Weak singularities of 3-D Euler equations and restricted regularity of Navier Stokes equation solutions with time dependent force terms

Jörg Kampen

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Abstract

Classical vorticity solution branches of the three dimensional incompressible Euler equation are constructed where a velocity component blows up at some point after finite time for regular velocity component data in $H^3 \cap C^3$ corresponding to vorticity data in $H^2 \cap C^2$. Furthermore, there are classical solution branches with $C^k \cap H^{k-2}$-data for $k \geq 2$ which develop weak singularities at some point of space time in $C^{k-1} \setminus C^k$ for any $k \geq 2$ after finite time. The time-local solution branches of the Euler equation are obtained by viscosity limits of solutions of Navier-Stokes-type viscosity ($\nu$)-extensions of time-reversed Euler-type equations. The main observation is that for certain carefully chosen velocity component data in $H^{1,1}$ with singular vorticity at a point an iterative solution scheme of this Navier-Stokes type equation in terms of convolutions with the Gaussian density of dispersion $\nu$ preserves a certain order of spatial polynomial decay at spatial infinity while the increments of the scheme become rather regular after two iteration steps. The increment velocity component functions, i.e., the velocity component functions minus the data convoluted with the Gaussian probability density then have a representation in terms of an ‘second order’ initial increment (obtained after two iteration steps) plus higher order increments which are all in $H^3 \cap C^3$ spatially. Here we need two iteration steps in order to define the initial increment as the first natural increment of a natural iteration scheme has $\nu$-independent upper bounds only in $H^2 \cap C^1$. Lipschitz-continuity of (some derivatives of) these strong increment data can be used in order to obtain $\nu$-independent estimates for that representations of the velocity component functions. This leads to classical solutions of the time-inverted Euler equation in the viscosity limit by compactness arguments or by local time contraction of the higher order increments. Hence, short time (weak) singularities are initial values of local solution branches of the time-reversed Euler type equations, which are constructed via (not time-reversible) viscosity extensions. These solutions of the 3-D Euler equation have a straightforward interpretation to be solutions of incompressible Navier Stokes equation with time dependent force terms with restricted regularity or blow up of a velocity component at some point, as the time-dependent force term can be chosen such that it cancels the viscosity term of the incompressible Navier Stokes equation.

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1 Idea of construction and statement of weak singularity theorems

We determine data with weakly singular short-time solutions of the three dimensional incompressible Euler equation on the whole Euclidean space. Let \( v_i, 1 \leq i \leq D \) denote the velocity component functions in dimension \( D = 3 \), and let

\[
\omega = \text{curl}(v) = \left( \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right)
\]

(1)

denote the corresponding vorticity functions. The incompressible Euler equation in vorticity form is

\[
\frac{\partial \omega}{\partial \tau} + v \cdot \nabla \omega = \frac{1}{2} \left( \nabla v + \nabla v^T \right) \omega,
\]

(2)

and for the corresponding Cauchy problem this equation has to be solved with some initial data \( \text{curl}(h) \) at time \( \tau = 0 \) of some function \( h = (h_1, h_2, h_3)^T \). It is well-known that

\[
v(t, x) = \int_{\mathbb{R}^3} K_3(x - y) \omega(t, y) dy,
\]

where

\[
K_3(x) = \frac{1}{4\pi} x \times h |x|^3.
\]

(3)

In order to construct local-time and long-time singularities we consider first a related equation which is obtained from (2) by the simple time transformation

\[
\tau = -t, \quad \omega^-(t, .) = \omega(\tau, .), \quad v^-(t, .) = v_i(\tau, .), \quad 1 \leq i \leq D.
\]

(4)

Spatial derivatives are untouched, so multiplying the transformed equation by \(-1\) and writing the equation in coordinates we have

\[
\frac{\partial \omega^-}{\partial t} - \sum_{j=1}^{3} v_j^- \frac{\partial \omega^-}{\partial x_j} = - \sum_{j=1}^{3} \frac{1}{2} \left( \frac{\partial v^-}{\partial x_j} + \frac{\partial v_j^-}{\partial x_i} \right) \omega_j^- - \sum_{j=1}^{3} \frac{1}{2} \left( \frac{\partial v^-}{\partial x_j} + \frac{\partial v_j^-}{\partial x_i} \right) \omega_j^-,
\]

\[
1 \leq i \leq D = 3.
\]

(5)

We refer to the equation in (5) as the 'time-reversed Euler equation'. We shall observe that the 'Navier-Stokes type' extension of the latter equation with an additional viscosity term, i.e., the equation

\[
\frac{\partial \omega_i^\nu}{\partial t} - \nu \Delta \omega_i^\nu - \sum_{j=1}^{3} v_j^\nu \frac{\partial \omega_i^\nu}{\partial x_j} = - \sum_{j=1}^{3} \frac{1}{2} \left( \frac{\partial v_i^\nu}{\partial x_j} + \frac{\partial v_j^\nu}{\partial x_i} \right) \omega_j^\nu,
\]

for \( 1 \leq i \leq D \), and with a positive constant \( \nu > 0 \) has a regular solution on a time interval \([0, T]\) for some \( T > 0 \) for a certain class of data in \( H^1 \) which have a singular vorticity at some point. Regularity results for this specific class of data \( \omega_i(0, .) = \omega_i^\nu, 1 \leq i \leq D \) lead to the conclusion that short or long-time solution families with parameter \( \nu > 0 \) have a viscosity limit solution branch

\[
\omega_i^\nu(t, .) = \lim_{\nu \downarrow 0} \omega_i^\nu(t, .), \quad 1 \leq i \leq D
\]

(7)

of the reversed Euler equation with the same initial data, where some regularity is preserved. The initial data \( \omega_i^\nu, 1 \leq i \leq D \) of the the reversed Euler equation are then 'final data' of the original incompressible Euler equation which have
vorticity singularities after finite time. If the solution branch constructed for
the time-reversed Euler equation is short-time, then the solution for the original
Euler equation is short-time with a short-time singularity, and if the solution
branch constructed for the time-reversed Euler equation is long-time, then the
solution for the original Euler equation is long-time with a long-time singularity.
We state these results more precisely in the two theorems at the end of this
section. In this paper we shall prove the existence of short time singularities.
The proof of long-time singularities will be considered in a subsequent paper.
In the following as usual let $C^\infty$ be the function space of real-valued functions
on $\mathbb{R}^D$ with continuous derivatives of any order, and let $H^s$ denote the Sobolev
function space of order $s \geq 0$ (for real numbers $s$). Furthermore, for functions
on a domain $[0,T) \times \mathbb{R}^D$ we denote the space of functions with continuous
time derivatives up to order $m$ (nonnegative integer) and multivariate spatial
derivatives up to order $n$ (nonnegative integer) by $C^{m,n}([0,T) \times \mathbb{R}^D)$ or just
by $C^{m,n}$ if the reference to the domain is known form the context. We have

**Theorem 1.1.** Let $D = 3$. There exist data $h_i \in H^2 \cap C^2$, $1 \leq i \leq D$ and a vorticity solution $\omega_i$, $1 \leq i \leq D$ of the three dimensional incompressible Euler
equation Cauchy problem such that after some finite time $T > 0$ there is a
blow-up of the classical solution, i.e.,

i) for some $T > 0$ there is a solution function $\omega_i : [0,T) \times \mathbb{R}^D \to \mathbb{R}$, $1 \leq i \leq D$ in $C^1([0,T),H^2 \cap C^2)$ which satisfies the incompressible Euler
equation pointwise on the domain $[0,T) \times \mathbb{R}^D$ in a classical sense;

ii) for the solution in item i) we have

\[
\sup_{\tau \in [0,T)} |\omega_i(\tau, x)| = \infty, \tag{8}
\]

i.e., there is no finite upper bound for the left side of (8).

Even a long time singularity version of Theorem 1.1 holds. We have

**Theorem 1.2.** A stronger version of Theorem 1.1 holds with the same text
except for the replacement of the quantor 'after some small finite time $T > 0$'
by 'after any finite time $T > 0$'.

**Remark 1.3.** Regularity $h_i \in C^\infty_0$, $1 \leq i \leq D$ with $C^\infty$ the space of smooth
functions vanishing at infinity can be obtained as the argument below indicates.
However, $h_i \in H^3 \cap C^2$ is essential, and we consider further regularity results and
long-time kinks and singularities in a subsequent paper. Here we are interested
mainly in the connection to singular solution or kinks of the Navier Stokes
equation with regular time dependent force terms.

The extended statement of Theorem 1.2 is proved in a subsequent paper.
Theorem 1.2 can be extended in the sense that there are solution branches with
weak singularities of any integer order. In order to have a succinct statement
we introduce the concept of a 'spatial kink of order $k$' of a classical solution.

**Definition 1.4.** We say that a classical solution branch $\omega_i$, $1 \leq i \leq D$ of an
incompressible Euler Cauchy problem with data $\omega_i^0$, $1 \leq i \leq D$ with $\omega_i^0 \in C^m$
for $m \geq k$ has spatial a kink of order $k \geq 1$, if there is a space-time point $(\tau, x)$
with $\tau > 0$ and $x \in \mathbb{R}^D$ such that

\[
\omega_i \in C^k \setminus C^{k-1} \text{ at } (\tau, x) \tag{9}
\]
for some integer \( k \geq 1 \).

