Changeover phenomenon in randomly colored Potts models

Nir Schreiber, Reuven Cohen, Gideon Amir and Simi Haber

Department of Mathematics, Bar Ilan University, Ramat Gan, 5290002, Israel
E-mail: nir.schreiber@gmail.com, reuven@math.biu.ac.il, gidi.amir@gmail.com and simi@math.biu.ac.il

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Abstract. A hybrid Potts model where a random concentration $p$ of the spins assume $q_0$ states and a random concentration $1 - p$ of the spins assume $q > q_0$ states is introduced. It is known that when the system is homogeneous, with an integer spin number $q_0$ or $q$, it undergoes a second or a first order transition, respectively. It is argued that there is a concentration $p^*$ such that the transition nature of the model is changed at $p^*$. This idea is demonstrated analytically and by simulations for two different types of interaction: the usual square lattice nearest neighboring and mean field (MF) all-to-all. Exact expressions for the second order critical line in concentration-temperature parameter space of the MF model together with some other related critical properties, are derived.

Keywords: classical phase transitions, exact results, phase diagrams

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1. Introduction

The Potts model [1, 2] has been extensively studied in the past few decades and the amount of related literature is enormous. Though the model has some experimental realizations, it is mostly useful as a laboratory for analytical concepts and simulations methods [3–17]. In the context of temperature-driven phase transitions, the Potts model can be realized as a generalization to the celebrated Ising model with the typical ‘up–down’ symmetry, as it has a multi-fold symmetry that is spontaneously broken.

An interesting problem that has been first studied by Baxter [3] is the dependence of the transition nature (first versus second order) on the spin number \(q\). Baxter considered the square lattice ferromagnetic model with nearest neighbors interaction (henceforth referred to as the standard model) and obtained an exact expression for the latent heat. That expression remained finite for \(q > 4\) and vanished for \(q = 4\). Using an equivalence of the Potts model to a six-vertex model [18], Baxter has shown that the free energy of the latter near the critical temperature was negligibly small. Based on these findings, Baxter conjectured that the model in subject exhibits a continuous transition for \(q \leq q_c\) and discontinuous transition for \(q > q_c\) where \(q_c = 4\) is the changeover integer or the maximal integer for which the transition is continuous. Recently, Duminil-Copin et al [19, 20] have rigorously proven Baxter’s conjecture using the random cluster or the Fortuin–Kasteleyn [21] representation of the Potts model. Another interesting manifestation of \(q_c = 4\) being a changeover integer is the presence of a multiplicative logarithmic correction term to the leading order power law divergence at criticality of, e.g., the specific heat and the magnetic susceptibility [6, 22]. This is in contrast to \(q < 4\) where this term is absent.

Though the changeover integer \(q_c = 4\) has been found for the square lattice, it is expected, due to universality, to hold for other translation-invariant lattices [4]. More precisely, two dimensional systems accompanying the spontaneous breaking of the \(q\)-fold Potts symmetry, are expected to maintain the changeover behavior of the standard model for a wide range of local interaction patterns. There are, however, a few counterexamples [23–25].

In [24, 25] the authors introduced the so-called Potts model with invisible colors (ICs) where the spins assume \(q\) ‘visible’ colors (Potts states) as in the local interaction standard model and additional \(r\) ‘ICs’ that control the entropy of but do not affect the energy of the system. The model has a growing interest in recent years and problems such as its marginal dimensions [26] or its behavior on general Bethe lattices [27], thin graphs [28] or scale-free networks [29], have been addressed.

It has been demonstrated [24] using a mean field (MF) solution to the Blume–Emery–Griffiths model [30] together with simulations, and also rigorously proven [31, 32], that the ICs model exhibits a first order transition for any \(q\) when \(r\) is sufficiently large. A non-trivial outcome of the latter result is the occurrence of a first order
transition for $q \leq 4$. The reason is that, typically, in the continuous transition cases large fractal-like configurations buffer between order and disorder, and, even though the number of such fractals grows exponentially with their size, it does not compensate the entropic cost of their construction when the total number of microstates is large enough.

