Classification of Second Order Symmetric Tensors in 5-Dimensional Kaluza-Klein-Type Theories

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Abstract

An algebraic classification of second order symmetric tensors in 5-dimensional Kaluza-Klein-type Lorentzian spaces is presented by using Jordan matrices. We show that the possible Segre types are [1,1111], [2111], [311], [z̅ 111], and the degeneracies thereof. A set of canonical forms for each Segre type is found. The possible continuous groups of symmetry for each canonical form are also studied.
1 Introduction

It is well known that a coordinate-invariant characterization of the gravitational field in general relativity is best given in terms of the curvature tensor and a finite number of its covariant derivatives relative to a canonically chosen field of Lorentz frames \([1] – [3]\). The Riemann tensor itself is decomposable into three irreducible parts, namely the Weyl tensor (denoted by \(W_{abcd}\)), the traceless Ricci tensor (\(S_{ab} \equiv R_{ab} - \frac{1}{4} R g_{ab}\)) and the Ricci scalar (\(R \equiv R_{ab} g^{ab}\)). The algebraic classification of the Weyl part of the Riemann tensor, known as Petrov classification, has played a significant role in the study of various topics in general relativity. However, for full classification of curvature tensor of nonvacuum space-times one also has to consider the Ricci part of the curvature tensor, which by virtue of Einstein’s equations \(G_{ab} = \kappa T_{ab} + \Lambda g_{ab}\) clearly has the same algebraic classification of both the Einstein tensor \(G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab}\) and the energy-momentum tensor \(T_{ab}\).

The algebraic classification of a symmetric two-tensor defined on a four-dimensional Lorentzian manifold, known as Segre classification, has been discussed by several authors \([4]\) and is of interest in at least three contexts. One is in understanding some purely geometrical features of space-times (see, e.g., Churchill \([5]\), Plebański, \([6]\) Cormack and Hall \([7]\) ). The second one is in classifying and interpreting matter field distributions \([8] – [16]\). The third is as part of the procedure for checking whether apparently different space-times are in fact locally the same up to coordinate transformations (equivalence problem \([4, 8], [17] – [19]\)).

Over the past three decades there has been a resurgence in work on Kaluza-Klein-type theories in five and more dimensions. This has been motivated, on the one hand, by the quest for a unification of gravity with the other fundamental interactions. From a technical viewpoint, on the other hand, they have been used as a way of finding new solutions of Einstein's equations in four dimensions, without ascribing any physical
meaning to the additional components of the metric tensor \[20\].

Using the theory of Jordan matrices we discuss, in this paper, the algebraic classification of second order symmetric tensors defined on five-dimensional (5-D for short) Lorentzian manifolds \(M\), extending previous results on this issue \[22\] – \[23\]. We show that at a point \(p \in M\) the Ricci tensor \(R\) can be classified in four Segre types and their twenty-two degeneracies. Using real half-null pentad bases for the tangent vector space \(T_p(M)\) to \(M\) at \(p\) we derive a set of canonical forms for \(R_{ab}\), generalizing the canonical forms for a symmetric two-tensor on 3-D and 4-D space-times manifolds \[21, 22\]. The continuous groups that leave invariant each canonical form for \(R_{ab}\) are also discussed. Although the Ricci tensor is constantly referred to, the results of the following sections apply to any second order real symmetric tensor on Kaluza-Klein-type 5-D Lorentzian manifolds.

2 Segre Types in 5-D Space-times

The essential idea underlying all classification is the concept of equivalence. Clearly the objects may be grouped into different classes according to the criteria one chooses to classify them. In this section we shall classify \(R^a_b\) up to similarity, which is a criterion quite often used in mathematics and physics. This approach splits the Ricci tensor in 5-D space-times into four Segre types and their degeneracies.

Before proceeding to the classification of the Ricci tensor let us state our general setting. Throughout this paper \(M\) is a 5-D space-time manifold endowed with a Lorentzian metric \(g\) of signature \((-++++)\), \(T_p(M)\) denotes the tangent vector space to \(M\) at a point \(p \in M\), and any latin indices but \(p\) range from 0 to 4.

Let \(R_{ab}\) be the covariant components of a second order symmetric tensor \(R\) at \(p \in M\). Given \(R_{ab}\) we may use the metric tensor to have the mixed form \(R^a_b\) of \(R\) at \(T_p(M)\). In this form the symmetric two-tensor \(R\) may be looked upon as a real linear operator \(R : T_p(M) \rightarrow T_p(M)\). If one thinks of \(R\) as a matrix \(R^a_{\ b}\), one can formulate the
eigenvalue problem

\[ R^a_b V^b = \lambda \delta^a_b V^b , \quad (2.1) \]

where \( \lambda \) is scalar and \( V^b \) are the components of a generic eigenvector \( V \in T_p(M) \).

The fact that we have \( \delta^a_b \) rather than \( g_{ab} \) on the right hand side of equation (2.1) makes apparent that we have cast the non-standard eigenvalue problem involving the hyperbolic (real) metric \( R^a_b V^b = \lambda g_{ab} V^b \) into the standard form (2.1) well known in linear algebra textbooks. However, we pay a price for this because now \( R^a_b \) is no longer symmetric in general. We remind that in a space with positive (or negative) definite metric a real symmetric operator can always be diagonalized over the reals. Despite this problem we shall work with the eigenvalue problem for \( R^a_b \) in the standard form (2.1).

