CAUCHY-FANTAPPIE TYPE OPERATORS AND DUALITY ON POLETSKY-STESSIN HARDY SPACES OF COMPLEX ELLIPSOIDS

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Abstract. In the first part of this study we consider the boundedness and compactness properties of Cauchy-Fantappie type operators on Poletsky-Stessin Hardy spaces $H^p_u(B^p)$ of complex ellipsoids. We show that boundedness and compactness criteria are given by the Carleson conditions. In addition we give a basic compactness property for the subsets of $H^p_u(B^p)$ spaces and the characterization of weakly convergent sequences in $H^p_u(B^p)$. In the second part we will discuss the dual complement of the complex ellipsoid and we will give a duality result for $H^p_u(B^p)$ spaces in the sense of Grothendieck-Köthe-da Silva.

Introduction

The aim of this paper is to study the behavior of Cauchy-Fantappie type operators on Poletsky-Stessin Hardy spaces [6] and to give a duality result for these spaces in the sense of [2]. In [8], [9] the Cauchy-Fantappie projection associated with Monge-Ampère measure is considered in relation with boundary values of Poletsky-Stessin Hardy spaces and Carleson measures. In this study we will examine the general Cauchy-Fantappie type operators in relation with Carleson measures. An analogous study has been done in [1] for Toeplitz operators in the setting of classical Bergman spaces of strictly pseudoconvex domains. In this paper we will work with the much more general holomorphic function spaces, namely Poletsky-Stessin Hardy spaces and structurally much more complex domains, the complex ellipsoids.

The organization of this paper is as follows: In Section 1, we recall the Poletsky-Stessin Hardy spaces, $H^p_u(B^p)$, on the complex ellipsoid $B^p$ and we introduce the Cauchy-Fantappie integral associated with the Monge-Ampère measure $\mu_u$, together with an integral representation for $H^p_u(B^p)$. The main results of this study are given in the following sections: In Section 2, we first introduce general Cauchy-Fantappie type operator associated with a finite Borel measure $\mu$ and we give boundedness and compactness criteria for these operators in terms of Carleson measures. In addition we will give a discussion of a compactness property for subsets of $H^p_u(B^p)$ and a characterization of vanishing Carleson measures of $H^p_u(B^p)$. Finally, in Section 3, we first give a brief introduction about Grothendieck-Köthe-da Silva duality for the spaces of holomorphic functions defined in a convex domain and then using a general characterization of dual complements of Reinhardt domains [2] we give the dual complement of some special type of complex ellipsoids. Finally, we prove a duality result for Poletsky-Stessin Hardy space of complex ellipsoid.

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1. Preliminaries

In this section we will give the preliminary definitions and some important results that we will use throughout this study. Before proceeding with Poletsky-Stessin Hardy spaces let us first recall the classical Hardy spaces given by [10]. Let $\Omega$ be a smoothly bounded, hyperconvex domain in $\mathbb{C}^n$ and $\lambda$ be a characterizing function for $\Omega$ which is defined in a neighborhood of $\overline{\Omega}$ i.e. $\lambda$ is smooth, $\lambda(x) < 0$ if and only if $x \in \Omega$, $\partial \Omega = \{ \lambda(x) = 0 \}$ and $|\nabla \lambda(x)| > 0$ if $x \in \partial \Omega$. (The last condition is equivalent to $\partial \lambda / \partial \nu_x > 0$ where $\nu_x$ is the outward normal at $x$.) Let $\Omega_r = \{ z : \lambda(z) < r : r < 0 \}$ and $\partial \Omega_r = \{ z : \lambda(z) = r \}$.

In [10], E.M. Stein defines the class $H^p$ as:

$$H^p = \{ f | \ f \ \text{holomorphic in} \ \Omega, \ \sup_{r<0} \int_{\partial \Omega_r} |f|^p d\sigma_r < \infty \}$$

where $d\sigma_r$ is the induced surface area measure on $\partial \Omega_r$. This space is equipped with the norm

$$\|f\|_p^p = \sup_{r<0} \int_{\partial \Omega_r} |f|^p d\sigma_r.$$ 

The space $H^p(\Omega)$ does not depend on the characterizing function $\lambda$ used in order to define $\Omega$ and one gets equivalent norms for different characterizing functions. In [6], Poletsky & Stessin introduced new Hardy type classes of holomorphic functions on hyperconvex domains in $\mathbb{C}^n$. Before defining these new classes let us first give some preliminary definitions. Let $\phi : \Omega \to [−\infty, 0)$ be a negative, continuous, plurisubharmonic exhaustion function for $\Omega$. Following [4] we define the pseudoball:

$$B(r) = \{ z \in \Omega : \phi(z) < r \} , \ r \in [−\infty, 0),$$

and pseudosphere:

$$S(r) = \{ z \in \Omega : \phi(z) = r \} , \ r \in [−\infty, 0),$$

and set

$$\phi_r(z) = \max\{ \phi(z), r \} , \ r \in (−\infty, 0).$$

In [4], Demailly introduced the Monge-Ampère measures in the sense of currents as :

$$\mu_{\phi,r} = (dd^c \phi_r)^n - \chi_{\Omega \setminus B(r)}(dd^c \phi)^n \ r \in (−\infty, 0).$$

It is clear from the definition that these measures are supported on $S(r)$. Demailly in [5], proved the so-called Lelong-Jensen Formula which is stated as follows:

**Theorem 1.1.** Let $r < 0$ and $\phi$ be a plurisubharmonic function on $\Omega$ then for any negative, continuous, plurisubharmonic exhaustion function $u$

$$\int_{S_u(r)} \phi d\mu_{u,r} - \int_{B_u(r)} \phi (dd^c u)^n = \int_{B_u(r)} (r - u) dd^c \phi (dd^c u)^{n-1}. $$

In this study we will use the boundary value characterization of Poletsky-Stessin Hardy spaces in most of the results so let us mention boundary measures which were introduced by Demailly in [5]. Now let $\phi : \Omega \to [−\infty, 0)$ be a continuous,
plurisubharmonic exhaustion for \( \Omega \) and suppose that the total Monge-Amplère mass is finite that is, we assume that 

\[
MA(\varphi) = \int_{\Omega} (dd^c \varphi)^n < \infty.
\]

Then as \( r \) approaches to 0, \( \mu_{\varphi,r} \) converges to a positive measure \( \mu_{\varphi} \) weak*-ly on \( \Omega \) with total mass \( \int_{\Omega} (dd^c \varphi)^n \) and supported on \( \partial \Omega \). This measure \( \mu_{\varphi} \) is called the Monge-Amplère measure on the boundary associated with the exhaustion \( \varphi \).