The continuous Potts gas (CPG) is another manifestation of a system displaying a changeover phenomenon (CP), that is, switching from one type of transition to another. CPG is related to a class of models where interacting particles sit in a box that occupies some volume in a continuous space. The number of different particles may be thought of as the equivalent of the $q$ color variable in the lattice model [33–36]. External field as well as color-independent two-body interaction may be additionally applied. MF analysis of CPG [37] indicates a peculiar CP for that model. For instance, a proper choice of parameters leads to a discontinuous transition for a two-particle ($q = 2$) system [37, 38].

We present a hybrid Potts (HP) model that captures the concept of manipulating the transition order, while keeping the model’s parameters fixed, by introducing inhomogeneity in the number of colors. Strictly speaking, we consider a Potts Hamiltonian $H(\{\sigma\})$ where $\{\sigma\}$ is a configuration of $N$ spins, each can select one out of the colors $\{1, 2, \ldots, Q_i\}$ where

\[ Q_i = -X_i(q - q_0) + q, \quad i = 1, \ldots, N, \]  

are i.i.d. random variables with $X_i \sim \text{Ber}(p)$. In other words, each spin can be colored in $q_0 \leq q_c$ ‘strong’ colors with probability $p$ and in $q > q_c$ colors (containing the same $q_0$ colors and additional $q - q_0$ ‘weak’ colors), with probability $1 - p$. We say that spins chosen with probability $p$ or $1 - p$ belong to strong or weak regions, respectively.

Since the HP model assumes varying concentrations of spins with different types of colors, between limiting homogeneous instances that lead to a transition which is either continuous or discontinuous, one might expect that at some concentration, $0 < p^* < 1$, the model exhibits a CP for every combination of $q_0 \leq q_c$ strong colors and a total number of $q > q_c$ colors.

The rest of the paper is organized as follows. In section 2 we find $p^*$ for the standard model. We also present the results of Monte-Carlo (MC) simulations. A detailed analysis of the MF model equipped with the HP prescription (1) is reported in section 3. Finally, our observations are summarized and discussed in section 4.

2. The standard model

In the present section we detect $p^*$ using a simple energy-entropy first principle argument. To be more specific, we compute the entropy loss against the energy gain in generating long range order (LRO), where it is known that in a second order transition, LRO is usually established by monochromatically coloring fractal clusters that form an exponential family with positive geometric entropy, and in a first order transition LRO is associated with the formation of monochromatic clusters that are taken from a sub-exponential family having zero geometric entropy. By comparing the changes in the corresponding free energies, $p^*$ can be found.

1 Any Potts Hamiltonian known to describe a system undergoing a ferromagnetic phase transition may apply.
We start with a system of \( N \) spins situated on the vertices of the square lattice and interacting via the usual Hamiltonian
\[
H_B = -J \sum_{\langle i,j \rangle} \delta \sigma_i \sigma_j,
\]
(2)
where \( J > 0 \) is the ferromagnetic coupling constant (\( J \), together with Boltzmann’s constant, \( k_B \), is henceforth set to one for convenience) and the summation is taken over nearest neighboring spins.

We call a large cluster simple if its number of sites per bond is minimal, i.e., \( 1/2 + o(1) \). We term a snake, a fractal with a maximal number of \( 1 + o(1) \) sites per bond. Any fractal therefore grows no faster than a snake and has \( 1/2 + \delta + o(1) \) sites per bond, where \( \delta \) is a positive parameter no larger than \( 1/2 \).

Consider a fractal contour made of \( l = O(N) \) bonds. The contour has energy \( -l \).

