An Ordinal-Free Proof of the Cut-elimination Theorem for an Impredicative Subsystem of $\Pi^1_1$-Analysis with $\omega$-rule

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Abstract

The aim of this paper is to give a simple ordinal-free proof of the cut-elimination theorem for an impredicative subsystem of $\Pi^1_1$-analysis with $\omega$-rule containing $ID_1$. A derivation $d$ in this system is transformed into a derivation $g(d)$ with $\Omega$-rule introduced by W. Buchholz in such a way that one cut-elimination step performed on a derivation $d$ corresponds under $g$ to passing to a subderivation of a derivation $g(d)$. The termination of reduction steps follows by transfinite induction on the height of $g(d)$.

1 Introduction

The aim of this paper is to give a simple ordinal-free proof of the cut-elimination theorem for an impredicative subsystem of $\Pi^1_1$-analysis with $\omega$-rule containing $ID_1$.

We embed cut-elimination process for an ordinary formulation of an impredicative subsystem of $\Pi^1_1$-analysis with $\omega$-rule (denoted by BI) to the process due to W. Buchholz [2, 5, 6]: a derivation $d$ in BI (with cut) is translated into a derivation $g(d)$, using $\Omega$-rule introduced by W. Buchholz. The system with $\Omega$-rule (denoted by $BI^\Omega$), the cut-elimination procedure introduced by W. Buchholz and his termination proof for cut-elimination are described in Section 3.

The central part of the present paper is the proof (in section 6) that one cut-elimination step performed on a derivation $d$ in BI corresponds to passing to a subderivation of $g(d)$ having smaller ordinal height:

$$BI: d \xrightarrow{\text{red}} \text{red}(d)$$

$$g \downarrow g$$

$$BI^\Omega: g(d) \xrightarrow{\geq} g(\text{red}(d)) \quad (1)$$
where \( g(d) > g(\text{red}(d)) \) means that the height of \( g(d) \) is greater than the height of \( g(\text{red}(d)) \), and \( \text{red}(d) \) denotes the result of applying a (cut-elimination) reduction step \( \text{red} \) to \( d \). This diagram holds for all cases except when \( d \) essentially ends in \( \omega \)-rule: \( d = \land \forall x A(x)(d_n)_{n \in \omega} \). Then \( g(d) \) also ends in “the same” \( \omega \)-rule, and (with the same exception) \( g(d_n) < g(\text{red}(d_n)) \) for all \( n \in \omega \). Now the termination of \( \text{red} \) (suitably defined for \( \omega \)-rule) follows by transfinite induction on (the height of) \( g(d) \). Like other procedures using \( \Omega \)-rule, our cut-elimination result is established for derivations in BI of arithmetical formulas.

In fact reductions applied to \( g(d) \) produce derivations in a wider system BI allowing to record intermediate steps of cut-elimination using symbols \( R, E, \ldots \) for operations \( R \) (one step cut-reduction), \( E \) (reduction of cut-degree of all cuts by 1), etc. This apparatus was first introduced by W. Buchholz [3] to give a finite term rewriting system for continuous cut-elimination. Syntactically a rule like \( E \) looks like Repetition:

\[
(E)_{\Gamma} \quad (\text{Rep})_{\Gamma}
\]

This corresponds to the fact that the last sequent of \( E(d) \) is the same as the last sequent of \( d \). On the other hand, in the definition of the cut-degree \( dg \), we set \( dg(E(d)) := dg(d) - 1 \) since \( E \) reduces the cut-degree by 1.

Our approach avoids use of complicated notations systems for large proof-theoretic ordinals. In this respect, it is similar to cut-elimination proofs for the first-order logic, first-order arithmetic, and predicative subsystems of analysis. In first-order logic, the proof proceeds by induction on (finite) cut-degree plus induction on (finite) height of the derivation with cut. In the case of arithmetic, the latter of these two inductions can be also done on the height of the derivation, avoiding introduction of ordinals < \( \epsilon_0 \). The same works for the predicative systems. Introduction of \( \Omega \)-rule allows to extend this approach to \( \Pi_1^1 \)-analysis.

We use \( \Omega \)-rule introduced by W. Buchholz since ordinal-free formulations and proofs of the cut-elimination theorem for the systems including \( \Omega \)-rule were given in [2] [5]. We treat here the simplest fragment of \( \Pi_1^1 \)-analysis allowing \( \Pi_1^1 \)-CA only for parameter-free \( \Pi_1^1 \)-formulas. The extension to the full \( \Pi_1^1 \)-CA of the results in this paper will be reported by the first author in [1].

We might add some remarks on a background of this paper. Since G. Mints’s papers [8] [9] [10] [12], the relationship between finitary and infinitary derivations has been investigated. In particular the relationship between the Gentzen-Takeuti method and the Schütte method has been elucidated by W. Buchholz [4] [5], and S. Tupailo [16]. In these works the authors have developed the way to recover informations about finitary derivations from infinitary derivations. In particular we want to mention W. Buchholz’s papers [4] [5]. In the former paper, a precise explanation of Gentzen’s reduction steps in [7] in terms of infinitary proof theory was given and it was shown that Gentzen’s reduction steps and assignment of ordinals to finite derivations can be derived from infinitary proof theory. In the latter paper, W. Buchholz proved the same thing for Takeuti’s consistency proofs for \( \Pi_1^1 \)-CA in [14].
Our aim is not an explanation of proof of the cut-elimination theorem for finitary systems, but to give a simple and transparent ordinal-free proof of the cut-elimination theorem for impredicative systems with \( \omega \)-rule. Our work was inspired by W. W. Tait’s unpublished paper [14].

The present paper consists of 6 sections. After recalling basic definitions in Section 2, we introduce infintary systems \( \text{BI}^\Omega_0 \), \( \text{BI}^\Omega_0 \) in Section 3. \( \text{BI}^\Omega_0 \) is arithmetic with \( \omega \)-rule, rule for second-order universal quantifier, and “Repetition Rule”:

\[
(\bigwedge_{\forall x.A}) \frac{\ldots A(x/n) \ldots}{\forall x.A}
\]

\[
(\bigwedge_{\forall x.A}) \frac{A(X/Y)}{\forall X A} \Gamma \frac{(\text{Rep})}{\Gamma}
\]

The rule \( \text{Rep} \) is obviously redundant, but is used as a delaying device to make the cut-elimination operator \( R_C \) (Theorem 1) continuous. \( \text{BI}^\Omega_0 \) is obtained by adding \( \Omega \), \( \bar{\Omega} \)-rules to \( \text{BI}^\Omega_0 \):

\[
(\Omega, \forall X A) \frac{\ldots A \forall X A(x) \ldots}{\forall X A} \frac{A(Y) \ldots A \forall X A(x) \ldots}{\phi}
\]

In Section 4 we define operators \( R, E, E_\omega \) on derivations in \( \text{BI}^\Omega \). The operator \( R \) transforms impredicative cuts into \( \bar{\Omega} \) and does one-reduction for other cuts in the standard way. \( E \) reduces cut-degree by 1, and \( E_\omega \) reduces cut-degree to 0. The collapsing operator \( D \) eliminates \( \bar{\Omega} \) from the given derivation \( d \) if \( d \) is cut-free and the end-sequent is arithmetical. Then we can prove that such a \( d \) can be transformed into a cut-free derivation \( d' \) of the same end-sequent in \( \text{BI}^\Omega_0 \). Finally we define the substitution operator \( S^X_T \), which is used to embed \( \Pi_1 \text{-CA} \) into \( \text{BI}^\Omega \) using \( \Omega \)-rule.

In Section 5 we introduce systems \( \text{BI}^- \) as arithmetic with \( \omega \)-rule, and rules for second-order quantifiers:

\[
(\bigwedge_{\forall x.A}) \frac{\ldots A(x/n) \ldots}{\forall x.A}
\]

\[
(\bigwedge_{\forall x.A}) \frac{A(X/Y)}{\forall X A} \frac{\forall \neg \forall X A(T)}{\forall \forall X A}
\]

with a suitable restriction on \( T \). The system \( \text{BI}^- \) is a weak subsystem of \( \Pi_1 \text{-CA} \) because \( \forall \neg \forall X A \)-rule is equivalent to \( \Pi_1 \text{-CA} \) only for parameter-free \( \Pi_1 \) formulas. The system which is obtained from \( \text{BI}^- \) by replacing \( \omega \)-rule by induction axiom has the strength of the theory of non-iterated inductive definition \( \Pi_1 \text{-ID} \).

\( \text{BI} \) is obtained by adding the following rules to \( \text{BI}^- \):

\[
(RA) \frac{C}{\phi} \frac{\neg C}{\phi} \frac{E_\phi}{\phi} \frac{E_\omega_\phi}{\phi}
\]
These rules correspond to operations $R$, $E$, $E_\omega$, $D$, $Sub^X_T$ mentioned above. They are necessary in order to connect separate steps of cut-elimination in BI to cut-elimination in BI$^\Omega$.

In Section 6 we give an ordinal-free proof of the cut-elimination theorem for BI. We define an embedding function $g$ from derivations in BI into the derivations in BI$^\Omega$ (Section 6.1). $g$ replaces rules $\bigvee_{X\in A}$, $R$, $E$, $E_\omega$, $D$, $Sub^X_T$ by $\Omega$-rule, operations $R$, $E$, $E_\omega$, $D$ and $S^X_T$. Next we define for each derivation $d$ in BI functions $tp(d)$ and $d[i]$ such that $tp(d)$ is the last inference symbol of $g(d)$, and $g(d[i])$ is the $i$-th immediate subderivation of $g(d)$ (Section 6.2).

We give an ordinal-free proof of the cut-elimination theorem for BI in Section 6.3. Our main observation is that $g(\text{red}(d))$ is a proper subderivation of $g(d)$ if $\text{red}(d)$ can be obtained from $d$ by an essential proof theoretic-reduction for each derivation in BI satisfying natural conditions (which we call proper derivations). Therefore the termination of the reduction procedure follows by transfinite induction on the height of $g(d)$. The cut-elimination theorem for BI$^-\Omega$ is obtained as a corollary.

Finally we see how the definition of $\text{red}$ works in Section 6.4. We see that $\text{red}$ simulates Gentzen-Takeuti reduction steps. In particular, it is shown that $\text{red}(d)$ corresponds to Takeuti-style reduction steps if $d$ is impredicative cut-elimination, and $g(d)$ ends in $\Omega$ ($tp(d) = \Omega$). We see that $\Omega$-rule lists up all cuts encountered in the future cut-elimination steps, and the result of Takeuti-style reduction $g(\text{red}(d))$ is seen to be one of such cuts. Although a correspondence between Takeuti’s reduction steps and Buchholz’s reduction steps were given by W. Buchholz [5], this will provide an additional informal connection between them.

2 Preliminaries

First we define a language $L$ which is the formal language of all systems considered below.

**Definition 1 Language $L$**

1. 0 is a term.
2. If $t$ is a term, then $S(t)$ is a term.
3. If $R$ is an $n$-ary predicate symbol for an $n$-ary primitive recursive relation, and $t_1,...,t_n$ are terms, then $R(t_1,...,t_n)$ is a formula. If $X$ is unary predicate variable, and $t$ is a term, then $X(t)$ is a formula. These formulas are called atomic formulas.
4. If $A$ is an atomic formula, then $\neg A$ is a formula. $A$ and $\neg A$ where $A$ is atomic are called literals.
5. If $A$ and $B$ are formulas, then $A \land B$, $A \lor B$ are formulas.

6. If $A(0)$ is a formula, then $\forall x A(x)$, and $\exists x A(x)$ are formulas.

7. If $A$ is a formula, and $A$ does not contain any second order quantifier and any free set variable other than $X$, then $\forall X A$ and $\exists X A$ are formulas.

If $A$ is a formula which is not atomic, then its negation $\neg A$ is defined using De Morgan’s laws. The set of true literals is denoted by TRUE. Formulas $A_0 \land A_1$, $\forall x A$, and $\forall X A$ are called negative. Formulas $A_0 \lor A_1$, $\exists x A$, and $\exists X A$ are called positive. $T$ denotes an expression $\lambda x. A$ where $A(0)$ is a formula (called abstraction). Formulas which do not contain any second order quantifier are called arithmetical.

**Definition 2** $rk(A)$ is defined as follows.

1. $rk(A) := 0$ if $A$ is a literal, $\forall X A(X)$, or $\exists X A(X)$.
2. $rk(A \land B) := rk(A \lor B) = \max(rk(A), rk(B)) + 1$.
3. $rk(\forall x A(x)) := rk(\exists x A(x)) = rk(A(0)) + 1$.

