Comment on “Born’s rule for arbitrary Cauchy surfaces”

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Received: date / Accepted: date

Abstract A recent article has raised the question of how to generalize the Born rule from non-relativistic quantum theory to curved spacetimes and claimed to answer it for the special-relativistic case (Lienert and Tumulka, Lett. Math. Phys. 110, 753 (2019)). The proposed generalization originated in prior works on ‘hypersurface Bohm-Dirac models’ as well as approaches to relativistic quantum theory developed by Bohm and Hiley. In this comment, we raise three objections to the rule and the broader theory in which it is embedded. In particular, to address the underlying assertion that the Born rule is naturally formulated on a spacelike hypersurface, we provide an analytic example showing that a spacelike hypersurface need not remain spacelike under proper time evolution—even in the absence of curvature. We finish by proposing an alternative ‘curved Born rule’ for the one-body case on general spacetimes, which overcomes these objections, and in this instance indeed generalizes the one Lienert and Tumulka attempted to justify. The respective mathematical theory is almost analogous for the conservation of charge.
and mass, being two additional examples of physical quantities obtained from integrating a scalar field over particular hypersurfaces. Our approach can also be generalized to the many-body case, which shall be the subject of a future work.

**Keywords** Integral conservation laws · Continuity equation · Born rule · Detection probability · Multi-time wave function · Spacelike hypersurface

**Mathematics Subject Classification (2010)** 81P05 · 81P15 · 81P16 · 81T99 · 83C99 · 35Q75

1 **Introduction**

A recent article [23] has raised the question of how to generalize the Born rule from non-relativistic quantum theory to curved spacetimes and claimed to answer it for the special-relativistic case. In the non-relativistic case, the Born rule determines the probability of detecting one or several particles in a given region of configuration space for fixed time, and it holds both for distinguishable and non-distinguishable particles. The proposed generalization, at least in the one-body case, does not rely on the symmetries of Minkowski spacetime or any geometric structures tied to it. In this instance, a generalization to curved spacetime is immediate. This curved-spacetime version applies to spacelike hypersurfaces in globally hyperbolic spacetimes and makes use of the volume form induced by the pullback metric to set up the integral. Following Dürr et al. (cf. Chap. 9 in [15]), the aforementioned approach to the relativistic Born rule had been developed in various prior works [14,36,6,13] in an attempt to find a relativistic theory of Bohmian mechanics, which goes back to one of their own articles (cf. §8 to §12 in [13]) and works composed by Bohm and Hiley (cf. §10.4 and §10.5 in [8]). The question is of potential foundational importance not only for the enterprise of Bohmian mechanics but for the general formulation of a relativistic quantum theory on curved spacetime. Therefore, as Lienert and Tumulka also acknowledged with the publication of their work [23], the question deserves both attention and care. We shall take their recent attempt of justification as an opportunity to comment on the proposed generalization as well as to point out what we consider a fundamental conceptual flaw of such ‘hypersurface Bohm-Dirac models’. We finish by proposing what we believe to be the most general ‘curved Born rule’ for the one-body case (up to questions of regularity). This comment is mostly self-contained; readers only interested in our resolution to the problem are invited to skip to Sec. 4 referring to the prior sections and cited works as needed.

Throughout this work, we use the (+−−−) convention, set the velocity of light equal to 1, use d for the exterior derivative and \( L_X \) for the Lie derivative along the vector field \( X \). As regularity questions are not our primary concern here, we work in the category of smooth manifolds and mappings. If \( \varphi \) is such a mapping, its pullback is \( \varphi^* \), and \( \varphi|_A \) is the map with domain restricted to \( A \). A dot · denotes matrix multiplication and tensor contraction of adjacent entries.

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1 The discovery of the rule is often credited to Born [9,10,11], but it was Pauli [29] who first expressed it fully (cf. [4]).

