Yang-Baxter algebra and generation of quantum integrable models

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March 30, 2022

Abstract

An operator deformed quantum algebra is discovered exploiting the quantum Yang-Baxter equation with trigonometric R-matrix. This novel Hopf algebra along with its $q \to 1$ limit appear to be the most general Yang-Baxter algebra underlying quantum integrable systems. Three different directions of application of this algebra in integrable systems depending on different sets of values of deforming operators are identified. Fixed values on the whole lattice yield subalgebras linked to standard quantum integrable models, while the associated Lax operators generate and classify them in an unified way. Variable values construct a new series of quantum integrable inhomogeneous models. Fixed but different values at different lattice sites can produce a novel class of integrable hybrid models including integrable matter-radiation models and quantum field models with defects, in particular, a new quantum integrable sine-Gordon model with defect.

PACS numbers 02.30.Ik, 02.20.Uw, 05.50.+q, 03.65.Fd

Key Words: Operator deformed quantum algebra; unifying scheme for quantum integrable systems; new quantum integrable models: inhomogeneous, matter-radiation, sine-Gordon with defect.

1 Introduction

Quantum integrable systems (QIS) are nonlinear, interacting and nonperturbative quantum many body or field models having large symmetries with mutually commuting set of conserved operators $\{C_n\}, \ n = 1, 2, \ldots$. Such models usually allow exact solution of the eigenvalue problem through the algebraic Bethe ansatz method, for all conserved operators including the Hamiltonian of the system.

Every lattice QIS is represented by its own Lax operator $L_j(\lambda)$ at each lattice site $j = 1, 2, ..., N$ and for the integrability it should satisfy quantum Yang-Baxter equation (QYBE)

$$R(\lambda - \mu)L_j(\lambda) \otimes L_j(\mu) = (L_j(\mu) \otimes I)(L_j(\lambda) \otimes I)R(\lambda - \mu), \ j = 1, 2, ..., N$$  \hspace{1cm} (1.1)

together with the ultralocality condition $L_j(\lambda) \otimes L_k(\mu) = (L_k(\mu) \otimes I)(L_j(\lambda) \otimes I)$, at $k \neq j$. These two relations combined together lead to several important consequences.

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i) They provide sufficient condition for quantum integrability. Defining monodromy matrix \( T(\lambda) = \prod_j L_j(\lambda) \) on the whole lattice, one shows that the same QYBE holds also for this global object:

\[
R(\lambda - \mu)T(\lambda) \otimes T(\mu) = (T(\mu) \otimes I)(T(\lambda) \otimes I)R(\lambda - \mu).
\]  

Taking trace from both sides one derives the trace identity: \([\tau(\lambda), \tau(\mu)] = 0\), for \( \tau(\lambda) = Tr(T(\lambda)) \), yielding the crucial integrability condition \([C_n, C_m] = 0\), through \( \tau(\lambda) = \sum_n C_n \lambda^n \).

ii) QYBE (1.2) plays important role in exact solution through the algebraic Bethe ansatz method by providing commutation relations between diagonal (generator of conserved operators) and off-diagonal (‘creation’ / ‘annihilation’ operators) elements of \( T(\lambda) \)-matrix.

iii) Finally, QYBE (1.1) defines the commutation relations between operator-elements of the quantum Lax operator (Yang-Baxter (YB) algebra), specified by the quantum \( R \)-matrix, a -number matrix which fixes the the structure constants of the algebra. Interestingly, the very fact that the same QYBE is valid also at the global level (1.2), tells that this YB algebra must be a Hopf algebra \( A \) with defining properties:

- **coproduct**: \( \Delta : A \rightarrow A \otimes A \),  
  **product**: \( m : A \otimes A \rightarrow A \),  
  **unit**: \( e : k \rightarrow A \),  
  **counit**: \( \epsilon : A \rightarrow k \)  
  and **antipode** (inverse): \( S : A \rightarrow A \).

For some integrable systems when the last property is absent the algebra becomes simply a bialgebra (algebra +coalgebra). The underlying YB algebra classifies the quantum integrable models into three major classes: *rational, trigonometric* and *elliptic*, distinguished by such solutions of the associated quantum \( R \)-matrix. Here we will focus on the first two classes.

## 2 Known Yang-Baxter algebra

The simplest YB algebra is given by the standard Lie algebra \( sl(2) : [S^+, S^-] = 2S^3 \), \( [S^3, S^\pm] = \pm S^\pm \). This is a cocommutative Hopf algebra generated by the quantum \( R \)-matrix:

\[
R_{11}^{11} = R_{22}^{22} = a(\lambda), \quad R_{12}^{12} = R_{21}^{21} = b(\lambda), \quad R_{21}^{12} = R_{12}^{21} = c,
\]  

having solutions \( R^{rat} \) in rational functions of the spectral parameter: \( a(\lambda) = \lambda + \alpha \), \( b(\lambda) = \lambda \), \( c = \sin \alpha \). This algebra is linked to the well known quantum integrable models [1] like 1) Nonlinear Schrödinger (NLS) equation and the lattice NLS, 2) xxx-spin-\( \frac{1}{2} \) chain 3) rational Gaudin model etc.

