NOVEL CHARACTERIZATIONS OF REAL
FINITE-DIMENSIONAL UNITAL ALGEBRAS

FRED GREENSITE

Abstract. We provide novel characterizations of the inversion operation on
the units of real finite-dimensional unital associative algebras, based on the
ways an algebra’s inversion operation can be uncurled. Using an associated
functor, these augment the tools available for providing a negative answer
to the (recursively undecidable) question of whether two such algebras are
related by an epimorphism. Various other features are presented, arising from
the extended notion of an algebra norm resulting from these characterizations.

1. Introduction: naïve motivation and significance

For geometrical purposes, it has often proved useful to attach a real number
to each member of a vector space, where this association is intended to become
an intrinsic property of the space. For the case of $\mathbb{R}^n$, this entails identifying a
function $\ell : \mathbb{R}^n \rightarrow \mathbb{R}$ that encodes features we may wish to graft onto $\mathbb{R}^n$, such as
that which attends the (degree-1 positive) homogeneity expressed by $\ell(\alpha s) = \alpha \ell(s)$ for
$\alpha > 0$, and the concept of a unit sphere $\{s \in \mathbb{R}^n : \ell(s) = 1\}$ with respect to
which, e.g., isotropy can be defined. We might also insist that $\ell$ be continuously
differentiable on appropriate domains and (with respect to the standard basis)
have an exterior derivative $\nabla \ell(s) \cdot ds$, where $\nabla \ell(s)$ is defined as the ordered $n$-tuple
of coordinate-wise one-dimensional derivatives. Evidently, $s$ and $\nabla \ell(s)$ are both
members of $\mathbb{R}^n$. However, the latter behave very differently if $s$ is replaced by $\alpha s$.

By the Euler Homogeneous Function Theorem, degree-1 positive homogeneity of $\ell$
implies $s \cdot \nabla \ell(s) = \ell(s)$. For $\alpha > 0$, replacing $s$ by $\alpha s$ then leads to
$\alpha s \cdot \nabla \ell(\alpha s) = \ell(\alpha s) = \alpha \ell(s) = \alpha s \cdot \nabla \ell(s)$, implying $\nabla \ell(\alpha s) = \nabla \ell(s)$ (where $\nabla \ell(\alpha s)$ means that
$\nabla \ell$ is evaluated at $\alpha s$). On the other hand, for degree-1 positive homogeneous $\ell$,
the expressions $s$ and $\ell(s) \nabla \ell(s)$ do behave the same way when $s$ is replaced by $\alpha s$
(each expression simply being multiplied by $\alpha$). So a “simplest” candidate for $\ell$
presents itself as the solution to

\begin{equation}
\tag{1.1}
s = \ell(s) \nabla \ell(s),
\end{equation}

under the constraint,

\begin{equation}
\tag{1.2}
\ell(\nabla \ell(s)) = 1,
\end{equation}

since satisfaction of (1.2) is a necessary condition for a nonnegative solution of
(1.1) to be degree-1 positive homogenous (e.g., apply $\ell$ to both sides of (1.1)). It
is easily shown that $\ell(s)$ is then the Euclidean norm (indeed, derivation of (1.1),
(1.2) from a formulation of Euclidean geometry that incorporates $\mathbb{R}$, e.g., using the
four postulates of Birkhoff [1], thereby constitutes a new proof of the Pythagorean
Theorem [2]).
But without unduly compromising the simplicity of the rationale, we could alternatively propose that (1.1) be replaced by

\[ Ls = \ell(s) \nabla \ell(s), \]

for a linear transformation \( L : \mathbb{R}^n \to \mathbb{R}^n \), since we would still have \( Ls \) and \( \ell(s) \nabla \ell(s) \) behaving the same when \( s \) is replaced by \( \alpha s \), \( \alpha > 0 \). Applying the dot product with \( s \) to both sides of (1.3), the Euler Homogeneous Function Theorem then implies

\[ s' Ls = s \cdot Ls = \ell^2(s), \]

where on the left-hand-side \( L \) is understood to be the matrix associated with the above linear transformation with respect to the standard basis. According to (1.3), \( L \) must be a real symmetric matrix since \( Ls \) is a gradient (i.e., the right-hand-side of (1.3) is equal to \( \frac{1}{2} \nabla \ell^2(s) \)). According to (1.4), \( L \) must be positive semi-definite.

From the Polarization Identity, it is seen that the above leads to arbitrary inner product spaces on \( \mathbb{R}^n \) - a nice, if simple, generalization.

However, the above argument can easily be further generalized to the task of attaching a “norm” to \( \mathbb{R}^n \) when the latter is the vector space of elements of a unital associative algebra. That is, the basic rationale of the argument is easily modified to allow the new feature of the inverse of an element to influence selection of a norm - something not explicitly addressed by the well-recognized “usual norm” (aka the “non-reduced norm”) of a finite-dimensional associative algebra as the determinant of an element’s image under the left regular representation (e.g., the algebra’s norm operation as defined by Bourbaki [3], and whose norm format is also fundamental more generally, such as in Algebraic Number Theory).

Thus, we wish to exploit the existence of an element’s inverse to aid in construction of a “norm-like” real function \( U(s) \) on the units of the algebra. By analogy with the vector space argument in the first two paragraphs of this section, one recognizes that \( s^{-1} \) and \( \nabla U(s) \) are members of \( \mathbb{R}^n \) if \( s \) is a unit of the algebra, and \( \nabla U(\alpha s) = \nabla U(s) \) for \( \alpha > 0 \) if degree-1 positive homogeneity is again mandated. While the expressions \( s^{-1} \) and \( \nabla U(s) \) thereby behave differently when \( \alpha s \) replaces \( s \), this time it is \( s^{-1} \) and \( \frac{\nabla U(s)}{U(s)} \) that behave the same under that replacement, in that both expressions are simply multiplied by \( \frac{1}{\alpha} \) - mirroring the situation that pertains to the first paragraph of this section. So now, based on that latter treatment (and its “minimalist” approach for attaching a norm to a vector space), we might be tempted to equate \( s^{-1} \) and \( \frac{\nabla U(s)}{U(s)} \) - except that the latter is the gradient of \( \log U(s) \), while \( s^{-1} \) is in general not a gradient. That problem resolves if we can find a linear transformation \( L : \mathbb{R}^n \to \mathbb{R}^n \) such that \( Ls^{-1} \) is a gradient, i.e., satisfies the exterior derivative condition,

\[ d ([Ls^{-1}] \cdot ds) = d ((ds)' Ls^{-1}) = 0. \]

Here and in the sequel the standard basis on \( \mathbb{R}^n \) is assumed, so that \( L \) in (1.5) is the matrix associated with the above linear transformation with respect to the standard basis.

But there is also the Euler Homogeneous Function Theorem to deal with, wherein \( \frac{\nabla U(s)}{U(s)} = 1 \). Denoting the multiplicative identity of the algebra as \( 1 \), we can define \( \| 1 \|^2 \) to be the number of independent entries on the main diagonal of the matrices comprising the left regular representation of the algebra. We are free to constrain \( L \) by the requirement that \( 1 \cdot (L 1) = 1' L 1 = \| 1 \|^2 \). It follows that we can now
propose,

\[(1.6)\]

\[Ls^{-1} = \|1\|^2 \frac{\nabla U(s)}{U(s)} = \left(\frac{\|1\|^2}{U(s)}\right) \nabla U(s),\]

if \(L\) satisfies both (1.5) and

\[(1.7)\]

\[s'Ls^{-1} = s \cdot Ls^{-1} = \|1\|^2,\]

since constraint (1.7) is a necessary condition for a solution of (1.6) to be degree-1 positive homogeneous (e.g., take the dot product of both sides of (1.6) with \(s\) and apply the Euler Homogeneous Function Theorem). Note that (1.7) must hold if \(s\) is replaced by \(s^{-1}\) as both are units, so we expect \(L\) to be a real symmetric matrix. Indeed, \(L\) is akin to an inner product matrix, and an important feature is that \(L\) need not be positive semi-definite (exploitable in a relativistic physics context [4]). Thus, (1.7) can be interpreted to mean that multiplicative inverses \(s, s^{-1}\) also behave inversely as vector space members with respect to inner product \(L\). If \(L\) satisfying (1.5) and (1.7) can be found, then one can provide \(U(s)\) by integrating (1.6), at least in some simply connected neighborhood of \(1\).

Equation (1.6) is analogous to (1.3) (which we emphasize with the format of the second equality in (1.6)), and (1.7) is analogous to (1.4). But the marked difference is that for the “naked” vector space application (where (1.3) and (1.4) pertain), \(L\) can be any real symmetric \((n \times n)\)-matrix (allowing for possibly complex-valued \(\ell(s)\)), while in the unital algebra application (where (1.6) and (1.7) pertain) the additionally required (1.5) very significantly constrains real symmetric \(L\). It is this constraint (which also allows for the further constraint (1.7)) that makes things interesting, because different algebras have very different collections of admissible choices for \(L\), and these differences distinguish the algebras in novel ways.

We call \(U(s)\) a Unital Norm (or an incomplete Unital Norm if its domain constitutes only a portion of the units), we call a symmetric matrix \(L\) satisfying (1.5) a Proto-norm, and we call a Proto-norm satisfying (1.7) a normalized Proto-norm. In general there is an \(m\)-parameter family of symmetric matrices, \(m \geq 1\) (i.e., an \(m\)-dimensional space) satisfying (1.5), representing the “Proto-norm family” associated with a unital associative algebra whose vector space of elements is \(\mathbb{R}^n\). If such an algebra \(\tilde{A}\) has the feature that its left regular representation \(\tilde{A}\) is such that the transpose operation on \(\mathbb{R}^{n \times n}\) maps \(\tilde{A}\) to \(\tilde{A}\), then the Unital Norm family associated with \(\tilde{A}\) contains a member that is essentially equivalent to usual norm on \(\tilde{A}\), apart from an exponent (Theorem 3.2). Otherwise, there is a Unital Norm family member producing values analogously equivalent to the values produced by the usual norm on a subalgebra of \(\tilde{A}\) (Corollary 3.3). However, it is also the case that the Unital Norm family frequently contains members that are markedly different and in some ways more capable than the usual norm, e.g., in their sensitivity to components of an element to which the usual norm is insensitive (such as with the Dual Numbers, Section 4.4, and their higher dimensional analogues, Corollary 3.9), and in their descriptions of the non-units via their association with an algebra-dependent type of singularity reflecting the topology of the space of units (Table 1 in Section 6).

In light of the foregoing, it is a simple matter to show from (1.6) that an incomplete Unital Norm (at least one of which exists for each of the above algebras) is uniquely associated with a normalized Proto-norm \(L\) (Corollary 2.2), and thereby characterizes the equivalence class of unital associative algebras on \(\mathbb{R}^n\) sharing the mapping \(s \to s^{-1}\) for units of its domain that are in the orthogonal complement.
of the kernel of $L$. These are the characterizations referred to in the title of this paper. Their interest is greatly augmented by the fact that they lead to a useful functor relating algebras and Proto-norm families (Section 6).

Thus, we can construct a Proto-norm category, whose objects are symmetric parametrized matrices (the set of objects encompasses the Proto-norm families), and whose morphisms arise in a natural way (based on set inclusion as regards the sets of real matrices implied by symmetric parametrized matrices under transformations preserving matrix symmetry). There is a functor relating the category of unital associative algebras whose vector space of elements is $\mathbb{R}^n$ (with algebra epimorphisms as morphisms) to the Proto-norm category (Theorem 6.1). Thereby, an epimorphism between two algebras can exist only if there is a morphism between the corresponding Proto-norm families - something that can be independently investigated such as by examination of the respective Proto-norm family members’ dimensions and determinants (Corollary 6.3, Corollary 6.7, Corollary 6.8, Corollary 6.9). In other words, Proto-norm families may have a lot to say about the existence of ideals, among other things. For example,

- If an algebra’s Proto-norm family is one-dimensional and contains nonsingular members, then the algebra is simple (Corollary 6.4). For instance, this yields an alternative demonstration that the algebra of real $(n \times n)$-matrices is simple.

- More generally, an epimorphism from a first algebra to a second algebra can exist only if the dimension of the Proto-norm family of the first algebra is greater than or equal to the dimension of the Proto-norm family of the second algebra (Corollary 6.3).

- There is an interesting “similarity” between the Proto-norm category and rings of real matrices. That is, $M_1$ and $M_2$ as distinct members of these rings are similar if they are matrices representing the same linear transformation with respect to different bases. Analogously, we can say that $P_1$ and $P_2$ as distinct objects in the Proto-norm category are similar if they are Proto-norm families of isomorphic algebras. There is the fundamental linear algebra result that if $M_1$ and $M_2$ are similar matrices, then $M_1 = K^{-1}M_2K$ for some nonsingular real matrix $K$. And analogously, if $P_1$ and $P_2$ are similar Proto-norm families, then $P_1 = K'P_2K$ for some nonsingular real matrix $K$ (Corollary 6.6), where “$'$” denotes the matrix transpose operation.

- From the prior bullet point, it follows that the dimension of a Proto-norm family is an isomorphism invariant (Corollary 6.7), as are the minimum and maximum dimensions of the kernels of the members of an algebra’s family of normalized Proto-norms (Corollary 6.9).

- Some algebras are “essentially the same” as their Proto-norm families in a certain sense. For example, when the members of the left regular representation of the algebra of complex numbers, or of the algebra of dual numbers, or of the algebra of real upper triangular Toeplitz matrices, are each multiplied on the right by the exchange matrix (the matrix with entry value 1 on its antidiagonal and 0 everywhere else), the result in each case is the Proto-norm family of the algebra (Section 4.1, Section 4.4, and Proposition 3.7). In cases like these, every member of the algebra has dual roles in that it implies a first linear transformation as its image under the left regular
representation, along with a second linear transformation (again acting on the algebra itself) as a Proto-norm.

One obvious application of the characterizations afforded by this approach relates to the recursively undecidable Isomorphism Problem (the problem of determining whether two algebras are isomorphic [5]). Novel isomorphism invariants forwarded by the functor augment the tools available for excluding an isomorphism (Corollary 6.7, Corollary 6.8, Corollary 6.9).

Finally, some extraneous spinoffs based on work described in this paper include the cited new proof of the Pythagorean Theorem [2], an algebraic analysis of the Cosmological Constant [4], and the Inverse Problems regularization method that originally led to Proto-norms and Unital Norms [6].

2. Unital Norms

As indicated in the Introduction, given a unital associative algebra with \( \mathbb{R}^n \) as its vector space of elements, we are interested in constructing a degree-1 positive homogeneous function \( \hat{U} : \mathbb{R}^n \to \mathbb{R} \) satisfying (1.6). The usual topology on \( \mathbb{R}^n \) is always assumed.

The following collects and formalizes our terminology.