Suppose the fractal has \( n \) sites where \( k \) of them are positioned in strong regions and \( n/l = 1/2 + \delta + o(1) \). Let \( \#(k,n) \) be the number of such fractals. The expected number of these fractals is
\[
\langle \#(k,n) \rangle = \mu^n \binom{n}{k} p^k (1 - p)^{n-k},
\]
(3)
where \( \sum_k \#(k,n) \sim \mu^n \) and the growth number \( \mu \) in general depends on \( \delta \) [23]. In making such fractals monochromatic, the entropy of the system is changed in the amount of
\[
-\beta \Delta f_{\text{frac}} = 2 \beta \frac{1}{1 + 2\delta} \ln \mu - \ln q + \sum_k \left( \frac{1}{n} \ln \binom{n}{k} p^k (1 - p)^{n-k} + \frac{k}{n} \ln \left( \frac{q}{q_0} \right) \right).
\]
(4)
Under the assumption of NDAM, the sum in (4) can be replaced with the maximum of the summand obtained at \( k \) satisfying
\[
\kappa = \frac{pq}{pq + (1 - p)q_0},
\]
(5)
with \( \kappa = k/n \). This brings (4) into the form
\[
-\beta \Delta f_{\text{frac}} = 2 \beta \frac{1}{1 + 2\delta} + \ln \mu - \ln q - \kappa \ln \kappa - (1 - \kappa) \ln (1 - \kappa) + \kappa \ln \left( \frac{q}{q_0} \right).
\]
(6)
Since, unlike fractals, large simple clusters grow sub-exponentially with their size, they are randomly distributed across the lattice. Accordingly, the change in the free energy per site due to simple clusters with the same energy \( -l \) is given by
\[
-\beta \Delta f_{\text{sim}} = 2 \beta - p \ln q_0 - (1 - p) \ln q.
\]
(7)
It may be more constructive to form large monochromatic simple clusters rather than fractals. Specifically, within the framework of (6) and (7), if at $\beta$ solving $\Delta f_{\text{sim}} = 0$ we have $\Delta f_{\text{frac}} \geq 0$, then it is entropically disadvantageous for the system to possess large fractals at that temperature. Instead, large simple monochromatic clusters are formed and the system undergoes a first order transition [23] at

$$\beta_c \approx \frac{1}{2} \left( p \ln q_0 + (1 - p) \ln q \right).$$

The marginal concentration $p^*$ below which large simple clusters are entropically more favorable than fractals can be estimated by taking $\kappa^*$ to satisfy (5) at $p^*$, plugging it together with the rhs of (8) into (6) and solving

$$\sup_{\delta} \left( \frac{p^* \ln q_0 + (1 - p^*) \ln q}{1 + 2\delta} + \ln \mu - \ln q - \kappa^* \ln \kappa^* - (1 - \kappa^*) \ln(1 - \kappa^*) + \kappa^* \ln p^* 
+ (1 - \kappa^*) \ln(1 - p^*) + \kappa^* \ln \left( \frac{q}{q_0} \right) \right) = 0,$$

for $p^*$. It may be useful to write (9) as

$$\sup_{\delta} \varphi(\delta, p^*, q_0, q) = 0.$$  

Apart from uncovering the dependence of $p^*$ on the spin numbers $q_0, q$, (10) tells that $p^*$ is indeed marginal, since, at a temperature that is a candidate to be the first order critical point, $\Delta f_{\text{frac}} = 0$, even for fractals that are taken from the most probable exponential family.

For a large total number of colors, a first order transition occurs when the concentration of strong regions is small. The reason is that, for $q$ large, the construction of monochromatic exponential families of fractals is entropically too expensive when the system is too sparsely populated by $q_0$-type spins. The critical behavior in this case is determined by the simple clusters and the transition is of first order. In order to quantify how sparse the strong regions are, it may be useful to find $p^*$ in the large $q$ limit. To this end we expand $\varphi(\delta, p^*, q_0, q)$ up to $o(p^*)$ and then solve (10) for $p^*$, keeping the leading order large $q$ term. This results in a monotonically decreasing behavior of $p^*$.

In particular, if for varying $q$ the sup in (10) is determined by values of $\delta$ that are not arbitrarily small, i.e., satisfying $\inf q \delta > 0$, we have

$$p^* \sim \frac{\ln q}{q},$$

which means that a first order transition takes place for small concentrations, typically below $\frac{\ln q}{q}$, of strong regions.

