FIXED POINT PROPERTY FOR A CAT(0) SPACE WHICH ADMITS A PROPER COCOMPACT GROUP ACTION

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Abstract. We prove that a geodesically complete CAT(0) space which admits a proper cocompact isometric action of a group and a complete locally doubling CAT(0) space satisfy a certain geometric condition obtained in the author’s previous paper. Suppose that \( \mathcal{Y} = \{Y_1, \ldots, Y_n\} \) is a finite family of geodesically complete CAT(0) spaces each of which admits a proper cocompact isometric action of a group. Then, combining our result with a theorem due to Izeki, Kondo, and Nayatani, it follows that a random group of the graph model has a common fixed point when it acts isometrically on any (finite or infinite) Cartesian product of CAT(0) spaces each of which is isometric to some \( Y_i \in \mathcal{Y} \). It also follows from our result that a sequence of expanders does not embed coarsely into such a product. The same results for a Cartesian product of complete CAT(0) spaces each of which is locally doubling with a common doubling constant also follow.

1. Introduction

In [4], Gromov introduced random groups of the graph model, and showed that when a random group of the graph model acts isometrically on any finite or infinite dimensional Hadamard manifold, there exists a common fixed point. Silberman [10] provided a detailed description of its proof for the case of Hilbert spaces.

In [7], Izeki and Nayatani introduced an invariant \( 0 \leq \delta(Y) \leq 1 \) for a complete CAT(0) space \( Y \). For the definition of the invariant \( \delta \), see Definition 3.1. Recently Izeki, Kondo and Nayatani [6] showed that for any constant \( 0 \leq c < 1 \), a random group of the graph model has a common fixed point when it acts isometrically on any complete CAT(0) space \( Y \) with \( \delta(Y) \leq c \) (Theorem 4.5). Since all Hadamard manifolds and Hilbert spaces satisfy \( \delta = 0 \), it generalizes Gromov’s theorem. Although their approach has a major advantage that it quantifies CAT(0) spaces by the invariant and gives a sufficient condition for the fixed point property by an isometric group action, it is hard to compute or estimate the invariant.

In particular, it has been unclear for which class \( \mathcal{Y} \) of CAT(0) spaces, there exists a constant \( 0 \leq c < 1 \) satisfying \( \delta(Y) < c \) for all \( Y \in \mathcal{Y} \). It had been even unknown whether the class consisting of all complete CAT(0) spaces admits such a constant \( 0 \leq c < 1 \) or not, until Kondo [8] showed the existence of complete CAT(0) spaces with \( \delta = 1 \) recently. In [13], the author tried to remedy this situation and obtained a geometric condition for a complete CAT(0) space \( Y \) to be \( \delta(Y) \leq C \), where \( 0 \leq C < 1 \) is a constant determined by a geometric property of \( Y \) (Theorem 5.2).

2010 Mathematics Subject Classification. Primary 53C23; Secondary 20F65, 20P05, 51F99.

Key words and phrases. CAT(0) space, Izeki-Nayatani invariant, fixed point property, coarse embedding, ultralimit.
Let $X$ be a metric space, $x \in X$, and $r > 0$. In this paper, we denote the open ball of radius $r$ centered at $x$ by $B(x,r)$, and the closed ball of radius $r$ centered at $x$ by $\overline{B}(x,r)$. Recall the following definitions (see [1] Chapter I.8).

**Definition 1.1.** An isometric action of a group $\Gamma$ on a metric space $X$ is called **cocompact** if there exists a compact subset $K \subset X$ such that $X = \cup_{\gamma \in \Gamma} \gamma K$. An isometric action of $\Gamma$ on a metric space $X$ is called **proper** if for each $x \in X$ there exists $r > 0$ such that the cardinality of the set $\{\gamma \in \Gamma \mid \gamma B(x,r) \cap B(x,r) \neq \phi\}$ is finite.

In this paper, we prove the following by using the criterion obtained in the author’s previous paper [13].

**Theorem 1.2.** Let $Y_1, Y_2, \ldots, Y_n$ be geodesically complete CAT(0) spaces. Suppose that each $Y_i$ admits a proper cocompact isometric action of a group. Then, there exists a constant $0 \leq c < 1$ such that any CAT(0) space $Y$ which is isometric to a (finite or infinite) product of CAT(0) spaces each of which is isometric to $Y_1, Y_2, \ldots, Y_{n-1}$, or $Y_n$ satisfies $\delta(Y) \leq c$.

For example, any Bruhat-Tits building associated to a semi-simple algebraic group is a geodesically complete CAT(0) space which admits a proper cocompact isometric action of a group. Combining Theorem 1.2 with Izeki-Kondo-Nayatani’s theorem (Theorem [15]), it follows that any isometric action on such $Y$ as in the above theorem by a random group of the graph model has a common fixed point. To state it precisely, let us recall the definitions of random groups of the graph model introduced by Gromov [1] and sequences of expanders.

**Definition 1.3** (Gromov [1]). Let $G = (V,E)$ be a finite combinatorial graph, where $V$ is the set of vertices and $E$ is the set of edges. Orient the edges $E$ arbitrarily. Fix $k$ alphabets $s_1, \ldots, s_k$. For each $e \in E$, choose an element $a(e)$ independently, uniformly at random from $\{s_1, \ldots, s_k, s_1^{-1}, \ldots, s_k^{-1}\}$. Let $c = e_1^{\epsilon_1} \cdot \cdot \cdot e_n^{\epsilon_n}$, $\epsilon_i \in E$ be a cycle in $G$, where $\epsilon_i = 1$ or $-1$, and $e^{-1}$ means the edge $e$ with the orientation reversed. The cycle $c$ defines a random word $a(c) = a(e_1)^{\epsilon_1} \cdot \cdot \cdot a(e_n)^{\epsilon_n}$ on $\{s_1, \ldots, s_k, s_1^{-1}, \ldots, s_k^{-1}\}$. Let $R_G$ be the set of the random words $a(c)$ for all cycles $c$ in $G$. In this way, we obtain a probability distribution over groups $\Gamma(G) = \langle s_1, \ldots, s_k | R_G \rangle$. We say $\Gamma(G)$ is the **random group** associated to $G$ with $k$ generators.

Let $G = (V,E)$ be a finite combinatorial graph. We define $\lambda_1(G)$ to be the first positive eigenvalue of the **combinatorial Laplacian** $\Delta_G$ of $G$, which acts on each real-valued function $f$ on $V$ as

$$\Delta_G f(v) = f(v) - \sum_{u \in V \text{ such that } \{v,u\} \in E} \frac{1}{\deg(u)} f(u), \quad v \in V,$$

where $\deg(v)$ is the number of edges at vertex $v$. $\lambda_1(G)$ can be computed variationally as

$$\lambda_1(G) = \inf_{\phi} \frac{\sum_{\{u,v\} \in E} \|\phi(u) - \phi(v)\|^2}{\sum_{v \in V} \deg(v) \|\phi(v) - \bar{\phi}\|^2},$$
where
\[ \bar{\phi} = \sum_{v \in V} \frac{\deg(v)}{2|E|} \phi(v), \]
and the infimum is taken over all nonconstant maps \( \phi : V \to \mathbb{R} \).

**Definition 1.4.** A sequence of expanders is a sequence \( \{ G_n = (V_n, E_n) \} \) of finite graphs which satisfies the following properties:

1. The number of vertices of \( G_n \) goes to infinity as \( n \) goes to infinity.
2. There exists \( d \) such that \( \deg(v) \leq d \) for all \( v \in V_n \) and all \( n \).
3. There exists \( \lambda > 0 \) such that \( \lambda_1(G_n) \geq \lambda \) for all \( n \).

Recall that the **girth** of a graph is the minimal length of a cycle in the graph. Now we can state a corollary of Theorem 1.2 as follows.

**Corollary 1.5.** Let \( Y_1, Y_2, \ldots, Y_n \) be geodesically complete CAT(0) spaces. Suppose that each \( Y_i \) admits a proper cocompact isometric action of a group. Let \( \mathcal{Y} \) be a class of all CAT(0) spaces which are isometric to (finite or infinite) products of CAT(0) spaces each of which is isometric to \( Y_1, Y_2, \ldots, Y_{n-1}, \) or \( Y_n \). If \( \{ G_n = (V_n, E_n) \} \) is a sequence of expanders, \( 2 \leq \deg(u) \leq d \) for all \( u \in V_n \) and all \( n \), and the girth of \( G_n \) is large enough, then with high probability, any isometric action of the random group \( \Gamma(G_n) \) on any \( Y \in \mathcal{Y} \) has a common fixed point.

We record another consequence of Theorem 1.2.

**Definition 1.6.** Let \( (X, d_X) \) and \( (Y, d_Y) \) be metric spaces. A map \( f : X \to Y \) is called a **coarse embedding** if there exists unbounded non-decreasing functions \( \rho_1, \rho_2 : [0, \infty) \to [0, \infty) \) such that
\[ \rho_1(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq \rho_2(d_X(x, x')), \]
for all \( x, x' \in X \). For a sequence \( \{(X_n, d_{X_n})\} \) of metric spaces, we call a sequence of maps \( f_n : X_n \to Y \) a **coarse embedding** if there exists unbounded non-decreasing functions \( \rho_1, \rho_2 : [0, \infty) \to [0, \infty) \) such that
\[ \rho_1(d_{X_n}(x, x')) \leq d_Y(f_n(x), f_n(x')) \leq \rho_2(d_{X_n}(x, x')), \]
for all \( n \) and all \( x, x' \in X_n \).

Since it is known by Kondo [8] that a sequence of expanders does not embed coarsely into a complete CAT(0) space \( Y \) with \( \delta(Y) < 1 \) (Theorem 4.4), the following corollary also follows from Theorem 1.2.

**Corollary 1.7.** Let \( Y_1, Y_2, \ldots, Y_n \) be geodesically complete CAT(0) spaces. Suppose that each \( Y_i \) admits a proper cocompact isometric action of a group. Let \( Y \) be a CAT(0) space which is isometric to a (finite or infinite) product of CAT(0) spaces each of which is isometric to \( Y_1, Y_2, \ldots, Y_{n-1}, \) or \( Y_n \). Then any sequence of expanders does not embed coarsely into \( Y \).

Since Gromov [4] proved that a sequence of expanders does not embed coarsely into a Hilbert space (cf. Theorem 1.3), the coarse embeddability of a sequence of expanders into a metric space \( Y \) has been a main obstruction for \( Y \) to be embedded
coarsely into a Hilbert space. Coarse embeddability of metric spaces into a Hilbert space (or Banach space) is a widely researched topic. We refer the reader to Chapter 11 of [9].

In this paper, we also estimate $\delta$ for another type of CAT(0) spaces.