Now we can introduce the Poletsky-Stessin Hardy classes, which will be our main focus throughout this study. In [6], Poletsky & Stessin gave the definition of new Hardy spaces using Monge-Amplère measures as:

**Definition 1.** \( H^p_{\varphi}(\Omega) \) for \( p > 0 \), is the space of functions \( f \in \mathcal{O}(\Omega) \) such that

\[
\limsup_{r \to 0^-} \int_{S_{\varphi}(r)} |f|^p d\mu_{\varphi,r} < \infty.
\]

The norm on these spaces is given by:

\[
\|f\|_{H^p_{\varphi}} = \left( \lim_{r \to 0^-} \int_{S_{\varphi}(r)} |f|^p d\mu_{\varphi,r} \right)^{\frac{1}{p}}
\]

and with respect to these norm the spaces \( H^p_{\varphi}(\Omega) \) are Banach spaces [6].

The next theorem gives us the comparison between Poletsky-Stessin Hardy spaces and the classical Hardy spaces ([9], Theorem 1.2):

**Theorem 1.2.** Suppose that \( \Omega \) is a smoothly bounded, hyperconvex domain with a plurisubharmonic characterizing function \( \rho \). Then \( H^p(\Omega) \subseteq H^p_{\rho}(\Omega), 1 \leq p < \infty \).

From now on we will focus on Poletsky-Stessin Hardy spaces on the complex ellipsoids in \( \mathbb{C}^n \) which are considered as model cases for domains of finite type.

It should be noted that although complex ellipsoids are convex domains they are not strictly pseudoconvex since they have Levi flat points at the boundary. The complex ellipsoid \( \mathbb{B}^p \subset \mathbb{C}^n \) is given as

\[
\mathbb{B}^p = \{ z \in \mathbb{C}^n, \rho(z) = \sum_{j=1}^{n} |z_j|^{2p_j} - 1 < 0 \}
\]

where \( p = (p_1, p_2, \ldots, p_n) \in \mathbb{Z}^n \). One can easily see that \( u(z) = \log(|z_1|^{2p_1} + |z_2|^{2p_2} + \ldots + |z_n|^{2p_n}) \) is a continuous, plurisubharmonic exhaustion function for \( \mathbb{B}^p \) so we can consider the Poletsky-Stessin Hardy spaces \( H^p_{u}(\mathbb{B}^p) \) associated with this exhaustion function.

Let \( d(\xi, z) = |v(\xi, z)| + |v(z, \xi)| \) be the quasi-metric defined on \( \overline{\mathbb{B}^p} \) where \( v(\xi, z) = \langle \partial \rho(\xi), \xi - z \rangle \). Then explicitly \( v(\xi, z) = \sum_{j=1}^{n} p_j |\xi_j|^{2(p_j-1)} \xi_j (\xi_j - z_j) \) and define the boundary balls as \( B(\xi, \varepsilon) = \{ \xi \in \partial \overline{\mathbb{B}^p}, d(\xi, z) < \varepsilon \} \). It is shown that \((\partial \mathbb{B}^p, d, d\mu_u)\) is a space of homogenous type ([3], pg:1483) and \( \frac{1}{(v(\xi, z))^n} \) is a standard kernel.

The Cauchy-Fantappie integral (from now on we will refer as CF-integral) of an \( L^p(d\mu_u) \) function \( f^* \) is defined as

\[
Hf(z) = \left( \frac{1}{2\pi i} \right)^n \int_{\partial \overline{\mathbb{B}^p}} f^*(\xi) d\mu_u(\xi) \left( v(\xi, z)^n \right)^n
\]
In [3], Hansson considered the boundedness of Cauchy-Fantappie integral. In his work he showed the homogeneity of the boundary of the complex ellipsoid with respect to the quasi-metric $d$ and the boundary measure $\partial \rho \wedge (\partial \partial \rho)^{n-1}$ where the function $\rho$ is defined as $\rho(z) = \sum_{j=1}^{n} |z_j|^{2p_j} - 1$. In fact an easy calculation shows that this measure is the boundary Monge-Ampère measure associated with the exhaustion function $u(z) = \log(|z_1|^{2p_1} + |z_2|^{2p_2} + ... + |z_n|^{2p_n})$, $p = (p_1, p_2, ..., p_n) \in \mathbb{Z}^n$ of the complex ellipsoid $\mathbb{B}^p$.

Before proceeding to further results let us also briefly discuss the Cauchy-Fantappie kernel. In the theory of holomorphic functions in one variable a fundamental tool is Cauchy integral formula and in the case of several variables one wants a suitable generalization to Cauchy integral. One of the possible choices for the generalization is the so called Szegö kernel however except for a few domains Szegö kernel has no explicit formulation. One other choice is the well known Bochner-Martinelli kernel but the major shortcoming of this kernel is that it is not holomorphic in $z$ variable (For details see [7]). Contrary to Bochner-Martinelli kernel, Cauchy-Fantappie kernel is holomorphic in $z$ hence it is a natural generalization of Cauchy kernel to multivariable case and it has reproducing property for the functions in the algebra $A(\mathbb{B}^p)$ ([7], Theorem 3.4).

