EQUIVARIANT COHOMOLOGICAL RIGIDITY OF CERTAIN $T$-MANIFOLDS

SOUMEN SARKAR AND JONGBAEK SONG

Abstract. We introduce the category of locally $k$-standard $T$-manifolds which includes well-known classes of manifolds such as toric and quasitoric manifolds, good contact toric manifolds and moment-angle manifolds. They are smooth manifolds with well-behaved actions of tori. We study their topological properties, such as fundamental groups and equivariant cohomology algebras. Then, we discuss when the torus equivariant cohomology algebra distinguishes them up to weakly equivariant homeomorphism.

1. Introduction

The interaction among algebra, combinatorics, geometry and topology is blossoming in the garden of mathematics. It often comes through group actions on topological or geometric objects. Depending on the hypothesis on group actions on spaces, one associates various types of combinatorial objects to analyze their algebraic, geometric and topological properties. Amongst a variety of examples along the lines of this philosophy, the spaces equipped with torus actions have been intensively studied for the last few decades and the following interesting bridges have been found.

Symplectic manifolds with Hamiltonian torus actions have convex polytopes as their images of moment maps \cite{Ati82, GS82}. In addition, if the dimension of the torus is the half of the dimension of a given manifold, then the image of the moment map is given by a Delzant polytope, and such a correspondence is bijective up to some equivalent relations \cite{Del88}. GKM manifolds are related to certain graphs with labels on edges encoding the tangential representation at each fixed point \cite{GKM98, GZ01}. A normal separated toric variety defines a collection of cones in a vector space, which is called a fan, and vice versa \cite{CLS11, Ful93, Od88}. Also, the pioneering paper \cite{DJ91} initiated a topological generalization of smooth projective toric varieties and studies them using the combinatorics of orbit spaces and group action data. These are even dimensional spaces with effective torus actions with nonempty fixed points such that dimensions of tori are less than or equal to the half of the dimensions of the given spaces. We refer to \cite{BT19b, BT19a} for the theory of $(2n, k)$-manifolds which puts together a wide class of $2n$-dimensional manifolds $M$ endowed with actions of $k$-dimensional tori $T$ such that $2 \dim T \leq \dim M$. The manifolds listed above are examples of $(2n, k)$-manifolds.

On the other hand, there are classes of manifolds beyond the above list of examples. For instance, compact connected contact toric manifolds of dimension...
greater than 3 with non-free torus actions are classified by cones satisfying certain conditions \cite{L03}. Moment-angle complexes and partial quotients have strong relationships with simplicial complexes \cite{Fra19, BP15}. We note that these examples are spaces $M$ with effective actions of tori $T$ such that $2 \dim T > \dim M$ and they have no fixed points.

In this paper, we generalize some of the classes of manifolds listed above and call them \textit{locally $k$-standard $T$-manifolds}, which are $(2n+k)$-dimensional manifolds with effective actions of $(n+k)$-dimensional tori satisfying certain local property. The idea of a locally $k$-standard $T$-manifold is motivated by the work of Davis–Januszkiewicz \cite{DJ91}, where they consider the standard $T^n$-action on $\mathbb{C}^n$ as a local model to define toric manifolds which are also called quasitoric manifolds in recent literature. Here, we consider $T^{n+k}$-action on $\mathbb{C}^n \times T^k$ which is an invariant subset of $\mathbb{C}^{n+k}$ with respect to the standard $T^{n+k}$-action and adopt this as the local model to define a locally $k$-standard $T$-manifold, see Definition 2.1. This category of manifolds includes toric and quasitoric manifolds \cite{DJ91, BP02}, locally standard torus manifolds \cite{MP06}, compact connected contact toric manifolds associated with good cones \cite{L03} and moment-angle manifolds \cite{BP02}, as well as infinitely many objects outside these categories which might be interesting in nature.

In this paper, we are primarily interested in locally $k$-standard $T$-manifolds whose orbit spaces are simple polytopes. We study their topological properties and equivariant classification via equivariant cohomology (or Borel equivariant cohomology). For a $G$-space $X$, the equivariant cohomology $H^*_G(X)$ is defined by the cohomology of the homotopy quotient, namely

$$H^*_G(X) := H^*(EG \times_G X),$$

where $EG$ is the total space of the universal principal $G$-bundle and $EG \times_G X$ denotes the orbit space of the diagonal $G$-action on $EG \times X$. The equivariant collapsing map $X \to \{pt\}$ induces an $H^*(BG)$-algebra structure on $H^*_G(X)$, which carries various information about the $G$-action on $X$.

Indeed, for the case of quasitoric manifolds or toric hyper-Kähler manifolds, somewhat surprisingly, the equivariant cohomology algebra distinguishes them up to weakly equivariant homeomorphisms or diffeomorphisms, see \cite{K11, M08}. Here, a weakly equivariant homeomorphism means a homeomorphism $\Psi : X \to Y$ between two $G$-spaces $X$ and $Y$ together with an automorphism $\delta$ of $G$ such that $\Psi(g \cdot x) = \delta(g) \cdot \Psi(x)$ for any $g \in G$ and $x \in X$. A weak isomorphism between two $R$-algebras for a ring $R$ is defined similarly. Now, one may ask the following:

\textbf{Question 1.1.} What are $G$-spaces $X$ and $Y$ such that they are weakly equivariantly homeomorphic, whenever their equivariant cohomologies are weakly isomorphic as $H^*(BG)$-algebras?

We call this question the \textit{equivariant cohomological rigidity} and in this paper, we give an affirmative answer to this question for a wide class of locally $k$-standard $T$-manifolds.

We begin Section 2 with the axiomatic and the constructive definition of a locally $k$-standard $T$-manifold and show these two definitions are equivalent when its orbit space is a simple polytope, see Corollary 2.4. Then we observe when a locally $k$-standard $T$-manifold can be constructed as a quotient of a moment-angle manifold, see Proposition 2.7. We investigate their fundamental groups in Lemma 2.5, which extends the result of \cite[Theorem 1.1]{L04}. This lemma helps to prove the main
Section 4 is devoted to studying the equivariant cohomology algebra of a locally $k$-standard $T$-manifold $M$. If a $T$-manifold has fixed points like GKM-manifolds, then one can use the localization technique to analyze the equivariant cohomology. Indeed, this is the case for the equivariant cohomological rigidity theorems given in [Mas08] and [Kur11]. However, a locally $k$-standard $T$-manifold may not have fixed points if $k \geq 1$. Hence, it is difficult to use the classical localization results to analyze the equivariant cohomology of $M$. Fortunately, using the invariant submanifolds corresponding to the faces of the orbit space, we can determine the generators and relations for the equivariant cohomology ring $H^*_T(M)$. Then, under a mild hypothesis, we realize the ring $H^*_T(M)$ as the Stanley–Reisner ring of the orbit space of $M$, see Theorem 3.5. We also investigate $H^*(BT)$-algebra structure of $H^*_T(M)$ which turns out to encode the complete information about the $T$-action on $M$.

Finally, in Section 4 we give a concrete answer to the equivariant cohomological rigidity about locally $k$-standard $T$-manifolds which generalizes the result of [Mas08] Section 4.

For convenience, we often use the following well-known identifications without explicitly mentioning it.

- $\mathbb{Z}^{n+k} \cong t_{\mathbb{Z}} \cong H_2(BT^{n+k}) \cong \text{Hom}(S^1, T^{n+k})$;
- $\mathbb{Z}^{n+k}^* \cong t_{\mathbb{Z}}^* \cong H^2(BT^{n+k}) \cong \text{Hom}(T^{n+k}, S^1)$,

where $t_{\mathbb{Z}} = \text{ker}(\exp: t \rightarrow T^{n+k})$ and all cohomologies in this paper are considered with integer coefficients.

## 2. Locally $k$-standard $T$-manifolds

Let $T$ be a torus. We consider manifolds with $T$-actions, which are denoted by $T$-manifolds. In this section, we introduce the category of locally $k$-standard $T$-manifolds and study their essential properties which can be encoded by some combinatorial data. This category of manifolds generalizes the concept of toric manifolds introduced in [DJ91] by extending the idea of locally standard $T$-action for a toric manifold.