**Definition 1.8.** A metric space is called doubling with doubling constant $N \in [1, \infty)$, if every closed ball can be covered by at most $N$ closed balls of half the radius. We say that a metric space is locally doubling with doubling constant $N \in [1, \infty)$ if any point has a neighborhood which is doubling with doubling constant $N$.

In the final section, we prove the following.

**Theorem 1.9.** If a complete CAT(0) space $Y$ is isometric to a (finite or infinite) product of locally doubling CAT(0) spaces with a common doubling constant $N \in [1, \infty)$, then there exists a constant $c < 1$ depending only on $N$ such that $\delta(Y) \leq c$.

We prove this theorem by showing that such $Y$ satisfies the condition obtained in the author’s previous paper [13]. We do not assume geodesic completeness in this theorem. To prove it without assuming geodesic completeness, we use an argument including the notion of ultralimit. Especially, we prove that the ultralimit of a sequence of doubling length spaces with a common doubling constant is also doubling with the same doubling constant (Proposition 7.5). The author guesses this fact should have been already known, though he does not know any references mentioning it.

The following corollaries follow.

**Corollary 1.10.** Let $N \in [1, \infty)$. Let $Y_N$ be a class of all complete CAT(0) spaces which are isometric to (finite or infinite) products of locally doubling CAT(0) spaces with doubling constant $N$. If $\{G_n = (V_n, E_n)\}$ is a sequence of expanders, $2 \leq \deg(u) \leq d$ for all $u \in V_n$ and all $n$, and the girth of $G_n$ is large enough, then with high probability, any isometric action of the random group $\Gamma(G_n)$ on any $Y \in Y_N$ has a common fixed point.

**Corollary 1.11.** Let $Y$ be a complete CAT(0) space which is isometric to a (finite or infinite) product of locally doubling CAT(0) spaces with a common doubling constant. Then any sequence of expanders does not embed coarsely into $Y$.

The paper is organized as follows. In Section 2 we briefly review some of the basic notions we will use in this paper. In Section 3 we recall the definition of the invariant $\delta$ and its basic properties. In Section 4 we explain two consequences which follow from the condition $\delta(Y) < 1$ for a complete CAT(0) space $Y$. One is the fact that a sequence of expanders does not embed coarsely into a complete CAT(0) space $Y$ with $\delta(Y) < 1$. The other is the fixed point theorem due to Izeki, Kondo, and Nayatani which we have mentioned above. In Section 5 we present a geometric condition for a complete CAT(0) space $Y$ to be $\delta(Y) < 1$, which is obtained in the author’s previous paper [13]. In Section 6 we use this condition to prove Theorem 1.2. In Section 7 we prove that the ultralimit of a sequence of doubling length spaces with a common doubling constant is also doubling with the same constant, and we use it to prove Theorem 1.9.
**Acknowledgements.** I would like to thank S. Nayatani, K. Fujiwara, H. Izeki, and T. Kondo for helpful discussions.

2. Preliminaries

In this section, we briefly recall the definitions of length spaces, geodesic spaces, CAT(0) spaces, Euclidean cones, tangent cones, ($\ell^2$)-products, geodesic completeness, and barycenters. For a detailed exposition, we refer the reader to [1], [2], and [3].

Let $(Y, d_Y)$ be a metric space. A path in $Y$ is a continuous map from a closed interval $[a, b]$ to $Y$. The length $L(\gamma)$ of a path $\gamma : [a, b] \rightarrow Y$ is defined as

$$L(\gamma) = \sup \sum_{i=1}^{k} d_Y(\gamma(t_{i-1}), \gamma(t_i)),$$

where the supremum is taken over all subdivisions

$$a = t_0 \leq t_1 \leq \cdots \leq t_k = b.$$

A path $\gamma : [a, b] \rightarrow Y$ is called arc-length parametrized if $L(\gamma|_{[a,t]}) = |t - a|$ for all $t \in [a, b]$, where $\gamma|_{[a,t]}$ is the restriction of $\gamma$ to $[a,t]$. Any path in a metric space can be reparametrized to an arc-length parametrized path. $Y$ is called a length space if the distance $d_Y(p,q)$ between any two points $p,q \in Y$ is equal to the infimum of the lengths of paths joining $p$ to $q$. We call a path $\gamma : [a, b] \rightarrow Y$ a geodesic if it is an isometric embedding of the interval $[a,b]$ into $Y$. A metric space is called a geodesic space if every pair of points is joined by a geodesic.

**Definition 2.1.** A metric space $Y$ is called geodesically complete if it is complete and any geodesic $\gamma : [0,a] \rightarrow Y$ is a restriction of some geodesic $\tilde{\gamma} : [0,b] \rightarrow Y$ with $0 < a < b$.

A geodesic triangle in $Y$ is a triple $\Delta = (\gamma_1, \gamma_2, \gamma_3)$ of geodesics $\gamma_i : [a_i, b_i] \rightarrow Y$ such that

$$\gamma_1(b_1) = \gamma_2(a_2), \quad \gamma_2(b_2) = \gamma_3(a_3), \quad \gamma_3(b_3) = \gamma_1(a_1).$$

For a geodesic triangle $\Delta = (\gamma_1, \gamma_2, \gamma_3)$ there is a geodesic triangle

$$\overline{\Delta} = (\overline{\gamma_1}, \overline{\gamma_2}, \overline{\gamma_3}), \quad \overline{\gamma}_i : [a_i, b_i] \rightarrow \mathbb{R}^2$$

in $\mathbb{R}^2$ such that $L(\gamma_i) = L(\overline{\gamma}_i)$ for each $i$. This triangle $\overline{\Delta}$ is unique up to isometry of $\mathbb{R}^2$. We call it the comparison triangle of $\Delta$ in $\mathbb{R}^2$. A geodesic triangle $\Delta$ is said to be thin if

$$d_Y(\gamma_i(s), \gamma_j(t)) \leq d_\kappa(\overline{\gamma}_i(s), \overline{\gamma}_j(t))$$

whenever $i,j \in \{1, 2, 3\}$ and $s \in [a_i,b_i]$, and $t \in [a_j,b_j]$.

**Definition 2.2.** A geodesic metric space is called a CAT(0) space if every geodesic triangle is thin.

A path $\gamma : I \rightarrow Y$ is called a local geodesic if for every $t \in I$ there exists a neighborhood $J$ of $t$ in $I$ such that the restriction $\gamma|_J : [c,d] \rightarrow Y$ is a geodesic. It is known that if $Y$ is geodesically complete then every geodesic $\gamma : [a,b] \rightarrow Y$
is a restriction of a local geodesic \( \tilde{\gamma} : \mathbb{R} \to Y \) defined on all real numbers (see [2 Corollary 9.1.28]). It is also known that on a CAT(0) space, every local geodesic is a geodesic (see [11, Chapter II, Proposition 1.4]). Thus we have the following.

**Proposition 2.3.** Let \( Y \) be a geodesically complete CAT(0) space. Then every geodesic \( \gamma : [a, b] \to Y \) is a restriction of a geodesic \( \tilde{\gamma} : \mathbb{R} \to Y \).

Suppose that \( Y \) is a CAT(0) space. Then by the definition of CAT(0) space, there is a unique geodesic joining any pair of points in \( Y \). Therefore for any triple of points \((p, q, r)\) in \( Y \), it makes sense to denote by \( \Delta(p, q, r) \) the geodesic triangle consisting of three geodesics joining each pair of the three points. Let \( \gamma : [a, b] \to Y \), \( \gamma' : [a', b'] \to Y \) be two geodesics on a CAT(0) space \( Y \) such that
\[
\gamma(a) = \gamma'(a') = p \in Y.
\]
We define the angle \( \angle_p(\gamma, \gamma') \) between \( \gamma, \gamma' \) as
\[
\angle_p(\gamma, \gamma') = \lim_{t \to a, t' \to a'} \angle_p^0(\gamma(t), \gamma'(t')),
\]
where \( \angle_p^0(\gamma(t), \gamma'(t')) \) is the corresponding angle of the comparison triangle of \( \Delta(p, \gamma(t), \gamma'(t')) \) in \( \mathbb{R}^2 \). The existence of the limit is guaranteed by the definition of CAT(0) space. The law of cosines on a Euclidean space yields
\[
\cos \angle_p(\gamma, \gamma') = \lim_{t \to a, t' \to a'} \frac{d_Y(p, \gamma(t))^2 + d_Y(p, \gamma'(t'))^2 - d_Y(\gamma(t), \gamma'(t'))^2}{2d_Y(p, \gamma(t))d_Y(p, \gamma'(t'))}.
\]

**Definition 2.4.** Let \((X, d_X)\) be a metric space. The cone \( \text{Cone}(X) \) over \( X \) is the quotient of the product \( X \times [0, \infty) \) obtained by identifying all points in \( X \times \{0\} \subset X \times [0, \infty) \). The point represented by \((x, 0)\) is called the origin of \( \text{Cone}(X) \) and we denote this point by \( O_{\text{Cone}(X)} \) in this paper. The cone distance \( d_{\text{Cone}(X)}(v, w) \) between two points \( v, w \in \text{Cone}(X) \) represented by \((x, t), (y, s) \in X \times [0, \infty) \) respectively, is defined by
\[
d_{\text{Cone}(X)}(v, w) = \sqrt{t^2 + s^2 - 2ts \cos(\min\{t, d_X(x, y)\})}.
\]
Then \((\text{Cone}(X), d_{\text{Cone}(X)})\) becomes a metric space. We call this metric space the Euclidean cone over \((X, d_X)\).

For an element \( v \in \text{Cone}(X) \) represented by \((x, r) \in X \times [0, \infty) \) and \( c > 0 \), we denote by \( cv \) the element represented by \((x, cr) \). We claim that
\[
d_{\text{Cone}(X)}(cv, cw) = cd_{\text{Cone}(X)}(v, w)
\]
holds for any \( v, w \in \text{Cone}(X) \)

**Definition 2.5.** Let \((Y, d_Y)\) be a CAT(0) space, and let \( p \in Y \). We denote by \((S_pY)° \) the quotient set of all nontrivial geodesics starting from \( p \) by the equivalence relation \( \sim \) defined by \( \gamma \sim \gamma' \iff \angle_p(\gamma, \gamma') = 0 \). Then the angle \( \angle_p \) induces a distance on \((S_pY)° \), which we denote by the same symbol \( \angle_p \). The space of directions \( S_pY \) at \( p \) is the metric completion of the metric space \((S_pY)°, \angle_p \). The tangent cone \( TC_pY \) of \( Y \) at \( p \) is the Euclidean cone \( \text{Cone}(S_pY) \) over the space of directions at \( p \). Define a map \( \pi_p : Y \to TC_pY \) by \( \pi_p(q) = ([\gamma], d_Y(p, q)) \) where \([\gamma]\) is the equivalence class represented by the unique geodesic \( \gamma \) joining \( p \) and \( q \).
It is easily seen that the map \( \pi_p \) defined above is 1-Lipschitz. It is also seen that each tangent cone \( TC_p Y \) is the metric completion of the Euclidean cone \( \text{Cone}((S_p Y)') \).