We first consider the cases when all eigenvalues of \( R \) are real. For these cases similarity transformations possibly exist \([24]\) under which \( R^a_b \) takes at \( p \) either one of the following Jordan canonical forms (JCF for short):

(a) Segre type [5]

\[
\begin{pmatrix}
\lambda_1 & 1 & 0 & 0 & 0 \\
0 & \lambda_1 & 1 & 0 & 0 \\
0 & 0 & \lambda_1 & 1 & 0 \\
0 & 0 & 0 & \lambda_1 & 1 \\
0 & 0 & 0 & 0 & \lambda_1
\end{pmatrix},
\]

(b) Segre type [41]

\[
\begin{pmatrix}
\lambda_1 & 1 & 0 & 0 & 0 \\
0 & \lambda_1 & 1 & 0 & 0 \\
0 & 0 & \lambda_1 & 1 & 0 \\
0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & \lambda_2
\end{pmatrix},
\]

(c) Segre type [32]

\[
\begin{pmatrix}
\lambda_1 & 1 & 0 & 0 & 0 \\
0 & \lambda_1 & 1 & 0 & 0 \\
0 & 0 & \lambda_1 & 0 & 0 \\
0 & 0 & 0 & \lambda_2 & 0 \\
0 & 0 & 0 & 0 & \lambda_3
\end{pmatrix},
\]

(d) Segre type [311]

\[
\begin{pmatrix}
\lambda_1 & 1 & 0 & 0 & 0 \\
0 & \lambda_1 & 1 & 0 & 0 \\
0 & 0 & \lambda_1 & 0 & 0 \\
0 & 0 & 0 & \lambda_2 & 0 \\
0 & 0 & 0 & 0 & \lambda_3
\end{pmatrix},
\]

(e) Segre type [221]

\[
\begin{pmatrix}
\lambda_1 & 1 & 0 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 & 0 \\
0 & 0 & \lambda_2 & 1 & 0 \\
0 & 0 & 0 & \lambda_2 & 0 \\
0 & 0 & 0 & 0 & \lambda_4
\end{pmatrix},
\]

(f) Segre type [2111]
or one of the possible block-degenerated Jordan matrices. Here \( \lambda_1, \cdots, \lambda_5 \in \mathcal{R} \). In the Jordan canonical form, a matrix consists of Jordan blocks (Jordan submatrices) along the principal diagonal. The above Segre types are nothing but their Segre characteristic, well known in linear algebra and algebraic geometry. The Segre characteristic is a list of digits inside square brackets, where each digit refers to the multiplicity of the corresponding eigenvalue, which clearly is equal to the dimension of the corresponding Jordan block. A Jordan matrix \( J \) is determined from \( R \) through similarity transformations \( X^{-1}RX = J \), and it is uniquely defined up to the ordering of the Jordan blocks. Further, regardless of the dimension of a Jordan block there is only one eigenvector associated to each block. Degeneracy amongst eigenvalues in different Jordan blocks will be indicated by enclosing the corresponding digits inside round brackets. For each degeneracy of this type it is easy to show that there exists an invariant subspace of eigenvectors (eigenspace), whose dimension is equal to the number of distinct Jordan submatrices with same eigenvalue. Thus, e.g., in the above case (d) if \( \lambda_1 = \lambda_2 \) then the Segre type is \([311]1\), and besides the obvious one-dimensional invariant subspace defined by an eigenvector associated to the eigenvalue \( \lambda_3 \), there is a two-dimensional invariant subspace of eigenvectors (2-eigenspace) with eigenvalue \( \lambda_1 \). Finally, it is worth noting that the Segre type \([1,1111]\) and its degeneracies are the only types that admit a timelike eigenvector. We shall clarify this point in the next section. The comma in these cases is used to separate timelike from spacelike eigenvectors.
We shall show now that the Lorentzian character of the metric $g$ on $M$, together with the symmetry of $R_{ab}$, rule out the above cases (a), (b), (c) and (e). Actually as the procedure for eliminating all these cases are similar, and a proof for the case (a) was already briefly outlined in Santos et al. [25], for the sake of brevity we shall discuss here in details how to eliminate the cases (c) and (e), leaving to the reader to work out the details for the case (b).

From linear algebra we learn that to bring $R^a_b$ to a Jordan canonical $J^a_b$ form there must exist a nonsingular matrix $X$ such that

$$X^{-1}RX = J.$$  \hfill (2.2)

We shall first consider the above case (c), where $J$ is the Jordan canonical matrix for the Segre type [32]. Multiplying the matricial equation (2.2) from the left by $X$ and equating the columns of both sides of the resulting matricial equation we have the Jordan chain relations

\begin{align*}
RX_1 &= \lambda_1 X_1, \quad (2.3) \\
RX_2 &= \lambda_1 X_2 + X_1, \quad (2.4) \\
RX_3 &= \lambda_1 X_3 + X_2, \quad (2.5) \\
RX_4 &= \lambda_2 X_4, \quad (2.6) \\
RX_5 &= \lambda_2 X_5 + X_4, \quad (2.7)
\end{align*}

where we have denoted by $X_A$ ($A = 1, \cdots, 5$) the column vectors of the matrix $X$ and beared in mind that here the matrix $J$ is that of case (c). As $R$ is a symmetric two-tensor, from equations (2.3) and (2.4) one easily obtains

$$\lambda_1 X_1.X_2 = \lambda_1 X_2.X_1 + X_1.X_1, \quad (2.8)$$

where the scalar products defined by a Lorentzian metric $g$ are indicated by a dot between two vectors. Equation (2.8) implies that $X_1$ is a null vector. Similarly eqs. (2.6) and
\[(2.7)\] imply that \(X_4\) is also a null vector. Moreover, if \(\lambda_1 \neq \lambda_2\) from eqs. \((2.3)\) and \((2.6)\) one finds that \(X_1.X_4 = 0\). If \(\lambda_1 = \lambda_2\) equations \((2.3)\), \((2.6)\) and \((2.7)\) imply again that \(X_1.X_4 = 0\). In short, \(X_1\) and \(X_4\) are both null and orthogonal to each other. As the metric \(g\) on \(M\) is locally Lorentzian the null vectors \(X_1\) and \(X_4\) must be collinear, i.e., they are proportional. Hence \(X\) is a singular matrix, which contradicts our initial assumption regardless of whether \(\lambda_1 = \lambda_2\) or \(\lambda_1 \neq \lambda_2\). So, there is no nonsingular matrix \(X\) such that equation \((2.3)\) holds for \(J\) in the JCF given in case (c). In other words, at a point \(p \in M\) the Ricci tensor \(R\) defined on 5-D Lorentzian manifolds \(M\) cannot be Segre types \([32]\) and its degeneracy \([(32)]\).

