\( \mathcal{N} = 2 \) supersymmetric sigma-models and duality

Sergei M. Kuzenko

School of Physics M013, The University of Western Australia
35 Stirling Highway, Crawley W.A. 6009, Australia

Abstract

For two families of four-dimensional off-shell \( \mathcal{N} = 2 \) supersymmetric nonlinear \( \sigma \)-models constructed originally in projective superspace, we develop their formulation in terms of \( \mathcal{N} = 1 \) chiral superfields. Specifically, these theories are: (i) \( \sigma \)-models on cotangent bundles \( T^*M \) of arbitrary real analytic Kähler manifolds \( M \); (ii) general superconformal \( \sigma \)-models described by weight-one polar supermultiplets. Using superspace techniques, we obtain a universal expression for the holomorphic symplectic two-form \( \omega^{(2,0)} \) which determines the second supersymmetry transformation and is associated with the two complex structures of the hyperkähler space \( T^*M \) that are complimentary to the one induced from \( M \). This two-form is shown to coincide with the canonical holomorphic symplectic structure. In the case (ii), we demonstrate that \( \omega^{(2,0)} \) and the homothetic conformal Killing vector determine the explicit form of the superconformal transformations. At the heart of our construction is the duality (generalized Legendre transform) between off-shell \( \mathcal{N} = 2 \) supersymmetric nonlinear \( \sigma \)-models and their on-shell \( \mathcal{N} = 1 \) chiral realizations. We finally present the most general \( \mathcal{N} = 2 \) superconformal nonlinear \( \sigma \)-model formulated in terms of \( \mathcal{N} = 1 \) chiral superfields. The approach developed can naturally be generalized in order to describe 5D and 6D superconformal nonlinear \( \sigma \)-models in 4D \( \mathcal{N} = 1 \) superspace.

---

1Based in part on lectures given at the Center for Quantum Spacetime, Sogang University, Seoul, October 2009.

2kuzenko@cyllene.uwa.edu.au
1 Introduction

Four-dimensional $\mathcal{N} = 2$ supersymmetric nonlinear $\sigma$-models can be formulated in terms of component fields $[1, 2, 3]$ or $\mathcal{N} = 1$ chiral superfields $[4, 5, 6]$. These constructions are quite elegant and geometric, especially the one in $\mathcal{N} = 1$ superspace. However,
they both present just the existence theorems in the sense that their practical usefulness is extremely limited if one is interested in the explicit construction of general $\mathcal{N} = 2$ supersymmetric nonlinear $\sigma$-models (or, equivalently, hyperkähler metrics). Achieving such a goal requires the use of $\mathcal{N} = 2$ superspace techniques, and the appropriate setting turns out to be the so-called projective superspace approach \cite{7,8} (see also \cite{9} for a recent review). The power of this approach in the context of $\mathcal{N} = 2$ supersymmetric nonlinear $\sigma$-models is due to the following reasons:

(i) the $\mathcal{N} = 2$ supersymmetric action is generated by a Lagrangian that can be chosen to be an arbitrary function (modulo some mild restrictions) of several superfield dynamical variables – off-shell $\mathcal{N} = 2$ projective supermultiplets;

(ii) such supermultiplets are naturally decomposed into a set of standard $\mathcal{N} = 1$ superfields.

The latter property in fact implies an intimate connection between the projective superspace approach and the $\mathcal{N} = 1$ superspace construction of \cite{4,5,6}. It is pertinent here to elaborate on this point in some more detail. For simplicity, our discussion will be restricted to the case of a single hypermultiplet.

In the projective superspace setting, there are infinitely many off-shell realizations for a neutral hypermultiplet, each of which is characterized by a finite number of auxiliary fields. Such off-shell supermultiplets are labelled by a positive integer $n = 2, 3, \ldots$, and are called real $\mathcal{O}(2n)$ multiplets. For a charged hypermultiplet, there exists a single off-shell realization with an infinite number of auxiliary fields, which is called the polar hypermultiplet (the terminology follows \cite{16}). All of these $\mathcal{N} = 2$ multiplets can readily be decomposed into a set of standard $\mathcal{N} = 1$ superfields, and the corresponding content is the following: two physical superfields $\Phi$ and $\Sigma$ and their conjugates $\bar{\Phi}$ and $\bar{\Sigma}$, as well as some number of auxiliary superfields $U_\iota$, where the index $\iota$ may take a finite ($2n - 3$ in the case of $\mathcal{O}(2n)$ multiplets) or infinite number of values (for the polar multiplet). The physical superfields $\Phi$ and $\Sigma$ are chiral and complex linear, respectively,

$$\bar{D}_\alpha \Phi = 0, \quad \bar{D}^2 \Sigma = 0,$$

where $\bar{D}_\alpha = \bar{D}_\alpha^\dagger$ and $\bar{D}^2 = \bar{D}\bar{D}$.

Alternatively, the projective superspace can be derived from harmonic superspace \cite{10,11} in a singular limit \cite{12,13}. However, the two approaches are truly complementary. While the harmonic formalism is indispensable for quantum calculations in $\mathcal{N} = 2$ super Yang-Mills theories, the projective formalism is ideal for $\sigma$-model constructions. It should be remarked that both approaches make use of the isotwistor superspace $\mathbb{R}^{4|8} \times \mathbb{C}\mathbb{P}^1$ pioneered by Rosly \cite{14}.

The case $n = 1$ corresponds to the $\mathcal{N} = 2$ tensor multiplet \cite{15}. This multiplet is very special, because (i) it involves no auxiliary superfields $U_\iota$; and (ii) the physical linear superfield becomes real, $\Sigma = \bar{\Sigma}$. 

\footnote{Projective superspace can be derived from harmonic superspace \cite{10,11} in a singular limit \cite{12,13}. However, the two approaches are truly complementary. While the harmonic formalism is indispensable for quantum calculations in $\mathcal{N} = 2$ super Yang-Mills theories, the projective formalism is ideal for $\sigma$-model constructions. It should be remarked that both approaches make use of the isotwistor superspace $\mathbb{R}^{4|8} \times \mathbb{C}\mathbb{P}^1$ pioneered by Rosly \cite{14}.

\footnote{The case $n = 1$ corresponds to the $\mathcal{N} = 2$ tensor multiplet \cite{15}. This multiplet is very special, because (i) it involves no auxiliary superfields $U_\iota$; and (ii) the physical linear superfield becomes real, $\Sigma = \bar{\Sigma}$.}
while the auxiliary superfields $\mathcal{U}_i$ are unconstrained. Upon reduction to $\mathcal{N} = 1$ superspace, the action functional of an $\mathcal{N} = 2$ supersymmetric $\sigma$-model takes the form

$$S = \int d^4x d^4\theta L_{\text{off-shell}}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}, \mathcal{U}_i) \, ,$$

(1.2)

for some Lagrangian $L_{\text{off-shell}}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}, \mathcal{U}_i)$. Although this action is formulated in $\mathcal{N} = 1$ superspace, and thus only its $\mathcal{N} = 1$ supersymmetry is manifest, it is in fact invariant under off-shell $\mathcal{N} = 2$ supersymmetry transformations provided (1.2) is derived from a manifestly $\mathcal{N} = 2$ supersymmetric action in projective superspace [8]. The superfields $\mathcal{U}_i$ are auxiliary because they are unconstrained and appear in the Lagrangian without derivatives. They can be integrated out, at least in principle, using the corresponding equations of motion

$$\frac{\partial}{\partial \mathcal{U}_i} L_{\text{off-shell}}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}, \mathcal{U}_i) = 0 \implies \mathcal{U}_i = \mathcal{U}_i(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) \, .$$

(1.3)

As a result, one arrives at an action formulated in terms of the physical superfields only,

$$S = \int d^4x d^4\theta L_{\text{on-shell}}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) \, .$$

(1.4)

This action is of course $\mathcal{N} = 2$ supersymmetric; however the corresponding transformations form a closed algebra on the mass shell only. Since $\Sigma$ is complex linear, the formulation (1.4) cannot be used directly to link the theory under consideration with the results of [4, 5, 6]. In order to obtain a formulation in terms of chiral superfields only, one has to dualize $\Sigma$ and $\bar{\Sigma}$ into a chiral superfield and its conjugate.³ The action (1.4) is equivalent to the following first-order action:

$$S_{\text{first-order}} = \int d^4x d^4\theta \left\{ L_{\text{on-shell}}(\Phi, \bar{\Phi}, \Sigma) + \Psi \Sigma + \bar{\Psi} \bar{\Sigma} \right\} \, .$$

(1.5)

Here $\Sigma$ is complex unconstrained, while $\Psi$ is chiral, $D_\alpha \Psi = 0$. Integrating out $\Sigma$ and $\bar{\Sigma}$ leads to an action of the form

$$S_{\text{dual}} = \int d^4x d^4\theta H(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) \, .$$

(1.6)

By construction, this theory is $\mathcal{N} = 2$ supersymmetric. It is formulated in terms of $\mathcal{N} = 1$ chiral superfields, in the spirit of [4, 5, 6], and therefore the Lagrangian $H$ is

³The existence of duality between the chiral and the complex linear superfields was noticed for the first time by Zumino [17] (see also [18]). This observation naturally extended the duality between the chiral and the real linear superfields [19]. General aspects of duality in nonlinear $\sigma$-models in $\mathcal{N} = 1$ superspace were elaborated in [4].
the hyperkähler potential of the target space. What we have described here is known as
the generalized Legendre transform procedure formalized by Lindström and Roček twenty
years ago [8]. In the most interesting case of polar hypermultiplet self-couplings, nontrivial
examples of the generalized Legendre transform have been given over the last ten years
[12, 20, 21, 22, 23, 24, 25].

Presently, an interesting open problem is to formulate general \( \mathcal{N} = 2 \) superconformal
nonlinear \( \sigma \)-models in terms of \( \mathcal{N} = 1 \) chiral superfields. Its significance follows from
several fundamental results in supersymmetry and their implications. Quaternion Kähler
manifolds are of special importance for supersymmetric theories with eight supercharges,
for they present target spaces of matter hypermultiplets in \( \mathcal{N} = 2 \) supergravity [2]. There
exists a one-to-one correspondence \[26, 27\] between \( 4n \)-dimensional quaternion Kähler
manifolds and \( 4(n + 1) \)-dimensional hyperkähler spaces possessing a homothetic confor-
mal Killing vector, and hence an isometric action of SU(2) rotating the complex structures
[28]. Such hyperkähler spaces, known as "hyperkähler cones" in the physics literature,
turn out to be the target spaces for rigid \( \mathcal{N} = 2 \) superconformal \( \sigma \)-models [29, 30, 31].
The quaternion Kähler manifolds emerge as the \( \mathcal{N} = 2 \) superconformal quotient of the
corresponding hyperkähler cones [31]. The quotient construction (including the Kähler
reduction from the hyperkähler cone to the twistor space of the quaternion Kähler mani-
fold) can naturally be carried out [31] if the \( \mathcal{N} = 2 \) superconformal \( \sigma \)-model is realized
in terms of \( \mathcal{N} = 1 \) chiral superfields. The above consideration indicates that the problem
of generating arbitrary quaternion Kähler metrics is essentially equivalent to the follow-
ing two problems in rigid supersymmetry: (i) construction of general off-shell \( \mathcal{N} = 2 \)
superconformal \( \sigma \)-models in projective superspace; and (ii) their on-shell re-formulation
in terms of \( \mathcal{N} = 1 \) chiral superfields.

Four-dimensional off-shell \( \mathcal{N} = 2 \) superconformal multiplets in projective superspace
and their couplings were described in detail in [32], building on the earlier equivalent
results in five dimensions [33]. In particular, the most general \( \sigma \)-model couplings of
superconformal weight-one polar hypermultiplets were given in [32]. First steps toward
developing the chiral formulation in \( \mathcal{N} = 1 \) superspace for the \( \sigma \)-models given were also
undertaken in [32]. The analysis was based on the idea that the \( \mathcal{N} = 2 \) superconformal
\( \sigma \)-models of weight-one polar hypermultiplets form a subclass in the family of the off-shell
\( \mathcal{N} = 2 \) supersymmetric \( \sigma \)-models on cotangent bundles of Kähler manifolds [12, 20, 21].
In the present paper, we complete the chiral formulation in \( \mathcal{N} = 1 \) superspace for the
superconformal \( \sigma \)-models introduced in [32].

