SURGERY ON DISCRETE GROUPS

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ABSTRACT. We study constructions of groups, in particular of groups of intermediate rank, which are accessible to surgery techniques.

The present paper makes use of some fragments of the cobordism theory to study countable discrete groups, especially from the point of view of geometric group theory. The aim of the paper is to study a notion of group cobordism, which fits into a cobordism category whose constituents are 2-dimensional in nature.

Classical surgery is a topological operation on manifolds. It has had great achievements, starting with the construction of exotic differentiable structures on the sphere of dimension 7. We are looking to use surgery techniques to construct exotic countable groups. A concrete such construction was given in [1].

The classical cobordism category \( \text{Bord}_n \) is easy to describe in dimension \( n = 2 \). In the case of groups, the 2-dimensional spaces that need to be considered are "branching" simplicial complexes, which makes the category significantly more difficult to grasp, even with simple input data. In essence, the dimension is \( n = 4 \), since every finitely presented group can be written as the fundamental group of a closed manifold of dimension 4. The theory however is not captured by \( \text{Bord}_4 \). We are looking in particular for results that can be applied directly to the study of groups. The “ST lemma” formulated in §6 or Theorem 10.1 are concrete examples of such results. We note that surgery theory is expected to see further developments in the direction of coarse geometry (compare [14, §4] and the references therein).

The group cobordisms are the arrows of a category whose objects are called collars. The latter are defined in §3, and are reminiscent (given the fact that they are branching spaces) of the usual notion of collar for Riemann surfaces. For example, here is what the “centrepiece” of a collar might look like topologically

\[ \text{namely, it is a product space, in this case of the complete graph on 4 vertices with a segment. We will distinguish between metric and simplicial collars, and group cobordisms.} \]

The main applications that we have in mind in this paper are to the construction of groups of intermediate rank (for which subject matter we shall refer to the introduction of [2]). For example, we will use group cobordisms to construct “exotic” groups of rank \( \frac{7}{4} \), in the sense of [3]. This can be achieved because the surgeries are performed in a controlled way. In particular, they preserve the “type” of the group under consideration (for instance, a surgery operation starting with a group of rank \( \frac{7}{4} \) returns a group of rank \( \frac{7}{4} \)).
The “type” can be defined in a simple way by listing the constraints on the local data that define the given class of groups (see §2); to such a type $A$ is associated a category $\text{Bord}_A$ of group cobordisms of type $A$, which describes the possible surgeries for the groups of type $A$. The categories $\text{Bord}_A$ are our main objects of study. For the purposes of rank interpolation, the case study is the category $\text{Bord}_{\frac{7}{4}}$ that describes surgeries for the groups of rank $\frac{7}{4}$. We will obtain some partial information on this category in this paper.

In the course of exploring the subject (and fitting it to the requirements of rank interpolation), a few concepts and facts have emerged, some of which we mention now: a) in §1, we define a notion of model geometry, which are the models, or building blocks, that can be used to construct any group of a given type. They will provide the simplest group cobordisms in $\text{Bord}_A$; b) in §3, we discuss a notion of $2$-transitivity for group actions on simplicial 2-complexes. An action of a group on a simplicial 2-complex is said to be $2$-transitive if every triangle intersects at most two orbits of vertices. These actions provide the simplest examples of collars; c) the “ST lemma”, mentioned above, is a classification of the smallest collars that can connect two models geometries of nonpositive curvature. This is a useful result in particular for the study of groups of intermediate rank, which act on 2-complexes of nonpositive curvature, in the CAT(0) sense; d) double covers provide explicit collars, cf. §7; e) “collar surgery” leads to exotic groups, and to “fake double covers”, see §9; f) in §11, we construct a group of rank $\frac{13}{4}$ (in the sense of [2]), which serves to illustrate the main question about types—the type constructibility problem—raised in §2. This group is a strange mixture of a group of rank $\frac{7}{4}$ and a group of (Coxeter) type $\tilde{A}_2$; g) the main result on group cobordisms is Theorem 10.1, which concerns non filling cobordisms of type $\frac{7}{4}$, and is used to construct the new groups of rank $\frac{7}{4}$ mentioned above (explicit drawings of group cobordisms can be found in Figures 3 and 4).

The category $\text{Bord}_A$ is a global object associated with the type $A$; a different global object, which also contains important information about the type $A$, is introduced in §2. It is a dynamical system called the space of complexes of type $A$, and it is a generalization (in particular to complexes of intermediate rank) of the space of triangle buildings defined in §3, where the type $A$ was the (Coxeter) type $\tilde{A}_2$. A notion of “indicability” (existence of a surjective morphism to $\mathbb{Z}$) in this space is discussed in §5 in relation with the existence of sufficiently many $\frac{3}{2}$-transitive actions for groups of type $A$. This provides finiteness information on categories such as $\text{Bord}_{\tilde{A}_2}$, for which Kazhdan’s property T can be used (see Theorem 12). In the opposite direction, the surgery constructions in the category $\text{Bord}_{\frac{7}{4}}$ shed light on some dynamical properties of this space for complexes of rank $\frac{7}{4}$. This latter point is discussed in §12.

The interactions between geometric group theory and category theory look promising. The present paper raises a few general questions to which we hope to return elsewhere—for example, it would be desirable for us to a) have a deeper understanding of the categories $\text{Bord}_A$ associated with a type $A$ (including $\text{Bord}_{\frac{7}{4}}$), and b) study quantum invariants for groups of intermediate rank (including groups of rank $\frac{7}{4}$) arising from topological quantum field theory constructions over group cobordism categories.
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1. Model geometries

Definition 1.1. A model geometry (of dimension 2) is a connected 2-complex $M$ with a distinguished vertex (the center) satisfying the following two conditions:

(1) every vertex in $M$ distinct from the center is adjacent to it
(2) every loop is attached to the center

Let $M$ be a model geometry. There are two types of edges in $M$ besides the loops:
- the half edges, exactly one of whose extremities coincide with the center
- the boundary edges, none of whose extremities coincide with the center

The core of $M$ is the maximal subcomplex $\text{Co} M$ of $M$ whose vertex set contains only the center. It contains the loops and the faces attached (exclusively) to loops. Thus we have an homotopy equivalence,

$$ M \sim \text{Co} M, $$

using (2). The fundamental group $\pi_1(M) \simeq \pi_1(\text{Co} M)$ is called the model group associated with $M$.

The link of $M$ is the link $\text{Lk} M$ of its center. (Recall that the link at a point in a 2-complex is the sphere of small radius around that point.)

The boundary $\partial M$ of $M$ has as vertex set the vertices of $M$ distinct from the center, and as edge set the boundary edges.

The vertex set of $\partial M$ is a subset of the set of vertices of the link:

$$ \partial^0 M \subset \text{Lk}^0 M. $$
To every edge $e$ in $\partial M$ is attached a weight $w(e)$ defined by:

$$w(e) := 1 + \text{number of inner loops of the (unique) face } f \text{ containing } e$$

$$= |f| - 2, \text{ where } |f| := \text{number of edges of } f$$

The weight represents the “inner perimeter” of the face $f$. (The addition of 1 in the definition accounts for the two inner half edges that are attached to $e$.)

The simplest examples of model geometries are cones on graphs.

**Example 1.2.** In the Euclidean plane tessellated by squares, the model geometry at every vertex is a flat square (with dashed boundary in the drawing)

![Diagram of a square with dashed boundary]

with 4 half edges and 4 boundary edges.

Model geometries provide an explicit source of group cobordisms, and will be used as such later in the paper.

We say that a face in a 2-complex $X$ is *crossing* if it contains at least 2 distinct vertices of $X$. Every 2-complex can be decomposed into model geometries and crossing faces as follows.

Let $r$ be a crossing face in $X$. The vertex set of $r$ can be partitioned according to the equivalence relation generated by the relation

“being adjacent in $r$ and corresponding to the same vertex of $X$”.

For example, in the following figure

![Diagram of a face with 10 vertices and 4 classes of vertices]

we have a face with 10 vertices and four classes of vertices, and two of these classes (represented with dots) belonging to the same vertex of $X$.

The “middle points” of every edge $e$ of $r$ whose extremities are inequivalent form the vertex set of a polygon (with filled interior) inscribed in $r$, which is represented with a dashed boundary in the figure.

Assume that $X$ is finite. Let $r_1, \ldots, r_n$ denote the crossing faces of $X$ and $s_1, \ldots, s_n$ denote the respective inscribed polygons.
**Definition 1.3.** The model geometry of $X$ at $x$ is the 2-complex

$$M_x = \bigcup_{i=1}^{n}(r_i \setminus s_i)_x$$

where $r_i \setminus s_i$ denotes the pieces outside $s_i$ and $|r_i|$ selects the components corresponding to $x$ (which may not be connected). The fundamental group

$$G_x := \pi_1(\text{Co} \ M_x) \cong \pi_1(M_x)$$

is called the model group of $X$ at $x$.

The 2-complex $X$ can be reconstructed in an obvious way as a quotient

$$X \cong \left( \bigcup_{x \in X^0} M_x \cup \bigcup_{i=1}^{n} s_i \right) / \sim$$

where $s_i$ is a (filled) polygon inscribed in the crossing relation $r_i$, possibly reduced to a segment, and the relation $\sim$ attaches the pieces together along their boundaries.

Furthermore, the following combinatorial relation holds:

$$\sum_{r \in X^2} |r| = \sum_{x \in X \cap \partial M_x} \sum_{e \in \partial M_x} w(e)$$

(weight equation).

(This corresponds to two ways of counting the number of edges of the crossing faces in $X$.)

Note that in the decomposition

$$X \cong \left( \bigcup_{x \in X^0} M_x \cup \bigcup_{i=1}^{n} s_i \right) / \sim$$

the quotient 2-complex structure on the right is a subdivision of the 2-complex structure of $X$. We call the map $M_x \to X$ a germ embedding of $M_x$ in $X$. It is not, strictly speaking, a 2-complex complex map, but can be seen as one once $X$ is endowed with the quotient structure coming from the quotient structure on the right.

The above discussion holds for infinite complexes, if one includes infinite families of model geometries and shapes, and leaves out the weight equation.

We are especially interested, in particular for the purpose of rank interpolation, in metric versions of this decomposition.

By metric 2-complex we mean that the edges and faces are endowed with a compatible metric (and the attaching maps are isometries), and we shall assume (for simplicity) that the faces are flat, by which we mean isometric to a Euclidean disk with a polygonal boundary, which is strictly convex in the sense that the inner angle at every vertex is $< \pi$. We say that a metric 2-complex is nonpositively curved if it satisfies the link condition condition [9] (which is to say that the girth of its links is $\geq 2\pi$ with respect to the angular metric).

If $X$ is a metric 2-complex, the model geometry $M_x$ at a vertex $x$ of $X$, in the sense of Definition [1.3], is endowed with the induced metric, where by definition the half-edges extend up to the middle points of edges, and the boundary edges and faces are endowed with the metric induced from the face of $X$ they belong to.

**Definition 1.4.** Two metric model geometries $M_1$ and $M_2$ are isomorphic if they are “germ isometric” in the sense that there exist open sets $V_1$ and $V_2$, respectively containing the cores of $M_1$ and $M_2$, and an isometry $V_1 \xrightarrow{\sim} V_2$. Such an isometry is called an isomorphism (a “germ isometry”) from $M_1$ to $M_2$. 
Note that isomorphisms be composed in the obvious way (restricting domains and ranges appropriately, since the intersection of two neighbourhood of the core remains a neighbourhood of the core), which defines a category (in fact, a groupoid) of metric model geometries. The isomorphism classes of this groupoid are the “germs” of metric model geometry. A set of model geometries is finite if the set of germs (the quotient space of the groupoid) is finite. This is the case for geometries of finite type as discussed in the forthcoming section.

We note that it is possible for a given 2-complex to be the core of two non-isomorphic model geometries.

2. Types

Definition 2.1. A simplicial type (in dimension 2) is
a) a set of graphs, and
b) a set of shapes.