Regarding the Segre type \([221]\) case, by analogous calculations one can show that equation \((2.2)\) gives rise to the following Jordan chain:

\[
\begin{align*}
RX_1 & = \lambda_1 X_1, \\
RX_2 & = \lambda_1 X_2 + X_1, \\
RX_3 & = \lambda_2 X_3, \\
RX_4 & = \lambda_2 X_4 + X_3, \\
RX_5 & = \lambda_3 X_5.
\end{align*}
\]

As the two pairs of equations \((2.9)\)–\((2.10)\) and \((2.12)\)–\((2.13)\) have the same algebraic structure of the two pairs of equations \((2.3)\)–\((2.4)\) and \((2.6)\)–\((2.7)\) of the case (c), they can similarly be used to show that \(X_1\) and \(X_3\) are null vectors. Moreover, here for any \(\lambda_3\) and regardless of whether \(\lambda_1 = \lambda_2\) or \(\lambda_1 \neq \lambda_2\) one can show that \(X_1\) and \(X_3\) are orthogonal to one another. Thus, at a point \(p \in M\), the Ricci tensor defined on 5-D Lorentzian manifolds cannot be brought to the Jordan canonical forms corresponding to the Segre types \([221]\), \([(22)1]\), \([2(21)]\) and \([(221)]\).

In brief, for real eigenvalues we have been left solely with the Segre type cases (d), (f) and (g) and their degeneracies as JCF for \(R^a_b\). In other words, when all eigenvalues are real a second order symmetric tensor on a 5-D Lorentzian \(M\) manifold can be of one
of the following Segre types at \( p \in M \):

1. \([1,1111]\) and its degeneracies \([1,11(11)], [1,(11)(11)], [(1,1)(11)1], [1,1(111)], [(1,11)11], [(1,1)(111)], [(1,111)1], [(1,1111)]\);

2. \([2111]\) and its specializations \([21(11)], [(21)11], [(21)(11)], [2(111)], [(211)1] \) and \([(2111)]\);

3. \([311]\) and its degeneracies \([3(11)], [(31)1] \) and \([(311)]\).

As a matter of fact, to complete the classification for the cases where the characteristic equation corresponding to (2.1) has only real roots one needs to show that the above remaining Segre types are consistent with the symmetry of \( R_{ab} \) and the Lorentzian character of the metric \( g \). In the next section we will find a set of canonical forms of \( R_{ab} \) corresponding to the above Segre types, which makes apparent that these types fulfill both conditions indeed.

In the remainder of this section we shall discuss the cases when the \( R^a_b \) has complex eigenvalues. It is well known that if \( z_1 \) is complex root of a polynomial with real coefficients so is its complex conjugate \( \bar{z}_1 \). As the characteristic equation associated to the eigenvalue problem (2.1) is a fifth order polynomial with real coefficients it must have at least one and may have at most three real roots. Accordingly, the characteristic polynomial will have at most four and at least two complex roots. As far as the multiplicities are concerned one can easily realize that while the complex roots can be either single or double degenerated, the real ones can have multiplicity 1, 2 and 3. Taking into account these remarks it is easy to figure out that when complex eigenvalues occur the possible Jordan matrices \( J^a_b \) for \( R^a_b \) are
One might think at first sight that an analogous procedure to that applied to eliminate Jordan matrices with real eigenvalues could be used here again to rule out some of the above Segre types. However it turns out that the method does not work when there are complex eigenvalues because the basic result that two (real) orthogonal null vectors are necessarily proportional does not hold for the complex vectors. Thus, for the above Segre type case (vi), e.g., one can derive from the corresponding Jordan chain that the first and the third column vectors of the matrix $X$ are both null and orthogonal to each other. Nevertheless, this does not imply that $X$ is a singular matrix, inasmuch as $X_1$ and $X_3$ are not necessarily proportional. Thus, we shall use instead an approach similar to that employed by Hall \[1, 23\] when dealing with the complex Segre types in the classification of the Ricci tensor in 4-D space-times manifolds.

| \( z_1 \) | 0 | 0 | 0 | 0 | \( z_1 \) | 0 | 0 | 0 | 0 | \( z_1 \) | 0 | 0 | 0 | 0 |
|----------------|-----|-----|-----|-----|----------------|-----|-----|-----|-----|----------------|-----|-----|-----|-----|
| 0 \( \bar{z}_1 \) | 0 | 0 | 0 | 0 | 0 \( \bar{z}_1 \) | 0 | 0 | 0 | 0 | 0 \( \bar{z}_1 \) | 0 | 0 | 0 | 0 |
| 0 | 0 | \( \lambda_1 \) | 1 | 0 | 0 | 0 | \( \lambda_1 \) | 1 | 0 | 0 | 0 | \( \lambda_1 \) | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | \( \lambda_1 \) | 1 | 0 | 0 | 0 | \( \lambda_1 \) | 0 | 0 | 0 | \( \lambda_1 \) | 0 | 0 | \( \lambda_2 \) | 0 |
| 0 | 0 | 0 | 0 | \( \lambda_1 \) | 0 | 0 | 0 | 0 | \( \lambda_1 \) | 0 | 0 | 0 | \( \lambda_1 \) | 0 | 0 | \( \lambda_3 \) |

