Trace Formulae, Zeta Functions, Congruences and Reidemeister Torsion in Nielsen Theory

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Abstract

In this paper we prove trace formulae for the Reidemeister number of a group endomorphism. This result implies the rationality of the Reidemeister zeta function in the following cases: the group is a direct product of a finite group and a finitely generated Abelian group; the group is finitely generated, nilpotent and torsion free. We connect the Reidemeister zeta function of an endomorphism of a direct product of a finite group and a finitely generated free Abelian group with the Lefschetz zeta function of the induced map on the unitary dual of the group. As a consequence we obtain a relation between a special value of the Reidemeister zeta function and a certain Reidemeister torsion. We also prove congruences for Reidemeister numbers of iterates of an endomorphism of a direct product of a finite group and a finitely generated free Abelian group which are the same as those found by Dold for Lefschetz numbers.

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0 Introduction

We assume everywhere $X$ to be a connected, compact polyhedron and $f : X \to X$ to be a continuous map. Taking a dynamical point of view, we shall be interested in the iterates of $f$. In the theory of discrete dynamical systems the following zeta functions have been studied: the Artin-Mazur zeta function

$$\zeta_f(z) := \exp \left( \sum_{n=1}^{\infty} \frac{F(f^n)}{n} z^n \right),$$

where $F(f^n)$ is the number of isolated fixed points of $f^n$; the Ruelle zeta function

$$\zeta_{f^g}(z) := \exp \left( \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \text{Fix}(f^n)} \prod_{k=0}^{n-1} g(f^k(x)) \right),$$

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where \( g : X \to \mathcal{C} \) is a weight function (if \( g \) is identically 1 then this is simply the Artin-Mazur function); the Lefschetz zeta function

\[
L_f(z) := \exp \left( \sum_{n=1}^{\infty} \frac{L(f^n)}{n} z^n \right),
\]

where

\[
L(f^n) := \sum_{k=0}^{\dim X} (-1)^k \text{Tr} \left[ f_{sk}^n : H_k(X; \mathbb{Q}) \to H_k(X; \mathbb{Q}) \right]
\]

are the Lefschetz numbers of the iterates of \( f \); reduced modulo 2 Artin-Mazur and Lefschetz zeta functions [14]; twisted Artin-Mazur and Lefschetz zeta functions [15], which have coefficients in the group ring \( \mathbb{Z}H \) of an Abelian group \( H \).

The above zeta functions are analogous to the Hasse-Weil zeta function of an algebraic variety over a finite field [30]. Like the Hasse-Weil zeta function, the Lefschetz function is always a rational function of \( z \), and is given by the formula:

\[
L_f(z) = \prod_{k=0}^{\dim X} \det (I - f_{sk} \cdot z)^{(-1)^{k+1}}.
\]

This immediately follows from the trace formula for the Lefschetz numbers of the iterates of \( f \). The Artin-Mazur zeta function has a positive radius of convergence for a dense set in the space of smooth self-maps of a compact smooth manifold [3]. Manning proved the rationality of the Artin-Mazur zeta function for diffeomorphisms of a compact smooth manifold satisfying Smale’s Axiom A [23]. The value of knowing that a zeta function is rational is that it shows that the infinite sequence of coefficients is closely interconnected, and is given by the finite set of zeros and poles of zeta function.

The Artin-Mazur zeta function and its modifications count periodic points of a map geometrically, and the Lefschetz’s type zeta functions do the same thing algebraically (with weight defined by index theory). A third way of counting periodic points is to use Nielsen theory. Let \( p : \tilde{X} \to X \) be the universal covering of \( X \) and \( \tilde{f} : \tilde{X} \to \tilde{X} \) a lifting of \( f \), i.e. \( p \circ \tilde{f} = f \circ p \). Two liftings \( \tilde{f} \) and \( \tilde{f}' \) are said to be conjugate if there is a \( \gamma \in \Gamma \cong \pi_1(X) \) such that \( \tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1} \). The subset \( p(\text{Fix}(\tilde{f})) \subset \text{Fix}(f) \) is called the fixed point class of \( f \) determined by the lifting class \([\tilde{f}]\). A fixed point class is said to be essential if its index is non-zero.

The number of lifting classes of \( f \) (and hence the number of fixed point classes, empty or not) is called the Reidemeister number of \( f \), denoted \( R(f) \). This is a positive integer or infinity. The number of essential fixed point classes is called the Nielsen number of \( f \), denoted by \( N(f) \). The Nielsen number is always finite. Both Nielsen and Reidemeister numbers are homotopy type invariants. In the category of compact, connected polyhedra the Nielsen number of a map is a lower bound for the least number of fixed points of maps in the homotopy class of \( f \).

Let \( G \) be a group and \( \phi : G \to G \) an endomorphism. Two elements \( \alpha, \alpha' \in G \) are said to be \( \phi \)-conjugate if there exists \( \gamma \in G \) such that \( \alpha' = \gamma \cdot \alpha \cdot \phi(\gamma)^{-1} \). The number of \( \phi \)-conjugacy classes is called the Reidemeister number of \( \phi \), denoted by \( R(\phi) \). We shall also write \( \mathcal{R}(\phi) \) for the set of \( \phi \)-conjugacy classes of elements of \( G \). One easily
shows that if $\psi$ is an inner automorphism of $G$ then $R(\phi \circ \psi) = R(\psi \circ \phi) = R(\phi)$. The group-theoretical and topological Reidemeister numbers are related as follows. If $G$ is the fundamental group of $X$ and $\phi$ is the endomorphism of $G$ induced by $f$ (this is only defined modulo inner automorphisms) then $R(\phi) = R(f)$.

If we consider the iterates of $f$ and $\phi$, we may define several zeta functions associated with Nielsen fixed point theory (see [3, 4, 13]). We assume throughout this article that $R(f^n) < \infty$ and $R(\phi^n) < \infty$ for all $n > 0$. The Reidemeister zeta functions of $f$ and $\phi$ and the Nielsen zeta function of $f$ are defined as power series:

$$R_{\phi}(z) := \exp \left( \sum_{n=1}^{\infty} \frac{R(\phi^n)}{n} z^n \right),$$
$$R_f(z) := \exp \left( \sum_{n=1}^{\infty} \frac{R(f^n)}{n} z^n \right),$$
$$N_f(z) := \exp \left( \sum_{n=1}^{\infty} \frac{N(f^n)}{n} z^n \right).$$

These functions are homotopy invariants. The Nielsen zeta function $N_f(z)$ has a positive radius of convergence which has a sharp estimate in terms of the topological entropy of the map $f$ [13]. In section 2 we give another proof of positivity of the radius of convergence. We also obtain an exact algebraic lower bound for the radius using the Reidemeister trace formula for generalized Lefschetz numbers.

We begin the article by proving in section 1 trace formulae for Reidemeister numbers in the following cases: $G$ is finite; $G$ is a direct product of a finite group and a finitely generated free Abelian group; $G$ is a finitely generated torsion free nilpotent group. By this we mean that there are finite dimensional complex vector spaces $H^i$ for $i = 0, \ldots, N$ and linear maps $T_i : H^i \to H^i$ such that for every natural number $n$, the Reidemeister number of $\phi^n$ is equal to $\sum_{i=0}^{N}(-1)^n \text{Tr}(T_i^n)$. Thus the definition of the Lefschetz number is a trace formula. These results had previously been known only for the finitely generated free Abelian groups [3], although the case of finite groups is implicit in [10, 11].