This paper is organized as follows. Section 2 provides a review of the formulation for
$\mathcal{N} = 2$ supersymmetric nonlinear $\sigma$-models in terms of $\mathcal{N} = 1$ superfields, which was pioneered in [5]. In section 3 we develop the chiral formulation in $\mathcal{N} = 1$ superspace for general off-shell $\mathcal{N} = 2$ supersymmetric $\sigma$-models on cotangent bundles of Kähler manifolds. This analysis is extended in section 4 to the case of general off-shell $\mathcal{N} = 2$ superconformal $\sigma$-models of weight-one polar hypermultiplets. In section 5 we propose a chiral formulation in $\mathcal{N} = 1$ superspace for the most general $\mathcal{N} = 2$ superconformal nonlinear $\sigma$-model. A brief discussion of the results obtained and their possible extensions is given in section 6. In the appendix, we provide a purely superspace proof of the conditions given in [5] for the $\sigma$-model (2.1) to be invariant under the transformations (2.2) and (2.3). This appendix makes the present paper essentially self-contained.

2 $\mathcal{N} = 2$ supersymmetric nonlinear sigma-models in $\mathcal{N} = 1$ superspace

In this section we review the formulation for $\mathcal{N} = 2$ supersymmetric nonlinear $\sigma$-models in terms of $\mathcal{N} = 1$ superfields, which was given in [5].

We start with a general $\mathcal{N} = 1$ supersymmetric nonlinear $\sigma$-model

$$S = \int d^4x d^4\theta K(\phi^a, \bar{\phi}^5), \quad \bar{D}_5 \phi^a = 0,$$

(2.1)

with $K$ the Kähler potential of a Kähler manifold $\mathcal{M}$, and look for those restrictions on the target space geometry which make the theory be $\mathcal{N} = 2$ supersymmetric.

To describe the second supersymmetry, one makes [4, 5] the ansatz

$$\delta \phi^a = \frac{1}{2} \bar{D}^2 (\epsilon \Omega^a), \quad \delta \bar{\phi}^a = \frac{1}{2} D^2 (\epsilon \bar{\Omega}^a),$$

(2.2)

for some functions $\Omega^a = \Omega^a(\phi, \bar{\phi})$ associated with the Kähler manifold $\mathcal{M}$. The transformation parameter $\epsilon$ is constrained by

$$\bar{D}_a \epsilon = \partial_a \epsilon = D^2 \epsilon = 0 \quad \Longleftrightarrow \quad \epsilon = \epsilon(\theta) = \tau + \epsilon^a \theta_a, \quad \tau = \text{const,} \quad \epsilon^a = \text{const} \quad (2.3)$$

4In the linear $\sigma$-model case, the ansatz (2.2) has its origin in $\mathcal{N} = 2$ superspace. One should start by considering the Fayet-Sohnius hypermultiplet [35, 36] described by a $\mathcal{N} = 2$ isospinor superfield $q^i$ obeying the constraints $D_a (i q^j) = D_a (i q^j) = 0$. This multiplet and its supersymmetry transformation law can readily be reduced to $\mathcal{N} = 1$ superspace; in particular, $q^i$ generates two $\mathcal{N} = 1$ multiplets $\phi_+$ and $\phi_-$, with $\phi^a := (\phi_+, \phi_-)$ chiral superfields. For the second supersymmetry transformation, one gets eq. (2.2) in which $\bar{\Omega}^\pm = \pm \bar{\phi}^\mp$. This procedure was carried out explicitly in [37] and implicitly in [38].
Here $\epsilon^\alpha$ is the supersymmetry parameter, while $\tau$ corresponds to a central charge transformation. If the action is invariant under the second supersymmetry transformation described by $\epsilon_\alpha$, then the central charge symmetry is generated by commuting the first (manifestly realized) and the second supersymmetry transformations.

The action is invariant under the central charge transformation provided

$$\tilde{\omega}_{\bar{c} \bar{e}} := g_{\bar{a} \bar{b}} \tilde{\Omega}^\alpha_{\bar{a}} = -\tilde{\omega}_{\bar{b} \bar{e}} , \quad \tilde{\Omega}^\alpha_{\bar{c}} := \partial_\bar{c} \tilde{\Omega}^\alpha ,$$

with $g_{\bar{a} \bar{b}} = K_{\bar{a} \bar{b}} := \partial_{\bar{a}} \partial_{\bar{b}} K$ the Kähler metric. The action is invariant under the transformation generated by the parameter $\epsilon_\alpha$ if the two-form $\omega_{bc}$ and its conjugate $\bar{\omega}_{\bar{b} \bar{c}}$ are covariantly constant,

$$\nabla_a \tilde{\omega}_{\bar{c} \bar{e}} = \partial_a \tilde{\omega}_{\bar{c} \bar{e}} = 0 ,$$
$$\nabla_{\bar{a}} \tilde{\omega}_{\bar{c} \bar{e}} = 0 .$$

On the mass shell,

$$\bar{D}^2 K_a = 0 ,$$

and the first and the second supersymmetry transformations generate the $\mathcal{N} = 2$ super-Poincaré algebra without central charge provided

$$\tilde{\Omega}^\alpha_{\bar{c}} \Omega^\beta_{\bar{b}} = -\delta^\alpha_{\beta} .$$

In fact, the closure of the supersymmetry algebra requires two more conditions

$$\bar{D}^2 \tilde{\Omega}^\alpha = 0 ,$$
$$\tilde{\Omega}^a_{\bar{c}} \nabla_{\bar{a}} \tilde{\Omega}^\alpha_{\bar{b}} - \tilde{\Omega}^a_{\bar{c}} \nabla_{\bar{a}} \tilde{\Omega}^\alpha_{\bar{b}} = 0 .$$

They hold due to (2.5a) – (2.7).

On the mass shell, the supersymmetry transformation (2.2) takes the form:

$$\delta \phi^a = \tilde{\epsilon}_\alpha \tilde{\Omega}^\alpha_{\bar{a}} \bar{D}_{\bar{a}} \tilde{\phi}^\beta .$$

Since $\delta \phi^a$ should be a vector field on $\mathcal{M}$, we conclude that $\tilde{\Omega}^a_{\bar{b}}$ is a tensor field on $\mathcal{M}$, and therefore $\omega_{ab}$ is a two-form.

Let $J \equiv J_3$ be the complex structure chosen on the target space $\mathcal{M}$,

$$J_3 = \begin{pmatrix} i \delta^a_{\bar{b}} & 0 \\ 0 & -i \delta^a_{\bar{b}} \end{pmatrix}.$$ 

\footnote{It will be explained shortly why $\omega_{bc}$ has to be a globally defined two-form on $\mathcal{M}$.}
The above consideration shows that there are two more complex structures defined as

\[ J_1 = \begin{pmatrix} 0 & \Omega^a_{\, \bar{b}} \\ \Omega^a_{\, \bar{b}} & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & i \Omega^a_{\, \bar{b}} \\ -i \Omega^a_{\, \bar{b}} & 0 \end{pmatrix} \]

(2.12)
such that \( \mathcal{M} \) is Kähler with respect to all of them, and the operators \( J_A = (J_1, J_2, J_3) \) form the quaternionic algebra:

\[ J_A J_B = -\delta_{AB} \mathbb{1} + \varepsilon_{ABC} J_C . \]

(2.13)

As a result, it has been demonstrated that the target space \( \mathcal{M} \) is a hyperkähler manifold.

As is seen from (2.12), the complex structures are given in terms of the tensor fields \( \bar{\Omega}^a_{\, \bar{b}} \) and \( \Omega^a_{\, \bar{b}} \), while the supersymmetry transformation (2.2) involves \( \bar{\Omega}^a \) and \( \Omega^a \). The latter can be constructed using the Kähler potential [5]:

\[ \bar{\Omega}^a = \omega^{ab} K_b(\phi, \bar{\phi}) . \]

(2.14)

Under the Kähler transformations

\[ K(\phi, \bar{\phi}) \rightarrow K(\phi, \bar{\phi}) + \Lambda(\phi, \bar{\phi}) , \]

(2.15)

\( \bar{\Omega}^a \) changes as follows: \( \omega^{ab} K_b \rightarrow \omega^{ab} K_b + \omega^{ab} \Lambda_b \). However, the supersymmetry variation \( \delta \phi^a = \frac{1}{2} \bar{D}^2(\bar{\epsilon} \bar{\Omega}^a) \) in (2.2) is invariant under the Kähler transformations, as emphasized in [6]. This completes our review of [5].

Most of the above relations were given in [5] without proof. Their purely superspace proof turns out to be nontrivial and quite interesting in its own right. It is described in the Appendix.

It should be pointed out that not all restrictions (2.3), which were originally put forward in [5], are necessary. In fact, it is sufficient to restrict the parameter in (2.2) to obey the constraint:

\[ D_\alpha \epsilon = \text{const} . \]

(2.16)

This leaves the following freedom in the choice of \( \epsilon \) in (2.2):

\[ \epsilon = \epsilon(\theta) + \bar{\mu} , \quad D_\alpha \bar{\mu} = 0 , \]

(2.17)

with \( \epsilon(\theta) \) given in (2.3). Choosing \( \epsilon \) in (2.2) to be an antichiral superfield \( \bar{\mu} \) provides an example of trivial symmetries of the form

\[ \delta \varphi^i = \Gamma^{ij} \frac{\delta S[\varphi]}{\delta \varphi^j} , \quad \Gamma^{ij} = -\Gamma^{ji} \]

(2.18)
any theory $S[\varphi]$ of bosonic fields $\varphi^i$ possesses. In particular, one can use such a trivial invariance to modify the second supersymmetry transformation on the manner [6]:
\[
\delta \phi^a = \frac{1}{2} \bar{D}^2 \left( \left[ \epsilon(\theta) + \bar{\epsilon}(\bar{\theta}) \right] \Omega^a \right), \quad \delta \bar{\phi} = \frac{1}{2} D^2 \left( \left[ \epsilon(\theta) + \bar{\epsilon}(\bar{\theta}) \right] \Omega \right).
\]

This results in no gain at all in four space-time dimensions. However, such a form of supersymmetry transformation is very useful in five and six dimensions.

### 3 Non-superconformal nonlinear sigma-models

In this section we will investigate four-dimensional off-shell $\mathcal{N} = 2$ supersymmetric nonlinear $\sigma$-models that are described in ordinary $\mathcal{N} = 1$ superspace by the action
\[
S[\Upsilon, \bar{\Upsilon}] = \frac{1}{2\pi i} \oint \frac{d\zeta}{\zeta} \int d^4x \, d^4\theta \, K(\Upsilon^I(\zeta)) \, \bar{\Upsilon}^J(\zeta) \, .
\]

The arctic $\Upsilon(\zeta)$ and antarctic $\bar{\Upsilon}(\zeta)$ dynamical variables are generated by an infinite set of ordinary superfields:
\[
\Upsilon(\zeta) = \sum_{n=0}^{\infty} \Upsilon_n \zeta^n = \Phi + \Sigma \zeta + O(\zeta^2), \quad \bar{\Upsilon}(\zeta) = \sum_{n=0}^{\infty} \bar{\Upsilon}_n (-\zeta)^{-n} \, .
\]

Here $\Phi$ is chiral, $\Sigma$ complex linear, eq. (1.1), and the remaining component superfields are unconstrained complex superfields. The above theory is a minimal $\mathcal{N} = 2$ extension of the general four-dimensional $\mathcal{N} = 1$ supersymmetric nonlinear $\sigma$-model [34]
\[
S[\Phi, \bar{\Phi}] = \int d^4x \, d^4\theta \, K(\Phi^I, \bar{\Phi}^J) \, ,
\]

with $K$ the Kähler potential of a real analytic Kähler manifold $\mathcal{M}$.

