PERIODIC SOLUTIONS FOR A CLASS OF SECOND ORDER
ODES WITH A NAGUMO CUBIC TYPE NONLINEARITY

CHIARA ZANINI
Dipartimento di Scienze Matematiche, Politecnico di Torino
Corso Duca degli Abruzzi 24
10129 Torino, Italy

FABIO ZANOLIN
Dipartimento di Matematica e Informatica, Università di Udine
via delle Scienze 206
33100 Udine, Italy

Abstract. We prove the existence of multiple periodic solutions as well as the presence of complex profiles (for a certain range of the parameters) for the steady-state solutions of a class of reaction–diffusion equations with a FitzHugh–Nagumo cubic type nonlinearity. An application is given to a second order ODE related to a myelinated nerve axon model.

1. Introduction and main results.

1.1. Motivation about the model investigated. The study of the so called Nagumo (or FitzHugh–Nagumo) type equations has proven to be relevant not only for its significance from the point of view of the investigations of nerve fiber models but also from the theoretical point of view, since the peculiar nonlinearity which appears in these equations, as well as the various different features shown by the solutions, have stimulated the development of new theoretical tools from nonlinear functional analysis and the theory of dynamical systems.

Usually, according to [18], by a Nagumo equation, we mean an autonomous system of the form

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} - f(u) + w, \quad \frac{\partial w}{\partial t} = bu, \tag{1} \]

where \( f \) is a cubic nonlinearity like

\[ f(u) = u(1 - u)(u - a), \quad 0 < a < 1. \tag{2} \]

In some works the condition \( \int_0^1 f(s) \, ds > 0 \) is added. This corresponds to the case \( 0 < a < \frac{1}{2} \) for a function \( f \) as in (2).

Nonlinear ODEs or PDEs in which a Nagumo type cubic function plays a central role in the equation arise not only in mathematical models for neurobiology [11, 12] but...
they appear in population genetics, chemical reaction theory, combustion theory and in a wide range of physical systems where the presence of excitable media are taken into account (see, for instance, the Introduction and the References in [16, 22]).

![Figure 1](image_url)  
**Figure 1.** Example of a function \( f(u) \) as in (2) for \( a = 0.4 \).

Generalizations of these equations to PDEs systems of the form

\[
\begin{align*}
\begin{aligned}
    u_t &= D_1 \Delta u + f(u) - v \\
    v_t &= D_2 \Delta v + \varepsilon(u - \gamma v)
\end{aligned}
\end{align*}
\]

have been considered by various authors as well (see [8] and the references therein). The search of the steady states or the travelling wave solutions for equation (1) leads to the study of some autonomous systems (corresponding, typically, to some third order ODEs). Some authors, however, with the aim of developing suitable variants of the FitzHugh–Nagumo model, have modified the original system, in order to take into account some special physical (or physiological) features of the model under consideration. With this respect, we recall the contributions by Grindrod and Sleeman [9] for the study of a myelinated nerve axon and by Chen and Bell [7] for the case of a nerve fiber model with spines (see also [4]). In both of the above recalled models, the equation for the stationary solutions is a nonautonomous second order nonlinear ODE with periodic coefficients which are stepwise functions. The associated dynamics in the phase–plane is that of two superimposed first order autonomous systems which switch from one to the other in a periodic fashion. Both the models present a typical threshold behavior. In [9, 7] the authors investigated the range of the parameters for which the trivial solution is the only periodic one and also proved the stability of such equilibrium solutions.

Reaction–diffusion equations in a one–dimensional non–homogeneous environment (medium) have been investigated by several authors in the last decade. See, for instance, [2, 3, 16, 22] and the references therein. In the above quoted papers, a typical model equation takes the form

\[
\begin{align*}
    u_t &= u_{xx} + g(u, x),
\end{align*}
\]

where the function \( g(u, x) \) is defined piecewise as follows:

\[
g(u, x) = \begin{cases} 
    f(u), & -\infty < x \leq x_1, x_2 < x \leq x_3, \ldots, x_{2n} < x < +\infty \\
    g_0(u, x), & \text{otherwise}.
\end{cases}
\]

In [22] the authors consider the case \( g_0(u, x) = 0 \). Other choices, however, are interesting (depending on the model considered). For instance, in [16, 4.2: Leaky gap], the case \( g_0(u, x) = -\gamma u \) is investigated.
In our study, we consider a medium having an inhomogeneous structure which repeats periodically. A natural problem for this kind of models is the search of stationary periodic solutions. Such kind of researches have raised some interest and have been performed not only in the case of neural models, but also in different areas of investigation, for instance, in connection with problems in nonlinear optics dealing with nonlinear Schrödinger equations with periodic inhomogeneous terms [5, 14]. Periodic structures have been also considered in [6, 19, 21] in a different context, namely, the study of front propagation.

In some recent papers [23, 24] we studied the existence and multiplicity of non-trivial periodic steady states for the Chen and Bell equation

$$v_{xx} - gv + n(x)f(v) = 0, \quad (3)$$

where $g > 0$ is a constant such that $1/g$ represents a Ohmic resistance, $f(v)$ is a Nagumo type nonlinearity and $n(x)$ is a periodic stepwise positive coefficient. We subsequently also proved the presence of complex (i.e. chaotic–like) solutions for the same model [25], by assuming (for simplicity) $f$ negative and convex on $[0, a]$ and positive and concave on $[a, 1]$, with $\int_0^1 f(s) \, ds > 0$. The argument of the proof easily adapts to the case $f(s) = s(s-a)(1-s)$, like in (2), as well as to more general kind of functions $f(v)$ whose graph is a $\backslash$\-shaped curve satisfying $\int_0^1 f(s) \, ds > 0$, as in Figure 1.

In this article we consider as a sample model the one discussed by Grindrod and Sleeman in [9]. The analysis of such model “inspired” some arguments in the proofs of Chen and Bell’s paper [7, p. 394]. Grindrod and Sleeman in [9] studied the case of a myelinated nerve axon, thought as an infinitely long cylindrical membrane, covered by a sheath of lipoproteins (the myelin) which insulates the axon from the external ionic fluid. It is known that gaps in the sheath (named as nodes of Ranvier) occur at evenly-spaced intervals. This is taken into account in Grindrod and Sleeman’s model. In fact, the authors consider the resulting dynamics by superimposing two different equations (one for the myelinated part of the fiber, the other one for the node of Ranvier). The search of the stationary solutions leads to the following system.

$$\begin{cases} 
 v_{xx} + \mu f(v) = 0, & x \in [0, \theta] \mod (L), \quad \text{[node]} \\
 v_{xx} - \frac{v}{\theta} = 0, & x \in [\theta, L] \mod (L), \quad \text{[myelinated]} 
\end{cases} \quad (4)$$

where $\theta, \mu, L, R > 0$ are given constants.

The function $v(x) = u(x,t) = constant \ with \ respect \ to \ t$ in Grindrod and Sleeman’s model represents the transmembrane potential, that is the difference between the longitudinal axoplasmic and the external fluid potentials. The nonlinearity $f$ is a sufficiently smooth function having a $\backslash$\-shaped graph. For instance, the choice of a cubic nonlinearity, as the function defined in (2) and with $0 < a < \frac{1}{2}$, is considered as appropriate (see [9, p. 121]).

In the present work we can relax (at a certain extent) such kind of assumptions.

As a general hypothesis for $f$ we suppose

$$(H_1) \quad f : \mathbb{R} \to \mathbb{R} \ is \ locally \ Lipschitz, \ with \ f(0) = 0 \ and \ such \ that$$

$$f(s) > 0, \ for \ s < 0 \ and \ f(s) < 0, \ for \ s > 1.$$ 

In some results we will add more specific hypotheses which require a Nagumo type shape for the graph of $f$. In such a case we will replace condition $(H_1)$ with the following
(H$_1$) $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, there exists $a \in ]0, 1[$ such that
\[ f(0) = f(a) = f(1) = 0 \] (5)
and, moreover,
\[ f(s) < 0 \text{ for } 0 < s < a \text{ and } s > 1, \quad f(s) > 0 \text{ for } s < 0 \text{ and } a < s < 1. \] (6)
We also introduce the primitive
\[ F(s) := \int_0^s f(\xi) \, d\xi. \]

With respect to the myelinated nerve axon model in [9], the parameter $\theta \in ]0, L[$ (the case biologically relevant requires $\theta << L$) represents the width of each of the Ranvier nodes, which are assumed to be periodically distributed along the nerve fiber and located at distance $L - \theta$ apart. By a scaling, one could assume $L = 1$ like in Grindrod and Sleeman’s model [9, p. 123]. In [9] the authors also assumed $\mu = 1$. For technical reasons, we prefer, however, to let the parameters $\mu$ and $L$ free.

Like in [9] we are interested only in those stationary states $v(x)$ which satisfy the condition
\[ 0 \leq v(x) \leq 1, \quad \forall x \in \mathbb{R} \]
since it has been proved in [9] that $[0, 1]$ is a positively invariant set for the solutions of the associated reaction–diffusion equation. In particular, we shall focus our attention to the search of periodic solutions as well as to the detection of solutions presenting some kind of more complex structure which, in any case, will have range in $[0, 1]$.

