QUANTUM TOROIDAL ALGEBRA ASSOCIATED WITH $\mathfrak{gl}_{m|n}$

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Abstract. We introduce and study the quantum toroidal algebra $E_{m|n}(q_1, q_2, q_3)$ associated with the superalgebra $\mathfrak{gl}_{m|n}$ with $m \neq n$, where the parameters satisfy $q_1q_2q_3 = 1$.

We give an evaluation map. The evaluation map is a surjective homomorphism of algebras $E_{m|n}(q_1, q_2, q_3) \rightarrow \tilde{U}_q \hat{\mathfrak{gl}}_{m|n}$ to the quantum affine algebra associated with the superalgebra $\mathfrak{gl}_{m|n}$ at level $c$ completed with respect to the homogeneous grading, where $q_2 = q^2$ and $q_3^{m-n} = c^2$.

We also give a bosonic realization of level one $E_{m|n}(q_1, q_2, q_3)$-modules.

1. Introduction

Quantum toroidal algebras were introduced in [GKV] motivated by the study of Hecke operators in algebraic surfaces. Since that time, the quantum toroidal algebras, especially those associated with $\mathfrak{gl}_n$, were found to have many applications in geometry, algebra, and mathematical physics.

We list a few facts involving quantum toroidal algebras of type A. The quantum toroidal algebras appear as Hall algebras of elliptic curves, [BS], [SV1], they also act on equivariant K-groups of Hilbert schemes and Laumon moduli spaces, [FT], [SV2], [T1]. The quantum toroidal algebras are natural dual objects to double affine Hecke algebras, [VV1]. The quantum toroidal algebras provide integrable systems of XXY-type, among them is a deformation of quantum KdV flows, [FJM1]. Characters of representations of quantum toroidal algebras appear in topological field theory, [FJMM1], AGT conjecture, [AFS]. The full list is much longer.

In this paper, we introduce the quantum toroidal algebras $E_{m|n}(q_1, q_2, q_3)$ related to the superalgebras $\mathfrak{gl}_{m|n}$, with $m \neq n$ and standard parity, and initiate their study. In our mind, this subject is long overdue. We expect these algebras to have many properties similar to the quantum toroidal algebras $E_{m|0}(q_1, q_2, q_3)$ associated with $\mathfrak{gl}_m$ which can be used in similar way, but with various new features occurring due to the supersymmetry. In particular, our future goal is to study the corresponding integrable systems.

We start by introducing the algebras $E_{m|n}(q_1, q_2, q_3)$ with $m \neq n$ and standard parity. As in the even case, they depend on the complex parameters $q_1, q_2, q_3$ such that $q_1q_2q_3 = 1$. We require that the algebra $E_{m|n}(q_1, q_2, q_3)$ has a “vertical” quantum affine subalgebra $U_q \hat{\mathfrak{gl}}_{m|n}$ in the new Drinfeld realization, a “horizontal” quantum affine subalgebra $U_q \hat{\mathfrak{sl}}_{m|n}$ given in Chevalley generators, and a symmetry with respect to the parity change, see (2.23). We always have $q_2 = q^2$. This leads us to the generators and relations presentation of $E_{m|n}(q_1, q_2, q_3)$, see Definition 2.1. Naturally, the algebra $E_{m|n}(q_1, q_2, q_3)$ is generated by currents $E_i(z), F_i(z)$, and half currents $K_i^{\pm}(z), i = 0, \ldots, m + n - 1$, labeled by nodes of the affine Dynkin diagram of type $\hat{\mathfrak{sl}}_{m|n}$ and standard parity, see Figure 1 and the relations are written in terms of the corresponding Cartan matrix. Similar to the even case, the quantum toroidal algebra $E_{m|n}(q_1, q_2, q_3)$ has a two-dimensional center.
2.1. Let\( I = \{ 1, \ldots, m + n - 1 \} \). Let \( \hat{I}^+ = \{ 1, \ldots, m - 1 \} \), \( \hat{I}^- = \{ m + 1, \ldots, m + n - 1 \} \), \( \hat{I}^1 = \{ 0, m \} \) and \( \hat{I} = \hat{I}^+ \cup \hat{I}^- \cup \hat{I}^1 \). In particular, if \( n = 1 \), we have \( \hat{I}^- = \emptyset \), and if \( m = 1 \), \( \hat{I}^+ = \emptyset \). The elements in these sets are to be read modulo \( m + n \).
Let $\hat{A} = (A_{i,j})_{i,j\in I}$ be the Cartan matrix of $\mathfrak{sl}_{m|n}$, and $A = (A_{i,j})_{i,j\in I}$ be the Cartan matrix of $\mathfrak{sl}_{m|n}$, both with the standard choice of parity. Namely, the odd simple roots correspond to $i \in \hat{I}^1 = \{0, m\}$. We set $|i| = 1$, $i \in \hat{I}^1$, and $|i| = 0$ otherwise. We have $\det(A) = m - n \neq 0$, $\det(\hat{A}) = 0$.

For $i \in \hat{I}^+ \cup \{m\}$, let $M_{i-1,i} = -1, M_{i,i-1} = 1$. For $j \in \hat{I}^- \cup \{0\}$, let $M_{j-1,j} = 1, M_{j,j-1} = -1$. Let also $M_{i,j} = 0$, if $i \neq j \pm 1$. Define the matrix $\hat{M} = (M_{i,j})_{i,j\in I}$. We have

$$\hat{A} = \begin{pmatrix}
0 & -1 & \cdots & \cdots & -1 \\
-1 & 2 & -1 & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
-1 & 0 & 1 & \cdots & \cdots \\
1 & -2 & \cdots & \cdots & 1 \\
\vdots & & & & m
\end{pmatrix}, \quad \hat{M} = \begin{pmatrix}
0 & -1 & \cdots & \cdots & -1 \\
1 & 0 & -1 & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
-1 & 0 & \cdots & \cdots & 1 \\
1 & \cdots & & & m
\end{pmatrix}.$$

Note that $\hat{A}$ is symmetric and $\hat{M}$ is skew-symmetric.

**Definition 2.1.** The quantum toroidal algebra associated with $\mathfrak{gl}_{m|n}$ is the unital associative superalgebra $\mathcal{E}_{m|n} = \mathcal{E}_{m|n}(q_1, q_2, q_3)$ generated by $E_{i,k}, F_{i,k}, H_{i,r}$, and invertible elements $K_i, C$, where $i \in \hat{I}$, $k \in \mathbb{Z}$, $r \in \mathbb{Z}^\vee$, subject to the relations (2.1)-(2.17) below. The parity of the generators is given by $|E_{i,k}| = |F_{i,k}| = |i|$ and 0 in all other cases.

The defining relations are given in terms of generating series

$$E_i(z) = \sum_{k \in \mathbb{Z}} E_{i,k} z^{-k}, \quad F_i(z) = \sum_{k \in \mathbb{Z}} F_{i,k} z^{-k}, \quad K_i^\pm(z) = K_i^{\pm 1} \exp(\pm (q - q^{-1}) \sum_{r > 0} H_{i,\pm r} z^{\mp r}).$$

We use the notation $[X,Y]_a = XY - (-1)^{\|X\|\|Y\|} aYX$. For simplicity, we write $[X,Y]_1 = [X,Y]$.

Let also $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$.

The relations are as follows.

### C, K relations

(2.1) $C$ is central, $K_i K_j = K_j K_i, K_i E_j(z) K_i^{-1} = q^{A_{i,j}} E_j(z), \quad K_i F_j(z) K_i^{-1} = q^{-A_{i,j}} F_j(z)$. 


K-K, K-E and K-F relations

\[ K_+^i(z)K_+^j(w) = K_+^j(w)K_+^i(z), \]
\[ \frac{d^{M_{ij}}C^{-1}z - q^{A_{i,j}}w}{d^{M_{ij}}Cz - q^{A_{i,j}}w} K_+^i(z)K_+^j(w) = \frac{d^{M_{ij}}q^{A_{i,j}}C^{-1}z - w}{d^{M_{ij}}q^{A_{i,j}}Cz - w} K_+^j(w)K_+^i(z), \]
\[ (d^{M_{ij}}z - q^{A_{i,j}}w)K_+^i(C^{-(1\pm 1)/2}z)E_j(w) = (d^{M_{ij}}q^{A_{i,j}}z - w)E_j(w)K_+^i(C^{-(1\pm 1)/2}z), \]
\[ (d^{M_{ij}}z - q^{-A_{i,j}}w)K_+^i(C^{-(1\mp 1)/2}z)F_j(w) = (d^{M_{ij}}q^{-A_{i,j}}z - w)F_j(w)K_+^i(C^{-(1\mp 1)/2}z). \]

E-F relations

\[ [E_i(z), F_j(w)] = \frac{\delta_{i,j}}{q - q^{-1}}(\delta \left( C\frac{w}{z} \right) K_+^i(w) - \delta \left( C\frac{z}{w} \right) K_+^i(z)). \]

E-E and F-F relations

\[ [E_i(z), E_j(w)] = 0, \quad [F_i(z), F_j(w)] = 0 \quad (A_{i,j} = 0), \]
\[ (d^{M_{ij}}z - q^{A_{i,j}}w)E_i(z)E_j(w) = (-1)^{|i||j|}(d^{M_{ij}}q^{A_{i,j}}z - w)E_i(w)E_j(z) \quad (A_{i,j} \neq 0), \]
\[ (d^{M_{ij}}z - q^{-A_{i,j}}w)F_i(z)F_j(w) = (-1)^{|i||j|}(d^{M_{ij}}q^{-A_{i,j}}z - w)F_j(w)F_i(z) \quad (A_{i,j} \neq 0). \]

Serre relations

\[ \text{Sym}_{x_1,x_2}[E_i(z_1), [E_i(z_2), E_{i\pm 1}(w)]]_{q^{-1}} = 0 \quad (i \notin \hat{I}^1), \]
\[ \text{Sym}_{x_1,x_2}[F_i(z_1), [F_i(z_2), F_{i\pm 1}(w)]]_{q^{-1}} = 0 \quad (i \notin \hat{I}^1). \]

If \( mn \neq 2, \)
\[ \text{Sym}_{x_1,x_2}[E_i(z_1), [E_{i+1}(w_1), E_i(z_2), E_{i-1}(w_2)]]_{q^{-1}} = 0 \quad (i \in \hat{I}^1), \]
\[ \text{Sym}_{x_1,x_2}[F_i(z_1), [F_{i+1}(w_1), F_i(z_2), F_{i-1}(w_2)]]_{q^{-1}} = 0 \quad (i \in \hat{I}^1). \]

If \((m, n) = (2, 1), \)
\[ \text{Sym}_{x_1,x_2}\text{Sym}_{w_1,w_2}[E_0(z_1), [E_0(w_1), E_0(z_2), E_0(w_2), E_1(y)]]_{q^{-1}} = \]
\[ = \text{Sym}_{x_1,x_2}\text{Sym}_{w_1,w_2}[E_2(w_1), [E_0(z_1), E_2(w_2), E_0(z_2), E_1(y)]]_{q^{-1}}. \]

If \((m, n) = (1, 2), \)
\[ \text{Sym}_{x_1,x_2}\text{Sym}_{w_1,w_2}[F_0(z_1), [F_0(w_1), F_0(z_2), F_2(w_2), F_1(y)]]_{q^{-1}} = \]
\[ = \text{Sym}_{x_1,x_2}\text{Sym}_{w_1,w_2}[F_2(w_1), [F_0(z_1), F_2(w_2), F_0(z_2), F_1(y)]]_{q^{-1}}. \]
Note that
\[ K := \prod_{i \in \hat{I}} K_i \]
is a central element.

