Inner Regularization of Log-Concave Measures and Small-Ball Estimates

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Abstract

In the study of concentration properties of isotropic log-concave measures, it is often useful to first ensure that the measure has super-Gaussian marginals. To this end, a standard preprocessing step is to convolve with a Gaussian measure, but this has the disadvantage of destroying small-ball information. We propose an alternative preprocessing step for making the measure seem super-Gaussian, at least up to reasonably high moments, which does not suffer from this caveat: namely, convolving the measure with a random orthogonal image of itself. As an application of this “inner-thickening”, we recover Paouris’ small-ball estimates.

1 Introduction

Fix a Euclidean norm $|\cdot|$ on $\mathbb{R}^n$, and let $X$ denote an isotropic random vector in $\mathbb{R}^n$ with log-concave density $g$. Recall that a random vector $X$ in $\mathbb{R}^n$ (and its density) is called isotropic if $EX = 0$ and $EX \otimes X = Id$, i.e. its barycenter is at the origin and its covariance matrix is equal to the identity one. Taking traces, we observe that $\mathbb{E}|X|^2 = n$. Here and throughout we use $\mathbb{E}$ to denote expectation and $\mathbb{P}$ to denote probability. A function $g : \mathbb{R}^n \to \mathbb{R}_+$ is called log-concave if $-\log g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex. Throughout this work, $C, c, c_2, C'$, etc. denote universal positive numeric constants, independent of any other parameter and in particular the dimension $n$, whose value may change from one occurrence to the next.

Any high-dimensional probability distribution which is absolutely continuous has at least one super-Gaussian marginal (e.g. \cite{13}). Still, in the study of concentration properties of $X$ as above, it is many times advantageous to know that all of the one-dimensional marginals of $X$ are super-Gaussian, at least up to some level (see e.g. \cite{24}, \cite{9}, \cite{14}). By this we mean that for some $p_0 \geq 2$:

$$\forall 2 \leq p \leq p_0 \quad \forall \theta \in S^{n-1} \quad (\mathbb{E}|\langle X, \theta \rangle|^p)^{\frac{1}{p}} \geq c(\mathbb{E}|G_1|^p)^{\frac{1}{p}}, \quad (1.1)$$

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where $G_1$ denotes a one-dimensional standard Gaussian random variable and $S^{n-1}$ is the Euclidean unit sphere in $\mathbb{R}^n$. It is convenient to reformulate this using the language of $L_p$-centroid bodies, which were introduced by E. Lutwak and G. Zhang in [16] (under a different normalization). Given a random vector $X$ with density $g$ on $\mathbb{R}^n$ and $p \geq 1$, the $L_p$-centroid body $Z_p(X) = Z_p(g) \subset \mathbb{R}^n$ is the convex set defined via its support functional $h_{Z_p(X)}$ by:

$$h_{Z_p(X)}(y) = \left( \int_{\mathbb{R}^n} |\langle x, y \rangle|^p g(x) dx \right)^{1/p}, \quad y \in \mathbb{R}^n.$$ 

More generally, the one-sided $L_p$-centroid body, denoted $Z^+_p(X)$, was defined in [9] (cf. [11]) by:

$$h_{Z^+_p(X)}(y) = \left( 2 \int_{\mathbb{R}^n} \langle x, y \rangle_+^p g(x) dx \right)^{1/p}, \quad y \in \mathbb{R}^n,$$

where as usual $a_+ := \max(a, 0)$. Note that when $g$ is even then both definitions above coincide, and that when the barycenter of $X$ is at the origin, $Z_2(X)$ is the Euclidean ball $B^n_2$ if and only $X$ is isotropic. Observing that the right-hand side of (1.1) is of the order of $\sqrt{p}$, we would like to have:

$$\forall 2 \leq p \leq p_0 \quad Z^+_p(X) \supset c\sqrt{p}B^n_2,$$  

(1.2)

where $B^n_2 = \{ x \in \mathbb{R}^n; |x| \leq 1 \}$ is the unit Euclidean ball.

