On Euclidean spinors and Wick rotations

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Abstract

We propose a continuous Wick rotation for Dirac, Majorana and Weyl spinors from Minkowski spacetime to Euclidean space which treats fermions on the same footing as bosons. The result is a recipe to construct a supersymmetric Euclidean theory from any supersymmetric Minkowski theory. This Wick rotation is identified as a complex Lorentz boost in a five-dimensional space and acts uniformly on bosons and fermions. For Majorana and Weyl spinors our approach is reminiscent of the traditional Osterwalder Schrader approach in which spinors are “doubled” but the action is not hermitean. However, for Dirac spinors our work provides a link to the work of Schwinger and Zumino in which hermiticity is maintained but spinors are not doubled. Our work differs from recent work by Mehta since we introduce no external metric and transform only the basic fields.

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1 Introduction.

Euclidean quantum field theory for bosons is used in particle physics to justify certain manipulations in Minkowski quantum field theory, such as the claim that for $t \to \pm \infty$ only the vacuum contributes to the transition amplitude \[1\]. (Sometimes a term $i\varepsilon \phi^2$ is added to the action by hand to obtain the same result in Minkowski spacetime, but this approach is somewhat ad hoc \[2\]). Furthermore Euclidean field theory is crucial for the physics of instantons. If one couples fermions to instantons, one needs Euclidean field theory for spinors. In fact, an $N = 2$ supersymmetric field theory has been constructed in Euclidean space \[3\]. Also finite temperature field theory and lattice gauge theory \[4\] use Euclidean field theory. In Fujikawa’s approach \[5\] to anomalies in quantum field theories one must regulate the Jacobian with a Gaussian integral whose part quadratic in momenta should be converging in all directions. To find the subleading terms (for example, terms with $A_\mu \gamma_5$ or $iA_\mu \gamma_5$) one would like to know the continuation of the field theory to Euclidean space. Some mathematical applications of recent work by Seiberg and Witten are also based on supersymmetric theories continued to Euclidean space \[6\]. In this article we present a new interpretation of the Wick rotation to Euclidean field theory. To avoid misunderstanding, we stress that the object of our study is \textbf{not} the Wick rotation of the momentum variable $k_0$ in a Feynman amplitude; rather, we are interested in the Wick rotation of the underlying field theory.

For spinors a quantum field theory in Euclidean space whose Green’s functions coincide with those of Minkowski spacetime after analytic continuation in the time coordinate, was constructed by Osterwalder and Schrader (OS) in 1973 \[7\]. Their work was based on canonical quantization rather than path integrals and is, as a consequence, rather complicated. For example (i) a spinor field in four dimensional Euclidean space is expanded in creation and annihilation operators depending on Euclidean \textbf{four}-momenta $k_\mu^E$, rather than three momenta $\vec{k}$ as in the Minkowski case, (ii) fields do not (and cannot) obey field equations, (iii) instead of the familiar spinors $u_\alpha^r(\vec{k})$ and $v_\alpha^r(\vec{k})$ (where $r = \pm$ and $\alpha = 1,...,4$ is the spinor index) of Minkowski spacetime, one now has spinors $U_\alpha^R(k_\mu^E)$, $V_\alpha^R(-k_\mu^E)$, $X_\alpha^R(l_\mu^E)$ and $W_\alpha^R(-l_\mu^E)$ where $R = 1,...,4$.

However, if one rephrases their work in the language of path integrals,
then the only new aspect is that one should “double” the number of spinor fields in Euclidean space. For example, for a Dirac spinor the action in Minkowski spacetime and in Euclidean space, respectively, reads

\[
\mathcal{L}_M = -\psi^\dagger i\gamma^0 (\gamma^\mu \partial_\mu + m) \psi \quad ; \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad \mu, \nu = 0, \ldots, 3
\]

\[
\mathcal{L}_E = -\chi^\dagger_E (\gamma^E_\mu \partial_\mu + m) \psi_E \quad ; \quad \{\gamma^E_\mu, \gamma^E_\nu\} = 2\delta^{\mu\nu} \quad \mu, \nu = 1, \ldots, 4
\]

where \(\chi^\dagger_E\) is not related to \(\psi_E\) by complex or hermitean conjugation. We have written the word “double” in quotation marks because the number of Grassmann integration variables is the same: \(\psi\) and \(\psi^\dagger\) in one case, and \(\psi\) and \(\chi^\dagger\) in the other. The only property that has changed is hermiticity. The Euclidean action of OS is not hermitean.

There are other approaches to Euclidean spinors, due to Schwinger \[8\] and Zumino \[3\]; Fubini, Hanson and Jackiw \[9\]; and more recently Mehta \[10\]. Nicolai \[11\] extends the work of OS to Majorana spinors and defines the following action in Euclidean space

\[
\mathcal{L}_E(Maj) = -\frac{1}{2} \psi^\dagger_E C(\gamma^E_\mu \partial_\mu + m) \psi_E, \tag{3}
\]

where \(\psi_E\) no longer satisfies a reality condition and \(C\) is the charge conjugation matrix which can equally well be defined in Minkowski and Euclidean space (it satisfies \(C^\dagger = -C\) and \(C\gamma^\mu C^{-1} = -\gamma^\mu\)). In Minkowski spacetime one imposes the condition \(\psi^\dagger i\gamma^0 = \psi^\dagger C\gamma^0 = \psi^\dagger C\gamma^0\gamma^\dagger C \gamma_\mu C^{-1} \gamma^\mu = -\), which yields a hermitean action, but again \(\mathcal{L}_E(Maj)\) is not hermitean. Zumino required that the Euclidean action be hermitean (which is not necessary on physical grounds) and since no Euclidean Majorana spinors exist (by which we mean spinors in Euclidean space satisfying a reality condition), he considered an \(N = 2\) supersymmetric model with Dirac spinors, whose kinetic term for the spinor fields coincides with the original hermitean action proposed by Schwinger. Fubini et al. use “radial quantization”, in which the radius in Euclidean space plays the rôle of the time coordinate. (In string theory one usually continues the world-sheet

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4Notice that in the path integral \(\chi^\dagger\) and \(\psi\) are treated as independent variables while in the canonical approach they contain the same (doubled) creation and annihilation operators. This is just like \(\psi\) and \(\psi^\dagger\) in Minkowski space.

