Monte-Carlo model for nonlinear interactions of Alfvén eigenmodes with energetic ions

S Tholerus and T Hellsten
Division of Fusion Plasma Physics, KTH School of Electrical Engineering, SE-100 44 Stockholm, Sweden
E-mail: simon.tholerus@ee.kth.se

Abstract. A Monte-Carlo model for interactions between a single Alfvén eigenmode and energetic ions in a tokamak is presented. A phenomenological decorrelation of the wave-particle phase is introduced to mimic decorrelation by collisions or by other waves. Analysis is dedicated to how the strength of the phase decorrelation affects the nonlinear wave-particle interactions. Several of the phenomena that have been observed in some earlier models describing the nonlinear dynamics of Alfvén eigenmodes have been verified, such as the growth and saturation of the wave mode amplitude giving rise to a localized flattening of the distribution function, as well as the generation of coherent structures in the distribution function. The degree of phase decorrelation is shown to have a strong effect on the dynamics of the Alfvén eigenmode excitation.

1. Introduction
Populations of energetic ions are routinely produced in present day experiments on magnetic confinement fusion, e.g. by fusion nuclear reactions, ion cyclotron resonant heating (ICRH) and heating by neutral beam injection (NBI). Shear Alfvén waves, such as the toroidal Alfvén eigenmodes (TAEs), can be excited \[1, 2\] by these energetic and resonant ion populations, to such an extent that they eject a large amount of resonant ions from the plasma, resulting in a significant reduction of heating efficiency \[3–5\]. Theoretical understanding of the nonlinear dynamics of such waves is of importance for the development of viable fusion power plants.

Models successfully describing the excitation of TAEs have been developed earlier. Major contributions to the field of TAE dynamics have been based on the model by Berk and Breizman \[6–8\], with contributions also made by Pekker \[9\]. Their model is constructed such that it takes into account the lowest order contributions to the nonlinear wave growth rate. It has been shown that systems with marginal linear instability of the wave growth rate give a variety of wave amplitude dynamics, including regimes of stable saturation, periodic and chaotic modulation, and explosive amplitude evolution \[9–12\]. In particular, it was found that when dynamical friction (drag) is much larger than velocity space diffusion of particles the mode grows explosively \[11, 12\]. Other important results include the creation of phase space hole and clump pairs around the wave resonance \[12–14\].

A hybrid magnetohydrodynamic-gyrokinetic code (HMGC) \[15–17\] was developed by Vlad et al to simulate linear and nonlinear dynamics of Alfvén eigenmodes. It was shown that the growth rate of energetic particle continuum modes (EPMs) \[18\], with frequencies dependent on characteristic energetic particle motion, increases rapidly with increasing energetic particle
pressure gradient above a certain threshold of the pressure gradient. It was also shown that the saturation of EPMs was caused by an outward displacement of energetic ions, which could have significant consequences on the energetic ion confinement.

The HAGIS (HAMiltonian Guiding centre System) code [19], developed by Pinches et al, is a gyro-kinetic code using a δf method based on Hamiltonian formalism to simulate nonlinear interactions of an arbitrary distribution of energetic ions and a set of discrete Alfvén eigenmodes, which are allowed to vary both in amplitude and phase. In the limit of marginally unstable TAEs the code has shown the existence of frequency sweeping events with a sweeping rate in quantitative agreement with theory [13, 20].

A Monte-Carlo model in the quasilinear regime, developed by Bergkvist et al [21], has described the excitation of TAEs by superthermal ions and renewal of the distribution function by ICRH. The model gave the correct magnitude of the eigenmode frequency side bands [22] by ICRH and the time scale of the rapid damping of modes as the ICRH was instantly turned off.

In this paper, a one dimensional Monte-Carlo model for the nonlinear interaction between an oscillating wave field and an ensemble of charged particles is developed. The model attempts to mimic the scenario of a fusion plasma confined in a tokamak, in which a set of energetic ions interacts with a discrete Alfvén eigenmode, such as a TAE. The model is constructed ad hoc to include basic dynamics necessary for the nonlinear excitation of a wave mode, describing phenomena such as the growth and saturation of a wave mode, and the generation of coherent structures in the distribution function. One of the major issues to be treated in this paper is the regime of partial phase decorrelation, i.e., regimes in between those of complete correlation and quasilinear theory. The presented one dimensional model will be used as a basis for the construction of a more complete model, not presented in this paper.