In section 1 we prove arithmetical congruences:

$$\sum_{d|n} \mu(d) \cdot R(\phi^{n/d}) \equiv 0 \mod n,$$
for the Reidemeister numbers of the iterations of a group endomorphism \( \phi \) of a direct product of a finite group and a finitely generated free Abelian group and corresponding congruences

\[
\sum_{d \mid n} \mu(d) \cdot R(f^{n/d}) \equiv 0 \pmod{n}
\]

for the Reidemeister numbers of a continuous map \( f \). These congruences are the same as those found by Dold \[5\] for Lefschetz numbers.

Recently a connection between the Lefschetz type dynamical zeta functions and the Reidemeister torsion was established by D. Fried \[16, 17\]. The work of Milnor \[24\] was the first indication that such a connection exists. In section 3 we establish a connection between Reidemeister torsion and the Reidemeister zeta function. We obtain an expression for the Reidemeister torsion of the mapping torus of the dual map of an endomorphism of a direct product of a finite group and a finitely generated free Abelian group, in terms of a special value of the Reidemeister zeta function of the endomorphism. The result is obtained by expressing the Reidemeister zeta function in terms of the Lefschetz zeta function of the dual map, and then applying the theorem of D. Fried. This means that the Reidemeister torsion counts the periodic point classes of the map \( f \). These results had previously been known for the finitely generated Abelian groups and finite groups \[11\].

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## 1 Group Theoretical Reidemeister Numbers

Let \( \phi \) be an endomorphism of a group \( G \). We shall write the group law of \( G \) multiplicatively. We shall write \( \{g\}_\phi \) for the \( \phi \)-conjugacy class of an element \( g \in G \). We shall write \( <g> \) for the ordinary conjugacy class of \( g \) in \( G \). The endomorphism \( \phi \) maps conjugate elements to conjugate elements. It therefore induces an endomorphism of the set of conjugacy classes in \( G \).

**Theorem 1** (\[14\]) Let \( G \) be a finite group and let \( \phi : G \to G \) be an endomorphism. Then \( R(\phi) \) is the number of ordinary conjugacy classes \( <x> \) in \( G \) such that

\[
<\phi(x)> = <x>.
\]

We now rephrase this to give a trace formula. Suppose \( G \) is finite and let \( W \) be the vector space of complex valued class functions on the group \( G \). A class function is a function which takes the same value on every element of a usual conjugacy class. The map \( \phi \) induces a linear map \( B : W \to W \) defined by

\[
B(f) := f \circ \phi.
\]

We shall write \( \hat{G} \) for the set of isomorphism classes of irreducible unitary representations of \( G \). There is a multivalued map \( \hat{\phi} : \hat{G} \to \hat{G} \) defined as follows. Let
Let \( \phi : G \to G \) be an endomorphism of a finite group \( G \). Then we have

\[
R(\phi) = \#\text{Fix}(\hat{\phi}) = \text{Tr} B
\]  

(1)

In the case when \( \phi \) is the identity map, this is Burnside’s result equating the number of irreducible representation of a finite group with the number of conjugacy classes in the group.

**Proof.** We shall calculate the trace of \( B \) in two ways. The characteristic functions of the conjugacy classes in \( G \) form a basis of \( W \), and are mapped to one another by \( B \) (the map need not be a bijection). Therefore the trace of \( B \) is the number of elements of this basis which are fixed by \( B \). By Theorem 1, this is equal to the Reidemeister number of \( \phi \). Another basis of \( W \), which is also mapped to itself by \( B \) is the set of traces of irreducible representations of \( G \) (see [21] chapter XVIII). From this it follows that the trace of \( B \) is the number of irreducible representations \( \rho \) of \( G \) such that \( \rho \) has the same trace as \( \hat{\phi}(\rho) \). However, representations of finite groups are characterized up to equivalence by their traces. Therefore the trace of \( B \) is equal to the number of fixed points of \( \hat{\phi} \). \( \square \)

The following result is known for free Abelian groups:

**Theorem 3** ([11]) Let \( \phi : \mathbb{Z}^k \to \mathbb{Z}^k \) be a group endomorphism. Then

\[
R(\phi) = | \det(I - \phi) | = \#\text{Fix}(\hat{\phi}).
\]

Furthermore the following trace formula holds:

\[
R(\phi) = (-1)^{r+p} \sum_{i=0}^{k} (-1)^i \text{Tr} (\wedge^i \phi).
\]

(2)

where \( p \) the number of eigenvalues \( \mu \) of \( M \) such that \( \mu < -1 \), and \( r \) the number of real eigenvalues whose absolute value is \( > 1 \). Here \( \wedge^i \) denotes the exterior power.

Now let \( F \) be a finite group and \( k \) a natural number. We shall consider an endomorphism \( \phi \) of the group \( G = \mathbb{Z}^k \times F \). Our aim is to prove a trace formula for the Reidemeister number of such an endomorphism. The torsion elements of \( G \) are precisely the elements of the finite, normal subgroup \( F \). For this reason we have \( \phi(F) \subseteq F \). Let \( \phi_{\text{finite}} : F \to F \) be the restriction of \( \phi \) to \( F \), and let \( \phi^\infty : G/F \to G/F \) be the induced map on the quotient group. Let \( \text{pr}_{\mathbb{Z}^k} : G \to \mathbb{Z}^k \) and \( \text{pr}_F : G \to F \) denote the projections onto \( \mathbb{Z}^k \) and \( F \). Then the composition

\[
\text{pr}_{\mathbb{Z}^k} \circ \phi : \mathbb{Z}^k \to G \to \mathbb{Z}^k
\]
is an endomorphism of $\mathbb{Z}^k$, which is given by some matrix $M \in M_k(\mathbb{Z})$. We denote by $\psi: \mathbb{Z}^k \to F$ the other component of the restriction of $\phi$ to $\mathbb{Z}^k$, i.e.

$$\psi(v) = \text{pr}_F(\phi(v)).$$

We therefore have for any element $(v, f) \in G$

$$\phi(v, f) = (M \cdot v, \psi(v)\phi(f)) .$$

**Lemma 1** Let $G$ be as above. Two elements $g_1 = (v_1, f_1)$ and $g_2 = (v_2, f_2)$ of $G$. are $\phi$-conjugate if and only if

$$v_1 \equiv v_2 \mod (1 - M)\mathbb{Z}^k$$

and there is a $h \in F$ with

$$hf_1 = f_2\phi((1 - M)^{-1}(v_2 - v_1))\phi(h).$$

**Proof.** Suppose $g_1$ and $g_2$ are $\phi$-conjugate. Then there is a $g_3 = (w, h) \in G$ with $g_3g_1 = g_2\phi(g_3)$. Therefore

$$(w + v_1, hf_1) = (v_2 + M \cdot w, f_2\psi(w)\phi(h)).$$

Comparing the first components we obtain $(1 - M) \cdot w = v_2 - v_1$ from which it follows that $v_1$ is congruent to $v_2$ modulo $(1 - M)\mathbb{Z}^r$. Substituting $(1 - M)^{-1}(v_2 - v_1)$ for $w$ in the second component we obtain the second relation in the lemma. The argument can easily be reversed to give the converse. \(\square\)

