List-Recoloring of Sparse Graphs

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Abstract

Fix a graph \(G\), a list-assignment \(L\) for \(G\), and \(L\)-colorings \(\alpha\) and \(\beta\). An \(L\)-recoloring sequence, starting from \(\alpha\), recolors a single vertex at each step, so that each resulting intermediate coloring is a proper \(L\)-coloring. An \(L\)-recoloring sequence transforms \(\alpha\) to \(\beta\) if its initial coloring is \(\alpha\) and its final coloring is \(\beta\). We prove there exists an \(L\)-recoloring sequence that transforms \(\alpha\) to \(\beta\) and recolors each vertex at most a constant number of times if (i) \(G\) is triangle-free and planar and \(L\) is a 7-assignment, or (ii) \(\text{mad}(G) < 17/5\) and \(L\) is a 6-assignment or (iii) \(\text{mad}(G) < 22/9\) and \(L\) is a 4-assignment. Parts (i) and (ii) confirm conjectures of Dvořák and Feghali.

1 Introduction

A proper \(k\)-coloring of a graph \(G\) assigns each vertex \(v\) of \(G\) a “color” from \(\{1, \ldots, k\}\) so that the endpoints of each edge get distinct colors. A recoloring step changes the color of a single vertex, so that the resulting coloring is also proper. Given two proper \(k\)-colorings of \(G\), say \(\alpha\) and \(\beta\), we want to know if we can transform \(\alpha\) to \(\beta\) by a sequence of recoloring steps, again requiring that each intermediate coloring is a proper \(k\)-coloring. If we can, then we might ask for the “distance” from \(\alpha\) to \(\beta\), the length of a shortest sequence of recoloring steps that transforms \(\alpha\) to \(\beta\)? And, more generally, what is the maximum distance between any pair \(\alpha\) and \(\beta\)? Such questions have been studied extensively, under the name Glauber dynamics, due to their applications in statistical physics; for example, see [14], [8] and their references.

Before going further, we will rephrase the problem in a bit more generality. For brevity, we defer many well-known definitions to the end of Section 2. Rather than considering \(k\)-colorings of \(G\), we fix a list-assignment \(L\) such that \(|L(v)| = k\) for all vertices \(v\), and we consider \(L\)-colorings. Given a proper \(L\)-coloring of \(G\), an \(L\)-recoloring step changes the color of a single vertex, so that the resulting coloring is also a proper \(L\)-coloring. Now we ask the same questions as above.

Given \(L\)-colorings \(\alpha\) and \(\beta\), can we transform \(\alpha\) to \(\beta\) by a sequence of \(L\)-recoloring steps, requiring that each intermediate coloring is a proper \(L\)-coloring? If so, what is the length of a shortest such recoloring sequence, the “distance” from \(\alpha\) to \(\beta\)? And what is the maximum distance over all pairs \(\alpha\) and \(\beta\)? Formally, the \(L\)-recoloring graph of \(G\), denoted \(R_L(G)\), has as its vertices all \(L\)-colorings of \(G\), and two vertices of the \(L\)-recoloring graph are adjacent if the \(L\)-colorings differ on a single vertex of \(G\). That is, what is the diameter of \(R_L(G)\)? We can also go further and ask about the maximum over all \(k\)-assignments \(L\).

Intuitively, all pairs of \(L\)-colorings should be “near” to each other when \(k\) is sufficiently large, relative to \(G\). But what does near mean? Rather than focusing on a single graph \(G\), we typically consider families \(\mathcal{G}\) of graphs \(G\) and study the maximum of \(\text{diam}(R_L(G))\) over all \(n\)-vertex graphs \(G \in \mathcal{G}\) and all \(k\)-assignments \(L\) for \(G\). Specifically, we study how this maximum

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Theorem 1. Let $G$ be a planar triangle-free graph. Fix a 7-assignment $L$ for $G$ and L-colorings $\alpha$ and $\beta$. There exists an $L$-recoloring sequence that transforms $\alpha$ into $\beta$ such that each vertex is recolored at most 30 times.

Theorem 2. Fix a graph $G$ with $\text{mad}(G) < 17/5$, a 6-assignment $L$ for $G$, and $L$-colorings $\alpha$ and $\beta$. There exists an $L$-recoloring sequence that transforms $\alpha$ into $\beta$ such that each vertex is recolored at most 12 times. In particular, this holds for every planar graph of girth at least 5.

Theorem 3. Fix a graph $G$ with $\text{mad}(G) < 22/9$, a 4-assignment $L$ for $G$, and $L$-colorings $\alpha$ and $\beta$. There exists an $L$-recoloring sequence that transforms $\alpha$ into $\beta$ such that each vertex is recolored at most 14 times. In particular, this holds for every planar graph of girth at least 11.

Theorems 1 and 2 confirm conjectures of Dvořák and Feghali [11]. Before discussing the proofs of Theorems 1-3 we say more about the history of recoloring and related problems.

It is very natural to want to transform one instance of something to another instance, by a sequence of small steps, maintaining a valid instance at each step. We can apply this paradigm to colorings, dominating sets, independent sets, perfect matchings, spanning trees, or solutions to a SAT problem, to name a few. Once we have specified the type of instance we want, then we must specify the allowable modifications to move from one instance to the next. For SAT problems, we typically restrict to flipping the true/false value of a single variable.

But for some problems, the “best” choice of allowable modification is less clear. When moving from one (list-)coloring to the next, we might restrict to recoloring a single vertex; or we might allow the more powerful notion of a Kempe swap. When moving from one independent set $I$ to another, we might allow replacing any vertex in $I$ with any vertex not in $I$. Or we might require that the replaced vertex be adjacent to the vertex replacing it (so-called, token sliding). All of these choices give rise to interesting problems. And this general area of study is known as reconfiguration. An early reference on this topic is [10]. For a more recent survey, see [15].