**Corollary 1.5.** Let \( D = 3 \). For any \( k \geq 2 \) and \( s \geq 0 \) there exists data \( h_i \in H^m \cap C^m \) with \( m \geq k + 2 \) and a vorticity solution \( \omega_i \), \( 1 \leq i \leq D \) of the three dimensional incompressible Euler equation Cauchy problem with data \( h_i \), \( 1 \leq i \leq n \) such that after some finite time the solution has a kink of order \( k \).

**Remark 1.6.** We treat the latter result as a Corollary because the proof method is the same, and only the choice of data for the time reversed Euler-type Cauchy problem has to be adapted.

Next we draw consequences for the Navier Stokes equation. A classical solution
\[
\omega_i, \ 1 \leq i \leq D, \ \omega_i(\tau, \cdot) \in H^3 \cap C^2, \ \tau \in [0, T)
\] (10)
of the incompressible Euler equation (on the interval \([0, T)\)) with a blow-up of vorticity at time \( T \) satisfies the vorticity form of the Navier Stokes equation
\[
\frac{\partial \omega}{\partial \tau} - \nu \Delta \omega + v \cdot \nabla \omega = \frac{1}{2} (\nabla v + \nabla v^T) \omega + F,
\] (11)
with force term \( F = (F_1, F_2, F_3)^T \) (on the same time interval) if
\[
F_i = -\nu \Delta \omega_i \in L^2, \ 1 \leq i \leq 3.
\] (12)
The analysis below shows that \( F_i \) is also \( L^2 \) with respect to time on the time interval \([0, T]\), where \( T \) is the time where the vorticity of the Euler equation blows up (this is because we choose data of the time-reversed Euler equation in \( H^2 \)). Similarly a regular classical solution on the time interval \([0, T]\)
\[
\omega_i, \ 1 \leq i \leq D, \ \omega_i(\tau, \cdot) \in H^{2+k} \cap C^{2+k}
\] (13)
of the incompressible Euler equation with a kink of order \( k \) at time \( T \) such that
\[
\omega(T, \cdot) \in C^{k-1} \setminus C^k,
\] (14)
without the vorticity form of the Navier Stokes equation (11), where
\[
F_i = -\nu \Delta \omega_i \in H^k \cap C^k, \ 1 \leq i \leq 3.
\] (15)
We conclude

**Theorem 1.7.** For the Cauchy problem for the Navier Stokes equation (11) for some time \( T > 0 \) time dependent force terms with
\[
F_i(\tau, \cdot) \in H^k \cap C^k, \ k \geq 0, \ \tau \in [0, T]
\] (16)
and data \( \omega_i(0, \cdot) \in H^{2+k} \cap C^{2+k}, \ 1 \leq i \leq 3, \ k \geq 0 \) can be chosen such that a regular classical solution of the Navier Stokes equation on the time interval \([0, T]\) has a blow-up (case \( k = 0 \)) or a kink of order \( k \) at time \( T \).
2 Proof of Theorem 1.1 and Corollary 1.5

For the Gaussian fundamental solution $G_{\nu}$ of

\[ p_{t} - \nu \Delta p = 0 \]  

we consider the iteration scheme

\[ v_{\nu,-k,i}^{\nu} = v_{i}^{f} \ast_{sp} G_{\nu} + \sum_{j=1}^{D} \left( v_{\nu,-k-1,j}^{\nu} \frac{\partial v_{\nu,-k-1,i}}{\partial x_{j}} \right) \ast G_{\nu} \]

\[ - \left( \sum_{j,m=1}^{D} \int_{\mathbb{R}^{D}} \left( \frac{\partial}{\partial x_{j}} K_{D}(.,y) \right) \sum_{j,m=1}^{D} \left( \frac{\partial v_{\nu,-k-1}^{\nu}}{\partial x_{j}} \frac{\partial v_{\nu,-k-1}^{\nu}}{\partial x_{m}} \right) (.,y) dy \right) \ast G_{\nu}, \]

\[ v_{\nu,0}^{\nu}(\cdot) := v_{i}^{f}(\cdot). \]

Here, the symbol $\ast_{sp}$ denotes spatial convolution and $\ast$ denotes convolution with respect to the time variable and the spatial variables. We define for $k \geq 1$

\[ \delta v_{i}^{\text{init},\nu,-k} = v_{i}^{\nu,-k} - v_{i}^{f} \ast_{sp} G_{\nu}, \]  

and

\[ \delta v_{i}^{\nu,-k} = v_{i}^{\nu,-k} - v_{i}^{\nu,-k-1}. \]

We show that the local time solution function $v_{i}^{\nu,-}$, $1 \leq i \leq D$ of the Navier Stokes type extension of the time-reversed Euler equation has a representation of the form

\[ v_{i}^{\nu,-} = v_{i}^{f} \ast_{sp} G_{\nu} + \delta v_{i}^{\text{init},\nu,-2} + \sum_{k=3}^{\infty} \delta v_{i}^{\nu,-k}, \]

which is spatially in $H^{3} \cap C^{3}$ for evaluations at all time $t \in [0,T]$ for some small time horizon $T > 0$.

**Remark 2.1.** Note the choice $k = 2$ in (21) with respect to the initial increment in (19). Indeed the choice $k = 1$ is not sufficient as the first initial increment in (19) is of lower regularity.

First note that the data $v_{i}^{f}$, $1 \leq i \leq D$ are $\nu$-independent Hölder continuous data such that we have

\[ \lim_{\nu \downarrow 0} v_{i}^{f} \ast_{sp} G_{\nu} = v_{i}^{f}, \quad 1 \leq i \leq D. \]  

The convergence in (22) is in $H^{2}$ spatially. Hence we have Hölder continuous and pointwise convergence in case of dimension $D = 3$. However, our main interest concerns the regularity of the increment

\[ \delta v_{i}^{\nu,-} = \lim_{\nu \downarrow 0} v_{i}^{\nu,-} - v_{i}^{f}. \]

For this purpose we need $\nu$-independent estimates of convolutions

\[ g_{\nu} \ast G_{\nu}, \quad g \ast G_{\nu,i} \]

with data

\[ g_{\nu} \in S := \{ \delta v_{i}^{\text{init},\nu,-2}, \delta v_{i}^{\nu,-k}, \quad k \geq 3 \}. \]
for \( k \leq 0 \) 

\[
\lim_{k \to 0} g_\nu = \text{constant.}
\]

The hypothesis of an existing continuous viscosity limit 

\[
g = \lim_{k \to 0} g_\nu
\]

for \( g_\nu \in S \) implies that the viscosity limit 

\[
\lim_{k \to 0} g_\nu * G_\nu \text{ exists.}
\]

Indeed for \( g_\nu \in S \) for \( z_i = \frac{y_i}{\nu}, 1 \leq i \leq D \) we have 

\[
g_\nu * G_\nu = \int_0^1 \int_{\mathbb{R}^D} g_\nu(x-y) \frac{1}{\sqrt{4\pi vs}} \exp \left( -\frac{|w|^2}{4vs} \right) dyds = 
\]

\[
\int_0^1 \int_{\mathbb{R}^D} g_\nu(x-\sqrt{v}z) \frac{1}{\sqrt{4\pi vs}} \exp \left( -\frac{|w|^2}{4vs} \right) dzds \to g(x)t \text{ as } \nu \downarrow 0.
\]

Similar considerations can be applied for multivariate spatial derivatives of order \( \gamma \) if \( D^\gamma g_\nu \) is at least continuous and \( D^\gamma \lim_{k \to 0} D^\gamma g_\nu \) exists. In (27) for 

\[
k = 1 \text{ we have } v_1^{\nu, -k-1} = v_1^{\nu, 0} = v_1^f, 1 \leq i \leq D.
\]

Hence for \( k = 1 \) the data \( v_1^f \) in the convolutions \( v_1^f * G_\nu \) and \( v_1^f * \sp G_\nu \) are independent of \( \nu \) and there are natural viscosity limits of these terms. For \( k = 2 \) convolutions of the form 

\[
D^2 (v_1^f \sp G_\nu) \text{ appear in the recursive scheme (especially with } 0 \leq |\gamma| \leq 1) \text{ and as part of the function } v_1^{\nu, -1}. \text{ We observe that these functions } D^2 (v_1^f \sp G_\nu), 0 \leq |\gamma| \leq 1 \text{ have a } \nu \text{-independent upper bound.}
\]

If we apply spatial derivatives to the scheme in (23), then it is natural to consider convolutions with first order spatial derivatives of the Gaussian \( G_{\nu,i} \). \text{ We prove the existence of } \nu \text{-independent upper bounds } h_2, h_3, k \geq 3, \text{ where } 

\[
|\delta v^{\nu, -2} \sp G_{\nu,i}(t, \cdot)|_{g C^3} \leq h_2,
\]

and where for \( k \geq 3 \) 

\[
|\delta v^{\nu, -k} \sp G_{\nu,i}(t, \cdot)|_{g C^3} \leq h_k.
\]

Here we shall use spatial rather Lipschitz continuity of the convoluted data \( \delta v^{\nu, -k} \sp G_{\nu,i}, 1 \leq i \leq D \text{ for } k \geq 3 \). \text{ Local time } \nu \text{-independent contraction in spatial } |.|_{g C^3} \text{-norms of the higher order increments } \delta v^{\nu, -k} \leq 1 \leq D \text{ for } k \geq 3 \text{ leads to a regular limit for the functional series in (23)}. \text{ Furthermore strong polynomial decay of order } 2(D+1) \text{ is preserved for the increments, i.e., for } k \geq 3 \text{ there is a } C > 0 \text{ independent of } \nu \text{ such that for all } |x| \geq 1 \text{ and } 0 \leq |\gamma| \leq 3 \text{ and } 0 \leq |\beta| \leq 2

\[
|D^2 \delta v^{\nu, -k} \sp G_{\nu,i}(t, x)| \leq \frac{C}{1 + |x|^{2(D+1)}},
\]

and such that for all \( |x| \geq 1 \text{ and } 0 \leq |\gamma| \leq 3 \text{ and } 0 \leq |\beta| \leq 2 \) 

\[
|D^2 \delta v^{\nu, -k} \sp G_{\nu,i}(t, x)| \leq \frac{C}{1 + |x|^{2(D+1)}}.
\]
We can thus use compactness arguments or $\nu$-independent local time contraction in order to verify the regularity of the limit for the functional series in (31).