5 For anti-commuting variables we define \((\chi^\dagger \psi)^\dagger = \psi^\dagger \chi\). Our convention for the metric signature in Minkowski spacetime is \(\eta_{\mu\nu} = \text{diag}(-, +, +, +)\).

6 For a similar approach in two dimensions see \[12\].
time $t$ to $-i\tau$, and makes then a conformal map to the “$z$-plane”. This is not the radial quantization of [9]).

As far as we know, Mehta was the first to attempt to write down a continuous one-parameter family of actions $\mathcal{L}_\theta$ such that $\mathcal{L}_{\theta=0}$ is the Minkowski action and $\mathcal{L}_{\theta=\pi/2}$ is Schwinger’s Euclidean action (although he seems to be unaware of Schwinger’s work). In his approach, the dependence of the Dirac matrices on the parameter $\theta$ is not a unitary rotation, indeed it is even discontinuous at the midpoint $\theta = \pi/4$. An “interpolating metric” $(g^\theta)_{\mu\nu}$ is also introduced by hand such that the Euclidean action is hermitean, $SO(4)$ invariant and without fermion doubling.

For an application of Euclidean field theory to stochastic processes see [13], for a superfield formulation of Euclidean supersymmetry see [14] and for an approach that uses zero modes in Euclidean space to compute anomalies, see [15]. We shall later comment on the relation between some of the above approaches, but first we shall introduce ours.

2 A new Wick rotation for Dirac spinors.

In our construction we transform only the basic fields. Hence no extraneous metrics $g^\theta_{\mu\nu}$ are introduced. The transformation induced on the Dirac matrices is unitary. The basic observation from which we start is that for a complex vector field $W_\mu$ one continues the combination $W_\mu^* W^\mu = -W_0^* W_0 + W_i^* W_i$ (which appears in the action) from Minkowski to Euclidean space by giving both $W_\mu$ and $W_\mu^*$ the same Wick rotation phase $e^{i\theta}$. If $t \rightarrow e^{-i\theta} \tau$ with $\tau$ the Euclidean time coordinate at $\theta = \pi/2$,

\begin{align*}
W_0(t, \vec{x}) &\rightarrow e^{i\theta} W_4^\theta(\tau, \vec{x}), \\
W_0^*(t, \vec{x}) &\rightarrow e^{i\theta} W_4^{\theta*}(\tau, \vec{x}),
\end{align*}

(4)

(5)

At $\theta = \pi/2$, one then obtains $W_4^E W_4^E + W_i^E W_i^E$ where we call $W_4^{\theta=\pi/2} \equiv W_4^E$ the fourth component of the Euclidean field. We view this transformation for vector fields as a matrix which is diagonal with entries $(e^{i\theta}, 1, 1, 1)$. This

\footnote{A formal way to achieve this is to introduce “hyperbolic complex” numbers of the form $a + e b$ where $a$ and $b$ are real and $e$ satisfies $e^* = -e$, but $e^2 = +1$ [14]. If one then replaces the transformation law $W_0 \rightarrow e^{i\theta} W_4(\theta)$ by $W_0 \rightarrow e^{i2\theta} W_4(\theta) = (\cos \theta + ie \sin \theta) W_4(\theta)$, one finds $W_0^* W_0 \rightarrow e^{2ie\theta} W_4^*(\theta) W_4(\theta)$ so that at $\theta = \pi/2$ one obtains $-W_4^2 W_4$.}
suggests to define a Wick rotation for a general field as in the theory of induced representations: by transforming its coordinates (the orbital part) and its indices (the spin part). It is just an accident that for vector fields the spin matrix is diagonal, but in general the spin matrix may be more complicated.

These observations suggest that one should consider the following Wick rotation for a Dirac spinor

$$\psi(t, \vec{x}) \rightarrow S(\theta)\psi(\tau, \vec{x})$$  \hspace{1cm} (6)
$$\psi^\dagger(t, \vec{x}) \rightarrow \psi^\dagger(\tau, \vec{x})S(\theta)$$  \hspace{1cm} (7)

where \(\psi_{\theta=\pi/2} = \psi_E\) is the Euclidean Dirac spinor. \(S(\theta)\) is a \(4 \times 4\) matrix acting on spinor indices. Since the matrix \(S(\theta)\) is diagonal in the vector case, there is at this point an ambiguity whether we should use \(S(\theta)\) or \(S(\theta)^\dagger\) in (7). At first sight one might expect that the correct relation should involve \(S(\theta)^\dagger\) in order that the group property holds. However, since the group is abelian (there is only one parameter \(\theta\)), also \(S(\theta)\) is possible. In fact, the correct relation involves \(S(\theta)\) as in (7) and \(S(\theta)^\dagger\) is inconsistent, as we shall show.

In order that at \(\theta = \pi/2\) the Euclidean action contains the operator 

\[
\gamma^\mu E \partial_\mu = \gamma^4 E \partial_4 + \gamma^k E \partial_k \text{ where } x^4 = \tau \text{ and } \gamma^\mu_E \text{ satisfy the Clifford algebra in (2)}.
\]

we define a matrix \(M\) by

\[
S(\theta) \gamma^k = MS(\theta)\gamma^k - \gamma^k S(\theta).
\]

\[
S(\theta) \gamma^4 = MS(\theta)\gamma^4 = \gamma^4 \cos \theta + \gamma^5 \sin \theta,
\]

satisfy the Euclidean Clifford algebra. (The \(i\) in \(\partial_4 = i\partial_\tau\) converts \(\gamma^0\) into \(\gamma^4 = i\gamma^0\). In order that the action at \(\theta = \pi/2\) have an \(SO(4)\) rather than \(SO(3,1)\) symmetry we require that \(M\) commutes with the \(SO(4)\) generators \([\gamma^\mu_E, \gamma^\nu_E]\). Hence \(M\) is proportional to a linear combination of the unit matrix and \(\gamma^5_E\). It is natural to assume that the Wick rotation matrix \(S(\theta)\) should depend only on \(\gamma^4\) and \(\gamma^5\) because the Wick rotation does not act on the space sector. The general case \(M = \alpha I + \beta \gamma^5_E\) does not lead to an hermitean action and we continue with \(M \sim \gamma^5_E\). The solution for the spin matrix \(S(\theta)\) is given by

\[
S(\theta) = e^{\gamma^4 \gamma^5 \theta/2} \text{; } \gamma^5 = (\gamma^5)^\dagger, \text{ } (\gamma^5)^2 = 1; \text{ } \gamma^4 = (\gamma^4)^\dagger, \text{ } (\gamma^4)^2 = 1.
\]