A more general (and more interesting) YB algebra is associated with the trigonometric \( R^{trig} \)-matrix solution of (2.3) as \( a(\lambda) = \sin(\lambda + \alpha) \), \( b(\lambda) = \sin \lambda \), \( c = \sin \alpha \). and is given by the celebrated quantum algebra \( sl_q(2) \): \( [S_q^+, S_q^-] = \frac{\sin(\alpha 2 S^3)}{\sin \alpha} \), \( [S^3, S_q^\pm] = \pm S_q^\pm \), where \( q = e^{i\alpha} \) is the deformation parameter. FRT construction [2, 3] gives an elegant way of obtaining this quantum algebra together with its non-cocommuting Hopf algebra properties and the explicit form of the associated \( (2 \times 2) \) Lax operator. \( sl_q(2) \) was shown [3] to be the underlying YB algebra of well known quantum integrable systems like 1) sine-Gordon (SG) and lattice SG model, 2) xxxz-spin chain etc.

We however observe that there exists a series of other important quantum integrable models of diverse nature having completely different Lax operators but associated with the same quantum matrix: \( R^{trig} \) or \( R^{rat} \). Few such examples are the quantum Toda chain belonging to the rational class, and the relativistic Toda chain, Liouville model, derivative NLS etc. belonging to the trigonometric class. Therefore in spite of the far-reaching success of the quantum algebra \( sl_q(2) \) it is pertinent to ask
i) Does it exhaust all possible YB algebras associated with the trigonometric $R$-matrix? ii) What are the YB algebras underlying other integrable models? iii) Can they all be obtained from $sl_2(2)$ or $sl(2)$? iv) Can there be any deep relation among diverse integrable models sharing the same quantum $R$-matrix, despite of having completely different Lax operators?

In search for answers to these fundamental questions, we conjecture that there must exist a more general YB algebra and the related quantum Lax operator as associated with the same $R^{\text{trig}}$-matrix (and its $q \to 1$ limit linked with $R^{\text{rat}}$-matrix), which should cover all quantum integrable models with $(2 \times 2)$ Lax matrix. It should include the known quantum algebras (and its $q \to 1$ limit) along with related models as particular cases, while having the freedom to generate YB algebras underlying all other known and new quantum integrable models.

3 Operator-deformed quantum algebra

Supporting our conjecture we discover a conceptually novel YB algebra, an algebra deformed not only by some operators. Our idea is to follow closely the FRT construction for $sl_2(2)$ and the associated Lax operator [2], but deform the related Borel subalgebras further by operators $\hat{c}_a^\pm$, $a = 1, 2$, resulting a new Lax operator

$$L_{\text{anc}}^\text{trig}(\xi) = \left( \begin{array}{cc} \xi c_1^+e^{i\alpha s^3} + \xi^{-1}c_1^-e^{-i\alpha s^3} & 2\sin \alpha s_q^- \\ 2\sin \alpha s_q^+ & \xi c_2^+e^{-i\alpha s^3} + \xi^{-1}c_2^-e^{i\alpha s^3} \end{array} \right), \quad \xi = e^{i\lambda},$$

(3.4)

while the $R^{\text{trig}}(\lambda)$-matrix remains the same. Inserting (3.4) together with in QYBE (1.1) we derive the novel operator-deformed quantum algebra, we call Ancestor algebra [4], defined through the relations

$$[s_q^+, s_q^-] = (\hat{M}^+ \sin(2\alpha s^3) - i\hat{M}^- \cos(2\alpha s^3)) \frac{1}{\sin \alpha}, \quad [s^3, s_q^\pm] = \pm s_q^\pm, \quad [\hat{M}^\pm, c] = 0$$

(3.5)

where the deforming operators $\hat{M}^\pm = \frac{1}{2}(\hat{c}_1^+\hat{c}_2^- \pm \hat{c}_1^-\hat{c}_2^+) \pm$ and all operators $\hat{c}_a^\pm$, $a = 1, 2$ are mutually commuting as well as central (commuting with all other generators of algebra (3.5)). We believe that (3.4) is the most general form of the Lax operator, which can generate $2 \times 2$ Lax operators of all integrable systems belonging to this class and the operator-deformed quantum algebra (3.5) is the most general YB algebra allowed by the simplest $R^{\text{trig}}$-matrix. This ancestor algebra exhibits the following unique and distinguishing properties.

1) The operator deformed quantum algebra (3.5) is a bialgebra (algebra + coalgebra) with the Hopf algebra properties:

   i) Coproduct $\Delta$:

   $$\Delta(s_q^+) = s_q^+ \otimes c_1^- q^{-s^3} + c_2^- q^s \otimes s_q^+,$$

   $$\Delta(s_q^-) = s_q^- \otimes c_2^+ q^{-s^3} + c_1^+ q^s \otimes s_q^-,$$

   $$\Delta(s^3) = I \otimes s^3 + s^3 \otimes I,$$  \quad \Delta(\hat{c}_a^\pm) = \hat{c}_a^\pm \otimes \hat{c}_a^\pm, \quad a = 1, 2$$

Note that unlike deforming parameter $q$, deforming operators $\hat{c}_a^\pm$ and hence $\hat{M}^\pm$ have nontrivial coproduct.

   ii) Antipode $S$:

   $$S(s_q^+) = -c_1^+e^{-i\alpha s^3}s_q^-e^{i\alpha s^3}(\hat{c}_2^+)^{-1},$$

   $$S(s_q^-) = -(\hat{c}_2^-)^{-1}e^{-i\alpha s^3}S^+e^{i\alpha s^3}(\hat{c}_1^-)^{-1},$$

   $$S(\hat{c}_a^\pm) = (\hat{c}_a^\pm)^{-1},$$

   $$S(e^{\pm i\alpha s^3}) = e^{\mp i\alpha s^3}.$$


In a QIS if any $\hat{c}_a^\pm = 0$, the antipode disappears turning the Hopf algebra into a bialgebra.

iii) Counit $\epsilon$:

$$\epsilon(s_q^+) = 0, \quad \epsilon(e^{\pm i\alpha s^3}) = 1, \quad \epsilon(\hat{c}_i^\pm) = c_i^\pm$$

For product $m$ one can take the formal definition of multiplication in the algebra, while unit $\eta$ may be defined through the unital element 1 as $\eta(\xi) \to \xi 1$.

