COMPLEXITY OF WEAKLY NULL SEQUENCES

DALE E. ALSPACH AND SPIROS ARGYROS

Abstract. We introduce an ordinal index which measures the complexity of a weakly null sequence, and show that a construction due to J. Schreier can be iterated to produce for each \( \alpha < \omega_1 \), a weakly null sequence \( (x^n)_{\alpha} \) in \( C(\omega^\omega) \) with complexity \( \alpha \). As in the Schreier example each of these is a sequence of indicator functions which is a suppression-1 unconditional basic sequence. These sequences are used to construct Tsirelson-like spaces of large index. We also show that this new ordinal index is related to the Lavrentiev index of a Baire-1 function and use the index to sharpen some results of Alspach and Odell on averaging weakly null sequences.

0. Introduction

In this paper we investigate the oscillatory behavior of pointwise converging sequences. Our main tool is a new ordinal index which measures the oscillation of such a sequence. We show that there are weakly null sequences of indicator functions in \( C(K) \) with arbitrarily large oscillation index and that the oscillation index is smaller than other similar ordinal indices. In particular the oscillation index of a pointwise converging sequence is directly compared to the Lavrentiev index of its limit, the \( \ell^1 \)-index defined by Bourgain, and the averaging index. Many of the results are directly related to those in [A-O] where the averaging index is used and we extend some results of that paper.

The first example of a weakly null sequence with no subsequence with averages going to zero in norm was constructed by Schreier [Sch]. His example is a sequence of indicator functions and, as observed by Pelczynski and Szlenk [P-S], can be realized on the space of ordinals less than or equal to \( \omega^\omega \) in the order topology. Because these are indicator functions the failure of the Banach-Saks property is solely dependent on the intersection properties of the sets. Our construction is also based on intersection properties of sets. Thus one viewpoint on the constructions in this paper is that there are families of sets with very complicated intersection properties. The purpose of the ordinal index is to measure the complexity of these intersection properties. The examples of weakly null sequences that we construct, like Schreier’s example, are suppression-1 unconditional basic sequences and can be considered as generalizations of Schreier’s construction.

Let us note that there is a strong relationship between this work and some unpublished results of Rosenthal on the unconditional basic sequence problem. Rosenthal

1991 Mathematics Subject Classification. 46B20.

Key words and phrases. ordinal, index, complexity, unconditional, \( L_1 \)-predual, \( C(K) \) space.

Research of the first author was supported in part by NSF grant DMS-8602395.

Typeset by \texttt{AMS-\LaTeX}
showed that any weakly null sequence of indicator functions in $C(K)$ has a subsequence which is an unconditional basic sequence. In our work we found that Rosenthal’s notion of weakly independent sets fits naturally into our viewpoint and we have incorporated some of Rosenthal’s work into the exposition. While we believe that our work explains some difficulties with unconditionality, it is based on intersection properties which cannot provide a complete explanation. For example, in contrast to Rosenthal’s result on sequences of indicator functions, it is known that given a weakly null sequence one cannot always find a subsequence which is an unconditional basic sequence \[M-R\] and \[O\]. Thus understanding the unconditional basic sequence problem is more complicated than just understanding the oscillation properties of weakly null sequences. On the other hand it is clear that the oscillation properties do play a fundamental role in unconditionality. It would be interesting if there were some way to incorporate these more subtle properties of weakly null sequences into an ordinal index.

The paper is organized into six sections. In the first we recall the definitions of some ordinal indices and trees. In the second we introduce the oscillation index and an essentially equivalent index which we call the spreading model index. In the third we prove that the oscillation index is essentially smaller than the $\ell^1$-index and show that it is related to the Lavrentiev index of a Baire-1 function. In the fourth section we define for each countable ordinal $\alpha$ a weakly null sequence of indicator functions and compute the oscillation index of the sequence and the size of the smallest $C(\alpha)$ space which can contain the sequence. In the fifth section we use an idea of Odell to show that the construction in Section 4 can be used to construct reflexive spaces similar to Tsirelson space with large oscillation index. In the sixth section we show that the averaging index, which has a definition similar to that of the spreading model index, is not smaller than the $\ell^1$ index. In particular a space is constructed which does not contain $\ell^1$ but has averaging index $\omega_1$. We also extend some results from \[A-O\] in order to better characterize those sequences which can be averaged a predictable finite number of times in order to get a weakly null sequence.

We will use standard terminology and notation from Banach space theory as may be found in the books of Lindenstrauss and Tzafriri \[L-T, I\] and \[L-T, II\] and Diestel \[D1\] and \[D2\]. If $\alpha$ is an ordinal, we will use $\alpha$ rather than $\alpha + 1$ to denote the space of ordinals less than or equal to $\alpha$ in the order topology and $C(\alpha)$ to denote the space of continuous functions on $\alpha$.

1. Preliminaries

In this section we will recall the definitions of the $\ell^1$-index of Bourgain, the Szlenk index, and the averaging index. In order to define the Bourgain $\ell^1$-index we first need to define trees and some related notions.

**Definition.** Given a set $S$ a tree $T$ on $S$ is subset of $\cup_{n=1}^{\infty} S^n \cup S^\infty$ such that if $b \in T$ then any initial segment of $b$ is also in $T$, i.e., if $b \in T$ and $b = (s_1, s_2, \ldots, s_n, s_{n+1})$ or $b = (s_1, s_2, \ldots, s_n, s_{n+1}, \ldots)$, then $(s_1, s_2, \ldots, s_n) \in T$. We will call the elements of the tree nodes and say that the node $b$, as above, is a successor of or is below $(s_1, s_2, \ldots, s_n)$. If $b$ and $c$ are nodes, we will write $b > c$ to indicate that $b$ is below $c$. In particular $(s_1, s_2, \ldots, s_n, s_{n+1})$ is an immediate successor of $(s_1, s_2, \ldots, s_n)$. If a node $x \in S^n$ then $n$ is the length or level of $x$. A branch of a tree is a maximal linearly ordered subset of the tree under this natural initial segment ordering. A
subtree of a tree $T$ is a subset of $T$ which is a tree. A tree $T$ is finitely branching if the number of immediate successor nodes of any node is finite. It is dyadic if this number is at most two for all nodes. Finally a tree is well-founded if all of its branches are of finite cardinality.

For well-founded trees there is a standard way to define the ordinal index of the tree.

**Definition.** Suppose that $T$ is a well-founded tree on a set $X$. Let $T_0 = T$ and for each ordinal $\alpha$ define

$$T^{\alpha + 1} = \bigcup_{n=1}^{\infty} \{ (x_1, x_2, \ldots, x_n) \in T^\alpha : \text{there is an } x \in X \text{ such that } (x_1, x_2, \ldots, x_n, x) \in T^\alpha \}.$$ 

For a limit ordinal $\alpha$ let $T^\alpha = \bigcap_{\beta<\alpha} T^\beta$. Let $o(T)$ be the smallest ordinal $\gamma$ such that $T^\gamma = \emptyset$. This is the order of the tree $T$.

Note that if $x = (x_1, x_2, \ldots, x_j) \in T^\alpha \setminus T^{\alpha + 1}$, the tree $T_x = \bigcup_{n=1}^{\infty} \{ (y_1, y_2, \ldots, y_n) : x + (y_1, y_2, \ldots, y_n) \}$, where “+” indicates concatenation, is of order $\alpha$.

Now we are ready to define the $\ell_1$-index of Bourgain.

**Definition.** Let $X$ be a Banach space and $\delta > 0$. Let

$$T(X, \delta) = \bigcup_{n=1}^{\infty} \{ (x_1, x_2, \ldots, x_n) \in X^n : ||x_j|| \leq 1 \text{ for all } j \leq n \text{ and } ||\sum_j \lambda_j x_j|| \geq \delta \sum |\lambda_j| \text{ for all } (\lambda_j) \in \mathbb{R}^n \}.$$ 

The $\delta \ell_1$-index of $X$ is $\ell(X, \delta) = o(T(X, \delta))$. If $T(X, \delta)^\alpha \neq \emptyset$ for all countable $\alpha$ then $\ell(X, \delta) = \omega_1$.

It is easy to see that if $\ell_1$ is isomorphic to a subspace of $X$ then $\ell(X, \delta) = \omega_1$ for some $\delta > 0$ and thus by a result of James [J] for all $\delta$, $0 < \delta < 1$. The converse is also true but requires the fact that if $T^\alpha(X, \delta) \neq \emptyset$ for all countable $\alpha$ then the tree has an infinite branch. (See [Bo].) Bourgain observed that the examples employed by Szlenk [Sz] to show that there are separable reflexive spaces with Szlenk index greater than any given countable ordinal also have large $\ell_1$-index. Note that this indicates that the Baire-1 functions in $X^{**}$ provide only a very weak indication as to the $\ell_1$-index of $X$.

Next we will define Bourgain’s index for Boolean independence. This index is a technical convenience which we will use in establishing lower bounds on the $\ell_1$-index.

**Definition.** Let $K$ be a set and $\{(A_n, B_n)\}$ be a sequence of pairs of subsets of $K$. Let

$$T(\{(A_n, B_n)\}) = \bigcup_{k=1}^{\infty} \{ (n_1, n_2, \ldots, n_k) \in \mathbb{N}^k : \text{for all } (\epsilon_1, \epsilon_2, \ldots, \epsilon_k) \in \{-1, 1\}^k, \cap_{i=1}^{k} \epsilon_i A_{n_i} \neq \emptyset \},$$
where \((-1)A_i = B_i\). Let \(\mathcal{B}(\{(A_n, B_n)\})\) be the order of the tree \(\mathcal{T}(\{(A_n, B_n)\})\).

We will usually take pairs of disjoint sets \((A_n, B_n)\) when we employ this index. Note that by Rosenthal’s theorem on \(\ell^1, |R|\), a bounded sequence \((x_n)\) in a Banach space \(X\) has a subsequence equivalent to the unit vector basis of \(\ell^1\) if (and only if) it has no weak Cauchy subsequence. Moreover for any such sequence there is a subsequence \((x_n)_{n \in M}\) and two numbers \(\delta > 0\) and \(r \) such that if

\[
A_n = \{x^* \in X^*: x^*(x_n) \geq r + \delta, ||x^*|| \leq 1\}
\]

and

\[
B_n = \{x^* \in X^*: x^*(x_n) \leq r, ||x^*|| \leq 1\}
\]

then \(\mathcal{T}(\{(A_n, B_n): n \in M\}) = \bigcup_{k=1}^{\infty} M^k\) and thus \(\mathcal{B}(\{(A_n, B_n) : n \in \mathbb{N}\}) = \omega_1\). Hence if we consider such sets \((A_n, B_n)\), the index is \(\omega_1\) for some \(\delta > 0\) and \(r\) if and only if \((x_n)\) has a subsequence with no weak Cauchy subsequence. In particular if \((x_n)\) converges \(w^*\) sequentially to some \(x^{**} \in X^{**}\) the index is countable.

Now we wish to recall some other ordinal indices which are more classical in spirit. Each of these indices is defined in terms of the oscillation of sequences on the unit ball of the dual in the \(w^*\) topology. First we recall the Szlenk index. Let \(X\) be a Banach space and for each \(\alpha < \omega_1\) let

\[
P_{\alpha+1}(\epsilon, B_X, B_{X^*}) = \{x^*: \text{there exists } x^*_n \in P_{\alpha}(\epsilon, B_X, B_{X^*}) \text{ and } x_n \in B_X \text{ such that } x^*_n \overset{w^*}{\to} x^*, x_n \overset{w}{\to} 0, \text{ and } x^*_n(x_n) \geq \epsilon\}
\]

At a limit ordinal \(\beta\) we take the intersection, that is,

\[
P_{\beta}(\epsilon, B_X, B_{X^*}) = \cap_{\alpha < \beta} P_{\alpha}(\epsilon, B_X, B_{X^*}).
\]

Now fix \(\{e_j\}\) a normalized weakly null sequence in \(X\). The next two ordinal indices are defined for each such sequence. The first occurred in a paper of Zalczwasser [Z] and in a paper of Gillespie and Hurwitz [G-H] and was used to prove that a pointwise converging sequence of bounded continuous functions on a compact metric space has a sequence of convex combinations going to zero in norm. Let

\[
Z_{\alpha+1}(\epsilon, \{e_j\}, B_{X^*}) = \{x^*: \text{there exists } x^*_n \in Z_{\alpha}(\epsilon, \{e_j\}, B_{X^*}) \text{ and } e_{j_n} \text{ such that } x^*_n \overset{w^*}{\to} x^* \text{ and } \lim |x^*_n(e_{j_{2n}}) - x^*_n(e_{j_{2n-1}})| \geq \epsilon\}.
\]

As before \(Z_{\beta} = \cap_{\alpha < \beta} Z_{\alpha}\), for a limit ordinal \(\beta\).

In [A-O] the following index was introduced in order to obtain some more precise information about the nature of the convex combinations obtained in these early papers in the context of weakly null sequences in a Banach space. Let \(A_{\alpha+1}(\epsilon, \{e_j\}, B_{X^*}) = \{x^*: \text{for every neighborhood } \mathcal{N} \text{ of } x^* \text{ relative to } A_{\alpha}(\epsilon, \{e_j\}, B_{X^*}) \text{ there exists an infinite set } L \subset \mathbb{N} \text{ such that } \ell^1 - SP(\{e_{i|\mathcal{N}}\}_{i \in L}) \geq \epsilon\}
\]
where
\[ \ell^1 - SP \{ \{ e_i \} \} = \lim_k \lim_{m \to \infty} \left\{ \left| k^{-1} \sum_{i=1}^{k} e_{n_i} \right| : m \leq n_1 < n_2 < \ldots < n_k \right\} . \]

As before \( A_\beta = \cap_{\alpha < \beta} A_\alpha \), if \( \beta \) is a limit ordinal. We will refer to this as the averaging index.

In each case the successor set (if non-empty) is a \( w^* \) closed nowhere dense subset of the set. (For the case of the Szlenk sets we need to assume that \( X^* \) is separable.) Thus the Baire Theorem gives in each case a largest ordinal \( \alpha \) with the \( \alpha \)th set non-empty and the \( (\alpha + 1) \)th empty. Denote \( o(P, \epsilon) \), \( o(Z, \epsilon) \), and \( o(A, \epsilon) \) as the indices for the Szlenk, Zalcwasser, and average sets, respectively. Clearly \( o(P, \epsilon) \geq o(Z, 2\epsilon) \) and \( o(Z, \epsilon) \geq o(A, \epsilon) \).

2. Weakly Null Sequences and the \( \ell^1 \)-index

We wish to introduce an index similar to the Szlenk index which will measure \( \ell^1 \)-ness in a different way than the \( \ell^1 \)-index of Bourgain.

Definition. Let \((K, d)\) be a Polish space and let \((f_n)\) be a pointwise convergent sequence of continuous functions on \( K \). Fix \( \epsilon > 0 \) and let
\[ A^+_{n,m} = \{ k \in K : f_n(k) - f_m(k) > \epsilon \} \]
and
\[ A^-_{n,m} = \{ k \in K : f_n(k) - f_m(k) < -\epsilon \} , \]
For each countable ordinal \( \alpha \) we define inductively a subset of \( K \) by
\[ O^\alpha (\epsilon, (f_n), K) = K, \]
\[ O^{\alpha+1} (\epsilon, (f_n), K) = \{ k \in O^\alpha (\epsilon, (f_n), K) : \text{for every neighborhood } N \text{ of } k \text{ there is an } N \in \mathbb{N} \text{ such that for all } n \geq N \text{ there exists an } M \in \mathbb{N} \text{ such that} \]
\[ \cap_{m \geq M} A^+_{n,m} \cap O^\alpha (\epsilon, (f_n), K) \cap N \neq \emptyset \text{ or} \]
\[ \cap_{m \geq M} A^-_{n,m} \cap O^\alpha (\epsilon, (f_n), K) \cap N \neq \emptyset . \]

If \( \beta \) is a limit ordinal, \( O^\beta (\epsilon, (f_n), K) = \cap_{\alpha < \beta} O^\alpha (\epsilon, (f_n), K) \).

Note that if \((f_n)\) converges pointwise to 0 and \( \epsilon' < \epsilon \), then for large enough \( M \)
\[ \{ k : f_n(k) \geq \epsilon' \} \supset \cap_{m \geq M} A^+_{n,m} \supset \{ k : f_n(k) \geq \epsilon \} \]
Thus \( O^\alpha (\epsilon, (f_n), K) \) is essentially the Szlenk index set, \( P_\alpha (\epsilon, (f_n), K) \), except that we do not allow the use of subsequences. Thus \( P_\alpha (\epsilon', (f_n), K) \supset O^\alpha (\epsilon, (f_n), K) \) in this case for all \( \epsilon' < \epsilon \). Also it is easy to see that \( O^\alpha (\epsilon, (f_n), K) \) is always closed. If the limit \( f \) is not continuous, the relationship with the Szlenk sets is less clear. However in the definition of the Szlenk sets it is possible to use weak Cauchy sequences in place of weakly null sequences. The sets obtained in this way behave in essentially the same way as the original Szlenk sets provided the dual is separable. Thus if \([f_n]^*\) is separable \( O^\alpha (\epsilon, (f_n), K) \) will be empty for some \( \alpha < \omega_1 \). Actually the index is always countable but this will be a consequence of our result relating this index to the \( \ell^1 \)-index.

This discussion (and some later results) justifies the following.
Lemma 2.2. Suppose \((f_n)\) is a pointwise converging sequence of continuous functions on a Polish space \(K\). \(\mathcal{O}(\epsilon) = \mathcal{O}(\epsilon, (f_n), K)\) is the largest ordinal \(\alpha\) such that \(\mathcal{O}^\alpha(\epsilon, (f_n), K)\) is non-empty. We will refer to \(\mathcal{O}(\epsilon)\) as the \(\epsilon\) oscillation index of the sequence \((f_n)\). (Some authors, e.g., [K-L], use the term oscillation index for an ordinal index defined in terms of the oscillation of a fixed function near a point. Here we only apply the term to sequences so no confusion should result.)

