Algorithms for the numerical determination of coordinates of analogues of Fermat-Steiner point

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Abstract. There are several points arbitrarily located in space. The problem is – to determine the coordinates of a new point, the sum of all distances to all points from which is to be minimal. Initially Pierre de Fermat, Evangelista Torricelli, and Jacobi Steiner had solved this problem but only for three points on the plane and the distance function of Euclid. For more than four on the plane the exact analytical solution have not been found yet. In this paper the algorithms for numerical determination of coordinates of Fermat-Steiner point both on the plane and in three-dimensional space for arbitrary distance metric have been presented, along with the visualization of testing.

1. Introduction
Let \( M_k \) be the points in three-dimensional space with the known coordinates of \( M_k = (x_k, y_k, z_k) \), \( k = 1, 2, \ldots, n \). The problem is "to find a point in space (i.e. the coordinates of this point) the sum of distances from which to the given points \( M_k \) will be minimal". This is the Steiner tree problem, named after Pierre de Fermat (1601-1665) and Jacob Steiner (1726–1863). It has acquired great practical importance after the wide application of Steiner networks [1-2]. In various problems, the distance between objects has been measured in different metrics. The application of Steiner networks has spread widely (even to the field of forming new software languages). However, for the number of points \( n \geq 4 \) there is no analytical solution of defining the coordinates of F.Sh. point with the optimal sum of distances to the fixed \( n \) points (even for the plane).

The detailed review of A.Yu. Uteshev [3] contains a theorem about the existence and uniqueness of the F.Sh. point for the arbitrary number of points in space \( \mathbb{R}^n \) with Euclidean distance. This task is often called "The problem of the optimal location of station nodes".

**Theorem [3].** Let all the point \( \{P_i\}_{i=1}^K \subset \mathbb{R}^n \) be different.

Minimum of function \( F(P) = \sum_{j=1}^K m_j |PP_j| \) exist and the only one.

In paper [3] various conditions for realizing the construction of the F.Sh. points under the conditions of the theorem are given.

If the distance is calculated to some positive degree, then this problem is related to Potential theory. A complex numerical solution is based on the Weiszfeld algorithm [4] and is presented in [3].

In different applications, the distance between objects has been measured in different metrics [5-7]. In [7] a simple iterative algorithm has been obtained for the numerical solution of the
F.Sh. problem on the plane. Moreover various objective functions have been examined using different norms for distance determination.

To determine the metric the spaces considered are \( l^N_p \), \( 1 \leq p < \infty \), with norms:

\[
\|C\|_{l^p} = \left( \sum_{k=1}^{n} |c_k|^p \right)^{1/p} \quad \text{for} \quad 1 \leq p < \infty
\]

and \( \|C\|_{l^\infty} = \max |c_k| \).

Computer calculations with the help of the obtained algorithms have shown the fact that F.Sh. points, determined with the help of different distances, have different coordinates already on the plane (Fig. 1).

**Figure 1.** Computer drawing defining F.Sh. points: 1) metric \( l^2 \), 2) metric \( l^1 \), 3) metric \( l^\infty \)

For \( p = 1 \), the distance is determined by the formula:

\[
d_1(A, B) = |x - x_k| + |y - y_k|.
\]

Thus for \( p = 2 \), the Euclidean metric is obtained:

\[
d_2(A, B) = \sqrt{|x - x_k|^2 + |y - y_k|^2}.
\]

The inequalities:

\[
d_2(A, B) \leq d_1(a, B) \leq \sqrt{2}d_2(a, B).
\]
Note that the coefficient $\sqrt{2}$ is similar to coefficient of meandering — $\mu$, adopted to estimate distances when traveling by water. To switch over to problem solving in three-dimensional space, let's recall the difficulties that arise on the plane \[7\]. Let the coordinates of origin points be $(x_k, y_k), k = 1, 2, \ldots, n$; and the coordinates of the desired F.Sh. point be $(x, y)$. Thus, for the Euclidean space, the objective function takes the following form

