LOWER BOUND FOR THE BOLTZMANN EQUATION WHOSE REGULARITY GROWS TEMPERED WITH TIME

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Abstract. As a first step towards the general global-in-time stability for the Boltzmann equation with soft potentials, in the present work, we prove the quantitative lower bounds for the equation under the following two assumptions, which stem from the available energy estimates, i.e. (i) the hydrodynamic quantities (local mass, local energy, and local entropy density) are bounded (from below or from above) uniformly in time, (ii) the Sobolev regularity for the solution grows tempered with time.

1. Introduction. In the present work we consider the lower bound for the Boltzmann equation with soft potentials. We will address the problem under the following assumptions: (A) the hydrodynamic quantities (local mass, local energy, and local entropy density) are bounded (from below or from above) uniformly in time; (B) the Sobolev regularity of the solution grows tempered with time. We begin with the introduction of the Boltzmann equation.

1.1. The Boltzmann equation. The Boltzmann equation describes the behavior of a dilute gas. When we assume periodic boundary condition in spatial space, the equation reads

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f), \quad x \in T^3, \quad v \in \mathbb{R}^3, \quad t \in [0, \infty),
\]

where \( T^3 \) is the 3-dimensional torus, \( f = f(t,x,v) \) is a time-dependent distribution function on \( T^3 \times \mathbb{R}^3 \), and \( Q \) is the quadratic Boltzmann collision operator with the bilinear form

\[
Q(g,f) = \int_{\mathbb{R}^3} dv_x \int_{S^2} d\sigma \cdot B(|v - v_x|, \cos \theta)(g'_x f' - g_x f),
\]

Here we use the shorthands \( f' = f(v') \), \( g_x = g(v_x) \), and \( g'_x = g(v'_x) \), where \( (v, v_x) \) and \( (v', v'_x) \) are the velocities of particles before and after the collision. We use the

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σ-representation, that is, $v'$ and $v'_*$ are given by

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma.$$ \hfill (2.1)

Moreover, $\theta \in [0, \pi]$ is the deviation angle between $v' - v'_*$ and $v - v_*$, and $B$ is the Boltzmann collision kernel. In this work, we will impose the following assumptions:

(A1) $\sigma \geq 0$ and $\sigma$ takes the following product form

$$B(v - v_*, \sigma) = |v - v_*|^{\gamma} b(\cos \theta) \quad (2.2)$$

with $\gamma \in (-3, 0)$ and $b$ is a nonnegative function on $\theta \in (0, \pi]$ with

$$l_b := \inf_{\pi/4 \leq \theta \leq 3\pi/4} b(\cos \theta) > 0. \quad (2.3)$$

(A2) (cutoff models) We assume that $n_b := \int_{S^2} b(\cos \theta) d\sigma = |S^1| \int_0^{\pi} b(\cos \theta) \sin \theta d\theta < \infty. \quad (2.4)$

(A3) (non cutoff models) $b$ fulfills the condition

$$b(\cos \theta) \sin \theta \sim_{\theta \rightarrow 0^+} b_0 \theta^{1-\nu}$$

with $\nu \in (0, 2)$ and $\gamma + \nu > -1$, then we have

$$m_b := \int_{S^2} b(\cos \theta)(1 - \cos \theta) d\sigma = |S^1| \int_0^{\pi} b(\cos \theta)(1 - \cos \theta) \sin \theta d\theta < \infty. \quad (2.5)$$

1.2. Short review of the previous results. Before introducing the setup of our problem, let us give a brief review on the pointwise lower bound for the equation.

- Homogeneous equation. In this case, the equation (1.1) is reduced to

$$\partial_t f(t, v) = Q(f, f)(t, v). \quad (1.6)$$

It is known that this equation is well-posed, and that the solution conserves mass, momentum and energy. Moreover, the entropy is a decreasing function, i.e.

$$\int_{\mathbb{R}^3} f(t, v) \log f(t, v) dv \leq \int_{\mathbb{R}^3} f(0, v) \log f(0, v) dv.$$ \hfill (1.7)

The first mathematically rigorous results on the lower bound of the solution to (1.6) are due to Carleman [4, 5]. Under the additional conditions that the initial data are Holder continuous and satisfy a certain moment assumption, he showed that the radial solutions to (1.6) with hard potentials are bounded from below by exponential functions, $e^{-|v|^{2+\varepsilon}}$, with $\varepsilon$ arbitrarily small. As a result, Carleman used it to determine the rate of entropy dissipation, i.e.

$$\frac{d}{dt} \int_{\mathbb{R}^3} f(t, v) \log f(t, v) dv = \int_{\mathbb{R}^3} Q(f, f)(t, v) \log f(t, v) dv. \quad (1.7)$$

Pulvirenti and Wennberg [14] removed the restriction of the radial symmetry. They proved that any solution to (1.6) with finite mass, energy and entropy fulfills the Gaussian lower bound, i.e.

$$\forall t \geq t_0 > 0, \quad \forall v \in \mathbb{R}^3, \quad f(t, v) \geq C_1 e^{-C_2|v|^2}$$

for any fixed $t_0 > 0$. These lower bounds play an important role in the entropy method. We refer readers to references [6, 16] where the authors made a detailed study of the entropy production rate.
• Inhomogeneous equation. In the remarkable paper [8] to derive estimates on the rate of convergence to equilibrium for the equation (1.1), the solutions were still assumed to be bounded pointwise from below. This motivated the work [13] by Mouhot to show that the regularity could induce the pointwise lower bound. More precisely, he proved that for a large class of collision kernels, including hard spheres and inverse power law interactions, any solution of the Boltzmann equation in $T^3 \times \mathbb{R}^3$, satisfying (1.16) and (1.17) (with $\varepsilon = 0$) automatically satisfies a lower bound, namely

$$\forall 0 < t_0 \leq t \leq T, \forall x \in T^N, \forall v \in \mathbb{R}^N, \ f(t, x, v) \geq C_1(T)e^{-C_2|v|^K},$$

where $K > 2$ for non cutoff model and $K = 2$ for cutoff model. Later Briant (see [2] and [3]) showed that these results can be generalized to the convex bounded domains and $C^2$ open bounded domains with nowhere null normal vector.

Very recently the Gaussian lower bound for the non cutoff equation can be recovered. In [12], under the spatial periodic conditions and the sole assumption that hydrodynamic quantities remain bounded, authors proved that

$$\forall t \geq t_0 > 0, \forall x \in T^N, \forall v \in \mathbb{R}^N, \ f(t, x, v) \geq a(t)e^{-b(t)|v|^2},$$

where $a(t)$ and $b(t)$ depends on $t$ and the hydrodynamic quantities. In [11], Henderson et al. established pointwise lower bounds for the equation in the whole space $\mathbb{R}^3$ with the initial data that may contain vacuum regions. Namely, they obtained that

$$f(t, x, v) \geq \mu(t, x)e^{-\eta(t, x)|v|^2}, \ (t, x, v) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3,$$

where the functions $\mu, \eta$ are uniformly positive and bounded on any compact subset of $(0, T] \times \mathbb{R}^3$.

1.3. Setup of our problem. Our main motivation to reconsider the pointwise lower bound for (1.1) stems from the problems on the global dynamics and global-in-time stability for the inhomogeneous equation with soft potentials. To explain it in details, we begin with the technical issues on the growth of the Sobolev regularity for the solutions.

For the homogeneous equation (1.6), it was shown in [7] that the weak solution satisfies the following estimates on the propagation of $L^1$ moment, namely,

$$|f(t)|_{L^1} \leq C(l)(1 + t), \ \forall t \in \mathbb{R}^+.$$ (1.11)

This will bring the trouble to show the uniform-in-time propagation of the Sobolev regularity.

For the inhomogeneous equation (1.1), to show the global-in-time stability, it is natural to assume that the background solution $g(t, x, v)$ to (1.1) has the good properties, for instance, the density defined in (1.12) is bounded from below and above. To prove the global-in-time stability, the continuity argument will be used. In particular, we will assume that for a longtime period, the new solution $f(t, x, v)$ to (1.1), generated by the perturbation of the initial data $g(0, x, v)$, is still close to the background solution. In other words, the hydrodynamic quantities of $f$ defined in (1.12-1.14) have the good control. However, due to the technical restriction (for instance, see [10]), if we introduce the error function $h$ between the solutions $g$ and $f$, that is $h := f - g$, then

$$\frac{d}{dt}\|hW_i\|_{L^2_t L^2_x}^2 + \|hW_{i+\gamma/2}\|_{L^2_t H^\gamma_x}^2 \leq C(g)\|h\|_{L^2_t L^2_x}^2,$$
where $W_l(v) := (1 + |v|^2)^{l/2}$. Since now $\gamma < 0$, it seems very hard to derive the uniform-in-time estimate of $\|hW_l\|_{L^2_x L^2_v}$. Technically, what we can expect is that

$$
\|(hW_l)(t)\|_{L^2_x L^2_v}^2 \leq (1 + t)^a
$$

with $a > 0$. This growth will induces the difficulty to get the better control of the lower bound of the density of $f$ which will be used to close the continuity argument.

We conclude that in both cases, the solution satisfies the same properties: (i). the hydrodynamic quantities (local mass, local energy, and local entropy density) are bounded (from below or from above) uniformly in time, (ii). the Sobolev regularity for the solution grows tempered with time.

To prove our desired results, our main idea lies in the faith that $H$-theorem will dominate the behavior of the solution even the Sobolev regularity grows polynomially with respect to time. To implement the idea, we shall use the entropy method. Thus the first step is to show the quantitative estimates on the dependence of the pointwise lower bound on the growth rate of the Sobolev regularity. Unfortunately we cannot see the explicit dependence on the time variable in the lower bounds estimates (1.8-1.10). This motivates our work.

1.4. Notations. Before stating our main results, we first introduce the basic notations which will be used throughout the whole paper.

