AN OPTIMAL OSMOTIC CONTROL PROBLEM FOR A
CONCRETE DAM SYSTEM

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Abstract. In this paper, an optimal control problem for a concrete dam system is considered. First, a mathematical model on the optimal osmotic control for the basis of concrete dams is built up, and an optimal line-wise control of the system governed by the hybrid problem for elliptic partial differential equations is investigated. Then, the regularity of the generalized solution to the adjoint state equations, and the existence and uniqueness of the $L^2$-solution for state equations are discussed and examined. Subsequently, the existence and uniqueness of the optimal control for the system, and a necessary and sufficient conditions for a control to be optimal and the optimality system are claimed and derived. Finally, the applications of the penalty shifting method with calculation of the optimal control of the system are studied, and the convergence of the method on an appropriate Hilbert space is claimed and proved.

1. Introduction. We are concerned with a concrete dam that has been built and water has already been stored in the reservoir, the osmosis will emerge in the body of the dam, the basis of the dam and the mountain rocks on both sides of the dam. Here, the osmosis emerge in the mountain rocks on both sides of the dam is called the dam-detouring osmotic problem. This problem was discussed in our article [12]. It should be noted that the problem treated in [12] was only about its calculation. In the present paper, we shall deeply investigate the penalty shifting method with calculation of the optimal osmotic control, discuss the structure of an approximation program, and claim and prove the convergence of the method used for the dam system control in appropriate Hilbert space.

To treat an optimal osmotic control problem for the concrete dam, first of all, we shall build up a mathematical model of the practical osmotic control problem for the basis of the concrete dam in terms of the law of conservation of mass and Darcy Law, which is an optimal line-wise control system formulated by a hybrid problem for elliptic partial differential equations with boundary conditions. Then, we shall utilize prior estimates and compactness theorem, and claim the regularity

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of the generalized solution for adjoint state equations, and prove the existence and uniqueness of the $L^2$-solution for state equations. By virtue of Gâteaux differentiation and Lions’s theory of variational inequalities, we shall subsequently derive a necessary and sufficient conditions for a control to be optimal, and obtain the optimality system consisting of elliptic partial differential equations and variational inequalities, which determines the optimal control. Finally, we shall focus on application of the optimal control of osmotic systems, and examine structure of the approximation program and the convergence of the method used in an appropriate Hilbert space.

2. Mathematical model of practical control system. In this section, we shall establish a mathematical model for a concrete dam in which water has already been stored in the reservoir, the osmosis will emerge in the basis of the dam. Since what we see happens here produces a great osmotic pressure that may even destroy the basis of the dam, we are facing a significant control problem for the basis of the dam. In general, it is carried out by passing drainage through a group of discharge pores located in the basis of the dam (Fig. 1) in order to reduce osmotic pressure. The osmotic water from the discharge pores is concentrated to the drainpipe in the corridor setting in the dam, and then this osmotic water is drained.

Let’s consider that the basis of the dam is formed of permeable and non-permeable stratum. The discharge pore is set in the permeable stratum (Fig. 1). Let’s now introduce three-dimensional Cartesian coordinate system $Ox_1x_2x_3$, where the permeable stratum is denoted by the region $\Omega \subset \mathbb{R}^3$ (Fig. 2), the boundary of the region $\Omega$ is denoted by $\Gamma$, and the interface between the permeable stratum $\Omega$ and the non-permeable stratum is denoted by $\Gamma_1$ (see Figs. 1, 2). Setting $\Gamma_0 = \Gamma \setminus \Gamma_1$, we have $\Gamma = \Gamma_0 \cup \Gamma_1$.

Let $m$ denote the number of discharge pores in $\Omega$ (see Fig. 2). Since the area of the section of the discharge pore is smaller than the area of the section of the region $\Omega$, we can look on discharge pores as line segments in $\Omega$, denoted by $M_iN_i$, $i = 1, 2, \cdots, m$. Let $M_i = (a_i, b_i, 0)$ denote the coordinate of the point $M_i$ $(a_i, b_i, 0)$, and $N_i = (a_i, b_i, h)$ $(h \geq 0)$. We assume that the intensity of the sink flow at the point $x = (a_i, b_i, x_3)$ $(0 \leq x_3 \leq h)$ in $M_iN_i$ is $-v_i(x_3) \geq 0$, the osmosis is a steady-state laminar flow, the fluid is a non-compressible, the matrix of penetration coefficient at $x \in \Omega$ is $[k_{ij}(x)]$, the fluid density is the constant $\rho$, the acceleration of gravity is $g$, the fluid dynamo-viscosity is $\mu$, the osmotic pressure at $x \in \Omega$ is $p(x)$. We define the osmotic potential $y(x)$ by the equality

$$y(x) = -x_3 + p(x)/(\rho g).$$

Seeing [11, 2] for general introduction to hybrid system and fluid dynamics, we now claim by means of the law of conservation of mass and Darcy Law that the osmotic potential function $y(x)$ is defined as a solution of the following hybrid problem for the elliptic partial differential equation with boundary conditions of mixed type:

$$Ay = \sum_{i=1}^{m} v_i(x_3) \delta(x_1 - a_i, x_2 - b_i) \quad \text{in} \ \Omega,$$

$$y(x) = y_0(x) \quad \text{on} \ \Gamma_0, \quad \frac{\partial y(x)}{\partial \nu} = 0 \quad \text{on} \ \Gamma_1,$$
where \( y_0(x) \) is a function with \( x \in \Gamma_0 \), and \( A \) is the uniform elliptic differential operator:

\[
(H_1) \quad Ay = -\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial y}{\partial x_j} \right) + a(x)y, \quad x \in \Omega,
\]

\[
(H_2) \quad a_0 \sum_{i=1}^{3} \xi_i^2 \leq \sum_{i,j=1}^{3} a_{ij}(x) \xi_i \xi_j \leq \alpha_1 \sum_{i=1}^{3} \xi_i^2 \quad \forall \xi_i \in R^1, \alpha_0, \alpha_1 > 0,
\]

\[
a_{ij}(x) = k_{ij}(x) \rho^2 g/\mu \geq a_0, \quad a(x) \geq a_0 > 0,
\]

\[
\delta \left( x_1 - a_i, x_2 - b_i \right) \text{ is Dirac Mass concentrated at point } (a_i, b_i) \text{ on } \Gamma_1 \text{ ([p79, [4]]), that is,}
\]

\[
\int_{\Omega} \delta(x_1 - a_i, x_2 - b_i)\varphi(x)dx = \int_{h}^{0} \varphi(a_i, b_i, x_3)dx_3 \quad \forall \varphi \in L^2(0, h; C^2(\Omega_{1,2}))
\]

where \( C^2(\Omega_{1,2}) \) is the function space of all functions whose second derivatives are continuous on \( \Omega_{1,2} \),

where \( \Omega_{1,2} = \Omega \cap \{h = 0\} \), \( \frac{\partial y}{\partial \nu} \) is the “co-normal” derivative with respect to \( A \):

\[
\begin{align*}
\frac{\partial y}{\partial \nu}|_{\Gamma_1} &= \sum_{i,j=1}^{3} a_{ij}(x) \frac{\partial y}{\partial x_j} \cos(n, x_i)|_{\Gamma_1}, \\
\cos(n, x_i) &= \text{ith direction cosine of } n, \\
\text{being the normal at } \Gamma_1 \text{ exterior to } \Omega.
\end{align*}
\]

The solution \( y(x) \) of the problem \((2.1)–(2.2)\) is obviously dependent on \( v \), where \( v = v(x_3) = (v_1(x_3), v_2(x_3), \cdots, v_m(x_3)) \), and hence we write it as \( y(x, v) \) or \( y(v) \).

Let \( z_d \in L^2(\Omega) \) denote a perfect osmotic potential giving in the engineering design. Then, our optimal osmotic control problem for the basis of the dam can be stated as follows:

\[
\underbrace{\text{Find } v(x_3) \text{ such that not only } \|y(v) - z_d\|_{L^2(\Omega)} \text{ is as small as possible, but also } \|v\|_{\mathcal{U}} \text{ is as small as possible,}}}_{(2.8)}
\]

where

\[
\mathcal{U} = \{ v \mid \|v\| \leq 0, \ v \in (L^2(0, h))^m \}.
\]

where \( (L^2(0, h))^m \) is the \( m \) - Cartesian product of \( L^2(0, h) \). It is clear that \( \mathcal{U} \) is closed non empty convex subset of \( (L^2(0, h))^m \).
From the discussion above, we are naturally led to the following cost functional:

\[
I(v) = \|y(v) - z_d\|_{L^2(\Omega)}^2 + N\|v\|_{V}^2, \quad N > 0, \tag{2.10}
\]

where \(y(v)\) is the solution of the problem (2.1)-(2.2). Then we can now reduce the optimal control problem (2.8) to the following minimization problem:

Find \(u \in \mathcal{U}\) satisfying \(I(u) = \inf_{v \in \mathcal{U}} I(v)\). \tag{2.11}

For the hybrid problem (2.1)-(2.2), the cost functional (2.10) and the minimization problem (2.11) constitute the mathematical model on the optimal osmotic control problem of the basis of the dam, which is called the optimal line-wise control of the system governed by the hybrid problem for elliptic partial differential equations. In (2.11), the element \(u \in \mathcal{U}\) for which \(I(v)\) attains its minimum, is termed the optimal line-wise control of the system, or the optimal control for short, and \(\{u, y(u)\}\) is said to be the optimal pair.

