The symmetries of five-dimensional minimal supergravity reduced to three dimensions

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Abstract

The 14 Killing vectors of the target space for five-dimensional minimal supergravity reduced to three dimensions are explicitly constructed in terms of the original field variables. These vectors generate the Lie algebra of $G_2$. We also construct a symmetrical $7 \times 7$ matrix representative of the coset $G_{2(+2)}/((SL(2,R) \times SL(2,R))$ as a function of the same fields.

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1 Introduction

A number of self-gravitating field theories in $D$ dimensions can be dimensionally reduced to gravity-coupled sigma models in three dimensions [1, 2]. Such sigma models are harmonic maps from a 3-dimensional (Minkowskian or Lorentzian) base space to a $p$-dimensional target space $T$. The target space is generically a coset $G/H$, where $G$ is the group of global isometries of $T$, and $H \in G$ the local isotropy subgroup. It is then possible to generate new solutions by applying a finite group transformation to the coset representative of a seed solution [3, 4]. Another fruitful application is the construction of multi-center solutions as totally geodesic submanifolds of the target space [5].

The reduction of $D = 11$ supergravity to $D = 3$ leads to the $E_8(+8)/SO(16)$ sigma model [6, 7]. More recently, it was shown that the reduction of five-dimensional minimal supergravity [8, 9] leads to the $G_{2(+2)}/SO(4)$ sigma model [10, 11, 2, 12] in the case of a Lorentzian 3-space, or $G_{2(+2)}/((SL(2,R) \times SL(2,R))$ for a Minkowskian 3-space. Matrix representations of this coset were given in these papers in terms of the 14-dimensional adjoint representation. A representation of the same coset as a $7 \times 7$ matrix was given in [13], however the parametrisation used leads to a matrix which is too complicated to be used for solution generation. The purpose of this paper is two-fold. First, we shall give an analytical construction of the 14 infinitesimal isometries (or Killing vectors) of the target space for dimensionally reduced five-dimensional minimal supergravity, and check that they generate the Lie algebra of the exceptional group $G_2$. Then, we shall construct a coset representative as a symmetrical $7 \times 7$ matrix.

Our reduction from five to three dimensions follows essentially the same path as in [10]. With a view for the paper to be self-contained, we outline this reduction in Sect. 2. The result is the metric (2.12) for an eight-dimensional target space $T$. The procedure followed in [10] to identify $T$ as the $G_{2(+2)}/SO(4)$ coset was to construct a matrix representative of this coset in the adjoint representation of $G_2$, and to show that the resulting metric was isometric to (2.12). We shall instead follow a ‘bottom-up’ approach. In Sect. 3 we show that, taking into account the field-theoretical construction of (2.12), we can identify nine manifest infinitesimal symmetries of this metric. Combining this information with the assumption that the unknown symmetry group $G$ must contain $SL(3,R)$, which is the isom-

\footnote{These Killing vectors were previously determined, using a different parametrisation, by a computer-assisted solution of the Killing equations [14].}
etry group for the vacuum sector (five-dimensional Einstein gravity reduced to three dimensions)\cite{15}, we show that the minimal Lie algebra necessarily closes to that of $G_2$. We solve in Sect. 4 the Lie brackets involving the five unknown generators, and determine these up to a single integration constant. We then determine the value of this constant so that these five generators are indeed Killing vectors of the metric (2.12). In Sect. 5, a symmetrical $7 \times 7$ matrix coset representative is obtained by exponentiating a Borel subalgebra. An alternative construction of the same matrix using the nine manifest Killing vectors of Sect. 3 is sketched in the Appendix.

\section{Five-to-three dimensional reduction}

The bosonic sector of five-dimensional minimal supergravity is defined by the Einstein-Maxwell-Chern-Simons action

$$ S_5 = \frac{1}{16\pi G_5} \int d^5x \left[ \sqrt{|g(5)|} \left( -R(5) - \frac{1}{4} F_{\mu\nu}(5) F^{\mu\nu}(5) \right) - \frac{1}{12\sqrt{3}} \epsilon^{\mu\nu\rho\sigma\lambda} F_{\mu\nu}(5) F_{\rho\sigma}(5) A(5) A(5) \right], \quad (2.1) $$

where $F(5) = dA(5)$, $\mu, \nu, \cdots = 1, \cdots, 5$, and $\epsilon^{\mu\nu\rho\sigma\lambda}$ is the five-dimensional antisymmetric symbol. This leads to the five-dimensional Maxwell-Chern-Simons and Einstein equations

$$ \sqrt{|g(5)|} D_\mu F_{\mu\nu}^{(5)} = \frac{1}{4\sqrt{3}} \epsilon^{\nu\rho\sigma\tau\lambda} F_{\rho\sigma}(5) F_{\tau\lambda}(5), \quad (2.2) $$

$$ R^{\mu}_{\nu}(5) - \frac{1}{2} R(5) \delta^\mu_\nu = \frac{1}{2} F_{\mu\rho}(5) F_{\rho\nu} - \frac{1}{8} F_{(5)}^2 \delta^\mu_\nu, \quad (2.3) $$

where $D_\mu$ is the covariant derivative associated with the metric $g(5)^{\mu\nu}$.

Assuming the existence of two Killing vectors, one can choose adapted coordinates such that the five-dimensional metric and electromagnetic potential depend only on three coordinates $x_i$ ($i = 1, 2, 3$) and split according to:

$$ ds^2_{(5)} = \lambda_{ab} (dz^a + a_i^a dx^i)(dz^b + a_i^b dx^i) + \tau^{-1} h_{ij} dx^i dx^j, \quad (2.4) $$

$$ A_{(5)} = \sqrt{3}(\psi_a dz^a + A_i dx^i), \quad (2.5) $$

where $a, b = 0, 1$ ($x^4 = z^0, x^5 = z^1$), and $\tau \equiv |\text{det} \lambda|$. The reduced metric $h_{ij}$ or its inverse $h^{ij}$ will be used to lower or raise indices $i, j, k$, and we will
denote by $\nabla_j$ the associated covariant derivative. The $\nu = i$ components of the Maxwell-Chern-Simons equations (2.2) can be written as

$$\nabla_j \left( \tau [F^{ij} + a^a \partial^j \psi_a - a^a \partial^i \psi_a] \right) = \nabla_j \left( \frac{1}{\sqrt{h}} \epsilon^{ijk} \psi_a \partial_k \psi_b \right),$$

(2.6)

where $F^{ij} \equiv \partial_i A_j - \partial_j A_i$. This equation allows to dualize the vector magnetic potential $A_i$ to a scalar magnetic potential $\mu$ defined by

$$F^{ij} = \partial^a \partial^b \psi_a - \partial^a \partial^b \psi_a + \frac{1}{\tau \sqrt{h}} \epsilon^{ijk} \eta_k, \quad \eta_k = \partial_k \mu + \epsilon^a \psi_a \partial_k \psi_a.$$

(2.7)

Similarly, the $\mu = i$, $\nu = a$ components of the Einstein equations (2.3) read

$$\nabla_j \left( \tau \lambda_{ab} G^{bij} \right) = -\epsilon^{ijk} \nabla_j \left( \frac{1}{\sqrt{h}} \psi_a \left[ 3 \partial_k \mu + \epsilon^{bc} \psi_b \partial_k \psi_c \right] \right),$$

(2.8)

where $G^{bij} \equiv \partial_i a^j - \partial_j a^i$. This is integrated by

$$\lambda_{ab} G^{bij} = \frac{1}{\tau \sqrt{h}} \epsilon^{ijk} V_{ak}, \quad V_{ak} = \partial_k \omega_a - \psi_a \left( 3 \partial_k \mu + \epsilon^c \psi_b \partial_k \psi_c \right),$$