2). Note that algebra (3.5) is deformed by parameter $q$ as in the usual quantum algebra, but additionally it is deformed also by central operators $\hat{M}^\pm$, prompting our definition of this deformed algebra as a novel operator deformed quantum algebra. The deforming operators being central, they act simply as multiplicative parameters, which however can take positive/negative or zero values, generating different possible subalgebras as described below. It should be noted that for quadratic algebra (3.5) more sophisticated construction like quantum double, universal R-matrix or even finite-dimensional representations might be problematic and could be an interesting future problem to explore. However this does not create any problem in applying it as a YB algebra for generating and classifying QIS, which is the main purpose of the present investigation. Generically this algebra allows infinite-dimensional representation, which may be given through canonical operators $[u, p] = i$ as

$$s^3 = u, \quad s_q^+ = e^{-ip}g(u), \quad s_q^- = g(u)e^{ip}$$

(3.6)

where

$$g^2(u) = \frac{1}{2\sin^2\alpha}(\kappa + \sin\alpha(s - u)(M^+ \sin\alpha(u + s + 1) - iM^- \cos\alpha(u + s + 1)))$$

(3.7)

with $\kappa$ and spin variable $s$ being arbitrary constants. We drop hat from the deforming central operators $\hat{M}^\pm$ which act only multiplicatively and can also have zero values.

The integrable ancestor model, as obtained using (3.6, 3.7) in its Lax operator (3.4), represents a generalized lattice sine-Gordon (LSG) model. It reduces to the known LSG [5] in the absence of deforming operators (i.e. at $\hat{M}^+ = 1, \hat{M}^- = 0$), when we also recover from (3.5) the well known $sl_q(2)$ and from (3.4) the known Lax operator of [2]. In general (3.5) can yield nine major subalgebras for the range of values $M^+ > 0, < 0, = 0$ (up to scaling), for each of the sectors $M^- = 0, 0 >, < 0$. Among these subalgebras $M^+ > 0, M^- = 0$ yields the well known $sl_q(2)$, while $M^+ < 0, M^- = 0$ sector gives the corresponding noncompact case. All other cases give a variety of subalgebras describing underlying YB algebras of important quantum integrable systems, including new ones.

### 3.1 Operator-deformed algebra at $q \to 1$-limit

Consider now $q \to 1$ limit of the above operator-deformed algebra, when the related structures reduces to their corresponding rational limit: Quantum $R^{\text{trig}}$-matrix goes to the rational one $R^{\text{rat}}$. The operators $s_q^b \to s^b, b = 1, 2, 3, \hat{c}_a^\pm \to c_a^0 \pm 1, a = 1, 2$ (since they are also q-deformed!) and the spectral parameter $\xi \to \lambda$, while the ancestor Lax operator (3.4) reduces to its rational form

$$L^{\text{rat}}_{\text{anc}}(\lambda) = \begin{pmatrix} c_0^0(\lambda + s^3) + c_1^1 & s^- \\ s^+ & c_0^0(\lambda - s^3) - c_1^1 \end{pmatrix}.$$  

(3.8)

Finally algebra (3.5), with deforming operators $\hat{M}^\pm \to \hat{n}^\pm$, reduces to another novel operator-deformed (but $q \to 1$) bi-algebra

$$[s^+, s^-] = 2\hat{n}^+s^3 + \hat{n}^-, \quad [s^3, s^\pm] = \pm s^\pm, \quad [\hat{n}^\pm, \cdot] = 0$$

(3.9)
with central operators $\hat{m}^+ = \hat{c}_1^0 \hat{c}_2^0$, $\hat{m}^- = \hat{c}_1^1 \hat{c}_2^0 + \hat{c}_1^0 \hat{c}_2^1$, as expressed through mutually commuting central operators $\hat{c}_a^i$, $a = 0, 1; i = 1, 2$. Interestingly it still exhibits non-cocommutative coproduct given by $\Delta(s^+) = \hat{c}_1^0 \otimes s^+ + s^+ \otimes \hat{c}_2^0$, $\Delta(s^-) = \hat{c}_2^0 \otimes s^- + s^- \otimes \hat{c}_1^0$, etc. Note that for nontrivial $\hat{m}^-$ this is a new operator-deformed Hopf algebra with nine distinct subalgebras with $m^+ > 0$, $m^- = 0$, $m^+ < 0$, $m^- = 0$, for each of the combinations from $m = 0, > 0, < 0$. The case $m^+ > 0$, $m^- = 0$ yields the standard Lie algebra $sl(2)$, while $m^+ < 0$, $m^- = 0$ yields its noncompact variant $sl(1, 1)$. The rest of the cases gives the freedom of describing other integrable models.