With the aim of analyzing (4) as a single equation, we introduce two auxiliary coefficients $\alpha(x)$ and $\beta(x)$ and consider the new equation
\[ v_{xx} - \beta(x)v + \alpha(x)f(v) = 0. \] (7)

For the weights $\alpha, \beta$ we assume
(H$_2$) $\alpha, \beta : \mathbb{R} \to \mathbb{R}^+ := [0, +\infty)$ are $L$-periodic measurable functions, with $\alpha, \beta \in L^1([0, L])$, such that
\[ \alpha(t) + \beta(t) > 0, \quad \text{for a.e. } t \in [0, L]. \] (8)

We recover system (4) from (7), for the particular choice
\[
\alpha(x) = \begin{cases} 
\mu, & \text{for } x \in [0, \theta[ \\
0, & \text{for } x \in ]\theta, L[ 
\end{cases}, \quad \beta(x) = \begin{cases} 
0, & \text{for } x \in [0, \theta[ \\
1/R, & \text{for } x \in ]\theta, L[.
\end{cases}
\] (9)

Our goal, however, is to study equation (7), possibly for a more general choice of $\alpha$ and $\beta$.

We observe that equation (7) is general enough to include other Nagumo type equations considered in the literature. For instance the equation (3) from the Chen and Bell model [7] turns out to be a special case of (7) with the choice
\[ \beta(x) \equiv g > 0, \quad \alpha(x) = n(x) \geq 0. \]
1.2. A dynamical system approach. In our proofs, we follow a dynamical system approach and therefore we consider the spatial variable $x$ as a time. With the position $x \leftrightarrow t$, equation (7) becomes

$$\ddot{v} - \beta(t)v + \alpha(t)f(v) = 0,$$  \hfill (10)

where the dot denotes the differentiation with respect to $t$. From this perspective, the number $L$, representing the basic length structure pattern of the fiber which repeats periodically, from now on, will be indicated as the new constant $T$. In particular, in condition $(H_2)$, as well as in (4), the substitution

$$L \leftrightarrow T$$

is tacitly assumed.

Dealing with (7) we always suppose $f$ to be a locally Lipschitz function and $\alpha, \beta : \mathbb{R} \rightarrow \mathbb{R}^+$ are $T$-periodic measurable functions, with $\alpha, \beta \in L^1([0,T])$. We also write equation (10) in the phase–plane $(q,p) = (v,\dot{v})$, as the first order system

$$\begin{cases} 
\dot{q} = p \\
\dot{p} = \beta(t)q - \alpha(t)f(q).
\end{cases}$$  \hfill (11)

The fundamental theory of ODEs ensures that, for every $t_0 \in \mathbb{R}$ and $z_0 = (q_0, p_0) \in \mathbb{R}^2$, there exists a unique (noncontinuable) solution $\zeta(\cdot) = \zeta(\cdot; t_0, z_0)$ of (11) satisfying the initial condition $\zeta(t_0) = z_0$. When $t_0 = 0$, we usually take the simplified notation $\zeta(t, z_0) := \zeta(t; 0, z_0)$.

According to a classical approach [15], the search of periodic (harmonic and subharmonic) solutions for system (11) can be performed by looking for the fixed points and periodic points of the Poincaré’s map

$$\Psi : \mathbb{R}^2 \supset \text{dom}(\Psi) \rightarrow \mathbb{R}^2, \quad z_0 \mapsto \zeta(T, z_0) := \zeta(T; 0, z_0),$$

where $\text{dom}(\Psi)$ is an open set. By the continuous dependence of the solutions with respect to the initial data, $\Psi$ turns out to be a homeomorphism of its domain onto its image.

Since we are looking for solutions $v(\cdot)$ to (10) such that

$$0 \leq v(t) \leq 1, \quad \forall t \in \mathbb{R},$$  \hfill (12)

we notice that such solutions are the same also for any other equation where the function $f(s)$ coincides with the given one in $[0,1]$, but is possibly different elsewhere.

To make this claim more precise, we state the following lemma which also improves a similar result in [25, Lemma A.1].

**Lemma 1.1.** Let $\alpha, \beta$ satisfy $(H_2)$ and suppose that $F : \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz function such that $F(0) = 0$ and

$$F(s) > 0 \text{ for } s < 0, \quad F(s) < 0 \text{ for } s > 1.$$  \hfill (13)

Let $v(\cdot)$ be a solution of

$$\ddot{v} - \beta(t)v + \alpha(t)F(v) = 0,$$

defined on an interval $[t_0, t_1]$ and such that

$$0 \leq v(t_0), v(t_1) \leq 1.$$

Then,

$$0 \leq v(t) \leq 1, \quad \forall t \in [t_0, t_1].$$

Moreover, $v(t) > 0 \forall t \in [t_0, t_1]$, if $v(t_0), v(t_1) > 0$. Furthermore, any nontrivial periodic solution $\tilde{v}(\cdot)$ of (13) satisfies $0 < \tilde{v}(t) \leq 1, \forall t \in \mathbb{R}$.
Proof. The result is an easy variant of the maximum principle. We give the details for the reader convenience. Equation (13) is of the form

\[ \ddot{v} + h(t, v) = 0, \]

where \( h : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), is a Carathéodory function, that is, \( h(\cdot, s) \) is measurable for all \( s \in \mathbb{R} \), \( h(t, \cdot) \) is continuous for almost every \( t \in \mathbb{R} \) and, for every \( r > 0 \) and \( a < b \), there is a measurable function \( \rho_r \in L^1([a, b], \mathbb{R}^+) \) such that \( |h(t, s)| \leq \rho_r(t) \) for almost every \( t \in [a, b] \) and every \( s \in [-r, r] \). Solutions of (14) are considered in the Carathéodory sense too (cf. [10, p.28]).

By the assumptions on \( \mathcal{F} \) we have that \( h(t, s) \) is locally Lipschitz (with respect to \( s \)),

\[ h(t, 0) = 0 \]

and, for almost every \( t \in \mathbb{R} \),

\[ h(t, s) > 0 \quad \forall s < 0, \quad h(t, s) < 0 \quad \forall s > 1. \]  

(16)

Let \( v(\cdot) \) be a solution of (14) defined on \([t_0, t_1]\) and such that \( v(t_0), v(t_1) \geq 0 \). Assume, by contradiction that there exists \( t_\ast \in [t_0, t_1] \) such that \( v(t_\ast) < 0 \). Then, by continuity, we can find a maximal interval \([t', t''] \subseteq [t_0, t_1] \) such that \( v(t) < 0 \) for all \( t \in [t', t''] \), \( v(t_\ast) = v(t'') = 0 \) and \( \dot{v}(t') \leq 0 \leq \dot{v}(t'') \). Integrating \(-\dot{v}(t)v(t) = h(t, v(t))v(t)\) on \([t', t'']\) and using the first sign condition in (16), we have \( 0 \leq \int_{t'}^{t''} \dot{v}(t)^2 dt \). Integrating also the second sign condition in (16), we obtain \( \int_{t'}^{t''} h(t, v(t))v(t)\dot{v}(t)dt < 0 \), a contradiction. This shows that \( v(t_0), v(t_1) \geq 0 \) implies \( v(t) > 0 \), \( \forall t \in [t_0, t_1] \). With a similar argument we can prove, from the second sign condition in (16), that \( v(t_0), v(t_1) \leq 1 \) implies \( v(t) \leq 1 \), \( \forall t \in [t_0, t_1] \).

This gives the first assertion of the Lemma. Assume now that \( v(\cdot) : \mathbb{R} \to \mathbb{R} \) is a periodic solution (14) of period \( \tau > 0 \). Let \( c \in \mathbb{R} \) be a fixed constant. Integrating \(-\dot{v}(t)(v(t) - c) = h(t, v(t))v(t) - c\) on \([0, \tau]\), we obtain \( \int_0^\tau h(t, v(t))(v(t) - c)dt = \int_0^\tau \dot{v}(t)^2 dt \geq 0 \). Using the sign condition (16) and choosing, respectively, \( c = 0 \) and \( c = 1 \), we find that \( v \not\equiv 0 \) and \( v \not\equiv 1 \). Hence, there exist \( \hat{t}, \hat{t}_\ast \in [0, \tau] \) such that \( 0 \leq v(\hat{t}) = v(\hat{t} + \tau) \) and \( v(\hat{t}_\ast) = v(\hat{t}_\ast + \tau) \leq 1 \) and therefore, \( v(t) \geq 0 \), \( \forall t \in [\hat{t}, \hat{t} + \tau] \) as well as \( v(t) \leq 1 \), \( \forall t \in [\hat{t}_\ast, \hat{t}_\ast + \tau] \). Finally, by the \( \tau \)-periodicity of \( v(\cdot) \) we find that \( 0 \leq v(t) \leq 1 \), \( \forall t \in \mathbb{R} \). We notice that until now, only the sign condition (16) was assumed. To conclude the proof, consider again a solution \( v(\cdot) \) of (14) defined on \([t_0, t_1]\) such that \( v(t_0), v(t_1) > 0 \). From the first part of the Lemma, we already know that \( v(t) \geq 0 \), \( \forall t \in [t_0, t_1] \). If, by contradiction, there exists \( t_\ast \in [t_0, t_1] \) such that \( v(t_\ast) = 0 \), we also have \( \dot{v}(t_\ast) = 0 \) and hence, by (15) and the uniqueness of the solutions for the initial value problems we obtain \( v = 0 \) on \([t_0, t_1] \), a contradiction. Finally, assume again that \( v(\cdot) \) is \( \tau \)-periodic. We have already proved that \( 0 \leq v(t) \leq 1 \), for all \( t \in \mathbb{R} \). If \( v \neq 0 \), then \( v(t_0) = v(t_0 + \tau) > 0 \) for some \( t_0 \) and therefore \( v(t) > 0 \) on \([t_0, t_0 + \tau]\) (by the same argument as above). The proof is complete. \[ \square \]

Let us assume (H1) for \( f \), and fix \( \varepsilon > 0 \) and \( M > |f(1)| \) such that \( |f(s)| \leq M \) for every \( s \in [-\varepsilon, 0] \cup [1, 1 + \varepsilon] \). Then we define the truncated function

\[ f_\varepsilon(s) := \begin{cases} f(-\varepsilon), & \text{for } s \leq -\varepsilon \\ f(s), & \text{for } -\varepsilon \leq s \leq 1 + \varepsilon \\ f(1 + \varepsilon), & \text{for } s \geq 1 + \varepsilon. \end{cases} \]

The function \( f_\varepsilon \) is locally Lipschitz, uniformly bounded with \( |f_\varepsilon(s)| \leq M \), for every \( s \in \mathbb{R} \setminus [0, 1] \) and coincides with \( f \) on \([0, 1]\). Moreover, if \( f \) is smooth (of class \( C^1 \)) in
a neighborhood of any point \( s_0 \in [0, 1] \), then \( f_\varepsilon \) is also smooth in a neighborhood of the same point.