\footnote{Our Euclidean signature is (+ + + +) so Euclidean indices may be raised and lowered with impunity, i.e. \(x^4 = x_4\). We define \(\gamma^4 \equiv i\gamma^0\) and \(\gamma^5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4\), similarly \(\gamma^5_E = \gamma^1_E \gamma^2_E \gamma^3_E \gamma^4_E\).}
Then

\[ L_E = -\psi_E^\dagger(\tau, \vec{x})\gamma^4(\gamma_E^4 \partial_4 + \gamma_E^k \partial_k + m)\psi_E(\tau, \vec{x}) \]  

(11)

where

\[ \gamma_E^k = S^{-1}(\theta = \pi/2)\gamma^k S(\theta = \pi/2) \]  

(12)

\[ \gamma_E^4 = S^{-1}(\theta = \pi/2)\gamma^4 S(\theta = \pi/2). \]  

(13)

It is now clear that the choice \( \psi \rightarrow \psi^\dagger S^\dagger \) (where \( S \equiv S(\theta = \pi/2) = e^{\gamma^4 \gamma^5 \pi/4} \) is not possible since defining \( S^\dagger \gamma^4 = (\alpha I + \beta \gamma_E^5)S^{-1} \) and using \( \gamma_E^5 = S^{-1}\gamma^5 S \) leads to \( SS^\dagger = \alpha \gamma^4 + \beta \gamma^5 \gamma^4 \) which is not consistent as in a general representation the right hand side is not symmetric.

Clearly, \( S(\theta) \) is unitary for all real \( \theta \), and satisfies \( S(\theta)\gamma^4 = \gamma^4 S^{-1}(\theta) \). Hence \( \gamma^\mu_E \) is hermitean. We note that \( S(\theta) \) describes a rotation in the “plane” spanned by \( \gamma^4 \) and \( \gamma^5 \). In particular, at \( \theta = \pi/2 \),

\[ \gamma_E^k = \gamma^k, \quad \gamma_E^4 = \gamma^5. \]  

(14)

Furthermore \( \gamma_E^5 = -\gamma^4 \). It clearly satisfies

\[ (\gamma_E^5)^\dagger = \gamma_E^5, \quad (\gamma_E^5)^2 = 1, \quad \{\gamma_E^5, \gamma^\mu_E\} = 0 \]  

(15)

and quantities such as \((1 + \gamma^5)/2\) appearing in an action transform into \((1 + \gamma_E^5)/2 = (1 - \gamma^4)/2\). Our Euclidean action can then finally be written as

\[ L_E = \psi_E^\dagger(\tau, \vec{x})\gamma_E^5(\gamma_E^\mu \partial_\mu + m)\psi_E(\tau, \vec{x}). \]  

(16)

This is the hermitean action Schwinger wrote down in 1959, but for eight component spinors. He considered real four component spinors in Minkowski spacetime and obtained the Euclidean theory by directly rotating the Dirac matrices. In order to still be able to impose a reality condition he found it necessary to double the number of spinor components in Euclidean space.

We have given a derivation which starts from transformation rules of the fields, which is clearly more fundamental. We keep the number of components of spinors equal to four, but since a Dirac spinor is equivalent to an eight component real spinor the Euclidean action we obtain is equivalent to Schwinger’s action. Also in Zumino’s work, the \( \gamma_E^5 \) is not explicit since he considered massless spinors and worked with Dirac matrices \( \gamma_E^\mu \) satisfying \( \{\gamma_E^\mu, \gamma_E^\nu\} = 2\delta^{\mu\nu} \). From our perspective \( \gamma_Z^\mu = i\gamma_E^5\gamma_E^\mu \) rather than \( \gamma_Z^\mu = \gamma_E^\mu \).
The procedure we have followed to obtain Euclidean spinors from their Minkowski counterparts boils down to the analytic continuation \( t \rightarrow -i\tau \) and a simultaneous rotation of the spinor indices. In this respect we differ from the work of OS who continue the Minkowski theory in time only and then construct a corresponding canonical Euclidean field theory. For vector fields one rotates the indices exactly as if \( \tau = it \) were part of a general coordinate transformation in the \( t\tau \) plane (\( t' = it \) yields \( W'_0(t') \equiv W_4(t') = \frac{\partial}{\partial t}W_0(t) = -iW_0(t) \)). Requiring that the flat space vielbein field \( e^\mu_m = \delta^\mu_m \) does not change, induces a compensating Lorentz transformation in the 4–5 plane. Our Wick rotation matrix \( S(\theta) \) is just the spinor representation of this complex finite Lorentz boost. Further interesting questions are raised if we consider our Wick rotation in curved space.

