Simulation of two-flavor symmetry-locking phases in ultracold fermionic mixtures

LUCA LEPORI¹,², ANDREA TROMBETTONI³,⁴ and WALTER VINCI⁵,⁶

¹ Departament de Física, Universitat Autònoma de Barcelona - E-08193 Bellaterra, Spain
² IPCMS (UMR 7504) and ISIS (UMR 7006), Université de Strasbourg and CNRS - Strasbourg, France
³ CNR-IOM DEMOCRITOS Simulation Center - Via Bonomea 265, I-34136 Trieste, Italy
⁴ SISSA and INFN, Sezione di Trieste - Via Bonomea 265, I-34136 Trieste, Italy
⁵ London Centre for Nanotechnology and Computer Science, University College London - 17-19 Gordon Street, London, WC1H 0AH, UK
⁶ University of Pisa, Department of Physics “E. Fermi” and INFN - Pisa, Italy

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Abstract – We describe an ultracold fermionic set-up where it is possible to synthesize a superfluid phase with symmetry obtained by locking independent invariance groups of the normal state. In this phase, named two-flavor symmetry-locking (TFSL) phase, non-Abelian fractional vortices with semi-integer flux and gapless non-Abelian Goldstone modes localized on them appear. Considerations on the possible experimental realization of the TFSL are also provided.

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Introduction. – Trapped ultracold atoms provide an excellent tool to simulate strongly interacting quantum systems [1]. The main reason is the high level of tunability and the very precise measurements achievable on such systems. Their versatility is considerably enlarged by two further ingredients: optical lattices [2] and gauge potentials [3]. The possibility to synthesize Abelian and non-Abelian gauge fields, also in the presence of optical lattices [4–9], promises a better understanding of an increasing number of relevant physical systems and new phases of gauge field theories.

In the non-Abelian case, hyperfine levels of suitable atoms are typically used as internal degrees of freedom where the gauge potential is defined: an advantageous proposal is given by earth-alkaline atoms [10]. To date, only static gauge fields have been experimentally simulated, but proposals for dynamical fields recently appeared in the literature [11–15].

A further promising and challenging application of ultracold atoms and synthetic gauge fields is the emulation of relativistic models relevant to high-energy physics: recent proposals focused on 2D Dirac fermions [16–23] and 3D Wilson fermions and axions [24–27], neutrino oscillations [28] and extra dimensions [29]. Important experimental achievements in this direction are the recent synthesis of 2D Dirac fermions [30] and the Haldane model [31] on honeycomb-like lattices.

In the light of the developments mentioned above, ultracold atoms can become a precious tool for the investigation of important topics in high-energy physics, as strongly coupled non-Abelian gauge theories, like quantum chromodynamics (QCD), dynamics of nucleons, nuclear matter or quarks under extreme conditions [32,33].

A very important concept arising in high-energy physics is the locking of symmetries, meaning that independent symmetries (global or local) of the Hamiltonian are mixed when the system enters in a certain phase, generally superfluid. A prominent example of this mechanism is the color-flavor locking (CFL), characterized by a mixing of the (local) SU(3)c color and the (global) SU(3)f flavor symmetries [34], independent in the normal state. A CFL regime is predicted at very large densities, as in the core of ultradense neutron stars, where the huge pressure and temperature allow quarks to be deconfined [32,33].

The characterizing property of a symmetry-locked phase is the presence of a complex pattern of spontaneous breaking of non-Abelian symmetries (SSB) induced by a multi-component superfluid condensate. This is at the heart of
its remarkable properties: the appearance of vortices and monopoles with semi-integer flux and non-Abelian degrees of freedom confined on them [35–39], and the breaking of translational invariance with the presence of ordered structures (as crystals and nets) [52,33]. Vortices in CFL phases also play an important role in determining the magnetic behaviour of neutron stars [40–42] and the properties of confinement and CSB [43].

We describe here a realistic proposal for the synthesis of a superfluid phase with locking between two global symmetries. Our relatively simple set-up differs from a realistic quantum simulation of CFL in two aspects: i) fermions involved in CFL have both color and flavor quantum numbers at the same time; ii) CFL locks both local (color) and global (flavor) symmetries, while in our model both the symmetries are global. Indeed we name the obtained symmetry-locked phase as two-flavor symmetry-locking (TFSL) phase. However, concerning i) our simplified model is still able to capture the effects of the various symmetries and their spontaneous breaking, independently from the internal space they are realized on. Moreover, the experimental implementation of the scheme proposed here provides a first and (in our opinion) necessary step towards the quantum simulation of a CFL phase featuring symmetry breaking of local symmetries, for which the use of dynamical gauge fields for ultracold atoms is required.