The relations (2.2)–(2.5) are equivalent to
\[
[H_{i,r}, E_j(z)] = \left[ rA_{i,j} \right] r d^{-rM_{i,j}} C^{-(r+|r|)/2} z^r E_j(z),
\]
\[
[H_{i,r}, F_j(z)] = -\left[ rA_{i,j} \right] r d^{-rM_{i,j}} C^{-(r-|r|)/2} z^r F_j(z),
\]
\[
[H_{i,r}, H_{j,s}] = \delta_{r+s,0} \left[ rA_{i,j} \right] r d^{-rM_{i,j}} C^r - C^{-r},
\]
for all \( r \in \mathbb{Z}^\times, i, j \in \hat{I} \), where \( [k] = \frac{q^k - q^{-k}}{q - q^{-1}} \).

The poles of the correlation functions of currents \( E_i(z) \) are depicted in Figure 1. For example, the correlation function of \( E_0(z)E_1(w) \) has a pole at \( z = q_1 w \), while the correlation function of \( E_1(z)E_0(w) \) has a pole at \( z = q_3 w \). The poles of the correlation functions of the currents \( F_i(z) \) are obtained from the Figure 1 replacing \( q \) by \( q^{-1} \), i.e., \( q_1 \) is replaced by \( q_3^{-1} \), and \( q_3 \) by \( q_1^{-1} \).

\[ \begin{array}{c}
q_1 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & m-1 & \rightarrow & q_3 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & q_1 & \rightarrow & 2 & \rightarrow & m-1 & \rightarrow & q_3 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
q_3^{-1} & \rightarrow & m+n-1 & \rightarrow & m+2 & \rightarrow & m+1 & \rightarrow & q_1^{-1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& \rightarrow & q_1^{-1} & \rightarrow & q_3^{-1} & \rightarrow & & & \\
\end{array} \]

**Figure 1.** Standard Dynkin diagram of type \( \widehat{sl}_{m|n} \).

2.2. **Horizontal and vertical subalgebras.** In this section, we define the horizontal and vertical subalgebras of \( \mathcal{E}_{m|n} \).

See Appendix B for the definitions of the quantum affine algebras \( U_q \widehat{sl}_{m|n} \) and \( U_q \widehat{gl}_{m|n} \).

We denote the subalgebra of \( \mathcal{E}_{m|n} \) generated by \( E_i(z), F_i(z), K_i(z), C, i \in I \), by \( U_q^{\text{ver}} \widehat{sl}_{m|n} \) and we call it the *vertical quantum affine* \( \widehat{sl}_{m|n} \). If \( K_0(z) \) is also included, the resulting subalgebra is denoted by \( U_q^{\text{ver}} \widehat{gl}_{m|n} \) and called the *vertical quantum affine* \( \widehat{gl}_{m|n} \). Note that \( U_q^{\text{ver}} \widehat{sl}_{m|n}, U_q^{\text{ver}} \widehat{gl}_{m|n} \) are given in new Drinfeld realization.

The current \( K_0(z) \) does not commute with \( U_q^{\text{ver}} \widehat{sl}_{m|n} \). To obtain a current in \( U_q^{\text{ver}} \widehat{gl}_{m|n} \) commuting with \( U_q^{\text{ver}} \widehat{sl}_{m|n} \) we proceed as follows.
For each \( r \in \mathbb{Z}^\times \), \( \det([rA_{i,j}]d^{-rM_{i,j}})_{i,j\in I} = [r]^{m+n} (d^{r(m-n)} + d^{r(n-m)} - q^{r(m-n)} - q^{r(n-m)}) \neq 0 \). Thus, the system

\[
\sum_{i\in I} \gamma_{i,r}[rA_{i,j}]d^{-rM_{i,j}} = 0 \quad (j \in I),
\]

has a one-dimensional space of solutions. The element \( H^\text{ver}_r = \sum_{i\in I} \gamma_{i,r} H_{i,r} \in U^\text{ver}_q \hat{\mathfrak{gl}}_{m|n} \) commutes with \( U^\text{ver}_q \hat{\mathfrak{sl}}_{m|n} \subset U^\text{ver}_q \hat{\mathfrak{gl}}_{m|n} \). Such element is unique up to scalar. We fix a normalization by requiring \( \gamma_0, r = 1, r \in \mathbb{Z}_{<0} \), and

\[
[H^\text{ver}_r, H^\text{ver}_s] = \delta_{r+s,0}[(n-m)r] \frac{1}{r} \frac{C^r - C^{-r}}{q - q^{-1}}.
\]

Set \( H^\text{ver}(z) = \sum_{r\in \mathbb{Z}^\times} H^\text{ver}_r z^{-r} \).

**Lemma 2.2.** The subalgebra \( U^\text{ver}_q \hat{\mathfrak{gl}}_{m|n} \) is isomorphic to \( U^\text{ver}_q \hat{\mathfrak{sl}}_{m|n} \).

**Proof.** We have a homomorphism \( v : U^\text{ver}_q \hat{\mathfrak{gl}}_{m|n} \to \mathcal{E}_{m|n} \) given by

\[
x^+_i(z) \mapsto E_i(d^{-i}z), \quad x^-_i(z) \mapsto F_i(d^{-i}z), \quad k_i(z) \mapsto K_i(d^{-i}z) \quad (i \in \hat{I} \cup \{m\}),
\]

\[
x^+_j(z) \mapsto E_j(d^{-2m+j}z), \quad x^-_j(z) \mapsto F_j(d^{-2m+j}z), \quad k_j(z) \mapsto K_j(d^{-2m+j}z) \quad (j \in \hat{I}^-),
\]

\[
c \mapsto C, \quad h(z) \mapsto H^\text{ver}(z).
\]

The evaluation map constructed in Theorem 3.3 produces a left-inverse of \( v \), see Lemma B.1. Thus, \( v \) is an embedding with image \( U^\text{ver}_q \hat{\mathfrak{gl}}_{m|n} \).

We denote the subalgebra of \( \mathcal{E}_{m|n} \) generated by \( E_{i,0}, F_{i,0}, K_i, i \in \hat{I} \), by \( U^\text{hor}_q \hat{\mathfrak{sl}}_{m|n} \) and we call it the horizontal quantum affine \( \mathfrak{sl}_{m|n} \). Note that \( U^\text{hor}_q \hat{\mathfrak{sl}}_{m|n} \) is given in Drinfeld-Jimbo realization.

We have a homomorphism \( h : U^\text{hor}_q \hat{\mathfrak{sl}}_{m|n} \to \mathcal{E}_{m|n} \) given by

\[
e_i \mapsto E_{i,0}, \quad f_i \mapsto F_{i,0}, \quad t_i \mapsto K_i \quad (i \in \hat{I}),
\]

and its image is \( U^\text{hor}_q \hat{\mathfrak{sl}}_{m|n} \).

**Conjecture 2.3.** The homomorphism \( h : U^\text{hor}_q \hat{\mathfrak{sl}}_{m|n} \to \mathcal{E}_{m|n} \) is injective. In particular, \( U^\text{hor}_q \hat{\mathfrak{sl}}_{m|n} \) is isomorphic to \( U^\text{hor}_q \hat{\mathfrak{sl}}_{m|n} \).

### 2.3. Isomorphisms

In this section, we list some isomorphisms involving superalgebra \( \mathcal{E}_{m|n} \). In all cases, it is easy to check that the maps are even, invertible and respect the defining relations.

For \( a \in \mathbb{C}^\times \), the shift of spectral parameter

\[
s_a : \mathcal{E}_{m|n} \to \mathcal{E}_{m|n},
\]

is defined by

\[
s_a(E_i(z)) = E_i(az), \quad s_a(F_i(z)) = F_i(az), \quad s_a(K_i^\pm(z)) = K_i^\pm(az), \quad s_a(C) = C \quad (i \in \hat{I}).
\]

For each \( j \in \hat{I} \), we have

\[
\chi_j : \mathcal{E}_{m|n} \to \mathcal{E}_{m|n},
\]
defined by
\[ \chi_j(E_i(z)) = E_i(z)z^{-\delta_{i,j}}, \quad \chi_j(F_i(z)) = F_i(z)z^{\delta_{i,j}}, \quad \chi_j(K_i^\pm(z)) = C^\pm \delta_{i,j} K_i^\pm(z), \quad \chi_j(C) = C \quad (i \in \hat{I}). \]

The following isomorphisms change the parameters of the algebra.

The diagram isomorphism
\[ (2.22) \quad \sigma : \mathcal{E}_{m|n}(q_1, q_2, q_3) \to \mathcal{E}_{m|n}(q_3, q_2, q_1), \]
defined by
\[ \sigma(E_i(z)) = E_{m-i}(z), \quad \sigma(F_i(z)) = F_{m-i}(z), \quad \sigma(K_i^\pm(z)) = K_{m-i}^\pm(z), \quad \sigma(C) = C \quad (i \in \hat{I}), \]
changes \(d\) to \(d^{-1}\).

The change of parity isomorphism
\[ (2.23) \quad \tau : \mathcal{E}_{m|n}(q_1, q_2, q_3) \to \mathcal{E}_{m|n}(q_3^{-1}, q_2^{-1}, q_1^{-1}), \]
defined by
\[ \tau(E_i(z)) = E_{-i}(z), \quad \tau(F_i(z)) = F_{-i}(z), \quad \tau(K_i^\pm(z)) = -K_{-i}^\pm(z), \quad \tau(C) = C \quad (i \in \hat{I}), \]
changes \(q\) to \(q^{-1}\).

We have \(\sigma^2 = \tau^2 = Id\).