3.2 Application to quantum integrable systems

Since only the ancestor Lax operator (3.4) contains the deforming operators but not the associated $R^{\text{trig}}$-matrix, we can construct Lax operators of all integrable systems belonging to the same trigonometric class together with their YB algebras, by suitable choices of the deforming operators. Similarly at $q \to 1$ limit we would get the corresponding result for the models belonging to the rational class all sharing the same $R^{\text{rat}}$-matrix. We identify three types of integrable models that we can construct following our scheme, depending on different range of values of the deforming operators.

i) For fixed values of the deforming operators on the whole lattice we get from (3.5) its different subalgebras which interestingly represent the underlying YB algebras of original integrable models. At the same time the ancestor $L^{\text{anc}}(\xi)$-operator (3.4) and its rational limit $L^{\text{rat}}_{\text{anc}}$ (3.8) generate in a unified way the representative Lax operators of known as well as new quantum integrable models as $R(xxz) : L^{\text{anc}}_{\text{anc}} \Rightarrow \text{i)xxz - spin chain, ii}sine-Gordon (lattice +field)}$, $\text{iii)}$derivative NLS (lattice +field ) $\text{iv)}$q-bosonic $\text{vii)}$Liouville (lattice +field $\text{v)}$massive Thirring (bosonic)

And at $q \to 1$ (rational limit) $R(xxx) : L^{\text{rat}}_{\text{anc}} \Rightarrow \text{i)xxx - spin chain, ii) NLS (lattice +field)}$, $\text{iii)}$ simple lattice NLS $\text{iv) Toda chain}$

This systematic generation of quantum integrable models from a single ancestor model also answers to some intriguing questions raised above. Our finding shows clearly that all these integrable models are related deeply to each other as descendants of the same ancestor and therefore they share the same inherited R-matrix. Moreover their underlying YB algebras are obtained as different subalgebras of the same ancestor algebra.

ii) For variable values of the deforming operators on the lattice however we obtain a new series of quantum integrable inhomogeneous models.

iii) Choosing different but fixed values of the deforming operators at different sectors of the lattice, we can combine different interacting integrable models and generate a novel class of integrable hybrid models.

We present below details of the YB algebras as subalgebras of our ancestor algebra and its major applications in quantum integrable systems as mentioned above.

4 Generation of integrable models and underlying YB algebras

Starting from the ancestor Lax operators (3.4) and (3.8) and the related operator deformed algebras (3.5) and (3.9) we generate quantum integrable models, known as well as new, together with their underlying YB algebras in a unified way.
4.1 q-deformed algebra and related integrable models

We first focus on models belonging to the trigonometric class with $R^{\text{trig}}$.

1) At q-deformed but operator-undeformed limit: $c_1^+ = c_2^+ = c_1^- = c_2^- = 1$ giving $M^+ = 1, M^- = 0$, our general algebra (3.5) reduces to the known quantum algebra $sl_q(2)$ and the ancestor Lax operator (3.4) to the known $L(\lambda)$-operator of [2]. Consequently the spin-$\frac{1}{2}$ representation would generate $xxz$-spin chain [3], while realization (3.7) reproduces $g(u) = \frac{1}{2\sin \alpha} [1 + \cos \alpha(2u + 1)]^{\frac{1}{2}}$ related to the known lattice SG model [5].

2) $M^+ = -1, M^- = 0$ case reduces (3.5) to the corresponding noncompact quantum algebra $sl_q(1,1)$ and reproduces the related integrable model, e.g. q-deformed Buck-Sukumar(BS) model.

3) Taking the deforming operators as $\hat{c}_1^+ = (\hat{c}_2^+)^{-1} = p^c$, we get $p,q$-parameter-deformed Hopf algebra $gl_{p,q}(2)$, which is represented by the same $sl_q(2)$ algebra, but is $p^\delta$-deformed as a co-algebra. This YB algebra is linked with the quantum integrable Ablowitz-Ladik model.

It is worth noting that, such an algebra is believed to be obtainable only through twisting transformation with an external parameter $p = e^{i\theta}$ [6]. Under such a transformation the $R$-matrix is also deformed to a twisted one: $R^{\text{trig}}(\lambda,\theta)$-matrix. However, contrary to this belief we obtain the same bi-algebra without changing the $R^{\text{trig}}(\lambda)$-matrix, since in our case the deforming operators replaces the role of twisting parameter as mentioned above.

4) Choosing the values of the operators as $M^+ = \sin \alpha, M^- = i \cos \alpha$, compatible with $c_1^+ = c_2^+ = 1, c_1^- = -iq, c_2^- = \frac{i}{q}$ one gets the q-bosonic algebra

$$[\psi_q, N] = \psi_q, \quad [\psi_q^\dagger, N] = -\psi_q^\dagger, \quad [\psi_q, \psi_q^\dagger] = \cos(\alpha(2N + 1)), \quad (4.10)$$

by assuming $s^+_q = \psi_q, \quad s^-_q = \psi_q^\dagger, \quad s^3 = N$. This opens up an interesting possibility of constructing quantum integrable models involving q-boson [7]. In this case (3.7) simplifies to $g^2(u) = [-2u]_q$.

A q-boson can be mapped into a standard boson $\psi$ on the lattice with commutation relation $[\psi, \psi^\dagger] = \frac{h}{\Delta},$ as $\psi_q = \psi(\frac{[2N]_q}{2N \cos \alpha})^{\frac{1}{2}}, \quad N = \psi^\dagger \psi$. This can construct from the above q-bosonic model an exact lattice version of the quantum integrable derivative NLS (with Lax operator obtained directly from (3.4)), which at the continuum limit $\Delta \to 0$ yields the quantum DNLS model [8] represented by the equation $i\psi_t = \psi_{xx} - i(\psi^\dagger \psi) \psi_x$. This QFT model in the N-particle sector is equivalent to the interacting Bose gas with derivative $\delta$-function potential [9].