As a straightforward consequence of Lemma 1.1 and the definition of \( f_\varepsilon \), the following properties hold with respect to the auxiliary equation

\[ \ddot{v} - \beta(t)v + \alpha(t)f_\varepsilon(v) = 0, \]  

(17)

with \( \alpha, \beta \) satisfying \((H_2)\).

\( (a_1) \) Every initial value problem associated to (17) has a unique solution which is defined for every \( t \in \mathbb{R} \).

\( (a_2) \) The sets of periodic solutions (of any period) of (10) and (17) coincide.

\( (a_3) \) Let \( v(\cdot) \) be a solution of (17) and assume that there exists a two-sided sequence \( (t_i)_{i \in \mathbb{Z}} \) of real numbers, with \( t_i \to \pm\infty \) for \( i \to \pm\infty \), such that \( v(t_i) \in [0, 1], \forall i \in \mathbb{Z} \). Then, \( v(t) \in [0, 1], \forall t \in \mathbb{R} \) and therefore \( v(\cdot) \) is a solution of (10).

In view of the above properties, in the sequel, when dealing with equation (10), besides condition \((H_1)\) we also assume

\[ |f(s)| \leq M, \quad \forall s \in \mathbb{R} \setminus [0, 1] \]  

(18)

(which implies that all the solutions of (10) are defined for every \( t \in \mathbb{R} \)) and

\[ \liminf_{s \to -\infty} f(s) > 0 > \limsup_{s \to +\infty} f(s). \]  

(19)

Such extra hypotheses, which are satisfied by \( f_\varepsilon \), are harmless with respect to the search of periodic solutions or solutions \( v(\cdot) \) with \( v(t_i) \in [0, 1] \) for a sequence \( t_i \) which is unbounded from above and from below (see \((a_2)\) and \((a_3)\), respectively).

Note that, according to \((a_1)\), when we replace \( f \) with \( f_\varepsilon \) we can assume that all the solutions of system (11) are globally defined and thus the associated Poincaré’s map turns out to be a global homeomorphism of the plane onto itself.

After these preliminary remarks, we are now in position to start the study of equation (10). More precisely, we are interested in finding nontrivial periodic solutions, as well as we want to prove the existence of solutions which have some kind of complex behavior. With this respect, our paper is organized as follows. In Section 2 we prove a result of existence and multiplicity of \( T \)-periodic solutions under general assumptions on the coefficients. Our main results (Proposition 1 and Proposition 2) when applied to the particular case of Grindrod and Sleeman’s model (4) require a suitable lower bound for \( \theta \mu \). This is consistent with the existence of only the trivial solution proved in [9] for \( \mu = 1 \) and \( \theta \ll T \). In Section 3 we confine our attention to (4) which is analyzed by means of phase–plane techniques. In Theorem 3.1 our argument follows closely an analysis performed in [9, pp.124-130], with the substantial difference that in [9] the goal was that of proving the nonexistence of nontrivial solutions, whence here we are addressed to the opposite case (i.e., the existence of nontrivial solutions) for a different range of parameters. In Theorem 3.2 we consider also the problem of multiplicity of periodic solutions which are characterized by their oscillatory properties on \([0, \theta]\). Finally, in Section 4 we prove the existence of infinitely many subharmonic solutions and chaotic behavior for (4). To this aim we apply recent results from the theory of topological horseshoes combined with some technical estimates for the time–mappings associated to (4).

2. Existence of nontrivial periodic solutions. In this section, following a variational approach like in [23], we prove the existence of nontrivial periodic solutions
for (10) under general assumptions on $\alpha$, $\beta$ and $f$. First of all, we define

$$r(t, s) := \beta(t)s - \alpha(t)f(s)$$

and

$$R(t, s) := \int_0^s r(t, \xi) \, d\xi = \frac{1}{2} s^2 \beta(t) - F(s)\alpha(t)$$

(recall that $F' = f$ with $F(0) = 0$).

We look for the existence of nontrivial critical points of the functional

$$I(u) := \int_0^T \left( \frac{1}{2} \dot{u}(t)^2 + R(t, u(t)) \right) \, dt$$

(20)
on the Hilbert space $H^1_T$. As usual, by $H^1_T$ we mean the Sobolev space of functions $u \in L^2([0, T])$ having a weak derivative $\dot{u} \in L^2([0, T])$. For our purposes in the definition of weak derivative, as in [17], we assume that $v$ is the weak derivative of $u$ if

$$\int_0^T u(t)\phi(t) \, dt = - \int_0^T v(t)\phi(t) \, dt, \quad \forall \phi \in C^\infty_T,$$

where $C^\infty_T$ is the space of indefinitely differentiable $T$-periodic functions. Note that, according to this convention, every map $u \in H^1_T$ is an absolutely continuous function with $\dot{u} \in L^2([0, T])$, such that $u(T) = u(0)$ (see [17, p.6–7]). The norm $\| \cdot \|$ for $H^1_T$ is the standard one $\| u \| := (\| u \|_2^2 + \| u' \|_2^2)^{1/2}$ associated to the inner product

$$\langle u, v \rangle := \int_0^T (\dot{u}(t)\dot{v}(t) + u(t)v(t)) \, dt.$$

The functional $\mathcal{I} : H^1_T \rightarrow \mathbb{R}$ is continuously differentiable with

$$d\mathcal{I}(u)[w] = \langle \mathcal{I}'(u), w \rangle = \int_0^T (\dot{u}(t)\dot{w}(t) + r(t, u(t))w(t)) \, dt, \quad \forall w \in H^1_T$$

and, moreover, if $v(\cdot)$ is a critical point of $\mathcal{I}$ (that is a solution of the corresponding Euler equation), then $\dot{v}$ has a weak derivative $\ddot{v}$ which is a solution of the periodic problem

$$\ddot{v} - \beta(t)v + \alpha(t)f(v) = 0, \quad v(T) - v(0) = \dot{v}(T) - \dot{v}(0) = 0$$

(21)

(see [17, Corollary 1.1]). Notice that if $v(\cdot)$ is a solution of (21), then it is the restriction to $[0, T]$ of a $T$-periodic solution of (10). If we assume now conditions $(H_1)$ and $(H_2)$, we have that any critical point of $\mathcal{I}$ takes its values in the interval $[0, 1]$ (see also Lemma 1.1). Accordingly, the behavior of $s \mapsto \alpha(t)f(s) - \beta(t)s$ outside this interval is not relevant with respect to the search of critical points of $\mathcal{I}$. Hence, it is not restrictive if we assume, together with

$$r(t, 0) \equiv 0$$

(19)

and

$$r(t, s) < 0, \forall s < 0, \quad r(t, s) > 0, \forall s > 1 \quad \text{for a.e. } t \in [0, T],$$

(which follows from $(H_1)$ and $(H_2)$) also

$$\lim_{s \rightarrow -\infty} r(t, s) < 0, \quad \lim_{s \rightarrow +\infty} r(t, s) > 0,$$

(19)

(which follows from (19)), where the limits are considered uniformly in $t \in [0, T]$.

The proof of the next result can be obtained by using a known argument, (see, for instance, [17, §1.5 and §4.2]). However, we give the details for completeness.
Lemma 2.1. Assume \((r_3)\). Then \(I\) is bounded from below and satisfies the Palais-Smale condition.

**Proof.** According to \((r_3)\), let \(\nu, N > 0\) be such that
\[ r(t,s) \leq -\nu, \quad \forall s \leq -N \quad \text{and} \quad r(t,s) \geq \nu, \quad \forall s \geq N \]
hold for a.e. \(t \in [0,T]\). Let also \(\rho \in L^1([0,T],[\mathbb{R}^+])\) be such that
\[ |r(t,s)| \leq \rho(t), \quad \forall s \in [-N,N] \quad \text{and for a.e.} \quad t \in [0,T]. \]
For \(R(t,s) := \int_0^s r(t,\xi) \, d\xi\), we have that
\[ R(t,s) \geq (|s| - N)^+ \nu - N \rho(t) \geq \nu|s| - N(\nu + \rho(t)) \]
which implies that the functional \(I\) is bounded from below, since
\[ I(u) \geq \frac{1}{2} |\dot{u}|_2^2 + \nu |u|_1 - c, \]
where \(c > 0\) is a suitable constant independent of \(u\).
Let \((u_j)_j\) be a Palais-Smale sequence in \(H^1_T\), that is we suppose that \(I(u_j)\) is bounded with \(I'(u_j) \to 0\). The boundedness of \(I(u_j)\) implies that there exists a constant \(K > 0\) such that
\[ |u_j|_2, \ |u_j|_1 \leq K, \quad \forall j. \]
By standard properties of Sobolev spaces, we find that \((u_j)_j\) is bounded in \(H^1_T\) and therefore, there exists a point \(u \in H^1_T\) such that, passing if necessary to a subsequence,
\[ u_j \to u \quad \text{in} \quad H^1_T \quad \text{and} \quad u_j \to u \quad \text{in} \quad C([0,T],\mathbb{R}). \]
Hence,
\[ \langle I'(u_j) - I'(u), u_j - u \rangle \to 0 \quad \text{as} \quad j \to +\infty. \]
On the other hand, we have that
\[ \langle I'(u_j) - I'(u), u_j - u \rangle = |\dot{u}_j - \dot{u}|_2^2 + \int_0^T (r(t,u_j(t)) - r(t,u(t))) (u_j(t) - u(t)) \, dt, \]
with
\[ \int_0^T |r(t,u_j(t)) - r(t,u(t))| \cdot |u_j(t) - u(t)| \, dt \to 0, \quad \text{as} \quad j \to +\infty \]
(by the Lebesgue dominated convergence theorem). Thus we conclude that \(\dot{u}_j \to \dot{u}\) in the \(L^2\)-norm and therefore \(u_j \to u\) in \(H^1_T\). This proves the Palais-Smale condition.

