Thermodynamics of Classical Systems on Noncommutative Phase Space

Mojtaba Najafizadeh
Department of Physics, Faculty of Sciences, Tarbiat Modares University,
P.O.Box: 14115-111, Tehran, Iran.
mnajafizadeh@gmail.com

and

Mehdi Saadat
Department of Physics, Faculty of Sciences, Shahid Rajaee Teacher Training University,
P.O.Box: 16785-163, Tehran, Iran.
saadat@srttu.edu

April 9, 2013

Abstract
We study the formulation of statistical mechanics on noncommutative classical phase space, and construct the corresponding canonical ensemble theory. For illustration, some basic and important examples are considered in the framework of noncommutative statistical mechanics: such as the ideal gas, the extreme relativistic gas and the 3-dimensional harmonic oscillator.

PACS numbers: 02.40.Gh. 05.20.-y

keywords: Noncommutative Phase-Space; Statistical Mechanics; Partition Function.
1 Introduction

In recent years, there has been an increasing interest in the study of physics on noncommutative (NC) spaces, because the effects of the space noncommutativity may become significant under extreme conditions, such as at energies above the TeV scale, or even at the string scale. There are many papers devoted to the study of various aspects of quantum field theory and quantum mechanics on NC spaces, where space coordinates are noncommuting with each other, but the momenta are commuting, or on NC phase space, where both “space-space” and “momentum-momentum” commutation relations can be nonvanishing. For references, see [1]-[10]. The Bose-Einstein statistics of noncommutative quantum mechanics requires both space-space and momentum-momentum noncommutativity [11, 12]. On the NC phase space, the NC algebra can be written as

\[
\begin{align*}
[x_i, x_j] &= i\hbar\theta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = i\hbar\bar{\theta}_{ij}, \quad [\hat{x}_i, \hat{p}_j] = i\hbar(\delta_{ij} - \frac{1}{4}\theta_{ik}\bar{\theta}_{kj}), \\
\end{align*}
\]

where \(\theta_{ij}\) is related to the noncommutativity of the space coordinates, while \(\bar{\theta}_{ij}\) reflects the noncommutativity of the momenta, and both of them are antisymmetric matrices with real constant elements. From the relations above, one can obtain a generalized Bopp’s shift as

\[
\begin{align*}
\hat{x}_i &= x_i - \frac{1}{2}\theta_{ij}p_j, \\
\hat{p}_i &= p_i + \frac{1}{2}\bar{\theta}_{ij}x_j,
\end{align*}
\]

where \(x_i\) and \(p_i\) are the coordinate and momentum operators on the usual (commutative) phase space, and \(i, j = 1, 2, 3\). After applying this shift, the effect caused by phase space noncommutativity can be calculated in the usual phase space [13]. In NC quantum mechanics and NC quantum field theory, the star product between two fields on NC phase space can be replaced by the generalized Bopp’s shift (2) for coordinates and (3) for momenta. The star product on the NC phase space can be defined as [13]

\[
(f \ast g)(x, p) = f(x, p) e^{\frac{i\hbar}{2} \theta_{ij} \bar{\theta}_{ji}} e^{\frac{i\hbar}{2} \bar{\theta}_{ij} \theta_{ji}} g(x, p)
\]

\[
= f(x, p)g(x, p) + \frac{i\hbar}{2} \theta_{ij} \partial_i f(x, p) \partial_j g(x, p) + \frac{i\hbar}{2} \bar{\theta}_{ij} \partial_i g(x, p) \partial_j f(x, p) + O(\theta^2) + O(\bar{\theta}^2).
\]
In this work, in Section 2, we present the uncertainty relations in the NC phase space and work out the NC deformation of Planck’s constant, which leads to a dimensionless classical partition function. In Section 3, assuming the existence of a symplectic structure consistent with the commutation rules (1), the corresponding classical partition function is derived. In Sections 4 to 6, three concrete examples are presented: the classical ideal gas, the extreme relativistic gas and the classical harmonic oscillator in a 3-dimensional NC phase space, and the new features that arise are discussed.