It will be convenient to use a more restrictive index at times. Let 
\[
\mathcal{O}^0_+(\epsilon, (f_n), K) = K.
\]
For each countable ordinal \(\alpha\) define inductively a subset of \(K\) by
\[
\mathcal{O}^{\alpha+1}_+(\epsilon, (f_n), K) = \{k \in \mathcal{O}^\alpha_+(\epsilon, (f_n), K) : \text{ for every neighborhood } \mathcal{N} \text{ of } k \text{ there is an } N \in \mathbb{N} \text{ such that for all } n \geq N \text{ there exists } \}
\]
\[
\text{an } M \in \mathbb{N} \text{ such that } \cap_{m \geq M} A_{n, m}^+ \cap \mathcal{O}^\alpha_+(\epsilon, (f_n), K) \cap \mathcal{N} \neq \emptyset \}.
\]
If \(\beta\) is a limit ordinal, \(\mathcal{O}^\beta_+(\epsilon, (f_n), K) = \cap_{\alpha < \beta} \mathcal{O}^\alpha_+(\epsilon, (f_n), K)\). As above the positive \(\epsilon\) oscillation index \(\mathcal{O}^\alpha_+(\epsilon)\) is the largest ordinal \(\alpha\) such that \(\mathcal{O}^\alpha_+(\epsilon, (f_n), K) \neq \emptyset\). In a similar way we define \(\mathcal{O}^\alpha_-(\epsilon, (f_n), K)\) and \(\mathcal{O}^\alpha_-(\epsilon, (f_n), K)\).

Remark 2.1. We do not need to have the sequence converging pointwise for the definition of the index to make sense. However, if \((f_n)\) has no pointwise convergent subsequence then, by Rosenthal’s \(\ell^1\) theorem [R], there is an infinite set \(M\) and real numbers \(\delta > 0\) and \(\tau\) such that \(\{k : f_n(k) \geq \tau + \delta\}, \{k : f_n(k) \leq \tau\}\) is Boolean independent. From this it follows that there is a Cantor set \(C \subseteq K\) so that relative to \(C\) and taking \(n\) and \(m\) from \(M\), \(\cup_{n \geq M} A_{n, m}\) is an \(\epsilon_n\)-net in \(C\) and \(\epsilon_n \to 0\) as \(n \to \infty\). Hence \(\mathcal{O}^1(\delta/2, (f_n)_{n \in M}, C) = C\). Thus for this subsequence the oscillation index is \(\omega_1\).

Definition. The \(\epsilon\)-oscillation index of a Banach space is the supremum of the \(\epsilon\)-oscillation index over all weak Cauchy sequences in the unit ball of the space as functions on the dual ball with the \(w^*\) topology.

The next lemma follows easily from the definitions but we state it for future reference.

Lemma 2.2. If \(K\) is a \(w^*\) closed subset of the dual ball of \(X\), then \(\mathcal{O}^\alpha_+(\epsilon, (f_n), K) \subseteq \mathcal{O}^\alpha_+(\epsilon, (f_n), B_X^*)\) for all \(\alpha\) and \(\epsilon\). In particular if \(T\) is a bounded operator from a Banach space \(X\) to a Banach space \(Y\), then
\[
T^*\mathcal{O}^\alpha_+(\epsilon, (Tf_n), B_Y^*) = \mathcal{O}^\alpha_+(\epsilon, (f_n), T^*B_Y^*) \subseteq \mathcal{O}^\alpha_+(\epsilon/||T||, (f_n), B_X^*)
\]
and consequently
\[
\mathcal{O}_+(\epsilon, (Tf_n), B_Y^*) \subseteq \mathcal{O}(\epsilon/||T||, (f_n), B_X^*).
\]

On the other hand we do not know if there is an essential equivalence between \(\sup_M \mathcal{O}(\epsilon, (f_n)_{n \in M}, B_X^*)\) and \(\sup_M \mathcal{O}(\epsilon, \lambda, (f_n)_{n \in M}, K)\) if \(K\) is a \(\lambda\) norming subset of \(B_X^*\). Simple examples show that considering subsequences is necessary in order for the indices to be approximately the same size.

Next we will present some examples in which we can compute some bounds on the oscillation index and compare this index to the Bourgain \(\ell^1\)-index. The first nontrivial example we wish to present is the Schreier sequence, [Sch].
Example 1. Let $\mathcal{F}_1 = \{F \subset \mathbb{N} : \min F \geq \text{card} F \} \cup \{\emptyset\}$. We will identify $\mathcal{F}_1$ with $\{1_F : F \in \mathcal{F}_1\}$ and note that the later is a countable compact (metric) space in the topology of pointwise convergence on $\mathbb{N}$. Define $f_n(F) = 1_F(n)$. Because each $F$ is finite $(f_n)$ converges pointwise to 0 on $\mathcal{F}_1$. It is easy to see that each $f_n$ is continuous on $\mathcal{F}_1$. A computation shows that

$$O^j(1, (f_n), \mathcal{F}_1) = \bigcup_{m > j} \{F \in \mathcal{F}_1 : \min F \geq m \text{ and card } F \leq m - j\} \cup \{\emptyset\}.$$ 

Thus $O^\omega(1, (f_n), K) = \{\emptyset\}$ and $O(1) = \omega$. Because these are indicator functions the same is true for all $\epsilon, 1 \geq \epsilon > 0$. Note that this is the maximal index because $\mathcal{F}_1$ is homeomorphic to $\omega^\omega$ in the order topology and hence there are only $\omega$ non-empty topological derived sets.

It follows from the combinatorial properties of $\mathcal{F}_1$ that $[f_n]$ contains $\ell_n^1$’s uniformly and thus $\ell([f_n]) \geq \omega$. (See [Pel-Sz] or Section 4.) Also it is not hard to see that the $\epsilon$-oscillation index of $C(\omega^\omega)$ is essentially the same as the Szlenk index, $\omega^{\frac{1}{2}}$.

Example 2. Tsirelson space $T$. (See [C-S] or Section 5.) The natural unit vector basis of $T$ is a weakly null sequence. $T$ is reflexive and contains many $\ell_n^1$’s. Thus $\ell(T, \delta) \geq \omega^*$, for $\delta = \frac{1}{4}$, for example. The computation of $O^1(\epsilon, (e_n), B_{T^1})$ does not seem to be easy. However note that the functionals $1_F$ for $F \in \mathcal{F}_1$ in example 1 above act on $T$ by $1_f \left(\sum a_n e_n\right) = \sum a_n$ and all $\|1_F\|_T \leq 2$. Moreover if $1_{F_1} \rightarrow 1_F$ pointwise on $\mathbb{N}$, then $1_{F_1}(x) \rightarrow 1_F(x)$ for all $x \in T$. Therefore the map $S : T \rightarrow C(\mathcal{F}_1) = C(\omega^\omega)$ defined by $Sx(F) = 1_F(x)$ is well defined and bounded by 2 and $Te_n = f_n$. It follows from Example 1 and Lemma 2.2 that $O^\omega(\frac{1}{2}, (e_n), B_{X^*}) \neq \emptyset$.

Example 3. $\ell^1$. There are no non-trivial weakly null sequences in $\ell^1$ and thus the oscillation index is 0. On the other hand the $\ell^1$-index is $\omega_1$.

The anomalous behavior of the index for $\ell^1$ can be corrected if we allow general sequences and modify the definition of $O^{\alpha + 1}$. However this seems pointless in view of Rosenthal’s characterization of $\ell^1$.

Remark 2.3. Suppose that $(f_n)$ is a pointwise converging sequence of uniformly bounded continuous functions on a compact metric space $K$. Then the mapping $S$ from $K$ into $c$ (the space of convergent sequences under the sup norm) $S(k) = (f_n(k))$ is continuous if the range is given the $\sigma(c, \ell^1(N))$ topology. If the functions $(f_n)$ separate points on $K$ then $S$ is also one-to-one. From this viewpoint we are investigating $\sigma(c, \ell^1(N))$ compact subsets of the ball of $c$. The oscillation set $O^{\alpha}(\epsilon, (f_n), K)$ is mapped by $S$ to the set $O^{\alpha}(\epsilon, (e_n), S(K))$, where $e_n$ denotes the functional evaluation at $n$, and thus the index may be computed in $c$. If the sequence $(f_n)$ converges to 0 then the sets are actually in $c_0$ and the topology $\sigma(c_0, \ell^1(N))$ is the weak topology.

3. Comparison with the $\ell^1$-index

The $\ell^1$-index defined by Bourgain [Bo] measures the degree to which $\ell^1$ isomorphically embeds in the space. Bourgain showed that this index is related to an ordinal index of Baire-1 functions in the second dual. In the next section we will
show that the presence of certain weakly null sequences can also raise the \( \ell^1 \)-index. Thus in contrast to the Baire-1 case the pointwise limit itself provides no information but rather the sequence carries the information. Our method is to use the oscillation index defined in the previous section. In this section we will show that this oscillation index is essentially bounded above by the \( \ell^1 \)-index, and prove that any sequence of continuous functions converging pointwise to a Baire-1 function of index \( \alpha \) must have a large oscillation index. We then get Bourgain’s result as a corollary. These ideas are also related to some work of Haydon, Odell, and Rosenthal [H-O-R] on what they term Baire-1/2 and Baire-1/4 functions in the second dual.

Now we will prove that a large oscillation index implies a large \( \ell^1 \)-index. For an ordinal \( \alpha = \omega^\gamma k + \beta \) with \( \beta < \omega^\gamma \) and \( k \in \mathbb{N} \), we define \( \alpha/2 = \omega^\gamma [(k+1)/2] \), where \( [\cdot] \) denotes the greatest integer function.

**Theorem 3.1.** If \( (f_n) \) is a pointwise converging sequence on a compact metric space \( K \) and \( O_\alpha (\epsilon, (f_n), K) \neq \emptyset \) then if \( \epsilon' < \epsilon \), \( \ell([f_n], \epsilon'/2) \geq \alpha/2 \). Moreover there is an \( \ell^1 \)-index tree on \( (f_n) \) with index \( \alpha/2 \).

The proof is easier if we assume that \( O_\alpha (\epsilon, (f_n), K) \neq \emptyset \), and in this case we get \( \alpha \) rather than \( \alpha/2 \) for the \( \ell^1 \)-index. The proof naturally divides into two parts: first a reduction to the case \( O_\alpha^{1/2} \neq \emptyset \) and then the proof of the result in this case. The reduction is proved as Lemma 3.4. The main idea in the remainder of the argument is to construct a tree of Boolean independent pairs of sets with large order where the pairs of sets are subsets of the \( A_{n,m} \)'s. Consequently we will actually construct our \( \ell^1 \) tree on \( (f_n - f_m)_{n,m \in \mathbb{N}} \) with constant \( \epsilon/2 \). A final argument is needed to get a \( \ell^1 \) tree on \( (f_n) \).

Before we proceed to the proof we need a few lemmas which describe some sufficient conditions for a tree to be a \( \ell^1 \)-tree. The following is an unpublished lemma of Rosenthal. (We actually only use the weaker version in which \( r \) does not depend on \( n \))

**Lemma 3.2.** Suppose \( (f_n)_{n=1}^m \) is a finite sequence of norm one functions on \( K \), \( \delta > 0 \), and for each \( n \) there is a number \( r_n \) such that if \( A_n = \{ f_n \geq r_n + \delta \} \) and \( B_n = \{ f_n \leq r_n \} \), \( \{(A_n, B_n)\} \) is Boolean independent. Then \( (f_n) \) is \( 2/\delta \) equivalent to the unit vector basis of \( \ell^1 \).

**Proof.** Suppose that \( a_n \in \mathbb{R} \) for all \( n \). Let \( F = \{ n : a_n \geq 0 \} \) and \( G = \{ n : a_n < 0 \} \). Let \( t \in \cap_{n \in F} A_n \cap \cap_{n \in G} B_n \) and \( t' \in \cap_{n \in F} A_n \cap \cap_{n \in G} B_n \). Then

\[
| \sum a_n f_n(t) - \sum a_n f_n(t') \| \geq \sum a_n |f_n(t) - f_n(t')| \geq \sum |a_n| \delta.
\]

Indeed, if \( n \in F \), \( a_n \geq 0 \), \( f_n(t) \geq r_n + \delta \) and \( f_n(t') \leq r_n \) and if \( n \in G \), \( a_n < 0 \), and \( f_n(t) \leq r_n \) and \( f_n(t') \geq r_n + \delta \). Hence either \( | \sum a_n f_n(t) | \geq \sum |a_n| \delta/2 \) or \( | \sum a_n f_n(t') | \geq \sum |a_n| \delta/2. \)

We would like to construct a Boolean independent tree of pairs of subsets of \( K \) where the pairs are related to the elements of the sequence \( (f_n) \). If we could, for example, choose sets of the form \( A_n \) and \( B_n \) as in the lemma above, we would immediately get a \( \ell^1 \) tree on \( (f_n) \). Actually somewhat less is sufficient.
Lemma 3.3. Suppose that \((x_n)\) is a sequence of functions on a set \(K\) with values in \([-1,1]\) and \(T\) is a tree on \((x_n)\) of order \(\alpha < \omega_1\). Further assume that there is a \(\delta > 0\) such that for each branch \(B\) of \(T\) there is a mapping \(\rho_B : B \to 2^K \times 2^K\) such that

a) if \(\rho_B(x_1, x_2, \ldots, x_n) = (A, B)\) then \(A \subseteq \{x_n \geq r + \delta\}\) and \(B \subseteq \{x_n \leq r\}\) for some \(r \in \mathbb{R}\). (\(r\) may depend on \(n\) and \(B\).)

b) \(\rho_B(B)\) is a Boolean independent sequence of sets.

Then \(T\) is an \(\ell^1\) tree of order \(\alpha\) with constant \(\delta/2\).

Proof. According to the previous lemma if \(B = \{(x_1), (x_1, x_2), \ldots\}\) then \((x_i)\) is \(2/\delta\) equivalent to the unit vector basis of \(\ell^1\). Thus each branch of \(T\) also satisfies the requirements for an \(\ell^1\) tree, and thus we have an \(\ell^1\)-tree of order \(\alpha\). \(\Box\)

Our next lemma allows us to reduce to the case of positive oscillation index. Below \(p(\alpha) = \inf \{\beta + \rho : \rho + \beta = \alpha\}\). In particular if \(\alpha = \omega^\gamma k + \beta\), where \(\beta < \omega^\gamma\), then \(p(\alpha) = \omega^\gamma k\).

Lemma 3.4. If \(\mathcal{O}^\alpha(\delta, (f_n), K) \neq \emptyset\) and \(\gamma \leq \alpha/2\), then for \(\epsilon = +\) or \(-\), \(\mathcal{O}_{\gamma}^\alpha(\delta, (f_n), K) \neq \emptyset\), for some subsequence \((f_n)\).

Proof. The proof is by induction on \(\alpha\). We will actually prove that if \(t \in \mathcal{O}^\alpha(\epsilon, (f_n), K)\) then there is an infinite set \(L \subseteq \mathbb{N}\) and ordinals \(\gamma\) and \(\lambda\) such that

\[
\min(\gamma + \lambda, \lambda + \gamma) \geq p(\alpha)
\]

and

\[
t \in \mathcal{O}_{\gamma}^\alpha(\delta, (f_n)_{n \in L}, K) \cap \mathcal{O}_{\lambda}^\alpha(\delta, (f_n)_{n \in L}, K).
\]

Suppose that \(p(\alpha) = \omega^n \cdot k\). Because the result depends only on \(p(\alpha)\), we need only consider ordinals \(\alpha\) of the form \(\omega^n \cdot k\) in the induction.

If \(\alpha = 1\), let \(t \in \mathcal{O}^\alpha(\delta, (f_n), K)\) and for \(\epsilon = +\) or \(-\) and \(i \in \mathbb{N}\) consider the set

\[
N_i^\epsilon = \{n : \text{there exists an } M \in \mathbb{N} \text{ such that } \cap_{m \geq M} A_{nm}^\epsilon \cap N_i \neq \emptyset\}
\]

where \(N_i\) is a decreasing sequence of neighborhoods with intersection \(\{t\}\). For at least one choice of \(\epsilon\), \(N_i^\epsilon\) is infinite for all \(i\). For that \(\epsilon\) let \(L\) be an infinite subset of \(\mathbb{N}\) such that \(L \setminus N_i^\epsilon\) is finite for each \(i\). Clearly \(t \in \mathcal{O}_{\gamma_i}^\alpha(\delta, (f_n)_{n \in L}, K)\).

