ON THE LOCAL VERSION OF THE SEVERI PROBLEM

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Abstract. For a given singularity of a plane curve we consider the locus of nodal deformations of the singularity with the given number of nodes and describe possible components of the locus. As applications, we solve the local symplectic isotopy for nodal curves in a neighborhood of a given pseudoholomorphic curve without multiple components and prove the uniqueness of the symplectic isotopy class for nodal pseudoholomorphic curves of low genus in \( \mathbb{C}P^2 \) and \( \mathbb{C}P^2 \# \mathbb{C}P^2 \).

0. Introduction

In the famous Anhang F of his book “Vorlesungen über algebraische Geometrie” [Sev], F. Severi offered a proof of the statement that the locus of irreducible plane curves of degree \( d \) having the prescribed number \( \nu \) nodes and no other singularities is connected. However, his argument, which involved degenerating the curve into \( d \) lines, is not correct. The problem was attacked by several authors, see review of Fulton [Ful], and the correct proof was given by Harris [Ha], following original ideas of Severi.

In this paper we consider the local version of the Severi problem. Let \( C^* \) be a germ of a holomorphic plane curve at the origin \( 0 \in \mathbb{C}^2 \) such that \( C^* \) has an isolated singularity at 0. In particular, \( C^* \) can be reducible but has no multiple components. In this case there exists a versal family \( \{C_s\}_{s \in \text{Def}(C^*,0)} \) of deformations of \( C^* \) with a non-singular finite-dimensional base \( \text{Def}(C^*,0) \). Here, as the curve \( C^* \) itself, the family \( \{C_s\}_{s \in \text{Def}(C^*,0)} \) and the ambient plane \( \mathbb{C}^2 \) are understood in the sense of germs of analytic spaces.

Fix an integer \( \nu \). Denote by \( \text{Def}_{\nu}(C^*,0) \) the locus of the curves in \( \text{Def}(C^*,0) \) having exactly \( \nu \) nodes and no other singularities, and by \( \text{Def}_{\nu}(C^*,0) \) its closure. One can show that \( \text{Def}_{\nu}(C^*,0) \) consists of deformations of \( C^* \) whose total virtual number of nodes is at least \( \nu \). Further, let \( \delta = \delta(C^*,0) \) be the virtual number of nodes of \( C^* \) at 0. It is easy to show that \( \text{Def}_0(C^*,0) = \text{Def}(C^*,0) \), \( \text{Def}_\nu(C^*,0) \) has pure codimension \( \nu \) in \( \text{Def}(C^*,0) \), is empty for \( \nu > \delta \), while non-empty and irreducible for \( \nu = \delta \).

The subject of the local Severi problem is description of irreducible components of \( \text{Def}_\nu(C^*,0) \) in the remaining case \( 0 < \nu < \delta \). The principle result of the present paper is

Main Theorem. For \( \nu < \delta \), every irreducible component of \( \text{Def}_\nu(C^*,0) \) contains a nodal curve with \( \delta \) nodes.

More precisely, we show inductively that every irreducible component of \( \text{Def}_\nu(C^*,0) \) contains a component of \( \text{Def}_{\nu+1}(C^*,0) \). In other words, every nodal curve in \( \text{Def}_{\nu}(C^*,0) \) can be degenerated inside the same component of \( \text{Def}_\nu(C^*,0) \) into a nodal curve with exactly one additional node.

The meaning of Main Theorem is that there are no “unexpected” components of \( \text{Def}_\nu(C^*,0) \), different from “expected” ones obtained by the following construction. First...
one deforms $C^*$ into a nodal curve $C'$ with $\delta$ nodes, this is a generic curve in the family \( \text{Def}_\delta(C^*, 0) \), and then smooths $\delta - \nu$ nodes of $C'$. In particular, Main Theorem implies that there exist not more than \( \binom{\delta}{\nu} \) irreducible components of \( \text{Def}_\nu(C^*, 0) \). Of course, this bound is very rough. However, a precise description of components of \( \text{Def}_\nu(C^*, 0) \) requires a description of the action of the monodromy group of \( \text{Def}_\delta(C^*, 0) \) on the set of nodes of the curve $C'$.

Author’s motivation for study of the local Severi problem was applications to the symplectic isotopy problem. It was pointed out in the paper \[SH\] that Main Theorem would imply the solution of the local isotopy problem for nodal pseudoholomorphic curves, which is a version of the local Severi problem for pseudoholomorphic curves. This result and its application to the symplectic isotopy problem are presented in Section 2.

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0.1. Scheme of the proof. The main idea of the proof is to trace the ramification locus of the projections of deformed curves onto a fixed coordinate axis $Oz \subset \mathbb{C}^2$, $Oz \cong \mathbb{C}$. This leads to another deformation problem. In this case too, there exists a semi-universal family \( \{ (C_s, f_s) \} \) of pairs “curve + projection” with a non-singular finite-dimensional base \( \text{Def}(C^*/\Delta) \). Let \( \text{Def}_\nu(C^*/\Delta) \) be the preimage of \( \text{Def}_\nu(C^*, 0) \) with respect to the natural “forgetful map” \( \text{Def}(C^*/\Delta) \to \text{Def}(C^*, 0) \). For a generic $s \in \text{Def}_\nu(C^*/\Delta)$, denote by $B_\nu(s)$ the branching divisor of the projection $f_s : C_s \to \Delta$. Then $B_\nu(s)$ depends holomorphically on a generic $s$, and the family \( \{ B_\nu(s) \} \) can be holomorphically extended to the whole germ \( \text{Def}_\nu(C^*/\Delta) \).

The idea of the proof can be now reformulated in study of the loci \( \mathcal{V}_k \subset \text{Def}_\nu(C^*/\Delta) \) given by the condition “the multiplicity of $0 \in \Delta$ in $B_\nu(s)$ is at least $k$”. This means that we study specializations of projections $f_s : C_s \to \Delta$, proceeding successively along the special strata of the discriminant locus in the space of branching divisors $B_\nu(s)$. We show inductively that at each step of the specialization $k \mapsto k + 1$, for generic $s \in \mathcal{V}_{k+1}$, the curve $C_s$ has simple branchings and nodes lying apart the vertical axis $Ow$, and there is the following alternative for the structure of $C_s$ at the axis $Ow$:

- either $C_s$ is non-singular at $Ow$, and in this case we can proceed to the next inductive step;
- or $C_s$ has exactly one singular points at the the axis $Ow$, at which $C_s$ has two branches, both non-singular.

We show that in the latter case one can produce exactly one desired node. Since the total tangency order can not exceed the degree $d$, the latter case must occur and the inductive procedure will terminate.

Observe that essentially the same construction was used in the Harris’ proof \[Ha\] of the global Severi problem. Namely, he studied the varieties $V_{d,g,k}$ of irreducible nodal curves of genus $g$ and degree $d$ in $\mathbb{CP}^2$ having tangency of order $k$ with some (not fixed) line $\ell$ at some point $p$. He showed inductively in $k$, that if $V_{d,g,k}$ is non-empty, then a generic curve in every irreducible component of $V_{d,g,k}$ has no other incidences and admits a degeneration either into a generic curve in $V_{d,g,k+1}$, or into a curve having exactly one extra node. To
adapt his point of view to our case, we must simply rotate $\ell$ into the axis $Ow$ and consider the projection $f : C \to \Delta$ from the infinity point of $Ow$. Note that his condition on the projection $f : C \to \Delta$ is stronger than our, he requires also a single ramification point of $f : C \to \Delta$ over $0 \in \Delta$.

1. Deformation of isolated singularities of plane curves

1.1. Isolated singularities of plane curves. We recall the standard definitions of the deformation theory, see also [Pal-1], [Pal-2], or [Tju].

**Definition 1.1.** An isolated singularity of a plane curve is a germ of a curve $(C,0)$ in $\mathbb{C}^2$ at the origin $0$, such that $C$ is non-singular at any $z \neq 0 \in C$. In particular, this means that there are no multiple components of $C$ at $0$.

A deformation of such an isolated singularity $(C,0)$ is given by an analytic map $\pi_S : \mathcal{C}_S \to S$ between germs of analytic sets $(\mathcal{C}_S,0)$ and $(S,s_0)$ such that $\pi_S$ is flat and the fiber $\pi_S^{-1}(s_0)$ is the germ $(C,0)$. The germs $(S,s_0)$ and $(\mathcal{C}_S,s_0)$ are the base and the total germ of the deformation, respectively. Such a deformation $\pi_S : \mathcal{C}_S \to S$ is also called a family of deformations of $(C,0)$.

Two deformations $(\mathcal{C}_S,0)$ and $(\mathcal{C}'_S,0)$ of $(C,0)$ with the same base $(S,s_0)$ are isomorphic, if there exists a germ biholomorphism $\varphi : (\mathcal{C}_S,0) \to (\mathcal{C}'_S,0)$ compatible with projections $\pi_S : \mathcal{C}_S \to S$ and $\pi'_S : \mathcal{C}'_S \to S$, respectively. The notion of as isomorphism of isolated singularities of plane curves is defined similarly.

If $\pi_S : \mathcal{C}_S \to S$ is a family of deformations of $(C,0)$, $(T,t_0)$ a germ of an analytic set, and $\varphi : (T,t_0) \to (S,s_0)$ an analytic map, then $\mathcal{C}_T := \varphi^*\mathcal{C}_S := \mathcal{C}_S \times_S T$ is also a deformation of $(C,0)$ with respect to the natural projection $\pi_T : \mathcal{C}_T \to T$. In this case $\pi_T : \mathcal{C}_T \to T$ is called the pulled-back family or a deformation obtained by the base change, and $\varphi : (T,t_0) \to (S,s_0)$ is called the base change map.

By G. Tjurina [Tju] (see also [Don], [Pal-1] and [Pal-2]), there exists a semi-universal family of deformations of any given isolated singularity of an analytic space. In our case we have

**Proposition 1.1.** Let $(C,0)$ be an isolated singularity of a plane curve. Then there exists a family $\pi_S : \mathcal{C}_S \to S$ of deformations of $(C,0)$ with the following properties:

i) Any deformation family $\pi_T : \mathcal{C}_T \to T$ of $(C,0)$ is isomorphic to the pulled-back family $\varphi^*\mathcal{C}_S \to T$ for an appropriate base change map $\varphi : (T,t_0) \to (S,s_0)$.  

References
ii) Any morphism \( \varphi : (S,s_0) \to (S,s_0) \), such that the pulled-back family \( \varphi^* \mathcal{C}_S \to S \) is isomorphic to \( \pi_S : \mathcal{C}_S \to S \), is an isomorphism.

Furthermore, assume that \( C^* \) is the zero divisor of the germ of a holomorphic function \( f(z,w) \) at \( 0 \in \mathbb{C}^2 \). Let \( \mathcal{F}^1(C^*,0) := \mathcal{O}_{\mathbb{C}^2,0}/(f, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial w}) \). Then \( \text{supp}(\mathcal{F}^1(C^*,0)) = \{0\} \) and \( S \) is smooth \( n \)-dimensional with \( n = \text{length}(\mathcal{F}^1(C^*,0)) \).

Moreover, let the germs \( \varphi_1(z,w), \ldots, \varphi_n(z,w) \) of holomorphic functions generate the basis of \( \mathcal{F}^1(C^*,0) \) over \( \mathbb{C} \). Set \( \Phi(z,w;s_1, \ldots, s_n) := f(z,w) + \sum_i s_i \varphi_i(z,w) \), \( S := (\mathbb{C}^n,0) \), and let \( \mathcal{C}_S \) be the germ at \( 0 \) of the zero divisor of \( \Phi \), equipped with the projection \( \pi_S : \mathcal{C}_S \to S \) given by \((z,w;s_1, \ldots, s_n) \mapsto (s_1, \ldots, s_n)\). Then map \( \pi_S : \mathcal{C}_S \to S \) is a deformation family of \( (C^*,0) \) with the desired properties.

The properties i) and ii) are completeness and minimality of the family \( \pi_S : \mathcal{C}_S \to S \), respectively. Notice also that even if we deform \( (C^*,0) \) as an abstract complex space, the whole deformation consists of plane curves.

1.2. Deformation of plane curves with projection. Now we give an explicit description of deformation of plane curves. Instead of germs, we shall work with closed analytic subsets in the bi-disc, a priori with multiplicities.

The following notations are used. \( \Delta^2 \) denotes the bi-disc with the standard complex structure and complex coordinates \((z,w), \text{pr}_1 : \Delta^2 \to \Delta \) is the projection on the first factor. For a complex (i.e. holomorphic) manifold \( X \), compact and with a piecewise smooth boundary \( \partial X \), we denote by \( \mathcal{H}(X) \) the space of holomorphic function which are continuous up boundary \( \partial X \). Similar notation \( \mathcal{H}(C) \) is used in the case when \( C \) is a nodal complex curve. Further, we denote by \( \mathcal{H}(C,X) \) the space of holomorphic maps which are continuous up boundary \( \partial C \) and have image in the interior of \( X \).

Let us start with some standard facts about holomorphic curves in bi-disc.

Lemma 1.2. i) Let \( C \) be a holomorphic curve in \( \Delta^2 \), possibly with multiple components. Assume that the projection \( \text{pr}_1 : C \to \Delta \) on the first factor \( \Delta \) is proper. Then \( C \) is the zero divisor of the uniquely defined unitary Weierstraß polynomial \( P_f(z,w) := w^d + \sum_{i=1}^d f_i(z)w^{d-i} \) whose coefficients \( f_1(z), \ldots, f_d(z) \) are bounded holomorphic functions, \( f_1(z), \ldots, f_d(z) \in \mathcal{O}(\Delta) \).

ii) Let \( S \) be a (Banach) analytic set and \( F(z,w;s) \) a holomorphic function on \( \Delta^2 \times S \), such that for every \( s \in S \) the projection \( \text{pr}_1 \) from zero divisor \( C_s \) of \( F(z,w;s) \) onto the first factor \( \Delta \) is proper. Then \( F(z,w;s) \) can be uniquely decomposed into the product \( F(z,w;s) = G(z,w;s) \cdot P(z,w;s) \) where \( G(z,w;s) \) is a holomorphic invertible function on \( \Delta^2 \times S \) and \( P(z,w;s) \) is a Weierstraß polynomial of the form \( P(z,w;s) = w^d + \sum_{i=1}^d f_i(z;s)w^{d-i} \) whose coefficients \( f_1(z;s), \ldots, f_d(z;s) \) are bounded holomorphic functions on \( \Delta \times S \).

The result is classical and follows essentially from the Weierstraß theorems, see e.g. [Gr-Ha], Chapter 0. The coefficients \( f_i(z) \in \mathcal{O}(\Delta) \) of the Weierstraß polynomial are used as natural coordinates on the space of curves in \( \Delta^2 \) with the cycle topology, when a curve is considered as a divisor.

Definition 1.2. Denote by \( \mathcal{P}^d(\Delta^2) \) the space of \( f = f(z) = (f_1(z), \ldots, f_d(z)) \in (\mathcal{H}(\Delta))^d \), for which the zero divisor \( C_f \) of the Weierstraß polynomial \( P_f(z,w) := w^d + \sum_{i=1}^d f_i(z)w^{d-i} \) lies in \( \Delta \times \Delta(r) \) for some \( r = r(f) < 1 \) and has no singularities at the boundary. This is a Banach manifold parameterizing curves \( C \) in \( \Delta^2 \) for which the projection \( \text{pr}_1 : C \to \Delta \)
is proper and has degree $d$. The curve corresponding to $f \in \mathcal{L}^d(\Delta^2)$ will be denoted by $C_f$. We shall identify $C_f$ with $f$ and write $C_f \in \mathcal{L}^d(\Delta^2)$.

The space $\mathcal{L}^d(\Delta^2)$ is too large to work with. We shall replace it by a finite dimensional moduli space of pairs “curve + projection”. The latter is defined by dividing out holomorphic “slidings” along vertical fibers in $\Delta^2$.

**Definition 1.3.** Let $C^* \in \mathcal{L}^d(\Delta^2)$ be a holomorphic curve which has no multiple components and singularities on the boundary, $P_0(z,w)$ its Weierstraß polynomial, and $F_0(z,w) \in \mathcal{H}(\Delta^2)$ a holomorphic function of the form $F_0(z,w) = G_0(z,w) \cdot P_0(z,w)$ with a non-vanishing $G_0(z,w) \in \mathcal{H}(\Delta^2)$. Define the sheaf $\mathcal{T}^1(C^*/\Delta) := \mathcal{O}(\Delta^2) / \langle F_0, \frac{\partial}{\partial w} F_0 \rangle$, where $(F_0, \frac{\partial}{\partial w} F_0)$ states for the ideal generated by $F_0$ and its derivative.

The set of the singular points of $C^*$ and the set of critical points of the projection $\text{pr}_1 : C^* \to \Delta$ is called the **singular set of** $\text{pr} : C^* \to \Delta$ or and denoted by $\text{sing}(C^*/\Delta)$.

**Lemma 1.3.** i) The support of the sheaf $\mathcal{T}^1(C^*/\Delta)$ is the set $\text{sing}(C^*/\Delta)$.

ii) Let $\varphi_1(z,w), \ldots, \varphi_n(z,w) \in \mathcal{H}(\Delta^2)$ be functions generating a basis of $\mathcal{T}^1(C^*/\Delta)$ over $\mathbb{C}$. Then every $F(z,w) \in \mathcal{H}(\Delta^2)$ sufficiently close to $F_0$ can be uniquely represented in the form

$$F(z,w) = G(z,w) \cdot (P_0(z,w + g(z,w)) + \sum_{i=1}^n s_i \varphi_i(z,w))$$

with a holomorphic function $G(z,w) \in \mathcal{H}(\Delta^2)$, constants $(s_1, \ldots, s_n) \in \mathbb{C}^n$, and a Weierstraß polynomial $g(z,w) = \sum_{i=0}^d w^i b_i(z)$ of degree $d$ with holomorphic coefficients $b_i(z) \in \mathcal{H}(\Delta)$.

