Simple closed geodesics
on most Alexandrov surfaces

Joël Rouyer       Costin Vîlcu

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Abstract

We study the existence of simple closed geodesics on most (in the
sense of Baire category) Alexandrov surfaces with curvature bounded
below, compact and without boundary. We show that it depends on
both the curvature bound and the topology of the surfaces.

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1 Introduction

The existence of closed geodesics is of certain interest in the geometry of
Riemannian surfaces, and was studied in many articles. We mention here
only a very few facts, related to our topic. In this paper, whenever we
consider several geodesics they are geometrically distinct.

In the late nineteenth century, J. Hadamard [14] showed that every non-
trivial homotopy class of closed curves on a closed Riemannian manifold \( M \)
contains geodesics.

It is a famous result of L. A. Lusternik and L. G. Schnirelman that for
every Riemannian metric on the 2-sphere there exist at least three simple
closed geodesics (and sometimes exactly three, \( e.g. \) for ellipsoids with distinct
axes) [17]. This was completed by a combined result of J. Franks and V.
Bangert [9], [4], stating that for every metric on such a surface there exist infinitely many closed geodesics.

On the other hand, for a given upper bound on the length, the number of closed geodesics is usually finite.

M. Mirzakhani [18] showed that the number $n_X(L)$, of simple closed geodesics of length $\leq L$ on a hyperbolic Riemannian surface $X$ of genus $g$, is asymptotic to $c_X L^{6g-6}$ as $L \to \infty$, where $c_X$ is a constant depending on $X$.

G. Contreras [8] proved that for every closed manifold $M$ of dimension at least two, there is an open and dense subset of the space of $C^\infty$ Riemannian metrics on $M$, any metric on which satisfies $\lim_{L \to \infty} \frac{\log p(L)}{L} > 0$, where $p(L)$ is the number of closed geodesics of length $\leq L$.

Recall that Baire categories were previously employed in the study of geodesics in the framework of Riemannian geometry. Improving previous results of several authors, H. Rademacher [22] proved that a $C^r$ typical metric on a compact simply connected manifold carries infinitely many (not necessarily simple) closed geodesics ($2 \leq r \leq \infty$).

In this paper we consider the Baire space $A(\kappa)$ of Alexandrov surfaces (definitions below), in which smooth Riemannian surfaces form a set of first category, even though dense. In this space, we study the existence of simple closed geodesics on a typical surface, and show that it depends on both the curvature bound and the topology of the surface.

Formally, we denote by $A(\kappa)$ the set of all compact Alexandrov surfaces with curvature bounded below by $\kappa$, without boundary. We refer to [7] or [25] for the precise definition and basic facts about such spaces.

It is known that these surfaces are 2-dimensional topological manifolds. Closed Riemannian surfaces with Gauss curvature at least $\kappa$ and $\kappa$-polyhedra (see §2 for the definition) are important examples of such surfaces.

It is also known that, endowed with topology induced by the Gromov-Hausdorff distance, $A(\kappa)$ is a Baire space [15]. In any Baire space, one says that most elements or a typical element enjoys a property $P$ if the set of those elements which do not satisfy $P$ it is of first category.

Let $A(\kappa, \chi)$ denote the set of those surface in $A(\kappa)$ whose Euler-Poincaré characteristic is $\chi$. The connected components of $A(\kappa)$ are the sets of those surfaces of a given topological type [24]. Therefore, $A(\kappa, \chi)$ (if non-empty) is a connected component of $A(\kappa)$ if $\chi$ is positive or odd, and is the union of two components otherwise.
The space of all convex surfaces in \( \mathbb{R}^3 \) is naturally endowed with the Pompeiu-Hausdorff metric. By celebrated results of Alexandrov (for existence, see [2] p. 362) and Pogorelov (for rigidity, see [21] p. 167), each surface \( A \in \mathcal{A}(0,2) \) can be realized as a convex surface in \( \mathbb{R}^3 \), unique up to an isometry of the ambient space. Therefore, the intrinsic geometry of convex surfaces is a particular case of the geometry of Alexandrov surfaces.

P. Gruber proved that most convex surfaces have no simple closed geodesics [11], and later improved this result by dropping the simpleness assumption [12]. His result strongly contrasts the mentioned result of L. A. Lusternik and L. G. Schnirelman. Nevertheless, on any convex surface there exist three simple closed quasi-geodesics [20] (see for example [2] p. 373 for the definition).

Adapted to our framework, P. Gruber’s result states that most Alexandrov surfaces in \( \mathcal{A}(0,2) \) have no (simple) closed geodesics. In this paper we investigate the typical existence – or non-existence – of simple closed geodesics for the other values of \( \kappa \) and \( \chi \).

Notice that it suffices to study the curvature bounds \( \kappa \in \{-1, 0, 1\} \), because there is a natural homothety from \( \mathcal{A}(\kappa) \) to \( \mathcal{A}(1) \) if \( \kappa > 0 \), and to \( \mathcal{A}(-1) \) if \( \kappa < 0 \). Also notice that \( \mathcal{A}(\kappa') \) is nowhere dense in \( \mathcal{A}(\kappa) \) for \( \kappa' > \kappa \), so a typical element in \( \mathcal{A}(\kappa') \) is not typical in \( \mathcal{A}(\kappa) \).

Since the total curvature of a surface of \( \mathcal{A}(\kappa, 0) \) vanishes, the space \( \mathcal{A}(0,0) \) contains only flat tori and flat Klein bottles (see [24], Lemma 4]). It follows that each \( A \in \mathcal{A}(0,0) \) is union of simple closed geodesics.

In Section 3 we prove that most surfaces in \( \mathcal{A}(-1) \) admit infinitely many, non-intersecting, simple closed geodesics, and in Section 4 we prove that most surfaces in \( \mathcal{A}(\kappa, 1) \) admit infinitely many simple closed geodesics, all of bounded length. This contrasts the mentioned result of M. Mirzakhani.

In Section 5 we treat the remaining case – \( \mathcal{A}(1,2) \) – and prove that a typical surface there has no simple closed geodesic.

Many properties of most convex surfaces have been investigated (see for example the surveys [13] and [28]), but only a few of them have been hitherto generalized to Alexandrov surfaces (see [1], [15]). In particular, most surfaces in \( \mathcal{A}(\kappa) \) if \( \kappa \neq 0 \), and most surfaces in \( \mathcal{A}(0) \setminus \mathcal{A}(0,0) \), are not Riemannian manifolds of class \( \mathcal{C}^2 \).
2 Preliminaries

Let $H$ and $K$ be compact subsets of a metric space $Z$; we denote by $d_Z^H(H,K)$ the usual Pompeiu-Hausdorff distance between them. We shall omit the superscript $Z$ whenever no confusion is possible.