The study of $\sigma$-models of the form (3.1) was initiated in [12, 20, 21] because of their interesting geometric properties. They form a subset in the family of most general hypermultiplet theories in projective superspace [8] obtained by replacing $K(\Upsilon, \bar{\Upsilon})$ in (3.1) with a Lagrangian $K(\Upsilon, \bar{\Upsilon}, \zeta)$ with explicit dependence on $\zeta$ (geometric aspects of these most general $\sigma$-models are briefly discussed in [9]). Our primary interest in such theories in the present paper is motivated by the fact that the off-shell $\mathcal{N} = 2$ superconformal $\sigma$-models of weight-one polar hypermultiplets [32] constitute a subclass in the family of actions (3.1).
3.1 General properties

The $\mathcal{N} = 2$ supersymmetric nonlinear $\sigma$-model (3.1) inherits all the geometric features of its $\mathcal{N} = 1$ predecessor (3.3). The Kähler invariance of the latter, $K(\Phi, \bar{\Phi}) \rightarrow K(\Phi, \bar{\Phi}) + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi})$, turns into

$$K(\Upsilon, \bar{\Upsilon}) \rightarrow K(\Upsilon, \bar{\Upsilon}) + \Lambda(\Upsilon) + \bar{\Lambda}(\bar{\Upsilon})$$

(3.4) for the model (3.1). A holomorphic reparametrization of the Kähler manifold, $\Phi^I \rightarrow \Phi'^I = f_I(\Phi)$, has the following counterpart

$$\Upsilon^I(\zeta) \rightarrow \Upsilon'^I(\zeta) = f_I(\Upsilon(\zeta))$$

(3.5) in the $\mathcal{N} = 2$ case. Therefore, the physical superfields of the $\mathcal{N} = 2$ theory

$$\Upsilon^I(\zeta)|_{\zeta=0} = \Phi^I, \quad \frac{d\Upsilon^I(\zeta)}{d\zeta}|_{\zeta=0} = \Sigma^I,$$

(3.6) should be regarded, respectively, as coordinates of a point in the Kähler manifold and a tangent vector at the same point. Thus the variables $(\Phi^I, \Sigma^I)$ parametrize the holomorphic tangent bundle $TM$ of the Kähler manifold $M$ [12].

To describe the theory in terms of the physical superfields $\Phi$ and $\Sigma$ only, all the auxiliary superfields have to be eliminated with the aid of the corresponding algebraic equations of motion

$$\oint \frac{d\zeta}{\zeta} \zeta^n \frac{\partial K(\Upsilon, \bar{\Upsilon})}{\partial \Upsilon^I} = \oint \frac{d\zeta}{\zeta} \zeta^{-n} \frac{\partial K(\Upsilon, \bar{\Upsilon})}{\partial \bar{\Upsilon}^J} = 0, \quad n \geq 2. \quad (3.7)$$

Let $\Upsilon_*(\zeta) \equiv \Upsilon_*(\zeta; \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ denote a unique solution subject to the initial conditions

$$\Upsilon_*(0) = \Phi, \quad \dot{\Upsilon}_*(0) = \Sigma. \quad (3.8)$$

The auxiliary superfields $\Upsilon_2, \Upsilon_3, \ldots$, and their conjugates, can be eliminated in perturbation theory using the ansatz [39]

$$\Upsilon'^I_n = \sum_{p=0}^{\infty} G^I_{J_1 \ldots J_{n+p}, \bar{L}_1 \ldots \bar{L}_p}(\Phi, \bar{\Phi}) \Sigma^{J_1} \ldots \Sigma^{J_{n+p}} \bar{\Sigma}^{L_1} \ldots \bar{\Sigma}^{L_p}, \quad n \geq 2. \quad (3.9)$$

Assuming that the auxiliary superfields have been eliminated, the action (3.1) should take the form [20, 21]:

$$S_{tb}[\Phi, \Sigma] = \frac{1}{2\pi i} \oint \frac{d\zeta}{\zeta} \int d^4 x d^4 \theta \{ K(\Upsilon_*(\zeta), \bar{\Upsilon}_*(\zeta)) + \mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) \}$$

$$\mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) = \sum_{n=1}^{\infty} \mathcal{L}_{J_1 \ldots J_n}(\Phi, \bar{\Phi}) \Sigma^{J_1} \ldots \Sigma^{J_n} \Sigma^{J_n} \ldots \Sigma^{J_n} \equiv \sum_{n=1}^{\infty} \mathcal{L}^{(n)} \quad (3.10)$$
where $\mathcal{L}_{IJ} = -g_{IJ}(\Phi, \bar{\Phi})$ and the coefficients $\mathcal{L}_{I_1\ldots I_n\bar{J}_1\ldots \bar{J}_n}$, for $n > 1$, are tensor functions of the Kähler metric $g_{IJ}(\Phi, \bar{\Phi}) = \partial_I \partial_{\bar{J}} K(\Phi, \bar{\Phi})$, the Riemann curvature $R_{IJK\bar{L}}(\Phi, \bar{\Phi})$ and its covariant derivatives. Each term in the action contains equal powers of $\Sigma$ and $\bar{\Sigma}$, since the original model (3.1) is invariant under rigid U(1) transformations \[20\]

\[\Upsilon(\zeta) \mapsto \Upsilon(e^{i\alpha} \zeta) \iff \Upsilon_n(z) \mapsto e^{in\alpha} \Upsilon_n(z). \quad (3.11)\]

For illustration, we give the explicit expressions \[32\] for two next-to-leading terms appearing in the expansion of $\mathcal{L}$:

\[\begin{align*}
\mathcal{L}^{(2)} &= \frac{1}{4} R_{I_1J_1I_2J_2} \Sigma^{I_1} \Sigma^{J_1} \Sigma^{I_2} \Sigma^{J_2}, \\
\mathcal{L}^{(3)} &= -\frac{1}{12} \left\{ \mathcal{N}_{I_3J_3} R_{I_1J_1I_2J_2} + R_{I_1J_1L_1J_2L_2J_3} \right\} \Sigma^{I_1} \Sigma^{I_2} \Sigma^{J_1} \Sigma^{J_2} \Sigma^{J_3}. \quad (3.12a) \end{align*}\]

The expression for $\mathcal{L}^{(4)}$, which is somewhat messy, can be found given in \[32\].

3.2 Considerations of extended supersymmetry

The action (3.1) is manifestly $\mathcal{N} = 1$ supersymmetric, and is also invariant under the off-shell second supersymmetry transformation \[8\] (see also \[32\] for a detailed derivation):

\[\begin{align*}
\delta \Upsilon_0 &= \bar{\xi}_a \hat{D}^\alpha \Upsilon_1, \\
\delta \Upsilon_1 &= -\bar{\varepsilon}^a \hat{D}_\alpha \Upsilon_0 + \bar{\xi}_a \hat{D}^\alpha \Upsilon_2, \\
\delta \Upsilon_k &= -\bar{\varepsilon}^a \hat{D}_\alpha \Upsilon_{k-1} + \bar{\xi}_a \hat{D}^\alpha \Upsilon_{k+1}, \quad k > 1. \quad (3.13a) \end{align*}\]

Upon elimination of the auxiliary superfields, this symmetry turns into the following:

\[\delta \Phi = \bar{\varepsilon}_a \hat{D}^\alpha \Sigma, \quad \delta \Sigma = -\bar{\varepsilon}^a \hat{D}_\alpha \Phi + \bar{\xi}_a \hat{D}^\alpha \Upsilon_2(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}), \quad (3.14)\]

where $\Upsilon_2$ now a composite field of the general form given in (3.9). Since $\Upsilon_2$ transforms as a connection under the holomorphic reparametrizations (3.5),

\[\Upsilon'_2 \mapsto \Upsilon''_2 = \frac{1}{2} \frac{\partial^2 f(\Phi)}{\partial \Phi^J \partial \Phi^K} \Sigma^J \Sigma^K + \frac{\partial f(\Phi)}{\partial \Phi^J} \Upsilon'_2, \quad (3.15)\]

we can rewrite $\Upsilon_2$ in more specific form \[32\]:

\[\begin{align*}
\Upsilon'_2(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) &= -\frac{1}{2} \hat{\Gamma}'_{JK}(\Phi, \bar{\Phi}) \Sigma^J \Sigma^K + G'(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}), \\
G'(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) := \sum_{p=1}^{\infty} G'(J_1\ldots J_{p+2}, L_1\ldots L_p)(\Phi, \bar{\Phi}) \Sigma^{J_1} \ldots \Sigma^{J_{p+2}} \Sigma^{L_1} \ldots \Sigma^{L_p}, \quad (3.16)\end{align*}\]
with $\Gamma^I_{JK}(\Phi, \bar{\Phi})$ the Christoffel symbols for the Kähler metric $g_{IJ}(\Phi, \bar{\Phi})$. Here the coefficients $G^I_{J_1 \ldots J_p \bar{L}_1 \ldots \bar{L}_p}(\Phi, \bar{\Phi})$ are tensor functions of the Kähler metric, the Riemann curvature $R_{IJKL}(\Phi, \bar{\Phi})$ and its covariant derivatives. To leading order, $G^I$ is \[32:\]

$$G^I(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) = \frac{1}{6} \nabla_L R_{IJL} (\Phi, \bar{\Phi}) \Sigma^J \Sigma^L + \mathcal{O}(\Sigma^4 \bar{\Sigma}^2). \quad (3.17)$$

For the action \[3.10\] to be invariant under the supersymmetry transformations \[3.14\] and \[3.16\], it can be shown that there should exist a function

$$\Xi(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) := \sum_{n=2}^{\infty} \Xi_{\Sigma_{I_1 \ldots I_n}} \Sigma_{J_1 \ldots J_n}(\Phi, \bar{\Phi}) \Sigma^{J_1} \ldots \Sigma^{J_n}, \quad (3.18)$$

with tensor coefficients $\Xi_{I_1 \ldots I_n} \Sigma_{J_1 \ldots J_n}(\Phi, \bar{\Phi})$, such that the following equations hold:

\[3.19a\]

$$\frac{\partial L}{\partial \Sigma^J} \frac{\partial G^I}{\partial \Sigma^J} = \frac{\partial \Xi}{\partial \Sigma^J},$$

\[3.19b\]

$$\nabla_I L + \frac{\partial L}{\partial \Sigma^J} \frac{\partial G^I}{\partial \Sigma^J} = \frac{\partial \Xi}{\partial \Sigma^J},$$

\[3.19c\]

$$\frac{1}{2} g_{IJ} \partial L \Sigma^J + \frac{1}{2} \frac{\partial L}{\partial \Sigma^J} \Sigma^J + g_{IJ} \Sigma^J \nabla_I G^J = -\nabla_I \Xi.$$

Here we have defined

$$\nabla_I L := \sum_{n=1}^{\infty} \left( \nabla_{I_1 \ldots I_n} \Sigma_{J_1 \ldots J_n}^{I_1 \ldots I_n}(\Phi, \bar{\Phi}) \right) \Sigma^{J_1} \ldots \Sigma^{J_n} \Sigma^{L_1} \ldots \Sigma^{L_n} = \frac{\partial L}{\partial \Phi^I} - \frac{\partial L}{\partial \Sigma^K} \Gamma^K_{IJ} \Sigma^J, \quad (3.20)$$

and similarly for $\nabla_I G^J$ and $\nabla_I \Xi$.