The reference [4] discussed the optimal boundary control of the system governed by the semi-linear elliptic equations with Dirichlet boundary conditions. The reference [10] studied same optimal Venttsel boundary control problems of elliptic equations with the Venttsel boundary conditions. We shall investigate an optimal line-wise control of the osmotic system governed by a hybrid problem for elliptic equations in this paper. It should be emphasized that there is an quite difference between the two cases. Actually, the control problem we shall discuss in this paper is more peculiar than the problems discussed in the both references [12, 10], and hence the results in this paper have both theoretical and practical significance for research on optimal control problems.

3. Regularity of solution for adjoint hybrid problem. Let’s start this section with investigation of regularity of the solutions of the problem (2.1) and (2.2) in order to find a \(L^2\) solution of the problem. So, we need to find a regular solution for the adjoint hybrid problem of (2.1)-(2.2) in \(H^2(\Omega) \cap V_1\), where \(V_1 = \{\varphi | \varphi \in H^1(\Omega), \varphi|_{\Gamma_0} = 0\}\). Eventually, we are led to the following adjoint hybrid problem

\[
A^*\varphi = \psi \quad \text{in} \in \Omega, \tag{3.1}
\]

\[
\varphi = 0 \quad \text{on} \Gamma_0, \quad \frac{\partial \varphi}{\partial \nu^*} = 0 \quad \text{on} \Gamma_1. \tag{3.2}
\]
From \((H_1)\) in the section 2 we see that
\[
(H_1^*) \quad A^* \varphi = - \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}(a_{ji}(x) \frac{\partial \varphi}{\partial x_j}) + a(x) \varphi. \tag{3.3}
\]

For the problem of the equation (3.1) with Dirichlet boundary condition, the regularity of the generalized solution was discussed by [7]. It should be emphasized that the article [7] did not discuss the regularity of the generalized solution of the hybrid problem for the equation (3.1) with boundary condition (3.2) of mixed type. Now, we shall study the regularity of the generalized solution of the hybrid problem (3.1)-(3.2).

We propose the following hypotheses:

\[
(H_3) \quad a_{ij} \in H^1(\Omega), \quad a \in L^2(\Omega), \quad \psi \in L^2(\Omega); \tag{3.4}
\]

\((H_4)\) \(\Omega\) is a pre-parallelepiped in \(\mathbb{R}^3\) with boundary \(\Gamma\), where \(\partial \Omega = \Gamma = \Gamma_0 \cup \Gamma_1\) is a 2-dimensional variety, \(\Gamma \in C^2(\Gamma), \Gamma_1, \Gamma_2\) and \(\Gamma_0 \setminus \Gamma_2\) are a base plane, upper base plane and lateral face of \(\Omega\) respectively, \(S = \Gamma_1 \cap \Gamma_0\) (Fig. 2).

Let \(H^r(E)\) denote the usual Sobolev Space of order \(r\) over \(E\), where \(r\) is a read number and \(E = \Omega\) or \(\Gamma_i, i = 0, 1, 2\), et al ([9]). Set
\[
\|\cdot\|_E = \|\cdot\|_{H^r(E)}, \quad \|\cdot\|_E = \|\cdot\|_{L^2(E)}, \tag{3.5}
\]
and let’s introduce and prove some Lemmas. In view of the Example 3.3 of Chapter 1 in Lions [8], pp.23-25], we can obtain the following lemma:

**Lemma 3.1.** [7] Assume that \((H_1)-(H_4)\) and \((H_1^*)\) hold, and that \(f \in L^2(\Omega), g \in H^{-\frac{1}{2}}(\Gamma)\). There exists a unique \(\varphi \in H^1(\Omega)\) such that
\[
A^* \varphi = f \quad \text{in} \ \Omega, \tag{3.6}
\]
\[
\frac{\partial \varphi}{\partial \nu} = g \quad \text{on} \ \Gamma. \tag{3.7}
\]
This function \(\varphi \in H^1(\Omega)\) satisfying (3.6)-(3.7) is called the generalized solution of the problem (3.6)-(3.7).

**Remark 1.** Let \(f \in L^2(\Omega), g \in L^2(\Omega)\). For the solution \(\varphi \in H^1(\Omega)\) of (3.6)-(3.7), utilizing prior estimates, we may deduce
\[
(\|\varphi\|_o^{(1)})^2 \leq C_1(\|f\|_O^2 + \|g\|_P^2), \tag{3.8}
\]
where constant \(C_1\) is independent of \(\varphi\).

With the following hypothesis \((H_5)\) \(\Omega' \subset \Omega, \partial \Omega' = \Gamma', \mes(\Omega')\) is a sufficiently small such that the operator \(A^*\) in \(\Omega'\) satisfies the inequality (4.26) of Chapter 3 in [7], i.e.
\[
\delta \equiv [2c_1 + \frac{8}{\alpha_0}c_2]c_0 \mes(\Omega') < 1,
\]
where
\[
c_1 = \frac{1}{3}[\frac{12M(2\alpha_0 + 1)}{\alpha_0^2}]^4, \quad \alpha_0 = \frac{1}{\mes(\Omega')} \int_{\Omega'} a(x)dx,
\]
\[
M = \|a - a_0\|_{\Omega'}, \quad \text{constant} \ c_0 \ \text{is independent of} \ \Omega'.
\]
we shall have another lemma as follows:
Lemma 3.2. Assume that \((H_1) - (H_5), (H_5^*)\) hold and \(\Omega' \subset \Omega\).

1) If \(\xi(x)\) belongs to \(C^2(\Omega')\) and \(\xi(x) = 0\) on \(\partial \Omega' = \Gamma'\). Then, for any function \(\varphi\) in \(H^2(\Omega)\), we have

\[
(\|\varphi\xi^2\|_{L^2(\Gamma')}^2)^2 \leq \frac{2}{a_0} \|\xi^2 A^* \varphi\|^2_{L^2(\Gamma')} + C_2 \int_{\Omega'} \varphi^2 (\xi^4 + \xi^2 \xi_{xx}) \, dx,
\]

where the constant \(C_2\) is independent of \(\varphi\) and \(\xi\).

2) If \(\Gamma' = \Gamma_0 + \Gamma_1\) and \(\Gamma_1 = \Gamma_1\), and \(\xi(x)\) belongs to \(C^2(\Omega')\) and \(\xi(x) = 0\) in the neighborhood of \((\Gamma' \setminus \Gamma_1)\). Then, for any function \(\varphi\) in \(H^2(\Omega)\) with \(\frac{\partial \varphi}{\partial n^*}|_{\Gamma'} = 0\), the inequality (3.9) holds.

Proof. Part 1) of Lemma is the formula (8.6) of Chapter 3 in [7]. Now, let's discuss part 2) of Lemma. Repeating the process of the proof for Lemma 8.2 of Chapter 3 in [7], we can arrive at the estimate:

\[
(\|\varphi\xi^2\|_{L^2(\Gamma')}^2)^2 \leq \frac{2}{a_0} \|\xi^2 A^* \varphi\|^2_{L^2(\Gamma')} + C_2 \int_{\Omega'} \varphi^2 (\xi^4 + \xi^2 \xi_{xx}) \, dx + I,
\]

where

\[
I = \int_{\Gamma_1} \xi^4 a_{ij} \varphi x_j [a_{lk} \varphi x_l \cos(n, x_i) - (a_{lk} \varphi x_l) x_i \cos(n, x_k)] \, d\Gamma = I_1 + I_2.
\]

We see from the hypothesis \(\frac{\partial \varphi}{\partial n^*}|_{\Gamma'} = 0\) and Definition (2.7) that

\[
I_1 = \int_{\Gamma_1} \xi^4 (a_{lk} \varphi x_l) x_i \frac{1}{a_{ji}} a_{kj} \varphi x_j \cos(n, x_k) \, d\Gamma = \int_{\Gamma_1} \xi^4 (a_{lk} \varphi x_l) x_i \frac{\partial \varphi}{\partial n^*} \, d\Gamma = 0.
\]

Considering \(a_{ij}(x) \geq a_0 > 0\) of the hypothesis \((H_2)\) and (2.7) leads to the following inequalities

\[
|I_2| \leq \int_{\Gamma'} |\xi^4 (a_{lk} \varphi x_l) x_i a_{ji} a_{kj} \varphi x_j \cos(n, x_k)| \, d\Gamma \\
\leq C_3 \int_{\Gamma'} |a_{kj} \varphi x_j \cos(n, x_k)| \, d\Gamma = C_3 \int_{\Gamma'} |\frac{\partial \varphi}{\partial n^*}| \, d\Gamma = 0.
\]

Applying the resulting expression above and (3.10) implies (3.9), and the Lemma 3.2 is established now.