Unfortunately, we cannot in general expect to satisfy (1.2) for $p_0$ which grows with the dimension $n$. This is witnessed by $X$ which is uniformly distributed on the $n$-dimensional cube $[-\sqrt{3}, \sqrt{3}]^n$ (the normalization ensures that $X$ is isotropic), whose marginals in the directions of the axes are uniform on a constant-sized interval. Consequently, some preprocessing on $X$ is required, which on one hand transforms it into another random variable $Y$ whose density $g$ satisfies (1.2), and on the other enables deducing back the desired concentration properties of $X$ from those of $Y$.

A very common such construction is to convolve with a Gaussian, i.e. define $Y := (X + G_n) / \sqrt{2}$, where $G_n$ denotes an independent standard Gaussian random vector in $\mathbb{R}^n$. In [11] (and in subsequent works like [12, 5]), the Gaussian played more of a regularizing role, but in [9], its purpose was to “thicken from inside” the distribution of $X$, ensuring that (1.2) is satisfied for all $p \geq 2$ (see [9, Lemma 2.3]). Regarding the transference of concentration properties, it follows from the argument in the proof of [11 Proposition 4.1] that:

$$\mathbb{P}(|X| \geq (1 + t)\sqrt{n}) \leq C\mathbb{P} \left( |Y| \geq \sqrt{\frac{(1+t)^2+1}{2}} \sqrt{n} \right) \quad \forall t \geq 0,$$  

(1.3)

and:

$$\mathbb{P}(|X| \leq (1 - t)\sqrt{n}) \leq C\mathbb{P} \left( |Y| \leq \sqrt{\frac{(1-t)^2+1}{2}} \sqrt{n} \right) \quad \forall t \in [0, 1],$$  

(1.4)
for some universal constant $C > 1$. The estimate (1.3) is perfectly satisfactory for transferring (after an adjustment of constants) deviation estimates above the expectation from $|Y|$ to $|X|$. However, note that the right-hand side of (1.4) is bounded below by $P(|Y| \leq \sqrt{n}/2)$ (and in particular does not decay to 0 when $t \to 1$), and so (1.4) is meaningless for transferring small-ball estimates from $|Y|$ to $|X|$. Consequently, the strategies employed in [11, 12, 5, 9] did not and could not deduce the concentration properties of $|X|$ in the small-ball regime. This seems an inherent problem of adding an independent Gaussian: small-ball information is lost due to the “Gaussian-thickening”.

The purpose of this note is to introduce a different inner-thickening step, which does not have the above mentioned drawback. Before formulating it, recall that $X$ (or its density) is said to be “$\psi_\alpha$ with constant $D > 0$” if:

$$Z_p(X) \subset D^{1/\alpha}Z_2(X) \quad \forall p \geq 2. \quad (1.5)$$

We will simply say that “$X$ is $\psi_\alpha$”, if it is $\psi_\alpha$ with constant $D \leq C$, and not specify explicitly the dependence of the estimates on the parameter $D$. By a result of Berwald [1] (or applying Borell’s Lemma [3] as in [21, Appendix III]), it is well known that any $X$ with log-concave density satisfies:

$$1 \leq p \leq q \Rightarrow Z_p(X) \subset Z_q(X) \subset C_q \frac{q}{p} Z_p(X). \quad (1.6)$$

In particular, such an $X$ is always $\psi_1$ with some universal constant, and so we only gain additional information when $\alpha > 1$.

**Theorem 1.1.** Let $X$ denote an isotropic random vector in $\mathbb{R}^n$ with a log-concave density, which is in addition $\psi_\alpha$ ($\alpha \in [1,2]$), and let $X'$ denote an independent copy of $X$. Given $U \in O(n)$, the group of orthogonal linear maps in $\mathbb{R}^n$, denote:

$$Y_u^\pm := \frac{X \pm U(X')}{\sqrt{2}}.$$

Then:

1. For any $U \in O(n)$, the concentration properties of $|Y_u^\pm|$ are transferred to $|X|$ as follows:

$$P(|X| \geq (1+t)\sqrt{n}) \leq (2 \max (P(|Y_u^+| \geq (1+t)\sqrt{n}), P(|Y_u^-| \geq (1+t)\sqrt{n})))^{1/2} \quad \forall t \geq 0,$$

and:

$$P(|X| \leq (1-t)\sqrt{n}) \leq (2 \max (P(|Y_u^+| \leq (1-t)\sqrt{n}), P(|Y_u^-| \leq (1-t)\sqrt{n})))^{1/2} \quad \forall t \in [0,1].$$