**Proof.** i) First, let us observe that the definition of $\mathcal{T}^1(C^*/\Delta)$ is independent of the particular choice of the function defining $C^*$. In particular, $\mathcal{T}^1(C^*/\Delta) = \mathcal{O}(\Delta^2) / \langle P_0, \frac{\partial}{\partial w} P_0 \rangle$. Further, it is clear that $\mathcal{T}^1(C^*/\Delta)$ vanishes outside $C^*$ and at regular points of $C^*$ which are not critical points of the projection $\text{pr}_1 : C^* \to \Delta$. Since $F_0$ must vanish at least quadratically at every singular point of $C^*$, $\frac{\partial}{\partial w} F_0$ must also vanish at singular points of $C^*$. So $\mathcal{T}^1(C^*/\Delta)$ is non-trivial at such points. Finally, observe that the vertical vector field $\frac{\partial}{\partial w}$ is tangent to $C^*$ at critical points of the projection $\text{pr}_1 : C^* \to \Delta$, and hence $\frac{\partial}{\partial w} F_0$ vanish at such points. This yields the first assertion of the lemma.

ii) This assertion will follow from the implicit function theorem provided we solve the corresponding linearized problem. Differentiating (1.1) we obtain the equation

$$\hat{F}(z,w) = \hat{G}(z,w) \cdot P_0(z,w) + G_0(z,w) \cdot (\frac{\partial}{\partial w} P_0(z,w) \cdot \hat{g}(z,w) + \sum_{i=1}^n \hat{s}_i \varphi_i(z,w))$$

where dotted symbols state for tangent vectors to the corresponding spaces. The latter equation is equivalent to

$$\hat{F} = \hat{G} \cdot P_0 + \frac{\partial}{\partial w} P_0 \cdot \hat{g} + \sum_{i=1}^n \hat{s}_i \varphi_i,$$

where dotted objects vary in the same Banach spaces as above. Application of the Weierstraß' division theorem shows that it is sufficient to consider the special case where $\hat{F}$ is a Weierstraß polynomial of degree $d - 1$ of the form $\sum_{i=0}^{d-1} w^i a_i(z)$ with holomorphic coefficients $a_i(z) \in \mathcal{H}(\Delta)$. Another application of the Weierstraß' division theorem shows that after replacing the functions $\varphi_i(z,w)$ by its remainders $\hat{\varphi}_i(z,w)$ after the division on $P_0(z,w)$ we obtain an equivalent problem. Observe also that the remainders $\hat{\varphi}_i(z,w)$
are also Weierstraß polynomials of degree $d - 1$. It follows that in a solution of the new problem
\begin{equation}
\hat{F} = \hat{G} \cdot P_0 + \frac{\partial}{\partial w} P_0 \cdot \hat{g} + \sum_{i=1}^{n} \hat{s}_i \hat{\varphi}_i
\end{equation}
the function $\hat{G}$ must be also a Weierstraß polynomial of degree $d - 1$.

Now consider \ref{1.4} as a system of linear equations on the coefficients of Weierstraß polynomial $\hat{G}$ and $\hat{g}$ so that $\hat{F} - \sum_{i=1}^{n} \hat{s}_i \hat{\varphi}_i$ is the inhomogeneous part. Then the matrix of coefficients of the linear system is the Sylvester matrix of the polynomials $P_0$ and $\frac{\partial}{\partial w} P_0$, so that its determinant is the resultant of the polynomials $P_0$ and $\frac{\partial}{\partial w} P_0$, i.e. the discriminant of $P_0$ with respect to the variable $w$. Let us denote this discriminant by $D(z)$. Then $D(z) \in \mathcal{H}(\Delta)$ and the zero set of $D(z)$ is exactly the projection of the support of $\mathcal{T}^1(C^*/\Delta)$. Since $D(z)$ is not vanishing identically, it follows the uniqueness of the solution of \ref{1.4} with given $\hat{F}$. By the hypotheses of the lemma, for a given $\hat{F}$ there exists a unique collection of parameters $(\hat{s}_1, \ldots, \hat{s}_n)$ such that $\hat{F} - \sum_{i=1}^{n} \hat{s}_i \hat{\varphi}_i$ lies in the ideal generated by $P_0$ and $\frac{\partial}{\partial w} P_0$. It follows then the solvability of the linear problem \ref{1.2}.

\noindent \textbf{Corollary 1.4.} \noindent i) The length $n$ of the sheaf $\mathcal{T}^1(C^*/\Delta)$ equals to the total vanishing order of the discriminant of the Weierstraß polynomial of $C^*$.

ii) The length of the sheaf $\mathcal{T}^1(C^*/\Delta)$ is constant under small deformations of $C^*$.

\noindent \textbf{Proof.} i) We maintain the notation used in the proof of \textbf{Lemma 1.3}. Let us apply the elementary ideals theory to the Sylvester matrix of $P_0$ and $\frac{\partial}{\partial w} P_0$. Since every ideal of $\mathcal{H}(\Delta)$ containing $D(z)$ is principle, we can bring the Sylvester matrix in the diagonal form, so that the product of the diagonal elements is $D(z)$. Now it is clear that the minimal number of the correction terms $\hat{s}_i \hat{\varphi}_i$, needed to solve \ref{1.4} with given $\hat{F}$ is the sum of total vanishing orders of the obtained diagonal elements.

The second assertion follows from the first one.

\noindent \textbf{Corollary 1.5.} \noindent i) Every curve $C^* \in \mathcal{X}^d(\Delta^2)$ is isomorphic to a curve $C \in \mathcal{X}^d(\Delta^2)$ defined by a polynomial.

ii) The deformation space $\text{Def}(C^*/\Delta)$ has natural algebraic structure.

\noindent \textbf{Proof.} i) By \textbf{Lemma 1.3}, it is sufficient to approximate the Weierstraß polynomial $P_0$ of $C^*$ by a polynomial $\hat{P}$ lying in the ideal generated by $P_0$ and $\frac{\partial}{\partial w} P_0$.

ii) By \textbf{Part i}), we may assume that $C^*$ is algebraic, i.e. the Weierstraß polynomial $P_0(z, w)$ of $C^*$ is a polynomial in the usual sense. Let $\varphi_1(z, w), \ldots, \varphi_n(z, w)$ be polynomials inducing a basis of $\mathcal{T}^1(C^*/\Delta) = \mathcal{O}(\Delta^2)/(\hat{P}_0, \frac{\partial}{\partial w} P_0)$ and $F(z, w; t)$ a polynomial in variables $z, w$, and $t = (t_1, \ldots, t_k)$, such that $F(z, w; 0)$ is a defining polynomial for $C^*$. We assert that the functions $G(z, w)$ and $g(z, w)$ solving the equation \ref{1.1} with r.h.s. $F(z, w; t)$ are polynomials in variables $z$ and $w$, and that the dependence of the parameters $s = (s_1, \ldots, s_n)$ and coefficients of $G(z, w)$ and $g(z, w)$ on $t = (t_1, \ldots, t_k)$ is algebraic. The first assertion means that the degree of $G(z, w)$ and $g(z, w)$ with respect to variables $z$ and $w$ is bounded uniformly in $t$. This fact follows from the linearization of \ref{1.1} given by \ref{1.2}. The second assertion is simply reformulation of the fact that \ref{1.1} is a system of algebraic equations on coefficients. The corollary follows.

\noindent \textbf{Definition 1.4.} Let $C^* \in \mathcal{X}^d(\Delta^2)$ be a curve defined by a polynomial $P_0(z, w)$. Fix polynomials $\varphi_1(z, w), \ldots, \varphi_n(z, w)$ generating a basis of $\mathcal{T}^1(C^*/\Delta)$. Define $\text{Def}(C^*/\Delta)$
as the germ of \( s = (s_1, \ldots, s_n) \in \mathbb{C}^n \) at \( s = 0 \), \( \mathcal{C} = \mathcal{C}(\mathbb{C}^*/\Delta) \) as the divisor of \( P(z,w; s) := P_0(z,w) + \sum_i s_i \varphi_i(z,w) \), \( \mathcal{C}_s \subset \Delta^2 \) as the fiber over \( s \) of the projection \( \pi_{\text{Def}} : \mathcal{C} \to \text{Def}(\mathbb{C}^*/\Delta) \), and \( \text{pr}_1 : \mathcal{C} \to \Delta \) as the projection on the \( z \)-disc.

It follows from Lemma 1.3 that \( \text{Def}(\mathbb{C}^*/\Delta) \overset{\text{Def}}{\longrightarrow} \mathcal{C} \overset{\text{pr}_1}{\longrightarrow} \Delta \) is a universal deformation family of the curve \( C^* \) equipped with the proper projection onto \( \Delta \). In particular, for another choice of \( \varphi_1(z,w), \ldots, \varphi_n(z,w) \) we obtain an isomorphic family. As usually, we identify the germ \( \text{Def}(\mathbb{C}^*/\Delta) \) with a small neighborhood of \( s = 0 \) in \( \mathbb{C}^n \) representing it.

**Lemma 1.6.** Let \( C^* \in \mathcal{P}^d(\Delta^2) \) be a curve and \( \{p_1, \ldots, p_l\} = \text{sing}(\mathbb{C}^*/\Delta) \) the set of singular points of \( \text{pr}_1 : C^* \to \Delta \). Denote by \( C_j^* \) the germ of \( C^* \) at \( p_j \). Then there exist a natural isomorphism \( \psi : \prod_j \text{Def}(C_j^*/\Delta) \cong \text{Def}(\mathbb{C}^*/\Delta) \) and a natural imbedding \( \prod_j \mathcal{C}(C_j^*/\Delta) \hookrightarrow \mathcal{C}(\mathbb{C}^*/\Delta) \) compatible with the isomorphism \( \psi \) and the projections \( \pi_{\text{Def}} : \mathcal{C}(\mathbb{C}^*/\Delta) \to \text{Def}(\mathbb{C}^*/\Delta) \).

**Proof.** Let \( P(z,w; s) = P_0(z,w) + \sum_i s_i \varphi_i(z,w) \) be the polynomial defining a family realizing \( \pi_{\text{Def}} : \mathcal{C}(\mathbb{C}^*/\Delta) \to \text{Def}(\mathbb{C}^*/\Delta) \). Choose disjoint neighborhoods \( U_j \) of \( p_j \) which are small bi-discs with sides parallel to \( \Delta^2 \), such that \( C \cap U_j \) lie in \( \mathcal{P}^d(U_j) \) for the corresponding degree \( d_j \). Counting parameters, we conclude that the restrictions of deformation family \( \mathcal{C}(\mathbb{C}^*/\Delta) \) to \( U_j \) induce the desired isomorphism. \( \square \)

Now let us describe deformation families \( \text{Def}(\mathbb{C}^*/\Delta) \) of lower dimension.

**Lemma 1.7.** Let \( C^* \in \mathcal{P}^d(\Delta^2) \) be a curve, and let \( n \) be the dimension of \( \text{Def}(\mathbb{C}^*/\Delta) \).

i) If \( n = 0 \), then \( C^* \) consists of \( d \) disjoint discs and the projection \( \text{pr}_1 : C^* \to \Delta \) is a trivial \( d \)-sheeted covering.

ii) If \( n = 1 \), then \( C^* \) consists of \( d-1 \) disjoint discs and the projection \( \text{pr}_1 : C^* \to \Delta \) is a trivial covering on \( d-2 \) of the discs, and a \( 2 \)-sheeted covering with one simple branching on the remaining disc.

iii) If \( n = 2 \), then the following cases are possible.

(a-b) \( C^* \) consists of \( d-2 \) disjoint discs, the projection \( \text{pr}_1 : C^* \to \Delta \) is a trivial covering on \( d-4 \) of the discs, and a \( 2 \)-sheeted covering with one simple ramification on each of the remaining \( 2 \) discs. The ramification points can be projected onto \( 2 \) distinct points on \( \Delta \) (case (a)) or onto a single point (case (b)).

(c) \( C^* \) consists of \( d-2 \) disjoint discs, the projection \( \text{pr}_1 : C^* \to \Delta \) is a trivial covering on \( d-3 \) of the discs, and a \( 3 \)-sheeted covering with \( 2 \) simple branching on the remaining disc.

(d) \( C^* \) consists of \( d-2 \) disjoint discs, the projection \( \text{pr}_1 : C^* \to \Delta \) is a trivial covering on \( d-3 \) of the discs, and a \( 3 \)-sheeted covering with one simple branching on the remaining disc.

(e) \( C^* \) consists of an annulus and \( d-2 \) discs, the components are disjoint, the projection \( \text{pr}_1 : C^* \to \Delta \) is a trivial covering on the discs and a \( 2 \)-sheeted covering with \( 2 \) simple branching on the annulus.

(f) \( C^* \) consists of \( d \) discs, \( 2 \) of them meets transversally at one point, the remaining \( d-2 \) are disjoint, the projection \( \text{pr}_1 : C^* \to \Delta \) is a trivial covering on every disc.

The proof of the lemma is straightforward. Let us observe that in the cases (a), (c), and (e) we have \( 2 \) simple ramifications over \( 2 \) distinct points \( z_1 \) and \( z_2 \) on the disc \( \Delta \), whereas the cases (b), (d), and (f), respectively, correspond to the case when the points \( z_1 \) and \( z_2 \) collapse.
Definition 1.5. Let \( C^* \in \mathscr{A}^d(\Delta^2) \) be a curve. Set \( n := \dim \Def(C^*/\Delta) \). Define \( \Def_\nu(C^*/\Delta) \) as the locus of those \( s \in \Def(C^*/\Delta) \) for which the curve \( C_s \) has exactly \( \nu \) nodes and no other singularities and the projection \( \pr_1 : C_s \to \Delta \) has \( n - 2\nu \) simple branchings distinct from the projections of nodes. Let \( \Def_\nu(C^*/\Delta) \) be the closure of \( \Def_\nu(C^*/\Delta) \) in \( \Def(C^*/\Delta) \).

\( \def \ii {\text{i)} } \) For \( s \in \Def_\nu(C^*/\Delta) \), denote by \( B_\nu(s) \) the branching divisor of the projection \( \pr_1 : C_s \to \Delta \) and by \( \sigma_{\nu,i}(s), i = 1, \ldots, n - 2\nu \), the \( i \)-th symmetric polynomial of points of \( B_\nu(s) \). Set \( \mathcal{D}_\nu(s)(z) := z^{n-2\nu} + \sum_{i=1}^{n-2\nu} (-1)^i \sigma_{\nu,i}(s) z^{n-2\nu-i} \), so that \( \mathcal{D}_\nu(s)(z) \) is the unitary polynomial in \( z \) with zero divisor \( B_\nu(s) \). Denote \( \mathcal{D}_\nu(s) := (\sigma_{\nu,1}(s), \ldots, \sigma_{\nu,n-2\nu}(s)) \).

Lemma 1.8. i) \( \Def_\nu(C^*/\Delta) \) is an algebraic subset of \( \Def(C^*/\Delta) \) of codimension \( \nu \).

ii) The functions \( \sigma_{\nu,i}(s), i = 1, \ldots, n - 2\nu \), are holomorphic on \( \Def_\nu(C^*/\Delta) \) and extend holomorphically on \( \Def_\nu(C^*/\Delta) \).

Proof. First, let us observe that the functions \( \sigma_{0,i}, i = 1, \ldots, n \), are well-defined and holomorphic on the neighborhood of \( C^* \) in the whole space \( \mathscr{A}^d(\Delta^2) \). This follows from the construction of the functions \( \sigma_{0,i} \), which is as follows. Starting from a curve \( C \in \mathscr{A}^d(\Delta^2) \) close to \( C^* \), we take its Weierstraß polynomial \( P_C(z,w) = w^d + \sum_i w^{d-i} a_i(z) \); compute the discriminant \( D_C(z) \) of \( P_C \) with respect to the variable \( w \), this is a polynomial in coefficients \( a_i(z) \) of \( P_C \); and then represent the discriminant in the form \( D_C(z) = \mathcal{D}_C(z) \cdot h_C(z) \) with a non-vanishing holomorphic function \( h_C(z) \in \mathcal{H}(\Delta) \) and a unitary polynomial \( \mathcal{D}_C(z) \) in the variable \( z \). Then \( \mathcal{D}_C(z) \) is the desired polynomial defining the branching divisor of the projection \( \pr_1 : C \to \Delta \) for a non-singular curve \( C \). In particular, \( \mathcal{D}_0(s)(z) = \mathcal{D}_{C_s}(z) \).

Now observe that for generic \( s \in \Def_\nu(C^*/\Delta) \) the polynomial \( \mathcal{D}_0(s)(z) \) has the following structure: it has \( n - 2\nu \) simple zeros \( z_1', \ldots, z_{n-2\nu} \) and \( \nu \) double zeros \( z_1'', \ldots, z_{\nu}'' \). We contend that the set of unitary polynomials \( p(z) \) of degree \( n \) having this structure is given by a quasi-affine set \( \mathbb{A}_p \) in \( \mathbb{C}^n \). To show this let us consider the map \( \Sym_\nu : \mathbb{C}^n \to \mathbb{C}^n \) associating to each \( n \)-tuple \( (z_1, \ldots, z_n) \) its elementary symmetric polynomials \( \sigma_1(z_1, \ldots, z_n), \ldots, \sigma_n(z_1, \ldots, z_n) \). Let \( A_\nu \) be the \( \Sym_\nu \)-image of the set given by equations \( z_1 = z_2, z_3 = z_4, \ldots, z_{2\nu-1} = z_{2\nu} \). Since \( \Sym_\nu \) is algebraic and proper, \( A_\nu \) is Zariski closed in \( \mathbb{C}^n \), and \( A_\nu^\circ \) is Zariski open subset of \( A_\nu \). The latter follows from the fact that the complement \( A_\nu \setminus A_\nu^\circ \) describes further incidences among the zeros \( z_1, \ldots, z_n \) of the polynomial \( p(z) \), so that it can be defined as the image of a union of appropriate linear subspaces on \( \mathbb{C}^n \) with respect to the map \( \Sym_\nu \).

As we shall show later, for \( \nu \leq \delta(C^*) \) the families \( \Def_\nu(C^*/\Delta) \) are non-empty and contains \( C^* \). By Lemmas 1.4 and 1.7, \( \Def_\nu(C^*/\Delta) \) has codimension \( \nu \) and \( \mathcal{D}_0(\Def_\nu(C^*/\Delta)) \subset A_\nu^\circ \). By continuity, we obtain \( \mathcal{D}_0(\Def(C^*/\Delta)) \subset A_\nu \). Thus \( \Def_\nu(C^*/\Delta) \subset \mathcal{D}_0^{-1}(A_\nu) \). Comparing codimension we conclude that \( \Def_\nu(C^*/\Delta) \) is a union of some irreducible components of \( \mathcal{D}_0^{-1}(A_\nu) \). Lemma 1.7 shows which components of \( \mathcal{D}_0^{-1}(A_\nu) \) belong to \( \Def_\nu(C^*/\Delta) \): exactly those ones which meet \( \Def_\nu(C^*/\Delta) \). This proves the first part of the lemma.

