Equivariant Kählerian extensions of contact manifolds

Ayşe Kurtdere

Abstract

For contact manifolds \((M, η)\) a complexification \(M^c\) is constructed to which the contact form \(η\) extends such that the exterior derivative of the extended form is Kählerian. In the case of a proper action of an extendable Lie group this construction is realized in an equivariant way. In a simultaneous stratification of \(M\) and \(M^c\) according to the istropy type, it is shown that the Kählerian reduction of the complexification can be seen as the complexification of the contact reduction.

1 Introduction

Manifolds with additional structure can sometimes be understood better if the structure extends to a complexification of the manifold. By a result of Whitney ([Wh1]) and Shuštric ([Sh]) a differentiable manifold \(M\) can be embedded as a closed, real analytic and totally real submanifold of a complex manifold \(M^c\) with the dimension \(\dim_C M^c = \dim_R M\). Using his solution of the Levi problem Grauert ([Gr]) proved that the complexification \(M^c\) can be realized as a Stein manifold, in particular, it can be holomorphically and properly embedded in some \(\mathbb{C}^N\). During the last two decades, complexifications of real manifolds with additional structure achieved some attention. An equivariant version for proper actions has been shown in [H2]. Stratmann ([St]) considers proper actions of Lie groups \(G\) on symplectic manifolds \((M, ω)\) and shows that there is a Stein complexification \(M^c\) of \(M\) with a \(G\)-invariant Kähler form \(τ\) such that \(ω = τ^*_M(τ)\) where \(τ_M : M \hookrightarrow M^c\) is the embedding of \(M\) in \(M^c\).

In this paper a similar extension result is shown for 1-forms. Contact manifolds \((M, η)\) on which a Lie group \(G\) acts properly by contact transformations are of particular interest. As a general assumption in this work the Lie group \(G\) has finitely many connected components, is extendable, i.e., the natural homomorphism \(G \rightarrow G^C\) is injective, and is acting properly on \(M\) as a group of diffeomorphisms. In Section 2 an equivariant complexification \(i_M : M \hookrightarrow M^c\) such that \(G\) acts on \(M^c\) properly by holomorphic transformations and a strictly plurisubharmonic, \(G\)-invariant function \(ρ : M^c \rightarrow \mathbb{R}\) are constructed with the property that \(η = i^*_M(ρ)\). For this, a slice construction for \(M = G \times^K S\), \(K\) maximal compact in \(G\), is used to construct a complexification \(M^c\) of \(M\) by a complexification \(G^C \times S^C\) of \(G \times S\):

\[
\begin{align*}
G \times S &\hookrightarrow \Omega \subset G^C \times S^C \\
\downarrow \pi_K &\quad \downarrow \pi_{K^C} \\
M = G \times^K S &\hookrightarrow M^C = G^C \times^K S^C.
\end{align*}
\]
The interplay of complexifications and contact reductions and Kählerian reductions is discussed in the remaining section. Roughly speaking, the complexifications of contact reductions of a $G$-contact manifold $(M, \eta)$ can be seen as the Kählerian reduction of the complexification. This is shown along a simultaneous stratification of both the contact manifold and its complexification.

2 Extension of forms

Let $G$ be an extendable Lie group. Let $M$ be a real analytic $G$-manifold with a 1-form $\eta$. In this section the form is extended to a complexification $M^c$ of $M$. This is done equivariantly for groups acting on $M$ and leaving $\eta$ invariant.

2.1 Equivariant extensions in the case of compact groups

Let $K$ be a compact transformation group and let $M$ be a $K$-manifold with a $K$-invariant 1-form $\eta$. In the following, an equivariant complexification $M^c$ of $M$ is constructed to which $\eta$ is extended equivariantly.

**Proposition 2.1.** Let $K$ be a compact Lie group, $M$ a $K$-manifold and $\eta$ a $K$-invariant 1-form. Then there is a $K$-equivariant complexification $M^c$ of $M$ and a $K$-invariant strictly plurisubharmonic function $\varphi : M^c \to \mathbb{R}$ such that

$$t_M^*(d^c \varphi) = \eta.$$

**Proof.** First, the local situation without the presence of symmetries is considered. By a theorem of Whitney ([Wh2, Theorem 1]) $M$ can be given an atlas with a real analytic structure. It can also be assumed that the action map $K \times M \to M$ is real analytic ([MS]). Let $X$ be a complexification of $M$ such that $t_M : M \hookrightarrow X$ is a real analytic, closed embedding ([WhBR]). It can be assumed that $X$ is a Stein manifold ([GR]). Let $(U_\alpha, \varphi_\alpha)_{\alpha \in I}$ be an atlas of real analytic charts $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{C}^{n}$. Every map $\varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to U_\alpha$ extends biholomorphically to a map $(\varphi_\alpha^{-1})^C : (\varphi_\alpha)^C(W_\alpha) \to W_\alpha$, where $W_\alpha$ is an open and connected neighbourhood of $U_\alpha$ in $X$ and $(\varphi_\alpha^{-1})^C(W_\alpha)$ is open in $U_\alpha \times i\mathbb{R}^n \subset \mathbb{C}^n$. Then $\bigcup_{\alpha} W_\alpha$ is an open submanifold of $X$ containing $M$. After shrinking, this set can be chosen as a Stein neighbourhood $M^c$ of $M$ in $X$ ([GR]). The biholomorphic maps $(\varphi_\alpha^{-1})^C$ have inverse biholomorphic maps, denoted here by $\varphi_\alpha^C$, which give an atlas $(W_\alpha, \varphi_\alpha^C)$ of $M^c$. Note that $\varphi_\alpha^C(W_\alpha) \subset U_\alpha \times i\mathbb{R}^n$ is an open neighbourhood of $\varphi_\alpha^C(W_\alpha \cap M) = U_\alpha \times \{0\}$ in $\mathbb{C}^n$. Let $x_1, \ldots, x_n$ be coordinates on $\varphi_\alpha^C(W_\alpha \cap M) = U_\alpha \times \{0\}$ and $x_1 + iy_1, \ldots, x_n + iy_n$ coordinates on $\varphi_\alpha^C(W_\alpha) \subset U_\alpha \times i\mathbb{R}^n$. There are uniquely defined smooth functions $f_1, \ldots, f_n : U_\alpha \to \mathbb{R}$ such that

$$\eta|_{U_\alpha} = f_1(x_1, \ldots, x_n)dx_1 + \ldots + f_n(x_1, \ldots, x_n)dx_n.$$

On $U_\alpha \times i\mathbb{R}^n \subset \mathbb{C}^n$ the function $\varrho_\alpha : U_\alpha \times i\mathbb{R}^n \to \mathbb{R}$, defined by

$$\varrho_\alpha((x_1 + iy_1, \ldots, x_n + iy_n)) = f_1(x_1, \ldots, x_n)y_1 + \ldots + f_n(x_1, \ldots, x_n)y_n,$$

satisfies $\varrho_\alpha|_{U_\alpha \times \{0\}} \equiv 0$ and this implies that

$$(U_\alpha)^*(d^c \varrho_\alpha)(x_1, \ldots, x_n) = \sum_{j=1}^n f_j(x_1, \ldots, x_n)dx_j = \eta|_{U_\alpha}(x_1, \ldots, x_n).$$
Denote by \( \tilde{\varphi}_\alpha : \varphi^C_\alpha(W_\alpha) \to \mathbb{R} \) the restriction \( \varphi_\alpha \mid \varphi^C_\alpha(W_\alpha) \). Then \( \varphi_\alpha := (\varphi^C_\alpha)^*(\tilde{\varphi}_\alpha) = \tilde{\varphi}_\alpha \circ \varphi^C_\alpha : W_\alpha \to \mathbb{R} \) has the properties \( \varphi_\alpha|_{U_\alpha \times \{0\}} \equiv 0 \) and \( (U_\alpha)^*(d^c \varphi_\alpha) = \eta|_{U_\alpha} \) for the embedding \( U_\alpha : U_\alpha \hookrightarrow W_\alpha \). If \( p \in U_\alpha \), the sets \( U(p) := U_\alpha \) and \( W(p) := W_\alpha \) have the desired properties. These locally defined function can be patched together to obtain a function \( \varphi : M^c \to \mathbb{R} \) such that \( \eta = \iota_M^*(d^c \varphi) \). So far, symmetries are not yet considered. There is a Stein complexification \( M^c \) of \( M \) and an atlas \((W_\alpha, \varphi^C_\alpha)\) such that \((U_\alpha := M \cap W_\alpha, \varphi^C_\alpha(U_\alpha))\) is an atlas \((U_\alpha, \varphi^C_\alpha(U_\alpha))\) and a function \( \varphi_\alpha : W_\alpha \to \mathbb{R} \) with the properties
\[
\varphi_\alpha|_{U_\alpha \times \{0\}} \equiv 0 \quad \text{and} \quad (U_\alpha)^*(d^c \varphi_\alpha) = \eta|_{U_\alpha} \tag{1}
\]
for the embedding \( U_\alpha : U_\alpha \hookrightarrow W_\alpha \). After shrinking and refining there is an atlas \((V_\beta, \varphi_\beta)_{\beta \in J} \) of \( M^c \) with a partition of unity \((\chi_\beta)_{\beta \in J} \). Since every \( V_\beta \subset W_{\alpha(\beta)} \) for some \( \alpha(\beta) \), it is possible to define \( \varphi_\beta := \varphi_{\alpha(\beta)}|_{V_\beta} \). Property \( \text{[1]} \) implies that for the case that \( M \cap V_\beta \) is non-empty, \((\iota_{M^c \cap V_\beta} \circ \iota_{M^c \cap V_\beta})^*(d^c \varphi_\beta) = \eta|_{M^c \cap V_\beta} \) and
\[
\varphi_\beta|_{M^c \cap V_\beta} \equiv 0. \tag{2}
\]
Define now for every function \( \varphi_\beta \) the smooth functions
\[
\chi_\beta \cdot \varphi_\beta : M^c \to \mathbb{R} \quad x \mapsto \begin{cases} (\chi_\beta \cdot \varphi_\beta)(x) & \text{for } x \in V_\beta \\ 0 & \text{for } x \in M^c \setminus V_\beta. \end{cases}
\]
The function \( \varphi : M^c \to \mathbb{R}, x \mapsto \sum_\beta (\chi_\beta \cdot \varphi_\beta)(x) \), is well-defined by local finiteness and smooth, too. Property \( \text{[2]} \) implies
\[
\iota_M^*(d^c \varphi) = \sum_\beta \chi_\beta \cdot \iota_M \cdot \iota_M^*(d^c \varphi_\beta) + 0 \cdot d^c \chi_\beta = \sum_\beta \chi_\beta \cdot \iota_M \cdot \eta = \eta.
\]
Possibly after shrinking \( M^c \) a strictly plurisubharmonic function \( \nu : M^c \to \mathbb{R} \) with the property \( \iota_M^*(d^c \nu) = 0 \) such that \( \varphi - \nu \) is strictly plurisubharmonic on an open neighbourhood of \( M \) and still satisfies \( \iota_M^*(d^c(\varphi - \nu)) = 0 \). The construction of a function \( \nu \) with these properties can be found in \([\text{HHL}], \text{Lemma 2}\). Now let \( M \) be a \( K \)-manifold and let \( \eta \) be a \( K \)-invariant 1-form. There is a \( K \)-equivariant complexification \( M^c \) of \( M \) \([\text{HHL}], \text{MS}, \text{Theorem 1.3}\). In particular \( K \) acts on \( M^c \) by holomorphic transformations and \( \iota_M : M \hookrightarrow M^c \) is a \( K \)-equivariant embedding. Perhaps after shrinking to a smaller \( K \)-invariant complexification \( M^c \) has a smooth strictly plurisubharmonic function \( \tilde{\varphi} : M^c \to \mathbb{R} \) such that \( \iota_M^*(d^c \tilde{\varphi}) = \eta \). Then \( \varphi(x) := \int_K \tilde{\varphi}(k^{-1} \cdot x) dk \) defines a \( K \)-invariant strictly plurisubharmonic function on \( M^c \) such that
\[
\iota_M^*(d^c \varphi) = \iota_M^*(\int_K \tilde{\varphi} \circ \psi_k^{-1} dk) = \iota_M^*(\int_K \psi_k^{-1} (\iota_M^*(d^c \tilde{\varphi})) dk) = \int_K \psi_k^{-1} (\iota_M^*(d^c \tilde{\varphi})) dk = \int_K \eta dk = \eta,
\]
where \( \psi_k : M^c \to M^c, \psi_k(x) = k \cdot x \), for all \( k \in K \).