Much work has focused on coloring reconfiguration. Recently, [11] completed the characterization of pairs $(g,k)$ such that for all planar graphs $G$ of girth at least $g$, every $k$-coloring of $G$ can be transformed to every other. Similar problems have been studied for graphs of bounded maximum average degree [13], graphs of bounded treewidth [2], and particularly $d$-degenerate graphs [7]. Now much emphasis has shifted to bounding the diameter of these recoloring graphs [6, 4, 5, 12].

The present paper is motivated by work of Dvořák and Feghali [10, 11]. They showed that if an $n$-vertex graph $G$ is planar, then any 10-coloring can be transformed to any other and, more strongly, that the corresponding recoloring graph has diameter at most $8n$. They proved an analogous result for 7-colorings of planar triangle-free graphs. Further, they conjectured that the same result holds more generally for $L$-colorings from an arbitrary list-assignment $L$ where, for all $v$, we have $|L(v)| \geq 7$. Theorem 1 confirms this conjecture. They also conjectured an analogous result for 6-colorings of planar graphs with girth at least 5. Theorem 2 confirms this latter conjecture. Bonamy et al. [3] showed that the 3-recoloring graph of the $n$-vertex path $P_n$...
has diameter $\Theta(n^2)$. Thus, if we are trying to prove linear upper bounds on the distance between any two $L$-colorings of $G$, we typically consider list-assignments $L$ where always $|L(v)| \geq 4$. Thus, Theorem 3 is the natural continuation of this line of study to sparser graphs.

2 Preliminaries

An $L$-recoloring sequence consists of an $L$-coloring and a sequence of $L$-recoloring steps, so that each later coloring is a proper $L$-coloring. An $L$-recoloring sequence is $k$-good if each vertex is recolored at most $k$ times. An $L$-coloring $\alpha$ of $G$ restricted to a subgraph $G'$ is denoted $\alpha_{G'}$. We will often extend an $L$-recoloring sequence $\sigma'$ that transforms $\alpha_{G'}$ to $\beta_{G'}$ to an $L$-recoloring sequence $\sigma$ that transforms $\alpha$ to $\beta$. This means that restricting $\sigma$ to $V(G')$ results in $\sigma'$.

The following Key Lemma is simple but powerful. It has been used implicitly in many papers, and first appeared explicitly in [1]. We phrase it in the slightly more general language of list-coloring, although the proof is identical. In fact, the reader familiar with correspondence coloring will note that the proof works equally well in that, still more general, context, too. The same is true of all the proofs in this paper.

**Key Lemma.** Fix a graph $G$, a list-assignment $L$ for $G$, and $L$-colorings $\alpha$ and $\beta$. Fix a vertex $v$ with $|L(v)| \geq d(v) + 1$ and let $G' := G - v$. Fix an $L$-recoloring sequence $\sigma'$ for $G'$ transforming $\alpha_{G'}$ to $\beta_{G'}$. If $\sigma'$ recolors $N_G(v)$ a total of $t$ times, then we can extend $\sigma'$ to an $L$-recoloring sequence for $G$ transforming $\alpha$ into $\beta$ and recoloring $v$ at most $\lceil t/(|L(v)| - d_G(v) - 1) \rceil + 1$ times.

**Proof.** Let $c_1, c_2, \ldots$ denote the sequence of colors used by $\sigma'$ to recolor vertices in $N(v)$, in order and possibly with repetition. Let $s := |L(v)| - d_G(v) - 1$. Immediately before $\sigma'$ is to recolor a vertex in $N(v)$ with $c_1$, we recolor $v$ with a color distinct from $c_1, \ldots, c_s$ and also distinct from all colors currently used on $N(v)$. This is possible precisely because $|L(v)| = s + d_G(v) + 1$. Immediately before $\sigma$ is to recolor a vertex in $N(v)$ with $c_{s+1}$, we recolor $v$ with a color distinct from $c_{s+1}, \ldots, c_{2s}$ (and those currently used on $N(v)$). Continuing in this way, we extend $\sigma'$ to an $L$-recoloring sequence that recolors $v$ at most $\lceil t/s \rceil$ times and transforms $\alpha$ to a coloring $\beta'$ that agrees with $\beta$ everywhere except possibly on $v$. Finally, if needed, we recolor $v$ with $\beta(v)$. So $v$ is recolored at most $\lceil t/s \rceil + 1$ times.

For easy reference we include the following lemma, which is folklore.

**Mad Lemma.** If $G$ is a planar graph with girth at least $g$, then $\text{mad}(G) < 2g/(g - 2)$.

For completeness we conclude this short section with some standard definitions. We denote the degree of a vertex $v$ by $d(v)$. For a planar graph $G$ and a face $f$ in a plane embedding of $G$, we denote the length of $f$ by $\ell(f)$. A $k$-vertex is a vertex of degree $k$. A $k^+$-vertex (resp. $k^-$-vertex) is one of degree at least (resp. at most) $k$. A $k/k^+ / k^- $-neighbor of a given vertex is an adjacent $k/k^+ / k^-$-vertex. A $L$-assignment $L$ for a graph $G$ gives each vertex $v$ a list $L(v)$ of allowable colors. If $|L(v)| = k$ for all $v$, then $L$ is a $k$-assignment. An $L$-coloring is a proper coloring $\alpha$ such that $\alpha(v) \in L(v)$ for all $v$. The maximum average degree of a graph $G$ is the maximum, over all nonempty subgraphs $H$ of $G$, of the average degree of $H$; it is denoted $\text{mad}(G)$. That is, $\text{mad}(G) := \max_{H \subseteq G} 2|E(H)|/|V(H)|$.

3 Triangle-free Planar Graphs: Lists of Size 7

In this section we prove Theorem 3 to simplify the proof, it is convenient to extract the following structural lemma about triangle-free planar graphs.

