1 Introduction

We consider the problem of autonomously driving a mobile robot through an unknown and unstructured 3D tunnel-like environment. This task is elemental for many applications of mobile robotics such as inspection and servicing of storm runoff networks, channelized aquifers, mines, bypass tunnels for dams, pipelines in sewer networks, power plants, factories, petrochemical, water supply and fluid transportation industries, underwater environmental studies and archeology, exploring flooded cenotes and ancient cisterns, indoor and city exploration, rescue, and surveillance by micro air vehicles (MAV), etc., to name just a few.

We focus on the case where the robot does not touch the perimeter of the tunnel, should respect a certain safety margin to it, and subject to this limitation, is free to move in all three dimensions. This is typical in aerial and underwater robotics, which are growing fields with ever-extending application possibilities, where they often hold promise of more effective alternative technology. To acquire this benefit, the robots should be equipped with navigation algorithms that enable them to operate autonomously through long time and distance scales. Meanwhile, the studied case does not concern most specialized in-pipe inspection robots manufactured up to now, whose locomotion relies on a firm contact with the tunnel boundary surface. For them, the proposed navigation solutions substantially benefit from the fact that the workspace is typically structured (is composed of standard element like straight pipe, T-junction, etc.) and may be partly or fully known.

This paper is targeted at another situation of an unstructured and unknown tunnel with a generic local geometry and a robot whose sensory data is confined to a close small patch of tunnel’s boundary, whereas the “opposite patch” may be out of the sensing distance. Such confined and overhead unstructured environments coupled with the lack of an exterior navigation assistance still constitute a real challenge for underwater autonomous navigation so that most respective missions (e.g., in underwater caves or shipwrecks) are performed by human divers or remotely teleoperated robotized vehicles up to now. Meanwhile, typical approaches to navigation of MAV’s have various limitations due to e.g., reliance on known, including pre-installed, landmarks, salient features or environmental patterns, maps, a priori image-databases, external positioning systems, or a partially structured environment to enable incremental motion planning. Some of these methods, including simultaneous 3D localization and mapping (SLAM), are computationally expensive, hardly match capacities of on-board processors and so use extensive off-board computations, which require reliable communication and may cause undesirable feedback delays.

This paper is aimed at showing that even in the face of nonholonomy, under-actuation, finite control range, poor knowledge of the scene, and limited sensory capacity, long-distance fully autonomous advancement through a generic tunnel is feasible at a little computational cost: control signal is generated via a direct reflex-like conversion of the current observation. No complex or dubious operations, like building a map or depositing marks in the scene, are employed for navigation.

An extended introduction and discussion of the control law that achieves the stated objective are given in a manuscript submitted by the authors to the IFAC journal Automatica. The subsequent text basically contains the proofs of the technical facts underlying justification of the convergence and performance of the algorithm proposed in that manuscript, which were not included in it because of the paper length limit. This text mainly focuses on geometric issues related to the concept of “tunnel”, its main content may be attributed to the area of differential geometry. The assumptions of the study are borrowed from that manuscript and are partly motivated by navigation and control concerns.

The body of the paper is organized as follows. Sections 2 and 3 introduce the tunnel environment and the assumptions of theoretical analysis, respectively. Section 4 presents a particular way of access to local features of tunnel’s surface that can be employed for global navigation. Section 5 discusses some properties of tunnels, the main result of Section 4 is proved in Section 6.
Throughout the paper, the following notations are used:

- $\langle A; B \rangle$ and $A \times B$, inner and cross product, respectively;
- $\| \cdot \|$, Euclidean norm of a vector and spectral norm of a matrix and linear operator;
- $\pi(r)$, set of all projections of point $r \in \mathbb{R}^3$ onto a regular surface $S \subset \mathbb{R}^3$, i.e., points $s \in S$ such that
  \[ \| r - s \| = d_S[r] := \min_{s' \in S} \| s' - r \|; \]  
  (1.1)
- $\mathcal{T}_s(S)$, plane tangent to $S$ at $s \in S$;
- $N(s)$, unit normal to tunnel $S$ directed inside the tunnel;
- $D_V W$, derivative of the field $W$ in direction of $V$;
- $S_s(V) = -D_V N$, shape operator (Weingarten map);
- $\Pi_s[V; W] := \langle S_s(V); W \rangle$, $V, W \in \mathcal{T}_s(S)$, second fundamental form (shape tensor);
- $\kappa_+(s) \leq \kappa_+(s)$, principal curvatures, i.e., the eigenvalues of the quadratic form $\Pi_s[\cdot; \cdot]$;
- $E_{\pm}(s)$, respective unit eigenvectors (principal eigenvectors) continuously depending on $s \in S$;
- $p_{\pm}(s)$, lines spanned by $E_{\pm}(s)$ (principal directions);
- $\tilde{r}(s) \in \mathcal{T}_s(S)$, smooth unit vector-field tangent to the meridian $M[b(s)]$ that passes through $s$;
- $\tilde{e}(s_\ast) \in \mathcal{T}_{S_\ast}[S(d_\ast)]$, similar vector-field on $S(d_\ast)$;
- $S(d_\ast) := \{ p = s + d_\ast N(s) \in S \setminus \partial S \}$, locus of the desired locations of the robot;
- $\mathcal{T}_{S_\ast}[S(d_\ast)]$, plane tangent to $S(d_\ast)$ at $s \in S(d_\ast)$;
- $N_\ast(s)$, unit normal to $S(d_\ast)$ directed outward $S$;
- $\Pr_L$, orthogonal projection onto a subspace $L \subset \mathbb{R}^3$;
- $\nabla_V W$, Levy-Civita covariant derivative, i.e., $\nabla_V W = \Pr_T D_V W$, where $T$ is the tangent plane;
- $I_d$, identity operator acting in $\mathcal{T}_s$;
- $[A, B, C] := (A; B \times C)$, triple product;
- $\langle A, B \rangle$, angle from $A$ to $B$, where $A, B \in \mathcal{T}_s(S)$ and positive angles are countered counterclockwise when looking from the side of $N(s)$;
- $\Re_\ast(\theta)$, rotation of the plane $\mathcal{T}_s(S)$ through angle $\theta$.

2 Tunnel environment

We consider a mobile robot that operates in a tunnel-like environment bounded by a regular surface $S \subset \mathbb{R}^3$. It is required to advance the robot through the tunnel and, in the case of a long operation, to eventually approach a certain "comfortable" value $d_\ast$ of the distance $d$ to $S$. A troubling trait of the robot is that its sensing capacity is limited. In its local frame, it identifies the direction $\hat{d}$ to a minimum-distance point $\pi(r)$ of $S$ and the distance to $S$ along any ray emitted from $r$ at an angle $\leq \alpha_\ast$ with respect to $\hat{d}$, where $\alpha_\ast > 0$ is a constant. Thus the robot "sees" only a small patch of $S$ around $\pi(r)$ but has no access to any global direction, including the desired direction of motion that corresponds to "advancing along the tunnel". Thus the robot has no other option but to recognize this direction from features of the sensed small patches of $S$.

As a result, the control objective is realistic not for any surface $S$: "advancement along the tunnel" should be of sense and its direction should be recognizable from observing small patches of $S$. A prototypical sample where these are satisfied is a circular cylinder. Then the "direction of advancement" is that of its generatrix, it can be determined as that in which any patch of the boundary has zero curvature. Inspired by this sample, we will deal with surfaces for which contortion of any patch is minimal in an acceptable direction of motion.

Now we pass to a rigorous definition.

Let $B$ be either a) the real line $\mathbb{R}$, or b) the unit circle $S^1 \subset \mathbb{R}^2$, or c) an interval $[b_-, b_+] \subset \mathbb{R}$. 

Definition 2.1 A tunnel with the basis $B$ is a $C^3$-smooth regular surface $S$ (with a one-dimensional boundary $\partial S = B^{-1}[b_−] \cup B^{-1}[b_+]$ in the case c)) equipped with a projection $B : S \to \mathbb{B}$ such that

i) The projection is a proper surjective submersion: it is $C^3$-smooth, maps $S$ onto $\mathbb{B}$, its differential has rank 1 everywhere, and the inverse image of any compact subset of $\mathbb{B}$ is compact;

ii) Any meridian $M(b) := B^{-1}[b]$ is a simple closed curve and the set $S$ is closed.

If $B = [b_−, b_+]$, the tunnel is said to be open, and closed otherwise. The (multivalued in general) basic coordinate $b(r) := \{B(s) : s \in \pi(r)\}$ of $r$ assesses progression of robot $r$ through the tunnel: Progression is associated with evolution of $b$ in a certain direction. The case of $B = \mathbb{R}$ is a model for a “very long tunnel”.

A prototypical sample of a tunnel is a right cylindrical surface, where $b$ is the coordinate along the axis of symmetry and the meridians are the perpendicular sections. A closed compact tunnel is exemplified by a torus $\varphi$ associated with evolution of $b$ in a certain direction. The case of $B = \mathbb{R}$ is a model for a “very long tunnel”.

