Biclique Graphs of $K_3$-free Graphs and Bipartite Graphs

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Abstract

A biclique of a graph is a maximal complete bipartite subgraph. The biclique graph of a graph $G$, $KB(G)$, defined as the intersection graph of the bicliques of $G$, was introduced and characterized in 2010. However, this characterization does not lead to polynomial time recognition algorithms. The time complexity of its recognition problem remains open. There are some works on this problem when restricted to some classes. In this work we give a characterization of the biclique graph of a $K_3$-free graph $G$. We prove that $KB(G)$ is the square graph of a particular graph which we call Mutually Included Biclique Graph of $G$ ($KB_m(G)$). Although it does not lead to a polynomial time recognition algorithm, it gives a new tool to prove properties of biclique graphs (restricted to $K_3$-free graphs) using known properties of square graphs. For instance we generalize a property about induced $P_3$'s in biclique graphs to a property about stars and proved a conjecture posted by Groshaus and Montero, when restricted to $K_3$-free graphs. Also we characterize the class of biclique graphs of bipartite graphs. We prove that $KB$(bipartite) = (IIC-comparability)$^2$, where IIC-comparability is a subclass of comparability graphs that we call Interval Intersection Closed Comparability.

Keywords: Bicliques; Biclique graphs; Triangle-free graphs; Bipartite graphs; Comparability graphs; Power of a graph

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1 Introduction

A biclique of a graph is a vertex set that induces a maximal complete bipartite subgraph. The biclique graph of a graph $G$, denoted by $KB(G)$, is the intersection graph of the bicliques of $G$. The biclique graph was introduced by Groshaus and Szwarcfiter [11], based on the concept of clique graphs. They gave a characterization of biclique graphs (in general) and a characterization of biclique graphs of bipartite graphs. The time complexity of the problem of recognizing biclique graphs remains open.

Bicliques in graphs have applications in various fields, for example, biology: protein-protein interaction networks [4], social networks: web community discovery [15], medicine [20], information theory [14]. More applications (including some of these) can be found in the work of Liu, Sim and Li [18].

The biclique graph can be considered as a graph operator: given a graph $G$, the operator $KB$ returns the biclique graph of $G$, $KB(G)$ [9]. Some problems related to graph operators are studied in relation to some classes of graphs. Given a graph operator $\mathcal{H}$ and a class $\mathcal{A}$, it is studied the problem of recognizing the class $\mathcal{H}(\mathcal{A})$, or the problem of recognizing the class $\mathcal{H}^{-1}(\mathcal{A})$. These problems have been studied in the context of the clique graph, $K$ (clique operator). There are works about $K(\mathcal{A})$, where $\mathcal{A}$ is clique-Helly, chordal, interval, split, diamond-free, dismantable graphs, arc-circular graphs etc [2, 6, 16, 17, 21].

There are few works in the literature about recognizing the biclique graphs of some graph classes [5, 7, 12, 22].

In this work we prove that every biclique graph of a $K_3$-free graph (triangle-free) is the square of some graph, that is, $KB(K_3\text{-free}) \subset (\mathcal{G})^2$, where $\mathcal{G}$ denote the class of all graphs and $(\mathcal{A})^2$ denote the class of the square graphs of the graphs of the class $\mathcal{A}$. This result gives a tool for studying other classes of biclique graphs. Known results on square graphs follow directly for biclique graphs (using known properties of square graphs). Also, some results on biclique graphs of graphs, when restricted to $K_3$-free graphs, can be easily proven using the fact that it is a square graph. For example, the fact that every $P_3$ is contained in a diamond or a 3-fan [11], the fact that the number of vertices of degree 2 is at most $n/2$ and the fact that the family of neighbourhoods of vertices of degree 2 satisfy the Helly property [10].

Groshaus and Montero presented a conjecture stating that a certain structure is forbidden in biclique graphs [10] Conjecture 6.2. We prove that this conjecture holds when $G$ is a $K_3$-free graph, using the fact that $H$ is the square of some particular graph.

Conjecture 1 ([10]). If $H = KB(G)$ for some graph $G$, where $H$ is not isomorphic to the diamond then there do not exist $v_1, v_2, \ldots, v_n \in V(H)$ such that $N_H(v_1) = N_H(v_2) = \cdots = N_H(v_n)$ and their neighbours are contained in a $K_n$ for $n \geq 2$.