It should be noted that even when the concentration of strong spins is well below the percolation threshold, the presence of these spins can affect the transition nature. Specifically, when $q$ is large, sparsely populated strong spins may lead to a second order transition since they considerably reduce the entropic cost of forming a monochromatic cluster having one of the strong colors.
Table 1. PC temperatures, $T_L$, and maximal values of the specific heat, $\text{max} C_L$, for different concentrations. The spin numbers are $q_0 = 2$, $q = 50$ and the lattice linear size is $L = 40$. The uniform error bar (0.001) of the temperatures is roughly estimated using ten WL runs. The jackknife method is employed to compute the error bars of $\text{max} C_L$.

| $p$   | $T_L$       | $\text{max} C_L$ |
|-------|-------------|------------------|
| 0     | 0.4793      | 3989 ± 16        |
| 0.05  | 0.4977      | 3405 ± 9         |
| 0.1   | 0.5132      | 173.29 ± 0.01    |
| 0.15  | 0.5316      | 61.498 ± 0.001   |

Our next goal is to numerically support the theory presented up to this point. For that purpose, we performed extensive MC simulations using the Wang–Landau (WL) method [16, 17] to calculate the density of states with energy $E$, $\Omega(E)$. Being equipped with the quantities $\Omega(E)$ enables us to compute the energy probability-distribution function (PDF) at any desired temperature. In particular, we are interested in the PDF at pseudo-criticality, e.g., at a temperature where the specific heat $C_L = L^{-d} \beta^2 (\langle E^2 \rangle - \langle E \rangle^2)$ with the energy moments given by $\langle E^n \rangle = \sum E^n e^{-\beta E} \sum E e^{-\beta E}$, is maximal [23, 40]. The pseudo-critical (PC) temperatures, $T_L$, together with the maximal values of the specific heat for different values of $p$, for a sample with a linear size $L = 40$ ($L = \sqrt{N}$) and spin numbers $q_0 = 2$, $q = 50$, are given in table 1. In order to roughly estimate the error bars of $T_L$ we ran the WL simulations ten times for a concentration $p = 1$ and $L = 40$. Taking the std of the PC temperatures obtained from these runs, where the temperature resolution in the canonical ensemble is 0.0001, gives 0.0008, thus, a plausible estimate of the error bars would be 0.001. The error bars of the maxima are computed using the jackknife method.

In figure 1, a plot of the specific heat against temperature is given for different concentrations. It can be verified that the peaks for the concentrations $p = 0$ and $p = 0.05$ are similar in magnitude and width. On the other hand, the peaks of the larger concentrations are (at least) an order of magnitude smaller and also broader. This may qualitatively indicate different scaling behaviors and thus different types of transitions.

In figure 2 we plot the PC energy PDF against the energy (per site). The temperatures at which the PDFs are computed are those presented in table 1. For $p = 0$, the values of the ordered and disordered energies (the positions of the two peaks) are $\varepsilon_o \simeq -1.9275$ and $\varepsilon_d \simeq -0.3544$, respectively, yielding $-(\varepsilon_o + \varepsilon_d) \simeq 2.2819$, in excellent agreement with $-(\varepsilon_o + \varepsilon_d) = 2(1 + 1/\sqrt{50}) = 2.282842 \ldots$ [41]. The excellent agreement between the PC temperature and the exact $T_c = 1/\ln(1 + \sqrt{50}) = 0.478862 \ldots$ [42] should be also noted. In particular, since $L = 40$ is larger than the correlation length [43], it is likely that the typical first order scaling law $|T_L - T_c| = O(L^{-d})$ ($|T_L - T_c| \simeq 0.0004$, cf $40^{-2} \simeq 0.0006$), is satisfied.

In a second order transition, the PC energy PDF is expected to display a single peak centered around the PC energy. On the other hand, in a first order transition, a
Figure 1. Specific heat as a function of temperature, for a sample with a linear size $L = 40$, and spin numbers $q_0 = 2$, $q = 50$, displayed on a semilogarithmic scale. The peaks are positioned at temperatures given in table 1. Note the differences in the peaks’ magnitude between the two lowest concentrations and the other concentrations.