2 The Uncertainty Relation on NC Phase Space

In the 3-dimensional classical partition function for a single particle, we put $\frac{1}{\hbar}$ as the quantity which makes the volume of phase space dimensionless. In this section, we derive the analogous factor, that makes the volume of NC phase space dimensionless. From the relation (1), one can write

$$[\hat{x}_i, \hat{p}_j] = i\hbar_{ij},$$

where $\hbar_{ij}$ is a tensor playing the role of the deformation of Planck’s constant on a NC phase space. By setting $\bar{\theta}_3 = \bar{\theta} = \frac{\theta}{2}$, and the rest of the $\theta$ and $\bar{\theta}$ components to zero (which can be done by a rotation or a redefinition of coordinates), Eq. (5) can be rewritten as

$$[\hat{x}_i, \hat{p}_j] = i\hbar \begin{pmatrix} 1 + \frac{\theta \bar{\theta}}{16} & 0 & 0 \\ 0 & 1 + \frac{\theta \bar{\theta}}{16} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad i, j = 1, 2, 3$$

where $\theta_{ij} = \frac{1}{2} \epsilon_{ijk} \theta_k$ and $\bar{\theta}_{ij} = \frac{1}{2} \epsilon_{ijk} \bar{\theta}_k$. The following equations

$$[\hat{x}_1, \hat{p}_1] = i\hbar_{11} = i\hbar \left( 1 + \frac{\theta \bar{\theta}}{16} \right),$$

$$[\hat{x}_2, \hat{p}_2] = i\hbar_{22} = i\hbar \left( 1 + \frac{\theta \bar{\theta}}{16} \right),$$

$$[\hat{x}_3, \hat{p}_3] = i\hbar_{33} = i\hbar ,$$

lead to the uncertainty relations on NC phase space.
\[ \Delta \hat{x}_1 \Delta \hat{p}_1 \sim \tilde{h}_{11} = \hbar \left( 1 + \frac{\theta \bar{\theta}}{16} \right), \quad (10) \]
\[ \Delta \hat{x}_2 \Delta \hat{p}_2 \sim \tilde{h}_{22} = \hbar \left( 1 + \frac{\theta \bar{\theta}}{16} \right), \quad (11) \]
\[ \Delta \hat{x}_3 \Delta \hat{p}_3 \sim \tilde{h}_{33} = \hbar. \quad (12) \]

From these relations, it is obvious that the following factor makes the volume of NC phase space dimensionless

\[ \frac{1}{\tilde{h}_{11} \tilde{h}_{22} \tilde{h}_{33}} = \frac{1}{\hbar^3} = \frac{1}{\hbar^3 \left( 1 + \frac{\theta \bar{\theta}}{8} \right)}. \quad (13) \]

In this work, we considered an expansion of the denominator up to the second order in the NC parameters. It should be mentioned that the deformation of the Planck constant, on a 2-dimensional NC phase space, has the form [13]

\[ \tilde{h} = \hbar (1 + \frac{\theta \bar{\theta}}{4}), \quad (14) \]

where \( \theta_{ij} = \epsilon_{ij} \theta \) and \( \bar{\theta}_{ij} = \epsilon_{ij} \bar{\theta} \). Thus, the appropriate factor, that makes the volume of the NC phase space dimensionless, can be written as

\[ \frac{1}{\hbar^2} = \frac{1}{\hbar^2 \left( 1 + \frac{\theta \bar{\theta}}{2} \right)}. \quad (15) \]

