Rigidity and the Lower Bound Theorem for Doubly Cohen-Macaulay Complexes

Eran Nevo *

November 2, 2018

Abstract

We prove that for $d \geq 3$, the 1-skeleton of any $(d-1)$-dimensional doubly Cohen-Macaulay (abbreviated 2-CM) complex is generically $d$-rigid. This implies that Barnette’s lower bound inequalities for boundary complexes of simplicial polytopes ([4],[3]) hold for every 2-CM complex of dimension $\geq 2$ (see Kalai [8]). Moreover, the initial part $(g_0, g_1, g_2)$ of the $g$-vector of a 2-CM complex (of dimension $\geq 3$) is an $M$-sequence. It was conjectured by Björner and Swartz [14] that the entire $g$-vector of a 2-CM complex is an $M$-sequence.

1 Introduction

The $g$-theorem gives a complete characterization of the $f$-vectors of boundary complexes of simplicial polytopes. It was conjectured by McMullen in 1970 and proved by Billera and Lee [5] (sufficiency) and by Stanley [13] (necessity) in 1980. A major open problem in $f$-vector theory is the $g$-conjecture, which asserts that this characterization holds for all homology spheres. The open part of this conjecture is to show that the $g$-vector of every homology sphere is an $M$-sequence, i.e. it is the $f$-vector of some order ideal of monomials. Based on the fact that homology spheres are doubly Cohen-Macaulay (abbreviated 2-CM) and that the $g$-vector of some other classes of 2-CM complexes is known to be an $M$-sequence (e.g. [14]), Björner and Swartz [14] recently suspected that

Conjecture 1.1 ([14], a weakening of Problem 4.2.) The $g$-vector of any 2-CM complex is an $M$-sequence.

We prove a first step in this direction, namely:

*Institute of Mathematics, The Hebrew University, Jerusalem Israel, E-mail address: eranevo@math.huji.ac.il
**Theorem 1.2** Let $K$ be a $(d-1)$-dimensional 2-CM simplicial complex (over some field) where $d \geq 4$. Then $(g_0(K), g_1(K), g_2(K))$ is an $M$-sequence.

This theorem follows from the following theorem, combined with an interpretation of rigidity in terms of the face ring (Stanley-Reisner ring), due (implicitly) to Lee [10].

**Theorem 1.3** Let $K$ be a $(d-1)$-dimensional 2-CM simplicial complex (over some field) where $d \geq 3$. Then $K$ has a generically $d$-rigid 1-skeleton.

Kalai [8] showed that if a simplicial complex $K$ of dimension $\geq 2$ satisfies the following conditions then it satisfies Barnette’s lower bound inequalities:

(a) $K$ has a generically $(\dim(K) + 1)$-rigid 1-skeleton.

(b) For each face $F$ of $K$ of codimension $> 2$, its link $lk_K(F)$ has a generically $(\dim(lk_K(F)) + 1)$-rigid 1-skeleton.

(c) For each face $F$ of $K$ of codimension 2, its link $lk_K(F)$ (which is a graph) has at least as many edges as vertices.

Kalai used this observation to prove that Barnette’s inequalities hold for a large class of simplicial complexes.

Observe that the link of a vertex in a 2-CM simplicial complex is 2-CM, and that a 2-CM graph is 2-connected. Combining it with Theorem 1.3 and the above result of Kalai we conclude:

**Corollary 1.4** Let $K$ be a $(d-1)$-dimensional 2-CM simplicial complex where $d \geq 3$. For all $0 \leq i \leq d-1$ $f_i(K) \geq f_i(n, d)$ where $f_i(n, d)$ is the number of $i$-faces in a (equivalently every) stacked $d$-polytope on $n$ vertices. (Explicitly, $f_{d-1}(n, d) = (d-1)n - (d+1)(d-2)$ and $f_i(n, d) = \binom{d}{i}n - \binom{d+1}{i+1}$ for $1 \leq i \leq d-2$.) □

Theorem 1.3 is proved by decomposing $K$ into a union of minimal $(d-1)$-cycle complexes (Fogelsanger’s notion [6]). Each of these pieces has a generically $d$-rigid 1-skeleton ([6]), and the decomposition is such that gluing the pieces together results in a complex with a generically $d$-rigid 1-skeleton. The decomposition is detailed in Theorem 3.4.

