Generating multi-scaling networks with different types of nodes

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A variety of scale-free networks have been created since the pioneer work by A.-L. Barabási and R. Albert. All these networks are homogeneous since they are composed of the same kind of nodes. In the realistic world, however, one element (node or vertex) in the network may play different roles and hence has different functions. In this Letter, we develop a new kind of network to account for this property. In our model, each type of nodes may exhibit a scaling law in the degree distribution and the scaling exponents are adjustable. As a consequence, the whole network lacks of such scaling characteristics, which indicates that many previous statistical results based on empirical data that claimed to be scale-free networks may need to be reexamined. This model poses an alternative way of the network division other than the module method. Besides, one can expect that this new network will exhibit some interesting properties concerning the dynamical processes on it.

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In recent years, an increasing interest on complex networks has risen in physical society because it shows as one of the most promising tools to describe various social, biological, and technical systems, such as the Internet$^1$, $^2$, $^3$, the World Wide Web (WWW)$^4$ or collaboration networks$^5$, $^6$. In the models, vertices or nodes may represent people, proteins, species, routers, or html documents while the links between the nodes correspond to acquaintance, physical interactions, predation relationships, cable connections, or hyper-links, respectively. In real-world networks, two outstanding phenomena are mostly presented. One is the small-world phenomenon$^7$, $^8$, which refers to the degree of the nodes (the number of links connected to it) obeying a power-law distribution as $P(k) \sim k^{-\gamma}$. To explain the scale-free property of a network, A.-L. Barabási and R. Albert (BA)$^9$ designed a model by introducing the concepts of growing network and of preferential attachment. The network are growing at a constant rate and new nodes are attached to the older ones with a probability which is proportional to the degrees of the preexisting nodes,

$$\Pi(k_i) = \frac{k_i}{\sum_j (k_j)}.$$  

(1)

The network generated in this way has a fixed scaling exponent $\gamma = 3$. Variant models that assign a fitness to every newly-added node, which accounts for the importance of the node for attracting new links, can generate networks with adjustable scaling exponents$^{10}$, $^{11}$, $^{12}$. Besides, many other network models and methods are proposed to probe the various properties in real-world networks$^{13}$, $^{14}$, $^{15}$, $^{16}$, $^{17}$, $^{18}$, $^{19}$.

Nevertheless, all the previous works are focused on networks in which all the nodes are identical and play a unique function in the homogeneous (or inhomogeneous) network. As one explicitly examines into a real-world network, however, it can be found that a node is generally not play only one role in the network. It may play multi-roles or couple with nodes in other systems. As show in Fig.1, nodes of type A or type B respectively form two subsystems with links between some of them. From another point of view, if we regard the whole object as two different networks which are respectively consisted of type A or type B nodes, then a node in one network may couple with a node in the other network even though this two networks may have completely different topology or function. These nodes may behavior quite differently as they present in different environments. For example, a researcher may work in two or more fields and cooperate with different authors. He writes papers in condensed matter physics with collaborators who are always engaged in this field while he may also contribute to the high energy physics by participating another group. He may also have an independent friendship network$^{20}$.

Many works have been carried out to deal with the community or module structure of the network, where nodes is liable to interconnect within one community$^{10}$, $^{18}$, $^{21}$, $^{22}$. In this Letter, we introduce another way to catalogue the network nodes according to their functions as relative independent subsystems. We will construct a heterogeneous network consisted of two types of nodes, type A and type B. As in a biochemical reaction processes, the reactants and their relationship form a complex network. One type of reactant preferentially combines with some other specific reactants. The attractance or binding energy $\alpha_1$ between the same type of nodes (A-A or B-B) differs from the attractance $\alpha_2$ between the different types of nodes (A-B). When a new node (type A or type B) joins the growing network, two rules govern its attachment to the preexisting nodes: i) The Barabási and Albert preferential attachment, and ii) The selection of partners according to its genus. The whole network is such composed of two types of nodes or vertices, i.e., it is a heterogeneous network with two connected subgraphs.
of $G_A$ and $G_B$. Figure 1 is a sample network generated by this algorithm for 30 nodes with $\alpha_2/\alpha_1 = 0.1$ and the proportion of type A nodes $p_A = 0.6$.

It is found that our model network exhibits a multi-scaling structure in which the degree distribution for each subgraph $G_A$ or $G_B$ shows a scaling law. As a consequence, the total degree distribution lacks of such scaling characteristics except for some special cases. The result is instructive for many recent statistics on empirical data that claimed to have the scaling law distribution. They are generally not reliable if the nodes play multi-roles in the generating process or the network is composed of several subsystems and need to be carefully reexamined. Many of them may show as the pseudo-scaling laws.