2. Wave-particle interaction model

The interaction of energetic particles with a single discrete wave mode is considered, with particle dynamics treated along one dimension. The component of the wave mode along this dimension has the form

$$E(\phi, t) = \hat{E}(t) \cos(n\phi - \omega_w t - \phi_0(t)) = E_1(t) \cos(n\phi - \omega_w t) + E_2(t) \sin(n\phi - \omega_w t). \quad (1)$$

The amplitudes $E_1(t)$ and $E_2(t)$ are reparameterizations of the two quantities $\hat{E}(t)$ and $\phi_0(t)$. The Lorentz force equation for particle interaction with this mode is

$$\frac{d\phi_i}{dt} = \frac{q}{m_i} \left[ E_1(t) \cos(n\phi_i - \omega_w t) + E_2(t) \sin(n\phi_i - \omega_w t) \right], \quad (2)$$

where $v_i$ and $\phi_i$ are the velocity and phase of particle $i$.

Interactions with magnetically trapped particles\(^1\) are considered in this paper. Note that passing particles in resonance with the wave field may as well excite TAEs. The restriction of the study to trapped particles is motivated partly by simplicity and partly by the fact that trapped particles in the MeV range are more likely to be resonant with TAEs. For this reason the conceptual study of trapped particle interactions with TAEs may be of interest.

The velocity and phase ($v_i$ and $\phi_i$) are the parallel velocity and toroidal phase, respectively, at some particular angle of the trapped orbit. A good approximation for magnetically trapped particles is that the precession frequency is linearly dependent on the particle kinetic energy. A first order expansion around the wave mode resonance would be

$$\frac{d\phi_i}{dt} = \omega_i = \frac{\omega_w}{n} + \frac{\partial \omega}{\partial W} \left( \frac{mv_i^2}{2} - W_{\text{res}} \right). \quad (3)$$

\(^1\) Trapped particles can refer to two different phenomena in this paper. Either the particles are toroidally trapped by the equilibrium magnetic field (referred to as magnetically trapped particles) or they may be trapped by the field of the wave mode (wave field trapped particles).
where $\partial\omega/\partial W$ is constant, and $W_{\text{res}} = mv_i^2/2$ is the resonant energy. Note that $W_{\text{res}}\partial\omega/\partial W \neq \omega_w/n$ can be a most valid assumption locally around the resonance $mv_i^2/2 = W_{\text{res}}$. This assumption becomes nonphysical close to zero kinetic energy, though, since $v_i = 0$ then results in $d\phi_i/dt \neq 0$. This fact should be kept in mind by ensuring that the wave mode only interacts with particles close to the resonance.

Using the coordinate substitution $\phi_i \rightarrow \phi_i + \omega_w t/n$ and $\omega_i \rightarrow \omega_i + \omega_w/n$ simplifies the problem. Equations (2) and (3) then become

$$\frac{dv_i}{dt} = \frac{q}{m} \left[ E_1(t) \cos(n\phi_i) + E_2(t) \sin(n\phi_i) \right],$$

$$\frac{d\phi_i}{dt} = \omega_i = \frac{\partial\omega}{\partial W} \left( \frac{mv_i^2}{2} - W_{\text{res}} \right).$$

It should be noted that the actual value of $\omega_w$ does not influence this particular problem, since the dynamics of the particles only depend on their relative phase velocity compared to the wave. The coefficient $\partial\omega/\partial W$ depends on the characteristics of the magnetically trapped orbit.

Considering that the resonant interacting particles are localized in the same region of phase space, the assumption that $\partial\omega/\partial W$ is constant and the same for all particles is valid.

The total wave mode energy is assumed to be of the form

$$W_w = \epsilon \langle E_2^2 \rangle_{\phi} = \frac{\epsilon (E_1^2 + E_2^2)}{2},$$

with $\epsilon$ being a proportionality factor depending on the characteristics of the wave mode and on plasma parameters. Energy balance for wave-particle interaction yields

$$\frac{dE_1}{dt} = -\frac{q}{\epsilon} \sum_{i}^{N_0} v_i \cos(n\phi_i) - \gamma_d E_1 \approx -\frac{qs}{\epsilon} \sum_{i}^{N_0/s} v_i \cos(n\phi_i) - \gamma_d E_1,$$