2. For any $U \in O(n)$:

$$Z_p^+(Y_u^\pm) \subset C_p^{1/\alpha}B_2^n \quad \forall p \geq 2. \quad (1.7)$$

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3. There exists a subset $A \subset O(n)$ with:

$$\mu_{O(n)}(A) \geq 1 - \exp(-cn) ,$$

where $\mu_{O(n)}$ denotes the Haar measure on $O(n)$ normalized to have total mass 1, so that if $U \in A$ then:

$$Z_p^+(Y_U^\pm) \supset c_1 \sqrt{p} B_2^n \quad \forall p \in [2, c_2 n^\frac{\alpha}{2}] . \tag{1.8}$$

Remark 1.2. Note that when the density of $X$ is even, then $Y_U^+$ and $Y_U^-$ in Theorem 1.1 are identically distributed, which renders the formulation of the conclusion more natural. However, we do not know how to make the formulation simpler in the non-even case.

Remark 1.3. Also note that $Y_U^\pm$ are isotropic random vectors, and that by the Prékopa–Leindler Theorem (e.g. [7]), they have log-concave densities.

As our main application, we manage to extend the strategy in the second named author’s previous work with O. Guédon [9] to the small-ball regime, and obtain:

Corollary 1.4. Let $X$ denote an isotropic random vector in $\mathbb{R}^n$ with log-concave density, which is in addition $\psi_\alpha$ ($\alpha \in [1, 2]$). Then:

$$P(|X| - \sqrt{n} \geq t \sqrt{n}) \leq C \exp(-cn^\frac{\alpha}{2} \min(t^{2+\alpha}, t)) \quad \forall t \geq 0 , \tag{1.9}$$

and:

$$P(|X| \leq \epsilon \sqrt{n}) \leq (C\epsilon)^{cn^\frac{\alpha}{2}} \quad \forall \epsilon \in [0, 1/C] . \tag{1.10}$$

Corollary 1.4 is an immediate consequence of Theorem 1.1 and the following result, which is the content of [9, Theorem 4.1] (our formulation below is slightly more general, but this is what the proof gives):

Theorem (Guédon–Milman). Let $Y$ denote an isotropic random vector in $\mathbb{R}^n$ with a log-concave density, so that in addition:

$$c_1 \sqrt{p} B_2^n \subset Z_p^+(Y) \subset c_2 p^{1/\alpha} B_2^n \quad \forall p \in [2, c_3 n^\frac{\alpha}{2}] , \tag{1.11}$$

for some $\alpha \in [1, 2]$. Then (1.9) and (1.10) hold with $X = Y$ (and perhaps different constants $C, c > 0$).

We thus obtain a preprocessing step which fuses perfectly with the approach in [9], allowing us to treat all deviation regimes simultaneously in a single unified framework. We point out that Corollary 1.4 by itself is not new. The large positive-deviation estimate:

$$P(|X| \geq (1 + t) \sqrt{n}) \leq \exp(-cn^\frac{\alpha}{2} t) \quad \forall t \geq C ,$$

was first obtained by G. Paouris in [22]; it is known to be sharp, up to the value of the constants. The more general deviation estimate (1.9) was obtained in [9], improving
when $t \in [0, C]$ all previously known results due to the first named author and to Fleury \cite{11, 12, 5} (we refer to \cite{9} for a more detailed account of these previous estimates). In that work, the convolution with Gaussian preprocessing was used, and so it was not possible to independently deduce the small-ball estimate (1.10). The latter estimate was first obtained by Paouris in \cite{23}, using the reverse Blaschke-Santaló inequality of J. Bourgain and V. Milman \cite{4}. In comparison, our main tool in the proof of Theorem 1.1 is a covering argument in the spirit of V. Milman’s M-position \cite{17, 19, 18} (see also \cite{25}), together with a recent lower-bound on the volume of $Z_p$ bodies obtained in our previous joint work \cite{14}.

