Concentration Properties of Restricted Measures with Applications to Non-Lipschitz Functions

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Abstract

We show that for any metric probability space \((M,d,\mu)\) with a subgaussian constant \(\sigma^2(\mu)\) and any set \(A \subset M\) we have \(\sigma^2(\mu_A) \leq c \log (e/\mu(A)) \sigma^2(\mu)\), where \(\mu_A\) is a restriction of \(\mu\) to the set \(A\) and \(c\) is a universal constant. As a consequence we deduce concentration inequalities for non-Lipschitz functions.

1 Introduction

It is known that many high-dimensional probability distributions \(\mu\) on the Euclidean space \(\mathbb{R}^n\) (and other metric spaces, including graphs) possess strong concentration properties. In a functional language, this may informally be stated as the assertion that any sufficiently smooth function \(f\) on \(\mathbb{R}^n\), e.g., having a bounded Lipschitz semi-norm, is almost a constant on almost all space. There are several ways to quantify such a property. One natural approach proposed by N. Alon, R. Boppana and J. Spencer [A-B-S] associates with a given metric probability space \((M,d,\mu)\) its spread constant,

\[ s^2(\mu) = \sup \text{Var}_\mu(f) = \sup \int (f - m)^2 \, d\mu, \]

where \(m = \int f \, d\mu\), and the sup is taken over all functions \(f\) on \(M\) with \(\|f\|_{\text{Lip}} \leq 1\). More information is contained in the so-called subgaussian constant \(\sigma^2 = \sigma^2(\mu)\) which is defined as the infimum over all \(\sigma^2\) such that

\[ \int e^{tf} \, d\mu \leq e^{\sigma^2 t^2/2}, \quad \text{for all } t \in \mathbb{R}, \tag{1.1} \]

for any \(f\) on \(M\) with \(m = 0\) and \(\|f\|_{\text{Lip}} \leq 1\) (cf. [B-G-H]). This quantity may also be introduced via the transport-entropy inequality relating the classical Kantorovich distance

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and the relative entropy from an arbitrary probability measure on $M$ to the measure $\mu$ (cf. [B-G]).

While in general $s^2 \leq \sigma^2$, the latter characteristic allows one to control subgaussian tails under the probability measure $\mu$ uniformly in the entire class of Lipschitz functions on $M$. More generally, when $\|f\|_{\text{Lip}} \leq L$, (1.1) yields

$$\mu\{|f - m| \geq t\} \leq 2e^{-t^2/(\sigma^2 L^2)}, \quad t > 0. \quad (1.2)$$

Classical and well-known examples include the standard Gaussian measure on $M = \mathbb{R}^n$ in which case $s^2 = \sigma^2 = 1$, and the normalized Lebesgue measure on the unit sphere $M = S^{n-1}$ with $s^2 = \sigma^2 = \frac{1}{n-1}$. The last example was a starting point in the study of the concentration of measure phenomena, a fruitful direction initiated in the early 1970s by V. D. Milman.

Other examples come often after verification that $\mu$ satisfies certain Sobolev-type inequalities such as Poincaré-type inequalities

$$\lambda_1 \text{Var}_\mu(u) \leq \int |\nabla u|^2 \, d\mu,$$

and logarithmic Sobolev inequalities

$$\rho \text{Ent}_\mu(u^2) = \rho \left[ \int u^2 \log u^2 \, d\mu - \int u^2 \, d\mu \log \int u^2 \, d\mu \right] \leq 2 \int |\nabla u|^2 \, d\mu,$$

where $u$ may be any locally Lipschitz function on $M$, and the constants $\lambda_1 > 0$ and $\rho > 0$ do not depend on $u$. Here the modulus of the gradient may be understood in the generalized sense as the function

$$|\nabla u(x)| = \limsup_{y \to x} \frac{|u(x) - u(y)|}{d(x, y)}, \quad x \in M$$

(this is the so-called “continuous setting”), while in the discrete spaces, e.g., graphs, we deal with other naturally defined gradients. In both cases, one has respectively the well-known upper bounds

$$s^2(\mu) \leq \frac{1}{\lambda_1}, \quad \sigma^2(\mu) \leq \frac{1}{\rho}. \quad (1.3)$$

For example, $\lambda_1 = \rho = n - 1$ on the unit sphere (best possible values, [M-W]), which can be used to make a corresponding statement about the spread and Gaussian constants.

One of the purposes of this note is to give new examples by involving the family of the normalized restricted measures

$$\mu_A(B) = \frac{\mu(A \cap B)}{\mu(A)}, \quad B \subset M \text{ (Borel)},$$

where a set $A \subset M$ is fixed and has a positive measure. As an example, returning to the standard Gaussian measure $\mu$ on $\mathbb{R}^n$, it is known that $\sigma^2(\mu_A) \leq 1$ for any convex body $A \subset \mathbb{R}^n$. This remarkable property, discovered by D. Bakry and M. Ledoux [B-L] in a sharper form of a Gaussian-type isoperimetric inequality, has nowadays several proofs and generalizations, cf. [B1] [B2]. Of course, in general, the set $A$ may have a rather disordered structure, for example, to be disconnected. And then there is no hope for validity of a Poincaré-type inequality for the measure $\mu_A$. Nevertheless, it turns out that the concentration property of $\mu_A$ is inherited from $\mu$, unless the measure of $A$ is too small. In particular, we have the following observation about abstract metric probability spaces.

**Theorem 1.1.** For any measurable set $A \subset M$ with $\mu(A) > 0$, the subgaussian constant $\sigma^2(\mu_A)$ of the normalized restricted measure satisfies

$$\sigma^2(\mu_A) \leq c \log \left( \frac{e}{\mu(A)} \right) \sigma^2(\mu), \quad (1.4)$$

where $c$ is an absolute constant.
One may further generalize this assertion by defining the subgaussian constant $\sigma^2_F(\mu)$ within a given fixed subclass $F$ of functions on $M$, by using the same bound (1.1) on the Laplace transform. This is motivated by a possible different level of concentration for different classes; indeed, in case of $M = \mathbb{R}^n$, the concentration property may considerably be strengthened for the class $F$ of all convex Lipschitz functions. In particular, one result of M. Talagrand [T1, T2] provides a dimension-free bound $\sigma^2_F(\mu) \leq C$ for an arbitrary product probability measure $\mu$ on the $n$-dimensional cube $[-1,1]^n$. Hence, a more general version of Theorem 1.1 yields the bound

$$\sigma^2_F(\mu_A) \leq c \log \left( \frac{e}{\mu(A)} \right)$$

with some absolute constant $c$, which holds for any Borel subset $A$ of $[-1,1]^n$ (cf. Section 6 below).

According to the very definition, the quantities $\sigma^2(\mu)$ and $\sigma^2(\mu_A)$ might seem to be responsible for deviations of only Lipschitz functions $f$ on $M$ and $A$, respectively. However, the inequality (1.4) may also be used to control deviations of non-Lipschitz $f$ – on large parts of the space and under certain regularity hypotheses. Assume, for example, $\int |\nabla f| d\mu \leq 1$ (which is kind of a normalization condition) and consider

$$A = \{ x \in M : |\nabla f(x)| \leq L \}.$$  \hspace{1cm} (1.5)

If $L \geq 2$, this set has the measure $\mu(A) \geq 1 - \frac{1}{L} \geq \frac{1}{2}$, and hence, $\sigma^2(\mu_A) \leq c \sigma^2(\mu)$ with some absolute constant $c$. If we assume that $f$ has a Lipschitz semi-norm $\leq L$ on $A$, then, according to (1.2),

$$\mu_A \{ x \in A : |f - m| \geq t \} \leq 2e^{-t^2/(c \sigma^2(\mu)L^2)}, \hspace{1cm} t > 0,$$

where $m$ is the mean of $f$ with respect to $\mu_A$. It is in this sense one may say that $f$ is almost a constant on the set $A$.

