Study of Harper’s Equation for the 2-D Systems of Antiferromagnetically Correlated Electrons in an External Magnetic Field

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Abstract

Considering interacting (antiferromagnetically correlated) electrons, we derive a generalized Harper’s equation for the square lattice of infinite size. We obtain an analytic expression for the density of states from the newly derived Harper’s equation. We present a predicted phase diagram of staggered magnetization in the plane of temperature vs doping rate and discover a possibility of reentrant behavior of the staggered magnetization even in the presence of applied magnetic field. It is shown that below a critical electron correlation strength (Coulomb repulsion) the staggered magnetization in the presence of magnetic field vanishes at an even denominator $q$ value but not at odd $q$ of a given magnetic flux quantum per plaquette, $p/q$.

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I. INTRODUCTION

Since the discovery of high temperature superconductors and the related insulating materials, there has been a lot of interest in two-dimensional spin-$\frac{1}{2}$ magnetic or antiferromagnetically correlated electron systems. However not much attention has been paid to the magnetic properties of these systems coupled to an external magnetic field. The original Harper’s equation is concerned with the energy dispersion involving the systems of non-interacting electrons under the magnetic field. Hence it is of great interest to study how the systems of interacting (antiferromagnetically correlated) electrons behave under an external magnetic field. We derive a generalized Harper’s equation which describes the dispersion of antiferromagnetically correlated electrons under the applied magnetic field. Earlier we paid attention to the dispersion of the antiferromagnetically correlated electrons only at half filling (and thus with no doping) and at zero temperature, by considering the square lattice of finite size. On the other hand, in the present study we derive a generalized Harper’s equation for the square lattice of the infinite size, and examine the dispersion relation as a function of temperature and doping rate. An analytic expression for the density of states is obtained from the generalized Harper’s equation for the system of antiferromagnetically correlated electrons. We present a phase diagram of staggered magnetization in the plane of temperature vs doping rate and find the hitherto-unnoticed reentrant behavior of the staggered magnetization even in the presence of external magnetic field. Finally it is shown that below a critical electron correlation strength the staggered magnetization disappears at an even denominator value of $q$ but not at odd $q$ of a given magnetic flux quantum per plaquette, $p/q$.

II. GENERALIZED HARPER’S EQUATION AND DENSITY OF STATES

We write the Hubbard model Hamiltonian describing the two-dimensional system of antiferromagnetically correlated electrons under an external magnetic field.
\[
H = -t \sum_{\langle ij \rangle} \left[ \exp \left( -i \frac{2\pi}{\phi_0} \int_{j}^{i} A \cdot d\ell \right) c_{i\sigma}^\dagger c_{j\sigma} + \text{H.c.} \right] + U \sum_{i} n_{i\uparrow} n_{i\downarrow} - \mu \sum_{i\sigma} c_{i\sigma}^\dagger c_{i\sigma},
\]

where \( t \) is the hopping integral; \( A \), the electromagnetic vector potential; \( \phi_0 = \frac{hc}{e} \), the elementary flux quantum; \( U \), the on-site Coulomb repulsion energy, and \( \mu \), the chemical potential. \( \langle ij \rangle \) stands for summation over nearest neighbor sites \( i \) and \( j \). \( c_{i\sigma}^\dagger \) (\( c_{i\sigma} \)) is the creation (annihilation) operator of an electron of spin \( \sigma \) at site \( i \), and \( n_{i\uparrow} \) (\( n_{i\downarrow} \)), the number operator of an up-spin (down-spin) electron at site \( i \).