**Lemma 1.** If $G$ is triangle-free and planar and $\delta(G) \geq 3$, then $G$ contains one of the following four configurations:
(a) a 5-vertex with at least three 3-neighbors,
(b) a path with each endpoint of degree 3 and at most three internal vertices, all of degree 4,
(c) a 4-face with an incident 3-vertex and three incident 4-vertices, or
(d) a 4-face \( f \) with degree sequence \( (3,4,5,4) \) or \( (3,4,4,5) \) such that the 5-vertex on \( f \) is also adjacent to a 3-vertex not on \( f \).

**Proof.** Assume the lemma is false, and let \( G \) be a counterexample. We use discharging, giving each vertex \( v \) initial charge \( 2d(v) - 6 \) and each face \( f \) initial charge \( \ell(f) - 6 \). By Euler’s formula, the sum of all initial charges is \(-12\). We use the following three discharging rules.

(R1) Each 4-vertex gives \( 1/2 \) to each incident face.
(R2) Each 6\(^+\)-vertex gives 1 to each incident face.
(R3) After receiving charge from (R1) and (R2), if any face \( f \) still needs charge, then it takes that charge equally from all incident 5-vertices.

Now we show that each vertex and face ends with nonnegative charge. This contradicts that the sum of the initial charges is \(-12\), and proves the lemma. We write \( \text{ch}^*(v) \) to denote the final charge of \( v \), that is, the charge of \( v \) after applying each of (R1)–(R3).

By assumption, \( G \) has no 2\(^-\)-vertices. Each 3-vertex starts and ends with charge 0. Each 4-vertex \( v \) has \( \text{ch}^*(v) = 2d(v) - 6 - 4(1/2) = 0 \). And each 6\(^+\)-vertex \( v \) has \( \text{ch}^*(v) \geq 2d(v) - 6 - d(v) \geq 0 \), since \( d(v) \geq 6 \). Clearly each 6\(^+\)-face finishes with charge nonnegative. By (R1), (R2), and (b), the same is true for each 5-face; and by (R3), the same is true for each face with an incident 5-vertex. So assume that \( f \) is a 4-face with no incident 5-vertex. By (R1), (R2), and (b), it is easy to check that \( f \) finishes with charge nonnegative; here we use that (c) is forbidden.

Thus, what remains is to show that each 5-vertex finishes with charge nonnegative. First, we classify the faces (and degrees of their incident vertices) that take more than \( 1/2 \) from each incident 5-vertex. It is easy to check that if \( f \) takes more than \( 1/2 \) from an incident 5-vertex, then \( f \) has one of the six types shown in Figure 1. Further, if a 5-vertex \( v \) is incident to at most one face of types 1–3, then each other face incident to \( v \) takes at most \( 3/4 \) from \( v \), so \( \text{ch}^*(v) \geq 2(5) - 6 - 1 - 4(3/4) = 0 \). Thus, below we assume that \( v \) is incident to at least two faces of types 1–3.

![Figure 1: The six types of 4-faces, up to rotation and reflection, that take charge more than 1/2 from each incident 5-vertex v.

In this rule and those following, if a vertex \( v \) is a cut-vertex and is incident to a face \( f \) at multiple points on a walk along the boundary of \( f \), then \( v \) gives \( f \) charge for each incidence.

\[ 4 \]
Suppose that a 5-vertex \( v \) is incident to a 4-face of type 1. By (d), \( v \) is not incident to any 4-face of type 2 or type 3. And by (a), \( v \) is not incident to another 4-face of type 1. Thus, \( v \) is incident with at most one face of type 1–3, a contradiction.

Suppose that a 5-vertex \( v \) is incident to a 4-face of type 2. By (d), \( v \) is not incident to any 4-face of type 3. So assume that \( v \) is incident to (exactly) two 4-faces of type 2. By (b), these faces do not share a 4-vertex. Let \( f_1, \ldots, f_5 \) denote the 5 faces incident to \( v \) (with multiplicity, if \( v \) is a cut-vertex) in cyclic order. By symmetry, we assume that \( f_1 \) and \( f_2 \) are type 2. However, now \( f_2 \) receives at most 1/2 from \( v \). Thus, \( \text{ch}^*(v) \geq 2(5) - 6 - 2(1) - 1/2 - 2(3/4) = 0 \).

Suppose that a 5-vertex \( v \) is incident to a 4-face of type 3. In fact, \( v \) must be incident to two such faces. By (d), these type 3 faces must share a common 3-vertex. So, by symmetry, we assume that \( f_2 \) and \( f_3 \) are type 3, and they share a 3-vertex. By (d), \( v \) has no other 3-neighbor. And by (b), each of \( f_1 \) and \( f_4 \) is either a 5\(^{\ast}\)-face or a 4-face with no incident 3-vertex. Thus, \( f_1 \) and \( f_4 \) each take at most 1/2 from \( v \). So \( \text{ch}^*(v) \geq 2(5) - 6 - 2(1) - 3/4 - 2(1/2) > 0 \). \qed

**Theorem 1.** Let \( G \) be a triangle-free planar graph, and let \( L \) be a 7-assignment for \( G \). If \( \alpha \) and \( \beta \) are \( L \)-colorings of \( G \), then \( \alpha \) can be transformed to \( \beta \) by recoloring each vertex at most 30 times, so that every intermediate coloring is a proper \( L \)-coloring.

**Proof.** Suppose the theorem is false, and let \( G \) be a smallest counterexample. We show that \( \delta(G) \geq 3 \), so Lemma 1 applies. We then show that each of configurations (a)–(d) in Lemma 1 is reducible for the present theorem. Thus, no counterexample exists, and the theorem is true.