(2.9)

where $\omega_a$ is the ‘twist’ or gravimagnetic two-potential [13]. It is then straightforward to show that the $\mu = i$, $\nu = j$ components of the five-dimensional Einstein equations (2.3) lead to the following three-dimensional Einstein equations:

$$R_{ij} = \frac{1}{4} Tr(\lambda^{-1} \partial_i \lambda \lambda^{-1} \partial_j \lambda) + \frac{1}{4} \tau^{-2} \partial_i \tau \partial_j \tau - \frac{1}{2} \tau^{-1} V_i^T \lambda^{-1} V_j$$

$$+ \frac{3}{2} \left( \partial_i \psi^T \lambda^{-1} \partial_j \psi - \tau^{-1} \eta_i \eta_j \right),$$

(2.10)

where $\lambda$ is the $2 \times 2$ matrix of elements $\lambda_{ab}$, $V_i$ the column matrix of elements $V_{ai}$, and $R_{ij}$ the Ricci tensor built out of the reduced metric $h_{ij}$. These equations, together with the other field equations arising from the dimensional reduction of the original five-dimensional field equations, derive from the reduced action (up to a multiplicative constant)

$$S_3 = \int d^3 x \sqrt{h} \left( -R + \frac{1}{2} G_{AB} \frac{\partial \Phi^A}{\partial x^i} \frac{\partial \Phi^B}{\partial x^j} h^{ij} \right),$$

(2.11)

where the $\Phi^A$ ($A = 1, \cdots, 8$) are the eight scalar fields $\lambda_{ab}$, $\omega_a$, $\psi_a$, and $\mu$. The action (2.11) describes the three-dimensional gravity coupled sigma model for the eight-dimensional target space $T$ with metric:

$$dS^2 \equiv G_{AB} d\Phi^A d\Phi^B = \frac{1}{2} Tr(\lambda^{-1} d\lambda \lambda^{-1} d\lambda) + \frac{1}{2} \tau^{-2} d\tau^2 - \tau^{-1} V^T \lambda^{-1} V$$

$$+ 3 \left( d\psi^T \lambda^{-1} d\psi - \tau^{-1} \eta^2 \right),$$

(2.12)
where
\[ \eta = d\mu + e^{ab}\psi_a d\psi_b , \quad V_a = d\omega_a - \psi_a \left( 3d\mu + e^{bc}\psi_b d\psi_c \right) . \] (2.13)

3 The symmetry algebra

The Killing equations
\[ J_{A;B} + J_{B;A} = 0 \] (3.1)
for the metric (2.12) constitute a system of 36 partial derivative equations for eight unknown functions of eight variables. The analytical solution of this system is possible in principle, but represents a formidable task which is best addressed by computer [14]. However it is possible to find the Killing vectors of (2.12) without explicitly solving (3.1) by using information on the manifest symmetries coming from the field-theoretical construction of (2.12), combined with information about the hidden symmetries of five-dimensional pure gravity (the vacuum sector of five-dimensional minimal supergravity) and with the Jacobi identities.

The manifest symmetries of \( \mathcal{T} \) have two origins. First, the original gauge invariances of (2.1) — diffeomorphism invariance for the five-dimensional tensor fields \( g(5)_{\mu\nu} \) and \( A(5)_\mu \), and gauge invariance for the gauge field \( A(5)_a \) — are broken by the dimensional reduction down to the corresponding diffeomorphism invariance for the three-dimensional tensor, vector and scalar fields, together with \( GL(2, R) \) invariance (freedom of choice of two basis vectors in the two-plane \((z^0, z^1)\)) and residual gauge invariance for the ‘electric’ potentials \( \psi_a \). Second, the duality equations (2.7) and (2.9) define the cyclic coordinates \( \mu \) and \( \omega_a \) only up to translations.

The corresponding infinitesimal symmetries lead to nine Killing vectors, a \( GL(2, R) \) tensor, two vectors, and a scalar. The four components of the mixed tensor
\[ M^b_a = 2\lambda_{ac} \frac{\partial}{\partial \lambda_{cb}} + \omega_a \frac{\partial}{\partial \omega_b} + \delta^b_a \omega_c \frac{\partial}{\partial \omega_c} + \psi_a \frac{\partial}{\partial \psi_b} + \delta^b_a \mu \frac{\partial}{\partial \mu} \] (3.2)
generate linear transformations in the \((z^0, z^1)\) plane obeying the \( gl(2, R) \) subalgebra,
\[ [M^b_a, M^d_c] = \delta^b_c M^d_a - \delta^d_a M^b_c . \] (3.3)
The two-vector and the scalar associated with the three cyclic ‘magnetic’ coordinates:
\[ N^a = \frac{\partial}{\partial \omega_a} . \] (3.4)
follow the commutation relations

\[
\begin{align*}
\left[ M^b_a, N^c \right] & = -(\delta^c_a N^b + \delta^b_a N^c), \\
\left[ M^b_a, Q \right] & = -\delta^b_a Q, \\
\left[ N^a, N^b \right] & = 0, \\
\left[ Q, N^a \right] & = 0.
\end{align*}
\]

Infinitesimal gauge transformations of the \( \psi_a \) are generated by the two-vector

\[
R^a = \frac{\partial}{\partial \psi_a} + 3\mu \frac{\partial}{\partial \omega_a} - \epsilon^{ab}_c \psi_b \left( \frac{\partial}{\partial \mu} + \psi_c \frac{\partial}{\partial \omega_c} \right)
\]

with the commutation relations

\[
\begin{align*}
\left[ M^b_a, R^c \right] & = -\delta^c_b R^a, \\
\left[ N^a, R^b \right] & = 0, \\
\left[ Q, R^a \right] & = 3N^a, \\
\left[ R^a, R^b \right] & = 2\epsilon^{ab}Q.
\end{align*}
\]

An exact solution of the five-dimensional field equations \((2.2), (2.3)\) is \( A^{(5)} = 0 \), corresponding to five-dimensional Einstein gravity. It is natural to assume that the isometry group \( SL(3, R) \) \cite{15} of the corresponding target space \((2.12) \text{ with } \psi = \mu = 0\) is a subgroup of the isometry group \( G \) of the full target space. This means that there must exist two more ‘hidden’ Killing vectors \( L_a \) completing the subalgebra \( sl(3, R) \):

\[
\begin{align*}
\left[ M^b_a, L^c \right] & = (\delta^c_b L_a + \delta^b_a L_c), \\
\left[ N^a, L^b \right] & = M^b_a, \\
\left[ L_a, L_b \right] & = 0.
\end{align*}
\]

Adding to the known form of the \( SL(3, R) \) for five-dimensional Einstein gravity the information from \((3.16)\), we know that

\[
L_a = \omega_a \omega_b \frac{\partial}{\partial \omega_b} + 2\omega_b \lambda_{ac} \frac{\partial}{\partial \lambda_{bc}} + \omega_c \psi_b \frac{\partial}{\partial \psi_b} + \omega_a \mu \frac{\partial}{\partial \mu} + \tau \lambda_{ab} \frac{\partial}{\partial \omega_b} + \cdots
\]

(3.18)
(the omitted terms are of order 0 in $\omega_a$). Commutation with $Q$,

$$[Q, L_a] = P_a,$$

(3.19)
gives two more generators

$$P_a = \omega_a \frac{\partial}{\partial \mu} + \cdots,$$

(3.20)
which now adds up to 13 generators. Finally commutation with the $R^a$ should lead in principle to four more generators, a traceless tensor $S_a^b$ and a scalar $T$,

$$[R^a, L_b] = S_a^b + \delta^a_b T.\hspace{1cm}(3.21)$$

At this stage we make the second, crucial, assumption that the algebra $Lie(G)$ is minimal and closes with a single scalar generator $T$ ($S_a^b = 0$):

$$[R^a, L_b] = \delta^a_b T.\hspace{1cm}(3.22)$$

This gives

$$T = \omega_c \frac{\partial}{\partial \psi_c} + 3\mu \omega_c \frac{\partial}{\partial \omega_c} + \cdots.$$

(3.23)

Now the full algebra can be found using the following three constraints:

1) Commutators must respect the Jacobi identities.