The second part of the lemma is straightforward. \( \square \)

Lemma 1.9. Let \( C^* \in \mathscr{A}^d(\Delta^2) \) be a curve represented by a Weierstraß polynomial \( P^*(z,w) \) such that \( \operatorname{sing}(C^*/\Delta) = \{0\} \). Let \( \delta \) be the nodal number of \( C^* \) at \( 0 \) and \( b \) the number of boundary components of \( C^* \). Then

\( a) \) For any \( \nu \) with \( 0 \leq \nu \leq \delta \) the space \( \Def_\nu(C^*/\Delta) \) contains \( C^* \);
b) for any $\nu$ with $0 \leq \nu \leq \delta$ and $s \in \text{Def}_\nu^b(C^*/\Delta)$ the normalization $\tilde{C}_s$ of $C_s$ has the Euler characteristic

\begin{equation}
\chi(\tilde{C}_s) = b + 2(\nu - \delta); \tag{1.5}
\end{equation}

c) $\text{Def}_\delta^b(C^*/\Delta)$ is irreducible at $C^*$. In particular, any two nodal curves $C_0, C_1 \in \text{Def}_\delta^b(C^*/\Delta)$, sufficiently close to $C^*$, can be connected by a path $C_t$ in $\text{Def}_\delta^b(C^*/\Delta)$, also close enough to $C^*$.

**Proof.** Part a). It follows from the hypothesis of the lemma and the Riemann-Hurwitz formula that $C^*$ consists of $b$ discs passing through origin $0 \in \Delta^2$. Let $C^*_j$, $j = 1, \ldots, b$, be the irreducible components of $C^*$ and $u^*_j : \Delta \to \Delta^2$ their parameterizations, i.e. holomorphic maps such that $u^*_j(\Delta) = C^*_j$. We may assume that the first component of each $u^*_j$ is given by the formula $\text{pr}_1 \circ u^*_j(\zeta_j) = \zeta_j^{d_j}$ where $d_j$ is the degree of the projection $\text{pr}_1 : C^*_j \to \Delta$ and $\zeta_j$ is a complex coordinate on the parameterizing disc $\Delta$. Making an appropriate approximation, we may additionally assume that every map $u^*_j : \Delta \to \Delta^2$ is polynomial, so that each $C^*_j \subset \Delta^2$ extends to a rational affine algebraic curve $u^*_j(\mathbb{C}) \subset \mathbb{C}^2$.

Let $(z, w)$ denote the complex coordinates on $\mathbb{C}^2$. Consider small perturbations $u^*_j : \mathbb{C} \to \mathbb{C}^2$ of the maps $u^*_j : \mathbb{C} \to \mathbb{C}^2$ in which the $z$-component of each $u^*_j$ varies in the space of unitary polynomials of degree $d_j$ whereas the $w$-component remains unchanged. Let $C^*_j := u^*_j(\mathbb{C}) \subset \mathbb{C}^2$, $j = 1, \ldots, b$, be the corresponding rational curves, and $C' := \bigcup_{j=1}^b C^*_j$ their union. Then, for a generic choice of $u^*_j$, the obtained curve $C'$ must be also generic enough. In particular, the only singularities of $C'$ are nodes, the projections $\text{pr}_1 : C^*_j \to \mathbb{C}_z$ onto the $z$-axis have only simple branchings, $d_j - 1$ for each $C^*_j$, the branchings are disjoint from each other and from the projection of nodes. The number of nodes of $C'$ is $\delta$, this is essentially the definition of the virtual number of nodes $\delta(C^*, 0)$. By Lemma [1.3], $C'$ can be transformed into a curve $C_s \in \text{Def}_\delta^b(C^*/\Delta)$, sufficiently close to $C^*$. To obtain the assertion for arbitrary $\nu = 0, \ldots, \delta$, one needs to smooth $\delta - \nu$ nodes on $C_s$. The needed construction is provided by Lemma [1.4].

**Part b)**. Let $n := \dim \text{Def}(C^*/\Delta)$. Then $\dim \text{Def}_\nu^b(C^*/\Delta) = n - \nu$. On the other hand, it follows from Lemmas [1.2] and [1.7] that for $s \in \text{Def}_\nu^b(C^*/\Delta)$ the branchings of the projection $\text{pr}_1 : C_s \to \Delta$ and the projections of nodes of $C_s$ form a local coordinate system on $\text{Def}_\nu^b(C^*/\Delta)$. Thus $n - \nu = b - \chi(\tilde{C}_s) + \nu$ by the Riemann-Hurwitz formula. Finally note that by the definition of $\delta = \delta(C^*, 0)$ the $s \in \text{Def}_\delta^b(C^*/\Delta)$ the curve $C_s$ consists of $b$ discs. Thus $\chi(\tilde{C}_s) = b$ in this case. The assertion follows.

**Part c)**. The irreducibility of $\text{Def}_\delta^b(C^*/\Delta)$ is equivalent to connectedness of $\text{Def}_\delta^b(C^*/\Delta)$ in a neighborhood of $C^*$, which is the second assertion of **Part a**. Let $C_0, C_1$ be two curves in $\text{Def}_\delta^b(C^*/\Delta)$ sufficiently close to $C^*$. As it is shown in the proof of **Part a**, we may assume that both $C_0$ and $C_1$ extend to rational affine algebraic curves in $\mathbb{C}^2$, $\bigcup_{j=1}^b u_{0,j}(\mathbb{C})$ and $\bigcup_{j=1}^b u_{1,j}(\mathbb{C})$ respectively, where the maps $u_{0,j} : \mathbb{C} \to \mathbb{C}^2$ are polynomial. Moreover, we may assume that $u_{i,j}(\Delta)$ are the components of $C_i$, and that for every $j = 1, \ldots, b$ the maps $u_{i,j}$ and $u_{1,j}$ are close on $\Delta(2)$, i.e. $\|u_{0,j}(\zeta) - u_{1,j}(\zeta)\| \ll 1$ for $|\zeta| < 2$.

Consider the polynomial maps $u_{\lambda,j}(\zeta) := (1 - \lambda)u_{0,j}(\zeta) + \lambda u_{1,j}(\zeta)$ where $\lambda$ varies in the unit disc $\Delta$. Set $C_{\lambda} := \Delta^2 \cap (\bigcup_{j=1}^b u_{\lambda,j}(\mathbb{C}))$. Then we obtain a family of curves in $\mathcal{Z}^b(\Delta)$ sufficiently close to $C^*$. It follows that there exists a family $\{C_{s(\lambda)}\}$ in $\text{Def}(C^*/\Delta)$ such that every $C_{s(\lambda)}$ is isomorphic to $C_{\lambda}$. Moreover, the dependence $s(\lambda)$ is holomorphic
since so is the family \( \{ C_\lambda \} \). Hence for all by finitely many \( \lambda \in \Delta \) the curve \( C_{s(\lambda)} \) lies in \( \text{Def}_\delta(C^*/\Delta) \). The lemma follows.

1.3. Proof of Main Theorem. An analogue of Main Theorem for pairs “curve + projection” is

**Theorem 1.10.** Let \( C^* \in \mathcal{D}^d(\Delta^2) \) be a curve whose unique singular point is the origin \( 0 \in \Delta^2 \). Let \( \delta = \delta(C^*,0) \) be the corresponding virtual number of nodes.

Then for every \( \nu < \delta \), every irreducible component of \( \text{Def}_\nu(C^*/\Delta) \) contains a component of \( \text{Def}_{\nu+1}(C^*/\Delta) \).

It is obvious that Theorem 1.10 implies Main Theorem.

Let us explain the ideas lying behind the proof of Theorem 1.10. A trivial but important observation is that the correspondence \( (s \in \text{Def}_\nu(C^*/\Delta)) \overset{F_H}{\mapsto} (\text{pr}_1 : C_s \to \Delta) \) defines a holomorphic map \( F_H \) between \( \text{Def}_\nu(C^*/\Delta) \) and the Hurwitz scheme \( H_{d,m} \) of simply branched coverings \( f : C \to Oz \) over the axis \( Oz \) of degree \( d \) with \( m \) branchings, \( m := n - 2\nu \). More precisely, the image of \( C_s \) under \( F_H \) is the trivial extension of the ramified covering \( \text{pr}_1 : C_s \to \Delta \) to the covering \( f' : \tilde{C}_s \to Oz \). Obviously, the map \( \mathcal{D}_\nu : \text{Def}_\nu(C^*/\Delta) \to \mathbb{C}^m \) factors in the composition \( \pi_H \circ F_H \), where \( \pi_H : H_{d,m} \to \mathbb{C}^m \setminus D_m \) is the natural map associating to each covering \( f : C \to Oz \) its branching divisor and \( D_m \subset \mathbb{C}^m \) is the discriminant locus.

The map \( \pi_H : H_{d,m} \to \mathbb{C}^m \setminus D_m \) is a non-ramified covering and its monodromy, the subject of the Hurwitz problem, is understood well enough so that one can show the following: In the case \( m \geq d \), every branched covering \( f : C \to Oz \) of degree \( d \) with \( m \) simple branchings can be degenerated in that way that two simple branchings “collapse” yielding a node. More precisely, one uses the monodromy of \( \pi_H : H_{d,m} \to \mathbb{C}^m \setminus D_m \) to attain to a covering \( f : C \to Oz \) which have the same monodromy at two simple branchings, say \( z_1, z_2 \in Oz \), and then contracts \( z_1 \) with \( z_2 \) producing the desired node. Notice that the whole construction can be realized by moving a single branching of \( f : C \to Oz \), say \( z_1 \), along an appropriate path \( \gamma \) in \( Oz \) winding around remaining branchings \( z_2, \ldots, z_m \). Remark that a similar argument is used in [C-H-S].

To realize this construction in our setting, it is necessary to have enough room for maneuvering with branch points of the covering. However, it is not so, and the reason for the failure is that the map \( s \in \text{Def}_\nu(C^*/\Delta) \mapsto \mathcal{D}_\nu(s) \in \mathbb{C}^m \) is, in general, not proper.

As an example, let us consider deformation of the ordinary triple point. Shifting the components of the singularity \( C^* \) we obtain three nodes which, after smoothing, yield six branch points \( z_1, \ldots, z_6 \) of the projection \( \text{pr}_1 : C_s \to \Delta \). It follows from Lemma 1.7 that if \( z_1, \ldots, z_6 \) are pairwisely distinct, then any small movement of \( z_1, \ldots, z_6 \) can be realized by an appropriate deformation of \( \text{pr}_1 : C_s \to \Delta \). This means that the image of \( \text{Def}(C^*/\Delta) \) in \( \mathbb{C}^6 \) contains an open set. However, there exists no deformation of \( C^* \) for which five branch points, say \( z_2, \ldots, z_6 \), collapse and the sixth point \( z_1 \) remains distinct. Let us assume the contrary and denote the collapsed points \( z_2 = \cdots = z_6 \) by \( z^* \). The monodromy at the points \( z_1 \) and \( z^* \) in the symmetric group \( \text{Sym}_3 \) must be a transposition and a product of 5 transpositions, respectively. However, since the total monodromy of \( \text{pr}_1 : C_s \to \Delta \) is trivial, we must have the same monodromy at \( z_1 \) and at \( z^* \). Thus we conclude that \( C_s \) has two components, say \( C'_s \) and \( C''_s \), and the projection \( \text{pr}_1 : C_s \to \Delta \) is an isomorphism on \( C'_s \) and 2 sheeted covering with 2 simple branchings on \( C''_s \). Now, remembering meaning of the points \( z_1 \) and \( z^* \), we see that \( C'_s \) and \( C''_s \) must have a single intersection point \( p \).
with intersection index 2, whose projection on $\Delta$ is the point $z^*$. But this implies that $C_s'$ must be vertical at $p$ and have ramification over $z^*$, a contradiction.

Since we have no possibility to collapse the branchings of the projection $\text{pr}_1 : C_s \to \Delta$ in the desired way, we study what type of a collapse can be reached. This is the next idea of the proof. For this purpose we fix an irreducible component of $\text{Def}_\nu(C^*/\Delta)$ and consider the loci of the parameters $s$ in this component for which the divisor $B_\nu(s)$ has multiplicity at least $k$ in the origin $0 \in \Delta$ with $k = 1, 2, \ldots$.

**Definition 1.6.** Define

\begin{equation}
\text{Def}_{\nu,k}(C^*/\Delta) := \{s \in \text{Def}_\nu(C^*/\Delta) : \mathcal{D}_\nu(s)(z) \text{ is divisible by } z^k \}
\end{equation}

\begin{equation}
\text{Def}^*(C^*/\Delta) := \{s \in \text{Def}(C^*/\Delta) : \mathcal{D}_\nu(s)(z) = z^n \}
\end{equation}

Besides, we fix an irreducible component $\mathcal{Y}$ of $\text{Def}_\nu(C^*/\Delta)$ through $C^*$ and set

\begin{equation}
\mathcal{Y}_k := \mathcal{Y} \cap \text{Def}_{\nu,k}(C^*/\Delta)
\end{equation}

\begin{equation}
\mathcal{Y}^* := \mathcal{Y} \cap \text{Def}^*(C^*/\Delta)
\end{equation}

Thus the loci $\text{Def}^*(C^*/\Delta) \subset \text{Def}(C^*/\Delta)$ and $\mathcal{Y}^* \subset \mathcal{Y}$ are given by the condition “$B_\nu(s)$ is supported in $0 \in \Delta$”.

**Lemma 1.11.** i) $\text{Def}_{\nu,k}(C^*/\Delta) \subset \text{Def}_\nu(C^*/\Delta)$ and $\mathcal{Y}_k \subset \mathcal{Y}$ are analytic subsets of codimension at most $k$.

ii) For $s \in \text{Def}^*(C^*/\Delta)$, the curve $C_s$ consists of discs and has a unique singular point lying on the axis $Ow$. Moreover, $\text{Def}^*(C^*/\Delta) \subset \text{Def}_\delta(C^*/\Delta)$.

iii) Assume that the generic curve in the family $\mathcal{Y}$ is irreducible. Then the codimension of $\mathcal{Y}^*$ in $\mathcal{Y}$ is at least $d + 1$ except the following cases when the codimension is $d$:

1. $C^*$ has a node at $0 \in \Delta^2$;
2. $C^*$ consists of two smooth discs which are vertical at the origin $0 \in \Delta^2$.

**Remark.** Part ii) of the lemma states that the locus $\text{Def}^*(C^*/\Delta)$ consists of curves which have the same topological type of the singularity as $C^*$.

**Proof.** i) By definition, $\text{Def}_{\nu,k}(C^*/\Delta) \subset \text{Def}_\nu(C^*/\Delta)$ and $\mathcal{Y}_k \subset \mathcal{Y}$ are given by $k$ holomorphic equations $\sigma_{\nu,i}(s) = 0$ with $i = n - 2\nu, n - 2\nu - 1, \ldots, n - 2\nu - k + 1$.

ii) By the definition of $\mathcal{D}_\nu$, for every $s \in \text{Def}^*(C^*/\Delta)$, all the singular points of $C_s$ and all the ramification points of the projection $\text{pr}_1 : C_s \to \Delta$ lie on the axis $Ow$. Hence irreducible components of $C_s$ are discs. Moreover, since $C_s$ is connected, there must be a unique intersection point of the irreducible components of $C_s$. Indeed, otherwise one would have either a ramification or a crossing over some $z' \neq 0 \in \Delta$, which contradicts the condition $\mathcal{D}_\nu(s)(z) = z^n$. Thus $C_s$ has a unique singular point $p_s'$. This implies that $\delta(C_s, p_s') = \delta = \delta(C^*, 0)$, which means $\text{Def}^*(C^*/\Delta) \subset \text{Def}_\delta(C^*/\Delta)$.

iii) We shall compare the codimensions of $\mathcal{Y}$ and $\mathcal{Y}^*$ in $\text{Def}(C^*/\Delta)$. By definition, $\mathcal{Y}$ has codimension $\nu$ in $\text{Def}(C^*/\Delta)$. To compute $\text{codim}(\mathcal{Y}^* \subset \text{Def}(C^*/\Delta))$, we first observe that $\mathcal{Y}^*$ is contained in an irreducible analytic set $\text{Def}_\delta(C^*/\Delta)$ which has codimension $\delta$ in $\text{Def}(C^*/\Delta)$. So $\text{codim}(\mathcal{Y}^* \subset \text{Def}(C^*/\Delta)) = \text{codim}(\mathcal{Y}^* \subset \text{Def}_\delta(C^*/\Delta)) + \delta$. Furthermore, since $\mathcal{Y}^* \subset \text{Def}^*(C^*/\Delta)$ it is enough to estimate $\text{codim}(\text{Def}^*(C^*/\Delta) \subset \text{Def}_\delta(C^*/\Delta))$.

In order to estimate the latter it is sufficient to construct a complex manifold $\mathcal{V}$ with a holomorphic map $f : \mathcal{V} \to \text{Def}_\delta(C^*/\Delta)$ such that $f(\mathcal{V}) \ni C^*$, and then estimate $\text{codim}(f^{-1}(\text{Def}^*(C^*/\Delta)) \subset \mathcal{V})$. By the universality of $\text{Def}(C^*/\Delta)$, such a map $f : \mathcal{V} \to \text{Def}_\delta(C^*/\Delta)$ corresponds to certain family of deformations of $C^*$. Recall that the
family $\text{Def}_\delta(C^*/\Delta)$ parameterizes those deformations of $C^*$ which consists of discs, and the number of these discs—denoted by $b$—is the number of irreducible components of $C^*$.

The desired family is constructed as follows. Let $C_i^*$ be the irreducible components of $C^*$ and $u_i^*: \Delta \to \Delta^2$ parameterizations of $C_i^*$. Since every projection $\text{pr}_i: C_i^* \to \Delta$ has a unique branching at the origin $0 \in \Delta$, $u_i^*$ can be chosen in the form $(\zeta^{d_i}, \varphi_i^*(\zeta))$, where $d_i$ is the degree of the projection $\text{pr}_i: C_i^* \to \Delta$. In particular, $\sum_{i=1}^{b} d_i = d$, the degree of $\text{pr}_1: C^* \to \Delta$. By Corollary 1.5 we may assume that every $\varphi_i^*$ is holomorphic in some larger disc $\Delta(r)$, $r > 1$. Consider holomorphic maps $u_i: \Delta(r) \to \mathbb{C}^2$ given by

$$u_i(\zeta) = (\zeta^{d_i} + p_i(\zeta), \varphi_i^*(\zeta) + q_i(\zeta))$$

where $p_i(\zeta)$ is a polynomial of degree $d_i - 1$ with zero free term and $q_i(\zeta)$ is a polynomial of degree at most 1. Let $\mathcal{V}$ be a sufficiently small ball in the space of the coefficients of the polynomials $p_i$ and $q_i$. We write $p_{i,v} = p_{i,v}(\zeta)$, $q_{i,v} = q_{i,v}(\zeta)$, and $u_{i,v} = u_{i,v}(\zeta)$ for the polynomials and holomorphic maps corresponding to the parameter $v \in \mathcal{V}$. The curve $C_v$ is then $(\bigcup_{i=1}^{b} u_i(v(\Delta(r)))) \cap \Delta^2$. Since the family $\{C_v\}$ depends holomorphically on $v \in \mathcal{V}$ it is given by a holomorphically map $f: \mathcal{V} \to \text{Def}(C^*/\Delta)$. Moreover, $f(\mathcal{V}) \subset \text{Def}_\delta(C^*/\Delta)$ by Lemma 1.4. Set $\mathcal{V}^*: = f^{-1}(\text{Def}(C^*/\Delta))$. Then, as it was already noted, $\text{codim}(\mathcal{V}^* \subset \mathcal{V}) \leq \text{codim}(\text{Def}(C^*/\Delta) \subset \text{Def}_\delta(C^*/\Delta))$. This means that $\text{codim}(\mathcal{Y}^* \subset \mathcal{Y}) = \text{codim}(\mathcal{Y}^* \subset \text{Def}(C^*/\Delta)) - \nu \geq \text{codim}(\text{Def}(C^*/\Delta) \subset \text{Def}_\delta(C^*/\Delta)) + \delta - \nu \geq \text{codim}(\mathcal{V}^* \subset \mathcal{V}) + \delta - \nu$.