$$\frac{dE_2}{dt} = -\frac{q}{\epsilon} \sum_{i}^{N_0} v_i \sin(n\phi_i) - \gamma_d E_2 \approx -\frac{qs}{\epsilon} \sum_{i}^{N_0/s} v_i \sin(n\phi_i) - \gamma_d E_2,$$

where the contributions to the orthogonal components of the wave mode have been separated. The parameter $\gamma_d$ is the wave field net damping rate, which can be used e.g. to model interactions with a thermal background of particles. The parameter $N_0$ can be considered as the physical number of particles in the system, whereas $s$ (typically $\gg 1$) is a resolution parameter that can be used to simulate $N_p \equiv N_0/s$ macroparticles. The ratio $\epsilon/s$ is a parameter quantifying the interaction strength between the wave mode and the macroparticles.

An important property of the model is the inclusion of a phenomenological decorrelation of the wave-particle phase, which mimics decorrelation by collisions between particles or by interactions with other waves. Mathematically, decorrelation is modeled using a Monte-Carlo approach. Specifically, it is introduced as a Wiener process in the toroidal angular position of the particles according to

$$\Delta\phi_i = \frac{\partial\omega}{\partial W} \left( \frac{mv_i^2}{2} - W_{\text{res}} \right) \Delta t + \pi \zeta_i \sqrt{\frac{\Delta t}{t_c}},$$

where $\zeta_i$ is a random variable of unit variance, and $t_c$ is the phase decorrelation time. In numerical simulations, the differentials $\Delta\phi$ and $\Delta t$ are small compared to macroscopic phase and time scales, but finite. Partial wave-particle phase decorrelation is defined by the condition
Figure 1. Phase space trajectories for trapped particles, with $n = 1$, $\phi_0 = 0$, and $q > 0$. The separatrices are marked with red curves.

$t \lesssim t_c$ for a finite $t_c$. The stochastic phase variation will in turn give rise to velocity space diffusion via interactions with the wave mode. Quasilinear theory is valid in the limit of $t_c$ much less than any other macroscopic time scale of the system. Variation of $t_c$ allows complete study of the transition between the correlated and the quasilinear regimes.

In the regime of a quasi-stationary wave, i.e., in the limit where the wave amplitude and phase are in principle unaffected by the transfer of energy from the particles, the dynamics of the particle interactions with the wave mode are similar to that of a pendulum interacting with the gravitational field. The parameter

$$\lambda \equiv 2 m \sqrt{\frac{|q| E_{\text{norm}}}{n v_{\text{res}}^2} \frac{\partial \omega}{\partial W}},$$

(10)

quantifies the width of the potential well where particles are trapped by the wave field, and $E_{\text{norm}} \equiv \lambda^2$, referred to as the normalized field strength, is a dimensionless quantification of the wave field amplitude. For $\lambda \ll 1$ the parameter is approximately the separatrix width in $\hat{v}$ space (see figure 1.a), where $\hat{v} \equiv v/v_{\text{res}}$ is the normalized velocity. If, on the other hand, $\lambda \gtrsim 1$ the phase space trajectories become asymmetric around the resonance, as can be seen in figure 1.b. In this regime, almost all closed phase space orbits pass through a region of $v \approx 0$. Hence, the model becomes invalid in this limit. The parameters have to be set such that the wave never grows to such large amplitudes. Another characteristic quantity is

$$\omega_B = \sqrt{n v_{\text{res}} |q| E_{\text{norm}} \frac{\partial \omega}{\partial W}},$$

(11)

which is identified as the bounce frequency for particles deeply trapped by the wave field.

The values of $\lambda$ and $\omega_B$ change as $\dot{E}$ varies with time. A useful parameter to derive would be the time scale at which $\dot{E}$ varies initially. Consider starting from a distribution function

$$f(\phi, v, t = 0) = \frac{1}{2\pi} F(v),$$

(12)

where $F(v)$ is the initial distribution of macroparticles in velocity space. Since the distribution function is initially flat and uncorrelated in phase $\phi$, $d\dot{E}/dt = 0$ to first order in time. Specifically,

$$\Delta \dot{E} = -\frac{q^2 N_0 E \Delta t^2}{em} + O(\Delta t^3),$$

(13)
where \( N_0 = s \int dv F(v) \) is the total number of energetic particles. The time scale for field amplitude variation can be identified as

\[
t_E = \frac{1}{q|q|} \sqrt{\frac{em}{N_0}}.
\]  

(14)

The inclusion of a collision operator would modify the second order time scale, and a phase dependent distribution function would give a nonzero first order variation of \( \dot{E} \). Equation (13) implies an initial decrease of the amplitude. However, if a considerably non-flat phase distribution is built up during time scales \( \lesssim t_E \), the amplitude may instead increase, initially.

