Some Properties Subclasses of Analytic Functions

BASEM AREF FRASIN
Faculty of Science, Department of Mathematics, Al al-Bayt University, P. O. Box: 130095 Mafraq, Jordan
e-mail: bafrasin@yahoo.com

ABSTRACT. The object of the present paper is to discuss some interesting properties of analytic functions $f(z)$ associated with the subclasses $D(\beta_1, \beta_2, \beta_3; \lambda)$, $\mathcal{S}(\theta, \alpha)$ and $\mathcal{U}(\theta, \alpha)$. Also, radius problems of $\frac{1}{2} f(\delta z)$ for $f(z)$ in the class $D(\beta_1, \beta_2, \beta_3; \lambda)$, $\mathcal{S}(\theta, \alpha)$ and $\mathcal{U}(\theta, \alpha)$ are considered.

1. Introduction and Definitions

Let $A$ denote the class of the normalized functions of the form

\begin{equation}
    f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\end{equation}

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For a function $f(z)$ in the class $A$, Sălăgean [1] defined the differential operator $D^k$, by

\begin{align*}
    D^0 f(z) &= f(z), \\
    D^1 f(z) &= Df(z) = zf'(z), \\
    D^2 f(z) &= D(D^1 f(z)) = z(zf'(z))',
\end{align*}

and

\begin{equation}
    D^k f(z) = D(D^{k-1} f(z)), \quad (k \in \mathbb{N}).
\end{equation}

Thus

\begin{equation}
    D^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n, \quad (k \in \mathbb{N} \cup \{0\}).
\end{equation}

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Let $\mathcal{G}(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

\[(3)\quad \text{Re} \left\{ \frac{D^3 f(z)}{z} \right\} > \alpha\]

for some $\alpha(0 \leq \alpha < 1)$ and for all $z \in \mathbb{U}$. Also, Let $\mathcal{Q}(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

\[
\text{Re} \left\{ \frac{D^2 f(z)}{D^1 f(z)} \right\} > \alpha
\]

for some $\alpha(0 \leq \alpha < 1)$ and for all $z \in \mathbb{U}$.

For analytic functions $f(z)$, Uyanik and Owa [2], obtained some interesting properties for analytic functions in the subclass $A(\beta_1, \beta_2, \beta_3; \lambda)$ defined by

\[
\beta_1 z \left( \frac{f(z)}{z} \right) ' + \beta_2 z^2 \left( \frac{f(z)}{z} \right) '' + \beta_3 z^3 \left( \frac{f(z)}{z} \right) ''' \leq \lambda
\]

for some complex numbers $\beta_1, \beta_2, \beta_3$, and for some real $\lambda > 0$; $z \in \mathbb{U}$, associated with close-to-convex functions and starlike functions of order $\alpha$.

In this paper, we define the following subclass of analytic functions.

**Definition 1.1.** A function $f(z)$ belonging to $\mathcal{A}$ is said to be in the class $\mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$ if it satisfies

\[(4)\quad \left| \beta_1 z \left( \frac{D^2 f(z)}{z} \right) ' + \beta_2 z^2 \left( \frac{D^2 f(z)}{z} \right) '' + \beta_3 z^3 \left( \frac{D^2 f(z)}{z} \right) ''' \right| \leq \lambda \quad (z \in \mathbb{U}),
\]

for some complex numbers $\beta_1, \beta_2, \beta_3$, and for some real $\lambda > 0$.

**Example 1.2.** Let us consider the function $f_\gamma(z)$, $\gamma \in \mathbb{R}$, given by

\[
f_\gamma(z) = z(1 + z)\gamma,
\]

then, we have

\[(5)\quad D^2 f_\gamma(z) = z + \sum_{n=2}^{\infty} n^2 \binom{\gamma}{n-1} z^n
\]

where

\[
\binom{\gamma}{n-1} = \frac{\gamma(\gamma-1)(\gamma-2)...(\gamma-n+2)}{(n-1)!}.
\]
From (5), it follows that
\[
\left| \beta_1 z \left( \frac{D^2 f_\gamma(z)}{z} \right)' + \beta_2 z^2 \left( \frac{D^2 f_\gamma(z)}{z} \right)'' + \beta_3 z^3 \left( \frac{D^2 f_\gamma(z)}{z} \right)''' \right| = \sum_{n=2}^{\infty} n^2 (n-1) \left( \frac{\gamma}{n-1} \right) (\beta_1 + (n-2)\beta_2 + (n-2)(n-3)\beta_3) z^{n-1}.
\]