2.4. Hopf superalgebra structure. The superalgebra \(\mathcal{E}_{m|n}\) has a topological Hopf superalgebra structure given on generators by
\[
\begin{align*}
\Delta E_i(z) &= E_i(z) \otimes 1 + K_i^-(z) \otimes E_i(C_1 z), \\
\Delta F_i(z) &= F_i(C_2 z) \otimes K_i^+(z) + 1 \otimes F_i(z), \\
\Delta K_i^+(z) &= K_i^+(C_2 z) \otimes K_i^+(z), \\
\Delta K_i^-(z) &= K_i^-(z) \otimes K_i^-(C_1 z), \\
\Delta C &= C \otimes C, \\
\varepsilon(E_i(z)) &= \varepsilon(F_i(z)) = 0, \quad \varepsilon(K_i^\pm(z)) = \varepsilon(C) = 1, \\
S(E_i(z)) &= -(K_i^-(C^{-1} z))^{-1} E_i(C^{-1} z), \\
S(F_i(z)) &= -F_i(C^{-1} z) (K_i^+(C^{-1} z))^{-1}, \\
S(K_i^\pm(z)) &= (K_i^\pm(C^{-1} z))^{-1}, \quad S(C) = C^{-1},
\end{align*}
\]
where \(C_1 = C \otimes 1,\ C_2 = 1 \otimes C\). The maps \(\Delta\) and \(\varepsilon\) are extended to algebra homomorphisms, and the map \(S\) to a superalgebra anti-homomorphism, \(S(xy) = (-1)^{|x||y|} S(y) S(x)\). Note that the tensor product multiplication is defined for homogeneous elements \(x_1, x_2, y_1, y_2 \in \mathcal{E}_{m|n}\) by \((x_1 \otimes y_1)(x_2 \otimes y_2) = (-1)^{|y_1||x_2|} x_1 x_2 \otimes y_1 y_2\) and extended to the whole algebra by linearity.

The vertical subalgebras \(U_{q}^{\text{ver}} \hat{\mathfrak{sl}}_{m|n}\) and \(U_{q}^{\text{ver}} \hat{\mathfrak{gl}}_{m|n}\) are Hopf subalgebras of \(\mathcal{E}_{m|n}\).
2.5. **Grading.** For each \( i \in \hat{I} \), the superalgebra \( \mathcal{E}_{m|n} \) has a \( \mathbb{Z} \)-grading given by

\[
\deg_i(E_{j,k}) = \delta_{i,j}, \quad \deg_i(F_{j,k}) = -\delta_{i,j},
\]

\[
\deg_i(H_{j,r}) = \deg_i K_j = \deg_i(C) = 0 \quad (j \in \hat{I}, k \in \mathbb{Z}, r \in \mathbb{Z}^\times).
\]

There is also the **homogeneous** \( \mathbb{Z} \)-grading given by

\[
\deg_\delta(E_{j,k}) = \deg_\delta(F_{j,k}) = k, \quad \deg_\delta(H_{j,r}) = r, \quad \deg_\delta(K_j) = \deg_\delta(C) = 0 \quad (j \in \hat{I}, k \in \mathbb{Z}, r \in \mathbb{Z}^\times).
\]

Thus the superalgebra \( \mathcal{E}_{m|n} \) has a \( \mathbb{Z}^{m+n+1} \)-grading given by

\[
\deg(X) = (\deg_0(X), \deg_1(X), \ldots, \deg_{m+n-1}(X); \deg_\delta(X)), \quad X \in \mathcal{E}_{m|n}.
\]

We call an \( \mathcal{E}_{m|n} \)-module \( V \) **admissible** if for any \( v \in V \) there exists an integer \( N = N_v \) such that \( Xv = 0 \) for all \( X \in \mathcal{E}_{m|n} \) with \( \deg_\delta(X) > N \).

We say an \( \mathcal{E}_{m|n} \)-module has level \((k^{\text{ver}}, k^{\text{hor}})\) if it has level \( k^{\text{ver}} \) as a \( U_q^{\text{ver}} \tilde{\mathfrak{sl}}_{m|n} \)-module, and level \( k^{\text{hor}} \) as a \( U_q^{\text{hor}} \tilde{\mathfrak{sl}}_{m|n} \)-module, i.e., if \((C, K)\) acts as \((q^{k^{\text{ver}}}, q^{k^{\text{hor}}})\).

3. **Level (1,0) modules, bosonic picture**

In this section, we construct \( \mathcal{E}_{m|n} \)-modules of level (1,0) using vertex operators.

3.1. **Heisenberg algebra.** Let \( \mathcal{H} \) be the associative algebra generated by \( H_{i,r}, c_{j,r}, i \in \hat{I}, j \in \hat{I}^- \cup \{m\}, r \in \mathbb{Z}^\times \), satisfying

\[
[H_{i,r}, H_{j,s}] = \delta_{r+s,0} \cdot \frac{[rA_{i,j}][r]}{r} d^{-rM_{i,j}},
\]

\[
[c_{i,r}, c_{j,s}] = \delta_{i,j} \delta_{r+s,0} \cdot \frac{[r]_2}{r},
\]

\[
[H_{i,r}, c_{j,s}] = 0.
\]

Note that (3.1) is equivalent to equation (2.20) with \( C = q \).

Denote by \( \mathcal{H}^\pm \) the (commutative) subalgebra generated by \( H_{i,r}, c_{j,r} \) with \( \pm r > 0, i \in \hat{I}, j \in \hat{I}^- \cup \{m\} \).

Let \( \mathcal{F} \) be the Fock space generated by a vector \( v_0 \) satisfying \( \mathcal{H}^+ v_0 = 0 \). Thus, \( \mathcal{F} \) is a free \( \mathcal{H}^- \)-module of rank 1

\[
\mathcal{F} = \mathcal{H} v_0 = \mathcal{H}^- v_0.
\]

Moreover, since \( \det([rA_{i,j}d^{-rM_{i,j}}])_{i,j} \neq 0 \), \( \mathcal{F} \) is an irreducible \( \mathcal{H} \)-module.

3.2. **Level (1,0) \( \mathcal{E}_{m|n} \)-modules.** Let \( Q_{m|n} \) be the \( \mathfrak{sl}_{m|n} \) root lattice and let \( \mathbb{C}\{Q_{m|n}\} \) be a twisted group algebra of \( Q_{m|n} \) generated by invertible elements \( e^{\alpha_i}, i \in I \), satisfying the relations

\[
e^{\alpha_i} e^{\alpha_j} = \begin{cases} 
(-1)^{\langle \alpha_i | \bar{\alpha}_j \rangle} e^{\bar{\alpha}_j} e^{\alpha_i} & (i, j \in \hat{I}^+ \cup \{m\}), \\
e^{\alpha_j} e^{\alpha_i} & (i \text{ or } j \in \hat{I}^-).
\end{cases}
\]
Define $\varepsilon : \hat{I} \times \hat{I} \to \{\pm 1\}$ by

$$
\varepsilon(i, j) = \begin{cases} 
(-1)^{\langle \hat{a}_i | \hat{a}_j \rangle} & (i > j \in \hat{I}^+ \cup \{m\}), \\
(-1)^{1+\delta_{m,1}} & (i = 0, j \in \hat{I}^+ \cup \{m\}), \\
1 & (\text{otherwise}),
\end{cases}
$$

Let $Q_c$ be the integral lattice generated by elements $c_i, i \in \hat{I}^- \cup \{m\}$, with bilinear form given by

$$
\langle c_i | c_j \rangle = \delta_{i,j}.
$$

Define $Q = Q_{m|n} \oplus Q_c$ and extend the bilinear forms on $Q_{m|n}$ and $Q_c$ to $Q$ by requiring $\langle \hat{a}_i | c_j \rangle = 0$. Set also $\langle \hat{A} | c_j \rangle = 0$, for any $\mathfrak{gl}_{m|n}$ weight $\hat{A}$.

Let $\mathbb{C}[Q_c]$ be the (commutative) group algebra of $Q_c$ and define $\mathbb{C}\{Q\} = \mathbb{C}\{Q_{m|n}\} \otimes \mathbb{C}[Q_c]$.

For $\alpha = \sum_{i \in I} r_i \hat{a}_i + \sum_{k \in \hat{I}^- \cup \{m\}} s_k c_k \in Q$, define

$$
e^\alpha = (e^{\hat{a}_1})^{r_1} \ldots (e^{\hat{a}_{m+n-1}})^{r_{m+n-1}} (e^c)^{s_m} \ldots (e^c)^{s_{m+n-1}}.
$$

Then, $\{e^\alpha, \alpha \in Q\}$ is a basis of $\mathbb{C}\{Q\}$.

Let $\hat{Q} \subset Q$ be the sublattice of rank $m+n$ generated by $\hat{a}_i, \hat{a}_m + c_m, \hat{a}_j + c_j - c_{j-1}, i \in \hat{I}^+, j \in \hat{I}^-$, and let $\mathbb{C}\{\hat{Q}\}$ be the subalgebra of $\mathbb{C}\{Q\}$ spanned by $e^\alpha, \alpha \in \hat{Q}$.

Following [KW], a $\mathfrak{sl}_{m|n}$ weight $\Lambda$ is a level 1 partially integrable weight if and only if $\Lambda = \Lambda_i, i \notin \hat{I}$, or $\Lambda = (1-a)\Lambda_0 + a\Lambda_m, a \in \mathbb{C}$.

Set

$$
\tilde{\Lambda} = \tilde{\Lambda}_i \quad \text{ (} \Lambda = \Lambda_i, \ i \in \hat{I}^+\text{)},
$$

$$
\tilde{\Lambda} = \tilde{\Lambda}_j - \sum_{i=j}^{m+n-1} c_i \quad \text{ (} \Lambda = \Lambda_j, \ j \in \hat{I}^-\text{)},
$$

$$
\tilde{\Lambda} = a\tilde{\Lambda}_m - a\sum_{i=m}^{m+n-1} c_i \quad \text{ (} \Lambda = (1-a)\Lambda_0 + a\Lambda_m, \ a \in \mathbb{C}\).
$$

Given a level 1 partially integrable weight $\Lambda$, define the space

$$
\mathcal{F}_\Lambda := \mathcal{F} \otimes \mathbb{C}\{\hat{Q}\}e^{\tilde{\Lambda}}.
$$

Define an action of the algebras $\mathcal{H}$ and $\mathbb{C}\{\hat{Q}\}$ on $\mathcal{F}_\Lambda$ as follows.

For $v \in \mathcal{F}$, $\alpha \in \hat{Q}$, set

$$x(v \otimes e^\alpha e^{\tilde{\Lambda}}) = (xv) \otimes e^\alpha e^{\tilde{\Lambda}} \quad \text{ (} x \in \mathcal{H}\),
$$

$$e^\beta (v \otimes e^\alpha e^{\tilde{\Lambda}}) = v \otimes (e^\beta e^\alpha e^{\tilde{\Lambda}}) \quad \text{ (} \beta \in \hat{Q}\).
$$

In particular, $\mathcal{F}_\Lambda$ is a free $\mathcal{H}^- \otimes \mathbb{C}\{\hat{Q}\}$-module of rank 1.

