Coulomb drag in mesoscopic rings

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Abstract

We develop a Luttinger liquid theory of the Coulomb drag of persistent currents flowing in concentric mesoscopic rings, by incorporating non-linear corrections to the electron dispersion relation. We demonstrate that at low temperatures, interactions between electrons in different rings generate an additional phase and thus alter the period of Aharonov-Bohm oscillations. The resulting nondissipative drag depends strongly on the relative parity of the electron numbers. We also show that interactions set a new temperature scale below which the linear response theory does not apply at certain values of external flux.

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During the last decade, persistent currents (PC’s) in mesoscopic rings have attracted significant interest both theoretically [1-3] and experimentally [7,8]. Much of this attention was due to a large discrepancy between experimentally observed current amplitudes in disordered metallic rings [7,8], and theoretical predictions based on a single-particle picture [2]. Yet unresolved, this puzzle has generated a number of theoretical works [3] on the role of electron-electron interactions in multi-channel disordered rings.

At the same time, in clean single-channel rings the theory predicts that PC’s at low temperatures should not be affected by interactions [10,4,6], and exhibit Aharonov-Bohm oscillations as a function of flux with the same period and amplitude as for non-interacting electrons. Results of a recent experiment on a single semiconductor ring with low number of channels are in agreement with these predictions [9].

On the other hand, interactions should become essential in a system consisting of a pair of clean 1D rings with different radii, placed concentrically, as is shown in Fig. 1. If the rings were isolated, PC’s in each ring would oscillate with a period determined by its radius. Inter-ring interactions will change the oscillation pattern by causing Coulomb drag of PC’s, as we show below.

There are, in general, two physical mechanisms for the current drag. The first mechanism [11], which originates from “friction” between two subsystems caused by scattering of carriers in one subsystem by density fluctuations in the other, has been widely studied during recent years [12], following experimental observation of the Coulomb drag [13]. At low temperatures, the resulting transresistance behaves as $T^2 (T^2 \ln T$ in a disordered system) and vanishes then with decreasing $T$ as the phase space for scattering shrinks. The second, nondissipative mechanism was pointed out by Rojo and Mahan [14], and is based on the observation that the van der Waals interaction between two current carrying subsystems is modified, resulting in a finite current drag at zero temperature.

The latter mechanism is, in fact, responsible for the PC drag in mesoscopic rings. Inter-ring interactions change the ground state energy of the system. As a result, PC in each ring, being a thermodynamic quantity, acquires a dependence on the flux penetrating the other ring. With decreasing temperature, this dependence should become sharper because of a singular (“sawtooth”) shape of the PC as a function of flux at zero temperature. In the vicinity of a “tooth”, even a small perturbation may affect strongly the amplitude of the PC. For this reason, an adequate theory of the PC drag must be valid beyond linear response.

In this paper we develop such a theory using the Luttinger liquid (LL) approach to PC in 1D rings. This approach is based on Haldane’s concept of topological excitations in finite-size 1D systems [15], and was recently extended by Loss [4] to account for parity effects [10] in the presence of external flux. However, in its standard formulation, the LL approach does not allow for current drag, which arises from the asymmetry between electrons and holes. With electron dispersion linearized, the asymmetry is lost and the electron and hole drags completely compensate each other. In LL formalism, the electron-hole symmetry manifests itself in a complete separation between zero modes, which carry PC, and bosonic fields, which are responsible for inter-ring interactions. We derive an interaction-induced correction to the zero mode spectrum by incorporating the lowest order non-linear correction to the electron dispersion relation. Further, we show that at low temperatures, the entire effect of inter-ring interactions is to generate a phase which, in turn, leads to PC drag by changing the flux seen by the electrons.
We begin by recalling the LL approach to interacting (spinless) electrons in a 1D ring with circumference \( L_i \) \((i = 1, 2)\) threaded by magnetic flux \( \phi_i \). By linearizing the spectrum near the Fermi points, the electrons are decomposed into right and left moving fermions with \( \psi_{i\alpha}(x) \) \((\alpha = \pm)\) satisfying twisted boundary conditions \( \psi_{i\alpha}(x + L_i) = (-1)^{N_i} e^{2\pi i \phi_i / \phi_0} \psi_{i\alpha}(x) \), where the sign factor reflects the dependence of the ground state on the parity of the particle number \( N_i \) and \( \phi_0 \) is the flux quantum. The fermion operator can be expressed through bosonic fields as \( \psi_{i\alpha}(x) = (2\epsilon L_i)^{-1/2} e^{-i\phi_{i\alpha}(x)} \), with \( \phi_{i\alpha}(x) = [\theta_i(x) - \alpha \varphi_i(x)]/2 \). The fields \( \varphi_i(x) \) and \( \theta_i(x) \) can be presented as