Another example is a surface of revolution, where the generatrix is a graph of a positive function defined on the $\mathbb{R}$ along $\mathbb{A}$, and there exists time $t_0$ such that

$$\Pi_a[\tau(s); \tau(t)] \geq \kappa - (s) + \Delta_t.$$  

The right cylindrical surface is a regular tunnel since $\kappa = 0$ and the meridian curvatures are positive. For the torus, the curvature of any meridian is $r^{-1}$ and $\kappa = \frac{\cos u}{R + r \cos u}$ [15, p. 157], where $R > r$ and the torus is parametrized by $[(R + r \cos u) \cos v, (R + r \cos u) \sin v, r \sin u], u, v \in [0, 2\pi]$. Hence the torus is a regular tunnel since $r^{-1} > \kappa$.

From now on, we deal with a regular tunnel.

In conclusion of the section, we discover a geometric sense of i) in Definition 2.2 by using the map

$$s \in S \setminus \partial S \mapsto J(s) := s + d_+N(s) \in S(d_+).$$
which diffeomorphically maps $S \setminus \partial S$ onto $S(d_*)$ (under the assumptions of this paper, see Corollary 5.1). This map gives rise to the meridians $\mathcal{M}_b = J[\mathcal{M}(b)]$ on $S(d_*)$. Their tangents can be arranged into the smooth unit vector-field $\vec{\tau}_s$ on $S(d_*)$:

$$\vec{\tau}_s(r) := \frac{J'(s)\vec{\tau}(s)}{\|J'(s)\vec{\tau}(s)\|} \quad \text{where} \ s := J^{-1}(r). \quad (2.4)$$

In Definition 2.2, i) means that while moving nearly over $S(d_*)$, the robot transverses the meridians in an unaltered direction at angles that are separated from zero and eventually crosses any meridian that lies in the direction of motion with respect to the initial one.

### 3 Assumptions of theoretical analysis

**Assumption 3.1** For any $r \in S(d_*)$, the set $\pi(r)$ of projections of $r$ onto $S$ contains a unique point $s(r)$.

For convenience’s sake, we define the operational zone $Z_{op}$ of the robot in terms of the extreme values $d_- < d_+$ that are assumed by the distance $d$ to the surface $S$ in this zone:

$$Z_{op} := \{ r = s + dN(s) : s \in S \setminus \partial S, d \in (d_-, d_+) \}, \quad \text{where} \ 0 < d_- < d_* < d_+. \quad (3.1)$$

**Assumption 3.2** Assumption 3.1 is valid with any $d_*$ from $[0, d_+]$ and there exists $\Delta_\kappa \in (0, 1]$ such that

$$1 - d_+ \kappa(s) \geq \Delta_\kappa \quad \forall s \in S. \quad (3.2)$$

Whenever $\kappa_+(s) \leq 0$, (3.2) is certainly true.

The last assumption is purely technical.

**Assumption 3.3** The maps $N, \kappa_{\pm}, E_{\pm}, \nabla B$ are Lipschitz: there exist constants $L_N, L_\kappa, L_E, L_B > 0$ such that

$$\|N(s_1) - N(s_2)\| \leq L_N \|s_1 - s_2\|, \quad (3.3)$$

$$\|\kappa_+(s_1) - \kappa_+(s_2)\| \leq L_\kappa \|s_1 - s_2\|,$$

$$\|E_+(s_1) - E_+(s_2)\| \leq L_E \|s_1 - s_2\|,$$

$$\|\nabla B(s_1) - \nabla B(s_2)\| \leq L_B \|s_1 - s_2\| \quad \forall s_1, s_2 \in S.$$

Furthermore, there exist $\Delta_B^+ > 0$ such that $\Delta_B^- \leq \|\nabla B(s)\| \leq \Delta_B^+ \forall s \in S$.

For a compact tunnel, this assumption necessarily holds.

### 4 Direction estimator

Its role is to generate the direction of motion along the tunnel: This direction should be transversal to the meridian passing through $s$ to meet i) in Definition 2.2. The premise is the sensory data, i.e., the direction estimator (DE) maps location dependent $O = O(r)$ sensory data into a straight line $p(r) \subset \pi (r)$ such that the associated map $r \in Z_{op}^{\pm} \rightarrow p(r) \in \mathcal{G}(1, 3)$ is continuous. Its exactness is an upper estimate $\beta$ of the angle between the lines $p(r)$ and $p_+[\pi(r)]$ that holds for any $r \in Z_{op}^{\pm}$.

**Definition 4.1** Direction estimator (DE) maps location dependent $O = O(r)$ sensory data into a straight line $p(r) \subset \pi (r)$ such that the associated map $r \in Z_{op}^{\pm} \rightarrow p(r) \in \mathcal{G}(1, 3)$ is continuous. Its exactness is an upper estimate $\beta > 0$ of the angle between the lines $p(r)$ and $p_+[\pi(r)]$ that holds for any $r \in Z_{op}^{\pm}$.

Here $\mathcal{G}(1, 3)$ is the smooth Grassmanian manifold of all one-dimensional linear subspaces of $\mathbb{R}^3$ [23] pp. 42-44. We assume that the output $p(O)$ of DE is given in the local frame of the robot. For a DE to be useful, its exactness $\beta$ should be less than the minimal angular discrepancy between $p_-[\pi(r)]$ and the meridian direction at point $\pi(r)$ since then $p(r)$ is transversal to the meridian, as is required.
Now we discuss a particular design of DE under which arbitrarily high exactness can be achieved.

**Most-distant-point-based estimator (MDPBE)** with parameter \( \alpha_e \in (0, \alpha_s) \) finds the tangential directions (given by \( \phi \)) of the ray \( \mathfrak{R} \) that are associated with the local maxima of the distance \( d(\alpha_e, \varphi) \), shifts every of the found \( \varphi \)'s into \((-\pi/2, \pi/2]\) by adding, if necessary, an integer multiple of \( \pi \), computes the arithmetic mean \( \varphi_\ast \) of the resultant angles, and returns the straight line \( p \) that goes in the normal plane \( d^\perp \) in the direction of \( \varphi_\ast \). MDPBE is well-posed if the number of local maxima is finite.

**Proposition 4.1** Let Assumptions 3.2 and 3.3 hold. For any \( \beta > 0 \), there is \( \bar{\pi} \in (0, \alpha_s) \) such that MDPBE is well-posed (with two local maxima) in the active operational zone and is a direction estimator with exactness \( \beta \) whenever \( \alpha_e \in (0, \bar{\pi}) \).

The proof of this proposition is given in Section 4.

5 **General properties of tunnels and technical facts**

Let Assumptions 3.2, 3.3 hold and let \( L_N \) be taken from 3.3.

**Lemma 5.1** The norm of the operator \( S_s : \mathcal{I}_s(S) \to \mathcal{I}_s(S) \) and the principal curvatures obey the estimates

\[
\| S_s \| \leq L_N \quad \text{and} \quad |\kappa_{\pm}(s)| \leq L_N \quad \forall s \in S. \tag{5.1}
\]

There exists \( \varepsilon > 0 \) such that \( (d, s) \mapsto h(d, s) := s + dN(s) \) is a \( C^2 \)-diffeomorphism of \( T := (-\varepsilon, d_+) \times (S \setminus \partial S) \) onto an open neighborhood of \((5.1)\), and \( h^{-1}(p) = \{d_s(p), \pi(p)\}, \pi(p) \notin \partial S \forall p \in \mathcal{Z}_{\mathrm{op}} \).

**Proof:** Inequalities (5.1) hold since \( \| S_s V \| = \| D_h N \| \leq L_N \| V \| \) for all \( V \in \mathcal{I}_s(S) \) by 3.3 and \( \kappa_{\pm} \) are the eigenvalues of \( S_s \). By (3.2) and (5.1), there exists \( \varepsilon \in (0, d_+/2) \) such that \( 1 - d\kappa_+(s) \geq \Delta_G \forall s \in S, d \in [0, d_+] \) and \( 1 - d\kappa_-(s) \geq \Delta_G \forall s \in S, d \in [-\varepsilon, 0] \). Hence for \( d \in [-\varepsilon, d_+] \) and \( s \in S \), the symmetric operator \( \mathbf{I}_d - d \cdot S_s : \mathcal{I}_s(S) \to \mathcal{I}_s(S) \) is positively definite, its least eigenvalue is no less than \( \Delta_G \) and so

\[
\|(\mathbf{I}_d - d \cdot S_s)^{-1}\| \leq \Delta_G^{-1}. \tag{5.2}
\]

The differential of \( h \) maps \( (\eta, V) \in \mathbb{R} \times \mathcal{I}_s(S) \) into \( W = \eta N + (\mathbf{I}_d - d\mathbf{S})V \in \mathbb{R}^3 \) and is invertible:

\[
\eta = \langle W; N \rangle, \quad V = (\mathbf{I}_d - d\mathbf{S})^{-1}[W - \eta N]. \tag{5.3}
\]

Hence \( h \) is a local diffeomorphism. So the image \( h(T) \) is open and the inverse image \( h^{-1}(p) \) is a discrete set for any point \( p \in h(T) \). For any \( (d, s) \in h^{-1}(p) \), we have \( p = h(d, s) = s + dN(s), |d| \leq d_+ \Rightarrow \|s\| \leq \|p\| + d_+ \) and so the inverse image \( h^{-1}(p) \) is bounded. Suppose that it is infinite. Then there exists an infinite sequence \( \{(d_j, s_j)\} \subset h^{-1}(p) \) such that \( d_j \to d, s_j \to s \) as \( j \to \infty \) and \( \zeta_j := (d_j, s_j) \neq \zeta \subset \mathcal{Z}_{\mathrm{op}} \forall j \). Here \( d \in [-\varepsilon, d_+] \) and \( s \in S \) and the differential \( h(d, s) \) is invertible by the foregoing. Meanwhile for any limit point \( V \) of the bounded sequence \( \zeta_j \), we have \( \|V\| = 1 \) and \( h'(d, s)V = 0 \Rightarrow V = 0 \Rightarrow \|V\| = 0 \). The contradiction obtained proves that the set \( h^{-1}(p) \) is finite.