We assume that the newly-added node presents as type A with probability $p_A$ and as type B with probability $p_B = 1 - p_A$, respectively. Each node has $m$ feet to be connected to the existing network. The vertices of type A and type B respectively form two vertex sets of $V_A = \{v_1, v_2, \ldots, v_{N_A}\}$ and $V_B = \{w_1, w_2, \ldots, w_{N_B}\}$. Here $N_A$ and $N_B$ are the total numbers of node A and node B, respectively. From the above rules, we obtain the continuously growing equations

$$\frac{\partial k_A(i)}{\partial t} = \frac{mp_A\alpha_1k_A(i)}{\sum_{j \in V_A} \alpha_1k_A(j) + \sum_{j \in W_B} \alpha_2k_B(j)} + \frac{mp_B\alpha_2k_A(i)}{\sum_{j \in V_A} \alpha_2k_A(j) + \sum_{j \in W_B} \alpha_1k_B(j)},$$

$$\frac{\partial k_B(i)}{\partial t} = \frac{mp_A\alpha_2k_B(i)}{\sum_{j \in V_A} \alpha_1k_A(j) + \sum_{j \in W_B} \alpha_2k_B(j)} + \frac{mp_B\alpha_1k_B(i)}{\sum_{j \in V_A} \alpha_2k_A(j) + \sum_{j \in W_B} \alpha_1k_B(j)}. \tag{2}$$

where $k_A(i)$ and $k_B(i)$ are the degree of the $i$-th node of either type A or type B, respectively. The first line of the above equations describes the growth rate of type A nodes in the network. The first term on the right-handed side represents a newly-added type A node, which is generated with probability $p_A$, is attached to a type A node in the preexisting network while the second term is a newly-added type B node to be attached also to a type A node in the network. Analogously, the second line describes the growth rate of type B nodes in the network.

To solve the above equations, we consider the thermodynamical approximation. Suppose the system has multi-scaling law, namely, the dynamic exponents depend on the attractance $\alpha_1$ and $\alpha_2$,

$$k_A(t, t_0, i) = m(t-t_0)^{\beta_A},$$

$$k_B(t, t_0, i) = m(t-t_0)^{\beta_B}. \tag{3}$$

where $t_0$ is the time at which the node $i$ was born. The dynamic exponents $\beta_A$ and $\beta_B$ are bounded, i.e. $0 < \beta_A, \beta_B < 1$ because a node always increases the number of links in time ($\beta_A, \beta_B > 0$) and $k_A, k_B$ cannot increase faster than $t$ ($\beta_A, \beta_B < 1$). We calculate the sum over $j$ in Eqs. (2) by writing them in the integral forms. By noting that the nodes respectively belong to sets $V_A$ and $W_B$ with probability $p_A$ and $p_B$, one has

$$\sum_{j \in V_A} k_A(j) = m \int_{t_0}^{t} p_A(t-t_0)^{\beta_A} \quad t \to \infty \quad \frac{mp_A t}{1 - \beta_A},$$

$$\sum_{j \in W_B} k_B(j) = m \int_{t_0}^{t} p_B(t-t_0)^{\beta_B} \quad t \to \infty \quad \frac{mp_B t}{1 - \beta_B}. \tag{4}$$

Here we have dropped the $t^0$ term for it becomes less important as $t \to \infty$.

After some calculations by introducing a variable $z = (1 - \beta_A)/(1 - \beta_B)$, we obtain a third order equation as

$$z^3 + \frac{p_A}{p_B} \left( \frac{2\alpha_1}{\alpha_2} + \frac{\alpha_2}{\alpha_1} - 1 \right) z^2 + \left[ \frac{p_A}{p_B} \left( -\frac{2\alpha_1}{\alpha_2} - \frac{\alpha_2}{\alpha_1} + 1 \right) + \frac{p_A^2}{p_B^2} \left( 2 - \frac{\alpha_2}{\alpha_1} \right) \right] z - \frac{p_A^3}{p_B^3} = 0, \tag{5}$$

$$1 - \frac{p_A \alpha_1}{p_A \alpha_1 + p_B \alpha_2} + \frac{1}{p_B \alpha_2 + p_A \alpha_1} = \beta_A. \tag{6}$$

Therefore, it demonstrates that in the thermodynamical limit, the degree distributions of the constituent nodes respectively obey the power law. The power exponents of each type of nodes are generally different. Obviously, the total degree distribution that comprises both type A and type B nodes will not exhibit the scaling characteristics.