A comparability graph $G$ is such that there is a partially ordered set (poset) $(V(G), \leq_G)$ where $uv \in E(G)$ if and only if $u$ and $v$ are comparable by $\leq_G$ [3].
We define a subclass of comparability graphs, the class of *interval intersection closed comparability* (IIC-comparability) graphs, and we prove that the class of biclique graphs of bipartite graphs is equal to the class of the square graphs of IIC-comparability graphs, that is, \( KB(\text{bipartite}) = (\text{IIC-comparability})^2 \), giving another characterization of biclique graphs of bipartite graphs.

### 1.1 Some Definitions and Notations

Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). Denote the set of *neighbours* of a vertex \( v \in V(G) \) as \( N_G(v) \). Let \( S \subseteq V(G) \) and define \( N^*_G(S) \) to be the set of vertices of \( G \) that are incident to every vertex of \( S \). That is, \( N^*_G(S) = \bigcap_{v \in S} N_G(v) \). Note that if \( U \subseteq S \), \( N^*_G(S) \subseteq N^*_G(U) \).

Given a poset \( P = (C, \leq) \) and \( x \in C \), let \( I^+_P(x) = \{ y \in C \mid y \leq x \} \) and \( I^-_P(x) = \{ y \in C \mid x \leq y \} \) be, respectively, the *predecessors interval* and *successors interval* of \( x \) in \( P \). We say that a poset \( P = (C, \leq) \) is *interval intersection closed (IIC)* if the sets \( I^-_P = \{ I^-_P(x) \mid x \in C \} \) and \( I^+_P = \{ I^+_P(x) \mid x \in C \} \) are closed under intersection. That is, for every pair \( u, v \in C \), the following sentences are true:

- if \( I^-_P(u) \cap I^-_P(v) \neq \emptyset \) then there is a \( w \in C \) such that \( I^-_P(w) = I^-_P(u) \cap I^-_P(v) \); and
- if \( I^+_P(u) \cap I^+_P(v) \neq \emptyset \) then there is a \( w \in C \) such that \( I^+_P(w) = I^+_P(u) \cap I^+_P(v) \).

Let the graph class *IIC-comparability* (Interval Intersection Closed Comparability) be the class of comparability graphs with posets that are IIC.

### 2 Mutually Included Biclique Graph

Denote a biclique \( P \) of \( G \), with bipartition \((X,Y)\), as \( XY \). That is, \( XY = X \cup Y \). Given a biclique \( P = XY \) and a vertex \( v \notin P \) then (i) \( v \notin N^*_G(X) \cup N^*_G(Y) \) or (ii) there are vertices \( x \in X \) and \( y \in Y \) such that \( x, y \in N_G(v) \). Note that in case (ii) the vertices \( x, y, \) and \( v \) form a \( K_3 \). So, if \( G \) is a \( K_3 \)-free graph (ii) is always false and if \( v \notin P \) then (i) holds.

**Observation 1.** *Given a \( K_3 \)-free graph \( G \), two independent sets, \( X \) and \( Y \), of \( G \) form a biclique \( XY \) of \( G \) iff \( N^*_G(X) = Y \) and \( N^*_G(Y) = X \).*

Note that no part of a biclique intersects both parts of another biclique. That is, if \( P = X_PY_P \) and \( Q = X_QY_Q \) are two bicliques of a graph \( G \), \( X_P \cap X_Q = \emptyset \) or \( X_P \cap Y_Q = \emptyset \). So, assume that \( X_P \cap Y_Q = \emptyset \) and \( X_Q \cap Y_P = \emptyset \). We say that \( P \) and \( Q \) are *mutually included* if \( X_Q \subset X_P \) and \( Y_P \subset Y_Q \). See Figure 1.

Define \( KB_m(G) \), the *mutually included biclique graph* of \( G \), as the graph with the bicliques of \( G \) as its vertex set and \( PQ \) is an edge iff \( P \) and \( Q \) are mutually included. Note that \( KB_m(G) \subseteq KB(G) \) (with the same vertex set).
Lemma 1. If $P = X_PY_P$ and $Q = X_QY_Q$ are two bicliques of a $K_3$-free graph $G$, that are not mutually included, such that $X_P \cap X_Q \neq \emptyset$ then there is a biclique $R = X_RY_R$ such that $X_R = X_P \cap X_Q$ and $Y_R \supseteq Y_P \cup Y_Q$.