This and local diffeomorphic property of \( h(\cdot) \) imply that \( h(\cdot) \) is a covering map [5] p. 67. Definition 2.1 and Assumption 3.3 imply that \( S, S \setminus \partial S \) and so \( T \) are arcwise connected. Hence all fibres \( F_p := h^{-1}(p), p \in h(T) \) have the same size (the multiplicity of the covering) [5] Th. 2.4.4. We are going to show that \( M = 1 \), possibly after properly decreasing \( \varepsilon \).

To this end, we first show this for the restriction \( h_{[\varepsilon, \varepsilon]}(S \setminus \partial S) \), which evidently also is a local diffeomorphism and a covering map. Suppose to the contrary that its multiplicity is greater than 1 for \( \varepsilon = \varepsilon_j \forall j \), where \( \varepsilon_j \in (0, \varepsilon) \) and \( \varepsilon_j \to 0 \) as \( j \to \infty \). We also pick \( s \in S \setminus \partial S \). For any \( j \), the fibre \( F_{s_j} \) contains \((0, s)\) and some other point \((\eta_j, s_j) \in \varepsilon_j) \times (S \setminus \partial S) \), where \( s = h(0, s) = h(\eta_j, s_j) = s_j + \eta_j N[s_j] \) and so \( \eta_j \neq 0 \). Hence

\[
\|s_j - s\| = \eta_j \leq \varepsilon_j, \quad \left\langle \eta_j^{-1}(s - s_j); N(s) \right\rangle = \langle N[s_j]; N[s] \rangle.
\]

As \( j \to \infty \), the unit vector \( \eta_j^{-1}(s - s_j) \) converges to the tangent plane \( \mathcal{Z}_s(S) \) and so the left hand side of the last equation goes to zero, whereas its right hand side goes to 1. This contradiction proves that for all small enough \( \varepsilon > 0 \), the multiplicity of the map \( h_{[\varepsilon, \varepsilon]}(S \setminus \partial S) \) is 1 and so this map is a diffeomorphism onto its image. We assume that \( \varepsilon \) is decreased (if necessary) to this range.

Now suppose that the multiplicity of \( h(\cdot) \) on \( T \) is greater than 1, pick \( s_1 \in S \setminus \partial S \subset h(T) \), and enumerate \((d_1 = 0, s_1), (d_2, s_2), \ldots, (d_{M}, s_M) \) the elements of the fibre \( F_p \), where \( p := h(0, s_1) = s_1 \). Here

\[
s_1 = h(d_1, s_1) = s_1 + d_1 N(s_1) \tag{5.4}
\]

and \( d_i \in [\varepsilon, d_+] \) \( \forall i \geq 2 \) since \( d_i \in (-\varepsilon, \varepsilon) \) is excluded by the just established property of \( h_{[-\varepsilon, \varepsilon]}(S \setminus \partial S) \). For \( i \geq 2 \), the straight-line segment connecting \( s_i \) and \( s_1 \) is a subset of \( h([0, d_i] \times \{s_i\}) \subset h(T) \) and so the point
\( (d_i, s_i) \in F_p \) gives rise to a lift of the path \( \mathbf{p}(\theta) = (1 - \theta)s_1 + \theta s_2 \in h(T), \theta \in [0, 1] \) that starts from this point, i.e., to a continuous map \( L_i(\cdot) = [d_i(\cdot), s_i(\cdot)] : [0, 1] \rightarrow T \) such that
\[
h[L_i(\theta)] = \mathbf{p}(\theta) \quad \forall \theta
\]  
(5.5) and \([d_i(0), s_i(0)] = [d_i, s_i] \forall i\). Since the path \( \mathbf{p}(\cdot) \) is smooth and \( h(\cdot) \) is locally diffeomorphic, any lift \( L_i(\cdot) \) is also smooth. For \( \theta \in [0, 1] \), the points \( L_1(\theta), \ldots, L_M(\theta) \) are pair-wise distinct and exhaust the fibre \( h^{-1}[\mathbf{p}(\theta)] = F_{\mathbf{p}(\theta)} \).

By Assumption 5.2 (in the part that refers to Assumption 5.1), the projection \( s(\theta) \) of \( \mathbf{p}(\theta) \) onto \( S \) is unique for all \( \theta \in [0, 1] \). So not only \( \mathbf{d}_S[\mathbf{p}(\theta)] \) but also \( s(\theta) \) continuously depend on \( \theta \). Furthermore,
\[
s(0) = s_1, \quad s(1) = s_2, \quad \text{and} \quad \mathbf{p}(\theta) = s(\theta) + \mathbf{d}_S[\mathbf{p}(\theta)] N[s(\theta)] \quad \forall s(\theta) \notin \partial S.
\]  
(5.6)
Since \( 0 \leq \mathbf{d}_S[\mathbf{p}(\theta)] < \varepsilon, -\varepsilon < d_i(\theta) < \varepsilon, \) and \( s(\theta) \approx s_1 \in S \setminus \partial S \Rightarrow s(\theta) \in S \setminus \partial S \) for \( \theta \approx 0 \), the diffeomorphic property of \( h_{(–\varepsilon, \varepsilon) \times (S \setminus \partial S)} \) and (5.3) (with \( i := 1 \)) yield that
\[
d_1(\theta) = \mathbf{d}_S[\mathbf{p}(\theta)], \quad s_1(\theta) = s(\theta) \quad \forall \theta \in [0, \tau]
\]  
(5.7)
with sufficiently small \( \tau \in (0, 1) \). We are going to show that in fact this is true with \( \tau = 1 \).

Suppose to the contrary that the maximal such \( \tau \) is less than 1. Then \( \mathbf{d}_S[\mathbf{p}(\theta)], s(\theta) \notin [d_1(\theta), s_1(\theta)] \) for \( \theta = \theta_j \forall j \), where \( \theta_j \in (\tau, 1) \) and \( \theta_j \rightarrow \tau \) as \( j \rightarrow \infty \). So (5.5), (5.6), and the diffeomorphic property of \( h_{(–\varepsilon, \varepsilon) \times (S \setminus \partial S)} \) exclude the case where \( \mathbf{d}_S[\mathbf{p}(\tau)] = 0 \). Thus
\[
d_1(\tau) = \mathbf{d}_S[\mathbf{p}(\tau)] > 0
\]  
(5.8)
and so \( \mathbf{d}_S[\mathbf{p}(\theta_j)], d_i(\theta_j) \in (0, d_+) \) and \( s(\theta_j) \notin \partial S \) for \( j \approx \infty \). From now on, we focus on only such \( j \)'s.

Suppose that there are arbitrarily large \( j \)'s for which \( \theta := \theta_j \) holds with the sign +. Then passing to a proper subsequence ensures that this is true for all \( j \). In this case, \( \mathbf{d}_S[\mathbf{p}(\theta_j)], s(\theta_j) \in \mathbb{T}, h[\mathbf{d}_S[\mathbf{p}(\theta_j)], s(\theta_j)] = \mathbf{p}(\theta_j), \) i.e., \( \mathbf{d}_S[\mathbf{p}(\theta_j)], s(\theta_j) \) \( \in F_p(\mathbf{p}(\theta_j)) \). Hence \( \mathbf{d}_S[\mathbf{p}(\theta_j)], s(\theta_j) = [d_1(\theta_j), s_1(\theta_j)] \) for some \( i = i(j) = 1, \ldots, M \).

Here \( i(j) \neq 1 \) since \( \mathbf{d}_S[\mathbf{p}(\theta_j)], s(\theta_j) = [d_1(\theta_j), s_1(\theta_j)] \). Then by passing to a subsequence once more, it can be ensured that \( i(j) \) does not alter with \( j \), i.e., \( i(j) \equiv i \geq 2 \forall j \). Then passing to the limit as \( j \rightarrow \infty \) in the equation \( \mathbf{d}_S[\mathbf{p}(\theta_j)] = d_1(\tau), \) yields \( \mathbf{d}_S[\mathbf{p}(\tau)] = d_1(\tau) \). It follows that the distance \( \mathbf{d}_S[\mathbf{p}(\tau)] \) from the point \( \mathbf{p}(\tau) \) to the surface \( S \) is attained at both \( s_1(\tau) \) and \( s_1(\tau) \), where \( [d_1(\tau), s_1(\tau)] \neq [d_1(\tau), s_1(\tau)] \) and \( d_1(\tau) = \mathbf{d}_S[\mathbf{p}(\tau)] \) imply that \( s_1(\tau) \neq s_1(\tau) \), in violation of Assumption 5.2.