2 A common approach to this issue is ‘quantum field theory on curved spacetime’, an offspring of algebraic quantum field theory [19]. See e.g. [21] and [12] for an introduction to the former. Recently, Miller et al. [26] have proposed their own version of such a quantum theory. However, the discussion here is not explicitly tied to any of these research programs.
2 The Born rule in ‘Hypersurface Bohm-Dirac Models’

For coherence, we shall first recapitulate the proposed generalization of the Born rule for the one-body case, which is part of the theory of ‘Hypersurface Bohm-Dirac Models’ (cf. [6,13] and Rem. 6 in [23]). Though the many-body case was also considered in [23] and our criticism refers to it as well, it is sufficient to consider the one-body case to clarify our central concerns. Furthermore, we believe that outlining the rule in the general-relativistic setting expresses the underlying assumptions more coherently, which is why, contrary to the original article [23], we do not restrict ourselves to the special-relativistic setting here. This choice will also ease comparison of the rule outlined here and the one we suggest in Sec. 4.

In order to formulate the rule on a 4-spacetime \( Q \) with metric \( g \), one first requires a ‘Lorentz frame’. Though neither Dürr et al. nor Lienert and Tumulka give a mathematical definition thereof and only consider Minkowski spacetime explicitly, the context reveals that the general construction they have in mind assumes that there exists a function \( \tau \in C^\infty (Q, \mathbb{R}) \) with future-directed timelike gradient \( \nabla \tau \) such that the level sets \( \Sigma_{\tau_0} := \tau^{-1} (\tau_0) \) foliate \( Q \) into orientable Cauchy surfaces. Note that the regular value theorem assures that all integral manifolds \( \Sigma_{\tau_0} \) are embedded hypersurfaces, and timelikeness of \( \nabla \tau \) implies that they are spacelike. Taking space-orientedness of \( Q \) as given in order to equip the integral manifolds with a natural orientation, the only topological restriction on \( Q \) is therefore that it be globally hyperbolic (cf. [27,5]).

The Born rule is now formulated on any given \( \Sigma_{\tau_0} \). One requires the following quantities: The vector field

\[
\mathbf{n} := \frac{\nabla \tau}{\sqrt{g(\nabla \tau, \nabla \tau)}},
\]

restricted to \( \Sigma_{\tau_0} \), is the respective future-directed normal vector field. Using the natural inclusion \( \iota_{\tau_0} : \Sigma_{\tau_0} \to Q \), one equips \( \Sigma_{\tau_0} \) with the Riemannian (negative pullback) metric \( -\iota_{\tau_0}^* g \). Denote by \( \nu_{\tau_0} \) the volume form induced by \( -\iota_{\tau_0}^* g \). In the one-body case, the respective quantum theory provides a current density vector field \( J \) (e.g. the Dirac current in Minkowski spacetime) satisfying the normalization condition

\[
\int_{\Sigma_{\tau_0}} g(J, n) \, \nu_{\tau_0} = 1.
\]

So, formally, the quantity

\[
\rho_0 := \left. g(J, n) \right|_{\Sigma_{\tau_0}}
\]

is a probability density with respect to the measure \( A \mapsto \int_A \nu_{\tau_0} \) on the Borel sets of \( \Sigma_{\tau_0} \) (cf. §3.1 in [23]). We can therefore ask for the probability to find a particle in a given ‘region’ on \( \Sigma_{\tau_0} \) (i.e. a given Borel set). At least for the one body case, this is the Born rule the title article attempted to justify.

But is there any straightforward justification for this choice? Indeed, if the divergence \( \text{div}(J) \) vanishes identically on \( Q \), which is for instance the case for the one-body Dirac theory in Minkowski spacetime (cf. §12.2 in [20]), then we may apply the divergence theorem to obtain probability conservation: For simplicity, assume \( \Sigma_{\tau_1} \) and \( \Sigma_{\tau_0} \) with \( \tau_1 > \tau_0 \) are 3-manifolds with corners (see
The two spacelike integral 3-manifolds $\Sigma_{\tau_0}$ and $\Sigma_{\tau_1}$ (with corners) are connected by the 3-manifold $\mathcal{N}$ (with corners). Together they form the boundary of a compact 4-manifold $\mathcal{M}$ (with corners), depicted here as a deformed cylinder. The arrows indicate the vector field $J$, which is tangent to $\mathcal{N}$. Vanishing divergence of $J$ assures that a probability measure on $\Sigma_{\tau_0}$ induces a probability measure on any $\Sigma_{\tau_1}$ obtained in this manner.