\[\square\]

### 2.2 Equivariant extensions for the case of proper actions

An extendable Lie group is characterized by the injectivity of the canonical \( G \)-equivariant homomorphism \( \iota_G : G \to G^C \), where \( G^C \) is the universal complexification. The aim of this subsection is the following result.
Theorem 2.2. Let $G$ be an extendable Lie group with finitely many connected components that acts properly on a manifold $M$ and let $K$ be a maximal compact subgroup of $G$. Let $\eta$ be a smooth $G$-invariant 1-form on $M$. The slice $S \subset M$ is embedded in a Stein $K^C$-manifold $S^C$ such that $M = G \times^K S$ is complexified by a $G \times K$-invariant Stein domain $M^c \subset G^C \times^K S^C$. Then there is a $G$-invariant strictly plurisubharmonic function $\varphi : M^c \to \mathbb{R}$ such that

$$\iota_M^* \varphi = \eta.$$ 

The proof of this at the end of the section needs some preparation. Let $G$ be an extendable Lie group with finitely many connected components and let $K$ be a maximal compact subgroup of $G$. By a theorem of Abels [Ah] there is a $K$-invariant submanifold $S$ in $M$ such that the map

$$G \times^K S \to M$$

$$[g, s] \mapsto g \cdot s$$

is a diffeomorphism. Here, $G \times^K S$ denotes the geometric quotient of $G \times S$ with respect to the free $K$-action

$$K \times (G \times S) \to G \times S$$

$$(k, (g, s)) \mapsto (g k^{-1}, k \cdot s)$$

and $[g, s] := \pi_K(g, s)$, where $\pi_K : G \times S \to G \times^K S$ is the canonical projection onto their geometric quotient. There is a real analytic structure on $M \cong G \times^K S$ such that the action map $G \times M \to M$, the slice $S$ and the $K$-action on $S$ may be assumed to be real analytic ([H], [IK]). In ([HIK], Section 7, Proposition 4, 4' and 5) a complexification of $G \times^K S$ is constructed with the help of a $G$-complexification $G^C$ of $G$ and a $K$-complexification $S^C$ of $S$ as the quotient $G^C \times S^C \to^K S^C$. In this quotient two points $p_1$ and $p_2$ are identified if $f(p_1) = f(p_2)$ for every $K$-invariant holomorphic function $f$. Since $G$ is assumed to be extendable here, a $G$-complexification can be realized as a $G^C$-manifold as in [HIK]. The proof is included here for the readers’ convenience.

Proposition 2.3. Let an extendable Lie group $G$ act properly and real analytically on a manifold $M = G \times^K S$, where $K$ is a maximal compact subgroup of $G$. Then there is a $K^C$-manifold $S^C$ such that a $G$-invariant domain $\Omega$ in $M^C = G^C \times^K S^C$ is a $G$-complexification of $M = G \times^K S$.

Proof. The slice $S$ can be $K$-equivariantly complexified in a Stein $K^C$-space $S^C$ ([HI], Section 6.6). Since $G$ is extendable, it can be complexified $G$-equivariantly to a $G$-invariant open domain $G^C$ in $G^C$. Then $M \cong G \times^K S$ can be $G$-equivariantly embedded in $M^C := G^C \times^K S^C$ as a totally real submanifold. If $M^c$ is a $G$-complexification of $M$, a $G$-invariant domain $\Omega$ containing $M$ in $M^C$ can be $G$-equivariantly, holomorphically and openly embedded in a neighbourhood of $M$ in $M^C$ ([HIK], Corollary 7). 

Following the notation introduced above consider a proper and real analytic action on $M$. Then $M = G \times^K S$. Let $\eta$ be a $G$-invariant smooth 1-form on $M$. Denote by $\pi_G : G \times S \to G$ and $\pi_S : G \times S \to S$ the projections on the first and on the second factor respectively.
Comparing coefficients implies that there are smooth functions $f_1, \ldots, f_n : S \to \mathbb{R}$ and a $K$-invariant 1-form $\sigma_S$ on $S$ such that

$$\pi_K^* \eta = \sum_{j=1}^n \pi_S^*(f_j) \cdot \pi_G^*(\beta_j) + \pi_S^*(\sigma_S)$$

such that $\sum_{j=1}^n \pi_S^*(f_j) \cdot \pi_G^*(\beta_j)$ is a $G \times K$-invariant 1-form.

**Proof.** The form $\pi_K^* \eta$ is a $G \times K$-invariant, smooth 1-form on $G \times S$. Let $\beta_1, \ldots, \beta_n$ be a basis of $G$-invariant 1-forms on $G$; then $\pi_G^*(\beta_1), \ldots, \pi_G^*(\beta_n)$ are their trivial extensions to $G \times S$. The embedding $\iota_S : S \to G \times S, s \mapsto (e, s)$, and the projection $\pi_S : G \times S \to S, (g, s) \mapsto s$, are $K$-equivariant if $K$ acts diagonally on $G \times S$ by $k \cdot (g, s) = (gk^{-1}, ks)$. Let $\sigma_S = \iota^*_S(\pi_K^* \eta)$; then $\pi_S^*(\sigma_S)$ is $G \times K$-invariant and for every tangent vector $(0, v) \in T_{(g, s)}(G \times S) \cong T_gG \times T_sS$,

$$(\pi_S^*(\sigma_S))(0, v) = \sigma_S(D\pi_S(0, v)) = (\pi_K^* \eta)(D\iota_S(D\pi_S(0, v))) = (\pi_K^* \eta)(0, v).$$

It follows that $(\pi_K^* \eta - \pi_S^*(\sigma_S))(h, v)(0, v) = 0$ for every $v \in T_sS$ and every $h \in G$. In other words $(\pi_K^* \eta - \pi_S^*(\sigma_S))(h, s) \in T_h^*G \oplus \{0\} \subset T_h^*G \oplus T_s^*S \cong T_{(h, s)}G \times S$. This implies that there are smooth functions $f_1, \ldots, f_n$ on $G \times S$ such that

$$(\pi_K^* \eta - \pi_S^*(\sigma_S))(h, s) = \sum_{j=1}^n f_j(h, s) \cdot \pi_G^*(\beta_j).$$

Comparing coefficients implies that there are smooth functions $f_1, \ldots, f_n$ on $S$ such that

$$(\pi_K^* \eta - \pi_S^*(\sigma_S))(h, s) = \sum_{j=1}^n \pi_S^*(f_j)(s) \cdot \pi_G^*(\beta_j).$$

\qed

Proposition [2.4] implies that there is a $K$-invariant strictly plurisubharmonic function $g_S : S^c \to \mathbb{R}$ on an equivariant $K$-complexification $S^c$ of $S$ such that $\sigma_S = (i_{S\to S^c})^*(d^c g_S)$. Assume that the situation is arranged as in Proposition [2.3]. Let $S^c$ be openly and $K$-equivariantly embedded in a $K^c$-manifold $S^C$ and let $G^c$ be a Stein $G$-complexification of $G$ which is $G$-equivariantly and openly embedded in $G^C$. In the next proposition, $G \times K$-invariant 1-forms on $G \times S$ are going to be extended equivariantly to $G^c \times S^c$.