If $X$ and $Y$ are compact metric spaces, we denote by $d_{GH}(X,Y)$ the Gromov-Hausdorff distance between $X$ and $Y$. For its definition and basic properties, we refer to [10] or [6]. Recall that we have $d_{GH}(H,K) \leq d_Z^H(H,K)$ for any compact subsets $H, K$ of a given metric space $Z$; moreover, we have the following lemma.

Lemma 1. Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of compact metric spaces converging to $X$ with respect to the Gromov-Hausdorff metric, and let $\{\varepsilon_n\}_{n \in \mathbb{N}}$ be a sequence of positive numbers. Then there exist a compact metric space $Z$, an isometric embedding $\varphi: X \to Z$ and, for each positive integer $n$, an isometric embedding $\varphi_n: X_n \to Z$, such that

$$d_Z^H(\varphi_n(X_n), \varphi(Y)) < d_{GH}(X_n, X) + \varepsilon_n.$$

A more sophisticated fact is the famous Perel’mann’s theorem of stability. The reader will find a complete proof in [16], or in the original manuscript [19]; in our case (2-dimensional spaces without boundary) the proof admits large simplifications. In order to give its statement, recall the definition of distortion. If $f: X \to Y$ is a map between metric spaces then

$$\text{dis}(f) = \sup_{x, x' \in X} |d(x, x') - d(f(x), f(x'))|.$$

Lemma 2 (Perel’mann’s stability theorem). Let $A_n, A \in \mathcal{A}(\kappa)$ and suppose that there exist functions $f_n: A \to A_n$ such that $\text{dis}(f_n) \to 0$. Then, for $n$ large enough, there exists homeomorphisms $h_n: A \to A_n$ such that $\sup_{x \in A} d(f_n(x), h_n(x)) \to 0$.

Consider two surfaces $S$ and $S'$ with boundaries $\partial S$ and $\partial S'$; assume there exist arcs $I \subset \partial S$ and $I' \subset \partial S'$ having the same length. By gluing $S$ to $S'$ along $I$ we mean identifying the points $x \in I$ and $\iota(x) \in I'$, where $\iota: I \to I'$ is a length preserving map between $I$ and $I'$.

Lemma 3 (Alexandrov’s gluing theorem). Let $S$ be a closed topological surface obtained by gluing finitely many geodesic polygons cut out from
surfaces in $A(\kappa)$, in such a way that the sum of the angles glued together at each point is at most $2\pi$. Then, endowed with the induced metric, $S$ belongs to $A(\kappa)$.

Let $M_\kappa$ denote the simply-connected and complete Riemannian surface of constant curvature $\kappa$.

A $\kappa$-polyhedron is an Alexandrov surface obtained by gluing finitely many geodesic polygons from $M_\kappa$. Let $P(\kappa)$ denote the set of all $\kappa$-polyhedra.

A formal proof for the next result can be found, for example, in [15].

**Lemma 4.** The subset of $\kappa$-polyhedra, and the subset of closed Riemannian surfaces with Gauss curvature at least $\kappa$, are both dense in $A(\kappa)$.

The length of a curve $\gamma$ will be denoted by $\ell(\gamma)$.

**Lemma 5.** Let $X$ be a compact metric space, and for each $n \in \mathbb{N}$ let $\gamma_n : [0,1] \to X$ be a rectifiable arc parametrized proportionally to the arc-length. Assume that the sequence $\{\ell(\gamma_n)\}_n$ is bounded. Then one can extract from it a subsequence converging uniformly to a rectifiable arc $\gamma : [0,1] \to X$. Moreover, $\ell(\gamma) \leq \lim\inf \ell(\gamma_n)$ and $\gamma_n([0,1])$ converges to $\gamma([0,1])$ for the Pompeiu-Hausdorff metric.

**Proof.** The choice of the parameter and the fact that $\{\ell(\gamma_n)\}_n$ is bounded imply that $\gamma_n$ are equi-continuous, hence the first statement follows from Ascoli’s theorem. The second statement is nothing but the semi-continuity of length (see for example [6, 2.3.4.iv]). The third statement is an obvious consequence of the first one.

If $P$ is a subset of a metric space $Z$ and $\rho$ a positive number, we denote by $N_\rho(P)$ the $\rho$-neighbourhood of $P$ in $Z$, namely

$$N_\rho(P) = \{ x \in Z | \exists y \in P \ d^Z(x,y) \leq \rho \}.$$

We end this section with a notion of stability for simple closed geodesics, which is essential in our proofs.

**Definition.** Let $A \in A(\kappa)$. A simple closed geodesic $G$ of $A$ is said to be stable if for any isometric embedding $\phi : A \to Z$ in any metric space $Z$, and for any positive number $\delta$, there exists $\eta > 0$ such that for any $A' \in A(\kappa)$ included in $Z$, if $d^Z_H(\phi(A), A') \leq \eta$ then there exists a simple closed geodesic $G'$ in $A'$ such that $d^Z_H(\phi(G), G') \leq \delta$. 

5
3 A curvature argument

We recall first the Poincaré’s disc model of $\mathbb{H}_1$. It consists of the standard open disk
\[ \mathbb{P} = \{(x, y) \mid x^2 + y^2 < 1\} \]
endowed with the distance
\[ d_\mathbb{P}(u, v) = \arccosh (1 + p(u, v)), \]
where
\[ p(u, v) = \frac{2 \|u - v\|^2}{(1 - \|u\|^2)(1 - \|v\|^2)}, \tag{1} \]
and $\| \|$ is the standard Euclidean norm. In this model, the geodesics are exactly the circular arcs normal to the disk boundary.

Lemma 6. Let $Q = Q(\lambda, \varepsilon)$ ($\lambda > 0$, $\varepsilon > 0$) be the geodesic quadrilateral of $\mathbb{P}$ whose vertices are $(\pm a, \pm b)$, where $a$, $b$ are chosen such that the distance (in $\mathbb{P}$) between the midpoint of the upper side $U$ of $Q$ (from $(a, b)$ to $(-a, b)$) and the midpoint of the lower side $L$ of $Q$ (from $(a, -b)$ to $(-a, -b)$) is $\lambda$, and the distance between the midpoints of the other two sides of $Q$ is $\varepsilon$.

i) The unique shortest path $\gamma_0$ in $Q$ from $L$ to $U$ is a segment of the $y$-axis.

ii) There exists a positive number $\alpha = \alpha(\lambda, \varepsilon)$ such that any path $\gamma$ from $L$ to $U$ intersecting either the left or the right side of $Q$ satisfies $\ell(\gamma) \geq \lambda + \alpha$.