Therefore, if $\gamma = 1$, then
\[
\left| \beta_1 z \left( \frac{D^2 f_1(z)}{z} \right)' + \beta_2 z^2 \left( \frac{D^2 f_1(z)}{z} \right)'' + \beta_3 z^3 \left( \frac{D^2 f_1(z)}{z} \right)''' \right| = |4\beta_1 z| \leq 4 |\beta_1|.
\]

This implies that $f_1(z) \in D(\beta_1, \beta_2, \beta_3; \lambda)$ for $\lambda \geq 4 |\beta_1|$. If $\gamma = 2$, then
\[
\left| \beta_1 z \left( \frac{D^2 f_2(z)}{z} \right)' + \beta_2 z^2 \left( \frac{D^2 f_2(z)}{z} \right)'' + \beta_3 z^3 \left( \frac{D^2 f_2(z)}{z} \right)''' \right| = \left| 8\beta_1 z + 18(\beta_1 + \beta_2) z^2 \right| \leq 26 |\beta_1| + 18 |\beta_2|.
\]

Therefore, $f_2(z) \in D(\beta_1, \beta_2, \beta_3; \lambda)$ for $\lambda \geq 26 |\beta_1| + 18 |\beta_2|$. Further, if $\gamma = 3$, then we have
\[
\left| \beta_1 z \left( \frac{D^2 f_3(z)}{z} \right)' + \beta_2 z^2 \left( \frac{D^2 f_3(z)}{z} \right)'' + \beta_3 z^3 \left( \frac{D^2 f_3(z)}{z} \right)''' \right| = \left| 12\beta_1 z + 54(\beta_1 + \beta_2) z^2 + 48(\beta_1 + 2\beta_2 + 2\beta_3) z^3 \right| \leq 114 |\beta_1| + 150 |\beta_2| + 96 |\beta_3|.
\]

Thus, $f_3(z) \in D(\beta_1, \beta_2, \beta_3; \lambda)$ for $\lambda \geq 114 |\beta_1| + 150 |\beta_2| + 96 |\beta_3|$.  

Now, let $A_\theta$ denote the subclass of $A$ consisting of functions $f(z)$ with
\[
a_n = |a_n| e^{i((n-1)\theta + \pi)} \quad (n = 2, 3, ...).
\]

Also, we introduce the subclasses $G(\theta, \alpha)$ and $Q(\theta, \alpha)$ of $A_\theta$ as follows:
\[
G(\theta, \alpha) = A_\theta \cap G(\alpha) \quad \text{and} \quad Q(\theta, \alpha) = A_\theta \cap Q(\alpha).
\]
2. Properties of the Class $\mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$

We first prove

**Theorem 2.1.** If $f(z) \in \mathcal{A}$ satisfies

\[
\sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|) |a_n| \leq \lambda
\]

for some complex numbers $\beta_1, \beta_2, \beta_3$ and for some real $\lambda > 0$, then $f(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$.

**Proof.** We observe that

\[
\left| \beta_1 z \left( \frac{D^2 f(z)}{z} \right)' + \beta_2 z^2 \left( \frac{D^2 f(z)}{z} \right)'' + \beta_3 z^3 \left( \frac{D^2 f(z)}{z} \right)''' \right|
\]

\[
= \left| \sum_{n=2}^{\infty} n^2(n-1)(\beta_1 + (n-2)\beta_2 + (n-2)(n-3)\beta_3) a_n z^{n-1} \right|
\]

\[
\leq \sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|) |a_n| |z|^{n-1}
\]

\[
< \sum_{n=2}^{\infty} n^2(n-1)(\beta_1 + (n-2)\beta_2 + (n-2)(n-3)\beta_3) |a_n|.
\]

Therefore, if $f(z)$ satisfies the inequality (6), then $f(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$.

