ON THE OCCURRENCE OF LARGE GAPS
IN SMALL CONTINGENCY TABLES

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ABSTRACT. Examples of small contingency tables on binary random variables with large integer programming gaps on the lower bounds of cell entries were constructed by Sullivant. We argue here that the margins for which these constructed large gaps occur are rarely encountered, thus reopening the question of whether linear programming is an effective heuristic for detecting disclosures when releasing margins of multi-way tables. The notion of rarely encountered is made precise through the language of standard pairs.

1. INTRODUCTION

When governmental bodies like census bureaus collect data there is a tension between publicly releasing as much information as possible while at the same time striving to ensure that any one individual’s private data cannot be discerned from the released information. One such case of this is when each piece of private data is recorded in a cell in a multi-dimensional contingency table and the released data is a collection of smaller margin tables (or simply margins) which portray the interactions between some subsets of the variables. These margins are simply higher dimensional analogues of row sums and column sums on the contingency table. So how would we know that a person’s individual data is protected despite the release of possibly many margins?

There are many criteria that may have to be accounted for in limiting the disclosure of private data: see for example [7] on issues with small cell counts in sparse contingency tables, or see [9] for privacy concerns in genetic databases (see [2, Page 5] for an introductory discussion). One such measure is the practical difficulty in attaining bounds for cell entries that can be discerned from the released margins. One way in which these bounds can be found is from standard methods in integer programming but solving the general integer program has a theoretical complexity of NP-complete [14, §18.1] and, practically speaking, are very challenging to solve. On the other hand, linear programs can be solved in polynomial time and there is significant high-powered software designed specifically to practically solve linear programs. So one must know if the linear relaxations of the integer programs associated to disclosure limitation are always good approximations for bounding cells in contingency tables.

Based on theoretical results for 2-way tables and practical experience on higher dimensional contingency tables, it was thought ([3], [5]) that the linear approximation was always reliable for bounding cells. However, using Gröbner basis techniques, Sullivant [17, Theorem 1] constructed
a family of contingency tables on \( n \geq 4 \) binary random variables with a specified collection of margins, denoted by \( \Delta_n \), such that the gap between the linear programming approximation of the cell bounds and the true integer programming cell bound for one of these margins is \( 2^{n-3} - 1 \). Thus, we are not entitled to always assume that the linear approximation of the bounds on cell entries is a faithful approximation to the true bounds on these cell entries.

All that said, perhaps the reason for [3] & [5] thinking that the linear approximation was reliable for bounding cell entries was that, in their practical experience, margins with large gaps were never encountered. Instead, perhaps the right claim for the practitioners to make is that the linear relaxations of the integer programs associated to disclosure limitation are almost always good approximations for bounding cells in contingency tables. In other words, while it is the case that Sullivant’s specified margins do indeed provide examples of the linear approximation being poor, these margins may be rarely encountered in practice. We argue here that this is the case for Sullivant’s large gaps and thus reopening the question of whether linear programming is an effective heuristic for detecting disclosures when releasing margins of multi-way tables:

**Theorem 1.1.** The margins for Sullivant’s \((2^{n-3} - 1)\)-gaps on the models \( \Delta^n \) are rare.

We will make the notion of rare precise through standard pairs. In the sections that follow we will first review and fortify Sullivant’s construction. Next, after defining standard pairs and arguing that they are the right tool to use for measuring rarity, we will prove Theorem 1.1 using a series of propositions regarding standard pairs specific to the strengthened Sullivant construction. We will see that the strengthened construction is not just made for its own sake but is crucial to proving Theorem 1.1. Finally, the detection of the \((2^{n-3} - 1)\)-gaps are made possible through Gröbner basis theory and this has been used to provide other examples of margins with large gaps [10, Corollary 4.3]. We will close with computational results showing that these other instances of large gaps are also rare in the same sense of Theorem 1.1.

### 2. An Alternative Construction Of Sullivant’s Large Gaps

Let \( A \) be a fixed matrix with \( N \) columns and \( c \) a real cost vector with \( N \) entries. Then for every fixed \( b \) that’s a non-negative integral combination of the columns of \( A \) (i.e. \( b \in \mathbb{N}A \)) we have the integer program \( \text{IP}_{A,c}(b) := \min \{ c \cdot u : Au = b, u \in \mathbb{N}^N \} \). The linear relaxation \( \text{LP}_{A,c}(b) \) of \( \text{IP}_{A,c}(b) \) is simply the same problem with the constraint \( u \in \mathbb{N}^N \) replaced by \( u \geq 0 \) and \( u \) real. For each fixed \( b \in \mathbb{N}A \) we have the quantity \( \text{gap}_{A,c}(b) := \text{optimal value of } \text{IP}_{A,c}(b) - \text{optimal value of } \text{LP}_{A,c}(b) \) and the maximum of all these is the integer programming gap [10] \( \text{gap}_c(A) = \max_{b \in \mathbb{N}A} \{ \text{gap}_{A,c}(b) \} \).

