Cohomology of Line Bundles: Proof of the Algorithm

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Abstract

We present a proof of the algorithm for computing line bundle valued cohomology classes over toric varieties conjectured by R. Blumenhagen, B. Jurke and the authors (arXiv:1003.5217) and suggest a kind of Serre duality for combinatorial Betti numbers that we observed when computing examples.

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1 Introduction

In a recent publication [BJRR1] we conjectured an algorithm\footnote{The speed-optimized implementation in C++ can be downloaded from \url{http://wwwth.mppmu.mpg.de/members/blumenha/cohomcalg} and is regularly updated. To get a first experience of the calculations possible, one can also have a quick start with a short Mathematica script that is also available there.} facilitating the computation of sheaf cohomology of line bundles over toric ambient spaces, quantities that are often required by string theorists since these data encode much of the phenomenological content of a chosen string compactification on an embedded Calabi-Yau manifold. The arising new computational possibilities in heterotic model building or F-theory will be investigated in a separate paper on applications [BJRR2], but before this we would like to give the algorithm a firm mathematical basis in order to dispel any lurking doubts about therewith achieved results in advance.

Before we blithely plunge into the abstract world of combinatorial algebra, we would first like to emphasize the advantages that showed up when we first made use of our conjectured algorithm. First of all, using Serre duality as a consistency check, we could often get rid of explicit maps of complexes – usually, one has to determine the ranks of sparse matrices whose dimensions may be huge for high-dimensional varieties. This sped up computations by an estimated (and palpable) factor of $10^6$ in comparison with existing routines that are used e.g. in Macaulay2 \cite{GS}. For most of the examples of toric ambient spaces common to F-theory or other parts of theoretical physics computations are instantaneous (at least if one restricts oneself to divisor charges of a sensible size), whereas on the same platform Macaulay2 needs some seconds already in the case of 2-dimensional varieties like the del-Pezzo surfaces.

Another important advantage is the restriction to contributions coming from the power set of the Stanley-Reisner ideal of the toric variety. Hence, our algorithm also outstrips the “chamber algorithm” that is presented in \cite{CLS} and implemented in \cite{CGH}. As it turns out, in many examples only $\sim 10\%$ of all possible denominators of representatives for Čech cohomology appear in the power set, so about $90\%$ of the complexes (with explicit matrices) that are evaluated in the chamber algorithm will not contribute from the very beginning, which also accounts for the lower performance of this method. Nevertheless, the chamber algorithm served as a motivation for our conjecture, since it conveys some intuitive feeling of how local sections fit together and how Čech cohomology arises.\footnote{In order to do sheaf cohomology computations on general toric varieties, the additional package “NormalToricVarieties.m2” written by Gregory Smith is needed. Since this is still work in progress, it is not yet included in the official distribution, but the package content can be copied from his homepage, and then separately loaded into Macaulay2.}

The aim of this paper is to substantiate the mathematical background of the conjectured algorithm and then give a rigorous proof in the language of Čech cohomology on an open cover of a toric variety can be shown to be isomorphic to sheaf cohomology, see Theorem 9.0.4 in \cite{CLS}.\footnote{Note that it can be shown that Čech cohomology on an open cover of a toric variety can be shown to be isomorphic to sheaf cohomology, see Theorem 9.0.4 in \cite{CLS}.}
combinatorial commutative algebra. In fact, when we set out for a mathematical
understanding, we found that many of the appearing structures could be given
a quick formulation in algebraic terms, and some features of the algorithm can
be derived quite easily in this manner. Therefore, Section 2 will be devoted to
the introduction of some concepts from commutative algebra and the beauty and
concreteness of these will then enable us to reformulate the original conjecture
and prove it in Section 3.

In writing the last sentences of this work, another proof of our conjecture was
published by S.Y. Jow in [Jow]. It puts more emphasis on elements of simplicial
topology, so we hope that the combinatorial counterpart that we present is of
interest on its own. For completeness, we will make the equivalence of both
approaches explicit right after our main theorem. In fact, the key to the meaning
of our “remnant cohomology” was also anticipated by Jow.

We finish this work by making some comments on further possible simplifica-
tions that may speed up computations by another of magnitude and in particular
point out the possibility of “Serre duality for Betti numbers” emerging from the
established relationships between toric and combinatorial algebra.

2 Mathematical Preliminaries

2.1 Normal Toric Varieties and Alexander duality

Since an introduction to toric varieties was already given in [BJRR1], we confine
ourselves to a short review of the setup. Let $X$ be a complete simplicial smooth
normal toric variety\footnote{in the sense of Chapter 3 of [CLS]} that is given in terms of $n$ vertices $\nu_i$ in $\mathbb{Z}^d$, a triangulation
of this vertex scheme yielding the fan $\Sigma$ of the variety $X$ and weights (often called
charges by the physicists) coming from relations among the vertices. The homoge-
neous coordinate ring or \textit{Cox ring} belonging to $X$ is

$$S = \mathbb{C}[x_1, x_2, \ldots, x_n]$$

and the \textit{irrelevant ideal} $B_{\Sigma}$ in $S$ is generated by the monomials

$$\{x_{j_1} \cdots x_{j_s} \mid \{\nu_{j_1}, \ldots, \nu_{j_s}\} \text{ spans a cone in } \Sigma\}, \quad (1)$$

so a minimal generating set of $B_{\Sigma}$ is given by the monomials corresponding to
complements of the maximal cones of $\Sigma$. The \textit{Stanley-Reisner ideal} $I_{\Sigma}$ in $S$ is generated by the monomials

$$\{x_{i_1} \cdots x_{i_s} \mid \nu_{i_1}, \ldots, \nu_{i_s} \text{ do not lie in a common cone of } \Sigma\}, \quad (2)$$

so a minimal set of generators would be given by monomials corresponding to
minimal sets of vertices that do \textit{not} span a cone in $\Sigma$.