Now suppose that the lemma is true for all \(\beta < \alpha\) and let \(t \in \mathcal{O}^\alpha(\delta, (f_n), K)\). If \(\alpha = \omega^n\), let \(\alpha_i \uparrow \alpha\). The inductive assumption implies that there are sequences \((\gamma_i)\) and \((\lambda_i)\) in \(L_i \subseteq \mathbb{N}\) such that

\[
\min(\gamma_i + \lambda_i, \gamma_i + \lambda_i) \geq p(\alpha_i)
\]

and

\[
t \in \mathcal{O}_{\gamma_i}^\alpha(\delta, (f_n)_{n \in L_i}, K) \cap \mathcal{O}_{\lambda_i}^\alpha(\delta, (f_n)_{n \in L_i}, K).
\]

We may assume that \(L_i \subseteq L_{i-1}\) for all \(i\) and that the sequences \((\gamma_i)\) and \((\lambda_i)\) are non-decreasing. It now follows that if \(L\) is an infinite subset of \(\mathbb{N}\) such that \(L \setminus L_i\) is finite for all \(i\) then

\[
t \in \mathcal{O}_{\gamma}^\alpha(\delta, (f_n)_{n \in L}, K) \cap \mathcal{O}_{\lambda}^\alpha(\delta, (f_n)_{n \in L}, K),
\]

where \(\gamma = \lim \gamma_i\) and \(\lambda = \lim \lambda_i\). We have that \(\min(\gamma + \lambda, \lambda + \gamma) \geq p(\alpha_i)\) for all \(i\) and \(\min(\gamma + \lambda, \lambda + \gamma) \geq \lim p(\alpha_i)\). Thus if \(p(\alpha_i)\) is increasing to \(p(\alpha)\), we are done.
Now we may assume that \( p(\alpha_i) = \omega^\epsilon \cdot k \) for all \( i \) and \( p(\alpha) = \omega^\epsilon \cdot (k + 1) \). (This argument will also apply to the successor ordinal case.) We have that for each \( s \in O^\omega \cdot k (\delta, (f_n), K) \) there is an infinite set \( L_s \) and ordinals \( \gamma_s, \lambda_s \) such that
\[
s \in O^\gamma_s \cdot (\delta, (f_n)_{n \in L_s}, K) \cap O^\lambda_s \cdot (\delta, (f_n)_{n \in L_s}, K)
\]
and
\[
\min (\gamma_s + \lambda_s, \lambda_s + \gamma_s) \geq \omega^\epsilon \cdot k.
\]
Let \( S = \{ s(m) : m \in \mathbb{N} \} \) be a countable dense subset of \( O^\omega \cdot k (\delta, (f_n), K) \). For each \( m \in \mathbb{N} \) there is an infinite set \( L_m \subset L_{m-1} \subset \mathbb{N} \) and ordinals \( \gamma_m \) and \( \lambda_m \) such that
\[
s(m) \in O^\gamma_m \cdot (\delta, (f_n)_{n \in L_m}, K) \cap O^\lambda_m \cdot (\delta, (f_n)_{n \in L_m}, K)
\]
and
\[
\min (\gamma_m + \lambda_m, \lambda_m + \gamma_m) \geq \omega^\epsilon \cdot k.
\]
By a diagonalization argument we may assume that the set \( L \) does not depend on \( m \) and that \( \gamma_m = \omega^\epsilon \cdot k_m \) and \( \lambda_m = \omega^\epsilon \cdot j_m \). Hence
\[
O^\omega \cdot k (\delta, (f_n), K) \subset \bigcup_{i+j=k} O^\omega \cdot i (\delta, (f_n)_{n \in L}, K) \cap O^\omega \cdot j (\delta, (f_n)_{n \in L}, K).
\]
Because each of these sets is closed, we need only consider those sets which contain \( t \). Moreover observe that if
\[
t \in O^\omega \cdot i (\delta, (f_n)_{n \in L}, K) \cap O^\omega \cdot j (\delta, (f_n)_{n \in L}, K)
\]
for two different pairs \((i, j)\) and \((i', j')\) then
\[
t \in O^\omega \cdot i'' (\delta, (f_n)_{n \in L}, K) \cap O^\omega \cdot j'' (\delta, (f_n)_{n \in L}, K)
\]
where \( i'' = \max (i, i') \) and \( j'' = \max (j, j') \) and \( \omega^\epsilon \cdot i'' + \omega^\epsilon \cdot j'' \geq p(\alpha) \), as required. If there is only one such pair \((i, j)\), we may assume that a neighborhood of \( t \) (relative to \( O^\omega \cdot k (\delta, (f_n), K) \)) is contained in \( O^\omega \cdot i''' (\delta, (f_n)_{n \in L}, K) \cap O^\omega \cdot j''' (\delta, (f_n)_{n \in L}, K) \). We have that
\[
t \in O^\omega \cdot (\delta, (f_n), K) \cap O^\omega \cdot i''' (\delta, (f_n)_{n \in L}, K) \cap O^\omega \cdot j''' (\delta, (f_n)_{n \in L}, K)
\]
and thus the case \( \alpha = \omega^\epsilon \) gives that
\[
t \in O^\omega \cdot (\delta, (f_n)_{n \in L'}, K) \cap O^\omega \cdot i''' (\delta, (f_n)_{n \in L'}, K) \cap O^\omega \cdot j''' (\delta, (f_n)_{n \in L'}, K)
\]
for some infinite \( L' \subset L \) and \( \epsilon = + \) or \( - \). Hence
\[
t \in O^\omega \cdot i'' (\delta, (f_n)_{n \in L'}, K) \cap O^\omega \cdot j'' (\delta, (f_n)_{n \in L'}, K),
\]
where \( i = i'' + 1 \) and \( j = j'' \) if \( \epsilon = + \), and \( i = i'' \) and \( j = j'' + 1 \) if \( \epsilon = - \), as required. \( \square \)

Now we are ready to begin the proof of Theorem 3.1.
Proof. By Lemma 3.4 we need only prove it in the simpler case indicated above, i.e., suppose that $O_n(\epsilon) \geq 2$. Let $f(t) = \lim f_n(t)$. Fix $\rho$, $\epsilon/2 > \rho > 0$. For any $g \in C(K)$ let $C(g, k_0) = \{ k : |g(k) - f(k_0)| < \rho \}$.

The proof is by induction on $\alpha$. As usual we must actually prove a little stronger statement to make the induction work.

INDUCTIVE HYPOTHESIS: Suppose that $k_1, k_2, \ldots, k_j$ are a finite number of points in $O_+^n(\epsilon, (f_n), K)$ and $N_{i1}, N_{i2}, \ldots, N_{ij}$ are neighborhoods of $k_1, k_2, \ldots, k_j$, respectively; then for every $\beta \leq \alpha$ there is a tree $T$ on $(f_n - f_m)$ of order $\beta$ such that for $i = 1, 2, \ldots, j$,

$$T_{Ni} = \left\{ (x_{1|N_i}, x_{2|N_i}, \ldots, x_{n|N_i}) : (x_1, x_2, \ldots, x_n) \in T \right\},$$

is an $\ell^1$ tree of order $\beta$ as in Lemma 3.3, i.e., for each $i$ and branch there is a mapping into the pairs of subsets of $K$ satisfying a) and b) with $\delta = \epsilon/2$ and $r = \epsilon/4$. Moreover we may assume that for each pair of sets

$$\rho_B(f_n - f_m, \ldots, f_n - f_m) = (A_j, B_j),$$

$$A_j \subset \bigcap_{i=1}^{j-1} C(f_n, k) \cap C(f_m, k) \cap C(f_m, k) \cap \{ s : |f_n(s) - f_n(k)| < \rho \},$$

and

$$B_j \subset \bigcap_{i=1}^{j} C(f_n, k') \cap C(f_m, k') \text{ for some } k \text{ and } k' \text{ in } K.$$ 

Assume the inductive hypothesis holds for all $\beta < \alpha$. First suppose that $\alpha$ is not a limit ordinal and that $O_+^n(\epsilon, (f_n), K) \neq \emptyset$. Let $(k_i)$ and $(N_i)$ be as above. For each $i$ there is an $N_i \in \mathbb{N}$ such that for each $n \geq N_i$, there is an $M_i^n$ such that

$$\cap_{m \geq M_i^n} A_{n,m}^+ \cap O_+^{\alpha-1}(\epsilon, (f_n), K) \cap N_i \neq \emptyset.$$ 

Let $n \geq \max \{ N_i \}$ such that $|f_n(k_i) - f(k_i)| < \rho$, let $M = \max \{ M_i^n \}$ and choose a point $k_i' \in \cap_{m \geq M} A_{n,m}^+ \cap O_+^{\alpha-1}(\epsilon, (f_n), K) \cap N_i$, for each $i$. We may also assume that for all $p \geq M$,

$$|f_p(k_i) - f(k_i)| < \rho$$

and

$$|f_p(k_i') - f(k_i')| < \rho,$$

for $i = 1, 2, \ldots, j$. Fix $m \geq M$ and for each $i$ let $N_i''$ be a neighborhood of $k'_i$ contained in

$$N_i \cap A_{n,m}^+ \cap C(f_m, k_i') \cap \{ k : |f_n(k) - f_n(k_i')| < \epsilon/8 \}$$

and let $N_i'''$ be a neighborhood of $k_i$ contained in

$$N_i \cap C(f_m, k_i) \cap C(f_n, k_i).$$ 

Now by the inductive hypothesis for every $\beta < \alpha$ there is a tree $T$ on $(f_p - f_q)$ such that $T_{N_i'}$ and $T_{N_i''}$ are $\ell^1$ trees of order $\beta$ for $i = 1, 2, \ldots, j$, and satisfy a) and b) of the lemma with $\delta = \epsilon/2$ and $r = \epsilon/4$. Let

$$T' = \{(f_n - f_m, x_1, x_2, \ldots, x_n) : (x_1, x_2, \ldots, x_n) \in T \}.$$
Clearly this is a \((\beta + 1)\) tree on \((f_n)\). We need to check the hypothesis of the lemma for \(T_{N_i}^\ell\), \(i = 1, 2, \ldots, j\). To define \(\rho_{B_i}^\ell\) on a branch

\[
B_i^\ell = \{(f_n - f_m), (f_n - f_m, x_1), (f_n - f_m, x_1, x_2)\} \text{ of } T_{N_i}^\ell,
\]

we let

\[
\rho_{B_i}^\ell (f_n - f_m, x_1, x_2, \ldots, x_n) = \rho_B((x_1, x_2, \ldots, x_n))
\]

and

\[
\rho_{B_i}^\ell ((f_n - f_m)) = (N_i^n, N_i^n)\,
\]

where \(\rho_B\) denotes the mapping from the branch \(B = \{(x_1), (x_1, x_2), \ldots\} \) of \(T_{N_i}^\ell\). If \(r = \epsilon/4\) and \(\delta = \epsilon/2\), the hypothesis of the lemma is satisfied and thus \(T_{N_i}^\ell\) is an \(\ell^1\)-tree of order \(\beta + 1\). Because this is true for all \(\beta < \alpha\) we get an \((\alpha + 1)\) \(\ell^1\)-tree.

For limit ordinals the conclusion is obvious. (Note we can actually get an \((\alpha + 1)\) \(\ell^1\)-tree in this case as well.)

The moreover assertion allows us to conclude that the we can construct a \(\ell^1\)-tree on \((f_n)\) of the same order. Indeed we claim that the tree obtained by replacing in each coordinate \(f_n - f_m\) by \(f_n\) is the required tree. First if \(f\) is continuous we can choose a neighborhood \(N\) of the point \(k_0 \in O^n(\epsilon, (f_n), K)\) so that \(N \subseteq \{k : |f(k) - c| < \rho\}, \) where \(c = f(k_0)\), and restrict all of the functions to this set \(N\). The proof then shows that the sets \(\{k : f_n \geq c + \epsilon - \rho\}\) and \(\{k : f_n \leq c + \rho\}\) are Boolean independent for any \((f_{n_1} - f_{m_1}, f_{n_2} - f_{m_2}, \ldots, f_{n_j} - f_{m_j})\) in the tree constructed.

If \(f\) is not continuous we use the following lemma.

**Lemma 3.5.** If \((x_n)\) is a uniformly bounded sequence in a Banach space \(X\) such that \(|\sum a_n x_n| \geq \delta \sum |a_n|\) for all \((a_n) \in \mathbb{R}^N\) and \(y \in X\), \(|y| intervenes, then

\[
||\sum a_n (x_n + y) || \geq \delta' \sum |a_n|,
\]

where \(\delta' = \max \{d(y, |x_n|)/2, \delta - ||y||\}\).

We omit the simple proof.

Our tree was actually constructed using sequences \((f_{n_i} - f_{m_i})_{i=1}^N\) and sets \((A_{n_i}, B_{n_i})\) where \(|f_{n_i}(k) - f(k')| < \rho\) for all \(k \in \cap_{i=1}^N A_{n_i}^{k_i}\), for all \(j\), for some \(k(\epsilon_1, \epsilon_2, \ldots, \epsilon_N) \subseteq \cap_{i=1}^N A_{n_i}^{k_i}\). Thus we can replace each \(f_{m_i}\) by \(g\) where

\[
g(k) = \begin{cases} f(k(\epsilon_1, \epsilon_2, \ldots, \epsilon_N)) & \text{if } k \in \cap_{i=1}^N A_{n_i}^{k_i}, \\ 0 & \text{otherwise}\end{cases}
\]

with a loss of at most \(\rho\). Observe that because \(\lim_{N} f_n = f\), \(\liminf_{N} d(f, [f_n - f : n \geq N]) \geq ||f||\). Thus we get the estimate

\[
\delta' \geq \max \left\{ \left(\frac{\epsilon}{2} - \rho\right) ||f||, \epsilon - ||f|| - \rho \right\}
\]

from the lemma. \(\square\)

Let us now examine the relationship between the oscillation index and \(\ell^1\)-spreading models.
Proposition 3.6. Let \((f_n)\) be a pointwise converging sequence on a Polish space \((K,d)\) with limit \(0\). If \(k \in O^\omega(\epsilon,(f_n),K)\) and \(N^\prime\) is a neighborhood of \(k\), then for every infinite \(M \subset \mathbb{N}\) there is an \(L \subset M\) such that \((f_n|N^\prime)_{n \in L}\) has \(\ell^1\) spreading model constant at least \(\epsilon\).

Proof. Fix \(j \in \mathbb{N}\) and choose \(L \subset M\) such that \((f_n|N)_{n \in L}\) has a spreading model. Because \(k \in O^\omega(\epsilon,(f_n),K,\delta)\), a careful examination of the proof of the previous theorem shows that there is an \(\epsilon = + or -\) such that given any \(N\) there is an \(L \subset \{N,N+1,\ldots\}\) with cardinality \(j\) with \(N \cap \cap_{\ell \in L} A_{\ell,m} = \emptyset\), for all large \(m\). Therefore \((f_n|N)_{n \in L}\) has \(\ell^1\) spreading model constant at least \(\epsilon\). \(\square\)

Next we will show that the oscillation index and the spreading model index are closely related to the oscillation index. This index is defined in terms of the spreading model constant and should be compared with the averaging index.

We wish to present two more examples, but before doing so we will introduce another ordinal index which we call the spreading model index and show that it is closely related to the oscillation index. Let \(S^\alpha(\epsilon,(f_n),K)\) be defined and assuming that \(S^\alpha(\epsilon,(f_n),K)\) has been defined let

\[
S^{\alpha+1}(\epsilon,(f_n),K) = \{x^*:\text{ for every neighborhood } N \text{ of } x^* \text{ relative to } S^\alpha(\epsilon,(f_n),K) \text{ and infinite } M \subset \mathbb{N} \text{ there is an } L \subset M \text{ with } \ell^1 - SP((f_n|N))_{n \in L} \geq \epsilon\}.
\]

As usual if \(\alpha\) is a limit ordinal \(S^\alpha(\epsilon,(f_n),K) = \cap_{\beta < \alpha} S^\beta(\epsilon,(f_n),K)\) and the ordinal index will be denoted by \(o(S,\epsilon)\).

Next we will show that the oscillation index and the spreading model index measure essentially the same thing.

Proposition 3.7. Suppose that \((f_n)\) is a weakly null sequence in \(B_{C(K)}\) for some compact metric space \(K\). Then

i) if \(S^1(\epsilon,(f_n),K) \neq \emptyset, O^1(\epsilon',(f_n),K) \neq \emptyset\) for every \(\epsilon' < \epsilon\).

ii) if \(O^\omega(\epsilon,(f_n),K) \neq \emptyset\), for some \(\epsilon > 0\), \(S^1(\epsilon,(f_n),K) \neq \emptyset\).

Proof. Suppose that \(O^1(\epsilon',(f_n),K) = \emptyset\). Then for each \(k \in K\) there is a neighborhood \(N^\prime_k\) of \(k\) and a subsequence \((f_{n_k})_{n \in M_k}\) such that \(\|f_{n_k}|N^\prime_k\| \leq \epsilon'\) for all \(n \in M_k\). Clearly \(\|\sum_{n \in F} f_{n_k}|N^\prime_k\| \leq |F|\epsilon'\) for every finite subset \(F\) of \(M_k\). Hence \(S^1(\epsilon,(f_n),K) = \emptyset\) for every \(\epsilon > \epsilon'\).

The second assertion follows from Proposition 3.6. \(\square\)

Corollary 3.8. Suppose that \((f_n)\) is a weakly null sequence on a compact metric space \(K\). Then

i) If \(o(S,\epsilon) \geq \alpha\), then \(O(\epsilon') \geq \alpha\), for every \(\epsilon' < \epsilon\).

ii) If \(O(\epsilon) \geq \omega^{1+\alpha}\), then \(o(S,\epsilon) \geq \omega^{\alpha}\).

Example 4. \(L^1\). Of course \(\ell(L^1,1) = \omega_1\). However if \((f_n)\) is a weakly convergent sequence in \(L^1\), then \((f_n)\) is uniformly integrable and hence by Dor’s Theorem [Dor] for every \(K < \infty\) there is an \(n\) such that if \(F\) is a set of integers of cardinality \(n\), \((f_j)_{j \in F}\) is not \(K\) equivalent to the unit vector basis of \(\ell^1_n\). Therefore \(\ell^1 - SP((f_n)_{n \in L}) = \emptyset\) for all \(L\) and thus \(O^\omega(\epsilon,(f_n),K) = \emptyset\) for every \(\epsilon > 0\).
Example 5. The spaces Szlenk [Sz] used to show that there are reflexive spaces with arbitrarily large (countable) Szlenk index are defined inductively as $X_1 = \ell^2$, $X_{n+1} = (X_n \oplus \ell^2)_1$, and for a limit ordinal $\alpha$, $X_\alpha = (\sum_{\beta < \alpha} X_\beta)_2$. The $\ell^1$-index of $X_\alpha = \alpha$ for $\varepsilon = 1$ and increases to $\alpha \omega$ as $\varepsilon$ goes to 0.

Now let us consider the oscillation index. First suppose that $\alpha$ is a limit ordinal. If $(f_n)$ is a weakly null sequence in $X_\alpha$, then by passing to a subsequence (which can only increase the index) we may assume that $f_n = g_n + h_n$ where $g_n \in \sum_{\beta \leq \lambda} X_\beta$ for all $n$ for some $\lambda < \alpha$ and $h_n \in \sum_{\beta \in B_n} X_\beta$ where $B_n = \{\beta : \beta_n < \beta \leq \beta_{n+1}\}$ for all $n$ and $\beta_1 > \lambda$. Note that $|h_n| = \ell^2$ and thus the $\ell^1$ spreading model index of $X_\alpha$ is the supremum of the indices of $X_\beta$, $\beta < \alpha$. Now suppose that $\alpha = \beta + k$ for some integer $n$. Then $X_\alpha = \left( X_\beta \oplus \ell^2 \oplus \ell^2 \oplus \cdots \ell^2 \right)_1$, and we may write $f_n = g_n + h_n$ where $g_n \in X_\beta$ and $h_n \in \sum_{i=1}^k \oplus \ell^2$ for all $n$. However we again have that $\ell^1 - SP(f_n) = \ell^1 - SP(g_n)$ and hence the spreading model index of $(f_n)$ is the spreading model index of $(g_n)$. Thus the spreading model index of $X_\alpha$ is the same as the index of $\ell^2$, for all $\alpha$, namely 0.