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2 Key Proposition

In this section, we prove the following key proposition:

Proposition 2.1. Let $X, X'$ be as in Theorem 1.1, let $U$ be uniformly distributed on $O(n)$, and set:

$$Y := \frac{X + U(X')}{\sqrt{2}}.$$ 

Then there exists a $c > 0$, so that:

$$\forall C_1 > 0 \quad \exists c_1 > 0 \quad \forall p \in [2, cn^{\alpha/2}] \quad \mathbb{P}(Z_p^+(Y) \subset c_1\sqrt{p}B_2^n) \geq 1 - \exp(-C_1 n).$$

Here, as elsewhere, “uniformly distributed on $O(n)$” is with respect to the probability measure $\mu_{O(n)}$.

We begin with the following estimate due to Grünbaum \cite{8} (see also \cite{6, Formula (10)} or \cite{2, Lemma 3.3} for simplified proofs):

Lemma 2.2 (Grünbaum). Let $X_1$ denote a random variable on $\mathbb{R}$ with log-concave density and barycenter at the origin. Then $\frac{1}{e} \leq \mathbb{P}(X_1 \geq 0) \leq 1 - \frac{1}{e}$.

Recall that the Minkowski sum $K + L$ of two compact sets $K, L \subset \mathbb{R}^n$ is defined as the compact set given by $\{x + y; x \in K, y \in L\}$. When $K, L$ are convex, the support functional satisfies $h_{K+L} = h_K + h_L$.

Lemma 2.3. With the same notations as in Proposition 2.1,

$$Z_p^+(Y) \supset \frac{1}{2\sqrt{2e^{1/p}}} (Z_p^+(X) + U(Z_p^+(X))).$$

Proof. Given $\theta \in S^{n-1}$, denote $Y_1 = \langle Y, \theta \rangle$, $X_1 = \langle X, \theta \rangle$ and $X'_1 = \langle U(X'), \theta \rangle$. By the Prékopa–Leindler theorem (e.g. \cite{7}), all these one-dimensional random variables have log-concave densities, and since their barycenter is at the origin, we obtain by Lemma 2.2:

$$h_{Z_p^+(Y)}^p(\theta) = 2\mathbb{E}(Y_1)^p = \frac{2}{2p/2} \mathbb{E}(X_1 + X'_1)^p \geq \frac{2}{2p/2} \mathbb{E}(X_1)^p \mathbb{P}(X'_1 \geq 0) \geq \frac{2}{e2^{p/2}} \mathbb{E}(X_1)^p.$$

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Exchanging the roles of $X_1$ and $X'_1$ above, we obtain:

$$h_{Z_p^+(Y)}(\theta) \geq \frac{1}{e^{2p/2}} \max \left( h_{Z_p^+(X)}(\theta), h_{Z_p^+(U(X'))}(\theta) \right).$$

Consequently:

$$h_{Z_p^+(Y)}(\theta) \geq \frac{1}{\sqrt{2e^{1/p}}} \frac{h_{Z_p^+(X)}(\theta) + h_{Z_p^+(U(X'))}(\theta)}{2},$$

and since $Z_p^+(U(X')) = U(Z_p^+(X')) = U(Z_p^+(X))$, the assertion follows. \hfill \Box

Next, recall that given two compact subsets $K, L \subset \mathbb{R}^n$, the covering number $N(K, L)$ is defined as the minimum number of translates of $L$ required to cover $K$. The volume-radius of a compact set $K \subset \mathbb{R}^n$ is defined as:

$$\text{V.Rad.}(K) = \left( \frac{\text{Vol}(K)}{\text{Vol}(B_n^2)} \right)^{\frac{1}{n}},$$

measuring the radius of the Euclidean ball whose volume equals the volume of $K$. A convex compact set with non-empty interior is called a convex body, and given a convex body $K$ with the origin in its interior, its polar $K^\circ$ is the convex body given by:

$$K^\circ := \{ y \in \mathbb{R}^n; \langle x, y \rangle \leq 1 \ \forall x \in K \}.$$

Finally, the mean-width of a convex body $K$, denoted $W(K)$, is defined as $W(K) = 2 \int_{S^{n-1}} h_K(\theta) d\mu_{S^{n-1}}(\theta)$, where $\mu_{S^{n-1}}$ denotes the Haar probability measure on $S^{n-1}$.

The following two lemmas are certainly well-known; we provide a proof for completeness.

