The relativistic mechanism of the Thomas–Wigner rotation and Thomas precession

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Abstract

We consider the Thomas–Wigner rotation of coordinate systems under successive Lorentz transformations of inertial reference frames, and disclose its physical mechanism on the basis of the relativistic contraction of moving scale, and the relativity of the simultaneity of events for different inertial observers. This result allows us to better understand the physical meaning of the Thomas precession, and to indicate some overlooked aspects of the physical interpretation of this effect, as related to two specific examples: the circular motion of a classical electron around a heavy nucleus, and the motion of a classical electron along an open path, where its initial velocity and acceleration are mutually orthogonal to each other.

Keywords: special relativity, Thomas–Wigner rotation, Thomas precession

(Some figures may appear in colour only in the online journal)

1. Introduction

It is understood that successive Lorentz transformations with non-collinear relative velocities between inertial reference frames entail a spatial rotation of coordinate axes of these frames, known as Thomas–Wigner rotation \cite{1, 2}. It is further understood that the Thomas–Wigner rotation of the coordinate axes of instantaneously co-moving frames of an electron, in orbit around an immovable nucleus, induces the Thomas precession of electron spin. Historically, this effect has played an important role in the semi-classical explanation of spin–orbit interaction in atoms \cite{1, 3}, as well as in the support of the entire hypothesis regarding spin \cite{4}.

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For these reasons, numerous papers have been published on this subject to date (see references [5–14], already published so far in the 21st century). One such recent paper on the subject was a previous publication from the authors of this paper [15], where for the first time we applied a correct expression for classical force on a moving dipole [16]. Using this expression, we have found that, for a laboratory observer, all electromagnetic contributions to the fine splitting of atomic energy levels mutually cancel each other, so that the measured spin-orbit coupling in hydrogen-like atoms represents a purely mechanical energy component, associated with the Thomas precession of electron spin. These results clearly indicate a requirement for more light to be shed on the physical interpretation of the Thomas precession, and its fundamental origin, Thomas–Wigner rotation.

It is widely recognized that both of these effects have a purely kinematical origin (see e.g. [17, 18]), and for this reason, many physicists seem to believe that this circumstance obviates the need to seek concrete physical mechanisms for the given effects. The available publications with respect to the derivation of the Thomas–Wigner rotation, and the Thomas precession, which apply either an analytical approach [1–3, 17–21], or a geometrical approach [22–27], are, in fact, no longer aiming to provide any physical interpretation in terms of known relativistic effects in Minkowskian space-time. As a result, many researchers remain satisfied with the general statement that Thomas–Wigner rotation reflects the general properties of the Lorentz group, where, as we know, spatial rotations constitute a sub-group [28].

This general statement should be kept in mind in any case, when dealing with the Thomas–Wigner rotation. At the same time, in section 2, we intend to demonstrate that the Thomas–Wigner rotation, like any other effect of classical physics, can be understood with the involvement of the known effects of relativity theory; specifically, the contraction of moving scale along the direction of its velocity, and the relativity of simultaneity of events for different inertial observers. In this way, we specify the notion of parallel coordinate systems in terms of relativity theory, and seek an answer to the question; why, under successive rotation-free space-time transformations between three inertial frames, do two of them become non-parallel to each other?

We provide answers to this question via simple particular problems involving special Lorentz transformations, where the direction of relative velocity is collinear to one of the spatial coordinate axes. In section 3, we also emphasize that the notion of parallel coordinate systems does not, in general, imply the spatial parallelism of the corresponding coordinate axes of two inertial frames, which can thus be seen as non-Cartesian for each other. Moreover, when a relative velocity between two parallel systems of coordinates becomes time-dependent, a spatial orientation of the coordinate axes of parallel systems also varies with time, due to the contraction of scale along the corresponding velocity vector, which gives a corresponding contribution to the observed spatial direction of particle spin undergoing Thomas precession (section 3). To the best of our knowledge, the latter effect has never been considered in any previous analysis of Thomas precession. At the same time, in sub-section 3.1, we show that the scale contraction effect for coordinate axes of a system co-moving with an orbiting classical electron around a heavy nucleus does not contribute to the averaged mechanical energy associated with the Thomas precession. Finally, in sub-section 3.2, we consider the motion of a classical electron with spin along an open curved path, and demonstrate some features of the Thomas precession of its spin, which seem to have been overlooked to date. We draw our conclusion in section 4.

In order to simplify the mathematical aspect of our analysis in the sections below, we consider the relative motion of inertial reference frames in the $xy$ plane only.

The paper will be interesting to undergraduates, and graduate students and to specialists as well.
2. The relativistic mechanism of Thomas–Wigner rotation as the result of the scale contraction effect, and the relativity of the simultaneity of events in three different inertial frames

One of the simplest and most familiar examples [10, 30] to demonstrate Thomas–Wigner rotation is presented in figure 1, where an inertial reference frame K₁ is moving with respect to a laboratory frame K₀ at constant velocity \(-u\{u,0,0\}\) along the axis \(x\), while an inertial reference frame K₂ is moving in the frame K₀ at constant velocity \(v\{0,v,0\}\) along the axis \(y\). Thus, the pairs of frames K₀, K₁, and K₀, K₂ are related to each other via special Lorentz transformations, and the corresponding coordinate axes of the frames remain mutually parallel to each other. Next, we consider the remaining pair of K₁, K₂ frames, and find the angles between the axes \(x₁, x₂\), and \(y₁, y₂\), correspondingly.

One way to solve this problem is to determine the relative velocity between frames K₁ and K₂, and to compare its spatial components on the coordinate axes of both frames, as undertaken in the aforementioned references [10, 30]. First, we calculate the velocity \(V_{12}\) of the frame K₂ with respect to K₁. Its components can be found via the Lorentz transformation between K₁, and K₀, taking into account the fact that for an observer in K₀, the velocity of K₂ has the components \(V_{2x} = 0\), \(V_{2y} = v\). Hence, using Einstein’s law of speed composition (see, for example, reference [17]), we obtain for an observer in K₁:

\[
V_{12x} = u, \quad V_{12y} = v \sqrt{1 - u^2/c^2},
\]

where \(c\) is the velocity of light in vacuum.

