Multiplicity and regularity of large periodic solutions with rational frequency for a class of semilinear monotone wave equations.

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Abstract

Nous démontrons l’existence d’une infinité de solutions fortes, de norme grande, pour une classe d’équations semilinéaires avec des conditions périodiques sur le bord:

\[ u_{tt} - u_{xx} = f(x, u), \]
\[ u(0, t) = u(\pi, t), \quad u_x(0, t) = u_x(\pi, t). \]

Notre méthode est basée sur de nouvelles estimations pour le problème linéaire avec conditions périodiques sur le bord, en combinant les méthodes de Littlewood-Paley, le théorème de Hausdorff-Young et une formulation variationnelle de Rabinowitz, [22],[23]. Nous construisons une nouvelle approche pour la régularité des solutions au sens des distributions en dérivant les équations et en utilisant les estimations de type Gagliardo-Nirenberg.

We prove the existence of infinitely many classical large periodic solutions for a class of semilinear wave equations with periodic boundary conditions:

\[ u_{tt} - u_{xx} = f(x, u), \]
\[ u(0, t) = u(\pi, t), \quad u_x(0, t) = u_x(\pi, t). \]

Our argument relies on some new estimates for the linear problem with periodic boundary conditions, by combining Littlewood-Paley techniques, the Hausdorff-Young theorem of harmonic analysis, and a variational formulation due to Rabinowitz [22],[23]. We also develop a new approach to the regularity of the distributional solutions by differentiating the equations and employing Gagliardo-Nirenberg estimates.

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1 Introduction

In this paper we construct infinitely many large classical time-periodic solutions for the following semilinear wave equation:

\[ u_{tt} - u_{xx} - f(x, u) = 0 \] (1.1)

\[ u(0, t) = u(\pi, t), \ u_x(0, t) = u_x(\pi, t) \] (1.2)

where \( f \) is \( C^{2,1} \), has polynomial growth and depends on \( x, u \). The existence of large periodic solutions with periodic boundary conditions is not well understood. As \( u = 0 \) is a trivial solution we seek here nontrivial solutions of (1.1), (1.2). When the frequency is irrational the method of Craig and Wayne in [13], extended to higher dimension by Bourgain [9] and Berti and Bolle [3] proves the existence of small periodic solutions for typical potentials but the existence of classical periodic solutions for rational frequency is not known. Note that typical constant potentials in [13], [9], [3] are satisfied for

\[ u_{tt} - \Delta u - m - f(x, u) = 0 \] (1.3)

for typical \( m \) which exclude \( m = 0 \). The so-called resonant case \( m = 0 \) and with \( f(x, u) \) independent of \( x \) for periodic boundary conditions has been studied by Berti and Procesi in [8]. The lack of \( x \) dependence in [8] allows to employ ordinary differential equations techniques and they showed the existence of quasi-periodic solutions where the frequency vector depends on two frequencies \( (\omega_1, \omega_2(\epsilon)) \). While they consider \( \omega_1 \in \mathbb{Q} \), their results do not imply the existence of periodic solutions with rational frequency as \( \omega_2(\epsilon) \) there is never rational. Chierchia and You in [11] study the problem with periodic boundary conditions and a potential:

\[ u_{tt} - u_{xx} - v(x)u - f(u) = 0 \] (1.4)

where \( f \) only depends on \( u \), however their method excludes the constants potentials \( v(x) = m \). Bricmont, Kupiainen and Schenkel in [6] prove the existence of quasi-periodic solutions with periodic boundary conditions in the non-resonant case \( m > 0 \) and \( f \) depending only on \( u \). In [6] they find quasi-periodic solutions for a set of positive measure of frequencies hence prove the existence of quasi-periodic solutions for irrational frequencies.

On the other hand there exists a substantial amount of literature for semilinear wave equations with Dirichlet boundary conditions see for instance [23], [22], [7] for rational frequencies and the proofs of existence of classical solutions with \( f \) having some spatial dependence rely on a fundamental solution discovered by Lovicarova in [19]. The existence of periodic solutions, with irrational frequencies with Dirichlet boundary conditions in the resonant case \( (m = 0) \) was shown by Lidskii and Schulman [18], by Bambusi in [1], Bambusi and Paleari in [2], Berti and Bolle [5], [4] and for quasi-periodic solutions in Yuan [26]. Quasi-periodic solutions with Dirichlet boundary conditions via KAM techniques has been shown by Pöschel [21], Kuksin [17] and Wayne [27].
De Simon and Torelli in [14] do not employ Lovicarova’s formula but their $C^0$ estimate relies on $L^2$ a priori estimates on $f(x, u)$ which are not readily available for distributional solutions of (1.1). The difficulty in proving regularity of distributional solution of (1.1) stems from the kernel of the d’Alembertian, which is infinite dimensional. In absence of a fundamental solution for the d’Alembertian under periodic boundary conditions problem we develop an approach based on tools from harmonic analysis such as Littlewood-Paley techniques, the Hausdorff-Young theorem and Gagliardo-Nirenberg estimates. The Hausdorff-Young theorem had been employed earlier by Willem in [28] to get a $L^\infty$ a priori estimate on solutions, by Coron to prove a Sobolev embedding in [12] and by Zhou in [29]. The argument we give here to prove the Sobolev embedding in [12] follows the Fourier approach to the Sobolev embedding as in the notes by Chemin, [10]. We do prove a stronger estimate than the one in [12], which provides information about the best constant of the Sobolev embedding. Our argument also shows that the Sobolev embedding in [12] is compact. In this paper the existence of classical solution for time periodic solutions with periodic boundary conditions of the semilinear wave equation (1.1) will be shown by proving the stronger $C^r$ Hölder estimates than the $L^\infty$ in [28], and our approach also gives an alternative proof of the existence of classical periodic solutions in the case of Dirichlet boundary conditions with semilinear term with some spatial dependence for $f(x, u)$ sufficiently smooth in both arguments $x$ and $u$.

In section 1 we prove the linear estimates we need to prove the regularity of the solution. In section 2 we follow the scheme of [23] and [22] to construct weak solutions and in section 3 we show the regularity of the solution by repeated differentiation of the equations, the linear estimates proved in section 1 and Gagliardo-Nirenberg inequalities.

