A SQP Based Line Search Method for Multi-objective Optimization Problems

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Abstract

In this paper a globally convergent sequential quadratic programming (SQP) method is developed for multi-objective optimization problems with inequality type constraints. A feasible descent direction is obtained using a linear approximation of all objective functions as well as constraint functions with a quadratic restriction. A non differentiable penalty function is used to restrict the constraint violations. A descent sequence is generated which converges to a critical point under the Mangasarian-Fromovitz constraint qualification along with some other mild assumptions. The method is verified and compared with existing methods using a set of test problems.

Keywords: multi-objective optimization, SQP method, critical point, Mangasarian-Fromovitz constraint qualification, purity metric, spread metrics

Subclass 90C26, 49M05, 97N40, 90B99

1 Introduction

Developing line search techniques for multi-objective optimization problems is an important research area. Since 2000 many researchers have been developing line search methods for unconstrained multi-objective optimization problems as an extension of single-objective line search techniques. These approaches are possible extensions of the steepest descent method, the Newton method, variations of this last method and trust region method which are explained in [10] [9] [30] [54] [1] [31]. A widely used line search technique for solving constrained single-objective optimization problems is sequential quadratic programming (SQP), which was developed by Wilson [35] and modified by several researchers (see [32] [13] [15]) in several directions. Recently Fliege and Vaz [11] [12] [14] [16].

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and Bennet et al. [14] have developed line search techniques for solving multi-objective optimization problems based on the SQP method. Similar to single-objective SQP method, a serious limitation of these methods is the inconstancy of the quadratic subproblem. In the single-objective case, Powell [29] suggested a modified subproblem to overcome this restriction, which was further modified in [36, 8, 21, 27] for better efficiency. In this paper a globally convergent SQP method is developed for constrained multi-objective optimization problems based on these ideas. Some portions of the theory of this paper can be regarded as extension of the theories of [27] to multi-objective case.

The outline of this paper is as follows. Some preliminaries on the existence of a solution of a multi-objective optimization problems are discussed in Section 2. A modified SQP scheme for inequality constrained multi-objective optimization problems is developed in Section 3 and global convergence of this scheme is proved in Section 4. In Section 5, the proposed method is compared with the existing algorithms using a set of test problems.

2 Preliminaries

Consider the multi-objective optimization problem:

\[
\text{MOP} : \quad \min (f_1(x), f_2(x), ..., f_m(x))
\]

subject to \(g_i(x) \leq 0, \quad i = 1, 2, ..., p\),

where \(f_j, g_i : \mathbb{R}^n \to \mathbb{R}\) are continuously differentiable for \(j \in \{1, 2, ..., m\}\) and \(i \in \{1, 2, ..., p\}\). Denote \(\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x_i \geq 0, \quad i = 1, 2, ..., n\}\), \(\Lambda_n = \{1, 2, ..., n\}\) for any \(n \in \mathbb{N}\), and the feasible set of the MOP as \(X = \{x \in \mathbb{R}^n : g_i(x) \leq 0, \quad i \in \Lambda_p\}\). Inequality in \(\mathbb{R}^n\) is understood componentwise. If there exists \(x \in X\) such that \(x\) minimizes all objective functions simultaneously then it is an ideal solution. But in practice, decrease of one objective function may cause increase of another objective function. So, in the theory of multi-objective optimization, optimality is replaced by efficiency. A feasible point \(x^* \in X\) is said to be an efficient solution of the MOP if there does not exist \(x \in X\) such that \(f(x) \leq f(x^*)\) and \(f(x) \neq f(x^*)\) hold where \(f(x) = (f_1(x), f_2(x), ..., f_m(x))\). A feasible point \(x^* \in X\) is said to be a weak efficient solution of MOP if there does not exist \(x \in X\) such that \(f(x) < f(x^*)\) holds. For \(x, y \in X\), we say \(x\) dominates \(y\), if and only if \(f(x) \leq f(y), \quad f(x) \neq f(y)\). A point \(x \in X\) is said to be non dominated if there does not exist any \(y \in X\) such that \(y\) dominates \(x\). If \(X^*\) is the set of all efficient solutions of the MOP, then the image of \(X^*\) under \(f\), i.e. \(f(X^*)\) is said to be the Pareto front of the MOP.