We want to point out that there is a nice duality operation for monomial
ideals that connects $B_{\Sigma}$ and $I_{\Sigma}$. Since it will be a key ingredient to account for the
appearance of $I_{\Sigma}$ in the computation of sheaf cohomology, we introduce it more
formally. First note that the fans we discern are all simplicial, i.e. for all cones in \( \Sigma \) the generating vertices are linearly independent vectors in \( \mathbb{Z}^d \). This means that instead of the fan \( \Sigma \), we may also work with the corresponding simplicial complex \( \Delta \) of the variety defined on the set \( [n] = \{1, \ldots, n\} \), where the \( i \)-dimensional faces of \( \Delta \) are in one-to-one correspondence with the \((i+1)\)-dimensional cones of \( \Sigma \). In the remainder of the paper, we will sometimes assume this perspective of things and restrict ourselves to the language of simplicial complexes. That means that we also write \( I_\Delta, B_\Delta, \ldots \), but one should keep in mind that any statements in this language can in principle be given a topological meaning when talking about the geometry of polyhedral cones rather than the algebra of simplicial complexes.

Write \( x^\sigma = \prod_{i \in \sigma} x_i \) for some subset \( \sigma \subseteq [n] \) and
\[
\mathfrak{m}^\sigma = \langle x_i \mid i \in \sigma \rangle
\]
for the monomial prime ideal corresponding to \( \sigma \). Then the (squarefree) Alexander dual of some monomial ideal \( J = \langle x^{\sigma_1}, \ldots, x^{\sigma_r} \rangle \) is
\[
J^* = \mathfrak{m}^{\sigma_1} \cap \cdots \cap \mathfrak{m}^{\sigma_r}.
\]
It is easy to show that for any simplicial complex we have \( B_\Delta = I_\Delta^* \). Since there is a bijection between simplicial complexes and squarefree monomial ideals this means that we can also have a look at the (Alexander) dual complex \( \Delta^* \), which is determined by \( I_{\Delta^*} = I_\Delta^* = B_\Delta \), i.e. the original irrelevant ideal becomes the Stanley-Reisner ideal of the dual complex. It turns out that these considerations are indeed relevant when it comes to the computation of (reduced) simplicial (co)homology by Hochster’s formula.

### 2.2 Sheaf-Module-Correspondence and Local Cohomology

In order to make use of algebraic concepts, we have to reformulate the computation of sheaf cohomology\(^6\) on the variety \( X \) in terms of module theory of the Cox coordinate ring \( S \). A first step towards this is the sheaf-module correspondence, which enables us to construct quasicoherent sheaves on \( X \) from any module \( M \) over \( S \) that is graded by the class group
\[
\text{Cl}(X) \cong \mathbb{Z}^{n-d}.
\]
For the details of the construction, see e.g. §5.3 of [CLS]. Since we deal with line bundles, the only important observation is that the coordinate ring \( S \) itself is \( \text{Cl}(X) \)-graded, i.e. it has a decomposition
\[
S = \bigoplus_{\alpha \in \text{Cl}(X)} S_\alpha, \quad S_\alpha \cdot S_\beta \subset S_{\alpha+\beta}
\]
\(^5\)A condensed introduction to simplicial complexes meeting our requirements is given e.g. by the first chapter of [MiS].
\(^6\)For a short review of sheaf theory and sheaf cohomology have a look at the appendix of [BJRRT].
\(^7\)Note that we always identify Picard group and class group of \( X \), since in the smooth case all Weil divisors are already Cartier.
and that the graded pieces $S_\alpha$ are naturally isomorphic to the sections of twisted line bundles, namely

$$S_\alpha \cong \Gamma(X, \mathcal{O}_X(\alpha)). \quad (6)$$

This means that the shift $S(\alpha)$ of $S$ defined by the grading $S(\alpha)_\beta = S_{\alpha+\beta}$ gives rise to the line bundle $\mathcal{O}_X(\alpha)$ and therefore sheaf cohomology should also be computable from $S(\alpha)$. As it turns out, the algebraic equivalent of sheaf cohomology is local cohomology with support on the irrelevant ideal. The reason for this lies in the fact that the map from modules to sheaves is not injective, since starting with $S$-modules $M$ that fulfill $(B_\Sigma)^l M = 0$ for $l \gg 0$ leads to trivial sheaves. Taking global sections on the sheaf side therefore in a certain way corresponds to looking at elements with support on the irrelevant ideal on the module side. Local cohomology is then defined completely analogous to sheaf cohomology as the right-derived functor of the operation of taking supports.