**Proposition 1** In the notation described above, $R(\phi) = R(\phi^{finite}) \times R(\phi^{\infty})$.

**Proof.** We partition the set $\mathcal{R}(\phi)$ of $\phi$-conjugacy classes of elements of $G$ into smaller sets:

$$\mathcal{R}(\phi) = \cup_{v \in \mathbb{Z}^k/(1 - M)\mathbb{Z}^k} \mathcal{R}(v),$$

where $\mathcal{R}(v)$ is the set of $\phi$-conjugacy classes $\{(w, f)\}_{\phi}$ for which $w$ is congruent to $v$ modulo $(1 - M)\mathbb{Z}^k$. It follows from the previous lemma that this is a partition. Now suppose $\{(w, f)\}_{\phi} \in \mathcal{R}(v)$. We will show that $\{(w, f)\}_{\phi} = \{(v, f)\}_{\phi}$ for some $f^* \in F$. This follows by setting $f^* = f\psi((1 - M)^{-1}(w - v))$ and applying the previous lemma with $h = id$. Therefore $\mathcal{R}(v)$ is the set of $\phi$-conjugacy classes $\{(v, f)\}_{\phi}$ with $f \in F$. From the previous lemma it follows that $(v, f_1)$ and $(v, f_2)$ are $\phi$-conjugate iff there is a $h \in F$ with

$$hf_1 = f_2\psi(0)\phi(h) = f_2\phi(h).$$

This just means that $f_1$ and $f_2$ are $\phi^{finite}$-conjugate as elements of $F$. From this it follows that $\mathcal{R}(v)$ has cardinality $R(\phi^{finite})$. Since this is independent of $v$, we have

$$R(\phi) = \sum_v R(\phi^{finite}) = \det(1 - M) \times R(\phi^{finite}).$$
Now consider the map $\phi^\infty : G/F \to G/F$. We have

$$\phi^\infty ((v, F)) = (M \cdot v, \psi(v) F) = (M \cdot v, F).$$

From this it follows that $\phi^\infty$ is equivalent to the map $M : \mathbb{Z}^k \to \mathbb{Z}^k$. This implies

$$R(\phi^\infty) = R(M : \mathbb{Z}^k \to \mathbb{Z}^k).$$

However by Theorem 3 we have $R(M : \mathbb{Z}^k \to \mathbb{Z}^k) = | \det(1 - M) |$. Therefore $R(\phi) = R(\phi^{\text{finite}}) \times R(\phi^\infty)$, proving Proposition 1.

Proposition 2 In the notation described above

$$\# \text{Fix} (\hat{\phi}) = \# \text{Fix} (\hat{\phi}^{\text{finite}}) \times \# \text{Fix} (\hat{\phi}^\infty).$$

Proof. Consider the dual $\hat{G}$. This is cartesian product of the duals of $\mathbb{Z}^k$ and $F$:

$$\hat{G} = \hat{\mathbb{Z}}^k \times \hat{F}, \rho = \rho_1 \otimes \rho_2$$

where $\rho_1$ is an irreducible representation of $\mathbb{Z}^k$ and $\rho_2$ is an irreducible representation of $F$. Since $\mathbb{Z}^r$ is Abelian, all of its irreducible representations are 1-dimensional, so $\rho_1(v)$ for $v \in \mathbb{Z}^k$ is always a scalar matrix, and $\rho_2$ is the restriction of $\rho$ to $F$. If $\hat{\phi}(\rho) = \rho$ then there is a matrix $T$ such that

$$\rho \circ \phi = T \cdot \rho \cdot T^{-1}.$$ 

This implies

$$\rho^{\text{finite}} \circ \phi^{\text{finite}} = T \cdot \rho^{\text{finite}} \cdot T^{-1},$$

so $\rho_2 = \rho^{\text{finite}}$ is fixed by $\phi^{\text{finite}}$. For any fixed $\rho_2 \in S(\phi^{\text{finite}})$, the set of $\rho_1$ such that $\rho_1 \otimes \rho_2$ is fixed by $\hat{\phi}$ is the set of $\rho_1$ satisfying

$$\rho_1(M \cdot v) \rho_2(\psi(v)) = T \cdot \rho_1(v) \cdot T^{-1}$$

for some matrix $T$ independent of $v \in \mathbb{Z}^k$. Since $\rho_1(v)$ is a scalar matrix, the equation is equivalent to

$$\rho_1(M \cdot v) \rho_2(\psi(v)) = \rho_1(v),$$

i.e.

$$\rho_1((1 - M)v) = \rho_2(\psi(v)).$$

Note that $\hat{\mathbb{Z}}^k$ is isomorphic to the torus $T^k$, and the transformation $\rho_1 \to \rho_1 \circ (1 - M)$ is given by the action of the matrix $1 - M$ on the torus $T^k$. Therefore the number of $\rho_1$ satisfying the last equation is the degree of the map $(1 - M)$ on the torus, i.e. $| \det(1 - M) |$. From this it follows that

$$\# \text{Fix} (\hat{\phi}) = \# \text{Fix} (\hat{\phi}^{\text{finite}}) \times | \det(1 - M) |.$$

As in the proof of Proposition 1 we have $R(\phi^\infty) = | \det(1 - M) |$. Since $\phi^\infty$ is an endomorphism of an Abelian group we have $\# \text{Fix} (\hat{\phi}^\infty) = R(\phi^\infty)$. Therefore

$$\# \text{Fix} (\hat{\phi}) = \# \text{Fix} (\hat{\phi}^{\text{finite}}) \times \# \text{Fix} (\hat{\phi}^\infty).$$

As a consequence we have the following:
Theorem 4  If \( \phi \) be any endomorphism of \( G \) where \( G \) is the direct product of a finite group \( F \) with a finitely generated free Abelian group, then
\[
R(\phi) = \#\text{Fix}(\hat{\phi})
\]

Proof. Since \( \phi^{\text{finite}} \) is an endomorphism of a finite group, by Theorem 2 we have \( R(\phi^{\text{finite}}) = \#\text{Fix}(\hat{\phi}^{\text{finite}}) \). Since \( \phi^{\infty} \) is an endomorphism of the finitely generated free Abelian group we have \( R(\phi^{\infty}) = \#\text{Fix}(\hat{\phi}^{\infty}) \) (see Theorem 2 in [11]). It follows from Propositions 1 and 2 that \( R(\phi) = \#\text{Fix}(\hat{\phi}) \). \( \square \)

Let \( W \) be the vector space of complex valued class functions on \( F \). The map \( \phi \) induces a linear map \( B : W \to W \) defined as above in Theorem 2.

Theorem 5  If \( G \) is the direct product of a free Abelian and a finite group and \( \phi \) an endomorphism of \( G \). Then the following trace formula holds:
\[
R(\phi) = (-1)^r + p \sum_{i=0}^{k} (-1)^i \text{Tr}(\wedge^i \phi^{\infty} \otimes B),
\]
where \( k \) is the rank of the free Abelian group \( G/F \); \( p \) is the number of \( \mu \in \text{Spec} \phi^{\infty} \) such that \( \mu < -1 \), and \( r \) the number of real eigenvalues of \( \phi^{\infty} \) whose absolute value is > 1.