Estimating $\text{codim}(\mathcal{V}^* \subset \mathcal{V})$ we first note that $p_{i,v}(\zeta) = 0$ if $v \in \mathcal{V}^*$ since otherwise the projection $\text{pr}_i: C_{i,v} \to \Delta$ would have branching outside $0 \in \Delta$. This define a linear subspace $\mathcal{V}'$ in $\mathcal{V}$ of codimension $\sum_{i=1}^{b} (d_i - 1) = d - b$. Perturbation of the free term of the polynomial $q_i$ corresponds to a vertical shift of $C_{i,v}$. This means that we must impose further $b - 1$ conditions to insure that $C_{i,v}$ pass through the same point on the axis $ Ow$. Together we obtain a linear subspace $\mathcal{V}''$ of $\mathcal{V}$ of codimension $d - 1$ parameterizing linear maps $q_i(\zeta) = a_i\zeta$. Since $\mathcal{V}^* \subset \mathcal{V}''$, the codimension of $\mathcal{Y}^*$ in $\mathcal{Y}$ is not less than $d$ and strictly larger if $\nu \leq \delta - 2$.

Assume that at least one component of $C^*$, say $C_1^*$, is singular at $0 \in \Delta^2$. Then the parameterizing map $u_1^*$ has the form $u_1^*(\zeta) = (\zeta^{d_1}, a_1\zeta^l + a_{l+1}\zeta^{l+1} + \cdots)$ with $d_1 \geq 2$ and $l \geq 2$. Then for every perturbation of $v$ in $\mathcal{V}''$ by means of a non-zero linear term $q_1(\zeta) = a_1\zeta$ the component $C_{1,v}$ must have a node outside the axis $ Ow$. This means that in this case the set $\mathcal{Y}^*$ is contained in the subspace of $\mathcal{V}''$ given by the condition $q_1(\zeta) = 0$, and hence $\text{codim}(\mathcal{Y}^* \subset \mathcal{Y}) \geq d + 1$.

It remains to consider the case when $\nu = \delta - 1$ and every component of $C^*$ at $0 \in \Delta^2$ is non-singular. Formula (1.5) and the irreducibility of a generic curve in $\mathcal{Y}$ imply that the number $b$ of the components of $C^*$ must be 1 of 2. However, the possibility $b = 1$ is excluded since otherwise $C^*$ must consist of a single non-singular component $0 \in \Delta^2$. Thus $C^*$ consists of two non-singular components, $C_1^*$ and $C_2^*$. Here we must distinguish the following three special subcases according to the degrees $d_1$ and $d_2$ of the projections $\text{pr}_1: C_1^* \to \Delta$ and $\text{pr}_2: C_2^* \to \Delta$, respectively:

(a) both $d_1$ and $d_2$ equal 1, i.e. both $C_1^*$ and $C_2^*$ project isomorphically onto $\Delta$;
(b) $d_1 = 1$ and $d_2 > 1$;
(c) $d_1 > 1$ and $d_2 > 1$.

Obviously, the subcases (b) and (c) correspond to the subcases (1) and (2) of the lemma, respectively. In the subcases a) the degree $d$ of $\text{pr}_1: C^* \to \Delta$ is 2, the Weierstraß polynomial has the form $P(z,w) = w^2 + a(z)w + b(z)$, and its discriminant is $a^2(z) - $
4b(z). This implies that the map $\mathcal{D}_0(s)$ is surjective in a neighborhood of the value $s^* \in \text{Def}(C^*/\Delta)$ corresponding to the curve $C^*$, and $\text{codim}(\mathcal{Y}^* \subset \text{Def}(C^*/\Delta)) = 2\delta$. Thus $\text{codim}(\mathcal{Y}^* \subset \text{Def}_{\delta-1}(C^*/\Delta)) = \delta + 1 \geq d + 1 = 3$ except the case $\delta = 1$ when $C^*$ has a nodal singularity at $0 \in \Delta^2$. This finishes the proof. \hfill $\square$

Let us give a proof Theorem 1.10 for the special cases which appear in Lemma 1.11 iii).

Lemma 1.12. Assume that the curve $C^*$ has two irreducible components at $0$ which are non-singular. Let $\delta = \delta(C^*, 0)$ be the corresponding virtual number of nodes.

Then for every $\nu < \delta$, every irreducible component of $\text{Def}_\nu(C^*/\Delta)$ contains a component of $\text{Def}_{\nu+1}(C^*/\Delta)$.

Proof. Consider first the special case when for every component $C_i$ of $C^*$ the projection $\text{pr}_1 : C_i^* \to \Delta$ has degree 1. Then $\text{pr}_1 : C^* \to \Delta$ has degree 2 and the Weierstrass polynomial of $C^*$ is of the form $P(z, w) = w^2 + \varphi(z)w + \psi(z)$. Hence the discriminant $D(z)$ of $P$ in the variable $w$ is $D(z) = \varphi^2(z) - 4\psi(z)$, which is linear in $\psi(z)$. This implies that the map $\mathcal{D}_0 : \text{Def}(C^*/\Delta) \to \mathbb{C}^n$, $n = 2\delta$, is a biholomorphism on the image. Via zeroes of the discriminant $D(z)$ we have a complete control on what happens in $\text{Def}(C^*/\Delta)$. In particular, a curve $C$ lies in $\text{Def}_0(C^*/\Delta)$ if and only if the discriminant $D(z)$ has exactly $\nu$ double zeroes and $2(\delta - \nu)$ simple ones. Moreover, such a curve $C$ can be holomorphically degenerated into a curve lying in $\text{Def}_{\nu+1}(C^*/\Delta)$. This implies the assertion of the lemma for the special case.

In the remaining case the curve $C^*$ has two irreducible components at the origin $0 \in \Delta^2$, both non-singular, such that at least one of them is vertical at $0$. Choose local holomorphic coordinate $(\tilde{z}, \tilde{w})$ at the origin $0 \in \Delta^2$ such that the corresponding projection $\tilde{\text{pr}}_1 : C^* \to \Delta$ has degree 2 at the origin. Then in a neighborhood of the origin every sufficiently small deformation $C_s$ of $C^*$ is given by the Weierstrass polynomial $\tilde{D}_s(z, \tilde{w}) = \tilde{w}^2 + \tilde{\varphi}_s(z)\tilde{w} + \tilde{\psi}_s(z)$. Let $\tilde{D}(s)(\tilde{z})$ be the polynomial of the degree $n$ in $\tilde{z}$, $n := 2\delta$, whose zero divisor is the zero divisor of the discriminant $\tilde{D}_s(z) = \tilde{\varphi}_s^2(z) - 4\tilde{\psi}_s(z)$. Then the coefficients of $\tilde{D}(s)(\tilde{z})$ define a holomorphic map $\tilde{\mathcal{D}} : \text{Def}(C^*/\Delta) \to C^n$ which has maximal rank at the base point $s^*$ corresponding to $C^*$. One uses the projection $\tilde{\text{pr}}_1 : C^* \to \Delta$ to produce the desired additional node on a curve $C$ from $\text{Def}_\nu^0(C^*/\Delta)$. \hfill $\square$

Now we proceed to the proof of Theorem 1.10. Assume that the singularity of the curve $C^*$ is not of the type treated in Lemma 1.12, i.e. that $C^*$ has at least 3 components or that at least one irreducible component of $C^*$ is singular at the origin $0$.

For every index $k = 1, \ldots, d$, we fix a decreasing sequence of irreducible components $\mathcal{Y}^*_k$ of $\mathcal{Y}^*_k$ at $C^*$ so that $\mathcal{Y} = \mathcal{Y}^*_0 \supset \mathcal{Y}^*_1 \supset \mathcal{Y}^*_2 \ldots$

Proposition 1.13. There exists index $k^* \in \{2, \ldots, d\}$ such that:

i) For $k = 0, \ldots, k^* - 1$ a generic curve $C$ of the family $\mathcal{Y}^*_k$ has the following structure:
   - $C$ is non-singular at the axis $Ow$;
   - the projection $\text{pr}_1 : C \to \Delta$ has branching order $k$ at the origin $0 \in \Delta$ and only simple branchings outside the origin;
   - $C$ is nodal with exactly $\nu$ nodes.

ii) A generic curve $C$ of the family $\mathcal{Y}^*_k$, has the following structure:
   - outside the axis $Ow$, $C$ is nodal with exactly $\nu$ nodes;
   - on the axis $Ow$, $C$ has a unique singular point which either is a node or consists of two non-singular vertical branches.
It follows from the second assertion of the proposition and Lemma 1.12 that a generic curve $C$ from $\mathcal{Y}'_k$ lies in the family $\text{Def}_{k+1}(C^*/\Delta)$ and is a non-singular point there. This in turn implies Theorem 1.10 and hence Main Theorem.

**Proof.** We use induction in $k$ showing that if for some $k = 0, 1, \ldots$ the structure of component $\mathcal{Y}'_k$ is given by $i$), then the structure of $\mathcal{Y}'_{k+1}$ is given by either $i$) or $\bar{i})$. We take the case $k = 0$ as the base since $\mathcal{Y}'_0$ has the property $i$) of the proposition. Observe also that the maximal possible branching order of the projection $\text{pr}_1 : C \to \Delta$ over $0 \in \Delta$ is $d - 1$. Thus for some $k^* \leq d$ we must obtain the case $\bar{i})$, and the induction will terminate.

So now we suppose that for some given $k < d$ the component $\mathcal{Y}'_k$ has the properties listed in $i)$. Let $\chi_k$ (resp. $\chi_{k+1}$) denote the Euler characteristic of (the normalization of) a generic curve from $\mathcal{Y}'_k$ (resp. $\mathcal{Y}'_{k+1}$). It follows from the assumption that $\chi_k = b + 2(\nu - \delta)$, whereas there are following two possibilities for $\chi_{k+1}$: $\chi_{k+1} = \chi_k$ and $\chi_{k+1} > \chi_k$. We consider these two cases separately.

**Case** $\chi_{k+1} = \chi_k$. To every curve $C$ from $\text{Def}_{\nu}(C^*/\Delta)$ whose normalization $\tilde{C}$ satisfies $\chi(\tilde{C}) = \chi_k$ we shall associate the following data: The zero divisor $Z_C = \sum_i m_i \zeta_i$ of the composition $\tilde{C} \to C \to \Delta$, denoted by $\text{pr}_1 : \tilde{C} \to \Delta$ and considered as a holomorphic function, and the collection of the **multiplicities** $(m_i)$ of the zero divisor $Z_C$, defined up to reordering.

Observe that the ramification points of the map $\text{pr}_1 : \tilde{C} \to \Delta$ are exactly those $\zeta_i \in \tilde{C}$ for which $m_i \geq 2$, and the branching index of $C$ at $0 \in \Delta$ is $\sum_i (m_i - 1)$. Thus $\sum_i (m_i - 1) = k$ (resp. $> k + 1$) for a generic curve $C$ in $\mathcal{Y}'_k$ (resp. in $\mathcal{Y}'_{k+1}$). Moreover, the multiplicities $(m_i)$ are the same for two generic curves in $\mathcal{Y}'_{k+1}$.

Recall that by Corollary 1.3 a generic curve $C$ in $\mathcal{Y}'_{k+1}$ can be extended to a holomorphic curve $C^+$ in a larger bi-disc $\Delta(r) \times \Delta$ with $r > 1$, such that the Euler characteristic of the normalization $\tilde{C^+}$ of $C^+$ is still $\chi_k$. Let $f : \tilde{C^+} \to \Delta(r) \times \Delta$ be the composition $\tilde{C^+} \to C^+ \to \Delta(r) \times \Delta$. Then every family $f_s : \tilde{C^+} \to \Delta(r) \times \Delta$ of sufficiently small perturbations of $f$ parameterized by $s \in \Delta$ induces a deformation family $C_s$ of $C$ defined by $C_s := f_s(\tilde{C^+}) \cap \Delta^2$. Observe that under condition $\sum_j (m_j - 1) \geq k + 1$ on multiplicities the curves $C_s$ remain in $\mathcal{Y}_{k+1}$ and hence in $\mathcal{Y}'_{k+1}$ by irreducibility reason. On the other hand, for an appropriate choice of the family $f_s$, $s \in \Delta$, the multiplicities of the curves $C_s$ will satisfy the condition $\sum_j (m_j - 1) = k + 1$ for any $s \neq 0$. Thus $\sum_j (m_j - 1) = k + 1$ for a generic curve $C$ from $\mathcal{Y}'_{k+1}$. In a similar way one shows that a generic curve $C$ from $\mathcal{Y}'_{k+1}$ must have the properties $i)$ of the proposition.

**Case** $\chi_{k+1} > \chi_k$. Let $C^\dagger$ be a generic curve from $\mathcal{Y}'_{k+1}$ and $p^i_1, \ldots, p^i_\ell \in C^\dagger$ the singular points of $C^\dagger$ and the ramification points of the projection $\text{pr}_1 : C^\dagger \to \Delta$. Choose sufficiently small bi-discs $\Delta^2_j$ centered at $p_j$ such that for the curves $C^\dagger_j := \Delta^2_j \cap C^\dagger$ the projections $\text{pr}_1 : C^\dagger_j \to \Delta_j$ on the $z$-component are proper. In particular, $\Delta^2_j$ are mutually disjoint.

Then by Lemma 1.6 we obtain a natural decomposition

$$
(1.10) \quad \text{Def}(C^\dagger/\Delta) = \prod_j \text{Def}(C^\dagger_j/\Delta_j).
$$

More precisely, this should be understand as a natural isomorphism of the germs (and hence of small neighborhoods) of the spaces at the points corresponding to the curve $C^\dagger$.

It follows from Corollary 1.4 that we can consider the space $\text{Def}(C^\dagger/\Delta)$ as an open subset of $\text{Def}(C^*/\Delta)$. In particular, the loci $\mathcal{Y}'_k \cap \text{Def}(C^\dagger/\Delta)$ and $\mathcal{Y}'_{k+1} \cap \text{Def}(C^\dagger/\Delta)$
describe the behavior of curves of the families \( \mathcal{Y}_k \) and \( \mathcal{Y}_{k+1} \) near \( C^\dagger \). We contend that the decomposition \( (\ref{1.10}) \) is compatible with the families \( \mathcal{Y}_k \) and \( \mathcal{Y}_{k+1} \).

To show this let us take a generic curve \( C \) in \( \text{Def}_{\nu,k}(C^\dagger/\Delta) \) sufficiently close to \( C^\dagger \). Considering the pieces \( C_j := \Delta^2_j \cap C \) of \( C \), we can “decompose” the numerical invariants characterizing \( \text{Def}_{\nu,k}(C^*/\Delta) \). Namely, we obtain the following decompositions:

(a) \( \nu = \sum_j \nu_j \) where \( \nu_j \) is the number of nodes of \( C_j \);
(b) \( k = \sum_j k_j \) where \( k_j \) is the branching degree of the projection \( \text{pr}_1 : C_j \rightarrow \Delta_j \) over \( z = 0 \) if the point \( p_j \) lies on the \( Ow \)-axis and \( k_j = 0 \) otherwise.

Using this we obtain further four natural decompositions:

(c) \( \text{Def}_{\nu}(C^\dagger/\Delta) = \bigcup_{\nu_j = \nu} \prod_j \text{Def}_{\nu_j}(C_j^\dagger/\Delta_j) \);
(d) \( \mathcal{Y}_\nu(s)(z) = \prod_j \mathcal{Y}_{\nu_j}(s_j)(z) \) for \( s = (s_j) \in \prod_j \text{Def}_{\nu_j}(C_j^\dagger/\Delta_j) \);
(f) \( \text{Def}_{\nu,k}(C^\dagger/\Delta) = \bigcup_{\nu_j = \nu} \prod_{j \neq k} \text{Def}_{\nu_j,k_j}(C_j^\dagger/\Delta_j) \);
(g) \( \text{Def}_{\nu,k+1}(C^\dagger/\Delta) = \bigcup_{\nu_j = \nu} \prod_{j \neq k} \text{Def}_{\nu_j,k_j}(C_j^\dagger/\Delta_j) \)

where the union in (f) and (g) is made only over those decompositions \( k = \sum_j k_j \) or \( k + 1 = \sum_j k_j \), respectively, which can appear in (b), i.e. for which the component \( k_j \) is zero if \( p_j^\dagger \) does not lies on the axis \( Ow \).

Decomposition (c) follows from the definition of the families \( \text{Def}_{\nu} \) and Lemma \( (1.6) \), decomposition (d) from Lemma \( (1.3) \), whereas decompositions (f) and (g) from (d) and also Lemma \( (1.3) \).

Since \( \mathcal{Y}_k' \) and \( \mathcal{Y}_{k+1}' \) are irreducible, there exist uniquely defined decompositions \( \nu = \sum_j \nu_j^\dagger, k = \sum_j k_j^\dagger \) and \( k + 1 = \sum_j k_j^\dagger \), such that \( \mathcal{Y}_k' \cap \text{Def}(C^\dagger/\Delta) \) lies in \( \prod_j \text{Def}_{\nu_j^\dagger,k_j^\dagger}(C_j^\dagger/\Delta_j) \) and \( \mathcal{Y}_{k+1}' \cap \text{Def}(C^\dagger/\Delta) \) lies in \( \prod_j \text{Def}_{\nu_j^\dagger,k_j^\dagger}(C_j^\dagger/\Delta_j) \). Moreover, there exists the unique index \( j_0 \), say \( j_0 = 1 \), such that \( k_j^\dagger = k_j' \) for \( j \neq j_0 = 1 \) and \( k_1^\dagger = k_1' + 1 \). Observe that the corresponding point \( p_1^\dagger \) lies on the axis \( Ow \).

