Confined phases of one-dimensional spinless fermions coupled to $Z_2$ gauge theory

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We investigate a quantum many-body lattice system of one-dimensional spinless fermions interacting with a dynamical $Z_2$ gauge field. The gauge field mediates long-range attraction between fermions resulting in their confinement into bosonic dimers. At strong coupling we develop an exactly solvable effective theory of such dimers with emergent constraints. Even at generic coupling and fermion density, the model can be rewritten as a local spin chain. Using the Density Matrix Renormalization Group the system is shown to form a Luttinger liquid, indicating the emergence of fermionic fractionalized excitations despite the confinement of lattice fermions. In a finite chain we observe the doubling of the period of Friedel oscillations which paves the way towards experimental detection of confinement in this system. We discuss the possibility of a Mott phase at the commensurate filling 2/3.

Introduction.— Lattice gauge theories, originally introduced by Wilson in high-energy physics [1], turned out to emerge in many condensed matter problems as low-energy effective theories of exotic quantum phases of matter [2–4]. They provide a natural description of fractionalization of elementary excitations and deconfined quantum critical points. The simplest lattice gauge theory that has $Z_2$ Ising degrees of freedom was invented by Wegner [5]. Being the first example of a system that exhibits topological order, it provided a paradigm shift in our understanding of phase transitions [6]. Its exactly solvable limit, the toric code model [7], gave the first impetus towards topological quantum computation.

Coupling to gapless dynamical matter can qualitatively change the phase diagram of a lattice gauge theory. The coupling of bosonic (Ising) matter to the Wegner’s $Z_2$ gauge theory was undertaken by Fradkin and Shenker already in 1979 [8], who demonstrated that in this case the system exhibits two phases akin to the pure $Z_2$ gauge theory. Models where a $Z_2$ gauge field couples to fermionic matter were studied much later. First, motivated by high-$T_c$ cuprate phenomenology, Senhul and Fisher introduced a two-dimensional $Z_2$ gauge theory minimally coupled to fractionalized fermions and bosons at finite density [9]. The fractionalized non-Fermi liquid phase called orthogonal fermions realizes another example of a $Z_2$ gauge theory coupled to fermions and Ising spins [10]. More recently, Gazit and collaborators investigated in detail the quantum phase diagram of two-dimensional spinful fermions coupled to the standard $Z_2$ gauge theory using sign-problem-free quantum Monte Carlo simulations [11]. In addition, using the Kramers-Wannier duality, Prosko and collaborators found several exactly soluble models of a non-standard $Z_2$ gauge theory coupled to fermionic matter [12]. Fermions at finite density coupled to Ising gauge theory without the Gauss law were studied in [13, 14]. For related recent work on unconstrained $Z_2$ gauge theories coupled to bosonic matter, see [15].

Studying models with dynamical matter coupled to gauge fields numerically is a challenging task. Recently this has motivated analogue quantum simulations of such problems, using ultracold atoms platforms in particular [16]. Pioneering experimental work in ultracold ions [17] has led to the first simulation of string breaking in the Schwinger model (QED$_2$) [18]. More recently, a Floquet implementation for $Z_2$ lattice gauge theories coupled to dynamical matter has been proposed [19] and a proof-of-principle experiment on a two-component mixture of ultracold bosons has been performed [20].

In this Letter we present a study of a one-dimensional quantum $Z_2$ lattice gauge theory coupled to spinless fermions at finite density. We demonstrate that a local change of basis recasts the model as a spin-1/2 chain without gauge redundancy. In addition to analytic arguments, we rely on the numerical solution using density matrix renormalization group (DMRG) approach in both finite and infinite geometries. At finite coupling the Ising gauge field mediates a linear attractive potential between fermions resulting in the confinement of pairs of fermions into bosonic dimer molecules [21]. As a result, we find that the gauge-invariant fermionic two-point correlation function decays exponentially. On the other hand, our findings suggest that at a generic fermion density a Luttinger liquid of dimers is formed. The dimer picture becomes especially tractable in the limit of strong coupling, where the dimers are tightly bound hard-core bosonic objects. In this regime, using second-order degenerate perturbation theory we are able to map the problem to a constrained model of bosons with short-range repulsive interactions which was solved analytically via the Bethe ansatz [22]. Recent experiments with one-dimensional chains of Rydberg atoms [23] reignited theoretical work on lattice models with extended hard-core constraints [24]. Our work demonstrates that extended hard-core constraints emerge naturally at low energies also in discrete lattice gauge theories with confined fermions.

