Subspace Embedding and Linear Regression with Orlicz Norm

Alexandr Andoni 1 Chengyu Lin 1 Ying Sheng 1 Peilin Zhong 1 Ruiqi Zhong 1

Abstract
We consider a generalization of the classic linear regression problem to the case when the loss is an Orlicz norm. An Orlicz norm is parameterized by a non-negative convex function \( G : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( G(0) = 0 \): the Orlicz norm of a vector \( x \in \mathbb{R}^n \) is defined as
\[
\|x\|_G = \inf \{ \alpha > 0 | \sum_{i=1}^n G(|x_i|/\alpha) \leq 1 \}.
\]
We consider the cases where the function \( G(\cdot) \) grows subquadratically. Our main result is based on a new oblivious embedding which embeds the column space of a given matrix \( A \in \mathbb{R}^{n \times d} \) with Orlicz norm into a lower dimensional space with \( \ell_2 \) norm. Specifically, we show how to efficiently find an embedding matrix \( S \in \mathbb{R}^{n \times m} \), \( m < n \) such that \( \forall x \in \mathbb{R}^d \), \( \Omega(1/(d \log n)) \cdot \|Ax\|_G \leq \|SAx\|_2 \leq O(d \log^2 n) \cdot \|Ax\|_G \). By applying this subspace embedding technique, we show an approximation algorithm for the regression problem \( \min_{x \in \mathbb{R}^d} \|Ax-b\|_G \), up to a \( O(d \log^2 n) \) factor. As a further application of our techniques, we show how to also use them to improve on the algorithm for the \( \ell_p \) low rank matrix approximation problem for \( 1 \leq p < 2 \).

1. Introduction
Numerical linear algebra problems play a significant role in machine learning, data mining, and statistics. One of the most important such problems is the regression problem, see some recent advancements in, e.g., (Zhong et al. 2016; Bhatia et al. 2015; Jain & Tewari, 2013; Liu et al. 2014; Dhillon et al. 2013). In a linear regression problem, given a data matrix \( A \in \mathbb{R}^{n \times d} \) with \( n \) data points \( A^1, A^2, \cdots, A^n \) in \( \mathbb{R}^d \) and the response vector \( b \in \mathbb{R}^n \), the goal is to find a set of coefficients \( x^* \in \mathbb{R}^d \) such that
\[
x^* = \arg \min_{x \in \mathbb{R}^d} l(Ax - b),
\]
where \( l : \mathbb{R}^n \to \mathbb{R}_+ \) is the loss function. When \( l(y) = \|y\|^2_2 = \sum_{i=1}^n y_i^2 \), then the problem is the classic least square regression problem (\( \ell_2 \) regression). While there has been extensive research on efficient algorithms for solving \( \ell_2 \) regression, it is not always a suitable loss function to use. In many settings, alternative choices for a loss function lead to qualitatively better solutions \( x^* \). For example, a popular such alternative is the least absolute deviation (\( \ell_1 \)) regression — with \( l(y) = \|y\|_1 = \sum_{i=1}^n |y_i| \) — which leads to solutions that are more robust than those of \( \ell_2 \) regression (see (Wikipedia, Gorard, 2005). In a nutshell, the \( \ell_2 \) regression is suitable when the data contains Gaussian noise, whereas \( \ell_1 \) — when the noise is Laplacian or sparse.

A further popular class of loss functions \( l(\cdot) \) arises from \( M \)-estimators, defined as \( l(y) = \sum_{i=1}^n M(y_i) \) where \( M(\cdot) \) is an \( M \)-estimator function (see (Zhang, 1997) for a list of \( M \)-estimators). The benefit of (some) \( M \)-estimators is that they enjoy advantages of both \( \ell_1 \) and \( \ell_2 \) regression. For example, when \( M(\cdot) \) is the Huber function (Huber et al. 1964), then the regression looks like \( \ell_2 \) regression when \( y_i \) is small, and looks like \( \ell_1 \) regression otherwise. However, these loss functions come with a downside: they depend on the scale, and rescaling the data may give a completely different solution!

Our contributions. We introduce a generic algorithmic technique for solving regression for an entire class of loss functions that includes the aforementioned examples, and in particular, a “scale-invariant” version of \( M \)-estimators. Specifically, our class consists of loss functions \( l(y) \) that are Orlicz norms, defined as follows: given a non-negative convex function \( G : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( G(0) = 0 \), for \( x \in \mathbb{R}^n \), we can define \( \|x\|_G \) to be an Orlicz norm with respect to \( G(\cdot) \): 
\[
\|x\|_G \triangleq \inf \{ \alpha > 0 | \sum_{i=1}^n G(|x_i|/\alpha) \leq 1 \}.
\]
Note that \( \ell_p \) norm, for \( p \in [1, \infty) \), is a special case of Orlicz norm with \( G(x) = x^p \). Another important example is the following “scale-free” version of \( M \)-estimator. Taking \( f(\cdot) \) to be a Huber function, i.e.
\[
f(x) = \begin{cases} 
x^2/2 & |x| \leq \delta \\
\delta(|x| - \delta/2) & \text{otherwise}
\end{cases}
\]
for some constant \( \delta \), we take \( G(x) = f(f^{-1}(1)x) \). Then the norm \( \|x\|_G \) looks like \( \ell_2 \) norm when \( x \) is flat, and looks like \( \ell_1 \) norm when \( x \) is sparse. Figure 1 shows the unit norm ball of this kind of Orlicz norm.
Our main result is a generic algorithm for solving any regression problem Eqn. (1) with any loss function that is a “nice” Orlicz norm; see Section 2 for a formal definition of “nice”, and think of it as subquadratic for now.

Our main result employs the concept of subspace embeddings, which is a powerful tool for solving numerical linear algebra problems, and as such has many applications beyond regression. We say that a subspace embedding matrix $S \in \mathbb{R}^{m \times n}$ embeds the column space of $A \in \mathbb{R}^{n \times d}$ with $u$-norm into a subspace with $v$-norm, if $\forall x \in \mathbb{R}^d$, we have $\|Ax\|_u / \alpha \leq \|SAx\|_v \leq \beta \|Ax\|_u$ where $\alpha \beta$ is called distortion (approximation). A long line of work studied $\ell_2$ regression problem based on $\ell_2$ subspace embedding techniques; see, e.g., (Clarkson & Woodruff, 2009; Clarkson et al., 2013; Mølgaard et al., 2019). Furthermore, there are works on $\ell_p$ regression problem based on $\ell_p$ subspace embedding techniques (see, e.g., (Sohler & Woodruff, 2011; Mølgaard et al., 2019), and similarly for $M$-estimators (Clarkson & Woodruff, 2015).

Our overall results are composed of four parts:

1. We develop the first subspace embedding method for all “nice” Orlicz norms. The embedding obtains a distortion factor polynomial in $d$, which was recently shown necessary (Wang & Woodruff, 2018).
2. Using the above subspace embedding, we obtain the first approximation algorithm for solving the linear regression problem with any “nice” Orlicz norm loss.
3. As a further illustration of the power of the subspace embedding method, we employ it towards improving on the best known result for another problem: $\ell_p$ low rank approximation for $1 \leq p < 2$ from (Song et al., 2017), which is the “$\ell_p$-version of PCA”.
4. Finally, we complement our theoretical results with experimental evaluation of our algorithms. Our experiments reveal that that the solution of regression under the Orlicz norm induced by Huber loss is much better than the solution given by regression under $\ell_1$ or $\ell_2$ norms, under natural noise distributions in practice. We also perform experiments for Orlicz regression with different Orlicz functions $G$ and show their behavior under different noise settings, thus exhibiting the flexibility of our framework.

To the best of our knowledge, our algorithms are the first low distortion embedding and regression algorithms for general Orlicz norm. For the problem of low rank approximation under $\ell_p$ norm, $p \in (1, 2)$, our algorithms achieve simultaneously the best approximation and the best running time. In contrast, all the previous algorithms achieve either the best approximation, or the best running time, but not both at the same time.

Our algorithms for subspace embedding and regression are simple, and in particular are not iterative. In particular, for the subspace embedding, the embedding matrix $S$ is generated independently of the data. In the regression problem, we multiply the input with the embedding matrix, and thus reduce to the $\ell_2$ regression problem, for which we can use any of the known algorithm.

Technical discussion. Next we highlight some of our techniques used to obtain the theoretical results.

Subspace embedding. Our starting point is a technique introduced in (Andoni et al., 2017) for the Orlicz norms, which can be seen as an embedding that has guarantees for a fixed vector only. In contrast, our main challenge here is to obtain an embedding for all vectors $x \in \mathbb{R}^d$ in a certain $d$-dimensional subspace. Consider a random diagonal matrix $D \in \mathbb{R}^{n \times n}$ with each diagonal entry is a “generalized exponential” random variable, i.e., drawn from a distribution with cumulative distribution function $1 - e^{-G(x)}$. Then, for a fixed vector $x \in \mathbb{R}^d$, (Andoni et al., 2017) show that $\|D^{-1}Ax\|_\infty$ is not too small with high probability. We can combine this statement together with a net argument and the dilation bound on $\|D^{-1}Ax\|_G$, to argue that $\forall x \in \mathbb{R}^d$, $\|D^{-1}Ax\|_\infty$ is not too small.

The other direction is more challenging — to show that for a given matrix $A \in \mathbb{R}^{n \times d}$, and any fixed vector $x \in \mathbb{R}^d$, $\|D^{-1}Ax\|_G$ cannot be too large. Once we show this “dilation bound”, we combine it with the well-conditioned basis argument (similar to (Dasgupta et al., 2009)), and prove that $\forall x \in \mathbb{R}^d$, $\|D^{-1}Ax\|_G$ cannot be too large. Overall, we have that $\forall x \in \mathbb{R}^d$, $\|D^{-1}Ax\|_G \leq O(d^2 \log n) \cdot \|Ax\|_G$, and $\|D^{-1}Ax\|_\infty \geq \Omega(1/(d \log n)) \cdot \|Ax\|_G$. Since $\ell_2$ norm is sandwiched by $\| \cdot \|_G$ and $\ell_\infty$ norm, we have that $\forall x \in \mathbb{R}^d$, $\Omega(1/(d \log n)) \cdot \|Ax\|_G \leq \|D^{-1}Ax\|_2 \leq O(d^2 \log n) \cdot \|Ax\|_G$. Then, the remaining part is to use standard techniques (Woodruff & Zhang, 2013; Woodruff, 2014) to perform the $\ell_2$ subspace embedding for the column space of $D^{-1}A$. See Theorem 16 for details.

