MASSEY PRODUCTS FOR GRAPH HOMOLOGY

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Abstract. This paper shows that the operad encoding modular operads is Koszul. Using this result we construct higher composition operations on (hairy) graph homology which characterize its rational homotopy type.

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1. Introduction.

Graph complexes are combinatorial objects which can be used to compute invariants of topological spaces. They were introduced by Kontsevich [Kon93], [Kon94] building on earlier combinatorial models for moduli spaces [Pen87] and incorporating influence from Feynman diagrams in quantum field theory. Depending on the particular combinatorics of the graphs involved, graph complexes may be used to calculate cohomology of moduli spaces of Riemann surfaces [Kon98], moduli spaces of tropical curves [CGP19], or embedding spaces of manifolds [ALV07], [AT14]. Further variants of graph complexes may be used to study the Grothendieck-Teichmüller Lie algebra [Wil15] or automorphisms of free groups [Kon93], [CV03].

To encode the construction of graph complexes, Getzler and Kapranov introduced in [GK98] the notions of modular operads and the Feynman transform, which we denote by \( \text{FT} \). With this notion, graph homology with labels in a cyclic operad \( \mathcal{O} \) may be defined as:

\[
G_{\mathcal{O}} := H_{\ast}(\text{FT}(\iota ! \mathcal{O}))
\]

where \( \iota ! \mathcal{O} \) is the extension of \( \mathcal{O} \) to higher genus by 0. The functor \( \text{FT} \) is homotopy involutive and preserves quasi-isomorphisms. This means that if \( \text{FT}(\iota ! \mathcal{O}) \) were equivalent to its own homology we could conclude \( \text{FT}(G_{\mathcal{O}}) \sim \iota ! \mathcal{O} \). This would be a powerful computational tool because it would imply, among other things, that the graph homology was generated in genus 0. Alas this formality property is rarely present, and the act of taking homology loses information about the homotopy type of the Feynman transform.

In this paper we give a way to systematically account for this loss of information by constructing an analog of Massey products for modular operads. Taking a suitable generalization of the Feynman transform which incorporates these higher operations, which we denote \( \text{ft} \), we do indeed find

\[
\text{ft}(G_{\mathcal{O}}) \sim \iota ! \mathcal{O}.
\]

In constructing these higher operations, our model to follow comes from rational homotopy theory. The cochain complex of a simply connected topological space need not be equivalent to its cohomology in the category of commutative algebras, but there are higher cohomology operations which fit together to form a homotopy commutative algebra such that the cochains and cohomology are equivalent in this larger category (see [Kad09], after [Sul77]). This fundamental example motivated the successful homotopy transfer theory for algebras over operads [LV12], [Ber14].
in which the property of Koszulity plays a fundamental role in furnishing small cofibrant resolutions encoding homotopy invariant structure. In this paper we endeavor to resolve the composition in modular operads, and to this end our first needed result is:

**Theorem A.** The operad encoding modular operads is Koszul.

The notion of Koszulity used in Theorem A requires a modest generalization of the state-of-the-art. In order to encode modular operads as algebras over a quadratic operad, we use a notion of colored operads where the colors form not just a set but a groupoid. Generalizing Koszul duality and homotopy transfer theory to this context is a necessary but straight-forward exercise and is carried out in Section 2.

With this generalization of Koszul duality in hand, we first define a quadratic groupoid colored operad \( \mathbb{M} \) whose algebras are modular operads, we then identify its quadratic dual \( \mathbb{M}^! \) to be a suspension of the operad encoding \( K \)-twisted modular operads, (denoted \( M_k \)), and we prove Theorem A by showing that the natural map

\[
\Omega(M^*_k) \xrightarrow{\sim} \mathbb{M}
\]

is a levelwise quasi-isomorphism (here \( \Omega \) denotes the cobar construction). We define a weak modular operad to be an algebra over \( \Omega(M^*_k) \).

Returning to the example of graph homology, we apply our results in the following way. First we use homotopy transfer theory to give \( G_O \) the structure of a weak modular operad such that

\[
G_O \sim \text{FT}(\iota_!(O))
\]

(where the right hand side has its (strong) modular operad structure). This endows graph homology with operations of the form:

\[
mp_\gamma : \left( \bigotimes_{i=1}^r G_O(v_i) \right) \rightarrow G_O(v_0)
\]

where \( \gamma \) is a (modular) graph of vertex type (color) \( (v_1, \ldots, v_r; v_0) \) having at least one edge. This operation has degree one less than the number of edges of \( \gamma \). We then prove the following structural result:

**Theorem B.** Let \( O \) be a Koszul cyclic operad with Koszul dual \( O^! \). Every graph homology class in \( G_O \) is either a generator of \( O^! \) or is in the image of some Massey product.

To prove this result, as well as to the organize this family of new operations, we generalize the Feynman transform to this setting. More precisely we define the weak Feynman transform as a pair of functors

\[
\text{ft} : \{\text{weak modular operads}\} \xrightarrow{\sim} \{\text{weak \& modular operads}\} : \text{ft}
\]

such that \( \text{ft}^2 \sim \text{id} \) and that \( \text{ft} \) preserves \( \infty \)-quasi-isomorphisms. In particular, since \( \text{ft}(G_O) \sim \iota_! O \), every graph homology class in genus \( g \geq 1 \) which is not a boundary is also not a cycle.

Since graph homology is not just a weak modular operad, but a (strong) modular operad as well, the differential in its weak Feynman transform is a sum of two differentials; the classical Feynman transform differential and a differential corresponding to Massey products of 2 edges or more. Filtering \( \text{ft}(G_O) \) by internal degree has the effect of isolating the (classical) Feynman transform differential, thus:

**Theorem C.** There is a spectral sequence whose first page is the homology of \( \text{FT}(G_O) \) and which converges to \( \iota_! O \). The higher differentials correspond to sums of linear duals of Massey products.

This is the analog of the Milnor-Moore spectral sequence from rational homotopy theory \cite{FHT01}. The computational facility of this and a related spectral sequence considered in subsection 4.5 is that it allows us to, roughly speaking, pair classes in \( H_*(\text{FT}(\iota_! O)) \) with classes in \( H_*(\text{FT}(\iota_!(O^!))) \) for \( O \) a Koszul cyclic operad, and \( \iota_* \) denoting the modular envelope.

For example, when \( O \) is the Lie operad we may compare Lie graph homology, denoted \( H_*(\Gamma_{g,n}) \) after \cite{CHKV16}, with a graph complex computing the top weight homology of the moduli space of punctured Riemann surfaces, denoted \( H_*(\Delta_{g,n}) \) after \cite{CGP19}, via:
Corollary 1.1. Fix \((g,n)\), a pair of natural numbers with \(g \geq 1\) and \(2g + n \geq 3\).

1. There is an upper half plane spectral sequence whose 0-page is \(\text{FT}(H_\ast(\Gamma_{g,n}))(g,n)\), whose bottom row computes \(H_\ast(\Delta_{g,n})\) and which converges to \(0 \cong u(Lie)(g,n)\).

2. There is an upper half plane spectral sequence whose 0-page coincides with \(\text{FT}(H_\ast(\Delta_{g,n}))\) as graded vector spaces, whose bottom row computes \(H_\ast(\Gamma_{g,n})\) and which converges to \(k \cong \iota_\ast(\text{Com})(g,n)\).

In the latter case the differential \(d^0\) consists of the loop-free terms of the Feynman transform differential, see subsection 4.5.2.

Although the results of this paper were developed with graph homology in mind, the homotopy transfer theory, weak Feynman transform and associated spectral sequences may of course be used to study other examples of dg modular operads. Let me conclude this introduction by highlighting one interesting future direction of this work. It involves modular operads built from the moduli spaces of Riemann surfaces of genus \(g\) one interesting future direction of this work. It involves modular operads built from the moduli spaces of Riemann surfaces of genus \(g\) and \(n\) punctures \(M_{g,n}\) and their Deligne-Mumford compactifications \(\overline{M}_{g,n}\). In particular \(H_\ast(M_{g,n})\) forms a modular operad which is formal [GSNPR05], while \(H_\ast(M_{g,n})\) forms a \(\mathfrak{g}\)-twisted modular operad which is not formal [APT]. In this example the spectral sequence by edge filtration may also be constructed topologically and may used to address these formality questions, and [KSV96] should be highlighted as one important and early source of these ideas.

The notion of weak modular operads, and in particular \(\infty\)-quasi-isomorphisms of such, suggest reinterpreting the results of [GK98] Proposition 6.11 as the existence of a weak \(\mathfrak{g}\)-twisted modular operad structure on \(H_\ast(M_{g,n})\) such that there exists an \(\infty\)-quasi-isomorphism of weak \(\mathfrak{g}\)-twisted modular operads:

\[
\text{FT}(H_\ast(\overline{M}_{g,n})) \sim H_\ast(M_{g,n})
\]

or equivalently a (strong) morphism of modular operads:

\[
\text{ft}(H_\ast(M_{g,n})) \sim H_\ast(\overline{M}_{g,n}).
\]

It would be interesting to give an explicit description of the weak \(\mathfrak{g}\)-twisted modular operad structure on \(H_\ast(M_{g,\leq 1,n})\). The underlying cyclic operad is formal, while the genus 1 spaces should receive higher operations corresponding to Getzler’s elliptic relation [Get97] on \(\overline{M}_{1,4}\). In particular, understanding the Massey products on \(H_\ast(M_{g,\leq 1,n})\) corresponds to understanding the differential on \(\text{ft}(H_\ast(M_{g,\leq 1,n}))\) which corresponds to understanding relations in \(H_\ast(\overline{M}_{1,n})\), and so the results of [Pet13] suggest this is a tractable problem.

Acknowledgement. I would like to thank Dan Petersen for sharing with me his observation that non-formality of the modular operad \(H_\ast(\Gamma_{g,n})\) may be seen as a consequence of the non-surjectivity of the assembly map. I would also like to thank Ralph Kaufmann for the insight that using groupoid colors/Feynman categories allows one to encode modular operads via a homogeneous-quadratic presentation. This paper has also benefited from helpful conversations with Alexander Berglund, Martin Markl and Bruno Vallette.

2. Koszul duality for groupoid colored operads.

In this section we:

- Give the definition of an operad colored by a groupoid.
- Define bar-cobar duality, quadratic duality and Koszulity for such operads.
- Give the bar-cobar construction and homotopy transfer theory for algebras over Koszul objects.

Our starting point is the work of Van der Laan [VdL03] who resolves the colored operad encoding non-symmetric operads using Koszul duality. This colored operad is homogeneous quadratic and its colors form a set. In order to encode modular operads as homogeneous quadratic it will be necessary to allow operads whose colors form a groupoid.

Groupoid colored operads were introduced by Petersen [Pet13] and simultaneously developed from the perspective of symmetric monoidal categories in [KW17]. It has been shown [BKW18] that
groupoid colored operads may also be viewed as set colored operads in which the automorphisms in the groupoid are viewed as unary operations in the operad. However, viewing our operads of interest as groupoid colored will be desirable for two reasons. First the passage from groupoid colored operads to set colored operads does not preserve quadraticity. By viewing our operads as groupoid colored they may be presented as (homogeneous) quadratic, allowing us to apply a direct generalization of the classical theory of Koszul duality. Second, in characteristic zero, any chain complex with a finite group action may be viewed as an equivariant direct generalization of the classical theory of Koszul duality. Second, in characteristic zero, any interest as groupoid colored will be desirable for two reasons. First the passage from groupoid in the groupoid are viewed as unary operations in the operad. However, viewing our operads of groupoid colored operads may also be viewed as set colored operads in which the automorphisms

\[ V \]

V_{\text{object of this category a}} \]

V_{\text{C}} \]

Groupoid-colored operads.\]

2.1. Monoidal definition. We define a monoidal product \( \circ \) in the category of \( V \)-colored sequences as follows. On objects:

\[ A \circ B(v_1, \ldots, v_r; v_0) = \left( \prod A(w_1, \ldots, w_s; v_0) \otimes_{\pi_j \text{Aut}(w_{\sigma(j)})} \otimes_{j=1}^s B(v_{I_j}; v_{\sigma(j)}) \right)_{S_r} \]

Here the coproduct is taken over all non-empty partitions \( \{ I_j \}_{j=1}^r \) of \( \{ 1, \ldots, r \} \) and all \( s \)-tuples \( (w_1, \ldots, w_s) \) of objects in \( V \), and the \( S_r \) action permutes these indices. The functor \( A \circ B \) is defined on morphisms by letting automorphisms operate on the respective factor(s) of \( B(..., v_j, ...; -) \) and \( A(-; v_0) \) and letting \( S_r \) operate on the partition \( I_j \) (changing the source). The product \( \circ \) has monoidal unit \( I \) with \( I(v; v) \) equal to the monoidal unit and \( I(\vec{v}) = 0 \) (the initial object) otherwise.

Definition 2.1. \( \text{Pet13} \) A \( V \)-colored operad is a monoid in the monoidal category of \( V \)-colored sequences.

In this paper \( V \)-colored operads will often be referred to simply as operads. We remark that considering c-monoids in this monoidal category gives us notions of \( V \)-colored cooperads.