This also yields a corresponding deviation bound on the whole space,

$$\mu \{ x \in M : |f - m| \geq t \} \leq 2e^{-t^2/(c \sigma^2(\mu)L^2)} + \frac{1}{L}.$$  \hspace{1cm} (1.6)

Stronger integrability conditions posed on $|\nabla f|$ can considerably sharpen the conclusion. By a similar argument, Theorem 1.1 yields, for example, the following exponential bound, known in the presence of a logarithmic Sobolev inequality for the space $(M,d,\mu)$, and with $\sigma^2$ replaced by $1/\rho$ (cf. [B-G]).

**Corollary 1.2.** Let $f$ be a locally Lipschitz function on $M$ with Lipschitz semi-norms $\leq L$ on the sets (1.5). If $\int e^{\psi_1 \nabla f}^2\ d\mu \leq 2$, then $f$ is $\mu$-integrable, and moreover,

$$\mu \{ x \in M : |f - m| \geq t \} \leq 2e^{-t/(c \sigma(\mu))}, \hspace{1cm} t > 0,$$

where $m$ is the $\mu$-mean of $f$ and $c$ is an absolute constant.

Equivalently (up to an absolute factor), we have a Sobolev-type inequality

$$\|f - m\|_{\psi_1} \leq c \sigma(\mu) \|\nabla f\|_{\psi_2},$$

connecting the $\psi_1$-norm of $f - m$ with the $\psi_2$-norm of the modulus of the gradient of $f$.

We prove a more general version of this corollary in Section 6 (cf. Theorem 6.1). As will be explained in the same section, similar assertions may also be made about convex $f$ and product measures $\mu$ on $M = [-1,1]^n$, thus extending Talagrand’s theorem to the class of non-Lipschitz functions.
In view of the right bound in (1.3) and (1.4), the spread and subgaussian constants for restricted measures can be controlled in terms of the logarithmic Sobolev constant $\rho$ via

$$s^2(\mu_A) \leq c^2(\mu_A) \leq c \log \left( \frac{e}{\mu(A)} \right) \frac{1}{\rho}.$$ 

However, it may happen that $\rho = 0$ and $\sigma^2(\mu) = \infty$, while $\lambda_1 > 0$ (e.g., for the product exponential distribution on $\mathbb{R}^n$). Then one may wonder whether one can estimate the spread constant of a restricted measure in terms of the spectral gap. In that case there is a bound similar to (1.4).

**Theorem 1.3.** Assume the metric probability space $(M,d,\mu)$ satisfies a Poincaré-type inequality with $\lambda_1 > 0$. For any $A \subset M$ with $\mu(A) > 0$, with some absolute constant $c$

$$s^2(\mu_A) \leq c \log^2 \left( \frac{e}{\mu(A)} \right) \frac{1}{\lambda_1}.$$ 

(1.7)

It should be mentioned that the logarithmic terms in (1.4) and (1.7) may not be removed and are actually asymptotically optimal as functions of $\mu(A)$, as $\mu(A)$ is getting small, see Section 7.

Our contribution below is organized into sections as follows:

2. Bounds on $\psi_\alpha$-norms for restricted measures.
3. Proof of Theorem 1.1. Transport-entropy formulation.
4. Proof of Theorem 1.3. Spectral gap.
5. Examples.
6. Deviations for non-Lipschitz functions.
7. Optimality.
8. Appendix.

**2 Bounds on $\psi_\alpha$-norms for restricted measures**

A measurable function $f$ on the probability space $(M,\mu)$ is said to have a finite $\psi_\alpha$-norm, $\alpha \geq 1$, if for some $r > 0$,

$$\int e^{(|f|/r)^\alpha} d\mu \leq 2.$$ 

The infimum over all such $r$ represents the $\psi_\alpha$-norm $\|f\|_{\psi_\alpha}$ or $\|f\|_{L^{\psi_\alpha}(\mu)}$, which is just the Orlicz norm associated with the Young function $\psi_\alpha(t) = e^{t^\alpha} - 1$.

We are mostly interested in the particular cases $\alpha = 1$ and $\alpha = 2$. In this section we recall well-known relations between the $\psi_1$ and $\psi_2$-norms and the usual $L^p$-norms $\|f\|_p = \|f\|_{L^p(\mu)} = (\int |f|^p d\mu)^{1/p}$. For the readers’ convenience, we include the proof in the appendix.

**Lemma 2.1.** We have

$$\sup_{p \geq 1} \frac{\|f\|_p}{p} \leq \|f\|_{L^{\psi_2}(\mu)} \leq 4 \sup_{p \geq 1} \frac{\|f\|_p}{\sqrt{p}},$$ 

(2.1)

$$\sup_{p \geq 1} \frac{\|f\|_p}{p} \leq \|f\|_{L^{\psi_1}(\mu)} \leq 6 \sup_{p \geq 1} \frac{\|f\|_p}{p}.$$ 

(2.2)

Given a measurable subset $A$ of $M$ with $\mu(A) > 0$, we consider the normalized restricted measure $\mu_A$ on $M$, i.e.,

$$\mu_A(B) = \frac{\mu(A \cap B)}{\mu(A)}, \quad B \subset M.$$ 

Our basic tool leading to Theorem 1.1 will be the following assertion.
Proposition 2.2. For any measurable function $f$ on $M$,

$$
\|f\|_{L^{p}(\mu \otimes \nu)} \leq 4e \log^{1/2} \left( \frac{e}{\mu(A)} \right) \|f\|_{L^{p}(\mu)}.
$$

(2.3)

Proof. Assume that $\|f\|_{L^{p}(\mu)} = 1$ and fix $p \geq 1$. By the left inequality in (2.1), for any $q \geq 1$,

$$
q^{p/2} \geq \int |f|^q d\mu \geq \mu(A) \int |f|^q d\mu_A.
$$

so

$$
\frac{\|f\|_{L^{q}(\mu)}}{\sqrt{q}} \leq \left( \frac{1}{\mu(A)} \right)^{1/q}.
$$

But by the right inequality in (2.1),

$$
\|f\|_{L^{q}(\mu)} \leq 4 \sup_{q \geq 1} \frac{\|f\|_q}{\sqrt{q}} \leq 4\sqrt{p} \sup_{q \geq p} \frac{\|f\|_q}{\sqrt{q}}.
$$

Applying it on the space $(M, \mu)$, we then get

$$
\|f\|_{L^{p}(\mu \otimes \nu)} \leq 4\sqrt{p} \sup_{q \geq p} \frac{\|f\|_{L^{q}(\mu \otimes \nu)}}{\sqrt{q}}
$$

$$
\leq 4\sqrt{p} \sup_{q \geq p} \left( \frac{1}{\mu(A)} \right)^{1/q} = 4\sqrt{p} \left( \frac{1}{\mu(A)} \right)^{1/p}.
$$

The obtained inequality,

$$
\|f\|_{L^{p}(\mu \otimes \nu)} \leq 4\sqrt{p} \left( \frac{1}{\mu(A)} \right)^{1/p},
$$

holds true for any $p \geq 1$ and therefore may be optimized over $p$. Choosing $p = \log \left( \frac{e}{\mu(A)} \right)$, we arrive at (2.3). \qed

A possible weak point in the bound (2.3) is that the means of $f$ are not involved. For example, in applications, if $f$ were defined only on $A$ and had $\mu_A$-mean zero, we might need to find an extension of $f$ to the whole space $M$ keeping the mean zero with respect to $\mu$. In fact, this should not create any difficulty, since one may work with the symmetrization of $f$.