The model.— We consider a one-dimensional chain, where spinless fermion operators $c_i$ live on sites and $Z_2$ Ising gauge fields operate on links, see Fig. 1. The quantum Hamiltonian
FIG. 1. Summary of the model: fermions (red) that occupy sites interact with the $Z_2$ gauge field defined on links. Any physical configuration must satisfy the Gauss law.

of the system is

$$H = -t \sum_i \left( c_i^\dagger \sigma^x_{i,i+1} c_{i+1} + \text{h.c.} \right) - h \sum_i \sigma^\tau_{i,i+1}. \quad (1)$$

Fermions are coupled minimally to the gauge field via the Ising version of the Peierls substitution. The second term in Eq. (1), which is the discrete version of the electric term in electrodynamics, induces transitions between the gauge Ising spins.

The Hamiltonian commutes with local $Z_2$ generators $G_i = \sigma^x_{i-1,i}(1-n_i^f)$, where $n_i^f = c_i^\dagger c_i$ is the number operator. This is the Ising version of the Gauss law and the physical Hilbert space is chosen to be invariant under all generators $G_i$. As a result, the Hamiltonian (1) must be minimized under the set of constraints $G_i = 1$. Importantly, regardless of the concrete form of the Hamiltonian, in a closed chain (periodic boundary conditions) the gauge constraint implies that the fermion parity $P = (-1)^{\sum_i n_i^f}$ of any state in the physical Hilbert space is even which removes all fermionic excitations from the energy spectrum. In a finite chain the allowed fermion parity depends on whether the chain ends with a site or a link [25].

For an arbitrary chain, the energy spectrum is symmetric with respect to the transformation $t \rightarrow -t$ since one can perform a unitary rotation acting on the links as $\sigma^x \rightarrow \sigma^x$, $\sigma^y \rightarrow -\sigma^y$, $\sigma^z \rightarrow -\sigma_z$, which flips the sign of $t$ in the Hamiltonian but preserves the Gauss law. A similar argument implies that in a periodic chain the spectrum is invariant also under $h \rightarrow -h$ [26]. In the following we will therefore only consider $t, h \geq 0$.

In addition to local $Z_2$ gauge invariance the model exhibits a global $U(1)$ symmetry which acts only on fermions $c_i \rightarrow e^{i\phi} c_i$. This allows one to work in the grand canonical ensemble with the Hamiltonian $H - \mu \sum_i n_i^f$ and change the fermion density by tuning the chemical potential $\mu$. While the Hamiltonian is invariant under the particle-hole transformation $c_i \rightarrow (-)^i c_i^\dagger$, $\mu \rightarrow -\mu$, the Gauss law transforms non-trivially as $G_i \rightarrow -G_i$. As a result, the physical energy spectrum of (1) is not symmetric under $\mu \rightarrow -\mu$.

At $h = 0$ the gauge field decouples and the model reduces to free fermions. Formally this can be demonstrated by introducing $Z_2$ gauge-invariant but non-local fermion operators $f_i = c_i \prod_{j \geq i} \sigma^+_{j,j+1}$. In terms of $f_i$ the Hamiltonian (1) at $h = 0$ reduces to the canonical model of non-interacting fermions, whose ground state at finite filling is a free Fermi gas. Note however that even in this case, due to the $Z_2$ Gauss law, a single $f_i$ does not create a physical excitation in a closed chain. We say that the model has emergent fermions $f_i$, which can only be created in pairs.

Confinement.— The gauge constraint ensures that bare $c_i$ fermions must be connected by electric strings with $\sigma^\tau = -1$. Since for $h > 0$ the electric term introduces an energy cost for such lines, one expects that the bare fermions are confined into gauge-invariant dimers. Later, we confirm this by showing that for any $h \neq 0$, the two-point correlator $\langle f_i f_{i+d} \rangle$ decays exponentially fast with $d$. In the limit $h \rightarrow \infty$, we will derive an effective integrable Hamiltonian for the bound states, which will also turn out to form a Luttinger liquid.

Remarkably, the low-energy theory of Eq. (1) is a Luttinger liquid for a generic value of $h$ with a smoothly varying Luttinger liquid parameter. At first sight, this might seem in contradiction with the observation that the bare $f_i$ fermions are confined and have exponentially decaying correlators when $h \neq 0$. However, in the universal low-energy regime, there will be new emergent deconfined fermions.

