Critical Dynamics: multiplicative noise fixed point in two dimensional systems

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We study the critical dynamics of a real scalar field in two dimensions near a continuous phase transition. We have built up and solved Dynamical Renormalization Group equations at one-loop approximation. We have found that, different form the case $d \lesssim 4$, characterized by a Wilson-Fisher fixed point with dynamical critical exponent $z = 2 + O(\epsilon^2)$, the critical dynamics is dominated by a novel multiplicative noise fixed point. The zeroes of the beta function depend on the stochastic prescription used to define the Wiener integrals. However, the critical exponents and the anomalous dimension do not depend on the prescription used. Thus, even though each stochastic prescription produces different dynamical evolutions, all of them are in the same universality class.

Out-of-equilibrium evolution near continuous phase transitions is a fascinating subject. While equilibrium properties are strongly constrained by symmetry and dimensionality, the dynamics is much more involved and generally depends on conserved quantities and other details of the system. The interest in critical dynamics is rapidly growing up in part due to the wide range of multidisciplinary applications in which criticality has deeply impacted. For instance, the collective behavior of different biological systems has critical properties, displaying space-time correlation functions with non-trivial scaling laws\textsuperscript{[1,2]}. Other interesting examples come from epidemic spreading models where dynamic percolation is observed near multicritical points\textsuperscript{[3]}. Moreover, strongly correlated systems, such as antiferromagnets in transition-metal oxides\textsuperscript{[4,5]} present a very rich phase diagram including ordered as well as topological phases. These compounds are generally described by dimmer models or related quantum field theory models\textsuperscript{[6]}, that seem to have anomalous critical dynamics\textsuperscript{[7]}

From a theoretical point of view, the standard approach to critical dynamics is the “Dynamical Renormalization Group (DRG)”\textsuperscript{[8]}, distinctly developed in a seminal paper by Hohenberg and Halperin\textsuperscript{[9]}. The simplest starting point is to assume that, near a critical point, the dynamics of the order parameter is governed by a dissipative processes driven by an overdamped additive noise Langevin equation. Then, the critical point is approached by integrating out short distance (high momentum) degrees of freedom in order to obtain the dynamics of an equivalent effective long distance (small momentum) model. From this procedure, it is possible to read, for instance, the typical relaxation time near a fixed point, given by $\tau \sim \xi^z$, where $\xi$ is the correlation length and $z$ is the dynamical critical exponent. At a critical point, $\xi \to \infty$ and therefore $\tau \to \infty$, meaning that the system does not reach the equilibrium at criticality. Together with usual equilibrium exponents, $z$ defines the universality class of the transition. Interestingly, since the symmetry of the model does not constrain dynamics, there are different dynamic universality classes for the same critical point.

As usual in renormalization group theory (RG)\textsuperscript{[10]}, the integration over higher momentum modes generates all kind of couplings, compatible with the symmetry of the problem. For this reason a consistent study of a RG flux should begin, at least formally, with the most general Hamiltonian containing all couplings compatible with symmetry. Interestingly, in a similar way, DRG transformations generate couplings that modify the probability distribution of the original stochastic process. In particular, we will show that one-loop perturbative corrections generate couplings compatible with a multiplicative noise stochastic processes\textsuperscript{[11]}, even in the case of assuming an additive processes as a starting point. In order to understand the fate of these couplings, we decided to analyze a more general dynamics for the order parameter near criticality, assuming a dissipative process driven by a multiplicative noise Langevin equation.

For concreteness, we present a simple model of a not conserved real scalar order parameter, $\phi(x, t)$ with quartic coupling $\phi^4(x, t)$ (model A of Ref.\textsuperscript{[9]}). We assume a multiplicative noise Langevin equation, modeled by a general dissipation function $G(\phi)$, with the same symmetry of the Hamiltonian. The upper critical dimension of this model is $d_c = 4$. For $d > 4$, the Gaussian fixed point with $z = 2$ correctly describes the phase transition. For $d \lesssim 4$, the Gaussian fixed point turns out to be unstable and the Wilson-Fisher fixed point\textsuperscript{[12]} shows up in a first order expansion around $\epsilon = 4 - d$. In this case, the dynamics is governed by $z = 2 + O(\epsilon^2)$ and all multiplicative noise coupling constants are irrelevant. In this sense, we recover the very well known results of Ref.\textsuperscript{[9]}. However, the dynamical behavior dramatically changes at $d = 2$. The main result of this letter is that, at $d = 2$, all multiplicative noise couplings are marginally relevant, flowing to a novel stable fixed point dominated by a multiplicative stochastic process. In figure\textsuperscript{[4]} we show the novel fixed point as well as the DRG flux. In the following, we present the model, we give details of the calculations and finally we discuss our results.