**Lemma 2.4.** Let $K \subset \mathbb{R}^n$ be a convex body with barycenter at the origin, so that:

$$N(K, B_n^2) \leq \exp(A_1 n) \quad \text{and} \quad \text{V.Rad.}(K) \geq a_1 > 0.$$

Then:

$$N(K^\circ, B_n^2) \leq \exp(A_2 n),$$

where $A_2 \leq A_1 + \log(C/a_1)$, and $C > 0$ is a universal constant.

**Proof.** Set $K_s = K \cap -K$. By the covering estimate of H. König and V. Milman [15], it follows that:

$$N(K^\circ, B_n^2) \leq N(K_s^\circ, B_n^2) \leq C_n N(B_n^2, K_s) .$$

Using standard volumetric covering estimates (e.g. [25], Chapter 7], we deduce:

$$N(K^\circ, B_n^2) \leq C_n \left( \frac{\text{Vol}(B_n^2 + K_s/2)}{\text{Vol}(K_s/2)} \right) \leq C_n N(K_s/2, B_n^2) \frac{\text{Vol}(2B_n^2)}{\text{Vol}(K_s/2)} .$$

By a result of V. Milman and A. Pajor [20], it is known that $\text{Vol}(K_s) \geq 2^{-n} \text{Vol}(K)$, and hence:

$$N(K^\circ, B_n^2) \leq (8C)^n N(K, B_n^2) \text{V.Rad.}(K)^{-n} \leq (8C/a_1)^n \exp(A_1 n),$$

as required. \hfill \Box
Lemma 2.5. Let $L$ denote any compact set in $\mathbb{R}^n$ $(n \geq 2)$, so that $N(L, B^n_2) \leq \exp(A_1 n)$. If $U$ is uniformly distributed on $O(n)$, then:

$$P(L \cap U(L) \subset A_3 B^n_2) \geq 1 - \exp(-A_2 n) ,$$

where $A_2 = A_1 + (\log 2)/2$ and $A_3 = C' \exp(6A_1)$, for some universal constant $C' > 0$.

Proof Sketch. Assume that $L \subset \bigcup_{i=1}^{\exp(A_1 n)} (x_i + B^n_2)$. Set $R = 4C \exp(6A_1)$, for some large enough constant $C > 0$, and without loss of generality, assume that among all translates $\{x_i\}$, $\{x_i\}_{i=1}^N$ are precisely those points lying outside of $R B^n_2$. Observe that for each $i = 1, \ldots, N$, the cone $\{t(x_i + B^n_2); t \geq 0\}$ carves a spherical cap of Euclidean radius at most $1/R$ on $S^{n-1}$. By the invariance of the Haar measures on $S^{n-1}$ and $O(n)$ under the action of $O(n)$, it follows that for every $i, j \in \{1, \ldots, N\}$:

$$P(U(x_i + B^n_2) \cap (x_j + B^n_2) \neq \emptyset) \leq \mu_{S^{n-1}}(B_2/R) ,$$

where $B_\varepsilon$ denotes a spherical cap on $S^{n-1}$ of Euclidean radius $\varepsilon$, and recall $\mu_{S^{n-1}}$ denotes the normalized Haar measure on $S^{n-1}$. When $\varepsilon < 1/(2C)$, it is easy to verify that:

$$\mu_{S^{n-1}}(B_\varepsilon) \leq (C\varepsilon)^{n-1} ,$$

and so it follows by the union-bound that:

$$P(L \cap U(L) \subset (R+1)B^n_2) \geq P(\forall i, j \in \{1, \ldots, N\} \quad U(x_i+B^n_2)\cap(x_j+B^n_2) = \emptyset) \geq 1-N^2(2C/R)^{n-1} .$$

Since $N \leq \exp(2A_1 (n-1))$, our choice of $R$ yields the desired assertion with $C' = 5C$. □

It is also useful to state:

Lemma 2.6. For any density $g$ on $\mathbb{R}^n$ and $p \geq 1$:

$$Z_p^+(g) \subset 2^{1/p}Z_p(g) \subset Z_p(g) - Z_p^+(g) . \quad (2.1)$$

Proof. The first inclusion is trivial. The second follows since $a^{1/p} + b^{1/p} \geq (a+b)^{1/p}$ for $a, b \geq 0$, and hence for all $\theta \in S^{n-1}$:

$$h_{Z_p^+(g)-Z_p^+(g)}(\theta) = h_{Z_p^+(g)}(\theta) + h_{Z_p^+(g)}(-\theta) \geq 2^{1/p}h_{Z_p(g)}(\theta) .$$

