RESOLUTION AND REGULARITY OF COVER IDEALS OF CERTAIN
MULTIPARTITE GRAPHS

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Abstract. Let $G$ be a finite simple graph on $n$ vertices. Let $J_G \subset \mathbb{K}[x_1, \ldots, x_n]$ be the cover ideal of $G$. In this article, we obtain the graded minimal free resolution and the Castelnuovo-Mumford regularity of $J_G^s$ for all $s \geq 1$ for certain multipartite graphs $G$.

1. Introduction

Recently there have been a lot of research on various properties of powers of homogeneous ideals. In particular, there have been a lot of interest on Castelnuovo-Mumford regularity of powers of ideals. It was shown by Kodiyalam [19] and independently by Cutkosky, Herzog and Trung [5] that if $I$ is a homogeneous ideal in a polynomial ring, then there exist non-negative integers $d, e$ and $s_0$ such that $\text{reg}(I^s) = ds + e$ for all $s \geq s_0$, where $\text{reg}(\cdot)$ denote the Castelnuovo-Mumford regularity. Kodiyalam proved that $d \leq \deg(I)$, where $\deg(I)$ denotes the largest degree of a homogeneous minimal generator of $I$. In general, the stability index $s_0$ and the constant term $e$ are hard to compute. There have been discrete attempts in identifying $s_0$ and $e$ for certain classes of ideals. Given a finite simple graph, one can identify the vertices with indeterminates and associate ideals in a polynomial ring, for example, edge ideals and cover ideals (see Section 2 for definition) corresponding to the given graph. For edge ideals of various classes of graphs, $e$ and $s_0$ have been computed, see for example [1, 6, 7, 10, 15, 17, 18, 20, 21, 23, 28, 29, 30, 31]. In the case of edge ideals, $d = 2$ and in the known cases, $e$ is connected to combinatorial invariants associated with the graph $G$. Not much is known about the regularity of powers of cover ideals. Since cover ideal is the Alexander dual of the edge ideal, its regularity is equal to the projective dimension of the edge ideal, [25]. Seyed Fakhari studied certain homological properties of symbolic powers of cover ideals of very well-covered and bipartite graphs. For a finite simple graph $G$ on the vertex set $\{x_1, \ldots, x_n\}$, let $J_G \subset \mathbb{R} = \mathbb{K}[x_1, \ldots, x_n]$, where $\mathbb{K}$ is a field, denote the cover ideal of $G$. It was shown that if $G$ is a very well-covered graph and $J_G$ has a linear resolution, then $J_G^s$ has a linear resolution for all $s \geq 1$. Furthermore, it was proved that if $G$ is a bipartite graph with $n$ vertices, then for $s \geq 1$,

$$\text{reg}(J_G^s) \leq s \deg(J_G) + \text{reg}(J_G) + 1.$$ 

Hang and Trung, in [26], studied unimodular hypergraphs and proved that if $\mathcal{H}$ is a unimodular hypergraph on $n$ vertices and rank $r$ and $J_{\mathcal{H}}$ is the cover ideal of $\mathcal{H}$, then there exists a non-negative integer $e \leq \dim(\mathbb{R}/J_{\mathcal{H}}) - \deg(J_{\mathcal{H}}) + 1$ such that

$$\text{reg}J_{\mathcal{H}}^s = \deg(J_{\mathcal{H}})s + e$$

for all $s \geq \frac{n}{r} + 1$. Since bipartite graphs are unimodular, their results hold true in the case of bipartite graphs as well. While the first result gives an upper bound for the constant term, the later result gives the upper bound for both the stability index and the constant term.

The Betti numbers and regularity are classical invariants associated to a module which can be computed from the resolution. However, one may be able to compute them without completely describing the resolution. To compute the syzygies explicitly is a challenging task. In this article, we obtain the complete description of the minimal free resolution, including the syzygies, of $J_G^s$. 

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for some classes of multipartite graphs, thereby obtaining a precise expression for \( \text{reg}(J^s_G) \). The paper is organized as follows. In Section 2, we collect the preliminaries required for the rest of the paper. We study the resolution of powers of cover ideals of certain bipartite graphs in Section 3. If \( G \) is a complete bipartite graph, then \( J_G \) is a regular sequence and hence the minimal graded free resolution of \( J_G^s \) can be obtained from [9, Theorem 2.1]. It can be seen that, in this case the index of stability, \( s_0 = 1 \) and the constant term is one less than the size of the minimum vertex cover. We then move on to study some classes of bipartite graphs which are not complete. We obtain the resolution and precise expressions for the regularity of powers of cover ideals of certain bipartite graphs.

Section 4 is devoted to the study of resolution and regularity of powers of cover ideals of certain complete multipartite graphs. When \( G \) is a complete multipartite graph if \( G \) can be obtained from \([9, \text{Theorem 2.1}]\). It can be seen that, in this case the index of stability, \( s_0 = 1 \) and the constant term is one less than the size of the minimum vertex cover.

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\section{Preliminaries}

In this section, we set the notation for the rest of the paper. All the graphs that we consider in this article are finite, simple and without isolated vertices. For a graph \( G \), \( V(G) \) denotes the set of all vertices of \( G \) and \( E(G) \) denotes the set of all edges of \( G \). A graph \( G \) is said to be a complete multipartite graph if \( V(G) \) can be partitioned into sets \( V_1, \ldots, V_k \) for some \( k \geq 2 \) such that \( \{x, y\} \in E(G) \) if and only if \( x \in V_i \) and \( y \in V_j \) for \( i \neq j \). When \( k = 2 \), the graph is called a complete bipartite graph. If \( k = 2 \) with \( |V_1| = m \) and \( |V_2| = n \), we denote the corresponding complete bipartite graph by \( K_{m,n} \) or by \( K_{V_1,V_2} \). If \( G \) and \( H \) are graphs, then \( G \cup H \) denote the graph on the vertex set \( V(G) \cup V(H) \) with \( E(G \cup H) = E(G) \cup E(H) \). A graph \( G \) is called a bipartite graph if \( V(G) = V_1 \cup V_2 \) such that \( \{x, y\} \in E(G) \) only if \( x \in V_1 \) and \( y \in V_2 \). A subset \( w = \{x_{i_1}, \ldots, x_{i_r}\} \) of \( V(G) \) is said to be a vertex cover of \( G \) if \( w \cap e \neq \emptyset \) for every \( e \in E(G) \). A vertex cover is said to be minimal if it is minimal with respect to inclusion.

Let \( G \) be a graph with \( V(G) = \{x_1, \ldots, x_n\} \). Let \( K \) be a field and \( R = K[x_1, \ldots, x_n] \). The ideal \( I(G) = \langle \{x_i x_j : \{x_i, x_j\} \in E(G)\} \rangle \subseteq S \) is called the edge ideal of \( G \) and the ideal \( J_G = \langle \{x_{i_1} \cdots x_{i_r} : \{x_{i_1}, \ldots, x_{i_r}\} \text{ is a vertex cover of } G \} \rangle \) is called the cover ideal of \( G \). It can also be seen that \( J_G \) is the Alexander dual of \( I(G) \).

Let \( S = R/I \), where \( R \) is a polynomial ring over \( K \) and \( I \) a homogeneous ideal of \( R \). For a finitely generated graded \( S \)-module \( M = \oplus M_i \), set

\[
\text{reg}_S M = \max\{i : \text{Tor}_i^S(M, K)_j \neq 0\},
\]

with \( t^S(M) = -\infty \) if \( \text{Tor}_i^S(M, K)_j = 0 \). The Castelnuovo-Mumford regularity, denoted by \( \text{reg}_S(M) \), of an \( S \)-module \( M \) is defined to be

\[
\text{reg}_S M = \max\{t^S_i(M) - i : i \geq 0\}.
\]

\section{Bipartite Graphs}

In this section, we study the regularity of powers of cover ideals of certain bipartite graphs. We begin with a simple observation concerning the vertex covers of a bipartite graph.
Proposition 3.1. Let $G$ be a bipartite graph on $n + m$ vertices. Then $G$ is a complete bipartite graph if and only if $J_G$ is generated by a regular sequence.

Proof. Let $V(G) = X \sqcup Y$ be the partition of the vertex set of $G$ with $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_{n+1}, \ldots, y_{n+m}\}$. First note that $J_G$ is generated by a regular sequence if and only if for any two minimal vertex covers $w, w'$, $w \cap w' = \emptyset$. If $G = K_{n,m}$, then $J_G = (x_1 \cdots x_n, y_{n+1} \cdots y_{n+m})$ which is a regular sequence. Conversely, suppose $G$ is not a complete bipartite graph. Since $G$ is a bipartite graph, note that $\prod_{x \in X} x, \prod_{y \in Y} y \in J_G$ are minimal generators of $J_G$. Therefore, there exist $x_{i_0} \in X$ and $y_{i_0} \in Y$ such that $\{x_{i_0}, y_{i_0}\} \notin E(G)$. Then $w = \{x_i, y_j : i \neq i_0 \text{ and } y_j \in N_G(x_{i_0})\}$ is a minimal vertex cover of $G$ that intersects $X$ as well as $Y$ non-trivially. Therefore $J_G$ is not a complete intersection. $\square$

First we discuss the regularity of powers of cover ideals of complete bipartite graphs. Since the cover ideal of a complete bipartite graph is a complete intersection, the result is a consequence of [10 Theorem 2.1].

Theorem 3.2. Let $J = J_{K_{m,n}}$ be the cover ideal of the complete bipartite graph $K_{m,n}$, $m \leq n$. Then $\text{reg}(J^s) = sn + m - 1$ for all $s \geq 1$.