2) It follows from the Jacobi identities involving the traceless part of $M_a^b$ that the commutators of tensorial operators are tensors. The only constant tensors are the Kronecker symbol $\delta^a_b$ and the antisymmetric symbols $\epsilon_{ab}$.

3) It also follows from the Jacobi identities involving the trace $Tr(M) \equiv M_{cc}$ that commutators must respect dimension. The degrees (logarithmic dimensions) of the various fields are, in a scale such that $\omega_a$ has degree 1,

$$[\psi_a] = 1/3,\quad [\mu] = [\lambda_{ab}] = 2/3,\quad [\omega_a] = 1.\hspace{1cm}(3.24)$$

This leads to the degrees of the various Killing vectors

$$[M_a^b] = 0,$$

$$[P_a] = 1/3,\quad [R^a] = -1/3,$$

$$[T] = 2/3,\quad [Q] = -2/3,$$

$$[L_a] = 1,\quad [N^a] = -1.$$

(3.25)
The full algebra consists of the above commutation relations (3.3), (3.6), (3.7), (3.8), (3.9), (3.11), (3.12), (3.13), (3.14), (3.15), (3.16), (3.17), (3.19), (3.22) together with

\[
\begin{align*}
[M_{ab}, P_c] &= \delta^b_c P_a, \quad (3.26) \\
[M_{ab}, T] &= \delta^b_a T, \quad (3.27) \\
[N^a, P_b] &= \delta^a_b Q, \quad (3.28) \\
[N^a, T] &= R^a, \quad (3.29) \\
[Q, P_a] &= -2\epsilon_{ab} R^b, \quad (3.30) \\
[Q, T] &= Tr(M), \quad (3.31) \\
[R^a, P_b] &= -3M^a_0 + \delta^a_b Tr(M), \quad (3.32) \\
[R^a, T] &= 2\epsilon^{ab} P_b, \quad (3.33) \\
[L_a, P_b] &= 0, \quad (3.34) \\
[L_a, T] &= 0, \quad (3.35) \\
[P_a, P_b] &= 2\epsilon_{ab} T, \quad (3.36) \\
[P_a, T] &= 3L_a. \quad (3.37)
\end{align*}
\]

This is a rank 2 algebra which can be put in the Cartan-Weyl form, with

\[
\begin{align*}
H_1 &= \frac{M_0^0 + M_1^1}{\sqrt{6}}, \quad H_2 = \frac{M_0^0 - M_1^1}{\sqrt{2}}, \\
E_1 &= M_0^1, \quad E_{-1} = M_1^1, \quad \alpha_1 = (0, \sqrt{2}), \\
E_2 &= \frac{1}{\sqrt{3}} P_0, \quad E_{-2} = \frac{1}{\sqrt{3}} R^0, \quad \alpha_2 = (\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}), \\
E_3 &= \frac{1}{\sqrt{3}} P_1, \quad E_{-3} = \frac{1}{\sqrt{3}} R^1, \quad \alpha_3 = (\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{2}}), \\
E_4 &= \frac{1}{\sqrt{3}} T, \quad E_{-4} = \frac{1}{\sqrt{3}} Q, \quad \alpha_4 = (\frac{\sqrt{2}}{\sqrt{3}}, 0), \\
E_5 &= L_0, \quad E_{-5} = -N^0, \quad \alpha_5 = (\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{2}}), \\
E_6 &= L_1, \quad E_{-6} = -N^1, \quad \alpha_6 = (\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{2}}),
\end{align*}
\]

with \(\alpha_4 = \alpha_2 + \alpha_3, \alpha_1 = \alpha_2 - \alpha_3, \alpha_6 = \alpha_3 + \alpha_4, \alpha_5 = \alpha_2 + \alpha_4 = \alpha_1 + \alpha_6\). The root space diagram is that of the 14-dimensional algebra \(Lie(G_2)\). Note that the roots are arranged in order of increasing degree, and that the hidden symmetry generators \((P_a, T, L_a)\) correspond to the five roots with positive abscissa.
4 Determination of the five hidden symmetries

Our strategy is to exploit the above commutation relations to determine the uncompletely known Killing vectors \( L^a, \) \( P^a \), and \( T \). First, the commutation relations (3.29) and (3.31) can be solved to yield

\[
T = \omega_b R^b + \mu \left[ 2\lambda_{bc} \frac{\partial}{\partial \lambda^{cb}} + \psi_b \frac{\partial}{\partial \psi^b} + \mu \frac{\partial}{\partial \mu} \right] + X ,
\]

where

\[
\frac{\partial X}{\partial \omega_a} = 0 , \quad \frac{\partial X}{\partial \mu} = 0 .
\]

The unknown \( X \) is parametrized by

\[
X = X_A \partial A \equiv X_{\lambda^{cd}} \frac{\partial}{\partial \lambda^{dc}} + X_{\omega} \frac{\partial}{\partial \omega} + \psi \frac{\partial}{\partial \psi^c} + X_{\mu} \frac{\partial}{\partial \mu} .
\]

Then, relation (3.32) gives

\[
P_a = -3\mu^2 \epsilon^{ab} \frac{\partial}{\partial \omega^b} - 2\mu \epsilon^{ab} \frac{\partial}{\partial \psi^b} - 2\mu \psi_a \left( \frac{\partial}{\partial \mu} + \psi_b \frac{\partial}{\partial \psi^b} \right)
+ \omega_a \frac{\partial}{\partial \mu} - \psi_a \left( \lambda_{bc} \frac{\partial}{\partial \lambda^{cb}} + \psi_b \frac{\partial}{\partial \psi^b} \right) - \epsilon^{ab} \left( \frac{\partial X_A}{\partial \psi^b} - X_{\mu} \frac{\partial}{\partial \omega^b} \right)
+ \frac{1}{2} \psi_a X_{\psi} \frac{\partial}{\partial \psi^b} + \frac{1}{2} \psi_a X_{\psi} \frac{\partial}{\partial \omega^b} .
\]

Inserting this in relation (3.32) gives a system of eight second order differential equations for eight unknown functions \( X_A \) of two variables \( \psi^c \) (also depending on the three “parameters” \( \lambda^{cd} \)):

\[
[R^a, P_b] = -3\omega^b \frac{\partial}{\partial \omega_a} - 3\psi^b \frac{\partial}{\partial \psi_a} - \delta^a_b \left( \lambda_{cd} \frac{\partial}{\partial \lambda^{dc}} - \psi^d \frac{\partial}{\partial \psi^d} + \mu \frac{\partial}{\partial \mu} \right)
- \frac{1}{2} \epsilon^{bc} \left( \frac{\partial^2 X_A}{\partial \psi^b \partial \psi^c} - \frac{\partial X_A}{\partial \psi^b} \frac{\partial}{\partial \omega_a} - \frac{\partial X_A}{\partial \psi^c} \frac{\partial}{\partial \psi^a} \frac{\partial}{\partial \omega^c} \right)
+ \left( \frac{\partial X_{\psi^b}}{\partial \psi^a} - \frac{1}{2} \delta_{\psi^b} X_{\psi^a} \frac{\partial}{\partial \omega^d} \right) \left( \frac{\partial}{\partial \mu} + \psi_d \frac{\partial}{\partial \psi^d} \right)
+ \frac{\partial X_{\psi^c}}{\partial \psi^d} \left( \delta^a_d \psi_b - \frac{1}{2} \delta^a_b \psi^d \right) \frac{\partial}{\partial \omega^c} - \left( X_{\psi^b} \frac{\partial}{\partial \omega} + \frac{1}{2} \delta^a_{\psi^b} X_{\psi^a} \frac{\partial}{\partial \omega^c} \right)
= -6\lambda_{bc} \frac{\partial}{\partial \lambda^{ca}} - 3\omega^b \frac{\partial}{\partial \omega^a} - 3\psi^b \frac{\partial}{\partial \psi^a} + \delta^a_b \left( 2\lambda_{cd} \frac{\partial}{\partial \lambda^{dc}} + \psi^c \frac{\partial}{\partial \psi^c} - \mu \frac{\partial}{\partial \mu} \right) .
\]