### 2.1. Axiomatic definition

Consider the action $\alpha: T^{n+k} \times \mathbb{C}^{n+k} \rightarrow \mathbb{C}^{n+k}$ of $(n+k)$-dimensional torus $T^{n+k}$ on $\mathbb{C}^{n+k}$ defined by

$$\alpha((t_1, \ldots, t_n, \ldots, t_{n+k}),(z_1, \ldots, z_n, \ldots, z_{n+k})) = (t_1z_1, \ldots, t_nz_n, \ldots, t_{n+k}z_{n+k}).$$

Then the set $\mathbb{C}^n \times T^k$ is a $T^{n+k}$-invariant subset of $\mathbb{C}^{n+k}$, and the orbit space $(\mathbb{C}^n \times T^k)/T^{n+k}$ is $\mathbb{R}_+^n$, the positive orthant. We call the restriction $\alpha|_{T^{n+k} \times (\mathbb{C}^n \times T^k)}$ the $k$-standard $T^{n+k}$-action on $\mathbb{C}^n \times T^k$.

**Definition 2.1.** A $(2n + k)$-dimensional smooth manifold $M$ with an effective $T^{n+k}$-action is called a locally $k$-standard $T$-manifold if it is locally isomorphic to $\mathbb{C}^n \times T^k$ with the $k$-standard $T^{n+k}$-action. Here, 'locally isomorphic' means for each point $p$ of $M$, there is

1. an automorphism $\theta_p \in \text{Aut}(T^{n+k})$;
2. a $T^{n+k}$-invariant neighborhood $U \subseteq M$ of $p$ which is $\theta_p$-equivariantly diffeomorphic to a $T^{n+k}$-invariant subset $V \subseteq \mathbb{C}^n \times T^k$. 

-
Since $T^{n+k}$-action is locally $k$-standard and transversality is a local property, we get that the orbit space $M/T^{n+k}$ is a nice manifold with corners of dimension $n$. In this paper, we are primarily interested in the locally $k$-standard $T$-manifolds whose orbit spaces are simple polytopes. Basic properties of simple polytopes can be found in [Zie95, BP15].

Let $q: M \to P$ be the orbit map where $P$ is an $n$-dimensional simple polytope. Let $F(P) := \{F_1, \ldots, F_m\}$ be the set of codimension-1 faces, called facets, of $P$. Then each $M_i := π^{-1}(F_i)$ is a $(2(n-1)+k)$-dimensional $T^{n+k}$-invariant submanifold of $M$. From the locally $k$-standardness and [DJ91 Lemma 1.3], we can show that $M_i$ is a $(2(n-1)+k)$-dimensional locally $k$-standard $T$-manifold over $F_i$. Therefore, the isotropy subgroup of $M_i$ is a circle subgroup $T_i$ of $T^{n+k}$. The group $T_i$ is uniquely determined by a primitive vector $λ_i \in \mathbb{Z}^{n+k}$. That is, we get a natural function

\begin{equation}
λ: \{F_1, \ldots, F_m\} \to \mathbb{Z}^{n+k}
\end{equation}

defined by $λ(F_i) = λ_i$.

Since each vertex $v$ of $P$ is the transversal intersection of $n$ facets $\{F_{i_1}, \ldots, F_{i_n}\}$, the manifolds $M_{i_1}, \ldots, M_{i_n}$ intersect transversely by the locally $k$-standardness. This implies that the submodule $A$ of $\mathbb{Z}^{n+k}$ generated by $\{λ_{i_1}, \ldots, λ_{i_n}\}$ corresponding to the $n$-dimensional subtorus $T_{i_1} \times \cdots \times T_{i_n}$ of $T^{n+k}$ is a direct summand of $\mathbb{Z}^{n+k}$. Indeed, there exists a primitive vectors $λ_{i_{n+1}}, \ldots, λ_{i_{n+k}} \in \mathbb{Z}^{n+k}$ such that the rank of $A \oplus \{λ_{i_{n+1}}, \ldots, λ_{i_{n+k}}\}$ is $n + k$ and the volume determined by $\{λ_{i_1}, \ldots, λ_{i_n}, λ_{i_{n+1}}, \ldots, λ_{i_{n+k}}\}$ in $\mathbb{R}^{n+k}$ is equal to the volume determined by $\{λ_{i_1}, \ldots, λ_{i_n}\}$ in $A \otimes \mathbb{R}$, since the vectors $λ_{i_{n+1}}, \ldots, λ_{i_{n+k}}$ determines the orbit of a point in $q^{-1}(v)$ which is a $k$-dimensional torus diffeomorphic to $0 \times T^k \subset \mathbb{C}^n \times T^k$.

Therefore, the locally $k$-standardness of $M$ implies that the set $\{λ_{i_1}, \ldots, λ_{i_n}\}$ is a part of a $\mathbb{Z}$-basis in $\mathbb{Z}^{n+k}$.

2.2. Constructive definition. In this subsection, we introduce a generalized hyper characteristic function $ξ$ on a simple polytope $P$ of dimension $n$ and discuss the construction of a $T^{n+k}$-manifold associated with $P$ and $ξ$. We denote by $F(P)$ the set of facets and $V(P)$ the set of vertices of $P$.

**Definition 2.2.** A function $ξ: F(P) \to \mathbb{Z}^{n+k}$ is called a generalized hyper characteristic function if $ξ$ satisfies the following:

\[\{ξ_{j_1}, \ldots, ξ_{j_m}\}\]

\[\{ξ_{j_1}, \ldots, ξ_{j_m}\}\]

\[\{ξ_{j_1}, \ldots, ξ_{j_m}\}\]

\[\{ξ_{j_1}, \ldots, ξ_{j_m}\}\]

where $ξ_j := ξ(F_j)$ for $j = 1, \ldots, m$.

We denote by $\text{im}(ξ)$ the module generated by $\{ξ_1, \ldots, ξ_m\}$ and by $\text{rk}(ξ)$ its rank. In this case, $n \leq \text{rk}(ξ) \leq n + k$ because of the hypothesis (2.1). We remark that the function $ξ$ is called a characteristic function for the case $k = 0$ and hyper characteristic function if $k = 1$. We refer to [DJ91] and [SS18] respectively. For simplicity, we call $ξ$ in Definition 2.2 a hyper characteristic function and call $(P, ξ)$ a hyper characteristic pair. We note that the map $λ$ in (2.1) is a hyper characteristic function. As it is constructed from a locally $k$-standard $T$-manifold, we prefer to distinguish $λ$ and $ξ$ until Corollary 2.4.

Now, we construct a $(2n+k)$-dimensional manifold with $T^{n+k}$-action as follows. For a point $x ∈ P$, let $F_{j_1} \cap \cdots \cap F_{j_k}$ be the face of $P$ containing $x$ in its relative interior. Then, we denote by $T_x$ the subgroup of $T^{n+k}$ determined by $\{ξ_{j_1}, \ldots, ξ_{j_k}\}$. 

If $x$ belongs to the relative interior of $P$, we define $T_x$ to be the identity in $T^{n+k}$.

We consider the following identification space

\[(2.2) \quad M(P, \xi) := (T^{n+k} \times P)/\sim,\]

where

\[(2.3) \quad (t, p) \sim (s, q) \text{ if and only if } p = q \text{ and } t^{-1}s \in T_p.\]

Here, $T^{n+k}$ acts on $M(P, \xi)$ induced by the multiplication on the first factor of $T^{n+k} \times P$.

**Proposition 2.3.** Let $(P, \xi)$ be a hyper characteristic pair. Then the space $M(P, \xi)$ as in (2.2) is locally isomorphic to $\mathbb{C}^n \times T^k$.

**Proof.** Let $q: M \to P$ be the orbit map. Let $U_v$ be the open subset of $P$ obtained by deleting all faces of $P$ not containing the vertex $v \in V(P)$. Let $v = F_1 \cap \cdots \cap F_m$ for a unique collection of facets $\{F_1, \ldots, F_m\}$ of $P$. Since $P$ is a simple polytope, there is a homeomorphism

\[ f: \mathbb{R}^n_+) \to U_v \]

as manifold with corners such that the facet $\{x_\ell = 0\}$ of $\mathbb{R}^n_+$ maps onto $F_\ell$ for $\ell = 1, \ldots, n$. Then the set $\{\xi_1, \ldots, \xi_n\}$ is a part of a basis by the definition of $\xi$.