To see this, it is useful to consider the bosonization of the electric term of the Hamiltonian (1). Due to the Gauss law, at any site $i$ we have $\sigma^\tau_{i-1,i} = (-1)^{n_i^f} \sigma^\tau_{i,i+1}$. Applying this relation successively at each site on the right of site $i$, and assuming that $\sigma^x$ at infinity (or at the boundary) is $+1$ we find that $\sigma_x$ equals the Jordan-Wigner string operator

$$\sigma^x_{i-1,i} = (-1)^{\sum_{j \geq i} n_j^f} \approx e^{i\pi \sum_{j \geq i} n_j^f} \approx e^{i\pi \int_{x \leq i} \rho^f dy}, \quad (2)$$

where in the last step we took the continuum limit. From this we learn that $\sigma^x$ can be removed from the Hamiltonian and replaced with a non-local operator that only depends on the density of fermions. Using bosonization $\rho^f(x) \rightarrow \rho^0_F(x) - \partial_y \phi(x)/\pi$ [27], the electric term of the Hamiltonian (1) can now be written as

$$-\hbar \int dx e^{i\pi \int_{y \leq x} \rho^f_0 dy} \phi(x) \rightarrow -\hbar \int dx \cos (k_F x - \phi(x)), \quad (3)$$

where the Fermi momentum $k_F = \pi \rho^0_F$. One might seemingly argue that this cosine energetically punishes $\pi$ kinks—which are exactly the fermionic excitations—and favors $2\pi$ kinks—giving rise to dimers as the effective degree of freedom. However, the spatial dependence of Eq. (3) implies that the perturbation is not RG-relevant: near the free-fermion point, the $U(1)$-symmetric relevant perturbations are $\cos(\phi)$ and $\cos(2\phi)$, with momentum $k = k_F$ and $k = 2k_F$, respectively. Due to the lattice translation symmetry of the Hamiltonian, neither of these terms can be generated at any finite filling fraction (i.e., $0 < k_F < \pi$) [28]. As a result, the system remains a Luttinger liquid as we tune $h$, with a flowing Luttinger liquid parameter due to the symmetry-allowed marginal perturbation $(\partial \phi)^2$.

Any Luttinger liquid has (emergent) deconfined fermionic excitations $e^{i(\phi \pm \theta)}$. The special property of $h = 0$ is that these emergent excitations of the Luttinger liquid coincide with the
lattice fermions \( f_i \). When \( h \neq 0 \), the lattice-continuum correspondence between \( f_i \) and the low-energy field operators \( \phi, \theta \) is modified [25]. While the bare fermions \( f_i \) are confined into dimers, we will argue below that the interactions between the latter lead to the formation of a collective Luttinger liquid phase whose elementary excitations are fractionalized. In fact, as we will now show, the model is equivalent to a spin-1/2 chain for any value of \( h \), illustrating that even for \( h = 0 \), the fermionic excitations can be thought of as collective deconfined fermionic spinons—in this limiting case created by \( f_i \).

To rewrite the model (1) as a local spin-1/2 chain, we introduce Majorana operators \( \gamma_i = c_i^\dagger + c_i \) and \( \tilde{\gamma}_i = i(c_i^\dagger - c_i) \). After some algebra [25], the Hamiltonian (1) is transformed to

\[
H = -\frac{t}{2} \sum_i (1 - X_{i-1,i}X_{i,i+2}) Z_{i,i+1} - \frac{h}{2} \sum_i X_{i,i+1},
\]

where \( X_{i,i+1} = \sigma_{i,i+1}^x + \gamma_i \gamma_{i+1} \) and \( Z_{i,i+1} = -i \tilde{\gamma}_i \sigma_{i,i+1}^y + \gamma_{i+1} \gamma_i \) are gauge-invariant local operators [29]. After including \( Y_{i,i+1} = -i \tilde{\gamma}_i \sigma_{i,i+1}^y + \gamma_{i+1} \gamma_i \), these obey the Pauli algebra. Since \( n_i^b = (1 - X_{i-1,i}X_{i,i+1})/2 \), confinement of fermions in our problem manifests itself in this formulation as confinement of domain walls, whose hopping is governed by the first term in Eq. (4). The gauge-invariant Majorana fermions correspond to such operators as \( X_n Z_{n+1} Z_{n+2} \cdots \). We emphasize that Eq. (4) is obtained by a local change of variables without any gauge-fixing; this is consistent with the field-theoretic perspective that a \( Z_2 \) gauged Dirac fermion is a compact boson [30].

Using the TeNPy Library [31], DMRG simulations presented in this Letter were performed for either the original constrained model (1) or the unconstrained model (4).

**FIG. 2.** Effective dimer model from the second order perturbation theory: (a) effective hopping of a dimer, (b) dimer length fluctuation that decreases the energy of a dimer.