Similarly, a metric type (in dimension 2) is
a) a set of metric graphs, and
b) a set of flat shapes.

In this definition the convention is that graphs have no orientation and that multiple edges are allowed. By “shape” we mean a disk with a fixed polygonal boundary, considered up to simplicial homeomorphism. By “flat shape” we mean a convex polygonal disk in the Euclidean plane \( \mathbb{R}^2 \) endowed with the induced metric, up to isometry.

Example 2.2 (Simplicial types).

1. “2-complexes”:
   a) all graphs
   b) all shapes
2. “simplicial 2-complexes”:
   a) all graphs
   b) \( \{ \Delta \} \) (one triangle)
3. “simplicial manifolds”:
   a) \( \{ n \text{-gons for all } n \geq 2 \} \)
   b) \( \{ \Delta \} \)

Example 2.3 (Metric types).

1. type \( \tilde{A}_2 \) (see [13]):
   a) incidence graphs of finite projective planes, all edges have length \( \pi/3 \)
   b) one equilateral triangle
2. rank \( \frac{7}{4} \) (see [3]):
   a) the Moebius-Kantor graph, all edges have length \( \pi/3 \)
   b) one equilateral triangle

The standard types for Bruhat–Tits buildings, in the Euclidean rank 2 case, namely, \( \tilde{A}_2, \tilde{B}_2 \) and \( \tilde{G}_2 \), are easily defined in this way. One can also, for example, consider the type “Tits geometries”, using a) \{spherical buildings\} and b) \{\( \Delta \)\}.

Definition 2.4. A type is finite if the two sets a) and b) are finite, when considered up to isomorphism (up to isometry in the metric case), and every graph in a) is finite.
Every 2-complex \( X \) is naturally affiliated with a type \( A_X \), called the type of \( X \), and defined by

a) the set of links at vertices in \( X \) not in the topological boundary of \( X \), and
b) the set of shapes \( X \) contains

This is a metric type if \( X \) is metric, where the links are endowed with the angular metric.

We will say that:

- \( X \) is of strict type \( A \) if \( A_X = A \)
- \( X \) is of type \( A \) if \( A_X \subset A \).
- \( X \) is of finite type if \( A_X \) is a finite type.

Note that one can consider abstractly the union of two types. For example, one defines the metric type \( \frac{7}{4} \vee \tilde{A}_2 \) as follows:

a) \{incidence graphs of projective planes\} \cup \{Moebius-Kantor graph\}, and
b) one equilateral triangle.

Every complex of type \( \tilde{A}_2 \) is in particular of type \( \frac{7}{4} \vee \tilde{A}_2 \).

**Definition 2.5.** Let \( A \) be a type. The *space of complexes of type \( A \)* is defined by

\[
\Omega_A := \text{the set of isomorphism classes of connected complexes of type } A \text{ without boundary}
\]

The space of complexes of finite types is:

\[
\Omega_{\text{ft}} := \text{the set of isomorphism classes of connected complexes of finite type without boundary}
\]

It is filtered by the lattice (with respect to inclusion) of finite types: if \( A \subset B \) are finite types then \( \Omega_A \subset \Omega_B \subset \Omega_{\text{ft}} \).

Note that there are countably many simplicial types, and uncountably many metric types up to type isomorphism. Accordingly, there is a metric and a simplicial space of complexes of finite types, together with a forgetful map \( \Omega_{\text{ft}}^{\text{metr}} \to \Omega_{\text{ft}}^{\text{simp}} \) whose fibres describe the possible metrizations.

The so-called pointed Gromov–Hausdorff topology provides a natural topology on “desingularized” versions \( \Lambda_A \) of \( \Omega_A \):

\[
\begin{array}{c}
\Lambda_A \\
\downarrow \\
\Omega_A
\end{array}
\]

It is easy to find such “desingularizations” of \( \Omega_A \) by *marking* the complexes. Thus, marking at vertices gives

\( \Lambda_A := \text{the set of base point preserving isomorphism classes of pairs (} X, * ) \)

where \( X \) runs over the complexes of type \( A \) and \( * \) (the base point) runs over the vertices of \( X \). One can also choose larger balls or compact sets as “base points”, if one wants to add local control to the isotropy groups.

One checks readily (by a standard diagonal argument) that

**Proposition 2.6.** If \( A \) is a finite type (metric or simplicial), then \( \Lambda_A \) is a compact Hausdorff space.
Remark 2.7. If \( \tilde{A}_{2,q} \) denotes the type “\( \tilde{A}_2 \) and order \( q \)”, then \( \Omega^{\infty}_{\tilde{A}_{2,q}} \), namely the subset of \( \Omega_{\tilde{A}_{2,q}} \) consisting of simply connected spaces, coincides with the space of triangle buildings \( E_q \) from [4, 5, 12].

Since model geometries are in particular 2-complexes, they have a well-defined affiliated type (both simplicial and metric). However, if a model geometry \( M \) sits in an ambient 2-complex \( X \), the type of \( M \) differs from that of \( X \) (due to the presence of additional faces lying outside the core). We shall refer to the type of a 2-complex \( X \) in which \( M \) embeds (germ simplicially or germ isometrically) as an embedded type for \( M \).

A type, if finite, can be “precomputed”, in the sense that one can give (at least in principle), an exhaustive list of all the model geometries it contains, thereby listing the basic buildings blocks for groups of a this type:

Proposition 2.8. If \( A \) is a finite (simplicial or metric) type, then the set of model geometries of embedded type \( A \) is finite up to isomorphism.

Proof. For \( X \) of type \( A \) and \( x \in X \) a vertex, let \( M_x \) denote the minimal subcomplex containing all faces containing \( x \). Since \( A \), either simplicial or metric, is finite, the set of all complexes \( M_x \) when \( X \) runs over the complexes of type \( A \) and \( x \) over the vertices of \( X \) is finite, respectively up to simplicial isomorphism or isometry. If \( M \) is a model geometry of embedded type \( A \), it embeds in \( M_x \) for some \( x \in X \). There are only finitely many such embeddings up to isomorphism. \( \square \)

In practice, the precomputation of all model geometries can only be achieved for the smallest types (a computer program can be written that outputs the model geometries for a given type, but we do not have an efficient algorithm—see [3, §4]).

Furthermore, even when if given type can be fully precomputed, the question remains to understand what complexes can be built from this finite list of model geometries. The decomposition

\[
X = \left( \bigcup_{x \in X^0} M_x \cup \bigcup_{i=1}^n s_i \right) / \sim
\]

described in §1 formulates the issue, but by no means addresses it. In a §4 we introduce a notion of \( \frac{1}{2} \)-transitivity, that goes one step further.

The problem that arises here (the “type constructibility problem”) is to find general conditions on a given type \( A \) that insures the existence of sufficiently many complexes (in particular, at least one) of strict type \( A \). For example, is there a connected 2-complex of strict type \( A \lor B \) provided that there exist connected 2-complexes of strict types \( A \) and \( B \) respectively, assuming minimal compatibility between \( A \) and \( B \)? This is a non trivial question already for \( A = \tilde{A}_2 \) and \( B = \frac{7}{4} \), which are combinatorially compatible, and in which case the resulting spaces are typical examples of “spaces of intermediate rank” in the spirit of [3].

3. Collars

Definition 3.1. A collar is a topological space of the form \( H \times (0,1) \) where \( H \) is a graph (not necessarily connected).

In the metric case, \( H \times (0,1) \) is also assumed to be endowed with a metric, which may not be a product metric, but fibers over the graph \( H \). The graph \( H \) is called the nerve of the collar. (From the point of view of topological quantum field theory,
the assumption that the nerve is a graph corresponds to restricting our attention to “closed strings”—it is possible but more complicated to also include “open strings” in the discussion.)

**Definition 3.2.** A collar in a 2-complex $X$ is an embedding $C : H \times (0,1) \rightarrow X$.

We shall refer to the domain $H \times (0,1)$ as the abstract collar defining $C$. The call dual of a collar is the collar $C' : H \times (0,1) \rightarrow X$ defined by $C'(x,t) := C(x,1-t)$.

**Definition 3.3.** Let $C$ be a collar in a 2-complex $X$. The collar closure of $C$ is the topological closure $\overline{C}$ of the image of $C$ in $X$.

In general, collar closures are not homeomorphic to product spaces.

**Definition 3.4.** The span of a collar $C$ in $X$ is the set $\text{span}(C)$ of vertices of $X$ contained in collar closure of $C$.

Let $C$ be a collar in a 2-complex $X$. The simplicial closure of $C$ is the union of all the open edges and open faces it intersects.

We shall only consider collars in $X$ that are:

1. simplicially closed, in the sense that the image of the map $C$ coincide with the simplicial closure, and
2. vertex free, in the sense that they do not intersect vertex set of $X$.

From now on, by collar in a 2-complex $X$, we will mean an embedding $C : H \times (0,1) \rightarrow X$ that satisfies these two conditions.

**Definition 3.5.** We say that two collars $C$ and $C'$ in $X$ and $X'$ are isomorphic if there is a simplicial isomorphism between their respective collar closures $\overline{C}$ and $\overline{C'}$.

In the metric case, the collars in $X$ are naturally endowed with the induced metric, and we further assume in the previous definition that the simplicial isomorphism is a simplicial isometry.

**Definition 3.6.** A collar $C$ in a 2-complex $X$ is conical if its collar closure is a cone over its nerve.

**Example 3.7.** Vertex neighborhoods provide simple examples of conical collars. For instance, let $r$ be a face of perimeter $n$, and consider the $n$ triangle faces associated with the barycentric subdivision of $r$. Then the union of all open triangles associated with these relations together with their open boundary edges, form a collar with corresponding abstract collar $S^1 \times (0,1)$.

A collar $C$ in $X$ with nerve $H$ and collar closure $\overline{C} \subset X$ can be extended to a map

$$H \times [0,1] \downarrow \overline{C}$$

The images of $H \times \{0\}$ (resp. $H \times \{1\}$) in $\overline{C}$ are subgraphs of the 1-skeleton of $X$ denoted $\partial^- C$ and $\partial^+ C$ respectively.

**Definition 3.8.** We call $\partial^- C$ (resp. $\partial^+ C$) the left (resp. right) boundary of $C$. 
Observe that in general the graphs $\partial^- C$ and $\partial^+ C$ are not isomorphic to the nerve. In fact, they need not be homotopy equivalent to it. Edges in $H$ may disappear, or may be turned into bouquets of circles.

**Definition 3.9.** We say that a collar $C$ in $X$ is *acylindrical* if $\partial^- C$ and $\partial^+ C$ have no common edge.

**Definition 3.10.** We say that a collar $C$ in $X$ is *boundary injective* if no edge in $\partial^- C$ (resp. $\partial^+ C$) belongs to two faces of $C$.

**Definition 3.11.** We say that a collar $C$ in $X$ is *open* if it is open as a subset of $X$.

(Therefore, if $C$ is an open collar in $X$, then all disk adjacent to an open edge $e$ of $C$ are contained in $C$.)

**Definition 3.12.** We say that a collar $C$ in $X$ is an *$h$-collar* if the maps $[0,1] \ni t \mapsto C(\cdot, t)$ is a homotopy equivalence between $\partial^- C$ and $\partial^+ C$.

We will turn our attention to the following types of collars:

**Definition 3.13.** If $X$ is a 2-complex, we write $\text{Col}(X)$ for the set of (simplicially closed and vertex-free) collars $C: H \times (0,1) \to X$ in $X$ which are open, acylindrical and nonconical. We write $\text{Col}_h(X)$ for the elements of $\text{Col}(X)$ which are $h$-collars.

**Example 3.14.** Assume that $G \acts X$ acts freely with exactly two orbits of vertices. Let $\text{Co} M$ and $\text{Co} N$ denote the (closed) cores of the two model geometries $M$ and $N$ in $X/G$. Then $C := X/G \setminus (\text{Co} M \cup \text{Co} N)$ is a collar in $\text{Col}(X/G)$.

