ON COMPLETELY NON-BAIRE UNION IN CATEGORY BASES

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Abstract. In this paper, we intend to show that under not too restrictive conditions, results much stronger than the one obtained earlier by Hejduk could be established in category bases.

1. INTRODUCTION

Hejduk [2] proved the following theorem

Theorem 1.1. Let \((X, \mathcal{S})\) be any arbitrary category base and \(\mathcal{M}(\mathcal{S})\) be the \(\sigma\)-ideal of all meager sets in the base \((X, \mathcal{S})\), satisfying the following conditions:

1. For an arbitrary cardinal \(\alpha < \text{car}(X)\), the family \(\mathcal{M}(\mathcal{S})\) is \(\alpha\)-additive, i.e., this family is closed under the union of arbitrary \(\alpha\)-sequence of sets.
2. There exists a base \(P\) of \(\mathcal{M}(\mathcal{S})\) of cardinality not greater than that of \(X\).

Thus if \(X \notin \mathcal{M}(\mathcal{S})\), then for an arbitrary family \(\{X_i\}_{i \in T}\) of meager sets, being a partition of \(X\), there exists a set \(T' \subseteq T\) such that \(\bigcup_{i \in T'} X_i\) is not a Baire set.

For our purpose, we need the following Definitions and Theorems. For references, the reader may see [3].

Definition 1.2. A category base is a pair \((X, \mathcal{C})\) where \(X\) is a non-empty set and \(\mathcal{C}\) is a family of subsets of \(X\), called regions satisfying the following set of axioms:

1. Every point of \(X\) belongs to some region; i.e., \(X = \bigcup \mathcal{C}\).
2. Let \(A\) be a region and \(\mathcal{D}\) be a non-empty family of disjoint regions having cardinality less than the cardinality of \(\mathcal{C}\). 
   i) If \(A \cap (\bigcup \mathcal{D})\) contains a region, then there is a region \(D \in \mathcal{D}\) such that \(A \cap D\) contains a region.

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ii) If $A \cap (\bigcup D)$ contains no region, then there is a region $B \subseteq A$ that is disjoint from every region in $D$.

**Definition 1.3.** In a category base $(X, \mathcal{C})$, a set is called singular if every region contains a subregion which is disjoint from the set. Any set which can be expressed as countable union of singular sets is called meager. We denote by $\mathcal{M}$ the class of all meager sets in $(X, \mathcal{C})$ which is obviously a $\sigma$-ideal. A set which is not meager is called abundant. A category base $(X, \mathcal{C})$ is called Baire base if every member of $\mathcal{C}$ is an abundant set.

**Definition 1.4.** In a category base $(X, \mathcal{C})$, a set $S$ is called Baire if in every region, there is a subregion in which either $S$ or its complement $X - S$ is meager. We denote by $\mathcal{B}$ the class of all Baire sets in $(X, \mathcal{C})$ which is obviously a $\sigma$-algebra.

**Theorem 1.5.** The intersection of two regions either contains a region or is a singular set.

**Proposition 1.6.** If $(X, \mathcal{C})$ is a category base, $\mathcal{N}$ is a subfamily of $\mathcal{C}$ with the property that each region in $\mathcal{C}$ contains a region in $\mathcal{N}$ and $Y = \bigcup \mathcal{N}$, then $(Y, \mathcal{N})$ is also a category base and the $\mathcal{N}$-singular sets coincide with the $\mathcal{C}$-singular subsets of $Y$. In addition, if $U$ is a subset of $Y$ and $Y - U$ is $\mathcal{N}$-singular, then $X - U$ is $\mathcal{C}$-singular.

**Proposition 1.7.** If $(Y, \mathcal{N})$ is a category base, then there exists a disjoint family $\mathcal{M}$ of $\mathcal{N}$ such that $Y - \bigcup \mathcal{M}$ is a singular set. Moreover, $\mathcal{M}$ can be so selected that for every region $N$, there exists a region $M \in \mathcal{M}$ such that $N \cap M$ contains a region.

**Theorem 1.8.** (The Fundamental Theorem) Every abundant set in a category base $(X, \mathcal{C})$ is abundant everywhere in some region. This means that for any abundant set $A$, there exists a region $C$ in every subregion $D$ of which $A$ is abundant.