- We define the weighted Lebesgue space $L^p_q(\mathbb{R}^3)(p \in [1, +\infty], q \in \mathbb{R})$ by the norm
  $$
  \|f\|_{L^p_q(\mathbb{R}^3)} = \left[ \int_{\mathbb{R}^3} |f(v)|^p \langle v \rangle^q dv \right]^{1/p},
  $$
  where $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$. When $p = +\infty$,
  $$
  \|f\|_{L^\infty_q(\mathbb{R}^3)} = \sup_{v \in \mathbb{R}^3} |f(v)| \langle v \rangle^q,
  $$

- The Sobolev space $W^{k,p}(\mathbb{R}^3)(p \in [1, +\infty])$ and $k \in \mathbb{N}$ is defined by
  $$
  \|f\|_{W^{k,p}(\mathbb{R}^3)} = \left[ \sum_{|\alpha| \leq k} \|\partial^\alpha f(v)\|_{L^p}^p \right]^{1/p}.
  $$

- “cst” shall denote any constants depending only on the parameters $\gamma, \nu$ and $b_0$. For the real $x$, we shall denote the positive part of $x$ by $x^+$ and use the shorthand $\tilde{\gamma} = (\gamma + 2)^+$.

1.5. Main results. Suppose the nonnegative function $f(t, x, v)$ is the solution to Boltzmann equation (1.1). Then the hydrodynamic quantities such as local mass, energy and the entropy are defined as follows:

$$
\rho_f(t, x) = \int_{\mathbb{R}^3} f(t, x, v) dv, \quad (1.12)
$$

$$
e_f(t, x) = \int_{\mathbb{R}^3} f(t, x, v)|v|^2 dv, \quad (1.13)
$$

$$
h_f(t, x) = -\int_{\mathbb{R}^3} f(t, x, v) \log f(t, x, v) dv. \quad (1.14)
$$

By setting

$$
l^p_f(t, x) = \|f(t, x, \cdot)\|_{L^p(\mathbb{R}^3_v)} \quad \text{and} \quad w_f(t, x) = \|f(t, x, \cdot)\|_{W^{2,\infty}(\mathbb{R}^3_v)}, \quad (1.15)
$$

...
we introduce
\[ D_f := \inf_{(t,x) \in [0,\infty) \times \mathbb{T}^3} \rho_f(t,x), \quad E_f := \sup_{(t,x) \in [0,\infty) \times \mathbb{T}^3} (\rho_f(t,x) + \epsilon_f(t,x)), \]
\[ H_f := \sup_{(t,x) \in [0,\infty) \times \mathbb{T}^3} h_f(t,x), \quad L^p_f(t) := \sup_{x \in \mathbb{T}^3} l^p_f(t,x), \quad W_f(t) := \sup_{x \in \mathbb{T}^3} w_f(t,x). \]

Our basic assumptions on the solution are as follows:

- **Bounds for the hydrodynamic quantities:**
  \[ D_f \geq D_0 > 0, \quad E_f \leq E_0 < \infty, \quad H_f \leq H_0 < \infty. \] (1.16)

- **Growth of Sobolev regularity:**
  \[ (i). \quad W_f(t) \leq (1 + t)\varepsilon, \quad (ii). \quad L^p_f(t) \leq (1 + t)^\varepsilon \] (1.17)
  with \( p > \frac{3}{\gamma + 7} \) and \( 0 \leq \varepsilon \leq 1 \).

We now state our main theorems. The first one deals with cutoff collision kernels.

**Theorem 1.1** (Cutoff case). Suppose that the kernel \( B \) verifies (A1)-(A2). Let \( f(t,x,v) \) be a mild solution of the Boltzmann equation in the torus such that (1.16) and (1.17)(ii) hold. Then for any fixed \( \tau_0 > 0 \), the solution \( f \) satisfies the following lower bound
\[ \forall t \in (\tau_0, +\infty), \forall x \in \mathbb{T}^3, \forall v \in \mathbb{R}^3, \quad f(t,x,v) \geq C_1 e^{-C_2 (1 + t)^{(1 + |v|^2)}}, \]
where \( C_1 = \text{cst} \ \tau_0 \) and
\[ C_2 = \text{cst} \ (1 + R_0^2) \]
\[ \times \left( \frac{\delta_0}{\xi} \prod_{k=1}^{\infty} (1 - \xi^k) \right)^{-2} \left( - \ln \left( \left( \frac{\delta_0}{\xi} \prod_{k=1}^{\infty} (1 - \xi^k) \right)^{3 + \gamma} \xi^{1/2} \tau_0 \right) + \tau_0 n_b (E_0 + 1) \right) \]
with \( \xi = \min \{ 16(\sqrt{2} \delta_0)^{-6 - 2\gamma}, 1/2 \} \) and \( \delta_0, R_0 \) depending on \( D_0, E_0 \) and \( H_0 \).

The second one is concerned with non cutoff models.

**Theorem 1.2** (Non cutoff case). Suppose that the kernel \( B \) verifies (A1)-(A3). Let \( f(t,x,v) \) be a mild solution of the Boltzmann equation in the torus such that (1.16) and (1.17)(i)(ii) hold. Then for any fixed \( 0 < \tau_0 < 1 \) and any \( K > 2\log \left( \frac{2 + \frac{1}{\gamma + 7}}{\log 2} \right) \), the solution \( f \) satisfies the following lower bound
\[ \forall t \in (\tau_0, +\infty), \forall x \in \mathbb{T}^3, \forall v \in \mathbb{R}^3, \quad f(t,x,v) \geq C_3 e^{-C_4 (1 + |v|^2)}, \]
where \( C_3 = \text{cst} \ \tau_0 \) and
\[ C_4(t) = C_4 (1 + t)^\varepsilon \left( \frac{2}{\gamma + 7} (\gamma + 1) + \frac{2 + \gamma}{2 + \gamma} + K \right) \]
with \( C_4 \) depending on \( D_0, E_0, H_0, \tau_0 \) and \( K \).

**Remark 1.1.** Since \( H \)-theorem will defeat the growth of the Sobolev regularity at the end of the day, the uniform-in-time bounds of the Sobolev regularity are expected. By (1.8), it means that the pointwise lower bound obtained here are the intermediate results. Therefore we do not follow the strategy used in [11, 12] to sharp our result.
Remark 1.2. The results give the quantitative estimates on the dependence of the pointwise lower bound on the growth rate of the Sobolev regularity. In particular, it indicates that the pointwise lower bound will become degenerated when \( t \) goes to infinity. However, in our forthcoming work ([9]) we show that such degeneration property (w.r.t. time) is harmless to the entropy method if \( \varepsilon \) is sufficiently small.

1.6. Comment on our proof. The proof mainly consists of two steps:

(i). Establish a lower bound on a ball in the considered time interval;
(ii). Spread this lower bound by iteration scheme thanks to the collision process.

In our case, in order to keep track of the dependence of the lower bound on the growth of the Sobolev regularity, we modify iteration scheme by designing the special time interval. This is quite different from the previous work. To be more precise, let us specify the well-constructed time interval:

(1). For the cutoff case, we restrict our consideration on \([\tau_0 + \bar{t} - 2^{-n}\tau_0, \tau_0 + \bar{t}]\) with some fixed \( \tau_0 > 0 \). Here \( 2^{-n}\tau_0 \) is the \( n \)-th iteration time stepping. In comparison to the time interval constructed in [13], the additional parameter \( \bar{t} \) is used to track the dependence on Sobolev regularity.

(2). For the non cutoff case, additional terms will be appeared in the Duhamel formulation of the equation due to the collisions which are close to be grazing. To balance it, we construct an important quantity \( r(\bar{t}) := \text{cst} \tau_0 (2 + \bar{t})^{2 - \nu} - \varepsilon (2 + \nu) \) for \( \bar{t} \geq \tau_0 \) and restrict our attention on the time interval \([\sum_{k=0}^{n} \Delta_k \tau(\bar{t}) + \bar{t}, \tau(\bar{t}) + \bar{t}]\) where \( \{\Delta_k\} \) is any sequence verifying that \( (\sum_{k=0}^{n} \Delta_k) \leq 1 \). We remark that \( \Delta_n \tau(\bar{t}) \) is the \( n \)-th iteration time stepping. Again the parameter \( \bar{t} \) is used to track the dependence on Sobolev regularity. It is worth to mention that \( r(\bar{t}) \) is continuous which will be used in the final step of the proof of Theorem 1.2.

2. Proof of Theorem 1.1. We first deal with the cutoff models. Let us introduce Grad’s splitting \( Q(g, f) := Q^+(g, f) - Q^-(g, f) \), where

\[
Q^+(g, f) := \int_{\mathbb{R}^3} dv* \int_{S^2} d\sigma B(|v - v_*|, \cos \theta) g'_* f',
\]
\[
Q^-(g, f) := \int_{\mathbb{R}^3} dv* \int_{S^2} d\sigma B(|v - v_*|, \cos \theta) g_* f.
\]

Generally \( Q^+ \) is called the gain term and \( Q^- \) is called the loss term. We write the loss term as

\[
Q^-(g, f) = L[g] f \tag{2.18}
\]

with

\[
L[g(t, x, \cdot)] := n_b \int_{\mathbb{R}^3} |v - v_*|^\gamma g(t, x, v_*) dv_*,
\]

where \( n_b \) is defined in (1.4). From Lemma 5.1, we obtain that

\[
\forall v \in \mathbb{R}^3, |L[g(t, x, \cdot)](v)| \leq \text{cst} \ n_b [e_\gamma + L_p^g] := C_L(t) \tag{2.19}
\]

with \( p > 3/(3 + \gamma) \) and \( e_\gamma, L_p^g \) defined in (1.13) and (1.15).
2.1. Proof of Theorem 1.1. We shall use the following Duhamel representation formula, i.e.,\[∀\]
v.2.1.
Proof of Theorem 1.1.

We remark that \(L(T)\) of the Boltzmann equation to the initial datum \(Hn\), we shall use in the cutoff case, i.e. the concept of mild solutions.

We first consider the case \(n\) depend on the same quantities plus \(\xi\) is any sequence in \(0,0,\bar{\delta}0,\tau0,\varepsilon,\bar{\delta}0,\tau0,\varepsilon\) by \(f0(t,x,v) = f(0,0,\bar{\xi})\). We define the concept of solution we shall use in the cutoff case, i.e. the concept of mild solutions.

**Definition 2.1.** Let \(f0\) be a measurable function nonnegative almost everywhere on \(T^3 \times R^3\). A measurable function \(f = f(t,x,v)\) on \([0,\infty) \times T^3 \times R^3\) is a mild solution of the Boltzmann equation to the initial datum \(f0(x,v)\) if for almost everywhere \((x,v)\) in \(T^3 \times R^3\):

\[
t \mapsto L[f(t,x+vt,\cdot)](v), \quad t \mapsto Q^+[f(t,x+vt,\cdot),f(t,x+vt,\cdot)](v)
\]

are in \(L^1_{loc}([0,\infty))\), and for each \(t \in [0,\infty)\), the equation (2.20) is satisfied and \(f(t,x,v)\) is nonnegative for almost every \((t,x,v)\).