In our analysis, we use the \(L_{\infty}\) non differentiable penalty function

\[
\Phi(x) = \max\{0, g_i(x), i \in \Lambda_p\}.
\]

In order to obtain a feasible descent direction, the penalty function for the MOP is used as the following merit function \(\Psi_{j,\sigma}(x)\), with a penalty parameter \(\sigma > 0\)
as

\[ \Psi_{j,\sigma}(x) = f_j(x) + \sigma \Phi(x), \quad j \in \Lambda_m. \]

Let \( I_0(x) = \{ i \in \Lambda_p : g_i(x) = \Phi(x) \} \) be the set of active or most violated constraints. The directional derivative of \( \Phi(x) \) in any direction \( d \in \mathbb{R}^n \) is

\[ \Phi'(x; d) = \max_{i \in I_0(x)} \{ \nabla g_i(x)^T d \}, \]

In general \( \Phi'(x; d) \) is not continuous. A continuous approximation of \( \Phi'(x; d) \) (see [2]) is

\[ \Phi^*(x; d) = \max_{i \in I_0(x)} \{ g_i(x) + \nabla g_i(x)^T d, 0 \} - \Phi(x). \]

Thus, an approximation of the directional derivative of \( \Psi_{j,\sigma}(x) \) is

\[ \theta_{j,\sigma}(x; d) = \nabla f_j(x)^T d + \sigma \Phi^*(x; d), \quad j \in \Lambda_m. \]

If all \( f_j, g_i \) are continuously differentiable then the necessary condition of weak efficiency can be derived using Motzkin’s theorem as follows.

**Theorem 1.** *(Fritz John necessary condition [Theorem 3.1.1, [26]](#))*

Suppose \( f_j, j \in \Lambda_m \) and \( g_i, i \in \Lambda_p \) are continuously differentiable at \( x^* \in X \). If \( x^* \) is a weak efficient solution of the MOP then there exist \( (\lambda, \mu) \in \mathbb{R}^m_+ \times \mathbb{R}^p_+ \), \( (\lambda, \mu) \neq 0^{m+p} \) satisfying

\[ \sum_{j \in \Lambda_m} \lambda_j \nabla f_j(x^*) + \sum_{i \in \Lambda_p} \mu_i \nabla g_i(x^*) = 0 \quad (2) \]
\[ \mu_i g_i(x^*) = 0 \quad \forall \quad i \in \Lambda_p. \quad (3) \]

The set of the vector \( (\lambda, \mu) \in \mathbb{R}^m_+ \times \mathbb{R}^p_+ \setminus \{0^{m+p}\} \) satisfying \( (2) \) and \( (3) \) are called Fritz John multipliers associated with \( x^* \). But the Fritz John necessary condition does not guarantee \( \lambda_j > 0 \), for at least one \( j \). So some constraint qualifications or regularity conditions should hold to ensure it.

Several constraint qualifications or regularity conditions are defined and discussed in [24, 23]. Through the discussion of this paper we consider the Mangasarian-Fromovitz constraint qualification (MFCQ).

**Definition 1.** *(27)* The Mangasarian-Fromovitz constraint qualification (MFCQ) is said to be satisfied at a point \( x \in \mathbb{R}^n \), if there is a \( z \in \mathbb{R}^n \) such that \( \nabla g_i(x)^T z < 0 \) for \( i \in I_0(x) \).

Suppose MFCQ holds at \( x \). Then the system of inequalities \( \nabla g_i(x)^T z < 0 \) for \( i \in I_0(x) \) has a nonzero solution \( z \in \mathbb{R}^n \). Hence by Gordan’s theorem of alternative \( \sum_{i \in \Lambda_0(x)} \mu_i \nabla g_i(x) = 0, \quad \mu_i \geq 0 \) has no nonzero solution. That is, \( \mu_i = 0 \) \( \forall \ i \in I_0(x) \).

Conversely suppose the system \( \sum_{i \in \Lambda_0(x)} \mu_i \nabla g_i(x) = 0, \quad \mu_i \geq 0 \) at \( x \) has no nonzero
solution \( \mu \). Then by Gordon’s theorem of alternative the system of inequalities 
\( \nabla g_i(x)^T z < 0 \) for \( i \in I_0(x) \) has some nonzero solution \( z \in \mathbb{R}^n \).

Above discussion concludes that MFCQ holds at \( x \) iff
\[
\sum_{i \in I_0(x)} \mu_i \nabla g_i(x) = 0, \quad \mu_i \geq 0 \implies \mu_i = 0 \quad \forall \ i \in I_0(x).
\] (4)

Strong and weak stationary points for single-objective optimization problems are defined in Definition 1 of [27]. In the light of this definition, strong and weak critical point of the MOP can be defined, taking care all objective functions together as follows.

**Definition 2.** A feasible point \( x \) of the MOP is said to be

(1) a strong critical point of the MOP if there exist vectors \( \lambda \in \mathbb{R}^m_+ \setminus \{0^m\} \) and \( \mu \in \mathbb{R}^p_+ \) satisfying (2) and (3).

