SCHRÖDINGER MEANS IN HIGHER DIMENSIONS

PER SJÖLIN AND JAN-OLOV STRÖMBERG

Abstract. Maximal estimates for Schrödinger means and convergence almost everywhere of sequences of Schrödinger means are studied.

1. Introduction

For \( f \in L^2(\mathbb{R}^n) \), \( n \geq 1 \) and \( a > 0 \) we set

\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) \, dx, \xi \in \mathbb{R}^n,
\]

and

\[
S_t f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi \cdot x} e^{it|\xi|^a} \hat{f}(\xi) \, d\xi, \quad x \in \mathbb{R}^n, \quad t \geq 0.
\]

We introduce Sobolev spaces \( H_s = H_s(\mathbb{R}^n) \) by setting

\[
H_s = \{ f \in \mathcal{S}'; \| f \|_{H_s} < \infty \}, \quad s \in \mathbb{R},
\]

where

\[
\| f \|_{H_s} = \left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^{s} |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2}
\]

and \( \mathcal{S} = \mathcal{S}(\mathbb{R}^n) \) denotes the Schwartz class. We let the sequence \( \{ t_m \}_1^\infty \) have the properties that

\[
1 > t_1 > t_2 > t_3 > \cdots > 0 \quad \text{and} \quad \lim_{m \to \infty} t_m = 0.
\]

We shall study the problem of deciding for which sequences \( \{ t_m \}_1^\infty \) and functions \( f \) one has

\[
\lim_{m \to \infty} S_{t_m} f(x) = f(x) \quad \text{almost everywhere.}
\]

For \( r > 0 \) we say that \( \{ t_m \}_1^\infty \in l^r \) if \( \sum_{m=1}^\infty t_m^r < \infty \), and that \( \{ t_m \}_1^\infty \in l^{r,\infty} \) if \( \# \{ m; t_m > b \} \lesssim b^{-r} \) for \( b > 0 \).

In Sjölin and Strömbäck [2] we proved that if \( a > 0, n \geq 1, 0 < s < a/2 \), and \( \{ t_m \}_1^\infty \in l^r \) for some \( r < 2s/(a-2s) \), then (3) holds if \( f \in H_s(\mathbb{R}^n) \).

In the case \( n = 1 \) Dimou and Seeger [1] have proved that if \( a > 0, a \neq 1, 0 < s < a/4 \) and \( \{ t_m \}_1^\infty \in l^r \), where \( r = 2s/(a-4s) \), then (3) holds if \( f \in H_s(\mathbb{R}) \). They also proved that here \( r \) cannot be replaced by \( r_1 \) if \( r_1 > r \).

In Section 3 of this paper we shall improve the result from [2] mentioned above in the case \( n \geq 2 \). We shall prove that (3) holds for \( f \in H_s(\mathbb{R}^n) \) if \( a > 0, a \neq 1, n \geq 2, 0 < s < a/2 \),

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and \( \{t_m\}_1^\infty \in l^{r,\infty} \) where \( r = 2s/(a-2s) \). In the proof of this result we shall use the following theorem.

**Theorem 1.** Assume \( a > 0, n \geq 1, \lambda \geq 1, \) and let the interval \( J \subset [0,1] \). Assume also that \( f \in L^2(\mathbb{R}^n) \) and \( \text{supp} \hat{f} \subset B(0;\lambda) = \{ \xi \in \mathbb{R}^n; |\xi| \leq \lambda \} \). Then one has

\[
\| \sup_{t \in J} |S_t f| \|_2 \leq (1 + C|J|^{1/2} \lambda^{a/2}) \|f\|_2.
\]

Let the ball \( B \) be a subset of \( \mathbb{R}^n \) and let the interval \( J \subset [0,1] \). Set \( E = B \times J \). In Section 2 we shall also study maximal functions of the type

\[
S_E^* f(x) = \sup_{(y,t) \in E} |S_t f(x+y)|, \ x \in \mathbb{R}^n, f \in \mathcal{S}(\mathbb{R}^n).
\]

In Section 4 we study relations between maximal estimates in one variable and maximal estimates in dimension \( n \geq 2 \).

In Section 5 finally we give a counter-example which shows that in the case \( n \geq 2, 0 < s < a/4, r = 2s/(a-4s), \{t_m\}_1^\infty \in l^{r,\infty}, \{t_n-t_{n+1}\}_1^\infty \) decreasing, there is no estimate

\[
\| \sup_m |S_{t_m} f| \|_2 \lesssim \|f\|_{H^s}
\]

for all radial functions \( f \in \mathcal{S}(\mathbb{R}^n) \).

We write \( A \lesssim B \) if there is a positive constant \( C \) such that \( A \leq CB \), and we write \( A \sim B \) if \( A \lesssim B \) and \( B \lesssim A \).

2. **Maximal estimates**

The following theorem was proved in Sjölin and Strömberg [2]. We shall here give a new proof which is simpler than the proof in [2].

**Theorem A.** (Sjölin-Strömberg) Assume \( a > 0, n \geq 1, \lambda \geq 1 \), and \( J \) interval in \( \mathbb{R} \). Assume also that \( f \in L^2(\mathbb{R}^n) \) and \( \text{supp} \hat{f} \subset B(0;\lambda) \).

Then one has

\[
\| \sup_{t \in J} |S_t f| \|_2 \leq (1 + C|J|^{1/2} \lambda^a) \|f\|_2.
\]

**Proof.** We can write

\[
S_t f(x) = c \int e^{i\xi \cdot x} e^{it|\xi|^a} \hat{f}(\xi) \, d\xi
\]

and \( J = [t_0, t_0 + r] \) where \( r = |J| \).

We have

\[
e^{it|\xi|^a} = \Delta + e^{it_0|\xi|^a}
\]

where \( \Delta = e^{it|\xi|^a} - e^{it_0|\xi|^a} \). It follows that

\[
\Delta = i|\xi|^a \int_{t_0}^t e^{i|\xi|^a s} \, ds,
\]
and
\[ S_1 f(x) = c \int_{\mathbb{R}^n} \int_{t_0}^t e^{i\xi \cdot x} |\xi|^a \xi^a \hat{f}(\xi) \, d\xi \, ds \]
\[ + c \int_{\mathbb{R}^n} e^{i\xi \cdot x} e^{i t_0 |\xi|^a} \hat{f}(\xi) \, d\xi \]
\[ = S_1(x, t) + S_2(x, t). \]

Hence
\[ |S_1(x, t)| \lesssim \int_{t_0}^t \left| \int_{\mathbb{R}^n} e^{i\xi \cdot x} |\xi|^a \xi^a \hat{f}(\xi) \, d\xi \right| \, ds \]
and
\[ \sup_{t \in J} |S_1(x, t)| \lesssim \int_{t_0}^{t_0 + r} \left| \int_{\mathbb{R}^n} e^{i\xi \cdot x} |\xi|^a \xi^a \hat{f}(\xi) \, d\xi \right| \, ds. \]

Using Minkovski’s inequality and Plancherel’s theorem we obtain
\[ \left( \int \sup_{t \in J} |S_1(x, t)|^2 \, dx \right)^{1/2} \lesssim \int_{t_0}^{t_0 + r} \|\xi|^a \hat{f}(\xi)\|_2 \, ds \]
\[ \leq r \lambda^a \left( \int_{\mathcal{B}(0, \lambda)} |\hat{f}(\xi)|^2 \right)^{1/2} \lesssim r \lambda^a \|f\|_2. \]

Also
\[ \left( \int \sup_{t \in J} |S_2(x, t)|^2 \, dx \right)^{1/2} = \left( \int |S_2(x, t_0)|^2 \, dx \right)^{1/2} = \|f\|_2, \]
and we obtain
\[ \|\sup_{t \in J} |S_1 f|\|_2 \leq (1 + Cr \lambda^a) \|f\|_2, \]
which proves the theorem. \( \square \)

In the following theorems let \( J \) denote an interval. In the one-dimensional case Dimou and Seeger obtained the following result.

**Theorem B.** (Dimou-Seeger) Assume \( a > 0, a \neq 1, n = 1, \lambda \geq 1 \). and \( J \subset [0, 1] \). Assume also that \( f \in L^2(\mathbb{R}) \) and \( \text{supp} \hat{f} \subset \{ \xi; \lambda/2 \leq |\xi| \leq \lambda \} \). Then one has
\[ \|\sup_{t \in J} |S_1 f|\|_2 \lesssim \left( 1 + |J|^{1/4} \lambda^{a/4} \right) \|f\|_2. \]

We have the following result.

**Theorem 1.** Assume \( a > 0, n \geq 1, \lambda \geq 1 \), and let the interval \( J \subset [0, 1] \). Assume also that \( f \in L^2(\mathbb{R}^n) \) and \( \text{supp} \hat{f} \subset B(0; \lambda) = \{ \xi \in \mathbb{R}^n; |\xi| \leq \lambda \} \). Then one has
\[ \|\sup_{t \in J} |S_1 f|\|_2 \leq (1 + C|J|^{1/2} \lambda^{a/2}) \|f\|_2. \]

**Proof.** First we study the case \( |J| \lambda^a \leq 1 \). Then \((1 + C|J| \lambda^a) \leq (1 + C|J|^{1/2} \lambda^{a/2}).\) Applying Theorem A we get
\[ \|\sup_{t \in J} |S_1 f|\|_2 \leq (1 + C|J| \lambda^a) \|f\|_2 \leq (1 + C|J|^{1/2} \lambda^{a/2}) \|f\|_2. \]
It remains to study the case $|J|\lambda^a > 1$. 
Cover $J$ with intervals $J_i, i = 1, 2, \ldots, N$, of length $\lambda^{-a}$. We may take $N \leq |J|\lambda^a + 1$. 
Using Theorem A we obtain

$$
\| \sup_{t \in J} |S_t f| \|_2 \leq \sum_{i=1}^N \| \sup_{t \in J_i} |S_t |f| \|_2 \leq \sum_{i=1}^N (1 + |J_i|\lambda^a) \| f \|_2 \lesssim N \| f \|_2 \leq (|J|\lambda^a + 1) \| f \|_2
$$

and the inequality in the theorem follows.

We have the following extension of Theorem B.

**Theorem 2.** Assume $a > 0, a \neq 1, \lambda \geq 1$ and $J \subset [0, 1]$. Assume also that $f \in L^2(\mathbb{R})$ with supp $\hat{f} \subset B(0, \lambda)$. Then one has

$$
\| \sup_{t \in J} |S_t f| \|_2 \lesssim C \left( |J|^{1/4} \lambda^{a/4} + 1 \right) \| f \|_2.
$$

**Proof.** First we study the case $|J|\lambda^a \leq 1$. From Theorem A we obtain

$$
\| \sup_{t \in J} |S_t f| \|_2 \lesssim \| f \|_2.
$$

which proves the theorem in this case.

We then consider the case $|J|\lambda^a > 1$, i.e. $\lambda > |J|^{-1/a}$.

We choose $k$ such that $2^{-k-1}\lambda < |J|^{-1/a} \leq 2^{-k}\lambda$ and then write $f = \sum_{j=0}^k f_j + g$ where supp $\hat{f}_j \subset \{ \xi; 2^{-j-1}\lambda \leq |\xi| \leq 2^{-j}\lambda \}$ for $j = 0, 1, \ldots, k$, and supp $\hat{g} \subset B(0, 2^{-k-1}\lambda)$. 
From Theorem A we conclude that

$$
\| \sup_{t \in J} |S_t g| \|_2 \lesssim (1 + |J||J|^{-1}) \| g \|_2 \lesssim \| f \|_2
$$

and it follows from Theorem B that

$$
\| \sup_{t \in J} |S_t f_j| \|_2 \lesssim \left( 1 + |J|^{1/4}2^{-ja/4}\lambda^{a/4} \right) \| f \|_2 \lesssim |J|^{1/4}2^{-ja/4}\lambda^{a/4} \| f \|_2,
$$

since $|J|2^{-ja/4}\lambda^a \geq 1$. Hence we have

$$
\sum_{j=0}^k \| \sup_{t \in J} |S_t f_j| \|_2 \lesssim |J|^{1/4}\lambda^{a/4} \| f \|_2
$$

and the theorem follows from the above estimates. 

The method of Dimou and Seeger to prove Theorem B can be extended to all dimensions $n$ an gives the following result.

**Lemma 1.** Assume $n \geq 1, a > 0$ and $\lambda \geq 1$. Let the interval $J \subset [0, 1]$, let $f \in L^2(\mathbb{R}^n)$ with supp $\hat{f} \subset \{ \lambda/2 \leq |\xi| \leq \lambda \}$.

Then one has

$$
\| \sup_{t \in J} |S_t f| \|_2 \leq C \left( |J|^{n/4}\lambda^{na/4} + 1 \right) \| f \|_2 \text{when } a \neq 1
$$
First we have the trivial estimate
\[ \| S_t f \|_2 \leq C \left( |J|^{(n+1)/4} \lambda^{(n+1)/4} + 1 \right) \| f \|_2 \text{ when } a = 1 \]
We shall then study a more general problem. Let \( E \) denote a bounded set in \( \mathbb{R}^{n+1} \). For \( f \in \mathcal{S}(\mathbb{R}^n) \) we introduce the maximal function
\[ S_E^* f(x) = \sup_{(y,t) \in E} |S_t f(x + y)|, \quad x \in \mathbb{R}^n. \]
The method used to prove Lemma 1 can also be used to prove the following result.