Proof. (i) $Q$ is convex in $\mathbb{P}$, whence $\gamma_0$ is a geodesic segment from $l \in L$ to $u \in U$. In order to maximize the denominator of $p(l, u)$ in Formula (1), we have to choose $l$ and $u$ on the $y$-axis. This condition also minimizes the numerator, whence the conclusion.

(ii) Assume the conclusion fails. So there exists a sequence $\{\gamma_n\}_n$ of curves from $l_n \in L$ to $u_n \in U$ via a point $r_n$ on (say) the right side $R$, such that $\ell(\gamma_n) \to \ell(\gamma_0)$. Let $m$ be the minimal value of the function $f : L \times U \times R \to \mathbb{R}$ given by $(l, u, r) \mapsto d_\mathbb{P}(l, r) + d_\mathbb{P}(r, u)$. By (i), $m > \ell(\gamma_0)$. On the other hand, $\ell(\gamma_n) \geq f(l_n, r_n, u_n) \geq m$, whence $\ell(\gamma_0) \geq m$ and we get a contradiction.

We shall denote by $C(\lambda, \varepsilon)$ the manifold with boundary obtained from the quadrilateral $Q(\lambda, \varepsilon)$ in Lemma 6 by gluing $L$ onto $U$, right onto right, and left onto left. The segment which was the $y$-axis in $\mathbb{P}$ becomes after gluing a simple closed geodesic. We call it the soul of $C(\lambda, \varepsilon)$.
Lemma 7. If $A \in \mathcal{A}(-1)$ contains a region $C$ isometric to $C(\lambda, \varepsilon)$ for some $\lambda, \varepsilon > 0$, then the soul of $C$ is a stable simple closed geodesic.

Proof. Let $G$ be the soul of $C = C(\varepsilon, l)$, with $C \subset A$. Let $\phi : A \rightarrow Z$ be an isometric embedding of $A$ in some metric space $Z$ and put $B = \phi(A)$. Choose $\delta > 0$. Assume that the result does not hold, hence there exists a sequence $\{B_n\}$ of Alexandrov surfaces isometrically embedded in $Z$ such that $\nu_n \overset{\text{def}}{=} d^Z_H(B, B_n)$ tends to 0, and $B_n$ has no simple closed geodesic $G'$ with $d^Z_H(G, G') \leq \delta$. Define functions $f_n : B \rightarrow B_n$ (not necessarily continuous) such that $d(x, f_n(x)) \leq \nu_n$; this is possible, by the definition of the Pompeiu-Hausdorff distance.

By Lemma 2, there exists a sequence of positive numbers $o_n$ convergent to 0 such that, for large $n$, a homeomorphism $h_n : B \rightarrow B_n$ exists and satisfies $d(h_n(x), f_n(x)) \leq o_n$. Hence $d_n \overset{\text{def}}{=} \text{dis}(h_n) \leq \nu_n + 2o_n \rightarrow 0$, and for all $x \in Z$ we have $d^Z(h_n(x), x) \leq \nu_n + o_n \rightarrow 0$.

Let $\varepsilon'$ be small enough to ensure that $C' \overset{\text{def}}{=} C(\varepsilon', l)$ is included in $N_{\delta/2}(G)$. For $n$ large enough, $h_n(C') \subset N_\delta(G)$.

Define two closed subset $C^1$, $C^2$ of $C'$, delimited by geodesics normal to $G$, such that $C' = C_1 \cup C_2$ and $C_1 \cap C_2$ is homeomorphic to the union of two closed ball, say $V$ and $W$ (see Figure 1). Let $\mathcal{K}$ (resp. $\mathcal{K}_n$) be the set of those closed curves $\mathbb{R}/\mathbb{Z} \rightarrow B$ (resp. $\mathbb{R}/\mathbb{Z} \rightarrow B_n$), parametrized proportionally to the arc-length, of length less than $2\ell(G)$, and union of two arcs from $v \in V$ (resp. $v_n \in h_n(V)$) to $w \in W$ (resp. $w_n \in h_n(W)$), one of them lying in

![Figure 1: Definition of $C_1$, $C_2$, $V$ and $W$ in the proof of Lemma 7.](image-url)
$C^1$ (resp. $h_n(C^1)$) and the other in $C^2$ (resp. $h_n(C^2)$). By Lemma 5, $K_n$ is compact and there exists a shortest curve $S_n : \mathbb{R}/\mathbb{Z} \rightarrow h_n(C')$ in $K_n$. It is clear that, for $n$ large enough, $d_H^{K_n}(\text{Im} S_n, G) \leq \delta$. By our assumption, $S_n$ is not a geodesic, and therefore intersects the boundary of $h_n(C')$.

Assume (by passing to a subsequence, if necessary) that $S_n$ converges to some closed curve $S \in K$; then $S$ touches $\partial C'$ and is not contractible in $C$. It follows (by Lemma 6) that $\ell(S) \geq \ell(G) + \alpha(\lambda', \varepsilon')$, and (by Lemma 5) that $\ell(S_n) \geq \ell(G) + \alpha(\lambda, \varepsilon')/2$ for $n$ large enough.

Let $v$ be the midpoint of $G \cap V$ and $w$ be the midpoint of $G \cap W$. Let $G_i$ ($i = 1, 2$) be the part of $G$ delimited by $u$ and $v$ which is contained in $C_i$. Take points $x_0 = u, x_1, \ldots, x_N = v$ on $G^1$ such that

$$\max_i d(x_i, x_{i+1}) \leq \frac{1}{2} d(G, \partial C^1).$$

Let $G_n^1 \subset B_n$ be the union of segments from $h_n(x_{i-1})$ to $h_n(x_i)$ ($1 \leq i \leq N^1$); for large $n$, $G_n^1 \subset h_n(C^1)$. Moreover,

$$\ell(G^1_n) = \sum_{i=1}^{N} d(h_n(x_{i-1}), h_n(x_i)) \leq \sum_{i=1}^{N} d(x_{i-1}, x_i) + N^1 d_n \leq \ell(G^1) + N^1 d_n.$$

Similarly, one can construct $G_n^2 \subset h_n(C^2)$. The length of $G_n \overset{\text{def}}{=} G_n^1 \cup G_n^2$ is at most $\ell(G) + (N^1 + N^2) d_n$. On the other hand, $G_n \in K_n$, whence $\ell(G_n) \geq \ell(S_n) \geq \ell(G) + \alpha(\lambda, \varepsilon')/2$, and we get a contradiction. □

**Theorem 8.** Most surfaces in $\mathcal{A}(-1)$ have infinitely many, non-intersecting, simple closed geodesics of bounded length.