Next, we prove

**Theorem 2.2.** If $f(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$ with $\arg \beta_1 = \arg \beta_2 = \arg \beta_3 = \phi$ and $a_n = |a_n| e^{i((n-1)\theta - \phi)}$ ($n = 2, 3, \ldots$), then we have

\[
\sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|) |a_n| \leq \lambda.
\]

**Proof.** For $f(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$, we see that

\[
\left| \beta_1 z \left( \frac{D^2 f(z)}{z} \right)' + \beta_2 z^2 \left( \frac{D^2 f(z)}{z} \right)'' + \beta_3 z^3 \left( \frac{D^2 f(z)}{z} \right)''' \right|
\]

\[
= \left| \sum_{n=2}^{\infty} n^2(n-1)(\beta_1 + (n-2)\beta_2 + (n-2)(n-3)\beta_3) a_n z^{n-1} \right|
\]

\[
= \left| \sum_{n=2}^{\infty} n^2(n-1)(\beta_1 + (n-2)\beta_2 + (n-2)(n-3)\beta_3) e^{i((n-1)\theta - \phi)} z^{n-1} \right|
\]

\[
\leq \sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|) |a_n| e^{i((n-1)\theta - \phi)} z^{n-1}
\]

\[
\leq \lambda.
\]
for all \( z \in \mathbb{U} \). Let us consider a point \( z \in \mathbb{U} \) such that \( z = |z| e^{-i\theta} \). Then we have
\[
\sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n - 2) |\beta_2| + (n - 2)(n - 3) |\beta_3|) |a_n| |z|^{n-1} \leq \lambda.
\]
Letting \( |z| \to 1^- \), we obtain
\[
\sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n - 2) |\beta_2| + (n - 2)(n - 3) |\beta_3|) |a_n| \leq \lambda.
\]

\[\square\]

**Corollary 2.3.** If \( f(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda) \) with \( \arg \beta_1 = \arg \beta_2 = \arg \beta_3 = \phi \) and \( a_n = |a_n| e^{i((n-1)\theta - \phi)} \ (n = 2, 3, \ldots) \), then we have
\[
|a_n| \leq \frac{\lambda}{n^2(n-1)(|\beta_1| + (n - 2) |\beta_2| + (n - 2)(n - 3) |\beta_3|)} \ (n = 2, 3, \ldots).
\]

**Example 2.4.** Let us consider the function \( f(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda) \) with \( \arg \beta_1 = \arg \beta_2 = \arg \beta_3 = \phi \) and
\[
a_n = \frac{\lambda e^{i((n-1)\theta - \phi)}}{n^2(n-1)^2(|\beta_1| + (n - 2) |\beta_2| + (n - 2)(n - 3) |\beta_3|)} \ (n = 2, 3, \ldots).
\]
Then we see that
\[
\sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n - 2) |\beta_2| + (n - 2)(n - 3) |\beta_3|) |a_n| = \lambda \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \lambda \sum_{n=2}^{\infty} \left( \frac{1}{n-1} - \frac{1}{n} \right) = \lambda.
\]

**Corollary 2.5.** If \( f(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda) \) with \( \arg \beta_1 = \arg \beta_2 = \arg \beta_3 = \phi \) and \( a_n = |a_n| e^{i((n-1)\theta - \phi)} \ (n = 2, 3, \ldots) \), then we have
\[
|z| - \sum_{n=2}^{j} |a_n| |z|^n - A_j |z|^{j+1} \leq |f(z)| \leq |z| + \sum_{n=2}^{j} |a_n| |z|^n + A_j |z|^{j+1}
\]
with
\[
A_j = \frac{\lambda - \sum_{n=2}^{j} n^2(n-1)(|\beta_1| + (n - 2) |\beta_2| + (n - 2)(n - 3) |\beta_3|) |a_n|}{j(j+1)^2(|\beta_1| + (j - 1) |\beta_2| + (j - 1)(j - 2) |\beta_3|)}
\]
and
\[ 1 - \sum_{n=2}^{j} |a_n| |z|^{n-1} - B_j |z|^j \leq |f'(z)| \leq 1 + \sum_{n=2}^{j} |a_n| |z|^{n-1} + B_j |z|^j \]

with
\[ B_j = \frac{\left( \lambda - \sum_{n=2}^{j} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|) |a_n| \right)}{j(j+1)(|\beta_1| + (j-1)|\beta_2| + (j-1)(j-2)|\beta_3|)} . \]