**Remark 1** We stress that $rk(A) = 0$ if $A$ is $\forall X A(X)$, or $\exists X A(X)$.

3 The Systems $\text{BI}_0^\Omega$, $\text{BI}^\Omega$

In this section, we define the systems $\text{BI}_0^\Omega$, $\text{BI}^\Omega$ using W. Buchholz’s notation from [5]. Only the minor formulas (which occur in the premises of the rule) and the principal formulas (which occur in the conclusion of the rule) are shown explicitly in inference rules. Any rule below is supposed to be closed under weakening, and contains contraction. For example the inference

$$
\frac{\Gamma, A_0 \quad \Gamma, A_1}{\Gamma, A_0 \land A_1}
$$

is indicated by $\bigland_{A_0 \land A_1}(d_0, d_1)$ where $d_i$ is a derivation of $\Gamma, A_i$ for $i \in \{0, 1\}$.

Let $I$ be an inference symbol of a system. Then we write $\Delta(I)$, and $|I|$ in order to indicate the set of principal formulas of $I$, and the index set of $I$. Moreover, $\bigcup_{i \in |I|}(\Delta_i(I))$ is the set of the minor formulas of $I$. For example the $\bigland_{A_0 \land A_1}$ inference can be of the form:

$$
\frac{d_0 : \Gamma, \Sigma, A_0, A_0 \land A_1 \quad d_1 : \Gamma, \Pi, A_1, A_0 \land A_1}{d : \Gamma, \Sigma, \Pi, A_0 \land A_1}
$$

Here $\Delta(\bigland_{A_0 \land A_1}) = \{A_0 \land A_1\}$, $|\bigland_{A_0 \land A_1}| = \{0, 1\}$, $\Delta_0(\bigland_{A_0 \land A_1}) = \{A_0\}$, and $\Delta_1(\bigland_{A_0 \land A_1}) = \{A_1\}$. If $d = I(d_i)|I|$, then $d_i$ denotes the subderivation of $d$ indexed by $i$. If $d$ is a derivation, $\Gamma(d)$ denotes its last sequent. Eigenvariables of the rules $\bigland_{\forall X A(X)}$, and $\Omega$ may occur free only in the premises, but not in the conclusion. Formulas which do not contain any subformula of the form $\exists X A(X)$ are called $\Pi^1$-formulas. Note that any arithmetical formula is $\Pi^1$-formula.
Definition 3  The systems $BI^0$, $BI^1$.

1. $BI^0$ consists of the following inference rules.

   \[(\text{Ax}\Delta) \quad \Delta \text{ where } \Delta = \{A\} \subseteq \text{TRUE or } \Delta = \{C, \neg C\}\]

   \[(\wedge_{A_0 \wedge A_1}) \quad \frac{A_0 \quad A_1}{A_0 \wedge A_1}\]

   \[(\vee^k_{A_0 \vee A_1}) \quad \frac{A_k}{A_0 \vee A_1} \text{ where } k \in \{0, 1\}\]

   \[(\forall_{\forall A}) \quad \frac{\ldots A(x/n) \ldots \text{ for all } n \in \omega}{\forall x A}\]

   \[(\forall^k_{\exists x A}) \quad \frac{A(x/k)}{\exists x A} \text{ where } k \in \omega}\]

   \[(\wedge_{\forall X A}) \quad \frac{A(X/Y)}{\forall X A} \text{ where } Y \text{ is an eigenvariable}\]

   \[(\text{Cut}_A) \quad \frac{A \quad \neg A}{\phi}\]

   \[(\text{Rep}_A) \quad \frac{\phi}{\phi}\]

2. $BI^1$ is obtained by adding the following rules to $BI^0$.

   \[(\Omega_{\neg \forall X A}) \quad \frac{\ldots \Delta^\forall X A(X) \ldots (q \in |\forall X A(X)|)}{\neg \forall X A}\]

   \[(\bar{\Omega}_{\neg \forall X A}) \quad \frac{A(Y) \quad \ldots \Delta^\forall X A(X) \ldots (q \in |\forall X A(X)|)}{\phi} \text{ where } Y \text{ is an eigenvariable}\]

with

(a) $|\forall X A(X)| := \{(d, X) | d \text{ is a cut-free derivation in } BI^0, X \notin FV(\Delta^\forall X A(X))\}$

(b) $\Delta^\forall X A(X) := \Gamma(d) \setminus \{A(X)\}$,
(c) $\Gamma(d)$ (the last sequent of $d$) is a $\Pi^1$-sequent.

**Definition 4** $| I |

Let $I$ be an inference symbol in $\text{BI}^\Omega$.

1. $|\text{Ax}| := \phi$.
2. $|\bigwedge_{A_0 \land A_1}| = |\text{Cut}| := \{0, 1\}$.
3. $|\bigvee_{\exists xA}| = |\text{Rep}| = |\bigwedge_{\forall XA(X)}| := \{0\}$.
4. $|\bigwedge_{\forall xA}| := \omega$.
5. $|\Omega_{\forall XA(X)}| := |\forall XA|$.
6. $|\tilde{\Omega}_{\forall XA(X)}| := \{0\} \cup |\forall XA(X)|$.

**Remark 2**

1. In the above definitions of $\Omega_{\forall XA(X)}$, and $\tilde{\Omega}_{\forall XA(X)}$, each $(d, X) = q$ represents implicitly a cut-free derivation

   \[
   \frac{\Delta_q, A(X)}{\Delta_q, \forall XA(X)} \bigwedge_{\forall XA}
   \]

2. $\Omega_{\forall XA(X)}$-rule comes from a constructive-reading of the formula $\neg \forall XA(X)$ as asserting the existence of a function which transforms any proof of $\forall XA(X)$ (say from the assumption $\Gamma$) into a proof of $\bot$ from the same assumption. In our situation, it turns out to be sufficient to consider only direct (cut-free) proof ending in $\bigwedge_{\forall XA}$ and possibly containing additional assumptions $\Delta_q$ where $\Delta_q$ is a $\Pi^1$-sequent.

   \[
   \frac{q : \Delta_q, A(X)}{\frac{\ldots, \Gamma, \Delta_q, \neg \forall XA(X) \ldots}{\Gamma, \neg \forall XA(X)} \Omega_{\forall XA(X)}
   \]

$d_q$ denotes the derivation of $\Gamma, \Delta_q$ for each $q \in |\forall XA(X)|$. A derivation $q$ of $\Delta_q, A(X)$ that determines the $q$-th premise $d_q$ of the derivation $d$ ending in $\Omega_{\forall XA(X)}$-rule is not by itself a part of $\Omega_{\forall XA(X)}$-rule. The $q$-th premise is the transformed derivation $d_q$ of $\Gamma, \Delta_q$. 

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3. The leftmost subderivation $d_0$ in $\tilde{\Omega}_{\forall XA(X)}$ denotes the derivation of $\Gamma, A(X)$. Therefore $\Omega_{\forall XA(X)}$ can be seen as a combination of $\land_{\forall XA}, \Omega_{\forall XA(X)}$, and $\text{Cut}_{\forall XA}$:

$$\begin{align*}
\{ q : \Delta_q, A(X) \} \\
\frac{\Gamma, \land Y \forall XA(X)}{\Gamma, \forall XA(X)} & (\land_{\forall XA}) \\
\frac{\ldots, \Gamma, \Delta_q \ldots}{\Omega_{\forall XA(X)}} & (\text{Cut}_{\forall XA(X)})
\end{align*}$$

The $\Omega_{\forall XA(X)}$-rule incorporates a hidden cut, but formally is not counted as a cut.

4. We write $\land Y \forall XA(X), \Omega_{\forall XA(X)}$ in order to indicate that $Y$ is the eigenvariable of these rules.

5. We drop sometimes subscripts in $\Delta_q, \Omega_{\forall XA(X)}, \tilde{\Omega}_{\forall XA(X)}$ when no confusion occurs.

6. An inference symbol $I$ is written as $I_{\Delta}$ to indicate that $\Delta$ is the set of principal formulas of $I$.

### 4 Cut-elimination Theorem for $\text{BI}^{\Omega}$

**Definition 5** *Cut-degree*

Let $I$ be an inference symbol, and $d$ be a derivation in $\text{BI}^{\Omega}$.

1. $dg(I) := \begin{cases} 
  rk(C) + 1 & \text{if } I = \text{Cut}_C; \\
  0 & \text{otherwise.}
\end{cases}$

2. $dg(I(d_{\tau \in I})) := \sup(\{dg(I)\} \cup \{dg(d_{\tau})| \tau \in I\}).$

**Remark 3** Notice that a derivation $d$ with $dg(d) = 0$ which may contain $\tilde{\Omega}$.

Let $d$ be a derivation in $\text{BI}^{\Omega}$. As usual, we write

$$d \vdash_m \Gamma$$

if $\Gamma(d) = \Gamma$, and $dg(d) \leq m$. We define an operator $\mathcal{R}_C$ which transforms impredicative cut into $\tilde{\Omega}$ and does one-step reduction for other cuts in the standard way.
\textbf{Theorem 1} There is an operator $\mathcal{R}_C$ on derivations in $\text{BI}^\Omega$ such that

If $d_0 \vdash_m \Gamma, C$, $d_1 \vdash_m \Gamma, \neg C$ and $rk(C) \leq m$, then $\mathcal{R}_C(d_0, d_1) \vdash_m \Gamma$.

\textbf{Proof.} We define $\mathcal{R}_C$, and prove its properties by double induction on $d_0$ and $d_1$. We consider subcases depending on $C$ or $\neg C$ being the principal formula of the last inference symbol of $d_0$ or $d_1$ respectively. Let $I_0$ and $I_1$ be the last inference symbols of $d_0$ and $d_1$.

1. $C \notin \Delta(I_0)$.

We define $\mathcal{R}_C(d_0, d_1) := I_0(\mathcal{R}_C(d_0, d_1))_{i \in \omega}$ using the induction hypothesis (IH). For example let $I_0$ be $\land \forall x \theta(x)$. Then $d_0 = \land \forall x \theta(x)(d_{0n})_{n \in \omega}$, and $d_{0n} \vdash_m \Gamma, A_0(n), C$. By IH, $\mathcal{R}_C(d_{0n}, d_1) \vdash_m \Gamma, A_0(n)$ for all $n \in \omega$. Therefore $\land \forall x \theta(x)(\mathcal{R}_C(d_{0n}, d_1))_{n \in \omega} \vdash_m \Gamma$.

2. $\neg C \notin \Delta(I_1)$.

Similar to 1, that is, $\mathcal{R}_C(d_0, d_1) := I_1(\mathcal{R}_C(d_0, d_1))_{i \in \omega}$.

3. $d_0$ is an axiom $C, \neg C$.

It follows that $\neg C \in \Gamma$. We define $\mathcal{R}_C(d_0, d_1) := \text{Rep}(d_1)$.

4. $d_1$ is an axiom $C, \neg C$.

Similarly to 3, we define $\mathcal{R}_C(d_0, d_1) := \text{Rep}(d_0)$.

5. $C \notin \Delta(I_0)$, and $\neg C \notin \Delta(I_1)$.

First, it is impossible that both $C$ and $\neg C$ are true literals. Second we may assume that $d_0$ or $d_1$ is not an axiom $C, \neg C$ (cf. the cases 3 and 4).

We consider subcases according to the last inference of $d_0$. We assume that $C$ is a negative formula. Other cases are treated similarly.

(a) $C \equiv C_0 \land C_1$.

Then $\neg C \equiv \neg C_0 \lor \neg C_1$, $d_0 = \land \neg C_0 \lor \neg C_1(d_{0n}, d_{01})$, and $d_1 = \lor k \neg C_0 \lor \neg C_1(d_{10})$ for some $k \in \{0, 1\}$. By IH, $\mathcal{R}_C(d_{0k}, d_1) \vdash_m \Gamma, C_k$, and $\mathcal{R}_C(d_{0}, d_{10}) \vdash_m \Gamma, \neg C_k$. We define $\mathcal{R}_C(d_0, d_1) := \text{Cut}_{C_k}(\mathcal{R}_C(d_{0k}, d_1), \mathcal{R}_C(d_0, d_{10}))$:

\[
\begin{array}{c}
\mathcal{R}_C(d_{0k}, d_1) \vdash_m \Gamma, C_k \\
\mathcal{R}_C(d_0, d_{10}) \vdash_m \Gamma, \neg C_k \\
\hline
\mathcal{R}_C(d_0, d_1) \vdash_m \Gamma
\end{array}
\]

(b) $C \equiv \forall x C_0$.

Then $d_0 = \land \forall x C_0(x)(d_{0n})_{n \in \omega}$, and $d_1 = \lor k \forall x \neg C_0(x)(d_{10})$ for $k \in \omega$. By IH, $\mathcal{R}_C(d_{0}, d_{10}) \vdash_m \Gamma, C_0(k)$, and $\mathcal{R}_C(d_{0n}, d_1) \vdash_m \Gamma, C_0(n)$ for all $n \in \omega$. We define $\mathcal{R}_C(d_0, d_1) := \text{Cut}_{C_0(k)}(\mathcal{R}_C(d_{0k}, d_1), \mathcal{R}_C(d_0, d_{10}))$:
Theorem 2. There is an operator \( \mathcal{E} \) symbol of \( \text{Cut} \). \( \mathcal{R}_C(d_0, d_1) \vdash_m \Gamma, C_0(k) \quad \mathcal{R}_C(d_0, d_1) \vdash_m \Gamma, \neg C_0(k) \quad \text{Cut}_{C_0(k)} \)

\[
\frac{\mathcal{R}_C(d_0, d_1) \vdash_m \Gamma, C_0(k) \quad \mathcal{R}_C(d_0, d_1) \vdash_m \Gamma, \neg C_0(k) }{\mathcal{R}_C(d_0, d_1) \vdash_m \Gamma} \]

c. \( C \equiv \forall X C_0(X) \).