More precisely, we may apply Proposition 2.2 on the product space $(M \times M, \mu \otimes \mu)$ to the product sets $A \times A$ and functions of the form $f(x) - f(y)$. Then we get

$$
\|f(x) - f(y)\|_{L^{p}(\mu_A \otimes \mu_A)} \leq 4e \log^{1/2} \left( \frac{e}{\mu(A)^2} \right) \|f(x) - f(y)\|_{L^{p}(\mu \otimes \mu)}.
$$

Since $\log \left( \frac{e}{\mu(A)^2} \right) \leq 2 \log \left( \frac{e}{\mu(A)} \right)$, we arrive at:

Corollary 2.3. For any measurable function $f$ on $M$,

$$
\|f(x) - f(y)\|_{L^{p}(\mu_A \otimes \mu_A)} \leq 4e\sqrt{2} \log^{1/2} \left( \frac{e}{\mu(A)} \right) \|f(x) - f(y)\|_{L^{p}(\mu \otimes \mu)}.
$$

Let us now derive an analog of Proposition 2.2 for the $\psi_1$-norm, using similar arguments. Assume that $\|f\|_{L^{p}(\mu)} = 1$ and fix $p \geq 1$. By the left inequality in (2.2), for any $q \geq 1$,

$$
q^q \geq \int |f|^q d\mu \geq \mu(A) \int |f|^q d\mu_A,
$$

(2.4)
so
\[
\frac{\|f\|_{L^q(\mu_A)}}{q} \leq \left( \frac{1}{\mu(A)} \right)^{1/q}.
\]
But, by the inequality (2.2),
\[
\|f\|_{L^q} \leq 6 \sup_{q \geq 1} \frac{\|f\|_q}{q} \leq 6p \sup_{q \geq p} \frac{\|f\|_q}{q}.
\]
Applying it on the space \((M, \mu_A)\), we get
\[
\|f\|_{L^q(\mu_A)} \leq 6p \sup_{q \geq p} \left( \frac{1}{\mu(A)} \right)^{1/q} = 6p \left( \frac{1}{\mu(A)} \right)^{1/p}.
\]
The obtained inequality,
\[
\|f\|_{L^q(\mu_A)} \leq 6p \left( \frac{1}{\mu(A)} \right)^{1/p},
\]
holds true for any \(p \geq 1\) and therefore may be optimized over \(p\). Choosing \(p = \log \left( \frac{e}{\mu(A)} \right)\), we arrive at:

**Proposition 2.4.** For any measurable function \(f\) on \(M\), we have
\[
\|f\|_{L^q(\mu_A)} \leq 6e \log \left( \frac{e}{\mu(A)} \right) \|f\|_{L^q(\mu)}.
\]

Similarly to Corollary 2.3 one may write down this relation on the product probability space \((M \times M, \mu \otimes \mu)\) with the functions of the form \(\tilde{f}(x, y) = f(x) - f(y)\) and the product sets \(\tilde{A} = A \times A\). Then we get
\[
\|f(x) - f(y)\|_{L^q(\mu_A \otimes \mu_A)} \leq 12e \log \left( \frac{e}{\mu(A)} \right) \|f(x) - f(y)\|_{L^q(\mu \otimes \mu)}.
\] (2.4)

3 Proof of Theorem 1.1. Transport-entropy formulation

The finiteness of the subgaussian constant for a given metric probability space \((M, d, \mu)\) means that \(\psi_2\)-norms of Lipschitz functions on \(M\) with mean zero are uniformly bounded. Equivalently, for any (for all) \(x_0 \in M\), we have that, for some \(\lambda > 0\),
\[
\int e^{d(x, x_0)^2/\lambda^2} \, d\mu(x) < \infty.
\]

The definition (1.1) of \(\sigma^2(\mu)\) inspires to consider another norm-like quantity
\[
\sigma_f^2 = \sup_{t \neq 0} \left[ \frac{1}{t^2/2} \log \int e^{tf} \, d\mu \right].
\]
Here is a well-known relation (with explicit numerical constants) which holds in the setting of an abstract probability space \((M, \mu)\). Once again, we include a proof in the appendix for completeness.

**Lemma 3.1.** If \(f\) has mean zero and finite \(\psi_2\)-norm, then
\[
\frac{1}{\sqrt{6}} \|f\|_{\psi_2}^2 \leq \sigma_f^2 \leq 4 \|f\|_{\psi_2}^2.
\]
One can now relate the subgaussian constant of the restricted measure to the subgaussian constant of the original measure. Let now $(M, d, \mu)$ be a metric probability space. First, Lemma 3.1 immediately yields an equivalent description in terms of $\psi_2$-norms, namely

$$\frac{1}{\sqrt{6}} \sup_{f} \| f \|_{\psi_2}^2 \leq \sigma^2(\mu) \leq 4 \sup_{f} \| f \|_{\psi_2}^2,$$

(3.1)

where the supremum is running over all $f : M \to \mathbb{R}$ with $\mu$-mean zero and $\|f\|_{\text{Lip}} \leq 1$. Here, one can get rid of the mean zero assumption by considering functions of the form $f(x) - f(y)$ on the product space $(M \times M, \mu \otimes \mu)$. If $f$ has mean zero, then, by Jensen’s inequality,

$$\iint e^{(f(x) - f(y))^2/\sigma^2} \, d\mu(x) \, d\mu(y) \geq \int e^{f(x)^2/\sigma^2} \, d\mu(x),$$

which implies that

$$\|f(x) - f(y)\|_{L^2(\mu \otimes \mu)} \geq \|f\|_{L^2(\mu)}.$$

On the other hand, by the triangle inequality,

$$\|f(x) - f(y)\|_{L^2(\mu \otimes \mu)} \leq 2 \|f\|_{L^2(\mu)}.$$

Hence, we arrive at another, more flexible relation, where the mean zero assumption may be removed.

**Lemma 3.2.** We have

$$\frac{1}{4\sqrt{6}} \sup_{f \in \mathcal{F}} \|f(x) - f(y)\|_{L^2(\mu \otimes \mu)}^2 \leq \sigma^2(\mu) \leq 4 \sup_{f \in \mathcal{F}} \|f(x) - f(y)\|_{L^2(\mu \otimes \mu)}^2,$$

where the supremum is running over all functions $f$ on $M$ with $\|f\|_{\text{Lip}} \leq 1$.

**Proof of Theorem 1.1.** We are prepared to make last steps for the proof of the inequality (1.3). We use the well-known Kirszbraun’s theorem: Any function $f : A \to \mathbb{R}$ with Lipschitz semi-norm $\|f\|_{\text{Lip}} \leq 1$ on $A$ admits a Lipschitz extension to the whole space $(K, \mathcal{M})$. Namely, one may put

$$\tilde{f}(x) = \inf_{a \in A} \left[ f(a) + d(a, x) \right], \quad x \in M.$$

Applying first Corollary 2.3 and then the left inequality of Lemma 3.2 to $\tilde{f}$, we get

$$\|f(x) - f(y)\|_{L^2(\mu \otimes \mu)} = \|\tilde{f}(x) - \tilde{f}(y)\|_{L^2(\mu \otimes \mu)} \leq \left(4\sqrt{2}\right)^2 \log \left(\frac{e}{\mu(A)}\right) \|\tilde{f}(x) - \tilde{f}(y)\|_{L^2(\mu \otimes \mu)} \leq \left(4\sqrt{2}\right)^2 \log \left(\frac{e}{\mu(A)}\right) \cdot (4\sqrt{2})^2 \sigma^2(\mu).$$

Another application of Lemma 3.2 in the space $(A, d, \mu_A)$ (now the right inequality) yields

$$\sigma^2(\mu_A) \leq 4 \cdot (4\sqrt{2})^2 \log \left(\frac{e}{\mu(A)}\right) \cdot (4\sqrt{6})^2 \sigma^2(\mu).$$

This is exactly (1.3) with constant $c = 4 \cdot (4\sqrt{2})^2 (4\sqrt{6})^2 = 3 \cdot 2^{12} e^2 = 90796.72...$.$\square$

**Remark 3.3.** Let us also record the following natural generalization of Theorem 1.1, which is obtained along the same arguments. Given a collection $\mathcal{F}$ of (integrable) functions on the probability space $(M, \mu)$, define $\sigma_2^2(\mu)$ as the infimum over all $\sigma^2$ such that

$$\int e^{(f-m)^2} \, d\mu \leq e^{(\sigma_2^2/2)}, \quad \text{for all } t \in \mathbb{R},$$
for any \( f \in \mathcal{F} \), where \( m = \int f \, d\mu \). Then with the same constant \( c \) as in Theorem 1.1, for any measurable \( A \subset M \), \( \mu(A) > 0 \), we have

\[
\sigma^2_{\mathcal{F}_A}(\mu_A) \leq c \log \left( \frac{e}{\mu(A)} \right) \sigma^2(\mu),
\]

where \( \mathcal{F}_A \) denotes the collection of restrictions of functions \( f \) from \( \mathcal{F} \) to the set \( A \).