### 3 Classical Partition Function on NC Phase Space

The purpose of this paper is precisely to study noncommutative classical systems. The passage between NC classical mechanics and NC quantum mechanics is assumed to be realized via the following generalized Dirac quantization condition:

\[ \{ f, g \} \rightarrow \frac{1}{i\hbar} [O_f, O_g] \quad (16) \]

where we denote the operator associated with a classical observable \( f \) as \( O_f \). This quantization generalizes the relations (1) in the following way:

\[ \{ \hat{q}_i, \hat{q}_j \} = \theta_{ij} , \quad \{ \hat{p}_i, \hat{p}_j \} = \bar{\theta}_{ij} , \quad \{ \hat{q}_i, \hat{p}_j \} = \delta_{ij} - \frac{1}{4} \theta_{ik} \bar{\theta}_{kj}, \quad (17) \]
where $\hat{q}_i$ and $\hat{p}_i$ are the coordinate and momentum classical observables in NC phase space. Moreover, the dimensions of $\theta_{ij}$ and $\bar{\theta}_{ij}$ are $(\text{length})^2/\hbar$ and $(\text{momentum})^2/\hbar$ respectively, with $i, j = 1, 2, 3$. From the relations above, one can derive an expression for NC classical observables as

$$\hat{q}_i = q_i - \frac{1}{2} \theta_{ij} p_j, \quad \hat{p}_i = p_i + \frac{1}{2} \bar{\theta}_{ij} q_j,$$

(18)

(19)

where $q_i = (x, y, z)$ and $p_i = (p_x, p_y, p_z)$ are the classical observables in the usual (commutative) phase space. As one can see, in the classical limit, the symplectic structure (17) will not depend on $\hbar$, as expected.

To obtain the classical partition function, in the canonical ensemble in NC phase space, it is possible to consider the following formula

$$Q_{NC}^1 = \frac{1}{\hbar^3} \int e^{-\beta H_{NC}(\hat{q}, \hat{p})} d^3\hat{q} d^3\hat{p},$$

(20)

which is written for a single particle, and which includes the $1/\hbar^3$ factor, that was derived in Section 2. $H_{NC}(\hat{q}, \hat{p})$ is the Hamiltonian of a NC classical system, expressed in terms of noncanonical coordinates and momenta. According to the relations (18) and (19), and considering $\theta_{ij} = \frac{1}{2} \epsilon_{ijk} \theta_k$ and $\bar{\theta}_{ij} = \frac{1}{2} \epsilon_{ijk} \bar{\theta}_k$, it is easy to write the integration measure in (20), up to the second order in the NC parameters, as

$$d^3\hat{q} d^3\hat{p} = \left(1 - \frac{\bar{\theta}}{8}\right) d^3q d^3p,$$

(21)

having assumed $\bar{\theta} = (0, 0, \theta)$ and $\bar{\sigma} = (0, 0, \bar{\sigma})$. If one writes $H_{NC}(\hat{q}, \hat{p})$ in terms of canonical coordinates and momenta (which can be done by applying (18) and (19) to the Hamiltonian), then the appropriate expression for the partition function turns out to be

$$Q_{NC}^1 = \frac{1}{\hbar^3(1 + \frac{\theta}{8})} \int e^{-\beta H_{NC}(q, p)} \left(1 - \frac{\bar{\theta}}{8}\right) d^3q d^3p$$

$$= \frac{1}{\hbar^3} \int e^{-\beta H_{NC}(q, p)} \left(1 - \frac{\theta_{ij}}{4}\right) d^3q d^3p.$$

(22)

where we used (13) and (21). $H_{NC}(q, p)$ is the Hamiltonian of a NC classical system, expressed in terms of canonical coordinates and momenta. Thus,
to find the partition function of a single particle of a NC classical system, it is enough to rewrite the Hamiltonian of the system in terms of canonical classical observables, and plug it into the relation (22). In the case when the basic constituents of the system are non-interacting, one can write the classical partition function for a system of $N$ particles in a 3-dimensional NC phase space as

$$Q_{NC}^N = \frac{1}{N!} [Q_{NC}^1]^N,$$  \hspace{1cm} (23)

where $1/N!$ is the Gibbs correction factor.