Proof. Consider the ideal $I = (T_1, T_2) \subset R = K[T_1, T_2]$ with $\deg T_1 = m$ and $\deg T_2 = n$. It follows from [10 Theorem 2.1] that the resolution of $I^s$ is

$$0 \rightarrow \bigoplus_{a_1 + a_2 = s+1, a_i \geq 1} R(-a_1m - a_2n) \rightarrow \bigoplus_{a_1 + a_2 = s} R(-a_1m - a_2n) \rightarrow I^s \rightarrow 0. \quad (1)$$

Note that $J = (x_1 \cdots x_m, y_{m+1} \cdots y_{m+n})$. Set $x_1 \cdots x_m = T_1$ and $y_{m+1} \cdots y_{m+n} = T_2$. Then $J^s$ has the minimal free resolution as in (1). If $m \leq n$, then $\text{reg}(J^s) = sn + m - 1$. $\square$

It follows from Theorem 3.2 that in the case of the complete bipartite graph $K_{m,n}$, the stability index is 1 and the constant term is $\tau - 1$, where $\tau$ is the size of a minimum vertex cover.

We now move on to study the cover ideals of bipartite graphs that are not complete. Since this is a huge class, we do not expect that a single expression may represent the regularity of powers of their cover ideals. Therefore, we restrict our attention to some of the structured subclasses of bipartite graphs.

We begin our investigation by describing certain properties of the cover ideals of a certain class of Cohen-Macaulay bipartite graphs.

Theorem 3.3. Let $G$ be the graph with $V(G) = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ and $E(G) = \{\{x_1, y_j\}, \{x_i, y_i\} : 1 \leq j \leq n, 2 \leq i \leq n\}$. Let $J_G \subset R = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the cover ideal of $G$. Then

1. $\mu(J_G) = 2^{n-1} + 1$;
2. $J_G$ is generated in degree $n$ and has a linear quotient;
3. $J_G$ has an $n$-linear resolution and hence $\text{reg} J_G = n$;
4. $J_G^s$ has an $ns$-linear resolution and hence $\text{reg} J_G^s = ns$;
5. $\text{pd} R/J_G = n$.

Proof. (1): Let $w$ denote a minimal vertex cover of $G$. If $x_i \in w$, then one and only one of $\{x_i, y_j\}$ is in $w$. And, if $x_i \notin w$, then $w = \{y_1, \ldots, y_n\}$. Therefore, $J_G$ is minimally generated by elements of the form $y_1 \cdots y_n$ and $x_1x_i \cdots x_jy_{j+1} \cdots y_{n-1}$, where $0 \leq j \leq n - 1$ and $2 \leq i_k \leq n$ for all $k$. Therefore, $\mu(J_G) = 1 + \sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1} + 1$.

(2): Clearly a minimal vertex cover has $n$ elements so that $J_G$ is generated in degree $n$. Write $J_G = (f_1, \ldots, f_{2^{n-1}+1})$, where $f_1 < f_2 < \cdots < f_{2^{n-1}+1}$ written in the graded reverse lexicographic ordering induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$. We prove that $J_G$ has linear quotients. Note that it is enough to prove that for an $i$ and $j < i$, there exists $k < i$ such that $f_k : f_i = u_t$ and $u_t|f_j$, where
where \( u_i = x_i \) or \( y_i \). \[16\]. For \( f_i \), let \( A_i = \{ i_j \in [n] : \deg x_{i_j} f_i = 1 \} \). Then \( f_i = \prod_{r \in A} x_r \prod_{s \in A^c} y_s \).

Let \( f_j < f_i \) be given. Let \( r \in A_i \setminus A_j \). Let \( f_k = \frac{y_r}{x_r} f_i \). Then \( f_k < f_i \) and \( f_k : f_i = y_r \). Since \( r \notin A_j \), \( y_i/f_j \). Hence \( J_G \) has linear quotients.

(3)\&(4): Since \( J_G \) has linear quotients, \( J^*_G \) has a linear resolution for all \( s \geq 1 \). Since \( J^*_G \) is generated in degree \( ns \), \( \deg(J^*_G) = ns \).

(5): It follows from \[12\] Theorem 3.4 that \( G \) is a Cohen-Macaulay bipartite graph. Therefore, \( \deg(I(G)) = \nu(G) + 1 \), where \( \nu(G) \) denotes the independence matching number of \( G \). \[20\]. It is easy to see that \( \nu(G) = n - 1 \). Therefore, \( \deg(I(G)) = 2n - (n - 1) - 1 = n \). Since \( J_G = I(G)^\nu \), by \[25\] pdim\((R/J_G) = n \).

In a personal communication, we have been informed that Seyed Fakhari has proved that if \( G \) is a bipartite graph, then for all \( s \geq 1 \),

\[
s \deg(J_G) \leq \deg(J^*_G) \leq (s - 1) \deg(J_G) + |V(G)| - 1.
\]

We note that the class of bipartite graphs discussed in Theorem \[13\] attain the lower bound.

Let \( U_1, \ldots, U_n \) and \( V_1, \ldots, V_n \) be sets of vertices. Set \( U = U_1 \cup \cdots \cup U_n \) and \( V = V_1 \cup \cdots \cup V_n \). Let \( G = K_{U_1, V} \cup K_{U_2, V_2} \cup \cdots \cup K_{U_n, V_n} \). For \( 1 \leq i, j \leq n \), set \( U_i = \{ x_{i_1}, \ldots, x_{i_m} \} \) and \( V_j = \{ y_{j_1}, \ldots, y_{j_m} \} \). Let \( J_G \) denote the cover ideal of \( G \). Then \( J_G \) can be identified with the cover ideal described in Theorem \[13\] by taking \( x_i = \prod_{x_{i_j} \in U_i} x_{i_j} \) and \( y_j = \prod_{y_{j_j} \in V_j} y_{j_j} \). But in this case, since \( \deg x_i \) is not necessarily one, the resolution need not be linear and hence the computation of regularity is non-trivial. We now study the resolution and the regularity for \( n = 2 \).

**Theorem 3.4.** Let \( U = U_1 \sqcup U_2 \) and \( V = V_1 \sqcup V_2 \) be a collection of vertices with \( |U| = n \), \( |U_i| = n_i \), \( |V| = m \), \( |V_i| = m_i \) and \( 1 \leq n_i, m_i \) for \( i = 1, 2 \). Let \( G \) be the bipartite graph \( K_{U_1, V} \cup K_{U_2, V_2} \). Let \( R = K[x_1, \ldots, x_n, y_1, \ldots, y_m] \). Let \( J_G \subset R \) denote the cover ideal of \( G \). Then the graded minimal free resolution of \( R/J_G \) is of the form:

\[
0 \longrightarrow R(-n - m_2) \oplus R(-m - n_1) \longrightarrow R(-(n_1 + m_2)) \oplus R(-m) \oplus R(-n) \longrightarrow R \longrightarrow 0.
\]

In particular,

\[
\deg(J_G) = \max\{n + m_2 - 1, m + n_1 - 1\}
\]

**Proof.** It can easily be seen that the cover ideal \( J_G \) is generated by \( g_1 = x_1 \cdots x_n \), \( g_2 = y_1 \cdots y_m \), and \( g_3 = x_1 \cdots x_{n_1} y_{m_1+1} \cdots y_m \). Set \( X_1 = x_1 \cdots x_{n_1} \), \( X_2 = x_{n_1+1} \cdots x_n \), \( Y_1 = y_1 \cdots y_{m_1} \), and \( Y_2 = y_{m_1+1} \cdots y_m \). Then we can write \( g_1 = X_1 X_2 \), \( g_2 = Y_1 Y_2 \) and \( g_3 = X_1 Y_2 \).

Consider the minimal graded free resolution of \( R/J_G \) over \( R \):

\[
\cdots \longrightarrow F \xrightarrow{\partial_2} R(-n) \oplus R(-m) \oplus R(-(n_1 + m_2)) \xrightarrow{\partial_1} R \longrightarrow 0,
\]

where \( \partial_1(e_1) = g_1 \), \( \partial_1(e_2) = g_2 \), and \( \partial_1(e_3) = g_3 \).

Let \( ae_1 + be_2 + ce_3 \in \ker \partial_1 \). Then \( aX_1 X_2 + bY_1 Y_2 + cX_1 Y_2 = 0 \). Solving the above equation, it can be seen that,

\[
\ker \partial_1 = \text{Span}_R\{Y_2 e_1 - X_2 e_3, X_1 e_2 - Y_1 e_3\}.
\]

Also, it is easily verified that these two generators are \( R \)-linearly independent. Hence \( \ker \partial_1 \cong R^2 \).

Note that \( \deg(Y_1 e_1 - X_2 e_3) = n + m_1 \) and \( \deg(X_1 e_2 - Y_1 e_3) = n_1 + m \). Therefore, we get the minimal free resolution of \( R/J_G \) as:

\[
0 \longrightarrow R(-n - m_2) \oplus R(-m - n_1) \xrightarrow{\partial_2} R(-(n_1 + m_2)) \oplus R(-m) \oplus R(-n) \xrightarrow{\partial_1} R \longrightarrow 0,
\]

where \( \partial_2(a, b) = a(Y_1 e_1 - X_2 e_3) + b(X_1 e_2 - Y_1 e_3) \). The regularity assertion follows immediately from the resolution. \[\square\]
Now, our aim is to compute the resolution and regularity of $J^*_G$, where $J_G$ is the cover ideal discussed in Theorem 3.4. For this, we first study the resolution of powers of the ideal $(X_1X_2, X_1Y_2, Y_1Y_2)$ and obtain the resolution and regularity of the cover ideal as a consequence.

**Theorem 3.5.** Let $R = K[X_1, X_2, Y_1, Y_2]$ and $J = (X_1X_2, X_1Y_2, Y_1Y_2)$ be an ideal of $R$. Then, for $s \geq 2$, the minimal free resolution of $R/J^s$ is of the form

$$0 \longrightarrow R_s(2) \longrightarrow R^{2\binom{s+1}{2}}(2) \longrightarrow R^{\binom{s+2}{2}} \longrightarrow R \longrightarrow 0,$$

and $\operatorname{reg}(J^s) = 2s$.