Figure 2. PC energy PDF for the bond-interaction model on the square lattice. The ‘strong’ Potts variable is $q_0 = 2$ and the total number of colors is $q = 50$. The lattice linear size is $L = 40$.

double peaked distribution centered around the energies of the coexisting ordered and disordered states is conventional [41]. Evidently, while the broad PDF for $p = 0.1$ is a manifestation of either a weak first order transition or a single second order peak that suffers from finite size effects, two pronounced peaks are observed for concentrations lower than $p = 0.1$ and a clear single peak is present for $p = 0.15$. The marginal concentration for a first order transition is therefore expected to be $0.05 < p^* \leq 0.15$, in reasonable agreement with (11) yielding $p^* \sim \frac{\ln 50}{50} \approx 0.08$. 

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3. MF model

In the present section we adopt the Bragg–Williams approach to solve the MF HP model. The MF Hamiltonian is given by

\[ H_{\text{MF}} = -\frac{1}{N} \sum_{i<j} \delta_{\sigma_i, \sigma_j}, \]  

(12)

where the normalization factor \( N^{-1} \) assures that the total energy is extensive. Let \( \xi_i \) and \( \eta_i \) be the fractions of spins with color \( i \in \{0, 1, \ldots, q_0-1, \ldots, q-1\} \), in strong and weak regions, respectively. The total energy can be expressed as

\[ \mathcal{E} = -\frac{1}{2} N \sum_i (p \xi_i + (1-p) \eta_i)^2 \]  

and the number of states with this energy is given by

\[ W = \left( p N \xi_0, \ldots, p N \xi_{q-1} \right) \left( (1-p) N \eta_0, \ldots, (1-p) N \eta_{q-1} \right). \]  

Thus, introducing the Lagrange multipliers \( a, b \), the free energy per site takes the form

\[ \beta f = \lim_{N \to \infty} \frac{1}{N} (\beta \mathcal{E} - \ln W) + \text{constraints} = \sum_i \left( p \xi_i \ln \xi_i + (1-p) \eta_i \ln \eta_i - \frac{1}{2} \beta (p \xi_i + (1-p) \eta_i)^2 \right) + a \left( \sum_i \xi_i - 1 \right) + b \left( \sum_i \eta_i - 1 \right). \]  

(13)

In order to find the quantities that minimize the free energy, we set \( \xi_j = 0, j = q_0, \ldots, q-1 \), differentiate (13) with respect to \( \xi_j, \eta_j \) and equate to zero. This gives

\[ \ln \xi_j - \beta (p \xi_j + (1-p) \eta_j) + a/p + 1 = 0, \ j = 0, \ldots, q_0-1, \]

\[ \ln \eta_j - \beta (p \xi_j + (1-p) \eta_j) + b/(1-p) + 1 = 0, \ j = 0, \ldots, q-1. \]  

(14)

The minimizing fractions \( \xi_j, \eta_j \) therefore satisfy the coupled equations

\[ \xi_j = \begin{cases} e^{\beta (p \xi_j + (1-p) \eta_j)} / Z_0 & , \ j = 0, \ldots, q_0-1 \\ 0 & , \ j = q_0, \ldots, q-1, \end{cases} \]  

(15)

and

\[ \eta_j = \begin{cases} e^{\beta (p \xi_j + (1-p) \eta_j)} / Z & , \ j = 0, \ldots, q_0-1 \\ e^{\beta (1-p) \eta_j} / Z & , \ j = q_0, \ldots, q-1, \end{cases} \]  

(16)

where \( Z_0 = e^{a+1} = \sum_{i=0}^{q_0-1} e^{\beta (p \xi_i + (1-p) \eta_i)} \) and \( Z = e^{b+1} = Z_0 + \sum_{i=q_0}^{q-1} e^{\beta (1-p) \eta_i} \) are the colors partition functions in strong and weak regions, respectively. The quantities in (15) and
Figure 3. Strong magnetization component $m_0$ as a function of temperature, for different concentrations. The total number of colors is $q = 6$. It should be noted that the symbols composing the graphs for $p \geq 0.4$ are more dense than for $p \leq 0.2$, in some neighborhood of $T_0$. A combination of continuous and discontinuous transitions at critical temperatures near $T_0 = 0.1518$ and $T_0 = 0.1021$, respectively, for $q = 1000$ and $p = 0.3$, is presented in the inset.