The CAT(0) condition is known to be preserved by the following operation.

**Definition 2.6.** Let \( (X_1, d_1), (X_2, d_2), \ldots \) be metric spaces with basepoints \( o_1 \in X_1, o_2 \in X_2, \ldots \), respectively. The \((\ell^2)-\)product \( X \) of \( X_1, X_2, \ldots \) with respect to the basepoints \( o_1, o_2, \ldots \) consists of all sequences \( (x_n)_n \) with \( x_n \in X_n \), satisfying \( \sum_n d_n(o_n, x_n)^2 < \infty \), and is equipped with the metric function \( d \) defined by

\[
d(x, y)^2 = \sum_{n=1}^{\infty} d_n(x_n, y_n)^2
\]

for any elements \( x = (x_1, x_2, \ldots) \in X \) and \( y = (y_1, y_2, \ldots) \in X \).

In this paper, a probability measure on a metric space is always finitely supported.

**Definition 2.7.** For a finitely supported probability measure \( \mu \) on a complete CAT(0) space \( (Y, d_Y) \), the point called the barycenter of \( \mu \) is defined to be the point which minimizes the function \( y \mapsto \int_Y d_Y(y, z)^2 \mu(dz) \). We denote the barycenter of \( \mu \) by \( \text{bar}(\mu) \).

The barycenter exists uniquely for any finitely supported probability measure \( \mu \) (see [11]). The following variance inequality holds (see [11, Proposition 4.4]).

**Proposition 2.8.** Let \( (Y, d_Y) \) be a complete CAT(0) space, and \( \mu \) be a finitely supported probability measure on \( Y \). Then we have

\[
\int_Y \{d_Y(y, z)^2 - d_Y(\text{bar}(\mu), z)^2\} \mu(dz) \geq d_Y(\text{bar}(\mu), y)^2
\]

for any \( y \in Y \).

A finitely supported probability measure \( \mu \) on a metric space \( Y \) is often written in the form

\[
\mu = \sum_{i=1}^{m} t_i \text{Dirac}_{y_i},
\]

where \( \text{Dirac}_{y_i} \) is the Dirac measure at \( y_i \in Y \) and each \( t_i \) is the weight \( \mu(\{y_i\}) \) at \( y_i \).

3. Izeki-Nayatani Invariant

In this section, we recall the definition of the Izeki-Nayatani invariant \( \delta \) and its basic properties.

**Definition 3.1** (Izeki-Nayatani [7]). Let \( Y \) be a complete CAT(0) space containing at least two points, and \( \mathcal{P}(Y) \) be the space of all finitely supported probability measures \( \mu \) on \( Y \) whose supports \( \text{supp}(\mu) \) contain at least two points. For \( \mu \in \mathcal{P}(Y) \), we define

\[
0 \leq \delta(\mu) = \inf_{\phi: \text{supp}(\mu) \rightarrow \mathcal{H}} \| \int_Y \phi(p)\mu(dp) \|^2 \leq 1,
\]
where the infimum is taken over all maps $\phi : \text{supp}(\mu) \to \mathcal{H}$ with $\mathcal{H}$ a Hilbert space such that
\begin{align}
\|\phi(p)\| &= d(p, \text{bar}(\mu)), \\
\|\phi(p) - \phi(q)\| &\leq d(p, q) \tag{3.2}
\end{align}
for all $p, q \in \text{supp}(\mu)$. We call such a map $\phi$ a realization of $\mu$. Notice that a realization $\phi$ of $\mu$ always exists. To see that, fix a unit vector $e \in \mathcal{H}$. Define $\phi(p) = d(p, \text{bar}(\mu))e$. Then by the triangle inequality, (3.2) is satisfied. We define the Izeki-Nayatani invariant $\delta(Y)$ of $Y$ as
\[0 \leq \delta(Y) = \sup_{\mu \in \mathcal{P}(Y)} \delta(\mu) \leq 1.\]

Followings are examples of spaces for which we know some estimations of the Izeki-Nayatani invariant:

- Assume that $Y$ is a finite or infinite dimensional Hadamard manifold or an $\mathbb{R}$-tree. Then we have $\delta(Y) = 0$.
- Assume that $Y_p$ is the building $\text{PSL}(3, \mathbb{Q}_p)/\text{PSL}(3, \mathbb{Z}_p)$. Then $\delta(Y_p) \geq \frac{(\sqrt{p} - 1)^2}{2(p - \sqrt{p} + 1)}$. When $p = 2$, we have $\delta(Y_2) \leq 0.4122\ldots$
- Assume that $Y$ is any complete CAT(0) cube complex. Then $\delta(Y) \leq \frac{1}{2}$.

The first two examples are estimated by Izeki and Nayatani \cite{7}, and the last one is estimated by Fujiwara and the present author \cite{3}. But generally computation or estimation of the Izeki-Nayatani invariant is difficult.

As we explain in Section 4, it is important to estimate $\delta$ from above by a constant less than 1. Recently, Kondo \cite{8} showed the existence of CAT(0) spaces with $\delta = 1$. In \cite{13}, the present author obtained a geometric condition for a complete CAT(0) space $Y$ to be $\delta(Y) < 1$ (see Section 5), which plays a central role in this paper.

Although the Izeki-Nayatani invariant is defined as a global invariant of the space, it can be estimated by the local property of the space. To see this, we define the following notation, which is introduced in \cite{7}.

**Definition 3.2 (Izeki-Nayatani \cite{7}).** Let $Y$ be a complete CAT(0) space, and $p \in Y$. We define $\delta(Y, p) \in [0, 1]$ to be
\[\delta(Y, p) = \sup \{\delta(\nu) \mid \nu \in \mathcal{P}(Y), \text{bar}(\nu) = p\},\]
where $\mathcal{P}(Y)$ is the space of all finitely supported probability measures on $Y$ whose supports contain at least two points. If no such $\nu$ exists, we define $\delta(Y, p) = -\infty$.

The following proposition is basic and important. However, there are no references containing its complete proof. So we present its detailed proof here.

**Proposition 3.3.** Suppose that $Y$ is a complete CAT(0) space. Then we have
\[\delta(Y) = \sup \{\delta(TC_pY, O) \mid p \in Y\} = \sup \{\delta(TC_pY) \mid p \in Y\},\]
where $O$ denotes the origin of the tangent cone $TC_pY$.

We use the following lemma to prove Proposition 3.3.
**Lemma 3.4.** Let $m \geq 2$ be a positive integer, $t_1, t_2, \ldots, t_m \geq 0$ be positive real numbers with $\sum_{i=1}^{m} t_i = 1$, and $\{\alpha_n\}$ be a sequence of positive real numbers. Let $\nu = \sum_{i=1}^{m} t_i \delta_{q_i}$ be an $m$-point supported probability measure on a complete CAT(0) space $(Y, d)$. For each $n \in \mathbb{N}$, let $(Y_n, d_n)$ be a complete CAT(0) space, and $\nu^{(n)} = \sum_{i=1}^{m} t_i \delta_{q_i^{(n)}}$ be an $m$-point supported probability measure on $Y_n$. If we have

$$\lim_{n \to \infty} \alpha_n d_n \left( q_i^{(n)}, q_j^{(n)} \right) = d(q_i, q_j), \quad \lim_{n \to \infty} \alpha_n d_n \left( q_i^{(n)}, \text{bar}(\nu^{(n)}) \right) = d(q_i, \text{bar}(\nu))$$

for any $1 \leq i, j \leq m$, then we have $\lim_{n \to \infty} \delta(\nu^{(n)}) = \delta(\nu)$.

We can prove this lemma by exactly the same argument as in the proof of Theorem 3.1 of [12], where the continuity of $\delta$ on the space of finitely supported probability measures with respect to a certain topology is established. Here we just present the proof of the inequality

$$\lim_{n \to \infty} \delta(\nu^{(n)}) \leq \delta(\nu).$$

This inequality suffices to prove Proposition 3.3. The opposite inequality also follows in the similar manner, but we omit its description here for avoiding tediousness.

**Proof.** For two finitely supported probability measures $\mu = \sum_{i=1}^{m} t_i \delta_{p_i}$ and $\mu' = \sum_{i=1}^{m} t_i \delta_{p_i'}$ on possibly different complete CAT(0) spaces, $(Y, d)$ and $(Y', d')$, respectively, it is immediate from the definition of $\delta$ that we have $\delta(\mu) = \delta(\mu')$ if there exists some constant $C > 0$ such that

$$d(p_i, p_j) = Cd'(p'_i, p'_j), \quad d(p_i, \text{bar}(\mu)) = Cd'(p'_i, \text{bar}(\mu')).$$

for any $1 \leq i, j \leq m$. So it suffices to prove only the case when $\alpha_n = 1$ for all $n$.

Let $L_{\nu}$ be the smallest positive value of

$$d(p, q)^2 - \{d(p, \text{bar}(\nu)) - d(q, \text{bar}(\nu))\}^2$$

for all $p, q \in \text{supp}(\nu)$. By a triangle inequality, the value of (3.4) is positive when $p$ is not on a geodesic joining bar($\nu$) to $q$ and $q$ is not on a geodesic joining bar($\nu$) to $p$. And such $p, q \in \text{supp}(\nu)$ should exist by the definition of barycenter. We define $L_{\nu^{(n)}}$ in the same manner for each $n \in \mathbb{N}$.

We assume that $n$ is large enough such that $L_{\nu^{(n)}} \geq \frac{1}{2} L_{\nu}$, and

$$d_n(q_i^{(n)}, q_j^{(n)})^2 - \{d_n(q_i^{(n)}, \text{bar}(\nu^{(n)}) - d_n(q_i^{(n)}, \text{bar}(\nu^{(n)}))\}^2 > 0$$

whenever

$$d(q_i, q_j)^2 - \{d(q_i, \text{bar}(\nu)) - d(q_i, \text{bar}(\nu))\}^2 > 0.$$

Let $\phi : \text{supp}(\nu) \to \mathcal{H}$ be an arbitrary realization of $\nu$. Define a map $\phi^{(n)} : \text{supp}(\nu^{(n)}) \to \mathcal{H}$ by declaring $\phi(q_i^{(n)}) = 0$ if $q_i = \text{bar}(\nu)$, and

$$\phi^{(n)}(q_i^{(n)}) = \frac{d_n(\text{bar}(\nu^{(n)}), q_i^{(n)})}{d(\text{bar}(\nu), q_i)} \phi(q_i)$$

otherwise. Let

$$R^{(n)} = \max \left\{ \|\phi^{(n)}(q_i^{(n)}) - \phi^{(n)}(q_j^{(n)})\|^2 - d_Y(q_i^{(n)}, q_j^{(n)})^2 \mid 0 \leq i, j \leq n \right\}.$$
We have $R^{(n)} \geq 0$ since the value is 0 if $i = j$. By the assumption of the lemma and the property of a realization, $R^{(n)}$ tends to 0 when $n$ goes to $\infty$.

