AN ALGORITHMIC APPROACH TO ENTANGLEMENT-ASSISTED QUANTUM ERROR-CORRECTING CODES FROM THE HERMITIAN CURVE

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Abstract. We study entanglement-assisted quantum error-correcting codes (EAQECCs) arising from classical one-point algebraic geometry codes from the Hermitian curve with respect to the Hermitian inner product. Their only unknown parameter is c, the number of required maximally entangled quantum states since the Hermitian dual of an AG code is unknown. In this article, we present an efficient algorithmic approach for computing c for this family of EAQECCs. As a result, this algorithm allows us to provide EAQECCs with excellent parameters over any field size.

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1. Introduction

Quantum error correcting codes (QECCs) can be constructed by means of classical linear codes with the use of the CSS method [5, 8, 13]. QECCs over $\mathbb{F}_q$, the finite field with $q$ elements, are constructed from self-orthogonal classical linear codes over $\mathbb{F}_q$ if we consider the Euclidean metric, and over $\mathbb{F}_{q^2}$ if we consider the Hermitian metric. In general the Hermitian metric produces QECCs with better parameters since we may consider linear codes over a bigger field, the downside being, however, that the computations are more involved.

Brun et al. proposed in [4] a pre-sharing entanglement protocol between encoder and decoder which increases the communication capacity and simplifies the theory of quantum error-correction since it is not restricted to self-orthogonal classical linear codes. These quantum codes are known as entanglement-assisted quantum error-correcting codes (EAQECCs). They were introduced over the binary field and then extended for an arbitrary finite field in [7]. As for QECCs, we may consider classical linear codes over $\mathbb{F}_q$ if we consider the Euclidean metric, and over $\mathbb{F}_{q^2}$ if we consider the Hermitian metric. EAQECCs have four fundamental parameters $[[n, k, d; c]]$: length, dimension, minimum distance and the number of required pairs of maximally entangled quantum states, respectively.

Among the classical codes used to construct quantum codes by means of the CSS construction, codes arising from algebraic geometry resources and tools (the so-called algebraic geometry codes or AG codes for short) have revealed themselves as good candidates to obtain quantum codes with good parameters [2, 3, 14, 17]. In particular, we should highlight AG codes from Castle curves (the so-called Castle codes) [11, 18], whose prototype is the Hermitian curve, which has traditionally played an important role in the theory of AG codes. Therefore, it is natural to consider the Hermitian curve to obtain EAQECCs with good parameters.

EAQECCs from AG codes have been addressed in [19, 20], where the Euclidean inner product is mainly considered. In this work, we consider Hermitian codes [12] – that is, one-point AG codes from the Hermitian curve over $\mathbb{F}_{q^2}$ – to obtain EAQECCs considering the Hermitian inner product. The length, dimension, and minimum distance of the EAQECCs from the Hermitian curve are known. Only one parameter remains: the number of required maximally entangled quantum states, $c$. Unlike the other parameters, however, $c$ is unknown if we consider the Hermitian inner product. Thus, the main task of this article is to compute $c$.

We may compute $c$ as the rank of a product of generator matrices, specifically the generator matrix of the classical linear code and the generator matrix of its Hermitian dual (see [7, Proposition 3]). However, the Hermitian dual of an AG code (and of a Hermitian code) is unknown in general and, hence, there is no formula for computing $c$ in the Hermitian metric case. In this work, we present an algorithmic approach for computing $c$. The
algorithm is based on a careful analysis of reduced polynomials modulo the Hermitian curve equation and on the valuation given by the pole order at the infinity point. This allows us to compute the parameters of EAQECCs from the Hermitian curve and illustrate that they have excellent parameters. The computational complexity of our algorithm is $O(q^5)$, which is significantly better than computing a rank of a matrix using methods from linear algebra since the length of a Hermitian code is $q^3$.

Finally, by considering $q$-adic expansions and Lucas’ theorem, we are able to reduce the number of iterations in our algorithm by a careful analysis of the $q$-th powers of Hermitian codes. The proofs are different for $q$ prime and non-prime, but they are developed in a parallel way. Naturally, the proof for the non-prime case is more involved and technical.

The structure of this paper is as follows. In Section 2, we present some standard results on classical and quantum codes. In Section 3, we review Hermitian codes and introduce the idea of reduced polynomials that is used in the following section. Section 4 shows the first main result: we describe an algorithmic approach to the problem of determining the parameters of EAQECCs derived from a one-point Hermitian code. Moreover, we provide codes with excellent parameters at the end of the section. Finally, in Section 5, we analyze the $q$-th powers of Hermitian codes, which allow us to improve the number of iterations of the algorithm in Section 4. We also compute the computational complexity of our algorithm and show, computationally, that the parameters of most of the EAQECCs in this family exceed the Gilbert-Varshamov bound [7].

The ideas and methods developed in this article can serve as a paradigm to study EAQECCs coming from of other AG codes obtained from different curves, and especially from Castle-type curves.

2. Preliminaries

In this section we recall some preliminary and well known results about linear vector spaces over a finite field, the Frobenius map and entanglement-assisted quantum codes. A more complete study of these topics can be found in [7, 9, 13, 15].

2.1. The Frobenius map and the Hermitian dual of a subspace. Let $q = p^r$ be a prime power, and let $\mathbb{F}_{q^2}$ be the finite field with $q^2$ elements. Additionally, let $\sigma : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}$ be the Frobenius map, $\sigma(x) = x^q$. It is well known that $\sigma$ is an $\mathbb{F}_q$-automorphism of $\mathbb{F}_{q^2}$ and an involution, that is $\sigma^2 = \text{id}$. Furthermore, considering the same notation, $\sigma$ may be extended to the map $\sigma : \mathbb{F}_{q^n} \to \mathbb{F}_{q^n}$, by $\sigma(v) = (v_1^q, \ldots, v_n^q)$. This map is $\mathbb{F}_q$-semilinear and bijective, thus it preserves intersections and linear dependence: given $V, W \subset \mathbb{F}_{q^2}$, we have $\sigma(V \cap W) = \sigma(V) \cap \sigma(W)$ and $\dim(\langle V \rangle) = \dim(\langle \sigma(V) \rangle)$, where $\langle V \rangle$ denotes the $\mathbb{F}_{q^2}$-linear subspace spanned by $V$. Following the usual notation in linear coding theory, we often write $v^q = \sigma(v)$ and $V^q = \sigma(V) = \{v^q : v \in V\}$. 
The Euclidean and Hermitian inner products in $\mathbb{F}_q^n$ are defined as follows: given $\mathbf{v}, \mathbf{w} \in \mathbb{F}_q^n$,

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \cdots + v_n w_n$$

and

$$\mathbf{v} \cdot_H \mathbf{w} = \sigma(\mathbf{v}) \cdot \mathbf{w}.$$

Given a linear subspace $V \subset \mathbb{F}_q^n$, its Euclidean and Hermitian duals are, respectively,

$$V^\perp = \{ x \in \mathbb{F}_q^n : \mathbf{v} \cdot x = 0 \text{ for all } \mathbf{v} \in V \},$$

$$V^\perp_H = \{ x \in \mathbb{F}_q^n : \mathbf{v} \cdot_H x = 0 \text{ for all } \mathbf{v} \in V \}.$$

We define $\Delta(V) = \dim(V \cap V^\perp_H)$.

**Proposition 1.** Let $V \subset \mathbb{F}_q^n$ be a linear subspace. We have

(a) $V^\perp_H = (V^q)^\perp = (V^\perp)^q$;

(b) $\Delta(V) = \dim(V^q \cap V^\perp)$;

(c) $\Delta(V) = \Delta(V^\perp)$.

**Proof.** (a) The first equality is clear. The second one follows from the fact that $\mathbf{v}^q \cdot x = 0$ if and only if $\mathbf{v} \cdot x^q = 0$. Write $\mathbf{y} = x^q$. Then $V^\perp = \{ \mathbf{y}^q \in \mathbb{F}_q^n : \mathbf{v} \cdot \mathbf{y} = 0 \text{ for all } \mathbf{v} \in V \} = (V^\perp)^q$, and the second equality of (a) holds. (b) Applying $\sigma$ we have $\Delta(V) = \dim(V \cap (V^\perp)^q) = \dim(V^q \cap V^\perp)$ which proves the statement in (b). (c) Since $(V^\perp)^\perp = V$ we get $\Delta(V^\perp) = \dim((V^\perp)^q \cap (V^\perp)^\perp) = \Delta(V)$.

In what follows, we use the expression given by Proposition 1(b) to compute $\Delta(V)$.

### 2.2. Entanglement-Assisted Quantum Error-Correcting Codes

A quantum code $Q$ is called an $[[n,k,d];c]$ entanglement-assisted quantum error-correcting code (EAQECC) if it encodes $k$ logical qudits into $n$ physical qudits using $c$ copies of maximally entangled quantum states and it can correct $\lfloor (d-1)/2 \rfloor$ quantum errors. The rate of an EAQECC is $k/n$, its relative distance is $d/n$, and its entanglement-assisted rate is $c/n$. When $c = n - k$ then the EAQECC is said to have maximal entanglement.