We will be especially interested in scenarios where \( c := (1,0)(= e_1 \in \mathbb{R}^N) \) but even in this case [10, §4] computing the gap precisely can be challenging. However, using Gröbner basis theory, we have the following proposition which is a quick way to get a lower bound on the integer programming gap. We repeat its proof here as we need its content for later results. From here on we replace \( \text{gap}_{e_1}(A) \) by simply \( \text{gap}_-(A) \).

**Proposition 2.1.** [10, Corollary 4.3] Let \( A \) be an integer matrix, let \( e_1 \) be a cost vector and let \( \succ \) be any term order. Suppose \( g := u - v \) is a reduced Gröbner basis element of \( A \) (with respect to the weight order induced by \( e_1 \) and \( \succ \) ) with \( u_1 = \alpha \geq 2 \). Then \( \text{gap}_-(A) \geq \alpha - 1 \).
Proof. (this proof due to Seth Sullivant) By our choice of $g$, $u$ is a non-optimal solution for the integer program with constraint vector $Au$. Now consider the vector $u - e_1$ where $e_1$ is the first standard unit vector. By the reducedness of $g$, $u - e_1$ must be an optimal solution for the integer program $\min \{ w_1 : Aw = A(u - e_1), w \text{ integral} \}$. On the other hand, the vector $w^* := (u - e_1) - \frac{(\alpha - 1)}{\alpha}(u - v)$ is a solution to the linear relaxation of the integer program with cost equal to zero, and so it must be an optimal solution to the linear relaxation of the above program. Thus, this gives an instance showing the gap is greater than or equal to $\alpha - 1$. □

We can rephrase Proposition 2.1 as saying that $\text{gap}_{A,e_1}(b) = \alpha - 1$ where $b = A(u - e_1)$. Note too, by the same argument, $u - (\alpha - \beta)e_1$ is an optimal integer solution for the integer program whose linear relaxation has an optimal solution of $\frac{\beta}{\alpha}(u - v)$.

**Corollary 2.2.** With the hypothesis of Proposition 2.1, for every integer $1 \leq \beta \leq \alpha - 1$, there exists a $b$ such that $\text{gap}_{A,e_1}(b) = \beta$.

We will be interested in scenarios like the figure below. The problem of bounding cell entries is precisely that of placing lower & upper bounds on, without loss of generality, the entry $u_{000}$ given that all the $u_i$’s are non-negative and integral and that they sum in a manner described by the margins below. The linear relaxation of this problem is approximating the bounds by permitting the $u_i$’s to be real valued and bounding the entry $u_{000}$ accordingly. We will focus on the discrepancy between the true lower bound and its linear approximation. In what follows we will review how the problem of finding the minimum value of the $u_0$ cell entry given the $S$ margins $b$ is equivalent to solving an integer program IP$_{A(S),(1,0)}(b)$ with the linear approximation being the linear relaxation of this program. The largest possible discrepancy for $S$ is precisely the integer programming gap$_-(S)$.

These margins can be described formally as follows: A hierarchical model $S$ on a ground set $[n] = \{1, 2, \ldots, n\}$ together with an integer vector $d = (d_1, d_2, \ldots, d_n)$. The quantity $n$ will be the dimension of our multiway contingency table and the $d_i$ is the number of levels in the $i^{th}$ direction of the table. Every facet $F$ of $S$ indicates a margin to be released and we can always construct a matrix $A(S)$ to describe these margins. From here on we will assume that our contingency tables are binary ($d_1 = d_2 = \cdots = d_n = 2$). See [11] for further discussion.

Using Proposition 2.1, Sullivant [17, Theorem 8] showed that for every $n \geq 4$ there is a model $\Delta_n$ with margin $b$ such that gap$_-(\Delta_n) \geq 2^{n-3} - 1$. Sullivant constructs a Graver basis element $\hat{f}_n$ with 0-th entry equal to $2^{n-3}$ for $A(\Delta_n)$ and, because of the algebraic interpretation of $\Delta_n$ could then claim that this Graver basis element is part of a reduced Gröbner basis.
In the remainder of this section we will show something a little stronger while simultaneously giving a slightly easier proof of [17, Theorem 8]: the constructed element  is in fact a circuit (a minimal linear dependence) for . While the proof is similar to that of [17, Theorem 8] it has the added benefit of avoiding the difficult primitive condition needed for Graver basis elements and, at the same time, showing that Sullivant’s construction is really a modified version of the well known checkerboard vector. More importantly, we will need the stronger circuit property in the next section where we prove Theorem 1.1. Let’s now build up Sullivant’s construction.

Example 2.3. Suppose we have a $2 \times 2 \times 2$ contingency table with released margins specified by the simplicial complex $B = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ (see Figure). If we replace the cell entries in the 3-dimensional binary table as follows: $u = (u_{000}, u_{001}, u_{011}, u_{100}, u_{101}, u_{110}, u_{111}) = (5, 3, 8, 6, 5, 10, 7)$ we then get the values in the released margins as specified in the figure.

