A GLOBAL REGULARITY RESULT FOR THE 2D BOUSSINESQ EQUATIONS WITH CRITICAL DISSIPATION

ATANAS STEFANOV\textsuperscript{1} AND JIAHONG WU\textsuperscript{2}

Abstract. This paper examines the global regularity problem on the two-dimensional incompressible Boussinesq equations with fractional dissipation, given by $\Lambda^\alpha u$ in the velocity equation and by $\Lambda^\beta \theta$ in the temperature equation, where $\Lambda = \sqrt{-\Delta}$ denotes the Zygmund operator. We establish the global existence and smoothness of classical solutions when $(\alpha, \beta)$ is in the critical range: 

$$\alpha > \frac{\sqrt{1777} - 23}{24} \approx 0.798103, \quad \beta > 0 \quad \text{and} \quad \alpha + \beta = 1.$$ 

This result improves the previous work of [25] which obtained the global regularity for $\alpha > \frac{23 - \sqrt{145}}{12} \approx 0.9132, \beta > 0 \quad \text{and} \quad \alpha + \beta = 1$.

1. Introduction

This paper aims at the global regularity issue on the two-dimensional (2D) Boussinesq equations with fractional dissipation

$$\begin{cases}
\partial_t u + u \cdot \nabla u + \nu \Lambda^\alpha u = -\nabla p + \theta e_2, & x \in \mathbb{R}^2, \ t > 0, \\
\nabla \cdot u = 0, & x \in \mathbb{R}^2, \ t > 0, \\
\partial_t \theta + u \cdot \nabla \theta + \kappa \Lambda^\beta \theta = 0, & x \in \mathbb{R}^2, \ t > 0, \\
u(u(x, 0) = u_0(x), \ \theta(x, 0) = \theta_0(x), & x \in \mathbb{R}^2,
\end{cases}
(1.1)$$

where $u = u(x, t)$ denotes the 2D velocity, $p = p(x, t)$ the pressure, $\theta = \theta(x, t)$ the temperature, $e_2$ the unit vector in the vertical direction, and $\nu > 0, \ \kappa > 0, \ 0 < \alpha \leq 2$ and $0 < \beta \leq 2$ are real parameters. Here $\Lambda = \sqrt{-\Delta}$ represents the Zygmund operator with $\Lambda^\alpha$ being defined through the Fourier transform,

$$\hat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi),$$

where the Fourier transform is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x) \, dx.$$When $\alpha = \beta = 2$, (1.1) reduces to the standard 2D Boussinesq equations with Laplacian dissipation. The standard Boussinesq equations model geophysical flows such as atmospheric fronts and oceanic circulation, and also play an important role in the study of Raleigh-Bernard convection (see, e.g., [11, 18, 32, 37, 42, 43]).

Mathematically the 2D Boussinesq equations serve as a lower dimensional model of the 3D hydrodynamics equations. In fact, the 2D Boussinesq equations retain some key features of the 3D Navier-Stokes and the Euler equations such as the vortex stretching mechanism. As pointed out in [33], the inviscid Boussinesq equations can be identified

\textit{2010 Mathematics Subject Classification.} 35Q35, 35B65, 76B03.

\textit{Key words and phrases.} Boussinesq equations, fractional dissipation, global well-posedness.
with the 3D Euler equations for axisymmetric flows. It is hoped that the study on the 2D Boussinesq equations will shed light on the mysterious global existence and smoothness problem on the 3D Navier-Stokes and Euler equations.

One main pursuit in the study of (1.1) has been to obtain the global regularity of its solutions for the smallest $\alpha$ and $\beta$. Intuitively, the smaller $\alpha$ and $\beta$ are, the harder the global regularity problem is. When there is no dissipation, namely $\nu = \kappa = 0$ in (1.1), the global regularity problem remains open. The standard idea of proving the global a priori bounds in Sobolev spaces fails. Potential finite time singularities have been explored from different perspectives including boundary effects and 1D models [31, 9, 8, 39].

At the other extremum, when $\nu > 0$, $\kappa > 0$, $\alpha = \beta = 2$, the global regularity can be easily obtained, in a similar fashion as for the 2D Navier-Stokes equations [17, 33]). It is natural to examine (1.1) with intermediate dissipation, which has attracted considerable attention in the last few years [1, 2, 3, 6, 7, 12, 24, 14, 15, 16, 20, 21, 22, 23, 25, 26, 27, 29, 30, 33, 36, 41, 45, 47, 19, 50]. The work of Chae [7] and of Hou and Li [23] shows that one full Laplacian dissipation in (1.1) is sufficient for the global regularity. More precisely, (1.1) with $\alpha = 2$ and $\kappa = 0$ or with $\beta = 2$ and $\nu = 0$ always possesses global classical solutions.

More recent work further reduces the values of $\alpha$ and $\beta$ and existing research appears to indicate that one-derivative dissipation is critical. Here one-derivative dissipation refers to the case when $\alpha + \beta = 1$ in (1.1). For the convenience of description, $\alpha + \beta = 1$ is referred to as the critical case, $\alpha + \beta > 1$ as the subcritical case while $\alpha + \beta < 1$ as the supercritical case. To position our work in a suitable context, we describe some recent results for these three cases.

We start with the subcritical case. Even this case is not easy. The global regularity has so far been established only for three subcritical cases [12, 34, 48]. The global existence and regularity problem for the critical case is more difficult. Two particular critical cases, $\alpha = 1$, $\kappa = 0$ and $\beta = 1$, $\nu = 0$, were studied by Hmidi, Keraani and Rousset, who introduced a combined quantity of the vorticity and the Riesz transform of the temperature and were able to establish the global regularity for both cases [21, 22]. For the more general critical case when the one derivative dissipation is split between the velocity equation and the temperature equation, the situation becomes more complex. The general critical case was recently dealt with by Jiu, Miao, Wu and Zhang and a global regularity result was obtained [25]. By reducing the global regularity issue on the critical Boussinesq system to a parallel problem for an active scalar equation with critical dissipation or, more precisely, the critical surface quasi-geostrophic (SQG) equation and taking advantage of the recent advances on the SQG equation, Jiu, Miao, Wu and Zhang obtained the global regularity in the critical regime: $\alpha + \beta = 1$ and $\alpha > \alpha_0$, where $\alpha_0 = \frac{3}{16} - \frac{\sqrt{1 - 16\nu}}{12} \approx 0.9132$. Attempts have also been made to go beyond the critical case and the global regularity has been established when the dissipation is logarithmically more singular than the critical case [20, 27]. The global well-posedness problem for the supercritical case $\alpha + \beta < 1$ is completely open. The only result currently available is the eventual regularity of weak solutions to (1.1) with $\alpha + \beta < 1$ and $\alpha > \alpha_0$ [26].
This paper establishes the global existence and regularity of classical solutions to (1.1) when $\alpha$ and $\beta$ are in the critical range: $\alpha + \beta = 1$ and $1 > \alpha > \frac{\sqrt{1777} - 23}{24} = 0.798103...$

This result improves the work of Jiu, Miao, Wu and Zhang [25] by allowing $\alpha$ to vary in a bigger interval, by keeping the relation $\alpha + \beta = 1$. The precise statement of our result is given in the following theorem.

**Theorem 1.1.** Consider (1.1) with $(u_0, \theta_0) \in H^\sigma(\mathbb{R}^2)$ for $\sigma > 2$. If the parameters in (1.1) satisfy

$$\nu > 0, \quad \kappa > 0, \quad 0.798103.. = \frac{\sqrt{1777} - 23}{24} < \alpha < 1, \quad \alpha + \beta = 1,$$

then (1.1) has a unique global solution $(u, \theta)$ satisfying, for any $T > 0$,

$$(u, \theta) \in C([0, T]; H^\sigma(\mathbb{R}^2)).$$

We outline the main idea in the proof of this theorem and explain how we improved [25]. A large portion of the efforts are devoted to obtaining global *a priori* bounds for $(u, \theta)$. Due to $\nabla \cdot u = 0$, the $L^2$-level global bounds for $(u, \theta)$ follow from easy energy estimates,

$$
\|\theta(t)\|_{L^q} \leq \|\theta_0\|_{L^q} \text{ for } q \in [1, \infty],
$$

$$(1.2)\quad \|\theta(t)\|_{L^2}^2 + 2\kappa \int_0^t \|\Lambda^{\frac{\alpha}{2}} \theta(\tau)\|_{L^2}^2 d\tau = \|\theta_0\|_{L^2}^2,$$

$$\|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\Lambda^{\frac{\alpha}{2}} u(\tau)\|_{L^2}^2 d\tau \leq (\|u_0\|_{L^2} + t\|\theta_0\|_{L^2})^2.$$ 

Naturally the next target is the global $H^1$-bound for $u$ or equivalently global $L^2$-bound for the vorticity $\omega = \nabla \times u$, which satisfies

$$
\begin{cases}
\partial_t \omega + u \cdot \nabla \omega + \nu \Lambda^{\alpha} \omega = \partial_1 \theta, \\
u = \nabla^\perp \psi, \quad \Delta \psi = \omega \quad \text{or} \quad u = \nabla^\perp \Delta^{-1} \omega.
\end{cases}
$$

(1.3)

Due to the presence of the “vortex stretching” term $\partial_1 \theta$, direct energy estimates do not yield the desired global bound for $0 < \alpha < 1$. For notational convenience, we set $\nu = \kappa = 1$ in (1.1) throughout the rest of this paper. The strategy is to hide $\partial_1 \theta$ by considering the combined quantity

$$G = \omega - \mathcal{R}_\alpha \theta \quad \text{with} \quad \mathcal{R}_\alpha = \Lambda^{-\alpha} \partial_1.$$

It is easy to check that $G$ satisfies

$$
\partial_t G + u \cdot \nabla G + \Lambda^{\alpha} G = [\mathcal{R}_\alpha, u \cdot \nabla] \theta + \Lambda^{\beta - \alpha} \partial_1 \theta.
$$

(1.4)

Here we have used the standard commutator notation

$$[\mathcal{R}_\alpha, u \cdot \nabla] \theta = \mathcal{R}_\alpha (u \cdot \nabla \theta) - u \cdot \nabla \mathcal{R}_\alpha \theta.$$

Although (1.4) appears to be more complicated than the vorticity equation, the commutator term $[\mathcal{R}_\alpha, u \cdot \nabla] \theta$ is less singular than $\partial_1 \theta$ in the vorticity equation. In fact, we are able to show the global bound for $\|G\|_{L^2}$ whenever $\alpha > \frac{3}{4}$ and $\alpha + \beta = 1$. The major
contribution of this paper is on the global $L^6$-bound for $G$. Previously, for $\alpha > \frac{4}{5}$ and $\alpha + \beta = 1$, \cite{25} obtained a global bound for $\|G\|_{L^6}$ for $q$ in the range
\begin{equation}
2 \leq q < q_0 \equiv \frac{8 - 4\alpha}{8 - 7\alpha}.
\end{equation}

Obviously $q_0 \in (2, 4)$ when $\alpha \in \left(\frac{4}{5}, 1\right)$. We are able to significantly enlarge the range of $q$. More precisely, we prove the following proposition.