Next, we calculate the components of velocity K₁ in K₂, designating it as \(V_{21}\). It is worth pointing out that for an observer in K₀, the velocity of K₁ delineates the components \(V_{1x} = -u\), \(V_{1y} = 0\), and using Einstein’s law of speed composition between K₂ and K₀, we obtain for an observer in K₂:

\[
V_{21x} = u, \quad V_{21y} = v \sqrt{1 - u^2/c^2},
\]
\[ V_{21x} = -u \sqrt{1 - \frac{v^2}{c^2}}, \quad (2a) \]
\[ V_{21y} = -v. \quad (2b) \]

Comparing equations (1a), (1b), (2a) and (2b), we reveal that

\[ V_{21x} \neq -V_{12x}, \quad (3a) \]
\[ V_{21y} \neq -V_{12y}, \quad (3b) \]
and

\[ V_{21}^2 = V_{12}^2. \quad (3c) \]

The fact of equal modulus of relative velocity for observers in the frames \( K_1, K_2 \) (see equation (3c)), with the inequality of their coordinate components (see equations (3a) and (3b)), proves the presence of spatial rotation in the coordinate axes of these frames. The value of the rotational angle \( \theta_{TW} \) (where the subscript ‘TW’ specifies that we deal with the Thomas–Wigner rotation) of frames \( K_1, K_2 \) can be found from the following equalities:

\[ |V_{12x}| = |V_{21x}| \cos \theta_{TW} - |V_{21y}| \sin \theta_{TW}, \quad (4a) \]
\[ |V_{12y}| = |V_{21x}| \sin \theta_{TW} + |V_{21y}| \cos \theta_{TW}. \quad (4b) \]

Substituting the components of relative velocities from equations (1) and (2) into equation (4), we obtain accurate calculations for \( \theta_{TW} \):

\[ \theta_{TW} \approx -\frac{uv}{2c^2}. \quad (5) \]

Hereinafter we adopt the sign ‘minus’ to denote clockwise rotation, and the sign ‘plus’ to denote counterclockwise rotation.

One can check that the general expression for the Thomas–Wigner angle (see [17]), as applied to the motional diagram in figure 1, yields the same result (5) within the adopted accuracy of calculations.

This problem demonstrates the emergence of a relative spatial rotation of coordinate systems \( K_1, K_2 \) under successive Lorentz transformations \( K_1 \to K_0 \to K_2 \) with non-collinear relative velocities. At the same time, equations (3)–(5) do not shed any light on the mechanism of Thomas Wigner rotation between the axes of \( K_1 \) and \( K_2 \); they show only that the Thomas–Wigner rotation must emerge, and do not satisfy the question, how does this rotation effectively emerge? What is more, the foregoing equality (3c) for the modulus of the relative velocity of frames \( K_1 \) and \( K_2 \) (which reflects the validity of the reciprocity principle [31]) gives the impression that an observer in \( K_1 \) sees frame \( K_2 \) as Cartesian and, conversely, an observer in \( K_2 \) sees frame \( K_1 \) as Cartesian. However, as we will see below, this is generally not the case.

Thus, when seeking an answer as to why the observers in frames \( K_1 \) and \( K_2 \) see the mutual rotation of their coordinate axes for the motional diagram in figure 1, we must first specify the definition of parallel coordinate systems in terms of relativity theory. Specifically, we define two coordinate systems, \( K \) and \( K' \), to be parallel to each other, if their relative velocities have equal and opposite in sign components on their corresponding coordinate axes (see [20]).

In other words, if an observer in \( K \) sees the velocity of \( K' \) as equal to \( V\{u,v,w\} \), then an observer in \( K' \) sees the velocity of \( K \) as equal to \( -V\{-u,-v,-w\} \). In this case, the Lorentz transformation from \( K \) to \( K' \) (\( L(V) \)), and the Lorentz transformation from \( K' \) to \( K \) (\( L(-V)=L^{-1}(V) \)) are classified as rotation-free.
Figure 2. The relative motion of frames K_1 and K_2 at constant velocity V, constituting the angle $\alpha$ with the axis x. (a) View for an observer in frame K_1; (b) view for an observer in frame K_2. The inertial systems K_1 and K_2 are parallel to each other according to the common definition, but the pairs of axes x_2 and y_2 are not parallel, and constitute the angles $\vartheta_{x_2}$ and $\vartheta_{y_2}$, respectively.

For example, special Lorentz transformations maintain the parallelism of coordinate axes of corresponding inertial frames, and thus they are always rotation-free. Therefore, in figure 1, the coordinate systems K_0, K_1 and coordinate systems K_0, K_2 are mutually parallel to each other.

However, one should stress that in the general case (where the relative velocity V between two inertial frames is not collinear to either coordinate axis), the relativistic definition of parallel coordinate systems (which only requires the equality of component of their mutual velocities with the reverse sign (i.e. $V\{u, v, w\}$ and $-V\{-u, -v, -w\}$) does not generally imply the parallelism of corresponding axes of these systems. More specifically, only under special Lorentz transformations does the notion of parallel coordinate systems of the frames K and K' imply the parallelism of their corresponding coordinate axes. However, in the general case, where the relative velocity V of two inertial frames K and K' is not parallel to either coordinate axis, the axes {x, y, z} might be non-parallel to the corresponding axes {x', y', z'}. This means, in particular, that both frames can be seen as non-Cartesian for each other. Nevertheless, the frames K and K' may remain parallel to each other according to the relativistic definition presented above.

Indeed, consider the relative motion of some inertial reference frames K_1 and K_2 with the constant velocity V, lying in the plane xy (see figure 2, adapted from [32]). This figure demonstrates that for an observer in frame K_1 (figure 2(a)), the angle between axes x_2 and y_2 of frame K_2 is no longer equal to $\pi/2$, and, analogously, for an observer in frame K_2, the angle between axes x_1 and y_1 of K_1 is not equal to $\pi/2$ (figure 2(b)). The change of spatial orientation of
coordinate axes of a moving system, as seen by the observer in their own resting system, is caused by the known relativistic effect of contraction of a moving scale along the direction of its motion.

In order to calculate the angle between the axes $x_2$ and $y_2$ for an observer in $K_1$, we first determine the spatial orientation of axis $x_2$ in frame $K_1$. For this purpose we may select any segment $X_2$ belonging to axis $x_2$ and point out that its projection onto the direction, orthogonal to $V$, remains unchanged, while its projection onto the vector $V$ contracts by $1/\gamma$ times, where $\gamma = (1 - \frac{V^2}{c^2})^{-1/2}$ is the Lorentz factor. Thus segment $X_2$, and therefore the entirety of axis $x_2$, is rotated by angle $\vartheta_{x_2}$ with respect to axis $x_1$. (Hereinafter the subscript ‘$\gamma$’ is introduced as a reminder that the angle originates from the scale contraction effect for the axis in question, unlike angle $\vartheta_{TW}$ introduced above in equations (4a) and (4b), which describes the common Thomas–Wigner rotation of appropriate coordinate axes.)