Another motivation for the implementation of a symmetry-locked phase with ultracold fermionic mixtures as proposed here is that —even with only global symmetries— non-Abelian fractional vortices (NAFV) appear. Remarkably the NAFV studied in this paper differ from the previously observed ones [44–46] in the origin of their non-Abelianity and fractionally. As a consequence their braiding properties are also different. The simulation of TFSL and CFL phases by using ultracold atoms would be then important to detect and study such exotic objects, especially due to the possibility of directly measuring various correlation functions (including those involving the density operator) [47].

The model. — We consider an optical lattice loaded with a mixture of four ultracold fermionic species: e.g., two different atoms, each one trapped in two hyperfine levels. We conventionally denote the four types of fermions by two different quantum labels (each one spanning two values) \( c = \{r, g\} \) and \( f = \{u, d\} \). The \( c \) and \( f \) indices are in turn collectively denoted as flavors since, in the model that we are going to adopt, each of them is associated with a global non-Abelian symmetry.

We assume for simplicity that the considered Fermi mixture is loaded in an optical lattice (say cubic), but the main conclusions of the paper and the occurrence of the TFSL do not depend on the presence of the lattice.

We assume an Hubbard Hamiltonian of the form \( \hat{H} = \hat{H}_{\text{kin}} + \hat{H}_{\text{int}} \) [48], where \( \hat{H}_{\text{kin}} \) is the usual tight-binding hopping Hamiltonian (the hopping coefficient \( t \) between nearest-neighbor sites is assumed for simplicity to be independent of \( c \) and \( f \) indices) and

\[
\hat{H}_{\text{int}} = -U_c \sum_{i,c \neq \prime} n_{i,c} n_{i,c} - U_f \sum_{i,f \neq \prime} n_{i,f} n_{i,f} - U_{c,f} \sum_{i,c,f} n_{i,c} n_{i,f},
\]

where \( n_{i,\sigma} = \hat{c}_{i,\sigma} \hat{c}_{i,\sigma} \) is the number operator of the fermion \( \sigma = r, g, u, d \) on the site \( i \). We also consider at the beginning balanced mixtures: the atomic numbers for the \( c \) and \( f \) species coincide \( (N_c = N_f \) with \( N_g = N_r = N_c / 2 \) and \( N_u = N_d = N_f / 2 \)). In (1) it is \( U_c = -\frac{4\pi \hbar^2}{m} a_c \int d\vec{r} \phi^*_c(\vec{r}) \phi_c(\vec{r}) \) and \( U_f = -\frac{4\pi \hbar^2}{m} a_f \int d\vec{r} \phi^*_f(\vec{r}) \phi_f(\vec{r}) \), where \( a_c \) and \( a_f \) are, respectively, the s-wave scattering lengths between two \( c \) and \( f \) fermions (the \( \phi \)s are the appropriate Wannier functions). Similarly \( -U_{c,f} \) is proportional to the s-wave scattering length \( a_{c,f} \) between one \( c \) fermion and a \( f \) one. The Hamiltonian in eq. (1) has an \( U(2) \times U(2) \equiv G \) symmetry enlarged to \( U(4) \) if \( U_c = U_f = U_{c,f} \).

Non-TFSL and TFSL phases. — We discuss for the moment the simplest model, where attraction arises between all the fermions: \( U_c, U_f, U_{c,f} > 0 \). Moreover we set for simplicity \( U_c = U_f = U \). Repulsive intra-c or intra-f interactions \( U_c < 0 \) or \( U_f < 0 \) are detrimental for the occurrence of the TFSL phase: for instance, this is the case for \( ^{173}\text{Yb} - ^{174}\text{Yb} \) mixtures. We will discuss in detail this situation later, proposing solutions to obtain still the TFSL phase.

For \( U > 0 \) and \( U_{c,f} > 0 \) the possible SF phases are parametrized by the pairings

\[
\langle c_i, u c_i, d \rangle \equiv \Delta_f, \quad \langle c_i, r c_i, g \rangle \equiv \Delta_c, \quad \langle c_i, c c_i, f \rangle \equiv \Delta_{c,f},
\]

with \( \Delta_c \), \( \Delta_f \) and \( \Delta_{c,f} \) being, respectively, two generic complex numbers and a \( 2 \times 2 \) complex matrix. We derive the free energy \( F \) in the mean-field approximation, similarly to the case of two-component mixtures [49]. It is convenient to define the order parameter \( \Delta_0 \) as \( 2|\Delta_0|^2 = |\Delta_c|^2 + |\Delta_f|^2 \) and to introduce the \( U(2)_{c,f} \) invariants \( \Delta^+_c = \text{Tr} \left( \Delta_{c,f}^\dagger \Delta_{c,f} \right) \), \( \Delta^+_{c,f} = 2 \text{det} \Delta_{c,f} \).