To the above Definitions and Theorems, we further add that in any category base $(X, \mathcal{C})$,

**Definition 1.9.** A subfamily $\mathcal{C}'$ of $\mathcal{C}$ is called a $\pi$-base [2] if every region $D \in \mathcal{C}$ contains a region $E \in \mathcal{C}'$.

**Definition 1.10.** A set $A$ is called completely non-Baire in a region $D$ if for every $B \in \mathcal{B}$ such that $B \cap D$ is abundant, both $A \cap B$ and $(D - A) \cap B$ are abundant. A set which is completely non-Baire in every region is called completely non-Baire.

The above definition is analogous to the notion of a ‘completely I-nonmeasurable set’ given in [4].
2. RESULTS

To start with, we make the following assumptions:

(i) The cardinality $(X)$ of $X$ is regular.

(ii) Our category base $(X, C)$ is a Baire base and possesses a $\pi$-base $C'$ with $(C')$ not exceeding $(X)$.

(iii) The system $(X, B, M)$ is $(X)$-additive which means that $\bigcup E \in M$ whenever $E \subseteq M$ and $(E) < (X)$ [1].

(iv) $(X) = \text{Min} \{\chi : C$ is $\chi$-saturated$\}$ where by the phrase “$C$ is $\chi$-saturated” we mean that if $E \subseteq C$ such that $(E) = \chi$, there are two distinct members $E, F \in E$ such that $E \cap F \neq \phi$.

We also use the combinatorial theorem [2] stated below

**Theorem 2.1.** If $X$ is an infinite set and $\Phi_1$ is a family of subsets of $X$ such that

(i) $(\Phi_1) \leq (X)$.
(ii) $(Z) = (X)$ for all $Z \in \Phi_1$.

then there exists a family $\Phi_2$ of subsets of $X$ such that

(a) $(\Phi_2) > (X)$.
(b) $Z_1 \neq Z_2$ implies that $(Z_1 \cap Z_2) < (X)$ for all $Z_1, Z_2 \in \Phi_2$.
(c) $(Z \cap Y) = (X)$ for all $Y \in \Phi_1, Z \in \Phi_2$.

Let $\mathcal{K}$ denote the family of all sets whose complements are members of $C'$. Let $\bigcap_{\sigma < (X)} \mathcal{K}$ : all sets representable as intersection of subfamilies of $\mathcal{K}$ whose cardinality is less than $(X)$.

$\bigcup_{\sigma < (X)} \bigcap_{\mathcal{K}}$ : all sets representable as countable union of sets from $\bigcap_{\mathcal{K}}$.

**Theorem 2.2.** Any singular set is contained in a singular set which belongs to the family $\bigcap_{\mathcal{K}}$. Any set in $M$ is a subset of some set in $\mathcal{M} \cap \bigcup_{\sigma < (X)} \bigcap_{\mathcal{K}}$.

**Proof.** Let $A$ be any singular set in $(X, C)$. Since $C'$ is a $\pi$-base, for every $C \in C$, there is a set $D \in C'$ such that $D$ is disjoint from $A$. This constitutes a subfamily $\mathcal{N}$ of $C$ satisfying the above property such that every member of $C$ contains a member of $\mathcal{N}$. Let $Y = \cup \mathcal{N}$. Then $(Y, \mathcal{N})$ is a category base and by Proposition 1.7, a subfamily $\mathcal{M}$ of
\( N \) can be so selected that for every \( N \in N \), there exists \( M \in M \) such that \( N \cap M \) contains a region. This makes \( Y - (\cup M) \) \( N \)-singular and consequently by Proposition 1.6, \( X - (\cup M) \) is \( C \)-singular or singular. Moreover, \( A \subseteq X - (\cup M) \).

Now by our assumption (iv), \( X - (\cup M) \) belongs to the family \( \bigcap_{\sigma < \#(X)} K \). Hence \( A \) is contained in a singular set which belongs to the family \( \bigcap_{\sigma < \#(X)} K \). Therefore, by definition, any member of \( M \) is a subset of some set in \( M \cap (\bigcup_{\sigma < \#(X)} K) \). □

Since \( \#(X) \) is a regular cardinal (by assumption (i)), \( \#(M \cap (\bigcup_{\sigma < \#(X)} K)) \) does not exceed \( \#(X) \).