**Proposition 1.2.** Let $0 < \beta < 1$ and $\alpha + \beta = 1$, with
\begin{equation}
1 > \alpha > \frac{\sqrt{1777} - 23}{24} = 0.798103.
\end{equation}

Let $(u_0, \theta_0)$ be as specified in Theorem 1.1 and let $(u, \theta)$ be the corresponding smooth solution of (1.1). Assume $G$ satisfies (1.4). Then, for any $T > 0$ and $t \leq T$,
\begin{equation}
\|G(t)\|_{L^q} \leq C \quad \text{for} \quad 2 \leq q \leq 6,
\end{equation}
where $C$ is a constant depending on $T$ and the initial data $(u_0, \theta_0)$.

The proof of Proposition 1.2 involves the decomposition of the velocity field
\begin{equation}
\begin{split}
u = \nabla \perp \Delta^{-1} \omega = \nabla \perp \Delta^{-1} G + \nabla \perp \Delta^{-1} R \theta \equiv \nu_G + \nu_R,
\end{split}
\end{equation}
commutator estimates and various functional inequalities. Proposition 1.2 is crucial in further showing that $G$ is actually globally regular in the sense that
\begin{equation}
\|G(t)\|_{B^{s}_{q,\infty}(\mathbb{R}^2)} \leq C \quad \text{for} \quad 0 \leq s \leq 3\alpha - 2 \quad \text{and} \quad 2 \leq q \leq 6,
\end{equation}
and for any $T > 0$ and $t \leq T$, where $C$ is a constant depending on $T$ and the initial data $(u_0, \theta_0)$. Here $B^{s}_{q,\infty}$ denotes an inhomogeneous Besov space (see Section 2 for more details on Besov spaces). Once Proposition 1.2 is established, the proof of (1.7) is similar to Proposition 7.1 in \cite{25}. A special consequence of (1.7) is that $G \in B^{0}_{\infty,1}(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$, which in turn implies that $u_G$ defined in (1.6) is Lipschitz,
\begin{equation}
\|\nabla u_G(t)\|_{L^\infty} \leq C.
\end{equation}

Then (1.6) with the equation of $\theta$ can be treated as a generalized critical SQG equation, which leads to the global regularity of $(u, \theta)$ following a similar approach as in \cite{25}.

The rest of this paper is divided into two main sections. Section 2 recalls the Littlewood-Paley decomposition, the definition of Besov spaces and some other related relevant facts. It also presents several commutator estimates and a global $L^2$-bound for $G$, which serve as a preparation for the proof of Proposition 1.2. Section 3 contains the proof of Proposition 1.2. In addition, the proof for Theorem 1.1 is also given in this section.

## 2. Preliminaries

This section includes several parts. It recalls the Littlewood-Paley theory, introduces the Besov spaces, provides Bernstein inequalities and Kato-Ponce estimates, proves several commutator estimates and a global $L^2$-bound for $G$. We start with the definitions of some of the functional spaces and related facts that will be used in the subsequent sections. Materials on Besov space and related facts presented here can be found in several books and many papers (see, e.g., \cite{41, 5, 35, 38, 41}).
2.1. Fourier transform and the Littlewood-Paley theory. We start with several notations. $\mathcal{S}$ denotes the usual Schwarz class and $\mathcal{S}'$ its dual, the space of tempered distributions. $\mathcal{S}_0$ denotes a subspace of $\mathcal{S}$ defined by

$$
\mathcal{S}_0 = \left\{ \phi \in \mathcal{S} : \int_{\mathbb{R}^d} \phi(x) x^\gamma dx = 0, |\gamma| = 0, 1, 2, \cdots \right\}
$$

and $\mathcal{S}_0'$ denotes its dual. $\mathcal{S}_0'$ can be identified as

$$
\mathcal{S}_0' = \mathcal{S}' / \mathcal{S}_0^\perp = \mathcal{S}' / \mathcal{P},
$$

where $\mathcal{P}$ denotes the space of multinomials. On the Schwartz class, we can define the Fourier transform and its inverse via

$$
\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx, \quad f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{ix\xi} d\xi.
$$

To introduce the Littlewood-Paley decomposition, we write for each $j \in \mathbb{Z}$

$$
A_j = \{ \xi \in \mathbb{R}^d : 2^{j-1} \leq |\xi| < 2^{j+1} \}.
$$

The Littlewood-Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j \in \mathbb{Z}} \in \mathcal{S}$ such that

$$
\text{supp} \hat{\Phi}_j \subset A_j, \quad \hat{\Phi}_j(\xi) = \hat{\Phi}_0(2^{-j} \xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd} \Phi_0(2^j x),
$$

and

$$
\sum_{j=-\infty}^{\infty} \hat{\Phi}_j(\xi) = \left\{ \begin{array}{ll} 1, & \text{if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0, & \text{if } \xi = 0. \end{array} \right.
$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$
\sum_{j=-\infty}^{\infty} \hat{\Phi}_j(\xi) \hat{\psi}(\xi) = \hat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.
$$

In addition, if $\psi \in \mathcal{S}_0$, then

$$
\sum_{j=-\infty}^{\infty} \hat{\Phi}_j(\xi) \hat{\psi}(\xi) = \hat{\psi}(\xi) \quad \text{for any } \xi \in \mathbb{R}^d.
$$

That is, for $\psi \in \mathcal{S}_0$,

$$
\sum_{j=-\infty}^{\infty} \Phi_j * \psi = \psi
$$

and hence

$$
\sum_{j=-\infty}^{\infty} \Phi_j * f = f, \quad f \in \mathcal{S}_0'
$$

in the sense of weak-* topology of $\mathcal{S}_0'$. For notational convenience, we define

(2.1) \quad \hat{\Delta}_j f = \Phi_j * f, \quad j \in \mathbb{Z}.
2.2. Besov spaces.

Definition 2.1. For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, the homogeneous Besov space $\dot{B}^s_{p,q}$ consists of $f \in \mathcal{S}_0'$ satisfying

$$
\|f\|_{\dot{B}^s_{p,q}} \equiv \|2^{js}\|\Delta_j f\|_{L^p}\|_q < \infty.
$$

We now choose $\Psi \in \mathcal{S}$ such that

$$
\hat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \hat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.
$$

Then, for any $\psi \in \mathcal{S}$,

$$
\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi
$$

and hence

$$
\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f
$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. To define the inhomogeneous Besov space, we set

$$
(2.2) \quad \Delta_j f = \begin{cases} 
0, & \text{if } j \leq -2, \\
\Psi * f, & \text{if } j = -1, \\
\Phi_j * f, & \text{if } j = 0, 1, 2, \ldots \n\end{cases}
$$

Definition 2.2. The inhomogeneous Besov space $B^s_{p,q}$ with $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$ consists of functions $f \in \mathcal{S}'$ satisfying

$$
\|f\|_{B^s_{p,q}} \equiv \|2^{js}\|\Delta_j f\|_{L^p}\|_q < \infty.
$$

The Besov spaces $\dot{B}^s_{p,q}$ and $B^s_{p,q}$ with $s \in (0,1)$ and $1 \leq p, q \leq \infty$ can be equivalently defined by the norms

$$
\|f\|_{\dot{B}^s_{p,q}} = \left( \int_{\mathbb{R}^d} \left( \frac{\|f(x+t) - f(x)\|_{L^p}^q}{|t|^{d+sq}} \right) \frac{dt}{t} \right)^{1/q},
$$

$$
\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \left( \int_{\mathbb{R}^d} \left( \frac{\|f(x+t) - f(x)\|_{L^p}^q}{|t|^{d+sq}} \right) \frac{dt}{t} \right)^{1/q}.
$$

When $q = \infty$, the expressions are interpreted as suprema instead of integrals.

Many frequently used function spaces are special cases of Besov spaces. The following proposition lists some useful equivalence and embedding relations.

Proposition 2.3. For any $s \in \mathbb{R}$,

$$
\dot{H}^s \sim \dot{B}_{2,2}^s, \quad H^s \sim B_{2,2}^s.
$$

For any $s \in \mathbb{R}$ and $1 < q < \infty$,

$$
\dot{B}_{q,\min\{q,2\}}^s \hookrightarrow W^s_q \hookrightarrow \dot{B}_{q,\max\{q,2\}}^s.
$$

In particular, $\dot{B}_{q,\min\{q,2\}}^0 \hookrightarrow L^q \hookrightarrow \dot{B}_{q,\max\{q,2\}}^0$. 
For notational convenience, we write $\Delta_j$ for $\hat{\Delta}_j$. There will be no confusion if we keep in mind that $\Delta_j$’s associated with the homogeneous Besov spaces is defined in (2.1) while those associated with the inhomogeneous Besov spaces are defined in (2.2). Besides the Fourier localization operators $\Delta_j$, the partial sum $S_j$ is also a useful notation. For an integer $j$,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k,$$

where $\Delta_k$ is given by (2.2). For any $f \in S'$, the Fourier transform of $S_j f$ is supported on the ball of radius $2^j$.

2.3. Bernstein inequalities and Kato-Ponce estimates. Bernstein’s inequalities are useful tools in dealing with Fourier localized functions and these inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives.