We now define a function spaces interrelating with \(A^*\) as follows:

\[
K^* \equiv \{ \varphi \in H^2(\Omega), A^* \varphi = 0, \frac{\partial \varphi}{\partial n^*}|_{\Gamma} = 0 \},
\]

\[
K \equiv \{ \psi_1 | \psi_1 \in H^2(\Omega), A \psi_1 = 0, \frac{\partial \psi_1}{\partial n}|_{\Gamma} = 0 \},
\]

\[
\int_{\Omega} f \psi_1 \, dx + \int_{\Gamma} g \psi_1 \, d\Gamma = 0, \forall \psi_1 \in K,
\]

\[
\mathcal{H}^0(\Omega) \times \mathcal{H}^{\frac{1}{2}}(\Gamma) \equiv \{(f, g) | (f, g) \in L^2(\Omega) \times H^{\frac{1}{2}}(\Gamma) \text{ with } (f, g) \text{ satisfying } (3.13)\}
\]

By means of the Theorem 5.3 of the Chapter 2 in [pp.198-199, [9]], we can obtain the following lemma:

Lemma 3.3 ([9]). Assume that \((H_1) - (H_4)\) and \((H_5^*)\) hold. Then, for any given \((f, g) \in \mathcal{H}^0(\Omega) \times \mathcal{H}^{\frac{1}{2}}(\Gamma)\), there exists a unique solution \(\varphi(x)\) of the problem (3.6)-(3.7) in quotient space \(H^2(\Omega)/K^*\), where \(H^2(\Omega)/K^*\) is the quotient space of \(H^2(\Omega)\)
by \( K^* \), that is, \( \varphi = \varphi_1 + \varphi_2 \), where \( \varphi_1 \) is determined uniquely in \( H^2(\Omega) \), and \( \varphi_2 \) is in \( K^* \).

**Lemma 3.4.** Suppose that the hypotheses of Lemma 3.3 hold. Then for any given \((f, g) \in L^2(\Omega) \times H^{\frac{1}{2}}(\Gamma)\), there exists a unique solution \( \varphi \) of the problem (3.6)-(3.7) in \( H^2(\Omega) \).

**Proof.** From Lemma 3.1, there exists a unique generalized solution \( \varphi \) of (3.6)-(3.7) in \( H^1(\Omega) \). On the other hand, we can claim and prove that

\[
K^* = \{0\}. \tag{3.15}
\]

It can be seen from the Definition (3.11) that the related expression (3.15) is equivalent to the fact that following problem

\[
A^* \varphi = 0 \text{ in } \Omega, \tag{3.16}
\]

\[
\frac{\partial \varphi}{\partial \nu^*} = 0 \text{ on } \Gamma \tag{3.17}
\]

has only zero solution. Multiplying (3.16) by \( \varphi \) and integrating it by parts yield over \( \Omega \) yield

\[
\int_{\Omega} [a_{ji} \varphi x_i \varphi x_j + a \varphi^2] \, dx = \int_{\Gamma} \varphi \frac{\partial \varphi}{\partial \nu^*} \, d\Gamma. \tag{3.18}
\]

Eventually, we have from (3.17) and (3.18) that

\[
\int_{\Omega} (a_{ji} \varphi x_i \varphi x_j + a \varphi^2) \, dx = 0. \tag{3.19}
\]

From (\( H_2) \) and (\( H_1^* \)) we deduce that

\[
\int_{\Omega} (a_{ji} \varphi x_i \varphi x_j + a \varphi^2) \, dx \geq \alpha_0 \sum_{i=1}^{3} (\varphi_{x_i}^2 + \varphi^2) \, dx \geq 0. \tag{3.20}
\]

Combining (3.19) and (3.20) with \( \alpha_0 > 0 \), we see that \( \varphi = 0 \) a.e. in \( \Omega \), and hence we obtain (3.15). Carrying on the same discussion, we can conclude that \( K = \{0\} \), and therefore for any \((f, g) \in L^2(\Omega) \times H^{\frac{1}{2}}(\Gamma)\), the related expression (2.13) is valid.

Thus, we obtain the result: for any \((f, g) \in L^2(\Omega) \times H^{\frac{1}{2}}(\Gamma)\), there exists a unique solution \( \varphi \) of (3.6)-(3.7) in \( H^2(\Omega) \). The Lemma 3.4 is established.

From Lemma 3.4 we can derive the following Theorem:

**Theorem 3.5.** Assume that (\( H_1 \)) – (\( H_4 \)) and (\( H_1^* \)) hold. Let \((f, g) \in L^2(\Omega) \times H^{\frac{1}{2}}(\Gamma)\). Then, the problem (3.6)-(3.7) has a unique generalized solution \( \varphi \) in \( H^1(\Omega) \), and this solution \( \varphi \) will belong to \( H^2(\Omega) \), which satisfies equation (3.6) in \( \Omega \) and the boundary condition (3.7) on \( \Gamma \).

Now, let’s discuss the generalized solution of the problem (3.1)-(3.2). Let

\[
V_1 = \{ \varphi | \varphi \in H^1(\Omega), \varphi|_{\Gamma_0} = 0 \} \tag{3.21}
\]

\[
\pi^*(\varphi, \eta) = \sum_{i,j=1}^{3} \int_{\Omega} (a_{ji} \frac{\partial \varphi}{\partial x_i} \frac{\partial \eta}{\partial x_j} + a \varphi \eta) \, dx, \tag{3.22}
\]

and

\[
L(\eta) = \int_{\Omega} \psi \eta \, dx. \tag{3.23}
\]
Theorem 3.6. Assume that \((H_1) - (H_4)\) and \((H^*_1)\) hold. Then there exists a unique \(\varphi \in V_1\) such that
\[
\pi^*(\varphi, \eta) = L(\eta) \quad \forall \eta \in V_1. \tag{3.24}
\]
the function \(\varphi \in V_1\) satisfying (3.24) is called the generalized solution of the problem (3.1)-(3.2) in \(V_1\).

Proof. From \((H_1), (H_2)\) and \((H^*_1)\), we can deduce that the continuous bi-linear form \(\pi^*\) on \(V_1\) is coercive:
\[
\pi^*(\varphi, \varphi) \geq \alpha_0 \|\varphi\|_{V_1}^2 \quad \forall \varphi \in V_1, \alpha_0 > 0.
\]
Then, based on Lax-Milgram Lemma (see [8]), for the continuous linear form \(L(\eta)\), there exists a unique \(\varphi \in V_1\) such that the equality (3.24) is valid. (3.24) is equivalent to (3.1)-(3.2), provided we use the definition of the derivative in the distribution sense or the dual sense. Theorem 3.2 is established now. \(\square\)

Theorem 3.7. Assume that \((H_1) - (H_2)\) and \((H^*_1)\) hold. Let \(\varphi \in V_1\) be the generalized solution of (3.1)-(3.2). Let \(\Omega_2 \subset \Omega\). Then this solution \(\varphi \in V_1\) belong to \(H^2(\Omega_2)\) and satisfies the equation (3.1) in \(\Omega_2\).

Proof. Let \(x_0\) be arbitrary given point in \(\Omega_2\), and let
\[K_\rho(x_0) = \{x \mid |x - x_0| < \rho\},\]
which is a boll in \(\Omega_2\), where \(\rho > 0\) is sufficiently small such that \(\overline{K_\rho(x_0)} \subset \Omega_2\), and \(K_\rho(x_0)\) as \(\Omega\) in \((H_5)\), it satisfies conditions in \((H_5)\). Since \(C^2(K_\rho)\) is dense in \(H^1(K_\rho)\) and \(L^2(K_\rho)\) respectively (cf.Ref.[9]), there exists the sequence \(\{\varphi_n(x)\} \subset C^2(K_\rho)\) such that
\[
\|\varphi_m - \varphi\|_{K_\rho}^{(1)} \to 0, \quad \|\varphi_m - \varphi\|_{K_\rho} \to 0 \quad \text{as} \quad m \to \infty, \tag{3.25}
\]
where \(\varphi \in V_1\) is the solution of (3.1)-(3.2).