The actual proof of the dilation bound is the most technically intricate result. Traditionally, since the $p^b$ power of the $\ell_p$ norm is the sum of the $p^b$ power of all the entries, it is easy to bound the expectation by using linearity of the expectation. However it is impossible to apply this analysis to Orlicz norm directly since Orlicz norm is not an “entrywise” norm. Instead, we exploit a key observation that the Orlicz norm of vectors which are on the unit ball can be

Figure 1. Unit norm balls of Orlicz norm induced by normalized Huber functions with different $\delta$. 
Subspace Embedding and Linear Regression with Orlicz Norm

2. Notations and preliminaries

In this paper, we denote \( \mathbb{R}_+ \) to be the set of nonnegative reals. Define \( [n] = \{1, 2, \ldots, n\} \). Given a matrix \( A \in \mathbb{R}^{n \times d} \), \( \forall i \in [n], j \in [d], A^i \) and \( A_j \) denotes the \( i \)-th row and the \( j \)-th column of \( A \) respectively. \( \text{nnz}(A) \) denotes the number of nonzero entries of \( A \). The column space of \( A \in \mathbb{R}^d \) is \( \{y \mid \exists x \in \mathbb{R}^d, y = Ax\} \). \( \forall i \neq j, \|A_i\|_p \overset{\text{def}}{=} (\sum |A_{ij}|^p)^{1/p} \), i.e. entrywise \( p \)-norm. \( \|A\|_p \) defines the Frobenius norm of \( A \), i.e. \((\sum A_{ij}^2)^{1/2} \). \( A^t \) denotes the Moore-Penrose pseudoinverse of \( A \). Given an invertible function \( f(\cdot) \), let \( f^{-1}(\cdot) \) be the inverse function of \( f(\cdot) \). If \( f(\cdot) \) is not invertible in \((\infty, +\infty)\) but it is invertible in \([0, +\infty)\), then we denote \( f^{-1}(\cdot) \) to be the inverse function of \( f(\cdot) \) in domain \([0, +\infty)\). \( \inf \) and \( \sup \) denote the infimum and supremum respectively. \( f'(x), f''(x), f(\cdot) \) denote the derivative, right derivative and left derivative of \( f(x) \) respectively. Similarly, define \( f''(x) \) for the second derivatives, and we define \( f''(x) = \lim_{h \to 0^+} \frac{f'(x + h) - f'(x)}{h} \).

In the following, we give the definition of Orlicz norm.

Definition 1 (Orlicz norm). For any nonzero monotone nondecreasing convex function \( G : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( G(0) = 0 \). Define Orlicz norm \( \| \cdot \|_G \) as: \( \forall n \in \mathbb{Z}, n \geq 1, x \in \mathbb{R}^n, \|x\|_G = \inf \{\alpha > 0 \mid \sum_{i=1}^{n} G(|x_i|/\alpha) \leq 1\} \). For any function \( G_1(\cdot) \) which is valid to define an Orlicz norm, we can always “simplify/normalize” the function to get another function \( G_2(\cdot) \) such that computing \( \|\cdot\|_{G_1} \) is equivalent to computing \( \|\cdot\|_{G_2} \).

Fact 2. Given a function \( G_1 : \mathbb{R}_+ \to \mathbb{R}_+ \) which can induce an Orlicz norm \( \| \cdot \|_{G_1} \) (Definition 1), define function \( G_2 : \mathbb{R}_+ \to \mathbb{R}_+ \) as the following: \( G_2(x) = \left\{ \begin{array}{ll} G_1^{-1}(1)x & 0 \leq x \leq 1 \\ s & x > 1 \end{array} \right. \) where \( s = \sup \{ (G_2(y) - G_2(x))/(y-x) \mid 0 \leq x \leq y \leq 1 \} \). Then \( \| \cdot \|_{G_2} \) is a valid Orlicz norm. Furthermore, \( \forall n \in \mathbb{Z}, n \geq 1, x \in \mathbb{R}^n \), we have \( \|x\|_{G_2} = \|x\|_{G_1}^{(1)}(1) \).

Thus, without loss of generality, in this paper we consider the Orlicz norm induced by function \( G(\cdot) \) which satisfies \( G(1) = 1 \), and \( G(x) \) is a linear function for \( x > 1 \). In addition, we also require that \( G(x) \) grows no faster than quadratically in \( x \). Thus, we define the property \( P \) of a function \( G : \mathbb{R} \to \mathbb{R}_+ \) as the following: 1) \( G \) is a nonzero monotone nondecreasing convex function in \((0, \infty)\); 2) \( G(0) = 0 \), \( G(1) = 1 \), \( \forall x \in \mathbb{R}, G(x) = G(-x) \); 3) \( G(x) \) is a linear function for \( x > 1 \), i.e. \( \exists \delta > 0, \forall x > 1, G(x) = sx + (1 - s) \); 4) \( \exists \delta > 0 \) such that \( G \) is twice differentiable on interval \((0, \delta)\). Furthermore, \( G_2''(0) \) and \( G_2''(0) \) exist, and either \( G_2''(0) > 0 \) or \( G_2''(0) > 0 \); 5) \( \exists \delta > 0, \forall 0 < x < y, G(y)/G(x) \leq C_G(y/x)^2 \).

The condition 1 is required to define an Orlicz norm. The conditions 2,3 are required because we can always do the simplification/normalization (see Fact 2). The condition 4 is required for the smoothness of \( G \). The condition 5 is due to the subquadratic growth condition. Subquadratic
growth condition is necessary for sketching \( \sum_{i=1}^{n} G(x_i) \) with sketch size sub-polynomial in the dimension \( n \), as shown by Braverman & Ostrovsky, 2010. For example, if \( G(x) = x^p \) for some \( p > 2 \), then \( \| \cdot \|_G \) is the same as \( \| \cdot \|_p \). It is necessary to take \( \Omega(\gamma^{1-2/p}) \) space to sketch \( \ell_p \) norm in \( n \)-dimensional space. Condition 5 is also necessary for 2-concave property. Kwapie & Schuett, 1985 shows that \( \| \cdot \|_G \) can be embedded into \( \ell_1 \) space if and only if \( G \) is 2-concave. Although Schuett, 1995 gives an explicit embedding into \( \ell_1 \), it cannot be computed efficiently.

There are many potential choices of \( G(\cdot) \) which satisfies property \( P \), the following are some examples: 1) \( G(x) = |x|^p \) for some \( 1 \leq p \leq 2 \). In this case \( \| \cdot \|_G \) is exactly the \( \ell_p \) norm \( \| \cdot \|_p \).

Lemma 4. Given a function \( G(\cdot) \) with property \( P \), then \( \forall x \in \mathbb{R}^n, \|x\|_2/\sqrt{n}G \leq \|G(x)\| \leq \|x\|_2 \).

Lemma 5. Given a function \( G(\cdot) \) with property \( P \), then \( \forall 0 < x < y \), we have \( y/x \leq G(y)/G(x) \).

Lemma 6. Given a function \( G(\cdot) \) with property \( P \), there exist a constant \( \alpha_G > 0 \) which may depend on \( G \), such that \( \forall 0 \leq a, b, \text{ if } ab \leq 1, \text{ then } G(a)G(b) \leq \alpha_G G(ab) \).

3. Subspace embedding for Orlicz norm using exponential random variables

In this section, we develop the subspace embedding under the Orlicz norms which are induced by functions \( G \) with the property \( P \). We first show how to embed the subspace with \( \| \cdot \|_G \) norm into a subspace with \( \ell_2 \) norm, and then we use dimensionality reduction techniques for the \( \ell_2 \) norm. Overall, we will prove Theorem 9 stated at the end of this section. Before discussing the details, we give formal definitions of subspace embedding.

Definition 7 (Subspace embedding for Orlicz norm). Given a matrix \( A \in \mathbb{R}^{m \times n} \), if \( S \in \mathbb{R}^{m \times n} \) satisfies \( \forall x \in \mathbb{R}^d, \|Ax\|_2/\alpha \leq \|S Ax\|_G \leq \beta \|Ax\|_G \) where \( \alpha, \beta \geq 1, \| \cdot \|_v \) is a norm (can still be \( \| \cdot \|_G \)), then we say \( S \) embeds the column space of \( A \) with Orlicz norm into the column space of \( SA \) with \( v \)-norm. The distortion is \( \alpha \beta \).

If the distortion and the \( v \)-norm are clear from the context, we just say \( S \) is a subspace embedding matrix for \( A \).

Definition 8 (Subspace embedding for \( \ell_2 \) norm). Given a matrix \( A \in \mathbb{R}^{n \times d} \) and \( \ell_2 \) norm, the following are some examples: 1) \( G \) is a sub-Gaussian distribution.

Table 1. Some of M-estimators.

| M-estimator | Formulation |
|-------------|-------------|
| HUBER       | \( x^2/2 \) |
|             | \( c(x-x^2/2) \) |

Before discussing the details, we give formal definitions of subspace embedding.

The main technical thrust is to embed \( \| \cdot \|_G \) into \( \ell_2 \) norm. As the embedding matrix, we use \( S = \Pi D^{-1} \) where \( \Pi \) is one of the above \( \ell_2 \) embedding matrices and \( D \) is a diagonal matrix of which diagonal entries are i.i.d. random variables drawn from the distribution with CDF \( 1 - e^{-G(t)} \). Equivalently, each entry on the diagonal of \( D \) is \( G^{-1}(u) \), where \( u \) is an i.i.d. sample from the standard exponential distribution, i.e. CDF is \( 1 - e^{-t} \). In Section 3.1, we will prove that \( \forall x \in \mathbb{R}^d, \|D^{-1}Ax\|_G \) will not be too large. In Section 3.2, we will show that \( \forall x \in \mathbb{R}^d, \|D^{-1}Ax\|_\infty \) cannot be too small. Then due to Lemma 9, we know that \( \|D^{-1}Ax\|_2 \) is a good estimator to \( \|Ax\|_G \). In Section 3.3, we show how to put all the ingredients together.

3.1. Dilation bound

We construct a randomized linear map \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \): 

\[ (x_1, x_2, ..., x_n) \xrightarrow{\text{random}} (x_1/u_1, x_2/u_2, ..., x_n/u_n) \]

where each \( u_i \) is drawn from a distribution with CDF \( 1 - e^{-G(t)} \). Notice that for proving the dilation bound, we do not need to assume \( u_i \) are independent.

Theorem 9. Given \( x \in \mathbb{R}^n \), let \( \| \cdot \|_G \) be an Orlicz norm induced by function \( G(\cdot) \) which has property \( P \), then \( f(x) = (x_1/u_1, x_2/u_2, ..., x_n/u_n) \), where each \( u_i \) is drawn from a distribution with CDF \( 1 - e^{-G(t)} \). Then with probability at least \( 1 - \delta - O(1/n^{19}) \), \( \|f(x)\|_G \leq O(\alpha_G \delta^{-1} \log(n)) \|x\|_G \), where \( \alpha_G \) is a constant may depend on function \( G(\cdot) \).

Proof sketch: By taking union bound, we have \( \forall i \in \{1, u_i \geq G^{-1}(1/n^{20}) \} \) with high probability. Let \( \alpha = \|x\|_G \). For \( \gamma \geq 1 \), we want to upper bound the probability that \( \|f(x)\|_G \geq \gamma \alpha \). This is equivalent to upper bound the probability that \( \|f(x)/(\gamma \alpha)\|_G \geq 1 \). Notice that \( \Pr(\|f(x)/(\gamma \alpha)\|_G \geq 1) = \Pr(\sum G(x_i/\alpha) \geq 1) \). By Markov inequality, it suffices to bound the expectation of \( \sum G(x_i/\alpha) \) conditioned on \( u_i \) not too small. By lemma 6, \( \sum G(x_i/\alpha) \leq \alpha_G \gamma / \sum G(x_i/\alpha) \leq 1/(u_i) \). Because \( u_i \) is not too small, the conditional expectation of \( 1/(u_i) \) is roughly \( O(\log n) \). So the probabil-
ity that $\|f(x)\|_G \geq \gamma \alpha$ is bounded by $O(\alpha_G \log n / \gamma)$, set $\gamma = O(\log n) \alpha_G / \delta$, we can complete the proof. See appendix for the details of the whole proof.

The final step is to use a well-conditioned basis; see details in appendix. We then obtain the following theorem.