**Proof.** Let $\mathcal{G}_p \subset \mathcal{A}(-1)$ be the set of all Alexandrov surfaces which admit at least $p$ non-intersecting simple closed geodesics, and let $\mathcal{S}_p \subset \mathcal{G}_p$ be the set of all Alexandrov surfaces which admit at least $p$ non-intersecting stable simple closed geodesics.

We claim that $\mathcal{S}_p \subset \text{int}\mathcal{G}_p$. Choose $A \in \mathcal{S}_p$; we have to prove that for any sequence $A_n \in \mathcal{A}(-1)$ converging to $A$, $A_n$ belongs to $\mathcal{G}_p$ for large $n$. By Lemma 7 we can assume that the surfaces $A_n, A$ are all included in the same metric space $Z$. Let $G_1, \ldots, G_p$ be $p$ non-intersecting and stable simple closed geodesics of $A$. Put

$$\delta = \frac{1}{3} \min_{1 \leq i < j < p} \min_{(x, y) \in G_i \times G_j} d(x, y).$$

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For $n$ large enough, there exists on $A_n$ a simple closed geodesic $G^n_i$, lying in $N_\delta (G_i) \ (i = 1, \ldots, p)$, and the geodesics $G^n_1, G^n_2, \ldots, G^n_p$ are non-intersecting by the choice of $\delta$. This proves the claim.

We now claim that $S_p$ is dense in $\mathcal{A}(-1)$. By Lemma 4, it suffices to approximate every Riemannian surface $R$ with surfaces in $S_p$. $R$ admits at least one simple closed geodesic $G$. A small neighbourhood of $G$ is homeomorphic to either a cylinder or a Möbius strip.

Assume first that we are in the former case. Cutting $R$ along $G$ yields a manifold $R'$ whose boundary $\partial R'$ consists of two topological circles. For small $\varepsilon > 0$, one can choose $\lambda$ such that the boundary of $C(\lambda, \varepsilon)$ is isometric to the boundary of $R'$. Hence we can glue $p$ copies of $C_1, \ldots, C_p$ of $C(\lambda, \varepsilon)$ between the two circles of $\partial R'$: one circle of $\partial R'$ is glued to the left side of $C_1$, the right side of $C_i \ (1 \leq i < p)$ is glued on the left side of $C_{i+1}$, and the right side of $C_p$ is glued on the other circle of $\partial R'$. By Lemmas 3 and 7, the obtained surface belongs to $S_p$, and for small $\varepsilon$ it is close to $R$.

Assume now that a neighbourhood of $G$ is a Möbius strip, hence $\partial R'$ consists of one topological circle of length $2\ell(G)$. For small $\varepsilon$, we can choose $\lambda$ such that each boundary component of $C(\lambda, \varepsilon)$ has length $2\ell(G)$. Glue successively $p$ copies of $C_1, \ldots, C_p$ of $C(\lambda, \varepsilon)$ onto $\partial R'$: the left side of $C_1$ on $\partial R'$ and the right side of $C_i$ on the left side of $C_{i+1} \ (1 \leq i < p)$. The obtained surface still has a boundary, namely the right side of $C_p$. Glue it on itself by identifying pairs of “opposite points” (i.e., points which are separating the boundary in two arcs of length $\ell(G)$). The obtained surface belongs to $S_p$ (by Lemmas 3 and 7) and is closed to $R$. This proves the second claim.

It follows that $\text{int} \ (G_p)$ is open and dense in $\mathcal{A}(-1)$, and

$$
\mathcal{G} = \left\{ A \in \mathcal{A}(-1) \mid A \text{ has infinitely many simple closed geodesics pairwise non-intersecting} \right\}
\supset \bigcap_{p \in \mathbb{N}} \text{int} \ (G_p)
$$

is residual in $\mathcal{A}(-1)$.

It is obvious from the above argument that the lengths of geodesics are bounded. \hfill \Box
4 A topological argument

In the previous section we have proven the existence of simple closed geodesics using a topology-free argument. In this section we shall use a topology-based argument, which essentially does not depend on the curvature bound. The case of $A(-1,1)$ is covered by both Section 3 and Section 4.

The proof of the following easy lemma is left to the reader.

**Lemma 9.** Let $Q_\kappa = Q_\kappa(\lambda,\varepsilon)$ be a geodesic quadrilateral in $\mathbb{M}_\kappa$ defined as in Figure 2 ($\kappa = 0, \pm 1$), let $L$ be its left side and $R$ be its right side. If $\kappa = 1$ assume, moreover, that $\lambda < \pi$. Denote by $s$ the symmetry with respect of its center.

The shortest curve from $x \in L$ to $s(x)$ is the segment between the midpoints of $L$ and $R$. Moreover, there exists a positive number $\beta = \beta(\lambda,\varepsilon)$ such that any curve from $x$ to $s(x)$ which touches either the upper or the lower side of $Q_\kappa$ has a length of at least $\lambda + \beta$.

Let $M_\kappa(\lambda,\varepsilon)$ be the compact Möbius strip obtained from the quadrilateral $Q_\kappa(\lambda,\varepsilon)$ in Lemma 9 by gluing the two $\varepsilon$ long sides. The segment joining the midpoints of the $\varepsilon$ long sides becomes a simple closed geodesic in $M_\kappa(\lambda,\varepsilon)$; call it the soul of $M_\kappa(\lambda,\varepsilon)$.

**Lemma 10.** If $A \in A(\kappa)$ contains a subset $M$ isometric to some $M_\kappa(\lambda,\varepsilon)$ then its soul is stable.

**Proof.** By Lemma 9 there exist $\beta = \beta(\lambda,\varepsilon)$ such that each non-contractible curve $\gamma \subset M$ intersecting $\partial M$ is longer that $\lambda + \beta$. From now on, the proof is the same as the proof of Lemma 7.