**Lemma 2.** Assume \( a > 0, n \geq 1 \) and \( \lambda \geq 1 \). Let the interval \( J \subset [0,1] \). Let \( B \) be a ball in \( \mathbb{R}^n \) with radius \( r \), let \( E = B \times J = \{(x,t); x \in B, t \in J\} \) and let \( f \in L^2(\mathbb{R}^n) \) with \( \text{supp } \hat{f} \subset \{ \lambda/2 \leq |\xi| \leq \lambda \} \).

Then one has
\[ \| S^*_E f \|_2 \leq C \left( |J|^{n/4} \lambda^{n/4} + r^{n/2} \lambda^{n/2} + 1 \right) \| f \|_2 \text{ when } a \neq 1, \]
and
\[ \| S^*_E f \|_2 \leq C \left( |J|^{(n+1)/4} \lambda^{(n+1)/4} + r^{n/2} \lambda^{n/2} + 1 \right) \| f \|_2 \text{ when } a = 1. \]

We observe that Lemma 1 is a special case of Lemma 2 by taking \( B = B(0, \epsilon) \) with \( \epsilon > 0 \) small enough.

**Proof of Lemma 2.** We let \( \chi \) denote a smooth non-negative function on \( \mathbb{R} \) supported on \([1/3, 4/3]\) and identically 1 on \([1/2, 1]\). We also use the same notation for the radial function on \( \mathbb{R}^n \) with \( \chi(\xi) = \chi(|\xi|) \).

We set
\[ S_{t} f(x,t) = \int_{\mathbb{R}^n} e^{i(x+\xi)} \hat{f}(\xi) \chi(\xi/\lambda) d\xi \quad x \in \mathbb{R}^n, t \in \mathbb{R}. \]

We introduce measurable functions \( t : \mathbb{R}^n \to J \) and \( b : \mathbb{R}^n \to B \) and then have
\[ S_{t} f(x + b(x), t(x)) = \int_{\mathbb{R}^n} e^{i(x+b(x)\cdot \xi + t(x))} \hat{f}(\xi) \chi(\xi/\lambda) d\xi \]
\[ = |\eta = \xi/\lambda| = \int_{\mathbb{R}^n} e^{i(\lambda(x+b(\eta) - \eta + t(x)\lambda^n |\eta|))} \hat{f}(\lambda \eta) \chi(\eta) d\eta \lambda^n \]
\[ = \lambda^n T_{\lambda}(\hat{f}(\cdot))(x) \]
where
\[ T_{\lambda} g(x) = \int_{\mathbb{R}^n} e^{i(\lambda(x+b(\eta) - \eta + t(x))\chi(\xi/\lambda))} g(\xi) d\xi. \]
We have \( \| \hat{f}(\cdot) \|_2 = c\lambda^{-n/2} \| f \|_2 \) and \( T_{\lambda} T^\ast_{\lambda} \) has kernel
\[ K_{\lambda}(x, y) = \int_{\mathbb{R}^n} e^{i(\lambda(x-y+b(\eta) - b(y)) - \xi + \lambda^n(t(x)-t(y))\xi)} \chi(\xi) d\xi. \]
We will majorize the kernel \( K_{\lambda} \) by a convolution kernel \( G_{\lambda} \), that is \( |K_{\lambda}(x,y)| \lesssim G(x-y) \).

One then has the \( L^2 \)-operator norm \( \| T_{\lambda} T^\ast_{\lambda} \| \lesssim \| G \|_1 \).

First we have the trivial estimate
\[ |K_{\lambda}(x,y)| \lesssim 1 \]
holds for all \(x\) and \(y\). We shall use this estimate when \(\lambda|x - y| \leq C_0 + 2\lambda d\) where \(d = 2r\).

For \(\lambda|x - y| > C_0 + 2\lambda d\) we have

\[
|x - y| \left(1 - \frac{d}{|x - y|}\right) = |x - y| - d < |x - y + b(x) - b(y)| < |x - y| + d = |x - y| \left(1 + \frac{d}{|x - y|}\right)
\]

and

\[
(1/2)|x - y| < |x - y + b(x) - b(y)| < (3/2)|x - y|.
\]

Introducing polar coordinates we have

\[
K_{\lambda}(x, y) = \int_0^\infty e^{i\lambda(x - y)t} r a \chi(r)^2 \left(\sum_{n=1}^\infty e^{i\lambda x - b(x) - b(y)} \int_{\mathbb{R}^{n-1}} e^{i\lambda x - b(x) - b(y)} j_{\lambda} d\sigma(\xi)\right) dr (5)
\]

We observe that inner integral is \(\hat{\sigma}(\lambda|x - y + b(x) - b(y)|r)\). According to Stein ([4], p. 347) one has

\[
\hat{\sigma}(\xi) = c|\xi|^{1-n/2} J_{(n-2)/2}(|\xi|).
\]

We take \(C_0\) large so that

\[
J_{(n-2)/2}(r) = a_0 \frac{e^{ir}}{r^{1/2}} + a_1 \frac{e^{ir}}{r^{3/2}} + \ldots + a_N \frac{e^{ir}}{r^{N+1/2}} + b_0 \frac{e^{-ir}}{r^{1/2}} + b_1 \frac{e^{-ir}}{r^{3/2}} + \ldots + b_N \frac{e^{-ir}}{r^{N+1/2}} + R(r) \text{ for } r \geq C_0,
\]

where \(|R(r)| \lesssim \frac{1}{r^{N+3/2}}\) for \(r \geq C_0\) (See [4] p. 338).

From this we get

\[
K_{\lambda}(x, y) = \int_0^\infty e^{i\lambda x - b(x) - b(y)} r a \chi(r)^2 r^{n-1} \left(a_0 \frac{e^{i|\lambda x - b(x) - b(y)|r}}{(\lambda|x - y + b(x) - b(y)|r)^{n+1/2}} + \ldots + b_N \frac{e^{i|\lambda x - b(x) - b(y)|r}}{(\lambda|x - y + b(x) - b(y)|r)^{N+1/2}} + R_1(\lambda|x - y + b(x) - b(y)|r) \right) dr (5)
\]

where \(R_1(r) = r^{1-n/2} R(r)\). It follows from ([4]) that

\[
R_1(\lambda|x - y + b(x) - b(y)|r) \lesssim \frac{1}{(\lambda|x - y + b(x) - b(y)|r)^{N+1/2}} \leq \frac{1}{(\lambda|x - y|/2)^{N+1/2}}.
\]

The remainder term contribute with a remainder part of the kernel

\[
K_{\lambda, \text{rem}}(x, y) \lesssim (\lambda|x - y|)^{-N-n/2-1/2}
\]

Set \(\Phi_{\lambda}(r) = \lambda^a (t(x) - t(y)) r^a + |\lambda x - y + b(x) - b(y)| r\)

The main term in ([5]) gives the following contribution to \(K_{\lambda}(x - y)\):

\[
K_{\lambda, \text{rem}}(x, y) = \int_0^\infty e^{i\Phi_{\lambda}(r)} \chi(r)^2 r^{n-1} r^{1/2} dr (5)
\]

We consider two cases:

Case 1: \(|x - y| >> \lambda^{-1}|t(x) - t(y)|\) gives \(\Psi_{\lambda} \gtrsim |\lambda x - y|\) and integrations by parts gives \(|K_{\lambda, \text{rem}}(x, y)| \lesssim (\lambda|x - y|)^{-N}\) for any large \(N\).

Case 2: \(|x - y| \lesssim \lambda^{-1}|t(x) - t(y)|\). In this case we have

\[
\Phi_{\lambda}'(r) = \lambda^a (t(x) - t(y)) a(a - 1)r^{a-2}
\]
and in the case \(a \neq 1\) van der Corput gives
\[
|K_{\lambda,0}(x, y)| \lesssim \lambda^{-a/2} |t(x) - t(y)|^{-1/2} (\lambda|x - y|)^{1/2 - 2a} \lesssim (\lambda|x - y|)^{-n/2}.
\]
In Case 2a with \(a = 1\) we have the trivial estimate estimate
\[
|K_{\lambda,0}(x, y)| \lesssim (\lambda|x - y|)^{(1-n)/2}
\]
and the other terms in \([5]\) can be estimated in the same way.

We note that Case 2 is contained in the set \(|x - y| \lesssim \lambda^{a-1}|J|\)

The other terms in \([2]\) can be estimated in the same way.

To summarize the estimates we see that \(|K_{\lambda}(x, y)| \lesssim G_{\lambda}(x - y)\) where
when \(a \neq 1\):
\[
G_{\lambda}(x) = \chi_{\{|x| < C_0 \lambda^{a-1+2d}\}}(x) + \chi_{\{|x| \geq \lambda^{-1}\}}(\lambda^{-N}|x|^N + \chi_{\{|x| \leq C\lambda^{a-1}|J|\}}(\lambda^{-n/2}|x|^{-n/2})
\]
and when \(a = 1\):
\[
G_{\lambda}(x) = \chi_{\{|x| < C_0 \lambda^{a-1+2d}\}}(x) + \chi_{\{|x| \geq \lambda^{-1}\}}(\lambda^{-N}|x|^N + \chi_{\{|x| \leq C|J|\}}(\lambda^{1-n/2}|x|^{1-n/2})
\]
In the case when \(a \neq 1\) we have
\[
\|G\|_1 \lesssim (\lambda^{-1} + d^n + \lambda^{-n} + \int_{|x| \leq C\lambda^{a-1}|J|} \lambda^{-n/2}|x|^{-n/2} d\lambda
\]
and the above integral is majorized by
\[
\lambda^{-n/2} \int_0^{C\lambda^{a-1}|J|} r^{n/2-1} dr \lesssim \lambda^{-n/2}(a-1)n/2|J|^{n/2} = \lambda^{\frac{a}{2}(a-2)}|J|^{n/2},
\]
Hence
\[
\|G\|_1 \lesssim \lambda^{-n} + d^n + \lambda^{\frac{a}{2}(a-2)}|J|^{n/2}
\]
in the case \(a \neq 1\). In the case \(a = 1\) we get
\[
\|G\|_1 \lesssim \lambda^{-n} + d^n + \lambda^{(1-n)/2} \int_0^{C|J|} r^{1/2+n/2} dr \lesssim \lambda^{-n} + d^n + \lambda^{(1-n)/2}|J|^{(n+1)/2}.
\]
In the case \(a \neq 1\) we obtain
\[
\|T_\lambda T_\lambda^*\| \lesssim \lambda^{-n} + d^n + \lambda^{\frac{a}{2}(a-2)}|J|^{n/2},
\]
and
\[
\|T_\lambda\| \lesssim \lambda^{-n/2} + d^{n/2} + \lambda^{\frac{a}{2}(a-2)}|J|^{n/4}.
\]
From this we get
\[
\|S_\lambda f(x + b(x), t(x))\|_2 \leq \lambda^n \|T_\lambda [\hat{f}(\lambda \cdot)]\|_2 \leq \lambda^n \|T_\lambda \| \|\hat{f}(\lambda \cdot)\|_2
\]
\[
\lesssim \lambda^n \left( \lambda^{-n/2} + d^{n/2} + \lambda^{\frac{a}{2}(a-2)}|J|^{n/4} \right)^{\lambda^{-n/2} \|f\|_2}
\]
\[
= \left( 1 + d^{n/2} \lambda^{n/2} + \lambda^{na/4}|J|^{n/4} \right) \|f\|_2.
\]
Finally in the case \(a = 1\) we obtain
\[
\|T_\lambda T_\lambda^*\| \lesssim \lambda^{-n} + d^n + \lambda^{(1-n)/2}|J|^{(n+1)/2},
\]
\[
\|T_\lambda\| \lesssim \lambda^{-n/2} + d^{n/2} + \lambda^{(1-n)/4}|J|^{(n+1)/4},
\]
and
\[
\|S_\lambda f(x + b(x), t(x))\|_2 \leq \lambda^n \|T_\lambda [\hat{f}(\lambda)]\|_2 \leq \lambda^n \|T_\lambda \cdot \|\hat{f}(\lambda)\|_2
\]
\[
\lesssim \lambda^n \left( \lambda^{-n/2} + a^{n/2} + \lambda^{(1-n)/4} |J|^{(n+1)/4} \right) \lambda^{-n/2} \|f\|_2
\]
\[
= \left( 1 + a^{n/2} \lambda^{n/2} + \lambda^{(n+1)/4} |J|^{(n+1)/4} \right) \|f\|_2.
\]

This completes the proof of Lemma 2. \qed

We shall then extend Lemma 2.