According to Lemma 2.1 and [17, Theorem 4.4], we have that \(I\) achieves its minimum on \(H^1_T\). This leads to the following result, where we introduce the mean values
\[ \bar{\alpha} := \frac{1}{T} \int_0^T \alpha(t) \, dt, \quad \bar{\beta} := \frac{1}{T} \int_0^T \beta(t) \, dt. \]

**Proposition 1.** Assume \((H_1)\) and \((H_2)\). Suppose also that there is a constant \(e \in [0,1]\) such that
\[ \int_0^e (\bar{\alpha} f(s) - \bar{\beta} s) \, ds = \bar{\alpha} F(e) - \bar{\beta} \frac{e^2}{2} > 0. \]  
(22)
Then, problem (21) has a nontrivial solution.
Proof. Consider the functional $I$ defined in (20) where we assume that the map $r(t,s)$, besides $(r_1)$ and $(r_2)$, satisfies also $(r_3)$ (thanks to the previously discussed harmless modification of $f$ outside a neighborhood of $[0, 1]$). By the assumption (22) we have that
\[
\min I \leq I(e) = \int_0^T R(t,e) \, dt = -\int_0^T \left( \int_0^e (\alpha(t)f(s) - \beta(t)s) \, ds \right) \, dt
\]
\[
= -\int_0^e \left( \int_0^T (\alpha(t)f(s) - \beta(t)s) \, dt \right) ds < 0 = I(0).
\]
This implies that the minimum of $I$ on $H^1_T$ is achieved at a nontrivial function. \qed

**Proposition 2.** Besides the assumptions (H1), (H2) and (22) of Proposition 1 suppose also that $f(s)$ is of class $C^1$ in a neighborhood of $s = 0$ with $f'(0) < 0$. Then there exist at least two nontrivial solutions for problem (21).

Proof. We define $\eta_0(t) := \alpha(t)f'(0) - \beta(t)$. The assumption (8) implies that $\eta_0(t) < 0$ for a.e. $t \in [0, T]$. Hence, the quadratic form
\[
u \mapsto \int_0^T \left( \dot{\nu}(t)^2 - \eta_0(t)\nu(t)^2 \right) dt
\]
(23)
is positive definite on $H^1_T$. Using the fact that
\[
I''(0)[\nu]^2 = \int_0^T \left[ \ddot{\nu}(t)^2 + \frac{\partial^2 \eta_0}{\partial s}(t,0)\dot{\nu}(t)^2 \right] dt = \int_0^T \left( \ddot{\nu}(t)^2 - \eta_0(t)\dot{\nu}(t)^2 \right) dt
\]
we find that $u = 0$ is a strict local minimum for the functional $I$. This implies that there are $\rho > 0$ and $\eta > 0$ (and we can also take $\rho < ||e||$ ) such that
\[
I(u) \geq \eta \quad \text{for} \quad ||u|| = \rho.
\]
On the other hand, by (22), we have that $I(e) < 0 < \eta$ with $||e|| > \rho$. Having the Palais-Smale condition satisfied (see Lemma 2.1), the Ambrosetti - Rabinowitz mountain pass theorem [1] ensures the existence of a critical point of $I$ at a level $c > 0$, where
\[
c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \quad \text{with} \quad \Gamma = \{ \gamma \in C([0, 1], H^1_T) : \gamma(0) = 0, \gamma(1) = e \}.
\]
We already know that $I$ achieves its absolute minimum at some level $d \leq I(e) < 0 < \eta \leq c$. In this manner, we can prove the existence of at least two nontrivial critical points which are positive $T$-periodic solutions of (10) with range on $[0, 1]$. \qed

**Example 1.** If we consider equation (4), with $f$ satisfying (H1), we enter in the setting of Proposition 1 and have that condition (22) is satisfied if there is a constant $e \in [0, 1]$ such that
\[
\theta \mu F(e) > \frac{e^2}{2} \frac{T - \theta}{R}.
\]
(24)
It may be worth to observe that in order to have condition (24) fulfilled, it is sufficient to assume $\int_0^1 f(s) \, ds > 0$ and choose a number $e$ close to 1, provided that the quotient $\frac{\mu R \theta}{T - \theta}$ is sufficiently large. For $\mu$ and $R$ fixed constants, the condition for the existence of nontrivial solutions will be satisfied for $\theta$ not too close to zero, a result which is consistent with that of Grindrod and Sleeman [9] who proved the existence of only the trivial periodic solution for $\theta$ small.
If we further suppose that \( f(s) \) is continuously differentiable in a neighborhood of \( s = 0 \) with \( f'(0) < 0 \), then, according to Proposition 2, we conclude that (4) has at least three \( T \)-periodic solutions, one is the trivial one and the other two take values in \([0, 1]\). Observe that for the usual choice of

\[
f(s) = s(s - a)(1 - s), \quad 0 < a < 1,
\]

we have \( f'(0) < 0 \) and the condition \( \int_0^1 f(s) \, ds > 0 \) corresponds to the case \( a < \frac{1}{2} \).

3. Phase–plane analysis of the Grindrod - Sleeman model. In the preceding section we have considered the general equation (10). From now on, we focus our attention on a more specific equation which is strictly related to the Grindrod - Sleeman model. Accordingly, we assume condition \( (H^*_1) \) and also

\[
\int_0^1 f(s) \, ds > 0.
\]  (25)

If we suppose that the coefficients \( \alpha, \beta \) are like in (9), we can describe the dynamics associated to (10) in the phase–plane as the superposition of the two planar systems

\[
\dot{q} = p, \quad \dot{p} = -\mu f(q) \quad \text{(26)}
\]

\[
\dot{q} = p, \quad \dot{p} = \frac{q}{R} \quad \text{(27)}
\]

When dealing with the two autonomous systems separately, we consider the dynamical systems associated to them. Using the language of dynamical systems, for any point \( P \in \mathbb{R}^2 \), we denote by \( \gamma^+(P), \gamma^−(P) \) and \( \gamma(P) \), the positive, negative semi–orbit and the orbit of the system passing through \( P \).

We follow the orbits of the first system for \( t \in [0, \theta] \) and those of the second one for \( t \in [\theta, T] \) and then repeat the switching from one system to the other by \( T \)-periodicity. Hence, the associated Poincaré’s map \( \Psi \) can be represented like

\[
\Psi := M \circ \Phi,
\]

where \( \Phi \) is the map that associates to any initial point \((q(0), p(0)) \in \mathbb{R}^2\) the solution of (26) evaluated at the time \( t = \theta \) and

\[
M := Y(T - \theta),
\]

where \( Y(\cdot) \) is the fundamental matrix of the linear system (27).

This remark suggests, like in [9, pp.124–130], to analyze separately the phase–portraits of the two systems. The geometry associated to (26) is that of a center at \((a, 0)\) surrounded by a homoclinic orbit at zero that we denote by \( \Gamma_0 \). The phase–portrait of system (27) is that of a standard saddle with \( p = -q/\sqrt{R} \) and \( p = q/\sqrt{R} \) as stable and unstable manifolds, respectively. Both systems (26) and (27) are conservative with energies

\[
E_1(q, p) := \frac{1}{2}p^2 + \mu F(q), \quad E_2(q, p) := \frac{1}{2}p^2 - \frac{1}{2R}q^2,
\]

respectively.

Taking a constant \( c \) with

\[
F(a) < c \leq F(1),
\]

we define the set

\[
\Gamma_c := \{(q, p) \in \mathbb{R}^2 : 0 \leq q \leq 1, \quad E_1(q, p) = \mu c\}.
\]

For \( c < 0 \), we see that \( \Gamma_c \) is a periodic orbit surrounding \((a, 0)\). Larger values of \( \mu \) produce the effect of decreasing the fundamental period of such a periodic orbit.
and, correspondingly, to increase the number of turns (rotation number) around the equilibrium point \((a,0)\) during the time interval \([0,T]\). The set \(\Gamma_0 \setminus \{(0,0)\}\) is an orbit which is homoclinic to \((0,0)\). For \(0 < c < F(1)\), the set \(\Gamma_c\) is an orbit path which intersects the vertical axis \(q = 0\) at the points \(0, \pm \sqrt{2\mu c}\) and the horizontal axis \(p = 0\) at a point \((q_c,0)\) with \(a < q_c < 1\) and \(F(q_c) = c\). Finally, \(\Gamma_c\) for \(c = F(1)\) consists of the equilibrium point \((1,0)\) and the intersection of its stable and unstable manifolds with the strip \([0,1] \times \mathbb{R}\).