Since our proof is of variational nature it is natural to ask if there is a notion of critical exponent or critical growth for this equation. An open question is then whether there are semilinear terms $f(x, u)$ of say exponential or super exponential type (as this paper deals with semilinear terms of polynomial type) for which there are large amplitude distributional solutions which are not classical ($f(x, u)$ being assumed to be smooth).

We seek time-periodic solutions satisfying periodic boundary conditions so we seek functions $u \in \mathbb{R}$ with expansions of the form
\[ u(x, t) = \sum_{(j, k) \in \mathbb{Z} \times \mathbb{Z}} \hat{u}(j, k)e^{i 2jx}e^{ikt} \]
and define the function space $E$:
\[ ||u||_E^2 = \sum_{2j \neq \pm k} \frac{|Q|}{4} |k^2 - 4j^2||\hat{u}(j, k)||^2 + \sum_{2j = \pm k} |4j^2||\hat{u}(j, k)||^2 + ||\hat{u}(0, 0)||^2 \]
where $Q = [0, \pi] \times [0, 2\pi]$ and define the functions spaces $E^+, E^-, N$ as follows:
\[ N = \{ u \in E, \hat{u}(j, k) = 0 \text{ for } 2|j| \neq |k| \}. \]
Note that in the case of periodic boundary conditions the structure of the kernel \(N\) of \(\Box\) is slightly different than in the case of Dirichlet boundary conditions. Here \(v \in N\) we have

\[
v(x, t) = \sum_{j=\pm k} v(j, k)e^{i2jx+ikt} = \sum_{j \neq 0} v(j, 2j)e^{i2j(x+t)} + \sum_{j} v(j, -2j)e^{i2j(x-t)}
\]

and

\[
v(x, t) = p_1(x + t) + p_2(x - t) = v^+(x, t) + v^-(x, t)
\]

where \(v^+(x, t) = p_1(x, t), v^-(x, t) = p_2(x, t),\) where the \(p_1, p_2 \in H^1(0, \pi)\) \(\pi\)-periodic functions and defined as \(p_1(s) = \sum_j p_1(j)e^{i2js}, p_2(s) = \sum_j p_2(j)e^{i2js}\) and \(p_1(0) = 0, p_1(0) = v(j, 2j), p_2(j) = v(j, -2j).\)

\[E^+ = \{u \in E, \hat{u}(j, k) = 0 \text{ for } |k| \leq 2|j|\}\]
\[E^- = \{u \in E, \hat{u}(j, k) = 0 \text{ for } |k| \geq 2|j|\}\]

\(u = v + w, w = w^+ + w^-\) where \(w \in E, w^+ \in E^+, w^- \in E^-\) and \(v \in N\) and define the norm on \(E \oplus N\)

\[
||u||_{\beta, E} = ||w^+||_{L^2}^2 + ||w^-||_{L^2}^2 + \beta ||v||_{H^1}^2.
\]

\[
I_\beta(u) = \int_Q 1/2(u_t^2 - u_x^2 - \beta(v^2 + v_x^2) - F(x, u))dxdt.
\]

When \(u\) is trigonometric polynomial, \(I_\beta\) can also be represented in \(E^m \oplus N^m\) as:

\[
I_\beta(u) = \frac{1}{2}((||w^+||_{L^2}^2 - ||w^-||_{L^2}^2) - \beta(||v||_{L^2}^2 + ||v_t||_{L^2}^2) - \int_Q F(x, u)dxdt.
\]

where \(\frac{\partial F(x, u)}{\partial u} = f(x, u),\) first seek weak solution of the modified equation:

\[
\Box u = \beta v_t - f(x, u) - \beta v
\]

and then send the parameter \(\beta\) to zero.

**Assumptions on \(f(u):\)**
we assume that there are positive constants \(c_0^1 \leq c_0^2, c_1^1, c_1^1\) such that

\[
c_0^1 ||u||_{L^2}^2 + c_1^1 \leq f(x, u) \leq c_0^2 ||u||_{L^2}^2 + c_1^2
\]

with \(c_0^1 > \frac{c_0^2}{\beta^2} \frac{1}{\pi^2} .\) These assumptions are satisfied by some nonlinearities of polynomial type. \(f(x, u)\) must also be strongly monotone increasing:

\[
\frac{\partial f(x, u)}{\partial u} \geq \alpha > 0
\]

**Theorem 1.1.** *Under assumptions (1.9), (1.10) and \(f \in C^{2,1}, (1.11), (1.12)\) admits infinitely many classical solutions.*
2 Estimates

Define $l^q = \{ \hat{u}(j,k) s.t. \sum_{(j,k) \in \mathbb{Z} \times \mathbb{Z}} |\hat{u}(j,k)|^q < +\infty \}$.

**Theorem 2.1.** The function $u = \sum_{2j \neq \pm k} \hat{u}(j,k) e^{2 ij x + j k t} \in C^{\gamma}$ where $\gamma < 1 - \frac{1}{p}$ if

$$\hat{u}(j,k) = \frac{\hat{f}(j,k)}{4j^2 - k^2} \quad (2.11)$$

for $2j \neq \pm k$, $\hat{f} \in l^q$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof:
Let $B_m$ the set

$$B_m = \{(j,k) \in \mathbb{Z} \times \mathbb{Z} \mid 2|j| + |k| \leq 2.2^m\}$$

and $\Delta_m$

$$\Delta_m = B_m \setminus B_{m-1}$$

so we have in $\Delta_m$

$$2^m \leq 2|j| + |k| \leq 2.2^m$$

and the $C^{\gamma}$ norm will be estimated by

$$\sup_m 2^m \|\Delta_m\|_{C^{0}}$$

see [25] or [16].

$$2^m \|\Delta_m\|_{C^{0}} = 2^m \| \sum_{(j,k) \in \Delta_m} \hat{u}(j,k) e^{2 ij x e^{ikt}} \|_{C^{0}}$$

$$= \| \sum_{(j,k) \in \Delta_m} 2^m \frac{\hat{f}(j,k)}{4j^2 - k^2} e^{2 ij x e^{ikt}} \|_{C^{0}}$$

$$\leq \sum_{(j,k) \in \Delta_m} \frac{2^m \hat{f}(j,k)}{4j^2 - k^2} \| e^{2 ij x e^{ikt}} \|_{C^{0}}$$

$$\leq \sum_{(j,k) \in \Delta_m} \frac{2^m \hat{f}(j,k)}{4j^2 - k^2} \| e^{2 ij x e^{ikt}} \|_{C^{0}}$$

$$\leq \sum_{(j,k) \in \Delta_m} \frac{2^m \hat{f}(j,k)}{4j^2 - k^2} \| e^{2 ij x e^{ikt}} \|_{C^{0}}$$

$$\leq c \| \hat{f} \|_{l^q} \leq c \| \hat{f} \|_{L^p} \quad (2.12)$$

as long as $\gamma < 1 - \frac{1}{p}$ and the last inequality follows from the Hausdorff-Young theorem.