\[
\begin{align*}
\varphi_i(x) &= \varphi_{iJ} + M_i 2\pi x / L_i + \varphi_i(x), \\
\theta_i(x) &= \theta_{iM} + (J_i - 2\Phi_i) 2\pi (x + L_i/2) / L_i + \theta_i(x),
\end{align*}
\]

where \( M_i \) and \( J_i \) are fermion number and (topological) current operators, and \( \theta_{iM} \) and \( \varphi_{iJ} \) are their conjugates: \([\theta_{iM}, M_i] = [\varphi_{iJ}, J_i] = 2\beta \delta_{ij} \). For further convenience, we have incorporated both the flux and parity dependence into a single quantity \( \Phi_i \), which is defined as \( \Phi_i = \phi_i/\phi_0 \) for \( N_i \) odd, and \( \Phi_i = \phi_i/\phi_0 - 1/2 \) for \( N_i \) even. Owing to the boundary condition, the eigenvalues of \( J_i \) and \( M_i \) satisfy the selection rule that their sum is even. The periodic fields \( \varphi_i(x) \) and \( \theta_i(x) \) have the usual representation in terms of bosonic creation and annihilation operators

\[
\begin{align*}
\varphi_i(x) &= \sum_{k_i \neq 0} \frac{2\pi}{L_i k_i} \left| \frac{1}{2} e^{ik_i x} (a_{ik_i}^\dagger + a_{ik_i}) \right|, \\
\theta_i(x) &= \sum_{k_i \neq 0} \frac{2\pi}{L_i k_i} \left| \frac{1}{2} \text{sgn}(k_i) e^{ik_i x} (a_{ik_i}^\dagger - a_{ik_i}) \right|
\end{align*}
\]

where \( k_i = 2\pi p / L_i \) \((p \) being an integer), and the regularization factor \( \exp(-\epsilon |k_i| L_i/2\pi) \) is implicit. In the absence of backscattering, \( \varphi_i \) is related to the charge density \((\) relative to the background density \( N_i / L_i)\) by \( \rho_i = \partial_x \varphi_i / 2\pi \). The eigenvalues of \( M_i \) represent the numbers of extra electrons added to the ring while that of \( J_i \) are the excess of left over right moving fermions \([13]\).

In terms of the bosonic fields \( \varphi_i \), the (normal ordered) Hamiltonian of free fermions with linear dispersion reads

\[
H_0 = \frac{v_i}{8\pi} \int_0^{L_i} dx : (4\pi \Pi_i)^2 + (\partial_x \varphi_i)^2 :,
\]

where \( v_i \) is the Fermi velocity and \( \Pi_i \equiv P_i - \Phi_i / L_i = \partial_x \theta_i / 4\pi \), \( P_i \) being the canonical momentum.

In calculating PC’s it is convenient to go to the Lagrangian formulation and present the partition function \( Z \) in terms of a functional integral. The free (Euclidian) action has the form

\[
S_0 = \sum_i \int_0^\beta d\tau \int_0^{L_i} dx \left\{ \frac{1}{8\pi v_i} [(\partial_\tau \varphi_i)^2 + v_i^2 (\partial_x \varphi_i)^2] - i \frac{\beta}{L_i} (\Phi_i + \delta_i) \partial_\tau \varphi_i \right\},
\]

where \( \beta \) is the inverse temperature and \( \varphi(\tau, x) \) has a decomposition \((\) up to a constant\)
Here zero modes \( n_i \) and \( m_i \) are winding numbers in \( \tau \) and \( x \) directions, respectively, and \( \delta_i \) in (6) enforces the selection rule by taking values \( \delta_i = 0 \) (1/2) for \( m_i \) even (odd) [4]. Note that \( m_i \) coincides with the eigenvalue of \( M_i \). In the following we assume the number of electrons in the rings fixed and restrict ourselves to the \( m_i = 0 \) (with \( \delta_i = 0 \)) sector.