**Definition 2.1.** For a unital associative algebra whose vector space of elements is \( \mathbb{R}^n \),

- A **unital neighborhood** \( \mathcal{N} \) is an open simply connected neighborhood of \( 1 \) consisting only of units. The **principal unital neighborhood** is the largest unit ball around \( 1 \) devoid of a non-unit.
- With respect to the standard basis, a **Proto-norm** is a real symmetric matrix \( L \) such that for \( s \in \mathcal{N} \),
  \[
  d ([Ls^{-1}] \cdot ds) = d ((s^{-1})'L(ds)) = 0.
  \]
- \( L \) is a **normalized Proto-norm** if for \( s \in \mathcal{N} \) it satisfies both (1.5) and
  \[
  s'Ln^{-1} = ||1||^2,
  \]
  where \( ||1||^2 \) is the number of independent entries on the main diagonal of the matrices comprising the left regular representation of the algebra.
- An **incomplete Unital Norm** is a function, \( U : \mathcal{N} \to \mathbb{R} \), with \( U(1) = 1 \), and for which there is a normalized Proto-norm \( L \) such that
  \[
  Ls^{-1} = ||1||^2 \frac{\nabla U(s)}{U(s)}.
  \]
- An incomplete Unital Norm is **singular** if the associated normalized Proto-norm is singular.
- \( U(s) \) is a **Unital Norm** if (1.6) holds when \( \mathcal{N} \) is replaced by the entire space of units of the algebra.
- For \( s \) in the space of units, if \( U(\hat{s}) \equiv \lim_{s \to \hat{s}} U(s) \) exists for each member \( \hat{s} \) of the closure of the space of units, then \( U \) is a **closed Unital Norm**.
- The algebra’s **Unital Norm family** is the set of all incomplete Unital Norms implied by the algebra having their domain as the principal unital neighborhood.
- The algebra’s **Proto-norm family** is the set of all Proto-norms implied by the algebra, and similarly for its **normalized Proto-norm family**.
Note that Definition 2.1 would still make sense if instead of requiring associativity of the algebra we only require that the units have unique inverses. The latter “extended” version of the definition will be exploited later on.

We now have the easy,

**Theorem 2.1.** A unital neighborhood \( N \) exists for any unital associative algebra whose vector space of elements is \( \mathbb{R}^n \). A normalized Proto-norm \( L \) implies an incomplete Unital Norm on \( N \) as the nonnegative function \( U : N \rightarrow \mathbb{R} \), with

\[
U(s) = e^{\frac{1}{\|1\|^2} \int_1^s [L^{-1} \cdot] dt}.
\]

For \( \alpha > 0 \), if \( s \) and \( \alpha s \) are both in \( N \), and the line segment with endpoints \( 1 \) and \( \frac{1}{\alpha} 1 \) is also in \( N \), then

\[
U(\alpha s) = \alpha U(s).
\]

Suppose there is a smooth path \( P \) from \( 1 \) to \( s \) which is inside \( N \) such that \( P^{-1} \) is also inside \( N \), where \( P^{-1} \) is the set consisting of the inverses of the members of \( P \). Then

\[
U(s)U(s^{-1}) = 1.
\]

**Proof.** To see that a unital neighborhood exists, it is sufficient to observe that such a neighborhood clearly exists for the left regular representation of the algebra (where one works within the matrix algebra \( \mathbb{R}^{n \times n} \), and the inverse of an element is simply the reciprocal of the determinant multiplied by the adjugate matrix).

Existence of normalized Proto-norm \( L \) implies (1.5) and (1.7). Satisfaction of the first of those equations along with the required \( U(1) = 1 \), results in the path-independent integral expression,

\[
\log U(s) = \frac{1}{\|1\|^2} \int_1^s [L^{-1} \cdot] dt,
\]

which yields (2.1).

For \( s \) and \( \alpha s \) in \( N \), and the line segment from \( 1 \) to \( \frac{1}{\alpha} 1 \) also in \( N \), with \( \alpha > 0 \), we have

\[
\log U(\alpha s) = \frac{1}{\|1\|^2} \int_1^s [L^{-1} \cdot] dt = \frac{1}{\|1\|^2} \int_1^{\frac{1}{\alpha} 1} [L(\alpha u)^{-1}] \cdot d(\alpha u)
\]

\[= \frac{1}{\|1\|^2} \int_1^s [L^{-1} \cdot] du + \frac{1}{\|1\|^2} \int_{\frac{1}{\alpha}}^1 [L^{-1} \cdot] du = \log U(s) + \log \alpha,
\]

where the integral from \( \frac{1}{\alpha} 1 \) to \( 1 \) can be easily evaluated along the line segment with those endpoints, and we have used (1.7) in evaluation of that integral to obtain the term \( \log \alpha \) on the right-hand-side of the final equation. Thus, we have \( U(\alpha s) = \alpha U(s) \).
Now suppose there is a smooth path \( P \) from 1 to \( s \) which is inside \( \mathcal{N} \) such that \( P^{-1} \) is also inside \( \mathcal{N} \). We then have,

\[
\log \mathcal{U}(s^{-1}) = \frac{1}{\|1\|^2} \int_1^{s^{-1}} [Lt^{-1}] \cdot dt
\]

\[
= \frac{1}{\|1\|^2} \int_1^s [Ly] \cdot d(y^{-1})
\]

\[
= \frac{1}{\|1\|^2} \left( (y^{-1} \cdot [Ly]) \right|_1^s - \int_1^s y^{-1} \cdot d(Ly)
\]

\[
= -\frac{1}{\|1\|^2} \int_1^s y^{-1} \cdot L(dy) = -\frac{1}{\|1\|^2} \int_1^s [Ly^{-1}] \cdot dy
\]

\[
= -\log \mathcal{U}(s),
\]

where we have used a change of variable (contemplating the left regular representation, it is appreciated that \( y = r^{-1} \) is a diffeomorphism on \( \mathcal{N} \), commutativity of a linear transformation and a differential, the property that \( L \) is a real symmetric matrix, and (1.7). Equation (2.2) then follows.

We can now indicate how the above leads to the promised characterizations of classes of real finite dimensional unital associative algebras. First, for any unital neighborhood, Theorem 2.1 demonstrates that each normalized Proto-norm uniquely implies an incomplete Unital Norm given by (2.1). Conversely, Corollary 2.2. An incomplete Unital Norm uniquely implies a normalized Proto-norm.

**Proof.** For a unital neighborhood \( \mathcal{N} \), let \( (\mathcal{N})^{-1} \) be the set of inverses of the members of \( \mathcal{N} \). The algebra’s left regular representation, and the matrix inverse formula from Laplace expansion, indicate that \( (\mathcal{N})^{-1} \) is homeomorphic to \( \mathcal{N} \). Thus, \( (\mathcal{N})^{-1} \) is an open set containing 1. Based on the usual topology of \( \mathbb{R}^n \), we can take \( \mathcal{N} \) to be an open ball around 1. Invoking the left regular representation, it is easy to see that by making \( \mathcal{N} \) smaller, the open set \( (\mathcal{N})^{-1} \) can be contained in an open ball around 1 of arbitrarily small radius, and where we can take this latter open ball to be unital neighborhood \( \mathcal{N} \). It then follows from (1.6) that a Proto-norm associated with an incomplete Unital Norm \( \mathcal{U}(s) \) on \( \mathcal{N} \) must be unique. This is because (1.6) implies that two such Proto-norms \( L_1, L_2 \) would be such that \( (L_1 - L_2)s^{-1} \) is zero, where \( s^{-1} \) can be an arbitrary member of \( (\mathcal{N})^{-1} \subset \mathcal{N} \), and (as previously noted) \( (\mathcal{N})^{-1} \) is an open set containing 1. □

It will soon be shown that a unital associative algebra with vector space of elements as \( \mathbb{R}^n \) always has a non-empty Unital Norm family (Corollary 3.4). Thus, the above indicate that any member of its Unital Norm family characterizes the equivalence class of real finite-dimensional unital associative algebras sharing the algebra’s mapping \( s \rightarrow s^{-1} \) on the orthogonal complement of the kernel of the Proto-norm associated with the Unital Norm family member. It will also be shown that it is always the case that at least one member of a Unital Norm family is multiplicative, i.e., \( \mathcal{U}(s_1s_2) = \mathcal{U}(s_1)\mathcal{U}(s_2) \). But in general, Unital Norms are not multiplicative. Equation (2.2) substitutes for that property.

Nonsingular closed Unital Norms can embody well known invariants such as the determinant, Euclidean norm, and Minkowski norm. However, singular Unital
Norms, un-closed Unital Norms, and incomplete Unital Norms in general, can also represent novel structures worthy of interest, as will be seen in the sequel.

3. Some basic results regarding characterization with Unital Norms and Proto-norms

In this section we will show that a unital associative algebra \( \mathfrak{A} \) with vector space of elements \( \mathbb{R}^n \) always has a Proto-norm family that is at least one-dimensional, and its Unital Norm family is always nonempty. Further, if \( \mathfrak{A} \) has the feature that its left regular representation \( \mathfrak{A} \) is such that the transpose operation on \( \mathbb{R}^{n \times n} \) maps \( A \) to \( A \), then the Unital Norm family associated with \( \mathfrak{A} \) has a member equivalent to the algebra’s usual norm on a unital neighborhood, apart from an exponent. Otherwise, there is a Unital Norm family member that is essentially equivalent to the usual norm on one of its subalgebras. Section 5.5 and Section 5.6.3 present examples of the latter situation.

The other results in this section concern the algebra of real matrices \( \mathbb{R}^{n \times n} \) and what can be viewed as something like its antithesis, the algebra of real upper triangular Toeplitz matrices \( \mathfrak{u}T_n \). These two algebras contrast in the following way:

- While almost all members of \( \mathbb{R}^{n \times n} \) have \( n \) distinct eigenvalues, this algebra has only a single normalized Proto-norm. In a neighborhood of the identity, its usual norm and solitary incomplete Unital Norm are both sensitive to every component of an element, and differ only by an exponent (\( n \) versus \( \frac{1}{n} \)).
- While the members of \( \mathfrak{u}T_n \) each have only a single eigenvalue, this algebra’s Proto-norm family is “as large as can be” - in that every element of this algebra in matrix format defines a unique Proto-norm member through the product of that matrix with the exchange matrix (the matrix with value 1 entries on the anti-diagonal and value 0 entries everywhere else). Furthermore, almost all of the resulting incomplete Unital Norms are sensitive to every component of an element, while the usual norm is sensitive to only a single component of an element.

We begin with \( \mathbb{R}^{n \times n} \). For an invertible element \( s \), with \( \text{det}(s) \) being its determinant, we have the well-known

\[
\nabla \text{det}(s) = [\text{Adj}(s)]' = [s^{-1}]' \text{det}(s),
\]

where \( \text{Adj}(s) \) is the adjugate matrix and “’” denotes the matrix transpose operation. The first of these equalities is Jacobi’s formula, and second follows from the classical matrix inverse expression resulting from Laplace expansion. The gradient notation on the left-hand-side is employed with the understanding that an element of the algebra (including the full expression on the right-hand-side), as a member of the vector space of elements, is identified as the \( n^2 \)-tuple obtained from sequentially appending the columns of its matrix format (and vice-versa), with a compatible understanding regarding the transpose operation as a particular involutive \( (n^2 \times n^2) \)-matrix. It then easily follows from (3.1) that the transpose operation as the involutive \( (n^2 \times n^2) \)-matrix is a normalized Proto-norm, and \( (\text{det}(s))^{\frac{1}{n}} \) is an incomplete Unital Norm, based on satisfaction of the requirements of Definition 2.1.
In the sequel, when only one algebra is under consideration we will often denote its multiplicative identity as 1. But when appropriate, 1_A will denote the multiplicative identity of algebra A.

Theorem 3.1. The normalized Proto-norm family associated with algebra \( \mathbb{R}^{n \times n} \) has only one member, this being the \((n^2 \times n^2)\)-matrix corresponding to the transpose operation on \((n \times n)\)-matrices. The resulting incomplete Unital Norm extends to the nonsingular closed Unital Norm \(|\det(s)|^{\frac{1}{n}}\).

Proof. The vector space of members of \( \mathbb{R}^{n \times n} \) consists of vectors with \( n^2 \) components. Using the sequentially appended matrix columns convention, we can equivalently represent a member \( s \in \mathbb{R}^{n \times n} \) in either matrix or column vector format. A Proto-norm for this algebra will be a matrix \( L \in \mathbb{R}^{n^2 \times n^2} \), and must satisfy (1.5), i.e., \( d ([Ls^{-1}] \cdot ds) = 0 \). This constitutes a condition on every pair of distinct components of \( Ls^{-1} \) (the latter having \( n^2 \) components - one for every entry of matrix \( s \)). Thus, the partial derivative of any component of \( Ls^{-1} \), e.g., the component corresponding to the \((i, j)\) entry of matrix \( s \), taken with respect to, e.g., the \((k, m)\) component of matrix \( s \) ((\(k, m\) \(\neq (i, j)\)), minus the partial derivative of the \((k, m)\) component of \( Ls^{-1} \) taken with respect to the \((i, j)\) component of matrix \( s \), must be zero.

To examine this constraint, we recall that Matrix Calculus supplies the derivative of the inverse of a matrix, \( \frac{\partial(s^{-1})}{\partial u} \) (with \( u \) a matrix), as follows. Differentiating both sides of the expression \( ss^{-1} = 1 \), we obtain \( \frac{\partial(s^{-1})}{\partial u} = \frac{\partial(s^{-1})}{\partial u} s + s^{-1} \frac{\partial s}{\partial u} = 0 \) (where 0 is the matrix of all zero entries). Hence,

\[
\frac{\partial(s^{-1})}{\partial u} = -s^{-1} \frac{\partial s}{\partial u} s^{-1},
\]

which is a \((n \times n)\)-matrix. Let \( \{e_{pq}\} \) be the canonical basis for \( \mathbb{R}^{n \times n} \), so that a basis member \( e_{rw} \) is the matrix whose \((r, w)\) entry is 1, but with all other entries zero. It follows that \( -s^{-1} \frac{\partial s_{pq}}{\partial e_{pq}} s^{-1} = -s^{-1} e_{pq} s^{-1} \) is a matrix whose entries are the \( n^2 \) components of the derivative of \( s^{-1} \) in the \( e_{pq} \) direction (i.e., the full derivative of \( s^{-1} \) is an array of all such matrices in each of the basis members of \( \{e_{pq}\} \)). Thus, the \( e_{km} \)-direction derivative matrix of \( s^{-1} \) is,

\[
-s^{-1} e_{km} s^{-1} = -(s^{-1})_{:k}(s^{-1})_{m:},
\]

where \((s^{-1})_{m:}\) indicates the \( m\)-th column of \( s^{-1} \), and \((s^{-1})_{k:}\) indicates the \( k\)-th row of \( s^{-1} \). Similarly, the \( e_{ij} \)-direction derivative matrix of \( s^{-1} \) is

\[
-s^{-1} e_{ij} s^{-1} = -(s^{-1})_{:i}(s^{-1})_{j:}.
\]

It is thus clear that the sets of entries of the \( e_{km} \)-direction derivative matrix and the \( e_{ij} \)-direction derivative matrix share the member \(- (s^{-1})_{mi}(s^{-1})_{jk}\), and they can share no other member because \(\{(s^{-1})_{pq}\} = \{\frac{C_{pq}}{|\det(s)|}\}\), where the right-hand-side of this equation is clearly a set of linearly independent functions since \(C_{pq}\) is the matrix of cofactors of \( s \). The latter observation also indicates that the union of the entries of the \( e_{km} \)-direction derivative matrix with the entries of the \( e_{ij} \)-direction derivative matrix is a set of linearly independent functions.

Therefore, the partial derivative in the \( e_{km} \)-direction of the \( e_{ij} \)-component of \( Ls^{-1} \) is a linear combination of the entries of the matrix on the right-hand-side of (3.2), and the partial derivative in the \( e_{ij} \)-direction of the \( e_{km} \)-component of
\(\text{Theorem 3.2.}\) Let \(\bar{A}\) be a unital associative algebra with \(\mathbb{R}^n\) as its vector space of elements, and let \(A\) be its left regular representation. If \(A\) is such that the transpose operation on \(\mathbb{R}^{n \times n}\) maps \(A\) to itself, then the Unital Norm family of \(\bar{A}\) has a member given by the usual norm raised to the \(\frac{1}{n}\)-power.