**Theorem 10.** Let $G(\cdot)$ be a function which has property $\mathcal{P}$. Given a matrix $A \in \mathbb{R}^{m \times n}$ with rank $d \leq n$, let $D \in \mathbb{R}^{n \times n}$ be a diagonal matrix of which each entry on the diagonal is an i.i.d. random variable drawn from the distribution with $CDF 1 - e^{-G(i)}$. Then, with probability at least 0.99, $\forall x \in \mathbb{R}^m$, $\|D^{-1}Ax\|_G \leq O(\alpha^2_G d^2 \log n)\|Ax\|_G$, where $\alpha_G^+ \geq 1$ are two constants which only depend on $G(\cdot)$.

The above theorem successfully embeds $\|\cdot\|_G$ into $\ell_2$ space. We now use $\ell_2$ subspace embedding to reduce the dimension. The following two theorems provide efficient $\ell_2$ subspace embeddings.

**Theorem 14.** (Clarkson & Woodruff, 2013). Given matrix $A \in \mathbb{R}^{m \times n}$ with rank $d$, let $t = \Theta(d^2 / \varepsilon^2)$, $S = \Phi Y \in \mathbb{R}^{t \times n}$, where $Y \in \mathbb{R}^{n \times n}$ is a diagonal matrix with each diagonal entry independently uniformly chosen to be $\pm 1$, $\Phi \in \mathbb{R}^{t \times n}$ is a binary matrix with $\Phi_{hi} = 1, \forall i \in [n]$, and remaining entries 0. Here $b : \{n\} \to \{t\}$ is a random hashing function such that for each $i \in [n]$, $h(i)$ is uniformly distributed in $[t]$. Then with probability at least 0.99, $\forall x \in \mathbb{R}^m$, $(1 - \varepsilon)\|Ax\|_2^2 \leq \|SAx\|_2^2 \leq (1 + \varepsilon)\|Ax\|_2^2$. Furthermore, $SA$ can be computed in $\text{nnz}(A)$ time.

**Theorem 15.** (See e.g. Woodruff, 2014). Given matrix $A \in \mathbb{R}^{m \times n}$ with rank $d$. Let $t = \Theta(d / \varepsilon^2)$, $S \in \mathbb{R}^{t \times n}$ be a random matrix of i.i.d. standard Gaussian variables scaled by $1 / \sqrt{t}$. Then with probability at least 0.99, $\forall x \in \mathbb{R}^m$, $(1 - \varepsilon)\|Ax\|_2^2 \leq \|SAx\|_2^2 \leq (1 + \varepsilon)\|Ax\|_2^2$.

We conclude the full theorem for our subspace embedding:

**Theorem 16.** Let $G(\cdot)$ be a function which has property $\mathcal{P}$. Given a matrix $A \in \mathbb{R}^{m \times n}$, let $D \in \mathbb{R}^{n \times n}$ be a diagonal matrix of which each entry on the diagonal is an i.i.d. random variable drawn from the distribution with $CDF 1 - e^{-G(i)}$. Let $\Pi_1 \in \mathbb{R}^{t \times m}$ be a sparse embedding matrix (see Theorem 12) and let $\Pi_2 \in \mathbb{R}^{t \times t}$ be a random Gaussian matrix (see Theorem 15) where $t_1 = \Theta(d^2)$, $t_2 = \Theta(d)$. Then, with probability at least 0.9, $\forall x \in \mathbb{R}^m$, $\Omega(1 / \alpha'_G d \log n)\|Ax\|_G^2 \leq \|\Pi_1 D^{-1}Ax\|_2^2 \leq O(\alpha''_G d^2 \log n)\|Ax\|_G$, where $\alpha'_G, \alpha''_G \geq 1$ are two constants which only depend on $G(\cdot)$. Furthermore, $\Pi_2 \Pi_1 D^{-1}A$ can be computed in $\text{nnz}(A) + \text{poly}(d)$ time.

### 3.3. Putting it all together

We now combine Theorem 12, Theorem 10, and Lemma 4 to get the following theorem.

**Theorem 13.** Let $G(\cdot)$ be a function which has property $\mathcal{P}$. Given a matrix $A \in \mathbb{R}^{n \times m}$ with rank $d \leq n$, let $D \in \mathbb{R}^{n \times n}$ be a diagonal matrix of which each entry on the diagonal is an i.i.d. random variable drawn from the distribution with $CDF 1 - e^{-G(i)}$. Then, with probability at least 0.98, $\forall x \in \mathbb{R}^m$, $\Omega(1 / (\alpha'_G d \log n))\|Ax\|_G^2 \leq \|D^{-1}Ax\|_2 \leq O(\alpha''_G d^2 \log n)\|Ax\|_G$, where $\alpha'_G, \alpha''_G \geq 1$ are two constants which only depend on $G(\cdot)$.

### 4. Applications

In this section, we discuss regression problem with Orlicz norm error measure, and low rank approximation problem with $\ell_p$ norm, which is a special case of the Orlicz norms.

**Definition 17.** Function $G(\cdot)$ has property $\mathcal{P}$. Given $A \in \mathbb{R}^{n \times d}, b \in \mathbb{R}^n$, the goal is to solve the following minimization problem $\min_{x \in \mathbb{R}^n} \|Ax - b\|_G$.

**Theorem 18.** Let $G(\cdot)$ have property $\mathcal{P}$. Given $A \in \mathbb{R}^{n \times d}, b \in \mathbb{R}^n$, Algorithm 1 can output a solution $\hat{x} \in \mathbb{R}^d$
such that with probability at least 0.8, $\|A\hat{x} - b\|_G \leq O(\beta_G d \log^2 n) \min_{x \in \mathbb{R}^d} \|Ax - b\|_G$, where $\beta_G$ is a constant which may depend on $G(\cdot)$. In addition, the running time of Algorithm 1 is $\text{nnz}(A) + poly(d)$.

**Proof sketch:** Let $S = \Pi_2 \Pi_1 D^{-1}$ be the subspace embedding for column space of $[\begin{array}{c} A \end{array}]$. Let $x^* = \arg \min_{x \in \mathbb{R}^d} \|Ax - b\|_G$. Due to Theorem 16, $\|A\hat{x} - b\|_G$ is bounded by $O(d \log n) \|(S\hat{x} - b)\|_2 \leq O(d \log n) \|S(Ax - b)\|_2 \leq O(d \log n) \|D^{-1}(Ax - b)\|_2$. Due to Theorem 3, $\|D^{-1}(Ax - b)\|_2 \leq O(1) \|D^{-1}(Ax - b)\|_G \leq O(d \log n) \|Ax - b\|_G$.

### 4.2. Regression with combined loss function

In this section, we want to point out that our technique can be used on solving regression problem with more general cost function. Recall that the goal is to solve the minimization problem $\min_{x \in \mathbb{R}^d} \|Ax - b\|_G$. Now, we consider there are multiple goals, and we want to minimize a linear combination of the costs. Now we give the definition of regression problem with combined cost function.

**Definition 19.** Suppose function $G_1(\cdot), G_2(\cdot), ..., G_k(\cdot)$ satisfies property $\mathcal{P}$. Given $A_1 \in \mathbb{R}^{n_1 \times d}, A_2 \in \mathbb{R}^{n_2 \times d}, ..., A_k \in \mathbb{R}^{n_k \times d}, b_1 \in \mathbb{R}^{n_1}, b_2 \in \mathbb{R}^{n_2}, ..., b_k \in \mathbb{R}^{n_k}$, the goal is to solve the following minimization problem $\min_{x \in \mathbb{R}^d} \sum_{i=1}^k \|A_i x - b_i\|_{G_i}$.

The idea of solving this problem is that we can embed every term into $l_1$ space, and then merge them into one term. By the standard technique, there is a way to embed $l_2$ space to $l_1$ space. We show the embedding as below. For the completeness, we put the proof of this lemma to the appendix.

**Lemma 20.** Let $Q \in \mathbb{R}^{n \times n}$ be a random matrix with each entry drawn uniformly from i.i.d. $\mathcal{N}(0, 1)$ Gaussian distribution. Let $B = (\sqrt{\pi}/2t) \cdot Q$. If $t = O(\epsilon^{-2} n \log (ne^{-1}))$, then with probability at least 0.98, $\forall x \in \mathbb{R}^n, \|Bx\|_1 \in ((1 - \epsilon)\|x\|_2, (1 + \epsilon)\|x\|_2)$.

**Theorem 21.** Let $k > 0$ be a constant, and $G_1(\cdot), G_2(\cdot), ..., G_k(\cdot)$ satisfy property $\mathcal{P}$. Given $A_1 \in \mathbb{R}^{n_1 \times d}, A_2 \in \mathbb{R}^{n_2 \times d}, ..., A_k \in \mathbb{R}^{n_k \times d}, b_1 \in \mathbb{R}^{n_1}, b_2 \in \mathbb{R}^{n_2}, ..., b_k \in \mathbb{R}^{n_k}$, Algorithm 2 can output a solution $\hat{x} \in \mathbb{R}^d$ such that with probability at least 0.7, $\sum_{i=1}^k \|A_i \hat{x} - b_i\|_{G_i} \leq O(\beta_G d \log^2 n) \min_{x \in \mathbb{R}^d} \sum_{i=1}^k \|A_i x - b_i\|_{G_i}$, where $\beta_G$ is a constant which may depend on $G_1(\cdot), G_2(\cdot), ..., G_k(\cdot)$. In addition, the running time of Algorithm 2 is $\sum_{i=1}^k \text{nnz}(A_i) + poly(d)$.

**Proof Sketch:** Let $A = [A_1^\top, A_2^\top, ..., A_k^\top]^\top, b = [b_1^\top, b_2^\top, ..., b_k^\top]^\top$, and $S = B\Pi_2 \Pi_1 D^{-1}$ be the subspace embedding for column space of $[\begin{array}{c} A \end{array}]$. Let $S_i = B(i)^\top \Pi_2 \Pi_1 (D(i)^{-1})$. Notice that $\forall x, \|S(Ax - b)\|_1 = \sum_{i=1}^k \|S_i(A_i x - b_i)\|_1$. Let $x^* = \arg \min_{x \in \mathbb{R}^d} \sum_{i=1}^k \|A_i x - b_i\|_{G_i}$. Due to Theorem 20, $\sum_{i=1}^k \|A_i \hat{x} - b_i\|_{G_i}$ is bounded by $O(d \log n) \sum_{i=1}^k \|S_i(A_i \hat{x} - b_i)\|_1 = O(d \log n) \|S(Ax - b)\|_1 \leq O(d \log n) \|S(Ax^* - b)\|_1 = \sum_{i=1}^k \|S_i(A_i x^* - b_i)\|_1$. Due to Theorem 21, $\sum_{i=1}^k \|S_i(A_i x^* - b_i)\|_1 \leq O(d \log n) \sum_{i=1}^k \|A_i x^* - b_i\|_{G_i}$.

One application of the above Theorem is to solve the LASSO (Least Absolute Shrinkage Sector Operator) regression. In LASSO regression problem, the goal is to minimize $\|Ax - b\|_2^2 + \lambda \|x\|_1$, where $\lambda$ is a parameter of regularizer. It is easy to show that it is equivalent to minimize $\|Ax - b\|_2^2 + \lambda' \|x\|_1$ for some other parameter $\lambda'$. When we look at $\|Ax - b\|_2 + \lambda \|x\|_1$, we can set $A_1 = A, b_1 = b, A_2 = \lambda' I, b_2 = 0, G_1(\cdot) = \lambda^2, G_2(\cdot) \equiv x$, then we are able to apply Theorem 21 to give a good approximation.