Proof. This follows from Theorems 3 and 2, Proposition 1 and the formula
\[
\text{Tr}(\wedge^i \phi^{\infty}) \cdot \text{Tr}(B) = \text{Tr}(\wedge^i \phi^{\infty} \otimes B).
\]
\( \square \)

Finally let \( \Gamma \) be a finitely generated, torsion free, nilpotent group. It is known [22] that any such group is a uniform, discrete subgroup of some simply connected nilpotent Lie group \( G \) (uniform means that the coset space \( G/\Gamma \) is compact). The space \( M = G/\Gamma \) is called a nil-manifold. Since \( \Gamma = \pi_1(M) \) and \( M \) is a \( K(\Gamma, 1) \), every endomorphism \( \phi : \Gamma \to \Gamma \) may be realized by a self-map \( f : M \to M \) such that \( f_\ast = \phi \) and thus \( R(f) = R(\phi) \). Any endomorphism \( \phi : \Gamma \to \Gamma \) may be uniquely extended to an endomorphism \( F : G \to G \). Let \( \tilde{F} : \tilde{G} \to \tilde{G} \) be the corresponding Lie algebra endomorphism induced from \( F \).

Theorem 6  If \( \phi \) an endomorphism of a finitely generated torsion free nilpotent group \( \Gamma \) then
\[
R(\phi) = (-1)^r + p \sum_{i=0}^{m} (-1)^i \text{Tr} \wedge^i \tilde{F},
\]
where \( m \) is Rank \( \Gamma = \dim M \); \( p \) is the number of eigenvalues \( \mu \) of \( \tilde{F} \) such that \( \mu < -1 \), and \( r \) the number of real eigenvalues of \( \tilde{F} \) whose absolute value is > 1.
Proof. Let \( f : M \to M \) be a map realizing \( \phi \) on a compact nil-manifold \( M \) of dimension \( m \). We are assuming throughout this article that the Reidemeister number \( R(f) = R(\phi) \) is finite. The finiteness of \( R(f) \) implies the non-vanishing of the Lefschetz number \( L(f) \) \([12]\). A strengthened version of Anosov’s Theorem \([1]\) is proven in \([25]\) which states in particular, that if \( L(f) \neq 0 \) then \( N(f) = |L(f)| = R(f) \). However it is known that \( L(f) = \det(\tilde{F} - 1) \) \([1]\). From this we have

\[
R(\phi) = R(f) = |L(f)| = |\det(1 - \tilde{F})| = (-1)^{r+p} \det(1 - \tilde{F})
\]

\[
= (-1)^{r+p} \sum_{i=0}^{m} (-1)^i \text{Tr}^i \tilde{F}.
\]

\[\blacksquare\]

Theorem 7 \([12]\) Let \( \Gamma \) be a finitely generated torsion free nilpotent group of rank \( k \). For any endomorphism \( \phi : \Gamma \to \Gamma \) such that \( R(\phi) \) is finite, there exists an endomorphism \( \psi : \mathbb{Z}^k \to \mathbb{Z}^k \) such that for all \( n \in \mathbb{N} \), \( R(\phi^n) = \#\text{Fix} \hat{\psi}^n \).

The following lemma is also useful for calculating Reidemeister numbers; it often allows one to reduce to the case that \( \phi \) is an isomorphism.

Lemma 2 Let \( \phi : G \to G \) be any endomorphism of any group \( G \), and let \( H \) be a subgroup of \( G \) with the properties

\[
\phi(H) \subset H
\]

\[
\forall x \in G \exists n \in \mathbb{N} \text{ such that } \phi^n(x) \in H.
\]

Then

\[
R(\phi) = R(\phi_H),
\]

where \( \phi_H : H \to H \) is the restriction of \( \phi \) to \( H \).

Proof. Let \( x \in G \). Then there is an \( n \) such that \( \phi^n(x) \in H \). It is known that \( x \) is \( \phi \)-conjugate to \( \phi^n(x) \) (see for example \([19]\)). This means that the \( \phi \)-conjugacy class \( \{x\}_\phi \) of \( x \) has non-empty intersection with \( H \).

Now suppose that \( x, y \in H \) are \( \phi \)-conjugate, ie. there is a \( g \in G \) such that

\[
gx = y\phi(g).
\]

We shall show that \( x \) and \( y \) are \( \phi_H \)-conjugate, ie. we can find a \( g \in H \) with the above property. First let \( n \) be large enough that \( \phi^n(g) \in H \). Then applying \( \phi^n \) to the above equation we obtain

\[
\phi^n(g)\phi^n(x) = \phi^n(y)\phi^{n+1}(g).
\]

This shows that \( \phi^n(x) \) and \( \phi^n(y) \) are \( \phi_H \)-conjugate. On the other hand, one knows that \( x \) and \( \phi^n(x) \) are \( \phi_H \)-conjugate, and \( y \) and \( \phi^n(y) \) are \( \phi_H \) conjugate, so \( x \) and \( y \) must be \( \phi_H \)-conjugate.
We have shown that the intersection with $H$ of a $\phi$-conjugacy class in $G$ is a $\phi_H$-conjugacy class in $H$. We therefore have a map

$\text{Rest} : \mathcal{R}(\phi) \to \mathcal{R}(\phi_H)$

$\{x\}_\phi \mapsto \{x\}_\phi \cap H$

This clearly has the two-sided inverse

$\{x\}_\phi \cap H \mapsto \{x\}_\phi$.

Therefore $\text{Rest}$ is a bijection and $R(\phi) = R(\phi_H)$. \hfill \Box

**Corollary 1** Let $H = \phi^n(G)$. Then $R(\phi) = R(\phi_H)$.

As an application of this we prove congruences for Reidemeister numbers. Let $\mu(d), d \in \mathbb{N}$ be the Möbius function, i.e.

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1, \\ (-1)^k & \text{if } d \text{ is a product of } k \text{ distinct primes,} \\ 0 & \text{if } d \text{ is not square-free.} \end{cases}$$

**Theorem 8** Let $\phi : G \to G$ be an endomorphism of the group $G$ such that all numbers $R(\phi^n)$ are finite, and let $H$ be a subgroup of $G$ with the properties

$$\phi(H) \subset H$$

$$\forall x \in G \exists n \in \mathbb{N} \text{ such that } \phi^n(x) \in H.$$ 

If one of the following conditions is satisfied:

(I) $H$ is a direct product of a finite group and a finitely generated free Abelian group, or

(II) $H$ is finitely generated, nilpotent and torsion free, then one has for all natural numbers $n$,

$$\sum_{d|n} \mu(d) \cdot R(\phi^{n/d}) \equiv 0 \mod n.$$

**Proof.** From Theorem 4 and Lemma 2 it follows immediately that, in case I, for every $n$

$$R(\phi^n) = \# \text{Fix } \left[ \hat{\phi}_H^n : \hat{H} \to \hat{H} \right].$$

Let $P_n$ denote the number of periodic points of $\hat{\phi}_H$ of least period $n$. One sees immediately that

$$R(\phi^n) = \# \text{Fix } \left[ \hat{\phi}_H^n \right] = \sum_{d|n} P_d.$$ 

Applying Möbius’ inversion formula, we have,

$$P_n = \sum_{d|n} \mu(d) R(\phi^{n/d}).$$

On the other hand, we know that $P_n$ is always divisible be $n$, because $P_n$ is exactly $n$ times the number of $\hat{\phi}_H$-orbits in $\hat{H}$ of length $n$. In the case II when $H$ is finitely generated, nilpotent and torsion free, we know from Theorem 7 that there exists an endomorphism $\psi : \mathbb{Z}^n \to \mathbb{Z}^n$ such that $R(\phi^n) = \# \text{Fix } \hat{\psi}^n$. The proof then follows as in previous case. \hfill \Box
2 Reidemeister and Nielsen zeta functions

In this section we reinterpret the results of the previous section in terms of the Reidemeister zeta function. We show in the cases that we have considered that the zeta function is a rational function with a functional equation.

