Nonextensive thermodynamic formalism for chaotic dynamical systems

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A nonextensive thermostatic approach to chaotic dynamical systems is developed by expressing generalized Tsallis distribution as escort distribution. We explicitly show the thermodynamic limit and also derive the Legendre Transform structure. As an application, bit variance is calculated for ergodic logistic map. Consistency of the formalism demands a relation between box size (ε) and degree of nonextensivity, given as (1 − q) ∼ −1/ln ε. This relation is numerically verified for the case of bit variance as well as using basic definition of Tsallis entropy.

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1. INTRODUCTION

Techniques borrowed from Gibbs-Boltzmann or extensive thermodynamics play important role in the characterization of complex behaviour exhibited by dynamical systems. Analogous notions of entropy, temperature, pressure and free energy can be applied to quantify the fractal or multifractal attractors of chaotic nonlinear mappings [1]-[3]. In fact, thermodynamic theory of multifractals is applicable to any arbitrary probability distribution irrespective of its mode of generation. A useful strategy here is the following: for the given probability distribution \( p_i \) with respect to subsystems whose joint probabilities factorise: if \( -\Psi = \ln Z = \sum_i p_i \exp(-\beta E_i) \), where \( b_i \) is bit number, form another set of normalized distributions \( P_i = p_i^\beta / \sum_i p_i^\beta \), called escort distributions of order \( \beta \). For different \( \beta \) values, the new distributions are better able to scan different features of the original distribution \( p_i \). Now in standard thermodynamics, maximizing the Shannon-Boltzmann entropy \( S = -\sum_i p_i \ln p_i \) under the constraint of given mean value of internal energy, one obtains the canonical distribution, \( P_i = \exp(\Psi - \beta E_i) \), where \( \Psi = -\ln Z(\beta) = -\ln \left( \sum_i \exp(-\beta E_i) \right) \) is equivalent to Helmholtz free energy and \( Z(\beta) \) is the partition function. In chaos theory, the escort distribution is also sought in a ‘generalized’ canonical form \( P_i = \exp(\Psi - \beta b_i) \), where \( \Psi = -\ln Z(\beta) \) and \( Z(\beta) = \sum_i p_i^\beta = \sum_i \exp(-\beta b_i) \). Thus in thermodynamic analogy, positive bit number \( b_i \) is equivalent to energy \( E_i \), \( \beta \) plays the role of inverse temperature and \( \Psi \) is a generalized free energy. Knowing the partition function is the first step to the study of analogous thermodynamic quantities and relations in chaos theory. The analogy goes deeper and interesting phenomena like phase transitions have been highlighted in dynamical systems [3].

On the other hand, it has been realized in recent years that the extensive formalism of thermodynamics fails to yield testable results (such as finite valued expressions for response functions of the various thermodynamic quantities) for systems with long-range interactions or which evolve with multifractal space-time constraints or have long term memory effects [4]. To deal with such cases, a nonextensive formalism of statistical thermodynamics was proposed by Tsallis [3]. This formalism has many successful applications to its credit by now, and the importance of nonextensive behaviour has been shown for diverse kinds of systems: Lévy anomalous diffusion [5], [6], [7], stellar polytropes [8], pure-electron plasma two dimensional turbulence [9], [10], solar neutrinos [11], [12], inverse bremsstrahlung in plasma [12], to name a few [3]. Generally, this formalism is based on the proposal of a non-logarithmic entropy

\[
S_q(p) = \frac{1 - \sum_i p_i^q}{q - 1} = \sum_i [b_i] p_i^q.
\]

Here, \([b_i]\) may just be considered as notation for the generalized bit number within the nonextensive framework. For \( q \rightarrow 1 \), \([b_i] \rightarrow b_i = -\ln p_i \) and Eq. (*8) gives the Shannon entropy. Tsallis entropy is pseudo-additive (nonextensive) with respect to subsystems whose joint probabilities factorise: if \( p_{i+I} = p_i^I p_j^I \), then \( s_q(p^{I+I}) = s_q(p^I) + s_q(p^I) + \)

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Exercise 1.9

(1 − q)S_q(p')S_q(p'') = \sum_{i} E_i p_i \beta \frac{1}{1-\beta}. Thus (1 − q) represents the deviation from extensivity. Maximising Tsallis entropy under the internal energy constraint \( U = \sum_i E_i p_i^\beta \) yields the equilibrium Tsallis distribution

\[
p_i^{(eq)} = \frac{\{1 - (1 - q)\beta E_i\}^{1/(1-q)}}{\sum_i \{1 - (1 - q)\beta E_i\}^{1/(1-q)}},
\]

where \( \beta \) is the inverse temperature. For \( q \to 1 \), we get back the canonical distribution.