To handle the polynomial growth of \(L^p\) norm of solutions, we consider \(\tilde{t} > 0\) as the initial time. Denote \(f(\tilde{t},x,v)\) by \(f(\tilde{t})\), from Duhamel formula (2.20), we have

\[
f(t,x+v(t-\tilde{t}),v) = f(\tilde{t}) \exp \left( - \int_\tilde{t}^t L[f(s,x+v(s-\tilde{t}),\cdot)](v) ds \right) + \int_\tilde{t}^t \exp \left( - \int_s^t L[f(s',x+v(s'-\tilde{t}),\cdot)](v) ds' \right) \times Q^+[f(s,x+v(s-\tilde{t}),\cdot),f(s,x+\bar{\varepsilon})](v) ds.
\]

**Proposition 2.1.** Suppose that the kernel \(B\) verifies (A1)(A2). Let \(f(t,x,v)\) be a mild solution of Boltzmann equation such that (1.16) and (1.17)(ii) hold. Then for any fixed \(\tau0 \in (0,\infty), x \in T^3\), there exists some \(R0 > 0\) and \(\varepsilon \in B(0,R0)\) such that

\[
\forall n \geq 0, \forall t \in \left[\tau0 + \frac{\tau0}{2^n+1}, \tau0 + \tilde{\tau}0\right], \forall v \in B(\varepsilon,\delta0), f(t,x+v(t-\tilde{t}),v) \geq an_{|B(\varepsilon,\delta0)},
\]

where the sequences \(\{an\}\) and \(\{\delta0\}\) satisfy the following induction formula

\[
a_{n+1} := \text{cst } b_{n0} \tau0 \exp \left( -\frac{\tau0}{2^n+1} A(\tau0 + \tilde{\tau}) \right) a_n^2 \frac{1}{2^n+2} \delta0^{3+\gamma} \xi_n^{1/2};
\]

\[
a_0 := e^{-\tau0 A(\tau0 + \tilde{\tau})} \left( 1 - e^{-\frac{\tau0}{2^n} A(\tau0 + \tilde{\tau})} \right)^2
\]

\[
\frac{2(\tau0 + \tilde{\tau})^2}{A(\tau0 + \tilde{\tau})^2} \delta0;
\]

\[
\delta_{n+1} := \sqrt{2} \delta0 (1 - \xi_n);
\]

\[
A(\tau0 + \tilde{\tau}) := \text{cst } n0 (E0 + \frac{L^p(\tau0 + \tilde{\tau}))}. \]

We remark that \(\{\xi_n\}_{n \geq 0}\) is any sequence in \((0,1), R0, \eta0, \delta0 > 0\) depend on \(D0, E0\) and \(H0\), and \(\varepsilon \in B(0,R0)\) depend on the same quantities plus \(x\).

**Proof.** We first consider the case \(n = 0\). Let \(t \in \left[\frac{\tau0}{2^n+1}, \tau0 + \tilde{\tau}\right]\). Thanks to Lemma 5.1, we have \(|L[f(t,x,\cdot)](v)| \leq C_L(t)\), where \(C_L(t)\) defined in (2.19). According to
We conclude the result for \( \forall \) By Lemma 5.2, we can choose \( R \) from (2.21), we have
\[
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\]
and from (2.19),
\[
C_L(s) \leq \text{cst } n_b \left( e^{f(s,x+v(s-t),\cdot)} + l_p f(s,x+v(s-t),\cdot) \right)
\leq \text{cst } n_b (E f(s) + L_p(s))
\leq \text{cst } n_b (E_0 + L_p(\tau_0 + \bar{t})) := A(\tau_0 + \bar{t}).
\]
Then we get
\[
f(t, x + v(t - \bar{t}), v) \geq f(\bar{t}) \exp(-(t - \bar{t}) A(\tau_0 + \bar{t})). \tag{2.22}
\]
By iterating (2.21), we have
\[
f(t, x + v(t - \bar{t}), v) \geq \int_{\bar{t}}^t \exp(-(t - s) A(\tau_0 + \bar{t})) Q^+ \left[ \int_{\bar{t}}^s \exp(-(s - s') A(\tau_0 + \bar{t})) \times Q^+ [f(s', x + v(s' - \bar{t}), \cdot)](v)\right] ds.
\]
Let us denote by \( \varphi^{R_0} \) the truncation \( \varphi 1_{|v| \leq R_0}. \) Thanks to (2.22), we can estimate the solution from below, that is, \( \forall v \in \mathbb{R}^3, \ |v| \leq R_0, \)
\[
f(t, x + v(t - \bar{t}), v) \geq Q^+ \left( Q^+ (f^{R_0} (\bar{t}), f^{R_0} (\bar{t})), f^{R_0} (\bar{t}) \right)(v)
\times \int_{\bar{t}}^t \exp(-(t - s) A(\tau_0 + \bar{t})) \exp(-(s - \bar{t}) A(\tau_0 + \bar{t}))
\left( \int_{\bar{t}}^s \exp(-(s - s') A(\tau_0 + \bar{t})) \exp(-2(s' - \bar{t}) A(\tau_0 + \bar{t}) ds' \right) ds
= e^{-(t-\bar{t}) A(\tau_0 + \bar{t})} \frac{1 - e^{-(t-\bar{t}) A(\tau_0 + \bar{t})}}{2(A(\tau_0 + \bar{t}))^2} Q^+ \left( Q^+ (f^{R_0} (\bar{t}), f^{R_0} (\bar{t})), f^{R_0} (\bar{t}) \right)(v).
\]
By Lemma 5.2, we can choose \( R_0 \) such that
\[
Q^+ \left( Q^+ (f^{R_0} (\bar{t}), f^{R_0} (\bar{t})), f^{R_0} (\bar{t}) \right)(v) \geq \eta_0 1_{B(\bar{v}, \delta_0)};
\]
where \( \bar{v} \in B(0, R_0). \) Note that the constants \( R_0, \eta_0 \) and \( \delta_0 \) only depend on \( D_0, E_0 \) and \( H_0. \) So we obtain \( \forall t \in [\bar{t}, t, \tau_0 + \bar{t}], \ |v| \leq R_0, \)
\[
f(t, x + v(t - \bar{t}), v) \geq e^{-(t-\bar{t}) A(\tau_0 + \bar{t})} \frac{1 - e^{-(t-\bar{t}) A(\tau_0 + \bar{t})}}{2(A(\tau_0 + \bar{t}))^2} \eta_0 1_{B(\bar{v}, \delta_0)}.
\]
This yields that \( \forall t \in [\bar{t}, t, \tau_0 + \bar{t}], \ |v| \leq B(\bar{v}, \delta_0), \)
\[
f(t, x + v(t - \bar{t}), v) \geq e^{-\tau_0 A(\tau_0 + \bar{t})} \frac{1 - e^{-\tau_0 A(\tau_0 + \bar{t})}}{2(A(\tau_0 + \bar{t}))^2} \eta_0 1_{B(\bar{v}, \delta_0)}.
\]
We conclude the result for \( n = 0 \) with
\[
a_0 := e^{-\tau_0 A(\tau_0 + \bar{t})} \frac{1 - e^{-\tau_0 A(\tau_0 + \bar{t})}}{2(A(\tau_0 + \bar{t}))^2} \eta_0.
\]
By the induction method, we assume that it holds for \( n: \)
\[
f(t, x + v(t - \bar{t}), v) \geq a_n I_{B(\bar{v}, \delta_n)}, \ \forall t \in [\tau_0 - \frac{\tau_0}{2^n+1} + \bar{t}, \tau_0 + \bar{t}], \ |v| \leq B(\bar{v}, \delta_n).
\]
According to Lemma 5.3, \(\forall t \in [\tau_0 - \frac{\tau_0}{2^{n+1}} + \bar{\ell}, \tau_0 + \bar{\ell}]\), we have
\[
f(t, x + v(t - \bar{\ell}), v) \geq \int_{t_0}^{t} \exp \left( - \int_{t}^{t_0} L[f(s', x + v(s' - \bar{\ell}), \cdot)](v) ds' \right) \\
\times Q^+[f(s, x + v(s - \bar{\ell}), \cdot), f(s, x + v(s - \bar{\ell}), \cdot)](v) ds \\
\geq \int_{t_0 - \frac{\tau_0}{2^{n+1}} + \bar{\ell}}^{t} \exp \left( - \int_{t}^{t_0} L[f(s', x + v(s' - \bar{\ell}), \cdot)](v) ds' \right) \\
\times Q^+[f(s, x + v(s - \bar{\ell}), \cdot), f(s, x + v(s - \bar{\ell}), \cdot)](v) ds \\
\geq \frac{1}{A(\tau_0 + \bar{\ell})} \left( 1 - \exp \left( - \left( \tau_0 + \frac{\tau_0}{2^{n+1}} - \bar{\ell} \right) A(\tau_0 + \bar{\ell}) \right) \right) \\
\times Q^+(a_n 1_{B(\bar{v}, \delta_n)}, a_n 1_{B(\bar{v}, \delta_n)})(v).
\]

Using the inequality \(1 - e^{-z} - ze^{-z} \geq 0, \forall z \geq 0\), we derive that
\[
f(t, x + v(t - \bar{\ell}), v) \geq \left( \tau_0 + \frac{\tau_0}{2^{n+1}} - \bar{\ell} \right) \exp \left( - \left( \tau_0 + \frac{\tau_0}{2^{n+1}} - \bar{\ell} \right) A(\tau_0 + \bar{\ell}) \right) \\
\times Q^+(a_n 1_{B(\bar{v}, \delta_n)}, a_n 1_{B(\bar{v}, \delta_n)})(v) \\
\geq \frac{\tau_0}{2^{n+2}} \exp \left( - \tau_0 A(\tau_0 + \bar{\ell}) \right) Q^+(a_n 1_{B(\bar{v}, \delta_n)}, a_n 1_{B(\bar{v}, \delta_n)})(v).
\]