Proof. A real \(n\)-dimensional unital associative algebra \(\bar{A}\) is isomorphic to its left regular representation \(A\), which is a subalgebra of \(\mathbb{R}^{n \times n}\). We denote the multiplicative identity of \(\mathbb{R}^{n \times n}\) as \(I\), and observe that \(1_A = I\). However, \(\|1_A\|^2 = 1_{\bar{A}}^2\), while \(\|1\|^2 = n\).

As noted shortly prior to the statement of Theorem 3.1, the transpose operation on \(\mathbb{R}^{n \times n}\) is a linear transformation \(T : \mathbb{R}^n \to \mathbb{R}^n\) and, also notating it as \(T\) with respect to the standard basis, matrix \(T\) represents a normalized Proto-norm associated with algebra \(\mathbb{R}^{n \times n}\). Being a Proto-norm means that integration of \(Ts^{-1}\) is path-independent in some simply connected neighborhood of the multiplicative identity of \(\mathbb{R}^{n \times n}\). Hence, \(\int_1^2 (T t^{-1}) \cdot dt\) is path-independent in the latter neighborhood and, in particular, is path-independent for paths in that neighborhood restricted to be in \(A\). By assumption, \(T\) maps \(A\) to \(A\). Thus, \(Ts^{-1}\) is curl-free when restricted to \(A\), i.e., \(d((Ts^{-1}) \cdot ds) = 0\) in a unital neighborhood of \(1_A\) in \(A\), since this is a necessary condition for \(T\) to be a Proto-norm on \(\mathbb{R}^{n \times n}\).

Accordingly, we can use \(T\) to define an inner product on \(A\) (possibly indefinite and/or degenerate) as \(\langle s_1, s_2 \rangle_A = \frac{\|1_A\|^2}{n} s_1^T s_2\). Since \(A\) and \(\bar{A}\) are isomorphic, the above inner product on \(A\) can be employed to produce an inner product on \(\bar{A}\) so that the two inner product spaces have the same geometries. Specifically, for \(s_1, s_2 \in A\), let \(I : A \to \bar{A}\) be the isomorphism, with \(\bar{s}_1 = Is_1\) and \(\bar{s}_2 = Is_2\). Then we define the induced inner product on \(\bar{A}\) via the self-adjoint

\[
(3.4) \quad T = \frac{\|1_{\bar{A}}\|^2}{\|1\|^2} (I^{-1})' T I^{-1} = \frac{\|1_{\bar{A}}\|^2}{n} (I^{-1})' T I^{-1},
\]
i.e., \( \langle s_1, s_2 \rangle_{\hat{A}} = s_1^* T s_2 \). Applying (3.4) and \( \|1_A\|^2 = \|\hat{1}_A\|^2 \), we have

\[
\langle s_1, s_2 \rangle_{\hat{A}} = \frac{\|1_A\|^2}{n} (s_1^* T s_2) = \langle s_1, s_2 \rangle_{\hat{A}}.
\]

The fraction \( \frac{\|1_A\|^2}{n} \) is employed in (3.4) in order that \( \langle 1_A, 1_A \rangle_{\hat{A}} = \|1_A\|^2 \), given that we have \( 1^T T 1 = n = \|1\|^2 \). Note that with respect to these inner products, the angle between two members of \( \hat{A} \) is equal to the angle between their two images in \( A \). Hence, in the unital neighborhoods of the respective multiplicative identities \( 1_A \) and \( \hat{1}_A \), the previously demonstrated \( d((T s^{-1}) \cdot ds) = d((ds)^T T s^{-1}) = 0 \) then implies \( d((\hat{T}(\hat{s})^{-1}) \cdot d\hat{s}) = d((ds)^T \hat{T}(\hat{s})^{-1}) = 0 \).

From (3.5) and the fact that \( T \) satisfies (1.7), we have

\[
\hat{s}^* \check{T}(\hat{s})^{-1} = \frac{\|1_A\|^2}{n} (s^* T s^{-1}) = \|1_A\|^2,
\]

which in this context is (1.7). Since \( d((\hat{T}(\hat{s})^{-1}) \cdot d\hat{s}) = 0 \) has already been established in the prior paragraph (i.e., (1.5) is satisfied), this means that the requirements of Definition 2.1 are satisfied. Hence, an incomplete Unital Norm on \( \hat{A} \) results from Theorem 2.1 via the normalized Proto-norm \( L = \check{T} \), as derived from the matrix transpose operation \( T \). Again applying (3.5), we then obtain

\[
\log U(s) = \frac{1}{\|1_A\|^2} \int_1^s (\hat{T}(\hat{s})^{-1}) \cdot d\hat{s} = \frac{1}{\|1_A\|^2} \int_1^s \frac{\|1_A\|^2}{n} (T s^{-1}) \cdot ds = \frac{1}{n} \log \det(s).
\]

Since the usual norm of \( \hat{s} \) is \((\det(s))^n\), the theorem follows. \( \square \)

**Corollary 3.3.** Let \( \hat{A} \) be a unital associative algebra with \( \mathbb{R}^n \) as its vector space of elements, let \( A \) be its left regular representation, and suppose \( \hat{A} \) is a subalgebra of \( A \) for which the \( \mathbb{R}^n \times \mathbb{R}^n \) transpose operation maps \( \hat{A} \) to \( A \). Further, let \( \underline{A} \) be the inverse image of \( \hat{A} \) with respect to the left regular representation isomorphism, and suppose that \( \hat{A} \) is an \( n \)-dimensional unital subalgebra of \( A \), \( m \geq 1 \). Let \( s \in A \) be the image of \( \hat{s} \in \hat{A} \) under the left regular representation, and let \( \hat{s} \) be the component of \( s \) that is in \( \hat{A} \). Then there is an \( n \) member of the Unital Norm family of \( \hat{A} \) given by \( U(s) = \det(\hat{s})^{\frac{1}{n}} \).

**Proof.** By hypothesis, the left regular representation of \( n \)-dimensional \( \hat{A} \) implies a representation of \( n \)-dimensional \( \underline{A} \) as a set of \( (n \times n) \)-matrices \( \hat{A} \) such that \( \underline{A} \) is isomorphic to \( \hat{A} \), and the \( (n \times n) \)-transpose operation \( T \) maps \( \hat{A} \) to \( A \). A trivial modification of the argument in the first three paragraphs of the proof of Theorem 3.2 indicates that \( T \) implies a linear transformation \( \underline{T} : \underline{A} \to \underline{A} \) such that (1.5) is satisfied on \( A \) with \( L = T \). This extends to a mapping with domain all of \( A \) by composing it with the orthogonal projection of \( \hat{A} \) to \( A \). This composition, \( T \), maps elements of \( \hat{A} \) in the orthogonal complement of \( A \) to zero. It is then easily verified that for \( L = \check{T} \) there is satisfaction of (1.5) on all of \( A \). Analogous to the fourth paragraph of the proof of Theorem 3.2, \( s^* \hat{T}(\hat{s})^{-1} = \|1_A\|^2 \). Thus, \( L = \|1_A\|^2 \hat{T} \) is a normalized Proto-norm for \( \hat{A} \). Integrating \( L(\hat{s})^{-1} = \|1_A\|^2 \frac{\|\hat{s}\|^2}{\det(s)} \) leads to the incomplete Unital Norm given in the last sentence of the theorem statement. \( \square \)

**Corollary 3.4.** Every unital associative algebra with \( \mathbb{R}^n \) as its vector space of elements has a nonempty Unital Norm family.
Proof. We need only identify a nonzero algebra \( \hat{A} \) satisfying the specifications of Corollary 3.3 such that \( \hat{A} \) is at least a one-dimensional algebra. In fact, it is clear that \( \hat{A} = \{ \alpha 1 : \alpha \in \mathbb{R} \} \) always satisfies the requirements. \( \square \)

Section 5.5 and Section 5.6.3 each present an example of an algebra \( \bar{A} \) such that the transpose operation applied to its left regular representation \( A \) does not map to \( A \), thereby illustrating Corollary 3.3.

We now turn to upper triangular Toeplitz algebras.

**Definition 3.1.** \( uT_n \) is the \( n \)-dimensional vector space of real upper triangular \((n \times n)\)-matrices. For \( s \in uT_n \), the notation \( s = (x_1, \ldots, x_n) \) indicates that \( s \) corresponds to the Toeplitz upper triangular \((n \times n)\)-matrix whose entries on the successive diagonals, beginning with the main diagonal and proceeding to the right, are given by the successive vector component values \( x_1, \ldots, x_n \).

In the sequel, for \( s \in uT_n \), \( s \) can refer either to its format as an \( n \)-vector or its format as an \((n \times n)\)-matrix, depending on the context.

The following result is well known, but we include the proof for completeness, and to introduce some notation.

**Proposition 3.5.** Under the usual matrix product, \( uT_n \) is a real \( n \)-dimensional unital algebra.

**Proof.** Let \( z \) be the \((n \times n)\)-matrix with entries having value 1 on its superdiagonal (the diagonal immediately to the right of the main diagonal) and value 0 otherwise. It is easily shown that for \( j < n \) its \( j \)-th power, \( z^j \), is the \((n \times n)\)-matrix with entries 1 on the \( j \)-th diagonal to the right of the main diagonal, with all other entries being 0. Thus, we can express the matrix format of \( s \in uT_n \) as a polynomial in \( z \), such that

\[
(3.6) \quad s = x_1 1 + x_2 z + x_3 z^2 + \cdots + x_n z^{n-1}.
\]

It is similarly easy to show that \( z^k \) is the matrix of all zero entries if \( k \geq n \). It follows that for \( s, t \in uT_n \), we have \( st \in uT_n \). Finally, for \( x_1 \neq 0 \), we have

\[
(3.7) \quad s^{-1} = (x_1 1 + x_2 z + x_3 z^2 + \cdots + x_n z^{n-1})^{-1}.
\]

The matrix adding to \( x_1 1 \) inside the parentheses on the right-hand-side of (3.7) has only zero as an eigenvalue, so the full expression on the right-hand-side of (3.7) has a power series expansion in \( z \). Since \( z^k \) is the zero matrix for \( k \geq n \), the power series expansion is a \((n-1)\)-degree polynomial in \( z \), which indicates that \( s^{-1} \in uT_n \). \( \square \)

We will show that upper triangular Toeplitz matrix algebras have Unital Norms sensitive to all element components, providing a great contrast with the usual norm for these algebras, which is sensitive to only one element component. One can easily generate other algebras with Unital Norms “similarly dominating” the usual norm, as in Section 5.

To approach the problem of deriving the Proto-norm family for \( uT_n \), the following will be important.

**Lemma 3.6.** For a unit \( s \in uT_n, n > 1 \), let \( (s^{-1})_p \) be the \( p \)-th component of \( s^{-1} \) in the latter’s \( n \)-vector format. Given \( k \leq n \), then for all \( j < k \) we have

\[
(3.8) \quad \frac{\partial (s^{-1})_k}{\partial x_j} = \frac{\partial (s^{-1})_{k-j+1}}{\partial x_1}.
\]
Proof. The proof is by induction. Equation (3.8) is a tautology for the \( k = 2 \) case. We thus assume the inductive hypothesis that (3.8) is true for all cases up through \( k - 1 \), and then establish its truth for the \( k \)-th case.

We must first express \( s^{-1} \) in terms of the components of \( s = (x_1, \ldots, x_n) \in \mathbb{U}_{T_n} \) with \( x_1 \neq 0 \). From the proof of Proposition 3.5, \( s^{-1} \) is an upper triangular Toeplitz matrix. But as an \( n \)-vector, it is easily seen that we can write it as

\[
(3.9) \quad s^{-1} = \left( \frac{1}{x_1}, \frac{y_1}{x_1}, \ldots, \frac{y_{n-1}}{x_1} \right),
\]

where the \( y_i \) are to be determined. Based on the notation introduced in the proof of Proposition 3.5, we can use (3.6) and (3.7) to write

\[
(3.10) \quad s = x_1 \left( 1 + \frac{x_2}{x_1} z + \frac{x_3}{x_1} z^2 + \cdots + \frac{x_n}{x_1} z^{n-1} \right)
\]

Substituting the above two equations into \( s^{-1} s = 1 \) leads to following set of relations, which allow the \( y_i \) to be computed recursively:

\[
(3.11) \quad -y_1 = \frac{x_2}{x_1} \]
\[
-y_2 = \frac{x_2}{x_1} y_1 + \frac{x_3}{x_1} \]
\[
\vdots \]
\[
-y_{k-1} = \frac{x_2}{x_1} y_{k-2} + \frac{x_3}{x_1} y_{k-3} + \cdots + \frac{x_k}{x_1} \]
\[
\vdots \]

For \( k > 1 \), (3.10) implies \( (s^{-1})_k = \frac{y_{k-1}}{x_1} \). Applying this to (3.11) with \( k > 1 \), we find

\[
(3.12) \quad (s^{-1})_k = -\frac{x_2}{x_1} (s^{-1})_{k-1} - \frac{x_3}{x_1} (s^{-1})_{k-2} - \cdots - \frac{x_k}{x_1} (s^{-1})_1
\]

Differentiating with respect to \( x_j \), we find

\[
(3.13) \quad \frac{\partial (s^{-1})_k}{\partial x_j} = -\frac{1}{x_1} \sum_{m=1}^{k-j} x_{m+1} \frac{\partial (s^{-1})_{k-m}}{\partial x_j} - \frac{1}{x_1} (s^{-1})_{k-j+1},
\]

where the upper limit on the sum on the right-hand-side is \( (k-j) \) rather than \( (k-1) \) because the partial derivatives in the sum are zero for \( m > k-j \) (i.e., \((s^{-1})_p\) is not a function of \( x_j \) for \( p < j \)).
Next, substituting \((k - j + 1)\) for \(k\) in (3.12) and then differentiating with respect to \(x_1\), we have
\[
\frac{\partial \left( s^{-1} \right)_{k-j+1}}{\partial x_1} = \frac{1}{x_1^2} \sum_{m=1}^{k-j} x_{m+1} \left( s^{-1} \right)_{k-j-m+1} \frac{\partial \left( s^{-1} \right)_{k-j-m+1}}{\partial x_1} \tag{3.14}
\]
where the equality of the first terms on the right-hand-sides of the two above equalities comes from (3.12) itself (with \((k - j + 1)\) substituting for \(k\)). Now, according to the induction hypothesis, \(\frac{\partial \left( s^{-1} \right)_{k-j+1}}{\partial x_1} = \frac{\partial \left( s^{-1} \right)_{k-j-m+1}}{\partial x_1}\). Inserting this into (3.14), we see that the left-hand-sides of (3.13) and (3.14) are equal, as was required to be shown.

Let \(I\) be the exchange matrix associated with a real \(n\)-dimensional unital associative algebra, i.e., the \((n \times n)\)-matrix with entries having value 1 on the anti-diagonal and zero everywhere else (relevant to the algebra’s isomorphic representation as a subalgebra of \(\mathbb{R}^{n \times n}\)).

**Proposition 3.7.** For any \(t \in \mathfrak{uT_n}\) let \(\hat{t}\) be its \((n \times n)\)-matrix format, i.e., its left regular representation. Then \(\hat{t} I\) is a Proto-norm. Conversely, each member of the Proto-norm family associated with \(\mathfrak{uT_n}\) can be expressed as \(\hat{t} I\) for some \(t \in \mathfrak{uT_n}\).