For any \(c\) with \(F(a) < c \leq F(1)\), and \((y_1,z_1),(y_2,z_2) \in \Gamma_c\) with \(y_1 < y_2 < 1\) and \(z_1, z_2 \geq 0\) \((z_i := \sqrt{\mu/2(c - F(y_i))}, i = 1,2)\), we can compute the time \(\sigma_c(y_1,y_2)\) needed to move from \((y_1,z_1)\) to \((y_2,z_2)\) along \(\Gamma_c\) in the upper half-plane, using the time-mapping formula

\[
\sigma_c(y_1,y_2) := \frac{1}{\sqrt{\mu}} \int_{y_1}^{y_2} \frac{ds}{\sqrt{2(c - F(s))}} = \frac{1}{\sqrt{\mu}} \int_{y_1}^{y_2} \frac{ds}{\sqrt{2(F(y_c^+) - F(s))}},
\]

where \(y_c^+\) is the solution of the equation \(F(y) = c\), with \(a < y_c^+ \leq 1\), so that \((y_c^+,0)\) is the intersection of the set \(\Gamma_c\) with the abscissa. Actually, for \(0 < c < F(1)\) we have \(y_c^+ = q_c\) (according to a notation introduced above). On the other hand, for \(F(a) < c < 0\), the closed orbit \(\Gamma_c\) intersects the abscissa at another point \((y_c^-,0)\) with \(0 < y_c^- < a\) such that \(F(y_c^-) = c\). Notice that by \((H_1^+)\), we know that \(F(s)\) is strictly decreasing on \([0,a]\) and strictly increasing on \([a,1]\). Therefore, the constants \(y_c^\pm\) are well defined.

The number \(\tau(c) := 2\sigma_c(y_c^-,y_c^+)\) is also equal to the time needed to move from \((y_2,-z_2)\) to \((y_1,-z_1)\) along \(\Gamma_c\) in the lower half-plane. In particular, for every \(c \in ]F(a),0[\), we have that

\[
\tau(c) := 2\sigma_c(y_c^-,y_c^+)
\]

is the fundamental period of the periodic orbit \(\Gamma_c\). Observe that

\[
\lim_{c \to 0^-} \tau(c) = +\infty.
\]

In fact, for any fixed \(\mu > 0\), as \(c\) tends to \(0^-\), the closed orbit \(\Gamma_c\) approaches the homoclinic \(\Gamma_0\). We also notice that the period \(\tau(c)\) depends upon the parameter \(\mu\). When we want to put in evidence this fact, we’ll denote it as \(\tau_\mu(c)\). From the time-mapping formula it is clear that, for any fixed \(c \in ]F(a),0[\), it follows that \(\tau_\mu(c) \to 0\) as \(\mu \to +\infty\).

Consider now the linear system (27). Taking a constant \(d\) with

\[
-\frac{1}{2R} < d < 0,
\]

we consider the set

\[
\Upsilon_d := \{(q,p) \in \mathbb{R}^2 : 0 \leq q \leq 1, \ E_2(q,p) = d\}.
\]

The set \(\Upsilon_d\) is the part of the trajectory of (27) passing through the point

\[
P_d := (yd,0), \quad \text{for} \ yd := \sqrt{2Rd},
\]

and contained in the strip \([0,1] \times \mathbb{R}\). For any \((w,z) \in \Upsilon_d\), with \(w \in [yd,1]\) and \(z = \sqrt{2d + R^{-1}w^2}\), we can compute the time \(\varsigma(yd,w)\) needed to move in the upper half-plane along \(\gamma^+(P_d)\) from \(P_d\) to the point \((w,z)\), which is the point on which the semi-orbit \(\gamma^+(P_d)\) intersects the vertical line \(q = w\). The same time is needed
to move from \((w, -z)\) to \(P_d\) along \(\gamma^{-}(P_d)\) contained in the lower half–plane. Using a time–mapping formula, we find the following integral expression for \(\varsigma(y_d, w)\).

\[
\varsigma(y_d, w) := \sqrt{R} \int_{y_d}^{w} \frac{ds}{\sqrt{s^2 - y_d^2}}.
\]

As described in [9], the strategy to find a periodic solution is based on the following steps: We choose a point \(P := (w, z)\) in the region

\[
\{(q, p) \in \mathbb{R}^2 : 0 < q < 1, 0 < p < q/\sqrt{R}, E_1(q, p) < \mu F(1)\}
\]

and consider the orbit path of (27) from \(P' := (w, -z)\) to \(P\). Such orbit path is contained in \(\Upsilon_d\) for \(d := E_2(w, z)\) and crosses the abscissa at the point \(P_d := (y_d, 0)\) for \(y_d := \sqrt{2R[E_2(w, z)]}\). Next, we consider the orbit path of (26) from \(P\) to \(P'\) contained in \(\Gamma_c\) for \(c := E_1(w, z)/\mu\). If \(0 \leq c < F(1)\) such orbit path belongs to the strip \(\{(q, p) : q \geq w\}\), while, if \(c < 0\), we have different possibilities to connect \(P\) to \(P'\) along \(\Gamma_c\) because, in this situation, \(\Gamma_c\) is a closed orbit. More precisely, we can go from \(P\) to \(P'\) along \(\Gamma_c\) remaining (as above) in the strip \(\{(q, p) : q \geq w\}\), or we can start in \(P\) and stop at \(P'\) after a certain number of turns around \((a, 0)\). In [9], the authors ruled out this latter possibility and focused their attention only upon the \(T\)-periodic solutions \((q(t), p(t))\) of (4) which satisfy

**Condition A**: \(q(t) \geq a, \forall t \in [0, \theta]\).

In order to obtain a \(T\)-periodic trajectory of (4) passing through \(P\) and following the procedure explained above, we shall require that

\[
\frac{T - \theta}{2} = \varsigma(y_d, w), \quad y_d := \sqrt{2R[E_2(w, z)]}
\]

and also, simultaneously,

\[
\frac{\theta}{2} = \sigma_c(w, y_d^+) + j\tau(c), \quad c := E_1(w, z)/\mu,
\]

for some nonnegative integer \(j\). More in detail, we assume \(j = 0\) if \(0 \leq E_1(w, z) < \mu F(1)\) and whenever Condition A is imposed.

In [9, Theorem 1.3] and for \(\mu = 1, T = 1\), Grindrod and Sleeman proved that for each fixed \(R > 0\) there exists \(\theta_0 > 0\) such that if \(0 < \theta < \theta_0\) then there are no periodic solutions of (4) satisfying Condition A. As a complementary result, we prove the existence of periodic solutions for \(\theta\) close to \(T\).

**Theorem 3.1.** Assume \((H_1)\) and (25). For any given \(T, \mu, R > 0\), there exists \(\theta_0 \in [0, T]\) such that system (4) has a \(T\)-periodic solution \(v(\cdot)\), satisfying Condition A, for every \(\theta \in [\theta_0, T]\).

**Proof.** From the definition of \(\varsigma\) and the energy relation

\[
\frac{1}{2} z^2 - \frac{1}{2R} w^2 = -\frac{1}{2R} y_d^2,
\]

we can write (28), as

\[
\frac{T - \theta}{2} = \sqrt{R} \cosh^{-1}\left(\left(1 - R\left(z/w\right)^2\right)^{-1/2}\right)
\]

and therefore (28) is satisfied provided that the point \(P = (w, z)\) is chosen with

\[
z = m(\theta) w, \quad \text{with} \quad m(\theta) := \frac{1}{\sqrt{R}} \tanh\left(\frac{T - \theta}{2\sqrt{R}}\right)
\]
(a relation already given in [9, Equation (1.8)]). Observe that

$$0 < m(\theta) < m^\# := \frac{1}{\sqrt{R}} \tanh \left( \frac{T}{2\sqrt{R}} \right), \quad \text{for every } \theta \in \mathbb{R}. $$

Since we look for solutions satisfying Condition A, we take $P = (w, z)$ with $w \in [a, 1]$ and study (29) with $j = 0$.

First of all, we introduce the constant

$$m^* = m^*_\mu := \frac{\sqrt{\mu}}{a} \sqrt{2(F(1) - F(a))}. $$

By the definition of $m^*$ it follows that, for every slope $m$ with $0 < m < m^*$, the line $p = mq$ intersects the stable manifold of $(1, 0)$ at a point $Q_m := (q_m, p_m)$ with $a < q_m < 1$. On the line $p = mq$ we consider the segment $[B_m Q_m]$ joining the point $B_m := (a, ma)$ with $Q_m$ and, for every $(w, z) \in [B_m Q_m] \setminus \{Q_m\}$ we evaluate the time $\sigma_c(w, y^+_c)$, where $c = E_1(w, z)/\mu$. More in detail, we have that $w \in [a, q_m]$, $z = mw$ and $y^+_c$ is the unique solution of the equation

$$F(x) = \frac{m^2 w^2}{2\mu} + F(w), \quad x \in [a, 1[ $$

(see also Figure 2 for a graphical explanation).

According to the above positions, the time $\sigma_c(w, y^+_c)$ is given by the function

$$\varphi(w, m, \mu) := \frac{1}{\sqrt{\mu}} \int_{w}^{F^{-1}\left(\frac{m^2 w^2}{2\mu} + F(w)\right)} \frac{ds}{\sqrt{2\left(\frac{m^2 w^2}{2\mu} + F(w) - F(s)\right)}}, \quad 0 < m < m^*, \ a \leq w < q_m, $$

where we have denoted by $F^{-1}$ the inverse of $F|_{[a, 1]}$. By construction, for every fixed $\mu > 0$ and $0 < m < m^*$ we have that $\varphi(w, m, \mu) \to +\infty$ as $w \nearrow q_m$. Therefore, the minimum value

$$\varphi_*(m, \mu) := \min_{w \in [a, q_m]} \varphi(w, m, \mu) $$

is well defined. By the properties of the time–mapping and structure of the orbits of (26) in the phase–plane we easily prove that

$$\lim_{\mu \to +\infty} \varphi_*(m, \mu) = \ell, $$

with $\varphi_*(m, \mu)$ strictly increasing as a function of $m$.