The partition function is given by

\[
Z = \int D\phi e^{-S},
\]

with \( S = S_0 + S_{int} \). Here the term \( S_{int} = \frac{1}{8\pi^2} \sum_{ij} \int_0^L d\tau \int_0^{L_i} dx \int_0^{L_j} dx' \partial_x \varphi_i V_{ij}(x,x') \partial_x' \varphi_j, \)

\[
(8)
\]

with \( V_{ij} \) being the Coulomb potential, describes intra- and inter-ring interactions, and the measure \( D\phi \) includes the sum over zero modes together with the integral over periodic fields \( \varphi_i \). The PC is found by separating out the contribution from the zero-modes. With \( \varphi_i(\tau, x) \) of the form (7) (with \( m_i = 0 \)), the latter are completely decoupled and the action takes the form

\[
S = \sum_i \left( \frac{n_i^2}{\beta T_i} - 2\pi m_i \Phi_i \right) + \mathcal{S},
\]

\[
(9)
\]

where \( \mathcal{S} = S_0[\varphi] + S_{int}[\varphi] \) is flux-independent and \( T_i = 2v_i/\pi L_i \) is the temperature scale set by the finite size of each ring. Thus, the PC, \( J_i = \beta^{-1} \partial_\eta \ln Z \), is carried by zero modes whose spectrum is unaltered by inter-ring interactions. This is a consequence of the linearization of the electron dispersion and the resulting quadratic form of the bosonic action \( S_0 \). Physically, linearization lifts the electron-hole asymmetry, which is responsible for the current drag, so that PC in each ring is sustained by its own flux \( \Phi_i \).

Let us now consider the lowest non-linear correction to the fermion dispersion relation. Following [15], we take the correction to the linearized fermion Hamiltonian as

\[
H_{nl} = -\frac{1}{2m_e} \sum_{\alpha} \int_0^{L_i} dx : \psi_{i\alpha}^\dagger(x) \partial_x^2 \psi_{i\alpha}(x) :,
\]

\[
(10)
\]

where \( m_e \) is electron mass. Transforming (10) with the help of bosonic representation of \( \psi_{i\alpha}(x) \), we obtain

\[
H_{nl} = \frac{1}{48\pi m_e} \sum_i \int_0^{L_i} dx : (\partial_x \varphi_i)^3 + 3\partial_x \varphi_i (4\pi \Pi_i)^2 :,
\]

\[
(11)
\]

which means that the corresponding bosonic theory is no longer free. Regarding \( H_{nl} \) as a small term, we can obtain a correction to the action by adding (11) to (5) and repeating the steps leading to the functional-integral representation of \( Z \) [17]. To first order in \( 1/m \), the correction takes the form

\[
S_{nl} = \frac{1}{48\pi m_e} \sum_i \int_0^\beta d\tau \int_0^{L_i} dx \left[ (\partial_x \varphi_i)^3 - \frac{3}{v_i^2} \partial_x \varphi_i (\partial_x \varphi_i)^2 \right].
\]

\[
(12)
\]

Using (7) (with \( m_i = 0 \)), we separate out the zero-mode contribution in \( S_{nl} \) by writing
where \( S_{nl} \) depends only on fields \( \varphi_i \), which are now coupled to the zero modes via the first term. Adding (13) to (12), we then perform the functional integral \( e^{-\beta \overline{S}} = \int D\varphi e^{-\overline{S} - S_{nl}} \).

Note that after the zero modes are separated out, the small term \( S_{nl} \) can be neglected and the remaining gaussian integral explicitly evaluated. The resulting \( \overline{F}(n_1, n_2) \) admits an expansion in terms of \( 1/m \) with the odd orders vanishing due to translational invariance. Thus, to the first non-vanishing order

\[
\overline{F}(n_1, n_2) = \overline{F}_0 + \sum_{ij} \overline{T}_{ij} n_i n_j,
\]

where \( \overline{F}_0 \) is \( n_i \) and \( \phi_i \) independent. The second term in (14), which is to be combined with the first term in (13), represents the correction sought to the zero-mode spectrum. The diagonal coefficients \( F_{ii} \) can be absorbed into \( T_i \) via renormalization of the Fermi velocities and do not play any role in the following [15]. Factorizing out the zero-mode part of the partition function, \( Z_0 \equiv e^{-\beta \overline{F}_0} = e^{\beta \overline{F}_0} Z \), we finally arrive at