**Definition 3.15.** If $A$ is a type, we let $\text{Col}_A$ be the set of all collars in $\text{Col}(X)$, considered up to isomorphism, where $X$ runs over the 2-complexes of type $A$.

This set is the object set of the group cobordism category $\text{Bord}_A$.

Observe that in general (even in the metric case) the embedding

$$C: H \times (0,1) \to X$$

does not embed the graph $H \times \{t_0\}$, for some $t_0 \in (0,1)$, as a totally geodesic subset of $X$.

**Definition 3.16.** We say that a metric collar $C$ in $\text{Col}(X)$ is *totally geodesic collar* if $C(H, t_0)$ is a totally geodesic subset of $X$ for some $t_0 \in (0,1)$.

Totally geodesic collars provide more options for metric surgery, including, for example, blow-ups of the graph $C(H, t_0)$ into a metric product $C(H, t_0) \times [0,1]$ ("collar dilatation") that preserves non positive curvature (but possibly alters the metric type of the complex).

**Example 3.17.** The union of all bowties for the group $\Gamma_\infty$ considered in [1] defines a totally geodesic collar.

Associated with $A$ is a set $\text{Ner}_A$ of topological (as opposed to metric) graphs, called the nerve space of $A$, which is the image of $\text{Col}_A$ under the nerve map $C \mapsto H$. The latter provides a forgetful map

$$\text{Col}_A \to \text{Ner}_A.$$

Note that even in the metric case, the graphs in $\text{Ner}_A$ need not be endowed with a natural metric, except in a few cases:
Definition 3.18. A type \( A \) splits if a metric can be found on every graph in \( \text{Ner}_A \) such that every elements in \( \text{Col}_A \) is a metric product, for a suitable metric on \((0,1)\).

Example 3.19. The type \( A \) defined by
a) 4-cycles with edges of length \( \pi/2 \)
b) Euclidean unit square
splits. The nerve space \( \text{Ner}_A \) consists of metric graphs which are finite disjoint unions of circles of integer length.

4. \( \frac{2}{3} \)-Transitivity

Let \( X \) be simplicial of dimension 2.

Definition 4.1. A group action \( G \) acts \( \frac{2}{3} \)-transitively if every (triangle) face of \( X \) intersects at most two orbits of vertices.

Every vertex-transitive action is \( \frac{2}{3} \)-transitive, as is every action with two orbits of vertices.

There are easy examples with arbitrarily large compact quotients \( X/G \). (Let \( X \) be the Euclidean plane tessellated by equilateral triangles, then the three simplicial actions \( \mathbb{Z} \to X \) translating the three directions are \( \frac{2}{3} \)-transitive.)

Furthermore, every group admits a \( \frac{2}{3} \)-transitive action on a simplicial complex of dimension 2, and, if finitely presented, a \( \frac{2}{3} \)-transitive action with compact quotient. (Let \( G \) be a group and \( X \) be a Cayley complex for \( G \), let \( X_\Delta \) be a barycentric subdivision of \( X \), then the action \( G \to X_\Delta \) is \( \frac{2}{3} \)-transitive.)

However, for every finite type \( A \) as defined in §2, \( \frac{2}{3} \)-transitivity is related to some global form of indicability (existence of an infinite abelian quotient \( G \to \mathbb{Z} \)). While not being strict indicability in the usual sense, this notion can still be profitably combined with some uniform version of the standard properties, such as uniform property \( \text{T} \).

We have, for instance, that:

Theorem 4.2. The family of groups which admit a free \( \frac{2}{3} \)-transitively on a Bruhat-Tits building of type \( \tilde{A}_2 \) and order \( q \) is finite.

We prove this in §5.

Remark 4.3. A complete classification of the groups defined by Theorem 4.2, for every fixed prime power \( q \), seems out of reach. It is also non trivial to find an asymptotic estimate of their number as \( q \to \infty \).

Definition 4.4. Two vertices \( x \) and \( y \) of \( X \) are said to be separated by a collar \( C \) if there exists a neighbourhood \( V \) containing \( x, y \) and \( C \) such that \( V \setminus C \) has exactly two connected components, one containing \( x \) and one containing \( y \).

For example, in a simplicialized 2-torus, the simplicial closure of an embedded circle not intersecting the vertex set defines a collar \( C \) separating the vertices of \( \partial^- C \) from the vertices in \( \partial^+ C \).

Lemma 4.5. If \( C \) is an open collar in \( X \) with disjoint and connected boundaries, and \( x, y \) are vertices in \( \partial^- C \) and \( \partial^+ C \) respectively, then \( C \) is separating \( x \) and \( y \).
Proof. Since $C$ has compact closure and $\partial^- C$ and $\partial^+ C$ are disjoint, we may choose disjoint neighborhoods $V^-$ and $V^+$ of $\partial^- C$ and $\partial^+ C$, respectively. Furthermore, since $\partial^- C$ and $\partial^+ C$ are connected, we may assume that $V^-$ and $V^+$ are. Since $C$ is open, the set $V = V^- \cup C \cup V^+$ is a neighborhood of $C$ containing $x$ and $y$. By construction, $V \setminus C$ has exactly two connected components containing $x$ and $y$ respectively.

Definition 4.6. We say that $C$ is a separating collar if it is open and if it has disjoint connected boundaries.

Lemma 4.7. Let $G \acts X$ be free and $\frac{2}{3}$-transitive. Any two distinct adjacent vertices $x, y$ of $X/G$ are separated by a collar, namely, the union of all open edges and open triangles whose vertex set closure coincides with $\{x, y\}$.

Proof. By definition, if $G \acts X$ is $\frac{2}{3}$-transitive then every triangle in $X/G$ contains at most two vertices, and therefore the following property holds: every inscribed polygon (in the sense of §1) in a crossing face of $X/G$ is degenerate and reduced to a segment. The union of these segments over all the triangles containing both $x$ and $y$ defines a graph $H$ in $X/G$ (which may not be connected). The resulting collar $C$ is the union of these triangles. Each triangle meets two points $x$ and $y$ of $X$, and the collar is separating $x$ from $y$ in the sense of the previous definition, by Lemma 4.5, since it is open, with disjoint connected boundaries.

Let $G \acts X$ be a $\frac{2}{3}$-transitive action. We define a graph $Z$ as follows:

- Vertex set of $Z :=$ vertex set of $X/G$
- Edge set of $Z :=$ pairs $(x, y)$ of adjacent vertices in $X/G$.

and consider the map

$$\pi: X/G \to Z$$

by sending every simplex of the model geometry over $x$ to $x$, and every simplex in the collar over $(x, y)$ to the open edge $(x, y)$.

Definition 4.8. The map $\pi: X/G \to Z$ (resp. the graph $Z$) is called the stack (resp. base) of a $\frac{2}{3}$-transitive action $G \acts X$.

The map $\pi$ is a stack in the sense of the Bass–Serre theory [12], as discussed in [8, Chap. 6], where the nerves provide the edge fibers of the stack.

The stack $\pi: X/G \to Z$ fails to provide a graph of groups decomposition for the group $G$ in general, and accordingly, the groups from Bass–Serre theory bear little resemblance to groups of intermediate rank, in general, due to the missing $\pi_1$-injectivity assumption. Algebraically, the stack map $\pi$ provides little information beyond that which is already contained in the Seifert–van Kampen theorem—namely, that of a (categorical) push-out diagram, where the arrows may fail to be injective.

Definition 4.9. We say that a collar $C$ in $X$ separating two vertices $x$ and $y$ has $\pi_1$-injective boundary if the maps $\partial^- C \to M_x$ and $\partial^+ C \to M_y$ into the model geometry at $x$ and $y$ respectively are $\pi_1$-injective.

5. INDICABILITY CRITERION

In this section we describe a “global indicability” criterion, which concerns the existence of an infinite abelian quotient $G \to \mathbb{Z}$ that takes place $\Omega_A$, where the group $G$ may vary and is obtained from surgery.
Definition 5.1. We say that a group $G$ is of type $A$ if it admits a free action $G \sim X$ with compact quotient on a complex $X$ of type $A$.

(In some cases it would certainly be appropriate to include proper actions in this definition.)

Theorem 5.2. Let $A$ be a finite simplicial type. Assume that there exist infinitely many pairwise nonconjugate free $\frac{2}{3}$-transitive actions with compact quotients on simplicial complexes of type $A$. Then there exists an indicable group of type $A$.

Proof. Let $G \sim X$ be free and $\frac{2}{3}$-transitive on a simplicial complex of type $A$ with stack

$$\pi: X/G \to Z.$$ 

For every pair $(x,y)$ of distinct adjacent vertices in $X/G$, let $C_{x,y}$ be the separating collar between $x$ and $y$ (Lemma 4.7), and, for every vertex $x$ in $X/G$, let $M_x$ be the model geometry at $x$ in $X/G$. (The collars $C_{x,y}$ are not assumed to have $\pi_1$-injective boundaries.)

Let us write

$$M_x \xleftarrow{C_{x,y}} M_y$$

for the corresponding configuration in quotient space $X/G$ (which corresponds to a closed edge in $Z$ under $\pi$). By the finiteness of $A$, the total number of all such collar configurations is finite up to isomorphism.

Assume that the set of conjugacy classes of free $\frac{2}{3}$-transitive actions $G \sim X$, where $X$ is a simplicial complex of type $A$ with $X/G$ compact, is infinite. Notice that the stacks of conjugate are equivariant, and the bases are isomorphic. Furthermore, since $A$ is finite, the bases $Z$ are uniformly locally bounded, and therefore we can find arbitrary long non-backtracking segments in the bases $Z$.

By the box principle, for one of these actions, say $G_0 \sim X_0$, one can find three disjoint consecutive edges $e, f, g$ in a segment included $Z_0$, whose collar configurations in $X_0/G_0$ are isomorphic to a given configuration. Two of the three edges, say $e$ and $f$, are oriented in the same direction, and we can find an isomorphism between the configuration

$$M_x \xleftarrow{C_{x,y}} M_y \quad \text{and} \quad M_x' \xleftarrow{C'_{x,y'}} M_y',$$

corresponding to $e$ and $f$ respectively, that takes $x$ to $x'$ and $y$ to $y'$.

Consider, then, infinitely many copies $G_p \sim X_p$, indexed by $p \in \mathbb{Z}$, of this action $G_0 \sim X_0$. The complex $G \sim X$ with $G \to \mathbb{Z}$ will be obtained by doing a simple surgery on the quotient spaces $X_p/G_p$ which respects the type $A$.

The surgery starts by “duplicating the collars” in order to be able to glue them together. Write $X_p/G_p$ (for every $p \in \mathbb{Z}$) as a quotient of a connected space $\hat{X}_p$

$$\pi_p: \hat{X}_p \to X_p/G_p$$

where $\pi_p$ is the identity map outside $C_{x_p,y_p}$ and $C'_{x_p',y_p'}$, and a two sheeted covered over $C_{x_p,y_p}$ and $C'_{x_p',y_p'}$, resulting in a space with four copies the same collar (up to isomorphism) in the neighborhood in of its boundary, say

$$C^-_{x_p,y_p}, C^+_{x_p,y_p}, C^-_{x_p',y_p'}, C^+_{x_p',y_p'}.$$

We distinguish three cases, according to the number of connected components of $Z_0 \setminus \{e, f\}$.

Note that $C^+_{x_p,y_p}$ and $C^+_{x_p',y_p'}$ are in the same component.
(1) If \( Z_0 \setminus \{ e, f \} \) is connected, then we identify (for every \( p \in \mathbb{Z} \))
\[
C^+_{x_p, y_p} \text{ with } C^-_{x_{p+1}, y'_{p+1}}, \quad \text{and} \quad
C^+_{x'_p, y'_{p}} \text{ with } C^-_{x_p, y_p}.
\]

(2) If \( Z_0 \setminus \{ e, f \} \) has two connected components, then \( \hat{X}_p \) has now two connected components. If the component containing \( C^+_{x_p, y_p} \) and \( C^-_{x'_{p}, y'_{p}} \) contains only these two collars, then we identify (for every \( p \in \mathbb{Z} \))
\[
C^+_{x_p, y_p} \text{ with } C^-_{x_{p+1}, y'_{p+1}}
\]
and discard the other component. Otherwise, one of the two components contains 3 collars, and then we repeat the steps described in the case where \( Z_0 \setminus \{ e, f \} \) is connected.