2.1.2. Monadic definition/interpretation. We first define the notion of a \( V \)-colored tree. This will be a leaf labeled, rooted tree along with a vertex of \( V \) labeling every edge, both internal and external. We furthermore require our vertices have arity \( \geq 1 \) – we do not allow vertices of arity 0. See Figure 1.

\[ ^1 \text{Resolving automorphisms is of interest in finite characteristic, but requires more sophistication; see [DV15].} \]
A $\mathbb{V}$-colored tree (pictured) is a rooted tree with leaves labeled by $\{1, \ldots, n\}$ and edges labeled by $v_i \in \text{ob}(\mathbb{V})$. Terminology is also diagrammed. Leaves and the root are called external edges, other edges are called internal. This tree is of “type” $(v_6, v_1, v_2; v_4)$.

To such a tree $T$ we associate the space

$$A(T) := \bigotimes_{w \in \text{vert}(T)} A(\vec{v}) / \sim$$

where $\vec{v}$ is the color scheme associated to the labels of the edges adjacent to the vertex $w$ and the equivalence relation is generated by saying we can move an automorphism along an internal edge. More precisely, each internal edge $e$ of $T$ is labeled by an object $v_e$ of $\mathbb{V}$ so the vertex above said edge has a right $\text{Aut}(v_e)$ action and the vertex below has a left $\text{Aut}(v_e)$ action and we take coinvariants with respect to this action for all internal edges.

A $\mathbb{V}$-colored operad structure on $A$ is then a rule which allows us to contract edges in $\mathbb{V}$-colored trees which are leveled and whose vertices are labeled by $A$. In the presence of a unit, this is the same as contracting edges in any tree labeled by $A$ (see [MSS02] for the one color analog). This observation gives the following alternate description of groupoid colored operads.

Define an adjunction

$$F : \{\mathbb{V}\text{-colored sequences in } \mathcal{C}\} \leftrightarrow \{\mathbb{V}\text{-colored operads in } \mathcal{C}\} : G$$

as follows. The right adjoint $G$ simply takes the underlying sequence (forgetting the monoid structure). The left adjoint $F$ is defined by

$$F(A)(\vec{v}) = \bigoplus_{\mathbb{V}\text{ colored trees of type } \vec{v}} A(T)$$

(2.1)

where “type” refers to the labels of the external edges of $\mathbb{V}$. The operad structure for $F(A)$ is given by grafting $\mathbb{V}$-colored trees. The unit is given by formally allowing the “empty trees” with no vertices and one edge, colored by any $v \in \text{ob}(\mathbb{V})$ in line 2.1. This contributes a summand $A(\mid_v) \cong k$ in $F(A)(v, v)$ for each such $v$. It is then a straightforward exercise to show that groupoid colored operads are algebras over the monad $GF$.

2.1.3. Non-unital and augmented variants. Since we will be interested in the bar construction, it will be convenient to be able to discard the unit in our operads. This is done via the following definition and assumption.

Definition 2.2. An augmentation of a groupoid colored operad is a morphism $P \to I$. An augmented operad is an operad along with an augmentation. The augmentation ideal is the kernel of this map, and is denoted $P$.

Assumption 2.3. From now on, all (unital) groupoid colored operads are augmented unless stated otherwise.
The augmentation ideal forms a non-unital groupoid colored operad, i.e. it is an algebras over the analogous monad formed by not allowing the empty colored trees in line 2. Formally adjoining a unit and taking the augmentation ideal give an equivalence of categories between non-unital operads and augmented operads which is compatible with the free operad construction and with their respective notions of algebras. In this paper it will sometimes be convenient to work with non-unital operads, keeping this equivalence in mind. In particular we implicitly use the unital or their respective notions of algebras. In this paper it will sometimes be convenient to work with non-unital operads, keeping this equivalence in mind. In particular we implicitly use the unital or non-unital variant of the free operad construction $F$ which is appropriate to the context.

2.1.4. Endomorphism objects. Let $X$ be a functor $\mathcal{V} \to \mathcal{C}$. To such an $X$ we associate the endomorphism $\mathcal{V}$-colored sequence:

$$End_{\mathcal{X}}(v_1, \ldots, v_n; v_0) = Hom_{\mathcal{C}}(X(v_1) \otimes \ldots \otimes X(v_n), X(v_0))$$

with the operad structure by composition of functions. We use the terminology “$\mathcal{V}$-module” to refer to such a functor. (This terminology should not be confused with $\mathcal{V}$-colored sequence).

As usual, we define a $\mathcal{P}$-algebra structure on $X$ to be a $\mathcal{V}$-colored operad map $\mathcal{P} \to End_{\mathcal{X}}$. We likewise define a (naive) morphism of $\mathcal{P}$-algebras to be a morphism of $\mathcal{V}$-modules which commutes with the adjoint form of the $\mathcal{P}$-algebra structure maps. The forgetful functor from $\mathcal{P}$-algebras to $\mathcal{V}$-modules has a left adjoint $F_{\mathcal{P}}$ defined level-wise by:

$$F_{\mathcal{P}}(X)(v_0) = \bigoplus_{\bar{v} = (-, \ldots, v_0)} \mathcal{P}(\bar{v}) \otimes_{\mathcal{V}} X(in(\bar{v}))$$

where $X(in(\bar{v})) = \otimes_{i \geq 1} X(v_i)$ and where $\otimes_{\mathcal{V}}$ means coinvariants with respect to the $Aut(v_i)$ actions.

The operad structure is by grafting, as usual.

In the case where $\mathcal{C}$ is the category of differential graded vector spaces over a field $k$, we define $\Sigma k$ to be the constant $\mathcal{V}$-module with target $\Sigma k$ – this means the field $k$ in degree 1 as a graded vector space and every automorphism acts by the identity. Then we define $End_{\Sigma \mathcal{C}} =: \Lambda$ and $End_{\Sigma^{-1} \mathcal{C}} =: \Lambda$. We define $s^+ := \Lambda^+ \otimes -$ and $s^- := \Lambda^\perp$. In particular $s$ raises degrees and $s^{-1}$ lowers degrees.

2.2. Koszul duality for groupoid-colored operads. At this point we leave the general discussion of groupoid colored operads and restrict to the case $\mathcal{C}$ is the category of dg vector spaces over a field $k$ of characteristic 0.

2.2.1. Quadratic objects. Observe that $F(E)$ is “weight” graded by the number of internal edges. A quadratic presentation of an operad $\mathcal{P}$ is an isomorphism $\mathcal{P} \cong F(E)/\langle R \rangle$, where $R$ is in weight 1 (write $R(\bar{v}) \subset F^1(E)(\bar{v})$). Given such a $\mathcal{P}$ we define the quadratic dual $\mathcal{P}^!$ as having generators:

$$E^!(v_1, \ldots, v_r; v_0) := \Sigma^{2-r} R(\bar{v}) \otimes_{\mathcal{V}} \langle sgn_r \rangle$$

(here $\ast$ means linear dual) and having relations, $R^\perp$, given by the aritywise orthogonal complement in the weight 1 subspaces of each color-scheme of the free operad. So

$$R^\perp(\bar{v}) := \{ \phi \in F^1(E^!)(\bar{v}) : \phi(R) = 0 \}$$

2.2.2. Finiteness assumptions. Before studying Koszul duality for groupoid colored operads, we impose several finiteness restrictions out of both necessity and convenience.

First, several constructions below require summing over automorphism groups. We therefore now assert:

**Assumption 2.4.** The group $Aut(v)$ is finite for every $v \in ob(\mathcal{V})$.

Note that this implies $Aut(\bar{v})$ is finite for every color scheme $\bar{v}$.

**Definition 2.5.** A non-unital $\mathcal{V}$-colored operad $\mathcal{P}$ is called reduced if for each color scheme $\bar{v}$, there are only finitely many $\mathcal{V}$-colored trees $T$ of type $\bar{v}$, such that $\mathcal{P}(T)$ is non-zero. A unital, augmented $\mathcal{V}$-colored operad is called reduced if its augmentation ideal is reduced.
Note that a non-colored operad is reduced iff $P(1) = k$ in the unital context, or $P(1) = 0$ in the non-unital context. This is because the number of trees with $n$ leaves is finite if and only if one disregards unary vertices. (Recall we do not allow vertices of arity 0.) We remark that if $P$ is reduced, then for each object $v_0$ there are only finitely many color schemes $\vec{v}$ with output $v_0$ such that $P(\vec{v})$ is non-zero, which follows by taking $T$ to be a corolla.

**Assumption 2.6.** From now on we assume that all groupoid colored operads are reduced.

**Definition 2.7.** We say a groupoid colored operad $P$ is finite dimensional if each $P(\vec{v})$ is finite dimensional. (Note the objects of $\mathcal{V}$ needn’t be a finite set.)

**Assumption 2.8.** From now on we assume that all groupoid colored operads are finite dimensional.

The reason for making this assumption is that it will reduce the level of technical detail, while not excluding our pertinent examples. These examples include the groupoid colored operad encoding modular operads, as well as the modular operads encoding graph homology and homology of moduli spaces. In particular, by making this assumption we circumvent, for the most part, the discussion of coalgebras, cooperads, conilpotence and cofreeness. Dropping Assumption 2.8 would be a straightforward exercise, following [LV12].

2.2.3. **Dual dg operad $D$.** We now define the dual dg operad $D$ following [GK94]. This construction is a particular case of Definition 7.4.1 of [KW17], so we merely unpack the definition in this case. We first give the definition in the non-unital case. For a $\mathbb{V}$-colored operad $P$, we define

$$D(\mathcal{P}) := (s^{-1}F(\Sigma \mathcal{P}^*), \partial_P)$$

In detail, if $\vec{v} = (v_1, \ldots, v_r; v_0)$ then

$$D(\mathcal{P})(\vec{v}) = s^{-1}F(\Sigma \mathcal{P}^*)(\vec{v}) = \bigoplus_{\mathcal{V} \text{ colored trees of type } \vec{v}} \Sigma^{1-r}(\Sigma \mathcal{P}^*)(T) \otimes sgn_r$$

The (external) degree of a homogeneous vector depends on the tree $T$. In particular if $T$ has $v(T)$ vertices then the degree is $1 - r + v(T)$ and corollas are in degree $-r$. A priori there can be things in positive degrees, since we allow 1-ary vertices, but Assumption 2.6 assures that for each $\vec{v}$, this complex is bounded above. The differential is, as in [GK94], defined over all ways to blow up a colored edge and has degree +1. Here Assumption 2.6 ensures this results in a finite sum.

**Lemma 2.9.** There exists a quasi-isomorphism\(^2\) $D^2(\mathcal{P}) \xrightarrow{\sim} \mathcal{P}$.

**Proof.** This is a straightforward generalization of Theorem 3.2.16 of [GK94] and a specific case of Theorem 7.4.3 of [KW17]. \qed

Finally we remark that when $\mathcal{P}$ is unital we define $D(\mathcal{P})$ by applying the analogous construction to the augmentation ideal (see subsection 2.1.3).

2.2.4. **Koszulity.** Let $\mathcal{P} \cong F(E)/\langle R \rangle$ be quadratic. From $E \hookrightarrow \mathcal{P}$, form $F(\Sigma \mathcal{P}^*) \to F(\Sigma E^*)$. We then identify $s^{-1}F(\Sigma E^*) \cong F(E^\vee)$ which induces a morphism of dg operads $D(\mathcal{P}) \to F(E^\vee)/R^\perp$.

**Definition 2.10.** A quadratic $\mathcal{V}$-colored operad $\mathcal{P}$ is Koszul if the map $D(\mathcal{P}^t) \to \mathcal{P}$ is a quasi-isomorphism. In this case we also define $P_\infty := D(\mathcal{P}^t)$.

This map may be described informally by saying a $\mathcal{V}$-colored tree labeled by elements which are indecomposable is mapped to the composition of those (generating) elements along the given tree in the operad $\mathcal{P}^t$. If the labels are not all indecomposable, this map sends such a labeled tree to zero.

\(^2\)By a quasi-isomorphism of $\mathcal{V}$-modules, $\mathcal{V}$-colored sequences or $\mathcal{V}$-colored operads we mean a map of such which is a level-wise (i.e. for each color resp. color scheme) quasi-isomorphism.

\(^3\)Using $\text{Det}(\mathcal{V} \oplus \mathcal{W}) \cong \text{Det}(\mathcal{V}) \otimes \text{Det}(\mathcal{W})$, see Lemma 4.7 of [GK95] and/or Definition 3.4 below.
2.3. Lie theoretic interpretation. We now give the \( V \)-colored analog of the Lie structure on the deformation complex of a morphism of operads and the representing object for the associated Maurer-Cartan functor.

For a \( V \)-colored operad \( P \) we define \( \lim(P) = (\prod_i \mathcal{P}(\vec{v})^{Aut(\vec{v})})_{S_i} \), where \( \mathcal{P}(\vec{v})^{Aut(\vec{v})} \) denotes the elements which are invariant under the right \( Aut(v_i) \) actions and the left \( Aut(v_0) \) action. The notation is taken from [KV17], which realizes this space as a categorical limit.

**Proposition 2.11.** Let \( O \) and \( P \) be a \( V \)-colored operads. The graded vector space \( \lim(sO \otimes P) \) carries the structure of a Lie algebra whose Maurer-Cartan elements are in bijective correspondence with the set of \( V \)-colored operad maps \( D(O) \to P \).

**Proof.** This is an example of Theorem 7.5.3 of [KV17]. The Lie bracket is defined as the commutator of a pre-Lie operations given by summing over all one edged colored compositions, analogously to the one colored case.