\[
Z_0 = \sum_{n_1 n_2} \exp \left( -\frac{1}{\beta} \sum_{ij} a_{ij} n_i n_j + 2\pi i \sum_i n_i \Phi_i \right),
\]

where \( a_{ii} = T_i^{-1} \), and the parameter \( a_{12} = \beta^2 \overline{T}_{12} \), given by

\[
a_{12} = \frac{-1}{16 m_e^2 v_f^2 v_2^2 \beta} \int d\tau_1 \int d\tau_2 \partial_{\tau_1} \partial_{\tau_2} \overline{D}_{12}(\tau_1, \tau_2) \partial_{\tau_1} \partial_{\tau_2} \overline{D}_{21}(\tau_2, \tau_1),
\]

describes the coupling between zero modes in different rings. Here \( \tau_i = (\tau_i, x_i) \) and \( \overline{D}_{ij}(\tau_1, \tau_2) \) is the nondiagonal part of the boson Green function, \( \overline{D}_{ij}(z_1, z_2) = \langle \varphi_i(z_1) \varphi_j(z_2) \rangle \), calculated from the action \( \overline{S} \). Due to the azimuthal symmetry of the system, a convenient basis for \( \overline{D}_{ij} \) is given by angular-momentum eigenfunctions, in which the Fourier–transform of the inverse Green function reads

\[
\overline{\mathcal{G}}_{ij}^{-1}(\omega, p) = \frac{\delta_{ij}}{4\pi v_i L_i} \left[ \omega^2 + \frac{2\pi p}{L_i} \left( \frac{2\pi p}{L_i} \right)^2 \right] + \frac{p^2}{L_i L_j} V_{ij}(p),
\]

where \( V_{ij}(p) \) is the Fourier-transform of the Coulomb interaction. For \( d \ll L_i \) the interaction is given by

\[
V_{ij}(p) \simeq \frac{2e^2}{\kappa(L_i L_j)^{1/2}} K_0 \left[ \frac{2\pi p (p + 1/2)d_{ij}}{(L_i L_j)^{1/2}} \right],
\]

where \( \kappa \) is the dielectric constant, \( K_0 \) is the modified Bessel function, and we used the notation \( d_{ij} = d \) for \( i \neq j \), and \( d_{ii} = w \), \( w \) being the width of the rings. The expression for \( a_{12} \) then takes the form

\[
a_{12} = \frac{-\pi^2}{4 m_e^2 v_1^2 v_2^2 L_i L_j^2 \beta} \sum_{\omega p} \omega^2 p^2 |\overline{D}_{12}(\omega, p)|^2.
\]
The frequency sum in (19) is straightforward and yields
\[ a_{12} = \frac{-2\pi^2}{m_e v_1 v_2 L_1 L_2 (T_1 T_2)^{1/2}} \sum_{p>0} pv_{12} \frac{\partial}{\partial v_{12}} \left[ \left( \frac{Q_+}{t_+} - \frac{Q_-}{t_-} \right) \frac{1}{Q_+^2 - Q_-^2} \right], \quad (20) \]
where
\[ t_\pm = \tanh \left[ \frac{\pi^2}{2} \left( \frac{T_1 T_2}{T} \right)^{1/2} p Q_\pm \right], \quad Q_\pm(p) = \sqrt{A(p) \pm B(p)}, \quad (21) \]
\[ A(p) = [r(1 + v_{11}) + r^{-1}(1 + v_{22})]/2, \quad B(p) = \sqrt{[r(1 + v_{11}) - r^{-1}(1 + v_{22})]^2/4 + v_{12}^2}, \quad (22) \]
\[ v_{ij}(p) = \frac{2}{\pi^2} (T_1 T_2)^{-1/2} V_{ij}(p) = \frac{2e^2}{\kappa \pi (v_i v_j)^{1/2}} K_0 \left[ \frac{2\pi(p + 1/2)d_{ij}}{(L_i L_j)^{1/2}} \right], \quad (23) \]
and \( r = (T_1/T_2)^{1/2} \) characterizes the asymmetry between rings. Let us obtain an estimate of \( a_{12} \) for identical rings, with \( v_i = v, T_i = T_0, \) and \( L_i = L. \) The analysis of (20) shows that for \( T \ll T_d \equiv T_0 (L/d)^{1/\alpha \ln(2d/w)}, \) where \( \alpha = 2e^2/\kappa \pi v \) is dimensionless interaction constant, the coupling \( a_{12} \) is temperature independent. Since for \( d \ll L \) one has \( T_d \gg T_0, \) and \( T_0 \) is the crossover temperature above which the PC is exponentially suppressed, \( a_{12} \) can be regarded as a constant in the entire range of relevant temperatures. For \( d \gg w \) the screening is strong and \( a_{12} \) is determined by the first order in the inter-ring interaction. The rhs of (20) then gives
\[ a_{12} \sim -\frac{\alpha^2}{32 T_0 d^2 k_F^2 [1 + \alpha \ln(2d/w)]^{5/2}}, \quad (24) \]
where \( k_F \) if the Fermi momentum. A \( d^{-2} \) dependence (without logarithmic factor) was obtained previously for infinite wires [14].