5) For another choice $M^+ = M^- = 1$ we derive a novel exponentially deformed algebra

$$[s^+_q, s^-_q] = \frac{e^{2i\alpha s^3}}{2i \sin \alpha}, \quad (4.11)$$

which turns out to be the underlying YB algebra of the quantum integrable exact lattice Liouville model, with its Lax operator obtained from (3.4) with realization (3.7) reducing to $g(u) = \frac{(1 + e^{\alpha(2u + 1)})^{\frac{1}{2}}}{\sqrt{2 \sin \alpha}}$. At the continuum limit it gives the quantum Liouville field model given by the equation $u_{xt} - u_{xx} = e^{\alpha u}$.

It is important to note that the present values of $M^\pm$ remain same even with $c^-_1 \neq 0$, which would give the same algebra and the same realization, but a different Lax operator. This is indeed an intriguing possibility of constructing different useful Lax operators for the same model, in a systematic way. For example, a second Liouville Lax operator we can construct here so easily recovers that invented by Faddeev through many innovative tricks [11].

6) In a similar way the particular case $M^\pm = 0$ can be achieved with different sets of choices:

i) $c^-_a = 0, c^+_a = 1, \ a = 1, 2$ ii) $c^+_2 = 0, c^-_1 = -c^+_1 = 1, iii) c^+_1 = 1, \ rest \ of \ c's = 0,$
all of which lead to the same algebra

\[ [s^+_q, s^-_q] = 0, \ [s^3_q, s^\pm_q] = \pm s^\pm_q. \]  

(4.12)

However, they may generate different Lax operators from (3.4), corresponding even to different models, though with the same underlying algebra. In particular, case i) leads to the light-cone SG model, while ii) and iii) give two different Lax operators found in [12] and [13] for the same relativistic Toda chain. Since here we get \( g(u) = \text{const.} \), interchanging \( u \to -ip, p \to -iu \), in (3.7) yields simply \( s^3 = -ip, s^\pm_q = \alpha e^{\mp u} \), which generates a quantum integrable discrete-time or relativistic quantum Toda chain [12].

Remarkably, all above algebras are obtained as subalgebras from our ancestor algebra, but can not be obtained from \( sl_q(2) \), without invoking singular limits. Note also that all the descendant models listed above have the same trigonometric \( R^{\text{trig}} \)-matrix, inherited from the ancestor model and similar is true for the rational class, as we will see below. This unveils the mystery why a wide range of models found to share the same \( R \)-matrix, while their \( L \)-operators are obtained from the same ancestor Lax operator (3.4) and their underlying YB algebras from the same ancestor algebra (3.5) at various reductions.

### 4.2 q-undeformed algebra and related integrable models

We focus now on the \( q \to 1 \) limit of algebra (3.5), which would still be operator deformed through \( \hat{m}^\pm \). It is interesting to find that the bosonic representation (3.7) at this limit reduces to a generalized Holstein-Primakov transformation (HPT)

\[ s^3 = s - N, \ s^+ = g_0(N)\psi, \ s^- = \psi^\dagger g_0(N), \ g_0^2(N) = \hat{m}^- + \hat{m}^+(2s - N), \ N = \psi^\dagger \psi. \]  

(4.13)

with the central deforming operators acting multiplicatively. This is an exact realization of (3.9), associated with the Lax operator (3.8). This rational ancestor model, representing a quantum integrable generalized lattice NLS model, can generate in a systematic way all other integrable models of the rational class through different choices for the values of the deforming operators. This constructs at the same time the Lax operators of the models as reductions of (3.8) and their underlying YB algebras as subalgebras of (3.9).

1) When the deforming operators vanish with the choice \( m^+ = 1, m^- = 0 \), (3.9) gives clearly the standard \( su(2) \) and for the spin \( \frac{1}{2} \) representation one recovers the \( xxx \) spin chain [3].

2) A bosonic realization of the same algebra simplifies (4.13) to the standard HPT and (3.8), reproducing the well known lattice NLS model [5].

3) On the other hand for \( m^+ = -1, m^- = 0 \), (3.9) reduces to \( su(1,1) \) and gives the related integrable models, one of which is the integrable BS type model constructed in sect. 7.

4) For the complementary choice \( m^+ = 0, m^- = 1 \), (3.9) reduces to a non semi-simple algebra

\[ [s^\pm, s^3] = \mp s^\pm, \ [s^+, s^-] = 1 \]  

(4.14)

and (4.13) with \( g_0(N) = 1 \) gives a direct bosonic realization \( s^+ = \psi, s^- = \psi^\dagger, s^3 = s - N \). Using this reduction in (3.8) we discover another quantum integrable simple lattice NLS model [14].

5) A trivial choice \( m^\pm = 0 \) reduces (3.9) to the same algebra (4.12)

\[ [s^+, s^-] = 0, \ [s^3, s^\pm] = \pm s^\pm \]  

(4.15)
and therefore we can take the same realization \( s^3 = -ip, s^\pm = \alpha e^{\pm u} \) found for the relativistic case, but now with ancestor Lax operator (3.8) and \( R^{rat} \)-matrix. This gives quantum integrable nonrelativistic Toda chain \([1]\).