Let’s turn to consider the following problem:
\[
\begin{cases}
A^* \varphi = \psi & \text{in } K_\rho, \\
\frac{\partial \varphi}{\partial \nu^*} |_{\Gamma_\rho} = \frac{\partial \varphi_m}{\partial \nu^*} |_{\Gamma_\rho}, \quad m = 1, 2, \ldots, \tag{3.26}
\end{cases}
\]
where \(\Gamma_\rho = \partial K_\rho\). According to Trace theorem ([Theorem 3.2 in Chapter 1, p.22, [8]]), we see that \(\frac{\partial \varphi_m}{\partial \nu^*} |_{\Gamma_\rho} \in H^{\frac{1}{2}}(\Gamma_\rho)\). By virtue of Lemma 3.4 and Theorem 3.1,
\[
\text{for given every integer } m > 0, \text{there exists a unique solution } \overline{\varphi}_m \text{ of the problem (3.26) in } H^2(K_\rho). \tag{3.27}
\]
Consequently, with the inequality (3.8) in the Remark 3.1 we see that
\[
\left(\|\overline{\varphi}_m\|_{K_\rho}^{(1)}\right)^2 \leq C_4 (\|\psi\|_{K_\rho}^2 + \|\frac{\partial \varphi_m}{\partial \nu^*}\|_{\Gamma_\rho}^2). \tag{3.28}
\]
By virtue of (3.25), it is easy to see that the right side of (3.28) is uniformly bounded, and then we have that
\[
\|\overline{\varphi}_m\|_{K_\rho}^{(1)} \leq C_5, \quad m = 1, 2, \ldots, \tag{3.29}
\]
where the constant \(C_5 > 0\) is independent of \(\overline{\varphi}_m\). As for the function \(\overline{\varphi}_m \equiv \varphi\), in light of (3.27), the inequality (3.9) in Lemma 3.2 follows.

Based on (3.29) and (3.28), it can be seen that the right side of (3.9) is uniformly bounded, and
\[
\|\overline{\varphi}_m \xi^2\|_{K_\rho}^{(2)} \leq C_6(\xi), \quad m = 1, 2, \ldots, \tag{3.30}
\]
where \( \xi(x) \) belong to \( C^2(K_\rho) \) and \( \xi(x) \) is equal to zero on \( \Gamma_\rho, C_6 > 0 \) is independent of \( \varphi_m \).

In view of the Theorem 6.2 (Rellich-Kondrachov’s Theorem) of Chapter 6 in [7], we can deduce that the injection map of \( H^1(K_\rho) \) into \( L^2(K_\rho) \) is compact. Then, from (3.29) we may extract a subsequence of the sequence \( \{ \varphi_m \} \), still denoted \( \{ \varphi_m \} \), such that as \( m \to +\infty \)

\[
\varphi_m \to \varphi \text{ in } L^2(K_\rho) \text{ strongly.} \tag{3.31}
\]

By virtue of the result (1.3.12) in [p.33, [3]], we may extract a subsequence of the sequence \( \{ \varphi_m \} \), still denoted by \( \{ \varphi_m \} \) from (3.29)-(3.20) such that

\[
\varphi_m \to \varphi \text{ in } H^1(K_\rho) \text{ weakly and in } H^2(K_{\rho-\rho/\tau}), \quad \tau = 2, 3, \cdots, \text{ weakly.} \tag{3.32}
\]

According to the Theorem 1 of the Section 1 in the Chapter 5 [11, p.120], we see that \( \varphi \) satisfies (3.29)-(3.30), and hence

\[
\varphi \in H^1(K_\rho), \quad \varphi \in H^2(K_{\rho-\rho/\tau}). \tag{3.33}
\]

From (3.27) and (3.31)-(3.33), it follows that \( \varphi \) satisfies the equation (3.26) in \( K_{\rho-\rho/\tau} \).

First, we shall prove that \( \varphi = \varphi_m \) in \( K_\rho \). For arbitrary function \( \eta \) in \( H^1(\Omega) \), we have that

\[
L_\rho^*(\varphi, \eta) = \int_{K_\rho} \psi \eta dx + \int_{\Gamma_\rho} \frac{\partial \varphi}{\partial \nu} \eta d\Gamma,
\]

\[
L_\rho^*(\varphi, \eta) = \int_{K_\rho} \psi \eta dx + \int_{\Gamma_\rho} \frac{\partial \varphi_m}{\partial \nu} \eta d\Gamma.
\]

Since the trace map of \( \varphi \in H^2(K_\rho) \to \varphi \in H^2(\Gamma_\rho) \) is a continuous map ([8, Theorem 4.4 in Chapter 1]), we see that as \( m \to +\infty \)

\[
L_\rho^*(\varphi - \varphi_m, \eta) = \int_{\Gamma_\rho} \frac{\partial}{\partial \nu} \varphi_m \eta d\Gamma \to 0.
\]

Hence, if \( \varphi - \varphi_m = (\varphi - \varphi_m) + (\varphi_m - \varphi_m) \) and \( \eta = \varphi_m - \varphi_m \), \( \varphi_m \to \varphi \) as \( m \to +\infty \), we have

\[
L_\rho^*(\varphi - \varphi_m, \eta) = L_\rho^*(\varphi - \varphi_m, \eta) + L_\rho^*(\varphi_m - \varphi_m, \eta) = L_\rho^*(\varphi - \varphi_m, \varphi_m - \varphi_m) \to 0 \text{ as } m \to +\infty.
\]

By virtue of (3.35)-(3.36), it follows that

\[
L_\rho^*(\varphi_m - \varphi_m, \varphi_m - \varphi_m) \to 0 \text{ as } m \to +\infty.
\]

From (3.37) and (3.34), we see that

\[
\|\varphi_m - \varphi_m\|_{K_\rho} \to 0 \text{ as } m \to +\infty.
\]

Combining (3.25),(3.32) and (3.38) leads to the fact that \( \varphi = \varphi_m \) in \( K_\rho \), and

\[
0 \leq \|\varphi - \varphi\|_{K_\rho} \leq \|\varphi - \varphi_m\|_{K_\rho} + \|\varphi_m - \varphi_m\|_{K_\rho} + \|\varphi_m - \varphi\|_{K_\rho} \to 0 \text{ as } m \to +\infty.
\]

Since \( \|\varphi - \varphi\|_{K_\rho} \) is independent of \( m \), it follows that \( \|\varphi - \varphi\|_{K_\rho} = 0 \), i.e. \( \varphi = \varphi_m \) in \( K_{\rho}(x_0) \). Then one can see from (3.33) that \( \varphi \in H^2(K_{\rho-\rho/\tau}(x_0)) \) and \( \varphi \) satisfies (3.26) in \( K_{\rho-\rho/\tau}(x_0) \). Because of arbitrariness of the point \( x_0 \in \Omega_2 \), we can now
assert that \( \varphi \in H^2(\Omega_2) \) and \( \varphi \) satisfies the equation (3.1) in \( \Omega_2 \). Theorem 3.3 is established now.

**Theorem 3.8.** Suppose that (H\(_1\)) (H\(_{1i}\)) and (H\(_{1i}\)) hold. If \( \varphi \in V_1 \) is the generalized solution of the problem (3.1)-(3.2), and the domain \( \Omega_1 \subset \Omega, \Gamma_1' \) is the boundary of \( \Omega_1, \Gamma_1' \subset C^2(\Gamma_1'), \Gamma_1' \cap \Gamma = \Gamma_1 \) (Fig. 3, Fig. 4). Then, the solution \( \varphi \) belongs to \( H^2(\Omega_1) \) and \( \varphi \) satisfies the equation (3.1) in \( \Omega_1 \) with the boundary condition \( \frac{\partial \varphi}{\partial \nu'}|_{\Gamma_1'} = 0 \).