The staggered magnetization (antiferromagnetic order) at site \( i \) is written as \( m_i = e^{iQ \cdot r_i} \sum_{\sigma} \langle c_{i\sigma}^\dagger c_{i\sigma} \rangle \), where \( Q = (\pi, \pi) \) and \( r_i = (i_x, i_y) \) with \( i_x \) and \( i_y \) being integers with the lattice spacing of unity. Introducing a uniform staggered magnetization \( m \) and a uniform doping rate \( \delta \), i.e.,

\[
m = \frac{1}{N} \sum_{i\sigma} e^{iQ \cdot r_i} \langle c_{i\sigma}^\dagger c_{i\sigma} \rangle ,
\]

\[
\delta = 1 - \frac{1}{N} \sum_{i} \langle n_{i} \rangle ,
\]

with the number of lattice sites \( N \), and using the Landau gauge \( A = B(0, x, 0) \), we obtain the mean field (Hartree-Fock) Hamiltonian in the momentum space,

\[
H = -t \sum_{k\sigma} \left[ 2 \cos k_x c_{k\sigma}^\dagger c_{k\sigma} + e^{-ik_y} c_{k-g,\sigma}^\dagger c_{k\sigma} + e^{ik_y} c_{k+g,\sigma}^\dagger c_{k\sigma} \right] - \frac{mU}{2} \sum_{k\sigma} \sigma c_{k+Q,\sigma}^\dagger c_{k\sigma} + \left[ \frac{U}{2} (1 - \delta) - \mu \right] \sum_{k\sigma} c_{k\sigma}^\dagger c_{k\sigma},
\]

where \( g \equiv \left( 2\pi \frac{p}{m_0}, 0 \right) = \left( 2\pi \frac{p}{q}, 0 \right) \) with \( \frac{p}{q} \), the number of flux quanta per plaquette. The first bracketed term in (3) represents hopping processes; the first term in the bracket represents the nearest neighbor hopping in the \( x \)-direction and the last two terms in the bracket, the nearest neighbor hopping in the \( y \)-direction. Because of the choice of the Landau gauge \( A = B(0, x, 0) \), the electron acquires no phase when it hops in the \( x \)-direction, while it acquires a phase when it hops in the \( y \)-direction, and the electromagnetic vector potential \( A \) shifts the wave vector of electron in the \( k_x \)-direction by \( g \equiv |g| = 2\pi \frac{p}{q} \). The second term results from the nature of the antiferromagnetic spin order of correlated electrons, which
causes the wave vector to shift by $\mathbf{Q}$. The last term represents energy shift by $\frac{U}{2}(1-\delta)$ as a result of hole doping.

The Hamiltonian (3) can be written as

$$H = H_0 + H_1,$$

$$H_0 = \frac{U}{2}(1-\delta) - \mu \sum_{k\sigma} c_{k\sigma}^\dagger c_{k\sigma},$$

$$H_1 = \sum_{k\sigma} \sum_{\mathbf{g},\sigma} C_{k\sigma}^\dagger H_{\mathbf{g} \sigma} C_{k\sigma},$$

where

$$C_{k\sigma} = \begin{bmatrix} c_{k+g,\sigma} \\ \vdots \\ c_{k+(q-1)g,\sigma} \\ c_{k\sigma} \\ c_{k+g+Q,\sigma} \\ \vdots \\ c_{k+(q-1)g+Q,\sigma} \\ c_{k+Q,\sigma} \end{bmatrix},$$

$$H_{\mathbf{g} \sigma} = \begin{bmatrix} T_k & V_{\sigma} \\ V_{\sigma} & -T_k \end{bmatrix},$$

$$T_k = -t \begin{bmatrix} M_1 & e^{-ik_y} & 0 & 0 & e^{ik_y} \\ e^{ik_y} & M_2 & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & M_{q-1} & e^{-ik_y} \\ e^{-ik_y} & 0 & 0 & e^{ik_y} & M_q \end{bmatrix},$$
and

$$V_\sigma = \begin{bmatrix}
-\frac{\sigma m U}{2} & 0 & 0 & 0 & 0 \\
0 & -\frac{\sigma m U}{2} & 0 & 0 & 0 \\
0 & 0 & \ddots & 0 & 0 \\
0 & 0 & 0 & -\frac{\sigma m U}{2} & 0 \\
0 & 0 & 0 & 0 & -\frac{\sigma m U}{2}
\end{bmatrix}$$  \tag{4e}