Recall that an \( L \)-recoloring sequence is 30-good if it recolors each vertex at most 30 times. If \( G \) contains a 2\(^{\ast}\)-vertex \( v \), then by minimality \( G - v \) has a 30-good recoloring sequence \( \sigma' \) to transforms \( \alpha|_{G-v} \) to \( \beta|_{G-v} \). By the Key Lemma, we extend \( \sigma' \) to a 30-good recoloring sequence \( G \) that recolors \( v \) at most \((30 + 30)/4 + 1 = 16 \) times. So \( \delta(G) \geq 3 \), and Lemma 1 applies.

(a) Suppose \( G \) contains a 5-vertex \( v \) with neighbors \( w_1, \ldots, w_5 \) such that \( d(w_1) = d(w_2) = d(w_3) = 3 \). Let \( G' := G - \{w_1, w_2, w_3\} \) and \( G'' := G' - v \). By minimality, \( G'' \) has a 30-good recoloring sequence \( \sigma'' \) transforming \( \alpha|_{G''} \) to \( \beta|_{G''} \). By the Key Lemma, we can extend \( \sigma'' \) to a 30-good recoloring sequence \( \sigma' \) for \( G' \) transforming \( \alpha|_{G'} \) to \( \beta|_{G'} \) that recolors \( v \) at most \((30 + 30)/4 + 1 = 16 \) times. By applying the Key Lemma to each of \( w_1, w_2, w_3 \) (in succession), we can extend \( \sigma' \) to a 30-good recoloring sequence \( \sigma \) for \( G \) that recolors each of \( w_1, w_2, w_3 \) at most \([(30 + 30 + 16)/3] + 1 = 27 \) times.

(b) Suppose \( G \) contains a path \( v_1v_2v_3v_4v_5 \) with \( d(v_1) = d(v_5) = 3 \) and \( d(v_2) = d(v_3) = d(v_4) = 4 \). Let \( G' := G - \{v_1, v_5\} \), let \( G'' := G' - \{v_2, v_4\} \), and let \( G''' := G'' - \{v_3\} \). By minimality, \( G''' \) has a 30-good \( L \)-recoloring sequence \( \sigma''' \) transforming \( \alpha|_{G'''} \) to \( \beta|_{G'''} \). By the Key Lemma, we extend \( \sigma''' \) to a 30-good \( L \)-recoloring sequence \( \sigma'' \) for \( G'' \) transforming \( \alpha|_{G''} \) to \( \beta|_{G''} \) that recolors \( v_3 \) at most \((30 + 30)/4 + 1 = 16 \) times. By using the Key Lemma twice, we extend \( \sigma'' \) to a 30-good \( L \)-recoloring sequence \( \sigma' \) for \( G' \) transforming \( \alpha|_{G'} \) to \( \beta|_{G'} \) that recolors each of \( v_2 \) and \( v_4 \) at most \([(30 + 30 + 16)/3] + 1 = 27 \) times. Finally, by using the Key Lemma twice more, we extend \( \sigma' \) to a 30-good \( L \)-recoloring sequence \( \sigma \) for \( G \) transforming \( \alpha \) to \( \beta \); note that we recolor each of \( v_1 \) and \( v_5 \), and at most \((30 + 30 + 27)/3 + 1 = 30 \) times. When the path has length \( t \), for a \( t \leq 3 \), the analysis is similar, but ends after only extending \( \sigma''' \) to \( t + 1 \) vertices.

(c) Suppose \( G \) contains a 4-face \( f \) with degree sequence (3,4,4,4). The proof is nearly identical to that of (b). Only now the 3-vertex is recolored at most \((30 + 27 + 27)/3 + 1 = 29 \) times.

(d) Suppose \( G \) contains a 4-face \( f \) with degree sequence (3,4,5,4) or (3,4,4,5) and the 5-vertex \( v \) on \( f \) is also adjacent to a 3-vertex \( w \) not on \( f \). Form \( G' \) from \( G \) by deleting \( w \), and form \( G'' \) from \( G' \) by deleting the vertices of \( f \). As in (c) and (b), \( G'' \) has a 30-good \( L \)-recoloring sequence, and we can extend it a 30-good \( L \)-recoloring sequence for \( G' \) that transforms \( \alpha|_{G'} \) to \( \beta|_{G'} \) so that each vertex other than the 3-vertex is recolored at most 27 times. (We extend to the vertices of \( f \) in order of decreasing distance from its 3-vertex.) By the Key Lemma we can extend \( \sigma' \) to a 30-good \( L \)-recoloring sequence for \( G \) that transforms \( \alpha \) to \( \beta \). This is because one neighbor of \( w \) is recolored by \( \sigma' \) at most 27 times. \qed
4 Graphs with Mad < 17/5: Lists of Size 6

The goal of this section is to prove Theorem 2. Very generally, the proof is similar to that of Theorem 1. However, the Key Lemma is not as well suited to working with lists of even size. So, rather than applying it directly, we will typically prove more specialized results that give better bounds in our particular cases. However, the proofs will be quite similar to that of the Key Lemma. This remark also applies to Section 5 since it considers lists of size 4.

**Theorem 2.** Let $G$ be a graph with $\text{mad}(G) < 17/5$. If $L$ is a $6$-assignment for $G$ and $\alpha$ and $\beta$ are $L$-colorings of $G$, then $\alpha$ can be transformed to $\beta$ by recoloring each vertex at most 12 times, so that every intermediate coloring is a proper $L$-coloring. In particular, this holds for every planar graph of girth at least 5.

**Proof.** By the Mad Lemma if $G$ is planar with girth at least 5, then $\text{mad}(G) < 10/3 < 17/5$. So, the second statement follows from the first. Now we prove the first.