In the case that \( O = D(P) \) we see that the natural morphism \( D^2(P) \to P \) specifies a Maurer-Cartan element in \( \lim(sP \otimes D(P)) \). Likewise, in the case that \( O = P^1 \) we see that the natural morphism \( D(P^1) \to P \) specifies a Maurer-Cartan element in \( \lim(sP \otimes P^1) \).

**Proposition 2.11** encodes adjointness for the cobar-bar constructions. In comparing these formulas with [KV17] note that our \( D \) was defined using cohomological conventions.

**Definition 2.12.** For an operad \( P \) and cooperad \( Q \) we define:

\[
\Omega(Q) = D(s^{-1}Q^*) \quad \text{and} \quad B(P) = s^{-1}D(P)^*
\]

**Corollary 2.13.** \((\Omega, B)\) is an adjoint pair.

**Proof.** From the definition it suffices to show that morphisms \( D(Q^*) \to P \) are in natural bijective correspondence with morphisms \( D(P) \to Q^* \). This is an immediate consequence of Proposition 2.11.

2.4. Bar construction for \( P^\infty \)-algebras. Fix \( P \) a Koszul \( V \)-colored operad. In this section we define a contravariant functor

\[
D: \{P^\infty\text{-algebras}\} \to \{sP^1\text{-algebras}\}
\]

We emphasize that in this subsection we are considering the source category with its naive morphisms, and we will extend the functor to \( \infty \)-morphisms in the subsequent subsection.

We define \( D(A) \) in two steps. We first define \( D(A) := F_{sP^1}(A^*) \) as underlying graded \( sP^1 \)-algebras, and then endow this space with a differential as follows. First consider the map of \( V \)-modules given levelwise by:

\[
A^*(v_0) \to P^\infty(\vec{v})^* \otimes^\vec{e} A^*(in(\vec{v})) \to sP^1(\vec{v}) \otimes^\vec{e} A^*(in(\vec{v}))
\]

The first map in this sequence is the dual of the \( P^\infty \) structure map and the second is given by the adjoint of the canonical Maurer-Cartan element in \( \lim(P^\infty \otimes sP^1) \) (which specifies a morphism of \( V \)-colored sequences \( P^\infty \to sP^1 \)). In particular this composite is degree 1 and \( Aut(v_0) \)-equivariant.

For each \( v_0 \in V \) we may form the sum of such composites over all \( \vec{v} \) with output \( v_0 \). Each of these sums is finite by Assumption 2.6. This collection of maps forms a morphism of \( V \)-modules. Since the target of this composite is the free \( sP^1 \)-algebra on the \( V \)-module \( A^* \), these maps extend to a unique degree 1 map of \( P^1 \)-algebras which we define to be \( \partial: D(A) \to D(A) \). Translating the Maurer-Cartan equation shows:

**Lemma 2.14.** \( \partial: D(A) \to D(A) \) is square zero.

This defines \( D \) on objects, namely: \( D(A) := (F_{sP^1}(A^*), \partial) \). On morphisms we observe that for each color scheme \( \vec{v} \), a (naive) \( P^\infty \)-algebra map \( A \to B \) determines a sequence

\[
B^*(v_0) \to P^\infty(\vec{v})^* \otimes^\vec{e} A^*(in(\vec{v})) \to sP^1(\vec{v}) \otimes^\vec{e} A^*(in(\vec{v}))
\]

which in turn induces a morphism of \( sP^1 \)-algebras \( D(B) \to D(A) \).
By viewing \( \{sP^i\text{-algebras}\} \subset \{sP^i_\infty\text{-algebras}\} \) we may consider \( D \) as a functor

\[
D : \{P_\infty\text{-algebras}\} \rightarrow \{sP^i_\infty\text{-algebras}\}
\]

(2.5)

We may thus iterate \( D \). Explicitly \( D^2(-) := D_{sP^i_\infty}(D_{sP^i_\infty}(-)) \) defines an endofunctor on the category of \( P_\infty \)-algebras under the identification \( (sP)^i \cong s^{-1}P^i \).

**Lemma 2.15.** Let \( A \) be a \( P_\infty \)-algebra. There is natural quasi-isomorphism \( D^2(A) \rightarrow A \).

**Proof.** This follows as in \([LY12]\) Theorem 11.3.3, so we merely sketch the ingredients. For \( v \in \text{ob}(V) \) the complex \( D^2(A)(v) \) may be identified as two layer colored trees with root vertex labeled by \( P \), top layer vertices labeled by \( P^i := (sP^i)^* \) and leaves labeled by \( A \). This complex may be described in terms of the monoidal product encoding groupoid colored operads as \( P \circ P^i \circ A \), where we abuse notation by letting the \( V \)-module \( A \) also denote the \( V \)-color scheme defined by \( A(v;v) := A(v) \) and \( A(\vec{v}) = 0 \) for those \( \vec{v} \neq (v;v) \). This complex may be filtered by the largest weight of \( P \) and \( P^i \). Considering the spectral sequence associated to this filtration at the \( E^{10} \) page only sees the differential coming from the MC element in \( \text{lim}(sP^i \circ P) \).

This differential restricted to the complex \( P \circ P^i \) is acyclic, which may be seen by comparing it to the corresponding induced differential on \( P \circ sD(P) \), for which an explicit null homotopy may be constructed. The result then follows from the convergence of this spectral sequence along with the fact that \( H_n(P \circ P^i \circ A) \cong H_n(P \circ P^i) \circ H_n(A) \). This last fact follows by employing Maschke’s theorem to conclude that each \( k[\text{Aut}(\vec{v})] \) is semi-simple thus \( k[\text{Aut}(\vec{v})]-\text{modules} \) are projective, and so here we have used Assumption 2.4 implying each \( \text{Aut}(\vec{v}) \) is a finite group. \( \square \)

2.5. **Infinity morphisms.**

**Definition 2.16.** Let \( A \) and \( B \) be \( P_\infty \)-algebras. An \( \infty \)-morphism from \( A \) to \( B \), denoted \( A \rightsquigarrow B \), is a \( sP^i \)-algebra map \( D(B) \rightarrow D(A) \). An \( \infty \)-quasi-isomorphism \( A \Rightarrow B \) is a \( sP^i \)-algebra map \( \phi : D(B) \rightarrow D(A) \) having the property that \( D(\phi) : D^2(A) \rightarrow D^2(B) \) is a quasi-isomorphism.

A few remarks about this definition are in order. An \( \infty \)-morphism \( f : A \rightarrow B \) has an underlying map \( A \rightarrow B \), corresponding to the dual of line 2.4 composed with the augmentation, in the case \( \vec{v} = (v_0, v_0) \). It follows from Lemma 2.15 that \( A \rightsquigarrow B \) is an \( \infty \)-quasi-isomorphism if and only if this underlying map is a quasi-isomorphism. In this way, a naive morphism is an example of an \( \infty \)-morphism and a naive quasi-isomorphism is an example of an \( \infty \)-quasi-isomorphism. More generally an \( \infty \)-morphism has an operation \( A(\text{in}(\vec{v})) \rightarrow B(\vec{v}_0) \) for every element in \( P^i(\vec{v}) \). The differential of such an operation is a sum over ways to partially compose in \( A \) and then apply \( f \) with ways to apply \( f \) to subsets of factors and then compose in \( B \).

Following \([LY12]\) we denote the category of \( P_\infty \)-algebras with their \( \infty \)-morphisms by \( \{\infty - P_\infty \text{-algebras}\} \). In particular \( \{P_\infty \text{-algebras}\} \subset \{\infty - P_\infty \text{-algebras}\} \) is a full subcategory. The functor \( D \) extends tautologically to a contravariant functor \( D : \{\infty - P_\infty \text{-algebras}\} \rightarrow \{sP^i\text{-algebras}\} \). Composing with inclusions we may also view \( D \) as a contravariant functor

\[
D : \{\infty - P_\infty \text{-algebras}\} \rightarrow \{\infty - sP^i_\infty \text{-algebras}\}
\]

(2.6)

which in turn may be iterated as above. To conclude we observe that \( D \) satisfies:

**Lemma 2.17.** \( D \) takes \( \infty \)-quasi-isomorphisms to quasi-isomorphisms.

**Proof.** This may be seen by applying Lemma 2.15 to the natural transformation \( D^3 \Rightarrow D \). \( \square \)

2.6. **Homotopy transfer theorem.** The purpose of this section is to show that any \( P \)-algebra is \( \infty \)-quasi-isomorphic to its homology.

**Definition 2.18.** A deformation retract of a \( P \)-algebra \( A \) onto a \( P \)-algebra \( B \) is a family of deformation retracts:

\[
h_v \bigcup A(v) \xrightarrow{\imath_v} B(v) \xrightarrow{\pi_v} \]

(2.7)

indexed by \( \text{ob}(V) \), such that \( h_v, \imath_v, \pi_v \) are \( \text{Aut}(v) \)-equivariant.
Lemma 2.19. Any $\mathcal{V}$-module admits its homology as a deformation retract.

Proof. Let $A$ be a $\mathcal{V}$-module and fix an arbitrary object $v$. In particular $(A(v),d)$ is a chain complex with an action of a group $G := \text{Aut}(v)$ which is finite by Assumption 2.4. Define a map

$$
\phi: H_*(A(v)) \rightarrow A(v)
$$

as follows. First pick a basis $h_i$ of $H_*(A(v))$. Then choose cycles $\gamma_i$ with $[\gamma_i] = h_i$. For each $g \in G$, the set $\{gh_i\}$ is still a basis, so there is a unique linear map $\phi_g$ such that $\phi_g(gh_i) = g\gamma_i$. Then define,

$$
\phi := \sum_{g \in G} \frac{\phi_g(h_i)}{|G|}.
$$

The map $\phi$ is $G$-equivariant. To see this, first observe that $g_0\phi_g(x) = \phi_{g_0g}(g_0x)$, which can be seen by writing $x$ in the basis $gh_i$. Then use this fact to conclude:

$$
g_0\phi(x) = g_0 \sum_{g \in G} \left[ \frac{\phi_g(x)}{|G|} \right] = \sum_{g \in G} \left[ \frac{\phi_{g_0g}(g_0x)}{|G|} \right] = \sum_{g \in G} \left[ \frac{\phi_g(g_0x)}{|G|} \right] = \phi(g_0x).
$$

The map $\phi$ is also injective. To see this, we observe that $\phi(h_i) = x$ for each homology class $x$ and each $g \in G$, again by writing $x$ in the basis $gh_i$, and therefore $[\phi(x)] = x$. It follows that $\phi$ specifies a $G$-closed subspace of $A(v)$ isomorphic to $H_*(A(v))$.

We then extend the linearly independent set of cycles $\phi(h_i)$ to a basis for all cycles in $A(v)$ by choosing boundaries $d(a_i)$. Then extend the basis $\{\phi(h_i)\} \cup \{d(a_i)\}$ for the cycles to a basis for all of $A(v)$ by adding $a_i$. Since $d$ is injective on the span of the $a_i$, we may identify the span of $a_i$ in a given degree with the boundaries in the degree of $d(a_i)$. These choices specify a $G$-equivariant decomposition $A_0 \cong H_0 \oplus B_0 \oplus B_{n-1}$ (under homological grading conventions) in which the differential is of the form $d(x,dy,z) = (0,dz,0)$.

Using this decomposition we define the projection $\pi: A(v) \rightarrow H_*(A(v))$ by $\pi(x,dy,z) = x$, which is $G$-equivariant. Using this decomposition we also define the homotopy $h: A_n \rightarrow A_{n+1}$ by $h(x,dy,z) = (0,0,y)$. This implies $dh(x,dy,z) = (0,dy,0)$ and $hd(x,dy,z) = (0,0,z)$. Therefore $dh + hd = id - \phi \circ \pi$, and so $h,\pi,\phi$ form a deformation retract as desired.

Using Lemma 2.19 we now record the homotopy transfer theorem in this context.

Theorem 2.20. Let $P$ be a Koszul operad and let $A$ be a $P_\infty$-algebra. There is a $P_\infty$ structure on $H_*(A)$ extending the induced $P$-algebra structure, for which $H_*(A)$ and $A$ are $\infty$-quasi-isomorphic.

Proof. This result may be proven in the uncolored case via explicit combinatorial formulas involving trees labeled by deformation retract data. These formulas are equally valid for $\mathcal{V}$-colored trees. The one difference worth emphasizing is that in the groupoid colored context, elements in the free operad are not represented uniquely by colored trees, since we may move automorphisms along internal edges. Therefore, for such formulas to be well defined it is essential that the maps $h$ decorating internal edges are equivariant. Lemma 2.19 ensures this is the case. With this in mind we simply sketch the proof, following [LV12] in the uncolored case.

First use Lemma 2.19 to fix a deformation retract between $A$ and $H_*(A)$:

$$
\begin{array}{ccc}
\mathcal{V} & \xrightarrow{\iota} & A \\
\pi \downarrow & & H_*(A) \\
\mathcal{V} & \xleftarrow{\epsilon} & A
\end{array}
$$

We then use this deformation retract to construct a map of dg operads $D\text{End}_{H(A)} \rightarrow D\text{End}_A$. This is equivalent to a map of dg cooperads $B\text{End}_A \rightarrow B\text{End}_{H(A)}$, which will be induced by a map of $\mathcal{V}$-colored sequences:

$$
F(\Sigma^{-1}\text{End}_A) \rightarrow \Sigma^{-1}\text{End}_{H(A)}.
$$

The map may be described diagrammatically following [LV12] p.378. Starting with a $\mathcal{V}$-colored tree whose vertices of type $\vec{v}$ are labeled by functions $A(\text{in}(\vec{v})) \rightarrow A(\text{out}(\vec{v}))$, we label the (internal) edges of the tree by $h$, the leaves by $\iota$ and the root by $\pi$. This labeling may then be read as a flow
chart to construct a linear map $H_\ast(A)(\text{leaves}(T)) \to H_\ast(A)(\text{root}(T))$. Notice degrees are preserved since the internal edges contribute degree +1, while the vertices contribute degree −1 resulting in the degree −1 shift in the target. The fact that $\pi$ and $\iota$ are equivariant ensures that this is a map of $V$-colored sequences. It remains to verify that the induced map respects the differential in the bar construction. This follows, as in Proposition 10.3.2 of [LV12], from the deformation retract equation.