**Proof.** For \((\gamma_n, \ldots, \gamma_1) \in \mathfrak{uT_n}\), let \(G_n\) be its Toeplitz \((n \times n)\)-matrix format (its left regular representation), and consider the resulting Hankel matrix,
\[
H_n \equiv G_n I = \begin{bmatrix}
\gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_n \\
\gamma_2 & \gamma_3 & \cdots & \cdots & 0 \\
\gamma_3 & \cdots & \cdots & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots & \vdots \\
\gamma_n & 0 & \cdots & \cdots & 0 
\end{bmatrix}.
\tag{3.15}
\]
For \(\left( s^{-1} \right)_p\) as defined in the statement of Lemma 3.6, we then have
\[
\left( H_n s^{-1} \right)_p = \left( \gamma_1 \left( s^{-1} \right)_1 + \cdots + \gamma_n \left( s^{-1} \right)_n \right)_p,
\tag{3.16}
\]
We will first demonstrate that \(H_n\) is a Proto-norm. According to (1.5) as referred to by Definition 2.1, we are required to demonstrate
\[
d \left( [H_n s^{-1}] \cdot ds \right) = 0.
\tag{3.17}
\]
To verify (3.17), we apply an induction argument. The \(n = 2\) case is easily demonstrated to satisfy the above equation (e.g., see Section 4.4). We assume the induction hypothesis that (3.17) holds for all cases less than \(n\), and now need to show that it holds for the \(n\)-th case. We begin by noting that the partial derivative of the first component of \(H_n s^{-1}\) with respect to any \(x_j\), with \(j > 1\), minus the partial derivative of the \(j\)-th component of \(H_n s^{-1}\) with respect to \(x_1\), is zero. We will then show that the analogous difference of partial derivatives of any two different components of \(H_n s^{-1}\) is also zero.
So, from (3.16),
\[
\frac{\partial (H_n s^{-1})}{\partial x_1} = \sum_{i=j}^{n} \gamma_i \frac{\partial (s^{-1})}{\partial x_1}.
\]
(3.18)

But,
\[
\frac{\partial (H_n s^{-1})}{\partial x_j} = \sum_{i=j}^{n} \gamma_i \frac{\partial (s^{-1})}{\partial x_j} = \sum_{i=j}^{n} \gamma_i \frac{\partial (s^{-1})}{\partial x_{j+1}},
\]
(3.19)

where the second equality follows (3.9) and (3.11), which indicate that for \(i < j\), and the third equality follows from Lemma 3.6. Equations (3.18) and (3.19) demonstrate that the partial derivative of the first component of \(H_n s^{-1}\) with respect to any \(x_j\), with \(j > 1\), minus the partial derivative of the \(j\)-th component of \(H_n s^{-1}\) with respect to \(x_1\), is equal to zero.

Now we will consider the vanishing of the partial derivative of the \(k\)-th component of \(H_n s^{-1}\) with respect to \(x_j\), minus the partial derivative of the \(j\)-th component of \(H_n s^{-1}\) with respect to \(x_k\), for all possible \(k > j > 1\) (i.e., all choices relevant to (3.17) other than considered in the prior paragraph). In fact, based on the telescoping format of (3.16), we already have the result that this difference vanishes, because it merely reduces to an instance of the theorem for the case of a matrix of the form \(H_k s^{-1}\) with \(k < n\), and this case is assumed to hold according to the induction hypothesis. So, (3.17) is established.

Furthermore, a Hankel matrix of the form (3.15) is the only possibility for a Proto-norm \(L\) because any other choice resulting in the coefficients of the terms in any two components of \(Ls^{-1}\) cannot lead to a null value for the difference of the two requisite partial derivatives (as is necessary for \(d([Ls^{-1}] \cdot ds) = 0\) due to linear independence of the individual terms summing to the components. That is, replacing \(H_n\) with matrix \(L = (a_{p,q})_{p=q=1}^{n}\) in (3.18) and (3.19), and invoking Lemma 3.6, we obtain
\[
\frac{\partial (L s^{-1})}{\partial x_1} - \frac{\partial (L s^{-1})}{\partial x_j} = n \sum_{i=1}^{n} a_{j,i} \frac{\partial (s^{-1})}{\partial x_1} - n \sum_{i=j}^{n} a_{1,i} \frac{\partial (s^{-1})}{\partial x_j}.
\]
(3.20)

This indicates that we must have \(a_{j,i} = 0\) for \(i > n - j + 1\), and otherwise have \(a_{j,i} = a_{1,i+j-1}\). Since this must hold for any \(j > 1\), \(L\) must be a Hankel matrix of the form (3.15). Thus, the \(H_n\) are the only possibilities for Proto-norms.

Since \(t = (\gamma_n, \ldots, \gamma_1)\) is an arbitrary member of \(u\mathbb{T}_n\), the proposition follows. □

**Corollary 3.8.** For any \(t \in u\mathbb{T}_n\), denote the value of the \((1,n)\) entry of its matrix format \(t\) as \(t_{1,n}\). Then \(\frac{1}{t_{1,n}}I\) is a normalized Proto-norm if \(t_{1,n} \neq 0\), and all normalized Proto-norms can be written in this form for some \(t \in u\mathbb{T}_n\).

**Proof.** Proposition 3.7 has shown that all Proto-norms of \(u\mathbb{T}_n\) have the form (3.15). We now consider constraint (1.7) of Definition 2.1. Let \(H_n^{(1)}\) be matrix \(H_n\) except...
with the (1, 1)-entry having the value 1. Since \( \|1\|^2 = 1 \) for this algebra, we must show that 
\[
 s'H_n^{(1)}s^{-1} = s \cdot [H_n^{(1)}s^{-1}] = 1.
\]
From (3.9) and (3.15), we have 
\[
 s'H_n s^{-1} = (x_1, \ldots, x_n) \cdot \left[ H_n \left( \frac{1}{x_1}, \frac{y_1}{x_1}, \ldots, \frac{y_{n-1}}{x_1} \right) \right].
\]
\[
 = \gamma_1 + \gamma_2 y_1 + \gamma_3 y_2 + \gamma_4 y_3 + \ldots + \gamma_n y_{n-1} \]
\[
+ \gamma_2 \frac{x_2}{x_1} y_1 + \gamma_4 \frac{x_2}{x_1} y_2 + \ldots + \gamma_n \frac{x_2}{x_1} y_{n-2} \]
\[
+ \gamma_3 \frac{x_3}{x_1} y_1 + \gamma_4 \frac{x_3}{x_1} y_2 + \ldots + \gamma_n \frac{x_3}{x_1} y_{n-3} \]
\[
+ \vdots \]
\[
+ \gamma_{n-1} \frac{x_{n-1}}{x_1} y_1 \]
\[
+ \gamma_n \frac{x_n}{x_1} \tag{3.21}
\]
For any \( i > 1 \), if we collect all the terms on the right-hand-side above that each have \( \gamma_i \) as a coefficient, and then apply (3.11), it is quickly seen that the sum of all terms having \( \gamma_i \) as a coefficient vanishes. It follows that \( s'H_n s^{-1} = \gamma_1 \). Thus, the requisite matrix satisfying constraint (1.7) of Definition 2.1 will be the right-hand-side of (3.15) with \( \gamma_1 = 1 \). Setting \( \frac{1}{t_1, n} t = (\gamma_n, \ldots, \gamma_2, 1) \), the corollary follows. \( \square \)

**Corollary 3.9.** For \( s = (x_1, \ldots, x_n) \in \mathfrak{u} \mathfrak{T}_n \) with \( x_1 \neq 0 \), the usual norm is \( x_1^n \).

The Unital Norm family of \( \mathfrak{u} \mathfrak{T}_n \) is the \( (n-1) \)-parameter family of incomplete Unital Norms,
\[
U_{\gamma_2, \ldots, \gamma_n}(x_1, \ldots, x_n) = x_1^{P_{\gamma_2, \ldots, \gamma_n}} \left( \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1} \right),
\]
where \( P_{\gamma_2, \ldots, \gamma_n}(w_1, \ldots, w_{n-1}) \) is a real \((n-1)\)-degree multivariate polynomial with coefficients dependent on the \((n-1)\) subscripted real parameters.

**Proof.** The usual norm of \( s = (x_1, \ldots, x_n) \in \mathfrak{u} \mathfrak{T}_n \) is immediately seen to be \( x_1^n \).

With respect to the notation \( G_n, H_n \) introduced at the beginning of the proof of Proposition 3.7, each \( (\gamma_n, \ldots, \gamma_2, 1) \in \mathfrak{u} \mathfrak{T}_n \) is associated with a Toeplitz matrix \( G_n^{(1)} \) having its \((1, n)\)-entry equal to 1. Corollary 3.8 indicates that \( H_n^{(1)} = G_n^{(1)} \mathbf{1} \) is a normalized Proto-norm. From Theorem 2.1, the associated Unital Norm is 
\[
\exp \left[ \int_1^s (H_n^{(1)} u^{-1}) \cdot du \right].
\]
Elementary integration and a simple induction argument quickly results in (3.22) and the \((n-1)\)-degree of the multivariate polynomial in question. \( \square \)

4. **Proto-norm and Unital Norm families for the real unital two-dimensional algebras**

Among other things, it will be seen that the Proto-norm families and Unital Norm families each fully characterize these algebras.

Algebra \( \mathbb{R} \) is the only one-dimensional real unital algebra, and it is easily shown that \( \mathcal{U}(s) = s \) is its unique incomplete Unital Norm (indeed, this is the \( n = 1 \) case of Theorem 3.1).
There are only three two-dimensional real unital algebras (up to an isomorphism), and each is commutative and associative. We will actually consider four such algebras, but two of them are isomorphic, though not isometrically isomorphic.

For all of the real unital two-dimensional algebras, the incomplete Unital Norms are multiplicative, i.e., $U(s_1 s_2) = U(s_1) U(s_2)$. It will also be seen that for each of these algebras, the members’ left regular representation multiplied the exchange matrix or alternatively the identity matrix results in the algebra’s Proto-norm family.

4.1. **Algebra $C$.** With respect to a unital neighborhood $N$, all of the incomplete Unital Norms associated with $C$ can be obtained in the following way.

The vector space of elements is $\mathbb{R}^2$, and $(x, y)^{-1} = \frac{(x, -y)}{x^2 + y^2}$. $C$ is isomorphic to the matrix algebra $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$, so that

$$L(x, y)^{-1} = \frac{1}{x^2 + y^2} \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} (\alpha x - \beta y) \\ \beta x - \gamma y \end{bmatrix} \begin{bmatrix} x^2 + y^2 \\ x^2 + y^2 \end{bmatrix}.$$

Satisfaction of (1.5) requires $\gamma = -\alpha$, so the Proto-norm family is two-dimensional and determined by the parametrized matrix $\begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix}$. Since $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix} = \begin{bmatrix} \beta & -\alpha \\ \alpha & \beta \end{bmatrix}$, the members of the left regular representation multiplied on the left by the exchange matrix yield the Proto-norm family.

Satisfaction of (1.7) requires $\alpha = 1$. Thus, a normalizedProto-norm is of the form $L = \begin{bmatrix} 1 & \beta \\ \beta & -1 \end{bmatrix}$. The incomplete Unital Norms are now given by (2.1). Suppose we take $N$ to be all of $\mathbb{C}$ exclusive of the origin and the negative real axis - an open simply connected domain of units containing $1_C = (1, 0)$, and such that if an element is in this domain then so is its inverse. To evaluate the integral in (2.1), we must thereby avoid the negative real axis. It is then convenient to specify the path of integration as the line segment from $(1, 0)$ to $(1, y)$, in union with the line segment from $(1, y)$ to $(x, y)$. This results in

$$\frac{1}{\|1\|^2} \int_{(1, 0)}^{(x, y)} [L(x', y')^{-1}] \cdot (dx', dy') = \int_0^y \frac{\beta + y'}{1 + \beta y'} dy' + \int_1^x \frac{x' - \beta y'}{x^2 + \beta y'^2} dx'$$

$$= \frac{1}{2} \log(x^2 + y^2) + \beta \left[ \arctan(y) - \arctan \left( \frac{x}{y} \right) + \arctan \left( \frac{1}{y} \right) \right],$$

where the values on the positive real axis are obtained through the appropriate limit procedure, so that the expression on the right-hand-side leads to definite finite values everywhere except at the origin and the negative real axis. We thus have the corresponding 1-parameter family of incomplete Unital Norms,

$$(4.1) \quad U_\beta(x, y) = \sqrt{x^2 + y^2} e^{\beta \left[ \arctan(y) - \arctan \left( \frac{x}{y} \right) + \arctan \left( \frac{1}{y} \right) \right]}.$$
incomplete) Unital Norm derived from the one-dimensional family of normalized Proto-norms. Indeed, for \( \beta \neq 0 \), the units on the negative real axis are in the closure of \( \mathcal{N} \), but \( \lim_{s \to \hat{s}} U_{\beta}(s) \) does not exist for \( s \in \mathcal{N} \) and \( \hat{s} \) on the negative real axis, where there is a finite jump discontinuity at each of those points. This feature reflects the nontrivial topology of the group of units as the plane exclusive of the origin.

An alternative path of integration is the union of the directed line segments from \((1, 0)\) to \((x, 0)\) and from \((x, 0)\) to \((x, y)\). Then instead of (4.1), we get

\[
U_{\beta}(x, y) = \sqrt{x^2 + y^2} e^{\beta \arctan(y/x)}.
\]

Unlike the case of (4.1), the integration producing (4.2) is only valid in the right half-plane (i.e., for \( x > 0 \)), but on that domain it of course gives the same result as (4.1) (easily verified by taking the gradients of the right-hand-sides of (4.1) and (4.2)).

The Unital Norms are functions of the eigenvalues of the left multiplication endomorphism associated with the element (thus, \( x^2 + y^2 \) is the product of the eigenvalues, \( x \) is half the sum of the eigenvalues, and \( y \) is obtained from a difference of the eigenvalues divided by \( 2i \)). In this regard, the (incomplete) Unital Norms mimic the usual norm. But as indicated above, for \( \beta \neq 0 \), the Unital Norms deviate from the usual norm in that they have a dependence on the topology of the group of units (the fact that it is not simply connected). This is manifested in the finite jump discontinuity of the Unital Norm across the negative real axis (for \( \beta \neq 0 \)). The usual norm is not sensitive to this topology.

It is also of interest that, for the \( \beta \neq 0 \) case, the circular symmetry (isotropy) of the norm is lost. As regards multiplication, the real axis “couldn’t be more different” than the imaginary axis. So, significantly, the fundamental distinction of the real and imaginary axes is reflected in the Unital Norm family.

If \((x, y)\) \(\in\) \(\mathbb{C}\) is written in polar form, (4.2) becomes

\[
U_{\beta}(re^{i\theta}) = re^{\beta \theta},
\]

valid on \(\mathbb{R}^2\) exclusive of the origin and negative real axis. The usual norm is \(r^2\), i.e., explicitly the square of an element’s Euclidean norm, while all but one incomplete Unital norm is sensitive to both the Euclidean norm and the phase. Indeed, on a small enough unital neighborhood the mapping defined by \((x, y) \rightarrow (U_{\beta}(x, y), U_{\beta'}(x, y))\) is a diffeomorphism for chosen \(\beta \neq \beta'\). For example,

\[
(r, \theta) = \left( \sqrt{U_{1}(x, y)U_{-1}(x, y)}, \log \sqrt{\frac{U_{1}(x, y)}{U_{-1}(x, y)}} \right).
\]

The prior paragraph leads to the following cute observation. As noted earlier, we have the unique incomplete Unital norm

\[
U(s) = s, \text{ for algebra } \mathbb{R}.
\]

It is thus rather intriguing that “according to (4.3)”,

\[
U_i(s) = s, \text{ for the (real) algebra } \mathbb{C}.
\]

We hasten to add that the latter is true only in a formal sense, as there has been no preparation for introduction of complex matrices as Proto-norms.