**Proposition 2.5.** For the smooth $G \times K$-invariant 1-form $\pi_K^* \eta$ on $G \times S$ there is a $G \times K$-invariant strictly plurisubharmonic function $g$ on some $G \times K$-invariant complexification $G^c \times S^c$ such that on a $G \times K$-invariant Stein domain $\Omega$ in $G^c \times S^c$

$$(i_{G \times S})^*(d^c g) = \pi_K^* \eta.$$

**Proof.** Let $\beta_1, \ldots, \beta_n$ be a basis of $G$-invariant 1-forms on $G$ and $f_1, \ldots, f_n \in C^\infty(S)$ and $\pi_K^* \eta = \sum_{j=1}^n \pi_S^*(f_j) \cdot \pi_G^*(\beta_j) + \pi_S^*(\sigma_S)$ be as above in Proposition [2.3]. Let $G^c$ be a Stein complexification of $G$ which is $G$-equivariant with respect to the left $G$-multiplication. Shrinking $G^c$ if necessary, Lemma 3.3 in [ST] and Theorem 1 in [VW] imply that there are $G$-invariant functions $g_1, \ldots, g_n$ on $G^c$ such that $i_G^*(d^c g_j) = \beta_j$ for $j = 1, \ldots, n$. It can be assumed that $g_j|G \equiv 0$. There is a complexification $S^c$ of
S with functions $F_1, \ldots, F_n : S^c \to \mathbb{R}$ such that $f_j = (\iota_{\mathcal{S} \to \mathcal{S}^c})^*(F_j)$ for $j = 1, \ldots, n$. The $G$-invariant function

$$\Theta : G^c \times S^c \to \mathbb{R}$$

$$(h, s) \mapsto F_1(s)(h) + \ldots + F_n(s)(h)$$

satisfies $d^c\Theta = \sum_{j=1}^n \pi_{S^c}(F_j) \cdot \pi_{G^c}^*(d^c\theta_j) + \pi_{S^c}^*(\eta_j) \cdot \pi_{G^c}^*(d^c\theta_j)$ with the projections $\pi_{G^c} : G^c \times S^c \to G^c$ and $\pi_{S^c} : G^c \times S^c \to S^c$. Now, the property $g_j|_{G} \equiv 0$ implies

$$(\iota_{G \times S})^*(d^c\Theta) = \sum_{j=1}^n \pi_{S^c}^*(f_j) \cdot \pi_{G^c}^*(\beta_j).$$

After shrinking $S^c$ Proposition 2.1 shows that there is a strictly plurisubharmonic $K$-invariant function $\theta : S^c \to \mathbb{R}$ such that $(\iota_{S \to S^c})^*(d^c\theta) = \sigma_S$. Define the $G \times K$-invariant function $\varrho$ by

$$\varrho(g, s) := \int_K (\Theta(\xi g^{-1}, ks) + (\pi_{S^c})^*(\theta)(ks))\,dk.$$

Then $(\iota_{G \times S})^*(d^c\varrho) = \pi_K^*\eta$, and $\varrho$ is a $G \times K$-invariant function. A partition of unity argument, worked out in Lemma 3.10 in [ST], which is e.g. shows that $\varrho$ can be assumed both $G \times K$-invariant and strictly plurisubharmonic.

Recall the original goal to extend a $G$-invariant 1-form $\eta$ to an equivariant complexification of $G \times K \text{S}$. This will be achieved with the help of Kählerian reduction of $G^c \times S^c$ with respect to the freely acting compact group $K$. Details on the momentum map geometry and on Kählerian reduction can be found e.g. in [H1], [HL], [ST]. The basic properties needed here are mentioned briefly in the remaining section.

Let $(\Omega, \omega)$ be a Kähler manifold and let $L$ be a Lie group which acts symplectically and by holomorphic transformations on $\Omega$, i.e., $(\psi_g)^*\omega = \omega$ for every $g \in L$, where $\psi : L \times \Omega \to \Omega$ is the action map. The action is called Hamiltonian if there is a moment map $\mu : \Omega \to \text{Lie}(L)^*$, where $\text{Lie}(L)^*$ is the dual vector space to the Lie algebra of $L$, with the following properties:

a) The map $\mu$ is $L$-equivariant with respect to the given action on $\Omega$ and the coadjoint action of $L$ on $\text{Lie}(L)^*$.

b) For every $\xi \in \text{Lie}(L)$ the function $\mu_\xi : \Omega \to \mathbb{R}$, $x \mapsto \langle \mu(x), \xi(x) \rangle$, satisfies

$$\iota_{\xi_\Omega}^*\omega = d\mu_\xi,$$

where $\xi_\Omega(x) = \frac{d}{dt}\exp(t\xi)\cdot x |_{t=0}$ and $\iota_{\xi_\Omega}^*\omega$ is the 1-form given by $(\iota_{\xi_\Omega}^*\omega)(v) = \omega(\xi_\Omega, v)$ for every $v \in TM$.

If $\Omega$ carries a differentiable, $L$-invariant strictly plurisubharmonic function $\varrho : \Omega \to \mathbb{R}$, the action is Hamiltonian with respect to the Kähler metric $\omega = -d^c\varrho$. In this case, a moment map is given by $\mu_\xi(x) = (d^c\varrho)(\xi_\Omega(x))$. For Hamiltonian actions, the momentum zero level $\mu^{-1}(0) = \{x \in \Omega | \mu_\xi(x) = 0 \text{ for all } \xi \in \text{Lie}(L)\}$ allows one to define the reduced space $\mu^{-1}(0)/L$.

**Proposition 2.6.** Let $L$ act freely and properly by holomorphic and symplectic transformations on a Kähler manifold $(\Omega, \omega)$. Assume that the action is Hamiltonian with moment map $\mu : \Omega \to \text{Lie}(L)^*$. Let $\Omega \hookrightarrow X$ be openly, holomorphically and $L$-equivariantly embedded in a complex $L^c$-manifold $X$ on which $L^c$ acts freely.
such that $X/_{L^c}$ is a smooth complex manifold and $\pi_{L^c} : X \to X/_{L^c}$ a submersion. Then the map

$$\kappa : \mu^{-1}(0) \times L \to X/_{L^c}, \quad Lx_0 \mapsto \pi_{L^c}(\iota_{\mu^{-1}(0)}(x_0))$$

is a local diffeomorphism and defines a unique complex structure on $\mu^{-1}(0) \times L$ such that $\kappa$ is a locally biholomorphic map of complex manifolds.

**Proof.** Since $\ker d\mu(x) = (T_xL \cdot x)^{\perp_\omega} = \{v \in T_xX | \omega(v,w) = 0 \ \forall w \in T_xL \cdot x\}$, $x \in X$, implies that $\text{rank}(\mu) = \text{dim} \text{Lie}(L)$ everywhere, $\mu^{-1}(0)$ is a smooth submanifold of $\Omega$. The fact that $L$ acts freely and properly implies that $\mu^{-1}(0) \times L$ is likewise a differentiable manifold. Furthermore, for a point $x_0 \in \mu^{-1}(0)$,

$$T_{x_0}\mu^{-1}(0) = T_{x_0}(L \cdot x_0) \oplus T_{x_0}(L^c \cdot x_0)^{\perp_\omega}.$$

In the commutative diagram

$$\begin{array}{ccc}
\mu^{-1}(0) & \xrightarrow{\iota_{\mu^{-1}(0)}} & X \\
\downarrow \pi_{\mu^{-1}(0)} & & \downarrow \pi_{L^c} \\
\mu^{-1}(0) \times L & \xleftarrow{\kappa} & X/_{L^c},
\end{array}$$

the maps $\pi_{\mu^{-1}(0)} : \mu^{-1}(0) \to \mu^{-1}(0) \times L$ and $\pi_{L^c} : X \to X/_{L^c}$ are submersions with kernels $\ker(D\pi_{\mu^{-1}(0)}(x)) = T_x(L \cdot x)$ and $\ker(D\pi_{L^c})(x) = T_x(L^c \cdot x)$ respectively. It follows that $D(\pi_{L^c} \circ \iota_{\mu^{-1}(0)})(x_0) = D(\kappa \circ \pi_{\mu^{-1}(0)})(x_0)$ maps $T_{x_0}(L^c \cdot x_0)^{\perp_\omega}$ bijectively onto $T_{x_0}(\pi_{L^c}(x_0))(X/_{L^c})$ and $D\pi_{\mu^{-1}(0)}$ maps bijectively onto $T_{x_0}(\pi_{L^c}(x_0))(\mu^{-1}(0) \times L)$. This implies that $D\kappa(\pi_{\mu^{-1}(0)}(x_0))$ is everywhere an isomorphism.

Let $\Omega \subset G^c \times S^c$ be a $G \times K$-invariant Stein domain, $G \times S \subset \Omega$, and $\rho : \Omega \to \mathbb{R}$ a $G \times K$-invariant strictly plurisubharmonic function such that $\langle \iota_{G \times S} \rangle^*(d\rho) = \pi_{K^c}^* \rho$. 

**Proposition 2.7.** There are $G$-invariant Stein domains $\Omega_1 \subset \mu^{-1}(0) \times L$, containing $G \times K$ and $\Omega_2 \subset G^c \times K^c \times S^c$ containing $G \times K$ and $S$ which are $G$-equivariantly biholomorphic.