**Effective theory of compact dimers at \( h \gg t \).**— When the electric coupling \( h \) is much larger than the hopping \( t \), extended dimers have large energy, and the fundamental low-energy degrees of freedom are tightly bound pairs of neighboring fermions. Such bosonic molecules live on the dual lattice formed by the links \( \ast (i, i + 1) \) of the original lattice, and are created by the gauge-invariant operator \( b_{i,\ast} = c_i^\dagger \sigma_i^y c_{i+1}^\dagger \). As a result of Pauli’s principle, the dimers are hardcore bosons which are also not allowed to occupy neighboring sites: the effective theory in this regime therefore has a constrained Hilbert space. Here we write the corresponding low-energy model that follows from the second order degenerate perturbation theory in the hopping parameter \( t \). First, a hopping of a dimer between neighboring sites of strength \( t_B = t^2/\hbar \) is generated by the process where one of the fermions forming the pair hops in one direction, and the other one follows (Fig. 2 (a)). Second, a pair can reduce its energy if one of the fermions hops once, and then hops back to the initial state (Fig. 2 (b)). Since such a process is inhibited when two pairs sit next to each other, a next-nearest-neighbor repulsion of dimers is induced. The strength of this repulsion is equal to \( U_B = 2t_B \). The effective Hamiltonian is therefore

\[
H_B = \sum_j \mathcal{P}_1 \left[ -t_B \left( b_{j,\ast} b_{j+1,\ast} + h.c. \right) + U_B n_j^B n_{j+2}^B \right] \mathcal{P}_1,
\]

where \( n_j^B = b_{j,\ast} b_{j,\ast} \) and \( \mathcal{P}_1 \) is a projector that enforces the hard-core constraints \( b_{j,\ast}^2 = b_{j,\ast} b_{j,\ast+1} = 0 \).

The model (5) can be mapped onto a constrained XXZ spin chain [25] which was solved analytically using the Bethe ansatz [22]. From that solution one finds that at a generic filling the boson-boson correlator \( \langle b_{j,\ast} b_{j,\ast+1} \rangle \) decays algebraically as \( |i^* - j^*|^{-\alpha} \) with a density-dependent exponent

\[
\alpha = \frac{1}{2(1 - \rho_B)^2 \eta_{\rho B}}
\]

which fixes the Luttinger parameter \( K = 1/(2\alpha) \) for the effective model (5). Here \( \rho_B = \rho t/2 \leq 1/2 \) is the boson density and the density-dependent parameter \( \eta_{\rho B} \) is determined by solving a certain system of integral equations [22, 32], see [25] for a summary. At low densities \( \eta_{\rho B} \to 1 \) leading to \( \alpha = 1/2 \) as expected for non-interacting hardcore-bosons \((K = 1)\). As the density increases towards \( \rho_B = 1/3 \), the particles start to feel more repulsive interactions, and the exponent \( \alpha \) increases monotonically. Due to the constraint the density \( \rho_B = 1/3 \) is special and can be considered as “half filling” since in this case there is one boson on every other bond allowed by the constraint. At this filling, for weak repulsion \( U_B/t_B < 2 \) the model (5) is a gapless Luttinger liquid, but it turns into a \( Z_3 \) Mott insulator as the repulsion parameter is increased to \( U_B/t_B > 2 \) [25]. Hence, at leading order in the \( t/\hbar \) expansion, this model lies exactly at the interface between these two phases. Note that at \( \rho_B = 1/2 \) the model (5) is always a \( Z_2 \) insulator which is enforced by the constraints.

As explained in detail in [25], the effective model for the dimers (5) can be related to the \( SU(2) \) invariant spin-1/2 Heisenberg chain in squeezed space whose local excitations are known to fractionalize into deconfined fermionic spinons [4, 33].

**Friedel oscillations.**— Another clear indication that the fundamental emergent degrees of freedom of our system are bosonic dimers follows from the Friedel oscillations observed in a finite chain with open boundary conditions. For free fermions \((h = 0)\) we find oscillations of the density with the inverse period equal to \( 2k_F \) [34]. In the limit \( h \gg t \) one expects a doubling of the period of Friedel oscillations because the density of hardcore dimers is one half of the density of fermions. Our DMRG results suggest that the doubling occurs...
for any $h \neq 0$ (see Fig. 3). Cold atom experiments can directly measure the periodicity of Friedel oscillations and thus detect signatures of confined fermions in this model.

**Correlation functions.**— Due to Elitzur’s theorem, in any gauge theory only gauge-invariant observables can have non-zero expectation values. In our model we construct such quantities by working with the gauge-invariant fermions $f_i$, or equivalently by putting strings of $\sigma_z$ between $c_i$ operators.

From our DMRG simulation we first extract the equal-time fermionic correlator $\langle f_i^\dagger f_j \rangle$. As illustrated in Fig. 4 the power-law behavior at $h = 0$ changes to an exponential decay for $h > 0$. We attribute this behavior to the long range confining interaction between $Z_2$ charges.

Next, we compute the dimer-dimer equal-time correlation function $\langle b_i^\dagger b_j \rangle$. Our DMRG results indicate that at a generic gauge coupling $h$ and filling $\rho_F$ the correlation function falls off algebraically with exponents shown in Fig. 5. In the regime $h \gg t$ our findings are in a good agreement with the predictions from the exactly-solvable bosonic model (5).