It is believed that Tsallis formalism is a natural frame to study systems with fractal structures [15]. Indeed the form of Tsallis entropy was inspired from multifractal ideas [5]. Also its connection with a scale-invariant thermostatistics has been conjectured [16]. Simple relations exist between of Tsallis entropy was inspired from multifractal ideas [5]. Also its connection with a scale-invariant thermostatistics and anomalous diffusion [17]. Recently, a precise relation was proposed between \( q \) and multifractal scaling properties [18]. For such cases, at the onset of chaos, the exponential sensitivity to initial conditions is replaced by power law sensitivity, and generalizaton of Kolmogorov-Sinai entropy has to be invoked on the lines of Tsallis formalism [19].

Realizing the clear signatures of nonextensivity in chaotic dynamical systems, we feel a nonextensive thermodynamic approach needs to be developed for such systems. In this paper, we develop the thermostatistics of such a formalism. The plan of the paper is as follows: in section II, we present our formalism by first writing the partition function, then seeking its thermodynamic limit. By generalizing the Rényi dimensions, we establish the Legendre Transform structure. In section III, we present as an application, the calculation of bit variance within nonextensive framework. In section IV, we discuss the relation between box size (degree of coarse graining) and degree of nonextensivity (1-q). Section V is devoted to conclusions and outlook.

II. GENERALIZED FORMALISM

A. Basic ingredients

Our approach is to work with escort distributions which are now sought not in canonical form, but as generalized Tsallis distribution. For this purpose, we consider a general, nonextensive statistics given by

\[
p_i = \{1 - (1 - q)\beta [b_i]\}^{1/(1-q)\beta}.
\]

For \( \beta = 1 \), this gives the relation between \( p_i \) and generalized bit number \( [b_i] \) in Tsallis statistics (see Eq. (1)). Also for \( q \to 1 \), \( p_i = \exp(-b_i) \), which corresponds to extensive statistics. Now the escort distribution for the given distribution of Eq. (3) is

\[
P_i = \frac{p_i^\beta}{\sum_i p_i^\beta} = \frac{\{1 - (1 - q)\beta [b_i]\}^{1/(1-q)\beta}}{\sum_i \{1 - (1 - q)\beta [b_i]\}^{1/(1-q)\beta}},
\]

which is a generalized Tsallis distribution (see Eq. (2)), where now \( [b_i] \) plays the role of energy \( E_i \). The generalized partition function, \( Z_q(\beta) = \sum_i p_i^\beta = \sum_i \{1 - (1 - q)\beta [b_i]\}^{1/(1-q)\beta} \). Again for \( \beta = 1 \), escort distribution \( P_i \) is identical to original distribution \( p_i \). Note that in general, both the original distribution and the escort distribution depend on \( \beta \).

Now an important feature of Tsallis formalism is that it retains much of the formal structure of standard thermodynamics, including Legendre Transform structure [14]. This is facilitated by a generalization of the logarithm function as

\[
\Psi_q = -\ln_q Z_q(\beta) = -\frac{Z_q^{1-q} - 1}{1-q}.
\]

With the aid of such modified logarithm (or conversely, the exponential function), we develop the nonextensive thermostatistics of multifractals. Let us consider a certain nonlinear mapping over one-dimensional (for simplicity) phase space which has been partitioned into boxes of equal size \( \epsilon \). Let \( p_i \) represent visiting frequency of the map for a box \( i \). Then we define local crowding indices as

\[
\alpha_i(\epsilon, q) = \frac{p_i^{(1-q)\beta} - 1}{(1-q)\beta \ln \epsilon},
\]

which for \( q \to 1 \), can be written as \( \alpha_i(\epsilon) = \frac{\ln p_i}{\ln \epsilon} \). From Eqs. (5) and (6), we may write

\[
[b_i] = \alpha_i(\epsilon, q)V,
\]

where \( V = -\ln \epsilon \), is the extensive volume parameter in thermodynamic analogy.
B. Thermodynamic limit

To derive $\epsilon \to 0$ or $V \to \infty$ limit of the generalized partition function, we replace the sum in $Z_q = \sum_i p_i^\beta$ by an integral over $\alpha$,

$$Z_q = \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \gamma(\alpha) \left(1 - (1 - q)\beta\alpha V\right)^{1/(1-q)},$$