In order to obtain $\nu$-independent estimates for convolutions of type $D_2^\gamma g_0 \ast G_{\nu,i}$ for $g_\nu \in S$ we may use Lipschitz continuity of the convoluted data. For the first increment member $\delta_{\nu,i}^{\text{init},\nu,-2}$ in (31) this means that for multiindices $0 \leq |\beta| \leq 2$ and $\forall y, y' \in \mathbb{R}^D$ we have

$$\left| D^\beta \delta_{\nu,i}^{\text{init},\nu,-2}(t, y) - D^\beta \delta_{\nu,i}^{\text{init},\nu,-2}(t, y') \right| \leq c |y - y'|^{\delta_0} \quad (33)$$

for some finite constant $c > 0$ and where $\delta_0 \geq 1$. Lipschitz continuity turns out to hold for the higher order increments $D^\beta \delta_{\nu,i}^{\text{init},\nu,-k}$, $k \geq 3$ for $0 \leq |\beta| \leq 2$ as well. We can have $c$ independent of $\nu$. Here we observe that the convolution with the Gaussian and the first order derivative of the Gaussian degenerates outside a ball $B_{\nu,0.5-\epsilon}(x)$ of radius $\nu^{0.5-\epsilon}$ for $0 < \epsilon < 0.5$ as $\nu$ becomes small. Indeed this holds for all spatial derivatives of the Gaussian as we have

$$\left| D^\gamma G_\nu(t, x; s, y) \right| \leq \frac{H_{\gamma, \nu(t-s)}(x-y)}{\sqrt{4\pi(t-s)}} \exp \left( -\frac{|x-y|^2}{4\nu(t-s)} \right) \quad \downarrow 0 \text{ as } \nu \downarrow 0,$$

where $H_{\gamma, \nu(t-s)}(x-y)$ is a Hermite-type polynomial of order $|\gamma|$ parameterized by $1/\nu(t-s)$. If the viscosity limit $\lim_{\nu \downarrow 0} g_\nu \ast D^\gamma G_\nu(t, x)$ exists (is finite for some $x$), then the 'mass' is concentrated in a ball of radius $B_{\nu,0.5-\epsilon}(x)$, i.e., we have

$$\lim_{\nu \downarrow 0} g_\nu \ast D_2^\gamma G_\nu(t, x) = \lim_{\nu \downarrow 0} \int_0^1 \int_{B_{\nu,0.5-\epsilon}(x)} g_\nu(s, y) D_2^\gamma G_\nu(t, x, s, y) dyds \quad (35)$$

Locally, we use simple Gaussian estimates of the form

$$\left| D^\gamma G_\nu(t, x; s, y) \right| \leq \frac{H_{\gamma, \nu(t-s)}(x-y)}{\sqrt{4\pi}} \frac{1}{\nu(t-s)} \left( |x-y|^2 \right)^{\frac{\gamma}{2} - \frac{3}{2}} \exp \left( -\frac{|x-y|^2}{4\nu(t-s)} \right) \quad (36)$$

for $\delta \in (0, 1)$. We remark that for $0 \leq |\gamma| \leq 1$ the constant $C_{\gamma}$ in (36) can be chosen to be

$$C_0 = \sup_{z > 0, \delta \in (0, 1)} z^{\frac{\gamma}{2} - \delta} \exp \left( -\frac{z^2}{4} \right) \quad (37)$$

and

$$C_1 = \sup_{z > 0, \delta \in (0, 1)} z^{\frac{\gamma}{2} - \delta} \exp \left( -\frac{z^2}{4} \right). \quad (38)$$

Note that $C_0, C_1$ in (37), (38) are independent of $\nu$. For our purposes it is sufficient to consider convolutions with the probability density, i.e., convolutions of the form

$$D_2^\gamma g_\nu \ast G_\nu. \quad (39)$$
for $0 \leq |\gamma| \leq 3$.

Remark 2.2. If

$$\sup_{\nu > 0} |D^2 g_\nu(s, y)| \leq C_{\gamma} \nu < \infty,$$  \hspace{1cm} (40)

then

$$\begin{align*}
|D^2 g_\nu * G_\nu(t, x)| &= \left| \int_0^t \int_{|x-y| \geq \sqrt{\nu}} D^2 g_\nu(s, x-y) \frac{1}{\sqrt{2\pi \nu(t-s)}} \exp \left( -\frac{|y|^2}{\nu(t-s)} \right) \, dy \, ds \right| \\
+ \int_0^t \int_{B_{\sqrt{\nu}}(x)} D^2 g_\nu(s, x-y) \frac{C_\nu}{\nu(t-s)^{3/4}} \, dy \, ds &\leq C_{\gamma} \nu \sqrt{t^{1-\delta} s^{3/2}} \leq C_{\gamma} C_0 t^{1-\delta}.
\end{align*}$$

The first integral converges to $D^2 g_\nu(t, x)$ as $\nu \downarrow 0$ (cf. (25)). Hence there is a $\nu$-independent upper bound if there is a finite upper bound $C_{\gamma}$ in (10).

For first order spatial derivatives and regular data $h$ with finite upper bound $C_h$ we observe that

$$\begin{align*}
\left| \int_{y \in B^3_{\sqrt{\nu}}} h(y) \frac{(x-y)_3}{\sqrt{2\pi \nu(t-s)}} \exp \left( -\frac{1}{\nu(t-s)} \right) \, dx \right| &\leq \int_{y \in B^3_{\sqrt{\nu}}} C_h \frac{1}{\sqrt{2\pi \nu(t-s)}} \exp \left( -\frac{1}{\nu(t-s)} \right) \Pi_{i \neq j} \, dy_i \\
 &\leq \int_{y \in B^3_{\sqrt{\nu}}} C_h \frac{1}{\sqrt{2\pi \nu(t-s)}} \exp \left( -\frac{1}{\nu(t-s)} \right) \exp \left( -\frac{1}{\nu(t-s)} \right) \Pi_{i \neq j} \, dy_i \\
 &\leq \frac{C_h}{\sqrt{2\pi \nu(t-s)}} \exp \left( -\frac{1}{\nu(t-s)} \right) \downarrow 0 \text{ as } \nu \downarrow 0.
\end{align*}$$

Here $\epsilon \in (0,5)$ may be arbitrarily small and $c > 0$ is a finite constant (indeed it would be equal to one if the parameter $\nu$ is adjusted such that $G_\nu$ is an exact probability density). The observation in (42) tells us that any mass of a convolution with $G_{\nu,j}$ evaluated at $(t, x)$ is concentrated in a ball $B_{\sqrt{\nu}}(x)$ of radius $\sqrt{\nu}$ around $x$ as $\nu$ becomes small. In order to obtain $\nu$-independent estimates we may Lipschitz continuity for terms

$$h * G_{\nu,k} \text{ (which are equivalent for } \nu > 0)$$

with data $h = D^2 g_\nu$ for $g_\nu \in S$ and $0 \leq |\beta| \leq 3$. Observe the symmetry

$$G_{\nu,j}(t, x; s, y) = -G_{\nu,j}(t, x^{i-}; s, y^{j-}),$$

where $x = (x_1, \ldots, x_n)$, $x^{i-} = (x_1^{i-}, \cdots, x_{i-}^{i-})$ and $x^{j-} = x_j - 2\delta_{ij} x_j$ with the Kronecker $\delta_{ij}$ and where the symbol $y^{i-}$ is defined analogously. Then Lipschitz continuity (or Hölder continuity with exponent $\delta_1 = 1$) of the shifted convoluted function $h(x, \cdot) = h(x - \cdot)$ and transformation $z = x - y$ (and $r = \sqrt{z_1^2 + z_2^2 + z_3^2}$
for brevity) leads to
\[ \int_{B_{\rho_0}} |h(y) - h(0)| \frac{2(\rho_0 - \rho_1)}{4\rho_1 \sqrt{4\rho_1 \rho_0}} \exp \left( -\frac{|x - y|^2}{4\rho_0} \right) dy \]
\[ \leq \int_{B_{\rho_0}} \left| h_x(z) - h_x(z_{\rho_1}) \right| \frac{2z_1}{4\rho_1 \sqrt{4\rho_1 \rho_0}} \exp \left( -\frac{|z|^2}{4\rho_0} \right) dz \]
\[ \leq \int_{B_{\rho_0}} 4|r|^2 |z|^{\frac{\delta_1 + 4}{2}} \nu^2 dr \]
\[ 4 \frac{r}{\nu^2} |z|^{\delta_1 + 25 - 1} = 4 \frac{r}{\nu} |(0.5)(\delta_1 + 25 - 1) - \delta_1 = 1. \]
Here, \( \delta \in (0.5, 1) \) and the latter equation holds even for \( \nu = 0 \) if the usual definition \( 0^0 = 1 \) is used.