(3) Finally, if \( Z_0 \setminus \{ e, f \} \) has three connected components, then the component containing \( C^+_{x_p, y_p} \) and \( C^-_{x'_{p}, y'_{p}} \) contains only these two collars, and we discard two connected components and repeat the steps in the case where \( Z_0 \setminus \{ e, f \} \) has 2 connected components.

In all three cases, the resulting space \( Y \) can be represented symbolically as an infinite chain:
\[
\ldots \leftrightarrow M_{x_p} - M_{y_p} \leftrightarrow M_{x'_p} - M_{y'_{p}} \leftrightarrow M_{x_{p+1}} - M_{y_{p+1}} \leftrightarrow \ldots
\]
\( \leftrightarrow \) indicates the surgery operation.

Note that the only new geometric configurations in \( Y \) are of the form
\[
M_x \leftrightarrow M_{y'} \quad \text{and} \quad M_{x'} \leftrightarrow M_y
\]
which fit into chains of the form
\[
M_x \overset{C_{x,y}}{\leftrightarrow} M_y \equiv M_{x'} \overset{C'_{x',y'}}{\leftrightarrow} M_y'
\]
where the type is preserved by definition of the identification. Therefore, \( Y \) is of type \( A \).

Furthermore, the space \( Y \) admits a free action of \( \mathbb{Z} \) by translations, whose quotient \( Y/\mathbb{Z} \) is compact and isomorphic to a "double" of \( X_0/G_0 \) (which needs not be a double cover). Namely, the quotient space \( Y/\mathbb{Z} \) can be represented symbolically as follows:

\[
Y/\mathbb{Z} = M_x \leftrightarrow M_y \leftrightarrow M_{x'} \leftrightarrow M_{y'}
\]

Let
\[
G := \pi_1(Y/\mathbb{Z}) \quad \text{and} \quad X := \hat{Y}/\mathbb{Z}.
\]

Since the type is clearly preserved by taking universal covers, the complex \( X \) is of type \( A \). Furthermore, by construction, \( G \rightarrow X \) is a free \( \frac{2}{3} \)-transitive action with compact quotient \( X/G = Y/\mathbb{Z} \) admits an infinite abelian cover. In particular \( G \rightarrow \mathbb{Z} \).

(Comment that the action of the group of Galois transformations of \( X \rightarrow Y \) itself is free and \( \frac{2}{3} \)-transitive.)
Proof of Theorem 4.2. Since the type $\tilde{\mathbb{A}}_{2,q} := \mathbb{A}_2$ of order $q$ is finite, if there are infinitely many $\frac{2\pi}{3}$-transitive free actions, then we can find an indicable group $G$ of type $\tilde{\mathbb{A}}_2$. This contradicts property T—In fact, this contradicts either Garland’s theorem directly, or indeed the Cartwright–Mlotkowski–Steger theorem that such a group has Kazhdan’s property T. We recall that Garland’s theorem is the statement that the first cohomology group $H^1(G, \pi)$ with coefficient in a finite dimensional unitary representation $\pi$ vanishes. In particular, $H^1(G, \mathbb{C}) = 0$ (trivial coefficients) so that $H^1(G, \mathbb{Z}) = \text{Hom}(G, \mathbb{Z})$ is torsion. Therefore, the number of groups, and for every such group, the number of its $\frac{2\pi}{3}$-transitive free action, is finite. □

We note that these constructions extend beyond simplicial complexes, using the following version of $\frac{2\pi}{3}$-transitivity:

**Definition 5.3.** A free action $G \sim X$ on a 2-complex is mildly transitive if every inscribed polygon in a crossing face of $G/X$ is reduced to a segment.

Equivalently, the equivalence relations defined in the proof of Proposition ?? has only two classes for every crossing face. A simplicial separating collar between any two distinct adjacent vertices can defined in the same way for mildly transitive actions, and for simplicial complexes, mild transitivity is equivalent to $\frac{2\pi}{3}$-transitivity.

The following straightforward generalization of Theorem 5.2 holds.

**Theorem 5.4.** If $A$ is finite and there are infinitely many free mildly transitive actions $G \sim X$, with $X$ of type $A$ and $X/G$ compact, then there exists an indicable group of type $A$.

6. The ST lemma

**Definition 6.1.** A collar is thick if every vertex of the nerve is adjacent to at least three edges.

**Definition 6.2.** A metric complex $X$ is $\theta$-convex if the angle of every face is $< \theta$. We let $A(2\pi, \theta)$ denote the metric type which describes the simplicial metric $\theta$-convex 2-complexes of nonpositive curvature.

(The type $A(2\pi, \theta)$ is uncountable.)

**Lemma 6.3** (ST lemma). The nerve of a minimal thick collar of $\text{Col}_{A(2\pi, 2\pi/3)}$ is isomorphic to one of the following two graphs:

a) the cylinder — or “thickened square”

$$S = \begin{array}{c} \includegraphics{cylinder.png} \end{array}$$

or,

b) the tetrahedron:

$$T = \begin{array}{c} \includegraphics{tetrahedron.png} \end{array}$$

(We view $S$ as a square with two opposite double edges.)

The proof is given below. Let $X$ be a nonpositively curved $\frac{2\pi}{3}$-convex metric simplicial 2-complex.

Let $C$ be a thick collar in $\text{Col}(X)$. Since $C$ is thick, the path $\gamma_t : t \mapsto C(v, t)$
in \(X\) is contained in the 1-skeleton of \(X\) for every vertex \(v\) of \(H\). Since \(C\) is vertex-free, \(\gamma_v\) does not intersect the vertex set of \(X\).

It follows that every edge \(e\) in \(H\) between two vertices \(u, v\) is attached to a unique vertex in \(\text{span}(C)\), defined as the intersection of the two edges \(e_u\) and \(e_v\) containing \(\gamma_u\) and \(\gamma_v\).

This gives a surjective map \(H^1 \rightarrow \text{span}(C)\).

**Definition 6.4.** This map is called the span decomposition of the nerve \(H\).

The span decomposition consists of \(|\text{span}(C)|\) subgraphs of \(H\), denoted \(H_x\) for \(x \in \text{span}(C)\), partitioning the edge set. Since every edge in \(H_x\) corresponds to a triangle attached to \(x\) we have

**Lemma 6.5.** For every \(x \in \text{span}(C)\), the graph \(H_x\) is isomorphic to subgraph of the link of \(x\).

Pictorially, we have \(|\text{span}(C)|\) subgraphs in the links of \(x \in \text{span}(C)\) that “move towards the centerpiece” of the collar to recombine into the nerve \(H\) of \(C\).

We note that:

**Lemma 6.6.** If \(C\) is a collar and \(\text{span}(C) = \{x\}\), then \(H\) is isomorphic to a union of connected components of the link of \(x\).

**Proof.** The nerve \(H\) is isomorphic to the subgraph \(H_x\) of the link \(L_x\). The collar \(C\) being open, if \(H_x\) contains a vertex of \(L_x\), it contains all the edges attached to \(x\). This implies that \(H_x\) is a union of connected components of \(L_x\).

Since we have assume that collars in \(\text{Col}(X)\) be non conical, this shows that \(|\text{span}(C)| \neq 1\) for every \(C \in \text{Col}(X)\).

**Lemma 6.7.** Under the assumptions of the ST lemma, the nerve \(H\) contains at least 4 vertices and 6 edges.

**Proof.** Let \(H\) denote the nerve of \(C\). If \(v\) denotes the number of vertices, then since \(C\) is thick, \(H\) has \(e \geq \frac{3}{2}v\) edges. The three minimal cases are:

a) \(v = 1\) and \(e \geq 2\)
b) \(v = 2\) and \(e \geq 3\)
c) \(v = 3\) and \(e \geq 5\).

which we will now show are not possible.

Note that if \(|\text{span}(C)| \geq 3\), then \(v \geq 4\), since the union of three edges corresponding to different span vertex cannot form a loop in \(H\). Therefore in the three above cases \(|\text{span}(C)| = 2\).

The graph \(H\) does not contain a loop, for if it does then so does one of the graphs \(H_x\) in the span decomposition, and since \(H_x \subset L_x\), this contradicts nonpositive curvature as link edges have length < \(\pi\) by strict convexity. This takes care of case a), and shows that every edge in the other cases has distinct extremities.

In the second case b), at least one of the two graphs \(H_x\) or \(H_y\), \(x, y \in \text{span}(C)\), must also contain a double edge, which in turn forces the link of \(x\), or that of \(y\), to have a double edge. The same strict convexity argument then applies.

In the last case, the graph \(H\) cannot contain a triple edge, for otherwise since \(|\text{span}(C)| = 2\) one of the two graphs \(H_x\) or \(H_y\), \(x, y \in \text{span}(C)\) would contain a double edge. Therefore we have either \(v = 3\) and \(e = 5\), or \(v = 3\), and \(e = 6\),
corresponding to a triangle with, respectively, two or three double edges, and at least one of the two span graphs, say \( H_x \), is a triangle (with simple edges). This contradicts the link condition by \( \frac{2\pi}{3} \)-convexity.

Therefore, \( v \geq 4 \) and \( e \geq 6 \).

**Proof of the ST lemma.** It follows from the previous lemma that if \( H \) is as in the ST lemma, and \( H \) is a minimal solution, then \( H \) is a connected cubic graph with 4 vertices and 6 edges. There are precisely two such graphs, \( T \) and \( S \). We will see in the forthcoming sections that these graphs indeed occur as nerves of collars.

**Remark 6.8.** It is clear from the proof that the ST lemma remains true when nonpositive curvature and \( \frac{2\pi}{3} \)-convexity are replaced by a rather weak systolic condition, namely, that \( X \) is a simplicial 2-complex whose links have girth \( \geq 4 \). (Note that the corresponding type \( A(4) \) is countably infinite.) For the purpose of rank interpolation, which is primarily concerned with questions on nonpositively curved spaces and CAT(0) groups, the metric version is directly useful (and in the situations we have in mind, the \( \frac{2\pi}{3} \)-convexity assumption is always satisfied). There are also more general versions of the lemma, with a modified classification, when the systolic assumption is further relaxed.

We will also need the following result.

**Definition 6.9.** A collar in \( X \) is of type \( S \) (resp. \( T \), resp. \( ST \)) if its nerve is isomorphic to \( S \) (resp. \( T \), resp. \( ST \)).

**Lemma 6.10** (Span decomposition of minimal collars.). If \( C \) is a collar of \( X \) of type \( ST \) with span \( \text{span}(C) = \{x, y\} \) and if \( X \) is \( \frac{\pi}{2} \)-convex, then in the span decomposition the two graphs \( H = H_x \cup H_y \) are isomorphic to a path of length 3.

Furthermore, the span decompositions are given by:

a)

\[
S = \begin{array}{c}
\bullet \\
\bullet
\end{array}
\]

b)

\[
T = \begin{array}{c}
\bullet \\
\bullet
\end{array}
\]

where (say) the dotted subgraph corresponds to \( H_x \) and its complement to \( H_y \).

**Proof.** Since \( H \) contains either a double edge or a cycle of length 3, an argument similar to that in the previous lemma shows, using \( \frac{2\pi}{3} \)-convexity, that \( |\text{span}(C)| \neq 1 \), i.e. \( x \neq y \).

Let \( H_x \) and \( H_y \) be the corresponding graphs, and consider the case of \( T \) first. By \( \frac{2\pi}{3} \)-convexity, neither \( H_x \) nor \( H_y \) can contain a cycle of \( T \), and they must therefore be (a priori possibly disconnected) trees. However, it is easy to check that in that case both \( H_x \) and \( H_y \) must have exactly 3 edges, and must be connected. The drawing shows the only possible embeddings up to graph isomorphism.