The minimization of \( F \) with respect to \( \Delta_c \) and \( \Delta_0 \) gives

\[
|\Delta_+| = |\Delta_-| \quad \text{and} \quad |\Delta_c| = |\Delta_f|.
\]

We find that for \( U_{c,f} \neq 0 \) the gap equations are not consistent if both \( \Delta_+ \) and \( \Delta_0 \) are non-zero both \( T = 0 \) and finite temperature and two phases are found as follows (see fig. 1): i) Non-TFSL phase: for \( U_{c,f} < U \) it is \( \Delta_+ = 0 \) and \( \Delta_0 \neq 0 \); ii) TFSL phase: for \( U_{c,f} > U \) it is \( \Delta_0 = 0 \) and \( \Delta_+ \neq 0 \).

Notice that \( |\Delta_+| \neq 0 \) corresponds to a TFSL phase, irrespectively of the value of \( \Delta_c \) and \( \Delta_f \) [50].

The findings in i) and ii) are consistent with the results discussed in literature (e.g., see [49,51]) that show how many-species superfluidity tends to avoid configurations with multiple pairings having different symmetries and competing with each others. The gap equations and the equations for the particle numbers (not written here) are the same in the TFSL phase and in the non-TFSL...
phase, but with $\Delta_+$ and $U_{cf}$ instead of $\Delta_0$ and $U$ (both at vanishing and at finite temperature). These equations have in turn the same functional form of Leggett’s equations for a two-component Fermi mixture across the BCS-BEC crossover [52]. Indeed they coincide for the diagonal pairing $\Delta_{cf} = \Delta \delta_{cf}$.

We notice at the end that a one-dimensional realization of the four-species model (1) was considered in [53], finding qualitative discrepancies from the behavior described above; such a mismatch is motivated by the enhanced effect of quantum fluctuations in that case, even at weak interactions.

**Properties of the TFSF phase.** – The key difference between the TFSF and the non-TFSF phase lies in the residual non-Abelian symmetry of the TFSF phase. In fact, the non-TFSF pairings induce a SSB of the Abelian factors: $U(2)_c \times U(2)_f \rightarrow SU(2)_c \times SU(2)_f$. At variance, the TFSF pairing $\Delta_{cf}$ satisfying $|\Delta_-| = |\Delta_+| \neq 0$ induces the following SSB pattern $G \rightarrow H$:

$$U(2)_c \times U(2)_f = U(2)_{c+f} \times U(2)_{c-f} \rightarrow U(2)_{c+f}.$$  \hfill (3)

This means that the SF phase (as well as $\Delta_{cf}$) has a residual symmetry group $H = U(2)_{c+f}$ given by the set of elements $(U_c, U_f) = (U_c, U_f^{-1}) = (U_{c+f}, U_f)$, $U_c$ and $U_f$ belonging respectively to $U(2)_c$ and $U(2)_f$. We define also $U(2)_{c-f} = (U_c, U_f) = (U_{c-f}, U_{c-f})$. $U(2)_{c+f}$, acts on $\Delta_{cf}$ as $U_{c+f} \Delta_{cf} U_{c+f}^{-1}$, and thus involves at the same time $c$ and $f$ transformations, originally independent. This mechanism is called symmetry locking [34].

The most general form of $\Delta_{cf}$ compatible with the gap equations for the TFSF phase is $\Delta_{cf} = \Delta \delta_{cf} \bf{1}$, with $\delta_{cf}$ being the Kronecker delta, or $\Delta_{cf} = \Delta \sigma_z$ ($\sigma_z$ are as usual the Pauli matrices) [54].

The SSB at the heart of TFSF implies the existence (both with and without vortices) of non-Abelian gapless Goldstone modes propagating in the whole condensate and described by the coset $G/H \sim S^2$. Such modes are related to spatial and time fluctuations of the order parameter $\Delta_{cf}$ in the general set described above. This is analogous to the presence of spin waves in the B phase of superfluid $^3$He [55]. The spectrum of gapless excitations is completely different from the non-TFSF scenario, where only two Abelian Goldstone modes appear.