The contradiction obtained proves that (5.6) with \( \theta := \theta_j \) holds with the sign + for all large enough \( j \). Then by (5.5) and (5.6), we have
\[
s(\theta_j) - \mathbf{d}_S[\mathbf{p}(\theta_j)] N[s(\theta_j)] = \mathbf{p}(\theta_j) = \mathbf{s}_1(\theta_j) + d_1(\theta_j) N[s_1(\theta_j)]
\]
\[
\downarrow \quad j \rightarrow \infty
\]
\[
s(\tau) - \mathbf{d}_S[\mathbf{p}(\tau)] N[s(\tau)] = \mathbf{p}(\tau) = s_1(\tau) + d_1(\tau) N[s_1(\tau)] \Rightarrow \mathbf{d}_S[\mathbf{p}(\tau)] = 0,
\]
in violation of (5.8). The contradiction obtained proves that (5.7) holds with \( \tau = 1 \).

It follows that \( d_1(1) = 0 \Rightarrow s_2 = \mathbf{p}(1) = s_1(1) + d_1(1) N[s_1(1)] = s_1(1) \). Hence
\[
s_1(\theta) + d_1(\theta) N[s_1(\theta)] = h[d_1(\theta), s_1(\theta)] = \mathbf{p}(\theta) = (1 - \theta)s_1 + \theta s_2 \quad \text{GeV (1 - \theta)s_2 + d_2 N[s_2] + \theta s_2
\]
\[
\downarrow
\]
\[
d_1'(\theta) N[s_1(\theta)] + \left[ \mathbf{I} d_1'(\theta) S_{s_1(\theta)} \right] s_1'(\theta) = \frac{d}{d\theta} \left( s_2 + d_2 (1 - \theta) N[s_2] \right) = -d_2 N[s_1(1)].
\]

By putting \( \theta := 1 \) here and taking into account that both \( s_1'(\theta) \) and \( \left[ \mathbf{I} d_1'(\theta) S_{s_1(\theta)} \right] s_1'(\theta) \) lie in the tangential plane, whereas \( N[s_1(\theta)] \) is normal to it, we see that \( d_2 = -d_1' (1) \). Thus
\[
\left[ \mathbf{I} d_1'(\theta) S_{s_1(\theta)} \right] s_1'(\theta) = d_1'(1) N[s_1(1)] - d_1'(\theta) N[s_1(\theta)]
\]
\[
= d_1'(1) \left[ N[s_1(1)] - N[s_1(\theta)] \right] - d_1'(\theta) N[s_1(\theta)]
\]

Then applying the orthogonal projection onto the tangent plane \( \mathcal{F}_{s_1(\theta)}(S) \), we see that
\[
\left\| \left[ \mathbf{I} d_1'(\theta) S_{s_1(\theta)} \right] s_1'(\theta) \right\| \leq d_2 \left\| N[s_1(1)] - N[s_1(\theta)] \right\| \leq L_N d_2 \left\| s_1(1) - s_1(\theta) \right\|.
\]

On the other hand, (5.2) implies that the left hand side is no less than \( \left\| s_1'(\theta) \right\| / \Delta_s \). As a result,
\[
\left\| s_1'(\theta) \right\| \leq L_N \Delta_s d_2 \left\| s_1(1) - s_1(\theta) \right\|
\]
and so $s_1(\theta) = s_1(1) \forall \theta$ by Gröwall’s inequality. Putting $\theta := 0$ here yields $s_1 = s_1(0) = s_1(1) = s_2$, in violation of the foregoing. This contradiction proves that the multiplicity $M = 1$ and so $h(\cdot)$ is a diffeomorphism.

To prove the last claim of the lemma, we consider $p \in \mathbb{Z}_{ap}$. By (5.1), $p = s + d\partial N(s)$, where $s \in S \setminus \partial S, d \in (d_-, d_+)$. We put

$$
p(\theta) := s + d\theta N(s) \quad \theta \in [0, 1]
$$

(5.9) and partly retrace the foregoing arguments. Specifically, we first establish that $s(\theta) = s$ and $d_s[p(\theta)] = d\theta$ for all small enough $\theta$ by using the diffeomorphic property of the restriction $h|_{(-\infty, \infty) \times (S \setminus \partial S)}$. As before, we then introduce the maximal interval $[0, \tau], \tau \leq 1$ on which (5.6) holds with $d_1(\theta) := d\theta$ and $s_1(\theta) := s$:

$$
\tau d = d_s[p(\theta)], \quad s = s(\theta) \quad \forall \theta \in [0, \tau].
$$

(5.10)

It remains to show that $\tau = 1$. Suppose to the contrary that $\tau < 1$. Then there there exists an infinite sequence $\{\theta_j\} \subset (\tau, 1]$ such that

$$
[d_s[p(\theta_j)], s(\theta_j)] \neq [\theta_j, d, s], \quad d_s[p(\theta_j)] \in [0, d_+), \quad s(\theta_j) \not\in \partial S \quad \forall j
$$

(5.11)

and $\theta_j \to \tau$ as $j \to \infty$. By passing to a subsequence, if necessary, it can be ensured that $[5.8]$ holds either with the sign $+$ for all $j$ or with the sign $-$ for all $j$. However in the first case, we have $h(d_s[p(\theta_j)], s(\theta_j)) = h(\theta_j, [d, s], [5.8])$ thanks to the established diffeomorphic property of $h(\cdot)$. So the second option holds: $p(\theta_j) = s(\theta_j) - d_s[p(\theta_j)] N[s(\theta_j)]$. Letting $j \to \infty$ yields that

$$
s + d\tau N(s) = p(\tau) = s(\tau) - d_s[p(\tau)] N[s(\tau)] \Rightarrow s - \tau dN(s) \Rightarrow d = 0,
$$

in violation of $d \in (d_-, d_+)$. This contradiction completes the proof. □

**Corollary 5.1** The map $\mathbb{L}_A$ is diffeomorphic, the norms of the differentials of $J$ and $J_{-1} := J^{-1}$ do not exceed $1 + d_s L_N$ and $1/\Delta_s$, respectively, and $S(d_s)$ is a smooth surface.

**Lemma 5.2** Let $Q : \mathbb{T}_s(S) \to \mathbb{T}_s(S)$ be a linear symmetric positively definite operator with the eigenvalues $0 < q_- \leq q_+$. Then

$$
sin\langle QA, QB \rangle = \xi \sin\langle A, B \rangle, \quad \xi \in [q_-/q_+, q_+/q_-].
$$

**Proof:** For all tangent vectors $C, D \in \mathbb{T}_s(S)$, we have

$$
sin\langle C, D \rangle = |N, C, D|/\|C\|\|D\|).
$$

(5.12)

Let $e_-$ and $e_+$ be orthonormal eigenvectors of $Q$ associated with $q_-$ and $q_+$, respectively. Since $QC = q_- (C; e_-) e_- + q_+ (C; e_+) e_+$, we have

$$
sin\langle QA, QB \rangle = \frac{|N, QA, QB|}{\|QA\|\|QB\|} = \frac{|N, QA - Q B|}{\|QA\|\|QB\|}.
$$

It remains to note that $q_- \|C\| \leq \|QA\| \leq q_+ \|C\|$. □

For any two vectors $A, B$, we denote by $B_A := B - (B; A) A/\|A\|^2$ the orthogonal projection of $B$ onto the plane normal to $A$. We also note that if $A, B \in \mathbb{T}_{s_+}(S(d_s)) (r \in S(d_s))$, then $B_A \in \mathbb{T}_{s_+}(S(d_s))$ is the orthogonal projection in the tangent plane $\mathbb{T}_{s_+}(S(d_s))$ of vector $B$ onto the line normal to $A$.