**4.3. $\ell_p$ norm low rank approximation using exponential random variables**

We discuss a special case of Orlicz norm $\| \cdot \|_{G_p}$, $\ell_p$ norm, i.e. $G(x) \equiv x^p$ for $p \in [1, 2]$. When rank pa-
rameter $k$ is $\omega(\log n + \log d)$, by using exponential random variables, we can significantly improve the approximation ratio of input sparsity time algorithms shown by (Song et al., 2017). The high level ideas combine the results of (Woodruff & Zhang, 2013; Song et al., 2017) and the dilation bound in Section 3. We define the problem in the following. See Appendix for the proof of Theorem 23.

**Definition 22.** Let $p \in [1, 2]$. Given $A \in \mathbb{R}^{n \times d}, n \geq d, k \in \mathbb{Z}, 1 \leq k \leq \min (n, d)$, the goal is to solve the following minimization problem: $\min_{U \in \mathbb{R}^{n \times k}, V \in \mathbb{R}^{k \times d}} \| UV - A \|_p^p$.

**Algorithm 3** $\ell_p$ norm low rank approximation using exponential random variables.

1. **Input:** $A \in \mathbb{R}^{n \times d}, k \in \mathbb{Z}, \min (n, d) \geq k \geq 1$.
2. **Output:** $\hat{U} \in \mathbb{R}^{n \times k}, \hat{V} \in \mathbb{R}^{k \times d}$.
3. Let $t_1 = \Theta(k^2), t_2 = \Theta(k), t_3 = \Theta(k \log k)$.
4. Let $P_1, S_1 \in \mathbb{R}^{k \times n}$ be two random sparse embedding matrices, $P_2, S_2 \in \mathbb{R}^{k \times k}$ be two random gaussian matrices, and $D_1, D_2 \in \mathbb{R}^{k \times k}$ be two random diagonal matrices with each diagonal entry independently drawn from distribution whose CDF is $1 - e^{-t_3}$. (See Theorem 16).
5. Let $T_2, R \in \mathbb{R}^{d \times k}$ be two random matrix, with i.i.d. entries drawn from standard $p$-stable distribution.
6. Let $S = S_2 S_1 D_1^{-1}, T_1 = P_2 P_1 D_2^{-1}$.
7. Solve $\tilde{X}, \tilde{Y} = \text{arg min}_{X \in \mathbb{R}^{n \times k}, Y \in \mathbb{R}^{k \times d}} \| T_1 A \tilde{X} Y S T_2 - T_1 A \tilde{T} \|_2^2$.
8. $\hat{U} = A \tilde{X}, \hat{V} = \tilde{Y} \tilde{S}$.

**Theorem 23.** Let $1 \leq p \leq 2$. Given $A \in \mathbb{R}^{n \times d}, n \geq d, k \in \mathbb{Z}, 1 \leq k \leq \min (n, d)$, with probability at least $2/3$, $\hat{U}, \hat{V}$ outputted by Algorithm 3 satisfies: $\| \hat{U} \hat{V} - A \|_p \leq \alpha \min_{U \in \mathbb{R}^{n \times k}, V \in \mathbb{R}^{k \times d}} \| U V - A \|_p$, where $\alpha = O(\min((k \log k)^{4-p} \log^{2p+2} n, (k \log k)^{-2p} \log^{4+p} n))$. In addition, the running time of Algorithm 3 is $\text{nnz}(A) + (n + d) \text{poly}(k)$.

5. **Experiments**

Implementation setups can be seen in appendix.

5.1. Orlicz Linear Regression

In this section, we show that our algorithm i) has reasonable and predictable performance under different scenarios and ii) is flexible, general and easy to use. We perform 3 sets of experiments. The first is to compare its performance with the standard $\ell_1$ and $\ell_2$ regression under different noise assumptions and dimensions of the regression problem; the second is to compare the performance of Orlicz regression with different $G$ under different noise assumptions; the third is to compare with Orlicz function $G$ that is different from standard $\ell_p$ and Huber function. We evaluate the performance of our Orlicz norm linear regression algorithm on simulated data.

**Comparison with $\ell_1$ and $\ell_2$ regression** We would like to see whether Orlicz norm linear regression leads to expected performance relative to $\ell_1$ and $\ell_2$ regression. We choose our Orlicz norm $| \cdot |_G$ to be induced by the normalized Huber function where the Huber function is defined as $f(x) = \begin{cases} x^2/2 & |x| \leq \delta \\ \delta \cdot (|x| - \delta/2) & \text{o.w.} \end{cases}$. We chose the parameter $\delta$ to be 0.75. Intuitively, it is between $\ell_1$ and $\ell_2$ norm (see Figure 1). In all the simulations, we generate matrix $A \in \mathbb{R}^{n \times d}$, ground truth $x^* \in \mathbb{R}^d$, and $b$ to be $Ax^*$ plus some particular noise. We evaluate the performance of each algorithm by the $\ell_2$ distances between the output $x$ and the ground truth $x^*$. In terms of algorithm details, since $n, d$ are not too large in our simulation, we did not apply the $\ell_2$ subspace embedding to reduce the dimension; we only use reciprocal exponential random diagonal embedding matrix to embed $\| \cdot \|_G$ to $\ell_2$ norm (see Theorem 13).

We experiment with two $n, d$ combinations, i) $n = 200, d = 10$ ii) $n = 100, d = 75$, and 3 noise setting with i) Gaussian noise ii) sparse noise and iii) mixed noise (addition of i) and ii), altogether $2 \times 3 = 6$ setting. The detail of data simulation can be seen in appendix. For each experiment we repeat 50 times and compute the mean. The results are shown in Table 2. Orlicz norm regression has better performance than $\ell_1$ and $\ell_2$ when the noise is mixed. When the noise is Gaussian or spare, Orlicz norm regression works better than $\ell_1$ and $\ell_2$ respectively. We did not experiment with Huber loss regression, since if we rescale the data and make it small/large in absolute values, the Huber regression will degenerate into respectively $\ell_2/\ell_1$ regression (see Introduction). See appendix for results on approximation ratio.

**Choice of $\delta$ for $G$ as a normalized Huber function** We compare the performance of Orlicz norm regression induced by $G$ as normalized Huber loss function with different $\delta$ under different noise assumptions. We fix $n = 500, d = 30$ and generate $A$ and $x$ as in the first set of experiments (see appendix). The noise is a mixture of $N(0, 5)$ Gaussian noise and sparse noise on 1% entries with different scale of uniform noise from $[-s, |Ax^*|_2, s, |Ax^*|_2]$, where scale $s$ is chosen from $[0, 0.5, 1, 2]$. Under each noise assumptions with different scale $s$, we compare the performance of Orlicz norm regression induced by $G$ with $\delta$ from $[0.05, 0.1, 0.2, 0.4, 1, 2]$. We repeat each experiment 50 times and report the mean of the $\ell_2$ distance between output $x$ and the ground truth $x^*$. The result is shown in Figure 2. When the scale is 0, the noise is almost Gaussian/sparse and we expect $\ell_2/\ell_1$ norm and thus large/small

| Table 2. Comparisons of different regressions in different noise and dimension settings; each entry is the error of $\ell_1, \ell_2, \ell_1$ norm regression. As expected, $\ell_2/\ell_1$ regression lead to best performance under Sparse/Gaussian noise setting, and the performance of Orlicz norm regression lies in between. |
|---|---|---|---|
| Gaussian | Sparse | Mixed |
| balance | 211.2/194.5/197.3 | 25.3/30.7/30.0 | 37.9/37.8/37.5 |
| overconstraint | 25.3/20.8/24.9 | 2e-9/1.6/1.5 | 8.7/7.6/7.5 |

\footnote{We use MATLAB’s \textit{linprog} to solve $\ell_1$ regression.}
Table 3. Orlicz regression with different choices of $G$, mean of the $\ell_2$ distances between the output and the ground truth in 50 repetitions of experiments.

| $G_{\delta=0.25}$ | $G_{\delta=0.75}$ | $G_{\ell_1,5}$ |
|-------------------|-------------------|-----------------|
| 17.0 | 45.0 | 909.8 |
| 60.2 | 405.7 | 14.7 |

We find that the modified $\delta$ to perform the best; anything scale lying in between these extremes will have an optimal $\delta$ in between. We observe the expected trend: as $s$ increases, the performance is optimal with smaller $\delta$.

### Beyond Huber function - A General Framework

We explore a variant Orlicz function $G$ and evaluate it under a particular setting; the evaluation criteria is the same as the first set of the experiments. The $G$ is of the same form aforementioned, except that it now grows at the order of $x^{1.5}$ when $x$ is small. We denote it by $G_{\ell_1,5}$, which is the normalization of function $f$, and $f$ is defined as:

$$ f(x) = \begin{cases} \frac{x^{1.5}}{1.5} & x \leq \delta \\ \delta^{0.5} \cdot (|x| - \delta^3) / 3 & \text{o.w.} \end{cases} $$

We generate a 500 x 30 matrix $A$ and the ground truth vector $x^*$ in the same way as before, and then add $N(0,5)$ Gaussian noises and 1 sparse outlier with scale $s = 100$. We find that the modified $G_{\ell_1,5}$ under this settings outperforms $\ell_1, \ell_2, \ell_1, G_{\delta=0.25}, G_{\delta=0.75}$ regression by a significant amount where $G_{\delta=0.25}, G_{\delta=0.75}$ are Orlicz norm induced by regular normalized Huber function with $\delta = 0.25, 0.75$ respectively. The results are shown in Table 3.

This experiment demonstrates that our algorithm is i) flexible enough to combine the advantage of norm functions, ii) general for any function that satisfies the nice property, and iii) easy to experiment with different settings, as long as we can compute $G$ and $G^{-1}$.

#### 5.2. $\ell_1$ low rank matrix approximation

In this section, we evaluate the performance of the $\ell_1$ low rank matrix approximation algorithm. We mainly compare the $\ell_1$ norm error of our algorithm with the error of Song et al. [2017] and standard PCA. Inputs are a matrix $A \in \mathbb{R}^{n \times d}$ and a rank parameter $k$; the goal is to output a rank $k$ matrix $B$ such that $\|A - B\|_1$ is as small as possible. The details of implementations are in the appendix. For each input, we run the algorithm 50 times and pick the best solution.

Datasets. We first run experiment on synthetic data: we randomly choose two matrices $U \in \mathbb{R}^{2000 \times 5}, V \in \mathbb{R}^{5 \times 2000}$ with each entry drawn uniformly from $(0,1)$. Then we randomly choose 100 entries of $UV$, and add random outliers uniformly drawn from $(-100,100)$ on those entries, thus we can get a matrix $A \in \mathbb{R}^{2000 \times 2000}$. In our experiment, $\|A\|_1$ is about $5.0 \times 10^9$. Then, we run experiments on real datasets diabetes and glass in UCI repository (Bache & Lichman, 2013). The data matrix of diabetes has size $768 \times 8$, and the data matrix of glass has size $214 \times 9$. For each data matrix, we randomly add outliers on 1% number of entries.

For each dataset, we evaluate the $\|A - B\|_1$. The result for the experiment on synthetic data is shown in Table 4 and the results for diabetes and glass are shown in Figure 3.