**Proof.** First, it has to be shown that $G \times S \subset \mu^{-1}(0)$. For this, the following calculation proves that for every $\zeta \in \text{Lie}(K)$ and every $(g,s) \in G \times S$,

$$\frac{d}{dt} \rho(\exp(it\zeta) \cdot (g,s))|_{t=0} = \langle \iota_{G \times S} \rangle^*(d\rho) \left( \frac{d}{dt} \rho(\exp(-t\zeta) \cdot (g \exp(t\zeta) \cdot s))|_{t=0} \right)$$

$$= \langle \pi_{K^c} \rangle \left( \frac{d}{dt} (g \exp(-t\zeta) \cdot (g \exp(t\zeta) \cdot s))|_{t=0} \right)$$

$$= \eta \left( \frac{d}{dt} \pi_K (g,s)|_{t=0} \right)$$

$$= 0.$$
Since $K$ acts freely and commutes with the $G$-action, $\mu^{-1}(0)/K$ is a $G$-manifold and, by Proposition 2.3, obtains a complex structure by the map

$$\kappa : \mu^{-1}(0)/K \to G^C \times K^C S^C,$$

which is a local diffeomorphism. Since the restriction $\kappa|_{G \times K}\eta$ defines a real analytic, $G$-equivariant isomorphism between two copies of $G \times K S$ in $\mu^{-1}(0)/K$ and in $G^C \times K^C S^C$ respectively, there are $G$-invariant and biholomorphic Stein neighbourhoods $\Omega_1$ of $G \times K S$ in $\mu^{-1}(0)/K$ and $\Omega_2$ of $G \times K S$ in $G^C \times K^C S^C$ ([HKK], Corollary 7).

It remains to show how to use the extension of $\pi_K^*\eta$ on $G^C \times S^C$ for an extension of $\eta$ on $G \times K S$. This is carried out in the proof of the Theorem 2.2 which can now be carried out.

**Proof of Theorem 2.2.** By Proposition 2.3 there is a Stein $G \times K$-complexification $\Omega \subset G^C \times S^C$ of $G \times S$ and a strictly plurisubharmonic $G \times K$-invariant function $\varrho : \Omega \to \mathbb{R}$ such that $\pi_K^*\eta = (\iota_{G \times S})^*(d^c \varrho)$. The moment map $\mu : \Omega \to \text{Lie}(K)^*$, $x \mapsto (\xi \mapsto d^c \varrho(\xi(x)))$, is defined for the $K$-action on $\Omega$. For the existence of the following quotients, note that the relevant groups $K$ and $K^C$ respectively act freely. Thanks to Proposition 2.4 the diagram

$$\begin{array}{ccc}
\mu^{-1}(0) & \xrightarrow{\iota_\mu} & \Omega \subset G^C \times S^C \\
\pi_{\mu^{-1}(0)} & \downarrow & \pi_K^C \\
\mu^{-1}(0)/K & \xrightarrow{\kappa} & G^C \times K^C S^C
\end{array}$$

commutes. It shows that there is a canonically defined complex structure on $M^c := \mu^{-1}(0)/K$. Note that the $G$-action on $\Omega$ induces a natural $G$-action by holomorphic transformations on the quotient $M^c = \mu^{-1}(0)/K$, because the $G$-action and the $K$-action on $\Omega$ commute. The function $\varrho_{\text{red}} : M^c = \mu^{-1}(0)/K \to \mathbb{R}$ which is induced by $(\iota_\mu)^*\varrho$ is $G$-invariant and has the property

$$(\iota_\mu)^*\varrho = (\pi_{\mu^{-1}(0)})^*(\varrho_{\text{red}}).$$

Then the $G \times K$-equivariant embedding $\iota_{G \times S} : G \times S \to \Omega \subset G^C \times S^C$ induces a $G$-equivariant embedding $\iota_M : M = G \times K S \to G^C \times K^C S^C$. The strictly plurisubharmonic function $\varrho_{\text{red}}$ on $\mu^{-1}(0)/K$ has the property that

$$(\iota_M)^*(d^c \varrho_{\text{red}}) = \eta.$$

To see this, consider the following commutative diagram:

$$\begin{array}{ccc}
G \times S & \xrightarrow{\iota_{G \times S}} & \mu^{-1}(0) & \xrightarrow{\iota_\mu} & \Omega \subset G^C \times S^C \\
\downarrow \pi_K & & \downarrow \pi_{\mu^{-1}(0)} & & \downarrow \pi_K^C \\
M = G \times K S & \xrightarrow{\iota_M} & M^c = \mu^{-1}(0)/K & \xrightarrow{\kappa} & M^C = G^C \times K^C S^C
\end{array}$$

Since $(\pi_K)^*\eta = (\iota_{G \times S})^*(d^c \varrho) = (\pi_K)^*((\iota_M)^*(d^c \varrho_{\text{red}}))$ surjectivity of $\pi_K$ implies that $\eta = (\iota_M)^*(d^c \varrho_{\text{red}})$.
2.3 Complexifications of contact and symplectic manifolds

In the case where $M$ is a contact manifold Theorem 2.2 can be reformulated in the sense that the 1-form $\eta$ can be extended to a 1-form $\eta^c$, e.g. $\eta^c := d^c \varrho$, on a Stein $G$-complexification $M^c$:

Every contact manifold $(M, \eta)$ with a proper $G$-action of a Lie group $G$ with finitely many connected components can be complexified equivariantly to a Stein $G$-complexification $M^c$ with a $G$-invariant 1-form $\eta^c$ such that $\iota_M^*(\eta^c) = \eta$ for the embedding $\iota_M : M \hookrightarrow M^c$.

A similar result for symplectic manifolds is proved by Stratmann (ST).

A contact manifold $(M, \eta)$ can be symplectified, i.e., it can be extended naturally to a symplectic manifold: If $(M, \eta)$ is a $(2n+1)$-dimensional contact manifold, the two-form

$$d(e^t \eta + dt) = e^t d\eta + e^t dt$$

on $M \times \mathbb{R}$ is symplectic. Here, $t$ denotes the standard coordinate on the $\mathbb{R}$-factor of $M \times \mathbb{R}$. A contact-form $\eta$ on $M$ induces a symplectic form $\omega = d(e^t \cdot (\pi_{M \times \mathbb{R} \to M})^*(\eta))$, where $t$ is the coordinate on $\mathbb{R}$ and $\pi_{M \times \mathbb{R} \to M}$ projects on the first factor. The complex extension of $M$ to $M^c$ induces a complex extension of $M \times \mathbb{R}$ to $M^c \times \mathbb{C}$. This means that $\eta$ extends to $(\iota_M)^*(d^c \varrho)$ on $M^c$ and $d(e^t \eta + dt)$ extends to $d^c \varrho$ on $\Omega$.

The symplectification is compatible with the extension to complexifications in the following sense.

**Proposition 2.8.** Let $(M, \eta)$ be a smooth contact manifold. Then there is a Stein complexification $M^c$ of $M$ and an open neighbourhood $\Omega$ of $M \times \mathbb{R}$ in $M^c \times \mathbb{C}$ such that there exists a strictly plurisubharmonic function $\varrho : \Omega \to \mathbb{R}$ for which

$$(\iota_{M \times \mathbb{R} \to M^c \times \mathbb{C}})^*(d^c \varrho) = d(e^t \eta + dt)$$

for the embeddings $\iota_{M^c} : M^c \hookrightarrow M^c \times \mathbb{C}, z \mapsto (z, 0)$, and $\iota_M : M \hookrightarrow M^c$

$$\iota_M^*((\iota_{M^c})^*(d^c \varrho)) = \eta.$$

**Proof.** There is a complexification $M^c$ of $M$ and a strictly plurisubharmonic function $\varrho_M : M^c \to \mathbb{R}$ such that $\iota_M^*(d^c \varrho_M) = \eta$. Then the function

$$\varrho : M^c \times \mathbb{C} \to \mathbb{R}, (m, z) \mapsto \text{Re}(z) \cdot \varrho_M(m)$$

has the property $(d^c \varrho) = e^t \cdot d^c \varrho_M - \varrho_M \cdot e^t ds$ where $z = t + is$. In particular,

$$\iota_M^*((\iota_M^*)^*(d^c \varrho)) = \iota_M^*(d^c \varrho_M) = \eta$$

and

$$(\iota_{M \times \mathbb{R}})^*(d^c \varrho) = (\iota_{M \times \mathbb{R}})^*(e^t \cdot d^c \varrho_M - \varrho_M \cdot e^t ds) = e^t (\iota_{M \times \mathbb{R}})^*(d^c \varrho_M) = e^t \eta.$$

If $\nu : M^c \to \mathbb{R}$ is a strictly plurisubharmonic function with the property $\iota_M^*(d^c \nu) = 0$ and $\iota_M^*(d\nu) = 0$, Proposition 2.1 can be applied to

$$\tilde{\nu} : M^c \times \mathbb{C} \to \mathbb{R}, (m, z) \mapsto \nu(z) + |z|^2$$

and to $\varrho$ to obtain a strictly plurisubharmonic function $\varrho : \Omega \to \mathbb{R}$ on a Stein neighbourhood $\Omega$ of $M \times \mathbb{R}$ in $M^c \times \mathbb{C}$. □
Proposition 2.8 also has an equivariant version:

**Corollary 2.9.** If $G \times M \to M$ is a proper $G$-action, there is a proper extension to $M^c$: The trivial extension to an action on $M^c \times \mathbb{C}$ defines equivariant embeddings

$$\iota_M : M^c \hookrightarrow M^c \times \mathbb{C}$$

such that $\varrho : \Omega \to \mathbb{R}$ can be chosen to be strictly plurisubharmonic and $G$-invariant on $\Omega \subset M^c \times \mathbb{C}$.

**Proof.** It has just to be observed that in the proof of Proposition 2.8 the function $\varrho : M^c \to \mathbb{R}$ can be chosen to be $G$-invariant by Theorem 2.2 and as a strictly plurisubharmonic function. 