This paper is organized as follows: In Section 2 we give the necessary background from rigidity theory, explain the connection between rigidity and the face ring, and reduce the results mentioned in the Introduction to Theorem 3.4. In Section 3 we give the necessary background on 2-CM complexes, prove Theorem 3.4 and discuss related problems and results.

## 2 Rigidity

The presentation of rigidity here is based mainly on the one in Kalai [8]. Let $G = (V, E)$ be a graph. A map $f : V \rightarrow \mathbb{R}^d$ is called a $d-$embedding. It
is rigid if any small enough perturbation of it which preserves the lengths of the edges is induced by an isometry of \( \mathbb{R}^d \). Formally, \( f \) is called rigid if there exists an \( \varepsilon > 0 \) such that if \( g : V \to \mathbb{R}^d \) satisfies \( d(f(v), g(v)) < \varepsilon \) for every \( v \in V \) and \( d(g(u), g(w)) = d(f(u), f(w)) \) for every \( \{u, w\} \in E \), then \( d(g(u), g(w)) = d(f(u), f(w)) \) for every \( u, w \in V \) (where \( d(a, b) \) denotes the Euclidean distance between the points \( a \) and \( b \)).

\( G \) is called generically \( d \)-rigid if the set of its rigid \( d \)-embeddings is open and dense in the topological vector space of all of its \( d \)-embeddings.

Let \( V = [n] \), and let \( \text{Rig}(G, f) \) be the \( dn \times |E| \) matrix which is defined as follows: for its column corresponding to \( \{v < u\} \in E \) put the vector \( f(v) - f(u) \) (resp. \( f(u) - f(v) \)) at the entries of the \( d \) rows corresponding to \( v \) (resp. \( u \)) and zero otherwise. \( G \) is generically \( d \)-rigid iff \( \text{Im} (\text{Rig}(G, f)) = \text{Im} (\text{Rig}(K_V, f)) \) for a generic \( f \), where \( K_V \) is the complete graph on \( V \). \( \text{Rig}(G, f) \) is called the rigidity matrix of \( G \) (its rank is independent of the generic \( f \) that we choose).

Let \( G \) be the 1-skeleton of a \((d-1)\)-dimensional simplicial complex \( K \). We define \( d \) generic degree-one elements in the polynomial ring \( A = \mathbb{R}[x_1, \ldots, x_n] \) as follows: \( \Theta_i = \sum_{v \in [n]} f(v)_i x_v \) where \( f(v)_i \) is the projection of \( f(v) \) on the \( i \)-th coordinate, \( 1 \leq i \leq d \). Then the sequence \( \Theta = (\Theta_1, \ldots, \Theta_d) \) is an l.s.o.p. for the face ring \( \mathbb{R}[K] = A/\mathcal{I}_K \) (\( \mathcal{I}_K \) is the ideal in \( A \) generated by the monomials whose support is not an element of \( K \)). Let \( H(K) = \mathbb{R}[K]/(\Theta) = H(K)_0 \oplus H(K)_1 \oplus \ldots \) where \( (\Theta) \) is the ideal in \( A \) generated by the elements of \( \Theta \) and the grading is induced by the degree grading in \( A \). Consider the multiplication map \( \omega : H(K)_1 \to H(K)_2 \), \( m \to \omega m \) where \( \omega = \sum_{v \in [n]} x_v \).

Lee [10] proved that

\[
\dim_{\mathbb{R}} \ker (\text{Rig}(G, f)) = \dim_{\mathbb{R}} H(K)_2 - \dim_{\mathbb{R}} \omega (H(K)_1). \quad (1)
\]

Assume that \( G \) is generically \( d \)-rigid. Then \( \dim_{\mathbb{R}} \ker (\text{Rig}(G, f)) = f_1(K) - \text{rank} (\text{Rig}(K_V, f)) = g_2(K) = \dim_{\mathbb{R}} H(K)_2 - \dim_{\mathbb{R}} H(K)_1 \). Combining with \( \boxed{1} \), the map \( \omega \) is injective, and hence \( \dim_{\mathbb{R}} H(K)/(\omega) = g_i(K) \) for \( i = 2 \); clearly this holds for \( i = 0, 1 \) as well. Hence \( (g_0(K), g_1(K), g_2(K)) \) is an M-sequence. We conclude that Theorem \( \boxed{1.3} \) implies Theorem \( \boxed{1.2} \) via the following algebraic result:

**Theorem 2.1** Let \( K \) be a \((d-1)\)-dimensional 2-CM simplicial complex (over some field) where \( d \geq 3 \). Then the multiplication map \( \omega : H(K)_1 \to H(K)_2 \) is injective. \( \square \)

In order to prove Theorem \( \boxed{1.3} \) we need the concept of minimal cycle complexes, introduced by Fogelsanger [9]. We summarize his theory below.

Fix a field \( k \) (or more generally, any abelian group) and consider the formal chain complex on a ground set \([n]\), \( C = (\oplus \{k T : T \subseteq [n]\}, \partial) \), where \( \partial(1T) = \sum_{t \in T} \text{sign}(t, T) T \setminus \{t\} \) and \( \text{sign}(t, T) = (-1)^{|\{s \in T : s < t\}|} \). Define subchain, minimal \( d \)-cycle and minimal \( d \)-cycle complex as follows:
\[ c' = \sum \{ b_T : T \subseteq [n], |T| = d + 1 \} \]
is a subchain of a \( d \)-chain \( c = \sum \{ a_T : T \subseteq [n], |T| = d + 1 \} \) iff for every such \( T, b_T = a_T \) or \( b_T = 0 \). A \( d \)-chain \( c \) is a \( d - \)cycle if \( \partial(c) = 0 \), and is a minimal \( d - \)cycle if its only subchains which are cycles are \( c \) and 0. A simplicial complex \( K \) which is spanned by the support of a minimal \( d - \)cycle is called a minimal \( d - \)cycle complex (over \( k \)), i.e. \( K = \{ S : \exists T S \subseteq T, a_T \neq 0 \} \) for some minimal \( d - \)cycle \( c \) as above. For example, triangulations of connected manifolds without boundary are minimal cycle complexes - fix \( k = \mathbb{Z}_2 \) and let the cycle be the sum of all facets.

The following is the main result in Fogelsanger’s thesis.

**Theorem 2.2 (Fogelsanger [6])** For \( d \geq 3 \), every minimal \((d - 1)-\)cycle complex has a generically \( d \)-rigid 1-skeleton.

We will need the following gluing lemma, due of Asimov and Roth, who introduced the concept of generic rigidity of graphs [1].

**Theorem 2.3 (Asimov and Roth [2])** Let \( G_1 \) and \( G_2 \) be generically \( d \)-rigid graphs. If \( G_1 \cap G_2 \) contains at least \( d \) vertices, then \( G_1 \cup G_2 \) is generically \( d \)-rigid.

Now we are ready to conclude Theorem 1.3 from the decomposition theorem, Theorem 3.4.

**Proof of Theorem 1.3** Consider a decomposition sequence of \( K \) as guaranteed by Theorem 3.4 \( K = \bigcup_{i=1}^m S_i \). By Theorem 2.2 each \( S_i \) has a generically \( d \)-rigid 1-skeleton. By Theorem 2.3 for all \( 2 \leq i \leq m \bigcup_{j=1}^i S_j \) has a generically \( d \)-rigid 1-skeleton, in particular \( K \) has a generically \( d \)-rigid 1-skeleton \((i = m)\). \( \Box \)

**Remark:** One can verify that Theorems 2.2 and 2.3 and hence also Theorem 1.3 continue to hold when replacing “generically \( d \)-rigid” by the notion “\( d \)-hyperconnected”, introduced by Kalai [7]. Both of these assertions have an interpretation in terms of algebraic shifting, introduced by Kalai (see e.g. his survey [8]), namely: for both the exterior and symmetric shifting operators over the field \( \mathbb{R} \), denoted by \( \Delta \), \( \{d, n\} \in \Delta(K) \). The existence of this edge in the shifted complex implies the non-negativity of \( g_2(K) \).