This last family of examples illustrates the fact that the oscillation index and the spreading model index are really measuring something stronger than the existence of many $\ell^1$’s in a space.

Next we will examine the ordinal index of a Baire-1 function $f$ on a Polish space $(K, d)$. The index is defined by considering two real numbers $c$ and $d$, $c \leq d$, and the disjoint $G_\beta$ sets,

$$C = \{k \in K : f(k) \leq c\} \text{ and } D = \{k \in K : f(k) \geq d\}.$$  

$L(f, c, d)$ is the smallest ordinal $\alpha$ such that there is a decreasing family of closed sets $F_\beta$, $\beta \leq \alpha$, with $F_0 = K$, $F_\alpha = \emptyset$, and for all $\beta < \alpha$, $F_\beta \setminus F_{\beta+1}$ is disjoint from $C$ or from $D$ and at a limit ordinal $\gamma$, $F_\gamma = \bigcap_{\beta < \gamma} F_\beta$. (See [Bo] where the definition is given in complementary terms or [K,p452].)

Bourgain shows that if $(f_n)$ is a pointwise converging sequence of continuous functions with limit $f$ and $\epsilon < (d - c)/2$ then $\omega^\varepsilon(f_n, f, \varepsilon) + 1$ is greater than $L(f, c, d)$. (Actually his result gives a slightly smaller bound.) We wish to show that in fact the oscillation index is also large. The proof of the following proposition is similar to that of Lemma 5, [H-O-R].

**Proposition 3.9.** Suppose that $(f_n)$ is a pointwise converging sequence of continuous function on a compact metric space $K$. Then if $d - c > \varepsilon$ and if $L(f, c, d) = \beta + m$, where $\beta$ is a limit ordinal and $m < \omega$, then $\mathcal{O}_+ \epsilon(f_n, K) \geq \beta + (m - 1)/2$.

**Proof.** Consider the following family of closed sets where $C = \{k : f(k) \leq c\}$ and $D = \{k : f(k) \geq d\}$, $F_0 = K$, $F_1 = \overline{D}$, $F_2 = F_1 \cap C$, $F_3 = F_2 \cap D$, and in general, $F_{\alpha+2n+1} = F_{\alpha+2n} \cap \overline{D}$ and $F_{\alpha+2n+2} = F_{\alpha+2n+1} \cap C$ if $\alpha$ is even and $n \in \mathbb{N}$. (Limit ordinals are even.) If $\alpha$ is a limit ordinal $F_\alpha = \cap_{\gamma < \alpha} F_\gamma$. 
It is easy to see that if \( \alpha \) is even \((F_{\alpha} \setminus F_{\alpha+1}) \cap D = \emptyset \) and \((F_{\alpha+1} \setminus F_{\alpha+2}) \cap C = \emptyset \). Next we will show that \( F_{\alpha+2} \cap D \subset O_+^1 (\epsilon, (f_n), F_{\alpha+1}) \) for all \( \alpha \) even.

Let \( d' \in F_{\alpha+2} \cap D \). This implies that \( d' \in F_{\alpha+1} \cap C \cap D \). \( F_{\alpha+1} \cap C \cap D \) contains points from \( D \) which are (non-trivial) limits of points in \( F_{\alpha+1} \cap C \) and hence there exists a sequence \((c_k)\) in \( F_{\alpha+1} \cap C \) with limit \( d' \). Moreover we may assume that no \( c_k \) is in \( O_+^1 (\epsilon, (f_n), F_{\alpha}) \) (else \( d' \) would be also). Choose \( N \in \mathbb{N} \) such that \(|f_n(d') - f(d')|\) is less than \( \delta/4 \) for all \( n \in N \), where \( 0 < \delta < (d - c - \epsilon) \). For each \( k \in \mathbb{N} \) there is an \( M_k \) such that \( f_m(c_k) < \delta/4 + f(c_k) \) for all \( m \geq M_k \). Now if \( \mathcal{N} \) is a neighborhood of \( d' \) and \( n \geq N \) then for some \( L \in \mathbb{N}, c_k \in \mathcal{N} \) and \( f_n(c_k) > f(d') - \delta/4 \), for all \( k \geq L \). Because

\[
f_n(c_k) - f_m(c_k) > f(d') - \delta/4 - f(c_k) - \delta/4 \geq d - c - \delta/2,
\]

for all \( k \geq L \) and \( m \geq M_k \), \( c_k \in \mathcal{N} \cap \cap_{m \geq M_k} A_{\beta, m}^+ \). Hence \( d' \in O_+^1 (\epsilon, (f_n), F_{\alpha+1}) \) and \( F_{\alpha+3} = F_{\alpha+2} \cap D \subset O_+^1 (\epsilon, (f_n), F_{\alpha+1}) \), for all \( \alpha \) even.

Because \( O_{\beta+1}^+ (\epsilon, (f_n), K) = O_+^1 (\epsilon, (f_n), O_+^\beta (\epsilon, (f_n), K)) \). A simple induction argument shows that \( F_{\alpha+2m+1} \subset O_+^m (\epsilon, (f_n), F_{\alpha+1}) \) for any integer \( m \) and even ordinal \( \alpha \). It follows then that if \( \beta \) is a limit ordinal \( F_{\beta} \subset O_+^\beta (\epsilon, (f_n), K) \) and for any \( m \in \mathbb{N}, F_{\beta+2m+1} \subset O_+^{\beta+m} (\epsilon, (f_n), K) \). \( \square \)

**Corollary 3.10.** Suppose that \( f \) is a Baire-1 function on a compact metric space \( K \) and that \( (f_n) \) is a sequence of continuous functions on \( K \) with \(|f_n| = 1\) which converge to \( f \) pointwise. If \( L(f, c, d) = \beta + m \), where \( c < d \), \( \beta \) is a limit ordinal and \( m \in \mathbb{N} \), then for any \( \epsilon < d - c \) there is an \( \epsilon/2 \) \( \ell^1 \)-tree of order \( \beta + m/2 \) on \((f_n - f_m)\), and there is an \( \epsilon/2 \) \( \ell^1 \)-tree on \((f_n)\) of the same order.

**Proof.** From the above proposition we get that \( O_+^{\beta+(m-1)/2} (\epsilon, (f_n), K) \neq \emptyset \). The proof of Theorem 3.1 shows that there is an \( \ell^1 \)-tree on \((f_n - f_m)\) of order \( \beta + (m - 1)/2 + 1 \) with lower estimate \( \epsilon \). The second assertion follows from an examination of the proofs of Proposition 3.9 and Theorem 3.1. It is easy to see that in the proof of the theorem we can replace \( A_{\beta, m}^+ \) by \( \{k : f_n(k) - c > \epsilon + \rho\} \), where \( \rho < d - c - \epsilon \), and always choose the points \( k_i \) and \( k_i' \) from \( C \). We then get that the sets \( \{k : f_n(k) > c + \epsilon + \rho\}, \{k : f_n(k) < c + \rho\}\}_{s=1} \) are Boolean independent for any node \((f_m, f_{m+1}, \ldots, f_n)\) of the tree constructed. \( \square \)

### 4. Construction of weakly null sequences with large oscillation index

In this section we wish to generalize the construction of Schreier of a weakly null sequence \((x_n)\) with no subsequence having the Banach-Saks property. As observed by Pelczynski and Siszlen [P-S] the Schreier sequence is a 1-suppression unconditional basis in \( C(\omega^\omega) \). Our goal is to prove

**Theorem 4.1.** For every \( \alpha < \omega_1 \) there is a weakly null sequence \((x_n^\alpha)\) in \( C(\omega^\omega) \) with \( O^\omega (1 - \epsilon, (x_n^\alpha), \omega^\omega) \neq \emptyset \) for every \( \epsilon > 0 \). Moreover for each \( \alpha, (x_n^\alpha) \) is a sequence of indicator functions and \((x_n^\alpha)\) is a 1-suppression unconditional basic sequence.

To construct these sequences and verify their properties it is useful to have several different viewpoints. The first viewpoint is contained in the following result.
Proposition 4.2. Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$ such that

i) if $F \in \mathcal{F}$ and $G \subset F$ then $G \in \mathcal{F}$, i.e., the family is adequate.

ii) $\{n\} \in \mathcal{F}$ for all $n \in \mathbb{N}$.

iii) if $F_j \in \mathcal{F}$ for $j = 1, 2, \ldots$ and $1_F$ converges pointwise to $1_F$ then $F \in \mathcal{F}$.

Let $x_n = 1_{\{F \in \mathcal{F} : n \in F\}}$ for $n = 1, 2, \ldots$.

Then $\mathcal{F}$ is a countable compact metric space under the topology induced by identifying $\mathcal{F}$ with $\{1_F : F \in \mathcal{F}\}$ under the topology of pointwise convergence. $(x_n)$ is a weakly null 1-suppression unconditional basic sequence in $C(\mathcal{F})$.

Proof. The first assertion is an easy consequence of iii) and we omit the argument. Note that $x_n(F) \neq 0$ if and only if $n \in F$. Hence

$$\|\sum a_n x_n\| = \sup \left\{ \|\sum a_n\| : F \in \mathcal{F} \right\}.$$  

If $G \subset \mathbb{N}$, then by i)

$$\|\sum_{n \in G} a_n x_n\| = \sup \left\{ \|\sum_{n \in G \cap F} a_n\| : F \in \mathcal{F} \right\} = \sup \left\{ \|\sum_{n \in F} a_n\| : F \in \mathcal{F} \text{ and } F \subset G \right\}$$

Clearly this is not greater than $\|\sum a_n x_n\|$. Because each $F \in \mathcal{F}$ is finite, $x_n(F) \neq 0$ for only finitely many $n$. Thus $(x_n)$ is weakly null. □

This proposition gives us an easy way of defining and verifying the properties of the Schreier sequence. As in the previous section let

$$\mathcal{F}_1 = \{F \subset \mathbb{N} : \min F \geq \text{card } F\}$$

(We consider $\emptyset$ to be in $\mathcal{F}_1$.) If $(x_n)$ is defined as in the proposition then it follows that $(x_n)$ is a 1-unconditional basic sequence and is weakly null. Finally if $L \subset \mathbb{N}$ is infinite and for each $k \in \mathbb{N}$, we let $L_k$ be the first $k$ elements of $L$

$$\|\sum_{n \in L_{2k}} x_n\| \geq k$$

because $L_{2k} \setminus L_k \in \mathcal{F}_1$ and

$$\|\sum_{n \in L_{2k+1}} x_n\| \geq k$$

because $L_{2k+1} \setminus L_{k+1} \in \mathcal{F}_1$. Hence

$$\|\sum_{n \in L_k} x_n\|/k \geq (k - 1)/(2k) \text{ for all } k,$$

and therefore $(x_n)_{n \in L}$ fails the Banach-Saks property. (See [D1,p.78].)

The major drawback to this representation of the Schreier sequence is that it is difficult to understand the topology of $\mathcal{F}_1$ and its relation to $(x_n)$. In this section we will prove some results twice. First we will give proofs based on representations like that above for the Schreier sequence. The second will be given using trees. We
have found that this second viewpoint is more intuitive (It is easy to draw pictures of the trees.) and we used it to establish these results originally.

Our next goal then is to use trees to describe the Schreier sequence and the underlying topological space and in particular to show that this sequence and its generalizations could be obtained by beginning with the coordinate functions on the Cantor set, and then essentially restricting them to suitable subsets of the Cantor set. More precisely, if we let $C = \{-1, 1\}^\mathbb{N}$ and $r_n ((\epsilon_i)) = \epsilon_n$, then $x_n = (r_n + 1)/2$ is a sequence of indicator functions. If $K$ is a compact subset of $C$, then the sequence $(x_n|K)$ is a sequence of indicator functions in $C(K)$ which will be equivalent to the Schreier sequence if $K$ is properly chosen.

Now let us work backwards from a sequence of indicator functions to find an appropriate minimal underlying topological space. Suppose that $(x_n)$ is a sequence of indicator functions on a set $K$. Define a tree $T$ on $\{-1, 1\}$ by

$$T = \bigcup_{n=1}^{\infty} \{(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) : \cap_{i=1}^{n} (\text{supp } x_i)^{\epsilon_i} \neq \emptyset\}$$

where

$$\text{supp } x_i = (\text{supp } x_i)^1 = \{k \in K : x_i(k) = 1\}$$

and

$$(\text{supp } x_i)^{-1} = K \setminus (\text{supp } x_i)^1.$$

Let $\overline{T} = T \cup \{(\epsilon_1, \epsilon_2, \ldots) : \cap_{n=1}^{\infty} (\text{supp } x_i)^{\epsilon_i} \neq \emptyset \text{ for all } n\}$. $\overline{T}$ is also a tree and both are subtrees of the full dyadic tree

$$D = \bigcup_{n=1}^{\infty} \{-1, 1\}^n \cup \{-1, 1\}^\mathbb{N}.$$ 

$D$ has a natural topology given by coordinate-wise convergence. Of course in this topology $\{-1, 1\}^\mathbb{N}$ is just the Cantor set and each finite sequence is an isolated point. Also any infinite sequence is the limit of its restrictions to the first $n$ coordinates, i.e., the nodes above it. Note also that this tree is closely related to the Boolean independence tree $T \left\{ (\text{supp } x_n)^1, (\text{supp } x_n)^{-1}\right\}$.

**Lemma 4.3.** $\overline{T}$ is the closure of $T$ in $D$.

**Proof.** Obvious. □

Throughout the remainder of this section we will assume that the sequence $(x_n)$ is pointwise convergent to 0 on $K$, $K$ is a compact metric space and each $x_n$ is continuous.

Given such a sequence $(x_n)$ the tree $T((x_n))$ defined above will be called the tree associated to $(x_n)$. Because $K$ is compact and $(x_n)$ is weakly null, $T$ contains no elements with infinitely many coordinates equal to 1 and every node is on an infinite branch. Hence $T$ is countable and therefore homeomorphic to some countable ordinal in the order topology. Unfortunately because the tree is not well founded, one cannot use the order of the tree $T$ to determine the topological type. To get around this problem we must study the relationship between the topology of $T$ and the structure of the tree.

Before we embark on the study of the topology let us consider a property of trees is analogous to property i) of Proposition 4.2. In an unpublished paper H.P. Rosenthal introduced the following notation.
**Definition.** Let us say that a tree $T$ on $\{-1,1\}$ is *weakly independent* if $(\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_j) \in T$ implies that for all $(\epsilon'_1, \epsilon'_2, \epsilon'_3, \ldots, \epsilon'_j)$ such that $\epsilon'_i = -1$ if $\epsilon_i = -1$ and $\epsilon'_i = 1$ or -1 if $\epsilon_i = 1$, $(\epsilon'_1, \epsilon'_2, \epsilon'_3, \ldots, \epsilon'_j) \in T$. Say that a sequence of indicator functions is *weakly independent* if the associated tree is.

Rosenthal also proved the following result in a slightly different form.

**Proposition 4.4.** A weakly null sequence of (non-zero) indicator functions on a compact metric space $K$ determines a weakly independent tree if and only if it is a 1-suppression unconditional basic sequence.

**Proof.** First suppose that $(x_n)$ is a sequence of indicator functions which is a 1-suppression unconditional basic sequence and $T$ is the associated tree. Then for any sequence of real numbers $(a_n)$ we have

$$||\sum a_n x_n|| = \sup \left\{ \left| \sum a_n x_n(k) \right| : k \in K \right\} = \sup \left\{ \left| \sum_{n: k \in \text{supp } x_n} a_n \right| : k \in K \right\}.$$ 

In particular suppose that $F \subset \{1,2,\ldots, j\}$ with

$$\bigcap_{n \in F} (\text{supp } x_n)^{-1} \cap \bigcap_{n \notin F, n \leq j} \text{supp } x_n \neq \emptyset,$$

i.e., the node $(\epsilon_1, \epsilon_2, \ldots, \epsilon_j) \in T$, where $\epsilon_i = 1$ if $n \notin F$ and $n \leq j$, and $\epsilon_i = -1$ if $n \in F$. In order to check weak independence it is sufficient to check the condition on a lower node. Hence we may assume that if $\{1,2,\ldots,j\} \supset G \supset F$, then $G \neq \{1,2,\ldots,j\}$, and we must show that

$$\bigcap_{n \notin G} \bigcap_{n \leq j} \text{supp } x_n \cap \bigcap_{n \in G} (\text{supp } x_n)^{-1} \neq \emptyset.$$

Because $(x_n)$ is 1-suppression unconditional

$$(j - \text{card } G)(1 + \text{card } G) = ||(1 + \text{card } G) \sum_{n \notin G} x_n + \sum_{n \notin G} (-1)x_n||$$

and thus there exists

$$k \in \bigcap_{n \notin G} \bigcap_{n \leq j} (\text{supp } x_n) \cap \bigcap_{n \in G} (\text{supp } x_n)^{-1}.$$

This implies that the node $(\epsilon_1, \epsilon_2, \ldots, \epsilon_j) \in T$ where $\epsilon_i = 1$ if $i \notin G$ and $i \leq j$ and $\epsilon_i = -1$ if $i \in G$. Thus $T$ is weakly independent.

Conversely suppose that $(x_n)$ is a weakly null sequence of indicator functions and that the associated tree $T$ is weakly independent. Then for any sequence of real numbers $(a_n)$ and finite subset $F$ of $\mathbb{N}$, we claim that

$$||\sum a_n x_n|| = \sup \left\{ \left| \sum_{n: k \in \text{supp } x_n} a_n \right| : k \in K \right\}.$$
\begin{align*}
\geq \sup \left\{ \left| \sum_{n \in F, k \in \text{supp } x_n} a_n \right| : k \in K \right\} = \left\| \sum_{n \in F} a_n x_n \right\|.
\end{align*}

To see the inequality suppose that \( k \) is any point in \( K \) and \( H = \{ n : x_n(k) = 1 \} \). Because \( T \) is weakly independent there is a point \( k' \) in \( K \) such that \( \{ n : x_n(k') = 1 \} = F \cap H \). Hence each sum on the righthand side of the inequality also occurs on the left. \( \Box \)

**Corollary 4.5.** Suppose that \((x_n)\) is a weakly null sequence of non-zero indicator functions in \( C(K) \) for some compact metric space \( K \). Then the following are equivalent.

i) The tree associated to \((x_n)\) is weakly independent.

ii) \( F = \{ F \subset \mathbb{N} : F = \{ n : x_n(k) = 1 \} \text{ for some } k \in K \} \) is adequate.

iii) \((x_n)\) is a 1-suppression unconditional basic sequence.