**Proof.** Denote the generators of $J$ by $g_1 = X_1X_2$, $g_2 = X_1Y_2$, and $g_3 = Y_1Y_2$. Note that $J$ is the edge ideal of $P_4$, the path graph on the vertices $\{X_1, X_2, Y_1, Y_2\}$. Since $P_4$ is chordal, it follows that $R/J^s$ has a linear resolution for all $s \geq 1$, [14]. Now we compute the Betti number of the $R/J^s$.

Write

$$(g_1, g_2, g_3)^s = (g_1^s, g_1^{s-1}g_2, g_1^{s-2}g_2, \ldots, g_1g_2^{s-1}, g_2^s, g_3^s)$$

where

$$(g_2, g_3)^t = (g_2^t, g_2^{t-1}g_3, g_2^{t-2}g_3^2, \ldots, g_2g_3^{t-1}, g_3^t).$$

For $i \geq j$, set $M_{i,j} = g_1^{s-i}g_2^{i-j}g_3 = (X_1X_2)^{s-i}Y_2^jX_1^{i-j}$. It follows that $\mu(J^s) = \binom{s+1}{2}(s+2) - 2$. Let $t_1 = \binom{s+1}{2}(s+2)$. Let $\partial_1 : R^{t_1} \longrightarrow R$ be the map $\partial_1(e_{p,q}) = M_{p,q}$. Since $g_i$’s are monomials, the kernel is generated by binomials of the form $M_{i,j}M_{i,j} - m_{k,l}M_{k,l}$, where $m_{p,q}$’s are monomials in $R$. Since the resolution of $R/J^s$ is linear, it is enough to find the linear syzygy relations among the generators of $\ker(\partial_1)$.

To find these linear syzgies, we need to find conditions on $i, j, k, l$ such that $M_{i,j}M_{i,j}$ is equal to $X_p^2$ or $Y_q^2$ for some $p, q$. First of all, note that for such linear syzgies, $|i - k|, |j - l| \leq 1$. If $i = k$ and $j = l + 1$, then $M_{i,j} = g_1^s = Y_1$. We get the same relation if $i = k$ and $j = l - 1$. If $i = k + 1$ and $j = l$, then $M_{i,j} = g_2^s = Y_2$. As before, $i = k - 1$ yields the same relation. Therefore, the kernel is minimally generated by

$$\{Y_1e_{i,j} - X_1e_{i,j+1}, X_2e_{i,j} - Y_2e_{i-1,j} \mid 0 \leq j < i \leq s\}.$$ 

Hence, $\mu(\ker \partial_1) = 2\binom{s+1}{2}$. Write the basis elements of $R^{2\binom{s+1}{2}}$ as

$$\{e_{i,p}, e_{2,i,q} \mid 1 \leq i \leq s, 0 \leq p < i \text{ and } 0 \leq q < i\}$$

and define $\partial_2 : R^{2\binom{s+1}{2}} \longrightarrow R^{t_1}$ by

$$\partial_2(e_{i,p}) = Y_1e_{i,p} - X_1e_{i+1,p+1}$$

$$\partial_2(e_{2,i,q}) = X_2e_{i,q} - Y_2e_{i-1,q}.$$ 

By [22] Proposition 3.2, $\operatorname{pdim}(R/J^s) = 3$ for all $s \geq 2$. Hence we conclude that the minimal graded free resolution of $R/J^s$ is of the form

$$0 \longrightarrow R^{t_1} \longrightarrow R^{2\binom{s+1}{2}} \longrightarrow R^{\binom{s+2}{2}} \longrightarrow R \longrightarrow 0.$$ 

Therefore,

$$\beta_s - 2\binom{s+1}{2} + \binom{s+2}{2} - 1 = 0,$$

so that $\beta_s = \binom{s+1}{2}$. Now we compute the generators of the second syzygy. Again, since the resolution is linear, it is enough to compute linear generators. First, note that

$$B = \{Y_2e_{1,i,j} - X_2e_{1,i+1,j} + Y_1e_{2,i+1,j} - X_1e_{2,i+1,j+1} \mid 0 \leq j < i < s\} \subseteq \ker \partial_2.$$
It can easily be verified that \( B \) is \( R \)-linearly independent. Since \( \mu(\ker \partial_2) = \binom{s}{2} \), \( B \) generates \( \ker \partial_2 \).

Let \( \{E_{i,j} \mid 0 \leq j < i < s\} \) denote the standard basis for \( R^{(s)} \). Define \( \partial_3 : R^{(s)} \rightarrow R^{(s+1)} \) by

\[
\partial_3(E_{i,j}) = Y_2e_{1,i,j} - X_2e_{1,i+1,j} + X_1e_{2,i+1,j} - Y_1e_{2,i,j+1}.
\]

Therefore, we get the minimal free resolution of \( R/J^s \) as

\[
0 \rightarrow R^{(s)} \xrightarrow{\partial_3} R^{(s+1)} \xrightarrow{\partial_2} R^{(s+2)} \xrightarrow{\partial_1} R \rightarrow 0.
\]

Since the resolution is linear, \( \text{reg} \, J^s = 2s \). \( \square \)

It is to be noted that the resolution and the projective dimension of the powers of the edge ideals of path graphs are known in the literature. In Theorem 3.4, we obtain the syzygies and the Betti numbers explicitly. The knowledge of syzygies is crucial in obtaining the regularity of powers of edge ideals of certain bipartite graphs as seen below:

**Theorem 3.6.** Let \( U = U_1 \cup U_2 \) and \( V = V_1 \cup V_2 \) be a collection of vertices with \( |U| = n \), \( |U_i| = n_i \), \( |V| = m \), \( |V_i| = m_i \) and \( 1 \leq n_i, m_i \) for \( i = 1, 2 \). Let \( G \) be the bipartite graph \( K_{U_i, V} \cup K_{U_2, V_2} \). Let \( R = K[x_1, \ldots, x_n, y_1, \ldots, y_m] \). Let \( J_G \subset R \) denote the cover ideal of \( G \). Then the minimal free resolution of \( R/J_G^s \) is of the form:

\[
0 \rightarrow R^{(s)} \rightarrow R^{(s+1)} \rightarrow R^{(s+2)} \rightarrow R \rightarrow 0.
\]

Moreover,

\[
\text{reg} \, J_G^s = \max \left\{ \begin{array}{ll}
(s-j)n_1 + (s-i)n_2 + jm_1 + im_2 & \text{for } 0 \leq j \leq i \leq s \\
(s-j)n_1 + (s-i)n_2 + (j+1)m_1 + im_2 - 1 & \text{for } 0 \leq j < i \leq s \\
(s-j)n_1 + (s-i)n_2 + jm_1 + im_2 - 1 & \text{for } 0 \leq j < i \leq s \\
(s-j)n_1 + (s-i)n_2 + (j+1)m_1 + (i+1)m_2 - 2 & \text{for } 0 \leq j < i < s
\end{array} \right.
\]

**Proof.** Let \( R = K[x_1, \ldots, x_n, y_1, \ldots, y_m] \). Then \( J_G = (x_1 \cdots x_n, y_1 \cdots y_m, x_1 \cdots x_{n_1}y_{m_1+1} \cdots y_m) \). Set \( X_1 = x_1 \cdots x_{n_1}, X_2 = x_{n_1+1} \cdots x_n, Y_1 = y_1 \cdots y_{m_1} \) and \( Y_2 = y_{m_1+1} \cdots y_m \). Then \( J_G = (X_1X_2, Y_1Y_2, X_1Y_2) \). Therefore, it follows from Theorem 3.4 that \( J_G^s \) has the given minimal free resolution.

To compute the regularity, we need to obtain the degrees of the syzygies. These shifts are given by the degrees of the minimal generators of the syzygies. Following the notation in the proof of Theorem 3.4, we can see that

\[
\begin{align*}
\text{deg} \, e_{i,j} &= (s-j)n_1 + (s-i)n_2 + jm_1 + im_2, \\
\text{deg} \, e_{1,i,j} &= (s-j)n_1 + (s-i)n_2 + (j+1)m_1 + im_2, \\
\text{deg} \, e_{2,i,j} &= (s-j)n_1 + (s-i+1)n_2 + jm_1 + im_2, \\
\text{deg} \, E_{i,j} &= (s-j)n_1 + (s-i)n_2 + (j+1)m_1 + (i+1)m_2.
\end{align*}
\]

Therefore, the assertion on the regularity follows. \( \square \)

### 3.1. Discussion

It has been proved by Hang and Trung, [27], that if \( G \) is a bipartite graph on \( n \) vertices and \( J_G \) is the cover ideal of \( G \), then there exists a non-negative integer \( e \) such that for \( s \geq n+2 \), \( \text{reg}(J_G^s) = \text{deg}(J_G^s) + e \), where \( \text{deg}(J_G) \) denote the maximal degree of minimal monomial generators of \( J_G \). It follows from Theorem 3.2 that if \( G = K_{n,m} \) with \( n \geq m \), then \( e = m-1 \) and the index of stability is 1. If the graph is not a complete bipartite graph, then \( e \) does not uniformly represent a combinatorial invariant associated to the graph as can be seen in the computations below. We compute the polynomial \( \text{reg}(J_G^s) \) for some classes of bipartite graphs that are considered in Corollary 3.6. We see that \( e \) depends on the relation between the integers \( n_1, n_2, m_1 \) and \( m_2 \). To find the polynomial \( \text{reg}(J_G^s) \), we need to identify which of those relations give rise to the regularity. If we assume that all those integers are at least one, then
Remark 3.7. If $G = K_{U_i,V} \cup K_{U_j,V_j}$ for some set of vertices $U_i, V_i$, then one can still describe the complete resolution and the regularity of $J_G^s$ using a similar approach. However, the resulting syzygies are not so easy to describe though the generating sets are similar. Therefore, we restrict ourselves to the above discussion.
4. Complete Multipartite Graphs

In this section our goal is to understand the resolution and regularity of the powers of cover ideals of complete multipartite graphs. Let $G$ be a complete $m$-partite graph and let $J_G$ be the cover ideal of $G$. The main idea in constructing the resolution of $J_G$ is to identify $J_G$ with the cover ideal of the complete graph $K_m$ on $m$ vertices. Then one replaces the variables by the product of a set of variables from each partitions of $m$-partite graphs to get the cover ideal $J_G$.