**Lemma 3.** In Lemma 1 and Lemma 2 above the condition \(\text{supp } \hat{f} \subset \{\xi; \lambda/2 \leq |\xi| \leq \lambda\} \) can be replaced by the weaker condition \(\text{supp } \hat{f} \subset \{\xi; |\xi| \leq \lambda\} \).

In the proof of Lemma 3 we shall use a result in Sjölin and Strömberg [3]. Let \(y_0 \in \mathbb{R}^n, t_0 \in \mathbb{R}^n, 0 < r \leq 1\), and let \(f \in L^2(\mathbb{R}^n) \) with \(\text{supp } \hat{f} \subset B(0; \lambda)\) and \(\lambda > 1\). Set \(F = \{(y, t); y_{0,j} \leq y_j \leq y_{0,j} + r \text{ for } 1 \leq j \leq n\}, t_0 \leq t \leq t_0 + r^a\} \).

It is proved that
\[
\|S^*_E f\|_2 \lesssim (1 + r^a \lambda^n)(1 + r \lambda)^n \|f\|_2,
\]
and the method in [3] can be used to prove that
\[
\|S^*_E f\|_2 \lesssim (1 + |J|\lambda^a)(1 + r \lambda)^n \tag{6}
\]
if \(E = B \times J\) with \(B\) and \(J\) as in Lemma 2.

The method to prove this is a generalisation of the method we used to prove Theorem A.

**Proof of Lemma 3.** Let \(N\) be the smallest integer such that \(|J|2^{-aN} \lambda^a + r^{2-N} \lambda < 2\).

We write \(f = \sum_{j=0}^N f_j\) where \(\text{supp } \hat{f}_j \subset \{2^{-j-1} \lambda \leq |\xi| \leq 2^{-j} \lambda\} \) for \(0 \leq j < N\) and \(\text{supp } \hat{f}_N \subset B(0; 2^{-N} \lambda)\). It follows from (3) that
\[
\|S^*_E f_N\|_2 \lesssim (1 + |J|2^{-aN} \lambda^a)(1 + r^{2-N} \lambda)^n \|f_N\|_2 \lesssim \|f\|_2.
\]

Also
\[
\|S^*_E f\|_2 \leq \sum_{j=0}^N \|S^*_E f_j\|_2.
\]

and according to Lemma 2 we have for \(a \neq 1\)
\[
\|S^*_E f_j\|_2 \leq C(2^{-j+1/4}|J|^{n/4} \lambda^{na/4} + r^{n/2} \lambda^{n/2-jn/2}) \|f_j\|_2.
\]

for \(0 \leq j < N\) It follows that
\[
\left\| S^*_E \left( \sum_{j=0}^{N-1} f_j \right) \right\|_2 \lesssim (|J|^{n/4} \lambda^{na/4} + r^{n/2} \lambda^{n/2}) \|f\|_2.
\]

and we obtain
\[
\|S^*_E f\|_2 \lesssim (|J|^{n/4} \lambda^{na/4} + r^{n/2} \lambda^{n/2} + 1) \|f\|_2
\]
for \(a \neq 1\).

The same proof works also for \(a = 1\) and this completes the proof of Lemma 3. \qed

We shall then prove the following theorem.
Theorem 3. Assume $a > 0, n \geq 1$ and $\lambda \geq 1$. Let the interval $J \subset [0,1]$, let $B$ be a ball in $\mathbb{R}^n$ with radius $r$ and set $E = B \times J$. Let $f$ be an function in $L^2(\mathbb{R}^n)$ with supp $\hat{f} \subset B(0;\lambda)$.

Then the following holds when $n = 1$ and $a \neq 1$:

$$
\|S^n_E f\|_2 \leq \left( |J|^{1/4} \lambda^{a/2} + r^{1/2} \lambda^{1/2} + 1 \right) \|f\|_2,
$$

when $n = 1$ and $a = 1$:

$$
\|S^n_E f\|_2 \leq \left( |J|^{1/2} \lambda^{a/2} + r^{1/2} \lambda^{1/2} + 1 \right) \|f\|_2,
$$

when $n \geq 2$ and $a \neq 1$:

$$
\|S^n_E f\|_2 \leq \left( |J|^{1/2} \lambda^{a/2} + r \lambda + 1 \right) (r \lambda + 1)^{(n-2)/2} \|f\|_2,
$$

when $n \geq 2$ and $a = 1$:

$$
\|S^n_E f\|_2 \leq \left( |J|^{1/2} \lambda^{a/2} + r^{n/(n+1)} \lambda^{n/(n+1)} + 1 \right) (r^{n/(n+1)} \lambda^{n/(n+1)} + 1)^{(n-1)/2} \|f\|_2.
$$

Proof. The cases with $n = 1$ in Theorem 3 follow directly from Lemma 3.

In the cases with $n \geq 2$ we shall use an argument similar to the proof of Theorem 1 by covering the Interval $J$ with intervals $J_i$ of equal length.

In the case $a \neq 1$ we have by Lemma 3 the estimate

$$
\|S^n_E f\|_2 \leq C \left( |J|^{n/2} \lambda^{na/2} + r^n \lambda^n + 1 \right) \|f\|_2^2.
$$

We have

$$
|J|^{n/2} \lambda^{na/2} + r^n \lambda^n + 1 \sim (|J| \lambda^a)^{n/2} + (r^2 \lambda^2 + 1)^{n/2}.
$$

Cover $J$ with intervals $J_i, i = 1, 2, \ldots, N$, of intervals of length $|J_i|$ such that $|J_i| \lambda^a = r^2 \lambda^2 + 1$ with $N \leq |J|/|J_i| + 1 = |J| \lambda^a (r^2 \lambda^2 + 1)^{-1} + 1$ Set $E_i = B \times J_i$ then we have

$$
\|S^n_{E_i} f\|_2 \leq \left( (|J_i| \lambda^a)^{n/2} + (r^2 \lambda^2 + 1)^{n/2} \right) \|f\|_2^2 = 2 \left( r^2 \lambda^2 + 1 \right)^{n/2} \|f\|_2^2
$$

and

$$
\|S^n_E f\|_2 \leq \sum_{i=1}^N \|S^n_{E_i} f\|_2 \lesssim N \left( r^2 \lambda^2 + 1 \right)^{n/2} \|f\|_2^2
$$

$$
\leq \left( |J| \lambda^a \left( r^2 \lambda^2 + 1 \right)^{-1} + 1 \right) \left( r^2 \lambda^2 + 1 \right)^{n/2} \|f\|_2^2
$$

$$
= \left( |J| \lambda^a + r^2 \lambda^2 + 1 \right) \left( r^2 \lambda^2 + 1 \right)^{(n-2)/2} \|f\|_2^2
$$

$$
\leq \left( |J|^{1/2} \lambda^{a/2} + r^{n-1} \lambda^{n-1} + 1 \right) \left( r^{n-1} \lambda^{n-1} + 1 \right)^{n-2} \|f\|_2^2,
$$

which gives the desired estimate in this case.

In the case $a = 1$ we have by Lemma 3 the estimate

$$
\|S^n_E f\|_2 \leq C \left( |J|^{(n+1)/2} \lambda^{(n+1)a/2} + r^n \lambda^n + 1 \right) \|f\|_2^2.
$$

We have

$$
|J|^{(n+1)/2} \lambda^{(n+1)a/2} + r^n \lambda^n + 1 \sim (|J| \lambda^a)^{(n+1)/2} + \left( r^{2n/(n+1)} \lambda^{2n/(n+1)} + 1 \right)^{(n+1)/2}.
$$
Cover $J$ with intervals $J_i, i = 1, 2, \ldots, N$, of intervals of length $|J_i|$ such that $|J_i|\lambda^a = r^{2n/(n+1)}\lambda^{2n/(n+1)} + 1$ and $N \leq |J|/|J_i| + 1 = |J|\lambda^a (r^{2n/(n+1)}\lambda^{2n/(n+1)} + 1)^{-1} + 1$ Set $E_i = B \times J_i$ then we have

$$\|S_{E_i}^*f\|_2^2 \lesssim \left((|J_i|\lambda^a)^{(n+1)/2} + \left(r^{2n/(n+1)}\lambda^{2n/(n+1)} + 1\right)^{(n+1)/2}\right) \|f\|_2^2$$

$$= 2\left(r^{2n/(n+1)}\lambda^{2n/(n+1)} + 1\right)^{(n+1)/2} \|f\|_2^2,$$

and

$$\|S_{E_i}^*f\|_2^2 \leq \sum_{i=1}^N \|S_{E_i}^*f\|_2^2 \lesssim N \left(r^{2n/(n+1)}\lambda^{2n/(n+1)} + 1\right)^{(n+1)/2} \|f\|_2^2$$

$$\leq (|J|\lambda^a \left(r^{2n/(n+1)}\lambda^{2n/(n+1)} + 1\right)^{-1} + 1) \left(r^{2n/(n+1)}\lambda^{2n/(n+1)} + 1\right)^{(n+1)/2} \|f\|_2^2$$

$$= \left(|J|\lambda^a + r^{2n/(n+1)}\lambda^{2n/(n+1)} + 1\right) \left(r^{2n/(n+1)}\lambda^{2n/(n+1)} + 1\right)^{(n+1)/2} \|f\|_2^2$$

$$\leq \left(|J|^{1/2}\lambda^{a/2} + r^{n/(n+1)}\lambda^{n/(n+1)} + 1\right)^2 \left(r^{n/(n+1)}\lambda^{n/(n+1)} + 1\right)^{n-1} \|f\|_2^2,$$

which gives the desired estimate in this case.

\[ \square \]

3. A Convergence Result

We shall here prove a convergence result for function in $H_\delta(\mathbb{R}^n), n \geq 2$, and begin with two lemmas.

**Lemma 4.** Assume $a > 0, a \neq 1, n \geq 2, \lambda \geq 1$ and $0 < s < a/2$. Also let $(t_m)_\infty \in l^{r,\infty}$, where $r = 2s/(a - 2s)$ Then one has

$$\|\sup_m |S_{t_m}f|\|_2 \lesssim \lambda^s \|f\|_2,$$

if $f \in L^2(\mathbb{R}^n)$ and supp $\hat{f} \subset \{\xi; \lambda/2 \leq |\xi| \leq \lambda\}$.

**Proof.** Let $0 < b < 1$. We have

$$\|\sup_m |S_{t_m}f|\|_2 \leq \|\sup_{t_m \leq b} |S_{t_m}f|\|_2 + \|\sup_{t_m > b} |S_{t_m}f|\|_2 = T_1 + T_2.$$

Theorem 1 gives the estimate

$$T_1 \lesssim \left(b^{1/2}\lambda^{a/2} + 1\right) \|f\|_2.$$

We also have $\#\{m; t_m > b\} \lesssim b^{-r}$ and it follows that

$$T_2 \lesssim b^{-r/2} \|f\|_2.$$

We then choose $b$ such that

$$b^{1/2}\lambda^{a/2} = b^{-r/2}.$$

One gets

$$b^{1/2 + r/2} = \lambda^{-a/2},$$
and \[ b = \lambda^{-a/(1+r)} < 1. \]

Hence \( b^{-r/2} \geq 1 \) and \[ T_1 \lesssim b^{1/2+r/2} \|f\|_2 = b^{-r/2} \|f\|_2. \]

We have shown that \[ \|\sup_m |S_m f|\|_2 \lesssim b^{-r/2} \|f\|_2, \]

where \( b^{-r/2} = \lambda^{ra/2(1+r)} = \lambda^s. \)

This completes the proof of the lemma. \( \square \)

We shall then improve Lemma 4 by proving the following lemma

**Lemma 5.** Let \( a, n, s, r, \) and \((t_m)^\infty_1\) have the same properties as in Lemma 4, and assume that \( f \in H_s(\mathbb{R}^n). \) Then

\[ \|\sup_m S_m f\|_2 \lesssim \|f\|_{H^s} \text{ for } f \in S(\mathbb{R}^n). \]

Before proving Lemma 5 we remark that the following theorem follows from Lemma 5 (see proof of Corollary 4 in Sjölin and Strömberg [2])

**Theorem 4.** Let \( a, n, s, r, \) and \((t_m)^\infty_1\) have the same properties as in Lemma 4 and assume that \( f \in H_s(\mathbb{R}^n). \) Then

\[ \lim_{m \to \infty} S_{t_m} f(x) = f(x) \text{ for almost every } x. \]

**Proof of Lemma 5.** It follows from the theorem on monotone convergence that instead of estimating \( \sup_m |S_m f| \) it is sufficient to estimate \( \sup_{m \leq M} |S_m f| \) for large integers \( M \) (as long as the estimates do not depend on \( M \)). We can find a measurable function \( t(x) \) such that

\[ \sup_{m \leq M} |S_m f| = |S_{t(x)} f(x)|, \]

and \( t(x) \) takes only finitely many values. We then define intervals \( I_k = (2^{-k-1}, 2^{-k}], \)