**Proof.** We have already shown that ii) \( \implies \) iii) and i) \( \iff \) iii). i) \( \implies \) ii) is immediate from the definition of weakly independent. \( \Box \)

**Remark 4.6.** Rosenthal showed (unpublished) that any weakly null sequence of indicator functions in a \( C(K) \) space has a subsequence which is an unconditional basic sequence by showing that there is a weakly independent subsequence.

**Remark 4.7.** Note that if \((x_n)\) is weakly independent and \( T \) is the associated tree then

\[
\bigcup_k \{ (n_1, n_2, \ldots, n_k) : (\epsilon_1, \epsilon_2, \ldots, \epsilon_m) \in T \text{ where } \epsilon_{n_i} = 1 \text{ for } i = 1, 2, \ldots, k \}
\]

\[
= T \left( \left( \text{supp } x_n, (\text{supp } x_n)^{-1} \right) \right),
\]

the Boolean independence tree.

In order to write tree elements more efficiently let us introduce the following notational conventions: If \( \pi = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k) \) and \( \pi' = (\epsilon'_1, \epsilon'_2, \ldots, \epsilon'_l) \) are two elements in \( \bigcup_{n=1}^{\infty} S^n \), for some set \( S \), then \( \pi \uparrow \pi' = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k, \epsilon'_1, \epsilon'_2, \ldots, \epsilon'_l) \), the concatenation of \( \pi \) and \( \pi' \). Let \( \pi_0 \) be the empty tuple, \( \pi_1 = (-1) \), and inductively define \( \pi_{n+1} = \pi_n \uparrow \pi_1 \), \( n = 1, 2, \ldots \). Also let \( \pi_\omega = (-1, -1, \ldots) \).

We will next introduce a derivation on a tree \( T \subset D \). Let

\[
\delta^1(T) = \{ \bar{t} \in T : \text{ there are infinitely many } \pi \in \overline{T \setminus T} \text{ with } \bar{t} \prec \pi \}
\]

and if \( \delta^\alpha(T) \) has been defined let \( \delta^\alpha+1(T) = \delta^1(\delta^\alpha(T)) \). If \( \beta \) is a limit ordinal let \( \delta^\beta(T) = \cap_{\alpha < \beta} \delta^\alpha(T) \). Finally define \( \delta(T) = \inf \{ \alpha : \delta^\alpha(T) = \emptyset \} \).

**Proposition 4.8.** Suppose that \( T \) is a weakly independent subtree of the dyadic tree with no infinite nodes, no infinite nodes in \( \overline{T} \) with infinitely many coordinates equal to 1, and every node of \( T \) is on some infinite branch. Then \( \delta(T) \) determines the homeomorphic type of \( \overline{T} \) up to the number of points in the last derived set, i.e., \( \delta^\alpha(T) = \emptyset \) if and only if \( \overline{T}^{(1+\alpha)} = \emptyset \).

**Proof.** First note that each element of \( \overline{T \setminus T} \) is in the first derived set of \( \overline{T} \) and, in fact, \( \overline{T \setminus T} \) is the first derived set. Now let us set up a correspondence between
the derived sets of $\mathcal{T}$ and the subscripts of $\mathcal{T}$. If $C$ is any closed subset of $\mathcal{T}\setminus \mathcal{T}$ then there is a tree $\mathcal{T}(C)$ with $\mathcal{T}(C)\setminus \mathcal{T}(C) = C$. Indeed let $\mathcal{T}(C)$ be the set of all nodes $\mathcal{x}$ of $\mathcal{T}$ for which there is some element $c$ of $C$ below $\mathcal{x}$.

As above let $C$ be a closed subset of $\mathcal{T}\setminus \mathcal{T}$. We claim that $\mathcal{T}(C^{(1)}) = \delta^1(\mathcal{T}(C))$. Suppose that $\mathcal{x} \in \mathcal{T}(C^{(1)})$. Then $\mathcal{x}$ is above some element $\mathcal{c}$ of $C^{(1)}$. Say $\mathcal{x} = \mathcal{y} + \mathcal{z}_k + \mathcal{w}_k$ where $\mathcal{y}$ is possibly the empty tuple. Hence there are distinct points $\mathcal{z}_k$ in $C$ which converge to $\mathcal{y}$ and therefore for large $k$, $\mathcal{z}_k = \mathcal{y} + \mathcal{z}_k + \mathcal{w}_k$, where the $\mathcal{z}_k$'s are distinct. This implies that $\mathcal{x} \in \delta^1(\mathcal{T}(C))$. Conversely suppose that $\mathcal{x} \in \delta^1(\mathcal{T}(C))$. Then there is a sequence of distinct points $(\mathcal{x}_i)$ in $\mathcal{T}(C)\setminus \mathcal{T}(C) = C$ such that $\mathcal{x}_i > \mathcal{y}$ for all $i$. By passing to a subsequence we may assume that $\mathcal{x}_i \to \mathcal{x} \in C$. Clearly $\mathcal{x} \in C^{(1)}$ and $\mathcal{x} > \mathcal{y}$, therefore $\mathcal{x} \in \mathcal{T}(C^{(1)})$.

Because $\mathcal{T}\setminus \mathcal{T} = \mathcal{T}^{(1)}$ and $\mathcal{T}(\mathcal{T}\setminus \mathcal{T}) = \mathcal{T}$, we have that

$$\delta^1(\mathcal{T}) = \mathcal{T}((\mathcal{T}\setminus \mathcal{T})^{(1)}) = \mathcal{T}(\mathcal{T}^{(1)^{(1)})} = \mathcal{T}(\mathcal{T}^{(2)}).$$

We claim that for every $\alpha < \omega_1$,

$$\delta^\alpha(\mathcal{T}) = \mathcal{T}(\mathcal{T}^{(1)^{(\alpha)})} = \mathcal{T}(\mathcal{T}^{(1+\alpha)})..$$

Indeed, if this is true for $\alpha$ we have by the previous claim that

$$\delta^{\alpha+1}(\mathcal{T}) = \delta^1(\mathcal{T}^{(1)^{(\alpha)}}) = \mathcal{T}(\mathcal{T}^{(1)^{(\alpha)^{(1)}}}) = \mathcal{T}(\mathcal{T}^{(1)^{(\alpha+1)}}).$$

Also observe that if $(C_i)$ is a decreasing family of closed subsets of $\mathcal{T}\setminus \mathcal{T}$ then $\mathcal{T}(\cap_{i=1}^\infty C_i) = \cap_{i=1}^\infty \mathcal{T}(C_i)$. Therefore if $\alpha_i \uparrow \alpha$ and the claim is true for each $\alpha_i$ then

$$\delta^\alpha(\mathcal{T}) = \cap_{\alpha_i}(\mathcal{T}^{(1)^{(\alpha_i)}}) = \mathcal{T}(\cap_{\alpha_i}(\mathcal{T}^{(1)^{(\alpha_i)}})) = \mathcal{T}(\mathcal{T}^{(1)^{(\alpha)}}),$$

establishing the claim. Hence $\delta^\alpha(\mathcal{T}) = \emptyset$ if and only if $\mathcal{T}^{(1+\alpha)} = \emptyset$. □

Let us now return to the Schreier sequence and examine the associated tree $S_i$.

Let $S_0 = \cup_{n=0}^\infty \{ (\epsilon_1, \epsilon_2, \ldots, \epsilon_n) : \epsilon_i = 1 \text{ for at most one } i, \epsilon_i = -1 \text{ otherwise, } 1 \leq i \leq n \}$. To define $S_i$ we need introduce the extension of one tree by another tree. If $\mathcal{T}$ and $\mathcal{S}$ are trees on the same set let $\mathcal{T} \boxplus \mathcal{S}$ denote $\{ \mathcal{x} + \mathcal{y} : \mathcal{x} \in \mathcal{T} \text{ and } \mathcal{y} \in \mathcal{S} \text{ where } n \text{ is the length of } \mathcal{x} \} \cup \mathcal{T}$. If $(\mathcal{T}_i)$ is a sequence of trees, we inductively define $\boxplus^\infty \mathcal{T}_i = [\boxplus^{n-1} \mathcal{T}_i] \boxplus \mathcal{T}_n$. If $n = 0$, let $\boxplus^0 \mathcal{T}_i = \{ \mathcal{e}_j : j = 1, 2, \ldots \}$. Also we will use the notation $\mathcal{L}(\mathcal{T}, n)$ for the subtree which is the union of $\{ \mathcal{e}_j : j = 0, 1, 2, \ldots, n-1 \}$ and the set of nodes of $\mathcal{T}$ equal to or below $\mathcal{e}_{n-1} + (1)$. In particular $\mathcal{L}(\mathcal{T}, n) \subset \{ \mathcal{e}_j : j = 1, 2, \ldots, n-1 \} \boxplus \mathcal{T}$. We need a way of forming a new tree $\mathcal{T}$ out of an infinite sequence of trees $\mathcal{T}_i$, $i = 1, 2, \ldots$, on $\{ -1, 1 \}$.

Define $\Sigma_{i=1}^\infty \mathcal{T}_i = \cup_{i=1}^\infty \mathcal{L}(\mathcal{T}_i, i)$. Clearly the resulting tree will depend on the order of the $\mathcal{T}_i$'s.
We claim that the tree associated to the Schreier sequence is
\[ S_1 = \sum_{i=1}^{\infty} \bigoplus^i S_0. \]
Indeed, the Schreier sequence \((x_n)\) is defined by the property that if \(k, N \in \mathbb{N}\) and \(k \leq n_1 < n_2 < \ldots < n_k \leq N\), then
\[ \bigcap_{i=1}^{k} \text{supp } x_{n_i} \cap \bigcap_{n \neq n_i, n < N} (\text{supp } x_n)^{-1} \neq \emptyset, \]
and these are the only non-empty intersections. Note that this is equivalent to saying that if \(m \leq k\) and \(k = n_1 < n_2 < \ldots < n_m, \bigcap_{i=1}^{m} \text{supp } x_{n_i} \cap \bigcap_{n \neq n_i, n < N} (\text{supp } x_n)^{-1} \neq \emptyset. \]
Hence the tree associated to \((x_n)\) contains for each \(k \in \mathbb{N}\), exactly those nodes of the dyadic tree of the form \(e_k\), where \(x\) has at most \(k\) coordinates equal to 1.

It is easy to see that \(\bigoplus^k S_0\) is exactly the nodes with at most \(k\) coordinates equal to 1. Thus \(S_1\) is the required tree.

To see what the topological type of \(S_1\) is we need only compute the order of \(\delta [\bigoplus^n S_0]\). Clearly \(\delta (S_0) = 2\). A straight-forward induction argument shows that \(\delta^n (L(\bigoplus^n S_0, n) = \{e_\omega\}\). It then follows easily that \(\delta (S_1) = \omega + 1\). Therefore \(S_1\) has exactly \(\omega\) non-empty derived sets. Actually it is not hard to see that it is homeomorphic to \(\omega^\omega\).

Now we are ready to generalize the Schreier example. Our goal is to build for each \(\alpha < \omega_1\) a weakly null sequence of indicator functions on a compact metric space (homeomorphic to \(\omega^\alpha\)) with oscillation index at least \(\omega^\alpha\). The sequence will also be a 1-unconditional basic sequence.

We will begin by defining the sequences in terms of subsets of \(\mathbb{N}\) as we did with the Schreier sequence. \(F_1\) has been defined. Suppose that \(F_\beta\) has been defined for all \(\beta < \alpha\). Let \(\alpha_i = \alpha - 1\) if \(\alpha\) is not a limit ordinal and \(\alpha_i \uparrow \alpha\) if \(\alpha\) is a limit ordinal. Define
\[ F_\alpha = \bigcup_{n=1}^{\infty} \{ \bigcup_{i=1}^{n} F_i : F_i \in F_{\alpha_i}, \text{ for } i \leq n, k \leq F_1 < \ldots < F_i < F_{i+1} < \ldots F_n \}\]
where the notation \(k \leq F_1 < \ldots F_i < F_{i+1} < \ldots F_n\) means that if \(F_i\) and \(F_j\) are nonempty and \(i < j\), then \(k \leq \min F_i\) and \(\max F_i < \min F_j\).

There is some ambiguity here but it will not be of any significance as long as the choice of the sequence \((\alpha_i)\) is fixed for each limit ordinal. For each \(\alpha < \omega_1\) let \((x_\alpha^n)\) denote the standard sequence of indicator functions on \(F_\alpha\), i.e., \(x_\alpha^n (F) = 1\) if \(n \in F\) and 0 otherwise.

**Proposition 4.9.** For each \(\alpha < \omega_1\), \(F_\alpha\) is a countable compact metric space under the topology induced by identifying \(F_\alpha\) with \(\{1_F : F \in F_\alpha\}\) under the topology of pointwise convergence and \((x_\alpha^n)\) is a weakly null 1-suppression unconditional basic sequence in \(C(F_\alpha)\).

**Proof.** It is sufficient to verify that \(F_\alpha\) satisfies the hypothesis of Proposition 4.2. Property ii) is obviously inherited by each \(F_\alpha\) from \(F_1\). For i) use induction and
note that if $G \subseteq F_1 \cup F_2 \cup \ldots \cup F_n \in \mathcal{F}_\alpha$, as in the definition, then $G \cap F_i \in \mathcal{F}_\alpha$, for each $i$. Hence $G = (G \cap F_1) \cup (G \cap F_2) \cup \ldots \cup (G \cap F_n) \in \mathcal{F}_\alpha$. Finally for iii) suppose that iii) holds for all $\beta < \alpha$, and that for each $k$, $F_k = \bigcup_{j=1}^n F_{kj}$ is an element of $\mathcal{F}_\alpha$, as in the definition. If $F_k$ converges to a non-empty set $F$, let $n = \min F$. Then $n \in F_k$ for all large $k$ and thus $n_k \leq n$. We may assume by passing to subsequences that for each $j \leq n$, $F_{kj}$ converges to some $F_j$. Hence, by induction, because we have that $F_{kj} \in \mathcal{F}_{\alpha j}$, for all $k$, $F_j \in \mathcal{F}_{\alpha j}$. The other properties are obvious and thus $F = \bigcup F_j \in \mathcal{F}_\alpha$, as claimed. \[\square\]

Now we want to consider the size of the underlying topological space. First we will compute the size directly using the families $\mathcal{F}_\alpha$. For each $\alpha < \omega_1$ and $n$, $n \in \mathbb{N}$ with $n \leq k$ let

$$\mathcal{F}_{\alpha,n,k} = \left\{ \bigcup_{i=1}^n F_i : F_i \in \mathcal{F}_\alpha, k \leq F_1 < \ldots < F_i < F_{i+1} < \ldots < F_n, \text{ and } F_n \in \mathcal{F}_{\alpha n} \right\}.$$ 

where $\alpha_i = \alpha - 1$ if $\alpha$ is not a limit ordinal and $\alpha_i \uparrow \alpha$ if $\alpha$ is a limit ordinal.

**Proposition 4.10.** For each $\alpha < \omega_1$, $\mathcal{F}_{\alpha}^{(\omega^\alpha)} = \{\emptyset\}$.

**Proof.** The result will follow from

CLAIM: If $\rho \leq \omega^{\alpha n}$, then

$$\mathcal{F}_{\alpha,n,k}^{(\rho)} = \left\{ \bigcup_{i=1}^n F_i : F_i \in \mathcal{F}_\alpha, k \leq F_1 < \ldots < F_i < F_{i+1} < \ldots < F_n, \text{ and } F_n \in \mathcal{F}_{\alpha n}^{(\rho)} \right\}.$$ 

The proof of the claim is by induction on $\alpha$, $n$, and $\rho$.

If $\alpha = 0$, $\mathcal{F}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$. Clearly $\mathcal{F}_0^{(1)} = \{\emptyset\}$.

Let $\alpha = 1$ and $k \geq n \geq 1$. Then if $G \in \mathcal{F}_{1,n,k}^{(1)}$, there is a non-trivial sequence $(G_m)$ in $\mathcal{F}_{1,n,k}$ which converges to $F$. Observe that we need only show that card $G < n$. However this is obvious because card $G_m \leq n$ for all $m$ and some portion of the $G_m$’s must go to $\infty$. Conversely, if $G \in \mathcal{F}_{1,n,k}$ and card $G < n$, then $G \cup \{m\} \in \mathcal{F}_{1,n}$ for $m = k, k+1, \ldots$. Hence $G \in \mathcal{F}_{1,n,k}^{(1)}$. Finally note that each $\mathcal{F}_{1,n,k}$ is closed, $\mathcal{F}_{1,n,k}^{(1)} = \mathcal{F}_{1,n-1,k}$, and if $G_m \in \mathcal{F}_{1,m,m}$ then $G_m \rightarrow \emptyset$. Therefore $\mathcal{F}_1^{(1)} = \{\emptyset\}$.