Introduce the zero-mode linear operators $z^{\pm H_{i,0}}, q^{\pm \alpha_{i,0}}, z^{\pm c_{j,0}}, \ i \in \hat{I}, \ j \in \hat{I}^- \cup \{m\}$, acting on $\mathcal{F}_\Lambda$ as follows.
For \( v \otimes e^\alpha e^{\hat{\Lambda}} \in \mathcal{F}_\Lambda \), with \( \alpha = \sum_{i \in I} r_i \alpha_i + \sum_{k \in I^c} s_k c_k \), set
\[
\begin{align*}
z^{\pm H_{i,0}}(v \otimes e^\alpha e^{\hat{\Lambda}}) &= z^{\pm (\alpha_i + \hat{\Lambda})} q^{\pm \frac{1}{2} \sum_{i \in I} r_i A_{i,i} M_{i,i} v} \otimes e^\alpha e^{\hat{\Lambda}}, \\
q^{\pm \alpha_{i,0}}(v \otimes e^\alpha e^{\hat{\Lambda}}) &= q^{\pm (\alpha_i + \hat{\Lambda})} v \otimes e^\alpha e^{\hat{\Lambda}}, \\
z^{\pm c_{j,0}}(v \otimes e^\alpha e^{\hat{\Lambda}}) &= z^{\pm (c_j + \hat{\Lambda})} v \otimes e^\alpha e^{\hat{\Lambda}}.
\end{align*}
\]

For \( i \in \hat{I} \) and \( j \in \hat{I}^c \cup \{m\} \), let
\[
\begin{align*}
H^+_i(z) &= \sum_{r > 0} \frac{H_{i,r} z^{r}}{r}, \\
c^+_j(z) &= \sum_{r > 0} \frac{c_{j,r} z^{r}}{r},
\end{align*}
\]

and define
\[
\begin{align*}
\Gamma^+_i(z) &= z \exp(H^+_i(q^{-1} z)) \exp(-H^+_i(z)) e^{\bar{\alpha}_i} z^{H_{i,0}}, \\
\Gamma^-_i(z) &= z \exp(-H^-_i(z)) \exp(H^-_i(q^{-1} z)) e^{-\bar{\alpha}_i} z^{-H_{i,0}}, \\
C^+_j(z) &= \exp(\pm c^+_j(z)) \exp(\mp c^-_j(z)) e^{\pm c_j} z^{\mp c_{j,0}}.
\end{align*}
\]

The following is proved by a direct computation.

**Lemma 3.1.** For \( i, j \in \hat{I}, r \in \mathbb{Z}^+ \), we have
\[
[H_{i,r}, \Gamma^+_j(z)] = \pm \frac{[r A_{i,j}]}{r} d^{-r M_{i,j}} q^{-(r \pm |r|)/2} z^{r} \Gamma^+_j(z).
\]

Define the normal ordering by
\[
\begin{align*}
x_r y_s &= \begin{cases} x_r y_s & (r < 0) \\ y_s x_r & (r \geq 0) \end{cases} & (x_r, y_s \in \{H_{i,r}, c_{i,r}\}), \\
A e^\alpha &= : e^\alpha A : = e^\alpha A & (\alpha \in Q, A \in \{z^{\pm H_{i,0}}, q^{\pm \alpha_{i,0}}, z^{\pm c_{j,0}}\}), \\
e^{\alpha_i} e^{\bar{\alpha}_j} &= \epsilon(i, j) e^{\alpha_i + \bar{\alpha}_j} & (i, j \in \hat{I}),
\end{align*}
\]

and extended inductively from right to left on larger products, e.g., \(: abc := a( : bc : ) :.\)

Given two currents \( X(z), Y(w) \), we say that \( X(z)Y(w) \) has contraction \( \langle X(z)Y(w) \rangle \)
if
\[
X(z)Y(w) = \langle X(z)Y(w) \rangle : X(z)Y(w) :.
\]

In this text, all contractions \( \langle X(z)Y(w) \rangle \) are Laurent series converging to rational functions in the
region \(|z| \gg |w|\).
Lemma 3.2. For $i, j \in \hat{I}$, $k, l \in \hat{I}^{-} \cup \{m\}$ we have

\begin{align*}
(3.3) & \quad \langle \Gamma_i^{\pm}(z) \Gamma_j^{\pm}(w) \rangle = ((z - w)(z - q^{\mp 2} w))^{A_{i,j} / 2}, \\
(3.4) & \quad \langle \Gamma_i^{\pm}(z) \Gamma_j^{\mp}(w) \rangle = \varepsilon(i, j) (z - d^{-M_{i,j}} q^{\mp 1} w)^{A_{i,j}} d^{A_{i,j} M_{i,j} / 2} \quad (i \neq j), \\
(3.5) & \quad \langle \Gamma_i^{\pm}(z) \Gamma_j^{\mp}(w) \rangle = ((z - qw)(z - q^{-1} w))^{-A_{i,j} / 2}, \\
(3.6) & \quad \langle \Gamma_i^{\pm}(z) \Gamma_j^{\mp}(w) \rangle = \varepsilon(i, j) (z - d^{-M_{i,j}} w)^{-A_{i,j}} d^{-A_{i,j} M_{i,j} / 2} \quad (i \neq j), \\
(3.7) & \quad \langle C_k^{\pm}(z) C_l^{\pm}(w) \rangle = (z - w)^{\delta_{k,l}}, \\
(3.8) & \quad \langle C_k^{\pm}(z) C_l^{\mp}(w) \rangle = (z - w)^{-\delta_{k,l}}.
\end{align*}

Proof. Let $\varepsilon \in \{ \pm 1 \}$.

Equations (3.3)-(3.6) follow from

$$
\langle z^{\pm \varepsilon H_{i,0}} e^{\pm \delta_{i,j}} \rangle = z^{\varepsilon A_{i,j}} d^{\pm A_{i,j} M_{i,j} / 2},
$$

$$
\langle \exp(\pm \varepsilon H_{i}^{+}(z)) \exp(\varepsilon H_{i}^{-}(w)) \rangle = \left(1 - \frac{w}{z}\right)^{\mp A_{i,j} / 2} \left(1 - \frac{q^{-1} w}{z}\right)^{\mp A_{i,j} / 2},
$$

$$
\langle \exp(\pm \varepsilon H_{i}^{+}(z)) \exp(\varepsilon H_{i}^{-}(w)) \rangle = \left(1 - d^{-M_{i,j}} w\right)^{\mp A_{i,j} / 2},
$$

for $(i \neq j)$. The equations (3.7) and (3.8) follow from

$$
\langle z^{\pm \varepsilon c_{k,0}} e^{\pm \varepsilon c_{l}} \rangle = z^{\pm \delta_{k,l}},
$$

$$
\langle \exp(\pm \varepsilon c_{k}^{+}(z)) \exp(\varepsilon c_{l}^{-}(w)) \rangle = \left(1 - \frac{w}{z}\right)^{\mp \delta_{k,l}}.
$$

These contractions are checked by a straightforward computation. \qed

Let $\partial_z$ be the $q$-difference operator

$$
\partial_z f(z) = \frac{f(qz) - f(q^{-1} z)}{(q - q^{-1}) z}.
$$

Theorem 3.3. The following expressions define a graded admissible $E_{m|n}$-module structure of level $(1,0)$ on $\mathcal{F}_A$.

\begin{align*}
C &= q, \quad K_i^{\pm 1} = q^{\pm \alpha_i.0}, \quad H_{i,r} = H_{i,r} \quad \hspace{1cm} (i \in \hat{I}), \\
E_i(z) &= \Gamma_i^{+}(z) \quad \hspace{1cm} (i \in \hat{I}^+), \\
F_i(z) &= \Gamma_i^{-}(z) \quad \hspace{1cm} (i \in \hat{I}^-), \\
E_m(z) &= d^m : \Gamma_m^{+}(z) C_m^{+}(d^m z) :; \\
F_m(z) &= : \Gamma_m^{-}(z) \partial_z [C_m^{-}(d^m z)] :, \\
E_i(z) &= d^{2m-i} : \Gamma_i^{+}(z) C_i^{+}(d^{2m-i} z) \partial_z [C_{i-1}^{-}(d^{2m-i} z)] : \quad (i \in \hat{I}^-), \\
F_i(z) &= d^{2m-i} : \Gamma_i^{-}(z) C_{i-1}^{+}(d^{2m-i} z) \partial_z [C_i^{-}(d^{2m-i} z)] : \quad (i \in \hat{I}^-), \\
E_0(z) &= : \Gamma_0^{+}(z) \partial_z [C_{m+n-1}^{-}(d^{m-n} z)] :, \\
F_0(z) &= d^{m-n} : \Gamma_0^{-}(z) C_{m+n-1}^{+}(d^{m-n} z) :.
\end{align*}
ordered summands. However, the coefficient of each summand is a Laurent polynomial. Thus, which is equivalent to the E-E relation.

Proof. For all possible $i, j$, we have

$$\partial \Gamma_i^+(z) \Gamma_j^+(w) = \Gamma_j^+(w) \Gamma_i^+(z) \left( \frac{q^{-1} w}{w - d_{i,j} q^{-1}} \right)^{A_{i,j}},$$

which is equivalent to the E-E relation.

If $i \in \hat{I}^+$, $j \in \hat{I}$ and $i \neq j$ we use (3.4) to get

$$\Gamma_i^+(z) \Gamma_j^+(w) = \Gamma_j^+(w) \Gamma_i^+(z) \left( \frac{q^{-1} w}{w - d_{i,j} q^{-1}} \right)^{A_{i,j}},$$

but in this case $\varepsilon(i, j) = (-1)^{A_{i,j}} \varepsilon(j, i)$, which is the needed sign.

The cases with $i \in \hat{I}^-$ or $i = j \in \hat{I}$ follow from the above equations noting that $\varepsilon(i, j) = \varepsilon(j, i)$ and by (3.7), (3.8),

$$\Gamma_i^+(z) \Gamma_j^+(w) = \Gamma_j^+(w) \Gamma_i^+(z) \left( \frac{q^{-1} w}{w - d_{i,j} q^{-1}} \right)^{A_{i,j}} \varepsilon(i, j) \varepsilon(j, i),$$

we have

$$E_i(z) E_j(w) = (-1)^{\delta_{m,1} + 1} E_j(w) E_i(z) \left( \frac{d_{i,j} q^{-1}}{w - d_{i,j} q^{-1}} \right)^{A_{i,j}}.$$

Therefore, the E-E relations hold for all $i, j \in \hat{I}$.

