Overall observational constraints on the running parameter $\lambda$ of Hořava-Lifshitz gravity

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We use observational data from Type Ia Supernovae (SNIa), Baryon Acoustic Oscillations (BAO), and Cosmic Microwave Background (CMB), along with requirements of Big Bang Nucleosynthesis (BBN), to constrain the running parameter $\lambda$ of Hořava-Lifshitz gravity, which determines the flow between the Ultra-Violet and the Infra-Red. We consider both the detailed and non-detailed balance versions of the gravitational sector, and we include the matter and radiation sectors. Allowing for variation of all the parameters of the theory, we construct the likelihood contours and we conclude that in $1\sigma$ confidence $\lambda$ is restricted to $|\lambda - 1| \lesssim 0.02$, while its best fit value is $|\lambda_0 - 1| \approx 0.002$. Although this observational analysis restricts the running parameter $\lambda$ very close to its IR value 1, it does not enlighten the discussion about the theory’s possible conceptual and theoretical problems.

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I. INTRODUCTION

Hořava recently proposed a power-counting renormalizable, Ultra-Violet (UV) complete theory of gravity [1–4]. Although presenting an Infra-Red (IR) fixed point, namely General Relativity, in the UV the theory possesses a fixed point with an anisotropic, Lifshitz scaling between time and space. Since then there has been a significant progress in examining the properties of the theory itself, including various extensions of the original basic version [5–30]. Additionally, application of Hořava-Lifshitz gravity as a cosmological framework gives rise to Hořava-Lifshitz cosmology [31–32], and in particular one can study specific solution subclasses [33–47], the phase-space behavior [48–52], the gravitational wave production [53–58], the perturbation spectrum [59–68], the matter bounce [69–74], the black hole properties [75–80], the dark energy phenomenology [81–82], the astrophysical phenomenology [83–101], the thermodynamic properties [102–103] etc. However, despite this extended research, there are still many ambiguities if Hořava-Lifshitz gravity is reliable and capable of a successful description of the gravitational background of our world, as well as of the cosmological behavior of the universe [11–13, 26, 104–108].

Although the discussion about the foundations and the possible conceptual and phenomenological problems of Hořava-Lifshitz gravity and cosmology is still open in the literature, it is worthy to examine the possible constraints that observational and cosmological data could impose on the parameters of the scenario (of the basic as well as of the various extended versions). In such an investigation a reasonable assumption is to set the running parameter $\lambda$ to its IR value 1, since all observations lie deep inside the IR. Thus, under this assumption in [99–98] the authors used Solar System observations in order to constrain some of the remaining model parameters of the basic version of Hořava-Lifshitz cosmology under detailed balance, while in [100] the analysis was performed including a soft detailed-balance breaking. Similarly, in [109] we used cosmological observations (Type Ia Supernovae (SNIa), Baryon Acoustic Oscillations (BAO) and Cosmic Microwave Background (CMB) ones, together with Big Bang Nucleosynthesis conditions) in order to impose complete constraints on all the parameters of the basic version of Hořava-Lifshitz cosmology and construct the corresponding contour plots, with or without the detailed-balance condition.

Although setting $\lambda$ to its IR value is a first and reasonable assumption, one could go beyond it, and examine the observational constraints that the data could impose on $\lambda$ itself. However, allowing $\lambda$ varying, that is preserving the Lorentz invariance breaking (which is restored in the exact IR value $\lambda = 1$), one must take into account that in theories with Lorentz invariance breaking the “gravitational” Newton’s constant $G_{\text{grav}}$, that is the one that is present in the gravitational action, does not coincide with the “cosmological” Newton’s constant $G_{\text{cosmo}}$, that is the one that is present in Friedmann equations [110]. Thus, in the case of Hořava-Lifshitz gravity one could use the deviation between $G_{\text{grav}}$ and $G_{\text{cosmo}}$, as it is constrained by measurements of the primordial abundance of He$^4$ [110], in order to extract an upper bound on $|\lambda - 1|$. This approach was followed in [20, 106] with the result $0 < |\lambda - 1| \lesssim 0.1$. However, it is obvious that such an approach can only provide a crude upper bound on $|\lambda - 1|$, since it considers that all the other parameters of the theory remain constant. The correct approach should be to perform a systematic investigation, allowing for simultaneous variations of all model parameters, and constrain all of them using observations.