§2 in [33] and references therein) and their boundaries are connected by another 3-manifold with corners $\mathcal{N}$ tangent to $J$, so that the union

$$\partial \mathcal{M} = \Sigma_{\tau_1} \cup \mathcal{N} \cup \Sigma_{\tau_0}$$

forms the boundary of a compact 4-manifold with corners $\mathcal{M}$. The situation is depicted in Fig. 1. Denoting by $\mu$ the volume form on $Q$ induced by $g$, the divergence theorem yields

$$0 = \int_{\mathcal{M}} \text{div}(J) \, \mu = \int_{\Sigma_{\tau_1}} g(J, n) \nu_{\tau_1} - \int_{\Sigma_{\tau_0}} g(J, n) \nu_{\tau_0}.$$ 

Thus we also have a probability measure on $\Sigma_{\tau_1}$.

Furthermore, in Minkowski spacetime and standard coordinates, as it is commonly done, we may set

$$J^0 = \rho \quad \text{and} \quad J^a = \rho v^a$$

for $a \in \{1, 2, 3\}$, so that div $J = 0$ is simply the non-relativistic continuity equation (cf. p. 2063 in [6]).

Superficially, the overall approach appears physically natural and conceptually consistent.

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3 A time-orientable Lorentzian manifold is space-orientable if and only if it is orientable (cf. p. 30 in [32]). Of course, a manifold admits a volume form if and only if it is orientable. Thus, no additional assumption is needed here.

4 See e.g. Thm. 16.32 in [22] for the Riemannian case applying to manifolds with boundary. The proof for the case of manifolds with corners in Lorentzian geometry is analogous (excluding the lightlike case, of course). A related discussion on the matter is provided in §3.0.2 in [35].

5 Subtleties regarding the compactness assumption on $\mathcal{M}$ and the most suitable $\sigma$-algebra on the hypersurfaces shall not concern us here, as they have no impact on the main argument.
3 Three objections

‘Lorentz frames’ In textbook treatments of the special theory of relativity one often encounters the notion of an ‘inertial frame (of reference)’, referring, modulo Poincaré transformations, to a choice of standard coordinates on Minkowski spacetime. In the general-relativistic case, an adequate mathematical implementation of the concept of a ‘frame of reference’ was already given by Walker in 1935 (cf. §2 [37]). For details we refer to the original articles on the Fermi-Walker derivative [17,37], the article by Mast and Strathdee [24], as well as the first author’s thesis [32] (§3.2 and §3.4.2 in particular).

Our objection is that the special-relativistic textbook treatment of inertial frames of reference – which the employed concept of a ‘Lorentz frame’ intends to generalize – is actually derived from the aforementioned general concept (via the use of normal coordinates), so that one may not assume without justification that this is an adequate point of departure for generalization. Indeed, a frame of reference (e.g. at a point) does in general not yield a foliation of the spacetime into spacelike hypersurfaces without further choices and assumptions. Clearly, this is not a mere criticism of choice of terminology, but one of underlying concepts.

Spacelike hypersurfaces In the previous section it was shown that an underlying assumption of the proposed Born rule is that the hypersurfaces, on which the particle is found, be spacelike. Naive attempts to relax this assumption run into trouble, since the pullback metric $-\iota^*\tau_0g$ is degenerate whenever the tangent space $T_{q}\Sigma_{\tau_0}$ is lightlike and thus the integrand becomes ill-defined. But is a generalization to the non-spacelike case necessary?