Thanks to Lemma 5.3, we have
\[
f(t, x + v(t - \bar{\ell}), v) \geq \text{cst} \frac{\tau_0}{2^{n+2}} \exp \left( - \frac{\tau_0}{2^{n+1}} A(\tau_0 + \bar{\ell}) \right) a_n^2 b_0^2 \delta_n^{3+\gamma} \xi_n^{1/2} 1_{B(\bar{v}, 2(1 - \xi_n))}.
\]

Let \(a_{n+1} := \text{cst} \frac{\tau_0}{2^{n+2}} \exp \left( - \frac{\tau_0}{2^{n+1}} A(\tau_0 + \bar{\ell}) \right) a_n^2 b_0^2 \delta_n^{3+\gamma} \xi_n^{1/2}\) and then we complete the proof of this proposition. \(\square\)

**Proposition 2.2.** Under the same assumptions in Proposition 2.1, for fixed \(\tau_0, \bar{\ell} > 0\), we have

\[
\forall x \in T^3, \forall v \in \mathbb{R}^3, f(\tau_0 + \bar{\ell}, x, v) \geq \rho e^{-\frac{(1 + R_0)(1 + |v|^2)}{2 \tau_0}},
\]

where \(\rho = \min(a_0, 1)\) and \(\phi = -\frac{c_3}{4 \ln(\alpha)}\) with

\[
c_4 = \delta_0 \prod_{k=1}^{\infty} (1 - \xi_k), \quad \xi = \min\{16(\sqrt{2} \delta_0)^{-6-2\gamma}, 1/2\};
\]
\[
\alpha = F(\tau_0 + \bar{\ell}) \lambda a_0, \quad F(\tau_0 + \bar{\ell}) = \text{cst} \ b_0 \tau_0 \exp \left( - \frac{\tau_0}{2} A(\tau_0 + \bar{\ell}) \right), \quad \lambda = \frac{(c_4 \sqrt{2})^{3+\gamma} \xi^{1/2}}{8}.
\]

We remark that \(a_0\) is defined in Proposition 2.1 and \(R_0, \delta_0, \eta_0\) depend on \(E_0, D_0\) and \(H_0\).

**Proof.** We choose the sequence \(\{\xi_n\}\) such that \(\xi_n = \xi^n\) with \(\xi \in (0, 1)\). Then from Proposition 2.1, we can obtain that

\[
\delta_n = \delta_0 2^{n/2} (1 - \xi)(1 - \xi^2) \cdots (1 - \xi^n) = \delta_0 2^{n/2} \prod_{k=1}^{n} (1 - \xi^k).
\]
Let $c_δ := δ_0 \prod_{k=1}^{∞}(1 - \xi^k) > 0$, then $δ_n > c_δ 2^{n/2}$. We have

$$∀n ≥ 0, \forall t ∈ \left[0 - \frac{7_0}{2^{n+1}} + \bar{l}, \tau_0 + \bar{l}\right], \forall v \in B(\bar{v}, c_δ 2^{n/2}), f(t, x + v(t - \bar{l}), v) ≥ a_n.$$  

In particular, if $t = \tau_0 + \bar{l}$, we get

$$∀n ≥ 0, \forall v \in B(\bar{v}, c_δ 2^{n/2}), f(\bar{l} + \tau_0, x + v\tau_0, v) ≥ a_n.$$  

We claim that $a_n ≥ α 2^n$ for some $α ∈ (0, 1)$. Let us postpone the proof and suppose that it holds at hand. Let $φ$ verify

$$\exp\left(-\frac{c_3^2}{4φ}\right) = α.$$

We get that

$$∀|v - \bar{v}| < c_δ 2^{n/2}, f(\bar{l} + \tau_0, x + v\tau_0, v) ≥ \exp\left(-\frac{1}{4φ}(2^{n/2}c_δ)^2\right).$$

It is easy to see that if $n ≥ 1$ and $|v - \bar{v}| ∈ [c_δ 2^{(n-1)/2}, c_δ 2^{n/2}]$, then

$$f(\bar{l} + \tau_0, x + v\tau_0, v) ≥ \exp\left(-\frac{1}{2φ}(2^{(n-1)/2}c_δ)^2\right) ≥ \exp\left(-\frac{1}{2φ}|v - \bar{v}|^2\right),$$

which holds independently of $n$. Thanks to this property, we conclude that

$$∀|v - \bar{v}| ≥ c_δ, f(\bar{l} + \tau_0, x + v\tau_0, v) ≥ \exp\left(-\frac{1}{2φ}|v - \bar{v}|^2\right).$$

Since $|v - \bar{v}| < c_δ$, we have $f(\bar{l} + \tau_0, x + v\tau_0, v) ≥ a_0$. Choose $ρ = \min(a_0, 1)$, then we obtain that $∀v \in \mathbb{R}^3$,

$$f(\bar{l} + \tau_0, x + v\tau_0, v) ≥ ρe^{-\frac{|v - \bar{v}|^2}{2φ}}.$$  

We recall the fact $|\bar{v}| < R_0$, which implies $e^{-\frac{|v - \bar{v}|^2}{2φ}} ≥ e^{-\frac{1 + R_0^2}{2φ}(1 + |v|^2)}$. Thus we obtain that

$$∀v \in \mathbb{R}^3, f(\tau_0 + \bar{l}, x + v\tau_0, v) ≥ ρe^{-\frac{1 + R_0^2}{2φ}(1 + |v|^2)}. \tag{2.23}$$

Notice that the above estimate does not depend on $\bar{v}$ and variable $x$. We finally get that

$$∀x ∈ \mathbb{T}^3, ∀v \in \mathbb{R}^3, f(\tau_0 + \bar{l}, x + v\tau_0, v) ≥ ρe^{-\frac{1 + R_0^2}{2φ}(1 + |v|^2)}.$$  

Now we prove the claim. By proposition 2.1, we have

$$a_{n+1} = \text{cst } b_0\tau_0 \exp\left(-\frac{7_0}{2^{n+1}} A(\tau_0 + \bar{l})\right)a_2^2 \frac{1}{2^{n+2}} δ_n^{3+\gamma} ξ_{n/2}^{(1/2)}.$$  

Let $F(\tau_0 + \bar{l}) := \text{cst } b_0\tau_0 \exp(-\frac{7_0}{2} A(\tau_0 + \bar{l}))$ and assume $F(\tau_0 + \bar{l}) < 1/2$ (otherwise, we can modify the constant if necessary). Let $λ_n = \frac{1}{2^{n+2}} δ_n^{3+\gamma} ξ_{n/2}^{(1/2)}$, then we have $a_{n+1} ≥ F(\tau_0 + \bar{l})λ_n a_0^2$. Moreover

$$a_n ≥ F(\tau_0 + \bar{l})^{2^n-1} \left[λ_{n-1}λ_{n-2} \cdots λ_0^2\right] a_0^{2^n}.$$  

Let $\lambda = \min\left\{\frac{c_α \sqrt{2}}{8} 3+\gamma 1/2, 1\right\}$, then $λ_n > λ^n$ and

$$a_n ≥ F(\tau_0 + \bar{l})^{2^n-1} λ^{2^n-1-n} a_0^{2^n} ≥ F(\tau_0 + \bar{l})^{2^n} λ^n a_0^{2^n}.$$  

Choose $α = F(\tau_0 + \bar{l})λ a_0$, then $α$ satisfies $a_n ≥ α 2^n$. This ends the proof of this proposition.  

□
Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 2.2, we have
\[ \forall \tau_0, t > 0, \quad f(\tau_0 + t, x, v) \geq \rho e^{-\frac{(1 + R_0^2)(1 + |v|^2)}{\phi}}, \]
where
\[ \phi = -\frac{c^2}{4 \ln(\alpha)}, \quad \rho = \min(a_0, 1), \]
\[ \alpha = F(\tau_0 + \bar{t}) \lambda a_0, \quad F(\tau_0 + \bar{t}) = \text{cst} \, L_0 \exp(-\frac{\tau_0 A(\tau_0 + \bar{t})}{2}), \quad \lambda = \frac{(c_\delta \sqrt{2})^{3+\gamma} \xi^{1/2}}{8}, \]
\[ a_0 = e^{-\tau_0 A(\tau_0 + \bar{t})} (1 - e^{-\frac{\tau_0 A(\tau_0 + \bar{t})^2}{2}}), \quad A(\tau_0 + \bar{t}) = \text{cst} \, n_0(E_0 + L_f^p(\tau_0 + \bar{t})). \]
By the condition (1.17)(ii), we obtain that
\[ A(\tau_0 + \bar{t}) \leq \text{cst} \, n_0(E_0 + (1 + \tau_0 + \bar{t})^\gamma); \]
\[ a_0 \geq \text{cst} \, A(\tau_0 + \bar{t}) \geq \text{cst} \, \tau_0 n_0(E_0 + (1 + \tau_0 + \bar{t})^\gamma); \]
\[ F(\tau_0 + \bar{t}) \geq \text{cst} \, \tau_0 n_0(E_0 + (1 + \tau_0 + \bar{t})^\gamma); \]
\[ \alpha \geq \frac{(c_\delta \sqrt{2})^{3+\gamma} \xi^{1/2}}{8} \text{cst} \, \tau_0 n_0(E_0 + (1 + \tau_0 + \bar{t})^\gamma); \]
\[ -\frac{\lambda}{2} = \frac{2 \ln \alpha}{c^2} \geq 2 \left( \delta_0 \prod_{k=1}^{\infty} (1 - \xi^k) \right)^{-2} \times \left( \ln \left( \frac{(c_\delta \sqrt{2})^{3+\gamma} \xi^{1/2}}{8} \text{cst} \, \tau_0 n_0(E_0 + (1 + \tau_0 + \bar{t})^\gamma) \right) \right) \]
\[ \geq -2 \left( \delta_0 \prod_{k=1}^{\infty} (1 - \xi^k) \right)^{-2} \left( -\ln \left( \frac{(c_\delta \sqrt{2})^{3+\gamma} \xi^{1/2}}{8} \text{cst} \, \tau_0 n_0(E_0 + 1) \right) \right) \]
\[ + \text{cst} \, \tau_0 n_0(E_0 + 1)(1 + \tau_0 + \bar{t})^\gamma. \]
Moreover, we can assume \( a_0 < 1 \) so that \( \rho = a_0 \). It holds that
\[ \forall x \in \mathbb{T}^3, \forall v \in \mathbb{R}^3, \quad f(\bar{t} + \tau_0, x, v) \geq C_1 e^{-C_2(1 + \tau_0 + \bar{t})^\gamma(1 + |v|^2)}, \]
where \( C_1 = \text{cst} \, \tau_0 \) and
\[ C_2 = \text{cst} \, (1 + R_0^2) \left( \delta_0 \prod_{k=1}^{\infty} (1 - \xi^k) \right)^{-2} \left( -\ln \left( \frac{(c_\delta \sqrt{2})^{3+\gamma} \xi^{1/2} \text{cst} \, \tau_0 n_0(E_0 + 1) \right) \right) \]
with \( \xi = \min \{16(\sqrt{2})^{6-2\gamma}, 1/2\} \). Finally, let \( t = \tau_0 + \bar{t} \), we derive that
\[ \forall x \in \mathbb{T}^3, \forall v \in \mathbb{R}^3, \quad f(t, x, v) \geq C_1 e^{-C_2(1 + t)^\gamma(1 + |v|^2)}, t > \tau_0. \]
We complete the proof of Theorem Theorem 1.1. \qed