The only inhomogeneous equation is

\[
\frac{\partial^2 X_{\lambda^{cd}}}{\partial \psi^a \partial \psi^b} = 6 \epsilon^{be} \left( \delta^a_{\psi^c} \lambda^{de} - \delta^a_{\psi^c} \lambda^{de} - \delta^a_d \lambda^{ce} \right) .
\]
This is solved by

\[ X_{\lambda}{}^{ab} = 6\epsilon^{cd}\lambda{}_{(ac}\psi_{b)}\psi_d + G_{ab}, \tag{4.7} \]

where the symmetric tensor \( G_{ab} \) depends only on the \( \lambda_{cd} \) (a component linear in the \( \psi_d \) will not lead to a second order tensor). The only possibility is \( G_{ab} = f(\tau)\lambda_{ab} \), and \([f] = [X] = 2/3\) means that necessarily \( f(\tau) \propto \tau^{1/2} \).

Such a fractional power can be reasonably excluded to occur in the Killings of (2.12), leading to

\[ G_{ab} = 0. \tag{4.8} \]

Next, we turn to the component equation along \( \partial/\partial \psi_b \),

\[ \frac{\partial^2 X_{\psi_b}}{\partial \psi_c \partial \psi_d} = 0, \tag{4.9} \]

which is solved by

\[ X_{\psi_b} = F^c_b(\lambda)\psi_c. \tag{4.10} \]

(a term of degree 0 in the \( \psi_d \) would not lead to a vector). Only one mixed tensor of the correct dimension can be constructed from \( \lambda_{cd} \) (without involving fractional powers of \( \tau \)), this is \( F^c_b = \alpha\epsilon^{cd}\lambda_{bd} \) (\( \alpha \) constant). Thus,

\[ X_{\psi_b} = \alpha\epsilon^{cd}\lambda_{bd}\psi_c. \tag{4.11} \]

Inserting this into the component along \( \partial/\partial \mu \) and using the identity

\[ \epsilon^{ac}\lambda_{bc} = \tau\epsilon_{bc}\lambda^{ac} \tag{4.12} \]

(where the \( \lambda^{ac} \) are the elements of the matrix \( \lambda^{-1} \)) leads to the equation

\[ \frac{\partial^2 X_{\mu}}{\partial \psi_a \partial \psi_b} = 2\alpha\tau\lambda^{ab}, \tag{4.13} \]

which is solved by

\[ X_{\mu} = \alpha\tau\lambda^{ab}\psi_a\psi_b + \beta\tau, \tag{4.14} \]

with \( \beta \) a new integration constant. Finally, the component along \( \partial/\partial \omega_c \)

reads

\[ \frac{\partial^2 X_{\omega_c}}{\partial \psi_a \partial \psi_b} = 4\alpha\tau \left[ \left( \delta^b_c\lambda^{ad} + \delta^a_c\lambda^{bd} \right) \psi_d + \lambda^{ab}\psi_c \right], \tag{4.15} \]

which is solved by

\[ X_{\omega_c} = 2\alpha\tau\lambda^{ab}\psi_a\psi_b\psi_c + \gamma\tau\psi_c, \tag{4.16} \]

with \( \gamma \) a third integration constant.
Now, the generators $T$ and $P_a$ are known up to three undetermined constants:

$$T = \left[ 2\mu\lambda_{bc} + 6\epsilon_{de}\lambda_{bd}\psi_c\psi_e \right] \frac{\partial}{\partial\lambda_{bc}}$$
$$+ \left[ 3\mu\omega_b + \gamma\tau\psi_b - \epsilon^{ce}\omega_c\psi_d + 2\alpha\tau\lambda^c\psi_b\psi_c\psi_d \right] \frac{\partial}{\partial\omega_b}$$
$$+ \left[ \omega_b + \mu\psi_b + \alpha\epsilon^{cd}\lambda_{bd}\psi_c \right] \frac{\partial}{\partial\psi_b}$$
$$+ \left[ \mu^2 + \beta\tau - \epsilon^{bc}\omega_b\psi_c + \alpha\tau\lambda^b\psi_b \psi_c \right] \frac{\partial}{\partial\mu},$$

(4.17)

$$P_a = \left[ 2\lambda_{bc}\psi_a - 6\lambda_{ab}\psi_c \right] \frac{\partial}{\partial\lambda_{bc}}$$
$$+ \left[ -3\mu^2\epsilon_{ab} + \frac{3\beta - \gamma}{2}\tau\epsilon_{ab} - 2\mu\psi_a\psi_b - \alpha\epsilon^{cd}\lambda_{ad}\psi_b\psi_c \right] \frac{\partial}{\partial\omega_b}$$
$$+ \left[ -2\mu\epsilon_{ab} + \frac{\alpha}{2}\lambda_{ab} - \psi_a\psi_b \right] \frac{\partial}{\partial\psi_b}$$
$$+ \left[ \omega_a - 2\mu\psi_a - \frac{\alpha}{2}\epsilon^{cd}\lambda_{ad}\psi_c \right] \frac{\partial}{\partial\mu}. \quad (4.18)$$

The remaining two generators $L_a$ can then be computed from (3.37):

$$L_a = \left[ 2\omega_b\lambda_{ac} + 2\mu (\lambda_{bc}\psi_a - 3\lambda_{ab}\psi_c) + 2\epsilon_{de}\lambda_{bd}\psi_a\psi_c \right] \frac{\partial}{\partial\lambda_{bc}}$$
$$+ \left[ \omega_a\omega_b + \frac{\alpha\gamma}{6}\tau\lambda_{ab} - \mu^2\epsilon_{ab} + \frac{3\beta - \gamma}{2}\tau\epsilon_{ab} - \mu^2\psi_a\psi_b \right. - \alpha\epsilon^{cd}\lambda_{ad}\psi_b\psi_c + \frac{8\alpha^2 + 7(\beta - \gamma)}{6}\tau\psi_a\psi_b + \alpha\tau\lambda^c\psi_a\psi_b\psi_c\psi_d \right] \frac{\partial}{\partial\omega_b}$$
$$\left. + \left[ -\mu^2\epsilon_{ab} + \frac{\alpha}{2}\mu\lambda_{ab} + \frac{7\beta - \gamma}{6}\tau\epsilon_{ab} + \omega_b\psi_a \right] \frac{\partial}{\partial\psi_b} \right. - \mu\psi_a\psi_b + \frac{\alpha}{2}\epsilon^{cd}\lambda_{bd}\psi_a\psi_c \right] \frac{\partial}{\partial\psi_b}$$
$$+ \left[ \mu\omega_a - \mu^2\psi_a - \frac{\alpha}{2}\epsilon^{cd}\lambda_{ad}\psi_c + \frac{\alpha^2 + \beta - \gamma}{2}\tau\psi_a + \frac{\alpha}{2}\tau\lambda^b\psi_a\psi_b\psi_c \right] \frac{\partial}{\partial\mu}, \quad (4.19)$$

consistent with (3.15) ($L_a$ is a vector of degree 1), (3.16) and (3.19). Computation of the commutator $[R^a, L_b]$ leads to

$$[R^a, L_b] = \delta^a_b \left\{ 2\mu\lambda_{cd} + 6\epsilon_{ef}\lambda_{ce}\psi_d\psi_f \right\} \frac{\partial}{\partial\lambda_{cd}}$$
\[ + \left[ 3\mu \omega_c + \left( \alpha^2 + 2\beta - \gamma \right) \tau \psi_c \right. \]
\[ - \epsilon^{de} \omega_d \psi_c \psi_e + 2\alpha \tau \lambda^{de} \psi_c \psi_d \psi_e \left. \frac{\partial}{\partial \omega_c} \right] \]
\[ + \left[ \omega_c + \mu \psi_c + \alpha \epsilon^{de} \lambda_{ce} \psi_d \right. \frac{\partial}{\partial \psi_c} \]
\[ + \left[ \mu^2 + \frac{\alpha^2 + 5\beta - 2\gamma}{3} \tau \right. \]
\[ - \epsilon^{cd} \omega_c \psi_d + \alpha \tau \lambda^{cd} \psi_c \psi_d \left. \right] \frac{\partial}{\partial \mu} \right) , \]
(4.20)

consistent with (3.22) provided
\[ \alpha^2 + 2(\beta - \gamma) = 0 . \]  
(4.21)