(16) can take the form

$$
\xi_j = \begin{cases} 
\frac{1}{q}(1 + (q_0 - 1)m_0) & , \quad j = 0 \\
\frac{1}{q_0}(1 - m_0) & , \quad j = 1, \ldots, q_0 - 1 \\
0 & , \quad j = q_0, \ldots, q - 1,
\end{cases}
$$

(17)

and

$$
\eta_j = \begin{cases} 
\frac{1}{q}(1 + (q_0 - 1)m_0)(1 + (q/q_0 - 1)m_1) & , \quad j = 0 \\
\frac{1}{q}(1 - m_0)(1 + (q/q_0 - 1)m_1) & , \quad j = 1, \ldots, q_0 - 1 \\
\frac{1}{q}(1 - m_1) & , \quad j = q_0, \ldots, q - 1,
\end{cases}
$$

(18)

where $m_0, m_1$ are the components of a two-fold magnetization $m$. Indeed, there are favored fractions $\xi_0 \geq \xi_j$ and $\eta_0 \geq \eta_j$ for all $1 \leq j \leq q - 1$, assuring the existence of LRO. Equations (17) and (18) also make sure that the ratio between the favored $j = 0$ color and the other $q_0 - 1$ strong colors in both regions is preserved.

Above the critical point, at the disordered state, strong colors are uniformly distributed so that (17) and (18) imply

$$
m_0 = 0.
$$

(19)
Figure 4. MC time variation of the simulated MF magnetization given by (27). The simulated system has $N = 500$ spins and a total spin number $q = 6$. For clarity, the presented observables capture a portion of 10% of the total number of sweeps. Two different concentrations corresponding to a first or a second order transition together with temperatures in the vicinity of $T_0$ (to be precise $T = T_0 + \frac{5}{N} = T_0 + 0.01$), are chosen. (a) $p = 0.1$ and $T = 0.2661$. (b) $p = 0.6$ and $T = 0.4199$. The magnetization is averaged over ten different runs. Energy per site (also averaged over ten different runs) against MC time is plotted in the inset. The energy fluctuates around $\langle\text{energy}\rangle \simeq -0.1822$ (std(energy) $\simeq 0.0037$), in plausible agreement with $\varepsilon_c = -0.172011 \ldots$ due to (26).

In the case that the transition is continuous, when plugging (17) and (18) into (13), the magnetization at the critical point, $m^*$, must satisfy $\nabla_m f = 0$ where (19) holds. It is known [2, 44] that for the homogeneous model $q_c = 2$. Thus, in the following we set $q_0 = 2$. Denoting the gradient components by $(g_0, g_1)$, the latter read

$$g_0 = -\frac{1}{2q^2} \left( (1-p)q (2 + m_1(q-2)) \ln \left( \frac{1-m_0}{1+m_0} \right) + \beta m_0 (m_1(1-p)(q-2) + p(q-2) + 2) \right) + p \tanh^{-1}(m_0),$$

(20)

$$g_1 = -\frac{(1-p)(q-2)}{2q^2} \left( \beta (1+m_0^2) q(m_1(1-p) + p) + 2\beta m_0^2 (1-m_1)(1-p) + q \left( m_0 \ln \left( \frac{1-m_0}{1+m_0} \right) - 2 \ln \left( \frac{2 + m_1(q-2)}{2q} \right) - \ln \left( 1-m_0^2 \right) \right) \right).$$

(21)
Substituting (19) in (20), $g_0$ indeed vanishes. The critical temperature is obtained by the further condition that the magnetization at criticality is an unstable point of the free energy. This can be formulated by considering the Hessian matrix $H(m, \beta, p, q)$ given by

\[
H(m, \beta, p, q) = \begin{pmatrix}
-\beta (1-m^2_0)(q-2) - 2m_1(1-p)(q-2) - 2p \\
(1-q)(q-2) \left[ 2\beta m_0 - 2p - m_1(1-p)(q-2) - 2q \ln \left( \frac{1+m_1}{1-m_0} \right) \right]
\end{pmatrix}
\]

computed at $m^*$ satisfying $m^*_0 = 0$, $g_1(m^*, \beta_c, p, q) = 0$, where the Hessian has a vanishing eigenvalue, or, equivalently, obeys

\[
\det H(m^*(\beta_c, p, q), \beta_c, p, q) = 0.
\]  