The running time of algorithm in Song et al. [2017] on diabetes and on glass are 5.69 and 11.97 seconds respectively, with ours being 3.18 and 3.74 seconds respectively. We also find that our algorithm consistently outperforms the other two alternatives (the y-coordinates are at log scale with base 10).

6. Conclusion and Future Work

We presented an efficient subspace embedding algorithm for orlicz norm and demonstrated its usefulness in regression/low rank approximation problem on synthetic and real datasets. Nevertheless, $O(d \log^2 n)$ is still a large theoretical approximation factor, and hence it is worth i) investigating whether the theoretical approximation ratio can be smaller if input are under some statistical distribution ii) calculating the actual approximation ratio with ground truth obtained by some slower but more accurate optimization algorithm. It is also worth examining whether our exponential embedding sketching method preserves the statistical properties of the regression error, since we assumed a different noise distribution from Gaussian/double-exponential as a starting point (Raskuti & Mahoney, 2014; Lopes et al., 2018).
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### A. Related Works

Existing literature studied the robust regression with respect to Huber loss function (Mangasarian & Musicant, 2000; Owen, 2007). Such regression can be applied to solve many problems like the people counting problem (Cavazza & Murino). To speed up the regression process, some dimensional reduction techniques can be used to reduce the number of observations (Geppert et al., 2015), also faster algorithms have been proposed to address the robust regression with reasonable assumption (Bhatia et al., 2015). Besides, different models of regression were explored, such as Gaussian process regression (Rasmussen, 2006), active regression with adaptive Huber loss (Cavazza & Murino, 2016).

Recent years, there are lots of randomized sketching and embedding techniques developed for solving numerical linear algebra problems. There is a long line of works, e.g. (Achlioptas, 2003; Clarkson & Woodruff, 2013; Nelson & Nguyê̂n, 2013) for ℓ2 subspace embedding, and works, e.g. (Sohler & Woodruff, 2011; Meng & Mahoney, 2013; Woodruff & Zhang, 2013; Wang & Woodruff, 2018) for ℓp subspace embedding. For more related works, we refer readers to the book (Woodruff, 2014). Based on sketching/embedding techniques, there is a line of works studied ℓ2 and ℓp regressions, e.g. (Clarkson & Woodruff).
With out loss of generality, we can assume (Mahoney et al., 2011) for more details. The fastest algorithm is shown by (Clarkson & Woodruff, 2013). For the entrywise $\ell_2$ norm low rank approximation problem, there is no known algorithm with theoretical guarantee until the work (Song et al., 2017). (Song et al., 2017) works only for $1 \leq p \leq 2$. Recently, (Chierichetti et al., 2017) gives algorithms for all $p \geq 1$. But either the running time is not in polynomial or the rank of the output is not exact $k$.

B. Proof of Fact 2.

Proof. Notice that $G_1$ is a nonzero nondecreasing convex function on $\mathbb{R}_+$, thus $G_1^{-1}(1)$ exists, and $G_2$ is a nonzero nondecreasing function. In addition because $s = \sup \{y/x \mid (G_2(y) - G_2(x)) / (0 \leq x \leq y \leq 1)\}$, $G_2$ is also convex. Thus $\|s\|_2$ is Orlicz norm. Let $x \in \mathbb{R}^n$. Notice that if $\alpha > 0$ satisfies $\sum_{i=1}^n G_1(|x_i|/\alpha) \leq 1$, then $\forall i \in [n], G_1(|x_i|/\alpha) \leq 1$. It means that $|x_i| \leq G_1^{-1}(1, \alpha)$, thus $\sum_{i=1}^n G_2(|x_i|/(G_2^{-1}(1, \alpha))) = \sum_{i=1}^n G_1(|x_i|/\alpha) \leq 1$. Similarly if $\alpha$ satisfies $\sum_{i=1}^n G_2(|x_i|/(G_2^{-1}(1, \alpha))) \leq 1$, then $\sum_{i=1}^n G_1(G_2^{-1}(1)|x_i|/\alpha) = \sum_{i=1}^n G_2(|x_i|/\alpha) \leq 1$. Therefore, $\|x\|_{G_1} = \|x\|_{G_2} / G_2^{-1}(1)$.

C. Proof of Lemma 3.

Due to convexity of $G$ and $G(1) = 1, G(0) = 0, \forall x \in [0, 1], G(x) \leq xG(1) + (1 - x)G(0) = x$. Since $x \leq 1$, $G(1)/G(x) \leq C_G(1)/x^2$, we have $G(x) \geq x^2/C_G$.

D. Proof of Lemma 4.

With out loss of generality, we can assume $\forall i \in [n], x_i \geq 0$. Let $x \in \mathbb{R}^n, \alpha = \|x\|_G$. We have $\sum_{i=1}^n G(x_i/\alpha) = 1$. If $x_i/\alpha \leq 1$, due to the convexity of $G$, $G(x_i/\alpha) \leq G(1) \cdot x_i/\alpha + G(0) \cdot (1 - x_i/\alpha) = G(1) \cdot x_i/\alpha = x_i/\alpha$. If $x_i/\alpha > 1$, then $G(x_i/\alpha) > 1$ which contradicts to $\sum_{i=1}^n G(x_i/\alpha) = 1$. Thus, $\|x\|_G \leq \|x\|_1$.

$\|x\|_1^2 = \sum_{i=1}^n (x_i/\alpha)^2 \leq \sum_{i=1}^n C_G G(x_i/\alpha) = C_G \alpha$. Then $\|x\|_2 \leq \sqrt{C_G} \alpha$.

E. Proof of Lemma 5.

Due to the convexity of $G(\cdot)$ and $G(0) = 0, \forall 0 < x < y$, we have $G(x) \leq G(y)x/y + G(0)(1 - x/y) = G(y)x/y$. Thus, $y/x \leq G(y)/G(x)$.

F. Proof of Lemma 6.

It is easy to see that $\forall x > 0, G(x) \neq 0$, since otherwise for $y > x$, the condition $G(y)/G(x) < C_G(x/y)^2$ would be violated. Let $s = G_1'(1)$. There are several cases.

If $a \geq 1$ or $b \geq 1$. Without loss of generality, assume $a \geq 1$. (a) $G(a)G(ab) = G(a)/G(ab) \leq saG(b)/G(ab)$, since $ab > b, G(ab)/G(b) \geq a$. Therefore, $G(a)G(b)/G(ab) \leq saG(b)/G(ab) \leq s$.

If $a, b \leq 1$, $0.5 \leq a \leq 1$ or $0.5 \leq b \leq 1$, we want to show $G(a)G(b)/G(ab)$ is still bounded. Without loss of generality, assume that $0.5 \leq a \leq 1$. Then $G(a)G(ab)/G(ab) = G(a)G(b)/G(ab) \leq G(1)/G(ab) \leq C_G/a^2 \leq 4C_G$.

If $a, b \leq 0.5$ and $G'(0) > 0$. Let $G'(0) = c > 0$. Therefore, there is a constant $\delta_3$ which may depend on $G$ such that $\forall x \in (0, \delta_3], \frac{G(x) - G(0)}{G(x) - G(0)} < \frac{c}{G(0)}$. Therefore, $\forall x \in (0, \delta_3], G(x) > \frac{\delta_3}{c}$. Due to Lemma 5, $\forall x \geq \delta_1, G(x)/G(\delta_1/G(\delta_1)) > c/2$. Therefore, $\forall x, G(x) > \frac{\delta_3}{c}/2$. Since $b \leq 0.5$, $ab \leq a \leq 1$. Since $G$ is convex, $G(a) \leq \frac{G(ab)}{G(ab)}G(ab) = \frac{G(ab)}{G(ab)}G(ab) = \frac{G(ab)}{G(ab)}G(ab) \leq \frac{G(ab)}{G(ab)}G(ab) \leq ab + G(ab) \leq 2a + G(ab) \leq 2(c + 2) + G(ab) \leq 2(c + 2) + 2 = 2c + 1 < c + 3$.

If $a, b \leq 0.5, G'(0) = 0, G''(0) = c_2 > 0$. Let $\epsilon = c_2/4$. Since $G$ is twice differentiable in $(0, \delta_2)$ and $G''(0), G''(0)$ exist, by Taylor’s Theorem, there is a constant $\delta_2 > 0$ which may depend on $G$ such that $|G(x) - G(0) + G''(0)x + c_2x^2/2| \leq \epsilon x^2$. Therefore, $\forall x \in (0, \delta_2], G(x) \geq c_2x^2/2, G(x) \leq c_2x^2$. Hence, $\forall a, b \in (0, \delta_2], G(a)G(b)/G(ab) \leq \frac{G(ab)}{G(ab)}G(ab) \leq \frac{G(ab)}{G(ab)}G(ab) \leq \frac{G(ab)}{G(ab)}G(ab) \leq \frac{G(ab)}{G(ab)}G(ab) \leq C_G\delta_2^2/(2a + 2b) + G(ab) \leq 2/c + 2 + 1 < c + 3$.

If $a, b > 0.5$, $G''(0) = 0, G''(0) = c_2 > 0$. Let $\epsilon = c_2/4$. Since $G$ is twice differentiable in $(0, \delta_2)$ and $G''(0), G''(0)$ exist, by Taylor’s Theorem, there is a constant $\delta_2 > 0$ which may depend on $G$ such that $|G(x) - G(0) + G''(0)x + c_2x^2/2| \leq \epsilon x^2$. Therefore, $\forall x \in (0, \delta_2], G(x) \geq c_2x^2/2, G(x) \leq c_2x^2$. Hence, $\forall a, b \in (0, \delta_2], G(a)G(b)/G(ab) \leq \frac{G(ab)}{G(ab)}G(ab) \leq \frac{G(ab)}{G(ab)}G(ab) \leq \frac{G(ab)}{G(ab)}G(ab) \leq C_G\delta_2^2/(2a + 2b) + G(ab) \leq 2/c + 2 + 1 < c + 3$.

G. Proof of Theorem 9.

Without loss of generality, we assume $\forall i \in [n], x_i \geq 0$. Fix $i \in [n]$, we have $\Pr(u_i \geq G^{-1}(1/n^{20})) = e^{-G(G^{-1}(1/n^{20}))} \geq 1 - 1/n^{20}$. Define $\mathcal{E}$ to be the event that $\forall i \in [n], u_i \geq G^{-1}(1/n^{20})$. By taking union bound over $n$ coordinates, $\mathcal{E}$ happens with probability at least $1 - 1/n^{10}$.
Let $\alpha = \|x\|_G$. Then, for any $\gamma \geq 1$, we have
\[
\Pr(\|f(x)\|_G \geq \gamma \alpha) = \Pr(\|f(x)\|_G \geq \gamma \alpha | \mathcal{E}) \Pr(\mathcal{E}) + \Pr(\mathcal{E}) \\
\leq \gamma \alpha \frac{1}{\gamma} \int_{\mathbb{R}} e^{-G(u)} du \\
= \frac{1}{\gamma} G(x_\alpha) + \frac{1}{\gamma} \int_{\mathbb{R}} G\left(\frac{x_\alpha}{u}\right) e^{-G(u)} du \\
\leq \frac{1}{\gamma} G(x_\alpha) + \frac{1}{\gamma} \int_{\mathbb{R}} G\left(\frac{x_\alpha}{u}\right) e^{-G(u)} du \\
\leq O(\log n) \frac{\alpha_G}{\gamma} G(x_\alpha).
\]

where $\alpha_G$ is a constant may depend on $G()$. The first inequality follows by $G(x_\alpha/1/(\gamma u)) \leq 1/\gamma \cdot G(x_\alpha/1/u) + (1 - 1/\gamma) \cdot G(0) \leq G(x_\alpha/1/u)/\gamma \leq G(x_\alpha/\gamma)$. The second inequality follows by $\int_{\mathbb{R}} e^{-G(u)} du \leq 1$. The third inequality follows by Lemma 6. Since $x_\alpha \leq 1$, then there is an $\alpha_G$ such that $G(u)G(x_\alpha/1/u) \leq \alpha_G G(x_\alpha/\gamma)$. The last inequality follows by $\int_{\mathbb{R}} e^{-x} dx = O(\log n)$.

