Stochastic resonance in bistable systems: The effect of simultaneous additive and multiplicative correlated noises

Claudio J. Tessone * and Horacio S. Wio†

Comisión Nacional de Energía Atómica,
Centro Atómico Bariloche and Instituto Balseiro (CNEA and UNC),
8400-San Carlos de Bariloche, Argentina

Abstract

We analyze the effect of the simultaneous presence of correlated additive and multiplicative noises on the stochastic resonance response of a modulated bistable system. We find that when the correlation parameter is also modulated, the system’s response, measured through the output signal-to-noise ratio, becomes largely independent of the additive noise intensity.

*Electronic address: tessonec@cab.cnea.edu.ar, Fellow of CNEA

†Electronic address: wio@cab.cnea.edu.ar, Member of CONICET, Argentina and Regular Associate ICTP. http://www.cab.cnea.edu.ar/Cab/invbasica/FisEstad/estadis.htm
The phenomenon of *stochastic resonance* (SR) has attracted considerable interest in the last decade due, among other aspects, to its potential technological applications for optimizing the output signal-to-noise ratio (SNR) in nonlinear dynamical systems. The phenomenon shows the counterintuitive role played by noise in nonlinear systems as it contributes to enhance the response of a system subject to a weak external signal. There is a wealth of papers, conference proceedings and reviews on this subject, Ref. [1] being the most recent one, showing the large number of applications in science and technology, ranging from paleoclimatology, to electronic circuits, lasers, and noise-induced information flow in sensory neurons in living systems.

A tendency shown in recent papers, and determined by the possible technological applications, points towards achieving an enhancement of the system response (that is obtaining a larger output SNR) by means of the coupling of several SR units [2–6] in what conforms an “extended medium” [7]. Another aspect that has also attracted interest is finding a system where the SNR becomes largely independent of external parameters such as noise intensity [4,8].

In this work we present some results related to the latter aspect. In order to reach such a goal we study a bistable system subject to both an additive and a multiplicative noise but, at variance with the work in Ref. [9], we consider that both noises are correlated. In addition to the modulation of the bistable potential by a weak external signal, we also consider the effect of a modulation of the correlation between both noises. We show that the result of this contribution is to widen the maximum of the SNR as a function of the additive noise intensity, making the detection of the signal less sensitive to the actual value of that noise. It is worth remarking here that on one hand, the additive (external) noise can be (naturally) assumed as white, while on the other hand, the multiplicative (internal) one, that generally involves characteristic time scales of the system, would not necessarily be white. However, as indicated in [10], as a first step in this study, we can approximate the multiplicative colored noise by a white one. Here we exploit the results of Madureira *et al.* [10] where it was shown that the activation rate in a bistable system subject to correlated...
additive and multiplicative noises is dramatically suppressed or enhanced according to the sign of the correlation. The model equation we consider corresponds to a onedimensional bistable system described by the Langevin equation

\[ \dot{x} = \varepsilon(t) + x - x^3 + x\xi_1(t) + \xi_2(t), \] (1)

where \(\varepsilon(t) = \varepsilon_0 \cos(\Omega t)\), \(\varepsilon_0\) and \(\Omega\) are the intensity and frequency of the potential modulation respectively. The additive and multiplicative Gaussian white noises, indicated by \(\xi_1(t)\) and \(\xi_2(t)\), satisfy

\[
\langle \xi_i(t) \rangle = 0, \quad i = 1, 2 \\
\langle \xi_1(t)\xi_1(t') \rangle = 2\delta(t-t')Q \\
\langle \xi_2(t)\xi_2(t') \rangle = 2\delta(t-t')D \\
\langle \xi_1(t)\xi_2(t') \rangle = 2\rho(t)\delta(t-t')\sqrt{DQ}.
\] (2)

The correlation intensity between both noises is indicated by the parameter \(\rho(t)\), that fulfills the condition \(|\rho| \leq 1\). The associated the Fokker-Planck equation (Stratonovich prescription)

\[
\partial_t P(x, t) = -\partial_x \left[ \left( x - x^3 + Qx + \rho(t)\sqrt{QD} - \varepsilon(t) \right) P(x, t) \right] \\
+ D \partial^2_{xx} \left[ \left( 1 + Rx^2 + 2\rho(t)\sqrt{R} \right) P(x, t) \right].
\] (3)

In what follows we assume that \(\rho\) is a time dependent function having the form \(\rho(t) = \rho_0 \cos(\Omega_\rho t)\). Here, and in order to simplify the analysis, we consider the case when \(\Omega_\rho = \Omega\), the most general case will be discussed elsewhere [11].