**Corollary 2.10.** Let $M$ be a real analytic manifold with a contact form $\eta$. Then there is a Stein complexification $M^c$ of $M$ and an open neighbourhood $\Omega$ of $M \times \mathbb{R}$ in $M^c \times \mathbb{C}$ such that the symplectic form $\omega := d(e^t \eta + dt)$ is the pull-back $(\iota_M \times \mathbb{R})^*(\beta)$ of a Kähler form $\beta$ on $\Omega$.

**Proof.** This is a consequence of Proposition 2.8 because for a strictly plurisubharmonic function $\varrho : M^c \to \mathbb{R}$, $\beta := dd^c \varrho$ is a Kähler form with the properties stated in Corollary 2.10.

**Remark.** Similarly to the equivariant statement in Corollary 2.9, an equivariant version of Corollary 2.10 can be formulated.

### 3 Compatibility of reductions

In this section, the compatibility of the complexification with reductions by symmetries is discussed. Roughly speaking, the guiding question is whether the Kählerian reduction of a complexification of a contact manifold can be regarded as the complexification of the contact reduction.

Throughout this section $(M, \eta)$ is assumed to be a contact manifold on which an extendable Lie group $G$ with finitely many connected components acts properly by contact transformations, i.e., by leaving $\eta$ invariant. Fix a $G$-invariant smooth Stein complexification $M^c$ of $M$ such that $\eta = \iota_M^*(dF\varrho)$ holds for some smooth $G$-invariant strictly plurisubharmonic function $\varrho : M^c \to \mathbb{R}$ (see Theorem 2.2). Furthermore, assume that there is a globalization $M^G$ of the local $G^c$-action on $M^c$ such that $M^c$ is openly and $G$-equivariantly embedded in the $G^c$-manifold $M^G$.

#### 3.1 Compatibility of moment maps for free actions

Under the assumptions stated at the beginning of the section, there exists a moment map on the contact manifold

$$\mu_M : M \to \mathfrak{g}^\ast$$

$$m \mapsto \xi \mapsto \eta(\xi_M(m)) = \eta(\frac{d}{dt}\exp(t\xi)\cdot m|_{t=0})$$
and a moment map on the Kähler manifold
\[ \mu_{M^c} : M^c \to \mathfrak{g}^* \]
\[ x \mapsto (\xi \mapsto d^c \varrho(\xi_{M^c}(x))). \]
The relation \( \eta = \iota_M^*(d^c \varrho) \) implies that the Kählerian moment map extends the contact moment map, i.e., \( \mu_{M^c} \circ \iota_M = \mu_M \). Cauchy-Riemann geometry enters the picture, because the hypersurface \( M^{CR} = \varrho^{-1}(0) \) plays a role as it contains \( M \). This fact makes use of the assumption that the Kählerian moment map is defined by the potential \( \varrho \).

**Lemma 3.1.** Let \((M^c, d^c \varrho)\) be a complexification of a contact manifold \((M, \eta)\) and \( \varrho : M^c \to \mathbb{R} \) be a strictly plurisubharmonic function with \( M \subseteq \varrho^{-1}(0) \) such that \( d^c \varrho \) extends \( \eta \) in the sense that \( \eta = \iota_M^*(d^c \varrho) \).

a) Then possibly after shrinking \( M^c \) to a smaller neighbourhood of \( M \), \( \varrho^{-1}(0) \) is a smooth hypersurface in \( M^c \).

b) The smooth hypersurface \( M^{CR} := \varrho^{-1}(0) \) is a strongly pseudoconvex hypersurface.

**Proof.** Since \( \eta \) is nowhere vanishing on \( M \) and it is the pull-back of \( d^c \varrho \), a) follows, because it is immediate that \( d\varrho \) vanishes nowhere in a neighbourhood of \( M \). The statement b) is just a matter of definitions.

The action of \( G \) leaves \( M^{CR} = \varrho^{-1}(0) \) invariant and the inclusions
\[ (M, \eta) \hookrightarrow (M^{CR}, d^c \varrho|_{M^{CR}}) \hookrightarrow (M^c, d^c \varrho) \]
are all \( G \)-equivariant. Assume that the Lie subgroup \( L \) of \( G \) acts freely (and properly) on the contact manifold \((M, \eta)\) and leaves \( \eta \) invariant. In the following proposition it is shown that in the setting of this work, the restriction \( \mu_{M^{CR}|_{M^{CR}}} \) can be regarded as the Cauchy-Riemann moment map defined in [LL]. This involves the natural projection \( \alpha_c : T_p M^{CR} \to T_p^c M^{CR}/H_p \), where \( H_p = T_p M \cap J(T_p M) \). It follows from the definition of the operator \( d^c \) that the Cauchy-Riemann tangent space can be described by \( H_p = \{ v \in T_p(M^{CR})| d^c \varrho(v) = 0 \} \). Let \( H = \cup_{p \in M^{CR}} H_p \) be the Cauchy-Riemann bundle of hyperplanes and \( B \) denote the (real) line bundle \( B = TM^{CR}/H \). Then \( \alpha \) can be considered as a \( B \)-valued 1-form which defines the Cauchy-Riemann moment map
\[ \mu_{M^{CR}} : M^{CR} \to \text{Lie}(L)^* \otimes B \]
\[ p \mapsto (\xi \mapsto \alpha_p(\xi_{M^{CR}}(p))) \]
for every \( \xi \in \text{Lie}(L) \).

**Proposition 3.2.** Let \( L \) act freely and properly on \( M \) and \( M^c \). For the inclusions
\[ (M, \eta) \hookrightarrow (M^{CR}, d^c \varrho|_{M^{CR}}) \hookrightarrow (M^c, \eta^c = d^c \varrho) \]
the Kählerian moment map
\[ \mu_{M^c} : M^c \to \text{Lie}(L)^* \]
\[ p \mapsto (\xi \mapsto (d^c \varrho)(\xi_{M^c}(p))) = \eta^c(\xi_{M^c}(p))) \]
has the property that its restriction \( \mu_{M^{CR}} : M^{CR} \to \text{Lie}(L)^* \otimes B \) is the Cauchy-Riemann moment map for the \( L \)-action on \( M^{CR} \), if \( TM^{CR}/H \) is trivialized by the mapping \( TM^{CR}/H \to \mathbb{R}, \alpha_p(v) \mapsto (d^c \varrho)(v) \).
Proof. The line bundle $B = TM^{\text{CR}}/H$ is trivializable in this situation by the map
\[
TM^{\text{CR}}/H \to \mathbb{R}, \quad \alpha_p(v) \mapsto d^c g(v).
\]
It is well-defined because if $\alpha_p(v) = \alpha_p(w)$, $v - w \in H$ and $(d^c g)(v - w) = 0$ and therefore $(d^c g)(v) = (d^c g)(v - w) + (d^c g)(w) = (d^c g)(w)$. Under this trivialization, $\text{Lie}(L)^* \otimes B \cong \text{Lie}(L)^*$ with the identification
\[
\text{Lie}(L)^* \otimes B \to \text{Lie}(L)^*, \quad (\xi \mapsto \alpha_p(\xi_{M^{\text{CR}}})) \mapsto (\xi \mapsto d^c g(\xi_{M^{\text{cr}}}(p)))
\]
defines the Cauchy-Riemann moment map $\mu_{M^{\text{cr}},\xi} : M^{\text{CR}} \to \text{Lie}(L)^*$ by
\[
\mu_{M^{\text{cr}},\xi}(p) = (\iota_{M^{\text{cr}}}^*)(d^c g)(\xi_{M^{\text{cr}}}(p)) = d^c g(\xi_{M^{\text{cr}}}(p))
\]
for $\xi \in \text{Lie}(L)$, where $\iota_{M^{\text{cr}}} : M^{\text{CR}} \hookrightarrow M^c$ embeds $M^{\text{CR}}$ into $M^c$. \qed

3.2 Cauchy-Riemann, contact and Kählerian reductions

It will be shown later that the reduction along suitable strata of orbit types can be described by quotients of free actions on certain submanifolds. This is why in this subsection, the case of a freely acting Lie group $L$ is considered. The properties $\mu_M = \mu_{M^{\text{cr}}|_{M^c}}$ and $\mu_{M^{\text{cr}}} = \mu_{M^c|_{M^{\text{cr}}}}$ yield the inclusions of the momentum zero levels
\[
(\mu_M)^{-1}(0) \hookrightarrow (\mu_{M^{\text{cr}}})^{-1}(0) \hookrightarrow (\mu_{M^c})^{-1}(0).
\]
The inclusion of $(\mu_M)^{-1}(0)$ and of $(\mu_{M^{\text{cr}}})^{-1}(0)$ in $(\mu_{M^c})^{-1}(0)$ will be examined more closely in the following.

Contact and Kählerian reduction

Now the connection between the contact reduction of $M$ and the Kählerian reduction of $M^c$ with respect to a freely and properly acting group $L$ is studied. The situation for the embedding of $M$ in $M^c$ can be summarized in the following diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{\iota_M} & M^c \\
\uparrow \iota_{(\mu_M)^{-1}(0)} & & \uparrow \iota_{(\mu_{M^c})^{-1}(0)} \\
(\mu_M)^{-1}(0) & \xrightarrow{\iota_M|_{(\mu_M)^{-1}(0)}} & (\mu_{M^c})^{-1}(0) \\
\downarrow \pi_{(\mu_M)^{-1}(0)} & & \downarrow \pi_{(\mu_{M^c})^{-1}(0)} \\
(\mu_M)^{-1}(0)/L & \xrightarrow{\iota_{(\mu_M)^{-1}(0)/L}} & (\mu_{M^c})^{-1}(0)/L.
\end{array}
\]

Note that Proposition 2.3 shows that for this case described here, $(\mu_{M^c})^{-1}(0)/L$ is a complex manifold and the function $g_{\text{red}}$, defined by $g_{\text{red}} \circ \pi_{(\mu_{M^c})^{-1}(0)} = g \circ \iota_{(\mu_{M^c})^{-1}(0)}$,
is a Kählerian potential. It can be checked that the mapping \( t_{(\mu_M)}^{-1}(0)/L \) is well-defined. In Proposition 3.3 it is shown that \((t_{(\mu_M)}^{-1}(0)/L)^* (d\vartheta_{\text{red}})\) is the unique 1-form \( \eta_{\text{red}} \) on \((\mu_M)^{-1}(0)/L\) with the property

\[
(\pi_{(\mu_M)}^{-1}(0))^* (\eta_{\text{red}}) = (t_{(\mu_M)}^{-1}(0))^* (\eta).
\]

(3)

The manifold \((\mu_M)^{-1}(0)/L\) with the unique 1-form \( \eta_{\text{red}} \) such that (3) holds is called the contact reduction of \((M, \eta)\) as defined in \([L2]\) and \([W]\).