**Proof.** It is clear from (3.2) that \( \frac{\partial \varphi}{\partial \nu'}|_{\Gamma_1'} = 0 \). Take a sequence \( \{ \varphi_m \} \subset H^2(\Omega_1) \) with \( \frac{\partial \varphi_m}{\partial \nu'}|_{\Gamma_1'} = 0 \) such that

\[
\varphi_m \rightarrow \varphi \quad \text{in} \quad H^1(\Omega_1), \quad m = 1, 2, \ldots . \tag{3.39}
\]

and consider the following problem:

\[
A' = \varphi \quad \text{in} \quad \Omega_1, \tag{3.40}
\]

\[
\frac{\partial \varphi}{\partial \nu'}|_{\Gamma_1'} = \frac{\partial \varphi_m}{\partial \nu'}|_{\Gamma_1'}, \quad m = 1, 2, \ldots . \tag{3.41}
\]

It implies from the Trace Theorem in [9] that \( \frac{\partial \varphi_m}{\partial \nu'}|_{\Gamma_1'} \in H^{\frac{1}{2}}(\Gamma_1') \). The Lemma 3.4 and Theorem 3.1 show that there exists a unique solution \( \varphi_m \) of the problem (3.40)-(3.41) in \( H^2(\Omega_1) \) for given every integer \( m > 0 \), and \( \varphi_m \) is the generalized solution of (3.40)-(3.41) in \( H^1(\Omega_1) \). By virtue of (3.8), we can obtain that

\[
\| \varphi_m \|_{H^2(\Omega_1)} \leq C_7, \tag{3.42}
\]

with a similar manner with the deduction of (3.29), where the constant \( C_7 > 0 \) is independent of \( \varphi_m \). Without loss of generality, we might assume that \( \text{mes}(\Omega_1) \), and \( \text{mes}(\Omega') \) satisfies conditions in (H\(_3\)). Then, it follows directly from (3.9) in Part 2) of Lemma 3.2 that

\[
\left( \| \varphi_m \|^2_{1, \Omega_1} \right) \leq C_8(\xi), \tag{3.43}
\]

where \( C_8(\xi) \) is independent of \( \varphi_m, \xi(x) \in C^2(\Omega_1) \), and \( \xi(x) \) is equal to zero on \( \Gamma_1' \setminus \Gamma_1 \). Let \( \varphi \) be the limit function of \( \varphi_m \) as \( m \to +\infty \). Similarly to the deduction of Theorem 3.3, we can show that \( \varphi \) belongs to \( H^2(\Omega_1) \), \( \varphi \) satisfies the equation (3.1) in \( \Omega_1 \) and \( \varphi = \varphi \in \Omega_1 \). Thus, the conclusion that \( \varphi \in H^2(\Omega_1) \) and \( \varphi \) satisfies the equation (3.1) in \( \Omega_1 \).

Now, let’s claim and prove that \( \frac{\partial \varphi}{\partial \nu'}|_{\Gamma_1} = \frac{\partial \varphi_m}{\partial \nu'}|_{\Gamma_1} = 0 \). From \( \frac{\partial \varphi_m}{\partial \nu'}|_{\Gamma_1} = 0 \) and the continuity of trace [8], we have that

\[
\frac{\partial \varphi_m}{\partial \nu'}|_{\Gamma_1} = (\lim_{m \to +\infty} \frac{\partial \varphi_m}{\partial \nu'})(|_{\Gamma_1}) = \lim_{m \to +\infty} (\frac{\partial \varphi_m}{\partial \nu'})(|_{\Gamma_1}) = 0.
\]

On the basis of the equalities above and \( \varphi = \varphi \in \Omega_1 \), it is easy to see that \( \frac{\partial \varphi}{\partial \nu'}|_{\Gamma_1} = \frac{\partial \varphi_m}{\partial \nu'}|_{\Gamma_1} = 0 \), and Theorem 3.4 is established.

**Theorem 3.9.** Suppose that (H\(_1\)) (H\(_{1i}\)) and (H\(_{1i}\)) hold. If \( \Omega \) satisfies (H\(_1\)) and \( \varphi \) is the generalized solution of (3.1)-(3.2) in \( V_1 \). Then, \( \varphi \) belongs to \( H^2(\Omega) \) and it satisfies equations (3.1)-(3.2), i.e. the problem (3.1)-(3.2) has a unique regular solution \( \varphi \) in \( H^2(\Omega) \cap V_1 \), where \( V_1 \) is defined by (3.21).

**Proof.** Since the domain \( \Omega \) is divided into three piece \( \Omega_1, \Omega_2 \) and \( \Omega_0 \), i.e. \( \Omega = \Omega_1 \cup \Omega_2 \cup \Omega_0 \) such that domains \( \Omega_1, \Omega_2 \) and \( \Omega_0 \) have all conical property and their boundaries are all piecewise smooth surfaces; moreover, \( \Omega_1 \subset \Omega, \Omega_2 \subset \Omega \) boundaries of \( \Omega, \Omega_1 \) and \( \Omega_0 \) are denoted by \( \Gamma, \Gamma_1 \) and \( \Gamma_0 \) respectively, the common piece between
Γ′₀ and the boundary Γ of Ω is Γ₀, i.e. Γ′₀ ∩ Γ = Γ₀(= Γ\Γ₁), and the common piece between Γ′₁ and Γ is Γ₁, i.e. Γ′₁ ∩ Γ = Γ₁(Figs. 3, 4).

1) For Ω₁, in light of Theorem 3.4, we see that φ ∈ H²(Ω₁) and φ satisfies the equation (3.1) in Ω₁ with the boundary condition ∂φ/∂ν|Γ₁ = 0.

2) For Ω₂, by means of Theorem 3.3, we derive that φ ∈ H²(Ω₂) and φ satisfies the equation (3.1) in Ω₂.

3) For Ω₀, by virtue of Theorem 10.1 of Chapter 3 in [7], we assert that φ ∈ H²(Ω₀) and φ satisfies the equation (3.1) in Ω₀ with the boundary condition φ|Γ₀ = 0.

With the hypothesis Ω = Ω₁∪Ω₂∪Ω₀ and the results 1),2),3) above, the desired conclusion that for ψ ∈ L²(Ω), the problem (3.1)-(3.2) has a unique regular solution in H²(Ω) ∩ V₁ follows. The proof of the Theorem 3.5 is complete.

The result of Theorem 3.5 above shows the regularity of the generalized solution for the adjoint hybrid problem (3.1)-(3.2).

4. Existence and uniqueness of L²-solutions for the state equations. As we know from the section 2, the state y(x) of the osmotic control system is defined
by the solution of the problem (2.1)-(2.2). In this section, we shall define a \( L^2 \)-solution \( y \) of state equations (2.1)-(2.2) in terms of the regular solution of adjoint equations (3.1)-(3.2) in Section 3.

Without loss of generality, let's discuss the problem (2.1)-(2.2) with \( m = 1 \):

\[ Ay = v(x_3)\delta(x_1 - a, x_2 - b) \text{ in } \Omega, \quad (4.1) \]
\[ y = y_0(x) \text{ on } \Gamma_0, \quad \frac{\partial y}{\partial \nu} = 0 \text{ on } \Gamma_1, \quad (4.2) \]

where \((a, b, x_3)(0 \leq x_3 \leq h) \in \Omega \) and \((y_0, v)\) satisfies hypothesis \((H_6)\):

\[(H_6) \quad v \in \mathcal{V} = \{ v \mid v \geq 0, v \in L^2(0, h) \}, \quad (4.3) \]
\[ y_0 \in L^2(\Gamma_0) \subset H^{-\frac{1}{2}}(\Gamma_0), \quad (y_0 + x_3) > 0. \quad (4.4) \]

With the hypothesis \((H_1) - (H_4)\) and \((H_6^*)\), we see from the Theorem 3.5 that the adjoint problem

\[ A^* \varphi = \psi \text{ in } \Omega, \quad \psi \in L^2(\Omega), \quad (4.5) \]
\[ \varphi = 0 \text{ on } \Gamma_0, \quad \frac{\partial \varphi}{\partial \nu^*} = 0 \text{ on } \Gamma_1 \quad (4.6) \]

has a unique regular solution \( \varphi \) in \( V \), where

\[ V = H^2(\Omega) \cap V_1, \quad (4.7) \]

which indicates that

\[ A^* \text{ is an isomorphism of } V \text{ onto } L^2(\Omega); \quad (4.8) \]
\[ \psi \to \varphi \text{ is a continuous linear map of } L^2(\Omega) \to V, \text{ denoted by } \varphi = \varphi(x; \psi). \quad (4.9) \]

Let \( \varphi \in V \) is the solution of (4.5)-(4.6). Multiplying the equation (4.1) by \( \varphi \), utilizing Green’s formula and noting (4.2) and (4.6) we obtain

\[ \int_\Omega y A^* \varphi dx = \int_0^h v(x_3)\varphi(a, b, x_3)dx_3 + \int_{\Gamma_0} y_0 \frac{\partial \varphi}{\partial \nu^*}d\Gamma \quad \forall \varphi \in V. \quad (4.10) \]

Conversely, if \( y \) satisfies (4.10) for “all” functions \( \varphi \in V \), (4.1)-(4.2) is obtained in the sense of distributions on \( \Omega \). Thus, we have

**Definition 4.1.** For arbitrary given \((v, y_0)\) satisfying conditions (4.3)-(4.4) in \((H_6)\), the function \( y \in L^2(\Omega) \) is called a \( L^2 \)-solution of the problem (4.1)-(4.2), if \( y \) satisfies the identity (4.10) for all \( \varphi \in V \).