3. Dynamics of the wave-particle interactions

3.1. Amplitude saturation

Assuming an initial distribution function \( f_{\text{init}}(\phi, v) \) inverted in velocity space (\( \text{sgn}(v) \frac{df_{\text{init}}}{dv} > 0 \)) the wave mode is expected to saturate at a level corresponding to the energy gain by a local flattening of the distribution function within the separatrices of the wave. In the limit of low amplitude (\( \lambda \ll 1 \)), the theoretical saturation level is

\[
E_{\text{norm}}^{(\text{sat})} = \left( \frac{16}{3} \frac{q}{nv_{\text{res}}} \left[ \frac{\partial \omega}{\partial W} \right]^{-1} \sqrt{\frac{2}{\epsilon m^3}} \frac{\partial f_{\text{init}}}{\partial v} \right)^4.
\]  

(15)

This expression assumes that \( \frac{\partial f_{\text{init}}}{\partial v} \) is constant within the trapped region of the saturated wave mode, and that \( \frac{\partial f_{\text{init}}}{\partial \phi} = 0 \). A simulation of the nonlinear growth and saturation of the wave mode amplitude using our model compared with the theoretical saturation level is shown in figure 2.a, indicating a good agreement between the numerical and theoretical saturation levels, given by (15).

From the assumption of a local flattening of the distribution function the final velocity distribution function will be on the form

\[
F_{\text{final}}(v) = \begin{cases} 
F_0 + \frac{dF}{dv}(v - v_{\text{res}}) & \text{for } |v - v_{\text{res}}| > \lambda v_{\text{res}} \\
F_0 + \frac{dF}{dv}(v - v_{\text{res}}) \times \frac{2}{\pi} \sin^{-1} \left( \frac{|v - v_{\text{res}}|}{\lambda v_{\text{res}}} \right) & \text{for } |v - v_{\text{res}}| \leq \lambda v_{\text{res}}
\end{cases}
\]  

(16)

**Figure 2.** a) Saturation of a wave mode with the corresponding theoretical prediction \( (\gamma_d = 0) \). b) A comparison between the theoretical and the numerical distribution function after wave saturation \( (t = 16t_E) \).
Figure 3. a and b) The time evolution of $E_{\text{norm}}$ for different values of $t_c$. c – e) The time evolution of the distribution function in velocity space for c) $t_c = 32t_E$, d) $t_c = 8t_E$, e) $t_c = t_E/2$. The solid vertical lines show the separatrix width at $t = 16t_E$.

where $F(v)$ is the integral of $f(\phi,v)$ over phase $\phi$. A comparison of the model equation (16) and the numerical distribution function after saturation was reached is shown in figure 2.b. Again, a good agreement with the model can be observed. Deviations from the predicted distribution function may come from asymmetries of the phase space trajectories (see figure 1), as well as from fluctuations due to discrete particle effects.

When including decorrelation of the wave-particle phase the saturation level and saturation time changes as can be seen in figure 3.a. Three additional regimes can be observed for different degrees of decorrelation, referred to as the weakly, intermediately and strongly decorrelated regime. In figure 3, $t_E\omega_B = 40\sqrt{E_{\text{norm}}} \sim 1$, implying that time scales for evolution of particles in phase space and for wave dynamics are similar.

In the weakly decorrelated regime, characterized by finite $t_c \gg t_E$, the amplitude grows until saturation at time scales much longer than in the correlated regime ($t_c = \infty$), and to a higher level of saturation. Amplitude oscillations around the saturated level are also damped by decorrelation. In the intermediately decorrelated regime, approximately identified as $2t_E \leq t_c \leq 8t_E$, both the saturation level and saturation time are rather insensitive to variations of $t_c$, and the saturation time is similar to that in the correlated regime. It can be viewed as a transition regime between the weakly and strongly decorrelated regime. The strongly decorrelated regime can be found below a relatively sharp limit of $t_c \approx t_E$. It can be identified as the quasilinear regime if $t_c$ is much less than any macroscopic time scale. Here, the growth of the wave mode amplitude becomes strongly damped on investigated time scales as compared to less decorrelated regimes. The mode will eventually grow and saturate even in the strongly decorrelated regime, but on much longer time scales ($t \sim 50t_E$ for $t_c = t_E/2$ and $t \sim 500t_E$ for $t_c = t_E/4$). However, there is an upper time limit for the validity of the model due to the neglect of e.g. dissipative mechanisms.