3  Dirac Spinors.

Zumino [3] constructed a supersymmetric Euclidean field theory in 1977 which follows Schwinger’s approach. The aim was to incorporate instantons into supersymmetric theories. The \( N = 2 \) supersymmetric action in Euclidean space obtained by him contains the fields \( A \) (scalar), \( B \) (pseudoscalar), \( V_\mu \) (real vector) and \( \psi \) (Dirac spinor) just as for the Minkowski spacetime action. However the actions differ in crucial parts. In Minkowski spacetime the \( N = 2 \) supersymmetric Yang Mills action reads

\[
\mathcal{L}_M = -\frac{1}{4}F^2_{\mu\nu} - \frac{1}{2}(D_\mu A)^2 - \frac{1}{2}(D_\mu B)^2 - \frac{1}{2}g^2(A \times B)^2 - \bar{\psi}^\dagger \gamma_4 \cdot D / A \psi - g\bar{\psi}^\dagger \gamma^4 \cdot (iA + B \gamma^5) \times \psi,
\]

where \( D_\mu A = \partial_\mu A + gV_\mu \times A \), idem \( D_\mu B \) and \( D_\mu \psi \), and \( F^2_{\mu\nu} = -2F^2_0 + F^2_5 \). The Minkowski \( N = 2 \) supersymmetry transformations are

\[
\delta \psi = \bar{D} A \epsilon + i\gamma^5 \bar{D} B \epsilon - \frac{i}{2} \gamma^\mu \gamma^\nu F_{\mu\nu} \epsilon - g(A \times B)\gamma^5 \epsilon \quad (18)
\]
\[
\delta \bar{\psi} = -\bar{\tau} \bar{D} A + i\bar{\tau} \gamma^5 \bar{D} B - \frac{i}{2} \bar{\tau} \gamma^\mu \gamma^\nu F_{\mu\nu} + g\bar{\tau} \gamma^5 (A \times B) \quad (19)
\]
\[
\delta A = \bar{\tau} \psi + \bar{\psi} \epsilon \quad (20)
\]
\[
\delta B = -i(\bar{\tau} \gamma^5 \psi + \bar{\psi} \gamma^5 \epsilon) \quad (21)
\]
\[
\delta V_\mu = -i(\bar{\tau} \gamma_\mu \psi + \bar{\psi} \gamma_\mu \epsilon). \quad (22)
\]
where $\bar{\psi} \equiv \psi^\dagger \gamma^4$.

In Euclidean space the following hermitean and supersymmetric action was constructed

$$L_E = -\frac{1}{2} F_{\mu}^{E2} - \frac{1}{4} F_{ij}^{E2} - \frac{1}{2} (D_\mu A_E) + \frac{1}{2} (D_\mu B_E) + \frac{1}{2} g^2 (A_E \times B_E)^2$$

$$+ \psi_E^\dagger \gamma_5 \cdot \nabla_E \psi_E + ig \psi_E^\dagger \cdot (\gamma_5 A_E + B_E) \times \psi_E,$$

(23)

which is invariant under the following transformation rules

$$\delta \psi_E = \nabla_E A_E \epsilon_E - \gamma_5 \nabla_E B_E \epsilon_E - \hat{\gamma}_E \nu \gamma_\mu F_{\mu\nu} \epsilon - ig \gamma_5 A_E \times B_E \epsilon_E$$

(24)

$$\delta \psi_E^\dagger = \epsilon_E^\dagger \nabla_E \psi_E - \epsilon_E^\dagger \nabla_E B_E \gamma_5 \epsilon_E - \frac{i}{2} \epsilon_E^\dagger \gamma_5 \gamma_\mu F_{\mu\nu} \epsilon + ig \epsilon_E^\dagger \gamma_5 (A_E \times B_E)$$

(25)

$$\delta A_E = -\epsilon_E^\dagger \gamma_5 \psi_E - \psi_E^\dagger \gamma_5 \epsilon_E$$

(26)

$$\delta B_E = \epsilon_E^\dagger \psi_E + \psi_E^\dagger \epsilon_E$$

(27)

$$\delta V_\mu^E = i(\epsilon_E^\dagger \gamma_\mu \gamma_5 \psi_E + \psi_E^\dagger \gamma_\mu \gamma_5 \epsilon_E).$$

(28)

Note that the transformations $\delta A_E$, $\delta B_E$ and $\delta V_\mu^E$ are all hermitean, while $\delta \psi_E$ and $\delta \psi_E^\dagger$ are related by hermitean conjugation.

We claim that this result follows unambiguously from our Wick rotation. To see this we make the following transformations in the Minkowski action (we give only the finite transformations but one may reinstate the Wick rotation angle $\theta$ as outlined above to obtain a continuous Wick rotation)

$$\psi \rightarrow e^{\gamma_4 \gamma_5 \pi/4} \psi_E; \quad \psi^\dagger \rightarrow \psi_E^\dagger e^{-\gamma_4 \gamma_5 \pi/4};$$

$$V_\mu = (V_0, \tilde{V})_\mu \rightarrow (iV_4^E, \tilde{V}_E^E)_\mu; \quad d^4x \rightarrow -id^4x;$$

$$A \rightarrow A_E; \quad B \rightarrow iB_E.$$  

(29)

The $i$ in the rule for $V_0$ is the $i$ from equation (5), but the $i$ in the rule for the pseudoscalar $B$ needs a short explanation. If a pseudoscalar is represented in Minkowski spacetime in the form $e^{\mu\nu\rho\sigma} \partial_\mu \phi_1 \partial_\nu \phi_2 \partial_\rho \phi_3 \partial_\sigma \phi_4$, then under the Wick rotation $t \rightarrow -i\tau$, the field $B$ goes over into $iB_E$ since only fields, not constants such as $e^{\mu\nu\rho\sigma}$ are transformed in our approach.

\footnote{To match the notations in (23)-(28) with those of [3], replace $A_E \rightarrow B$, $B_E \rightarrow A$ (the reason for this permutation will soon become clear), $\gamma_5^E \rightarrow \gamma^\mu$, $-\gamma_5^E \rightarrow \gamma_5 (\gamma_5^E \rightarrow i\gamma_5 \gamma^\mu)$ and the Euclidean supersymmetry parameter $\epsilon_E \rightarrow -i\epsilon$.}
It is straightforward to check that the substitution of the transformations (29) into the Minkowski action (17) yield the Euclidean action (23). Under the same Wick rotation rules, the Minkowski supersymmetry transformations go over to those of the Euclidean theory provided we at the same time also replace $\epsilon \rightarrow e^{\gamma^4 \gamma^5 \pi/4} e_E$ and $\epsilon^\dagger \rightarrow \epsilon_E^\dagger e^{\gamma^4 \gamma^5 \pi/4}$. This rule is clearly compatible with the Wick rotation rules of $\psi$ and $\psi^\dagger$ since the $N = 2$ supersymmetry parameter $\epsilon$ itself is a Dirac spinor. Further we note that our Wick rotation is consistent with the hermiticity assignments of the continued Euclidean fields. This is a non-trivial statement, consider the supersymmetry transformation rule $\delta \psi = M\epsilon$ for some matrix $M$. To obtain the rule for $\delta \psi^\dagger$ one can proceed in two ways. Either one first constructs $\delta \psi_E$ and then takes the hermitean conjugate, or one first takes the hermitean conjugate of $\delta \psi$ and then continues. In the latter case one should use (7). Hence
\begin{align}
\delta \psi = M\epsilon \Rightarrow S\delta \psi_E = (M)_E S\epsilon_E \Rightarrow \delta \psi_E = S^{-1}(M)_E S \epsilon_E \\
\Rightarrow \delta \psi^\dagger_E = \epsilon_E^\dagger S^{-1}(M)_E^\dagger S \equiv S^{-1}M\epsilon S \Rightarrow \delta \psi^\dagger_E = \epsilon_E^\dagger S(M^\dagger)_E S^{-1} \\
(30)
\end{align}

