Black-hole perturbation theory: The asymptotic spectrum of the prolate spin-weighted spheroidal harmonics

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(Dated: May 11, 2014)

Prolate spin-weighted spheroidal harmonics play a key role in black-hole perturbation theory. In particular, the highly damped quasinormal resonances of rotating Kerr black holes are closely related to the asymptotic eigenvalues of these important functions. We here present a novel and compact derivation of the asymptotic eigenvalues of the prolate spin-weighted spheroidal harmonics. Our analysis is based on a simple trick which transforms the corresponding spin-weighted spheroidal angular equation into a Schrödinger-like wave equation which is amenable to a standard WKB analysis. Our analytical results for the prolate asymptotic spectrum agree with previous numerical computations of the eigenvalues which appear in the literature.

I. INTRODUCTION.

The characteristic dynamics of test fields in black-hole spacetimes has been studied extensively since the pioneering work of Regge and Wheeler [1], see also [2–4] and references therein. An astrophysically realistic model of wave dynamics in black-hole spacetimes should involve a non-spherical background geometry with angular momentum. In terms of the Boyer-Lindquist coordinates, the spacetime of a rotating Kerr black hole is described by the line-element [5, 6]

\[
ds^2 = -(1 - \frac{2Mr}{\rho^2})dt^2 - \frac{4Mr \sin^2 \theta}{\rho^2} dtd\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(\frac{r^2 + a^2 + \frac{2Ma^2r \sin^2 \theta}{\rho^2}}{\rho^2}\right) \sin^2 \theta d\phi^2,
\]

where \( M \) and \( a \) are the mass and angular momentum per unit mass of the black hole, respectively. (We use gravitational units in which \( G = c = 1 \).) Here \( \Delta \equiv r^2 - 2Mr + a^2 \) and \( \rho \equiv r^2 + a^2 \cos^2 \theta \).

In this paper we consider perturbations of the non-spherical Kerr spacetime. The dynamics of a test field \( \Psi \) in the rotating Kerr spacetime is governed by the well-known Teukolsky master equation [7]. One may decompose the field as

\[
s\Psi_{lm}(t, r, \theta, \phi) = e^{i\omega_t} \psi_{lm}(r) e^{-i\omega t},
\]

where \( \omega \) is the (conserved) frequency of the mode, \( l \) is the spheroidal harmonic index, and \( m \) is the azimuthal harmonic index. The parameter \( s \) is called the spin weight of the field, and is given by \( s = \pm 2 \) for gravitational perturbations, \( s = \pm 1 \) for electromagnetic perturbations, \( s = \pm \frac{1}{2} \) for massless neutrino perturbations, and \( s = 0 \) for scalar perturbations [7].

With the decomposition [2], \( \psi \) and \( S \) obey radial and angular equations, both of confluent Heun type [7–11], coupled by a separation constant \( A(\omega) \). The radial Teukolsky equation is given by [7]

\[
\Delta^{-s} \frac{d}{dr} \left( \Delta^{s+1} \frac{d\psi}{dr} \right) + \left[ \frac{K^2 - 2i\omega(r - M)K}{\Delta} - a^2 \omega^2 + 2ma\omega - A + 4i\omega r \right] \psi = 0,
\]

where \( K \equiv (r^2 + a^2)\omega - am \).

The angular functions \( S(\theta; \omega) \) are the spin-weighted spheroidal harmonics which are solutions of the angular equation [7–11]

\[
\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial S}{\partial \theta} \right) + \left[ c^2 \cos^2 \theta - 2cs \cos \theta \frac{(m + s \cos \theta)^2}{\sin^2 \theta} + s + A \right] S = 0,
\]

where \( c \equiv \omega \). The angular functions are required to be regular at the poles \( \theta = 0 \) and \( \theta = \pi \). These boundary conditions pick out a discrete set of eigenvalues \( \{a_{lm} \} \) labeled by the integers \( m \) and \( l \). [In the \( c \to 0 \) limit these angular functions become the familiar spin-weighted spherical harmonics with the corresponding angular eigenvalues \( A = l(l+1) - s(s+1) + O(\omega^2) \).]

The spin-weighted spheroidal harmonics \( S(\theta; c) \) and their corresponding eigenvalues \( \{a_{lm} \} \) have attracted much attention over the years from both physicists and mathematicians [11–17]. It is worth emphasizing that in order...
to compute the characteristic resonances of black holes, one must first compute the (closely related) angular eigenvalues \( \{ \gamma_{lm} \} \) [see Eq. (3)]. In particular, in the framework of semi-classical general relativity, it has been conjectured that the highly damped resonances may shed light on the quantum properties of black holes. For rotating black holes, these highly damped resonances are characterized by \( c_I = \Im \) (where \( c_I \equiv \Im c \)).