Thus we have shown how the ancestor model (3.8) with (4.13) can generate quantum integrable models, all sharing the same rational \( R \)-matrix in a unified way and their YB algebras like \( sl(2), (4.14), (4.15) \) are obtained as subalgebras from operator-deformed ancestor algebra (3.9).

It is important to note that Lax operators like (3.4) and (3.8) in their bosonic realization appeared in some earlier work \([10]\) and they were shown to be the most general possible form in their respective class.

Apart from the discrete models obtained above, one can construct integrable QFT models from their exact lattice versions at the continuum limit. For such construction we have to scale the operators like \( \Delta p_j, \Delta \hat{c}_a^\pm, \Delta \hat{\psi}_j \), with lattice spacing \( \Delta \), which at continuum limit \( \Delta \to 0 \) give \( p_j \to p(x), \psi_j \to \psi(x) \) etc. In this process Lax operator \( L(x, \lambda) \) for the continuum model is obtained from its discrete counterpart as \( L_j(\lambda) \to I + \Delta L(x) \). The associated \( R \)-matrix however remains the same since it does not contain \( \Delta \). Thus integrable quantum field models like sine-Gordon, Liouville, NLS or the derivative NLS can be obtained from their discrete variants constructed above.

### 5 Quantum integrable inhomogeneous models

In spite of intensive study of integrable inhomogeneous classical models, their quantum versions seem to remain unexplored, except perhaps the recent work \([15]\). We can construct a novel class of quantum integrable models as inhomogeneous versions of the original integrable models listed above. The idea of construction is to take the values of the deforming operators to be site dependent functions, which would replace \( \hat{M}^\pm \) in \( g(u) \) (3.7) by site dependent operators \( \hat{M}_j^\pm \). Consequently all \( \hat{c} \)'s in Lax operator (3.4) would be site-dependent \( \hat{c}_j \)'s.

However the YB algebra in inhomogeneous lattice models remains the same as in their original models with the same quantum \( R \)-matrix. Physical interpretation of such inhomogeneities may be as impurities, varying external fields, incommensurate-Ness etc. Few examples of such models are

1. Inhomogeneous sine-Gordon model

Taking the values of deforming operators different at different lattice sites: \( c_1^\pm = \Delta m_j e^{i\alpha \theta_j}, \quad c_2^\pm = \Delta m_j e^{-i\alpha \theta_j} \), which yields \( M_j^\pm = (\Delta m_j)^2, M_j^- = 0 \), one can construct a novel variable mass discrete SG model without spoiling its integrability. In the field limit \( \Delta \to 0 \) it would yield a quantum integrable inhomogeneous sine-Gordon model with variable mass \( m(x) \) in an external gauge field \( \theta(x) \).

2. Inhomogeneous lattice NLS model

Consider site-dependent (in general also time-dependent) deforming operators in (3.8) and in (4.13) as

\[
\begin{align*}
    c_0^1 &= c_0^2 \equiv g_j(t), \\
    c_1^1 &= -c_1^2 = f_j(t), \quad \text{giving} \quad m^+ = g_j^2, \quad m^- = 0, \\
\end{align*}
\]

with \( f_j, g_j \) time dependent arbitrary discrete functions. This gives from (4.13) \( g_j^3(N) = g_j^2(2s_j - N_j \Delta) \), which reduces (3.8) to the \( L \)-operator of the quantum integrable inhomogeneous exact lattice NLS. At continuum and high spin limit \( s_j \to \frac{1}{2} g_j^{-1} \) limit we get a quantum integrable inhomogeneous NLS field model with two arbitrary functions \( f(x, t), g(x, t) \) \([15]\). Note that unlike conventional inhomogeneous classical models, our construction is iso-spectral, since inhomogeneity enters here solely through the deforming operators in the Lax operator without affecting the spectral parameter and hence the \( R \)-matrix.
3). Quantum integrable inhomogeneous Toda chain

We choose inhomogeneity as $c_{1j}, c_{0j}$ with $c_2 = c_0 = 0$, resulting $m^+ = 0$. This gives from (3.8) quantum integrable inhomogeneous Toda chain with the Hamiltonian

$$H = \sum_j \left( p_j + \frac{c_{1j}}{c_{0j}} \right)^2 + \frac{1}{e^{u_{j+1} - u_j}}$$

In a similar way inhomogeneous versions of Liouville model, relativistic Toda, Ablowitz-Ladik model etc. can be constructed.

6 Quantum integrable hybrid models

A new class of integrable models may be constructed combining different integrable models which interact between themselves and preserve their integrability as a hybrid model.

The idea of construction is to take Lax operators $L^{(a)}_j$ of different descendant models $a = 1, 2, 3\ldots$, having the same $R$-matrix (as listed above) and insert them at different sites $j$ through $T(\lambda) = L^{(1)}_1 L^{(2)}_2 \cdots L^{(c)}_N$, which satisfies the QYBE and hence represents a quantum integrable model. Few interesting examples are:

1). Massive Thirring model: A spectacular example is a new bosonic version of quantum integrable massive Thirring model [8, 4]. It can be constructed as a hybrid of two integrable DNLS models by fusing two DNLS Lax operators: $L_{DNLS_1}(\xi, \psi_1) \otimes L_{DNLS_2}^{-1}(\xi, \psi_2)$ yielding the Lax operator of the massive Thirring model involving two bosonic fields ($\psi_1, \psi_2$).

2). Integrable matter radiation models: Following the above idea we may construct a series of important matter-radiation models and their q-deformations as quantum integrable hybrid models [16]. We furnish more details in the next section.