We now introduce the necessary notions and then state the relevant special case of Theorem 9.5.7 in [CLS]. For an $S$-module $M$ and an ideal $J \subset S$ one defines the $J$-torsion submodule or submodule supported on $J$ by

$$\Gamma_J(M) = \{ y \in M \mid J^l y = 0 \text{ for some } l \in \mathbb{N} \}. \quad (7)$$

The $i$-th local cohomology module of $M$ with support on $J$ is then defined to be the module $H^i_J(M)$ obtained from any injective resolution $0 \to I^0 \to I^1 \to \cdots$ of $M$ by taking the $i$-th cohomology of the subcomplex $0 \to \Gamma_J(I^0) \to \Gamma_J(I^1) \to \cdots$. In particular, if $M$ is graded by $\text{Cl}(X)$, then also $\Gamma_J(M)$ and all $H^i_J(M)$ will inherit this grading. The precise connection between line bundle cohomology and local cohomology is then given by

$$H^i(X, \mathcal{O}_X(\alpha)) \cong H^{i+1}_{B\Sigma}(S)_\alpha \quad \text{for } i \geq 1, \alpha \in \text{Cl}(X). \quad (8)$$

Furthermore, there is an exact sequence

$$0 \to H^0_{B\Sigma}(S)_\alpha \to S_\alpha \to H^0(X, \mathcal{O}_X(\alpha)) \to H^1_{B\Sigma}(S)_\alpha \to 0, \quad (9)$$

which is necessary to determine the 0-th rank of sheaf cohomology. Because of eq. (8) and $H^0(X, \mathcal{O}_X(\alpha)) = \Gamma(X, \mathcal{O}_X(\alpha))$, the middle map is an isomorphism. Furthermore $H^0_{B\Sigma}(S) = 0$, since $S$ has no zero divisors and so in the special case of line bundles we get

$$H^i(X, \mathcal{O}_X(\alpha)) \cong H^{i+1}_{B\Sigma}(S)_\alpha \quad \text{for } i \geq 0, \alpha \in \text{Cl}(X). \quad (10)$$

Before we finish this section, we want to introduce a finer grading of the Cox coordinate ring $S$, namely the $\mathbb{Z}^n$-grading that is given by $\deg x_i = e_i \in \mathbb{Z}^n$. The connection to the class group grading is given by the map

$$f : \mathbb{Z}^n \to \text{Cl}(X) \cong \mathbb{Z}^{n-d}, \quad e_i \mapsto [D_i] = (Q^{(1)}_i, \ldots, Q^{(n-d)}_i), \quad (11)$$

8 the shift in the rank comes from a shift between the ordinary and the local Čech complex, see also Theorem 9.5.7 in [CLS].
where the $Q_i^{(r)}$ denote the charges (or weights) of the coordinate divisor $D_i$ belonging to the hypersurface $\{x_i = 0\}$. Since this finer grading is also inherited by the local cohomology modules, we may write eq. (10) as

$$H^i(X, \mathcal{O}_X(\alpha)) = \bigoplus_{\substack{u \in \mathbb{Z}^n \colon f(u) = \alpha}} H_{B_\mathcal{E}^i(S)}^u$$

for any $\alpha \in \text{Cl}(X)$. So the procedure would be to try and compute all $\mathbb{Z}^n$-graded pieces of local cohomology and at the end determine sheaf cohomology of $\mathcal{O}_X(\alpha)$ by summing up the contributions fulfilling $f(u) = \alpha$, which is a matrix equation over the integers solvable by standard techniques using Ehrhart polynomials. In fact, this is nothing but the rationom counting procedure of our algorithm, but we will state this later in a more precise form.

### 2.3 Simplicial Complexes and Free Resolutions

The crucial link between sheaf cohomology over $X$ and the combinatorics of the associated Stanley-Reisner ideal $\mathcal{I}_\Delta$ will be a theorem by Mustaţă that expresses the dimensions of fine-graded local cohomology modules by Betti numbers of $\mathcal{I}_\Delta$. But before we state this result, we want to use this section to introduce some additional notions from homological algebra.

Let $\Delta$ be a simplicial complex on $[n]$. For each $i \geq -1$ denote by $F_i$ the set of $i$-dimensional faces (subsets $\sigma \subseteq [n]$ of cardinality $i + 1$) and let $\mathbb{C}^{F_i}$ be the complex vector space whose basis elements $e_\sigma$ correspond to all $\sigma \in F_i$. The reduced chain complex of $\Delta$ is the complex

$$\widetilde{C}_\bullet(\Delta) : 0 \leftarrow \mathbb{C}^{F_{-1}} \xleftarrow{\partial_{-1}} \mathbb{C}^{F_0} \xleftarrow{\partial_0} \cdots \xleftarrow{\partial_{n-1}} \mathbb{C}^{F_n} \xrightarrow{\text{boundary}} 0.$$  \hspace{1cm} (13)

The boundary maps $\partial_i$ are defined by setting $\text{sign}(j, \sigma) = (-1)^{r-1}$ when $j$ is the $r$-th element of $\sigma \subseteq [n]$ written in increasing order, and

$$\partial_i(e_\sigma) = \sum_{j \in \sigma} \text{sign}(j, \sigma)e_{\sigma \setminus j}.$$  \hspace{1cm} (14)

One has $\partial_i \circ \partial_{i+1} = 0$ and therefore defines the $i$-th reduced homology of $\Delta$ (with coefficients in $\mathbb{C}$) as

$$\widetilde{H}_i(\Delta) = \ker(\partial_i) / \text{im}(\partial_{i+1}).$$  \hspace{1cm} (15)

Note that there is a distinction between the void complex $\{\}$ and the irrelevant complex $\{\emptyset\}$. One has $\widetilde{H}_{-1}(\{\emptyset\}) \cong \mathbb{C}$, since the empty set is a face of dimension $-1$.