**Theorem 9** Let $G$ be the direct product of a free Abelian and a finite group and $\phi$ an endomorphism of $G$. Then $R_\phi(z)$ is a rational function and is equal to

$$R_\phi(z) = \left( \prod_{i=0}^{k} \det(1 - \wedge^i \phi^\infty \otimes B \cdot \sigma \cdot z)^{(-1)^{i+1}} \right)^{(-1)^r}$$  \hspace{1cm} (5)

where matrix $B$ is as defined in Theorem 2, $\sigma = (-1)^p$, $p$, $r$ and $k$ are the constants described in Theorem 3.

**Proof.** From Proposition 1 it follows that $R(\phi^n) = R((\phi^\infty)^n \cdot R((\phi^{finite})^n))$. From this formula, Theorem 2 and 3 we have the trace formula for $R(\phi^n)$:

$$R(\phi^n) = (-1)^{r+pm} \sum_{i=0}^{k} (-1)^i \text{Tr} \ \wedge^i (\phi^\infty)^n \cdot \text{Tr} \ B^n$$

$$= (-1)^{r+pm} \sum_{i=0}^{k} (-1)^i \text{Tr} \ (\wedge^i (\phi^\infty)^n \otimes B^n)$$

$$= (-1)^{r+pm} \sum_{i=0}^{k} (-1)^i \text{Tr} \ (\wedge^i \phi^\infty \otimes B)^n.$$

We now calculate directly

$$R_\phi(z) = \exp \left( \sum_{n=1}^{\infty} \frac{R(\phi^n)}{n} z^n \right)$$

$$= \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^r \sum_{i=0}^{k} (-1)^i \text{Tr} \ (\wedge^i \phi^\infty \otimes B)^n (\sigma \cdot z)^n}{n} \right)$$

$$= \left( \prod_{i=0}^{k} \left( \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} \ (\wedge^i \phi^\infty \otimes B)^n \cdot (\sigma \cdot z)^n \right) \right)^{(-1)^i} \right)^{(-1)^r}$$

$$= \left( \prod_{i=0}^{k} \det \left( 1 - \wedge^i \phi^\infty \otimes B \cdot \sigma \cdot z \right)^{(-1)^{i+1}} \right)^{(-1)^r}.$$  \hspace{1cm} \square

**Theorem 10** If $\Gamma$ is a finitely generated torsion free nilpotent group and $\phi$ an endomorphism of $\Gamma$. Then $R_\phi(z)$ is a rational function and is equal to

$$R_\phi(z) = \left( \prod_{i=0}^{m} \det(1 - \wedge^i \tilde{F} \cdot \sigma \cdot z)^{(-1)^{i+1}} \right)^{(-1)^r}$$  \hspace{1cm} (6)

where $\sigma = (-1)^p$, $p$, $r$, $m$ and $\tilde{F}$ is defined in theorem 6.
Proof. If we repeat the proof of the Theorem 4 for $\phi^n$ instead of $\phi$ we obtain that
\[ R(\phi^n) = (-1)^{r+m} \det(1 - \tilde{F})( \text{we suppose that Reidemeister numbers } R(\phi^n) \text{ are finite for all } n). \]
Last formula implies the trace formula for $R(\phi^n)$:
\[ R(\phi^n) = (-1)^r \sum_{i=0}^m (-1)^i \text{Tr} \left( \wedge^i \tilde{F} \right)^n \]
From this we have formula 6 immediately by direct calculation as in Theorem 9.  \(\square\)

Corollary 2. Under the assumptions of Theorem 6, the zeros and poles of the Reidemeister zeta function are reciprocals of eigenvalues of one of the maps
\[ \wedge^i \tilde{F} : \wedge^i \tilde{G} \to \wedge^i \tilde{G} \quad 0 \leq i \leq \text{Rank } \Gamma \]

We now turn our attention to the functional equations satisfied by these zeta functions. These equations are consequences of a duality on the vector spaces with which we have obtained trace formulae.

Theorem 11. Let $\Gamma$ be a finitely generated, torsion free, nilpotent group and $\phi$ an endomorphism of $\Gamma$. The Reidemeister zeta function $R_\phi(z)$ satisfies the following functional equation:
\[ R_\phi \left( \frac{1}{dz} \right) = \epsilon_1 \cdot R_\phi(z)^{(-1)^{\text{Rank } \Gamma}}, \tag{7} \]
where $d = \det \tilde{F}$ and $\epsilon_1$ is a non-zero complex constant.

Proof. Via the natural non-singular pairing $(\wedge^i \tilde{F}) \otimes (\wedge^{m-i} \tilde{F}) \to \mathcal{C}$ the operators $\wedge^{m-i} \tilde{F}$ and $d \cdot (\wedge^i \tilde{F})^{-1}$ are adjoint to each other.

Consider an eigenvalue $\lambda$ of $\wedge^i \tilde{F}$. By Theorem 10, This contributes a term
\[ \left( 1 - \frac{\lambda \sigma}{dz} \right)^{(-1)^{i+1}} \left( \frac{1}{dz} \right)^{(-1)^r} \]
to $R_\phi \left( \frac{1}{dz} \right)$. This may be rewritten as follows:
\[ \left( 1 - \frac{d \sigma z}{\lambda} \right)^{(-1)^{i+1}} \cdot \left( \frac{-dz}{\lambda \sigma} \right)^{(-1)^r}. \]
Note that $\frac{d}{\lambda}$ is an eigenvalue of $\wedge^{m-i} \tilde{F}$. Multiplying these terms together we obtain,
\[ R_\phi \left( \frac{1}{dz} \right) = \left( \prod_{i=1}^m \left( \prod_{\wedge^i \tilde{F} \in \text{Spec } \wedge^i \tilde{F}} \left( \frac{1}{\wedge^i \tilde{F} \sigma} \right)^{(-1)^r} \right) \right) \times R_\phi(z)^{(-1)^m}. \]
The variable $z$ has disappeared because
\[ \sum_{i=0}^m (-1)^i \dim \wedge^i \tilde{G} = \sum_{i=0}^m (-1)^i C_m^i = 0. \]
\(\square\)
2.1 The Reidemeister zeta functions of a continuous map.

Now suppose $X$ is a compact, connected polyhedron with universal cover $\tilde{X}$ and let $f : X \to X$ is a continuous self-map. We choose a lifting $\tilde{f} : \tilde{X} \to \tilde{X}$ of $f$ as reference. Let $\Gamma$ be the group of covering translations of $\tilde{X}$ over $X$. Then every lifting of $f$ can be written uniquely as $\gamma \circ \tilde{f}$, with $\gamma \in \Gamma$. So elements of $\Gamma$ serve as coordinates of liftings with respect to the reference $\tilde{f}$. Now for every $\gamma \in \Gamma$ the composition $\tilde{f} \circ \gamma$ is a lifting of $f$ so there is a unique $\gamma' \in \Gamma$ such that $\gamma' \circ \tilde{f} = \tilde{f} \circ \gamma$. This correspondence $\gamma \mapsto \gamma'$ is an endomorphism of $\Gamma$. As such it is dependent on the choice of $\tilde{f}$. However the class of the endomorphism modulo inner endomorphisms of $\Gamma$ depends only on the map $f$.