Now we choose $\zeta_1, \ldots, \zeta_k \in \mathbb{Z}^{n+k}$ such that $\{\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_k\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^{n+k}$. Then, we get a diffeomorphism $\theta: T^{n+k} \to T^{n+k}$ determined by the linear map sending $e_\ell$ to $\xi_\ell$ for $\ell = 1, \ldots, n$ and $e_{n+r}$ to $\zeta_r$ for $r = 1, \ldots, k$, where $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+k}\}$ is the standard basis of $\mathbb{Z}^{n+k}$. Therefore we get the following commutative diagram

\[
\begin{array}{ccc}
T^{n+k} \times \mathbb{R}^n_+ & \xrightarrow{\theta \times 1} & T^{n+k} \times U_v \\
\downarrow & & \downarrow \\
(T^{n+k} \times \mathbb{R}^n_+)/\sim_s & \xrightarrow{\hat{f}} & (T^{n+k} \times U_v)/\sim_s
\end{array}
\]

where $\sim_s$ is similarly defined as the relation $\sim$ in (2.3) using $\{e_1, \ldots, e_n\}$ as a hyper characteristic function on facets of $\mathbb{R}^n_+$. So the map $\hat{f}$ is a homeomorphism. The space $(T^{n+k} \times \mathbb{R}^n_+)/\sim_s$ is homeomorphic to $\mathbb{C}^n \times T^k$. Therefore, $M(P, \xi)$ is a topological manifold locally isomorphic to $\mathbb{C}^n \times T^k$, since $P = \bigcup_{v \in V(P)} U_v$. \qed

Note that the function $\lambda$ defined in (2.1) satisfies the condition of a hyper characteristic function. Proposition 2.3 together with similar arguments in the proofs of [DJ91, Lemma 1.4, Proposition 1.8] gives the following corollary.

**Corollary 2.4.** Let $M$ be a locally $k$-standard $T$-manifold with dim $M = 2n + k$ such that $M/T^{n+k}$ is a simple polytope $P$ and $\lambda$ is a function as defined in (2.1). Then, $M$ is equivariantly homeomorphic to $M(P, \lambda)$ as defined in (2.2).

We remark that $M$ induces a smooth structure on $M(P, \lambda)$ due to Corollary 2.4. Therefore, $M(P, \lambda)$ is diffeomorphic to a locally $k$-standard $T$-manifold. We also note that the orientation of $M(P, \lambda)$ can be induced from the orientation of $P$ and $T^{n+k}$.

**Lemma 2.5.** Let $M$ be a locally $k$-standard $T$-manifold with a hyper characteristic function $\lambda$ as defined in (2.1). If $\text{rk}(\lambda) = n + k$, then the fundamental group $\pi_1(M)$ is a finite abelian group.
Proof. Let \( q: T^{n+k} \times P \to M \) be the quotient map for the equivalence relation \( \sim \) in (2.3). Then it follows that \( q^{-1}(x) \) is connected for all \( x \in M \). Also we have that \( T^{n+k} \times P \) is locally path-connected and \( M \) is semi-locally simply connected since it is locally \( \mathbb{C}^n \times T^k \). Then \cite[Theorem 1.1]{CGM12} gives a surjective map

\[
\pi_1(q): \pi_1(T^{n+1} \times P) \to \pi_1(M).
\]

Since \( P \) is contractible, \( \pi_1(T^{n+k} \times P) = \pi_1(T^{n+k}) \).

Let \( S^1(\lambda_i) \) be the circle subgroup of \( T^{n+k} \) determined by \( \lambda_i \) for \( i = 1, \ldots, m \). So each \( S^1(\lambda_i) \) is a loop in \( T^{n+k} \) containing the identity. Let \( \omega_i \in \pi_1(T^{n+k}) \) represent this loop for \( i = 1, \ldots, m \). Then, \( \omega_i = \lambda_{i,1} e_1 + \cdots + \lambda_{i,n+k} e_{n+k} \) with respect to the standard generators \( \{ e_1, \ldots, e_{n+k} \} \) of \( \pi_1(T^{n+k}) \), where we denote \( \lambda_i := (\lambda_{i,1}, \ldots, \lambda_{i,n+k}) \in \mathbb{Z}^{n+k} \). Since \( \text{rk}(\lambda) = n+k \), the quotient \( \pi_1(T^{n+k})/\langle \omega_1, \ldots, \omega_m \rangle \) is a finite abelian group. Under the quotient map \( q \), the circle \( S^1(\lambda_i) \) collapses to a point in \( M \). So \( \pi_1(q)(\omega_i) \) is the identity in \( \pi_1(M) \) for \( i = 1, \ldots, m \). Hence, the lemma follows. \( \square \)

Above lemma extends \cite[Theorem 1.1]{Ler04} which is about the fundamental group of contact toric manifolds of Reeb type.

**Proposition 2.6.** Let \( P \) and \( \xi \) be as above. Then, \( \text{rk}(\xi) = n+r \) for some \( 0 \leq r < k \) if and only if \( M(P, \xi) \) is homeomorphic to \( N(P, \xi) \times T^{k-r} \) for some locally \( r \)-standard \( T \)-manifold \( N(P, \xi) \) of dimension \( 2n+r \) with \( \text{rk}(\xi) = n+r \).

**Proof.** Assume that \( \text{rk}(\xi) = n+r \) for some \( 0 \leq r < k \). Let

\[
M(\xi) := (\langle \xi_1, \ldots, \xi_m \rangle \otimes \mathbb{R}) \cap \mathbb{Z}^{n+k}.
\]

Then, \( M(\xi) \) is a direct summand of \( \mathbb{Z}^{n+k} \), hence we have \( \mathbb{Z}^{n+k} \cong M(\xi) \oplus M(\xi)^\perp \). This induces a decomposition \( T^{n+k} \cong T^{n+r}_\xi \times T^\perp_\xi \), where \( T^{n+r}_\xi \) and \( T^\perp_\xi \) are tori of dimension \( n+r \) and \( k-r \), respectively.

Now, we have

\[
M(P, \xi) = (T \times P)/\sim \cong ((T^{n+r}_\xi \times T^\perp_\xi) \times P)/\sim = (T^{n+r}_\xi \times P)/\sim \times T^\perp_\xi,
\]

where the last equality follows because the equivalent relation \( \sim \) does not identify \( T^\perp_\xi \). Here, we define \( \xi \) by the composition

\[
\begin{align*}
\mathcal{F}(P) \xrightarrow{\xi} \mathbb{Z}^{n+k} \xrightarrow{\cong} M(\xi) \oplus M(\xi)^\perp \xrightarrow{pr_1} M(\xi).
\end{align*}
\]

This establishes “only if” part of the statement.

Conversely, assume that \( M(P, \xi) \) is equivariantly homeomorphic to \( T^{k-r} \times N(P, \xi) \) for some locally \( r \)-standard \( T \)-manifold \( N(P, \xi) \) with \( \text{rk}(\xi) = n+r \). Observe that

\[
\pi_1(T^{k-r} \times N(P, \xi)) \cong \mathbb{Z}^{k-r} \times G
\]

for \( G = \pi_1(N(P, \xi)) \) which is a finite abelian group by Lemma \ref{lem:2.5}. Therefore, if \( \text{rk}(\xi) \neq n+r \), then the proof of “only if” part and Lemma \ref{lem:2.5} contradicts (2.3). Hence, we conclude \( \text{rk}(\xi) = n+r \). \( \square \)
Because of Proposition 2.6 it suffices to assume \(k \in \{0, \ldots, m-n\}\), since the maximum of \(\text{rk}(\xi)\) is \(m\), namely the number of facets of \(P\). In particular, if the hyper characteristic function \(\xi\) is given by \(\xi(F_i) = e_i\) which is the \(i\)-th standard unit vector in \(\mathbb{Z}^m\), the resulting space \(M(P, \xi)\) is called the moment-angle manifold which we discuss below.