In the present work we are interested in performing such an holistic observational constraining of all the pa-
rameters of the basic version of Hořava-Lifshitz cosmology, and especially of the running parameter $\lambda$, using SNIa, BAO and CMB cosmological observations in order to construct the corresponding probability contour-plots. Furthermore, in order to be general and model-independent, we perform our analysis with and without the detailed-balance condition. The plan of the work is the following: in section II we present the basic ingredients of Hořava-Lifshitz cosmology, extracting the Friedmann equations, and describing the dark matter and dark energy dynamics. In section III we constrain both the detailed-balance and the beyond-detailed-balance formulations using cosmological observations, and we present the corresponding likelihood contours. Finally, in section IV we summarize the obtained results.

II. HOŘAVA-LIFSHITZ COSMOLOGY

In this section we briefly review the scenario where the cosmological evolution is governed by Hořava-Lifshitz gravity \[31,32\]. The dynamical variables are the lapse and shift functions, $N$ and $N_i$, respectively, and the spatial metric $g_{ij}$ (roman letters indicate spatial indices). In terms of these fields the full metric is written as:

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt),$$

where indices are raised and lowered using $g^{ij}$. The scaling transformation of the coordinates reads: $t \to \tilde{t}^\lambda$ and $x^i \to \tilde{x}^i$.

A. Detailed Balance

The gravitational action is decomposed into a kinetic and a potential part as $S_g = \int dt d^3x \sqrt{|g|} N (L_K + L_V)$. The assumption of detailed balance \[3\] reduces the possible terms in the Lagrangian, and it allows for a quantum inheritance principle \[1\], since the $(D+1)$-dimensional theory acquires the renormalization properties of the $D$-dimensional one. Under the detailed balance condition the full action of Hořava-Lifshitz gravity is given by

$$S_g = \int dt d^3x \sqrt{|g|} N \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \frac{\kappa^2}{2w} C_{ij} C^{ij} + \frac{\kappa^2 \mu}{2w^2 \sqrt{g}} R_{ij} \nabla_j R^i + \frac{\kappa^2 \mu^2}{8} R_{ij} R^{ij} + \frac{\kappa^2 \mu^2}{8(3\Lambda - 1)} \left[ 1 - \frac{4\Lambda}{4} R^2 + \lambda R - 3\Lambda^2 \right] \right\},$$

where

$$K_{ij} = \frac{1}{2N} (g_{ij} - \nabla_i N_j - \nabla_j N_i)$$

is the extrinsic curvature and

$$C^{ij} = \frac{\epsilon^{ijk}}{\sqrt{g}} \nabla_k (R^j_i - \frac{1}{4} R \delta^j_i)$$

the Cotton tensor, and the covariant derivatives are defined with respect to the spatial metric $g_{ij}$. $\epsilon^{ijk}$ is the totally antisymmetric unit tensor, $\lambda$ is a dimensionless constant and the variables $\kappa$, $w$ and $\mu$ are constants with mass dimensions $-1$, $0$ and $1$, respectively. Finally, we mention that in action \[2\] we have already performed the usual analytic continuation of the parameters $\mu$ and $w$ of the original version of Hořava-Lifshitz gravity, since such a procedure is required in order to obtain a realistic cosmology \[33,37,81,101\] (although it could fatally affect the gravitational theory itself). Therefore, in the present work $\Lambda$ is a positive constant, which as usual is related to the cosmological constant in the IR limit.