Define $\tilde{\phi}^{(n)} : \text{supp}(\nu^{(n)}) \to H \oplus \mathbb{R}$ by declaring

$$\tilde{\phi}^{(n)}(q_i^{(n)}) = \left( \cos \theta \phi(q_i^{(n)}), \sin \theta d_n(\text{bar}(\nu^{(n)}), q_i^{(n)}) \right) \in H \oplus \mathbb{R}$$

for each $i$, where $\theta = \sin^{-1} \sqrt{\frac{2R^{(n)}}{L^2}}$.

Now we will show that $\tilde{\phi}^{(n)}(q_i^{(n)})$ is a realization of $\mu_n$. The condition (3.1) in Definition 3.1 is trivially holds. So we will show the 1-Lipschitz condition

$$\|\tilde{\phi}^{(n)}(q_i^{(n)}) - \tilde{\phi}^{(n)}(q_j^{(n)})\| \leq d_n\left(q_i^{(n)}, q_j^{(n)}\right),$$

by considering three cases separately. First, we consider the case when

$$d_n(q_i^{(n)}, q_j^{(n)})^2 - \{d_n(q_i^{(n)}, \text{bar}(\nu^{(n)})) - d_n(q_j^{(n)}, \text{bar}(\nu^{(n)}))\}^2 = 0.$$

In this case, we also have

$$d(q_i, q_j)^2 - \{d(q_i, \text{bar}(\nu)) - d(q_j, \text{bar}(\nu))\}^2 = 0$$

by an assumption. Thus an inequality (3.5) holds as equality. The second case is when

$$d_n(q_i^{(n)}, q_j^{(n)})^2 - \{d_n(q_i^{(n)}, \text{bar}(\nu^{(n)})) - d_n(q_j^{(n)}, \text{bar}(\nu^{(n)}))\}^2 \neq 0,$$

and

$$\|\phi^{(n)}(q_i^{(n)}) - \phi^{(n)}(q_j^{(n)})\|^2 - \{d_n(q_i^{(n)}, \text{bar}(\nu^{(n)})) - d_n(q_j^{(n)}, \text{bar}(\nu^{(n)}))\}^2 \leq L_{\nu^{(n)}}$$

holds. In this case, we have

$$\|\tilde{\phi}^{(n)}(q_i^{(n)}) - \tilde{\phi}^{(n)}(q_j^{(n)})\| \leq \|\phi^{(n)}(q_i^{(n)}) - \phi^{(n)}(q_j^{(n)})\| \leq d_n\left(q_i^{(n)}, q_j^{(n)}\right)$$

by the definition of $L_{\nu^{(n)}}$. The final case is when

$$\|\phi^{(n)}(q_i^{(n)}) - \phi^{(n)}(q_j^{(n)})\|^2 - \{d(q_i^{(n)}, \text{bar}(\nu^{(n)})) - d(q_j^{(n)}, \text{bar}(\nu^{(n)}))\}^2 > L_{\nu^{(n)}}.$$

In this case, we have

$$\|\phi^{(n)}(q_i^{(n)}) - \phi^{(n)}(q_j^{(n)})\|^2 - \|\tilde{\phi}^{(n)}(q_i^{(n)}) - \tilde{\phi}^{(n)}(q_j^{(n)})\|^2$$

$$= \sin^2 \theta \left\{ \|\phi^{(n)}(q_i^{(n)}) - \phi^{(n)}(q_j^{(n)})\|^2 - \left(\|\phi^{(n)}(q_i^{(n)})\| - \|\phi^{(n)}(q_j^{(n)})\|\right)^2 \right\}$$

$$= \frac{2R^{(n)}}{L^2} \left\{ \|\phi^{(n)}(q_i^{(n)}) - \phi^{(n)}(q_j^{(n)})\|^2 - \{d_n(q_i^{(n)}, \text{bar}(\nu^{(n)})) - d_n(q_j^{(n)}, \text{bar}(\nu^{(n)}))\}^2 \right\}$$

$$\geq \frac{R^{(n)}}{L^2} \left\{ \|\phi^{(n)}(q_i^{(n)}) - \phi^{(n)}(q_j^{(n)})\|^2 - \{d_n(q_i^{(n)}, \text{bar}(\nu^{(n)})) - d_n(q_j^{(n)}, \text{bar}(\nu^{(n)}))\}^2 \right\}$$

$$\geq R^{(n)}.$$

Thus the inequality (3.5) also holds. Hence we have proved that $\tilde{\phi}^{(n)}$ is a realization of $\mu^{(n)}$. 
The value of
\[ \left\| \int_{Y} \tilde{\phi}(p) \nu(p) (dp) \right\|^2 / \int_{Y} \|\tilde{\phi}(p)\|^2 \nu(p) (dp) \]
tends to
\[ \left\| \int_{Y} \phi(p) \nu(p) (dp) \right\|^2 / \int_{Y} \|\phi(p)\|^2 \nu(p) (dp) \]
as \( n \) goes to \( \infty \) because \( \lim_{n \to \infty} \tilde{\phi}(\tilde{q}_i^n) = \phi(q_i) \) for each \( i \). Since a realization \( \phi \) is taken arbitrarily, we have proved that
\[ \lim_{n \to \infty} \delta (\nu^n) \leq \delta (\nu). \]
□

**Proof of Lemma 3.3.** The inequality
\[ \delta (Y) \leq \sup \{ \delta (TC_p Y, O) \mid p \in Y \} \]
was proved in [7, Lemma 6.2], and the inequality
\[ \sup \{ \delta (TC_p Y, O) \mid p \in Y \} \leq \sup \{ \delta (TC_p Y) \mid p \in Y \} \]
is trivial from the definition. So we need only to prove the inequality
\[ \text{(3.6)} \sup \{ \delta (TC_p Y) \mid p \in Y \} \leq \delta (Y). \]
To this end, it suffices to show that \( \delta (TC_p Y) \leq \delta (Y) \) for any \( p \in Y \). Let \( \nu' \) be an arbitrary finitely supported probability measure on \( TC_p Y \) whose support contains at least two points. Since \( \text{Cone} ((S_p Y)^{\circ}) \) is dense in \( TC_p Y \), there exists a finitely supported probability measure \( \nu \) on \( \text{Cone} ((S_p Y)^{\circ}) \) such that \( \delta (\nu) \) is sufficiently close to \( \delta (\nu') \) by the continuity of \( \delta \) established in Theorem 3.1 of [12] (or Lemma 3.4 in the present paper). Set
\[ \nu = \sum_{i=1}^{m} t_i \text{Dirac}_{v_i}, \]
and
\[ v_0 = \text{bar}(\nu). \]
For each \( i = 0, \ldots, m \), put \( v_i = ([\gamma_i], r_i) \), where \( \gamma_i : [0, l_i] \to Y \) is a geodesic starting from \( p \) and \([\gamma_i] \) denotes the direction represented by \( \gamma_i \).

We assume that \( n \) is large enough such that \( \frac{1}{n} r_i \leq l_i \) for all \( i = 0, \ldots, m \). For every \( n \in \mathbb{N} \) and every \( i = 0, \ldots, m \), we define a point \( p_i^{(n)} \in Y \) by
\[ p_i^{(n)} = \gamma_i \left( \frac{1}{n} r_i \right). \]
We define a probability measure \( \mu_n \) on \( Y \) to be
\[ \mu_n = \sum_{i=1}^{m} t_i \text{Dirac}_{p_i^{(n)}}. \]
Now it suffices to show that \( \delta(\nu) = \lim_{n \to \infty} \delta(\mu_n) \). First, by the definition of the distance on a tangent cone, we have
\[
(3.7) \quad \lim_{n \to \infty} \left| nd\left( p_i^{(n)}, p_j^{(n)} \right) - d(v_i, v_j) \right| = 0
\]
for each \( 0 \leq i, j \leq m \). Then it follows that
\[
(3.8) \quad \lim_{n \to \infty} \left| n^2 \int_Y d(p_0^{(n)}, y)^2 \mu_n(dy) - \int_{TC_pY} d(v_0, v)^2 \nu(dv) \right| = 0.
\]
Put \( v_0^{(n)} = n\pi_p(\text{bar}(\mu_n)) \in TC_pY \) for each \( n \). Then we have
\[
\begin{align*}
n^2 \left( \int_Y d(p_0^{(n)}, y)^2 \mu_n(dy) - \int_Y d(\text{bar}(\mu_n), y)^2 \mu_n(dy) \right) \\
\leq n^2 \int_Y d(p_0^{(n)}, y)^2 \mu_n(dy) - \int_{TC_pY} d(v_0^{(n)}, v)^2 \nu(dv) \\
\leq \left( n^2 \int_Y d(p_0^{(n)}, y)^2 \mu_n(dy) - \int_{TC_pY} d(v_0^{(n)}, v)^2 \nu(dv) \right) \\
+ \left( \int_{TC_pY} d(v_0, v)^2 \nu(dv) - \int_{TC_pY} d(v_0^{(n)}, v)^2 \nu(dv) \right) \\
\leq n^2 \int_Y d(p_0^{(n)}, y)^2 \mu_n(dy) - \int_{TC_pY} d(v_0, v)^2 \nu(dv).
\end{align*}
\]
The first inequality follows from the fact that the map \( q \mapsto n\pi_p(q) \) is \( n \)-Lipschitz. The third inequality follows from \( v_0 = \text{bar}(\nu) \). Since the right-hand side of the above inequality tends to 0 by \((3.8)\), Proposition 2.8 yields
\[
\lim_{n \to \infty} nd\left( \text{bar}(\mu_n), p_0^{(n)} \right) = 0.
\]
Thus, since
\[
(3.9) \quad \left| nd\left( \text{bar}(\mu_n), p_i^{(n)} \right) - d(v_0, v_i) \right| \\
\leq \left| nd\left( \text{bar}(\mu_n), p_i^{(n)} \right) - nd\left( p_0^{(n)}, p_i^{(n)} \right) \right| + \left| nd\left( p_0^{(n)}, p_i^{(n)} \right) - d(v_0, v_i) \right| \\
\leq nd\left( \text{bar}(\mu_n), p_0^{(n)} \right) + \left| nd\left( p_0^{(n)}, p_i^{(n)} \right) - d(v_0, v_i) \right| ,
\]
we have
\[
\lim_{n \to \infty} nd\left( \text{bar}(\mu_n), p_i^{(n)} \right) = d(\text{bar}(\nu), v_i).
\]
for all \( i = 1, \ldots, m \). We also have
\[
\lim_{n \to \infty} nd\left( p_i^{(n)}, p_j^{(n)} \right) = d(v_i, v_j)
\]
for all \( 1 \leq i, j \leq m \) by \((3.7)\).
These inequalities yield
\[
\delta(\nu) = \lim_{n \to \infty} \delta(\mu_n)
\]
by Lemma 3.4.

\[ \square \]

\textbf{Remark 3.5.} If we use the notion of ultralimit, the proof of Proposition 3.3 becomes very simple. By Proposition 4.2 of [3], if \( \{Y_n\}_{n \in \mathbb{N}} \) is a sequence of complete CAT(0) spaces, \( \omega \) is a nonprincipal ultralimit on \( N \), and \( Y_\omega \) is the ultralimit of \( \{Y_n\} \) with respect to \( \omega \), then \( \delta(Y_\omega) \leq \omega\text{-}\lim_n \delta(Y_n) \) holds. Combining this with Proposition 7.6 in Section 7, the inequality (3.6) follows immediately.