\( d_0 = \bigwedge \forall X C_0(X)(d_{00}), \) and \( d_1 = \Omega(d_{1q})_{q \in [\forall X C_0(X)]} \). We had \( d_{00} \vdash_m \Gamma, C_0(Y), \forall X C_0(X), \) and \( d_{1q} \vdash_m \Gamma, \Delta_q^{\forall X C_0(X)}, \neg \forall X C_0(X) \). We just push \( \mathcal{R}_C \) into the premises, and introduce \( \Omega^Y \).

\( \mathcal{R}_C(d_0, d_1) := \)

\[
\mathcal{R}_C\left(\mathcal{R}_C(d_0, d_1), \mathcal{R}_C(d_0, d_{1q})_{q \in [\forall X C_0(X)]}\right)
\]

Let us check that all provisos are satisfied for the new \( \Omega^Y \)-inference. \( d_{00} \vdash_m \Gamma, C_0(Y), \forall X C_0(X), \) and \( d_{1q} \vdash_m \Gamma, \Delta_q^{\forall X C_0(X)} \), for each \( q \in [\forall X C_0(X)] \). Since \( Y \) is an eigenvariable, \( Y \) does not occur free in \( \Gamma \). By IH, \( \mathcal{R}_C(d_0, d_{1q}) \vdash_m \Gamma, \Delta_q^{\forall X C_0(X)} \) for all \( q \in [\forall X C_0(X)] \), and \( \mathcal{R}_C(d_0, d_1) \vdash_m \Gamma, C_0(Y) \).

Thus

\[
\frac{\mathcal{R}_C\left(\mathcal{R}_C(d_0, d_1), \mathcal{R}_C(d_0, d_{1q})_{q \in [\forall X C_0(X)]}\right) }{\mathcal{R}_C(d_0, d_1) \vdash_m \Gamma} \]

\( \square \)

Iterating \( \mathcal{R}_C \), we define an operator \( \mathcal{E} \) which reduces cut-degree by 1.

**Theorem 2**. There is an operator \( \mathcal{E} \) on derivations in \( \mathcal{B} \mathcal{D}^3 \) such that

1. if \( d \vdash_{m+1} \Gamma \), then \( \mathcal{E}(d) \vdash_m \Gamma \),
2. \( \Gamma(d) = \Gamma(\mathcal{E}(d)) \).

**Proof.** By Induction on \( d \). We consider two cases. Let \( I \) be the last inference symbol of \( d \).

1. \( I = \text{Cut}_C \).

   By IH, \( \mathcal{E}(d_0) \vdash_m \Gamma, C \), and \( \mathcal{E}(d_1) \vdash_m \Gamma, \neg C \). We define \( \mathcal{E}(d) := \text{Rep}(\mathcal{R}_C(\mathcal{E}(d_0), \mathcal{E}(d_1))) \):

\[
\frac{\mathcal{E}(d_0) \vdash_m \Gamma, C \quad \mathcal{E}(d_1) \vdash_m \Gamma, \neg C }{\mathcal{R}_C(\mathcal{E}(d_0), \mathcal{E}(d_1)) \vdash_m \Gamma} \quad \text{Rep}
\]
2. \( I \neq \text{Cut}_C \).

We define \( E(d) := \Pi \omega \) for example, let \( I = \Omega \), and \( d = \Omega \omega \).

\[
\forall X C_0(X) \quad \text{for all } q \in \forall X C_0(X).
\]

Therefore \( \Omega(d_q) \vdash \Pi \omega \) for all \( q \in |\forall X C_0(X)| \).

Using \( \mathcal{E} \), we can define an operator \( E_\omega \) which reduces cut-degree to 0.

**Theorem 3** There is an operator \( E_\omega \) on derivations in \( BI^\Omega \) such that

1. if \( d \vdash \omega \Gamma \), then \( E_\omega(d) \vdash 0 \Gamma \),
2. \( \Gamma(d) = \Gamma(E_\omega(d)) \).

**Proof.** By induction on \( d \). We consider subcases according to the last inference symbol of \( d \).

1. \( I = \text{Cut}_C \).

Then \( d_0 \vdash \omega \Gamma, C, \) and \( d_1 \vdash \omega \Gamma, \neg C \). By IH, \( \mathcal{E}(E_\omega(d_0)) \vdash 0 \Gamma, C \) and \( \mathcal{E}(E_\omega(d_1)) \vdash 0 \Gamma, \neg C \). Let \( mh(C) := m \), then \( \text{Cut}_C(E_\omega(d_0), E_\omega(d_1)) \vdash m+1 \Gamma \). Let \( \mathcal{E}^m \) denote \( m+1 \) applications of the operator \( \mathcal{E} \). We define

\[
\mathcal{E}(d) := \text{Rep}(\mathcal{E}^m(\text{Cut}_C(E_\omega(d_0), E_\omega(d_1)))) \vdash 0 \Gamma:
\]

\[
\mathcal{E}_\omega(d) := I(\mathcal{E}_\omega(d))_{i \in |I|}.
\]

2. \( I \neq \text{Cut}_C \).

Now define the collapsing operator \( D \) which eliminates \( \Omega \) if \( dg(d) = 0 \), and \( \Gamma(d) \) is a \( \Pi^1 \)-sequent.

**Theorem 4** There is an operator \( D \) such that

1. if \( d \vdash 0 \Gamma \) and \( \Gamma \) is a \( \Pi^1 \)-sequent, then \( BI^\Omega \ni D(d) \vdash 0 \Gamma \),
2. \( \Gamma(d) = \Gamma(D(d)) \).
**Proof.** By induction on \(d\).

First note that \(d\) does not contain \(\Omega\) since \(d\) is cut-free and \(\Gamma\) is a \(\Pi^1\)-sequent. Let \(I\) be the last inference symbol of \(d\). We consider subcases according to \(I\).

1. \(I = \tilde{\Omega}^Y\).

Let \(d = \tilde{\Omega}^Y(d_{\tau})_{\tau \in \{0\} \cup \{\forall X A(X)\}}\). Then \(Y \not\in FV(\Gamma)\), \(d_0 \vdash_0 \Gamma, A(Y)\), and \(d_{q} \vdash_0 \Gamma, \Delta_q^{\forall X A(X)}\) for all \(q \in \{\forall X A(X)\}:

\[
\frac{d_0 \vdash_0 \Gamma, A(Y) \ldots d_q \vdash_0 \Gamma, \Delta_q^{\forall X A(X)} \ldots}{d \vdash_0 \Gamma} \tilde{\Omega}^Y
\]

Note that \(\Gamma, A(Y)\) is a \(\Pi^1\)-sequent. We have \(B_0^{\Omega} \ni \mathcal{D}(d_0) \vdash \Gamma, A(Y)\) by IH, and \(Y \not\in FV(\Gamma(\mathcal{D}(d_0))\setminus\{A(Y)\})\). Define \(q_0 := (\mathcal{D}(d_0), Y)\). Then \(q_0 \in \{\forall X A(X)\}\). Since the last sequent of \(d_0\) is \(\Gamma, A(Y)\), the last sequent of \(d_{q_0}\) is \(\Gamma\). Hence, using IH again, we define

\[
\mathcal{D}(d) := \text{Rep}(\mathcal{D}(d_{q_0})) \in B_0^{\Omega}.
\]

2. \(I \not= \tilde{\Omega}^Y\).

We may write \(d\) as \(I(d_i)_{i \in |I|}\). By IH, \(\mathcal{D}(d_i) \ni B_0^{\Omega}\). Then, noting that \(I\) cannot be \(\Omega\) or \(\text{Cut}\), we define \(\mathcal{D}(d) := I(\mathcal{D}(d_i))_{i \in |I|} \in B_0^{\Omega}\).

**Corollary 1** If \(d \in B_0^{\Omega}\) and \(\Gamma(d)\) is a \(\Pi^1\)-sequent, then there exists \(d'\) such that \(\Gamma(d) = \Gamma(d')\) and \(d' \in B_0^{\Omega}\).

**Proof.**

Assume that \(d \vdash \Gamma\). Then \(E_\omega(d) \vdash_0 \Gamma\) by Theorem 3. Therefore \(B_0^{\Omega} \ni \mathcal{D}(E_\omega(d)) \vdash \Gamma\) by Theorem 4.

Finally we define an operator \(S^X_T\) such that \(S^X_T(d) \vdash \Gamma(d)[X/T]\) for \(d \in B_0^{\Omega}\).

**Theorem 5** There exists an operator \(S^X_T\) such that

if \(B_0^{\Omega} \ni d \vdash_0 \Gamma\), then \(B_0^{\Omega} \ni S^X_T(d) \vdash_0 \Gamma[X/T]\).

**Proof.** By induction on \(d\). Let \(d\) be \(I(d_i)_{i \in |I|}\). Note that \(I\) cannot be \(\text{Cut}_C\). If \(I_A\) is an inference symbol in \(B_0^{\Omega}\), then \(I_A[X/T]\) is also an inference symbol in \(B_0^{\Omega}\) with the index set \(|I_A[X/T]| = |I|\). In particular for \(A = \forall Y B\) we have \(A[X/T] = A\) because \(\forall Y B\) does not contain any free predicate variable by Definition 1. Thus we define \(B_0^{\Omega} \ni S^X_T(d) := I_A[X/T](S^X_T(d_i))_{i \in |I|}\). □
5 The Systems BI\(^{-}\), BI

In this section, we introduce a system BI\(^{-}\) which is an impredicative subsystem of \(\Pi^1_1\)-analysis with \(\omega\)-rule, and a system BI which is obtained by adding the rules \(R_A, E, E_\omega, D, Sub^T\) to BI\(^{-}\). These rules \(R_A, E, E_\omega, D, Sub^T\) correspond to the operators \(R_A, E, E_\omega, D, S^T\) which have been defined in the previous section.

We do not define \(\Delta(Sub^T), \Delta_0(Sub^T)\) where \(Sub^T\) is the substitution rule (cf. Remark 4 below). Eigenvariables may occur only in the premises, but not in the conclusion.

**Definition 6** The systems BI\(^{-}\) and BI

1. BI\(^{-}\) consists of the following rules.

   \[(Ax\Delta) \quad \Delta where \Delta = \{A\} \subseteq TRUE or \Delta = \{C, \neg C\}\]

   \[(\land_{A_0 \land A_1}) \quad \frac{A_0 \quad A_1}{A_0 \land A_1}\]

   \[(\lor_{A_0 \lor A_1}) \quad \frac{A_k}{A_0 \lor A_1} where k \in \{0, 1\}\]

   \[(\forall_x A) \quad \frac{\ldots A(x/n) \ldots}{\forall_x A} for all n \in \omega\]

   \[(\exists_x A) \quad \frac{A(x/k)}{\exists_x A} where k \in \omega\]

   \[(\land_{\forall X A}) \quad \frac{A(Y)}{\forall X A} where Y is an eigenvariable\]

   \[(\lor_{\forall X A}) \quad \frac{\neg A(X/T)}{\forall X A}\]

2. BI is obtained by adding the following rules to BI\(^{-}\).

   \[(R_A) \quad \frac{A \quad \neg A}{\phi}\]
Definition 7 \(|I|\)

Let \(I\) be an inference symbol in BI.

1. \(|\text{Ax}| := \phi.\)

2. \(|\wedge_{A_0 \wedge A_1}| = |\text{Cut}| = |\text{R}| := \{0, 1\}.\)

3. \(|\vee_{A_0 \vee A_1}^k| = |\vee_{3x A}^k| = |\wedge_{\forall X A (X)}^T| = |\wedge_{\forall X A}| = |E| = |E_\omega| = |D| = |\text{Sub}_{T X}| := \{0\}.\)

4. \(|\forall X A| := \omega.\)

Remark 4

1. As said before, we do not define \(\Delta(\text{Sub}_{T X}), \Delta_i(\text{Sub}_{T X})\), but we define directly \(\Gamma(\text{Sub}_{T X}(d_0)) := \Gamma(d_0)[X/T].\)

2. The rules \(R_A, E, E_\omega, D, \text{Sub}[X/T]\) allow to record intermediate steps of the cut-elimination process for BI. As we will see below, \(R_A, E, E_\omega, D, \text{Sub}[X/T]\) are replaced by the corresponding operations \(\mathcal{R}_A, \mathcal{E}, \mathcal{E}_\omega, \mathcal{D}, \mathcal{S}_{T X}\) in the translation of BI into BI\(^{-}\) (cf. Definition 9).

6 Cut-elimination Theorem for BI

In this section, we will give an ordinal-free proof of the cut-elimination theorem for BI using cut-elimination for BI\(^{-}\) proved in the previous section.