Finally there remains to check the commutation relations (3.17) (the remaining commutation relations will then be satisfied by virtue of the Bianchi identities). A lengthy computation leads to
\[ [L_a, L_b] = (3\beta - \gamma) \tau \begin{cases} \lambda_{[ac} \psi_{b]} \left( 8\psi_d \partial_{\lambda_{cd}} - \frac{2\alpha}{3} \partial_{\psi_c} \right) \\
- \epsilon_{ab} \left[ \frac{7}{12} \alpha \tau (\alpha - \lambda^{de} \psi_d \psi_e) + \frac{1}{4} \mu^2 \right] \psi_c \partial_{\omega_c} \\
+ \frac{\alpha}{2} \tau \epsilon_{ab} \left( \frac{\alpha}{4} + \lambda^{de} \psi_d \psi_e \right) \partial_{\mu} \end{cases} , \]
(4.22)

where we have taken (4.21) into account. So (3.17) is satisfied provided
\[ \beta = \gamma = \frac{\alpha^2}{3} . \]  
(4.23)

We have thus obtained a realisation of \( \text{Lie}(G_2) \) depending on an arbitrary real parameter \( \alpha \). The reason for this arbitrariness may be traced to the fact that the algebra (3.3), (3.6)-(3.9), (3.11)-(3.14) of the manifest symmetries is invariant under the combined scale transformation \( \Phi^A \rightarrow \Phi'^A \) with
\[ \lambda'_{ab} = \lambda_{ab} , \quad \psi'_a = k^2 \psi_a , \quad \mu' = k^2 \mu , \quad \omega'_a = k^3 \omega_a , \]
(4.24)

depending on a real parameter \( k \). The five hidden symmetries transform according to their appropriate scales \( (P'_a = kP_a , \ T' = k^2 T , \ L'_a = k^3 L_a) \) provided the integration constant \( \alpha \) is also rescaled to
\[ \alpha' = k^2 \alpha . \]
(4.25)
However the target space metric (2.12) is not invariant under the transformation (4.24):

\[
\begin{align*}
\frac{1}{4}T (\lambda'^{-1} d\lambda' \lambda'^{-1} d\lambda') + \frac{1}{4} \tau'^{-2} d\tau'^2 - \frac{1}{2} k^6 \tau'^{-1} V'^T \lambda'^{-1} V' \\
+ \frac{3}{2} \left( k^2 d\psi'^T \lambda'^{-1} d\psi' - k^4 \tau'^{-1} \eta'^2 \right).
\end{align*}
\] (4.26)

Let us determine the value of \(\alpha\) such that \(T\) is a Killing vector of the target space metric (2.12) (relations (3.33) and (3.37) then imply that the \(L_a\) and \(P_a\) are also Killing vectors). The action of \(T\) leads to the first order variations (written in matrix notation)

\[
\begin{align*}
\delta \lambda &= 2\mu \lambda + 3(\lambda J \psi \cdot \psi - \psi \cdot \psi J \lambda), \\
\delta \omega &= 3\mu \omega + \frac{3\alpha^2}{4} \tau \psi - (\omega J \psi) \psi + 2\alpha \tau (\psi \lambda^{-1} \psi) \psi, \\
\delta \psi &= \omega + \mu \psi - \alpha \lambda J \psi, \\
\delta \mu &= \mu^2 + \frac{\alpha^2}{4} \tau - (\omega J \psi) + \alpha \tau (\psi \lambda^{-1} \psi),
\end{align*}
\] (4.27-4.30)

with \(J^{ab} \equiv \epsilon^{ab}\). This leads to

\[
\begin{align*}
\frac{1}{4} \delta (\lambda^{-1} d\lambda)^2 &= \tau^{-1} d\tau d\mu + 3 \left( \psi \lambda^{-1} d\lambda J \psi + d\psi \lambda^{-1} d\lambda J \psi \right), \\
\frac{1}{4} \delta (\tau^{-1} d\tau)^2 &= 2 \tau^{-1} d\tau d\mu, \\
\frac{1}{2} \delta (d\psi \lambda^{-1} d\psi) &= 3 \left( \psi J d\psi \right) \left( \psi \lambda^{-1} d\psi \right) + \left( d\omega \lambda^{-1} d\psi \right) \\
&\quad + d\mu \left( \psi \lambda^{-1} d\psi \right) - \alpha \left( d\psi \lambda^{-1} d\lambda J \psi \right), \\
- \frac{1}{2} \delta (\tau^{-1} \eta^2) &= - \left[ d\mu + \psi J d\psi \right] \left[ \frac{\alpha^2}{4} \tau^{-1} d\tau \\
&\quad + 2 \tau^{-1} (\psi J d\omega) + 2 \alpha \left( \psi \lambda^{-1} d\psi \right) \right], \\
- \frac{1}{2} \delta (\tau^{-1} V \lambda^{-1} V) &= \tau^{-1} \left[ 3d\mu + (\psi J d\psi) \right] \left[ \alpha (\psi J d\omega) + \frac{3\alpha^2}{4} \tau \left( \psi \lambda^{-1} d\psi \right) \right] \\
&\quad - \frac{3\alpha^2}{4} \left( d\psi \lambda^{-1} d\omega \right) + 2 \alpha \left( \psi \lambda^{-1} d\psi \right) \left( \psi \lambda^{-1} d\omega \right) \\
&\quad - \left( \psi \lambda^{-1} \psi \right) \left( d\psi \lambda^{-1} d\omega \right). \quad (4.31-4.35)
\end{align*}
\]
Collecting these, we obtain

\[
\delta(dS^2) = 3\left(1 - \frac{\alpha^2}{4}\right)\tau^{-1}d\tau \left[d\mu + \left(\psi Jd\psi\right)\right] + 3\left(1 - \frac{\alpha^2}{4}\right)\left(d\omega \lambda^{-1}d\psi\right)
\]

\[
+ 3(\alpha - 2)\tau^{-1}\left[d\mu + \left(\psi Jd\psi\right)\right] \left(\psi Jd\omega\right)
\]

\[
+ 3\left(3 - 2\alpha + \frac{\alpha^2}{4}\right)\left(\psi Jd\psi\right) \left(\psi \lambda^{-1}d\psi\right)
\]

\[
+ 3\left[ (\psi \lambda^{-1}d\lambda Jd\psi) - d\psi \lambda^{-1}d\lambda J\psi \right] - \tau^{-1}d\tau \left(\psi Jd\psi\right)
\]

\[
+ 2\alpha \left[ \left(\psi \lambda^{-1}d\psi\right) \left(\psi \lambda^{-1}d\omega\right) - \left(\psi \lambda^{-1}\psi\right) \left(d\psi \lambda^{-1}d\omega\right) \right]
\]

\[
- \tau^{-1}\left(\psi Jd\psi\right) \left(\psi Jd\omega\right) \right].
\]