Remark: The argument here provides an alternate proof of the Hölder continuity of weak solutions of $\square w = f$ where $f \in L^p \cap N^\perp$ where $N^\perp$ denotes the weak orthogonal of the kernel of $\square$ with Dirichlet boundary conditions, proved by
Brezis and Coron and Nirenberg in [7] via Lovicarova’s fundamental solution, for $1 < p \leq 2$.
In the case that $p = 2$ we have $u \in C^{0,\gamma}$ or similarly $f \in H^{\alpha}$ implies $u \in C^{0+\frac{1}{2}}$.
Define
\[ u_{h_1,h_2}(x,t) = u(x + h_1,t + h_2) \] (2.13)
and
\[ \Delta_{m}^{++} = \{(j,k) \in \mathbb{Z} \times \mathbb{Z} : (j,k) \in \Delta_m, j \geq 0, k \geq 0\} \] (2.14)
\[ \Delta_{m}^{+-} = \{(j,k) \in \mathbb{Z} \times \mathbb{Z} : (j,k) \in \Delta_m, j \geq 0, k < 0\} \] (2.15)
\[ \Delta_{m}^{-+} = \{(j,k) \in \mathbb{Z} \times \mathbb{Z} : (j,k) \in \Delta_m, j < 0, k \geq 0\} \] (2.16)
\[ \Delta_{m}^{--} = \{(j,k) \in \mathbb{Z} \times \mathbb{Z} : (j,k) \in \Delta_m, j < 0, k < 0\} \] (2.17)
and define $u^{--}, u^{++}, u^{-+}, u^{+-}$ as
\[ \hat{u}^{++}(j,k) = \hat{u}(j,k) \text{ if } j \geq 0, k \geq 0 \]
\[ = 0 \text{ otherwise} \] (2.18)
\[ \hat{u}^{+-}(j,k) = \hat{u}(j,k) \text{ if } j \geq 0, k < 0 \]
\[ = 0 \text{ otherwise} \] (2.19)
\[ \hat{u}^{-+}(j,k) = \hat{u}(j,k) \text{ if } j < 0, k \geq 0 \]
\[ = 0 \text{ otherwise} \] (2.20)
\[ \hat{u}^{--}(j,k) = \hat{u}(j,k) \text{ if } j < 0, k < 0 \]
\[ = 0 \text{ otherwise} \] (2.21)
Lemma 2.1. If $u^{++} \in C^{0,\gamma}$ then $u^{++} \in H^{\gamma'}$ if $\gamma' < \gamma$.
The analogue is also true for $u^{-+}, u^{+-}, u^{--}$.
Proof:
\[ \|u^{++}\|_{H^{\gamma'}}^2 = \sum_{m} \sum_{(j,k) \in \Delta_{m}^{++}} (|2|j| + |k|)^{2\gamma'}|\hat{u}(j,k)|^2 \]
\[ \leq \sum_{m} \sum_{(j,k) \in \Delta_{m}^{++}} 2^{2(m+1)\gamma'}|\hat{u}(j,k)|^2 \]
\[ \leq \sum_{m} 2^{(m+1)\gamma'} \sum_{(j,k) \in \Delta_{m}^{++}} |e^{2(|j| + |k|)h - 1}|^2|\hat{u}(j,k)|^2 \]
with $h = h(m) = \frac{2\pi}{3} 2^{-m}$.
Remark: in the next line the sum over $\Delta_{m}^{++}$ is extended to the whole series but
Depending on Parseval.

Then

\[ \|u^+\|^2_{H^{2(\gamma')}} \leq \sum_m 2^{(m+1)\gamma'} \|u_{h(m),h(m)} - u\|^2 \tag{2.22} \]

\[ \leq \sum_m 2^{(m+1)\gamma'} \|u_{h(m),h(m)} - u\|_{C^0} \]

\[ \leq \sum_m 2^{(m+1)\gamma'} \|u\|_{C^{\gamma},\gamma}^2 |h(m)|^{2\gamma} \]

\[ \leq \sum_m 2^{(m+1)\gamma'} \|u\|_{C^{\gamma},\gamma}^2 \frac{2\pi}{3} 2^{-m \gamma} \]

\[ \leq c \|u\|_{C^{\gamma},\gamma}^2 \sum_m 2^{m(\gamma' - \gamma)} \]

\[ \leq c(\gamma - \gamma') \|u\|_{C^{\gamma},\gamma}^2 \]

The estimates for \( u^-, u^+, u^+ \) follow similarly by replacing \( u_{h,h} \) in the preceding argument (2.23) by \( u_{h,-h}, u_{-h,h}, u_{h,-h} \). We can conclude by noting:

\[ \|u\|_{H^{2(\gamma')}}^2 = \|u^+\|_{H^{2(\gamma')}}^2 + \|u^-\|_{H^{2(\gamma')}}^2 + \|u^+\|_{H^{2(\gamma')}}^2 + \|u^-\|_{H^{2(\gamma')}}^2 \tag{2.23} \]

We prove a bootstrapping estimate in the next lemma. It follows from the proof theorem 4 in [24] established for Dirichlet boundary conditions.

**Lemma 2.2.** Let \( f, w \in L^2(Q) \) such that

\[ \hat{f}(j,k) = 0 = \hat{w}(j,k) \text{ for } 2j = \pm k \tag{2.24} \]

and

\[ (-k^2 + 4j^2)\hat{w}(j,k) = \hat{f}(j,k), \text{ for } 2j \neq \pm k \tag{2.25} \]

then \( w \in H^1 \)

Proof:

\[ \|w\|_{H^1}^2 = \sum_{2j \neq \pm k} \frac{4j^2 + k^2}{4j^2 - k^2} |\hat{f}(j,k)|^2 \]

\[ = \sum_{2j \neq \pm k} \frac{1}{2} \frac{(2j - k)^2 + (2j + k)^2}{(2j - k)^2(2j + k)^2} |\hat{f}(j,k)|^2 \]

\[ \leq \sum_{2j \neq \pm k} |\hat{f}(j,k)|^2 \]

\[ \leq \|f\|_{L^2}^2. \tag{2.26} \]

Let \( E^* \) be the closure of \( \{ e^{2jx + ikt}, \ 2j \neq \pm k \} \) under the norm

\[ \|u\|_{E^*}^2 = \sum_{2j \neq \pm k} |\tilde{u}(j,k)|^2 |k^2 - 4j|^s \]

then we have the Sobolev estimate:
Theorem 2.2. \(0 < s < 1\) the space \(E^s\) is continuously embedded in \(L^p\) where \(p = \frac{2}{1-s}\).