Let \(x\) be a point in the relative interior of \(F_{j_1} \cap \cdots \cap F_{j_s}\). We write

\[
\tilde{T}_x := \{(t_1, \ldots, t_m) \in T^m \mid t_i = 1 \text{ for } i \notin \{j_1, \ldots, j_s\}\}.
\]

When \(x\) belongs to the relative interior of \(P\), we define \(\tilde{T}_x\) to be the identity in \(T^m\). One can define the identification space

\[
Z_P := (T^m \times P)/_{\sim_z}
\]

which is called the moment-angle manifold associated to \(P\), see for instance [BP15 Chapter 6]. Here, \((t, p) \sim_z (s, q)\) if and only if \(p = q\) and \(t^{-1}s \in \tilde{T}_p\). Notice that \(Z_P\) is equipped with a \(T^m\)-action given by the coordinate multiplication on the first factor of \(T^m \times P\). This makes \(Z_P\) a locally \((m-n)\)-standard \(T\)-manifold.

Assume that \(\text{rk}(\xi) = n + k\). Then, regarding \(\xi\) as a matrix of size \((n + k) \times m\) by listing \(\xi_1, \ldots, \xi_m\) as its column vectors, we have a short exact sequence

\[
1 \longrightarrow \ker(\exp \xi) \longrightarrow T^m \overset{\exp\xi}{\longrightarrow} T^{n+k} \longrightarrow 1.
\]

Now, we consider the space

\[(2.5)\]

\[
X(P, \xi) := Z_P / \ker(\exp \xi),
\]

where the action of \(\ker(\exp \xi)\) factors through \(T^m\). The torus \(T^{n+k} \cong T^m / \ker(\exp \xi)\) acts on \(X(P, \xi)\) residually. Note that the condition \((\ast)\) in Definition 2.2 implies that \(\ker(\exp \xi) \cap T^m_p = \{id\}\) for every \(p \in P\). Hence, \(\ker(\exp \xi)\) acts freely on \(Z_P\).

The proof of following proposition is same as the standard argument in toric topology, for instance see [BP15 Proposition 7.2.1].

**Proposition 2.7.** Let \((P, \xi)\) be a hyper characteristic pair with \(\text{rk}(\xi) = n + k\). Then there is a weakly equivariant homeomorphism between \(M(P, \xi)\) in (2.2) and \(X(P, \xi)\) in (2.5).

Here the “weakness” of equivariant homeomorphism arises by the existence of automorphism between the standard torus \(T^{n+k}\) with residual torus \(T^m / \ker(\exp \xi)\).

We now exhibit several classes of examples of locally \(k\)-standard \(T\)-manifolds. When \(k = 0\) in Definition 2.1 the resulting category of manifolds is introduced in [DJ91] which are called quasitoric manifolds. The case where \(k = |F(P)| - \dim P = m - n\) contains all moment angle manifolds associated to simple polytopes as we discussed above. When \(k = 1\), the category of \(M(P, \xi)\) contains good contact toric manifolds introduced in [Ler03]. See Example 2.8.

**Example 2.8 (Good contact toric manifolds).** Let \(P\) be an \(n\)-dimensional simple lattice polytope embedded in \(\mathbb{R}^{n+1} \setminus \{0\}\). Consider the cone \(C(P)\) on \(P\) with apex \(0 \in \mathbb{R}^{n+1}\), and the set \(\{\tilde{F} \mid F \in F(P)\}\) of facets of \(C(P)\), where \(\tilde{F} := C(F) \setminus \{0\}\). Now, define a function \(\xi: F(P) \rightarrow Z^{n+1}\) by \(\xi(F)\) to be the primitive outward normal vectors of \(\tilde{F}\), and assume that \(\xi\) satisfies the condition \((\mathfrak{c})\). Then, the resulting space \(M(P, \xi)\) is \(T^{n+1}\)-equivariantly homeomorphic to a good contact toric manifold whose moment cone is \(C(P)\).
8 S. SARKAR AND J. SONG

Figure 1. Hyperplane cut of a quasitoric manifold; (A) Characteristic pair of a quasitoric manifold, (B) Hyper characteristic pair of a locally 1-standard $T$-manifold of dimension 5.

Example 2.9 (Hyperplane cut of a quasitoric manifold). Let $X$ be a $2n$-dimensional quasitoric manifold which is a locally 0-standard $T^n$-manifold and $q: X \to Q$ be the associated orbit map. Let $H$ be a hyperplane in $\mathbb{R}^n$ which does not contain any vertex of $Q$. Since $Q$ is an $n$-dimensional simple polytope, $P := Q \cap H$ is an $(n-1)$-dimensional simple polytope. Then $q^{-1}(P)$ is a $T^n$-invariant subspace of $X$.

Note that if $v$ is a vertex of $P$, then $v$ is the intersection of an edge $e$ and $H$. Let $P_v$ be the open subset of $P$ obtained by deleting all faces of $P$ not containing $v$ and $Q_v$ be the open subset of $Q$ obtained by deleting all faces of $Q$ not containing the edge $e$. Then $Q_v$ is homeomorphic to $P_v \times \hat{e}$ as manifold with corners where $\hat{e}$ is the relative interior of $e$. So $q^{-1}(Q_v) = q^{-1}(P_v) \times \hat{e}$. Since $P_v$ and $\hat{e}$ intersect transversally, $q^{-1}(P_v)$ is a codimension-1 submanifold of $q^{-1}(Q_v)$. Therefore, $q^{-1}(P)$ is a $(2n-1)$-dimensional manifold with an effective $T^n$-action which satisfies the condition of Definition 2.1. Hence $q^{-1}(P)$ is a locally 1-standard $T$-manifold of dimension $2n-1$. We observe this in the following particular case.

Consider $Q$ and $P$ as described in Figure 1(A) and (B) respectively. The space $q^{-1}(P)$ is a $S^1$-bundle over $\mathbb{C}P^2 \# \mathbb{C}P^2$. We recall that good contact toric manifolds bijectively correspond to the moment cones see [Ler03]. Here, hyper characteristic vectors on $P$ do not satisfy the condition of a moment cone. So $q^{-1}(P)$ is not a contact toric manifold, but in the category of the case where $k = 1$.

A similar procedure as in Example 2.9 applies to a locally $k$-standard $T$-manifold, which provides a new locally $(k+1)$-standard $T$-manifold which is of codimension 1 in the original manifold.

3. Equivariant cohomology of locally $k$-standard $T$-manifolds

In this section, we study the equivariant cohomology algebra of a locally $k$-standard $T$-manifold $M(P, \xi)$. For simplicity, we write $M := M(P, \xi)$ and $T := T^{n+k}$ throughout the remaining part of this paper.

3.1. $H_T^*(M)$ as a ring. Consider the set $V(P) := \{v_1, \ldots, v_\ell\}$ of vertices of $P$ and write $v_i := F_{i_1} \cap \cdots \cap F_{i_n}$, the intersection of $n$ facets $F_{i_1}, \ldots, F_{i_n}$. We denote by
For a vertex \( v_i \in V(P) \) the constructive definition of \( M \) shows \( \pi^{-1}(v_i) = T_{v_i}^\perp \times v_i \).

Identifying \( H_2(BT) \) with the lattice \( \langle e_1, \ldots, e_{n+k} \rangle \) generated by the standard unit vectors \( \{e_1, \ldots, e_{n+k}\} \) of \( \mathbb{Z}^{n+k} \), we have \( H_2(BT_{v_i}) \cong \langle \xi_1, \ldots, \xi_n \rangle \), the sublattice of \( \mathbb{Z}^{n+k} \) generated by \( \{\xi_1, \ldots, \xi_n\} \). It is a direct summand of \( \mathbb{Z}^{n+k} \) by the hypothesis (\( \clubsuit \)). Hence, we get

\[
H_2(BT) \cong H_2(BT_{v_i}) \oplus N_{v_i}
\]

for some \( k \)-dimensional \( \mathbb{Z} \)-submodule \( N_{v_i} \) of \( H_2(BT) \). This gives us the following identification of cohomology groups

\[
H^2(BT_{v_i}) \cong H^2(BT)/\text{Ann}(H_2(BT_{v_i})),
\]

which is also isomorphic to the annihilator \( \text{Ann}(N_{v_i}) \) of \( N_{v_i} \).