In order to add the matter component (including both dark and baryonic matter) in the theory one can follow two equivalent approaches. The first is to introduce a scalar field \[51,32\] and thus attribute to dark matter a dynamical behavior, with its energy density $\rho_m$ and pressure $p_m$ defined through the field kinetic and potential energy. Although such an approach is theoretically robust, it is not suitable from the phenomenological point of view since it requires specially-designed matter-potentials in order to acquire an almost constant matter equation-of-state parameter $(w_m = p_m/\rho_m)$ as it is suggested by observations. In the second approach one adds a cosmological stress-energy tensor to the gravitational field equations, by demanding to recover the usual general relativity formulation in the low-energy limit \[13,40,44\]. Thus, this matter-tensor is a hydrodynamical approximation with $\rho_m$ and $p_m$ (or $\rho_r$ and $w_r$). Similarly, one can additionally include the standard-model-radiation component (corresponding to photons and neutrinos), with the additional parameters $\rho_r$ and $p_r$ (or $\rho_r$ and $w_r$). Such an approach, although not fundamental, is better for a phenomenological analysis, such the one performed in this work.

In order to investigate cosmological frameworks, we impose the projectability condition \[11\] and we use an FRW metric

$$N = 1, \quad g_{ij} = a^2(t) \gamma_{ij}, \quad N^i = 0,$$

with

$$\gamma_{ij} dx^i dx^j = \frac{dr^2}{1 - K r^2} + r^2 d\Omega^2,$$

where $K <,=,>$ 0 corresponding to open, flat, and closed universe respectively (we have adopted the convention of taking the scale factor $a(t)$ to be dimensionless and the curvature constant $K$ to have mass dimension 2).

By varying $N$ and $g_{ij}$, we extract the Friedmann equations:

$$H^2 = \frac{\kappa^2}{6(3\Lambda - 1)} \left( \rho_m + \rho_r + \frac{3\kappa^2 \mu^2 K^2}{8(3\Lambda - 1) a^2} + \frac{3\kappa^2 \mu^2 \Lambda^2}{8(3\Lambda - 1)} \right) - \frac{\kappa^4 \mu^2 \Lambda K}{8(3\Lambda - 1)^2 a^2},$$
\[ \dot{H} + \frac{3}{2} H^2 = -\frac{\kappa^2}{4(3\lambda - 1)} \left( w_m \rho_m + w_r \rho_r \right) - \kappa^2 \left[ \frac{\kappa^2 \mu^2 K^2}{8(3\lambda - 1)a^4} - \frac{3\kappa^2 \mu^2 \Lambda^2}{8(3\lambda - 1)} \right] - \frac{\kappa^4 \mu^2 K}{16(3\lambda - 1)^2a^2}, \]  

where \( H \equiv \frac{\dot{a}}{a} \) is the Hubble parameter. As usual, \( \rho_m \) (dark plus baryonic matter) follows the standard evolution equation

\[ \dot{\rho}_m + 3H(\rho_m + p_m) = 0, \]  

while \( \rho_r \) (standard-model radiation) follows

\[ \dot{\rho}_r + 3H(\rho_r + p_r) = 0. \]  

Lastly, concerning the dark-energy sector we can define

\[ \rho_{DE} = \frac{3\kappa^2 \mu^2 K^2}{8(3\lambda - 1)a^4} + \frac{3\kappa^2 \mu^2 \Lambda^2}{8(3\lambda - 1)} \]  

\[ p_{DE} = \frac{\kappa^2 \mu^2 K^2}{8(3\lambda - 1)a^4} - \frac{3\kappa^2 \mu^2 \Lambda^2}{8(3\lambda - 1)} \]  

The term proportional to \( a^{-4} \) is the usual “dark radiation term”, present in Hořava-Lifshitz cosmology \([31, 32]\), while the constant term is just the explicit cosmological constant. Therefore, in expressions \([11, 12]\) we have defined the energy density and pressure for the effective dark energy, which incorporates the aforementioned contributions. Finally, note that using \([11, 12]\) it is straightforward to show that these dark energy quantities satisfy the standard evolution equation:

\[ \dot{\rho}_{DE} + 3H(\rho_{DE} + p_{DE}) = 0. \]  