The condition of genericity of \( C^\dagger \) in \( \mathcal{Y}_{k+1}' \) implies that \( \mathcal{Y}_{k+1}' \) is non-singular at \( C^\dagger \). Thus every family \( \text{Def}_{\nu_j^\dagger,k_j^\dagger}(C_j^\dagger/\Delta_j) \) is non-singular and generic at \( C_j^\dagger \). This means that \( \text{Def}_{\nu_j^\dagger,k_j^\dagger}(C_j^\dagger/\Delta_j) \) are non-singular and generic at \( C_j^\dagger \) for every \( j \neq 1 \). Consequently, there exists irreducible components \( \mathcal{Y}_1^\dagger \) of \( \text{Def}_{\nu_1^\dagger,k_1^\dagger}(C_1^\dagger/\Delta_1) \) and \( \mathcal{Y}_1^\dagger \) of \( \text{Def}_{\nu_1^\dagger,k_1^\dagger}(C_1^\dagger/\Delta_1) \) at \( C_1^\dagger \) such that

\[
(1.11) \quad \mathcal{Y}_k' \cap \text{Def}_{\nu,k}(C^\dagger/\Delta) = \mathcal{Y}_1^\dagger \times \prod_{j > 1} \text{Def}_{\nu_j^\dagger,k_j^\dagger}(C_j^\dagger/\Delta_j)
\]
\[
(1.12) \quad \mathcal{Y}_{k+1}' \cap \text{Def}_{\nu,k}(C^\dagger/\Delta) = \mathcal{Y}_1^\dagger \times \prod_{j > 1} \text{Def}_{\nu_j^\dagger,k_j^\dagger}(C_j^\dagger/\Delta_j)
\]

Moreover, every factor in \( (1.11) \) represents a family of curves which satisfies the conditions listed in the part i) of the hypothesis of the proposition.

We contend that the decompositions \( (1.11) \) and \( (1.12) \) are non-trivial in the sense that the dimension of every factor \( \text{Def}_{\nu_j^\dagger,k_j^\dagger}(C_j^\dagger/\Delta_j) \) is positive and the number of the factors—which is the number \( l \) of the points \( p_j^\dagger \)—is at least 2. The latter follows from Lemma
Moreover, we have shown that at least one point \( p_j \) lies not on the axis \( Ow \). This implies that the dimension of every factor \( \text{Def}_{\nu_j,k^j}(C^j_j/\Delta_j) \) is strictly less that the dimension of \( Y'_k \).

This provides that now we can use the induction in the dimension of the family \( Y'_k \). This means that since the dimension of \( Y'_k \) is strictly less that the dimension of \( Y'_k' \) and since the family \( Y'_k' \) has the properties listed in the part i) of the proposition, the family \( Y'_k'_{k+1} \) must be either of type i) or type ii) of the proposition. In view of properties of the decomposition (1.12), the same dichotomy holds also for \( Y_k'_{k+1} \).

\[ \square \]

2. Application to the symplectic isotopy problem

2.1. The symplectic isotopy problem for nodal surfaces. We consider a version of the symplectic isotopy problem for surfaces in a symplectic 4-manifold with positive ordinary double points. As an introduction to the problem we refer to author’s paper [Sh].

Definition 2.1. Let \((X,\omega)\) be a symplectic 4-manifold. A nodal symplectic surface in \(X\) is an immersed surface \( \Sigma \subset X \) such that the restriction \( \omega|_{\Sigma} \) never vanishes and the only singularities of \( \Sigma \) are positive ordinary double points, called nodes.

Note that the restriction \( \omega|_{\Sigma} \) induces the orientation on \( \Sigma \). Recall that an ordinary double point of an immersed oriented surface in a 4-fold is positive if the self intersection number at this point is +1.

Two closed nodal symplectic surfaces \( \Sigma_0 \) and \( \Sigma_1 \) in \((X,\omega)\) are symplectically isotopic if they can be connected by an isotopy \( \Sigma_t \) consisting of nodal symplectic surfaces. Such an isotopy \( \Sigma_t \) is called a symplectic isotopy between \( \Sigma_0 \) and \( \Sigma_1 \).

Now the symplectic isotopy problem can be formulated as follows: Given a symplectic 4-manifold \((X,\omega)\) and closed irreducible nodal symplectic surfaces \( \Sigma_0 \), \( \Sigma_1 \subset X \) lying in the same integer homology class and having the same genus \( g \), does there exists a symplectic isotopy between \( \Sigma_0 \) and \( \Sigma_1 \)?

Note that the genus of a closed irreducible nodal symplectic surface \( \Sigma \) in a symplectic 4-manifold \((X,\omega)\) can be computed by the genus formula

\[ g(\Sigma) = \frac{[\Sigma]^2 - c_1(X,\omega) \cdot [\Sigma]}{2} + 1 - \delta(\Sigma), \]

where \( \delta(\Sigma) \) is the number of nodes on \( \Sigma \) and \( c_1(X,\omega) \) is the first Chern class of \((X,\omega)\) (see e.g. [Gro], [McD-Sa-1], or [Sh]). Thus in the situation of the symplectic isotopy problem the number of nodes on \( \Sigma_0 \) and \( \Sigma_1 \) is the same.

In the paper [Fi-St] Fintushel and Stern exhibited a class of symplectic 4-folds \((X,\omega)\) with the following property. There exists an infinite number of symplectic imbeddings \( \Sigma_i \hookrightarrow X \), such that all \( \Sigma_i \) are homologous but pairwise non-isotopic, even smoothly. So the answer to the symplectic isotopy problem can be negative in general. On the other hand, the results of the paper [Sh] give reason to hope that the answer might be positive for special symplectic 4-folds. Namely, in [Sh] the author formulated the following

Conjecture. Let \( \Sigma_0 \) and \( \Sigma_1 \) be closed irreducible nodal symplectic surfaces in a closed symplectic 4-manifold \((X,\omega)\) lying in the same integer homology class and having the same genus \( g \). Then a symplectic isotopy between \( \Sigma_0 \) and \( \Sigma_1 \) exists provided \( c_1(X) \cdot [\Sigma_0] > 0 \).
As it was mentioned *Introduction*, the solution of the local Severi problem in the form of *Main Theorem* implies a solution of the local symplectic isotopy problem for the case of immersed surfaces with (positive) nodes. In order to explain this relation, let us make an overview of the method used for constructing a symplectic isotopy.

First, recall that there exists a complete classification of compact symplectic 4-folds $X$ which come in question.

**Proposition 2.1.** Let $(X, \omega)$ be a compact symplectic 4-fold and $\Sigma \subset X$ a closed symplectic nodal surface with $\langle c_1(X), [\Sigma] \rangle > 0$. Assume that $\Sigma$ is not an exceptional sphere. Then $X$ is either $\mathbb{CP}^2$, or a ruled complex surface, or its blow-up.

For the precise description of the blow-up procedure in symplectic category we refer to \[McD-3\] and \[Gi-St\].

**Proof.** For the case of imbedded $\Sigma$, this proposition is proved in \[McD-Sa-2\], Corollary 1.5. The general case follows from the fact that every symplectic nodal surface $\Sigma$ in a symplectic 4-fold can be “symplectically smoothed”, i.e. deformed into an imbedded symplectic surface. □

The complete description of possible symplectic structures on such $X$ was given in \[McD-2\], \[La-McD\], and \[McD-Sa-2\], see also \[Li-Liu\], \[Liu\].

**Proposition 2.2.** i) Every symplectic form $\omega$ on $\mathbb{CP}^2$ is isotopic to a multiple of the Fubuni-Study form $\omega_{st}$.

ii) Every symplectic form $\omega$ on a (minimal) ruled complex surface $X$ is compatible with some genuine complex structure $J$.

The minimality is understood in the sense of ruled complex surfaces so that $X$ is not a blow-up of another ruled complex surface. Thus the $\mathbb{CP}^2$ blown-up once is minimal in this sense. The compatibility of $J$ and $\omega$ means that they define a Kähler structure on $X$.

Now we recall main features of Gromov’s theory of pseudoholomorphic curves which is for now the most effective approach to the symplectic isotopy problem.

**Definition 2.2.** An *almost complex structure* on a manifold $X$ is an endomorphism $J$ of the tangent bundle $TX$ such that $J^2 = -\text{Id}$. The pair $(X, J)$ is called an *almost complex manifold*.

An almost complex structure $J$ on a symplectic manifold $(X, \omega)$ is called $\omega$-tame if $\omega(v, Jv) > 0$ for any non-zero tangent vector $v$. The set of $\omega$-tame almost complex structures on $X$ is denoted by $\mathcal{J}_\omega$.

**Definition 2.3.** A *parameterized $J$-holomorphic curve* in an almost complex manifold $(X, J)$ is given by a Riemann surface $S$ with a complex structure $J_S$ on $S$ and a (non-constant) $C^1$-map $u : S \to X$ satisfying the *Cauchy-Riemann equation*

$$(2.2) \quad du + J \circ du \circ J_S = 0.$$  

In this case we call $u$ a $(J_S, J)$-*holomorphic map*, or simply $J$-*holomorphic map*. Here we use the fact that if $u$ is not constant, then the structure $J_S$ is unique. In particular, such a map $u$ equips $S$ with a complex structure $J_S$.

A *non-parameterized $J$-holomorphic curve* is the image $C = u(S)$ of a non-constant $J$-holomorphic map $u : S \to X$. Since the map $u$ equips $S$ with a complex structure $J_S$ we
obtain a Riemann surface \((S,J_S)\) which can be seen as the \textit{normalization} \(\widetilde{C}\) of \(C = u(S)\) provided \(C\) is non-multiple.

The structure of \(J\)-holomorphic maps and curves is very similar to that of usual holomorphic objects, for details see e.g. \cite{M-W}, \cite{Si}, \cite{Si-2}, and \cite{Sh}. In particular, the notions of an \textit{irreducible component} and the \textit{multiplicity} of a component have the usual meaning.

The notion \textit{\(J\)-holomorphic curve} or simply \textit{\(J\)-curve} means either parameterized or non-parameterized curve. We say about \textit{pseudoholomorphic} maps and curves if the structure \(J\) is clear from the context or not specified.

We always assume that the parameterizing surface \(S\) is compact but not necessary closed, so that the boundary \(\partial S\) of \(S\) can be non-empty. In this case we assume that \(\partial S\) consists of finitely many smooth circles and that both the structure \(J_S\) and the parameterizing map \(u\) are \(C^1\)-smooth up to boundary \(\partial S\). The \textit{boundary} \(\partial C\) of a pseudoholomorphic curve \(C\) parameterized by \(u : S \to X\) is the set \(u(\partial S)\). We say that a curve \(C\) is \textit{non-singular} at the boundary \(\partial C\) if \(u\) is an imbedding near \(\partial S\).

Applying Gromov’s theory to the symplectic isotopy problem, one uses the following argumentation. It is well-known that the set \(\mathcal{J}_\omega\) of tame almost complex structures in a symplectic manifold \((X,\omega)\) is non-empty and contractible (see e.g. \cite{Gr}, \cite{Mc-D-Sa}). In particular, any two \(\omega\)-tame almost complex structures \(J_0\) and \(J_1\) can be connected by a homotopy (path) \(J_t\), \(t \in [0,1]\), inside \(\mathcal{J}_\omega\). Furthermore, every immersed surface \(\Sigma\) in a symplectic 4-fold \((X,\omega)\) with ordinary double points is \(J\)-holomorphic curve with respect to some \(\omega\)-tame structure \(J\) if and only if \(\Sigma\) is a nodal \(\omega\)-symplectic surface.

Now let \((X,J_1)\) be a (compact) ruled complex surface with a Kähler form \(\omega\) and \(\Sigma\) a nodal \(\omega\)-symplectic closed surface in \(X\). Find an \(\omega\)-tame almost complex structure \(J_0\) making \(\Sigma\) a \(J_0\)-holomorphic curve. Find a path \(h : [0,1] \to \mathcal{J}_\omega\) such that \(h(0) = J_0\) and \(h(1) = J_1\), so that \(J_t := h(t)\) is a homotopy between \(J_0\) and \(J_1\). Fix points \(x = (x_1,\ldots,x_k)\) on \(X\) and consider the spaces

\[
\mathcal{M}_{h,x} := \left\{ (C,t) : \begin{array}{l}
t \in [0,1], \ C \text{ is a non-multiple irreducible } h(t)\text{-}\text{holomorphic curve of geometric genus } g \\
\text{homology class } [\Sigma] \text{ passing through } x_1,\ldots,x_k\end{array} \right\},
\]

\[
\mathcal{M}^0_{h,x} := \{ (C,t) \in \mathcal{M}_{h,x} : \ C \text{ is nodal} \}
\]

together with the projection \(\text{pr}_{h,x} : \mathcal{M}_{h,x} \to [0,1]\).

The reason for introducing the points \(x_1,\ldots,x_k\) will be explained later. For a while, we may assume that \(k = 0\) and there is no constrain on curves to pass through given points.

It is known that for a \textit{generic} path \(h : [0,1] \to \mathcal{J}_\omega\) the space \(\mathcal{M}_{h,x}\) has a natural structure of a smooth manifold of the expected dimension

\[
\dim_{\mathbb{R}} \mathcal{M}_{h,x} = 1 + 2(c_1(X) \cdot [\Sigma] + g - 1 - k)
\]
such that the projection \(\text{pr}^0_{h,x}\) is smooth, and \(\mathcal{M}^0_{h,x}\) is open in \(\mathcal{M}_{h,x}\). Let us denote by \(\text{pr}_{h,x}\) the restriction of \(\text{pr}^0_{h,x}\) onto \(\mathcal{M}^0_{h,x}\).

A crucial observation is that a section \(s(t) = (C_t,t)\) of the projection \(\text{pr}^0_{h,x}\) with \(C_0 = \Sigma\), if exists, would give a symplectic isotopy between \(\Sigma\) and a holomorphic curve \(C_1\). Furthermore, since the moduli space of nodal \(J_1\)-holomorphic (and hence algebraic) curves of the given geometric genus \(g\) and homology class \([\Sigma]\) in \(X\) is quasi-projective, it has finitely many components. This would reduce the symplectic isotopy problem to the \textit{Severi problem} of \((X,J_1)\): the description of components of the space \(\mathcal{M}_{J_1}\) of nodal irreducible
curves in \((X,J_1)\) of given homology class and genus. The case of primary interest for the symplectic isotopy problem is the one with \(c_1(X)\cdot [C] > 0\). There is a certain progress in this direction after Harris’ paper, see \([\text{Ran}]\) and \([G-L-Sh]\). However, the answer to the Severi problem in the case \(c_1(X)\cdot [C] > 0\) in general is still unknown.

Constructing of a section \(s(t)\) of the projection \(\text{pr}_{h,x}\) one challenges two principal difficulties. The first one is that the projection \(\text{pr}_{h,x}: \mathcal{M}_{h,x} \to [0,1]\) considered as a real function can have local maxima. However, as it was shown in Section 4 of \([\text{Sh}]\), this difficulty does not occur if \(c_1(X)\cdot [\Sigma] > 0\). More precisely, it is proved that

- \(S1\) the complement of \(\mathcal{M}_{h,x}^o\) in \(\mathcal{M}_{h,x}\) has Hausdorff codimension \(\geq 2\);
- \(S2\) if the number \(k\) of fixed points \(x\) is strictly less than \(c_1(X)\cdot [\Sigma]\), then (for a generic \(h\)) every critical point of the projection \(\text{pr}_{h,x}: \mathcal{M}_{h,x} \to [0,1]\) is saddle.

This insures that a section of \(\text{pr}_{h,x}\) over \([0,t_0]\) can be continued to a bigger interval \([0,t_1]\), \(t_1 > t_0\).

The second difficulty comes from the fact that the space \(\mathcal{M}_{h,x}\) is not compact and the projection \(\text{pr}_{h,x}: \mathcal{M}_{h,x} \to [0,1]\) is not proper. Gromov’s compactness theorem provides that there exists a nice compactification \(\overline{\mathcal{M}}_{h,x}\) of \(\mathcal{M}_{h,x}\) such that

- \(\overline{\mathcal{M}}_{h,x}\) is a compact Hausdorff topological space;
- it has a natural stratification whose strata are smooth for a generic \(h\);
- \(\text{pr}_{h,x}: \mathcal{M}_{h,x} \to [0,1]\) extends to a proper projection \(\overline{\text{pr}}_{h,x}: \overline{\mathcal{M}}_{h,x} \to [0,1]\);
- \(\overline{\text{pr}}_{h,x}\) is smooth on every stratum of \(\overline{\mathcal{M}}_{h,x}\).

More precisely, every stratum of \(\overline{\mathcal{M}}_{h,x}\) consists of pairs \((C,t)\) such that \(C\) is possibly reducible and not reduced \(h(t)\)-holomorphic curve in the homology class \([\Sigma]\) passing through \(x\). Thus every \(C\) is a formal sum \(C = \sum_i m_i C_i\) of closed irreducible \(h(t)\)-holomorphic curves with positive integer multiplicities \(m_i\), such that \([\Sigma] = \sum_i m_i [C_i]\) and \(x_1,\ldots,x_k \in \text{supp}(C) = \cup_i C_i\). The strata are indexed by obvious combinatorial data: homology classes, genera, multiplicities of single components, and the distribution of the points \(x_1,\ldots,x_k\) on the components. The smooth structure on the strata describes deformation of components in terms of solutions of the equation \((2.2)\). The topology on the whole compactification \(\overline{\mathcal{M}}_{h,x}\) is the cycle topology in which every curve \(C = \sum_i m_i C_i\) is considered as a closed 2-current on \(X\), see below for details. We refer to \([\text{Sh}]\) for more details on the structure of \(\overline{\mathcal{M}}_{h,x}\).

**Definition 2.4.** Let \(C_n\) be a sequence of pseudoholomorphic curves in a manifold \(X\) with parameterizations \(u_n: S_n \to X\). It converges to a pseudoholomorphic curve \(C^*\) with a parameterization \(u^*: S^* \to X\) in the cycle topology if

- \(CT1\) the boundaries \(\partial S_n\) and \(\partial S^*\) have the same number of circles; moreover, there exists diffeomorphisms \(\varphi_n : \partial S^* \to \partial S_n\) such that the maps \(u_n \circ \varphi_n : \partial S^* \to X\) converge to \(u^*|_{\partial S^*}: \partial S^* \to X\) in the \(C^1\)-topology;
- \(CT2\) for any continuous 2-form \(\psi\) on \(X\) the integrals \(\int_{u_n(S_n)} \psi\) converge to \(\int_{u^*(S^*)} \psi\);
- \(CT3\) curves \(C_n\) and \(C^*\) are holomorphic with respect to almost complex structures \(J_n\) and \(J^*\) on \(X\) respectively, such that \(J_n\) converge to \(J^*\) in the \(C^0\)-topology.