We answer this question in the affirmative based on two observations:

i) First, let us consider the related question on how one would generally define the ‘time evolution’ of a hypersurface $\Sigma_0$. For an individual point this is done via an observer curve, i.e. a curve with future-directed, timelike unit tangent vector at each parameter value $\tau$. At $\tau = 0$ the curve intersects the given point. Hence, if we generalize this to hypersurfaces – a set of points – and do not allow self-intersections or tearing, we have to consider a vector field $X$ whose integral curves are observer curves, i.e. observer (vector) fields. If we then apply the flow $\Phi$ of $X$ to the hypersurface (and assume the respective integral curves are defined for the given parameter $\tau$), we obtain a new hypersurface $\Sigma_{\tau} := \Phi_{\tau}(\Sigma_0)$ diffeomorphic to $\Sigma_0$. One may now ask the following question:

Does the flow of an observer vector field preserve the causal character of a hypersurface, e.g. does an initially spacelike hypersurface remain spacelike under time evolution?

As the following counterexample shows, this is not true, even in the absence of curvature.

6 Lienert and Tumulka only made this assumption up to sets of (Borel) measure zero, yet the argument we make here does not change in any substantial manner by this complication. They also made a remark on the timelike case (Rem. 8), but did not provide any full account. Regarding the latter, we will show in Sec. that there is no need for any causality assumption on the hypersurface.
Example 1 In Minkowski 3-spacetime and standard coordinates \((t,x,y)\), consider the observer field with values

\[
\vec{X}(t,x,y) = \omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \tag{7}
\]

\[
X^0(t,x,y) = \sqrt{1 + \left(\vec{X}(t,x,y)\right)^2} \tag{8}
\]

for some \(\omega > 0\). This describes a rotation about the central axis \(\mathbb{R} \times \{0,0\}\). As \((7)\) describes a linear vector field (cf. §1.1 in [18] and Ex. 3.2.8 in [34]) and the respective matrix factor is a rotation, we find that

\[
\Phi_\tau(t_0,x_0,y_0) = \begin{pmatrix} \sqrt{1 + \omega^2(x_0^2 + y_0^2)} \tau + t_0 \\ x_0 \cos(\omega \tau) - y_0 \sin(\omega \tau) \\ x_0 \sin(\omega \tau) + y_0 \cos(\omega \tau) \end{pmatrix} \tag{9}
\]

Let us start with the \((t_0 = 0)\)-hypersurface \(\Sigma_0\) and set \(r_0 := \sqrt{x_0^2 + y_0^2}\). We then see that the (pushforward of the) radial vector field

\[
\frac{\partial}{\partial r} \bigg|_{(t_0,x_0,y_0)} = \frac{1}{r_0} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{10}
\]

initially tangent to \(\Sigma_0\), becomes lightlike at

\[
\tau = \frac{1 + \omega^2 r_0^2}{\omega^2 r_0} \quad \text{for} \quad r_0 > 0 \tag{11}
\]

and stays timelike afterwards. The time evolution of \(\Sigma_0\) is illustrated qualitatively in Fig. 2.

The simplicity of Ex. 1 suggests that the behavior is generic. Indeed, we recently gave another example (Ex. 3) in [33] where this effect was also exhibited in a curved spacetime. Together with our first point in this section, there is hence no physical reason to exclude such time evolutions for the initial hypersurface.

ii) This argument also concerns the assumption of global hyperbolicity on the spacetime. In 1965 Penrose [30] discovered the following property of plane-wave spacetimes:

No spacelike hypersurface exists in the space-time which is adequate for the global specification of Cauchy data.

See also [31] and Chap. 13 in [2] for a discussion. Though the argument that such spacetimes are strong idealizations of actual gravitational radiation does have merit, it nonetheless constitutes a weak, physical argument of why one ought not be able to set up a quantum theory on such spacetimes. In short, global hyperbolicity constitutes an overly restrictive assumption.