Figure 2 plots the dependence of the exponents $\gamma_A$ and $\gamma_B$ ($\gamma = 1 + 1/\beta$) on the relative attractive strength
between the nodes $\alpha_2/\alpha_1$. It is seen that the exponents can be either larger than 3 or in the regime of $2 < \gamma < 3$, depending on the parameter value of $\alpha_2/\alpha_1$. The upper panel shows that if $\gamma_A > 3$, then $\gamma_B < 3$ whereas if $\gamma_A < 3$, then $\gamma_B > 3$. For smaller $\alpha_2/\alpha_1$, the newly-added node will preferentially attaches to the same type of preexisting nodes. In the case of $p_A > 0.5$, type A nodes with larger degrees are continuing to attract more links and thus the exponent is relative small. As $\alpha_2/\alpha_1 \to 0$, the scaling exponents for both genuses of nodes tend to 3. In this case, the coupling between the two types of nodes are weak and the resultant network nearly divides into two isolated subsystems. At the point of $\alpha_2/\alpha_1 = 1$, which corresponds to the case that A and B nodes are completely identical, our model degenerates into the ordinary BA model with a fixed exponent $\gamma = 3$. As a special case of $p_A = p_B = 0.5$, the exponents $\gamma_A = \gamma_B \equiv 3$, regardless of the detailed values of $\alpha_2/\alpha_1$.

For larger $\alpha_2/\alpha_1$ ($\alpha_2/\alpha_1 > 1$, see the lower panel of Fig.2) where one type of node will preferentially attach to the different type of nodes (A to B or B to A), the network seems to be interweaved alternatively by node A and node B. Hence for $p_A > 0.5$, it becomes hard for the type A nodes that already have more links to get even more links, i.e., the degree growth is damped. Most nodes of this type will remain few links and so the scaling exponent becomes larger. On the other hand, for $p_B < 0.5$, those nodes of type B that already have more links will continue to attract more links from type A nodes and so the scaling exponent decreases as $\alpha_2/\alpha_1$ increases.

To check the above theoretical results, we simulate the growth process on the computer. We start with $N_0 = 4$ interconnected nodes consisted of 2 type A and 2 type B nodes, respectively. Figure 3 displays the simulated degree distribution $P(k)$ for $p_A = 0.9$ and $\alpha_2/\alpha_1 = 0.5$. A total of 200,000 nodes are involved. The solid lines are respectively the theoretical results for type A ($\gamma_A = 2.95$) and type B ($\gamma_B = 4.06$) nodes. It is seen that the simulating data coincide with the theoretical result quite well. Here we point out an important fact. Just from the simulating data (see the inset of Fig.3), one may misinterpret that the total distribution indiscriminating the node genuses also show a power law relation. However, it is incorrect from the theoretical considerations. This fact precautions us that it should be wary when dealing with empirical data that include several types of nodes to draw out a scaling law.

Figure 4 is the same as in Fig.3 for $\alpha_2/\alpha_1 = 3.0$. The two solid lines correspond to the theoretical results with $\gamma_A = 3.96$ and $\gamma_B = 2.18$, respectively. It shows that for type B nodes, the data deviate from the theoretical lines. This is because we have dropped in Eqs. a term which behaviors as $t^{\beta}$. When $\beta \to 1$ (or $\gamma \to 2$), as in the present case, this term becomes increasingly important for limited number of simulating nodes and hence the approximation becomes poor. With the increase of node number, the coincidence should improve correspondingly.

In summary, we have developed a bipartite network in which the nodes are divided into two genuses according to the interactions between them. There is an important difference between our model and the fitness model. In the fitness model, each node has fixed fitness while in our model, the interaction between a node with others is dependent on the types of its partners. Just as in a library, one classifies the books into catalogues and the books are connected by the cross-index table. One can reach a specific book through different ways by following the classification method. Two major conclusions can be reached in our model: i) The network is catalogued by the genuses or functions of the nodes while most previous works divide the network by communities or modules. Our model is in fact heterogenous. ii) There are multi-scaling characteristics for each type of nodes, which implies that the total degree lacks of a power law distribution and many previous empirical statistics may need to be reexamined. Our model may provide a prototype to discuss couplings between two or even more subsystems which have scaling properties in degree distributions. It is also interesting to explore various dynamical processes such as searching processes on this network. We expect that new algorithms based on our model or its possible variants will largely promote the searching efficiency on this kind of networks.

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Figure Captions

Figure 1 (Color online) A heterogeneous network generated from the rules described in the context. The dotted lines represent interactions between different types of nodes.

Figure 2 (Color online) Dependence of the scaling exponents of the degree distributions for either type A ($\gamma_A$) or type B ($\gamma_B$) nodes on parameter ratio $\alpha_2/\alpha_1$. Upper panel: $p_A = 0.9$. Lower panel: $\gamma_A$ versus $\alpha_2/\alpha_1$ for $p_A = 0.05, 0.20, 0.50, 0.70, 0.90$.

Figure 3 (Color online) Comparison of simulating degree distributions for type A and type B nodes with theoretical results. $\alpha_2/\alpha_1 = 0.5$. The total number of nodes are 200,000 and $p_A = 0.9$. Inset: The degree distribution indiscriminating the node genus.

Figure 4 (Color online) Same as in figure 3 for $\alpha_2/\alpha_1 = 3.0$. 