**Proposition 3.3.** Let the extendable Lie group \( L \) with finitely many connected components act freely and properly on \( M \). Then \((t_{(\mu_M)}^{-1}(0)/L)^* (d\vartheta_{\text{red}})\) is the unique 1-form \( \eta_{\text{red}} \) on \((\mu_M)^{-1}(0)/L\) such that

\[
(\pi_{(\mu_M)}^{-1}(0))^* (\eta_{\text{red}}) = (t_{(\mu_M)}^{-1}(0))^* (\eta).
\]

**Proof.** The assumptions on the action of \( L \) imply that the geometric quotients \((\mu_M)^{-1}(0)/G \) and \((\mu_M^{-1}(0)/G\) are manifolds. The function \( \vartheta_{\text{red}} : (\mu_M^{-1}(0)/L) \to \mathbb{R} \) is defined by \( \vartheta_{\text{red}} \circ \pi_{(\mu_M)}^{-1}(0) = \vartheta \circ t_{(\mu_M)}^{-1}(0) \). The desired result follows from the identity

\[
(\pi_{(\mu_M)}^{-1}(0))^* (t_{(\mu_M)}^{-1}(0)/L)^* (d\vartheta_{\text{red}}) = (t_{(\mu_M)}^{-1}(0))^* (\eta),
\]

(4)

because the uniqueness of the contact reduction implies that the reduced contact structure is defined by the 1-form \((t_{(\mu_M)}^{-1}(0)/L)^* (d\vartheta_{\text{red}})\). Since property (3) holds for the unique 1-form \( \eta_{\text{red}} \), the 1-form \((t_{(\mu_M)}^{-1}(0)/L)^* (d\vartheta_{\text{red}})\) agrees with \( \eta_{\text{red}} \) which provides the contact reduction \((\mu_M)^{-1}(0)/L, \eta_{\text{red}}\). \( \square \)

**Corollary 3.4.** The 2-form \( \omega_{\text{red}} := d\eta_{\text{red}} \) on \((\mu_M)^{-1}(0)/L\) satisfies

\[
(\pi_{(\mu_M)}^{-1}(0))^* \omega_{\text{red}} = (t_{(\mu_M)}^{-1}(0))^* d\eta.
\]

**Proof.** It follows from Proposition 3.3 that

\[
(t_{(\mu_M)}^{-1}(0))^* d\eta = d(t_{(\mu_M)}^{-1}(0))^* (\eta) = (\pi_{(\mu_M)}^{-1}(0))^* ((t_{(\mu_M)}^{-1}(0)/L)^* (d\vartheta_{\text{red}})).
\]

(\( \square \))

**Cauchy-Riemann and Kählerian reduction**

The following result characterizes both the contact reduction and the Cauchy-Riemann reduction of \( M^{\text{CR}} \) as the hypersurface \((\vartheta_{\text{red}})^{-1}(0)\) in the Kählerian reduced space \((\mu_M)^{-1}(0)/L\). The following sketch illustrates the setting:

\[
\begin{array}{ccc}
M^{\text{CR}} & \xrightarrow{\ i_{M^{\text{CR}}} \ } & M^c \\
\downarrow \ & \ & \downarrow \ \\
(\mu_{M^{\text{CR}}})^{-1}(0) & \xrightarrow{\ i_{(\mu_{M^{\text{CR}}})^{-1}(0)} \ } & (\mu_{M^c})^{-1}(0) \\
\downarrow \pi_{(\mu_{M^{\text{CR}}})^{-1}(0)} & \ & \downarrow \pi_{(\mu_{M^c})^{-1}(0)} \\
(\mu_{M^{\text{CR}}})^{-1}(0)/L & \xrightarrow{\ i_{(\mu_{M^{\text{CR}}})^{-1}(0)/L} \ } & (\mu_{M^c})^{-1}(0)/L.
\end{array}
\]
Proposition 3.5. The hypersurface \((\varrho_{\text{red}})^{-1}(0)) \subset (\mu_{M^c})^{-1}(0)/_L\) can be regarded in two ways:

a) The pull-back of the 1-form \(d^*\varrho_{\text{red}}\) to \((\varrho_{\text{red}})^{-1}(0)\) gives \((\varrho_{\text{red}})^{-1}(0)\) the structure of a contact manifold which is isomorphic to the contact reduced space for the \(L\)-action on \((M^{CR}, (\iota_{M^{CR}})^* (d^*\varrho)))\).

b) The hypersurface \((\varrho_{\text{red}})^{-1}(0)\) is isomorphic as a Cauchy-Riemann manifold to the Cauchy-Riemann reduction of \(M^{CR}\) with respect to \(L\).

**Remark.** Since \(\varrho_{\text{red}}\) is strictly plurisubharmonic, the form \(d^*\varrho_{\text{red}}\) pulled back to \((\varrho_{\text{red}})^{-1}(0)\) is a contact form.

**Proof.**

a) As mentioned in Proposition 3.3 there is a unique contact structure \(\eta_{\text{red}}\) on the reduced space \((\mu_{M^{CR}})^{-1}(0)/_L\) such that the identity

\[
(\iota_{(\mu_{M^{CR}})^{-1}(0)})^* (\iota_{M^{CR}})^* (d^*\varrho)) = (\pi_{(\mu_{M^{CR}})^{-1}(0)})^* (\eta_{\text{red}})
\]

holds. The commutativity of the diagram above shows that

\[
(\pi_{(\mu_{M^{CR}})^{-1}(0)})^* (\iota_{(\mu_{M^{CR}})^{-1}(0)/L}^* (d^*\varrho)) = (\iota_{M^{CR}} \circ \iota_{(\mu_{M^{CR}})^{-1}(0)})^* (d^*\varrho)
\]

and therefore \((\pi_{(\mu_{M^{CR}})^{-1}(0)})^* (\iota_{M^{CR}})^* (d^*\varrho)) = (\iota_{M^{CR}} \circ \iota_{(\mu_{M^{CR}})^{-1}(0)})^* (d^*\varrho)).\]

Since the reduced form is the unique 1-form with this property it follows that \((\iota_{(\mu_{M^{CR}})^{-1}(0)/L})^* (d^*\varrho_{\text{red}}))\) gives the contact structure.

b) Let \(\varrho_{\text{red}}\) be the function on the Kählerian reduction \((\mu_{M^c})^{-1}(0)/_L\) which is induced by the restriction \(\varrho_{(\mu_{M^c})^{-1}(0)}\). This is a strictly plurisubharmonic function, and if 0 is a regular value of \(\varrho\), 0 remains a regular value of \(\varrho_{\text{red}}\). The map \((\mu_{M^{CR}})^{-1}(0)/_L \rightarrow (\mu_{M^c})^{-1}(0)\) induces a bijection between \((\mu_{M^{CR}})^{-1}(0)/_L\) and \((\varrho_{\text{red}})^{-1}(0))\). Since the group action on \(M^c\) is by holomorphic transformations and leaves the Cauchy-Riemann hypersurface \(M^{CR}\) invariant, the induced action on \(M^{CR}\) is by Cauchy-Riemann diffeomorphisms. The strictly plurisubharmonic function \(\varrho\) defines a Cauchy-Riemann submanifold \(\varrho^{-1}(0) \cap (\mu_{M^c})^{-1}(0)\) which is mapped to \((\varrho_{\text{red}})^{-1}(0) \subset (\mu_{M^c})^{-1}(0)/_L\) by the Cauchy-Riemann map \(\pi_{(\mu_{M^{CR}})^{-1}(0)}\). Since Loose ([13], Theorem 1.2) proves that the projection \(\pi_{(\mu_{M^{CR}})^{-1}(0)}\) defines a unique Cauchy-Riemann structure on \((\mu_{M^{CR}})^{-1}(0)/_L\), \((\varrho_{\text{red}})^{-1}(0))\) can be regarded as the Cauchy-Riemann reduction of \(\varrho^{-1}(0)\) with respect to \(L\).

**Remark.** In particular, the contact manifold \((M, \eta)\) is embedded in the \((2n-1)\)-dimensional contact and Cauchy-Riemann manifold \((M^{CR}, \eta^{CR})\) with the contact form \(\eta^{CR} = (\iota_{M^{CR}})^* (d^*\varrho))\).

The following proposition summarizes the results on the compatibility of the respective reductions.
Proposition 3.6. Let $L$ be an extendable Lie group and $L \times M^c \to M^c$ a free and proper action that extends $L \times M \to M$ and leaves $g : M^c \to \mathbb{R}$ invariant. Then there is the following commutative diagram

$$
\begin{array}{ccc}
(\mu_M)^{-1}(0) & \hookrightarrow & (\mu_{M^c\mathbb{R}})^{-1}(0) \\
\downarrow \pi_{(\mu_M)^{-1}(0)} & & \downarrow \pi_{(\mu_{M^c\mathbb{R}})^{-1}(0)} \\
(\mu_M)^{-1}(0)/L & \hookrightarrow & (\mu_{M^c\mathbb{R}})^{-1}(0)/L
\end{array}
$$

of smooth maps.