(i) Segre type \([z \bar{z} 3]\)  
(ii) Segre type \([z \bar{z} 21]\)  
(iii) Segre type \([z \bar{z} 111]\)  
(iv) Segre type \([z \bar{z} w \bar{w} 1]\)  
(v) Segre type \([2z \bar{z} \bar{z} 1]\)  
(vi) Segre type \([2z 2\bar{z} 1]\)
Suppose that the \( R^a_b \) has a complex eigenvalue \( z_1 = \alpha + i\beta \ (\alpha, \beta \in \mathbb{R}, \beta \neq 0) \) associated to the eigenvector \( V = Y + iZ \), with components \( V^a = Y^a + iZ^a \) relative to a basis in which \( R^a_b \) is real. The eigenvalue equation

\[
R^a_b V^b = z_1 V^a
\]  

implies

\[
R^a_b \tilde{V}^b = \bar{z}_1 \tilde{V}^a,
\]

where the eigenvalue \( \bar{z}_1 = \alpha - i\beta \) and where \( \tilde{V}^a = Y^a - iZ^a \) are the components of a second eigenvector \( \tilde{V} \). Since \( R_{ab} \) is symmetric we have \( \tilde{V}_a R^a_b V^b = V_a R^a_b \tilde{V}^b \), which together with eqs. (2.14) and (2.15) yield

\[
Y_a Y^a + Z_a Z^a = 0.
\]

From this last equation it follows that either one of the vectors \( Y \) or \( Z \) is timelike (and the other spacelike) or both are null and, since \( \beta \neq 0 \), not proportional to each other. Regardless of whether \( Y \) and \( Z \) are both null vectors or one timelike and the other spacelike, the real and the imaginary part of (2.14) give

\[
\begin{align*}
R^a_b Y^b &= \alpha Y^a - \beta Z^a, \\
R^a_b Z^b &= \beta Y^a + \alpha Z^a.
\end{align*}
\]

Therefore, in either case the vectors \( Y \) and \( Z \) span a timelike invariant 2-space of \( T_p(M) \) under \( R^a_b \). The 3-space orthogonal to this timelike 2-space is spacelike and must have three spacelike orthogonal eigenvectors (\( x, y \) and \( z \), say) of \( R^a_b \) with real eigenvalues (see next section for more details about this point). These spacelike eigenvectors together with \( V \) and \( \tilde{V} \) complete a set of five linearly independent eigenvectors of \( R^a_b \) at \( p \in M \). Therefore, when there exists a complex eigenvalue \( R^a_b \) is necessarily diagonalizable over the complex field and possesses three real eigenvalues. In other words, among the possible JCF for \( R^a_b \) with complex eigenvalues only that of case (iii) and its degeneracies are
allowed, i.e., only the Segre type \([z \bar{z} 111]\) and its specializations \([z \bar{z} 1(11)]\) and \([z \bar{z} (111)]\) are permitted.

We can summarize the results of the present section by stating the following theorem:

**Theorem 1** Let \(M\) be a real five-dimensional manifold endowed with a Lorentzian metric \(g\) of signature \((-++++)\). Let \(R^a_b\) be the mixed form of a second order symmetric tensor \(R\) defined at any point \(p \in M\). Then \(R^a_b\) takes one of the following Segre types: \([1,1111]\), \([2111]\), \([311]\), \([222z \bar{z} 111]\), or some degeneracy thereof.

### 3 Canonical Forms

In the previous section we have classified the Ricci tensor up to similarity transformations. A further refinement to that classification is to choose exactly one element, as simple as possible, from each class of equivalent objects. The collection of all such samples constitutes a set of canonical forms for \(R^a_b\). In this section we shall obtain such a set for the symmetric Ricci tensor in terms of semi-null pentad bases of vectors \(B = \{l, m, x, y, z\}\), whose non-vanishing inner products are

\[
l^a m_a = x^a x_a = y^a y_a = z^a z_a = 1.
\]  

(3.1)

At a point \(p \in M\) the set of second order symmetric tensors on a 5-D manifold \(M\) constitutes a 15-dimensional vector space \(V\), which can be spanned by the following 15 basis symmetric tensors:

\[
\begin{align*}
l_a l_b, & \quad m_a m_b, \quad x_a x_b, \quad y_a y_b, \quad z_a z_b, \quad 2 l_a m_b, \quad 2 l_a x_b, \quad 2 l_a y_b, \quad 2 l_a z_b, \\
& \quad 2 m_a x_b, \quad 2 m_a y_b, \quad 2 m_a z_b, \quad 2 x_a y_b, \quad 2 x_a z_b, \quad 2 y_a z_b.
\end{align*}
\]  

(3.2)

Clearly any symmetric two-tensor at \(p \in M\) is a vector in \(V\) and can be written as a linear combination of the basis elements (3.2). So, for example, bearing in mind
the non-vanishing inner products given by \((3.1)\), one can easily work out the following
decomposition (completeness relation) for the metric:

\[ g_{ab} = 2 l(a_m b) + x_a x_b + y_a y_b + z_a z_b. \]  

(3.3)

As far as the Ricci tensor is concerned the most general decomposition of \(R_{ab}\) in terms
of the above semi-null basis is manifestly given by

\[ R_{ab} = 2 \rho_1 l(a_m b) + \rho_2 l_a l_b + \rho_3 x_a x_b + \rho_4 y_a y_b + \rho_5 z_a z_b + \rho_6 m_a m_b + 2 \rho_7 l_a x_b + 2 \rho_8 l_a y_b + 2 \rho_9 l_a z_b + 2 \rho_{10} m(a x_b) + 2 \rho_{11} m(a y_b) + 2 \rho_{12} m(a z_b) + 2 \rho_{13} x(a y_b) + 2 \rho_{14} x(a z_b) + 2 \rho_{15} y(a z_b), \]  

(3.4)

where the coefficients \(\rho_1, \ldots, \rho_{15} \in \mathbb{R}\).