**Proposition 2.** [7, Proposition 3 and Corollary 1] Let $C$ be a linear code over $\mathbb{F}_q$ with parameters $[[n,k,d];d']_q$. Then there is an EAQECC $Q$ with parameters $[[n,2k-n+c,d';c]]_q$, where $d'$ is the minimum Hamming weight of a vector in the set $C \setminus (C \cap C^\perp)$, and

$$c = \dim(C^\perp - \dim(C^\perp \cap C))$$

is the number of required maximally entangled quantum states.

From Propositions 1 and 2, the amount of entanglement of a code $Q$ given by Eq. (1) can be computed as $c = \dim(C^\perp) - \Delta(C) = n - \dim(C) - \Delta(C)$. In this paper we are going to use this formula to determine $c$. Furthermore, we clearly have that $d' \geq d(C)$.

In order to show the performance of the EAQECCs given in this article we will consider the Gilbert-Varshamov bound [7, Theorem 5] for EAQECCs considering the Hermitian inner product.
Theorem 3. Assume the existence of positive integers \( n, k \leq n, d, c \leq (n - k)/2 \) such that

\[
q^{n+k} - q^{n-k-2c} \sum_{i=1}^{d-1} \binom{n}{i} (q^2 - 1)^i < 1.
\]

Then there exists an EAQECC \( \mathcal{Q} \) with parameters \([n, k - c, d; c]\).q.

Furthermore, we will consider the entanglement-assisted analog of the Singleton bound given in [1, Theorem A.5]: if there exists an EAQECC with parameters \([n, k, d; c]\)q and \( d \leq (n + 2)/2 \), then \( 2d \leq n + 2 - k + c \). Note that the previous Singleton type bound does not hold for all range of parameters of an EAQECC. In the light of this bound, the following defect has been used to evaluate the performance of EAQECCs [1, Notation 3.7].

Definition 1. Let \( \mathcal{Q} \) be an \([n, k, d; c]\)q EAQECC. Then the entanglement-assisted quantum Singleton defect of \( \mathcal{Q} \) is the integer

\[
n + 2 - k + c - 2d.
\]

Note that the entanglement-assisted quantum Singleton defect is non-negative if \( d \leq (n + 2)/2 \), and otherwise it may be negative.

3. Reduced polynomials for Hermitian codes

In this section we recall some basic facts about the Hermitian curve, the classical codes derived from it, Hermitian codes, and some tools we use in subsequent computations. For a complete treatment see [12].

3.1. Hermitian curves and codes. Let \( \mathcal{X} \) be the Hermitian curve defined over \( \mathbb{F}_{q^2} \) by the affine equation \( x^{q+1} = y^q + y \). This is a nonsingular plane curve of genus \( g = q(q - 1)/2 \), with \( n = q^3 \) rational affine points, \( P_1, \ldots, P_n \), plus one point at infinity, \( Q \). Let \( \nu_Q \) be the valuation of \( \mathbb{F}_{q^2}(\mathcal{X}) \) given by the order at \( Q \) and \( \nu = -\nu_Q \). From the equation of \( \mathcal{X} \) we have \( \nu(x) = q \) and \( \nu(y) = q + 1 \). The following property of \( \nu \) will be widely used in what follows.

Proposition 4. [12, Theorem 2.16(iii)] Given functions \( f_1, f_2 \in \mathbb{F}_{q^2}(\mathcal{X})^* \), we have \( \nu(f_1 + f_2) \leq \max\{\nu(f_1), \nu(f_2)\} \) with equality if \( \nu(f_1) \neq \nu(f_2) \).

For \( m \geq 0 \), we consider the Riemann-Roch space

\[
\mathcal{L}(m) = \mathcal{L}(mQ) = \{ f \in \mathbb{F}_{q^2}(\mathcal{X}) : f = 0 \text{ or } \div(f) \geq -mQ \}.
\]

This is a linear space, whose dimension is denoted by \( \ell(m) \). Recall that \( \ell(m) = m + 1 - g \) when \( m \geq 2g - 1 \), according to the Riemann-Roch theorem [12]. Let \( \mathcal{L}(\infty) = \bigcup_{m=0}^{\infty} \mathcal{L}(m) \) and denote by \( S \) the Weierstrass semigroup of \( Q \), then \( S = \{ \nu(f) : f \in \mathcal{L}(\infty), f \neq 0 \} = \)
(q, q + 1). Since $\mathcal{L}(\infty)$ is a finitely generated $\mathbb{F}_{q^2}$-algebra, we have that $\mathcal{L}(\infty) = \mathbb{F}_{q^2}[x, y]$. Abusing the notation, the elements of $\mathcal{L}(\infty)$ will be called polynomials.

**Lemma 5.** If $f_1, \ldots, f_t \in \mathcal{L}(\infty)$ are nonzero polynomials of pairwise different orders, then they are linearly independent.

**Proof.** Suppose $\lambda_1 f_1 + \cdots + \lambda_t f_t = 0$, then Proposition 4 ensures that $\nu(0) = \nu(\lambda_1 f_1 + \cdots + \lambda_t f_t) = \max\{\nu(f_i) : \lambda_i \neq 0\}$. Hence $\lambda_1 = \cdots = \lambda_t = 0$. □

Consider the evaluation map $ev : \mathcal{L}(\infty) \rightarrow \mathbb{F}_{q^2}^n, ev(f) = (f(P_1), \ldots, f(P_n))$. The algebraic geometry code $C(m)$ is defined as $C(m) = ev(\mathcal{L}(m))$. Since $C(n + 2g - 1) = \mathbb{F}_{q^2}^n$ – see eg. [22] – we can restrict to $0 \leq m \leq n + 2g - 1$. In this range, we can consider the quantum code $Q(m)$ over $\mathbb{C}^q$ obtained from $C(m)$ by the construction described in Proposition 2. The code $Q(m)$ has parameters $[[n, 2k - n + c, \geq d, c]]_q$, where $n = q^3, k = k(m)$ and $d = d(m)$ are, respectively, the length, dimension and minimum distance of $C(m)$, and $c = n - k(m) - \Delta(C(m))$ is the minimum number of maximally entangled quantum states consumed by the code. Recall that the dimension of $C(m)$ is $k(m) = \ell(m) - \ell(m - n)$, and its exact minimum distance $d(m)$ has been determined in [24]. Therefore, our main task is to compute $\Delta(m) = \Delta(C(m)) = \dim(C(m)^\perp \cap C(m)^\perp)$, which is the only unknown value among the previous parameters. Please note that the dual code $C(m)^\perp$ satisfies the well known relation $C(m)^\perp = C(m^\perp)$, where $m^\perp = n + 2g - 2 - m$, see [12].

### 3.2. Reduced polynomials.

A monomial $x^a y^b$ is called reduced if $0 \leq a < q^2, 0 \leq b < q$. Thus, the number of reduced monomials is at most $n = q^3$. A polynomial $f \in \mathcal{L}(\infty)$ is reduced if $f = 0$ or it is the sum of reduced monomials. We denote by $\mathcal{R}$ the set of all reduced polynomials and $\mathcal{R}^* = \mathcal{R} \setminus \{0\}$. Consider the reduction $\mathcal{r} : \mathcal{L}(\infty) \rightarrow \mathcal{R}$ as follows. Given a polynomial $f$, $\mathcal{r}(f)$ is obtained by performing the substitutions

$$(R1) \quad \mathcal{r}(y^q) = x^{q+1} - y \quad \text{and} \quad (R2) \quad \mathcal{r}(x^{q^2}) = x$$

as many times as possible. In other words, $\mathcal{r}(f)$ is the unique (up to multiplication by a constant $\lambda \in \mathbb{F}_{q^2}$) reduced polynomial in the coset of $f$ modulo the ideal $(x^{q+1} - y^q - y, x^{q^2} - x) \subset \mathcal{L}(\infty)$. This comes from considering the ideal generated from the Hermitian equation, the field equations $x^{q^2} - x$ and $y^{q^2} - y$, and the lexicographical ordering with $y > x$.

The next lemma gives some properties of reduced polynomials that we shall use later.

**Lemma 6.** The following statements hold.

(a) Two distinct reduced monomials are linearly independent;

(b) For any $f \in \mathcal{L}(\infty)$, we have $ev(f) = ev(\mathcal{r}(f))$;

(c) The evaluation map $ev : \mathcal{R} \rightarrow \mathbb{F}_{q^2}^n$ is an isomorphism of vector spaces. Thus, any code $C \subset \mathbb{F}_{q^2}^n$ can be obtained by evaluating reduced polynomials.
Proof. (a) follows from the fact that \( \nu(x^ay^b) = aq + b(q+1) \) and Lemma 5. (b) is a consequence of \( y^n = x^{q+1} - y \) in \( \mathcal{L}(\infty) \) and \( ev(x^y) = ev(x) \). For (c), note that the order of a reduced polynomial is at most \( q^3 + q^2 - q - 1 = n + 2g - 1 \) and that \( ev: \mathcal{L}(q^3 + q^2 - q - 1) \to \mathbb{F}_q^n \) is surjective. From this, it follows that \( ev: \mathcal{R} \to \mathbb{F}_q^n \) must be surjective as well. Since the dimension of \( \mathcal{R} \) is at most \( n \), we get the isomorphism. \( \square \)

In view of property (a) in Lemma 6, any nonzero reduced polynomial has a unique leading monomial with respect to \( \nu \). We define the normalization of \( f \in \mathcal{R} \) as \( n(f) = 0 \) if \( f = 0 \), and \( n(f) = \lambda f \), where \( \lambda \in \mathbb{F}_q^* \) is chosen such that \( n(f) \) has leading coefficient equal to 1 (with respect to \( \nu \)), if \( f \in \mathcal{R}^* \). The normalized reduction of \( f \in \mathcal{L}(\infty) \) will be \( \mathrm{v}(f) = n(\mathrm{v}'(f)) \).