3. Proof of Theorem 1.2. In non cutoff cases, instead of Grad’s splitting, we use the following splitting:
\[ Q(g, f) = \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} d\sigma B g'_s(f' - f) + f(\int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} d\sigma B(g'_* - g_*)) = Q^1 + Q^2. \] (3.24)
Thanks to the cancellation Lemma (see [1]), the operator \( Q^2 \) can be written as
\[ Q^2(g, f) = S[g]f \]
3.1. Proof of Theorem 1.2. In this section, we shall prove Theorem 1.2. Recall that $\nu \in (0, 2)$ and we make the following splitting for any $\eta \in (0, \pi/4)$:

$$Q = Q^+ - Q^- + Q_1^+ + Q_1^-,$$

(3.26)

where $Q^+_\eta$ and $Q^-_\eta$ are the gain and loss terms corresponding to the kernel $B^\eta = |v - v_s|\gamma b(\cos \theta)1_{|\theta| \geq \eta} = |v - v_s|\gamma b(\cos \theta)$. $Q_1^+$ and $Q_1^-$ are the operators defined in the above which correspond to kernel $B_\eta = |v - v_s|\gamma b(\cos \theta)1_{|\theta| < \eta} = |v - v_s|\gamma b(\cos \theta)$. It is straightforward to check that $b^\eta \geq l_0$ on $[\pi/4, 3\pi/4]$. Since $b^\eta = b$ for $\theta \in [\pi/4, 3\pi/4]$ and thus the constants given in Lemma 5.3 on $Q^+_\eta$ are uniform to $\eta$. Moreover, we have

$$m_{b^\eta} \sim_{\eta \to 0} \frac{b_0}{(2 - \nu)} \eta^{2 - \nu}, n_{b^\eta} \sim_{\eta \to 0} \frac{b_0}{\nu} \eta^{-\nu}.$$  

(3.27)

By Duhamel formula, we derive that $\forall t \in [0, \infty), \forall x \in \mathbb{T}^3, \forall v \in \mathbb{R}^3$:

$$f(t, x + vt, v) = f_0(x, v) \exp \left( - \int_0^t \left( S_\eta + L_\eta \right) [f(s, x + vs, \cdot)](v) ds \right)$$

(3.28)

$$+ \int_0^t \left\{ \exp \left( - \int_s^t \left( S_\eta + L_\eta \right) [f(s', x + vs', \cdot)](v) ds' \right) \times (Q^+_\eta + Q^+_1)[f(s, x + vs, \cdot), f(s, x + vs, \cdot)](v) \right\} ds,$$

where $L_\eta$ and $S_\eta$ are the operators defined in (2.18) and (3.25) corresponding respectively to $Q^-_\eta$ and $Q^2_\eta$. Let us define the concept of mild solution we shall use in the non cutoff case:

**Definition 3.1.** Let $f_0$ be a measurable function nonnegative almost everywhere on $\mathbb{T}^3 \times \mathbb{R}^3$. A measurable function $f = f(t, x, v)$ on $[0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3$ is a mild solution of the Boltzmann equation to the initial datum $f_0(x, v)$ if there exists a $\eta_0 > 0$ such that for all $0 < \eta < \eta_0$, for almost every $(x, v) \in \mathbb{T}^3 \times \mathbb{R}^3$:

$$t \mapsto Q^+_\eta[f(t, x + vt, \cdot), f(t, x + vt, \cdot)](v), \quad t \mapsto Q^+_1[f(t, x + vt, \cdot), f(t, x + vt, \cdot)](v),$$

$$t \mapsto L_\eta[f(t, x + vt, \cdot)](v), \quad t \mapsto S_\eta[f(t, x + vt, \cdot)](v)$$

are in $L^1_{loc}([0, \infty))$, and for each $t \in [0, \infty)$, the equation (3.28) is satisfied and $f(t, x, v)$ is nonnegative for almost every $(x, v)$.
We consider \( \bar{t} \geq \tau_0 > 0 \) as the initial time and denote \( f(\bar{t}, x, v) \) by \( f(\bar{t}) \). By (3.28), \( \forall t \in [\bar{t}, \infty), \forall x \in \mathbb{T}^3, \forall v \in \mathbb{R}^3 \), we have

\[
f(t, x + v(t - \bar{t}), v) = f(\bar{t}) \exp \left( -\int_{\bar{t}}^{t} (S_n + L_n)[f(s, x + v(s - \bar{t}), \cdot)](v)ds \right) + \int_{\bar{t}}^{t} \left\{ \exp \left( -\int_{s}^{t} (S_n + L_n)[f(s', x + v(s' - \bar{t}), \cdot)](v)ds' \right) \times (Q_n^+ + Q_n^1)[f(s, x + v(s - \bar{t}), \cdot), f(s, x + v(s - \bar{t}), \cdot)](v) \right\}ds.
\]

Thanks to Lemma 5.1 and Lemma 5.4, we have

\[
|L_n[f]| \leq C_f m_{\nu, 0}(1 + t)^\varepsilon, \quad |S_n[f]| \leq C_f m_{\nu, 0}(1 + t)^\varepsilon, \\
|Q_n^i(f, f)| \leq C_f m_{\nu, 0}(1 + t)^{2\varepsilon} \langle v \rangle^5
\]

with a constant \( C_f \) depending on the uniform bounds on \( f \). Recall that \( \nu, \nu, \) and \( m_{\nu} \) are defined in (1.4) and (1.5) with respect to kernels \( B^0 \) and \( B_n \).

**Proposition 3.1.** Suppose that the kernel \( B \) verifies (A1)(A3). Let \( f(t, x, v) \) be a mild solution of Boltzmann equation such that (1.16) and (1.17)(i)(ii) hold. Fix \( \tau_0 \in (0, 1) \), then for any \( \bar{t} \geq \tau_0 \) and \( \bar{v}(\bar{t}) = \text{cst} C_f \tau_0 (2 + \bar{t})^{-\frac{1}{2}\frac{2}{1+2\varepsilon}} \), any sequence \( \{\Delta_n\}_{n \geq 0} \) of positive numbers such that \( \sum_{n \geq 0} \Delta_n = 1 \), there exists some \( R_0 \) and \( \bar{v} \in B(0, R_0) \) such that

\[
\forall n \geq 0, \forall t \in \left[ \left( \sum_{k=0}^{n} \Delta_k \right) \bar{t} + \bar{t}, \bar{t} + t \right], \forall v \in \mathbb{R}^3, \forall x \in \mathbb{T}^3, \ f(t, x + v(t - \bar{t}), v) \geq a_n \mathbb{B}(v, \delta_n),
\]

where the sequences \( \{a_n\} \) and \( \{\delta_n\} \) satisfy the induction formula

\[
a_{n+1} = \text{cst} \bar{v} \Delta_{n+1} a_n^2 \delta_n \frac{\varepsilon + 3}{\varepsilon} \xi^\frac{1}{n} \times \exp \left\{ - \left[ C_f a_n^2 \delta_n^{3+\gamma} (\delta_n)^{-\frac{\gamma}{n}} \xi^\frac{1}{n} \right] (\sum_{k=0}^{n+1} \Delta_k) (2 + \bar{t})^{\frac{2+\varepsilon}{2\varepsilon}} \right\},
\]

and the sequence \( \delta_n \) satisfies the induction formula

\[
\delta_{n+1} = \sqrt{2} \delta_n (1 - \xi^n), \quad \delta_0 = C_f (1 + \bar{t})^{-\varepsilon}.
\]

We remark that \( \xi \in (0, 1) \), the constants \( R_0, a_0 > 0 \) and \( C_f \) depend on \( D_0, E_0 \) and \( H_0 \), and \( \bar{v} \in B(0, R_0) \) depend on the same quantities plus \( x \).

