Sum rules for hadronic elastic scattering for the Tevatron, RHIC and LHC

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Under the assumption of total absorption and dominance of the imaginary part of the scattering amplitude, we present a sum rule for any hadronic elastic differential cross-section \( \frac{d\sigma}{dt} \): the dimensionless quantity \( \frac{1}{\pi} \int dt \sqrt{\frac{dt}{\pi}} \frac{d\sigma}{dt} \to 1 \), at asymptotic energies. Experimental data from ISR and Tevatron confirm a trend towards its saturation and some estimates are presented for LHC. Its universality and further consequences for the nature of absorption in QCD based models for elastic and total cross-sections are explored.

Ab initio calculations of hadronic elastic amplitudes and total cross-sections (in QCD) are presently difficult due to our meager understanding of “soft” physics, that is, the non-perturbative and confinement region of QCD. Hence, the need to invoke general principles such as analyticity and unitarity to obtain bounds and restrictions on these amplitudes [1–3]. Analyticity and unitarity are expected to hold for finite-ranged hadron dynamics, only massive hadrons being the bound states of quarks and glue. In the following, we find that, under rather mild assumptions, a universal behavior for all hadrons is likely to emerge at asymptotic energies.

Consider the amplitude for an elastic process \( A(p_a) + B(p_b) \to A(p_c) + B(p_d) \). Let \( s = (p_a + p_b)^2 \), be the square of the CM energy, \( t = (p_a - p_c)^2 \), be the momentum transfer and let us normalize the amplitude so that the differential and total cross-sections are given by

\[
d\sigma/dt = \pi |F(s,t)|^2; \quad \sigma_{tot}(s) = 4\pi \Im F(s,t=0).
\]

(1)

To enforce (direct or \( s \)-channel) unitarity and incorporate the knowledge that most of the hadronic scatterings at high energies are peaked in the forward direction, an eikonal formalism is convenient. The elastic amplitude may be expanded in the impact parameter (\( b \)-space) in the usual fashion

\[
F(s,t) = i \int_0^\infty (b db) J_0(b\sqrt{-t}) \tilde{F}(s,b),
\]

(2a)

in terms of the “partial \( b \)-wave” amplitudes

\[
\tilde{F}(s,b) = 1 - \eta(s,b)e^{2i\delta_R(s,b)},
\]

(2b)

where the inelasticity factor \( \eta(s,b) \) lies between 0 and 1, and \( \delta_R(s,b) \) is the real part of the phase shift. The dimensionless “\( b \)-wave cross-sections” are given by

\[
\frac{d^2\sigma_{el}}{db^2} = 1 - 2\eta(s,b) \cos(2\delta_R(s,b)) + \eta^2(s,b),
\]

(3a)

\[
\frac{d^2\sigma_{inel}}{db^2} = 1 - \eta^2(s,b),
\]

(3b)

and

\[
\frac{d^2\sigma_{tot}}{db^2} = 2[1 - \eta(s,b) \cos(2\delta_R(s,b))].
\]

(3c)

Eqs.(3) show explicitly the maximum permissible rise for the different cross-sections due to unitarity. For complete absorption of “low” partial waves at asymptotic energies (which translates into \( \eta(s,b) \to 0 \) for \( b \to 0 \) and \( s \to \infty \)), one obtains the geometric limit (including the contribution from shadow scattering):

\[
\frac{d^2\sigma_{el}}{db^2} = \frac{d^2\sigma_{inel}}{db^2} = \frac{1}{2} \frac{d^2\sigma_{tot}}{db^2} \to 1 \quad \text{for} \quad b \to 0 \quad \text{and} \quad s \to \infty.
\]

(4)
Evidence for such a maximum rise (i.e., the validity of Eq.(4)) has been provided through various models, such as the resummed soft gluon models [4–7] and other models [8–10], all of whom incorporate the observed rise in pp and p\bar{p} total cross-sections.

Our objective in the present paper is to provide model independent predictions through sum rules over experimentally measurable quantities such as \(\frac{d\sigma}{dt}\). We shall first derive a lower bound for the dimensionless integral \(I_0(s)\), defined as

\[
I_0(s) = \frac{1}{2} \int_{-\infty}^{0} (dt) \sqrt{\frac{d\sigma}{\pi dt}} .
\]  

Using Eqs.(1) and (2), it is easy to show that

\[
I_0(s) \geq 1 - \eta(s, 0).
\]