In the remainder of this section we shall show that for each of the Segre types of
the theorem \([\text{3.1}]\) a semi-null pentad basis with non-vanishing inner products \((3.1)\) can be
introduced at \(p \in M\) such that \(R_{ab}\) takes one and only one of the following canonical
forms:

\[ [1, 111] \quad R_{ab} = 2 \rho_1 l(a_m b) + \rho_2 (l_a l_b + m_a m_b) + \rho_3 x_a x_b + \rho_4 y_a y_b + \rho_5 z_a z_b, \]  

(3.5)

\[ [2111] \quad R_{ab} = 2 \rho_1 l(a_m b) + l_a l_b + \rho_3 x_a x_b + \rho_4 y_a y_b + \rho_5 z_a z_b, \]  

(3.6)

\[ [311] \quad R_{ab} = 2 \rho_1 l(a_m b) + 2 l(a x_b) + \rho_1 x_a x_b + \rho_4 y_a y_b + \rho_5 z_a z_b, \]  

(3.7)

\[ [z z 111] \quad R_{ab} = 2 \rho_1 l(a_m b) + \rho_2 (l_a l_b - m_a m_b) + \rho_3 x_a x_b + \rho_4 y_a y_b + \rho_5 z_a z_b, \]  

(3.8)

where \(\rho_1, \ldots, \rho_5 \in \mathbb{R}\) and \(\rho_2 \neq 0\) in \((3.8)\).

Before proceeding to the individual analysis of the above Segre types it is worth
noticing that since \(R_{ab}\) is symmetric the condition

\[ g_{ac} \, R^c_{\ b} = g_{bc} \, R^c_{\ a} \]  

(3.9)

must hold for each possible Segre type, where the metric tensor \(g\) is assumed to have a
Lorentzian signature \((-++++)\).
Segre type $[1,1111]$. For this type if one writes down a general symmetric real matrix $g_{ab}$, uses eq. (3.9) and the corresponding Jordan matrix (case (g) of the previous section) one obtains

$$g_{ab} = \text{diag} (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5),$$

(3.10)

where $\mu_1, \cdots, \mu_5 \in \mathcal{R}$. As $\det g_{ab} < 0$, then all $\mu_a \neq 0$ ($a = 1, \cdots, 5$). Moreover, at least one $\mu_a < 0$. As a matter of fact, owing to the Lorentzian signature $(-++++)$ of the metric one and only one $\mu_a$ is negative. On the other hand $R^a_b$ is real, symmetric and has five different eigenvalues. So, bearing in mind that the scalar product on $T_p(M)$ is defined by $g_{ab}$, the associated eigenvectors are orthogonal to each other and their norms are equal to $\mu_a \neq 0$. Thus they are either spacelike or timelike. Since one $\mu_a < 0$, one of the eigenvectors must be timelike. Then the others are necessarily spacelike. Hence, without loss of generality one can always choose a basis with five orthonormal vectors $\vec{B} = \{t, w, x, y, z\}$ defined along the invariant directions so that

$$R_{ab} = -\tilde{\rho}_1 t_a t_b + \tilde{\rho}_2 w_a w_b + \rho_3 x_a x_b + \rho_4 y_a y_b + \rho_5 z_a z_b,$$

(3.11)

where $-t^a t_a = w^a w_a = x^a x_a = y^a y_a = z^a z_a = 1$.

It should be noticed that the above form for the Segre type $[1,1111]$ makes apparent that this type as well as its degeneracies are consistent with both the symmetry of $R_{ab}$ and the Lorentzian signature of the metric tensor $g$.

Finally, we introduce two null vectors $l = \frac{1}{\sqrt{2}}(t + w)$ and $m = \frac{1}{\sqrt{2}}(t - w)$ to form a semi-null basis $\mathcal{B} = \{l, m, x, y, z\}$, in terms of which $R_{ab}$ takes the canonical form (3.5), with

$$\rho_1 = \frac{1}{2}(\tilde{\rho}_1 + \tilde{\rho}_2) \quad \text{and} \quad \rho_2 = \frac{1}{2}(\tilde{\rho}_2 - \tilde{\rho}_1).$$

(3.12)

We remark that according to the form (3.11) this type may degenerate into eleven other types in agreement with the previous section. Further, they are all diagonalizable with five linearly independent real eigenvectors, one of which is necessarily timelike. In the non-degenerated case $[1,1111]$ a semi-null pentad basis $\mathcal{B}$ is uniquely determined.
When degeneracies exist, though, the form (3.11) (or alternatively (3.5)) is invariant under appropriate continuous transformations of the pentad basis of vectors. Hereafter, in dealing with the symmetries of any Segre types we shall consider only continuous transformations of the pentad basis of vectors. Thus, for example, in the type [(1, 11)11] the vectors \( t, w \) and \( x \) are determined up to 3-dimensional Lorentz rotations \( SO(1, 2) \) in the invariant 3-space of eigenvectors which contains these vectors. Similarly, for the Segre type \([1, 1(11)]\) the vectors \( x, y \) and \( z \) can be fixed up to 3-D spatial rotations \( SO(3) \), whereas the type \([1, 11(11)]\) allows spatial rotations in the plane \((y, z)\). Finally, the invariance group for the type \([(1, 1111)]\) is the full generalized Lorentz group.

**Segre type [2111].** If one writes down a general symmetric matrix \( g_{ab} \) and uses the symmetry condition (3.9) and the corresponding Jordan matrix \( J^a_b \) for this type (case (f) in section 2) one obtains

\[
g_{ab} = \begin{pmatrix}
0 & \epsilon & 0 & 0 & 0 \\
\epsilon & \gamma & 0 & 0 & 0 \\
0 & 0 & \epsilon_1 & 0 & 0 \\
0 & 0 & 0 & \epsilon_2 & 0 \\
0 & 0 & 0 & 0 & \epsilon_3
\end{pmatrix},
\]