The following is a basic property of \( \delta \). It is a slight generalization of Proposition 6.5 of [7] and quite similar to Lemma 4.3 of [13]. However, we include its proof for the sake of completeness.

\textbf{Proposition 3.6.} Let \( Y_1, Y_2, Y_3, \ldots \) be complete CAT(0) spaces. Let \( Y \) be a product of \( Y_1, Y_2, Y_3, \ldots \) (with respect to some basepoints). Then we have

\[ \delta(Y) = \sup \{ \delta(Y_i) \mid i = 1, 2, 3, \ldots \}. \]

\textbf{Proof.} The inequality \( \delta(Y) \geq \sup \{ \delta(Y_i) \mid i = 1, 2, 3, \ldots \} \) is obvious since every \( Y_i \) is isometrically embedded into \( Y \). Let \( \mu = \sum_{i=1}^{m} t_i \text{Dirac}_{p_i} \) be an arbitrary probability measure on \( Y \) whose support contains at least two points. For each \( i \), set \( p_i = (p_i^{(1)}, p_i^{(2)}, p_i^{(3)}, \ldots) \), where \( p_i^{(n)} \in Y_n \) for each \( n \). Define a probability measure \( \mu_n \) on \( Y_n \) to be

\[ \mu_n = \sum_{i=1}^{m} t_i \text{Dirac}_{p_i(n)} \]

for each \( n \). Set \( \text{bar}(\mu) = (b_1, b_2, b_3, \ldots) \), where \( b_n \in Y_n \) for each \( n \). Then we have \( \text{bar}(\mu_n) = b_n \) for every \( n \). That is because if we had \( \text{bar}(\mu_n) \neq b_n \) for some \( n \), then it would follow that

\[ \int_Y d_Y(p, B)^2 \mu(dp) < \int_Y d_Y(p, \text{bar}(\mu))^2 \mu(dp), \]

where \( B \) is a point on \( Y \) such that its \( n \)-th component is \( b_n \) and its \( i \)-th component is \( \text{bar}(\mu_i) \) for every \( i \neq n \).

For each \( n \), let \( \phi_n : \text{supp}(\mu_n) \to \mathcal{H}_n \) be a realization of \( \mu_n \) with

\[ \delta(\mu_n) = \frac{\| \int_{Y_n} \phi_n(p) \mu_n(dp) \|^2}{\int_{Y_n} \| \phi_n(p) \|^2 \mu_n(dp)}. \]

Existence of such \( \phi_n \) follows from the compactness of the space of all realization of \( \mu_n \). Define a map \( \phi : \text{supp}(\mu) \to \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \cdots \) as

\[ \phi(p_i) = \left( \phi_1(p_i^{(1)}), \phi_2(p_i^{(2)}), \phi_3(p_i^{(3)}), \ldots \right), \quad i = 1, \ldots, m. \]

It is easily seen that \( \phi \) is a realization of \( \mu \). And it follows that

\[ \delta(\mu) \leq \frac{\| \int_Y \phi(p) \mu(dp) \|^2}{\int_Y \| \phi(p) \|^2 \mu(dp)} = \sum_{n=1}^{\infty} \frac{\| \sum_{i=1}^{m} t_i \phi_1(p_i^{(n)}) \|^2}{\sum_{n=1}^{\infty} \sum_{i=1}^{m} t_i \| \phi_1(p_i^{(n)}) \|^2} \leq \sup_n \frac{\| \sum_{i=1}^{m} t_i \phi_1(p_i^{(n)}) \|^2}{\sum_{i=1}^{m} t_i \| \phi_1(p_i^{(n)}) \|^2} \leq \sup_n \delta(\mu_n). \]
Thus the inequality \( \delta(Y) \leq \sup \{ \delta(Y_i) \mid i = 1, 2, 3, \ldots \} \) follows since \( \mu \) is taken arbitrarily.

\[ \square \]

4. Two Consequences of \( \delta < 1 \)

In this section, we explain two consequences of the condition \( \delta(Y) < 1 \) for a complete CAT(0) space \( Y \). One is the fact mentioned by Kondo \[8\] that a sequence of expanders does not embed coarsely into a complete CAT(0) space \( Y \) with \( \delta(Y) < 1 \). The other is a theorem due to Izeki, Kondo and Nayatani which states that a random group of the graph model has a common fixed point when it acts isometrically on a complete CAT(0) space \( Y \) with \( \delta(Y) < 1 \).

For a finite graph \( G \) and a complete CAT(0) space \( Y \), Wang \[14\] defined the following analog of \( \lambda_1(G) \).

**Definition 4.1.** Let \( G = (V, E) \) be a finite graph, and \( Y \) be a complete CAT(0) space. We assume that \( Y \) contains at least two points. The Wang’s invariant \( \lambda_1(G, Y) \) is defined by

\[
\lambda_1(G, Y) = \inf_{\phi} \frac{\sum_{(u, v) \in E} d_Y(\phi(u), \phi(v))^2}{\sum_{v \in V} \deg(v) d_Y(\phi(v), \phi)^2},
\]

where the infimum is taken over all nonconstant maps \( \phi : V \to Y \), and \( \overline{\phi} \) denotes the barycenter of the probability measure \( \sum_{v \in V} \frac{\deg(v)}{2|E|} \text{Dirac}_{\phi(v)} \) on \( Y \).

If we see \( \mathbb{R} \) as a CAT(0) space, \( \lambda_1(G) = \lambda_1(G, \mathbb{R}) \) holds. If we take a Hilbert space \( \mathcal{H} \), it is not hard to show from the definition \( \lambda_1(G, \mathcal{H}) = \lambda_1(G) \). Originally, the invariant \( \delta(Y) \) was introduced to give an estimate of \( \lambda_1(G, Y) \).

**Theorem 4.2 (Izeki-Nayatani \[7, Proposition 6.3\]).** Let \( G \) be a finite graph and \( Y \) a complete CAT(0) space. Then we have

\[
(1 - \delta(Y))\lambda_1(G) \leq \lambda_1(G, Y) \leq \lambda_1(G).
\]

In \[4\], Gromov showed that a sequence of expanders does not embed coarsely into Hilbert spaces, and it is straightforward to see that his argument applies to the following generalized form (see \[8\] or \[3\]).

**Proposition 4.3.** Let \( Y \) be a complete CAT(0) space. Suppose that sequence \( \{ G_n = (V_n, E_n) \} \) of finite graphs satisfies the following properties:

1. The number of vertices of \( G_n \) goes to infinity as \( n \) goes to infinity.
2. There exists \( d \) such that \( \deg(v) \leq d \) for all \( v \in V_n \) and all \( n \), where \( \deg(v) \) is the number of edges at vertex \( v \).
3. There exists \( \lambda > 0 \) such that \( \lambda_1(G_n, Y) \geq \lambda \) for all \( n \).

Then the sequence \( \{ G_n = (V_n, E_n) \} \) does not embed coarsely into \( Y \).

A sequence of graphs as in the above proposition is one we may call a sequence of “\( Y \)-expanders”. Theorem \[1,2\] guarantees that if a complete CAT(0) space \( Y \) satisfies \( \delta(Y) < 1 \), then an ordinary sequence of expanders is also a sequence of “\( Y \)-expanders”. Thus the following theorem follows from Proposition \[1,3\] as observed in \[8\].
**Theorem 4.4.** If a complete CAT(0) space \( Y \) satisfies \( \delta(Y) < 1 \), any sequence of expanders does not embed coarsely into \( Y \).

Now we proceed to the other consequence. In [7], [5] and [6], Izeki, Kondo and Nayatani have proved that certain classes of groups must have fixed points when they isometrically act on a complete CAT(0) space \( Y \) whenever \( \delta(Y) \) is bounded from above by the corresponding constants. Among these, they proved the following.

**Theorem 4.5 (Izeki-Kondo-Nayatani [6]).** Let \( 0 \leq C < 1 \). If \( \{G_n = (V_n, E_n)\} \) is a sequence of expanders, \( 2 \leq \deg(u) \leq d \) for all \( u \in V_n \) and all \( n \), and the girth of \( G_n \) is large enough, then with high probability, any isometric action of the random group \( \Gamma(G_n) \) on a complete CAT(0) space \( Y \) with \( \delta(Y) \leq C \) has a common fixed point.

Formally they showed that given \( k \in \mathbb{N}, d \in \mathbb{N}, \lambda > 0 \), there exists an explicit constant \( g = g(k, \lambda) \) such that if \( \{G_n = (V_n, E_n)\} \) is a sequence of expanders such that for all \( n, \lambda \leq \lambda_1(G_n), \) the girth of \( G_n \) is at least \( g \), and \( 3 \leq \deg(u) \leq d \) for all \( u \in V_n \), then the probability for the random group \( \Gamma(G_n) \) generated by \( k \) elements to have a common fixed point when it acts on a complete CAT(0) space with \( \delta(Y) < 1 \) is at least \( 1 - ae^{-b|V_n|} \), where \( a, b \) are explicit and only depend on the parameters \( k, d \) and \( \lambda \). The statements in Corollary 1.5 and Corollary 1.10 can be understood similarly.

5. A criterion for estimating the Izeki-Nayatani invariant

In this section, we present a criterion for a complete CAT(0) space \( Y \) to be \( \delta(Y) < 1 \), which is obtained in the author’s previous paper [13]. To describe the criterion, we prepare the following property for metric spaces.