Hardy spaces which are examined in [3] are exactly the Poletsky-Stessin Hardy spaces $H^p_u(\mathbb{B}^p)$ that are generated by the exhaustion function $u$. In [8], [9] it is shown that for the functions in $H^p_u(\mathbb{B}^p)$ the boundary value function $f^* \in L^p(d\mu_u)$ exists so the CF-integral of $f^*$ is well-defined. In [8] we show that CF-integral has reproducing property for the functions in $H^p_u(\mathbb{B}^p)$:

**Proposition 1.1.** Let $f \in H^p_u(\mathbb{B}^p)$ be a holomorphic function then

$$f(z) = Hf(z) = \left(\frac{1}{2\pi i}\right)^n \int_{\partial \mathbb{B}^p} \frac{f^*(\xi)d\mu_u(\xi)}{(v(\xi, z))^n}$$

2. **CAUCHY-FANTAPPIE TYPE OPERATORS AND CARLESON MEASURES**

In this section we will consider the boundedness and compactness conditions of Cauchy-Fantappie type operators on Poletsky-Stessin Hardy spaces $H^p_u(\mathbb{B}^p)$ of the complex ellipsoid. In [1], Abate et al. considered the relation between the Toeplitz operators on classical Bergman spaces and Carleson measures on strongly pseudoconvex domains. In this section of our study we will follow an analogous approach to [1] and we will give boundedness and compactness criteria for Cauchy-Fantappie operators in relation with Carleson measures. Moreover, we will give a compactness criterion for the subsets of Poletsky-Stessin Hardy spaces $H^p_u(\mathbb{B}^p)$ and a characterization of vanishing Carleson measures on $H^p_u(\mathbb{B}^p)$.

In this study we will deal with Cauchy-Fantappie type operators which are somehow different and more general than the classical Cauchy-Fantappie integral operators because instead of using the classical Cauchy-Fantappie forms we just combine the Cauchy-Fantappie kernel with an arbitrary positive Borel measure. Via this method we aim to understand the inclusion relations between Poletsky-Stessin Hardy spaces and other classes of holomorphic functions boundary values of which are controlled in the Lebesgue spaces of the boundary.
Definition 2. Let $\mu$ be a finite, positive Borel measure on $\mathbb{B}^p$. For $f \in H_u^p(\mathbb{B}^p)$, the Cauchy-Fantappie operator associated to $\mu$ is given by

$$H_\mu(f) = \int_{\mathbb{B}^p} \frac{f(\xi)}{(v(z,\xi))^n} d\mu(\xi)$$

Note that $H_\mu$ is loosely defined here because it is not clear when the integral above will converge, even if the measure $\mu$ is finite as the kernel itself can be unbounded. We should have assumed that all measures in consideration satisfy the condition:

$$\int_{\mathbb{B}^p} \frac{1}{(v(z,\xi))^n} d\mu < \infty$$

However in this study we work with the Carleson measures and this allows us not to take the above condition into consideration. Let us now introduce these special measures:

Definition 3. Let $H_u^p(\mathbb{B}^p)$ be the Poletsky-Stessin Hardy space on complex ellipsoid $\mathbb{B}^p$. Given $p > 1$, a finite Borel measure $\mu$ on $\mathbb{B}^p$ is a Carleson measure of $H_u^p(\mathbb{B}^p)$ if there is a continuous inclusion $H_u^p(\mathbb{B}^p) \hookrightarrow L^p(\mu)$ that is, there is a constant $C > 0$ such that

$$\forall f \in H_u^p(\mathbb{B}^p), \quad \int_{\mathbb{B}^p} |f|^p d\mu \leq C \|f\|_{H_u^p(\mathbb{B}^p)}^p$$

Furthermore, say that $\mu$ is vanishing Carleson measure of $H_u^p(\mathbb{B}^p)$ if the inclusion $H_u^p(\mathbb{B}^p) \hookrightarrow L^p(\mu)$ is compact.

It is important to note that as result of the Lelong-Jensen formula we can easily see that the most canonical examples of Carleson measures of $H_u^p(\mathbb{B}^p)$ are the Monge-Ampère measures $(dd^cu)^n$ themselves so we have a considerable class of measures to work with.

Theorem 2.1. Let $\mathbb{B}^p$ be the complex ellipsoid. Given $1 \leq p < \infty$,

1. $H_\mu(f) : H_u^p(\mathbb{B}^p) \rightarrow H_u^p(\mathbb{B}^p)$ is a continuous operator if and only if $\mu$ is a Carleson measure of $H_u^p(\mathbb{B}^p)$.

2. $H_\mu(f) : H_u^p(\mathbb{B}^p) \rightarrow H_u^p(\mathbb{B}^p)$ is a compact operator if and only if $\mu$ is a vanishing Carleson measure of $H_u^p(\mathbb{B}^p)$.