As an illustration of Theorem 3.2, we observe that the transpose operation applied to the above matrix subalgebra of \(\mathbb{R}^{2 \times 2}\) isomorphic to \(\mathbb{C}\) does indeed lead to
a normalized Proto-norm. Thus, \( \mathbb{C} \) is isomorphic to the matrix subalgebra of \( \mathbb{R}^{2 \times 2} \) defined by elements of the form \( \begin{bmatrix} x & -y \\ y & x \end{bmatrix}, x, y \in \mathbb{R} \). The transpose operation on that isomorphic matrix algebra obviously corresponds to taking a conjugate as defined on the algebra of complex numbers. But with respect to \( \mathbb{C} \) as having a vector space of elements \( \mathbb{R}^2 \), we notice that \( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \), so the normalized Proto-norm family member (with \( \beta = 0 \)) acting on \( (x, y) \) in the prior equation produces an element’s conjugate - just like the transpose operation on the matrix subalgebra format of \( \mathbb{C} \). That is, in reference to the notation of the proof of Theorem 3.2, transpose \( T \) on the matrix subalgebra of \( \mathbb{R}^{2 \times 2} \) implies \( T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \) relevant to the \( \mathbb{R}^2 \)-vector space of \( \mathbb{C} \). Thus, the transpose operation on the matrix algebra isomorphic to \( \mathbb{C} \) leads to a Proto-norm for \( \mathbb{C} \). It is also easy to verify \( (x, y) T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right)^{-1} = 1 \) (i.e., (1.7)), so the incomplete Unital Norm \( \sqrt{x^2 + y^2} \) follows from Theorem 2.1 applied with \( L = T \).

4.2. Algebra \( \mathcal{C} \) (the split-complex numbers). Applying the analogous arguments as used in Section 4.1, the Proto-norm family in this case is determined by the parametrized matrix \( \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} \), representing a set identical to the elements of \( \mathcal{C} \) in its left regular representation as a subalgebra of \( \mathbb{R}^{2 \times 2} \).

The normalized Proto-norm family is given by \( \left\{ \begin{bmatrix} 1 & \beta \\ \beta & 1 \end{bmatrix} : \beta \in \mathbb{R} \right\} \). Let us take \( \mathcal{N} \) to be the simply connected domain consisting of the open subset of the right half-plane containing \( 1 = (1, 0) \) that is bounded by the lines \( y = x \) and \( y = -x \) (this is uniquely forward as the largest possible connected neighborhood of \( 1_{\mathcal{C}} \) containing only units, and any member of the set is such that its inverse is also in the set). Evaluation of the integral in (2.1) for the points of \( \mathcal{N} \) leads to the 1-parameter family of incomplete Unital Norms,

\[
\mathcal{U}_\beta(x, y) = \sqrt{x^2 - y^2} \left( \frac{x + y}{x - y} \right)^{\frac{\beta}{2}} = \sqrt{x^2 - y^2} e^{\beta \arctanh \left( \frac{x}{y} \right)} = \left( x + y \right)^{\frac{1 + \beta}{2}} \left( x - y \right)^{\frac{1 - \beta}{2}}.
\]

(4.4)

It follows that \( \mathcal{U}_\beta(s) = |x + y|^{\frac{1 + \beta}{2}} |x - y|^{\frac{1 - \beta}{2}} \) is the Unital Norm extension of every incomplete Unital Norm. The hyperbolic norm, \( \mathcal{U}_0(x, y) = \sqrt{x^2 - y^2} \), is a non-singular closed Unital Norm (and is comparable to the usual norm on this algebra, apart from an exponent and the absolute value operation). The only singular Unital Norms are those for which \( \beta = \pm 1 \). These cases are of some interest because \( \mathcal{U}_{\pm 1}(x, y) = |x \pm y| \).

As was the case with \( \mathcal{C} \), the Unital Norm family associated with the algebra of split complex numbers is sensitive to the topology of the set of units (in the present case, the the set of units is not path-connected), due to the Unital Norm singularities at the points \( (x, y) \) for which \( x^2 = y^2 \) when \( \beta \neq 0 \). But unlike the case of \( \mathcal{C} \), the discontinuity in \( \mathcal{U}_\beta(x, y) \), \( \beta \neq 0 \), is unbounded.

As was also the case with \( \mathcal{C} \), \( (x, y) \rightarrow (\mathcal{U}_\beta(x, y), \mathcal{U}_{\beta'}(x, y)) \) is a diffeomorphism in the requisite incomplete Unital Norm neighborhood of \( 1 \) for \( \beta \neq \beta' \).
This algebra is isomorphic to $\mathbb{R} \oplus \mathbb{R}$ (though not isometrically isomorphic), and more relevant features can be inferred from the presentation in the next subsection.

As was done with $\mathbb{C}$ at the end of Section 4.1, we can again illustrate Theorem 3.2 by first noting that $\mathbb{Q}$ is isomorphic to the subalgebra of $\mathbb{R}^{2 \times 2}$ consisting of elements of the form
\[
\begin{bmatrix}
x & y \\
y & x
\end{bmatrix}, x, y \in \mathbb{R}.
\]
In this case, the transpose operation $T$ is simply the identity. This corresponds to the Proto-norm $\overline{T} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ relevant to $\mathbb{Q}$.

Also note that the left regular representation is identical to the Proto-norm family.

4.3. Algebra $\mathbb{R} \oplus \mathbb{R}$. In the expression $\mathbb{R} \oplus \mathbb{R}$, “$\oplus$” signifies the direct sum of algebras. $\mathbb{R} \oplus \mathbb{R}$ is isomorphic to the subalgebra of algebra $\mathbb{R}^{2 \times 2}$ whose members are diagonal matrices. In this example, $\mathbb{1}_{\mathbb{R} \oplus \mathbb{R}} = (1, 1)$ and $\|\mathbb{1}_{\mathbb{R} \oplus \mathbb{R}}\|^2 = 2$. Satisfaction of (1.5) leads to the two-dimensional Proto-norm family given by the parametrized matrix $\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$. Equation (2.1) indicates that the members of the incomplete Unital Norm family are given by
\[
(4.5) \quad U_\sigma(x, y) = x^{\frac{\sigma y}{\sigma y - x}} y^{\frac{x}{x - \sigma y}} = (xy)^{\frac{\sigma}{\sigma y - x}}.
\]
Hence, for this algebra we have a 1-parameter normalized Proto-norm family, all members of which lead to incomplete Unital Norms that can be made into Unital Norms by application of the absolute value operation. $U_0(x, y)$ is the geometric mean, which extends to a closed Unital Norm.

Using (4.5), we have $U_1(x, y) = |x|$, and $U_{-1}(x, y) = |y|$, as singular Unital Norms. Though not a Unital Norm, we obtain the taxicab norm as
\[
U_1(x, y) + U_{-1}(x, y) = |x| + |y|.
\]

The usual norm for this algebra is $xy$, but according to (4.5) all but two incomplete Unital Norms are sensitive to both $xy$ and $\frac{y}{x}$. Thus, as was the case with $\mathbb{C}$ and $\mathbb{Q}$, the mapping defined by $(x, y) \to (U_\beta(x, y), U_{\beta'}(x, y))$ is a diffeomorphism in some unital neighborhood for $\beta \neq \beta'$. For example,$$
(x, y) = \left(U_1(x, y), U_{-1}(x, y)\right).
$$

4.4. Algebra $\mathbb{D}$ (the dual numbers). The algebra of dual numbers is a two-dimensional real algebra, often presented as composed of elements of the form $x + ye, x, y \in \mathbb{R}, e^2 = 0$. The left regular representation maps $(x, y)$ to $\begin{bmatrix} x & y \\ 0 & x \end{bmatrix}$, and so $\|1\|^2 = 1$. This is the $n = 2$ case of Corollary 3.9, but we’ll treat it explicitly here.

We have $(x, y)^{-1} = \left(\frac{x}{x^2}, -\frac{y}{x^2}\right)$ since $\begin{bmatrix} x & y \\ 0 & x \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{x} & -\frac{y}{x^2} \\ 0 & \frac{1}{x} \end{bmatrix}$. We must now find $L = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$ such that (1.5) and (1.7) are satisfied. We have $L \left[(x, y)^{-1}\right] =$...
\[ (\frac{\alpha}{x} - \frac{\beta y}{x^2} - \frac{\gamma y}{x}). \] The partial derivative of the second component with respect to \( x \) minus the partial derivative of the first component with respect to \( y \) is \( \frac{2\beta y}{x^2} \).

Equation (1.5) requires this to vanish uniformly in a unital neighborhood of \((1,0)\). Thus, we must have \( \gamma = 0 \). Hence, the Proto-norm family is \( \begin{bmatrix} \alpha & \beta \\ \beta & 0 \end{bmatrix} \), \( \alpha, \beta \in \mathbb{R} \).

Note that the Proto-norm family is given by the members of the algebra’s left regular representation when each is multiplied on the right by the exchange matrix \( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).

Satisfaction of (1.7) requires \( \alpha = 1 \). Substituting the resulting \( L \) into (2.3) along with \( U(1,0) = 1 \), we find that

\[
\log U(x, y) = \int_1^x [Lt^{-1}] \cdot dt = \int_{(1,0)}^{(x,y)} \left( \frac{1}{x'} - \frac{\beta y'}{x'^2} \frac{\beta}{x'} \right) \cdot (dx', dy').
\]

Taking \( N \) as the open right half-plane, we have a 1-parameter family of incomplete Unital Norms given by

\[
U_\beta(x, y) = xe^{\beta\frac{y}{x}},
\]

which extend to Unital Norms by replacing the right-hand-side above with \( |x|e^{\beta\frac{y}{x}} \).

It is seen that \( U_0(x, y) = |x| \) is a closed singular Unital Norm (and apart from the exponent is comparable to the result forwarded by the usual norm, which is \( x^2 \)). All other members of the family are nonsingular but “unclosed”. These Unital Norms characterize the non-units \((0, v)\), being associated with the singularity defined by \( xe^{\frac{y}{x}} \) as \( x \) tends to 0, which is of a different type (i.e., has different behavior) than the singularities associated with the non-units of the other two-dimensional algebras (i.e., of the types defined by (4.2) and (4.5)).

Furthermore, the nonsingular Unital Norms (i.e., those for which the Proto-norm is nonsingular and which correspond here to \( \beta \neq 0 \)) have additional interest because they are not purely dependent on eigenvalues of an algebra element’s associated left multiplication endomorphism - unlike \( U_0(s) \) and the usual norm. That is, \( x \) is the only eigenvalue of the matrix format of the element \((x, y)\) and determines the usual norm. But from (4.6), \( U_\beta(x, y) \) for \( \beta \neq 0 \) is also dependent on \( y \), which is entirely independent of \( x \).

As was the case with \( \mathbb{C}, \mathbb{Q}, \) and \( \mathbb{R} \oplus \mathbb{R}, (x, y) \to (U_\beta(x, y), U_\beta'(x, y)) \) is a diffeomorphism in the requisite incomplete Unital Norm neighborhood of \( 1 \) for \( \beta \neq \beta' \). For example,

\[
(x, y) = \left( U_0(x, y), U_0(x, y) \log \left[ \frac{U_1(x, y)}{U_0(x, y)} \right] \right).
\]

Before leaving \( \mathbb{D} \), it is worthwhile again highlighting a new feature present with this algebra: for a unit \((x, y)\), all but one incomplete Unital Norm is sensitive to both \( x \) and \( y \), while the usual norm is sensitive only to \( x \).

5. Examples of Proto-norm and Unital Norm families for real unital algebras of higher dimension, including some not associative \(*\)-algebras

The two-dimensional real unital algebras have generalizations as the Cayley-Dixon sequence algebras (\( \mathbb{R} \) and \( \mathbb{C} \) being its lowest dimensional members), the Spin Factor Jordan algebras (generalizing \( \mathbb{Q} \)), and real upper triangular matrix
algebras with at least some redundant diagonal components (generalizing the lowest dimensional case, \(D\)). Their Unital Norm families will be the subject of this section.

Spin Factor Jordan Algebras and Cayley-Dixon Algebras after the quaternions are power-associative but not associative. While the usual norm is thereby inapplicable, there is an “Algebraic Norm” replacement [7] (based on the minimal polynomial, and thus analogous to the usual norm as given by the constant term of a characteristic polynomial). For the Spin Factor Jordan Algebras and Cayley-Dixon Algebras, the Algebraic Norms are essentially the Minkowski norm and Euclidean norm, respectively. Thus, useful norms are not restricted to associative (or alternative) algebras, even though the multiplicative property (i.e., \(N(s_1s_2) = N(s_1)N(s_2)\)) must then be sacrificed [8]. So, despite a strong bias against a norm not satisfying the multiplicative property, these examples indicate that “algebraic” norms that are not multiplicative can nevertheless be very significant [9].

Of course, apart from the omnipresent Unital Norm family member comparable to the usual norm as guaranteed by Corollary 3.3 and Corollary 3.4 (as well as all Unital Norms resulting from two-dimensional real unital algebras), Unital Norms are not typically multiplicative. The multiplicative property is instead replaced by \(U(s)U(s^{-1}) = 1\). The latter equation also replaces the multiplicative property for the Algebraic Norm in the cases of the not associative Spin Factor Jordan Algebras and Cayley-Dixon Algebras. Like the Algebraic Norm, the Unital Norm/Proto-norm constructions remain valid for these latter two classes of algebras, as we now show.

5.1. A closed Unital Norm for \(*\)-algebras whose Hermitian elements are real. Consider an algebra associated with an involutive antiautomorphism “\(*\)” (known as a \(*\)-algebra, star-algebra, or involutive algebra) such that the Hermitian elements are members of \(\mathbb{R}\) (thus, \(s + s^*\) and \(ss^*\) are both real). Then

\[
X^2 - (s + s^*)X + ss^*,
\]

is the minimal polynomial of any not real element \(s\), and annihilates all elements, real and not real. Consistent with comments in the prior paragraphs, we define its constant term to be the “Algebraic Norm” of an element \(s\), by analogy with the characteristic polynomial defined for an associative algebra, which supplies the norm from the constant term of that element-associated polynomial annihilating all elements.

On these \(*\)-algebras, the multiplicative identity \(1\) is 1 (where the latter is the algebra element that is the multiplicative identity on the subalgebra \(\mathbb{R}\) rather than a field element). Accordingly, for these algebras, we define the expression \(\|1\|^2\) (e.g., that appears in Definition 2.1) to be 1. We also have,

**Lemma 5.1.** The inverse of a unit \(s\) in a finite-dimensional \(*\)-algebra whose Hermitian elements are real is unique, and given by \(s^{-1} = s^*ss^*\).