**Theorem 2.3.** Every abundant Baire set, i.e., every set in the family \( B - M \) contains a set of the form \( F - G \) where \( F \in C' \) and \( G \in M \cap (\bigcup_{\sigma < \#(X)} K) \).

**Proof.** Let \( S \in B - M \). Then there exists \( C \in C \) such that \( C - S \subseteq M \). By theorem 2.1, \( C - S \subseteq G \in M \cap (\bigcup_{\sigma < \#(X)} K) \). Let \( F \in C' \) such that \( F \subseteq C \). Choose the set \( F - G \) which proves the theorem. □

Let \( \mathcal{H} = \{ F - G : F \in C', G \in M \cap (\bigcup_{\sigma < \#(X)} K) \} \). From the definition of the class, it follows that \( \#(\mathcal{H}) \leq \#(X) \) and for any \( H \in \mathcal{H} \), let \( Y(H) = \{ t \in T : X_t \cap H \neq \phi \} \). Since the system \( (X, B, M) \) is \( \#(X) \)-additive (by assumption (iii)) and the family \( \{ X_t \}_{t \in T} \) gives rise to a partition of \( X \), so \( \#(Y(H)) = \#(X) \).

Let \( \Phi_1 = \{ Y(H) : H \in \mathcal{H} \} \). Then \( \#(\Phi_1) \leq \#(X) \). We now use the combinatorial theorem (Theorem 2.1) stated above by virtue of which there exists a family \( \Phi_2 \) of subsets of \( T \) satisfying the properties stated in Theorem 2.1.

For any \( Z \in \Phi_2 \), let \( X(Z) = \bigcup \{ X_t : t \in Z \} \). We claim that \( X(Z) \) is completely non-Baire. Since every set from the family \( B - M \) contains a set which belongs to the family \( \mathcal{H} \), the claim will be settled if we can show that for any \( Z \in \Phi_2 \) and any \( H \in \mathcal{H} \), both \( H \cap X(Z) \) and \( H - X(Z) \) are abundant. The first one is obvious since \( \#(Z \cap Y) = \#(X) \) for all \( Y \in \Phi_1 \) and \( Z \in \Phi_2 \) (c of Theorem 2.1) and the second one follows from (b) of Theorem 2.1 and the fact that \( (X, B, M) \) is \( \#(X) \)-additive (assumption (iii)). Thus

**Theorem 2.4.** If conditions (i), (ii), (iii) and (iv) are satisfied for any category base \( (X, C) \) and \( X = \bigcup_{t \in T} X_t \) is a partition of \( X \) into meager
sets, then there exists a family $\Phi_2$ of subsets of $T$ such that $\#(\Phi_2) > \#(X)$ and for every $Z \in \Phi_2$, $X(Z)$ is completely non-Baire. Moreover, for $Z_1, Z_2 \in \Phi_2(Z_1 \neq Z_2), X(Z_1) \cap X(Z_2) \in \mathcal{M}$.

The proof of the last part of the above theorem follows from (b) of Theorem 2.1 and assumption (iii).

If we prefer to call two sets $A$ and $B$ as almost disjoint in a category base if $A \cap B$ is meager, then what the above theorem indicates is this that under certain circumstances, it is possible to obtain from a non-Baire unions which are mutually almost disjoint.

In the next theorem, we show that even if \{X_t\}_{t \in T} is a family of disjoint sets but not a partition of $X$, still it is possible to obtain at least $\#(X)$ number of mutually disjoint completely non-Baire unions by slightly changing the hypothesis. Here we have adapted the technique used to prove Theorem 2.1 [4].

**Theorem 2.5.** If conditions (i), (ii), (iii) and (iv) are satisfied for any category base $(X, \mathcal{C})$ and \{X_t\}_{t \in T} is a family of disjoint meager sets such that for every $C \in \mathcal{C}$, $(\bigcup_{t \in T} X_t) \cap C$ is abundant, then there are at least $\#(X)$ number of completely non-Baire unions which are mutually disjoint.