7
Lemma 5.3 Let a point $r = r(t)$ smoothly move over $S(d_\ast)$ and let $V(t), W(t) \in \mathfrak{T}_{r(t),s}[S(d_\ast)]$ be smooth non-vanishing vector-fields defined on its trajectory. For $\phi := \langle V, W \rangle$, we have

$$\dot{\phi} \cos \phi = \frac{\langle (\nabla_r V)_{V_{\perp}}; \mathcal{R}(\nabla_r W) \rangle}{\|V\| \|W\|}. $$

**Proof:** We put $V_0 = V/\|V\|, W_0 = W/\|W\|$. By (5.12),

$$\dot{\phi} \cos \phi = \frac{d}{dt}[N_\ast, V_0, W_0] = [N_\ast, V_0, W_0] + [N_\ast, \dot{V}_0, W_0] + [N_\ast, V_0, \dot{W}_0]. $$

Here $\|N_\ast\| \equiv 1 \Rightarrow \langle \dot{N}_\ast; N_\ast \rangle = 0 \Rightarrow \dot{N}_\ast \in \mathfrak{T}_s[S(d_\ast)]$. Hence the three vectors $N_\ast, V_0, W_0$ lie in a common (tangent) plane. Thus they are linearly dependent and so $[N_\ast, V_0, W_0] = 0$. Meanwhile,

$$\nabla_r V = \text{Pr}_{T_s[S(d_\ast)]} \dot{V} = \dot{V} - \xi N_\ast$$

with a properly chosen $\xi \in \mathbb{R}$, whereas

$$\|V\| \dot{V}_0 = \dot{V}_{V_{\perp}} = \langle \nabla_r V + \xi N_\ast \rangle_{V_{\perp}} \langle \nabla_r V \rangle_{V_{\perp}} + \xi N_\ast,$$

where (a) holds since $N_\ast$ is perpendicular to $V$. Therefore,

$$\|V\| \|W\| \|N_\ast, V_0, W_0\| = [N_\ast, (\nabla_r V)_{V_{\perp}}, W] = \langle \langle (\nabla_r V)_{V_{\perp}}; \mathcal{R}(-\pi/2)W \rangle \rangle.$$

The proof is completed by handling $[N_\ast, V_0, W_0]$ likewise. \hfill \Box

**Lemma 5.4** Let $s \in S$ and let $\vartheta \in (0, \pi/2]$ be the angle between the principal line $p_\ast(s)$ and the line $l_s = \{g\tau(s) + s : g \in \mathbb{R}\}$ tangent to the meridian. The following inequalities hold:

$$\Delta_\vartheta \leq 2L_N \text{ and } \vartheta \geq \arcsin \sqrt{\frac{\Delta_\vartheta}{2L_N}},$$

where $\Delta_\vartheta$ and $L_N$ are taken from (5.9) and (5.8), respectively.

**Proof:** It suffices to note that

$$\Pi_s[V; V] = \varphi_- \langle V; E_- \rangle^2 + \varphi_+ \langle V; E_+ \rangle^2,$$

$$\varphi_- + \Delta_\vartheta \leq \varphi_- (\vartheta, E_-)^2 + \varphi_+ (\vartheta, E_+)^2 = \varphi_- \cos^2 \vartheta + \varphi_+ \sin^2 \vartheta,$$

$$\Rightarrow \Delta_\vartheta \leq (\varphi_- - \varphi_+ \sin^2 \vartheta \leq 2L_N \sin^2 \vartheta \Rightarrow (5.13).$$

**Lemma 5.5** There exists $L_\tau > 0$ such that $\|\nabla_V \tau_\ast \| \leq L_\tau \|V\|$ for any vector $V \in \mathfrak{T}_s[S(d_\ast)]$ and $s \ast S(d_\ast)$.

**Proof:** By (2.1), $\tau_\ast(r) = \tau_0[J_{\perp}(r)]$ for $\tau_0(s) := \frac{\tilde{\tau}(s)}{\|\tilde{\tau}(s)\|}$ and $\tau_\ast(s) := J'(s)\tilde{\tau}(s)$. Hence by Corollary (5.1)

$$\|\nabla_V \tau_\ast \| \leq \|D_V \tau_0\| = \|D_W \tau_0\|,$$

$$1 = \|\tilde{\tau}\| = \|J_{\perp} \tilde{\tau}\| \leq \|\tilde{\tau}\|/\|\tau_\ast\|, \quad \|D_W \tilde{\tau}_0\| = \frac{1}{\|\tilde{\tau}\|} \|D_W \tilde{\tau}_0 - (D_W \tau_0) \tilde{\tau}_0\| \leq \frac{\|D_W \tau_0\|}{\|\tau_\ast\|}.$$ (5.16)

Now we invoke Assumption (5.3) and similarly see that $\|D_W \tilde{b}\| \leq L_B \|W\|/\|\mathcal{D}_B\|$ for $\tilde{b} := \nabla B/\|\nabla B\|$. Since $\tilde{\tau} = \pm \left[ E_{\perp} \langle \tilde{b}, E_{\perp} \rangle - E_{\perp} \langle \tilde{b}, E_{\perp} \rangle \right]$, we also get the following

$$\|D_W E_{\pm} \| \leq L_E \|W\|, \quad \|D_W \varphi_{\pm} \| \leq L_\varphi \|W\|,$$

$$\|D_W E_{\pm} \langle \tilde{b}, E_{\perp} \rangle \| \leq \|D_W \tilde{b}\| + \|D_W E_{\pm}\| + \|D_W E_{\pm}\| \leq (L_B/\mathcal{D}_B + 2L_E) \|W\|,$$

$$\|D_W \tau_\ast \| \leq 2(L_B/\mathcal{D}_B + 2L_E) \|W\|,$$

$$\|D_W (\varphi_{\pm} \langle E_{\perp}; \tilde{\tau}\rangle)\| \leq \{L_\varphi + |\varphi_{\pm}| \|E_{\perp} + 2(\mathcal{D}_B/\mathcal{D}_B + 2L_E)\| \} \|W\|.$$ (5.14)

$$\tilde{\tau}_0 := \tilde{\tau} + d_\ast \xi_{\ast} \tilde{\tau} = \tilde{\tau} - d_\ast [\varphi_{\pm} \langle E_{\perp}; \tilde{\tau}\rangle E_{\perp} + \varphi_{\pm} \langle E_{\perp}; \tilde{\tau}\rangle E_+],$$

$$\Rightarrow \|D_W \tilde{\tau}\| \leq 2(L_B/\mathcal{D}_B + 2L_E) \|W\| + 2d_\ast [L_\varphi + L_N (6L_E + 2L_B/\mathcal{D}_B)] \|W\|. \quad (5.17)$$

The proof is completed by gathering (5.14) -- (5.17). \hfill \Box
6 Proof of Proposition 4.1

Let \( D^c_0 \subset \Sigma_c(S) \) stands for the disc in the tangent plane \( \Sigma_c(S) \) with a radius of \( \eta > 0 \) centered at \( c \). Formula \[5.11\] makes sense for any vector \( V \in \mathbb{R}^3 \). By using this, we extend the quadratic form \( \Pi_a[:,:] \) and the associated symmetric bilinear form from the tangential plane to the entire space \( \mathbb{R}^3 \). If the tunnel is open, we put \( S^0 := \{ s \in S : s \text{ is not in the } \delta_s\text{-vicinity of } \partial S \} \); otherwise, \( S^0 := S \).

**Lemma 6.1** There exists \( \eta > 0 \) such that for any point \( c \in S^0 \), a patch \( \mathcal{P}_c(S) \) of \( S \) around \( c \) is a graph of a \( C^2 \)-smooth function \( g_c : D^c_0 \to \mathbb{R} \):

\[
\mathcal{P}_c(S) = \{ s = \Gamma_c(p) := p + g_c(p)N(c) : p \in D^c_0 \}. \tag{6.1}
\]

**Proof:** Let \( \gamma \) be the normal section of \( S \) by the plane \( \mathcal{P} \) that contains \( c \) and is coplanar with \( N := N(c) \) and a unit vector \( V \in \Sigma_c(S) \). There is a smooth function \( f(\theta) : E \to \mathbb{R}, E := (-\varepsilon_-, \varepsilon_+), \varepsilon_\pm > 0 \) such that its graph is a part of \( \gamma, \) i.e., \( \gamma(\theta) := c + \theta V + f(\theta)N \in \gamma, \theta \in E \). Since \( f'(0) = 0 \), reducing \( \varepsilon_\pm \), if necessary, ensures that for \( L_N \) from \(6.3\):

\[
L_N\|\gamma(\theta) - \gamma(0)\|^2 + |f'(\theta)|^2 < 1/2, \quad \|\gamma(\theta) - \gamma(0)\| < \delta_s/2 \quad \forall \theta \in E. \tag{6.2}
\]

From now on, we consider the maximal such an interval \( E \).