**Proof.** We can write \(s = \frac{s + s^*}{2} + \frac{s - s^*}{2}\), where the first term on the right-hand-side is real and the second term on the right-hand-side is antisymmetric with respect to application of the \(*\)-operation. We define \(\sigma\) to be the first term on the right-hand-side, and \(\bar{s}\) to be the second term on the right-hand-side. Suppose \(x = \chi + \bar{x}\) is an inverse of \(s\), where \(\chi\) is real, and \(\bar{x}\) is antisymmetric. Then

\[
(5.1) \quad sx = \sigma\chi + \sigma\bar{x} + \chi\bar{s} + \bar{s}\bar{x} = 1,
\]
and \(xs = \sigma \chi + \sigma \bar{x} + \chi s + \bar{x} s = 1\). Thus, \(\bar{x} = \bar{x} s\). Since \(s^* = (\frac{\bar{x} + s}{\sigma})^* = -\frac{\bar{x} + s}{\sigma}\), we have \((\bar{x} s)^* = \bar{x}^* s^* = \bar{x} s = \bar{x} s\). Hence, \(\bar{x} = \bar{x} s\) is real. It follows from (5.1) that \(\sigma \chi + \bar{x} = 1\) and \(\sigma \bar{x} + \chi = 0\). The latter has has a unique solution, yielding \(\chi + \bar{x} = \frac{\sigma \bar{x} - \chi}{\sigma^2 - ss^*} = \frac{s^*}{ss^*}\), where the second equality follows since the right-hand-side is obviously an inverse \(1 = s \frac{\bar{x}}{ss^*} = \frac{\bar{x}}{ss^*} s\) because \(s s^*\) is real and thus self-adjoint), and the inverse has just now been seen to be unique. \(\square\)

Applying the “extended” version of Definition 2.1 (as specified immediately following the statement of that definition) to this class of \(*\)-algebras, we can now provide a Unital Norm by identifying a linear transformation \(L\) satisfying (1.5), (1.7) and then integrating (1.6).

**Proposition 5.2.** Given a \(*\)-algebra with vector space of elements \(\mathbb{R}^n\) and whose Hermitian elements are members of \(\mathbb{R}\), the Algebraic Norm on the \(*\)-algebra in a sufficiently small neighborhood of 1 is the square of an incomplete Unital Norm that extends to a closed Unital Norm.

**Proof.** By definition, the Algebraic Norm is
\[
N(s) = ss^* = N(s^*) \in \mathbb{R}.
\]
We then have \(N(\alpha s) = \alpha^2 N(s)\) for \(\alpha \in \mathbb{R}\), i.e., degree-2 homogeneity of \(N(s)\). For \(N(s) \neq 0\), Lemma 5.1 implies
\[
s^{-1} = \frac{s^*}{N(s)}.
\]
Applying \(N(\cdot)\) to both sides of (5.3), the degree-2 homogeneity and (5.2) imply \(N(s)N(s^{-1}) = 1\), so that
\[
\sqrt{N(s)}\sqrt{N(s^{-1})} = 1.
\]
We are required to show that \(\sqrt{N(s)}\) is an incomplete Unital Norm that extends to a closed Unital Norm. Thus, we must first verify that \(\sqrt{N(s)}\) satisfies the requirements of the extended version of Definition 2.1 as indicated prior to the statement of this proposition.

To begin with, (5.4) already satisfies (2.2). Equation (5.2) indicates that \(N(s)\) is a quadratic form, so there is an orthonormal basis with respect to which the members of the algebra can be written as \(s = (s_1, \ldots, s_i, \ldots)\) with \(N(s) = \sum_i \delta_i s_i^2\) and \(\delta_i = -1, 0, 1\), since \(\star\) is an involution. Let \(K_1 \equiv \text{diag}\{\delta_i\}\). Then,
\[
N(s) = s \cdot (K_1 s).
\]
If \(s^{-1}\) exists (i.e., \(N(s) \neq 0\)), we then have
\[
\nabla \left[ \sqrt{N(s)} \right] = \frac{K_1 s}{\sqrt{N(s)}},
\]
Define \(K_2\) to be the algebra’s involutive antiautomorphism operation \(^*\). It follows that \(K_1 K_2 \frac{s}{\sqrt{N(s)}} = \nabla \left[ \sqrt{N(s)} \right]\). Defining \(L \equiv K_1 K_2\), dividing both sides of the equation in the last sentence by \(\sqrt{N(s)}\), and using (5.2) and (5.3), we then have
\[
L s^{-1} = \frac{\nabla \left[ \sqrt{N(s)} \right]}{\sqrt{N(s)}}.
\]
This equation is valid for $N(s) > 0$, and the latter pertains in a sufficiently small neighborhood of 1. Thus, (1.6) is satisfied. From (5.5), (5.6), and (5.7), we have

$$s \cdot [Ls^{-1}] = s \cdot K_1 \left[ \frac{s}{N(s)} \right] = 1,$$

which is (1.7).

To establish that $\sqrt{N(s)}$ is an incomplete Unital Norm, we now only need to verify that (1.5) is satisfied, i.e., $d ([Ls^{-1}] \cdot ds) = 0$. According to (5.6), (5.7), and the definition of $K_1$, we have $d ([Ls^{-1}] \cdot ds) = d \left( \sum_i \frac{\delta_i s_i ds_i}{N(s)} \right)$. Performing the requisite differentiation to obtain the component of the exterior derivative associated with a choice of $i$ and $j$, we find

$$\frac{\partial [Ls^{-1}]}{\partial s_i} - \frac{\partial [Ls^{-1}]}{\partial s_j} = -2 \frac{\delta_i s_j (\delta_i s_i)}{N^2(s)} + 2 \frac{\delta_i s_i (\delta_j s_j)}{N^2(s)} = 0,$$

where $\frac{\partial [Ls^{-1}]}{\partial s_i}$ is the partial derivative of the $j$-th component of $[Ls^{-1}]$ with respect to the $i$-th component of $s$, etc. Thus, $d ([Ls^{-1}] \cdot ds) = 0$.

Consequently, $\sqrt{N(s)}$ satisfies the incomplete Unital Norm requirements of the appropriately modified Definition 2.1. It is then easy to verify that $\sqrt{|N(s)|}$ is a closed Unital Norm.

5.2. The Cayley-Dixon sequence. From the proof of Proposition 5.2, it is easily shown that a normalized Proto-norm associated with an algebra of the Cayley-Dixon sequence (all of whose members are $\ast$-algebras whose Hermitian elements are real) is given by diag\{1, −1, −1, . . . , −1\}, with the associated Unital Norm seen to be the Euclidean norm. Although it is seen in Section 4.1 that $C$ is associated with a 1-parameter family of normalized Proto-norms, it can be shown that all other algebras of the sequence have only the above normalized Proto-norm, i.e., the Unital Norm family has only one member.

5.3. Spin Factor Jordan Algebras. Consider the bilinear product “$\cdot$” such that for $\alpha, \beta \in \mathbb{R}$, $a, b \in \mathbb{R}^m$,

$$(\alpha + a) \cdot (\beta + b) = [\alpha \beta + a \cdot b] + [\beta a + \alpha b] \in \mathbb{R} \oplus \mathbb{R}^m.$$  

(5.8)

This makes the vector space $\mathbb{R} \oplus \mathbb{R}^m$ a unital algebra with the multiplicative identity 1 as simply 1 (this is an abuse of notation, since 1 is an algebra element while 1 is a field element; to be precise, in (5.8) the term $(\alpha + a)$ could be instead written as $a = (\alpha 1 + a)$, so that algebra elements are presented in boldface). For $m \geq 2$, this is known as a Spin Factor Jordan Algebra [9].

There are two involutive antiautomorphisms on each of these commutative algebras, one of which is the identity and the other is the conjugate operation, $(\alpha + a)^* = (\alpha - a)$. The latter is the only one of the two involutive antiautomorphisms for which the Hermitian elements are real. According to Proposition 5.2, the Unital Norm is the Minkowski norm.

5.4. The algebra with vector space of elements $\mathbb{R}^n$ and component-wise multiplication. A higher dimensional analogue of the example of Section 4.3, taken as the algebraic direct sum of $n$ copies of $\mathbb{R}$, is associated with an $(n - 1)$-parameter family of Unital norms.
This algebra is isomorphic to the subalgebra of $\mathbb{R}^{n \times n}$ whose members are diagonal matrices. In this example, $1 = (1, \ldots, 1)$ and $\|1\|^2 = n$. The two constraints controlling the Unital Norm family are (1.5) and (1.7). It is easily appreciated that (1.5) forwards the Proto-norm family $\text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$, so the Proto-norm family is the same as the members of the left regular representation of the algebra. Equation (1.7) introduces the constraint $\sum_1^n \sigma_i = n$, but otherwise arbitrary $\sigma_i \in \mathbb{R}$. This means that for $s = (x_1, \ldots, x_n)$, we must have $L[(x_1, x_2, \ldots, x_n)^{-1}] = (\frac{\alpha_1}{x_1}, \frac{\alpha_2}{x_2}, \ldots, \frac{\alpha_n}{x_n})$. Evaluation of (2.1) indicates that the members of the Unital Norm family are given by

$$U_{\sigma_1, \ldots, \sigma_n}(x_1, \ldots, x_n) = (x_1^{\sigma_1} x_2^{\sigma_2} \ldots x_n^{\sigma_n})^\frac{1}{n},$$

with $\sum_1^n \sigma_i = n$. Thus, for this algebra we have a $(n - 1)$-dimensional Unital Norm family, all members of which can be made into Unital Norms by application of the absolute value operation. The choices where one or more of the $\sigma_i$ vanish yield singular Unital Norms (some of which are closed).

Similar to the analogous case in two dimensions (Section 4.3), members of the Unital Norm family exist that are sensitive only to any desired subset of an element’s components, and all units can potentially be reconstructed from combinations of different Unital Norm values - unlike with the usual norm.

The geometric mean is the incomplete Unital Norm given by (5.9) with $\sigma_1 = \sigma_2 = \cdots = \sigma_n = 1$ with $\mathcal{N}$ as the open first orthant.

5.5. Upper triangular matrix algebras with independent entries on the main diagonal. Consider the real three-dimensional algebra $(x, y, z) \leftrightarrow \begin{bmatrix} x & z \\ 0 & y \end{bmatrix}$, where identification of a vector space member with a matrix via “$\leftrightarrow$” indicates the manner in which multiplication is performed. Or in other words, the algebra is

$$\left\{ \begin{bmatrix} x & z \\ 0 & y \end{bmatrix} \right\}, \; x, y, z \in \mathbb{R}.$$ 

We have $(x, y, z)^{-1} = \left( \frac{1}{x}, \frac{1}{y}, -\frac{x}{xy} \right)$, and $1 = (1, 1, 0)$, and $\|1\|^2 = 2$. Starting with a generic self-adjoint $L = \begin{bmatrix} \alpha & \beta & \gamma \\ \beta & \delta & \epsilon \\ \gamma & \epsilon & \xi \end{bmatrix}$, the curl of $L(x, y, z)^{-1}$ is then $\left( \frac{x(y-x)+\xi z}{xy^2}, \frac{\gamma(y-x)-\xi z}{xy^2}, \frac{\beta(x^2-y^2)-\gamma xy z}{x^2y^2} \right)$. Since this is required to vanish in a neighborhood of $(1, 1, 0)$, all entries of the matrix $L$ must be zero except for $\alpha$ and $\delta$. Thus, $L(x, y, z)^{-1} = (\frac{x}{y}, \frac{y}{x}, 0)$. From (1.7), we then have the requirement that

$$(x, y, z) \cdot [L(x, y, z)^{-1}] = \alpha + \delta = 2.$$ 

Equivalently, we can alternatively specify $\alpha \equiv 1 + \sigma$ and $\delta \equiv 1 - \sigma$, for arbitrary $\sigma \in \mathbb{R}$. We thus have the normalized Proto-norm family composed of all

$$L = \begin{bmatrix} 1 + \sigma & 0 & 0 \\ 0 & 1 - \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$\sigma \in \mathbb{R}$. Equation (2.1) then leads to the 1-parameter family of incomplete Unital Norms

$$U_\sigma(x, y, z) = x^{\frac{1+\sigma}{2}} y^{\frac{1-\sigma}{2}} = (xy)^{\frac{\sigma}{2}} \left( \frac{x}{y} \right)^{\frac{\sigma}{2}}.$$
Each member of the family is evidently singular (i.e., $L$ is singular). Nevertheless, every member of the family extends to a Unital Norm by exchanging the right-hand-side of (5.11) for $|x|^{1+\sigma} |y|^{1-\sigma}$ - and the case of $\sigma = 0$ corresponds to a closed Unital Norm.

This analysis easily generalizes to an $n \times n$ triangular matrix algebra whose elements can have distinct eigenvalues. The domain of the Unital Norm has dimension $\frac{n(n+1)}{2}$, but the value assigned the Unital Norm is the same as the right-hand-side of (5.9).

The left regular representation of an element $(x, y, z)$ is given by

$$
\begin{bmatrix}
x & 0 & 0 \\
0 & y & 0 \\
0 & 0 & z
\end{bmatrix}.
$$

Thus, the $\mathbb{R}^{3 \times 3}$ transpose operation maps outside the algebra’s representation. However, for the subalgebra defined by elements $(x, y, 0)$, the left regular representation is

$$
\begin{bmatrix}
x & 0 & 0 \\
0 & y & 0
\end{bmatrix}.
$$

The usual norm of the subalgebra is thus $xy$, and the incomplete Unital norm on the subalgebra is $(xy)^{\frac{1}{2}}$, and these differ only by an exponent - consistent with Theorem (3.2). Note that the latter incomplete Unital Norm on the subalgebra provides the values for an incomplete Unital Norm on the full algebra as in (5.11), corresponding to the choice of normalized Proto-norm in (5.10) given by $\sigma = 0$, i.e., $\mathcal{U}(x, y, z) = (xy)^{\frac{1}{2}}$, consistent with Corollary 3.3. The fact that the transpose operation on $\mathbb{R}^{3 \times 3}$ does not map the left regular representation of

$$
\left\{ \begin{bmatrix} x & z \\ 0 & y \end{bmatrix} : (x, y, z) \in \mathbb{R}^3 \right\}
$$

to itself explains why there is no incomplete Unital Norm that is a power of the usual norm $x^2 y$.

5.6. Some higher dimensional algebras with Unital Norms that are sensitive to more element components than is usual norm. The usual norm on $\mathbb{R}^{n \times n}$ is sensitive to all components of an element of the algebra (because the determinant of an element of $\mathbb{R}^{n \times n}$ is sensitive to the values of all entries of its matrix). However, this is not true in general for associative algebras. For example, for any algebra isomorphic to a triangular matrix algebra, the usual norm is only sensitive to the components on the main diagonal of the triangular matrix representation. This is also true for the Unital Norm if all entries on the main diagonal of the triangular matrix representation are allowed to be distinct. But if at least two entries on the main diagonal always take equivalent values for the particular algebra, then almost all of its Unital Norms family members will be sensitive to one or more components not on the main diagonal, in addition to all of the components on the main diagonal (these are matrix algebras for which an element that is not a diagonal matrix is never diagonalizable).

The dual numbers have already been presented as a two-dimensional example of this in Section 4.4 and this fully generalizes according to Corollary 3.9. We will now highlight an additional feature using the next three examples.

5.6.1. A three-dimensional algebra. Consider the real algebra with elements

$$(x, z, w) \leftrightarrow \begin{bmatrix} x & z & w \\ 0 & x & z \\ 0 & 0 & x \end{bmatrix}.$$
The constraints (1.5) and (1.7) lead to the normalized Proto-norms,

\[ L = \begin{bmatrix} 1 & \beta & \gamma \\ \beta & 0 & 0 \\ \gamma & 0 & 0 \end{bmatrix}, \quad \beta, \gamma \in \mathbb{R} \]

which is just the \( n = 3 \) case of Proposition 3.7. Evaluating (2.1), we now have a 2-parameter family of incomplete Unital Norms,

\[(5.12) \quad U_{\beta,\gamma}(x, z, w) = x e^{\beta \frac{z}{2} + \gamma \frac{z}{2} - \frac{1}{4} (\frac{z}{2})^2}.\]

The choice \( \beta = \gamma = 0 \) produces a singular closed Unital Norm, \( U_{0,0}(x, z, w) = |x| \).

The other Unital Norms in the family are nonsingular, but more importantly are distinguished by their dependence on \( z \) and \( w \), i.e., unlike the usual norm. The non-units are associated with a different type of singularity than in any of the other examples we have considered.