**Lemma 3.1.** \( H^*_T(T_{v_i}^\perp \times v_i) \cong H^*(BT_{v_i}) \).

**Proof.** Hypothesis (\( \clubsuit \)) gives that (3.1) is a split exact sequence, which yields an identification \( T \cong T_{v_i} \times T_{v_i}^\perp \). Hence, we have

\[
H^*_T(T_{v_i}^\perp \times v_i) = H^*(ET \times_T T_{v_i}^\perp) = H^*(ET_{v_i} \times_{T_{v_i}} (ET_{v_i}^\perp \times_T T_{v_i}^\perp)) = H^*(BT_{v_i}),
\]

where the third equality holds, because \( ET_{v_i}^\perp \) is contractible and \( T_{v_i}^\perp \) acts freely on \( T_{v_i} \). \( \square \)

Recall from Section[2] that each facet \( F_j \in \mathcal{F}(P) \) is associated with a codimension 2 submanifold \( M_j \) which is fixed by the circle subgroup of \( T \) generated by \( \xi_j \in \mathbb{Z}^{n+k} \cong \text{Hom}(S^1, T) \). Let \( \tau_j \in H_T^2(M) \) be the equivariant Thom class of the normal bundle \( \nu(M_j) \) of \( M_j \) in \( M \). To be more precise, it is the image of the identity in \( H_T^2(M_j) \) via the equivariant Gysin homomorphism \( H_T^2(M_j) \to H_T^{*+2}(M) \).

Now, we consider the map

\[
f_{v_i} : H_T^*(M) \to H_T^*(T_{v_i}^\perp)
\]

induced from the inclusion \( T_{v_i}^\perp \times v_i \hookrightarrow M \). Here we denote by \( \tau_j|_{v_i} \) the image of \( \tau_j \) via \( f_{v_i} \). It satisfies the following property:

**Lemma 3.2.** For a vertex \( v_i = F_{i_1} \cap \cdots \cap F_{i_n} \) of \( P \),

1. \( \tau_j|_{v_i} = 0 \) if \( j \notin \{i_1, \ldots, i_n\} \);
2. \( \{\tau_{i_1}|_{v_i}, \ldots, \tau_{i_n}|_{v_i}\} \) is a linearly independent set.

**Proof.** For \( j \notin \{i_1, \ldots, i_n\} \), two subspaces \( \pi^{-1}(F_j) = M_j \) and \( \pi^{-1}(v_i) = T_{v_i}^\perp \times v_i \) do not intersect. Hence, the assertion (1) follows.
To show the second assertion, let \( \nu(T_{v_i}^+) \) be the normal bundle of \( T_{v_i}^+ \times v_i \) in \( M \) and consider its \( T \)-equivariant decomposition

\[
\nu(T_{v_i}^+) \cong \bigoplus_{k=1}^{n} \nu(M_{i_k})|_{T_{v_i}^+}.
\]

(3.4)

Here, \( \nu(M_{i_k})|_{T_{v_i}^+} \) denotes the restriction of \( \nu(M_{i_k}) \) to \( T_{v_i}^+ \times v_i \). The equivariant Euler class of \( \nu(M_{i_k})|_{T_{v_i}^+} \) agrees with \( \tau_{i_k}|_{v_i} \in H_2^2(T_{v_i}^+) \) for each \( k = 1, \ldots, n \) by the naturality. Hence, the claim follows from the equivariant decomposition in (3.4).

We now discuss more properties on equivariant Thom classes for the preparation of the study in Section 4. Let \( x_{j,i} \in H^2(BT) \) be a representative of \( \tau_j|_{v_i} \in H_2^2(T_{v_i}^+) \approx H^2(BT)/\text{Ann}(H_2(BT_{v_i})) \) via (3.2) and Lemma 3.1. Furthermore, we may take \( x_{i_k} = 0 \) for \( j \notin \{i_1, \ldots, i_n\} \) by Lemma 3.2. Hence, we write \( x_{i_k} \) for simplicity. Identifying \( H_2(BT) \) and \( H_2(BT_{v_i}) \) with \( \mathbb{Z}^{n+k} \) and \( \langle \xi_{i_1}, \ldots, \xi_{i_n} \rangle \subset \mathbb{Z}^{n+k} \) respectively, we regard \( x_{i_1}, \ldots, x_{i_n} \) as elements of \( (\mathbb{Z}^{n+k})^* \).

**Lemma 3.3.** For each vertex \( v_i = F_{i_1} \cap \cdots \cap F_{i_n} \), any representative \( x_{i_1} \in (\mathbb{Z}^{n+k})^* \) of \( \tau_{i_1}|_{v_i} \) is an element in

\[
\text{Ann} \langle \xi_{i_1}, \ldots, \xi_{i_{s-1}}, \xi_{i_{s+1}}, \ldots, \xi_{i_n} \rangle \setminus \text{Ann} \langle \xi_{i_1}, \ldots, \xi_{i_n} \rangle,
\]

i.e., \( \langle \xi_{i_1}, x_{i_1} \rangle = 0 \) for all \( r \in \{1, \ldots, n\} \setminus \{s\} \) and \( \langle \xi_{i_s}, x_{i_s} \rangle \neq 0 \).

**Proof.** Recall the following identifications:

1. \( H_2(BT) \cong \mathbb{Z}^{n+k} \cong \text{Hom}(S^1, T) \). We denote by \( \lambda_\xi \in \text{Hom}(S^1, T) \) the weight corresponding to \( \xi \in \mathbb{Z}^{n+k} \cong H_2(BT) \).
2. \( H^2(BT) \cong (\mathbb{Z}^{n+k})^* \cong \text{Hom}(T, S^1) \). We denote by \( \chi_\xi \in \text{Hom}(T, S^1) \) the character corresponding to \( \xi \in (\mathbb{Z}^{n+k})^* \cong H^2(BT) \).
3. For each \( t \in S^1 \), we have

\[
(\chi_\xi \circ \lambda_\xi)(t) = t^{\langle \xi, \xi \rangle},
\]

where \( \langle \cdot, \cdot \rangle \) denotes the standard paring between elements in \( \mathbb{Z}^{n+k} \) and its dual \( (\mathbb{Z}^{n+k})^* \).

Now, the equivariant decomposition of \( \nu(T_{v_i}^+) \) as in (3.4) together with (3.5) implies that \( \langle \xi_{i_s}, x_{i_s} \rangle = 0 \) for all \( r \in \{1, \ldots, n\} \setminus \{s\} \) and \( \langle \xi_{i_s}, x_{i_s} \rangle \neq 0 \). \( \Box \)

In addition to Lemma 3.3, one can choose a particular representative \( x_{i_1} \in (\mathbb{Z}^{n+k})^* \) of \( \tau_{i_1}|_{v_i} \) such that \( \langle \xi_{i_1}, x_{i_1} \rangle = 1 \). Indeed, we extend hyper characteristic vectors \( \{\xi_{i_1}, \ldots, \xi_{i_n}\} \) around a vertex \( v_1 \) to a basis \( \{\xi_{i_1}, \ldots, \xi_{i_n}, \eta_1, \ldots, \eta_k\} \) of \( \mathbb{Z}^{n+k} \). Then, we take first \( n \) elements of \( \{x_{i_1}, \ldots, x_{i_n}, x_{i_{n+1}}, \ldots, x_{i_{n+k}}\} \subset (\mathbb{Z}^{n+k})^* \) which is dual to \( \{\xi_{i_1}, \ldots, \xi_{i_n}, \eta_1, \ldots, \eta_k\} \). With this observation, we get the following conclusion.

**Corollary 3.4.** For each vertex \( v_i = F_{i_1} \cap \cdots \cap F_{i_n} \) and associated set

\[
\{\tau_{i_1}|_{v_1}, \ldots, \tau_{i_n}|_{v_1}\} \subset H_2^2(T_{v_1}^+) \cong (\mathbb{Z}^{n+k})^*/\text{Ann}(\xi_{i_1}, \ldots, \xi_{i_n})
\]

of restrictions of equivariant Thom classes, there is a set \( \{x_{i_1}, \ldots, x_{i_n}\} \subset (\mathbb{Z}^{n+k})^* \) of representatives of (3.6) such that

\[
\langle \xi_{i_r}, x_{i_s} \rangle = \begin{cases} 
0 & \text{if } r \neq s; \\
1 & \text{if } r = s.
\end{cases}
\]

(3.7)
In particular, the submodule generated by such a set \( \{x_i, \ldots, x_n\} \) is a direct summand of \((\mathbb{Z}^{n+k})^*\).