**Definition 1** The endomorphism $f_* : \Gamma \to \Gamma$ determined by the lifting $\tilde{f}$ of $f$ is defined by

$$f_*(\gamma) \circ \tilde{f} = \tilde{f} \circ \gamma.$$

**Lemma 3** ([19]) Lifting classes of $f$ are in one to one correspondence with $f_*$-conjugacy classes in $\pi$, the lifting class $[\gamma \circ \tilde{f}]$ corresponding to the $f_*$-conjugacy class of $\gamma$. We therefore have $R(f) = R(f_*)$.

Using this lemma we may apply all previous theorems to the Reidemeister zeta functions of continuous maps.

2.2 Convergence of the Nielsen Zeta Function

We denote by $R$ the radius of convergence of the Nielsen zeta function $N_f(z)$, and by $h(f)$ the topological entropy of a continuous map $f$. Let $h$ be the infimum of $h(g)$ where $g$ ranges over maps with the same homotopy type as $f$. The following is known.

**Theorem 12** ([13]) With the above notation $R \geq \exp(-h) > 0$.

The radius of convergence $R$ is directly connected with asymptotic Nielsen number which is defined to be the growth rate of the sequence $\{N(f^n)\}$ of the Nielsen numbers of the iterates of $f$. In this section we give a new proof that $R$ is positive, and give an exact algebraic lower bound for $R$ using the trace formulae (13) and (14) for generalized Lefschetz numbers.

We begin by recalling some known facts (see [22],[29],[1],[10]). The fundamental group $\pi = \pi_1(X,x_0)$ splits into $f_*$-conjugacy classes. Let $\pi_f$ denote the set of $f_*$-conjugacy classes, and $\mathbb{Z}\pi_f$ denote the Abelian group freely generated by $\pi_f$. We shall use the bracket notation $a \mapsto [a]$ for both projections $\pi \to \pi_f$ and $\mathbb{Z}\pi \to \mathbb{Z}\pi_f$. Let $x$ be a fixed point of $f$. Choose a path $c$ from $x_0$ to $x$. The $f_*$-conjugacy class in $\pi$ of the loop $c \cdot (f \circ c)^{-1}$, which is evidently independent of the choice of $c$, is called the coordinate of $x$. Two fixed points are in the same fixed point class $F$ iff they have the same coordinates. This $f_*$-conjugacy class is thus called the coordinate of
the fixed point class $F$ and denoted $cd_{\pi}(F, f)$. The generalized Lefschetz number or the Reidemeister trace [23] is defined as

$$L_{\pi}(f) := \sum_F \text{Index } (F, f) \cdot cd_{\pi}(F, f) \in \mathbb{Z}\pi_f,$$

the summation being over all essential fixed point classes $F$ of $f$. The Nielsen number $N(f)$ is the number of non-zero terms in $L_{\pi}(f)$, and the indices of the essential fixed point classes appear as the coefficients in $L_{\pi}(f)$. This invariant used to be called the Reidemeister trace because it can be computed as an alternating sum of traces on the chain level as follows ([23], [24]). Assume that $X$ is a finite cell complex and $f : X \to X$ is a cellular map.

A cellular decomposition $\{e^d_j\}$ of $X$ lifts to a $\pi$-invariant cellular structure on the universal covering $\tilde{X}$. Choose an arbitrary lift $\tilde{e}^d_j$ for each $e^d_j$. These lifts constitute a free $\mathbb{Z}\pi$-basis for the cellular chain complex of $\tilde{X}$. The lift $\tilde{f}$ of $f$ is also a cellular map. In every dimension $d$, the cellular chain map $\tilde{f}$ gives rise to a $\mathbb{Z}\pi$-matrix $\tilde{F}_d$ with respect to the above basis, i.e. $\tilde{F}_d = (a_{ij})$ if $\tilde{f}(\tilde{e}^d_i) = \sum_j a_{ij}\tilde{e}^d_j$, where $a_{ij} \in \mathbb{Z}\pi$. Then we have the Reidemeister trace formula

$$L_{\pi}(f) = \sum_d (-1)^d[\text{Tr } \tilde{F}_d] \in \mathbb{Z}\pi_f.$$  

We now describe alternative to the Reidemeister trace formula proposed by Jiang [20]. This approach is useful when we study the periodic points of $f$. The mapping torus $T_f$ of $f : X \to X$ is the space obtained from $X \times [0, \infty)$ by identifying $(x, s + 1)$ with $(f(x), s)$ for all $x \in X, s \in [0, \infty)$. On $T_f$ there is a natural semi-flow $\phi : T_f \times [0, \infty) \to T_f, \phi_t(x, s) = (x, s + t)$ for all $t \geq 0$. The function $f : X \to X$ is the return map of the semi-flow $\phi$. A point $x \in X$ and a positive number $\tau$ determine the orbit curve $\phi_{(x, \tau)} := \phi_t(x)_{0 \leq t \leq \tau}$ in $T_f$. Take the base point $x_0$ of $X$ as the base point of $T_f$. It is known that the fundamental group $H := \pi_1(T_f, x_0)$ is obtained from $\pi$ by adding a new generator $z$ and adding the relations $z^{-1}gz = f_* (g)$ for all $g \in \pi = \pi_1(X, x_0)$. Let $H_c$ denote the set of conjugacy classes in $H$ and let $\mathbb{Z}H$ be the integral group ring of $H$. Define also $\mathbb{Z}H_c$ to be the free Abelian group with basis $H_c$. We again use the bracket notation $a \to [a]$ for both projections $H \to H_c$ and $\mathbb{Z}H \to \mathbb{Z}H_c$. If $F^n$ is a fixed point class of $f^n$, then $f(F^n)$ is also fixed point class of $f^n$ and Index $(f(F^n), f^n) = \text{Index } (F^n, f^n)$. Thus $f$ acts as an index-preserving permutation among fixed point classes of $f^n$. We define an $n$-orbit class $O^n$ of $f$ to be the union of elements of an orbit of this action. In other words, two points $x, x' \in \text{Fix}(f^n)$ are said to be in the same $n$-orbit class of $f$ if and only if some $f^i(x)$ and some $f^j(x')$ are in the same fixed point class of $f^n$. The set $\text{Fix}(f^n)$ is a disjoint union of $n$-orbit classes. A point $x$ is a fixed point of $f^n$ if and only if $\phi_{(x, n)}$ is a closed curve. The free homotopy class of the closed curve $\phi_{(x, n)}$ will be called the $H$-coordinate of point $x$, written $cd_H(x, n) = [\phi_{(x, n)}] \in H_c$. It follows that periodic points $x$ of period $n$ and $x'$ of period $n'$ have the same $H$-coordinate if and only if $n = n'$, and $x$ and $x'$ belong to the same $n$-orbits class of $f$. 


Recently, Jiang \cite{20} has considered generalized Lefschetz number with respect to $H$:

$$L_H(f^n) := \sum_{O^n} \text{Index}(O^n, f^n) \cdot cd_H(O^n) \in \mathbb{Z}H_c.$$ \hspace{1cm} (10)

He proved the following trace formula:

$$L_H(f^n) = \sum_d (-1)^d [\text{Tr}(z \tilde{F}_d)^n] \in \mathbb{Z}H_c,$$ \hspace{1cm} (11)

where $\tilde{F}_d$ are the $\mathbb{Z}\pi$-matrices defined in (9) and $z \tilde{F}_d$ is regarded as a $\mathbb{Z}H$-matrix.