(4.36)

The last two terms in square brackets vanish identically, while the remaining terms vanish provided

\[
\alpha = 2, \quad \beta = 1, \quad \gamma = 3.
\]

(4.37)

5 Coset representative

In this section we assume for definiteness that the original five-dimensional metric \(g_{(5)\mu\nu}\) is Lorentzian, one of the two Killing vectors of the original five-dimensional theory (2.1) being timelike (the corresponding solutions are stationary) and the other being spacelike. The reduced metric \(h_{ij}\) is then Euclidean and the matrix field \(\lambda\) has signature (+−). In the vacuum sector (\(\psi_a = 0, \mu = 0\)), the target space metric (2.12) reduces to that of the five-dimensional symmetric space \(SL(3, R)/SL(2, R) \) [15], with signature (+ + + − −). The full eight-dimensional metric (2.12), with signature (+ + + − − − −), is that of the symmetric space \(G_{2(2)}/(SL(2, R) \times SL(2, R))\). This coset was constructed in [10] in terms of the 14-dimensional adjoint representation of \(G_{2(2)}\). We present here a more convenient representation (previously published without details in [14]) in terms of symmetrical \(7 \times 7\) matrices.

The matrix representatives \(j_M\) (\(M = 1, \cdots, 14\)) of the real form of \(Lie(G_2)\) may be derived from the \(Z\) matrices of [16] by omitting \(i\)’s. Their
generic block decomposition is

\[ j = \begin{pmatrix} S & \tilde{V} & \sqrt{2}U \\ -\tilde{U} & -S^T & \sqrt{2}V \\ \sqrt{2}V^T & \sqrt{2}U^T & 0 \end{pmatrix}, \quad (5.1) \]

where \( S \) is a \( 3 \times 3 \) matrix, \( U \) and \( V \) are 3-component column matrices, \( U^T \) and \( V^T \) the corresponding transposed row matrices, and \( \tilde{U}, \tilde{V} \) are the \( 3 \times 3 \) dual matrices \( \tilde{U}_{ij} = \epsilon_{ijk}U_k \). The matrices \( m_a^b, n^a \) and \( \ell_a \) generating \( SL(3, R) \) are of type \( S \), the corresponding \( 3 \times 3 \) blocks being

\[ S_{m_0^0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S_{m_0^1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]
\[ S_{m_1^0} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{m_1^1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]
\[ S_{n_0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad S_{n_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \]
\[ S_{\ell_0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{\ell_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \quad (5.2) \]

The matrices \( p_a \) and \( q \) are of type \( U \), the corresponding \( 1 \times 3 \) blocks being

\[ U_{p_0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad U_{p_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad U_q = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}. \quad (5.3) \]

The matrices \( r^a \) and \( t \) are of type \( V \), the corresponding \( 1 \times 3 \) blocks being

\[ V_{r_0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad V_{r_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad V_t = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (5.4) \]

Due to the form of (5.1), the transposed matrices \( j_A^T \) are related to the original matrices \( j_A \) by

\[ j_A^T = -K_j A K, \quad (5.5) \]
where the involution $K$ has the block structure
\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}.
\tag{5.6}
\]

A representative $N(\Phi^A)$ of the coset $G/H$ (here $G = G_{2(2)}$ and $H = SL(2, R) \times SL(2, R)$) transforms by global right action of $G$ and local right action of $H$:
\[
N(\Phi) \rightarrow h(\Phi)N(\Phi)g
\tag{5.7}
\]
($g \in G$, $h(\Phi) \in H$). The corresponding infinitesimal transformation is
\[
J_MN(\Phi) = N(\Phi)j_M + q_M^\alpha(\Phi)k_\alpha N(\Phi),
\tag{5.8}
\]
where $J_M$, $M = 1, \cdots, 14$ are Killing vectors acting as differential operators, $j_M$ are the corresponding matrices of the Lie($G$) algebra, $y_\alpha$, $\alpha = 1, \cdots, 6$ are generators of the isotropy subalgebra Lie($H$) and $q_M^\alpha(\Phi)$ are gauge functions. The gauge can be fixed so that for a suitably chosen Borel subalgebra $M = A = 1, \cdots, 8$ the functions $q_A^\alpha(\Phi)$ vanish, and the corresponding subset of the equations (5.8) reduces to
\[
J_A N(\Phi) = N(\Phi)j_A.
\tag{5.9}
\]
From a solution $N(\Phi)$ of (5.9), one then constructs the gauge-independent symmetrical matrix,
\[
M = N^T \eta N,
\tag{5.10}
\]
where $\eta$ is a constant symmetrical matrix invariant under the isotropy subgroup $H$,
\[
\eta^T = \eta, \quad h^T \eta h = \eta.
\tag{5.11}
\]
The matrix $M(\Phi)$ is invariant under the local action of $H$ and transforms tensorially under the global action of $G$,
\[
M(\Phi) \rightarrow g^T M(\Phi) g.
\tag{5.12}
\]
The $\sigma$-model current
\[
\mathcal{J} = M^{-1}dM
\tag{5.13}
\]
constructed from the coset representative $M(\Phi)$ is invariant under the action of $G$. The target space metric (2.12) is given in terms of this current by
\[
dS^2 = \frac{1}{4} \text{Tr}(\mathcal{J}^2).
\tag{5.14}
Consequently, the current $\mathcal{J}$ is conserved by virtue of the field equations deriving from the gravitating $\sigma$-model action (2.11)

$$\frac{1}{\sqrt{|h|}} \partial_i \left( \sqrt{|h|} h^{ij} \mathcal{J}_j \right) = 0.$$  (5.15)

Note that the definition of the matrix $M(\Phi)$ is not unique, as the current (5.13) is invariant under $M(\Phi) \rightarrow PM(\Phi)$ ($P \in G$). For instance a group equivalent coset representative is

$$M' \equiv KM = N^{-1}\eta'N \quad (\eta' = K\eta = \eta K),$$  (5.16)

using $N^T = KN^{-1}K$ which follows from exponentiating (5.5).

It is convenient to choose as generators of the Borel subalgebra eight of the manifest symmetry generators, i.e. three independent components of $M_{ab}$ together with the two $N^a$, $Q$ and the two $R^a$. A covariant solution of the first three equations (5.9) would involve trading the two-metric $\lambda_{ab}$ for a zweibein $\epsilon^i_a$. We shall bypass this by noting that, due to the structure (5.1) of the matrix generators, the vacuum ($\psi = \mu = 0$) matrix $M$ is of the form

$$M_1 = \begin{pmatrix} \chi & 0 & 0 \\ 0 & \chi^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$  (5.17)

with $\chi$ the $SL(3,R)/SL(2,R)$ coset representative

$$\chi = \begin{pmatrix} \lambda - \tau^{-1}\omega \omega^T & \tau^{-1}\omega \\ \tau^{-1}\omega^T & -\tau \end{pmatrix},$$  (5.18)

where $\lambda$ is a $2 \times 2$ block, and $\omega$ a 2-component column matrix. Thus the static ($\omega = 0$) vacuum matrix $M$ is

$$M_0(\lambda) = N(\epsilon)^T \eta N(\epsilon) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & -\tau^{-1} & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & -\tau \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$  (5.19)

The next three equations

$$\frac{\partial}{\partial \omega_a} N = Nn_a, \quad \frac{\partial}{\partial \mu} N = Nq$$  (5.20)
are readily integrated to:

\[ N(\epsilon, \psi, \mu, \omega) = N(\epsilon, \psi)e^{\mu q}e^{\omega a n^a}. \]  \tag{5.21}

