text{for any } N_0 \in [0, B].
\]
Therefore, \( aN^*_2 > 1 \) and conditions in Theorem 2.2 hold implying that system (3) is permanent in \( \mathbb{R}^+_2 \).

6.7. Proof of Theorem 3.4

Proof According to Lemma 2.4, we see that system (3) is mapped into the compact set \([0, e^{-1/r}] \times [0, e^{-1/r}]\) after the first two iterations. From proof of Theorem 3.1, we know that system (3) has only the trivial boundary equilibrium \( E_0 = (0,0) \) which is locally asymptotically stable in the case that \( 4bm > (b+1)^2 \). Therefore, the host population of Equation (3) has the following property:
\[
N_{t+1} = N_t e^{r(1-N_t-m/(1+bN_t))} e^{-aP_t} \\
\leq N_t e^{r(1-N_t-m/(1+bN_t))} e^{-aP_t} \\
\leq N_t e^{\max_{0 \leq N \leq e^{-1/r}} [r(1-N-m/(1+bN))]} \\
\leq N_t e^{-h} \leq N_0 e^{-th} \to 0 \text{ as } t \to \infty.
\]
This indicates that for system (3), we have
\[
P_{t+1} = N_t e^{r(1-N_t-m/(1+bN_t))} (1 - e^{-aP_t}) \\
\leq N_t e^{r(1-N_t-m/(1+bN_t))} (1 - e^{-aP_t}) \\
\leq N_t e^{-h} \to 0 \quad \text{as } N_t \to 0.
\]
Therefore, if \( 4mb > (b+1)^2 \), then we have \( \lim_{t \to \infty} \max \{N_t, P_t\} = 0 \), i.e. system (3) has global stability at \((0,0)\).

From proof of Theorem 3.1, we know that system (3) has three boundary equilibrium \( E_0 = (0,0), E_1 = (N^*_1,0) \) and \( E_2 = (N^*_2,0) \) if \( b > 1, 1 < m < (b+1)^2/4b \). Let \( F(N) = \)
\(N \epsilon^{r(1-N-m/(1+bN_t))}\), then \(F(N_t) = N_t^*\) and there exists a unique \(\tilde{N} \in (N_1, N_2)\) such that \(F'(\tilde{N}) = 0\). This indicates that \(F(N)\) is increasing on \((0, \tilde{N})\) and \(F(N) < N\) for any \(N \in (0, N^*_1)\). Therefore,

\[
N_2 = N_1 \epsilon^{r(1-N_1-m/(1+bN_1))-aP_1} \\
\leq N_1 \epsilon^{r(1-N_1-m/(1+bN_1))} < N_1 < N_0.
\]

This implies that \(\Omega_1 = \{(N, P) \in \mathbb{R}^2_+: N < N_1^*\}\) is positively invariant for system (3). In addition, by induction, we have \(N_t \to 0\) as \(t \to \infty\), thus, \(P_t \to 0\) as \(t \to \infty\). Therefore, we can conclude that if \(N_0 < N_1^*\) then the solution of system (3) converges to \((0, 0)\).

If \(ae^{r-1}/r < 1\), then for system (3), we have

\[
P_{t+1} = N_t \epsilon^{r(1-N_t-m/(1+bN_t))}[1 - e^{-aP_t}] \\
\leq \frac{e^{r-1}}{r} \frac{1 - e^{-aP_t}}{P_t} \leq aP_t e^{r-1} \\
\leq P_0 \left(\frac{ae^{r-1}}{r}\right)^t \to 0 \quad \text{as} \quad t \to \infty.
\]

Therefore, the population of parasitism of system (3) converges to 0 if \(ae^{r-1}/r < 1\). \(\blacksquare\)

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