For well chosen initial data \( v_i \), \( 1 \leq i \leq 3 \) (for the viscosity extension of the time reversed problem) we observe that \( \delta v_i^{\text{init}, \nu, -2} \) and \( \delta v_i^{\nu, -2} \) gain regularity independently of \( \nu \) as \( k \) increases from 1 to 2 and observe that for all \( t \geq 0 \) the function
\[ v_i^{\nu, -}(t, .) - v_i^f *_{sp} G_\nu = \delta v_i^{\text{init}, \nu, -2}(t, .) + \sum_{l=3}^{\infty} \delta v_i^{\nu, -l}(t, .) \]
have \( \nu \)-independent spatial regular upper bounds with respect to a \( H^3 \cap C^3 \)-norm. Moreover we shall observe that the functional series in (44) can be differentiated twice with respect to the spatial variable member by member, where we use strong polynomial decay at spatial infinity. The latter property of strong polynomial decay at spatial infinity (indeed a spatial polynomial decay at spatial infinity of order \( 2(D + 1) \)) is preserved by the increments in (45) and for spatial derivatives of these increments up to order 3. Moreover we shall observe that
\[ \lim_{\nu \to 0} \nu \Delta v_i^{\nu, -} \equiv 0, \]
and conclude that \( v_i^{\nu, -}, 1 \leq i \leq D \) is a regular classical solution of the viscosity extension of the time-reversed incompressible Euler equation which has a regular viscosity limit which solves the time reversed incompressible Euler equation.

**Remark 2.3.** Note that the viscosity limit \( \nu \downarrow 0 \) for \( D_2^2 v_i^{\nu, -}, 1 \leq i \leq D, 0 \leq |\beta| \leq 2 \) can be obtained by limits
\[ D_2^2 v_i^{\nu, -} - v_i^f *_{sp} D_2^2 G_\nu = D_2^2 \delta v_i^{\text{init}, \nu, -2} + \sum_{l=3}^{\infty} D_2^2 \delta v_i^{\nu, -l}(t, .), \]
where for \( 0 \leq |\beta| \leq 2 \) we can use
\[ \delta v_i^{\text{init}, \nu, -2} = v_i^{\nu, -2} - v_i^f *_{sp} G_\nu \]
\[ = \sum_{j=1}^{D} \left( D_2^\beta \left( v_j^{\nu, -2, j} \partial_{x_j} v_j^{\nu, -2, j} \right) \right) * G_\nu \]
\[ - \left( \sum_{j, m=1}^{D} \int_{\mathbb{R}^D} \left( D_2^\beta \left( \frac{\partial}{\partial x_j} K_D (., y) \right) \sum_{m=1}^{D} \left( \frac{\partial v_j^{\nu, -2, m}}{\partial x_m} \partial_{x_m} v_j^{\nu, -2, m} \right) (., y) dy \right) * G_\nu. \]
and for $k \geq 3$

\[
D^βδv_1^{ν−k} = + \sum_{j=1}^{D} \left(D^β_2 \left(v_j^{ν−k−1} \frac{∂v_1^{ν−k−1}}{∂x_j}\right)\right) \ast G_ν
\]

\[
- \left(\sum_{j,m=1}^{D} \int_{\mathbb{R}^D} \left(\frac{∂}{∂x_j} K_D(., y)\right) \sum_{j,m=1}^{D} \left(D^β_2 \left(v_j^{ν−k−2} \frac{∂v_1^{ν−k−2}}{∂x_m}\right)\right) (., y) dy\right) \ast G_ν
\]

\[
- \sum_{j,m=1}^{D} \left(D^β_2 \left(v_j^{ν−k−2} \frac{∂v_1^{ν−k−2}}{∂x_j}\right)\right) \ast G_ν
\]

\[
+ \left(\sum_{j,m=1}^{D} \int_{\mathbb{R}^D} \left(\frac{∂}{∂x_j} K_D(., y)\right) \sum_{j,m=1}^{D} \left(D^β_2 \left(v_j^{ν−k−2} \frac{∂v_1^{ν−k−2}}{∂x_m}\right)\right) (., y) dy\right) \ast G_ν.
\]

(50)

For $0 ≤ |β| ≤ 2$ we can consider the viscosity limit of a functional series of convolutions with the probability density $G_ν$. There is no degeneracy issue for this classical limit.

We summarize the five steps of the proof.

i) As usual let $H^{2,1}$ be the Sobolev space where weak derivatives up to second order are in $L^1$. We choose data $v_i^f$, $1 ≤ i ≤ D$, which determine the vorticity data $ω_i^f = ω_i^{ν^−}(0, .), 1 ≤ i ≤ D$ by the curl operation. Recall that the velocity data can be recovered from the vorticity data by the Biot-Savart law. For a positive number $β_0 ∈ (2, 2 + α_0)$, and $α_0 ∈ (0, \frac{1}{2})$ we consider velocity component data $v_i^f ∈ H^{2,1}, 1 ≤ i ≤ 3$ where for one index $i_0 ∈ \{1, 2, 3\}$ we have $v_i^f(x) = g_0(r)$ for some univariate function $g_0$ which we are going to define next. Here the function $g_0$ is dependent on the radial component of spherical polar coordinates $r = √x_1^2 + x_2^2 + x_3^2 ∈ \mathbb{R}_+$, where $\mathbb{R}_+$ denotes the set of positive real numbers. This function $g_0 : \mathbb{R}_+ \to \mathbb{R}$ satisfies

\[
g_0(r) = φ_1(r)r^{β_0} \sin \left(\frac{1}{r^{1+α_0}}\right), \tag{51}\]

where $φ ∈ C_0^{∞}$ is defined by

\[
φ_1(r) = \begin{cases} 
1 & \text{if } r ≤ 1, \\
φ_1(r) = α_+(r) & \text{if } 1 ≤ r ≤ 2, \\
0 & \text{if } r ≥ 2. 
\end{cases} \tag{52}\]

Furthermore, $α_+$ is a smooth function with bounded derivatives for $1 ≤ r ≤ 2$. Such functions are well-known in the context of partitions of unity. As usual, $C_0^{∞}$ denotes the function space of smooth functions with compact support.

Remark 2.4. In order to prove the existence of kinks or weak singularities we may consider for $k ≥ 2$ and $β_0^k ∈ (k + 1, k + 1 + α_0)$, $α_0 ∈ (0, \frac{1}{2})$ velocity component data $v_i^f ∈ H^{2+k}, 1 ≤ i ≤ 3$ where for one index $i_0 ∈ \{1, 2, 3\}$ we have $v_i^f(x) = g_{(k)}(r)$ and where $g_{(k)} : \mathbb{R}_+ \to \mathbb{R}$ is defined by

\[
g_{(k)}(r) = φ_1(r)r^{β_0^k} \sin \left(\frac{1}{r^{1+α_0}}\right). \tag{53}\]
For $j \in \{1, 2, 3\} \setminus \{i_0\}$ velocity component data and corresponding vorticity component data are assumed to satisfy

$$v_j^i, \omega_j^i \in C_\infty^\infty,$$  \hspace{1cm} (54)

Note that for $k = 0$ and $\beta_0 = \beta_0^0$ close to $1 + \alpha_0$ we have $v_j^i \in H^{2,1}$ \hspace{1cm} (55)

Note that for $k = 0$ and $\beta_0 = \beta_0^0$ close to $1 + \alpha_0$ we have $v_j^i \in H^{1,1}$ for all $1 \leq i \leq D = 3$ as the singularity of $v_j^i = v_j^i$ (resp. $\omega_j^i = \omega_j^i$) at $r = 0$ (polar coordinates) is of order $|\beta_0^0 - 4 - 2\alpha_0| < 2.5$ for $\beta_0^0$ close to $2 + \alpha_0$ as $\alpha_0 \in (0, 0.5)$. Similarly for any $l \geq 0$ we have choices $\beta_0^0 \in (k + 1, k + 1 + \alpha_0)$ such that $c \in (0, \frac{4}{3})$ such that

$$g_{(k,l)}(\cdot) \in C^{l-1} \cap C^l.$$ \hspace{1cm} (55)

We just have to choose $l = \min \{m \geq 0 | \beta_0^0 - m(2 + \alpha_0) < 0\}$. For this choice of $l$ we have $g_{(k,l)}(\cdot) \in H^{l,1}$.

ii) For $1 \leq i \leq 3$ and for all $\alpha, x \in \mathbb{R}^3 \setminus \{0\}$, $r = \sqrt{x_1^2 + x_2^2 + x_3^2} \in \mathbb{R} \setminus \{0\}$ we have for each iteration $(k = 1)$

$$|D_\gamma^2 \delta v_i^{\text{init.}, \nu, -1}(t, x)| = |D_\gamma^2 \delta v_i^{\text{init.}, \nu, -1}(t, x)| \leq C_{r^1 + 2(\beta_0^0 - (4 + 2\alpha_0)) - |\gamma|},$$ \hspace{1cm} (56)

and for the second iteration $(k = 2)$ and all $\delta \in (0.5, 1)$

$$|D_\gamma^2 \delta v_i^{\text{init.}, \nu, -2}(t, x)| \leq C_{r^{2(\beta_0^0 - 1) + 1 - |\gamma|}.}$$ \hspace{1cm} (57)

For the higher order increments and for $k \geq 3$ we also have for all $\delta \in (0.5, 1)$

$$|D_\gamma^2 \delta v_i^{\text{init.}, \nu, -k}(t, x)| \leq C_{r^{2(\beta_0^0 - 1) + 1 - |\gamma|}.}$$ \hspace{1cm} (58)

Indeed these increments gain regularity at each iteration step, but we shall not consider this here.