Let us now mention an interesting connection of the subgaussian constants with the Kantorovich distances

\[
W_1(\mu, \nu) = \inf \int \int d(x, y) \pi(x, y)
\]

and the relative entropies

\[
D(\nu||\mu) = \int \log \frac{d\nu}{d\mu} d\nu
\]

(called also Kullback-Leibler’s distances or informational divergences). Here, \( \nu \) is a probability measure on \( M \), which is absolutely continuous with respect to \( \mu \) (for short, \( \nu \ll \mu \)), and the infimum in the definition of \( W_1 \) is running over all probability measures \( \pi \) on the product space \( M \times M \) with marginal distributions \( \mu \) and \( \nu \), i.e., such that

\[
\pi(B \times M) = \mu(B), \quad \pi(M \times B) = \nu(B) \quad \text{(Borel } B \subset M).\]

As was shown in [B-G], if \((M, d)\) is a Polish space (complete separable), the subgaussian constant \( \sigma^2 = \sigma^2(\mu) \) may be described as an optimal value in the transport-entropy inequality

\[
W_1(\mu, \nu) \leq \sqrt{2\sigma^2 D(\nu||\mu)}. \tag{3.2}
\]

Hence, we obtain from the inequality (1.4) a similar relation for measures \( \nu \) supported on given subsets of \( M \).

**Corollary 3.4.** Given a Borel probability measure \( \mu \) on a Polish space \((M, d)\) and a closed set \( A \) in \( M \) such that \( \mu(A) > 0 \), for any Borel probability measure \( \nu \) supported on \( A \),

\[
W_1^2(\mu_A, \nu) \leq c \sigma^2(\mu) \log \left( \frac{e}{\mu(A)} \right) D(\nu||\mu_A),
\]

where \( c \) is an absolute constant.

This assertion is actually equivalent to Theorem 1.1. Note that, for \( \nu \) supported on \( A \), there is an identity \( D(\nu||\mu_A) = \log \mu(A) + D(\nu||\mu) \). In particular, \( D(\nu||\mu_A) \leq D(\nu||\mu) \), so the relative entropies decrease when turning to restricted measures.

### 4 Proof of Theorem 1.3. Spectral gap

Theorem 1.1 insures, in particular, that, for any function \( f \) on the metric probability space \((M, d, \mu)\) with Lipschitz semi-norm \( \|f\|_{\text{Lip}} \leq 1 \),

\[
\text{Var}_{\mu_A}(f) \leq c \log \left( \frac{e}{\mu(A)} \right) \sigma^2(\mu)
\]

up to some absolute constant \( c \). In fact, in order to reach a similar concentration property of the restricted measures, it is enough to start with a Poincaré-type inequality on \( M \),

\[
\lambda_1 \text{Var}_{\mu}(f) \leq \int |\nabla f|^2 \, d\mu.
\]
Under this hypothesis, a well-known theorem due to Gromov-Milman and Borovkov-Utev asserts that mean zero Lipschitz functions \( f \) have bounded \( \psi_1 \)-norm. One may use a variant of this theorem proposed by Aida and Strook [A-S], who showed that

\[
\int e^{\sqrt{\lambda} f} \, d\mu \leq K_0 = 1.720102... \quad (\| f \|_{\text{Lip}} \leq 1).
\]

Hence

\[
\int e^{\sqrt{\lambda} |f|} \, d\mu \leq 2K_0 \quad \text{and} \quad \int e^{\frac{1}{2} \sqrt{\lambda} |f|} \, d\mu \leq \sqrt{2K_0} < 2,
\]

thus implying that \( \| f \|_{\psi_1} \leq \frac{2}{\sqrt{\lambda_1}} \). In addition,

\[
\int e^{\sqrt{\lambda} (f(x) - f(y))} \, d\mu(x) \, d\mu(y) \leq K_0^2, \quad \int e^{\sqrt{\lambda} |f(x) - f(y)|} \, d\mu(x) \, d\mu(y) \leq 2K_0^2 < 6.
\]

From this,

\[
\int e^{\frac{1}{2} \sqrt{\lambda} |f(x) - f(y)|} \, d\mu(x) \, d\mu(y) < 6^{1/3} < 2,
\]

which means that \( \| f(x) - f(y) \|_{\psi_1} \leq \frac{3}{\sqrt{\lambda_1}} \) with respect to the product measure \( \mu \otimes \mu \) on the product space \( M \times M \). This inequality is translation invariant, so the mean zero assumption may be removed. Thus, we arrive at:

**Lemma 4.1.** Under the Poincaré-type inequality with spectral gap \( \lambda_1 > 0 \), for any mean zero function \( f \) on \( (M, d, \mu) \) with \( \| f \|_{\text{Lip}} \leq 1 \),

\[
\| f \|_{\psi_1} \leq \frac{2}{\sqrt{\lambda_1}}.
\]

Moreover, for any \( f \) with \( \| f \|_{\text{Lip}} \leq 1 \),

\[
\| f(x) - f(y) \|_{L^{\psi_1}(\mu \otimes \mu)} \leq \frac{3}{\sqrt{\lambda_1}} \tag{4.1}
\]

This is a version of the concentration of measure phenomenon (with exponential integrability) in presence of a Poincaré-type inequality. Our goal is therefore to extend this property to the normalized restricted measures \( \mu_A \). This can be achieved by virtue of the inequality (2.4) which when combined with (4.1) yields an upper bound

\[
\| f(x) - f(y) \|_{L^{\psi_1}(\mu_A \otimes \mu_A)} \leq 36 e \log \left( \frac{e}{\mu(A)} \right) \frac{1}{\sqrt{\lambda_1}}.
\]

Moreover, if \( f \) has \( \mu_A \)-mean zero, the left norm dominates \( \| f \|_{L^{\psi_1}(\mu_A)} \) (by Jensen’s inequality). We can summarize, taking into account once again Kirszbraun’s theorem, as we did in the proof of Theorem 1.1.

**Proposition 4.2.** Assume the metric probability space \( (M, d, \mu) \) satisfies a Poincaré-type inequality with constant \( \lambda_1 > 0 \). Given a measurable set \( A \subset M \) with \( \mu(A) > 0 \), for any function \( f : A \to \mathbb{R} \) with \( \mu_A \)-mean zero and such that \( \| f \|_{\text{Lip}} \leq 1 \) on \( A \),

\[
\| f \|_{L^{\psi_1}(\mu_A)} \leq 36 e \log \left( \frac{e}{\mu(A)} \right) \frac{1}{\sqrt{\lambda_1}}.
\]

Theorem 1.3 is now easily obtained with constant \( e = 2 (36 e)^2 \) by noting that \( L^2 \)-norms are dominated by \( L^{\psi_1} \)-norms. More precisely, since \( e|t| - 1 \geq \frac{1}{2} t^2 \), one has \( \| f \|_{\psi_1}^2 \geq \frac{1}{2} \| f \|_2^2 \).
5 Examples

Theorems 1.1 and 1.3 involve a lot of interesting examples. Here are a few obvious cases.