![FIG. 3. Friedel oscillations in a chain of length $L = 160$ at filling $\rho_F = 1/8$. At $h = 0$ the oscillations have the wave-vector $2k_F$, but for $h > 0$ the wave-vector is reduced to $k_F$ resulting in the doubling of the period.](image)

![FIG. 4. The absolute value of the fermion correlator $\langle f_i^\dagger f_j \rangle$ for different values of the gauge coupling $h$. At $h = 0$ the correlators decay inversely proportional to the distance. For $h > 0$ the decay is exponential, indicating confinement of fermions $f_i$.](image)

![FIG. 5. Exponent of the power-law decay of the pair-pair correlator $\langle b_i^\dagger b_j \rangle$. The numerical results at $h = 10$ are in a good agreement with the predictions from the exactly-solvable bosonic model (5).](image)

Outlook—Does the Mott state appear at the filling $2/3$? We found above that second order perturbation theory fixes the ratio $U_B/t_B = 2$ in the Hamiltonian (5), which at the filling $\rho_B = 1/3$ places the effective model exactly at the transition point between the Luttinger liquid and the $Z_2$ Mott state. Higher-order perturbation theory can modify this ratio and in addition will generate beyond nearest-neighbor hoppings and interactions. Such calculation will shed some light on the fate of the filling $\rho_F = 2/3$ in the regime $h \gg t$. In the regime $h \sim t$ it would be interesting to investigate the nature of the
ground state at this commensurate filling numerically.

The model studied in this Letter can be realized experimentally using ultracold atoms in optical lattices. One possibility to implement the coupling of fermions to a $Z_2$ lattice gauge field is to use the Floquet scheme proposed and demonstrated in Refs. [19, 20]. An alternative implementation is in the context of doped quantum magnets: Consider a 1D Fermi-Hubbard model at strong coupling $U \gg t$ and in the presence of a staggered Zeeman field $h \sum_j (\pm 1)^j \hat{S}_j^z$ with $h \gg J$ exceeding the super-exchange energy $J = 4t^2/U$. This model can be implemented by confining two pseudospin states with different magnetic moments (e.g., 40K [35]) to a zig-zag optical lattice and adding a perpendicular magnetic gradient. At half-filling the ground state is a classical Néel state. By adiabatically decreasing the density of the system, mobile holes can be introduced and the ground state can be adiabatically prepared [36]. Because the strong Zeeman field $h \gg J$ suppresses fluctuations of the spins in the $x- y$ plane, pairs of holes are connected by a string of misaligned spins which can be mapped onto the $Z_2$ effective field lines considered in our model. In the limit $U \to \infty$ this model is equivalent to Eq. (1), with holes corresponding to the fermionic $Z_2$ charges.

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SUPPLEMENTAL MATERIAL: CONFINED PHASES OF ONE-DIMENSIONAL SPINLESS FERMIONS COUPLED TO $Z_2$ GAUGE THEORY

Fermion parity in a finite chain

In a closed chain the Gauss law enforces that the physical Hilbert space contains only states with even number of fermions, i.e., the fermion parity $P = (-1)^{\sum_i n_i} = 1$. In a finite chain the allowed fermion parity depends on whether the chain ends with a site or a link. In the case of a link-boundary, an odd number of fermions is not prohibited by the Gauss law (see Fig. S1 a) and the fermion parity can be either odd or even. On the other hand, for the site-boundary the Gauss law constraint must be modified at the edge. If one imposes at the left boundary $G_1 = (-1)^{n_i} \sigma_{1,2}^\epsilon = 1$ and right boundary $G_L = (-1)^{n_L} \sigma_{L-1,L}^\epsilon = 1$, the fermions parity must be even (see Fig. S1 b).

FIG. S1. Chain with link-boundary (a) and site-boundary (b) on the right. In the first case, an unpaired fermion can be accommodated, since it is possible to connect it to the boundary with an electric string that terminates on the last link.

Lattice-continuum correspondence

For the free-fermion case, there is a well-known correspondence between the lattice fermion operator and the continuum fields. This correspondence continues to hold in the well-explored scenario where local quartic interactions are introduced [33]. However, if the fermion is coupled to a gauge field, there is a qualitatively new perturbation possible, considered in the model (1) of the main text. We claim that if $h \neq 0$, the lattice-continuum correspondence changes, such that the lattice fermion $c_i$ (or better, its gauge-invariant non-local version $f_i$) no longer corresponds to $e^{\imath \phi(x)}$. Indeed, this is already evidenced by the fact that the former lattice operator is numerically observed to have exponentially decaying correlations, see Fig. 4, whereas $\langle e^{-\imath \phi(x) + \theta(x)} e^{\imath \phi(0) + \theta(0)} \rangle$ is algebraically decaying within the Luttinger liquid framework.