In fact, in the assertions below we shall have even a little bit finer version of the cycle topology. Namely, the convergence of the structures \(J_n \to J^*\) will be in the Hölder \(C^{0,\alpha}\)-topology with some \(0 < \alpha < 1\) except sufficiently small neighborhoods of the singular points of \(C^*\).
Using the saddle property $S2)$ one can show that under condition $c_1(X) \cdot [\Sigma] > k$ there exists a continuous piecewise smooth section $s(t) = (C(t), t)$ section of $[\mathcal{M}_{h,x}^\circ: \mathcal{M}_{h,x} \to [0, 1]$.

One would obtain the desired symplectic isotopy if one manages to “push” such a section $s$ into $\mathcal{M}_{h,x}^\circ$, i.e. deform $s$ into a section $s'(t)$ with values in $\mathcal{M}_{h,x}^\circ$, or even in $\mathcal{M}_{h,x}$.

To understand whether such a deformation exists one needs a description how different strata of $\mathcal{M}_{h,x}^\circ$ are attached to each other. Thus we are led to the question of description of possible symplectic isotopy classes of nodal curves in a neighborhood of a given singular pseudoholomorphic curve $C^*$. This question is often related to as the local symplectic isotopy problem.

As it was noticed in [Sh], Main Theorem provides a sufficiently complete solution of local symplectic isotopy problem for nodal curves in a neighborhood of a reduced pseudoholomorphic curve $C^*$, i.e. in the case when every irreducible component of $C^*$ is non-multiple. For a precise statement we need a generalization of some notions for the case of pseudoholomorphic curves.

**Definition 2.5.** Let $X$ be a 4-manifold, $J_0$ an almost complex structure on $X$, and $C_0$ a $J_0$-holomorphic curve with a parameterization $u_0 : S \to X$. Assume that $C_0$ has no multiple component and that the boundary $\partial C$ is non-singular or empty.

An **equigeneric deformation** $C_t$ of $C_0$ is given by a family $J_t$ of almost complex structures on $X$ and a family $u_t : S \to X$ of $J_t$-holomorphic maps such that $C_t = u_t(S)$ and such that every $u_t$ is an imbedding near the boundary $\partial S$. We assume that the structures $J_t$ and the parameterization maps $u_t$ depend continuously on $t$. Every pseudoholomorphic curve $C_1$ which appears in this way is also called an equigeneric deformation of $C_0$.

A **maximal nodal deformation** of $C_0$ is a nodal curve $C_1$ which is an equigeneric deformation of $C_0$. As in the usual holomorphic case, every singular point $p$ of $C_0$ “splits” under maximal nodal deformation into certain number of nodes. This number is called the (virtual) **nodal number** of $C_0$ at $p$ and denoted usually by $\delta(C^*, p)$. The sum $\delta(C^*) := \sum \delta(C^*, p_i)$ over all singular points of $C^*$ is the maximal number of nodes which can be obtained by a deformation of $C^*$ which is small in the cycle topology.

A **nodal deformation** of $C_0$ is given by a family $J_t$ of almost complex structures on $X$ and a family $C_t$ of nodal $J_t$-holomorphic curves such that

- the structures $J_t$ depend continuously on $t$;
- the curves $C_t$ depend continuously on $t$ with respect to the cycle topology;
- every $C_t$ is imbedded near the boundary $\partial C_t$; moreover, the boundaries $\partial C_t$ depend continuously on $t$ with respect to the $C^1$-topology.

As in the holomorphic case, every small deformation $C_1$ of a nodal curve $C_0$ is nodal again; however, some nodes of $C_0$ disappear and some persist. We say that $C_1$ is obtained from a nodal curve $C_0$ by **smoothing the nodes** $p_1, \ldots, p_l$ of $C_0$ if $C_1$ is a small nodal deformation of $C_0$ and the set of nodes which disappear is $\{p_1, \ldots, p_l\}$.

The following result about the uniqueness of maximal nodal deformation and smoothing of a prescribed set of nodes is proved in [Sh].

**Proposition 2.3.** i) Let $X$ be a 4-manifold and $C^*$ be a pseudoholomorphic curve whose boundary is either empty or smooth imbedded. Then two sufficiently small maximal nodal deformations $C_0$ and $C_1$ of $C^*$ can be connected by an isotopy $C_t$ which is close to $C^*$ in the cycle topology.

ii) Let $X$ be a 4-manifold, $C^*$ be a nodal pseudoholomorphic curve whose boundary is either empty or smooth imbedded, and $\{p_1, \ldots, p_l\}$ a prescribed subset of the set of nodes...
of $C^*$. Then two sufficiently small deformations $C_0$ and $C_1$ of $C^*$ obtained by smoothing
the prescribed nodes $p_1, \ldots, p_l$ can be connected by an isotopy $C_t$ which is close to $C^*$ in
the cycle topology.

In both cases, if $C^*$ is $J^*$-holomorphic and the structure $J^*$ is tamed by a symplectic
form $\omega$, then the isotopy $C_t$ can be chosen $\omega$-symplectic.

2.2. Existence of symplectic isotopy between nodal surfaces. The first application
of Main Theorem is the positive solution of the local symplectic isotopy problem for nodal
pseudoholomorphic curves without multiple components.

**Theorem 2.4.** Let $X$ be a 4-manifold, $J^*$ an almost complex structure on $X$, and $C^*$ a
$J^*$-holomorphic curve. Assume that $C^*$ has no multiple component and that the boundary
$\partial C$ is smooth imbedded or empty.

Let $C$ be some nodal deformation of $C^*$ and $C^\dagger$ a maximal nodal deformation of $C^*$,
both sufficiently close to $C^*$ in the cycle topology. Then there exists an isotopy $C_t$ between
$C$ and a small deformation $C^\ddagger$ of $C^\dagger$ obtained by smoothing an appropriate set of nodes
of $C^\dagger$. Moreover, the isotopy $C_t$ can be realized sufficiently close to $C^*$

Moreover, if the structure $J^*$ is tamed by a symplectic form $\omega$ on $X$, then the isotopy
$C_t$ can be made $\omega$-symplectic.

**Proof.** As it was already indicated, the assertion follows from Main Theorem and the
techniques developed in [Sh], especially in Subsection 6.2. Let us outline the modifications
needed to adapt the argumentation used there to our situation.

**Special case.** Assume that $X$ is the unit ball in $\mathbb{C}^2$, the structure $J^*$ is sufficiently close
to the standard structure in $\mathbb{C}^2$, and $C^*$ has a single singularity at the origin $0 \in B$.

**Preparatory construction.** Performing an appropriate isotopy, one can reduce the problem
to the situation when $C^*$ is holomorphic. The construction of such an isotopy used in [Sh]
apply here with minor modification.

**Induction by complexity of singularities.** In [Mi-Wh], Micallef and White has proved that the
local behavior of pseudoholomorphic curves is essentially the same as the one of genuine
holomorphic curves. In particular, one obtains well-defined notions of the topological
type of the singularity and of codimension of a singularity of a given topological type.
The latter is the codimension of the space of curves with the singularity of the given
topological type in the whole space of curves.

A parameter version of the result of Micallef and White was proved in [Sh], Section
3. In particular, the actual codimension of the set of pseudoholomorphic curves with a
singularity of a given topological type is the expected one, see [Sh] for details. Inductively,
we may assume that the assertion of the theorem holds for all pseudoholomorphic curves
whose singularities have smaller codimension than that of $C^* \subset B$.

**Main construction.** One tries to find an isotopy $C_t$ between $C =: C_0$ and a holomorphic
curve $C_1$ controlling the behavior of $C_t$ near the boundary so that $C_t$’s remain close to
$C^*$. It is proved in [Sh], Subsection 6.2, that there exists an isotopy $C_t$ such that

- $C_t$ is parameterized by $t \in [0, t^+]$ and remains close to $C^*$;
- for some increasing sequence $t_n$ converging to $t^+$ the sequence $C_{t_n}$ converges to a
curve $C^+$;
- the curve $C^+$ either is holomorphic in the usual sense or has singularities of codi-
mension strictly smaller than that of $C^* \subset B$. 
If the obtained curve is $C^+$ is holomorphic, then the assertion of the theorem for the special case of a single singularity follows from **Main Theorem**. Otherwise, the assertion follows by induction.

**General case.** One performs appropriate constructions in a neighborhood of every singular point of $C^*$ and then extend the obtained local families of deformations of $C^*$ to a global family $C_t$. Such a family $C_t$ can be made $J_t$-holomorphic since there are no integrability condition on the structures $J_t$. \(\square\)

Our second application is the positive solution of the (global) symplectic isotopy problem for nodal surfaces of lower genus.

**Theorem 2.5.** i) Let $(X,\omega)$ be a $\mathbb{CP}^2$ with the Fubini-Study form. Then every two symplectic nodal irreducible surfaces $\Sigma_0, \Sigma_1$ of the same degree and the same genus $g \leq 4$ are symplectically isotopic.

\[\Sigma_0, \Sigma_1 \text{ of the same degree and the same genus } g \leq 4, \text{ are symplectically isotopic.}\]

ii) Let $X$ be $\mathbb{CP}^2$ blown-up at one point, and $\omega$ a symplectic form on $X$. Then every two symplectic nodal irreducible surfaces $\Sigma_0, \Sigma_1$ of the same homology class and the same genus $g \leq 2$ are symplectically isotopic.

iii) Let $X = S^2 \times S^2$ and $\omega$ a product symplectic form on $X$. Then every symplectic nodal irreducible surface $\Sigma$ of genus $g \leq 3$ is symplectically isotopic to an algebraic curve. In particular, there exist finitely many symplectic isotopy classes of nodal irreducible surface $\Sigma$ of a given genus $g \leq 3$ in a given homology class on $S^2 \times S^2$.

The general idea of the proof is as follows. In all three cases there exists the standard complex structure $J_{st}$ on $X$ tamed by the symplectic form $\omega$. This means that $(X,J_{st})$ is isomorphic to $\mathbb{CP}^2$, the blown-up $\mathbb{CP}^2$, or $\mathbb{CP}^1 \times \mathbb{CP}^1$, respectively. We shall show that every symplectic nodal surface $\Sigma \subset X$ satisfying the hypotheses of the theorem is symplectically isotopic to a $J_{st}$-holomorphic curve. The uniqueness of the symplectic isotopy class in the case of the (blown-up) $\mathbb{CP}^2$ will follow then from the irreducibility of the Severi variety $V_g(X, [\Sigma])$ of irreducible nodal $J_{st}$-holomorphic curves in $X$ of genus $g$ in the homology class $[\Sigma]$. This result is proved by Harris [14] for $\mathbb{CP}^2$ and by Ziv Ran [15] for $\mathbb{CP}^2$ blown-up at one point.

Now let $\Sigma \subset X$ be a symplectic nodal surface satisfying the hypotheses of the theorem. In particular, $\Sigma$ is irreducible and has genus $g$ at most 4, 3, or 2 according to $X$. To find a symplectic isotopy between $\Sigma$ and a $J_{st}$-holomorphic curve we repeat the construction which was used in [14], Subsection 6.3, and exposed in Subsection [2,7].

First we establish possible values of the “anti-canonical degree” $c_1(X) \cdot [\Sigma]$ for nodal symplectic surfaces satisfying the hypotheses of **Theorem 2.5**.

**Lemma 2.6.** Let $\Sigma$ be a nodal symplectic surface in a symplectic 4-fold $(X,\omega)$. Then “anti-canonical degree” $c_1(X) \cdot [\Sigma]$ is at least 1 if $X$ is the blown-up $\mathbb{CP}^2$ and $\Sigma$ is $J$-holomorphic for some structure $J$ which can be included in a generic 1-parameter family of structures $J_t$; at least 2 if $X$ is $S^2 \times S^2$; and at least 3 if $X$ is $\mathbb{CP}^2$.

Moreover, if the equality holds then, according to the case, $\Sigma$ is

1. an exceptional sphere, if $X$ is the blown-up $\mathbb{CP}^2$;
2. a “horizontal” or “vertical” line representing the homology class $[S^2 \times pt]$ or $[pt \times S^2]$, respectively, if $X$ is $S^2 \times S^2$;
3. a “line” i.e. a sphere of degree 1, if $X$ is $\mathbb{CP}^2$. 

Remark. The same assertion holds in the case when \( \Sigma \) is an algebraic curve in the (blown-up) \( \mathbb{CP}^2 \) or \( \mathbb{CP}^1 \times \mathbb{CP}^1 \), respectively. This classical result follows also from the proof of the lemma.

Proof. Case \( X = \mathbb{CP}^2 \). In this case \( (X, \omega) \) is symplectomorphic to \( \mathbb{CP}^2 \) equipped with some positive multiple of the Fubini-Study form \( \omega_{FS} \). The group \( \mathbb{H}_2(\mathbb{CP}^2, \mathbb{Z}) \) is \( \mathbb{Z} \) and every \( \omega \)-symplectic nodal surface must have positive degree \( d \). Then \( c_1(X) \cdot [\Sigma] = 3d \). The genus formula for symplectic nodal surfaces insures that \( \Sigma \) is an imbedded sphere in the case \( d = 1 \).

Case \( X \) is the \( \mathbb{CP}^2 \) blown-up at one point. We use basic properties of symplectic blown-up in dimension 4 and symplectic exceptional spheres, see e.g. [McD-3]. Assume that \( J \) is a generic \( \omega \)-tame almost complex structure on \( X \) and \( \Sigma \subset X \) an irreducible nodal \( J \)-holomorphic curve. Furthermore, we assume that \( \Sigma \) is not an exceptional sphere since otherwise \( c_1(X) \cdot [\Sigma] = 1 \). Then that there exists a \( J \)-holomorphic exceptional sphere \( E \subset X \). It follows from the genericity of \( J \) that \( E \) meets \( \Sigma \) only at smooth points and transversally. Perturbing \( J \), we can make \( J \) integrable near \( E \). Denote \( d_E := [\Sigma] \cdot [E] \). Then \( d_E \) is a non-negative integer. Contracting \( E \) we obtain

- a compact 4-manifold \( X' \) diffeomorphic to \( \mathbb{CP}^2 \);
- a point \( p_E \) which appears instead of the exceptional sphere \( E \), such that \( X' \backslash \{p_E\} \) is canonically identified with \( X \backslash E \);
- the symplectic form \( \omega' \) on \( X' \) whose restriction on \( X' \backslash \{p_E\} \) coincides with \( \omega|_{X \backslash E} \);
- an \( \omega' \)-tame almost complex structure \( J' \) on \( X' \) which is integrable near \( p_E \);
- a \( J' \)-holomorphic curve \( \Sigma' \) in \( X' \) such that \( \Sigma' \backslash \{p_E\} \) coincides with \( \Sigma \backslash E \) and such that \( \Sigma' \) has \( d_E \) non-singular transversal branches at \( p_E \).

Note that \( \Sigma' \) is irreducible since \( \Sigma \) is assumed to be so. Denote by \( d \) the degree of \( \Sigma' \) in \( \mathbb{CP}^2 \). Then the homology class of \( \Sigma \) is \( [\Sigma] = dL - d_EL \) where \( L \) denotes a “line in \( X \)”, i.e. the lift to \( X \) of a generic \( J' \)-holomorphic line in \( X' \). In particular, \( c_1(X) \cdot [\Sigma] = 3d - d_E \).

We assert that \( d_E \leq d \) and the equality holds if and only if \( d_E = d = 1 \). Indeed, perturbing \( \Sigma' \) at \( p_E \) we obtain a nodal symplectic surface with \( \frac{d_E(d_E - 1)}{2} \) new nodes instead the singularity of \( \Sigma' \) at \( p_E \). The genus formula for this perturbation reads

\[
g(\Sigma) = g(\Sigma') = \frac{(d - 1)(d - 2)}{2} - \delta(\Sigma) - \frac{d_E(d_E - 1)}{2}
\]

where \( \delta(\Sigma) \) is the number of nodes of \( \Sigma \). This implies the desired inequality \( d_E \leq d \) and shows that the equality holds in the unique case \( d_E = d = 1 \). This case corresponds to the \( J' \)-holomorphic line in \( X' \) passing through \( p_E \).

Now, the inequality \( d_E \leq d \) together with the formula \( c_1(X) \cdot [\Sigma] = 3d - d_E \) yield the desired inequality \( c_1(X) \cdot [\Sigma] \geq 2 \).

Remark. Observe that as the consequence of the argumentation above we obtain that the equality \( c_1(X) \cdot [\Sigma] = 2 \) holds in the unique case when \( \Sigma \) is an imbedded sphere with trivial normal bundle meeting the exceptional sphere \( E \) at a single point. This means that \( \Sigma \) is a fiber of a \( J \)-holomorphic ruling on \( X \), see [McD-2] and [McD-Sa-2] for details.

Case \( X = S^2 \times S^2 \). In this case \( \mathbb{H}_2(X, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \). We use the “almost complex” geometry of ruled symplectic 4-manifold, see [McD-2] and [McD-Sa-2] for details. It provides the existence of an \( \omega \)-tame almost complex structure \( J \) on \( X \) with the following properties:

- \( \Sigma \subset X \) is a \( J \)-holomorphic curve;
the existence of \( J \)-holomorphic curves \( L_h \) and \( L_v \) which represent the “horizontal” and “vertical” homology classes \([S^2 \times pt]\) and \([pt \times S^2]\), respectively.

It follows that \( [\Sigma] = a[L_h] + b[L_v] \) with non-negative integers \( a = [\Sigma] \cdot [L_v] \) and \( b = [\Sigma] \cdot [L_h] \), and that \( c_1(X) \cdot [\Sigma] = 2a + 2b \). Thus \( c_1(X) \cdot [\Sigma] \geq 2 \), and the equality holds if and only if \( \Sigma \) is either “horizontal” or “vertical” line as above. \( \square \)

Turn back to the proof of Theorem 2.5. Recall that \( J_w \) denotes an \( \omega \)-tame integrable structure such that \((X, J_w)\) is isomorphic to \( \mathbb{CP}^2 \) or \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) or the blown-up \( \mathbb{CP}^2 \) according to the case we have. Find an \( \omega \)-tame almost complex structure \( J_0 \) making \( \Sigma \) a \( J_0 \)-holomorphic curve, denoted by \( C_0 \). Set \( k := c_1(X) \cdot [\Sigma] - 1 \). Fix \( k \) distinct points \( x = (x_1, \ldots, x_k) \) on \( C_0 \). Perturbing \( C_0 \) and the points, we may assume that \( x_1, \ldots, x_k \) are in general position with respect to the structure \( J_w \) in the following sense. For any closed surface \( S \), not necessary connected, the moduli space \( \mathcal{M}_{J_w, x}(S, X, [\Sigma]) \) of \( J_w \)-holomorphic (and hence algebraic) curves of the homology class \([\Sigma]\) with normalization \( S \) passing through \( x \) is either empty or a complex space of the expected dimension.