The objects under consideration have several useful properties:

\[3.21a\]

$$\Sigma^J \frac{\partial L}{\partial \Sigma^J} = \Sigma^J \frac{\partial L}{\partial \Sigma^J},$$

\[3.21b\]

$$\Sigma^J \frac{\partial G^I}{\partial \Sigma^J} = \Sigma^J \frac{\partial G^I}{\partial \Sigma^J} + 2G^I,$$

\[3.21c\]

$$\Sigma^J \frac{\partial \Xi}{\partial \Sigma^J} = \Sigma^J \frac{\partial \Xi}{\partial \Sigma^J} + \Xi.$$

These properties and the equations \[3.19a\] and \[3.19b\] allow us to obtain the following expression for $\Xi$:

$$\Xi = \Sigma^I \nabla_I L + 2G^I \frac{\partial L}{\partial \Sigma^I}. \quad (3.22)$$
We see that $\Xi$ is uniquely determined in terms of the Lagrangian $\mathcal{L}$ and the vector field $G^I$ appearing in the second supersymmetry transformation (3.14), (3.16).

It is of interest to discuss the special case when $\mathcal{M}$ is a Hermitian symmetric space and thus the curvature tensor is covariantly constant,

$$\nabla_L R_{I_1J_1J_2J_3} = \nabla_{\bar{L}} R_{I_1J_1J_2J_3} = 0 \ .$$

(3.23)

This case has been studied in detail in [24, 25], and therefore we can compare the above results with those obtained in [24, 25]. First of all, since the curvature is covariantly constant, we have

$$\nabla_i \mathcal{L} = 0 \ , \quad G^I = 0 \quad \Rightarrow \quad \Xi = 0 \ ,$$

(3.24)

as a consequence of (3.22). The fact that the conditions (3.23) imply $G^I = 0$ was noticed in [32] and can be explained as follows. The tensor fields $G^I_{J_1...J_{p+2}} L_{I_1...I_p} K(\Phi, \bar{\Phi})$ in (3.16) have an odd number of indices. On the other hand, the Kähler metric, its inverse and the Riemann tensor are the only algebraically independent tensors in the case (3.23). It is therefore easy to understand that any tensor descendant of these geometric objects must carry an even number of indices, and thus $G^I = 0$. If the relations (3.24) hold, eq. (3.19c) reduces to the equation

$$\frac{1}{2} R_{K IL}^J \frac{\partial \mathcal{L}}{\partial \Sigma^J} \Sigma^K \Sigma^L + \frac{\partial \mathcal{L}}{\partial \bar{\Sigma}^J} g_{JJ} \Sigma^J = 0 \quad (3.25)$$

derived in [24]. This equation allows one to uniquely reconstruct $\mathcal{L}$ in the case of covariantly constant curvature [24, 25]. Similarly, in the general case, eqs. (3.19a) – (3.19c) can be used to determine the functional form of $\mathcal{L}$ and $G^I$.

### 3.3 Dual formulation

To construct a dual formulation of the theory (3.10), we follow [20, 21] and consider the first-order action

$$S_{\text{first-order}} = \int d^4x \ d^4\theta \left\{ K(\Phi, \bar{\Phi}) + \mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) + \Psi_I \Sigma^I + \bar{\Psi}_I \bar{\Sigma}^I \right\} \quad (3.26)$$

Here the tangent vector $\Sigma^I$ is complex unconstrained, while the one-form $\Psi_I$ is chiral, $\bar{D}_a \Psi_I = 0$. Eliminating $\Sigma$’s and their conjugates, by using their equations of motion

$$\frac{\partial}{\partial \Sigma^I} \mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) + \Psi_I = 0 \ ,$$

(3.27)
leads to the dual action

\[ S_{\text{ctb}}[\Phi, \Psi] = \int d^4x d^4\theta \left\{ K(\Phi, \bar{\Phi}) + \mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) \right\}, \quad (3.28) \]

where

\[ \mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = \sum_{n=1}^{\infty} \mathcal{H}^{I_1 \cdots I_n J_1 \cdots J_n}(\Phi, \bar{\Phi}) \Psi_{I_1} \cdots \Psi_{I_n} \bar{\Psi}_{J_1} \cdots \bar{\Psi}_{J_n}, \]

\[ \mathcal{H}^{IJ}(\Phi, \bar{\Phi}) = g^{IJ}(\Phi, \bar{\Phi}). \quad (3.29) \]

The variables \((\Phi^I, \Psi_J)\) parametrize the cotangent bundle \(T^*M\) of the Kähler manifold \(M\) \(^{(20)}\). The superfield Lagrangian

\[ \mathbb{K}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) := K(\Phi, \bar{\Phi}) + \mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) \quad (3.30) \]

is the hyperkähler potential of the target space.

It should be noted that eq. (3.21a) is equivalent to the relation

\[ \Psi_I \frac{\partial \mathcal{H}}{\partial \Psi_I} = \bar{\Psi}_I \frac{\partial \mathcal{H}}{\partial \bar{\Psi}_I}, \quad (3.31) \]

which will be used in what follows. It follows from (3.31) that the group \(U(1)\) acts on the hyperkähler manifold \(T^*M\) by holomorphic transformations

\[ \Phi^I \rightarrow \Phi^I, \quad \Psi_I \rightarrow e^{-i\alpha} \Psi_I, \quad \alpha \in \mathbb{R}, \quad (3.32) \]

and this action is isometric with respect to the Kähler metric (see (3.38) for notation)

\[ g_{ab} = \frac{\partial^2 \mathbb{K}}{\partial \phi^a \partial \phi^b} = \left( \begin{array}{cc} \frac{\partial^2 \mathbb{K}}{\partial \phi^I \partial \phi^J} & \frac{\partial^2 \mathbb{K}}{\partial \phi^I \partial \bar{\phi}^J} \\ \frac{\partial^2 \mathbb{K}}{\partial \bar{\phi}^I \partial \phi^J} & \frac{\partial^2 \mathbb{K}}{\partial \bar{\phi}^I \partial \bar{\phi}^J} \end{array} \right). \quad (3.33) \]

This agrees with results in the mathematical literature \(^{[40, 41]}\).

The first-order action (3.26) can be shown to be invariant under the following second supersymmetry transformation:

\[ \delta \Phi^I = \frac{1}{2} D^2 \{ \bar{\theta} \Sigma^I \}, \quad (3.34a) \]

\[ \delta \Sigma^I = -\epsilon D \Phi^I - \frac{1}{2} \bar{\epsilon} D \left\{ \Gamma^I_{JK} \Sigma^J \Sigma^K \right\} + \frac{1}{2} \bar{\epsilon} \theta \Gamma^I_{JK} \Sigma^J \bar{D}^2 \Sigma^K \]

\[ \quad + \bar{\epsilon} \bar{D} G^I + \frac{1}{2} \bar{\epsilon} \theta \frac{\partial G^I}{\partial \Sigma^J} \Sigma^J \bar{D}^2 \Sigma^J, \quad (3.34b) \]

\[ \delta \Psi_I = -\frac{1}{2} \bar{D}^2 \left\{ \bar{\theta} \left( K_I - \Gamma^I_{JK} \Psi_K \Sigma^J + \frac{\partial G^I}{\partial \Sigma^J} \Psi_J + \frac{\partial \Xi}{\partial \Sigma^I} \right) \right\}. \quad (3.34c) \]
These results, in conjunction with eq. (3.19b) and the standard properties of Legendre transform, lead to the supersymmetry invariance of the dual theory (3.28):

\[
\delta \Phi^I = \frac{1}{2} \bar{D}^2 \left\{ \overline{\epsilon \theta} \frac{\partial H}{\partial \bar{\Psi}^I} \right\}, \quad (3.35a)
\]

\[
\delta \Psi_I = -\frac{1}{2} \bar{D}^2 \left\{ \overline{\epsilon \theta} \left( K_I - \Gamma^K_{IJ} \Psi^K \frac{\partial H}{\partial \bar{\Psi}^J} + \nabla_I H \right) \right\}. \quad (3.35b)
\]

This form of the second supersymmetry is useful in the special case when the Kähler manifold is Hermitian symmetric, eq. (3.23); then \( \nabla_I H = 0 \), and the transformations (3.35a) and (3.35b) reduce to those obtained in [24]. A different form for the second supersymmetry follows from the identity

\[
\nabla_I H = \frac{\partial H}{\partial \Phi^I} + \Gamma^K_{IJ} \Psi^K \frac{\partial H}{\partial \bar{\Psi}^J} \quad (3.36)
\]

and the explicit expression for the hyperkähler potential, eq. (3.30). These observations lead to

\[
\delta \Phi^I = \frac{1}{2} \bar{D}^2 \left\{ \overline{\epsilon \theta} \frac{\partial K}{\partial \bar{\Psi}^I} \right\}, \quad \delta \Psi_I = -\frac{1}{2} \bar{D}^2 \left\{ \overline{\epsilon \theta} \frac{\partial \bar{K}}{\partial \bar{\Phi}^I} \right\}. \quad (3.37)
\]

Finally, if we introduce the condensed notation

\[
\phi^a := (\Phi^I, \Psi_I), \quad \bar{\phi}^\bar{a} = (\bar{\Phi}^I, \bar{\Psi}_I), \quad (3.38)
\]

as well as the standard symplectic matrix \( J = (J^{ab}) \), its inverse \( J^{-1} = (-J_{ab}) \) and their complex conjugates,

\[
J^{ab} = J^{\bar{a} \bar{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_{ab} = J_{\bar{a} \bar{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (3.39)
\]

then the supersymmetry transformation (3.37) can be rewritten as

\[
\delta \phi^a = \frac{1}{2} \bar{D}^2 \left\{ \overline{\epsilon \theta} \frac{\partial \bar{K}}{\partial \bar{\phi}^b} \right\} = \frac{1}{2} \bar{D}^2 \left\{ \overline{\epsilon \theta} \bar{\Omega}^a \right\}, \quad \bar{\Omega}^a := J^{ab} \frac{\partial \bar{K}}{\partial \bar{\phi}^b}. \quad (3.40)
\]

The universal form of this transformation law is quite remarkable.

### 3.4 Holomorphic two-form

By definition, the anti-holomorphic two-form is

\[
\bar{\omega}_{\bar{b} \bar{c}} = g_{\bar{a} \bar{b}} \bar{\Omega}^a_{\bar{c}}, \quad (3.41)
\]
with $g_{ab}$ the Kähler metric. As is seen from (3.40), $\bar{\omega}_{bc}$ is indeed antisymmetric,

$$\bar{\omega}_{ab} = g_{bc} J^{cd} g_{db} .$$  \hspace{1cm} (3.42)

Direct calculations, based on the explicit structure of the hyperkähler potential $K$, show that the two-form $\omega_{ab}$ looks like

$$\omega_{ab} = \mathcal{J}_{ab} + \mathcal{O}(\Psi \bar{\Psi}) .$$

Since $\omega_{ab}$ should be holomorphic, we immediately conclude that

$$\omega_{ab} = \mathcal{J}_{ab} , \quad \bar{\omega}_{\bar{a} \bar{b}} = \mathcal{J}_{\bar{a} \bar{b}} .$$  \hspace{1cm} (3.43)

We see that the holomorphic symplectic two-form $\omega^{(2,0)}$ of the hyperkähler manifold $T^*M$ coincides with the canonical holomorphic symplectic two-form,

$$\omega^{(2,0)} := \frac{1}{2} \omega_{ab} d\phi^a \wedge d\bar{\phi}^b = d\Phi^I \wedge d\bar{\Psi}_I .$$  \hspace{1cm} (3.44)

This agrees with results in the mathematical literature [40, 41].

Next, since the metric is Hermitian with respect to each of the complex structures, we conclude

$$\omega^{ab} = g^{ac} g^{bd} \omega_{cd} = \mathcal{J}^{ab} , \quad \bar{\omega}^{\bar{a} \bar{b}} = g^{\bar{a} \bar{c}} g^{\bar{b} \bar{d}} \omega_{\bar{c} \bar{d}} = \mathcal{J}^{\bar{a} \bar{b}} .$$  \hspace{1cm} (3.45)

Since $\omega^{ab}$ has been shown to coincide with the symplectic matrix $\mathcal{J}^{ab}$, the expression for $\bar{\Omega}^a$ in our supersymmetry transformation law (3.40) takes the form (2.14).