## 3 Decomposing a 2-CM complex

**Definition 3.1** A simplicial complex \( K \) is 2 – CM (over a fixed field \( k \)) if it is Cohen-Macaulay and for every vertex \( v \in K \), \( K - v \) is Cohen-Macaulay of the same dimension as \( K \).

Here \( K - v \) is the simplicial complex \( \{T \in K : v \notin T\} \). By a theorem of Reisner [11], a simplicial complex \( L \) is Cohen-Macaulay iff it is pure and for every face \( T \in L \) (including the empty set) and every \( i < \text{dim}(\text{lk}_L(T)) \),
$\tilde{H}_i(lk_L(T); k) = 0$ where $lk_L(T) = \{S \in L : T \cap S = \emptyset, T \cup S \in L\}$ and $H_i(M; k)$ is the reduced $i$-th homology of $M$ over $k$. The proof of Theorem \ref{thm:main} is by induction on $\text{dim}(K)$. Let us first consider the case where $K$ is 1-dimensional.

A (simple finite) graph is 2-connected if after a deletion of any vertex from it, the remaining graph is connected and non trivial (i.e. is not a single vertex nor empty). Note that a graph is 2-CM iff it is 2-connected.

**Lemma 3.2** A graph $G$ is 2-connected iff there exists a decomposition $G = \bigcup_{i=1}^m C_i$ such that each $C_i$ is a simple cycle and for every $1 < i \leq m$, $C_i \cap (\bigcup_{j<i} C_j)$ contains an edge.

Moreover, for each $i_0 \in [m]$ the $C_i$’s can be reordered by a permutation $\sigma : [m] \rightarrow [m]$ such that $\sigma^{-1}(1) = i_0$ and for every $i > 1$, $C_{\sigma^{-1}(i)} \cap (\bigcup_{j<i} C_{\sigma^{-1}(j)})$ contains an edge.

**Proof**: Whitney \cite{Whitney} showed that a graph $G$ is 2-connected if it has an open ear decomposition, i.e. there exists a decomposition $G = \bigcup_{i=0}^m P_i$ such that each $P_i$ is a simple open path, $P_0$ is an edge, $P_0 \cup P_1$ is a simple cycle and for every $1 < i \leq m P_i \cap (\bigcup_{j<i} P_j)$ equals the 2 end vertices of $P_i$.

Assume that $G$ is 2-connected and consider an open ear decomposition as above. Let $C_1 = P_0 \cup P_1$. For $i > 1$ choose a simple path $P_i$ in $\bigcup_{j<i} P_j$ that connects the 2 end vertices of $P_i$, and let $C_i = P_i \cup P_i$. $(C_1, \ldots, C_m)$ is the desired decomposition sequence of $G$.

Let $C$ be the graph whose vertices are the $C_i$’s and two of them are neighbors iff they have an edge in common. Thus, $C$ is connected, and hence the ‘Moreover’ part of the Lemma is proved.

The other implication, that such a decomposition implies 2-connectivity, will not be used in the sequel, and its proof is omitted. \hfill $\Box$

For the induction step we need the following cone lemma. For $v$ a vertex not in the support of a $(d-1)$-chain $c$, let $v \ast c$ denote the following $d$-chain: if $c = \sum\{a_T T : v \notin T \subseteq [n], |T| = d\}$ where $a_T \in k$ for all $T$, then $v \ast c = \sum\{\text{sign}(v, T)a_T T \cup \{v\} : v \notin T \subseteq [n], |T| = d\}$ where $\text{sign}(v, T) = (-1)^{|\{t \notin T : t < v\}|}$.

**Lemma 3.3** Let $s$ be a minimal $(d-1)$-cycle and let $c$ be a minimal $d$-chain such that $\partial(c) = s$, i.e. $c$ has no proper subchain $c'$ such that $\partial(c') = s$. For $v$ a vertex not in any face in supp$(c)$, the support of $c$, define $\tilde{s} = c - v \ast s$. Then $\tilde{s}$ is a minimal $d$-cycle.

**Proof**: $\partial(\tilde{s}) = \partial(c) - \partial(v \ast s) = s - (s - v \ast \partial(s)) = 0$ hence $\tilde{s}$ is a $d$-cycle. To show that it is minimal, let $\tilde{s}$ be a subchain of $\tilde{s}$ such that $\partial(\tilde{s}) = 0$. Note that supp$(c) \cap$ supp$(v \ast s) = \emptyset$.