**Proposition 2.4.** Let $\alpha \geq 0$. Let $1 \leq p \leq q \leq \infty$.

1) If $f$ satisfies

$$\text{supp} \hat{f} \subset \{ \xi \in \mathbb{R}^d : |\xi| \leq K 2^j \},$$

for some integer $j$ and a constant $K > 0$, then

$$\| \Lambda^\alpha f \|_{L^q(\mathbb{R}^d)} \leq C_1 2^{\alpha j + j d (\frac{1}{p} - \frac{1}{q})} \| f \|_{L^p(\mathbb{R}^d)},$$

where $C_1$ is a constant depending on $K, \alpha, p$ and $q$ only.

2) If $f$ satisfies

$$\text{supp} \hat{f} \subset \{ \xi \in \mathbb{R}^d : K_1 2^j \leq |\xi| \leq K_2 2^j \}$$

for some integer $j$ and constants $0 < K_1 \leq K_2$, then

$$C_1 2^{\alpha j} \| f \|_{L^q(\mathbb{R}^d)} \leq \| \Lambda^\alpha f \|_{L^q(\mathbb{R}^d)} \leq C_2 2^{\alpha j + j d (\frac{1}{p} - \frac{1}{q})} \| f \|_{L^p(\mathbb{R}^d)},$$

where $C_2$ is a constant depending on $K_1, K_2, \alpha, p$ and $q$ only.

The following Kato-Ponce type estimate (also known as fractional Leibnitz rule) will be used extensively,

$$\| \Lambda^s (fg) \|_{L^p} \leq C(\| \Lambda^s f \|_{L^{p_1}} \| g \|_{L^{r_1}} + \| \Lambda^s g \|_{L^{q_1}} \| f \|_{L^{r_2}}),$$

whenever $s > 0, 1 < p, q_1, q_2 < \infty, 1 < r_1, r_2 \leq \infty, \frac{1}{p} = \frac{1}{q_1} + \frac{1}{r_1} = \frac{1}{q_2} + \frac{1}{r_2}$. This inequality can be found in many references (see [28] and [19] for recent results and survey of the literature on the topic). As a corollary, we can deduce the following estimate, at least for all integers $m$

$$\| \Lambda^s (f^m) \|_{L^p} \leq C \| \Lambda^s f \|_{L^q} \| f \|_{L^r}^{m-1},$$

whenever $1 < p, q, r < \infty$ and $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. 


2.4. Commutator estimates. As we have seen already, the equation (1.4) involves commutators. Thus, we shall need to develop the corresponding estimates, so that we can bound them suitably.

Lemma 2.5. Let \( 1 > \alpha > 1/2 \) and \( 1 < p < \infty \), \( 1 < p_1, p_2 \leq \infty \) : \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \). For any integer \( k \), we have for the \( u_G \) and \( u_\theta \) defined in (1.6),

\[
\| \Delta_k([R_{\alpha}, u_G \cdot \nabla] \theta) \|_{L^p} \leq C 2^{(1-\alpha)k} \| G \|_{L^{p_1}} \| \theta \|_{L^{p_2}};
\]

(2.5)

\[
\| \Delta_k([R_{\alpha}, u_\theta \cdot \nabla] \psi) \|_{L^p} \leq C 2^{(2-2\alpha)k} \| \theta \|_{L^{p_1}} \| \psi \|_{L^{p_2}}.
\]

(2.6)

More generally, for \( 0 \leq s \leq 1 - \alpha \), we have

\[
\| \Delta_k([R_{\alpha}, u_G \cdot \nabla] \theta) \|_{L^p} \leq C_s 2^{(1-\alpha-s)k} \| G \|_{L^{p_1}} \| \Lambda^s \theta \|_{L^{p_2}};
\]

(2.7)

\[
\| \Delta_k([R_{\alpha}, u_\theta \cdot \nabla] \psi) \|_{L^p} \leq C_s 2^{(2-2\alpha-s)k} \| \Lambda^s \theta \|_{L^{p_1}} \| \psi \|_{L^{p_2}}.
\]

(2.8)

Proof. Most of the proof will be concerned with (2.5), (2.6). At the end, we will indicate the (small) modifications needed for (2.7) and (2.8).

In order to simplify notations, we denote \( u_j := \Delta_j u \), \( u_{<k} := S_k u \), \( u_{[k-A,k+B]} := \sum_{j=k-A}^{k+B} u_j \). If \( A, B < 10 \), we use \( u_{\sim k} \) to denote \( u_{[k-A,k+B]} \), etc. We use the following paraproduct decomposition for the product of two functions

\[
\Delta_k(fg) = \Delta_k(f_{<k-10} g_{\sim k}) + \Delta_k(f_{\sim k} g_{<k+10}) + \Delta_k\left( \sum_{l=k+10}^{\infty} f_l g_{\sim l} \right).
\]

We refer to the first term as low-high interaction, the second term is high-low interaction and the third term is high-high interaction. Only the low-high interaction term is not straightforward and requires the commutator structure. Note that according to the definition (1.6), we have that \( u_G \sim \Lambda^{-1} G \), \( u_\theta \sim \Lambda^{-\alpha} \theta \).

More precisely, for any \( 1 \leq p \leq \infty \),

\[
\| \Delta_l(u_G) \|_{L^p} \sim 2^{-l} \| \Delta_l G \|_{L^p} \leq C 2^{-l} \| G \|_{L^p};
\]

\[
\| \Delta_l(u_\theta) \|_{L^p} \sim 2^{-\alpha l} \| \Delta_l \theta \|_{L^p} \leq C 2^{-\alpha l} \| \theta \|_{L^p}.
\]

High-low interactions

For (2.5), we have by Hölder’s inequality,

\[
\| \Delta_k([R_{\alpha}, (u_G)_{\sim k} \cdot \nabla] \theta_{<k+10}) \|_{L^p} \leq C \| R_{\alpha} \Delta_k[(u_G)_{\sim k} \cdot \nabla \theta_{<k+10}] \|_{L^p}
\]

\[
+ \| (u_G)_{\sim k} \cdot \nabla |R_{\alpha} \theta_{<k+10} |_{L^p} \| \leq C 2^{k(1-\alpha)} \| (u_G)_{\sim k} \|_{L^{p_1}} \| \nabla \theta_{<k+10} \|_{L^{p_2}}
\]

\[
+ C \| (u_G)_{\sim k} \|_{L^{p_1}} \| \Lambda^{2-\alpha} \theta_{<k+10} \|_{L^{p_2}}.
\]

But \( \| (u_G)_{\sim k} \|_{L^{p_1}} \leq C 2^{-k} \| G \|_{L^{p_1}} \), and

\[
\| \nabla \theta_{<k+10} \|_{L^{p_2}} \leq C 2^{k} \| \theta \|_{L^{p_2}}, \quad \| \Lambda^{2-\alpha} \theta_{<k+10} \|_{L^{p_2}} \leq C 2^{(2-\alpha)k} \| \theta \|_{L^{p_2}}.
\]

Putting everything together yields the desired inequality

\[
\| \Delta_k([R_{\alpha}, (u_G)_{\sim k} \cdot \nabla] \theta_{<k+10}) \|_{L^p} \leq C 2^{k(1-\alpha)} \| G \|_{L^{p_1}} \| \theta \|_{L^{p_2}}.
\]

Similarly, for (2.6), we have by Hölder’s inequality,

\[
\| \Delta_k([R_{\alpha}, (u_\theta)_{\sim k} \cdot \nabla] \psi_{<k+10}) \|_{L^p} \leq C \| R_{\alpha} \Delta_k[(u_\theta)_{\sim k} \cdot \nabla \psi_{<k+10}] \|_{L^p}
\]

\[
+ \| (u_\theta)_{\sim k} \cdot \nabla |R_{\alpha} \psi_{<k+10} |_{L^p} \| \leq C 2^{k(1-\alpha)} \| (u_\theta)_{\sim k} \|_{L^{p_1}} \| \nabla \psi_{<k+10} \|_{L^{p_2}}
\]

\[
+ C \| (u_\theta)_{\sim k} \|_{L^{p_1}} \| \Lambda^{2-\alpha} \psi_{<k+10} \|_{L^{p_2}}.
\]
Again, \( \|(u_\theta)_{\sim k}\|_{L^p_1} \leq C 2^{-k\alpha} \|\theta\|_{L^p_1} \), in conjunction with (2.9), yields
\[
\|\Delta_k([\mathcal{R}_\alpha, (u_\theta)_{\sim k} \cdot \nabla] \psi_{\sim k+10})\|_{L^p} \leq C 2^{2(1-\alpha)k} \|\theta\|_{L^p_1} \|\psi\|_{L^p_2}.
\]

**High-high interactions**

For (2.6), we have that \( u \cdot \nabla \theta = \nabla \cdot [u \theta] \) (since \( \nabla \cdot u = 0 \)). Thus, we have by Hölder’s inequality,
\[
\sum_{l=k+10}^\infty \Delta_k([\mathcal{R}_\alpha, (u_\theta)_l \cdot \nabla] \psi_{\sim l}) L^p \leq C \sum_{l=k+10}^\infty \|\mathcal{R}_\alpha \nabla \Delta_k[(u_\theta)_l \cdot \psi_{\sim l}]\|_{L^p}
+ C \sum_{l=k+10}^\infty \|\nabla \Delta_k[(u_\theta)_l \cdot \mathcal{R}_\alpha \psi_{\sim l}]\|_{L^p} \leq C 2^{k(2-\alpha)} \sum_{l=k+10}^\infty \|(u_\theta)_l\|_{L^p_1} \|\psi_{\sim l}\|_{L^p_2}
+ C 2^k \sum_{l=k+10}^\infty \|(u_\theta)_l\|_{L^p_1} \|\Lambda^{1-\alpha} \psi_{\sim l}\|_{L^p_2}
\leq C \|G\|_{L^p_1} \|\theta\|_{L^p_2} \sum_{l=k+10}^\infty (2^{k(2-\alpha)} 2^{-l} + 2^k 2^{-\alpha l}) \leq C 2^{k(1-\alpha)} \|G\|_{L^p_1} \|\theta\|_{L^p_2}.
\]