The above formulation of Hořava-Lifshitz cosmology has been performed under the imposition of the detailed-balance condition. However, in the literature there is a discussion whether this condition leads to reliable results or if it is able to reveal the full information of Hořava-Lifshitz gravity \([31, 32]\). Thus, one should study also the Friedmann equations in the case where detailed balance is relaxed. In such a case one can in general write \([11, 13, 15, 48, 49]\):

\[ H^2 = \frac{2\sigma_0}{(3\lambda - 1)} \left( \rho_m + \rho_r \right) + \frac{2}{(3\lambda - 1)} \left[ \frac{\sigma_1}{6} + \frac{\sigma_3 K^2}{6a^6} + \frac{\sigma_4 K}{6a^6} \right] + \frac{\sigma_2 K}{3(3\lambda - 1)a^2} \]  

\[ \dot{H} + \frac{3}{2} H^2 = -\frac{3\sigma_0}{(3\lambda - 1)} \left( w_m \rho_m + w_r \rho_r \right) - \frac{3}{(3\lambda - 1)} \left[ \frac{\sigma_1}{6} + \frac{\sigma_3 K^2}{18a^4} + \frac{\sigma_4 K}{6a^6} \right] + \frac{\sigma_2 K}{6(3\lambda - 1)a^2} \]  

where \( \sigma_0 \equiv \kappa^2/12 \), and the constants \( \sigma_i \) are arbitrary (with \( \sigma_2 \) being negative). Note that one could absorb the factor of 6 in redefined parameters, but we prefer to keep it in order to coincide with the notation of \([13, 48]\). As we observe, the effect of the detailed-balance relaxation is the decoupling of the coefficients, together with the appearance of a term proportional to \( a^{-6} \). In this case the corresponding quantities for dark energy are generalized to

\[ \rho_{DE|_{non-db}} = \frac{\sigma_1}{6} + \frac{\sigma_3 K^2}{6a^6} + \frac{\sigma_4 K}{6a^6} \]  

\[ p_{DE|_{non-db}} = -\frac{\sigma_1}{6} + \frac{\sigma_3 K^2}{18a^4} + \frac{\sigma_4 K}{6a^6}. \]  

Again, it is easy to show that

\[ \dot{\rho}_{DE|_{non-db}} + 3H(\rho_{DE|_{non-db}} + p_{DE|_{non-db}}) = 0. \]
Finally, if we force \([12], [19]\) to coincide with the standard Friedmann equations, we result to:

\[
G_{\text{cosmo}} = \frac{6\sigma_0}{8\pi(3\lambda - 1)}
\]

\[
\sigma_2 = -3(3\lambda - 1),
\]

while in this case the “gravitational” Newton’s constant \(G_{\text{grav}}\) reads \([13]\):

\[
G_{\text{grav}} = \frac{6\sigma_0}{16\pi}.
\]

### III. OBSERVATIONAL CONSTRAINTS

Having presented the cosmological equations of a universe governed by Hořava-Lifshitz gravity, both with and without the detailed-balance condition, we now proceed to study the observational constraints on the model parameters. This is performed in the following two subsections, for the detailed and non-detailed balance scenarios separately. We mention that, contrary to \([109]\), in this work we allow the running parameter \(\lambda\) to vary too, and in order to be general enough we do not use any theoretical argument to restrict it in any specific interval, handling it as completely free.