with $M_n = 2 \cos(k_x + ng)$. The summation $\sum'_k$ is over the reduced Brillouin zone, $\{(k_x, k_y) \mid -\frac{\pi}{q} \leq k_x \leq \frac{\pi}{q}, -\frac{\pi}{2} \leq k_y \leq \frac{\pi}{2}\}$. The Brillouin zone is reduced by $1/q$ because there exist $q$ plaquettes per magnetic unit cell, and is further reduced by $1/2$ as a result of staggered magnetization (or antiferromagnetic order). The diagonal matrix $T_k$ is associated with electron hopping and contains information on the phase modulation of hopping electrons under the influence of the external field. The off-diagonal matrix $V_\sigma$ represents the antiferromagnetic electron correlation.

From the eigenvalue equation of the Hamiltonian matrix $H_{k\sigma}$ with the identity matrix $I$,

$$\det(H_{k\sigma} - E_k I) = 0 ,$$  \tag{5}

we obtain the quasiparticle energy dispersion $E_k$ of antiferromagnetically correlated electrons in the presence of magnetic field. In the limiting case of noninteracting electrons ($U = 0$) the ‘generalized’ Harper’s equation \cite{5} above is reduced to the original Harper’s equation derived by Hasegawa et al.\cite{8}, that is,

$$\det(T_k - \varepsilon_k I) = 0 ,$$  \tag{6}

where $\varepsilon_k$ is the energy dispersion of noninteracting electrons in the presence of magnetic field. Following Hasegawa et al.\cite{8}, Eq. \cite{3} can be rewritten in a further simplified form,

$$\gamma(\varepsilon) = \cos(qk_x) + \cos(qk_y) ,$$  \tag{7}

where $\gamma(\varepsilon)$ is given in Table \ref{table:1} for various values of $p/q$ including Hasegawa et al.’s results.\cite{8}

Now for the case of antiferromagnetically correlated electrons we obtain from the diagonalization of the Hamiltonian matrix $H_{k\sigma}$ in Eq. \cite{4c} above,
\[ E_k = \sqrt{\varepsilon_k + \Delta^2}, \quad (8) \]

with the band gap, \(2\Delta = mU\). The band gap \(2\Delta\) is seen to depend on the magnitude of both the staggered magnetization \(m\) and the electron correlation strength (Coulomb repulsion) \(U\).

The energy dispersion relation Eq. (8) above leads to the density of states,

\[
g(E) = 4 \int' \frac{d^2k}{(2\pi)^2} \delta(E - E_k) = \frac{2}{q\pi^2} \left| \frac{d\gamma(\varepsilon)}{d\varepsilon} \right| K \left( \sqrt{1 - \left( \frac{\gamma(\varepsilon)}{2} \right)^2} \right) \left| \frac{E}{\varepsilon} \right|, \quad (9)\]

where \(|\varepsilon| = \sqrt{E^2 - \left( \frac{mt^*}{2} \right)^2}\) and \(K\) is the complete elliptic integral of the first kind.\(^{11}\)

Eq. (8) above represents the density of states for the systems of correlated electrons in the presence of magnetic field. In the limiting case of vanishing electron correlation, i.e., \(U = 0\), it becomes exact. In Fig. 1 we display the density of states with \(U = 0\) predicted from the analytic expression of the density of states in Eq. (9). Encouragingly the analytic result is in excellent agreement with the numerical results obtained by Hasegawa et al.\(^8\) In the presence of magnetic field with the flux quanta per plaquette \(p/q\), a single band splits into \(q\) subbands. This is analogous to the energy level splitting into Landau levels for electrons embedded in a continuum state under a magnetic field.

### III. PHASE DIAGRAM AND REENTRANT BEHAVIOR OF STAGGERED MAGNETIZATION IN APPLIED MAGNETIC FIELDS

The staggered magnetization \(m\) and the chemical potential \(\mu\) vary with both the temperature \(T\) and the doping rate \(\delta\); they are obtained from the following self-consistent mean field equations, which are derived from the use of Eqs. (2),

\[
1 = \int dE \frac{g(E)}{4} \frac{U}{2E} \left[ \tanh \frac{E^{(+)} - E^{(-)}}{2T} - \tanh \frac{E^{(-)}}{2T} \right], \quad (10a)
\]
\[ \delta = \int dE \frac{g(E)}{4} \left[ \tanh \frac{E^+}{2T} + \tanh \frac{E^-}{2T} \right] , \tag{10b} \]

where \( E^{(\pm)} = \pm E + \frac{U}{2} (1 - \delta) - \mu \).