We define an embedding function \(g\) from derivations in BI into the derivations in BI\(^{-}\) (Section 6.1). Next we define functions \(tp(d), d[i]\) where \(d\) is an derivation in BI, such that \(tp(d)\) is the last inference symbol, and \(g(d[i])\) is the \(i\)-th immediate subderivation of \(g(d)\) (Section 6.2).

In Section 6.3, we first define reduction steps \(\text{red}(d)\). We give an ordinal-free proof of the cut-elimination theorem for BI. The cut-elimination theorem for BI\(^{-}\) is obtained as a corollary.
Finally, we see how the definition of red works (Section 6.4). In particular, red simulates Gentzen-Takeuti’s steps. See remark 15 in Section 6.4 which illustrates this in detail when \( g(d) \) ends in \( \Omega \). In this case, an informal connection between Takeuti’s reduction in [15] and Buchholz’s cut-reduction for \( \Omega \)-systems is observed.

### 6.1 Interpretation of BI in BI\(^{\Omega} \)

We define a function \( g \) which maps derivations in BI into the derivations in BI\(^{\Omega} \). Next we show that every \( D, Sub^{\Omega}_T \) can be interpreted as \( D, S^{\Omega}_T \) if \( d \) is a proper derivation (Definition 10).

We define \( dg(d) \) where \( d \) is a derivation in BI in such a way that \( dg(d) \leq dg(g(d)) \).

**Definition 8** \( dg(d) \)

Let \( d \) be a derivation in BI.

\[
dg(d) := \begin{cases} 
\text{max}(rk(A(T)), dg(d_0)) & \text{if } I = \bigwedge_{\forall XA(X)}; \\
\text{max}(rk(C) + 1, dg(d_0), dg(d_1)) & \text{if } I = \text{Cut}_C; \\
dg(d) := \text{max}(rk(C), dg(d_0), dg(d_1)) & \text{if } I = R_C; \\
dg(d_0) - 1 & \text{if } I = E; \\
0 & \text{if } I = E^\omega; \\
\sup\{dg(d_\tau) | \tau \in I\} & \text{otherwise}. 
\end{cases}
\]

As usual, we write \( d \vdash_m \Gamma \) for a derivation in BI if \( \Gamma(d) = \Gamma \) and \( dg(d) \leq m \).

**Definition 9** Embedding function \( g \) from BI into BI\(^{\Omega} \).

Let \( d \) be a derivation in BI. We define the function \( g \) by induction on \( d \).

1. \( g(Ax_\Delta) := Ax_\Delta \).
2. \( g(\bigwedge_{A_0 \land A_1}(d_0, d_1)) := \bigwedge_{A_0 \land A_1}(g(d_0), g(d_1)) \).
3. \( g(\bigvee_{A_0 \lor A_1}^k(d_0)) := \bigvee_{A_0 \lor A_1}^k(g(d_0)) \).
4. \( g(\bigwedge_{\forall y A}(d_n)_{n \in \omega}) := \bigwedge_{\forall y A}(g(d_n)_{n \in \omega}) \).
5. \( g(\bigvee_{\exists y A}^k(d_0)) := \bigvee_{\exists y A}^k(g(d_0)) \).
6. \( g(\bigwedge_{\forall X A}(d_0)) := \bigwedge_{\forall X A}(g(d_0)) \).
7. \( g(\bigvee_{\forall X A}^T(d_0)) := \Omega(\mathcal{R}_{A(T)}(S_2^X(d_q), g(d_0)))_{q \in \forall X A(X)} \),
   where \( (d_q, X) = q \in [\forall X A(X)] \).
8. \(g(Cut_C(d_0, d_1)) := Cut_C(g(d_0), g(d_1))\).

9. \(g(E(d_0)) := E(g(d_0))\).

10. \(g(E_\omega(d_0)) := E_\omega(g(d_0))\).

11. \(g(D(d_0)) := \begin{cases} D(g(d_0)) & \text{if } g(d_0) \vdash_0 \Gamma \text{ and } \Gamma \text{ is a } \Pi^1 \text{-sequent (cf. Theorem 4)}; \\ g(d_0) & \text{otherwise}. \end{cases}\)

12. \(g(Sub^X_{T}(d_0)) := \begin{cases} S^X_T(g(d_0)) & \text{if } BI^\Omega_0 \ni g(d_0) \vdash_0 \Gamma \text{ (cf. Theorem 5)}; \\ g(d_0) & \text{otherwise}. \end{cases}\)

13. \(g(R_C(d_0, d_1)) := R_C(g(d_0), g(d_1))\).

Remark 5

1. Let \(d = \bigvee_{\forall X A(X)}^X(d_0)\). Then \(d\) is transformed into \(g(d)\) in BI as follows:

2. Let \(\text{red}\) be a reduction relation in BI which will be defined later. Although our intention will become clear later, we note that the definition of \(\text{red}\) (cf. Definition 13) comes from the commutative diagram presented in the Introduction. For example, the following diagram holds (cf. Theorem 9):

Now we define the notion of proper derivation so that the operations \(D\) and \(S^X_T\) are applied only to subderivations satisfying the conditions in Theorems 4, 5 respectively.

Definition 10 Proper derivation

A derivation \(d\) in BI is called proper if for every subderivation \(h\) of \(d\),

1. if \(h = D(h_0)\), then \(dg(h_0) = 0\) and \(\Gamma(h_0)\) is a \(\Pi^1\)-sequent;
2. if \(h = Sub^X_T(h_0)\), then \(h_0 = D(h_{00})\).
Remark 6 Let \( d \) be a derivation in \( \text{BI}^- \). Then it is clear that \( d \) is a proper derivation because such a \( d \) does not contain \( D, \text{Sub}^X \).

Theorem 6 Let \( d \) be a proper derivation in \( \text{BI} \). Then \( g(d) \vdash_{dg(d)} \Gamma(d) \).

Proof. By induction on \( d \). Let \( \Gamma := \Gamma(d) \).

1. \( d = \text{Ax}_\Delta \). Trivial.

2. \( d = \bigvee_{A \in \Delta}^k (d_0) \).

   \[ d_0 \vdash \Gamma', A_k, A_0 \land A_1 \text{ for } k \in \{0, 1\} \]. By IH, \( g(d_0) \vdash_{dg(d_0)} \Gamma', A_k, A_0 \land A_1 \).

   Therefore \( \bigvee_{A \in \Delta}^k (g(d_0)) \vdash_{dg(d)} \Gamma \).

3. \( d = \bigwedge_{A \in \Delta}^k (d_0, d_1), \bigwedge_{\forall x A}^k (d_i), \bigvee_{\exists x A}^k (d_0), \text{ or } \bigwedge_{\forall x A}^k (d_0) \).

   Similar to 2.

4. \( d = \text{Cut}_A (d_0, d_1) \).

   \[ d_0 \vdash \Sigma, A, \text{ and } d_1 \vdash \Pi, \neg A \text{ with } \Sigma \cup \Pi = \Gamma \]. By IH, \( g(d_0) \vdash_{dg(d_0)} \Sigma, A, \text{ and } g(d_1) \vdash_{dg(d_1)} \Pi, \neg A \).

   By cut-rule, we obtain \( \text{Cut}_A (d_0, d_1) \vdash_{dg(d)} \Gamma \) since \( dg(d) = \max (rk(A) + 1, dg(d_0), dg(d_1)) \).

5. \( d = E(d_0) \).

   By IH, \( g(d_0) \vdash_{dg(d_0)} \Gamma \). By Theorem 2, we obtain \( E(g(d_0)) \vdash_{dg(d)} \Gamma \).

6. \( d = E_\omega (d_0) \).

   By IH, \( g(d_0) \vdash_{dg(d_0)} \Gamma \). By Theorem 3, we obtain \( E_\omega (g(d_0)) \vdash 0 \Gamma \). Note that \( dg(d) = 0 \).

8. \( d = D(d_0) \).

   \[ \Gamma(d_0) \] is a \( \Pi^1 \)-sequent, and \( dg(d_0) = 0 \) since \( d \) is proper. By IH, \( g(d_0) \vdash 0 \Gamma \).

   By Theorem 4, \( D(g(d_0)) \vdash_{dg(d)} \Gamma \). Note that \( dg(d) = 0 \).

9. \( d = \text{Sub}^X (d_0) \).

   \[ d_0 \] is of the form \( D(d_0) \) since \( d \) is proper. By IH and Theorem 4, \( \text{BI}_1^0 \ni \text{D}(g(d_0)) \vdash 0 \Gamma' \).

   Therefore, by Theorem 5, \( \text{Sub}^X (D(g(d_0))) \vdash_{dg(d)} \Gamma' [X/T] \).

   Note that \( dg(d) = 0 \).

10. \( d = R_A (d_0, d_1) \).

    \[ d_0 \vdash \Sigma, A, \text{ and } d_1 \vdash \Pi, \neg A \text{ with } \Sigma \cup \Pi = \Gamma \]. By IH, \( g(d_0) \vdash \Sigma, A, \text{ and } g(d_1) \vdash \Pi, \neg A \).

    By Theorem 1, we obtain \( R_A (d_0, d_1) \vdash_{dg(d)} \Gamma \) since \( dg(d) = \max (rk(A), dg(d_0), dg(d_1)) \). \( \square \)
6.2 Definition of $tp(d)$, and $d[i]$

We define below (Definition 12) $tp(d)$, and $d[i]$ where $i \in |tp(d)|$ for each proper derivation $d \in BI$ such that

1. $tp(d)$ is the last inference symbol of $g(d)$.
2. $d[i]$ is also a proper derivation in $BI$.
3. $g(d[i])$ is the $i$-th immediate subderivation of $g(d)$.

In fact, for $d$ with $tp(d) = \Omega$ or $\bar{\Omega}$, an element of the index set may be themselves a pair $q = (h, X)$. Therefore we need a preliminary definition.

**Definition 11** $|\forall X A|^{+}, |I|^{+}, g(q)$

We define $|\forall X A|^{+}$, $|I|^{+}$ where $|I|$ is an inference symbol of $BI^{\Omega}$ and $g(q)$ where $q = (h, X) \in |\forall X A|^{+}$ as follows:

1. $|\forall X A|^{+} := \{(h, X)| h$ is of the form $D(h_0)$ satisfying (a)-(c) below\}
   
   (a) $h$ is a proper derivation in $BI$,
   
   (b) $\Delta_{\forall X A(h, X)}^{\forall X A(X)} \cong \Gamma(h) \setminus \{A(X)\}$, and
   
   (c) $X /\notin FV(\Delta_{\forall X A(h, X)}^{\forall X A(X)})$.

2. $|I|^{+} := \begin{cases} |\forall X A|^{+} & \text{if } I = \Omega_{\forall X A(X)}; \\
\{0\} \cup |\forall X A|^{+} & \text{if } I = \bar{\Omega}_{\forall X A(X)}; \\
I & \text{otherwise}. \end{cases}$

3. $g(q) := (g(d), X)$ where $q = (d, X) \in |\forall X A|^{+}$.

**Remark 7** If $q \in |\forall X A|^{+}$, then we have $g(q) \in |\forall X A|$ by Theorem 6.

**Definition 12** $tp(d), d[i]$

By primitive recursion on $d$ we define $tp(d)$, and derivations $d[i]$ where $i \in |tp(d)|$ such that

1. $tp(d)$ is the last inference symbol of $g(d)$, and

2. $g(d[i]) = \begin{cases} g(d)_{g(i)} & \text{if } tp(d) \in \{\Omega_{\forall X A}, \bar{\Omega}_{\forall X A(X)}\}, \text{and } i \neq 0; \\
g(d)_{i} & \text{otherwise.} \end{cases}$

We assume *separation of eigenvariables*: all eigenvariables in $d$ are distinct and none of them occurs below the inference in which it is used as an eigenvariable.

1. $d = Ax_{\Delta}; tp(d) := Ax_{\Delta}$. 