(8)

where $\gamma(\alpha)d\alpha$ are the number of boxes with $\alpha_i$ in the interval $[\alpha, \alpha + d\alpha]$. Now in extensive case, $\gamma(\alpha)$ is assumed to scale as $\sim \epsilon^{-f(\alpha)} = \exp(f(\alpha)V)$. In the nonextensive case, this can be generalized as $\gamma(\alpha) \sim (1 + (1 - q)f(\alpha)V)^{1/(1-q)}$. Thus we finally obtain

$$Z_q = \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \left\{\left(1 + (1 - q)f(\alpha)V\right)\left(1 - (1 - q)\beta\alpha V\right)\right\}^{1/(1-q)}.$$

(9)

It is useful to remember that the integrand above is in a sense, a nonextensive generalization of the function $\exp((f(\alpha) - \beta\alpha)V)$ in the extensive case. Now applying the saddle point method for the case of large $V$, we may write the above integral as

$$Z_q = \left\{\left(1 + (1 - q)f(\alpha)V\right)\left(1 - (1 - q)\beta\alpha V\right)\right\}^{1/(1-q)},$$

(10)

where $\alpha$ corresponds to the maximum value of the integrand for given $\beta$ value. Then using Eq. (5), we can write

$$\Psi_q = (\beta\alpha - f(\alpha)) - (1 - q)\beta\alpha f(\alpha)V.$$

(11)

Now in the limit $V \to \infty$, the last term above diverges. To keep the generalized free energy density finite, we assume that

$$(1 - q)\beta \sim \frac{1}{V}.$$

(12)

Then Eq. (11) is given by

$$\lim_{\epsilon \to 0} \frac{\Psi_q}{V} \sim (\beta\alpha - f(\alpha)) - \alpha f(\alpha).$$

(13)

Note that the first term in parentheses is same as obtained for the extensive case. The second term arises due to the nonextensive nature of the present formalism. Also it may be noted that if we assume volume parameter $V$ to be positive definite, then $(1 - q)\beta \geq 0$.

C. Generalized Rényi Dimensions

Next, we propose to generalize the usual Rényi dimensions [21]. This is required to discuss Legendre Transform structure of the present formalism. We define

$$D_q(\beta) = \lim_{\epsilon \to 0} \frac{1}{(1 - \beta)V} \ln q \left(\sum_i p_i^\beta\right).$$

(14)

Again assuming the validity of Eq. (12) for $\beta \neq 0$, in the $\epsilon \to 0$ limit, we get

$$D_q(\beta) = \frac{\beta}{1 - \beta} \left(\left(\sum_i p_i^\beta\right)^{1-q} - 1\right).$$

(15)

For $\beta = 0$, the last term of the Eq. (14) vanishes and thus, Eq. (12) cannot be inferred. Then using Eq. (14) and $V = -\ln \epsilon$, we get $D_q(0) = -\lim_{\epsilon \to 0} \frac{\ln N}{\ln \epsilon}$, where $N$ is the number of nonempty boxes. Also for $\beta = 1$, Eq. (15) gives $D_q(1) = -(1 - q)\sum_i p_i \ln p_i$. To see the behaviour for large $\beta$, we write from Eqs. (3) and (7)
$p_i^\beta = \{1 - (1 - q)\beta \alpha_i V\}^{1/(1-q)}.$  \hspace{0.5cm} (16)

Again in the limit of box size $\epsilon \to 0$, we assume the scaling form Eq. (12) and write

$p_i^\beta = \{1 - \alpha_i\}^{1/(1-q)}$.  \hspace{0.5cm} (17)

For very large values of $\beta$, the above expression is dominated by minimum value of $\alpha$. Using this condition in Eq. (15), we can write

$D_q(\beta) = \frac{\beta}{\beta - 1} \alpha_{\text{min}}$,  \hspace{0.5cm} (18)

which for $\beta \to +\infty$ yields $D_q(\infty) = \alpha_{\text{min}}$. Similarly, for $\beta \to -\infty$, one can show $D_q(-\infty) = \alpha_{\text{max}}$. These results are thus identical to those obtained for the usual Rényi dimensions.