For (2.6), we proceed in the same way, but note that towards the end, we need that \( \alpha > 1/2 \). We have
\[
\sum_{l=k+10}^\infty \Delta_k([\mathcal{R}_\alpha, (u_\theta)_l \cdot \nabla] \psi_{\sim l}) L^p \leq C \sum_{l=k+10}^\infty \|\mathcal{R}_\alpha \nabla \Delta_k[(u_\theta)_l \cdot \psi_{\sim l}]\|_{L^p}
+ C \sum_{l=k+10}^\infty \|\nabla \Delta_k[(u_\theta)_l \cdot \mathcal{R}_\alpha \psi_{\sim l}]\|_{L^p} \leq C 2^{k(2-\alpha)} \sum_{l=k+10}^\infty \|(u_\theta)_l\|_{L^p_1} \|\psi_{\sim l}\|_{L^p_2}
+ C 2^k \sum_{l=k+10}^\infty \|(u_\theta)_l\|_{L^p_1} \|\Lambda^{1-\alpha} \psi_{\sim l}\|_{L^p_2}
\leq C \|G\|_{L^p_1} \|\theta\|_{L^p_2} \sum_{l=k+10}^\infty (2^{k(2-\alpha)} 2^{-\alpha l} + 2^k 2^{-l(2\alpha-1)}) \leq C 2^{2k(1-\alpha)} \|G\|_{L^p_1} \|\theta\|_{L^p_2}.
\]

**Low high interactions**

Now we need to estimate \( \|\Delta_k([\mathcal{R}_\alpha, (u_\theta)_{k-10} \cdot \nabla] \psi_{\sim k})\|_{L^p} \). As before, we use the divergence free condition to reduce to
\[
\|\Delta_k([\mathcal{R}_\alpha, (u_\theta)_{k-10} \cdot \nabla] \psi_{\sim k})\|_{L^p} = \|\nabla \Delta_k([\mathcal{R}_\alpha, (u_\theta)_{k-10}] \theta_{\sim k})\|_{L^p} \leq C 2^{k} \|\Delta_k([\mathcal{R}_\alpha, (u_\theta)_{k-10}] \theta_{\sim k})\|_{L^p},
\]
so that now, we need to check
\[
(2.10) \quad \|\Delta_k([\mathcal{R}_\alpha, (u_\theta)_{k-10}] \theta_{\sim k})\|_{L^p} \leq C 2^{-k\alpha} \|G\|_{L^p_1} \|\theta\|_{L^p_2}.
\]

In addition, \( \mathcal{R}_\alpha = \partial_1 \Lambda^{-\alpha} \), so we use the (standard) product rule to conclude
\[
[\mathcal{R}_\alpha, (u_\theta)_{k-10}] \theta_{\sim k} = \mathcal{R}_\alpha[(u_\theta)_{k-10} \cdot \theta_{\sim k}] - (u_\theta)_{k-10} \cdot \mathcal{R}_\alpha \theta_{\sim k}
= \Lambda^{-\alpha} [\partial_1 (u_\theta)_{k-10} \cdot \theta_{\sim k}] + [\Lambda^{-\alpha}, (u_\theta)_{k-10}] \partial_1 \theta_{\sim k}.
\]

Clearly, the first term satisfies the required estimates, since
\[
\|\Delta_k \Lambda^{-\alpha} [\partial_1 (u_\theta)_{k-10} \cdot \theta_{\sim k}]\|_{L^p} \leq 2^{-k\alpha} \|\partial_1 (u_\theta)_{k-10}\|_{L^p_1} \|\theta_{\sim k}\|_{L^p_2} \leq C 2^{-k\alpha} \|G\|_{L^p_1} \|\theta\|_{L^p_2},
\]
which is (2.10). It then remains to show
\begin{equation}
\|\Delta_k[\Lambda^{-\alpha}, f_{<k-10}g_{\sim k}]\|_{L^p} \leq C2^{-k(1+\alpha)}\|\nabla f\|_{L^p} \|g\|_{L^{p_2}}.
\end{equation}
Indeed, if we show that, we apply it to \(f = u_G, g = \partial_\theta\) and we obtain the result.

Write
\[
\Delta_k[\Lambda^{-\alpha}, f_{<k-10}g_{\sim k}] = \Delta_k\Lambda^{-\alpha}[f_{<k-10}g_{\sim k}] - \Delta_k[f_{<k-10}\Lambda^{-\alpha}g_{\sim k}]
\]
Denote the multiplier of \(g_{\sim k}\) by \(\tilde{\Delta}_k\). Note that by the support properties of the corresponding multipliers, we have that \(\tilde{\Delta}_k\|\Delta_k\|_\infty = \|\Lambda^{-\alpha}\|_\infty \|\Delta_k\|_\infty\). Thus,
\[
\Delta_k[\Lambda^{-\alpha}, f_{<k-10}g_{\sim k}] = \Delta_k\tilde{\Delta}_k\Lambda^{-\alpha}[f_{<k-10}g_{\sim k}] - \Delta_k[f_{<k-10}\Lambda^{-\alpha}g_{\sim k}]
\]
Thus, it suffices to estimate \(\|\tilde{\Delta}_k\Lambda^{-\alpha}, f_{<k-10}g_{\sim k}\|_{L^p}\). Furthermore, we have that
\[
\tilde{\Delta}_k\Lambda^{-\alpha} = 2^{-k\alpha}P_k,
\]
where \(\hat{P}_k\hat{f}(\xi) = \hat{\chi}(2^{-k}\xi)\hat{f}(\xi)\), where \(\hat{\chi}\) is a \(C^\infty\) function, supported in \(\{\xi : |\xi| \in (1/2, 2)\}\).
Thus, we need to show
\begin{equation}
\|\tilde{\Delta}_k\Lambda^{-\alpha}, f_{<k-10}g_{\sim k}\|_{L^p} \leq C2^{-k}\|\nabla f\|_{L^p} \|g\|_{L^{p_2}}.
\end{equation}
But this is a standard result in harmonic analysis. Here is the easy proof for completeness
\[
[P_k, f]g(x) = 2^{kd}\int_{R^d} \hat{\chi}(2^k(x-y))(f(y) - f(x))g(y)dy
\]
\[= 2^{kd}\int_{R^d} \hat{\chi}(2^k(x-y))g(y)(\int_0^1 \langle y - x, \nabla f(x - \rho(x-y)) \rangle d\rho)dy
\]
It follows that
\[
\|\tilde{\Delta}_k\Lambda^{-\alpha}, f_{<k-10}g_{\sim k}\|_{L^p} \leq C2^{-k}\|\nabla f\|_{L^p} \|g\|_{L^{p_2}}.
\]
By Hölder’s
\[
\|\tilde{\Delta}_k\Lambda^{-\alpha}, f_{<k-10}g_{\sim k}\|_{L^p} \leq C\|\nabla f\|_{L^p} \|g\|_{L^{p_2}} \int_{R^d} 2^{kd}|z|\|\hat{\chi}(2^kz)\|dzd\rho = C2^{-k}\|\nabla f\|_{L^p} \|g\|_{L^{p_2}}.
\]
This finishes the proof of (2.12) and hence of (2.5).

For the low-high interaction term of (2.6), we reduce similarly. More precisely, by the divergence free condition
\[
\|\Delta_k([\mathcal{R}_\alpha, (u_\theta)_{<k-10} \cdot \nabla]\psi_{\sim k})\|_{L^p} = \|\nabla \Delta_k([\mathcal{R}_\alpha, (u_\theta)_{<k-10} \cdot \nabla]\psi_{\sim k})\|_{L^p} \leq C2^k\|\Delta_k([\mathcal{R}_\alpha, (u_\theta)_{<k-10} \cdot \nabla]\psi_{\sim k})\|_{L^p},
\]
Next,
\[
[\mathcal{R}_\alpha, (u_\theta)_{<k-10} \cdot \nabla]\psi_{\sim k} = \mathcal{R}_\alpha[(u_\theta)_{<k-10} \cdot \nabla]\psi_{\sim k} - (u_\theta)_{<k-10} \cdot \mathcal{R}_\alpha\psi_{\sim k} = \Lambda^{-\alpha}[\partial_\theta(u_\theta)_{<k-10} \cdot \psi_{\sim k}] + [\Lambda^{-\alpha}, (u_\theta)_{<k-10} \cdot \partial_\theta]\psi_{\sim k}.
\]
For the first term, since \(\|\partial_\theta(u_\theta)_{<k-10}\|_{L^{p_1}} \leq C2^{(1+\alpha)}\|\theta\|_{L^{p_1}},\)
\[
\|\Delta_k\Lambda^{-\alpha}[\partial_\theta(u_\theta)_{<k-10} \cdot \psi_{\sim k}]\|_{L^p} \leq 2^{-k\alpha}\|\partial_\theta(u_\theta)_{<k-10}\|_{L^{p_1}} \|\psi_{\sim k}\|_{L^{p_2}},
\]
For the second, we can reduce, in a similar way, to proving an estimate in the form
\[
\|\tilde{\Delta}_k\Lambda^{-\alpha}, (u_\theta)_{<k-10}\psi_{\sim k}\|_{L^p} \leq 2^{-2k\alpha}\|\theta\|_{L^{p_1}} \|\psi\|_{L^{p_2}}.
\]
Recalling $\Delta_k\Lambda^{-\alpha} = 2^{-k\alpha}P_k$, we have by (2.12) and $\|\nabla(u_\theta)_{<k-10}\|_{L^p} \leq C2^{k(1-\alpha)}\|\theta\|_{L^p}$,

$$
\|[\Delta_k\Lambda^{-\alpha}(u_\theta)_{<k-10}]\|_{L^p} = 2^{-k\alpha}\|P_k(u_\theta)_{<k-10}\|_{L^p}
\leq C2^{-k(1+\alpha)}\|\nabla(u_\theta)_{<k-10}\|_{L^p} \psi_{<k}\|_{L^p} \leq C2^{-2k\alpha}\|\theta\|_{L^p}\|\psi\|_{L^{p_2}}.
$$

Regarding the proofs of (2.7) and (2.8), one just needs to go back to the arguments presented above and trace the derivatives. More precisely, for (2.7), things are clear in the high-high and the low high interaction cases, since we use (2.5) and the inequality

$$
2^{k(1-\alpha)}\|\theta_{\geq k-10}\|_{L^{p_2}} \leq C2^{k(1-\alpha-s)}\|\Lambda^s\theta\|_{L^{p_2}}.
$$

In the high-low interaction case, note that we have used (2.13). The proof of (2.14) is similar.