Let $R = K[x_1, \ldots, x_m]$. It is known that the cover ideal $J_G$ of complete graph $G = K_m$ is generated by all squarefree monomials $x_1 x_2 \cdots \hat{x}_i \cdots x_m$ of degree $m-1$. Moreover one can also identify this cover ideal with the squarefree Veronese ideal $I = I_{m,m-1}$, and hence it is a polymatroidal ideal, \cite{11}.

Let $I \subset R$ be an ideal of $R$. The ideal $I$ is said to have linear quotients if there exists an ordered system of homogeneous generators $f_1, \ldots, f_\ell$ of $I$ such that for all $j = 1, \ldots, \ell$ the colon ideals $(f_1, \ldots, f_{j-1}) : f_j$ are generated by a subset of $\{x_1, \ldots, x_n\}$. It was proved by Conca and Herzog that a polymatroidal ideal has linear quotients with respect to the reverse lexicographical order of the generators \cite[Proposition 5.2]{14}. Herzog and Hibi proved that if $I$ is generated in degree $d$ and has linear quotients, then $I$ has a $d$-linear resolution, \cite[Proposition 8.2.1]{13}. Furthermore product of polymatroidal monomial ideals are polymatroidal \cite[Theorem 5.3]{14}. Hence powers of polymatroidal monomial ideals are polymatroidal. Therefore we have the following:

**Remark 4.1.** The cover ideal $J_G$ of complete graph $G = K_m$ has linear quotients and hence has linear resolution. Moreover $J_G^s$ has linear resolution for all $s \geq 1$.

If $I$ is an ideal of $R$ all of whose powers have linear resolution, then depth $R/I^k$ is a non-increasing function of $k$ and depth $R/I^k$ is constant for all $k \gg 0$. \cite[Proposition 2.1]{11}. Further, we have:

**Remark 4.2.** \cite[Corollary 3.4]{11} Let $R = K[x_1, \ldots, x_m]$ and $J_G$ be the cover ideal of $G = K_m$. Then

\[
\text{depth } R/J_G^s = \max\{0, m - s - 1\}.
\]

In particular, depth $R/J_G^s = 0$ for all $s \geq m - 1$.

We begin our investigation by studying the resolution of powers of cover ideal of $K_3$.

4.1. **Complete tripartite:** It has been shown by Beyarslan et al., \cite[Theorem 5.2]{2}, that $\text{reg}(I(C_3)^s) = 2s$ for all $s \geq 2$. We first describe the graded minimal free resolution of $I(C_3)^s$ for all $s \geq 1$. We also obtain the Hilbert series of the powers.

**Theorem 4.3.** Let $R = K[x_1, x_2, x_3]$ and $I = (x_1 x_2, x_1 x_3, x_2 x_3)$. Then the graded minimal free resolution of $R/I$ is of the form:

\[
0 \to R(-3)^2 \to R(-2)^3 \to R \to 0.
\]

For $s \geq 2$, the graded minimal free resolution of $R/I^s$ is of the form:

\[
0 \to R(-2s - 2)^\binom{s}{2} \to R(-2s - 1)^{2\binom{s+1}{2}} \to R(-2s)^{\binom{s+2}{2}} \to R \to 0
\]

so that $\text{reg}_R(I^s) = 2s$. Moreover, the Hilbert series of $R/I^s$ is given by

\[
H(R/I^s, t) = \frac{1 + 2t + 3t^2 + \cdots + 2st^{2s-1} - (\binom{s+2}{2} - 2s - 1)t^{2s}}{(1 - t)}.
\]

**Proof.** It is clear that the resolution of $I$ is as given in the assertion of the theorem. Therefore $\text{reg}(R/I) = 1$. It follows from \cite{14}, that $I^s$ has a linear minimal free resolution for all $s \geq 1$. 


Note that by \cite[Lemma 3.1]{ref2}, depth $R/I^s = 0$ for all $s \geq 2$ so that $\text{pdim} R/I^s = 3$ for all $s \geq 2$. Hence, the minimal free resolution of $R/I^s$ is of the form:

$$0 \to R(-2s-2)^{\beta_3} \xrightarrow{\partial_3} R(-2s-1)^{\beta_2} \xrightarrow{\partial_2} R(-2s)^{\beta_1} \xrightarrow{\partial_1} R \to 0$$

Now we describe completely the minimal free resolution of $R/I^s$ for $s \geq 2$. Let $g_1 = x_1 x_2, g_2 = x_1 x_3$ and $g_3 = x_2 x_3$. Write $I^s = (g_1^s, g_2^s, g_3^s, \ldots, g_1(g_2, g_3)^{s-1}, (g_2, g_3)^{s})$. The generators of $I^s$ are of the form $g_1^{\ell_1} g_2^{\ell_2} g_3^{\ell_3}$, where $0 \leq \ell_1, \ell_2, \ell_3 \leq s$ and $\ell_1 + \ell_2 + \ell_3 = s$. Denote by $f_{\ell_1,\ell_2,\ell_3} = g_1^{\ell_1} g_2^{\ell_2} g_3^{\ell_3}$. Then $f_{\ell_1,\ell_2,\ell_3} = x_1^{\ell_1} x_2^{\ell_2} x_3^{\ell_3}$. It is easy to see that $\mu(I^s) = \binom{s+2}{2}$, since the minimal number of generating elements of the monomial ideal $I^s$ is same as the total number of non-negative integral solution of $\ell_1 + \ell_2 + \ell_3 = s$, which is $\binom{s+2}{2}$.

Let \{\(e_{\ell_1,\ell_2,\ell_3} : 0 \leq \ell_1, \ell_2, \ell_3 \leq s; \ell_1 + \ell_2 + \ell_3 = s\)\} denote the standard basis for $R(\binom{s+2}{2})$ and consider the map $\partial_1 : R(\binom{s+2}{2}) \to R$ defined by $\partial_1(e_{\ell_1,\ell_2,\ell_3}) = f_{\ell_1,\ell_2,\ell_3}$. We find the minimal generators for $\ker \partial_1$. Since $f_{\ell_1,\ell_2,\ell_3}$’s are monomials, the kernel is generated by binomials of the form $m_{\ell_1,\ell_2,\ell_3} f_{\ell_1,\ell_2,\ell_3} - m_{\ell_1,\ell_2,\ell_3} f_{\ell_1,\ell_2,\ell_3}$, where $m_{i,j,k}$’s are monomials in $R$. Since the minimal free resolution is linear, the kernel is generated in degree 1. Note that $\frac{f_{\ell_1,\ell_2,\ell_3}}{f_{\ell_1,\ell_2,\ell_3}} = x_1^{\ell_1-\ell} x_2^{\ell_2-\ell} x_3^{\ell_3-\ell}$.

Hence, for $f_{\ell_1,\ell_2,\ell_3}$ to be a linear fraction, $|t_i - \ell_i| \leq 1$ for $i = 1, 2, 3$.

Let $t_3 = \ell_3, t_2 = \ell_2 + 1$ and $t_1 = \ell_1 - 1$. The corresponding linear syzygy relation is

$$x_3 \cdot f_{\ell_1,\ell_2,\ell_3} - x_2 \cdot f_{\ell_1-1,\ell_2+1,\ell_3} = 0,$$

where $1 \leq \ell_1 \leq s$, and $0 \leq \ell_2, \ell_3 \leq s - 1$. Note that the number of such relations is equal to the number of integral solution to $(\ell_1 - 1) + \ell_2 + \ell_3 = s$, i.e., $\binom{s+1}{2}$. Similarly, if $t_3 = \ell_3 + 1, t_2 = \ell_2 - 1$ and $t_1 = \ell_1$, then we get the corresponding linear syzygy relation as

$$x_2 \cdot f_{\ell_1,\ell_2,\ell_3} - x_1 \cdot f_{\ell_1,\ell_2-1,\ell_3+1} = 0,$$

where $0 \leq \ell_1, \ell_3 \leq s - 1$, and $1 \leq \ell_2 \leq s$. Note that the linear syzygy relation obtained by fixing $\ell_2$ and taking $|t_i - \ell_i| = 1$ for $i = 1, 2$

$$x_3 \cdot f_{\ell_1,\ell_2,\ell_3} - x_2 \cdot f_{\ell_1-1,\ell_2,\ell_3+1} = 0$$

can be obtained from Equations \cite{ref1}, \cite{ref2} by setting the $\ell_i$’s appropriately. Therefore,\n
$\ker \partial_1 = \langle x_3 \cdot e_{\ell_1,\ell_2,\ell_3} - x_2 \cdot e_{\ell_1-1,\ell_2+1,\ell_3}; x_2 \cdot e_{\ell_1-1,\ell_2+1,\ell_3} - x_1 \cdot e_{\ell_1-1,\ell_2,\ell_3+1} : 1 \leq \ell_1 \leq s, 0 \leq \ell_2, \ell_3 \leq s - 1 \rangle$.

Since there are $\binom{s+2}{2}$ minimal generators of each type in the list above, $\mu(\ker \partial_1) = \beta_2 = 2\binom{s+1}{2}$.