### 4 Superconformal nonlinear sigma-models I

It was demonstrated in [32] that the action (3.1) is $N = 2$ superconformal provided:

(i) the arctic variables $Y^I(\zeta)$ transform as superconformal weight-one arctic multiplets;
(ii) the Kähler potential obeys the homogeneity condition

$$\Phi^I \frac{\partial}{\partial \Phi^I} K(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}) .$$  \hspace{1cm} (4.1)

With the homogeneity condition imposed, no Kähler invariance survives. The geometric interpretation of such $\sigma$-models, albeit formulated in a slightly different form, was given in [42]. This interpretation will be briefly discussed in section 6.
4.1 Off-shell superconformal transformations

It was also shown in \[32\] that the general $\mathcal{N} = 2$ superconformal transformation decomposes, upon reduction to $\mathcal{N} = 1$ superspace, into three types of $\mathcal{N} = 1$ transformations. The latter are the following:

1. An arbitrary $\mathcal{N} = 1$ superconformal transformation generated by

$$\xi = \bar{\xi} = \xi^a(z) \partial_a + \xi^\alpha(z) \bar{D}^\alpha$$

such that

$$[\xi, D_\alpha] = \omega_{\alpha}{}^\beta D_\beta + \left(\sigma - 2\bar{\sigma}\right) D_\alpha ,$$

see \[43\] for more details. This transformation acts on the components $\Upsilon_k$ of an weight-one arctic multiplet by the rule:

$$\delta \Upsilon_k = -\xi \Upsilon_k - 2k(\bar{\sigma} - \sigma)\Upsilon_k - 2\sigma \Upsilon_k .$$

2. An extended superconformal transformation generated by a spinor parameter $\rho^\alpha$ constrained as

$$D_\alpha \rho^\beta = 0 , \quad D^{(\alpha} \rho^{\beta)} = 0 ,$$

and hence

$$\partial^{(\alpha} \rho^{\beta)} = D^2 \rho^\beta = 0 .$$

The general solution for (4.5) is

$$\rho^\alpha(x_{(+)}, \theta) = \epsilon^\alpha + \lambda \theta^\alpha - i \eta_{\bar{\alpha}} x_{(+)}^{\bar{\alpha}} , \quad x_{(+)}^{\alpha} = x^\alpha + i \theta^\alpha \bar{\theta} .$$

Here the constant parameters $\epsilon^\alpha$, $\lambda$ and $\eta_{\bar{\alpha}}$ correspond to (i) a second Q-supersymmetry transformation ($\epsilon^\alpha$); (ii) an off–diagonal SU(2)-transformation ($\lambda$); and (iii) a second S-supersymmetry transformation ($\eta_{\bar{\alpha}}$). The extended superconformal transformation acts on the components $\Upsilon_k$ of an weight-one arctic multiplet by the rule:

$$\delta \Upsilon_0 = \bar{\rho}_{\bar{\alpha}} \bar{D}^{\bar{\alpha}} \Upsilon_1 + \frac{1}{2} (\bar{D}_{\bar{\alpha}} \rho^{\bar{\alpha}}) \Upsilon_1 ,$$

$$\delta \Upsilon_1 = -\rho^\alpha D_\alpha \Upsilon_0 + \bar{D}^{\bar{\alpha}} (\rho_{\bar{\alpha}} \Upsilon_2) - \frac{1}{2} (D^\alpha \rho_\alpha) \Upsilon_0 ,$$

$$\delta \Upsilon_k = -\rho^\alpha D_\alpha \Upsilon_{k-1} + \bar{\rho}_{\bar{\alpha}} \bar{D}^{\bar{\alpha}} \Upsilon_{k+1}$$

$$+ \frac{1}{2}(k - 2)(D^\alpha \rho_\alpha) \Upsilon_{k-1} + \frac{1}{2}(k + 1)(\bar{D}_{\bar{\alpha}} \rho^{\bar{\alpha}}) \Upsilon_{k+1} , \quad k > 1 .$$

---

In the standard $\mathcal{N} = 2$ superspace parametrized by variables $z^A = (x^a, \theta_1^\alpha, \bar{\theta}_2^{\bar{\alpha}})$, this transformation rotates the Grassmann variable $\theta^\alpha_1$ into $\theta^\alpha_2$ and vice versa.
3. A shadow chiral rotation. In $\mathcal{N} = 2$ superspace, this is a phase transformation of $\theta_2^\alpha$ only, with $\theta_1^\alpha$ kept unchanged. Its action on the arctic weight-one multiplet is

$$\Upsilon(\zeta) \rightarrow \Upsilon'(\zeta) = e^{-i(\zeta/2)\alpha} \Upsilon(e^{i\alpha}\zeta) .$$ (4.9)

4.2 Homogeneity conditions

Suppose we have eliminated all the auxiliary superfields $\Upsilon_2, \Upsilon_3, \ldots$, and their conjugates. Let $\Upsilon_*(\zeta) \equiv \Upsilon_*(\zeta; \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ be the unique solution to the auxiliary field equations (3.7) under the initial conditions (3.8). Due to (4.1), any rescaled holomorphic function

$$\Upsilon_\bullet(\zeta) := c \Upsilon_*(\zeta) , \quad c \in \mathbb{C} - \{0\}$$ (4.10)

also solves the auxiliary field equations (3.7) and is characterized by the initial conditions

$$\Upsilon_\bullet(0) = c \Phi , \quad \dot{\Upsilon}_\bullet(0) = c \Sigma .$$ (4.11)

This means that the Lagrangian $\mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ in (3.10) obeys the homogeneity condition

$$\left( \Phi^I \frac{\partial}{\partial \Phi^I} + \Sigma^I \frac{\partial}{\partial \Sigma^I} \right) \mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) = \mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) .$$ (4.12)

Similar considerations imply that the functions $G^I(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ and $\Xi(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$, which were introduced in subsection 3.2 also obey certain homogeneity conditions. In particular, one can notice that

$$\left( \bar{\Phi}^J \frac{\partial}{\partial \bar{\Phi}^J} + \bar{\Sigma}^J \frac{\partial}{\partial \bar{\Sigma}^J} \right) G^I(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) = 0 ,$$ (4.13a)

$$\left( \bar{\Phi}^J \frac{\partial}{\partial \bar{\Phi}^J} + \bar{\Sigma}^J \frac{\partial}{\partial \bar{\Sigma}^J} \right) \Xi(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) = \Xi(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) .$$ (4.13b)

These relations can be used to obtain a new representation for $\Xi$. It is

$$\Xi = -\frac{\partial \mathcal{L}}{\partial \Sigma^I} \Phi^I - K_I \Sigma^I ,$$ (4.14)

compare with (3.22).

4.3 On-shell superconformal transformations

Upon elimination of the auxiliary superfields, the action (3.10) must be invariant under on-shell $\mathcal{N} = 2$ superconformal transformations of the physical superfields. It is instructive to check this invariance explicitly.
Let us consider the $\mathcal{N} = 1$ superconformal transformation (4.2), (4.3). The physical chiral $\Phi := \Upsilon_0$ and complex linear $\Sigma := \Upsilon_1$ superfields transform as
\begin{align}
\delta_{\xi} \Phi &= -\xi \Phi - 2\sigma \Phi , \\
\delta_{\xi} \Sigma &= -\xi \Sigma - 2\bar{\sigma} \Sigma .
\end{align}
These transformations are consistent with the off-shell constraints $\bar{D}_\alpha \Phi = 0$ and $\bar{D}^2 \Sigma = 0$. Now, using the relations (3.21a), (4.1) and (4.12), one readily check that the action is invariant under the $\mathcal{N} = 1$ superconformal transformations,
\begin{equation}
\delta_{\xi} S = -\int d^4x d^4\theta \left( \xi + 2(\sigma + \bar{\sigma}) \right) \left\{ K(\Phi, \bar{\Phi}) + \mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) \right\} = 0 .
\end{equation}

We next turn to the extended superconformal transformation
\begin{align}
\delta \Phi &= \bar{\rho}_\alpha \bar{D}_\alpha \Sigma + \frac{1}{2} \left( D_\alpha \bar{\rho}^\alpha \right) \Sigma , \\
\delta \Sigma &= -\rho^\alpha D_\alpha \Phi - \frac{1}{2} \left( D^\alpha \rho_\alpha \right) \Phi + \bar{D}_\alpha \left( \rho^\alpha \Upsilon_2(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) \right) ,
\end{align}
with $\Upsilon_2(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ given by eq. (3.16). It should be recalled that the parameter $\rho^\alpha$ obeys the constraints (4.5) and its explicit form is given by (4.7). Since the parameters $\rho^\alpha$ and $\bar{\rho}^\delta$ are independent, modulo complex conjugation, it is sufficient to analyze the variation of the action in the case when $\rho^\alpha = 0$ and $\bar{\rho}^\delta \neq 0$. It is
\begin{equation}
\delta_{\bar{\rho}} S = \int d^4x d^4\theta \bar{\rho}_\alpha \bar{D}_\alpha \bar{\Sigma} \\
+ \frac{1}{2} \int d^4x d^4\theta \left( \bar{D}_\alpha \bar{\rho}^\alpha \right) \left\{ \Sigma^I \nabla_I \mathcal{L} + 2 \frac{\partial \mathcal{L}}{\partial \bar{\Sigma}^I} G^I - \frac{\partial \mathcal{L}}{\partial \Sigma^I} \bar{\Phi}^I - K_I \Sigma^I \right\} .
\end{equation}
Making use of the relations (3.22) and (4.14) gives
\begin{equation}
\delta_{\bar{\rho}} S = \int d^4x d^4\theta \bar{D}_\alpha \left( \rho^\delta \bar{\Sigma} \right) = 0 .
\end{equation}
Finally, it remains to consider the shadow chiral rotation (4.9)
\begin{equation}
\Phi' = e^{-(i/2)\alpha} \Phi , \quad \Sigma' = e^{(i/2)\alpha} \Sigma .
\end{equation}
The action is invariant under such transformations, as a consequence of eq. (4.1), (3.21a) and (4.12).
4.4 Dual formulation

We now turn to considering the superconformal symmetries within the dual formulation (3.28). First of all, we should discuss the homogeneity properties of $H(\Phi, \bar{\Phi}, \Psi, \bar{\Psi})$. Using the homogeneity condition (4.12) and the standard properties of the Legendre transformation, one obtains

$$\left( \frac{\partial \Phi_I}{\partial \Phi^I} + \frac{\partial \Psi_I}{\partial \Psi^I} \right) H(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = H(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) .$$

(4.21)

Taking into account eq. (3.31), this is equivalent to

$$\left( \frac{\partial \Phi_I}{\partial \Phi^I} + \frac{\partial \Psi_I}{\partial \Psi^I} \right) H(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = H(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) .$$

(4.22)

The explicit structure of the first-order action, eq. (3.26), as well as the $\mathcal{N} = 1$ superconformal transformation law of $\Sigma$, eq. (4.15b), imply that the corresponding transformation of $\Psi$ is

$$\delta \xi \Psi = -\xi \Psi - 2\sigma \Psi$$

(4.23)

which coincides with that of $\Phi$, eq. (4.15b). Using the homogeneity conditions (4.1) and (4.22), one immediately sees that the action (3.28) is $\mathcal{N} = 1$ superconformal.