The signed curvature \( \varkappa(\theta) \) of \( \gamma \) at point \( \gamma(\theta) \) is

\[
\varkappa(\theta) = \frac{f''(\theta)}{(1 + |f'(\theta)|^2)^{3/2}} \overset{(a)}{=} \frac{\varkappa_S(\theta)}{\cos \beta}. \tag{6.3}
\]

Here \( \beta \) is the angle between \( V \times N \) and \( V(\theta) \times N(\gamma(\theta)) \), whereas \( \varkappa_S(\theta) \) is the normal curvature of \( S \) at \( \gamma(\theta) \) in direction of \( V(\theta) := \gamma'(\theta) = V + f'(\theta)N \), and \( (a) \) holds by Meusnier's theorem \[15 \text{ p. } 142\]. Here \( \| V \times N \| = 1 \) and \( \| V(\theta) \times N(\gamma(\theta)) \| = \| V(\theta) \| = \sqrt{1 + |f'(\theta)|^2} \). So \( \| V(\theta) \| \cos \beta \) is the quadruple product of \( V, N, V(\theta), N(\gamma(\theta)) \). By using formula (25) in \[6 \text{ p. } 76\]

\[
\langle A \times B; C \times D \rangle = \langle A; C \rangle \langle B; D \rangle - \langle A; D \rangle \langle B; C \rangle,
\]

we see that

\[
\sqrt{1 + |f'(\theta)|^2} \cos \beta = \langle V; V(\theta) \rangle \langle N; N(\gamma(\theta)) \rangle - \langle V; N(\gamma(\theta)) \rangle \langle V(\theta); N \rangle. \tag{6.4}
\]

\[
1 + \langle N; N(\gamma(\theta)) \rangle - N - f'(\theta) \langle V - V(\theta); N(\gamma(\theta)) \rangle \geq 1 - \|N(\gamma(\theta)) \|-N - |f'(\theta)||V - V(\theta)|| \geq 1 - L_N\|\gamma(\theta) - \gamma(0)\| - |f'(\theta)|^2 \geq 1/2;
\]

\[
\varkappa_S(\theta) = \frac{\Pi_{\gamma(\theta)}[\gamma'(\theta); \gamma'(\theta)]}{\|\gamma'(\theta)\|^2}. \tag{6.5}
\]

Hence due to \(6.3\), \(6.4\), \(6.6\), and \(6.7\):

\[
\frac{|f''(\theta)|}{(1 + |f'(\theta)|^2)^{3/2}} = |\varkappa(\theta)| \leq 2\frac{\Pi_{\gamma(\theta)}[\gamma'(\theta); \gamma'(\theta)]}{\|\gamma'(\theta)\|^2} \Rightarrow |f''(\theta)| \leq 2\Pi_{\gamma(\theta)}[\gamma'(\theta); \gamma'(\theta)]\. \tag{6.8}
\]

So thanks to \(5.4\), \(1.10\), \(1.12\), and Assumption \(5.3\):

\[
\Pi_{\gamma_0}[V; V] - \Pi_{\gamma_0}[V; V] \leq 2(L_{\gamma_0} + 2L_EL_N)\|s_2 - s_1\|\|V\|^2;
\]

\[
|f''(\theta)| \leq 2L_N\|\gamma'(\theta)\|^2 = 2L_N(1 + |f'(\theta)|^2). \tag{6.9}
\]

For some \( \theta_+ = \theta_+(L_N) > 0 \), the solution \( q(\theta) = \tan(2L_N\theta) \) of the Cauchy problem \( q(0) = 0 \) for the ode \( q' = \lambda(q) := 2L_N(1 + q^2) \) is defined on \([0, \theta_+]\) and

\[
\int_0^{\theta_+} \sqrt{1 + q(\xi)^2} \, d\xi \leq \delta_s/3, \quad \int_0^{\theta_+} \sqrt{1 + q(\xi)^2} \, d\xi + q(\theta_+) \leq 1/3. \tag{6.10}
\]

By \(6.10\), \( q'_+ \leq \lambda(q_+) \) and \( q_+(0) = 0 \) for \( q_+ := \pm f'(\theta) \), \( \pm f'(-\theta) \). Hence by \[11 \text{ Th. } 4.1 \text{ p. } 26\], every of these \( q_+ \)'s does not exceed \( q(\theta) \) whenever \( |\theta| < \min\{\varepsilon_-, \varepsilon_+\} \) and \( \theta \leq \theta_+ \). For such \( \theta \), we thus have

\[
|f'(\theta)| \leq q(|\theta|), \quad \|\gamma'(\theta)\| \leq \sqrt{1 + |q(\theta)|^2}, \quad \|\gamma''(\theta)\| \leq \lambda(q(|\theta|)), \quad \|\gamma(\theta) - \gamma(0)\| \leq \int_0^{|	heta|} \sqrt{1 + |q(\xi)|^2} \, d\xi. \tag{6.11}
\]
We are going to show that \( \min\{\varepsilon_-, \varepsilon_+\} > \theta_+ \). Suppose to the contrary that \( \varepsilon_0 = \min\{\varepsilon_-, \varepsilon_+\} \leq \theta_+ \), where \( \sigma = \pm \). Let \( \sigma = + \), the case \( \sigma = - \) is treated likewise. For \( \theta \in [0, \varepsilon_+] \), we have
\[
L_N \|\gamma(\theta) - \gamma(0)\| + |f'(\theta)|^2 \leq L_N \int_0^\theta \sqrt{1 + q(s)^2} \, ds + q(\theta)^2 \leq 1/3, \tag{6.13}
\]
\[
\|\gamma(\theta) - \gamma(0)\| \leq \int_0^\theta \sqrt{1 + q(s)^2} \, ds \leq \delta_+/3. \tag{6.14}
\]
Since \( \|\gamma'(\theta)\| \) and \( \|\gamma''(\theta)\| \) stay bounded as \( \theta \to \varepsilon_+ \) by (6.12), there exist \( s_{im} := \lim_{\theta \to \varepsilon_+} \gamma'(\theta) \) and \( W := \lim_{\theta \to \varepsilon_+} \gamma''(\theta) \) and the vector \( W \) is not aligned with \( N \). Meanwhile, \( s_{im} \notin \partial S \), thanks to (6.14) since \( \gamma(0) = c \) is not in the \( \delta_n \)-vicinity of \( \partial S \). It follows that the normal section \( \gamma \) extends as a graph of a smooth function to the right of \( \varepsilon_+ \). Meanwhile, (6.13) and (6.14) imply that (6.12) holds for \( \theta = \varepsilon_+ \) and so for \( \theta > \varepsilon_+, \theta \approx \varepsilon_+ \). These violate the definition of \( \varepsilon_+ \) as an end of the maximal interval. The contradiction obtained proves that \( \min\{\varepsilon_-, \varepsilon_+\} > \theta_+ \).

Now we put \( \eta := \theta_+, V_p := \frac{p - c}{\|p-c\|} \) for all \( p \in D^p_\varepsilon, p \neq c \) and emphasize the dependence of \( f(\cdot) \) on \( V \) by adding the index \( V \) to \( f \), thus obtaining \( f_{V}(\cdot) \). Then the function
\[
ge_c(p) := \begin{cases}
   f_{V_p}(\|p-c\|) & \text{whenever } p \neq c, \\
   0 & \text{if } p = c.
\end{cases}
\]
is well-defined and meets (6.1). Since \( \|N\gamma(\theta) - N\| \leq L_N \|\gamma(\theta) - \gamma(0)\| < 1/2 \) by (6.8) and (6.2), the vectors \( N \) and \( N\gamma(\theta) \) are not aligned. It follows that the function \( g_c(\cdot) \) is \( C^2 \)-smooth on \( D^p_\varepsilon \).

Lemma 6.2 After properly reducing \( \eta > 0 \), if necessary, the following inequalities hold for any \( p \in D^p_\varepsilon, c \in S^0 \):
\[
\|\nabla g_c(p)\| \leq 2L_N \|p - c\|, \tag{6.15}
\]
\[
|g_c(p)| \leq L_N \|p - c\|^2. \tag{6.16}
\]

Proof: For any unit vector \( V \in \mathfrak{T}_\varepsilon(S) \) and \( p \in D^p_\varepsilon \), we have \( \zeta(\theta) := \Gamma_c[p + \theta V] \in S \forall \theta \approx 0 \Rightarrow \mathfrak{T}_\varepsilon(S) \ni \zeta(0) = V + (\nabla g_c(p); V) N(c) \). So \( \zeta'(0) \) is normal to \( N(p) \) and
\[
\langle V; N(p) \rangle = - \langle N(c); N(p) \rangle \langle \nabla g_c(p); V \rangle. \tag{6.17}
\]
Here \(|\langle V; N(p) \rangle| = |\langle V; N(p) - N(c) \rangle| \leq \|N(p) - N(c)\| \leq L_N \|p - c\| \) by (5.3). Meanwhile, \( \langle N(c); N(p) \rangle = 1 + \langle N(c); N(p) - N(c) \rangle \leq \|N(p) - N(c)\| \leq L_N \|p - c\| \). So by properly reducing \( \eta > 0 \), if necessary, we can ensure that \( \langle N(c); N(p) \rangle \geq 1/2 \). Then (6.17) implies that \(|\langle \nabla g_c(p); V \rangle| \leq 2L_N \|p - c\| \). Maximization over \( V \in \mathfrak{T}_\varepsilon, \|V\| = 1 \) yields (6.15). Also,
\[
|g_c(p)| = \left| \int_0^1 \langle \nabla g_c[p + (1 - \theta)c]; p - c \rangle \, d\theta \right| \leq 2L_N \|p - c\|^2 \int_0^1 \theta \, d\theta \Rightarrow (6.10). \tag{6.10}
\]
Based on (6.11) and (6.10), we see that
\[
\|\Gamma_c(p) - c\| \leq \|p - c\| + L_N \|p - c\|^2. \tag{6.18}
\]

Lemma 6.3 There is a non-decaying function \( \zeta(\cdot) \) such that \( \zeta(\eta) \to 0 \) as \( \eta \to 0^+ \) and the Hessian \( g_c'' \) obeys the estimate
\[
\|g_c''(p) - g_c''(c)\| \leq \zeta(\|p - c\|) \quad \forall p \in D^p_\varepsilon, c \in S^0.
\]