#### A. Constraints on Detailed-Balance scenario

We work in the usual units suitable for observational comparisons, namely setting \(8\pi G_{\text{grav}} = 1\) (we have already set \(c = 1\) in order to obtain \([10]\)). This allows us to reduce the parameter space, since in this case \([17]\) gives

\[
\kappa^2 = 4,
\]

and thus \([16]\) lead to:

\[
G_{\text{cosmo}} = \frac{1}{4\pi(3\lambda - 1)}
\]

\[
\mu^2\Lambda = \frac{(3\lambda - 1)^2}{2}.
\]

Inserting these relations into Friedmann equation \([7]\) we obtain

\[
H^2 = \frac{2}{3(3\lambda - 1)} \left( \rho_m + \rho_r \right) + \frac{1}{3} \left( \frac{3K^2}{2\Lambda a^2} + \frac{3\Lambda}{2} \right) - \frac{K}{a^2}.
\]

In terms of the usual density parameters \((\Omega_m \equiv \rho_m/(3H^2), \Omega_K \equiv -K/(H^2a^2), \Omega_r \equiv \rho_r/(3H^2))\) this expression becomes:

\[
1 - \frac{2}{(3\lambda - 1)} (\Omega_m + \Omega_r) - \Omega_K = \frac{1}{H^2} \left( \frac{K^2}{2\Lambda a^4} + \frac{\Lambda}{2} \right).
\]

Applying this relation at present time and setting the current scale factor \(a_0 = 1\) we obtain:

\[
1 - \frac{2}{(3\lambda - 1)} (\Omega_m + \Omega_r) - \Omega_K = \frac{1}{H_0^2} \left( \frac{K^2}{2\Lambda} + \frac{\Lambda}{2} \right),
\]

where a 0-subscript denotes the present value of the corresponding quantity. Note that \(\Omega_m\) includes contributions from both baryons \(\Omega_{b0}\) as well as dark matter \(\Omega_{\text{DM0}}\).

In order to proceed to the elaboration of observational data, we consider as usual the matter (dark plus baryonic) component to be dust, that is \(w_m = 0\), and similarly for the standard-model radiation we consider \(w_r = 1/3\), where both assumptions are valid in the epochs in which observations focus. Therefore, the corresponding evolution equations \((9), (10)\) give \(\rho_m = \rho_{m0}/a^3\) and \(\rho_r = \rho_{r0}/a^4\) respectively. Finally, it proves convenient to use the redshift \(z\) as the independent variable instead of the scale factor \((1 + z = a_0/a = 1/a)\). Inserting these into Friedmann equation \(27\) we obtain

\[
H^2 = H_0^2 \left\{ \frac{2}{(3\lambda - 1)} \left[ \Omega_m(1 + z)^3 + \Omega_r(1 + z)^4 \right] + \Omega_K(1 + z)^2 + \left[ \omega + \frac{\Omega_{K0}}{4\omega}(1 + z)^4 \right] \right\},
\]

where we have also introduced the dimensionless parameter

\[
\omega = \frac{\Lambda}{2H_0^2}.
\]

Thus, the constraint \(29\) can be rewritten as:

\[
\frac{2}{(3\lambda - 1)} (\Omega_m + \Omega_r) + \Omega_K + \omega + \frac{\Omega_{K0}}{4\omega} = 1.
\]

We remind that the term \(\Omega_{K0}^2/(4\omega)\) is the coefficient of the dark radiation term, which is a characteristic feature of the Hořava-Lifshitz gravitational background. Since this dark radiation component has been present also during the time of nucleosynthesis, it is subject to bounds from Big Bang Nucleosynthesis (BBN). As discussed in more details in the Appendix of \([109]\), if the upper limit on the total amount of dark radiation allowed during BBN is expressed through the parameter \(\Delta N_\nu\) of the effective neutrino species \([111]-[114]\), then we obtain the following constraint :

\[
\frac{\Omega_{K0}^2}{4\omega} = 0.135\Delta N_\nu\Omega_{r0}.
\]

In this work, in order to ensure consistency with BBN, we adopt an upper limit of \(\Delta N_\nu \leq 2.0\) following \([113]\).