(2) a weak critical point of the MOP if there exists an infeasible sequence \( \{x^k\} \) converging to \( x \in X \) such that
\[
\lim_{k \to \infty} \frac{\max_{d \in D(x^k)} \max_{i \in \Lambda_p} \{g_i(x^k) + \nabla g_i(x^k)^T d; 0\}}{\Phi(x^k)} = 1,
\] (5)

where \( D(x^k) = \{d : \nabla f_j(x^k)^T d \leq 0, \ j \in \Lambda_m\} \).

One may observe that a strong critical point is a KKT point of the MOP.

### 3 A SQP based line search method for MOP

In order to obtain a feasible descent direction at \( x \), we solve a quadratic programming subproblem \( QP(x) \) at \( x \) as
\[
(QP(x)) : \quad \text{min} \ t + \frac{1}{2} d^T d \quad \text{subject to} \quad \begin{align*}
\nabla f_j(x)^T d & \leq t \quad j \in \Lambda_m, \\
g_i(x) + \nabla g_i(x)^T d & \leq t \quad i \in \Lambda_p.
\end{align*}
\] (6) (7)

Here the quadratic term \( \frac{1}{2} d^T d \) is used to avoid unboundedness of the solutions. Advantage of \( (QP(x)) \) over the subproblems in [11 14] is, feasibility of \( QP(x) \) is guaranteed. One may observe that \( t = \Phi(x) \), \( d = 0 \) is a feasible solution of \( QP(x) \). The solutions of \( QP(x) \) satisfy the MFCQ since the system
\[
- \sum_{j \in \Lambda_m} \lambda_j - \sum_{i \in \Lambda_p} \mu_i = 0 \\
\sum_{j \in \Lambda_m} \lambda_j f_j(x) + \sum_{i \in \Lambda_p} \mu_i g_i(x) = 0 \\
\lambda_j \geq 0, \mu_i \geq 0
\]
implies $\lambda_j = 0 \ \forall j$ and $\mu_i = 0 \ \forall i$. Hence there exist $\lambda \in \mathbb{R}^m_+$, $\mu \in \mathbb{R}^p_+$, $(\lambda, \mu) \neq 0^{m+p}$ satisfying the KKT optimality conditions. As a result,

$$d + \sum_{j \in \Lambda_m} \lambda_j \nabla f_j(x) + \sum_{i \in \Lambda_p} \mu_i \nabla g_i(x) = 0,$$

(8)

$$1 - \sum_{j \in \Lambda_m} \lambda_j - \sum_{i \in \Lambda_p} \mu_i = 0,$$

(9)

$$\lambda_j \geq 0 \quad \lambda_j (\nabla f_j(x)^T d - t) = 0, \quad j \in \Lambda_m,$$

(10)

$$\mu_i \geq 0 \quad \mu_i (g_i(x) + \nabla g_i(x)^T d - t) = 0 \quad i \in \Lambda_p,$$

(11)

$$\nabla f_j(x)^T d - t \leq 0, \quad j \in \Lambda_m,$$

(12)

$$g_i(x) + \nabla g_i(x)^T d - t \leq 0 \quad i \in \Lambda_p.$$

(13)

**Lemma 1.** Suppose that $(t, d)$ is the solution of the QP$(x)$.

(I) Then

$$t \leq \Phi(x) - \frac{1}{2} d^T d.$$  

(14)

(II) If $d = 0$ and the MFCQ holds at $x$ then $x$ is a strong critical point of MOP.

(III) If $d \neq 0$ then $d$ is a descent direction of $\Psi_{j, \sigma}(x)$ at $x$ for $\sigma$ sufficiently large.

**Proof:**

(I) Since the constraint set of MOP and the single-objective optimization problem in [27] are same so (14) follows from Lemma 2(I) of [27].

(II) Let $(t, 0)$ be the solution of the QP$(x)$. Hence replacing $d$ by $0$ in (8)-(13), we get

$$\sum_{j \in \Lambda_m} \lambda_j \nabla f_j(x) + \sum_{i \in \Lambda_p} \mu_i \nabla g_i(x) = 0$$

(15)

$$1 - \sum_{j \in \Lambda_m} \lambda_j - \sum_{i \in \Lambda_p} \mu_i = 0$$

(16)

$$\lambda_j \geq 0 \quad \lambda_j t = 0, \quad j \in \Lambda_m$$

(17)

$$\mu_i \geq 0 \quad \mu_i (g_i(x) - t) = 0 \quad i \in \Lambda_p$$

(18)

$$0 \leq t, \quad g_i(x) \leq t \quad i \in \Lambda_p.$$  

(19)