□

The next two theorems play a crucial role in our argument. The first is due to Paouris [22], and the second to the authors [14]:

Theorem (Paouris). With the same assumptions as in Theorem 1.1:

$$W(Z_p(X)) \leq C\sqrt{p} \quad \forall p \in [2, cn^{3/2}] . \quad (2.2)$$
Theorem (Klartag–Milman). With the same assumptions as in Theorem 1.1:

\[ \text{V.Rad.}(Z_p(X)) \geq c\sqrt{p} \quad \forall p \in [2, cn^{\alpha/2}] \quad \text{(2.3)} \]

We are finally ready to provide a proof of Proposition 2.1:

Proof of Proposition 2.1. Let \( p \in [2, cn^{\alpha/2}] \), where \( c > 0 \) is some small enough constant so that (2.2) and (2.3) hold. We will ensure that \( c \leq 1 \), so there is nothing to prove if \( n = 1 \). By (2.1), Sudakov’s entropy estimate (e.g. [25]) and (2.2), we have:

\[ N(Z_p(X)/\sqrt{p}, B^2_2) \leq N(2^{1/p}Z_p(X)/\sqrt{p}, B^2_2) \leq \exp(\tilde{C}nW(2^{1/p}Z_p(X)/\sqrt{p})^2) \leq \exp(Cn) \quad \text{(2.4)} \]

Note that by (2.1) and the Rogers–Shephard inequality [26], we have:

\[ 2^{n/p}\text{Vol}(Z_p(X)) \leq \text{Vol}(Z_p(X) - Z_p(X)) \leq 4^n\text{Vol}(Z_p(X)) \]

Consequently, the volume bound in (2.3) also applies to \( Z_p(X) \):

\[ \text{V.Rad.}(Z^+_p(X)) \geq c_1\sqrt{p} \quad \text{(2.5)} \]

By Lemma 2.4, (2.4) and (2.5) imply that:

\[ N(\sqrt{p}(Z^+_p(X)^o), B^2_2) \leq \exp(C_2n) \]

Consequently, Lemma 2.3 implies that if \( U \) is uniformly distributed on \( O(n) \), then for any \( C_1 \geq C_2 + (\log 2)/2 \), there exists a \( C_3 > 0 \), so that:

\[ \mathbb{P}\left( Z^+_p(X)^o \cap U(Z^+_p(X)^o) \subset \frac{C_3}{\sqrt{p}}B^2_2 \right) \geq 1 - \exp(-C_1n) \]

or by duality (since \( T(K)^o = (T^{-1})^*(K^o) \) for any linear map \( T \) of full rank), that:

\[ \mathbb{P}\left( Z^+_p(X) + U(Z^+_p(X)) \supset \frac{C_3^{-1}}{\sqrt{p}}B^2_2 \right) \geq \mathbb{P}\left( \text{conv}(Z^+_p(X) \cup U(Z^+_p(X))) \supset \frac{C_3^{-1}}{\sqrt{p}}B^2_2 \right) \geq 1 - \exp(-C_1n) \]

Lemma 2.3 now concludes the proof.

3 Remaining Details

We now complete the remaining (standard) details in the proof of Theorem 1.1.

Proof of Theorem 1.1.
1. For any $U \in O(n)$ and $t \geq 0$, observe that:

$$2 \max \left( \mathbb{P} \left( \left| \frac{X + U(\theta)}{\sqrt{2}} \right| \leq t \right) , \mathbb{P} \left( \left| \frac{X - U(\theta)}{\sqrt{2}} \right| \leq t \right) \right)$$

$$\geq \mathbb{P} \left( \left| \frac{X + U(\theta)}{\sqrt{2}} \right| \leq t \right) + \mathbb{P} \left( \left| \frac{X - U(\theta)}{\sqrt{2}} \right| \leq t \right)$$