We can find the value of the angle $\vartheta_{x_2}$, by applying the relativistic transformation of length under the rotation-free Lorentz transformation [17]

$$r_1 = r_2 - \frac{(\gamma - 1)}{\gamma} (V \cdot r_2) V, \quad (6)$$

and choosing the components of vector $r_2 = r_2(X_2,0,0)$ in frame $K_2$. Hence, we obtain the components of this vector for an observer in frame $K_1$:

$$X_{1x} = X_{2x} - \frac{(\gamma - 1)}{\gamma^2} (X_2 V \cos \alpha) V \cos \alpha = \frac{X_2}{\gamma} (\gamma \sin^2 \alpha + \cos^2 \alpha), \quad (7a)$$

$$X_{1y} = \frac{(\gamma - 1)}{\gamma^2} (X_2 V \cos \alpha) V \sin \alpha = -\frac{(\gamma - 1)}{\gamma} \sin \alpha \cos \alpha, \quad (7b)$$

and

$$\tan \vartheta_{x_2} = \frac{X_{1y}}{X_{1x}} = -\frac{(\gamma - 1) \sin \alpha \cos \alpha}{\gamma \sin^2 \alpha + \cos^2 \alpha}. \quad (8)$$

Thus, to the accuracy of calculations $c^{-2}$, equation (8) yields

$$\hat{\vartheta}_{x_2} \approx \frac{V^2}{2c^2} \sin \alpha \cos \alpha \approx -\frac{V_x V_y}{2c^2} = -\frac{uv}{2c^2}, \quad (9)$$

with the reminder that the sign ‘minus’ corresponds to clockwise rotation.

Similarly, designating the segment $Y_2$ on axis $y_2$, and determining its components for an observer in frame $K_1$, we find the angle between axes $y_1$ and $y_2$ due to the scale contraction effect:

$$\hat{\vartheta}_{y_2} \approx \frac{V^2}{2c^2} \sin \alpha \cos \alpha = \frac{V_x V_y}{2c^2} = \frac{uv}{2c^2}, \quad (10)$$

which corresponds to the rotation in the counter clockwise direction.

Thus, we see that axes $x_1$, $x_2$, and $y_1$, $y_2$ are not parallel to each other, and that the angle between axes $x_2$ and $y_2$, as seen in $K_1$, is equal to $(\pi/2 + |\hat{\vartheta}_{x_2}| + |\hat{\vartheta}_{y_2}|)$.

The symmetric situation is realized for an observer in frame $K_2$: looking at frame $K_1$, he sees the angle between axes $x_1$ and $y_1$ to be equal to $(\pi/2 + |\hat{\vartheta}_{x_2}| + |\hat{\vartheta}_{y_2}|)$, as shown in figure 2(b).

At the same time, we emphasize that systems $K_1$ and $K_2$ remain parallel to each other, according to the relativistic definition presented above. Indeed, for an observer in frame $K_1$, the components of velocity of frame $K_2$ on axes $x_1$ and $y_1$ are equal to $V \cos \alpha, V \sin \alpha$ (figure 2(a)), while for an observer in $K_2$, the components of velocity of $K_1$ on axes $x_2$ and $y_2$ are equal to
Thus, the relativistic definition of parallel coordinate systems is indeed fulfilled with respect to frames $K_1$ and $K_2$, in spite of the non-parallelism of their corresponding coordinate axes.

Now we return to the motional diagram of the three inertial reference frames $K_0$, $K_1$, and $K_2$, specified in figure 1, but this time we consider this diagram in frame $K_1$ (figure 3). One can see that in this figure, we have drawn axes $y_1$ and $y_2$ to be parallel to each other, and axes $x_1$ and $x_2$ to be non-parallel to each other. Let us demonstrate that this is actually the case for an observer in $K_1$, and concurrently determine the angle $\Omega$ between axes $x_1$ and $x_2$.

First of all, we point out that the motional diagrams in figures 1 and 3 imply the special Lorentz transformations between the pairs of frames $K_0$, $K_1$, and $K_0$, $K_2$, which thus are rotation-free. At the same time, the transformation between frames $K_1$ and $K_2$ is more complicated, so that we cannot assert a priori that this transformation should be a rotation-free transformation, either. Therefore equation (6), as well as the subsequent equations (7)–(10), which are valid with respect to rotation-free Lorentz transformations in general, are not applicable to frames $K_1$ and $K_2$ in figure 3. In these conditions, we propose the following way out, for the determination of mutual spatial orientation of the $x$- and $y$-axes of the two frames.

Suppose that during the motion of $K_2$, a designated segment $X_2$ on the axis $x_2$ at the appropriate time interval does intersect the $x$-axis of $K_1$, and, analogously, a designated segment $Y_2$ on the axis $y_2$ at the appropriate time interval, does intersect the $y$-axis of $K_1$. Concurrently, an observer in $K_1$ measures the time moments of intersection of the edge points of segment $X_2$ with axis $x_1$, as well as the time moments of intersection of the edge points of segment $Y_2$ with axis $y_1$. It is obvious that in the case where intersection of the edges of the moving segment $X_2$ with the axis $x_1$ happens simultaneously in the frame $K_1$, then this observer concludes that axis $x_1$ and segment $X_2$ (thus the entirety of axis $x_2$) are parallel to each other. An analogous result is derived with respect to the intersection of the moving segment $Y_2$ with axis $y_1$. Alternatively, if in frame $K_1$ the time moments of intersection of the edge points of segment $X_2$ with the axis $x_1$ differ from each other, then segment $X_2$ and thus the entirety of axis $x_2$ is not parallel to axis $x_1$ in the considered frame $K_1$. In this case, the inclination angle between axes $x_1$, $x_2$, can be calculated straightforwardly, if one knows the $x$- and $y$-components of the velocity of segment $X_2$ in $K_1$, and the corresponding time moments of intersection of its edges with axis $x_1$. 

![Figure 3. Diagram of motion of the inertial reference frames $K_0$ and $K_2$, as viewed by an observer in the inertial reference frame $K_1$.](image-url)
By analogy, one can designate segment $Y_2$ on axis $y_2$, and use a similar method to determine the inclination angle between axes $y_1$, $y_2$ for an observer in frame $K_1$.

Using this method, we show that with reference to the diagram in figure 3, axes $y_1$ and $y_2$ remain parallel to each other for an observer in $K_1$. The may be satisfactorily proved by demonstrating that for any segment $Y_2$ belonging to axis $y_2$, the intersection of its edge points with axis $y_1$ happens simultaneously for an observer in $K_1$.

As a first step to this proof, we notice the trivial parallelism of axes $y_0$ and $y_2$ for an observer in $K_0$. Next, we note that the relative motion of frames $K_1$ and $K_0$ happens along the $x$-axis, and hence, the simultaneous intersection of all points of segment $Y_2$ with axis $y_1$ is trivial for an observer in $K_1$. Thus, we conclude that, based on the diagram in figure 3, axes $y_1$ and $y_2$ remain parallel to each other in all frames under consideration.