$\Phi(x) \leq t$ follows from definition of $\Phi(x)$ and (19). Then $t$ satisfying (14) with $d = 0$ implies $\Phi(x) \geq t$. Hence $\Phi(x) = t$. From (18), it follows that $\mu_i = 0 \ \forall i \notin I_0(x)$. Also, $\lambda_j > 0$ holds for at least one $j$. Otherwise (15) and (16) will imply $\sum_{i \in \Lambda_p} \mu_i \nabla g_i(x) = 0$, $\mu_i > 0$ for at least one $i$, which violates the MFCQ. This implies that $t = 0 = \Phi(x)$ (from (17)) i.e., $x$ is a feasible point. Then from (18),

$$\mu_i g_i(x) = 0, \quad \mu_i \geq 0 \quad i \in \Lambda_p.$$
Therefore, $x$ is a strong critical point of $MOP$, which follows from (15).

(III) Suppose $d \neq 0$, then the following two cases could arise:  
Case-1: Let $\Phi(x) > 0$. Applying (14) in (7) we get
\[
   g_i(x) + \nabla g_i(x)^T d \leq t \leq \Phi(x) - \frac{1}{2} d^T d < \Phi(x).
\]
Since $0 < \Phi(x)$, we have $\max_{i \in I_0(x)} \{g_i(x) + \nabla g_i(x)^T d, 0\} - \Phi(x) < 0$, from the inequalities above. That is, $\Phi^*(x; d) < 0$. If $\sigma$ is chosen in such way that
\[
   \nabla f_j(x)^T d + \sigma \Phi^*(x; d) \leq -\frac{1}{2} d^T d < 0
\]
holds for all $j$ then $d$ will be a descent direction of $\Psi_{j,\sigma}(x)$ for all $j$ (from Lemma 2.1(1) of [3]).

Case-2: If $\Phi(x) = 0$, then $t = 0$, $d = 0$ is a feasible solution of $QP(x)$. So $g_i(x) + \nabla g_i(x)^T d \leq t \leq 0$ holds for all $i \in I_0(x)$. So
\[
   \Phi^*(x; d) = \max_{i \in I_0(x)} \{g_i(x) + \nabla g_i(x)^T d, 0\} - \Phi(x) = 0.
\]
Also, $d \neq 0$ implies $t < 0$. Hence from (6), we have $\nabla f_j(x)^T d \leq t < 0$ and consequently $\nabla f_j(x)^T d + \sigma \Phi^*(x; d) < 0$. 

Let $(t^k, d^k)$ be the solution of the $QP(x^k)$. Following the arguments of the proof of Lemma 1(III), the penalty parameter $\sigma_k$ is updated to force $d^k$ to remain a descent direction for all $\Psi_{j,\sigma_k}(x^k)$. At $k$-th iteration $\sigma_k$ is unchanged if $d^k$ is descent direction; otherwise, $\sigma_k$ is updated as
\[
   \sigma_{k+1} = \max \left\{ \frac{\nabla f_j(x^k)^T d^k + \frac{1}{2} d^k d^k}{-\Phi^*(x^k; d^k)}, 2\sigma_k, \ j \in \Lambda_m \right\} .
\]
The theoretical results developed so far are summarized in the following algorithm.

**Algorithm 1.** (A SQP based algorithm for the MOP)

*Step 1.* (Initialization) Choose $x^0 \in \mathbb{R}^n$, some scalars $r \in (0, 1)$, $\beta \in (0, 1)$, the initial penalty parameter $\sigma_0 > 0$, and an error tolerance $\epsilon$. Set $k := 0$.

*Step 2.* Solve the $QP(x^k)$ to find the descent direction $(t^k; d^k)$ with Lagrange multipliers $(\lambda_i^k, \mu_i^k)$. If $\|d^k\| < \epsilon$, then stop, otherwise go to Step 3.

*Step 3.* If $\theta_j, \sigma_k(x^k; d^k) \leq -\frac{1}{2} d^k d^k$ for all $j$, let $\sigma_{k+1} = \sigma_k$. Otherwise, $\sigma_{k+1}$ is updated using (20).

*Step 4.* Compute step length $\alpha_k$ as the first number in the sequence \{1, $r$, $r^2$, ...\} satisfying
\[
   \Psi_{j,\sigma_{k+1}}(x^k + \alpha_k d^k) - \Psi_{j,\sigma_{k+1}}(x^k) \leq \alpha_k \beta \theta_j, \sigma_{k+1}(x^k; d^k) \quad \forall \ j \in \Lambda_m.
\]

(21)
Step 5. Update $x^{k+1} = x^k + \alpha_k d^k$. Set $k := k + 1$ and go to Step 2.

4 Convergence

In this section the global convergence of Algorithm 1 is proved. The following lemma is used to establish that Step 4 is well-defined. The extension to the multi-objective case justifies the convergence analysis.