Proof. Let $G$ be a $K_3$-free graph and let $P = X_PY_P$ and $Q = X_QY_Q$ be two bicliques of $G$ that are not mutually included and that $X_P \cap X_Q \neq \emptyset$. Let $X_R = X_P \cap X_Q$, $Y_R = N_G^c(X_R)$. Note that $Y_R$ is an independent set (as $G$ is a $K_3$-free graph) and that $(Y_P \cup Y_Q) \subseteq Y_R$ (by definition of $N_G^c(X_R)$). See Figure 1b.

Suppose $R = X_RY_R$ is not a biclique of $G$. Then, by Observation \[ N_G^c(Y_R) \neq X_R. \] By definition, $X_R \subseteq N_G^c(Y_R)$, so there is a vertex $x \in N_G^c(Y_R) \setminus X_R$. As $x$ is neighbour of every vertex of $Y_R$, it is also neighbour of every vertex of $Y_P$, then $P \cup \{x\}$ induces a complete bipartite subgraph and $P$ is not a biclique. So, there is no such vertex $x$, $N_G^c(Y_R) = X_R$ and $R = X_RY_R$ is a biclique of $G$. $\square$

Corollary 1. If $P$ and $Q$ are two intersecting bicliques of a $K_3$-free graph $G$ then $P$ and $Q$ are mutually included or there is a biclique $R$ that is mutually included both with $P$ and $Q$.

Lemma 2. If $P$ and $Q$ are two bicliques such that there is a biclique mutually included with both $P$ and $Q$, then $P \cap Q \neq \emptyset$.

Proof. Suppose $P = X_PY_P$ and $Q = X_QY_Q$. Let $R = X_RY_R$ be a biclique mutually include with $P$ and $Q$. Suppose w.l.o.g. that $X_P \subseteq X_R$ and $Y_R \subseteq Y_P$. Also suppose that $X_Q \cap Y_R = \emptyset$. If $X_R \subseteq X_Q$, then $X_P \subseteq X_Q$ and $P \cap Q \neq \emptyset$. On the other hand, if $X_Q \subseteq X_R$, then $Y_R \subseteq Y_Q$ and $Y_R \subseteq Y_P \cap Y_Q$ and also $P \cap Q \neq \emptyset$. $\square$

Theorem 1. If $G$ is a $K_3$-free graph, then $KB(G) = (KB_{in}(G))^2$. 

Figure 1: (a) Mutually included bicliques (b) Two intersecting not mutually included bicliques, $P$ and $Q$ and a biclique $R$ mutually included with both.
Proof. Let \( G \) be a \( K_3 \)-free graph and let \( P \) and \( Q \) be two bicliques.

If \( P \) and \( Q \) intersect, by Corollary 1, \( P \) and \( Q \) are mutually included or there is a biclique \( R \) that is mutually included with both. That is, \( P \) and \( Q \) are at distance at most 2 in \( KB_m(G) \).

If \( P \) and \( Q \) do not intersect, by Lemma 2, there is no biclique that is mutually included with both. That is, \( P \) and \( Q \) are at distance at least 3 in \( KB_m(G) \).

So \( KB(G) = (KB_m(G))^2 \).

Corollary 2. \( KB(K_3\text{-free}) \subsetneq (G)^2 \).

Proof. By Theorem 1, \( KB(K_3\text{-free}) \subseteq (G)^2 \).

Observe that \( net^2 \in (G)^2 \) but \( net^2 \notin KB(G) \) (from the work of Montero [19], by inspection). See Figure 2a.

- (a) \( net \) graph (b) co-domino graph, the complement of the domino graph (c) 4-wheel graph

In general, it is not the case that the biclique graph is the square of some graph. For instance, consider the co-domino graph of Figure 2b. The biclique graph of co-domino, is the 4-wheel graph of Figure 2c, that is \( KB(co-domino) = 4-wheel \). But 4-wheel is not the square of any graph (by inspection).

3 Properties

In this section we present properties of mutually included biclique graphs, square graphs, and therefore, for biclique graphs of triangle-free graphs.

Lemma 3. Let \( P = X_P Y_P \) and \( Q = X_Q Y_Q \) be two bicliques of a \( K_3 \)-free graph \( G \) such that \( X_P \cap Y_Q = X_P \cap X_Q = \emptyset \). \( P \) and \( Q \) are mutually included iff \( X_P \subset X_Q \) or \( X_Q \subset X_P \).