\( k = 0, 1, \ldots, \) and sets \( F_k = \{ x \in \mathbb{R}^n, t(x) \in I_k \}. \) The Sets \( F_k \) are disjoint and \( \mathbb{R}^n = \bigcup_{k \geq 0} F_k. \) We let \( X_k \) denote the characteristic function of \( F_k \) and then have \( \sum_{k \geq 0} \chi_k = 1 \) and

\[ S_{t(x)} f(x) = \sum_{k \geq 0} X_k(x) S_{t(x)} f(x). \] (7)

We write a function \( f = \sum_{j \geq 0} f_j \) by splitting its Fourier transform \( \hat{f} = \sum_{j \geq 0} \hat{f}_j \),

where \( \hat{f}_j \) is supported in \( \Omega_j \) where \( \Omega_0 = \{ |\xi| \leq 1 \} \) and \( \Omega_j = \{ 2^{-j-1} |\xi| \leq 2^j \} \) for \( j > 0. \)

We shall then split the sum (7) into three parts. For \( j \geq 0 \) set

\[ k(j) = (a - 2s) j = j 2s/r, \]

\[ b(j) = 2^{-k(j)}, \]

\[ b_1(j) = 2^{-k(j) - \epsilon_1 j} \]

and

\[ b_2(j) = 2^{-k(j) + \epsilon_2 j}, \]

where \( \epsilon_1 = 2\epsilon \) and \( \epsilon_2 = 2\epsilon/r \) and \( \epsilon \) is a small positive number.
We have

\[ S(t(x))f(x) = \sum_{k \geq 0} \sum_{j \geq 0} X_k(x)S^0_{t(x)}f_j(x) = S^1_{t(x)}f(x) + S^2_{t(x)}f(x) + S^3_{t(x)}f(x), \]

where

\[ S^1_{t(x)}f(x) = \sum_{j \geq 0} \sum_{k \geq 0} X_k(x)S_{t(x)}f_j(x), \]
\[ S^2_{t(x)}f(x) = \sum_{j \geq 0} \sum_{k \geq k(j) - \epsilon_j} X_k(x)S_{t(x)}f_j(x), \]
\[ S^3_{t(x)}f(x) = \sum_{j \geq 0} \sum_{k \geq k(j) - \epsilon_j} X_k(x)S_{t(x)}f_j(x). \]

Invoking Theorem 1 we obtain

\[ |S^1_{t(x)}f(x)| \leq \sum_{j \geq 0} \sum_{k \geq k(j) + \epsilon_j} X_k(x)|S_{t(x)}f_j(x)| \leq \sum_{j \geq 0} \sup_{t_m \leq b_1(j)} |S_{t_m}f_j(x)|, \]
and

\[ \|S^1_{t(x)}f\|_2 \leq \sum_{j \geq 0} \| \sup_{t_m \leq b_1(j)} |S_{t_m}f_j(x)| \|_2 \lesssim \sum_{j \geq 0} (1 + b_1(j)^{1/2}2^{aj/2}) \|f_j\|_2. \]

We have

\[ b_1(j)^{1/2}2^{aj/2} = 2^{-1/2}k(j) - \frac{1}{4}j2^{aj/2} = 2^{-1/2}(a - 2s + \epsilon_j - a)j = 2^{j(s - \epsilon)}, \]
and it follows that

\[ \|S^1_{t(x)}f\|_2 \lesssim \sum_{j \geq 0} 2^{j(s - \epsilon)} \|f_j\|_2 \leq \|f\|_{H_s}. \]

We shall then estimate \( S^2_{t(x)}f(x) \). One has

\[ |S^2_{t(x)}f(x)| \leq \sum_{j \geq 0} \sum_{k \leq k(j) - \epsilon_j} X_k(x)|S_{t(x)}f_j(x)| \leq \sum_{j \geq 0} \sup_{t_m \geq b_2(j)} |S_{t_m}f_j(x)|, \]
and we obtain

\[ \| \sup_{t_m \geq b_2(j)} |S_{t_m}f_j(x)| \|_2 \lesssim \sum_{t_m > b_2(j)/2} \|f_j\|_2^2 = \# \{m; t_m > b_2(j)/2\} \|f_j\|_2^2 \lesssim b_2(j)^{-r} \|f_j\|_2^2 \]

We also have

\[ b_2(j)^{-r} = 2^{rk(j) - \epsilon_2rj}, \]
and

\[ rk(j) - \epsilon_2rj = 2j(s - \epsilon). \]

It follows that

\[ \|S^2_{t(x)}f\|_2 \lesssim \sum_{j \geq 0} 2^{j(s - \epsilon)} \|f_j\|_2 \leq \|f\|_{H_s}. \]
It remains to study $S_{t(x)}^3 f(x)$. We let $[k(j)]$ denote the integral part of $k(j)$. and setting $l = k - [k(j)]$ we obtain

$$|S_{t(x)}^3 f(x)| \leq \sum_{j \geq 0} \sum_{k \geq 0} X_k(x)|S_{t(x)} f_j(x)|$$

$$= \sum_{l=-\infty}^{\infty} j > \max \{(l-1)/\epsilon_1, -1/l_2\} X_{k(j)]+l}(x)|S_{t(x)} f_j(x)|.$$

Using the fact that $X_k = X_k^2$ and applying Cauchy-Schwarz inequality one obtains

$$\left( \sum_{j > \max \{(l-1)/\epsilon_1, -1/l_2\}} X_{k(j)]+l}(x)|S_{t(x)} f_j(x)| \right)^2 \leq \left( \sum_{j > \max \{(l-1)/\epsilon_1, -1/l_2\}} X_{k(j)]+l}(x) \left( \sum_{j > \max \{(l-1)/\epsilon_1, -1/l_2\}} X_{k(j)]+l}(x)|S_{t(x)} f_j(x)|^2 \right) \right) .$$

The first sum on the second line is majorized by

$$C_0 \max_k \# \{ j; |k(j)| = k \} \leq 1,$$

and it follows that

$$\| \sum_{j > \max \{(l-1)/\epsilon_1, -1/l_2\}} X_{k(j)]+l}(x)|S_{t(x)} f_j(x)| \|^2 \leq \sum_{j > \max \{(l-1)/\epsilon_1, -1/l_2\}} \int X_{k(j)]+l}(x)|S_{t(x)} f_j(x)|^2 dx$$

$$\leq \sum_{j > \max \{(l-1)/\epsilon_1, -1/l_2\}} \int_{t_m \in I_{k(j)]+l}} \sup_{t_m \in I_{k(j)]+l}} |S_t f_j(x)|^2 dx.$$

Invoking Minkovski’s inequality we then obtain

$$\|S_{t(x)}^3 f\|_2 \leq \sum_{l=-\infty}^{\infty} \left( \sum_{j > \max \{(l-1)/\epsilon_1, -1/l_2\}} \| \sup_{t_m \in I_{k(j)]+l}} |S_{t_m} f_j\|_2 \right)^{1/2} .$$

Furthermore for $l \geq 0$ we have by Theorem 1.

$$\| \sup_{t_m \in I_{k(j)]+l}} |S_{t_m} f_j\|_2^2 \leq (1 + 2^{-(k(j)]-l+2\phi_j}) \|f_j\|_2^2 \leq (1 + 2^{-(k(j)]-l+2\phi_j}) \|f_j\|_2^2$$

$$\leq (1 + 2^{2\phi_j} 2^{-l}) \|f_j\|_2^2 \leq 2^{l} 2^{2\phi_j} \|f_j\|_2^2 .$$

For $l \leq 0$ we have

$$\| \sup_{t_m \in I_{k(j)]+l}} |S_{t_m} f_j\|_2^2 \leq (\{ m; t_m \in I_{k(j)]+l} \}) \|f_j\|_2^2 \leq 2^{-r-(k(j)]-l)} \|f_j\|_2^2 \leq 2^{r} 2^{2\phi_j} \|f_j\|_2^2 .$$
We conclude that
\[
\|S^3_{(x)} f\|_2 \lesssim \sum_{l=-\infty}^{0} \left( \sum_{j=-l/\epsilon_2}^{0} 2^l 2^{2s_j} \|f_j\|_2^2 \right)^{1/2} + \sum_{l=1}^{\infty} \left( \sum_{j>(l-1)/\epsilon_1} \sum_{j>(l-1)/\epsilon_1} 2^{-l/2} 2^{2s_j} \|f_j\|_2^2 \right)^{1/2} 
\]
\[
\leq \left( \sum_{l=-\infty}^{0} 2^{l/2} + \sum_{l=1}^{\infty} 2^{-l/2} \right)^{1/2} \left( \sum_{j>0} 2^{2s_j} \|f_j\|_2^2 \right)^{1/2} \lesssim \|f\|_{H_s}.
\]
This completes the proof of Lemma 5.

4. Relations between maximal estimates in one variable and maximal estimates in dimension $n \geq 2$.

Next we shall consider the Schrödinger equation on radial or symmetric functions on $\mathbb{R}^n$ and will see how it can be reduced to a one-dimensional problem.

Remark. In this paper we have the Fourier transform $\hat{f}$ of a function on $\mathbb{R}^n$ defined by (1), and then yields $\|\hat{f}\|_{L(\mathbb{R}^n)} = (2\pi)^{n/2} \|f\|_{L(\mathbb{R}^n)}$. We set
\[
\alpha_n = (2\pi)^{n/2}, \text{ for } n \geq 1
\]
in this section.

Let $S^{(k)}_{(x)}$ denote the $k$-dimensional Schrödinger operator (with a given $a > 0$ in its definition) and let
\[
S^{(k)}_{E}(x) = \sup_{t \in E} \|S^{(k)}_{(x)} f(x)\|
\]
where $f$ is a function on $\mathbb{R}^k$ and the supremum is taken over a set $E \subset [0,1]$.

Remark. We may in this section replace the Fourier multiplier functions $\{e^{i|\xi|^a}\}_t$ with any family of radial Fourier multiplier functions $\{\hat{k}_t(|\xi|)\}_t$ satisfying $|\hat{k}_t(|\xi|)| \leq 1$.

We shall prove the following theorem.

Theorem 5. Let $s \geq 0$, let $n \geq 2$ and let $E$ be a given subset of the interval $[0,1]$.

If
\[
\|S^{(1)}_{E}(x) f\|_2 \leq C \|f\|_{H_s}
\]
for all functions $f$ in $\mathcal{S}(\mathbb{R})$,

then
\[
\|S^{(n)}_{E}(x) f\|_2 \leq C_{n,k} \|f\|_{H_s}
\]
for all functions $f$ in $\mathcal{S}(\mathbb{R}^n)$ of the form $f(x) = f_0(x)P(x)$, where $f_0 \in \mathcal{S}(\mathbb{R})$ and is radial and $P$ is a solid spherical harmonic on $\mathbb{R}^n$ of degree $k \geq 0$.

Theorem 5 will follow with some approximation arguments from the following theorems.
When $k_2$, It $f\in L^2(\mathbb{R}^n)$ with surfaces, where the $f$ is a solid spherical harmonic on $\mathbb{R}^n$ of degree $k_2$ and normalised so that $\|P\|_{L^2(\mathbb{S}^{n-1})} = 1$.

Let $f_P$ be the symmetric function on $\mathbb{R}^n$ defined by its Fourier transform
\[
\hat{f}_P(\xi) = P(\xi')f_1(|\xi|)|\xi|^{1/2-n/2} \quad \text{for } \xi = \xi'|\xi| \in \mathbb{R}^n.
\]

Let $\hat{f}_1$ be the inverse Fourier transform of $f_1$ on $\mathbb{R}$. Note that $\alpha_1\|\hat{f}\|_{L^2(\mathbb{R}^n)} = \|f_1\|_{L^2(\mathbb{R}^n)}$.

Assume that $E$ is subset of the interval $[0, 1]$. Then there is a constant $C_{n,k}$ dependent only on $n, k$ but independent of $a > 0, E$ and $f_1$ such that
\[
\alpha_n\|S^{(n)}_E f_P\|_{L^2(\mathbb{R}^n)} \leq \alpha_1\sqrt{2}\|S^{(1)}_E \hat{f}_1\|_{L^2(\mathbb{R})} + C_{n,k}\|f_1\|_{L^2(\mathbb{R}^+)}.
\]

Remark. Most of the results in this section can also be formulated with norms on spheres, where the $L^2(\mathbb{R}^n)$ is replaced by the norms $L^2(\mathbb{S}^{n-1}(r))$ for $r > 0$.

Theorem 6. Let the functions $f_1, P, f_P$ and the set $E$ be as in Theorem 6 and let $Q$ be a solid spherical harmonic on $\mathbb{R}^n$ of degree $k_1$ and normalised so that $\|Q\|_{L^2(\mathbb{S}^{n-1})} = 1$ and define $f_Q$ by its Fourier transform
\[
\hat{f}_Q(\xi) = Q(\xi')f_1(|\xi|)|\xi|^{1/2-n/2} \quad \text{for } \xi = \xi'|\xi| \in \mathbb{R}^n.
\]

Set $v = k + n/2 - 1$ and $v_1 = k_1 + n_1/2 - 1$.