Proof. In view of Theorem 2.1, we know that
\[ \sum_{n=j+1}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|) |a_n| \]
\[ \leq \lambda - \sum_{n=2}^{j} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|) |a_n| . \]

Further, we note that
\[ j(j+1)^2(|\beta_1| + (j-1)|\beta_2| + (j-1)(j-2)|\beta_3|) \sum_{n=j+1}^{\infty} |a_n| \]
\[ \leq \sum_{n=j+1}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|) |a_n| , \]

which is equivalent to
\[ \sum_{n=j+1}^{\infty} |a_n| \leq \frac{\left( \lambda - \sum_{n=2}^{j} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|) |a_n| \right)}{j(j+1)^2(|\beta_1| + (j-1)|\beta_2| + (j-1)(j-2)|\beta_3|)} = A_j . \]

Thus, we have
\[ |f(z)| \leq |z| + \sum_{n=2}^{j} |a_n| |z|^n + \sum_{n=j+1}^{\infty} |a_n| |z|^n \leq |z| + \sum_{n=2}^{j} |a_n| |z|^n + A_j |z|^{j+1} \]

and
\[ |f(z)| \geq |z| - \sum_{n=2}^{j} |a_n| |z|^n - \sum_{n=j+1}^{\infty} |a_n| |z|^n \geq |z| - \sum_{n=2}^{j} |a_n| |z|^n - A_j |z|^{j+1} . \]
Next, we observe that
\[
j(j + 1)(|\beta_1| + (j - 1) |\beta_2| + (j - 2) |\beta_3|) \sum_{n=j+1}^{\infty} n |a_n|
\]
\[
\leq \sum_{n=j+1}^{\infty} n^2(n - 1)(|\beta_1| + (n - 2) |\beta_2| + (n - 2)(n - 3) |\beta_3|) |a_n|
\]
\[
\leq \lambda - \sum_{n=2}^{j} n^2(n - 1)(|\beta_1| + (n - 2) |\beta_2| + (n - 2)(n - 3) |\beta_3|) |a_n|
\]
which implies that
\[
\sum_{n=j+1}^{\infty} n |a_n| \leq \frac{\left(\lambda - \sum_{n=2}^{j} n^2(n - 1)(|\beta_1| + (n - 2) |\beta_2| + (n - 2)(n - 3) |\beta_3|) |a_n|\right)}{j(j + 1)(|\beta_1| + (j - 1) |\beta_2| + (j - 1)(j - 2) |\beta_3|)} = B_j.
\]
Therefore, we obtain that
\[
|f'(z)| \leq 1 + \sum_{n=2}^{j} n |a_n| |z|^{n-1} + \sum_{n=j+1}^{\infty} n |a_n| |z|^{n-1} \leq 1 + \sum_{n=2}^{j} n |a_n| |z|^{n-1} + B_j |z|^j
\]
and
\[
|f'(z)| \geq 1 - \sum_{n=2}^{j} n |a_n| |z|^{n-1} - \sum_{n=j+1}^{\infty} n |a_n| |z|^{n-1} \geq 1 - \sum_{n=2}^{j} n |a_n| |z|^{n-1} - B_j |z|^j.
\]

3. Radius Problem for the Class \(\mathcal{G}(\theta, \alpha)\)

To obtain the radius problem for the class \(\mathcal{G}(\theta, \alpha)\), we need the following lemma.

Lemma 3.1. If \(f(z) \in \mathcal{G}(\theta, \alpha)\), then
\[
|f'(z)| \leq 1 + \sum_{n=2}^{j} n |a_n| |z|^{n-1} + \sum_{n=j+1}^{\infty} n |a_n| |z|^{n-1} \leq 1 + \sum_{n=2}^{j} n |a_n| |z|^{n-1} + B_j |z|^j
\]
and
\[
|f'(z)| \geq 1 - \sum_{n=2}^{j} n |a_n| |z|^{n-1} - \sum_{n=j+1}^{\infty} n |a_n| |z|^{n-1} \geq 1 - \sum_{n=2}^{j} n |a_n| |z|^{n-1} - B_j |z|^j.
\]

\(\square\)
for all $z \in \mathcal{U}$. Let us consider a point $z \in \mathcal{U}$ such that $z = |z| e^{-i\theta}$. Then we have

$$1 - \sum_{n=2}^{\infty} n^3 |a_n| |z|^{n-1} > \alpha$$

Letting $|z| \to 1^-$, we obtain the inequality (7).