Proof. Let \( P = X_P Y_P \) and \( Q = X_Q Y_Q \) be two bicliques such that \( X_P \cap Y_Q = Y_P \cap X_Q = \emptyset \).

By definition if \( P \) and \( Q \) are mutually included then \( X_P \subset X_Q \) or \( X_Q \subset X_P \).

Now suppose \( X_P \subset X_Q \). Then \( N^*_G(X_Q) \subseteq N^*_G(X_P) \), by the definition. By Observation 1, \( Y_P = N^*_G(X_P), Y_Q = N^*_G(X_Q) \) and \( Y_P \neq Y_Q \). So, \( Y_Q \subset Y_P \). Consequently \( P \) and \( Q \) are mutually included.
Changing roles of \( P \) and \( Q \), we conclude that \( P \) and \( Q \) are also mutually included in the case when \( X_Q \subset X_P \).

Note that, for any graph, the union of two mutually included bicliques induces a bipartite graph and a set of mutually included bicliques are “nested”. In Figure 3 are presented a set of “nested” mutually included bicliques.

\[
\begin{align*}
\text{Figure 3: Nested mutually included bicliques.}
\end{align*}
\]

**Lemma 4.** For any graph \( G \), and a set \( C \) of bicliques of \( G \) such that every two of them are mutually included, then there is an ordering of \( C \), \((P_1, \ldots, P_k)\) such that \( X_1 \subset \cdots \subset X_k \) (and \( Y_k \subset \cdots \subset Y_1 \)), assuming that \( P_i = X_iY_i \), for \( 1 \leq i \leq k \).

**Proof.** Let \( G \) be any graph and \( C \) be a set of bicliques of \( G \) such that every two of them are mutually included. Recall that the relation \( \subset \) is a partial order and note that the bicliques of \( C \) has parts \( X_1, \ldots, X_k \) that are all comparable under that partial order. So there is an ordering of \( C \), \((P_1, \ldots, P_k)\) such that \( X_1 \subset \cdots \subset X_k \) (and \( Y_k \subset \cdots \subset Y_1 \)), assuming that \( P_i = X_iY_i \), for \( 1 \leq i \leq k \).

Any other biclique \( Q = X_QY_Q \) that is mutually included with \( P_i \), for some \( 1 \leq i \leq k \) is also mutually included with \( P_j \), for every \( 1 \leq j \leq i - 1 \) or every \( i + 1 \leq j \leq k \).

**Theorem 2.** For every graph \( G \) and for every clique \( C \) of \( H = KB_m(G) \) there is an ordering of the vertices of \( C \) such that for every vertex \( Q \in V(H) \), with \( N_H(Q) \cap C \neq \emptyset \), the vertices of \( N_H(Q) \cap C \) are consecutive in that ordering and include the first or the last vertex of that ordering.

**Proof.** Let \( G \) be a graph and \( C \) a clique of \( H = KB_m(G) \). That is, \( C \) is a set of bicliques of \( G \) such that every two of them are mutually included. By Lemma 4, there is an ordering \((P_1, \ldots, P_k)\) of the bicliques of \( C \). Let \( Q \in V(H) \), such that \( N_H(Q) \cap C \neq \emptyset \). Suppose that \( Q = X_QY_Q \) and \( P_i = X_iY_i \), for \( 1 \leq i \leq k \).

Suppose that \( X_Q \subset X_\ell \) for some \( 1 \leq \ell \leq k \) and that \( \ell \) is minimum. Then \( X_Q \subset X_i \) for every \( \ell \leq i \leq k \). That is, \( Q \) is mutually included with \( P_i \), for \( \ell \leq i \leq k \).

Now suppose that \( X_r \subset X_Q \) for some \( 1 \leq r \leq k \) and that \( r \) is maximum. By the same argument, \( Q \) is mutually included with \( P_i \), for \( 1 \leq i \leq r \).
Using Theorem 2, can be proved that some structures do not occur in $KB_m(G)$.

Let $T_{net} = \{x_1, x_2, x_3\}$ be the vertices of the triangle of the net graph (Figure 2a) and let $S_{net} = \{s_1, s_2, s_3\}$ be the other vertices such that each $x_is_i$ is an edge. Let $net^*$ be any graph generated by a net graph with 0 or more edges added connecting only vertices of $S_{net}$. See Figure 4.