Proof. (1) Let $p > 1$ and $\mu$ be a Carleson measure of $H_u^p(\mathbb{B}^p)$, then

$$|H_\mu(f)| = \left| \int_{\mathbb{B}^p} \frac{f(z)}{(v(z,\xi))^n} d\mu(\xi) \right| \leq \int_{\mathbb{B}^p} \left| \frac{f(\xi)}{(v(\xi,z))^n} \right| d\mu(\xi)$$

so if we integrate both sides with respect to $d\mu_{u,r}$ we have the following

$$\int_{S_u(r)} |H_\mu(f)|^p d\mu_{u,r} \leq \int_{S_u(r)} \int_{\mathbb{B}^p} \left| \frac{f(\xi)}{(v(\xi,z))^n} \right|^p d\mu(\xi) d\mu_{u,r}(z)$$

$$= \int_{\mathbb{B}^p} |f(\xi)|^p \left( \int_{S_u(r)} \frac{1}{(v(\xi,z))^n} d\mu_{u,r}(z) \right) d\mu(\xi)$$

Now since $\int_{S_u(r)} \frac{1}{(v(\xi,z))^n} d\mu_{u,r}(z)$ increases we have

$$\int_{S_u(r)} \frac{1}{(v(\xi,z))^n} d\mu_{u,r}(z) \leq \int_{\partial \mathbb{B}^p} \frac{1}{(v(\xi,z))^n} d\mu_u(z)$$
First of all from (3) we see that
\[ \int_{S_{u}(r)} |H_{\mu}(f)|^{p}d\mu_{u,r} \leq C \int_{\mathbb{B}^{p}} |f|^{p}d\mu(\xi) \]
however \( \mu \) is a Carleson measure for \( H_{\mu}^{p}(\mathbb{B}^{p}) \) so we have
\[ \int_{S_{u}(r)} |H_{\mu}(f)|^{p}d\mu_{u,r} \leq A\|f\|_{H_{\mu}^{p}(\mathbb{B}^{p})}^{p} \]
hence as \( r \to 0 \) we have \( \|H_{\mu}(f)\|_{H_{\mu}^{p}(\mathbb{B}^{p})} \leq A\|f\|_{H_{\mu}^{p}(\mathbb{B}^{p})} \) so \( H_{\mu}(f) : H_{\mu}^{p}(\mathbb{B}^{p}) \to H_{\mu}^{p}(\mathbb{B}^{p}) \) continuously.
Conversely, suppose \( H_{\mu}(f) : H_{\mu}^{p}(\mathbb{B}^{p}) \to H_{\mu}^{p}(\mathbb{B}^{p}) \) continuously and let \( f \in H_{\mu}^{p}(\mathbb{B}^{p}) \) be arbitrary,
\[ \int_{\mathbb{B}^{p}} |f|^{p}d\mu = \int_{\mathbb{B}^{p}} \left( \int_{\partial \mathbb{B}^{p}} \frac{f(z)}{|v(\xi,z)|^{n}}d\mu_{u}(\xi) \right)^{p}d\mu \leq \int_{\mathbb{B}^{p}} \left( \int_{\partial \mathbb{B}^{p}} \frac{1}{|v(\xi,z)|^{n_{p}}}d\mu_{u}(\xi) \right)\|H_{\mu}(f)\|_{H_{\mu}^{p}(\mathbb{B}^{p})}^{p}d\mu_{u} \]
but from the continuity assumption we have \( \|H_{\mu}(1)\|_{H_{\mu}^{p}(\mathbb{B}^{p})} \leq C\|1\|_{H_{\mu}^{p}(\mathbb{B}^{p})}^{p} \)
and thus for some \( C > 0 \),
\[ \int_{\mathbb{B}^{p}} |f|^{p}d\mu \leq C \int_{\partial \mathbb{B}^{p}} |f(\xi)|^{p}d\mu_{u} \]
which gives that \( \mu \) is a Carleson measure for \( H_{\mu}^{p}(\mathbb{B}^{p}) \).

(2) First of all from (3) we see that \( H_{\mu} \) maps \( L^{p}(\mu) \) into \( L^{p}(d\mu_{u,r}) \) continuously (independent from \( r \)) and since \( \mu \) is a vanishing Carleson measure we have the inclusion \( I : H_{\mu}^{p}(\mathbb{B}^{p}) \to L^{p}(\mu) \) compact. Therefore, \( H_{\mu}(f) : H_{\mu}^{p}(\mathbb{B}^{p}) \to H_{\mu}^{p}(\mathbb{B}^{p}) \) is obtained as a composition of a bounded operator with a compact operator, so it is compact.
For the converse direction let us take \( \psi \) as identity on \( \mathbb{B}^{p} \) in (9), Theorem 3.3, then we see that a Borel measure \( \mu \) is vanishing Carleson if and only if
\[ \frac{\mu(B(\xi, \varepsilon))}{\mu(B(\xi, \varepsilon))} \to 0 \]
as \( \varepsilon \to 0 \) uniformly on \( \xi \in \partial \mathbb{B}^{p} \) where \( B(\xi, \varepsilon) \) is the boundary ball given in (9), Section 3. Now take \( f^{*} = \chi_{B(\xi, \varepsilon)} \) then since \( H_{\mu} \) is compact we have the following
\[ \frac{\|H_{\mu}(hf^{*})\|_{H_{\mu}^{p}(\mathbb{B}^{p})}}{\|hf^{*}\|_{H_{\mu}^{p}(\mathbb{B}^{p})}} \to 0 \]
as \( \varepsilon \to 0 \). If we write this explicitly we obtain that
\[ \int_{S_{u}(r)} \left( \int_{\mathbb{B}^{p}} \frac{\chi_{B(\xi, \varepsilon)}(v(\xi,z))^{n}}{|v(\xi,z)|^{n_{p}}}d\mu_{u,r} \right)^{p}d\mu_{u,r} = \int_{S_{u}(r)} \left( \int_{B(\xi, \varepsilon)} \frac{1}{|v(\xi,z)|^{n_{p}}}d\mu_{u,r} \right)^{p}d\mu_{u,r} \approx \int_{B(\xi, \varepsilon)} \left( \int_{S_{u}(r)} \frac{1}{|v(\xi,z)|^{n_{p}}}d\mu_{u,r} \right) d\mu = C \int_{B(\xi, \varepsilon)} d\mu = C \mu(B(\xi, \varepsilon)) \]
so as \( r \) approaches to 0 we obtain \( \|H_{\mu}(hf^{*})\|_{H_{\mu}^{p}(\mathbb{B}^{p})} \approx C \mu(B(\xi, \varepsilon)) \) and then
\[ \frac{\|H_{\mu}(hf^{*})\|_{H_{\mu}^{p}(\mathbb{B}^{p})}}{\|hf^{*}\|_{H_{\mu}^{p}(\mathbb{B}^{p})}} \approx \frac{C \mu(B(\xi, \varepsilon))}{\mu(B(\xi, \varepsilon))} \to 0 \]
Hence $\mu$ is a vanishing Carleson measure for $H^p_u(\mathbb{B}^p)$.