Now from Proposition 2.11 we know that a $\mathcal{P}_\infty$ algebra structure may be recast as a wrong-way morphism $\mathcal{D}\mathcal{E}\mathcal{N}\mathcal{D} \to \mathcal{P}_\iota$. Therefore if $A$ is a $\mathcal{P}_\infty$ algebra we may compose $\mathcal{D}\mathcal{E}\mathcal{N}\mathcal{D}_{H_\ast(A)} \to \mathcal{D}\mathcal{E}\mathcal{N}\mathcal{D}_A \to \mathcal{P}_\iota$ to endow $H_\ast(A)$ with the structure of a $\mathcal{P}_\infty$ algebra. It remains to observe that $\iota$ may be extended to an $\infty$-morphism compatible with this transferred structure. Such an $\infty$-morphism is given by maps

$$\mathcal{P}_\iota(\vec{v}) \to Hom(\otimes_{i=1}^n H_\ast(A(v_i)), A(v_0))$$

Following [LV12] Theorem 10.3.6, these are defined by decomposing $\mathcal{P}_\iota$ and using the same flow chart formula as above, except we label the root by $h$ in place of $\pi$. We remark it is also possible to extend $\pi$ to an $\infty$-quasi-isomorphism, see [LV12] Proposition 10.3.9.

3. Weak modular operads.

In this section we will

1. Define the colored operad encoding modular operads, call it $\mathcal{M}$, via a quadratic presentation.
2. Interpret $\mathcal{M}$ and its quadratic presentation in terms of (nested) graphs.
3. Calculate the quadratic dual of $\mathcal{M}$, specifically $\mathcal{M}^! = s^{-1}M_{s!}$.
4. Prove $D(M) \xrightarrow{s^{-1}M_{s!}}$ and conclude $\mathcal{M}$ is Koszul.
5. Introduce the category of weak modular operads as the category of $D(s^{-1}M_{s!})$-algebras.

3.1. Modular Operads. Modular operads were introduced in [GK98]. They are generalizations of operads allowing composition across all connected graphs — not just rooted trees. Modular operads may be defined as algebras over a particular groupoid colored operad which we will denote by $M$. In this section we will present $M$ quadratically as $M = F(E)/(S)$.

To this end we now specialize to a particular groupoid $\mathcal{V}$ as follows. Its objects are pairs of non-negative integers $(g, n)$ such that $n + 2g - 3 \geq 0$. Its only morphisms are automorphisms and $\text{Aut}((g, n)) = S_n$. We fix this $\mathcal{V}$ for the remainder of the paper.

3.1.1. Modular generators. Define the $\mathcal{V}$-colored sequence $E$ in sets as follows. First, we let

$$E((g, n); (g + 1, n - 2)) := \{\xi_{i,j} : \{i \neq j\} \subset \{1, \ldots, n\}\} \times S_{n-2}$$

The right $S_{n-2}$-action is given by multiplication on the right hand factor and the left $S_n$-action is given by the opposite of $\sigma\xi_{i,j} = \xi_{\sigma(i), \sigma(j)}\sigma$, where $\sigma \in S_n$ and $\sigma'$ is the composite

$$\{1, \ldots, n - 2\} \to \{1, \ldots, n\} \setminus \{i, j\} \xrightarrow{\sigma} \{1, \ldots, n\} \setminus \{\sigma(i), \sigma(j)\} \to \{1, \ldots, n - 2\}$$

where the first and last arrows are the unique order preserving bijection.

Second we let

$$E((g_1, n), (g_2, m); (g_1 + g_2, n + m - 2)) := \{\sigma_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\} \times S_{n+m-2}$$

Given $i, j$ we define a total order on the disjoint union $(\{1, \ldots, n\} \setminus \{i\}) \cup (\{1, \ldots, m\} \setminus \{j\})$ by

$$\{1, \ldots, i - 1\}^l < \{j + 1, \ldots, m\}^r < \{1, \ldots, j - 1\}^r < \{i + 1, \ldots, n\}^l$$

Here $l$ and $r$ mean left and right and we take the subsets to be totally ordered as usual. Note this total order is not symmetric, in the sense that switching the order of $n$ and $m$ in its construction gives a different total order to this set. However these two orderings have the same underlying cyclic order and are related by $t^{m-j+i-1}$, where $t$ is the generator of the cyclic group of order $n + m - 2$. 

11
The right $S_{n+m-2}$ action is given by multiplication. The left $S_n \times S_m$ action is defined to be the opposite of $\sigma'_r \circ_j = \sigma_1(i) \circ_{\sigma_2(j)}(\sigma_1, \sigma_2)$ where $\sigma' \in S_{n+m-2}$ is defined by

$$\{1, \ldots, n + m - 2\} \to (\{1, \ldots, n\} \setminus \{i\}) \sqcup (\{1, \ldots, m\} \setminus \{j\})^{\sigma_1 \sigma_2}$$

$$(\{1, \ldots, n\} \setminus \{\sigma_1(i)\}) \sqcup (\{1, \ldots, m\} \setminus \{\sigma_2(j)\}) \to \{1, \ldots, n + m - 2\}$$

where the first map is the unique order preserving bijection with respect to the order $i, j$ and the last map is the unique order preserving bijection with respect to the order $\sigma_1(i), \sigma_2(j)$.

The $S_2$ action

$$E((g_1, n_1), (g_2, n_2); (g_1 + g_2, n_1 + n_2 - 2)) \to E((g_2, n_2), (g_1, n_1); (g_1 + g_2, n_1 + n_2 - 2))$$

is then defined by $i \circ_j \times id \mapsto j \circ_i \times i^{m-j-i-1}$. Finally we define $E(else) = \emptyset$. It will be shown below that the elements of $E$ correspond bijectively to certain labeled graphs with 1 edge, see subsection 3.1.2.

3.1.2. Modular relations. We now define the operadic ideal of relations, denoted $S$. These relations come in three families. The first is all expressions of the form:

$$\xi_{i,j}$$

where $i, j$ corresponds to $\xi_{i,j}$ above. By applying permutations to these relations (and moving automorphisms along edges) we may use these relations to produce relations of the form $\xi_{i,j} \xi_{l,k} = \xi_{l',k'} \xi_{i',j'}$ where the $'$ notation indicates a relabeling which may be deciphered from the commutation relations of subsection 3.1.1.

The second family is all expressions of the form:

$$\eta_{i,j}$$

where $i|j$ corresponds to $i \circ_j$. We again may apply permutations to find further relations of the form $\xi_{i,j} \ \xi_{l,k} = \xi_{l',k'} \ \xi_{i,j'}$.

The final family is all expressions of the form:
here we suppress some edge labels for readability. We may again apply permutations to decipher relations of the form \( \rho_k \circ \rho_j = \rho_k' \circ \rho_j' \).

Informally generators will correspond to graphs with one edge and relations will correspond to graphs with 2 edges, and may be interpreted as saying both ways to assemble such a graph yield the same result. In this language, the first family of relations corresponds to graphs with two loops, the third family is composition along a tree with two edges and the second family corresponds to graphs with a two edged circuit.

3.1.3. Modular Operads. Define \( M \) to be the \( V \)-colored operad in sets presented by \( F(E)/\langle S \rangle \). We define the category of algebras over \( M \) to the category of modular operads in sets. Define \( \mathbb{M} \) to be \( V \)-colored operad in chain complexes given by the linearization of \( M \). We define the category of algebras over \( \mathbb{M} \) to be the category of dg modular operads.

3.2. Graphs and nested graphs. An abstract graph is a 4-tuple \( \Gamma = (V, F, a, i) \) of a finite set \( V \) (whose elements are called vertices), a finite set \( F \) (whose elements are called half-edges or flags), a function \( a : F \rightarrow V \) (specifying the vertex adjacent to a flag) and an involution \( i : F \rightarrow F \). The fixed points of this involution are called the legs of the graph and the orbits of this involution are called the edges of the graph. A subgraph is a pair of subsets of \( V \) and \( F \) which are closed under \( a \) and \( i \). We say a subgraph is proper if its set of edges is a nonempty proper subset of the set of edges of \( \Gamma \).

In this paper we consider graphs with additional labellings. A leg labeled graph is a graph along with a bijection between \( \{1, \ldots, n\} \) and the set of legs. A genus labeled graph is a graph along with a function from \( V \) to the non-negative integers. A genus labeled graph is stable if each vertex \( v \) has \( n + 2g - 3 \geq 0 \), where \( n \) is the number of adjacent flags (i.e. \( |a^{-1}(v)| \)) and \( g \) is the genus label of \( v \).

To a graph we may associate a 1-dimensional CW complex whose one cells correspond to the set of edges and legs (we add 0-cells to the ends of legs) and we say a graph is connected if this CW complex is connected. The internal genus of a graph is the rank of the first homology of this CW complex. We now define the class of graphs that will be considered in this paper:

**Definition 3.1.** From now on in this paper the word “graph” refers to an abstract graph which is leg labeled, genus labeled, stable and connected and which has at least one edge. We also use the terminology “modular graph” to emphasize this list of criteria.

We say a vertex of a graph is of type \( (g,n) \) if it has \( n \) adjacent flags and genus label \( g \). The total genus of a graph is the sum of the internal genus and the genus labels of all vertices. We say a graph is of type \( (g,n) \) if it has \( n \) legs and total genus \( g \).

3.2.1. Nested graphs.

**Definition 3.2.** A nest of a graph is a proper, connected subgraph containing no legs. Two nests are compatible if one is contained in the other (i.e they are nested) or if they are disjoint. Disjoint means they share no edges and share no vertices. A collection of compatible nests is called a nesting. Every nesting is a poset by containment. A nested graph is a graph along with a choice of nesting. The set of all nestings of a graph is itself a poset by adding or removing nests. A
maximal element in this poset is said to be fully nested. By convention we allow the empty nesting consisting of no nests.

We typically denote nests by \( N_i \) and a nesting by \( \mathcal{N} = \{N_1, \ldots, N_r\} \). The number of nests in a nesting is denoted \(|\mathcal{N}|\). Let us record the following immediate consequence of the definition for future use:

**Lemma 3.3.** Let \((\Gamma, \mathcal{N})\) be a nested graph and let \( e \) be an edge of \( \Gamma \). The subset of the poset \( \mathcal{N} \) consisting of those nests containing \( e \) is totally ordered.

**Proof.** Since any two nests in this subset share an edge, they are not disjoint. Thus any two nests in this subset are comparable, hence the claim. \( \square \)

**Definition 3.4.** For a finite set \( X \) we let \( \text{Det}(X) \) be the top exterior power of the span of \( X \). This is a one-dimensional vector space concentrated in degree \(|X|\) with an action of \( S_X \). We also write \( \text{det}(X) \) for \( \Sigma^{-|X|} \text{Det}(X) \), concentrated in degree 0. We refer to an element of \( \text{det}(X) \) as a mod 2 order of the set \( X \).

In a fully nested graph there is a bijection between the set of nests union-ed with another point \(*\) and the set of edges – for we can associate an edge to the nest of minimal depth it is contained in, with the outside edge associated to \(*\). Consequently there is an isomorphism

\[ \Sigma \text{Det}(\mathcal{N}_\gamma) \cong \text{Det}(\text{Edges}(\Gamma)) \] (3.1)

**Lemma 3.5.** There exist bijective correspondences:

- \( E \leftrightarrow \) modular graphs with one edge,
- \( F(E) \leftrightarrow \) fully nested modular graphs,
- \( S \leftrightarrow \) (two ways to nest) graphs with two edges,
- \( M \leftrightarrow \) modular graphs.

**Proof.** The statement encodes the existence of such bijections for each vertex type. For example, the set \( E((g,n),(h,m);(g+h,n+m-2)) \) corresponds to graphs with one edge formed by attaching two flags adjacent to vertices of type \((g,n)\) and \((h,m)\) respectively. The right symmetric group actions permute the labels after attaching, the left symmetric group actions permute the labels before attaching.

The second correspondence is seen by considering \( F(E) \) to be trees labeled by \( E \). The leaves of such trees correspond to the vertices of the graph; the vertices of the tree correspond to edges in the graph; the (internal) edges of the tree correspond to nestings in the graph. The groupoid colored operad condition allows us to transport isomorphisms along the edges of the tree – this corresponds to permuting matched labels inside and outside a nest before attaching them. In summary, we have sub-correspondences:

| rooted trees | graphs |
|--------------|--------|
| leaves       | vertices|
| vertices     | edges   |
| edges        | nests   |

(3.2)

For the third correspondence, pairs of composible generators give graphs with two edges. The relations in \( S \) then correspond to the (two) orders in which these two edge graphs can be glued together from generators, which in turn correspond to the two ways to nest them.