Let $G$ be a graph with $\text{mad}(G) < 17/5$. We claim that $G$ contains either (i) a $2$-vertex or (ii) a $3$-vertex or (iii) a $4$-vertex with four $3$-neighbors. We assume that $G$ is a counterexample to the claim, and use discharging to reach a contradiction. We give each $G$ a $3$-vertex with at least two $3$-neighbors or (iii) a $4$-vertex with four $3$-neighbors. The key lemma is in order. By minimality, $\delta(G) \geq 3$. If $v$ is a $3$-vertex, then the absence of (ii) implies that $v$ has at least two $4$-neighbors. We claim that $\text{ch}^*(v) \geq 3 + 2(1/5) = 17/5$. If $v$ is a $4$-vertex, then the absence of (iii) implies that $v$ has at most three $3$-neighbors. So $\text{ch}^*(v) \geq 4 - 3(1/5) = 17/5$. If $v$ is a $5^+$-vertex, then $\text{ch}^*(v) \geq d(v) - d(v)/5 = 4d(v)/5 \geq 4$, since $d(v) \geq 5$.

We show that none of (i), (ii), and (iii) appears in a minimal counterexample to the theorem.

(i) Suppose $G$ has a $2^-$-vertex $v$. Let $G' := G - v$. By minimality, $G'$ has a 12-good $L$-coloring $\sigma'$ transforming $\alpha_{1|G'}$ to $\beta_{1|G'}$. By the Key Lemma, we extend $\sigma'$ to a 12-good $L$-coloring $\sigma''$ of $G$.

(ii) Suppose $G$ has a 3-vertex $v$ with at least two $3$-neighbors, $w_1$ and $w_2$. Let $G' := G - \{v, w_1, w_2\}$. By minimality, $G'$ has a 12-good $L$-coloring $\sigma'$ that transforms $\alpha_{1|G'}$ to $\beta_{1|G'}$. Let $a_1, a_2, \ldots$ denote the colors that are used by $\sigma'$ to recolor $N(w_1) \setminus \{v\}$, and $b_1, b_2, \ldots$ denote the colors that are used by $\sigma'$ to recolor $N(w_1) \setminus \{v, w_2\}$. (Each of $a_1, \ldots$ and $b_1, \ldots$ is in order and with repetition.) The proof is similar to that of the Key Lemma, but a bit more involved. For simplicity, we assume that $w_1 w_2 \notin E(G)$, since the other case is similar but easier.

When $\sigma'$ is about to recolor $N(w_1) \setminus \{v\}$ with $a_1$, we first recolor $w_1$ to avoid $\{a_1, a_2, a_3\}$, as well as the two colors currently used on $N(w_1) \setminus \{v\}$. If this requires coloring $w_1$ with the color currently on $v$, then before recoloring $w_1$ we recolor $v$ (as we will explain shortly). When $\sigma'$ is about to recolor $N(w_1) \setminus \{v\}$ with $a_4$, we first recolor $w_1$ to avoid $\{a_4, a_5, a_6\}$, again recoloring $v$ beforehand if needed. When $\sigma'$ is about to recolor $N(w_1) \setminus \{v\}$ with $a_7$, we first recolor $w_1$ to avoid $\{a_7, a_8\}$ and all colors currently used on $N(w_1)$. Thereafter, each time that $\sigma'$ specifies recoloring $N(w_1) \setminus \{v\}$ with the color currently on $w_1$, we first recolor $w_1$ to avoid the next two colors to be used by $\sigma'$ on $N(w_1) \setminus \{v\}$, as well as the three colors currently used on $N(w_1)$.

The number of times that $w_1$ is recolored is at most $2 + (24 - 6)/2 + 1 = 12$. We treat $w_2$ in exactly the same way as $w_1$. So what remains is to specify how we handle $v$.

Denote $N(v) \setminus \{w_1, w_2\}$ by $\{x\}$. Each time that $\sigma'$ is about to recolor $x$ with a color currently used on $v$, we first recolor $v$ to avoid the next two colors used by $\sigma'$ on $x$, as well as the three colors currently used on $N(v)$. Each time that some $w_i$ requires $v$ to be recolored (at most twice for each $w_i$ near the start, and at most once total near the end, for up to five times total, as we
explain below), before \( w_i \) is recolored, we recolor \( v \) to avoid its current color, the up to three colors currently used on \( N(v) \), as well as the next color that will be used on \( x \), i.e., the next \( c_j \).

Finally, assume that \( G \) has been recolored from \( \alpha \) to some \( \beta' \) that agrees with \( \beta \) except possibly on \( \{v, w_1, w_2\} \). Recolor \( v \) to avoid \( \{\beta'(w_1), \beta'(w_2), \beta(w_1), \beta(w_2), \beta(x)\} \), recolor each \( w_i \) to \( \beta(w_i) \), then recolor \( v \) to \( \beta(v) \). So how many times is \( v \) recolored? It might be recolored due to each \( w_i \) at most twice early on. It might be recolored \( 12/2 \) times due to colors used by \( \sigma^* \) on \( x \). It is subtle, but true, that recoloring \( v \) due to some \( w_i \) does not cause us to “lose ground” with respect to upcoming colors to be used on \( x \). Each time that we recolor \( v \) due to \( x \), we do so to avoid the next two colors to be used on \( x \). However, immediately after we recolor \( v \), we use the first of those colors on \( x \). So the color of \( v \) is chosen to avoid at most one upcoming color on \( x \). Whenever \( v \) is recolored due to some \( w_i \), the color for \( v \) is again chosen to avoid the next upcoming color used on \( x \). So the total number of times that \( v \) is recolored is \( 2 + 2 + 12/2 + 2 = 12 \).

(iii) Suppose \( G \) contains a 4-vertex \( v \) with four 3-neighbors, \( w_1, w_2, w_3, w_4 \). (Again, for simplicity we assume that \( w_i, w_j \not\in E(G) \) for all \( i, j \in \{1, 2, 3, 4\} \); the other cases are similar.) The proof is nearly identical to that for (ii) above, with each \( w_i \) being treated as \( w_1 \) and \( w_2 \) were above. Suppose that \( \alpha \) is transformed to some \( \beta' \) that agrees with \( \beta \) except on \( \{v, w_1, w_2, w_3, w_4\} \). Now each \( w_i \) such that \( \beta(w_i) \neq \beta'(v) \) is recolored to \( \beta(w_i) \). Next \( v \) is recolored to avoid its current color and those currently on \( N(v) \). Afterward, each remaining \( w_i \) is recolored to \( \beta(w_i) \). Finally, \( v \) is recolored to \( \beta(v) \). The analysis for each \( w_i \) is identical to that for (ii). And \( v \) is recolored at most \( 2(4) + 2 = 10 \) times.