\textbf{Model–} We consider the following Hamiltonian of a real scalar field $\phi(x)$:

\[
H = \int d^d x \left\{ \frac{1}{2} |\nabla \phi|^2 + \frac{r}{2} \phi^2 + \frac{u}{4!} \phi^4 \right\} \tag{1}
\]

where $\Lambda$ is an ultraviolet momentum cut-off. $\{r, u\}$ are the
quadratic and quartic coupling constants respectively. The dynamic evolution is given by the Langevin equation,
\[ \frac{\partial \phi(x, t)}{\partial t} = -\Gamma \frac{\delta H}{\delta \phi(x, t)} + G(\phi^2) \eta(x, t) \tag{2} \]
where \( \eta(x, t) \) is a Gaussian white noise field: \( \langle \eta(x, t) \rangle = 0 \) and \( \langle \eta(x, t) \eta(x', t') \rangle = 2 \Gamma \delta(x - x') \delta(t - t') \). \( \Gamma \) is a diffusion constant and \( G(\phi^2) \) is a diffusion function characterizing the multiplicative noise distribution. \( \langle \ldots \rangle \) represents stochastic mean value. In order to correctly define Eq. (2), we need to fix a stochastic prescription to properly compute the time Wiener integrals. In this paper, we adopted the so called Generalized Stratonovich prescription that is parametrized by a kernel \( \alpha \) of \( \alpha \) is the Itô prescription, \( \alpha = 1/2 \) is the Stratonovich one, while \( \alpha = 1 \) is the Hanggi-Klimontovich or anti-Itô convention. 

To compute dynamical correlation functions, we use a generalized Martin-Siggia-Rose-Janssen-DeDominicis formalism (MSRJD) \[14–16\], that represents the Langevin dynamics by a field theory. The generating functional \[13\] \[17\] \[18\]
\[ Z[J_\phi, J_\varphi] = \int D\phi D\varphi D\xi D\bar{\xi} e^{-S[\phi, \varphi; \xi, \bar{\xi}]} \int d^dx dt \{ J_\phi \phi + J_\varphi \varphi \} \tag{3} \]
is written as a functional integral over four fields, where \( \phi(x, t) \) and \( \varphi(x, t) \) are two real scalar fields essentially representing a local order parameter and a response field respectively. \( \xi(x, t) \) and \( \bar{\xi}(x, t) \) are anticommuting Grassmann fields.

The “action” is given by,
\[ S = \int \frac{d^d x dt}{\lambda} i \varphi \left\{ \frac{\partial \phi}{\partial t} + \frac{\Gamma}{2} \frac{\delta H}{\delta \phi} + \Gamma G \frac{\delta G}{\delta \phi} \xi \right\} + \frac{\Gamma}{2} G^2 \varphi^2 \]
\[ - \int \frac{d^d x dt d^d x' dt'}{\lambda} \xi(x, t) K(x - x', t - t') \bar{\xi}(x', t') \xi(x, t) \tag{4} \]
where the kernel
\[ K(x - x', t - t') = \frac{d \delta(t - t')}{dt} + \frac{\Gamma}{2} \frac{\delta^2 H}{\delta \phi(x, t) \delta \phi(x', t')} \tag{5} \]

Without loose generality, we chose a diffusion function satisfying \( G(0) = 1 \), which is the usual additive noise value. Then, we can write
\[ G^2(\phi^2) = 1 + \sum_{n=1}^{\infty} g_n \phi^{2n} \tag{6} \]
where \( g_n \), with \( n = 1, 2, \ldots \), are coupling constants defining the multiplicative noise distribution.

By trivial power counting, we immediately verify that \( r = 2 \), \( u = 4 - d \), \( \Gamma = z - 2 \) and \( [g_n] = n(2 - d) \), where the notation \( \ldots \rangle \) indicates dimension in powers of momentum. Of course, the dimensions of \( r \) and \( u \) are the usual ones in the equilibrium theory. This defines the upper-critical dimension \( d_c = 4 \). The dynamical critical exponent \( z \) is fixed by demanding \( [\Gamma] = 0 \). For \( d > 2 \), \( [g_n] < 0 \), therefore, all the multiplicative noise couplings are irrelevant at tree level. However, at \( d = 2 \), \( [g_n] = 0 \), \( \forall n \). In this case, the entire function \( G(\phi) \) produces marginal couplings at tree level. In order to study the fate of the multiplicative coupling, it is necessary to compute fluctuations.