Case 1: $v$ is contained in a face in supp$(\tilde{s})$. By the minimality of $s$, supp$(v \ast s) \subseteq$ supp$(\tilde{s})$. Thus, by the minimality of $c$ also supp$(c) \subseteq$ supp$(\tilde{s})$ and
for every induction hypothesis, for every $i > 1$, $S_i \cap (\cup_{j<i} S_j)$ contains a $d$-face.

Moreover, for each $i_0 \in [m]$ the $S_i$’s can be reordered by a permutation $\sigma : [m] \to [m]$ such that $\sigma^{-1}(1) = i_0$ and for every $i > 1$, $S_{\sigma^{-1}(i)} \cap (\cup_{j<i} S_{\sigma^{-1}(j)})$ contains a $d$-face.

**proof:** The proof is by induction on $d$. For $d = 1$, by Lemma 3.2, $K = \cup_{i=1}^{m(K)} C_i$ such that each $C_i$ is a simple cycle and for every $i > 1$ $C_i \cap (\cup_{j<i} C_j)$ contains an edge. Define $s_i = \sum \{\text{sign}(i)e : e \in (C_i)_1\}$, then $s_i$ is a minimal 1-cycle (orient the edges properly: $\text{sign}(i)$ equals 1 or $-1$ accordingly) whose support spans the simplicial complex $C_i$. Moreover, by Lemma 3.2 each $C_{i_0}$, $i_0 \in [m(K)]$, can be chosen to be the first in such a decomposition sequence.

For $d > 1$, note that the link of every vertex in a 2-CM simplicial complex is 2-CM. For a vertex $v \in K$, as $lk_K(v)$ is 2-CM then by the induction hypothesis $lk_K(v) = \cup_{i=1}^{m(v)} C_i$ such that each $C_i$ is a minimal $(d-1)$-cycle complex and for every $i > 1$ $C_i \cap (\cup_{j<i} C_j)$ contains a $(d-1)$-face. Let $s_i$ be a minimal $(d-1)$-cycle whose support spans $C_i$. As $K - v$ is CM of dimension $d$, $\tilde{H}_{d-1}(K - v; k) = 0$. Hence there exists a $d$-chain $c$ such that $\partial(c) = s_i$ and $\text{supp}(c) \subseteq K - v$.

Take $c_i$ to be such a chain with a support of minimal cardinality. By Lemma 3.2, $\tilde{s}_i = c_i - v \ast s_i$ is a minimal $d$-cycle. Let $S_i(v)$ by the simplicial complex spanned by $\text{supp}(\tilde{s}_i)$; it is a minimal $d$-cycle complex. By the induction hypothesis, for every $i > 1$ $S_i(v) \cap (\cup_{j<i} S_j(v))$ contains a $d$-face (containing $v$). Thus, $K(v) := \cup_{i=1}^{m(v)} S_i(v)$ has the desired decomposition for every $v \in K$. $K = \cup_{v \in \text{Ver}(K)} K(v)$ as $\text{st}_K(v) \subseteq K(v)$ for every $v$, where $\text{st}_K(v) = \{T \in K : T \cup \{v\} \in K\}$.

Let $v$ be any vertex of $K$. Since the 1-skeleton of $K$ is connected, we can order the vertices of $K$ such that $v_1 = v$ and for every $i > 1$ $v_i$ is a neighbor of some $v_j$ where $1 \leq j < i$. Let $v_{l(i)}$ be such a neighbor of $v_i$. By the induction hypothesis we can order the $S_j(v_i)$’s such that $S_1(v_i)$ will contain $v_{l(i)}$, and hence, as $K$ is pure, will contain a $d$-face which appears in $K(v_{l(i)})$ (this face contains the edge $\{v_i, v_{l(i)}\}$). The resulting decomposition sequence $(S_1(v_1), ..., S_{m(v_i)}(v_1), S_1(v_2), ..., S_{m(v_2)}(v_2), ..., S_{m(v_n)}(v_n))$ is as desired.