Let \( \mathcal{M} = \{1, x, y, \ldots \} = \{f_1, f_2, \ldots, f_n\} \) be the set of reduced monomials ordered according to the values of \( \nu \) and let \( \mathcal{M}_m = \{f_1, \ldots, f_{\ell(m)}\} \) for \( 0 \leq m \leq q^3 + q^2 - q - 1 \). From Lemma 6 it holds that \( \{ev(f_i) : f_i \in \mathcal{M}\} \) is a basis of \( \mathcal{C}(q^3 + q^2 - q - 1) \cong \mathbb{F}_q^n \), and \( \{ev(f_i) : f_i \in \mathcal{M}_m\} \) is a basis of \( \mathcal{C}(m) \). In what follows, we deal with codes in terms of reduced polynomials.

Example 1. As mentioned before, the Hermitian curve over \( \mathbb{F}_q \) has \( q^3 \) rational affine points and one point at infinity, which is also rational. The genus of the curve is \( g = q(q-1)/2 \). A basis for \( \mathcal{L}(m) \) is given by the set \( \{x^iy^j : i \geq 0, 0 \leq j \leq q-1, iq+j(q+1) \leq m\} \).

Let \( m = 22 \). Then, \( B = \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, x^4, x^3y, x^2y^2, x^5, x^4y, x^3y^2, x^6, x^5y, x^4y^2, x^7, x^6y\} \) is a basis for \( \mathcal{L}(m) \). Consider \( B^q = \{f^q : f \in B\} \). Since \( q = 3 \), the reductions are given by \( \mathrm{v}'(x^3) = x \) and \( \mathrm{v}'(y^3) = x^4 - y \). Thus, the normalized reduction \( \mathrm{v} \) on \( B^q \) gives the polynomials listed in Table 1. Lastly, one can check that \( \langle ev(B^q) \rangle = \langle ev(\mathrm{v}(B^q)) \rangle \).

4. The parameters of EAQECs from Hermitian codes

In this section we shall compute the parameters of the EAQEC \( \mathcal{Q}(m) \) obtained from the Hermitian code \( \mathcal{C}(m) \). Let us remember that this problem leads us to the computation of \( \Delta(m) = \dim(\mathcal{C}(m)^{1/\ell(m) \cap C(m)}) = \dim(\mathcal{C}(m)^q \cap C(m^{1/2})) \), where \( m^{1/2} = n + 2g - 2 - m \). In some cases, this number is easy to determine, see [21].

Proposition 7. If \( m \leq q^2 - 2 \), then \( \Delta(m) = \ell(m) \).

Proof. Note that \( \mathcal{L}(m)^q \subseteq \mathcal{L}(qm) \), where \( \mathcal{L}(m)^q = \{f^q : f \in \mathcal{L}(m)\} \). If \( m \leq q^2 - 2 \), then \( qm \leq m^{1/2} \) and thus \( \mathcal{C}(m)^q \subseteq C(m^{1/2}) \), so \( \Delta(m) = \dim(\mathcal{C}(m)^q) = \dim(\mathcal{C}(m)) \). \( \square \)

For \( m < q^3 \) the evaluation map \( ev : \mathcal{L}(m)^q \to \mathbb{F}_q^m \) is injective since \( ev(f^q) = ev(f)^q \) and \( ev(\mathcal{L}(m)) \) is one-to-one. For \( m \geq q^3 \) this is no longer true. Thus, in this case we cannot expect a formula for \( \Delta(m) \) in terms of \( \ell(m) \), as in Proposition 7. In the sequel, we develop a procedure that performs such a computation. We can restrict to \( m \geq q^2 - 1 \) since
We then set $\phi$. This reduction may be described step-by-step as follows. Define $\phi$ where $\phi$. Notice that the functions $\phi$ are linearly independent, as $\dim(C(m)^q) = \dim(C(m))$. In general, however, they do not have pairwise different orders. We shall construct a new set of functions $\phi_1, \ldots, \phi_t$ such that $\langle \phi_1, \ldots, \phi_t \rangle = \langle \tau(f_1^q), \ldots, \tau(f_t^q) \rangle$ and all $\phi_i$’s have different orders. To that end, we define a second reduction on the $\tau(f_i^q)$’s, denoted by $s$ and described iteratively via the rule

$$s(\tau(f_i^q)) := 1 \quad \text{and} \quad s(\tau(f_i^q)) := \tau(f_i^q) \mod \langle s(\tau(f_1^q)), \ldots, s(\tau(f_{i-1}^q)) \rangle,$$

We then set $\phi_i = s(\tau(f_i^q))$ for $i = 1, \ldots, t$.

This reduction may be described step-by-step as follows. Define $\phi_1 := 1$. Since $f_1 = 1$, it is obvious that $\langle \phi_1 \rangle = \langle \tau(f_1^q) \rangle$. Once normalized reductions $\phi_1, \ldots, \phi_{t-1}$ such that $\langle \phi_1, \ldots, \phi_{t-1} \rangle = \langle \tau(f_1^q), \ldots, \tau(f_{t-1}^q) \rangle$ are computed, set $\phi = \tau(f_t^q)$ and do:

| $i$ | $f_i$ | $\tau'(f_i^q)$ | $\tau(f_i^q) = n(\tau(f_i^q))$ |
|-----|-------|----------------|-------------------------------|
| 1   | 1     | 1              | 1                            |
| 2   | $x$   | $x^3$          | $x^3$                        |
| 3   | $y$   | $x^4 + 2y$    | $x^4 + 2y$                   |
| 4   | $x^2$ | $x^6$         | $x^6$                       |
| 5   | $xy$  | $x^7 + 2x^3y$ | $x^7 + 2x^3y$                |
| 6   | $y^2$ | $x^8 + x^4y + y^2$ | $x^8 + x^4y + y^2$ |
| 7   | $x^3$ | $x$            | $x$                         |
| 8   | $x^2y$ | $2x^6y + x^2$ | $x^6y + 2x^2$                |
| 9   | $xy^2$ | $x^7y + x^3y^2 + x^3$ | $x^7y + x^3y^2 + x^3$ |
| 10  | $x^4$ | $x^4$         | $x^4$                       |
| 11  | $x^3y$ | $x^5 + 2xy$  | $x^5 + 2xy$                  |
| 12  | $x^2y^2$ | $x^6y^2 + x^6 + x^2y$ | $x^6y^2 + x^6 + x^2y$ |
| 13  | $x^5$ | $x^7$         | $x^7$                       |
| 14  | $x^4y$ | $x^8 + 2x^4y$ | $x^8 + 2x^4y$                |
| 15  | $x^3y^2$ | $x^5y + xy^2 + x$ | $x^5y + xy^2 + x$ |
| 16  | $x^6$ | $x^2$         | $x^2$                       |
| 17  | $x^5y$ | $2x^7y + x^3$ | $x^7y + 2x^3$                |
| 18  | $x^4y^2$ | $x^8y + x^4y^2 + x^4$ | $x^8y + x^4y^2 + x^4$ |
| 19  | $x^7$ | $x^5$         | $x^5$                       |
| 20  | $x^6y$ | $x^6 + 2x^2y$ | $x^6 + 2x^2y$                |

Table 1. Normalized reductions for $q = 3$ and $m = 22$

Proposition 7 covers the remaining cases. On the other hand, from Proposition 1(c), we can also assume $m \leq m^*$, that is, we can restrict to $m \leq m^* = \lfloor n/2 + g - 1 \rfloor$ [23]. We write $M_* = M_{m^*}$.
(S1): If $\nu(\phi) \neq \nu(\phi_i)$ for all $i < t$, then set $\phi_i \leftarrow \phi$. End.

(S2): If $\nu(\phi) = \nu(\phi_i)$ for some $i < t$, then set $\phi \leftarrow \phi - n(\phi - \phi_i)$. Repeat from (S1).

Note that when $\nu(\phi) = \nu(\phi_i)$ for some $i < t$, then we have $\langle \phi_1, \ldots, \phi_{t-1}, n(\phi - \phi_i) \rangle = \langle \phi_1, \ldots, \phi_{t-1}, \phi \rangle$. Then, by the initial choice $\phi = r(f^q_i)$ and the induction hypothesis $\langle \phi_1, \ldots, \phi_{t-1} \rangle = \langle r(f^q_1), \ldots, r(f^q_{t-1}) \rangle$, in each step of the previous procedure it holds that $\langle \phi_1, \ldots, \phi_{t-1}, \phi \rangle = \langle r(f^q_1), \ldots, r(f^q_t) \rangle$. In particular, the functions $r(f^q_1), \ldots, r(f^q_t)$ being linearly independent, we deduce that $\phi \neq 0$. Note also that, in the case $\nu(\phi) = \nu(\phi_i)$ for some $i < t$, the leading terms of $\phi$ and $\phi_i$ coincide by Lemma 6. Hence, $\nu(\phi)$ decreases in each iteration, and after at most $t$ such iterations, we obtain $\phi_t$. Furthermore, the $\phi_i$’s are reduced since they are linear combinations of reduced polynomials. Thus, finally we obtain reduced functions $\phi_1, \ldots, \phi_t$, of pairwise different orders, such that $\langle \phi_1, \ldots, \phi_t \rangle = \langle r(f^q_1), \ldots, r(f^q_t) \rangle$.