To better understand the failure of the usual correspondence, it is instructive to consider the equivalent bosonic formulation of the model as shown in Eq. (4). As mentioned in the main text, the lattice fermionic operator is given by $X_n Z_{n+1} Z_{n+2} \cdots$, corresponding to the aforementioned continuum field if $h = 0$. Equivalently, the domain wall operator $Z_n Z_{n+1} Z_{n+2} \cdots$ corresponds to $e^{\imath \phi(x)}$ (again, if $h = 0$); we now clarify why this correspondence breaks down for $h \neq 0$. The reason that they should correspond for $h = 0$ is dictated by very general symmetry properties: the lattice $Z_2$ symmetry is generated by the global operator $\prod_n Z_n$, whereas the continuum $Z_2$ symmetry $\theta \rightarrow -\theta$ is generated by $e^{\imath \int \partial_x \phi(x) dx}$ (this is a simple consequence of the fact that $\theta(x)$ and $\partial_x \phi(x)$ are conjugate fields). Such symmetry matching is a general and powerful method for creating lattice-continuum correspondences. This also explains why things change for $h \neq 0$: the lattice $Z_2$ symmetry is explicitly broken by the last term in Eq. (4); instead, the system develops an emergent $Z_2$ symmetry at low energies (due to the perturbation not being relevant, as discussed in the main text). There is thus no more reason that the lattice domain wall operator should correspond to the continuum field $e^{\imath \phi(x)}$.

From $Z_2$ gauge theory with fermions to a local spin $1/2$ model

The Hamiltonian (1) is written in terms of fermionic degrees of freedom that are not $Z_2$ gauge invariant. Here we demonstrate how to cast the Hamiltonian into a local form in terms of gauge-invariant observables.

First, it is convenient to introduce Majorana variables $\gamma_i = c^\dagger_i + c_i$ and $\tilde{\gamma}_i = i(c^\dagger_i - c_i)$, or equivalently, $c_i = (\gamma_i + i\tilde{\gamma}_i)/2$ and $c^\dagger_i = (\gamma_i - i\tilde{\gamma}_i)/2$. In terms of these $\Pi_i = (-1)^{n_i} = i\tilde{\gamma}_i\gamma_i$. Now, using the gauge condition $G_i = +1$, we can rewrite the kinetic term as
Using the Gauss law, the chemical potential term $-\mu$ becomes bilocal in this formulation $-\mu \sum_i (1 - X_{i-1,i}X_{i,i+1})/2$.

### From constrained XXZ chain to constrained bosons

We start from the Hamiltonian of the constrained XXZ chain analyzed in [22]

$$H_{XXZ} = -\frac{1}{2} \sum_i P_i \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z - h \sigma_i^z \right) P_i,$$

where $P_i$ projects out the states, where two spins up are at a distance that is smaller or equal to $l$ lattice spacings. We introduce hard-core boson operators

$$b_i^\dagger = \frac{1}{2} \left( \sigma_i^x + i \sigma_i^y \right), \quad b_i = \frac{1}{2} \left( \sigma_i^x - i \sigma_i^y \right)$$

which implies $n_i^B = b_i^\dagger b_i = (1 + \sigma_i^z)/2$. Up to a constant shift, in terms of these operators the Hamiltonian (S4) reads

$$H = -\sum_i P_i \left[ (b_i^\dagger b_{i+1} + h.c.) + 2\Delta n_i^B n_{i+1}^B + 2(h - \Delta) n_i^B \right] P_i,$$

where the projector $P_i$ forbids two bosons at distances smaller or equal to $l$. For $l = 1$ the resulting constrained bosonic model has a unit nearest-neighbor hopping $t_B = 1$, the next-nearest-neighbor interaction of strength $U_B/t_B = -2\Delta$ and the chemical potential $\mu_B/t_B = 2(h - \Delta)$.

### How to determine $\eta_{\rho_B}$ in Eq. (6)

As demonstrated in the previous subsection, the constrained bosonic model (5) introduced in the main text can be mapped onto the constrained XXZ chain (S4) which was solved using the Bethe ansatz [22]. Employing this mapping, for $U_B = 0$ (that corresponds to $\Delta = 0$ in the constrained XXZ chain) the boson-boson correlator decays as $\langle b_i^\dagger b_j \rangle \sim |i^x - j^x|^{-\alpha}$ with $\alpha = (1 - \rho_B)^{-2}/2$. At $U_B \neq 0$ ($\Delta \neq 0$) the exponent is

$$\alpha = \frac{1}{2} (1 - \rho_B)^{-2} \eta_{\rho_B}^{-2},$$

(S7)
where the density-dependent parameter $\eta_{\rho_B}$ can be obtained by solving a system of integral equations derived in [22, 32]. In particular, for $U_B > 2t_B$ ($\Delta < -1$) one can use the parametrization $\Delta = -\cosh(\lambda)$ and the equations to solve are