**Proof.** Without loss of generality, we may assume that $\#(\mathcal{H}) = \#(X)$. Also as $(\bigcup_{t \in T} X_t) \cap C$ is abundant for every $C \in \mathcal{C}$, so for any $H \in \mathcal{H}$, $\#(\{t \in T : X_t \cap H \neq \emptyset\}) = \#(X)$ by condition (iii). Let $\alpha$ denote the smallest ordinal representing $\#(X)$ and we enumerate all the set in $\mathcal{H} = \{H_\beta : \beta < \alpha\}$. By transfinite induction we will construct a sequence \{(X_\xi, \eta, d_\xi) \in \{X_t\}_{t \in T} \times H_\xi : \xi, \eta < \alpha\}$ such that

1. $X_\xi, \eta \cap H_\xi \neq \emptyset$ for all $\xi, \eta < \alpha$.
2. $\bigcup_{\xi, \eta < \alpha} X_\xi, \eta \cap \{d_\xi : \xi < \alpha\} = \phi$.
3. $X_\xi, \eta \neq X_{\xi', \eta'}$ for all $\xi, \xi' < \alpha$ and for all $\eta, \eta' < \alpha$ ($\eta \neq \eta'$).

We fix $\beta < \alpha$ and suppose that we have already defined the sequence \{(X_\xi, \eta, d_\xi) \in \{X_t\}_{t \in T} \times H_\xi : \xi, \eta < \beta\}$ satisfying the conditions:

1. $X_\xi, \eta \cap H_\xi \neq \emptyset$ for all $\xi, \eta < \beta$.
2. $\bigcup_{\xi, \eta < \beta} X_\xi, \eta \cap \{d_\xi : \xi < \beta\} = \phi$.
3. $X_\xi, \eta \neq X_{\xi', \eta'}$ for all $\xi, \xi' < \beta$ and for all $\eta, \eta' < \beta$ ($\eta \neq \eta'$).

Then let $X(d_\xi)$ denote that unique $X_t$ such that $d_\xi \in X_t$ and choose $X_{\beta, \eta} \in \{X_t\}_{t \in T}$ such that

1. $X_{\beta, \eta} \neq X_{\beta, \eta'}$ for all $\eta, \eta' < \beta (\eta \neq \eta')$.
2. $X_{\beta, \eta} \cap H_\beta \neq \emptyset$ for all $\eta < \beta$. 
\[ (3) \bigcup_{\eta<\beta} X_{\beta,\eta} \cap (\bigcup \{X(d_\xi) : \xi < \beta\}) = \phi. \]

The choice is justified by virtue of the fact stated at the beginning of the proof. Also, owing to the same reasoning, we can further choose \(X_{\xi,\beta} \in \{X_t\}(\xi \leq \beta)\) and \(d_\beta \in H_\beta\) such that
\begin{enumerate}
  \item \(X_{\xi,\beta} \neq X_{\xi',\beta}\) for all \(\xi, \xi' \leq \beta\).
  \item \(X_{\xi,\beta} \cap H_\xi \neq \phi\) for all \(\xi \leq \beta\).
  \item \(X_{\xi,\beta} \cap \left(\bigcup \{X(d_\xi') : \xi' < \beta\}\right) = \phi\) for all \(\xi \leq \beta\).
  \item \(\bigcup_{\xi,\eta \leq \beta} X_{\xi,\eta} \cap \{d_\beta\} = \phi\).
\end{enumerate}

It is not hard to check that each member in the family \(\bigcup_{\xi<\alpha} X_{\xi,\eta} : \eta < \alpha\) of mutually disjoint sets is a completely non-Baire union. \(\square\)

In any Baire base \((X, \mathcal{C})\), \((\bigcup X_t) \cap C\) is abundant for every \(C \in \mathcal{C}\) provided \(X = \bigcup_{t \in T} X_t\) is a partition of \(X\). So by the above theorem in any category base \((X, \mathcal{C})\) if conditions (i)–(iv) are satisfied, then given a partition \(X = \bigcup_{t \in T} X_t\) of \(X\) into a family of meager sets, it is possible to construct at least \(\#(X)\) number of completely non-Baire unions which are mutually disjoint. In Theorem 2.4, we have theoretically established the existence of more than \(\#(X)\) number of completely non-Baire unions which are mutually almost disjoint. But we are not certain whether they are mutually disjoint. However, we can construct at least \(\#(X)\) number of completely non-Baire unions which are mutually disjoint.

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