**Proof.** We divide the proof into two steps.

**Step 1.** We consider the case \( n = 0 \). Since \( Q_n^1(f, f) \geq 0 \), then from (3.29), we have that \( \forall t \in [\bar{t}, \bar{t} + t], \)

\[
f(t, x + v(t - \bar{t}), v) \geq f(\bar{t}, x, v) \exp \left( -\int_{\bar{t}}^{t} (S_n + L_n)[f(s, x + v(s - \bar{t}), \cdot)](v)ds \right) + \int_{\bar{t}}^{t} \left\{ \exp \left( -\int_{s}^{t} (S_n + L_n)[f(s', x + v(s' - \bar{t}), \cdot)](v)ds' \right) \times Q_n^1[f(s, x + (s - \bar{t}), \cdot), f(s, x + (s - \bar{t}), \cdot)](v) \right\}ds.
\]
By Lemma 5.5, we know that \( f(\bar{t}, x, v) \geq a1_{B(\bar{v}, \delta_0)} \), where \( \bar{v} \in B(0, R_0) \), \( \delta_0 = C_f(1 + t)^{-\varepsilon} \) and \( R_0, a \) do not depend on \( t \). Then

\[
\begin{align*}
 f(t, x + v(t - \bar{t}), v) &\geq a1_{B(\bar{v}, \delta_0)} \exp \left( - \int_0^t (S_\eta + L_\eta)[f(v)] ds \right) \\
 &\quad + \int_0^t \left\{ \exp \left( - \int_s^t (S_\eta + L_\eta)[f(v)] ds' \right) \times Q^1_\eta[f, f](v) \right\} ds.
\end{align*}
\]

Assume that \( v \in B(\bar{v}, \delta_0) \subset B(0, R_0 + \delta_0) \). By the upper bounds of \( L_\eta, S_\eta, Q^1_\eta \), we obtain that \( \forall t \in [\bar{t}, \bar{t} + \varepsilon], \forall v \in B(\bar{v}, \delta_0) \),

\[
f(t, x + v(t - \bar{t}), v) \geq a1_{B(\bar{v}, \delta_0)} e^{-C_f(\eta + m_v + m_b_\eta)(1 + t)^{\varepsilon}} - C_f m_{b_\eta} \tau(1 + t)^{2\varepsilon}. \quad (3.31)
\]

Choose \( \eta := \bar{\eta}(\bar{t}) = cst \ C_f (2 + \bar{t})^{-\frac{2\varepsilon}{\varepsilon}} \) and \( \tau := \tau(\bar{t}) = cst \ C_f \tau(2 + \bar{t})^{-\frac{\varepsilon(2 + \varepsilon)}{2 + \varepsilon}} < \frac{1}{2} \tau_0 < \frac{1}{2} \), then by (3.27), it holds that

\[
C_f m_{b_\eta} (2 + \bar{t})^{2\varepsilon} \leq \frac{a}{2}, \quad e^{-C_f(\eta + m_v + m_b_\eta)(\bar{t} + t)^{\varepsilon}} \geq \frac{1}{2}.
\]

For \( t \in [\bar{t}, \bar{t} + \varepsilon(\bar{t})] \) and \( v \in B(\bar{v}, \delta_0(\bar{t})) \), we have

\[
f(t, x + v(t - \bar{t}), v) \geq \frac{1}{4} a1_{B(\bar{v}, \delta_0(\bar{t}))}.
\]

It ends the proof by letting \( a_0 = \frac{1}{4} a \).

**Step 2.** By inductive method, we suppose that the \( n \)-th step is satisfied:

\[
\begin{align*}
&\forall t \in \left[ \left( \sum_{k=0}^n 1 \right) \tau + \bar{t}, \bar{t} + \bar{t} \right], \forall v \in \mathbb{R}^3, \\
&f(t, x + v(t - \bar{t}), v) \geq a_n 1_{B(\bar{v}, \delta_n)}.
\end{align*}
\]

From Duhamel formula (3.29) and (3.30), we obtain that

\[
f(t, x + v(t - \bar{t}), v) \geq \int_{\sum_{k=0}^n 1}^t e^{-C_f(\eta + m_v + m_b_\eta)(t-s)(1+t)^{\varepsilon}} \\
\times [Q^1_\eta(a_n 1_{B(\bar{v}, \delta_n)}, a_n 1_{B(\bar{v}, \delta_n)}) - C_f m_{b_\eta} \tau(1 + t)^{2\varepsilon} (v)] ds.
\]

Restrict \( v \) to \( B(\bar{v}, \delta_{n+1}) \subset (0, R_0 + \delta_{n+1}) \), then from Lemma 5.3, for \( t \in \left[ \left( \sum_{k=0}^{n+1} 1 \right) \tau + \bar{t}, \bar{t} + \bar{t} \right] \) and \( v \in B(\bar{v}, \delta_{n+1}) \), we have

\[
f(t, x + v(t - \bar{t}), v) \geq \int_{\sum_{k=0}^{n+1} 1}^t e^{-C_f(\eta + m_v + m_b_\eta)(t-s)(1+t)^{\varepsilon}} \\
\times \left[ cst \ a_n^2 \ \delta_{n+\gamma}^{\frac{1}{2}n} 1_{B(\bar{v}, \delta_{n+1})} - C_f m_{b_\eta} \tau(1 + t)^{2\varepsilon} (\delta_{n+1})^{\frac{1}{2}n} \right] ds.
\]

Since \( \lim_{\eta \to 0} m_{b_\eta} = 0 \), choose \( \eta(\bar{t}) = cst \ \left[ C_f a_n^2 \delta_{n+\gamma}^{\frac{1}{2}n} (\delta_{n+1})^{\frac{1}{2}n} \right]^{\frac{1}{2}} \) and then

\[
C_f m_{b_\eta} \tau(1 + t)^{2\varepsilon} (\delta_{n+1})^{\frac{1}{2}n} \leq \frac{1}{2} cst \ a_n^2 \delta_{n+\gamma}^{\frac{1}{2}n} \xi_{\delta_{n+1}}^{\frac{1}{2}n},
\]
where we use the fact $\langle \delta_n \rangle \sim \langle \delta_{n+1} \rangle$. Together with (3.27), we derived that

$$f(t, x + v(t - \bar{t}), v) \geq \text{cst} \int_{\sum_{k=0}^{n+1} \Delta_k} e^{-C_f(t\nu_0 + m_0)(t-s)(1+\varepsilon)}$$

$$\times \int_{\sum_{k=0}^{n+1} \Delta_k} \sqrt{\alpha^2 \delta_n^3 + \gamma \xi^2 n^2} \times 1_B(\bar{v}, \delta_{n+1})$$

$$\geq \text{cst} \varepsilon \Delta_{n+1} e^{-C_f(t\nu_0 + m_0)(\sum_{k \geq n+1} \Delta_k)(2+\varepsilon)}$$

$$\times \int_{\sum_{k=0}^{n+1} \Delta_k} \sqrt{\alpha^2 \delta_n^3 + \gamma \xi^2 n^2} \times 1_B(\bar{v}, \delta_{n+1}).$$

Plugging $\eta(\bar{t})$ into the above formula and by (3.27), we obtain that

$$a_{n+1} \geq \text{cst} \varepsilon \Delta_{n+1} \alpha_n^2 \delta_n^3 + \gamma \xi^2 n^2 e^{-C_f(t\nu_0 + m_0)(\delta_n - \gamma \xi^2 n^2)} \int_{\sum_{k \geq n+1} \Delta_k} \sum_{k \geq n+1} \Delta_k = 0$$

It ends the proof. □

**Proposition 3.2.** Under the above assumption in Proposition 3.1, $\forall \tilde{t} \in [\tau_0, \infty)$ and $\varepsilon(\tilde{t}) = \text{cst} C_f(2 + \tilde{t}) \frac{(2+\varepsilon)}{2+\varepsilon} + \frac{2+\varepsilon}{2+\varepsilon}$, we have

$$\forall x \in \mathbb{T}^3, \forall v \in \mathbb{R}^3, f(x + \bar{t}, x + v, v) \geq C_5 e^{-C_f(t)(1+|v|^2)}$$

where $C_5 = \text{cst} \tau_0$ and

$$C(\tilde{t}) = C_4(2 + \tilde{t})^{(2+\varepsilon) + \frac{2+\varepsilon}{2+\varepsilon} + \frac{2+\varepsilon}{2+\varepsilon} + K).$$

We remark that $C_4$ depends on $D_0, E_0, H_0, \tau_0$ and $K$.

**Proof.** By Proposition 3.1, we note that

$$\forall n \geq 0, \forall t \in \left(\sum_{k=0}^{n} \Delta_k\right) x + \bar{t}, x + \bar{t} \right], \forall v \in \mathbb{R}^3, \forall x \in \mathbb{T}^3, f(t, x + v(t - \bar{t}), v) \geq a_n 1_B(\bar{v}, \delta_n).$$

Suppose that $t = \bar{t} + \varepsilon$, then we have

$$\forall n \geq 0, f(x + \bar{t}, x + v, v) \geq a_n 1_B(\bar{v}, \delta_n),$$

where $a_{n+1} = \text{cst} \varepsilon \Delta_{n+1} \alpha_n^2 \delta_n^3 + \gamma \xi^2 n^2 e^{-C_f(t\nu_0 + m_0)(\delta_n - \gamma \xi^2 n^2)} \int_{\sum_{k \geq n+1} \Delta_k} \sum_{k \geq n+1} \Delta_k = 0$ and $\delta_{n+1} = \sqrt{2} \delta_n(1 - \xi^n)$, $\delta_0 = C_f(1 + \tilde{t})^{-\varepsilon}$. Then $\delta_n$ satisfies

$$\text{cst} C_f(1 + \tilde{t})^{-\varepsilon} 2^n/2 \leq \delta_n \leq C_f(1 + \tilde{t})^{-\varepsilon} 2^n/2.$$

We set the sequences

$$\Delta_{n+1} = \alpha \beta \kappa \alpha^{\kappa \alpha^{\Psi}}$$

where $\kappa > 2 + \frac{2\nu}{2 - \nu}$, $\beta \in \left(\frac{2\nu}{2 - \nu}, \kappa - 2\right)$, $\alpha \in (0, 1)$ and $\Psi(\tilde{t}) \geq 1$. A simple computation yields that

$$\sum_{k \geq n+1} \Delta_k \leq \text{cst} \alpha^{\beta \kappa \alpha^{\Psi}} \Delta_{n+1} \geq \text{cst} \alpha^{\beta \Psi(\tilde{t})} \alpha^{\beta \kappa \alpha^{\Psi}}.$$

We claim that

$$a_n \geq \tau_0 \alpha^{\kappa \alpha^{\Psi}}.$$
In fact, when \( n = 0, \) \( a_0 = \frac{\log \kappa}{\log \sqrt{2}} \), we choose \( \alpha < \min \{ 1, \frac{1}{2} a \} \). Suppose that it holds for \( n \). Let us first consider the exponential component of \( a_{n+1} \). Since that \( \langle \delta_n \rangle \leq C_f 2^{n/2} \), we have

\[
\begin{align*}
[C_f \alpha_n^2 \delta^{3+\gamma} \langle \delta_n \rangle^{-\varepsilon} \xi^{\frac{3}{2} n}]^{\frac{2}{3+\gamma}} & \left( \sum_{k \geq n+1} \Delta_k \right) \\
& \leq C_f \left[ \frac{2(3+\gamma)/2 \xi^{1/2}}{2} \right]^{\frac{2}{3+\gamma}} \frac{\tau_0}{r_0} \alpha (\beta - \frac{2\nu}{3+\gamma}) \kappa^n \langle \delta \rangle \left( (2 + \tilde{t})^\varepsilon \right)^{\frac{\kappa}{3+\gamma}} (3+\gamma).
\end{align*}
\]