Taking the vector space dual of the chain complex (and the transpose of all maps) one gets the cochain complex of $\Delta$ as $\widetilde{C}_\bullet^*(\Delta) = (\widetilde{C}_\bullet(\Delta))^*$ with coboundary maps $\partial^i = \partial_i^*$. One similarly defines the $i$-th reduced cohomology of $\Delta$ as

$$\widetilde{H}^i(\Delta) = \ker(\partial^{i+1}) / \text{im}(\partial^i).$$  \hspace{1cm} (16)
Since we have coefficients in \( \mathbb{C} \), there is a canonical isomorphism \( \tilde{H}^i(\Delta) \cong (\tilde{H}(\Delta))^* \) and thus
\[
\text{dim}_\mathbb{C} \tilde{H}^i(\Delta) = \text{dim}_\mathbb{C} \tilde{H}(\Delta).
\] (17)

We now shortly review the definitions of (minimal) free resolutions, Taylor resolution, Betti numbers of an \( S \)-module \( M \) and for completeness also write down (both versions of) the Hochster formula. Details and proofs can be found in [MiS].

We begin with the definition of a free resolution. Let \( V_i, 0 \leq i \leq \ell \) be a collection of free \( S \)-modules, i.e. all the \( V_i \) have the form \( \bigoplus q \mathbb{Z} S(-a_q) \) with all \( a_q \in \mathbb{Z} \) and \( S(a) \) denoting the degree shift of \( S \) by \( a \). A sequence
\[
F_\bullet : 0 \leftarrow V_0 \xleftarrow{\phi_1} V_1 \leftarrow \cdots \leftarrow V_{\ell-1} \xleftarrow{\phi_{\ell}} V_\ell \leftarrow 0
\] (18)
of free modules with maps fulfilling \( \phi_i \circ \phi_{i+1} = 0 \) is called a complex. This complex is \( \mathbb{Z}^n \)-graded if each homomorphism is of degree 0, i.e. for all elements \( r_i \in V_i \) one has \( \text{deg} r_i = \text{deg} \phi_i(r_i) \) in \( \mathbb{Z}^n \). Assuming \( V_\ell \neq 0 \), the length of the complex \( F_\bullet \) equals \( \ell \). A complex \( F_\bullet \) is called a free resolution of an \( S \)-module \( M \) if \( F_\bullet \) is acyclic (i.e. exact everywhere except in homological degree 0), where \( M = V_0 / \text{im}(\phi_1) \). The Hilbert Syzygy Theorem tells us that every \( S \)-module has a free resolution with length at most \( n \). Since we will only look at modules \( M \) of the form \( I \) or \( S/I \), where \( I \) is a monomial ideal of \( S \), we always get free resolutions that are naturally \( \mathbb{Z}^n \)-graded.

Define a partial order on \( \mathbb{Z}^n \) by letting \( a \leq b \) if the components fulfil \( a_i \leq b_i \) for all \( i \in [n] \). To state when a free resolution is minimal, we introduce the concept of monomial matrices. These represent maps
\[
\bigoplus_q S(-a_q) \xleftarrow{\phi} \bigoplus_p S(-a_p)
\] (19)
between two free \( S \)-modules. A monomial matrix consists of entries \( \lambda_{qp} \in \mathbb{C} \) arranged in columns labeled by the source degrees \( a_p \) and rows labeled by the target degrees \( a_q \) and whose entry \( \lambda_{qp} \) is zero unless \( a_p \geq a_q \) in the partial order of \( \mathbb{Z}^n \). The map \( \phi \) will then send the basis vector \( 1_{a_q} \) of \( S(-a_q) \) to the element \( \lambda_{qp} x^{a_p-a_q} \cdot 1_{a_q} \) in \( S(-a_q) \). The condition \( a_p \geq a_q \) then just guarantees that \( x^{a_p-a_q} \in S \) and the image of \( \phi \) makes sense.

Such a monomial matrix is called minimal if \( \lambda_{qp} = 0 \) whenever \( a_p = a_q \). Similarly, a free resolution of some module \( M \) is called a minimal free resolution if all the maps in the resolution can be represented by minimal monomial matrices. This means that the ranks of the free modules \( V_i \) in a resolution are all simultaneously minimized. In particular, any free resolution of \( M \) contains the (unique up to isomorphism) minimal resolution as a subcomplex.

As an example of a free resolution, take a monomial ideal \( I = \langle m_1, \ldots, m_t \rangle \) in \( S \) and write \( m_\tau = \text{lcm}\{m_j | j \in \tau\} \) for any \( \tau \subseteq [t] \). Furthermore, set \( a_\tau = \)
deg \( m_\tau \) \( \in \mathbb{Z}^n \). The full Taylor resolution of \( I \) is based on the reduced chain complex

\[
\mathcal{T}_\bullet(t) := \widetilde{\mathcal{C}}_\bullet(\Delta[t])
\]

of the full simplex \( \Delta[t] \) consisting of all subsets of \([t]\). To arrive at the Taylor resolution, one substitutes all \( \mathbb{C}^{F_j} \) in \( \mathcal{T}_\bullet(t) \) by \( \bigoplus_{\tau \in F_j} S(-a_\tau) \) and puts the boundary maps \( \partial_j \) into a sequence of monomial matrices \( M(\partial_j) \) with source and target labels \( a_\tau \) corresponding to faces \( \tau \in \Delta[t] \) and entries \( \lambda_{\tau, \tau \downarrow k} = \text{sign}(k, \tau) \) equal to the sign factors from eq. (14). One arrives at an acyclic complex of the form

\[
\mathcal{F}_\bullet^T : 0 \leftarrow S \xleftarrow{M(\partial_0)} \bigoplus_{\tau \in F_0} S(-a_\tau) \xleftarrow{M(\partial_1)} \cdots \xleftarrow{M(\partial_{t-1})} S(-a_{[t]}) \leftarrow 0,
\]

whose 0-th homology equals \( S/I \), so this is a free resolution of \( S/I \) of length \( t \).