**Corollary 3.2.** If $f(z) \in \mathcal{G}(\theta, \alpha)$, then

$$|a_n| \leq \frac{1 - \alpha}{n^3} \quad (n = 2, 3, \ldots).$$

**Remark 3.3.** By Lemma 3.1, we observe that if $f(z) \in \mathcal{G}(\theta, \alpha)$, then

$$\sum_{n=2}^{\infty} n^2(n-1)|a_n| \leq \sum_{n=2}^{\infty} n^3|a_n| \leq 1 - \alpha.$$

Applying Theorem 2.1 and Lemma 3.1, we derive

**Theorem 3.4.** If $f(z) \in \mathcal{G}(\theta, \alpha)$, and $\delta \in \mathbb{C}$ ($0 < |\delta| < 1$). Then the function $\frac{1}{\delta} f(\delta z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$ for $(0 < |\delta| \leq |\delta_0(\lambda)|$, where $|\delta_0(\lambda)|$ is the smallest positive root of the equation

$$|\beta_1| = \frac{\delta \sqrt{2(|\delta|^2 + 2)(1 - \alpha)}}{(1 - |\delta|^2)^2}$$

$$+ \frac{\delta^2 \sqrt{6(3|\delta|^4 + 14|\delta|^2 + 3)(1 - \alpha - 4|a_2|^2)}}{(1 - |\delta|^2)^3}$$

$$+ \frac{\delta^3 \sqrt{48(6|\delta|^6 + 52|\delta|^4 + 43|\delta|^2 + 4)(1 - \alpha - 4|a_2|^2 - 18|a_3|^2)}}{(1 - |\delta|^2)^4} = \lambda$$

in $0 < |\delta| < 1$.

**Proof.** For $f(z) \in \mathcal{G}(\theta, \alpha)$, we see that

$$\frac{1}{\delta} f(\delta z) = z + \sum_{n=2}^{\infty} \delta^{n-1} a_n z^n$$

and

$$\sum_{n=2}^{\infty} n^2(n-1)|a_n|^2 \leq 1 - \alpha.$$
Thus, to show that \( \frac{1}{2} f(\delta z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda) \), from Theorem 2.1, it is sufficient to prove that
\[
\sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|\delta|^{n-1}|a_n| \leq \lambda.
\]

Applying Cauchy-Schwarz inequality, we note that
\[
\sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|\delta|^{n-1}|a_n|
\]
\[
\leq \frac{|\beta_1|}{|\delta|} \left( \sum_{n=2}^{\infty} n^2(n-1)|\delta|^{2n} \right)^{\frac{1}{2}} \left( \sum_{n=2}^{\infty} n^2(n-1)|a_n|^2 \right)^{\frac{1}{2}}
\]
\[
+ \frac{|\beta_2|}{|\delta|} \left( \sum_{n=3}^{\infty} n^2(n-1)(n-2)^2|\delta|^{2n} \right)^{\frac{1}{2}} \left( \sum_{n=3}^{\infty} n^2(n-1)|a_n|^2 \right)^{\frac{1}{2}}
\]
\[
+ \frac{|\beta_3|}{|\delta|} \left( \sum_{n=4}^{\infty} n^2(n-1)(n-2)^2(n-3)^2|\delta|^{2n} \right)^{\frac{1}{2}} \left( \sum_{n=4}^{\infty} n^2(n-1)|a_n|^2 \right)^{\frac{1}{2}}
\]
\[
\leq \frac{|\beta_1|}{|\delta|} \left( \sum_{n=2}^{\infty} n^2(n-1)|\delta|^{2n} \right)^{\frac{1}{2}} \sqrt{1 - \alpha}
\]
\[
+ \frac{|\beta_2|}{|\delta|} \left( \sum_{n=3}^{\infty} n^2(n-1)(n-2)^2|\delta|^{2n} \right)^{\frac{1}{2}} \sqrt{1 - \alpha - 4|a_2|^2}
\]
\[
+ \frac{|\beta_3|}{|\delta|} \left( \sum_{n=4}^{\infty} n^2(n-1)(n-2)^2(n-3)^2|\delta|^{2n} \right)^{\frac{1}{2}} \sqrt{1 - \alpha - 4|a_2|^2 - 18|a_3|^2}.
\]