This theorem implies that the embedding in \(E^1 \subset L^p\) is compact, as \(E^1 \subset E^s\) is compact for \(s < 1\). We will show that it also implies a Gagliardo-Nirenberg inequality of the type:

\[
||u||_{L^p} \leq c(p)||u||_{L^2}^{1-s(p)}||u||_{E^1}^{s(p)}
\] (2.27)

where \(c(p)\) will be computed explicitly.

Proof:

\[
f = f_{1,A} + f_{2,A}
\] (2.28)

where

\[
f_{1,A} = \sum_{2j \neq \pm k, |j| + |k| \leq A} \hat{f}(j, k)e^{2\pi j x}e^{ikt}
\] (2.29)

and

\[
f_{2,A} = \sum_{2j \neq \pm k, |j| + |k| > A} \hat{f}(j, k)e^{2\pi j x}e^{ikt}
\] (2.30)

\[
|f_{1,A}| \leq \sum_{2j \neq \pm k, |j| + |k| \leq A} |\hat{f}(j, k)|
\]

\[
\leq \sum_{2j \neq \pm k, |j| + |k| \leq A} |4j^2 - k^2|^{-\frac{s}{2}}|4j^2|^{-\frac{s}{2}}|\hat{f}(j, k)|
\] (2.31)

and applying Cauchy-Schwarz we have

\[
|f_{1,A}| \leq \left( \sum_{2j \neq \pm k, |j| + |k| \leq A} \frac{1}{|4j^2 - k^2|^s} \right)^{\frac{1}{2}} \left( \sum_{2j \neq \pm k, |j| + |k| \leq A} |4j^2 - k^2|^s |\hat{f}(j, k)|^2 \right)^{\frac{1}{2}}
\]

\[
\leq ||f||_{E^s} \left( \sum_{m,n \leq A} \frac{1}{m^n} \right)^{\frac{1}{2}}
\]

\[
\leq e||f||_{E^s} \left( \int_1^A \frac{dm}{m^s} \int_1^A \frac{dn}{n^s} \right)^{\frac{1}{2}}
\]

\[
\leq e||f||_{E^s} A^{-s+1}.
\] (2.32)

Now we seek \(A_\lambda\) such that

\[
|f_{1,A}| \leq \frac{\lambda}{4}.
\] (2.33)

So we require the estimate

\[
c||f||_{E^s} A^{-s} \leq \frac{\lambda}{4}
\] (2.34)

this leads to the inequality

\[
A^{-s} \leq \frac{\lambda}{4c||f||_{E^s}}.
\]
So let $A_\lambda$:

\[A_\lambda = \left(\frac{\lambda}{4c||f||_{E^s}}\right)^{\frac{1}{p}}.\]

Now

\[
\int_{[0,\pi][0, 2\pi]} |f(x, t)|^p dx dt = p \int_0^\infty y^{p-1} w(y) dy
\]

where $w_f(y) = |\{ (x, t) \in [0, \pi][0, 2\pi] : |f(x, t)| > y\}|$. Now $|f(x, t)| > \lambda$ implies $|f_{1, A}| > \frac{\lambda}{2}$ or $|f_{2, A}| > \frac{\lambda}{2}$. Recalling (2.33) and the definition of $A_\lambda$ conclude that

\[|f_{2, A}| > \frac{\lambda}{2}\]  (2.35)

and

\[w_f(\lambda) \leq w_{f, A}(\frac{\lambda}{2})\]  (2.36)

hence

\[
\int_{[0,\pi][0, 2\pi]} |f(x, t)|^p dx dt = p \int_0^\infty \lambda^{p-1} w_f(\lambda) d\lambda \\
\leq p \int_0^\infty \lambda^{p-1} w_{f, A}(\frac{\lambda}{2}) d\lambda.
\]

Since

\[w(\lambda) \leq \frac{1}{\lambda^2} \int_{|f| \geq \lambda} |f(x, t)| dx dt\]  (2.37)

\[
\int_{[0,\pi][0, 2\pi]} |f(x, t)|^p dx dt \leq \int_0^\infty \lambda^{p-3} \int_{|f_{2, A}(x, t)| > \frac{\lambda}{2}} |f_{2, A}(x, t)|^2 dx dt d\lambda \\
\leq \int_0^\infty \lambda^{p-3} \int_{[0,\pi][0, 2\pi]} |f_{2, A}(x, t)|^2 dx dt d\lambda.  (2.38)
\]

Then we can invoke Parseval formula to deduce

\[
\int_{[0,\pi][0, 2\pi]} |f(x, t)|^p dx dt \leq \int_0^\lambda \lambda^{p-3} \sum_{2j \neq \pm k} |\hat{f}_{2, A}(j, k)|^2 d\lambda \\
= \int_0^\lambda \lambda^{p-3} \sum_{2j \neq \pm k, 2|j| + |k| > A_\lambda} |\hat{f}(j, k)|^2 d\lambda (2.39)
\]

Now

\[2|j| + |k| \geq A_\lambda = \left(\frac{\lambda}{4c||f||_{E^s}}\right)^{\frac{1}{p-1}}\]

implies

\[\lambda \leq 4c||f||_{E^s}((2|j| + |k|))^{1-s}\]
We continue the estimate from (2.39):

\[
\int_{[0,\pi][0,2\pi]} |f(x,t)|^p \, dx \, dt \, \lambda \leq \sum_{2j \neq k} \int_0^\infty \lambda^{p-3} \hat{f}(j,k)^2 \cdot 1_{\{(\lambda,j,k) \in 2|j|+|k| \geq A\lambda\}} \, d\lambda \\
\leq \sum_{2j \neq k} \int_0^4 \hat{f}(j,k)^2 \cdot \lambda^{p-3} \, d\lambda \\
\leq \sum_{2j \neq k} |\hat{f}(j,k)|^2 \int_0^4 |\hat{f}(j,k)|^{1-\epsilon} \, d\lambda \\
\leq \sum_{2j \neq k} |\hat{f}(j,k)|^2 \frac{1}{p-2} [4c|f|_{L^p}((2|j|+|k|)^{1-\epsilon})]^{p-2}
\]