Consider then the case of collars of type \( S \). By strict convexity, neither \( H_x \) nor \( H_y \) can contain a double edge, so both of them have at least two edges. We
distinguish two cases. In the first case, $H_x$ has two edges and $H_y$ four. It follows
that $H_y$ corresponds to a circle of length 4 in $H$, which contradicts $\pi_2$-convexity.
Therefore both $H_x$ and $H_y$ must have 3 edges. Again, the drawing shows the only possible embeddings up to graph isomorphism. □

Definition 6.11. We say that a collar is treeable if the span decomposition of its edge set consists of maximal subtrees.

Lemma 6.10 shows that collars of type $ST$ are treeable.

Proposition 6.12. Let $X$ be a nonpositively curved $\pi_2$-convex metric simplicial 2-complex and $C$ be a boundary injective treeable collar in $X$ spanning two vertices. Then $C$ is a $h$-collar.

This provides an easy criterion for checking if a collar of type $ST$ is a $h$-collar in a given complex $X$ satisfying the assumptions.

Proof. Let us write $\text{span}(C) = \{x, y\}$ and let $H := H_x \cup H_y$ be the span decomposition of the nerve $H$. By assumption, both $H_x$ and $H_y$ are subtrees of $H$.

Since $C$ is treeable, $H \times [0, 1]$ is a homotopy between $H/H_x$ and $H/H_y$, where $H/H_x$ and $H/H_y$ denotes respectively the retract of $H$ along $H_x$ and $H_y$.

The extension of $C$ to $H \times [0, 1]$, namely the map

$$H \times [0, 1] \xrightarrow{\text{C}}$$

whose image is the collar closure $\overline{C}$ of $C$ in $X$, induces two maps $H/H_x \rightarrow \partial^- C$ and $H/H_y \rightarrow \partial^+ C$. Since $X$ is simplicial, these maps send edges to edges, and do so injectively, by boundary injectivity. Furthermore, since $H_x$ and $H_y$ are maximal subtrees, they are graph isomorphisms, and both $\partial^- C$ and $\partial^+ C$ are bouquets of circles. Thus $H \times [0, 1] \rightarrow \overline{C}$ is a homotopy between $\partial^- C$ and $\partial^+ C$. □

Finally, we observe:

Lemma 6.13. Under the assumptions of Lemma 6.10,

(1) there is a graph involution $\theta: S \rightarrow S$ that respects the span decomposition, namely, such that

$H_x \leftrightarrow H_y$.

(2) there is no graph involution $T \rightarrow T$ that respects the span decomposition.

(3) there exists an element $\sigma: T \rightarrow T$ of order 4 that respects the span decomposition, namely, such that

$H_x \leftrightarrow H_y$.

Proof. (1) is clear. For (2), note that the symmetry group being the symmetric group on the 4 vertices, no involution respects the span decomposition. However, if the external vertices in $T$ are labelled 1,2,3, oriented counterclock wise, with 1 on top and 4 in the center, then the 4-cycle $\sigma = (1243)$ provides the desired transformation of order 4. □
Remark 6.14. If the involution $\theta : S \to S$ described in (1) extends to the collar $C$ of type $S$ (for example if $X$ is made of equilateral triangles), then $C$ is self-dual.

7. Double covers

In this section we show that collars of type $T$ do not occur among double covers in complexes of type $\frac{7}{4}$ or $\tilde{A}_{2,2}$. Double covers are a convenient way to produce filling cobordisms of the given type.

Using a simple idea explained in §9 below, this observation can be used to construct “fake double covers”, by substituting collars of type $S$ by collars of type $T$ in plain double covers.

Collars of type $T$ can be ruled out by a simple symmetry argument using the involution acting on double covers.

In fact, the argument leads naturally to new collars in such complexes, including collars of the following two types:

**Definition 7.1.** Let us call $\Theta$-nerve (resp. $\Theta'$-nerve) the graph

$$\Theta := \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot
\end{array}$$

(resp. $\Theta' := \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot
\end{array}$)

with the indicated span decomposition. A collar is of type $\Theta$ (resp. $\Theta'$) if its nerve is the $\Theta$-nerve (resp. $\Theta'$-nerve).

These are larger collars which behave similarly to collars of type $ST$ in this context. We will see that collars of type $\Theta'$ also do not appear among double covers in complexes of type $\frac{7}{4}$ or $\tilde{A}_{2,2}$.

Let us fix some notation for this section:

- $X$ denotes a metric 2-complex which is either of type $\frac{7}{4}$ or $\tilde{A}_{2,2}$ with one vertex.
- $X'$ denotes an arbitrary 2-cover of $X$
- $C$ denotes the collar separating the two vertices $x$ and $y$ in $X'$, and $H$ refers to the nerve of $C$ with its nerve decomposition.

**Lemma 7.2.** The nerve $H$ has either 6 or 12 vertices. Furthermore, $\text{Aut}(H)$ contains an involution $s$ exchanging the two components of the span decomposition.

**Proof.** The nerve $H$ has $v$ vertices and $e = \frac{3}{2}v$ edges. Furthermore, if $e_x$ and $e_y$ denotes the number of edges of $H_x$ and $H_y$ respectively, so $e = e_x + e_y$, then we have

$$2e_x + e_y \leq |L_y| \leq 24$$
$$2e_y + e_x \leq |L_x| \leq 24$$

by assumption, therefore $e \leq 16$. Since $e$ is a multiple of 3, it follows that $e = 6, 9, 12, 15$.

By assumption, $\pi_1(X)$ admits an index 2 subgroup acting $\frac{2}{3}$-transitively on $X$ with quotient $X'$, which gives the desired involution

$$s : H \to H.$$
It is clear that this involution takes $H_x$ to $H_y$. In particular, $e_x = e_y$ so $e$ is even, narrowing down the options to $e = 6$ or $e = 12$. □

This proves our first claim:

**Proposition 7.3.** The collar $C$ is not of type $T$.

**Proof.** Assume that that nerve $H$ contains 6 edges. By Lemma 6.3, $H$ as a nerve coincides with $S$ or with $T$ with the given span decomposition. The second case is ruled out by the fact that the involution $s$ must permute the two copies of the path of length 3 in the span decomposition. (Compare Lemma 6.13). □

We are naturally lead to consider the case of collars with 12 edges. In that case, we assume furthermore that $H$ is connected. A priori, disconnected collars may appear, and the connected components, being collars themselves, must have at least 6 edges by Lemma 6.3. In particular both connected components must be of type $ST$. We will come back to this interesting situation later.

We need to explain now how the collars of type $\Theta$ and $\Theta'$ arise. Then collars of type $\Theta'$ can be ruled out using symmetry.

**Definition 7.4.** We say that a collar is cubic if its nerve is isomorphic to the cube.

The following proves our second claim, that collars of type $\Theta'$ do not appear.

**Lemma 7.5.** Assume that $H$ has 12 edges, and that the components of the nerve decomposition are connected. Then $C$ is either cubic or of type $\Theta$. In particular, $C$ is not of type $\Theta'$.

**Proof.** Using the notation of Lemma 7.2 if $e = 12$, then $e_x = e_y = 6$, and using the symmetry $s$ we see that $H_x$ and $H_y$ are either

- two circles of length 6, or
- two isomorphic trees with 6 edges.

Furthermore, by our assumption, the two trees in the second case are connected.

The first case is in fact not possible, for one of the vertices of $H$ would have order 4. This same argument also works in the second case to show that both trees can't be straight segments.

Note that by definition the involution $s$ cannot fix an edge of $H$. Neither can it fix a vertex, since $H$ is cubic. Let $a, b, c$ be the number of vertices of order 1, 2, 3 respectively in $H_x$ and $H_y$. Note that $a = b$ since since every $x$-vertex of order 1 can be paired with a $y$-vertex of order 2. Furthermore,

$$a + 2b + 3c = 12$$

(counting edges in $H_x$ or $H_y$) so

$$a + c = 4.$$  

If $H_x$ is connected then

$$a + b + c = 7$$

so $a = b = 3$ and $c = 1$. Therefore $H_x$ and $H_y$ are tripods of height 2. It is not hard to check that there are three ways to combine these tripods together to give a cubic graph on 8 vertices. The three possibilities are the cube, and the two graphs $\Theta$ and $\Theta'$. However, the graph $\Theta'$ does not admit an involution that permutes $H_x$ and $H_y$. □
We now consider the case where the components of the span decomposition are disconnected.

**Lemma 7.6.** If the components of the span decomposition are disconnected, then they have two connected components, both of which being segments of length 3.

**Proof.** In the notation of the previous lemma, if \( H_x \) has \( k \geq 2 \) connected components connected then

\[
a + b + c = 6 + k
\]

so \( a = b = 2 + k \) and \( c = 2 - k \geq 0 \). This forces \( k = 2 \), \( a = b = 4 \) and \( c = 0 \). In that case we have either that

\[H_x \simeq H_y \simeq P_3 \cup P_3\]

or that

\[H_x \simeq H_y \simeq P_3 \cup P_1\]

where \( P_n \) denotes the segment with \( n \) edges. An argument similar to the case of a single segment \( \simeq P_6 \) in Lemma 7.8 shows that this is not possible in a connected graph \( H \). \(\square\)

**Definition 7.7.** A collar is said to be octagonal if its nerve is isomorphic to the following graph

![Octagonal Collar](image)

with the given span decomposition.

**Lemma 7.6** implies:

**Lemma 7.8.** Assume that \( H \) has 12 edges, and that the components of the nerve decomposition are disconnected. Then \( C \) is octagonal.

To summarize, we have shown that:

**Proposition 7.9.** In a 2-cover (with connected nerve) of a 2-complex of type \( \tilde{T} \) or \( \tilde{A}_{2, 2} \) with one vertex, the collar is either:

- of type \( S \)
- cubic
- of type \( \Theta \)
- octagonal

In particular, the collars of types \( T \) and \( \Theta' \) do not appear.

### 8. Group Cobordisms

**Definition 8.1.** A group cobordism is a 2-complex \( X \) together with a pair \((C, D)\) of collars in \( \text{Col}(X) \) whose boundaries \( \partial^- C \) and \( \partial^+ D \) form a partition of the topological boundary of \( X \):

\[
\partial X = \partial^- C \cup \partial^+ D.
\]
We call \( C \) and \( D \) respectively the domain and range of \( X \), and use the notation \( X: C \to D \) to denote cobordisms of domain \( C \) and range \( D \). We view \( C \) and \( D \) as subcomplexes of \( X \). Namely, we assume that \( X \) is endowed with fixed injective 2-complex morphisms \( L_X: C \to X \) and \( R_X: D \to X \). The collar boundary of \( X \) is defined by

\[
\partial_{c} X := C + D.
\]

Here and below, + indicates set-theoretic union. If either \( C \) or \( D \) is empty, we say that \( X \) is a null-cobordism or a filling.

We shall call dual of a group cobordism \( X: C \to D \) is the group cobordism \( X': D' \to C' \) where \( X \) coincide with \( X' \) as a topological space, and \( C', D' \) are the respective dual of \( C \) and \( D \). We have a partition

\[
\partial X' = \partial^{-} D' \cup \partial^{+} C'.
\]

**Definition 8.2.** A morphism (resp. isomorphism) of group cobordisms is 2-complex morphism (resp. isomorphism) that takes collar boundaries to collar boundaries.

Let \( A \) be a type (simplicial or metric). We define a small category \( \text{Bord}_A \) as follows:

- **Object set:** the set \( \text{Col}_A \) of all isomorphism classes of collars in 2-complexes \( X \) of type \( A \), which we view as 2-complexes \( C, D, E, \ldots \).
- **Arrow set:** the set of all equivalence classes of group cobordisms of type \( A \), where \( X: C \to D \) and \( Y: C \to D \) are said to be equivalent if there is a 2-complex isomorphism \( \varphi: X \to Y \), such that \( \varphi \circ L_X = L_Y \) and \( \varphi \circ R_X = R_Y \).