The fourth correspondence follows from the second and third, since in \( M \) we can exchange a nest with an adjacent edge of lesser depth and in this way identify all maximal nestings. Observe that, with respect to this correspondence, the operad structure of \( M \) is given by graph insertion: one inserts a graph of type \( v \) into a vertex of type \( v \). \( \square \)

We emphasize that in table 3.2 we mean the internal edges of the tree in the terminology of Figure 1. From now on we refer to the internal edges simply as the edges of a colored tree.
Proof. Pick a color scheme a mod 2 order on the set of nests (or equivalently graph edges) and a mod 2 order on the set of
Lemma 3.7. We first describe the free operad generated by
\( E \) only types of vertices appearing in
\(|\varepsilon|\) of the graph, viewed as a 1-dimensional CW complex, reveals that
\( e \) edges is equal to
\( e \) on the set of edges in the graph.
\( (\Gamma,\mathcal{N}) \) det
The degree is such a vector is the number of edges of
\( \det \) the
\( E \) identified with its linear dual. In particular
\( \text{sgn} \) u vertex
\( \vec{v} \) of
\( T \) may be identified with the span of a set of
\( E^* \)\( (\vec{v}) \) may be identified with the span of \( \vec{v} \)-graphs, along with a mod 2 order on the set of nests (or equivalently graph edges) and a mod 2 order on the set of vertices. The degree of a vertex corresponding to a given graph is the rank of its first homology.

**Proof.** Pick a color scheme \( \vec{v} \) of length \( r \). For a \( \vec{v} \)-tree \( T \) we let \( U_T \) and \( B_T \) be its set of unary and binary vertices respectively. Recall that since \( E^\gamma \) is concentrated in length 1 and 2, these are the only types of vertices appearing in \( F(E^\gamma) \). Now a priori,
\[
F(E^\gamma)(\vec{v}) = \bigoplus_{\vec{v}\text{-trees } T} E^\gamma(T) = \bigoplus_{\vec{v}\text{-trees } T} (\otimes_{u\in U_T} E^\gamma(u)) \otimes (\otimes_{b\in B_T} E^\gamma(b)) / \sim \\
= \bigoplus_{\vec{v}\text{-trees } T} (\otimes_{u\in U_T} \Sigma E^*(u)) \otimes (\otimes_{b\in B_T} E^*(b) \otimes \text{sgn}_b) / \sim \\
= \bigoplus_{\vec{v}\text{-trees } T} E^*(T) \otimes \Sigma |U_T| (\text{det}(\text{Edges}(T)) \otimes \text{sgn}_r)
\]
Here we abuse notation by writing \( E(u) \) when we mean \( E \) evaluated at the color scheme of the vertex \( u \). We write \( \text{sgn}_b \) to mean the alternating representation of \( \text{in}(b) \). The last equality follows by identifying every vertex with the edge or leaf above it – the \( \text{sgn}_r \) corresponds to the flags.

Observe that since \( E \) was described via a specific basis (of one edges graphs) it is canonically identified with its linear dual. In particular \( E^*(T) \) may be identified with the span of a set of nested graphs via the correspondence in line 3.2. Thus, \( F(E^\gamma)(\vec{v}) \) may be identified with the span of \( \vec{v} \)-graphs, along with a mod 2 order on both its vertices (from the \( \text{sgn}_r \) term) and its nests (from the \( \text{det}(\text{Edges}(T)) \) term). Applying equation 3.1 we see this is the same as having a mod 2 order on the set of edges in the graph.

Finally we observe that the number of unary vertices in a tree with \( r \) leaves and \( e_T \) (internal) edges is equal to \( e_T + 2 - r \) (inductively on \( U_T \) say). Therefore in the corresponding nested graph \((\Gamma,\mathcal{N})\) the degree is measured by \(|\mathcal{N}| + 2 - v_T\), but since this graph is fully nested the number of nests is one less than the number of edges and so the degree is \( e_T + 1 - v_T \). The Euler characteristic of the graph, viewed as a 1-dimensional CW complex, reveals that \( v_T - e_T = 1 - |H_1(\Gamma)| \), and so the degree is \(|H_1(\Gamma)|\) as desired. ∎
Let me remark that from this proof we also see:

\[ F(E^\vee)(\vec{v}) = s^{-1} \bigoplus_{\vec{v}\text{-trees } T} E^*(T) \otimes \text{Det}(\text{Edges}(T)) \]

where capital Det means no longer in degree 0 (Definition 3.4).

Having described the free operad \( F(E^\vee) \), we now look to describe the relations \( S^\perp \) and the quotient \( \mathbb{M}^\perp = F(E^\vee)/(S^\perp) \). First observe/recall that if \( \Gamma_1 \) and \( \Gamma_2 \) are two nestings of a graph \( \Gamma \in F(E)^1 \) (so with two edges), then \( \Gamma_1 - \Gamma_2 \in S \). It follows that if we denote their characteristic functionals \( \eta_{\Gamma_1} \), then \((\eta_{\Gamma_1} + \eta_{\Gamma_2})(\Gamma_1 - \Gamma_2) = 0 \). Clearly this functional vanishes, then, on any relation and so \( \eta_{\Gamma_1} + \eta_{\Gamma_2} \) lives in \( S^\perp \). Since the graph edge order in Lemma 3.7 was induced by the order of nestings, this shows that we may view \( S^\perp \) as consisting of the sum of two ways to order the edges in a two edged graph (with a given mod 2 vertex order). Iterating this, we find:

**Lemma 3.8.** (Description of \( \mathbb{M}^\perp \)) The groupoid colored operad \( \mathbb{M}^\perp \) has the following description. \( \mathbb{M}^\perp(\vec{v}) \) is the span of graphs of type \( \vec{v} \) along with a mod 2 order on both the set of edges and the set of vertices. The degree of such a graph \( \Gamma \) is the rank of \( H_1(\Gamma) \). The operad structure corresponds to insertion of graphs, analogously to \( \mathbb{M} \) – with the convention that the new order takes outside before inside.

**Proof.** From Lemma 3.7 we may view \( F(E^\vee) \) as the span of fully nested graphs along with a mod two order on the set of vertices and edges. The degrees were also verified in the proof of that lemma.

Since \( S^\perp \) is the sum of two ways to nest such a graph with two edges, we see that every homogeneous vector may be represented (non-uniquely) by a tree which is a left comb. This is done by renesting until the poset of nestings is fully ordered. Such a nesting induces an order on the set of edges of the graph, and switching two adjacent edges corresponds to switching a nest as in \( S^\perp \), and so gives back the negative of the input. Since the set of adjacent permutations \( (i,i+1) \) generates the symmetric group, any two such left combs represent the same element, up to a sign corresponding to permutation of edges. Thus, different nestings represent the same element, up to this sign, and so we may view \( \mathbb{M}^\perp \) as unnested graphs with a mod 2 edge order. The mod 2 vertex order is simply inherited since the relations aren’t sensitive to the vertex orders in the free operad.

**Corollary 3.9.** \( \mathbb{M}^\perp = s^{-1}\mathbb{M}_R \).

**Proof.** Taking \( s\mathbb{M}^\perp \) we find that the degree of a graph is \((v - 1) + (e + 1 - v) = e \) and graphs have odd edges, but no longer odd vertices due to the \( sgn_r \) acting on \( \mathbb{M}^\perp(v_1,\ldots,v_r;v_0) \). This is precisely \( \mathbb{M}_R \), hence the claim.

We note that this corollary was proven independently by Batanin and Markl (Theorem 12.10 of BMIS) as part of a broader study of Koszul duality for the authors’ notion of operadic categories.

**3.5. \( \mathbb{M} \) is Koszul.** We now turn to the main technical result of this paper.

**Theorem 3.10.** The \( V \)-colored operad \( \mathbb{M} \) is Koszul.

**Proof.** For this we will show that the natural map \( D(\mathbb{M}) \to \mathbb{M}^\perp \) is a levelwise quasi-isomorphism. From Corollary 3.9 it is enough to show that for each \( \vec{v} \), the induced map \( sD(\mathbb{M})(\vec{v}) \to \mathbb{M}_R(\vec{v}) \) is a quasi-isomorphism. By definition, the chain complex \( sD(\mathbb{M})(\vec{v}) \) has underlying graded vector space \( F(\Sigma\mathbb{M}^*)(\vec{v}) \). Under the identification \( (\Sigma\mathbb{M}^*)(T) \cong (\Sigma\mathbb{M})(T) \cong \Sigma(\text{Det}(\text{Edges}(T))) \otimes \mathbb{M}(T) \), every element in this vector space may be written as a span of nested modular \( \vec{v} \)-graphs along with a mod 2 order on the set of nests. We call such vectors homogeneous.

This correspondence between nested graphs and labeled trees is similar to that above, except now the vertices of trees are labeled with graphs that can have more than one edge. We call such \(^4\)Strictly speaking \( S^\perp \subset F(E^\vee) \), and so must carry a vector in the component \( \text{det}(\text{Edge}(T)) \otimes \text{det}(\text{leaf}(T)) \). This component is determined by composition in the tree by the convention that top goes on the right and bottom goes on the left.
graphs the layer of the nested graph. Homogeneous vectors of $M(T)$ correspond to nested graphs via:

| rooted tree | graph |
| leaves | vertices |
| vertices | layers |
| edges | nests |

Note that we allow the empty nesting, which corresponds to corollas on the tree side. With this description, the differential is given by summing over ways to add a nest. With regard to the mod 2 order, the new nest is placed in the last position. The degree of a homogeneous vector is one greater than the number of nests or equivalently the number of layers.

Every homogeneous vector determines a homogeneous element in $\bar{M}(\vec{v})$; just forget the nesting. This determines a splitting of complexes:

$$sD(M)(\vec{v}) = \bigoplus_{\gamma \in M(\vec{v})} C^*_{\overrightarrow{\gamma}}$$

Here $C^*_{\overrightarrow{\gamma}}$ is a cochain complex spanned by nestings of the $\vec{v}$-graph $\gamma$. This complex is concentrated between degrees 1 and $|\gamma|$ (the number of edges of $\gamma$), inclusive. To specify a cochain we need not only a nesting of the graph, but also a mod 2 order on the set of nests.

We prove this by induction on the number of edges of $\gamma$. For the base step of the induction we observe that the complex $C^*_{\overrightarrow{\gamma}} \cong \Sigma k$ if $\gamma$ has one edge. So now suppose that $\gamma$ has more than one edge and choose an edge $e$ such that removing $e$ either does not disconnect the graph, or disconnects the graph into two components, one of which is a lone vertex. (Clearly such an $e$ always exists.) We define $\gamma \setminus e$ to be the graph formed by removing $e$ from $\gamma$ – in the case that this disconnects the graph, we take it to mean the non-trivial component. In particular the graph $\gamma \setminus e$ has one less edge then $\gamma$, so for the induction step it suffices to show that the complexes $C^*_{\overrightarrow{\gamma}}$ and $\Sigma C^*_{\overrightarrow{\gamma \setminus e}}$ are chain homotopic.

To this end, the remainder of the proof will be dedicated to constructing a deformation retract:

$$\Sigma C^*_{\overrightarrow{\gamma \setminus e}} \ x \ C^*_{\overrightarrow{\gamma \setminus e}} \ x \ H \quad (3.3)$$

**Notation**: We use the following notation for dealing with nested graphs. A nest will be denoted capital $\mathcal{N}$. Let $\mathcal{N}_e$ be the nest on $\gamma$ which contains only the edge $e$. When $e$ is fixed we let $\mathcal{N}_{\max}$ be the nest on $\gamma$ which contains all edges except $e$.

A cochain, or equivalently a mod 2 ordered nesting, will be denoted $\mathfrak{N}$. So we may write $N \in \mathfrak{N}$, using a subscript to denote the graph if needed. We often refer to a cochain as a nesting, keeping the mod 2 order implicit. To a nesting $\mathfrak{N}_{\gamma \setminus e}$ there is a nesting $\mathfrak{N}_\gamma$, given by taking just the same nests. There is also a nesting $\mathfrak{N}_\gamma \cup \mathcal{N}_{\max}$ with the induced order (so $\mathcal{N}_{\max}$ placed in the last position).

The idea behind the proof will be to realize $C^*_{\overrightarrow{\gamma}}$ as a “cylinder” whose top is given by those nestings containing $\mathcal{N}_e$ and whose bottom is given by those nestings containing $\mathcal{N}_{\max}$. Some intuition behind this idea is given below in Remark 3.11 and Figure 3.3.

**Definition of $\nu$**: To a nesting of the graph $\gamma \setminus e$ we get a nesting of $\gamma$ by keeping all the nests we started with and adding the nest containing all the edges of $\gamma$ except $e$ in the last position.

\footnote{By definition a layer of a nested graph is the graph formed by first choosing a nest and then collapsing all of its subnests to vertices. In particular the edges in a layer are the edges which are in a nest but not any of its subnests. We also refer to the graph outside of all nests, collapsing maximal nests to vertices, as a layer. In particular the number of layers is one greater than the number of nests.}
the notation above:
\[ \iota(\mathcal{N}_\gamma e) = \mathcal{N}_e \cup N_{\text{max}} \]

Observe that because \( \iota \) adds one nest and has its source shifted by one (see line 3.3), it is a degree 0 map. To see that \( \iota \) commutes with the differential observe if I first apply \( \iota \), then sum over all ways to add a nest, none of these nests can contain the edge \( e \), because \( e \) is already outside a nest of maximal size. Each of these nests then could be added to the nesting on \( \gamma \setminus e \) before applying \( \iota \), which is in turn the definition of \( d \) before \( \iota \).