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2. \( d = \bigwedge_{A_0 \land A_1}(d_0, d_1); tp(d) := \bigwedge_{A_0 \land A_1}, d[i] := d_i. \)

3. \( d = \bigvee_{A_0 \lor A_1}^k(d_0); tp(d) := \bigvee_{A_0 \lor A_1}^k, d[0] := d_0. \)

4. \( d = \bigwedge_{\forall x A}(d_i); tp(d) := \bigwedge_{\forall x A}, d[i] := d_i. \)

5. \( d = \bigvee_{\exists x A}^k(d_0); tp(d) := \bigvee_{\exists x A}, d[0] := d_0. \)

6. \( d = \bigwedge_{\forall x A}(d_0); tp(d) := \bigwedge_{\forall x A}, d[0] := d_0. \)

7. \( d = \bigvee_{\forall x A(X)}^k(d_0); tp(d) := \Omega_{\forall x A}, d[(h, X)] := R_{A(T)}(Sub_X^X(h), d_0). \)

8. \( d = \text{Cut}_A(d_0, d_1); tp(d) := \text{Cut}_A, d[i] := d_i. \)

9. \( d = E(d_0); \)
   (a) \( tp(d_0) = \text{Cut}_C; tp(d) := \text{Rep}, d[0] := R_C(E(d_0[0]), E(d_0[1])). \)
   (b) otherwise \( ; tp(d) = tp(d_0), d[i] := E(d_0[i]). \)

10. \( d = E_{\omega}(d_0); \)
    (a) \( tp(d_0) = \text{Cut}_C; tp(d) := \text{Rep}, d[0] := E^{n+1}(\text{Cut}_C(E_{\omega}(d_0[0]), E_{\omega}(d_0[1]))) \)
        where \( rk(C) = n, \) and \( E^{n+1} \) is \( n + 1 \) applications of \( E \)-rule.
    (b) otherwise \( ; tp(d) := tp(d_0), d[i] := E_{\omega}(d_0[i]). \)

11. \( d = D(d_0); \)
    (a) \( tp(d_0) = \text{Cut}_C; tp(d) := \text{Rep}, d[0] := D(d_0[(D(d_0[0]), Y)]). \)
    (b) otherwise \( ; tp(d) := tp(d_0), d[i] := D(d_0[i]). \)

12. \( d = \text{Sub}_X^X(d_0); tp(d) := \text{tp}(d_0), d[i] := \text{Sub}_X^X(d_0[i]). \)

13. \( d = R_A(d_0, d_1); \)
    (a) \( A \notin \Delta(tp(d_0)); tp(d) := tp(d_0), d[i] := R_A(d_0[i], d_1). \)
    (b) \( \neg A \notin \Delta(tp(d_1)); tp(d) := tp(d_1), d[i] := R_A(d_0, d_1[i]). \)
    (c) \( A \in \Delta(tp(d_0)), \) and \( \neg A \in \Delta(tp(d_1)); \)
        i. \( tp(d_0) = A_{X \Delta}; tp(d) := \text{Rep}, d[0] := d_1. \)
        ii. \( tp(d_1) = A_{X \Delta}; tp(d) := \text{Rep}, d[0] := d_0. \)
        iii. \( A = A_0 \land A_1; \) Then \( tp(d_0) = A_{X \Delta}, \) and \( tp(d_1) = \bigvee_{A_0 \lor \neg A_1}^k \)
            for \( k \in \{0, 1\}. \)
            \( tp(d) := \text{Cut}_{A_0}, d[0] := R_A(d_0[0], d_1[1]), d[1] := R_A(d_0, d_1[0]). \)
    iv. \( A = A_0 \lor A_1, \forall x A, \) or \( \exists x A : \) similarly to the case of \( A_0 \land A_1. \)
    v. \( A = \forall x A; \) Then \( tp(d_0) = \bigwedge_{\forall x A}^Y, \) and \( tp(d_1) = \Omega_{\forall x A(X)} \)
        \( tp(d) := \bigwedge_{\forall x A(X)}^Y, d[0] := R_{\forall x A}(d_0[0], d_1), d[q] := R_{\forall x A}(d_0, d_1[q]) \)
        for \( q \in [\forall x A]^+. \)
    vi. \( A = \exists x A; \) similarly to the case of \( \forall x A. \)
Remark 8

1. In the case 9-(a), we have defined $tp(d) = Rep$. This is explained as follows. If $tp(d_0) = Cut_C$, then $g(d_0) = Cuts C \cdot (g(d_0)_0, g(d_0)_1)$. Then, by the definition $E$ (cf. Theorem 2),

$$E(g(d_0)) = Rep(R_C(E(g(d_0)_0), E(g(d_0)_1))).$$

The case 10-(a) can be explained similarly.

2. The case 11-(a) is explained as follows. If $tp(d_0) = \tilde{\Omega_Y \neg \forall X A(X))$, then $g(d_0)$ is the following derivation:

$$\begin{array}{c}
\{ g : \Delta_q, A(X) \} \\
\vdots \\
\Gamma, A(Y) \quad \ldots \quad \Gamma, \Delta_q \ldots \\
\vdots \\
\tilde{\Omega_Y \neg \forall X A(X)}
\end{array}$$

By Theorem 4, $D(g(d_0)) = Rep(D(g(d_0)_0))$ with $q_0 = (D(g(d_0)_0), Y)$. Note that $D(g(d_0)) \vdash \Gamma, A(Y)$. Thus we have defined $tp(d) = Rep$.

3. In the case 13-(c)-v, $tp(d_0) = \tilde{\Omega_Y \forall X A(X)}$, and $tp(d_1) = \Omega_Y \forall X A(X))$. Then, by the definition of $R$, we introduce $\tilde{\Omega_Y \forall X A(X)}$ (cf. Theorem 1). Thus we have defined $tp(d) = \tilde{\Omega_Y \forall X A(X)}$.

Theorem 7 Let $d$ be a proper derivation in BI with $d \vdash_m \Gamma$. Then the following conditions are satisfied:

1. If $Y$ is an eigenvariable of $tp(d)$, then $Y$ is an eigenvariable in $d$, and $Y \notin FV(\Gamma(d))$.

2. $d[i]$ is also a proper derivation in BI for all $i \in |tp(d)|$.

3. $g(d[i]) = \begin{cases} g(d)_g(i) & \text{if } tp(d) \in \{ \Omega_Y \forall X A(X), \tilde{\Omega_Y \forall X A(X)} \}, \text{and } i \neq 0. \\ g(d)_i & \text{otherwise.} \end{cases}$

4. $d[i] \vdash_m \Gamma, \Delta_i(tp(d))$ for all $i \in |tp(d)|$.

5. If $tp(d) = Cut_A$, then $rk(A) < dg(d)$.

Proof.

First we show 1. Assume that $Y$ is an eigenvariable of $tp(d)$. Then we can prove that $Y$ is an eigenvariable in $d$ by induction on $d$. Thus $Y \notin FV(\Gamma(d))$ follows from separation of eigenvariables. 2-5 are proved simultaneously by induction on $d$. We consider only the interesting cases.
1. \( d = \bigwedge_{A_0, A_1} (d_0, d_1) \) Then \( tp(d) = \bigwedge_{A_0, A_1}, d[i] = d_i. \)

Let \( i \in \{0, 1\} \). Then \( d[i] = d_i \) is proper since \( d \) is proper. \( g(d[i]) = g(d_i) \) is the \( i \)-th immediate subderivation of \( g(d) = \bigwedge_{A_0, A_1} (g(d_0), g(d_1)). \) \( d[i] = d_i \vdash_m \Gamma, A_i, \bigwedge_{A_0, A_1} (d_0, d_1). \) Thus the claims hold for this case.

2. \( d = \bigvee_{X \subseteq A(X)} (d_0). \)

Then \( tp(d) = \Omega \), and \( d[(h, X)] = R_{A(T)}(Sub^X_T (h), d_0) \) for \( (h, X) \in \forall X A(X)^+. \)

To show that \( d[(h, X)] \) is proper, it suffices to show that \( Sub^X_T (h) \) is proper because \( d_0 \) is proper. By the definition, \( h \) is a proper derivation of the form \( D(h_0). \) Thus \( Sub^X_T (h) \) is also proper.

Next we show \( g(d[(h, X)]) = g(d) g((h, X))). \) We have

\[
\begin{align*}
g(d) &= \Omega (R_{A(T)}(S^X_T (g), g(d)))_{g \in \forall X A(X)}. \\
g(d[(h, X)]) &= R_{A(T)}(S^X_T (g(h)), g(d_0)).
\end{align*}
\]

Therefore \( g(d[(h, X)]) = g(d) g((h, X))) \) because \( g((h, X)) \in \forall X A(X) \) (cf. Remark 7).

Finally we prove \( d[(h, X)] \vdash_m \Gamma, \neg \forall X A(X), \Delta_{(h, X)} \). It is easy to see that \( Sub^X_T (h) \vdash_0 \Delta_{(h, X)}, A(T) \) since \( h \vdash_0 \Delta_{(h, X)} \), \( A(X) \) and \( X \notin FV(\Delta_{(h, X)}). \)

Thus \( d[(h, X)] = R_{A(T)}(Sub^X_T (h), d_0) \vdash_m \Gamma, \neg \forall X A(X), \Delta_{(h, X)}. \)

3. \( d = Cut_{A} (d_0, d_1). \)

By the definition, \( tp(d) = Cut_{A} \), and \( d[i] = d_i. \) \( d[i] \) is proper because \( d \) is proper. It easy to see that \( g(d[i]) = g(d_i) \) is the \( i \)-th immediate subderivation of \( g(d) \) because \( g(d) = Cut_{A}(g(d_0), g(d_1)). \) It is clear that \( d[i] = d_i \vdash_m \Gamma, (\neg)A \) where \( (\neg)A \) denotes \( A \) or \( \neg A \). Finally \( dg(d) > rk(A) \) by Definition 8.

4. \( d = E(d_0). \) We consider two subcases.

(a) \( tp(d_0) = Cut_C. \)

\( tp(d) = Rep, d[0] = R_C(E(d_0[0]), E(d_0[1])). \) Then \( d[0] \) is proper because \( d_0[0], d_0[1] \) are proper by IH.

Next we show that \( g(d[0]) = g(d_0). \) Observe that

\[
\begin{align*}
g(d) &= g(E(d_0)) = E(Cut_C(g(d_0_0), g(d_0_1))) = Rep(R_C(E(g(d_0_0)), E(g(d_0_1)))). \\
\end{align*}
\]

On the other hand, using IH,

\[
\begin{align*}
g(d[0]) &= R_C(E(g(d[0])), E(g(d_0[1]))) = R_C(E(g(d_0_0)), E(g(d_0_1))).
\end{align*}
\]

We have \( d_0[0] \vdash_m \Gamma, C, \) and \( d_0[1] \vdash_m \Gamma, \neg C \) by IH. Thus \( E(d_0[0]) \vdash_m \Gamma, C, \) and \( E(d_0[1]) \vdash_m \Gamma, \neg C. \) By IH, \( rk(C) < dg(d_0) \) \( \leq m + 1. \) Therefore \( d[0] \vdash_m \Gamma. \)
5. \( d = D(d_0) \).

We consider only the interesting subcase in which \( tp(d_0) = \bar{\Omega} \) (cf. Remark 8.2). Since \( d \) is proper, \( tp(d_0) \not\in \{ \Omega, \text{Cut}_C \} \). By Theorem 6, \( g(d_0) \vdash_0 \Gamma(d_0) \). Thus \( tp(d_0) \) cannot be \( \text{Cut}_C \). Since \( \Gamma(d) \) is a \( \Pi^1 \)-sequent, \( tp(d_0) \) cannot be \( \Omega \).

First we show that \( d[0] = D(d_0([D(d_0[0]), Y])) \) is a proper derivation in BI. Indeed by IH, \( d_0[0] \) is a proper derivation with \( d_0[0] \vdash_0 \Gamma \), \( A(Y) \). Note that \( \Gamma, A(Y) \) is a \( \Pi^1 \)-sequent. Then \( d_0([D(d_0[0]), Y]) \vdash_0 \Gamma \) by IH because \( (D(d_0[0]), Y) \in |\forall X A(X)|^+ \). Thus \( d[0] \) is also a proper derivation. This argument shows that \( d[0] \vdash_0 \Gamma \) too.

Next we prove \( g(d[0]) = g(d_0) \). By Theorem 4,

\[
g(d) = D_0(\bar{\Omega}(g(d_0)_i)) = \text{Rep}(D_0(g(d_0)_q))
\]

with \( q = (D_0(g(d_0)_0), Y) \). On the other hand, using IH,

\[
g(d[0]) = D_0(g(d_0([D_0(d_0[0]), Y]))) = D_0(g(d_0)) = D_0(g(d_0)) = D_0(g(d_0)) \text{ by IH}.
\]

6. \( d = \text{Sub}^X_T(d_0) \).

In this case, \( g(d) = S^X_T(g(d_0)), tp(d) = tp(d_0)_Y[A(X)], d[i] = \text{Sub}^X_T(d_0[i]) \). \( d_0 \) is of the form \( D(d_{00}) \) and is proper since \( d \) is proper. We consider subcases according to \( tp(d_0) \). By Theorems 4, 6 we have \( \text{BI}^0_0 \vdash g(d_0) = D(g(d_{00})) \vdash_0 \Gamma \). Thus we have \( tp(d_0) \not\in \{ \Omega, \bar{\Omega}, \text{Cut}_C \} \).

The claims follow now from IH. For example, let \( tp(d_0) = \bigwedge_{\forall Z A(Z)}^Y \). Then

\[
\text{tp} d = \text{tp}(d_0)_{\Delta[X/T]} = \bigwedge_{\forall Z A(Z)}^Y \text{because} \forall Z A(Z) \text{ does not contain any free predicate variable.}
\]

Note that \( d_0[0] = D(d_{00}[0]) \). \( d_0[0] \) is proper, and is of the form \( D(h) \) for some \( h \) by IH, and hence \( d[0] \) is proper.