(31)

(where $S \equiv e^{\gamma^4 \gamma^5 \pi/4} = S(\theta = \pi/2)$). Since the matrix $M$ may depend on fields and/or derivatives its continuation $M_E$ is in general not equal to $M$. Consistency requires
\begin{align}
S^{-1}(M)_E^\dagger S = S(M^\dagger)_E S^{-1}. \\
(32)
\end{align}

One may check that the matrix $M$ of the Minkowski supersymmetry transformation rule $\delta \psi$ in (18) satisfies this consistency condition.

It is interesting to note that just as non-compact $SO(3,1)$ Minkowski spacetime Lorentz transformations become compact $SO(4)$ rotations in Euclidean space, in contradistinction the compact axial $U(1)$ transformations of the Minkowski theory become non-compact scale transformations in the Euclidean case [3][10].

The requirement of hermiticity of the Euclidean action is less justified than the property which Osterwalder and Schrader (and Nicolai) imposed, namely that the Greens functions in Minkowski and Euclidean space are each other’s analytic continuation. We have shown that our approach happens to satisfy both requirements simultaneously for Dirac spinors. The way we
have achieved this is by not only continuing analytically the time coordinate as in OS, but also rotating spinor indices (with the matrix $S$). Finally let us observe that Zumino’s Euclidean pseudoscalar kinetic action $+1/2(\partial_\mu B)^2$ is not damped in the Euclidean path integral (recall that $\exp i \int d^4x \mathcal{L}_M \to \exp \int d^4x \mathcal{L}_E$). In this light one may prefer to trade the requirement of hermiticity (of the action and supersymmetry transformations) for damping by introducing a field $B'_E = i B_E$.

## 4 Majorana spinors.

Although we now have obtained a satisfactory treatment for Dirac spinors, the fundamental supersymmetry models in Minkowski spacetime are $N = 1$ supersymmetric with Majorana spinors. The question now arises what our approach yields in this case.

It is well known that the existence of Majorana spinors depends on the dimension and metric signature of spacetime and on whether one considers the massive or massless case [17]. Already in 1959 Schwinger [8] observed that no massive Majorana spinors can exist in four dimensional Euclidean space if one imposes a reality condition on the spinors themselves and hermiticity of the action. Within a path integral approach, Nicolai [11] obtained an action depending only on a single spinor $\psi$ (and not $\psi^\dagger$) without any reality condition,

$$\mathcal{L}_E(Maj) = -\frac{1}{2} \psi_E^\dagger C (\gamma^\mu E \partial_\mu + m) \psi_E$$

where $C$ is the Minkowski conjugation matrix and his Dirac matrices $\gamma_E^\mu = (\gamma^1, \gamma^2, \gamma^3, \gamma^4)$ undergo no rotation. In this framework, he is still able to write down a supersymmetric action since, within an $N = 1$ supersymmetric model, the reality of $\psi_E$ is not needed to verify invariance under supersymmetry transformations. Further, the various Fierz identities required are unchanged since the charge conjugation matrix $C$ is still that of the Minkowski theory.

In our approach we begin with the usual Minkowski action

$$\mathcal{L}_M(Maj) = -\frac{1}{2} \psi^\dagger C (\gamma^\mu \partial_\mu + m) \psi.$$  

Dropping the reality constraint, we can now unambiguously perform our
Wick rotation on $\psi$ as given above. We find

$$L_{E}(\text{Maj}) = -\frac{1}{2} \psi_{E}^{\top} S^{\top} CSS^{-1} (\gamma^{\mu} \partial_{\mu} + m) S \psi_{E}$$

and the matrix $C_{E} = S^{\top} CS$ also satisfies the relations $C_{E} = -C_{E}^{\top}$ and $C_{E} \gamma_{E}^{\mu} C_{E}^{-1} = -\gamma_{E}^{\mu}$. Formally this action is the same as that of Nicolai except that Nicolai performs no rotation on spinor indices (and therefore his charge conjugation matrix is the original Minkowski charge conjugation matrix). Neither action is hermitean and in neither case are the classical fields $\psi$ and $\psi^{\dagger}$ related by a reality condition.

Let us now turn our attention to the $N = 1$ Wess Zumino model \[18\] with Minkowski action

$$L_{M} = -\frac{1}{2} \left[ \left( \partial_{\mu} A \right)^{2} + m^{2} A^{2} + \left( \partial_{\mu} B \right)^{2} + m^{2} B^{2} + \psi^{\dagger} C (\vartheta + m) \psi \right]$$

$$-mgA(A^{2} + B^{2}) - g\psi^{\dagger} C (A + i\gamma^{5} B) \psi - \frac{1}{2} g^{2} (A^{2} + B^{2})^{2}, \quad (36)$$

invariant under the transformations

$$\delta A = \epsilon^{\dagger} C \psi; \quad \delta B = -i \epsilon^{\dagger} C \gamma^{5} \psi; \quad (37)$$

$$\delta \psi = (\vartheta - m)(A - i\gamma^{5} B) \epsilon - g(A - i\gamma^{5} B)^{2} \epsilon. \quad (38)$$