Lemma 2. Suppose $\nabla f_j(x)$ and $\nabla g_i(x)$ are Lipschitz continuous for every $j \in \Lambda_m$ and $i \in \Lambda_p$ with Lipschitz constant $L$ and let $(t^k, d^k)$ be the solution of the $QP(x^k)$ with $d^k \neq 0$. Then (21) holds for $\alpha$ sufficiently small.

Proof: Since $\nabla f_j(x)$ and $\nabla g_i(x)$ are Lipschitz continuous for every $j \in \Lambda_m$ and $i \in \Lambda_p$, from Lemma 2.1(3) of [2], there exists $L > 0$ such that

$$\Psi_{j,\sigma_{k+1}}(x^k + \alpha d^k) \leq \Psi_{j,\sigma_{k+1}}(x^k) + \alpha \theta_{j,\sigma_{k+1}}(x^k; d^k) + \frac{1}{2}(1 + \sigma_{k+1})La^2 \|d^k\|^2$$

holds for every $\alpha \in [0, 1]$. Hence for every $\alpha \in [0, 1]$ and $\beta \in (0, 1)$ (initialized in Step 1 of Algorithm 1) we have,

$$\Psi_{j,\sigma_{k+1}}(x^k + \alpha d^k) - \Psi_{j,\sigma_{k+1}}(x^k) - \beta \alpha \theta_{j,\sigma_{k+1}}(x^k; d^k)$$

$$\leq (1 - \beta)\alpha \theta_{j,\sigma_{k+1}}(x^k; d^k) + \frac{1}{2}(1 + \sigma_{k+1})La^2 \|d^k\|^2. \quad (22)$$

Since $d^k \neq 0$, from Step 3 of Algorithm 1,

$$(1 - \beta)\alpha \theta_{j,\sigma_{k+1}}(x^k; d^k) \leq \frac{1}{2}(1 - \beta)\|d^k\|^2 < 0.$$ 

Hence from (22), (21) holds for $\alpha > 0$ sufficiently small.

Lemma 3. Let $(t^k, d^k)$ be the solution of the subproblem $QP(x^k)$ and assume that the sequences $\{x^k\}$ and $\{(t^k, d^k)\}$ are bounded. If $x^k \to x^*$ as $k \to \infty$, then $\{(t^k, d^k)\}$ converges to $(t^*, d^*)$, where $(t^*, d^*)$ is the solution of $QP(x^*)$.

In particular, if $d^k$ converges to 0 and the MFCQ holds at every $x^k$ then $x^*$ is a strong critical point of the MOP.

Proof: If possible let $\{x^k\}$ converges to $x^*$ but $\{(t^k, d^k)\}$ does not converge to $(t^*, d^*)$. Since $\{(t^k, d^k)\}$ is bounded, there exists a subsequence $\{(t^k, d^k)\}_{k \in K}$ converging to $(t, d) \neq (t^*, d^*)$. Since $(t^k, d^k)$ is the optimal solution of $QP(x^k)$, there exists $(\lambda^k, \mu^k)$ such that $(t^k, d^k; \lambda^k, \mu^k)$ satisfies the KKT optimality conditions (8)-(13). Now (9) implies $\{\lambda^k\}$ and $\{\mu^k\}$ are bounded. Hence there exists a converging sub sequence of the subsequence $\{(\lambda^k, \mu^k)\}_{k \in K}$. Without loss of generality we may assume $\lambda^k \to \lambda^*$ and $\mu^k \to \mu^*$ as $k \to \infty$ and $k \in K$. 

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Hence in [8]-[13], taking limit \( k \to \infty \), \( k \in K \), we have
\[
\bar{d} + \sum_{j \in \Lambda_m} \lambda^*_j \nabla f_j(x^*) + \sum_{i \in \Lambda_p} \mu^*_i \nabla g_i(x^*) = 0,
\]
\[
1 - \sum_{j \in \Lambda_m} \lambda^*_j - \sum_{i \in \Lambda_p} \mu^*_i = 0,
\]
\[
\lambda^*_j \geq 0 \quad \lambda^*_j (\nabla f_j(x^*)^T \bar{d} - \bar{i}) = 0, \quad j \in \Lambda_m,
\]
\[
\mu^*_i \geq 0 \quad \mu^*_i (g_i(x^*) + \nabla g_i(x^*)^T \bar{d} - \bar{i}) = 0 \quad i \in \Lambda_p,
\]
\[
\nabla f_j(x^*)^T \bar{d} - \bar{i} \leq 0, \quad j \in \Lambda_m,
\]
\[
\nabla g_i(x^*)^T \bar{d} - \bar{i} \leq 0 \quad i \in \Lambda_p.
\]
These imply that \((\bar{t}, \bar{d}; \lambda^*, \mu^*)\) satisfies first order necessary conditions of the convex quadratic programming subproblem \(QP(x^*)\). Hence \((\bar{t}, \bar{d})\) is an optimal solution of \(QP(x^*)\). This contradicts the fact that \((t^*, d^*)\) is the optimal solution of \(QP(x^*)\), since \(QP(x^*)\) has unique solution. Hence \(\{(t^k, d^k)\}\) converges to \((t^*, d^*)\).