Frame-dependent dynamics Our final point concerns the notion of a ‘preferred Lorentz frame’ (cf. Rem. 6 and §1.2.3 in [23]). In [23] one reads the following:

For several particles (with or without interaction), Bohmian mechanics postulates that one foliation \(\mathcal{F}\) of [Minkowski spacetime] into Cauchy surfaces (not necessarily horizontal) is singled out in nature, and the law of motion [Ref. [13]] depends on it.
Indeed, Münch-Berndl et al. [6] themselves acknowledged the problematic nature of such an approach:

It might be argued that such a structure violates the spirit of relativity, and regardless of whether or not we agree with this, it must be admitted that achieving relativistic invariance in a realistic (i.e. precise) version of quantum theory without the invocation of such structure seems much more difficult;
[citations omitted]

To us, it is, however, quite apparent that such a construction violates not only the 'spirit of relativity', but one of its core pillars: The general principle of relativity. As Einstein himself put it (cf. [16]):

We shall be true to the principle of relativity in its broadest sense if we give such a form to the laws that they are valid in every such four-dimensional system of co-ordinates, that is, if the equations expressing the laws are co-variant with respect to arbitrary transformations.

As the above statement does not just refer to the field equations themselves but to any fundamental, dynamical law, we cannot evade the conclusion that the proposed construction amounts to nothing less than a de facto reintroduction of the 'ether'—whose dismissal is among the very hallmarks of relativity theory. Though we do not wish to downplay the conceptual importance of Bell’s theorem [3,25] in this context, we do wish to warn against the resurrection of widely discredited concepts on the sole basis of non-rigorous, heuristic arguments (see Question 5 in [25] in particular).
4 The general ‘curved Born rule’ for one body

In order to be able to answer the general question on how to formulate the ‘curved Born rule’ for many bodies or even the case that the number of bodies is not conserved, one first has to understand the simpler one-body case. The argument that a relativistic theory cannot support such a description is void, for it is only a structural aspect of the theory we consider here, which is a priori independent of the question under which physical conditions the one-body case is realized in practice. Therefore, we will focus on the one-body case here.

In the article [23] it was rightly acknowledged that for the one-body case the Born rule has to amount to an integral over a sufficiently regular, orientable hypersurface in the spacetime. This becomes evident when one considers the analogue of the one-body Schrödinger theory. As discussed above, assuming spacelikeness of this hypersurface is, however, too restrictive. It is therefore necessary to find an integrand that does not require this assumption. Indeed, in the special case of a spacelike hypersurface $\Sigma_{\tau_0}$, we may re-express the suggested integrand by

$$g(J, n) \nu_{\tau_0} = \iota_{\tau_0}^* (J \cdot \mu).$$

(12)

The proof of this identity is analogous to the Riemannian case, see e.g. Lem. 16.30 in [22]. The right hand side of (12) is well-defined, even if $\Sigma_{\tau_0}$ is not spacelike.

In the spacelike case, the orientation on $\Sigma_{\tau_0}$ was also rightly acknowledged to be the one induced by the future-directed normal vector field $n$. For a lightlike hypersurface, however, the respective normal vector field is tangent to the hypersurface, hence it cannot be used to induce an orientation on it. Similarly, for a timelike hypersurface no canonical choice of normal vector field or orientation exists. Nonetheless, if we assume $\Sigma_{\tau_0}$ to be nowhere tangent to $J$, we may use $J$ to induce an orientation on $\Sigma_{\tau_0}$ instead. First, this coincides with the orientation chosen by Lienert and Tumulka for the spacelike case. Second, it prevents the hypersurface from ‘bending over’, see Fig. 3. Last, the function

$$P_{\Sigma_{\tau_0}} : \ A \mapsto P_{\Sigma_{\tau_0}}(A) := \int_A J \cdot \mu$$

(13)

with $P_{\Sigma_{\tau_0}}(\Sigma_{\tau_0}) = 1$ is a probability measure on, say, Lebesgue subsets of the hypersurface $\Sigma_{\tau_0}$. That $\Sigma_{\tau_0}$ be nowhere tangent to $J$ then assures that $P_{\Sigma_{\tau_0}}(A)$ is strictly positive whenever $A$ is not of measure zero.