For planar graphs of girth at least 5, we can improve the above result to a 10-good \( L \)-recoloring sequence. This is because such graphs contain either a 2

### 5 Graphs with Mad < 22/9: Lists of Size 4

The goal of this section is to prove the following result.

**Theorem 3.** Let \( G \) be a graph with \( \text{mad}(G) < 22/9 \). If \( L \) is a \( 4 \)-assignment for \( G \) and \( \alpha \) and \( \beta \) are \( L \)-colorings of \( G \), then \( \alpha \) be can be transformed to \( \beta \) by recoloring each vertex at most 14 times, so that every intermediate coloring is a proper \( L \)-coloring. In particular, this holds for every planar graph with girth at least 11.

Note that the latter statement follows from the former by the Mad Lemma so we prove the former. We assume \( G \) is a smallest counterexample, and again use discharging. This time we show that \( G \) has average degree at least \( 22/9 \), contradicting the hypothesis. Generally speaking, we send charge from \( 3^+ - \text{vertices} \), which begin with extra charge, to \( 2 \)-vertices, which begin needing more charge. To prove that each \( 3 \)-vertex ends with sufficient charge, we show that various configurations (see Section 5.1) cannot appear in our minimum counterexample \( G \).

A thread in \( G \) is a path\(^2\) with all internal vertices of degree 2. A \( k \)-thread is a thread with \( k \) internal vertices. A \( 3_{a,b,c} \)-vertex is a \( 3 \)-vertex that is the endpoint of a maximal \( a \)-thread, a maximal \( b \)-thread, and a maximal \( c \)-thread, all distinct. A weak neighbor of a \( 3^+ \)-vertex \( v \) is a \( k \)-thread

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\(^2\)We also allow a thread to be a cycle with a single \( 3^+ \)-vertex (which serves as both endpoints of the thread) and all other vertices of degree 2.
3^2-vertex w such that v and w are endpoints of a common thread. A 2-vertex v is nearby a 3^2-vertex w if v is an interior vertex of a thread with w as one endpoint.

Since the proof is longer than our previous proofs, we have one subsection for reducibility and another for discharging. In both subsections, G is a minimum counterexample to Theorem 1.

5.1 Reducibility

Lemma 2. G is connected and δ(G) ≥ 2.

Proof. If G is disconnected, then each component has a 14-good L-recoloring sequence, by minimality. Combining these gives a 14-good L-recoloring sequence for G. So G is connected. Suppose instead that G contains a 1-vertex v and let G' := G − v. By minimality, G' has a 14-good L-recoloring sequence σ' that transforms α_{G'} to β_{G'}. Since the neighborhood of v is recolored at most 14 times, by the Key Lemma we can extend σ' to a 14-good recoloring sequence for G that recolors v at most 14/2 + 1 = 8 times.

Lemma 3. Let v_1, v_2, v_3, v_4 be a 2-thread in some subgraph H of G. Let H' := H − {v_2, v_3}. Let σ' be a 14-good L-recoloring sequence for H' that transforms α_{H'} to β_{H'}. If σ' recolors v_4 at most s times, and s ≤ 11, then H has a 14-good L-recoloring sequence that recolors v_3 at most s + 3 times and transforms α to β.

Proof. Let a_1, . . . denote the sequence of new colors used by σ' on v_1 and let b_1, . . . denote the sequence of new colors used by σ' on v_4. For convenience, let a_0 := α(v_1) and b_0 := α(v_4). Each time that σ' recolors v_4 from b_i to b_{i+1}, we first, if needed, recolor v_3 to avoid b_i, b_{i+1}, and the color currently used on v_2. Since L is a 4-assignment, v_3 always has an available color. When σ' recolors v_1 from a_0 to a_1, we first recolor v_2 to avoid a_0, a_1, a_2; if this requires recoloring v_2 with the color currently used on v_3, then beforehand we recolor v_3 to avoid the colors currently on v_2, v_3, v_4. Afterward, we recolor v_2 to avoid a_0, a_1, a_2. Thereafter, each time that σ' recolors v_1 from a_i to a_{i+1}, we first recolor v_2 to avoid a_i, a_{i+1}, and the color currently used on v_3.

This process recolors α to an L-coloring β' that agrees with β everywhere except possibly on v_2 and v_3. Now, if needed, we recolor v_2 with β(v_2). If β(v_2) = β'(v_3), then we first recolor v_3 to some other arbitrary color, currently unused on v_2, v_3, and v_4. Finally, if needed, we recolor v_3 with β(v_3). The number of recolorings of v_2 is at most 1 + (14 − 2) + 1 = 14. The number of recolorings of v_3 is at most 1 + s + 2 ≤ 14.

Lemma 4. Let v_1, v_2, v_3, v_4, v_5 be a 3-thread in some subgraph H of G (not necessarily proper). Let H' := H − {v_2, v_3, v_4}. Let σ' be a 14-good L-recoloring sequence for H' that transforms α_{H'} to β_{H'}. We can extend σ' to a 14-good L-recoloring sequence for H that transforms α_{H} to β_{H} and recolors v_3 at most 4 times. In particular, G has no 3-threads.

Proof. See the left of Figure 2. The proof is very similar to that of Lemma 3 so we just sketch the details. We essentially treat both v_2 and v_3 in the way that we treated v_2 in the proof of Lemma 3. We extend the recoloring sequence to v_2 and v_4 so that in total they require v_3 to be recolored at most three times (once each near the start and once together near the end) to recolor v_2 and v_4 with β(v_2) and β(v_4)). Finally, we may need to recolor v_3 at the end, so that it uses the color β(v_3). Thus, v_3 is recolored at most 2(1) + 1 + 1 = 4 times.