In particular, if \(d^k\) converges to 0 and the MFCQ holds at every \(x^k\) then replacing \(d^*\) by 0 in [8]-[13] at \((x^k, t^*, d^*, \lambda^*, \mu^*)\) and proceeding as in Lemma [11] it is easy to prove that \(x^*\) is a strong critical point of the MOP.

**Lemma 4.** Suppose that \(\sigma_k = \sigma > 0\) for \(k\) large enough, the sequences \(\{x^k\}\) and \(\{(t^k, d^k)\}\) are bounded, \(\nabla f_j(x)\) and \(\nabla g_i(x)\) are Lipschitz continuous for every \(j \in \Lambda_m\) and \(i \in \Lambda_p\) with Lipschitz constant \(L\), and \(\{x^k\}_{k \in K}\) is a convergent subsequence. Then \(d^k \to 0\) as \(k \to \infty\) and \(k \in K\).

**Proof:** Without loss of generality, assume that \(\sigma_k = \sigma\) for all \(k \in K\). If possible suppose that there exists an infinite subset \(K' \subset K\) and a positive constant \(\eta\) such that
\[
\|d^k\| \geq \eta, \quad k \in K'.
\]
First we will show that there exists \(\delta > 0\) such that \(\alpha_k \geq \delta\) holds for every \(k\), where \(\alpha_k\) is the step length obtained in Step [3] of Algorithm [1]. From this step either \(\alpha_k = 1\) or \(\alpha_k = r^k\) holds for some \(k_1 \in \mathbb{N}\). If \(\alpha_k = r^k\) holds then there exists \(j \in \Lambda_m\) satisfying,
\[
\Psi_{j, \sigma}(x^k + r^k \cdot d^k) - \Psi_{j, \sigma}(x^k) > r^k \cdot \beta \theta_{j, \sigma}(x^k; d^k).
\]
Then from (22),
\[
\frac{1}{2} (1 + \sigma) L r^2 (k_1 - 1) \|d^k\|^2 \geq -r^k \cdot (1 - \beta) \theta_{j, \sigma}(x^k; d^k).
\]
From Step [3] of Algorithm [1]
\[
\frac{1}{2} (1 + \sigma) L r^k \cdot \|d^k\|^2 \geq \frac{1}{2} (1 - \beta) \|d^k\|^2
\]
\[
\Rightarrow r^k \geq \frac{r (1 - \beta)}{(1 + \sigma) L}
\]
Choose $\delta = \min\{1, \frac{r(1-\beta)}{1+\sigma} L\}$. Then $\alpha_k \geq \delta$ holds for every $k$. Now

$$\Psi_{j,\sigma}(x^{k+1}) - \Psi_{j,\sigma}(x^0) = \sum_{l=0}^k \Psi_{j,\sigma}(x^l + \alpha_l d^l) - \Psi_{j,\sigma}(x^l) = \sum_{l=0}^k \beta \alpha_k \|d^k\|^2 \leq -(k+1)\delta \beta \eta^2$$

for $\forall k \in K'$ (from (23)).

This implies $\Psi_{j,\sigma}(x^k + \alpha_k d^k) \to -\infty$ as $k \to \infty$ and $k \in K'$ (since $\alpha_0 \beta \eta^2 > 0$).

This contradicts the assumption that $\{x^k\}$, $\{(t^k, d^k)\}$ are bounded as $\Psi_{j,\sigma}$ is a continuous function. So there does not exist any $K' \subset K$ and $\eta > 0$ such that (23) holds. Hence the lemma follows.

**Lemma 5.** If $\sigma_k \to \infty$ and the sequences $\{x^k\}$, $\{(t^k, d^k)\}$ are bounded then $\lim_{k \to \infty} \Phi(x^k) = 0$.

**Proof:** Proof of this result is similar to the proof of Lemma 7, [27].

**Theorem 2.** Let $\{x^k\}$ be a sequence generated by Algorithm 1, the sequences $\{x^k\}$ and $\{(t^k, d^k)\}$ are bounded, $\nabla f_j(x)$ and $\nabla g_i(x)$ are Lipschitz continuous for every $j \in \Lambda_m$ and $i \in \Lambda_p$ with Lipschitz constant $L$, and the MFCQ is satisfied at every $x^k$. Then any accumulation point of $\{x^k\}$ is a critical point (either weak or strong critical point) of the MOP.