If $v = v_1$ then
\[
\alpha_n\|S^{(n)}_E f_P\|_{L^2(\mathbb{R}^n)} = \alpha_n\|S^{(n)}_E f_Q\|_{L^2(\mathbb{R}^n)}.
\]

If $2v_1 = 2v \mod (4)$ then
\[
\left|\alpha_n\|S^{(n)}_E f_P\|_{L^2(\mathbb{R}^n)} - \alpha_n\|S^{(n)}_E f_Q\|_{L^2(\mathbb{R}^n)}\right| \leq (C_{n,k} + C_{n_1,k_1})\|f_1\|_{L^2(\mathbb{R}^+)}.
\]

Remark. In the special case $n = 1$ we have $k = 0, 1$ and use a special definition of $f_P$.

When $k = 0$ the even function $f_e = f_P$ is defined by
\[
\hat{f}_e(\xi) = \begin{cases} \frac{1}{\sqrt{2}}f_1(\xi) & \text{for } \xi \geq 0, \\ \frac{1}{\sqrt{2}}f_1(-\xi) & \text{for } \xi < 0, \end{cases}
\]

and when $k = 1$ the odd function $f_o = f_P$ is defined by
\[
\hat{f}_o(\xi) = \begin{cases} \frac{1}{\sqrt{2}}f_1(\xi) & \text{for } \xi \geq 0, \\ \frac{1}{\sqrt{2}}f_1(-\xi) & \text{for } \xi < 0. \end{cases}
\]

Estimates in the opposite direction of (8) in Theorem 6 are somewhat more complicated.
We will state the key estimate in this section. First let us define the complex unit vectors

\[ \gamma(v) = e^{-i \frac{\bar{v} + \bar{\bar{v}}}{2}}. \]

We have the following

**Proposition 1.** Let \( f \) be a function in \( L^2(\mathbb{R}) \) whose Fourier transform \( \hat{f} \) is equal to a function in \( \mathcal{F}(\mathbb{R}) \) on \((-\infty, -1]\) and on \([1, \infty)\).

Let \( n \geq 1, k \geq 0, v = n/2 + k - 1 \) and \( \gamma(v) \) as above. Assume that \( \hat{f} \) satisfies the symmetry

\[ \gamma(v) \hat{f}(-r) = \gamma(v) \hat{f}(r), \text{ for all } r > 0. \]

Let \( P \) be a solid spherical harmonic on \( \mathbb{R}^n \) of degree \( k \) normalised so that \( \|P\|_{L^2(\mathbb{R}^n)} = 1. \)

Let \( f_P \) be the symmetric function on \( \mathbb{R}^n \) defined by its Fourier transform

\[ \hat{f}_P(\xi) = P(\xi) \hat{f}(|\xi|^{1/2-n/2}) \text{ for } \xi = \xi'|\xi| \in \mathbb{R}^n. \]

Let \( E \) be any subset of the interval \([0, 1]\) containing 0. Then there is a constant \( C_v \) dependent only on \( v \) but independent of \( a > 0, E \) and \( f \) such that

\[ \alpha_n \|S^{(n)}_E f_P\|_{L^2(\mathbb{R}^n)} = \alpha_1 \left( \int_0^\infty |S^{(1)}_E f(x)|^2 \, dx \right)^{1/2} + R(f, v) \]

where the restterm

\[ |R(f, v)| \leq C_v \|f\|_{L^2(\mathbb{R})}. \]

The restterm \( R(f, v) \) depend on the parameter \( a > 0 \) in the definition of \( S_t \) and the set \( E. \)

Proposition 1 follows directly from Proposition 2 with Corollary 3 and Proposition 3 which are stated and proved at the end of this section.

Using Proposition 1 it is easy to prove Theorems 6 and 7.

**Proofs of Theorem 6 and Theorem 7.** When \( 2v_1 = 2v \) mod \( 4 \) then \( \gamma(v_1) = \pm \gamma v. \) Define the function \( f \) by its Fourier transform

\[ c \hat{f}(\xi) = \left\{ \begin{array}{ll} f_1(\xi), & \xi > 0, \\
(\gamma(v))^2 f_1(-\xi) = (\gamma(v_1))^2 f_1(-\xi), & \xi < 0. \end{array} \right. \quad (9) \]

By Proposition 1

\[ \alpha_n \|S^{(n)}_E f_P\|_{L^2(\mathbb{R}^n)} = \alpha_1 \left( \int_0^\infty |S^{(1)}_E f(x)|^2 \, dx \right)^{1/2} + R(f, v) \]

\[ \alpha_n \|S^{(n)}_E f_P\|_{L^2(\mathbb{R}^{n+1})} = \alpha_1 \left( \int_0^\infty |S^{(1)}_E f(x)|^2 \, dx \right)^{1/2} + R(f, v_1) \]

we conclude that

\[ \left| \alpha_n \|S^{(n)}_E f_P\|_{L^2(\mathbb{R}^n)} - \alpha_n \|S^{(n)}_E f_P\|_{L^2(\mathbb{R}^{n+1})} \right| = |R(f, v) - R(f, v_1)| \]

\[ \leq |R(f, v)| + |R(f, v_1)| \leq (C_v + C_{v_1}) \|f_1\|_{L^2(\mathbb{R}^n)}. \]

In the special case \( v = v_1 \) we get

\[ \alpha_n \|S^{(n)}_E f_P\|_{L^2(\mathbb{R}^n)} = \alpha_n \|S^{(n)}_E f_P\|_{L^2(\mathbb{R}^{n+1})}. \]
This completes the proof of Theorem 7.

For the proof of Theorem 6 we let \( f \) be defined by (3). We will use following property of the Fourier transform on \( L^2(\mathbb{R}^n) \) Let the operator \( g \to g_\cdot \) is defined by \( g_\cdot (x) = g(-x) \). This operator commutes with the Fourier transform and also with the inverse Fourier transform. We have

\[
(g_\cdot \hat{g})_\cdot = \alpha_n \hat{g} \quad \text{and} \quad (g_\cdot \hat{g})_\cdot = (\alpha_n)^{-1} \hat{g}.
\]

From this we get

\[
|S^f(x)|^2 = |S_f^f(x) + \nabla \cdot S^f(x) - e^{x} S^f(x) - \nabla \cdot S^f(x)|^2 \leq 2(|S_f^f(x)|^2 + |S^f(x)|^2) \quad \text{for} \quad x > 0
\]

and hence

\[
|S^f(x)|^2 \leq 2(|S_f^f(x)|^2 + |S^f(x)|^2) \quad \text{for} \quad x > 0.
\]

Integrating over the positive interval we get

\[
\int_0^\infty |S^f(x)|^2 \, dx \leq 2\|S^f\|_{L^2(\mathbb{R})}^2.
\]

From this and Proposition 1 we get the desired estimate in Theorem 6. \( \square \)

Now we consider estimates in the opposite direction of the estimate in Theorem 6. First a lemma which follows directly from Proposition 1

**Lemma 6.** Let \( f \) be a function in \( L^2(\mathbb{R}) \) equal to a function in \( \mathcal{S}(\mathbb{R}) \) on \((-\infty, -1)\) and on \([1, \infty)\). Let \( \hat{f} \) be the Fourier transform of \( f \). Let \( f_P \) be the symmetric function on \( \mathbb{R}^n \) defined by its Fourier transform

\[
\hat{f}_P(\xi) = P(\xi') \hat{f}(\xi)|\xi|^{1/2-n/2} \quad \text{for} \quad \xi = \xi' |\xi| \in \mathbb{R}^n.
\]

If \( \hat{f} \) satisfy the symmetry \( \gamma(v) \hat{f}(-r) = \gamma(v) \hat{f}(r) \) for all \( r > 0 \), then

\[
\alpha_1 \left( \int_0^\infty |S^f(x)|^2 \, dx \right)^{1/2} \leq \alpha_n \|S_E^f\|_{L^2(\mathbb{R})} + C_{n,k} \|f\|_{L^2(\mathbb{R})}^2,
\]

**Proof of Lemma 6.** From Proposition 1 we get

\[
\alpha_1 \left( \int_0^\infty |S^f(x)|^2 \, dx \right)^{1/2} = \alpha_n \|S_E^f\|_{L^2(\mathbb{R})} - R(f, v)
\]

with the restterm satisfying \( |R(f, v)| \leq C_v \|f\|_{L^2(\mathbb{R})} \). This completes the proof of the lemma. \( \square \)

Next we will see how we can combine several estimates like those in Lemma 7. We have

**Lemma 7.** Let \( f \) be a function in \( L^2(\mathbb{R}) \) equal to a function in \( \mathcal{S}(\mathbb{R}) \) on \((-\infty, -1)\) and on \([1, \infty)\). Let \( \hat{f} \) be the Fourier transform of \( f \). Let \( n \geq 1, k \geq 0 \) and \( n_1 \geq 1, k_1 \geq 0 \) and let \( P \) be a solid spherical harmonic on \( \mathbb{R}^n \) of degree \( k \) and \( Q \) be a solid spherical harmonic on \( \mathbb{R}^k \) of degree \( k_1 \) and normalised so that \( \|P\|_{L^2(S^{n-1})} = \|Q\|_{L^2(S^{k_1-1})} = 1 \). Let \( f_P \) be the symmetric function on \( \mathbb{R}^n \) defined by its Fourier transform

\[
\hat{f}_P(\xi) = P(\xi') \left(-\gamma(v_1) \hat{f}(\xi') + \gamma(v_1) \hat{f}(-\xi')\right)|\xi|^{1/2-n/2} \quad \text{for} \quad \xi = \xi' |\xi| \in \mathbb{R}^n,
\]
and let $f_Q$ be the symmetric function on $\mathbb{R}^{n_1}$ defined by its Fourier transform

$$\hat{f}_Q(\xi) = Q(\xi') \left( \overline{\gamma(v)} \hat{f}(|\xi|) - \gamma(v) \hat{f}(-|\xi|) \right)|\xi|^{1/2-n_1/2} \text{ for } \xi = \xi'|\xi| \in \mathbb{R}^{n_1},$$

where $v = k + n/2 - 1$ and $v_1 = k_1 + n_1/2 - 1$ and assume that $2v_1 \neq 2v \text{ mod } (4)$, i.e. $\gamma(v_1) \neq \gamma(v)$.

Let $E$ be any subset of the interval $[0,1)$. Then there is a constant $C_{n,k}$ and a constant $C_{n_1,k_1}$ not dependent of $a > 0$, $E$ and $f$ such that

$$c\alpha_1 \left( \int_0^\infty |S_E^{(1)} f(x)|^2 dx \right)^{1/2} \leq \alpha_n \|S_E^{(n)} f_P\|_{L^2(\mathbb{R}^n)} + \alpha_n \|S_E^{(n)} f_Q\|_{L^2(\mathbb{R}^{n_1})} + (C_{n,k} + C_{n_1,k_1}) \|f\|_{L^2(\mathbb{R})},$$

where $c = 2|\sin \frac{\pi(v-v_1)}{2}| \in \{2, \sqrt{2}\}$.

As a special case of Lemma 6 we have

**Corollary 1.** Assume $f$ satisfy the symmetry

$$\hat{f}(\xi) = \begin{cases} c_1 g(\xi) & \text{for } \xi > 0, \\ c_2 g(-\xi) & \text{for } \xi < 0, \end{cases} \text{ with } (c_1, c_2) \in \mathbb{C}^2, (c_1, c_2) \neq 0,$$

and set $c = 2|\sin \frac{\pi(v-v_1)}{2}| \in \{2, \sqrt{2}\}$, $c'_1 = c_1 \gamma(v_1) - c_2 \gamma(v_1)$ and $c'_2 = c_1 \gamma(v) - c_2 \gamma(v)$.

Then

$$c\alpha_1 \left( \int_0^\infty |S_E^{(1)} f(x)|^2 dx \right)^{1/2} \leq |c'_1| \alpha_n \|S_E^{(n)} f_P\|_{L^2(\mathbb{R}^n)} + |c'_2| \alpha_n \|S_E^{(n)} f_Q\|_{L^2(\mathbb{R}^{n_1})} + (C_{n,k} + C_{n_1,k_1}) \|f\|_{L^2(\mathbb{R})},$$

where

$$\hat{f}_P(\xi) = P(\xi') g(|\xi|)|\xi|^{1/2-n/2} \text{ for } \xi = \xi'|\xi| \in \mathbb{R}^n,$$

and

$$\hat{f}_Q(\xi) = Q(\xi') g(|\xi|)|\xi|^{1/2-n_1/2} \text{ for } \xi = \xi'|x| \in \mathbb{R}^{n_1}.$$

We are mostly interested in the cases when $(c_1, c_2)$ is $(1,1)$, $(1,-1)$, $(1,0)$ or $(0,1)$ which corresponds to even or odd functions or functions with Fourier transform supported on the positive or negative half-axis.