We take now $m$ such that $0 < m < \min\{m^\#, m^*_\mu\}$ and observe that, in order to have (28) and (29) satisfied we need to find a suitable $m$ such that

$$T - 2\sqrt{R} \cosh^{-1} \left( \left(1 - Rm^2\right)^{-1/2} \right) > \varphi_*(m, \mu). \quad (30) $$

Such inequality is always satisfied for $m > 0$ sufficiently small (that is for $m$ in a right neighborhood of 0) as the left-hand side of (30) tends to $T$ while the right-hand side tends to 0 as $m \to 0^+$. If we denote by $m_1$ the infimum of the $m > 0$ such that (30) does not hold, we obtain, on the contrary, that (30) will be satisfied for every $m \in [0, m_1]$. To conclude the proof, we define

$$\theta^*_0 := T - 2\sqrt{R} \cosh^{-1} \left( \left(1 - Rm^2\right)^{-1/2} \right).$$
For every $\theta \in [\theta^*_0, T]$, we determine a line $p = m(\theta)q$ with slope $m = m(\theta)$ such that $m < m^*_\mu$. For all the points $(w, z)$ on the segment $[B_mQ_m]$ we have equation (28) satisfied. On the other hand, by the choice of $m(\theta) < m_1$ we also have that $\theta > \phi_\mu(m, \mu)$. Therefore, there exists at least one point $(w, z) \in [B_mQ_m]\setminus \{Q_m\}$ such that (29) holds (with $j = 0$). This gives the existence of at least one solution for system (4) satisfying Condition A.

Every $T$-periodic solution $v(\cdot)$, satisfying Condition A, that we find in the proof of Theorem 3.1 presents the following behavior: $v(t)$ is concave on $[0, \theta]$, strictly increasing on $[0, \theta/2]$, strictly decreasing on $[\theta/2, \theta]$ and, moreover, $v(t) \geq a$, $\forall t \in [0, \theta]$. On the interval $[\theta, T]$ the solution $v(t)$ satisfies the linear equation $v'' - R^{-1}v = 0$ and therefore it is convex.

To conclude this section, we now look for solutions of (4) which are not necessarily concave on $[0, \theta]$, in the sense that they present a certain number of oscillations in such interval. To this aim, we look for the solvability of system (28)-(29) in the case when (29) holds for some integer $j \geq 1$. Accordingly, we have:

**Theorem 3.2.** Assume $(H^*_1)$ and (25). Let $T, R > 0$, be fixed. For every $\theta \in [0, T]$ and each integer $j \geq 1$, there exists $\mu^* > 0$ such that for all $\mu > \mu^*$ system (4) has a $T$-periodic solution $v(\cdot)$ with $v(t) - a$ having exactly $2j$-zeros in $[0, \theta]$.

**Proof.** Along the proof, we keep unaltered all the notation introduced for the proof of Theorem 3.1.

First of all, we take $\theta \in [0, T]$ and, consequently, we have the slope $m(\theta) \in [0, m^*_\theta]$ fixed. By this choice, we have condition (28) satisfied for all the points $(w, z)$ lying...
on the segment \(L_\theta := \{(w, z) : z = m(\theta)w, 0 < w < 1, z > 0\}\).

Next, we choose \(\mu > \mu^\#(\theta) := \frac{a^2 m(\theta)^2}{2|F(a)|}\)

so that the point \(B := (a, m(\theta)a)\) lies in the interior of the region bounded by the homoclinic \(\Gamma_0\), because the energy of the level line of (26) passing through \(B\) satisfies \(E_1(B) < 0\). The fact that \(F\) is strictly increasing on \([a, 1]\), guarantees that there exists a unique point \(Q_\mu := (q_\mu, p_\mu) \in \Gamma_0 \cap L_\theta\) for \(a < q_\mu < y_0^+ < 1\). Every point \((w, z)\) on the segment \([BQ_\mu] \subset L_\theta\) belongs to a closed orbit \(\Gamma_c\) of (26), for \(c = E_1(w, z)/\mu\), having a fundamental period that we have already denoted by \(\tau_\mu(c)\).

Suppose now that we have fixed an integer \(j \geq 1\) which represents the number of turns around the center \((a, 0)\). As we already observed, \(\tau_\mu(c) \to 0\) as \(\mu \to +\infty\), hence we can find \(\mu^* = \mu^*(\theta, j) \geq \mu^\#(\theta)\) such that \(\tau_\mu(c) < \theta/(j + 1)\) for all \(\mu > \mu^*\), with \(c = E_1(B)/\mu\). In this manner, we obtain

\[
\frac{\theta}{2} > \sigma_c(w, y_0^+) + j \tau_\mu(c), \quad \text{for } c = E_1(B)/\mu \ \forall \mu > \mu^*.
\]

On the other hand, we already know that for any \(\mu > \mu^\#(\theta)\), it holds that \(\tau_\mu(c) \to +\infty\) as \(c \to 0^+\). The continuity of the maps \(\sigma\) and \(\tau\) with respect to their arguments, allows us to claim that there exists a point \(\tilde{P} = (\tilde{w}, \tilde{z}) = (\tilde{w}, m(\theta)\tilde{w}) \in [B, Q_\mu]\) for which (29) is satisfied (see also Figure 3 for a graphical explanation).

![Figure 3](image-url)

**Figure 3.** The figure visualizes the construction described in the proof of Theorem 3.2. In this specific example, we have taken \(f(s) = s(1-s)(s-a)\) with \(a = 0.4\). The level lines of energy \(E_1\) are displayed with \(\mu = 4\). The segment \(L_\theta\) on the line \(p = m(\theta)q\) on which we select the subset \([BQ_\mu]\) is drawn for \(m(\theta) \approx 0.34\).

For the solution \(\tilde{v}(\cdot)\) of (4) departing from \(\tilde{P}\) at \(t = 0\), there exists \(0 < t_1^- < t_1^0 < t_1^- < s_1^0 < \cdots < t_j^+ < t_j^0 < t_j^- < s_j^0 < \theta\) such that \(\tilde{v}(t_j^0) = \tilde{v}(s_j^0) = a\). Moreover,
\( \hat{v}(\cdot) \) is strictly concave on \([0,t_0]\) and each of the intervals \([s_i^0, t_{i+1}^0]\), achieving its maximum for \( t = t_i^0 \) and for \( t = t_{i+1}^0 \), respectively. The same solution is strictly convex in each of the intervals \([s_i^0, s_{i+1}^0]\), achieving its minimum for \( t = t_i^- \). For \( t \in [\theta, T] \), the solution satisfies the linear equation \( v'' - R^{-1}v = 0 \) and therefore it is convex.

We conclude this section with a short comment about the main differences between the results obtained here and those of Section 2. In Propositions 1-2, by means of functional analytic tools, we have obtained results for the existence of at least one (respectively two) nontrivial periodic solutions for a general equation including Grindrod and Sleeman model as a special case. On the contrary, in Theorems 3.1-3.2, by means of a more direct phase–plane analysis approach which concerns only the case of system (4), we are able to achieve more information about the oscillatory behavior of the solutions and also to improve the multiplicity result (at least for certain values of the parameters).

Theorem 3.2 suggests the fact that the dynamical properties of the solutions become more and more complex as the parameter \( \mu \) increases. The presence of some kind of chaotic behaviour is investigated in the next section, with the aid of some topological tools.

4. Solutions with a complex behavior. Throughout this section we continue to use, without any change, the notation of Section 3.

**Theorem 4.1.** Assume \((H_1)\) and (25) and let \( T, R \) and \( \theta \in [0, T] \) be fixed. Then there is a constant \( \omega^* > 0 \) such that, for every \( \mu > \omega^* \), there exists a compact invariant set \( \Lambda \subseteq [a, 1] \times [0, +\infty) \) for which \( \Psi|\Lambda \) is semiconjugate via a continuous surjection \( h \) to a two-sided Bernoulli shift on two symbols. Moreover, for each periodic sequence of two symbols \( (i_k)_{k \in \mathbb{Z}} \) there is a point \( z \in \Lambda \) which is periodic and such that \( h(z) = (i_k)_{k \in \mathbb{Z}} \).

We remark that the assertion of Theorem 4.1 about the semiconjugation of \( \Psi|\Lambda \) has the following interpretation in terms of the solutions to (4): Consider a two-sided sequence of two symbols \( \xi := (i_k)_{k \in \mathbb{Z}} \) with \( i_k \in \{1, 2\} \), \( \forall \, k \in \mathbb{Z} \). Then, there is at least one solution \( \xi = v_\xi(x) \) of (4) which is defined for every \( x \in \mathbb{R} \) and satisfies \( 0 < v_\xi(x) < 1 \), \( \forall \, x \in \mathbb{R} \) as well as \( (v_\xi(0), \frac{1}{2}\pi v_\xi(0)) \in \Lambda \). Moreover, the symbol \( i_k = 1 \) means that \( v_\xi(x) \) has precisely two strict maximum points separated by one strict minimum point along the time interval \( [(k-1)T, (k-1)T + \theta) \), while, the symbol \( i_k = 2 \) means that \( v_\xi(x) \) has precisely three strict maximum points separated by two strict minimum points along the time interval \( [(k-1)T, (k-1)T + \theta) \). In both situations, \( v_\xi(x) \) is convex in the interval \( [(k-1)T + \theta, kT] \), with \( v'_\xi((k-1)T + \theta) < 0 \) and \( v'_\xi(kT) > 0 \). If the sequence \( (i_k)_{k \in \mathbb{Z}} \) is periodic, that is \( i_k = i_{k+\ell} \) for some \( \ell \geq 1 \), then we can take \( v_\xi(\cdot) \) as a \( \ell T \)-periodic solution as well.