Turning to the partition function (13), we observe that since the zero modes in different rings are now coupled, the PC in, say, ring 1, \( J_1 = -\partial F_0/\partial \phi_1, \) depends also on the flux through ring 2. Since the coupling \( a_{12} \) is small, one could try to obtain a correction to the current by expanding the free energy to first order in \( a_{12}. \) It is important to realize, however, that at low temperatures such an expansion does not exist for all values of \( \Phi_1. \) In order to make this point clear, let us rewrite (13) in a different form using Poisson’s formula (omitting constant prefactor)
\[ Z_0 = \sum_{p_1 p_2} \exp \left[ -\beta \sum_{ij} (p_i - \Phi_i) c_{ij} (p_j - \Phi_j) \right], \quad (25) \]
where \( \hat{c} = \pi^2 \hat{a}^{-1}. \) For \( a_{12} \ll a_{ii} \) one has for diagonal elements \( c_{ii} = \pi^2 T_i \) which, in the absence of interactions, is the level spacing at the Fermi level. With such form of \( Z_0, \) the PC is given by
\[ J_1(\Phi_1, \Phi_2) = \frac{2}{\phi_0} \left[ c_{11}(\langle p_1 \rangle - \Phi_1) + c_{12}(\langle p_2 \rangle - \Phi_2) \right], \quad (26) \]
where \( \langle p_i \rangle \) stands for the average of \( p_i \) calculated from the partition function (25). Note that \( \Phi_i \) in (26) takes values in the interval \( 0 < \Phi_i < 1, \) and \( J_i \) is periodically continued outside
of this interval. For further analysis, it is convenient to obtain an equation for $\langle p_i \rangle$. This can be done by making use of the following identity

$$\langle p_i \rangle = \frac{1}{2} - \frac{1}{2} \sum_{n=-\infty}^{\infty} \left\langle \tanh \beta \left[ \sum_{j} c_{ij} (p_j - \Phi_j) + c_{ii} (n + 1/2 - p_i) \right] \right\rangle,$$

(27)

which can be readily derived by substituting Jacoby’s product formula for the $\theta_3$-function in place of the sum over variable $p_i$ in the double sum (24) (one notices, for example, that for $c_{12} = 0$ the rhs of (27) is $p$-independent and $\langle p_i \rangle - 1$ reduces to the log-derivative of the $\theta_3$-function). For low temperatures, $T \ll c_{ii}$, the fluctuations of $p_i$ are suppressed and all moments factorize, $\langle p_i^m \rangle = (\langle p_i \rangle)^m$, which allows one to replace $p_i$ by $\langle p_i \rangle$ in the rhs of (27). It is also easy to see that for such temperatures, all terms in the sum with $n \neq 0$ cancel each other out (up to exponentially small corrections). Then (27) simplifies to the system

$$\langle p_1 \rangle = f_0 \left[ c_{11} (1 - 2 \Phi_1) + 2 c_{12} (\langle p_2 \rangle - \Phi_2) \right],$$

$$\langle p_2 \rangle = f_0 \left[ c_{22} (1 - 2 \Phi_2) + 2 c_{12} (\langle p_1 \rangle - \Phi_1) \right],$$

(28a)

(28b)

where $f_0(x) = (e^{\beta x} + 1)^{-1}$ is the Fermi function. This system, together with (24) determines the PC at low temperatures, $T \ll c_{ii} = \pi^2 T_i$. In the absence of inter-ring coupling ($c_{12} = 0$) one recovers the PC for an isolated ring,

$$J_i^0(\Phi_i) = 2I_i \left\{ f_0 \left[ c_{ii} (1 - 2 \Phi_i) \right] - \Phi_i \right\},$$

(29)

where $I_i = c_{ii}/\phi_0 = e v_i / L_i$ is the current amplitude. The current $J_i^0$ is a linear function of flux except in the interval $\Phi_i - 1/2 \sim T/c_{ii} \ll 1$ in which it rapidly changes from $-I_i$ to $I_i$. Note that $\Phi_i$ depends on the parity of the total number of electrons, $N_i$, resulting in diamagnetic (paramagnetic) current for $N_i$ odd (even) [10,4].