Now assume the result holds for all $\gamma < \alpha$. Fix $\rho < \omega^{\alpha n}$ and $k \geq n \geq 1$ and assume that the result has been proved for $\rho$. (Note that it always holds for $\rho = 0$.) Let $\mathcal{F} = \mathcal{F}_{\alpha,n,k}^{(\rho)}$. Suppose that $(G_m)$ is a non-trivial sequence in $\mathcal{F}$ which converges to $G$. Suppose that $G_m = \bigcup_{i=1}^n G_{m,i}$. By passing to a subsequence if necessary we may assume that for each $i$, $G_{m,i} \rightarrow G_i$. If $(G_{m,n})$ is a non-trivial sequence, then $G_n \in \mathcal{F}_{\alpha n}^{(\rho+1)}$ and thus $G = \bigcup_{i=1}^n G_i$ where $G_i \in \mathcal{F}_\alpha$, for $i = 1, 2, \ldots, n$ and $G_n \in \mathcal{F}_{\alpha n}^{(\rho+1)}$, that is,

$$G \in \left\{ \bigcup_{i=1}^n F_i : F_i \in \mathcal{F}_\alpha, k \leq F_1 < \ldots < F_i < F_{i+1} < \ldots < F_n, \text{ and } F_n \in \mathcal{F}_{\alpha n}^{(\rho+1)} \right\}.$$ 

If $(G_{m,n})$ is trivial (eventually constant) then $(G_{i,n})$ is eventually constant for all $i$. However this contradicts the non-triviality of the sequence $(G_m)$.

Conversely if

$$G \in \left\{ \bigcup_{i=1}^n F_i : F_i \in \mathcal{F}_\alpha, k \leq F_1 < \ldots < F_i < F_{i+1} < \ldots < F_n, \text{ and } F_n \in \mathcal{F}_{\alpha n}^{(\rho+1)} \right\}.$$
Then there is a non-trivial sequence \((G_m)\) in \(\mathcal{F}_{\alpha}^{(\rho)}\) which converges to \(F_n\). Clearly we may assume that \(\min G_m > \max_{i=1}^{n-1} F_i\) for all \(m\). Then \(G'_m = \cup_{i=1}^{n-1} F_i \cup G_m \in \mathcal{F}_{\alpha,n,k}^{(\rho)}\) for all \(m\) and \((G'_m)\) converges to \(G\). Therefore \(G \in \mathcal{F}_{\alpha,n,k}^{(\rho+1)}\).

Clearly if \(\rho_j \uparrow \rho\), \(\mathcal{F}_{\alpha,n,k}^{(\rho)} = \cap_{j=1}^{\infty} \left\{ \cup_{i=1}^{n} F_i : F_i \in \mathcal{F}_{\alpha_1}, k \leq F_1 < \ldots < F_i < F_{i+1} < \ldots < F_n, \text{ and } F_n \in \mathcal{F}_{\alpha,n,k}^{(\rho_j)} \right\}\)

Thus the formula holds for all ordinals \(\rho \leq \omega^\alpha\).

Finally observe that by induction we have that \(\mathcal{F}_{\alpha,n,k}^{(\omega^\alpha)} = \mathcal{F}_{\alpha,n-1,k}\) and hence that \(\mathcal{F}_{\alpha,n,k}^{(\omega^\alpha+\omega^\alpha+\ldots+\omega^\alpha)} = \{\emptyset\}\). To see that \(\mathcal{F}_{\alpha}^{(\omega^\alpha)} = \{\emptyset\}\) note that \(\mathcal{F}_{\alpha} = \cup_{n=1}^{\infty} \mathcal{F}_{\alpha,n,n}\) and that if \(G_n \in \mathcal{F}_{\alpha,n,n}\) for all \(n\) then \(G_n \to \emptyset\). □

Next we will prove the result again but using trees. First we will translate the construction into the tree representation.

Observe that there is a simple correspondence between

\[\{\cup_{i=1}^{n} F_i : F_i \in \mathcal{F}_{\alpha_1}, k \leq F_1 < \ldots < F_i < F_{i+1} < \ldots < F_n\}\]

and \(\bigoplus_{i=1}^{n} \mathcal{S}_{\alpha_i}\), where \(\mathcal{S}_{\alpha_i}\) is the tree corresponding to \(\mathcal{F}_{\alpha_i}\). Indeed,

\[\cap_{n \in F_i} \operatorname{supp} x_n^\alpha \cap \cap_{n \notin F_i} (\operatorname{supp} x_n^\alpha)^{-1} \neq \emptyset,\]

for all \(m\), if and only if \(F_i \in \mathcal{F}_{\alpha}\). Thus if \(f_j = (\epsilon_{m+1}, \epsilon_{m+2}, \ldots, \epsilon_{m+1})\), where \(m_i = \min F_i - 1\), and \(\epsilon_j = 1\) if \(j \in F_i\), \(\epsilon_j = 0\) otherwise, then we have \(f_1 + f_2 + \ldots + f_n + m\) in \(\bigoplus \mathcal{S}_{\alpha}\), for all \(m\). Conversely if we have \(f_1 + f_2 + \ldots + f_n\) in \(\bigoplus \mathcal{S}_{\alpha}\), let \(m_i\) be the sum of the lengths of the \(f_j\)'s, \(j = 1, 2, \ldots, i - 1\), and \(F_i = \{m_i + k : \epsilon_k = 1\}\) for \(f_j = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{m_i+1-1})\). Then \(F_i \in \mathcal{F}_{\alpha}\), and \(\max F_i < \min F_{i+1}\), for \(i = 1, 2, \ldots, n\).

Next we need to take care of the \(n \leq \min F_1\) condition. Observe that we could define

\[\mathcal{F}_{\alpha} = \cup_{n=1}^{\infty} \{\cup_{n=1}^{\infty} F_i : F_i \in \mathcal{F}_{\alpha_1}, n \leq F_1 < \ldots < F_i < F_{i+1} < \ldots < F_n \text{ and } n \in F_1\}\]

and the set would remain the same. We claim that the tree associated to \(x_n^{\alpha+1}\) is

\[\mathcal{S}_{\alpha+1} = \bigoplus_{i=1}^{\infty} \mathcal{S}_i\]

We have already shown above that \(\bigoplus \mathcal{S}_{\alpha}\) corresponds to the union of \(i\) ordered sets from \(\mathcal{F}_{\alpha}\). Now note that the nodes corresponding to \(\bigoplus \mathcal{S}_i\) in the sum are all below or equal to \(\overline{f}_{i-1}\) + (1), and thus correspond exactly to the unions of \(i\) orders sets from \(\mathcal{F}_{\alpha}\) with smallest element of the first set equal to \(i\). Thus \(\mathcal{S}_{\alpha+1}\) is the correct tree. A similar argument shows that for a limit ordinal \(\alpha\)

\[\mathcal{S}_{\alpha} = \bigoplus_{i=1}^{\infty} \mathcal{S}_i\]
where \(\alpha_i\) is the defining sequence for \(F_\alpha\).

Now that we have the trees \(S_\alpha, \alpha < \omega_1\), we can determine the underlying topological spaces by using Proposition 4.7.

**Proof of Proposition 4.10.** Inductively assume that \(\delta(L(S_\beta, j))\) is \(\omega^\beta\), for all \(\beta < \alpha\) and \(j \in \mathbb{N}\). Let \(\alpha_i = \alpha - 1\) if \(\alpha\) is a successor and \(\alpha_i \uparrow \alpha\) otherwise.

CLAIM: \(\delta(L(\bigcup_{i=1}^n S_\alpha_i, j)) = \omega^\alpha_n + \omega^\alpha_{n-1} + \ldots + \omega^\alpha_1 + 1\) for all \(j \in \mathbb{N}\).

Indeed if \(\overline{x} \in L(\bigcup_{i=1}^n S_\alpha_i, j)\), \(\overline{x} = \overline{y}_1 + \overline{y}_2 + \ldots + \overline{y}_{n-1}\) and the length of \(\overline{x}\) is \(k \geq j\), then

\[
\{\overline{t}_k + \overline{y} : \overline{x} = \overline{y}_1 + \overline{y}_2 + \ldots + \overline{y}_{n-1}\} \cup \{\overline{t}_i : i = 1, 2, \ldots, k\} \supset L(S_\alpha_i, k + 1)
\]

Thus

\[
\overline{x} \in \delta^\alpha_n(L(\bigcup_{i=1}^n S_\alpha_i, j)),
\]

by the inductive hypothesis. Because \(k\) is arbitrary the claim follows by induction on \(n\). (Obviously \(\delta(L(\bigcup_{i=1}^n S_\alpha_i, j)) \leq \omega^\alpha_n + \omega^\alpha_{n-1} + \ldots + \omega^\alpha_1 + 1\))

It now is easy to see that \(\delta(L(S_\alpha, j)) = \omega^\alpha + 1\), because the order is larger than \(\lambda_n = \omega^\alpha_n + \omega^\alpha_{n-1} + \ldots + \omega^\alpha_1\), for all \(n\) and the elements \(e_i, i \in \mathbb{N}\), are in \(\delta^\lambda_n(L(\bigcup_{i=1}^n S_\alpha_i, j))\) for all \(\lambda\).

Our next task is to compute the oscillation index of \((x^\alpha_\lambda)\). As before we will compute this in two ways first by using the family \(F_\alpha\) and then by using trees.

**Proposition 4.11.** \(O^{\omega^\alpha}(\cdot, (x^\alpha_\lambda), F_\alpha) \neq \emptyset\), for all \(\alpha < \omega_1, \epsilon < 1\).

**Proof.** We will show that \(O^{\lambda}(\cdot, (x^\alpha_\lambda), F_\alpha) = F_\alpha^{(\lambda)}\) for all \(\lambda\). In view of Proposition 4.10, it is sufficient to show that if \(F \in F_\alpha^{(\lambda + 1)}\) then there is a \(N \in \mathbb{N}\) such that for all \(n \geq N, F \cup \{n\} \in F_\alpha^{(\lambda)}\). We will use induction on \(\alpha\) and \(\lambda\).

If \(\alpha = 1\), then for any \(\lambda \in \mathbb{N}\) the claim is immediate from the definition of \(F_1\) and Proposition 4.10.

Now assume that \(\alpha > 1\) and that the claim is true for all \(\beta < \alpha\) and all \(\lambda\). If \(F \in F_\alpha^{(1)}\) then either \(F = \emptyset\) and the claim is obvious or \(F \in F_\alpha^{(1)}\) for some \(k \in \mathbb{N}\). By Proposition 4.10,

\[
F_\alpha^{(1)} = \left\{ \bigcup_{i=1}^k F_i : F_i \in F_\alpha, k \leq F_1 < \ldots < F_i < F_{i+1} < \ldots < F_k, \text{ and } F_k \in F_\alpha^{(1)} \right\}
\]

Suppose that \(F = \bigcup_{i=1}^k F_i\) as above. By the inductive hypothesis \(F_k \cup \{n\} \in F_\alpha^{(1)}\) for all \(n \geq N\), for some \(N \in \mathbb{N}\), and thus for \(n > \max\{N\} \cup F_k, \cup_{i=1}^{k-1} F_i \cup F_k \cup \{n\} \in F_\alpha^{(0)}\).

Next assume the claim for all \(\gamma \leq \lambda\) and let \(F \in F_\alpha^{(\lambda + 1)}\). If \(F = \emptyset\) then \(\{n\} \in F_\alpha^{(\lambda)}\) for all sufficiently large \(n\). Otherwise \(F \in F_\alpha^{(\lambda + 1)}\) for some \(k\). Let \(\lambda = \omega^{\alpha_k} + \omega^{\alpha_{k-1}} + \ldots + \omega^{\alpha_j} + \rho\) for some \(j \leq k + 1\) and \(\rho < \omega^{\alpha_j - 1}\). By the claim in the proof of Proposition 4.10

\[
F_\alpha^{(\lambda + 1)} = F_\alpha^{(\lambda + 1)} = \left\{ \bigcup_{i=1}^{\rho+1} F_i : F_i \in F_\alpha, k \leq F_1 < \ldots < F_i < F_{i+1} < \ldots < F_j, \text{ and } F_j \in F_\alpha^{(\rho+1)} \right\}
\]
Suppose that \( F = \bigcup_{n=1}^{j-1} F_i \) as above. By the inductive hypothesis \( F_{j-1} \cup \{n\} \in F_{\alpha_j-1} \) for all \( n \geq N \), for some \( N \in \mathbb{N} \), and thus for \( n > \max\{N\} \cup F_{j-1}, \bigcup_{n=1}^{j-1} F_i \cup F_{j-1} \cup \{n\} \in F_{\alpha_j-1, k}. \) \( \square \)

In the proof above we established that we can always use the special sequence \( (F \cup \{n\})_{n \geq N} \) to reach a set \( F \) from a smaller derived set. The next definition describes this same property for the associated tree.

**Definition.** Let \( T \) be a tree on \( \{-1, 1\} \) with no nodes in \( \overline{T} \) with infinitely many coordinates equal to 1. We will say that \( T \) has property FB (fully branching) if \( \mathfrak{T} + \mathfrak{T} \in \overline{T} \) implies that there is an \( N \in \mathbb{N} \) such that either

1. \( \mathfrak{T} + \mathfrak{T} + (1) \in T \) for all \( j \geq N \) or
2. \( \mathfrak{T} + \mathfrak{T} + (1) \notin T \) for all \( j \geq N \).

**Lemma 4.12.** Suppose that \( T \) and \( U \) are weakly independent trees with property FB and all nodes of \( T \) and \( U \) are on branches with limit of the form \( x + e_\omega \), then \( T \oplus U \) has property FB, and for all \( n, \mathcal{L}(T, n) \) has property FB.

**Proof.** Suppose that \( \mathfrak{T} + \mathfrak{T} \in \overline{T} \). Because \( T \oplus U \supseteq T \), if i) occurs in \( T \), the same is true in \( T \oplus U \). If ii) occurs in \( T \) but not in \( T \oplus U \), then there is a sequence of incomparable nodes of the form \( \mathfrak{T}_n + \mathfrak{T}_j \) in \( U \) where \( n \) is greater than the length of \( \mathfrak{T} \) and does not depend on \( j \) and \( \mathfrak{T}_j \) has at least one coordinate equal to one. By passing to a subsequence we may assume that \( \mathfrak{T}_n + \mathfrak{T}_j \) converges to \( \mathfrak{T}_n + \mathfrak{T} \). The assumption that the \( \mathfrak{T}_n + \mathfrak{T}_j \)'s are incomparable guarantees that \( (\mathfrak{T}_n + \mathfrak{T}) + \mathfrak{T} \) does not satisfy ii). Because \( U \) has property FB there is an \( N \) such that \( \mathfrak{T}_n + \mathfrak{T} + \mathfrak{T} + (1) \notin U \) for all \( j > N \). Because \( U \) is weakly independent this implies that \( \mathfrak{T}_n + \mathfrak{T} + (1) \notin U \) for all \( j > N \), where \( k \) equals \( n \) plus the length of \( \mathfrak{T} \). Hence \( \mathfrak{T} + \mathfrak{T}_n + (1) \in T \oplus U \) for all \( m > N + k \).\( \square \)

If \( \mathfrak{T} + \mathfrak{T} + \mathfrak{T} \in \overline{T} \oplus U \), where \( \mathfrak{T}_n + \mathfrak{T} \in T \), \( n \) is the length of \( \mathfrak{T} \) and \( \mathfrak{T} \in T \), then \( \mathfrak{T} + \mathfrak{T} + \mathfrak{T} \) will satisfy i), respectively ii), if \( \mathfrak{T}_n + \mathfrak{T} + \mathfrak{T} \) satisfies i), respectively ii).

The second assertion is obvious. \( \square \)

**Proposition 4.13.** Suppose that \( (x_n) \) is a weakly independent sequence of continuous indicator functions on a compact metric space \( K \) which converge pointwise to 0 and that the associated tree \( T \) has property FB. Then for any \( \epsilon < 1 \), and \( \alpha < \omega_1 \), \( O^\alpha(\epsilon, (x_n), K) \neq \emptyset \) if and only if \( \overline{T}^{(1+\alpha)} \neq \emptyset \).

**Proof.** For each \( n \) and \( \epsilon_n \) in \( T \setminus T \), let \( \hat{x}_n \) such that \( \epsilon_n = 1 \), and 0, otherwise. In this way we have defined a sequence of indicator functions on \( Q = T \setminus T \) with span isometric to the span of \( (x_n) \).\( \square \)

Clearly \( O^\alpha(\epsilon, (x_n), K) \neq \emptyset \) if and only if \( O^\alpha(\epsilon, (\hat{x}_n), Q) \neq \emptyset \). Now observe that \( (\epsilon_n) + \mathfrak{T}_w \in O^1(\epsilon, (\hat{x}_n), Q) \) if and only if there is an \( N \in \mathbb{N} \) such that for all \( j \geq N \), \( (\epsilon_n) + \mathfrak{T} + (1) \in T \). (Use the weak independence of \( (\hat{x}_n) \).) Clearly the latter condition implies that \( (\epsilon_n) + \mathfrak{T} \in \overline{T}^{(2)} = Q^{(1)} \). Conversely if \( (\epsilon_n) \in Q^{(1)} \), then weak independence and property FB imply that \( (\epsilon_n) + \mathfrak{T} + (1) \in T \), for all \( j \geq N \), for some \( N \in \mathbb{N} \). Finally note that weak independence and property FB are inherited by \( T(Q^{(1)}) \) which is the tree associated to \( (\hat{x}_n) \). Also if \( \alpha_k \uparrow \alpha \), \( \cap T(Q^{(\alpha)}) = T(Q^{(\alpha)}) \) and it is straight-forward to check that weak independence and property FB are inherited by the intersection. Thus transfinite induction may be used to complete the proof. \( \square \)
Proposition 4.11 follows as a corollary of this result, that is, \( \mathcal{O}^{\omega^\alpha}(\epsilon, (x_n^\alpha), K) \neq \emptyset \), for all \( \alpha < \omega_1, \epsilon < 1 \).

Because the underlying space for \( (x_n^\alpha) \) is \( \omega^\alpha \), \( \omega^\alpha \) is the maximal possible oscillation index. Also note that because each \( (x_n^\alpha) \) is an unconditional basic sequence, all of the spaces are isomorphic to complemented subspaces of the Pelczynski universal space \( U_1 \), [L-T,I,p92].

5. Reflexive spaces with large oscillation index

In the previous section we constructed weakly null sequences with oscillation index \( \omega^\alpha \). Because these sequences were in \( C(\mathcal{F}_n) \) it follows that the span of \( (x_n^\alpha) \) contains \( c_0 \) and thus is not reflexive. In this section we will explore an idea of E. Odell for constructing Tsirelson-like spaces with large oscillation index.