Proof. The momentum zero levels $(\mu_M)^{-1}(0)$, $(\mu_{M^c\mathbb{R}})^{-1}(0)$ and $(\mu_{M^c})^{-1}(0)$ are smooth because the $L$-orbits have constant dimensions. Since the three actions of $L$ are proper and free, the three quotients

$$(\mu_M)^{-1}(0)/L \text{ and } (\mu_{M^c\mathbb{R}})^{-1}(0)/L \text{ and } (\mu_{M^c})^{-1}(0)/L$$

are differentiable manifolds and the natural projections

$$\pi_{(\mu_M)^{-1}(0)} \text{ and } \pi_{(\mu_{M^c\mathbb{R}})^{-1}(0)} \text{ and } \pi_{(\mu_{M^c})^{-1}(0)}$$

are differentiable maps as well as the induced inclusions

$$(\mu_M)^{-1}(0)/L \hookrightarrow (\mu_{M^c\mathbb{R}})^{-1}(0)/L \hookrightarrow (\mu_{M^c})^{-1}(0)/L.$$

$\square$

3.3 Compatibility of reduced strata

The results of the Subsections 3.1 and 3.2 are now applied to general proper actions on contact manifolds $(M, \eta)$ and their complexifications $M^c$. Let $H$ be a compact subgroup of $G$. The isotropy types of $H$ define a stratification of $M$ ([SI]). The stratum $(M^c)_{(H)} = \{ x \in M^c | g_0 \in G : g_0 G_x g_0^{-1} = H \}$ of points in $M^c$ with isotropy type $H$ is $G$-invariant and contains the complex submanifold $M^c_H = \{ x \in M^c | G_x = H \}$. Then

$$(\mu_{M^c})^{-1}(0) \cap M^c_H/\sim (\mu_{M^c})^{-1}(0) \cap M^c_H/L,$$

where $L = N_G(H)/H$ acts freely ([CH], [SI]). Proposition 3.6 implies that it is a Kähler manifold. To abbreviate, define

$$\mathcal{M}(M^c_{(H)}) := (\mu_{M^c})^{-1}(0) \cap M^c_{(H)} \text{ and } \mathcal{M}(M^c_H) := (\mu_{M^c})^{-1}(0) \cap M^c_H$$

and similarly, in the contact case,

$$\mathcal{M}(M_{(H)}) := (\mu_M)^{-1}(0) \cap M_{(H)} \text{ and } \mathcal{M}(M_H) := (\mu_M)^{-1}(0) \cap M_H.$$

For future reference, the necessary facts for the Kählerian reduction along the strata $\mathcal{M}(M^c_{(H)})$ are summarized here; they are well known ([HHL], [LAW], [W], [SI]).
a) Let \( x_0 \in (\mu_M)^{-1}(0) \) and let \( V \) be the orthogonal complement to the tangent space of the local \( G^c \)-orbit through \( x_0 \) with respect to the Kählerian metric. The momentum zero level along the stratum \( (M^c)_{(H)} \), i.e., \( \mathcal{M}(M^c_{(H)}) \), is locally and equivariantly isomorphic to \( G \times^H V_H \) where \( V_H = \{ v \in V | h \cdot v = v \) for all \( h \in H \}. The Kählerian reduced space \( \mathcal{M}(M^c_{(H)})_G \) is locally homeomorphic to \( V_H \) (\[S3\]).

b) The Kempf-Ness reduced space \( (\mu_M)^{-1}(0) \), can be stratified into the strata \( \mathcal{M}(M^c_{(H)})_G \), which inherit a natural symplectic and complex structure.

Proposition 3.3 can be applied to the free action of \( L := N_G(H)_{/H} \) on \( \mathcal{M}(M_H) \), where \( N_G(H) \) is the normalizer of \( H \) in \( G \). In the case of a contact manifold \( (M, \eta) \) on which \( G \) acts in a proper fashion by contact transformations, recall the following facts ([W], [LAV]):

c) The stratum \( M_H = \{ m \in M | G_m = H \} \) is a contact manifold and for the stratum \( M_{(H)} = \{ m \in M | G_m \text{ is conjugate to } H \} \) the quotients

\[
\mathcal{M}(M_H)_{/L} = \mathcal{M}(M_{(H)})_G
\]

are naturally isomorphic manifolds, where \( L = N_G(H)_{/H} \) acts freely and properly.

d) The manifold \( \mathcal{M}(M_H)_{/L} \) carries a uniquely induced contact form \( \eta_{\text{red}} \) with the property

\[
(\iota_{\mathcal{M}(M_H)})^*(\eta) = (\pi_{\mathcal{M}(M_H)})^*(\eta_{\text{red}}),
\]

where \( \iota_{\mathcal{M}(M_H)} : \mathcal{M}(M_H) \hookrightarrow M \) and \( \pi_{\mathcal{M}(M_H)} \) is the projection of \( \mathcal{M}(M_H) \) to \( \mathcal{M}(M_H)_{/L} \).

Note that these facts treat every stratum independently and one obtains for each stratum \( \mathcal{M}(M_{(H)}) \) of \( (\mu_M)^{-1}(0) \) a reduced contact space; there is no condition that links the various contact structures.

**Lemma 3.7.** If \( x_0 \in (\mu_M)^{-1}(0) \) and \( H = G_{x_0} \) then \( (M^c)_H \) complexifies \( M_H \).

**Proof.** Let \( V \) be the complex vector subspace in \( T_{x_0}M^c \), which is the complement with respect to the Kählerian metric of the local \( G^c \)-orbit through \( x_0 \). There is a \( G \)-invariant neighbourhood \( U(x_0) \) of \( x_0 \) which is openly and \( G \)-equivariantly embedded in the complex \( G^c \)-manifold \( G^c \times^H V \) ([HI], [K]). Since \( x_0 \in (\mu_M)^{-1}(0) \subset (\mu_M)^{-1}(0) \subset (\mu_M)^{-1}(0) \), \( H^C = (G_{x_0})^C = (G^C)_{x_0} \) and it is possible to assume in addition that \( V = W^C \), where \( W \subset T_{x_0}M \) is an \( H \)-invariant subspace such that \( M \cap U(x_0) \) embeds openly in \( G \times^H W \). If \( W_{(H)} = \{ w \in W | h \cdot w = w \) for all \( h \in H \}, it follows that for \( V_{(H)} = \{ v \in V | h \cdot v = v \) for all \( h \in H \} \subset (W_{(H)})^C \). Finally \( U(x_0) \cap M_H \hookrightarrow G \times^H W_{(H)} \) and \( U(x_0) \cap M_H \hookrightarrow G^C \times^H (W_{(H)})^C \) are open embeddings. Since \( G^C \times^H (W_{(H)})^C \) can be regarded as the complexification of \( G \times^H W_{(H)} \), this proves the claim.

The set \( (M^c)_H = \{ z \in M^c | G_z = H \} \) is a complex submanifold of \( M^c \) ([S3]). The normalizer \( N_G(H) \) of \( H \) in \( G \) acts naturally on \( (M^c)_H \). The induced action of \( L := N_G(H)_{/H} \) on \( (M^c)_H \) is free. If \( x_0 \in (\mu_M)^{-1}(0) \) and \( H = G_{x_0} \), it follows that \( M_H \hookrightarrow \)
Proposition 3.8. Let \( \varphi_{\text{red}} : (\mu c)^{-1}(0)_G \to \mathbb{R} \) be the function defined by \( \varphi_{\text{red}} \circ \pi_{(\mu c)^{-1}(0)} = \varphi \circ \pi_{(\mu c)^{-1}(0)} \). Let \( x_0 \in (\mu M)^{-1}(0) \) and \( H = Gx_0 \) be the isotropy group. The contact manifold \( (M^c)_H, (\iota_{M^c_H})^*(\eta) \) embeds in the Kähler manifold \( ((M^c)_H, -dd^c \varrho((M^c)_H)) \) such that \( (\iota_{M^c_H})^*(\eta) = \varrho((M^c)_H) \). The reduced spaces

\[
(M^c)_L / \varphi_{\text{red}} \big| (M^c)_L / (M^c)_H
\]

are related by the property that for the induced embedding

\[
(M^c)_H / L \hookrightarrow (M^c)_L / \varphi_{\text{red}} | (M^c)_L / (M^c)_H
\]

\[
(\iota_{M^c_H})^*(\eta) = \varphi_{\text{red}} \big| (M^c)_L / (M^c)_H
\]

Proof. Since \( L = N_G(H) / H \) acts freely on \( (M^c)_H \), Proposition 3.3 applies. It follows that the contact moment map \( \mu_{M^c_H} : M^c_H \to \text{Lie}(L)^* \) and the Kählerian moment map \( \mu_{M^c_H} : (M^c)_H \to \text{Lie}(L)^* \) define smooth momentum zero levels \( M^{c_H} \) and \( M^{c_H} / L \), because \( L \) acts freely. For the same reason \( M^{c_H} / L \) and \( M^{c_H} / L \) are smooth, and the restriction of \( \varphi | (M^c)_L / (M^c)_H \) defines a strictly plurisubharmonic function \( \varphi_{\text{red}} | (M^c)_L / (M^c)_H \) such that the embedding \( \iota_{M^c_H} / L : (M^c)_H / L \hookrightarrow (M^c)_L / L \)

satisfies \( (\iota_{M^c_H})^*(\eta) \). \Box

In the same way, the strata \( M^c_H / G \) and \( (M^c)^c_H / G \) are compatible. For every closed subgroup \( H \) of \( G \) the strata \( M^c_H / G \) of \( (\mu M)^{-1}(0)_G \) and \( M^c_H / G \) of \( (\mu M)^{-1}(0)_G \) are compatible by the function \( \varphi_{\text{red}} | (\mu M)^{-1}(0)_G \to \mathbb{R} \) induced by the restriction \( \varphi | (\mu M)^{-1}(0)_G \) in the following sense:

Proposition 3.9. The embedding \( M \hookrightarrow M^c \) induces embeddings

\[
M^c_H / G \hookrightarrow M^c_H / G \text{ and } M^c_H / G \hookrightarrow M^c_H / G
\]

such that a contact manifold \( M^c_H / G \) embeds in \( M^c_H / G \) and has the property

\[
(\iota_{M^c_H})^*(\eta) = \varphi_{\text{red}} | (M^c)_L / (M^c)_H
\]

where \( \eta_{\text{red}} \) is the reduced contact form on \( M^c_H / G \).
Proof. The quotients $\mathcal{M}(M_H)/_L$ and $\mathcal{M}(M_H)/_G$ are naturally diffeomorphic ([W]). Similarly $\mathcal{M}(M_H^c)/_L$ and $\mathcal{M}(M_H^c)/_G$ are naturally diffeomorphic. Proposition 3.8 and the commutativity of the diagram

\[
\begin{array}{ccc}
\mathcal{M}(M_H)/_L & \hookrightarrow & \mathcal{M}(M_H^c)/_L \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{M}(M_H)/_G & \hookrightarrow & \mathcal{M}(M_H^c)/_G \\
\end{array}
\]

prove the claim.