A variant of Theorem 4.1 is the following.

**Theorem 4.2.** Assume \((H_1)\) and (25). Let \( R, \delta := T - \theta > 0 \) and

\[ \mu > \frac{a^2}{2R|F(a)|} \]

be fixed. Then there exists a constant \( \theta^* = \theta^*(\mu) > 0 \) such that for every \( \theta > \theta^* \) and \( T = \theta + \delta \), the same conclusion of Theorem 4.1 holds.
In [9] the authors focused their attention to the case of a “healthy” nerve fiber with $\theta << T$ and proved the nonexistence of nontrivial periodic solutions. Theorem 4.2 is, in some sense, complementary as we give more relevance to the nonlinear part of the equation which gives account of the unmyelinated part of the fiber.

We present only the proof of Theorem 4.1. A similar argument can be adapted to get a proof of Theorem 4.2. For the full details how to apply the abstract results about chaotic–like dynamics to our setting, we refer to [25] where the Chen and Bell model (instead of the Grindrod–Sleeman one) was discussed.

Proof. By ($H^*_1$), as we already observed, we know that $F$ is negative and strictly decreasing on $]0,a[$ and strictly increasing on $[a,1]$. Hence, (25) implies that there exists a unique point in $]a,1[$ where $F$ vanishes. Let $b \in ]a,1[$ be such that

$$F(b) = \int_0^b f(s) \, ds = 0.$$  

Consider for a moment the trajectories of the subsystem (27) and we choose $w_+ \in ]a,b[$ such that

$$\varsigma(w_+, b) = \sqrt{R} \int_{w_+}^b \frac{ds}{\sqrt{s^2 - w_+^2}} = \sqrt{R} \cosh^{-1} \left( \frac{b}{w_+} \right) < \frac{T - \theta}{2}. \quad (31)$$

We also fix $w_- \in ]0,a[$ such that

$$\varsigma(w_-, a) = \sqrt{R} \int_{w_-}^a \frac{ds}{\sqrt{s^2 - w_-^2}} = \sqrt{R} \cosh^{-1} \left( \frac{a}{w_-} \right) > \frac{T - \theta}{2}. \quad (32)$$

We look at the level sets $\Upsilon_d$ for $d = d_+ := E_2(w_+,0)$ and for $d = d_- := E_2(w_-,0)$. From (31) we have that for every $P \in S_+ := S_{d_+} \cap \{(w,z) : w_+ \leq w \leq b, z \leq 0\}$, its image $M(P)$ through the Poincaré’s map after the time $T - \theta$ lies in the region $B := ]b, +\infty[ \times ]0, +\infty[$. Analogously, from (32) we have that the image through the Poincaré’s map of the set $S_- := \Upsilon_{d_-} \cap \{(w,z) : a \leq w \leq b, z \leq 0\}$, is contained in $S_- \cup A$, for

$$A := \{(w,z) : 0 < w < a, |z| < w/\sqrt{R}\}$$

and therefore it does not cross the vertical half–line $\{(a,z) : z \geq 0\}$. See Figure 4 for a visual description.

As a next step, we focus our attention on the subsystem (26) and choose a point $w_+^0 \in ]w_+, b[$ with $F(w_+^0) > F(w_-)$. We also denote by $w_-^0$ the unique solution of $F(x) = F(w_+^0)$ with $0 < x < a$. By the choice of $w_-^0$ and the properties of $F$ we know that $0 < w_-^0 < w_-$. Within the strip $[0,1] \times \mathbb{R}$, consider the level line $\Gamma_{c_0}$ for (26) defined by

$$E_1(q,p) = \mu c_0, \quad \text{for} \ c_0 := F(w_+^0) = F(w_-^0) < 0.$$
The set $\Gamma_{c_0}$ is a closed curve contained in the region bounded by the homoclinic which is strictly star-shaped around $(a,0)$. It represents a periodic orbit of system (26) with fundamental period

$$\tau_{\mu}(c_0) = \frac{2}{\sqrt{\mu}} \int_{w_0^-}^{w_0^+} \frac{ds}{\sqrt{2(c_0 - F(s))}}.$$

$\Gamma_{c_0}$ intersects the vertical line $q = a$ at the points $(a, \pm \sqrt{\mu} \sqrt{2(F(w_0^0) - F(a))})$. For our geometry, we need to have the intersection of $\Gamma_{c_0}$ with the positive vertical half-line through $(a,0)$ above the intersection of the same vertical line with the unstable manifold $p = \frac{q}{\sqrt{R}}$ of (27). Accordingly, we take

$$\mu > \omega_1 := \frac{a^2}{2R(F(w_0^0) - F(a))}.$$
Now we define the following regions
\[ \mathcal{M} := \{(q,p) : 0 \leq q \leq b, \mu c_{0} \leq E_{1}(q,p) \leq 0\} \]
and
\[ \mathcal{W} := \{(q,p) : 0 \leq q \leq b, E_2(w_+,0) \leq E_2(q,p) \leq E_2(w_-,0)\}. \]
The sets \( \mathcal{M} \) and \( \mathcal{W} \) intersects into two components, which are rectangular sets (that is subsets of the plane which are homeomorphic to the closed unit square). We call \( \mathcal{U} \) the upper intersection and \( \mathcal{V} \) the lower one (see Figure 5 where we have put in evidence \( \mathcal{U} \) and \( \mathcal{V} \) with a lighter and a darker color, respectively).

![Figure 5. Example of an annular region \( \mathcal{M} \) between the level lines \( \Gamma_{c_{0}} \) and \( \Gamma_{0} \) which intersects the set \( \mathcal{W} \) (which is the part of the strip \([0,b] \times \mathbb{R}\) between the orbits of (27) passing through \((w_-,0)\) and \((w_+,0)\)). We have put in evidence, using different colors, the two intersections \( \mathcal{U} \) and \( \mathcal{V} \). In this specific example, we have taken \( f(s) = s(1-s)(s-a) \) with \( a = 0.4, R = 1 \) and \( \mu = 20 \).]

We fix an orientation on \( \mathcal{U} \) and on \( \mathcal{V} \). Namely, we select two disjoint compact arcs \( \mathcal{U}_i^- \) and \( \mathcal{U}_r^- \) on \( \partial \mathcal{U} \) which are the intersections of \( \mathcal{U} \) with the inner and the outer boundary of \( \mathcal{M} \). Similarly, we orientate the set \( \mathcal{V} \) by taking as \( \mathcal{V}_i^- \) and \( \mathcal{V}_r^- \) the intersections of \( \mathcal{V} \) with the outer and the inner boundaries of \( \mathcal{W} \). We claim now that for \( \mu > 0 \) sufficiently large (or for a \( \theta > 0 \) sufficiently large), a topological horseshoe structure like in [13] and [20] is associated to the Poincaré map \( \Psi \) restricted to the set \( \mathcal{U} \).

To check this assertion, we proceed as follows. We consider a (continuous) path \( \gamma = \gamma(s) : [0,1] \to \mathcal{U} \) with \( \gamma(0) \in \mathcal{U}_i^- \) and \( \gamma(1) \in \mathcal{U}_r^- \) and we activate system (26) for \( 0 \leq t \leq \theta \). Observe that the trajectories are run in the clockwise sense.

Since \( \gamma(1) \) belongs to the homoclinic, it is clear that the angular displacement between \( \Phi(\gamma(1)) \) and \( \gamma(1) \) is less than \( 3\pi/2 \). On the other hand, if we take \( \mu > 0 \) large enough, we may let the point \( \gamma(0) \) (which belongs to a periodic orbit) to run
an angle larger than $5\pi$ in the given time interval. This fact implies that the image of the path $\gamma(s)$ through $\Phi$ winds in a spiral–like fashion along the annulus $\mathcal{M}$ and it crosses at least twice the set $\mathcal{V}$ from $\mathcal{V}_r^-$ to $\mathcal{V}_l^-$. To put this assertion in a more formal way, we introduce polar coordinates with center at $(a,0)$ and denote by $\varphi(t,z)$ the angular coordinate associate to a solution

$$(q(t), p(t)) = (a + \varrho(t) \cos \varphi(t), \varrho(t) \sin \varphi(t))$$

of (26) with

$$(q(0), p(0)) = z \in \mathcal{U}.$$ 

It is easy to check that, for every $z \in \mathcal{U}$, $\varphi(t, z)$ is strictly decreasing in the $t$-variable. Moreover,

$$z \in \mathcal{U}_r^- \implies \varphi(t, z) > -\pi, \quad \forall t \geq 0$$ 

since the points in $\mathcal{U}_r^-$ belong to the homoclinic. On the other hand, the points of $\mathcal{U}_l^-$ lie on a periodic orbit of period $\tau_\mu(c_0)$. Therefore for every positive integer $k$, we find that

$$z \in \mathcal{U}_l^- \implies \varphi(t, z) - \varphi(0, z) \leq -2k\pi \text{ if and only if } t \geq k\tau_\mu(c_0).$$

For the point $P$ such that $\{P\} = \mathcal{U}_l^- \cap \Upsilon_d$, we have a well defined time $t^* > 0$ such that

$$\varphi(t^*, P) = -5\pi$$

and it follows that

$$\varphi(t, z) \leq -5\pi, \quad \forall z \in \mathcal{U}_l^- \text{ and } t \geq t^*.$$ 

By the previous discussion, we easily find an upper bound for $t^*$, that is:

$$t^* < 3\tau_\mu(c_0).$$
In view of these preliminaries, we claim that a symbolic dynamics of two symbols is obtained if
\[ \theta > 3 \tau_\mu(c_0), \]
which corresponds to the case in which \( \mu \) is sufficiently large (or \( \theta \) is the large parameter).