The asymptotic limit \( c_I \to \infty \) of the angular equation (4) was studied in [11, 12] for the \( s = 0 \) case. It was found that the asymptotic scalar eigenvalues are given by

\[
\gamma_A_{lm} = [2(l + |m|) + 1]c_I + O(1) .
\]  

(5)

As pointed out in [16], the analysis of [11, 12] for the spin-0 case is somewhat incomplete. The analysis of [11, 12] requires the knowledge of the number of zeros of the scalar harmonics in the interval \([0, \pi]\). However, as emphasized in [16], the approximated solution found in [11, 12] for the \( c_I \to \infty \) limit is only valid in a region far from the end-points \( \theta = 0, \pi \). Thus, the analysis of [11, 12] cannot rule out the possible existence of additional zeros of the angular eigenfunctions near the end-points. The possible omission of such zeros would lead to a wrong asymptotic behavior of the prolate (with \( c = ic_I \)) eigenvalues, see [16] for details. In this respect the analysis of [11, 12] is not complete. One of the goals of the present paper is to present a more rigorous proof of the formula (5) for the asymptotic scalar eigenvalues.

The asymptotic spectrum of the prolate eigenvalues for the general spin case was first studied in [13]. However, as emphasized in [16], the analysis of [11, 12] requires the knowledge of the number of zeros of the scalar harmonics in the interval \([0, \pi]\). Thus, the analysis of [11, 12] cannot rule out the possible existence of additional zeros of the angular eigenfunctions near the end-points. The possible omission of such zeros would lead to a wrong asymptotic behavior of the prolate (with \( c = ic_I \)) eigenvalues, see [16] for details. In this respect the analysis of [11, 12] is not complete. One of the goals of the present paper is to provide a (simple) analytical proof for the prolate formula (5) in the general spin case.

II. A COORDINATE TRANSFORMATION

It proves useful to introduce the coordinate \( x \) defined by [17, 21]

\[
x \equiv \ln \left( \tan \left( \frac{\theta}{2} \right) \right) ,
\]

(7)
in terms of which the angular equation (4) becomes a Schrödinger-like wave equation of the form [22]

\[
\frac{d^2 S}{dx^2} - U S = 0 ,
\]

(8)

where

\[
U(\theta) = (m + s \cos \theta)^2 - \sin^2 \theta(c^2 \cos^2 \theta - 2cs \cos \theta + s + A) .
\]

(9)

Note that the interval \( \theta \in [0, \pi] \) maps into \( x \in [-\infty, \infty] \). The Schrödinger-type angular equation (8) is now in a form that is amenable to a standard WKB analysis.

III. THE SPIN-0 (SCALAR) CASE.

We shall first consider the spin-0 (scalar) case. In this case the effective potential \( U(\theta) \) is in the form of a symmetric (invariant under the transformation \( \theta \to \pi - \theta \)) potential well: in the \( c_I \to \infty \) limit it has a local minimum at

\[
\theta_{\text{min}} = \frac{\pi}{2} \quad \text{with} \quad U(\theta_{\text{min}}) = -A + O(1) .
\]

(10)

Regions where \( U(\theta) < 0 \) are characterized by an oscillatory behavior of the wave-function \( S \) (the ‘classically allowed regions’), while regions with \( U(\theta) > 0 \) (the ‘classically forbidden regions’) are characterized by an exponential behavior (evanescent waves). The ‘classical turning points’ are characterized by \( U = 0 \). There is a pair \( \{ 0^+ \theta, 0^- \theta \} \) of such turning points (with \( 0^- \theta < 0^+ \theta \)) which in the \( c_I \to \infty \) limit are located in the near vicinity of \( 0^+ \theta_{\text{min}} \):

\[
0^+ \theta = \frac{\pi}{2} \pm \frac{\sqrt{A}}{|c_I|} + O(c_I^{-3/2}) ,
\]

(11)
A standard textbook second-order WKB approximation for the bound-state ‘energies’ of a Schrödinger-like wave equation of the form \( (5) \) yields the well-known quantization condition \( (23,27) \)

\[
\int_{x^-}^{x^+} dx \sqrt{-U(x)} = (N + \frac{1}{2})\pi \quad ; \quad N = 0, 1, 2, \ldots ,
\]

(12)

where \( x^\pm \) are the turning points [with \( U(x^\pm) = 0 \)] of the potential well, and \( N \) is a non-negative integer.