In most studies of dark energy models it is customary to ignore curvature (e.g. \([116]-[123]\)), especially concerning observational constraints. This practice is well motivated...
since most inflationary scenarios predict a high degree of spatial flatness, and furthermore the CMB data impose stringent constraints on spatial flatness in the context of constant-w models (for example a combination of WMAP+BAO+SN Ia data [115] provides the tight simultaneous constraints $-0.0179 \leq \Omega_{K0} \leq 0.0081$ and $-0.12 \leq 1 + w \leq 0.14$, both at 95% confidence).

However, it is important to keep in mind that due to degeneracies in the CMB power spectrum (see [124] and references therein), the limits on curvature depend on assumptions regarding the underlying dark energy scenario. For instance, if instead of a constant $w$ one assumes a linearly varying $w$ (that is $w(a) = w_0 + (1 - a) w_a$), the error on $\Omega_{K0}$ is of the order of a few percent, that is much larger [125–127] (see [128–130] for the constraints on curvature for different parameterizations). The authors of [130] showed that for some models of dark energy the constraint on the curvature is at the level of 5% around a flat universe, whereas for others the data are consistent with an open universe with $\Omega_{K0} \sim 0.2$. According to [128], geometrical tests such as the combination of the Hubble parameter $H(z)$ and the angular diameter distance $D_A(z)$, using (future) data up to sufficiently high redshifts $z \sim 2$, might be able to disentangle curvature from dark energy evolution, though not in a model-independent way. Furthermore, in [131, 132] the authors highlighted the pitfalls arising from ignoring curvature in studies of dynamical dark energy, and recommended to treat $\Omega_{K0}$ as a free parameter to be fitted along with the other model parameters. Lastly, note that in the present work the spatial curvature plays a very crucial role, since Hořava-Lifshitz cosmology coincides completely with ΛCDM if one ignores curvature [31, 52]. Therefore, and following the discussion above, we choose to treat $\Omega_{K0}$ as a free parameter.

In summary, the scenario at hand involves seven model parameters namely the cosmological parameters $\Omega_{m0}$, $\Omega_{c0}$, $\Omega_{r0}$ and $H_0$ and the model parameters $\lambda$, $\omega$ and $\Delta N_\nu$, subject to constraint equations (32) and (33). We marginalize over the cosmological parameters $\Omega_{m0}$, $\Omega_{c0}$, $\Omega_{r0}$ and $H_0$. Of the four remaining parameters, only two are independent. We choose $\lambda$ and $\Delta N_\nu$ as our free parameters. Once these are chosen, and for a given choice of curvature, $\Omega_{K0}$ and $\omega$ are immediately fixed from the constraint equations. In particular, $\omega$ can be determined by eliminating $\Omega_{K0}$ from relations (32) and (33):

$$\omega = -2 \text{sgn}(\Omega_{K0}) \sqrt{0.135 \Delta N_\nu \Omega_{r0} \omega + 0.135 \Delta N_\nu \Omega_{r0}} + 2 \left[ \frac{\Omega_{m0} + \Omega_{r0}}{3\lambda - 1} \right] - 1 = 0.$$  

(34)

This fractional-order equation in $\omega$ can have one or two roots. In the latter case we use the larger one, since our goal is to set an upper limit on $\omega$ (and hence on $\Lambda$). $\Omega_{K0}$ can then be found from $\omega$ using (33).

In Fig. 1 we display the likelihood-contours for the free parameters $\lambda$ vs $\Delta N_\nu$ for both positive and negative curvature. All other parameters have been marginalized over. The details and the techniques of the fitting procedure can be found in the Appendix of [10]. Additionally, in Table I we summarize the 1σ limits on the parameter values for the detailed-balance scenario. These bounds are in agreement with the corresponding ones of our previous work [10]. Furthermore, as we observe, the data
constrain $\lambda$ to roughly $\lambda = 1^{+0.01}_{-0.02}$ at the 1$\sigma$ level, while its best fit value is very close to 1 ($\lambda_{b,f} = 0.006$) for both positive and negative curvature.