Let us study in detail the simpler model of just one marginal coupling constant \( g_1 = g \), and \( g_n = 0 \) for \( n \neq 1 \). In this case, \( G^2 = 1 + g \phi^2 \). We can split the action into two terms, \( S = S_0 + S_I \), where \( S_0 \) is the quadratic part and \( S_I \) codifies interaction terms. We find,
\[ S_0 = \int \frac{d^d x dt}{\lambda} \left\{ i \varphi \Delta \phi + \frac{\Gamma}{2} \varphi^2 - \xi \delta \xi \right\} \tag{7} \]
where the differential operator \( \Delta = \partial_t + (\Gamma/2)(-\nabla^2 + r) \). The interacting part of the action reads,
\[ S_I = \int \frac{d^d x dt}{\lambda} \times \frac{\eta_1}{3! \omega^2} \bar{\varphi} \varphi^2 \xi \xi \]
\[ + \frac{\eta_2}{4} \varphi^2 \varphi^2 \phi^2 + i \eta_3 \varphi^2 \phi^2 \xi \xi \xi \xi \]

The last two terms codify the effect of multiplicative noise dynamics.

**DRG transformation**— In order to built a DRG transformation we firstly lower the ultraviolet cut-off to \( \Lambda/b \), with \( b > 1 \) and split the fields: \( \phi = \phi'^{<} + \phi'^{>}, \varphi = \varphi'^{<} + \varphi'^{>}, \xi = \xi'^{<} + \xi'^{>} \) and \( \bar{\xi} = \bar{\xi}'^{<} + \bar{\xi}'^{>} \), in such a way that the Fourier transformed fields with superscript “<” have support within the sphere \( k < \Lambda/b \), while the fields labeled by “>” have support on a spherical shell \( \Lambda/b < k < \Lambda \). We define the transformed action \( S'^{<}= \phi'^{<} \varphi'^{<} \xi'^{<} \bar{\xi}'^{<} \), by integrating out the fields with momentum higher than \( \Lambda/b \) in the following way
\[ e^{-S'^{<}/\lambda} = \int D\phi D\varphi D\xi D\bar{\xi} e^{-S_{\lambda}/\phi, \varphi, \xi, \bar{\xi}} \tag{9} \]

Since the transformed action, \( S'^{<}/\lambda \), has a momentum cut-off \( \Lambda/b \), in order to compare it with the original one, \( S_{\lambda} \), with cut-off \( \Lambda \), we re-scale momentum, frequency and fields as
\[ k' = bk, \omega' = b^2 \omega \]
\[ \phi'^{<}(k'/b, \omega'/b^2) = b^{d-2} \eta_1 \phi(k', \omega') \]
\[ \phi'^{>}(k'/b, \omega'/b^2) = b^{d+2} \eta_2 \phi(k', \omega') \]
\[ \xi'^{<}(k'/b, \omega'/b^2) = b^{d+2} \eta_3 \xi(k', \omega') \]
\[ \xi'^{>}(k'/b, \omega'/b^2) = b^{d+2} \eta_4 \xi(k', \omega') \]
with the substitution of the fields 

\[ \tilde{\phi} \]

read, \[ -R \]

the \[ \phi \]

FIG. 1. Vertex as read from the action \( S_f \) of Eq. (8). The dashed line represents the field \( \phi(x, t) \) while the solid line \( \phi'(x, t) \). The wiggle lines correspond to the Grassmann fields \( \{ \xi(x, t), \xi'(x, t) \} \). The last two vertex characterize the multiplicative noise distribution.

\[ \text{FIG. 2. First order corrections. Internal continuous lines represent the } G^> \text{ propagator, while internal arrowed lines correspond to the response function } R^<. \text{ Wiggle internal lines represent Grassmann propagators. The diagrams I, II, III, V and VI are contributions that essentially renormalize the } r \text{ parameter. Diagram IV, contribute to renormalize } \Gamma, \text{ been responsible for the anomalous dimension } \eta. \]

\[ \text{Perturbative one-loop DRG– The main difficulty of this procedure is to compute the functional integral of Eq. (9). To do this, we implement a perturbative calculation. To second order in the couplings we find,} \]

\[ S'_{\Lambda/0} = S^< + (S_I)_{S_0} - \frac{1}{2} \left( (S^2_{S_0}) - (S_I)^2_{S_0} \right) \]  