We examine the dependence of staggered magnetization on the external magnetic field in the \( T-\delta \) plane by numerically solving Eqs. (10) for \( m \) and \( \mu \) at each temperature \( T \) and doping rate \( \delta \). The phase diagram of the staggered magnetization in the \( T-\delta \) plane is displayed in Fig. for several values of \( p/q \). Interestingly we find that the reentrant behavior of the staggered magnetization appears even in the presence of the external magnetic field. However, the reentrant behavior of staggered magnetization in the absence of magnetic field has earlier been discovered by other investigators. Halvorsen et al. found the reentrant behavior using the Hubbard model within the self-consistent second-order weak \( U \)-perturbation treatment. Their studies are limited to a narrow range with small \( U \) compared to our present approach which can deal with the entire range of \( U \). Recently Inaba et al. found a similar reentrant behavior using the \( t-J \) Hamiltonian in the slave-boson representation. However their studies refer to the case of a large \( U \) limit due to the use of the \( t-J \) Hamiltonian. Unlike our present study their studies above refer to the reentrant behavior in the absence of the external magnetic field.

We now investigate the reentrant behavior of the staggered magnetization in detail. In Fig. we display the staggered magnetization \( m \) at \( p/q = 1/2 \) as a function of temperature \( T \) at various doping rates \( \delta \). At half filling \( (\delta = 0) \) the staggered magnetization reaches a maximum value at zero temperature. On the other hand, away from half filling \( (\delta \neq 0) \) the predicted staggered magnetization shows a maximum at a finite temperature. Above this temperature the reentrant behavior of a paramagnetic state is predicted. The cause of the reentrant behavior can be explained from the nesting property of the energy surface at saddle points. In Fig. we display the variation of energy dispersion of the highest occupied subband with the external magnetic field (or \( p/q \)). The saddle points in the Brillouin zone are denoted by black dots at the bottom of the graph. The adjacent saddle points are separated by the well-defined nesting vector of \( Q/q \), by which the staggered magnetization
is defined. In Fig. 5 we display the variation of Fermi surface (thick solid lines) with the magnetic field at finite doping rates. As the temperature increases, the Fermi surface tends to smear out, which causes increment in the number of nesting channels, and consequently the staggered magnetization arises. As the temperature still increases, further smearing of the Fermi surface opens other channels than the nesting channels. As a result the staggered magnetization will eventually disappear to allow transition to a paramagnetic state. This feature is well depicted in Figs. 2 and 3.

IV. STAGGERED MAGNETIZATION AT ZERO TEMPERATURE AND AT HALF FILLING IN A MAGNETIC FIELD

Now we investigate the staggered magnetization at half filling $\delta = 0$ and at $T = 0K$ in a magnetic field. The chemical potential is given by $\mu = U/2$, and Eq. (10b) is trivially identified since the right hand side of the equation becomes zero, thus satisfying the condition of half filling, that is, $\delta = 0$ as it should be. Eq. (10a) can be expressed as

$$\frac{1}{U} = \int_{-\infty}^{0} d\varepsilon \frac{g_0(\varepsilon)}{2} \frac{1}{\sqrt{\varepsilon^2 + \left(\frac{mt}{2}\right)^2}}.$$  \hfill (11)

Here $g_0(\varepsilon)$ is the density of states of noninteracting electrons in the presence of magnetic field. It is easily obtained from Eq. (9). Eq. (11) is in a similar form to the gap equation that appears in the spin density wave theory of cuprate materials. We will use it for the determination of the staggered magnetization $m$ for the system of interacting electrons (with correlation strength $U$) in the presence of magnetic field. In Fig. 6 the oscillatory staggered magnetization is displayed as a function of magnetic field (specifically $p/q$), for several chosen values of correlation strength $U$. The solid lines are the results of self-consistent calculations for a $20 \times 20$ finite square lattice. Various other symbols represent the results from the newly derived analytic relation (11) above. Encouragingly they are in good agreement with the self-consistent calculations for the finite size square lattice.