D. Legendre Transform structure

Now from Eq. (15), we write

$Z_q = \sum_i p_i^\beta = \left\{1 + \left(\frac{1 - \beta}{\beta}\right)D_q(\beta)\right\}^{1/(1-q)}$.  \hspace{0.5cm} (19)

Using Eq. (10) with $\epsilon = 0$, and Eq. (19), we get

$(\beta - 1)D_q(\beta) = \beta \alpha - (1 - \alpha)f(\alpha)$,  \hspace{0.5cm} (20)

which we choose to write as

$\tau_q(\beta) = \beta \alpha - f_q(\alpha)$,  \hspace{0.5cm} (21)

where we define $\tau_q(\beta) = (\beta - 1)D_q(\beta)$ and $f_q(\alpha) = (1 - \alpha)f(\alpha)$. We find that

$\frac{\partial f_q(\alpha)}{\partial \alpha} = \beta$,  \hspace{0.5cm} $\frac{\partial \tau_q(\beta)}{\partial \beta} = \alpha$.  \hspace{0.5cm} (22)

We note $\tau_q(\beta)$ being equivalent to the generalized free energy $\Psi_q$ (that already obeys Legendre Transform structure), is a concave function of its argument. Then from Eq. (21), we infer that $f_q(\alpha)$ is also a concave function, implying that the Legendre Transform structure is well defined.

III. GENERALIZED BIT VARIANCE: AN EXAMPLE

An alternative method to characterize multifractal statistics is using bit moments and bit cumulants of escort distributions [22]. Particularly, the second bit cumulant ($\Gamma_2$) measures the variance of bit number. In the following, we derive expression for bit variance within the generalized statistical framework and apply it to ergodic logistic map. For nonextensive case, we write generalized second bit cumulant for escort distribution $P$ as

$\Gamma_2^{(q)}(P) = -\beta^2 \frac{\partial^2 \Psi_q}{\partial \beta^2} = q\left\{\left(\sum_i p_i^\beta\right)^{-q} \left(\sum_i [b_i]^2 p_i^\beta(2q-1)\right)\right.$

$\left. - \left(\sum_i p_i^\beta\right)^{-q-1} \left(\sum_i [b_i] p_i^\beta q\right)^2\right\}$.  \hspace{0.5cm} (23)

The corresponding quantity for the original distribution ($\beta = 1$) is given by

$\Gamma_2^{(q)}(p) = -\frac{\partial^2 \Psi_q}{\partial \beta^2}\bigg|_{\beta=1} = q\left\{\sum_i [b_i]^2 p_i^{(2q-1)} - \left(\sum_i [b_i] p_i^q\right)^2\right\}$.  \hspace{0.5cm} (24)
Note that for $q \to 1$, we have $\Gamma_2(p) = \sum p_i (\ln p_i)^2 - (\sum p_i \ln p_i)^2$. For the case of ergodic maps, bit variance density which is equivalent to heat capacity is given as $C_2 = (\langle (\ln \rho)^2 \rangle - \langle \ln \rho \rangle^2)$, where $\rho$ is the natural invariant density of the map. For the nonextensive case, we find

$$C_2(q) = \frac{q}{(q-1)^2}(\langle \rho^{(2q-1)} \rangle - \langle \rho^q \rangle^2).$$

(25)

In Fig. 1, we plot $C_2$ and $C_2(q)$ against the nonlinearity parameter $r$ for the logistic map, $x_{n+1} = rx_n(1-x_n)$. We find that for $0 < q < 1$, the bit variance (which represents size of fluctuations in bit number) $C_2(q)$ is larger than the quantity $C_2$ of extensive case. We present an interpretation of this result in the next section.

We also find quite an interesting feature i.e. the general trend of $C_2$ vs. $r$ curve, evaluated at smaller box size can be matched by $C_2(q)$ evaluated at arbitrarily larger box size and appropriate value of $q < 1$. In Fig. 2, we show two cases to corroborate this point. Quite remarkably, the two quantities show matching for a wide range of the parameter $r$. To be precise, we note that decreasing the box size, lifts the heat capacity curve $C_2$. Again, $q < 1$ also results in a general increase in heat capacity $C_2(q)$ (Fig. 1). Thus we can expect that heat capacity results with $q = 1$ and smaller box size are reproduced with $q < 1$ and larger box size. This implies that a range of $q$ and $\epsilon$ values exist, which yield the same value for $C_2(q)$. In the following, we make use of this possibility.

**IV. CONNECTION BETWEEN $\epsilon$ AND $(1-Q)$**

An important assumption in the above which makes the whole thermodynamic formalism consistent, is the relation between box size $\epsilon$ and the degree of nonextensivity $(1-q)$, as given by Eq. (12). Note that this relation is for an escort distribution of order $\beta$. For the original distribution, $\beta = 1$ and we have $(1-q) \sim -\frac{1}{\ln \epsilon}$. In the following, we numerically check the validity of this scaling form. Again we consider the example of section III. For exactness, we focus on a fixed value of nonlinearity parameter $r = 3.81$ in the chaotic region. The way we go about this exercise is as follows: for a given value of $q$ and box size $\epsilon$, we calculate $C_2(q)$. Then we vary both $q$ and $\epsilon$, so that we obtain same value of $C_2(q)$ within good approximation. Fig. 3 shows that a direct proportionality in fact exists between $(1-q)$ and $-1/\ln \epsilon$, for very small $\epsilon$.