We finish this subsection by showing the following theorem about the ring structure of the equivariant cohomology of \( M \). We recall that the equivariant cohomology \( H^*_T(M) \) is isomorphic to

\[
\text{SR}(P) := \mathbb{Z}[y_1, \ldots, y_m]/\langle y_{i_1} \cdots y_{i_r} | F_{i_1} \cap \cdots \cap F_{i_r} = \emptyset \rangle,
\]

which is called the Stanley–Reisner ring of \( P \), see [DJ91] Section 4. Here, \( y_i \)'s are indeterminates of degree 2.

**Theorem 3.5.** Let \( M := M(P, \xi) \) be a locally \( k \)-standard \( T \)-manifold of dimension \( 2n + k \) such that \( \text{im}(\xi) = \mathbb{Z}^{n+k} \). Let \( \tau_j \) be the equivariant Thom class of \( \nu(M_j) \) for \( j = 1, \ldots, m \). Then the equivariant cohomology \( H^*_T(M) \) is isomorphic to

\[
\mathbb{Z}[\tau_1, \ldots, \tau_m]/\langle \tau_{i_1} \cdots \tau_{i_r} | F_{j_1} \cap \cdots \cap F_{j_r} = \emptyset \rangle
\]
as rings.

**Proof.** A hyper characteristic function \( \xi \) of rank \( n + k \), regarded as a matrix of size \((n + k) \times m\) whose column vectors are indexed by facets of \( P \), yields a short exact sequence

\[
0 \longrightarrow \ker \xi \longrightarrow \mathbb{Z}^m \longrightarrow \mathbb{Z}^{n+k} \longrightarrow 0.
\]

Since \( \mathbb{Z}^{n+k} \) is a free \( \mathbb{Z} \)-module, the above short exact sequence splits. This also implies that the following short exact sequence of tori

\[
1 \longrightarrow K \longrightarrow \mathbb{Z}^m \xrightarrow{\exp \xi} T \longrightarrow 0
\]
splits, where \( K := \ker(\exp \xi) \) which is also equal to \( \ker(\exp \xi) \) because \( \text{im}(\xi) = \mathbb{Z}^{n+k} \).

Now, consider the following identifications

\[
\begin{align*}
ET^m \times T^m \mathbb{Z}_P &\cong (EK \times ET) \times_{K \times T} \mathbb{Z}_P \\
\cong (EK \times ET) \times_T \mathbb{Z}_P/K \\
\cong ET \times_T (ET \times_T \mathbb{Z}_P/K)
\end{align*}
\]

Hence, we conclude that

\[
H^*_T(M) \cong \text{SR}(P).
\]

Now, it remains to verify the relation between Thom classes \( \tau_j \)'s of (3.9) and \( y_i \)'s of \( \text{SR}(P) \). First, note that \( \{\tau_1, \ldots, \tau_m\} \) is linearly independent. Indeed,

\[
f_{v_i} \left( \sum_{j=1}^m a_j \tau_j \right) = \sum_{j=1}^m a_j \tau_j |_{v_i} = \sum_{r=1}^n a_{i_r} \tau_{r} |_{v_i} \in H^2_T(T^r_{v_i}),
\]

where \( f_{v_i} \) is defined in (3.3) and the second equality follows from Lemma 3.2.1. Hence, if \( \sum_{j=1}^m a_j \tau_j = 0 \), then \( \sum_{k=1}^n a_{i_k} \tau_{i_k} |_{v_i} = 0 \). This implies that \( a_{i_1} = \cdots = a_{i_m} = 0 \) by Lemma 3.2.2. The same procedures for other vertices establish \( a_1 = \cdots = a_m = 0 \).

Next, the cup product \( \tau_{i_1} \cap \cdots \cap \tau_{i_r} \) vanishes whenever \( F_{i_1} \cap \cdots \cap F_{i_r} = \emptyset \), because \( \tau_{i_1} \cap \cdots \cap \tau_{i_r} \) represents the equivariant Poincare dual of the intersection.
The hyper characteristic function $\xi_H$ is a subgroup of homomorphism $\pi$. The following lemma allows us to see $\xi_H$ that the generators of $\text{SR}(\tilde{\tau})$ for some linear map $\tilde{\tau}$ satisfying $\tilde{\tau}(\xi_H(u)) = \tilde{\tau}(u)$ for each element $u$. Hence, one has an isomorphism between $\xi_H$ and $\text{SR}(\tilde{\tau})$ by sending $\tau_j$ to $y_j$ for $j = 1, \ldots, m$. □

The authors of [DJ91] showed that the Borel construction of a quasitoric manifold and the Borel construction of the corresponding moment angle manifold are same as in (3.10). Then they use a certain decomposition of $\mathcal{Z}_P$ arising from the cubical decomposition of $P$ and Mayer–Vietories sequence to prove that the Equivariant cohomology of $\mathcal{Z}_P$ is isomorphic to $\text{SR}(\tilde{\tau})$. In addition to it we showed in Theorem 3.5 that the generators of $\text{SR}(\tilde{\tau})$ can be identified with the equivariant Thom classes $\tau_j$’s which holds for a class of locally $k$-standard $T$-manifolds containing quasitoric manifolds and moment angle manifolds.

3.2. Algebra structure. Note that $H^*_T(M)$ is equipped with $H^*(BT)$-algebra structure induced from the Borel fibration

$$M \xrightarrow{\eta} ET \times_T M \xrightarrow{\pi} BT.$$ (3.11)

The following lemma allows us to see $H^2(BT)$ as a subset of $H^*_T(M)$.

**Lemma 3.6.** Let $M$ be a locally $k$-standard $T$-manifold of dimension $2n + k$ with the hyper characteristic function $\xi$ and assume that $\text{rk}(\xi) = n + k$. Then, $H^2(BT)$ is a subgroup of $H^*_T(M)$. Moreover, the inclusion of $H^2(BT)$ into $H^*_T(M)$ is the homomorphism $\pi^*: H^2(BT) \to H^*_T(M)$.

**Proof.** The cohomology Leray–Serre spectral sequence for the fibration (3.11) gives us

$$E^{p,q}_2 = H^p(BT; H^q(M; \mathbb{Z})),$$

where the system of local coefficients is simple, because $BT$ is simply connected. We refer to [McC01] Proposition 5.20. Notice that $E^{2,0}_2 = H^2(BT; \mathbb{Z})$ and $E^{0,1}_2 = 0$ by Lemma 2.5 together with Hurewicz theorem and the universal coefficient theorem, see for instance [Bre93] Theorem 3.4, Corollary 7.3. Therefore, the differential

$$d^{1,0}_2: E^{0,1}_2 \to E^{2,0}_2$$

is a zero map, which implies that

$$H^2(BT) = E^{2,0}_2 = E^{3,0}_3 = \cdots = E^{2,0}_\infty \subset H^*_T(M).$$

Now, the second assertion directly follows from [McC01] Theorem 5.9. □

Hence, for each element $u \in H^2(BT)$, we have

$$\pi^*(u) = u = \sum_{j=1}^m w_j(u) \cdot \tau_j = \sum_{j=1}^m \langle \tilde{w}_j, u \rangle \tau_j$$ (3.12)

for some linear map

$$w_j: H^2(BT) \to \mathbb{Z}.$$ 

Here, $\tilde{w}_j \in \mathbb{Z}^{n+k}$ and $\langle \cdot, \cdot \rangle$ denote the element corresponding to $w_i$ via the identifications of $\text{Hom}(H^2(BT), \mathbb{Z}) \cong \text{Hom}((\mathbb{Z}^{n+k})^*, \mathbb{Z}) \cong \mathbb{Z}^{n+k}$ and the standard paring between $\mathbb{Z}^{n+k}$ and $(\mathbb{Z}^{n+k})^*$, respectively.