We show \( g(d[0]) = g(d_0) \). By IH

\[
g(d[0]) = S^X_T(g(d_0[0])) = S^X_T(g(d_0)).
\]

Indeed by Theorem 5,

\[
g(d) = S^X_T(g(d_0)) = S^X_T(\bigwedge_{\forall Z A(Z)}^Y (g(d_0))) = \bigwedge_{\forall Z A(Z)}^Y (S^X_T(g(d_0))).
\]

Next we show \( d[0] \vdash_m \Gamma[X/T], \forall Z A(Z), A(Y) \). By IH, we have \( d_0[0] \vdash_m \Gamma, \forall Z A(Z), A(Y) \). As remarked above, \( \forall Z A(Z) \) does not contain any free predicate variable. Since \( Y \) is an eigenvariable in \( d \) by Theorem 7.1, it follows that \( X \neq Y \) by separation of eigenvariables. Therefore \( d[0] = \text{Sub}^X_T(d_0[0]) \vdash_m \Gamma[X/T], \forall Z A(Z), A(Y) \).
7. \( d = R_A(d_0, d_1) \).

\( d_0 \) and \( d_1 \) are proper because \( d \) is proper. We assume that \( A \in \Delta(tp(d_0)) \), and \( \neg A \in tp(d_1) \). We consider only few cases.

(a) \( A = A_0 \land A_1 \).

\[ tp(d_0) = \bigwedge_{A_0 \land A_1} \land \bigvee_{\neg A_0 \lor \neg A_1} \text{ for some } k \in \{0, 1\}, \quad \text{tp}(d) = \text{Cut}_{\Delta_k}, \quad d[0] = R_A(d_0[k], d_1), \quad \text{and } d[1] = R_A(d_0, d_1[0]). \]

It is easy to show that \( d[0], d[1] \) are proper, and \( g(d[i]) = g(d) \) for \( i \in \{0, 1\} \) by IH, and Theorem 1.

By IH, we have \( d_0[k] \vdash_m \Gamma, A_k, A_0 \land A_1 \), and \( d_1[0] \vdash_m \Gamma, \neg A_k, A_0 \land A_1 \).

Thus \( d[0] \vdash_m \Gamma, A_k \), and \( d[1] \vdash_m \Gamma, \neg A_k \). Note that \( \text{rk}(A) \leq dg(d) \leq m \) by Definition 8.

Since \( \text{rk}(A_k) < \text{rk}(A_0 \land A_1) \), it follows that \( \text{rk}(A_k) < dg(d) = \max(\text{rk}(A), dg(d_0), dg(d_1)). \)

(b) \( A = \forall X A \).

\[ tp(d_0) = \bigwedge_{\forall X A} \land \bigvee_{\neg \forall X A}, \quad \text{tp}(d_1) = \bigwedge_{\forall X A} \land \bigvee_{\neg \forall X A}. \]

Let \( B = \forall X A(X) \). Then \( g(d) \) is of the following form (cf. Theorem 1):

\[
\frac{q : \Delta_q, A(X)}{\Gamma, A(Y) \vdash X A(X) \quad \Gamma, \neg \forall X A(X)} \quad \frac{R_B \Gamma, \forall X A(X) \quad \Gamma, \neg \forall X A(X)}{\Gamma} \quad \frac{\Gamma, A(Y) \vdash Y A(X)}{\Gamma} \quad \frac{\Gamma, \neg \forall X A(X)}{\neg \forall X A(X)} \quad \frac{R_B \Gamma, \forall X A(X) \quad \Gamma, \neg \forall X A(X)}{\Gamma}
\]

where \( g(d_0) \vdash \Gamma, A(Y) \), and \( g(d_0) \vdash \Gamma, \Delta_q, \neg \forall X(X) \).

\( d[0] = \text{Re}_{\forall X A}(d_0[0], d_1), \quad d[q] = \text{Re}_{\forall X A}(d_0, d_1[q]) \) for \( q \in [\forall X A]^+ \). It is easy to show that \( d[0], d[1] \) are proper.

We show that \( g(d[i]) = g(d) \) for \( i \in \{0\} \cup [\forall X A] \). On the one hand,

\[ g(d) = \bigwedge_{\forall X A} \bigwedge_{\forall X A}(g(d_0), g(d_1), \ldots) \text{ for } q \in [\forall X A]. \]

On the other hand, using IH,

\[ g(d[0]) = \text{Re}_{\forall X A}(g(d_0), g(d_1)), \quad g(d[q]) = \text{Re}_{\forall X A}(g(d_0), g(d_1[q])) \text{ for } q \in [\forall X A]^+. \]

Therefore \( g(d[i]) = g(d) \) for \( i \in \{0\} \cup [\forall X A] \).

By IH, we have \( d_0[0] \vdash_m \Gamma, \forall X A, A(Y) \), and \( d_1[q] \vdash_m \Gamma, \neg \forall X A(X), \Delta_q \) for all \( q \in [\forall X A]^+ \). Thus \( d[0] \vdash_m \Gamma, A(Y) \), and \( d[1] \vdash_m \Gamma, \Delta_q \) for all \( q \in [\forall X A]^+ \). \( \Box \)
6.3 Proof of the Cut-elimination Theorem for BI

In this section we prove the cut-elimination theorem for BI. Assume that

\[ d \text{ is a proper, derivation in BI, } \Gamma(d) \text{ is a } \Pi^1 \text{-sequent and } dg(d) = 0. \] \hfill (2)

Now recall the diagram (1) presented in the Introduction:

\[
\begin{align*}
\text{BI} : d & \xrightarrow{\text{red}} \text{red}(d) \\
g & \downarrow g \\
\text{Bi}^\Omega : g(d) & \xrightarrow{\geq} g(\text{red}(d)) 
\end{align*}
\]

We define this cut-reduction relation \( \text{red} \) between derivations satisfying (2) such that \( g(\text{red}(d)) \) is the result of taking an immediate subderivation of \( g(d) \), and \( \text{red}(d) \) corresponds to the result of applying the Gentzen-Takeuti reduction steps to \( d \). The first condition says that the relation \( \text{red} \) satisfies the diagram (1). In order to define \( \text{red} \), we need the following lemma:

**Lemma 1** If \( d \) is a derivation in BI satisfying (2), then \( tp(d) \notin \{\text{Cut}_C, \Omega\} \).

**Proof.** By Theorem 6, \( g(d) \vdash_0 \Gamma(d) \). Thus \( tp(d) \notin \{\text{Cut}_C, \Omega\} \). \( \square \)

Let \( \text{last}(d) \) denote the last inference symbol of \( d \).

**Definition 13** \( \text{red}(d) \)

Let \( d \) be a derivation in BI satisfying (2). We define \( \text{red}(d) \) such that \( \text{red}(d) \) is also a proper derivations in BI, and \( \Gamma(d) = \Gamma(\text{red}(d)) \) by induction on \( d \). We consider subcases according to \( tp(d) \). By Lemma 1, \( tp(d) \notin \{\text{Cut}_C, \Omega\} \). Let \( I = \text{last}(d) \). We consider separately the cases according to whether \( I = \text{tp}(d) \) or not.

\[
\text{red}(d) = \begin{cases} 
\text{Ax}_{\Delta} & \text{if } \text{tp}(d) = \text{Ax}_{\Delta}; \\
I(\text{red}(d_0)) & \text{if } \text{tp}(d) = I \in \{ \bigvee_{A_0 \land A_1}^k, \bigwedge^k_{X \land A}, \bigvee_{X \land A}^Y \}; \\
\bigwedge_{X \land A}(\text{red}(d_0), \text{red}(d_1)) & \text{if } \text{tp}(d) = I = \bigwedge_{X \land A}; \\
\text{tp}(d)(d[0]) & \text{if } I \neq \text{tp}(d) \in \{ \bigvee_{A_0 \land A_1}^k, \bigwedge_{X \land A}^Y \}; \\
\text{tp}(d)(d[0], d[1]) & \text{if } I \neq \text{tp}(d) = \bigwedge_{A_0 \land A_1}; \\
\text{tp}(d)(d[1])_{i \in \omega} & \text{if } I \neq \text{tp}(d) = \bigwedge_{X A}; \\
\text{tp}(d)[d[0]] & \text{if } \text{tp}(d) = \text{Rep}; \\
\text{tp}(d)[D(d[0]), Y] & \text{if } \text{tp}(d) = \tilde{\Omega}^Y_{X \land A(X)};
\end{cases}
\]

**Remark 9**

1. We explain the last clause. If \( \text{tp}(d) = \tilde{\Omega}^Y \), then \( g(d) \) is of the following form:
Then \( d[0] \vdash_0 \Gamma, A(Y) \). Since \( \Gamma \) is a \( \Pi^1 \)-sequent, \( \Gamma, A(Y) \) is also a \( \Pi^1 \)-sequent. Therefore we see that \( q = (D(d[0]), Y) \in |\forall X A(X)|^+ \), and hence \( d[q] \vdash_0 \Gamma(d) \) by Theorem 7.4.

2. If \( d \) satisfies (2), then \( E \), or \( E_\omega \)-rule has to occur under every cut-rule in \( d \).

3. We see how Definition 13 works in the next section (Theorem 9-16).

**Definition 14** Let \( d \) be a derivation \( I(d_i)_{i \in |I|} \) in \( \text{BI}^{\Pi^1} \). Then \( |d| \) (the ordinal height of \( d \)) is defined by \( |d| := \sup(|d_i| + 1)_{i \in |I|} \).

Then \( |d'| < |d| \) if \( |d'| \) is one of the immediate subderivations of \( d \). In the "essential" cases \( tp(d) \in \{ \text{Rep}, \tilde{\Omega}^Y \} \), the height of \( g(red(d)) \) is smaller than one of \( g(d) \) (Lemma 2).

**Lemma 2** Assume that \( d \) satisfies (2), and \( tp(d) \in \{ \text{Rep}, \tilde{\Omega}^Y_{\neg \forall X A(X)} \} \). Then \( |g(d)| > |g(red(d))|, dg(d) \geq dg(red(d)) \) and \( \Gamma(d) = \Gamma(red(d)) \).

**Proof.** Assume that \( tp(d) \in \{ \text{Rep}, \tilde{\Omega}^Y_{\neg \forall X A(X)} \} \). We show first that \( |g(d)| > |g(red(d))| \). By the assumption and Definition 13, \( red(d) = d[0] \) or \( d[q] \) for \( q = (D(d[0]), Y) \). Note that \( g(red(d)) \) is one of the immediate subderivation of \( g(d) \) by Theorem 7.3. Thus it follows that \( |g(d)| > |g(red(d))| \).

Next we prove that \( dg(d) \geq dg(red(d)) \) and \( \Gamma(d) = \Gamma(red(d)) \). By Theorem 7.4, \( d[i] \vdash_0 \Gamma, \Delta_{i}(tp(d)) \) for all \( i \in |tp(d)| \). Note that \( dg(d) = 0 \). If \( tp(d) = \text{Rep} \), then \( red(d) = d[0] \vdash_0 \Gamma \) because \( \Delta_0(\text{Rep}) = \phi \).

If \( tp(d) = \tilde{\Omega}^Y_{\neg \forall X A(X)} \), we have \( d[q] \vdash_0 \Gamma, \Delta_{q}(\tilde{\Omega}^Y_{\neg \forall X A(X)}) \) where \( q = (D(d[0]), Y) \). Again by Theorem 7.3, \( D(d[0]) \vdash_0 \Gamma, A(Y) \). Thus \( \Delta_{q}(\tilde{\Omega}^Y_{\neg \forall X A(X)}) = \{ \Gamma, A(Y) \} \). Therefore \( d[q] \vdash_0 \Gamma \). \( \Box \)

Call a derivation \( d \) in \( \text{BI} \) is cut-free if \( d \) does not contain \( \text{Cut}, R \), and \( \text{red}(d) = d \).

**Theorem 8** Let \( d \) be a proper derivation in \( \text{BI} \) such that \( \Gamma(d) \) is a \( \Pi^1 \)-sequent, and \( dg(d) = 0 \). Then there exists a cut-free derivation \( d' \) in \( \text{BI} \) such that \( \Gamma(d) = \Gamma(d') \).

**Proof.** By induction on \( |g(d)| \). We consider subcases according to \( tp(d) \).

By Lemma 1, \( tp(d) \neq \text{Cut}_{\mathcal{C}}, \Omega \).
1. \( tp(d) = \text{Rep.} \)

By Lemma 2, \( |g(\text{red}(d))| < |g(d)| \), and \( \Gamma(\text{red}(d)) = \Gamma(d) \) where \( \text{red}(d) = d[0] \). By IH, there exists a cut-free derivation \( d[0]' \) such that \( \Gamma(d[0]') = \Gamma(\text{red}(d)) \). Take \( d' := d[0]' \). Note that \( \text{red}(d') = \text{red}(d[0]') = d[0]' = d' \).

\( \Gamma(d') = \Gamma(d[0]') = \Gamma(\text{red}(d)) = \Gamma(d) \).

2. \( tp(d) = \text{Ax}_\Delta \)

We take \( \text{Ax}_\Delta \) as \( d' \).