(23)

Fixing $q$, (23) implicitly determines the second order critical line in concentration-temperature plane by

\[
\frac{1}{2} \exp \left( \frac{\beta_c - q}{q-2} \right) = \frac{\beta_c - 2}{(q-2)(2-\beta_c p)}.
\]  

(24)

The critical magnetization reads

\[
m^* = \left( 0, \frac{q(2-\beta_c p)}{\beta_c (1-p)(q-2)} - \frac{2}{q-2} \right).
\]  

(25)

Plugging (25) into the energy part of (13) yields the following expression for the critical energy

\[
\varepsilon_c = -\frac{(\beta_c - 4)\beta_c + 2q}{2\beta_c^2(q-2)}.
\]  

(26)
It is expected that the critical temperature, $\beta_c$, behaves as a continuous function of the concentration $p \in [0, 1]$. However, the solution to (24) at $p = 0$ gives $\beta_c = q$, whereas the homogeneous system undergoes a first order transition at $\beta_c = \frac{2(q-1)\ln(q-1)}{q-2}$ [2]. Thus, continuity implies that there exists a finite concentration $p^*$ where for $p < p^*$ (24) does not hold, and the transition becomes a first order transition. It should be noted that $p^*$ is expected to vary with $q$.

A first order transition is associated with a discontinuity of the magnetization at the critical point. In other words, there are two points $m^*$ and $m^{**}$ that simultaneously minimize the free energy, that is, simultaneously solve $\nabla m f = 0, f(m^*) = f(m^{**})$.

The nature of the transition and, consequently, the presence of $p^*$, can be phenomenologically detected by fixing the concentration and examining the behavior of $m$ that numerically minimizes (13) when the temperature is varied. In figure 3, the ‘strong’ component $m_0$ is plotted against temperature, for $q = 6$ and different concentrations. Apparently, for $p \leq 0.2$ the magnetization has a discontinuity in the vicinity of $T_0$ (taken to be the point where the numerical derivative $\Delta m_0 / \Delta T$ is ‘large’), indicating the presence of $m_0^* = 0, m_0^{**} > 0$, whereas for $p \geq 0.4$ it is continuous at $T_0$. It is therefore expected that $0.2 < p^* \leq 0.4$ for this spin number.

Interestingly, as can be verified from the inset of figure 3, for $q = 1000$ there is a concentration ($p = 0.3$) such that, when cooled, the system first exhibits a continuous transition and then a discontinuous transition at a lower temperature. A similar behavior has been found for other concentrations. For $q = 6$ this phenomenon has not been clearly observed. Thus, it is expected that for small $q$, the dual transition is either a weak effect or does not exist. This may exemplify the observation that in the case where in addition to the second order line, a first order branch is launched, $p^*$ being the concentration where the second order line terminates, is well defined.

We next discuss the results of Metropolis [44] MC simulations performed to capture numerically the concentration-dependent CP for the MF model. We consider the simulated magnetization (SM)

$$\text{SM} = p \left( \frac{q_0 z_0 - 1}{q_0 - 1} \right) + (1 - p) \left( \frac{q z - 1}{q - 1} \right),$$

where $z_0$ and $z$ are the maximal fractions of monochromatic spins in strong and weak locations, respectively, and $q_0 = 2$. A sample of $N = 500$ spins is used in the simulations where the SM together with the energy (per site) proportional to (12) are observed over a total number of $5 \times 10^5$ MC sweeps. In figure 4(a) it is seen that the SM performs multiple ‘flip-flops’ between two values of ordered and disordered states, as expected from a system undergoing a first order transition. On the other hand, in figure 4(b) the system exhibits a typical second order transition by means of fluctuations around a single value.