The composition in \( \text{Bord}_A \) is defined by amalgamation as in a standard cobordism category. If \( X: C \to D \) and \( Y: D \to E \) are group cobordisms, then

\[
Y \circ X: C \to E
\]

is a group cobordism, with domain \( C \) and range \( E \), given by

\[
Y \circ X = (X \sqcup Y)/\sim_P
\]

where \( \sim_P \) identifies the two copies of \( D \) using \( L_Y \circ R_X^{-1} \), and

\[
\partial(Y \circ X) = \partial^{-} C' \cup \partial^{+} E'.
\]

This gives a well-defined arrow from \( C \) to \( E \), and composition is strict. The following lemma holds because our notion of type is local.

**Lemma 8.3.** If \( X \) and \( Y \) are group cobordisms of type \( A \) then \( Y \circ X \) is a group cobordism of type \( A \).

**Proof.** Since both \( X \) and \( Y \) are of embedded type \( A \), and \( \sim_D \) identifies the two copies of \( D \), the set of shapes in \( Y \circ X \) is indeed described by \( A \). To check the link condition, we only need to consider the vertices in the boundary of \( D \) (viewed as sitting in \( Y \circ X \)) which do not belong to the boundary of \( Y \circ X \). If \( x \in \partial^{-} D \) (resp. \( y \in \partial^{+} D' \)) is such a vertex, then it is an interior vertex of \( X \) (resp. of \( Y \)), and therefore its link is indeed described by \( A \). \( \square \)

Note that the collar closure of \( C: H \times (0,1) \to X \) defines itself a group cobordism of type \( A \), which corresponds to the unit arrow in the category \( \text{Bord}_A \).

**Definition 8.4.** We call \( \text{Bord}_A \) the (unoriented) group cobordism category of type \( A \).
As in the classical case, the category $\text{Bord}_A$ is a symmetric monoidal category, even if it is not, strictly speaking, a cobordism category in the classical sense [16]. Nevertheless, the abstract cobordism relation (see [16, Chapter I]) defines an equivalence relation on $\text{Bord}_A$.

**Definition 8.5.** Two objects $C, D$ in $\text{Bord}_A$ are cobordant, in symbols, $C \equiv D$, if there exists two arrows $X, Y$ in $\text{Bord}_A$ such that

$$C + \partial_c X \simeq D + \partial_c Y$$

where $\simeq$ denotes collar isomorphism.

Note that if $C \equiv D$ and $E \equiv F$ then $C + D \equiv E + F$.

**Lemma 8.6.** The relation $\equiv$ is an equivalence relation.

**Proof.** $X = Y = \emptyset$ provides reflexivity, symmetry is obvious, and if $C \equiv D$ and $D \equiv E$, then choosing arrows $X, Y, Z, T$ such that

$$C + \partial_c X \simeq D + \partial_c Y$$

$$D + \partial_c Z \simeq E + \partial_c T$$

then using the arrows $X + Z$ and $Y + T$ we see that

$$C + \partial_c X + \partial_c Z \simeq D + \partial_c Y + \partial_c Z \simeq E + \partial_c T + \partial_c Y$$

so $C \equiv E$. $\square$

**Definition 8.7.** Let $A$ be a type (topological or metric). The (unoriented) group cobordism monoid of type $A$ is the set

$$\Omega_A := \text{Bord}_A^0 / \equiv$$

where $\text{Bord}_A^0$ denotes the object set and the monoid structure is given by

$$[C] + [D] := [C + D]$$

with neutral element $[\emptyset]$.

It is obvious that:

**Lemma 8.8.** $\Omega_A$ is an abelian monoid.

The following elementary proposition, combined with Prop. 7.9 allows to show that some classes vanish in $\Omega_A$ (and $\Omega_{\tilde{A}_2}$).

**Proposition 8.9.** Let $A$ be a type. If $C$ be the separating collar in a complex of type $A$ with two vertices, then

$$[C] = [C'] = 0$$

in the group cobordism monoid $\Omega_A$. (Here $C'$ denotes the dual collar.)

**Proof.** By assumption we can write $X = (X^- \sqcup X^+)/\sim$, where $\partial_c X^- = \overline{C}$ and $\partial_c X^+ = \overline{C'}$, which proves the assertion. $\square$
9. Fake double covers

In this section we construct “fake” double covers for complexes of rank $\frac{7}{4}$ using a simple surgery in the following way:

1. Assume that $X' \to X$ is a double cover of a complex $X$ of rank $\frac{7}{4}$ with 1 vertex, whose separating collar $C$ is (say, connected) of type $S$.
2. Choose a collar $D$ of type $T$ whose boundary is isomorphic to $\partial C$:
   \[ \partial^+ D = \partial^- C \text{ and } \partial^- D = \partial^+ D, \]
3. Substitute $D$ to $C$ in $X'$.

This leads to a new complex $X''$, which is “of fake rank $\frac{7}{4}$”. Here is an explicit construction.

The classification in [3, §4] provides several complexes $X$ of rank $\frac{7}{4}$ with one vertex such that $H_1(X, \mathbb{Z}/2\mathbb{Z}) \neq 0$. One of the simplest examples is the complex (denoted $V_3^3$ in [3, §4]) defined by

\[ X := [(1, 1, 3), [2, 2, 4), [1, 5, 2), [3, 6, 4), [3, 7, 6), [4, 6, 8), [5, 7, 8), [5, 8, 7)] \]

whose rational homology is reduced to $\mathbb{Z}$. The corresponding group of rank $\frac{7}{4}$ has a presentation of the following form:

\[ \Gamma := \langle s, t \mid s^2 t^3 s t^2 = t^2 s^2, t^2 = s^2 t^2 s^2 t^{-2} t s \rangle. \]

The group $\Gamma$ admits an index 2 subgroup $\Gamma'$, which appears as the fundamental group of a covering space $X' \to X$ of degree 2 and can be explicitly computed to be:

\[ X' := [(1, 1, 3), [2, 12, 4), [1, 15, 12), [3, 6, 4), [3, 7, 6), [4, 6, 8), [5, 7, 8), [5, 8, 7), [11, 1, 13), [12, 2, 14), [11, 5, 2), [13, 16, 14), [13, 17, 16), [14, 16, 18), [15, 17, 18), [15, 18, 17)] \]

where the new edges are labeled $1x$ for every edge with label $x$ in $X$.

A direct computation shows:

**Lemma 9.1.** The separating collar $C$ in $X'$ is of type $S$.

In order to construct a “fake” double cover, we have to substitute to $C$ a collar $D$ of type $T$.

The collar closure of $C$ in $X'$ is the 2-complex given by:

\[ [[1, 11, 3), [2, 12, 4), [1, 15, 12), [11, 1, 13), [12, 2, 14), [11, 5, 2)]. \]

We will consider instead the following collar

\[ [[[1, -2, 3), [-11, 12, 4), [1, 15, 12), [11, 1, 13), [12, 2, 14), [11, 5, 2)]. \]

where $-x$ indicate that the edge $x$ is opposite in the triangle. Note that, by definition, this construction flips the appropriate edges in the nerve of the collar $C$, which will therefore become a collar of type $T$.

In other words, we define a new complex $X''$ as follows:

\[ X'' := [[1, -2, 3), [-11, 12, 4), [1, 15, 12), [3, 6, 4), [3, 7, 6), [4, 6, 8), [5, 7, 8), [5, 8, 7), [11, 1, 13), [12, 2, 14), [11, 5, 2), [13, 16, 14), [13, 17, 16), [14, 16, 18), [15, 17, 18), [15, 18, 17)] \]

By construction we have:

**Lemma 9.2.** The separating collar $D$ in $X''$ is of type $T$. 
The flip in the nerve of $X'$ corresponds to a transformation of its links which exchanges the extremities of two edges. This transformation can easily be computed explicitly:

**Lemma 9.3.** The links in $X''$ are isomorphic to:

$$
\begin{array}{c}
\text{Figure 1. The fake Moebius–Kantor graph}
\end{array}
$$

Note that the fake Moebius–Kantor graph has girth 5, and therefore the space $X''$ admits sections of positive curvature; at the same time, part of the flatness is turned into negative curvature. A simplicial 2-complex whose link are isomorphic to the fake Moebius–Kantor graph is said to be of fake rank $\frac{7}{4}$.

The lack of symmetries of collars of type $T$ (compare the proof of Prop. 7.3) shows that:

**Proposition 9.4.** The complex $X''$ is not a double cover.

In general, a 2-complex with two vertices fails to be a double cover for various reasons, which can be detected in balls of sufficiently large radius around the vertices; the most obvious reason is to have non isomorphic links at the vertices (see §11 for the existence of such complexes).

10. NON FILLING GROUP COBORDISMS OF RANK $\frac{7}{4}$

Model geometries, as defined in §1, can be classified, and give rise to the simplest group cobordisms. In the present section we classify the nonfilling such cobordisms, for groups of rank $\frac{7}{4}$. In turn, these cobordisms can be used to construct new groups of intermediate rank.

The main result is that:

**Theorem 10.1.** There exist precisely two non filling group cobordisms of rank $\frac{7}{4}$ with one vertex whose boundary collars are connected of type $ST$. Furthermore, the two cobordisms are self-dual, and their collars are pairwise isomorphic, self-dual, and of type $S$.

Observe that such cobordisms do not exist, for example, for groups (of rank 2) of type $A_2$.

**Proof of Theorem 10.1.** Let $X: C \to D$ be such a group cobordism, and $L$ be the link of the unique vertex in $X$ not in the boundary. By assumption, $L$ is isomorphic to the Moebius–Kantor graph (link of rank $\frac{7}{4}$).
By definition of the collar of type $S$ (see Lemma 10.3), we have two disjoint embeddings of the path of length 3 in $L$, given by the span decomposition. Let $\alpha$ and $\beta$ denote the two copies of these two paths.

Observe that $\alpha$ and $\beta$ are roots in $L$ in the sense of [2]. Since $L$ is the graph of rank $\frac{7}{4}$, these roots have rational rank $\text{rk}(\alpha), \text{rk}(\beta) \in \{\frac{3}{2}, 2\}$.

Lemma 10.2. The automorphism group of $L$ has exactly two orbits of roots, which are the set of roots of rank 2 and the set of roots of rank $\frac{3}{2}$.

Proof. The argument can be extracted from the proof of Proposition 41 in [3]. Namely, the automorphism group $G$ is transitive on the flags $A \subset \gamma$ where $\gamma$ is a simplicial path of length 2 and $A$ is an extremity of $\gamma$, from which it follows that there are at most 2 orbits of roots. It is clear on the other hand that roots of rank 2 and roots of rank $\frac{3}{2}$ belong to distinct $G$-orbits. $\square$

Lemma 10.3. The open 1-neighborhoods of $\alpha$ and $\beta$ are disjoint: $N_1(\alpha) \cap N_1(\beta) = \emptyset$.

Proof. If not, an edge connecting $\alpha$ and $\beta$ would correspond to a triangle whose vertices are distinct embedded vertices of $X$, i.e., form three distinct vertices in a complex of rank $\frac{7}{4}$ in which $X$ embeds. Therefore such an edge corresponds to an edge in the boundary of $X$ that connects the collars $C$ and $D$, which contradicts the fact that $\partial X = \partial^- C \cup \partial^+ D$. $\square$

Lemma 10.4. The loops in $X$ (viewed as a model geometry, as defined in §1) can be of three types:

- disjoint from $N_1(\alpha \cup \beta)$
- connecting two vertices in $N_1(\alpha)$
- connecting two vertices in $N_1(\beta)$

Proof. This follows from the fact that if a loop intersects $N_1(\alpha)$, then it is the boundary to an embedded triangle of $X$ having an angle in $N_1(\alpha)$. Such a triangle belongs to $C$, and its boundary loop connects two vertices in $N_1(\alpha)$. $\square$

By Lemma 10.2 it is enough to consider the two cases

a) $\text{rk}(\alpha) = \frac{3}{2}$, and,

b) $\text{rk}(\alpha) = 2$.

We start with b). The following lemma is straightforward.

Lemma 10.5. If $\text{rk}(\alpha) = 2$, then $L \setminus N_2(\alpha)$ is a segment of simplicial length 5.