Another way of saying this is as follows: The computation of these margins is equivalent to the computation of $A(B)u = (b_{000}, b_{001}, b_{011}, b_{110}, b_{100}, \ldots, b_{111})$, where the columns of the 0/1-matrix $A(B)$ have indices $i$ which are ordered lexicographically $(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)$. In turn, the columns of $A(B)$ are labelled $e_i$ in order. For each face $F$ of $B$ we have a row matrix with $2^{|F|}$ rows and $2^n = 2^3$ columns: one row for each $k \in \{0, 1\}^{|F|}$ with the $i^{th}$ entry in that row being equal to 1 (and 0 otherwise) if and only if $i$ restricted to $F$, $i_F$ equals $k$. For example, the row matrix for $\{1, 2\}$ of $B$ is

$$A(B)|_{\{1,2\}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

This proof of $A(B)$ being the matrix that describes precisely the margins of $S$ can be seen in [11, Equations (3) & (4)] and holds for any (binary or otherwise) hierarchical model. Also, the computational package 4ti2 [1, 11] can be used to compute the matrix of margins for any (binary or otherwise) model.

The binary hierarchical model in the previous example is simply the boundary of the 3-simplex. The first step in Sullivant’s construction is based on the model $B_n = \{S \subseteq [n - 2]\}$. The following lemma is a well known folklore result but we prove it here for the sake of completeness:

Lemma 2.4. The kernel of the matrix $A(B_n)$ is 1-dimensional. Letting $e_i$ be the standard unit vector in index position $i$, the unique basis element (up to scalar multiplication) for this kernel is the checkerboard vector $\text{checker} = \sum_{i:1 \text{-even}} e_i - \sum_{i:1 \text{-odd}} e_i$

Proof. From [11, Theorem 2.6] the dimension of the kernel of $A(B_n)$ is exactly the number of elements in $2^{[n-2]}$ that are not in $B_n$. There is only one such element, $[n - 2]$ itself, hence the kernel of $A(B_n)$ has dimension 1. Since every subset of the columns of size $2^{n-1}$ is a linearly independent set then every column of $A(B_n)$ must be used non-trivially in the unique dependence, say $w$, of $A(B_n)$. i.e., $w_i \neq 0$ for every index $i \in \{0, 1\}^{n-2}$. Let the last component of $w$ be $w_1 = 1$.

From [11, Lemma 2.1] this dependence must equal zero on every facet of the model $B_n$ and all the facets of $B_n$ are of the form $[n-2] \setminus \{j\}$ where $j \in [n-2]$. For each facet $S := [n-2] \setminus \{j\}$,
there is precisely only one other index that creates a column with a non-zero entry in its $k = 1$
row of facet $S$ and this index is $1 - e_j$. Furthermore, since each of these entries equals 1,
then $w_{1-e_j}$ must equal $-1$ for every $j \in [n-2]$. In other words, the index vectors $i$ with
$1 \cdot i = n - 3$ must take the value $-1$ in $w$. Repeating this argument on each of the $i$ with
$1 \cdot i = n - 3$, we get that $w$ must also take the values of $+1$ on each of the positions indexed
by $i$ with $1 \cdot i = n - 4$. Repeating recursively we get the vector $w = \text{checker}$ (up to possible
sign change) as claimed.

The checkerboard vector is so called because of the alternating $+1/-1$'s depending on parity.
Next consider for each $n \geq 4$ the model $\Gamma_n := B_n \cup \{\{n-1\}\}$. It is not too difficult to see that
its matrix of margins is $A(\Gamma_n) = \begin{bmatrix} A(B_n) & A(B_n) \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ where the columns are indexed by all
the $(i|0)$'s first followed by the $(i|1)$'s. The bottom two row vectors come from the row matrix
for the face $\{n-1\}$ of $\Gamma_n$. Permitting an abuse of notation, we write $\Gamma^n$ in place of $A(\Gamma_n)$.

Next, let $\sigma$ be the following index subset of the columns of $\Gamma^n$
$$
\sigma = \{(0|0), (0|1)\} \cup \{(i|0) : 1 \cdot i \text{ odd}\} \cup \{(i|1) : i \neq 0, 1 \cdot i \text{ even}\}.
$$

Then the submatrix of $\Gamma^n$ indexed in order by these columns is $
\Gamma^n_\sigma = \begin{bmatrix} e_0 & e_0 & A(B_n)_{\overline{0}} \\ 1 & 0 & r \\ 0 & 1 & 1 - r \end{bmatrix}
$
where $r$ equals 1 in its first $2^{n-3}$ entries and 0 in the remaining $2^{n-3} - 1$ entries, $e_0$ is the first
column of $A(B_n)$ and $A(B_n)_{\overline{0}}$ is simply $A(B_n)$ with its first column indexed by 0 removed.