In order to perform the evaluation of the correlation function and power spectral density needed to obtain the SNR, we exploit the results of the two-state model [12,13]. Those authors have reduced the problem of obtaining the SNR of a nonlinear and essentially bistable system subject to a weak periodic signal, to a description where the transitions occur between the two minima of the deterministic potential. The main approximation is an adiabatic-like one, corresponding to the assumption that the relaxation times around each
minima are much shorter than any other characteristic time, such as the Kramers time for transitions between the two stable points or the inverse of the signal frequency $\Omega$. It is also important to note here that, if the modulation is done around $\rho \neq 0$, it becomes necessary to extend the two state approach in order to include the asymmetry of the potential.

In the absence of any signal, the deterministic potential of the system has two minima located at the points $x_\pm = \pm 1$. It has been shown that such minima are not coincident with the maxima of the stationary probability distribution [13].

Let us call $W_+(t)$ and $W_-(t)$ the nonstationary transition rates from the state $x_+$ to $x_-$ and from the state $x_-$ to $x_+$ respectively. As indicated before, the signal is introduced through a simultaneous modulation in the potential and in the correlation. We may note here that the last contribution does not modify the deterministic potential barrier. However, what we find is that the rates $W_\pm$ do change, i.e. in [10] it was shown that if $\rho > 0$ the rate $W_+$ increases while $W_-$ decreases, and viceversa.

If both modulations are small in comparison with the barrier height, that is $\rho_0 \ll 1$ and $\varepsilon_0 \ll V(0) - V(\pm 1)$, then it is possible to make a Taylor expansion of the functions $W_\pm(t)$. We thus obtain

$$W_\pm(t) = \frac{1}{2} \left( W_o \mp (\alpha_\rho + \alpha_\varepsilon) \cos(\Omega t) + \mathcal{O}(\varepsilon_0^2) + \mathcal{O}(\rho_0^2) \cdots \right),$$

where $W_o$ is the transition rate evaluated in absence of modulation, $\rho_0 = \varepsilon_0 = 0$. The terms $\alpha_\rho$ and $\alpha_\varepsilon$ are given by

$$\frac{\alpha_\varepsilon}{2} = - \frac{dW_o}{d\varepsilon} \bigg|_{\varepsilon=0} \varepsilon_0,$$

$$\frac{\alpha_\rho}{2} = - \frac{dW_o}{d\rho} \bigg|_{\rho=0} \rho_0.$$

The transition rate $W_o$ could be obtained calculating the mean-first-passage-time $T(R, \rho)$ [10], yielding

$$\frac{1}{W_o} = T(R, \rho) = \frac{1}{D} \int_{-1}^{1} dx \, H(x) \exp[\Phi(x)/D] \int_{-\infty}^{x} dy \, H(y) \exp[-\Phi(y)/D].$$
where \( R \equiv Q/D \), and the function \( H(x) \) is
\[
H(x) = \sqrt{\frac{1}{1 + Rx^2 + 2\rho\sqrt{R}x}}.
\]
The determination of the effective potential \( \Phi(x) \) is made studying the stationary probability density, which is given by
\[
\Phi(x) = \int_{-\infty}^{x} H(x') \left( 1 + Rx'^2 + \left( \varepsilon_0 \cos(\Omega t) + 2\rho\sqrt{R} \right) x' \right) dx'.
\]