This gives us 3 possibilities for the position of $\beta$. One corresponds to a root of rank 2, and the two other cases, which are permuted by a graph automorphism, to a root of rank $\frac{3}{2}$.

In fact, the position of $\beta$ in $L$ is uniquely determined relative to $\alpha$:

Lemma 10.6. If $\text{rk}(\alpha) = 2$, then $\beta$ is the unique root of rank 2 included in $L \setminus N_2(\alpha)$. 
Proof. Assume that $\beta$ is one of the two roots of rank $\frac{3}{2}$. Let $u$ denote the unique vertex of $L$ which is at distance 2 from both $\alpha$ and $\beta$. The unique loop $\gamma$ of $M$ through $x$ intersect $L$ at a vertex $v$, which must be at distance 1 from either $\alpha$ or $\beta$, which is a contradiction. \hfill \Box

Let us now consider a). We have the following analog of Lemma 10.5:

**Lemma 10.7.** If $\text{rk}(\alpha) = \frac{3}{2}$, then $L \setminus N_{2}(\alpha)$ is the disjoint union of a root and an edge.

In particular, the root $\beta$ is uniquely determined relative to $\alpha$. Note that in both cases a) and b) we have:

**Lemma 10.8.** $\text{rk}(\alpha) = \text{rk}(\beta)$.

We shall refer to a) as the rank $\frac{3}{2}$ case and to b) as the rank 2 case. They are represented on the left hand side and respectively the right hand side in the figure below.

In order to prove Theorem 10.1 we have to show that there exists precisely one group cobordism for each of these two cases. We consider the rank $\frac{3}{2}$ case (left) first.

By Lemma 10.4 the loops connect points in the 1-neighborhood of the roots, and there exists a unique edge at distance 2 from both roots (edge $(7, 12)$ in the figure). This edge must have its two extremities in a loop, say $\gamma_1$.

We have to show that:

**Lemma 10.9.** There exists a unique choice for the three remaining loops $\gamma_2, \gamma_3, \gamma_4$ in $X$, up to permutation of the symbols.

**Proof.** A triangle in the core of $X$ can either be glued on 2 or on 3 distinct loops. Since $\gamma_1$ connects two adjacent edges of $L$, it must contains a core triangle of each type. We denote $\gamma_2$ the loop connecting the edges of the first triangle face (say $f$) and $\gamma_3, \gamma_4$ the loops connecting the edges of the second triangle face (say $g$).

There are two solutions for $\gamma_2$, namely $(6, 11)$ or $(8, 13)$, which are symmetric.
The link support of \( g \) is therefore, respectively,
\[
[(12, 13), (7, 8), (4, 15)] \text{ or } [(11, 12), (6, 7), (4, 15)].
\]
We have to show that these two configurations exclude collars of type \( T \), but are compatible with collars of type \( S \). We will show that there is a unique way to extend the first configuration, and by symmetry, and unique way to extend the second, which are therefore isomorphic as group cobordisms. The unique extension is represented in Figure 3.

A direct computation shows that the collars of type \( S \) and \( T \) induce the following local geometry in \( X \):

![Figure 2. Local geometry induced by a collar of type \( S \) (left) and a collar of type \( T \) (right)](image)

In view of the local geometry, for both types of collars, the loop \( \gamma_3 \) is determined by the given configuration to have support (4, 13), which in turn, forces \( \gamma_4 \) (up to permutation of indices) to have support (8, 15).

Lemma 10.9 determines the faces \( a, b, c \) of the left collar. They are shown in Figure 3. Note that \( a \) and \( c \) are adjacent in the root \( \alpha \), which proves that the left hand side collar is of type \( S \). Furthermore, it turns out that it is indeed possible to extend this configuration by adding three more faces \( a', b', c' \) belonging to the right hand side collar. Again, \( a' \) and \( c' \) are adjacent, and the right hand side collar must be of type \( S \).

We summarize these results as follows.

It remains to show that:

**Lemma 10.10.** The first group cobordism of rank \( \frac{7}{4} \) is self-dual.
Proof. Assuming that the duality automorphism exist, it has to exchange the roots and in particular it can only send 3 to 5 or to 14. Since it must also preserve the loops, so in particular it must fix \{7, 12\} as a set. Since \(d(3, 7) = 2\) and \(d(3, 12) = 3\), we have either \(3 \to 5, 7 \to 7\) or \(3 \to 14, 7 \to 12\). The former case implies \(16 \to 14, 12 \to 12\), so 15 is fixed, it follows that 8 is also fixed, and so is the tripod at 7, which is a contradiction. Therefore, the partial permutation

\[(3, 2, 1, 16, 7) \leftrightarrow (14, 9, 10, 5, 12)\]

is determined, and particular, \(8 \leftrightarrow 13\) and \(6 \leftrightarrow 11\), so \(\gamma_1\) and \(\gamma_2\) are preserved (reversing the orientation), while \(\gamma_3\) and \(\gamma_4\) are permuted. It follows that the automorphism can be written as the following product of eight transpositions:

\[(3, 14)(2, 9)(1, 10)(16, 5)(7, 12)(8, 13)(6, 11)(4, 15)\]

Conversely, one checks that this indeed defines an automorphism of the given group cobordism (in the sense of Def. 8.2).

We shall now turn to the case \(\text{rk}(\alpha) = 2\).

By Lemma 10.5 \(L \setminus N_2(\alpha)\) is a segment of simplicial length 5 and, by Lemma 10.6 the same is true of \(L \setminus N_2(\beta)\). Let us call \(\beta\) and \(\bar{\beta}\) these two segments, respectively. The extremities of \(\bar{\beta}\) and at distance 1 from \(\alpha\) and 2 from \(\beta\), and similarly for that of \(\beta\). By Lemma 10.4 there must therefore exist a loop \(\gamma_1\) joining the extremities of \(\bar{\beta}\), and similarly a loop \(\gamma_2\) joining the extremities of \(\beta\).

Therefore, the two other loops must connects vertices pairwise in the set \{3, 6, 11, 14\}.

There is at most one possibility that is compatible with the model geometries in Figure 2.

Conversely, it is easy to show that this configuration extends to a model geometry of type \(\frac{3}{2}\), by completing the two remaining faces \(f\) and \(g\), as shown in Figure 4.

Finally note that, as in the rank \(\frac{3}{2}\) case:

Lemma 10.11. The second group cobordism of rank \(\frac{7}{4}\) is self-dual.

This concludes the proof of Theorem 10.1. □
11. Type constructibility: an example

The aim of the present section is, a) to construct a complex $X$ of rank $15/8$ (in the sense of [2]) with two vertices, whose links are respectively of rank $7/4$ (corresponding to the Moebius–Kantor graph) and of rank 2 (corresponding to the incidence graph of the Fano plane), as announced in §9, and, b) to illustrate the constructibility problem concluding the discussion of types in §2.

The Moebius–Kantor graph has 16 vertices and 24 edges, and the double cover $X$ constructed in §9 therefore has 16 faces (and Euler characteristic 2)

The incidence graph of the Fano plane $P_{2}F_{2}$, on the other hand, has 14 vertices and 21 edges. Both graphs are transitive and cubic, and there is no a priori obstruction—whether it be combinatorial or of homogeneity—to the existence of a complex $X$ as described in a) above. Such a complex, if it exists, must have 15 faces (and Euler characteristic 2).

It is unclear however that the lack of combinatorial and homogeneity obstructions ensures that a 2-complex of the given (strict) type actually exists. This is the theme of the type constructibility problem from Section 2 mentioned in b).

Our goal is to construct a complex of strict metric type $15/8$, which is defined by:

- the Moebius-Kantor graph and the incidence graph of the Fano plane $P_{2}F_{2}$, where all edges have length $\pi/3$
- one equilateral triangle

Consider the complex $X$ defined by the following 15 faces:

$X := \{(1,2,3), (1,8,13), (1,12,4), (2,13,10), (2,12,7), (3,7,6), (3,14,7),\$

$\{(4,4,5), (5,15,14), (5,14,15), (6,9,11), (6,11,9), (8,8,9), (10,13,15), (10,11,12)\}$

It is not hard to check that the corresponding group $G$ admits a presentation with 3 generators and 4 relations:

$G := \langle a, b, c \mid a^2 b^2 = b^2 a^2, cac^{-1} = b^3 a^{-2}, a^2 c^{-1} = ba^{-1} c^2 b^{-1}, bab^{-1} = ca^{-1} c^2 a^{-1} \rangle.$

Observe that $\mathbb{Z}^2 \to G$. Furthermore, we have:

$H_1(X, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.$
A direct computation shows that $X$ solves the constructibility problem for groups of type $\frac{15}{8}$.

**Proposition 11.1.** The 2-complex $X$ is of strict type $\frac{15}{8}$.

It is obvious that $X$ is not the double cover of a complex. As for the fake double cover in Section 9, one can describe the collar closure explicitly, which consists of 9 faces:

$$[[1, 2, 3], [1, 8, 13], [1, 12, 4], [2, 13, 10],
[2, 12, 7], [3, 7, 6], [3, 14, 7], [10, 13, 15], [10, 11, 12]].$$

The nerve of this collar, with 6 vertices and 9 edges, is represented in the following figure.

![Collar Closure Diagram]

12. **Type preserving constructions**

Theorem 10.1 suggests to construct groups using a surgery similar to the collar surgery in §9 by substituting cobordisms instead of collars. The point of these constructions is that the type of the given group (here type $\frac{7}{4}$) is preserved by the surgery.

Here is what a portion of such a group would look like:

$$\cdots \xrightarrow{\text{cobordism } X} \xrightarrow{\text{cobordism } Y} \cdots$$

where “$\leq$” and “$\geq$” respectively symbolize the first and the second cobordism from Theorem 10.1 and “−” refers to the collar of type $S$.

More precisely, the construction starts with two fixed nonisomorphic group cobordisms $X: C \to C$ and $Y: C \to C$, where $C$ is a collar of type $S$, and the resulting complex of rank $\frac{7}{4}$ corresponds to compositions in $\text{Bord}_{\frac{7}{4}}$ as prescribed by the given sequence. In the above example, the complex is $\cdots \circ Y \circ X \circ Y \circ X \circ Y \circ \cdots$

In addition, the 2-cover described in §7 provides filling group cobordisms of type $\frac{7}{4}$, whose collars are also isomorphic to the collars (of type $S$) appearing the cobordisms of Theorem 10.1.

Using these cobordisms as fillings, one can construct four families of groups of type $\frac{7}{4}$:

1. the segment groups,
2. the circle groups,
3. the N-groups,
4. the Z-groups,

which are parametrized, respectively, by

1. $\{\frac{3}{2}, 2\}^{(0,1,\ldots,n)}$, for $n \geq 0$,
2. $\{\frac{3}{2}, 2\}^{\mathbb{Z}/n\mathbb{Z}}$, for $n \geq 2$,
3. $\{\frac{3}{2}, 2\}^\mathbb{N}$,
4. $\{\frac{3}{2}, 2\}^\mathbb{Z}$.

The corresponding groups are finitely presented in case (1) and (2), and infinitely presented in case (3) and (4).
For example, the group $G_\omega$ in case (1) parametrized by $\omega = (\frac{3}{2}, 2, 2)$ in $(\frac{3}{2}, 2)^{\{0,1,2\}}$ is associated with the complex $X_\omega$ of rank $\frac{7}{4}$ defined by

$$\bullet - \frac{3}{2} - 2 - 2 - \bullet$$

One can write down explicit presentations for these groups, at least in cases (1) and (2). It is clear that:

**Proposition 12.1.** The groups $G_\omega$ for $\omega \in (\frac{3}{2}, 2)^{\{0,1,2\}}$, where $S = \{0, 1, \ldots, n\}$, $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{N}$, or $\mathbb{Z}$, admit a free $\frac{2}{3}$-transitive action on a $\text{CAT}(0)$ complex of rank $\frac{7}{4}$.

We now turn to the space $\Lambda_\frac{7}{4}$ of complexes of type $\frac{7}{4}$.