The F-F relations are analogously.

The E-F relations are trivial for $|i - j| > 1$. For $i$ or $j \in \hat{I}$ with $i \neq j$, it follows directly from (3.6).

If $i \in \hat{I}^- \cup \{m\}$ and $j = i + 1$, we have

$$\langle E_i(z) F_{i+1}(w) \rangle = \langle F_{i+1}(w) E_i(z) \rangle$$

$$= d^{m-2i-1} \langle \Gamma_i^+(z) \Gamma_i^+(w) \rangle \langle C_i^+(d^{2m-2i} z) C_i^+(d^{2m-2i} w) \rangle = d^{6m-3i-3/2}.$$

Thus, $[E_i(z), F_{i+1}(w)] = 0$.

The case $i \in \hat{I}^- \cup \{0\}$ and $j = i - 1$ is treated similarly. Due to the presence of the q-difference operators $\partial_z$ and $\partial_w$ in a non-trivial contraction, the expansion of $E_i(z) F_j(w)$ has four normal-ordered summands. However, the coefficient of each summand is a Laurent polynomial. Thus, $[E_i(z), F_{i-1}(w)] = 0$.

If $i = j \in \hat{I}^+$, we have

$$\langle \Gamma_i^+(z) \Gamma_i^-(w) \rangle = \left( \frac{1}{(z - q w)(z - q^{-1} w)} \right) (|z| \gg |w|),$$

$$\langle \Gamma_i^-(w) \Gamma_i^+(z) \rangle = \left( \frac{1}{(z - q w)(z - q^{-1} w)} \right) (|w| \gg |z|).$$
We can change the region where the second rational function is expanded to the same region as the first one by adding \(\delta\)-functions

\[
\begin{align*}
(3.9) \quad & \frac{1}{(z - qw)(z - q^{-1}w)} = \\
(3.10) \quad & \left( \frac{1}{(z - qw)(z - q^{-1}w)} + \frac{1}{qw^2(q - q^{-1})} \right) \delta \left( \frac{w}{z} \right) + \frac{1}{qz^2(q - q^{-1})} \delta \left( \frac{z}{w} \right) \quad (|z| \gg |w|).
\end{align*}
\]

Now,

\[
\begin{align*}
\frac{1}{qw^2(q - q^{-1})} \delta \left( \frac{w}{z} \right) : \Gamma^+_i(z) \Gamma^-_i(w) := & \frac{1}{(q - q^{-1})} \delta \left( \frac{w}{z} \right) K^+(w), \\
\frac{1}{qz^2(q - q^{-1})} \delta \left( \frac{z}{w} \right) : \Gamma^+_i(z) \Gamma^-_i(w) := & -\frac{1}{(q - q^{-1})} \delta \left( \frac{z}{w} \right) K^-(z).
\end{align*}
\]

Therefore, the \(E\)-\(F\) relations follow for \(i = j \in \hat{I}^+\).

For \(i = j = 0\) we have

\[
\begin{align*}
E_0(z) F_0(w) &= d^{m-n} : \Gamma^+_0(z) \Gamma^-_0(w) : \partial_z [C^+_{m-n-1}(d^{m-n}z)] C^+_{m-n-1}(d^{m-n}w), \\
F_0(w) E_0(z) &= d^{m-n} : \Gamma^-_0(w) \Gamma^+_0(z) : C^+_{m-n-1}(d^{m-n}w) \partial_z [C^+_{m-n-1}(d^{m-n}z)].
\end{align*}
\]

Then,

\[
\begin{align*}
[E_0(z), F_0(w)] := & \Gamma^+_0(z) \Gamma^-_0(w) : \partial_z \left[ \frac{1}{w} \delta \left( \frac{w}{z} \right) \right] \\
= & \frac{1}{zw(q - q^{-1})} \left( \delta \left( \frac{w}{z} \right) - \delta \left( \frac{z}{w} \right) \right) : \Gamma^+_0(z) \Gamma^-_0(w) :.
\end{align*}
\]

Thus, the \(E\)-\(F\) relations also follow for \(i = 0\). The case \(i = m\) is analogous and the case \(i \in \hat{I}^-\) is longer, but checked by the same procedure.

In any admissible representation, it is enough to check the quadratic relations, then the Serre relations follow automatically. Namely, the Serre relations are checked by commuting each summand and passing to a common region of convergence of the rational functions by adding suitable \(\delta\)-functions. We check the quartic relation (2.12) with \(i = m\) as an example.

Write the ten summands in (2.12) as follows. Let

\[
(3.11) \quad E_m(z_1) E_{m+1}(w_1) E_m(z_2) E_{m-1}(w_2) = E.
\]

Then, using the \(E\)-\(E\) relations, write the remaining terms of (2.12) in the form

\[
\begin{align*}
(3.12) \quad & E_{m+1}(w_1) E_m(z_2) E_{m-1}(w_2) E_m(z_1) = \frac{(dz_1 - q^{-1}w_2)(dz_1 - qw_1)}{(dq^{-1}z_1 - w_2)(dz_1 - w_1)} E, \\
(3.13) \quad & E_m(z_1) E_{m-1}(w_2) E_m(z_2) E_{m+1}(w_1) = \frac{(dz_2 - q^{-1}w_2)(dqz_2 - w_1)}{(dz_2 - qw_1)(dq^{-1}z_2 - w_2)} E, \\
(3.14) \quad & E_{m+1}(w_1) E_m(z_1) E_{m-1}(w_2) E_m(z_2) = \frac{(dz_2 - q^{-1}w_2)(dz_1 - qw_1)}{(dq^{-1}z_2 - w_2)(dz_1 - w_1)} E, \\
(3.15) \quad & E_{m-1}(w_2) E_m(z_1) E_{m+1}(w_1) E_m(z_2) = \frac{(dz_2 - q^{-1}w_2)(dz_1 - q^{-1}w_2)}{(dq^{-1}z_2 - w_2)(dq^{-1}z_1 - w_2)} E,
\end{align*}
\]
\[(3.16)\] \[E_{m-1}(w_2)E_m(z_2)E_{m+1}(w_1)E_m(z_1) = \frac{(dz_1 - q^{-1}w_2)(dz_1 - qw_1)(dqz_2 - w_1)(dz_2 - q^{-1}w_2)}{(dqz_1 - z_2)(dqz_1 - w_1)(dz_2 - q^{-1}w_1)(dq^{-1}z_2 - w_1)} E,\]

\[(3.17)\] \[E_m(z_2)E_{m+1}(w_1)E_m(z_1)E_{m-1}(w_2) = \frac{(dz_1 - q^{-1}w_2)(dz_1 - qw_1)(dqz_2 - w_1)}{(dqz_1 - w_1)(dz_2 - q^{-1}w_1)(dq^{-1}z_1 - w_1)} E,\]

\[(3.18)\] \[E_m(z_2)E_{m-1}(w_2)E_m(z_1)E_{m+1}(w_1) = \frac{(dz_1 - q^{-1}w_2)(dz_2 - w_1)(dqz_1 - w_1)}{(dq^{-1}z_1 - w_2)(dqz_1 - w_1)(dz_2 - qw_1)} E,\]

\[(3.19)\] \[-2\] \[E_m(z_1)E_{m+1}(w_1)E_{m-1}(w_2)E_m(z_2) = -2 \frac{(dz_2 - q^{-1}w_2)}{(dq^{-1}z_2 - w_2)} E,\]

\[(3.20)\] \[-2\] \[E_m(z_2)E_{m+1}(w_1)E_{m-1}(w_2)E_m(z_1) = 2 \frac{(dz_1 - q^{-1}w_2)(dz_1 - qw_1)(dqz_2 - w_1)}{(dq^{-1}z_1 - w_2)(dqz_1 - w_1)(dz_2 - qw_1)} E.\]

The rational functions of the r.h.s. of the equations \((3.12)-(3.20)\) are expanded in the region given by the increasing order of appearance of the coordinates in the l.h.s. For example, the rational function in equation \((3.12)\) is expanded in the region \(\lvert w_1 \rvert \gg \lvert z_2 \rvert \gg \lvert w_2 \rvert \gg \lvert z_1 \rvert \).

Summing up l.h.s. of equations \((3.11)-(3.20)\) we get the expansion of \((2.12)\). The sum of the rational functions in the r.h.s. vanishes as a rational function. However, similar to \(E-F\) relation, we must switch to the common convergence region and verify that the coefficients of the delta functions yielded also vanish at the respective support, cf. \((3.9)\).

For example, we choose \(\lvert z_2 \rvert \gg \lvert w_2 \rvert \gg \lvert z_1 \rvert \gg \lvert w_1 \rvert \) as a common region. The rational function in the r.h.s. \((3.14)\) in this region becomes

\[
\frac{(dz_2 - q^{-1}w_2)(dz_1 - qw_1)}{(dqz_1 - w_1)(dqz_1 - w_1)} + (1 - q^{-2})(1 - q^2)\delta \left( \frac{dqz_1}{w_1} \right) \delta \left( \frac{dz_2}{qw_2} \right) - \frac{(dz_2 - q^{-1}w_2)(1 - q^2)}{q(dq^{-1}z_2 - w_2)} \delta \left( \frac{dqz_1}{w_1} \right) - (1 - q^{-2})(dz_1 - qw_1) \delta \left( \frac{dqz_1}{w_1} \right) \delta \left( \frac{dz_2}{qw_2} \right) .
\]

Other terms are similar. After changing the region of convergence of all rational functions the \(\delta\)-functions yielded are \(\delta \left( \frac{dz_2}{qw_2} \right), \delta \left( \frac{dqz_1}{w_1} \right), \delta \left( \frac{dqz_2}{w_2} \right), \delta \left( \frac{dqz_1}{w_1} \right) \delta \left( \frac{dqz_1}{w_1} \right)\) and \(\delta \left( \frac{dz_2}{qw_2} \right) \delta \left( \frac{dqz_1}{w_1} \right),\) and the coefficient of each one vanishes at the respective support.

Thus, \((2.12)\) with \(i = m\) is proved.

3.3. **Screenings.** The \(E_{m|n}\)-modules obtained in Theorem 3.3 are not irreducible in general. To find their irreducible quotient, we follow [K1] and [KSU], and introduce the following \(\xi - \eta\) system.