Thus, we have
\[
\sum_{i=1}^{n} \mathbb{E} \left( G\left(\frac{x_\alpha}{\gamma_{x_i}}\right) \right) \Pr(\mathcal{E}) \\
\leq O(\log n) \frac{\alpha_G}{\gamma} \sum_{i=1}^{n} G(x_\alpha) \leq O(\log n) \frac{\alpha_G}{\gamma}.
\]

Then,
\[
\Pr(\|f(x)\|_G \geq \gamma \alpha) \leq O(\log n) \frac{\alpha_G}{\gamma} + 1/n^{19}.
\]

It is equivalent to
\[
\Pr(\|f(x)\|_G \leq \gamma \alpha) \geq 1 - O(\log n) \frac{\alpha_G}{\gamma} - 1/n^{19}.
\]

Set $\gamma = O(\log n) \frac{\alpha_G}{\gamma}$, we complete the proof.

**H. Proof of Theorem 10**

Similar to (Dasgupta et al. [2009]), we can define a well conditioned basis for Orlicz norm.

**Definition 24** (Well conditioned basis for Orlicz norm). Given a matrix $A \in \mathbb{R}^{n \times m}$ with rank $d$, let $U \in \mathbb{R}^{n \times d}$ be a matrix which has the same column space of $A$. If $U$ satisfies $1. \forall x \in \mathbb{R}^d, \|x\|_G \leq \beta\|Ux\|_G$. 2. $\sum_{i=1}^{d} \|U_i\|_G \leq \alpha$, then $U$ is an $(\alpha, \beta, G)$-well conditioned basis of $A$.

Fortunately, the such good basis exists for Orlicz norm.

**Theorem 25** (See Connection to Auerbach basis in Section 3.1 of (Dasgupta et al., 2009)). Given a matrix $A \in \mathbb{R}^{n \times m}$ with rank $d$ and norm $\|\cdot\|_G$, there exist a matrix $U \in \mathbb{R}^{n \times d}$ which is a $(d, 1, G)$ well conditioned basis of $A$.

**Proof of Theorem 10.** Notice that $D^{-1} Ax$ is exactly the same as $f(Ax)$. There is a matrix $U \in \mathbb{R}^{n \times d}$ which is $(d, 1, G)$-well conditioned basis of $A$. Since $\forall x \in \mathbb{R}^m$, there is always a vector $y \in \mathbb{R}^d$ such that $Ax = Uy$, we only need to prove that with probability at least 0.99,
\[
\forall x \in \mathbb{R}^d, \|D^{-1} U x\|_G \leq O(\alpha_G d^2 \log n) \|U x\|_G,
\]

where $D, \alpha_G$ are the same as stated in the Theorem. According to Theorem 9, if we look at a fixed $i \in [d]$, then with probability at least $1 - 0.01/d$, $\|D^{-1} U_i\|_G \leq O(\alpha_G d \log(n))$. Now we have for any $x \in \mathbb{R}^d$,
\[
\|D^{-1} U x\|_G \leq \sum_{i=1}^{d} \|x_i\| \|D^{-1} U_i\|_G \\
\leq \|x\|_{\infty} \sum_{i=1}^{d} \|D^{-1} U_i\|_G \\
\leq O(\alpha_G d \log(n)) \|x\|_{\infty} \sum_{i=1}^{d} \|U_i\|_G \\
\leq O(\alpha_G d^2 \log(n)) \|U x\|_G.
\]

The first inequality follows by triangle inequality. The third inequality follows by $\forall i \in [d], \|D^{-1} U_i\|_G \leq O(\alpha_G d \log(n))$. The forth inequality follows by $(d, 1, G)$-well conditioned basis. \qed

**I. Proof of Theorem 12**

Now, in the following, we present the concept of $\varepsilon$-net.

**Definition 26** ($\varepsilon$-net for $\ell_2$ norm). Given $A \in \mathbb{R}^{n \times m}$ with rank $d$, let $S$ be the $\ell_2$ unit ball in the column space of $A$, i.e., $S = \{ y : \|y\|_2 = 1, \exists x \in R^m, y = Ax \}$. Let $N \subset S$, if $\forall x \in S, \exists y \in N$ such that $\|x - y\|_2 \leq \varepsilon$, then we say $N$ is an $\varepsilon$-net for $S$. 


The following theorem gives an upper bound of the size of \( \varepsilon \)-net.

**Theorem 27** (Lemma 2.2 of [Woodruff, 2014]). Given \( A \in \mathbb{R}^{n \times m} \) with rank \( d \), let \( S \) be the \( \ell_2 \) unit ball in the column space of \( A \). There exist an \( \varepsilon \)-net \( N \) for \( S \), such that \( |N| \leq (1 + 4/\varepsilon)^d \).

It suffices to prove \( \forall x \in \mathbb{R}^m, \|Ax\|_2 = 1 \) we have \( \Omega(1/(\alpha_G^2 d \log n)) \|Ax\|_G \leq \|D^{-1}Ax\|_\infty \). Let \( D \in \mathbb{R}^{n \times n} \) be a diagonal matrix of which each entry on the diagonal is an i.i.d. random variable drawn from the distribution with CDF \( 1 - e^{-G(t)} \). Let \( \alpha_G^2 \geq 1 \) be a sufficiently large constant. Let \( S \) be the \( \ell_2 \) unit ball in the column space of \( A \). Let \( t_1 = \Theta(\alpha_G^2 d \log n), t_2 = \Theta(\alpha_G^2 d \log n) \), where \( \alpha_G \) is the parameter stated in Theorem 10. Set \( \varepsilon = O(1/(\sqrt{n}C_G t_1 t_2)) \). There exist an \( \varepsilon \)-net \( N \) for \( S \), and

\[
|N| = e^{O(d(\log n + \log(C_G \alpha_G^2 \alpha_G)))}.
\]

By taking union bound over the net points, according to Theorem 11, with probability at least 0.99,

\[
\forall x \in N, \|D^{-1}x\|_\infty \geq \Omega(1/(\alpha_G^2 d \log n)) \|x\|_G. \tag{2}
\]

Also due to Theorem 10, with probability at least 0.99,

\[
\forall x \in S, \|D^{-1}x\|_G \leq O(\alpha_G^2 d^2 \log n) \|x\|_G. \tag{3}
\]

By taking union bound, with probability at least 0.98, the above two events will happen. Then, in this case, consider a \( y \in S \), let \( x \in N \) such that \( \|x - y\|_2 \leq \varepsilon \), let \( z = x - y \), we have

\[
\|D^{-1}y\|_\infty \geq \|D^{-1}x\|_\infty - \|D^{-1}z\|_\infty \\
\geq \frac{1}{t_1} \|y\|_G - t_2 \sqrt{C_G} \|z\|_G \\
\geq \frac{1}{t_1} \|y\|_G - t_2 \frac{t_1}{t_1} \|z\|_G - t_2 \sqrt{C_G} \|z\|_G \\
\geq \frac{1}{t_1} \|y\|_G - 2t_2 \sqrt{C_G} \|z\|_G \\
\geq \frac{1}{t_1} \|y\|_G - O\left(\frac{2}{\sqrt{C_G} t_1}\right) \\
\geq \Omega(1/t_1) \|y\|_G \\
= \Omega(1/(\alpha_G^2 d \log n)) \|y\|_G.
\]

The first inequality follows by triangle inequality. The second inequality follows by Equation (2) and Lemma 4, i.e. \( \|D^{-1}z\|_\infty \leq \|D^{-1}z\|_2 \leq \sqrt{C_G} \|D^{-1}z\|_G \). The third inequality follows by triangle inequality. The forth inequality follows by \( t_1, C_G \geq 1 \). The fifth inequality follows by Lemma 4: \( \|z\|_G \leq \|z\|_1 \leq \sqrt{n} \|z\|_2 = \sqrt{n} \varepsilon = O(1/(\sqrt{n}C_G t_1)) \). The sixth inequality follows by Lemma 4: \( \|y\|_G \leq \frac{1}{\sqrt{C_G}} \|y\|_2 = 1/\sqrt{C_G} \).

### J. Proof of Theorem 13

Due to Theorem 12 and Theorem 10, with probability at least 0.98, \( \forall x \in \mathbb{R}^m, \Omega(1/(\alpha_G^2 d \log n)) \|Ax\|_G \leq \|D^{-1}Ax\|_\infty \leq \|D^{-1}Ax\|_2 \). And \( \|D^{-1}Ax\|_2 \leq \sqrt{C_G} \|D^{-1}Ax\|_G \leq O(\sqrt{C_G} \alpha_G d^2 \log n) \|Ax\|_G \).

### K. Proof of Theorem 16

Due to Theorem 14 and Theorem 15, with probability at least 0.95, \( \forall x \in \mathbb{R}^m, \|\Pi_1 D^{-1} Ax\|_2 \) is a constant approximation to \( \|\Pi_1 D^{-1} Ax\|_2 \) and \( \|\Pi_2 D^{-1} Ax\|_2 \) is a constant approximation to \( \|D^{-1} Ax\|_2 \). Combining with Theorem 13, we complete the proof.

### L. Proof of Theorem 18

Let \( x^* = \arg \min_{x \in \mathbb{R}^d} \|Ax - b\|_G \). Due to Theorem 9, with probability at least 0.99,

\[
\|D^{-1}(Ax^* - b)\|_G \leq O(\alpha_G^2 d \log n) \|Ax^* - b\|_G. \tag{4}
\]

Now let \( A' = [A b] \in \mathbb{R}^{n \times (d+1)} \). Due to Theorem 16, with probability at least 0.9, we have

\[
\forall x \in \mathbb{R}^{d+1}, \Omega(1/(\alpha_G^2 d \log n)) \|A'x\|_G \leq \|\Pi_2 \Pi_1 D^{-1} A'x\|_2. \tag{5}
\]

Then,

\[
\|Ax - b\|_G \leq O(\alpha_G^2 d \log n) \|\Pi_2 \Pi_1 D^{-1} (Ax - b)\|_2 \\
\leq O(\alpha_G^2 d \log n) \|\Pi_2 \Pi_1 D^{-1} (Ax^* - b)\|_2 \\
\leq O(\alpha_G^2 d \log n) \|D^{-1} (Ax^* - b)\|_2 \\
\leq O(\alpha_G^2 \sqrt{C_G} d \log n) \|D^{-1} (Ax^* - b)\|_G \\
\leq O(\alpha_G^2 \alpha_G \sqrt{C_G} d \log^2 n) \|Ax^* - b\|_G.
\]

The first inequality follows by Equation (5). The second inequality follows by \( \hat{x} = (\Pi_2 \Pi_1 D^{-1} A)^\dagger \Pi_2 \Pi_1 D^{-1} b \), which is the optimal solution for \( \min_{x \in \mathbb{R}^d} \|\Pi_2 \Pi_1 D^{-1} (Ax - b)\|_2 \). The third inequality follows by Theorem 14 and Theorem 15. The forth inequality follows by Lemma 4. The last inequality follows by Equation (4). Let \( \beta_G = \alpha_G \sqrt{C_G} \), we complete the proof of the correctness of Algorithm 1.