**We would have obtained (2.7), instead of (2.5). The arguments for (2.8) are of similar nature and we omit them.**

We now can deduce the following

**Corollary 2.6.** Let $1 > \alpha > 1/2$ and $1 < p_2 < \infty, 1 < p_1, p_3 \leq \infty$, so that

$$
\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1.
$$

For every $s_1 : 0 \leq s_1 < 1 - \alpha$ and $s_2 : s_2 > 1 - \alpha - s_1$, there exists a $C = C(p_1, p_2, p_3, s_1, s_2)$, so that

$$
\|\nabla\theta_{<k+10}\|_{L^{p_2}} \leq 2^k\|\theta\|_{L^{p_2}}.
$$

Proof. Recall that we have $\sum_j \hat{\Delta}_j = I_d$. Take $\hat{\Delta}_j$ with similar properties, so that $\hat{\Delta}_j \hat{\Delta}_j = \hat{\Delta}_j$. For (2.13), we have by (2.7),

$$
|\int_{\mathbb{R}^d} F[\mathcal{R}_\alpha, u_\theta \cdot \nabla]\psi dx| \leq C\|\Lambda^{s_1}\theta\|_{L^{p_1}}\|F\|_{W^{s_2,p_2}} \|G\|_{L^{p_3}}
$$

Similarly, for every $s_1 : 0 \leq s_1 < 1 - \alpha$ and $s_2 : s_2 > 2 - 2\alpha - s_1$, we have

$$
|\int_{\mathbb{R}^d} F[\mathcal{R}_\alpha, u_\theta \cdot \nabla]\psi dx| \leq C\|\Lambda^{s_1}\theta\|_{L^{p_1}}\|F\|_{W^{s_2,p_2}} \|\psi\|_{L^{p_3}}
$$

**Proof.** Recall that we have $\sum_j \hat{\Delta}_j = I_d$. Take $\hat{\Delta}_j$ with similar properties, so that $\hat{\Delta}_j \hat{\Delta}_j = \hat{\Delta}_j$. For (2.13), we have by (2.7),

$$
|\int_{\mathbb{R}^d} F[\mathcal{R}_\alpha, u_\theta \cdot \nabla]\psi dx| \leq \sum_j |\hat{\Delta}_j F||\hat{\Delta}_j[\mathcal{R}_\alpha, u_\theta \cdot \nabla]\theta||dx| \leq C\sum_j 2^{j(1-\alpha-s_1)}\|\hat{\Delta}_j F\|_{L^{p_2}} \|G\|_{L^{p_3}} \|\Lambda^{s_1}\theta\|_{L^{p_1}} \leq C\|G\|_{L^{p_3}} \|\Lambda^{s_1}\theta\|_{L^{p_1}} \max(\|F_{<10}\|_{L^{p_2}}, \sup_{j \geq 0} 2^{j}s_2\|\hat{\Delta}_j F\|_{L^{p_2}}),
$$

which of course implies (2.13). The proof of (2.14) is similar.

2.5. $L^2$ bound for $G$. We present the global $L^2$-bound for $G$, which improves the corresponding $L^2$-bound in Theorem 5.1 of [25] by relaxing the condition from $\alpha > \frac{4}{5}$ to $\alpha > \frac{2}{3}$.

**Lemma 2.7.** Let $\alpha > \frac{2}{3}$ and $(u, \theta)$ be the solution of (1.1) in some interval $[0, T]$. Then, $G$ defined in (1.4) satisfies for every $0 \leq t \leq T$,

$$
\|G(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{2}{3}}G(\tau)\|_{L^2}^2 d\tau \leq C(T, u_0, \theta_0).
$$
Proof. The proof has some similarities to that for Theorem 5.1 in [25], but we make use of the global bound (see (1.2))
\[ \int_0^T \| \Lambda^{\frac{\theta}{2}} (\tau) \|_{L^2}^2 d\tau < C(T, \theta_0). \]
Taking the inner product of (1.4) with $G$, we obtain, after integration by parts,
\[ (2.15) \quad \frac{1}{2} \frac{d}{dt} \| G \|_{L^2}^2 + \| \Lambda^{\frac{2}{\theta}} G \|_{L^2}^2 = J_1 + J_2, \]
where
\[ J_1 = \int G \Lambda^{1-2\alpha} \partial_t \theta \, dx, \quad J_2 = \int G [\mathcal{R}_\alpha, u \cdot \nabla] \theta \, dx. \]
Applying Hölder’s inequality and noting that the Riesz transform $\Lambda^{-1} \partial_t$ is bounded in $L^q$ for any $1 < q < \infty$, we obtain, due to $\frac{3}{4} < \alpha < 1$,
\[ |J_1| \leq \| \Lambda^{2-\frac{s}{\theta}} \theta \|_{L^2} \| \Lambda^{\frac{2}{\theta}} G \|_{L^2} \leq \frac{1}{4} \| \Lambda^{\frac{2}{\theta}} G \|_{L^2}^2 + C \| \theta \|_{H^\frac{4}{\theta}}^2. \]
To bound $J_2$, we write $u = u_G + u_\theta$ as in (1.6). Again, due to $\frac{3}{4} < \alpha < 1$, we can choose $1 - \alpha < s < \alpha/2$ and then apply Corollary 2.6 to obtain
\begin{align*}
\left| \int G [\mathcal{R}_\alpha, u_G \cdot \nabla] \theta \, dx \right| & \leq C \| \theta \|_{L^\infty} \| G \|_{L^2} \| G \|_{H^s} \leq C \| \theta_0 \|_{L^\infty} \| G \|_{L^2} \| G \|_{H^\frac{4}{\theta}} \\
& \leq \frac{1}{4} \| \Lambda^{\frac{2}{\theta}} G \|_{L^2}^2 + C \| G \|_{L^2}^2.
\end{align*}
Applying Corollary 2.6 with $s_1 = \frac{(1-\alpha)}{2}$ and $\frac{3}{4}(1-\alpha) < s_2 < \frac{\alpha}{2}$, we have
\begin{align*}
\left| \int G [\mathcal{R}_\alpha, u_\theta \cdot \nabla] \theta \, dx \right| & \leq C \| \Lambda^{s_1} \theta \|_{L^2} \| \theta \|_{L^\infty} \| G \|_{H^{s_2}} \leq C \| \theta_0 \|_{L^\infty} \| \Lambda^{\frac{s_1}{\theta}} \theta \|_{L^2} \| G \|_{H^\frac{4}{\theta}} \\
& \leq \frac{1}{4} \| \Lambda^{\frac{s_1}{\theta}} G \|_{L^2}^2 + C \| \Lambda^{\frac{s_1}{\theta}} \theta \|_{L^2}^2.
\end{align*}
Therefore,
\[ |J_2| \leq \frac{1}{2} \| \Lambda^{\frac{s_1}{\theta}} G \|_{L^2}^2 + C \| G \|_{L^2}^2 + C \| \Lambda^{\frac{s_1}{\theta}} \theta \|_{L^2}^2. \]
Inserting the bounds for $J_1$ and $J_2$ to (2.15) and applying Gronwall’s inequality yield the desired bound. This completes the proof of Lemma 2.7. 
\[ \square \]

3. On the $L^6$ bound for $G$

This section proves Proposition 1.2, which provides a global $L^6$-bound for $G$. Once Proposition 1.2 is established, Theorem 1.1 can be proven similarly to [25]. We nevertheless give a brief outline of its proof here.

Before we start proving Proposition 1.2, let us prepare with the following observation. By the estimates for the evolution of $\theta$ in (1.2), we have control of $\| \theta(t) \|_{L^\infty}$ and $\| \Lambda^{(1-\alpha)/2} \theta \|_{L^3_t L^3_x}$. By Gagliardo-Nirenberg’s inequality, we have that for all $\gamma \in (0, 1/2)$,
\[ (3.1) \quad \| \Lambda^{\gamma(1-\alpha)} \theta \|_{L^{\frac{2}{1-2\gamma}}_t L^{\frac{4}{1-2\gamma}}_x} \leq \| \Lambda^{(1-\alpha)/2} \theta \|_{L^3_t L^3_x}^{2\gamma} \| \theta_0 \|_{L^\infty_t L^2_x}^{1-2\gamma}. \]
Proof of Proposition 1.3 We consider $\gamma \in (0, 1/2)$, to be fixed momentarily and also define $\alpha_{cr}$ be the solution to $(2 - \gamma)(1 - \alpha) = \frac{2}{3}$ or

$$\alpha_{cr} = \frac{4 - 2\gamma}{5 - 2\gamma}.$$  

Note that for each $\alpha > \alpha_{cr}$, we have that $(2 - \gamma)(1 - \alpha) < \frac{2}{3}$. We henceforth assume $\alpha > \alpha_{cr}$.

In view of Lemma 2.7 it suffices to consider the case $q = 6$. By multiplying (1.4) by $G|G|^4 = G^5$ and integrating in $x$, we obtain

$$\frac{1}{q} \partial_t \| G(t) \|_L^6 + \int G^5 \Lambda^\alpha G \, dx = \int G^5 [R_{\alpha}, u \cdot \nabla] \theta \, dx + \int G^5 \Lambda^{1-2\alpha} \partial_1 \theta \, dx.$$  

By the maximum principle of [13] and Sobolev embedding, we have

$$\int G^5 \Lambda^\alpha G \, dx \geq C \int |\Lambda^{\frac{s}{2}} G^3|^2 \, dx \geq C \| G \|_{L^6}^{\frac{s}{2}}.$$  

Next, we deal with the second term on the right hand side of (3.2). Note that $\partial_1 \Lambda^{-1}$ is the Riesz transform in the first variable, which is bounded on all $L^p$, $1 < p < \infty$ spaces. We have by Hölder’s inequality and the Kato-Ponce estimate (2.4)

$$\left| \int G^5 \Lambda^{1-2\alpha} \partial_1 \theta \, dx \right| \leq \| \Lambda^{\gamma(1-\alpha)} \theta \|_{L^{\frac{4}{3}}} \| \partial_1 \Lambda^{-1} \Lambda^{(2-\gamma)(1-\alpha)} (G^5) \|_{L^{\frac{4}{3}}}$$  

$$\leq C \| \Lambda^{\gamma(1-\alpha)} \theta \|_{L^4} \| \Lambda^{(2-\gamma)(1-\alpha)} G^5 \|_{L^4} \| G \|_{H^{\alpha/2}} \| G \|_{L^{\frac{4}{1-2\alpha}}}.$$  

where in the last line, we have used that $\alpha > \alpha_{cr}$.