To show that probability conservation also holds here, first factorize the current density

$$J = \rho X,$$

(14)

such that $\rho$ has physical dimensions of one over volume and $X$ is a future-directed timelike vector field. One may – but need not – require $X$ to be an observer field. While the notation $\rho$ was consciously chosen to suggest that it ought to be considered an invariant probability density – this becomes of relevance in the many-body theory – formally for the one-body case, (14) is only needed to obtain

\[ \text{A subset } A \text{ of a manifold } M \text{ is a Lebesgue set, if for every local chart } (U, \kappa) \text{ on } M \text{ the image } \kappa(U \cap A) \text{ is a Lebesgue set. As Lebesgue-measurability is a local property and invariant under coordinate transformations, it is sufficient to check this assumption for a collection of charts covering } A \text{ (cf. Chap. XII, Sec. 1 in [2]).} \]
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Figure 3  The sketch shows a hypersurface $\Sigma_{\tau_0}$ in $(2+1)$-dimensional Minkowski spacetime. At the longly dashed line $\Sigma_{\tau_0}$ is tangent to $\partial/\partial t$, so $\partial/\partial t$ cannot be used to define an orientation on $\Sigma_{\tau_0}$. In fact, as any Euclidean product of any future-directed timelike vector field $J$ with the respective Euclidean normal vector field on $\Sigma_{\tau_0}$ has to change sign and this is continuous by assumption, the Born rule cannot be formulated on $\Sigma_{\tau_0}$—independent of the particular choice of $J$. Regarding the general case, the example also shows that demanding $\Sigma_{\tau_0}$ to be nowhere tangent to $J$ only up to a set of measure zero is not sufficient to define an orientation on $\Sigma_{\tau_0}$.

a suitable velocity vector field $X$. Using the flow $\Phi$ of $X$ and setting $\tau_0 = 0$ for convenience, again set $\Sigma_{\tau} := \Phi_{\tau} (\Sigma_0)$ whenever defined. The identity

$$ (\Phi_{-\tau} \circ X)_{\Phi_{-\tau}(q)} = X_{\Phi_{-\tau}(q)} , $$

(15)

which holds for all $(-\tau, q) \in \text{dom} \Phi$, shows that the above non-tangency condition on $\Sigma_0$ is indeed preserved under time evolution. Now, from the invariant definition of divergence

$$ \mathcal{L}_J \mu = \text{div}(J) \mu $$(16)

(cf. Lem. 7.21 in [28]) and Cartan’s magic formula, one derives the identity

$$ \mathcal{L}_X (\rho X \cdot \mu) = \text{div} (\rho X) \cdot X \cdot \mu . $$

(17)

Using a generalized version of the Reynolds transport theorem, which we proved in a previous work (cf. Cor. 1 in [33]), we thus find

$$ \frac{d}{d\tau} \int_{\Sigma_{\tau}} \rho X \cdot \mu = \int_{\Sigma_{\tau}} \text{div} (\rho X) \cdot X \cdot \mu . $$

(18)

Therefore, $\text{div}(J) = 0$ again assures probability conservation.

It is also worthy of note that $J \cdot \mu$ is absolutely invariant with respect to $X$, so that probability conservation holds for any vector field $fX$ rescaled by a real-valued function $f$ (cf. p. 182 sq. in [24]). This property also shows that the choice of $\Sigma_0$ in

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8 The function $f$ may indeed take any values in $\mathbb{R}$, though strict positivity is required for $fX$ to be future-directed timelike. Also note that particular choices of $f$ may turn $fX$ incomplete.
this theory is arbitrary, as long as $X$ is nowhere tangent. In particular, it exposes the unnaturalness of restricting oneself to (subsets of) spacelike hypersurfaces as domains of integration.