As a next step, we should consider the extended superconformal transformation. Using the transformation laws (4.17a) and (4.17b), one can check that the first-order action (3.26) possesses the following invariance:

$$\delta \Phi^I = \frac{1}{2} \bar{D}^2 \{ \bar{\rho} \Sigma^I \} ,$$

(4.24a)

$$\delta \Sigma^I = -\rho D\Phi - \frac{1}{2} (D\rho) \Phi + \bar{D}_\alpha \left\{ \bar{\rho} \left( C^I - \frac{1}{2} \Gamma^I_{JK} \Sigma^J \Sigma^K \right) \right\}$$

$$- \frac{1}{2} \rho \Gamma^I_{JK} \Sigma^J \bar{D}^2 \Sigma^K + \frac{1}{2} \rho \frac{\partial G^I}{\partial \Sigma^J} \Sigma^J \bar{D}^2 \Sigma^K ,$$

(4.24b)

$$\delta \Psi_I = -\frac{1}{2} \bar{D}^2 \left\{ \bar{\rho} \left( K_I - \Gamma^K_{IJ} \Psi_K \Sigma^I \right) + \frac{\partial G^I}{\partial \Sigma^J} \Psi_J + \frac{\partial \Xi}{\partial \Sigma^I} \right\} .$$

(4.24c)

Here we have introduced the parameter $\bar{\rho}$ and its conjugate defined by

$$\bar{\rho}_\alpha = \bar{D}_\alpha \bar{\rho} , \quad \rho_\alpha = D_\alpha \rho .$$

(4.25)

Modulo redefinitions of the form

$$\bar{\rho} \rightarrow \bar{\rho} + \mu , \quad \bar{D}_\alpha \mu = 0 ,$$

(4.26)
the parameter $\rho$ can be chosen to be

$$\rho = \tau + \epsilon \theta + \frac{1}{2} \lambda \theta^2 + i x^a \theta \sigma_a \bar{\eta},$$  \hspace{1cm} (4.27)

with $\tau$ corresponding to a “central charge” transformation.

It follows from eqs. (4.24a) – (4.24c) that the dual theory (3.28) is invariant under the following extended superconformal transformation:

$$\delta \Phi^I = \frac{1}{2} \bar{D}^2 \left\{ \bar{\rho} \frac{\partial \mathcal{H}}{\partial \Psi^I} \right\},$$  \hspace{1cm} (4.28a)

$$\delta \Psi_I = -\frac{1}{2} \bar{D}^2 \left\{ \bar{\rho} \left( K_I - \Gamma^K_{IJ} \Psi^K \frac{\partial \mathcal{H}}{\partial \Psi^J} + \nabla_I \mathcal{H} \right) \right\}.$$  \hspace{1cm} (4.28b)

Similarly to the second supersymmetry transformation (3.40), this can be rewritten in terms of the hyperkähler potential as follows:

$$\delta \phi^a = \frac{1}{2} J^{ab} \bar{D}^2 \left\{ \bar{\rho} \frac{\partial K}{\partial \phi^b} \right\}.$$  \hspace{1cm} (4.29)

If only the second term in the parameter (4.27) is kept, $\rho \to \epsilon \theta$, then our transformation law (4.29) reduces to (3.40).

It remains to point out that the shadow chiral rotation (4.20) turns into

$$\phi^a \longrightarrow e^{-(i/2)\alpha} \phi^a.$$  \hspace{1cm} (4.30)

### 4.5 Homothetic conformal Killing vector

Before we continue, it is worth recalling the salient facts about homothetic conformal Killing vectors (see [28, 31] for more details). By definition, a homothetic conformal Killing vector $\chi$ on a Kähler manifold $(\mathcal{M}, g_{ab})$,

$$\chi = \chi^a \frac{\partial}{\partial \phi^a} + \chi^a \frac{\partial}{\partial \phi^b} \equiv \chi^\mu \frac{\partial}{\partial \phi^\mu},$$  \hspace{1cm} (4.31)

obeys the constraint

$$\nabla_\nu \chi^\mu = \delta_\nu^\mu \iff \nabla_b \chi^a = \delta_b^a, \quad \nabla_b \chi^a = \partial_b \chi^a = 0.$$  \hspace{1cm} (4.32)

In particular, $\chi$ is holomorphic. Its properties include:

$$g_{ab} \chi^a \bar{\chi}^b = K, \quad \chi_a := g_{ab} \bar{\chi}^b = \partial_a K.$$  \hspace{1cm} (4.33)
with $K$ the Kähler potential.

If $I^\mu_\nu$ is the complex structure on the Kähler manifold, then $\upsilon^\mu := I^\mu_\nu \chi^\nu$ is a holomorphic Killing vector. In the case that $\mathcal{M}$ is a hyperkähler cone, there are three complex structure, $(J_A)^\mu_\nu$, and the Killing vectors $\upsilon_A^\mu := (J_A)^\mu_\nu \chi^\nu$ generate the Lie algebra of the group SU(2).

Let us return to the hyperkähler cone studied in the main body of this section. Here the homothetic conformal Killing vector vector proves to be
\[
\chi^a = \phi^a, \quad \bar{\chi}^\bar{a} = \bar{\phi}^\bar{a}.
\] (4.34)
This can readily be checked using the homogeneity condition (4.22) and the explicit form of the Christoffel symbols on Kähler manifolds.

The extended superconformal transformation (4.29) can now be rewritten in terms of the homothetic conformal Killing vector:
\[
\delta \phi^a = \frac{1}{2} \bar{D}^2 \left\{ \bar{\rho} \omega^{ab} \chi_b \right\}.
\] (4.35)

## 5 Superconformal nonlinear sigma-models II

We now have all prerequisites available to develop a chiral formulation in $\mathcal{N} = 1$ superspace for the most general $\mathcal{N} = 2$ superconformal nonlinear $\sigma$-model. Given a hyperkähler cone $\mathcal{M}$, we pick one of its complex structures, say $J_3$, and introduce complex coordinates $\phi^a$ compatible with it. In these coordinates, $J_3$ has the form (2.11). Two other complex structures, $J_1$ and $J_2$, become
\[
J_1 = \begin{pmatrix} 0 & g^{ac} \omega_{cb} \\ g^{ac} \omega_{cb} & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & ig^{ac} \omega_{cb} \\ -i g^{ac} \omega_{cb} & 0 \end{pmatrix},
\] (5.1)
where $g_{ab}$ be the Kähler metric, and $\omega_{ab}$ the holomorphic symplectic two-form. Let $\chi$ be the homothetic conformal Killing vector, eq. (4.32). We then have $g_{\bar{a}b} = \partial_a \partial_b K$, where the potential $K$ is related to $\chi$ according to eq. (4.33). With this Kähler potential, the $\sigma$-model action (2.1) turns out to be $\mathcal{N} = 2$ superconformal, as we are going to prove.

Our first observation is that the action (2.1) is invariant under $\mathcal{N} = 1$ superconformal transformations of the form:
\[
\delta \xi \phi^a = -\xi \phi^a - 2\sigma \chi^a(\phi),
\] (5.2)
compare with eqs. (4.15a) and (4.23). The invariance follows from the identity
\[
\chi^a(\phi) \partial_a K(\phi, \bar{\phi}) = K(\phi, \bar{\phi})
\] (5.3)
and the standard properties of the \(\mathcal{N} = 1\) superconformal Killing vectors \[43\]. What we have actually demonstrated here, can be rephrased as follows. If a Kähler manifold \(\mathcal{M}\) possesses a homothetic conformal Killing vector \(\chi\), then the associated \(\mathcal{N} = 1\) nonlinear \(\sigma\)-model (2.1) is superconformal (see also \[28\] \[44\]).

Next, we define the extended superconformal transformation of \(\phi^a\):
\[
\delta \phi^a = \frac{1}{2} \bar{D}^2 \{ \bar{\rho} \omega^{ab} \chi_b \} ,
\] (5.4)
which should be compared with (4.35). We now prove that the action is invariant under (5.4). As before, it is sufficient to evaluate the variation \(\delta \rho S\) which corresponds to the choice \(\rho = 0\) and \(\bar{\rho} \neq 0\). The variation of the action is
\[
\delta \rho S = -\frac{1}{2} \int d^4x d^4\theta \left( \bar{D}_a \chi_a \right) \left( \bar{D}^a \rho \right) \omega^{ab} \chi_b = -\frac{1}{2} \int d^4x d^4\theta \bar{\rho}_\alpha \left( \bar{D}^a \bar{\phi}^\alpha \right) g_{\bar{c}a} \omega^{ab} \chi_b
\]
\[
= -\frac{1}{2} \int d^4x d^2\theta d^2\bar{\theta} \bar{\rho}_\alpha \left( \bar{D}^a \bar{\phi}^\alpha \right) \bar{\omega}_{\bar{a}b} \bar{\chi}^b .
\] (5.5)
Since the tensor fields \(\bar{\omega}_{\bar{a}b}\) and \(\bar{\chi}^b\) are anti-holomorphic, and the parameter \(\bar{\rho}_\alpha\) is antichiral, the combination \(\bar{\rho}_\alpha \bar{\omega}_{\bar{a}b} \bar{\chi}^b\) appearing in the integrand is antichiral. As a result, doing the Grassmann integral \(\int d^2\theta\) gives
\[
\delta \rho S = \frac{1}{8} \int d^4x d^2\theta \bar{\rho}_\alpha \bar{\omega}_{\bar{a}b} \bar{\chi}^b D^2 \bar{D}^a \bar{\phi}^\alpha = 0 ,
\] (5.6)
for \(D^2 \bar{D}^a \bar{\phi}\) is identically zero for any antichiral superfield \(\bar{\phi}\).

Finally, we define the infinitesimal shadow chiral rotation of \(\phi^a\):
\[
\delta \phi^a = -\frac{i}{2} \alpha \chi^a(\phi) , \quad \bar{\alpha} = \alpha ,
\] (5.7)
compare with (1.30). Because of the identity (5.3), this transformation leaves the action invariant. It should be remarked that the shadow chiral rotation is generated by the Killing vector
\[
v = i \chi^a(\phi) \frac{\partial}{\partial \phi^a} - i \bar{\chi}^\alpha(\bar{\phi}) \frac{\partial}{\partial \bar{\phi}^\alpha} .
\] (5.8)
6 Discussion and future directions

In this paper we developed the formulation in terms of $\mathcal{N} = 1$ chiral superfields for the following $\mathcal{N} = 2$ supersymmetric theories: (i) $\sigma$-models on cotangent bundles of Kähler manifolds; (ii) general superconformal $\sigma$-models described by weight-one polar supermultiplets. Using the considerations of duality, we demonstrated that the holomorphic symplectic two-form $\omega_{ab}$ of the hyperkähler space $T^*\mathcal{M}$ coincides with the canonical holomorphic symplectic structure, eq. (3.44). In the case (ii), we also determined the homothetic conformal Killing vector $\chi$, eq. (4.34). The explicit expressions for $\omega^{(2,0)}$ and $\chi$ are necessary in the context of the $\mathcal{N} = 2$ superconformal quotient construction [31] allowing one to reduce the hyperkähler cone to the corresponding quaternion Kähler space.

We also presented, in section 5, the most general $\mathcal{N} = 2$ superconformal nonlinear $\sigma$-model formulated in terms of $\mathcal{N} = 1$ chiral superfields. It would be interesting to compare the corresponding superconformal transformations, obtained using off-shell techniques, with those defined in the component approach to $\mathcal{N} = 2$ superconformal $\sigma$-models [30].

The off-shell $\mathcal{N} = 2$ superconformal $\sigma$-model defined by eqs. (3.1) and (4.1) possesses a slightly different realization [32]. It can be defined if the arctic weight-one hypermultiplets $\Upsilon^I(\zeta)$ include a compensator $\Upsilon(\zeta)$, that is an arctic multiplet with its lowest-order ($\zeta$-independent) component $\Phi$ is everywhere non-vanishing. Then, we can introduce new dynamical variables comprising the only weight-one multiplet $\Upsilon(z, \zeta)$ and some set of weight-zero arctic multiplets $\upsilon^i(z, \zeta)$.

\[
S[\Upsilon(\zeta), \upsilon(\zeta)] = \oint \frac{d\zeta}{2\pi i} \int d^4x d^4\theta \Upsilon \exp \left\{ K(\upsilon^i, \bar{\upsilon}^j) \right\}, \tag{6.1}
\]

with $K(\upsilon, \bar{\upsilon})$ the Kähler potential of a Kähler manifold $\mathcal{M}_K$. This action is invariant under Kähler transformations of the form

\[
\Upsilon \rightarrow e^{-\Lambda(\upsilon)} \Upsilon, \quad K(\upsilon, \bar{\upsilon}) \rightarrow K(\upsilon, \bar{\upsilon}) + \Lambda(\upsilon) + \bar{\Lambda}(\bar{\upsilon}), \tag{6.2}
\]

with $\Lambda$ a holomorphic function. In accordance with [42] (see also [43]), the space $\mathcal{M}_K$ is necessarily a Kähler-Hodge manifold, and the arctic variables $\Upsilon^I$ and $\Xi$ in (6.1) parametrize a holomorphic line bundle over $\mathcal{M}_K$.