Now we will give a compactness property for the subsets of Poletsky-Stessin Hardy spaces $H^p_u(\mathbb{B}^p)$ and a characterization of vanishing Carleson measures on $H^p_u(\mathbb{B}^p)$ but before that we need to recall the following from the leading work of Poletsky & Stessin:

**Definition 4.** Let $D$ be a hyperconvex domain in $\mathbb{C}^n$ and $v$ be a continuous, plurisubharmonic exhaustion function. For $\phi$ being a non-negative plurisubharmonic function on $D$, we have

$$\|\phi\|_v \leq \lim_{r \to 0} \int_{S_u(r)} \phi d\mu_{v,r}$$

**Theorem 2.2** ([6], Theorem 3.6). Let $v$ be a continuous, plurisubharmonic exhaustion function on a hyperconvex domain $D$. Then for any compact set $K \subset D$ there is a constant $C$ such that for all $w \in K$, and all non-negative plurisubharmonic functions $\varphi$ on $D$ we have

$$\varphi(w) \leq C \|\varphi\|_v$$

As an immediate consequence of the above result we have the following

**Corollary 2.1.** Let $1 \leq p \leq \infty$. Then for every relatively compact subdomain $D_0 \subset \subset \mathbb{B}^p$ we can find a constant $C = C(D_0, p) > 0$ such that

$$\sup_{z \in D_0} |f(z)| \leq C \|f\|_{H^p_u(\mathbb{B}^p)}$$

for all $f \in H^p_u(\mathbb{B}^p)$

Now using this we give a basic compactness property for the subsets of Poletsky-Stessin Hardy space $H^p_u(\mathbb{B}^p)$ on complex ellipsoid $\mathbb{B}^p$:

**Lemma 2.1.** Let $\mathbb{B}^p$ be the complex ellipsoid, $1 \leq p < \infty$. Then

(i) If $\{f_k\} \subset H^p_u(\mathbb{B}^p)$ is a norm bounded sequence converging uniformly on compact subsets to $h \in \mathcal{O}(\mathbb{B}^p)$, then $h \in H^p_u(\mathbb{B}^p)$.

(ii) The inclusion $H^p_u(\mathbb{B}^p) \hookrightarrow \mathcal{O}(\mathbb{B}^p)$ is compact, that is any norm bounded subset of $H^p_u(\mathbb{B}^p)$ is relatively compact in $\mathcal{O}(\mathbb{B}^p)$.

**Proof.** (i) Assume that $\{f_k\} \subset H^p_u(\mathbb{B}^p)$ is a norm bounded sequence converging uniformly on compact subsets to $h \in \mathcal{O}(\mathbb{B}^p)$. Then,

$$\int_{S_u(r)} |h|^p d\mu_{u,r} = \int_{S_u(r)} \lim_{k \to \infty} |f_k|^p d\mu_{u,r} \leq \lim_{k \to \infty} \int_{S_u(r)} |f_k|^p d\mu_{u,r} \leq \sup_k \|f_k\|_{H^p_u(\mathbb{B}^p)}$$

by Fatou’s Lemma and thus as $r \to 0$ we have $\|h\|_{H^p_u(\mathbb{B}^p)} \leq \sup_k \|f_k\|_{H^p_u(\mathbb{B}^p)} < \infty$ and $h \in H^p_u(\mathbb{B}^p)$ as claimed.

(ii) We have to prove that any norm bounded sequence in $H^p_u(\mathbb{B}^p)$ admits a subsequence converging uniformly on compact subsets. But indeed previous theorem says that sup-norm on a relatively compact subset $D_0 \subset \subset \mathbb{B}^p$ of any $f \in H^p_u(\mathbb{B}^p)$ is bounded by a constant times its $H^p_u(\mathbb{B}^p)$-norm. Therefore, if $\{f_k\} \subset H^p_u(\mathbb{B}^p)$ is norm bounded, by considering $B_u(r)$’s as an increasing
exhaustion and applying Montel’s Theorem to each $B_n(r)$ we obtain a subsequence $\{f_{k_n}\}$ converging uniformly on compacta to a holomorphic $h \in \mathcal{O} (\mathbb{B}^p)$ and moreover $h \in H_u^p (\mathbb{B}^p)$. 

\[ \square \]

**Proposition 2.1.** Let $\mu$ be a finite, positive Borel measure on $\overline{\mathbb{B}}^p$ and $1 < p < \infty$. Then $\mu$ is a vanishing Carleson measure of $H_u^p (\mathbb{B}^p)$ if and only if $\| f_k \|_{L^p (\mu)} \to 0$ for all norm bounded sequences $\{f_k\} \subset H_u^p (\mathbb{B}^p)$ converging to 0 uniformly on compacta.