3). Exotic Hybrid models: We can construct in principle hybrid integrable lattice models by combining Lax operators of any lattice models belonging to the same class. For example one can build hybrids of

i) Toda chain and discrete NLS model, ii) Relativistic Toda chain and the Ablowitz-Ladik model

iii) lattice sine-Gordon with Liouville model etc.

4) Integrable quantum field models with defect

It is challenging to extend the concept of hybrid models to quantum field models. A particular direction of this program is to construct integrable quantum field model with defect, e.g., to construct NLS, sine-Gordon (SG) or Liouville QFT model with defect.

Corrigan et al [17] have applied similar idea for constructing SG equation at $x < 0, x > 0$ and a defect at $x = 0$. Their approach is concentrated mostly on the soliton scattering problem in analogy with the classical case. It is however a challenge to construct and solve a genuine quantum integrable SG field model with defect. We could make such a breakthrough very recently by building a quantum integrable exact lattice version of the SG model with defect which satisfies the QYBE and allows exact solution through algebraic Bethe ansatz [18]. We furnish some details in sect. 8.
7 Quantum integrable multi-atom matter-radiation models and their q-deformations

The strategy of construction is to build the monodromy matrix of the system by taking a combination of different Lax operators: $T(\lambda) = L^s(\lambda) \prod_{j=1}^{N_a} L_j^S(\lambda)$, with $L^s(\lambda)$ linked either to the rational (3.8) or to the trigonometric (3.4) ancestor model, while $N_a$-number of $L_j^S(\lambda)$ are related to the spin or the q-spin model. By construction $T(\lambda)$ must satisfy the YB equation with $R^{rat}(\lambda)$ or the $R^{trig}(\lambda)$-matrix in the first or in the second case, respectively, representing a quantum integrable system in both the cases with $\tau(\lambda) = tr\, T(\lambda)$, generating a commuting set of conserved operators. In our matter-radiation models, the subsystem described by $L^s(\lambda)$ with the bosonic or the q-bosonic realization represents the single mode radiation field, while $L_j^S(\lambda)$ (in spin or q-spin realization) represents atoms, i.e. the matter part of the model.

Thus through different reductions of a general model we can generate the following integrable multi-atom MR models in a unified way [16].

1). Integrable multi-atom Buck-Sukumar (BS) model:
   The model is given by $H = H_d + H_{Ss} + H_{SS}$ with
   \[ H_{BS} = \omega f s^3 + \sum_{j} \left( \omega_{aj} S_j^z + \alpha (s^+ S_j^- + s^- S_j^+) \right) + \alpha \sum_{i<j} (S_i^- S_j^+ - S_i^+ S_j^-) , \] \hspace{1cm} (7.18)
   which with a bosonic realization of $su(1, 1)$: $s^+ = \sqrt{N} b^\dagger, s^- = b \sqrt{N}, s^3 = N + \frac{1}{2}$ and the spin-s operator $\vec{S} = \frac{1}{2} \sum_k \vec{\sigma}_k$, represents an integrable multi-atom BS model with inter-atomic interactions and nondegenerate atomic frequencies (AF). At $N_a = 1$, (7.18) recovers the known BS model [19].

2). Integrable multi-atom Jaynes-Cummings (JC) model:
   With bosonic realization $s^- = b, s^+ = b^\dagger, s^3 = b^\dagger b$, we get a similar multi-atom JC type model. The known JC model [20] is recovered at $N_a = 1$, when interatomic couplings vanish and all AF coincide.

3). Integrable trapped ion (TI) model
   with realization $s^\pm = e^{\mp i x\lambda}, \ s^3 = p + x$, we get a new quantum integrable trapped ion model with exponential nonlinearity, having the form (at $N_a = 1$)
   \[ H_{TI} = (\omega_{a} - \omega_f) S^z + S^{2\dagger} + \alpha (e^{-i x} S^+ + e^{i x} S^-) + H_{xp}, \quad H_{xp} = \frac{1}{2} (p^2 + x^2) + xp, \] \hspace{1cm} (7.19)
   \[ \vec{S} = \frac{1}{2} \sum_{k} \vec{\sigma}_k, \]

7.1 Integrable q-deformed MR models

The strategy here is the same, only one has to start now from the trigonometric ancestor Lax operator (3.4) and associated $R^{trig}$-matrix. The Hamiltonian for our q-deformed MR models takes the form (for $N_a = 1$)
   \[ H_{qMR} = H_d + (s_q^+ S_q^- + s_q^- S_q^+) \sin \alpha, \quad H_d = -i c_0 \cos (\alpha X) + c \sin (\alpha X), \] \hspace{1cm} (7.20)
   with $X = (s_q^3 - S_q^3 + \omega)$, which represents a new class of integrable MR models with $S_q \in su_q(2)$ and $s_q$ generating (3.5). We list below a new series of q-deformed integrable MR models which we obtain from the same (7.20) for different realizations of $s_q$. 

10
1. **Integrable q-deformed BS model**

This may be constructed from (7.20) at \( c_0 = 0 \), by realizing \( s_q \) through q-boson:

\[
s_q^+ = \sqrt{N} q b_q^+, s_q^- = b_q \sqrt{N} q, \quad s_q^3 = N + \frac{1}{2},
\]

and quantum spin operator \( S_q \) through its co-product:

\[
S_q^+ = \sum_j q^{-\delta^+} \sum_{\ell<j} \sigma^-_j \sigma^+_{\ell} q^{\delta^+} \Sigma_{\ell>j} \sigma^-_{\ell}, \quad S_q^z = \sum_j \sigma_j^z.
\]

Note that at \( s = 1 \), we get an integrable version of an earlier model [22].