Let \( X := C_f [2(3+\gamma)/2 \xi^{1/2}] \frac{\tau_0}{r_0} \frac{2\nu}{3+\gamma} \) and \( Y := \alpha (\beta - \frac{2\nu}{3+\gamma}) \kappa^n \langle \delta \rangle \). It is easy to see that \( XY \leq 1/2 \) if \( \alpha \) is small enough. It implies that

\[
a_{n+1} \geq \text{cst} \tau \Delta_{n+1} \alpha_n^2 \delta^{3+\gamma} \xi^{\frac{3}{2} n} \exp \left\{ - (1/2) \varepsilon (2 + \tilde{t})^\varepsilon \delta (\beta - \frac{2\nu}{3+\gamma}) \kappa^n \langle \delta \rangle \right\}
\]

\[
\geq C_f \tau_0 \alpha \kappa^n \langle \delta \rangle \left( (2 + \tilde{t})^\varepsilon \right)^{\frac{\kappa}{3+\gamma}} (3+\gamma)^n \exp \left\{ - (1/2) \varepsilon (2 + \tilde{t})^\varepsilon \delta (\beta - \frac{2\nu}{3+\gamma}) \kappa^n \langle \delta \rangle \right\}
\]

Due to the facts that \( \varepsilon (\tilde{t}) = \text{cst} C_f \tau_0 (2 + \tilde{t})^\varepsilon \frac{\tau_0}{r_0} \frac{2\nu}{3+\gamma} \) and \( \kappa > \beta + 2, \) for sufficiently small \( \alpha \), we derive that

\[
C_f \tau_0 \alpha \kappa^n \langle \delta \rangle \left( (2 + \tilde{t})^\varepsilon \right)^{\frac{\kappa}{3+\gamma}} (3+\gamma)^n \geq \alpha \kappa^n \langle \delta \rangle,
\]

which gives

\[
a_{n+1} \geq \text{cst} \tau \kappa^n \langle \delta \rangle \alpha^{-\beta \langle \delta \rangle} \left( (2 + \tilde{t})^\varepsilon \right)^{\frac{\kappa}{3+\gamma}} (3+\gamma)^n \exp \left\{ - (1/2) \varepsilon (2 + \tilde{t})^\varepsilon \delta (\beta - \frac{2\nu}{3+\gamma}) \kappa^n \langle \delta \rangle \right\}
\]

Let \( \langle \delta \rangle \left( (2 + \tilde{t})^\varepsilon \right)^{\frac{\kappa}{3+\gamma}} (3+\gamma)^n \). Since \(- (\log \alpha) \beta \) is sufficiently large (recalling that \( \alpha \) is sufficiently small), we obtain that

\[
a_{n+1} \geq \tau \kappa^n \langle \delta \rangle \alpha^{-\beta \langle \delta \rangle} \left( (2 + \tilde{t})^\varepsilon \right)^{\frac{\kappa}{3+\gamma}} (3+\gamma)^n \exp \left\{ - (1/2) \varepsilon (2 + \tilde{t})^\varepsilon \delta (\beta - \frac{2\nu}{3+\gamma}) \kappa^n \langle \delta \rangle \right\}
\]

where \( \alpha \) only depends on \( \kappa \) (or \( K \)), \( \tau_0 \) and \( a_0 \).

Now we are in a position to prove the proposition. Recall that \( \delta_n \leq C \delta 2^{n/2} \), where \( C = C_f (1 + \tilde{t})^{-\varepsilon} \). Choosing \( \phi \) such that \( \exp \left( \frac{(C_f)}{\kappa \phi} \right) = \alpha \), we have for \( |v - \tilde{v}| \in \left[ C \delta 2^{(n-1)/2}, C \delta 2^{n/2} \right] \),

\[
f(\tau + \tilde{t}, x + v \phi, v^2) \geq \tau_0 e^{\frac{\log \kappa}{\log \sqrt{2}} \kappa^n \langle \delta \rangle} \left( (2 + \tilde{t})^\varepsilon \right)^{\kappa^n \langle \delta \rangle} \left( (2 + \tilde{t})^\varepsilon \right)^{\frac{\kappa}{3+\gamma}} (3+\gamma)^n \exp \left\{ - (1/2) \varepsilon (2 + \tilde{t})^\varepsilon \delta (\beta - \frac{2\nu}{3+\gamma}) \kappa^n \langle \delta \rangle \right\}
\]

where \( K = \frac{\log \kappa}{\log \sqrt{2}} \). Due to the facts that \( \frac{1}{\phi} = \frac{k \log \alpha}{(C_f) \kappa} \) and \( C \delta = C_f (1 + \tilde{t})^{-\varepsilon} \), we have

\[
f(\tau + \tilde{t}, x + v \phi, v^2) \geq \tau_0 e^{\frac{\log \kappa}{\log \sqrt{2}} \kappa^n \langle \delta \rangle} (1 + |v|^K)
\]
where \( C(\mathcal{I}) = C_4(2 + \mathcal{I})^{\varepsilon/2 + (3+\gamma)/2 + 2\varepsilon + K} \) with \( C_4 \) depending on \( D_0, E_0, H_0, \tau_0 \) and \( K \). When \( |v - \bar{v}| < C_3 \), we have
\[
f(t + \mathcal{I}, x + v \cdot x, v) \geq \tau_0 e^{\mathcal{I}(\mathcal{Y}(\mathcal{I}))} \geq \tau_0 e^{(\log \alpha)\mathcal{Y}(\mathcal{I}) + |v - \bar{v}|^K} \geq \tau_0 e^{C(\mathcal{I})(1 + |v|^K)}.
\]
We remark that this lower bound does not depend on \( \bar{v} \). This ends the proof of the proposition.

We are now ready to give the proof to Theorem 1.2.

**Proof of Theorem 1.2.** Thanks to proposition 3.2, we have
\[
\forall x \in \mathbb{T}^3, \forall v \in \mathbb{R}^3, f(t + \mathcal{I}, x + v \cdot x, v) \geq \tau_0 e^{C(t)(1 + |v|^K)}.
\]
Notice that the right-hand term in the above estimate does not rely on \( x \), so we have
\[
f(t + \mathcal{I}, x, v) \geq \tau_0 e^{C(t)(1 + |v|^K)}.
\]
Recall that \( \mathcal{Y}(\mathcal{I}) = \text{cst } C_f \mathcal{P}_0(2 + \mathcal{I})^{-(\gamma + 2\varepsilon)} < \frac{1}{2} \tau_0 \) and \( \mathcal{I} \geq \tau_0 \) which implies \( t + \mathcal{I} \sim \mathcal{I} \), then we deduce that
\[
f(t + \mathcal{I}, x, v) \geq \tau_0 e^{C(\mathcal{I})(1 + |v|^K)}.
\]
Moreover, by continuity, it is easy to see that for any \( t > 2\tau_0 \), there exists \( \mathcal{I} \geq \tau_0 \) such that \( \mathcal{Y}(\mathcal{I}) + \mathcal{I} = t \). Thus
\[
\forall t \in (2\tau_0, \infty), \forall x \in \mathbb{T}^3, \forall v \in \mathbb{R}^3, f(t, x, v) \geq \tau_0 e^{C(t)(1 + |v|^K)}.
\]
We complete the proof of Theorem 1.2.

### 4. Generalization of the result to the cutoff approximation

In this section, we want to generalize our results to the Boltzmann equation in the cutoff approximation (see [10]). To do that, we introduce the truncated Boltzmann collision operator in the cutoff approximation:
\[
Q_{\varepsilon}(g, f) := \int_{\mathbb{R}^3} dv \int_{S^2} d\sigma B_{\varepsilon}(|v - v_*|, \cos \theta)(g^v f^v - g_* f),
\]
where \( B_{\varepsilon}(|v - v_*|, \cos \theta) = |v - v_*|^{\gamma} b(\cos \theta) 1_{|v| \geq \varepsilon} = |v - v_*|^{\gamma} b_\varepsilon(\cos \theta) \), \( \varepsilon \ll 1 \). We have the following theorem:

**Theorem 4.1.** Suppose that the kernel \( B \) verifies (A1)-(A3) and \( f_\varepsilon \) is a classical solution of Boltzmann equation with truncated operator \( Q_{\varepsilon} \). Then \( \forall t \in (\tau_0, +\infty), \forall x \in \mathbb{T}^3, \forall v \in \mathbb{R}^3, \)
\[
f_\varepsilon(t, x, v) \geq C_1 e^{-C_2(1 + t)^{\varepsilon} |v|^2 \varepsilon^{-\nu}} \quad \text{with} \quad C_1 := \left( \frac{C_2}{C_4} e^{-\nu} (1 + t)^{\varepsilon} \left( \frac{\gamma}{\nu} + \frac{2\varepsilon + 4K}{\nu} \right) \right)^{\frac{1}{1 - \varepsilon}},
\]
where \( 1_A \) is the characteristic function of set \( A \).

**Proof.** To prove the result, we only need to combine the previous theorems in a suitable way. We first note that \( Q_{\varepsilon} \) can be dealt with by the cutoff method. In this situation, one has \( b_\varepsilon = b_\varepsilon \) and \( n_\varepsilon = \varepsilon^{-\nu} \), by Theorem 1.1, we have
\[
\forall t \in (\tau_0, +\infty), \forall x \in \mathbb{T}^3, \forall v \in \mathbb{R}^3, f_\varepsilon(t, x, v) \geq C_1 e^{-C_2(1 + t)^{\varepsilon} |v|^2 \varepsilon^{-\nu}},
\]
where $C_1 = \text{cst} \; \tau_0$ and 
$$C_2 = \text{cst} \; (1 + R_0^2) \times \left( \delta_0 \prod_{k=1}^{\infty} (1 - \xi^k) \right)^{-2} \left( - \ln \left( \delta_0 \prod_{k=1}^{\infty} (1 - \xi^k) \right)^{3+\gamma} \xi^{1/2} l_0 \tau_0^2 \right) + \tau_0 (E_0 + 1)$$
with $\xi = \min \{ 16(\sqrt{2} \delta_0)^{-2}, 1/2 \}$.