**Definition of \( \pi \):** Informal definition: remove \( e \) from all nests.

More precisely: starting from a mod 2 ordered nesting \( \mathcal{N} = \{N_1, \ldots, N_r\} \) we consider the graphs
\[ N_i - e. \]
If \( e \notin N_i \) then \( N_i - e := N_i \). Else, \( N_i - e \) is defined to be the graph formed by removing the two flags of \( e \) from the subgraph \( N_i \). We then use the list of graphs \( \{N_i - e\} \) to form a nesting by the following procedure. If \( N_{\text{max}} - e = N_{\text{max}} \) or \( N_e - e = \emptyset \) appears on the list discard it. If \( N_i - e = N_j - e \) (which happens if \( N_i \cup e = N_j \) or vice versa), identify these (identical) sets in the list. Finally it may be the case that some \( N_i - e \) is disconnected in to two components, each with a non-zero number of edges. In this case, we split this entry into its two connected components which are added to the list. If one or both of these components already appears in the list it may be discarded/identified. Modifying the list \( \{N_i - e\} \) according to these rules yields a nesting on \( \mathcal{N} \). If this nesting has \( r - 1 \) nests we define it to be \( \pi(\mathcal{N}) \) (up to sign/orderings which are fixed below). Otherwise we define \( \pi(\mathcal{N}) = 0 \). Observe that since \( \pi \) always has one fewer nest in the target than in the source, it has degree 0.

To fix signs/orders we first characterize the nestings \( \mathcal{N} \) for which \( \pi(\mathcal{N}) \neq 0 \). For a fixed \( \mathcal{N} \) the following are mutually exclusive: \(^6\)

1. \( N_{\text{max}} \in \mathcal{N} \),
2. \( N_e \in \mathcal{N} \),
3. \( N_i = N_j \cup e \) for some \( i, j \),
4. \( N_i = N_j \cup e \cup N_l \) for some \( i, j, l \) with \( N_j \cap N_l = \emptyset \).

Moreover if (3) or (4) happens it does so for unique indices. \(^7\) Then \( \pi(\mathcal{N}) \neq 0 \) if and only if both one of (1-4) happens and each (other) time \( N - e \) is disconnected then exactly one of the two components already appears as a nest (“other” means excluding (4) above, which is the case where both components are nests).

We now assume \( \pi(\mathcal{N}) \neq 0 \) and fix the mod 2 ordering on \( \pi(\mathcal{N}) \) as follows. We first permute the elements of \( \mathcal{N} \) so that in case (1) \( N_{\text{max}} \) is in the last position, in case (2) \( N_e \) is in the penultimate position, in case (3) \( N_i \) is in the penultimate position, in case (4) \( N_i \) is in the penultimate position. Then \( \pi(\mathcal{N}) \) carries the order induced by removing the last entry in case (1) and the penultimate entry in cases (2,3,4). Note that if there exists some other nest \( N \in \mathcal{N} \) with \( N - e \) disconnected, then (as above) exactly one of these components already appears on the list. The induced order on the terms in \( \pi(\mathcal{N}) \) is given by replacing \( N \) with the new component, while keeping the redundant component in its original position. Finally, we note that in the special case that \( \mathcal{N} = \{N_e\} \), there is no penultimate position, but we simply declare \( \pi(\mathcal{N}) \) to be the negative of the empty nesting. Observe that with these sign conventions, \( \pi \circ \iota \) is the identity.

**Proof that \( \pi \) is a chain map:** Consider the cases 1, 2a, 2b, 3a, 3b, 4a, 4b which correspond to (1-4) above, but where “a” means \( \pi(\mathcal{N}) \neq 0 \) (so every time \( N - e \) is disconnected (if any), at least one of the components was already a nest) and case “b” means “not a”, and so in cases 2b, 3b, 4b, \( \pi(\mathcal{N}) = 0 \). Observe that in case (1), \( e \) is contained in no nest so there is only one case here. Since the terms in \( d(\mathcal{N}) \) add a nest, if (1-4) applies to \( \mathcal{N} \) then it also applies to every term in \( d(\mathcal{N}) \). We use this fact to show that \( \pi d = d \pi \) by looking at the possible cases.

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\(^6\)Proof of mutually exclusive: Say two nests “cross” if they are neither disjoint nor nested. If (1) then any nest containing \( e \) would cross \( N_{\text{max}} \), thus \( e \) is in no nest, hence not (2-4). If (2) then (3-4) would imply \( N_j \) and \( N_e \) cross, whence not (3) or (4). Finally (3) implies \( N_l \) would cross \( N_i \) (index from (3)) hence not (4).

\(^7\)Proof: In both (3) and (4), \( N_i \) is characterized as the smallest nest in \( \mathcal{N} \) containing \( e \), which is unique by Lemma 3.3.
Case 1: If $\mathfrak{R}$ contains $N_{\text{max}}$, then applying $d$ before $\pi$ we add nests, but they can’t contain $e$ without crossing $N_{\text{max}}$. These are the same nests we find if we apply $\pi$ before $d$.

Case 2a: The terms in $d(\mathfrak{R})$ land in case (2a) or case (2b). Since $\pi$ of the terms in (2b) is zero, it suffices to show that $\pi$ of the terms of type (2a) is $d\pi(\mathfrak{R})$. The terms in $d(\mathfrak{R})$ of type (2a) are formed by adding a nest which either contains $e$ or doesn’t contain $e$. Those terms which don’t contain $e$ correspond to adding nests in $\gamma \setminus e$, and so correspond to terms in $d\pi(\mathfrak{R})$. On the other hand, consider those terms in $d(\mathfrak{R})$ of type (2a) formed by adding a nest $N$ which does contain $e$. If $N - e$ is connected, it specifies a nest on $\gamma \setminus e$ which is not an element of $\pi(\mathfrak{R})$. If $N - e$ is disconnected, then the type (2a) assumption implies exactly one of the components is not an element of $\pi(\mathfrak{R})$. In either case adding this new nest gives the corresponding term in $d\pi(\mathfrak{R})$.

Case 2b: The terms in $d(\mathfrak{R})$ land in case (2b) unless the added nest is the union of $e$ with one of the components of a nest in $\mathfrak{R}$ which was disconnected by removing $e$, (then we land in case (2a)). These terms come in pairs, one for each such component, which cancel after applying $\pi$ (see Figure 2). To see that they carry opposite sign note that the differential places the nest corresponding to each component in the last position, while $\pi$ of each of these replaces the disconnected nest with the other component. To relate these two terms requires transposing the last term with the position of the disconnected term. On the other hand $d\pi(\mathfrak{R}) = 0$, whence this case.

Cases 3, 4: This works the same as case (2), except the role of $N_c$ is played by the smallest nest containing $e$ (denoted $N_i$ above). In particular, $\mathfrak{R}$ being of case 3 or 4 is a property closed under $d$, and added nests in $d(\mathfrak{R})$ can’t contain $e$ without also containing all of $N_i$.

Case 5: Suppose none of the above cases occur. Then $d\pi(\mathfrak{R}) = 0$ and we will show $\pi d(\mathfrak{R}) = 0$ as well. The only possible terms in $d(\mathfrak{R})$ which are not annihilated by $\pi$ are those which are of type (1-4) above. Case (1) is possible only if no nest of $\mathfrak{N}$ contains $e$. In this case the terms in $d(\mathfrak{R})$ which don’t vanish under $\pi$ correspond to adding $N_{\text{max}}$ and adding the smallest valid nest containing $e$. (The fact that there is a smallest valid nest containing $e$ is ensured since $\mathfrak{R}$ is not of type (1).) The term which adds and removes $N_{\text{max}}$ will occur with a + sign, while the term which adds the smallest valid nest containing $e$ is of type (2), (3) or (4), and so must be put in the penultimate position to apply $\pi$, thus occurring with opposite sign, or occurs with opposite sign by convention if $\mathfrak{R}$ was the empty nesting.

We now suppose there is a nest in $\mathfrak{R}$ which contains $e$, and we let $N_i$ be the smallest such nest. The terms in $d(\mathfrak{R})$ which are not immediately seen to vanish under $\pi$ are those which add the smallest valid nest containing $e$ and which adds a connected component of the layer graph of $N_i - e$. Recall here that the layer graph inside $N_i$ is the graph formed by collapsing its subnests to vertices. The fact that $N_i$ is the smallest nest containing $e$ means that $e$ is an edge in this layer graph. If the layer graph of $N_i$ is disconnected in to two nontrivial components when removing $e$, then adding the smallest valid nest containing $e$ lands in case (2b),(3b) or (4b) since $N_i - e$ is disconnected with neither component appearing in $d(\mathfrak{R})$. On the other hand a term in $d(\mathfrak{R})$ which adds a component of the layer graph will be of type (5), since there are two nontrivial connected components. So in this case $\pi$ of each term in the differential vanishes, and in particular $\pi d(\mathfrak{R}) = 0$. 

![Figure 2](image-url)
Let us now consider the case that the layer graph of $N_i$ is not disconnected by removing $e$. Then there are two terms in $d(\mathcal{R})$ which are sent to something non-zero by $\pi$, they correspond to adding the smallest sub-nest in $N_i$ containing $e$ and adding the nest corresponding to the layer graph of $N_i - e$. The fact that $\mathcal{R}$ is not of type (2–4) ensures these are valid new nests. Taking $\pi$ of these two differential terms removes the just added nests. These two nestings have the same list of nests and it remains to verify that they have opposite sign. Assume without loss of generality that $N_i$ was in the final position of $\mathcal{R}$. In the former term the sign is determined by transposing the new nest into the penultimate position before applying $\pi$. In the latter term the sign is determined by applying $\pi$ without any permutation, hence these terms have opposite signs, and so $\pi d(\mathcal{R}) = 0$ in this case as well. See Figure 3 for an example.

This completes our verification that $d\pi = \pi d$.

**Definition of $H$:** Informal description: Remove $e$ one nest at a time.

Precise definition: we define the chain homotopy $H : C^*(\gamma) \to C^{*+1}(\gamma)$ as follows. Fix a nesting of $\gamma$, denoted $\mathcal{R} = \{N_1, \ldots, N_r\}$. If $e$ is not contained in any nest of $\mathcal{R}$ then $H(\mathcal{R}) = 0$. Else we assume that $N_r < N_{r-1} < \ldots < N_{r-q}$ are the nests containing $e$ (after Lemma 3.3), and in particular $N_r$ is the smallest nest containing $e$. In this case we define $H$ in two steps. The first step is to remove $e$ from $N_r$. We may view this is a nest (or a pair of nests in the disconnected case) on $\gamma \setminus e$ (as above) and hence a nesting on $\gamma$. We say $H(\mathcal{R})$ is zero unless this reduces the number of nests by 1, in which case we call this new nesting the leading term of $H(\mathcal{R})$. It has nests $N_{r-1} < \ldots < N_{r-q}$ which contain $e$. We then proceed to remove $e$ from each of these nests one at a time in the nested order to form a new nesting at each stage. We continue until we reach the end, or until we get a nest of the wrong degree (which can only happen due to disconnectivity since $N_i \setminus e$ and $N_{i-1}$ are separated by at least two edges). $H(\mathcal{R})$ is defined to be the sum of these nestings. In particular $H(\mathcal{R})$ is a sum of at most $q + 1$ nestings.

Regarding signs: $H(\mathcal{R})$ is nonzero only if removing $e$ from the smallest nest containing it reduces the number of nests by 1. This happens if and only if one of the mutually exclusive cases (2),(3),(4) above occur. In these cases we follow the opposite reordering conventions as we did above when we defined $\pi$ – namely we assume the smallest nest containing $e$ is in the final position before removing/identifying it.

**Proof that $H$ is a chain homotopy.** It remains to verify that $dH + Hd = id - \iota \pi$. We will do this by analyzing the possible cases for $\mathcal{R}$ as considered above.

**Case 1:** If $N_{\max} \in \mathcal{R}$ then $(id - \iota \pi)(\mathcal{R}) = 0$, while $e$ is contained in no nest of $\mathcal{R}$ and no nest of $d(\mathcal{R})$. Hence $dH + Hd(\mathcal{R}) = 0$ as well.

**Case 2a:** This means $N_r = N_{r-1}$, and each time $N_i - e$ is disconnected, exactly one if its components appears in $\mathcal{R}$. Terms in $Hd$ are given by first adding a nest, call it $N_d$, and then applying $H$ which removes $e$ one nest at a time. Under case (2a), these terms appear in $dH$ by adding either $N_d$ or (a connected component of) $N_d - e$ to each nesting appearing in $H(\mathcal{R})$. Here there is the possibility that $N_d - e$ is disconnected in to two components, neither of which belong to the given nesting, but $H$ applied to such a term will vanish. Thus, the terms of $Hd \subset$ the terms in $dH$ (up to sign) in this case.
The signs in the above correspondence are opposite because \( dH \) removes the smallest nest containing \( e \) in the last position, while \( Hd \) does the same thing from the penultimate position. In particular \( dH + Hd \) is the sum of terms in \( dH \) which do not appear in \( Hd \). Let us describe these. There are (at most) two terms for each such \( N_i \) containing \( e \). The \( i^{th} \) term (reindexing to count right to left) in \( H(\mathcal{R}) \) has nests \( N_i - e \) and \( N_{i+1} \). Applying \( d \), the two distinguished terms are adding \( N_i \) or adding \( N_{i+1} - e \). These terms cancel in pairs, except for the first and last ones which are \( N_{\text{max}} \) and \( N_e \) respectively (interpreting the last stage as adding \( N_{\text{max}} \)). Adding back \( N_e \) gives \( id \) and adding in \( N_{\text{max}} \) after having removed all the \( e \) terms gives \( -\pi \). The minus sign occurs because \( \pi \) must transpose \( N_e \) into the penultimate position. See Figure 4 for an example of this.