Fix a generic path \( h(t) \) of \( \omega \)-tame almost complex structures \( J_t := h(t) \) connecting \( J_0 \) with \( J_w = J_1 \). Without loss of generality we may assume that \( J_t \) depend \( C^t \)-smoothly on \( x \in X \) and \( t \) for some \( \ell \gg 0 \). Our hope is to find an isotopy \( C_t \) between \( \Sigma = C_0 \) and a \( J_1 \)-holomorphic curve which consists of \( J_t \)-holomorphic curves. Trying to construct such a family \( C_t \) for maximal possible interval we obtain

**Proposition 2.7.** There exists a \( t^+ > 0 \) which is maximal with respect to the following condition:

- For any \( t < t^+ \) there exists a \( J_t \)-holomorphic curve \( C_t \) such that
  - i) \( C_t \) passes through the fixed points \( x = (x_1, \ldots, x_k) \);
  - ii) \( C_t \) is non-multiple and irreducible;
  - iii) the curve \( C_0 \) is symplectically isotopic to the curve obtained from some maximal nodal deformation \( C'_t \) of \( C_t \) by smoothing an appropriate set of nodes of \( C'_t \).

Let \( t_n \) be an increasing sequence converging to \( t^+ \). Fix \( J_{t_n} \)-holomorphic curves \( C_n \) with these properties. Property iii) implies that the \( C_n \) have the same homology class as \( C_0 \). Going to a subsequence we may assume that they converge to a \( J_{t^+} \)-holomorphic curve \( C^+ \) in the cycle topology.

**Proposition 2.8.** Under the hypotheses of Theorem 2.5, assume that \( C^+ \) has multiple components. Then \( C^+ \) has two irreducible components, \( C' \) of multiplicity 1 and \( L \) of multiplicity 2 such that, according to the case,

1. \( C' \) has genus 2 and \( L \) is an exceptional line, if \( X \) is the blown-up \( \mathbb{CP}^2 \);
2. \( C' \) has genus 3 and \( L \) is a horizontal or vertical line, if \( X \) is \( S^2 \times S^2 \);
3. \( C' \) has genus 4 and \( L \) is a line, if \( X \) is \( \mathbb{CP}^2 \).

Moreover, the curve \( L \cup C' \) is nodal and the marked points \( x \) are disjoint from the nodes of \( L \cup C' \).

The latter condition means that \( L \cup C' \) is in generic position and there are no further degeneration or incidences than those stipulated by the hypotheses of the proposition.

**Proof.** Let \( C^+ = \sum m_i C_i^+ \) be the decomposition of \( C^+ \) into irreducible components with multiplicities \( m_i \). Set \( \mu := c_1(X) \cdot [C^+] = c_1(X) \cdot [\Sigma] \) and \( \mu_i := c_1(X) \cdot [C_i^+] \). Let \( g_i \) be the (geometric) genus of \( C_i \) and \( k_i \) the number of the marked points \( x \) lying on \( C_i \). It follows
then that \( k_i \leq \mu_i + g_i - 1 \). The reason is that otherwise the expected dimension of the space of irreducible curves of genus \( g_i \) in the homology class \([C_i]\) passing through \( k_i \) points is negative; hence the existence of such a constellation would contradict the condition of the generality of \( h(t) \), see e.g. Subsection 2.4 of [11]. Besides, we have the obvious (in)equalities \( \mu = \sum \mu_i, \sum k_i \geq k = \mu - 1 \) and \( g := g(\Sigma) \leq \sum m_i g_i \). Taking into account the inequality \( \mu_i \geq 3, 2, \) or \( 1 \), according to the cases of Lemma 2.6 and distinguishing the case of equality, we see that multiple components are possible only in the cases described in the proposition.

The genericity properties of \( C' \) follows from the condition of the genericity of \( x \) and \( h(t) \). Namely, similarly to the usual holomorphic (and hence algebraic) case, every additional incidence or degeneration condition, such as appearance of a cusp or a triple point, makes the expected dimension of the corresponding constellation negative, which would again contradict the genericity, see [11].

Let us distinguish the cases according to the structure of the curve \( C^+ \).

**Case 1.** \( C^+ \) is irreducible. We claim that \( t^+ = 1 \) in this case. Assuming the contrary it is sufficient to show that for some \( t^{++} > t^+ \) there exists a \( J_{t^{++}} \)-holomorphic curve \( C^{++} \) with the properties given in Proposition 2.7. To do this we fix some parameterization \( u^+: S^+ \to C^+ \subset X \) and consider the relative moduli space \( \mathcal{M}_{h,x}(S^+, X) \) of \( J_t = h(t) \)-holomorphic curves which are parameterized by \( S^+ \), pass through \( x \) and lie in the homology class \([\Sigma]\). This space is non-empty because it contains \( C^+ \). It follows the from the results of [11], especially Subsection 4.5, that for some \( t^{++} > t^+ \) such a curve \( C^{++} \) does exist.

Now, since \( t^+ = 1 \), the structure \( J_{t^+} = J_{st} \), the standard one, and \( C^+ \) is an irreducible algebraic curve in \((X, J_{st})\). Let \( g^+ \) be the geometric genus of \( C^+ \). It follows now from Proposition 2.1 of [11] that every component of the variety \( V([C^+], g^+) \) of irreducible curves of geometric genus \( g \) in \([C^+]\) is of expected dimension \( c_1(X) \cdot [C^+] + g^+ - 1 \) and contains a nodal curve. Consequently, \( C^+ \) can be included in a 1-dimensional family \( \{C_\lambda\} \) whose generic member \( C_\lambda \) is an irreducible nodal curve of the same genus as \( C^+ \). Observe that such \( C_\lambda \) is a maximal nodal deformation of \( C^+ \). Then, smoothing an appropriate set of nodes of \( C_\lambda \), we obtain the desired algebraic curve which is symplectically isotopic to \( C_0 = \Sigma \) by Theorem 2.4. This yields the proof of Theorem 2.5 for the special Case 1 of irreducible \( C^+ \). The existence of the desired smoothing is provided by

**Lemma 2.9.** Let \( X \) be a non-singular complex projective surface and \( C \subset X \) a nodal curve without multiple components such that \( c_1(X) \cdot C_i \) is positive for every irreducible component \( C_i \) of \( C \). Then every prescribed set of nodes of \( C \) can be smoothed by some deformation of \( C \).

**Proof.** Let \( \tilde{C} \) be the normalization of \( C \), \( u: \tilde{C} \to X \) the induced immersion, \( \mathcal{I}_C \) the defining ideal of \( C \subset X \) and \( \mathcal{N}_C := (\mathcal{I}_C / \mathcal{I}_C^2)^* \) the normal sheaf of \( C \). Then there exists a natural projection map \( p: \mathcal{O}_X(TX) \to \mathcal{N}_C \) with the following properties:

- the kernel \( \text{Ker}(p) \) is naturally isomorphic to the sheaf \( \mathcal{O}_C(TC) \) of sections of the tangent bundle of (the normalization of) \( C \);
- the image \( \text{Im}(p) \) is naturally isomorphic to the sheaf \( \mathcal{O}_C(TX/du_x(TC)) \) of sections of the normal bundle \( N_C := TX/du_x(TC) \);
- the cokernel \( \mathcal{N}_C / \text{Im}(p) \) is isomorphic to the sum \( \sum_i \mathcal{O}_{x_i} \) over all nodal points \( x_i \) of \( C \).
More precisely, we construct appropriate sheaves on the normalization $\tilde{C}$ and then push them forward onto $C$ or $X$ by means of $u$.

Now let $x = \{x_1, \ldots, x_k\}$ be some set of nodes of $C$. Denote by $\mathcal{N}_C x$ the sheaf on $C$ which coincides with $\mathcal{N}_C$ at each smooth point of $C$ and each of the nodes $\{x_1, \ldots, x_k\}$, and with the image $\text{Im}(p) = \mathcal{O}_C(N_C)$ at each remaining node. The deformation theory (see e.g. [Pal-1] and [Pal-2]) insures that

- the space of deformations of $C$ which smooth the prescribed nodes $x$ is given by a Kuranishi model $\Phi : B \to H^1(C, \mathcal{N}_C x)$ for some holomorphic map $\Phi$ defined in some ball $B$ in $H^0(C, \mathcal{N}_C x)$;
- the natural projection $H^0(C, \mathcal{N}_C x) \to H^0(C, \sum_{i=1}^k \mathcal{O}_{x_i})$ describes the smoothing of nodes. In particular, a deformation with the tangent vector $v \in H^0(C, \mathcal{N}_C x)$ smooths the node $x_i$ if and only if the projection of $v$ in $H^0(C, \mathcal{O}_{x_i})$ does not vanish.

Let $C_i$ be an irreducible component of $C$, $g_i$ its geometric genus, and $N_{C_i} := TX/du_i(TC_i)$ the corresponding normal bundle. Then $c_1(N_{C_i}) = c_1(X) \cdot [C_i] + (2g_i - 2) > 2g_i - 2$ by the hypothesis of the lemma. Consequently, $H^1(C, \mathcal{O}(N_{C_i})) = 0$ for each single normal bundle. Thus the obstruction group $H^1(C, \mathcal{N}_C x)$ vanishes and every prescribed set of nodes $x$ can be smoothed. \qed

**Case 2.** $C^+$ is reducible but without multiple components. Let $C_i^+$ be the irreducible components of $C^+$. Then the (bi)degree of each $C_i^+$ is strictly less than the (bi)degree of $C^+$. Applying induction, we may assume that the assertion of Theorem 2.2 holds for every $C_i^+$. Moreover, we may also suppose that for $t \in [t^+, 1]$ there exist families $\{C_{i,t}^+\}$ of $J_t$-holomorphic curves with the following properties:

- $C_{i,t}^+ = C_i^+$, i.e. every family $\{C_{i,t}^+\}$ starts from $C_i^+$ at $t^+$;
- for every $t \in [t^+, 1]$ the curve $C_i^+ := \cup C_{i,t}^+$ is nodal;
- smoothing appropriate set of nodes on $C_i^+ = \cup_i C_{i,t}^+$ we obtain a curve which is symplectically isotopic to $C_0 = \Sigma$.

For the final value $t = 1$, the existence of nodal curve $C_i^+ = \cup_i C_{i,1}^+$ with the desired properties follows from Proposition 2.1 of [Ha].

In particular, smoothing appropriate set of nodes on the “final” curve $C_i^+ = \cup_i C_{i,1}^+$ gives the desired algebraic curve which is symplectically isotopic to $C_0 = \Sigma$.

It remains to consider

**Case 3.** $C^+$ has multiple components. Recall that $C^+$ was obtained as the limit of a sequence of $J_{t_n}$-holomorphic curves $C_{t_n}$. To simplify notation, we write $J^+$ instead of $J_{t^+}$, $C_n$ instead of $C_{t_n}$, and $J_n$ instead of $J_{t_n}$.

Notice that the limit $C_n \to C^+$ is understood in the cycle topology. However, we obtain more information about the behavior of $C_n$ near $C^+$ if we take the limit in the stable map topology instead of the cycle one.

For the definition of the stable map topology and related notions in full generality we refer to Section 5 of [Sh] and [v-Sh], as also to [Ha-Mo] and [Pu-Pa] for the algebraic setting. In our setting, the limit object is given by an abstract closed nodal curve $\tilde{C}^+$ equipped with $J^+$-holomorphic map $u^+: \tilde{C}^+ \to X$ which have the following properties:

St1) Let $\{\tilde{C}_\lambda\}$ be any semi-universal family of deformations of $\tilde{C}^+$ such that $\tilde{C}_\lambda^+$ is the curve points $\tilde{C}^+$ itself. Then there exists a sequence of parameters $\lambda_n$ converging to
\( \lambda^+ \) such that, after going to a subsequence, \( \hat{C}_{\lambda n} \) is isomorphic to the normalization \( \hat{C}_n \) of \( C_n \).

**St2)** the image \( u^+ (\hat{C}^+) \), counted with multiplicities, is \( C^+ \).

**St3)** If \( \hat{C}_i^+ \) is a rational irreducible component of \( \hat{C}^+ \) and the number of nodal points on \( \hat{C}_i^+ \), counted with multiplicities, is less than 3, then \( u^+ \) is non-constant on \( \hat{C}_i^+ \).

The first condition means that there we can imbed \( \hat{C}^+ \) and the normalizations \( \hat{C}_n \) in \( X \times \mathbb{CP}^N \), pseudoholomorphic with respect to the structures \( J^+ \times J_{\mathbb{CP}^N} \) and \( J_n \times J_{\mathbb{CP}^N} \) respectively, so that the images \( \hat{C}_n \subset X \times \mathbb{CP}^N \) will converge to \( \hat{C}^+ \subset X \times \mathbb{CP}^N \) in the cycle topology, and so that the projection of these images onto \( X \) gives the sequence \( C_n \) converging to \( C^+ \). In particular, the map \( u^+ \) can be obtained as the projection from \( \hat{C}^+ \subset X \times \mathbb{CP}^N \) onto \( C^+ \subset X \), and the second condition follows. The last condition excludes the appearance of redundant components and insures the uniqueness of the limit in the stable map topology. Observe also that by the second condition the arithmetic genus of \( \hat{C}^+ \) is the geometric genus of \( C_n \).

The crucial point in treating of **Case 3** is study of the deformation problem of the pair \( (\hat{C}^+, u^+) \) in the stable map topology. We start with establishing the possibilities for the structure of \( (\hat{C}^+, u^+) \). Obviously, we must have a component \( \hat{C}' \) mapped by \( u^+ \) onto the component \( C' \) of \( C^+ \) as in **Proposition 2.10**. Denote by \( \hat{C}'' \) the remaining part of \( \hat{C}^+ \).

**Lemma 2.10.** Under the hypotheses of **Theorem 2.3** and **Proposition 2.4**,

i) \( \hat{C}' \) is the normalization of \( C' \) and \( u^+ : \hat{C}' \to C' \) is the normalization map;

ii) there are the following possibilities for the remaining part \( \hat{C}'' \):

(A) \( \hat{C}'' \) consists of two rational components \( \hat{C}_1'' \) and \( \hat{C}_2'' \), each mapped by \( u^+ \) isomorphically onto the line \( L \) and attached to \( \hat{C}' \) at points \( z_1^X, z_2^X \in \hat{C}', \) respectively; the images \( u^+ (z_1^X) \) and \( u^+ (z_2^X) \) are two distinct intersection points of \( C' \) and \( L \).

(B1) \( \hat{C}'' \) is rational and attached to \( \hat{C}' \) at a point \( z_1^X \) whose image \( u^+ (z_1^X) \) is an intersection point of \( C' \) and \( L \); the map \( u^+ : \hat{C}'' \to X \) is a too shifted covering of \( L \subset X \) branched over two distinct points \( y_1, y_2 \in L \).

(B2) \( \hat{C}'' \) consists of two rational components \( \hat{C}_1'' \) and \( \hat{C}_2'' \), each mapped by \( u^+ \) isomorphically onto the line \( L \); \( \hat{C}_1'' \) is attached to \( \hat{C}' \) at a point \( z_1^X \in \hat{C}' \) and \( \hat{C}_2'' \) to \( \hat{C}' \) at a point \( z_2^X \in \hat{C}' \); the image \( u^+ (z_1^X) \) is an intersection point of \( C' \) and \( L \); the image \( u^+ (z_2^X) \) lies on \( L \) apart from \( u^+ (z_1^X) \).

(B3) \( \hat{C}'' \) consists of three rational components \( \hat{C}_0'', \hat{C}_1'', \), and \( \hat{C}_2'' \); \( \hat{C}_0'' \) is attached to \( \hat{C}' \) at a point \( z_0^X \in \hat{C}' \); \( \hat{C}_1'' \) and \( \hat{C}_2'' \) are attached to \( \hat{C}_0'' \) at two distinct points \( z_1^X, z_2^X \in \hat{C}_0'' \), which are distinct also from \( z_0^X \); \( u^+ \) maps \( \hat{C}_1'' \) and \( \hat{C}_2'' \) isomorphically onto \( L \) and \( \hat{C}_0'' \) constantly into the point \( u^+ (z_0^X) \) which is an intersection point of \( C' \) and \( L \).

iii) If \( X \) is the blown up \( \mathbb{CP}^2 \) and \( L \) is an exceptional line, then only case (A) is possible.

**Proof.** The first assertion follows by comparing the geometric genera of \( C' \) and \( \hat{C}' \). The same argument implies that the remaining part \( \hat{C}'' \) must consist of trees of rational curves. Thus \( L \) can be covered either by one or by two distinct rational curves. Elementary combinatorics shows that the cases (A) and (B1–B3) are the only possibilities for such trees of rational curves.

Now assume that \( X \) is the blown up \( \mathbb{CP}^2 \) and the sequence \( C_n \) converges to one of the the constellations (B1–B3). Then we can choose an appropriate piece \( C_n^* \) of each \( C_n \) such that
\( C_n^o \) are connected and the limit of \( C_n^o \) in the cycle topology consists of the exceptional line \( L \) with multiplicity 2 and a disc \( D \) transversal to \( L \). The intersection index of \( C_n^o \) with \( L \) must be \( \left| C_n^o \right| \cdot [L] = [D] \cdot [L] + 2 \cdot [L]^2 = -1 \). Now observe that \( C_n^o \) are holomorphic with respect to structures \( J_n \) converging to \( J^+ \) such that there exists a sequence of \( J_n \)-holomorphic exceptional lines \( L_n \) which converges to the line \( L \). Consequently, \( \left| C_n^o \right| \cdot [L] = \left| C_n^o \right| \cdot [L_n] \geq 0 \). With this contradiction the proof is finished. \( \Box \)

**Remark.** Observe that the constellations (A) and (B3) are rigid i.e. determined by the curve \( C^+ \) and the combinatorics. To the contrary, we obtain moduli in the constellations (B1) and (B2), namely, positions of the branching points \( y_1 \) and \( y_2 \) in the second constellation, and position of the point \( u^+(z_2^+) \) in the third one. The constellation (B1) degenerates in (B2) as \( y_1 \) and \( y_2 \) collapse apart from \( u^+(z_1^+) \), and in (B3) as \( y_1 \) and \( y_2 \) collapse with \( u^+(z_1^+) \). These combinatorial data and varying parameters is the additional information we obtain taking the limit in the stable map topology instead of the cycle one.

Trying to deform \( (\hat{C}^+, u^+) \) in the stable map topology into an irreducible curve we come to the gluing problem for pseudoholomorphic curves. Let us resume the results of Subsection 5.3 of [8] on this topic which we shall use.

**Definition 2.6.** A **pants** \( P \) is a complex curve which can be obtained from \( \mathbb{C}P^1 \) by removing 3 disjoint discs with smooth boundary. **Boundary annuli** in a pants \( P \) are disjoint annuli \( A_1, A_2, A_3 \subset P \) each adjacent to some boundary circle of \( P \).