More can be said about initial growth rates from figure 3.b. They are similar for $t_c \geq 8t_E$, and then decrease for increasing decorrelation until $t_c \leq t_E/2$, where they become more or less independent of $t_c$. Simulations with the SELFO code in quasilinear regime have demonstrated
an initially exponential growth of the mode in time [21]. However, statistical fluctuations are too large to tell whether the initial growth is exponential or not in figure 3.b for $t_c \leq t_e/2$.

Via interactions with the wave the wave-particle phase decorrelation translates into diffusion in velocity space, as seen in figure 3.c and d. The velocity diffusion explains the higher saturation levels in the weakly and intermediately decorrelated regimes, since it allows a wider range in the velocity distribution to become flattened. The final saturation level depends on the initial distribution function of the entire population of energetic particles, rather than on the distribution locally around the resonance. The velocity space diffusion increases for increasing wave-particle phase decorrelation, until a certain limit. In figure 3.e one can see that the diffusion almost vanishes at the investigated time scales for $t_c = t_e/2$. As the mode grows on longer time scales velocity diffusion will eventually kick in even in the strongly decorrelated regime.

3.2. Generation of coherent structures
When wave-particle phase decorrelations are negligible, coherent structures in phase space may appear. Formation of such structures strongly affects the dynamics of the wave mode amplitude. The irregular structures in figure 4.a are possible when the time scales of particle evolution in phase space (approximately the inverse bounce frequency of the particles deeply trapped by the wave field) are similar to the time scales of wave amplitude and phase evolution.

The growth of the mode is a nonlinear effect, requiring an initial build-up of phase correlation. In the strongly decorrelated regime the necessary build-up of correlation cannot occur. Velocity space diffusion is also a process that can transfer energy to the wave mode, since it results in a flattening of the velocity distribution. At low mode amplitudes the wave-particle phase decorrelation and the velocity diffusion are decoupled. Thus, velocity space diffusion is also damped in the strongly decorrelated regime.

In the intermediately decorrelated regime, where not all phase correlations are immediately smeared out, the wave interactions with the particles also give rise to diffusion in velocity space on time scales $t \gtrsim t_c$. This can be seen in figure 4.b. Even though the modes grow to almost the same level in figures 4.a and b, there are different physics behind the growth.

4. Conclusions and discussion
The presented model successfully describes several phenomena regarding the nonlinear interactions of Alfvén eigenmodes with energetic ions in a tokamak, such as the nonlinear growth
and saturation of the wave mode, due to a local flattening of an inverted distribution function in the region of phase space where particles are trapped by the wave field. The discrete particle picture that has been used allows the study of nonlinear structures in the distribution function.

The dynamics of the system are fundamentally different for different degrees of wave-particle phase decorrelation. In addition to the fully correlated regime, three regimes have been observed: the weakly, the intermediated, and the strongly decorrelated regime. In the weakly decorrelated regime the mode amplitude grows until saturation at time scales larger than in the fully correlated regime, although initial growth rates are similar. The intermediatedly decorrelated regime is a transition regime where velocity space diffusion is the largest. In this regime the initial growth rate decreases for increasing degree of decorrelation, although other dynamics of the wave mode are fairly unchanged. Velocity space diffusion may increase the saturation level of the mode depending on the initial distribution function. In the strongly decorrelated regime, both the generation of coherent structures and velocity space diffusion are suppressed relative to the more weakly decorrelated regimes, which heavily damps the nonlinear growth of the wave mode.

The purpose of the presented model was to investigate the basic principles needed to successfully describe the nonlinear dynamics of TAEs that include decorrelation of the wave-particle phase. The investigation was a first step in the development of a new model with a more complete description of the wave-particle interactions, e.g. by considering ICRF- or NBI-driven scenarios and dissipative mechanisms, such as wave damping and particle drag. Other possible generalizations are to consider a 3D toroidal plasma and more advanced, time dependent gyro center orbit geometries. Collisions can then be modeled e.g. as a decorrelation in the pitch angle.

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