One can look for the origin of the proposed relation between $(1-q)$ and $\epsilon$, in the definition of Tsallis entropy itself. Thus consider a unit interval divided into $W$ number of equal sized boxes. Assuming equiprobability, we have $p_i = \frac{1}{W} = \epsilon$. Then Tsallis entropy for this distribution is given by $S_q = \frac{1-\epsilon^{(q-1)}}{q-1}$. Again keeping $S_q$ fixed, we vary $q$ and $\epsilon$ and as shown in Fig. 4, find that the proposed scaling form $(1-q) \sim 1/V$ is justified for large $V$.

Finally, we make a remark on the results of Fig. 1. How do we interpret the higher values of $C_2(q)$ over $C_2$ for $q < 1$ in terms of effects of nonextensivity ? In the present context, we look at this issue from the viewpoint of inherent correlations due to nonextensivity. It is known that if the bit cumulants of subsystems, deviate from additivity, this indicates the presence of correlations. This happens naturally for the nonextensive case, where e.g. the first bit cumulant, $I_1^{(q)}$ is identical to Tsallis entropy and hence is nonadditive with respect to statistically independent subsystems. Tsallis entropy is superextensive for $0 < q < 1$, i.e. the entropy of composite system is larger than the sum of entropies of individual subsystems. This implies that negative correlations are set up for $0 < q < 1$. We propose that the overall increase in bit variance for the nonextensive case above, reflects these implicit correlations. In the thermodynamic limit of $V \to \infty$, the proportionality between $(1-q)$ and $1/V$ suggests that $q \to 1-0$. Thus these correlations can be expected to be suppressed in this limit. Such correlations have also been studied in the context of classical ideal gas model within nonextensive framework [23].

**V. CONCLUSION AND OUTLOOK**

In the present paper, we presented a nonextensive generalization of thermostatistics of dynamical systems. We have seen that in the thermodynamic limit, generalized free energy density is finite if we assume direct proportionality between $(1-q)$ and $1/V$. This relation also plays important role in exhibiting the Legendre Transform structure. We have numerically shown the validity of this relation for generalized bit variance and from the definition of Tsallis entropy itself. We know finite partitioning of the phase space is a fact of life in the study of chaotic systems. That this feature can be related to $(1-q)$, provides a new insight into the nature of nonextensivity. Further development of the formalism treating boxes with variable size, extension to dynamical aspects involving time evolution will be extremely welcome. It can be hoped that these possible developments will connect with the recent work [13], [3] on
low-dimensional dissipative systems and their power law sensitivity to initial conditions at bifurcation points or onset of chaos.

VI. ACKNOWLEDGEMENTS

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FIG. 1. Heat capacity equivalent to bit variance is plotted against control parameter $r$ of logistic map. Solid curve represents $C_2$ i.e. the extensive case, while dotted curve shows the behaviour of $C_2^{(q)}$. $q$ value is set at 0.97. Higher value of bit variance in the latter case reflect the presence of nonextensive correlations.
FIG. 2. $C_2$ (solid line) and $C_2^{(q)}$ (dashed line with symbols) can be matched for a wide range of $r$ values: (a) $C_2$ with $2^{11}$ boxes. $C_2^{(q)}$ at $q = 0.99$ with $2^{10}$ boxes. (b) $C_2$ with $2^{15}$ boxes. $C_2^{(q)}$ at $q = 0.98$ with $2^{10}$ boxes. The plots show that decreasing $q$ from unity in $C_2^{(q)}$, has the equivalent effect of decreasing the box size in $C_2$. Also note that $C_2$ increases in general, on decreasing the box size.
Heat capacity

(b)
FIG. 3. Plot between $(1-q)$ and $1/V$, keeping fixed $C_2^{(q)}$ value originally calculated at $r = 3.81$, $q = 0.9$ and box size equal to $1/5000$. The points can be fitted to straight line $1.1x - 0.032$. 
FIG. 4. For equiprobability, and $p_i = \epsilon$, the plot between $(1 - q)$ and $1/V$, keeping fixed Tsallis entropy originally calculated at $q = 0.9$ and box size = $1/5000$. Straight line fit is given by $2.26x - 0.17$. 