**Proof.** Assume that $\mu$ is a vanishing Carleson measure of $H_u^p (\mathbb{B}^p)$ then by definition, the inclusion $H_u^p (\mathbb{B}^p) \hookrightarrow L^p (\mu)$ is compact and take $\{f_k\} \subset H_u^p (\mathbb{B}^p)$ norm bounded and converging to 0 uniformly on compacta. In particular, $\{f_k\}$ is relatively compact in $L^p (\mu)$; we must prove that $f_k \to 0$ in $L^p (\mu)$. Now, for $0 < h_0 < 1$,

\[
\int_{\mathbb{B}^p} |f_k|^p d\mu = \int_{\mathbb{B}^p \setminus (1-h_0)\mathbb{B}^p} |f_k|^p d\mu + \int_{(1-h_0)\mathbb{B}^p} |f_k|^p d\mu
\]

and the second integral on the right can be made arbitrarily small since $f_k \to 0$ uniformly on compacta. For the first integral by ([9], proof of Theorem 3.3) we know that by choosing appropriate $h_0$ we have,

\[
\int_{\mathbb{B}^p \setminus (1-h_0)\mathbb{B}^p} |f_k|^p d\mu \leq C\varepsilon \| f_k \|_{H^p_u (\mathbb{B}^p)}
\]

for arbitrary $\varepsilon > 0$ since $\mu$ is a vanishing Carleson measure of $H_u^p (\mathbb{B}^p)$. Therefore

\[
\int_{\mathbb{B}^p} |f_k|^p d\mu \to 0
\]

as claimed.

Conversely assume that all norm bounded sequences in $H_u^p (\mathbb{B}^p)$ converging to 0 uniformly on compacta converge to 0 in $L^p (\mu)$. To prove that the inclusion $H_u^p (\mathbb{B}^p) \hookrightarrow L^p (\mu)$ is compact it suffices to show that if $\{f_k\}$ is norm bounded in $H_u^p (\mathbb{B}^p)$ then it admits a subsequence converging in $L^p (\mu)$. Lemma 2.1 yields a subsequence $\{f_{k_j}\}$ converging uniformly on compacta to $h \in H_u^p (\mathbb{B}^p)$. Then $\{f_{k_j} - h\}$ converges to 0 uniformly on compacta, by assumption this yields $\| f_{k_j} - h \|_{L^p (\mu)} \to 0$ and thus $\{f_{k_j}\} \to h$ in $L^p (\mu)$ as desired. \[ \square \]

Recall that a sequence $\{x_k\}$ in a normed space $X$ is called \textit{weakly convergent} if there is an $x \in X$ such that for every $\phi \in X^*$,

$$
\lim_{k \to \infty} \phi(x_k) = \phi(x).
$$

Now we will continue with a characterization of weakly convergent sequences in $H_u^p (\mathbb{B}^p)$ for $1 < p < \infty$ but before that we need the following lemma:

**Lemma 2.2.** Let $1 < p < \infty$. $H_u^p (\mathbb{B}^p)$ is reflexive and thus the unit ball of $H_u^p (\mathbb{B}^p)$ is weakly compact.

**Proof.** In the proof of Theorem 2.1 in ([9]), we have showed that $H_u^p (\mathbb{B}^p)$ is a closed subspace of the Lebesgue space $L^p (\partial \mathbb{B}^p)$ so $H_u^p (\mathbb{B}^p)$ is also reflexive. Hence the closed unit ball of $H_u^p (\mathbb{B}^p)$ is weakly compact. \[ \square \]

**Lemma 2.3.** Let $1 < p < \infty$. Then a sequence $\{f_k\} \subset H_u^p (\mathbb{B}^p)$ is norm bounded and converges uniformly on compacta to $h \in H_u^p (\mathbb{B}^p)$ if and only if it converges weakly to $h$. 

8 SİBEL ŞAHİN
Proof. Let \( \{f_k\} \) be a norm bounded sequence in \( H^p_u(\mathbb{B}^p) \) and converges uniformly to \( h \in H^p_u(\mathbb{B}^p) \) on compact subsets. We need to show that \( \Phi(f_k) \) converges to \( \Phi(h) \) for all \( \Phi \in (H^p_u(\mathbb{B}^p))^* \). Take an arbitrary subsequence \( \Phi(f_{k_j}) \) and by the previous lemma we know that the unit ball of \( H^p_u(\mathbb{B}^p) \) is weakly compact and by Eberlein-Šmuliand Theorem we can characterize this by sequential compactness (although the weak topology is not metrizable) so we have that there exists a subsequence \( f_{k_{j_l}} \) such that \( \Phi(f_{k_{j_l}}) \to \Phi(\gamma) \) for all \( \Phi \in (H^p_u(\mathbb{B}^p))^* \). Since this is true for all \( \Phi \in (H^p_u(\mathbb{B}^p))^* \), it is also true for point evaluations and \( f_{k_{j_l}}(x) = \gamma(x) = h(x) \). The last part is due to \( f_k \) converging to \( h \) uniformly on compacta and consequently it being convergent pointwise. Hence \( \Phi(f_{k_{j_l}}) \) converges \( \Phi(h) \) for all \( \Phi \in (H^p_u(\mathbb{B}^p))^* \). Therefore, every subsequence of \( \Phi(f_k) \) has a further subsequence converging to \( \Phi(h) \) hence \( \Phi(f_k) \to \Phi(h) \).

Conversely, assume that \( f_k \to 0 \) weakly in \( H^p_u(\mathbb{B}^p) \), in particular, is norm bounded in \( H^p_u(\mathbb{B}^p) \). Therefore by Lemma 2.1 (ii) to prove that \( f_k \to 0 \) uniformly on compacta it is sufficient to show that any converging (uniformly on compacta) subsequence must converge to 0. But if \( \{f_k\} \to h \in H^p_u(\mathbb{B}^p) \) uniformly on compacta, the previous argument shows that \( f_k \) converges weakly to \( h \), the uniqueness of weak limit then gives \( h = 0 \) and we are done. \( \square \)

Thus for \( 1 < p < \infty \), Proposition 2.1 is a particular case of the following well-known result (\( \textbf{I} \), Proposition 4.7)

**Theorem 2.3.** Let \( T : X \to Y \) be a linear operator between Banach spaces. Then,

(i) If \( T \) is compact then for any sequence \( \{x_k\} \subset X \) weakly converging to 0, the sequence \( \{Tx_k\} \) strongly converges to 0 in \( Y \).