5.6.2. A four-dimensional algebra. Consider the real unital algebra with elements

\[(x, v, z, w) \leftrightarrow \begin{bmatrix} x & z & w \\ 0 & x & v \\ 0 & 0 & x \end{bmatrix}.\]

Straightforward application of the constraints (1.5) and (1.7) leads to the normalized Proto-norm family

\[ \begin{bmatrix} 1 & \beta & \gamma & \delta \\ \beta & 0 & \frac{1}{2} \delta & 0 \\ \gamma & \frac{1}{2} \delta & 0 & 0 \\ \delta & 0 & 0 & 0 \end{bmatrix}, \quad \text{and the associated Unital Norms}, \]

\[ U_{\beta,\gamma,\delta}(x, v, z, w) = x e^{\beta \frac{z}{2} + \gamma \frac{z}{2} + \delta \frac{z}{2} - \frac{1}{4} (\frac{z}{2})^2}.\]

5.6.3. A five-dimensional algebra. Consider the real unital algebra with elements

\[(x, y, v, z, w) \leftrightarrow \begin{bmatrix} x & z & w \\ 0 & x & v \\ 0 & 0 & y \end{bmatrix}.\]

Straightforward application of the constraints (1.5) and (1.7) leads to the normalized Proto-norm family

\[ \begin{bmatrix} 1 + \sigma & 0 & \beta & 0 & 0 \\ 0 & 1 - \sigma & 0 & 0 & 0 \\ \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and the associated Unital Norms}, \]

\[ U_{\sigma,\beta}(x, y, v, z, w) = |xy|^\frac{1}{2} \left| \frac{x}{y} \right|^a e^{\frac{\beta z}{2}}.\]

In this example, almost all Unital Norms are influenced the element’s \( z \)-component, while this sensitivity is not present with the usual norm (given by \( x^4 y \)). However, like the usual norm, the Unital Norm remains insensitive to the \( w \) and \( v \) components. Once again, there is a new singularity type associated with the non-units.
Finally, the left regular representation of this algebra with elements \((x, y, v, z, w)\)
is given by
\[
\begin{bmatrix}
x & 0 & 0 & 0 & 0 \\
0 & y & 0 & 0 & 0 \\
v & x & 0 & 0 & 0 \\
z & 0 & 0 & x & 0 \\
w & z & 0 & 0 & x
\end{bmatrix}.
\]
The transpose operation on \(\mathbb{R}^{5 \times 5}\) does not map this representation algebra into itself, due to its effect on the last three columns and last three rows - which explains why there is no incomplete Unital Norm that is a power of the usual norm \(x^4y\). However, modifying the transpose operation to act only on the \((2 \times 2)\)-submatrix of values in the upper left corner of the representation (while suppressing the components \(v, z, w\)) produces the incomplete Unital Norm \((xy)^{\frac{1}{2}}\) on the full algebra, \(\mathcal{U}(x, y, v, z, w) = (xy)^{\frac{1}{2}}\), consistent with Corollary 3.3.

6. The functor associated with application of Proto-norm families

6.1. Summary of results up to this point. The usual norm is defined from the solitary mapping given by the left regular representation of the algebra. But an incomplete Unital Norm is defined by any mapping in the normalized Proto-norm family associated with the algebra - and one of these mappings supplies a Unital Norm that gives results that are essentially the same as the usual norm if the algebra given by the left regular representation has the feature that the transpose operation maps it to itself - Theorem 3.2 (otherwise, there is a Unital Norm family member producing values essentially equivalent to those produced by the usual norm on a subalgebra - Corollary 3.3 and Corollary 3.4). Furthermore, a Proto-norm family is always at least one-dimensional, and a Unital Norm family is always nonempty. The Proto-norm family is thereby a potentially much richer reflection of an algebra’s structure than the usual norm. Thus, when the family has more than one member, various novel features can emerge.

- As indicated by Corollary 3.9, the algebra of upper triangular Toeplitz matrices provides an instance where the Unital Norms are especially impressive due to their “dominance” as regards sensitivity to all algebra element components compared to the usual norm, which in this case is sensitive to only a single element component. Other algebras with this greater element component sensitivity of the Unital Norm can be easily constructed, as upper triangular matrix algebras almost all of whose elements are defective matrices.
- It may be possible to reconstruct a unit from its incomplete Unital Norm values. For example, a unit in any of the two-dimensional real unital algebras can be reconstructed from the two norm values determined by two of its incomplete Unital Norms, if the unit is in a unital neighborhood of 1.
- The non-units are associated with an algebra-dependent type of singularity reflecting the topology of the space of units.
- Some algebras are “essentially the same” as their Proto-norm families in a certain sense. For example, when the members of the left regular representation of the algebra of complex numbers, or of the algebra of dual numbers, or of the algebra of real upper triangular Toeplitz matrices, are each multiplied by the exchange matrix (the matrix with entry value 1 on its antidiagonal and 0 everywhere else), the result in each case is the Proto-norm family of the algebra. In cases like these, every member of the algebra has
dual roles in that it implies a first linear transformation as its image under
the left regular representation, along with a second linear transformation
(again acting on the algebra itself) as a Proto-norm.

Table I summarizes examples presented in Sections 4 and 5.

The notion that Proto-norm families and Unital Norm families characterize
the inversion operation of an algebra is further formalized in the functor presented in
Section 6.2. Section 6.3 uses Table I to illustrate the functor.

6.2. The functor. A superficial perusal of the Table I’s last column seems to
indicate that various of the algebras have relationships through their Proto-norm
families. That notion is confirmed in the construction of a functor, as accomplished
in this subsection.

From now on, “algebra” will be taken to mean a unital associative algebra whose
vector space of elements is \( \mathbb{R}^n \), unless indicated otherwise.

In practical terms, to compute a Proto-norm family of some algebra, whose
family members are thereby real symmetric \((n \times n)\)-matrices with respect to the
standard basis, we start with a generic symmetric \((n \times n)\)-matrix each of whose
entries is a real parameter (e.g., \( \alpha, \beta, \gamma, \ldots \)). The \( \frac{n(n+1)}{2} \) entries in the upper
triangular portion of the matrix are initially each a different parameter, and the
other entries of the matrix are determined by symmetry. The uncurling constraint
(1.5) is then applied, which leads to some matrix entries possibly being given the
value zero, and others being some other linear combination of the original
\( \frac{n(n+1)}{2} \) parameters. Thus, the Proto-norm family is described by a parametrized matrix,
whose entries are linear combinations of components of a \( m \)-dimensional parameter
vector, with \( 1 \leq m \leq \frac{n(n+1)}{2} \), where \( m \) is the number of distinct parameters
ultimately present in the parametrized matrix. We consider that the parametrized
matrix is equivalent to the set of real matrices given by all possible realizations
of the parameter vector as it ranges over \( \mathbb{R}^m \) (where a “realization” is a choice of
a real value for each component of the parameter vector), and in the sequel the
latter interpretation of any parametrized matrix is always understood. Thus, the
Proto-norm family is a \( m \)-dimensional space represented by the above parametrized
matrix.

We now define \( \mathcal{A} \) to be the category whose objects are the unital associative
algebras whose vector space of elements is \( \mathbb{R}^n \) where \( n \) is any particular positive
integer, and whose morphisms are algebra epimorphisms. Composition of mor-
phisms is simply composition of the epimorphisms, and thereby associative. The
identity morphisms obviously exist. We always assume the standard basis on \( \mathbb{R}^n \),
and with this understanding in the sequel we will use the same symbol, e.g., \( K \),
to denote an epimorphism and its associated matrix with respect to the standard
basis, depending on the context.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
algebra & Algebraic Norm & Unital Norm family & Proto-norm family \\
\hline
$\mathbb{R}^{n \times n}$ & $(\det(s))^n$ & $(\det(s))^\frac{1}{n}$ & $\alpha (n^2 \times n^2)$-“transpose matrix” \\
\hline
$\mathbb{C}$ & $x^2 + y^2 = r^2$ & $\sqrt{x^2 + y^2} e^{\beta \arctan(\frac{y}{x})} = re^{\beta \theta}$ & $\begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix}$ \\
\hline
$\mathbb{E}$ & $x^2 - y^2$ & $\sqrt{x^2 - y^2} e^{\beta \arctan(\frac{y}{x})} = (x^2 - y^2)\left(\frac{1}{2} \left(\frac{y}{x} + \frac{x}{y}\right)\right)$ & $\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$ \\
\hline
$\mathbb{R} \oplus \mathbb{R}$ & $xy$ & $(xy)^\frac{1}{2} \left(\frac{z}{y}\right)^\frac{2}{3}$ & $\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$ \\
\hline
$\mathbb{D}$ & $x^2$ & $x e^{\beta \frac{y}{x}}$ & $\begin{bmatrix} \alpha & \beta \\ \beta & 0 \end{bmatrix}$ \\
\hline
$\mathbb{H}$ & $(x^2 + y^2 + z^2 + w^2)^2$ & $\sqrt{x^2 + y^2 + z^2 + w^2}$ & diag\{a, -a, -a, -a\} \\
Cayley-Dixon seq., after $\mathbb{H}$ & (Euclidean norm)$^2$ & Euclidean Norm & diag\{a, -a, -a, ..., -a\} \\
Spin Factor Jordan & (Minkowski norm)$^2$ & Minkowski norm & diag\{a, a, ..., a\} \\
$\bigoplus_{i=1}^n \mathbb{R}$ & \(\prod x_i\) & \((\prod x_i^\sigma)^\frac{1}{\sigma}, \sum \sigma_i = n\) & diag\{\sigma_1, ..., \sigma_n\} \\
\hline
\end{tabular}
\caption{“Algebraic Norm” means the usual norm or its extension to not associative $^*$-algebras whose Hermitian elements are real.}
\end{table}
We define category \( \mathcal{P} \) as follows. An object in this category is a symmetric parametrized \((n \times n)\)-matrix where \( n \) is any particular positive integer, and where each matrix entry is a linear combination of the components of a \( m \)-parameter vector, \( m \geq 1 \), so that the set of real matrices implied by all realizations of the \( m \)-vector of parameters is an \( m \)-dimensional space. Consider two objects \( P_1, P_2 \) of \( \mathcal{P} \), where \( P_1 \) is a \((j \times j)\)-matrix parametrized by the components of a \( k \)-parameter vector and \( P_2 \) is a \((q \times q)\)-matrix parametrized by the components of a \( r \)-parameter vector. There is a morphism \( P_1 \to P_2 \) if and only if there exists a real \((q \times j)\)-matrix \( V \) such that \( VV' \) is nonsingular and \( V'P_2V \subseteq P_1 \), where the latter subset expression is interpreted to mean that the set of real matrices implied by \( V'P_2V \) through all realizations of the \( r \)-parameter vector associated with \( P_2 \), is a subset of the set of real matrices implied by \( P_1 \) through all realizations of the associated \( k \)-parameter vector. If it exists, the morphism is clearly unique. Note the requirement that \( VV' \) be nonsingular assures that the object \( V'P_2V \) will have the same number of parameters as \( P_2 \), since nonsingularity of \( VV' \) assures that \( V'P_2V V' \) has as many parameters as \( P_2 \), while \( V'P_2V \) cannot have a greater number of parameters than \( P_2 \). The identity morphisms obviously exist.

Now consider the expression \( P_1 \xrightarrow{p_1} P_2 \xrightarrow{p_2} P_3 \). It implies that there exists a real matrix \( V_2 \) such that \( V_2'P_3V_2 \subseteq P_2 \) with \( V_2V_2' \) nonsingular, and there exists a real matrix \( V_1 \) such that \( V_1'P_2V_1 \subseteq P_1 \) with \( V_1V_1' \) nonsingular. This implies \( V_1'V_2'P_3V_2V_1 \subseteq P_1 \), i.e., \( (V_2V_1)'(V_3V_2)(V_1V_1') \subseteq P_1 \). For \( P_1 \xrightarrow{f} P_3 \) to exist, it is then sufficient that \( (V_2V_1)'(V_3V_2)(V_1V_1')V_1' \) be nonsingular. To see that the latter is indeed nonsingular, first recall that \( V_1V_1' \) is nonsingular. This is the matrix of some linear transformation, so there is a basis such that \( V_1V_1' \) can be replaced by the identity matrix. With respect to this basis, \( V_2 \) is replaced by the matrix \( V_2V_2' \), and the matrix \( V_2V_2' \) is replaced by the matrix \( V_2V_2'' \), which is then evidently nonsingular since \( V_2V_2'' \) is nonsingular. But with respect to the new basis, \( V_2V_1V_2' \) becomes \( V_2V_2'' \), and is thus nonsingular, as was required to be shown. Thus, \( P_1 \xrightarrow{p_1} P_2 \xrightarrow{p_2} P_3 \) implies the existence of \( P_1 \xrightarrow{f} P_3 \), which is unique since all category \( \mathcal{P} \) morphisms are unique. We define \( p_1 \circ p_2 = f \). Associativity of morphism composition is obvious.

A mapping \( F \) from category \( \mathcal{A} \) to category \( \mathcal{P} \) is defined as follows.

\( \mathcal{F}[A] \) is the Proto-norm family of an object \( A \) in category \( \mathcal{A} \) as defined in Definition 2.1. \( \mathcal{F}[A] \) is obviously an object of category \( \mathcal{P} \). For an algebra epimorphism, implying the category \( \mathcal{A} \) morphism \( A_1 \xrightarrow{a} A_2 \), we define \( \mathcal{F}(a) \) to be the category \( \mathcal{P} \) morphism \( \mathcal{F}[A_1] \xrightarrow{\mathcal{F}(a)} \mathcal{F}[A_2] \). If this morphism exists, it is unique (a feature of category \( \mathcal{P} \) morphisms in general, as noted above). So, in order to demonstrate that \( \mathcal{F} \) is a functor, we only need to show that this morphism does indeed exist, and that \( \mathcal{F} \) satisfies the requisite commutative diagram.

**Theorem 6.1.** \( \mathcal{F} : \mathcal{A} \to \mathcal{P}, \) is a covariant functor.

**Proof.** We will first show that the category \( \mathcal{A} \) morphism \( A_1 \xrightarrow{a} A_2 \) implies the existence of category \( \mathcal{P} \) morphism \( \mathcal{F}[A_1] \xrightarrow{\mathcal{F}(a)} \mathcal{F}[A_2] \).

Morphism \( a \) results from an algebra epimorphism \( K : A_1 \to A_2 \). Let \( I_{A_2} \) be the multiplicative identity on \( A_2 \). Then \( I_{A_2} = K(ss^{-1}) = K(s)K(s^{-1}) \), so that \( (Ks)^{-1} = K(s^{-1}) \). By definition, given the Proto-norm family \( \mathcal{F}[A_2] \) associated with \( A_2 \) and an arbitrary real matrix \( \tilde{F}[A_2] \) in the set of real matrices implied by
\[ F[A_2], \text{ we have } d \left( (s_2^{-1})' F[A_2](ds_2) \right) = 0 \text{ in a unital neighborhood of } 1_{A_2}, \text{ where } s_2 \text{ is notation for a variable in } A_2. \] Taking \( s_1 \) to be notation for a variable in \( A_1 \), and with respect to appropriate unital neighborhoods in \( A_1 \) and \( A_2 \), we then have

\[
0 = d \left( (s_2^{-1})' F[A_2]ds_2 \right) = d \left( ((Ks_1)^{-1})' F[A_2] d(Ks_1) \right) = d \left( (s_1^{-1})' [K' F[A_2]K] ds_1 \right),
\]

using the equation at the end of the second sentence of this paragraph, and the fact that a differential commutes with a linear transformation. Equation (6.1) is simply an instance of (1.5), so it follows that symmetric \( K' F[A_2]K \) is a Proto-norm associated with \( A_1 \). Consequently, \( K' F[A_2] K \subset F[A_1] \). Furthermore, \( KK' \) is nonsingular since \( K \) is an epimorphism. Thus, criteria for the existence of morphism \( F(a) \) are fulfilled (with \( V = K \)).