Define the set $\Phi(m) = \{ \phi_i : 1 \leq i \leq \ell(m) \}$. The above properties can be summarized as follows:

**Proposition 8.** $\Phi(m)$ is a set of linearly independent reduced polynomials with pairwise different orders and $C(m)^q = ev(\langle \Phi(m) \rangle)$.

The set $\Phi(m)$ provides an efficient way to compute $\Delta(m)$.

**Lemma 9.** Let $m, m'$ be two nonnegative integers with $m \leq m^*$. The set $\{ \phi \in \langle \Phi(m) \rangle : \nu(\phi) \leq m' \}$ is a linear space whose basis is $\Phi(m, m') = \{ \phi_i \in \Phi(m) : \nu(\phi_i) \leq m' \}$.

**Proof.** It is a consequence of Proposition 4 and Lemma 5, taking into account that all functions of $\Phi(m)$ have different orders. \qed

**Proposition 10.** Let $0 \leq m \leq m^*$. We have $\Delta(m) = \# \{ \phi_i \in \Phi(m) : \nu(\phi_i) \leq m^\perp \} = \# \Phi(m, m^\perp)$.

**Proof.** Let $x \in C(m)^q \cap C(m^\perp)$. There exists a reduced polynomial $f \in R$ such that $\nu(f) \leq m^\perp$ and $ev(f) = x$. Furthermore, by Proposition 8, there exists $\phi \in \langle \Phi(m) \rangle$ such that $x = ev(\phi)$. Now, since both $f$ and $\phi$ are reduced, Lemma 6 gives $\phi = f$. Thus $\nu(\phi) \leq m^\perp$. Then the number of vectors $x$ in $C(m)^q \cap C(m^\perp)$ is exactly the number of $\phi$’s in $\Phi(m)^q$ such that $\nu(\phi) \leq m^\perp$. The conclusion follows from Lemma 9. \qed

The previous arguments lead directly to an algorithmic way of computing $\Delta(m)$. Note that $-\nu$ is the valuation given by the point at infinity and LC stands for leading coefficient (with respect to $\nu$).

The next remark clarifies some properties of Algorithm 1.

---

1We are showing in the following section that we can reduce the number of iterations. It is achieved by a careful analysis of the reduced monomials created in the present process.
Algorithm 1 A basis for $\Delta(m)$ (for computing $c$)

Input: $m$ with $q^2 - 1 \leq m \leq m^* = \lfloor n/2 + g - 1 \rfloor = \lfloor q^3/2 + (q^2 - q)/2 - 1 \rfloor$; $\mathcal{M}_m = \{f_1, \ldots, f_\ell\}$, where $\ell = m + 1 - (q^2 - q)/2$.

Output: A basis for $\Delta(m)$.

1: for $j = 1, \ldots, \ell$ do
2: \hline
3: \hline
4: \hline
5: \hline
6: \hline
7: \hline
8: \hline
9: \hline
10: \hline
11: \hline
12: \hline
13: \hline
14: \hline
15: \hline
16: \hline
17: \hline
18: \hline
19: \hline
20: \hline
21: \hline

\textbf{Remark 1.} (a) Note that all polynomials involved in the computations described above have coefficients in $\mathbb{F}_p$, where $p$ is the characteristic of $\mathbb{F}_q^2$. Therefore, the algorithm runs over $\mathbb{F}_p$, although both the Hermitian curve $\mathcal{X}$ and AG codes obtained from them are defined over $\mathbb{F}_q^2$.

(b) In view of Proposition 1(c), we have treated only the case $m \leq m^*$. For larger values of $m$, one can use the identity $\Delta(m) = \Delta(m^\perp) = \#\Phi(m^\perp, m)$.

(c) For $m \geq n + 2q - 2 - (q^2 - 2) = q^3 - q$, we have $m^\perp \leq q^2 - 2$. By Proposition 1 and Proposition 7 this implies $\Delta(m) = \Delta(m^\perp) = \ell(m^\perp)$. Hence, the EAQECC constructed from $\mathcal{C}(m)$ has entanglement $c = \ell(m^\perp) - \ell(m^\perp - n) - \Delta(m) = 0$, meaning that it is a standard quantum code. Similarly, if $m \leq q^2 - 2$ we have $\Delta(m) = \ell(m)$ and $\mathcal{C}(m)$ has dimension $k(m) = \ell(m)$. Thus, the resulting EAQECC has dimension 0. All these codes can be discarded if one is only interested in EAQECCs with $c > 0$ and $k > 0$. 

Example 2. Let $q = 3$. Here $g = 3$ and $m^* = 15$. Algorithm 1 gives the data shown in Table 2.

$$
\begin{array}{cccccc}
 f_i & \nu(f_i) & \nu(f_i^q) & \nu(\nu(f_i^q)) & \phi_i & \nu(\phi_i) \\
 1 & 0 & 1 & 0 & 1 & 0 \\
 x & 3 & x^3 & 9 & x^3 & 9 \\
 y & 4 & x^4 + 2y & 12 & x^4 + 2y & 12 \\
 x^2 & 6 & x^6 & 18 & x^6 & 18 \\
 xy & 7 & x^7 + 2x^3y & 21 & x^7 + 2x^3y & 21 \\
 y^2 & 8 & x^8 + x^4y + y^2 & 24 & x^8 + x^4y + y^2 & 24 \\
 x^3 & 9 & x & 3 & x & 3 \\
 x^2y & 10 & x^6y + 2x^2 & 22 & x^6y + 2x^2 & 22 \\
 xy^2 & 11 & x^7y + x^3y^2 + x^3 & 25 & x^7y + x^3y^2 + x^3 & 25 \\
 x^4 & 12 & x^4 & 12 & y & 4 \\
 x^3y & 13 & x^5 + 2xy & 15 & x^5 + 2xy & 15 \\
 x^2y^2 & 14 & x^6y^2 + x^6 + x^2y & 26 & x^6y^2 + x^6 + x^2y & 26 \\
 x^5 & 15 & x^7 & 21 & x^3y & 13 \\
\end{array}
$$

Table 2. Algorithm 1 for $q = 3$.

In the first column we list the monomials in $M_4$. The second and sixth columns of this table allows us to compute $\Delta(m)$ for all values of $m$. For example, $\Delta(10) = \#\Phi(10, 21) = 6$. If $m > 15$ we apply the duality property; for example $\Delta(21) = \#\Phi(21, 10) = \#\Phi(21, 10) = 6$.

We obtain quantum codes over $\mathbb{C}^3$ with parameters $[[27, 1, 19; 16]]_3$, $[[27, 4, 16; 13]]_3$, $[[27, 6, 13; 9]]_3$, $[[27, 9, 10; 6]]_3$, $[[27, 13, 7; 4]]_3$, and $[[27, 16, 4; 1]]_3$. All previous codes have entanglement-assisted quantum Singleton defect equal to 6. An important feature of the first three examples above is that there is no (quantum) stabilizer code attaining the respective parameters.

Continuing the analysis for EAQECCs over $\mathbb{C}^4$, we derive the codes $[[64, 1, 49; 45]]_4$, $[[64, 5, 42; 35]]_4$, $[[64, 16, 30; 22]]_4$, $[[64, 24, 21; 12]]_4$, $[[64, 33, 14; 7]]_4$, $[[64, 35, 12; 3]]_4$, and $[[64, 39, 8; 1]]_4$. They have entanglement-assisted quantum Singleton defect equal to 12, 12, 12, 12, 10, and 12. Lastly, considering EAQECCs over $\mathbb{C}^5$, the codes have parameters equal to $[[125, 1, 101; 96]]_5$, $[[125, 9, 91; 84]]_5$, $[[125, 15, 81; 70]]_5$, $[[125, 36, 56; 41]]_5$, $[[125, 54, 41; 29]]_5$, $[[125, 70, 26; 15]]_5$, and $[[125, 90, 10; 1]]_5$, which have entanglement-assisted quantum Singleton quantum defect equal to 20, 20, 20, 20, 20, 20, and 18. Regarding parameters attainability using stabilizer codes, we have that the first three and the first four EAQECCs over $\mathbb{C}^4$ and over $\mathbb{C}^5$, respectively, are unique. We summarize these results in Table 3, where GV stands for the Gilbert-Varshamov bound [7].