Under our Wick rotation rules $\psi \rightarrow S \psi - E$, $A \rightarrow A_{E}$, $B \rightarrow B_{E}$, $\epsilon \rightarrow S \epsilon_{E}$ and $t \rightarrow -i \tau$ we obtain

$$L_{E} = -\frac{1}{2} \left[ \left( \partial_{\mu} E A \right)^{2} + m^{2} A_{E}^{2} + \left( \partial_{\mu} E B \right)^{2} + m^{2} B_{E}^{2} + \psi_{E}^{\dagger} C_{E} (\vartheta^{E} + m) \psi_{E} \right]$$

$$-mgA_{E}(A_{E}^{2} + B_{E}^{2}) - g\psi_{E}^{\dagger} C_{E} (A_{E} + i\gamma^{5}_{E} B_{E}) \psi_{E} - \frac{1}{2} g^{2} (A_{E}^{2} + B_{E}^{2})^{2} \quad (39)$$

invariant under

$$\delta A_{E} = \epsilon_{E}^{\dagger} C_{E} \psi_{E}; \quad \delta B_{E} = i \epsilon_{E}^{\dagger} C_{E} \gamma^{5}_{E} \psi_{E}; \quad \quad (40)$$

$$\delta \psi_{E} = (\vartheta^{E} - m)(A_{E} - i\gamma^{5}_{E} B_{E}) \epsilon_{E} - g(A_{E} - i\gamma^{5}_{E} B_{E})^{2} \epsilon_{E}. \quad (41)$$

Let us make a few comments. (I) Since we no longer require reality we have not transformed the pseudoscalar field $B$ with a factor $i$. We could, however,
have done so. (II) The real fields $A$ and $B$ are still real in Euclidean space unlike $\psi$. Grassmann integration $D\psi$ sees no difference between real and complex spinors, but for bosons, Gaussian integration over real or complex scalars yields different results. However, the supersymmetry transformations $\delta A_E$ and $\delta B_E$ can no longer be real in the absence of a reality condition. In practical terms however, they still provide Ward identities for Greens functions. (III) In verifying the invariance of the $N = 1$ supersymmetric action we must use certain Fierz identities whose validity is unaffected by our Wick transformation since $C_{\psi}$ has the same properties as in the Minkowski case.

In summary, at the level of path integral quantization we obtain an action with the same essential features as the $N = 1$ supersymmetric action proposed by Nicolai. Namely a non-hermitean action depending on the spinor $\psi_E$ subject to no reality condition. In our approach however, Minkowski and Euclidean Greens functions are related not only by the continuation $t \to -i\tau$ but also by the rotation $S(\theta)$ of spinor indices.

## 5 Weyl spinors.

In Minkowski spacetime one may always rewrite a Weyl spinor as a Majorana spinor. However, in the previous section we found it necessary to drop the reality condition on Majorana spinors when continuing to Euclidean space. The Berezin integration of the path integral is not sensitive to this modification and in this way we were able to reproduce continued Greens functions.

We now consider the Wick rotation for Weyl spinors. In Minkowski spacetime $\psi_L = \frac{1}{2}(1 + \gamma^5)\psi$ and $\psi_L^\dagger = \psi_L^\dagger \frac{1}{2}(1 + \gamma^5)$ is the hermitean conjugate of $\psi_L$. As suggested by the analysis for Majorana spinors, we should first drop this hermiticity requirement and then perform the Wick rotation as for Dirac spinors. Therefore we consider spinors $\psi$ and $\chi^\dagger$, unrelated by hermitean or complex conjugation but nevertheless eigenstates of $\gamma^5$,

\[
\psi_{L,R} = \frac{1 \pm \gamma^5}{2}\psi \quad ; \quad \gamma^5\psi_{L,R} = \pm\psi_{L,R} ;
\]

\[
\chi_{L,R}^\dagger = \chi_{L,R}^\dagger \frac{1 \pm \gamma^5}{2} \quad ; \quad \chi_{L,R}^\dagger \gamma^5 = \pm\chi_{L,R}^\dagger ;
\]

(42) 

(43)
where \( \psi \) and \( \chi^\dagger \) are unconstrained Dirac spinors. Transforming \( \psi \to S\psi_E \) and \( \chi^\dagger \to \chi^\dagger_E S \) (as usual \( S = e^{i\gamma^5\pi/4} \)) in accordance with our Wick rotation rules, we obtain the transformation rules for Weyl spinors

\[
\psi_{L,R} \to S \frac{1 \pm \gamma_E^5}{2} \psi_E, \tag{44}
\]

\[
\chi^\dagger_{L,R} \to \chi^\dagger_E \frac{1 \mp \gamma_E^5}{2} S, \tag{45}
\]

where we used \( S^{-1} \frac{1\pm \gamma^5}{2} = \frac{1\pm \gamma_E^5}{2} \) (and similarly for \( S \) and \( S^{-1} \) interchanged).

Consider now the Minkowski action for a left-handed Weyl spinor (in which we have already relaxed hermiticity by replacing \( \psi^\dagger \to \chi^\dagger \))

\[
\mathcal{L}_M(\text{Weyl}, L) = -\chi^\dagger_L \gamma^4 \phi \psi_L = -\chi^\dagger \gamma^4 \phi \frac{1 + \gamma^5}{2} \psi. \tag{46}
\]

Under Wick rotation we obtain

\[
\mathcal{L}_E(\text{Weyl}, L) = \chi^\dagger_E \gamma^5 \phi \frac{1 + \gamma^5}{2} \psi_E = -\chi^\dagger_L \phi \psi^E_L. \tag{47}
\]

where in Euclidean space \( \psi^E_{L,R} = \frac{1 + \gamma_E^5}{2} \psi_E \) and \( \chi^\dagger_{L,R} = \chi^\dagger_E \frac{1 + \gamma_E^5}{2} \). As in the the Majorana case this action is not (and cannot be) hermitean. Further notice that the \( \chi^\dagger_L \leftrightarrow \psi_L \) coupling of Minkowski space becomes \( \chi^\dagger_R \leftrightarrow \psi_L \) in Euclidean space [10].