$$\Phi(x, y) = \sum_{k=1}^{n} \sqrt{(x - x_k)^2 + (y - y_k)^2}.$$ 

The task is to find a pair of numbers $(x, y)$ for which this function reaches its minimum value. The sign of the existence of an extremum has the form of a system of equations of a nonlinear type. The solution of this system is perhaps approximately:

$$\begin{cases}
\frac{\partial \Phi}{\partial x} = \sum_{k=1}^{n} \frac{(x - x_k)}{\sqrt{(x - x_k)^2 + (y - y_k)^2}}, \\
\frac{\partial \Phi}{\partial y} = \sum_{k=1}^{n} \frac{(y - y_k)}{\sqrt{(x - x_k)^2 + (y - y_k)^2}}.
\end{cases}$$

The appearance of various metrics in applied problems and urban planning is well known. For example, for urban settlement in a mountainous area the optimal service area from the center towards periphery will no longer be a circle, as on a flat plain, but an ellipse or diamond. When moving in areas with a large number of channels and rivers, the coefficient of meandering $\mu_k$ is used to determine the distance. Then the objective function, subject to the difficulty of delivery, may take the following form

$$\Phi_\mu(x, y) = \sum_{k=1}^{n} \mu_k \sqrt{(x - x_k)^2 + (y - y_k)^2}.$$ 

The existence of a minimum of this function even in space is justified in the above-mentioned theorem \[3\]. This raises a natural question: will the coordinates of the F.Sh. points be the same for the different metrics?

The distance measured by the metric $d_1$ is often used in cities with a grid of streets, similar to checkbook in a cage, thus sometimes it is called the "Manhattan metric".

In Figure 2, the program shows the location of F.Sh. point in this metric.

The present paper is devoted to the construction of a new simpler iterative algorithm for the numerical determination of coordinates of F.Sh. point in three-dimensional space.

The presented calculations are restricted to the Euclidean metric. However, the algorithm works for an arbitrary objective function and for the space metrics $l_p$ considered above. Furthermore, the question of the uniqueness of solution for various metrics is not considered in this paper in view of its ambiguity \[8\]. The objective function has been built for the set of points $M_k = (x_k, y_k, z_k), k = 1, 2, \ldots, n$ has the form:

$$\Phi(M) = \Phi(x, y, z) = \sum_{k=1}^{n} \sqrt{(x - x_k)^2 + (y - y_k)^2 + (z - z_k)^2} \quad (1)$$
2. Algorithm for the numerical decision of the problem

Let’s split the algorithm into sequential actions.

1. The first action — scaling the primary data. Let $M_k = (x_k, y_k, z_k)$, $k = 1, 2, \ldots, n$ be the points in three-dimensional space. Now let’s define the boundaries of these points location. Let assume that all the points located in the first quadrant and

\[
\begin{align*}
\min x_k &= a, & \min y_k &= b, & \min z_k &= c, \\
\max (x_k - a) &= A, & \max (y_k - b) &= B, & \max (z_k - c) &= C
\end{align*}
\] (2)

Then the coordinate transformation

\[
\hat{x} = \frac{x - a}{A}, \quad \hat{y} = \frac{y - b}{B}, \quad \hat{z} = \frac{z - c}{C}
\] (3)

converts all points to a single cube $[0, 1]^3$. In this cube the directed search will be conducted. The inverse transformation gives the coordinates necessary for problem solving.

2. The second action — calculation of the values of objective function at the nodes of a grid of cube partition $[0, 1]^3$. Let’s divide each side of the cube into two equal intervals. This will divide the cube itself into eight equal cubes. Let’s then consider a network consisting of centers of these eight cubes of smaller size:

\[
m_{i,j,k}(1) = \left( \frac{1}{4} + \Delta \times i, \frac{1}{4} + \Delta \times j, \frac{1}{4} + \Delta \times k \right),
\]

where $i = j = k = 0.1$.