Commutativity of the following diagram is then evident from the uniqueness of category \( \mathcal{P} \) morphisms, and the way composition of category \( \mathcal{P} \) morphisms is defined,

\[
\begin{array}{ccc}
A_1 & \xrightarrow{a_1} & A_2 & \xrightarrow{a_2} & A_3 \\
\downarrow & & \downarrow & & \downarrow \\
F[A_1] & \xrightarrow{F(a_1)} & F[A_2] & \xrightarrow{F(a_2)} & F[A_3]
\end{array}
\]

Thus, \( F \) is a covariant functor. \( \square \)

We then have the following useful results. The first is immediate.

**Corollary 6.2.** An epimorphism from one algebra to another algebra can exist only if there is a category \( \mathcal{P} \) morphism associated with their respective Proto-norm families.

**Corollary 6.3.** An epimorphism from a first algebra to a second algebra can exist only if the dimension of the Proto-norm family of the first algebra is greater than or equal to the dimension of the Proto-norm family of the second algebra.

*Proof.* Given algebra epimorphism \( K : A_1 \to A_2 \), suppose \( A_1 \) is a \( j \)-dimensional algebra with a Proto-norm family utilizing a \( k \)-parameter vector (and thereby the Proto-norm family is a \( k \)-dimensional space), and \( A_2 \) is a \( q \)-dimensional algebra with a Proto-norm family utilizing a \( r \)-parameter vector (and thereby the Proto-norm family is a \( r \)-dimensional space). Evidently, \( K \) is a \((q \times j)\)-matrix with \( q \leq j \) and \( KK' \) is nonsingular (since \( K \) is an epimorphism). The proof of Theorem 6.1 has shown that \( K' F[A_2]K \subset F[A_1] \). Thus, \( K' F[A_2]K \) cannot have more parameters than \( F[A_1] \) (i.e., the dimension of \( K' F[A_2]K \) cannot exceed the dimension of \( F[A_1] \)). Furthermore, \( K' F[A_2]K \) has at least as many parameters as \( KK' F[A_2]K' \), but it has less than or equal to the number of parameters of \( F[A_2] \). However, \( KK' F[A_2]K' \) has the same number of parameters as \( F[A_2] \) since \( KK' \) is nonsingular. Hence, the dimension of \( F[A_2] \) cannot exceed the dimension of \( F[A_1] \). \( \square \)

**Corollary 6.4.** If the Proto-norm family of an algebra is one-dimensional and contains a nonsingular member, then the algebra is simple.
Proof. Suppose algebra $A_1$ has a non-trivial ideal and its Proto-norm family is one-dimensional and has a nonsingular member. It follows that $F[A_1]$ is represented by $\alpha L_1$, $\alpha \in \mathbb{R}$, where $L_1$ is a real nonsingular $(j \times j)$-matrix. Since $A_1$ has a non-trivial ideal, there exists an algebra epimorphism $K : A_1 \to A_2$ where $A_2$ has smaller dimension than $A_1$. Corollary 6.3 indicates that the Proto-norm family of $A_2$ cannot have dimension greater than one, and thus $F[A_2]$ is represented by $\alpha L_2$ where $L_2$ is a real $(q \times q)$-matrix with $q < j$. From the proof of Theorem 6.1, we must then have that the nonzero matrix $K' L_2 K$ is a member of the Proto-norm family $F[A_1]$. But the determinant of the nonzero matrix $K' L_2 K$ is zero since $K$ has a nontrivial kernel, while the only member of the Proto-norm family $F[A_1]$ having zero determinant is the zero matrix (corresponding to $\alpha = 0$), since $L_1$ is necessarily nonsingular. It follows that $K$ cannot exist, and so $A_1$ is simple. \(\square\)

In view of Theorem 3.1, an easy consequence of Corollary 6.4 is an alternative demonstration that the algebra of real $(n \times n)$-matrices $\mathbb{R}^{n \times n}$ is simple.

Note that the converse of Corollary 6.4 is not true. For example, $\mathbb{C}$ is simple but has a two-dimensional Proto-norm family (see Section 4.1).

In the proof of Theorem 6.1, the transformation $F[A_2] \to K'F[A_2]K$ plays a key role. The transformation is invertible since $K$ is an algebra epimorphism from $A_1$ to $A_2$. That is, for $P \equiv F[A_2]$ and $\hat{P} \equiv K'F[A_2]K = K'PK$, we have $P = (KK')^{-1}KPK(KK')^{-1}$. Despite fundamental differences, the format of the transformation of Proto-norm families in the first sentence of this paragraph is reminiscent of a similarity transformation of matrices, with $K'$ substituting for $K^{-1}$. In the case where $K$ is invertible (i.e., the epimorphism is an isomorphism), the analogy can be made explicit as follows.

**Definition 6.1.** $M_1$ and $M_2$ as members of matrix rings are similar if they are matrices representing the same linear transformation with respect to different bases.

**Definition 6.2.** $P_1$ and $P_2$ as objects in category $\mathcal{P}$ are similar if they are Proto-norm families of isomorphic algebras.

**Theorem 6.5.** If $M_1$ and $M_2$ are similar matrices as in Definition 6.1, then $M_1 = K^{-1}M_2K$ for some real nonsingular matrix $K$.

**Theorem 6.6.** If $P_1$ and $P_2$ are similar Proto-norm families as in Definition 6.2, then $P_1 = K'P_2K$ for some real nonsingular matrix $K$.

**Proof.** If $P_1$ and $P_2$ are similar objects of category $\mathcal{P}$, then by definition there exists an algebra isomorphism $K : A_1 \to A_2$ where $A_1, A_2$ are the algebras whose Proto-norm families are $P_1, P_2$, respectively. Using the same argument as in the second paragraph of the proof of Theorem 6.1, we have both $K'P_2K \subset P_1$ and $(K^{-1})'P_1K^{-1} = (K')^{-1}P_1K^{-1} \subset P_2$. It follows that $P_1 = K'P_2K$. \(\square\)

The above proof immediately implies,

**Corollary 6.7.** The dimension of a Proto-norm family is invariant under an isomorphism of algebras.

Despite Theorem 6.6, important distinctions between the Proto-norm family transformation and the matrix similarity transformation are that $K$ in the theorem statement is not unique (there is an equivalence class of such matrices) and there is no analogue of group notions related to conjugacy classes.

The following are used in Section 6.3.2.
Definition 6.3. An object of $P$ is nonnegative if the determinants of all of its implied real matrices are nonnegative, it is nonpositive if none of the determinants of its implied real matrices are positive, and it is otherwise indefinite.

Corollary 6.8. If there is an isomorphism between two algebras then their Proto-norm families are either both nonnegative, both nonpositive, or both indefinite. Furthermore, the determinants of the respective Proto-norm families are dependent on the same number of parameters.

Proof. If $K : A_1 \rightarrow A_2$ is the algebra isomorphism, then it follows from Theorem 6.6 that $\det(F[A_1]) = \det(K'F[A_2]K) = \det(K'K)\det(F[A_2])$. Corollary 6.8 immediately follows. □

Similarly,

Corollary 6.9. For a given algebra, let $k_{\text{min}}$ be the minimum dimension of the kernels of the members of the algebra’s normalized Proto-norm family, and let $k_{\text{max}}$ be the maximum dimension of the kernels of the members of the algebra’s normalized Proto-norm family. Then $k_{\text{min}}$ and $k_{\text{max}}$ are isomorphism invariants.

6.3. Table I examples of the functor in action. Table I in Section 6.1 readily provides examples of application of the functor. We denote the algebras in rows 1 through 14 (the first column of Table I) as $A_1, A_2, \ldots, A_{14}$.

6.3.1. Row 1. We have already noted that since $\mathbb{R}^{n \times n}$ is associated with a 1-parameter Proto-norm family (Theorem 3.1), the functor indicates that this algebra is simple (Corollary 6.4).

6.3.2. Rows 3 and 4. The algebras pertaining to these rows, $\mathbb{C}$ and $\mathbb{R} \oplus \mathbb{R}$, are isomorphic. Their Proto-norm family members are given by the parametrized matrices as the final entry in row 3 and the final entry in row 4 of Table I. If we set $\sigma_1 = \alpha - \beta$ and $\sigma_2 = \alpha + \beta$, defining a new set of two independent parameters, then the parametrized matrices $\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$ and $\begin{bmatrix} \alpha - \beta & 0 \\ 0 & \alpha + \beta \end{bmatrix}$ represent the same set of real matrices, i.e., they are the same Proto-norm family. For $V = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, we have $\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} = V' \begin{bmatrix} \alpha - \beta & 0 \\ 0 & \alpha + \beta \end{bmatrix} V$. That is, the morphism of Proto-norm families exists, as required. As indicated in remarks following the statement of Corollary 6.7, other satisfactory choices for $V$ exist, and these include the matrix defining the algebra isomorphism between $\mathbb{R} \oplus \mathbb{R}$ and $\mathbb{C}$ (as expected from the proof of Theorem 6.6).

While $\mathbb{C}$ and $\mathbb{R} \oplus \mathbb{R}$, are isomorphic, an isomorphism between any two of $\mathbb{C}, \mathbb{C}$, and $\mathbb{D}$ is excluded by Corollary 6.8.

6.3.3. Rows 1, 2, 3, 4, and 5. The $n = 1$ case of row 1 refers to the algebra $\mathbb{R}$, and its Proto-norm family, i.e., the category $P$ object $F[\mathbb{R}]$, is represented by the parametrized matrix $[\sigma]$. The row 2 algebra is $\mathbb{C}$, and its Proto-norm family $F[\mathbb{C}]$ is represented by the category $P$ object $\begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix}$. There is no morphism pointing from $F(\mathbb{C})$ to $F(\mathbb{R})$. That is, there is clearly no nonzero real matrix $V$ such that $V'F[\sigma]V \subseteq \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix}$, since all the real matrix members of $V'F[\sigma]V$ must have
determinant zero - while \( \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix} \) has determinant zero if and only if \( \alpha, \beta = 0 \) and is thereby the zero matrix.

On the other hand, consider the algebra in row 4, \( \mathbb{R} \oplus \mathbb{R} \). There is clearly a morphism from \( \mathbb{F}[\mathbb{R} \oplus \mathbb{R}] \) to \( \mathbb{F}[\mathbb{R}] \) - i.e., take \( V = [1, 0] \), so that \( V'\sigma V = \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix} \subset \mathbb{F}[\mathbb{R} \oplus \mathbb{R}] \). Thus, unlike the case of \( \mathbb{C} \), an epimorphism from \( \mathbb{R} \oplus \mathbb{R} \) to \( \mathbb{R} \) is not precluded. And indeed, the epimorphism does exist, resulting from the ideal \( \{0\} \oplus \mathbb{R} \). A similar argument applies to existence of an epimorphism from \( \mathbb{C} \) (row 3) to \( \mathbb{R} \), and existence of an epimorphism from \( \mathbb{D} \) (row 5) to \( \mathbb{R} \).

6.3.4. *Rows 10 and 4.* There is a morphism between \( A_{10} \) and \( \mathbb{R} \oplus \mathbb{R} \) associated with the ideal \( \left\{ \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} : z \in \mathbb{R} \right\} \). Corollary 6.2 then implies that there must exist a morphism from \( \mathbb{F}(A_{10}) \) and \( \mathbb{F}(A_4) \). Indeed, existence of the latter morphism is obvious from the last entries in rows 10 and 4 of Table I.

6.3.5. *Rows 13, 12, and 4.* There is an epimorphism from \( A_{13} \) to \( \mathbb{R} \oplus \mathbb{R} \) associated with the ideal \( \left\{ \begin{bmatrix} 0 & w \\ 0 & 0 \end{bmatrix} : w, v \in \mathbb{R} \right\} \). Corollary 6.2 then implies that there must exist a morphism from \( \mathbb{F}(A_{13}) \) and \( \mathbb{F}(A_4) \). Indeed, existence of the latter morphism is obvious from the last entries in rows 13 and 4 of Table I.

On the other hand, as regards \( A_{13} \) and its subalgebra \( A_{12} \), applying Corollary 6.3 to the relevant Proto-norm families in Table I indicates that there can be no morphism from \( \mathbb{F}(A_{13}) \) to \( \mathbb{F}(A_{12}) \) - precluding an epimorphism of the respective algebras.

6.3.6. *Rows 12, and 5.* There is an epimorphism from \( A_{12} \) to \( \mathbb{D} \) associated with the ideal \( \left\{ \begin{bmatrix} 0 & w \\ 0 & 0 \end{bmatrix} : w, v \in \mathbb{R} \right\} \). Corollary 6.2 then implies that there must exist a morphism from \( \mathbb{F}(A_{12}) \) to \( \mathbb{F}(A_5) \). Indeed, existence of the latter morphism is obvious from the last entries in rows 12 and 5 of Table I.

6.3.7. *Rows 11 and 5.* There is an epimorphism from \( A_{11} \) to \( A_{5} \) associated with the ideal \( \left\{ \begin{bmatrix} 0 & w \\ 0 & 0 \end{bmatrix} : w \in \mathbb{R} \right\} \). Accordingly, the morphism \( \mathbb{F}(A_{11}) \to \mathbb{F}(A_5) \) must exist - and it clearly does by inspection of the depicted Proto-norm families at the end of rows 11 and 5.

6.3.8. *Row 6, the Quaternion algebra \( \mathbb{H} \).* A simple calculation shows that \( \mathbb{F}(\mathbb{H}) \) is given by the parametrized matrix \( \text{diag}\{\alpha, -\alpha, -\alpha, -\alpha\} \). Corollary 6.4 then indicates that \( \mathbb{H} \) is simple. This is an alternative to the ideal-based argument that \( \mathbb{H} \) is simple because it can be shown to be a division algebra.

Independent of the above, Theorem 6.1 indicates that there cannot be an epimorphism from \( \mathbb{H} \) to its subalgebra \( \mathbb{C} \) since their Proto-norm families depicted in row 6 and row 2 preclude a morphism pointing from \( \mathbb{F}(\mathbb{H}) \) to \( \mathbb{F}(\mathbb{C}) \), due to Corollary 6.3.
References

[1] Birkhoff, GD (1932), A Set of Postulates for Plane Geometry (Based on Scale and Protractors), Annals of Mathematics, 33: 329–345.
[2] Greensite FS (2022), A new proof of the Pythagorean Theorem inspired by novel characterizations of unital algebras. arXiv:2209.14119
[3] Bourbaki N (1989) Algebra I. Springer, Berlin, p. 543.
[4] Greensite FS (2022), Cosmology from a non-physical standpoint: an algebraic analysis. arXiv:2209.14127
[5] Prest M (1991) Wild representation type and undecidability. Communications in Algebra, 19(3):919-929.
[6] Greensite FS (2022), Origin of the Proto-norm functor from Inverse Problems. arXiv:2209.14137
[7] Jacobson N (1963) Generic norm of an algebra, Osaka mathematics Journal, 15(1):25-50.
[8] Schafer R (1963) On forms of degree $n$ permitting composition, Journal of Mathematics and Mechanics, 12:777-792.
[9] McCrimmon K (2004) A taste of Jordan algebras, Springer, New York.