**Theorem 4.2.** Assume that \((H_1) - (H_4), (H_6)\) and \((H_6^*)\) hold. Then, there exists a unique function \( y(v) \in L^2(\Omega) \) satisfying the identity (4.10), that is, \( y(v) \) is a unique \( L^2 \)-solution of the problem (4.1)-(4.2), and

\[ \text{mapping } v \to y(v) \text{ is a continuous affine map of } \mathcal{V} \to L^2(\Omega). \quad (4.11) \]

**Proof.** From the Sobolev embedding theorem ([[1], Theorem 6.2]), we have that

\[ H^2(\Omega) \hookrightarrow C^0([0, h]) \hookrightarrow L^2(0, h), \]

and then for \( \varphi \in V \),

\[ \varphi \to \varphi(a, b, x_3) \text{ is a continuous linear map of } V \to L^2(0, h). \quad (4.12) \]
On the base of the Trace Theorem in [8], we have
\[
\begin{cases}
\varphi \to \frac{\partial \varphi}{\partial \nu^*}|_{\Gamma_0} \text{ is a continuous linear map} \\
of V \to H^2(\Gamma_0).
\end{cases}
\] (4.13)

Set
\[
\mathcal{L}_1(\varphi) = \int_0^h v(x_3)\varphi(a, b, x_3)dx_3 + \int_{\Gamma_0} y_0 \frac{\partial \varphi}{\partial \nu^*}d\Gamma.
\] (4.14)

From (4.9) and (4.12)-(4.14) it follows that
\[
\begin{cases}
\text{setting } \mathcal{L}(\psi) \equiv \mathcal{L}_1(\varphi(x; \psi)) = \int_0^h v(x_3)\varphi(a, b, x_3; \psi)dx_3 + \int_{\Gamma_0} y_0 \frac{\partial \varphi(x; \psi)}{\partial \nu^*}d\Gamma,
\mathcal{L}(\psi) \text{ is a continuous linear functional on } L^2(\Omega).
\end{cases}
\] (4.15)

Applying Riesz’s Representation Theorem with (4.15) implies that there exists a unique \( y \in L^2(\Omega) \) such that identity
\[
\int_\Omega y\varphi dx = \mathcal{L}(\psi)
\] (4.16)

is true. In light of (4.15) and \( \psi = A^*\varphi \) in (4.5), it can be seen that (4.16) becomes (4.10), and there exists a unique \( L^2 \)-solution \( y \) of the problem (4.1)-(4.2) in \( L^2(\Omega) \).

In light of the structure of the right part for the identity (4.10), we may subsequently assert that the mapping \( v \to y(v) \) is a continuous affine map of \( \mathcal{W} \to L^2(\Omega) \), and the Theorem 4.1 is established.

\( \square \)

5. Existence and uniqueness of optimal control and optimality system.

We now start this section with the following result:

**Theorem 5.1.** Assume that \((H_1) - (H_4)\) and \((H_6)^*\) and \((H_6)\) hold. Let the cost function \( I(v) \) be given by (2.10), where \( y(v) \in L^2(\Omega) \) is the solution of (4.1)-(4.2). Then the optimal control problem (2.11) admits a unique solution \( u \in \mathcal{W} \), which is characterized by

\[
\int_\Omega (y(v) - z_d)(y(v) - y(u))dx + N \int_0^h u(v - u)dx_3 \geq 0 \quad \forall v \in \mathcal{W}, \; u \in \mathcal{W},
\] (5.1)

that is, the necessary and sufficient condition for a control to be optimal is \( u \) satisfies the variational inequality (5.1).

**Proof.** In view of the definition of \( I(v) \) in (2.10) and Theorem 4.1, it follows that the functional \( v \to I(v) \) is continuous from \( \mathcal{W} \) to \( R \), which is strictly convex with \( I(v) \geq N\|v\|^2_{\mathcal{W}}, N > 0 \), hence \( I(v) \to +\infty \) as \( \|v\|_{\mathcal{W}} \to +\infty \). By virtue of the Remark 1.1, the Theorem 1.3 of the Section 1 and the Chapter 1 in [6], we see that there exists a unique element \( u \in \mathcal{W} \) such that the equality \( I(u) = \inf_{v \in \mathcal{W}} I(v) \) is true and \( u \) is characterized by

\[
I'(u)(v - u) \geq 0 \quad \forall v \in \mathcal{W},
\] (5.2)

where \( I'(u)(v - u) \) is Gâteaux differential value of \( I(v) \), evaluated at \( u \) along the direction \( (v - u) \) [10].

A simple calculation in terms of (2.10) shows that

\[
I'(u)(v - u) = 2 \int_\Omega (y(u) - z_d)(y(v) - y(u))dx + 2N \int_0^h u(v - u)dx_3.
\] (5.3)
Consequently, the inequality (5.2) becomes (5.1) in light of (5.3). The proof of the Theorem 5.1 is complete now.

Next, we shall transform (5.1) by utilizing the adjoint state, and introduce the adjoint state $p(u)$ by
\[
A^*p(u) = y(u) - z_d \text{ in } \Omega, \tag{5.4}
\]
\[
p(u) = 0 \text{ on } \Gamma_0, \quad \frac{\partial p(u)}{\partial \nu^*} = 0 \text{ on } \Gamma_1. \tag{5.5}
\]
According to the Theorem 3.5, the problem (5.4)-(5.5) admits a unique solution $p(u)$ in $V$. Hence, applying Green's formula with (4.1)-(4.2) and (5.4)-(5.5) yields
\[
\int_{\Omega} (y(u) - z_d)(y(v) - y(u)) \, dx = \int_{0}^{h} p(a, b, x_3; u)(v - u) \, dx_3. \tag{5.6}
\]
and so the inequality (5.1) becomes
\[
\int_{0}^{h} [p(a, b, x_3; u) + Nu](v - u) \, dx_3 \geq 0 \quad \forall v \in \mathcal{U} \tag{5.7}
\]
by virtue of (5.6).

As a result discussed above, we actually obtained the following significant result.

**Theorem 5.2.** The state $y(v)$ is defined by the solution of (4.1)-(4.2). The optimal control $u \in \mathcal{U}$ corresponding the cost functional (2.10) is determined by the optimality system consisting of equations (4.1)-(4.2) (where $v = u$) and (5.4)-(5.5) with the variational inequality (5.7).

6. **Penalty shifting method for digital approximation.** This section is devoted to investigating applications of the penalty shifting method [12, 13] to calculate the optimal control of osmotic systems for the basis of the dam. It has been seen that the optimal control problem (2.11) can be written as the minimization problem as follows:

\[
\begin{cases}
\text{with respect to } (y(v), v) \text{ under constraints (4.1)-(4.2) and } v \in \mathcal{U}, \\
\text{find } u \in \mathcal{U} \text{ satisfying the equality } \inf_{v \in \mathcal{U}} I(y(v), v) = I(y(u), u), \\
\text{where } I(y(v), v) = I(v) \text{ in (2.10)}. \tag{6.1}
\end{cases}
\]

The main idea here for application is to approximate the solution $(y(u), u)$ of the constrained minimization problem (6.1) by a family $\{y_k, u_k\}$ of solutions to the non-constrained minimization problem in which variables $y$ and $v$ are mutually independent.

First, let’s introduce a space for investigation:
\[
Y = \left\{y | y \in L^2(\Omega), A\, y \in W'(\Omega), y|_{\Gamma_0} \in L^2(\Gamma_0), \left. \frac{\partial y}{\partial \nu} \right|_{\Gamma_1} = 0 \right\}. \tag{6.2}
\]
where $W = L^2(0, h; H_0^2(\Omega_{1,2}))$, $W' = L^2(0, h; H^{-2}(\Omega_{1,2}))$ is the dual space of $W$, $\Omega_{1,2} = \Omega \cap \{x_3 = 0\}$. In light of the fact that $\delta(-a, \cdot - b) \in H^{-2}(\Omega_{1,2})$ (see (2.6)) $\delta \cdot (-a, -b) \in L^2(0, h; H^{-2}(\Omega_{1,2})) \equiv W'$.