Write the standard basis of $R(\binom{s+1}{2})$ as

$$B_2 = \left\{ e_{\ell_1,\ell_2,\ell_3} : 1 \leq \ell_1 \leq s, 0 \leq \ell_2, \ell_3 \leq s - 1 \right\}$$

and define $\partial_2 : R(\binom{s+1}{2}) \to R(\binom{s+2}{2})$ by

$$\partial_2(e_{\ell_1,\ell_2+1,\ell_3}) = x_2 \cdot e_{\ell_1-1,\ell_2+1,\ell_3} - x_1 \cdot e_{\ell_1-1,\ell_2,\ell_3+1},$$

$$\partial_2(e_{\ell_1,\ell_2,\ell_3}) = x_3 \cdot e_{\ell_1,\ell_2,\ell_3} - x_2 \cdot e_{\ell_1-1,\ell_2+1,\ell_3}.$$

From the equation, $\beta_3 - \binom{2(s+1)}{2} - \binom{s+2}{2} = 0$, it follows that $\beta_3 = \binom{s}{2}$. Now, we describe $\ker \partial_2$. For $2 \leq \ell_1 \leq s$ and $0 \leq \ell_2, \ell_3 \leq s - 2$, set

$$E_{\ell_1,\ell_2,\ell_3} = -x_3 \cdot e_{\ell_1,\ell_2-1,\ell_3}, x_3 \cdot e_{\ell_1,\ell_2+1,\ell_3}; x_2 \cdot e_{\ell_1,\ell_2+2,\ell_3}, x_2 \cdot e_{\ell_1,\ell_2+1,\ell_3}, \ell_1 - 1 - x_3 \cdot e_{\ell_1,\ell_2+1,\ell_3}, \ell_1 - 1.$$

Note that

$$B_3 = \left\{ E_{\ell_1,\ell_2,\ell_3} : 2 \leq \ell_1 \leq s, 0 \leq \ell_2, \ell_3 \leq s - 2 \right\} \subset \ker \partial_2.$$
Since \( B_3 \) is \( R \)-linearly independent and \( |B_3| = \binom{s}{3} \), \( \ker \partial_2 = \langle B_3 \rangle \). Let \( \{ H_{t_1,t_2,t_3} : 2 \leq t_1 \leq s, 0 \leq t_2, t_3 \leq s - 2 \} \) denote the standard basis of \( R(\binom{s}{3}) \). Define \( \partial_3 : R(\binom{s}{3}) \to R(\binom{2s+1}{2}) \) by \( \partial_3(H_{t_1,t_2,t_3}) = E_{t_1,t_2,t_3} \) which completes the description of the resolution.

It can also be seen that \( \deg \epsilon_{t_1,t_2,t_3} = 2s \), \( \deg \epsilon_{1,(t_2+1,t_3),t_1-1} = \deg \epsilon_{2,(t_1,t_2),t_3} = 2s + 1 \) and \( \deg H_{t_1,t_2,t_3} = 2s + 2 \). Therefore we get the minimal graded free resolution of \( R/I^s \) as

\[
0 \to R(-2s-2)(\binom{s}{2}) \xrightarrow{\partial_3} R(-2s-1)(\binom{2s+1}{2}) \xrightarrow{\partial_2} R(-2s)(\binom{s+2}{2}) \xrightarrow{\partial_1} R \to 0.
\]

Therefore, the Hilbert series of \( R/I^s \) is given by

\[
H(R/I^s,t) = \frac{1 - \left(\frac{s+2}{2}\right)t^{2s} + 2\left(\frac{s+1}{2}\right)t^{2s+1} - \left(\frac{s}{2}\right)t^{2s+2}}{(1-t)^3} = (1-t)^{-2}\left(1 - \left(\frac{s+2}{2}\right)t^{2s} + 2\left(\frac{s+1}{2}\right)t^{2s+1} - \left(\frac{s}{2}\right)t^{2s+2}\right).
\]

By expanding \((1-t)^{-2}\) in the power series form and multiplying with the numerator, we get the required expression. \( \square \)

We now proceed to compute the minimal graded free resolution of powers of complete tripartite graphs.

**Notation 4.4.** Let \( G \) denote a complete tripartite graph with \( V(G) = V_1 \sqcup V_2 \sqcup V_3 \) and \( E(G) = \{\{a,b\} : b \in V_i, b \in V_j, i \neq j\} \). Set \( V_1 = \{x_1, \ldots, x_{m_1}\}, V_2 = \{y_1, \ldots, y_{m_2}\} \) and \( V_3 = \{z_1, \ldots, z_{m_3}\} \). Let \( J_G \) denote the vertex cover ideal of \( G \). Let \( X = \prod_{i=1}^{m_1} x_i, Y = \prod_{j=1}^{m_2} y_i \) and \( Z = \prod_{k=1}^{m_3} z_i \). It can be seen that \( J_G = (XY, XZ, YZ) \).

**Theorem 4.5.** Let \( R = K[x_1, \ldots, x_{m_1}, y_1, \ldots, y_{m_2}, z_1, \ldots, z_{m_3}] \). Let \( G \) be a complete tripartite graph as in Notation 4.4. Let \( J_G \subset R \) denote the cover ideal of \( G \). Then for all \( s \geq 2 \), the minimal free resolution of \( R/J_G^s \) is of the form:

\[
0 \to R(\binom{s}{2}) \xrightarrow{\partial_3} R(\binom{2s+1}{2}) \xrightarrow{\partial_2} R(\binom{s+2}{2}) \xrightarrow{\partial_1} R \to 0.
\]

Set \( \alpha = (s - \ell_3)m_1 + (s - \ell_2)m_2 + (s - \ell_1)m_3 \). Then for all \( s \geq 2 \),

\[
\text{reg}(J_G^s) = \max \left\{ \alpha, \alpha + m_3 - 1, \alpha + 2m_3 - 2 \right\},
\]

for \( 0 \leq \ell_1, \ell_2, \ell_3 \leq s \).

**Proof.** Taking \( X = x_1, Y = x_2 \) and \( Z = x_3 \) in Theorem 4.4, it follows that the minimal free resolution of \( S/J_G^s \) is of the given form:

\[
0 \to R(\binom{s}{2}) \xrightarrow{\partial_3} R(\binom{2s+1}{2}) \xrightarrow{\partial_2} R(\binom{s+2}{2}) \xrightarrow{\partial_1} R \to 0.
\]

We now compute the degrees of the generators and hence obtain the regularity. Let \( \deg X_1 = m_1, \deg X_2 = m_2 \), and \( \deg X_3 = m_3 \). Then it follows that

\[
\deg \epsilon_{t_1,t_2,t_3} = \deg \left( X_1^{s-\ell_3} X_2^{s-\ell_3} X_3^{s-\ell_1} \right) = (s - \ell_3)m_1 + (s - \ell_2)m_2 + (s - \ell_1)m_3.
\]

We observe that

\[
\deg \epsilon_{1,(t_2+1,t_3),t_1-1} = \deg \epsilon_{2,(t_1,t_2),t_3} = \deg \epsilon_{t_1,t_2,t_3} + \deg X_3,
\]

\[
\deg H_{t_1,t_2,t_3} = \deg \epsilon_{t_1,t_2,t_3} + 2 \deg X_3.
\]
Let \( s \geq \) the ideal of the complete graph cover ideals of 4-partite graphs. For this purpose, we first study the resolution of powers of cover 4.2. Complete 4-partite graphs.

For all \( m \), therefore, \( \ell \) can derive this form from the above expression. Depending on the size of partitions, the leading term of the minimum value to

\[
\ell = (3.1), \text{one can derive various expressions for } \text{reg}(J^s) \text{ for different cases as well. Consider the arithmetic progression } m_1 = m + 2r, m_2 = m, \text{ and } m_3 = m + r:
\]

Corollary 4.6. Let \( m, r \) be any two positive integers. Let \( m_1 = m + 2r, m_2 = m, \text{ and } m_3 = m + r \) in Theorem 4.5. Then for all \( s \geq 2 \), we have

\[
\text{reg}(J^s) = s(2m + 3r) + 2m - 2.
\]

Proof. By Theorem 4.5, \( \alpha = (s - \ell_3)m_1 + (s - \ell_2)m_2 + (s - \ell_1)m_3 \). Hence we get

\[
\alpha = s(3m + 3r) - \ell_3(m + 2r) - \ell_2(m) - \ell_1(m + r).
\]

Using Theorem 4.5 we have for all \( s \geq 2 \),

\[
\text{reg}(J^s) = \max \left\{ \begin{array}{ll}
\alpha, & \text{for } 0 \leq \ell_1, \ell_2, \ell_3 \leq s, \\
\alpha + (m + r) - 1, & \text{for } 1 \leq \ell_1 \leq s, 0 \leq \ell_2, \ell_3 \leq s - 1, \\
\alpha + 2(m + r) - 2, & \text{for } 2 \leq \ell_1 \leq s, 0 \leq \ell_2, \ell_3 \leq s - 2,
\end{array} \right.
\]

where \( \ell_1 + \ell_2 + \ell_3 = s \). Since regularity \( \text{reg}(J^s) \) is maximum of all the numbers, we need to maximize the value of \( \alpha \). For this to happen, negative terms in \( \alpha \) should be minimum. The coefficient of \( \ell_3 \) is largest among the negative terms in \( \alpha \), so \( \ell_3 \) should be assigned the least value. After \( \ell_3 \), assign the minimum value to \( \ell_1 \), and finally take \( \ell_2 = s - \ell_1 - \ell_3 \). For example, to get the maximum of \( \alpha \) when \( 2 \leq \ell_1 \leq s, 0 \leq \ell_2, \ell_3 \leq s - 2 \), put \( \ell_3 = 0, \ell_1 = 2, \text{ and } \ell_2 = s - 2 \). We get for all \( s \geq 2 \)

\[
\text{reg}(J^s) = \max \left\{ \begin{array}{ll}
s(2m + 3r), & \text{for } 0 \leq \ell_1, \ell_2, \ell_3 \leq s, \\
s(2m + 3r) + m - 1, & \text{for } 1 \leq \ell_1 \leq s, 0 \leq \ell_2, \ell_3, \ell_4 \leq s - 1, \\
s(2m + 3r) + 2m - 2, & \text{for } 2 \leq \ell_1 \leq s, 0 \leq \ell_2, \ell_3, \ell_4 \leq s - 2.
\end{array} \right.
\]

For all \( m \geq 1 \), and for all \( s \geq 2 \), we get

\[
\text{reg}(J^s) = s(2m + 3r) + 2m - 2.
\]

\[ \square \]

4.2. Complete 4-partite graphs. We now describe the resolution and regularity of powers of cover ideals of 4-partite graphs. For this purpose, we first study the resolution of powers of cover ideal of the complete graph \( K_4 \).