**Proof of Corollary 1.** The functions $f_P$ and $f_Q$ defined in the Lemma 7 have Fourier transforms

$$\hat{f}_P(\xi) = c'_1 P(\xi') g(|\xi|)|\xi|^{1/2-n/2} \text{ for } \xi = \xi'|\xi| \in \mathbb{R}^n,$$

and

$$\hat{f}_Q(\xi) = c'_2 Q(\xi') g(|\xi|)|\xi|^{1/2-n_1/2} \text{ for } \xi = \xi'|x| \in \mathbb{R}^{n_1},$$

for $c'_1 = c_1 \gamma(v_1) - c_2 \gamma(v_1)$ and $c'_2 = c_1 \gamma(v) - c_2 \gamma(v)$. The desired estimates follows directly from Lemma 7. $\square$
Proof of Lemma 7. We will essentially only do elementary calculation in $\mathbb{C}^2$, and use the triangle inequality for $L^2$ norms. Some details are left to the reader.

Set
\[ f_{P,1}(r) = -\gamma(v_1)\hat{f}(r) + \gamma(v_1)\hat{f}(-r) \text{ for } r > 0, \]
and
\[ f_{Q,1}(r) = \gamma(v)\hat{f}(r) - \gamma(v)\hat{f}(-r) \text{ for } r > 0. \]

Then
\[ \| \sup_{t \in E} \tilde{S}_t f_{P,1} \|_{L^2(\mathbb{R}^+)} = \alpha_n \| S^{(n)}_{E} f_P \|_{L^2(\mathbb{R}^n)} \]
and
\[ \| \sup_{t \in E} \tilde{S}_t f_{Q,1} \|_{L^2(\mathbb{R}^+)} = \alpha_{n_1} \| S^{(n_1)}_{E} f_Q \|_{L^2(\mathbb{R}^{n_1})}, \]
where $\tilde{S}_t$ is defined as in (10). Let the functions $f_{P,2}$ and $f_{Q,2}$ on $\mathbb{R}$ be defined by
\[ f_{P,2}(r) = \begin{cases} \gamma(v)f_{P,1}(r) & \text{for } r \geq 0, \\ \gamma(v)f_{P,1}(-r) & \text{for } r < 0, \end{cases} \]
and
\[ f_{Q,2}(r) = \begin{cases} \gamma(v_1)f_{Q,1}(r) & \text{for } r \geq 0, \\ \gamma(v_1)f_{Q,1}(-r) & \text{for } r < 0. \end{cases} \]

We have
\[ f_{P,2} + f_{Q,2} = \begin{cases} (-\gamma(v_1)\gamma(v) + \gamma(v)\gamma(v_1))f(r) + (\gamma(v_1)\gamma(v) - \gamma(v)\gamma(v_1))f(-r) & \text{for } r \geq 0, \\ (-\gamma(v_1)\gamma(v) + \gamma(v)\gamma(v_1))f(r) + (\gamma(v_1)\gamma(v) - \gamma(v)\gamma(v_1))f(-r) & \text{for } r < 0, \end{cases} \]
\[ = (\gamma(v_1)\gamma(v) - \gamma(v)\gamma(v_1))\hat{f}(r) \text{ for } r \in \mathbb{R}. \]

By the assumption $2v_1 \neq 2v \mod (4)$, we have
\[ |\gamma(v_1)\gamma(v) - \gamma(v)\gamma(v_1)| = |e^{-i\frac{\pi}{2} + \frac{\pi}{4}}e^{i\frac{\pi}{2} + \frac{\pi}{4}} - e^{-i\frac{\pi}{2} + \frac{\pi}{4}}e^{i\frac{\pi}{2} + \frac{\pi}{4}}| \\
= |e^{-i\frac{\pi}{2}(v_1-v)} - e^{i\frac{\pi}{2}(v_1-v)}| = |2\sin(\pi(v_1-v)/2)| = c \in \{2, \sqrt{2}\} \]

Lemma 7 now follows from Lemma 6 and Proposition 3. \hfill \square

Next we observe that $S^{(1)}_t f(-x) = S^{(1)}_t f_-(x)$, where $\hat{f}_-(\xi) = \hat{f}(-\xi)$. Thus if $f$ is an even or an odd function on $L^2(\mathbb{R}^n)$ then $S^{(1)}_{E} f$ is even and we obtain
\[ \| S^{(1)}_{E} f \|_{L^2(\mathbb{R})} = \sqrt{2} \left( \int_0^\infty |S^{(1)}_{E} f(x)|^2 dx \right)^{1/2}. \]

We also observe that any function $f$ on $\mathbb{R}$ can be written as a sum of an even and an odd function $f = f_e + f_o$. Then also the Fourier transforms $\hat{f}_e$ and $\hat{f}_o$ are even respective odd and we have $\hat{f} = \hat{f}_e + \hat{f}_o$, and that $\|f\|^2 = \|f_e\|^2 + \|f_o\|^2$.

Using Corollary 1 of Lemma 7 we get the following estimates in different cases.
Theorem 8. Let $f$ be a function in $L^2(\mathbb{R})$ equal to a function in $\mathcal{S}(\mathbb{R})$ on $(-\infty, -1)$ and on $[1, \infty)$. Decompose $f$ into even and odd functions $f = f_e + f_o$ with Fourier transforms $\hat{f}_e$ and $\hat{f}_o$. Let $n \geq 1, k \geq 0$ and $n_1 \geq 1, k_1 \geq 0$ and let $P$ be a solid spherical harmonic on $\mathbb{R}^n$ of degree $k$ and and $Q$ be a solid spherical harmonic on $\mathbb{R}^{n_1}$ of degree $k_1$ and normalised so that $\|P\|_{L^2(S^{n-1})} = \|Q\|_{L^2(S^{n_1-1})} = 1$. Define the symmetric functions

$$
\hat{f}_e, P(\xi) = \hat{f}_e(|\xi|) P(|\xi'|)|\xi|^{1/2-n/2} \text{ for } \xi = \xi' |\xi| \in \mathbb{R}^n,
$$

$$
\hat{f}_e, Q(\xi) = \hat{f}_e(|\xi|) Q(|\xi'|)|\xi|^{1/2-n_1/2} \text{ for } \xi = \xi' |\xi| \in \mathbb{R}^{n_1},
$$

$$
\hat{f}_o, P(\xi) = \hat{f}_o(|\xi|) P(|\xi'|)|\xi|^{1/2-n/2} \text{ for } \xi = \xi' |\xi| \in \mathbb{R}^n,
$$

$$
\hat{f}_o, Q(\xi) = \hat{f}_o(|\xi|) Q(|\xi'|)|\xi|^{1/2-n_1/2} \text{ for } \xi = \xi' |\xi| \in \mathbb{R}^{n_1}.
$$

Let $\gamma(v) = e^{-i\frac{2\pi}{v} + \frac{\pi}{4}}$, $v = k + n/2 - 1$ and $v_1 = k_1 + n_1/2 - 1$ and assume that $2v_1 \neq 2v$ mod (4).

Then we have in different cases.

**Case 1.** Assume $\gamma(v)/\gamma(v) = 1$ and $\gamma(v_1)/\gamma(v_1) = -1$.

Then

$$
\alpha_1 \|S_E^{(n)} f_e\|_{L^2(\mathbb{R})} \leq \alpha_n \sqrt{2} \|S_E^{(n)} f_e, P\|_{L^2(\mathbb{R}^n)} + C_{n,k}' \|f_e\|
$$

$$
\alpha_1 \|S_E^{(n)} f_o\|_{L^2(\mathbb{R})} \leq \alpha_n \sqrt{2} \|S_E^{(n)} f_o, Q\|_{L^2(\mathbb{R}^{n_1})} + C_{n,k}' \|f_o\|
$$

and

$$
\alpha_1 \|S_E^{(n)} f\|_{L^2(\mathbb{R})} \leq \alpha_n \sqrt{2} \|S_E^{(n)} f_e, P\|_{L^2(\mathbb{R}^n)} + \alpha_n \sqrt{2} \|S_E^{(n)} f_o, Q\|_{L^2(\mathbb{R}^{n_1})} + (C_{n,k}' + C_{n_1,k_1}') \|f\|_{L^2(\mathbb{R})}.
$$

**Case 2.** Assume that $\gamma(v)/\gamma(v) = i$ and $\gamma(v_1)/\gamma(v_1) = -i$.

Then

$$
\alpha_1 \|S_E^{(n)} f_e\|_{L^2(\mathbb{R})} \leq \alpha_n \|S_E^{(n)} f_e, P\|_{L^2(\mathbb{R}^n)} + \alpha_n \|S_E^{(n)} f_e, Q\|_{L^2(\mathbb{R}^{n_1})} + (C_{n,k}' + C_{n_1,k_1}') \|f_e\|_{L^2(\mathbb{R})},
$$

$$
\alpha_1 \|S_E^{(n)} f_o\|_{L^2(\mathbb{R})} \leq \alpha_n \|S_E^{(n)} f_o, P\|_{L^2(\mathbb{R}^n)} + \alpha_n \|S_E^{(n)} f_o, Q\|_{L^2(\mathbb{R}^{n_1})} + (C_{n,k}' + C_{n_1,k_1}') \|f_o\|_{L^2(\mathbb{R})},
$$

and

$$
\alpha_1 \|S_E^{(n)} f\|_{L^2(\mathbb{R})} \leq \alpha_n \|S_E^{(n)} f_e, P\|_{L^2(\mathbb{R}^n)} + \alpha_n \|S_E^{(n)} f_e, Q\|_{L^2(\mathbb{R}^{n_1})} + (C_{n,k}' + C_{n_1,k_1}') \|f\|_{L^2(\mathbb{R})}.
$$
Case 3: Assume $\gamma(v)/\gamma(v) = 1$ and $\gamma(v_1)/\gamma(v_1) = i$.

Then

\[
\|S_E^{s(n)} f\|_{L^2(\mathbb{R})} \leq \alpha_n \sqrt{2} \|S_E^{s(n)} f\|_{L^2(\mathbb{R}^n)} + C_{n,k} \|f\|_{L^2(\mathbb{R})},
\]

\[
\|S_E^{s(n)} f\|_{L^2(\mathbb{R})} \leq \alpha_n \sqrt{2} \|S_E^{s(n)} f\|_{L^2(\mathbb{R}^n)} + 2\alpha_n \|S_E^{s(n)} f\|_{L^2(\mathbb{R}^n)}
\]

\[+(C_{n,k} + C_{n_1,k}) \|f\|_{L^2(\mathbb{R})}\]

and

\[
\|S_E^{s(n)} f\|_{L^2(\mathbb{R})} \leq \alpha_n \sqrt{2} \|S_E^{s(n)} f\|_{L^2(\mathbb{R}^n)} + \alpha_n \sqrt{2} \|S_E^{s(n)} f\|_{L^2(\mathbb{R}^n)}
\]

\[+(\alpha_n + 2\alpha_n) \|S_E^{s(n)} f\|_{L^2(\mathbb{R}^n)} + (C_{n,k} + C_{n_1,k}) \|f\|_{L^2(\mathbb{R})}\]

We have similar statements with $\gamma(v)/\gamma(v) = -1$ and $\gamma(v_1)/\gamma(v_1) = -i$.

It remains to prove Proposition 1. For this we will use Proposition 2 and Proposition 3 below.

First we define the operator $\tilde{S}_t$ for $t > 0$ by

\[
\tilde{S}_t g(r) = \int_0^\infty J_m(rs)(rs)^{1/2}g(s)e^{its} ds
\]

(10)

for $g \in L^2(\mathbb{R}^+)$. Here $J_m$ is the Bessel function of order $m$ with integer or half-integer $m > -1$. Then we have

Proposition 2. Let $f_1$ be a function in $L^2(\mathbb{R}^+)$ . Let $P$ be a solid spherical harmonic on $\mathbb{R}^n$ of degree $k$ normalised so that $\|P\|_{L^2(\mathbb{S}^{n-1})} = 1$. Let $\tilde{f}$ be the Fourier transform of $f$ and Let $f_P$ be the symmetric function on $\mathbb{R}^n$ defined by its Fourier transform

\[
\tilde{f}_P(\xi) = f_1(\xi)|P(\xi^\prime)|\xi^{1/2-n/2} \text{ for } \xi = \xi^\prime|\xi| \in \mathbb{R}^n
\]

Let $\tilde{S}_t$ defined as in (10) with $v = n/2 + k - 1$. Then

\[
\tilde{S}_t^{(n)} f_P(x) = c_{k,n} \alpha_n^{-1} |x|^{1/2-n/2} \tilde{S}_t f_1(|x|)P(-x') \text{ for } x = x'|x| \in \mathbb{R}^n,
\]

with $|c_{k,n}| = 1$.

We get the corollaries.

Corollary 2. $\tilde{S}_t$ has norm 1 on $L^2(\mathbb{R}^n)$.