The exponentials in (5.21) are easily computed as the matrices \( \mu q \) and \( \omega n \equiv \omega a n^a \) are nilpotent,

\[
e^{\mu q} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \mu^2 & -\sqrt{2}\mu \\ \mu J & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\sqrt{2}\mu & 1 \end{pmatrix}, \quad e^{\omega n} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\omega^T & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \omega & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},
\]  \tag{5.22}

where again the first and third rows and columns are double, and

\[
J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]  \tag{5.23}

Let us now show that the last two equations

\[ R^aN = N r^a \]  \tag{5.24}

are solved by

\[ N(\Phi) = N(\epsilon)e^{\psi_a r^a}e^{\mu q}e^{\omega a n^a}. \]  \tag{5.25}

Again, the matrix \( \psi_a r^a \) is nilpotent, so that

\[
e^{\psi r} = \begin{pmatrix} 1 & 0 & 0 & -J \psi & 0 \\ 0 & 1 & -\psi^T J & 0 & 0 \\ \psi \psi^T & 0 & 1 & 0 & \sqrt{2}\psi \\ 0 & 0 & 0 & 1 & 0 \\ \sqrt{2}\psi^T & 0 & 0 & 0 & 1 \end{pmatrix}.
\]  \tag{5.26}

This can be used to show that

\[
[R^a, e^{\psi r}] = \frac{\partial}{\partial \psi a}(e^{\psi r}) = e^{\psi r} \left[ r^a + \epsilon^{abc} \psi_b (q + \psi_c n^c) \right].
\]  \tag{5.27}

Using this, together with the commutator,

\[
[r^a, e^{\mu q}] = -3e^{\mu q} \mu n^a,
\]  \tag{5.28}
we obtain successively
\[
R^a e^{\psi r} e^{\mu q} e^{\omega n} = e^{\psi r} \left[ r^a + \epsilon^{ab} \psi_b (q + \psi, n^c) + R^a \right] e^{\mu q} e^{\omega n}
\]
\[
= e^{\psi r} \left[ r^a + 3\mu N^a - \epsilon^{ab} \psi_b (Q - q + \psi_c (N^c - n^c)) \right] e^{\mu q} e^{\omega n}
\]
\[
= e^{\psi r} e^{\mu q} \left[ r^a + 3\mu (N^a - n^a) - \epsilon^{ab} \psi_b \psi_c (N^c - n^c) \right] e^{\omega n}
\]
\[
= e^{\psi r} e^{\mu q} e^{\omega n} r^a.
\]
(5.29)

The final result for the matrix \(M\) is
\[
M(\Phi) = e^{\omega n T} e^{\mu q T} e^{\psi r T} M_0(\lambda) e^{\psi r} e^{\mu q} e^{\omega n}.
\]
(5.30)

It follows from (5.1) and (5.19) that the matrix \(M\) has the symmetrical block structure
\[
M = \begin{pmatrix}
A & B & \sqrt{2}U \\
B^T & C & \sqrt{2}V \\
\sqrt{2}U^T & \sqrt{2}V^T & S
\end{pmatrix},
\]
(5.31)

where \(A\) and \(C\) are symmetrical \(3 \times 3\) matrices, \(B\) is a \(3 \times 3\) matrix, \(U\) and \(V\) are 3-component column matrices, and \(S\) a scalar. It also follows from (5.5) that the inverse matrix is given by
\[
M^{-1} = KMK = \begin{pmatrix}
C & B^T & -\sqrt{2}V \\
B & A & -\sqrt{2}U \\
-\sqrt{2}V^T & -\sqrt{2}U^T & S
\end{pmatrix},
\]
(5.32)

Computation of the product (5.30), with the matrices (5.19), (5.22) and (5.26) gives the coset matrix \(M\) in the form (5.31), with
\[
A = \begin{pmatrix}
(1 - y)\lambda + (2 + x)\psi\psi^T - \tau^{-1}\tilde{\omega}\tilde{\omega}^T \\
+\mu(\psi\psi^T\lambda - J)(-\lambda^{-1}\tilde{\omega})^T \\
\tau^{-1}\tilde{\omega}^T
\end{pmatrix},
\]
\[
B = \begin{pmatrix}
(\psi\psi^T - \mu J)\lambda^{-1} - \tau^{-1}\tilde{\omega}\tilde{\omega}^T J \\
\tau^{-1}\tilde{\omega}^T J \\
\tau^{-1}\psi^T J
\end{pmatrix},
\]
\[
C = \begin{pmatrix}
(1 + x)\lambda^{-1} - \lambda^{-1}\psi\psi^T \lambda^{-1} \\
\tilde{\omega}^T\lambda^{-1} + \psi^T (z + \mu \lambda^{-1} J) \\
\lambda^{-1}\tilde{\omega} - J(z - \mu J\lambda^{-1})\psi
\end{pmatrix},
\]

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\[
U = \left\{
\begin{array}{l}
(1 + x - \mu J \lambda^{-1}) \psi - \mu \tau^{-1} \tilde{\omega} \\
\mu \tau^{-1}
\end{array}
\right),
\]
\[
V = \left\{
\begin{array}{l}
(\lambda^{-1} + \mu \tau^{-1} J) \psi \\
\psi^T \lambda^{-1} \tilde{\omega} - \mu (1 + x - z)
\end{array}
\right),
\]
\[
S = 1 + 2(x - y),
\]
\[
(5.33)
\]
with
\[
\tilde{\omega} = \omega - \mu \psi, \quad x = \psi^T \lambda^{-1} \psi, \quad y = \tau^{-1} \mu^2, \quad z = y - \tau^{-1} \psi^T J \tilde{\omega}.
\]
\[
(5.34)
\]

6 Conclusion

We have shown that the isometry algebra of the target space for five-dimensional supergravity reduced to three dimensions is that of $G_2$ by combining knowledge about the manifest symmetries (gauge invariances) of the theory with the $SL(3, R)$ invariance of the vacuum sector. We have then solved the Lie brackets to determine the generators of the hidden symmetries in terms of the field variables, and constructed a symmetrical $7 \times 7$ matrix representative of the coset $G_{2(+2)}/((SL(2, R) \times SL(2, R))$ as a function of the same fields.

This coset representative was used in [14] to generate a doubly-rotating charged black ring through the action of a group transformation (5.12) on a neutral 5D black ring with two angular momenta [17][18]. After completion of the present work, several related papers appeared. The geometry of the symmetric space $G_{2(2)}/SO(4)$ was studied in great detail in [19], where in particular the Iwasawa parametrization of the coset and the Killing vectors were also given explicitly. This approach was applied in [20] to the analysis of the supersymmetry constraints associated with a number of black hole solutions to gauged and ungauged 5D supergravity. BPS and non-BPS multi-centered attractor flows were constructed in [21], following the procedure advocated in [5].

These works certainly do not exhaust the potentialities of the sigma-model approach for generating solutions of five-dimensional supergravity. In [22], it was shown that stationary solutions to the four-dimensional Einstein-Maxwell equations can be generated from static solutions by a combination of $SU(2,1)$ group transformations and global coordinate transformations. This procedure can be extended to generate new stationary solutions of five-dimensional supergravity, which contains a four-dimensional Einstein-Maxwell sector [23]. In unrelated recent work, $SL(3, R)$ transformations were also used to generate stationary solutions of five-dimensional gravity.
from static solutions [24], and to construct a static black ring with Kaluza-Klein monopole charge [25]. It would be interesting to extend these techniques to the case of five-dimensional supergravity.