To obtain an upper bound for \(I_0(s)\), via the obvious inequality

\[
I_0(s) \leq \int_{0}^{\infty} (qdq) \left[ |ImF(s, q^2)| + |ReF(s, q^2)| \right],
\]

further input are needed. Since \(F(s, q^2)\) is analytic in \(q^2 = -t\) until \(q^2 = -\mu_0^2\) (the lowest mass exchanged in the t-channel), the \(b\)-expansion must converge until this imaginary \(q = i\mu_0\). For positive values of \(t\), in Eq.(2) the Bessel functions \(J_0(b\sqrt{-t})\) become \(I_0(b\sqrt{-t})\). It is easily shown that the convergence of the \(b\)-expansion requires that \(|\eta(s, b)|\) does not change sign. Using Eqs.(7-8) and the lack of change of sign assumption, we obtain

\[
I_0(s) \leq 1 + 2 |\tan(2\delta_R(s, 0))| .
\]

We can relate \(|\tan(2\delta_R(s, 0))|\) to the asymptotic value of the \(\rho\)-parameter and obtain finally the upper bound

\[
I_0(s) \leq 1 + \frac{K}{\ln(s/s_0)},
\]

where \(K\) is a positive constant. Combining Eqs.(6) and (10), we have the asymptotic bounds

\[
1 - \eta(s, 0) \leq I_0(s) \leq 1 + \frac{K}{\ln(s/s_0)}. \tag{11}
\]

The above bounds have been obtained incorporating (i) positivity, (ii) unitarity, (iii) correct behavior near \(b = 0\) and (iv) the asymptotic behavior \(b \to \infty\). The lack of oscillations implies that the physics of the inelasticity factor \(\eta\) and the real part of the phase shift \(\delta_R\) becomes very simple at high energies. \(\eta\) starts out being small in the central region (\(b\) near zero) and then monotonically increases to its asymptotic value 1 (for large \(b\)), whereas the real part of the phase shift has a maximum in the central region and dies out monotonically in the peripheral region. Under these premises and provisos, the bounds only require values of these quantities in the central region and any explicit dependences on the mass parameters \(\mu_{1,2}\) disappear. As we shall discuss later, it is due to unitarity relating how fast an amplitude decreases with momentum transfer (the “slope parameter”) to the strength of the amplitude (“coupling constants” or the prefactors). The above becomes an equality and in fact \(I_0(s)\) equals 1 if, as argued previously, at high energies, \(\eta(s, 0)\) approaches zero. Our sum rule then reads

\[
I_0(s) = \frac{1}{2} \int_{-\infty}^{0} (dt) \sqrt{\frac{d\sigma}{\pi dt}} \to 1, \text{ for } s \to \infty. \tag{12}
\]
$I_0(s)$ should rise from its threshold value $2|a_0|k \to 0$, where $a_0$ is the S-wave scattering length (complex for $p\bar{p}$) and $k$ is the CM 3-momentum, to its asymptotic value 1 as $s$ goes to infinity. In Fig.(1), we show a plot of this integral for available data [11–18] on $pp$ and $p\bar{p}$ elastic scattering for high energies [19]. Highest energy data at $\sqrt{s} = 1.8$ TeV for $p\bar{p}$ from the Fermilab Tevatron [11], give an encouraging value of $0.98 \pm 0.03$ demonstrating that indeed the integral is close to its asymptotic value of 1. We expect it to be even closer to 1 at the LHC (our extrapolation gives the value $0.99 \pm 0.03$ for LHC).

![Plot of $I_0(s)$ vs. $\sqrt{s}$ using experimental data [11–18]. The last point is our extrapolation for LHC.](image)

**FIG. 1.** A plot of $I_0(s)$ vs. $\sqrt{s}$ using experimental data [11–18]. The last point is our extrapolation for LHC.

Having found the trend to its asymptotic value at the highest available energies, we may return to ask whether the two assumptions made to obtain the sum rule are really necessary. Presently, we know theoretically that in the forward direction ($t = 0$), the imaginary part must dominate the real part if the cross-section saturates the Froissart-Martin bound. That is, if at high energies [20]

$$\sigma_{tot}(s) \rightarrow \text{Constant} \times \ln^2(s/s_0), \quad \text{then } \rho(s, 0) = \frac{\Re F(s, 0)}{3m F(s, 0)} \rightarrow \frac{\pi}{\ln(s/s_0)} \rightarrow 0. \tag{13}$$

We can make a crude estimate of $\rho(s, 0)$ through Eq.(13). For the highest Tevatron energy, we find it to be about 0.2. Since the contribution of the real part to the integral only enters as $(1/2)\rho^2$, the neglect of the real part in the forward direction would affect the integrand by about 2%, i.e., well within the experimental errors which are about 3%.