(3.13)

where \( \gamma, \epsilon, \epsilon_1, \epsilon_2, \epsilon_3 \in \mathcal{R} \). Again from the corresponding Jordan matrix it also follows that the column vectors \( l^a = (1, 0, 0, 0, 0) \), \( \tilde{x}^a = (0, 0, 1, 0, 0) \), \( \tilde{y}^a = (0, 0, 0, 1, 0) \) and \( \tilde{z}^a = (0, 0, 0, 0, 1) \) are the only independent eigenvectors of \( J^a_b \), and have associated eigenvalues \( \lambda_1, \lambda_3, \lambda_4 \) and \( \lambda_5 \), respectively. These four vectors together with the column null vector \( m^a = (-\gamma/(2\epsilon^2), 1/\epsilon, 0, 0, 0) \) constitute a basis of \( T_p(M) \). Clearly the spacelike vectors \((\tilde{x}, \tilde{y}, \tilde{z})\) can be suitably normalized to form a semi-null pentad basis \( B = \{l, m, x, y, z\} \) such that both equations (3.1) and (3.3) hold. Using now that \( l, x, y \) and \( z \) are eigenvectors, eq. (3.4) simplifies to

\[
R_{ab} = 2 \rho_1 l(a m_b) + \rho_2 l(a l_b) + \rho_3 x_a x_b + \rho_4 y_a y_b + \rho_5 z_a z_b,
\]

(3.14)
where the condition $\rho_2 \neq 0$ must be imposed otherwise $m$ would be a fifth linearly independent eigenvector. A transformation (null rotation) of the form $l^a \rightarrow \zeta l^a$, $m^a \rightarrow m^a/\zeta$, $\zeta \in \mathbb{R}^+$ can be used to set $\rho_2 = 1$ if $\rho_2 > 0$, and $\rho_2 = -1$ if $\rho_2 < 0$, making apparent that (3.4) are the canonical forms for the Segre type [2111].

It is worth noticing that the above derivation of (3.4) and the canonical form itself make apparent that this Segre type as well as its degeneracies are consistent with the Lorentzian signature of the metric tensor and the symmetry of $R_{ab}$.

From the canonical form (3.4) one learns that this type may degenerate into the six types we have enumerated in the previous section. Although for this type we only have four invariant directions intrinsically defined by $R_{a\ b}$, for the non-degenerated case [2111] a semi-null pentad basis $\mathcal{B}$ used in (3.4) can be uniquely fixed [26]. However, when there are degeneracies the canonical form (3.4) allows some freedom in the choice of a semi-null pentad basis, i.e., the canonical form is invariant under some appropriate continuous transformation of the basis of vectors. Thus, e.g., the type [21(11)] permits local rotational symmetry (LRS for short) in the plane ($y$, $z$). Similarly a local null rotation symmetry (LNRS) is allowed in the degenerated types [(21)11] and [(21)(11)]. Clearly this latter type admits both local isotropies LRS and LNRS. Similar assertions can obviously be made about other specializations of the type [2111], we shall not discuss them all here for the sake of brevity, though.

**Segre type [311].** This case can be treated similarly to the Segre type [2111]. From the associated Jordan matrix $J^a_{\ b}$ it follows that the column vectors $l^a = (1, 0, 0, 0, 0)$ $\tilde{y}^a = (0, 0, 1, 0, 0)$ and $\tilde{z}^a = (0, 0, 0, 1, 0)$ are the only independent eigenvectors of $J^a_{\ b}$ with eigenvalues $\lambda_1$, $\lambda_2$ and $\lambda_3$, respectively. Further, the restrictions upon $g_{ab}$ imposed by eq. (3.9) give $l^a l_a = l^a \tilde{y}_a = l^a \tilde{z}_a = 0$. So, $l$ is a null vector and $(\tilde{y}, \tilde{z})$ are spacelike vectors. One can suitably normalize these spacelike vectors and then select a semi-null pentad basis of $T_p(M)$ containing $l$, the normalized vectors $(y, z)$ and two other vectors $x$ and $m$ with the orthonormality relations given by equation (3.11). Obviously this semi-
null basis satisfies the completeness relation (3.3). Now the fact that \( l, y \) and \( z \) are eigenvectors of \( R^a_b \) can be used to reduce the general decomposition (3.4) to

\[
R_{ab} = 2 \rho_1 l(a)m_b + 2 \rho_7 l(a)x_b + \rho_2 l(y_b) + \rho_3 x_a x_b + \rho_4 y_a y_b + \rho_5 z_a z_b , \tag{3.15}
\]

where the condition \( \rho_7 \neq 0 \) must be imposed otherwise \( x \) would be a fourth linearly independent eigenvector. Besides, as any linear combination of the form \( \kappa_1 l + \kappa_2 m + \kappa_3 x \) (with \( \kappa_1, \kappa_2, \kappa_3 \in \mathbb{R} \) and \( \kappa_2, \kappa_3 \neq 0 \)) must not be an eigenvector one finds that \( \rho_3 = \rho_1 \).

The transformation

\[
x^a \rightarrow x^a + 2 \xi l^a , \tag{3.16}
\]

\[
m^a \rightarrow m^a - 2 \xi x^a - 2 \xi^2 l^a , \tag{3.17}
\]

where \( \xi = -\rho_2/(4 \rho_7) \), yields \( \rho_2 = 0 \). Finally a transformation (null rotation) \( l^a \rightarrow l^a/\rho_7, m^a \rightarrow \rho_7 m^a \) can now be used to set \( \rho_7 = 1 \), therefore reducing (3.15) to the canonical form (3.7).

Here again it is worth noting that the canonical form (3.7) itself and the method used to find it make clear that the Segre type [311] and its degeneracies are consistent with the Lorentzian signature of the metric tensor and the symmetry of \( R_{ab} \).

From the canonical form (3.4) one obtains that this Segre type gives rise to three degenerated types in agreement with the section 2. Here again for the non-degenerated type [311] a semi-null pentad basis \( \mathcal{B} \) used in (3.7) can be fixed [29], but when there are degeneracies the canonical form is invariant under some appropriate continuous transformation of the basis of vectors. So, for example, the type [3(11)] permits LRS, the Segre type [(31)1] admits LNRS, and the type [(311)] allows both LRS and LNRS.