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Appendix

We first constructed the coset representative matrix \( M(\Phi) \) using a different procedure. Dualise the Killing vectors \( J_M = J_M^A \partial / \partial \Phi^A \) to the Killing one-forms

\[
\bar{J}_M = G_{AB} J_M^A d\Phi^B,
\]

where \( G_{AB} \) is the target space metric (2.12). The matrix current (5.13) is proportional to the Killing product of these one-forms with the matrices \( j_M \):

\[
M^{-1} dM = J \equiv 8 \sum_M \eta^{MN} \bar{J}_M j_N,
\]

where \( \eta^{MN} \) is the inverse of the Killing metric \( \eta_{MN} = 4Tr(j_M j_N) \). In the present case, the current (A.2) is given by

\[
J = \left( \bar{M}_a^b - \frac{1}{3} \delta^a_b Tr \bar{M} \right) m_a^b + \bar{N} a^a n^a_T + \bar{L} a^a_T + \frac{1}{3} \left( \bar{R}^a a^a_T + \bar{P} a^a_T + \bar{Q} q_T + \bar{T} t_T \right).
\]

The target space metric can be read off from (2.12):

\[
G_{\lambda a \lambda b} = \frac{1}{2} (\lambda^{(ab} \chi^{c)} + \chi^{ac} \lambda^{bd} ),
\]

\[
G_{\omega a \omega b} = -\tau^{-1} \chi^{ab} ,
\]

\[
G_{\omega a \mu} = 3\tau^{-1} \chi^{ab} \psi_b ,
\]

\[
G_{\mu \mu} = -3\tau^{-1} (1 + 3\lambda^{ab} \psi_a \psi_b ) ,
\]

\[
G_{\omega a \psi b} = -\tau^{-1} \chi^{ac} \psi^d c \psi_d ,
\]

\[
G_{\mu \psi a} = 3\tau^{-1} (1 + \chi^{cd} \psi^d c \psi_d ) \psi^a_b ,
\]

\[
G_{\psi a \psi b} = 3 \left[ \lambda^{ab} - \tau^{-1} (1 + \frac{1}{3} \chi^{cd} \psi^d c \psi_d ) \psi^a e \psi^b f \right].
\]
This leads to the Killing one-forms:

\[
\tilde{M}_a^b = b \left( \lambda^{-1} d\lambda + \tau^{-1} d\tau - \tau^{-1} \left[ \lambda^{-1} d\omega^T + \omega^T \lambda^{-1} d\omega \right] \right)
- (3\mu - J\psi\psi^T)\psi^T \lambda^{-1} d\omega + 3\tau^{-1} \left[ \lambda^{-1} \psi^T + \omega^T \lambda^{-1} \psi \right] d\mu
- \mu (1 + 3\psi^T \lambda^{-1} \psi) + J\psi\psi^T (1 + \psi^T \lambda^{-1} \psi) d\mu
+ 3\lambda^{-1} d\psi^T \psi^T + \tau^{-1} \left[ (\lambda^{-1} \psi^T + \omega^T \lambda^{-1} \psi) \right] 
- 3\mu (1 + \psi^T \lambda^{-1} \psi) + 3J\psi\psi^T (1 + \frac{1}{3} \psi^T \lambda^{-1} \psi) \psi^T Jd\psi, \right)
\]

\[
\tilde{N}_a = a \left( -\tau^{-1} \lambda^{-1} \left[ d\omega - \psi (3d\mu + \psi^T Jd\psi) \right] \right), \quad (A.5)
\]

\[
\tilde{Q} = 3\tau^{-1} \left[ - (1 + 3\psi^T \lambda^{-1} \psi) d\mu + \psi^T \lambda^{-1} d\omega - (1 + \psi^T \lambda^{-1} \psi) \psi^T Jd\psi \right], \quad (A.6)
\]

\[
\tilde{R}_a = a \left( 3\lambda^{-1} \left[ d\psi - \tau^{-1} \mu (d\omega - \psi (3d\mu + \psi^T Jd\psi)) \right] \right)
+ 3\tau^{-1} J\psi \left[ (2 + 3\psi^T \lambda^{-1} \psi) d\mu - \psi^T \lambda^{-1} d\omega \right]
+ 3(2 + \psi^T \lambda^{-1} \psi) \psi^T Jd\psi, \right)
\]

(the last five Killing one-forms will not be used in the following).

We solved the system of partial differential equations \[A.2\] in special cases. In the vacuum sector \((\mu = \psi = 0)\), the symmetrical solution is the Maison matrix \[5.18\]

\[
M_1(\lambda, \omega) = e^{\omega n^T} M_0(\lambda) e^{\omega n}. \quad (A.6)
\]

In the magnetostatic sector \((\omega = \psi = 0)\), the symmetrical solution is

\[
M_2(\lambda, \mu) = e^{\mu q^T} M_0(\lambda) e^{\mu q}. \quad (A.7)
\]

We only give here details on the solution in the electrostatic sector \((\omega = \mu = 0)\). The symmetrical matrix \(M_3\) solves the equation

\[
M_3^{-1} dM_3 = J_3, \quad (A.8)
\]

where \(J_3\) is obtained from \(J\) by setting \(\omega\) and \(\mu\) as well as \(d\omega\) and \(d\mu\) to zero. This equation constrains the tensorial characters and degrees of the
various matrix elements of $M$ to be (in $5 \times 5$ notation)

$$M = \begin{pmatrix}
M_{ab} & M_{a3} & M_{a\alpha} & M_{a3} & M_{a}
M_{3b} & M_{33} & M_{3\beta} & M_{33} & M_{3}
M_{a\alpha} & M_{a3} & M_{a\beta} & M_{a3} & M_{a}
M_{3\beta} & M_{33} & M_{3\beta} & M_{33} & M_{3}
M_{b} & M_{3} & M_{b} & M_{3} & M
\end{pmatrix} \quad \text{(A.9)}$$

(there should be no confusion between these matrix elements and the Killing vectors $M_{a\beta}$), and

$$[M] = \begin{pmatrix}
2/3 & -1/3 & 0 & 1 & 1/3 \\
-1/3 & -4/3 & -1 & 0 & -2/3 \\
0 & -1 & -2/3 & 1/3 & -1/3 \\
1 & 0 & 1/3 & 4/3 & 2/3 \\
1/3 & -2/3 & -1/3 & 2/3 & 0
\end{pmatrix}. \quad \text{(A.10)}$$

These in turn severely constrain the possible dependence of the matrix elements which must be built from the fields $\lambda_{ab}$, $\psi_a$, $\tau = \det \lambda$ (recall $[\lambda] = 2/3$ and $[\psi] = 1/3$), and the constant tensors $\epsilon_{ab}$, $\delta^b_a$ and $\epsilon^{ab}$, with dimensionless coefficients depending on the only dimensionless scalar $x \equiv \psi^T \lambda^{-1} \psi$.

Combining this information with the constraint that the matrix $M_3$ and its inverse are related by (5.32), we can determine this matrix by solving only part of the equations (A.8). The following matrix solves the equations (A.8) for the components $m_{a\beta}$, $n^{a\alpha}$, $r^{aT}$ and $q^{T}$:

$$M_3 = \begin{pmatrix}
\lambda + (2 + x)\psi^T \psi & 0 & \psi \psi^T \lambda^{-1} & -\lambda J \psi & \sqrt{2} (1 + x) \psi \\
0 & \lambda^{-1} \psi^T \psi & -\tau^{-1} \psi^T J & 0 & 0 \\
\psi^T J \lambda & 0 & (1 + x) \lambda^{-1} - \lambda^{-1} \psi \psi^T \lambda^{-1} & 0 & \sqrt{2} \lambda^{-1} \psi \\
\sqrt{2} (1 + x) \psi^T & 0 & \sqrt{2} \psi^T \lambda^{-1} & 0 & 1 + 2x
\end{pmatrix}. \quad \text{(A.11)}$$

This is of the form (5.31) with the blocks given by (5.33), and can be split up as the product

$$M_3(\lambda, \psi) = e^{\psi r T} M_0(\lambda) e^{\psi r}, \quad \text{(A.12)}$$

where the matrix $e^{\psi r}$ is given in (5.26).

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