3. \( d = I(d_0) \), and \( I = \bigvee_{A_0 \wedge A_1} \bigvee_{\exists x A} \bigwedge_{\forall x A} \).

Note that \( d_0 \) is a proper derivation. Using IH, take \( d' := I(d_0) \). Note that \( \text{red}(d') = \text{red}(I(d_0)) \). \( I(d_0) = d' \). \( \Gamma(d') = \Gamma(I(d_0)) = \Gamma(d) \).

4. \( d = \Delta_{\forall x A} \)

\( d' := \Delta_{\forall x A}(d'_i)_{i \in \omega} \).

5. \( d = \bigwedge_{A_0 \wedge A_1} (d_0, d_1) \)

\( d' := \bigwedge_{A_0 \wedge A_1} (d'_0, d'_1) \).

6. \( \text{last}(d) \neq \text{tp}(d) \) where \( \text{tp}(d) = \bigvee_{A_0 \wedge A_1} \bigvee_{\exists x A} \bigwedge_{\forall x A} \).

By Theorem 7.3, \( |g(d[0])| < |g(d)| \). Using IH, take \( d' := \text{tp}(d)(d[0]') \). Note that \( \text{red}(\text{tp}(d)(d[0]')) = \text{tp}(\text{red}(d[0]')) = \text{tp}(d)(d[0]') = d' \). \( \Gamma(d') = \Gamma(\text{tp}(d)(d[0]')) = \Gamma(d) \) by Theorem 7.4.

7. \( \text{last}(d) \neq \text{tp}(d) \) where \( \text{tp}(d) = \bigwedge_{A_0 \wedge A_1} \).

Take \( d' := \bigwedge_{A_0 \wedge A_1} (d[0]', d[1]') \).

8. \( \text{last}(d) \neq \text{tp}(d) \) where \( \text{tp}(d) = \Delta_{\forall x A} \).

Take \( d' := \Delta_{\forall x A}(d[i])_{i \in \omega} \).

9. \( \text{tp}(d) = \Omega^X \).

|\( g(\text{red}(d)) \) \( |g(d)\) \( \Gamma(\text{red}(d)) = \Gamma(d) \) by Lemma 2. Recall that \( \text{red}(d) = d[q] \) with \( q = (D(d[0]), X) \). By IH, there exists a cut-free derivation \( d[q]' \) such that \( \Gamma(d[q]') = \Gamma(\text{red}(d)) \). Take \( d' := d[q]' \). Note that \( \text{red}(d') = \text{red}(d[q]') = d[q]' = d' \). \( \Gamma(d') = \Gamma(d[q]') = \Gamma(\text{red}(d)) = \Gamma(d) \).

Corollary 2 Let \( d \) be a proper derivation in BI such that \( \Gamma(d) \) is a \( \Pi^1 \)-sequent. Then there exists a cut-free derivation \( d' \) in BI such that \( \Gamma(d) = \Gamma(d') \).

**Proof.** Since \( dg,E_\omega(d) = 0 \), there exists a cut-free derivation \( d' \) such that \( \Gamma(d) = \Gamma(d') \) by Theorem 8. \( \square \)

Recall that BI\(^-\) is obtained from BI by deleting \( R_C \), \( E \), \( E_\omega \), \( D \), \( \text{Sub} \). A derivation \( d \) in BI\(^-\) is cut-free if \( d \) does not contain \( \text{Cut} \). The cut-elimination theorem for BI\(^-\) now follows.
Corollary 3 Let $d$ be a derivation in $BI$ such that $\Gamma(d)$ is a $\Pi^1$-sequent. Then there exists a cut-free derivation $d'$ in $BI$ such that $\Gamma(d) = \Gamma(d')$.

Proof. Note that $E_\omega(d)$ is a proper derivation in $BI$ with $dg(d) = 0$. Therefore, by Corollary 2, there exists a cut-free derivation $d' \in BI$ such that $\Gamma(d) = \Gamma(d')$. By deleting $E, E_\omega, D, Sub$ from $d'$, we obtain the required derivation. □

Remark 10 Corollary 2, and 3 are not just normal form theorems because we can record intermediate steps of the cut-elimination as we will see in Theorems 9-16 in the next section.

6.4 Connection with Gentzen-Takeuti’s Reduction

In this section, we see that $red$ simulates Gentzen-Takeuti’s steps. Especially, an informal connection between Takeuti’s reduction in [15] and Buchholz’s cut-reduction for $\Omega$-systems is observed (cf. Theorem 16 and Remark 15).

Theorem 9 If $d = E(Cut_C(d_0, d_1))$, then $red(d) = R_C(E(d_0), E(d_1))$, and $|g(d)| > |g(red(d))|$.

Proof. First

$$g(d) = g(E(Cut_C(d_0, d_1))) = E(Cut_C(g(d_0), g(d_1))) = Rep(R_C(E(g(d_0)), E(g(d_1))))$$

Thus $tp(d) = Rep$. Thus $|g(d)| > |g(red(d))|$ follows from Lemma 2. $red(d) = d[0] = R_C(E(d_0), E(d_1))$. □

Remark 11 Since $g(red(d)) = R_C(E(g(d_0)), E(g(d_1)))$, the following diagram commutes:

$$
\begin{array}{ccc}
E(Cut_C(d_0, d_1)) & \xrightarrow{red} & R_C(E(d_0), E(d_1)) \\
g \downarrow & & \downarrow g \\
R_C(E(g(d_0)), E(g(d_1))) & \xrightarrow{\sim} & R_C(E(g(d_0)), E(g(d_1)))
\end{array}
$$

Theorem 10 If $d = R_C(d_0, d_1)$, and $d_0 = Ax_{\Delta}$, then $red(d) = d_1$, and $|g(d)| > |g(red(d))|$.

Proof. First

$$g(R_C(d_0, d_1)) = R_C(g(d_0), g(d_1)) = R_C(Ax_{\Delta}, g(d_1)) = Rep(g(d_1)).$$

Hence $tp(d) = Rep$. Thus $|g(d)| > |g(red(d))|$ follows from Lemma 2. We have $red(d) = d[0] = d_1$ by Definition 13. □

Theorem 11 Assume that $d = E(R_{C_\omega \land C_1}(\land_{C_\omega \land C_1}(d_{000}, d_{001}), \lor^{k}_{C_\omega \lor C_1}(d_{010})))$. Then
1. \( \text{red}(d) = R_{C_k}(E(R_C(g(d_{00k}), g(d_{01}))), E(R_C(g(d_{00}), g(d_{010})))) \), and
\[ |g(d)| > |g(\text{red}(d))| \]

**Proof.** First note that
\[
g(d) = g(E(R_C(\bigwedge_{C_0 \land C_1} (d_{000}, d_{001}), \bigvee_{C_0 \lor \neg C_1}^k (d_{010}))))
\]
\[
= \mathcal{E}(g(R_C(\bigwedge_{C_0 \land C_1} (d_{000}, d_{001}), \bigvee_{C_0 \lor \neg C_1}^k (d_{010}))))
\]
\[
= \mathcal{E}(R_C(\bigwedge_{C_0 \land C_1} (g(d_{000}), g(d_{001})), \bigvee_{C_0 \lor \neg C_1}^k (g(d_{010}))))
\]
\[
= \mathcal{E}(\text{Cut}_{C_k}(R_C(g(d_{00k}), g(d_{01})), R_C(g(d_{00}), g(d_{010}))))
\]
\[
= \text{Rep}(R_{C_k}(\mathcal{E}(R_C(g(d_{00k}), g(d_{01}))), \mathcal{E}(R_C(g(d_{00}), g(d_{010}))))).
\]
Thus \( \text{tp}(d) = \text{Rep} \), and \( |g(d)| > |g(\text{red}(d))| \) follows from Lemma 2. We have
\[
\text{red}(d) = d[0] = R_{C_k}(E(R_C(g(d_{00k}), g(d_{01}))), E(R_C(g(d_{00}), g(d_{010})))).
\]

\[ \square \]

**Remark 12** Note that \( \text{red}(d) \) corresponds to the result of Gentzen’s reduction in G. Gentzen [7]. This is essentially observed by W. Buchholz [4].

**Theorem 12** Assume that \( d = E_\omega(R_C(\bigwedge_{C_0 \land C_1} (d_{000}, d_{001}), \bigvee_{C_0 \lor \neg C_1}^k (d_{010}))). \) Then

1. \( \text{red}(d) = E^{m+1}(\text{Cut}_{C_k}(E_\omega(R_C(g(d_{00k}), g(d_{01}))), E_\omega(R_C(g(d_{00}), g(d_{010})))) \)
   for \( m = \text{rk}(C_k) \), and
2. \( |g(d)| > |g(\text{red}(d))| \).

**Proof.**
\[
g(d) = g(E_\omega(R_C(\bigwedge_{C_0 \land C_1} (d_{000}, d_{001}), \bigvee_{C_0 \lor \neg C_1}^k (d_{010}))))
\]
\[
= \mathcal{E}_\omega(g(R_C(\bigwedge_{C_0 \land C_1} (d_{000}, d_{001}), \bigvee_{C_0 \lor \neg C_1}^k (d_{010}))))
\]
\[
= \mathcal{E}_\omega(R_C(\bigwedge_{C_0 \land C_1} (g(d_{000}), g(d_{001})), \bigvee_{C_0 \lor \neg C_1}^k (g(d_{010}))))
\]
\[
= \mathcal{E}_\omega(\text{Cut}_{C_k}(R_C(g(d_{00k}), g(d_{01})), R_C(g(d_{00}), g(d_{010}))))
\]
\[
= \text{Rep}(E^{m+1}(\text{Cut}_{C_k}(E_\omega(R_C(g(d_{00k}), g(d_{01}))), E_\omega(R_C(g(d_{00}), g(d_{010}))))).
\]
Thus \( \text{tp}(d) = \text{Rep} \), and \( |g(d)| > |g(\text{red}(d))| \) follows from Lemma 2. \( \text{red}(d) = d[0] \)
where
\[
d[0] = E^{m+1}(\text{Cut}_{C_k}(E_\omega(R_C(g(d_{00k}), g(d_{01}))), E_\omega(R_C(g(d_{00}), g(d_{010}))))
\]

\[ \square \]
Theorem 13 Assume that $d = E(R_C(\bigwedge_{x \in \omega} (d_{00n})_{n \in \omega}, \bigvee_{x} C_0(x)(d_{010}))$.
Then

1. $\text{red}(d) = R_{C_0(k)}(E(R_C(d_{00k}, d_{01})), E(R_C(d_{00}, d_{010})))$, and
2. $|g(d)| > |g(\text{red}(d))|$.

Proof.

\begin{align*}
g(d) &= g(E(R_C(\bigwedge_{x \in \omega} (d_{00n})_{n \in \omega}, \bigvee_{x} C_0(x)(d_{010})))) \\
&= E(g(R_C(\bigwedge_{x \in \omega} (d_{00n})_{n \in \omega}, \bigvee_{x} C_0(x)(d_{010})))) \\
&= E(R_C(\bigwedge_{x \in \omega} C_0(x)(g(d_{00n}))_{n \in \omega}, \bigvee_{x} C_0(x)(g(d_{010})))) \\
&= E(Cut_{C_0(k)}(E(R_C(g(d_{00k}), g(d_{01})), R_C(g(d_{00}), g(d_{010})))) \\
&= \text{Rep}(R_{C_0(k)}(E(R_C(g(d_{00k}), g(d_{01})), R_C(g(d_{00}), g(d_{010}))))).
\end{align*}

Thus $tp(d) = \text{Rep}$, hence $|g(d)| > |g(\text{red}(d))|$ follows from Lemma 2. We have $\text{red}(d) = d[0] = R_{C_0(k)}(E(R_C(d_{00k}, d_{01})), E(R_C(d_{00}, d_{010})))$. □

Remark 13 Here $\text{red}(d)$ corresponds to Schütte’s cut-elimination step for $\omega$-arithmetic.

Theorem 14 Assume that $d = E_\omega(R_C(\bigwedge_{x \in \omega} (d_{00n})_{n \in \omega}, \bigvee_{x} C_0(x)(d_{010}))$.
Then

1. $\text{red}(d) = E^{m+1}(Cut_{C_0(k)}(E_\omega(R_C(g(d_{00k}), g(d_{01}))), E_\omega(R_C(g(d_{00}), g(d_{010}))))$ for $m = \text{rk}(C_k)$, and
2. $|g(d)| > |g(\text{red}(d))|$.