We obtained the derivatives \( dW_o(R, \rho)/d\varepsilon \) and \( dW_o(R, \rho)/d\rho \) from the evaluation of \( dT(R, \rho)/d\varepsilon \) and \( dT(R, \rho)/d\rho \) respectively, through
\[
\frac{dW_o(R, \rho)}{d\varepsilon} = -\frac{1}{T(R, \rho)^2} \frac{dT(R, \rho)}{d\varepsilon}, \tag{8}
\]
\[
\frac{dW_o(R, \rho)}{d\rho} = -\frac{1}{T(R, \rho)^2} \frac{dT(R, \rho)}{d\rho}. \tag{9}
\]
The results for the derivatives of \( T(R, \rho) \), after a long but straightforward calculation, are
\[
D^2 \frac{dT}{d\varepsilon} = -\int_{-1}^{1} dx H(x) \exp \left[ \frac{\Phi(x)}{D} \right] \int_{-1}^{x} dy H(y) \exp \left[ \frac{-\Phi(y)}{D} \right] \int_{-1}^{x} dy H(y)^2 y
+ \int_{-1}^{x} dy H(y) \exp \left[ \frac{-\Phi(y)}{D} \right] \int_{-1}^{y} dz H(z)^2 z, \tag{10}
\]
for the contribution of the potential modulation, while for the contribution of the correlation modulation we obtain
\[
\frac{D^2}{\sqrt{QD}} \frac{dT}{d\rho} = -\int_{-1}^{1} dx H(x) \exp \left[ \frac{\Phi(x)}{D} \right] \times H(x)^2 x \int_{-1}^{x} dy H(y) \exp \left[ \frac{-\Phi(y)}{D} \right]
+ \frac{2}{D} \int_{-1}^{x} dy H(y) \exp \left[ \frac{-\Phi(y)}{D} \right] \int_{-1}^{x} dy y^2(1 - y^2)H(y)^4
- \int_{-1}^{x} dy H(y)^3 y + \frac{2}{D} \int_{-1}^{x} dy H(y) \exp \left[ \frac{-\Phi(y)}{D} \right] \int_{-1}^{y} z^2(1 - z^2)H(z)^4. \tag{11}
\]

Once \( \alpha_\varepsilon \) and \( \alpha_\rho \) are determined, we can obtain the auto-correlation function, which is given by
\[
\langle x(t)|x(t+\tau) \rangle = e^{-W_o|\tau|} \left[ 1 - \frac{(\alpha_\varepsilon + \alpha_\rho)^2 \cos(\Omega t - \phi)}{\alpha + \Omega} \right]
+ \frac{(\alpha_\varepsilon + \alpha_\rho)^2 (\cos(\Omega \tau) + \cos [\Omega(2t + \tau) + 2\phi])}{2((\alpha_\varepsilon + \alpha_\rho)^2 + \Omega^2)}. \tag{12}
\]
From $\langle x(t)|x(t+\tau) \rangle$ the power-spectrum density (psd) $\langle S(\omega) \rangle$ is readily obtained as

$$\langle S(\omega) \rangle_t = \left[ 1 - \frac{(\alpha_\rho + \alpha_\varepsilon)^2}{2(W_0^2 + \Omega^2)} \right] \left[ \frac{2W_0}{W_0^2 + \omega^2} \right] + \frac{\pi(\alpha_\rho + \alpha_\varepsilon)^2}{2(W_0^2 + \omega^2)} \delta(\omega - \Omega)$$

where we have written only the expression for positive values of $\omega$ [13].

Then, to determine the output SNR $\mathcal{R}$, we use the standard definition

$$\mathcal{R} = 10 \log_{10} \left( \frac{S_s}{S_n \Delta} + 1 \right),$$

where $S_s$ and $S_n$ are the (integrated) psd with and without signal respectively, evaluated at the modulation frequency. Here, the parameter $\Delta$ is introduced in order to tune the theoretical result when compared with a numerical simulation or an experiment, $\Delta$ being related to the bandwidth of the sampling frequencies. It is clear that the evaluation of $\mathcal{R}$ is critically dependent on the evaluation of $dW_0/d\varepsilon$ and $dW_0/d\rho$. For this reason the results of Eqs. (10, 11) have been numerically tested for a large range of values of the different parameters.