In the following we explain the oscillatory behavior of staggered magnetization observed
in our earlier work. At even denominator values of $q$ in $p/q$ the staggered magnetization is predicted to disappear (e.g., see the case of $p/q = 1/2$). This feature is well depicted in Fig. 6. We now define the critical electron correlation strength (Coulomb repulsion) $U_{p/q}$ as a value below which the staggered magnetization vanishes, i.e., $m = 0$. The predicted staggered magnetization from Eq. (11) vanishes at the even $q$ values below a critical value $U_{p/q}$, i.e., $U < U_{p/q}$. On the other hand in the absence of magnetic field the staggered magnetization tends to appear even at small values of $U$. The critical correlation strength $U_{p/q}$ is obtained by substituting $m = 0$ in Eq. (11), and is shown for various values of $p/q$ in Table II. For odd $q$ in $p/q$, the integral in Eq. (11) is logarithmically divergent, and thus $U_{p/q}$ does not exist. Although not numerically precise for the case of finite size calculations (solid lines) with the even denominator values of $q$ in $p/q$ a propensity of vanishing staggered magnetization is seen below $U_{p/q}$ as shown in Fig. 6. Thus the oscillatory behavior of the staggered magnetization is found to occur owing to its disappearance distinctively at the even denominator values of $q$ in $p/q$.

V. CONCLUSION

We derived a generalized Harper’s equation for the energy dispersion relation of interacting (antiferromagnetically correlated) electrons in an external magnetic field. Unlike the original Harper’s equation which deals only with noninteracting electron systems, the generalized Harper’s equation derived in Eq. (11) (with Eq. (4c)) has a definite merit of studying the physical properties of correlated electron systems in the presence of the external magnetic field. From this Harper’s equation we derived an analytic formula for the density of states of the antiferromagnetically correlated electrons in the magnetic field. For the limiting case of noninteracting electrons, the analytic equation for the density of states is found to be in good agreement with the numerical work of Hasegawa et al. Further we presented the phase diagram of staggered magnetization in the plane of temperature $T$ vs doping rate $\delta$ as a function of magnetic field (specifically, flux quanta per plaquette $p/q$).
and correlation strength $U$. From this study we demonstrated a possibility of reentrant behavior of staggered magnetization even in the presence of the applied magnetic field. A more accurate account of electron correlations beyond the mean field approximations may not alter the qualitative nature of the reentrant behavior that we discovered in this study. Although not reported here, we find from the exact diagonalization study of Hubbard model that the accurate account of correlations does not affect the qualitative finding here. In the present study we neglected the Zeeman coupling. At such large Coulomb repulsion energies as $U = 8t$ which we used in our calculations, we find that the Zeeman effect does not substantially alter the observed staggered magnetization. For the case of half filling at zero temperature, we obtained a gap equation (11) for the determination of staggered magnetization for antiferromagnetically correlated electron systems at a given correlation strength $U$ in the presence of magnetic field. From this derivation we were able to determine a critical correlation strength $U_{p/q}$ below which staggered magnetization disappears at even denominator values of $q$ (but not at odd denominator values of $q$) of the magnetic flux quanta per plaquette, $p/q$.

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TABLE CAPTIONS

TABLE I. $\gamma(\varepsilon)$ for various values of magnetic flux quanta per plaquette $p/q$. The energy dispersion of noninteracting electrons at a given $p/q$ is determined from the Harper’s equation, $\gamma(\varepsilon) = \cos(qk_x + qk_y)$.

TABLE II. Critical correlation strength $U_{p/q}$ as a function of magnetic flux quanta per plaquette $p/q$. 
FIGURE CAPTIONS

FIG. 1. Density of states of noninteracting electrons in the presence of applied magnetic field for various flux quanta per plaquette $p/q$ based on the analytic expression (9).