It is easy to see that the space $Y$ equipped with the norm
\[
\|y\|_Y = (\|y\|_{H^2} + \|Ay\|_{W'} + \|y\|_{L^2})^2, \tag{6.3}
\]
forms a Hilbert space. Suppose that $c \geq 0$, $\mu = (\lambda, \eta)$, $\lambda \in W$, $\eta \in L^2(\Gamma_0)$, and define an augmented Lagrangian:

$$J(y, v, c, \mu) = I(y, v) + c[|Ay - v\delta|^2_{\alpha} + |y - y_0|^2_{\eta}] + \langle \lambda, Ay - v\delta \rangle + [\eta, y - y_0]_{\Gamma_0}$$  \hspace{1cm} (6.4)

on the space $Y \times \mathcal{W}$, where $|||e||| = |||e|||_{\mathcal{L}^2(\mathcal{E})}$, $[\alpha, \beta]_{\mathcal{E}}$ denotes the scalar product in $L^2(E)$, $(\alpha_1, \beta_1)$ denotes the dual product between $W$ and $W'$, $\alpha_1 \in W$, $\beta_1 \in W'$. In $J(y, v, c, \mu)$, the variables $y$ and $v$ are two mutually independent. Since $W$ is a Hilbert space (see [8]), we see from the self-conjugality for a Hilbert space that there exists a unique function $(\psi_A - v^0) \in W$ satisfying the follow equalities:

\[
\begin{align*}
(\lambda, Ay - v\delta)_{W'} &= [\lambda, \psi_A - v^0]_W,
(\lambda, Ay - v\delta)(\lambda) &= [\lambda, \psi_A - v^0]_W, \forall \lambda \in W, (\psi_A - v^0) \in W, \\
|||Ay - v\delta|||_{W'} &= |||\psi_A - v^0|||_W.
\end{align*}
\]

Lemma 6.1. The minimization problem

$$\inf_{y \in Y, v \in \mathcal{W}} J(y, v, c, \mu)$$  \hspace{1cm} (6.6)

admits a unique solution $(\hat{y}, \hat{u}) \in Y \times \mathcal{W}$, for any given $\mu$ and $c > 0$.

Proof. Assume that

$$g = (y, v), \quad G = Y \times \mathcal{W}, \quad |||g|||_G^2 = |||y|||_Y^2 + |||v|||_{\mathcal{W}}^2.$$  \hspace{1cm} (6.7)

Then the problem (6.6) becomes

$$\inf_{g \in G} J(g, c, \mu).$$  \hspace{1cm} (6.8)

From the definition (6.4) and notations (6.7), we may assert that $J(g, c, \mu) \to +\infty$ as $|||g|||_G \to +\infty$. Moreover, it can be easily verified from (6.7),(6.4),(2.10) and (4.11) that $J(g, c, \mu)$ is also strictly convex and strongly continuous. With the Remark 1.2 of Chapter 1 in [8], there exists a unique element $\hat{g} \in G$ such that

$$J(\hat{g}, c, \mu) = \inf_{g \in G} J(g, c, \mu).$$  \hspace{1cm} (6.9)

Eventually, we can conclude from the notation (6.7) that the problem (6.6) admits a unique solution $\hat{g} = (\hat{y}, \hat{u}) \in Y \times \mathcal{W} = G$. The proof of Lemma 6.1 is complete now. \hfill $\square$

Let $J'(\bar{y}, \bar{v}, c, \mu, y - \bar{y}, v - \bar{v})$ be Gâteaux differential value of $J(y, v, c, \mu)$ evaluated at $(\bar{y}, \bar{v})$ along the direction $(y - \bar{y}, v - \bar{v})$ (see [9]).

Lemma 6.2. Let arbitrary point $(\bar{y}, \bar{v})$ be given in $Y \times \mathcal{W}$. Then for any given $\mu$ and $c > 0$, we have :

\[
J(y, v, c, \mu) = J(\bar{y}, \bar{v}, c, \mu) + |||y - \bar{y}|||_{\alpha}^2 + N||v - \bar{v}||_{\beta}^2 \\
+ c[\|A(y - \bar{y}) - (v - \bar{v})\delta\|_{W'}^2 + ||y - \bar{y}||_{\eta}^2] \\
+ J'(\bar{y}, \bar{v}, c, \mu, y - \bar{y}, v - \bar{v}) \quad \forall (y, v) \in Y \times \mathcal{W}. \hspace{1cm} (6.10)
\]

Proof. Since

$$||\alpha||_{E}^2 - ||\bar{\alpha}||_{E}^2 = ||\alpha - \bar{\alpha}||_{E}^2 + 2[\alpha - \bar{\alpha}, \bar{\alpha}]_E,$$  \hspace{1cm} (6.11)

We see from (6.4), (2.10), (6.5) and (6.11) that

$$J(y, v, c, \mu) - J(\bar{y}, \bar{v}, c, \mu)$$

\[
= ||y - \bar{y}||_\alpha^2 + N||v - \bar{v}||_\beta^2 + c[\|\psi_A - \bar{\psi}_A - (v^0 - \bar{v}^0)\|_{W'}^2 + ||y - \bar{y}||_{\eta}^2]
\]
the optimality condition (5.7), we may arrive at the following result:

Since \((\bar{v}, \bar{v})_h = [v - \bar{v}, v - \bar{v}]_{L^2(0,h)}\), In view of Gâteaux differentiation in [3] and (6.4)-(6.5), we have

\[
J'(\bar{\gamma}, \bar{v}, c, \mu, y - \bar{\gamma}, v - \bar{v}) = \lim_{\theta \to 0^+} \frac{1}{\theta} \left( J(\bar{\gamma} + \theta(y - \bar{\gamma}), \bar{v} + \theta(v - \bar{v}), c, \mu) - J(\bar{\gamma}, \bar{v}, c, \mu) \right)
\]

\[
= 2[y - \bar{\gamma}, \bar{v} - z_d]_\Omega + 2N[v - \bar{v}, v]_h + 2c[(\psi_A - \bar{\psi}_A) - (v^0 - \bar{v}^0)]_W + 2c[y - \bar{\gamma}, y - \bar{\gamma}]_h + [\eta, y - \bar{\gamma}]_{\Gamma_0}.
\]

Subtracting (6.13) from (6.12) yields

\[
J(y, v, c, \mu) - J(\bar{\gamma}, \bar{v}, c, \mu,) = J'(\bar{\gamma}, \bar{v}, c, \mu, y - \bar{\gamma}, v - \bar{v})
\]

\[
= \|y - \bar{\gamma}\|_\Omega^2 + N\|v - \bar{v}\|_h^2 + c\|A(y - \bar{\gamma}) - (v - \bar{v})\|_W^2, \quad \forall (y, v) \in Y \times \mathcal{W}.
\]

\[
J'(\bar{\gamma}, \bar{v}, c, \mu, y - \bar{\gamma}, v - \bar{v}) \geq 0 \quad \forall (y, v) \in Y \times \mathcal{W},
\]

and therefore (6.10) holds. The Lemma 6.2 is established now. □

**Lemma 6.3.** Suppose that \((\bar{\gamma}, \bar{v})\) is the minimizing point of \(J(y, v, c, \mu)\) in \(Y \times \mathcal{W}\). Then for any \(\mu \geq 0\) and \(c > 0\), we have

\[
J(y, v, c, \mu) \geq J(\bar{\gamma}, \bar{v}, c, \mu) + \|y - \bar{\gamma}\|_\Omega^2 + N\|v - \bar{v}\|_h^2 + c\|A(y - \bar{\gamma}) - (v - \bar{v})\|_W^2, \quad \forall (y, v) \in Y \times \mathcal{W}.
\]

**Proof.** Since \((\bar{\gamma}, \bar{v})\) is the minimizing point of \(J(y, v, c, \mu)\), then (6.14) follows from (6.10)(where \(\bar{\gamma} = \gamma, \bar{v} = \bar{v}\)), and the necessary optimality condition is given by Theorem 1.3 of Chapter 1 in [8] as follows

\[
J'(\bar{\gamma}, \bar{v}, c, \mu, y - \bar{\gamma}, v - \bar{v}) \geq 0 \quad \forall (y, v) \in Y \times \mathcal{W},
\]

and the Lemma 6.3 is established. □

**Lemma 6.4.** Suppose that \((\bar{\gamma}, u)\) is the solution of the optimal problem (6.1), where \(\bar{\gamma} = \bar{\gamma}(x, u)\). Then there exists \(\mu = (\lambda, \eta)\) such that:

\[
J(y, v, 0, \mu) \geq J(\bar{\gamma}, u) + \|y - \bar{\gamma}\|_\Omega^2 + N\|v - u\|_h^2 \quad \forall (y, v) \in Y \times \mathcal{W}.
\]

**Proof.** Let \(p(x, u)\) denote the adjoint state given by the equations (5.4)-(5.5) (where \(y(u) = \bar{\gamma}(u)\)). From Theorem 3.5 and (4.12), we see that \(p(x, u) \in V \subset W\) and \(p(a, b, \cdot, u) \in L^2(0, h)\). Setting

\[
\lambda = -2p \quad \text{in} \quad \Omega, \quad \eta = 2\frac{\partial p}{\partial v^*} \quad \text{on} \quad \Gamma_0.
\]

and applying the Green's formula in [9] with the equations (4.1)-(4.2) and (5.4)-(5.5) (where \(y = \bar{\gamma}, u = u\)) yields:

\[
\langle p, A(y - \bar{\gamma}) \rangle_W = [y - z_d, y - \bar{\gamma}]_\Omega + \langle \frac{\partial p}{\partial v^*}, y - y_0 \rangle_{\Gamma_0},
\]