**Comments about experimental feasibility.** – An instance of experimental set-up realizing our scheme is provided (possibly without optical lattice, in the continuous) by $^{171}$Yb and $^{173}$Yb mixtures [56, 57], the first one having a $1/2$ hyperfine degeneracy and the second one a $5/2$ multiplet, - in the latter case only two levels have to be populated selectively. The main motivation for this choice is that these atoms have natural interspecies interactions not depending on the particular hyperfine levels considered [58]. Even if this property may hold in other fermionic mixtures (for instance involving earth-alkaline atoms), to the best of our knowledge $^{171}$Yb-$^{173}$Yb is at the present time the unique stable mixture having it experimentally realized.

The $c$ index indicates now the two hyperfine levels of $^{171}$Yb, while $f$ refers to two hyperfine levels of $^{172}$Yb.

The scattering lengths are, respectively, $a_{171} \equiv a_c = -3 a_0$, $a_{173} \equiv a_f = +200 a_0$ and $a_{171,173} \equiv a_{cf} = -578 a_0$ [59] ($a_0$ being the Bohr radius). Therefore, $U_c$ is positive and very small, $U_{cf}$ is positive and $U_f$ is negative and relatively large (corresponding to the sensible repulsion between $f$ atoms).

The main effect of the intraspecies interactions is to induce an effective imbalance between the chemical potentials of the two atomic isotopes: this competes with the formation of the TFSF state driven by the dominating interaction $\propto a_{171,173}$. In the mean-field Hamiltonian from (1) (and the same populations for all the hyperfine levels both of $^{171}$Yb and $^{173}$Yb), this imbalance results in a term $U_c \nu_c \sum_i c_i^\dagger c_i \nu_c + U_f \nu_f \sum_i c_i^\dagger f_i c_i$, being $\nu_c = \nu_f \equiv \nu$ the filling of each atomic species. In this way, a different phase instead of TFSF, as FFLO [60, 61], can be realized. While in this paper we are not strictly interested to analyze the phase diagram of Yb mixtures, we address briefly how to overcome this problem. Tuning the interactions by optical Feshbach resonance is presently not a realistic solution, due to significant atomic losses.

The first possibility is to unbalance the $c$ and $f$ populations in number, $N_c > N_f$, such that the difference of the Fermi energies $\Delta E_F \equiv E_F^{(c)} - E_F^{(f)} = 2t (\cos k_F^{(c)} - \cos k_F^{(f)})$ in the normal states approaches the quantity $(U_f \nu_f - U_c \nu_c)$.

In this way the two unbalances compensates each others, allowing the appearance of the TFSF phase. An alternative (but experimentally more difficult) way to create a compensating unbalance is to allow a difference in the hopping terms, $(t_c - t_f) > 0$.

A different possibility involves two additional species in the $f$ multiplet and it assumes possible to couple $f$-atoms in pairs (say $(1, 3)$ and $(2, 4)$) via two Raman pulses with amplitude $\Omega$. This model has a $U(2)_c \times U(4)_f$ global symmetry that is explicitly broken to $U(2)_c \times U(2)_f$ by the Raman pulses. The new eigenstates in the normal state, where the residual symmetry

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is exactly realized, are obtained introducing the operators $\eta_{13/24}\hat{\phi} = \frac{\hat{c}_{\uparrow} \hat{c}_{\downarrow} + \hat{c}_{\downarrow} \hat{c}_{\uparrow}}{\sqrt{2}}$ with energy $\varepsilon_\pm(k) = \varepsilon(k) \pm \Omega$. A direct mean-field calculation shows that tuning $\Omega \approx (U_f - U_c)\nu$ compensates the original effective imbalance between the $^{173}$Yb levels and $\eta_{13/24}^{-1}$ [62] and allows for the TFSL pairing between them. Instead the states $\eta_{13/24}^{(-)}$, largely (anyway more than the unmixxed levels of $^{173}$Yb) imbalanced with respect of $^{171}$Yb, may be challenging for $^{171}$Yb-$^{173}$Yb mixtures. However it could be applied to other fermionic mixtures where the population unbalance might be difficult to be realized.

**Experimental detection.** From the previous discussion of their pairing structures, it emerges how to experimentally discriminate between TFSL and non-TFSL superfluids. In the latter situation one finds two conventional superfluids with non-vanishing pairings $\Delta_c$ and $\Delta_f$, while in the TFSL phase a pairing between $c$ and $f$ fermions arises, in principle along all the possibilities allowed by the TFSL symmetry: $\Delta_{c,f} = \Delta U^c_{\text{synth}} U_f$ (see a schematic representation in fig. 1). Therefore by observing the atomic densities and the interference patterns created by the fermionic condensates below the critical temperature, for example when vortices are induced by rotation [47], it is possible to discriminate the non-TFSL and TFSL phases. Various experimental techniques are available for the required measurement, both destructive and non-destructive [2]. It would be also interesting to observe signature of the TFSL phase using RF spectroscopy (see [63] and references therein).