(ii) Suppose that the unit ball of \( X \) is weakly compact, then if for any sequence \( \{x_k\} \subset X \) weakly converging to 0 the sequence \( \{Tx_k\} \) strongly converges to 0 in \( Y \), it follows that \( T \) is compact.

Now as an immediate consequence of this we have the following:

**Corollary 2.2.** Let \( 1 < p < \infty \). Then a linear operator \( T : H^p_u(\mathbb{B}^p) \to X \) taking values in a Banach space \( X \) is compact if and only if for any norm-bounded sequence \( \{f_k\} \subset H^p_u(\mathbb{B}^p) \) converging uniformly on compacta to 0, the sequence \( \{Tf_k\} \) converges to 0 in \( X \).

3. **Duality**

In [2] L. Aizenberg et al. considered Grothendieck-Köthe-da Silva duality for the classical Hardy spaces of a convex domain and in this section we will give analogous results for Poletsky-Stessin Hardy spaces following their general idea. Before proceeding with the duality arguments, we will first consider the dual complement \( \mathbb{B}^p \) of the complex ellipsoid \( \mathbb{B}^p \) and then we will prove the duality relation for the Poletsky-Stessin Hardy spaces of complex ellipsoids. Now let us first give some basic facts about the dual complements following [2]:

**Definition 5.** A domain \( \Omega \subset \mathbb{C}^n \) is called linearly convex if for every \( \xi \in \partial \Omega \) there exists a complex hyperplane

\[
\alpha = \{z \in \mathbb{C}^n : \quad \alpha_1 z_1 + \ldots + \alpha_n z_n + \beta = 0\}
\]

through \( \xi \) and does not intersect \( \Omega \).
Let $\Omega$ be a linearly convex domain. If $0 \in \Omega$, then its dual complement

$$\tilde{\Omega} = \{w \in \mathbb{C}^n : w_1z_1 + \ldots + w_nz_n \neq 1, \ z \in \Omega\}$$

is the set of hyperplanes that do not intersect the domain $\Omega$. Now let us continue
with the main result given in [2] considering the duality of the classical Hardy spaces
on linearly convex domains. The classical Hardy space on the dual complement
of the domain $\Omega$ is defined as follows:

**Definition 6.** Let $0 \in \Omega$ be a linearly convex domain with $C^2$ boundary. By
Hardy space for $q \geq 1$ on the dual complement $\tilde{\Omega}$ we mean the space of functions
$g$, holomorphic in the open domain $\text{int}(\tilde{\Omega})$ so that

$$\limsup_{\epsilon \to 0} \int_{\partial \tilde{\Omega}} |g(\xi - \epsilon \nu_\xi)|^q d\sigma(\xi) < \infty$$

where the vector $\nu_\xi$ is the exterior normal unit vector at $\xi \in \tilde{\Omega}$. Since
$\partial \tilde{\Omega} = \partial \text{int}(\tilde{\Omega})$,
this definition is meaningful and this space is denoted by $H^q(\tilde{\Omega})$.

The duality result for the classical Hardy spaces is the following, ([2], Theorem
3.1, pp:1354):

**Theorem 3.1.** Let $\Omega = \{z \in \mathbb{C}^n : g(z, \overline{z}) < 0\}$ where $g \in C^3(\overline{\Omega})$
is its defining function, be bounded, strictly convex domain. If $0 \in \Omega$, then

$$(H^p(\Omega))' = H^q(\tilde{\Omega}),$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$. Furthermore the isomorphism is realized:

$$F(f) = F_\phi(f) = \int_{\partial \Omega} \phi(w)f(z)\omega(z, w),$$

where $\phi \in H^q(\tilde{\Omega})$ and $f \in H^p(\Omega)$.

As it can be seen, this theorem is valid on a strictly convex domain, now combining
our work in [8] and [9] with the idea given in [2], we can extend this result
to the Poletsky-Stessin Hardy spaces of the complex ellipsoids and this results
is important in two different aspects because first Poletsky-Stessin Hardy classes
is much more general than the classical Hardy spaces and the second is that the complex ellipsoids are not strictly convex domains infact they are the model domains
for pseudoconvex domains of finite type which is more general than the strictly
convex domains. Now let us first give the setting for this generalization:

As it is pointed out in [2]. in general it is quite complicated to describe the dual
complement of a domain $\Omega$, however for the case of Reinhardt domains with center
at the origin there are precise results. If $\Omega$ is a Reinhardt domain centered
at the origin then $F(\Omega) \subset \mathbb{R}_+^n$, where $\mathbb{R}_+^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \ x_i \geq 0\}$
and $F(z_1, z_2, \ldots, z_n) = (|z_1|, |z_2|, \ldots, |z_n|)$. For any $B \subset \mathbb{R}_+^n$, its inverse image by $F^{-1}$ is defined to be the set $F^{-1}(B) = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : \ F(z_1, z_2, \ldots, z_n) \in B\}$. Then one can verify that the domain $\Omega \subset \mathbb{C}^n$ is Reinhardt if and only if $\Omega = F^{-1}(F(\Omega))$. Hence any Reinhardt domain $\Omega$ is determined completely by its absolute image $F(\Omega)$. Thus we have the following definition:
Definition 7. Let \( \Omega \subset \mathbb{C}^n \) be a Reinhardt domain centered at the origin \( 0 \in \mathbb{C}^n \). We say that the point \((y_1, ..., y_n) \in \tilde{F}(\Omega) \in \mathbb{R}^n_+ \) if and only if \( \sum_{i=1}^n x_i y_i < 1 \) for every \((x_1, ..., x_n) \in F(\Omega)\). Then the dual complement of \( \Omega \) is the set \( \tilde{\Omega} = F^{-1}(\tilde{F}(\Omega)) \).

From [2], we have the following characterization of the dual complement of a Reinhardt domain centered at the origin \( 0 \in \mathbb{C}^n \), ([2], pg:1342).