![Figure 4: net* structure. The dotted edges are optional.](image)

**Corollary 3.** For every graph $G$, $KB_m(G)$ does not contain a net* as an induced subgraph.

**Proof.** Let $G$ be a graph and $H = KB_m(G)$. Suppose there is a net* graph as an induced subgraph of $H$. As each vertex of $S_{net}$ is adjacent to only one vertex of $T_{net}$, and $T_{net}$ is a triangle, by Theorem 2, there is an ordering of $T_{net}$. Considering the neighbourhoods of $s_1$ and $s_3$, $x_1$ and $x_3$ are the first and the last (or the inverse) of that ordering. Also by Theorem 2, the neighbourhood of $s_2$ should include $x_1$ or $x_3$, but $N_H(s_2) = x_2$. Therefore $H$ does not contain a net* as an induced subgraph.

Now we present a property about square graphs that implies in a property of biclique graphs of triangle-free graphs.

Let $D_n$ be a graph with vertex set $V(D_n) = \{v, u_1, \ldots, u_n, w_1, \ldots, w_n\}$ and edge set $E(D_n) = \{vu_i \mid 1 \leq i \leq n\} \cup \{u_iw_i \mid 1 \leq i \leq n\}$. Let $D_n^- = D_n - w_n$. See Figures 5a and 5b.

![Figure 5: (a) $D_n$ and (b) $D_n^-$ graphs.](image)

Call “old” edges the edges of $G^2$ (for some graph $G$) that are also edges of $G$ and call the other edges of $G^2$ as “new” edges. That is, “new” edges are the edges that are in $G^2$ and do not exist in $G$. 

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Theorem 3. Let $H = G^2$ for some graph $G$. Then, every induced $K_{1,n}$ of $H$ is contained in a $(D_n)^2$ or a $(D_n^-)^2$.

Proof. Let $H = G^2$ for some graph $G$. Suppose $S = \{v, w_1, \ldots, w_n\}$ induces a $K_{1,n}$ in $H$. Observe first that $H[S]$ can contain at most one old edge. If $e_i = vw_i$ is a new edge, then there is a vertex $v_{e_i}$ of $G$ that is adjacent in $G$ to $v$ and to $w_i$. Observe that $v_{e_i}$ can not be adjacent to any $w_j$, $j \neq i$ in $G$. Then, for every new edge $e_k = vw_k$ of $H[S]$ there is in $G$ a (different) vertex $v_{e_k}$ adjacent to $v$ and $w_k$.

If all edges of $H[S]$ are new edges, vertices $v, v_{e_1}, \ldots, v_{e_n}, w_1, \ldots, w_n$ induce a $D_n$ in $G$. Consequently $H[S]$ is contained in $(D_n)^2$.

Otherwise, w.l.o.g. suppose $e_n$ is an old edge of $H[S]$. Then, $G[\{v, v_{e_1}, \ldots, v_{e_{n-1}}, w_1, \ldots, w_n\}]$ is isomorphic to $(D_n^-)$ and $H[S]$ is included in $(D_n^-)^2$.

See in Figures 6(a) and 6(b) how a $K_{1,n}$ is included in a $(D_n)^2$ or a $(D_n^-)^2$.

![Diagram](image.png)

Figure 6: (a) $(D_n)^2$ and (b) $(D_n^-)^2$ graphs. The vertices in the grey area form a clique $(\{v, u_1, \ldots, u_n\})$. The edges of the $K_{1,n}$ are marked in dashed lines.

That property (of biclique graphs of triangle-free graphs) is a generalization of a very used property involving $P_3$ in biclique graphs. Also, we think that this new property can be a useful tool for solving the biclique graph recognition problem.

Corollary 4. Every induced $K_{1,n}$ of a biclique graph of triangle-free graph is contained in a $(D_n)^2$ or a $(D_n^-)^2$.

Note that a $P_3$ is a $K_{1,2}$, $(D_2)^2$ is a 3-fan and $(D_n^-)^2$ is a diamond. Then, the fact that every $P_3$ is contained in a diamond or a 3-fan, when restricted to biclique graphs of triangle-free graphs, is a particular case of Corollary 4.

We present the following result which generalizes the Conjecture 1 when restrict to biclique graph of triangle-free graphs.