Inheritance of polynomial decay (cf. item iii)) then implies for $1 \leq i \leq 3$ and for all $k \geq 3$

$$\delta v_i^{\nu, -k}(t, \cdot) \in H^3 \cap C^3,$$ \hspace{1cm} (59)

and

$$\delta v_i^{\nu, -2}(t, \cdot) \in H^3 \cap C^3.$$ \hspace{1cm} (60)

iii) Given $T > 0$ for all $\nu > 0$ polynomial decay of order $2(D + 1)$ at spatial infinity is inherited by the increments $\delta v_i^{\nu, -2}, \delta v_i^{\nu, -k}, k \geq 3, 0 \leq |\gamma| \leq 3$ and the corresponding value functions $\delta v_i^{\nu, -k}(t, \cdot)$ for $t > 0$ and $k \geq 1$. More precisely for $1 \leq i \leq 3$, and $k \geq 3$ there exists a finite constant $c > 0$ such that for all $t \in [0, T]$ and $|x| \geq 1$

$$\forall k \geq 1 \forall \alpha \leq |\gamma| \leq 3 \hspace{0.5cm} |D^2 \delta v_i^{\nu, -2}(t, x)| \leq \frac{c}{1 + |x|^{2(D + 1)}},$$ \hspace{1cm} (61)

and

$$\forall k \geq 1 \forall \alpha \leq |\gamma| \leq 3 \hspace{0.5cm} |D^2 \delta v_i^{\nu, -k}(t, x)| \leq \frac{c}{1 + |x|^{2(D + 1)}}. \hspace{1cm} (62)$$
For small $T > 0$ it is observed that the constant $c > 0$ can be chosen independently of the iteration index $k$. Furthermore, for $0 \leq |\gamma| \leq 4$

$$|v_i^f *_{sp} D_x^2 G_\nu(t,.)| \leq \frac{c}{1 + |x|^2 (s + 1)} \ t > 0 \text{ and } c \text{ depends on } t > 0, \ (63)$$

such that a corresponding statement holds for the value functions $v_i^{\nu,-k}$, $1 \leq i \leq 3$ and $k \geq 2$.

iv) For the parameter choices of $\beta_0, \alpha_0$ with $\beta_0 = 1.5 - \epsilon$ for small $\epsilon > 0$ of item i), for $1 \leq i \leq 3$, $\nu > 0$ and for $k \geq 3$ and some time $T > 0$

$$|\delta v_i^{\nu,-k+1}(t,.)|_{H^\infty \cap C^3} \leq \frac{1}{2} |\delta v_i^{\nu,-k}(t,.)|_{H^\infty \cap C^3}. \ (64)$$

Hence, for all $t \in [0, T]$

$$v_i^{\nu,-}(t,.) - v_i^f *_{sp} G_\nu(t,.) = \delta v_i^{\text{init,} \nu,-2} + \sum_{l=3}^{\infty} \delta v_i^{\nu,-l} \in H^3 \cap C^3. \ (65)$$

v) Choose a time horizon $T > 0$ as in the previous step such that contraction holds for the higher order increments $\delta v_i^{\nu,-k}$ with $k \geq 3$ as in (64). For an upper bound $C > 0$ independent of $\nu > 0$ we have

$$|\delta v_i^{\text{init,} \nu,-2}(t,.) + \sum_{l=3}^{\infty} \delta v_i^{\nu,-l}(t,.)|_{H^\infty \cap C^3}$$

$$= |v_i^{\nu,-}(t,.) - v_i^f *_{sp} G_\nu(t,.)|_{H^\infty \cap C^3} \leq C. \ (66)$$

For all $\nu > 0$ and for some $T > 0$ (independent of $\nu$) the function

$$v_i^{\nu,-}(t,.) = v_i^f *_{sp} G_\nu(t,.) + \delta v_i^{\text{init,} \nu,-2} + \sum_{l=3}^{\infty} \delta v_i^{\nu,-l} \ (67)$$

satisfies the Navier stokes equation on the time interval $[0, T]$ such that

$$\nu \Delta v_i^{\nu,-} = \frac{\partial v_i^{\nu,-}}{\partial t} - \sum_{j=1}^{D} v_j^{\nu,-} \frac{\partial v_i^{\nu,-}}{\partial x_j}$$

$$+ \sum_{j,m=1}^{D} \int_{R^D} \left( \frac{\partial}{\partial x_j} K_D(. - y) \right) \sum_{j,m=1}^{D} \left( \frac{\partial v_j^{\nu,-}}{\partial x_j} \frac{\partial v_m^{\nu,-}}{\partial x_m} \right) (t, y) dy, \ (68)$$

where the right side of (68) is the time-reversed Euler equation operator applied to $v_i^{\nu,-}$. We choose a sequence $(\nu_\mu)_{\mu\geq1}$ converging to zero and use probability density representations (convolutions with $G_\nu$ itself) of increments $D_2^\beta \delta v_i^{\text{init,} \nu,-2}$ and $D_2^\beta \delta v_i^{\nu,-l}$, $l \geq 3 \text{ for } 0 \leq |\beta| \leq 2$ in

$$v_i^{\nu,-} := v_i^f *_{sp} G_\nu + \delta v_i^{\nu,-} \ (69)$$

in order to show that the viscosity limit $v_i^-$, $1 \leq i \leq D$ is a classical solution of the time-reversed Euler equation exists. Here

$$\delta v_i^\nu := \lim_{\nu \downarrow 0} \delta v_i^{\text{init,} \nu,-2} + \sum_{l=3}^{\infty} \delta v_i^{\nu,-l} \in C^1 ((0, T], H^3 \cap C^3). \ (70)$$
We conclude that the original Euler equation develops in opposite time direction from data at time \( \tau = 0 \) (corresponding to time \( s = T \) of the reversed-time Euler equation) a weak singularity at time \( T > 0 \) (corresponding to data at time \( t = 0 \) the time-reversed Euler equation).

ad i) first we note that multiindices are denoted by \( \alpha = (\alpha_1, \ldots, \alpha_D), \beta = (\beta_1, \ldots, \beta_D), \ldots \) in this paper, and \( \alpha_0, \beta_0 \) just refer to positive real numbers.

For the derivative of the data \( v_{i_0}^f = v_1^f \) and \( k = 0 \) we compute for \( r \neq 0 \) and \( r \leq 1 \)

\[
g'(r) = \frac{d}{dr}r^{\beta_0} \sin \left( \frac{1}{r^{\tau + \alpha_0}} \right) = \beta_0 r^{\beta_0 - 1} \sin \left( \frac{1}{r^{\tau + \alpha_0}} \right) - (1 + \alpha_0) r^{\beta_0 - 2 - \alpha_0} \cos \left( \frac{1}{r^{\tau + \alpha_0}} \right). \tag{71}\]

The derivative \( g' \) of the function \( g \) at \( r = 0 \) is strongly singular for \( \beta \in (2, 2 + \alpha_0) \) and \( \alpha_0 \in (0, \frac{1}{2}) \). Note that it is 'oscillatory' singular bounded for \( \beta_0 = \alpha_0 \in \left( 0, \frac{1}{2} \right) \). Note that for data \( v_{i_0}^f (x_1, x_2, x_3) = g(r) \) we have (for \( r \neq 0 \))

\[
v_{i_0}^f = g'(r) \frac{\partial r}{\partial x_j} = g'(r) \frac{x_j}{r}, \tag{72}\]

In polar coordinates \( (r, \theta, \phi) \in [0, \infty) \times [0, \pi] \times [0, 2\pi] \) with

\[
x_1 = r \sin(\theta) \cos(\phi), \quad x_2 = r \sin(\theta) \sin(\phi), \quad x_3 = r \cos(\theta), \tag{73}\]