5. $q$-th powers of Hermitian codes

The computation of $\Delta(m)$ in the previous algorithm relies on the comparison of $\nu(f_i)$ and $\nu(\phi_i)$ for all $f_i \in M_4$. The hard part of this algorithm is the computation (and storage) of all reductions $\nu(f_i^q)$, $1 \leq i \leq m^*$. However, in practice, the algorithm only requires the
Table 3. Examples of code’s parameters and comparative analysis by means of coding bounds.

| Parameters          | Singleton defect | Exceeding GV |
|---------------------|------------------|--------------|
| $[27, 1, 19; 16]_3$ | 6                | ✓            |
| $[27, 4, 16; 13]_3$ | 6                | ✓            |
| $[27, 13, 7; 4]_3$  | 6                | ✓            |
| $[27, 16, 4; 1]_3$  | 6                | ✓            |
| $[64, 5, 42; 35]_4$ | 12               | ✓            |
| $[64, 16, 30; 22]_4$| 12               | ✓            |
| $[64, 35, 12; 3]_4$ | 10               | ✓            |
| $[64, 39, 8; 1]_4$  | 12               | ✓            |
| $[125, 1, 101; 96]_5$| 20              | ✓            |
| $[125, 9, 91; 84]_5$| 20              | ✓            |
| $[125, 36, 56; 41]_5$| 20            | ✓            |
| $[125, 70, 26; 15]_5$| 20          | ✓            |
| $[125, 90, 10; 1]_5$| 18             | ✓            |

5.1. Reducing $q$-th powers. Let us recall that the reduction $\nu'(f)$ of a polynomial $f \in \mathcal{L}(\infty) = \mathbb{F}_q[x, y]$ is obtained by performing the substitutions

$$(R1) \quad \nu'(y^q) = x^{q+1} - y \quad \text{and} \quad (R2) \quad \nu'(x^q) = x$$

as many times as possible. In other words, $\nu'(f)$ is the unique (up to multiplication by a constant $\lambda \in \mathbb{F}_q^*$) reduced polynomial in the coset of $f$ modulo the ideal $(x^{q+1} - y^q - y, x^q - x) \subset \mathcal{L}(\infty)$, with respect to $\nu$. The next lemma gives some properties of $\nu'$.

**Lemma 11.** Let $g_1, g_2 \in \mathbb{F}_q[x, y]$. The following properties hold.

(a) $\nu'(g_1 + g_2) = \nu'(g_1) + \nu'(g_2)$;
(b) $\nu'(g_1g_2) = \nu'(\nu'(g_1)\nu'(g_2))$.

**Proof.** Both properties follow from the fact that the natural map $\mathbb{F}_q[x, y] \to \mathbb{F}_q[x, y]/(x^{q+1} - y^q - y, x^q - x)$ is a ring homomorphism. \hfill $\square$
As a notation, given a non-negative integer $\delta$, we write

$$m(\delta) = \begin{cases} 0 & \text{if } \delta = 0 \\ \delta \mod (q^2 - 1) & \text{if } q^2 - 1 \nmid \delta \\ q^2 - 1 & \text{if } q^2 - 1 | \delta \text{ and } \delta \neq 0. \end{cases}$$

That is, for $\delta \neq 0$, $m(\delta)$ is the remainder of $\delta$ modulo $q^2 - 1$ in the interval $[1, q^2 - 1]$ rather than $[0, q^2 - 2]$ as usual. In particular $m(\delta) \leq q^2 - 1$.

**Lemma 12.** Let $\delta_1, \delta_2$ be nonnegative integers. The following properties hold.

(a) $m(\delta_1 + \delta_2) = m(m(\delta_1) + \delta_2) = m(m(\delta_1) + m(\delta_2))$.

(b) If $\delta_2 > 0$ then $m(q^2 - 1 + \delta_2) = m(\delta_2)$.

Given nonnegative integers $\delta, \mu$, we denote the binomial coefficient modulo $p$ as

$$\binom{\mu}{\delta}_p \equiv \binom{\mu}{\delta} \mod p$$

where $p$ is the characteristic of $F_q$.

**Proposition 13.** Let $f = x^a y^b$ with $0 \leq a$ and $0 \leq b < q$. Then we have

$$r'(f^q) = \sum_{j=0}^{b} (-1)^j \binom{b}{j} p^{m(f)-j(q+1)} y^j.$$

**Proof.** If $b = 0$ then only the reduction $\text{(R2)}$ is involved in $r'$. If $a = 0$, according to the Newton’s binomial formula and Lemma 11 we have

$$r'(y^b)^q = r'((x^{q+1} - y)^b) = r' \left( \sum_{j=0}^{b} (-1)^j \binom{b}{j} p^{x^{m(aq)+(b-j)(q+1)} y^j} \right) = \sum_{j=0}^{b} (-1)^j \binom{b}{j} p x^{(b-j)(q+1)} y^j.$$

If $ab > 0$, according to Lemma 11 and the previous computations we have

$$r'((x^a y^b)^q) = \sum_{j=0}^{b} (-1)^j \binom{b}{j} p^{x^{m(aq)+(b-j)(q+1)} y^j} = \sum_{j=0}^{b} (-1)^j \binom{b}{j} p x^{m(aq)+(b-j)(q+1)} y^j$$

since all summands in the last expression are reduced monomials. □
5.2. Computing $\nu(f^q)$). Case $q$ prime. From Proposition 13 we can deduce the values $\nu(f^q)$, for $f_i \in M_\ast$ we need to run our algorithm. When $q = p$ is a prime number, then all binomial coefficients in the formula of Proposition 13 are non-zero. When $q$ is not a prime then some of these binomial coefficients may vanish, and the description of $\nu(f^q)$ becomes more involved. We first study the case in which $q$ is a prime number. The case $q$ not a prime will be treated in the next subsection.

Let $f \in M_\ast$. We will write the order $\nu(f)$ as $uq^2 + sq + t$ with $0 \leq s, t < q$ and $0 \leq u \leq (q + 1)/2$. Note that this representation is unique.

**Proposition 14.** Let $q = p$ be a prime, and let $f \in M_\ast$. Write $\nu(f) = uq^2 + sq + t$ with $0 \leq u \leq (q + 1)/2$ and $0 \leq s, t < q$. Then

\[
\nu(f^q) = \begin{cases} 
  sq^2 + (u + t)q & \text{if } s \geq t \text{ and } u + t < q^2 - sq; \\
  q^3 - 2q^2 + (u + t)q + 1 & \text{if } s \geq t \text{ and } u + t \geq q^2 - sq; \\
  q^3 - q^2 + (u + t - 1)q + s + 1 & \text{if } s < t \text{ and } u + t \leq q + s + 1; \\
  q^3 - 2q^2 + (u + t - 1)q + s + 2 & \text{if } s < t \text{ and } u + t > q + s + 1.
\end{cases}
\]

**Proof.** The assumption $\nu(f) = uq^2 + sq + t$ implies $f = x^{uq^2+st}y^t$ and therefore

\[
\nu'(f^q) = \sum_{j=0}^{t} (-1)^j \binom{t}{j}_p x^{m(uq^2+sq+t-j(q+1))} y^j.
\]

from Proposition 13. Since $t < q = p$, all binomial coefficients are nonzero modulo $p$, so it is enough to find the highest order among all the summands in the previous expression. To do this, we use the following observation. According to Lemma 12 (b), when the condition $(S): sq + u + t > j(q + 1)$ holds, then $m(uq^2 + sq + t - j(q + 1)) = m(sq + t + u - j(q + 1))$. In particular $(S)$ is always satisfied for $j = 0$, and for all values of $j$ when $s \geq t$. Let us distinguish three separate cases.

**Case 1:** If $s \geq t$ and $sq + u + t < q^2$, as above we have $m(sq + u + t - j(q + 1)) = sq + u + t - j(q + 1)$. The maximum order among the summands of $\nu'(f^q)$ is $sq^2 + (u + t)q$, obtained for $j = 0$.

**Case 2:** If $s \geq t$ and $sq + u + t \geq q^2$, then $s = q - 1, t \geq 1$ and $sq + u + t \leq q^2 + q$, hence $m(sq + u + t - j(q + 1)) = sq + u + t - (q^2 - 1)$ if $j = 0$ and $m(sq + u + t - j(q + 1)) = sq + u + t - j(q + 1)$ if $j > 0$. Thus the maximum order in $\nu'(f^q)$ is $q^3 - 2q^2 + (u + t)q + 1$, which is obtained for $j = 1$.

**Case 3:** If $s < t$, then $u > 0$. Note that the condition $(S)$ is satisfied for all $j \leq s$. Since $sq + u + t < q^2$, in this range we have $m(sq + u + t - j(q + 1)) = sq + u + t - j(q + 1)$. So the maximum order among the summands of $\nu'(f^q)$ corresponding to $j \leq s$ is $sq^2 + (u + t)q$, which is obtained for $j = 0$. Consider now the summand corresponding to $j = t$. Since $uq^2 + sq + t - t(q + 1) = (u - 1)(q^2 - 1) + (q + s - t)q + u - 1$, we have $m(uq^2 + sq + t - t(q + 1)) = (q + s - t)q + u - 1$ and the corresponding monomial has order $(q + s - t)q^2 + (u - 1)q + t(q + 1)$, which is bigger than the orders we have obtained for
We now have a necessary condition for two functions \( f, f' \) which can be rearranged to obtain the result.

Proof. If \( \nu(q) = \nu(q') \) holds for two functions \( f, f' \in M \). However, we will postpone this until after Proposition 15, which concerns the \( \nu \)-values of all monomials in \( f^q \). To state this proposition, we use \( \text{supp} f \) to denote the monomial support of \( f \). That is, if \( f = \sum c_{a,b} x^a y^b \in \mathbb{F}_{q^2}[x, y] \), we define \( \text{supp} f = \{ x^a y^b | c_{a,b} \neq 0 \} \).

Proposition 15. Let \( f \in M \). Then

\[
\text{supp}(f^q) \subseteq \{ f_j \in M : \nu(f_j) \equiv q \nu(f) \pmod{q^2 - 1} \}.
\]

Furthermore, if \( q + 1 \) divides \( \nu(f) \), then \( \nu(f) \equiv q \nu(f) \pmod{q^2 - 1} \).