**Lemma 3.7.** Let $\{x_i, \ldots, x_n\} \subset \mathbb{Z}^{n+k}$ be the set as in Corollary 3.4. Then, $\{\tilde{w}_{i_s}, \ldots, \tilde{w}_{i_r}\}$ satisfies

$$\langle \tilde{w}_{i_r}, x_{i_s} \rangle = \begin{cases} 
0 & \text{if } r \neq s; \\
1 & \text{if } r = s.
\end{cases}$$ (3.13)
Proof. Consider the following commutative diagram,

\[
\begin{array}{ccc}
H^2(BT) & \xrightarrow{\pi^*} & H^2(M) \\
\downarrow{=} & & \downarrow{=} \\
\langle \mathbb{Z}^{n+k} \rangle^* & \xrightarrow{pr_{v_i}} & \langle \mathbb{Z}^{n+k} \rangle^*/\text{Ann} \langle \xi_1, \ldots, \xi_n \rangle
\end{array}
\]

where \(\pi^*\) and \(f_{v_i}\) are defined in (3.12) and (3.13), respectively and \(pr_{v_i}\) is the projection. Now, we evaluate \(x_{i_k} \in \langle \mathbb{Z}^{n+k} \rangle^*\). Then, we have

\[
f_{v_i}(\pi^*(x_{i_k})) = f_{v_i}(x_{i_k}) = f_{v_i} \left( \sum_{j=1}^{m} \langle \tilde{w}_j, x_{i_k} \rangle \tau_j \right) = \sum_{r=1}^{n} \langle \tilde{w}_{i_k}, x_{i_k} \rangle \tau_r |_{v_i}
\]

and \(pr_{v_i}(x_{i_k}) = \tau_{i_k} |_{v_i}\). Hence the commutativity of the diagram (3.14) shows

\[
\tau_{i_k} |_{v_i} = \sum_{r=1}^{n} \langle \tilde{w}_{i_k}, x_{i_k} \rangle \tau_r |_{v_i},
\]

which implies the desired relation (3.16), because the set \(\{\tau_{i_k} |_{v_i}, \ldots, \tau_{i_n} |_{v_i}\}\) is linearly independent, see Lemma 3.2.

We note that relations (3.16) and (3.13) are independent from the choice of a vertex of \(P\). To be more precise, if two distinct vertices \(v_i\) and \(v_\ell\) are contained in a common facet \(F_j\), then both \(\xi_j\) in (3.16) and \(\tilde{w}_j\) in (3.13) are related to two different set of dual elements \(B_1 := \{x_{i_1}, \ldots, x_{i_n}\}\) and \(B_2 := \{y_{i_1}, \ldots, y_{i_n}\}\) corresponding to restrictions of equivariant Thom classes around \(v_i\) and \(v_\ell\), respectively. Two relations (3.16) and (3.13) holds for both \(B_1\) and \(B_2\).

Now, we prove the following theorem.

**Theorem 3.8.** Let \(M\) be a locally \(k\)-standard \(T\)-manifold of dimension \(2n + k\) with \(\text{rk}(\xi) = n + k\). Let \(\tilde{w}_j\) be the vector as in (3.12). Then, \(\tilde{w}_j = \xi_j\) for each \(j = 1, \ldots, m\).

**Proof.** We claim that

\[
\langle \tilde{w}_j - \xi_j, x \rangle = 0, \text{ for all } x \in \langle \mathbb{Z}^{n+k} \rangle^*.
\]

Take a vertex \(v\) of given facet \(F_j\) and assume \(v = F_{i_1} \cap \cdots \cap F_{i_n}\), namely, \(j\) is an element of \(\{i_1, \ldots, i_n\}\). Then, Corollary 3.4 gives us a decomposition

\[
\langle \mathbb{Z}^{n+k} \rangle^* \cong \langle x_{i_1}, \ldots, x_{i_n} \rangle \oplus \text{Ann} \langle \xi_1, \ldots, \xi_{i_n} \rangle.
\]

For elements \(x \in \text{Ann} \langle \xi_{i_1}, \ldots, \xi_{i_n} \rangle\), we apply (3.15) to get

\[
0 = f_{v}(\pi^*(x)) = f_{v}(x) = f_{v} \left( \sum_{j=1}^{m} \langle \tilde{w}_j, x \rangle \tau_j \right) = \sum_{r=1}^{n} \langle \tilde{w}_{i_k}, x \rangle \tau_r |_{v_i}.
\]

Since the set \(\{\tau_{i_1} |_{v_i}, \ldots, \tau_{i_n} |_{v_i}\}\) is linearly independent, we have \(\langle \tilde{w}_{i_k}, x \rangle = 0\) for all \(r = 1, \ldots, n\). In particular, \(\langle \tilde{w}_j, x \rangle = 0\) because \(j\) is an element of \(\{i_1, \ldots, i_n\}\). Moreover, \(\langle \xi_j, x \rangle = 0\) because \(x \in \text{Ann} \langle \xi_1, \ldots, \xi_{i_n} \rangle\). Hence, the assertion (3.16) has been established for \(x \in \text{Ann} \langle \xi_{i_1}, \ldots, \xi_{i_n} \rangle\). For elements \(x \in \langle x_{i_1}, \ldots, x_{i_n} \rangle\), the claim follows from Corollary 3.4 and Lemma 3.2. Hence, the result follows. \(\square\)
4. Equivariant cohomological rigidity

In this section, we answer the equivariant cohomological rigidity problem for the category of locally $k$-standard $T$-manifolds. The main theorem (Theorem 4.2) states that the weak isomorphism classes of the equivariant cohomology distinguishes the weak homeomorphism classes of locally $k$-standard $T$-manifolds. Here, weak homeomorphism between two $T$-manifolds $M$ and $M'$ means a homeomorphism $\Psi: M \to M'$ such that $\Psi(t \cdot x) = \delta(t) \cdot \Psi(x)$ for arbitrary $t \in T$ and $x \in M$ and for some automorphism $\delta \in \text{Aut}(T)$. A weak isomorphism between two equivariant cohomology algebras are similarly defined with respect to an automorphism of $H^*(BT)$.

We first prepare the following proposition which tells about the combinatorics of the simple polytopes from their Stanley-Reisner rings. It can be deduced from the result of [BG96] as well as the relationship between simple convex polytope and its dual simplicial complex.

**Proposition 4.1.** Let $P$ and $P'$ be simple polytopes with facets $\mathcal{F}(P)$ and $\mathcal{F}(P')$ respectively. Suppose $\text{SR}(P)$ and $\text{SR}(P')$ are isomorphic as $\mathbb{Z}$-algebras (see (3.8)). Then there is a bijection $\phi: \mathcal{F}(P) \to \mathcal{F}(P')$ which induces a face preserving homeomorphism $\bar{\phi}: P \to P'$.

Recall from Corollary 2.4 and Proposition 2.7 that a locally $k$-standard $T$-manifold is determined by its hyper characteristic pair. Now, we introduce our main results.

**Theorem 4.2.** Let $M$ and $M'$ be locally $k$-standard $T$-manifolds associated with $(P, \xi)$ and $(P, \xi')$ respectively such that $\text{im}(\xi)$ and $\text{im}(\xi')$ are direct summands of $\mathbb{Z}^{n+k}$. Then, $M$ and $M'$ are weakly equivariantly homeomorphic if and only if their equivariant cohomology algebras are weakly isomorphic.

**Proof.** Suppose we have two locally $k$-standard $T$-manifolds $M$ and $M'$ of dimension $(2n + k)$. Let $\psi: H^*_T(M) \to H^*_T(M')$ be a weak isomorphism as $H^*(BT)$-algebras. We adhere notations discussed in Section 2.

**Case 1.** We first consider the case when $\xi$ and $\xi'$ are surjective. Theorem 3.5 implies that the equivariant cohomology ring of $M$ is isomorphic to the Stanley–Reisner ring $\text{SR}(P)$ of the orbit space $P$ of $M$. Hence, an algebra isomorphism $\psi$ induces an isomorphism between $\text{SR}(P)$ and $\text{SR}(P')$. Using Proposition 4.1, $\psi$ induces a bijection

$$\phi: \mathcal{F}(P) \to \mathcal{F}(P')$$

which also defines a face preserving homeomorphism $\bar{\phi}: P \to P'$. Moreover, by the proof of Theorem 3.5, we have a bijection between two sets of equivariant Thom classes of $M$ and $M'$ up to sign.