**Corollary 11.** Let $A \in P(\kappa)$ be homeomorphic to $\mathbb{RP}^2$ ($\kappa \in \{-1,0,1\}$), and let $G$ be a non-contractible simple closed geodesic in $A$. If $\kappa = 1$, assume moreover that $\ell(G) < \pi$. Then $G$ is stable.
A polyhedral disk $D$ is a 2-dimensional disk obtained by gluing a finite collection of geodesic triangles of $\mathbb{M}_\kappa$, in such a way that the sum of the angles glued together at each point is at most $2\pi$. By definition, an angle of $\partial D$ is a point whose space of directions has a length distinct from $\pi$. This length will be called the measure of the angle.

**Lemma 12.** Any polyhedral disk $D$ different from a half-sphere and whose boundary has no angles can be approximated (with respect to the Gromov-Hausdorff distance) by polyhedral disks whose boundary has two angles of measure less than $\pi$, separating it in two equally long curves.

**Proof.** We claim that $D$ has at least one vertex. If $\kappa \leq 0$, this follows from the Gauss-Bonnet Formula. If $\kappa = 1$ and $D$ had no vertices, then gluing two copies of it along its boundary would provide a simply connected 1-polyhedron without vertices. Such a polyhedron must be the standard sphere, in contradiction with the fact that $D$ is not a half-sphere. Hence $D$ contains at least one vertex $v$, say of singular curvature $\omega(v)$.

Choose two points $p, p' \in \partial D$ separating $\partial D$ into two arcs of equal length. Let $\sigma$ be a segment emanating from $p$ and normal to $\partial D$, and let $q$ be a point of $\sigma$ close to $p$. Let $\gamma$ be a segment between $q$ and $v$; $\gamma \cap \partial D = \emptyset$, because $q, v \not\in \partial D$ and $D$ is convex. Let $w$ be a point close to $v$ such that $\angle qvw = \frac{2\pi - \omega(v)}{2}$; it exists, because $D$ is polyhedral. Such $w$ is joined to $q$ by precisely two segments, say $\gamma_1, \gamma_2$. Cut out from $D$ the digon they are bounding and glue $\gamma_1$ onto $\gamma_2$. On the obtained disk $D'$, $q$ is a vertex of small singular curvature $\omega(q)$.

Consider a quadrilateral $abcb'$ in $\mathbb{M}_\kappa$ such that $d(a, b) = d(a, b') = d(p, q)$, $\angle abc = \angle ab'c = \pi/2$ and $\angle bab' \leq \omega(q)$. Note that, if $d(p, q) < \pi/2$, we have $\angle bcb' < \pi$.

Cut $D'$ along the arc $\sigma'$ of $\sigma$ from $p$ to $q$ and glue $abcb'$, $a$ at $q$ and the sides $ab, ab'$ along the two images of $\sigma'$. The resulting disk boundary has one angle at $c$.

Do the same construction starting at the point $p'$, to obtain the desired approximation of $D$. $\Box$

An almost-geodesic $G$ on a $\kappa$-polyhedron is a polygonal line admitting at each of its points $x$ (except its endpoints, if any) two tangent directions, dividing the space of directions at point $x$ in two curves, at least one of which has length $\pi$.

The proof of the next simple result is left to the reader.
Lemma 13. Let \( P \in \mathcal{A}(\kappa, 2) \) be a \( \kappa \)-polyhedron whose vertices have singular curvature less than \( \pi \). Let \( \{ \Gamma_n \} \) be a sequence of geodesics on \( P \) converging to \( \Gamma \subset P \) with respect to the Pompeiu-Hausdorff distance. Then \( \Gamma \) is an almost-geodesic.

We denote by \( S_\alpha \in \mathcal{P}(1) \) the orientable surface obtained by gluing the two sides of a digon in \( \mathbb{M}_1 \) of angle \( 2\pi - \alpha \).

Lemma 14. If \( P \in \mathcal{A}(1, 2) \) is a 1-polyhedron then \( \text{diam}(P) \leq \pi \), with equality if and only if \( P = S_\alpha \) for some \( \alpha \in [0, 2\pi] \).

Proof. The inequality \( \text{diam}(A) \leq \pi \) is well-known for any \( A \in \mathcal{A}(1) \) (see [7, Theorem 3.6]), and all surfaces \( S_\alpha \) have diameter \( \pi \).

Let \( u, v \in P \) such that \( \text{diam}(P) = \pi = d(u, v) \). Consider a triangle \( uvx \) in \( P \) and let \( \tilde{u}\tilde{v}\tilde{x} \) be a comparison triangle on the sphere \( \mathbb{M}_1 \). We have \( \angle uxv \geq \angle \tilde{u}\tilde{x}\tilde{v} = \pi \). It follows that the union of the segments \( ux \) and \( xv \) is a geodesic on \( P \), hence \( x \) is not a vertex. The conclusion follows from the fact that the only 1-polyhedra with at most 2 vertices are the surfaces \( S_\alpha \).

The following lemma is a variant of a result of V. A. Toponogov, see for example [26] or [17, p. 297].

Lemma 15. Let \( G \) be a simple closed almost-geodesic of length \( 2\pi \) on the 1-polyhedron \( P \in \mathcal{A}(1, 2) \). If the boundary of one of the two half-surfaces bounded by \( G \) has no angles then this half-surface is isometric to a half-sphere.

Proof. Let \( C \) be the half-surface of \( P \) whose boundary has no angles. If \( x \) is a point of \( G \), we denote by \( x' \) the point on \( G \) such that \( G \setminus \{ x, x' \} \) consists two equally long arcs. By the use of a (non trivial) comparison argument, it follows that \( G \) is the union of two segments between \( x \) and \( x' \in G \), see the proof of Theorem 3.4.10 in [17, p. 297].

Now choose \( p \in G \) and glue \( C \) on itself by identifying points \( x \in G \) and \( y \in G \) such that \( d(x, p) = d(y, p) \). Since \( G \) is the union of two segments, the diameter of the obtained surface is \( \pi \), hence this surface is \( S_\pi \) (by Lemma 14) and \( C \) is the standard half-sphere.

The following lemma follows directly from Lemma 15.
Lemma 16. The length of a simple closed geodesic $G$ on a $1$-polyhedron $A \in \mathcal{A}(1,1)$ satisfies $\ell(G) \leq \pi$, with equality if and only if $A$ is the projective space with constant curvature $1$.

Theorem 17. Most surfaces in $\mathcal{A}(\kappa,1)$ have infinitely many simple closed geodesics of bounded length.