**Lemma 2.5.** For each $n \geq 4$ there is a unique relation on the columns of the matrix
$\Gamma^n_\sigma$ given by
$$
f_n := u_n - v_n := 2^{n-3}e_{(0|0)} + \sum_{i\neq 0,1 \text{ even}} e_{(i|1)} - (2^{n-3} - 1)e_{(0|1)} - \sum_{i \cdot i \cdot i \text{ odd}} e_{(i|0)}. 
$$

**Proof.** We first observe that $\operatorname{rank}(\Gamma^n_\sigma) = \operatorname{rank}(A(B_n)) + 1$. In addition, $\Gamma^n_\sigma$
has only one column more than $A(B_n)$ and so the kernel of $\Gamma^n_\sigma$ has dimension equal to 1. The unique relation $f_n$ (up
to scalar multiplication) must also respect those relations on $A(B_n)$ and so $\Gamma^n_\sigma$ must respect the checkerboard relation of Lemma 2.4. But then there are also the last two rows of $\Gamma^n_\sigma$ to account
for, forcing the coefficients of $e_{(0|0)}$ to be $2^{n-3}$ and of $e_{(0|1)}$ to be $2^{n-3} - 1$ as claimed.

**Corollary 2.6.** The vector $f_n$ is a circuit (i.e. a minimal linear dependence) for the matrix
$\Gamma^n$.

To complete Sullivant’s construction, the logit model of $\Gamma_n$ is given by $\Delta_n := \logit(\Gamma_n) =
\{T \cup \{n\} : T \in \Gamma_n\} \cup 2^{[n-1]}$. The matrix $\Delta^n$ for the model $\Delta_n$ is the Lawrence lifting of
$\Gamma^n$ [13] and equals $\Delta^n = \begin{bmatrix} \Gamma^n & 0 \\ 0 & \Gamma^n \\ I & I \end{bmatrix}$ where $I$ is the $2^{n-1}$ identity matrix. Since $\Delta^n$ is the
Lawrence lifting of $\Gamma^n$ then [15] Chapter 7] we have the following property: $g$ is an (integer)
vector in the kernel of $\Gamma^n$ with $g \in \mathbb{R}^{2^{n-1}}$ if and only if the lifted $\hat{g} := g - g \in \mathbb{R}^{2^n}$ is in the
(integer) kernel of $\Delta^n$. Consequently, if we have a vector $g \in \mathbb{R}^{2^{n-1}}$ in the kernel of $\Gamma^n$ and if
the support of $g$ equals $\tau \subseteq [2^{n-1}]$ then we denote the support of $\hat{g}$ by $\hat{\tau} \subseteq [2^n]$ and we can
also describe $\hat{\tau}$ as follows: if $\eta \in \{0,1\}^{n-1}$ then $\eta \in \tau \iff (\eta|0), (\eta|1) \in \hat{\tau}$. Because of
the isomorphism $g \leftrightarrow (g, -g)$ between the kernels of matrices for any model and its logit, we have the first part of the following corollary. The second part follows from the first.

**Corollary 2.7.** (1) The lifted $\hat{f}_n$ is a circuit for $\Delta^n$. (2) (from [15] Prop. 4.11 & Thm. 7.1]) The lifted $\hat{f}_n$ is a reduced Gröbner basis element for $\Delta^n$.

The lifted $\hat{f}_n$’s are precisely what Sullivant constructed and argued that these were reduced Gröbner basis elements for $\Delta^n$ by showing that they were Graver basis elements, which is a weaker condition than (1) above but still sufficient to be able to apply condition (2). Consequently, applying Proposition 2.1 to $\hat{f}_n$, we get that gap$_- (\Delta_n) \geq 2^{n-3} - 1$ and that this gap is given by the margin $b := \Delta^n(\hat{u}_n - e_0)$. In addition, from this margin $b$, other margins that provide the same size gap are easy to construct. In the next section, via standard pairs, we show that for each $n \geq 4$ all these margins are very rare.

3. Rare Encounters With Large Gaps Via Standard Pairs

In order to examine the frequency with which the $(2^{n-3} - 1)$-gap occurs we need the notion of standard pairs [16]. Let $A$ be a fixed matrix with $N$ columns and $c$ a cost vector with $N$ entries. If $\gamma \in \mathbb{N}^N$ and $\tau \subseteq [N]$, we denote by $(\gamma, \tau)$ the set of vectors $(\gamma + \sum_{i \in \tau} n_i e_i : n_i \in \mathbb{N})$. We call $\gamma$ the root of the pair and $\tau$ the free directions of the pair. We say that the pair is associated for the family of integer programs $\text{IP}_{A, c} := \{\text{IP}_{A, c}(b) : b \in \mathbb{N}A\}$ if both (i) $\supp(\gamma) \cap \tau = \emptyset$ and (ii) every vector $p \in (\gamma, \tau)$ is an optimal solution for $\text{IP}_{A, c}(Ap)$. Furthermore, if there does not exist another associated pair $(\gamma', \tau')$ with $(\gamma, \tau) \subsetneq (\gamma', \tau')$ then we say that $(\gamma, \tau)$ is a standard pair for the family of integer programs.