(2.40)

now if \( s = (1-s)(p-2) \) i.e. \( s(p) = \frac{p-2}{p-1} \) then

\[
\int_{[0,\pi][0,2\pi]} |f(x,t)|^p \, dx \, dt \leq \frac{(4c)^{p-2}}{p-2} |f|^p_{L^p} \tag{2.41}
\]

and we have the following Gagliardo-Nirenberg inequality for \( p > 2 \):

\[
|u|_{L^p} \leq c(p)|u|_{L^p} \leq c(p)|u|^{1-s(p)}_{L^2} |u|^{s(p)}_{L^1} \tag{2.42}
\]

3 Construction of the weak solution

For the Galerkin procedure we define the spaces:

\[
E^m = \text{span}\{ \sin 2jx \cos kt, \sin 2jx \sin kt, \cos 2jx \cos kt, \cos 2jx \sin kt, \ 2j+k \leq m, 2j \neq k \},
\]

\[
E^{-m} = \text{span}\{ \sin 2jx \cos kt, \sin 2jx \sin kt, \cos 2jx \cos kt, \cos 2jx \sin kt, \ 2j+k \leq m, 2j < k \},
\]

\[
E^+l = \text{span}\{ \sin 2jx \cos kt, \sin 2jx \sin kt, \cos 2jx \cos kt, \cos 2jx \sin kt, \ 2j+k \leq l, 2j > k \},
\]

\[
N^m = \text{span}\{ \sin 2jx \cos kt, \sin 2jx \sin kt, \cos 2jx \cos kt, \cos 2jx \sin kt, \ 2j \leq m \}
\]

which are employed in the minimax procedure. We denote by \( P^m \) the projection of \( E \oplus N \) into \( E^m \oplus N^m \). The functional \( I_{\beta} \) satisfies the Palais-Smale condition. The arguments follow as in [23], we do not repeat them here.

**Lemma 3.1.** \( \forall u \in E^+l \), there is a constant \( C(l) \) independent of \( \beta, m \) such that

\[
I_{\beta}(u) \leq M(l) \tag{3.43}
\]
Proof:
Let \( u \in E^+ \)

\[
I_\beta(u) = \frac{1}{2}||w^+||_E^2 - \frac{1}{2}||w^-||_E^2 - \beta||v||_{H^1}^2 - \int_Q F(u) dx dt
\]

\[
\leq \frac{1}{2}||w^+||_E^2 - \frac{1}{2}||w^-||_E^2 - \beta||v||_{H^1}^2 - \left( c(s) + \sup_{u \in E^+} \frac{1}{2}||w^+||_E^2 - c(s, Q)||u||_{L^2}^{s+1} \right) \tag{3.44}
\]

Now in \( E^+ \)

\[
||u||_E^2 \leq l||u||_{L^2}^2 \tag{3.45}
\]

and on the other-hand

\[
\sup_{u \in E^+} \frac{1}{2}||w^+||_E^2 - c(s, Q)||u||_{L^2}^{s+1} > 0 \tag{3.46}
\]

while as \( ||u||_E \to +\infty \) in \( E^+ \) is dominated by \( ||u||_{L^2}^{s+1} \) as \( s + 1 > 2 \) and is attained at say \( \overline{u} \) hence we have

\[
c(s, Q)||\overline{u}||_{L^2}^{s+1} \leq ||\overline{u}||_E^2 \leq l||\overline{u}||_{L^2}^2 \tag{3.47}
\]

and we can conclude there is \( M(l) \) depending on \( l \) but independent of \( \beta \) such that

\[
I_\beta(u) \leq M(l). \tag{3.48}
\]

Also \( E^+ \) is finite dimensional hence there is \( R(l) \) such that for all \( u \in E^+ \oplus E^{-m} \oplus N^m \) and \( ||u||_{E, \beta} \geq R(l) \) implies \( I_\beta(u) \leq 0 \).

**Theorem 3.1.** Let \( f \) be \( C^1 \), for \( l \) large enough there is a distributional solution \( u = v + w \) of the modified problem \( (1.8) \).

Proof:
In this proof the constants may dependent on \( \beta \) and \( f \) but are independent of \( m \). The proof of this theorem here is slightly simpler from the one in [23] as we take advantage of the polynomial growth of the nonlinear term. We also employ Galerkin approximation.

Let \( u^m = w^m + v^m \in E^m \oplus N^m \) a distributional solution corresponding to the critical value \( c_1 \), and any \( \phi \in E^m \oplus N^m \):

\[
I'(u^m) \phi = 0 \tag{3.49}
\]

now taking \( \phi = v^m_{tt} \in N^m \) we have

\[
(\beta v^m_{tt}, v^m_{tt})_{L^2} = (f(x, u^m), v^m_{tt})_{L^2} + \beta(v^m_t, v^m_t)
\]

and by \( (1.3) \) there are constant positive \( c, d \) such that

\[
\beta||v^m_{tt}||_{L^2}^2 \leq c||u^s||_{L^2}||v^m_t||_{L^2} + d||v^m_{tt}||_{L^2}
\]
\[ \beta \| v_{tt}^m \|_{L^2} \leq c \| v_t^m \|_{L^2} \]

hence

\[ \| v_t^m \|_{L^2} \leq c(\beta) \]

we now have

\[ w_{tt}^m - w_{xx}^m = \beta v_t^m + P^m f(x, u_t^m) \in L^2 \]

hence \( w^m \in H^1 \cap C^\gamma \), \( \gamma < \frac{1}{2} \) by theorem 2.1 and lemma 2.2. This now implies \( w^m \in H^2 \), \( w^m \to w \) as \( m \to +\infty \) pointwise and \( w \in H^1 \cap C^\gamma \). Then if \( \phi = v_{ttt}^m \) then

\[ (\beta v_t^m, v_{ttt}^m)_{L^2} = (f(x, u_t^m), v_{ttt}^m)_{L^2} - \beta(v_t^m, v_{ttt}^m) \]

so there exists \( c \) independent of \( m \)

\[ (\beta v_t^m, v_{ttt}^m)_{L^2} = (f(x, u_t^m)u_{ttt}^m, v_{ttt}^m)_{L^2} - \beta(v_t^m, v_{ttt}^m) \]

and we deduce \( \| v_{tt}^m \|_{L^2} \leq c(\beta) \) hence \( v_{tt}^m \to v_t \in C^0 \) as \( m \to +\infty \) hence \( v \) is \( C^2 \) and \( w \) is \( C^\gamma \) by applying theorem 2.1 to 1.8. We now have

\[ u_l^m \to u \in C^\gamma \text{ as } m \to +\infty \]

and since (3.49) holds for any \( \phi \in E^m \oplus N^m \) we can deduce

\[ I'(u)\phi = 0 \quad \forall \phi \in E^m \oplus N^m, \quad (3.50) \]

now sending \( m \to \infty \), \( u \) is a weak solution of (1.8).