Proof. Denote by $\mathcal{S}_m$ the set of those surfaces in $\mathcal{A}(\kappa,1)$ which admit at least $m$ stable simple closed geodesics. We only need to prove that $\mathcal{S}_m$ is dense; afterwards the proof proceeds in the same way as the proof of Theorem 8.

Let $P_0 \in \mathcal{A}(\kappa,1)$ be the real projective plane of constant curvature. Choose $A \in \mathcal{A}(\kappa,1) \setminus \{P_0\}$ and approximate $A$ by a polyhedron $P \neq P_0$. The shortest non-contractible closed curve on $P$ is a geodesic $G$. Note that, by Lemma 16, if $\kappa = 1$ then $\ell(G) < \pi$. Cutting $P$ along $G$ provides a polyhedral disk $D$. By Lemma 12, $D$ can be approximated by polyhedral disks $D'$ whose boundary has two angles of measure $\pi - \alpha_0$, for small positive $\alpha_0$, separating it into two equally long arcs. One can adjust the parameters $\lambda$ and $\varepsilon$ such that the boundary length of $\Lambda(\lambda,\varepsilon)$ is exactly $2L$. Consider in $\mathbb{M}_\kappa$ the $(2m+2)$-gon $\Pi_\kappa(m,\lambda,\varepsilon) = a_0a_1 \ldots a_m a_0 b_1 \ldots b_m$ defined as in Figure 3, where $\varepsilon = d(a_i,a_{i-1}) = d(b_i,b_{i-1})$ ($i = 1, \ldots, m$) and $\lambda$ is the distance between mid-points of opposite edges (i.e., the length of a gray line in Figure 3). Glue the side $a_i a_{i-1}$ onto the side $b_i b_{i-1}$ ($i = 1, \ldots, m$), to obtain a surface $\Lambda(\lambda,\varepsilon)$ homeomorphic to a Möbius strip. Its boundary has two angles of measure $\pi + \alpha$ (with $\alpha > 0$ and tending to 0 when $\varepsilon$ tends to 0) separating it into two equally long arcs. One can adjust the parameters $\lambda$ and $\varepsilon$ such that the boundary length of $\Lambda(\lambda,\varepsilon)$ is exactly $2L$. 

Figure 3: Definition of $\Pi_\kappa(m,\varepsilon)$ in the proof of Theorem 17 (in the case $\kappa = 0$).
and such that $\alpha \leq \alpha_0$. So we can glue this $\Lambda_\kappa(\lambda, \varepsilon)$ to the boundary of $D'$. The resulting surface (which still belongs to $\mathcal{A}(1, 1) \cap \mathcal{P}(1)$) approaches $P$ when $\varepsilon \to 0$. It is clear that this surface admits at least $m$ non-contractible simple closed geodesics, corresponding to the gray lines in Figure 3. These geodesics are stable by Corollary 11, proving the density of $S_m$ in $\mathcal{A}(\kappa, 1)$.

It is clear from the above argument that the lengths of geodesics are bounded. The proof is complete.

5 Remaining case

P. Gruber proved that most convex surfaces have no simple closed geodesics [11], and his proof can be easily adapted for most surfaces in $\mathcal{A}(0, 2)$. An important step in his proof was to find a dense set of convex polyhedra without simple closed geodesics; this followed immediately from the Gauss-Bonnet formula, because the curvature of a convex polyhedron is concentrated at its vertices. This proof idea cannot be translated to polyhedra in $\mathcal{A}(1, 2)$, because, in our case, the curvature measure is no longer supported by vertices.

Lemma 18. For any $a < 2\pi$, any 1-polyhedron $P \in \mathcal{A}(1, 2)$ has at most finitely many closed almost-geodesics of length less than $a$.

Proof. A simple closed almost-geodesic which does not pass through any vertex is a simple closed geodesic. Two such geodesics are necessarily intersecting, for otherwise the topological cylinder they would bound would have to be flat by the Gauss-Bonnet formula.

Assume there are infinitely many simple closed geodesics of length less than $a$; by compactness (see Lemma 5), one can find a sequence $G_n$ (with $\ell(G_n) \leq a$) of distinct simple closed geodesics converging to an almost-geodesic $G$. For $n, m$ large enough, $G_n$ and $G_m$ are not separated by vertices. Hence each portion of $G_n$ between two points of $G_n \cap G_m$ measures $\pi$. It follows that $\ell(G_n) \geq 2\pi > a$.

Now choose a vertex $v$ and examine the simple closed almost-geodesics of length at most $a$ passing through $v$. As precedently, if there are infinitely many, one can find a sequence $G_n$ of such curves converging to $G$. Obviously $v$ also belongs to $G$. For $n, m$ large enough $G_n$ and $G_m$ are not separated by vertices, thus, if $G_n \cap G_m$ contains a second point, then the previous argument applies and $\ell(G_n) > a$. 

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Therefore, we can assume moreover that all curves $G_n$ lie in the same half-surface bounded by $G$. Hence one can extract from $G_n$ a subsequence such that $G_m$ lies between $G_n$ and $G$ for any $m > n$. Let $\alpha_n$ be the angle at point $v$ of the half surface bounded by $G_n$ and containing $G$; the sequence $\alpha_n$ is decreasing, in contradiction with the fact that all $G_n$ are supposed to be almost-geodesics.

\textbf{Remark 1.} We obtained a few properties of polyhedra in $A(1, 2)$, see Lemmas 13, 14, 15 and 18. Notice that our polyhedra are different from the ball-polyhedra, defined and studied in a series of papers by K. Bezdek and his collaborators, see e.g. [5].

\textbf{Lemma 19.} For any $a < 2\pi$, any surface $A \in A(1, 2)$ can be approximated by surfaces without simple closed geodesics of length at most $a$.

\textit{Proof.} First approximate $A$ by a 1-polyhedron $P \in A(1, 2)$. By Lemma 18, $P$ carries finitely many simple closed almost-geodesics of length at most $a$. On this polyhedron, choose on each simple closed geodesic $G$ of length at most $a$ a point $x_G$ which does not belong to any other simple closed almost-geodesic of length at most $a$.

Consider the surface $P_\varepsilon$ obtained from $P$ in the following way. First divide all distances on $P$ by $1 + \varepsilon$, to obtain a $(1 + \varepsilon)^2$-polyhedron. Then, for each chosen point $x_G$, cut out a small isosceles triangle $x_Gy_Gy'_G$, symmetric with respect to the geodesic normal to $G$ at $x_G$, such that $d(x_G, y_G) = d(x_G, y'_G) = \varepsilon$ and $\angle y_Gx_Gy_G = \frac{\pi}{2}$. Then, replace this triangle by a triangle $T_G$ of $\mathbb{M}_1$ with the same edge lengths.