**Proof:**

(i) Convergence to a strong critical point:

Let $K$ be an infinite index set such that $x^k \to x^*$ as $k \to \infty$ and $k \in K$. Let $\{(t^k, d^k)\}$ be the solution of the $QP(x^k)$. If $d^k \to 0$ as $k \to \infty$ then by Lemma 3, $x^*$ is a strong critical point of MOP.

(ii) Convergence to a weak critical point:

If there exists a constant $c_0 > 0$ such that $\|d^k\| \geq c_0$ for large $k \in K$, then from Lemma 4, $\sigma_k \to \infty$ as $k \to \infty$. Consider $D(x^k)$ in Definition 2 which takes care all objective functions. Proof of Theorem 1 of [27] is valid for this new $D(x^k)$. As a result $\{x^k\}$ converges to a weak critical point of MOP.

Hence any accumulation point of $\{x^k\}$ is either a strong critical point or a weak critical point.

5 Numerical illustration and discussion

In this section the proposed method (Algorithm 1) is compared with a classical method (weighted sum method) and the method developed by Fliege and Vaz [11]. In order to compare different methods we use the performance profiles presented in [11, 5, 38, 39] with respect to the purity metric and the $\Gamma$ and $\Delta$ spread metrics. (The readers may see the details in [11, 5]).

Test problems: A set of test problems, collected from different sources, are summarized in Tables 1 and 2. Bound constrained test problems are summarized in Table 1. Linear and nonlinear constrained test problems are summarized in
Table 2 In Table 2, ‘linear’ is the number of linear constraints except bound constraints, and ‘nonlinear’ is the number of nonlinear constraints. In both tables m is the number of objective functions and n represents the number of variables.

| Problem | Source | m | n | Linear | Nonlinear |
|---------|--------|---|---|--------|-----------|
| ABC2d1 | 127    | 2 | 2 | 1      | 2         |
| ABC2d2 | 127    | 2 | 2 | 0      | 2         |
| ABC2d3 | 127    | 1 | 1 | 1      | 1         |
| ABC2d4 | 127    | 1 | 1 | 1      | 1         |
| ABC2d5 | 127    | 2 | 2 | 0      | 2         |
| ABC2d6 | 127    | 2 | 2 | 0      | 2         |
| ABC2d7 | 127    | 2 | 2 | 0      | 2         |
| ABC2d8 | 127    | 2 | 2 | 0      | 2         |

Table 1: Multi-objective test problems with bound constraints

| Problem | Source | m | n | Linear | Nonlinear |
|---------|--------|---|---|--------|-----------|
| ABCComp |        | 2 | 2 | 1      | 2         |
| BC2d1   |        | 1 | 1 | 1      | 1         |
| BC2d2   |        | 1 | 1 | 1      | 1         |
| BC2d3   |        | 1 | 1 | 1      | 1         |
| BC2d4   |        | 1 | 1 | 1      | 1         |
| BC2d5   |        | 1 | 1 | 1      | 1         |
| BC2d6   |        | 1 | 1 | 1      | 1         |
| BC2d7   |        | 1 | 1 | 1      | 1         |
| BC2d8   |        | 1 | 1 | 1      | 1         |
| BC2d9   |        | 2 | 2 | 0      | 2         |

Table 2: Multi-objective test problems with linear and nonlinear constraints

**Implementation details:** Following three methods are compared here.

**Algorithm 1** MOSQP.

**Algorithm 2** Weighted sum method -MOS.

**Algorithm 3** SQP method [11]-MOSQP(F).

MATLAB code (2016a) is developed for all three methods. The MATLAB code of MOSQP(F) is available in public domain, which is not used here. For MOSQP(F), we have developed own code which uses only the Step 4 (third stage) of Algorithm 4.1 [11] since the convergence analysis of Algorithm 4.1 [11] is different from the convergence analysis of MOSQP. Multi start techniques, similar to MOSQP, is used to generate an approximated Pareto front for MOSQP(F).

- Quadratic subproblems are solved using MATLAB function ‘quadprog’ with ‘Algorithm’, ‘interior-point-convex’.
For MOS, the test problems are converted to single-objective optimization problems and solved using MATLAB function ‘fmincon’ with ‘Algorithm’ ‘sqp’, Specified ‘objective gradient’ and ‘constraint gradient’, and initial approximation as \((l+u)/2\), where \(l\) and \(u\) are used as in [11].

\(\|d^{k}\| < 10^{-5}\) or a maximum of 500 iterations are considered as stopping criteria.