For any set $S$ let $\mathbb{Z}S$ denote the free Abelian group with basis $S$. Define a norm on $\mathbb{Z}S$ by

$$\| \sum_i k_i s_i \| := \sum_i |k_i| \in \mathbb{Z},$$ \hspace{1cm} (12)

where the $s_i$ in $S$ are all different.

For a $\mathbb{Z}H$-matrix $A = (a_{ij})$, define its norm by $\|A\| := \sum_{i,j} |a_{ij}|$. Then we have inequalities $\|AB\| \leq \|A\| \cdot \|B\|$ when $A, B$ can be multiplied, and $\|\text{Tr} A\| \leq \|A\|$ when $A$ is a square matrix. For a matrix $A = (a_{ij})$ in $\mathbb{Z}S$, its matrix of norms is defined to be the matrix $A^{\text{norm}} := (\|a_{ij}\|)$ which is a matrix of non-negative integers. In what follows, the set $S$ will be $\pi$, $H$ or $H_c$. We denote by $s(A)$ the spectral radius of $A$, $s(A) = \lim\limits_n (\|A^n\|)^{1/n}$, which coincides with the largest modulus of an eigenvalue of $A$.

**Theorem 13** For any continuous map $f$ of any compact polyhedron $X$ into itself the Nielsen zeta function has positive radius of convergence $R$, which admits following estimations

$$R \geq \frac{1}{\max_d \|z \tilde{F}_d\|} > 0$$ \hspace{1cm} (13)

and

$$R \geq \frac{1}{\max_d s(\tilde{F}_d^{\text{norm}})} > 0,$$ \hspace{1cm} (14)

where $\tilde{F}_d$ is as in (9).

**Proof.** By the homotopy type invariance of the invariants we can suppose that $f$ is a cellular map of a finite cell complex. By definition the Nielsen number $N(f^n)$ is the number of non-zero terms in $L_\pi(f^n)$. The norm $\|L_H(f^n)\|$ is the sum of absolute values of the indices of all the $n$-orbits classes $O^n$. It equals $\|L_\pi(f^n)\|$, the sum of absolute values of the indices of all the fixed point classes of $f^n$, because any two fixed point classes of $f^n$ contained in the same $n$-orbit class $O^n$ must have the same index. From this we have $N(f^n) \leq \|L_\pi(f^n)\| = \|L_H(f^n)\| = \sum_d (-1)^d [\text{Tr}(z \tilde{F}_d)^n] \leq \sum_d \|\text{Tr}(z \tilde{F}_d)^n\| \leq \sum_d \|z \tilde{F}_d^n\| \leq \sum_d \|(z \tilde{F}_d)^n\|^\frac{1}{n}$ (see \cite{20}). The radius of convergence $R$ is given by the Cauchy-Hadamard formula:

$$\frac{1}{R} = \lim\sup\limits_n \left(\frac{N(f^n)}{n}\right)^\frac{1}{n} = \lim\sup\limits_n (N(f^n))^{\frac{1}{n}}.$$
Therefore we have:

\[ R = \frac{1}{\limsup_n (N(f^n))^{\frac{1}{n}}} \geq \frac{1}{\max_d \|z\tilde{F}_d\|} > 0. \]

The inequalities:

\[ N(f^n) \leq \|L_\pi(f^n)\| = \|L_H(f^n)\| = \|\sum_d (-1)^d [\Tr (z\tilde{F}_d)^n]\| \leq \sum_d \|\Tr (z\tilde{F}_d)^n]\|

\[ \leq \sum_d \|\Tr (z\tilde{F}_d)^n\| \leq \sum_d \Tr ((z\tilde{F}_d)^n)_{\text{norm}} \leq \sum_d \Tr ((\tilde{F}_d)^n)_{\text{norm}} \]

\[ \leq \sum_d \Tr ((\tilde{F}_d)^n)_{\text{norm}} \]

together with the definition of spectral radius give the bound:

\[ R = \frac{1}{\limsup_n (N(f^n))^{\frac{1}{n}}} \geq \frac{1}{\max_d s(F_d^\text{norm})} > 0. \]

\[ \square \]

**Example 1** Let \( X \) be a surface with boundary, and \( f : X \rightarrow X \) a continuous map. Fadell and Husseini [6] devised a method of computing the matrices of the lifted chain map for surface maps. Suppose \( \{a_1, \ldots, a_r\} \) is a free basis for \( \pi_1(X) \). Then \( X \) has the homotopy type of a bouquet \( B \) of \( r \) circles which can be decomposed into one 0-cell and \( r \) 1-cells corresponding to the \( a_i \), and \( f \) has the homotopy type of a cellular map \( g : B \rightarrow B \). By the homotopy invariance, we may replace \( f \) by \( g \) in computations. The homomorphism \( \tilde{f}_* : \pi_1(X) \rightarrow \pi_1(X) \) induced by \( f \) or \( g \) is determined by the images \( b_i = \tilde{f}_*(a_i) \), \( i = 1, \ldots, r \). The fundamental group \( \pi_1(T_f) \) has a presentation \( \pi_1(T_f) = \langle a_1, \ldots, a_r, z | a_i z = z b_i, i = 1, \ldots, r \rangle \). Let

\[ D = (\frac{\partial b_i}{\partial a_j}) \]

be the Jacobian in Fox calculus (see [4]). Then, as pointed out in [4], the matrices of the lifted chain map \( \tilde{g} \) are

\[ \tilde{F}_0 = (1), \quad \tilde{F}_1 = D = (\frac{\partial b_i}{\partial a_j}). \]

Now, we may find bounds for the radius \( R \) using the above theorem.

### 3 Reidemeister Torsion as a Special Value

Reidemeister torsion is an algebraically defined quantity associated to an acyclic cochain complex. It is used to distinguish between complexes with the same homology. Roughly speaking, if Euler characteristic is regarded as a graded version generalization of dimension, then Reidemeister torsion may be viewed as a graded version of the
absolute value of the determinant. More precisely, let \( d^i : C^i \to C^{i+1} \) be a bounded, acyclic cochain complex of finite dimensional complex vector spaces. There is a chain contraction \( \delta^i : C^i \to C^{i-1} \) i.e. a linear map such that \( d \circ \delta + \delta \circ d = id \). We have linear maps \((d + \delta)^+ : C^+ := \oplus C^{2i} \to C^- := \oplus C^{2i+1} \) and \((d + \delta)^- : C^- \to C^+ \). Since \((d+\delta)^2 = id + \delta^2 \) is unipotent, it follows that \((d+\delta)^+ \) is bijective. Given positive densities \( \Delta_i \) on \( C^n \), we may define with respect to these densities, \( \tau(C^n, \Delta_i) := | \det(d+\delta)^+ | \) (see \([17]\)).

Reidemeister torsion is used in the following geometric setting. Let \( K \) be a finite simplicial complex and \( E \) a flat, finite dimensional, complex vector bundle over \( K \) with fibre \( V \). Denote by \( \rho_E : \pi_1(K) \to GL(V) \) the holonomy of \( E \). Suppose now that one has on each fibre of \( E \) a positive density which is locally constant on \( K \).