$$1 = \eta(U) + \frac{1}{2\pi} \int_{-U_0}^{U_0} \frac{\sinh(2\lambda)\eta(U')}{\cosh(2\lambda) - \cos(U - U')} dU',$$  \hspace{1cm} (S8)

$$Q(U) = \frac{1}{2\pi} \sinh \lambda - \frac{1}{2\pi} \int_{-U_0}^{U_0} \frac{\sinh(2\lambda)Q(U')}{\cosh(2\lambda) - \cos(U - U')} dU',$$  \hspace{1cm} (S9)

$$\int_{-U_0}^{U_0} Q(U) dU = \left\{ \begin{array}{ll} \frac{\rho_B}{1-\rho_B}, & 0 \leq \rho_B \leq \frac{1}{3}, \\ \frac{\rho_B}{1-\rho_B}, & \frac{1}{3} \leq \rho_B \leq \frac{2}{3}, \end{array} \right.$$  \hspace{1cm} (S10)

where $U_0$ in Eq. (S8) is determined by solving Eqs. (S9) and (S10). Finally, the parameter $\eta_{\rho_B}$ that appears in Eq. (S7) can now be obtained by evaluating the function $\eta(U)$ at $U = U_0$.

For $2t_B > U_B > -2t_B$ ($-1 < \Delta < 1$) one uses instead the parametrization $\Delta = -\cos \gamma$ and obtains similar equations, with the hyperbolic functions replaced by their trigonometric counterparts.

$$\rho_B = 1/3 \text{ state in the constrained bosonic model (5)}$$

FIG. S2. Exponent for the power-law decay of the two-point correlator in the constrained model (S4) in the vicinity of the commensurate density $\rho_B = 1/3$.

After determining $\eta_{\rho_B}$ from Eq. (S8), (S9),(S10), one observes that for $U_B > 2t_B$ the exponent $\alpha(\rho_B)$ has a cusp-like peak at $\rho_B = 1/3$, see Fig. S2. The value of $\alpha$ at this point is independent of $U_B/t_B$ and equals to $9/2$. However as $U_B \to 2t_B$ from above, the width of the peaks goes to zero. On the other hand, as $U_B \to 2t_B$ from below, $\alpha \to 9/4$. As a result, at the filling $\rho_B = 1/3$ the value of the exponent $\alpha$ is a discontinuous function of the ratio $U_B/t_B$. Physically, at the density $\rho_B = 1/3$ the constrained bosonic model is in the $Z_3$ Mott state for $U_B > 2t_B$, while it forms a Luttinger liquid for $U_B \leq 2t_B$. The narrowing of the peak corresponds therefore to the closing of the Mott gap as the critical value $\bar{U}_B = 2t_B$ is approached from above. In the Luttinger liquid language, $\alpha = 9/2$ corresponds to a value $K = 1/9$ for the Luttinger parameter. This is exactly the value of $K$, where the commensurate-incommensurate (Mott-$\delta$) transition into a Mott phase of commensurability 3 takes place [33].

Mapping to spin-1/2 Heisenberg chain in squeezed space and emergent deconfined excitations at $h \gg t$

When $h = 0$, the bare fermions $f_i$ describe deconfined excitations carrying one unit of the $U(1)$ charge. Here we discuss the other extreme, $h \gg t$, where the bare fermions are confined into tightly bound dimers carrying two units of the $U(1)$ charge. The interactions between these dimers, in turn, lead to the formation of a collective Luttinger liquid phase which has fractionalized collective excitations carrying one unit of the $U(1)$ charge. We will demonstrate this now by establishing explicitly a mapping of our model with $h \gg t$ to the spin-1/2 Heisenberg model.
When $\hbar/t \to \infty$ we can restrict ourselves to the set of basis states with $\mathbb{Z}_2$ electric field lines ($\sigma^x_{i,i+1} = -1$) of length one, with one fermion at each end. We can label these basis states by the positions $x_1 < x_2 < ... < x_N$ of the fermions, subject to the constraint that

$$x_{2n} = x_{2n-1} + 1 \quad \text{for } n = 1...N. \quad (S11)$$

The $\mathbb{Z}_2$ field configuration follows from the Gauss law, noting that $\sigma^x_{j-1,j} = 1$ for all $j \leq x_1$ left of the first fermion. An example is illustrated in Fig. S3 (a).