Then we can define \( g_\theta(u) = u(x, t + \theta) \). Define:

\[ G = \{g_\theta \text{ s.t. } \theta \in [0, 2\pi)\} \]

(3.51)

\[ V_l = N^m \oplus E^{-m} \oplus E^{+l} \]

(3.52)

\[ G_l = \{h \in C(V_l, E^m) \text{ such that } h \text{ satisfies } \gamma_1 - \gamma_4\} \]

(3.53)

Fix\( G = \{u \in E \text{ s.t. } g(u) = u \forall g \in G\} = \text{span}\{\cos 2jx, \sin 2jx, \ j \in \mathbb{Z}\} \subset E^- \).

Define \( P_0^m, P^{-m} \) the orthogonal projections from \( E^m \oplus N^m \) onto respectively \( N^m (= E^{0m}) \), \( E^{-m} \) and \( P_l \) the orthogonal projection from \( E^m \oplus N^m \) onto \( V_l \).

\[
\begin{align*}
\gamma_1 & \text{ h is equivariant} \\
\gamma_2 & h(u) = u \text{ if } u \in \text{Fix}_G \\
\gamma_3 & \text{ There exists } r(h) \ h(u) = u \text{ if } u \in V_l \backslash B_{r(h)} \\
\gamma_4 & u = w^+ + w^- + v \in V_l \ (P_0^m + P^{-m})h(u) = \alpha(u)v + \alpha^-(u)w^- + \phi(u) \text{ where } \alpha, \overline{\alpha} \in C(V_l, [1, \overline{\alpha}])
\end{align*}
\]

and \( 1 < \overline{\alpha} \) depends on \( h, \phi \) continuous. Define

\[ c_l(\beta) = \inf_{h \in G_l} \sup_{u \in V_l} I_\beta(h(u)) \]

(3.54)

and \( c_l(\beta) \to +\infty \) as \( l \to \infty \) independently of \( m, \beta \).

**Lemma 3.2.** \( c_l(\beta) \to +\infty \) as \( l \to +\infty \)**
Proof:

\[ I_\beta(u) = \frac{1}{2} ||w^+||_E^2 - \frac{1}{2} ||w^-||_E^2 - \beta ||v||_{H^1}^2 - \int_Q F(u)dxdt \]  
(3.55)

and there exists by assumptions (1.9) \( c(s), d(s) > 0 \) such that
\[ I_\beta(u) \geq \frac{1}{2} ||w^+||_E^2 - \frac{1}{2} ||w^-||_E^2 - \beta ||v||_{H^1}^2 - c(s) \int_Q |u|^{s+1}dxdt - d(s) \]  
(3.56)

and if \( u \in \partial B \cap V_{l-1}^\perp \) we have
\[ I_\beta(u) \geq \frac{1}{2} ||w^+||_E^2 - c(s) \int_Q |u|^{s+1}dxdt - d(s). \]  
(3.57)

Now by the Sobolev embedding theorem 2.2 there is \( \theta(s) < 1 \) such that if \( \hat{u}(j,k) = 0 \) for \( 2j = \pm k \) we have
\[ ||u||_{L^{s+1}} \leq ||u||_{E^{\theta(s)}} \]  
(3.58)

hence
\[ I_\beta(u) \geq \frac{1}{2} ||w^+||_E^2 - c(s)(||u||_{L^2}^{1-\theta(s)} ||u||_{E^1}^{\theta(s)})^{s+1} - d(s) \]
\[ \geq \frac{1}{2} \rho^2 - \rho^{s+1}(1-\theta(s))^{s+1}. \]  
(3.59)

If we choose a constant \( C(s) \) large and \( \rho = \frac{1}{C(s)} t^{(1-\theta(s))^{\frac{s+1}{s+1}}} \)
\[ I_\beta(u) \geq \frac{1}{4} \rho^2 - d(s). \]  
(3.60)

Applying the corollary 2.4 in [15] to \( P_{l-1} h \in C(\partial B_{\rho^l}, V_{l-1}) \) we have
\[ h(V_l) \cap \partial B_{\rho^l} \cap V_{l-1}^{\perp} \neq \emptyset \]  
(3.61)

hence
\[ \sup_{V_l} I_\beta(h(u)) \geq \inf_{u \in \partial B_{\rho^l} \cap V_{l-1}^{\perp}} I(u) \geq \frac{1}{4} \rho^2 - d(s) \rightarrow +\infty \]  
(3.62)

The \( c_1(\beta) \) are critical values of \( I_\beta \) on \( E^m \). This is obtained by a standard argument see [23] propositions 2.33 and 2.37.

**Lemma 3.3.** If \( u \) is a critical point of \( I_\beta \) in \( E^m \oplus N^m \) then there are constants \( c_1, c_2 \) independent of \( m, \beta \) such that
\[ ||f(u)||^{\frac{s+1}{s+1}}_{L^\frac{s+1}{s+1}} \leq c_1 I(u) + c_2 \]  
(3.63)
Proof:
If \( u \) is a critical point of \( I_\beta \) then \( I'_\beta(u)\phi = 0 \) \( \forall \phi \in E^m \oplus N^m \) hence
\[
I_\beta(u) = I_\beta(u) - I'_\beta(u)u \\
= \int_Q \frac{1}{2} u f(u) - F(u) dxdt \geq a_1(s) \int_Q |u|^{s+1} dxdt - a_2(s) \tag{3.64}
\]
such constant \( a_1(s), a_2(s) \) exist because \( f \) satisfies (1.9). Then we have
\[
I_\beta(u) \geq c_1 \int_Q |f(u)|^{\frac{s+1}{s}} dxdt - c_2(s). \tag{3.65}
\]
Let \( u^m = w^m + v^m \) the approximate solution on \( E^m \oplus N^m \) then
\[
\hat{\Box} w^m(j,k) = \hat{f}(u^m)(j,k) \tag{3.66}
\]
\( \forall 2j \neq k \in E^m \), hence by lemma 2.1 and the Hausdorff-Young we have
\[
\|w^m\|_{C^\gamma} \leq c \tag{3.67}
\]
with \( c \) independent of \( m, \beta \). Hence we can conclude that \( w = \lim_{m \to +\infty} u^m \in C^\gamma \) for any \( \gamma < 1 - \frac{s}{s+1} \).