Corollary 3. We have

\[
\tilde{S}_t^{(n)} f_P(x) = \alpha_n^{-1} r^{1/2-n/2} \tilde{S}_t^{(n)} f_1(r)|P(-x')| \text{ for } x = x'|x| \in \mathbb{R}^n
\]

and

\[
\alpha_n \|S_E^{s(n)} f_P\|_{L^2(\mathbb{R}^n)} = \|\tilde{S}_t f_P\|_{L^2(\mathbb{R}^n)}.
\]

Proof of Corollary 2. We have

\[
\|\tilde{S}_t f_1\|_{L^2(\mathbb{R}^n)} = \alpha_n \|S_E^{s(n)} f_P\|_{L^2(\mathbb{R}^n)} = \|e^{it\xi}|\xi|^n \tilde{f}_P\|_{L^2(\mathbb{R}^n)} = \||\tilde{f}_P\|_{L^2(\mathbb{R}^n)} = \|f_1\|_{L^2(\mathbb{R}^n)}.
\]

Proof of Corollary 3. First part follows directly, as we may take out the factor $\alpha_n^{-1} r^{1/2-n/2} |P(-x')|$, from the the supremum on the left hand side. Second part is obtained by integration over $\mathbb{R}^n$. □
We will use the following estimate for the operator $\tilde{S}_t$

**Proposition 3.** Let $\tilde{S}_t$ be defined as in (10) with $v = n/2 + k - 1$ the and let $f$ be a function in $L^2(\mathbb{R})$ Let $f_1$ be a function in $L^2(\mathbb{R}^n)$ and let $f$ be the function in $L^2(\mathbb{R})$ with Fourier transform

$$f(\xi) = \begin{cases} \gamma(v)f_1(\xi) & \text{for } \xi > 0, \\ \overline{\gamma(v)}f_1(-\xi) & \text{for } \xi < 0. \end{cases}$$

Then

$$\tilde{S}_tf_1(x) = \alpha_1S^{(1)}_t f(x) + R_{t,v}f_1(x)$$

where the remainder terms satisfy

$$\left(\int_0^\infty \sup_{t \in E} |R_{t,v}f_1(x)|^2 \, dx\right)^{1/2} \leq C_2 \|f_1\|_{L^2(\mathbb{R}^n)}.$$

It remains to prove the propositions in the section. Proposition 1 follows directly from Proposition 2 with Corollary 3 and Proposition 3

**Proof of Proposition 2.** Assume $f_1(\xi) = \overline{\gamma(v)f(\xi)}$ for $\xi > 0$. We have

$$\alpha_1 \|S^{(n)}_E f\|_{L^2(\mathbb{R}^n)} = \|S^{(1)}_E f_1\|_{L^2(\mathbb{R}^n)} \leq \alpha_1 \|S^{(1)}_E f\|_{L^2(\mathbb{R}^n)} + \|R_{t,v}f_1\|_{L^2(\mathbb{R}^n)}$$

Set $R(f,v) = \int_0^\infty \sup_{t \in E} |R_{t,v}f_1(x)|^2 \, dx$. Then

$$R(f,v) = \left(\int_0^\infty \sup_{t \in E} |R_{t,v}f_1(x)|^2 \, dx\right)^{1/2} \leq C_2 \|f_1\|_{L^2(\mathbb{R}^n)} = \frac{\alpha_1}{\sqrt{2}} C_0 \|f\|_{L^2(\mathbb{R})}.$$

and we obtain the desired estimate. This completes the proof of Proposition 1. □

**Proof of Proposition 2.** Assume $f = f_0P$ as in the above theorem. It follows from Stein and Weiss [5], p. 158 that (when $\tilde{f}(x) = F_0(x)P(x)$ where

$$F_0(x) = c_{n,k} \alpha_1 r^{1-n/2-k} \int_0^\infty f_0(s) J_{n/2+k-1}(rs) s^{n/2+k} \, ds$$

for $r = |x|$. Here $J_m$ denotes Bessel functions of order $m$ and $c_{n,k}$ is a constant with $|c_{n,k}| = 1$. (The one-dimensional cases follow elementary as $J_{1/2}(r) = \sqrt{2} \cos(r)/\sqrt{\pi r}$ and $J_{1/2}(r) = \sqrt{2} \sin(r)/\sqrt{\pi r}$.)

Also

$$S_t f(x) = \alpha_1^{-1} c_{n,k} r^{1-n/2-k} \left(\int_0^\infty J_{n/2+k-1}(rs) F_0(s) e^{its} s^{1/2+k} \, ds\right) P(-x),$$

where $r = |x| > 0$. We set

$$f_1(\xi) = F_0(|\xi|)|\xi|^{-1/2+n/2+k},$$

then $\|f_1\|_{L^2(\mathbb{R}^n)} = \|\tilde{f}_p\|_{L^2(\mathbb{R}^n)}$.

We obtain

$$S_t f_p(x) = \alpha_1^{-1} c_{n,k} r^{1-n/2-k} \left(\int_0^\infty J_{n/2+k-1}(rs) f_1(s)(rs)^{1/2} e^{its} \, ds\right) P(-x/r).$$

We have

$$S_t f_1(x) = \alpha_1^{-1} c_{n,k} r^{1-n/2-k} \tilde{S}_t f_1(r) P(-x/r),$$
where the operator $\tilde{S}_t$ is defined by (10).
This completes the proof of Proposition 2.

For proof of Proposition 3 we need the following

Lemma 8. Let $K(r)$ be a non-negative function on $(0, \infty)$ satisfying

$$\int_0^\infty \frac{K(r)}{\sqrt{r}} \, dr = A < \infty,$$

and let

$$Tf(s) = \int_0^\infty K(rs)f(r) \, dr$$

for $f \in L^2(0, \infty)$.

Then we have

$$\|Tf\|_{L^2(\mathbb{R}_+)} \leq A \|f\|_{L^2(\mathbb{R}_+)}.$$

Proof of Proposition 3. We have the Bessel function

$$J_v(r) = \sqrt{2\pi r} \cos(r - \pi v/2 - \pi/4) + O(r^{-3/2}) \text{ as } r \to \infty.$$

See Stein and Weiss [5], p. 158. It follows that

$$r^{1/2} J_v(r) = \gamma_v e^{ir} + \bar{\gamma}_v e^{-ir} + K_v(r)$$

for $r > 0$, where

$$\gamma_v = \frac{1}{\sqrt{2\pi}} e^{-i(\pi v/2 + \pi/4)} = \alpha_1 \gamma(v) \frac{1}{2\pi}$$

and

$$|K_v(r)| \leq C_v \frac{1}{1 + r}.$$

We get

$$\tilde{S}_t f_1(r) = \gamma(v) \alpha_1 \frac{1}{2\pi} \int_0^\infty e^{irs} e^{its} f_1(s) \, ds + \bar{\gamma}(v) \alpha_1 \frac{1}{2\pi} \int_0^\infty e^{-irs} e^{its} f_1(s) \, ds$$

$$+ \int_0^\infty K_v(rs) e^{its} f_1(s) \, ds$$

$$= \alpha_1 \frac{1}{2\pi} \int_0^\infty e^{irs} e^{it|s|^a} \gamma(v) f_1(s) \, ds + \alpha_1 \frac{1}{2\pi} \int_{-\infty}^0 e^{irs} e^{it|s|^a} \bar{\gamma}(v) f_1(-s) \, ds + R_{t,v} f_1(r)$$

$$= \alpha_1 \frac{1}{2\pi} \int_{-\infty}^\infty e^{irs} e^{it|s|^a} \tilde{f}(s) \, ds = \alpha_1 S_t^{(1)} f(r) + R_{v,t} f_1(r)$$

for $r > 0$.

Let $\tilde{R}_v$ be the sublinear operator on $L^2[0, \infty)$ defined by

$$\tilde{R}_v g(r) = \int_0^\infty |K_v(rs)| |g(s)| \, ds.$$

Then

$$\sup_{t \geq 0} |R_{v,t} f_1(r)| \leq \tilde{R}_v f_1(r).$$
Since \( \int_0^\infty |K_v(r)|r^{-1/2} \, dr \leq C_v \int_0^\infty r^{-1/2}(1+r)^{-1/2} \, dr < \infty \) we obtain by Lemma 8
\[
\left( \int_0^\infty \sup_{t>0} |R_{v,t}f_1(r)|^2 \, dr \right)^{1/2} \leq \left( \int_0^\infty (\tilde{R}_v\hat{f}_1(r))^2 \, dr \right)^{1/2} \leq C_v \|f_1\|_2.
\]
which completes the proof of the Proposition 3.

In this section it remains only to prove Lemma 8.

**Proof of Lemma 8.** We have
\[
Tf(s) = \int_0^\infty K(u)f\left(\frac{u}{s}\right)\frac{1}{s} \, du, \quad s > 0,
\]
and we set
\[
h(s) = f\left(\frac{1}{s}\right)\frac{1}{s} \quad \text{and} \quad h_r(s) = h\left(\frac{s}{r}\right)\frac{1}{\sqrt{r}} \quad \text{for } r > 0 \text{ and } s > 0.
\]
It then follows that
\[
f\left(\frac{T}{s}\right)\frac{1}{s} = f\left(\frac{1}{s/r}\right)\frac{1}{s/r} = h\left(\frac{s}{r}\right)\frac{1}{\sqrt{r}} = h_r(s)\frac{1}{\sqrt{r}}.
\]
We have
\[
Tf(s) = \int_0^\infty K(r)h_r(s)\frac{1}{\sqrt{r}} \, dr,
\]
and an application of Minkovski’s inequality gives
\[
\|Tf\|_2 \leq \int_0^\infty K(r)\frac{1}{\sqrt{r}} \|h_r\|_2 \, dr.
\]
We observe that
\[
\|h_r\|_{L^2(\mathbb{R}^+)} = \|h\|_2 = \|f\|_{L^2(\mathbb{R}^+)},
\]
and get
\[
\|Tf\|_{L^2(\mathbb{R}^+)} \leq \int_0^\infty K(r)\frac{1}{\sqrt{r}} \|f\|_{L^2(\mathbb{R}^+)} = A\|f\|_{L^2(\mathbb{R}^+)},
\]
This completes the proof of Lemma 8. \(\square\)

5. A COUNTER-EXAMPLE

We shall give a counter-example in dimension \(n \geq 2\).

**Theorem 9.** Assume that \( (t_k)_{k=1}^\infty \) is decreasing and \( (t_k - t_{k+1})_{k=1}^\infty \) is decreasing and lim \( t_k \to 0 \). Assume \( a > 0, a \neq 1, \) and \( 0 < s < a/4, \) and \( n \geq 2, \) Set \( r(s) = 2s/(a - 4s) \) and assume that \( (t_k)_{k=1}^\infty \notin \mathcal{D}(s,\infty) \).

Then there is no estimate
\[
\|\sup_k |S_{t_k}f|\|_2 \lesssim \|f\|_{H_s}
\]
for all radial function \( f \in \mathcal{S}(\mathbb{R}^n) \).
Proof. In the case \( n = 1 \) this theorem is proved in Dimou and Seeger \([1]\) (with the exception that their counter-example is not radial) and we shall modify their proof. Assuming that \( (t_k)_{k=1}^\infty \notin \ell^n(\infty) \) Dimou and Seeger first construct sequences \((b_j)_{j=1}^\infty\) and \((M_j)_{j=1}^\infty\) of positive numbers such that \( \lim_{j \to \infty} b_j = 0 \) and \( \lim_{j \to \infty} M_j = \infty \). Taking \( \epsilon < 10^{-1}(a + 2)^{-1} \) they then set
\[
\lambda_j = M_j^2 b_j^{\frac{1}{a-4s}} \quad \text{and} \quad \rho_j = \epsilon b_j^{-1/2} \lambda_j^{1-a/2} = \epsilon M_j^{\frac{2-a}{a-4s}} b_j^{\frac{1-2s}{a-4s}}
\]
for \( j = 1, 2, 3, \ldots \). We shall consider these numbers for \( j \) large and observe that \( \rho_j / \lambda_j = \epsilon M_j^{1-2a/4s} \leq \epsilon \).

In \([\] the function
\[
\Phi_{\lambda, \rho}(\xi, x, t) = x(\rho \xi - \lambda) + t(\lambda - \rho \xi)^{a, n}, |\xi| \leq 1/2, x \in \mathbb{R}, t > 0,
\]
is studied and it is proved that for \( x \in I_j = [0, a \lambda_j^{a-1} b_j/2] \) there exists \( t_k(x, j) \) such that
\[
\max_{|\xi| \leq 1/2} |e^{i \Phi_{\lambda, \rho}(\xi, x, t_k(x, j))} - 1| \leq 1/2. \tag{11}
\]
We shall use the inequality \((11)\) in our proof and shall also use that
\[
\rho_j \lambda_j^{a-1} b_j \to \infty \quad \text{as} \quad j \to \infty \quad \tag{12}
\]
To prove \((12)\) observe that
\[
\rho_j \lambda_j^{a-1} b_j = \epsilon M_j^{\frac{2-a}{a-4s}} b_j^{\frac{1-2s}{a-4s}} M_j^{\frac{2(a-1)}{a}} b_j^{\frac{a-1}{a-4s}} b_j = \epsilon M_j^{\frac{2-a}{a-4s}} b_j^{\frac{1-2s}{a-4s}} = \epsilon M_j b_j^{-\frac{2a}{4s}}
\]
which implies \((12)\).