Proof. If \( f = 1 \) then both results are clear. Let us assume \( f \neq 1 \). We have \( \nu(f^q) = q \nu(f) \), so in particular, \( \nu(f^q) \equiv q \nu(f) \pmod{q^2 - 1} \). Thus, it suffices to show that when applying each of the reductions (R1) and (R2) to \( f^q \), the orders of all resulting monomials remain in the original equivalence class.

Hence, let \( f^q = x^a y^b \). If \( a < q^2 \) and \( b < q \), then the result is immediate since \( \nu(x^a y^b) = x^a y^b \). If \( a \geq q^2 \), we can apply (R2) once to obtain the monomial \( x^{a-(q^2-1)} y^b \). This has order \( \nu(x^a y^b) \) - \( q(q^2 - 1) \), meaning that the equivalence class modulo \( q^2 - 1 \) is preserved. If \( b \geq q \), applying (R1) gives two monomials \( x^{a+(q+1)} y^{b-q} \) and \( x^{a} y^{b-(q-1)} \). These have orders \( \nu(x^a y^b) \) and \( \nu(x^a y^b) \) - \( q(q^2 - 1) \), respectively. Again, the resulting orders are in the same equivalence class as \( \nu(x^a y^b) \). This proves the first claim of the proposition.

For the second claim, one has that \( q + 1 \mid \nu(f) \) implies \( (q - 1) \nu(f) \equiv 0 \pmod{q^2 - 1} \), which can be rearranged to obtain the result.

Corollary 16. Let \( f, f' \in M \). If \( \nu(f^q) = \nu(f'^q) \) then \( \nu(f) \equiv \nu(f') \pmod{q^2 - 1} \).

Proof. Proposition 15 tells us that \( \nu(f^q) \equiv q \nu(f) \pmod{q^2 - 1} \), and similarly for \( f'^q \). Combining these equivalences gives the result.

We now have a necessary condition for two functions \( f, f' \in M \) to satisfy \( \nu(f^q) = \nu(f'^q) \). This can be used to bound the number of reductions in our algorithm in Section 4, since the reduction \( s \) is only applied to the monomials \( f_i \) for which there exists
$f_j \in \mathcal{M}_*$ with $j < i$ and $\nu(\tau(f_j)) = \nu(\tau(f_i))$. The number of such $f_j$’s should be moderate, as the following proposition shows that the map $f \mapsto \nu(f) \mod (q^2 - 1)$ is quite uniformly distributed.

**Proposition 17.** (a) The map $\mathcal{M}_* \to \mathbb{Z}/(q^2 - 1)\mathbb{Z}$ given by $f \mapsto \nu(f) \mod (q^2 - 1)$ is surjective. 
(b) For $0 \leq k < q^2 - 1$, the number of monomials $f \in \mathcal{M}$ with $\nu(f) \equiv k \pmod{q^2 - 1}$ is

- $q + 2$ when $k = 0$
- $q + 1$ when $k \neq 0$ and $q + 1 \mid k$
- $q$ when $k \neq 0$ and $q + 1 \nmid k$

The proof of this result can be found in the Appendix. To illustrate the preceding results, we give the following example.

**Example 3.** Let $q = 5$, and consider $f = f_{24} = x^3y^3$ with $\nu(f_{24}) = 33 = q^2 + q + 3$. Thus, $\nu(f_{24}) \mod (q^2 - 1) = 9$. Turning to the $q$'th power, we see that $\tau(f_{24}^q) = 3x^{21}y^2 + 4x^{15}y^3 + x^9 + 2x^3y$. Table 4 lists the orders of the monomials in $\text{supp}(\tau(f_{24}^q))$ along with their orders modulo $q^2 - 1$. In each case, this remainder is 21, which is exactly $q\nu(f_{24}) \mod (q^2 - 1)$.

The remaining monomials $f_j$ with $\nu(f_j) \equiv 9 \pmod{q^2 - 1}$ are $f_{48} = x^9y^2$, $f_{72} = x^{15}y$, $f_{96} = x^{21}$, and $f_{119} = x^{21}y^4$. Thus, there are a total of 5 such monomials as predicted by Proposition 17. If we consider the support of their $q$'th powers, we obtain the four monomials in Table 4 and the monomial $f_{60} = x^9y^4$.

One may also note that there is a duality between these two sets of monomials. Namely, the monomial support of $\{f_{12}^q, f_{36}^q, f_{60}^q, f_{84}^q, f_{108}^q\}$ is exactly $\{f_{24}, f_{48}, f_{72}, f_{96}, f_{119}\}$.

| $j$ | $f_j$ | $\nu(f_j)$ | $\nu(f_j) \mod (q^2 - 1)$ |
|-----|-------|------------|-----------------------------|
| 108 | $x^{21}y^2$ | 117 | 21 |
| 84  | $x^{15}y^3$ | 93  | 21 |
| 36  | $x^9$      | 45  | 21 |
| 12  | $x^3y$    | 21  | 21 |

**Table 4.** Monomials in the support of $\tau(f_{24}^q)$ for $q = 5$

### 5.3. Computing $\nu(\tau(f_i^q))$. Case $q$ non-prime

In this section we describe $\nu(\tau(f_i^q))$ when $q = p^r$ is not a prime, analogous to what was done in Proposition 14 when $q$ is a prime number. Our study will rely on Lucas’ theorem, which relates the binomial coefficient $\binom{j}{i} \mod p$ to the $p$-ary representations of $t$ and $j$. In [10], the concept of a $p$-shadow is defined as follows. Let $t = \sum_{i=0}^{r-1} t_i p^i$ and $j = \sum_{i=0}^{r-1} j_i p^i$ be the $p$-ary representations of $t$ and $j$, where $0 \leq j \leq t < q$. If $j_i \leq t_i$ for all $i$, then $j$ is said to be in the $p$-shadow of $t$,
and we write \( j \leq_p t \). As a corollary to Lucas’ theorem \([6, 16]\), we have

\[
\binom{t}{j}_p \neq 0 \quad \text{if and only if} \quad j \leq_p t.
\]

Inspired by the \( p \)-shadow, we introduce the \( p \)-illumination.

**Definition 2.** Let \( q = p^r \) be a prime power and let \( 0 \leq j \leq t < q \) be integers. If \( t = \sum_{i=0}^{r-1} t_i p^i \) and \( j = \sum_{i=0}^{r-1} j_i p^i \) are the \( p \)-ary representations of \( t \) and \( j \), we let

\[
\rho_t(j) = \sum_{i=0}^{i^*} (j_i - t_i) p^i,
\]

where \( i^* \) is the largest index such that \( j_{i^*} > t_{i^*} \). If no such \( i^* \) exists, we let \( \rho_t(j) = 0 \). We call \( \rho_t(j) \) the \( p \)-illumination of \( j \) with respect to \( t \).

It is easy to verify that \( \rho_t(j) \) is a non-negative integer. Additionally, \( j - \rho_t(j) \) is in the \( p \)-shadow of \( t \), and it is the largest integer less than or equal to \( j \) that satisfies this property. This follows from the observation that \( j - \rho_t(j) \) has the same \( p \)-ary digits as \( t \) for indices \( 1, 2, \ldots, i^* \). Each increment up to \( j \) will have at least one of these digits greater than the corresponding \( p \)-digit of \( t \). These considerations yield the following lemma.

**Lemma 18.** Let \( q = p^r \) be a prime power, \( t < q \) and \( 0 \leq j \leq t \). If \( \rho_t(j) = 0 \), then \( \binom{t}{j}_p \neq 0 \). Otherwise, if \( \rho_t(j) > 0 \), then

\[
\binom{t}{j}_p = \binom{t}{j-1}_p = \cdots = \binom{t}{j-\rho_t(j)+1}_p = 0 \quad \text{and} \quad \binom{t}{j-\rho_t(j)}_p \neq 0.
\]

**Proof.** By (5), \( \binom{t}{j}_p \) is non-zero if and only if \( j' \leq_p t \). The observations immediately below Definition 2 imply that the first such \( j' \) below \( j \) is \( j - \rho_t(j) \), proving the lemma. \( \square \)

In other words, the largest integer \( j^* \leq j \) satisfying \( \binom{t}{j}_p \neq 0 \) is \( j^* = j - \rho_t(j) \). In addition, we can apply Lemma 18 to \( t - j \) and use the symmetry of the binomial coefficient, \( \binom{t}{j} = \binom{t}{t-j} \), to infer that \( \binom{t}{j}_p = \cdots = \binom{t}{j+\rho_t(t-j)-1}_p = 0 \) and \( \binom{t}{j+\rho_t(t-j)}_p \neq 0 \). That is, the smallest integer \( j^* \geq j \) satisfying \( \binom{t}{j}_p \neq 0 \) is \( j^* = j + \rho_t(t-j) \).