**Segre type \([z \bar{z} 111]\).** For this type, according the previous section \( R^a_b \) necessarily has two complex eigenvectors \( \mathbf{V} = \mathbf{Y} \pm i \mathbf{Z} \) with associated eigenvalues \( \alpha \pm i \beta \) (\( \alpha, \beta \in \mathbb{R}, \beta \neq 0 \)). Moreover, they are orthogonal, i.e. (2.16) holds, and the real vectors \( \mathbf{Y} \) and \( \mathbf{Z} \) span a timelike invariant 2-subspace of \( T_p(M) \) under \( R^a_b \), i.e. eqs. (2.17) and (2.18)
hold as well. This timelike invariant plane contains two distinct null directions, which
we shall use to fix a pair of real null vectors \( \mathbf{l} \) and \( \mathbf{m} \) of a semi-null pentad basis. When
\( \mathbf{Y} \) and \( \mathbf{Z} \) are both null one can choose their directions to fix two suitably normalized
null vectors \( \mathbf{l} \) and \( \mathbf{m} \), necessary to form a semi-null pentad basis. When they are not
null vectors one can, nevertheless, use them to find out the needed two null vectors as
follows.

If \( \mathbf{V} = \mathbf{Y} + i \mathbf{Z} \) is an eigenvector of \( R^a_b \) so is \( \mathbf{V}' = \rho e^{i\theta} (\mathbf{Y} + i \mathbf{Z}) \) whose components
are obviously given by

\[
Y'^a = \rho \cos \theta Y^a - \rho \sin \theta Z^a, \tag{3.18}
\]
\[
Z'^a = \rho \cos \theta Z^a + \rho \sin \theta Y^a, \tag{3.19}
\]

where \( 0 < \rho < \infty \) and \( 0 \leq \theta < \pi \). Now, to form a semi-null basis we first need a
pair of vectors \((\mathbf{Y}', \mathbf{Z}')\) such that

\[
Y'^a Y'_a = Z'^a Z'_a = 0, \tag{3.20}
\]
\[
Y'^a Z'_a = 1. \tag{3.21}
\]

Clearly from (2.16) if \( \mathbf{Y}' \) is a null vector, so is \( \mathbf{Z}' \). The substitution of (3.18) and (3.19)
into (3.20) and (3.21) gives rise to a pair of equations which can be solved in terms of \( \theta \)
and \( \rho \) to give

\[
\cot (2\theta) = \frac{Y'^a Z'_a}{Y'^a Y'_a}, \tag{3.22}
\]
\[
\rho^2 = \frac{\sin (2\theta)}{Y'^a Y'_a}, \tag{3.23}
\]

where \( Y'^a Y'_a \neq 0 \), as \( \mathbf{Y} \) is a non-null vector. We then choose the two null vectors we need
to form a semi-null basis by putting \( \mathbf{l} = \mathbf{Y}' \) and \( \mathbf{m} = \mathbf{Z}' \). In terms of these vectors the
eigenvalue equation (2.14) becomes

\[
R^a_b (l^b \pm i m^b) = (\alpha \pm i \beta) (l^a \pm i m^a), \tag{3.24}
\]
making clear that for this type one can always choose a pair of null vectors such that $l^a \pm im^a$ are eigenvectors with eigenvalues $\alpha \pm i\beta$.

Using now that $l^a \pm im^a$ are eigenvectors eq. (3.4) simplifies to

$$R_{ab} = 2\rho_1 l(a)m_b + \rho_2 (l_a l_b - m_a m_b) + \rho_3 x_a x_b + \rho_4 y_a y_b + \rho_5 z_a z_b$$

$$+ 2\rho_{13} x(a)y_b + 2\rho_{14} x(a)z_b + 2\rho_{15} y(a)z_b,$$

where $\rho_1 = \alpha$ and $\rho_2 = \beta \neq 0$. Clearly using eqs. (3.3) and (3.25) one finds that the corresponding mixed matrix $R^a_b$ takes a block diagonal form with two blocks. The first one $T^a_b$ is a $(2 \times 2)$ matrix acting on the timelike invariant 2-space. According to (3.24) it is diagonalizable over the complex field $\mathbb{C}$, with eigenvectors $l^a \pm im^a$ and associated eigenvalues $\alpha \pm i\beta$. The second block $E^a_b$ is a $(3 \times 3)$ symmetric matrix acting on the 3-space orthogonal to the above timelike invariant 2-space. Hence it can be diagonalized by spatial rotation of the basis vectors $(x, y$ and $z)$, i.e., through real similarity transformations. Thus, there exists an orthogonal basis $(\tilde{x}, \tilde{y}, \tilde{z})$ relative to which $E^a_b$ takes a diagonal form with real coefficients. In the basis $B = \{l, m, \tilde{x}, \tilde{y}, \tilde{z}\}$ one must have

$$R_{ab} = 2\alpha l(a)m_b + \beta (l_a l_b - m_a m_b) + \tilde{\rho}_3 \tilde{x}_a \tilde{x}_b + \tilde{\rho}_4 \tilde{y}_a \tilde{y}_b + \tilde{\rho}_5 \tilde{z}_a \tilde{z}_b,$$

rendering explicit that there are three spacelike orthogonal eigenvectors of $R^a_b$ with real eigenvalues, as we have pointed out in the previous section. Actually, the above three orthogonal spacelike eigenvectors $(\tilde{x}, \tilde{y}, \tilde{z})$ constitute a basis of the 3-space (3-eigenspace) orthogonal to the invariant timelike 2-subspace (2-eigenspace) of $T_p(M)$ spanned by either $(l, m)$ or $(Y, Z)$, in agreement with the section 2.