**Definition 5.1.** Let \( 0 < \theta < \frac{\pi}{2}, 0 < \alpha < 1 \) and \( \varepsilon > 0 \). Let \((X, d)\) be a metric space. We say that \( X \) has the property \( P(\theta, \alpha, \varepsilon) \) if there exists a finite subset \( S \subset X \) such that

\[
\# \{ s \in S \mid \|d(x, s) - d(y, s)\| \geq \varepsilon \} \geq \alpha \#S
\]

holds for all \( x, y \in X \) with \( d(x, y) \geq \theta \). Here, \( \#S \) is the cardinality of the subset \( S \).

**Theorem 5.2.** Let \( 0 < \theta < \frac{\pi}{2}, 0 < \alpha < 1 \) and \( \varepsilon > 0 \). Suppose that \( Y \) is a CAT(0) space such that each of its tangent cone \( TC_pY \) is isometric to a (finite or infinite) product of the Euclidean cones over metric spaces each of which has the property \( P(\theta, \alpha, \varepsilon) \). Then there exists a constant \( C(\theta, \alpha, \varepsilon) < 1 \) depending only on \( \theta, \alpha \) and \( \varepsilon \) such that

\[
\delta(Y) \leq C(\theta, \alpha, \varepsilon).
\]

This theorem is just a reformulation of Proposition 5.4 in the author’s previous paper [13]. So we do not repeat its proof here.

**Lemma 5.3.** Let \( \mathcal{X} \) be a Gromov-Hausdorff precompact family of metric spaces. Then there exist some constants \( 0 < \theta < \frac{\pi}{2}, 0 < \alpha < 1 \) and \( \varepsilon > 0 \) such that any \( X \in \mathcal{X} \) satisfies the property \( P(\theta, \alpha, \varepsilon) \).
Before proving this lemma, recall that the Gromov-Hausdorff precompactness is known to be equivalent to the uniform total boundedness.

**Definition 5.4.** The family $\mathcal{X}$ of metric spaces is known to be uniformly totally bounded if and only if the following two conditions are satisfied:

1. There is a constant $D$ such that $\text{diam}(X) \leq D$ for all $X \in \mathcal{X}$.
2. For any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that each $X \in \mathcal{X}$ contains a subset $S_{X,\varepsilon}$ with the following property: the cardinality of $S_{X,\varepsilon}$ is no greater than $N(\varepsilon)$ and $X$ is covered by the union of all open $\varepsilon$-balls whose centers are in $S_{X,\varepsilon}$.

**Proof of Lemma 5.3.** $\mathcal{X}$ is uniformly totally bounded since it is Gromov-Hausdorff precompact. Thus, there exists a positive integer $N$ such that each $X \in \mathcal{X}$ contains a subset $S_X \subset X$ with the following property: the cardinality of $S_X$ is no greater than $N$ and $X$ is covered by the union of all open $\varepsilon$-balls whose centers are in $S_X$.

By the definition of the subset $S_X$, for any $x, y \in X$ with $d_X(x, y) \geq \frac{\pi}{3}$, there exist $s_0, s_1 \in S_X$ such that

$$d_X(s_0, x) \geq \frac{\pi}{4}, \quad d_X(s_0, y) \leq \frac{\pi}{12},$$

$$d_X(s_1, y) \geq \frac{\pi}{4}, \quad d_X(s_1, x) \leq \frac{\pi}{12}.$$ 

Hence, there exist two distinct elements $s_0, s_1 \in S$ such that

$$\|d_X(x, s_0) - d_X(y, s_0)\| \geq \frac{\pi}{6},$$

$$\|d_X(x, s_1) - d_X(y, s_1)\| \geq \frac{\pi}{6},$$

for any $x, y \in X$ with $d_X(x, y) \geq \frac{\pi}{3}$. Thus each $X \in \mathcal{X}$ has the property $P(\frac{\pi}{3}, \frac{2}{N}, \frac{\pi}{6}).$ $\square$

### 6. CAT(0) Spaces which Admit Proper Cocompact Group Actions

In this section, we prove Theorem 1.2. Our proof consists of two lemmas.

**Lemma 6.1.** Let $Y$ be a geodesically complete CAT(0) space. If there exists a positive real number $r > 0$ such that the family $\{B(p, r)\}_{p \in Y}$ consisting of all $r$-balls in $Y$ is Gromov-Hausdorff precompact, then the family $\{S_p Y\}_{p \in Y}$ consisting of all spaces of directions is also Gromov-Hausdorff precompact.

**Proof.** Let $p \in Y$ be an arbitrary point on $Y$. We denote the canonical inclusion of $S_p Y$ into $TC_p Y$ by $\iota$. Then it is straightforward from the definition of the metric on Euclidean cones that we have

$$\frac{2}{\pi} d_S(x, y) \leq d_T(\iota(x), \iota(y)) \leq d_S(x, y) \quad \text{(6.1)}$$

for all $x, y \in S_p Y$, where $d_S$ and $d_T$ represent the distance function of $S_p Y$ and $TC_p Y$, respectively.

Fix some $0 < r' < r$. By the assumption, the family $\{B(p, r')\}_{p \in Y}$ is uniformly totally bounded. Hence, for any $\varepsilon > 0$, there exists a positive integer $N$ which is
independent of $p$ such that each $B(p, r)$ is covered by $N$ open balls of radius $\frac{r' \varepsilon}{2\pi}$. Then the metric sphere

$$S(p, r') = \{ q \in Y \mid d_Y(p, q) = r' \} \subset B(p, r)$$

is also covered by $N$ open balls of radius $\frac{r' \varepsilon}{2\pi}$ in $Y$.

Let us denote by $F : TC_pY \rightarrow TC_pY$ the map associating each element of $TC_pY$ represented by $(x, t) \in S_pY \times [0, \infty)$ to the element represented by $(x, \frac{1}{2}t)$. This map clearly satisfies

$$d_T(F(v), F(w)) = \frac{1}{r'} d_T(v, w)$$

for all $v, w \in TC_pY$.

Now, we have $F \circ \pi_p(S(p, r')) \subset i(S_pY)$, where $\pi_p : Y \rightarrow TC_pY$ is the $1$-Lipschitz map defined in Definition 2.5 and by (6.2), it can be covered by $N$ open balls of radius $\frac{\varepsilon}{2\pi}$ in $TC_pY$.

Since each geodesic starting from $p$ can be extended up to $S(p, r')$ by geodesic completeness of $Y$, $F \circ \pi_p(S(p, r'))$ is no other than $i((S_pY)^\circ)$. Hence $F \circ \pi_p(S(p, r'))$ is dense in $i(S_pY)$, and we see that $i(S_pY)$ is covered by $N$ open balls of radius $\frac{\varepsilon}{2\pi}$ in $TC_pY$. Let us denote these balls by $B_1, B_2, \ldots, B_N$. Then $\{i^{-1}(B_i)\}_{i=1}^N$ covers $S_pY$. By (6.1), each $i^{-1}(B_i)$ has diameter less than $\varepsilon$. Hence we can choose $N$ balls of radius $\varepsilon$ in $S_pY$, which cover $S_pY$. Since $\varepsilon > 0$ is arbitrary, we have proved that $\{S_pY\}_{p \in Y}$ is uniformly totally bounded. Hence it is Gromov-Hausdorff precompact. \hfill \qed

**Lemma 6.2.** Let $Y$ be a metric space. Assume that some group $\Gamma$ acts on $Y$ properly and cocompactly by isometries. Then there exists some positive real number $r > 0$ such that the family $\{B(p, r)\}_{p \in Y}$ consisting of all $r$-balls in $Y$ is a Gromov-Hausdorff precompact family of metric spaces.

**Proof.** Since $\Gamma$ acts on $Y$ cocompactly, there exists a compact subset $K \subset Y$ such that $K \cup_{\gamma \in \Gamma} \gamma K = Y$. Since $\Gamma$ acts on $Y$ properly, for any $x \in K$, there is $r_x > 0$ such that the cardinality of the set $\{\gamma \in \Gamma \mid \gamma B(x, 2r_x) \cap B(x, 2r_x) \neq \emptyset\}$ is finite. Let $\{B(x_i, r_i)\}_{i=1}^N$ be one of finite subcovers of the open cover $\{B(x, r_x)\}_{x \in K}$ of $K$.

Though it is a well-known fact, we first observe that $Y$ should be locally compact. Let $y \in Y$ be an arbitrary point, and let $r_0 = \min\{r_1, r_2, \ldots, r_N\}$. Observe that if there are infinite elements $\gamma \in \Gamma$ with $B(y, r_0) \cap \gamma K \neq \emptyset$, then there exists some $i \in \{1, \ldots, N\}$ with infinite elements $\gamma' \in \Gamma$ satisfying

$$B(y, r_0) \cap \gamma' B(x_i, r_i) \neq \emptyset. \tag{6.3}$$

Also, observe that if we can take $\gamma_1 \in \Gamma$ and $\gamma_2 \in \Gamma$ as $\gamma' \in (6.3)$, then the element $\gamma_0 = \gamma_2^{-1} \gamma_1$ satisfies

$$B(x_i, 2r_i) \cap \gamma_0 B(x_i, 2r_i) \neq \emptyset. \tag{6.4}$$

Thus if there were infinite elements $\gamma \in \Gamma$ with $B(y, r_0) \cap \gamma K \neq \emptyset$, there would be infinite $\gamma_0 \in G$ with $(6.3)$. It contradicts the definition of $r_i$. Thus there are only
finite elements $\gamma \in \Gamma$ with $B(y, r_0) \cap \gamma K \neq \phi$. Let $\{\gamma_j\}_{j=1}^M$ be the set of all such elements. By the definition of $K$, we have

$$B(y, r_0) \subset \bigcup_{j=1}^M \gamma_j K.$$  

Since the right-hand side is compact, any closed ball centered at $y$ with a radius less than $r_0$ is compact. Hence $Y$ is locally compact.

Therefore for any $x \in K$, there exists a precompact open ball $B_x \subset Y$ centered at $x$. Let $\{B_i\}$ be a finite subcover of the open cover $\{B_x\}_{x \in K}$ of $K$, and define $U = \bigcup_i B_i$. Then $U$ is a precompact open subset containing $K$.

For each point $x \in K$ we define $f(x) > 0$ to be $f(x) = \sup\{\alpha > 0 \mid B(x, \alpha) \subset U\}$. Let $y \in K$ be an arbitrary point, and let $\eta > 0$ be an arbitrary positive real number. Set $\kappa = \min\{f(y), \eta\}$. Then for any $y' \in B(y, \kappa)$, we have $f(y') \geq f(y) - \eta$. Hence $f$ is a lower semi-continuous function on $K$. Thus there exists $x_0 \in K$ on which $f$ attains the minimum value of $f$. Set $r = f(x_0)$. Then we have $B(x, r) \subset U$ for all $x \in U$.