In terms of \( \rho_E \) this assumption means \( | \det \rho_E | = 1 \). The cochain complex \( C^i(K; E) \) with coefficients in \( E \) may be identified with the direct sum of copies of \( V \) associated to each \( i \)-cell \( \sigma \) of \( K \). This identification is achieved by choosing a base point in each component of \( K \) and a base point in each \( i \)-cell. By choosing a flat density on \( E \) we obtain a preferred density \( \Delta_i \) on each \( C^i(K, E) \). The Reidemeister torsion \( \tau(K; E) \) is then defined to be the positive real number \( \tau(C^*(K; E), \Delta_i) \). This has many nice properties. It is invariant under subdivisions of \( K \). Thus for a smooth manifold, one may unambiguously define \( \tau(K; E, D_i) \) to be the torsion of any smooth triangulation of \( K \). However, Reidemeister torsion is not an invariant under a general homotopy equivalence. This was in fact the reason for its introduction.

In the case that \( K \) is the circle \( S^1 \), let \( A \) be the holonomy of a generator of its fundamental group \( \pi_1(S^1) \). Then \( E \) is acyclic if and only if \( I - A \) is invertible and then

\[
\tau(S^1; E) = | \det(I - A) | \tag{15}
\]

Note that the choice of generator is irrelevant as \( I - A^{-1} = (A^{-1})(I - A) \) and \( | \det(-A^{-1}) | = 1 \).

It might be expected that the Reidemeister torsion counts something geometric (like the Euler characteristic). D. Fried showed that it counts the periodic orbits of a flow and the periodic points of a map. We will show that Reidemeister torsion counts the periodic point classes of a map (fixed point classes of the iterations of the map).

Some further properties of \( \tau \) describe its behavior under bundles.

Suppose \( p : X \to B \) is a simplicial bundle with fiber \( F \), where \( F, B, X \) are all finite complexes and \( p^{-1} \) lifts subcomplexes of \( B \) to subcomplexes of \( X \). We assume here that \( E \) is a flat, complex vector bundle over \( B \). We form its pullback \( p^*E \) over \( X \). Note that the vector spaces \( H^i(p^{-1}(b), \mathcal{C}) \) with \( b \in B \) form a flat vector bundle over \( B \), which we denote \( H^i F \). The integral lattice in \( H^i(p^{-1}(b), \mathbb{R}) \) determines a flat density by the condition that the covolume of the lattice is 1. Suppose that the bundle \( E \otimes H^i F \) is acyclic for all \( i \). Under these conditions D. Fried \([17]\) has shown that the bundle \( p^*E \) is acyclic, and

\[
\tau(X; p^*E) = \prod_i \tau(B; E \otimes H^i F)^{(-1)^i}. \tag{16}
\]

Now let \( f : X \to X \) be a homeomorphism of a compact polyhedron \( X \), and let \( T_f := (X \times I)/(x, 0) \sim (f(x), 1) \) be the mapping torus of \( f \). We shall consider the
bundle \( p : T_f \to S^1 \) over the circle \( S^1 \). Given a flat, complex vector bundle \( E \) with finite dimensional fibre over the base \( S^1 \), we form its pullback \( p^*E \) over \( T_f \). The vector spaces \( H^i(p^{-1}(b), \mathbb{C}) \) with \( b \in S^1 \) also form flat vector bundles over \( S^1 \), which we denote \( H^iF \). The integral lattice in \( H^i(p^{-1}(b), \mathbb{R}) \) determines a flat density by the condition that the covolume of the lattice is 1. Suppose that the bundle \( E \otimes H^iF \) is acyclic for all \( i \). Under these conditions, D. Fried \( [17] \) has shown that \( p^*E \) is also acyclic and the following holds:

\[
\tau(T_f; p^*E) = \prod_i \tau(S^1; E \otimes H^iF)^{(-1)^i}.
\] (17)

Let \( g \) be the preferred generator of the group \( \pi_1(S^1) \) and let \( A = \rho(g) \), where \( \rho : \pi_1(S^1) \to GL(V) \). The holonomy around \( g \) of the bundle \( E \otimes H^iF \) is then \( A \otimes f_i^* \).

Since \( \tau(E) = |\det(I - A)| \) it follows from (24) that

\[
\tau(T_f; p^*E) = \prod_i |\det(I - A \otimes f_i^*)|^{(-1)^i}.
\] (18)

Now consider the special case in which \( E \) is one-dimensional, so the holonomy \( A \) is a complex scalar \( \lambda \) with absolute value one. In terms of the rational function \( L_f(z) \) we have \( [17] \):

\[
\tau(T_f; p^*E) = \prod_i |\det(I - \lambda f_i^*)|^{(-1)^i} = |L_f(\lambda)|^{-1}
\] (19)

**Theorem 14** Let \( \phi : G \to G \) be a group automorphism, where \( G \) is the direct sum of a finite group with a finitely generated Abelian group, then

\[
\tau \left( \hat{T}^n \phi; p^*E \right) = |L_\phi(\lambda)|^{-1} = |R_\phi(\sigma \lambda)|^{(-1)^r+1},
\]

where \( \lambda \) is the holonomy of \( E \) around \( S^1 \) and \( r \) and \( \sigma \) are the constants described in Theorem 3.

**Proof.** We know from Theorem 4 that \( R(\phi^n) \) is the number of fixed points of the map \( \hat{T}^n \). It remains to show that the number of fixed points of \( \hat{T}^n \phi \) is equal to the absolute value of its Lefschetz number. We assume without loss of generality that \( n = 1 \). We are assuming throughout that \( R(\phi) \) is finite, so the fixed points of \( \hat{T} \phi \) form a discrete set. We therefore have

\[
L(\phi) = \sum_{x \in \text{Fix } \hat{T} \phi} \text{Index } (\hat{T} \phi, x).
\]

Since \( \phi \) is a group endomorphism, the trivial representation \( x_0 \in \hat{G} \) is always fixed. Let \( x \) be any fixed point of \( \hat{T} \). Since \( \hat{G} \) is union of tori \( \hat{G}_0, \ldots, \hat{G}_t \) and \( \hat{T} \) is a linear map, we can shift any two fixed points onto one another without altering the map \( \hat{T} \). Therefore all fixed points have the same index. It is now sufficient to show that \( \text{Index } (\hat{T} \phi, x_0) = \pm 1 \). This follows because the map on the torus

\[
\hat{T} \phi : \hat{G}_0 \to \hat{G}_0
\]
lifts to a linear map of the universal cover, which is an euclidean space. The index is then the sign of the determinant of the identity map minus this lifted map. This determinant cannot be zero, because $1 - \hat{\phi}$ must have finite kernel by our assumption that the Reidemeister number of $\phi$ is finite (if $\det(1 - \hat{\phi}) = 0$ then the kernel of $1 - \hat{\phi}$ is a positive dimensional subspace of $\hat{G}$, and therefore infinite).

\[\square\]

### 4 Concluding remarks and open questions

For the case of almost nilpotent groups (ie. groups with polynomial growth, in view of Gromov’s Theorem [18]) we believe that some power of the Reidemeister zeta function is a rational function and that the congruences for the Reidemeister numbers are also true. We intend to prove this conjecture by identifying the Reidemeister number on the nilpotent part of the group with the number of fixed points in the direct sums of the duals of the quotients of successive terms in the central series. We then hope to show that the Reidemeister number of the whole endomorphism is a sum of numbers of orbits of such fixed points under the action of the finite quotient group (ie the quotient of the whole group by the nilpotent part). The situation for groups with exponential growth is very different. There one can expect the Reidemeister number to be infinite as long as the endomorphism is injective.

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