As discussed in detail in [42], the theory (6.1) possesses a dual formulation in which the arctic compensator $\Upsilon$ and its conjugate are dualized into an $O(2)$ multiplet $H(\zeta)$ [7] (or $\mathcal{N} = 2$ tensor multiplet [15])

\[
H(\zeta) = \frac{1}{\zeta} \phi + G - \zeta \phi, \quad D_\zeta \phi = 0, \quad D^2 G = 0, \quad G = G. \tag{6.3}
\]
The corresponding action, which is a rigid supersymmetric version of the theory introduced in [46], is

\[ S[H(\zeta), \upsilon(\zeta)] = -\oint \frac{d\zeta}{2\pi i} \int d^4x d^4\theta H \ln H + \oint \frac{d\zeta}{2\pi i} \int d^4x d^4\theta H K(\upsilon, \bar{\upsilon}^j). \quad (6.4) \]

Here the first term is the $\mathcal{N} = 2$ projective-superspace formulation [7] of the $\mathcal{N} = 2$ improved tensor multiplet model [47].

It would be very interesting to extend the analysis given in sections 3 and 4 to the case of the $\sigma$-models (6.1) and (6.4). The important feature of (6.1) is that the Kähler potential $K$ is essentially arbitrary. The structure of the action (6.1) is similar to that describing locally supersymmetric $\sigma$-models in $\mathcal{N} = 1$ supergravity, see e.g. [43] for a review. As to the formulation (6.4), it turns out to be useful for generating new hyperkähler cones [42].

Our derivation of the chiral formulation in $\mathcal{N} = 1$ superspace for general $\mathcal{N} = 2$ superconformal nonlinear $\sigma$-models can naturally be extended to five and six dimensions. In the 5D case, one has to use the formalism of superconformal Killing vectors and the off-shell superconformal $\sigma$-models introduced in [32]. In the 6D case, the general aspects of superconformal symmetry in superspace have been elaborated by Park [48]. General off-shell 6D $\mathcal{N} = (1, 0)$ superconformal nonlinear $\sigma$-models have not yet been described in the literature, however they can be constructed in complete analogy with the 5D construction of [32]. It should be emphasized that 5D and 6D rigid supersymmetric nonlinear $\sigma$-models with eight supercharges have been formulated in 4D $\mathcal{N} = 1$ superspace in Refs. [6] and [49], respectively. The work of Bagger and Xiong was based on the careful analysis of supersymmetry transformations in five dimensions. As to the six-dimensional construction of [49], the hyperkähler conditions on the target space geometry were derived by the authors on the basis of the requirement that the component action must be Lorentz invariant, without any analysis of supersymmetry.

In conclusion, we would like to raise an issue that is simplest to formulate in terms of the $\mathcal{N} = 2$ supersymmetric $\sigma$-models on cotangent bundles of Kähler manifolds. For these theories, we have considered the three different realizations: (i) the off-shell $\Upsilon$-formulation given by the action (3.1); (ii) the $\Phi\Sigma$-formulation (3.10) which emerges from (3.1) upon elimination of the auxiliary superfields; (iii) the $\Phi\Psi$-formulation (3.28) which is dual to (3.10). Deriving the latter formulation was the actual goal of our analysis. So, do we really need the off-shell realization (3.1) and/or its reduced version (3.10)? The answer is “Yes” in general. The point is that the “Hamiltonian” $\mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi})$ in the hyperkähler potential (3.30) must obey a highly nonlinear differential equation (this formulation is therefore not suitable to study deformations of the hyperkähler structure).
On the other hand, the tangent-bundle realization (3.10) is generated by two functions $\mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ and $G^I(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ which must satisfy the quadratic differential equations (3.19a) – (3.19c), with $\Xi$ given by eq. (3.22). The problem of solving these equations is technically much simpler than that of solving the nonlinear equation obeyed by $\mathcal{H}$. As an illustration, consider the case when $K(\Phi, \bar{\Phi})$ corresponds to a Hermitian symmetric space. In this case, $G^I = \Xi = 0$ and $\mathcal{L}$ obeys the linear differential equation (3.25) which is easy to solve [24, 25]. On the other hand, in the cotangent-bundle realization $\mathcal{H}$ obeys the quadratic differential equation [24]

$$\mathcal{H}^I g_{IJ} - \frac{1}{2} \mathcal{H}^K \mathcal{H}^L R_{KJL}^I \Psi_I = \bar{\Psi}_J, \quad \mathcal{H}^I = \frac{\partial \mathcal{H}}{\partial \Psi_I}. \quad (6.5)$$

Solving this equation is more challenging. Its solution was found in [25].

**Acknowledgements:**
The hospitality of the Center for Quantum Spacetime at Sogang University (grant number R11-2005-021) during the final stage of this project is gratefully acknowledged. The author is also grateful to Ian McArthur for reading the manuscript. This work is supported in part by the Australian Research Council.

### A Extended supersymmetry

Here we derive the conditions [5] for the $\sigma$-model (2.1) to be invariant under the transformations (2.2) and (2.3). For our consideration, it is handy to make use of the following condensed notation:

$$S = \int d^4x d^2\theta d^2\bar{\theta} K(\phi^a, \bar{\phi}^\bar{a}) \equiv \int K. \quad (A.1)$$

Since the superfield parameters $\epsilon$ and $\bar{\epsilon}$ in (2.2) are independent, modulo complex conjugation, it is sufficient to analyze only the variation $\delta_\epsilon S$ corresponding to the choice $\epsilon = 0$ and $\bar{\epsilon} \neq 0$.

Varying the action gives

$$\delta_\epsilon S = \frac{1}{2} \int K_a D^2 (\bar{\epsilon} \bar{\Omega}^a) = -\frac{1}{2} \int K_{ab} (\bar{D}_\alpha \bar{\phi}^\bar{b}) \bar{D}^\beta (\bar{\epsilon} \bar{\Omega}^a)$$

$$= -\frac{1}{2} \int g_{ab} (\bar{D}_\alpha \bar{\phi}^\bar{b}) \left\{ \bar{\Omega}^a \bar{D}^\beta \bar{\epsilon} + \bar{\epsilon} \bar{\Omega}^a \bar{\Omega}^\beta \bar{D}^\beta \phi^\bar{a} \right\}. \quad (A.2)$$

\(^7\)Of course, the off-shell $\sigma$-model (3.1) is generated by an arbitrary real analytic function $K(\Phi, \bar{\Phi})$, and therefore is most suitable if one is interested in deformations of the hyperkähler structure. But here we still have to address the problem of eliminating the auxiliary superfields.
Choose $\bar{e} = \tau = \text{const}$. Then $\delta_{\bar{e}}S$ reduces to

$$2\delta_{\bar{e}}S = -\tau \int g_{a\overline{b}} \Omega^a\Omega^\overline{b}(D_{\dot{\alpha}}\phi^\overline{b}) \dot{D}^\dot{\alpha}\phi^\overline{a} \equiv -\tau \int \bar{\omega}_{\overline{b}c}(D\phi^\overline{b})D\phi^\overline{a}. \quad (A.3)$$

Since $(D\phi^\overline{b})D\phi^\overline{a}$ is symmetric, the above variation vanishes if the equation (2.4) holds. As a result, the variation (A.2) becomes

$$\delta_{\bar{e}}S = -\frac{1}{2} \int g_{a\overline{b}} (D_{\dot{\alpha}}\phi^\overline{b}) \Omega^a \dot{D}^\dot{\alpha}. \quad (A.4)$$

One can now see that it is not necessary to constrain the parameter as in eq. (2.3), and instead it is sufficient to impose the weaker constraint (2.16). We see that $\delta_{\bar{e}}S$ vanishes if and only if the following functional

$$\Xi_{\dot{\alpha}}[\phi, \phi] := \int \Omega^a g_{a\overline{b}} (\dot{D}_{\dot{\alpha}}\phi^\overline{b}) = \int \dot{\Omega}^a \dot{D}_{\dot{\alpha}}K_a \quad (A.5)$$

is identically zero.

Let us vary (A.5) with respect to $\phi$:

$$\delta_{\phi} \Xi_{\dot{\alpha}}[\phi, \phi] = -2 \int \bar{\omega}_{\overline{b}c} \delta \phi^\overline{b} \dot{D}_{\dot{\alpha}}\phi^\overline{c}. \quad (A.6)$$

To make this variation vanish identically, one has to impose the equation (2.5a), that is $\bar{\omega}_{\overline{b}c} = \bar{\omega}_{\overline{b}c}(\phi)$. Indeed, since $\phi$ and $\delta\phi$ are antichiral, we then have

$$\int \bar{\omega}_{\overline{b}c} \delta \phi^\overline{b} \dot{D}_{\dot{\alpha}}\phi^\overline{c} = -\frac{1}{4} \int d^4x d^2\theta \bar{\omega}_{\overline{b}c} \delta \phi^\overline{b} D^2 \dot{D}_{\dot{\alpha}}\phi^\overline{c} = 0. \quad (A.7)$$

Return to the general analysis of the requirement that the functional (A.5) should vanish identically. We can represent

$$\Xi_{\dot{\alpha}}[\phi, \phi] = \frac{1}{16} \int d^4x D^2 \dot{D}^2 \left\{ \Omega^a \dot{D}_{\dot{\alpha}}K_a \right\}. \quad (A.8)$$

Here $\dot{D}^2 \left\{ \Omega^a \dot{D}_{\dot{\alpha}}K_a \right\}$ can be expressed in terms of the anti-holomorphic two-form $\bar{\omega}_{\overline{b}c}$, with the property that

$$D_{\dot{\alpha}}\bar{\omega}_{\overline{b}c} = 0. \quad (A.9)$$

One thus obtains

$$\dot{D}^2 \left\{ \Omega^a \dot{D}_{\dot{\alpha}}K_a \right\} = 2\bar{\omega}_{\overline{b}c} (\dot{D}_{\dot{\alpha}}\phi^\overline{b}) \dot{D}^2\phi^\overline{c} + \frac{1}{2} \mathcal{F}_{b\overline{c}\overline{d}} (\dot{D}_{\dot{\alpha}}\phi^\overline{b})(\dot{D}_{\dot{\beta}}\phi^\overline{c}) \dot{D}_{\dot{\gamma}}\phi^\overline{d}, \quad (A.10)$$

where

$$\mathcal{F}_{b\overline{c}\overline{d}} = \mathcal{F}_{b\overline{d}\overline{c}} := \nabla_e \bar{\omega}_{\overline{b}\overline{d}} + \nabla_{\overline{d}} \bar{\omega}_{\overline{b}e} + 4\bar{\omega}_{\overline{b}e} \Gamma^e_{\overline{c}\overline{d}}. \quad (A.10)$$
Upon plugging the expression (A.9) into (A.7), there occur several sectors that differ from each other by the number of superfields hit by derivatives. One particular sector, which is proportional to $\partial \bar{\Phi} \partial \bar{\Phi} \partial \bar{\Phi}$, can be seen to vanish if and only if the equation (2.5b). As a result, the above expression for $F_{\bar{c}\bar{d}}$ simplifies

$$F_{\bar{c}\bar{d}} = 4 \bar{\omega}_{\bar{b}\bar{e}} \Gamma_{\bar{c}\bar{d}}^e. \quad (A.11)$$