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2 Indeed, the implicit functions theorem guarantees that, for $q$ fixed, (24) defines a continuously differentiable function $\beta_c = \psi(p)$, $p \in (0, 1)$.

3 In practice we measure, e.g. for $p = 0.1$, $m_0(0.2560) = 0.9149$ associated with $m_0^*$ and $m_0(0.2561) = 2.337 \times 10^{-5}$ associated with $m_0^{**}$, such that $f(0.2560) = -2.0711 \simeq f(0.2561) = -2.0242$.

4 For instance, the $p = 0.2$ graph may obscure a small finite magnetization in the vicinity of a second order critical point. If such a finite magnetization exists, then $p^* < 0.2$. 

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We conclude this section by delving into the continuous regime of figure 3, first by solving (24) taking $q = 6$ and $p = 0.6$, to give $T_c = \beta_c^{-1} = 0.409809\ldots$. Next, from figure 5, where a plot of $m$ for $q = 6$ and $p = 0.6$ against temperature is presented, it can be deduced that since the value of $T_0$ agrees well with $T_c$, $m$ is continuous at $T_c$. It should be emphasized at this point that, unlike $m_0$, $m_1$ is actually not a proper order parameter as it does not vanish at any finite temperature. It has, however, a computational value. For instance, it can be verified from figure 5 that $m_1$ is not differentiable at $T_c$ which makes $T_c$ a unique temperature. Furthermore, substituting $q = 6$, $p = 0.6$, $\beta_c = 2.440161\ldots$ in (25), gives $m_1^* = 0.323567\ldots$ which is in excellent agreement with $m_1$ computed at $T_0$.

4. Summary and discussion

It is demonstrated on the standard Potts model that in a mixed system of spins randomly colored in $q_0$ or $q > q_c \geq q_0$ colors, the transition order depends on the concentration $p$ of $q_0$-type spins. For that system there is a critical concentration $0 < p^* < 1$ such that the transition is discontinuous when $p$ is below $p^*$. For a small concentration of strong regions and a large total number of colors, a CP is verified analytically and numerically. In the large $q$ limit, $p^*$ displays a monotonically decreasing behavior. This means that in the asymptotic regime, with increasing values of $q$, it is entropically beneficial to construct sparser monochromatic fractals that may have a declining growth constant, than simple large-scale clusters.

The Bragg–Williams approximation, applied to the MF model, uncovers the presence of the marginal concentration for that model, where the strong regions are occupied by Ising-like spins. Unlike in the standard model where it is argued that a first order transition occurs for concentrations below $p^*$, in the MF model we conclude the presence of $p^*$ from the other direction, that is, the second order critical temperature, as a function of $p$, is defined on some interval with a left endpoint which is equal to $p^*$.

It is inspected that for $q$ large there are concentrations hosting both continuous and discontinuous transitions. A similar phenomenon, namely, a sequential continuous and discontinuous transition occurring at some values of $r$, has been reported for the ICs model [26] where the Bragg–Williams approximation in the context of the MF model was employed.

A reasonable explanation for the large $q$ dual transition is that, at the higher temperature it is entropically favorable to typically leave the weak regions disordered, hence, Ising-like LRO is established in the strong regions. However, when the temperature is lowered, at some point it becomes energetically beneficial to expand the predetermined LRO by invading the weak regions. It is yet not clear whether the dual transition is manifested for any $q > 2$.

The simulations produce observables that have a different qualitative behavior for different concentrations. In the standard model case the PC energy PDF changes its shape from dual to single peaked, with increasing values of $p$ and in the MF case the magnetization has a typical single versus non-single valued temporal behavior, for different values of $p$. Thus, in both cases, the essence of $p^*$ as a changeover point is revealed.
Our model provides a rather general framework of governing the transition order of well known Potts systems, without interfering with the interaction content of such systems when no heterogeneity in the number of colors is introduced, in the sense that a CP is observed for \textit{any} setup of strong and weak colors. We believe that the HP machinery can be applied to other Potts systems such that a concentration-dependent CP is detected in those systems.

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