1) The standard Gaussian measure $\mu = \gamma$ on $\mathbb{R}^n$ satisfies a logarithmic Sobolev inequality on $M = \mathbb{R}^n$ with a dimension-free constant $\rho = 1$. Hence, from Theorem 1.1 we get:

**Corollary 5.1.** For any measurable set $A \subset \mathbb{R}^n$ with $\gamma(A) > 0$, the subgaussian constant $\sigma^2(\gamma_A)$ of the normalized restricted measure $\gamma_A$ satisfies

$$\sigma^2(\gamma_A) \leq c \log \left( \frac{e}{\gamma(A)} \right),$$

where $c$ is an absolute constant.

As it was already mentioned, if $A$ is convex, there is a sharper bound $\sigma^2(\gamma_A) \leq 1$. However, it may not hold without convexity assumption. Nevertheless, if $\gamma(A)$ is bounded away from zero, we obtain a more universal principle.

Clearly, Corollary 5.1 extends to all product measures $\mu = \nu^n$ on $\mathbb{R}^n$ such that $\nu$ satisfies a logarithmic Sobolev inequality on the real line, and with constants $c$ depending on $\rho$, only. A characterization of the property $\rho > 0$ in terms of the distribution function of the measure $\nu$ and the density of its absolutely continuous component may be found in [B-G].

2) Consider a uniform distribution $\nu$ on the shell

$$A_\varepsilon = \{ x \in \mathbb{R}^n : 1 - \varepsilon \leq |x| \leq 1 \}, \quad 0 \leq \varepsilon \leq 1 \ (n \geq 2).$$

**Corollary 5.2.** The subgaussian constant of $\nu$ satisfies $\sigma^2(\nu) \leq \frac{c}{n}$, up to some absolute constant $c$.

In other words, mean zero Lipschitz functions $f$ on $A_\varepsilon$ are such that $\sqrt{n}f$ are subgaussian. This property is well-known in the extreme cases – on the unit Euclidean ball $A = B_n$ ($\varepsilon = 1$) and on the unit sphere $A = S^{n-1}$ ($\varepsilon = 0$).

Let $\mu$ denote the normalized Lebesgue measure on $B_n$. In the case $\varepsilon \geq \frac{1}{n}$, the shell $A_\varepsilon$ represents the part of $B_n$ of measure

$$\mu(A_\varepsilon) = 1 - \left( 1 - \frac{1}{n} \right)^n \geq 1 - \frac{1}{e}.$$ 

Since the logarithmic Sobolev constant of the unit ball is of order $\frac{1}{n}$, and therefore $\sigma^2(\mu) \leq \frac{c}{n}$, the assertion of Corollary 5.1 immediately follows from Theorem 1.1. If $0 \leq \varepsilon \leq \frac{1}{n}$, the assertion follows from a similar concentration property of the uniform distribution on the unit sphere. Indeed, with every Lipschitz function $f$ on $A_\varepsilon$ one may associate its restriction to $S^{n-1}$, which is also Lipschitz (with respect to the Euclidean distance). On the other hand, for any $r \in [1 - \varepsilon, 1]$ and $\theta \in S^{n-1}$, we have $|f(r\theta) - f(\theta)| \leq |r - 1| \leq \varepsilon \leq \frac{1}{n}$, thus proving the claim.

3) The two-sliced product exponential measure $\mu$ on $\mathbb{R}^n$ with density $2^{-n}e^{-\|x_1\|_1+\cdots+\|x_n\|_1}$ satisfies a Poincaré-type inequality on $M = \mathbb{R}^n$ with a dimension-free constant $\lambda_1 = 1/4$. Hence, from Proposition 4.2 we get:

**Corollary 5.3.** For any measurable set $A \subset \mathbb{R}^n$ with $\mu(A) > 0$, and for any function $f : A \to \mathbb{R}$ with $\mu_A$-mean zero and $\|f\|_{L^1} \leq 1$, we have

$$\|f\|_{L^1(\mu_A)} \leq c \log \left( \frac{e}{\mu(A)} \right),$$

where $c$ is an absolute constant. In particular,

$$s^2(\mu_A) \leq c \log^2 \left( \frac{e}{\mu(A)} \right).$$
Clearly, Corollary 6.3 extends to all product measures $\mu = \nu^n$ on $\mathbb{R}^n$ such that $\nu$ satisfies a Poincaré-type inequality on the real line, and with constants $c$ depending on $\lambda_1$, only. A characterization of the property $\lambda_1 > 0$ may also be given in terms of the distribution function of $\nu$ and the density of its absolutely continuous component (cf. [B-G]).

4a) Let us take the metric probability space $(\{0,1\}^n, d_n, \mu)$, where $d_n$ is the Hamming distance, that is, $d_n(x,y) = \sharp\{i : x_i \neq y_i\}$, equipped with the uniform measure $\mu$. For this particular space, Marton established the transport-entropy inequality (3.2) with an optimal constant $\sigma^2 = \frac{n}{2}$, cf. [Mar]. Using the relation (3.2) as an equivalent definition of the subgaussian constant, we obtain from Theorem 1.1:

$$\sigma^2(\mu_A) \leq n \log \left( \frac{e}{\mu(A)} \right). \quad (5.1)$$

4b) Let us now assume that $A$ is monotone, i.e., $A$ satisfies the condition

$$(x_1, \ldots, x_n) \in A \quad \Rightarrow \quad (y_1, \ldots, y_n) \in A, \text{ whenever } y_i \geq x_i, \ i = 1, \ldots, n.$$ 

Recall that the discrete cube can be equipped with a natural graph structure: there is an edge between $x$ and $y$ whenever they are of Hamming distance $d_n(x,y) = 1$. For monotone sets $A$, the graph metric $d_A$ on the subgraph on $A$ is equal to the restriction of $d_n$ to $A \times A$. Indeed, we have:

$$d_n(x,y) \leq d_A(x,y) \leq d_A(x,x \wedge y) + d_A(y, x \wedge y) = d_n(x,x \wedge y) + d_n(y, x \wedge y) = d_n(x,y),$$

where $x \wedge y = (x_1 \wedge y_1, \ldots, x_n \wedge y_n)$. Thus,

$$s^2(\mu_A, d_A) \leq \sigma^2(\mu_A, d_A) \leq cn \log \left( \frac{e}{\mu(A)} \right).$$

This can be compared with what follows from a recent result of Ding and Mossel (see [D-M]). The authors proved that the conductance (Cheeger constant) of $(A, \mu_A)$ satisfies $\phi(A) \geq \frac{cn}{\lambda_1 \mu(A)}$. However, this type of isoperimetric results may not imply sharp concentration bounds. Indeed, by using Cheeger inequality, the above inequality leads to $\lambda_1 \geq c\mu(A)^2/n^2$ and $s^2(\mu_A, d_A) \leq 1/\lambda_1 \leq \sigma^2(\mu_A)^2/\mu(A)^2$, which is even worse than the trivial estimate $s^2(\mu_A, d_A) \leq \frac{1}{2} \text{diam}(A)^2 \leq n^2/2$.

5) Let $(M, d, \mu)$ be a (separable) metric probability space with finite subgaussian constant $\sigma^2(\mu)$. The previous example can be naturally generalized to the product space $(M^n, \mu^n)$, when it is equipped with the $\ell^1$-type metric

$$d_n(x,y) = \sum_{i=1}^n d(x_i, y_i), \quad x = (x_1, \ldots, x_n), \ y = (y_1, \ldots, y_n) \in M^n.$$ 

This can be done with the help of the following elementary observation.

**Proposition 5.5.** The subgaussian constant of the space $(M^n, d_n, \mu^n)$ is related to the subgaussian constant of $(M, d, \mu)$ by the equality $\sigma^2(\mu^n) = n \sigma^2(\mu)$.