We will now prove Theorem 1.1 relying on the verification of propositions that will follow. As in the previous section, $\hat{\sigma}$ denotes the support of $\hat{f}_n$, the Gröbner basis element of $\Delta^n$ which we showed was also a circuit of $\Delta^n$. As before, let the indices of the columns of $\Delta^n$ be denoted by $[i]l[l']$ where $i \in \{0, 1\}^{n-2}$ and $l, l' \in \{0, 1\}$. Finally, let $M(q)$ denote the set of margins $\{b : \Delta^n x = b$ and $1 \cdot x = q\}$. Note that $\Delta^n$ is graded i.e. the entries in each column of $\Delta^n$ sum to $n$ and so every margin $b$ belongs to a unique $M(q)$.

**Proof of Theorem 1.1.** From Proposition 2.1, Sullivant’s $(2^{n-3} - 1)$-gap was created by the margin $b = \Delta^n(\hat{u}_n - e_{(0|0|0)})$. In the remainder of this section we will show the following:

**Claim:** The margin $b = \Delta^n(\hat{u}_n - e_{(0|0|0)})$ belongs to the image (under $\Delta^n$) of the standard pair $((2^{n-3} - 1) \cdot e_{(0|0|0)}, \hat{\sigma} \setminus \{(0|0|0)\})$ and to no other standard pair.

Note too that most elements from the standard pair $((2^{n-3} - 1) \cdot e_{(0|0|0)}, \hat{\sigma} \setminus \{(0|0|0)\})$ are (coordinate-wise) greater than $\hat{u}_n - e_{(0|0|0)}$ and it follows easily that these elements also create gaps of size $2^{n-3} - 1$. We will call the non-negative integer image under $\Delta^n$ of this standard pair the Sullivant $(2^{n-3} - 1)$-margins.

These margins are rare in the following sense. By the gradedness of $\Delta^n$ each $M(q)$ is contained in the lattice points of a slice of the cone cone$(\Delta^n)$. This cone has dimension $2^{n-1} + 2^{n-2}$ [11] Theorem 2.6] and so each margin slice $M(q)$ of this cone is a collection of lattice points in a polytope of dimension $2^{n-1} + 2^{n-2} - 1$. On the other hand, the Sullivant $(2^{n-3} - 1)$-margins all live in the shifted cone $\Delta^n(\hat{u}_n - e_{(0|0|0)}) + \text{cone}(\Delta^n \setminus \{(0|0|0)\})$ which has dimension $|\hat{\sigma}| - 1 = 2(2^{n-2} + 1) - 1 = 2^{n-1} + 1$. Consequently for each $q$, the $(2^{n-3} - 1)$-margins sit in a $2^{n-1}$-dimensional slice of the $(2^{n-1} + 2^{n-2} - 1)$-dimensional $M(q)$. Hence, the
(2^{n-3} - 1)-margins sit in a relatively very small slice of $M(q)$ for every $q$ and would thus, in a random uniform choice of margin from $M(q)$, be rarely encountered. 

Before proving the central claim of the above proof there are a number of things to note from the above analysis of the $(2^{n-3} - 1)$-margins. A reasonable alternative approach to measuring the frequency of these $(2^{n-3} - 1)$-margins would be to ignore the standard pair analysis and instead ask how frequently these margins occur asymptotically. i.e. among all the margins with large 1-norms. In this case, regardless of the model $\mathcal{S}$, most gaps are 0. This makes reasonable sense when phrased as the linear relaxation of an integer program has an integer solution for most right hand sides $\mathbf{b}$ when $|\mathbf{b}| \gg 0$. A clean algebraic statement in terms of the Hilbert function of toric ideals can be seen in [15, Prop. 12.16]. However, given that released margins with large norms will come from tables with large cell counts (which are harder to bound tightly and are consequently more secure), the asymptotic results are not relevant thus justifying the need for the analysis above.

Also note that since the cone is shifted significantly from the origin then the $(2^{n-3} - 1)$-margins only start to appear in $M(q)$ slices for $q \geq 2^{n-3} - 1$ and so on contingency tables with mostly small counts, instances of these large gaps will never be encountered. Contingency tables with small cell counts are common in practice (see, for example, [3, 7]) which could give a further explanation as to why the large gaps are not encountered in practice.

Finally, we noted that a random uniform choice of margin in $M(q)$ is highly unlikely to pick out a $(2^{n-3} - 1)$-margin but there may be some prior distribution on the margins that is not uniform. A very reasonable assumption, based on sums being distributed normally, is that the margins that are most frequently encountered are those in the centre of cone$(\Delta^n)$ but in our instance the $(2^{n-3} - 1)$-margins appear in a shifted cone, shifted in a highly skewed fashion away from the centre of cone$(\Delta^n)$ along one of the extreme rays of that cone. So in fact, the uniform assumption in the proof of Theorem [11] may even be overly generous to the occurrence of the $(2^{n-3} - 1)$-margins.