4 Classical Ideal Gas on NC Phase Space

Let us consider a system of $N$ identical molecules, assumed to be monoatomic (so that there are no internal degrees of motion to be considered), confined to a space of volume $V (= L^3)$ and in equilibrium at a temperature $T$. Since there are no intermolecular interactions to be taken into account, the Hamiltonian of a single molecule of the system, in NC classical phase space, is simply given by

$$H_{NC}(\hat{q}, \hat{p}) = \frac{\hat{p}_i \hat{p}_i}{2m}.$$  \hspace{1cm} (24)

Thus, from the relations (18) and (19), the Hamiltonian takes the following form

$$H_{NC}(q, p) = \frac{1}{2m} \left( p^2 - \frac{1}{2} \bar{\theta} L_z + \frac{1}{16} \bar{\theta}^2(x^2 + y^2) \right),$$  \hspace{1cm} (25)

where $\bar{\theta} \cdot (\vec{q} \times \vec{p}) = \bar{\theta} L_z$ and $(\bar{\theta} \times \vec{q})^2 = \bar{\theta}^2(x^2 + y^2)$, with the assumption

$$\bar{\theta} = (0, 0, \theta), \quad \theta_{ij} = \frac{1}{2} \epsilon_{ijk} \theta_k,$$  \hspace{1cm} (26)

and

$$\bar{\theta} = (0, 0, \bar{\theta}), \quad \bar{\theta}_{ij} = \frac{1}{2} \epsilon_{ijk} \bar{\theta}_k.$$  \hspace{1cm} (27)

Therefore, the partition function reads

$$Q_{NC}^1 = \frac{1}{h^3} \int e^{-\frac{1}{2m}(\theta^2 - \frac{1}{4} \bar{\theta} L_z + \frac{1}{16} \bar{\theta}^2 x^2 + y^2)} \left( 1 - \frac{\theta \bar{\theta}}{4} \right) d^3q d^3p.$$  \hspace{1cm} (28)

By performing the integral, up to second order in the NC parameters, and using Eq. (23), one obtains

$$Q_{NC}^N = \frac{V^N}{N! h^{3N} (2\pi m KT)^{\frac{3N}{2}}} \left( 1 - \frac{\theta \bar{\theta}}{4} \right)^N.$$  \hspace{1cm} (29)
As a consequence, the Helmholtz free energy is given by

\[ A^{NC} = -KT \ln (Q_{NC}^N) = A + NKT \theta \bar{\theta}/4 , \]  

(30)

where \( A \) is the Helmholtz free energy in the ordinary (commutative) phase space. The complete thermodynamics of the ideal gas can be derived from (29) and (30) in a straightforward way. For instance,

\[ S^{NC} = -\left( \frac{\partial A^{NC}}{\partial T} \right)_{N,V} = S - NK\theta \bar{\theta}/4 , \]  

(31)

\[ \mu^{NC} = \left( \frac{\partial A^{NC}}{\partial N} \right)_{V,T} = \mu + KT \theta \bar{\theta}/4 , \]  

(32)

\[ P^{NC} = -\left( \frac{\partial A^{NC}}{\partial V} \right)_{N,T} = P , \]  

(33)

\[ U^{NC} = -\frac{\partial}{\partial \beta} \ln (Q_{NC}^N) = U , \]  

(34)

\[ C_{V}^{NC} = \left( \frac{\partial U^{NC}}{\partial T} \right)_{N,V} = C_V , \]  

(35)

where \( S, \mu, P, U \) and \( C_V \) are the usual thermodynamic quantities, which have been calculated in [14].