Indeed, one may argue by induction on $n$. Let $f$ be a function on $M^n$. The Lipschitz property $\|f\|_{1,p} \leq 1$ with respect to $d_n$ is equivalent to the assertion that $f$ is coordinatewise
Lipschitz, that is, any function of the form \( x_i \to f(x) \) has a Lipschitz semi-norm \( \leq 1 \) on \( M \) for all fixed coordinates \( x_j \in M \) \( (j \neq i) \). Hence, in this case, for all \( t \in \mathbb{R} \),
\[
\int_M e^{tf(x)} \, d\mu(x_n) \leq \exp \left\{ t \int_M f(x) \, d\mu(x_n) + \frac{\sigma^2 t^2}{2} \right\},
\]
where \( \sigma^2 = \sigma^2(\mu) \). Here the function \( (x_1, \ldots, x_{n-1}) \to \int_M f(x) \, d\mu(x_n) \) is also coordinatewise Lipschitz. Integrating the above inequality with respect to \( d\mu^{n-1}(x_1, \ldots, x_{n-1}) \) and applying the induction hypothesis, we thus get
\[
\int_M e^{tf(x)} \, d\mu^n(x) \leq \exp \left\{ t \int_M f(x) \, d\mu^n(x) + n \frac{\sigma^2 t^2}{2} \right\}.
\]

But this means that \( \sigma^2(\mu^n) \leq n\sigma^2(\mu) \).

For an opposite bound, it is sufficient to test (1.1) for \((M^n, d_n, \mu^n)\) in the class of all coordinatewise Lipschitz functions of the form \( f(x) = u(x_1) + \cdots + u(x_n) \) with \( \mu \)-mean zero functions \( u \) on \( M \) such that \( ||u||_{\text{Lip}} \leq 1 \).

**Corollary 5.6.** For any Borel set \( A \subset M^n \) such that \( \mu^n(A) > 0 \), the subgaussian constant of the normalized restricted measure \( \mu^n_A \) with respect to the \( \ell^1 \)-type metric \( d_n \) satisfies
\[
\sigma^2(\mu^n_A) \leq cn\sigma^2(\mu) \log \left( \frac{e}{\mu^n(A)} \right),
\]
where \( c \) is an absolute constant.

For example, if \( \mu \) is a probability measure on \( M = \mathbb{R} \) such that \( \int_\mathbb{R} e^{x^2/\lambda^2} \, d\mu(x) \leq 2 \) \( (\lambda > 0) \), then for the restricted product measures we have
\[
\sigma^2(\mu^n_A) \leq c n \lambda^2 \log \left( \frac{e}{\mu^n(A)} \right) \tag{5.2}
\]
with respect to the \( \ell^1 \)-norm \( ||x||_1 = |x_1| + \cdots + |x_n| \) on \( \mathbb{R}^n \).

Indeed, by the integral hypothesis on \( \mu \), for any \( f \) on \( \mathbb{R} \) with \( ||f||_{\text{Lip}} \leq 1 \),
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(f(x) - f(y))^2/2\lambda^2} \, d\mu(x) \, d\mu(y) 
\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(x-y)^2/2\lambda^2} \, d\mu(x) \, d\mu(y) 
\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(x+y)^2/\lambda^2} \, d\mu(x) \, d\mu(y) \leq 4.
\]

Hence, if \( f \) has \( \mu \)-mean zero, by Jensen’s inequality,
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{f(x)^2/4\lambda^2} \, d\mu(x) \, d\mu(y) \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(f(x) - f(y))^2/4\lambda^2} \, d\mu(x) \, d\mu(y) \leq 2,
\]
meaning that \( ||f||_{L^2(\mu)} \leq 2\lambda \). By Lemma 3.1 cf. (5.1), it follows that \( \sigma^2(\mu) \leq 16\lambda^2 \), so, (5.2) holds true by an application of Corollary 5.6.

## 6 Deviations for non-Lipschitz functions

Let us now turn to the interesting question on the relationship between the distribution of a locally Lipschitz function and the distribution of its modulus of the gradient. We still keep the setting of a metric probability space \((M, d, \mu)\) and assume it has a finite subgaussian constant \( \sigma^2 = \sigma^2(\mu) \) \( (\sigma \geq 0) \).

Let us say that a continuous function \( f \) on \( M \) is locally Lipschitz, if \( |\nabla f(x)| \) is finite for all \( x \in M \). Recall that we consider the sets
\[
A = \{ x \in M : |\nabla f(x)| \leq L \}, \quad L > 0. \tag{6.1}
\]
First we state a more general version of Corollary 1.2.
**Theorem 6.1.** Assume that a locally Lipschitz function $f$ on $M$ has Lipschitz semi-norms $\leq L$ on the sets of the form $(6.1)$. If $\mu\{|\nabla f| \geq L_0\} \leq \frac{1}{2}$, then for all $t > 0$,

$$(\mu \otimes \mu)\{|f(x) - f(y)| \geq t\} \leq 2 \inf_{L \geq L_0} \left[ e^{-t^2/\sigma^2 L^2} + \mu\{|\nabla f| > L\} \right], \quad (6.2)$$

where $c$ is an absolute constant.

**Proof.** Although the argument is already mentioned in Section 1, let us replace (1.6) with a slightly different bound. Applying Theorem 1.1, the definition (1.1) yields

$$\int \int e^{t|f(x) - f(y)|} d\mu_A(x) d\mu_A(y) \leq e^{\sigma^2 L^2 t^2/2}, \quad \text{for all } t \in \mathbb{R},$$

where $A$ is defined in (6.1) with $L \geq L_0$, and where $c$ is universal constant. From this, for any $t > 0$,

$$(\mu_A \otimes \mu_A)\{(x, y) \in A \times A : |f(x) - f(y)| \geq t\} \leq 2e^{-t^2/(2\sigma^2 L^2)},$$

and therefore

$$(\mu \otimes \mu)\{(x, y) \in A \times A : |f(x) - f(y)| \geq t\} \leq 2e^{-t^2/(2\sigma^2 L^2)}.$$

The product measure of the complement of $A \times A$ does not exceed $2\mu\{|\nabla f| > L\}$, and we obtain (6.2). \hfill \Box

If $\int e^{\nabla f^2} d\mu \leq 2$, we have, by Chebyshev’s inequality, $\mu\{|\nabla f| \geq L\} \leq 2e^{-L^2}$, so one may take $L_0 = \sqrt{\log 4}$. Theorem 6.1 then gives that, for any $L^2 \geq \log 4$,

$$(\mu \otimes \mu)\{|f(x) - f(y)| \geq t\} \leq 2e^{-t^2/\sigma^2 L^2} + 4e^{-L^2}.$$
For illustration, let \( \mu = \mu_1 \otimes \cdots \otimes \mu_n \) be an arbitrary product probability measure on the cube \([-1, 1]^n\). If \( f \) is convex and Lipschitz on \( \mathbb{R}^n \), thus with \( \| \nabla f \| \leq 1 \), then
\[
(\mu \otimes \mu) \{ |f(x) - f(y)| \geq t \} \leq 2e^{-t^2/c}. \tag{6.3}
\]
This is one of the forms of Talagrand’s concentration phenomenon for the family of convex sets/functions (cf. [T1, T2, M, L]). That is, the subgaussian constants \( \frac{1}{\sqrt{2}} \sigma_f(\mu) \) are bounded for the class \( \mathcal{F} \) of convex Lipschitz \( f \) and product measures \( \mu \) on the cube. Hence, using Theorem 6.3, Talagrand’s deviation inequality (6.3) admits a natural extension to the class of non-Lipschitz convex functions:

**Corollary 6.4.** Let \( \mu \) be a product probability measure on the cube, and let \( f \) be a convex function on \( \mathbb{R}^n \). If \( \mu \{ |\nabla f| \geq L_0 \} \leq \frac{1}{2} \), then for all \( t > 0 \),
\[
(\mu \otimes \mu) \{ |f(x) - f(y)| \geq t \} \leq 2 \inf_{L \geq L_0} \left[ e^{-t^2/cL^2} + \mu \{ |\nabla f| > L \} \right],
\]
where \( c \) is an absolute constant.

In particular, we have a statement similar to Corollary 1.2 for this family of functions, namely
\[
\| f - m \|_{L^{\psi_1}(\mu)} \leq c \| \nabla f \|_{L^{\psi_2}(\mu)},
\]
where \( m \) is the \( \mu \)-mean of \( f \).