Theorem 4. Let $G$ be a biclique graph of a triangle-free graph. If there exists an independent set $I$ in $G$ with $|I| = n \geq 3$, such that $|\bigcap_{v \in I} N_G(v)| \geq 3$. Then $|\bigcup_{v \in I} N_G(v)| > n$. 

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Proof. By Theorem 1, $G = (KB_m(H))^2$ for some triangle-free graph $H$. Suppose there is an independent set $I = \{v_1, \ldots, v_n\}$ in $G$ with $n \geq 3$, such that $|\bigcap_{v \in I} N_G(v)| \geq 3$.

Consider vertex $v_i$. We affirm that there exists an old edge $e_i = v_iw_j$, $w_j \in N_G(v_i)$. Let $w$ be a neighbour of $v_i$. If $wv_i$ is not an old edge, there exists $w_i \in N_G(v_i)$ such that $v_iw_i, v_iw$ are edges in $KB_m(H)$, that is $v_iw_i$ and $v_iw$ are old edges. Observe that $w_i \neq w_j$ when $i \neq j$, since two vertices of $I$ does not have a common neighbour in $KB_m(H)$, otherwise they would be adjacent in $G$. Therefore $|\bigcup_{v \in I} N_G(v)| \geq n$.

Suppose there are only one old edge incident to each vertex of $I$, otherwise $|\bigcup_{v \in I} N_G(v)| > n$. Suppose w.l.o.g. that $W = \{w_1, w_2, w_3\} \subseteq \bigcap_{v \in I} N_G(v)$. That is, $v_1w_1, v_2w_2$ and $v_3w_3$ are old edges. Since $v_1w_2$ exists and is a new edge then, there exists a vertex $w_{1.2} \in N_G(v_1)$ such that $v_1w_{1.2}$ and $w_1w_{1.2}$ are old edges. Note that $w_{1.2} = w_1$ as $v_1w_1$ is the unique old edge incident to $v_1$ and we conclude that $w_1w_2$ is an old edge. Now following the same arguments for edges $v_2w_3$ and $v_3w_1$ we obtain that $w_1w_2, w_2w_3$ and $w_1w_3$ are old edges. Finally, vertices $v_1, v_2, v_3, w_1, w_2, w_3$ induce a net in $KB_m(H)$ what leads to a contradiction according to Corollary 3. Consequently, $|\bigcup_{v \in I} N_G(v)| > n$.

Observe that in Theorem 4 we do not ask for vertices to have the same neighbourhood. Finally, we prove the Conjecture 1 for the case of biclique graphs of triangle-free graphs as a corollary of Theorem 4.

**Corollary 5.** Let $G$ be a graph not isomorphic to the diamond. Suppose $v_1, \ldots, v_n$ are vertices of $G$ with $n \geq 2$, such that $N_G(v_1) = N_G(v_2) = \cdots = N_G(v_n)$, with $|N_G(v_1)| = m$, for $m \leq n$. Then $G$ is not a biclique graph of a triangle-free graph $H$.

**Proof.** By contradiction, suppose $G$ is the biclique graph of some triangle-free graph. For $m = 2$, if $N_G(v_1) = K_2$, a contradiction is obtained by Corollary 4. The case $N_G(v_1) = K_2$ is proved by Groshaus and Montero [10] Proposition 4.6.

The case $m \geq 3$ follows directly by Theorem 4.

**4 Mutually Included Biclique Graphs of Bipartite Graphs**

In the case of bipartite graphs the parts of the bicliques are also bipartitioned, and we can establish a partial order of the bicliques based on one part of the bipartite graph.

Let $G = (A \cup B, E)$ be a bipartite graph. Let $B(G)$ be the set of bicliques of $G$. Define the relation $\prec_G$ over $B(G)$ such that $P \prec_G Q$ when $(P \cap A) \subset (Q \cap A)$. The reflexive closure of $\prec_G$ is the partial order $\preceq_G$.

**Lemma 5.** For every bipartite graph $G$ the poset $(B(G), \preceq_G)$ is IIC.
Proof. Let $G = (A \cup B, E)$ be a bipartite graph, with the poset $P = (B(G), \preceq_G)$ and let $P$ and $Q$ be two different bicliques of $G$.