In the rest of the proof we show that, for $\varepsilon$ small enough, $P_\varepsilon$ has no simple closed geodesic of length at most $a$. Suppose on the contrary that there exists a simple closed geodesic $G_\varepsilon \subset P_\varepsilon$ such that $\ell(G_\varepsilon) \leq a$. Since the points $x_G$ are (corresponding to) vertices of $P_\varepsilon$, $G_\varepsilon$ is not (corresponding to) a simple closed geodesic of $P$. Hence $G_\varepsilon$ should pass across at least one triangle $T_G$.

Denote by $G^-_\varepsilon$ the part of $G_\varepsilon$ outside the interior of all triangles $T_G$. By compactness, $G^-_\varepsilon$ admits (at least) a limit curve $G_0 \subset P$, when $\varepsilon$ tends to 0. Since $G^-_\varepsilon$ can be seen as a curve on $P$, Lemma 13 implies that $G_0$ is an almost-geodesic through $x_G$, hence $G_0 = G$. It follows that, for small $\varepsilon$, $G_\varepsilon$ is included in a neighbourhood $V_{G_\varepsilon}$ of $G$ in $P_\varepsilon$. Moreover, for distinct simple closed geodesics $F$ and $G$, $V_{G_\varepsilon} \cap x_Fy_Fy'_F = \emptyset$.

Let $x'_G$ be the point on $G$ which, together with $x_G$, divides $G$ into two equally-long arcs. Denote by $N$ (resp. $N'$) a geodesic arc normal to $G$ through
Notice that $V_{G,\varepsilon}$ may be chosen to be symmetrical with respect to $N$ (or, equivalently, with respect to $N'$); denote by $s$ this symmetry; we have $G = s(G)$, $N = s(N)$, $N' = s(N')$.

Assume first that $G_{\varepsilon} \neq s(G_{\varepsilon})$. Since $G_{\varepsilon} \cap (N \cup N') \subset G_{\varepsilon} \cap s(G_{\varepsilon})$, $G_{\varepsilon}$ and $s(G_{\varepsilon})$ intersect in at least two points, and so define at least two digons, symmetric to each other and of perimeter $2\Lambda_{\varepsilon}$. Now replace back $T_{\varepsilon}$ by a triangle of curvature $(1 + \varepsilon)^2$ and extend the remaining parts of $G_{\varepsilon}$ and $s(G_{\varepsilon})$ to complete the digons. This produces two spherical digons of perimeter $2\pi/(1 + \varepsilon)$, and thus contradicts the fact that $\lim \Lambda_{\varepsilon} \leq a$.

Therefore, we may assume that $G_{\varepsilon} = s(G_{\varepsilon})$. We claim that $G_{\varepsilon} \cap G \neq \emptyset$. Suppose on the contrary that $G_{\varepsilon}$ and $G$ are not intersecting. Then the boundary of the topological cylinder $C$ between them has only one angle (at $x_G$), of measure $\pi - \eta$, with $\eta > 0$. By the Gauss-Bonnet formula, the total curvature of $C$ should equal $-\eta$, which is obviously impossible, hence $G_{\varepsilon} \cap G \neq \emptyset$.

Notice that $G_{\varepsilon} \cap G \neq \emptyset$ contains precisely two points, because otherwise $G$ and $G_{\varepsilon}$ would determine at least three digons, two of which would have perimeter $2\pi/(1 + \varepsilon)$, and so the length $G$ would be at least $2\pi/(1 + \varepsilon)$, and its limit when $\varepsilon$ goes to 0 would be greater than $a$.

The next argument is illustrated by Figure 4. Put $G_{\varepsilon} \cap G = \{v_G, v'_G\}$ (with $v'_G = s(v_G)$). $G$ and $G_{\varepsilon}$ are delimitating two digons, one of which is spherical (because it doesn't intersect $T_G$) and has perimeter $2\pi/(1 + \varepsilon)$.

The geodesic $G_{\varepsilon}$ intersects the segments $x_Gy_G$ and $x'_Gy'_G$ at $z_G$ and $z'_G$ respectively. Let $\phi$ be the angle at $z_G$ of the geodesic triangle $x_Gz_Gz'_G$.

Now cut out $T_G$ and glue back a triangle of curvature $(1 + \varepsilon)^2$; extend $G_{\varepsilon}$ beyond $z_G$ and $z'_G$ until it self-intersects, say at $u_G$. Denote by $2\alpha$ the angle of the quadrilateral $x_Gz_Gu_Gz'_G$ at $u_G$. Put $\rho = 1 + \varepsilon$, $\lambda = d(x_G, z_G)/\varepsilon$.

The rest of the proof consists in computing (a Taylor expansion of) $d(v_G, x_G)$ as a function of $\varepsilon$, by means of spherical trigonometry.

Denote by $2\gamma$ the angle of $T_G$ at point $x_G$. Using twice the law of sines, one can compute

$$
\gamma = \arcsin \frac{\sin \left( \frac{1}{\rho} \arcsin \left( \sin \frac{\pi}{4} \sin \rho \varepsilon \right) \right)}{\sin \varepsilon}
= \frac{\pi}{4} - \frac{1}{6} \varepsilon^3 + O(\varepsilon^4).
$$

The law of cosines for angles in one half of the triangle $x_Gz_Gz'_G \subset T_G$
gives
\[ \cos \frac{\pi}{2} = -\cos \phi \cos \gamma + \sin \phi \sin \gamma \cos \lambda \varepsilon, \]
whence
\begin{align*}
\tan \phi &= \frac{1}{\tan \gamma \cos \lambda \varepsilon} \\
&= 1 + \frac{\lambda^2 \varepsilon^2}{2} + \frac{\varepsilon^3}{3} + O(\varepsilon^4). 
\end{align*}

By straightforward computations
\begin{align*}
\sin \phi &= \frac{\sqrt{2}}{2} \left( 1 + \frac{\lambda^2 \varepsilon^2}{4} + \frac{\varepsilon^3}{6} \right) + O(\varepsilon^4), \\
\cos \phi &= \frac{\sqrt{2}}{2} \left( 1 - \frac{\lambda^2 \varepsilon^2}{4} - \frac{\varepsilon^3}{6} \right) + O(\varepsilon^4). 
\end{align*}