FIG. 2. Phase diagram of staggered magnetization for several values of magnetic flux quanta per plaquette $p/q$. Each line represents a boundary between the antiferromagnetic (AF) phase and the paramagnetic (PM) phase.

FIG. 3. Variation of staggered magnetization as a function of temperature for various doping rates at $p/q = 1/2$ and $U = 8t$.

FIG. 4. Variation of quasiparticle energy dispersion surface of the highest occupied subband with $p/q$. Black dots denote the saddle points of the surface. Rectangles at the bottom of the graph represent the reduced Brillouin zones, and arrows, the nesting vectors between two adjacent saddle points.

FIG. 5. Variation of Fermi surfaces with $p/q$ at zero temperature and at hole doping rates, (a) $\delta = 0.33$ (b) $\delta = 0.30$ (c) $\delta = 0.10$ (d) $\delta = 0.17$. Black dots denote the saddle points; rectangles, the reduced Brillouin zones, and arrows, the nesting vectors between two adjacent saddle points.

FIG. 6. Staggered magnetization (antiferromagnetic order) at $T = 0K$ as a function of $p/q$, the magnetic flux quanta per plaquette. The solid lines denote the results from self-consistent calculations on a $20 \times 20$ finite square lattice, and various other symbols represent the results calculated from the analytic expression (11) corresponding to the square lattice of infinite size.
| $p$ | $q$ | $2\gamma(\varepsilon)$ |
|-----|-----|-------------------------|
| 1   | 2   | $\varepsilon^2 - 4$    |
| 1   | 3   | $-\varepsilon^3 + 6\varepsilon$ |
| 1   | 4   | $\varepsilon^4 - 8\varepsilon^2 + 4$ |
| 1   | 5   | $-\varepsilon^5 + 10\varepsilon^3 + \varepsilon \left( -15 + 10 \cos\left(\frac{2\pi}{7}\right) \right)$ |
| 2   | 5   | $-\varepsilon^5 + 10\varepsilon^3 + \varepsilon \left( -20 - 10 \cos\left(\frac{2\pi}{7}\right) \right)$ |
| 1   | 6   | $\varepsilon^6 - 12\varepsilon^4 + 24\varepsilon^2 - 4$ |
| 1   | 7   | $-\varepsilon^7 + 14\varepsilon^5 + \left( -49 + 14 \cos\left(\frac{2\pi}{7}\right) \right) \varepsilon^3 + \left( 42 - 28 \cos\left(\frac{2\pi}{7}\right) - 28 \cos\left(\frac{3\pi}{7}\right) \right) \varepsilon$ |
| 2   | 7   | $-\varepsilon^7 + 14\varepsilon^5 + \left( -49 - 14 \cos\left(\frac{2\pi}{7}\right) \right) \varepsilon^3 + \left( 28 - 28 \cos\left(\frac{2\pi}{7}\right) + 56 \cos\left(\frac{3\pi}{7}\right) \right) \varepsilon$ |
| 3   | 7   | $-\varepsilon^7 + 14\varepsilon^5 + \left( -56 - 14 \cos\left(\frac{2\pi}{7}\right) + 14 \cos\left(\frac{3\pi}{7}\right) \right) \varepsilon^3 + \left( 56 + 56 \cos\left(\frac{2\pi}{7}\right) - 28 \cos\left(\frac{3\pi}{7}\right) \right) \varepsilon$ |