To define these spaces we begin with the space \([x_n^0]\) in place of \( c_0 \) in the Tsirelson construction, [C-S,p 14]. Suppose that \( x = \sum a_n t_n^\alpha \) where \( (t_n^\alpha) \) is the unit vector basis of the space of sequences with only finitely many nonzero coordinates. Let

\[
||x||_0 = || \sum a_n x_n^\alpha ||
\]

and inductively define

\[
||x||_{m+1} = \max \left\{ ||x||_m, 2^{-1} \max_{\{p_i\} \in \mathcal{F}_1} \sum_{i=1}^k \sum_{n=p_i}^{p_i+1-1} a_n t_n^\alpha ||_m \right\},
\]

where \( k \) is the cardinality of \( \{p_i\} \). Let \( ||x|| = \lim ||x||_m \). Let \( T_\alpha \) be the completion of span \( \{t_n^\alpha\} \) under \( || \cdot || \). Observe that Tsirelson space is \( T_0 \) in this construction, i.e., let \( \mathcal{F}_0 = \{\{n\} : n \in \mathbb{N}\} \), then \( (x_n^0) = \{1(\{n\})\} \) is equivalent to the \( c_0 \) basis.

Also observe that for each \( \alpha \) the norm on \( T_\alpha \) satisfies

\[
(*) \quad ||x|| = \max \left\{ ||x||_0, 2^{-1} \max_{\{p_i\} \in \mathcal{F}_1} \sum_{i=1}^k \sum_{n=p_i}^{p_i+1-1} a_n t_n^\alpha ||_m \right\}.
\]

**Proposition 5.1.** For each \( \alpha < \omega_1 \), \( T_\alpha \) is a reflexive Banach space with unconditional basis with no subspace isomorphic to \( \ell^p \), \( 1 \leq p < \infty \), or \( c_0 \). Moreover \( \mathcal{O}^{\omega^\alpha}(\epsilon, (t_n^\alpha), B(T_\alpha)) \neq \emptyset \) for all \( \epsilon < 1 \).

**Proof.** Fix \( \alpha < \omega_1 \). Clearly \( ||x||_0 \geq ||x||_{c_0} \). Therefore \( ||x|| \geq ||x||_T \) for all \( x \in T_\alpha \) where \( || \cdot ||_T \) denotes the norm on Tsirelson space. (This is \( T_0 \) above.) It follows from (*) that if \( \{u_i\} \) is a sequence of \( k \) normalized blocks of the basis in \( T_\alpha \) with support beyond \( k \) that

\[
\sum_{i=1}^k |c_i| \geq \sum_{i=1}^k c_i u_i ||_T \geq 2^{-1} \sum_{i=1}^k |c_i|,
\]

for all choices of scalars \( (c_i) \). Therefore \( T_\alpha \) does not contain \( c_0 \) or \( \ell^p \), for any \( p > 1 \).

To see that \( \ell^1 \) is not isomorphic to a subspace of \( T_\alpha \) we need only examine the proof that \( T \) does not contain \( \ell^1 \) as given in [C-S,p 17]. The only properties of \( T \) that are used in the argument are that the norm satisfies equation (*) and that the \( c_0 \) norm of a long average is small. The proof then for \( T \) will carry over to \( T_\alpha \).
provided we use a sequence with small $|| \cdot ||_0$. To do this let $(u_i)$ be a normalized block basis of $(t^n_\alpha)$ such that

$$\sum |a_i| \geq || \sum a_i u_i || \geq \frac{8}{9} \sum |a_i|.$$ 

Because $(x^n_\alpha)$ is weakly null in $C(\mathcal{F}_\alpha)$, so is $(u_i)$ and thus there is a sequence of disjoint convex combinations of $(u_i)$,

$$y_j = \sum_{i \in E_j} \lambda_i u_i,$$

where $E_1 < E_2 < \ldots$, $\sum_i \lambda_i = 1$, and $\lambda_i \geq 0$ for all $i$, such that $||y_j||_0 \leq 2^{-j}$ for $j = 0, 1, \ldots$. It follows then that

$$\sum |a_i| \geq || \sum a_i y_i || \geq \frac{8}{9} \sum |a_i|,$$

and

$$||y_0 + r^{-1}(y_1 + y_2 + \ldots + y_r)||_0 \leq 1 + r^{-1}.$$ 

Using these $y_i$'s the remainder of the proof carries over without change to $T_\alpha$.

Obviously $T_\alpha$ has an unconditional basis and thus by a result of James [L-T, I, p. 97], $T_\alpha$ is reflexive.

Finally to see that the oscillation index is large we use the observation that the operator $S$ from $T_\alpha$ to $[x^n_\alpha]$ defined by $S(\sum c_n t_n) = \sum c_n x^n_\alpha$ is bounded by 1 and thus $O^\lambda (\epsilon, (t^n_\alpha), B_{T_\alpha}) \supset S^* O^\lambda (\epsilon, (x^n_\alpha), \mathcal{F}_\alpha) \neq \emptyset$, for every $\lambda \leq \omega^\alpha$, by Lemma 2.2. \qed

While these space $T_\alpha$ share important properties with $T$ let us note that they do not possess the property that every block basis dominates a subsequence of the basis. In particular

**Proposition 5.2.** For every $\alpha < \omega_1$ there is a block basis $(u_i)$ of $(t^n_\alpha)$ and an increasing sequence of integers $(k_i)$ such that

$$|| \sum a_i t_{k_i} ||_T \leq || \sum a_i u_i || \leq 2(1 + \epsilon) || \sum a_i t_{k_i} ||_T$$

for any sequence of scalars $(a_i)$.

**Proof.** Fix $\alpha < \omega_1$. Let $(v_i)$ be a normalized block basic sequence of $(x^n_\alpha)$ which is $(1 + \epsilon)$-equivalent to the usual unit vector basis of $c_0$ and let $u_i = v_i/||v_i||$, $i = 1, 2, \ldots$. The idea is to show that for any sequence of scalars $(a_i)$

$$(**) \quad || \sum a_i t_{k_i} ||_{T,m} \leq || \sum a_i u_i ||$$

and

$$(***) \quad || \sum a_i u_i ||_m \leq 2(1 + \epsilon) || \sum a_i t_{2k_i} ||_T$$
\[ m = 0, 1, \ldots, \text{ where} \]

\[ \| \sum a_i t_i \|_{T, m+1} = \max \left\{ \| \sum a_i t_i \|_{T, m}, 2^{-1} \max_{\{p_i\} \in \mathcal{F}_1} \sum_{i=1}^k \sum_{n=p_i+1}^{p_{i+1}-1} a_n t_n \|_{T, m} \right\}, \]

and

\[ \| \sum a_i t_i \|_{T, 0} = \sup |a_i|. \]

Once this is accomplished we use the fact that there is a constant \( K \), such that

\[ \| \sum a_i t_i \|_T \leq \| \sum a_i t_i \|_{T, m+1} \leq K \| \sum a_i t_i \|_T \]

and hence \( |u_i| \) is isomorphic to \( T \). (See [C-S,p.26]. In fact the argument given here is derived from the arguments of Casazza, Johnson and Tzafriri, [C-S,p.34-38].)

We will establish (***) by induction on \( m \). For \( m = 0 \), (***) is immediate. Now assume the inequality holds for \( m \) and we will prove it for \( m + 1 \).

Let \( k_n \) be the first element in the support of \( u_n \). Then

\[ \| \sum a_i t_i \|_{T, m+1} = \max \left\{ \| \sum a_i t_i \|_{T, m}, 2^{-1} \max_{\{p_i\} \in \mathcal{F}_1} \sum_{i=1}^k \sum_{n=p_i+1}^{p_{i+1}-1} a_n t_n \|_{T, m} \right\}, \]

\[ \leq \max \left\{ \| \sum a_i u_i \|, 2^{-1} \max_{\{p_i\} \in \mathcal{F}_1} \sum_{i=1}^k \sum_{n=p_i}^{p_{i+1}-1} a_n u_n \| \right\}, \]

\[ \leq \max \left\{ \| \sum a_i u_i \|, 2^{-1} \max_{\{p_i\} \in \mathcal{F}_1} \sum_{i=1}^k \| P_i \sum a_n u_n \| \right\}, \]

\[ = \| \sum a_i u_i \| \]

where \( P_i \sum b_n t_n = \sum_{n=q_i}^{q_{i+1}-1} b_n t_n \). The last inequality above holds because \( \{q_i\} = \{k_{p_i}\} \in \mathcal{F}_1 \text{ if } \{p_i\} \in \mathcal{F}_1 \).

For the inequality (***) we need to work a little harder.

\[ \| \sum a_n u_n \|_{m+1} = \max \left\{ \| \sum a_n u_n \|_{m}, 2^{-1} \max_{\{q_i\} \in \mathcal{F}_1} \sum_{i=1}^k \| P_i \sum a_n u_n \|_m \right\}, \]

where \( P_i \) denotes the basis projection onto \([t_j : q_i \leq j < q_{i+1}]\). Fix \( \{q_i\} \in \mathcal{F}_1 \) and for each \( i \) let \( G_i = \{ n : q_i \leq k_n < k_{n+1}, \text{ and } n \notin G_i \} \). For each \( n \) let

\[ H_n = \{ i : k_n \leq q_i < k_{n+1} \text{ or } k_n < q_{i+1} \leq k_{n+1}, \text{ and } n \notin G_i \}. \]

Consider the sum corresponding to the \( q_i \)'s.
2^{-1} \sum_{i=1}^{k} \|P_i \sum_{n \in G_i} a_n u_n\|_m \leq 2^{-1} \sum_{i=1}^{k} \left\| \sum_{n \in G_i} a_n u_n \right\|_m + 2^{-1} \sum_{n=1}^{\infty} \sum_{i \in H_n} \|P_i a_n u_n\|_m

\leq 2^{-1} \sum_{i=1}^{k} \left\| \sum_{n \in G_i} a_n u_n \right\|_m + 2^{-1} \sum_{n: H_n \neq \emptyset} 2^{|a_n u_n|}_{m+1}

\leq 2^{-1} \sum_{i=1}^{k} 2(1 + \varepsilon) \left\| \sum_{n \in G_i} a_n t_{2k_{n+1}} \right\|_T + \sum_{n: H_n \neq \emptyset} \|a_n t_{2k_{n+1}}\|_T

\leq (1 + \varepsilon) \left[ \sum_{i=1}^{k} \left\| \sum_{n \in G_i} a_n t_{2k_{n+1}} \right\|_T + \sum_{n: H_n \neq \emptyset} \|a_n t_{2k_{n+1}}\|_T \right]

Observe that there are at most \( k \) integers \( n \) such that \( H_n \neq \emptyset \) and that \( k \leq q_1 < k_{m+1} \), where \( m \) is the smallest integer such that \( P_1 u_m \neq 0 \). Let \( n_i = \min G_i \) for \( i = 1, 2, \ldots, k \). Then

\[ \{2k_{n_i+1} : i = 1, 2, \ldots, k\} \cup \{2k_{n+1} : H_n \neq \emptyset\} \]

is a set of at most \( 2k \) integers greater than \( 2k \). Hence this sum in brackets is at most \( 2 \| \sum a_n t_{2k_{n+1}} \|_T \). Therefore

\[
\max \left\{ \left\| \sum a_i u_i \right\|_m, 2^{-1} \max_{\{q_i\} \in F} \sum_{i=1}^{k} \left\| P_i \sum_{n \in G_i} a_n u_n \right\|_m \right\}
\]

\[
\leq \max \left\{ \left\| \sum a_i t_{2k_{n+1}} \right\|_T, 2(1 + \varepsilon) \max_{\{q_i\} \in F} \sum_{i=1}^{k} \left\| P_i \sum_{n \in G_i} a_n t_{2k_{n+1}} \right\|_T \right\}
\]

\[
= 2(1 + \varepsilon) \left\| \sum a_i t_{2k_{n+1}} \right\|_T,
\]

as claimed. □

Remark 5.3. Argyros [A] has modified the construction to obtain spaces \( X_{\alpha} \), \( \alpha < \omega_1 \), such that all of the subspaces of \( X_{\alpha} \) have index at least \( \alpha \). To accomplish this he uses the sets \( F_\alpha \) in the definition of the norm instead of starting with the space \([x^\alpha_\omega]\).

6. Comparison with the averaging index

In [A-O] the averaging index (See Section 1 for the definition.) was used to get a somewhat more constructive version of Mazur’s Theorem. In this section we will show that the averaging index is much larger than the spreading model index by showing that there exists a Banach space \( X \) such that for every \( \alpha < \omega_1 \) there is a weakly null sequence \( (x_n) \) in \( X \) with averaging index at least \( \alpha \), and yet \( \ell^1 \) is not isomorphic to a subspace of \( X \). We will also show that the spreading model index can be used to strengthen some of the results in [A-O].

The construction of the example will be based on the infinite branching James tree construction [J] in combination with \( \omega^\omega \). Let

\[ \mathcal{T} = \cup_{n=1}^{\infty} \{ (\alpha_1, \alpha_2, \ldots, \alpha_n) : \alpha_i < \omega^\omega \text{ for each } i \}. \]
Let $X_0$ be the linear subspace of the functions from $T$ into $C_0(\omega^\omega)$ which are nonzero at only finitely many points of $T$. We will use the notation $(f_\bar{t})$ where $f_\bar{t} \in C(\omega^\omega)$ for all $\bar{t}$ in $T$ to denote an element of $X_0$ with the understanding that the index $\bar{t}$ runs over $T$. We will also use $f_\bar{t}$ to denote the element of $X_0$ which is 0 except at $\bar{t}$ and $f_\bar{t}$ at $\bar{t}$.

Next we will introduce some linear functionals on $X_0$. For each $\bar{s} \in T$ and $i < |\bar{s}|$ (the length of $\bar{s}$) define

$$L(\bar{s},i)(f_\bar{t}) = \begin{cases} 0 & \text{if } \bar{s} \not> \bar{t} \text{ or } |\bar{t}| < i \\ f_\bar{t}(\bar{s}(|\bar{t}| + 1)) & \text{if } \bar{s} > \bar{t} \text{ and } i \leq |\bar{t}| < |\bar{s}| \end{cases},$$

where $\bar{s} > \bar{t}$ denotes that $\bar{s}$ is below $\bar{t}$, and extend linearly to $X_0$.

We will refer to a pair $(\bar{s},i)$ as a segment and define it to be the set of nodes of length at least $i$ which are above $\bar{s}$. We also want to have a notion of incomparable segments. Suppose that $j < i$ and that $(\bar{s},i)$ and $(\bar{t},j)$ are segments then $(\bar{s},i)$ and $(\bar{t},j)$ are incomparable if they are disjoint and $\bar{t}(m) \neq \bar{s}(m)$ for some $m \leq j$, i.e., they are on branches that split by level $j$. Note that if $\{(\bar{s}_k,i_k)\}$ are pairwise incomparable nodes then $L(\bar{s},i)(f_\bar{t}) \neq 0$ for at most one $k$ for each $\bar{t} \in T$.

Now we will introduce a norm on $X_0$. For $F \in X_0$ define

$$||F|| = \sup \left\{ \sum_j \left[ L(\bar{s}_j,i_j)(F) \right]^2 : \{ (\bar{s}_j,i_j) \text{ pairwise incomparable} \} \right\}^{\frac{1}{2}}.$$

Let $X$ be the completion of $X_0$ under this norm. Clearly $C_0(\omega^\omega)$ is isometric to $X_T = \{ f_\bar{t} : f \in C_0(\omega^\omega) \}$ for each fixed $\bar{t}$. For each $k \in \mathbb{N}$ define a projection $P_k$ on $X$ by

$$P_k((f_\bar{t})) = \begin{cases} f_\bar{t} & \text{if } |\bar{s}| \leq k \\ 0 & \text{if } |\bar{s}| > k \end{cases}.$$

It is easy to see that $||P_k|| = 1$ for all $k$.

**Proposition 6.1.** $\ell^1$ is not isomorphic to a subspace of $X$.

The example is similar to an example of Odell [O] and his arguments can be modified to prove Propositon 6.1. However we will give a slightly different proof which does not directly use the branch functionals.

**Proof.** Suppose that $(y_n)$ is a normalized sequence in $X$ which is $K$-equivalent to the usual unit vector basis of $\ell^1$. Because $\ell^1$ is not isomorphic to a subspace of $C_0(\omega^\omega)$ and range $(P_k - P_{k-1})$ is isometric to $[\sum C_0(\omega^\omega)]_2$, it can be shown by induction that $\ell^1$ is not isomorphic to a subspace of range $P_k$ for any $k$. Therefore by passing to a subsequence we may assume that $(P_k(y_n))_n$ is weakly Cauchy for each $k$. Because for each $k$ there are convex combinations of $(P_k(y_{2n} - y_{2n-1}))$ with small norm we can find a sequence of disjointly supported (relative to the $y_n$'s) convex combinations of $(y_{2n} - y_{2n-1})$, $(z_j)$, and an increasing sequence of intergers $(k_j)$ such that

$$\sum_{j=m}^{\infty} ||P_{k_j}z_j|| < 2^{-m},$$
for \( m = 1, 2, \ldots \). Moreover because \( X_0 \) is dense in \( X \) we may assume (by passing to a subsequence) that
\[
\sum_{j=1}^{m-1} ||(I - P_{k_m})z_j|| < 2^{-m},
\]
for \( m = 1, 2, \ldots \). In this way we get a sequence equivalent to the unit vector basis of \( \ell^1 \) which is essentially supported on disjoint levels of \( T \). By a standard perturbation argument we may assume that \((I - P_{k_m+1})z_m = 0 = P_{k_m}z_m\), and that \( z_{m|x} \neq 0 \) for only finitely many \( t \) for all \( m \).