If the total cross-section were to increase only as $\ln(s/s_0)$, the ratio of real to the imaginary part would still go down to zero as in Eq.(13), albeit with a smaller constant $\pi/2$. Thus, for rising cross-sections, we are assured of the correctness of our first assumption in the forward direction. For non-forward directions, we have no direct evidence experimentally. However, since the overall differential cross-section would be decreasing (for $t \neq 0$) as a function of $s$, their contribution to $I_0(s)$ is less important. It is for exactly the same reason that the second assumption, i.e. the absence of zeros in the imaginary part of the non-forward amplitude, is not really necessary. So long as any possible such zeroes remain at some finite values of (negative) $t$, they would not significantly upset our results. The satisfaction of our sum rule a posteriori justifies this claim.

There are some interesting and significant consequences which follow from the above analysis. First, since the asymptotic value is reached from below, we may bound $\eta$, the inelasticity in the central ($b = 0$) region. For example,
even at $\sqrt{s} = 100 \text{ GeV}$, absorption is not complete but only about 80%, giving us a quantitative understanding of where the onset of “high energy” lies.

Another deduction concerning universality of the above result may be made. That is, the central value of inelasticity should approach zero for the scattering of all hadrons (at least for all hadrons made of light quarks). Such a result follows naturally from QCD if we recall first the experimental fact that both for nucleons as well as for mesons [21,22], half the hadronic energy is carried by glue.

In QCD, such an equipartition of energy has been derived rigorously to hold for all hadrons which are bound states of massless quarks [23]. Since all available high energy elastic scattering data are for nucleons and light mesons, all of which are made of the very light quarks, we have excellent support from QCD for equipartition. If one couples this with the notion that the rise of the cross-section is through gluon-gluon scattering, which is flavour independent, the asymptotic equality of (the rise in) all hadronic cross-sections automatically emerges.

Of course, the approach to asymptotia would not be the same between nucleon-nucleon and meson-nucleon scatterings. It is unfortunate that data for $\pi N$ scattering are available only up to $\sqrt{s} = 20 \text{ GeV}$, which is far from asymptotic. In fact, for this channel, $I_0(s)$ is only about 0.6 at the highest energy measured so far. Since the same asymptotic value of 1 for this integral should be reached for all hadrons, the rise with energy must be even more dramatic for meson-nucleon scattering. In principle, such a test for RHIC and LHC may be feasible through Bjorken’s suggestion [24] of converting an incident proton into a pion by isolating the one pion exchange contribution via tagging or triggering on a leading neutron or $\Delta^{++}$.

It appears reasonable to extend our analysis to $NA$ or even $AB$ elastic scatterings, where $A, B$ are nuclei. Also for these processes, at very high energies, we expect from QCD that the central inelasticity should approach zero and hence (modulo possible complications were $\rho(s, t)$ to be anomalously large), $I_0(s)$ should again asymptote to 1. For illustrative purposes, let us consider the following very simple expression which incorporates the sum rule:

$$|F_{AB}(s, t)| = I_0(s) B_{AB}(s)e^{(1/2)\rho_{AB}(s)t}. \quad (14)$$

Parametrizations of the above form (which underestimate the large $t$ contributions by ignoring the secondary slopes) are routinely used. However, what is new here is that since $I_0(s)$ goes to 1 in the asymptotic limit, it is our prediction that the prefactor would, in the same limit, become equal to the diffraction width $B_{AB}(s)$. Physically, it says that unitarity correlates and limits how large the amplitude can be, as a function of the energy, to how fast it decreases, as a function of the momentum transfer. If Eq.(14) holds, we may use the optical theorem to obtain the approximate expression

$$I_0(s) = \frac{\sigma_{tot}(s)}{4\pi B_{AB}(s)} \left[1 + (1/2)\rho_{AB}(s, 0)\right]. \quad (15)$$

Under the same assumption, we would have

$$\frac{\sigma_{el}(s)}{\sigma_{tot}(s)} \approx \frac{I_0(s)}{4} \left[1 + \rho_{AB}(s, 0)\right]. \quad (16)$$

For the highest Tevatron energy $\sqrt{s} = 1.8 \text{ TeV}$, Eq.(16) would estimate the elastic to total ratio to be about 0.25 in excellent accord with the experimental value $0.25 \pm 0.02$ [25].

Future experiments from RHIC and LHC should be able to test our sum rule predictions for $pp, p\bar{p}$ and hopefully other elastic channels. For this purpose, it would be useful if experimentalists would present values of $I_0(s)$ directly from their experimental data, obviating thereby interpolations (such as those carried out by us to obtain FIG. 1).

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