For applications, this means that starting with some toric variety \( X \), the power set of its Stanley-Reisner ideal \( I \) contains all information about the full Taylor resolution of \( S/I \). Unfortunately, the Taylor resolution is almost never minimal. More precisely, the Taylor resolution is minimal if and only if for all faces \( \sigma \in \Delta[r] \) and all indices \( i \in \sigma \), the monomials \( m_\sigma \) and \( m_\sigma \downarrow i \) are different.

All information about the minimal free resolution of some \( S \)-module can be encoded in some integer numbers. If we take the complex \( \mathcal{F}_\bullet \) from (13) to be a minimal free resolution of an \( S \)-module \( M \) and write the \( V_i \) as

\[
V_i = \bigoplus_{a \in \mathbb{Z}^n} S(-a)^{\beta_i,a},
\]

then the \( i \)-th Betti number of \( M \) in degree \( a \) is the invariant

\[
\beta_{i,a}(M) = \beta_{i,a}.
\]

Betti numbers can also be characterized more categorically in terms of the Tor-functor. For two \( S \)-modules \( M \) and \( N \) one can describe the modules \( \text{Tor}_i^S(M, N) \) by applying the functor \( \_ \otimes_S N \) to a free resolution of \( M \) and taking homology of the resulting complex. Since all relevant notions have a generalization to the \( \mathbb{Z}^n \)-graded setting, also the Tor-modules can be given a natural \( \mathbb{Z}^n \)-grading.

Intuitively speaking, the Betti numbers of \( M \) then describe what survives when tensoring any free resolution with \( \mathbb{C} \) and taking homology:

\[
\beta_{i,a}(M) = \dim_{\mathbb{C}} \left( \text{Tor}_i^S(M, \mathbb{C})_a \right).
\]

---

9. Here, the term “power set of an ideal” stands for taking all possible unions of the generators. In fact, the sequences for “remnant cohomology” in the algorithm of [BJRR1] come from the combinatorics of this power set and the connection with the full Taylor resolution of \( S/I \) will be important for the proof.

10. For example, the Taylor resolution of the Stanley-Reisner ring of \( X = dP_3 \) is not minimal, since the subset \( \{m_1, m_2, m_3\} = \{x_1x_2, x_1x_3, x_2x_3\} \) is among the generators of its Stanley-Reisner ideal, cf. the examples in [BJRR1].

11. See [Wei] for more details on these categorical issues.
Note that the tensor product over $S$ of a shifted free module $S(-a)$ with the ground field $C \cong S/m$ is equal to a copy of the ground field in degree $a \in \mathbb{Z}^n$:

$$S(-a) \otimes_S C \cong C(-a)$$

This means that one has an easy description of the degree $a$ piece of a tensored resolution $F_\bullet \otimes_S C$. All copies of $S(-a)$ for some $a \in \mathbb{Z}^n$ that were present in the resolution become one-dimensional vector spaces $C(-a)$ and since all maps between source and target degrees with $a_p \neq a_q$ become zero, one can restrict to degree $a$ by just looking at the subcomplex of $F_\bullet \otimes_S C$ made up of the spaces $C(-a)$. These considerations will play a role in the proof of our theorem later.

The Betti numbers of a monomial ideal $I_\Delta \subseteq S$ may be calculated in different ways. One possibility is to take the (co)homology of certain simplicial subcomplexes of the associated complex $\Delta$ (resp. the Alexander dual $\Delta^*$). This is described by the Hochster formula (resp. its dual version).

For each $\sigma \subseteq [n]$ define the restriction of a simplicial complex $\Delta$ to $\sigma$ by

$$\Delta|_\sigma = \{ \tau \in \Delta \mid \tau \subseteq \sigma \}.$$ (26)

For $\sigma \subseteq [n]$ write $\tilde{\sigma} \in \mathbb{Z}^n$ for the (squarefree) degree with components $\tilde{\sigma}_i = 1$ if $i \in \sigma$ and $\tilde{\sigma}_i = 0$ otherwise. Since the meaning can always be inferred from the context, we subsequently omit the tilde and write $\sigma$ also for the element in $\mathbb{Z}^n$. Treating $I_\Delta$ and $S/I_\Delta$ as (graded) $S$-modules, their Betti numbers lie only in squarefree degrees $\sigma$ and can be calculated by the Hochster formula

$$\beta_{i-1,\sigma}(I_\Delta) = \beta_{i,\sigma}(S/I_\Delta) = \dim_C \tilde{H}^{i-1}(\Delta|_\sigma).$$ (27)

There is also a description of these Betti numbers in terms of the Alexander dual complex $\Delta^*$. For any $\sigma \subseteq [n]$ define the link of $\sigma$ inside the simplicial complex $\Delta$ to be

$$\text{link}_\Delta(\sigma) = \{ \tau \in \Delta \mid \tau \cup \sigma \in \Delta \text{ and } \tau \cap \sigma = \emptyset \}.$$ (28)

Furthermore, denote the complement of $\sigma \subseteq [n]$ by $\overline{\sigma} = [n] \setminus \sigma$. Then the dual Hochster formula states that

$$\beta_{i,\sigma}(I_\Delta) = \beta_{i+1,\sigma}(S/I_\Delta) = \dim_C \tilde{H}_{i-1}(\text{link}_{\Delta^*}(\overline{\sigma})).$$ (29)

Because of eq. (27), in all of these formulas simplicial homology may be treated for cohomology when computing Betti numbers.