If $P \preceq_G Q$ then $P \in I^+_P(P) \cap I^-_P(Q) = I^+_P(P) \neq \emptyset$ and $P$ is the maximum of $I^+_P(P) \cap I^-_P(Q)$. Also $Q \in I^+_P(P) \cap I^-_P(Q) = I^+_P(Q) \neq \emptyset$ and $Q$ is the minimum of $I^+_P(P) \cap I^-_P(Q)$.

Now suppose $P$ and $Q$ are not comparable.

If $P \cap Q \cap A = \emptyset$ then $I^+_P(P) \cap I^-_P(Q) = \emptyset$.

Suppose $P \cap Q \cap B = \emptyset$ and there is a biclique $R \in I^+_P(P) \cap I^-_P(Q)$. By definition, $(P \cap A) \subseteq (R \cap A)$ and $(Q \cap A) \subseteq (R \cap A)$, that is, $(P \cup Q) \cap A \subseteq (R \cap A)$. Then, $N^+_G(R \cap A) \subseteq (P \cap Q \cap B)$ and $(P \cap Q \cap B) \neq \emptyset$. So, if $P \cap Q \cap B = \emptyset$ then $I^+_P(P) \cap I^-_P(Q) = \emptyset$.

If $P \cap Q \cap A \neq \emptyset$ then, by Lemma 1, there is a biclique $R$ such that $R \in I^+_P(P) \cap I^-_P(Q) \neq \emptyset$ and $R$ is the maximum of $I^+_P(P) \cap I^-_P(Q)$. The same for part B and $I^+_P(P) \cap I^-_P(Q)$.

So, the poset $P = (B(G), \preceq_G)$ is IIC.

We show that $KB_m$(bipartite) is an IIC-comparability graph.

**Lemma 6.** If $G$ is a bipartite graph then, $KB_m(G)$ is an IIC-comparability graph.

Proof. Let $G = (A \cup B, E)$ be a bipartite graph with the poset $(B(G), \preceq_G)$. By Lemma 3, two different bicliques, $P$ and $Q$, of $G$ are mutually included iff $P$ and $Q$ are comparable by $\preceq_G$.

So, $KB_m(G)$ is the comparability graph of the poset $(B(G), \preceq_G)$.

Moreover, as $(B(G), \preceq_G)$ is IIC, by Lemma 5, $KB_m(G)$ is an IIC-comparability graph.

Let $P = (V, \preceq)$ be a poset. Define the *predecessors-successors bipartite graph* $G_P = (A \cup B, E)$ as follows: $A = \{a_v \mid v \in V\}$; $B = \{b_v \mid v \in V\}$; $E = \{a_u,b_v \mid u \preceq v, \text{ for } u,v \in V\}$.

Let $X_v = \{a_u \in A \mid u \in I^+_P(v)\}$ and $Y_v = \{b_w \in B \mid w \in I^-_P(v)\}$.

**Lemma 7.** Given a poset $P = (V, \preceq)$ and its predecessors-successors graph $G_P$. For every $v \in V$, $X_vY_v$ is a biclique of $G_P$.

Proof. Let $P = (V, \preceq)$ be a poset and its predecessors-successors graph $G_P$.

Let $v \in V$. As for every $a_u \in X_v$, $u \preceq v$ and for every $b_w \in Y_v$, $v \preceq w$, then $u \preceq w$ and $a_ub_w \in E(G_P)$. So $X_vY_v$ induces a complete bipartite subgraph of $G_P$.

Now suppose $X_vY_v$ is not maximal. Then w.l.o.g. there is a vertex $a_x \notin X_vY_v$ such that $X_vY_v \cup \{a_x\}$ induces a complete bipartite subgraph of $G_P$. By definition, $x \preceq w$, for every $w \in I^+_P(v)$. But $v \in I^+_P(v)$, and then $x \preceq v$. Consequently, $a_x \in X_v$ and $X_vY_v$ is a biclique of $G_P$.

For every subset $S \subseteq A \cup B$, define $V(S) = \{v \in V \mid a_v \in S \text{ or } b_v \in S\}$ to be the base set of $S$. 

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Lemma 8. Given an IIC poset $P = (V, \leq)$ and its predecessors-successors graph $G_P$. Every biclique of $G_P$ is equal to $X_vY_v$ for some $v \in V$.

Proof. Let $P = (V, \leq)$ be an IIC poset and its predecessors-successors graph $G_P$. And let $XY$ be any biclique $X_vY_v$ of $G_P$.