The law of cosines for angles in the triangle \( u_G x_G z_G \) gives
\begin{align*}
\cos \alpha &= -\cos \phi \cos \frac{\pi}{4} + \sin \phi \sin \frac{\pi}{4} \cos \rho \lambda \varepsilon \\
&= \left( \frac{1}{6} - \frac{1}{2} \lambda^2 \right) \varepsilon^3 + O(\varepsilon^4). 
\end{align*}
The law of cosines for angles in the triangle \( v_G x_G z_G \) gives

\[
\cos \beta = - \cos (\pi - \phi) \cos \frac{\pi}{4} + \sin (\pi - \phi) \sin \frac{\pi}{4} \cos \rho \varepsilon \\
= 1 - \lambda^2 \left( \frac{\varepsilon^2}{4} + \frac{\varepsilon^4}{2} \right) + O(\varepsilon^4),
\]

whence

\[
\sin \beta = \frac{\sqrt{2}}{2} \lambda \varepsilon (1 + \varepsilon) + O(\varepsilon^3).
\]

At last, the law of sines in the same triangle \( v_G x_G z_G \) yields

\[
\sin \rho d(v_G, x_G) = \frac{\sin \rho \lambda \varepsilon}{\sin \beta} \sin \phi \\
= 1 + O(\varepsilon).
\]

On the other hand, \( d(v_G x_G) \) does not depend on \( \varepsilon \), and so is equal to \( \pi/2 \). Hence the length of \( G \) is \( 2\pi \) and we get a contradiction. This ends the proof.

**Theorem 20.** Most \( A \in A(1,2) \) have no simple closed geodesic.

**Proof.** A closed geodesic on \( A \in A(1,2) \) is seen as a map from \( \mathbb{R}/\mathbb{Z} \) to \( A \); its parameter is assumed proportional to the arc-length. For a given surface \( A \), define \( \mathcal{H}_A(\varepsilon, \eta, a) \) as the set of all simple closed geodesics \( G \) of \( A \) such that

(i) for any \( t \in \mathbb{R}/\mathbb{Z} \) and any \( s \in [0, \varepsilon] \), \( d(\gamma(t), \gamma(t+s)) = s\ell(G) \),

(ii) for any points \( x, y \in G \) whose distance along \( G \) is at least \( \varepsilon \), we have \( d_A(x,y) \geq \eta \), and

(iii) \( \ell(G) \leq a \).

Denote by \( \mathcal{M}_{pqr} \) the set of all \( A \in A(1,2) \) such that \( \mathcal{H}_A \left( \frac{1}{p}, \frac{1}{q}, 2\pi - \frac{1}{r} \right) \) is nonempty.

We have to prove that the set

\[
\mathcal{M} \overset{\text{def}}{=} \{ A \in A(1,2) \mid A \text{ has a simple closed geodesic} \}
\]

is meager. By Lemma \([16]\), we have

\[
\mathcal{M} = \{ S_0 \} \cup \bigcup_{p,q,r \in \mathbb{N}^*} \mathcal{M}_{pqr}.
\]

Each set \( \mathcal{M}_{pqr} \) has empty interior by Lemma \([19]\); we show next that it is closed. Let \( A_n \in \mathcal{M}_{pqr} \) be a sequence converging to \( A \in A(1,2) \). By
Lemma 1, we can assume that $A_n$ and $A$ are embedded in the same compact metric space $Z$. Let $G_n$ be a geodesic in $\mathcal{H}_{A_n}\left(\frac{1}{p}, \frac{1}{q}, 2\pi - \frac{1}{r}\right)$. Notice that $\ell(G_n) < 2\pi$, hence by Ascoli's theorem we can extract from $G_n$ a converging subsequence; denote by $G : \mathbb{R}/\mathbb{Z} \to A$ its limit. Since $\ell$ is lower semi-continuous, $G$ belongs to $\mathcal{H}_A\left(\frac{1}{p}, \frac{1}{q}, 2\pi - \frac{1}{r}\right)$. This ends the proof. 

6 Conclusions

Gathering together Theorems 8, 17 and 20, we get

**Summarizing Theorem.** i) For $\kappa = 1$ we have:

i.1) most surfaces in $A(1, 1)$ have infinitely many simple closed geodesics;

i.2) most surfaces in $A(1, 2)$ have no simple closed geodesic.

ii) For $\kappa = 0$ we have:

ii.1) most surfaces in $A(0, 2)$ have no closed geodesic;

ii.2) most surfaces in $A(0, 1)$ have infinitely many simple closed geodesics;

ii.3) all surfaces in $A(0, 0)$ are unions of simple closed geodesics.

iii) Most surfaces in $A(-1)$ have infinitely many non-intersecting simple closed geodesics.

**Remark 2.** P. Gruber proved that most convex surfaces have no closed geodesics [12], and his proof yields the above result on most surfaces in $A(0, 2)$. Whether most surfaces in $A(1, 2)$ do not have non-simple closed geodesics remains an open question.

It is also an open question whether a typical surface in $A(-1)$ or in $A(\kappa, 1)$ also has infinitely many non-simple closed geodesics of a given “flat knot type” (with the terminology in [3]).

Our final remark concerns the length spectrum of Alexandrov surfaces.

**Remark 3.** One can also consider lengths in the statements of Theorems 8 and 17. Put $\mathcal{B}(-1) = A(-1)$, $\mathcal{B}(0) = A(0, 1)$ and $\mathcal{B}(1) = A(1, 1)$. With the very same proof ideas, but varying the parameters $\lambda$ and $\varepsilon$, one can prove the following statement.

Let $\kappa \in \{-1, 0, 1\}$; for any $\delta > 0$ there exists a residual set $\mathcal{C}$ in $\mathcal{B}(\kappa)$ such that, for any $A \in \mathcal{C}$, there exist $L > 0$ and infinitely many simple closed geodesics on $A$ whose lengths are pairwise different and belong to $[L, L + \delta]$. 

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Jöel Rouyer  
Institute of Mathematics “Simion Stoilow” of the Romanian Academy,  
P.O. Box 1-764, Bucharest 70700, ROMANIA  
Joel.Rouyer@ymail.com, Joel.Rouyer@imar.ro

Costin Vilcu  
Institute of Mathematics “Simion Stoilow” of the Romanian Academy,  
P.O. Box 1-764, Bucharest 70700, ROMANIA  
Costin.Vilcu@imar.ro