In the following lemma we follow closely the method of [22] to get an a priori estimate on \( \|v\|_{C^0} \) independently of \( \beta \).

**Lemma 3.4.** There is a constant \( c \) independent of \( \beta \) such that
\[
\|v(\beta)\|_{C^0} \leq c \tag{3.68}
\]
Prove:
First note that by (3.61):
\[
\|v(\beta)\|_{L^2}^2 \leq \|u\|_{L^{s+1}}^{s+1} \leq c(l) + a_2(s) \tag{3.69}
\]
so we already have a \( L^2 \) a priori estimate on \( v \) independently of \( \beta \). The point of this lemma is then to prove a \( C^0 \) estimate. We will discuss two cases:

**Case 1:**
\( \|v(\beta)\|_{C^0} \leq 8\|v(\beta)\|_{L^2} \). Then we have a \( C^0 \) estimate on \( v(\beta) \) independently of \( \beta \).

**Case 2:**
\( \|v(\beta)\|_{C^0} > 8\|Q\|\|v(\beta)\|_{L^2} \).

Let \( \phi \in N \) then we have
\[
\int_Q [-\beta v_{tt} + \beta v + f(v + w)] \phi dxdt = 0 \tag{3.70}
\]
or
\[
\int_Q \beta v \phi + \beta v_t \phi_t + [(f(v + w) - f(w)) \phi] dxdt = - \int_Q f(w) \phi dxdt \tag{3.71}
\]
and $q$ is the function defined as

$$ q(s) = \begin{cases} 
  s + M & s \geq M \\
  0 & -M \leq s \leq M \\
  s - M & s < M 
\end{cases} $$

and choose

$$ \phi(x, t) = q(v^+(x, t)) + q(v^-(x, t)). \tag{3.72} $$

$$ \int_Q v^{-} q^{+} dxdt = \frac{1}{|Q|} \sum_{j,k} \int_{Q} v^{-} (j,k) q^{+} (j,k) $$

$$ = \frac{1}{|Q|} v^{-} (0,0) q^{+} (0,0) $$

$$ \leq ||v^{-}||_{L^2} ||q^{+}||_{L^1} = ||v^{-}||_{L^2} ||q^{+}||_{L^1} \tag{3.73} $$

and $\int_Q v^{+} q^{-} dxdt = \frac{1}{|Q|} \int_{Q} v^{+} (0,0) q^{-} (0,0) = 0$ similarly.

$$ \int_Q v^{+} v^{-} dxdt = \int_Q q'(v^+) (v^{+}_t)^2 + q'(v^-) (v^{-}_t)^2 + \frac{\partial}{\partial t} (q(v^+)) v^{-}_t + \frac{\partial}{\partial t} (q(v^-)) v^{+}_t dxdt $$

$$ = \int_Q q'(v^+) (v^{+}_t)^2 + q'(v^-) (v^{-}_t)^2 dxdt, \tag{3.74} $$

we define

$$ \psi(z) = \begin{cases} 
  \min_{|\xi| \leq M_0} f(z + \xi) - f(\xi) & z \geq 0 \\
  \max_{|\xi| \leq M_0} f(z + \xi) - f(\xi) & z < 0 
\end{cases} $$

which is monotone in $z$ with $\psi(0) = 0$. $Q_\delta = \{(x, t) \in Q, |v(x, t)| \geq \delta\}$, $Q_\delta^+ = \{(x, t) \in Q, v(x, t) \geq \delta\}$, $Q_\delta^- = Q_\delta \setminus Q_\delta^+$. If $v \geq 0$ then

$$ \int_{Q_\delta^+} [f(v + w) - f(w)] [q^+ + q^-] dxdt \geq \frac{\psi(\delta)}{||v||_{C^0}} \int_{Q_\delta^+} v(q^+ + q^-) dxdt. $$

Now for $v \leq -\delta$ when $v < 0$ then $f(v + w) - f(w) \leq \psi(v)$ and also $q^+ + q^- \leq 0$ and similarly

$$ \int_{Q_\delta^-} (f(v + w) - f(w)) (q^+ + q^-) dxdt \geq \frac{-\psi(-\delta)}{||v||_{C^0}} \int_{Q_\delta^-} v(q^+ + q^-) dxdt $$

now define $\nu(z) = \min(\psi(z), \psi(-z))$ for $z \geq 0$, and $||v^\pm||_{C^0} = \max(||v^\pm||_{C^0}, ||v^-||_{C^0})$ then

$$ \int_{Q_\delta} [f(v + w) - f(w)] [q^+ + q^-] dxdt \geq \frac{\nu(\delta)}{||v||_{C^0}} \int_{Q} v^+ q^+ + v^- q^- dxdt - \delta \int_{Q} |q^+| + |q^-| dxdt $$

$$ - \frac{\nu(\delta)}{||v||_{C^0}} ||v^-||_{L^2} \int_{Q} |q^+| + |q^-| dxdt $$

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where we employ (3.73) to estimate the terms $\int_Q v^+ q^- + v^- q^+ \, dxdt$, and since $sq(s) \geq M|q(s)|$, then

$$\|f(w)\|_{C^0} \int_Q (|q^+| + |q^-|) \, dxdt \geq \frac{(M - ||v^-||_{L^2} - \delta)\nu(\delta)}{\|v\|_{C^0}} \int_Q (|q^+| + |q^-|) \, dxdt$$

(3.75)

then for arbitrary $M < ||v^\pm||_{C^0}$ and choosing $\delta = \frac{||v^\pm||_{C^0}}{2}$ we deduce

$$\nu\left(\frac{1}{2}||v^\pm||_{C^0}\right) \leq 8\|f(w)\|_{C^0}$$

(3.76)

hence $\|v\|_{C^0}$ is bounded independently of $\beta$.