Consider the sets $V(X)$ and $V(Y)$. Suppose $V(X)$ do not have a maximum element. Then there are at least two maximal elements in $V(S)$, $i$ and $j$. As $P$ is IIC and $I_P^+(i) \cap I_P^+(j) \neq \emptyset$ (as $a_i$ and $a_j$ have at least one neighbour in common in $Y$), $I_P^+(i) \cap I_P^+(j)$ has a minimum, $m$. As $V(Y) \subseteq I_P^+(i) \cap I_P^+(j)$, $a_m \in X$, consequently $m \in V(X)$. Moreover, $i \leq m$ and $j \leq m$, which implies that $i$ and $j$ are not maximal elements of $V(X)$. So, $V(X)$ has a maximum. Using a similar reasoning we can show that $V(Y)$ has a minimum. Let $v$ be the maximum element of $V(X)$. Then $v \in V(Y)$ and $v$ is the minimum element of $V(Y)$. That is, $XY = X_vY_v$. □ □

We show now that the class of IIC-comparability is exactly the class of $KB_m$(bipartite).

Theorem 5. $KB_m$(bipartite) = IIC-comparability.

Proof. By Lemma 6, $KB_m$(bipartite) ⊆ IIC-comparability.

Now let $H$ be an IIC-comparability graph, $H$, with its IIC poset, $P = (V(H), \leq)$, and its predecessors-successors graph $G_P$. Recall that $G_P$ is a bipartite graph.

By Lemmas 7 and 8 there is a bijection $\phi$ between the vertex set of $H$ and the set of bicliques of $G_P$, given by $\phi(v) = X_vY_v$.

Let $u, v \in V(H)$, with $u \neq v$.

Suppose $uv \in E(H)$. Then $u \leq v$ or $v \leq u$ (as $H$ is a comparability graph with poset $P$). So $X_u \subseteq X_v$ or $X_v \subseteq X_u$ and $X_uY_u$ and $X_vY_v$ are mutually included (by Lemma 5) and then $\phi(u)\phi(v) \in E(KB_m(G_P))$.

Now suppose $\phi(u)\phi(v) \in E(KB_m(G_P))$. Then $X_u \subseteq X_v$ or $X_v \subseteq X_u$. As $a_u \in X_u$ and $a_v \in X_v$, then $u \leq v$ or $v \leq u$ and $uv \in E(H)$.

So, $\phi$ is an isomorphism and $H \simeq KB_m(G_P)$. Consequently IIC-comparability $\subseteq KB_m$(bipartite).

Therefore, $KB_m$(bipartite) = IIC-comparability. □ □ □

Then, considering Theorems 1 and 5 we conclude the characterization of biclique graphs of bipartite graphs.

Corollary 6. $KB$(bipartite) = ($IIC$-comparability)$^2$.

Proof. By Theorems 1 and 5 □ □ □

5 Final Remarks

In this work we prove that the biclique graph of a $K_3$-free graph is the square of some particular graph called $KB_m$ extending all known properties of square graphs to biclique graphs of triangle-free graphs and providing a tool to prove
other properties. Some published properties about biclique graphs can be easily proved if we restrict to square graphs.

We prove that $KB(G) \not\subseteq (G)^2$ and $(G)^2 \not\subseteq KB(G)$ by presenting the graphs 4-wheel (Figure 2c), that is the biclique graph of some graph but is not the square of any graph, and the square graph of the net (Figure 2a), that is not a biclique graph of any graph. We conclude that the biclique graphs of $K_3$-free graphs are in the intersection $KB(G) \cap (G)^2$, but it is not known if $KB(K_3$-free) $\subseteq KB(G) \cap (G)^2$ or $KB(K_3$-free) $= KB(G) \cap (G)^2$.

Moreover, we give the first known property (which was conjectured by Groshaus and Montero) of biclique graphs that does not hold for square graphs.

We study properties of $KB_m$ graphs in terms of ordering of the vertices of their cliques and neighbourhood (Theorem 2). That property lead to a forbidden structure (net*) for mutually included biclique graphs. We need to find other properties of these graphs in order to better understand their square graphs.

We also show that the class of biclique graphs of bipartite graphs are exactly the square of a subclass of comparability graphs (IIC-comparability). These characterizations (partial in the case of $K_3$-free graphs) do not lead to polynomial time recognition algorithms. However, it gives another tool to study the problem of recognizing biclique graphs and their properties.

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