Considering now a relative spatial orientation of the $x$-axes of frames $K_1$, $K_2$ and $K_0$, we note that the axes $x_2$ and $x_0$ are parallel to each other in both frames $K_2$ and $K_0$, given that these frames are related via special Lorentz transformation. Hence, we conclude that for an observer in $K_0$, the intersection of the $x_2$ and $x_0$ axes happens simultaneously for all of their points. However, this is no longer the case for an observer in $K_1$, where frame $K_0$ is moving with a constant velocity $u$ along the $x_1$-axis. Specifically, for any segment $X_2$ belonging to axis $x_2$, the intersection of its edges with axis $x_0$, as seen by an observer in $K_1$, happens at different time moments separated by the interval

$$\Delta t (K_1) \approx -\frac{X_2u}{c^2}, \quad (11)$$

to the adopted accuracy of calculation $c^{-2}$. Equation (11) follows from the Lorentz transformations for the time intervals in frames $K_1$ and $K_0$ and, from the physical viewpoint, it reflects a relativity in the simultaneity of events in these frames. Taking into account that in frame $K_1$, segment $X_2$ has the $y$-component of its velocity $V_y = v\sqrt{1 - u^2/c^2}$, we conclude that the time difference (11) corresponds to the difference of $y$-coordinates for the opposite ends of this segment

$$\Delta y (K_1) \approx -\frac{X_2uv}{c^2}. \quad (12)$$

Therefore, for an observer in frame $K_1$, the angle between segment $X_2$ (and thus the entirety of axis $x_2$) and axis $x_1$ is equal to

$$\Omega_x \approx \frac{\Delta y (K_1)}{X_2} \approx -\frac{uv}{c^2}, \quad (13)$$

where we have used equations (1a) and (1b) within the accuracy of calculation $c^{-2}$.

Comparing the spatial orientation of axes $x_2$, $y_2$ of frame $K_2$ in figures 2(a) and 3 (see figure 4), we can now reveal the common spatial rotation of system $K_2$ in figure 3 with respect to the system $K_2$ in figure 2(a), at the angle

$$\frac{\Omega_x}{2} \approx -\frac{uv}{2c^2} = \theta_{TW} \quad (14)$$

in the clockwise direction, which coincides with the Thomas–Wigner angle $\theta_{TW}$ defined by (5). This happens in full agreement with the general group properties of Lorentz transformations, where the succession of two transformations with non-collinear relative velocities (in our case $K_1 \rightarrow K_0 \rightarrow K_2$) entails an additional spatial rotation of coordinate systems $K_1$ and $K_2$ in comparison with the case of direct rotation-free transformation $K_1 \rightarrow K_2$. 


Figure 4. The coordinate system $K_2$, being related to $K_1$ via the rotation-free Lorentz transformation (reproduced from figure 2(a) and drawn in red), and the same coordinate system, being related to $K_1$ via successive Lorentz transformation $K_1 \rightarrow K_0 \rightarrow K_2$ (reproduced from figure 3 in black), differ from each other for an observer in $K_1$ by a common spatial rotation of the axes $x_2$ and $y_2$ in the clock-wise direction at the Thomas–Wigner angle $\theta_{TW}$, based on the known result of special relativity theory.

At the same time, our approach to Thomas–Wigner rotation, which apparently has not been previously applied, allows the disclosure of its actual physical mechanism. Specifically, with reference to figures 3 and 4, we find that the spatial orientation of axes $x_2$, $y_2$ of frame $K_2$, with respect to the corresponding axes of frame $K_1$, is defined via the appropriate composition of the angles $\theta_{TW}$ and $\theta_c$, resulting from the common rotation of coordinate systems and the scale contraction effect for moving $x$- and $y$-axes, correspondingly.

In particular, with respect to the angle $\Omega_y$ between axes $y_1$, $y_2$ in figure 4, as shown in the frame $K_1$, we obtain the equality

\[ \Omega_y = \theta_{cy} + \theta_{TW} = 0, \]

(15a)

insofar as, in the adopted accuracy of calculations, the angles $\theta_{TW}$ and $\theta_c$ have equal magnitudes and opposite signs (compare equations (10) and (5)). Thus, in figures 3 and 4, the spatial orientation of axis $y_2$ with respect to axis $y_1$ remains unchanged in spite of the joint action of the scale contraction effect and of Thomas–Wigner rotation.

Further, determining the spatial orientation of axis $x_2$ with respect to axis $x_1$ in figure 4, we obtain the angle the between these axes

\[ \Omega_x = \theta_{cx} + \theta_{TW} = -\frac{uv}{c^2}, \]

(15b)

(see equations (9) and (5)).

Looking at equations (15a) and (15b), one can conjecture that these equations have the general character, in the case of planar motion, of three inertial reference frames (in our case, in the $xy$-plane), which tells us that a visible orientation of the axes $x$ and $y$ is always defined by the angle

\[ \Omega_{x(y)} = \theta_{cx(y)} + \theta_{TW}. \]

(16)
One can demonstrate that this is actually the case, though, for brevity, we omit the straightforward proof of equation (16), which is based on the fact that at the non-vanishing x- and y- components of a relative velocity between inertial frames K₁ and K₂, the angles \( \theta_{cx} \) and \( \theta_{cy} \) always have opposite signs and equal magnitudes under rotation-free Lorentz transformations between these frames. In these conditions, the validity of equation (16), where the Thomas–Wigner rotational angle can be written in two equivalent forms, i.e.

\[
\theta_{TW} = \Omega_x - \theta_{cx} = \Omega_y - \theta_{cy},
\]

is closely related to the relativity of the simultaneity of events in different inertial reference frames, as we have shown in the analysis of the motional diagrams in figures 3 and 4.

Equation (17) allows us to disclose the actual physical mechanism of Thomas–Wigner rotation, which is shaped by a joint action of two relativistic effects: the contraction of a moving scale along the vector of its velocity in a frame of observation, and the relativity of the simultaneity of events for different inertial observers.

Addressing to equations (15a), (15b) and (16), we see that, in general, \( \Omega_x \neq \Omega_y \). This inequality explains the reason why, in the motional diagram presented in figures 3 and 4, frame K₂ is seen as non-Cartesian by an observer in frame K₁, and vice versa. Concurrently, it can be observed that in the particular case where \( \theta_{cx} = \theta_{cy} = 0 \), equations (15) and (16) yield the equality

\[
\Omega_x = \Omega_y = \theta_{TW},
\]

which means that the Thomas–Wigner rotation leaves Cartesian for each other both inertial frames under consideration.