The **standard smoothing of a node** is the family

\[
\mathcal{A}_\lambda := \{(z_1, z_2) \in \Delta^2 : z_1 \cdot z_2 = \lambda \}
\]

with the parameter \( \lambda \) varying in a disc \( \Delta(\varepsilon) := \{|\lambda| < \varepsilon\} \) of radius \( \varepsilon < 1 \). It deforms the **standard node** \( \mathcal{A}_0 \), consisting of two discs \( \Delta_1 \) and \( \Delta_2 \) with the canonical coordinates \( z_1 \) and \( z_2 \) respectively, into annuli \( \mathcal{A}_\lambda \), \( \lambda \neq 0 \). The **boundary annuli** \( A_1, A_2 \subset \mathcal{A}_\lambda \) are given by

\[
A_1 := \{(z_1, z_2) \in \mathcal{A}_\lambda : 1 - \delta < |z_1| < 1\} \quad A_2 := \{(z_1, z_2) \in \mathcal{A}_\lambda : 1 - \delta < |z_2| < 1\}
\]

with \( \delta < \frac{1-\varepsilon}{2} \). We consider \( A_1 \) and \( A_2 \) with the canonical coordinates \( z_1 \) and \( z_2 \), respectively, as a “constant” part inside deforming curves \( \mathcal{A}_\lambda \).

For an almost complex manifold \( (X, J) \), we denote by \( \mathcal{P}(\mathcal{A}_\lambda, X, J) \) the space of \( J \)-holomorphic maps \( u : \mathcal{A}_\lambda \to X \) which are \( C^1 \)-smooth up to boundary. In the case of \( \mathcal{A}_0 \) such a map \( u : \mathcal{A}_0 \to X \) is given by its components \( u_1 : \Delta_1 \to X \) and \( u_2 : \Delta_2 \to X \), both \( J \)-holomorphic, such that \( u_1(0) = u_2(0) \). For any compact (nodal) curve \( C \) with the smooth boundary \( \partial C \), possibly empty, the space \( \mathcal{P}(C, X, J) \) is defined in a similar way.

**Proposition 2.11.** i) For any compact (nodal) curve \( C \) without closed components the space \( \mathcal{P}(C, X, J) \) has a natural structure of a Banach manifold.

ii) For any given structure \( J^* \), a compact (nodal) curve \( C \) without closed components, and a map \( u^* \in \mathcal{P}(C, X, J^*) \) there exists an open neighborhood \( \mathcal{U} \subset \mathcal{P}(C, X, J^*) \) of \( u^* \) and a map \( G = G(u, J) : \mathcal{U} \to \mathcal{P}(C, X, J) \) depending smoothly on \( u \in \mathcal{U} \) and on a structure \( J \) sufficiently \( C^1 \)-close to \( J^* \) such that, for \( J \) fixed, the map \( G_J : \mathcal{U} \to \mathcal{P}(C, X, J) \) is an open smooth imbedding.

iii) The restriction maps \( R_\lambda : \mathcal{P}(\mathcal{A}_\lambda, X, J) \to \mathcal{P}(A_1, X, J) \times \mathcal{P}(A_2, X, J) \) given by \( R_\lambda(u) := (u|_{A_1}, u|_{A_2}) \) are smooth closed imbeddings.
iv) For any given \(J^*\) and \(u^* \in \mathcal{P}(\mathcal{A}_0, X, J^*)\) there exists an open neighborhood \(\mathcal{U} \subset \mathcal{P}(\mathcal{A}_0, X, J^*)\) of \(u^*\) and a map \(G = G(\lambda, u, J) : \mathcal{U} \to \mathcal{P}(\mathcal{A}_\lambda, X, J)\) defined for \(u \in \mathcal{U}\), \(\lambda\) sufficiently close to 0, and for structures \(J\) sufficiently \(C^1\)-close to \(J^*\), such that:

- \(G\) is continuous in \(\lambda\) and \(C^1\)-smooth in \(u\) and \(J\);
- for \(\lambda\) and \(J\) fixed, the map \(G : \mathcal{U} \to \mathcal{P}(\mathcal{A}_\lambda, X, J)\) is an open \(C^1\)-smooth imbedding.

Moreover, in cases \(\text{iii) and iv)\) the \(C^1\)-smoothness is uniform in \((\lambda, u, J)\).

The meaning of the last part of the proposition is that we can “glue” the components \(u^*_{1,2}\) of any given pseudoholomorphic map \(u^* : \mathcal{A}_0 \to X\) into a pseudoholomorphic map \(u : \mathcal{A}_\lambda \to X\), also varying the almost complex structure. To apply the proposition in our situation we must decompose \(\widehat{C}^+\) into appropriate pieces. For the proof of the following assertion we refer to [IV-SI].

**Proposition 2.12.** There exist a covering \(\{V_a\}\) of \(\widehat{C}^+\) and families of deformations \(V_{a,\lambda_a}\) of some pieces \(V_a\) with the following properties:

- Every piece \(V_a\) is isomorphic to the standard node \(\mathcal{A}_0\), or the disc \(\Delta\), or an annulus \(\mathcal{A}_\lambda\), or a pants.
- Each intersection \(V_a \cap V_b\), if non-empty, is an annulus \(A_{ab}\) which is a boundary annulus for both \(V_a\) and \(V_b\).
- The pieces included in the deformation families are all nodal pieces \(V_a \cong \mathcal{A}_\lambda^+ = 0\) and some annular pieces \(V_a \cong \mathcal{A}_\lambda^{+ \neq 0}\). The deformation family for such a piece \(V_a \cong \mathcal{A}_\lambda^+\) is of the form \(V_{a,\lambda_a} = \mathcal{A}_\lambda\) with \(\lambda_a\) varying in a small neighborhood of \(\lambda^+_a\).
- Let \(\lambda\) be the system of all \(\lambda_a\)’s which appear as the parameter of the families \(V_{a,\lambda_a}\)’s, and let \(\widehat{C}_\lambda\) be the curve obtained by replacing each varying piece \(V_a\) by the piece \(V_{a,\lambda_a}\). Then \(\{\widehat{C}_\lambda\}\) is a semi-universal family of deformations of \(\widehat{C}^+\).

We divide the obtained parameters \(\lambda = (\lambda_1, \ldots, \lambda_l)\) into two groups: \(\lambda'' = (\lambda''_1, \ldots, \lambda''_{l''})\) each describing the smoothing of the corresponding node on \(\widehat{C}^+\), and the remaining \(\lambda' = (\lambda'_1, \ldots, \lambda'_l, l' \neq l)\). Thus we obtain \(\lambda'' = (\lambda''_1, \lambda''_2)\) in the cases (1) and (3) of **Lemma 2.10**, \(\lambda' = (\lambda'_l)\) in the case (2), and \(\lambda'' = (\lambda''_1, \lambda''_2, \lambda''_2)\) in the case (4). Let \(\lambda^+ = (\lambda'^+, \lambda''^+)\) be the set of parameters corresponding to the curve \(\widehat{C}^+\) so that \(\lambda''^+ = 0\).

Using the covering \(\{V_a\}\) we describe the problem of deformation of \((\widehat{C}^+, u^+)\) in terms of compatibility of deformations of single pieces \(V_a\) and the restrictions of \(u^+\) onto \(V_a\)’s. Namely, let us fix a small \(C^1\)-neighborhood \(\mathcal{U}_J\) of \(J^+\), small neighborhoods \(\mathcal{U}_\lambda'\) and \(\mathcal{U}_\lambda''\) of \(\lambda'^+\) and \(\lambda''^+\) in the spaces of parameters \(\lambda'\) and \(\lambda''\), respectively, and, for each \(V_a\), a small neighborhood \(\mathcal{U}_a\) of the restriction \(u_{1,a}^+ := u^+_a|_{V_a}\) in the space \(\mathcal{P}(V_a, X, J^+)\). Consider the map

\[
\mathcal{G} : (\prod_a \mathcal{U}_a) \times \mathcal{U}_\lambda' \times \mathcal{U}_\lambda'' \times \mathcal{U}_J \longrightarrow \prod_{a \neq b} \mathcal{P}(A_{ab}, X, J),
\]

where the product \(\prod_{a \neq b} \mathcal{P}(A_{ab}, X, J)\) is taken over all pairs \((a, b)\) for which the intersection \(V_a \cap V_b\) is a non-empty annulus \(A_{ab}\). For such a pair \((a, b)\), the component \(\mathcal{G}_{ab}\) of \(\mathcal{G}\) is defined as follows. We take the \(a\)-th component \(u_a\) of \(u \in \prod_a \mathcal{U}_a\), compute its deformation \(u'_a := G(u_a, \lambda_a, J)\) or \(u'_a := G(u_a, J)\) according to the type of \(V_a\), the obtained map \(u'_a\) lies in \(\mathcal{P}(V_{a,\lambda_a}, J)\) or \(\mathcal{P}(V_a, J)\), respectively, and then restrict \(u'_a\) onto \(A_{ab}\).

Observe that every annulus \(A_{ab}\) appears twice, as \(V_a \cap V_b\) and as \(V_b \cap V_a\), but the components \(\mathcal{G}_{ab}\) and \(\mathcal{G}_{ba}\) do not coincide in general. Moreover, the set of conditions

\[
\mathcal{G}_{ab}(u, \lambda, J) = \mathcal{G}_{ba}(u, \lambda, J)
\]

for each pair \((a, b)\)
is the compatibility condition on the pieces $G(u_a, \lambda_a, J)$ or $G(u_a, J)$ to be the restrictions on $V_a$ of a well-defined $J$-holomorphic map $u : \hat{C}_i \to X$. Thus, denoting by $\mathcal{D}_J \subset \prod_{a \neq b} \mathcal{P}(A_{ab}, X, J)$ the “diagonal set” given by the set of conditions $u_{ab} = u_{ba}$, we obtain the set-theoretic equality

$$\mathcal{P}(\hat{C}_i, X, J) = \mathcal{G}(\cdot, \lambda, J)^{-1}(\mathcal{D}_J),$$

which holds locally near $(\hat{C}^+, J^+)$. Observe also that $\mathcal{G}$ is only continuous in $\lambda''$ but still $C^1$-smooth in the remaining variables $u$, $\lambda'$, and $J$.

**Lemma 2.13.** Let $\lambda^*, J^*$, and $u^* \in \mathcal{P}(\hat{C}_i^*, X, J^*)$ be close to $\lambda^+, J^+$, and $u^+$ respectively. Set $u^*_a := u^*|_{V_a, \lambda_a^*}$ and $u^* := (u^*_a) \in \prod_a \mathcal{G}_a$.

Then the map $\mathcal{G}(\cdot, \lambda^*, \lambda''^*, J^*)$, with the arguments $u$ and $\lambda'$ varying and $\lambda''^*$ and $J^*$ fixed, is transversal to the submanifold $\mathcal{D}_J$ at the point $(u^*, \lambda^*, J^*)$.

**Proof.** The transversality means that the image of differential of the map $\mathcal{G}(\cdot, \lambda^*, \lambda''^*, J^*)$ at the point $(u^*, \lambda^*, J^*)$ is the compatibility condition on the pieces $G(\cdot, \lambda_a, J)$ between Banach spaces remains surjective after a small perturbation. We compute the compatibility condition on the pieces $G(\cdot, \lambda_a, J)$ by trees of rational curves. 

We may assume that $\lambda^* = \lambda^+$, $\hat{C}^* = \hat{C}^+$, $J^* = J^+$, and $u^* = u^+$. The general case follows from this special one by the following argument. A surjective linear Fredholm map between Banach spaces remains surjective after a small perturbation. We compute the deformation problem in two steps as follows: first, we consider the deformation problems for each component $\hat{C}'$ and $\hat{C}'_i$ of $\hat{C}^+$, and then impose the conditions of “attaching”.

**Step 1.** Observe that the parameters $\lambda'$ parameterize a complete family of deformations of $C'$. This follows from the fact that $\hat{C}^+$ differs from $C'$ by trees of rational curves. Consequently, the map $\mathcal{G}(\cdot, \lambda^*, \lambda''^*, J^*)$ describes the problem of deformation of $C'$ as a parameterized $J^+$-holomorphic curve of the given geometric genus $g = g(C')$. Observe also that the curve $C'$ is immersed and $c_1(X) \cdot [C'] > 0$. These two conditions imply that the deformation problem is unobstructed, see e.g. [H-L-S] or [S], Section 2. The same argument applies for the components $\hat{C}'_i$.

**Step 2.** After solving the problems of the first step, we obtain local deformations families of $J^+$-holomorphic maps: $u_{s'} : S \to X$, defined on a closed real surface $S$ of genus $g = g(C')$, and $u_{s'_i} : S^2 \to X$, one for each component $\hat{C}'_i$ and $\hat{C}'_i$, respectively. To fit together in a map of a connected curve $\hat{C}_X, \lambda''^+$, the maps $u_{s'}$ and $u_{s'_i}$ must satisfy certain “attaching conditions” defined as follows. Each nodal point $z^+_i$ on $\hat{C}_X, \lambda''^+$ has two pre-images on the components $\hat{C}_X$ and $\hat{C}'_i$, say $z^+_i$ and $z^-_i$, and the images of these points in $X$ must coincide. The transversality of this “attaching problem” is equivalent to the original transversality. For this purpose possibility to move arbitrarily the image of one of the points $z^+_i$ and $z^-_i$ is sufficient. The latter condition is equivalent to the transversality of the problems of deformations of the curves $C''$ and $\hat{C}'_i$ constrained by the condition of passing through given points.

We contend that this new deformation problem is unobstructed. Let us consider the special case when the curve $\hat{C}^+$ is as in the case (A) of Lemma 2.10 and $L$ is an exceptional line. In this case the component $C''$ of $C^+$ meets $L$ at two points at least. This implies that $c_1(X) \cdot [C''] > 3$ since otherwise $C''$ would meet $L$ at a single point, see the remark in
the proof of Lemma 2.6. Now, since $C'$ is immersed and $c_1(X) \cdot [C']$ is strictly larger than the number $k = 2$ of the constraining points, the problem of deformation of $C'$ constrained at $k = 2$ points is unobstructed. This yields the desired transversality for the special case we consider.

The other cases can be treated similarly. □

As a corollary of Lemma 2.13 we obtain the local symplectic isotopy in a neighborhood of $C^+$. 

**Corollary 2.14.** i) Let $(\hat{C}^+, u^+)$ be as in the case (A) of Lemma 2.10 and $(\hat{C}_0, u_0)$, $(\hat{C}_1, u_1)$ two small deformations of $(\hat{C}^+, u^+)$ in the stable map topology, such that $C_i := u_i(\hat{C}_i)$ are irreducible and nodal. Then there exists a symplectic isotopy $C_t$ between $C_0$ and $C_1$ close to $C^+$ in the cycle topology.

ii) Let $(\hat{C}_0^+, u_0^+)$ and $(\hat{C}_1^+, u_1^+)$ be as in the cases (B1–3) of Lemma 2.10 and $(\hat{C}_0, u_0)$, $(\hat{C}_1, u_1)$ two small deformations of $(\hat{C}_i^+, u_i^+)$ in the stable topology, $i = 0, 1$ respectively. Assume that $C_i := u_i(\hat{C}_i)$ are irreducible and nodal. Then there exists a symplectic isotopy $C_t$ between $C_0$ and $C_1$ close to $C^+$ in the cycle topology.

Observe that the almost complex structure can also vary.

**Proof.** Let $J^*$ be a structure close to $J^+$. Set $\mathcal{M}_{J^*} := \bigcup \mathcal{M}_{\mathcal{J}^*}(\cdot, \cdot, X^*, J^*)^{-1}(\mathcal{J}_*)$ and let $\mathcal{M}_{J^*}^{\text{sing}}$ be the set of parameters $(u, \lambda, J^*) \in \mathcal{M}_{J^*}$ where $u(\hat{C}_\lambda)$ is not nodal and irreducible. It follows from Lemma 2.13 that $\mathcal{M}_{J^*}$ is a topological manifold in a neighborhood of $(u^+, \lambda^+, J^*)$ and that $\mathcal{M}_{J^*}^{\text{sing}}$ has Hausdorff codimension $\geq 2$ in $\mathcal{M}_{J^*}$. This fact and Lemma 2.13 imply part i) of the corollary.

For part ii) we use an additional possibility to connect $(\hat{C}_0^+, u_0^+)$ and $(\hat{C}_1^+, u_1^+)$ by a path $(\hat{C}_t^+, u_t^+)$ continuous in the stable map topology such that $u_t^+(\hat{C}_t^+)$ is constantly $C^+$. □

Now we are ready to finish

**Proof of Theorem 2.7.** Recall that it remains to consider the following situation: There exists a sequence $C_n$ of pseudoholomorphic nodal curves such that each $C_n$ is symplectically isotopic to $\Sigma$ and such that there exists the limit $(\hat{C}^+, u^+)$ of $C_n$ in the stable map topology. Furthermore, the possibilities for the structure of $(\hat{C}^+, u^+)$ are given by Lemma 2.10, Lemma 2.13 and Corollary 2.14 insures the possibility of restoration of the symplectic isotopy class of $\Sigma$ by $C^+ = u^+(C^+)$ and the combinatorial data.

The scheme of the proof is the same as before: First, we show that there exists a symplectic isotopy $C^+_t$ between $C^+_0 := C^+$ and a holomorphic curve $C^+_1$, and then deform $C^+_1$ into a holomorphic nodal curve in the symplectic isotopy class of $\Sigma$.

Proving the existence of the desired symplectic isotopy $C^+_t$ we apply the induction in the “anti-canonical degree”. Namely, by Lemma 2.6 we have $c_1(X) \cdot [C'] < c_1(X) \cdot [\Sigma]$ for the component $C'$. Thus there exists a symplectic isotopy $C'_t$ between $C' = C^'_0$ and a holomorphic curve $C'_1$. The existence of a similar symplectic isotopy for $L$ is follows directly from the following fact: For a generic path of tame structures $J_t$ and a generic choice of points $x_1, \ldots, x_k$ with $k := c_1(X) \cdot [L] - 1$ there exists a unique path $L_t$ formed by $J_t$-holomorphic curves in the homology class $L$. This fact was exploited by several authors, see e.g. [Bar]. It follows then that both isotopies $C'_t$ and $L_t$ can be made $J_t$-holomorphic for the same path of tamed structures $J_t$. Then for a generic choice of isotopies $C^+_t$ and $L_t$ the curves $C^+_t := C^+_t \cup L_t$ will form the desired symplectic isotopy.
The combinatorial data are translated along the path $C^t$ onto the obtained holomorphic curve $C^+_1$. Since Lemma 2.13 and Corollary 2.14 hold also for the structure $J^t_1$, we can deform $C^+_1$ into a nodal $J^t$-holomorphic curve in the symplectic isotopy class of $\Sigma$. Theorem 2.5 follows. \hfill \square

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