Let us now apply this formalism to the \( N = 1 \) super Yang Mills theory with Minkowski action

\[
\mathcal{L}_M = -\psi_L^\dagger \gamma^4 \cdot \overrightarrow{\partial} \psi_L - \frac{1}{4} F^2_{\mu \nu}, \tag{48}
\]

which enjoys the supersymmetry

\[
\delta \psi_L = \frac{1}{2} \gamma^\mu \gamma^\nu F_{\mu \nu} \epsilon_L, \tag{49}
\]

\[
\delta \psi_{L,R}^\dagger \gamma^4 = -\frac{1}{2} \epsilon_{L,R} \gamma^4 \gamma^\mu \gamma^\nu F_{\mu \nu}, \tag{50}
\]

\[
\delta V_\mu = -\epsilon_L \gamma^4 \gamma_\mu \psi_L + \psi_{L,R}^\dagger \gamma^4 \gamma_\mu \epsilon_L. \tag{51}
\]

Again, \( D_\mu \psi = \partial_\mu \psi + g V_\mu \times \psi \), the fields \( \psi \) and \( V_\mu \) are in the adjoint representation of some non-abelian gauge group and the subscript L denotes
projection by $\frac{1+\gamma^5}{2}$. We now replace $\psi$ in (48) by the independent spinor $\chi^\dagger$ and perform our Wick rotation $t \rightarrow -i\tau$, $\psi \rightarrow S\psi_E$, $\chi^\dagger \rightarrow \chi^\dagger E S$ and $V_\mu \rightarrow \text{diag}(i, 1, 1, 1) \mu \nu V^E_\mu \nu$. The result reads

$$L_E = -\chi^E_R \cdot (\partial^E + gV^E \times)\psi^E_L - \frac{1}{4} F^E_{\mu \nu}.$$  

(52)

Applying the transformation rules (44) and (45) to the supersymmetry transformations (49)–(51) we obtain the Euclidean supersymmetry transformation rules under which the Euclidean action (52) is invariant

\begin{equation}
\delta \psi^E_L = \frac{1}{2} \gamma^\mu \gamma^\nu F^E_{\mu \nu} \psi^E_L, \tag{53}
\end{equation}

\begin{equation}
\delta \chi^E_R = -\frac{1}{2} \epsilon^\dagger E R \gamma^\mu \gamma^\nu F^E_{\mu \nu}, \tag{54}
\end{equation}

\begin{equation}
\delta V^E_\mu = -(\epsilon^\dagger E R \gamma^\mu \psi^E_L - \psi^E_L \gamma^\mu \epsilon^E_L). \tag{55}
\end{equation}

As in the Majorana case, the action and the supersymmetry transformation rules no longer respect hermiticity and the same comments as made in the previous section apply here. Finally, notice also that by taking $\psi$ and $\chi^\dagger$ unrelated we obviate the need to check that the continuation procedure satisfies the consistency condition (32).

6 Conclusion.

We have defined a continuous Wick rotation on Dirac spinors $\psi$ and their hermitean conjugates by means of a Lorentz transformation $S(\theta)$ in the Euclidean-Minkowski $t \tau$ plane. The resulting Euclidean Greens functions are not only obtained from their Minkowski counterparts by continuation in the time variable, but also by a rotation of spinor indices. In this respect our work agrees with work of Schwinger, who, however, did not implement this idea on the basic fields. For Majorana and Weyl spinors however, one is forced to drop the reality/hermiticity condition if one wants to perform the Wick rotation and hence our formalism, which still uses the Lorentz rotation $S$, now resembles the Osterwalder Schrader approach.

Our final rule for the Wick rotation on spinors is

\begin{equation}
\psi(t, \vec{x}) \rightarrow S(\theta)\psi(\tau, \vec{x}), \tag{56}
\end{equation}

\begin{equation}
\psi^\dagger(t, \vec{x}) \rightarrow \psi^\dagger(\tau, \vec{x})S(\theta) \tag{57}
\end{equation}
with \( S(\theta) = e^{\gamma^4 \gamma^5 \theta/2} \). Using \( S(\theta)\gamma^4 = \gamma^4 S^{-1}(\theta) \), the Dirac Minkowski action goes over into

\[ -\psi^\dagger \gamma^4 (\gamma^k \partial_k + \gamma^0 \partial_0 + m) \psi \rightarrow -\psi^\dagger E \gamma^4 (\gamma^E_k \partial_k + \gamma^4 \partial_4 + m) \psi_E \quad (58) \]

and since we found that \( \gamma^4 = -\gamma^5_E \), the action is clearly hermitean and \( SO(4) \) invariant. Thus the troublesome \( \gamma^4 \) is not omitted in Euclidean space as one might expect, but rather reinterpreted as \( \gamma^5_E \) in Euclidean space \[10\].

With the understanding of the Wick rotation provided by our formalism one can explicitly and continuously follow how vertices and propagators in interacting field theories change as one rotates from the Minkowski to the Euclidean case. This has practical advantages, for example it explains which terms with factors of \( \gamma^5 \) and axial vector fields in regulators for anomalies acquire factors of \( i \). Another useful application is supersymmetry; our rules automatically generate \( N = 1 \) and \( N = 2 \) theories in Euclidean space whenever they exist in Minkowski space.

Also, another obvious open question is how to generalize our Wick rotation to odd dimensions in which there is no \( \gamma^5 \). One possibility is to double the Dirac matrices, which suggests that the cases in which one is able to operate with undoubled spinor degrees of freedom in Euclidean space are the exception rather than the norm \[20\].

In this article we have discussed only the path integral aspects of the Wick rotation. The Osterwalder Schrader approach is entirely based on a canonical approach, and our Wick rotation can also be formulated in a canonical way, but this will be published elsewhere \[21\]. There we also discuss the Osterwalder Schrader reflection positivity axiom which is the main ingredient for establishing the existence of a positive semi-definite self-adjoint Hamiltonian\[10\].

The last comment we wish to make is perhaps the most interesting but certainly the least understood and speculative. Both in our work and also in that of Osterwalder Schrader, one keeps noticing five-dimensional aspects\[11\].