**Proposition 19.** Let \( q = p^r \) be a prime power, and let \( f \in \mathcal{M}_+ \). Write \( \nu(f) = uq^2 + sq + t \) with \( 0 \leq u \leq (q+1)/2 \) and \( 0 \leq s, t < q \). Then \( \nu(\tau(f^s)) \) equals

\[
\begin{cases}
  sq^2 + (u+t)q & \text{if } s \geq t, \; u+t < q^2 - sq; \\
  q^3 - 2q^2 + (u+t)q + 1 - (q^2 - 1)\rho_t(t - 1) & \text{if } s \geq t, \; u+t \geq q^2 - sq; \\
  q^3 - q^2 + (u+t - 1)q + s + 1 - (q^2 - 1)\rho_t(t - s - 1) & \text{if } s < t, \; u+t \leq q + s + 1; \\
  q^3 - 2q^2 + (u+t - 1)q + s + 2 - (q^2 - 1)\rho_t(t - s - 2) & \text{if } s < t, \; u+t > q + s + 1.
\end{cases}
\]
Proof. The proof of this result is similar to that of Proposition 14, in which we sought the summand $j$ providing the highest order in the writing of $r'(f^q)$ given in the Proposition 13. Let $j^*$ be the index of the summand providing the maximum order in our case $q$ not a prime. Following the proof of Proposition 14, in case 1 such maximum order is obtained for $j = 0$. Since $(\frac{t}{p}) \neq 0$, we have $j^* = 0$. In case 2, the maximum order comes from the summand corresponding to the smallest index $j^* \geq 1$ with $(\frac{t}{j^*}) \neq 0$, that is for $j^* = 1 + \rho_1(t - 1)$ according to Lemma 18. In case 3, the maximum order comes as well from the summand corresponding to the smallest index $j^* \geq j$ with $(\frac{t}{j^*}) \neq 0$, where $j = s + 1$ when $u + t \leq q + s + 1$ and $j = s + 2$ when $u + t > q + s + 1$. Such indices are $j^* = s + 1 + \rho_1(t - s - 1)$ and $j^* = s + 2 + \rho_1(t - s - 2)$ respectively. In all cases, it is enough to compute the orders of the summands corresponding to these $j^*$’s to obtain the stated formula. □

Note that when $\nu(f) \geq q^2$, the values $\nu(r'(f^q))$ are, in most cases, bigger when $q$ is a prime number than when it is not, $q = p^r$ with $r > 0$. Therefore, also $\Delta$ increases (and so the entanglement $c$ decreases) when $r$ increases.

5.4. Bounding the complexity of Algorithm 1. Having described the $q$’th powers given in the previous sections, we now use those results to bound the computational complexity of Algorithm 1. Since the most computationally costly part are the reductions $s$ – that is, lines 12 and 13 – we focus on bounding the number of such reductions as well as bounding the cost of each reduction.

**Proposition 20.** The total number of reductions $s$ in Algorithm 1 is $O(q^4)$, and each reduction requires $O(q)$ field operations.

Proof. First recall that $\phi_i = r(f_i^q)$. Thus, Proposition 15 ensures that all monomials $f_k$ in $\phi_i$ satisfy $\nu(f_k) \equiv q\nu(f_i) \pmod{q^2 - 1}$. If we consider some $\phi_j$ during the algorithm, similar arguments as in Corollary 16 show that the reduction $\phi_j = \phi_j - \phi_i$ can happen only if $\nu(f_j) \equiv \nu(f_i) \pmod{q^2 - 1}$. Thus, we bound the total number of reductions by an amortized analysis, grouping polynomials $\phi_i$ based on the equivalence class of $\nu(f_i)$.

Fix a $k \in \{0, 1, \ldots, q^2 - 2\}$, and consider all polynomials $\phi_i$ such that $\nu(f_i) \equiv k \pmod{q^2 - 1}$. By Proposition 17, there can be at most $q + 2$ such monomials. The first time the algorithm encounters such a $\phi_i$ no reduction will happen. The second time such a $\phi_i$ is found, there will be at most one reduction. The third time at most two and so forth. Thus, the total number of reductions required to process the $\phi_i$ with $\nu(f_i) \equiv k \pmod{q^2 - 1}$ cannot exceed

$$\sum_{j=1}^{q+2}(j - 1) = \frac{q(q + 1)}{2}.$$

Because there are $q^2 - 1$ possible values of $k$, the overall number of reductions during the algorithm is at most $\frac{1}{2}q(q + 1)(q^2 - 1)$, which is $O(q^4)$. 


To prove the claim on the number of field operations per reduction, note that each $\phi_i$ can contain at most $q + 2$ monomials by Propositions 15 and 17.

Since the reductions $s$ are what dominates the complexity of Algorithm 1, Proposition 20 implies that the number of field operations used during the algorithm is $O(q^5)$, which is significantly better than computing $c$ as in Proposition 2 using linear algebra methods. Namely, it is significantly better than computing the rank of $GG^*$, where $G$ is a generator matrix of the linear code and $G^*$ is the $q$-th power of the transpose matrix of $G$ (see [7, Proposition 3]).

We can optimize the algorithm even further by using the values of $\nu(r(f^q_i))$ found in Sections 5.2 and 5.3. Namely, the algorithm only requires the knowledge of $r(f^q_i)$ when $f_i$ is involved in some reduction $s$, which does not happen for all the values of $i$ (see for example the case $q = 3$ in Table 2).

Therefore, we can modify the algorithm as follows: we first compute the orders $\nu(r(f^q_i))$ from Propositions 14 and 19. If $f_i$ is not involved in any reduction $s$, then we will not calculate $r(f^q_i)$. Otherwise, when it is necessary to know this data, it is computed from Proposition 13, and the corresponding polynomials $\phi$’s from the reduction $s$. The obtained result is shown in Table 5 for $q = 3$. The reader can compare this table with Table 2, in which the unmodified process is shown.

| $f_i$ | $\nu(f_i)$ | $r(f^q_i)$ | $\nu(r(f^q_i))$ | $\phi_i$ | $\nu(\phi_i)$ |
|-------|-------------|-------------|------------------|-----------|----------------|
| 1     | 0           | $0$         | 0                | 0         | 0              |
| $x$   | 3           | $9$         | 0                | 0         | 0              |
| $y$   | 4           | $x^4 + 2y$  | 12               | $x^4 + 2y$ | 12             |
| $x^2$ | 6           | 18          | 0                | 0         | 0              |
| $xy$  | 7           | $x^7 + 2x^3y$ | 21          | $x^7 + 2x^3y$ | 21             |
| $y^2$ | 8           | 24          | 0                | 0         | 0              |
| $x^3$ | 9           | 3           | 0                | 0         | 0              |
| $x^2y$| 10          | 22          | 0                | 0         | 0              |
| $xy^2$| 11          | 25          | 0                | 0         | 0              |
| $x^4$ | 12          | $x^4$       | 12               | $y$       | 4              |
| $x^3y$| 13          | 15          | 0                | 0         | 0              |
| $x^2y^2$| 14      | 26          | 0                | 0         | 0              |
| $x^5$ | 15          | $x^7$       | 21               | $x^3y$    | 13             |

Table 5. Results of the modified algorithm for $q = 3$.

5.5. Computational results. We have computed the parameters of all Hermitian EAQECCs up to field size 16 using our algorithm. These lists of parameters reveal that many of the resulting EAQECCs exceed the Gilbert-Varshamov bound [7]. Once $C(m)$ is large...
enough to produce an EAQECC of non-zero dimension, all following values of \( m \) seem to produce codes exceeding the bound as well until the resulting entanglement \( c \) is ‘too small’ compared to the code length. More specifically, we have verified that the parameters of the EAQECCs exceed the Gilbert-Varshamov bound [7] for the values of \( c \) specified in Table 6. For lower values of \( c \), it is sometimes possible to find codes that exceed the bound, but it will not be true for all codes with this entanglement.

\[
\begin{array}{ccc}
q & n & c \\
2 & 8 & 0–3 \\
3 & 27 & 1–16 \\
4 & 64 & 3–45 \\
5 & 125 & 4–96 \\
7 & 343 & 10–288 \\
8 & 512 & 9–441 \\
9 & 729 & 14–640 \\
11 & 1331 & 38–1200 \\
13 & 2197 & 51–2016 \\
16 & 4096 & 45–3825 \\
\end{array}
\]

Table 6. Hermitian EAQECCs exceeding the GV bound.

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**References**

[1] A. Allahmadi, A. Alkenani, R. Hijazi, N. Muthana, F. Özbudak, and P. Solé. New constructions of entanglement-assisted quantum codes. *Cryptography and Communications*, 2021. doi:10.1007/s12095-021-00499-7.

[2] A. Ashikhmin, S. Litsyn, and M. Tsfasman. Asymptotically good quantum codes. *Physical Review A*, 63:032311, 2001. doi:10.1103/physreva.63.032311.

[3] D. Bartoli, M. Montanucci, and G. Zini. On certain self-orthogonal AG codes with applications to quantum error-correcting codes. *Designs, Codes and Cryptography*, 89:1221–1239, 2021. doi:10.1007/s10623-021-00870-y.

[4] T. Brun, I. Devetak, and M.-H. Hsieh. Correcting quantum errors with entanglement. *Science*, 314:436–439, 2006. doi:10.1126/science.1131563.

[5] A.R. Calderbank, E.M. Rains, P.M. Shor, and N.J.A. Sloane. Quantum error correction via codes over GF(4). *IEEE Transactions on Information Theory*, 44:1369–1387, 1998. doi:10.1109/18.681315.

[6] N. J. Fine. Binomial coefficients modulo a prime. *The American Mathematical Monthly*, 54:589, 1947. doi:10.2307/2304500.
[7] C. Galindo, F. Hernando, R. Matsumoto, and D. Ruano. Entanglement-assisted quantum error-correcting codes over arbitrary finite fields. *Quantum Information Processing*, 18:116, 2019. doi: 10.1007/s11128-019-2234-5.