In terms of (6.10), (6.4) (where \(c = 0\), (4.1)-(4.2)(where \(y = \bar{\gamma}, v = u\)) (6.18) and the optimality condition (5.7), we may arrive at the following result:

\[
J'(\bar{\gamma}, u, 0, \mu, y - \bar{\gamma}, v - u) = J(y, v, 0, \mu) - J(\bar{\gamma}, u, 0, \mu) - \|y - \bar{\gamma}\|_\Omega^2 - N\|v - u\|_h^2
\]

\[
= 2\langle p, A(y - \bar{\gamma}) \rangle_W + 2N[v - u, u]_h - 2\langle p, Ay - vu \rangle_W,
\]

\[
= 2\int_0^h (p(a, b, x; u) + Nu)(v - u)dx \geq 0.
\]
Since \((\bar{y}, u)\) is the solution of the problem (4.1)-(4.2)(where \(\bar{y}, v = u\)), we have
\[
J(\bar{y}, u, c, \mu) = I(\bar{y}, u).
\] (6.19)
In consideration of (6.10) (where \(y = \bar{y}, \tau = u, c = 0\)), we can subsequently obtain
\[
J(y, v, 0, \mu) = I(\bar{y}, u) + \|y - \bar{y}\|^2_{\Omega} + N\|v - u\|^2_{\Omega}
+ J'(\bar{y}, u, 0, \mu, y - \bar{y}, v - u), \quad \forall (y, v) \in Y \times \mathcal{U}.
\] (6.20)
Hence, the inequality (6.16) follows from (6.21) and (6.19), and the proof of Lemma 6.4 is complete. 

Let’s consider the minimizing sequence \(\{(y_k, u_k)\}\) obtained by employing the multiplier adjustment rule, we have
\[
\begin{aligned}
\lambda_{k+1} &= \lambda_k + \alpha c(\psi_{A_k} - u_k^0) \quad \text{in } \Omega, \\
\eta_{k+1} &= \eta_k + c(y_k - y_0) \quad \text{on } \Gamma_0,
\end{aligned}
\] (6.21)
where
\[
\|\psi_{A_k} - u_k^0\|_W = \|Ay - u_k\delta\|_W, \quad \langle \lambda_k, Ay_k - u_k\delta \rangle = \langle \lambda_k, \psi_{A_k} - u_k^0 \rangle_W.
\] (6.22)
and \((y_0, u_0^0)\) is the initial point of the minimizing sequence (see (6.5)). In (6.22), \(0 \leq \alpha \leq 2\), \(\mu_0 = (\lambda_0, \eta_0)\) is any given initial value in \(W \times L^2(\Gamma_0)\). Next, let’s propose and prove the following important result:

**Theorem 6.5.** The Sequence \(\{(y_k, u_k)\}\) converges strongly in \(Y \times \mathcal{U}\) to the optimal solution \((y(u), u)\) of the problem (6.1).

**Proof.** Let \(\mu = (\lambda, \eta)\) be multiplier introduced by (6.17) in the proof of Lemma 6.4.

It follows from (6.22)-(6.23) that
\[
\begin{aligned}
\|\lambda_k - \lambda\|^2_{\Omega} &= \|\lambda_{k+1} - \lambda\|^2_{\Omega} - \alpha^2 c^2\|\psi_{A_k} - u_k^0\|^2_\Omega - 2\alpha c\|\lambda_k - \lambda, \psi_{A_k} - u_k^0\|^2_W, \\
\|\eta_k - \eta\|^2_{\Gamma_0} &= \|\eta_{k+1} - \eta\|^2_{\Gamma_0} - \alpha^2 c^2\|y_k - y_0\|^2_{\Gamma_0} - 2\alpha \|\eta_k - \eta, y_k - y_0\|_{\Gamma_0}.
\end{aligned}
\] (6.23)
If we replace \(y\) by \(y(u)\), \(v\) by \(u\), \(\bar{y}\) by \(y_k\), \(\bar{u}\) by \(u_k\), and \(\mu\) by \(\mu_k\) in (6.14), then we obtain from Lemma 6.3, (6.20) and (6.23) that
\[
I(y(u), u) \geq J(y_k, u_k, c, \mu_k) + \|y(u) - y_k\|^2_\Omega + N\|u - u_k\|^2_\Omega
+ c\|\psi_{A_k} - u_k^0\|^2_W + \|y_k - y_0\|^2_{\Gamma_0}.
\] (6.24)
By similar manner, with replacing \(y\) by \(y_k\), \(v\) by \(u_k\), \(\bar{y}\) by \(y(u)\), and \(\mu\) by \(\mu_k\) in (6.16), we shall derive from Lemma 6.4 that
\[
J(y_k, u_k, 0, \mu_k) \geq J(y(u), u) + \|y_k - y(u)\|^2_\Omega + N\|u - u_k\|^2_\Omega.
\] (6.25)
Hence, adding the inequality (6.25) to (6.26) yields
\[
J(y_k, u_k, 0, \mu_k) \geq J(y_k, u_k, c, \mu_k) + 2\|y(u) - y_k\|^2_\Omega + 2N\|u - u_k\|^2_\Omega
+ c\|\psi_{A_k} - u_k^0\|^2_W + \|y_k - y_0\|^2_{\Gamma_0}.
\] (6.26)
On the other hand, by means of Definition (6.4) of \(J\) in (6.27) and (6.23), rearranging terms from (6.27) leads to the following inequalities
\[
\begin{aligned}
&\|\lambda, \psi_{A_k} - u_k^0\|^2_W + \|\eta, y_k - y_0\|_{\Gamma_0} + \|\lambda_k, \psi_{A_k} - u_k^0\|^2_W
+ 2\|y(u) - y_k\|^2_\Omega + 2N\|u - u_k\|^2_\Omega.
\end{aligned}
\] (6.27)
Consequently, we can assert from (6.32) and (6.35) that $k$ increasing and therefore it admits a limit, and now, it follows from (6.30) that the sequence $\{\|\lambda_k - \lambda\|_W^2 + \|\eta_k - \eta\|_{\Gamma_0}^2\}$ is non-increasing and therefore it admits a limit, and

$$
\|\lambda_k - \lambda\|_W^2 + \|\eta_k - \eta\|_{\Gamma_0}^2 \\
\geq \|\lambda_{k+1} - \lambda\|_W^2 + \|\eta_{k+1} - \eta\|_{\Gamma_0}^2 + 2\alpha^2 \|\lambda_k - \lambda, \psi_{\lambda_k} - u_k\|_W - 2\alpha^2 \|y_k - y_0\|_{\Gamma_0}^2
$$

Eventually, adding the inequality (6.28) to (6.29) and rearranging terms with (6.5), we obtain

$$
\|\lambda_k - \lambda\|_W^2 + \|\eta_k - \eta\|_{\Gamma_0}^2 \\
\geq \|\lambda_{k+1} - \lambda\|_W^2 + \|\eta_{k+1} - \eta\|_{\Gamma_0}^2 + 4\alpha^2 \|u - u_k\|_2^2 + 2\alpha^2 \|A y_k - u_k\delta\|_{\Omega_2}^2
$$

Now, it follows from (6.30) that the sequence $\{\|\lambda_k - \lambda\|_W^2 + \|\eta_k - \eta\|_{\Gamma_0}^2\}$ is non-increasing and therefore it admits a limit, and

$$
\|y(u) - y_k\|_{L^2(\Omega)} \to 0, \\
\|u - u_k\|_{L^2(0, T)} \to 0, \\
\|A y_k - u_k\delta\|_{L^2(0, T; H^{-2}(\Omega_2))} \to 0, \\
\|y_k - y_0\|_{L^2(\Gamma_0)} \to 0.
$$

as $k \to +\infty$. Since the equality $A y(u) = u\delta$ is valid, it implies from (6.33) that

$$
\|A(y(u) - y_k) - (u - u_k)\delta\|_{L^2(0, T; H^{-2}(\Omega_2))} \to 0, \quad k \to +\infty
$$

Consequently, we can assert from (6.32) and (6.35) that

$$
\|A(y(u) - y_k)\|_{L^2(0, T; H^{-2}(\Omega_2))} \to 0, \quad k \to +\infty.
$$

Since $y_k$ belongs to $Y$ defined by (6.2) and $y(u)$ satisfies the equations (4.1)-(4.2)(where $v = u$), we have

$$
\frac{\partial y(u)}{\partial n} = 0 \text{ on } \Gamma_1 \text{ and } \frac{\partial y_k}{\partial n} = 0 \text{ on } \Gamma_1.
$$

In light of the definition (6.2) - (6.3) together with (6.31), (6.36), (6.34), (6.37) and (6.32), we eventually arrive at the result that $(y_k, u_k) \to (y, u)$ strongly in $Y \times \mathbb{U}$ as $k \to \infty$, and the proof of Theorem 6.1 is complete now. □

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