**Proof of Lemma 6.3** An affine function \( l_{a,v}(x) = a + \langle x, v \rangle \) \( (v \in \mathbb{R}^n, a \in \mathbb{R}) \) may be called to be a tangent function to \( f \), if \( f \geq l \) on \( \mathbb{R}^n \) and \( f(x) = l_{a,v}(x) \) for at least one point \( x \). It is well-known that
\[
f(x) = \sup \{ l_{a,v}(x): l_{a,v} \in \mathcal{L} \},
\]
where \( \mathcal{L} \) denotes the collection of all tangent functions \( l_{a,v} \). Put,
\[
g(x) = \sup \{ l_{a,v}(x): l_{a,v} \in \mathcal{L}, |v| \leq L \}.
\]
By the construction, \( g \leq f \) on \( \mathbb{R}^n \) and, moreover,
\[
\| g \|_{\text{Lip}} \leq \sup \{ \| l_{a,v} \|_{\text{Lip}}: l_{a,v} \in \mathcal{L}, |v| \leq L \} = \sup \{ |v|: l_{a,v} \in \mathcal{L}, |v| \leq L \} \leq L.
\]
It remains to show that \( g = f \) on the set \( A = \{ |\nabla f| \leq L \} \). Let \( x \in A \) and let \( l_{a,v} \) be tangent to \( f \) and such that \( l_{a,v}(x) = f(x) \). This implies that \( f(y) - f(x) \geq \langle y - x, v \rangle \) for all \( y \in \mathbb{R}^n \) and hence
\[
|\nabla f(x)| = \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|} \geq \limsup_{y \to x} \frac{\langle y - x, v \rangle}{|y - x|} = v.
\]
Thus, \( |v| \leq L \), so that \( g(x) \geq l_{a,v}(x) = f(x) \).

\[ \Box \]

### 7 Optimality

Here we show that the logarithmic dependence in \( \mu(A) \) in Theorems 1.1 and 1.3 is optimal, up to the universal constant \( c \). We provide several examples.

**Example 1.** Let us return to Example 4. Section 5 of the hypercube \( M = \{0, 1\}^n \), which we equip with the Hamming distance \( d_h \) and the uniform measure \( \mu \). Let us test the inequality (5.1) of Corollary 5.4 on the set \( A \subset \{-1, 1\}^n \) consisting of \( n + 1 \) points
\[
(0, 0, 0, \ldots, 0), \ (1, 0, 0, \ldots, 0), \ (1, 1, 0, \ldots, 0), \ \ldots, \ (1, 1, 1, \ldots, 1).
\]
We have $\mu(A) = (n + 1)/2^n \geq 1/2^n$. The function $f : A \to \mathbb{R}$, defined by

$$f(x) = \sharp \{i : x_i = 1\} - \frac{n}{2},$$

has a Lipschitz semi-norm $\|f\|_{\text{Lip}} \leq 1$ with respect to $d$ and the $\mu_A$-mean zero. Moreover, $\int f^2 \, d\mu_A = \frac{n(n+1)}{12}$. Expanding the inequality $\int e^{tf} \, d\mu_A \leq e^{\sigma^2(\mu_A) t^2/2}$ at the origin yields $\int f^2 \, d\mu_A \leq \sigma^2(\mu_A)$. Hence, recalling that $\sigma^2(\mu) \leq \frac{n}{4}$, we get

$$\sigma^2(\mu_A) \geq \int f^2 \, d\mu_A \geq \frac{n^2}{12} \geq \frac{n}{3} \sigma^2(\mu) \geq \frac{1}{3} \log 2 \sigma^2(\mu) \log \left(\frac{1}{\mu(A)}\right).$$

This example shows the optimality of (5.1) in the regime $\mu(A) \to 0$.

**Example 2.** Let $\gamma_n$ be the standard Gaussian measure on $\mathbb{R}^n$ of dimension $n \geq 2$. We have $\sigma^2(\gamma_n) = 1$. Consider the normalized measure $\gamma_{A_R}$ on the set

$$A_R = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 \geq R^2\}, \quad R \geq 0.$$

Using the property that the function $\frac{1}{2} (x_1^2 + x_2^2)$ has a standard exponential distribution under the measure $\gamma_n$, we find that $\gamma_n(A_R) = e^{-R^2/2}$. Moreover,

$$s^2(\gamma_{A_R}) \geq \text{Var}_{\gamma_{A_R}}(x_1) = \int x_1^2 \, d\gamma_{A_R}(x) = \frac{1}{2} \int (x_1^2 + x_2^2) \, d\gamma_{A_R}(x) = \frac{1}{e^{-R^2/2}} \int_{R^2/2}^{\infty} re^{-r} \, dr = \frac{R^2}{2} + 1 = \log \left(\frac{e}{\gamma_n(A_R)}\right).$$

Therefore,

$$\sigma^2(\gamma_{A_R}) \geq s^2(\gamma_{A_R}) \geq \log \left(\frac{e}{\gamma_n(A_R)}\right),$$

showing that the inequality (1.3) of Theorem 1.1 is optimal, up to the universal constant, for any value of $\gamma_n(A) \in [0, 1]$.

**Example 3.** A similar conclusion can be made about the uniform probability measure $\mu$ on the Euclidean ball $B(0, \sqrt{n})$ of radius $\sqrt{n}$, centred at the origin (asymptotically for growing dimension $n$). To see this, it is sufficient to consider the cylinders

$$A_{\varepsilon} = \{(x_1, y) \in \mathbb{R} \times \mathbb{R}^{n-1} : |x_1| \leq \sqrt{n-\varepsilon^2} \text{ and } |y| \leq \varepsilon\}, \quad 0 < \varepsilon \leq \sqrt{n},$$

and the function $f(x) = x_1$. We leave to the readers corresponding computations.

**Example 4.** Let $\mu$ be the two-sided exponential measure on $\mathbb{R}$ with density $\frac{1}{2} e^{-|x|}$. In this case $\sigma^2(\mu) = \infty$, but, as easy to see, $2 \leq s^2(\mu) \leq 4$ (recall that $\lambda_1(\mu) = \frac{1}{2}$). We are going to test optimality of the inequality (1.7) on the sets $A_R = \{x \in \mathbb{R} : |x| \geq R\} \ (R \geq 0)$. Clearly, $\mu(A_R) = e^{-R}$, and we find that

$$s^2(\mu_{A_R}) \geq \text{Var}_{\mu_{A_R}}(x) = \int_{-\infty}^{\infty} x^2 \, d\mu_{A_R}(x) = \frac{1}{e^{-R}} \int_R^{\infty} r^2 e^{-r} \, dr = R^2 + 2R + 2 \geq (R + 1)^2 = \log^2 \left(\frac{e}{\mu(A_R)}\right).$$

Therefore,

$$s^2(\mu_{A_R}) \geq \log^2 \left(\frac{e}{\mu(A_R)}\right),$$

showing that the inequality (1.7) is optimal, up to the universal constant, for any value of $\mu(A) \in (0, 1]$. 