(where for \( r \neq 0 \) and \( x_1 \neq 0 \) we have \( r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \theta = \arccos \left( \frac{x_3}{r} \right), \quad \phi = \arctan \left( \frac{x_2}{x_1} \right) \)) we get

\[
v_{i_0}^{f,1} = g'(r) \frac{x_1}{r} = g'(r) \sin(\theta) \cos(\phi),
\]

\[
v_{i_0}^{f,2} = g'(r) \frac{x_2}{r} = g'(r) \sin(\theta) \sin(\phi),
\]

\[
v_{i_0}^{f,3} = g'(r) \frac{x_3}{r} = g'(r) \cos(\theta), \tag{74}\]

such that we have

\[
v_{i_0}^f \in H^{1,1} \text{ obviously.} \tag{75}\]

The second derivative of \( g \) is

\[
g''(r) = \frac{d^2}{dr^2}r^{\beta_0} \sin \left( \frac{1}{r^{\tau + \alpha_0}} \right)
\]

\[
= \frac{d}{dr} \left( \beta_0 r^{\beta_0 - 1} \sin \left( \frac{1}{r^{\tau + \alpha_0}} \right) - (1 + \alpha_0) r^{\beta_0 - 2 - \alpha_0} \cos \left( \frac{1}{r^{\tau + \alpha_0}} \right) \right)
\]

\[
= \beta_0 (\beta_0 - 1) r^{\beta_0 - 2} \sin \left( \frac{1}{r^{\tau + \alpha_0}} \right) - (1 + \alpha_0) \beta_0 r^{\beta_0 - 3 - \alpha_0} \cos \left( \frac{1}{r^{\tau + \alpha_0}} \right)
\]

\[\quad + (1 + \alpha_0)(2 + \alpha_0 - \beta_0) r^{\beta_0 - 2 - \alpha_0} \cos \left( \frac{1}{r^{\tau + \alpha_0}} \right)
\]

\[\quad - (1 + \alpha_0)^2 r^{\beta_0 - 4 - 2\alpha_0} \sin \left( \frac{1}{r^{\tau + \alpha_0}} \right). \tag{76}\]

We have \( v_{i_0}^f \in H^{2,1} \), since

\[
\beta_0 - 4 - 2\alpha_0 > -3. \tag{77}\]

Note that

\[
v_{i_0}^f (0, \cdot) C^6 (\mathbb{R}^3), \tag{78}\]

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for Hölder constants of order \( \delta \in (0, \frac{1}{2}) \). Similarly we have
\[
\omega_{t_0}^j = \omega_0^j = \omega_1(0, \cdot) \in H^{1,1}.
\] (79)

ad ii) For \( k = 1 \), (13) becomes
\[
v_i^{\nu,-1} = v_i^f(\cdot) \ast_{sp} G_\nu - \sum_{j=1}^{D} v_j^{f} \frac{\partial v_i^f}{\partial x_j} \ast G_\nu
\]
\[- \sum_{j,m=1}^{D} \int_{\mathbb{R}^D} \left( \frac{\partial}{\partial x_i} K_{D,i}(\cdot, -y) \right) \sum_{j,m=1}^{D} \left( \frac{\partial v_i^f}{\partial x_j} \frac{\partial v_i^f}{\partial x_m} \right) (t, y) dy \ast G_\nu
\]=: v_i^f(\cdot) \ast_{sp} G_\nu + B_0 \ast G_\nu - L_0 \ast G_\nu.
\] (80)

where \( B_0 \) and \( L_0 \) denote abbreviations of the lowest order approximations of the Burgers term and the Leray projection term. We have
\[
\left\| \left( v_j^{f} \frac{\partial v_i^f}{\partial x_j} \right) (t, \cdot) \right\| \leq c r^{2\beta_0 - (2+\alpha_0)}
\] (81)

and as \( \frac{\partial}{\partial x_i} K_{D,i}(\cdot, -y) \sim \frac{1}{r^2} \) for \( D = 3 \) we have
\[
\left\| \int_{\mathbb{R}^D} K_{D,i}(\cdot, -y) \sum_{j,m=1}^{D} \left( \frac{\partial v_i^f}{\partial x_j} \frac{\partial v_i^f}{\partial x_m} \right) (t, y) dy \right\| \leq c r^{2\beta_0 + 1 - (2+\alpha_0)}
\] (82)

Using the \( \nu \)-independent Gaussian estimates above with \( \Delta r := |x - y| \) for the Burgers term \( B^0 \) we get
\[
\left\| B^0 \ast D^2 G_\nu(t, \cdot) \right\| \leq c r^{2\beta_0 - (1+\alpha_0) - |\gamma|},
\] (83)

and as \( \frac{\partial}{\partial x_i} K_{D,i}(\cdot, -y) \sim \frac{1}{r} \) for \( D = 3 \) for the Leray projection term \( L^0 \) we have
\[
\left\| L^0 \ast D^2 G_\nu(t, \cdot) \right\| \leq c r^{2\beta_0 + 1 - (2+\alpha_0) - |\gamma|}
\] (84)

Hence the statement in (56) follows.

For \( k = 2 \), (13) becomes
\[
v_i^{\nu,-2} = v_i^f(\cdot) \ast_{sp} G_\nu - \sum_{j=1}^{D} v_j^{\nu,-1} \frac{\partial v_i^{\nu,-1}}{\partial x_j} \ast G_\nu
\]
\[- \sum_{j,m=1}^{D} \int_{\mathbb{R}^D} \left( \frac{\partial}{\partial x_i} K_{D,i}(\cdot, -y) \right) \sum_{j,m=1}^{D} \left( \frac{\partial v_i^{\nu,-1}}{\partial x_j} \frac{\partial v_i^{\nu,-1}}{\partial x_m} \right) (t, y) dy \ast G_\nu
\]=: v_i^f(\cdot) \ast_{sp} G_\nu + B_1 \ast G_\nu - L_1 \ast G_\nu.
\] (85)

where \( B_1 \) and \( L_1 \) denote abbreviations of the next order of approximation of the Burgers term and the Leray projection term. The weakest term in the recursive classical representation of
\[
\frac{\partial v_i^{\nu,-1}}{\partial x_j}(t, \cdot)
\] (86)
is
\[
v_{t,j}^f *_{sp} G_\nu = v_{t}^f *_{sp} G_{\nu,j}, \tag{87}
\]
which appears in a full (time and space) convolution in the next step. Hence we may use the \(\nu\)-independent upper bounds for \(G_{\nu,j}\). We have
\[
|v_{t,j}^f *_{sp} G_{\nu,j}| \leq c \beta_0^{-1}. \tag{88}
\]
Hence
\[
\left| \left( v_{t,j}^{\nu,-1} \frac{\partial v_{t,j}^{\nu,-1}}{\partial x_j} \right) (t,.) \right| \leq c t^{2\beta_0 - 1} \tag{89}
\]
and as \(\frac{\partial}{\partial x_j} K_{D,j}(\cdot, y) \sim \frac{1}{y^D}\) for \(D = 3\) we have
\[
\left| \int_{\mathbb{R}^D} K_{D,j}(\cdot, y) \sum_{j,m=1}^{D} \left( \frac{\partial v_{t,j}^{\nu,1}}{\partial x_j} \frac{\partial v_{t,m}^{\nu,1}}{\partial x_m} \right) (t,y)dy \right| \leq c t^{2(\beta_0 - 1) + 1}. \tag{90}
\]
Using the \(\nu\)-independent Gaussian estimates above with \(\Delta r := |x - y|\) for the Burgers term \(B^1\) we get
\[
\left| B^1 * D^2_\gamma G_{\nu}(t,.) \right| \leq c t^{2\beta_0 - 1 - |\gamma|}, \tag{91}
\]
and for the Leray projection term \(L^0\) we have
\[
\left| L^1 * D^2_\gamma G_{\nu}(t,.) \right| \leq c t^{2(\beta_0 - 1) + 1 - |\gamma|}. \tag{92}
\]
The parameter \(\beta_0\) can be chosen to be \(2.5 - \epsilon\) for \(\epsilon > 0\) small. Furthermore, if we use \(\nu\)-independent estimates of convolutions \(h * G_{\nu,j}\), then \(h\) is strongly Lipschitz continuous in the sense that \(|h(y) - h(y')| \leq c |y - y'|\). Similar estimates hold for the higher order increments \(\partial v_{t,\nu,j}^{\nu,1} \) for \(k \geq 3\).