Lemma 5. Let v be a 3-vertex with 2-neighbors w_1, w_2, w_3, and let x_1 be the other neighbor of w_1. Let G' := G − {v, w_1, w_2, w_3}, and let σ' be a 14-good L-recoloring sequence that transforms α_{G'} to β_{G'}. If σ' recolors x_1 at most 9 times, then G has a 14-good L-recoloring sequence that transforms α to β.
Figure 2: A 3-thread (left), as in Lemma 4, and a 3-vertex \( v \) with three 2-neighbors (right), as in Lemma 5. Here and throughout Section 5.1, round vertices have all incident edges drawn, but square vertices have some incident edges undrawn.

**Proof.** See Figure 2. By Lemma 4, we extend \( \sigma' \) to a 14-good \( L \)-recoloring sequence for \( G - \{ w_1 \} \) that recolors \( v \) at most 4 times. After this, we extend the 14-good \( L \)-recoloring sequence to \( w_1 \) by the Key Lemma, since the number of times its neighborhood is recolored is at most 4 + 9.

![Figure 3: A 4-vertex with 6 nearby 2-vertices; two cases in Lemma 6.](image)

**Lemma 6.** \( G \) does not contain any 3-vertex with 4 or more nearby 2-vertices, and \( G \) does not contain any 4-vertex with 6 or more nearby 2-vertices.

**Proof.** First suppose that \( G \) contains a 3-vertex or 4-vertex \( v \) that has \( d(v) - 1 \) incident 2-threads. For the case when \( d(v) = 4 \), see Figure 3 (left). Form \( G' \) from \( G \) by deleting the interior vertices of the \( d(v) - 1 \) incident 2-threads. And let \( G'' := G' - v \). By minimality, there exists a 14-good \( L \)-recoloring sequence \( \sigma'' \) for \( G'' \) that transforms \( \alpha_{G''} \) to \( \beta_{G''} \). By the Key Lemma, we can extend \( \sigma'' \) to a 14-good \( L \)-recoloring sequence \( \sigma' \) for \( G' \) that recolors \( v \) at most \( 1 + 14/2 = 8 \) times. By applying Lemma 4 to each 2-thread incident to \( v \), we can extend \( \sigma' \) to a 14-good \( L \)-recoloring sequence \( \sigma \) for \( G \) that transforms \( \alpha \) to \( \beta \).

Suppose instead that \( G \) contains a 3-vertex or 4-vertex \( v \) that has 2 incident 1-threads and \( d(v) - 2 \) incident 2-threads. See Figure 3 (right). Form \( G' \) from \( G \) by deleting the interior vertices of the 2-threads incident to \( v \). Form \( G'' \) from \( G' \) by deleting \( v \) and the interior vertices of the 1-threads incident to \( v \). Now the argument is very similar to that in the previous paragraph. By minimality, there exists a 14-good \( L \)-recoloring sequence \( \sigma'' \) for \( G'' \) that transforms \( \alpha_{G''} \) to \( \beta_{G''} \). By Lemma 4 we extend it to a 14-good \( L \)-recoloring sequence \( \sigma' \) for \( G' \) that transforms \( \alpha_{G'} \) to \( \beta_{G'} \) and recolors \( v \) at most 4 times. By applying Lemma 4 for each 2-thread incident to \( v \), we can extend \( \sigma' \) to a 14-good \( L \)-recoloring sequence \( \sigma \) for \( G \) that transforms \( \alpha \) to \( \beta \).

Since \( G \) has no 3-threads, by Lemma 4 all possible cases are handled above.

![Figure 4: A \( 3_{2,1,0} \)-vertex \( v \) adjacent to a \( 3_{1,1,0} \)-vertex \( w \) (left) and a \( 3_{2,1,0} \)-vertex \( v \) adjacent to a \( 3_{2,0,0} \)-vertex \( w \) (right); two cases of Lemma 7](image)

**Lemma 7.** No \( 3_{2,1,0} \)-vertex is adjacent to a \( 3_{1,1,0} \)-vertex or a \( 3_{2,0,0} \)-vertex or a \( 3_{2,1,0} \)-vertex.
Proof. Assume, to the contrary, that $G$ contains a $3_{2,1,0}$-vertex $v$ with a 3-neighbor $w$ that is a $3_{1,1,0}$-vertex or a $3_{0,0,0}$-vertex or a $3_{2,1,0}$-vertex. Figure 4 shows the first two of these cases. Form $G'$ from $G$ by deleting the interior vertices of a 2-thread incident to $v$. Form $G''$ from $G'$ by deleting $v$ and the interior vertex of an incident 1-thread. Form $G'''$ from $G''$ by deleting $w$ and all interior vertices of its incident threads. By minimality, $G'''$ has a 14-good $L$-recoloring sequence $\sigma'''$ that transforms $\alpha_{G'''}$ to $\beta_{G'''}$.

If $w$ is a $3_{2,1,0}$-vertex or a $3_{1,1,0}$-vertex, then by Lemma 4 we extend $\sigma'''$ to a 14-good $L$-recoloring sequence $\sigma''$ for $G''$ that recolors $w$ at most 4 times and transforms $\alpha_{G''}$ to $\beta_{G''}$. If instead $w$ is a $3_{2,0,0}$-vertex, then we first extend to $w$ by the Key Lemma, recoloring $w$ at most $14/2 + 1 = 8$ times, and then extend to its incident 2-thread by Lemma 8. Again by Lemma 8 we extend $\sigma''$ to a 14-good $L$-recoloring sequence $\sigma'$ for $G'$ that recolors $v$ at most $8 + 3 = 11$ times and transforms $\alpha_{G'}$ to $\beta_{G'}$. Now by Lemma 8 we extend $\sigma'$ to a 14-good $L$-recoloring sequence $\sigma$ for $G$ that transforms $\alpha$ to $\beta$. Note that $\sigma$ recolors the neighbor of $v$ on its incident 2-thread at most $11 + 3 = 14$ times.