This particular case is realized for the motional diagram in figure 5, where the pairs of frames K₀, K₁ and K₀, K₂ are related to each other via rotation-free Lorentz transformations, having velocities \( V_{01}\{u,0,0\} \) and \( V_{02}\{u,v,0\} \), respectively. In this case, the relative motion of frames K₁ and K₂ happens only along the y-axis. Then, according to the general rule, the transformation from K₁ to K₂ is composed from the special Lorentz transformation along the y-axis, which keeps frames K₁, K₂ Cartesian for each other, followed by a subsequent Thomas–Wigner rotation of coordinate axes of system K₂ at the angle \( \theta_{TW} \) defined by equation (5), which keeps the axes orthogonal to each other. Thus, under the special rotation-free Lorentz transformation from K₁ to K₂, we get \( \theta_{cx} = \theta_{cy} = 0 \), so that we actually arrive at equation (18).

Let us show that the result (18) for the motional diagram in figure 5 can be again understood via (i) the scale contraction effect along the relative velocity \( V_{02}\{u,v,0\} \) between frames K₀, K₂, as well as (ii) a relativity of the simultaneity of events between the frames K₀, K₁, moving at relative velocity \( V_{01}\{u,0,0\} \).

In this way, we can see that for an observer in frame K₀, the coordinate axes of K₂ are no longer orthogonal to each other, as explained above in the analysis of a similar motional diagram in figure 2. In particular, in frame K₀, axis \( x_2 \) experiences spatial rotation with respect to axis \( x_0 \) at angle \( \theta_{cx} \) in the clockwise direction, defined by equation (9), whereas axis \( y_2 \) is turned out with respect to axis \( y_0 \) at angle \( \theta_{cy} \) in the counter-clockwise direction, according to equation (10). The corresponding directions of axes \( x_2 \) and \( y_2 \) in frame K₀ are depicted in figure 5 as continuous black lines. We also note that equations (9) and (10) yield

\[
|\theta_{cx}| = |\theta_{cy}| = uv/2c^2.
\]

Further, on we point out that for the motional diagram in question, frame K₁ moves only along axis \( x_0 \) of K₀. Hence, within the adopted accuracy of calculations, the spatial orientation of the axis \( y_2 \), as seen in K₀, remains the same for an observer in frame K₁ and thus constitutes
Figure 5. The inertial reference frames $K_0$, $K_1$ and $K_2$ are related to each other by rotation-free Lorentz transformations having relative velocities $V_{01}(u,0,0)$ and $V_{02}(u,v,0)$, respectively, and frames $K_1$ and $K_2$ move with respect to each other only along axis $y$. The black continuous lines indicate the spatial orientation of the $x$- and $y$-axes of all three systems for an observer in frame $K_0$. The dotted red lines show the spatial orientation of the axes $x_2, y_2$, as seen in frame $K_1$. As a result, from the viewpoint of the observer in $K_1$, the Cartesian system $K_2$ experiences the common rotation in the counter-clockwise direction at the Thomas–Wigner angle $\theta_{TW}$.

In order to determine the spatial orientation of axis $x_2$ in frame $K_1$, we mark segment $X_2$ as belonging to axis $x_2$, and first determine the difference of time moments of intersection of the opposite ends of this segment with the axis $x_0$. One can see from figure 5 that this difference is defined by the expression

$$\Delta t_{K_0} = \frac{X_2 \theta_{cx}}{u} \approx -X_2 \frac{u}{2c^2},$$

where we have used equation (9).

Next, we determine the corresponding time difference in frame $K_1$ by applying the Lorentz transformation for the time intervals between $K_0$ and $K_1$. Taking into account that these frames move with respect to each other with a constant velocity $u$ along axis $x$, and using equation (20), we then obtain

$$\Delta t_{K_1} = \gamma \left( \Delta t_{K_0} + \frac{X_2 u}{c^2} \right) \approx X_2 \frac{u}{2c^2}.$$  

Thus, comparing equations (20) and (21), we see that the time intervals $\Delta t_{K_0}$, $\Delta t_{K_1}$ have equal values and opposite signs. This indicates that an observer in frame $K_0$ sees the clockwise rotation of axis $x_2$ at angle $\theta_{cx}$ with respect to axis $x_1$, whereas, due to a relativity of simultaneity of events in the frames $K_0$ and $K_1$, an observer in $K_1$ sees a counterclockwise rotation of axis $x_2$ with respect to axis $x_1$ at the same angle $\theta_{cx}$, which thus should be identified with the angle of Thomas–Wigner rotation (see figure 5, where the direction of the axis $x_2$ for an observer in $K_1$ is shown as a dashed red line). Indeed, for an observer in $K_1$, the rotational angles between
axes \( y_2, y_1 \) and between axes \( x_2, x_1 \) have the same sign and the same value, as defined by equation (19), indicative of Thomas–Wigner rotation between the Cartesian systems \( K_1 \) and \( K_2 \).

At the same time, we again emphasize that the motional diagram in figure 5 represents a rather specific case, characterized by a special rotation-free Lorentz transformation between frames \( K_1 \) and \( K_2 \). For the most part, in physical situations, this condition is not usually fulfilled; hence, in general, Thomas–Wigner rotation involves non-Cartesian inertial frames.

This remarks occurs significance in the physical interpretation of the Thomas precession, which is analyzed in the next section.

3. Thomas precession: some overlooked features

As is known, Thomas precession occurs for a particle with spin, moving along a curved path (where the particle should be considered as a point, in order to exclude any emergent relativistic effects for extended bodies [20]), and the frequency of Thomas precession in the laboratory frame is defined by the expression (e.g. [6])

\[
\omega_T = \left( 1 - \frac{1}{\gamma} \right) \frac{v \times \dot{v}}{v^2},
\]  

(22)

Here, \( v \) is the velocity of the particle, and \( \dot{v} = d\dot{v}/dt \) is its acceleration.

We would like to remind that in the derivation of this equation, it is tacitly assumed that at any time moment \( t \), the Lorentz transformation from the laboratory frame \( K_0 \) to frame \( K(t) \), co-moving with the particle, is rotation-free. Furthermore, it is adopted that the Lorentz transformation from frame \( K(t) \) to frame \( K(t + dt) \), co-moving with the particle at time moments \( t \) and \( t + dt \), respectively, is always rotation-free. In this case, the successive Lorentz transformations \( K_0 \rightarrow K(t) \rightarrow K(t + dt) \) entail the Thomas–Wigner rotation of system \( K(t + dt) \) with respect to system \( K_0 \), which represents the origin of the Thomas precession [1]. Designating, as before, the angle of Thomas–Wigner rotation between systems \( K_0 \) and \( K(t + dt) \) via \( \theta_{TW} \), we obtain the value of the frequency of Thomas precession as

\[
\omega_T = \frac{d\theta_{TW}}{dt},
\]  

(23)

which, at the relative velocity \( \dot{v} dt \) between frames \( K(t) \) and \( K(t + dt) \), and the known expression for the Thomas–Wigner angle \( \theta_{TW} \), straightforwardly yields equation (22) (see e.g. [20]).