![Diagram](attachment:image.png)

**FIG. S3.** Mapping of compact dimers in $\mathbb{Z}_2$ lattice gauge theory to spins in squeezed space.

Now we will introduce a new set of labels for these basis states $|x_1, ..., x_N\rangle$, by working in a squeezed space (this mapping is motivated by related constructions in doped spin chains [37, 38]): To this end we remove every second fermion from the chain, see Fig. S3 (b):

$$|x_1, ..., x_N\rangle \equiv |\tilde{x}_1, ..., \tilde{x}_{N/2}\rangle, \quad 0 \leq \tilde{x}_j \leq \tilde{L} = L - N/2. \quad (S12)$$

Here $L$ denotes the length of the chain and, without loss of generality, we consider the case when the total fermion number $N$ is even. Using the constraint Eq. (S11), the configuration $x_1, ..., x_N$ can be reconstructed from the squeezed space configuration $\tilde{x}_1 < ... < \tilde{x}_{N/2}$:

$$x_{2n} = \tilde{x}_n + n, \quad x_{2n-1} = x_{2n} - 1, \quad (S13)$$

with $n = 1...N/2$. The reverse is also true, establishing the one-to-one correspondence between the two basis representations.

It is more convenient to work in second quantization, so we introduce hard-core bosons $\tilde{b}^\dagger_j$ in squeezed space, with $j = 1...\tilde{L}$. The basis states thus read

$$|\tilde{x}_1, ..., \tilde{x}_{N/2}\rangle = \prod_{n=1}^{N/2} \tilde{b}^\dagger_{\tilde{x}_n} |0\rangle. \quad (S14)$$

Performing perturbation theory in $t/h$ up to second order, we arrive at the following effective Hamiltonian in squeezed space:

$$\hat{H}_{\text{eff}} = -t_B \sum_{j=1}^{\tilde{L}-1} \left( \tilde{b}^\dagger_{j+1} \tilde{b}_j + \text{h.c.} \right) + U_B \tilde{n}_j^b \tilde{n}_{j+1}^b. \quad (S15)$$

As in the main text, $t_B = t^2/2h$ and $U_B = 2t_B$, and we introduced $\tilde{n}_j^b = \tilde{b}_j^\dagger \tilde{b}_j$.

The hard-core bosons $\tilde{b}_j$ can be mapped to spin-1/2 operators $\tilde{S}_j$ in squeezed space in the usual way,

$$\tilde{S}_j^z = (1/2 - \tilde{n}_j), \quad (S16)$$

$$\tilde{S}_j^+ = \tilde{b}_j, \quad (S17)$$

$$\tilde{S}_j^- = \tilde{b}_j^\dagger. \quad (S18)$$
Therefore, up to a constant overall energy shift and after changing
\[ \tilde{S}_{2i}^{x,y} \rightarrow -\tilde{S}_{2i}^{x,y} \]
on even sites, we obtain
\[ \hat{H}_{\text{eff}} = U_B \sum_{j=1}^{L-1} \hat{S}_{j+1} \cdot \hat{S}_j. \]  
(S19)

Hence asymptotically when \( h \gg t \) the problem maps to the \( SU(2) \) invariant spin-1/2 Heisenberg model in squeezed space, with anti-ferromagnetic coupling \( U_B \).

It is well-known that the collective excitations of the spin-1/2 antiferromagnetic Heisenberg chain are fractionalized fermionic spinons [4, 33], carrying spin 1/2. Since the creation of a new dimer of two units of \( U(1) \) charge – by applying \( \tilde{b}^\dagger_j \) – corresponds to a spin-flip with \( \Delta \tilde{S}_z = \pm 1 \) in our mapping, it follows that the spinon excitations carrying \( \tilde{S}_z = \pm 1/2 \) correspond to unit-charged collective excitations in the original \( Z_2 \) lattice gauge theory.

### Average electric string length from two-particle Hamiltonian

In our problem two \( Z_2 \) charges are always connected by an electric string. Due to the competition between the energy cost \( \propto h \) of the string and the kinetic term, which tends to delocalize fermions, the string has an average length \( \langle l \rangle \) which is a decreasing function of \( h/t \). Since the center-of-mass motion of the two particles can be separated out, \( \langle l \rangle \) can be computed from the ground state of the following single-particle Hamiltonian
\[ H = -2t \sum_{l=1}^{\infty} (f_{l+1}^\dagger f_l + \text{h.c.}) + 2h \sum_{l=1}^{\infty} l f_{l}^\dagger f_{l}. \]  
(S20)

We find that \( \langle l \rangle \) is rather small (of order of units of the lattice spacing) even for moderate values of \( h/t \), see Fig. S4. The dimers can be thought as sufficiently separated molecules provided their size is much smaller than the average interparticle distance, i.e., \( \langle l \rangle \ll \rho_F^{-1} \).

![FIG. S4. The average string length \( \langle l \rangle \) as a function of \( h/t \) extracted from the Hamiltonian (S20).](image-url)