Indeed, for \( \theta > 3 \tau_\mu(c_0) \), the sets
\[ \mathcal{H}_1 := \{ z \in \mathcal{U} : \varphi(\theta, z) \in [-3\pi, -2\pi] \} \]
and
\[ \mathcal{H}_2 := \{ z \in \mathcal{U} : \varphi(\theta, z) \in [-5\pi, -4\pi] \} \]
are compact, disjoint and nonempty.

Consider the path \( \gamma \) as above and look at the map \([0,1] \ni s \rightarrow \varphi(\theta, \gamma(s))\). By the assumptions, we know that \( \varphi(\theta, \gamma(0)) < -5\pi \), while \( \varphi(\theta, \gamma(1)) > -\pi \). By continuity, there exist two disjoint subintervals \([s_0', s_1'] \subset [0,1]\) such that \( \gamma(s) \in \mathcal{H}_1, \forall s \in [s_0', s_1'] \) and \( \gamma(s) \in \mathcal{H}_2, \forall s \in [s_0'', s_1''] \). Moreover, \( \varphi(\theta, \gamma(s_0')) = -3\pi, \varphi(\theta, \gamma(s_0'')) = -5\pi, \varphi(\theta, \gamma(s_1')) = -2\pi \) and \( \varphi(\theta, \gamma(s_1'')) = -4\pi \). By the invariance of the region \( \mathcal{M} \) with respect to the solutions of (26), we can find two other subintervals \([r_0', r_1'] \subset [s_0', s_1']\) and \([r_0'', r_1''] \subset [s_0'', s_1'']\) such that
\[ \gamma(s) \in \mathcal{H}_1 \text{ and } \Phi(\gamma(s)) \in \mathcal{V}, \forall s \in [r_0', r_1'] \text{ with } \Phi(\gamma(r_0')) \in \mathcal{V}_- \text{ and } \Phi(\gamma(r_1')) \in \mathcal{V}_- \]
\[ \gamma(s) \in \mathcal{H}_2 \text{ and } \Phi(\gamma(s)) \in \mathcal{V}, \forall s \in [r_0'', r_1''] \text{ with } \Phi(\gamma(r_0'')) \in \mathcal{V}_- \text{ and } \Phi(\gamma(r_1'')) \in \mathcal{V}_- . \]

At the time \( t = \theta \), we switch to system (27) and, by the previous observations about the action of the matrix \( M \), we conclude that the each of the paths \([r_0', r_1'] \ni s \rightarrow \Phi(\gamma(s)) \) and \([r_0'', r_1''] \ni s \rightarrow \Phi(\gamma(s)) \) possesses sub-paths which are expanded by \( M \) into curves crossing again \( \mathcal{U} \). More precisely, there are two further subintervals \([l_0', l_1'] \subset [r_0', r_1']\) and \([l_0'', l_1''] \subset [r_0'', r_1'']\) such that the following property holds for \( \Psi = M \circ \Phi : \)
\[ \gamma(s) \in \mathcal{H}_1 \text{ and } \Psi(\gamma(s)) \in \mathcal{U}, \forall s \in [l_0', l_1'] \text{ with } \Psi(\gamma(l_0')) \in \mathcal{U}_- \text{ and } \Psi(\gamma(l_1')) \in \mathcal{U}_- \]
\[ \gamma(s) \in \mathcal{H}_2 \text{ and } \Psi(\gamma(s)) \in \mathcal{U}, \forall s \in [l_0'', l_1''] \text{ with } \Psi(\gamma(l_0'')) \in \mathcal{U}_- \text{ and } \Psi(\gamma(l_1'')) \in \mathcal{U}_- . \]

At this step, we enter in the frame of the theory of topological horseshoes as developed in [13, 20] and a direct application of a combination of those results allows us to conclude. In the language of [13] the horseshoe hypothesis \( \Omega \) is satisfied to the map \( \Psi \) and the set \( \mathcal{U} \) with crossing number \( M \geq 2 \).

We observe that (by minor modifications in the argument) it is possible to obtain extensions of Theorem 4.1 and Theorem 4.2 by producing symbolic dynamics on \( p \) objects with \( p > 2 \).

As a conclusion, we remark that even if in Sections 3 and 4 we have taken the Grindrod–Sleeman model as the basic equation for our study, we stress that our approach may be applied to other related equations with a Nagumo type nonlinearity and a stepwise periodic weight coefficient.
REFERENCES

[1] A. Ambrosetti and P. Rabinowitz, Dual variational methods in critical point theory and applications, J. Functional Analysis, 14 (1973), 349–381.

[2] D. G. Aronson, N. V. Mantzaris and H. G. Othmer, Wave propagation and blocking in inhomogeneous media, Discrete and Continuous Dynamical Systems, 13 (2005), 843–876.

[3] D. G. Aronson and V. Padron, Pattern formation in a model of an injured nerve fiber, SIAM J. Appl. Math., 70 (2009), 789–802.

[4] S. M. Baer and J. Rinzel, Propagation of dendritic spikes mediated by excitable spines: A continuum theory, J. Neurophys., 65 (1991), 874–890.

[5] J. Belmonte-Beitia and P. J. Torres, Existence of dark soliton of the cubic nonlinear Schrödinger equation with periodic inhomogeneous nonlinearity, J. Nonlinear Math. Phys., 15 (2008), 65–72.

[6] H. Berestycki and F. Hamel, Front propagation in periodic excitable media, Comm. Pure Appl. Math., 55 (2002), 949–1032.

[7] P.-L. Chen and J. Bell, Spine-density dependence of the qualitative behavior of a model of a nerve fiber with excitable spines, J. Math. Anal. Appl., 187 (1994), 384–410.

[8] E. N. Dancer and S. Yan, Interior peak solutions for an elliptic system of FitzHugh-Nagumo type, J. Differential Equations, 229 (2006), 654–679.

[9] P. Grindrod and B. D. Sleeman, A model of a myelinated nerve axon: Threshold behaviour and propagation, J. Math. Biol., 23 (1985), 119–135.

[10] J. K. Hale, “Ordinary Differential Equations,” Second edition, R. E. Krieger Publ. Co., Inc., Huntington, New York, 1980.

[11] S. Hastings, Some mathematical problems from neurobiology, Amer. Math. Monthly, 82 (1975), 881–895.

[12] S. Hastings, Some mathematical problems arising in neurobiology, in “Mathematics of Biology,” C.I.M.E. Summer Sch., 80, Springer, Heidelberg, (2010), 179–264.

[13] J. Kennedy and J. A. Yorke, Topological horseshoes, Trans. Amer. Math. Soc., 353 (2001), 2513–2530.

[14] Y. Kominis and K. Hizanidis, Lattice solitons in self-defocusing optical media: Analytical solutions of the nonlinear Kronig-Penney model, Optics Letters, 31 (2006), 2888–2890.

[15] M. A. Krasnosel’ski˘ı, “The Operator of Translation Along the Trajectories of Differential Equations,” Translations of Mathematical Monographs, 19, American Mathematical Society, Providence, R.I., 1968.

[16] T. J. Lewis and J. P. Keener, Wave-block in excitable media due to regions of depressed excitability, SIAM J. Appl. Math., 61 (2000), 293–316.

[17] J. Mawhin and M. Willem, “Critical Point Theory and Hamiltonian Systems,” Appl. Math. Sci., 74, Springer-Verlag, New York, 1989.

[18] H. P. McKean, Jr., Nagumo’s equation, Advances in Math., 4 (1970), 209–223.

[19] A. Mellet, J.-M. Roquejoffre and Y. Sire, Generalized fronts for one-dimensional reaction-diffusion equations, Discrete Contin. Dyn. Syst., 26 (2010), 303–312.

[20] D. Papini and F. Zanolin, On the periodic boundary value problem and chaotic-like dynamics for nonlinear Hill’s equations, Adv. Nonlinear Stud., 4 (2004), 71–91.

[21] J. X. Xin, Existence and nonexistence of traveling waves and reaction-diffusion front propagation in periodic media, J. Statist. Phys., 73 (1993), 893–926.

[22] J. Yang, S. Kalliadasis, J. H. Merkin and S. K. Scott, Wave propagation in spatially distributed excitable media, SIAM J. Appl. Math., 63 (2002), 485–509.

[23] C. Zanini and F. Zanolin, Positive periodic solutions for ordinary differential equations arising in the study of nerve fiber models, in “Applied and Industrial Mathematics in Italy,” Ser. Adv. Math. Appl. Sci., 69, World Sci. Publ., Hackensack, NJ, (2005), 564–575.

[24] C. Zanini and F. Zanolin, Multiplicity of periodic solutions for differential equations arising in the study of a nerve fiber model, Nonlinear Anal. Real World Appl., 9 (2008), 141–153.

[25] C. Zanini and F. Zanolin, Complex dynamics in a nerve fiber model with periodic coefficients, Nonlinear Anal. Real World Appl., 10 (2009), 1381–1400.

Received September 2011; revised November 2011.

E-mail address: chiara.zanini@polito.it
E-mail address: fabio.zanolin@uniud.it