Moreover, every $S_j(v_{i_0})$ where $i_0 \in [n]$ and $j \in [m(v_{i_0})]$ can be chosen to be the first in such a decomposition sequence. Indeed, by the induction hypothesis $S_j(v_{i_0})$ can be the first in the decomposition sequence of $K(v_{i_0})$, and as mentioned before, the connectivity of the 1-skeleton of $K$ guarantees
that each such prefix \((S_1(v_0), \ldots, S_m(v_0))(v_0)\) can be completed to a decomposition sequence of \(K\) on the same \(S_j(v_i)'s\). □

Theorem 3.3 follows also from the following corollary combined with Theorem 2.2.

**Corollary 3.5** Let \(K\) be a \(d\)-dimensional 2-CM simplicial complex over a field \(k\) (\(d \geq 1\)). Then \(K\) is a minimal cycle complex over the Abelian group \(\bar{k} = k(x_1, x_2, \ldots)\) whose elements are finite linear combinations of the \(x_i\)'s with coefficients in \(k\).

**Proof:** Consider a decomposition \(K = \bigcup_{i=1}^m S_i\) as guaranteed by Theorem 3.4 where \(S_i = \text{supp}(c_i)\) (the closure w.r.t. inclusion of \(\text{supp}(c_i)\)) for some minimal \(d\)-cycle \(c_i\) over \(k\). Define \(\tilde{c}_i = x_i c_i\), thus \(\tilde{c}_i\) is a minimal cycle over \(\bar{k}\). Define \(\tilde{c} = \sum_{i=1}^m \tilde{c}_i\). Clearly \(\tilde{c}\) is a cycle over \(\bar{k}\) whose support spans \(K\). It remains to show that \(\tilde{c}\) is minimal. Let \(\tilde{c}'\) be a subchain of \(\tilde{c}\) which is a cycle, \(\tilde{c}' \neq \tilde{c}\). We need to show that \(\tilde{c}' = 0\). Denote by \(\tilde{\alpha}_T(\tilde{c}'\tilde{c})\) the coefficient of the set \(T\) in \(\tilde{c}'\tilde{c}\) and by \(\tilde{\alpha}_T(i)\) the coefficient of the set \(T\) in \(\tilde{c}_i\). If \(\tilde{\alpha}_T(i) = 0\) then for every \(i\) such that \(\tilde{\alpha}_T(i) \neq 0\), the minimality of \(\tilde{c}_i\) implies that \(\tilde{\alpha}'_T = 0\) whenever \(\tilde{\alpha}'_T(i) \neq 0\). By assumption, there exists a set \(T_0\) such that \(\tilde{\alpha}'_{T_0} = 0 \neq \tilde{\alpha}'_{T_0}\). In particular, there exists an index \(i_0\) such that \(\tilde{\alpha}'_{T_{i_0}}(i_0) \neq 0\), hence \(\tilde{\alpha}'_{F} = 0\) whenever \(\tilde{\alpha}'_{F}(i_0) \neq 0\). As \(S_{i_0} \cap (\cup_{j<i_0} S_j)\) contains a \(d\)-face in case \(i_0 > 1\), repeated application of the above argument implies \(\tilde{\alpha}'_{F} = 0\) whenever \(\tilde{\alpha}'_{F}(1) \neq 0\). Repeated application of the fact that \(S_i \cap (\cup_{j<i} S_j)\) contains a \(d\)-face for \(i = 2, 3, \ldots\) and of the above argument shows that \(\tilde{\alpha}'_{F} = 0\) whenever \(\tilde{\alpha}'_{F}(i) \neq 0\) for some \(1 \leq i \leq m\), i.e. \(\tilde{c}' = 0\). □

A pure simplicial complex has a nowhere zero flow if there is an assignment of integer non-zero weights to all of its facets which forms a \(\mathbb{Z}\)-cycle. This generalizes the definition of a nowhere zero flow for graphs (e.g. [12] for a survey).

**Corollary 3.6** Let \(K\) be a \(d\)-dimensional 2-CM simplicial complex over \(\mathbb{Q}\) (\(d \geq 1\)). Then \(K\) has a nowhere zero flow.