In these points then, we consistently calculate the value of the objective function $\Phi (m_{i,j,k}(1))$ and find the lowest value of it. If the calculation accuracy is sufficient, then this point is acknowledged as a solution.
3. The third action — the partitioning and calculation here occurs in one of the eight small cubes (i.e. parts of the big partitioned cube), in which the objective function has reached its minimum. By applying formulas (2) and (3) we transit to new unit cube \([0,1]^3\) and repeat the second action. Let’s call the centers of new cubes of even smaller size \(-m_{i,j}(2)\) — the nodes of the network of partitioning of the new cube. In these points we consistently calculate the value of the objective function \(\Phi (m_{i,j}(2))\) and find the lowest value of it [12-15]. If the calculation accuracy is sufficient, then this point is acknowledged as a solution.

If not, then all actions are repeated.

Remark. Accuracy of the approximate calculation of coordinates increases with exponential velocity, as \(O\left(\frac{1}{2^m}\right)\), where \(m = 3n\).

3. Flowchart for the computer implementation of the problem solution

The scheme is an iterative scheme, containing an iterative cycle that works until the desired accuracy is achieved. The designed algorithm of search ensures the fast achievement of the desired accuracy.

![Flowchart](image)

**Figure 3.** Flowchart for the computer implementation of the problem solution.

4. Examples of testing visualization

In this section with the use of test tasks we will check the operational validity of the designed algorithm and the program obtained. From the programming point of view - this is the task
of program action visualization. The starting points have been chosen by two ways: 1) given arbitrarily, 2) given randomly by generating random numbers.

Images obtained with the help of the program are given below (Fig. 4, Fig. 5).

Figure 4. The starting points given arbitrarily

Figure 5. The starting points chosen given randomly by generating random numbers
5. Means of computer realization
As the implementation tools, the integrated development environment of Microsoft Visual Studio 2015 was chosen, which includes the C# language and the Net Framework virtual machine. This assembly allows you to write software that has a graphical interface. The programs allow you to set points arbitrarily or generate them randomly. The accuracy of the calculation is specified. As a result, the program will build a 3d visualization that shows all the given points and the found optimal point Φ, and calculates the coordinates of the point. For visualization, the library component [5] is used.

6. Ferma Steiner points in regular polygons
In this chapter the testing of the program obtained for regular polygons is shown. The distance is considered in Banach spaces $l_1$, $l_2$ and in space $l_\infty$. In space $l_\infty$ the norm is defined by the formula:

$$\|C\|_{l_\infty} = \max |c_k|.$$  

The Fermat-Torricelli-Steiner points for regular polygons with 5, 7, 8, 16 vertices are presented in the following figures:

Figure 6. The F.Sh. points of the regular pentagon are different
Figure 7. The F.Sh. points of the regular heptagon are different

Figure 8. The F.Sh. points of the regular octagon coincide
Figure 9. The F.Sh. points of the regular hexagon coincide

Visualization of the determination of coordinates of F.Sh. points in regular polygons allowed to put forward a hypothesis and prove the following (hitherto unknown).

**Statement.** For regular polygons with an even number of vertices \( n = 2p \) the F.Sh. points are determined for distances in spaces \( l_1, l_2 \) and \( l_\infty \) coincide. For regular polygons with an odd number of vertices \( n = 2p + 1 \) the F.Sh. points are determined for distances in spaces \( l_1, l_2 \) and \( l_\infty \) are distinct.

The above-mentioned proof uses the symmetry argument and takes into account the geometry of the unit balls in these spaces (not cited here due to its simplicity).

7. **Conclusions and practical relevance**

The determination of the coordinates of the F.Sh. points for various metrics has a variety of practical applications. For example, in the southern part of Vietnam, the traffic between small settlements is currently carried out along canals and rivers. These settlements are seasonally flooded and can be under water for up to 3-5 months [10]. The definition of the best coordinates for the construction of an enlarged dry settlement for the temporary stay of workers of rice plantations is becoming topical. In this case, the metric of space is useful [9-11].

A similar task is possible in urban planning where the organization of streets "in a cage" (for example, Manhattan) is needed.

If we locate the emitting sources in fixed points \( M_k \), then the intensity of the combined effect at the \( M \) point is maximal. The task of determining of such point is relevant for technical aesthetics, design and applied construction tasks, as well as for communications planning. If the light emitters are placed in the \( M_k \) points, then the Fermat-Steiner point will have maximum illumination. This effect will be valid and for the other emitters.
8. References

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