**B. Constraints on Beyond-Detailed-Balance scenario**

In units where $8\pi G_{\text{grav}} = 1$ relation (24) gives

$$\sigma_0 = 1/3. \quad (35)$$

Using this value and following the procedure of the previous subsection, the Friedmann equation (13) can be written as

$$H^2 = H_0^2 \left\{ \frac{2}{(3\lambda - 1)} \left[ \Omega_{m0}(1 + z)^3 + \Omega_{r0}(1 + z)^4 \right] + \Omega_{K0} (1 + z)^2 + \frac{2}{(3\lambda - 1)} \left[ \omega_1 + \omega_3 (1 + z)^4 + \omega_4 (1 + z)^6 \right] \right\}. \quad (36)$$

where we have introduced the dimensionless parameters $\omega_1$, $\omega_3$ and $\omega_4$, related to the model parameters $\sigma_1$, $\sigma_3$ and $\sigma_4$ through:

$$\omega_1 = \frac{\sigma_1}{6H_0^2}, \quad \omega_3 = \frac{\sigma_3 H_0^2 \Omega_{K0}^2}{6}, \quad \omega_4 = -\frac{\sigma_4 \Omega_{K0}}{6}. \quad (37)$$

Additionally, we consider the combination $\omega_4$ to be positive, in order to ensure that the Hubble parameter is real for all redshifts. $\omega_4 > 0$ is required also for the stability of the gravitational perturbations of the theory [13-15]. For convenience we moreover assume $\sigma_3 \geq 0$, that is $\omega_3 \geq 0$.

In summary, the present scenario involves the following parameters: the cosmological parameters $H_0$, $\Omega_{m0}$, $\Omega_{K0}$, $\Omega_{b0}$, $\Omega_{r0}$, and the model parameters $\lambda$, $\omega_1$, $\omega_3$ and $\omega_4$. Similarly to the detailed-balance section these are subject to two constraints. The first one arises from the Friedmann equation at $z = 0$, which leads to

$$\frac{2}{(3\lambda - 1)} \left[ \Omega_{m0} + \Omega_{r0} + \omega_1 + \omega_3 + \omega_4 \right] + \Omega_{K0} = 1. \quad (38)$$

This constraint eliminates the parameter $\omega_1$. The second constraint arises from BBN considerations. The term involving $\omega_3$ represents the usual dark-radiation component. In addition, the $\omega_4$-term represents a kination-like component (a quintessence field dominated by kinetic energy [132, 133]). If $\Delta N_r$ represents the BBN upper limit on the total energy density of the universe beyond standard model constituents, then as we show in the Appendix of [103], we acquire the following constraint at the time of BBN ($z = z_{\text{BBN}}$) [111-114]:

$$\omega_3 + \omega_4 (1 + z_{\text{BBN}})^2 = \omega_{3\text{max}} \equiv 0.135 \Delta N_r \Omega_{r0}. \quad (39)$$

It is clear that BBN imposes an extremely strong constraint on $\omega_4$, since its largest possible value (corresponding to $\omega_3 = 0$) is $\sim 10^{-24}$. Finally, $\omega_{3\text{max}}$ denotes the upper limit on $\omega_3$. In the following, we use expression (39) to eliminate $\omega_4$. For convenience, instead of $\omega_3$ we define a new parameter

$$\alpha = \frac{\omega_3}{\omega_{3\text{max}}}, \quad (40)$$

which has the interesting physical meaning of denoting the ratio of the energy density of the Hořava-Lifshitz dark radiation to the total energy density of Hořava-Lifshitz dark radiation and kination-like components at the time of BBN.