If we now recall that $\partial_a \Gamma_{\bar{c}\bar{d}}^e = R_{\bar{c}\bar{d}a}^e$ is the Riemann curvature, then (A.7) becomes

$$\Xi_a[\phi, \bar{\phi}] = \frac{1}{4} \int d^4x \, D^a \left[ (\bar{D}_a \bar{\phi}^b) (\bar{D}_b \bar{\phi}^c) (\bar{D}_c \bar{\phi}^d) \right] \bar{\omega}_{\bar{b}\bar{e}} R_{\bar{c}\bar{d}a}^e D\Phi^a$$

$$+ \frac{1}{8} \int d^4x \, (\bar{D}_a \bar{\phi}^b)(\bar{D}_b \bar{\phi}^c)(\bar{D}_c \bar{\phi}^d) \, D^a \left[ \bar{\omega}_{\bar{b}\bar{e}} R_{\bar{c}\bar{d}a}^e D\Phi^a \right]. \quad (A.12)$$

This expression vanishes due to the following two observations. First, it holds that

$$(\bar{D}_a \bar{\phi}^b)(\bar{D}_b \bar{\phi}^c)(\bar{D}_c \bar{\phi}^d) + (\bar{D}_a \bar{\phi}^c)(\bar{D}_c \bar{\phi}^d)(\bar{D}_d \bar{\phi}^a) + (\bar{D}_a \bar{\phi}^d)(\bar{D}_d \bar{\phi}^a)(\bar{D}_a \bar{\phi}^c) = 0. \quad (A.13)$$

Second, the covariant constancy of $\bar{\omega}_{\bar{b}\bar{e}}$ implies that the tensor $\bar{\omega}_{\bar{b}\bar{e}} R_{\bar{c}\bar{d}a}^e$ is completely symmetric in its “barred” indices,

$$T_{\bar{a}\bar{b}\bar{c}\bar{d}} := \bar{\omega}_{\bar{b}\bar{e}} R_{\bar{c}\bar{d}a}^e = T_{\bar{a}(\bar{b}\bar{c}\bar{d})}. \quad (A.14)$$

This completes the proof.

References

[1] T. L. Curtright and D. Z. Freedman, “Nonlinear sigma models with extended supersymmetry in four-dimensions,” Phys. Lett. B 90, 71 (1980) [Erratum-ibid. B 91, 487 (1980)]; L. Alvarez-Gaumé and D. Z. Freedman, “Geometrical structure and ultraviolet finiteness in the supersymmetric sigma model,” Commun. Math. Phys. 80, 443 (1981).

[2] J. Bagger and E. Witten, “Matter couplings in N = 2 supergravity,” Nucl. Phys. B 222, 1 (1983).

[3] J. De Jaegher, B. de Wit, B. Kleijn and S. Vandoren, “Special geometry in hypermultiplets,” Nucl. Phys. B 514, 553 (1998) [arXiv:hep-th/9707262].

[4] U. Lindström and M. Roček, “Scalar tensor duality and N = 1, 2 nonlinear sigma models,” Nucl. Phys. B 222, 285 (1983).

[5] C. M. Hull, A. Karlhede, U. Lindström and M. Roček, “Nonlinear sigma models and their gauging in and out of superspace,” Nucl. Phys. B 266, 1 (1986).
[6] J. Bagger and C. Xiong, “$N = 2$ nonlinear sigma models in $N = 1$ superspace: Four and five dimensions,” arXiv:hep-th/0601165.

[7] A. Karlhede, U. Lindström and M. Roček, “Self-interacting tensor multiplets in $N = 2$ superspace,” Phys. Lett. B 147, 297 (1984).

[8] U. Lindström and M. Roček, “New hyperkähler metrics and new supermultiplets,” Commun. Math. Phys. 115, 21 (1988); “$N = 2$ super Yang-Mills theory in projective superspace,” Commun. Math. Phys. 128, 191 (1990).

[9] U. Lindström and M. Roček, “Properties of hyperkähler manifolds and their twistor spaces,” arXiv:0807.1366 [hep-th].

[10] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev, “Unconstrained $N = 2$ matter, Yang-Mills and supergravity theories in harmonic superspace,” Class. Quant. Grav. 1, 469 (1984).

[11] A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky and E. S. Sokatchev, Harmonic Superspace, Cambridge University Press, 2001.

[12] S. M. Kuzenko, “Projective superspace as a double-punctured harmonic superspace,” Int. J. Mod. Phys. A 14, 1737 (1999) [arXiv:hep-th/9806147].

[13] D. Jain and W. Siegel, “Deriving projective hyperspace from harmonic,” Phys. Rev. D 80, 045024 (2009) [arXiv:0903.3588 [hep-th]].

[14] A. A. Rosly, “Super Yang-Mills constraints as integrability conditions,” in Proceedings of the International Seminar on Group Theoretical Methods in Physics,” (Zvenigorod, USSR, 1982), M. A. Markov (Ed.), Nauka, Moscow, 1983, Vol. 1, p. 263 (in Russian).

[15] J. Wess, “Supersymmetry and internal symmetry,” Acta Phys. Austriaca 41, 409 (1975).

[16] F. Gonzalez-Rey, M. Roček, S. Wiles, U. Lindström and R. von Unge, “Feynman rules in $N = 2$ projective superspace. I: Massless hypermultiplets,” Nucl. Phys. B 516, 426 (1998) [arXiv:hep-th/9710250].

[17] B. Zumino, “Superspace,” in Unification of the Fundamental Particle Interactions, S. Ferrara, J. Ellis and P. van Nieuwenhuizen (Eds.), Plenum Press, 1980, p. 101.

[18] S. J. Gates Jr. and W. Siegel, “Variant superfield representations,” Nucl. Phys. B 187, 389 (1981).

[19] W. Siegel, “Gauge spinor superfield as a scalar multiplet,” Phys. Lett. B 85, 333 (1979).
[20] S. J. Gates Jr. and S. M. Kuzenko, “The CNM-hypermultiplet nexus,” Nucl. Phys. B 543, 122 (1999) [arXiv:hep-th/9810137].

[21] S. J. Gates Jr. and S. M. Kuzenko, “4D N = 2 supersymmetric off-shell sigma models on the cotangent bundles of Kähler manifolds,” Fortsch. Phys. 48, 115 (2000) [arXiv:hep-th/9903013].

[22] M. Arai and M. Nitta, “Hyper-Kähler sigma models on (co)tangent bundles with SO(n) isometry,” Nucl. Phys. B 745, 208 (2006) [arXiv:hep-th/0602277].

[23] M. Arai, S. M. Kuzenko and U. Lindström, “Hyperkähler sigma models on cotangent bundles of Hermitian symmetric spaces using projective superspace,” JHEP 0702, 100 (2007) [arXiv:hep-th/0612174].

[24] M. Arai, S. M. Kuzenko and U. Lindström, “Polar supermultiplets, Hermitian symmetric spaces and hyperkähler metrics,” JHEP 0712, 008 (2007) [arXiv:0709.2633 [hep-th]].

[25] S. M. Kuzenko and J. Novak, “Chiral formulation for hyperkähler sigma-models on cotangent bundles of symmetric spaces,” JHEP 0812, 072 (2008) [arXiv:0811.0218 [hep-th]].

[26] A. Swann, “HyperKähler and quaternion Kähler geometry,” Math. Ann. 289, 421 (1991).

[27] K. Galicki, “Geometry of the scalar couplings in N = 2 supergravity models,” Class. Quant. Grav. 9, 27 (1992).

[28] G. W. Gibbons and P. Rychenkova, “Cones, tri-Sasaki structures and superconformal invariance,” Phys. Lett. B 443, 138 (1998) [arXiv:hep-th/9809158].

[29] B. de Wit, B. Kleijn and S. Vandoren, “Rigid N = 2 superconformal hypermultiplets,” in Supersymmetries and quantum symmetries, J. Wess and E. A. Ivanov (Eds.), Springer-Verlag, 1999, p. 37 (Lectures Notes in Physics, Vol. 524) arXiv:hep-th/9808160.

[30] B. de Wit, B. Kleijn and S. Vandoren, “Superconformal hypermultiplets,” Nucl. Phys. B 568, 475 (2000) [arXiv:hep-th/9909228].

[31] B. de Wit, M. Roček and S. Vandoren, “Hypermultiplets, hyperkähler cones and quaternion-Kähler geometry,” JHEP 0102, 039 (2001) [arXiv:hep-th/0101161].

[32] S. M. Kuzenko, “On superconformal projective hypermultiplets,” JHEP 0712, 010 (2007) [arXiv:0710.1479[hep-th]].

[33] S. M. Kuzenko, “On compactified harmonic / projective superspace, 5D superconformal theories, and all that,” Nucl. Phys. B 745, 176 (2006) [arXiv:hep-th/0601177].

[34] B. Zumino, “Supersymmetry and Kähler manifolds,” Phys. Lett. B 87, 203 (1979).
[35] P. Fayet, “Fermi-Bose hypersymmetry,” Nucl. Phys. B 113, 135 (1976).

[36] M. F. Sohnius, “Supersymmetry and central charges,” Nucl. Phys. B 138, 109 (1978).

[37] A. Galperin, E. Ivanov and V. Ogievetsky, “Superfield anatomy of the Fayet-Sohnius multiplet,” Sov. J. Nucl. Phys. 35, 458 (1982) [Yad. Fiz. 35, 790 (1982)].

[38] M. Roček and P. K. Townsend, “Three-loop finiteness of the N = 4 supersymmetric nonlinear sigma model,” Phys. Lett. B 96, 72 (1980).

[39] S. M. Kuzenko and W. D. Linch, “On five-dimensional superspaces,” JHEP 0602, 038 (2006) [arXiv:hep-th/0507176].

[40] D. Kaledin, “Hyperkähler structures on total spaces of holomorphic cotangent bundles,” in D. Kaledin and M. Verbitsky, Hyperkähler Manifolds, International Press, Cambridge MA, 1999 [alg-geom/9710026]; “A canonical hyperkähler metric on the total space of a cotangent bundle,” in Quaternionic Structures in Mathematics and Physics, S. Marchiafava, P. Piccinni and M. Pontecorvo (Eds.), World Scientific, 2001 [alg-geom/0011256].

[41] B. Feix, “Hyperkähler metrics on cotangent bundles,” Cambridge PhD thesis, 1999; J. reine angew. Math. 532, 33 (2001).

[42] S. M. Kuzenko, U. Lindström and R. von Unge, “New extended superconformal sigma models and quaternion Kähler manifolds,” JHEP 0909, 119 (2009) [arXiv:0906.4393 [hep-th]].

[43] I. L. Buchbinder and S. M. Kuzenko, Ideas and Methods of Supersymmetry and Supergravity or a Walk Through Superspace, IOP, Bristol, 1998.

[44] I. R. Klebanov and E. Witten, “Superconformal field theory on threebranes at a Calabi-Yau singularity,” Nucl. Phys. B 536, 199 (1998) [arXiv:hep-th/9807080].

[45] N. J. Hitchin, A. Karlhede, U. Lindström and M. Roček, “Hyperkähler metrics and supersymmetry,” Commun. Math. Phys. 108, 535 (1987).

[46] S. M. Kuzenko, “On N = 2 supergravity and projective superspace: Dual formulations,” Nucl. Phys. B 810, 135 (2009) [arXiv:0807.3381 [hep-th]].

[47] B. de Wit, R. Philippe and A. Van Proeyen, “The improved tensor multiplet in N = 2 supergravity,” Nucl. Phys. B 219, 143 (1983).

[48] J. H. Park, “Superconformal symmetry in six dimensions and its reduction to four,” Nucl. Phys. B 539, 599 (1999) [arXiv:hep-th/9807186].

[49] S. J. Gates Jr., S. Penati and G. Tartaglino-Mazzucchelli, “6D supersymmetric nonlinear sigma-models in 4D, N = 1 superspace,” JHEP 0609, 006 (2006) [arXiv:hep-th/0604042].