Theorem 4.7. Let \( R = K[x_1, x_2, x_3, x_4] \) and \( I = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4) \). Then, for \( s \geq 3 \), the minimal graded free resolution of \( R/I^s \) is of the form

\[
0 \rightarrow R(-3s - 3) \beta_3 \rightarrow R(-3s - 2) \beta_3 \rightarrow R(-3s - 1) \beta_2 \rightarrow R(-3s) \beta_2 \rightarrow R \rightarrow 0
\]

where

\[
\beta_1 = \binom{s + 3}{3}, \beta_2 = 3 \binom{s + 2}{3}, \beta_3 = 3 \binom{s + 1}{3}, \text{ and } \beta_4 = \binom{s}{3}.
\]
In particular, reg \( R(I^s) = 3s \). Moreover the Hilbert series of \( R/I^s \) is given by

\[
H(R/I^s,t) = \frac{1 + 2t + 3t^2 + 4t^3 + \cdots + 3s t^{3s-1} - (\binom{s+3}{3} - 3s - 1) t^{3s} + \binom{s}{3} t^{3s+1}}{(1-t)^2}
\]

Let \( 0 \leq \ell_i \leq s \) for all \( i \) and \( \ell_1 + \ell_2 + \ell_3 + \ell_4 = s \). The number of non-negative integral solution to the linear equation \( \ell_1 + \ell_2 + \ell_3 + \ell_4 = s \) is \( \binom{s+3}{3} \). Hence we have \( \mu(J^s) = \beta_1 = \binom{s+3}{3} \). Note that \( g_1 f_1 g_2 f_2 g_3 f_3 g_4 f_4 = x_1^{s-\ell_4} x_2^{s-\ell_3} x_3^{s-\ell_2} x_4^{s-\ell_1} \) with \( 0 \leq \ell_1, \ell_2, \ell_3, \ell_4 \leq s \). Set \( f_{\ell_1,\ell_2,\ell_3,\ell_4} = x_1^{s-\ell_4} x_2^{s-\ell_3} x_3^{s-\ell_2} x_4^{s-\ell_1} \). Let

\[
\{ e_{\ell_1,\ell_2,\ell_3,\ell_4} : 0 \leq \ell_1, \ell_2, \ell_3, \ell_4 \leq s \}
\]

denote the standard basis of \( R_{\binom{s+3}{3}} \) and consider the map \( \partial_1 : R_{\binom{s+3}{3}} \rightarrow R \) defined by \( \partial_1(e_{\ell_1,\ell_2,\ell_3,\ell_4}) = f_{\ell_1,\ell_2,\ell_3,\ell_4} \).

Suppose \( t_4 = \ell_4 + 1 \), \( t_3 = \ell_3 - 1 \) and \( t_j = \ell_j \) for \( j = 1, 2 \). Then we get the linear syzygy relation

\[
x_2 f_{t_1,t_2,t_3,t_4} - x_1 f_{t_1,t_2,t_3-1,t_4+1} = 0.
\]

Fixing \( t_3 \) and \( t_4 \), with \( 1 \leq t_3 + t_4 \leq s \), there are as many linear syzygies as there are the number of solutions of \( t_1 + t_2 = s - (t_3 + t_4) \). Therefore, for the pair \( (t_3,t_4) \), there are \( \binom{s+1}{2} + \binom{s}{2} + \cdots + \binom{2}{2} = \binom{s+2}{3} \) number of solutions. Similarly for each pair \( (t_4,t_2), (t_4,t_1), (t_3,t_2), (t_3,t_1) \) and \( (t_2,t_1) \), we get \( \binom{s+2}{3} \) linear syzygies. Note that the syzygies \( x_4 f_{t_1+1,t_2-1,t_3,t_4} - x_3 f_{t_1,t_2,t_3,t_4} \) and \( x_3 f_{t_1,t_2,t_3,t_4} - x_2 f_{t_1,t_2-1,t_3+1,t_4} \) give rise to another linear syzygy \( x_4 f_{t_1+1,t_2-1,t_3,t_4} - x_2 f_{t_1,t_2-1,t_3+1,t_4} \). The same linear syzygy can also be obtained from a combination of linear syzygies that arise out of the pairs \((t_1,t_4)\) and \((t_3,t_4)\). Therefore, to get a minimal generating set, we only need to consider linear syzygies corresponding to the pairs \((t_1,t_2)\), \((t_2,t_3)\) and \((t_3,t_4)\). For each such pair, we have \( \binom{s+2}{3} \) number of linear syzygies. Hence \( \beta_2 = 3 \binom{s+2}{3} \).

Write the basis elements of \( R_{3\binom{s+2}{3}} \) as

\[
B_2 = \left\{ e_{(1,\ell_1-1,\ell_2),\ell_3+1,\ell_4} : 1 \leq \ell_1 \leq s, \ 0 \leq \ell_2, \ell_3, \ell_4 \leq s-1 \right\}
\]
and define $\partial_2 : R^{3(\ell + 1)} \longrightarrow R^{3\ell}$ by

$$
\begin{align*}
\partial_2(e_1) &= x_1\ell_1 - \ell_2, \ell_3, \ell_4 + x_1\ell_1 - \ell_2, \ell_3, \ell_4 + 1; \\
\partial_2(e_2) &= x_3\ell_1 - 1, \ell_2 + 1, \ell_3, \ell_4 - x_2\ell_1 - \ell_2, \ell_3, \ell_4; \\
\partial_2(e_3) &= x_4\ell_1, \ell_2, \ell_3, \ell_4 - x_3\ell_1 - 1, \ell_2, \ell_3, \ell_4.
\end{align*}
$$

Now we decipher the Betti numbers $\beta_3$ and $\beta_4$ to complete the resolution. The Hilbert series of $R/I^s$ is

$$
H(R/I^s, t) = \frac{(1 - \beta_1 t^{3s} + \beta_2 t^{3s+1} - \beta_3 t^{3s+2} + \beta_4 t^{3s+3})}{(1 - t)^4}.
$$

Since $\dim R/I^s = 2$, the polynomial $p(t)$ has a factor $(1 - t)^2$. Note that $(1 - t)^2$ is a monic polynomial of degree 2, hence we can write $p(t) = (1 - t)^2 \cdot q(t)$ where

$$
q(t) = \beta_4 t^{3s+1} + (2\beta_4 - \beta_3) t^{3s} + a_{3s-1} t^{3s-1} + \cdots + a_1 t + 1.
$$

On the other hand, we also have

$$
\begin{align*}
&= \frac{(1 - \beta_1 t^{3s} + \beta_2 t^{3s+1} - \beta_3 t^{3s+2} + \beta_4 t^{3s+3})}{(1 - t)^2} \\
&= (1 + 2t + 3t^2 + 4t^3 + \cdots + (n + 1)t^n + \cdots) (1 - \beta_1 t^{3s} + \beta_2 t^{3s+1} - \beta_3 t^{3s+2} + \beta_4 t^{3s+3}) \\
&= (1 + 2t + 3t^2 + \cdots + (3s) t^{3s-1} + (3s + 1 - \beta_1) t^{3s} + (3s + 2 - 2\beta_1 + \beta_2) t^{3s+1} + a_j t^j + \cdots.
\end{align*}
$$

For the expressions in Equations 5 and 6 to be equal, they should agree coefficient-wise. In particular, we should have that

$$
\beta_4 = 3s + 2 - 2\beta_1 + \beta_2 \quad \text{and} \quad 2\beta_4 - \beta_3 = 3s + 1 - \beta_1.
$$

On substituting $\beta_1 = \binom{s+3}{3}$ and $\beta_2 = 3\binom{s+2}{3}$, we get $\beta_4 = \binom{s+1}{3}$ from the first equation and substituting the value in the second equation, we get $\beta_3 = 3\binom{s+1}{3}$. Now we verify that $a_j = 0$ for all $j \geq 3s + 2$. Note that for $r \geq 2$, the coefficient of $t^{3s+r}$ in Equation 6 is

$$
a_{3s+r} = (3s + r + 1) + (r - 2)\beta_4 - (r - 1)\beta_3 + r\beta_2 - (r + 1)\beta_1.
$$

Since, $1 - \beta_1 + \beta_2 - \beta_3 + \beta_4 = 0$, this equation is reduced to $a_{3s+r} = (3s + 3) - \beta_3 + 2\beta_2 - 3\beta_1$. Applying the binomial identity $\binom{n+1}{r+1} = \binom{n}{r+1} + \binom{n}{r}$ repeatedly, we get $a_{3s+r} = 0$ for all $r \geq 2$. Hence the Hilbert series of $R/I^s$ is

$$
H(R/I^s, t) = \frac{1 + 2t + 3t^2 + 4t^3 + \cdots + 3st^{3s-1} - \binom{s+3}{3} t^{3s} + \binom{s}{3} t^{3s+1}}{(1 - t)^2}.
$$