Finally, one can suitably normalize the above spacelike basis vectors $(\tilde{x}, \tilde{y}, \tilde{z})$, then select a semi-null pentad basis of $T_p(M)$ containing the above null vectors $(l, m)$ and the normalized new spacelike vectors $(x, y, z)$ with the orthonormality relations given by equation (3.21). Thus, in terms of this semi-null pentad basis $R_{ab}$ takes the canonical form (3.8) for this Segre type.
It should be noticed that the canonical form \((3.8)\) evince that this Segre type and its degeneracies are consistent with the symmetry \((3.9)\) and the Lorentzian signature of the metric tensor \(g\). Moreover, eq. \((3.8)\) gives rise to two degeneracies, namely \([z \bar{z} 1(11)]\) and \([z \bar{z} (111)]\).

Here again for the non-degenerated case \([z \bar{z} 111]\) a semi-null pentad basis \(\mathcal{B}\) used in \((3.8)\) can be fixed \([26]\), but when there are degeneracies the canonical form is invariant under some appropriate continuous transformation of the basis of vectors. So for the type \([z \bar{z} (111)]\) the vectors \(x, y\) and \(z\) can only be fixed up to 3-D spatial rotations \(SO(3)\), whereas the type \([z \bar{z} 1(11)]\) allows LRS.

It should be stressed that the classification we have discussed in this work applies to any second order symmetric tensor \(R\) at a point \(p \in M\) and can vary as \(p\) changes in \(M\).

To conclude, we should like to mention that the classification and the canonical forms we have studied in the present work generalize those discussed by Graham Hall and colaborators for symmetric two-tensors on 4-D and 3-D space-time manifolds \([22, 21]\).

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References

[1] E. Cartan, “Leçons sur la Géométrie des Éspaces de Riemann”, Gauthier-Villars, Paris (1951). Reprinted, Éditions Jacques Gabay, Paris, 1988. English translation by J. Glazebrook, Math. Sci. Press, Brookline (1983).

[2] A. Karlhede, Gen. Rel. Grav. 12, 693 (1980).
[3] See M. A. H. MacCallum and J. E. F. Skea, “SHEEP: A Computer Algebra System for General Relativity”, in *Algebraic Computing in General Relativity*, Lecture Notes from the First Brazilian School on Computer Algebra, Vol. II, edited by M. J. Rebouças and W. L. Roque. Oxford U. P., Oxford (1994); and also the fairly extensive literature therein quoted on the equivalence problem.

[4] G. S. Hall, *Diff. Geom.* **12**, 53 (1984). This reference contains an extensive bibliography on the classification of the Ricci tensor on 4-dimensional Lorentzian space-times manifolds.

[5] R. V. Churchill, *Trans. Amer. Math. Soc.* **34**, 784 (1932).

[6] J. Plebański, *Acta Phys. Pol.* **26**, 963 (1964).

[7] W. J. Cormack and G. S. Hall, *J. Phys. A* **12**, 55 (1979).

[8] G. S. Hall, *Arab. J. Sci. Eng.* **9**, 87 (1984).

[9] G. S. Hall and D. A. Negm, *Int. J. Theor. Phys.* **25**, 405 (1986).

[10] M. J. Rebouças, J. E. Áman and A. F. F. Teixeira, *J. Math. Phys.* **27**, 1370 (1986).

[11] M. J. Rebouças and J. E. Áman, *J. Math. Phys.* **28**, 888 (1987).

[12] M. O. Calvão, M. J. Rebouças, A. F. F. Teixeira and W. M. Silva Jr., *J. Math. Phys.* **29**, 1127 (1988).

[13] J. J. Ferrando, J.A. Morales and M. Portilla, *Gen. Rel. Grav.* **22**, 1021 (1990).

[14] M. J. Rebouças and A. F. F. Teixeira, *J. Math. Phys.* **32**, 1861 (1991).

[15] M. J. Rebouças and A. F. F. Teixeira, *J. Math. Phys.* **33**, 2855 (1992).

[16] J. Santos, M. J. Rebouças and A. F. F. Teixeira, *J. Math. Phys.* **34**, 186 (1993).
[17] M. A. H. MacCallum, “Classifying Metrics in Theory and Practice”, in *Unified Field Theory in More Than 4 Dimensions, Including Exact Solutions*, edited by V. de Sabbata and E. Schmutzer. World Scientific, Singapore (1983).

[18] M. A. H. MacCallum, “Algebraic Computing in General Relativity”, in *Classical General Relativity*, edited by W. B. Bonnor, J. N. Islam and M. A. H. MacCallum. Cambridge U. P., Cambridge (1984).

[19] M. A. H. MacCallum, “Computer-aided Classification of Exact Solutions in General Relativity”, in *General Relativity and Gravitational Physics (9th Italian Conference)*, edited by R. Cianci, R. de Ritis, M. Francaviglia, G. Marmo, C. Rubano and P. Scudellaro. World Scientific Publishing Co., Singapore (1991).

[20] See, for example, R. J. Gleiser and M. C. Diaz, *Phys. Rev. D* **37**, 3761 (1988) and references therein quoted on this subject.

[21] G. S. Hall, T. Morgan and Z. Perjés, *Gen. Rel. Grav.* **19**, 1137 (1987).

[22] G. S. Hall, *J. Phys. A* **9**, 541 (1976).

[23] G. S. Hall, “Physical and Geometrical Classification in General Relativity”, Brazilian Center for Physics Research Monograph, CBPF-MO-001/93 (1993).

[24] G.E. Shilov, *Linear Algebra* (Dover Publ. Inc., New York, 1977).

[25] J. Santos, M. J. Rebouças and A. F. F. Teixeira, “Classification of the Ricci Tensor in 5-Dimensional Space-times,” in *Gravitation: the Spacetime Structure*, proceeding of the “8th Latin American Symposium on Relativity and Gravitation,” edited by P. Letelier and W.A. Rodrigues. World Scientific Publishing Co., Singapore (1994).

[26] Actually the pentad basis can be fixed up to discrete transformations, as inversions and relabelling of the basis vectors.
[27] It should be noticed that the values of $\theta$ outside this range, i.e. $\theta \in [\pi, 2\pi)$, simply revert the directions of $Y'$ and $Z'$. Thus, they need not to be considered.