2. **Integrable q-deformed JC model**

The model can be constructed similarly from the same general model (7.20) with choice \( c_0 = i, c = 1 \) and q-bosonic realization \( s_q^+ = b_q^+, s_q^- = b_q, s^3 = p \),

3. **Integrable q-deformed TI model**

Under reduction \( c_0 = i, c = 0 \) and realizing through canonical operators: \( s_q^\pm = e^{\mp i x}, \quad s^3 = p \), we can construct a q-deformed TI model again from (7.20).

By taking higher \( N_a \) values multi-atom integrable variants of the above q-deformed matter-radiation models can be constructed.

8 **Quantum integrable sine-Gordon field model with defect**

At the discrete level, as discussed above, it is easy to build hybrid models inserting Lax operators of different integrable systems at different lattice sites. However when we try to take the continuum limit for constructing the corresponding field model, the above construction breaks down due to the lack of any overlapping smooth region. Therefore one needs certain mechanism similar to the classical formulation of the same problem [17].

We report here about some breakthrough in solving this problem, with details to be presented separately [18]. We construct our quantum integrable sine-Gordon model with defect first as an exact lattice version with its monodromy matrix defined as

\[
T^N_N(\xi) = (L_N(\xi, u_N^+) \cdots L_1(\xi, u_1^+)) F_0(\xi, u_0^+, u_0^-) \left( L_{-1}(\xi, u_{-1}) \cdots L_{-N}(\xi, u_{-N}) \right)
\]

(8.21)

where \( L_j(\xi, u_j^\pm), \quad j = \pm 1, \ldots, \pm N \) is the quantum Lax operator of exact lattice SG model [5] for the field \( u_j^\pm \).

At the defect point \( j = 0 \) we define an overlapping quantum Lax operator

\[
F^d(\xi, u_0^+, u_0^-) = \left( \begin{array}{cc} \xi e_{-1}(P^+)^{-1} & ae_{-1}P^- \xi e^{-1}(P^+)^{-1} \\ -ae_{+}(P^+)^{-1} & \xi e_{-1}P_+ \end{array} \right),
\]

(8.22)

where \( e_\pm = e^{\pm i \frac{\pi}{2} (u_0^+ + u_0^-)}, \quad P^\pm = e_\pm (p_0^+ \pm p_0^-) \), with \([u_k, p_j] = i\delta_{kj}\). Remarkably (8.22) is linked with the ancestor Lax operator (3.4) under reduction \( c_0 = 0, a = 1, 2, \) and therefore must satisfy the QYBE (1.1) with \( R^{tir} \) matrix, which is also the case with the sine-Gordon Lax operators \( L_j^\pm \) appearing in (8.21).

This proves the quantum integrability of the system with \( \tau(\xi) = tr(T^N_N(\xi)) \), generating the mutually commuting set of conserved operators.

It is crucial that this discrete integrable system has a smooth continuum limit: \( \Delta \to 0 \), with the fields \( u_j^\pm \to u^\pm(x), p_j^\pm \to \Delta p^\pm(x) \), having \([u^\pm(x), p^\pm(y)] = i\delta(x - y) \) and \( \sigma^\pm L_j^\pm \to (1 + \Delta U^{SG\pm}(x)) \), for \( x \in (0^+, +\infty) \) and \( x \in (-\infty, 0^-) \), respectively, with \( U^{SG\pm}(x) \) giving the Lax operator for the sine-Gordon field model with fields \( u^\pm(t, x) \).

At the defect point we get the transition: \( F^d_0 \to (1 + \Delta L(x)) F_0 \), at \( x \in [0^- , 0^+] \), where \( L(x) = (F_0^x + F^p(x)) F_0^{-1}(x) \), where \( F^p \) is obtained from \( F^d \) by replacing \( P^\pm \to \frac{1}{\xi}(p^+(x) \pm p^-(x)) \) and \( F_0 \) gives the known Backlund operator for the sine-Gordon model [23].

Starting from the Lax operators at different regions we can find all conserved quantities including Hamiltonian, momentum etc. defined on the whole axis, (including the defect point) by using the
standard Riccati equation technique. We can also find the corresponding eigenvalues exactly by applying the algebraic Bethe ansatz method [24] to the lattice regularised model (8.21) we have started with.

9 Concluding remarks

We have presented here a unified construction of quantum integrable models from a single ancestor model based on a general Yang-Baxter Hopf algebra, representing a new operator deformed quantum algebra. Though some of our results were published earlier, we present our whole scheme in a systematic way focusing on a new concept of operator-deformation of an algebra, which is our main concern. We also report on some completely new result on the quantum integrable sine-Gordon model with defect and other hybrid models like integrable q-deformation of a few matter-radiation models.

We emphasize that all integrable models listed here allow exact Bethe ansatz solution. Moreover, similar to their unified construction one can get their solution also in a unified and almost model-independent way. For this one has to start from the general solution of the ancestor model following the algebraic Bethe ansatz and subsequently obtain the results for all descendant models with the same $R$-matrix as various reductions of the general result.

10 acknowledgement

I express my sincere thanks to the organizers of the NLPTE4 conference and the AvH foundation (Germany) for logistic and financial support.

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