For the running time, according to Theorem 16, computing \( \Pi_2 \Pi_1 D^{-1} A \) and \( \Pi_2 \Pi_1 D^{-1} b \) needs \( \text{nnz}(A) + \text{poly}(d) \) time. Since \( \Pi_2 \Pi_1 D^{-1} A \) has size \( \text{poly}(d) \), computing \( \hat{x} = (\Pi_2 \Pi_1 D^{-1} A)^\dagger \Pi_2 \Pi_1 D^{-1} b \) needs \( \text{poly}(d) \) running time. The total running time is \( \text{nnz}(A) + \text{poly}(d) \).

### M. proof of Lemma 20

Before we prove the Lemma, we need following tools.
Lemma 28 (Concentration bound for sum of half normal random variables). For any $k$ i.i.d. random Gaussian variables $z_1, z_2, \cdots, z_k$, we have that

$$\Pr\left(\frac{1}{\sqrt{k}} \sum_{i=1}^{k} |z_i| \in \left(1 - \varepsilon \sqrt{\frac{2}{\pi}}, (1 + \varepsilon)\sqrt{\frac{2}{\pi}}\right)\right) \geq 1 - e^{-\Omega(k\varepsilon^2)}.$$ 

Lemma 29. Let $G \in \mathbb{R}^{k \times m}$ be a random matrix with each entry drawn uniformly from i.i.d. $N(0, 1)$ Gaussian distribution. With probability at least 0.99, $|G| \leq 10\sqrt{km}$.

Proof. Since $\mathbb{E}(\|G\|_F^2 = km)$, we have that $\Pr(\|G\|_F^2 \geq 100km) \leq 0.01$. Thus, with probability at least 0.99, we have $\|G\|_2 \leq \|G\|_F \leq 10\sqrt{km}$.

Now, let us prove the lemma.

Proof of Lemma 20. Without loss of generality, we only need to prove $\forall x \in \mathbb{R}^n$ with $\|x\|_2 = 1$, we have $\|Bx\|_1 \in (1- \varepsilon, 1 + \varepsilon)$. Let $S = \{v \mid v \in \mathbb{R}^n, \|v\|_2 = 1\}$. Due to Theorem 21 we can find a set $S \subseteq S$ which satisfies that $\forall u \in S$ there exists $v \in G$ such that $\|u - v\|_2 \leq \varepsilon/(1000n)$ and $|G| \leq (4000n/\varepsilon)^{20n}$. Let $k \geq c\varepsilon^{-2}n \ln(n/\varepsilon)$ where $c$ is a sufficiently large constant. By Lemma 29 we have that for a fixed $v \in G$, with probability at least $1 - e^{-980n\ln(4000n/\varepsilon)}$, $\|Bv\|_1 \in (1- \varepsilon, 1 + \varepsilon)$. By taking union bound over all the points in $G$, we have $\Pr(\forall v \in G, \|Bv\|_1 \in (1- \varepsilon, 1 + \varepsilon)) \geq 1 - e^{-980n\ln(4000n/\varepsilon)}$. These two inequalities imply that $\|B\|_2 \leq 10\sqrt{n} \cdot \sqrt{\pi/2}/t$.

By taking union bound, we have with probability at least 0.98,

$$\forall x \in S, \|Bx\|_1 \in (1 - 2\varepsilon, 1 + 2\varepsilon).$$

By adjusting the $\varepsilon$, we complete the proof.

N. Proof of Theorem 21

Without loss of generality, we assume constant $k \leq 2$. Otherwise, we can always adjust constants in all the related theorems and lemmas to make larger $k$ work. Let $x^* = \arg\min_{x \in \mathbb{R}^n} \sum_{i=1}^{k} \|A_i x - b_i\|_{G_i}$. By Theorem 9 and taking union bound, we have that with probability at least 0.98,

$$\forall i \in \{1, 2, \cdots, k\},
\|(D^{(i)})^{-1}(A_i x^* - b_i)\|_{G_i} \leq O(\alpha G_i \log n)\|A_i x^* - b_i\|_{G_i}.

(6)$$

Now let $A_i' = [A_i , b_i] \in \mathbb{R}^{n_i \times (d+1)}$. Due to Theorem 16 and union bound, with probability at least 0.8, we have

$$\forall x \in \mathbb{R}^{d+1},
\Omega(1/(\alpha_{G_i}' \log n_i)) A_i' x_{G_i} \leq \|\Pi_2^{(i)} \Pi_1^{(i)} (D^{(i)})^{-1} A_i' x\|_2.

(7)$$

Then,

$$\sum_{i=1}^{k} \|A_i \hat{x} - b_i\|_{G_i} \leq \sum_{i=1}^{k} O(\alpha_{G_i}' \log n)\|\Pi_2^{(i)} \Pi_1^{(i)} (D^{(i)})^{-1} A_i \hat{x} - b_i\|_2
$$

$$\leq \sum_{i=1}^{k} O(\alpha_{G_i}' \log n)\|B^{(i)} \Pi_2^{(i)} \Pi_1^{(i)} (D^{(i)})^{-1} A_i \hat{x} - b_i\|_1
$$

$$\leq O(\max_{i \in [k]} \alpha_{G_i}' \log n)\|B^{(i)} \Pi_2^{(i)} \Pi_1^{(i)} (D^{(i)})^{-1} (A_i x^* - b_i)\|_1
$$

$$\leq O(\max_{i \in [k]} \alpha_{G_i}' \log n)\|B^{(i)} \Pi_2^{(i)} \Pi_1^{(i)} (D^{(i)})^{-1} (A_i x^* - b_i)\|_1
$$

$$\leq O(\max_{i \in [k]} \alpha_{G_i}' \log n)\|B^{(i)} \Pi_2^{(i)} (D^{(i)})^{-1} (A_i x^* - b_i)\|_1
$$

$$\leq O(\max_{i \in [k]} \alpha_{G_i}' \log n)\|\Pi_2^{(i)} (D^{(i)})^{-1} (A_i x^* - b_i)\|_2
$$

$$\leq O(\max_{i \in [k]} \sqrt{C_{G_i}} \alpha_{G_i}' \log n)\|B^{(i)} \Pi_2^{(i)} (D^{(i)})^{-1} (A_i x^* - b_i)\|_2
$$

The first inequality follows by Equation 2. The second inequality follows by Lemma 20. The forth inequality follows by $\hat{x}$ is the optimal solution for
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\[
\min_{x \in \mathbb{R}^d} \|B^2 \Pi_2 \Pi_1 D^{-1}(Ax - b)\|_1. \quad \text{The sixth inequality follows by Lemma 20. The seventh inequality follows by Theorem 14 and Theorem 15. The eighth inequality follows by Lemma 4. The last inequality follows by Equation 6. Let } \beta_G = (\max_{d \in [k]} \sigma_G)(\max_{d \in [k]} \sqrt{\gamma_G})(\max_{d \in [k]} \alpha_G), \text{ we complete the proof of the correctness of Algorithm 2.}
\]

For the running time, according to Theorem 16, computing \( \Pi_2 \Pi_1 D^{-1}A \) and \( \Pi_2 \Pi_1 D^{-1}b \) needs \( \sum_{i=1}^k \min(A_i) + \text{poly}(d) \) time. Due to Lemma 20, the size of \( B \) is \( \text{poly}(d) \). To compute \( B^2 \Pi_2 \Pi_1 D^{-1}A \) and \( B^2 \Pi_2 \Pi_1 D^{-1}b \), we need additional \( \text{poly}(d) \) time. Since \( B^2 \Pi_2 \Pi_1 D^{-1}A \) has size \( \text{poly}(d) \), computing the optimal solution of \( \min_{x \in \mathbb{R}^d} \|B^2 \Pi_2 \Pi_1 D^{-1}(Ax - b)\|_1 \) by using linear programming needs \( \text{poly}(d) \) running time. The total running time is \( \sum_{i=1}^k \min(A_i) + \text{poly}(d) \).

**O. Proof of Theorem 23**

Before we prove the Theorem, we need to show following Lemmas.

**Lemma 30** (Song et al. 2017). Let \( A \in \mathbb{R}^{n \times d}, R \in \mathbb{R}^{d \times t_1}, k \) be the same as in the Algorithm 3, then with probability at least 0.9,

\[
\min_{X \in \mathbb{R}^{n \times k}, Y \in \mathbb{R}^{k \times d}} \|ARXY - A\|_p^p \leq O((k \log k)^{1-p/2} \log n) \min_{U \in \mathbb{R}^{n \times k}, \forall \epsilon \in \mathbb{R}^{k \times d}} \|UV - A\|_p^p.
\]

**Lemma 31** (Woodruff & Zhang 2013). Let \( 1 \leq p \leq 2 \). Given a matrix \( A \in \mathbb{R}^{n \times d}, d \leq n \), let \( D \in \mathbb{R}^{n \times n} \) be a diagonal matrix of which each entry on the diagonal is an i.i.d. random variable drawn from the distribution with \( \text{CDF} = e^{-t} \). Let \( \Pi_1 \in \mathbb{R}^{n \times n} \) be a sparse embedding matrix (see Theorem 18) and let \( \Pi_2 \in \mathbb{R}^{t_2 \times t_1} \) be a random Gaussian matrix (see Theorem 19) where \( t_1 = \Omega(d^3), t_2 = \Omega(d) \). Then, with probability at least 0.9,

\[
\forall x \in \mathbb{R}^d,
\Omega(1/ \min((d \log d)^{1/p}, (d \log d \log n)^{1-p+1/2}))\|Ax\|_p \leq \|\Pi_1 \Pi_2 D^{-1}Ax\|_2.
\]

**Lemma 32. Let \( A \in \mathbb{R}^{n \times d}, S \in \mathbb{R}^{t \times n}, R \in \mathbb{R}^{d \times t_3}, k \) be the same as in the Algorithm 3, then with probability at least 0.9,

\[
\min_{X \in \mathbb{R}^{n \times k}, Y \in \mathbb{R}^{k \times d}} \|ARXYSA - A\|_p^p \leq \beta \min_{U \in \mathbb{R}^{n \times k}, \forall \epsilon \in \mathbb{R}^{k \times d}} \|UV - A\|_p^p,
\]

where

\[
\beta = O((k \log k)^{2-p/2} \log^{p+1} n, (k \log k)^{2-p} \log^{2+p/2} n).\]

**Proof.** Let

\[
X^*, V^* = \arg \min_{X \in \mathbb{R}^{n \times k}, Y \in \mathbb{R}^{k \times d}} \|ARXY - A\|_p.
\]

Let

\[
U^* = ARX^*, \tilde{V} = (SU^*)^tSA. \quad \text{Let } \gamma = \min\{k \log k, (k \log k \log n)^{1-p/2}\}. \quad \text{We have}
\]

\[
\|U^* \tilde{V} - A\|_p^p \leq 2^{p-1} \|U^*(\tilde{V} - V^*)\|_p^p + 2^{p-1} \|U^*V^* - A\|_p^p
\]

\[
\leq O(\gamma) \sum_{i=1}^d \|SU^*(\tilde{V} - V^*)\|_p^p + 2^{p-1} \|U^*V^* - A\|_p^p
\]

\[
\leq O(\gamma) \sum_{i=1}^d \|(SU^* - A)\|_2 + \|SU^* - A\|_2^p
\]

\[
+ 2^{p-1} \|U^*V^* - A\|_p^p
\]

\[
\leq O(\gamma) \sum_{i=1}^d \|(SU^* - A)\|_2 + 2^{p-1} \|U^*V^* - A\|_p^p
\]

\[
\leq O(\gamma) \|D_{i^{-1}}(U^*V^* - A)\|_2 + 2^{p-1} \|U^*V^* - A\|_p^p
\]

\[
\leq O(\gamma) \log^p(nd) \|U^*V^* - A\|_p^p + 2^{p-1} \|U^*V^* - A\|_p^p
\]

\[
= O(\gamma) \log^p(n) \|U^*V^* - A\|_p^p.
\]

The first inequality follows by convexity of \( x^p \). The second inequality follows by Lemma 31. The third inequality follows by triangle inequality. The fourth inequality follows by \( \tilde{V} = (SU^*)^tSA \). The fifth inequality follows by Theorem 14 and Theorem 15. The sixth inequality follows by \( p \leq 2 \). The seventh inequality follows by Theorem 9.