Lemma 3.1. For \( r > 0 \), \( p > 1 \) and \( k_i \in \mathbb{R}^n_+ \setminus \{0\} \) fixed numbers, let

\[
\Omega = \{ z \in \mathbb{C}^n | \sum_{i=1}^n k_i |z_i|^p < r^p \}
\]

be a Reinhardt domain centered at origin. Then for \( q = \frac{p}{p-1} \), the dual complement is

\[
\tilde{\Omega} = \{ \xi \in \mathbb{C}^n | \sum_{i=1}^n (k_i)^{\frac{1}{1-p}} |\xi_i|^q < \frac{1}{r^q} \}
\]

Now since the complex ellipsoid \( \mathbb{B}_p \) is a Reinhardt domain centered at the origin, the above lemma allows us to deduce the following:

Corollary 3.1. Let \( \mathbb{B}_p = \{ z \in \mathbb{C}^n, \sum_{i=1}^n |z_i|^{2p} - 1 < 0 \}, p \in \mathbb{Z}_+ \), be the complex ellipsoid. Then for \( q \in \mathbb{R}_+ \) such that \( q = \frac{p}{2p-1} \), the dual complement of \( \mathbb{B}_p \) is

\[
\tilde{\mathbb{B}}_p = \{ \xi \in \mathbb{C}^n, \sum_{i=1}^n |\xi_i|^{2q} - 1 \leq 0 \}
\]

For \( \mathbb{B}_p \) and dual complement \( \tilde{\mathbb{B}}_p \) choose the exhaustion functions \( u \) and \( \tilde{u} \) respectively as follows:

\[
u(z) = \ln(|z_1|^{2p} + |z_2|^{2p} + ... + |z_n|^{2p})
\]

\[
\tilde{u}(z) = \ln(|z_1|^{2q} + |z_2|^{2q} + ... + |z_n|^{2q})
\]

where \( p \) and \( q \) are given as in the previous corollary.

Now define the Poletsky-Stessin Hardy space on the dual complement of a linearly convex domain following the classical definition given in [2]:

Definition 8. Let \( 0 \in \Omega \) be a linearly convex domain with \( C^2 \) boundary and \( \tilde{u} \) be a continuous, negative, plurisubharmonic exhaustion function for \( \tilde{\Omega} \). For \( 1 < p < \infty \), Poletsky-Stessin Hardy space on the dual complement \( \tilde{\Omega} \) is the space of functions \( f \) holomorphic in the open domain \( \text{int}(\tilde{\Omega}) \) so that

\[
\lim_{r \to 0^+} \int_{S_{\tilde{u},r}} |f|^p d\mu_{\tilde{u},r} < \infty.
\]

We will continue with the following duality argument for the Poletsky-Stessin Hardy spaces of the complex ellipsoids:

Theorem 3.2. \((H^r_u(\mathbb{B}_p))' = (H^s_{\tilde{u}}(\mathbb{B}_p))\), \( r > 1, \frac{1}{r} + \frac{1}{s} = 1 \). Furthermore the following isomorphism is realized:

\[
F(f) = F_\phi(f) = \int_{\partial \mathbb{B}_p} \phi f d\mu_{\tilde{u}}
\]

where \( \phi \in H^s_{\tilde{u}}(\mathbb{B}_p) \) and \( f \in H^r_u(\mathbb{B}_p) \).
Proof. Consider the space $L^r_u(\partial B^p)$. Then the space $H^r_u(B^p)$ is a closed subspace of $L^r_u(\partial B^p)$ with respect to the $L^r_u$-norm. Thus for every element $F \in (H^r_u(B^p))^\prime$, there exists a function $g \in L^r_u(\partial B^p)$ such that

$$F(f) = \int_{\partial B^p} f(z)g(z)\,d\mu_u(z)$$

Now using Cauchy-Fantappie representation of $H^r_u(B^p)$ functions, we write (3) again

$$F(f) = \lim_{\hat{r} \to 1} \int_{\partial B^p} f(z)\left(\int_{\partial B^p} \frac{g(z)\,d\mu_u(z)}{(v(z,\xi))^n}\right)\,d\mu_u(z)$$

Taking the limit outside the integral and changing the order of integration leads to

$$F(f) = \lim_{\hat{r} \to 1} \int_{\partial B^p} f(z)\left(\int_{\partial B^p} \frac{g(z)\,d\mu_u(z)}{(v(z,\xi))^n}\right)\,d\mu_u(z)$$

the convexity of the ellipsoid implies that $(\hat{B}^p) = B^p$. Thus in the inner integral we make a change of variables

$$w : \xi \in B^p \to w(\xi) \in (\hat{B}^p)$$

$$(\xi_1, \xi_2, \ldots, \xi_n) \to (\xi_1^\hat{r}, \xi_2^\hat{r}, \ldots, \xi_n^\hat{r}) = w$$

and deduce that

$$F(f) = \lim_{\hat{r} \to 1} \int_{\partial B^p} f(z)\left(\int_{\partial B^p} \frac{G(w)}{(v(w,\xi))^n}\,d\mu_u(w)\right)\,d\mu_u(\xi)$$

Now by using the boundary value characterization of Poletsky-Stessin Hardy spaces of complex ellipsoids (8, 9) and the fact that Cauchy-Fantappie integral operator is bounded on $L^r_u$ to $H^r_u$ (8, Theorem 1), the inner integral is a function from $H^r_u(\hat{B}^p)$. Now as $\hat{r} \to 1$ we have

$$F(f) = F_\phi(f) = \int_{\partial B^p} \phi f\,d\mu_u$$

where $\phi \in H^r_u(\hat{B}^p)$. Thus $(H^r_u(B^p))^\prime = (H^r_u(\hat{B}^p))^\prime$. \qed

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