It is essential to find a set of well distributed solutions of MOP. Spreading out an approximation to a Pareto set is a difficult problem. One simple technique may not work always in a satisfactory manner for all type of problems. Here, to generate an approximated Pareto front, we have selected the initial point with the strategies \textit{LINE} and \textit{RAND} and random parameters in the scalarization method. \textit{LINE} is considered only for bi-objective optimization problems and \textit{RAND} is considered for both bi-objective and more than two objective optimization problems.

- For bi-objective optimization problems we have considered the strategy for selecting initial points as \textit{LINE}. Here 100 initial points are chosen in the line segment joining \(l\) and \(u\), i.e. \(x^{0,k} = l + k \frac{u-l}{99}\), \(k = 0, 1, 2, ..., 99\) and for MOS we have solved problems of the form \(\min_{x \in X} w f_1(x) + (1-w) f_2(x)\) for \(w = k/99\), \(k = 0, 1, 2, ..., 99\).

- For every test (two or three objective) problem initial points selection strategy \textit{RAND} is considered. Here 100 random initial points are selected uniformly distributed in \(l\) and \(u\), and for MOS we have solved \(\min_{x \in X} \sum_{j \in \Lambda_m} w_j f_j(x) \geq 0 \ \ w \neq 0\), where \(w\) is a random vector. Every test problem is executed 10 times with random initial points and weights.

Restoration procedure is not used for MOSQP(F) if the quadratic subproblem is infeasible. These points are excluded. Quadratic subproblem \((QP(x^k))\) of Algorithm [1] always has a solution since this is a convex quadratic problem and has at least one feasible solution.

In strategy \textit{LINE}, Figures [1a] and [1b] correspond to the performance profiles for the purity metric comparing MOSQP with MOS and MOSQP with MOSQP(F), respectively. The performance profile using the \(\Gamma\) metric with the strategy \textit{LINE} between MOSQP and MOS is provided in Figure [2a] and between MOSQP and MOSQP(F) is provided in Figure [2b]. The performance profile using the \(\Delta\) metric with the strategy \textit{LINE} between MOSQP and MOS is provided in Figure [3a] and between MOSQP and MOSQP(F) is provided in Figure [3b].

In \textit{RAND}, the average of 10 purity metric values obtained in 10 different runs is denoted as the average purity metric value. Average \(\Gamma\) and \(\Delta\) metric values
Figure 1: Purity performance profile in **LINE**

(a) purity between MOSQP and MOS
(b) purity between MOSQP and MOSQP(F)

Figure 2: $\Gamma$ performance profile in **LINE**

(a) $\Gamma$ between MOSQP and MOS
(b) $\Gamma$ between MOSQP and MOSQP(F)

Figure 3: $\Delta$ performance profile in **LINE**

(a) $\Delta$ between MOSQP and MOS
(b) $\Delta$ between MOSQP and MOSQP(F)
are obtained similarly. Average purity performance profile between MOSQP and MOS is provided in Figure 4a and average purity performance profile between MOSQP and MOSQP(F) is provided in Figure 4b. Average Γ performance profile in **RAND**, between MOSQP and MOS and between MOSQP and MOSQP(F) are provided in Figures 5a and 5b respectively. Average Δ performance profile between MOSQP and MOS is provided in Figure 6a and average Δ performance profile between MOSQP and MOSQP(F) is provided in Figure 6b.

![Figure 4: Average purity performance profile in **RAND**](image)

(a) purity between MOSQP and MOS  
(b) purity between MOSQP and MOSQP(F)

![Figure 5: Average Γ performance profile in **RAND**](image)

(a) Γ between MOSQP and MOS  
(b) Γ between MOSQP and MOSQP(F)
Figure 6: Average $\Delta$ performance profile in $\textsc{RAND}$

One may observe from the above figures that using the initial point selection strategy $\textsc{LINE}$ the method (MOSQP) proposed in this paper gives better results than MOS and MOSQP(F) in purity and $\Gamma$ metrics. Similarly using initial point selection strategy $\textsc{RAND}$ MOSQP gives better results than MOS and MOSQP(F) in $\Gamma$ and $\Delta$ metrics.

6 Conclusion

In this paper we have developed a globally convergent modified SQP method for constrained multi-objective optimization problem. This method is free from any kind of a priori chosen parameters or ordering information of objective functions. To generate an approximated Pareto front, we have used the initial point selection strategies $\textsc{LINE}$ and $\textsc{RAND}$. There is no single spreading technique for line search methods that can work in a satisfactory manner for all types of multi-objective programming problems. Spreading out an approximation to a Pareto front is a difficult task. A well distributed spreading technique is discussed in Step 3 of Algorithm 1.4 of [11]. We keep the implementation of these techniques for future developments.

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