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\[10\] In more detail, this axiom requires that there exists an anti-linear mapping \( \Theta \) which transforms an arbitrary function \( F \) of the fields at positive times into a function \( \Theta F \) of fields at negative times such that \( \langle \theta FF \rangle \) is semi-positive. If such a mapping exists one can define a Hilbert space with positive norm and a positive transfer matrix. The choice of time axis with respect to which time is defined is arbitrary and breaks the four-dimensional symmetry but the results do not depend on the axis chosen \[21\].

\[11\] It may be interesting also to recall the work of \[22\] in which the four-dimensional Wess
For example, in the OS canonical approach, the expansion into creation and annihilation operators has a normalisation factor $(k_E^2 + m^2)^{-1/2}$ in Euclidean space, as opposed to $(\vec{k}^2 + m^2)^{-1/2}$ in Minkowski spacetime. The former is clearly the usual normalization of a five-dimensional theory at $t = 0$, with four space momenta $k_E^\mu$. Further Schwinger introduces new fields $B(x)$ for the original fields $A(x)$ which satisfy the commutation relations $[A(x), B(y)] = \delta^4(x - y)$, which again can be viewed as an equal time canonical commutation relation in five dimensions. We found a Wick rotation from Minkowski to Euclidean space which is a five-dimensional Lorentz rotation, but there are many more questions and ideas yet to be developed.

**Acknowledgements.**

It is a pleasure to thank R. Schrader, P. Breitenlohner, D. Maison, E. Seiler and C. Preitschopf for discussions.

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*Zumino model is obtained by dimensional reduction “by Legendre transformation” from the on-shell theory five dimensions*
References

[1] C. Itzykson and J. Zuber, *Quantum Field Theory*, McGraw Hill 1980. E. Abers and B. Lee, Phys. Rep. 2 (1973) 1.

[2] L. Ryder, *Quantum Theory of Fields*, Cambridge University Press 1985. P. Ramond, *Field Theory: a modern primer*, Addison Wesley 1990.

[3] B. Zumino, Phys. Lett. 69B (1977) 369.

[4] For an application of the Osterwalder Schrader formalism to lattice fermions see K. Osterwalder and E. Seiler, Ann. Phys. 110 (1978) 440 and E. Seiler, *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics*, Springer-Verlag, Berlin 1982.

[5] K. Fujikawa, Phys. Rev. Lett. 42 (1979) 1195, Phys. Rev. D21 (1980) 2848, D22 (1980) 1499, Phys. Rev. Lett. 44 (1989) 1733. 65.

[6] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19, Erratum-ibid. B430 (1994) 485; Nucl. Phys. B431 (1994) 484.

[7] K. Osterwalder and R. Schrader, Phys. Rev. Lett. 29 (1972) 1423; Helv. Phys. Acta 46 (1973) 277; CMP 31 (1973) 83 and CMP 42 (1975) 281. K. Osterwalder, *Euclidean Greens fuctions and Wightman distributions*; in G. Velo and A. Wightman (Eds.) *Constructive Field Theory - Erice lectures 1973*, Springer-Verlag Berlin 1973. For a more recent discussion see K. Osterwalder in *Advances in Dynamical Systems and Quantum Physics*, Capri conference, World Scientific 1993.

[8] J. Schwinger, Phys. Rev. 115 (1959) 721 and 117 (1960) 1407. This paper contains the action for a Dirac spinor with a $\gamma^5$ matrix, see page 727 and the functional $W$ on page 728. See also Proc. Nat. Ac. of Sc. 44 (1958) 956, where different analytic continuations of the time variable for time-ordered or anti-time-ordered Minkowski Greens functions are considered.

[9] S. Fubini, A. Hanson and R. Jackiw, Phys. Rev. D7 (1973) 1732. R. Jackiw in Erice 1973, Proceedings, *Laws Of Hadronic Matter*, New York 1975. In this work, fields are canonically quantized on equal proper time
hyperboloids and subsequently continued to imaginary time so that the quantization becomes radial.

[10] M. Mehta, Phys. Rev. Lett. 65 (1990) 1983, 66 (1991) 522, Mod. Phys. Lett. A30 (1991) 2811, Phys. Lett. B274 (1991) 53.

[11] H. Nicolai, Nucl. Phys. B140 (1978) 294; B156 (1979) 157; B156 (1979) 177; Phys. Lett. 89B (1980) 341. For earlier work on Euclidean Majorana spinors see also J. Fröhlich and K. Osterwalder, Helv. Phys. Acta, 47 (1974) 781.

[12] D. Borthwick, J. Math. Phys. 34 (1993) 2691. This paper applies Nicolai's Majorana version of the OS formalism to two dimensions. For results for the Yukawa model in two dimensions see E. Seiler, Commun. Math. Phys. 42 (1975) 163.

[13] J. Kupsch and W. D. Thacker, Fortschr. Phys. 38 (1990) 35. Z. Haba and J. Kupsch, Fortschr. Phys. 43 (1995) 41; J. Kupsch, J. Geom. and Phys. 11 (1993) 507, Ind. J. Phys. 79A (1996) 293, and Rev. Math. Phys. 2 (1990) 457.

[14] J. Lukierski and A. Novicki, J. Math. Phys. 25 (1984) 2545.

[15] H. Banerjee, Z. Phys. C62 (1994) 511, Resolution of the strong CP and U(1) problems, hep-th/9508065.

[16] G. Gibbons, M. Green and M. Perry, Phys. Lett. B370 (1996) 37.

[17] P. van Nieuwenhuizen, Simple supergravity, and the Kaluza-Klein program; in B. DeWitt, R. Stora (Eds.), Relativity, Groups and Topology II – Les Houches 1983; North-Holland, Amsterdam 1984.

[18] J. Wess and B. Zumino, Nucl. Phys. B70 (1974) 39.

[19] R. Streeter and A. Wightman, PCT spin, statistics and all that., Benjamin, New York 1964. .

[20] A. Waldron, to appear.

[21] I. Montvay and G. Münster, Quantum Fields on a Lattice, Cambridge University Press 1994.
[22] M. Sohnius, K. Stelle and P. West, Nucl. Phys. B173 (1980) 127. I. Martin-Hernandez and J. G. Taylor, Phys. Lett. B185 (1987) 99.