We now turn our attention to proving the main claim in the proof of Theorem [11]. We first need the following remark:

**Remark 3.1.** Given any matrix $A$ and a circuit $\mathbf{w}$ of $A$, every integer vector in the kernel of $A$ with support contained in the support of $\mathbf{w}$ must be an integer multiple of $\mathbf{w}$. In particular, if $w_1 \geq 1$ then $\mathbf{w} - \mathbf{e}_1$ cannot be in the kernel of the matrix $A$.

For our interests, where $\mathbf{c} := \mathbf{e}_0$ and $A = \Delta^n$, we can rename the family of integer programs as $\text{IP}_{\Delta^n,-}$ and the optimality condition (ii) in the associated pairs as follows:

$$(\text{ii})_- : \not \exists \text{ both } \{n_l \in \mathbb{N} : l \in \tau\} \text{ and } \mathbf{t} \in \mathbb{N}^{2^n} \text{ such that } t_{(0|0|0)} < p_{(0|0|0)} \text{ and } \Delta^n \mathbf{p} := \Delta^n(\gamma + \sum_{l \in \tau} n_l \mathbf{e}_l) = \Delta^n \mathbf{t}$$

**Proposition 3.2.** The pair $(k \cdot e(0|0|0), \hat{\sigma}\backslash\{(0|0|0)\})$ is an associated pair for $\text{IP}_{\Delta^n,-}$ for all $1 \leq k \leq 2^{n-3} - 1$.

**Proof.** Clearly each pair satisfies condition (i) above so all we need verify is the rephrased condition (ii)_-. We will first show that condition (ii)_- holds for $k = 2^{n-3} - 1$.

Let $\mathbf{p} = (2^{n-3} - 1) \cdot \mathbf{e}_{(0|0|0)} + \sum_{l \in \sigma} n_l \mathbf{e}_l$. If there were to exist a $\mathbf{t}$ such that $\Delta^n \mathbf{p} = \Delta^n \mathbf{t}$ with $t_{(0|0|0)} < 2^{n-3} - 1$ then $\mathbf{p} - \mathbf{t}$ would be an integer vector in the kernel of $\Delta^n$. Since both $\mathbf{p}$ and
t are non-negative, we can assume that their respective supports are disjoint. Since p is the positive part of the kernel element then an index element (η|0) (or (η|1)) is in the support of p if and only if (η|1) (or (η|0) respectively) is in the support of t. Therefore, by the construction of \( \hat{f}_n \) and since \( \text{supp}(p) \subseteq \hat{\sigma} \) we must have \( \text{supp}(t) \subseteq \hat{\sigma} \).

But this cannot occur: if the set \( \text{supp}(t) \) were contained in \( \hat{\sigma} \) then \( \text{supp}(p - t) \) would also be contained in \( \hat{\sigma} \) with the \((0|0|0)\)-entry of this integer vector being positive and less than \( 2^{n-3} - 1 \), which using the fact that \( \hat{f}_n \) forms a circuit, would contradict Remark 3.1. Hence \((\hat{\sigma}) \subseteq \hat{\sigma} \) is satisfied and so \(((2^{n-3} - 1) \cdot e(0|0|0)) \cdot \hat{\sigma}\) \(\subseteq\) \(\{(0|0|0)\}\) is an associated pair.

For the other values of \( k \), we know that for any family of integer programs, if we have a vector \( p \) that is an optimal solution and another non-negative integer vector \( p' \) with \( p' \leq p \) then \( p' \) is also an optimal solution for that family. This proves that we have an associated pair for the other values of \( k \) too.

Note that nothing special was used here about the matrix \( \Delta^n \), only that it was the matrix of margins for a logit model, so we have the following general result for identifying quick gaps for other logit models:

**Corollary 3.3.** If \( \hat{f} \) be a circuit for any model logit(\( S \)) with \( f_0 = \alpha \) and \( \hat{\sigma} = \text{supp}(\hat{f}) \). Then \((k \cdot e_0, \hat{\sigma}\} \{0\}) \) is an associated pair for IP_{logit(\( S \))} for all \( 1 \leq k \leq \alpha - 1 \).

The next proposition claims that each of the associated pairs from the previous proposition are in fact standard pairs.

**Proposition 3.4.** The pair \((k \cdot e_{(0|0|0)}, \hat{\sigma}\} \{0\}) \) is a standard pair for IP_{\( \Delta^n \)} for all \( 1 \leq k \leq 2^{n-3} - 1 \).