One of the goals of the present work was to look for a larger independence of the SNR from the external parameters, particularly the additive noise intensity. In Fig. 1 we show the results for $\mathcal{R}$ as a function of the noise intensities $D$ and $Q$, for a fixed value of $\rho_0$ and different values of $\varepsilon_0$. The widening of the peak as a function of $D$ for a region of values of $Q$ is apparent. However, this widening occurs at the expense of a (small) reduction in the maximum of $\mathcal{R}$. The limiting case for $Q \to 0$, reducing to the usual SR behaviour [13], should be compared with the case when $Q \neq 0$ to see clearly the widening effect. Also, the behaviour for $D \to 0$ (and $Q \neq 0$) reduces to the case studied in [3]. Such a widening of $\mathcal{R}$ is due to the additive dependence of the output signal on $\rho_0^2$.

Another novel aspect of our results is that when $\varepsilon_0 \to 0$, we have a SR phenomenon purely due to the modulation of the correlation parameter that is more localized in the $(D, Q)$ parameter space. When one or both of these parameters goes to 0 or $\infty$ we find that $\mathcal{R} \to 0$.

The results obtained by modifying only the intensity of correlation modulation amplitude
are depicted in Fig. 2. In this case, the results are essentially the same as in the previous figure.

It is worth to note here that in Figs. 1b,c and 2b, keeping constant $D$, the SNR grows as $Q$ increases. This remarkable fact implies that the additive SR (the case $Q = 0$) could be enhanced by adding a multiplicative noise.

We have also studied the effect of changing the frequency modulation $\Omega$ on SNR, observing the same dependence on this parameter, as in other SR systems. For very low frequencies the effect is slightly larger than for higher frequencies, but it is essentially frequency-independent.

Finally, in Fig. 3, we depict a contour plot of the SNR as a function of $\rho_0$ and $\varepsilon_0$. We fixed $D$, $Q$ and the frequency modulation $\Omega$. A paraboloid-like dependence on those variables is obtained, with $R$ increasing as any of the variables increases too.

In conclusion, we have shown that the simultaneous modulation of the potential and the correlation parameter between the additive and multiplicative noises induces a widening of the output SNR, making the system response less dependent on the precise value of the additive noise. It is worth noting that if the modulation frequencies for the potential and the correlation are too different, what we find is a superposition of the SR effect associated to each separated modulation, that is: two differentiated peaks in the SNR at the corresponding frequencies.

The present result could be of relevance for both technological and biological systems, as in electronic signal detectors or sensory systems one wishes that the detection capacity shall be as little dependent as possible on the (usually external) additive noise, the (usually internal) multiplicative noise being a kind of tuning parameter. A more complete study of the dependence of this effect on the modulation frequency of the correlation will be presented elsewhere [11].

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FIGURES

FIG. 1. Here we depict the SNR as function of the additive and multiplicative noise intensities $D$ and $Q$. On the left side we show the 3D plots associated with the contour plots on the right side. We have fixed $\rho_0 = 0.05$ and the frequency modulation $\Omega = 0.005$, while the different cases correspond to: (a) $\varepsilon_0 = 0.01$, (b) $= 0.05$, (c) $= 0.10$. For the sake of clarity, the point of view in the left side 3D plots, is from the upper left corner in the contour plots.

FIG. 2. We show the SNR in $(D, Q)$ space. Once again, the left side 3D plots correspond to the contour plots on the right side, and the point of view for the 3D plots corresponds to the upper left corner in the contour plots. We fixed the frequency modulation $\Omega = 0.005$, the amplitude of potential modulation $\varepsilon_0 = 0.05$, and varied the potential amplitude: (a) corresponds to $\rho_0 = 0.01$; (b) $= 0.10$.

FIG. 3. Contour of $R$ as a function of the modulation intensities $\rho_0$ and $\varepsilon_0$. The values for the noises are $D = 0.25$, $Q = 0.10$ and for the frequency modulation $\Omega = 0.005$. 
Figure 1
Stochastic Resonance in bistable systems: The effect of simultaneous additive and multiplicative noises
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Figure 2
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