(15)

where \( S^< = S^<_0 + S^<_I \) means the action of Eqs. (7) and (8) with the substitution of the fields \( \phi, \phi', \xi, \xi' \) by \( \phi^<, \phi'^<, \xi^<, \xi'^<. \) The notation \( \langle \cdots \rangle_{S_0} \) means the mean value computed with the quadratic action \( S_0 \) of Eq. (7) but with the momentum constrained to the small shell \( \Lambda/\mathbf{k} < k < \Lambda. \) The more efficient way to organize this expansion is by using Feynman diagrams and grouping terms by the number of internal loops. In fig. 1 we depicted the four vertex of our model. From \( S_0 \) we can read two propagators: \( G^>(x-x', t-t') = \langle \phi(x, t)\phi(x', t') \rangle \) represented by an internal continuous line and \( R^<(x-x', t-t') = \langle \phi(x, t)\phi(x', t') \rangle \Theta(t-t') \), represented by an internal continuous line with an arrow. The Grassmann field propagator is related with the response function \( \langle \xi(x, t)\xi(x', t') \rangle = iR^<_{\mathbf{k}}(x-x', t-t') \) and is represented by an internal wiggled line. The bare response fields are not correlated \( \langle \phi(x, t)\phi(x', t') \rangle = 0 \). In Fourier space, the propagators read, \( G^>(k, \omega) = \Gamma[\omega^2 + \frac{\pi^2}{\mathbf{k}^2}(k^2 + \mathbf{r}^2)]^{-1} \) and \( R^<(k, \omega) = -i[-\omega + \frac{\pi^2}{\mathbf{k}^2}(k^2 + \mathbf{r}^2)]^{-1}. \) In these expressions \( 1 - \mathbf{r}^2 \mathbf{k}/\Lambda < 1. \)

The first order corrections are depicted in fig. 2 while we show the relevant second order one loop diagrams in figure 3. We explicitly computed all diagrams and, after rescaling momenta, frequencies and fields, we can read the running coupling constants by comparing \( S'_{\Lambda} \) with the original action \( S_{\Lambda}. \)

The one-loop perturbative DRG equations are displayed in a simpler way by using dimensionless coupling constants. To this purpose we rescaled the couplings \( r \to r\Lambda^{-2} \), \( u \to (\Omega_d/(2\pi)^d)u\Lambda^{d-4} \) and \( g \to (\Omega_d/(2\pi)^d)g\Lambda^{2-d} \), where \( \Omega_d \) is the area of a \((d-1) \) dimensional sphere or radius one. We have found:

\[ \frac{d}{dt} = 2r + \frac{u}{2} + \frac{r}{1 + r} - 2\alpha g \]  

(16)

\[ \frac{du}{dt} = (4 - d)u - \frac{3}{2} \frac{u^2}{(1 + r)^2} + 4 \frac{ug}{1 + r} \]  

(17)

\[ \frac{d\Gamma}{dt} = \{ z - 2 \} \Gamma \]  

(18)

\[ \frac{dg}{dt} = (2 - d)g + \frac{g^2}{1 + r} - \frac{5}{2} \frac{gu}{(1 + r)^2} + \frac{1}{4} \frac{u^2}{(1 + r)^3} \]  

(19)

with the anomalous dimensions, \( \eta = \eta_1 = 0, \eta_2 = g/(2(1 + r)) \) and \( \eta_3 = 0. \) The first three equations, with \( g = 0 \), are the usual DRG equation at one-loop approximation. Multiplicative noise contributes with third term of Equation (16), coming from diagrams III and V of figure 2. In these diagrams, the self-correlated response function is singular, therefore, it is necessary to define it as \( R^<_{\text{retarded}}(0, 0) = -i\alpha, \) where the parameter \( \alpha \) corresponds with the stochastic prescription \( [17] \), used to defined the Langevin equation (2). Multiplicative noise also contribute to the evolution of \( u, \) as can be seen in Eq. (17). The last term, proportional to \( ug, \) is generated by diagram 4 of figure 3. Eq. (19) is the key equation that controls the fate of the multiplicative coupling. The last three terms come from diagrams 11, 12, 13 and 14 of figure 3.