ad iii) Given \(k \geq 1\) choose a number \(m\) is such that for \(t > 0\)
\[
\forall \ 0 \leq |\gamma| \leq m \ D^2_\gamma v_{t,j}^{\nu,1,\nu,k} (t,.) \text{is continuous and bounded.} \tag{93}
\]
For \(0 \leq |\gamma| \leq m\) and for \(|\beta| + 1 = |\gamma|\), \(\beta_j + 1 = \gamma_j\) (if \(|\gamma| \geq 1\)) we consider a representation of \(D^2_\gamma v_{t,j}^{\nu,1,\nu,k}\), \(1 \leq i \leq D\), \(k \geq 1\) of the form
\[
D^2_\gamma v_{t,j}^{\nu,1,\nu,k} = v_{t}^f *_{sp} D^2_\gamma G_{\nu} - D^2_\gamma \left( \sum_{j=1}^{D} v_{t,j}^{\nu,-k-1} \frac{\partial v_{t,j}^{\nu,-k-1}}{\partial x_j} \right) * G_{\nu,j} \tag{94}
\]
\[-D^2_\gamma \left( \sum_{j,m=1}^{D} \int_{\mathbb{R}^D} \left( \frac{\partial}{\partial x_j} K_{D,j}(\cdot, y) \right) \sum_{j,m=1}^{D} \left( \frac{\partial v_{t,j}^{\nu,1,\nu,k-1}}{\partial x_j} \frac{\partial v_{t,m}^{\nu,1,\nu,k-1}}{\partial x_m} \right) (t,y)dy \right) * G_{\nu,j}.\]
Here, recall \(G_{\nu}\) is the fundamental solution of the heat equation \(p_{t,\nu} - \nu \Delta p = 0\), \(*\) denotes the convolution, \(*_{sp}\) denotes the spatial convolution, and \(K_D\) denotes the fundamental solution of the Laplacian equation for dimension \(D \geq 3\). In the following the constant \(c > 0\) is generic. Note that for \(1 \leq i \leq D\) the initial data \(v_{t}^i\) have polynomial decay of any order at spatial infinity, i.e. we have for \(|x| \geq 1\)
\[
|v_{t}^i(x)| \leq \frac{c}{1 + |x|^2(D+1)+D+m} \tag{95}
\]
for some finite constant \( c > 0 \) and \( t \geq 0 \). Hence, for multiindices \( 0 \leq |\gamma| \leq m \) and \( t > 0 \) we have for some finite constant \( c > 0 \) and for \( |x| \geq 1 \)

\[
|v^f_t *_{sp} D^\gamma_x G_\nu(t, x)| \leq \frac{c}{1 + |x|^{2(D+1)}},
\]

(96)

Assuming inductively

\[
\forall \ l \leq k - 1 \ \forall 0 \leq |\gamma| \leq m \ \left| D^\gamma_x v^\nu_{t-j} \right| \leq \frac{c}{1 + |x|^{2(D+1)}}
\]

(97)

we have or some finite constant \( c > 0 \) and for \( |x| \geq 1 \)

\[
|D^\beta_x B^{k-1}| := \left| \sum_{j=1}^{D} D^\beta_x \left( v^\nu_{t-j} \frac{\partial v^\nu_{t-j}}{\partial x_j} \right)(t,.) \right| \leq \frac{c}{1 + |x|^{4(D+1)}},
\]

(98)

and

\[
|D^\beta_x L^{k-1}| \leq \frac{c}{1 + |x|^{4(D+3)}},
\]

(99)

where

\[
D^\beta_x L^{k-1} = \sum_{j,m=1}^{D} \int_{\mathbb{R}^D} \left( \frac{\partial}{\partial x_j} K_D(\cdot, - y) \right) \sum_{j,m=1}^{D} \left( D^\beta_x \left( \frac{\partial v^\nu_{t-j}}{\partial x_j} \frac{\partial v^\nu_{t-j}}{\partial x_m} \right)(t,y) dy.
\]

(100)

Convolutions with \( G_\nu \) or \( G_{\nu,1} \) weaken this polynomial decay by order \( D \) at most such that we (generously) get

\[
|D^\beta_x B^{k-1} * G_{\nu,j}| \leq \frac{c}{1 + |x|^{2(D+1)}}
\]

(101)

and

\[
|D^\beta_x L^{k-1} * G_{\nu,j}| \leq \frac{c}{1 + |x|^{2(D+1)}}.
\]

(102)

Hence using the representation (103) and (105), (101), (102) we get

\[
\forall \ l \leq k \ \forall 0 \leq |\gamma| \leq m \ \left| D^\beta_x v^\nu_{t-j} \right| \leq \frac{c}{1 + |x|^{2(D+1)}}
\]

(103)

and by (97) the same holds for the increments \( D^\beta_x \Delta v^\nu_{t-j} \).

Note that similar conclusions can be made using a vorticity iteration scheme \( \left( \omega^\nu_{t-k} \right)_{k \geq 0, \ 1 \leq i \leq D} \), where for \( k = 0 \)

\[
\omega^\nu_{t_0} = f_i, \text{ for } 1 \leq i \leq D,
\]

(104)

and where for \( k \geq 1 \) the function \( \omega^\nu_{t-k} \), \( 1 \leq i \leq D \) is determined as the time-local solution of the Cauchy problem

\[
\begin{cases}
\frac{\partial \omega^\nu_{t-k}}{\partial s} - \nu \Delta \omega^\nu_{t-k} = - \sum_{j=1}^{3} \frac{\partial v^\nu_{t-j}}{\partial x_j} \frac{\partial \omega^\nu_{t-k}}{\partial x_j}, \\
- \sum_{j=1}^{3} \frac{1}{2} \left( \frac{\partial v^\nu_{t-j}}{\partial x_j} + \frac{\partial v^\nu_{t-j}}{\partial x_k} \right) \omega^\nu_{t-k}, \\
\omega^\nu_{t-k}(0,.) = \omega^f_i,
\end{cases}
\]

(105)
on a domain \([0, T] \times \mathbb{R}^D\) for some time horizon \(T > 0\). Here, for \(k - 1 \geq 0\)

\[
u^{\nu, -k-1}(t, x) = \int_{\mathbb{R}^3} (K_3(y)\omega^{\nu, -k-1}(t, x - y)) \, dy,
\]  

(106)

by the Bio-Savart law.

Ad i) For \(k \geq 3\) we have \(\delta v_i^{\nu, -k}(t, \cdot) \in H^3_t \cap C^3\) corresponding to \(\delta \omega_i^{\nu, -k}(t, \cdot) \in H^2_t \cap C^2\). For convenience of the reader we do some explicit very simple computations concerning the recursive relation of increments. For the scheme \(\omega_i^{\nu, -k}(s, \cdot), k \geq 1\) with

\[
\omega_i^{\nu, -k}(s, \cdot) = f_i * \omega \nu + \left( \sum_{j=1}^{3} v_j^{\nu, -k-1} \frac{\partial \omega_i^{\nu, -k-1}}{\partial x_j} \right) * \omega \nu,
\]

(107)

we have

\[
\omega_i^{\nu, -k}(s, \cdot) - \omega_i^{\nu, -k-1}(s, \cdot) = \left( \sum_{j=1}^{3} v_j^{\nu, -k-1} \frac{\partial \omega_i^{\nu, -k-1}}{\partial x_j} \right) * \omega \nu
\]

\[
- \left( \sum_{j=1}^{3} v_j^{\nu, -k-1} \frac{\partial \omega_i^{\nu, -k-1}}{\partial x_j} \right) * \omega \nu
\]

\[
+ \left( \sum_{j=1}^{3} v_j^{\nu, -k-2} \frac{\partial \omega_i^{\nu, -k-1}}{\partial x_j} \right) * \omega \nu
\]

\[
+ \left( \sum_{j=1}^{3} v_j^{\nu, -k-1} \frac{\partial \omega_i^{\nu, -k-2}}{\partial x_j} \right) * \omega \nu
\]

\[
- \left( \sum_{j=1}^{3} v_j^{\nu, -k-2} \frac{\partial \omega_i^{\nu, -k-1}}{\partial x_j} \right) * \omega \nu
\]

\[
+ \left( \sum_{j=1}^{3} v_j^{\nu, -k-1} \frac{\partial \omega_i^{\nu, -k-2}}{\partial x_j} \right) * \omega \nu
\]

\[
= \left( \sum_{j=1}^{3} \left( v_j^{\nu, -k-1} - v_j^{\nu, -k-2} \right) \frac{\partial \omega_i^{\nu, -k-1}}{\partial x_j} \right) * \omega \nu
\]

\[
+ \sum_{j=1}^{3} v_j^{\nu, -k-2} \left( \omega_j^{\nu, -k-1} - \omega_j^{\nu, -k-1} \right) * \omega \nu
\]

\[
- \left( \sum_{j=1}^{3} \left( \frac{1}{2} \left( \omega_j^{\nu, -k-1} + \omega_j^{\nu, -k-1} - \delta \omega_j^{\nu, -k-1} \right) \frac{\partial \omega_i^{\nu, -k-1}}{\partial x_j} \right) * \omega \nu
\]

\[
- \left( \sum_{j=1}^{3} \frac{1}{2} \left( \omega_j^{\nu, -k-2} + \omega_j^{\nu, -k-2} \right) \frac{\partial \omega_i^{\nu, -k-1}}{\partial x_j} \right) * \omega \nu
\]

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Hence for the increments

\[ \delta \omega^\nu_{i,-k} := \omega^\nu_{i,-k} - \omega^\nu_{i,-k-1} \]  

(109)
satisfy the recursion

\[ \delta \omega^\nu_{i,-k} = \left( \sum_{j=1}^{n} \delta v^\nu_{i,-k-1}\frac{\partial \omega^\nu_{i,-k-1}}{\partial x_j} \right) * G_{ij} 
+ \left( \sum_{j=1}^{n} v^\nu_{i,-k-1}\frac{\partial \delta \omega^\nu_{i,-k-1}}{\partial x_j} \right) * G_{ij} 
- \left( \sum_{j=1}^{n} \frac{\partial v^\nu_{i,-k-1}}{\partial x_j} \right) \omega^\nu_{i,-k-1} * G_{ij} 
- \left( \sum_{j=1}^{n} \frac{\partial v^\nu_{i,-k-2}}{\partial x_j} \right) \delta \omega^\nu_{i,-k-1} * G_{ij}, \]  