We set \(\text{Res}_z(\sum_{i \in \mathbb{Z}} a_i z^{-i}) = a_1.\)

For \(i \in \hat{I}^- \cup \{m\},\) introduce the screening operators

\[
\xi_i = \text{Res}_z \left( z^{-1}C^i_-(z) \right),
\]

\[
\eta_i = \text{Res}_z C^i_+(z),
\]

acting on \(F_{\Lambda},\) with \(\Lambda = \Lambda_j, j \notin \hat{I},\) or \(\Lambda = (1 - a)\Lambda_0 + a\Lambda_m, a \in \mathbb{Z}.\)
The odd operators $\xi_i, \eta_i$, satisfy

\[
[\xi_i, \eta_j] = \delta_{i,j}, \\
[\xi_i, \xi_j] = [\eta_i, \eta_j] = 0, \\
\mathcal{F}_\Lambda = \xi_i \eta \mathcal{F}_\Lambda + \eta_i \xi \mathcal{F}_\Lambda,
\]

for all $i, j \in \hat{I} \cup \{m\}$.

Define

\[
\xi = \prod_{i \in \hat{I} \cup \{m\}} \xi_i, \quad \eta = \prod_{i \in \hat{I} \cup \{m\}} \eta_i.
\]

**Proposition 3.4.** If $\Lambda = \Lambda_i, i \not\in \hat{I}^1$ or $\Lambda = (1 - a)\Lambda_0 + a\Lambda_m, a \in \mathbb{Z}$, the screening operators $\eta_i, i \in \hat{I} \cup \{m\}$, (super)commute with the $\mathcal{E}_{m|n}$-action on $\mathcal{F}_\Lambda$ given by Theorem 3.3. Thus, ker $\eta$ and coker $\eta$ are $\mathcal{E}_{m|n}$-modules.

**Proof.** It is enough to show $[\eta_i, C_i^+(w)] = [\eta_i, \partial_w C_i^+(w)] = 0$.

Using (3.7) we have

\[
[\eta_i, C_i^+(w)] = \text{Res}_z [C_i^+(z), C_i^+(w)] = 0,
\]

and by (3.8)

\[
[\eta_i, \partial_w C_i^-(w)] = \partial_w (\text{Res}_z [C_i^+(z), C_i^-(w)]) = \partial_w (1) = 0.
\]

$\square$

Level 1 partially integrable representations of $U_q \hat{\mathfrak{sl}}_{m|n}$ with $m \neq n$ were constructed in [KSU] using the formulas in Theorem 3.3 for $i \in I$ and $d = 1$. Our space $\mathcal{F}_\Lambda$ differs from theirs by the extra current $H_0(z)$ present in $U_q^{\text{ver}} \hat{\mathfrak{g}}_{m|n}$. The conjectural identification given in [K2] and [KSU] in our context is the following.

**Conjecture 3.5.** We have the following identifications

\[
V(\Lambda_i) = \text{ker} \eta = \eta \xi \mathcal{F}_{\Lambda_i}, \quad (i \in I),
\]

\[
V((1 - a)\Lambda_0 + a\Lambda_m) = \mathcal{F}_{(1-a)\Lambda_0 + a\Lambda_m},
\]

\[
V((1 - a)\Lambda_0 + a\Lambda_m) = \text{coker} \eta = \xi \eta \mathcal{F}_{(1-a)\Lambda_0 + a\Lambda_m}, \quad (a \in \mathbb{Z}^\geq),
\]

\[
V((1 - a)\Lambda_0 + a\Lambda_m) = \text{ker} \eta = \eta \xi \mathcal{F}_{(1-a)\Lambda_0 + a\Lambda_m}, \quad (a \in \mathbb{Z}^\leq),
\]

where $V(\Lambda)$ is the irreducible highest weight $U_q^{\text{ver}} \hat{\mathfrak{g}}_{m|n}$-module with highest weight $\Lambda$.

## 4. Evaluation homomorphism

In this section, we construct an evaluation map from $\mathcal{E}_{m|n}$ to a suitable completion $\tilde{U}_q \hat{\mathfrak{g}}_{m|n}$ of $U_q \hat{\mathfrak{g}}_{m|n}$. See Appendix [B] We follow the strategy used in [FJM2].
4.1. Fused Currents. Introduce the following fused currents in \( \tilde{U}_q \mathfrak{gl}_{m|n} \)

\[
X^+(z) = \left[ \prod_{i=1}^{m+n-2} \left( 1 - \frac{z_i}{z_{i+1}} \right) \right] x_{m+n-1}^+(q^{-m+n-1}c^{-1}z_{m+n-1}) \cdots x_{m+i}^+(q^{-m+i}c^{-1}z_{m+i}) \cdots \times x_m^+(q^{-m}c^{-1}z_m) \cdots x_1^+(q^{-1}c^{-1}z_1) \bigg|_{z_1=\cdots=z_{m+n-1}=z},
\]

\[
X^-(z) = \left[ \prod_{i=1}^{m+n-2} \left( 1 - \frac{z_{i+1}}{z_i} \right) \right] x_1^-(q^{-1}c^{-1}z_1) \cdots x_i^-(q^{-i}c^{-1}z_i) \cdots x_m^-(q^{-m}c^{-1}z_m) \cdots x_{m+n-1}^-(q^{-m+n-1}c^{-1}z_{m+n-1}) \bigg|_{z_1=\cdots=z_{m+n-1}=z},
\]

\[
k^\pm(z) = \prod_{i=1}^{m} k_i^\pm(q^{-i}c^{-1}z) \prod_{j=m+1}^{m+n-1} k_j^\pm(q^{-2m+j}c^{-1}z).
\]

See [FJMM2] for the details on fused currents.

The homomorphism \( v \) defined in the Lemma 2.2 maps the element \( h_{0,r} \) in the following way

\[
v(h_{0,r}) = \frac{1}{\beta_{0,r}} \left( \gamma_{0,r}H_{0,r} + \sum_{i \in \bar{t} \cup \{m\}} (\gamma_{i,r} - \beta_{i,r}d^{ir})H_{i,r} + \sum_{j \in \bar{t}^-} (\gamma_{j,r} - \beta_{j,r}d^{2m-j}r)H_{j,r} \right),
\]

where \( \{\gamma_{i,r}\}_{i \in \bar{t}} \) and \( \{\beta_{i,r}\}_{i \in \bar{t}} \) are the fixed solutions of the systems (2.21) and (B.1), respectively. For each \( r \in \mathbb{Z}^x \), define

\[
\tilde{h}_{0,r} = \frac{1}{\gamma_{0,r}} \left( \beta_{0,r}h_{0,r} + \sum_{i \in \bar{t} \cup \{m\}} (\beta_{i,r} - \gamma_{i,r}d^{-ir})h_{i,r} + \sum_{j \in \bar{t}^-} (\beta_{j,r} - \gamma_{j,r}d^{-(2m-j)r})h_{j,r} \right).
\]

Thus, \( v(\tilde{h}_{0,r}) = H_{0,r} \) for all \( r \in \mathbb{Z}^x \).

Define \( A_{\pm r}, B_{\pm r}, r \in \mathbb{Z}_{>0} \) by

\[
A_r = -\frac{q - q^{-1}}{c^r - c^{-r}} \left( \tilde{h}_{0,r} + \sum_{i=1}^{m} (c^2 q^i)^r h_{i,r} + \sum_{j=m+1}^{m+n-1} (c^2 q^{2m-j})^r h_{j,r} \right),
\]

\[
A_{-r} = \frac{q - q^{-1}}{c^{-r} - c^r} c^{-r} \left( \tilde{h}_{0,-r} + \sum_{i=1}^{m} q^{-i r} h_{i,-r} + \sum_{j=m+1}^{m+n-1} q^{-2m+j}r h_{j,-r} \right),
\]

\[
B_r = \frac{q - q^{-1}}{c^r - c^{-r}} c^r \left( \tilde{h}_{0,r} + \sum_{i=1}^{m} q^{i r} h_{i,r} + \sum_{j=m+1}^{m+n-1} q^{2m-j}r h_{j,r} \right),
\]

\[
B_{-r} = -\frac{q - q^{-1}}{c^{-r} - c^r} \left( \tilde{h}_{0,-r} + \sum_{i=1}^{m} (c^{-2} q^{-i})^r h_{i,-r} + \sum_{j=m+1}^{m+n-1} (c^{-2} q^{-2m+j})^r h_{j,-r} \right),
\]

and let \( A^\pm(z) = \sum_{r>0} A_{\pm r} z^{\mp r}, B^\pm(z) = \sum_{r>0} B_{\pm r} z^{\mp r}. \)

Let also \( K = q^{-A_{m+n-1}} \). We have \( K x_i^\pm(z) K^{-1} = q^{(\delta_{i,1} + \delta_{-1,1})} x_i^\pm(z). \)
Theorem 4.1. Fix $u \in \mathbb{C}^\times$. The following map is a surjective homomorphism of superalgebras $\text{ev}_u : \mathcal{E}_{m|n} \to U_q \mathfrak{g}_{m|n}$ with $C^2 = q_3^{m-n}$:

$K \mapsto 1, \quad C \mapsto c, \quad H^{ver}(z) \mapsto h(z),$

$E_i(z) \mapsto x_i^+(d^i z), \quad F_i(z) \mapsto x_i^-(d^i z), \quad K^+_i(z) \mapsto k^+_i(d^i z) \quad (i \in \hat{I} \cup \{m\}),$

$E_j(z) \mapsto x_j^+(d^{2m-j} z), \quad F_j(z) \mapsto x_j^-(d^{2m-j} z), \quad K^+_j(z) \mapsto k^+_j(d^{2m-j} z) \quad (j \in \hat{I}^-),$

$E_0(z) \mapsto u^{-1} e^{A_-}(z) X^-(z) e^{A_+}(z) \mathcal{K},$

$F_0(z) \mapsto u \mathcal{K}^{-1} e^{B_-}(z) X^+(z) e^{B_+}(z).$

Moreover, the evaluation map $\text{ev}_u$ is graded: if $X \in \mathcal{E}_{m|n}$ and $\deg(X) = (d_0, d_1, \ldots, d_{m+n-1}; d_\delta)$, then $\deg(\text{ev}_u(X)) = (d_1 - d_0, \ldots, d_{m+n-1} - d_0; d_\delta)$.

Proof. For simplicity, we fix $u = 1$ and write $\text{ev}_1 = \text{ev}$. The relations with no index 0 are clear. For $i \in \hat{I}, r > 0$, we have

$$[h_{i,r}, e^{A_+}(z)] = 0,$$

$$[h_{i,r}, e^{A_-}(z)] = z^r e^{A_-}(z) c^{-r} \left( \sum_{j \in \hat{I} \cup \{m\}} \frac{[rA_{i,j}]}{r} q^{-jr} + \sum_{j \in \hat{I}^-} \frac{[rA_{i,j}]}{r} q^{-(2m-j)r} \right),$$

$$[h_{i,r}, X^-(z)] = -z^r X^-(z) c^{-r} \left( \sum_{j \in \hat{I} \cup \{m\}} \frac{[rA_{i,j}]}{r} q^{-jr} + \sum_{j \in \hat{I}^-} \frac{[rA_{i,j}]}{r} q^{-(2m-j)r} \right).$$

Thus,

$$\text{ev}([H_{i,r}, E_0(z)]) = z^r c^{-r} \frac{[rA_{i,0}]}{r} \text{ev}(E_0(z)).$$

The $H$-$E$ relations with $r < 0$ and the $H$-$F$ relations can be checked in the same way.