Let $\varepsilon > 0$ be an arbitrary positive real number. Observe that there exists a positive integer $N$ such that each $U$ is covered by $N$ open balls of radius $\varepsilon$, since $U$ is precompact. Let $y \in Y$ be an arbitrary point. There exists $\gamma \in \Gamma$ with $y \in \gamma K$. Then $B(y, r)$ can be covered by $N$ balls of radius $\varepsilon$ since $\gamma^{-1} B(y, r)$ is no other than $B(\gamma^{-1} y, r)$ which is contained in $U$. Hence the family $\{B(p, r)\}_{p \in Y}$ of $r$-balls is uniformly totally bounded. Thus it is Gromov-Hausdorff precompact. \hfill $\Box$

**Proof of Theorem 1.2.** If $Y_i$ is a geodesically complete CAT(0) space which admits a proper cocompact isometric action of a group, then the family $\{S_p Y_i\}_{p \in Y_i}$ consisting of all spaces of directions is Gromov-Hausdorff precompact by Lemma 6.1 and Lemma 6.2. Hence, by Theorem 5.2 and Lemma 5.3, there exists a constant $0 \leq c_i < 1$ such that $\delta(Y_i) \leq c_i$. Thus the theorem follows from Proposition 3.6. \hfill $\Box$

**Remark 6.3.** We remark that the geodesic completeness is essential. In [8], Kondo constructed a sequence of CAT(0) cones $T_1, T_2, T_3, \ldots$ with $\lim_{i \to \infty} \delta(T_i) = 1$. For each $i$, let $T_i' \subset T_i$ be a metric ball of radius $\frac{1}{2}$ centered at the origin. Glue $T_1', T_2', \ldots$ by identifying the origin of every $T_i'$. Then the resulting space $T'$ is not geodesically complete, but it is a compact CAT(0) space satisfying $\delta(T') = 1$.

Corollary 1.5 and Corollary 1.7 follow from Theorem 4.5 and Theorem 4.4, respectively.

7. **ULTRALIMITS AND DOUBLING CAT(0) SPACES**

In this section, we prove Theorem 1.9. To this end, we first prove that the ultralimit of a sequence of doubling length spaces with a common doubling constant is also doubling with the same doubling constant.

First, we recall the definitions of ultrafilters and ultralimits. Let $I$ be a set. A collection $\omega \subset 2^I$ of subsets of $I$ is called a filter on $I$ if it satisfies the following conditions:

(a): $\phi \notin \omega$,

(b): $A \in \omega, A \subset B \Rightarrow B \in \omega$. 


(c): \( A, B \in \omega \Rightarrow A \cap B \in \omega \).

An ultrafilter is a maximal filter. The maximality condition can be rephrased as the following condition:

(d): For any decomposition \( I = A_1 \cup \cdots \cup A_m \) of \( I \) into finitely many disjoint subsets, \( \omega \) contains exactly one of \( A_1, \ldots, A_m \).

An ultrafilter \( \omega \) on \( I \) is called nonprincipal if it satisfies

(e): For any finite subset \( F \subset I, F \notin \omega \).

Zorn’s lemma guarantees the existence of nonprincipal ultrafilters on any infinite set \( I \).

Let \( \omega \) be an ultrafilter on \( I \), \( f : I \to Y \) be a map from \( I \) to a topological space \( Y \), and \( y \in Y \). Then we write

\[ \omega \text{-} \lim_i f(i) = y \]

if for every neighborhood \( U \) of \( y \) the preimage \( f^{-1}(U) \) belongs to \( \omega \). If \( Y \) is compact and Hausdorff, such a point \( y \) exists uniquely for any map \( f : I \to Y \). We use the following lemma.

Lemma 7.1. Let \( \omega \) be an ultrafilter on a set \( I \), and let \( J \in \omega \). Then the set

\[ \omega_J = \{ K \in \omega \mid K \subset J \} \]

is an ultrafilter on \( J \). Moreover, if \( \omega_J \text{-} \lim_J f|_J(j) = y \) holds for a topological space \( Y \), a map \( f : I \to Y \), and a point \( y \in Y \), then \( \omega \text{-} \lim_i f(i) = y \) holds. Here, \( f|_J \) denotes the restriction of \( f \) to \( J \).

Proof. It is straightforward to see that \( \omega_J \) is an ultrafilter on \( J \). We only show the “moreover” part. Assume that \( \omega_J \text{-} \lim_J f|_J(j) = y \) holds. Let \( U \subset Y \) be an arbitrary neighborhood of \( y \). Then by the assumption, \( f|_J^{-1}(U) \in \omega_J \). Then \( f|_J^{-1}(U) \in \omega \) by the definition of \( \omega_J \). Thus \( f^{-1}(U) \in \omega \) since \( f|_J^{-1}(U) \subset f^{-1}(U) \). Hence we have shown that \( \omega \text{-} \lim_i f(i) = y \).

Let \( \omega \) be an ultrafilter on a set \( I \). Let \( \{(X_i, d_i)\}_{i \in I} \) be a sequence of metric spaces indexed by \( I \). Let \( \prod_{i \in I} X_i \) be the set of all sequences \( \{p_i\}_{i \in I} \) with \( p_i \in X_i \) for each \( i \in I \). Let \( \sim \) be a relation on \( \prod_{i \in I} X_i \) such that \( \{p_i\} \sim \{q_i\} \) holds if and only if \( \omega \text{-} \lim_i d_i(p_i, q_i) = 0 \). It is not difficult to see that it is an equivalence relation. We denote by \( \omega \text{-} \lim_i (X_i, d_i) \) the set of all equivalence classes of the equivalence relation \( \sim \). An equivalence class \( p \in \omega \text{-} \lim_i X_i \) represented by a sequence \( \{p_i\} \) is denoted by \( \omega \text{-} \lim_i p_i \). We define the distance \( d_\omega(p, q) \) between \( \omega \text{-} \lim_i p_i, \omega \text{-} \lim_i q_i \in \omega \text{-} \lim_i X_i \) by

\[ d_\omega(p, q) = \omega \text{-} \lim_i d_i(p_i, q_i) \in [0, \infty]. \]

Then \( (\omega \text{-} \lim_i (X_i, d_i), d_\omega) \) becomes a metric space whose distance function possibly takes the value \( \infty \).

Definition 7.2. Let \( \omega \) be an ultrafilter on a set \( I \). Let \( \{(X_i, d_i)\}_{i \in I} \) be a sequence of metric spaces indexed by \( I \). We call the metric space \( (\omega \text{-} \lim_i (X_i, d_i), d_\omega) \) defined above the ultralimit of \( \{(X_i, d_i)\}_{i \in I} \) with respect to \( \omega \).
An ultralimit \((\omega \lim_i (X_i, d_i), d_\omega)\) decomposes into components consisting of points of mutually finite distance. If we are given a basepoint \(x_i\) of every \(X_i\), we can pick out the component consisting of points which have finite distance from \(\omega \lim_i x_i\). This component is an ordinary metric space where the distance between any two points is finite, and we denote it by \(\omega \lim_i (X_i, d_i, x_i)\).

For a sequence \(\{A_i\}_{i \in I}\) of subsets with \(A_i \subset X_i\), we denote by \(\omega \lim_i A_i\) the subset of \(\omega \lim_i (X_i, d_i)\) consisting of all points which are represented by sequences in \(\prod_{i \in I} A_i\).

**Lemma 7.3.** Let \(\omega\) be an ultrafilter on a set \(I\), and \(\{(X_i, d_i)\}_{i \in I}\) be a sequence of metric spaces. Let \(\{A_i^{(1)}\}_{i \in I}, \ldots ,\{A_i^{(m)}\}_{i \in I}\) be sequences of subsets such that \(A_i^{(k)} \subset X_i\) for any \(k = 1, \ldots , m\) and any \(i \in I\). Then we have

\[
\omega \lim_i \left( \bigcup_{k=1}^m A_i^{(k)} \right) = \bigcup_{k=1}^m \omega \lim_i A_i^{(k)}.
\]

**Proof.** The right-hand side of (7.1) is trivially contained in the left-hand side. Let \(x\) be an arbitrary point in \(\omega \lim_i (\bigcup_{k=1}^m A_i^{(k)})\) and let \(\{x_i\}_{i \in I}\) be a sequence in \(\prod_{i \in I} (\bigcup_{k=1}^m A_i^{(k)})\) representing \(x\). For any \(k \in \{1, 2, \ldots , m\}\), set

\[I_k = \left\{ i \in I : k = \min\{l : x_i \in A_i^{(l)}\} \right\}.
\]

Then \(I = I_1 \cup \cdots \cup I_m\) is a decomposition of \(I\) into disjoint subsets. \(\omega\) contains exactly one of these subsets. Let \(I_l \in \omega\). Choose a sequence \(\{y_i\}\) in \(\prod_{i \in I} A_i^{(l)}\) such that \(y_i = x_i\) whenever \(i \in I_l\). Then we have

\[\omega \lim_i d_i(x_i, y_i) = \omega \lim_i d_i(x_i, y_i) = 0
\]

by Lemma 7.1. This yields that a sequence \(\{y_i\} \in \prod_{i \in I} A_i^{(l)}\) also represents \(x\). Hence \(x \in \omega \lim_i A_i^{(l)}\), which proves the lemma. \(\square\)

**Lemma 7.4.** Let \(\omega\) be an ultrafilter on a set \(I\), and \(\{(X_i, d_i)\}_{i \in I}\) be a sequence of length spaces. Let \(x = \omega \lim_i x_i\) be a point on the ultralimit \(\omega \lim_i (X_i, d_i)\). Then we have

\[
\overline{B}(x, r) = \omega \lim_i \overline{B}(x_i, r)
\]

for any \(r > 0\).