At this point, we must emphasize that the available attempts to provide a physical interpretation of the Thomas precession tacitly adopt that angle \( \theta_{TW} \) in equation (23) describes a visible rotation of the axes of the proper system of an electron with respect to the coordinate system of a laboratory frame. Clearly, this assumption implies that frames \( K(t) \) and \( K(t + dt) \) are both Cartesian; however, as we have shown in section 2, this is generally not the case, and a visible orientation of coordinate axes of frame \( K(t + dt) \) with respect to the corresponding coordinate axes of the laboratory frame \( K_0 \) is given instead by the angles \( \Omega_x \), and \( \Omega_y \), as in equation (16), which is based on the case of motion only in the \( xy \)-plane.

This signifies that the physical mechanism of the Thomas precession of a particle with spin should be reanalyzed in view of the results obtained above, based on equations (15)–(17). Below, we analyze the implications of these equations, based on two specific examples: the motion of a particle with spin along a circular orbit, and the motion of a particle along an open path in the \( xy \)-plane in a special case, where the \( x \)-component of velocity remains constant, while the \( y \)-component of velocity linearly increases with time.
3.1. Circular motion of particle with spin

In the classical approach, this case corresponds to the circular rotation in the $xy$-plane of a classical electron, having charge $e$ and spin $s$, around an immovable nucleus, with charge $Ze$, and practically infinite rest mass. It is well known that this problem was considered by Thomas in the derivation of the frequency of Thomas precession of electron’s spin and determination of spin–orbit coupling in the classical approach [1]. In the current analysis, in addition to Thomas’s approach, we would like to determine the spatial orientation of the coordinate axes of the proper frame of the electron, which influences the visible direction of spin of classical electrons with respect to the corresponding coordinate axes of laboratory frame $K_0$ (see figure 6).

In order to solve this problem, we introduce a rotational frequency $\omega$ for the electron, and determine the $x$- and $y$-components of its orbital velocity $v$, which for a laboratory observer have the values

\begin{align}
v_x &= -v \sin \omega t, \\
v_y &= v \cos \omega t,
\end{align}

where $t$ indicates the time remaining between $(n-1)T_e \leq t < nT_e$; here, $T_e$ is the rotational period of the electron around the nucleus, and $n$ is the number of revolutions of the electron around the nucleus during the total time of observation.

Further, using equation (22), we determine the frequency of Thomas precession of the electron’s spin, which for a circular motion (where $\dot{\mathbf{v}} = \mathbf{\omega} \times \mathbf{v}$) is equal to

$$|\omega_T| = \left(1 - \frac{1}{\gamma^2}\right) \frac{\mathbf{v} \times (\mathbf{\omega} \times \mathbf{v})}{v^2} = \left(1 - \frac{1}{\gamma^2}\right) \omega \approx \frac{v^2}{2c^2} \omega.$$  

Here, we have used the vector identity $\mathbf{v} \times (\mathbf{\omega} \times \mathbf{v}) = \mathbf{\omega} v^2 - \mathbf{v} (\mathbf{v} \cdot \mathbf{\omega}) = \mathbf{\omega} v^2$ and applied an approximation to the order $c^{-2}$. Thus, at $v \ll c$ (which is the case for the classical electron in the hydrogen atom), the frequency of Thomas precession $\omega_T \ll \omega$. Therefore, for each rotational period, we can consider the current angle of Thomas precession $\varphi_T \approx \omega_T n T_e$ as a constant value.
Thus, the velocity components (24a) and (24b), being defined with respect to the x- and y-axes of the laboratory frame, become equal to
\begin{align}
v_{xe} &= -v \sin(\omega t - \phi_T), \\
v_{ye} &= v \cos(\omega t - \phi_T),
\end{align}
with respect to the corresponding axes of the electron’s co-moving frame.

Substituting components (25a) and (25b) into equations (9) and (10), we obtain
\begin{align}
|\theta_{cx}| &= |\theta_{cy}| = \frac{v^2}{2c^2} \sin(\omega t - \phi_T) \cos(\omega t - \phi_T)
\end{align}
to the accuracy $e^{-2}$.

As an example, in figure 6 we show the current angle of Thomas precession $\phi_T = \omega_T n T_e$, as well as the angles $\theta_{cx}$ and $\theta_{cy}$, at the time moment $t$. A visible orientation of the axes $x_e$ and $y_e$ of the proper system of electron $K_e$, defined by the angles $\Omega_x$ and $\Omega_y$, via equation (16), is shown in red.

We see that, in general, the electron system $K_e$ is not Cartesian for an observer attached to the immovable nucleus, with the exception of time moment $t = \phi_T/\omega$, when both angles $\theta_{cx}$, and $\theta_{cy}$ become equal to zero (see equation (26)). Thus, at this time moment, a laboratory observer sees the Cartesian electron frame turned around the axis $z$ at the angle $\phi_T$ of Thomas precession.

Further, based on equation (16), we determine the precession frequency of visible orientations of the axes $x_e$ and $y_e$:
\begin{equation}
\frac{d\Omega_{x(y)}}{dt} = \frac{d\theta_{x(y)}}{dt} + \omega_T.
\end{equation}

Combining equations (25)–(27), and taking into account that angles $\theta_{cx}$ and $\theta_{cy}$ always bear opposite signs, in the case of the counterclockwise rotation of electron in figure 6:
\begin{align}
\frac{d\Omega_x}{dt} &= \omega_T (1 + \cos 2(\omega t - \phi_T)) , \\
\frac{d\Omega_y}{dt} &= \omega_T (1 - \cos 2(\omega t - \phi_T)) ,
\end{align}
where we consider the current angle of Thomas precession $\phi_T$ as the constant value within each rotational period of electron due to the inequality $\omega_T \ll \omega$.

Note that in the case of the clockwise rotation of the electron, the angles $\Omega_x$, and $\Omega_y$ replace each other in equation (28).