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Appendix

Proof of Lemma 2.1. Using the homogeneity, in order to derive the right-hand side inequality in (2.1), we may assume that $\sup_{p \geq 1} \frac{\|f\|_p}{\sqrt{p}} \leq 1$. Then $\int |f|^p \, d\mu \leq p^{p/2}$ for all $p \geq 1$, and by
Chebyshev’s inequality,

$$1 - F(t) \equiv \mu\{|f| \geq t\} \leq \left(\frac{\sqrt[p]{2}}{t}\right)^p, \text{ for all } t > 0.$$  

If $t \geq 2$, choose here $p = \frac{1}{4} t^2$, in which case $1 - F(t) \leq 2^{-\frac{1}{4}} t^2$. Integrating by parts, we have, for any $0 < \varepsilon < \frac{\log 2}{4}$,

$$\int e^{\varepsilon f^2} d\mu = -\int_0^\infty e^{\varepsilon t^2} d(1 - F(t))$$

$$= 1 + 2\varepsilon \int_0^2 t e^{\varepsilon t^2} (1 - F(t)) dt + 2\varepsilon \int_2^\infty t e^{\varepsilon t^2} (1 - F(t)) dt$$

$$\leq 1 + 2\varepsilon \int_0^2 t e^{\varepsilon t^2} dt + 2\varepsilon \int_2^\infty t e^{-\frac{\log 2}{4} t^2} dt$$

$$= e^{4\varepsilon} + \frac{\varepsilon}{4} e^{-(\log 2 - 4\varepsilon)} = e^{4\varepsilon} \left(1 + \frac{\varepsilon}{2(\log 2 - \varepsilon)}\right).$$

If $\varepsilon \leq \frac{\log 2}{8}$, the latter expression does not exceed $\frac{3}{2} e^{4\varepsilon}$ which does not exceed 2 for $\varepsilon \leq \frac{\log 2}{4}$. Both inequalities are fulfilled for $\varepsilon = \frac{\log 2}{10}$, and with this value $\int e^{\varepsilon f^2} d\mu \leq 2$. Hence

$$\|f\|_{L^2(\mu)} \leq \frac{1}{\sqrt{\varepsilon}} = \frac{10}{\sqrt{2}} < 4,$$  

which yields the right inequality in (2.1). Conversely, if $\|f\|_{L^2(\mu)} = 1$, then $\int e^{\varepsilon f^2} d\mu = 2$. Since $u(t) = t^p e^{-t^2}$ is maximized in $t > 0$ at $t_0 = \sqrt{\frac{2}{p}}$, we get

$$\|f\|_p = \int u(f) e^{\varepsilon f^2} d\mu \leq u(t_0) \cdot 2 = 2 \left(\frac{\sqrt[p]{2}}{\sqrt{2} e}\right)^p.$$

Hence, $\frac{\|f\|_p}{\sqrt[p]{2} e} \leq 2^{1/p} < 1$, which yields the left inequality.

Now, let us turn to (2.2) and assume that $\sup_{p \geq 1} \frac{\|f\|_p}{\sqrt[p]{2} e} = 1$. Then $\int |f|^p d\mu \leq p^p$ for all $p \geq 1$, and by Chebyshev’s inequality, for all $t > 0$,

$$1 - F(t) \equiv \mu\{|f| \geq t\} \leq \left(\frac{p}{t}\right)^p.$$  

If $t \geq 2$, we may choose here $p = \frac{1}{4} t^2$ in which case $1 - F(t) \leq 2^{-\frac{1}{4}} t^2$, while for $1 < t < 2$ we choose $p = 1$, so that $1 - F(t) \leq \frac{1}{t}$. Arguing as before, we have, for any $0 < \varepsilon < \frac{\log 2}{2}$,

$$\int e^{\varepsilon |f|} d\mu = 1 + \varepsilon \int_0^1 e^{\varepsilon t} (1 - F(t)) dt + \varepsilon \int_1^2 e^{\varepsilon t} (1 - F(t)) dt + \varepsilon \int_2^\infty e^{\varepsilon t} (1 - F(t)) dt$$

$$\leq 1 + \varepsilon \int_0^1 e^{\varepsilon t} dt + \varepsilon \int_1^2 \frac{e^{\varepsilon t}}{t} dt + \varepsilon \int_2^\infty e^{\varepsilon t} e^{-\frac{\log 2}{t} t} dt.$$  

The pre-last integral cannot be bounded by $\int_1^2 \frac{e^{\varepsilon t}}{t} dt = e^{2\varepsilon} \log 2$, so

$$\int e^{\varepsilon |f|} d\mu \leq e^\varepsilon + \varepsilon e^{2\varepsilon} \log 2 + \frac{\varepsilon}{\log 2 - \varepsilon} e^{-2(\log 2 - \varepsilon)}.$$  

For $\varepsilon = \frac{1}{6}$, the latter expression is equal to 1.98903902..., and thus $\int e^{\varepsilon |f|} d\mu < 2$. Hence

$$\|f\|_{L^1(\mu)} \leq \frac{1}{\varepsilon} = 6.$$
Conversely, if \(\|f\|_{L^p(\mu)} = 1\), then \(\int e^{\mu f} d\mu = 2\). Since \(u(t) = t^p e^{-t}\) is maximized at \(t_0 = p\), we get
\[
\|f\|_p^p = \int u(f) e^{\mu f} d\mu \leq u(t_0) \cdot 2 = 2 \left(\frac{p}{e}\right)^p.
\]
Hence, \(\|f\|_p \leq \frac{2^{1/p}}{e} < 1\), which yields the left inequality.

**Proof of Lemma 3.1** First assume that \(\|f\|_{\psi_2} = 1\), i.e., \(\int e^{\psi^2} d\mu = 2\). The function
\[
u(t) = \log \int e^{t f} d\mu
\]is smooth, convex, with \(\nu(0) = 0\) and
\[
\nu'(t) = \frac{\int \mu e^{t f} d\mu}{\int e^{t f} d\mu}.
\]
In particular, \(\nu'(0) = 0\). Note that, by Jensen’s inequality, \(\int e^{t f} d\mu \geq 1\), so \(\nu(t) \geq 0\). Further differentiation gives
\[
\nu''(t) = \frac{\int f^2 e^{t f} d\mu - (\int f e^{t f} d\mu)^2}{(\int e^{t f} d\mu)^2} \leq \int f^2 e^{t f} d\mu.
\]
Using \(tf \leq \frac{t^2 + f^2}{2}\) and the elementary inequality \(xe^{-x/2} \leq 2e^{-1}\), we get, for \(|t| \leq 1\),
\[
\int f^2 e^{t f} d\mu \leq \int f^2 e^{\frac{t^2 + f^2}{2}} d\mu = e^{t^2/2} \int f^2 e^{f^2/2} d\mu \leq e^{t^2/2} 2e^{-1} \int e^{f^2} d\mu \leq 4.
\]
Thus, \(\nu''(t) \leq 4\), and by Taylor’s formula, \(\nu(t) \leq 2t^2\).
On the hand, for \(|t| \geq 1\), by Cauchy’s inequality,
\[
\int e^{t f} d\mu \leq \int e^{\frac{t^2 + f^2}{2}} d\mu = e^{t^2/2} \int e^{f^2/2} d\mu \leq e^{t^2/2} \left(\int e^{f^2} d\mu\right)^{1/2} = \sqrt{2} e^{t^2/2} \leq e^{(1+\log 2)t^2/2}.
\]
Hence, in this case \(\nu(t) \leq \frac{1+\log 2}{2} t^2 < t^2\). Thus,
\[
\sigma_1^2 = \sup_{t \neq 0} \frac{u(t)}{t^2/2} \leq 4,
\]
proving the right inequality of Lemma 3.1.

For the left inequality, let \(\sigma_1^2 = 1\). Then \(\int e^{t f} d\mu \leq e^{t^2/2}\) for all \(t \in \mathbb{R}\), which implies
\[
1 - F(t) \equiv \mu\{|f| \geq t\} \leq 2e^{-t^2/2}, \quad t \geq 0.
\]
Form this, integrating by parts, we have, for any \(0 < \varepsilon < \frac{1}{2}\),
\[
\int e^f d\mu = \int_0^\infty e^{ft} dF(t) = -\int_0^\infty e^{ft} d(1-F(t)) = 1 + 2\varepsilon \int_0^\infty te^{t^2} (1-F(t)) \, dt \leq 1 + 4\varepsilon \int_0^\infty te^{t^2} e^{-t^2/2} \, dt = 1 + \frac{2\varepsilon}{\frac{1}{2} - \varepsilon}.
\]
The last expression is equal to 2 for \(\varepsilon = \frac{1}{3}\), which means that \(\|f\|_{\psi_2} \leq \sqrt{6}\). \(\square\)