[8] D. Gottesman. Class of quantum error-correcting codes saturating the quantum Hamming bound. *Physical Review A*, 54:1862–1868, 1996. doi:10.1103/physreva.54.1862.

[9] G. G. La Guardia. *Quantum Error Correction*. Springer International Publishing, 2020. doi:10.1007/978-3-030-48551-1.

[10] A. Guo, S. Kopparty, and M. Sudan. New affine-invariant codes from lifting. In *Proceedings of the 4th conference on Innovations in Theoretical Computer Science - ITCS’13*, pages 529–540. ACM Press, 2013. doi:10.1145/2422436.2422494.

[11] F. Hernando, G. McGuire, F. Monserrat, and J. J. Moyano-Fernández. Quantum codes from a new construction of self-orthogonal algebraic geometry codes. *Quantum Information Processing*, 19:117, 2020. doi:10.1007/s11128-020-2616-8.

[12] T. Høholdt, J. H. Van Lint, and R. Pellikaan. Algebraic geometry codes. In V. S. Pless, W. C. Huffman, and R. A. Brualdi, editors, *Handbook of coding theory*, pages 871–961. Elsevier, Amsterdam, 1998.

[13] A. Ketkar, A. Klappenecker, S. Kumar, and P.K. Sarvepalli. Nonbinary stabilizer codes over finite fields. *IEEE Transactions on Information Theory*, 52:4892–4914, 2006. doi:10.1109/tit.2006.883612.

[14] J.-L. Kim and G. L. Matthews. Quantum error-correcting codes from algebraic curves. In *Series on Coding Theory and Cryptology*, pages 419–444. World Scientific, 2008. doi:10.1142/9789812794017\_0012.

[15] R. Lidl and H. Niederreiter. *Finite Fields*. Cambridge University Press, 1996. doi:10.1017/cbo9780511525926.

[16] É. Lucas. *Théorie des nombres*. Gauthier-Villars et fils, Paris, 1891.

[17] R. Matsumoto. Improvement of Ashikhmin-Litsyn-Tsfasman bound for quantum codes. *IEEE Transactions on Information Theory*, 48:2122–2124, 2002. doi:10.1109/tit.2002.1013156.

[18] C. Munuera, W. Tenório, and F. Torres. Quantum error-correcting codes from algebraic geometry codes of Castle type. *Quantum Information Processing*, 15:4071–4088, 2016. doi:10.1007/s11128-016-1378-9.

[19] F. R. F. Pereira, R. Pellikaan, G. G. La Guardia, and F. M. de Assis. Application of complementary dual AG codes to entanglement-assisted quantum codes. In *2019 IEEE International Symposium on Information Theory (ISIT)*, pages 2559–2563. IEEE, 2019. doi:10.1109/isit.2019.8849685.

[20] Francisco Revson F. Pereira, Ruud Pellikaan, Giuliano Gadioli La Guardia, and Francisco Marcos de Assis. Entanglement-assisted quantum codes from algebraic geometry codes, 2019. arXiv:1907.06357.

[21] P. K. Sarvepalli and A. Klappenecker. Nonbinary quantum codes from Hermitian curves. In *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes*, pages 136–143. Springer Berlin Heidelberg, 2006. doi:10.1007/11617983\_\_13.

[22] Henning Stichtenoth. *Algebraic Function Fields and Codes*. Graduate Texts in Mathematics. Springer, 2 edition, 2009.

[23] H. Tiersma. Remarks on codes from Hermitian curves. *IEEE Transactions on Information Theory*, 33:605–609, 1987. doi:10.1109/tit.1987.1057327.

[24] K. Yang and P. V. Kumar. On the true minimum distance of Hermitian codes. In *Lecture Notes in Mathematics*, pages 99–107. Springer Berlin Heidelberg, 1992. doi:10.1007/bfb0087995.
Appendix A. Proof of Proposition 17

In this appendix we give the proof of Proposition 17, which relies on the following lemmata.

Lemma 21. Let $f = x^a y^b \in \mathcal{M}$, and assume that any $f_j \in \mathcal{M}$ with $\nu(f_j) \equiv \nu(f) \pmod{q^2 - 1}$ satisfies $\nu(f_j) \leq \nu(f)$. Then $(q - 2)(q + 1) < a \leq (q - 1)(q + 1)$ and $0 < b < q$.

Proof. Assume first that $b = 0$. Then $f' = x^a y^{q-1}$ satisfies $\nu(f') = \nu(f) + q^2 - 1$, meaning that $\nu(f') > \nu(f)$ and $\nu(f') \equiv \nu(f) \pmod{q^2 - 1}$. But this contradicts the choice of $f$. Thus, let $b > 0$, and consider the case $a \leq (q - 2)(q + 1)$. As before, we can find $f' = x^{a+(q+1)} y^{b-1} \in \mathcal{M}$ satisfying $\nu(f') = \nu(f) + q^2 - 1$, which is again a contradiction. □

Lemma 22. Let $f = x^a y^b \in \mathcal{M}$. Then $q + 1 \mid \nu(f)$ if and only if $q + 1 \mid a$.

Proof. We have $\nu(f) = aq + b(q + 1)$, which is divisible by $q + 1$ if and only if $q + 1 \mid a$ because $q$ and $q + 1$ are relatively prime. □

Proposition 17. (a) The map $\mathcal{M}_* \rightarrow \mathbb{Z}/(q^2 - 1)\mathbb{Z}$ given by $f \mapsto \nu(f) \pmod{q^2 - 1}$ is surjective.

(b) For $0 \leq k < q^2 - 1$, the number of monomials $f \in \mathcal{M}$ with $\nu(f) \equiv k \pmod{q^2 - 1}$ is

- $q + 2$ when $k = 0$
- $q + 1$ when $k \neq 0$ and $q + 1 \nmid k$
- $q$ when $k \neq 0$ and $q + 1 \mid k$

Proof. (a) is clear. For (b) we show how to construct the given number of monomials in each of the three cases. To see that there cannot be any additional monomials, note that there are $q - 2$ values of $k$ such that $k \neq 0$ and $q + 1 \mid k$ and $q^2 - q$ values such that $k \neq 0$ and $q + 1 \nmid k$. Hence, the total number of monomials considered is $(q + 2) + (q - 2)(q + 1) + (q^2 - q)q = q^3$ which is exactly the number of monomials in $\mathcal{M}$.

Denote by $f^* = x^{a^*} y^{b^*} \in \mathcal{M}$ the monomial of largest order such that $\nu(f^*) \equiv k \pmod{q^2 - 1}$. Consider now the following two sequences of rational functions

\begin{align*}
  (6) & \quad x^{a^* - i(q+1)} y^{b^* + i}, \quad i = 0, 1, \ldots, q - 1 - b^*
  \\
  (7) & \quad x^{a^* - (q-1-b^*)+i(q+1)} y^i, \quad i = 0, 1, \ldots, b^*.
\end{align*}

The reader may note that all expressions in (6) and (7) are distinct; setting $i = 0$ in (6) gives $f^*$, and all $f$ in (6) and (7) satisfy $\nu(f) \equiv \nu(f^*) \pmod{q^2 - 1}$. Thus, we only need determine which of these rational functions are in $\mathcal{M}$.

Consider first an $f = x^a y^b$ from (6). It is immediately clear that $0 \leq b < q$, so $f \in \mathcal{M}$ if and only if $0 \leq a < q^2$. That $a < q^2$ follows immediately from $a \leq a^*$ combined with the fact that $f^* \in \mathcal{M}$. To prove that $a$ is nonnegative, note that $f^*$ satisfies Lemma 21
by definition. This ensures that \( a^* = (q - 2)(q + 1) + t \) for some \( 0 < t \leq q + 1 \) and that \( b^* > 0 \). The latter implies that \( i \leq q - 2 \), and we obtain

\[
a = (q - 2)(q + 1) + t - i(q + 1) \geq (q - 2)(q + 1) + t - (q - 2)(q + 1) = t > 0.
\]

Hence, the rational functions in (6) are in \( M \) for all values of \( k \).

Considering now \( f = x^a y^b \) from (7), it is again straight-forward to check that \( 0 \leq b < q \) and \( a < a^* < q^2 \) as above. Additionally, if \( i \leq b^* - 1 \), we have

\[
a = (q - 2)(q + 1) + t - (q - 1 - b^* + i)(q + 1) \geq (q - 2)(q + 1) + t - (q - 2)(q + 1) > 0,
\]

so these are all in \( M \) regardless of the value of \( k \). If \( i = b^* \), however, we obtain \( a = (q - 2)(q + 1) + t - (q - 1)(q + 1) \), which is only non-negative if \( t = q + 1 \). This happens if and only if \( q + 1 \mid a \), which in turn happens if and only if \( q + 1 \mid k \) by Lemma 22.

Summing up, (6) and (7) gives \( (q - b^*) + b^* = q \) monomials if \( q + 1 \nmid k \) and \( (q - b^*) + (b^* + 1) = q + 1 \) monomials if \( q + 1 \mid k \).

The final part of the proof is the following observation. The monomial 1 is not contained in (6) or (7) for any \( k \). Thus, when \( k = \nu(1) = 0 \), we obtain one extra monomial in addition to the \( q + 1 \) from (6) and (7).\[\square\]