Equation (28) characterizes the rates of time variation for the angles $\Omega_x$ and $\Omega_y$, which lie between 0 and $2\omega_T$, and determine the momentary frequency of the Thomas precession for the visible orientation of electron spin. We note that the average frequency of the Thomas precession of electron spin is always equal to $\omega_T$, which follows from the equality
\begin{equation}
\frac{d\Omega_x}{dt} = \frac{d\Omega_y}{dt} = \omega_T.
\end{equation}

Thus, equation (29) demonstrates that the revealed fluctuation of the precession of the x- and y-axes of the proper frame of electron around the frequency $\omega_T$, based on equation (28), does not affect the average energy associated with the Thomas precession, which remains equal to
\begin{equation}
U_T = s \cdot \omega_T.
\end{equation}
The classical electron $e$ along with its proper reference frame $K_e$ is moving with respect to laboratory frame $K$ with a constant velocity $u$ along axis $x$, and is accelerated along axis $y$ with the acceleration value $a$. We aim to determine the frequency of Thomas precession of electron spin in frame $K$, as well as in an inertial frame $K'$, moving with respect to $K$ at constant velocity $u$ along the $x$-axis (the Bäcyr configuration [33]).

As we have recently have shown [15], the energy term alone (30) determines the observed spin–orbit interval in a hydrogen-like atom, insofar as, at the correct expression for the energy of the electric/magnetic dipole in an electromagnetic field [16], all energy terms of electromagnetic origin mutually cancel each other.

Thus, our analysis does not lead to any new consequences with respect to the dynamics of an orbiting electron with spin; at the same time, however, equations (25)–(29) obtained above reveal some kinematical features of Thomas precession of spin in classical electrons, which seems had been missed to the moment.

### 3.2. Motion of particle with spin along an open path

In this subsection, we consider the case of the motion of a particle with spin along an open curved path, where its velocity $u$ and acceleration $a$ are orthogonal to each other at each time moment; for example, $u$ represents a constant value and lies along axis $x$, while $a$ is collinear to axis $y$, as shown in figure 7. We further assume that the force acting on the particle along the $y$-axis is constant (e.g. the motion of the classical electron $e$ with the spin $s$ in the constant electric field $E$, which is collinear to the axis $y$). Note that in this configuration, in the absence of force on the electron along the $x$-axis, its velocity $u$ along this axis cannot remain constant. Indeed, due to the conservation of momentum along the axis $x$, we have the equality

$$\frac{d}{dt} \gamma mu = \frac{d\gamma}{dt} mu + \gamma m \frac{du}{dt} = 0,$$  \hspace{1cm} (31)

where $\gamma$ is the Lorentz factor.

Insofar as $d\gamma/dt$ increases with time, while the electron is accelerated along the $y$-axis, equation (31) shows that $du/dt$ is not equal to zero, and represents a negative value. Therefore, the assumed constancy of the velocity $u$ implies the presence of some appropriate force along the $x$-axis, which maintains the constancy of $u$, while the Lorentz factor of the electron $\gamma$ increases. We will not analyze the dynamical side of this problem any further, and instead focus our attention on the kinematics of the moving electron in figure 7.
In this way, we designate \( v(t) \) as the velocity of the electron along axis \( y \) at a given time moment \( t \), and determine the dependence of acceleration on time, applying the motional equation of the electron along axis \( y \), i.e.

\[
\frac{d}{dt}(\gamma mv) = eE. \tag{32}
\]

Assuming the constancy of \( u \), we obtain the following, based on equation (32):

\[
\frac{dv}{dt} = a = \frac{eE}{\gamma^3 m}, \tag{33}
\]

which shows that for the problem under consideration, the acceleration \( a \) along axis \( y \) decreases over time, with the increase of the Lorentz factor \( \gamma \).

This means that the frequency of the Thomas precession for electron spin, defined via equation (22) as

\[
(\omega_T)_e = \left(1 - \frac{1}{\gamma}\right) \frac{ua}{v^2 + u^2} = \left(1 - \frac{1}{\gamma}\right) \frac{eauE}{(v^2 + u^2) \gamma^3 m} \tag{34}
\]

also decreases with time, and tends to zero at large \( \gamma \).

For simplicity, we will not analyze the strong relativistic case any further, and, as above, will conduct our analysis in the weak relativistic limit, corresponding to the accuracy of calculations \( c^{-2} \). As a result, the velocities \( u \) and \( v(t) \) are both much smaller than \( c \).

Next, we determine a visible orientation of the \( x \)- and \( y \)-axes of the frame \( K_e \) for a laboratory observer \( K \) on the basis of equation (16), as well as the frequencies of spatial rotation for both axes, which contribute to the frequency of Thomas precession for electron spin \( s \).

Here, one should notice that the magnetic dipole moment \( m \) of the electron, associated with its spin \( s \), also involves the Larmor precession around the vector of magnetic field \( B \), existing in the frame, and co-moving with the electron. However, in the following, we consider only the Thomas precession of electron spin, defined by the precession of the coordinate axes of the co-moving frame of the electron.

In order to solve this problem, we introduce two Lorentz frames \( K_e(t) \) and \( K_e(t + dt) \) co-moving with the accelerating electron at time moments \( t \), and \( t + dt \), respectively, which thus move with respect to each other with a relative velocity of \( a dt \) along axis \( y \), and both move at constant velocity \( u \) along the \( x \)-axis of the laboratory frame \( K \). This motional diagram exactly reproduces the motional diagram in figure 3 considered above, where frame \( K_0 \) is associated with frame \( K_e(t) \), and frame \( K_2 \) with \( K_e(t + dt) \), while frame \( K_1 \) represents the laboratory frame of observation. Therefore, in the adopted accuracy of calculations \( c^{-2} \), we can directly use the results obtained for the diagram in figure 3. Hence, during the acceleration of the electron along the \( y \)-axis, the spatial orientation of this axis of its proper frame remains unaltered (see equation (15a)). From the physical viewpoint, this means that the spatial turn of axis \( y \) of \( K_e \), at the angle \( \theta_{y_e} \), caused by the scale contraction effect for this axis, as seen in laboratory frame \( K \), is exactly balanced by the angle of Thomas–Wigner rotation \( \theta_{TW} \) for an accelerated electron, insofar as for the configuration in figure 7, angles \( \theta_{y_e} \) and \( \theta_{TW} \) have equal values, but opposite signs.

In other words, at those time moments when electron spin \( s \) is collinear with axis \( y_e \), then the frequency of Thomas precession of system \( K_e \), as seen by a laboratory observer in \( K \), is exactly counterbalanced by the time derivative of the angle \( \theta_{y_e} \), caused by the scale contraction effect for the \( y \)-axis of frame \( K_e \) along the direction of its motion in \( K \). As a result, at such time moments, the laboratory observer sees a non-changed spatial orientation of axis \( y_e \), and measures only the Larmor precession of electron spin.
On the other hand, we note that with regard to axis $x_e$, the angle of Thomas–Wigner rotation $\theta_{TW}$, and angle $\theta_{cx}$ have the same value and the same sign. Therefore, for those time moments when electron spin $s$ is collinear with axis $x$, its rotational frequency due to Thomas precession, as seen in laboratory frame $K$, has twice the value $\omega_T$ estimated via equation (22).