### 2.4 Local Cohomology and Betti Numbers

Now we come to the connection between local cohomology and Betti numbers stated in [EMS]. Let $a \in \mathbb{Z}^n$ and define the set of indices with negative entries by $\text{neg}(a) = \{ i \in [n] \mid a_i < 0 \} \subseteq [n]$. As it turns out, the graded parts of local cohomology of $S$ with support on the irrelevant ideal of some toric variety
do only depend on the negative entries in the degree, i.e., for \( a, b \in \mathbb{Z}^n \) with \( \text{neg}(a) = \text{neg}(b) \), one has
\[
H^i_{B_\Sigma}(S)_a \cong H^i_{B_\Sigma}(S)_b \tag{30}
\]
or in the notation of the last section
\[
H^i_{B_\Sigma}(S)_a \cong H^i_{B_\Sigma}(S)_{-\sigma} \tag{31}
\]
when \( \text{neg}(a) = \sigma \). Therefore, one only has to compute \( H^i_{B_\Sigma}(S)_{-\sigma} \) for all \( \sigma \subseteq [n] \).

The decisive result is now that via Alexander duality there is a direct relation between these graded pieces of local cohomology and the Betti numbers of \( S/I_\Sigma \), since (cf. Corollary 1.2 in [EMS])
\[
H^i_{B_\Sigma}(S)_{-\sigma} \cong \text{Tor}_{|\sigma|+i+1}^S(S/I_\Sigma, \mathbb{C})_{\sigma}, \tag{32}
\]
and application of eq. (24) then gives
\[
\dim_{\mathbb{C}}(H^i_{B_\Sigma}(S)_{-\sigma}) = \beta_{|\sigma|+i+1,\sigma}(S/I_\Sigma). \tag{33}
\]
Inserting this into eq. (12), we finally get a closed formula for the line bundle cohomology
\[
h^i(\alpha) := \dim_{\mathbb{C}} H^i(X, \mathcal{O}_X(\alpha)) \]

as
\[
h^i(\alpha) = \sum_{u \in \mathbb{Z}^n \atop f(u) = \alpha} \dim_{\mathbb{C}} H^{i+1}_{B_\Sigma}(S)_{-\text{neg}(u)} = \sum_{\sigma \subseteq [n]} |(\alpha, \sigma)| \cdot \beta_{|\sigma|+i,\sigma}(S/I_\Sigma), \tag{34}
\]
where \( |(\alpha, \sigma)| \) counts the number of elements in the set
\[
(\alpha, \sigma) = \{u \in \mathbb{Z}^n \mid f(u) = \alpha, \text{neg}(u) = \sigma\}. \tag{35}
\]

This is the starting point for the proof of our algorithm to which we turn next.

3 Proof of the Conjecture

We first want to restate the conjecture from [BJRR1] in a more precise form. Therefore, we divide it into two parts, where the first part refers to the possible restriction to degrees in the power set of the Stanley-Reisner ideal by a vanishing of Betti numbers while the second gives a mathematically precise way to compute the “remnant cohomology”.

Let \( X \) be a complete simplicial smooth normal toric variety in the sense of [CLS] endowed with all necessary data listed in Section 2.1. In particular, let the Stanley-Reisner ideal \( I_\Sigma = \langle S_1, \ldots, S_t \rangle \) be generated by \( t \) different squarefree monomials \( S_i \) in the coordinate ring \( S = \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n] \). For all \( \tau \subseteq [t] \) set
\( S_\tau = \text{lcm}\{S_i \mid i \in \tau\} \) and denote by \( a_\tau = \deg(S_\tau) \in \mathbb{Z}^n \) its degree in the natural \( \mathbb{Z}^n \)-grading of \( S \). Furthermore, the degrees in the power set of \( I_\Sigma \) are given by
\[
\mathcal{P}(I_\Sigma) = \{a_\tau \mid \tau \in [t]\}.
\]

Recall from Section 2.3 that the full Taylor resolution \( F_\bullet(T) \) of \( S/I_\Sigma \) is based on the Taylor complex \( T_\bullet(t) \) which is just the reduced chain complex of the full simplex \( \Delta_{[t]} \). For some \( \sigma \subseteq [n] \) define the (relative) simplicial subcomplex
\[
\Gamma^\sigma = \{\tau \in [t] \mid a_\tau = \sigma\}.
\]
of the full simplex \( \Delta_{[t]} \). One can define maps between the sets \( F_j(\Gamma^\sigma) \) of \( j \)-dimensional faces of this subcomplex similar to eq. (14) by
\[
\phi_j : F_j(\Gamma^\sigma) \longrightarrow F_{j-1}(\Gamma^\sigma), \quad e_\tau \mapsto \sum_{k \in \tau} \text{sign}(k, \tau) e_{\tau \setminus k},
\]
where \( e_{\tau \setminus k} = 0 \) if \( \tau \setminus k \notin \Gamma^\sigma \) and \( \text{sign}(k, \tau) = (-1)^{s-1} \) when \( k \) is the \( s \)-th element of \( \tau \subseteq [t] \) written in increasing order. Since this is just the restriction of the boundary maps in \( T_\bullet(t) \), it is easy to see that this yields a well-defined complex \( \tilde{C}_\bullet(\Gamma^\sigma) \) with associated (reduced relative) homology \( \tilde{H}_\bullet(\Gamma^\sigma) \).