5 Extreme Relativistic Gas on NC Phase Space

Let us now consider an ideal extreme relativistic gas, consisting of \( N \) monoatomic molecules with energy-momentum relationship \( E = pc \), \( c \) being the speed of light, confined to a space of volume \( V (= L^3) \) and in equilibrium at temperature \( T \). In the NC classical phase space, the Hamiltonian of a single molecule of the system is given by

\[ H^{NC}(q,p) = c \sqrt{\hat{\theta}_i \hat{p}_i} \]

\[ = c \sqrt{\left( p_i + \frac{1}{2} \hat{\theta}_{ij} q_j \right) \left( p_i + \frac{1}{2} \hat{\theta}_{ik} q_k \right)} \]

\[ = pc - \frac{c \bar{\theta} L_z}{4p} + \frac{c \bar{\theta}^2}{32p} (x^2 + y^2) - \frac{c \bar{\theta}^2 L_z^2}{32p^3} + O(\bar{\theta}^3) , \]  

(36)
where we used the conditions (26) and (27). In view of (22), the partition function can be readily obtained as

\[
Q_{NC}^1 = \frac{1}{h^3} \int e^{-\beta \left( p c - \frac{\theta L_z}{4} + \frac{\theta^2}{32p}(x^2 + y^2) - \frac{\theta^2}{32p^3} \right) \left( 1 - \frac{\theta \tilde{\theta}}{4} \right)} d^3q d^3p
\]

\[
= 8\pi V \left( \frac{KT}{hc} \right)^3 \left( 1 - \frac{\theta \tilde{\theta}}{4} \right),
\]

whence

\[
Q_{NC}^N = \frac{1}{N!} \left( 8\pi V \right)^N \left( \frac{KT}{hc} \right)^{3N} \left( 1 - \frac{\theta \tilde{\theta}}{4} \right)^N.
\]

Finally, the complete thermodynamics of the ideal extreme relativistic gas, in NC classical phase space, would be similar to Eq. (30) to (35).

6 3-Dimensional Harmonic Oscillator on NC Phase Space

We shall now examine a system of \( N \), practically independent, 3-dimensional harmonic oscillators. The Hamiltonian of each of them, up to second order in the NC parameters, is then given by

\[
H_{NC}(\hat{q}, \hat{p}) = \frac{1}{2m} (\hat{p} \hat{p}) + \frac{1}{2} m \omega^2 (\hat{q} \hat{q}),
\]

or

\[
H_{NC}(q, p) = \frac{1}{2m} (p_i + \frac{1}{2} \theta_{ij} q_j)(p_i + \frac{1}{2} \theta_{ik} q_k) + \frac{1}{2} m \omega^2 (q_i - \frac{1}{2} \theta_{ij} p_j)(q_i - \frac{1}{2} \theta_{ik} p_k)
\]

\[
= \frac{1}{2m} \left( p^2 - \frac{1}{2} \theta L_z + \frac{1}{16} \theta^2 (x^2 + y^2) \right)
\]

\[
+ \frac{1}{2} m \omega^2 \left( q^2 - \frac{1}{2} \theta L_z + \frac{1}{16} \theta^2 (p_x^2 + p_y^2) \right),
\]

where we used Eq. (26) and (27). Thus, by analogy with Eq. (22), we readily obtain the partition function for the single oscillator as

\[
Q_{NC}^1 = \frac{1}{h^3} \int e^{-\beta \left( p c - \frac{\theta L_z}{4} + \frac{\theta^2}{32p}(x^2 + y^2) - \frac{\theta^2}{32p^3} \right) \left( 1 - \frac{\theta \tilde{\theta}}{4} \right)} d^3q d^3p
\]

\[
	imes (1 - \frac{\theta \tilde{\theta}}{4}) d^3q d^3p.
\]
\[ Q_N^{NC} = \frac{1}{(\beta \hbar \omega)^3} \left( 1 - \frac{\theta \bar{\theta}}{8} \right)^N. \]  