We again emphasize that this result is valid only in the weak relativistic limit, corresponding to the accuracy of calculation $c^{-2}$. When this limit is abandoned, we must take into account the decrease of the frequency of Thomas precession with the increase in the $\gamma$-factor of the electron (equation (35)), which tends to zero in the strong relativistic limit. Therefore, when the velocity of the electron in frame $K$ tends to $c$, a visible orientation of the $x$- and $y$-axes becomes fully determined by angles $\theta_{cx}$ and $\theta_{cy}$, regardless of any pre-history relating to the Thomas precession of system $K_e$. Specifically, one may straightforwardly derive that in such an ultra-relativistic case, where the components of velocity of the electron in frame $K$ tend to \( \left\{ u, c\sqrt{1 - \frac{u^2}{c^2}}, 0 \right\} \), then angles $\theta_{cx}$ and $\theta_{cy}$ acquire the values

\[
\theta_{cx} = -\frac{u}{c}, \quad \theta_{cy} = \frac{\pi}{2} - \frac{u}{c}
\]

(35)

at $u \ll c$. This equation shows that for a laboratory observer, both $x$- and $y$-axes become practically anti-parallel to each other, and constitute the angle $ulc$ with axis $x$.

Finally, it is interesting to introduce an inertial observer with a proper frame $K'$, moving at constant velocity $u$ in frame $K$ (see figure 7) during the entire process of acceleration of an electron. In frame $K'$, the electron always moves along axis $\gamma'$, so that frame $K_e$, co-moving with the electron, appears Cartesian at any time moment. Hence, in frame $K'$, both angles $\theta'_{cx}$, and $\theta'_{cy}$ are equal to zero at any time moment. This result, in general, is not surprising, and only reflects the relativity of the scale contraction effect. A more important observation is related to the fact that in frame $K'$, the velocity of electron $v'(t)$ and its acceleration $a'(t)$ are both directed along the $\gamma'$-axis. Hence, their vector product is equal to zero at any time moment, which means, according to equation (22), that the frequency of Thomas precession of electron spin should be identically equal to zero in frame $K'$, in spite of its finite value in $K$. This result seems at odds with classical causality, and constitutes the essence of the Bacry paradox [33].

The resolution of this paradox has already been presented in [29], and is based on the fact that the velocity parameter entering into an expression (22) for the frequency of Thomas precession should be associated only with the rotation-free Lorentz transformation from the frame of observation to the frame, co-moving with a particle with spin. For the problem sketched in figure 7, such a rotation-free Lorentz transformation by supposition is carried out between frames $K$ and $K_e$, and hence, in the case of rotation-free transformation from $K$ to $K'$, we obtain that the corresponding transformation from $K'$ to $K_e$ cannot be rotation-free due to the non-commutativity property of the Lorentz transformation. Therefore, a tacit assumption by Bacry with respect to the parallelism of the corresponding axes of the frames $K'$ and $K_e$ was erroneous, and by eliminating this mistake, the paradox disappears [29].

4. Conclusion

We have shown that the Thomas–Wigner rotation of the coordinate axes of inertial reference frames, emerging under successive space-time transformations with non-collinear velocities, finds its consistent physical interpretation on the basis of the well-known relativistic effects of (i) the contraction of moving scale and (ii) the relativity of the simultaneity of events for different inertial observers. We demonstrated the validity of this statement for the motional
diagram in figure 3, where two inertial frames $K_0$ and $K_2$ are moving in the third inertial frame $K_1$, with mutually orthogonal velocities. In this case, we have found that under successive rotation-free transformations $K_1 \rightarrow K_0 \rightarrow K_2$, the direction of coordinate axes of $K_2$ for an observer in $K_1$ is determined by two effects: the *contraction of moving scale* defined by the motion of $K_2$ in $K_1$, and the *relativity of simultaneity* of events, defined by the relative motion of $K_1$ and $K_0$. As the result of the combined action of these relativistic effects, an observer in $K_1$ fixes a rotation of coordinate systems $K_2$ at the Thomas–Wigner angle $\theta_{TW}$, as compared with the case where a direct rotation-free Lorentz transformation between $K_1$ and $K_2$ would be applied (see figure 4). We particularly emphasize the fact that the contraction effect for coordinate axes of frame $K_2$ along the direction of its motion in $K_1$ makes these frames mutually non-Cartesian for each other, as shown in figure 2, and the Thomas–Wigner rotation of the axes of $K_2$, with respect to the corresponding axes of $K_1$, also renders both systems non-Cartesian for each other. As a result, the spatial orientation of the $x$- and $y$-axes of $K_2$ for an observer in $K_1$ depends both on the scale contraction effect for each axis, different, in general, for each coordinate axis, as well as the Thomas–Wigner rotation, being common for both axes; see equation (16).

Based on these results, we reanalyzed the Thomas precession in two particular cases: (1) the circular motion of a classical electron around a heavy nucleus, and (2) the motion of a classical electron along an open path, where the electron’s velocity and its acceleration are mutually orthogonal to each other. In the case of circular motion in the plane $xy$, we have shown that the frequency of Thomas precession for visible spatial orientation of the $x$- and $y$-axes oscillates near the average value of $\omega_1$, and that this result is also valid for the visible orientation of the spin of the classical electron. In this case, the averaged energy associated with the Thomas precession is given by equation (30), and does not affect the value of spin-orbit coupling in hydrogen-like atoms, as originally estimated by Thomas [1].

In the case of motion of the classical electron along an open path, in the particular case where the electron’s initial velocity $u$ and its acceleration $a$ are mutually orthogonal to each other (figure 7), we obtain in the weak relativistic limit that a visible orientation of axis $y_e$ of the proper electron’s frame $K_e$ in the laboratory frame $K$ remains unchanged, whereas the orientation of axis $y_e$ changes over time at double the frequency of the Thomas precession estimated via equation (22). With further increases in the velocity of the electron $v(t)$ along axis $y$, when its Lorentz factor $\gamma$ becomes much greater than unity, the frequency of Thomas precession of the electron’s spin tends to zero (equation (34)), and the spatial orientations of the $x$- and $y$-axes become fixed, as defined by equation (35), from the perspective of a laboratory observer.

Thus, the analysis and discussion presented above allow us to better understand the physical implications of Thomas–Wigner rotation and the Thomas precession, and to cover some points which have to date been overlooked in the physical interpretation of these effects.

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