**Theorem.** Let \( \alpha \in \text{Cl}(X) \) and \( h^i(\alpha) := \dim_{\mathbb{C}} H^i(X, \mathcal{O}_X(\alpha)) \). Then we have:

(a) For all \( \sigma \subseteq [n] \) with \( \sigma \notin \mathcal{P}(I_\Sigma) \) the associated Betti numbers vanish, i.e.
\[
\beta_{r,\sigma}(S/I_\Sigma) = 0 \quad \text{for all } r \geq 0.
\]

Therefore, one may restrict the sum in eq. (34) to
\[
h^i(\alpha) = \sum_{\sigma \subseteq [n]} |(\alpha, \sigma)| \cdot \beta_{|\sigma|-i,\sigma}(S/I_\Sigma),
\]
where \( |(\alpha, \sigma)| \) counts the number of elements in the set
\[
(\alpha, \sigma) = \{u \in \mathbb{Z}^n \mid f(u) = \alpha, \text{neg}(u) = \sigma\}.
\]

(b) The Betti numbers \( \beta_{r,\sigma}(S/I_\Sigma) \) can be calculated from the degree \( \sigma \) part of the full Taylor resolution. This can be described in terms of the (relative) subcomplex \( \Gamma^\sigma \) as
\[
\beta_{r,\sigma}(S/I_\Sigma) = \dim_{\mathbb{C}} \tilde{H}_{r-1}(\Gamma^\sigma).
\]

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12 We want to note that the set \( (\alpha, \sigma) \) corresponds to all so-called *rationoms* \( x^u \) with \( f(u) = \alpha \) and precisely the coordinates \( x_i \) with \( i \in \sigma \) standing in the denominator. Intuitively, these rationoms can be interpreted as “representatives” of Čech cohomology on intersections of open sets in the toric variety, cf. Section 2.2 of [BJRR1].

13 This corresponds to the sequences for “remnant cohomology” in [BJRR1]. By counting the number of times that a fixed denominator \( x^\sigma \) appears in rank \( r \) of the Stanley-Reisner power set, one gets the number of \( (r-1) \)-faces of the complex \( \Gamma^\sigma \). If one also takes notice of the different combinations of Stanley-Reisner generators that lead to this denominator, one can write down the maps in \( \tilde{H}_\bullet \) and gets a well-defined complex.
Proof of the Theorem: To get the desired Betti numbers, we can tensor the Taylor resolution of \((S/I_\Sigma)\) with \(C \sim S/m\), extract the degree \(\sigma\) part and take homology, cf. \((24)\). As we have described around eq. \((25)\), the tensored resolution will just be made up of vector spaces \(C(-a)\) of degree \(a\) at the locations of \((S(-a))\) in the original resolution. Considering the maps of the tensored resolution, note that all entries \(\lambda_{\tau,\rho}\) with \(a_{\tau} \neq a_{\rho}\) become zero, since they correspond to multiplication by \(x^{a_{\rho} - a_{\tau}} = 0\) in \(S/m\). So we can easily extract graded parts. In particular, since we started with a Taylor complex, the restriction of the tensored resolution to its degree \(\sigma\) part will consist of all occurrences of \(C(-a_{\tau})\) with \(a_{\tau} = \sigma\) and therefore be equivalent to the (relative) complex \(\Gamma^\sigma\) with maps as in eq. \((38)\), i.e.

\[
(F \otimes_S C)_{\sigma} \cong \tilde{C}_{-1}(\Gamma^\sigma).
\]  

(43)

The shift by one in the homological degree on the right hand side comes from the fact that the empty set in the Taylor complex lies in homological degree \(-1\) while the corresponding free module \(S\) in the full Taylor resolution of eq. \((21)\) lies in homological degree \(0\). To get \(\operatorname{Tor}^S_r((S/I_\Sigma, C)_{\sigma})\), we still have to take homology, yielding

\[
\operatorname{Tor}^S_r((S/I_\Sigma, C)_{\sigma}) \cong \tilde{H}_{r-1}(\Gamma^\sigma),
\]  

(44)

which finally implies eq. \((12)\) by taking dimensions. If \(\sigma \notin \mathcal{P}(I_\Sigma)\), the complex \(\Gamma^\sigma\) is void and therefore has zero homology. This implies that the respective Betti numbers vanish and the expression for \(h^i(\alpha)\) then follows from eq. \((34)\). \(\square\)

We also want to point out the simple connection of our theorem to the statement in [Jow]. To see this, fix some \(\sigma \subseteq [n]\) and then arrange the generators of \(I_\Sigma = \langle S_1, \ldots, S_m, S_{m+1}, \ldots, S_t \rangle\) such that \(S_i\) divides \(x^\sigma\) for all \(i \leq m\) and \(m\) is maximal with this property. Now define a subcomplex of \(\Delta_{[m]}\) by

\[
\Lambda^\sigma = \{ \mu \in [m] \mid a_{\mu} < \sigma \}.
\]