**Lemma 12.2.** Let $S$ refer to either $\{0, 1, \ldots, n\}$, $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{N}$, or to $\mathbb{Z}$, and $\Lambda_\frac{7}{4}$ denote the space of complexes of rank $\frac{7}{4}$. The map

$$\rho : (\frac{3}{2}, 2)^{\{0,1,2\}} \to \Lambda_\frac{7}{4}$$

$$\omega \mapsto (X_\omega, 0)$$

where $(X_\omega, 0)$ means that the complex $X_\omega$ pointed at the unique vertex in the cobordism over $0 \in S$, is

- injective if $S = \{0, 1, 2, \ldots, n\}$ or $S = \mathbb{N}$, and,
- at most 2-to-1 if $S = \mathbb{Z}/n\mathbb{Z}$ or $S = \mathbb{Z}$.

Note that $\rho$ is continuous, and has as image a compact subset of $\Lambda_\frac{7}{4}$.

**Proof.** Assume first that $S = \{0, 1, 2, \ldots, n\}$ or $S = \mathbb{N}$. Let $\omega, \omega'$ be two elements of $(\frac{3}{2}, 2)^{\{0,1,2\}}$. If $\varphi : (X_\omega, 0) \to (X_{\omega'}, 0)$ is an isomorphism, then $\varphi$ sends the cobordism over $t \in S$ in $X_\omega$ to the cobordism over $t$ in $X_{\omega'}$. Indeed this is so by definition for $0 \in S$, and the result follows by induction. Therefore, for every $s \in S$ we have $\omega(s) = \omega'(s)$, so $\omega = \omega'$.

If $S = \mathbb{Z}/n\mathbb{Z}$ or $S = \mathbb{Z}$, induction (namely, the 1-dimensional nature of $S$) shows that either $\omega(k) = \omega'(k)$ or $\omega(k) = \omega'(-k)$. Therefore the map $\rho$ is at most 2-to-1. \qed

If $A$ is a type, then the space $\Lambda_A$ (of pointed isomorphism classes) of complexes of type $A$ is a dynamical system. It is endowed with the so-called “base point dynamic” is defined by the equivalence relation $R_A \subset \Lambda_A \times \Lambda_A$, given by

$$(X, \ast) \sim_R (X', \ast') \iff \exists \varphi : X \xrightarrow{\sim} X'$$

isometry.

The study of Euclidean buildings (of rank 2) from the point of view of dynamical systems was begun in [11, 5, 12], and the above extends these considerations to more general types.

An open problem raised in these papers (see e.g. Question 7.2 in [12]) was whether the space $\Lambda_A$, in the case $A = \hat{A}_2$ of Euclidean triangle buildings, supports a diffuse invariant measure. The original motivation was that if $\Lambda_A$ indeed admits such a measure, then it provides a new source of probability measure preserving (pmp) equivalence relations with the property T of Kazhdan—one which involves no group in the constructions. Observe in particular that the leaves defining $\Lambda_A$ are pairwise non isomorphic by definition of the base point dynamics.
It is desirable to formulate the absence of such a group in a precise way, and this leads to [12 Question 7.3]. Is there—in case an invariant measure exists—a nontrivial equivalence subrelation of $R_A$ which is the orbit partition of an essentially free action of some discrete countable group? A hypothetical such $G$ would be a “structure group” for the complexes of type $A$, which allows to construct pairwise non isomorphic (quasi-periodic) such complexes (namely, Euclidean buildings) by perturbing the geometry “above $G$” using it as a blueprint.

The first example of a pmp equivalence relation that cannot be written as the orbit partition of an essentially free action of a countable group was obtained by Furman [7, Theorem D].

The surgery constructions in the previous sections shed light on these questions for groups of intermediate rank—of type $A_2^-$—instead of rank $2$—of type $A_2$.

**Theorem 12.3.** The space $\Lambda_{\frac{3}{2}}$ supports a diffuse invariant probability measure.

**Proof.** Consider the semi-direct product $G = \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ acting on the Cantor set $X := \left(\frac{3}{2}, 2\right)^\mathbb{Z}$ as follows

$$s \cdot (\omega_k)_{k \in \mathbb{Z}} = (\omega_{k+1})_{k \in \mathbb{Z}}.$$ 

Endow $X$ with the Bernoulli measure $\mu := \left(\frac{1}{2}\delta_{\frac{3}{2}} + \frac{1}{2}\delta_2\right)^\mathbb{Z}$. It is well–known and easy to prove that:

**Lemma 12.4.** The action $G \sim (X, \mu)$ is an essentially free pmp action.

**Proof.** The fact that $\mu$ is invariant is clear. Let us explain why the action is essentially free. Let $e \neq s \in G$. Then $s\omega = \omega$ if and only if $\omega$ is constant on the $s$ orbit in $\mathbb{Z}$. Since $s \neq e$, we can find infinitely many pairwise disjoint pairs of integers (which depend on $s$), say $(k_i, l_i)_{i \in \mathbb{N}}$, such that $\omega_{k_i} = \omega_{l_i}$ for every fixed point $\omega$ of $s$. This gives infinitely many constraints on the coordinates of the elements $\omega$ in the fixed point set $X^s$, and since $\mu$ is a product measure, it follows that $\mu(X^s) = 0$. □

**Remark 12.5.** The action $G \sim (X, \mu)$ is sometimes called a generalized Bernoulli shift, and is useful to give examples of operator algebras.

The map

$$\rho \left(\frac{3}{2}, 2\right)^\mathbb{Z} \to \Lambda_{\frac{3}{2}}$$

$$\omega \mapsto (X_\omega, 0)$$

is an orbit map, namely $x \sim_G y \iff \rho(x) \sim_{R_A} \rho(y)$.

Indeed, if $s \in G$ and $\omega' = s\omega$, then by construction $(X_{\omega'}, 0) = (X_\omega, s^{-1}(0))$, so $(X_{\omega'}, 0) \sim_{R_A} (X_\omega, 0)$. Conversely if $(X_\omega, 0) \sim_{R_A} (X_\omega', 0)$ then let $\varphi: X_\omega \to X_\omega'$ be an isometry. Since the geometry encodes the sequences $\omega$ and $\omega'$ in the corbodisms at vertices, in an orderly fashion, there exists a $k_0 \in \mathbb{Z}$ such that either $\omega(k) = \omega'(k_0 + k)$ or $\omega(k) = \omega'(k_0 - k)$ for all $k \in \mathbb{Z}$. In both cases we can find $s \in G$ such that $\omega' = s\omega$.

**Lemma 12.6.** The map $\rho$ is essentially 2-to-1.

**Proof.** By essentially 2-to-1 we mean that there is a measurable set $X' \subset X$ of full measure $\mu(X') = 1$ and such that $\rho|_{X'}$ is 2-to-1. By Lemma 12.2 $\rho$ is at most 2-to-1. If $\rho(\omega) = \rho(\omega')$ and $\omega \neq \omega'$ then $\omega(k) = \omega'(-k)$ for all $k \in \mathbb{Z}$, that is, $\omega' = s\omega$ where $s \in G$ is the symmetry around 0. Therefore the restriction of $\rho$ to $X' := X \setminus X^s$ is a 2-to-1 map. By Lemma 12.4 $\mu(X^s) = 0$. □
Lemma 12.7. The measure $\rho_*(\mu)$ is invariant (and diffuse).

This lemma will be a consequence of Lemma 12.6 and Lemma 12.9 below, and it concludes the proof of Theorem 12.3. □

This suggests the following definition.

Definition 12.8. Let $A$ be a type. A structure group for $A$ is a countable group $G$ for which there exists an integer $n \geq 1$, an essentially free pmp action $G \curvearrowright X$ on standard probability space $X$, and an essentially $n$-to-1 Borel map $\rho : X \to \Lambda_A$

such that $x \sim_G y \iff \rho(x) \sim_R \rho(y)$ for every $x, y \in X$.

It may be useful to also consider the case of nonsingular actions (instead of pmp). The map $\rho$ allows to push-forward (diffuse, invariant) measures on $X$ to (diffuse, invariant) measures to the space $\Lambda_A$:

Lemma 12.9. Let $\rho : X \to Y$ be a Borel map between standard probability spaces. Assume that

a) $R \subset X \times X$ and $S \subset Y \times Y$ are standard Borel equivalence relations with countable classes

b) $\rho$ is an orbit map, in the sense that $x \sim y \iff \rho(x) \sim \rho(y)$ for all $x, y \in X$.

c) $\mu$ is an $R$-invariant probability measure on $X$

d) $n \geq 1$ is an integer such that $\rho$ is essentially $n$-to-1 with respect to $\mu$ (there exists a measurable set $X' \subset X$ with $\mu(X') = 1$ such that $\rho|_{X'}$ is $n$-to-1

Then the measure $\rho_*(\mu)$ is an $S$-invariant measure on $Y$.

This lemma does not seem to belong to the literature on orbit equivalence (see e.g. [11] or [10]), so we give a proof.

Proof. Consider the map

$\pi := \rho \times \rho : X \times X \to Y \times Y$

and its restriction $\pi : R \to S$. It is clear that

$\pi \circ \sigma = \sigma \circ \pi$

where $\sigma$ denotes the flip map $(x, y) \to (y, x)$ on both $R$ and $S$. The measure $\mu$ is invariant if and only if

$\sigma_* \nu = \nu$

where

$\nu(A) := \int_X |A \cap \{x\} \times X| d\mu(x)$

is the right-counting measure on $R$ (see [6] Theorem 2).

Since

$\sigma_*(\pi_* \nu) = (\sigma \circ \pi)_* \nu = (\pi \circ \sigma)_* \nu = \pi_*(\sigma_* \nu) = \pi_* \nu$,
the lemma follows from the fact that $\pi_*\nu$ is a multiple of the right-counting measure $\nu_\rho$ on $S$ associated with $\rho_*\mu$:

\[
\pi_*\nu(A) = \int_X |\pi^{-1}(A) \cap \{x\} \times X|d\mu(x) \\
= \int_X |\{(x, y) \in R, (\rho(x), \rho(y)) \in A\}|d\mu(x) \\
= \int_X |\{y \in X, (\rho(x), \rho(y)) \in A\}|d\mu(x) \\
= \int_X |\{y \in X, (z, \rho(y)) \in A\}|d\rho_*\mu(z) \\
= n \int_Y |\{t \in Y, (z, t) \in A\}|d\rho_*\mu(z) \\
= n\nu_\rho(A)
\]

for every Borel set $A \subset S$. □

**Remark 12.10.** There is a pointwise (less functorial) proof of Lemma 12.9. Let $\psi \subset S \times S$ be a partial automorphism of $S$. Then $\pi^{-1}(\psi)\subset R$ whose (left and right) fibers over $X$ are either empty, or they have cardinality $n$. Using the Lusin selection theorem (for Borel maps with countable fibers), one can find partitions:

\[
\rho^{-1}(\text{dom } \psi) = \bigsqcup_{i=1}^n A_i, \quad \rho^{-1}(\text{ran } \psi) = \bigsqcup_{j=1}^n B_j,
\]

and partial isomorphisms $\varphi_{i,j}: A_i \to B_j$ of $R$ such that

\[
\pi^{-1}(\psi) = \bigsqcup_{i,j=1}^n \varphi_{i,j}.
\]

Since $\mu$ is invariant, we have $\mu(A_i) = \mu(B_j)$ for all $i, j = 1, \ldots, n$, and

\[
\rho_*\mu(\text{dom } \psi) = n\mu(A_1) = n\mu(B_1) = \rho_*\mu(\text{ran } \psi)
\]

which shows that $\rho_*\mu$ is $S$-invariant.

Theorem 12.3 is, therefore, a consequence of:

**Theorem 12.11.** The dihedral group $D_\infty$ is a structure group for the type $A_\infty^4$.

This theorem shows that not only surgery for the type $A_\infty^4$ is possible, but that it can be done with great freedom. The questions raised in [12] for the type $\tilde{A}_2$ remain open.

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