Since $\psi$ is an algebra isomorphism, the algebra structures are compatible with $\psi$, i.e., $\pi'^* = \psi \circ \pi^*$, where $\pi^*$ and $\pi'^*$ are defined in (3.12) for $M$ and $M'$, respectively. Then,

$$\pi'^*(u) = \sum_{j=1}^m \langle \xi_j', u \rangle \tau'_j$$

and

$$\psi(\pi'^*(u)) = \sum_{j=1}^m \langle \xi_j, u \rangle \psi(\tau_j) = \sum_{j=1}^m \epsilon_j \langle \xi_{\phi(j)}, u \rangle \tau'_{\phi(j)}$$
We note that $H$ weakly homeomorphic.

Let $J := \{ j \mid \epsilon_j = -1 \}$ and define an automorphism $\delta: T^m \rightarrow T^m$ such that $\delta$ sends $j$-th coordinate of $(t_1, \ldots, t_m) \in T^m$ to its conjugate whenever $j \in J$. This implies that $\delta(\exp(\ker \xi)) = \exp(\ker \xi')$. Hence, we have the following commutative diagram

$$Z_P = (T^m \times P)/\sim \xrightarrow{\Psi} (T^m \times P')/\sim = Z_{P'}$$

$$Z_P/\exp(\ker \xi) \xrightarrow{\tilde{\Psi}} Z_{P'}/\exp(\ker \xi'),$$

where $\Psi$ is a $\delta$-equivariant homeomorphism induced from $\delta \times \tilde{\delta}$. Therefore, $\tilde{\Psi}$ is a weakly equivariantly homeomorphism with respect to an isomorphism between $T^m/\ker \xi$ and $T^m/\ker \xi'$ induced from $\delta$. Hence, the result follows in this case.

**Case 2.** Now, we consider the case where $\mathrm{rk}(\xi) < n+k$. In this case, by Proposition 2.6, $M$ is equivariantly homeomorphic to $T^{n+k-\mathrm{rk}(\xi)} \times N$ for some locally $r$-standard $T$-manifold $N$ with a surjective hyper characteristic function where $0 \leq r < k$. The torus $T$ is also decomposed into $T^{n+k-\mathrm{rk}(\xi)} \times T_r^{\mathrm{rk}(\xi)}$ accordingly. Observe that

$$H^*_T(M) = H^*((ET \times_T (T^{n+k-\mathrm{rk}(\xi)} \times N))$$

$$= H^*((ET^{n+k-\mathrm{rk}(\xi)} \times_{T^{n+k-\mathrm{rk}(\xi)}} T^{n+k-\mathrm{rk}(\xi)}) \times (E_{T_r^{\mathrm{rk}(\xi)}}(\xi) \times T_r^{\mathrm{rk}(\xi)} N))$$

$$= H^*_T(ET^r(\xi))(N).$$

We note that $H^*(BT_r^{\mathrm{rk}(\xi)}(\xi))$-algebra structure on $H^*_T(ET^r(\xi))(N)$ is induced from the $H^*(BT)$-algebra structure on $H^*_T(M)$ and the decomposition

$$H^*(BT) \cong H^*(BT^{n+k-\mathrm{rk}(\xi)}) \otimes H^*(BT_r^{\mathrm{rk}(\xi)}(\xi)),$$

where $H^*(BT^{n+k-\mathrm{rk}(\xi)})$ acts trivially on $H^*_T(ET^r(\xi))(N)$.

A weak isomorphism $\psi: H^*_T(M) \rightarrow H^*_T(M')$ induces a weak isomorphism

$$\psi^*: H^*_T(ET^r(\xi))(N) \rightarrow H^*_T(ET^r(\xi'))(N')$$

between two locally $r$-standard $T$-manifolds $N$ and $N'$, which are now in **Case 1**. Hence, $N$ and $N'$ are weakly homeomorphic, which implies that $M$ and $M'$ are also weakly homeomorphic.

We note that Theorem 4.2 generalizes the result of [Mas08] which deals with quasitoric manifolds.

**Acknowledgements.** The authors are grateful to IBS–CGP for the hospitality in July 2019 and grateful to KAIST and IIT-Madras for supporting their visits. They are also grateful to Anthony Bahri, Mikiya Masuda and Dong Youp Suh for helpful comments. The authors thank the anonymous referee for helpful comments to improve the manuscript.

The first author is supported by MATRICS grant MTR/2018/000963 of SERB India and International office of IIT Madras. The second author has been supported by Basic Science Research Program through the National Research Foundation of
Korea (NRF) funded by the Ministry of Education (NRF-2018R1D1A1B07048480) and a KIAS Individual Grant (MG076101) at Korea Institute for Advanced Study.

REFERENCES

[Ati82] M. F. Atiyah. Convexity and commuting Hamiltonians. Bull. London Math. Soc., 14(1):1–15, 1982.

[BG96] W. Bruns and J. Gubeladze. Combinatorial invariance of Stanley-Reisner rings. Georgian Math. J., 3(4):315–318, 1996.

[BP02] Victor M. Buchstaber and Taras E. Panov. Torus actions and their applications in topology and combinatorics, volume 21 of University Lecture Series. American Mathematical Society, Providence, RI, 2002.

[BP15] Victor M. Buchstaber and Taras E. Panov. Toric topology, volume 204 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2015.

[Bre93] Glen E. Bredon. Topology and geometry, volume 139 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1993.

[BT19a] Victor M. Buchstaber and Svjetlana Terzić. Toric topology of the complex Grassmann manifolds. Mosc. Math. J., 19(3):397–463, 2019.

[BT19b] V. M. Bukhshtaber and S. Terzich. The foundations of (2n, k)-manifolds. Mat. Sb., 210(4):41–86, 2019.

[CGM12] Jack S. Calcut, Robert E. Gompf, and John D. McCarthy. On fundamental groups of quotient spaces. Topology Appl., 159(1):322–330, 2012.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. Toric varieties, volume 124 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011.

[Del88] Thomas Delzant. Hamiltioniens périodiques et images convexes de l’application moment. Bull. Soc. Math. France, 116(3):315–339, 1988.

[DJ91] Michael W. Davis and Tadeusz Januszkiewicz. Convex polytopes, Coxeter orbifolds and torus actions. Duke Math. J., 62(2):417–451, 1991.

[Fra19] Matthias Franz. The cohomology rings of smooth toric varieties and quotients of moment-angle complexes. [arXiv:1907.04791v2 [math.AT], 2019.

[Ful93] William Fulton. Introduction to toric varieties, volume 131 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[GKM98] Mark Goresky, Robert Kottwitz, and Robert MacPherson. Equivariant cohomology, Koszul duality, and the localization theorem. Invent. Math., 131(1):25–83, 1998.

[GS82] V. Guillemin and S. Sternberg. Convexity properties of the moment mapping. Invent. Math., 67(3):491–513, 1982.

[GZ01] V. Guillemin and C. Zara. 1-skeleta, Betti numbers, and equivariant cohomology. Duke Math. J., 107(2):283–349, 2001.

[Kur11] Shintaro Kuroki. Equivariant cohomology distinguishes the geometric structures of toric hyperkahler manifolds. Tr. Mat. Inst. Steklova, 275:262–294, 2011.

[Ler03] Eugene Lerman. Contact toric manifolds. J. Symplectic Geom., 1(4):785–828, 2003.

[Ler04] Eugene Lerman. Homotopy groups of K-contact toric manifolds. Trans. Amer. Math. Soc., 356(10):4075–4083, 2004.

[Mas08] Mikiya Masuda. Equivariant cohomology distinguishes toric manifolds. Adv. Math., 218(6):2005–2012, 2008.

[McC01] John McCleary. A user’s guide to spectral sequences, volume 58 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, second edition, 2001.

[MP06] Mikiya Masuda and Taras Panov. On the cohomology of torus manifolds. Osaka J. Math., 43(3):711–746, 2006.

[Oda88] Tadao Oda. Convex bodies and algebraic geometry, volume 15 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1988. An introduction to the theory of toric varieties, Translated from the Japanese.

[SS18] Soumen Sarkar and Dong Youp Suh. A new construction of lens spaces. Topology Appl., 240:1–20, 2018.
[Zie95] Günter M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

Department of Mathematics, Indian Institute of Technology Madras, Chennai 600036, India

Email address: soumen@iitm.ac.in

School of Mathematics, KIAS, Seoul 02455, Republic of Korea

Email address: jongbaek@kias.re.kr