To check the $E$-$E$ relations we first use (A.1) and (A.2) to get

$$\text{ev}(E_0(z)E_i(w)) = e^{A_-}(z) X^-(z) \text{ev}(E_i(w)) e^{A_+}(z) \mathcal{K} \left( \frac{z - q_3^{-1}w}{z - q_1w} \right)^{\delta_{i,1}} q^{-\delta_{i,m+n-1}-\delta_{i,1}},$$

$$\text{ev}(E_i(w)E_0(z)) = e^{A_-}(z) \text{ev}(E_i(w)) X^-(z) e^{A_+}(z) \mathcal{K} \left( \frac{w - q_3 z}{w - q_1^{-1}z} \right)^{\delta_{i,m+n-1}}.$$

For $i \neq 1, m+n-1$, $\text{ev}([E_0(z), E_i(w)]) = 0$ by (A.13).

For $i = 1$, the $E$-$E$ relation reduces to

$$(q^{-1}z - dw) e^{A_-}(z) [X^-(z)x_i^+(dw)] e^{A_+}(z) \mathcal{K} = 0,$$

which follows from (A.18). The case $i = m+n-1$ is similar. The case $i = 0$ is checked using (A.3), (A.4) and (A.9).

The $F$-$F$ relations are verified by the same argument.

For the $E$-$F$ relations

$$\text{ev}([E_0(z), F_i(w)]) = 0 \quad (i \neq 0),$$
we proceed as in the $E$-$E$ case by bringing $A^-(z)$ to the left and $A^+(z)$ to the right using (A.3) and (A.4). The relations then follow from (A.13), (A.16) and (A.17). The same is done for $\text{ev}([E_i(z), F_0(w)]) = 0$ ($i \neq 0$).

For the $i = 0$ case, using (A.1),(A.6) and (A.10), we get

\[
ev(E_0(z)F_0(w)) = e^{A^-(z)}e^{B^-(w)}X^-(z)X^+(w)e^{A^+(z)}e^{B^+(w)},
\]

and similarly

\[
ev(F_0(w)E_0(z)) = e^{A^-(z)}e^{B^-(w)}X^+(w)X^-(z)e^{A^+(z)}e^{B^+(w)}.
\]

By (A.20),

\[
ev([E_0(z), F_0(w)]) = e^{A^-(z)}e^{B^-(w)}\frac{1}{q-q^{-1}}\left(\delta \left(\frac{w}{z}\right)k^-(w) - \delta \left(\frac{z}{w}\right)k^+(z)\right)e^{A^+(z)}e^{B^+(w)}.
\]

The relation (2.6) with $i = j = 0$ follows from

\[
e^{A^-(z)}e^{B^-(cz)} = \tilde{k}_0^-(z), \quad e^{A^-(cw)}e^{B^-(w)} = k_0^-(k^-(w))^{-1},
\]

\[
e^{A^+(cw)}e^{B^+(w)} = \tilde{k}_0^+(w), \quad e^{A^+(z)}e^{B^+(cz)} = k_0^-(k^+(z))^{-1},
\]

where $\tilde{k}_0^\pm = \exp\left(\pm(q-q^{-1})\sum_{r>0}\tilde{h}_{0,\pm r}z^{\pm r}\right)$.

Finally, we check the Serre relations. For the relation

\[\text{ev} \left(\text{Sym}_{z_1,z_2} [E_1(z_1), [E_1(z_2), E_0(w)], q], q^{-1}\right) = 0\]

we use (A.11) and (A.18) to obtain

\[-(q+q^{-1})\text{ev} \left(E_1(z_1)E_0(w)E_1(z_2)\right) = -q(q+q^{-1})\left(\frac{z_2-q_3w}{z_2-q_1^{-1}w}\right)\text{ev} \left(E_1(z_1)E_1(z_2)E_0(w)\right),
\]

\[\text{ev} \left(E_0(w)E_1(z_1)E_1(z_2)\right) = q^2\left(\frac{z_1-q_3w}{z_1-q_1^{-1}w}\right)\left(\frac{z_2-q_3w}{z_2-q_1^{-1}w}\right)\text{ev} \left(E_1(z_1)E_1(z_2)E_0(w)\right).
\]

Thus,

\[\text{Sym}_{z_1,z_2} \left(\text{ev} [E_1(z_1), [E_1(z_2), E_0(w)], q], q^{-1}\right) = \frac{(1-q^2)w}{q_3(w-q_1z_1)(w-q_1z_2)}\text{Sym}_{z_1,z_2} \left(\left(z_1-q^2z_2\right)\text{ev} \left(E_1(z_1)E_1(z_2)E_0(w)\right)\right) = 0,
\]

where the last equality follows from the quadratic relation for $x_1^+(dz_1)x_1^+(dz_2)$. The Serre relations in all the remaining cases are checked in the same way.

The statement about grading is straightforward.

By Theorem 4.1, any admissible $U_q \tilde{\mathfrak{gl}}_{m|n}$-module of any level $c$ can be pulled back by $\text{ev}_u$ to a representation of $\mathcal{E}_{m|n}$ with $q_3^{m-n} = c^2$ and $q_2 = q^2$. Such $\mathcal{E}_{m|n}$ modules are called evaluation modules.

There exist another evaluation map $\text{ev}_u^{(1)}$ obtained by composing the map $\text{ev}_u$ for $\mathcal{E}_{n|m}$ with the change of parity isomorphism (2.23). For this map we have $C^2 = q_1^{n-m}$.

The evaluation maps $\text{ev}_u$ and $\text{ev}_u^{(1)}$ prefer our choice of the vertical subalgebra related to the zero node of the Dynkin diagram. In fact, there are evaluation maps which prefer any node. The
ones related to odd node \( m \) are obtained by the diagram automorphism \([2, 22]\). However, the vertical subalgebras related to even nodes appear in non-standard parities, and we do not discuss the corresponding isomorphisms or evaluation maps in this paper.

### Appendix A.

In this Appendix, we collect some useful formulas for commutation relations of various currents.

**Lemma A.1.** For \( i \in I \), we have

\[
\begin{align*}
(A.1) & \quad e^{A^+(z)} x_i^+(dw) e^{-A^+(z)} = x_i^+(dw) \left( \frac{z - q_3^{-1} w}{z - q_1 w} \right)^{\delta_{i,1}}, \\
(A.2) & \quad e^{-A^+(z)} x_i^+(d^{m-n+1} w) e^{A^-(z)} = x_i^+(d^{m-n+1} w) \left( \frac{w - q_3 z}{w - q_1^{-1} z} \right)^{\delta_{i,m+n-1}}, \\
(A.3) & \quad e^{A^+(z)} x_i^- (dw) e^{-A^+(z)} = x_i^- (dw) \left( \frac{z - cq_1 w}{z - c q_3 w} \right)^{\delta_{i,1}}, \\
(A.4) & \quad e^{-A^-(z)} x_i^- (d^{m-n+1} w) e^{A^-(z)} = x_i^- (d^{m-n+1} w) \left( \frac{w - c q_1^{-1} z}{w - c q_3 z} \right)^{\delta_{i,m+n-1}}, \\
(A.5) & \quad e^{B^+(z)} x_i^+(d^{m-n+1} w) e^{-B^+(z)} = x_i^+(d^{m-n+1} w) \left( \frac{z - q_3^{-1} w}{z - c^{-1} q_3 w} \right)^{\delta_{i,1,m+n-1}}, \\
(A.6) & \quad e^{-B^+(z)} x_i^+(dw) e^{B^-(z)} = x_i^+(dw) \left( \frac{w - c^{-1} q_1^{-1} z}{w - c^{-1} q_3 z} \right)^{\delta_{i,1}}, \\
(A.7) & \quad e^{B^-(z)} x_i^- (d^{m-n+1} w) e^{-B^-(z)} = x_i^- (d^{m-n+1} w) \left( \frac{z - q_3^{-1} w}{z - q_1 w} \right)^{\delta_{i,1,m+n-1}}, \\
(A.8) & \quad e^{-B^-(z)} x_i^- (dw) e^{B^-(z)} = x_i^- (dw) \left( \frac{w - q_3 z}{w - q_1^{-1} z} \right)^{\delta_{i,1}}, \\
(A.9) & \quad e^{A^+(z)} e^{A^-(w)} = e^{A^-(w)} e^{A^+(z)} \frac{(z - w)^2}{(z - q_2 w)(z - q_2^{-1} w)}, \\
(A.10) & \quad e^{A^+(z)} e^{B^-(w)} = e^{B^-(w)} e^{A^+(z)} \frac{(z - c q_2 w)(z - c^{-1} q_2^{-1} w)}{(z - c w)(z - c^{-1} w)}, \\
(A.11) & \quad e^{B^+(z)} e^{B^-(w)} = e^{B^-(w)} e^{B^+(z)} \frac{(z - w)^2}{(z - q_2 w)(z - q_2^{-1} w)}, \\
(A.12) & \quad e^{B^+(w)} e^{A^-(z)} = e^{A^-(z)} e^{B^+(w)} \frac{(z - c^{-1} q_2 w)(z - c q_2^{-1} w)}{(z - c w)(z - c^{-1} w)}.
\]

**Lemma A.2.** The fused currents \( X^\pm(z) \) satisfy

\[
\begin{align*}
(A.13) & \quad [x_i^\pm(z), X^\pm(w)] = [x_i^\pm(z), X^\mp(w)] = 0 & (i \neq 1, m + n - 1), \\
(A.14) & \quad q(w - c^{-1} q_3 z) X^+(z) x_i^+(dw) = (w - c^{-1} q_1^{-1} z) x_i^+(dw) X^+(z),
\]

\[\Box\]
fundamental weight and
the symmetric bilinear form given by
\[ \langle \alpha \rangle \]
We recall them here using the standard choice of parity.
The relations are as follows.
The presentations of the superalgebra
\[ U \]
The element
\[ t \]
The remaining affine fundamental weights are \( \Lambda \).
The affine fundamental weights are \( \Lambda = \Lambda_i + \Lambda_0, \ i \in I \). The affine roots are \( \alpha_i = \bar{\alpha}_i, \ i \in I \), and \( \alpha_0 = \delta - \sum_{i \in I} \alpha_i \). Set also \( \bar{\alpha}_0 = -\sum_{i \in I} \bar{\alpha}_i \).