Turning the coupling on, PC’s can be found by iterating the system (28). First, in the argument of the Fermi function we substitute $\langle p_i \rangle$ expressed via $J_i$ by neglecting the second term in (28) (and in a similar expression for $J_2$). Substituting (28) back into (26), we finally obtain

$$J_i(\Phi_1, \Phi_2) = J_i^0(\Phi_1 - \delta_1), \quad \delta_1 = b J_2(\Phi_1, \Phi_2),$$

(30)

with $b = \phi_0 c_{12}/2 c_{11} c_{22} = -\phi_0 a_{12}/2\pi^2$. Eq. (30), which is our main result, describes the mutual dependence (Coulomb drag) of PC’s in coupled rings. It shows that electrons encircling the ring $i$ acquire additional (Berry’s) phase proportional to PC in the ring $j$. Thus, inter-ring interactions change the period of the Aharonov-Bohm oscillations. In particular, peak positions of PC get shifted by an amount $\delta_i$. Remarkably, the resulting oscillation pattern depends strongly on the relative parity of electron numbers $N_i$, as it can be seen in Fig. 2. We emphasize that in the vicinity of the peak, $\Phi_i \simeq 1/2$, PC (30) cannot be expanded in $\delta_i$ if the temperature is low enough: $T \lesssim T^* \equiv c_{12}$. Similarly, $J_j(\Phi_1, \Phi_2)$ in the expression for $\delta_i$ cannot be replaced by $J_j^0(\Phi_j)$ for $\Phi_j \simeq 1/2$. If $\Phi_i$ is not too close to 1/2, one can neglect $\delta_i$ in the argument of the Fermi function and the correction to PC takes the form $\delta J_i / I_i = 2b J_j$ (note that $b$ is positive).
Eq. (30) was obtained for temperatures lower than $T_i$. With increasing $T$, the PC amplitude gets damped due to the loss of phase coherence. Expanding the exponent in (25) in terms of $c_{12}$, the correction to the free energy can be written as

$$\delta F_0 = \phi_0 b J_0^0(\Phi_1) J_2^0(\Phi_2),$$

(31)

with $J_0^0(\Phi_i) = (4I_i/\pi)(T/T_i)e^{-T/T_i}\sin(2\pi\Phi_i)$ [4]. Correspondingly, the correction to PC for $T \gg T_i$ is suppressed by an additional damping factor.

Finally, let us address the experimental implications of our result. PC’s with amplitude of about 4 nA were observed [9] in a micron-size GaAlAs/GaAs ring with width $w \approx 160$ nm, and Fermi wavelength of electrons $\lambda_F \approx 40$ nm, which corresponds to $\alpha \approx 0.5$ and $T_0 \approx 200$ mK. Assuming that two concentric rings were fabricated on the same plane, the estimate of $a_{12}$ for the above parameters and $d \sim w$ yields a small phase shift $\delta \sim 10^{-4}$. However, this small value is due to the rather large distance between rings in the plane. The situation will be significantly improved if two concentric rings were fabricated on parallel 2D layers. Since the interlayer distance can be as small as 3 nm [21], this would allow one to decrease the actual separation between current carrying channels by 1–2 orders of magnitude. A reliable estimate of $a_{12}$ for $d \ll w$ would require careful treatment of the screened interaction between the two closely spaced rings and depend on details of the structure. The $d$–dependence (24) suggests, however, that in this case one might expect a phase shift of several percent, which would be within resolution of experimental observations.

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FIGURES

FIG. 1. Schematic picture of two concentrically placed rings.

FIG. 2. Example of PC in ring 1 at $\phi_2/\phi_1 = 1.1$ and $T_i/T = 5.0$ for different inter–ring couplings: $c_{12}/c_{ii} = 0$ (solid line), and $c_{12}/c_{ii} = 0.1$ with the same (dashed line) and opposite (dotted line) parities of electron numbers.
Fig. 1
Fig. 2