Making use of
\[
\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad (|x| < 1),
\]
we have
\[
\sum_{n=2}^{\infty} n^2(n-1)x^n = \frac{2x^2(x+2)}{(1-x)^4}.
\]
Since
\[
\sum_{n=3}^{\infty} (n-2)x^{n-1} = x^2 \left( \sum_{n=3}^{\infty} (n-2)x^{n-3} \right) = x^2 \left( \sum_{n=3}^{\infty} x^{n-2} \right) = \frac{x^2}{(1-x)^2},
\]
we see that
\[ \sum_{n=3}^{\infty} (n-1)(n-2)^2 x^n = x^3 \left( \frac{x^2}{(1-x)^2} \right)^\prime = \frac{2x^3 + 4x^4}{(1-x)^4}. \]
and thus, we obtain
\[ \sum_{n=3}^{\infty} n(n-1)(n-2)^2 x^n = \frac{6x^3 + 18x^4}{(1-x)^5} \]
which yields
\[
(11) \quad \sum_{n=3}^{\infty} n^2(n-1)(n-2)^2 x^n = \frac{6x^3(3x^2 + 14x + 3)}{(1-x)^6}.
\]
Furthermore, we have
\[ \sum_{n=4}^{\infty} (n-1)(n-2)^2(n-3)^2 x^n = x^4 \left( \sum_{n=4}^{\infty} (n-1)(n-2)^2(n-3)^2 x^{n-4} \right) \]
\[ = x^4 \left( \sum_{n=4}^{\infty} (n-2)(n-3)x^{n-1} \right)^\prime, \]
but
\[ \sum_{n=4}^{\infty} (n-2)(n-3)x^{n-1} = x^3 \left( \sum_{n=4}^{\infty} (n-2)(n-3)x^{n-4} \right) = \frac{2x^3}{(1-x)^3} \]
thus, we have
\[ \sum_{n=4}^{\infty} (n-1)(n-2)^2(n-3)^2 x^n = \frac{12x^4 + 72x^5 + 36x^6}{(1-x)^6}, \]
which yields
\[ \sum_{n=4}^{\infty} n(n-1)(n-2)^2(n-3)^2 x^n = \frac{48x^4 + 384x^5 + 288x^6}{(1-x)^7}. \]
This gives us that
\[
(12) \quad \sum_{n=4}^{\infty} n^2(n-1)(n-2)^2(n-3)^2 x^n = \frac{48x^4(6x^3 + 52x^2 + 43x + 4)}{(1-x)^8}.
\]
Therefore, from (9)- (12) with $|\delta|^2 = x$, we obtain

$$\sum_{n=2}^{\infty} n^2(n-1) (|\beta_1| + (n-2) |\beta_2| + (n-2)(n-3) |\beta_3|) |\delta|^{n-1} |a_n|$$

\[ \leq |\beta_1| |\delta|^2 \sqrt{2(|\delta|^2 + 2)(1-\alpha)} \left(1 - |\delta|^2\right)^2 + |\beta_2| |\delta|^2 \sqrt{6(3|\delta|^4 + 14|\delta|^2 + 3)(1-\alpha - 4 |a_2|^2)} \left(1 - |\delta|^2\right)^3 \]
\[ + |\beta_3| |\delta|^3 \sqrt{48(6|\delta|^6 + 52|\delta|^4 + 43|\delta|^2 + 4)(1-\alpha - 4 |a_2|^2 - 18 |a_3|^2)} \left(1 - |\delta|^2\right)^4 \]