(110)
where

\[ \delta v^\nu_{i,-k-1} := v^\nu_{i,-k-1} - v^\nu_{i,-k-2}. \]  

(111)

Elementary arguments (we considered similar arguments elsewhere) lead to

**Lemma 2.5.** There exists \( T > 0 \) and a constant \( c \in (0, 1) \) such that for all \( k \geq 3 \) we have

\[ \max_{1 \leq i \leq D, s \in [0, T]} |\delta \omega^\nu_{i,-k+1}(s, .)|_{H^3 \cap C^2} \leq c \max_{1 \leq i \leq D, s \in [0, T]} |\delta \omega^\nu_{i,-k}(s, .)|_{H^3 \cap C^2}, \]  

(112)
and for all \( k \geq 3 \)

\[ \max_{1 \leq i \leq D, s \in [0, T]} |\delta v^\nu_{i,-k+1}(s, .)|_{H^3 \cap C^3} \leq c \max_{1 \leq i \leq D, s \in [0, T]} |\delta v^\nu_{i,-k}(s, .)|_{H^3 \cap C^3}. \]  

(113)

ad v) Choose a time horizon \( T > 0 \) as in the previous step such that contraction holds for the higher order increments \( \delta v^\nu_{i,-k} \) with \( k \geq 3 \) as in (113). We choose the increment \( \delta v^\nu_{i,-2} \) obtained after \( k = 2 \) iterations of the scheme. We observed that there is an upper bound \( C > 0 \) independent of \( \nu \) such that

\[ |\delta v^\nu_{i,-2} + \sum_{l=3}^{\infty} \delta v^\nu_{i,-l}|_{H^3 \cap C^3} = |v^\nu_{i,-t}(., .) - v^\nu_{i,-t} *_{sp} G_{ij}(t, .)|_{H^3 \cap C^3} \leq C \]  

(114)
where \( C > 0 \) is independent of \( \nu > 0 \). For all \( \nu > 0 \) the function

\[ v^\nu_{i,-t}(., .) = v^\nu_{i,-t} *_{sp} G_{ij}(t, .) + \delta v^\nu_{i,-2} + \sum_{l=3}^{\infty} \delta v^\nu_{i,-l}, \]  

(115)
satisfies the Navier stokes equation on the time interval \([0, T]\) such that

\[ \nu \Delta v^\nu_{i,-} = \frac{\partial v^\nu_{i,-}}{\partial t} - \sum_{j=1}^{D} v^\nu_{i,-} \frac{\partial v^\nu_{i,-}}{\partial x_j} 
+ \sum_{j,m=1}^{D} \int_{R^d} \left( \frac{\partial}{\partial x_j} K_D(., y) \right) \sum_{j,m=1}^{D} \left( \frac{\partial v^\nu_{i,-}}{\partial x_j} \frac{\partial v^\nu_{i,-}}{\partial x_m} \right)(t, y)dy, \]  

(116)
where the right side of (116) is the time-reversed Euler equation operator applied to \( v_i^{\nu,-} \). Recall that the time horizon \( T > 0 \) is structurally independent of \( \nu \) in the contraction result. Next we may use the strong polynomial decay at spatial infinity in order to transform on a bounded domain. We choose a sequence \((\nu_p)_{p \geq 1}\) converging to zero and consider the spatial transformation

\[
\delta v_i^{e, \text{init}, -\nu_p, 2}(t, y) = \delta v_i^{\text{init}, -\nu_p, 2}(t, x)
\]

for \( y_j = \arctan(x_j), \ 1 \leq j \leq D \) and for all \( t \in [0, T] \). Note that for

\[
|\delta v_i^{\text{init}, -\nu_p, 2}(t, x)| \leq \frac{c}{1 + |x|^{2m}}
\]

and multiindices \( \gamma \) with \( 0 \leq |\gamma| \leq 3 + \epsilon \) for some \( \epsilon > 0 \) we have for all \( t \in [0, T] \) and all \( x \in \mathbb{R}^D \)

\[
|D_y \delta v_i^{e, \text{init}, -\nu_p, 2}(t, y)| \leq c_0(1 + |x|^{2m})|D_x \delta v_i^{\text{init}, -\nu_p, 2}(t, x)| \leq C
\]

for some finite constants \( c_0, C \). This implies

\[
\delta v_i^{e, \text{init}, -0, 2}(t, ) := \lim_{\nu_p \downarrow 0} \delta v_i^{e, \text{init}, -\nu_p, 2}(t, ) \in H^3 \cap C^3 \quad \text{for all } 0 \leq t \leq T,
\]

and as for some finite \( C > 0 \) independent of \( \nu_p \)

\[
\sup_{\nu_p > 0} \left(1 + |x|^{2m}\right) |D_y \delta v_i^{\text{init}, -\nu_p, 2}(t, x)| \leq C
\]

we conclude that for all \( t \in [0, T] \)

\[
\delta v_i^{e, \text{init}, -0, 2}(t, ) \in H^3 \cap C^3.
\]

The latter statement transfers to \( \delta v_i^{\text{init}, 0, -2}(t, ), \ 1 \leq i \leq D \). Similarly for the higher order increments \( \delta v_i^{\nu, -k}, \ 1 \leq i \leq D \) for \( k \geq 3 \). Hence the viscosity limit \( v_i^{-} \), \( 1 \leq i \leq D \) satisfies

\[
v_i^{-}(t, ) - v_i^f \ast_{sp} G_{\nu_p}(t, ) = \delta v_i^{-}(t, ) \in H^3 \cap C^3
\]

for all \( t \in [0, T], \) where indeed

\[
\delta v_i^{-} := \lim_{\nu_p \downarrow 0} \delta v_i^{\nu, -k} + \sum_{l=k}^{\infty} \delta v_i^{\nu, -l} \in C^1((0, T], H^3 \cap C^3).
\]

Next, we observe that the function \( v_i^{-}, \ 1 \leq i \leq D \) of the described regularity is a classical solution of the time-reversed Euler equation. Indeed, for some finite \( C > 0 \)

\[
\lim_{\nu_p \downarrow 0} \nu_p |\Delta \delta v_i^{-}(t, )|_{C^1 \cap H^1} \leq \lim_{\nu_p \downarrow 0} \nu_p \sup_{t} |\delta v_i^{\nu_p, -}(t, )|_{H^3 \cap C^3} \leq \nu_p C \downarrow 0 \text{ as } \nu_p \downarrow 0
\]

(125)
for the right side in \((116)\). Note that the set of continuous functions on \(\mathbb{R}^D\) which vanish at spatial infinite is closed. In our context of functions with very strong polynomial decay we may even transform to a bounded domain as we have observed. Hence we may consider norms \(\|f\|_{C^0} = \sup_x |f(x)|\) and similar norms \(\|f\|_{C^m}\) for derivatives up to order \(1 \leq m \leq 3\) in our context. Furthermore, we have

\[
\lim_{\nu_p \downarrow 0} \nu_p \|\Delta (v_i^f \ast_{sp} G_{\nu})(t, \cdot)\|_{C^0} = \lim_{\nu_p \downarrow 0} \nu_p^{1-\delta} \| \sum_j v_{i,j}^f \ast_{sp} \nu^\delta G_{\nu,j}(t, \cdot)\|_{C^0} \downarrow 0 \quad \text{as} \quad \nu_p \downarrow 0,
\]

where in the last step we may use the local upper bound

\[
|\int_{|x-y| \leq 1} (\sum_j v_{i,j}^f)(t, y) \nu^\delta G_{\nu,j}(t, x; 0, y)| \leq \int c_0 r x_{\nu}^{\delta_0 - (2 + \alpha_0)} \frac{c_1}{r |x-y|^{1-\delta}} dy 
\leq c_2 r^{\delta_0 + (2\delta - 1) - (2 + \alpha_0)} \quad \text{for} \quad 0 \leq r = \sqrt{x_1^2 + x_2^2 + x_3^2}
\]

in a ball \(B_1(x)\) of radius 1 around \(x\), \(\delta \in (0.5, 1)\), and for some finite constants \(c_0, c_1, c_2\) which are independent of \(\nu\). For \(\beta_0 \) close to \(2 + \alpha_0\) this integral is bounded. Note that this is also achieved for weaker parameter conditions (for \(\alpha_0, \beta_0\)) if \(\delta\) is chosen to be \(1 - \epsilon\) for small \(\epsilon > 0\). Note that the integral outside this ball clearly converges to zero as \(\nu_p \downarrow 0\). Here, note that we used the factor \(\nu^\delta\) for the latter conclusion in order to ensure that a standard estimate of the Gaussian is independent of \(\nu\). Similar arguments show that the velocity viscosity limit \(v^-\) has spatial regularity \(H^3 \cap C^4\) for positive time corresponding to a spatial vorticity regularity \(H^2 \cap C^2\) for positive time \(t > 0\). We conclude that the original Euler equation develops in opposite time direction from data at time \(t = 0\) (corresponding to time \(s = T\) of the reversed-time Euler equation) a weak singularity at time \(T > 0\) (corresponding to data at time \(s = 0\) the time-reversed Euler equation).

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