By writing $u = u_G + u_\theta$ as in (1.5), the first term on the right hand side of (3.2) is split into two terms. We have, according to (2.14), for every $s > (2 - \gamma)(1 - \alpha)$,

$$\left| \int G^5 [R_{\alpha}, u_G \cdot \nabla] \theta \, dx \right| \leq C_s \| \Lambda^{\gamma(1-\alpha)} \theta \|_{L^4} \| \theta \|_{L^\infty} \| G^5 \|_{W^{s, \infty}}$$  

$$\leq \| \Lambda^{\gamma(1-\alpha)} \theta \|_{L^4} \| \theta_0 \|_{L^\infty} \| G \|_{H^s} \| G \|_{L^{\frac{4}{1-2\alpha}}}.$$  

We now pick $s$ so close to $(2 - \gamma)(1 - \alpha)$, so that $(2 - \gamma)(1 - \alpha) < s < \alpha/2$. This is possible, because $\alpha > \alpha_{cr}$. Thus,

$$\left| \int G^5 [R_{\alpha}, u_G \cdot \nabla] \theta \, dx \right| \leq C \| \Lambda^{\gamma(1-\alpha)} \theta \|_{L^4} \| G \|_{H^{s/2}} \| G \|_{L^{\frac{4}{1-2\alpha}}}.$$  

To handle the term $\int G^5 [R_{\alpha}, u_\theta \cdot \nabla] \theta \, dx$, we apply (2.13) with $s_1 = 0$. We have for $s > 1 - \alpha$,

$$\frac{1}{p} = \frac{9}{10} - \frac{1}{10} (1 - \gamma)(1 - \alpha), \quad \frac{1}{q} = \frac{1 + (1 - \gamma)(1 - \alpha)}{10}, \quad \frac{1}{p_1} = \frac{1 - (1 - \gamma)(1 - \alpha)}{2},$$  

$$\left| \int G^5 [R_{\alpha}, u_G \cdot \nabla] \theta \, dx \right| \leq C \| G^5 \|_{W^{s, p}} \| G \|_{L^q} \| \theta \|_{L^\infty} \leq C \| G \|_{W^{s, p_1}} \| G^4 \|_{L^q/4} \| G \|_{L^q}$$  

$$\leq C \| G \|_{H^{s + (1 - \gamma)(1 - \alpha)}} \| G \|_{L^q/4}.$$
where we have applied the Sobolev embedding \( \|G\|_{W^{s,p}} \leq C \|G\|_{H^{s(1-\gamma)(1-\alpha)}}. \) Let now 1 \( \gg \alpha > 0 \) and choose \( s = 1 - \alpha + \sigma \). Note that \( s(1-\gamma)(1-\alpha) = (2-\gamma)(1-\alpha) + \sigma < \alpha/2 \), if \( \sigma \) is small enough. We further bound \( \|G\|_{H^{s(1-\gamma)(1-\alpha)}} \) as follows. If \( \alpha > \frac{5}{6} \), we invoke the Gagliardo-Nirenberg inequality
\[
\|G\|_{H^{s(1-\gamma)(1-\alpha)}} = \|G\|_{H^{2-\gamma(1-\alpha)+\sigma}} \leq C \|G\|_{H^{\alpha/2}}^\mu \|G\|_{L^2}^{1-\mu},
\]
where
\[
(3.3) \quad \mu = \frac{2((2-\gamma)(1-\alpha) + \sigma)}{\alpha}.
\]
If \( \frac{1}{2} < \alpha \leq \frac{5}{6} \), then we take \( \sigma > 0 \) small such that
\[
(3.4) \quad 0 < \frac{1}{2}(5 - 6\alpha + 3\sigma) < 1
\]
and use the Gagliardo-Nirenberg inequality
\[
\|G\|_{H^{2-\gamma(1-\alpha)+\sigma}} \leq C \|G\|_{H^{\alpha/2}}^{\nu_1} \|G\|_{L^2}^{\nu_2} \|G\|_{L^6}^{\frac{(5-6\alpha+3\sigma)}{2}},
\]
where \( \nu_1 + \nu_2 + \frac{1}{2}(5 - 6\alpha + 3\sigma) = 1 \) and
\[
\frac{1}{2} - \frac{(2-\gamma)(1-\alpha) + \sigma}{2} = \nu_1 \left( \frac{1}{2} - \frac{\alpha}{4} \right) + \frac{1}{2} \nu_2 + \frac{1}{12}(5 - 6\alpha + 3\sigma),
\]
or
\[
(3.5) \quad \nu_1 = \frac{2 - 6\gamma(1-\alpha)}{3\alpha}, \quad \nu_2 = \frac{3}{2} + 3\alpha - \frac{3\sigma}{2} - \frac{2 - 6\gamma(1-\alpha)}{3\alpha}.
\]
Note that our \( \gamma, \alpha \) need to be such that \( \nu_1, \nu_2 \in (0, 1) \). We return later in the proof to check this. Note that by Lemma 2.7 we have the control of \( \|G\|_{L^2} \). In the case of \( \alpha > \frac{5}{6} \), we collect all the estimates for the right hand side of (3.2) to obtain
\[
\partial_t \|G(t)\|_{L^6}^6 + C \|G(t)\|_{L^6}^{6/12} \leq C \|\Lambda^{\gamma(1-\alpha)}\theta\|_{L^{\mu}}^4 \|G\|_{H^{\alpha/2}} \|G\|_{L^2}^4\frac{5}{12} + C \|G\|_{H^{\alpha/2}}^r \|G\|_{L^2}^5,
\]
where \( \mu \) is given by (3.3). In the case of \( \frac{1}{2} < \alpha \leq \frac{5}{6} \), we have
\[
\partial_t \|G(t)\|_{L^6}^6 + C \|G(t)\|_{L^6}^{12/12} \leq C \|\Lambda^{\gamma(1-\alpha)}\theta\|_{L^{\mu}}^4 \|G\|_{H^{\alpha/2}} \|G\|_{L^2}^4\frac{5}{8} + C \|G\|_{H^{\alpha/2}}^{\nu_2} \|G\|_{L^2}^5 \|G\|_{L^6}^5,
\]
where \( \nu_1 \) is given by (3.5). By the Gagliardo-Nirenberg inequalities, we have
\[
\|G\|_{L^8}^{\frac{8}{2r}} \leq \|G\|_{L^6}^{\beta_1} \|G\|_{L^{\frac{12}{\alpha}}}^{1-\beta_1}, \quad \|G\|_{L^9}^9 \leq \|G\|_{L^6}^{\beta_2} \|G\|_{L^{\frac{12}{\alpha}}}^{1-\beta_2},
\]
where \( \beta_1 = \beta_1(\gamma, \alpha), \beta_2 = \beta_2(\gamma, \alpha) \) are determined from
\[
\frac{1-2\gamma}{8} = \beta_1 \frac{6}{12} + (1-\beta_1) \frac{2-\alpha}{12}, \quad \frac{1+(1-\gamma)(1-\alpha)}{10} = \frac{1}{q} = \beta_2 \frac{6}{12} + (1-\beta_2) \frac{2-\alpha}{12}.
\]
In short,

\[
\beta_1 = \frac{12}{\alpha} \left( \frac{1 - 2\gamma}{8} + \frac{\alpha - 2}{12} \right)
\]

(3.6)

\[
\beta_2 = \frac{12}{\alpha} \left( \frac{1}{10} \left( 1 + (1 - \gamma)(1 - \alpha) \right) + \frac{\alpha - 2}{12} \right)
\]

(3.7)

Note that our \( \gamma, \alpha \) need to be such that \( \beta_1(\gamma, \alpha), \beta_2(\gamma, \alpha) \in (0, 1) \). We return later in the proof to check this. We apply Young’s inequality to obtain for all small \( \delta > 0 \),

\[
\|G\|_{H^{\alpha/2}}^\mu \|G\|_{L^q}^\nu \leq \|G\|_{H^{\alpha/2}}^\mu \|G\|_{L^{\frac{12}{5(1 - \beta_2)}}}^{\frac{5\beta_2}{5(1 - \beta_2)}} \leq \delta \|G\|_{L^{\frac{12}{5(1 - \beta_2)}}}^6 + C_\delta \left( \|G\|_{H^{\alpha/2}}^\mu \|G\|_{L^6}^{\frac{5\beta_2}{5(1 - \beta_2)}} \right)^\nu,
\]

\[
\|G\|_{H^{\alpha/2}}^\mu \|G\|_{L^6}^{\frac{(5 - 6\alpha + 3\gamma)}{2(1 - \beta_1)}} \|G\|_{L^{\frac{12}{5(1 - \beta_2)}}}^5 \leq \delta \|G\|_{L^{\frac{12}{5(1 - \beta_2)}}}^6 + C_\delta \left( \|G\|_{H^{\alpha/2}}^\mu \|G\|_{L^6}^{\frac{(5 - 6\alpha + 3\gamma)}{2(1 - \beta_1)} + 5\beta_2} \right)^\nu,
\]

where \( \nu' = \frac{\nu}{\nu - 1} \) is the dual index. Similarly, applying Young’s inequality to the other terms yields the estimate

\[
\|\Lambda^{\gamma(1 - \alpha)} \theta\|_{L^4} \|G\|_{H^{\alpha/2}} \|G\|_{L^{\frac{12}{2(1 - \beta_1)}}}^4 \leq \delta \|G\|_{L^{\frac{12}{2(1 - \beta_1)}}}^6 + C_\delta \left( \|\Lambda^{\gamma(1 - \alpha)} \theta\|_{L^4} \|G\|_{H^{\alpha/2}}^\mu \|G\|_{L^6}^{\frac{4\beta_1}{2(1 - \beta_1)^2}} \right)^\nu.
\]