As a summary, the geometry of the contact, Cauchy-Riemann and Kählerian reductions can be described as follows.

Corollary 3.10. Let $G$ be an extendable Lie group and $(M, \eta)$ a proper $G$-contact manifold. There is an equivariant Stein complexification $M^c$ with a smooth strictly plurisubharmonic function $\varrho : M^c \to \mathbb{R}$ such that $d^c \varrho$ extends $\eta$. Let the function $\vartheta_{\text{red}} : (\mu_{M^c})^{-1}(0)/_G \to \mathbb{R}$ be defined by $\vartheta_{\text{red}} \circ \varrho \circ (\mu_{M^c})^{-1}(0) = \vartheta_{\text{red}} \circ \varrho \circ (\mu_{M^c})^{-1}(0)$. Let $H < G$ be a compact subgroup and $L = N_G(H)/H$. Then for every stratum $(M^c)_{(H)}$, the 1-form $d^c(\vartheta_{\text{red}}|_{(M^c)_{(H)}})_G$

a) provides the Kählerian reduction of $(M^c)_{(H)}$,

b) pulled-back to $(\vartheta_{\text{red}})^{-1}(0) \cap (\mathcal{M}(M^c)_{(H)})_G$ is equivalent to the contact and the Cauchy-Riemann reduction of $(M^\text{CR})_{(H)} := (\vartheta|_{(M^c)_{(H)}})^{-1}(0) \subset (M^c)_{(H)}$

c) and pulled back to $\mathcal{M}(M_H)/_G$ provides the contact reduction of $M_H$.

Proof. The compatibility of the reductions with the Kählerian reduction is shown in Proposition 5.6. The Cauchy-Riemann reduction is carried out in Proposition 3.5 and the contact reduction by Proposition 5.3. These results are applied to the stratifications described in Proposition 4.8 and Proposition 5.9.

Piecewise contact structures

For a proper action $G \times M \to M$ on a contact manifold $(M, \eta)$ by contact transformations $(\mu_M)^{-1}(0)/_G$, is stratified into smooth contact manifolds $\mathcal{M}(M_H)/_G$ ([W], [LW]). The respective contact structures are induced by the contact reductions. However, these contact structures are treated separately; the transition between two strata is not worked out in [W] and [LW]. In the symplectic setting, a Poisson structure can be defined which allows one to discuss the compatibility of the various strata. As a suitable tool in the case of contact manifolds, the following definition of a piecewise contact structure is suggested here to state a compatibility condition from stratum to stratum.

Definition 3.11. Let $M = \bigcup_{\alpha \in I} M_\alpha$ be a stratified topological space such that every stratum $M_\alpha$ is a differentiable manifold. A family of 1-forms $\eta_\alpha$ on $M_\alpha$, $\alpha \in I$, is called a piecewise contact structure if
a) each \((M_\alpha, \eta_\alpha)\) is a contact manifold,

b) there is a complex space \(M^c\) with a stratification \(M^c_\alpha\) into complex manifolds such that an embedding \(\iota : M \hookrightarrow M^c\) induces embeddings \(\iota_\alpha : M_\alpha \hookrightarrow M^c_\alpha\) as totally real submanifolds,

c) there is a strictly plurisubharmonic function \(\varphi : M^c \to \mathbb{R}\) such that for every \(\alpha\) the restricted function \(\varphi |_{M^c_\alpha}\) is smooth and satisfies

\[
\iota^*_\alpha (d\varphi |_{M^c_\alpha}) = \eta_\alpha.
\]

**Remark.** Theorem 2.2 shows that a smooth contact manifold is a piecewise contact manifold which consists of one stratum only.

In the case of an extendable Lie group \((HK, K)\) show that \((\mu_{M^c})^{-1}(0)\) /\(G\) inherits from \(M^c\) the structure of a complex space in a natural way on which \(\varphi_{\text{red}}\) is a strictly plurisubharmonic function. Then \((\mu_{M^c})^{-1}(0)\) /\(G\) = \(\bigcup_{H < G} M(M^c_H) /\(G\)) stratifies the reduced space \((\mu_{M^c})^{-1}(0)\) /\(G\) and Proposition 3.9 can now also be stated as follows:

**Theorem 3.12.** Let \(G\) be an extendable Lie group which acts properly on a contact manifold \((M, \eta)\) by contact transformations. Then there is a canonically defined structure of a piecewise contact manifold on the quotient \((\mu_M)^{-1}(0)\) /\(G\). \(\Box\)

**References**

[Ab] Abels, H. Parallelizability of proper actions, global K-slices and maximal compact subgroups. Math. Ann. 212 (1974), 1 – 19.

[Gr] Grauert, H. On Levi’s problem and the imbedding of real analytic manifolds. Ann. Math. Vol. 68, No. 2 (1958), 460 – 472.

[GH] Greb, D.; Heinzner, P. Kählerian Reduction in Steps. Symmetry and Spaces (In Honor of Gerry Schwarz), Progress in Mathematics 278, Bosten: Birkhäuser (2010), 63 – 82.

[H1] Heinzner, P. Geometric invariant theory on Stein spaces. Math. Ann. 289 (1991), 631 – 662.

[H2] Heinzner, P. Equivariant holomorphic extensions of real analytic manifolds. Bull. Soc. Math. France. 121 (1993), 101 – 119.

[HHK] Heinzner, P.; Huckleberry, A. T.; Kutzschebauch, F. A real analytic version of Abels’ theorem and complexifications of proper Lie group actions. Complex analysis and geometry (Trento, 1993), 229–273, Lecture Notes in Pure and Appl. Math., 173 (1996), Dekker, New York.

[HHL] Heinzner, P.; Huckleberry, A.T.; Loose, F. Kählerian extensions of the symplectic reduction. J. reine angew. Math. 455 (1994), 123 – 140.

[HK] Heinzner, P.; Kurtdere, A. Kählerian reductions for proper actions. In preparation (2010).
[HL] Heinzner, P.; Loose, F. Reduction of complex Hamiltonian $G$-spaces. Geometric and Functional Analysis. Vol. 4, No. 3 (1994), 288 – 297.

[I] Illman, S. Every proper smooth action of a Lie group is equivalent to a real analytic action: a contribution to Hilbert’s fifth problem. Ann. of Math. Stud. 138 (1995), 189–220.

[K] Kurtdere, A. Kählerian extensions of contact manifolds and their reductions. Thesis (2009). Ruhr-Universität Bochum. www-brs.ub.ruhr-uni-bochum.de/ntahtml/HSS/Diss/KurtdereAyse/diss.pdf

[KU] Kutzschebauch, F. On the uniqueness of the analyticity of a proper $G$-action. Manuscripta Math. 90 (1996), no. 1, 17–22.

[LW] Lerman, E.; Willett, C. Topological structures of contact and symplectic quotients. Int. Math. Res. Notices, No. 1 (2001), 34 – 52.

[L1] Loose, F. A Remark on the Reduction of Cauchy-Riemann Manifolds. Math. Nachr. 214 (2000), 39 – 51.

[L2] Loose, F. Reduction in contact geometry. Journal of Lie Theory. Vol. 11 (2001), 9 – 22.

[MS] Matumoto, T.; Shiota, M. Unique triangulation of the orbit space of a differentiable transformation group and its applications. Homotopy theory and related topics, Adv. Stud. Pure Math. 9 (1986), 41 – 55.

[Sh] Shutrick, H. B. Complex extensions. Quart. J. of Math. Series 2, t. 9 (1958), 189 – 201.

[Sj] Sjamaar, R. Holomorphic slices, symplectic reduction and multiplicities of representations. Ann. Math. (2) 141 (1995), 87 – 129.

[St] Stratmann, B. Complexification of Proper Hamiltonian $G$-spaces. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) Vol. 30 (2001), 515–534.

[Wh1] Whitney, H. Analytic extensions of differentiable functions defined in closed sets. Tr. American Math. Soc. Vol. 36, No. 1 (1934), 63 – 89.

[Wh2] Whitney, H. Differentiable manifolds. Ann. of. Math. 37 (1939), 645 – 680.

[WhBr] Whitney, H.; Bruhat, F. Quelques propriétés fondamentales des ensembles analytiques réels. Comment. Math. Helv. 33 (1959), 132 – 160.

[W] Willett, C. Contact reduction. Tr. American Math. Soc., Vol. 354, No. 10 (2002), 4245 – 4260.

[Wi] Winkelmann, J. Invariant hyperbolic Stein domains. Manus. Math. 79 (1993), 329 – 334.