Thus, we have

\[
\|U^*V^* - A\|_p^p \leq O((k \log k)^{1-p/2} \log n) \min_{U \in \mathbb{R}^{n \times k}, \forall \epsilon \in \mathbb{R}^{k \times d}} \|UV - A\|_p^p.
\]

Due to Lemma 32, we have

\[
\|U^*V^* - A\|_p^p \leq O((k \log k)^{1-p/2} \log n) \min_{U \in \mathbb{R}^{n \times k}, \forall \epsilon \in \mathbb{R}^{k \times d}} \|UV - A\|_p^p.
\]

**Lemma 33** (Song et al. 2017). Let \( A \in \mathbb{R}^{n \times d}, S \in \mathbb{R}^{t \times n}, R \in \mathbb{R}^{d \times t_3}, k, T_2 \in \mathbb{R}^{d \times t_2} \) be the same as in the Algorithm 3, then with probability at least 0.9, if \( \alpha \geq 1, X, Y \) satisfy

\[
\|ARXYSA - AT_2\|_p^p \leq \alpha \min_{X, Y} \|ARXYSA - AT_2\|_p^p,
\]

then

\[
\|ARXYSA - A\|_p^p \leq \alpha O(\log n) \min_{X, Y} \|ARXYSA - A\|_p^p.
\]

**Lemma 34. Let \( A \in \mathbb{R}^{n \times d}, S \in \mathbb{R}^{t \times n}, R \in \mathbb{R}^{d \times t_2}, k, T_1 \in \mathbb{R}^{t_2 \times n}, T_2 \in \mathbb{R}^{d \times t_3} \) be the same as in the
Algorithm 3, then with probability at least 0.9, if for \( \alpha \geq 1 \)
\[
\sum_{i=1}^{t_3} \|T_1(AR\hat{X}\hat{Y}SAT_2 - AT_2)\|_2^p \\
\leq \alpha \min_{X,Y} \sum_{i=1}^{t_3} \|T_1(ARXY SAT_2 - AT_2)\|_2^p,
\]
then
\[
\|AR\hat{X}\hat{Y} SAT_2 - AT_2\|_p^p \leq \alpha \beta \min_{X,Y} \|ARXY SAT_2 - AT_2\|_p^p,
\]
where \( \beta = O(\min(k \log k \log^p n, (k \log k)^{1-p/2} \log^{1+p/2} n)) \).

**Proof.** Let
\[
X^*, Y^* = \arg \min_{X,Y} \sum_{i=1}^{t_3} \|T_1(ARXY SAT_2 - AT_2)\|_2^p.
\]
Let \( L = AR, N = SAT_2, M = AT_2 \). Let \( \gamma = \min\{k \log k, (k \log k \log n)^{1-p/2}\} \). Let \( H = \hat{X}\hat{Y} \) and let \( H^* = X^*Y^* \). We have
\[
\|LHN - M\|_p^p \leq 2^{p-1} \|LHN - LH^*N\|_p^p + 2^{p-1} \|LH^*N - M\|_p^p \\
\leq O(\gamma) \sum_{i=1}^{t_3} \|T_1(LHN - LH^*N)\|_2^p + 2^{p-1} \|LH^*N - M\|_p^p \\
\leq O(\gamma) \sum_{i=1}^{t_3} (\|T_1(LHN - M)\|_2^p + \|T_1(LH^*N - M)\|_2^p) + 2^{p-1} \|LH^*N - M\|_p^p \\
\leq O(\gamma) (\sum_{i=1}^{t_3} \|T_1(LHN - M)\|_2^p + \|D_2^{-1}(LH^*N - M)\|_2^p) + 2^{p-1} \|LH^*N - M\|_p^p \\
\leq O(\gamma) (\sum_{i=1}^{t_3} \|D_2^{-1}(LH^*N - M)\|_2^p + \|D_3^{-1}(LH^*N - M)\|_2^p) + 2^{p-1} \|LH^*N - M\|_p^p \\
\leq O(\gamma) \alpha (\sum_{i=1}^{t_3} \|D_2^{-1}(LH^*N - M)\|_2^p + \|D_3^{-1}(LH^*N - M)\|_2^p) \\
\leq O(\gamma) \alpha (\sum_{i=1}^{t_3} \|D_2^{-1}(LH^*N - M)\|_2^p + 2^{p-1} \|LH^*N - M\|_p^p) \\
\leq O(\gamma) \alpha \|LH^*N - M\|_p^p.
\]
Now let us prove Theorem:

**Proof.** Notice that
\[
\hat{X}, \hat{Y} = \arg \min_{X,Y} \sum_{i=1}^{t_3} \|T_1(AR\hat{X}\hat{Y} SAT_2 - AT_2)\|_2^p,
\]
we have
\[
\sum_{i=1}^{t_3} \|T_1(AR\hat{X}\hat{Y} SAT_2 - AT_2)\|_2^p \leq O((k \log k)^{1-p/2} (\min_{X,Y} \sum_{i=1}^{t_3} \|T_1(ARXY SAT_2 - AT_2)\|_2^p)^{1/2}).
\]
It means
\[
\sum_{i=1}^{t_3} \|T_1(AR\hat{X}\hat{Y} SAT_2 - AT_2)\|_2^p \leq \beta_1 \min_{X,Y} \|ARXY SAT_2 - AT_2\|_p^p,
\]
where
\[
\beta_1 = O(\min((k \log k)^{2-p/2} \log^p n, (k \log k)^{2-p} \log^{1+p/2} n)).
\]
Due to Lemma 32 we have
\[
\|AR\hat{X}\hat{Y} SAT_2 - AT_2\|_p^p \leq \beta_1 \min_{X,Y} \|ARXY SAT_2 - AT_2\|_p^p.
\]
Then, according to Lemma 32 we have
\[
\|AR\hat{X}\hat{Y} SAT_2 - AT_2\|_p^p \leq \beta_2 \min_{U,V} \|UV - A\|_p^p,
\]
where
\[
\beta_2 = O(\min((k \log k)^{4-2p} \log^{2p+2} n, (k \log k)^{4-2p} \log^{4+p} n)).
\]

**P. Implementation Setups**

We implement all the algorithms in MATLAB. We ran experiments on a machine with 16G main memory and Intel Core i7-3720QM@2.60GHz CPU. The operating system is Ubuntu 14.04.5 LTS. All the experiments were in single threaded mode.
Q. Data Simulation for Comparison with $\ell_1$ and $\ell_2$ Regression

We generate a matrix $A \in \mathbb{R}^{n \times d}$, $x^* \in \mathbb{R}^d$ as following: set each entry of the first $d + 5$ rows of $A$ as i.i.d. standard random Gaussian variable, each entry of $x^*$ as i.i.d. standard random Gaussian variable. For $n \geq i \geq d + 6$, we uniformly choose $p \in [d + 5]$, and set $A_i = Ap_i, b_i = b_p$. We perform experiments under 3 different noise assumptions and 2 dimension combinations of $N, d$ and in total $3 \times 2 = 6$ experiments. The 3 different noise assumptions are, respectively i) $N(0, 50)$ Gaussian noise with on all the entries of $Ax^*$; ii) sparse noise, where we randomly pick $3\%$ number of entries of $Ax^*$, and add uniform random noise from $[-\|Ax^*\|_2, \|Ax^*\|_2]$ on each entry to get $b$; iii) mixed noise, which is $N(0, 5)$ Gaussian noise plus sparse noise. The 2 different dimension combinations are i) balance, where $n = 100 \approx d = 75$; ii) overconstraint, where $n = 200 \gg d = 10$.

R. Experiments on Approximation Ratio

Here is a documentation of our preliminary experiments on calculating the actual approximation ratio for the experiment settings mentioned in Section 5.1, Comparison with $\ell_1$ and $\ell_2$ regression. The approximation ratio of interest is calculated as follows: $\|Ax^* - b\|_G \|Ax^* - b\|_G$, where $x'$ is the output of our novel embedding based algorithm and $x^*$ is the optimal solution. Since $\| \cdot \|_G$ is convex, we can formulate this problem as a convex optimization problem and use a vanilla gradient descent algorithm to calculate the optimal solution. We heuristically stop our gradient descent algorithm when the one step brings less than $10^{-7}$ improvement on the loss function and set the learning rate to be 0.001. Admittedly, we have not yet thoroughly and rigidly examined the convergence of the vanilla gradient descent algorithm (a direction of future work), and hence such calculation of approximation ratio is only a preliminary attempt.

Under the mixed noise setting, we varied different scale $s$ of the uniform noise to be 0, 1, 2, 3 and delta to be 0.1, 0.25, 0.5, 0.75. With $n = 200, d = 10$, for each of these $4 \times 4 = 16$ settings, we run the algorithm repeatedly for 50 times, and the worst approximation ratio is 1.06 among these 800 runs. Experimentally, it is far below the theoretical guarantee $d \log^2(n) = 584 \gg 1.06$, and the approximation ratio is robust among different noise settings. For $n = 100, d = 75$, due to time limit, we only run each of the 16 settings for 5 times, and the worst approximation ratio is 1.31.

S. Implementation Detail for Low Rank Approximation

- For our algorithm, set $t_1 = 4k, t_2 = 8t_1$, set $S \in \mathbb{R}^{t_1 \times n}, T_1 \in \mathbb{R}^{t_2 \times n}$ to be two random cauchy matrices, and set $R \in \mathbb{R}^{d \times t_1}, T_2 \in \mathbb{R}^{d \times t_2}$ to be two embedding matrices with exponential random variables (see Theorem 16.) We solve the minimization problem $\min_{X,Y} \|T_1ARXYSAT_2 - T_1AT_2\|_F^2$, and set $B = ARXYSA$.

- For algorithm in [Song et al., 2017], we set $t_1 = 4k, t_2 = 8t_1$. We set $S \in \mathbb{R}^{t_1 \times n}, T_1 \in \mathbb{R}^{t_2 \times n}, R \in \mathbb{R}^{d \times t_1}, T_2 \in \mathbb{R}^{d \times t_2}$ to be four random cauchy matrices. We solve the minimization problem $\min_{X,Y} \|T_1ARXYSAT_2 - T_1AT_2\|_F^2$, and set $B = ARXYSA$.

- For PCA, we project $A$ onto the space spanned by top $k$ singular vectors to get $B$. 