We now complete the description of the resolution. Write the basis elements of $R^{3(\ell + 1)}$ as

$$
B_3 = \left\{ \begin{array}{c}
E_{1,\ell_1,\ell_2,\ell_3,\ell_4} \\
E_{2,\ell_1,\ell_2,\ell_3,\ell_4} \\
E_{3,\ell_1,\ell_2,\ell_3,\ell_4}
\end{array} \right\},
$$

Note that $|B_3| = 3\binom{s+1}{3}$. Now define the map $\partial_3 : R^{3(\ell + 1)} \longrightarrow R^{3(\ell + 2)}$ by

$$
\begin{align*}
\partial_3(E_{1,\ell_1,\ell_2,\ell_3,\ell_4}) &= x_4 e(2, \ell_1 - 1, \ell_4), \ell_2 + 1, \ell_3 + 1, \ell_4 + x_1 e(1, \ell_1 - 1, \ell_2), \ell_3 + 1, \ell_4 - x_3 e(3, \ell_1, \ell_4), \ell_1 - 1, \ell_2 + 1 \\
&\quad - x_3 e(2, \ell_1 - 2, \ell_4), \ell_2 + 2, \ell_3 - x_3 e(1, \ell_1 - 2, \ell_2 + 1), \ell_3 + 1, \ell_4 + x_1 e(3, \ell_1, \ell_4), \ell_1 - 1, \ell_2; \\
\partial_3(E_{2,\ell_1,\ell_2,\ell_3,\ell_4}) &= x_4 e(1, \ell_1 - 1, \ell_4), \ell_2 + 1, \ell_3 - x_3 e(1, \ell_1 - 2, \ell_2 + 1), \ell_3 + 1, \ell_4 - x_2 e(3, \ell_1, \ell_4), \ell_1 - 1, \ell_2 + 1 \\
&\quad + x_1 e(3, \ell_1, \ell_4), \ell_1 - 1, \ell_2; \\
\partial_3(E_{3,\ell_1,\ell_2,\ell_3,\ell_4}) &= x_3 e(1, \ell_1 - 2, \ell_2 + 1), \ell_3 + 1, \ell_4 - x_2 e(2, \ell_1 - 2, \ell_4), \ell_2 + 1, \ell_3 + 1 - x_2 e(1, \ell_1 - 2, \ell_2), \ell_3 + 2, \ell_4 + x_1 e(2, \ell_1 - 2, \ell_4), \ell_3 + 1, \ell_4.
\end{align*}
$$
We now compute the kernel of $\partial_3$. Consider the set
\[
A = \{ H_{\ell_1,\ell_2,\ell_3,\ell_4} : 3 \leq \ell_1 \leq s, \ 0 \leq \ell_2, \ell_3, \ell_4 \leq s - 3 \},
\]
where
\[
H_{\ell_1,\ell_2,\ell_3,\ell_4} = x_4E_{3,\ell_1,\ell_2,\ell_3,\ell_4} - x_3E_{2,\ell_1-1,\ell_2+1,\ell_3,\ell_4} - x_3E_{3,\ell_1-1,\ell_2+1,\ell_3,\ell_4} + x_2E_{1,\ell_1-1,\ell_2,\ell_3+1,\ell_4} - x_1E_{1,\ell_1-1,\ell_2,\ell_3,\ell_4+1} + x_1E_{2,\ell_1-1,\ell_2,\ell_3,\ell_4+1}.
\]
One can verify that $\partial_3(H_{\ell_1,\ell_2,\ell_3,\ell_4}) = 0$. Hence the set $A$ lies in the kernel of the map $\partial_3$. To say that the set $A$ is precisely the kernel of the map $\partial_3$, one needs to show that $\mu(A) = \binom{s}{3}$. Let $\ell_1' = 1 - 3$, then one has $\ell_1' + \ell_2 + \ell_3 + \ell_4 = s - 3$. The total number of non-negative integral solution of this linear equation is precisely $\binom{s}{3}$, hence $\mu(A) = \binom{s}{3}$. Write the basis elements of $R(\ell)$ as $B_4 = \{ G_{\ell_1,\ell_2,\ell_3,\ell_4} : 3 \leq \ell_1 \leq s, \ 0 \leq \ell_2, \ell_3, \ell_4 \leq s - 3 \}$ and define the map $\partial_4 : R(\ell) \to R^3(\ell)$ by
\[
\partial_4(G_{\ell_1,\ell_2,\ell_3,\ell_4}) = H_{\ell_1,\ell_2,\ell_3,\ell_4}.
\]
This is an injective map and hence we get the complete resolution:
\[
0 \to R(-3s - 3)^{\beta_4} \to R(-3s - 2)^{\beta_3} \to R(-3s - 1)^{\beta_2} \to R(-3s)^{\beta_1} \to R \to 0.
\]
\[\square\]

Note that in the above proof, we used $s \geq 3$ only to conclude that $\text{pd}(R/I^4) = 4$. By Remark 4.2, $\text{depth}(R/I) = 2$ and hence $\text{pd}(R/I) = 2$. Similarly, $\text{depth}(R/I^2) = 1$ and hence $\text{pd}(R/I^2) = 3$. This forces $\partial_2$ to be injective when $s = 1$ and $\partial_3$ to be injective when $s = 2$. The computations of syzygies in the cases of resolution of $R/I$ and $R/I^2$ remain the same as given in the above proof. Therefore, we get resolutions truncated at $R^{3^2}$ in the case of $R/I$ and truncated at $R^{3^3}$ in the case of $R/I^2$, with the expressions for $\beta_2$ and $\beta_3$ coinciding with the ones given in the proof. Therefore, we can conclude that in this case, $\text{reg}(I^s) = 3s$ for all $s \geq 1$.

As an immediate consequence, we obtain an expression for the asymptotic regularity of cover ideals of complete 4-partite graphs.

**Theorem 4.8.** Let $G$ denote a complete 4-partite graph with $V(G) = \bigcup_{i=1}^{4} V_i$ and $E(G) = \{ \{a, b\} : a \in V_i, b \in V_j, i \neq j \}$. Set $V_i = \{ x_{i1}, \ldots, x_{im_i} \}$ for $i = 1, \ldots, 4$. Let $J_G \subseteq R = K[x_{ij} : 1 \leq i \leq 4; 1 \leq j \leq m_i]$ denote the cover ideal of $G$. Then the minimal free resolution of $R/J_G^s$ is of the form:
\[
0 \to R(s) \to R^3(s^4) \to R^3(s^3) \to R(s^2) \to R(s) \to 0.
\]

Set $\alpha = (s - \ell_4)m_1 + (s - \ell_3)m_2 + (s - \ell_2)m_3 + (s - \ell_1)m_4$. Furthermore, we have
\[
\text{reg}(J_G^s) = \max \begin{cases} 
\alpha, & \text{for } 0 \leq \ell_1, \ell_2, \ell_3, \ell_4 \leq s, \\
\alpha + m_4 - 1, & \text{for } 1 \leq \ell_1 \leq s, \ 0 \leq \ell_2, \ell_3, \ell_4 \leq s - 1, \\
\alpha + 2m_4 - 2, & \text{for } 2 \leq \ell_1 \leq s, \ 0 \leq \ell_2, \ell_3, \ell_4 \leq s - 2, \\
\alpha + 3m_4 - 3, & \text{for } 3 \leq \ell_1 \leq s, \ 0 \leq \ell_2, \ell_3, \ell_4 \leq s - 3,
\end{cases}
\]
where $\ell_1 + \ell_2 + \ell_3 + \ell_4 = s$.

**Proof.** Let $X_i = \prod_{j=1}^{m_i} x_{ij}$. Then $J_G = (X_1X_2X_3, X_1X_2X_4, X_1X_3X_4, X_2X_3X_4)$. Set $X_1 = x_1$, $X_2 = x_2$, $X_3 = x_3$ and $X_4 = x_4$ in the previous theorem. Then it follows that the minimal free resolution of $R/J_G^s$ is of the given form
\[
0 \to R(s) \to R^3(s^4) \to R^3(s^3) \to R(s) \to 0.
\]
To compute the regularity of $R/J_G^s$, we first need to find the degree’s of the generators of the syzygies. Following the notation of the previous theorem, we have
\[
\deg e_{\ell_1,\ell_2,\ell_3,\ell_4} = \deg \left( X_1^{s-\ell_1} X_2^{s-\ell_2} X_3^{s-\ell_3} X_4^{s-\ell_4} \right) = (s - \ell_4) m_1 + (s - \ell_3) m_2 + (s - \ell_2) m_3 + (s - \ell_1) m_4
\]
and we have
\[
\begin{align*}
\deg e_{(1,\ell_1-1,\ell_2),\ell_3,\ell_4} &= \deg e_{(2,\ell_1-1,\ell_4),\ell_2+1,\ell_3} = \deg e_{(3,\ell_3,\ell_4),\ell_1,\ell_2} = \deg e_{\ell_1,\ell_2,\ell_3,\ell_4} + \deg(X_4); \\
\deg E_{\ell_1,\ell_2,\ell_3,\ell_4} &= \deg E_{\ell_2,\ell_1,\ell_3,\ell_4} = \deg E_{\ell_3,\ell_1,\ell_2,\ell_4} = \deg e_{\ell_1,\ell_2,\ell_3,\ell_4} + 2 \deg(X_4); \\
\deg G_{\ell_1,\ell_2,\ell_3,\ell_4} &= \deg e_{\ell_1,\ell_2,\ell_3,\ell_4} + 3 \deg(X_4).
\end{align*}
\]
Therefore, by setting $\alpha = \deg e_{\ell_1,\ell_2,\ell_3,\ell_4}$, we get
\[
\text{reg}(J_G^s) = \max \left\{ \alpha, \alpha + \deg(X_4) - 1, \alpha + 2 \deg(X_4) - 2, \alpha + 3 \deg(X_4) - 3 \right\}
\]
for $0 \leq \ell_1, \ell_2, \ell_3, \ell_4 \leq s$.