**Proof.** It will suffice to consider the case of \( k = 1 \). By the previous proposition, we know that the pair is associated. Recall that we have a containment of associated pairs \((\gamma, \tau) \subseteq (\gamma', \tau') \) if and only if \( \gamma' \leq \gamma \) and \( \text{supp}(\gamma - \gamma') \cup \tau \subseteq \tau' \). If \( \gamma = e_{(0|0|0)} \) and \( \tau = \hat{\sigma}\} \{0\} \) then such a \( \gamma' \) would equal 1 or \( e_{(0|0|0)} \). If \( \gamma' = 1 \) then \((0|0|0) \in \tau' \) and \((1, \hat{\sigma}) \) would have to be an associated pair, which cannot be the case since the non-optimal solution \( \hat{u}_n \) is in this pair.

The other alternative is that \( \gamma' = e_{(0|0|0)} \) and in this case we need to show there does not exist an \( l \notin \hat{\sigma} \) for which the pair \((e_{(0|0|0)}, \hat{\sigma}\} \{0\}) \) \(\cup \{l\} \) is an associated pair. Such \( l \)'s are of one of the following forms: (a) \((i|1|0)\), (b) \((i|0|1)\), (c) \((i|0|0)\) or (d) \((i|1|1)\) where \( 0 \neq i \in \{0, 1\}^{n-2} \) as before. Note that regardless of the value of \( i \neq 0 \), we always have the relation \( \Delta^n w^+ = \Delta^n w^- \) where \( w^+ := e_{(0|0)0} + e_{(i|1)0} + e_{(0|1)1} + e_{(i|0)1} \) and \( w^- := e_{(0|1)0} + e_{(i|0)0} + e_{(0|0)1} + e_{(i|1)1} \).

(a) By construction, \( l = (i|1|0) \) is equivalent to \((i|0|0)\) and \((i|0|1)\) both being elements of \( \hat{\sigma} \). We claim that the pair \((0|0|0), \hat{\sigma}\} \{0\} \cup \{i|1|0\}) \) violates condition \((\hat{\sigma}) \subseteq \hat{\sigma} \). To see this notice that the choices of \( p = w^+ \) and \( t = w^- \) satisfy the following: \( p \) has support in \( \hat{\sigma} \cup \{i|1|0\} \), that \( \Delta^n p = \Delta^n t \) and \( 0 = t_{(0|0|0)} < p_{(0|0|0)} = 1 \). Hence, this is such a choice for \( p \) and \( t \) violating \((\hat{\sigma}) \subseteq \hat{\sigma} \).

(b) The exact same choice of \( p \) and \( t \) can be made in this case as for part (a).

(c) Consider the integral vector \( \hat{f}_n - (2^{n-3} - 1)(w^+ - w^-) \). This vector is in the kernel of \( \Delta^n \) with \((0|0|0)\)-entry equal to 1 and with positive support wholly contained in \( \hat{\sigma} \cup \{l\} \).

Letting \( p \) and \( t \) be the positive part and negative parts respectively of \( f_n - (2^{n-3} - 1)(w^+ - w^-) \) we have a violation of condition \((\hat{\sigma}) \subseteq \hat{\sigma} \).
Thus we have shown the \( k = 1 \) case. For \( 2 \leq k \leq 2^{n-3} - 1 \) simply replace \( p \) and \( t \) by \( k \cdot p \) and \( k \cdot t \) respectively.

In the last proposition we created a set of standard pairs that contained the optimal solutions \( \hat{u}_n - (2^{n-3} - k)e_{(0|0|0)} \) for every \( 1 \leq k \leq 2^{n-3} - 1 \). We can now complete the proof of the central claim in Theorem 1.1.

Proposition 3.5. The optimal solution \( \hat{u}_n - e_{(0|0|0)} \) is contained in precisely one standard pair, namely, \((2^{n-3} - 1) \cdot e_{(0|0|0)}, \hat{\sigma} \setminus \{(0|0|0)\}\)

Proof. The analysis is very similar to that which was carried out in Proposition 3.4. We need to show that for any \( l \notin \hat{\sigma} \), there exists \( n_l \in \mathbb{N} \) such that \( \hat{u}_n - e_{(0|0|0)} + n_le_l \) is not optimal. Such \( l \)'s are of one of the following forms: (a) \((i|1|0)\), (b) \((i|0|1)\), (c) \((i|0|0)\) or (d) \((i|1|1)\) where \( 0 \neq i \in \{0, 1\}^{n-2} \) as before. The kernel element \( w^+ - w^- \) of \( \Delta^n \) is as above.

(a) By construction, \( l = (i|1|0) \notin \hat{\sigma} \) is equivalent to \( 1 \cdot i \) being odd. If \( \hat{u}_n - e_{(0|0|0)} + n_le_l \) were optimal for every \( n_l \in \mathbb{N} \) then every vector \( 0 \leq z \leq \hat{u}_n - e_{(0|0|0)} + n_le_l \) would also be optimal. But, since \( 1 \cdot i \) is odd, then \( z = w^+ \) is such a vector and we already know that this vector is not optimal.

(b) The case of \( l = (i|0|1) \notin \hat{\sigma} \) with \( 1 \cdot i \) even can be argued as in part (a).