$$= \mathbb{P} \left( \frac{|X|^2 + |\theta|^2}{2} + \langle X, U(\theta) \rangle \leq t^2 \right) + \mathbb{P} \left( \frac{|X|^2 + |\theta|^2}{2} - \langle X, U(\theta) \rangle \leq t^2 \right)$$

$$\geq \mathbb{P} (|X| \leq t \text{ and } |\theta| \leq t \text{ and } \langle X, U(\theta) \rangle \leq 0)$$

$$+ \mathbb{P} (|X| \leq t \text{ and } |\theta| \leq t \text{ and } \langle X, U(\theta) \rangle > 0)$$

$$= \mathbb{P} (|X| \leq t \text{ and } |\theta| \leq t) = \mathbb{P}(|X| \leq t) \cdot \mathbb{P}(|\theta| \leq t) .$$

Similarly:

$$2 \max \left( \mathbb{P} \left( \left| \frac{X + U(\theta)}{\sqrt{2}} \right| \geq t \right) , \mathbb{P} \left( \left| \frac{X - U(\theta)}{\sqrt{2}} \right| \geq t \right) \right) \geq \mathbb{P}(|X| \geq t)^2 .$$

This is precisely the content of the first assertion of Theorem 1.1

2. Given $\theta \in S^{n-1}$, denote $Y_1 = P_\theta Y_+^U$, $X_1 = P_\theta X$ and $X_2 = P_\theta U(X')$, where $P_\theta$ denotes orthogonal projection onto the one-dimensional subspace spanned by $\theta$. We have:

$$h_{Z_p(Y_+^U)} (\theta) = (\mathbb{E} |Y_1|^p)^{\frac{1}{p}} = \left( \mathbb{E} \left| \frac{X_1 + X_2}{\sqrt{2}} \right|^p \right)^{\frac{1}{p}}$$

$$\leq \frac{1}{\sqrt{2}} \left( (\mathbb{E} |X_1|^p)^{\frac{1}{p}} + (\mathbb{E} |X_2|^p)^{\frac{1}{p}} \right) = \frac{1}{\sqrt{2}} (h_{Z_p(X)}(\theta) + h_{Z_p(U(X))}(\theta)) .$$

Employing in addition (2.1), it follows that:

$$Z_p^+(Y_+^U) \subset 2^{1/p} Z_p(Y_+^U) \subset \frac{2^{1/p}}{\sqrt{2}} (Z_p(X) + U(Z_p(X))) ,$$

and the second assertion for $Y_+^U$ follows since $Z_p(X) \subset C p^{\frac{1}{n}} B_2^n$ by assumption. Similarly for $Y_-^U$.

3. Given a natural number $i$, set $p_i = 2^i$. Proposition 2.1 ensures the existence of a constant $c > 0$, so that for any $C_1 > 0$, there exists a constant $c_1 > 0$, so that for any $p_i \in [2, cn^2]$, there exists a subset $A_i \subset O(n)$ with:

$$\mu_{O(n)}(A_i) \geq 1 - \exp(-C_1 n) ,$$

so that:

$$\forall U \in A_i \quad Z_{p_i}(Y_+^U) \supset c_1 \sqrt{p_i} B_2^n .$$
Denoting $A_0 := \cap \left\{ A_i : p_i \in [2, cn^{\frac{1}{2}}] \right\}$, and setting $A = A_0 \cap -A_0$, where $-A_0 := \{-U \in O(n) ; U \in A_0\}$, it follow by the union-bound that:

$$\mu_{O(n)}(A) \geq 1 - 2 \log(C_2 + n) \exp(-C_1n) .$$

By choosing the constant $C_1 > 0$ large enough, we conclude that:

$$\mu_{O(n)}(A) \geq 1 - \exp(-C_3n) .$$

By construction, the set $A$ has the property that:

$$\forall U \in A \forall p_i \in [2, cn^{\frac{1}{2}}] \quad Z_{p_i}(Y^{U}_{\pm}) \supset c_1 \sqrt{p_i} B^n_2 .$$

Using (1.6), it follows that:

$$\forall U \in A \forall p \in [2, cn^{\frac{1}{2}}] \quad Z_p(Y^{U}_{\pm}) \supset \frac{c_1}{\sqrt{2}} \sqrt{p} B^n_2 ,$$

thereby concluding the proof of the third assertion.

$\square$

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