Proof.

\begin{align*}
g(d) &= g(E_\omega(R_C(\bigwedge_{x \in \omega} (d_{00n})_{n \in \omega}, \bigvee_{x} C_0(x)(d_{010})))) \\
&= E_\omega(g(R_C(\bigwedge_{x \in \omega} (d_{00n})_{n \in \omega}, \bigvee_{x} C_0(x)(d_{010})))) \\
&= E_\omega(R_C(\bigwedge_{x \in \omega} C_0(x)(g(d_{00n}))_{n \in \omega}, \bigvee_{x} C_0(x)(g(d_{010})))) \\
&= E_\omega(Cut_{C_0(k)}(R_C(g(d_{00k}), g(d_{01})), R_C(g(d_{00}), g(d_{010})))) \\
&= \text{Rep}(E^{m+1}(Cut_{C_0(k)}(E_\omega(R_C(g(d_{00k}), g(d_{01}))), E_\omega(R_C(g(d_{00}), g(d_{010}))))).
\end{align*}

Thus $tp(d) = \text{Rep}$, and $|g(d)| > |g(\text{red}(d))|$ follows from Lemma 2. We have $\text{red}(d) = d[0] = E^{m+1}(Cut_{C_0(k)}(E_\omega(R_C(g(d_{00k}), g(d_{01}))), E_\omega(R_C(g(d_{00}), g(d_{010}))))$.
□
Therefore we can conclude, using (3) and (4), that

\[ \text{Theorem 15} \quad \text{Assume that } d = D(E_\omega(R_C(\bigwedge_{\forall X C_0(X)}(d_{0000}), \bigvee_{\exists X \neg C_0(X)}(d_{0010})))) . \]

Then

1. \( \text{red}(d) = D(E_\omega(R_C(d_{0000}, R_{C_0(T)}(\text{Sub}_X^T(D(E_\omega(R_C(d_{0000}, d_{0001}))), d_{0010})))) , \)

and

2. \( |g(d)| > |g(\text{red}(d))| . \)

\[ \text{Proof.} \quad \text{Note that } d_{0000} = \bigwedge_{\forall X C_0(X)}(d_{0000}) , \text{ and } d_{001} = \bigvee_{\exists X \neg C_0(X)}(d_{0010}) . \]

Recall that \( g \) replaces \( R , E_\omega , \bigvee_{\exists X \neg C_0(X)} \) by \( R , E_\omega , \Omega \), and \( R_{\exists X \neg C_0(X)} \) transforms impredicative cuts into \( \Omega \). Let \( q = (d_{001}, X) \) be any element of \( |\forall X C_0(X)| \). Then

\[ g(d) = D(E_\omega(R_C(\bigwedge_{\forall X C_0(X)}(g(d_{0000})), \Omega(\ldots R_{C_0(T)}(S_X^T(d_{001}), g(d_{0010})))\ldots))) , \]

\[ = D(E_\omega(\bigwedge_{\forall X C_0(X)}(g(d_{0000})), g(d_{001})), R_C(g(d_{000}), R_{C_0(T)}(S_X^T(d_{001}), g(d_{0010})))) , \]

\[ = D(\bigwedge_{\forall X C_0(X)}(g(d_{000}), g(d_{001}))), \Omega(R_C(g(d_{000}), R_{C_0(T)}(S_X^T(d_{001})), g(d_{0010})))) , \]

\[ = \text{Rep}(D(E_\omega(R_C(g(d_{000}), R_{C_0(T)}(S_X^T(d_{001})), g(d_{0010})))) , g(d_{0010}))) , \]

Thus \( \text{tp}(d) = \text{Rep} \), hence \( |g(d)| > |g(\text{red}(d))| \) follows from Lemma 2. Note that

\[ d[0] = D(d_0[[D(d_0[0]), X]]). \]

(3)

Since \( \text{tp}(d_0) = \tilde{\Omega} \), we have \( d_0[0] = E_\omega(R_C(d_{0000}, d_{001})). \) It is possible to see that \( d_0[0] \vdash_0 \Gamma , A(X) \) and \( (D(d_0[0]), X) \in |\forall X A(X)|^+ \). Let \( q^+ = (D(d_0[0]), X) . \) Then

\[ d_0([D(d_0[0]), X]) = E_\omega(R_C(d_{0000}, d_{001} [[ q^+ ]])). \]

(4)

Now we have \( d_{001}[[ q^+ ]] = R_{C_0(T)}(\text{Sub}_X^T(D(E_\omega(R_C(d_{0000}, d_{001}))), d_{0010}))) . \) Let \( d_{q^+} := D(E_\omega(R_C(d_{0000}, d_{001}))). \) Then \( d_{001}[[ q^+ ]] = R_{C_0(T)}(\text{Sub}_X^T(d_{q^+}, d_{0010}))) . \)

Therefore we can conclude, using (3) and (4), that

\[ \text{red}(d) = d[0] = D(E_\omega(R_C(d_{0000}, R_{C_0(T)}(\text{Sub}_X^T(d_{q^+}, d_{0010})))) . \]

\[ \square \]

\[ \text{Remark 14} \]

1. Theorem 15 holds even if \( E_\omega \) in a given derivation \( d \) is replaced by \( E^{m+1}_\omega \) such that \( dg(E^{m+1}_\omega(R_C(\ldots))) = 0 \).

2. Theorem 15 shows that \( \text{red}(d) \) corresponds to Takeuti’s reduction step in the case of impredicative cut-elimination ending with \( D (\text{tp}(d) = \text{Rep}) \). This fact is essentially observed by W. Buchholz [5] and proved there for derivations of empty sequent (“inconsistency”) in a weak subsystem of \( \Pi^1 \)-analysis which has the strength of \( \text{ID}_1 \).

3. The next theorem gives an additional informal connection between Gentzen-Takeuti method and Buchholz’s method (Theorem 16).
4. In Theorems 9-15, use of Rep-rule has played a crucial role in establishing that $|g(d)| > |g(red(d))|$. Rep is also necessary in order to define $tp(d)$ and $d[i]$.

**Theorem 16** Assume that $d = E^{m+1}(RC(\forall_X C_0(X)(d_{000}), \bigvee^T_{X\neg C_0(X)}(d_{010})))$ for $m + 1 = rk(C_0(T))$.

1. $red(d) = E^{m+1}(RC(d_{000}, R_{C_0(T)}(Sub_X^T(D(E^{m+1}(RC(d_{000}, d_{01}))(d_{010}))))), d_{010}))$.

2. $|g(d)| > |g(red(d))|$.

**Proof.** See also Remark 15 below. Let $d_{000} = \forall_X C_0(X)(d_{000})$, and $q = (d_{01q}, X)$ be any element of $[\forall XC_0(X)]$. By Theorem 6,

$$g(d) = g(E^{m+1}(R_C(\exists_X C_0(X)(d_{000}), \bigvee^T_{X\neg C_0(X)}(d_{010}))))$$

$$= E^{m+1}(R_C(\exists_X C_0(X)(g(d_{000})), \exists_X C_0(T)(S_x^X(d_{01q}), g(d_{010}))))$$

Thus $tp(d) = \bar{\Omega}$. Hence $|g(d)| > |g(red(d))|$ follows from Lemma 2. Let $q^+ = (D(d[0]), X)$ as in Definition 13. Then

$$g^+ = (D(E^{m+1}(R_C(d_{000}, d_{01}))), X)) = (D(E^{m+1}(R_C(d_{000}, d_{01})))).$$

Let $d_{q^+} := D(E^{m+1}(R_C(d_{000}, d_{01})))$. Then we have

$red(d) = d[q^+] = E^{m+1}(d_0[q^+]) = E^{m+1}(RC(d_{000}, R_{C_0(T)}(Sub_X^T(d_{q^+}, d_{010}))))$.

$\square$

**Remark 15** Now we explain an informal connection between Takeuti’s reduction in [15] and Buchholz’s cut-reduction for $\Omega$-systems. For the sake of simplicity, we write $E^{m+1}$ as if it were a single rule. Using a traditional notation, $d$ in Theorem 15 is of the following form:

$$\frac{\Gamma, C_0(X), \forall X C_0(X)}{\Gamma, \forall X C_0(X)} \quad \frac{\Gamma, \neg C_0(T), \exists X \neg C_0(X)}{\Gamma, \exists X \neg C_0(X)} \quad \frac{\bigvee^T_{X\neg C_0(X)}}{RC}$$

Therefore $g(d)$ is of the following form:

$$\frac{\Gamma, C_0(X), \forall X C_0(X)}{\Gamma, \forall X C_0(X)} \quad \frac{\Gamma, \exists X \neg C_0(X)}{RC} \quad \frac{\bigvee^T_{X\neg C_0(X)}}{RC}$$

$$\frac{\Delta q, C_0(X)}{\Delta q, C_0(T)} \quad \frac{\Delta q, \exists X \neg C_0(X)}{RC} \quad \frac{\bigvee^T_{X\neg C_0(X)}}{RC}$$

$$\frac{\Gamma, \Delta q}{\Gamma} \quad \frac{\bigvee^T_{X\neg C_0(X)}}{RC}$$

$$\frac{\Gamma, \Delta q}{\Gamma} \quad \frac{\bigvee^T_{X\neg C_0(X)}}{RC}$$

$$\frac{\Gamma, \Delta q}{\Gamma} \quad \frac{\bigvee^T_{X\neg C_0(X)}}{RC}$$

$$\frac{\Gamma, \Delta q}{\Gamma} \quad \frac{\bigvee^T_{X\neg C_0(X)}}{RC}$$

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On the other hand, \(\text{red}(d)\) is of the following form:

\[
\begin{array}{c}
\vdash \Gamma, C_0(X), \forall X C_0(X) \quad \Gamma, \exists X \neg C_0(X) \\
\vdash \Gamma, C_0(X) \\
\vdash \Gamma, C_0(X) \\
\end{array}
\]

\[
\begin{array}{c}
\vdash \Gamma, C_0(X) \\
\vdash \Gamma, C_0(X) \\
\vdash \Gamma, C_0(T) \\
\end{array}
\]

\[
\begin{array}{c}
\vdash \Gamma, C_0(X), \forall X C_0(X) \\
\vdash \Gamma, \exists X \neg C_0(X) \\
\end{array}
\]

\[
\begin{array}{c}
\vdash \Gamma, C_0(X) \\
\vdash \Gamma, C_0(T) \\
\vdash \Gamma, \exists X \neg C_0(X) \\
\end{array}
\]

\[
\begin{array}{c}
\vdash \Gamma, C_0(X), \forall X C_0(X) \\
\vdash \Gamma, \exists X \neg C_0(X) \\
\end{array}
\]

\[
\begin{array}{c}
\vdash \Gamma, C_0(X) \\
\vdash \Gamma, C_0(T) \\
\vdash \Gamma, \exists X \neg C_0(X) \\
\end{array}
\]

Therefore \(g(\text{red}(d))\) is of the following form:

\[
\begin{array}{c}
\vdash \Gamma, C_0(X), \forall X C_0(X) \quad \Gamma, \exists X \neg C_0(X) \\
\vdash \Gamma, C_0(X) \\
\vdash \Gamma, C_0(X) \\
\end{array}
\]

\[
\begin{array}{c}
\vdash \Gamma, C_0(X) \\
\vdash \Gamma, C_0(T) \\
\vdash \Gamma, \exists X \neg C_0(X) \\
\end{array}
\]

\[
\begin{array}{c}
\vdash \Gamma, C_0(X), \forall X C_0(X) \\
\vdash \Gamma, \exists X \neg C_0(X) \\
\end{array}
\]

\[
\begin{array}{c}
\vdash \Gamma, C_0(X) \\
\vdash \Gamma, C_0(T) \\
\vdash \Gamma, \exists X \neg C_0(X) \\
\end{array}
\]

\[
\begin{array}{c}
\vdash \Gamma, C_0(X), \forall X C_0(X) \\
\vdash \Gamma, \exists X \neg C_0(X) \\
\end{array}
\]

\[
\begin{array}{c}
\vdash \Gamma, C_0(X) \\
\vdash \Gamma, C_0(T) \\
\vdash \Gamma, \exists X \neg C_0(X) \\
\end{array}
\]

In the \(\bar{\Omega}\)-rule at the end of \(g(d)\), we have a derivation of \(\Gamma, \Delta_q\) for each \(q \in |\forall X C_0(X)|\). Note that \(\Gamma\) is a \(\Pi^1\)-sequent, and \((D(E^{m+1}_\varepsilon \{RC(d_{000}, d_{01})\}), X) \in |\forall X C_0(X)|^+\). Let’s take \((D(E^{m+1}_\varepsilon \{RC(g(d_{000}), g(d_{011}))\}), X)\) as \(q\), and \(\Gamma\) as \(\Delta_q\). Then the \(q\)-th right premise of the \(\bar{\Omega}\)-rule is \(g(\text{red}(d))\), hence a subderivation of \(g(d)\). Thus \(|g(d)| > |g(\text{red}(d))|\) holds.

This is explained informally as follows. Introducing \(\bar{\Omega}\)-rule in the process of the cut-elimination, we list up every cuts which can possibly occur in the future cut-elimination steps, and the result of Takeuti’s reduction \(g(\text{red}(d))\) is seen to be one of such cuts.

The theorem 9-16 shows that the diagram (1) in the Introduction commutes for essential reductions mentioned in these theorems. In particular the size of derivation is strictly decreasing in essential reductions: \(|g(d)| > |g(\text{red}(d))|\). Therefore the reduction steps \(\text{red}\) terminate as shown in Theorem 8 because other cases can be reduced to these essential case, and \(g(d)\) is a well-founded tree for each \(d\) in BI.

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