**Non-Abelian fractional vortices.** In this section we study the behavior of the TFSL phase under the application of a synthetic Abelian magnetic field $\vec{B}_{\text{synth}}$ along $\hat{z}$: this can be obtained by putting under rotation the lattice [64] or by optical means [65], for example with the implementation of multipod schemes [3]. When $\vec{B}_{\text{synth}}$ is applied to the non-TFSL phase, vortices emerge in the spatial configuration of $r$-y and $u$-d condensates, while in the TFSL phase there is a locked spatial configuration involving both $c$ and $f$ atoms. The detection of vortices and the related spatial distributions of the fermionic species provides a first clear-cut characterization of the TFSL phase.

Vortices in the TFSL phase exhibit two remarkable properties: a) they host localized non-Abelian Goldstone modes (NAGM); b) they have fractional flux. To see the first property we start from an SF phase described at $\vec{B}_{\text{synth}} = 0$ by $\Delta_{c,f} = \Delta \hat{I}$. At $\vec{B}_{\text{synth}} \neq 0$, the energetically favored configurations host vortices in the condensates $\Delta_{yc}$ and $\Delta_{yd}$. In the following we assume that the vortices in the two components do not overlap completely. This implies that $\Delta_{c,f}(\vec{r})$ is not any longer a multiple of the identity matrix. Indeed let us consider the spatial dependence of the gap parameter around a vortex configuration, for example in the $\Delta_{yc}$ component:

$$\Delta_{c,f}(\vec{r}) = \Delta(r) (\mathbf{1} + \sigma_z \eta_z(r)) e^{i \frac{\pi}{4} \sigma_z \cdot \vec{\sigma} \cdot \eta_z(r)}$$

(4)

where $\theta = [0,2\pi)$ is the spatial angle around the vortex and $\vec{r}$ is the distance. Moreover we have $\eta_z(0) = 1$ at the core of the vortex while $\Delta(r) \to \Delta$ and $\eta_z(r) \to 0$ at distances $r$ large compared to the vortex typical size.

Equation (4) implies that the TFSL phase undergoes an additional SSB $H \to H_V$ along the following pattern:

$$U(2)_{c+f} \to U(1)_{c+f} \times U(1)_{\sigma^y}/Z_2$$

(5)

where $U(1)_{\sigma^y}$ (generated by $\sigma^y$) is contained in $SU(2)_{c+f}$. Two additional Goldstone modes then appear around the positions $\vec{r} \approx \vec{r}_i$ of the separated vortices, where $\eta_z \neq 0$. These modes can be made manifest by noticing that, once a solution of the type shown in eq. (4) is found, one can generate a continuous family of degenerate solutions by applying $c-f$ rotations:

$$U_{c+f} \Delta_{c,f}(r) U^{-1}_{c+f} = \Delta(r) (\mathbf{1} + \vec{S} \cdot \vec{\sigma} \eta_z(r)) e^{i \frac{\pi}{4} \vec{S} \cdot \vec{\sigma} \cdot \eta_z(r)}$$

(6)

with $\vec{\sigma} = (\sigma_x,\sigma_y,\sigma_z)$ and $|\vec{S}| = 1$ being a normalized vector parameterizing the Goldstone modes. The discussion above implies that vortices in the TFSL phase are endowed with NAGM forming a non-linear representation of the group $H$ [66] and spanning the target space $H/H_V = S^2$. A comparison with vortices in the non-TFSL phase helps clarification. In that case the lowest-energy configuration is

$$\Delta_{c,f}(\vec{r}) = \Delta_c(r) e^{i \theta}$$

(7)

with $\alpha = c,f$: these vortices are Abelian [40,67], since eq. (7) parameterizes $U(2)_{c+f}/SU(2)_{c+f}$, which as in (7) has finite energy also in the TFSL phase but this energy is bigger than for (6), then the latter one is selected in this case.

We comment now our assumption of vortex separation in the TFSL case. The issue can be tackled by explicit calculation of the Landau-Ginzburg functional (LGF) (not reported here) and via direct comparison to the LGF results for multi-component mixtures [44]. It turns out that vortices in the different components of $\Delta_{c,f}$ generally repel each other [68–70].