Then set \( J_j = [a \lambda_j^{a-1} b_j/4, a \lambda_j^{a-1} b_j/2] \) and let \( C_1 \) be a large constant. It follows from \((12)\) that \( \lambda_j^{a-1} b_j \to \infty \) as \( j \to \infty \) and hence
\[
2C_1 \leq a \lambda_j \lambda_j^{a-1} b_j / 4
\]
and
\[
\frac{2C_1}{\lambda_j} \leq a \lambda_j^{a-1} b_j / 4
\]
for large \( j \). We conclude that
\[
|x| \in J_j \text{ implies } \lambda_j |x| \geq 2C_1. \tag{13}
\]
Now let \( \sigma \) denote the surface measure on the unit sphere in \( \mathbb{R}^n \). We have
\[
\hat{\sigma}(y) = c_1 \frac{e^{i|y|}}{|y|^{n/2-1/2}} + c_2 \frac{e^{-i|y|}}{|y|^{n/2-1/2}} + R(y), \tag{14}
\]
where
\[
|R(y)| \leq \frac{1}{|y|^{n/2+1/2}} \leq \frac{\delta}{|y|^{n-2-1/2}} \text{ for } |y| \geq C_1 \tag{15}
\]
and \( \delta \) is small. (See Stein [1], p. 347).
Then assume that \( g \in C_0^\infty(\mathbb{R}) \), supp \( g \subset [-1/2, 1/2] \), \( g \geq 0 \), \( \int g \, dx = 1 \), and \( g \) even. We define a function \( f \in \mathcal{S}(\mathbb{R}^n) \) by setting
\[
\hat{f}(\xi) = \frac{1}{\rho} g \left( \frac{\lambda - \lambda}{\rho} \right) \quad \text{for} \quad \xi \in \mathbb{R}^n.
\]
Here \( \lambda = \lambda_j \), \( \rho = \rho_j \) and \( f = f_j \).
It is easy to see that \( \hat{f}(\xi) \neq 0 \) implies \( \lambda - \rho/2 \leq |\xi| \leq \lambda + \rho/2 \). We also have
\[
\int |\hat{f}(\xi)|^2 \lambda^{2s} \, d\xi \lesssim \int \rho^{-2} \lambda^{2s} r^{n-1} \, dr \lesssim \rho^{-1} \lambda^{2s+n-1}
\]
and
\[
\|f\|_{H_s} \lesssim \rho^{-1/2} \lambda^{s+n/2-1/2}
\]
Using polar coordinates we have
\[
S_t f(x) = c \int \hat{f}(\xi) e^{it|\xi|^2} \, d\xi = c \int e^{it|\xi|^2} \hat{f}(\xi) \, d\xi
\]
\[
= c \int_0^\infty \frac{1}{\rho} g \left( \frac{r - \lambda}{\rho} \right) e^{itr^2} \left( \int \left| e^{itr} \hat{\sigma}(\xi') \right| r^{n-1} \, dr \right) \, d\xi
\]
\[
= c \int_0^\infty \frac{1}{\rho} g \left( \frac{r - \lambda}{\rho} \right) e^{itr^2} \hat{\sigma}(r)x r^{n-1} \, dr
\]
where by (13)
\[
\hat{\sigma}(r)x = c_1 \frac{e^{ir|x|}}{(r|x|)^{n/2-1/2}} + c_2 \frac{e^{-ir|x|}}{(r|x|)^{n/2-1/2}} + R(rx).
\]
We assume \( |x| \in J \) and (13) gives \( \lambda |x| \geq 2C_1 \) and \( r |x| \geq C_1 \) in the above integral. Hence by (15)
\[
|R(rx)| \leq \delta(r|x|)^{-n/2+1/2}.
\]
It follows that
\[
S_t f(x) = c_1 \int_0^\infty \frac{1}{\rho} g \left( \frac{r - \lambda}{\rho} \right) e^{itr^2} \frac{e^{ir|x|}}{(r|x|)^{n/2-1/2}} r^{n-1} \, dr
\]
\[
+ c_2 \int_0^\infty \frac{1}{\rho} g \left( \frac{r - \lambda}{\rho} \right) e^{itr^2} \frac{e^{-ir|x|}}{(r|x|)^{n/2-1/2}} r^{n-1} \, dr + c_3 \int_0^\infty \frac{1}{\rho} g \left( \frac{r - \lambda}{\rho} \right) e^{itr^2} R(rx) r^{n-1} \, dr
\]
Setting \( \xi = (r - \lambda)/\rho \) so that \( r = \lambda + \rho \xi \) we obtain
\[
S_t f(x) = c_1 S^1_t f(x) + c_2 S^2_t f(x) + S^3_t f(x)
\]
where
\[
S^1_t f(x) = \left( c \int g(\xi) e^{it(\lambda + \rho \xi)^n + |x| (\lambda + \rho \xi)} (\lambda + \rho \xi)^{n/2-1/2} \, d\xi \right) |x|^{1/2-n/2},
\]
\[ S^2_t f(x) = \left( c \int g(\xi) e^{i[(\lambda + \rho \xi)^n - |x|(\lambda + \rho \xi)]} (\lambda + \rho \xi)^{n/2 - 1/2} \, d\xi \right) |x|^{1/2 - n/2}, \]

and

\[ |S^2_t f(x)| \lesssim \delta \int (\lambda + \rho \xi)^{n/2 - 1/2} g(\xi) \, d\xi \, |x|^{1/2 - n/2} \leq C\delta \lambda^{n/2 - 1/2} |x|^{1/2 - n/2} \]

In \( S^2_t f(x) \) and \( S^2_j f(x) \) we can replace \( \xi \) by \(-\xi\). We use that \( g \) is even and get the phase functions

\[ \Phi_1(\xi) = |x|(\lambda - \rho \xi) + t(\lambda - \rho \xi)^a \]

and

\[ \Phi_2(\xi) = -|x|(\lambda - \rho \xi) + t(\lambda - \rho \xi)^a \]

and replace \((\lambda + \rho \xi)^{n/2 + 1/2}\) by \((\lambda - \rho \xi)^{n/2 - 1/2}\).

We have

\[ \Phi_2(\xi) = |x|((\rho \xi - \lambda)) + t(\lambda - \rho \xi)^a = \Phi_{\lambda, \rho}(\xi, |x|, t) = \Phi(\xi) \]

and we also have

\[ S^2_t f(x) = \left( \int e^{i\Phi(\xi) \Lambda(\xi) g(\xi) \, d\xi} \right) |x|^{1/2 - n/2} \]

where \( \Lambda(\xi) = c(\lambda - \rho \xi)^{n/2 - 1/2} \). Choosing \( t = t_k(|x|, j) \) we obtain

\[ |x|^{n/2 - 1/2} |S^2_{tk(|x|, j)} f(x)| \geq \int g \Lambda \, d\xi - \int |e^{i\Phi} - 1| g \Lambda \, d\xi \]

\[ \geq \int g \Lambda \, d\xi - \max_{|\xi| \leq 1/2} |e^{i\Phi(\xi)} - 1| \int g \Lambda \, d\xi \geq \frac{1}{2} \int g \Lambda \, d\xi \]

for \(|x| \in J_j\) since \( \max_{|\xi| \leq 1/2} |e^{i\Phi(\xi)} - 1| \leq \frac{1}{2} \) according to inequality (11).

It follows that

\[ \sup_k |S^2_{tk} f(x)| \geq \frac{1}{2} \int g \Lambda \, d\xi |x|^{1/2 - n/2} \geq c\lambda^{n/2 - 1/2} |x|^{1/2 - n/2} \]

for \(|x| \in J_j\). It remains to study \( S^1_t f(x) \). We set \( h = g\Lambda \) an then have \( h \lesssim \lambda^{n/2 - 1/2} \) and \( |h'| \lesssim (\lambda + \rho)^{n/2 - 3/2} \leq 2\lambda^{n/2 - 1/2} \). Integrating by parts we obtain

\[ |x|^{n/2 - 1/2} S^1_t f(x) = \int e^{i\Phi_1} h \, d\xi = \int e^{i\Phi_1} i\Phi' h \, d\xi \]

\[ = - \int e^{i\Phi_1} \left( \frac{1}{\Phi'} h' + \frac{\Phi''}{2\Phi'} h \right) d\xi \]

We have

\[ \Phi_1' = -|x|\rho - \rho\alpha t(\lambda - \rho \xi)^{a-1} \]

and

\[ \Phi_1'' = \rho(a - 1)\rho^2(\lambda - \rho \xi)^{a-2} \]

and it follows that \( |\Phi_1'| \geq \rho|x| \) and \( |\Phi_1'| \geq \rho\alpha(\lambda - \rho \xi)^{a-1} \). We have

\[ \frac{1}{|\Phi_1'|} \leq \frac{1}{\rho|x|} \]
and
\[ \frac{|\Phi''|}{|\Phi'|^2} = \frac{1}{|\Phi'| |\Phi''|} \leq \frac{1}{\rho|x|} \frac{a|a-1|t_{\rho}^2(\lambda - \rho \xi)^{a-2}}{a(t(\lambda - \rho \xi)^{a-1})} \lesssim \frac{1}{\rho|x|} \frac{\rho}{\lambda - \rho \xi} \lesssim \frac{1}{\rho|x|}. \]

It follows that
\[ |x|^{n/2-1/2} |S^1 f(x)| \lesssim \frac{1}{\rho|x|} \int (|h| + |h'|) d\xi \lesssim \frac{1}{\rho|x|} \lambda^{n/2-1/2}. \]

and if \( |x| \in J_j \) we get
\[ |S^1 f(x)| \lesssim |x|^{1/2-n/2} \frac{1}{\rho_j \lambda_j^{-1}} b_j \lambda_j^{n/2-1/2} \lesssim \delta \lambda_j^{n/2-1/2} |x|^{1/2-n/2}, \]

where we have used (12). Hence we have
\[ S^1 f(x) = \sup_k |S_k f(x)| \geq c \lambda_j^{n/2-1/2} |x|^{1/2-n/2} \]

for \( |x| \in J_j \).

The theorem will follow if we show that with \( f = f_j \) we have
\[ \frac{\|S^1 f_j\|_2}{\|f_j\|_{H^s}} \to \infty \text{ as } j \to \infty. \]

We have
\[ \int_{|x| \in J_j} |S^1 f(x)|^2 dx \geq \int_{|x| \in J_j} |S^1 f(x)|^2 dx \gtrsim \int_{|x| \in J_j} \lambda_j^{n-1} |x|^{1-n} dx \geq |I_j| \lambda_j^{n-1} \]

and
\[ \|f\|_{H^s}^2 \lesssim \rho^{-1} \lambda_j^{2s+n-1}. \]

With \( f = f_j \) we get
\[ \left( \frac{\|S^1 f_j\|_2}{\|f_j\|_{H^s}} \right)^2 \gtrsim \frac{\lambda_j^{n-1} |I_j|}{\rho_j^{-1} \lambda_j^{2s+n-1}} = \rho_j \lambda_j^{-2s} |I_j|. \]

We have \( |I_j| = a \lambda_j^{-1} b_j / 2 \) and obtain
\[ \left( \frac{\|S^1 f_j\|_2}{\|f_j\|_{H^s}} \right)^2 \gtrsim \rho_j \lambda_j^{-2s} \lambda_j^{-1} b_j = \rho_j \lambda_j^{-1-2s} b_j \]
\[ = \epsilon M_j^{\frac{2-a}{a}} b_j \left( M_j^{\frac{2}{a}} b_j - \frac{1}{a-4} \right) \]
\[ = \epsilon M_j^{\frac{2-a}{a}} b_j \left( a+1+2s(a+4) \right) \]
\[ = \epsilon M_j^{\frac{2-a}{a}}. \]

Since \( a - 4s > 0 \) and \( M_j \to \infty \) as \( j \to \infty \) we conclude that
\[ \frac{\|S^1 f_j\|_2}{\|f_j\|_{H^s}} \to \infty \text{ as } j \to \infty, \]

This completes the proof of theorem. \( \square \)
Now let \( a > 0, a \neq 1, 0 < s < a/4 \) and \( r = 2s/(a - 4s) \). Also let \((t_m)_{m}^{\infty}\) satisfy (2) and let \((t_m - t_{m+1})_{m}^{\infty}\) be decreasing.

It is proved in Dimou and Seeger [1] that in the case \( n = 1 \) one has

\[
\| \sup |S_{t_m}f| \|_2 \lesssim \| f \|_{H^s}, \quad f \in \mathcal{S}(\mathbb{R})
\]

if \((t_m)_{m}^{\infty} \in l^{r,\infty}\).

It then follows from Theorems 5 and 9 that in the case \( n \geq 2 \) one has

\[
\| \sup |S_{t_m}f| \|_2 \lesssim \| f \|_{H^s}
\]

for all radial functions \( f \) in \( \mathcal{S}(\mathbb{R}^n) \), if and only if \((t_m)_{m}^{\infty} \in l^{r,\infty}\).

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Department of Mathematics, KTH Royal Institute of Technology, Stockholm, Sweden
E-mail addresses: persj@kth.se, jostromb@kth.se