In the Drinfeld-Jimbo realization, the algebra \( U_q \widehat{\mathfrak{sl}}_{m|n} \) is generated by Chevalley type elements \( e_i, f_i, t_i, \ i \in \hat{I} \). The parity of generators is given by \(|e_0| = |f_0| = |e_m| = |f_m| = 1\) and 0 otherwise. The relations are as follows.

\[
\begin{align*}
t_i t_j & = t_j t_i, \\
t_i e_j t_i^{-1} & = q^{A_{i,j}} e_j, \\
t_i f_j t_i^{-1} & = q^{-A_{i,j}} f_j, \\
[e_i, e_j] & = [f_i, f_j] = 0 \quad (A_{i,j} = 0), \\
[e_i, [e_i, e_{i+1}]_q]_{q^{-1}} & = [f_i, [f_i, f_{i+1}]_q]_{q^{-1}} = 0 \quad (i \notin \hat{I}), \\
[e_i, [e_{i+1}, [e_i, e_{i-1}]_q]_{q^{-1}}] & = [f_i, [f_{i+1}, [f_i, f_{i-1}]_q]_{q^{-1}}] = 0 \quad (i \in \hat{I}, mn \neq 2), \\
[e_2, [e_0, [e_2, [e_0, e_1]_q]]_{q^{-1}} & = [e_0, [e_2, [e_0, e_2]_q]]_{q^{-1}} = 0 \quad ((m, n) = (2, 1)), \\
[f_2, [f_0, [f_2, [f_0, f_2]_q]]_{q^{-1}} & = [f_0, [f_2, [f_0, f_2]_q]]_{q^{-1}} = 0 \quad ((m, n) = (2, 1)), \\
[e_1, [e_0, [e_1, [e_0, e_2]_q]]_{q^{-1}} & = [e_0, [e_1, [e_0, e_1]_q]]_{q^{-1}} = 0 \quad ((m, n) = (1, 2)), \\
[f_1, [f_0, [f_1, [f_0, f_2]_q]]_{q^{-1}} & = [f_0, [f_1, [f_0, f_2]_q]]_{q^{-1}} = 0 \quad ((m, n) = (1, 2)).
\end{align*}
\]

The element \( t_0 t_1 \cdots t_{m+n-1} \) is central.
In the new Drinfeld realization, the algebra $U_q\widehat{\mathfrak{sl}}_{m|n}$ is generated by current generators $x^\pm_{i,n}, h_{i,r}, k_i^{\pm 1}, c^{\pm 1}, i \in I, n \in \mathbb{Z}, r \in \mathbb{Z}^\times$, satisfying

\[
\begin{align*}
c \text{ is central, } k_i k_j &= k_j k_i, \quad k_i x^\pm_j(z) k_i^{-1} = q^{\pm A_{i,j}} x^\pm_j(z), \\
[h_{i,r}, h_{j,s}] &= \delta_{r+s,0} [r A_{i,j}] c^r - c^{-r} \quad (r) = q - q^{-1}, \\
[h_{i,r}, x^\pm_j(z)] &= \pm \frac{[r A_{i,j}]}{r} c^{-(r \pm |r|)/2} z^r x^\pm_j(z), \\
x^\pm_i(z), x^\pm_j(w) &= \frac{\delta_{i,j}}{q - q^{-1}} \left( \delta(c w) k^+_i(w) - \delta(c z) k^-_i(z) \right), \\
(z - q^{\pm A_{i,j}} w)x^\pm_i(z)x^\pm_j(w) + (w - q^{\pm A_{i,j}} z)x^\pm_j(w)x^\pm_i(z) &= 0, \quad (A_{i,j} \neq 0), \\
x^\pm_i(z), x^\pm_j(w) &= 0 \quad (A_{i,j} = 0), \\
\text{Sym}_{z_1, z_2}[x^\pm_i(z_1), x^\pm_j(z_2), x^\pm_j(w)]_{q^{-1}} &= 0 \quad (i \neq m), \\
\text{Sym}_{z_1, z_2}[x^\pm_m(z_1), x^\pm_{m+1}(w_1), x^\pm_{m}(z_2), x^\pm_{m-1}(w_2)]_{q^{-1}} &= 0 \quad (m, n > 1),
\end{align*}
\]

where $x^\pm_i(z) = \sum_{k \in \mathbb{Z}} x^\pm_{i,k} z^{-k}$, $k_i^{\pm 1} = k_i^{\pm 1} \exp \left( \pm (q - q^{-1}) \sum_{r > 0} h_{i,r} z^{r} \right)$.

An isomorphism between the two realizations is given by

\[
\begin{align*}
e_i &\mapsto x^+_i, \quad f_i \mapsto x^-_i, \quad t_i \mapsto k_i \quad (i \in I), \\
t_0 &\mapsto c(k_1 k_2 \cdots k_{m+n-1})^{-1}, \\
e_0 &\mapsto (-1)^n(k_1 k_2 \cdots k_{m+n-1})^{-1}[x^-_{m+1}, x^-_{m}, \cdots, [x^-_2, x^-_1] q^{-1} \cdots q^{-1}], \\
f_0 &\mapsto k_1 k_2 \cdots k_{m+n-1} \cdot \cdots \cdot [x^+_2 q, \cdots, x^+_1 q, x^+_0 q, x^+_m q, x^+_m q, \cdots, x^+_m q, q^{-1} \cdots q^{-1}].
\end{align*}
\]

Note that $t_0 t_1 \cdots t_{m+n-1} \mapsto c$.

The quantum affine superalgebra $U_q\widehat{\mathfrak{sl}}_{m|n}$ is obtained from $U_q\widehat{\mathfrak{sl}}_{m|n}$ in the new Drinfeld realization by including the currents $k_0^{\pm 1}(z)$ subject to the same relations.

For $r \in \mathbb{Z}^\times$, let

\[
\beta_{i,r} = \begin{cases} 
\frac{q^{(m-n-i)r} + q^{ir}}{q^r - q^{-r}} & (i \in \hat{I} \cup \{ m \}), \\
\frac{q^{(2m-n-i)r} + q^{(2m-i)r}}{q^{2r} - q^{-2r}} & (i \in \hat{I}^-).
\end{cases}
\]

Then, the elements $h_r = \sum_{i \in \hat{I}} \beta_{i,r} h_{i,r} \in U_q\widehat{\mathfrak{sl}}_{m|n}$ commute with $U_q\widehat{\mathfrak{sl}}_{m|n} \subset U_q\widehat{\mathfrak{gl}}_{m|n}$ and satisfy

\[
[h_r, h_s] = \delta_{r+s,0} [(n - m)r] \frac{1}{r} \frac{c^r - c^{-r}}{q - q^{-1}}.
\]

Set $h(z) = \sum_{k \in \mathbb{Z}^\times} h_k z^{-k}$.

We use a completion of $U_q\widehat{\mathfrak{gl}}_{m|n}$, denoted by $\widetilde{U}_q\widehat{\mathfrak{gl}}_{m|n}$, obtained by performing the following two steps.

The algebra $U_q\widehat{\mathfrak{sl}}_{m|n}$ contains the algebra of the root lattice $\mathbb{Z}Q$ generated by $k_i = q^{\alpha_i}, i \in \hat{I}$. As a first step, we extend it to the weight lattice in a straightforward way. Namely, let $P$ be the $\widehat{\mathfrak{sl}}_{m|n}$
weight lattice and $C\mathbb{P}$ the corresponding group algebra spanned by $q^A, A \in P$. We have an inclusion of algebras $CQ \subset C\mathbb{P}$. Let $U_P$ be the superalgebra $U_P = U_q \tilde{\mathfrak{gl}}_{m|n} \otimes CQ C\mathbb{P}$ with the relations
\[ q^A q^{A'} = q^{A+A'}, \quad q^0 = 1, \quad q^A x^\pm_i(z) q^{-A} = q^{\pm\langle A|\alpha_i \rangle} x^\pm_i(z) \quad (A, A' \in P).\]

For each $i \in I$, the superalgebra $U_P$ has a $\mathbb{Z}$-grading given by
\[ \deg_i(x^\pm_{j,k}) = \pm \delta_{i,j}, \quad \deg_i(h_{j,r}) = \deg_i(q^A) = \deg_i(c) = 0 \quad (j \in I, k \in \mathbb{Z}, r \in \mathbb{Z}, A \in P).\]

There is also the homogeneous $\mathbb{Z}$-grading given by
\[ \deg_\delta(x^\pm_{j,k}) = k, \quad \deg_\delta(h_{j,r}) = r, \quad \deg_\delta(q^A) = \deg_\delta(c) = 0 \quad (j \in I, k \in \mathbb{Z}, r \in \mathbb{Z}, A \in P).\]

Thus the superalgebra $U_P$ has a $\mathbb{Z}^{m+n}$-grading given by
\[ \deg(X) = (\deg_1(X), \ldots, \deg_{m+n-1}; \deg_\delta(X)), \quad X \in U_P.\]

As the second step, we define $\tilde{U}_q \tilde{\mathfrak{gl}}_{m|n}$ to be the completion of $U_P$ with respect to the homogeneous grading in the positive direction. The elements of $\tilde{U}_q \tilde{\mathfrak{gl}}_{m|n}$ are series of the form $\sum_{j=0}^\infty g_j$, with $g_j \in U_P$, $\deg_\delta g_j = j$.

**Lemma B.1.** We have an embedding 
\[ U_q \tilde{\mathfrak{gl}}_{m|n} \to \tilde{U}_q \tilde{\mathfrak{gl}}_{m|n}. \]

A $U_q \tilde{\mathfrak{gl}}_{m|n}$-module $V$ is admissible if for any $v \in V$ there exist $N = N_v > 0$ such that $xv = 0$ for all $x \in U_q \tilde{\mathfrak{gl}}_{m|n}$ with $\deg_\delta x > N$. Any admissible $U_q \tilde{\mathfrak{gl}}_{m|n}$-module is also an $\tilde{U}_q \tilde{\mathfrak{gl}}_{m|n}$-module.

A $U_q \tilde{\mathfrak{gl}}_{m|n}$-module $V$ is called highest weight module if $V$ is generated by the highest weight vector $v$:
\[ V = U_q \tilde{\mathfrak{gl}}_{m|n} v, \quad e_iv = 0, \quad k_0^+(z)v = q^{\lambda_0}v, \quad t_jv = q^{\lambda_j}v, \quad i \in \tilde{I}, j \in I.\]

Highest weight $U_q \tilde{\mathfrak{gl}}_{m|n}$-modules are admissible.

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