Thus, we may absorb the terms \( 2C\delta \|G\|_{L^{\frac{12}{2(1 - \beta_1)}}}^6 \) (for small enough \( \delta \)) by the corresponding term on the left hand side and we arrive at the inequality, in the case when \( \alpha > \frac{5}{6} \),

\[
\partial_t \|G(t)\|_{L^6}^6 \leq C \left( \|G\|_{H^{\alpha/2}}^\mu \|G\|_{L^{\frac{12}{2(1 - \beta_1)}}}^{5\beta_2} \left( \frac{12}{5(1 - \beta_2)} \right)^\nu \right) + C \left( \|\Lambda^{\gamma(1 - \alpha)} \theta\|_{L^4} \|G\|_{H^{\alpha/2}}^\mu \|G\|_{L^6}^{\frac{4\beta_1}{2(1 - \beta_1)^2}} \right)^\nu.
\]

and, for \( \frac{1}{2} < \alpha \leq \frac{5}{6} \),

\[
\partial_t \|G(t)\|_{L^6}^6 \leq C \left( \|G\|_{H^{\alpha/2}}^\mu \|G\|_{L^{\frac{12}{2(1 - \beta_1)}}}^{\frac{(5 - 6\alpha + 3\gamma)}{2(1 - \beta_1)} + 5\beta_2} \left( \frac{12}{5(1 - \beta_2)} \right)^\nu \right) + C \left( \|\Lambda^{\gamma(1 - \alpha)} \theta\|_{L^4} \|G\|_{H^{\alpha/2}}^\mu \|G\|_{L^6}^{\frac{4\beta_1}{2(1 - \beta_1)^2}} \right)^\nu.
\]

In order to close the argument, in the case of \( \alpha > \frac{5}{6} \), the indices should satisfy

\[
5\beta_2 \left( \frac{6}{5(1 - \beta_2)} \right)^\nu \leq 6; \quad 4\beta_1 \left( \frac{3}{2(1 - \beta_1)} \right)^\nu \leq 6,
\]

(3.8)

\[
\beta_1 \left( \frac{6}{5(1 - \beta_2)} \right)^\nu \leq 2; \quad \left( \frac{3}{2(1 - \beta_1)} \right)^\nu \leq \frac{2}{2\gamma + 1}.
\]

(3.9)
and, in the case of $\frac{1}{2} < \alpha \leq \frac{5}{6}$,
\begin{equation}
(3.10) \quad \left(\frac{5 - 6\alpha + 3\sigma}{2}\right) + 5\beta_2 \left(\frac{6}{5(1 - \beta_2)}\right)' \leq 6; \quad 4\beta_1 \left(\frac{3}{2(1 - \beta_1)}\right)' \leq 6,
\end{equation}
\begin{equation}
(3.11) \quad \nu_1 \left(\frac{6}{5(1 - \beta_2)}\right)' \leq 2; \quad \left(\frac{3}{2(1 - \beta_1)}\right)' \leq \frac{2}{2\gamma + 1}.
\end{equation}
Indeed, by Young’s inequality, we have
\[ \|G\|^\mu_{\left(\frac{6}{5(1 - \beta_2)}\right)}' \|G\|^\nu_1 \left(\frac{6}{5(1 - \beta_2)}\right)' \leq C(1 + \|G\|^2_{H^{\alpha/2}}) \]
and
\[ \|G\|_{L^6}^{5\beta_2 \left(\frac{6}{5(1 - \beta_2)}\right)'}, \quad \|G\|_{L^6}^{\left(\frac{5 - 6\alpha + 3\sigma}{2}\right) + 5\beta_2 \left(\frac{6}{5(1 - \beta_2)}\right)'} \leq 1 + \|G\|_{L^6}^6. \]
This implies the differential inequality
\[ \partial_t \|G(t)\|_{L^6}^6 \leq C(1 + \|G\|_{L^6}^6) \left(\|G\|_{H^{\alpha/2}}^{1/\alpha} \|G\|_{H^{\alpha/2}} \left(\frac{3}{2(1 - \beta_1)}\right)' + \|G\|^2_{H^{\alpha/2}}\right). \]
Applying Gronwall’s then yields
\[ \|G(T)\|_{L^6}^6 \leq (1 + \|G(0)\|_{L^6}^6) \exp(A(T)), \]
where
\[ A(T) = \int_0^T \left[\|G\|_{H^{\alpha/2}}^{1/\alpha} \|G\|_{H^{\alpha/2}} \left(\frac{3}{2(1 - \beta_1)}\right)' + \|G\|^2_{H^{\alpha/2}}\right] dt. \]
By Hölder’s inequality in time, we conclude that
\[ A(T) \leq C_T \|\Lambda^{(1 - \alpha)}\|_{L^\infty_t} \left\|\|G\|_{H^{\alpha/2}}^{1/\alpha} \|G\|_{H^{\alpha/2}} \right\|_{L^2_t}^{2/\alpha} + \|G\|^2_{L^3_t H^{\alpha/2}}. \]
Recall that we have control of all these norms in terms of a function of $T$, which finishes the argument.

It remains to analyze the inequalities (3.8), (3.9), (3.10) and (3.11). It turns out that (3.8) and (3.10) are always satisfied, because
\[ \frac{1}{6} \left(\frac{6}{5(1 - \beta_2)}\right)' = \frac{5\beta_2}{6} = 1 - \frac{5(1 - \beta_2)}{6} - \frac{5\beta_2}{6} = \frac{1}{6} > 0, \]
\[ \frac{1}{3} \left(\frac{3}{2(1 - \beta_1)}\right)' = \frac{2\beta_1}{3} = 1 - \frac{2(1 - \beta_1)}{3} - \frac{2\beta_1}{3} = \frac{1}{3} > 0, \]
\[ \frac{1}{6} \left(\frac{5 - 6\alpha + 3\sigma}{2} + 5\beta_2\right)' = 1 - \frac{5(1 - \beta_2)}{6} - \frac{1}{6} \left(\frac{5 - 6\alpha + 3\sigma}{2} + 5\beta_2\right) \]
\[ = \frac{1}{6} \left(1 - \frac{5 - 6\alpha + 3\sigma}{2}\right) > 0, \]
where we have used (3.7) to obtain the last inequality. The first condition in (3.9) is true due to $\alpha > \frac{5}{6}$. In fact, (3.9) is equivalent to
\[ 1 - \frac{5(1 - \beta_2)}{6} \geq \frac{(2 - \gamma)(1 - \alpha) + \sigma}{\alpha}, \]
which, by (3.7), is the same as
\[
\left(3.12\right) \quad \frac{1}{6} + \frac{10}{\alpha} \left[ \frac{1 + (1 - \gamma)(1 - \alpha)}{10} + \frac{\alpha - 2}{12} \right] \geq \frac{(2 - \gamma)(1 - \alpha) + \sigma}{\alpha}.
\]

It is easily checked that (3.12) always holds for \(\alpha > \frac{5}{6}\) and \(\sigma > 0\) small. The second condition in (3.9), namely, \(\left(\frac{3}{2(1 - \beta_1)}\right)' \leq \frac{2}{2\gamma + 1}\) is equivalent to \(\frac{3}{2(1 - \beta_1)} \geq \frac{2}{1 - 2\gamma}\) or \(\beta_1 \geq \frac{1 + 6\gamma}{4}\). Using (3.6) for \(\beta_1\) yields
\[
\alpha \geq \frac{12\gamma + 2}{3 - 6\gamma}.
\]

Finally, we turn to the conditions in (3.11). The first condition in (3.11), namely \(\nu_1\left(\frac{6}{(1 - \beta_2)}\right)' \leq 2\) is always true due to the definition of \(\nu_1\) in (3.5). In fact, \(\nu_1\left(\frac{6}{(1 - \beta_2)}\right)' \leq 2\) is equivalent to \(1 - \frac{5}{6}(1 - \beta_2) \geq \frac{1}{2}\nu_1\), or, by (3.5) and (3.7),
\[
\frac{1}{6} + \frac{10}{\alpha} \left[ \frac{1 + (1 - \gamma)(1 - \alpha)}{10} + \frac{\alpha - 2}{12} \right] \geq \frac{2 - 6\gamma(1 - \alpha)}{6\alpha},
\]
which is always true. The second condition in (3.11) is the same as the second condition in (3.9) and has already been dealt with.

Now, this means that \(\alpha\) needs to satisfy the following inequalities
\[
\alpha > \max \left(\frac{12\gamma + 2}{3 - 6\gamma}, \frac{4 - 2\gamma}{5 - 2\gamma}\right).
\]

The smallest value of this maximum is achieved for \(\gamma_0 = \frac{43 - \sqrt{1777}}{36}\) so that the value of \(\alpha_{cr}\) is minimized and we get
\[
\alpha_{cr} = \frac{4 - 2\gamma_0}{5 - 2\gamma_0} = \frac{12\gamma_0 + 2}{3 - 6\gamma_0} = \frac{\sqrt{1777} - 23}{24} = 0.798103\ldots
\]

Finally, recall that we also need to check that \(\nu_1, \nu_2, \beta_1(\gamma_0, \alpha), \beta_2(\gamma_0, \alpha) \in (0, 1)\) for \(\alpha \in (\alpha_{cr}, 1)\). The picture below verifies this. This completes the proof of Proposition 1.2.
We now briefly sketch the proof of Theorem 1.1.