**Fractional flux.** Remarkably, fundamental vortices in the TFSL phase have a fractional (half) flux compared to an Abelian vortex of a non-TFSL phase, as can be seen by a direct computation or by more general arguments [35]. Indeed, vortices are classified via their quanta of flux by the element of the first homotopy group $\pi_1 [71,72]$ defined on the SSB pattern (3), that is $\pi_1(\overline{U(2)_{c+f} \times U(2)_{c-f}}) \sim \pi_1(U(2)_{c-f}) = Z/2$ in a TFSL phase and $\pi_1(U(1)_{c,f}) = Z$ in a non-TFSL one.
The discussed properties lead us to refer to the obtained solitons as NAFV [73]. Various types of fractional vortices have already been studied in various inhomogeneous systems, e.g. in Josephson junctions systems [74–76], in spin-1 Bose condensates [44] and helium 3 [77]. However fractionality in these cases generally requires $\Delta(\vec{r})$ to be asymptotically $\propto e^{i\kappa \theta}$, with $\kappa = p/q$ a rational number: this implies in turn nontrivial braiding properties due to the presence of spatial branch-cuts in the definition of $\Delta(\vec{r})$ [78]. Conversely the braiding is trivial in our case since it involves no anyonic statistics (notice the integer numbers in front of $\theta$ in the phases appearing in (4) and (6)). Other examples of fractional vortices but with integer phase arise in multi-components Bose superfluids [79] or metallic liquid hydrogen [80]. There an explicit interaction between the pairings is needed, unlike in the present case, where this property is possible only thanks to the group structure of $U(2)_{c-f}$.

Outlook. – We proposed and discussed a set-up of a fermionic ultracold mixture in which it is possible to synthesize two-flavor symmetry-locking (TFSL) phases. Due to their symmetry, these TFSL phases host exotic non-Abelian vortices with semi-integer flux and localized gapless modes confined on them. The origin of the non-Abelianity, the braiding properties, the mechanism and the consequences of fractionality for such vortices are discussed. To the best of our knowledge, we predicted for the first time the existence of such solitons in an experimentally accessible set-up.

The effect of repulsive intraspecies interactions has been discussed, showing that they can destroy the TFSL phase: two different solutions have been proposed, based on the creation of a counter-unbalance compensating this effect. A discussion of the detection of the TFSL and non-TFSL phases has been also provided.

An important issue, not discussed here in full generality and to develop in the future, is the characterization of the phase diagram for the four-species mixture involved, varying the interspecies and intraspecies interactions.

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REFERENCES
[1] Bloch I., Dalibard J. and Zwerger W., Rev. Mod. Phys., 80 (2008) 885.
[2] Lewenstein M., Sanpera A. and Ahufinger V., Ultra-cold Atoms in Optical Lattices: Simulating Quantum Many-body Systems (Oxford University Press, Oxford) 2012.
[3] Dalibard J., Gerbier F., Juzeliūnas G. and Öhberg P., Rev. Mod. Phys., 83 (2011) 1523.
[4] Jaksch D. and Zoller P., New J. Phys., 5 (2003) 56.
[5] Österloh K., Baig M., Santos L., Zoller P. and Lewenstein M., Phys. Rev. Lett., 95 (2005) 010403.
[6] Adelsburger M., Atala M., Nascimbène S., Trotzky S., Chen Y.-A. and Bloch I., Phys. Rev. Lett., 107 (2011) 255301.
[7] Jimenez-Garcia K., Le Blanc L. J., Williams R. A., Beeler M. C., Perry A. R. and Spielman I. B., Phys. Rev. Lett., 108 (2012) 225303.
[8] Hauke P., Tieleman O., Celi A., Ölschläger C., Simonet J., Struck J., Weinberg M., Windpassinger P., Sengstock K., Lewenstein M. and Eckardt A., Phys. Rev. Lett., 109 (2012) 145301.
[9] Atala M., Adelsburger M., Lohse M., Barreiro J. T., Paredes B. and Bloch I., Nat. Phys., 10 (2014) 588, arXiv:1402.0819.
[10] Gerbier F. and Dalibard J., New J. Phys., 12 (2010) 033007.
[11] Zohar E., Cirac J. I. and Reznik B., Phys. Rev. Lett., 109 (2012) 125302; 110 (2013) 055302; 125304.
[12] Banerjee D., Dalmonte M., Müller M., Rico E., Stebler P., Wiese U.-J. and Zoller P., Phys. Rev. Lett., 109 (2012) 175302.
[13] Tagliacozzo L., Celi A., Zamora A. and Lewenstein M., Ann. Phys., 330 (2013) 160.
[14] Banerjee D., Bögli M., Dalmonte M., Rico E., Stebler P., Wiese U.-J. and Zoller P., Phys. Rev. Lett., 110 (2013) 125303.
[15] Edmonds M. J., Valiente M., Juzeliūnas G., Santos L. and Öhberg P., Phys. Rev. Lett., 110 (2013) 085301.
[16] Zhu S.-L., Wang B. and Duan L.-M., Phys. Rev. Lett., 98 (2007) 260402.
[17] Wunsch B., Guinea F. and Solis F., New J. Phys., 10 (2008) 103027.
[18] Wu C. and Das Sarma S., Phys. Rev. B, 77 (2008) 235107.
[19] Juzeliūnas G., Ruseckas J., Lindberg M., Santos L. and Öhberg P., Phys. Rev. A, 77 (2008) 011802(R).
[20] Lim L.-K., Smith C. M. and Hemmerich A., Phys. Rev. Lett., 100 (2008) 130402.
[21] Hou J.-M., Yang W.-X. and Liu X.-J., Phys. Rev. A, 79 (2009) 043621.
[22] Lee K. L., Grémaud B., Han R., Englert B.-G. and Miniatura C., Phys. Rev. A, 80 (2009) 043411.
[23] Alba E., Fernández-Gonzalvo X., Mur-Pettit J., Garcia-Ripoll J. J. and Pachos J. K., Ann. Phys., 325 (2013) 64.
[24] Lamata L., Léon J., Schätz T. and Solano E., Phys. Rev. Lett., 98 (2007) 253005.
[25] Bermudez A., Mazza L., Rizzi M., Goldman N., Lewenstein M. and Martin-Delgado M. A., Phys. Rev. Lett., 105 (2010) 190404.
[26] Lepori L., Mussardo G. and Trombettoni A., EPL, 92 (2010) 50003.
[27] Mazza L., Bermudez A., Goldman N., Rizzi M., Martin-Delgado M. A. and Lewenstein M., New J. Phys., 14 (2012) 015001.
[28] Lan Z., Celi A., Lu W., Öhberg P. and Lewenstein M., Phys. Rev. Lett., 107 (2011) 253001.
Auzzi R., Bolognesi S., Evslin J., Konishi K., and Lukin M. D., Phys. Rev. Lett., 106 (2012) 133001.