Figure 5: A $3_{1,1,1}$-vertex $v$ weakly adjacent to a $3_{1,1,1}$-vertex $w$ (left) and a $3_{1,1,1}$-vertex $v$ weakly adjacent to a $3_{2,1,0}$-vertex $w$ (right); the two cases of Lemma 8.

Lemma 8. Each $3_{1,1,1}$-vertex has no weak 3-neighbor that is a $3_{2,1,0}$-vertex and no weak 3-neighbor that is a $3_{1,1,1}$-vertex.

Proof. First suppose that $v$ and $w$ are both $3_{1,1,1}$-vertices and $v$ is a weak neighbor of $w$; let $x$ be their common neighbor. See Figure 5 (left). Let $G' := G - x$ and form $G''$ from $G$ by deleting $v$, $w$, and all interior vertices of incident 1-threads. By minimality, $G''$ has a 14-good $L$-recoloring sequence that transforms $\alpha_{G''}$ to $\beta_{G''}$. By applying Lemma 4 twice, we extend $\sigma''$ to a 14-good $L$-recoloring sequence $\sigma'$ that recolors each of $v$ and $w$ at most 4 times and transforms $\alpha_{G'}$ to $\beta_{G'}$. By the Key Lemma, we extend $\sigma'$ to a 14-good $L$-recoloring sequence for $G$ that transforms $\alpha$ to $\beta$. The number of times that $\sigma$ recolors $x$ is at most $4 + 4 + 1$.

Now suppose instead that $v$ is a $3_{1,1,1}$-vertex and $w$ is a $3_{2,1,0}$-vertex and that $v$ and $w$ have a common 2-neighbor, $x$. See Figure 5 (right). The proof is nearly identical. The difference is that to extend a 14-good recoloring sequence from $G''$ to $G'$, we use Lemma 4 for $v$, but for $w$ we first use the Key Lemma and then use Lemma 3 to note that $w$ is recolored at most $1 + 14/2 = 8$ times. Again, we can extend to a 14-good $L$-recoloring sequence for $G$ that transforms $\alpha$ to $\beta$. Now the number of times that $\sigma$ recolors $x$ is at most $4 + 8 + 1 = 13$.

5.2 Discharging

Lemma 9. If $G$ is a graph with $\delta(G) \geq 2$ and $\text{mad}(G) < 22/9$, then $G$ contains one of the configurations forbidden, by Lemmas 5 and 8, from appearing in a minimal counterexample to Theorem 3. Thus, Theorem 3 is true.

Proof. Each vertex $v$ begins with charge $d(v)$ and we discharge so that each vertex ends with charge at least $2 + 4/9$. We use the following three discharging rules.

(R1) Each $3^+$-vertex sends $2/9$ to each nearby $2$-vertex.
(R2) Each $4^+$-vertex and $3_{0,0,0}$-vertex and $3_{1,0,0}$-vertex sends 1/9 to each 3-neighbor and sends 1/18 to each weak 3-neighbor with a common 2-neighbor.

(R3) Each $3_{1,1,0}$-vertex sends 1/18 to each weak 3-neighbor with a common 2-neighbor.

Now we show that each vertex finishes with charge at least $2 + 4/9$. Recall, from Lemma 4 that $G$ has no 3-threads. And recall, from Lemma 2 that $\delta(G) \geq 2$. If $d(v) = 2$, then $\text{ch}^*(v) = 2 + 2(2/9) = 2 + 4/9$. If $d(v) \geq 5$, then $\text{ch}^*(v) \geq d(v) - 2(2/9)d(v) = 5d(v)/9 \geq 25/9 > 2 + 4/9$. If $d(v) = 4$, then, by Lemma 6, $v$ has at most 5 nearby 2-vertices. So $v$ gives away at most $5(2/9)$ by (R1) and at most $4(1/18)$ by (R2). Now $\text{ch}^*(v) \geq 4 - 5(2/9) - 4(1/18) = 36/9 - 12/9 = 24/9 = 2 + 6/9$. Thus, by Lemma 6 we assume that $v$ is a $3_{a,b,c}$-vertex, where $a + b + c \leq 3$.

If $v$ is a $3_{0,0,0}$-vertex, then $\text{ch}^*(v) \geq 3 - 3(1/9) = 2 + 6/9$.

If $v$ is a $3_{1,1,0}$-vertex, then $\text{ch}^*(v) \geq 3 - 2(2/9) - 2(1/18) = 2 + 1/2$.

If $v$ is a $3_{2,0,0}$-vertex, then $\text{ch}^*(v) \geq 3 - 2(2/9) = 2 + 5/9$.

If $v$ is a $3_{2,1,0}$-vertex, then $\text{ch}^*(v) \geq 3 - 3(2/9) + 1/9 = 2 + 4/9$, since $v$ receives at least 1/9 from its $3^*$-neighbor by (R2); this is because, by Lemma 7, the $3^*$-neighbor of $v$ cannot be a $3_{1,1,0}$-vertex or a $3_{2,0,0}$-vertex or a $3_{2,1,0}$-vertex.

If $v$ is a $3_{1,1,1}$-vertex, then $\text{ch}^*(v) \geq 3 - 3(2/9) + 3(1/18) = 2 + 1/2$, since $v$ receives at least 1/18 from each weak 3-neighbor by rule (R2) or (R3); this is because, by Lemma 8 a $3_{1,1,1}$-vertex has no weak neighbor that is a $3_{1,1,1}$-vertex or a $3_{2,1,0}$-vertex (or a $3_{2,0,0}$-vertex).

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