Now, let us consider the complex number $\delta$ ($0 < |\delta| < 1$) such that

\[ |\beta_1| |\delta|^2 \sqrt{2(|\delta|^2 + 2)(1-\alpha)} \left(1 - |\delta|^2\right)^2 + |\beta_2| |\delta|^2 \sqrt{6(3|\delta|^4 + 14|\delta|^2 + 3)(1-\alpha - 4 |a_2|^2)} \left(1 - |\delta|^2\right)^3 \]
\[ + |\beta_3| |\delta|^3 \sqrt{48(6|\delta|^6 + 52|\delta|^4 + 43|\delta|^2 + 4)(1-\alpha - 4 |a_2|^2 - 18 |a_3|^2)} \left(1 - |\delta|^2\right)^4 \]

\[ = \lambda \]

If we define the function $h(|\delta|)$ by

\[ h(|\delta|) = |\beta_1| |\delta| (1 - |\delta|^2)^2 \sqrt{2(|\delta|^2 + 2)(1-\alpha)} \]
\[ + |\beta_2| |\delta|^2 \left(1 - |\delta|^2\right) \sqrt{6(3|\delta|^4 + 14|\delta|^2 + 3)(1-\alpha - 4 |a_2|^2)} \]
\[ + |\beta_3| |\delta|^3 \sqrt{48(6|\delta|^6 + 52|\delta|^4 + 43|\delta|^2 + 4)(1-\alpha - 4 |a_2|^2 - 18 |a_3|^2)} \]
\[ - \lambda \left(1 - |\delta|^2\right)^4 , \]

then we have $h(0) = -\lambda < 0$ and $h(1) = 4\sqrt{3} |\beta_3| \sqrt{105(1-\alpha - 4 |a_2|^2 - 18 |a_3|^2)} > 0$. This means that there exists some $\delta_0$ such that $h(\delta_0) = 0$ ($0 < |\delta_0| < 1$). This completes the proof of the theorem.
4. Radius Problem for the Class $Q(\theta, \alpha)$

For the class $Q(\theta, \alpha)$, we prove the following lemma.

**Lemma 4.1.** If $f(z) \in Q(\theta, \alpha)$, then

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq 1 - \alpha. \quad (13)$$

**Proof.** Let $f(z) \in Q(\theta, \alpha)$. Then, we have

$$\text{Re}\left\{ \frac{D^2 f(z)}{D f(z)} \right\} = \text{Re}\left\{ \frac{1 + \sum_{n=2}^{\infty} n^2 a_n z^n}{1 + \sum_{n=2}^{\infty} n a_n z^n} \right\} = \text{Re}\left\{ \frac{1 - \sum_{n=2}^{\infty} n^2 |a_n| e^{i(n-1)\theta} z^{n-1}}{1 - \sum_{n=2}^{\infty} n |a_n| e^{i(n-1)\theta} z^{n-1}} \right\} > \alpha$$

for all $z \in U$. Let us consider a point $z \in U$ such that $z = |z| e^{-i\theta}$. Then we have

$$\frac{1 - \sum_{n=2}^{\infty} n^2 |a_n| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} n |a_n| |z|^{n-1}} > \alpha$$

Letting $|z| \to 1^-$, we obtain the inequality (13). $\square$

**Corollary 4.2.** If $f(z) \in Q(\theta, \alpha)$, then

$$|a_n| \leq \frac{1 - \alpha}{n(n-\alpha)} \quad (n = 2, 3, ...).$$

**Remark 4.3.** If $f(z) \in Q(\theta, \alpha)$, then

$$\sum_{n=2}^{\infty} n^2 (n-1) |a_n| \leq \sum_{n=2}^{\infty} n(n-\alpha) |a_n| \leq 1 - \alpha.$$
root of the equation

\[
|\beta_1| \left| \frac{\sqrt{2(|\delta|^2 + 2)(1 - \alpha)}}{1 - |\delta|^2} \right|^2 \\
+ |\beta_2| \left| \frac{\sqrt{6(|\delta|^4 + 14|\delta|^2 + 3)(1 - \alpha - 4|a_2|^2)}}{(1 - |\delta|^2)^3} \right|^2 \\
+ |\beta_3| \left| \frac{\sqrt{48(|\delta|^6 + 52|\delta|^4 + 43|\delta|^2 + 4)(1 - \alpha - 4|a_2|^2 - 18|a_3|^2)}}{(1 - |\delta|^2)^4} \right|^2
\]

\begin{align*}
\lambda &= \text{in } 0 < |\delta| < 1.
\end{align*}

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