Tarruell L., Greif D., Uehlinger T., Jotzu G. and Esslinger T., Nature, 483 (2012) 392.

Jotzu G., Messer M., Desbuquois R., Lebrat M. Uehlinger T., Greif D. and Esslinger T., Nature, 515 (2014) 237, arXiv:1406.7874.

Alford M., Schmitt A., Rajagopal K. and Schäfer T., Rev. Mod. Phys., 80 (2008) 1455.

Anglani R., Casalbuoni R., Cimmino M., Gatto R., and Ciminale M., Rev. Mod. Phys., 81 (2009) 185.

Kobayashi M., Kawaguchi Y., and Nitta M., Phys. Rev. D, (2013) 1191.

Hanany A. and Tong D., JHEP, 07 (2003) 037.

Eto M., Isozumi Y., Nitta M., Ohashi K. and Sakai N., Phys. Rev. Lett., 96 (2006) 161601.

Balachandran A. P., Digal S. and Matsupura T., Phys. Rev. D, 73 (2006) 074009.

Nakano E., Nitta M. and Matsupura T., Phys. Rev. D, 78 (2008) 045004.

Vinci W., Cipriani M. and Nitta M., Phys. Rev. D, 86 (2012) 065018.

Eto M., Nakano E. and Nitta M., Nucl. Phys. B, 821 (2009) 129.

Kasamatsu C., Tsubota M. and Ueda M., Int. J. Mod. Phys. B, 19 (2005) 1835.

Koraymash Y., Kawaguchi Y., Nitta M. and Ueda M., Phys. Rev. Lett., 103 (2009) 115301.

Stamper-Kurn D. M. and Ueda M., Rev. Mod. Phys., 85 (2013) 1191.

Fetter A. L., Rev. Mod. Phys., 81 (2009) 647.

Hofstetter W., Criac I., Zoller P., Demler E. and Lukin M. D., Phys. Rev. Lett., 89 (2002) 220407.

Annett J. F., Superconductivity, Superfluids and Condensates (Oxford University Press, Oxford) 2004, Chaps. 6, 7.

For $U = U_{cf}$, effects beyond the mean-field approximation should be considered to study the the existence and the properties of the TPSL phase.

Honerkamp C. and Hofstetter W., Phys. Rev. Lett., 92 (2004) 170403; Phys. Rev. B, 70 (2004) 094521.

Zwerger W. (Editor), The BCS-BEC Crossover and Unitary Fermi Gas (Springer, Heidelberg) 2012.