On the other hand, the operator $Q_\varepsilon$ can also be treated by the non cutoff method. In fact, we can make the following splitting for any $\eta \in (0, \pi/4)$:
$$Q_\varepsilon = Q_{\varepsilon, \eta}^+ - Q_{\varepsilon, \eta}^- + Q_{\varepsilon, \eta}^1 + Q_{\varepsilon, \eta}^2,$$
where $Q_{\varepsilon, \eta}^+$ and $Q_{\varepsilon, \eta}^-$ are the gain and the lost terms associated to the Grad’s cutoff kernel $B_{\varepsilon}^0 = |v - v_*|^\gamma b_\varepsilon (\cos \theta) 1_{|\theta| \geq \eta} = |v - v_*|^\gamma b_\varepsilon (\cos \theta)$. $Q_{\varepsilon, \eta}^1$ and $Q_{\varepsilon, \eta}^2$ are defined by (3.24) and correspond to the kernel $B_{\varepsilon, \eta} = |v - v_*|^\gamma b_\varepsilon (\cos \theta) 1_{|\theta| < \eta} = |v - v_*|^\gamma b_\varepsilon (\cos \theta)$. Notice that $B_{\varepsilon, \eta} = 0$, $B_{\varepsilon}^\eta = B_\varepsilon$ when $\eta \leq \varepsilon$. Moreover $(l_0)^\eta = l_0 = l_0$ and
$$m_{b_{\varepsilon, \eta}} \leq \eta \to 0 \text{ cst } b_0 (2 - \nu) \eta^{2-\nu}, \quad n_{b_{\varepsilon, \eta}} \leq \eta \to 0 \text{ cst } b_0 \nu^{1-\nu}.$$

By repeating the proof of Theorem 1.2, one may check that $C_3$ and $C_4$ in Theorem 1.2 are uniform with respect to $\varepsilon$, which implies that
$$\forall t \in (\tau_0, +\infty), \; \forall x \in \mathbb{T}^3, \; \forall v \in \mathbb{R}^3, \; \int_0 (t, x, v) \geq C_3 e^{-C_4 (1+\varepsilon) (\frac{1}{2} + \frac{2+\gamma}{3+\gamma}) (\nu)^K (v)^K},$$
where $C_3 = \text{cst} \; \tau_0$ and $C_4$ depends on $D_0, E_0, H_0, \tau_0$ and $K$.

5. Appendix.

**Lemma 5.1.** Let $g$ be a smooth function. Then
$$\forall v \in \mathbb{R}^3, \; |g * | \cdot |(v)| \leq \text{cst} \; \| g \|_{L^2} + \| g \|_{L^p}$$
with $p > 3/(3 + \gamma)$.

**Proof.** It is easy to see that
$$|g * | \cdot |(v)| = \int g * | \cdot |(v) dt, \; \int g * | \cdot |(v) dv_s = \int |v - v_*| |g(t, x, v)| dt, \; \int |v - v_*| |g(t, x, v)| dv_s = \int _{|v - v_*| \leq 1} |v - v_*|^\gamma g(t, x, v) dv_s + \int _{|v - v_*| > 1} |v - v_*|^\gamma g(t, x, v) dv_s = I + II,$$
On one hand, we have $|I(v)| \leq \| g \|_{L^1} \leq \| g \|_{L^2}$. On the other hand, by Cauchy-Schwartz inequality,
$$|I(v)| \leq \left( \int _{|v - v_*| \leq 1} |v - v_*|^\gamma g(t, x, v) dt \right)^{1/p'} \| g \|_{L^p} \leq \text{cst} \; \| g \|_{L^p}$$
with $p > 3/(3 + \gamma)$. Then the desired result follows. \qed

**Lemma 5.2.** (see [13]) Suppose the kernel $B$ verifying (A1)(A2) and $\rho_g, e_g, h_g$ defined in (1.12)(1.13) and (1.14). Let $g(v)$ be a nonnegative function on $\mathbb{R}^3$ with bounded energy $e_g$ and entropy $h_g$, moreover mass $\rho_g$ satisfies $0 < \rho_g < \infty$. Then there exists $R_0, \delta_0$ and $\eta_0 > 0$, and $v \in B(0, R_0)$ such that
$$Q^+ (Q^+ (g \chi_{B(0, R_0)}), g \chi_{B(0, R_0)}), g \chi_{B(0, R_0)}) \geq \eta_0 \chi_{B(0, \delta_0)},$$
where $R_0, \delta_0$ and $\eta_0$ depend on $B$, the lower bound of $\rho_g$ and the upper bound of $e_g$ and $h_g$. 

Lemma 5.4. Suppose the kernel \( B \) verifying (A1)(A2). Then for any \( \tilde{v} \in \mathbb{R}^3 \), \( 0 < r < R, \xi \in (0,1) \), we have
\[
Q^+(1_B(\tilde{v},R),1_B(\tilde{v},r)) \geq \text{cst} \ l_b R^3 \gamma \xi^{1/2} 1_{B(\tilde{v},\sqrt{r^2 + R^2} (1 - \xi))}.
\]
In particular case \( \delta = r = R \), we obtain
\[
Q^+(1_B(\tilde{v},R),1_B(\tilde{v},R)) \geq \text{cst} \ l_b^{3/2} \gamma R \xi^{3/2} 1_{B(\tilde{v},\sqrt{R} (1 - \xi))}
\]
for any \( \tilde{v} \in \mathbb{R}^3 \) and \( \xi \in (0,1) \).

Lemma 5.4. Suppose the kernel \( B \) verifying (A1)(A3) and \( Q^1 \) defined in (3.24). Let \( f, g \) be smooth functions on \( \mathbb{R}^3 \), then
\[
\left\{ \begin{array}{l}
|Q^1(g,f)(v)| \leq \text{cst} \ m_b \|g\|_{L^1_+} \|f\|_{W^{2,\infty}} \langle v \rangle^\gamma, \quad 2 + \gamma \geq 0, \\
|Q^1(g,f)(v)| \leq \text{cst} \ m_b \|g\|_{L^1_+} \|g\|_{L^\infty} \|f\|_{W^{2,\infty}}, \quad 2 + \gamma < 0
\end{array} \right.
\]
with \( p > 3/(5 + \gamma) \).

Proof. For any test function \( \varphi \), by change the variables, we have
\[
\int_{\mathbb{R}^3} Q^1(g,f)(v) \varphi(v) dv = \int_{\mathbb{R}^6 x \mathbb{S}^2} |v - v_*|^7 b(\cos \theta) g(v_*) (f(v) - f(v')) \varphi(v') d\sigma dv_* =: A'.
\]
Then by Taylor expansion, we have
\[
f(v) - f(v') = (v - v') \cdot \nabla f(v') + \frac{1}{2} \int_0^1 (1 - t)(v - v') \otimes (v - v') : \nabla^2 f(v' + t(v - v')) dt.
\]
Observe that
\[
\int_{\mathbb{R}^6 x \mathbb{S}^2} |v - v_*|^7 b \left( \frac{v - v_*}{|v - v_*|}, \sigma \right) (v - v') \cdot \nabla f(v') \varphi(v') d\sigma dv_* = 0
\]
where \( T_\sigma(v) \) represents the transform such that \( T_\sigma(v) = v \) (see [1]). Then we derive that
\[
|A'| \leq \int_0^1 \int_{\mathbb{R}^6 x \mathbb{S}^2} |v - v_*|^7 b(\cos \theta) g(v_*) |v - v'|^2 |\nabla^2 f(v' + t(v - v'))| |\varphi(v')| d\sigma dv_* dt.
\]
Since \( |v - v'| = |v - v_*| \sin(\theta/2) \), by change of variable \( v \rightarrow v' \), we have
\[
|A'| \leq \text{cst} \ |v - v_*| \sin^2(\theta/2) |v - v_*|^{2+\gamma} |g(v_*)| |\varphi(v')| d\sigma dv_*
\]
\[
\leq \text{cst} \ m_b \|g\|_{W^{2,\infty}} \|\varphi\|_{L^1} \int_{\mathbb{R}^3} |v - v_*|^{2+\gamma} |g(v_*)| d\sigma dv_*.
\]
If \( 2 + \gamma \geq 0 \), using the fact that \( |v - v_*|^{2+\gamma} \leq \langle v \rangle^{2+\gamma} |v_*|^{2+\gamma} \), we have
\[
|A'| \leq \text{cst} \ m_b \|g\|_{L^1_+} \|f\|_{W^{2,\infty}} \langle v \rangle^{3/2} \|\varphi\|_{L^1}.
\]
Then we get desired results by duality. For the other case, the proof is similar to Lemma 5.1 and we omit the details. This ends the proof.

Lemma 5.5. Let \( g \) be a nonnegative function on \( \mathbb{R}^3 \) and \( a, \rho_g, w_g \) are defined in (1.12), (1.13), (1.15). If \( e_g \) is bounded and \( \rho_g \) satisfies \( 0 < \rho_g < \infty \). Then there exists \( R_0, \delta_0 \) and \( a \) and \( \tilde{v} \in B(0, R_0) \) such that
\[
g(v) \geq a1_{B(\tilde{v}, \delta_0)},
\]
where
\[
R_0 = \sqrt{\frac{2\rho_g}{\rho_g}}, \quad a = \frac{\rho_g}{4 |B(0, R_0)|} \quad \text{and} \quad \delta_0 = \frac{\rho_g}{4 |B(0, R_0)| w_g}.
\]
Proof. It is not difficult to check that
\[
\int_{|v|>\sqrt{\frac{2a}{ρg}}} g(v)dv < \frac{ρg}{2e} \int_{\mathbb{R}^3} |v|^2 g(v)dv < \frac{ρg}{2},
\]
which implies that for \( R_0 = \sqrt{\frac{2a}{ρg}} \), \( \int_{|v|\leq R_0} g(v)dv > \frac{ρg}{2} \). Then there exists \( \bar{v} \in B(0, R_0) \) such that
\[
g(\bar{v}) \geq \frac{ρg}{2|B(0, R_0)|},
\]
where \( |B(0, R_0)| \) is volume of \( B(0, R_0) \). Observe that \( |g(v_1) - g(v_2)| \leq w_g|v_1 - v_2| \), so for \( δ = \frac{ρg}{4|B(0, R_0)||w_g|} \) and \( a = \frac{δ}{4|B(0, R_0)|} \), we have \( g(v) \geq a1_{B(\bar{v}, δ)} \).

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