4 Regularity of the solution

Here we prove that if $f \in C^{2,1}$ then the weak solution $u$ is $C^2$. Since $||v||_{C^0}, ||w||_{C^0}$ are bounded independently of $\beta$, $f(u) \in C^0$. We also have

$$(-4j^2 + k^2)\hat{w}(j, k) = f(x, v + w)(j, k) \ 2j \neq \pm k.$$  

(4.77)

Then by lemma 2.2 we have $w \in H^1$. Since $f$ is smooth then too $f(w + v) \in H^1$. Then (4.77) implies $w \in H^2$ and iterating once again leads to $w \in H^3$. Now going back to the original equation:

$$-\beta v_{tt} = \Box w - f(x, u) - \beta v$$

(4.78)

and recalling that $v \in C^2$ we deduce $v \in H^3$ which with (4.77) implies $w \in H^4$. Iterating once more implies $v \in H^4$ then again $w \in H^5$ and $v \in H^5$. Thus we can differentiating with refer to $t$ in the weak sense and we have

$$\Box w_t - \beta v_{ttt} = -f_u'(x, u)(w_t + v_t) - \beta v_t$$

(4.79)

in Fourier space. We now want to get estimates independently of $\beta$ pass to the limit and find solutions of (1.1). Now multiplying by $v_t(\beta)$ (this is possible since $v \in H^1$) and integrating we have

$$\beta(v_t, v_t) + \beta(v_{tt}, v_t) + (f_u'(x, u)v_t, v_t) = -(f_u'(x, u)w_t, v_t)$$

(4.80)

and

$$\alpha ||v_t||_{L^2}^2 < (f_u'(x, u)v_t, v_t) \leq -(f_u'(x, u)w_t, v_t)$$

(4.81)

Now since $f_u' > \alpha > 0$ and $||w||_{H^2} \leq c$ with $c$ independently of $\beta$ hence there is a constant $c$ independent of $\beta$ such that $||v_t||_{L^2} \leq c$. This combined with (4.77) implies $||w||_{H^2} \leq c$ where $c$ is independent of $\beta$. Differentiating (4.79) with refer to $t$ we get

$$\beta v_{tt} + \Box w_{tt} - \beta v_{ttt} + f_u'(x, u)v_{tt} = -f_u''(x, u)v_t^2 - f_u''(x, u)w_t^2 - 2f_u'''(x, u)w_t v_t - f_u''(x, u)w_{tt}$$

(4.82)
Now we multiply (4.82) by \( v_t \) and estimate the \( L^2 \) norm of the first term of the RHS.

\[
(f'' \times u, v_t^2, v_t) \leq c(f) \int_0^\pi \int_0^{2\pi} v_t^2 |v_t| dx dt
\]

\[
\leq c(f) \left( \int_0^\pi \int_0^{2\pi} v_t^2 dx dt \right)^{\frac{1}{2}} \left( \int_0^\pi \int_0^{2\pi} |v_t|^2 dx dt \right)^{\frac{1}{2}}
\]

we then deduce

\[
(f'' \times u, v_t^2, v_t) \leq c(f) \|v_t\|_{L^2} \|v_t\|_{H^\frac{1}{2}}
\]

(4.83)

where the constant \( c(f) \) is independent of \( \beta \) and the inequalities in the previous argument stems from the Gagliardo-Nirenberg inequality.

The \( L^1 \) norms of the terms in the RHS of (4.82) multiplied by \( v_t \) can be estimated by noting that \( f(u) \in H^1, w \in C^{\gamma, \gamma}, 0 < \gamma < \frac{1}{2} \), and that the respective norms can be estimated are independently of \( \beta \):

\[
(f'' \times u, w_t^2, v_t) \leq c \|v_t\|_{L^2}
\]

(4.84)

\[
(2f'' \times u, w_t, v_t) \leq c \|v_t\|_{L^2}
\]

(4.85)

\[
(f'' \times u, w_t, v_t) \leq c \|w_t\|_{L^2} \|v_t\|_{L^2}
\]

(4.86)

recalling (4.82), multiplying by \( v_t \)

\[
\beta(v_t, v_t) + \beta(v_t, v_t) + (f'' \times u, v_t, v_t) = (-f'' \times u, v_t, v_t) + (f'' \times u, u) v_t + v_t + v_t)
\]

(4.87)

We can now combine from (4.82), (4.83), (4.84), (4.85), (4.86) and we have

\[
\beta(v_t, v_t) + \beta(v_t, v_t) + (f'' \times u, u) v_t, v_t) \leq c \|v_t\|_{H^\frac{1}{2}}^2
\]

(4.88)

thus there exists \( c \) independent of \( \beta \) such that \( \|v_t\|_{L^2} \leq c \) where \( c \) is independent of \( \beta \). At this stage we can conclude that there is a constant \( c \) independent of \( \beta \) such that \( \|f(u)\|_{H^2} \leq c \). Combining this with (4.77) we have \( \|w\|_{H^3} \leq c \) with \( c \) is independent of \( \beta \), \( w \in C^{\gamma, \gamma} \) and \( v \in C^1 \) with upper bounds independent of \( \beta \). We have now proved that if \( f \) is \( C^2 \) then the solution is \( u \in H^2 \cap C^1 \) is a weak solution of the equation. We now differentiate (4.82) we have

\[
\beta v_t + \Box w_t + f'(u)v_t = -f'' \times u(x, u) v_t - f'' \times u(x, u)(v_t + w_t)v_t - f'' \times u(x, u)2v_t w_t - f'' \times u(x, u)w_t^2
\]

\[
-2f'' \times u(x, u)w_t v_t - f'' \times u(x, u)(v_t + w_t)v_t - 2f'' \times u(x, u)w_t v_t
\]

\[
-2f'' \times u(x, u)w_t v_t - f'' \times u(x, u)(v_t + w_t)v_t - f'' \times u(x, u)w_t v_t
\]

and multiplying both sides of the preceding equality by \( v_{ttt} \) and integrating we conclude that \( \|v_{ttt}\|_{L^2} \leq c \) where \( c \) is independent of \( \beta \) thus \( v \) is \( C^2 \). Now recalling the Holder regularity bootstrap and (4.77) we get \( w \in C^{\gamma, \gamma}, 0 < \gamma < \frac{1}{2} \).
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