We use relation (39) to eliminate $\omega_4$ in favor of $\alpha$ and $\Delta N_r$, and treat $\lambda$, $\alpha$, $\Omega_{K0}$ and $\Delta N_r$ as our free parameters, marginalizing over $H_0$, $\Omega_{m0}$, $\Omega_{b0}$ and $\Omega_{r0}$. Using the combined SNIa+CMB+BAO data, we construct likelihood contours for different combinations of the above parameters, which are presented in Fig. 2. In each case, the free parameters not included in the plot have been marginalized over. The details and the techniques of the construction have been described in the Appendix of [103], with the only difference being that in the present work WMAP 7-year data [133] and the more recent Constitution Supernovae dataset [134] have been used. In Table II we summarize the 1$\sigma$ limits on the parameter values for the beyond-detailed-balance scenario.

| $\Omega_{K0}$ | $\Delta N_r$ | $\alpha$ | $\lambda$ |
|-------------|-------------|--------|--------|
| (-0.01, 0.01) | (0.2, 0.3) | (0.98, 1.01) |

TABLE II: 1$\sigma$ limits on the free parameters of the beyond-detailed-balance scenario. The cosmological parameters $\Omega_{m0}$, $\Omega_{b0}$, $\Omega_{r0}$ and $H_0$ have been marginalized over.

Furthermore, as we observe, in 1$\sigma$ confidence the running parameter $\lambda$ of Hořava-Lifshitz gravity is restricted to the interval $|\lambda - 1| \lesssim 0.02$, for the entire allowed range of $\omega_3$ (that is of $\sigma_3$). Finally, the best fit value for $\lambda$ restricts $|\lambda - 1|$ to much more smaller values, namely $|\lambda_{b,f} - 1| \approx 0.002$.

**IV. CONCLUSIONS**

In this work we have constrained the running parameter $\lambda$ of Hořava-Lifshitz gravity using observational SNIa, BAO and CMB data as well as considerations from BBN. In order to be general enough we have not used any theoretical argument to a priori restrict $\lambda$ in any specific interval, handling it as a completely free parameter. Additionally, we have performed our investigation under the detailed-balance condition, as well as under its relaxation. Finally, we have included the matter and radiation sectors following the usual effective fluid approach. We
stress that we have let all the parameters of the theory to vary, performing an overall and holistic investigation, and we have not followed the naive approach, that is to keep everything fixed and vary only $\lambda$ in order to match observations.

As we showed, Hořava-Lifshitz cosmology, either with or without the detailed-balance condition, can be compatible with observations. We constructed the likelihood-contours for the involved free parameters, and we found that in $1\sigma$ level $\lambda$ is restricted to $|\lambda - 1| \lesssim 0.02$. As expected, these bounds are one order of magnitude tighter than the corresponding ones arising from the consideration of primordial He$^4$-abundance measurements in order to extract a crude upper bound $2\times 10^{-6}$. Finally, concerning the best fit value for $\lambda$ we obtain $|\lambda_{b.f} - 1| \approx 0.006$ for the detailed balance and $|\lambda_{b.f} - 1| \approx 0.002$ for the beyond-detailed-balance cases. The aforementioned features act as additional arguments in favor of the present holistic investigation, where all the parameters of the theory are allowed to vary, comparing to the naive or partial ones in which only $\lambda$ varies. Lastly, it is interesting to note that these results, arising from cosmological observations, are relatively close to the preliminary parametrized post-Newtonian (PPN) estimations for Hořava-Lifshitz gravity ($0 < |\lambda - 1| \lesssim 4 \times 10^{-7}$), despite the fact that the Solar System measurements (that lie at the basis of PPN parameters $[138, 141]$) are much more accurate than the cosmological ones.

In summary, as expected, we found that the value of the running parameter $\lambda$ of Hořava-Lifshitz gravity is restricted to a very tight window around its IR value.

Finally, we should mention that although the present analysis provides the bounds for the running parameter, it does not enlighten the discussion about the possible conceptual problems and instabilities of Hořava-Lifshitz gravity, nor can it address the questions concerning the validity of its theoretical background, which is the subject of interest of other studies. The present work just faces the problem from the phenomenological point of view, which is necessary but not sufficient, and thus its results can be taken into account only if Hořava-Lifshitz gravity passes successfully the aforementioned theoretical tests.

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