**Proof:** Consider a decomposition \(K = \bigcup_{i=1}^m S_i\) as guaranteed by Theorem 3.4. Multiplying by a common denominator, we may assume that each \(S_i = \text{supp}(c_i)\) for some minimal \(d\)-cycle \(c_i\) over \(\mathbb{Z}\) (instead of just over \(\mathbb{Q}\)). Let \(N\) be the maximal \(|\alpha|\) over all nonzero coefficients \(\alpha\) of the \(c_i\)'s, \(1 \leq i \leq m\). Let \(\tilde{c} = \sum_{i=1}^m (N^m)^i c_i\). \(\tilde{c}\) is a nowhere zero flow for \(K\); we omit the details. □

**Problem 3.7** Can the \(S_i\)'s in Theorem 3.4 be taken to be homology spheres?

Yhonatan Iron and I proved (unpublished) the following lemma:
Lemma 3.8 Let $K$, $L$ and $K \cap L$ be simplicial complexes of the same dimension $d-1$. Assume that $K$ and $L$ are weak-Lefschetz, i.e. that multiplication by a generic degree-one element $g$ in $H = H(K), H(L)$, $g : H_{i-1} \rightarrow H_i$, is injective for all $i \leq \lfloor d/2 \rfloor$. If $K \cap L$ is CM then $K \cup L$ is weak-Lefschetz.

In view of this lemma, if the intersections $S_i \cap (\cup_{j<i} S_j)$ in Theorem 3.4 can be taken to be CM, and the $S_i$’s can be taken to be homology spheres, then Conjecture 1.1 would be reduced to the long standing $g$-conjecture for homology spheres. Can the intersections be guaranteed to be CM?

Acknowledgments

I would like to thank my advisor Gil Kalai, Anders Björner and Ed Swartz for helpful discussions. This research was done during the author’s stay at Institut Mittag-Leffler, supported by the ACE network.

References

[1] L. Asimov and B. Roth, The rigidity of graphs, *Trans. Amer. Math. Soc.*, **245** (1978), 279-289.

[2] L. Asimov and B. Roth, The rigidity of graphs: part II, *J. Math. Anal. Appl.*, **68** (1979), 171-190.

[3] D. Barnette, The minimum number of vertices of a simple polytope, *Isr. J. Math.*, **10** (1971), 121-125.

[4] D. Barnette, A proof of the lower bound conjecture for convex polytopes, *Pac. J. Math.*, **46** (1973), 349-354.

[5] L. G. Billera and C. W. Lee, A proof of the sufficiency of McMullen conditions for $f$-vectors of simplicial convex polytopes, *J. Combi. Theory, Ser. A*, **31** (1981), 237-255.

[6] A. Fogelsanger, The generic rigidity of minimal cycles, PhD. Dissertation, Cornell University (1988). Also at [http://www.people.cornell.edu/pages/alf6/rigidity.htm](http://www.people.cornell.edu/pages/alf6/rigidity.htm).

[7] G. Kalai, Hyperconnectivity of graphs, *Graphs and Combi.*, **1**, (1985), 65-79.

[8] G. Kalai, Rigidity and the lower bound theorem, *Inven. Math.*, **88**, (1987), 125-151.

[9] G. Kalai, Algebraic Shifting, *Advanced Studies in Pure Math.*, **33** (2002), 121-163.
[10] K. W. Lee, Generalized stress and motion, in Polytopes: Abstract, Convex and Computational (T. Brztriczzy et al., eds.), (1995), pp.249-271.

[11] G. Reisner, Cohen-Macaulay quotients of polynomial rings, Advances in Math., 21, (1976), 30-49.

[12] P.D. Seymour, Nowhere-zero flows, in Handbook of Combinatorics (R. Graham et al., eds.), Elsevier, Amsterdam, (1995), pp. 289-299.

[13] R. P. Stanley, The number of faces of simplicial convex polytopes, Adv. Math., 35 (1980), 236-238.

[14] E. Swartz, $g$-elements, finite buildings and higher Cohen-Macaulay connectivity, [http://www.math.cornell.edu/~ebs/papers.html](http://www.math.cornell.edu/~ebs/papers.html) preprint.

[15] H. Whitney, Non-separable and planar graphs, Trans. Amer. Math. Soc., 34 (1932), 339-362.