**Remark 1**

i) It is indeed possible to derive the non-relativistic continuity equation with density $J_0$ for inertial frames of reference in Minkowski spacetime, but the question of the Newtonian limit requires more care than what was outlined at the end of Sec. 2 (see Chap. 4 in [32]).

ii) In practice, $\Sigma_0$ should be chosen to be ‘maximal’ in the sense that one is able to capture all possible particle positions while also excluding ‘same points at different times’. In the article [23] this problem was tackled by demanding that Minkowski spacetime be foliated by the $\Sigma_t$-s. Considering the general case of an arbitrary spacetime – which need not be contractible – we hold the following condition on $\Sigma_0$ to be more adequate, in addition to the one that it be nowhere tangent to $X$: The images of the connected components of $S_0$ under $\Phi$ should be mutually disjoint and for each such component $N_0$ there should not exist another connected hypersurface $N_0'$ nowhere tangent to $X$ with $N_0 \subset N_0'$, such that

$$S_0' = N_0' \cup S_0 \quad (19a)$$

still satisfies the former part of the condition.

Regarding the first part of the condition, recall that

$$\Phi_t \mid_{\Sigma_0} : \Sigma_0 \rightarrow \Sigma_t \quad (19b)$$

is a homeomorphism onto its image, provided it exists for given $\tau \in \mathbb{R}$. Thus, the number of connected components of $\Sigma_0$ is preserved, i.e. $\Sigma_\tau$ can neither ‘tear’ nor ‘merge’. Roughly speaking, this part of the condition assures that (a part of) one component of $\Sigma_0$ is not (a part of) another component ‘at different times’. The second part of the condition assures that $\Sigma_0$ is ‘inextendible’ in the demanded sense.

The seemingly opaque maximality condition allows one to set $S_0$ at ‘different times’, and, stated in this manner, it generalizes to the case that $S_0$ is a manifold with boundary (or with corners, etc.). It also does not require any additional topological or causality conditions to be imposed on the spacetime. Of course, in common situations one chooses $S_0$ to be connected and ‘inextendible’ in the above sense, as here $S_0$ is required for the setup of the problem and $\Phi$ is not known beforehand.

iii) If the flow $\Phi$ is incomplete, it may happen that $N_\tau := \Phi_\tau (N_0)$ exists for some $N_0 \subset S_0$ and $\tau \in \mathbb{R}$, while $\Sigma_\tau$ does not. Then $P_{\Sigma_\tau} (N_\tau)$ is still a physically meaningful probability and so is the probability of the complementary event $N^c_\tau$

$$P_{\Sigma_\tau} (N^c_\tau) = 1 - P_{\Sigma_\tau} (N_\tau) \quad (19c)$$

\footnote{Note that, by assumption, $Q$ is the domain of the timelike vector field $J = \rho X$ and hence $\rho$ vanishes nowhere. On a larger domain $\rho$ may have nodes, yet formally those are excluded from $Q$.}
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Up to questions of regularity, we hold the conclusion to be inevitable, that the construction provided here is the adequate ‘curved Born rule’ for the one-body case. In fact, the theory of the conservation of other scalar integral quantities such as mass or charge is analogous, see e.g. Ex. 3 in [33]. A treatment of the theory for \( N \) bodies for arbitrary \( N \in \mathbb{N} \) would go beyond the scope of this comment. Though it is possible to proceed along similar lines, the theory is more involved and we intend to make this the subject of a future work. Furthermore, while the construction here suggests that an invariant dynamics of \( J \) – one that respects the general principle of relativity – is to be preferred (so-called “serious Lorentz invariance”, see p. 226 in [15]), we believe that the question ought to be treated separately from the general framework.

Acknowledgements The authors would like to acknowledge support from The Robert A. Welch Foundation (D-1523). R. Tumulka deserves gratitude for helpful criticism. M. R. thanks S. Miret-Artés, Y. B. Suris, and G. Rudolph for their support in making this work possible.

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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