Proof: In \( \mathbb{R}^3 \), we pick a Cartesian coordinate system centered at \( c \) so that its \( x \) and \( y \)-axes lie in the tangent plane \( \mathfrak{T}_\varepsilon(S) \) and the \( z \)-axis is co-directed with \( N(c) \). We also identify any point \( p \in \mathfrak{T}_\varepsilon(S) \) with the pair \( (x, y) \) of its coordinates. Since the map \( \Gamma := \Gamma_c = \{x,y,z(x,y), z(x,y) := g_c(x,y) \} \) is a coordinate chart on \( S \), we have [15] p. 154
\[
\Pi_{\Gamma(p)}[V_p; V_p] = A(p)(dx)^2 + 2B(p)dy dx + C(p)(dy)^2, \tag{6.19}
\]
where \( V_p := \Gamma_p'(p)dx + \Gamma_p'(p)dy = \{dx, dy, x'_dx + z'_dy, y'_dz \}, \)
\[
A(p) = \langle N_{\Gamma(p)}; \Gamma''_{xx}(p) \rangle = \langle N_{\Gamma(p)}; N_c \rangle \xi''_{xx}(p),
\]
\[
B(p) = \langle N_{\Gamma(p)}; \Gamma''_{xy}(p) \rangle = \langle N_{\Gamma(p)}; N_c \rangle \xi''_{xy}(p),
\]
\[
C(p) = \langle N_{\Gamma(p)}; \Gamma''_{yy}(p) \rangle = \langle N_{\Gamma(p)}; N_c \rangle \xi''_{yy}(p).
\]
After reducing \( \eta \) so that \( \eta < 1 \), we have for \( p \in D^2 \),
\[
|\Pi_{\Gamma(p)} [V_p; V_p] - \Pi_{\Gamma(c)} [V_c; V_c]| \leq |\Pi_{\Gamma(p)} [V_p; V_p] - \Pi_{\Gamma(c)} [V_p; V_p]| + |\Pi_{\Gamma(c)} [V_p; V_p] - \Pi_{\Gamma(c)} [V_c; V_c]|
\]
\[
\leq 2(L_x + 2L_E L_N)|\Pi_{\Gamma(p)} - \Pi_{\Gamma(c)}| ||[V_p]||^2 + |\Pi_{\Gamma(c)} [V_p; V_c] - V_c - V_c| + 2|\Pi_{\Gamma(c)} [V_p - V_c; V_c]|
\]
\[
\leq 2(L_x + 2L_E L_N)\eta(1 + L_N^2)||[V_p]||^2 + 2L_N||V_p - V_c||^2 + 2||V_p - V_c|| ||V_c||.
\]
By noting that \( \nabla g_c(c) = 0 \) due to (6.15), we have \( ||V_p - V_c|| \leq ||\nabla g_c(p)||\xi_\delta \), where \( \xi_\delta := (\sqrt{(dx)^2 + (dy)^2}) \). So by Lemma 6.2, \( ||V_p - V_c|| \leq 2L_N^2\eta\xi_\delta \). Meanwhile \( ||V_c|| = \xi_\delta \). Thus
\[
|\Pi_{\Gamma(p)} [V_p; V_p] - \Pi_{\Gamma(c)} [V_c; V_c]| \leq M_\Pi \eta \xi_\delta^2,
\]
where \( M_\Pi := 2(L_x + 2L_E L_N)(1 + L_N^2)[1 + 2L_N^2] + 4L_N^2(L_N + 1) \).

Here \( \Pi_{\Gamma(p)} [V_p; V_p] - \Pi_{\Gamma(c)} [V_c; V_c] = [A(p) - A(c)](dx)^2 + [2|B(p) - B(c)|]dxdy + |C(p) - C(c)|(dy)^2 \). Hence
\[
|A(p) - A(c)| < M_\Pi \eta, \quad |C(p) - C(c)| < M_\Pi \eta, \quad |B(p) - B(c)| < M_\Pi \eta.
\]

At the same time,
\[
A(p) - A(c) = \langle N_{\Gamma(p)}; N_c \rangle z_{xx}(p) - z_{xx}(c) = \langle N_{\Gamma(p)}; N_c \rangle \left[ z''_{xx}(p) - z''_{xx}(c) \right] + \langle N_{\Gamma(p)}; N_c \rangle - 1 \right] z''_{xx}(c);
\]
\[
\left| \langle N_{\Gamma(p)}; N_c \rangle - 1 \right| = \left| \langle N_{\Gamma(p)}; N_c \rangle - \langle N_{\Gamma(c)}; N_c \rangle \right| \leq L_N ||\Pi_{\Gamma(p)} - e|| \leq L_N ||p - c|| + L_N ||p - c||^2
\]
\[
\leq L_N(1 + L_N\eta), \quad \left| \langle N_{\Gamma(p)}; N_c \rangle \right| \geq 1 - L_N\eta(1 + L_N\eta) \geq 1/2
\]

if \( \eta < (\sqrt{3} - 1)/(2L_N) \). Meanwhile (6.19) implies that
\[
\langle g''_c(V; V) \rangle = \Pi_c [V; V]
\]
and so \( \langle z''_{xx}(c)V; V \rangle \rangle = \langle g''_c(V; V) \rangle = \Pi_c [V; V] = \langle g_c(V; V) \rangle \leq L_N||V||^2 \) by (5.1). Hence \( z''_{xx}(c) \leq L_N, z''_{yy}(c) \leq L_N, z''_{xy}(c) \leq L_N \). As a result,
\[
|z''_{xx}(p) - z''_{xx}(c)| = \frac{|A(p) - A(c) - \langle N_{\Gamma(p)}; N_c \rangle - 1 \rangle z''_{xx}(c)|}{\langle N_{\Gamma(p)}; N_c \rangle} \leq 2(M_\Pi + L_N^2 + L_N^3\eta\eta).
\]
Similarly \( |z''_{yy}(p) - z''_{yy}(c)| \leq 2(M_\Pi + L_N^2 + L_N^3\eta\eta)\eta \leq 2(M_\Pi + L_N^2 + L_N^3\eta\eta) \eta \). It remains to note that for a given \( p \) from the basic disk \( D^2 \), the above estimates remain true with \( \eta \) being artificially reduced and made arbitrarily close to \( ||p - c|| \).

\textbf{Corollary 6.1} The following relations hold for any \( c \in S^0, p \in D^2_c \):
\[
ge_c(p) = 1/2\Pi_c[p - c; p - c] + \omega_c(p), \quad \text{where}
\]
\[
\omega_c(p) \leq |c||p - c||p - c||/2,
\]
\[
||\nabla \omega_c(p)|| \leq |c||p - c||/||p - c||,
\]
\[
||\nabla \omega_c(p)|| \leq |c||p - c||/||p - c||.
\]

The last inequality is immediate from Lemma 6.2 and (6.20). Then putting \( p(\theta) := (p - c)^\theta + c \), we have
\[
||\nabla \omega_c(p)|| = \left| \int_0^1 \omega_c '' (p(\theta)) \frac{d\theta}{||p(\theta) - c||} \right| \leq ||p - c|| \int_0^1 \omega_c '' (p(\theta)) \frac{d\theta}{||p(\theta) - c||} \leq |c||p - c||^2.
\]

Now we consider a point \( r \in Z_{\alpha}^{\text{op}} \) and its projection \( c = c(r) := \pi(r) \) onto \( S \); then \( c \in S^0 \) by the definition of \( Z_{\alpha}^{\text{op}} \). The rays \( R \) emitted from \( r \) at a given angle \( \alpha \in [0, \alpha] \) to \( d = c - r \) are parametrized \( \Re = \Re(\varphi) \) by angle \( \varphi \) so that \( \Re(\varphi) \) passes through the point \( c + d\varphi \) tan \( \alpha \), where \( d \) is the distance from \( r \) to \( S \) and \( c(\varphi) := E(\cos \varphi + E(\sin \varphi \varphi \sin \alpha) \cos \alpha \); see Fig. 1. The distance \( d(\alpha, \varphi) \) from \( r \) to the point \( p \) of intersection of \( \Re(\varphi) \) with the surface \( S \) is given by \( d(\alpha, \varphi) = (d + x^2 \sin^2 \alpha) / \cos \alpha \), where \( x = x(\alpha, \varphi) \) is defined in Fig. 1 and scaling of the distance from \( p \) to the tangent plane by the factor \( \sin \alpha \) is introduced for further convenience. It follows that maximization of \( d(\alpha, \varphi) \) over \( \varphi \) is equivalent to maximization of \( x \). Meanwhile, \( x \) is the root of the equation
\[
x = \lambda(x, \varphi; \alpha) := \frac{g_c(c + (d + x^2 \sin^2 \alpha)\sin \varphi \tan \alpha)}{\sin^2 \alpha}.
\]
For this equation to be well-posed, the argument of $g_c(\cdot)$ should be in the domain $D_c^\theta$ of definition of $g_c(\cdot)$, which is equivalent to

$$x \in I := \left[-\frac{\eta \cos \alpha}{\sin^3 \alpha} - \frac{d}{\sin^2 \alpha}, \frac{\eta \cos \alpha}{\sin^3 \alpha} - \frac{d}{\sin^2 \alpha}\right].$$  \hspace{1cm} (6.22)$$

**Lemma 6.4** There is $\alpha_1 = \alpha_1(\eta) \in (0, \alpha_2]$ such that whenever $\alpha \leq \alpha_1$, equation (6.21) has an unique root on the interval $[422]$ and this root smoothly depends on $\varphi$ and $r \in \mathbb{Z}^a_{op}$. 