**Proof.** The right-hand side of (7.2) is trivially contained in the left-hand side. Let \(y\) be an arbitrary point on the ball \(\overline{B}(x, r) \subset \omega \lim_i (X_i, d_i)\) and let \(\{y_i\}\) be a sequence representing \(y\). We define a new sequence \(\{y_i'\}\) as follows. For \(i \in I\) satisfying \(d_i(x_i, y_i) \leq r\), define \(y_i' = y_i\). For \(i \in I\) satisfying \(d_i(x_i, y_i) > r + 1\), define \(y_i' = x_i\). Now, let \(m\) be an arbitrary positive integer, and consider the case in which \(i \in I\) satisfies \(r + \frac{1}{m+1} < d_i(x_i, y_i) \leq r + \frac{1}{m}\). In this case, we can take an arc-length parametrized path \(\gamma : [0, L] \to X_i\) of length \(L \leq r + \frac{2}{m}\), joining \(x_i\) to \(y_i\), and we define \(y_i'\) to be a point \(\gamma(L - \frac{2}{m})\). In this case, \(y_i'\) satisfies \(d_i(x_i, y_i') \leq r\) and \(d_i(y_i, y_i') \leq \frac{2}{m}\).
To see that $y$ is contained in the right-hand side of (7.2), it suffices to show that the sequence $\{y'_i\}$ defined as above satisfies
\[(7.3) \quad \omega\text{-lim}_i d_i(y_i, y'_i) = 0.\]

Let $U \subset \mathbb{R}$ be an arbitrary neighborhood of a point $0$ in $\mathbb{R}$. Choose a positive integer $m$ large enough, such that $\overline{B}(0, \frac{2}{m}) \subset U$. Let $I_m \subset I$ be the subset
\[I_m = \left\{ i \in I \mid d_i(x_i, y_i) \leq r + \frac{1}{m} \right\}.
\]

Then we have $I_m \in \omega$ since $\omega$-lim, $d_i(x_i, y_i) \leq r$. On the other hand, by the definition of $y'_i$, if $i \in I_m$ then $d_i(y_i, y'_i) \in \overline{B}(0, \frac{2}{m})$ holds. Thus
\[I_m \subset \left\{ i \in I \mid d_i(y_i, y'_i) \in \overline{B} \left( 0, \frac{2}{m} \right) \right\} \subset \{ i \in I \mid d_i(y_i, y'_i) \in U \}.
\]
Hence, $\{ i \in I : d_i(y_i, y'_i) \in U \} \in \omega$. This yields (7.3).

We obtain the following proposition from Lemma 7.4 and Lemma 7.5.

**Proposition 7.5.** Let $\omega$ be an ultrafilter on a set $I$, $\{(X_i, d_i)\}_{i \in I}$ be a sequence of length spaces, and $x = \omega$-lim$x_i$ be a point on the ultralimit $\omega$-lim$_i(X_i, d_i)$. Let $N \in \mathbb{N}$, and $r > 0$. Suppose that for each $i \in I$ the closed $r$-ball $\overline{B}(x_i, r)$ in $X_i$ is covered by $N$ closed $\frac{r}{2}$-balls. Then the closed $r$-ball $\overline{B}(x, r)$ in the ultralimit is also covered by $N$ closed $\frac{r}{2}$-balls. In particular, if each $(X_i, d_i)$ is doubling with a common doubling constant, then $\omega$-lim$_i(X_i, d_i)$ is also doubling with the same constant.

**Proof.** By Lemma 7.4 we have
\[\overline{B}(x, r) = \omega$-lim$_i \overline{B}(x_i, r).\]

By the assumption, for each $i \in I$ there exists $N$ points $x_i^{(1)}, \ldots, x_i^{(N)}$ such that
\[\overline{B}(x_i, r) \subset \bigcup_{k=1}^N \overline{B} \left( x_i^{(k)}, \frac{r}{2} \right).\]

Then we have
\[\omega$-lim$_i \overline{B}(x_i, r) \subset \omega$-lim$_i \left( \bigcup_{k=1}^N \overline{B} \left( x_i^{(k)}, \frac{r}{2} \right) \right).\]

By Lemma 7.3, we have
\[\omega$-lim$_i \left( \bigcup_{k=1}^N \overline{B} \left( x_i^{(k)}, \frac{r}{2} \right) \right) = \bigcup_{k=1}^N \left( \omega$-lim$_i \overline{B}(x_i^{(k)}, \frac{r}{2}) \right).\]

If we denote the point $\omega$-lim$_i x_i^{(k)}$ in the ultralimit by $x^{(k)}$ for each $k \in \{1, \ldots, N\}$, then, according to Lemma 7.3, the right-hand side of the above equality is no other than $\bigcup_{k=1}^N \overline{B}(x^{(k)}, \frac{r}{2})$. Thus we obtain
\[\overline{B}(x, r) \subset \bigcup_{k=1}^N \overline{B} \left( x^{(k)}, \frac{r}{2} \right).\]
which proves the proposition.

\[ \text{Proof.} \] Let \( (Y, d) \) be a CAT(0) space, \( p \in Y \) and \( \omega \) be a nonprincipal ultrafilter on \( \mathbb{N} \). For \( n \in \mathbb{N} \) we define a new metric \( d_n \) on \( Y \) as

\[ d_n(p,q) = nd(p,q), \quad p,q \in Y. \]

Then the tangent cone \( TC_pY \) isometrically embeds into \( \omega \)-limit \( \omega \)-ball of \((Y, d_n, p)\).

**Proposition 7.6.** Let \( (Y, d) \) be a CAT(0) space, \( p \in Y \) and \( \omega \) be a nonprincipal ultrafilter on \( \mathbb{N} \). For \( n \in \mathbb{N} \) we define a new metric \( d_n \) on \( Y \) as

\[ d_n(p,q) = nd(p,q), \quad p,q \in Y. \]

Then the tangent cone \( TC_pY \) isometrically embeds into \( \omega \)-limit \( \omega \)-ball of \((Y, d_n, p)\).

**Proof.** We construct an embedding \( f \) of \( TC_pY \) into \( \omega \)-limit \( \omega \)-ball of \((Y, d_n, p)\). It suffices to construct an embedding of the Euclidean cone \( Cone((S_pY)^\circ) \) since \( TC_pY \) is the metric completion of \( Cone((S_pY)^\circ) \) and an ultralimit is always complete. We can construct an embedding \( f : Cone((S_pY)^\circ) \rightarrow \omega \)-limit \((Y, d_n, p)\) as follows. For the origin \( O \in Cone((S_pY)^\circ) \), define \( f(O) = \omega \)-limit \( p \). Assume that \( v \in Cone((S_pY)^\circ) \) is represented by \( ([\gamma], r) \in (S_pY)^\circ \times (0, \infty) \) where \([\gamma] \in (S_pY)^\circ \) is the direction represented by a nontrivial geodesic \( \gamma : [0, a] \rightarrow Y \) starting from \( p \). For \( n \in \mathbb{N} \) satisfying \( a \geq \frac{\pi}{n} \), define \( x_n = \gamma(\frac{\pi}{n}) \in Y \). For \( n \in \mathbb{N} \) satisfying \( a < \frac{\pi}{n} \), define \( x_n = p \). We define \( f(v) \in \omega \)-limit \((Y, d_n, p)\) to be the point represented by the sequence \( \{x_n\} \in \prod_{n \in \mathbb{N}} Y \). Then by (2.1) and the definition of the distance on Euclidean cones, it is easily seen that the map \( f \) defined here is well-defined and isometric.

If we use this proposition, the following corollary follows from Proposition 7.3.

**Corollary 7.7.** Let \( N \in [1, \infty) \). Suppose that a CAT(0) space \( (Y, d) \) is locally doubling with doubling constant \( N \). Then each tangent cone \( TC_pY \) of \( Y \) is doubling with doubling constant \( N \).

**Proof.** Let \( p \in Y \) be an arbitrary point on \( Y \), and \( \omega \) be a nonprincipal ultrafilter on \( \mathbb{N} \). For each \( n \in \mathbb{N} \), let \( d_n \) be a new metric on \( Y \) defined by

\[ d_n(p,q) = nd(p,q), \quad p,q \in Y. \]

Since \( TC_pY \) embeds isometrically into \( \omega \)-limit \( \omega \)-ball of \((Y, d_n, p)\) by Lemma 7.6, it suffices to show that for any \( r > 0 \) and \( y = \omega \)-limit \( y_n \in \omega \)-limit \((Y, d_n, p)\), the closed \( r \)-ball of the metric space \( \omega \)-limit \((Y, d_n, p)\) centered at \( y \) is covered by at most \( N \) closed balls of radius \( \frac{r}{2} \).

Let us denote by \( s \) the distance between \( y \) and \( \omega \)-limit \( p \) in the ultralimit. Without loss of generality, we can assume that the sequence \( \{y_n\} \) satisfies \( d_n(p, y_n) \leq 2s \) for every \( n \).

Since \( Y \) is locally doubling, there exists \( \varepsilon > 0 \) such that the closed \( \varepsilon \)-ball of \((Y, d)\) centered at \( p \) is doubling with doubling constant \( N \). Then, for each \( n \), the closed \( n\varepsilon \)-ball of \((Y, d_n)\) centered at \( p \) is doubling with doubling constant \( N \). Since for any \( n \geq \frac{r+2s}{2\varepsilon} \), the closed \( r \)-ball of \((Y, d_n)\) centered at \( y_n \) is contained in the closed \( n\varepsilon \)-ball of \((Y, d_n)\) centered at \( p \), it is covered by at most \( N \) closed balls of radius \( \frac{r}{2} \). Hence, by Proposition 7.3, the closed \( r \)-ball centered at \( y \) in the ultralimit is also covered by at most \( N \) closed balls of radius \( \frac{r}{2} \).

**Lemma 7.8.** Let \( Y \) be a CAT(0) space and \( p \in Y \). Assume that the tangent cone \( TC_pY \) is doubling with doubling constant \( N \in [0, \infty) \). Then there exist \( 0 \leq \theta < \frac{\pi}{2} \),
0 < \alpha < 1 and \varepsilon > 0 depending only on N such that the space of directions S_pY at p of Y has the property P(\theta, \alpha, \varepsilon).

Proof. We assume that N is a natural number. Since TC_pY is doubling with doubling constant N, there exist closed balls B_1, B_2, \ldots, B_{N^2} with diameter at most \frac{1}{4}, which cover the closed ball of radius 1 centered at the origin of the cone TC_pY. Hence S_pY is covered by \{\iota^{-1}(B_i)\}, where \iota : S_pY \to TC_pY is the canonical inclusion. By the inequality (6.1), each \iota^{-1}(B_i) has diameter at most \frac{\varepsilon}{2}. Thus the lemma follows from the similar argument as in the proof of Lemma 5.3. 

\hspace{\stretch{1}} \Box

Proof of Theorem 1.9. Let Y_1, Y_2, Y_3, \ldots be complete CAT(0) spaces. By Corollary 7.7 and Lemma 7.8 if each Y_i is locally doubling with doubling constant N, there exist 0 < \theta < \frac{\pi}{2}, 0 < \alpha < 1 and \varepsilon > 0 depending only on N such that every space of directions of Y_i has the property P(\theta, \alpha, \varepsilon). According to Theorem 5.2, this implies the existence of a constant 0 \le c < 1 depending only on N such that \delta(Y_i) < c. By Proposition 3.6 it immediately follows that the product Y of Y_1, Y_2, Y_3, \ldots also satisfies \delta(Y) < c. 

\hspace{\stretch{1}} \Box

Corollary 1.10 and Corollary 1.11 follow from Theorem 4.5 and Theorem 4.4 respectively.

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