(c) The next case is \( l = (i|0|0) \notin \hat{\sigma} \) with \( 1 \cdot i \) being even. Here the index vector \((i|0|0)\) is in \( \text{supp}(w^-) \) and \((i|0|1)\) is in \( \text{supp}(w^+) \). Similar to the proof of Proposition 3.4 part (c), let \( p \) and \( t \) be the positive part and negative parts respectively of \( \hat{f}_n - (w^+ - w^-) \). In this case, \( p \leq \hat{u}_n - e_{(0|0|0)} + e_i \) and so it would need to be optimal if the free direction \( l \) were to be allowed. However, \( 0 = t_{(0|0|0)} < p_{(0|0|0)} = 2^{n-3} - 1 \) and so we cannot have \( l = (i|0|0) \) with \( 1 \cdot i \) being even as a free direction in a standard pair that contains \( \hat{u}_n - e_{(0|0|0)} \).

(d) The case of \( l = (i|1|1) \notin \hat{\sigma} \) with \( 1 \cdot i \) being odd is the same as that made in part (c).\( \square \)

4. Closing Remarks

Rather than asserting that linear programming is an effective heuristic for detecting disclosures when releasing margins of multi-way tables our result reopens this possibility, proposing that indeed large gaps in small hierarchical models do exist but may only rarely be encountered in practice. We have not addressed what happens for the other large gaps from Corollary 2.2 that occur in the model \( \Delta_n \). Nor have we addressed the extent to which the rarity encountered here happens for other hierarchical models. We attempt to address this by briefly reporting on some computational results.

From Corollary 2.2 and the discussion preceding it there are standard pairs like those from Proposition 3.4 whose respective images contain the \( k \)-margins for \( 1 \leq k \leq 2^{n-3} - 2 \) respectively. Using Macaulay 2 [8] we were able to confirm for \( n = 4 \) and \( n = 5 \) that while we did not have the uniqueness property of Proposition 3.5 for these \( k \)-gaps it was the case that the standard pairs \((\hat{\gamma}, \hat{\tau})\) for all of these margins had \(|\hat{\tau}| = |\hat{\sigma}| - 1\). Hence, for \( n = 4, 5 \) the computational evidence suggests that each of these \( k \)-gaps were contained in a \( 2^{n-1}-\)dimensional slice of the \((2^{n-1} + 2^{n-2} - 1)\)-dimensional \( M(q) \)'s. Furthermore, they were all highly skewed along the
(0)(0)(0) ray in the same manner as discussed after the proof of Theorem 1.1 for the \((2^n - 3 - 1)\)-margins, again making these \(k\)-margins unlikely to be encountered.

Other instances of models with large gaps constructed from Proposition 2.1 can be found in [1, Prop. 2.7]. The binary model there is the collection of all edges of the complete graph with \(n \geq 4\) vertices and a Gröbner basis element (with respect to \((1, 0)\)) is found there that provides a lower bound on the gap that grows linearly in \(n\). In this case too the computational evidence using Macaulay 2 indicates that all margins attained from Proposition 2.1 occur rarely. In the course of this work the answer to the following question seemed to be “yes” and may be of independent interest for those interested in Markov moves (see for example [6, Ch. 1]):

**Question 4.1.** If \(g := u - v\) is a Gröbner basis element for the matrix of margins for the model \(S\) then is it true that (1) \(|\text{supp}(g)| > \text{rank}(A(S))\) implies that all entries of \(g\) belong to \([-1, 0, +1]\)? (2) Given any gap arising from Proposition 2.1 is it true that its standard pair is always of the form \((\gamma, \tau)\) where \(\tau \subset \text{supp}(g)\)?

Thomas Kahle computationally verified that (1) is true for all models recorded at [12]. Note that if both (1) and (2) are true then every gap attained from Proposition 2.1 would be rare in the sense of Theorem 1.1.

Finally, the gaps coming from Proposition 2.1 are not the only way that gaps can arise. The gap can be computed precisely [10] by solving a collection of group relaxations [14, Ch. 24] coming from the collection of standard pairs for IP \(\Delta_n\). Using Macaulay 2, in the case of \(n = 5\) there are 1280 such standard pairs \((\gamma, \hat{\tau})\) that need to be considered and 1013 produce a gap greater than 0. But when checked computationally for the \(n = 5\) case each of the standard pairs that had gap greater than or equal to 1 were exactly those that had the number of free directions strictly less (and considerably less) than the rank of \(\Delta^5\). Similarly for \(n = 4\) and \(n = 5\) in the case of the model studied in [4, Prop. 2.7].

In conclusion the computations using Macaulay 2 suggest that the results of Section 3 may be the typical scenario, that the gaps provided from Proposition 2.1 may always be rare and furthermore that other gaps greater than or equal to 1 may be equally rare. Thus the computations lend further support to linear programming being an effective heuristic for detecting disclosures when releasing margins of multi-way tables.

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