**Proof:** We first note that for $x \in I$,

$$|\lambda_x^\prime| \cos \alpha = |\langle \nabla g_c[c + (d + x \sin^2 \alpha)\tilde{c}(\varphi) \tan \alpha]; \tilde{c}(\varphi) \rangle \sin \alpha| \leq 2L_N|d + x \sin^2 \alpha| \tan \alpha \sin \alpha \leq 2L_N \eta \sin \alpha.$$ 

So $|\lambda_x^\prime| \leq 1/2 \forall x \in I,\alpha \in (0, \alpha_1]$ if $\alpha_1$ is chosen small enough. This implies that the root $x \in I$ of (6.21) (if exists) is unique. At the right end of $I$, the r.h.s. of (6.21) equals $-g_c[c + \eta \tilde{c}(\varphi)] \sin^2 \alpha$ and is bounded over $\varphi$ due to (6.10), whereas the l.h.s. is $|\eta \tilde{c}(\varphi) - d| \sin^2 \alpha \geq |\eta \tilde{c}(\varphi) - d_+| \sin^2 \alpha$, where $\eta \tilde{c}(\varphi) - d_+ \to \infty$ as $\alpha \to 0^+$. So by reducing $\alpha_1 > 0$, if necessary, it can be ensured that r.h.s. $< l.h.s.$ at the right end of $I$ for all $\varphi$ and $r \in \mathbb{Z}^a_{op}$. Similarly, it can be ensured that r.h.s. $> l.h.s.$ at the left end. Hence the root $x \in I$ of equation (6.21) does exist. By the implicit function theorem (applied to the function $x - \lambda(x, \varphi(\alpha))$) and Lemma 5.1 this root smoothly depends on $\varphi$ and $r \in \mathbb{Z}^a_{op}$. 

**Lemma 6.5** As $\alpha \to 0^+$, the function $\lambda(x, \varphi(\alpha))$ and its first and second derivatives in $x$ and $\varphi$ converge to the function $\lambda(x, \varphi(0)) = -\Pi^2_k[\tilde{c}(\varphi); \tilde{c}(\varphi)]/2$ and its respective derivatives uniformly over $r \in \mathbb{Z}^a_{op}, \varphi \in \mathbb{R}$, and $x \in [-\pi, \pi]$ for any $\pi > 0$. 

**Proof:** For $p : = c + (d + x \sin^2 \alpha)\tilde{c}(\varphi) \tan \alpha, x \in [-\pi, \pi]$, and $k := d_+ + \pi$, we have $||p - c|| = |x \sin^2 \alpha + d| \tan \alpha \leq k \tan \alpha$. Thanks to Corollary 6.3,

$$\lambda(x, \varphi(\alpha)) = A + B,$$

where

$$A := -\Pi_k[\langle d + x \sin^2 \alpha \rangle \tilde{c}(\varphi); (d + x \sin^2 \alpha) \tilde{c}(\varphi)],$$

$$B := -\frac{\omega_c(p)}{\sin^2 \alpha}; |B| \leq \zeta(k \tan \alpha) \frac{k^2}{2 \cos^2 \alpha},$$

$$|B_x^\prime| \leq \frac{|\langle \nabla \omega_c(p); \tilde{c}(\varphi) \rangle| |d + x \sin^2 \alpha| \sin \alpha \cos \alpha}{\sin \alpha \cos \alpha} \leq k \zeta(k \tan \alpha)/ \cos^2 \alpha,$$

$$|B_{x\varphi}^\prime| \leq \frac{|\langle \nabla \omega_c(p); \tilde{c}(\varphi) \rangle| |d + x \sin^2 \alpha| \sin \alpha \cos \alpha}{\sin \alpha \cos \alpha} \leq k \zeta(k \tan \alpha) \tan^2 \alpha \sin^2 \alpha,$$

$$|B_{x\varphi}^\prime| \leq \frac{|\langle \nabla \omega_c(p); \tilde{c}(\varphi) \rangle| |d + x \sin^2 \alpha| \sin \alpha \cos \alpha}{\sin \alpha \cos \alpha} \leq 2k \zeta(k \tan \alpha) \tan^2 \alpha \sin^2 \alpha,$$

$$|B_{x\varphi}^\prime| \leq \frac{|\langle \nabla \omega_c(p); \tilde{c}(\varphi) \rangle| |d + x \sin^2 \alpha| \sin \alpha \cos \alpha}{\sin \alpha \cos \alpha} \leq 2k \zeta(k \tan \alpha) \tan^2 \alpha \sin^2 \alpha.$$
Corollary 6.2  The function $x(\phi)$ is $C^2$-smooth. This function and its first and second derivatives are continuous in $r$ and converge to the root $y(\phi) = -d^2 \Pi [\phi]$ of the equation $y = \lambda(y, \phi)$ and its respective derivatives uniformly over $\phi \in \mathbb{R}, \ r \in Z_\alpha$ as $\alpha \to 0^+$.

Proof of Proposition 4.1. Due to the definition of $\bar{\phi}(\phi)$ (see Fig. 1) and (5.14),

$$4y(\phi) = -2d^2[\kappa_-(c) \cos^2 \phi + \kappa_+(c) \sin^2 \phi] = -d^2[\kappa_-(c) + \kappa_+(c)] + d^2[\kappa_+(c) - \kappa_-(c)] \cos(2 \phi).$$

Since $\Delta_r \leq \kappa_+ - \kappa_- \leq 2L_N$ due to (2.2) and (5.1), we have

$$y'(\phi) = 0 \iff \phi = 0, \pi, \mod 2 \pi,$$

$$y''(\phi) \leq -d^2 \Delta_r < 0 \forall \phi \in \pi, \mod 2 \pi,$$

$$y''(\phi) \geq d^2 \Delta_r > 0 \forall \phi \in \pi, \mod 2 \pi,$$

$$|y''(\phi)| \leq d^2 |\kappa_+ - \kappa_-| \leq 2d^2 L_N, \quad |y'''(\phi)| \leq 4d^2 L_N.$$ 

So putting $\delta := \frac{d^2 \Delta_r}{8d^2 L_N}$, we have $y(0) = y(\pi)$,

$$y''(\phi) \leq -d^2 \Delta_r / 2 \leq 0 \forall \phi \in (-\delta, \delta) \cup (\pi - \delta, \pi + \delta) \mod 2 \pi,$$

$$y''(\phi) \geq d^2 \Delta_r / 2 > 0 \forall \phi \in (\pi - \delta, \pi + \delta) \mod 2 \pi,$$

$$|y'''(\phi)| \geq d^2 \delta \Delta_r / 2 \quad \text{whenever } \phi \text{ is outside all of the listed intervals.}$$

Corollary 6.2 ensures that the following relation hold for all small enough $\alpha$:

$$x''(\phi) \leq -d^2 \Delta_r / 4 < 0 \forall \phi \in (-\delta, \delta) \cup (\pi - \delta, \pi + \delta) \mod 2 \pi,$$

$$x''(\phi) \geq d^2 \Delta_r / 4 > 0 \forall \phi \in (\pi - \delta, \pi + \delta) \mod 2 \pi,$$

and

$$|x'(\phi)| \geq d^2 \delta \Delta_r / 4 \quad \text{whenever } \phi \text{ is outside all of the listed intervals.}$$

By applying the implicit function theorem to the equation $x'(\phi) = 0$ and reducing $\delta$, if necessary, we see that this equation has a single root on every of the following $(\mod 2 \pi)$-intervals $(-\delta, \delta), (\pi - \delta, \pi + \delta), (-\pi/2 - \delta, \pi/2 - \delta), (\pi/2 - \delta, \pi/2 + \delta)$, this root continuously depends on $r \in Z_\alpha$ and uniformly goes to the center of the respective interval as $\alpha \to 0^+$. Meanwhile, the last displayed formulas assure that there are no other roots and $x(\phi)$ attains its local maxima at two points: at the roots $\varphi_0, \varphi_\pi \in (-\delta, \delta)$ and $\varphi_\pi \in (\pi - \delta, \pi + \delta)$. So MDPBE is well posed. It remains to note that it returns the line that goes in the direction of $\phi_* := 1/2(\varphi_0 + \varphi_\pi - \pi)$, where $\varphi_* \to 0$ as $\alpha \to 0^+$ uniformly over $r \in Z_\alpha$ and $\varphi = 0$ corresponds to the principal direction $p_-(c)$. □

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