Using the relation \( dx/d\theta = 1/\sin \theta \), one can write the WKB condition \( (12) \) in the form

\[
\int_{\theta^-}^{\theta^+} d\theta \sqrt{-U(\theta)} \frac{1}{\sin \theta} = (N + \frac{1}{2})\pi \quad ; \quad N = 0, 1, 2, \ldots .
\]

(13)

The WKB quantization condition \( (13) \) determines the eigenvalues \( \{A\} \) of the associated spin-weighted spheroidal harmonics in the large-\( c_I \) limit. The relation so obtained between the eigenvalues and the parameters \( c, m, s \) and \( N \) is rather complex and involves elliptic integrals. However, given the fact that in the \( c_I \to \infty \) limit the turning points \( \theta^\pm \) lie in the vicinity of \( \theta = \frac{\pi}{2} \) [see Eq. \( (11) \)], one can approximate the integral in \( (11) \) by \( (28) \)

\[
\int_{\theta^-}^{\theta^+} d\theta \sqrt{\frac{A}{\sin \theta} - c_I^2(\theta - \frac{\pi}{2})^2} + A = (N + \frac{1}{2})\pi \quad ; \quad N = 0, 1, 2, \ldots .
\]

(14)

Evaluating the integral in \( (14) \) is straightforward, and one finds

\[
A(N) = (2N + 1)|c_I| + O(1) \quad ; \quad N = 0, 1, 2, \ldots \]

(15)

for the quantized spectrum. This completes our proof for the prolate asymptotic spectrum in the scalar \( (s = 0) \) case \( (29) \).

IV. THE GENERAL SPIN CASE.

We shall now consider the general spin case. In this case the effective potential \( U(\theta) \) is complex-valued. Its minimum is located at \( s\theta_{\text{min}} = \frac{\pi}{2} + \frac{is}{c_I} + O(c_I^{-2}) \), while the two turning points are located at

\[
s\theta^\pm = \frac{\pi}{2} \pm \frac{\sqrt{A}}{|c_I|} + \frac{is}{c_I} + O(c_I^{-3/2}) .
\]

(16)

A natural generalization of the WKB analysis to the case of complex-valued potentials is provided in \( (30) \): the WKB quantization rule is given by the standard relation

\[
\int_{\theta^-}^{\theta^+} d\theta \sqrt{-U(\theta)} \frac{1}{\sin \theta} = (N + \frac{1}{2})\pi \quad ; \quad N = 0, 1, 2, \ldots ,
\]

(17)

which can be approximated near \( \theta = \pi/2 \) by \( (28) \)

\[
\int_{\theta^-}^{\theta^+} d\theta \sqrt{-c_I^2(\theta - \frac{\pi}{2})^2 - 2is(c_I - \frac{\theta}{2}) + A} = (N + \frac{1}{2})\pi \quad ; \quad N = 0, 1, 2, \ldots .
\]

(18)

As emphasized in \( (30) \), the integration path between the two complex turning points \( \{s\theta^-, s\theta^+\} \) should be chosen such that

\[
\Im \{\sqrt{-U(\theta)}\} = 0
\]

(19)

along the integration contour \( (30) \). In the \( c_I \to \infty \) limit this requirement is easily fulfilled by a straight line (parallel to the real \( \theta \)-axis) which connects the two turning points \( (16) \). Substituting \( \theta = \phi - \frac{\pi}{2} \) (where \( \phi \in \mathbb{R} \) runs from \( \theta^- \) to \( \theta^+ \)) into \( (15) \) and neglecting terms of order \( O(1) \), one finds that along the path \( (16) \) the integral \( (18) \) can be written as

\[
\int_{\theta^-}^{\theta^+} d\phi \sqrt{-c_I^2(\phi - \frac{\pi}{2})^2 + A} = (N + \frac{1}{2})\pi \quad ; \quad N = 0, 1, 2, \ldots .
\]

(20)
This yields
\[ A(N) = (2N + 1)|c_1| + O(1) \quad ; \quad N = 0, 1, 2, \ldots \]  
(21)
for the quantized spectrum. This completes our proof for the prolate asymptotic spectrum in the general spin case. It is worth emphasizing that the analytical formula (21) for the prolate asymptotic spectrum agrees with the numerical results presented in [16, 20].

ACKNOWLEDGMENTS

This research is supported by the Carmel Science Foundation. I thank Uri Keshet, Oded Hod, Yael Oren, Arbel M. Ongo and Ayelet B. Lata for helpful discussions.

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