| 1   | 8   | $\varepsilon^8 - 16\varepsilon^6 + \left( 72 - 8\sqrt{2} \right) \varepsilon^4 + \left( -96 + 32\sqrt{2} \right) \varepsilon^2 + 4$ |
| 3   | 8   | $\varepsilon^8 - 16\varepsilon^6 + \left( 72 + 8\sqrt{2} \right) \varepsilon^4 + \left( -96 - 32\sqrt{2} \right) \varepsilon^2 + 4$ |
| 1   | 9   | $-\varepsilon^9 + 18\varepsilon^7 + \left( -99 + 4\sqrt{3} \cos\left(\frac{\pi}{18}\right) + 6 \cos\left(\frac{2\pi}{9}\right) + 4\sqrt{3} \cos\left(\frac{7\pi}{18}\right) \right) \varepsilon^5$ |
|     |     | $+ \left( 186 - 24\sqrt{3} \cos\left(\frac{\pi}{18}\right) - 36 \cos\left(\frac{\pi}{9}\right) + 4\sqrt{3} \cos\left(\frac{7\pi}{18}\right) - 28 \sqrt{3} \cos\left(\frac{2\pi}{18}\right) + 24 \cos\left(\frac{4\pi}{9}\right) \right) \varepsilon^3$ |
|     |     | $+ \left( -126 + 36\sqrt{3} \cos\left(\frac{\pi}{18}\right) + 36 \cos\left(\frac{\pi}{9}\right) + 36 \sqrt{3} \cos\left(\frac{7\pi}{18}\right) - 54 \cos\left(\frac{4\pi}{9}\right) \right) \varepsilon$ |
| 2   | 9   | $-\varepsilon^9 + 18\varepsilon^7 + \left( -99 + 4\sqrt{3} \cos\left(\frac{5\pi}{18}\right) - 4\sqrt{3} \cos\left(\frac{7\pi}{18}\right) + 6 \cos\left(\frac{4\pi}{9}\right) \right) \varepsilon^5$ |
|     |     | $+ \left( 186 - 4\sqrt{3} \cos\left(\frac{\pi}{18}\right) - 24 \cos\left(\frac{2\pi}{9}\right) - 28 \sqrt{3} \cos\left(\frac{5\pi}{18}\right) + 24 \sqrt{3} \cos\left(\frac{7\pi}{18}\right) - 60 \cos\left(\frac{4\pi}{9}\right) \right) \varepsilon^3$ |
|     |     | $+ \left( -126 + 54 \cos\left(\frac{2\pi}{9}\right) + 36 \sqrt{3} \cos\left(\frac{2\pi}{18}\right) - 36 \sqrt{3} \cos\left(\frac{7\pi}{18}\right) + 90 \cos\left(\frac{4\pi}{9}\right) \right) \varepsilon$ |
| 4   | 9   | $-\varepsilon^9 + 18\varepsilon^7 + \left( -99 - 4\sqrt{3} \cos\left(\frac{\pi}{18}\right) - 6 \cos\left(\frac{2\pi}{9}\right) - 4\sqrt{3} \cos\left(\frac{5\pi}{18}\right) - 6 \cos\left(\frac{4\pi}{9}\right) \right) \varepsilon^5$ |
|     |     | $+ \left( 186 + 28\sqrt{3} \cos\left(\frac{\pi}{18}\right) + 60 \cos\left(\frac{\pi}{9}\right) + 24 \sqrt{3} \cos\left(\frac{5\pi}{18}\right) + 4\sqrt{3} \cos\left(\frac{7\pi}{18}\right) + 36 \cos\left(\frac{4\pi}{9}\right) \right) \varepsilon^3$ |
|     |     | $+ \left( -126 - 36 \sqrt{3} \cos\left(\frac{\pi}{18}\right) - 90 \cos\left(\frac{2\pi}{9}\right) - 36 \sqrt{3} \cos\left(\frac{5\pi}{18}\right) - 36 \cos\left(\frac{4\pi}{9}\right) \right) \varepsilon$ |
| $\frac{p}{q}$ | $\frac{1}{5}$ | $\frac{1}{6}$ | $\frac{1}{7}$ | $\frac{3}{5}$ | $\frac{1}{2}$ |
|-----|-----|-----|-----|-----|-----|
| $U_{p/q}$ | 1.29 | 1.49 | 1.87 | 0.897 | 3.11 |

**TABLE II.**
FIGURES

FIG. 1.
FIG. 2.
FIG. 3.
FIG. 4.
FIG. 5.
FIG. 6.