Therefore, the complete thermodynamic expressions for the NC 3-dimensional harmonic oscillator will be given by:

\[ A^{NC} = -KT \ln(Q_N^{NC}) = A + NK T \theta \bar{\theta} / 8, \]

\[ S^{NC} = -\left( \frac{\partial A^{NC}}{\partial T} \right)_{N,V} = S - NK \theta \bar{\theta} / 8, \]

\[ \mu^{NC} = \left( \frac{\partial A^{NC}}{\partial N} \right)_{V,T} = \mu + KT \theta \bar{\theta} / 8, \]

\[ P^{NC} = -\left( \frac{\partial A^{NC}}{\partial V} \right)_{N,T} = P, \]

\[ U^{NC} = -\frac{\partial}{\partial \beta} \ln(Q_N^{NC}) = U, \]

\[ C_V^{NC} = \left( \frac{\partial U^{NC}}{\partial T} \right)_{N,V} = C_V. \]

### 7 Conclusions

In this work, we studied the effects of a NC phase space on some classical statistical systems. We presented a formulation to calculate the classical partition function according to the canonical ensemble theory, up to the second order in the noncommutative parameters, Eq. (22). In order to make the volume of NC phase space dimensionless, we used the uncertainty relations ((10) - (12)) and, then, found the expressions (13) and (15) for the three- and two-dimensional NC phase space, respectively. In Sections 4, 5 and 6, we considered the generalization of three well-known classical systems to a
NC phase space. The results show that the terms arising from the noncommutativity of phase space emerge in thermodynamic quantities that are not measurable \((A, S\) and \(\mu)\). We conclude that, in classic statistical mechanics, the present experimental instruments cannot currently detect the effects of noncommutativity, at least up to the second order in the NC parameters. The final point that must be emphasized is about the entropy. It is known that the NC phase space is generally “more ordered” than the corresponding commutative phase space. The reason is that, for a physical problem defined in a NC phase space, the degeneracy of the system decreases \([15]\) (a related facet of this issue is the existence of “exotic” phases, such as striped phases, in NC models, which have been observed in numerical simulations of noncommutative scalar field theory: see Refs. \([16, 17]\) and references therein). Therefore, one expects the entropy of the NC system to be reduced with respect to the commutative phase space. This intuitive expectation is indeed confirmed by relations (31) and (46).

References

[1] N. Seiberg, and E. Witten, JHEP 032, 9909 (1999).

[2] M. Chaichian, M. M. Sheikh-Jabbari, and A. Tureanu, Phys. Rev. Lett. 86, 2716 (2001).

[3] M. Chaichian, A. Demichev, P. Presnajder, M. M. Sheikh-Jabbari, and A. Tureanu, Nucl. Phys. B 611, 383 (2001).

[4] M. Chaichian, P. Presnajder, M. M. Sheikh-Jabbari, and A. Tureanu, Phys. Lett. B 527, 149 (2002), p. 149-154.

[5] O. F. Dayi, and A. Jellal, J. Math. Phys. 43, 4592 (2002).

[6] H. Falomir, J. Gamboa, M. Loewe, F. Mendez, and J. C. Rojas, Phys. Rev. D 66, 045018 (2002).

[7] K. Li, and S. Dulat, Eur. Phys. J. C 46, 825 (2006).

[8] K. Li, and J. Wang, Eur. Phys. J. C 50, 1007 (2007).

[9] B. Mirza, and M. Zarei, Eur. Phys. J. C 32, 583 (2004).
[10] B. Mirza, R. Narimani, and M. Zarei, Eur. Phys. J. C 48, 641 (2006).
[11] M. Demetrian, and D. Kochan, Acta Phys. Slov. 52, 1 (2002), p. 1-9.
[12] Jian-zu Zhang, Phys. Lett. B 584, 204 (2004).
[13] K. Li, and S. Dulat, Chin. Phys. C 34, 944 (2010).
[14] R. K. Pathria, Statistical Mechanics, 2nd ed. (Butterworth-Heinemann, Oxford, 1996), Chap. 3.
[15] B. Muthukumar, and P. Mitra, Phys. Rev. D 66, 027701 (2002).
[16] M. Panero, JHEP 0705 082 (2007).
[17] M. Panero, SIGMA 2 081 (2006).