Next note that by a theorem of James [J] we may assume that \((z_j)\) is \((1 + \epsilon)\)-equivalent to the basis of \( \ell^1 \). For any node \( \pi \) let \( W(\pi) = \{ t : t > \pi \} \), the wedge determined by \( \pi \). For each \( i \) there are finitely many nodes \( \pi(i, j) \) of length \( k_i \) such that if \( t \notin \cup_j W(\pi(i, j)) \) then \( z_{i|t} = 0 \). Let \( N \) be the number of nodes in the support of \( z_1 \). We claim that for each \( i \) there is a set \( \mathcal{F} = \mathcal{F}(i) \) of cardinality at most \( N \) such that
\[
||z_i - z_1|_{\cup_{j \in \mathcal{F}} W(\pi(i, j))}| < 4\epsilon.
\]
Indeed, if not let \( S = \{ (\pi, i) \} \) be a family of incomparable segments which compute the norm of \( z_1 + z_i \). Let \( S' \) be the set of segments in \( S \) which intersect the support of \( z_1 \) and \( S'' = S \setminus S' \). Clearly \( S' \) contains at most \( N \) segments. We have that
\[
4(1 + \epsilon)^{-2} \leq ||z_1 + z_i||^2
\]
\[
= \sum_{(\pi, j) \in S'} \left[ L(\pi, j)(z_1 + z_i) \right]^2 + \sum_{(\pi, j) \in S''} \left[ L(\pi, j)(z_1 + z_i) \right]^2
\]
\[
\leq \left[ \sum_{(\pi, j) \in S'} \left[ L(\pi, j)(z_1) \right]^2 \right]^{\frac{1}{2}} + \left[ \sum_{(\pi, j) \in S''} \left[ L(\pi, j)(z_1) \right]^2 \right]^{\frac{1}{2}}
\]
\[
+ \sum_{(\pi, j) \in S''} \left[ L(\pi, j)(z_1 + z_i) \right]^2
\]
\[
\leq (1 + (1 - 4\epsilon)^2 + (4\epsilon)^2 = 4(1 - 4\epsilon + 8\epsilon^2).
\]
Clearly this is impossible for small enough \( \epsilon \).

It follows that by another perturbation argument that we may assume that each \( z_i \) is supported in at most \( N \) wedges. As above let \( \pi(i, j) \), \( j = 1, 2, \ldots, N \) be the nodes of length \( k_i \) so that \( z_i \) is supported in \( \cup_j W(\pi(i, j)) \). We will next refine our sequence \((z_j)\) to get a subsequence such that there are branches \( \pi_1, \ldots, \pi_k \), \( k \leq N \) such that if \( \pi \) is any branch then \( \pi > \pi(i, j) \) for at most \( N \) nodes not on some \( \pi_m \) and \( \pi_j > \pi(i, j) \) for \( j = 1, 2, \ldots, k \) for all \( i \). Such a subsequence is easily obtained by induction on \( N \). Indeed, if \( N = 1 \), either there are infinitely many \( i \) and incomparable branches \((\pi_i, k_i)\) such that \( \pi_i > \pi(i, 1) \) and \( \pi_i \neq \pi(m, 1) \) for any \( m \neq i \), or there is a branch \( \pi_1 \) which contains all but finitely many of the nodes \( \pi(i, 1) \). Now suppose that \( \pi_1, \ldots, \pi_k \) are branches such that if \( \pi \) is any branch then \( \pi > \pi(i, j) \) for at most \( N-1 \) nodes not on some \( \pi_m \), and \( \pi_j > \pi(i, j) \) for \( j = 1, 2, \ldots, k \) for all \( i \). As above if there is some branch which contains all but finitely many of the nodes \( \pi(i, N) \) we add that branch to our list as \( \pi_{k+1} \) and pass to a subsequence
Proposition 6.2. Let $\mathcal{S}$ be a well-founded subtree of $\mathcal{T}$ and for each $\pi \in \mathcal{S}$ let $[f_\pi]$ be a normalized weakly null sequence in $X_\pi$. Then $[f_\pi]_{\pi \in \mathcal{S}}$ (reordered) is a weakly null sequence in $X$.

Proof. $X^*$ is the closed linear span of the functionals $\left[ L_{(\pi, i)} \right]_{\pi \in \mathcal{T}, i \in \mathbb{N}}$. Because $\mathcal{S}$ is well founded, for any $\bar{\pi} \in \mathcal{T}$, $\pi < \bar{\pi}$ for only finitely many $\pi \in \mathcal{S}$. Hence for any $\epsilon > 0$, $\bar{\pi} \in \mathcal{T}$, and $i \in \mathbb{N}$, $|L_{(\pi, i)} \left[ f_\pi \right]| \geq \epsilon$ for only finitely many $\pi$ and $\pi$. Therefore $[f_\pi]_{\pi \in \mathcal{S}}$ is a weakly null sequence in $X$. □

In order to estimate the averaging index we need to have some information about the $w^*$ topology on the functionals $L_{(\pi, i)}$.

Lemma 6.3. Let $\bar{\pi} \in \mathcal{T}$ and let $(\alpha_i)$ be a sequence in $\omega^\omega$ which converges to $\alpha$. Then

i) If $\alpha < \omega^\omega$, $L_{(\bar{\pi}, (\alpha_i), j)} \xrightarrow{w^*} L_{(\bar{\pi}, (\alpha), j)}$

ii) If $\alpha = \omega^\omega$, $L_{(\bar{\pi}, (\alpha_i), j)} \xrightarrow{w^*} L_{(\bar{\pi}, j)}$.

Proof. We need only consider the values of the functionals at $f_\pi$ for those $\pi \in \mathcal{T}$ such that $\pi \leq \bar{\pi}$ and $|\pi| - 1 \leq |\pi| \leq |\bar{\pi}|$. For all others the values do not depend on $(\alpha_i)$. If $\pi = \bar{\pi}$ then

$L_{(\bar{\pi}, (\alpha_i), j)} [f_\pi] = f_\pi(\alpha_i) \rightarrow f_\pi(\alpha) = L_{(\bar{\pi}, (\alpha), j)} [f_\pi]$, if $\alpha_i \rightarrow \alpha < \omega^\omega$,

and

$L_{(\bar{\pi}, (\alpha_i), j)} [f_\pi] = f_\pi(\alpha_i) \rightarrow f_\pi(\alpha) = 0 = L_{(\bar{\pi}, j)} [f_\pi]$, if $\alpha_i \rightarrow \omega^\omega$.

If $\bar{\pi} = \bar{\pi} + (\beta)$, then

$L_{(\bar{\pi}, (\alpha_i), j)} [f_\pi] = f_\pi(\beta) \rightarrow f_\pi(\beta) = L_{(\bar{\pi}, (\alpha), j)} [f_\pi]$, if $\alpha_i \rightarrow \alpha < \omega^\omega$,

and

$L_{(\bar{\pi}, (\alpha_i), j)} [f_\pi] = f_\pi(\beta) \rightarrow f_\pi(\beta) = L_{(\bar{\pi}, j)} [f_\pi]$, if $\alpha_i \rightarrow \omega^\omega$.

□

We will need to use well-founded subtrees of $\mathcal{T}$ of a special type, so as a technical convenience we introduce the following.

$$(z_i)_{i \in M}$$ such that $\bar{b}_{k+1} > \pi(i, N)$ for all $i \in M$. If this is not the case then we can find a subsequence such that there is at most one of the nodes $\pi(i, N)$ on any branch. Clearly this subsequence has the required properties.

To complete the argument we need to make a few observations about the norm on $X$. First observe that if $\pi$ is any branch and for each node $\bar{\pi}$ on $\pi$, $f_{\bar{\pi}}$ is a fixed function in $C_0(\omega^\omega)$ then $[f_{\bar{\pi}}]_{\bar{\pi} \in \mathcal{T}, \pi}$ is isomorphic to $c_0$. Second if the norm of $\sum a_i z_i$ is computed using only segments which intersect at most $N$ of the supports of the $z_i$’s, i.e., $\pi(i, j)$ for at most $N$ $j$’s, then the norm is at most $\left( \sum a_i^2 \right)^{1/2} N^2$. Finally observe that for the sequence $(z_i)$ a segment can intersect more than $N$ of the supports of the $z_i$’s only if it lies along one of the branches $\bar{b}_k$. A straightforward computation using these observations shows that the $z_i$’s are not equivalent to the unit vector basis of $\ell^1$. □
Definition. A well-founded tree $S \subset T$ is said to be complete if
i) $\pi + (\beta) \in S$ for some $\beta < \omega^\omega$ and $\pi \in T$ then $\pi + (\alpha) \in S$ for all $\alpha < \omega^\omega$ and $\pi \in S$,
ii) $(\alpha) \in S$ for all $\alpha < \omega^\omega$.

Next we will verify that such things exist.

Proposition 6.4. For every $\alpha < \omega_1$ there exists a complete well-founded tree $S \subset T$ of order at least $\alpha$.

Proof. For $\alpha = 1$ this is obvious. Suppose for all $\beta < \alpha$ there is a complete well-founded subtree $S$ of $T$ with order $\beta$. If $\alpha = \beta + 1$, we define $U = \{ (\eta) + \pi : \pi \in S \cup \{(\eta)\} \}$ and $\eta < \omega^\omega$. Clearly $o(U) = o(S) + 1$ and it is obvious that $U$ is complete. If $\alpha$ is a limit ordinal and $\alpha_i \uparrow \alpha$, for each $i$ let $S_{\alpha_i}$ be a complete well-founded subtree of $T$ of order $\alpha_i$. Let $U = \{ (\eta) + \pi : \pi \in S_{\alpha_i} \cup \{(\eta)\} \}$ and $\eta < \omega^\omega$. Clearly $U$ is complete, $o(U) \geq o(S_{\alpha_i})$ for all $i$ and thus $o(U) \geq \alpha$. □

We are now ready to show that there are weakly null sequences in $X$ with large averaging index.

Proposition 6.5. Suppose that $S$ is a complete well-founded subtree of $T$ and for each $\pi \in S$, $[f_T^n]_{n \in \mathbb{N}}$ is the Schreier sequence. Then for every $\beta < o(S)$ and $\pi \in S^{(\beta)}$, $L(\pi, 1) \in A_\beta \left[ \frac{1}{2}, [f_T^n]_{n \in \mathbb{N}, \pi \in S}, B_{X^*} \right]$.

Proof. We proceed by induction on $\beta$. It is sufficient to prove the result for $\beta + 1$ assuming it for $\beta$. If $\pi \in S^{(\beta + 1)}$, then because $S$ is complete $\pi + (\alpha) \in S^{(\beta)}$ for all $\alpha < \omega^\omega$. Moreover $L(\pi + (\alpha), 1) [f_T^n] = f_T^n (\alpha)$. In Section 4 it was shown that $\omega^\omega \in \mathcal{O}^1 \left( \frac{1}{2}, (f^n) \right) \subset A_1 \left( \frac{1}{2}, (f^n), \omega^\omega \right)$ where $(f^n)$ is the Schreier sequence. Therefore by Lemma 6.3, $L(\pi, 1) \in A_1 \left[ \frac{1}{2}, [f_T^n]_{n \in \mathbb{N}}, A_\beta \left[ \frac{1}{2}, [f_T^n]_{n \in \mathbb{N}, \pi \in S}, B_{X^*} \right] \right]$. Because the definition of the averaging index only requires that a subsequence have $\ell^1 - SP \geq \frac{1}{2}$, it follows that

$$L(\pi, 1) \in A_1 \left[ \frac{1}{2}, [f_T^n]_{n \in \mathbb{N}, \pi \in S}, A_\beta \left[ \frac{1}{2}, [f_T^n]_{n \in \mathbb{N}, \pi \in S}, B_{X^*} \right] \right] = A_{\beta + 1} \left[ \frac{1}{2}, [f_T^n]_{n \in \mathbb{N}, \pi \in S}, B_{X^*} \right]. \quad \square$$

Corollary 6.6. For every $\alpha < \omega_1$ there is a weakly null sequence in $X$ such that $o(A_\frac{1}{2}) \leq \alpha$.

Next we want to consider the relationship between the spreading model index and the constructive version of Mazur’s theorem as presented in [A-O]. One purpose of that paper was to try to use the averaging index to determine if given a Banach space $X$ there is an integer $k$ such that any weakly null sequence needs to be averaged at most $k$ times in order to get a norm null sequence. In what follows we will use the notation of [A-O] and refer the reader there for the relevant definitions.

The definition of the spreading model index makes some arguments of the type used in [A-O] difficult because of its sensitivity to passing to subsequences. On the other hand we are going to consider all weakly null sequences in the space $X$ so it seems natural to look for sequences which are in some sense extremal.
Proposition 6.7. Suppose $X$ is a subspace of $C(K)$ for some compact metric space $K$ and that $(x_n)$ is a weakly null sequence in the unit ball of $X$. Then for every $\epsilon > 0$ there is a subsequence $(x_n)_{n \in M}$ of $(x_n)$ such that for any $L \subset M$, infinite, and $\alpha < \omega_1$,

$$S^\alpha (\epsilon, (x_n)_{n \in L}, K) = S^\alpha (\epsilon, (x_n)_{n \in M}, K).$$

Proof. The idea is to use repeatedly the following lemma [A-O].

Lemma 6.8. Let $K$ be a second countable compact Hausdorff space and let $(x_n)$ be a weakly null sequence in $C(K)$. Then there is a subsequence $(x_n)_{n \in M}$ such that for every $t \in K$ and every neighborhood $N$ of $t$ there is a neighborhood $N'$ of $t$, $N' \subset N$, such that $(x_n|_{N'})$ has a spreading model.

We will prove the proposition by induction on $\alpha$. Note that there is an ordinal $\alpha_0$ such that $S^{\alpha_0} (\epsilon, (y_n), K) = \emptyset$ for every $\epsilon > 0$ and weakly null sequence $(y_n)$ in $C(K)$, and thus the induction is over a countable set. Fix $\epsilon > 0$.

Induction Hypothesis. If $(x_n)$ is a weakly null sequence in the unit ball of $X$. Then for every $\epsilon > 0$ there is a subsequence $(x_n)_{n \in M}$ of $(x_n)$ such that for any $L \subset M$, infinite, and $\beta \leq \alpha$,

$$S^\beta (\epsilon, (x_n)_{n \in L}, K) = S^\beta (\epsilon, (x_n)_{n \in M}, K).$$

Observe that the lemma proves the result for $\alpha = 1$. Now suppose that the result is true for all $\beta < \alpha$. If $\alpha = \beta + 1$, for some $\beta$, then let $(x_n)_{n \in L}$ be a subsequence such that

$$S^\lambda (\epsilon, (x_n)_{n \in J}, K) = S^\lambda (\epsilon, (x_n)_{n \in L}, K),$$

for all infinite $J \subset L$ and $\lambda \leq \beta$. Apply the lemma to $(x_n|_{\beta^\beta (\epsilon, (x_n)_{n \in J}, K)})_{n \in L}$ to get a subsequence $(x_n)_{n \in M}$ and note that the case $\alpha = 1$ applies to show that

$$S^{\beta + 1} (\epsilon, (x_n)_{n \in J}, K) = S^1 \left( \epsilon, \left( x_n|_{\beta^\beta (\epsilon, (x_n)_{n \in L}, K)} \right)_{n \in J}, S^\beta (\epsilon, (x_n)_{n \in L}, K) \right)$$

$$= S^1 \left( \epsilon, \left( x_n|_{\beta^\beta (\epsilon, (x_n)_{n \in L}, K)} \right)_{n \in M}, S^\beta (\epsilon, (x_n)_{n \in L}, K) \right) = S^{\beta + 1} (\epsilon, (x_n)_{n \in M}, K).$$

for all infinite $J \subset M$.

If $\alpha$ is a limit ordinal, let $\alpha_i \uparrow \alpha$ and choose infinite sets $M_1 \subset M_2 \subset \ldots \subset M_i \subset \ldots$, such that

$$S^\lambda (\epsilon, (x_n)_{n \in J}, K) = S^\lambda (\epsilon, (x_n)_{n \in M_i}, K),$$

for all infinite $J \subset M_i$ and $\lambda \leq \alpha_i$. Then if $M$ is an infinite set such that $M \setminus M_i$ is finite for all $i$, it follows that

$$S^\lambda (\epsilon, (x_n)_{n \in J}, K) = S^\lambda (\epsilon, (x_n)_{n \in M_i}, K),$$

for all infinite $J \subset M$ and $\lambda < \alpha$. However because $S^\alpha = \cap_{\lambda < \alpha} S^\lambda$, the equality holds for $\lambda = \alpha$ as well. □
Corollary 6.9. Suppose that \((x_n)\) is a weakly null sequence in the ball of \(C(K)\) for some compact metric space \(K\). Then there is a subsequence \((x_n)_{n \in M}\) such that

\[
A^\lambda \left( \epsilon, (x_n)_{n \in M}, K \right) = S^\lambda \left( \epsilon, (x_n)_{n \in M}, K \right),
\]

for all \(\lambda \leq \alpha\) and \(\epsilon > 0\).

Proof. The proof above gives us a subsequence so that passing to subsequences has no effect on \(\ell^1 - SP (x_n|\mathcal{N})\) for some \(\mathcal{N} \downarrow \{t\}\) for all \(t\), therefore

\[
A^1 \left( \epsilon, (x_n)_{n \in M}, K \right) = S^1 \left( \epsilon, (x_n)_{n \in M}, K \right).
\]

Induction completes the proof. \(\square\)

Corollary 6.10. Let \(X\) be a subspace of \(C(K)\), \(K\) compact metric, such that

\[
\sup \{ o(S, (x_n), \epsilon) : \epsilon > 0 \text{ and } (x_n) \subset B_X \text{ is weakly null} \} < \omega^{k+1}, \text{ for some } k \in \{-1, 0, 1, 2, \ldots\}. \]

Then \(X\) has property-\(A(k+2)\).

Proof. To establish property \(A(m)\) it is sufficient to show that every weakly null sequence has a subsequence with a norm null convex block subsequence of \(m\)-Averages. According to the previous corollary every weakly null sequence has a subsequence for which the averaging index and spreading model index agree. Therefore by [A-O, Theorem 4.1], \(X\) has property-\(A(k+2)\). \(\square\)

Because the spreading model index is in general smaller than the averaging index this corollary gives a real strengthening of [A-O, Corollary 4.2]. In particular the space constructed above has averaging index \(\omega_1\) but spreading model index at most \(\omega\).

Acknowledgement. The authors were visitors at the University of Texas during the time that a portion of this work was done and would like to thank the Department of Mathematics at University of Texas for its hospitality. In particular we are grateful to H.P. Rosenthal and E. Odell for many helpful comments.

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**Department of Mathematics, Oklahoma State University, Stillwater, OK 74078-0613**

**Department of Mathematics, University of Crete, Heraklion, Crete**