Capponi S., Roux G., Lecheminant P., Azaria P., Boulat E. and White S. R., Phys. Rev. A, 77 (2008) 013624.

To see the $U(2)$ invariance when $\Delta_{cf} = \Delta \sigma_z$, it suffices to consider the following choice for the flavor transformations $U_f = \sigma_z U_f \sigma_z$, with $U_f$ chosen freely. It immediately follows that $\sigma_z = U_f \sigma_z U_f^{-1}$.

Volovik G. E., The Universe in a Helium Droplet (Oxford University Press, Oxford) 2003.

Dickerscheid D. B. M., Kawaguchi Y. and Ueda M., Phys. Rev. A, 77 (2008) 053605.

Yip S.-K., Phys. Rev. A, 83 (2011) 063607.

Gorskov A. V., Hermele M., Gurarie V., Xu C., Julienne P. S., Ye J., Zoller P., Demler E., Lukin M. D. and Rey A. M., Nat. Phys., 6 (2010) 289.

Tae S., Takasu Y., Sugawa S., Yamazaki R., Tsumoto T., Murakami R. and Takahashi Y., Phys. Rev. Lett., 105 (2010) 190401.

Fulde P. and Ferrell R. A., Phys. Rev., 135 (1964) 550.

Larkin A. L. and Ovchinnikov Y. N., Zh. Eksp. Teor. Fiz., 47 (1964) 1136.

Pethick C. J. and Smith H., Bose-Einstein Condensation in Dilute Gases, 2nd edition (Cambridge University Press) 2008, Chap. 16.

Schrotzke A., Shin Y.-il, Schuck C. H. and Ketterle W., Phys. Rev. Lett., 101 (2008) 140403.

Cooper N. R., Adv. Phys., 57 (2008) 539.

Lin Y. J., Compton R. J., Jimenez-Garcia K., Porto J. V. and Spielman I. B., Nature, 462 (2009) 628.

Weinberg S., The Quantum Theory of Fields, Vol. 2 (Cambridge University Press, Cambridge) 1996.

Manton N. and Sutcliffe P., Topological Solitons (Cambridge University Press, Cambridge) 2004.

Nakano E., Nitta M. and Matsupura T., Phys. Lett. B, 672 (2009) 61.

Eto M., Kasamatsu K., Nitta M., Takeuchi H. and Tsubota M., Phys. Rev. A, 83 (2011) 063603.

The only exceptions are the diagonal cases $\Delta_{cf} = \Delta I$ and $\Delta_{cf} = \sigma_\epsilon$, where no mean-field interaction terms arise between vortices in different condensates. In this scenario a more precise evaluation of LGF coefficients, taking into account quantum fluctuations, is required. However, even if not spontaneous, vortex separation can be driven artificially here in various ways, for instance by an additional soft perturbation $\delta H_{soft} \propto \bar{z} \sigma_z$ in the flavour space. This field induces an unbalance in the number of vortices in $\Delta_{cf}$ and $\Delta_{cf'}$, hence vortex separation.

Coleman S., The magnetic monopole fifty years later, in The Unity of the Fundamental Interactions, edited by Zichichi A. (Plenum, London) 1983.

Nakahara M., Geometry, Topology and Physics, 2nd edition (Institute of Physics, Bristol) 2003.

High-energy physics community refers to solitons like ours simply as non-Abelian vortices [35,40], since the patterns labeled by $\pi_1(U(2)_{cf})$ are partly in the non-Abelian group $SU(2)_{cf}$. We added here the word “fractional” to avoid confusion with another more common type of non-Abelian vortex in condensed-matter literature. This definition is in fact usually used to denote the non-Abelianity of the first homotopy group $\pi_1$ defining the set of vortex charges: this feature critically affects the behavior of the vortices under merging and braiding [45,46].

Ustinov A., Appl. Phys. Lett., 80 (2002) 3153.

Goldchin E., Sterck A., Gaber T., Koelle D. and Kleiner R., Phys. Rev. Lett., 92 (2004) 047005.

Pfeiffer J., Schuster M., Abdumalikov A. A. and Ustinov A. V., Phys. Rev. Lett., 96 (2006) 037003.

Vollhardt D. and Wölfle P., The Superfluid Phases of Helium 3 (Taylor & Francis, London) 1990.

Ivanov D. A., Phys. Rev. Lett., 86 (2001) 268.

Babaev E., Phys. Rev. Lett., 89 (2002) 067001.

Babaev E. and Ashcroft N. W., Nat. Phys., 3 (2007) 530.