**Case 2b:** Now suppose that \( \mathcal{R} \) is of type \((2b)\), and so \( \pi(\mathcal{R}) = 0 \). This happens due to disconnectivity of some \( N_{r-j} - e \) such that neither component appears for some \( j \geq 1 \). In this case, the terms of \( Hd \) which do not appear in \( dH \) correspond to adding one or the other component unioned with \( e \) via \( d \). Applying \( H \) identifies these terms with opposite sign, corresponding to permuting the components. Thus \( dH + Hd \) may again be described as the terms in \( dH \) which do not appear in \( Hd \). \( H \) consists of \( j \) nestings and there are \( 2j - 1 \) terms appearing in \( dH \) but not in \( Hd \). They correspond to adding nests \( N_{r-i+1} \) and \( N_{r-i} - e \) to the \( j^{th} \) term of \( H(\mathcal{R}) \) for \( 1 \leq i \leq j - 1 \), and \( N_{r-j+1} \) to the \( j^{th} \) term. These terms cancel in pairs, except for the first term which was the identity.

**Cases 3,4:** These cases again follow similarly to case 2, except the role of \( N_e \) is played by the smallest nest containing \( e \). See Figure 5 for an example of this.

**Case 5:** We now suppose cases (1-4) do not occur. If \( e \) is not contained in any nest, then \( \pi(\mathcal{R}) = 0 \) and \( H(\mathcal{R}) = 0 \), so it remains to analyze \( Hd(\mathcal{R}) \). The only non zero terms in \( Hd(\mathcal{R}) \) are given by adding \( e \) and then taking away \( e \). That is, there is a unique smallest nest which adds \( e \) appearing in the differential, and \( H \) of this term gives the identity back (with correct sign by convention). The other terms in the differential are annihilated by \( H \), and so \( Hd(\mathcal{R}) = \mathcal{R} \), which in this case implies \( Hd + dH = id - \pi \).

So now suppose \( e \) is contained in some \( N_e < ... < N_{r-q} \). Having excluded cases (2-4), we must have one of the following mutually exclusive situations:
The groupoid colored operad encoding (cyclic) operads is Koszul.

Definition 3.13. The category of algebras over $\mathbb{M}_\infty := D(\mathbb{M}^3)$ with its $\infty$-morphisms is the category of weak modular operads.
Figure 7. For $\gamma$, either of the 4 edged graphs above, $C^{-}_{\ast}(\gamma)$ is a 3 dimensional CW complex depicted below it. In both cases $C^{-}_{\ast}(\gamma \setminus e)$ is a 2 dimensional CW complex equal to chains on a solid pentagon. Here we depict the polyhedra in top down/annulus view – the three cells are not depicted. Only the codimension 1 cells are labeled. The colors indicate the flow of a contraction in which the inside/top cells (labeled by $e$) are contracted to the outside/bottom solid pentagons labeled by $abc$. This contraction encodes the maps $H, \iota, \pi$ constructed in the proof of Theorem 3.10 up to signs and degrees.

Let us unpack this definition. A weak modular operad has an operation parametrized by every modular graph $\Gamma$ carrying a mod 2 order on its set of edges. This operation is invariant with respect to automorphisms of the graph. The degree of this operation is $1 - e(\Gamma)$ (under cohomological conventions). These operations may be freely composed to specify an operation corresponding to every nested graph. These compositions are then subject to the differential constraint that $d$ of an operation corresponding to a nested graph $\mathcal{N}$ is the sum of operations corresponding to ways to add a nest to $\mathcal{N}$. Observe that a modular operad is a weak modular operad such that only one edged nests act in a non-zero way.

We consider the category of weak modular operads along with its $\infty$-morphisms. Explicitly, an $\infty$-morphism of weak modular operads $f: A \to B$ is a way to associate an element in $B(v_0)$ to every modular graph of color $(v_1, \ldots, v_n; v_0)$ carrying vertex labels in $A(v_i)$ and a mod 2 order on the set of edges. The differential of such a map is a signed sum over ways to partially compose along subgraphs in $A$ and then apply $f$ plus the ways to map along subgraphs via $f$ and then compose in $B$.

We may similarly consider weak $\mathcal{R}$-twisted modular operads, which we define as algebras over $(\mathcal{M}_\mathcal{R})_{\infty} := D(\mathcal{M}^1_{\mathcal{R}})$ In particular a weak $\mathcal{R}$-twisted modular operad has an operation of degree 1 associated to every graph.

3.6. Massey Products for modular operads. For us, the terminology “Massey product” refers to the transferred weak modular operad structure on the homology of a modular operad (resp. the $\mathcal{R}$-twisted analog). Let us unpack the definition here in the non-twisted case.
Let $A$ be a dg modular operad and choose a deformation retract of $\mathcal{V}$-modules:

\[ h \circlearrowleft A \xrightarrow{\iota} \xrightarrow{\pi} H_*(A) \]

Fix a modular graph $\gamma$ of type $v_0$ whose vertices are of type $(v_1, \ldots, v_r)$. The Massey product corresponding to this information is:

\[ mp_\gamma: \Sigma^{s(\gamma) - 1} (\otimes_{i=1} H_*(A)(v_i)) \to H_*(A)(v_0) \]

defined by summing over all full nestings of $\gamma$, where we apply $\iota$ at the vertices (to move to $A$), apply $h$ at each nesting, and finally $\pi$ to move back. This operation has degree equal to the number of nests, which for a full nesting is 1 fewer than the number of edges. Observe that if $\gamma$ has one edge, then the corresponding Massey product is degree 0 and coincides with the modular operad structure induced from the chain level. Since the differential on $H_*(A)$ is zero, the differential condition says that the sum over all ways to add a nest (viewed as composition in the free operad generated by the above operations) is ($d$ of something which is in turn) equal to 0.

Having endowed $H_*(A)$ with the structure of a weak modular operad in this way, the map $\iota$ extends to a morphism $\iota: H_*(A) \to A$. The component $\iota_\gamma$ of this morphism runs $\otimes_{i=1} H_*(A)(v_i) \to A(v_0)$ and may be defined as $h$ post-composed with the sum over full nestings whose vertices are labeled by $\iota$ and nests are labeled by $h$.

We remark that the $K$-twisted analog works similarly, with differing degrees.

4. Graph homology and the weak Feynman transform.

In this section we:

- Define the weak Feynman transform of a weak modular operad and use Theorem 3.10 to establish its homotopy theoretic properties.
- Define (hairy) graph homology and use the weak Feynman transform to show that Massey products hit all graph homology classes.
- Construct spectral sequences which compute the homology of weak Feynman transform from the homology of the classical Feynman transform.

4.1. The weak Feynman transform. In this section we consider the categories of weak modular operads with their $\infty$-morphisms.

**Definition 4.1.** The weak Feynman transform is a pair of functors

\[ \mathfrak{ft}^+ : \{\text{weak modular operads}\} \rightleftarrows \{\text{weak } K\text{-modular operads}\} : \mathfrak{ft}^- \]

defined by $\mathfrak{ft}^+ := D_{\mathcal{M}_\infty}$ and $\mathfrak{ft}^- := D_{(\mathcal{M}_K)_\infty}$.

Let’s unpack this definition. The weak Feynman transform of a weak modular operad $A$ may be identified with the free $s\mathcal{M}^! \cong M_K$-algebra on $A^*$. By definition this is given level-wise by the space:

\[ \mathfrak{ft}^+(A)(g, n) = \bigoplus_{\vec{v} \text{ such that } v_0 = (g, n)} \mathcal{M}_\mathcal{R}(\vec{v}) \otimes^{\vec{R}} A^*(\text{in}(\vec{v})) \]  

(4.1)

We may think of an element of this space as a graph whose vertices are labeled by elements of $A^*$. The implication of taking invariants with respect to automorphisms of $\vec{v}$ is two-fold. First, it identifies across permutation of vertex orders. This action is free on $\mathcal{M}_\mathcal{R}$ and has the effect of simply ignoring the vertex order of the graph. Second it ensures that permuting labels of flags which are then glued together to form edges has no effect. So if we denote by $\text{Aut}(\gamma)$ the group of flag permutations which preserve leaf labels and vertex adjacencies, then these elements are $\text{Aut}(\gamma)$-invariant.
Each summand in line 4.1 splits over the underlying graphs, so if we define $R(\gamma) := Det(Edges(\gamma))$, then we may write:

$$\text{ft}^+(A)(g,n) \cong \bigoplus_{(g,n)\text{-graphs } \gamma} (R(\gamma) \otimes A^*(\gamma))^{Aut(\gamma)}$$

where $A^*(\gamma)$ is the unordered tensor product of the inputs $A^*(v_i)$.

Let us now unpack the differential on these spaces. For each modular graph $\gamma$ of type $(g,n)$, the weak modular operad structure gives us a map:

$$(A(\gamma) \otimes R(\gamma))^{Aut(\gamma)} \rightarrow A(g,n)$$

of degree 1 – the structure map corresponding to $\gamma$ as defined above has degree $1 - e$ and the space $R(\gamma)$ is concentrated in degree $-e$. Define $d_\gamma$ to be the linear dual of this map. Notice this also has degree 1 (arrows switch directions and degrees switch sign). We then define

$$d: A^*(g,n) \rightarrow \bigoplus_{(g,n)\text{-graphs } \gamma} (A^*(\gamma) \otimes R(\gamma))^{Aut(\gamma)}$$

to be the sum over all $d_\gamma$. Due to the genus labeling, there are only finitely many $\gamma$ for a fixed $(g,n)$. This map extends uniquely to a degree 1 map of $\mathcal{R}$-twisted modular operads $d: \text{ft}^+(A) \rightarrow \text{ft}^+(A)$.

Conversely, since $s\mathcal{M}_\mathcal{R}^! \cong M$, the weak Feynman transform of a weak $\mathcal{R}$-modular operad $B$ may be identified with a free modular operad (not twisted):

$$\text{ft}^-(B)(g,n) = \bigoplus_{(g,n)\text{-graphs } \gamma} B^*(\gamma)^{Aut(\gamma)}.$$ 

In this case the differential may also be described as a sum over all graphs, although now the differential has degree 1 because all the operations had degree 1 to begin with. The comment about finiteness for terms in the differential still applies.

We remark that the above description may be compared with the original construction of Getzler and Kapranov to show that the Feynman transform of a (strict) modular operad (resp. strict $\mathcal{R}$-twisted modular operad) as defined in [GK98] agrees with Definition 4.1. This follows from the fact that for a strict modular operad, only the graphs with 1 edge act non-trivially and so $d$ coincides with the $\mathcal{GK98}$ differential, call it $d_{FT}$. The external differential in the weak Feynman transform is of the form $d = d_{FT} + \text{higher terms}$, but it is not in general true that $d_{FT}$ is square zero.

4.2. Parity. To formalize considerations of modular versus $\mathcal{R}$-twisted modular operads we define a category

$$\{\pm\text{modular operads}\} := \{\text{modular operads}\} \coprod \{\mathcal{R}\text{-twisted modular operads}\}$$

Here we take the disjoint union of both objects and morphisms. From now on we refer, by abuse of terminology, to objects of this category simply as modular operads, and we refer to their parity as odd (for those which are $\mathcal{R}$-twisted) or even (for those which are not) as needed. We repeat this construction to form the category of $\{\pm\text{weak modular operads}\}$, with their $\infty$-morphisms.

Note that with this definition $\text{ft}^\pm$ (resp. $\text{FT}^\pm$) combine to define parity reversing endofunctors on these categories. In particular, if we define $\text{ft} := \text{ft}^+ \sqcup \text{ft}^-$, then we may state the following immediate corollary of Theorem 3.10 without reference to parity:

**Corollary 4.2.** The functor $\text{ft}$ sends $\infty$-quasi-isomorphisms to quasi-isomorphisms. The functor $\text{ft}^2$ is level-wise quasi-isomorphic to the identity.

4.3. Definition of graph homology. Cyclic operads may be defined as the full subcategory consisting of those modular operads which have $A(g,n) = 0$ for $g \geq 1$. We similarly define the category of weak cyclic operads to be the full subcategory of weak modular operads which is 0 in genus $\geq 1$. Notice with our parity conventions, this gives us a notion of both even and odd (weak) cyclic operads.

We denote the inclusion functor from (weak) cyclic operads to (weak) modular operads by $\iota$. This functor is right adjoint to the functor which forgets higher genus, call this functor $\iota^*$. This functor is itself a right adjoint to a certain $\iota_*$, usually called the modular envelope. This
choice of notation for the triple of adjoint functors $(\iota_*, \iota^*, \iota)$ may be justified by considering their compatibility with the bar construction, see section 9.1 of [War19].

**Definition 4.3.** For a cyclic operad $O$, $O$-graph homology is the modular operad $G_O := H_*(\text{FT}(\iota_! O))$.