(45)

where “\(<\)” is the partial order of \(\mathbb{Z}^n\). It is easy to see that \(\Gamma^\sigma\) also lies in \(\Delta_{[m]}\) and actually is the relative complex

\[
\Gamma^\sigma = \Delta_{[m]} / \Lambda^\sigma.
\]

(46)

Equivalence of our theorem to the theorem of Jow is then provided by the Lemma. The homologies of the reduced chain complexes of \(\Lambda^\sigma\) and \(\Gamma^\sigma\) are isomorphic up to a shift in the homological degree, i.e.

\[
\tilde{H}_i(\Gamma^\sigma) \cong \tilde{H}_{i-1}(\Lambda^\sigma).
\]

(47)

Proof. Because of eq. \((10)\), one has an exact sequence of reduced chain complexes

\[
0 \to \tilde{C}_* (\Lambda^\sigma) \to \tilde{C}_* (\Delta_{[m]}) \to \tilde{C}_* (\Gamma^\sigma) \to 0,
\]

(48)

where the first map is the natural inclusion. Since \(\Delta_{[m]}\) is a full simplex, its homology vanishes and the long exact sequence in homology then yields

\[
\tilde{H}_i(\Gamma^\sigma) \cong \tilde{H}_{i-1}(\Lambda^\sigma).
\]

(49) \(\square\)

12
4 Conclusions and Open Questions

As one can already tell from the Hochster formulas, there are many different complexes from which the Betti numbers of a squarefree monomial ideal can be calculated. All of these have their computational advantages and disadvantages. We want to mention the version of Smith in [MaS], Proposition 3.2, which is implemented in the Macaulay2 package “NormalToricVarieties.m2”. It follows from an application of the Hochster formula (27) to the results of Section 2.4. Since this version of the Hochster formula contains a restriction of the complex \( \Delta \), it also requires the knowledge of the maximal faces of \( \Delta \) and therefore of the irrelevant ideal of the toric variety. In contrast to this, a nice feature of the new algorithm is that it is only based on the combinatorics of the Stanley-Reisner ideal, which also explains why the vanishing of Betti numbers with degrees not in the power set of \( I_\Sigma \) was not detected before.

Since we aim for an efficient computation of sheaf cohomology, it would be desirable to get rid of explicit maps in the complexes that are needed for the computation of Betti numbers. The perfect statement would be that for a fixed degree there can at most be one nontrivial Betti number, as then the alternating sum of dimensions and some considerations about the possible homological degree of the contribution would suffice to determine it. This statement is strikingly similar to the Eagon-Reiner-Theorem for Cohen-Macaulay Stanley-Reisner rings, see Theorem 5.56 in [MiS]. But for a normal toric variety, the fact that \( S/I_\Sigma \) is Cohen-Macaulay only implies that the Alexander dual \( B_\Sigma \) has the desired property. The question is now, if some of these simplifications carry over to the resolution of \( I_\Sigma \). There are some theorems concerning such a duality for resolutions, but the only precise statements can be made for so-called extremal Betti numbers, see [BCP]. It is not clear to the authors how many non-extremal Betti numbers are around in the general case that spoil the broth.

In fact, by running our program for all “boundary divisors” with charge

\[
\alpha_\sigma = f(-\sigma) \in Cl(X), \quad \sigma \subseteq [n]
\]

for certain (perhaps too special) examples\(^{14}\) of toric varieties, we learned that they always had at most one non-trivial cohomology and the number coincided with the only non-zero Betti number in degree \( \sigma \). This means that for these examples we not only found a certain uniqueness of the cohomological degree that a “denominator” can contribute to, but also that either there is only one non-empty “chamber”, namely \( |(\alpha, \sigma)| = 1 \), or all other solutions

\[
\{ \mathbf{u} \in \mathbb{Z}^n \mid f(\mathbf{u}) = \alpha_\sigma, \mathbf{u} \neq -\sigma \}
\]

lie in chambers with vanishing Betti numbers. This implies that in these cases one has some kind of “Serre duality for Betti numbers”, meaning that when starting\(^{14}\) For \( X = dP_3 \) the divisor \( D = -3H - X - Y - Z \) that we took as an example in [BJRR1] is such a boundary divisor with the same charge as \( x_1^{-1}x_2^{-1}x_3^{-1} \).
with Serre duality for line bundles
\[ h^i(\alpha_{\sigma}) = h^{d-i}(\alpha_{\overline{\sigma}}), \tag{52} \]
where \( \overline{\sigma} \) denotes the complement of \( \sigma \), one also gets
\[ \beta_{|\sigma|-1,\sigma}(S/I_{\Sigma}) = \beta_{n-d-(|\sigma|-1),\overline{\sigma}}(S/I_{\Sigma}). \tag{53} \]
In particular, by an application of our theorem, there is another vanishing of Betti numbers
\[ \beta_{r,\sigma}(S/I_{\Sigma}) = 0 \text{ whenever } \overline{\sigma} \notin P(I_{\Sigma}) \tag{54} \]
in these examples.

We would very much appreciate comments or explicit (counter-)examples concerning these issues and hope that the computational power of the algorithm will lead to some new insights in the diverse areas of mathematics and physics it can be applied to.

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