Proof of Theorem 1.1. The global existence and smoothness of solutions is proven via two steps. The first step is the local well-posedness of (1.1), which can be established through a standard procedure (see, e.g., [33, 44]). The second step extends the local solution of the first step into a global one through a priori estimates. Proposition 1.2 provides a global $L^q$-bound for $G$ for any $2 \leq q \leq 6$. As in Proposition 7.1 in [25], we can show that, for any $0 \leq s \leq 3\alpha - 2$,

$$\sup_{0 \leq t \leq T} \|G(t)\|_{B^{s}_{6,\infty}} \leq C(T, u_0, \theta_0).$$

Recall the embedding, $B^{3\alpha-2}_{6,\infty}(\mathbb{R}^2) \hookrightarrow B^0_{\infty,1}(\mathbb{R}^2)$ for $\alpha > \frac{7}{9}$. Therefore, for $\alpha > \frac{\sqrt{1777} - 23}{24} > \frac{7}{9}$

$$\|\nabla u_G\|_{L^\infty} = \|\nabla \nabla^\perp \Delta^{-1} G\|_{L^\infty} \leq \|G\|_{B^0_{\infty,1}} \leq \|G\|_{B^0_{6,\infty}}.$$ 

This yields a global Lipschitz bound on $u_G$. The rest of the proof is the same as in [25]. We thus omit further details. This completes the proof of Theorem 1.1.

Acknowledgements

Stefanov’s research is partially supported by NSF grant DMS 1313107. Wu is partially supported by NSF grant DMS 1209153 and the AT&T Foundation at Oklahoma State University. Wu thanks Professors Quansen Jiu, Changxing Miao and Zhifei Zhang for discussions and thanks Dr. Haifeng Shang for comments.

References

[1] D. Adhikari, C. Cao and J. Wu, The 2D Boussinesq equations with vertical viscosity and vertical diffusivity, *J. Differential Equations* **249** (2010), 1078–1088.
[2] D. Adhikari, C. Cao and J. Wu, Global regularity results for the 2D Boussinesq equations with vertical dissipation, *J. Differential Equations* **251** (2011), 1637–1655.
[3] D. Adhikari, C. Cao, J. Wu and X. Xu, Small global solutions to the damped two-dimensional Boussinesq equations, *J. Differential Equations* **256** (2014), 3594–3613.
[4] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Springer, 2011.
[5] J. Bergh and J. Löfström, *Interpolation Spaces, An Introduction*, Springer-Verlag, Berlin-Heidelberg-New York, 1976.
[6] C. Cao and J. Wu, Global regularity for the 2D anisotropic Boussinesq equations with vertical dissipation, *Arch. Ration. Mech. Anal.* **208** (2013), 985–1004.
[7] D. Chae, Global regularity for the 2D Boussinesq equations with partial viscosity terms, *Adv. Math.* **203** (2006), 497–513.
[8] K. Choi, T. Hou, A. Kiselev, G. Luo, V. Sverak and Y. Yao, On the finite-time blowup of a 1D model for the 3D axisymmetric Euler equations, arXiv:1407.4770 [math.AP] 17 Jul 2014.
[9] K. Choi, A. Kiselev and Y. Yao, Finite time blow up for a 1D model of 2D Boussinesq system, *Commun. Math. Phys.*, in press.
[10] P. Constantin, Euler equations, Navier-Stokes equations and turbulence, *Mathematical foundation of turbulent viscous flows*, 143, Lecture Notes in Math., **1871**, Springer, Berlin, 2006.
2D BOUSSINESQ EQUATIONS

[11] P. Constantin and C.R. Doering, Infinite Prandtl number convection, *J. Statistical Physics* **94** (1999), 159–172.
[12] P. Constantin and V. Vicol, Nonlinear maximum principles for dissipative linear nonlocal operators and applications, *Geom. Funct. Anal.* **22** (2012), 1289–1321.
[13] A. Córdoba and D. Córdoba, A maximum principle applied to quasi-geostrophic equations, *Comm. Math. Phys.* **249** (2004), 511–528.
[14] X. Cui, C. Dou and Q. Jiu, Local well-posedness and blow up criterion for the inviscid Boussinesq system in Hölder spaces, *J. Partial Differential Equations* **25** (2012), 220–238.
[15] R. Danchin, Remarks on the lifespan of the solutions to some models of incompressible fluid mechanics, *Proc. Amer. Math. Soc.* **141** (2013), 1979–1993.
[16] R. Danchin and M. Paicu, Global existence results for the anisotropic Boussinesq system in dimension two, *Math. Models Methods Appl. Sci.* **21** (2011), 421–457.
[17] C. Doering and J. Gibbon, *Applied analysis of the Navier-Stokes equations*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 1995.
[18] A.E. Gill, *Atmosphere-Ocean Dynamics*, Academic Press, London, 1982.
[19] L. Grafakos and S. Oh, The Kato-Ponce inequality. *Comm. Partial Differential Equations* **39** (2014), 1128–1157.
[20] T. Hmidi, On a maximum principle and its application to the logarithmically critical Boussinesq system, *Anal. Partial Differential Equations* **4** (2011), 247–284.
[21] T. Hmidi, S. Keraani and F. Rousset, Global well-posedness for a Boussinesq-Navier-Stokes system with critical dissipation, *J. Differential Equations* **249** (2010), 2147–2174.
[22] T. Hmidi, S. Keraani and F. Rousset, Global well-posedness for Euler-Boussinesq system with critical dissipation, *Comm. Partial Differential Equations* **36** (2011), 420–445.
[23] T. Hou and C. Li, Global well-posedness of the viscous Boussinesq equations, *Discrete and Cont. Dyn. Syst.* **12** (2005), 1–12.
[24] J. Jia, J. Peng and K. Li, On the global well-posedness of a generalized 2D Boussinesq equations, *arXiv:1410.8642v1 [math.AP]* 31 Oct 2014.
[25] Q. Jiu, C. Miao, J. Wu and Z. Zhang, The 2D incompressible Boussinesq equations with general critical dissipation, *SIAM J. Math. Anal.* **46** (2014), 3426-3454.
[26] Q. Jiu, J. Wu, and W. Yang, Eventual regularity of the two-dimensional Boussinesq equations with supercritical dissipation, *J. Nonlinear Science*, in press.
[27] D. KC, D. Regmi, L. Tao and J. Wu, The 2D Euler-Boussinesq equations with a singular velocity, *J. Differential Equations* **257** (2014), 82–108.
[28] C. E. Kenig, G. Ponce, and L. Vega, Well-posedness and scattering results for the generalized Korteweg-de-Vries equation via the contraction principle, *Comm. Pure App. Math.* **46** (1993), 527–620.
[29] M. Lai, R. Pan and K. Zhao, Initial boundary value problem for two-dimensional viscous Boussinesq equations, *Arch. Ration. Mech. Anal.* **199** (2011), 739–760.
[30] A. Larios, E. Lunasin and E.S. Titi, Global well-posedness for the 2D Boussinesq system with anisotropic viscosity and without heat diffusion, *J. Differential Equations* **255** (2013), 2636–2654.
[31] G. Luo and T. Hou, Potentially singular solutions of the 3D incompressible Euler equations, *arXiv:1310.0497v2 [physics.flu-dyn]* 8 Dec 2013.
[32] A.J. Majda, *Introduction to PDEs and Waves for the Atmosphere and Ocean*, Courant Lecture Notes in Mathematics **9**, AMS/CIMS, 2003.
[33] A.J. Majda and A.L. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge University Press, Cambridge, 2001.
[34] C. Miao and L. Xue, On the global well-posedness of a class of Boussinesq-Navier-Stokes systems, *NoDEA Nonlinear Differential Equations Appl.* **18** (2011), 707–735.
[35] C. Miao, J. Wu and Z. Zhang, Littlewood-Paley Theory and its Applications in Partial Differential Equations of Fluid Dynamics, Science Press, Beijing, China, 2012 (in Chinese).
[36] K. Ohkitani, Comparison between the Boussinesq and coupled Euler equations in two dimensions. Tosio Kato’s method and principle for evolution equations in mathematical physics (Sapporo, 2001). Surikaisekikenkyusho Kokyuroku No. 1234 (2001), 127–145.
[37] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer-Verlag, New York, 1987.
[38] T. Runst and W. Sickel, *Sobolev Spaces of fractional order, Nemytskij operators and Nonlinear Partial Differential Equations*, Walter de Gruyter, Berlin, New York, 1996.

[39] A. Sarria and J. Wu, Blowup in stagnation-point form solutions of the inviscid 2d Boussinesq equations, [arXiv:1408.6625 [math.AP]] 28 Aug 2014.

[40] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ, 1970.

[41] H. Triebel, *Theory of Function Spaces II*, Birkhauser Verlag, 1992.

[42] B. Wen, N. Dianati, G. Chini and C. Doering, New upper bounds and reduced dynamical modeling for Rayleigh-Bénard convection in a fluid saturated porous layer, *Commun. Nonlinear Sci. Numer. Simul.* **17** (2012), 2191–2199.

[43] J. Whitehead and C. Doering, Internal heating driven convection at infinite Prandtl number, *J. Math. Phys.* **52** (2011), 093101, 11 pp.

[44] J. Wu, The 2D Boussinesq equations with partial or fractional dissipation, Lectures on the analysis of nonlinear partial differential equations, Morningside Lectures in Mathematics, Edited by Fang-Hua Lin and Ping Zhang, International Press, Somerville, MA, 2014, in press.

[45] J. Wu and X. Xu, Well-posedness and inviscid limits of the Boussinesq equations with fractional Laplacian dissipation, *Nonlinearity* **27** (2014), 2215–2232.

[46] J. Wu, X. Xu and Z. Ye, Global smooth solutions to the n-dimensional damped models of incompressible fluid mechanics with small initial datum, *J. Nonlinear Science*, in press.

[47] X. Xu, Global regularity of solutions of 2D Boussinesq equations with fractional diffusion, *Nonlinear Analysis: TMA* **72** (2010), 677–681.

[48] W. Yang, Q. Jiu and J. Wu, Global well-posedness for a class of 2D Boussinesq systems with fractional dissipation, *J. Differential Equations* **257** (2014), 4188–4213.

[49] Z. Ye and X. Xu, Global regularity of the two-dimensional incompressible generalized magnetohydrodynamics system, *Nonlinear Anal.* **100** (2014), 86–96.

[50] K. Zhao, 2D inviscid heat conductive Boussinesq equations on a bounded domain, *Michigan Math. J.* **59** (2010), 329–352.

1 Department of Mathematics, University of Kansas, 1460 Jayhawk Blvd, Lawrence, Kansas 66045-7594

E-mail address: stefanov@ku.edu

2 Department of Mathematics, Oklahoma State University, Stillwater, OK 74078

E-mail address: jiahong.wu@okstate.edu