This system has several fixed points, depending essentially on dimensionality. For \( d > 4 \), the Gaussian fixed point, \( u^* = r^* = g^* = 0 \) with \( z = 2 \) correctly describes the phase transition. For \( d \leq 4 \), a Wilson-Fisher fixed point shows up at order \( \epsilon = 4 - d, u^* = (2/3)\epsilon, r^* = -(1/6)\epsilon \) with \( z = 2 + O(\epsilon^2) \) and \( g^* \sim O(\epsilon). \) At this level of approximation, \( g \) is an irrelevant scaling variable and the dynamics is driven by a usual
additive noise stochastic process. We have checked that this behavior remains the same at $O(\epsilon^2)$ thus, we recover the well known results of Ref. [9]. However, at $d = 2$ the dynamical behavior completely changes its character. In this case, $g$ is no longer irrelevant but turns out to be marginally relevant. The flux gets away from its additive value, even in the case of having an initial value $g = 0$. We have found a novel fixed point $r^* = [\alpha ((-4\alpha - 3\sqrt{2} + 1) + 2(\sqrt{2} + 3)))/(4(\alpha - 6)\alpha - 48), u^* = [\alpha ((49 + 11\sqrt{2})\alpha + 32\sqrt{2} + 168) + 36\sqrt{2} + 84)/(2((\alpha - 6)\alpha - 12)^2), g^* = [-\alpha + \sqrt{2} + 3]^{-1}].$ The position of this fixed point is weakly dependent on the stochastic prescription $\alpha$. In the Stratonovich prescription, $\alpha = 1/2$, the fixed point is located at $r^* = -0.268, u^* = 0.991$ and $g^* = 0.141$. Linearizing the DRG equations around this fixed point, we have one repulsive and two attractive directions defined by the eigenvalues $\lambda = \{2, -6.002, -1.926\}$, with the corresponding normalized eigenvectors $v_1 = \{0.594, 0.804, 0\}$, $v_2 = \{0.142, -0.915, 0.377\}$ and $v_3 = \{-0.119, 0.949, 0.290\}$. These results are more easily visualized in figure 4 where we show the DRG flux obtained by numerically solving equations (10)–(19). The relevant eigenvalue $\lambda = 2$ as well as the anomalous dimension $\eta = 2/(3\sqrt{2} + 7)$ do not depend on $\alpha$.

Summary and discussions. We have built up and solved the DRG equations of a real scalar field with quartic interactions whose dynamics is driven by a general overdamped Langevin equation. The DRG transformation generates couplings that modify the distribution probability of the stochastic process. In particular, multiplicative noise processes are generated by integrating out higher momentum degrees of freedom. The main result of the letter is that, at $d = 2$, multiplicative couplings are marginally relevant, driven the DRG flux to a novel fixed point dominated by a not trivial multiplicative noise process. The zeroes of the $\beta$–function explicitly depend on the stochastic prescription. However, and most importantly, the critical exponent $\nu = 1/2$ and the anomalous dimension $\eta \approx 0.096$ do not depend on the value of $\alpha$. This means that dynamical evolutions driven by different stochastic prescriptions are in the same universality class. It is worth noting that, at one-loop level, the anomalous dimension enters the fields $\phi$ and $\varphi$ with different sign. This fact produces non trivial power law decay of $\langle \phi \varphi \rangle$ correlations, however it does not modify the asymptotic behavior of the response function $\langle \phi \varphi \rangle$. This fact is modified at two loop level, in which $\eta_1 \neq \eta_2$.

At this point, some caveats are in order. Perturbation theory is not a quite controlled approximation in two-dimensions. Even though we expect substantial corrections in the position of the fixed points and the critical exponents, we strongly believe that the existence of the multiplicative noise fixed point and the topology of the DRG flux is robust. We have computed two-loop corrections to the DRG equations and have found similar qualitative results. We will present details of this calculation elsewhere. Moreover, several exhaustive calculations of the equilibrium RG in $d = 2$, up to five loops approximation [19–21] further support this claim. In order to get a deeper understanding on the fate of the multiplicative dynamical fixed point at higher orders and to precisely compute critical exponents, it could be interesting to implement a non-perturbative approach to the DRG [22].

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FIG. 4. DRG flux, obtained by numerically solving Equations 16 to 19. We have fixed the Stratonovich prescription $\alpha = 1/2$. In (a), we depict the flux in the $r - u$ plane at $g^* = 0.141$. It can be clearly seen a fix point $u^* = 0.991$, $r^* = -0.268$, with one attractive and one repulsive direction with a similar topology of a Wilson-Fischer fixed point. In (b) we show the $u - g$ plane at $r^* = -0.268$. We can observe a completely attractive fixed point at $u^* = 0.991$, $g^* = 0.141$. In (c), we show the fixed point in the $r - g$ plane $r^* = -0.268, g^* = 0.1411$ at the critical value $u^* = 0.991$. 