Here also, we have obtained an expression for $\text{reg}(J_G^s)$ not in the form of a linear polynomial. But, as we have demonstrated in the previous cases, this can always be derived for a given graph. Analyzing the interplay between the cardinalities of the partitions, one can obtain the polynomial expression. We discuss the unmixed case as an example. Let $m_1 = m_2 = m_3 = m_4 = m$. Then
\[
\alpha = (s - \ell_4) m_1 + (s - \ell_3) m_2 + (s - \ell_2) m_3 + (s - \ell_1) m_4
\]
\[
= (4s - (\ell_1 + \ell_2 + \ell_3 + \ell_4)) m = 3ms.
\]
Therefore $\text{reg}(J_G^s) = 3ms + (3m - 3)$ for all $s \geq 3$.

Let the partitions be in an arithmetic progression.

**Corollary 4.9.** Let $m, r$ be any two positive integers. Consider the arithmetic progression $m_1 = m$, $m_2 = m + r$, $m_3 = m + 2r$, and $m_4 = m + 3r$ in Theorem 4.8 Then, for all $s \geq 3$, we have
\[
\text{reg}(J_G^s) = s(3m + 6r) + 3m - 3.
\]

**Proof.** We have from Theorem 4.8 \[ \alpha = (s - \ell_4) m_1 + (s - \ell_3) m_2 + (s - \ell_2) m_3 + (s - \ell_1) m_4. \] On substituting the values of $m_i$’s in $\alpha$, we get
\[
\alpha = s(4m + 6r) - m\ell_4 - (m + r)\ell_3 - (m + 2r)\ell_2 - (m + 3r)\ell_1
\]
By Theorem 4.8 we have for all $s \geq 3$,
\[
\text{reg}(J_G^s) = \max \left\{ \alpha, \alpha + (m + 3r) - 1, \alpha + 2(m + 3r) - 2, \alpha + 3(m + 3r) - 3 \right\}
\]
for $0 \leq \ell_1, \ell_2, \ell_3, \ell_4 \leq s$.

where $\ell_1 + \ell_2 + \ell_3 + \ell_4 = s$. To achieve the maximum value of $\alpha$, negative terms in $\alpha$ should be minimum. The coefficient of $\ell_1$ in negative terms in $\alpha$ is largest, so $\ell_1$ should be assigned the minimum value. After assigning the minimum value to $\ell_1$, assign the minimum value to $\ell_2$, and similarly minimum value to $\ell_3$. Then assign $\ell_4 = s - \ell_2 - \ell_3 - \ell_4$. For instance, to get the maximum of $\alpha$ when $1 \leq \ell_1 \leq s$, $0 \leq \ell_2, \ell_3, \ell_4 \leq s - 1$, put $\ell_1 = 1, \ell_2 = 0, \ell_3 = 0$, and $\ell_4 = s - 1$. With appropriate substitution, we get for all $s \geq 3$
\[
\text{reg}(J_G^s) = \max \left\{ s(3m + 6r), s(3m + 6r) + m - 1, s(3m + 6r) + 2m - 2, s(3m + 6r) + 3m - 3 \right\}
\]
for $0 \leq \ell_1, \ell_2, \ell_3, \ell_4 \leq s$. 

Clearly for all \( m \geq 1 \), and for all \( s \geq 3 \), we get
\[
\text{reg}(J^s_G) = s(3m + 6r) + 3m - 3.
\]

4.3. Complete \( m \)-partite graphs. Let \( G \) be a complete graph on \( m \)-vertices. Then the cover ideal \( J_G \) of \( G \) is generated by \( \{x_1 \cdots x_i : 1 \leq i \leq m\} \). It follows from Remark 4.2 that depth \( R/J^s_G = 0 \) for all \( s \geq m - 1 \). Moreover, by Remark 4.1, we know that \( R/J^s_G \) has linear resolution for all \( s \geq 1 \). Therefore, the minimal graded free resolution of \( R/J^s_G \) for all \( s \geq m - 1 \) is of the form
\[
0 \rightarrow R(-(s(m - 1) - m + 1))^{\beta m} \rightarrow \cdots \rightarrow R(-(s(m - 1) - 1))^{\beta_2} \rightarrow R(-(s(m - 1)))^{\beta_1} \rightarrow R \rightarrow 0.
\]

Let \( g_1, g_2, \ldots, g_m \) be the minimal generators \( J_G \). Then the elements in \( J^s_G \) consists of elements \( T_{\ell_1, \ell_2, \ldots, \ell_m} \), where \( T_{\ell_1, \ell_2, \ldots, \ell_m} = g_{\ell_1} g_{\ell_2} \cdots g_{\ell_m} \) such that \( \ell_1 + \ell_2 + \cdots + \ell_m = s \) and \( 0 \leq \ell_i \leq s \). Therefore the total number of elements in \( J^s_G \) is same as the total number of non-negative integral solution to the linear equation \( \ell_1 + \ell_2 + \cdots + \ell_m = s \) which is \( \binom{s + m - 1}{m - 1} \). Hence \( \mu(J^s_G) = \binom{s + m - 1}{m - 1} \).

Therefore \( \beta_1 = \binom{s + m - 1}{m - 1} \).

Let \( \{e_{\ell_1, \ell_2, \ldots, \ell_m} : 0 \leq \ell_i \leq s; \text{ and } \ell_1 + \ell_2 + \cdots + \ell_m = s\} \) denote the standard basis for \( R^{\beta_1} \). Let \( \partial_i : R^{\beta_1} \rightarrow R \) be the map \( \partial_i(e_{\ell_1, \ell_2, \ldots, \ell_m}) = T_{\ell_1, \ell_2, \ldots, \ell_m} \). As done in the proofs of Theorems 4.3 and 4.7, we can see that the first syzygy is given by the relations of the form
\[
x_i \cdot T_{\ell_1, \ell_2, \ldots, \ell_i - 1, \ell_i + 1, \ell_i + 2, \ldots, \ell_m} - x_{i+1} \cdot T_{\ell_1, \ell_2, \ldots, \ell_i - 1, \ell_i, \ell_i + 1, \ldots, \ell_m} = 0
\]
for each \( 1 \leq i \leq m - 1 \). Set \( \ell_i - 1 = \ell'_i \) and \( \ell_{i+1} + 1 = \ell'_{i+1} \). Then it can be seen that, for each \( 1 \leq i \leq m - 1 \), there exist as many such relations as the number of non-negative integer solutions of \( \ell_1 + \cdots + \ell'_i + \cdots \ell_m = s - 1 \). Therefore, the total number of such linear relations is \((m - 1)\binom{s + m - 2}{m - 1}\). However it is not very difficult to realize that writing down the higher syzygy relations are quite challenging. Based on Theorems 4.3 and 4.7 and some of the experimental results using the computational commutative algebra package Macaulay 2 [3], we propose the following conjecture:

**Conjecture 4.10.** Let \( R = K[x_1, x_2, \ldots, x_m] \) and let \( J \) be the cover ideal of the complete graph \( K_n \). The minimal graded free resolution of \( R/I^s \) for all \( s \geq m - 1 \) is of the form
\[
0 \rightarrow R(-(s(m - 1) - m + 1))^{\beta m} \rightarrow \cdots \rightarrow R(-(s(m - 1) - 1))^{\beta_2} \rightarrow R(-(s(m - 1)))^{\beta_1} \rightarrow R \rightarrow 0,
\]
where
\[
\beta_i = \binom{m - 1}{i - 1} \binom{s + m - i}{m - 1}.
\]

Notice that proving the above conjecture will give the Betti numbers of powers of cover ideals of complete \( m \)-partite graphs. We conclude our article by proposing an expression for the regularity of powers of the cover ideals of complete \( m \)-partite graphs:

**Conjecture 4.11.** Let \( G \) denote a complete \( m \)-partite graph with \( V(G) = \bigcup_{i=1}^m V_i \) and \( E(G) = \{(a, b) : a \in V_i, b \in V_j, i \neq j\} \). Set \( V_i = \{x_{i1}, \ldots, x_{in_i}\} \) for \( i = 1, \ldots, m \). Let \( J_G \subset R = K[x_{ij} : 1 \leq i \leq m; 1 \leq j \leq n_i] \) denote the cover ideal of \( G \). Let \( 0 \leq \ell_1, \ell_2, \ldots, \ell_m \leq s \) be integers such that \( \ell_1 + \ell_2 + \cdots + \ell_m = s \). Set
\[
\alpha = s \cdot \left( \sum_{i=1}^m n_i \right) - \sum_{i=1}^m n_i \ell_{m+1-i}.
\]
Then for all \( s \geq m - 1 \), one has

\[
\operatorname{reg}(J_G^s) = \max \left\{ \begin{array}{ll}
\alpha, & \text{for } 0 \leq \ell_1, \ell_2, \ldots, \ell_m \leq s, \\
\alpha + n_m - 1, & \text{for } 1 \leq \ell_1 \leq s, 0 \leq \ell_2, \ldots, \ell_m \leq s - 1, \\
\alpha + 2(n_m - 1), & \text{for } 2 \leq \ell_1 \leq s, 0 \leq \ell_2, \ldots, \ell_m \leq s - 2, \\
\vdots \\
\alpha + (m - 1)(n_m - 1), & \text{for } m - 1 \leq \ell_1 \leq s, 0 \leq \ell_2, \ldots, \ell_m \leq s - (m - 1). 
\end{array} \right.
\]

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