We emphasize the parity change: if $O$ was even then its graph homology is odd and vice versa. We remark that the relationship between odd cyclic operads and cyclic operads was extensively studied in [KWZ15]. In particular there is an isomorphism of categories between even and odd cyclic operads given by the functor $\Sigma s^{-1}$, shift and desuspend. In this way we can form canonical odd cyclic operads associated to a given cyclic operad. Thus to any cyclic operad we can consider both its graph homology (the odd modular operad $G_O$) and the graph homology of its oddification, the even modular operad $G_{\Sigma s^{-1}O}$. It is this latter modular operad which is what is more often called (hairy) graph homology and which matches the examples of [Kon93], [Kon94].

4.4. Massey products for graph homology. Fix $O$ a cyclic operad and choose a deformation retract of $\text{V}$-modules between $G_O$ and $\text{FT}(\iota_! O)$ using Lemma 2.19. Via the homotopy transfer theory above we then endow $O$-graph homology with the structure of a weak modular operad such that, as weak modular operads $G_O \sim \text{FT}(\iota_! O)$.

**Lemma 4.4.** $\text{ft}(G_O) \sim \iota_! O$. In particular $\text{ft}(G_O)(g, n) \sim 0$ if $g > 0$.

**Proof.** Apply $\text{ft}$ to $G_O \sim \text{FT}(\iota_! O) = \text{ft}(\iota_! O)$ and use Corollary 4.2 to conclude $\text{ft}^2 \sim 0$ for genus $\geq 1$. \qed

**Corollary 4.5.** Every graph homology class in genus $\geq 1$ is in the image of some Massey product.

**Proof.** Fix a graph homology class $\eta \in G_O(g, n)$ with $g \geq 1$. This determines an element in the complex $\text{ft}(G_O)(g, n)$ corresponding to the corolla with $n$ flags, genus label $g$ and vertex label $\eta^*$. This element is not a boundary (for degree reasons, since it labels a corolla and since the internal differential is 0), so it can not be a cycle without violating Lemma 4.4. It follows that the sum over linear duals of Massey products with target $\eta$ is non-zero, hence the claim. \qed

We remark that if $O$ is a Koszul cyclic operad, then $G_O(0, -)$ is simply the Koszul dual cyclic operad $O^!$, and in particular all classes in genus 0 are generated by cyclic operadic compositions of the generators of $O^!$.

4.5. Filtrations of the weak Feynman transform. Given a weak modular operad whose internal differential is zero, we may consider the underlying (strong) modular operad and its (classical) Feynman transform. The goal of this subsection will be to come up with filtrations which isolate this differential and hence produce spectral sequences which compute the homology of the weak Feynman transform from the homology of the (classical) Feynman transform.

4.5.1. Internal degree filtration. Let $A$ be a weak modular operad of even parity with internal differential $d_A = 0$. A homogeneous element in $\text{ft}(A)(g, n)$ is an element of some $\mathfrak{R}(\beta) \otimes A^*(\beta)$, for some modular graph $\beta$. If $\beta$ has $s$ edges, then we say such an element of degree $m$ has internal degree $r = m - s$. With respect to the bigrading $(m, r)$, the part of the differential corresponding to a graph $\gamma$ takes:

$$
\text{ft}(A)(g, n)^{m,r} \xrightarrow{d_{\gamma}} \text{ft}(A)(g, n)^{m+1,r+1-e(\gamma)}
$$

In particular, since $d_A = 0$, the ft differential can not increase the internal grading. We may thus define a filtration on the chain complex $\text{ft}(A)(g, n)$ by defining $F^r(\text{ft}(A)(g, n)) \subset \text{ft}(A)(g, n)$ to be the subspace of elements of internal degree $\leq r$.

We define $(E_*A)^{\bullet \bullet}, d^*)$ to be the family of spectral sequences associated to this filtration at each $(g, n)$.

**Lemma 4.6.** The spectral sequences $(E_*A)^{\bullet \bullet}, d^*)$ take the following form:

- $(E_0(A), d^0)$ is the (classical) Feynman transform of the (strong) modular operad underlying $A$.
• $E_1(A)$ is the homology of the Feynman transform of the (strong) modular operad underlying $A$.
• $d^n$ is induced by blowing up graphs with $n + 1$ edges. In particular if $A$ is a strong modular operad then $d^n = 0$ for $n \geq 1$.
• $E_\infty(A) = H_*(ft(A))$.

**Proof.** We observe that with respect to this bigrading, $r$ is preserved by $d_\gamma$ for graphs $\gamma$ with 1 edge and decreases otherwise. Thus $d^0$ blows up one edge, which is precisely the classical Feynman transform differential of the underlying modular operad. The convergence of the spectral sequence follows from Assumption 2.8, in particular this filtration is both bounded below and exhaustive. □

**Remark 4.7.** There is a corresponding spectral sequence for $B$ of odd parity. In this case we filter by row degree $r - s$ so that $d_s$ lowers the row/filtration degree for graphs with more than one edge and preserves the row for graphs with one edge.

4.5.2. **Genus label filtration.** First we recall (subsection 3.2) that modular graphs come with genus labeling of the vertices. Given a modular graph $\gamma$, we let $\ell(\gamma)$ denote the sum of its vertex labels and $g(\gamma)$ denote its total genus. These natural numbers are related by the formula $g(\gamma) = \ell(\gamma) + \beta_1(\gamma)$, where $\beta_1(\gamma)$ is the first Betti number of $\gamma$ when viewed as a CW complex.

In this section we let $B$ be a weak modular modular operad (of either parity) and we do not assume that $d_B = 0$. Define $F^q(ft(B)(g,n)) \subset ft(B)(g,n)$ to be the subspace of homogeneous elements whose corresponding modular graph has $\ell \leq q$. Since each $d_\gamma$ decreases $\ell$ by a nonnegative integer (namely $\beta_1(\gamma)$), this filtration is compatible with the differential. In addition it is both bounded below and exhaustive:

$$0 \subset F^0(ft(B)(g,n)) \subset \cdots \subset F^q(ft(B)(g,n)) = ft(B)(g,n)$$

If we use the bigrading $(r-s,\ell)$, for $ft(B)$, then this is just filtration by rows. Denote the associated spectral sequences over all $(g,n)$ by $(L_s(B)**, d^s)$. These are upper half plane spectral sequences which converge level-wise to $H_*(ft(B))$.

The spectral sequences $(L_s(B)**, d^s)$ are most interesting when we can understand the differential $d^0$. To fix a context in which this is possible we first make the following observation: many modular operads which are not formal have underlying cyclic operads which are formal. This means that when applying homotopy transfer theory to construct weak modular operads, the underlying weak cyclic operads will often be (strong) cyclic operads.

**Lemma 4.8.** Suppose that the weak cyclic operad $\iota^*(B)$ is a (strong) cyclic operad. Then the upper half plane spectral sequences $(L_s(B)**, d^s)$ have bottom rows $(L_0(B)**, d^0) = (FT(\iota^*(B)), d^0)$.

**Proof.** For any weak modular operad we have $F^0(ft(B)) = ft(\iota^*(B))$. The condition that the weak cyclic operad $\iota^*(B)$ is strong ensures that the only non-zero compositions in $\iota^*(B)$ are generated by one-edged trees, from which the equality of the differentials follows. □

In particular, in this case we find $\iota^*B$-graph homology on row 0 of page $L_1$: $\mathcal{G}_{\iota^*B} \cong L_1(B)**.d_0$.

4.6. **The case $O = \Sigma s^{-1} Lie$, aka $\Gamma_{g,n}$.** We conclude this paper by performing some analysis of these spectral sequences and Massey products associated to Lie graph homology.

Lie graph homology may be used to study the group homology of the outer automorphism groups of free groups. Here we will use the notation $\Gamma_{g,n}$ of [CHKV16], which denotes a group whose homology satisfies $H_*(\Gamma_{g,n}) \cong \mathcal{G}_{\Sigma s^{-1}Lie}(g,n)$. We begin by recalling several results from [CHKV16].

• The homology of $\Gamma_{g,n}$ is concentrated between degrees 0 and $2g + n - 3$ inclusive – this upper bound is called the virtual cohomological dimension (VCD) of $\Gamma_{g,n}$.
• Classes in the VCD can not be in the image of the modular operad generated by $H_0(\Gamma_{0,3})$ for degree reasons.
• The class generating $H_0(\Gamma_{0,3})$ along with those in the VCD generate all classes under the modular operad structure in genera $\leq 2$. This is conjectured to be the case for all genera.
• The dimension of $H_i(\Gamma_{1,n})$ is $binom{n-1}{i}$ if $i$ is even and $0 \leq i \leq n - 1$, and is 0 otherwise.
Proposition 4.9. (Massey Products for $\Gamma_{1,m}$) The semi-classical weak modular operad $H_*(\Gamma_{g\leq 1,n})$ is generated by $H_0(\Gamma_{0,3})$.

Proof. By “semi-classical” we mean the restriction to genera $\leq 1$, after [Get98]. The results of [CHKV16] reduce the statement to showing that the classes of degree equal to the VCD are in the image of some Massey product with genus 0 source. In genus 1, the homology classes of degree equal to the VCD are spanned by some $g_{2m+1} \in H_{2m}(\Gamma_{1,2m+1})$. By Corollary 4.5 $g_{2m+1}$ must be in the image of some Massey product

$$mp_\gamma: (\otimes_i H_*(\Gamma_{g_i,n_i})) \to H_{2m}(\Gamma_{1,2m+1}),$$

such that $\beta_1(\gamma) + \sum g_i = 1$.

Suppose that $\beta_1(\gamma) = 0$. Then the modular graph $\gamma$ is a tree with $s \geq 1$ edges along with some distinguished vertex $v_i$ of type $(1,n_i)$. By stability considerations we must have $n_i + s \leq 2m + 1$, since vertices of type $(0,2)$ and $(0,1)$ are not allowed in $\gamma$. The degree of $mp_n$ is $s - 1$ and the degree of the input is determined by the genus 1 label and must be $\leq n_i - 1$. Thus, the maximum output degree of such an operation is $n_i - 1 + s - 1 \leq 2m - 1$, which would be a contradiction. We thus conclude $\beta_1(\gamma) = 1$ from which it follows that $g_i = 0$ for all $i$. The claim then follows from the fact that each $H_*(\Gamma_{0,n_i})$ is generated by $H_0(\Gamma_{0,3})$ under cyclic operadic compositions. $\square$

It is interesting to see how the two spectral sequences in subsection 4.5 can be played off each other in this example. First consider the internal degree filtration spectral sequence (subsection 4.5.1) applied to $A = H_*(\Gamma_{g,n})$. In genus 1 it takes the following form.

1. There are only even rows.
2. The non-zero rows are complexes of trees along with a distinguished vertex. Row $2m$ may be viewed as a bar construction of the cyclic module $H_{2m}(\Gamma_{1,n})$ over the commutative operad.
3. The bottom row is $FT(\iota_* Com)(1,n)$.
4. The spectral sequence converges to 0 since $g \geq 1$.

See Figure 8 for an example when $n = 5$.

We remark that the bottom row of the spectral sequence is of independent interest, as it computes the top weight homology of the moduli space of punctured Riemann surfaces [CGP19]. Let us adopt the notation (up to a shift in degree) of op. cit. by writing

$$H_*(\Delta_{g,n}) := H_*(FT(\iota_* Com))(g,n).$$

In genus 1 this homology has rank $(n-1)!/2$ in degree $n$ and 0 elsewhere (Theorem 1.2 of [CGP19]). It is generated by vertex labellings of an $n$-gon. We remark that by $S_n$ equivariance of the differentials in this spectral sequence, the differential on page $2m$ must take the alternating class $g_{2m+1}$ to a class represented by the sum of labels of a $2m + 1$-gon.
Next we consider the genus label filtration spectral sequence (subsection 4.5.2) applied to $B = H_\ast(\Delta_{g,n})$. In genus 1 it takes the form:

1. There are two rows: the top corresponds to trees with a distinguished vertex (labeled by 1).
2. The bottom row on page $L_0$ is $\text{FT}(\iota!(\Sigma s^{-1}\text{Lie}))$ (since $\iota^\ast(H_\ast(\text{FT}(\iota_*\text{Com}))) = \Sigma s^{-1}\text{Lie}$).
3. The bottom row on page $L_1$ is $H_\ast(\Gamma_{1,*})$.
4. It converges to $k$ in every bidegree.

See Figure 9 for an example.

As an immediate corollary we see that the row $\ell = 1$ of this spectral sequence computes the reduced homology $\tilde{H}_\ast(\Gamma_{1,n})$. We remark that this complex is substantially smaller than the $\ell = 0$ row; when $n = 4$ the bottom row is dimension 174, the top row is dimension 9; when $n = 3$ the bottom row has dimension 18, the top row has dimension 1. In particular the cycle in $\text{Lie}(2n+3)^{S_2}$ representing $g_{2n+1}$ may be found by applying the differential $d^1$ to the anti-invariant class of $H_\ast(\Delta_{1,2n+1})$, which is represented by a sum over labellings of a $2n + 1$-gon with a leg at each vertex.

To conclude we observe that the spectral sequences of subsection 4.5 may be applied to these examples for arbitrary genus. This in particular establishes Corollary 1. However, even in genus 2 a complete analysis of Massey products/differentials in these spectral sequences would be substantially more difficult, and I do not necessarily expect the analog of Proposition 4.9 to hold.

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